HARMONIOUS AND VERTEX GRACEFUL LABELING
ON PATH AND STAR RELATED GRAPHS

P. Selvaraju\textsuperscript{1}, P. Balaganesan\textsuperscript{2, \S}, J. Renuka\textsuperscript{3}, M.L. Suresh\textsuperscript{4}
\textsuperscript{1}\textsuperscript{}Department of Mathematics
Vel Tech. Multi Tech. Dr. Rangarajan Dr. Sankanthula
Engineering College
Avadi, Chennai, 600 062, INDIA
\textsuperscript{2,\textsuperscript{4}}Department of Mathematics
Hindustan University
Chennai, 603 103, INDIA
\textsuperscript{3}Departments of Mathematics
Sri Sai Ram Engineering College
INDIA

Abstract: In this paper, we show that $B^2(n, n)$ is harmonious \cite{4}, $P^n_i$ is harmonious \cite{3}, $P_n \times C_m$ is vertex graceful for $n \geq 2$, and $m \geq 5$, $m$ is odd, $B^2(n, n)$ is vertex graceful \cite{4}, $P^n_i$ is vertex graceful, \cite{3}, $L_n \circ K_1$ is vertex graceful $\forall n$ and $P_n \times P_2$ is vertex graceful, $n$ is odd \cite{3}.

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1. Introduction

Graph labeling, where the vertices are assigned values subject to certain conditions have often been motivated by practical problems. Labelled graphs serves as useful mathematical models for a broad range of applications such as coding
theory, including the design of good radar type codes, synch-set codes, missile guidance codes and convolution codes with optimal autocorrelation properties. They facilitate the optimal non standard encoding of integers.

All graphs in this paper are finite, simple graphs with no loops or multiple edges. The symbols \( V(G) \) and \( E(G) \) denote the vertex set and edge set of the graph \( G \). A graph with \( p \) vertices and \( q \) edges is called \( G(p,q) \) graph. Harmonious graphs naturally arose in the study by Graham and Sloane [1] of modular version of additive base problems stemming from error correction codes. They obtained some graphs are harmonious.

**Definition 1.1.** A Graph \( G \) is said to be harmonious if there exist an injection \( f : V(G) \to \mathbb{Z}_q \) such that the induced function \( f^* : E(G) \to \mathbb{Z}_q \) defined by \( f^*(uv) = (f(u) + f(v))(mod \ q) \) is a bijection from \( E(G) \) onto \( \mathbb{Z}_q \), then, \( f \) is said to be harmonious labeling of \( G \).

**Definition 1.2.** A graph \( G \) with \( p \) vertices and \( q \) edges is said to be vertex graceful if a labeling \( f : V(G) \to \{1, 2, 3 \ldots p\} \) exists in such a way that the induced labeling \( f^* : E(G) \to \mathbb{Z}_q \) defined by \( f^*(u,v) = f(u) + f(v)(mod \ q) \) is a bijection from \( E(G) \) onto \( \mathbb{Z}_q \). The concept of vertex graceful was introduced by Lee, Pan and Tsai in 2005.

**Definition 1.3.** For a simple connected graph \( G \) the Square of graph \( G \) is denoted by \( G^2 \) and defined as the graph with the same vertex set as of \( G \) and two vertices are adjacent in \( G^2 \) if they are at a distance 1 or 2 apart in \( G \).

2. **Main Results of Harmonious Labeling on Path and Star Related Graphs**

**Theorem 2.1.** The graph \( B^2(n, n) \) is harmonious \( \forall \ n \).

**Proof.** Consider \( B^2(n, n) \) with the vertex set \( \{u, v, u_i, v_i, 1 \leq i \leq n\} \) where \( u_i, v_i \) are the pendant vertices. Let \( G \) be the graph \( B^2(n, n) \), then \( |V(G)| = 2n + 2 \) and \( |E(G)| = 4n + 1 \). We define the vertex labeling \( f : V(G) \to \{0, 1, 2, 3 \ldots (q - 1)\} \) as follows:

\[
\begin{align*}
v &= 0, \quad u = 2n + 1 \\
v_i &= i, \quad 1 \leq i \leq n \\
u_i &= n + i, \quad 1 \leq i \leq n
\end{align*}
\]

Let \( A, B, C, D \) denote edge set.

\[
A = \{e_i = vv_i/e_i = i : 1 \leq i \leq n\}
\]
\[
B = \{e_i = uv_i/e_i = (2n + i + 1)(\text{mod } q) : 1 \leq i \leq n\}
\]
\[
C = \{e_i = vu_i/e_i = (n + i)(\text{mod } q) : 1 \leq i \leq n\}
\]
\[
D = \{e_i = uu_i/e_i = (3n + i + 1)(\text{mod } q) : 1 \leq i \leq n\}
\]

It is clear that vertex set labeling and edge set labeling are distinct. Hence, the \(B^2(n, n)\) is harmonious graph \(\forall n\). \(\square\)

**Illustration 2.2.** A harmonious graph \(B^2(7, 7)\) is shown in the figure 1.

![Figure 1: The graph \(B^2(7, 7)\)](image)

**Remark 2.3.** Let \(\alpha\) be the collection of paths \(P^m_i\), where \(n\) is odd, and \(P^m_i = u_1^i, u_2^i, ..., u_n^i, 1 \leq i \leq m\). Let \(G\) be the graph obtained from \(\alpha\) with \(V(G) = \bigcup_{i=1}^{m} V(P^m_i)\) and \(E(G) = \bigcup_{i=1}^{m} E(P^m_i) \cup (u_{n+1}^i u_{n+1}^{i+1})\)

**Theorem 2.4.** The graph \(P^m_i\) is harmonious graph, \(\forall n\).

**Proof.** Let \(G = P^m_i\) be a graph with \(p = 2n\) vertices and \(q = (2n - 1)\). The required vertex labeling \(f : V(G) \rightarrow \{0, 1, 2, \cdots, q - 1\}\) is as follows:

**case(i):** \(n\) is odd
\[
u_j^i = n(i - 1) + j - 1; 1 \leq j \leq n, i = 1, 2.
\]

Let \(A\) and \(B\) denote edge set.
\[
A = \{e_j^1 = u_j^1 u_{j+1}^2 / e_j^1 = (2n(i - 1) + 2j - 1)(\text{mod } q) : 1 \leq j \leq n - 1, i = 1, 2\}
\]
\[
B = \{e_{n+1}^1 = u_{n+1}^1 u_{n+1}^2 / e_{n+1}^1 = (2n + 1)(\text{mod } q)\}
\]

**case(ii):** \(n\) is even
\[
u_j^i = n(i - 1) + j - 1; 1 \leq j \leq n, 1 \leq i \leq m, i\) is odd
\[
u_j^i = ni - j; 1 \leq j \leq n, 1 \leq i \leq m, i\) is even

Let \(A, B, C\) are denote edge set.
\( A = \{ e^j_i = u^j_i u^{j+1}_i / e^j_i = (2n(i-1) + 2j -1) (mod\ q) : 1 \leq j \leq n-1, 1 \leq i \leq m, i \text{ is odd} \} \)

\( B = \{ e^j_i = u^j_i u^{i+1}_i / e^j_i = (2(ni - j) - 1) (mod\ q) : 1 \leq j \leq n-1, 1 \leq i \leq m, \ i \text{ is even} \} \)

\( C = \{ e^n_i = u^n_i u^{i+1}_i / e^n_i = (2ni - 1) (mod\ q) \} \)

It is clear that vertex set labeling and edge set labeling are distinct. Hence, the graph \( G \) is harmonious graph \( \forall n \).

**Illustration 2.5.** A harmonious graph \( P^5_i (i = 1, 2) \) is shown in the figure 2.

![Image](image.png)

Figure 2: The graph \( P^5_i (i = 1, 2) \)

3. Main Results of Vertex Graceful labeling on Path and Star Related Graphs

**Theorem 3.1.** The graph \( P_n \times C_m \) is a vertex graceful graph, \( \forall n, n \geq 2 \) and \( m \geq 5, m \) is odd.

**Proof.** Consider the graph \( G = P_n \times C_m \) with \( nm \) vertices and \( q = (2n-1)m \) edges. Suppose that the vertices \( v^j_i; 1 \leq i \leq m \) and \( j = 0, 1, 2, \cdots , n \) of the cycle \( C_m \) run consecutively with \( v^1_1 \) joined to \( v^n_m \). The required vertex labeling \( f : V(G) \to 1, 2, \cdots , p \) is as follows:

\[
 v^j_i = \begin{cases} 
\frac{n-1}{2} + \frac{1+i}{2} + nj; & 1 \leq i \leq n \text{ and } i \text{ is odd}, \ 1 \leq j \leq m \\
\frac{1+i}{2} + \frac{nj}{2}; & 1 \leq i \leq n \text{ and } i \text{ is odd and } j = 0, 2, 4, \cdots , n 
\end{cases}
\]
Let $A, B, C$ denote the edge set.

$A = \{e_i^j = v_i^j v_i^{j+1} = \left(3n + 1 \over 2 + 2n j + i\right) \pmod{q} | 1 \leq i \leq n, j = 0, 1, 2, \ldots, n-1\}$

$B = \{e_i^j = v_i^j v_k^j = \left(n + 1 \over 2 + 2n j + k\right) \pmod{q} / k = (1 + i)(\pmod{n}) | 1 \leq i \leq n, j = 0, 2, 4, \ldots, n\}$

$C = \{e_i^j = v_i^j v_i^{j+1} = \left(n + 1 \over 2 + 2n j + i\right) \pmod{q} / 1 \leq i \leq n, 1 \leq j \leq m and j is odd\}$

It is clear that vertex set labeling and edge set labeling are distinct. Hence, the $P_n \times C_m$ is a vertex graceful graph, for $n \geq 2$ and $m \geq 5$, $m$ is odd.

**Illustration 3.2.** A vertex graceful graph $P_3 \times C_5$ is shown in the figure 3.

![Figure 3: The graph $P_3 \times C_5$](image)

**Theorem 3.3.** The graph $B^2(n, n)$ is vertex graceful graph $\forall n$.

**Proof.** Consider $B^2(n, n)$ with the vertex set $\{u, v, u_i, v_i, 1 \leq i \leq n\}$ where $u_i, v_i$ are the pendant vertices. Let $G$ be the graph $B^2(n, n)$ then $|V(G)| = 2n + 2$ and $|E(G)| = 4n + 1$. We define the vertex labeling $f : V(G) \rightarrow \{1, 2, ..., p\}$ as follows.

**case(i):** $n$ is odd

$v = 1, u = 2n + 2$

$v_i = i + 1, 1 \leq i \leq n$
\[ u_i = n + i + 1, 1 \leq i \leq n. \]

Let \( A, B, C, D \) denote edge set.

\[ A = \{ e_i = vv_i/e_i = i + 2 : 1 \leq i \leq n \} \]

\[ B = \{ e_i = uu_i/e_i = (2n + i + 3)(mod \ q) : 1 \leq i \leq n \} \]

\[ C = \{ e_i = vu_i/e_i = (n + i + 2)(mod \ q) : 1 \leq i \leq n \} \]

\[ D = \{ e_i = uu_i/e_i = (3n + i + 3)(mod \ q) : 1 \leq i \leq n \} \]

**case (ii):** n is even

\[ u_i^j = n(i - 1) + j; 1 \leq j \leq n, 1 \leq i \leq m, \text{ i is odd} \]

\[ u_i^j = ni - j + 1; 1 \leq j \leq n, 1 \leq i \leq m, \text{ i is even} \]

Let \( A, B, C \) are denote edge set.

\[ A = \{ e_i^j = u_i^j u_{i+1}^j/e_i^j = (2n(i - 1) + 2j + 1)(mod \ q) : 1 \leq j \leq n - 1, 1 \leq i \leq m, \text{ i is odd} \} \]

\[ B = \{ e_i^j = u_i^j u_{i+1}^j/e_i^j = (2(ni - j) + 1)(mod \ q) : 1 \leq j \leq n - 1, 1 \leq i \leq m, \text{ i is even} \} \]

\[ C = \{ e_i^j = u_i^j u_{i+1}^j/e_i^j = (2ni + 1)(mod \ q) \} \]

It is clear that vertex set labeling and edge set labeling are distinct. Hence, the \( B^2(n, n) \) is vertex graceful graph. \( \square \)

**Illustration 3.4.** A vertex graceful graph \( B^2(7, 7) \) is shown in the figure 4.

![Figure 4: B^2(7, 7)](image)

**Theorem 3.5.** The graph \( P_n^i \) is vertex graceful graph \( \forall n, n \) is odd.

**Proof.** Let \( G = P_n^i \) be a graph with \( p = 2n \) vertices and \( q = (2n - 1) \). The required vertex labeling \( f : V(G) \to \{1, 2, ..., p\} \) is as follows:

**case (i):** n is odd
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\[ u_j^i = n(i - 1) + j; 1 \leq j \leq n, i = 1, 2. \]

Let \( A \) and \( B \) denote edge set.
\[ A = \{ e_{j}^{i} = u_{j+1}^{i}/e_{j}^{i} = (2n(i - 1) + 2j + 1)(mod \ q) : 1 \leq j \leq n - 1, \ i = 1, 2 \} \]
\[ B = \{ e_{n+1}^{i} = u_{n+1}^{i}/e_{n}^{i} = (2n + 1)(mod \ q) \} \]

It is clear that vertex set labeling and edge set labeling are distinct. Hence the graph \( G \) vertex graceful for all \( n \), \( n \) is odd.

**Case (ii):** \( n \) is even, \( 1 \leq i \leq m \)

\[ u_j^i = n(i - 1) + j; 1 \leq j \leq n, 1 \leq i \leq m, \ i \text{ is odd} \]
\[ u_j^i = ni - j + 1; 1 \leq j \leq n, 1 \leq i \leq m, \ i \text{ is even} \]

Let \( A, B, C \) denote edge set.
\[ A = \{ e_{j}^{i} = u_{j+1}^{i}/e_{j}^{i} = (2n(i - 1) + 2j + 1)(mod \ q) : 1 \leq j \leq n - 1, 1 \leq i \leq m, \ i \text{ is odd} \} \]
\[ B = \{ e_{n+1}^{i} = u_{n+1}^{i}/e_{n}^{i} = (2ni - j + 1)(mod \ q) : 1 \leq j \leq n - 1, 1 \leq i \leq m, \ i \text{ is even} \} \]
\[ C = \{ e_{n}^{i} = u_{n}^{i}/e_{n}^{i} = (2ni + 1)(mod \ q) \} \]

It is clear that vertex set labeling and edge set labeling are distinct. Hence, the graph \( G \) is vertex graceful graph \( \forall n \). \( \square \)

**Illustration 3.6.** A vertex graceful graph \( P_5^i \) \((i = 1, 2)\) is shown in the figure 5.

![Figure 5: P_5^i(i = 1, 2)](image)

**Definition 3.7.** The graph \( L_n = P_n \times P_2 \) is called the ladder.

**Theorem 3.8.** The graph \( L_n \circ K_1 \) is a vertex graceful \( \forall n \).

**Proof.** consider the graph \( G = L_n \circ K_1 \). Let \( V(L_n) = u_i, v_i : 1 \leq i \leq n \).
\[ E(L_n) = u_i v_i = 1 \leq i \leq n - 1 \cup u_i u_{i+1} : 1 \leq i \leq n - 1 \]
∪ \nu_i\nu_{i+1} : 1 \leq i \leq n - 1. Let \nu_i be pendent vertex adjacent to \nu_i and let \pi_i be the pendent vertex adjacent to \nu_i. The required vertex labeling \( f : V(G) \rightarrow \{1, 2, \ldots, p\} \) is as follows:

\[
\begin{align*}
\nu_i &= 5i - 4; 1 \leq i \leq n \\
\pi_i &= 5i - 3; 1 \leq i \leq n \\
\nu_i &= 5i - 5; 1 \leq i \leq n \\
\pi_i &= 5i - 2; 1 \leq i \leq n
\end{align*}
\]

Let \( A, B, C, D, E \) denote the edge set.

\[
\begin{align*}
A &= \{e_i = u_i\nu_i/e_i = 10i - 9(\mod q) : 1 \leq i \leq n\}, \\
B &= \{e_i = u_i\pi_i/e_i = 10i - 7(\mod q), 1 \leq i \leq n\} \\
C &= \{e_i = \nu_i\pi_i/e_i = 10i - 5(\mod q) : 1 \leq i \leq n\}, \\
D &= \{e_i = u_i\nu_{i+1}/e_i = 10i - 3(\mod q) : 1 \leq i \leq n\} \\
E &= \{e_i = \nu_i\nu_{i+1}/e_i = 10i - 1(\mod q) : 1 \leq i \leq n\}
\end{align*}
\]

It is clear that vertex set labeling and edge set labeling are distinct. Then the graph \( G = L_n \circ K_1 \) is vertex graceful \( \forall n \).

**Theorem 3.9.** The graph \( P_n \times P_2 \) is a vertex graceful \( \forall n, n \) is odd

**Proof.** Consider the graph \( G = P_n \times P_2 \) with \( 2n \) vertices and \( q = 3n - 2 \) edges. Let \( v_{1j} \) and \( v_{2j} \) be the first and second row vertices of \( G \) respectively for \( 1 \leq j \leq n \). The required vertex labeling \( f : V(G) \rightarrow \{1, 2, \cdots, p\} \) is as follows:

\[
\begin{align*}
v_{1j} &= \frac{j+1}{2}, 1 \leq j \leq n; j \text{ is odd} \\
v_{1j} &= \frac{n+j+1}{2}, 1 \leq j \leq n; j \text{ is even} \\
v_{2j} &= \frac{3n+j}{2}, 1 \leq j \leq n; j \text{ is odd} \\
v_{2j} &= \frac{2n+j}{2}, 1 \leq j \leq n; j \text{ is even}
\end{align*}
\]

Let \( A, B, C \) denote edge set.

\[
\begin{align*}
A &= \{e_j = v_{1j}v_{1j+1}/e_j = \frac{n+2j+3}{2}(\mod q) : 1 \leq j \leq n - 1\} \\
B &= \{e_j = v_{2j}v_{2j+1}/e_j = \frac{5n+2j+1}{2}(\mod q) : 1 \leq j \leq n - 1\} \\
C &= \{e_j = v_{1j}v_{2j}/e_j = \frac{3n+1+2j}{2}(\mod q) : 1 \leq j \leq n\}
\end{align*}
\]

It is clear that vertex set labeling and edge set labeling are distinct. Hence, the graph \( P_n \times P_2 \) is a vertex graceful \( \forall n, n \) is odd.

**Illustration 3.10.** A vertex graceful graph \( P_7 \times P_2 \) is shown in the figure 6.
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