The analysis of the laminar-turbulent transition onset for the Poiseuille flow in the flat channel with the velocity profile periodic background perturbation

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Abstract. The paper adopts an approach to predicting the laminar-turbulent transition onset, which is based on the hypothesis of a threshold “connection” of the transverse viscosity factor and the subsequent transverse velocity components “generation”. In the framework of this approach, certain empirical local transition conditions are proposed and imposed on the dimensionless invariant complexes that characterize the laminar flow under study. Based on these conditions, an analysis of the Poiseuille flow transition onset in the flat channel in the presence of the stationary velocity field periodic background perturbation is carried out. A comparison with the known experimental data is presented.

1. Introduction
At present, the main approaches to predicting the laminar-turbulent transition onset are based on the hydrodynamic stability theory. Such approaches are about the imposition of the velocity and pressure periodic background perturbations on the initially laminar flow satisfying the Navier-Stokes equations. The final forecast on the transition onset reduces to an analysis of the conditions under which such perturbations amplitudes or the perturbations' kinetic energy will subsequently increase without limit [1-5]. Nevertheless, the matter of the pulsating background origin, as a rule, does not become the subject of the discussion.

However, as noted in [6]: “One of the problems of the turbulence is the lack of understanding of the turbulent pulsations occurrence.” In other words, the question of the turbulence initiation and the turbulent pulsations origin remains open. This is one of the reasons why the turbulent transition issue is a long way from the final solution. Moreover, a large number of various factors contribute to the turbulence initiation.

Such factors include the existence of the initial pulsations or the flow perturbations at the entrance to the considered flow region [5, 7, 8], the presence of the roughness at the flow region's rigid boundaries [9], and many other circumstances [5, 10]. Each of these factors, to one degree or another, contributes to the transition formation process.

The paper [11] proposed that the turbulence occurrence mechanism was associated with the existence of the fluid or gas coherent density perturbations, when the molecules collective vibrations correspond to the elementary perturbations.

Another important factor is acoustic perturbations. Such approach was considered in [12, 13], where on the basis of this circumstance a rationale for the pulsation field formation mechanism was proposed.
In [14], the rheological factor was considered as a possible laminar – turbulent transition initiator. The proposed approach was based on the transverse viscosity threshold “addition” hypothesis that does not manifest itself in the laminar flows. As a result, an interpretation of the laminar-turbulent transition origination onset (the transverse velocity components “generation” with respect to the laminar flow streamlines) was given accounted for the rheological factor in a small neighborhood of some spatial point \( \vec{X} \).

In addition, such empirical conditions like transverse velocity components “generation”(with respect to the initial laminar streamlines)

\[
K_2(\vec{X}) > K_{2G}(\vec{X}) = k_0 + \frac{k_1}{K_3(\vec{X}) - k_2},
\]

and the unlimited increase in the “generated” transverse velocity components were proposed

\[
\max_{X \in G} \{ K_1(\vec{X}) \} > q_0 \cdot \left[ K_3(\vec{X}) \right]^{q_1}.
\]

In the last condition (2), \( G \) is a spatial zone inside the fluid flow region, for the points \( \vec{X} \) of which the condition (1) for the transverse velocity components “generation” is satisfied.

In (1), (2), the coefficients \( k_0, k_1, k_2, q_0, q_1 \) are the empirical constants whose numerical values are given in [15], and \( K_1, K_2, K_3 \) are the dimensionless complexes that represent the functions of the considered spatial point \( \vec{X} \) positions and that are determined exclusively through the invariant quantities via the following relations

\[
K_j = \frac{E_j}{D_j}; \quad K_2 = \frac{\rho^2 \cdot U_{2s}^3 \cdot I_{2s}}{E_{s}^2}; \quad K_3 = \frac{\rho^2 \cdot U_{s}^3}{\mu \cdot E_{s}};
\]

\[
\tilde{E} = \text{grad} \left\{ P + \frac{\rho \cdot U^2}{2} \right\}; \quad \tilde{D} = \text{grad} \left\{ 2 \cdot \mu \cdot \sqrt{I_2} \right\},
\]

where \( P \) is the pressure; \( U \) is the fluid velocity vector module; \( \rho, \mu \) are the density and the dynamic fluid viscosity, respectively; \( I_2 \) is the second invariant of the strain rate tensor; \( E, D \) are the vectors’ \( \tilde{E}, \tilde{D} \) modules, respectively.

In (3), the subscript \( s \) indicates that the corresponding functions are calculated at the considered point \( \vec{X} \).

In the case of simultaneous fulfillment of conditions (1), (2) at the corresponding point \( \vec{X} \) in the flow region, it is proposed to consider such point and its small neighborhood as the laminar-turbulent transition initiator.

Based on this approach, in [16], the “generation” of the transverse velocity components was analyzed for the cases of the plane one-dimensional flow and the further formation of the laminar-turbulent transition. In the same paper, the role of the inflection point on the viscous fluid velocity profile which is well known from the Rayleigh theorem for the ideal fluid flow case, was considered.
In this article, based on the approach described in [14], on a test example of the Poiseuille flow in the flat channel with the velocity field initial periodic background perturbation, we analyze it to predict the laminar-turbulent transition onset.

2. The initial state of the channel velocity field
Let us consider the Poiseuille flow in the flat channel in the presence of the velocity field stationary periodic background perturbation. In the traditional coordinate system $O'xy$, presented in Figure 1, the distribution of the “main”, longitudinal (excluding background perturbations) velocity component is described by the well-known parabolic relation

\[ V(y) = \frac{3}{2} V_{\text{aver}} \left( 1 - (y')^2 \right); \quad y' = \frac{y}{h}, \]  

where $y$ is the transverse coordinate off the channel symmetry axis; $h$ is half the channel width; $V_{\text{aver}}$ is the channel cross section average value for this fluid velocity component.

Let us choose a spatial point $O$ in the channel and associate with it, as with a new origin, another local coordinate system $Ox_1x_2$, as shown in Figure 1.

Suppose that at some point in time, which is conditionally accepted as the start time, a plane flow has formed in the channel, the velocity field of which in the local Cartesian coordinate system around the particular point in the dimensional record form is described by the following

\[ U_1 = V + u_I = V + u; \quad U_2 = u_2 = -\delta \cdot u; \]  

\[ V = V(x_2) = \frac{3}{2} V_{\text{aver}} \left( 1 - \left( \frac{y_0 + x_{2M}}{h} \right)^2 \right); \]

\[ u_I = u = u(x_{1M}, x_{2M}) = U_A \cdot \sin \left( \alpha_1 \cdot x_{1M} + \alpha_2 \cdot x_{2M} \right); \]

\[ \alpha_1 = \frac{2 \cdot \pi}{\lambda_1} ; \quad \alpha_2 = \frac{2 \cdot \pi}{\lambda_2} ; \quad \delta = \frac{\alpha_1}{\alpha_2} = \frac{\lambda_2}{\lambda_1}, \]

where $V(x_{2M})$ is the fluid flow rate longitudinal component presented taking into account (4) as a
function of the transverse coordinate \( x_{2M} \) in the local reference system; \( u_j \), are the velocity field background perturbation components; \( u(x_{1M}, x_{2M}) \) is the periodic function that determines velocity field background perturbation components in the local coordinate system; \( U_A \) is the periodic background perturbation amplitude; \( \lambda_1, \lambda_2 \) are the periodic background perturbation components wavelengths in the longitudinal and transverse directions, respectively.

The relations (5) identically satisfy the continuity condition. At the same time, note that they determine the velocity components at some spatial point \( M(x_{1M}, x_{2M}) \), located around the particular point \( O \), merely at the start time.

3. The main dimensionless complexes for characterizing the laminar-turbulent transition onset
   
   In the case under consideration, the initial stage of the flow evolution from the declared starting state in the point's \( O \) small neighborhood will be determined by the dimensionless complexes (3), which must be calculated at the point \( O \) (the local coordinate origin) for \( x_{1M} = 0 \) and \( x_{2M} = 0 \). After appropriate transformations, these dimensionless complexes, taking into account (5), can be written as

\[
K_1 = \left[ f(y_0') \cdot \frac{2 \cdot \delta \cdot K_{\text{int}}}{(1 + \delta^2) \cdot y_0' - (1 - \delta^2) \cdot K_{\text{int}}} \right]^2;
\]

\[
K_2 = \frac{9 \cdot Re^2 \cdot (1 - y_0'^2)^2}{64 \cdot f(y_0)} \left[ \frac{2 \cdot \delta \cdot K_{\text{int}}}{(1 + \delta^2)} \right]^2 + \left[ y_0' - \frac{1 - \delta^2}{1 + \delta^2} \cdot K_{\text{int}} \right]^2;
\]

\[
K_3 = \frac{9 \cdot Re^2 \cdot (1 - y_0'^2)^3}{32 \cdot \sqrt{f(y_0)}}.
\]

In the latter relations, for brevity, the following notation was applied

\[
f(y_0') = 1 + \left( \frac{3}{4} \cdot Re \cdot (1 - y_0'^2) \cdot (y_0' - K_{\text{int}}) \right)^2;
\]

\[
Re = \frac{2 \cdot \rho \cdot V_{\text{aver}} \cdot h}{\mu}; \quad K_{\text{int}} = \frac{\alpha_2 \cdot h \cdot (1 + \delta^2) \cdot U_A}{3 \cdot V_{\text{aver}}}; \quad y_0 = \frac{y_0'}{h}.
\]

where \( Re \) is the Reynolds number; \( K_{\text{int}} \) is a dimensionless complex characterizing the intensity of the velocity field initial background perturbations.

If we introduce the intensity degree of the velocity field initial periodic background perturbation using the relation

\[
\varepsilon = \frac{1}{2} \frac{\left( \langle u_1^2 \rangle + \langle u_2^2 \rangle \right)}{V_{\text{aver}}} = \frac{U_A}{V_{\text{aver}}} \sqrt{\frac{1}{4} (1 + \delta^2)}
\]

\[
\overline{(u_j^2)} = \frac{1}{\lambda_1 \cdot \lambda_2} \int_0^{\lambda_1} \int_0^{\lambda_2} u_j^2(x_1, x_2) \cdot dx_1 \cdot dx_2; \quad j = 1, 2,
\]
then the dimensionless complex $K_{int}$ can also be represented in the following form

$$K_{int} = \frac{2 \cdot \alpha_2 \cdot h \cdot e \cdot \sqrt{1 + \delta^2}}{3}.$$  

First of all, note that the disturbance background intensity degree is introduced in this case only for the initial velocity field. In this regard, the averaging is carried out strictly over the spatial coordinates. Nevertheless, it is proposed to associate it with the traditional intensity of the turbulent flows perturbations.

Now, taking into account (6), let us analyze the initial stage of the hydrodynamic process possible development around the particular point of the flow region from the starting state (5) from the standpoint of the conditions (1), (2).

4. The analysis of the laminar-turbulent transition conditions onset

The relations (6) allow, taking into account (1), (2), to establish the relationship between the parameter $K_{int}$ characterizing the velocity initial background perturbation and the Reynolds number from the point of view of initiating the laminar-turbulent transition onset at each specific point in the flow region.

As an example, let us consider the following, rather simple particular case, when $\delta = 1$ and the selected point $O$ in the flow region is located on the channel symmetry axis ($y_0 = 0$). We specially note that this selected point $O$ is also an inflection point, in particular along the transverse coordinate for the longitudinal component of the velocity background perturbation.

Then from (6) for the first dimensionless complex we immediately obtain

$$K_1 \to \infty.$$  

The last result means that condition (2) is certainly satisfied. Therefore, to start the transition, it is sufficient to ensure at this point only the fulfillment of condition (1) of the transverse velocity component “generation”.

The second and third dimensionless complexes from (6) for the particular case take the form

$$K_2 = K_2(Re, K_{int}) = \frac{9 \cdot Re^2 \cdot K_{int}^2}{4 \cdot (16 + 9 \cdot Re^2 \cdot K_{int}^2)}; \quad K_3 = K_3(Re, K_{int}) = \frac{9 \cdot Re^2}{8 \cdot \sqrt{16 + 9 \cdot Re^2 \cdot K_{int}^2}}.$$  

We emphasize that in the framework of the particular case in (8), it is assumed that $K_{int} \neq 0$.

Figure 2 presents, taking into account (8), a graphical interpretation of the fulfillment of condition (1) for the particular example of the Reynolds number fixed value. Analyzing the dependences presented in this figure, we can see that for the considered fixed value of Reynolds number there is a range of parameter values $K_{int,min} < K_{int} < K_{int,max}$ characterizing the velocity field periodic perturbations initial intensity for which condition (1) is satisfied. Taking into account the result of (7), we conclude that the laminar-turbulent transition initiation corresponds to such range of parameter values $K_{int}$.

Naturally, the boundary values $K_{int,min}$ and $K_{int,max}$ for the parameter $K_{int}$ in the framework of the particular case ($y_0^0 = 0; \delta = 1$) are represented by the Reynolds number functions

$$K_{int,min} = K_{int,min}(Re); \quad K_{int,max} = K_{int,max}(Re).$$  

5
and are evaluated from the solution to the equation
\[ K_2(Re, K_{int}) = K_{2G}(Re, K_{int}). \quad (10) \]

![Graph showing the relationship between \( K_{int} \) and \( K_2, K_{2G} \).](image)

**Figure 2.** The graphic interpretation of the fulfillment of the condition for the transverse velocity components “generation” and, accordingly, the initiation of the laminar-turbulent transition onset on the symmetry channel axis (\( y_0' = 0 \)) if \( Re = 180 \) и \( \delta = 1 \). 1 - \( K_2( K_{int}) \); 2 - \( K_{2G}( K_{int}) \); 3 - \( K_{int,min} = 0.0217 \); 4 - \( K_{int,max} = 0.1755 \).

Let us note that the numerical solution of equation (10) with respect to the parameter \( K_{int} \) at the fixed value \( Re \) shows that it has no real roots for the values \( Re < 97.4 \). In this case, the required parameters reach the limited value \( K_{int,max} = K_{int,min} \approx 0.0643 \) at this boundary (at the minimum value \( Re \approx 97.4 \)).

Such Reynolds number threshold value, below which condition (1) is not fulfilled and it is impossible to “generate” the transverse velocity component, is in quite satisfactory agreement with the test value \( Re_{min}^{(exp)} = 140 \) obtained in [7]. Note that a discussion of the experimental data from [7] with a view to comparing them with the results following from the energy theory was carried out in [3].

The roots dependences (9) of the equation (10) on the Reynolds number are presented in Figure 3.

![Graph showing the relationship between \( K_{int} \) and \( Re \).](image)

**Figure 3.** The possible “generation” region (shaded) of the transverse velocity components on the channel symmetry axis (\( y_0' = 0 \)) for the plane Poiseuille flow with the velocity initial background perturbation at \( \delta = 1 \). 1 - \( K_{int,min}(Re) \); 2 - \( K_{int,max}(Re) \).
An analysis of the lower graph, which limits the turbulence initiation region, allows us to draw the quite expected conclusion. As the Reynolds number increases, a monotonous decrease in the parameter characterizing the intensity degree of the velocity field periodic initial background perturbation, at which the laminar-turbulent transition initiation begins, is observed. This result is in qualitative agreement with many experimental results, for example [17], which indicate rather large critical Reynolds numbers (up to the values of 20,000 and higher) with special measures to reduce the initial pulsations at the channel inlet.

The boundary shape presented in Figure 3 for the turbulence initiation region in the plane of the parameters $Re$ and $K_{int}$ is, in a sense, qualitatively similar (although there are significant differences) to the neutral curve limiting the instability region in the hydrodynamic stability theory [1, 2, 5].

The considered particular case refers only to the spatial point located on the channel symmetry axis, taking into account the additional condition $\delta = 1$. Given that, the velocity initial background perturbation has an inflection profile at this point. Naturally, this is a rather narrow particular case. At the same time, we can trace from this example, within the framework of the proposed approach, some trends of influence on the laminar-turbulent transition on initial background set, such as the Reynolds number and the characteristics of the velocity initial background perturbation.

5. Conclusion

The analysis of the conditions of the laminar-turbulent transition onset based on the approach proposed in [14] has been carried out. As a test example, the Poiseuille flow in the flat channel with the velocity field initial periodic background perturbation has been considered. The obtained results, at least on a qualitative level, are consistent with the generally accepted ideas about the perturbation background influence on the critical Reynolds number value. The minimum (critical) value of the Reynolds number has been determined, which satisfactorily corresponds to the known experimental data [7].

The obtained results have showed that the proposed empirical conditions (1), (2), taking into account (3), can be used to predict the laminar-turbulent transition onset according to the velocity and pressure fields analysis results.

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