Disproving the Neighbourhood Conjecture

Heidi Gebauer *

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Abstract

We study the following Maker/Breaker game. Maker and Breaker take turns in choosing vertices from a given \(n\)-uniform hypergraph \(F\), with Maker going first. Maker’s goal is to completely occupy a hyperedge and Breaker tries to avoid this. Beck conjectures that if the maximum neighborhood size of \(F\) is at most \(2n-1\) then Breaker has a winning strategy. We disprove this conjecture by establishing an \(n\)-uniform hypergraph with maximum neighborhood size \(3 \cdot 2^{n-3}\) where Maker has a winning strategy. Moreover, we show how to construct an \(n\)-uniform hypergraph with maximum degree \(\frac{2^{n-1}}{n}\) where Maker has a winning strategy.

Finally we show that each \(n\)-uniform hypergraph with maximum degree at most \(\frac{2^{n-2}}{cn}\) has a proper halving 2-coloring, which solves another open problem posed by Beck related to the Neighbourhood Conjecture.

1 Introduction

A hypergraph is a pair \((V, E)\), where \(V\) is a finite set whose elements are called vertices and \(E\) is a family of subsets of \(V\), called hyperedges. We study the following Maker/Breaker game. Maker and Breaker take turns in claiming one previously unclaimed vertex of a given \(n\)-uniform hypergraph, with Maker going first. Maker wins if he claims all vertices of some hyperedge of \(F\), otherwise Breaker wins.

Let \(F\) be a \(n\)-uniform hypergraph. The degree \(d(v)\) of a vertex \(v\) is the number of hyperedges containing \(v\) and the maximum degree of \(F\) is the maximum degree of its vertices. The neighborhood \(N(e)\) of a hyperedge \(e\) is the set of hyperedges of \(F\) which intersect \(e\) and the maximum neighborhood size of \(F\) is the maximum of \(|N(e)|\) where \(e\) runs over all hyperedges of \(F\).

The famous Erdős-Selfridge Theorem [3] states that for each \(n\)-uniform hypergraph \(F\) with less than \(2^{n-1}\) hyperedges Breaker has a winning strategy. This upper bound on the number of hyperedges is best possible as the following example shows. Let \(T\) be a rooted binary tree with \(n\) levels and let \(G\) be the hypergraph whose hyperedges are exactly the sets \(\{v_0, \ldots, v_{n-1}\}\) such that \(v_0, v_1, \ldots, v_{n-1}\)

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*Institute of Theoretical Computer Science, ETH Zurich, CH-8092 Switzerland. Email: gebauerh@inf.ethz.ch.
is a path from the root to a leaf. Note that the number of hyperedges of $G$ is $2^{n-1}$. To win the game on $G$ Maker can use the following strategy. In his first move he claims the root $m_1$ of $T$. Let $b_1$ denote the vertex occupied by Breaker in his subsequent move. In his second move Maker claims the child $m_2$ of $m_1$ such that $m_2$ lies in the subtree of $m_1$ not containing $b_1$. More generally, in his $i$th move Maker selects the child $m_i$ of his previously occupied node $m_{i-1}$ such that the subtree rooted at $m_i$ contains no Breaker’s node. Note that such a child $m_i$ always exists since the vertex previously claimed by Breaker is either in the left or in the right subtree of $m_{i-1}$ (but not in both!). Using this strategy Maker can achieve to own some set $\{v_0, \ldots, v_{n-1}\}$ of vertices such that $v_0, v_1, \ldots, v_{n-1}$ is a path from the root to a leaf, which corresponds to some hyperedge of $G$. Hence Maker has a winning strategy on $G$.

Note that both the maximum neighborhood size and the maximum degree of $G$ are $2^{n-1}$, thus equally large as the number of hyperedges of $G$. This provides some evidence that in order to be a Maker’s win a hypergraph must have largely overlapping hyperedges. Moreover, Beck [2] conjectured that the main criterion for whether a hypergraph is a Breaker’s win is not the cardinality of the hyperedge set but rather the maximum neighborhood size, i.e. the actual reason why each hypergraph $H$ with less than $2^{n-1}$ edges is a Breaker’s win is that the maximum neighborhood of $H$ is smaller than $2^{n-1}$.

**Neighborhood Conjecture** (Open Problem 9.1(a), [2]) Assume that $F$ is an $n$-uniform hypergraph, and its maximum neighborhood size is smaller than $2^{n-1}$. Is it true that by playing on $F$ Breaker has a winning strategy?

Further motivation for the Neighborhood Conjecture is the well-known Erdős-Lovász 2-coloring Theorem – a direct consequence of the famous Lovász Local Lemma – which states that every $n$-uniform hypergraph with maximum neighborhood size at most $2^{n-3}$ has a proper 2-coloring. An interesting feature of this theorem is that the board size does not matter. In this paper we prove by applying again the Lovász Local Lemma that in addition every $n$-uniform hypergraph with maximum neighborhood size at most $2^{n-3} - \frac{3}{n}$ has a so called *proper halving* 2-coloring, i.e., a proper 2-coloring in which the number of red vertices and the number of blue vertices differ by at most 1 (see Theorem 1.3 for details). This guarantees the existence of a course of the game at whose end Breaker owns at least one vertex of each hyperedge and thus is the winner. This suggests that the game we study is a priori not completely hopeless for Breaker.

In our first theorem we prove that the Neighborhood Conjecture, in this strongest of its forms, is not true.

**Theorem 1.1** There is an $n$-uniform hypergraph $H$ with maximum neighborhood size $2^{n-2} + 2^{n-3}$ where Maker has a winning strategy.

In the hypergraph $H$ we will construct to prove Theorem 1.1 one vertex has degree $2^{n-2}$. How-
ever, the existence of vertices with high degree is not crucial. We can also establish a hypergraph
with maximum degree $\frac{2^{n-1}}{n}$ on which Maker has a winning strategy. In this case the maximum
neighborhood size is at most $2^{n-1} - n$, which is weaker than Theorem 1.1 but also disproving the
Neighborhood Conjecture.

**Theorem 1.2** There is an $n$-uniform hypergraph $H$ with maximum degree $\frac{2^{n-1}}{n}$ where Maker has a
winning strategy.

In his book [2] Beck also poses several weakenings of the Neighborhood Conjecture, i.e.

(i) (Open Problem 9.1(b), [2]) If the Neighborhood Conjecture is too difficult (or false) then how
about if the upper bound on the maximum neighborhood size is replaced by an upper bound
$\frac{2^{n-c}}{n}$ on the maximum degree where $c$ is a sufficiently large constant?

(ii) (Open Problem 9.1(c), [2]) If (i) is still too difficult, then how about a polynomially weaker
version where the upper bound on the maximum degree is replaced by $n^{-c} \cdot 2^n$, where $c > 1$ is
a positive absolute constant?

(iii) (Open Problem 9.1(d), [2]) If (ii) is still too difficult, then how about an exponentially weaker
version where the upper bound on the maximum degree is replaced by $c^n$, where $2 > c > 1$ is
an absolute constant?

(iv) (Open Problem 9.1(e), [2]) How about if we make the assumption that the hypergraph is almost
disjoint?

(v) (Open Problem 9.1(f), [2]) How about if we just want a proper halving 2-coloring?

Note that Theorem 1.2 disproves (i) for $c = 1$.

Finally we deal with (v). It is already known that the answer is positive if the maximum degree
is at most $(\frac{3}{2} - o(1))^n$. According to Beck [2] the real question in (v) is whether or not $\frac{3}{2}$ can be
replaced by 2. We prove that the answer is yes.

**Theorem 1.3** For every $n$-uniform hypergraph $F$ with maximum degree at most $\frac{2^{n-2}}{en}$ there is a
proper halving 2-coloring.

Before starting with the actual proofs we fix some notation. Let $T$ be a rooted binary tree of height
$h$. With a path of $T$ we denote an ordinary path $v_i, v_{i+1}, \ldots, v_j$ of $T$ where $v_k$ is on level $k$ for every
$k = i, \ldots, j$. A branch of $T$ is a path starting at the root of $T$. Finally, a full branch of $T$ is a branch
of length $h + 1$. The hypergraphs we will construct to prove Theorem 1.1 and Theorem 1.2 both
belong to the class $C$ of hypergraphs $H$ whose vertices can be arranged in a binary tree $T_H$ such
that each hyperedge of $H$ is a path of $T_H$. Depending on the context we consider a hyperedge $e$ of
a hypergraph $H$ either as a set or as a path in $T_H$. So we will sometimes speak of the start or end
node of a hyperedge.
2 Counterexample to the Neighborhood Conjecture

Proof of Theorem 1.1: Our goal is to construct an element $H \in C$ with the required maximum neighborhood size where Maker has a winning strategy. Before specifying $H$ we fix Maker’s strategy. In his first move he claims the root $m_1$ of $T_H$. In his $i$th move he then selects the child $m_i$ of his previously occupied node $m_{i-1}$ such that the subtree rooted at $m_i$ contains no Breaker’s vertex. Note that such a child $m_i$ always exists since the vertex previously claimed by Breaker is either in the left or in the right subtree of $m_{i-1}$ (but not in both!). This way Maker can achieve some full branch of $T_H$ by the end of the game. This directly implies the following.

Observation 2.1 Let $G \in C$ be an $n$-uniform hypergraph such that every full branch of $T_G$ contains a hyperedge. Then Maker has a winning strategy on $G$.

So in order to prove Theorem 1.1 it suffices to show the following claim.

Lemma 2.2 There is an $n$-uniform hypergraph $H \in C$ with maximum neighborhood $2n - 2 + 2n - 3$ such that each full branch of $T_H$ contains a hyperedge of $H$.

□

Proof of Lemma 2.2: We construct $H$ as follows. Let $T'$ be a binary tree with $n - 1$ levels. For each leaf $u$ of $T'$ we proceed as follows. Then we add two children $v, w$ to $u$ and let the full branch ending at $v$ be a hyperedge. Then we attach a subtree $S$ with $n - 2$ levels to $w$ (such that $w$ is the root of $S$). We need to achieve that each full branch containing $w$ contains a hyperedge. For each leaf $u'$ of $S$ we therefore do the following. We add two children $v', w'$ to $u'$ and let the path from $u$ to $v'$ be a hyperedge. Moreover, we attach a subtree $S'$ with $n - 1$ levels to $w'$ (such that $w'$ is the root of $S'$). We have to complete our tree in such a way that each full branch containing $w'$ contains a hyperedge. To this end we let each path from $u'$ to a leaf of $S'$ be a hyperedge. Figure 1 shows an illustration. It remains to show that the maximum neighborhood of the resulting hypergraph $H$ is at most $2n - 2 + 2n - 3$.

Proposition 2.3 Every hyperedge $e$ of $H$ intersects at most $2n - 2 + 2n - 3$ other hyperedges.

□

Proof of Proposition 2.3: We fix six vertices $u, u', v, v', w, w'$ according to the above description, i.e., $u$ is a node on level $n - 2$ whose children are $v$ and $w$, $u'$ is a descendant of $w$ on level $2n - 4$ whose children are $v'$ and $w'$. Let $e$ be a hyperedge of $H$. Note that the start node of $e$ is either the root $r$ of $T_H$, a node on the same level as $u$ or a node on the same level as $u'$. We now distinguish these cases.

Case (a): The start node of $e$ is $r$.

By symmetry we assume that $e$ ends at $v$. According to the construction of $T_H$ the hyperedge $e$ intersects the $2^{n-2} - 1$ other hyperedges starting at $r$ and the $2^{n-3}$ hyperedges starting at $u$. So altogether $e$ intersects $2^{n-2} + 2^{n-3} - 1$ hyperedges, as claimed.
Figure 1: An illustration of $H$. The marked paths represent exemplary hyperedges.

**Case (b):** The start node of $e$ is on the same level as $u$.

By symmetry we suppose that $e$ starts at $u$ and ends at $v'$. The hyperedges intersecting $e$ can be divided into the following three categories.

- The hyperedge starting at $u$ and ending at $v$,
- the $2^{n-3} - 1$ hyperedges different from $e$ starting at $u$, and
- the $2^{n-2}$ hyperedges starting at $u'$,

implying that $e$ intersects at most $2^{n-2} + 2^{n-3}$ hyperedges in total.

**Case (c):** The start node of $e$ is on the same level as $u'$

By symmetry we assume that $e$ starts at $u'$. Then $e$ intersects the $2^{n-2}$ other hyperedges starting at $u'$ and the hyperedge starting at $u$ and ending at $v'$, thus $2^{n-2} + 1$ hyperedges altogether.
3 A Degree-Regular hypergraph with small maximum degree which is a Maker’s win.

We need some notation first. Throughout this paper log will denote logarithm to the base 2. The vertex set and the hyperedge set of a hypergraph $G$ are denoted by $V(G)$ and $E(G)$, respectively. By a slight abuse of notation we consider $E(G)$ as a multiset, i.e. each hyperedge $e$ can have a multiplicity greater than 1. By a bottom hyperedge of a tree $T_G$ we denote a hyperedge covering a leaf of $T_G$. As in the previous section we only deal with hypergraphs of the class $\mathcal{C}$.

Before tackling the rather technical proof of Theorem 1.2 we show the following weaker claim.

3.1 A weaker statement

Theorem 3.1 There is a $n$-uniform hypergraph $H$ with maximum degree $\frac{2^{n+1}}{n}$ where Maker has a winning strategy.

Let $d = \frac{2^n}{n}$. For simplicity we assume that $n$ is a power of 2, implying that $d$ is power of 2 as well. Due to Observation 2.1 it suffices to show the following.

Lemma 3.2 There is an $n$-uniform hypergraph $G \in \mathcal{C}$ with maximum degree $2d$ such that every full branch of $T_G$ contains a hyperedge of $G$.

□

Proof of Lemma 3.2: To construct the required hypergraph $G$ we establish first a (not necessarily $n$-uniform) hypergraph $H$ and then successively modify its hyperedges and $T_H$. The following lemma is about the first step.

Lemma 3.3 There is a hypergraph $H \in \mathcal{C}$ with maximum degree $2d$ such that every full branch of $T_H$ has $2^i$ bottom hyperedges of size $\log d + 1 - i$ for every $i$ with $0 \leq i \leq \log d$.

Proof of Lemma 3.3: Let $T$ be a binary tree with $\log d + 1$ levels. In order to construct the desired hypergraph $H$ we proceed for each vertex $v$ of $T$ as follows. For each leaf descendant $w$ of $v$ we let the path from $v$ to $w$ be a hyperedge of multiplicity $2^{l(v)}$ where $l(v)$ denotes the level of $v$. Figure 2 shows an illustration. The construction yields that each full branch of $T_H$ has $2^i$ bottom hyperedges of size $\log d + 1 - i$ for every $i$ with $0 \leq i \leq \log d$. So it remains to show that $d(v) \leq 2d$ for every vertex of $v \in V(T)$. Note that every vertex $v$ has $2^{\log d - l(v)}$ leaf descendants in $T_H$, implying that $v$ is the start node of $2^{\log d - l(v)} \cdot 2^{l(v)} \leq d$ hyperedges. So the degree of the root is at most $d \leq 2d$. We then apply induction. Suppose that $d(u) \leq 2d$ for all nodes $u$ with $l(u) \leq i - 1$ for some $i$ with $1 \leq i \leq \log d$ and let $v$ be a vertex on level $i$. By construction exactly half of the hyperedges containing the ancestor of $v$ also contain $v$ itself. Hence $v$ occurs in at most $\frac{1}{2} \cdot 2d = d$ hyperedges as non-start node. Together with the fact that $v$ is the start node of at most $d$ hyperedges this implies that $d(v) \leq d + d \leq 2d$. □
Figure 2: An illustration of $\mathcal{H}$ for $d = 4$. The hyperedge $\{a, b, c\}$ has multiplicity 1, $\{b, c\}$ has multiplicity 2 and $\{c\}$ has multiplicity 4.

The next lemma deals with the second step of the construction of the required hypergraph $\mathcal{G}$.

**Lemma 3.4** There is a hypergraph $\mathcal{H}' \in \mathcal{C}$ with maximum degree $2d$ such that each full branch of $T_{\mathcal{H}'}$ has $2^i$ bottom hyperedges of size $\log d + 1 - i + \lfloor \log \log d \rfloor$ for some $i$ with $0 \leq i \leq \log d$.

**Proof:** Let $\mathcal{H} \in \mathcal{C}$ be a hypergraph with maximum degree $2d$ such that every leaf $u$ of $T_{\mathcal{H}}$ is the end node of a set $S_i(u)$ of $2^i$ hyperedges of size $\log d + 1 - i$ for every $i$ with $0 \leq i \leq \log d$. (Lemma 3.3 guarantees the existence of $\mathcal{H}$.) To each leaf $u$ of $T_{\mathcal{H}}$ we then attach a binary tree $T'_u$ of height $\lfloor \log \log d \rfloor$ in such a way that $u$ is the root of $T'_u$. Let $v_0, \ldots, v_{2^{\lfloor \log \log d \rfloor} - 1}$ denote the leaves of $T'_u$. For every $i$ with $0 \leq i \leq 2^{\lfloor \log \log d \rfloor} - 1$ we then augment every hyperedge of $S_i(u)$ with the set of vertices different from $u$ along the full branch of $T'_u$ ending at $v_i$.

After repeating this procedure for every leaf $u$ of $T_{\mathcal{H}}$ we get the desired hypergraph $\mathcal{H}'$. It remains to show that every vertex in $\mathcal{H}'$ has degree at most $2d$. To this end note first that during our construction the vertices of $\mathcal{H}$ did not change their degree. Secondly, let $u$ be a leaf of $T_{\mathcal{H}}$. By assumption $u$ has degree at most $2d$ and by construction $d(v) \leq d(u)$ for all vertices $v \in V(\mathcal{H}') \setminus V(\mathcal{H})$, which completes our proof. □

**Lemma 3.5** There is a hypergraph $\mathcal{H}'' \in \mathcal{C}$ with maximum degree $2d$ such that every full branch of $T_{\mathcal{H}''}$ has one bottom hyperedge of size $\log d + 1 + \lfloor \log \log d \rfloor$.

Note that due to our choice of $d$, Lemma 3.5 directly implies Lemma 3.2. □

**Proof of Lemma 3.5:** By Lemma 3.4 there is a hypergraph $\mathcal{H}' \in \mathcal{C}$ with maximum degree $2d$ such that each full branch of $T_{\mathcal{H}'}$ has $2^i$ bottom hyperedges of size $\log d + 1 - i + \lfloor \log \log d \rfloor$ for some $i$ with $0 \leq i \leq \log d$. For every leaf $u$ of $T_{\mathcal{H}'}$ we proceed as follows. Let $e_1, \ldots, e_{2^i}$ denote the bottom hyperedges of $\mathcal{H}'$ ending at $u$. We then attach a binary tree $T''$ of height $i$ to $u$ in such a way that $u$ is the root of $T''$. Let $p_1, \ldots, p_{2^i}$ denote the full branches of $T''$. We finally augment $e_j$ with the vertices along $p_j$, for $j = 1 \ldots 2^i$.

After repeating this procedure for every leaf $u$ of $T_{\mathcal{H}'}$ we get the resulting graph $\mathcal{H}''$. By construction every full path of $T_{\mathcal{H}''}$ has one bottom hyperedge of size $\log d + 1 + \lfloor \log \log d \rfloor$. A similar argument as in the proof of Lemma 3.4 shows that the maximum degree of $\mathcal{H}''$ is at most $2d$. □
To prove Theorem 1.2 we then use the same basic ideas, augmented with some refined analysis. To achieve the additional factor of $\frac{1}{4}$ in the bound on the maximum degree we however have to deal with many technical issues.

### 3.2 The actual Theorem

We fix some notation first. A unit is a set of $2^i$ hyperedges of size $\log d + 1 - i$ for some $i \leq \log(d) + 1$. Similarly, a unit of power $k$ denotes a set of $2^i$ hyperedges of size $\log d + 1 - i + k$ for some $i \leq \log(d) + 1$. Let $U$ be a unit. By a slight abuse of notation we let the length $l(U)$ of a unit $U$ denote the size of the hyperedges of $U$. Accordingly, a unit is called a bottom unit if all of its hyperedges are bottom hyperedges.

Note that we have already used the term of a unit implicitly in the proof of Theorem 3.1, e.g. the hypergraph $\mathcal{H}$ mentioned in Lemma 3.3 has the property that each full branch of $T_{\mathcal{H}}$ has $\log d + 1$ bottom units of length at most $\log d + 1$ each, the hypergraph $\mathcal{H}'$ of Lemma 3.4 corresponds to a tree $T_{\mathcal{H}'}$ where each full branch contains one bottom unit of power $\lfloor \log \log d \rfloor$ and, finally, in the tree $T_{\mathcal{H}''}$ of Lemma 3.5 every full branch contains a bottom unit of length $n$, which represents an ordinary hyperedge of size $n$.

**Proof of Theorem 1.2** Due to Observation 2.1 it suffices to show the following.

**Lemma 3.6** There is an $n$-uniform hypergraph $\mathcal{H} \in \mathcal{C}$ with maximum degree $\frac{2n-1}{n}$ such that every full branch of $T_{\mathcal{H}}$ contains a hyperedge of $\mathcal{H}$.

□

**Proof of Lemma 3.6**

Let $d = \frac{2n-2}{n}$. For simplicity we assume that $n$ is a power of 2, implying that $d$ is a power of 2. From now on by a hypergraph we mean an ordinary hypergraph of $\mathcal{C}$ with maximum degree $2d$.

We now state some technical lemmas.

### 3.2.1 General Facts

The basic operation we use in our construction will be denoted by node splitting. Let $\mathcal{G}$ be a hypergraph and let $u$ be a leaf of $T_{\mathcal{G}}$ such that there is a set $S$ of bottom hyperedges ending at $u$. Then splitting $u$ means that we add two children $v_1, v_2$ to $u$, partition $S$ into two subsets $S_1, S_2$ and augment every hyperedge of $S_i$ with $v_i$ for $i = 1, 2$. Possibly we also add new hyperedges of size 1 containing either $v_1$ or $v_2$. Figure 3 shows an illustration for $|S| = 2$. We will often apply a series of hyperedge splittings. By extending a hypergraph $\mathcal{G}$ at a leaf $u$ of $T_{\mathcal{G}}$ we denote the process of successively splitting one of the current leaves in the subtree of $u$; i.e., the resulting hypergraph can be obtained by adding to $u$ a left and a right subtree, modifying the hyperedges of $\mathcal{G}$ containing $u$ and possibly adding some new hyperedges starting at a descendant of $u$ (the other hyperedges remain as they are).
The next lemma is about another basic modification.

**Lemma 3.7** Let \( G \) be a hypergraph and let \( u \) be a leaf of \( T_G \) such that the full branch of \( T_G \) ending at \( u \) contains \( i \) bottom units \( U_1, \ldots, U_i \) with \( l(U_j) \leq \log d \). Then \( u \) can be split in such a way that each full branch containing \( u \) has \( i + 1 \) bottom units \( U_1', \ldots, U_{i+1}' \) with \( l(U_1') = 1 \) and \( l(U_{j+1}') = l(U_j) + 1 \) for \( j = 1 \ldots i \).

**Proof:** Let \( v_1, v_2 \) be the children of \( u \). For each \( U_i \) we proceed as follows. To half of the hyperedges of \( U_i \) we add \( v_1 \) and to the other half we add \( v_2 \). Finally, we let \( \{v_1\}, \{v_2\} \) be hyperedges occurring with multiplicity \( d \) each. Let \( G' \) denote the resulting hypergraph. By construction \( G' \) fulfills the requirements of **Lemma 3.7** as far as the bottom units \( U_1', \ldots, U_{i+1}' \) are concerned. It remains to show that \( G \) has maximum degree \( 2d \). To this end note that apart from \( v_1 \) and \( v_2 \) all vertices of \( G' \) have the same degree as in \( G \). The construction yields that \( d_G(v_1), d_G(v_2) \leq d + \frac{d_G(u)}{2} \). Since by assumption \( d_G(u) \leq 2d \) we are done. \( \Box \)

Note that **Lemma 3.3** states that there is a hypergraph \( H \in \mathcal{C} \) such that each full branch of \( T_H \) has \( \log d + 1 \) bottom units of length at most \( \log d + 1 \). We generalize this fact in the following two statements, which are both direct Corollaries of Lemma 3.7.

**Corollary 3.8** Let \( i \leq \log d + 1 \). Then there is a hypergraph \( G \) such that each full branch of \( T_G \) contains \( i \) bottom units \( U_1, \ldots, U_i \) with \( l(U_j) = j \) for \( j = 1 \ldots i \).

**Corollary 3.9** Let \( r \leq s \) be integers with \( s \leq \log d + 1 \). Let \( G \) be a hypergraph and let \( u \) be a leaf of \( T_G \) such that the full branch ending at \( u \) contains \( i \) bottom units \( U_1, \ldots, U_i \) with \( l(U_j) \leq r \) for every \( j = 1, \ldots, i \). Then \( G \) can be extended at \( u \) in such a way that in the tree \( T_{G'} \) corresponding to the resulting hypergraph \( G' \) each full branch containing \( u \) has \( i + s - r \) bottom units \( V_1, \ldots, V_{s-r}, V_1', \ldots, V_i' \) with \( l(V_j) = j \) for \( j = 1 \ldots s - r \) and \( l(V_j') = l(U_j) + s - r \) for \( j = 1 \ldots i \).

Next we describe how one can develop some units by giving up others. Let \( k \geq 0 \) and let \( i \) be an even number. Suppose there is a hypergraph \( G \) and a vertex \( u \in V(G) \) such that \( u \) is a leaf of \( T_G \) and the full branch ending at \( u \) contains \( i \) bottom units \( U_1, \ldots, U_i \) of power \( k \) each. Then \( u \) can be split in such a way that each full branch of containing \( u \) has \( \frac{i}{2} \) bottom units of power \( k + 1 \). Indeed, we just have to split \( u \) in such a way that one child \( v \) of \( u \) is added to all hyperedges of \( U_j \) for every
$j \leq \frac{c}{2}$ whereas the other child $w$ of $u$ is added to all hyperedges of $U_j$ for every $j \geq \frac{c}{2} + 1$. This directly implies the following.

**Proposition 3.10** Let $k \geq 0$ and let $i$ be a power of 2. Suppose that there is a hypergraph $G$ and a leaf $u$ of $T_G$ such that the full branch ending at $u$ contains $i$ bottom units $U_1, \ldots, U_i$ of power $k$ each. Then $G'$ can be extended at $u$ in such a way that in the tree $T_{G'}$ of the resulting hypergraph $G'$ each full branch containing $u$ has a bottom unit of power $k + \log i$.

We describe some other frequently applied modifications of hypergraphs. Let $k \geq 0$, let $G$ be a hypergraph and let $u$ be a leaf of $T_G$ such that the full branch ending at $u$ contains a bottom unit $U$ of power $k$ with $|U| \geq 2$. Similarly as above we can split $u$ in such a way that each full branch containing $u$ has a bottom unit $U'$ of power $k$ with $|U'| = \frac{|U|}{2}$. By successively splitting the descendants of $u$ in this way we obtain that finally (in the resulting tree) each full branch containing $u$ has a bottom unit of power $k$ with $|U| = 1$. Together with the fact that a unit $U$ of power $k$ with $|U| = 1$ must have length $\log d + k + 1$ this implies that to show Lemma 3.6 it is sufficient to establish a hypergraph $G$ where each full branch of $T_G$ contains one bottom unit of power $n - \log d - 1$. Together with Proposition 3.10 this implies the following.

**Observation 3.11** Suppose that there is a hypergraph $G$ where each full branch $P$ of $T_G$ contains $l_P$ bottom units of power $k_P$ such that $k_P + [\log l_P] \geq n - \log d - 1$. Then Lemma 3.6 holds.

We are now able to roughly describe the actual construction of $\mathcal{H}$.

### 3.2.2 Development of the game

Let $U$ be a unit and let $v$ be a vertex. By a slight abuse of notation we will sometimes say "$v$ is added to $U$" to express that $v$ is added to all hyperedges of $U$.

Our goal is to show the following.

**Lemma 3.12** There is a hypergraph $G$ such that every leaf $u$ of $T_G$ is the end node of $2 \log d - 6$ bottom units $U_1, \ldots, U_{2 \log d - 6}$ such that $l(U_j) \leq (1 - c) \log d$ for $j \leq \log d$ and some constant $c > 0$.

Before proving Lemma 3.12 we show that it implies Lemma 3.6. Let $c' = \frac{c}{4}$. For each leaf $u$ of $T_G$ we proceed as follows. We add two children $v, w$ to $u$ and then for $j = 1 \ldots 2 \log d - 6$ add to $U_j$ the node $v$ if $j \leq (1 - c') \log d$ and $w$, otherwise. Then the full branch ending at $w$ contains $(1 + c') \log d - 6 \geq (1 + c'') \log d$ bottom units of power 1 for some suitable constant $c'' > 0$. Our aim is to apply Observation 3.11 (Note that if the full branch ending at $v$ contained the same amount of bottom units as the full branch ending at $w$ then we would be done.) To this end we will split $v$. Note that the full branch ending at $v$ has $(1 - c') \log d$ units $V_1, \ldots, V_{(1-c') \log d}$ of power 1 with $l(V_j) = l(U_j) + 1 \leq (1 - c) \log d + 1$ for every $j = 1, \ldots, (1 - c') \log d$. Since $l(V_j) \leq \log d + 1$ we have $|V_j| \geq 2$ and therefore every $V_j$ can be partitioned into two units $V'_j, V''_j$ of
power 0 with $|V_j'|, |V_j''| = \frac{|V_j|}{2}$. By applying Corollary 3.10 for $i = 2(1 - c') \log d$, $r = (1 - c) \log d + 1$ and $s = \log d + 1$ we get that our current hypergraph can be extended at $v$ in such a way that each full branch containing $v$ has $(2 + \frac{c}{2}) \log d$ bottom units.

After repeating this procedure for every leaf $u$ of $T_G$ we can apply Observation 3.11 which completes our proof.

**Proof of Lemma 3.12**: For simplicity we assume that $\log d$ is even. We say that a full branch $P$ of a tree $T_G$ has property $\mathcal{P}$ if it contains $2 \log d - 6$ bottom units $U_1, \ldots, U_{2 \log d - 6}$ such that $l(U_j) \leq (1 - c) \log d$ for $j \leq \log d$ and some constant $c > 0$. Our construction of the desired hypergraph $\mathcal{G}$ will consist of two major steps. The next proposition is about the first step.

**Proposition 3.13** Let $i$ be an integer with $0 \leq i \leq \log d - 2$. Let $k_1 = \log d$, if $i = 0$ and $k_1 = \log d - i - 2$, otherwise. Then there is a hypergraph $\mathcal{G}$ such that each full branch of $T_G$ either has property $\mathcal{P}$ or contains $\log d + i$ bottom units $U_1, \ldots, U_{\log d + i}$ with

- $l(U_j) = j$ for $j \leq k_1$
- $l(U_{k_1 + 2r - 1}), l(U_{k_1 + 2r}) = k_1 + r + 1$ for $r \geq 1$

**Proof**: We proceed by induction. By Corollary 3.8 applied for $i = \log d$ the claim is true for $i = 0$. Suppose that it holds for $i \leq \frac{\log d}{2} - 2$. For each leaf $u$ of $T_G$ we then proceed as follows. If the full branch ending at $u$ has property $\mathcal{P}$ then we do nothing. Otherwise, induction yields that the full branch ending at $u$ contains $\log d + i$ bottom units $U_1, \ldots, U_{\log d + i}$ according to the description in Proposition 3.13. We then add two children $v, w$ to $u$. For $j = 1 \ldots \log d + i$ we then add to $U_j$ the vertex $v$ if $j \leq i + 2$ and $w$, otherwise. Note that the full branch ending at $w$ contains $\log d - 2$ bottom units $V_{i+3}, \ldots, V_{\log d + i}$ of power 1 with $l(V_j) = l(U_j) + 1$ for $j = i + 3 \ldots \log d + i$. Since each $V_j$ is of length at most $\log d + 1$ it contains at least two hyperedges and can thus be partitioned into two units $V_j', V_j''$ of power 0 with $l(V_j'), l(V_j'') = l(U_j) + 1$. Moreover, $l(V_j') \leq k_1 + \left\lceil \frac{r - k_1}{2} \right\rceil + 1$ (it can be checked that this is true both for $r \geq k_1$ and $r \leq k_1$). Hence $l(V_j') \leq k_1 + \left\lceil \frac{r + \log d - k_1}{2} \right\rceil + 1$. So $l(V_j') \leq \frac{3}{4} \log d + 1$ and thus the full branch ending at $w$ has property $\mathcal{P}$.

It remains to consider the full branch $P$ ending at $v$. $P$ contains $i + 2$ units $V_1, \ldots, V_{i+2}$ of power 1, which due to a similar argument as before correspond to $2(i + 2)$ units $V_1', V_1'', \ldots, V_{i+2}', V_{i+2}''$ with $l(V_j'), l(V_j'') = l(U_j) + 1 = j + 1$ (note that $i + 2 \leq k_1$) for $j = 1 \ldots i + 2$. By applying Corollary 3.9 for $r = i + 3$ and $s = \log d$ we get that our current hypergraph can be extended at $v$ in such a way that each full branch containing $v$ has the $\log d + i + 1$ required bottom hyperedges (considering the induction hypothesis for $i + 1$). After repeating this procedure for every leaf $u$ of $T_G$ the resulting hypergraph fulfills our hypothesis for $i + 1$. □

The following corollary specifies the result of our first step.

**Corollary 3.14** Let $k_1 = \frac{\log d}{2} - 1$. Then there is a hypergraph $\mathcal{G}$ such that each full branch of $T_G$ either has property $\mathcal{P}$ or contains $\frac{3}{4} \log d - 1$ units $U_1, \ldots, U_{\frac{1}{4} \log d - 1}$ such that
• \( l(U_j) = j \) for \( j \leq k_1 \)

• \( l(U_{k_1+2r-1}), l(U_{k_1+2r}) = k_1 + r + 1 \) for \( r \geq 1 \)

The next proposition deals with the second major step of our construction.

**Proposition 3.15** Let \( i \) be an integer with \( \log d - 1 \leq i \leq \log d - 6 \) and let \( k_1 = \frac{\log d}{2} - 1 \), if \( i = \frac{\log d}{2} - 1 \) and \( k_1 = \log d - i - 4 \), otherwise. Then there is a \( k_2 \geq 2 \) such that there is a hypergraph \( G \) where each full branch of \( T_G \) either has property \( P \) or contains \( \log d + i \) units \( U_1, \ldots, U_{\log d + i} \) with

- \( l(U_j) \leq j \) for \( j \leq k_1 \)
- \( l(U_{k_1+2r-1}), l(U_{k_1+2r}) \leq k_1 + r + 1 \) for \( 1 \leq r \leq k_2 \)
- \( l(U_{k_1+2k_2+2m-1}), l(U_{k_1+2k_2+2m}) \leq k_1 + k_2 + m + 2 \) for \( m \geq 1 \)

Note that Proposition 3.15 applied for \( i = \log d - 6 \) directly implies Lemma 3.12 \( \square \)

So it remains to show Proposition 3.15.

**Proof of Proposition 3.15:** Corollary 3.14 yields that our claim is true for \( i = \frac{\log d}{2} - 1 \) (with \( k_2 = \infty \)). Suppose that the claim holds for \( i \). For each leaf \( w \) of \( T_G \) we proceed as follows. If the full branch ending at \( u \) has property \( P \) we do nothing. Otherwise induction yields that the full branch ending at \( u \) contains \( \log d + i \) bottom units \( U_1, \ldots, U_{\log d + i} \) according to the description in Proposition 3.15. In this case we add two children \( v, w \) to \( u \) and for \( j = 1 \ldots \log d + i \) add to \( U_j \) the node \( v \), if \( j \leq i + 3 \) and \( w \), otherwise. The full branch \( P \) ending at \( w \) contains \( \log d - 3 \) units \( U_{i+4}, \ldots, U_{\log d+i} \) of power 1 with \( l(U'_j) = l(U_j) + 1 \). The induction hypothesis yields that for each \( U'_j \) we have \( l(U'_j) \leq \log d \), implying that \( |U'_j| \geq 2 \). So \( U'_j \) can be partitioned into two units \( V'_j, V''_j \) of power 0 with \( l(V'_j), l(V''_j) = l(U'_j) \). Due to our hypothesis \( l(V'_j) \) (and \( l(V''_j) \), respectively) is at most \( k_1 + 2 + \lceil \frac{m_i}{d} \rceil \) (note that this also holds for \( j \leq k_1 \)) and so for \( j \) with \( i+4 \leq j \leq i+3 + \frac{\log d}{2} \) we have \( l(V'_j) \leq k_1 + 2 + \frac{i+3}{2} + \frac{\log d}{4} \leq \frac{3}{2} \log d + 3 \). Since \( P \) contains \( V'_{i+4}, V''_{i+4}, \ldots, V'_{\log d+i}, V''_{\log d+i} \) it has property \( P \).

It remains to consider the full branch \( P \) ending at \( v \). \( P \) contains \( i + 3 \) units \( U'_{i+3} \) of power 1. For a similar reason as above they can be partitioned into \( 2(i + 3) \) units \( V^{(1)}_1, V^{(2)}_1, \ldots, V^{(1)}_{i+3}, V^{(2)}_{i+3} \) with \( l(V^{(s)}_j) = l(U_j) + 1 \) for \( s \in \{1, 2\} \). According to our assumption we have for \( s \in \{1, 2\} \)

- \( l(V^{(s)}_j) \leq j + 1 \) for \( j \leq k_1 \)
- \( l(V^{(s)}_{k_1+2r-1}), l(V^{(s)}_{k_1+2r}) \leq k_1 + r + 2 \) for \( 1 \leq r \leq k_2 \)
- \( l(V^{(s)}_{k_1+2k_2+2m-1}), l(V^{(s)}_{k_1+2k_2+2m}) \leq k_1 + k_2 + m + 3 \) for \( m \geq 1 \)

Note that for each \( V^{(s)}_j \) we have \( l(V^{(s)}_j) \leq j + 2 \leq i + 5 \) (this can be seen by considering each of the three possible intervals for \( j \) separately and using that \( k_2 \geq 1 \)). Let \( k'_i = \log d - i - 5 \). By applying Corollary 3.9 for \( r = i + 5 \) and \( s = \log d \) we obtain that our current graph can be extended at \( v \) in
such a way that each full branch of the tree $T_{G'}$ of the resulting graph $G'$ contains $\log d + i + 1$ units $X_1, \ldots, X_{k'_i}, W_1^{(1)}, W_1^{(2)}, \ldots, W_{i+3}^{(1)}, W_{i+3}^{(2)}$ with

- $l(X_j) \leq j$ for $j \leq k'_1$
- $l(W_j^{(s)}) \leq j + k'_1 + 1$ for $s \in \{1, 2\}$ and $j \leq k_1$
- $l(W_{k_1+2r-1}^{(s)}) \leq k_1 + k'_1 + r + 2$ for $s \in \{1, 2\}$ and $r \leq k_2$
- $l(W_{k_1+2k_2+2m-1}^{(s)}) \leq k_1 + k_2 + k'_1 + m + 3$ for $s \in \{1, 2\}$ and $m \geq 1$

Let $i' = i + 1$ and $k'_2 = k_1$. Note that $k'_1 = \log d - i' - 4$ and that $k'_2 \geq 2$ (due to the fact that by definition $k_1 \geq 2$). The fact that $k_2 \geq 2$ guarantees that after a suitable renaming the units $X_1, \ldots, X_{k'_i}, W_1^{(1)}, W_1^{(2)}, \ldots, W_{i+3}^{(1)}, W_{i+3}^{(2)}$ fulfill our hypothesis for $i', k'_1$ and $k'_2$. □

4 Establishing a Proper Halving 2-Coloring

**Proof of Theorem 1.3** For simplicity we only consider hypergraphs with an even number of vertices. We will show the following stronger claim.

**Proposition 4.1** Let $F$ be a $n$-uniform hypergraph with maximum degree at most $\frac{2^n}{dn}$. Then for each pairing $(v_1, w_1), (v_2, w_2), (v_3, w_3), \ldots$ of the vertices of $F$ there is a proper 2-coloring such that $v_k$ and $w_k$ have different colors for each $k$.

To prove Theorem 1.3 it suffices to prove Proposition 4.1. We adapt a proof by Kratochvíl, Savický and Tuza [4].

**Proof of Proposition 4.1** Our claim is a consequence of Lovász Local Lemma.

**Lemma 4.2 (Lovász Local Lemma.)** Let $A_1, \ldots, A_m$ be events in some probability space, and let $G$ be a graph with vertices $A_1, \ldots, A_m$ and edges $E$ such that each $A_i$ is mutually independent of all the events $\{A_j \mid \{A_i, A_j\} \notin E, i \neq j\}$. If there exist real numbers $0 < \gamma_i < 1$ for $i = 1, \ldots, m$ satisfying

$$Pr(A_i) \leq \gamma_i \prod_{j : \{A_i, A_j\} \in E} (1 - \gamma_i)$$

for all $i = 1, \ldots, m$ then

$$Pr(\neg A_1 \land \neg A_2 \land \cdots \land \neg A_m) > 0$$

For a proof of the Lovász Local Lemma and different versions, see e.g. [1]. Let $d = \frac{2^n}{dn}$. Note that each proper coloring of $F$ fulfilling the condition that $v_k$ and $w_k$ have different colors for each $k$ is a proper-2-coloring. In each edge of $F$ we then replace $w_k$ with $\bar{v}_k$, expressing that $w_k$ gets the "inverse" color of $v_k$. Let $F'$ denote the resulting hypergraph. Note that the maximum degree of $F'$ is at most $2d = \frac{2^n}{2en}$. Indeed, the degree of $v_k$ is bounded by the number of edges possessing
plus the number of edges possessing \( \bar{v}_k \). Since edges containing both \( v_k, \bar{v}_k \) get two colors in every coloring we can ignore those edges and assume that no edge of \( \mathcal{F}' \) contains both \( v_k, \bar{v}_k \) for some \( k \). Since every proper 2-coloring of \( \mathcal{F}' \) directly provides the desired proper halving 2-coloring, it suffices to show that \( \mathcal{F}' \) has a proper 2-coloring. To this end we apply the Lovász Local Lemma. Let the probability space be the set of all color assignments to the vertices of \( \mathcal{F} \) with the uniform distribution. Let \( E(\mathcal{F}') = \{ E_1, \ldots, E_m \} \) and let \( A_i \) be the event that \( E_i \) is monochromatic in a random 2-coloring. Let \( G \) be the graph where \( A_i \) and \( A_j \) are connected if they have a vertex in common. Since every vertex has degree at most \( 2d \) every \( A_i \) has degree at most \( n \cdot (2d - 1) \). Note that \( \Pr(A_i = 1) = 2 \cdot 2^{-n} \). We let \( \gamma_i = e \cdot \Pr(A_i = 1) = 2e \cdot 2^{-n} \) for each \( i \). Hence

\[
\frac{\gamma_i}{\Pr(A_i = 1)} \prod_{A_i, A_j \in E(G)} (1 - \gamma_j) \geq e \left( 1 - \frac{2e}{2^n} \right)^n \left( \frac{2^n}{2^n - 1} \right)^{-1} > e \left( 1 - \frac{2e}{2^n} \right)^{\frac{2^n}{2^n - 1}} > ee^{-1} = 1
\]

Hence \( Pr(\neg A_1 \land \neg A_2 \land \cdots \land \neg A_m) > 0 \) and therefore there is a proper 2-coloring on \( \mathcal{F}' \). □

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