Parallel Spinors on Pseudo-Riemannian $Spin^c$ Manifolds

Aziz Ikemakhen *

Abstract

We describe, by their holonomy groups, all complete simply connected irreducible non-locally symmetric pseudo-Riemannian $Spin^c$ manifolds which admit parallel spinors. So we generalize the Riemannian $Spin^c$ case ([8]) and the pseudo-Riemannian $Spin$ one ([1]).

Mathematics Subject Classifications: 53C50, 53C27.
Key words: holonomy groups, Pseudo-Riemannian $Spin^c$ manifolds, parallel spinors.

1 Introduction

In ([8]), Moroainu described all complete simply connected Riemannian $Spin^c$ manifolds admitting parallel spinors. Precisely, he showed the following result:

Theorem 1 A complete simply connected $Spin^c$ Riemannian manifolds $(M, g)$ admits a parallel spinor if and only if it is isometric to the Riemannian product $(M_1, g_1) \times (M_2, g_2)$ of a complete simply connected kähler manifold $(M_1, g_1)$ and a complete simply connected $Spin$ manifold $(M_2, g_2)$ admitting a parallel spinor. The $Spin^c$ structure of $(M, g)$ is then the product of the canonical $Spin^c$ structure of $(M_1, g_1)$ and the $Spin$ structure of $(M_2, g_2)$.

In ([1]), Baum and Kath characterized, by their holonomy group, all simply connected irreducible non-locally symmetric pseudo-Riemannian $Spin$ manifolds admitting parallel spinors. Precisely, they proved the following result:

Theorem 2 Let $(M, g)$ be a simply connected irreducible non-locally symmetric pseudo-Riemannian $Spin$ manifold of dimension $n = p + q$ and signature $(p, q)$. We denote by $N$ the dimension of the space of parallel spinors on $M$. Then $(M, g)$ admits a parallel spinors if and only if the holonomy group $H$ of $M$ is (up to conjugacy in $O(p, q)$) one in the following list :

*Faculté des sciences et techniques, B.P. 549, Gueliz, Marrakech, Maroc. E-mail: ikemakhen@fstg-marrakech.ac.ma
Our aim is to generalize this result for the simply connected irreducible non-locally symmetric pseudo-Riemannian $\text{Spin}^c$ manifolds. More precisely, we show that:

Theorem 3 Let $(M,g)$ be a simply connected irreducible non-locally symmetric pseudo-Riemannian manifold of dimension $n = p + q$ and signature $(p,q)$. Then the following conditions are equivalent

(i) $(M,g)$ is a $\text{Spin}^c$ manifold which admits a parallel spinor,
(ii) Either $(M,g)$ is a $\text{Spin}$ manifold which admit a parallel spinor, or $(M,g)$ is a kähler not special kähler manifold,
(iii) The holonomy group $H$ of $(M,g)$ is (up to conjugacy in $O(p,q)$) one in table 1 or $H = U(p',q')$, $p = 2p'$ and $q = 2q'$.

For $H = U(p',q')$ the dimension of the space of parallel spinors on $M$ is 2.

This theorem is a contribution to the resolution of the following problem :

(P) What are the possible holonomy groups of simply connected pseudo-Riemannian $\text{Spin}^c$ manifolds which admit parallel spinors?

Some partial answers to this problem have been given by M. Wang for the Riemannian $\text{Spin}$ case (2), by H. Baum and I. Kath for the irreducible pseudo-Riemannian $\text{Spin}$ one (11), by Th. Leistner for the Lorentzian $\text{Spin}$ one (6, 7), by A. Moroianu for the Riemannian $\text{Spin}^c$ one (Theorem 1), and by author for the totally reducible pseudo-Riemannian $\text{Spin}$ one and for Lorentzian $\text{Spin}^c$ one (3, 4). The problem remains open even though big progress have been made.

The proof of Theorem 3 is based on Theorem 2 and the technique used by Moroianu to prove Theorem 1 that we adapted to the pseudo-Riemannian case.

In paragraph 2 of this paper we define the group $\text{Spin}^c(p,q)$ and its spin representation. We also define the $\text{Spin}^c$-structure on pseudo-Riemannian manifolds and its associated spinor bundle. In paragraph 2 we give an algebraic characterization to the pseudo-Riemannian $\text{Spin}^c$ manifolds which admit parallel spinors and we prove Theorem 3.
2 Spinor representations and $Spin^{c}$- bundles

2.1 $Spin^{c}(p, q)$ groups

Let $<, >_{p,q}$ be the ordinary scalar product of signature $(p, q)$ on $\mathbb{R}^{m}$ ($m = p + q$). Let $Cl_{p,q}$ be the Clifford algebra of $\mathbb{R}^{p,q} := (\mathbb{R}^{m}, <, >_{p,q})$ and $Cl_{p,q}$ its complexification. We denote by $\cdot$ the Clifford multiplication of $Cl_{p,q}$. $Cl_{p,q}$ contains the groups $S^{1} := \{ z \in \mathbb{C}; \| z \| = 1 \}$ and $Spin(p, q) := \{ X_{1} \cdot \ldots \cdot X_{2k}; < X_{i}, X_{i} >_{p,q} = \pm 1; \ k \geq 0 \}$.

Since $S^{1} \cap Spin(p, q) = \{-1, 1\}$, we define the group $Spin^{c}(p, q)$ by

$$Spin^{c}(p, q) = Spin(p, q) \cdot S^{1} \div \{-1, 1\} = Spin(p, q) \times \mathbb{Z}_{2} S^{1}.$$ 

Consequently, the elements of $Spin^{c}(p, q)$ are the classes $[g, z]$ of pairs $(g, z) \in Spin(p, q) \times S^{1}$, under the equivalence relation $(g, z) \sim (-g, -z)$. The following suites are exact (see [5]):

$$1 \rightarrow \mathbb{Z}_{2} \rightarrow Spin(p, q) \xrightarrow{\lambda} SO(p, q) \rightarrow 1$$

$$1 \rightarrow \mathbb{Z}_{2} \rightarrow Spin^{c}(p, q) \xrightarrow{\xi} SO(p, q) \times S^{1} \rightarrow 1,$$

where $\lambda(g)(x) = g \cdot x \cdot g^{-1}$ for $x \in \mathbb{R}^{m}$ and $\xi([g, z]) = (\lambda(g), z^{2})$.

Let $(e_{i})_{1 \leq i \leq m}$ be an orthonormal basis of $\mathbb{R}^{p,q}$ ($< e_{i}, e_{j} > = \varepsilon_{i} \delta_{ij}$, $\varepsilon_{i} = -1$ for $1 \leq i \leq p$ and $\varepsilon_{i} = -1$ for $1 + p \leq i \leq m$ ). The Lie algebras of $Spin(p, q)$ and $Spin^{c}(p, q)$ are respectively

$$spin(p, q) := \{ e_{i} \cdot e_{j}; 1 \leq i < j \leq m \}$$

and

$$spin^{c}(p, q) := spin(p, q) \oplus i \mathbb{R}.$$ 

The derivative of $\xi$ is a Lie algebra isomorphism and it is given by

$$\xi_{*}(e_{i} \cdot e_{j}, it) = (2E_{ij}, 2it),$$

where $E_{ij} = -\varepsilon_{j}D_{ij} + \varepsilon_{i}D_{ji}$ and $D_{ij}$ is the standard basis of $gl(m, \mathbb{R})$ with the $(i, j)$-component equal 1 and all other zero.


2.2 Spin\(^c\) bundles

In this paper, we will use the following isomorphisms:

Let \( U = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \), \( V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), \( E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \), and \( \mathbb{C}(2^n) \) the complex algebra consisting of \( 2^n \times 2^n \)-matrices.

In case \( m = p + q = 2n \) is even, we define (see [1]) \( \Phi_{p,q} : \mathbb{C}(2^n) \rightarrow \mathbb{C}(2^n) \) by

\[
\Phi_{p,q}(e_{2j-1}) = \tau_{2j-1}E \otimes \ldots \otimes E \otimes U \otimes T \otimes \ldots \otimes T
\]

\[
\Phi_{p,q}(e_{2j}) = \tau_{2j}E \otimes \ldots \otimes E \otimes V \otimes T \otimes \ldots \otimes T \tag{1}
\]

Where \( \tau_j = i \) if \( \varepsilon_j = -1 \) and \( \tau_j = 1 \) if \( \varepsilon_j = 1 \).

In case \( m = 2n + 1 \) is odd, \( \Phi_{p,q} : \mathbb{C}(2^n) \rightarrow \mathbb{C}(2^n) \oplus \mathbb{C}(2^n) \) is defined by

\[
\Phi_{p,q}(e_k) = (\Phi_{p,q-1}(e_k), \Phi_{p,q-1}(e_k)), \quad k = 1, \ldots, m - 1;
\]

\[
\Phi_{p,q}(e_m) = (iT \otimes \ldots \otimes T, -iT \otimes \ldots \otimes T). \tag{2}
\]

This yields representations of the spin group \( Spin(p, q) \) in case \( m \) even by restriction and in case \( m \) odd by restriction and projection onto the first component. The module space of \( Spin(p, q) \)-representation is \( \Delta_{p,q} = \mathbb{C}2^n \). The Clifford multiplication is defined by

\[
X \cdot u := \Phi_{p,q}(X)(u) \quad \text{for} \quad X \in \mathbb{C}^m \quad \text{and} \quad u \in \Delta_{p,q}. \tag{3}
\]

A usual basis of \( \Delta_{p,q} \) is the following : \( u(\nu_n, \ldots, \nu_1) := u(\nu_n) \otimes \ldots \otimes u(\nu_1); \quad \nu_j = \pm 1, \)

where

\[
u(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u(-1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2. \]

The spin representation of the group \( Spin(p, q) \) extends to a \( Spin^c(p, q) \)-representation by :

\[
\Phi_{p,q}([g, z])(v) = z \Phi_{p,q}(g)(v), \tag{4}
\]

for \( v \in \Delta_{p,q} \) and \([g, z] \in Spin^c(p, q) \). Therefore \( \Delta_{p,q} \) becomes the module space of \( Spin^c(p, q) \)-representation (see [2]).

There exists a hermitian inner product \( <.,.>_{\Delta} \) on the spinor module \( \Delta_{p,q} \) defined by

\[
<v, w>_{\Delta} := \frac{1}{p^{p-1}}(e_1 \cdot \ldots \cdot e_p \cdot v, w); \quad \text{for} \quad v, w \in \Delta_{p,q},
\]

where \( (z, z') = \sum_{i=1}^{2^n} z_i \cdot \overline{z_i} \) is the standard hermitian product on \( \mathbb{C}2^n \).

\( <.,.>_{\Delta} \) satisfies the following properties :

\[
<X \cdot v, w>_{\Delta} = (-1)^{p+1} <v, X \cdot w>_{\Delta}, \tag{5}
\]

for \( X \in \mathbb{C}^m \).
2.3 Spinor bundles

Let \((M, g)\) be a connected pseudo-Riemannian manifold of signature \((p, q)\). And let \(P_{SO(p,q)}\) denote the bundle of oriented positively frames on \(M\).

**Definition 1** A structure \(Spin^c\) on \((M, g)\) is the data of a \(S^1\)-principal bundle \(P_L\) over \(M\) and a \(\xi\)-reduction \((P_{Spin^c(p,q)}, \Lambda_{p,q})\) of the product \((SO(p, q) \times S^1)\)-principal bundle \(P_{SO(p,q)} \times P_L\). i.e. \(\Lambda : P_{Spin^c(p,q)} \to (P \times P_L)\) is a 2-fold covering verifying:

i) \(P_{Spin^c(p,q)}\) is a \(Spin^c(p,q)\)-principal bundle over \(M\),

ii) \(\forall u \in Q, \forall a \in Spin(p, q)\),

\[\Lambda(ua) = \Lambda(a)\xi(a).\]

What means, the following diagram commutes:

\[
\begin{array}{ccc}
P_{Spin^c(p,q)} \times Spin^c(p,q) & \longrightarrow & P_{Spin^c(p,q)} \\
\Lambda \otimes \xi \downarrow & & \Lambda \downarrow \\
(P_{SO(p,q)} \times P_L) \times (SO(p, q) \times S^1) & \longrightarrow & P_{SO(p,q)} \times P_L
\end{array}
\]

Not that \((M, g)\) carries a \(Spin^c\)-structure if and only if the second Steifl-Whitney class of \(M\), \(w_2(M)\) is the mod reduction of an integral class \([5]\).

**Example 1** Every pseudo-Riemannian Spin manifold is canonically a \(Spin^c\) manifold. The \(Spin^c\)-manifold is obtained as

\[P_{Spin^c(p,q)} = P_{Spin(p,q)} \times_{Z_2} S^1,\]

where \(P_{Spin(p,q)}\) is the Spin-bundle and \(Z_2\) acts diagonally by \((-1, -1)\).

**Example 2** Any irreducible pseudo-Riemannian kähler manifold is canonically a \(Spin^c\) manifold.

Indeed The holonomy group \(H\) of \((M, g)\) is \(U(p', q')\), where \((p, q) = (2p', 2q')\) is the signature of \((M, g)\). Then \(P_{SO(p,q)}\) is reduced to \(U(p', q')\)-principal bundle \(P_{U(p', q')}\). Moreover there exists an \(<.\ldots>_p.q\)-orthogonal almost complex structure \(J\) which we can imbed \(U(p', q')\) in \(SO(p, q)\) by

\[A + iB = ((akl)_{1 \leq k,l \leq m} + i(bkl)_{1 \leq k,l \leq m}) \to \left( \begin{array}{cc} a_{kl} & b_{kl} \\ -b_{kl} & a_{kl} \end{array} \right)_{1 \leq k,l \leq m}.\]

and \((e_k, J e_k)_{k=1,..,p'+q'}\) is an orthogonal basis of \(\mathbb{R}^{p,q}\).

We consider the homomorphism

\[
\alpha : U(p', q') \leftrightarrow SO(p, q) \times S^1 \\
\quad \quad C \to (i(C), \det(C))
\]
Since the proper values of every element $C \in U(p', q')$ is in $S^1$ and
\[
\cos 2\theta + \varepsilon_k \sin 2\theta \ e_k \cdot J e_k = \varepsilon_k (\cos \theta \ e_k + \sin \theta \ J e_k) \cdot (\sin \theta \ e_k - \cos \theta \ J e_k),
\]
where $\varepsilon_k = \langle e_k, e_k \rangle_{p,q}$, then the following homomorphism
\[
\tilde{\alpha} : U(p', q') \to Spin^c(p, q) \times S^1
\]
\[
C \to \prod_{k=1}^m (\cos 2\theta_k + \varepsilon_k \sin 2\theta_k \ e_k \cdot J e_k) \times e^{\frac{2i}{\Delta} \sum \theta_k},
\]
is well defined, where $e^{i\theta_k}$ are the proper values of $C$. And it is easy to verifies that
\[
\xi \circ \tilde{\alpha} = \alpha.
\]
Consequently,
\[
P_{Spin^c(p, q)} = P_{U(p', q')} \times \tilde{\alpha} Spin^c(p, q).
\]
Now, let denote by $S := P_{Spin(p, q)} \times \phi_{p,q} \Delta_{p,q}$ the spinor bundle associated to the $Spin^c$-structure $P_{Spin(p,q)}$, by $L := P_L \times iC = P_{Spin(p,q)} \times _{S^1} C$ the complex line bundle associated to the auxiliary bundle $P_L$, where
\[
i : S^1 \to GL(C).
\]
The Clifford multiplication given by (3) defines a Clifford multiplication on $S$:
\[
TM \otimes S = (P_{Spin(p,q)} \times \phi_{p,q} \mathbb{R}^m) \otimes (P_{Spin(p,q)} \times \phi_{p,q} \Delta_{p,q}^\pm) \to S
\]
\[
(X \otimes \psi) = [q, x] \otimes [q, v] \to \ [q, x \cdot v] =: X \cdot \psi.
\]
Since the scalar product $\langle \ldots \rangle_\Delta$ is $Spin^c_0(p, q)$-invariant, it defines a scalar product on $S$ by :
\[
\langle \psi, \psi_1 \rangle_\Delta = \langle v, v_1 \rangle_\Delta, \text{ for } \psi = [q, v] \text{ and } \psi_1 = [q, v_1] \in \Gamma(S).
\]
According to (5), it is then easy to verify that
\[
\langle X \cdot \psi, \psi_1 \rangle_\Delta = (-1)^{p+1} < \psi, X \cdot \psi_1 >_\Delta,
\]
for $X \in \Gamma(M)$ and $\psi, \psi_1 \in \Gamma(S)$.
Now, as in the Riemannian case ( see [2]), if $(M, g)$ is a $Spin^c$ pseudo- Riemannian manifold, every connection form $A : TP_L \to i\mathbb{R}$ on the $S^1$- bundle $P_L$ defines ( to- gether with the Levi-Civita $D$ of $(M, g)$ ) a covariant derivative $\nabla^A$ on the spinor bundle $S$, called the spinor derivative associated to $(M, g, S, L, A)$.
Henceforth, a $Spin^c$- pseudo- Riemannian manifold will be the data of a set $(M, g, S, P_L, A)$, where $(M, g)$ is an oriented connected pseudo- Riemannian manifold, $S$ is a $Spin^c$ structure, $L$ is the complex line bundle associated to the auxiliary bundle of $S$ and $A$ is a connection form on $P_L$. Using (6) and by the same proof in the Riemannian case ( see [2]), we conclude that
Proposition 1 \( \forall X,Y \in \Gamma(M) \) and \( \forall \psi, \psi_1 \in \Gamma(S) \),
\[
\nabla^A_Y (X \cdot \psi) = X \cdot \nabla^A_Y (\psi) + D_Y X \cdot \psi. \tag{7}
\]
Let us denote by \( F_A := i\omega \) the curvature form of \( A \), seen as an imaginary-valued 2-form on \( M \), by \( R \) and \( Ric \) respectively the curvature and the Ricci tensor of \( (M, g) \) and by \( R^A \) the curvature tensor of \( \nabla^A \). Like Riemannian case ( see [2]), if we put \( A(X) := X \cdot \omega \) we have

Proposition 2 For \( \forall q = (e_1, \ldots, e_m) \) a local section of \( P_{Spin(p,q)} \), \( \forall X,Y \in \Gamma(M) \) and \( \forall \psi \in \Gamma(S) \),
\[
R^A(X,Y) \psi = \frac{1}{2} \sum_{1 \leq i < j \leq m} \varepsilon_i \varepsilon_j g(R(X,Y)e_i,e_j)e_i \cdot e_j \cdot \psi + i \frac{1}{2} \omega(X,Y) \cdot \psi. \tag{9}
\]
\[
\sum_{1 \leq i \leq m} \varepsilon_i R^A(X,e_i) \psi = -\frac{1}{2} Ric(X) \cdot \psi + i \frac{1}{2} A(X) \cdot \psi. \tag{10}
\]

Remark 1 According to Example 1, if \( (M,g) \) is \( Spin \) then it is \( Spin^c \). Moreover, the auxiliary bundle \( P_L \) is trivial and then there exists a global section \( \sigma : M \to P_L \). We choose the connection defined by \( A \) to be flat, and we denote \( \nabla^A \) by \( \nabla \). Conversely, if the auxiliary bundle \( P_L \) of a \( Spin^c \)-structure is trivial, it is canonically identified with a \( Spin \)-structure. Moreover, if the connection \( A \) is flat, by this identification, \( \nabla^A \) corresponds to the covariant derivative on the spinor bundle.

3 parallel Spinors

3.1 algebraic characterization

It is well known that there exists a bijection between the space \( \mathcal{PS} \) of all parallel spinors on \( (M, g) \) and the space
\[
V_{\tilde{H}} = \{ v \in \Delta_{p,q}; \Phi_{p,q}(\tilde{H})(v) := \tilde{H} \cdot v = v \}
\]
of all fixed spinors of \( \Delta_{p,q} \) with respect to the holonomy group \( \tilde{H} \) of the connection \( \nabla^A \). If \( (M, g) \) is supposed simply connected, then \( \mathcal{PS} \) is in bijection with
\[
V_{\tilde{H}} = \{ v \in \Delta_{p,q}; \tilde{H} \cdot v = 0 \},
\]
where \( \tilde{H} \) is the Lie algebra of \( \tilde{H} \). With the notations introduced in subsection 2.1, for \( B \in \text{spin}(p,q) \) and \( t \in \mathbb{R} \), we have
\[ \xi^{-1}(B,\bar{t}) = (\lambda^{-1}(B), \frac{1}{2}\bar{t}) = \left( \frac{1}{4} \sum_{i=1}^{m} \varepsilon_{i} e_{i} \cdot B(e_{i}), \frac{1}{2}\bar{t} \right). \]  

(11)

Moreover, \( \xi(\tilde{H}) = H \times H_{A} \), where \( H \) is the holonomy group of \((M,g)\) and \( H_{A} \) the one of \( A \). \( H_{A} = S^{1} \), if \( A \) is flat and \( H_{A} = \{1\} \) otherwise. Then

\[ V_{\tilde{H}} = \{ v \in \Delta_{p,q}; \xi^{-1}(\mathcal{H} \oplus H_{A}) \cdot v = 0 \}, \]

where \( \mathcal{H} \) is the Lie algebra of \( H \) and \( H_{A} \) the one of \( H_{A} \). \( \phi_{p,q} \) is linear, then if we differentiate the relation (4) we get :

\[ \phi_{p,q}(B,\bar{t})(v) = \text{id}v + \phi_{p,q}(B)(v) = \text{id}v + B \cdot v. \]

According to (11) we have ,

\[ \phi_{p,q}(\xi^{-1}(B,\bar{t}))(v) = \frac{1}{2} \text{id}v + \lambda^{-1}(B) \cdot v. \]

Therefore \((M,g)\) admits a parallel spinor if and only if there exists \( 0 \neq v \in \Delta_{p,q} \) such that

\[ \begin{cases} \mathcal{H} \cdot v := \lambda^{-1}(\mathcal{H}) \cdot v = \mathcal{H}_{A} v, \\ \mathcal{H}_{A} = \{0\} \text{ or } i\mathbb{R}. \end{cases} \]  

(12)

### 3.2 Proof of Theorem 3

Let \((M,g,S,P_{L},A)\) be a \( Spin^{c} \) structure where \((M,g)\) is a simply connected irreducible non-locally symmetric pseudo-Riemannian manifold of dimension \( n = p + q \) and signature \((p,q)\), which admits a non trivial parallel spinor \( \psi \).

We consider the two distributions \( T \) and \( E \) defined by

\[ T_{x} := \{ X \in T_{x}M; \ X \cdot \psi = 0 \}, \]
\[ E_{x} = \{ X \in T_{x}M; \ \exists Y \in T_{x}M; \ X \cdot \psi = iY \cdot \psi \}, \]

for \( x \in M \). Since \( \psi \) is parallel, By (7), \( T \) and \( E \) are parallel. Since \( T \) is isotropic and the manifold \((M,g)\) is supposed irreducible, by the holonomy principe, we have

\[ T = 0. \]  

(13)

Now denote by \( F \) the image of the Ricci tensor:

\[ F_{x} := \{ Ric(X); \ X \in T_{x}M \}. \]

Since \( \psi \) is parallel, (10) shows that

\[ Ric(X) \cdot \psi = iA(X) \cdot \psi. \]  

(14)
Then $F \subset E$. Consequently, from (13), we have
\[ E^\perp \subset F^\perp = \{ Y \in TM ; \ Ric(Y) = 0 \} = \{ Y \in TM ; A(Y) = 0 \}. \]

$(M,g)$ is supposed irreducible, by the holonomy principle, $E = 0$ or $E = TM$.
If $E = 0$, then $F = 0$. This gives $Ric = 0$ and $A = 0$. According to Remark 1, $(M,g)$ is Spin and $\psi$ is a parallel spinor on $M$.
If $E = TM$, we have a $(1,1)$-tensor $J$ definite by
\[ X \cdot \psi = iJ(X) \cdot \psi, \text{ where } X \in TM. \] (15)

**Lemma 1** If $(X + iY) \cdot \psi = 0$ then $g(X,Y) = 0$ and $g(X,X) = g(Y,Y)$.

**Proof.** See the proof in ([2], p. 65) for the Riemannian case who is valid for the pseudo-Riemannian case.

Lemma 1 implies that $J$ defines an orthogonal almost complex structure on $M$.
Moreover, from (7) and (15) we obtain $J$ is parallel, since $\psi$ is parallel. In consequence, $(M,g)$ is a kähler manifold.
Now if $(M,g)$ is a kähler manifold, then there exists a canonical $Spin^c$ structure of $(M,g)$. And from Remark 1, the following conditions are equivalent
(a) $(M,g)$ is not Spin,
(b) $H_A = S^1$,
(c) $(M,g)$ is not Ricci-flat,
(d) $H = U(p',q')$.
Then the equivalence between (i) and (ii) are proved. And from Theorem 2, we have the equivalence between (ii) and (iii). To finish the proof of Theorem 3, it remains to show for $H = U(p',q')$ that $N = 2$. For this, we remark that $U(p',q') = SU(p',q') \times U_{S^1}$, where
\[ U_{S^1} = \{ \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} ; \lambda \in S^1 \} \]
\[ u(p',q') = su(p',q') \oplus u_{S^1}, \text{ where } u_{S^1} \simeq i\mathbb{R} \text{ is the Lie algebra of } U_{S^1}. \]
If we consider the imbedding
\[ i : u(p',q') \rightarrow so(2p',2q') \]
\[ A + iB = ((a_{kl})_{1 \leq k,l \leq n} + i(b_{kl})_{1 \leq k,l \leq n}) \rightarrow \left( \begin{array}{cc} a_{kl} & b_{kl} \\ -b_{kl} & a_{kl} \end{array} \right)_{1 \leq k,l \leq n} \]
u_{S^1} is generated by $D_1 = E_{12}$. From [11], $u^+ := u(1,...,1)$ and $u^- := u(-1,...,-1)$ generate the space $V_{su(p',q')} = \{ v \in \Delta_{p,q} : su(p',q') \cdot v = 0 \}$. Moreover, by (1)
\[ D_1 \cdot u^+ = i u^+ \text{ and } D_1 \cdot u^- = -iu^-. \]
Then $u^+$ and $u^-$ generate the space

$$V_{u(p',q')} = \{ v \in \Delta_{p,q} : u(p',q') \cdot v = iR v \}.$$ 

And the proof of Theorem 3 is finished.

References

[1] Baum, H. and Kath, I.: Parallel spinors and holonomy groups on pseudo- Riemannian spin manifolds. Ann. Glob. Anal. Geom., 17:1-17 (1999).

[2] Friedrich, Th.: Dirac Operators in Riemannian Geometry. Graduate Studies in Mathematics. Volume 25. AMS Providence, Rhode Island 2000.

[3] Ikemakhen, A.: Groupes d’holonomie et spineurs paralleles sur les variétés pseudo-Riemanniennes complètement réductibles, C. R. Acad. Sci. Paris, Ser. I 339 (2004) 203-208.

[4] Ikemakhen, A.: Parallel Spinors on Lorentzian Spin$^c$ Manifolds, Préprint Université Cadi-Ayyad (2004).

[5] Lawson, H.B. and Michelsohn, M.-L.: Spin geometry Princeton Univ.Press 1989.

[6] Leistner, Th.: Holonomy and Parallel Spinors in Lorentzian Geometry, PhD thesis, Humbold-University of Berlin, (2003).

[7] Leistner, Th.: Towards a classification of Lorentzian holonomy groups. Part II: Semisimple, non-simple weak-Berger algebras, arXiv:math.DG/0309274 v1, (2003).

[8] Moroianu, A.: Parallel and Killing Spinors on Spin$^c$ Manifolds. Commun. Math. Phys. 187, 417-427 (1997).

[9] Wang, M.Y.: Parallel spinors and parallel forms, Ann. Global Anal. Geom. 7 (1989) 1, 59-68.