NONPERTURBATIVE INFRARED MULTIPLICATIVE RENORMALIZABILITY OF TWO-DIMENSIONAL COVARIANT GAUGE QCD

V. Gogohia, Gy. Kluge and J. Nyiri
HAS, CRIP, RMKI, Depart. Theor. Phys., Budapest 114, P.O.B. 49, H-1525, Hungary
email addresses: gogohia@rmki.kfki.hu, kluge@rmki.kfki.hu and nyiri@rmki.kfki.hu

Abstract

A nonperturbative approach to two-dimensional covariant gauge QCD is presented in the context of the Schwinger-Dyson equations and the corresponding Slavnov-Taylor identities. The distribution theory, complemented by the dimensional regularization method, is used in order to treat correctly the infrared singularities which inevitably appear in the theory. By working out the multiplicative renormalization program we remove them from the theory on a general ground and in a self-consistent way, proving thus the infrared multiplicative renormalizability of two-dimensional QCD within our approach. We also show explicitly how to formulate the bound-state problem and the Schwinger-Dyson equations for the gluon propagator and the triple gauge field proper vertex, all free from the infrared singularities.

PACS numbers: 11.15.Tk, 12.38.Aw, 12.38.Lg
I. INTRODUCTION

It is well known that two-dimensional (2D) QCD is an ultraviolet (UV), i.e., perturbatively (PT) super-renormalizable field theory [1,2]. However, our investigation of 2D covariant gauge QCD [3] within the distribution theory (DT) [4] clearly shows that this theory is infrared (IR) divergent as well since its free gluon propagator IR singularity is nonperturbative (NP), i.e., is a severe one. For that very reason, it becomes inevitable firstly to regularize it and secondly to prove its IR renormalizability, i.e., to prove that all the NP IR singularities can be removed from the theory on a general ground and in a self-consistent way. This is just the primary goal of this paper which is a direct continuation of our previous work [3]. In order to formulate the IR multiplicative renormalization (IRMR) program in 2D QCD, it is reasonable to start with the explanation within our approach why the IR structure of 2D QCD is so singular.

Let us consider the Schwinger-Dyson (SD) equation for the quark propagator (PT unrenormalized for simplicity in order not to complicate notations) in momentum space with Euclidean signature

\[ S^{-1}(p) = S_0^{-1}(p) - g^2 C_F i \int \frac{d^m q}{(2\pi)^n} \Gamma_\mu(p, q) S(p - q) \gamma_\mu D_{\mu\nu}^0(q), \]

where \( C_F \) is the eigenvalue of the quadratic Casimir operator in the fundamental representation (for \( SU(N_c) \), in general, \( C_F = (N_c^2 - 1)/2N_c = 4/3 \) and \( S_0^{-1}(p) = i(\hat{p} + m_0) \) with \( m_0 \) being the current ("bare") mass of a single quark. \( \Gamma_\mu(p, q) \) is the corresponding quark-gluon proper vertex function. Instead of the simplifications due to the limit \( N_c \to \infty \) at fixed \( g^2 N_c \) and light-cone gauge [5,6], we have used the free gluon propagator in the covariant gauge from the very beginning. This makes it possible to maintain the direct interaction of massless gluons which is the main dynamical effect in QCD of any dimensions. In the covariant gauge it is

\[ D_{\mu\nu}^0(q) = i(g_{\mu\nu} + (\xi - 1)\frac{q_\mu q_\nu}{q^2}) \frac{1}{q^2}, \]

where \( \xi \) is the gauge fixing parameter.

The important observation now is that for the free gluon propagator the exact singularity \( 1/q^2 \) at \( q^2 \to 0 \) in 2D QCD is severe and therefore it should be correctly treated within the DT [4,7] (in Ref. [7] some fundamental results of pure mathematical tractate on the DT [4] necessary for further purpose are presented in a suitable form). In order to actually define the system of SD equations in the IR region, it is convenient to apply the gauge-invariant dimensional regularization (DR) method of ’t Hooft and Veltman [8] in the limit \( D = 2 + 2\epsilon, \epsilon \to 0^+ \). Here and below \( \epsilon \) is the small IR regularization parameter which is to be set to zero at the end of computations. This severe singularity should be treated in the sense of the DT (i.e., under integrals, taking into account the smoothness properties of the corresponding test functions), where the relation [4,7]

\[ (q^2)^{-1} = \frac{\pi}{\epsilon} \delta^2(q) + \text{finite terms}, \quad \epsilon \to 0^+, \]

holds. We point out that after introducing this expansion here and everywhere one can fix the number of dimensions, i.e., put \( D = n = 2 \) without any further problems since there
will be no other severe IR singularities with respect to $\epsilon$ as $\epsilon \to 0^+$ in the corresponding SD equations but those explicitly shown in this expansion.

Let us make a few remarks. It is worth emphasizing that the IR singularity (1.3) is unique, the simplest IR singularity possible in 2D QCD, however, it is a NP (severe) singularity at the same time [4,7]. In other words, the free gluon propagator may serve as a rather good approximation to the full gluon propagator, at least in the deep IR region, since it exactly reproduces a possible IR singularity of the full gluon propagator. In this connection, let us remind that in 4D QCD the free gluon’s IR singularity is not severe, i.e., the Laurent expansion (1.3) does not exist in this case, so it is a PT singularity there. Secondly, the DT theory clearly shows that each skeleton diagram (which itself is the sum of infinite series of diagrams) explicitly diverges as $1/\epsilon$ (see Eqs. (1.2)-(1.3)). This is true even for skeleton diagrams containing the scattering kernels which in their turn are defined by their own skeleton expansions (see below).

II. IRMR PROGRAM IN THE QUARK-GHOST SECTOR

In the presence of such a severe singularity (1.3) all Green’s functions and coupling constant (which in 2D QCD has the dimensions of mass) become generally dependent on the IR regularization parameter $\epsilon$, i.e., they become IR regularized (for simplicity, this dependence is not shown explicitly in what follows). The relations between the IR regularized and the IR renormalized (denoted with bars and which exist as $\epsilon$ goes to zero, by definition) quantities determine the corresponding IRMR constants. Let us remind some of these relations introduced in Ref. [3]: $\Gamma_{\mu}(p, q) = Z_1^{-1}(\epsilon)\bar{\Gamma}_{\mu}(p, q)$, $g^2 = X(\epsilon)\bar{g}^2$ and $S(p) = Z_2(\epsilon)\bar{S}(p)$. Here and below $Z_1(\epsilon)$, $Z_2(\epsilon)$ and $X(\epsilon)$ are the above-mentioned IRMR constants of the vertex, quark propagator and coupling constant squared, respectively. The $\epsilon$-dependence is indicated explicitly in order to distinguish them from the usual ultraviolet (UV) renormalization constants. In all relations containing the IRMR constants, the $\epsilon \to 0^+$ limit is always assumed at the final stage. The similar relations should be introduced for the ghost degrees of freedom [3].

In their turn, the relations between these constants (established by passing to the IR renormalized quantities in the corresponding SD equations and Slavnov-Taylor (ST) identities) determine the corresponding IR convergence conditions. The system of the corresponding IR convergence conditions in the quark-ghost sector consists of the quark SD, the ghost self-energy and the quark ST identity conditions. They are [3]:

$$X(\epsilon)Z_2^2(\epsilon)Z_1^{-1}(\epsilon) = \epsilon,$$

$$X(\epsilon)\bar{Z}_1(\epsilon)\bar{Z}_2^{-1}(\epsilon) = \epsilon, \quad \epsilon \to 0^+,$$

$$Z_1^{-1}(\epsilon)\bar{Z}(\epsilon) = Z_2^{-1}(\epsilon).$$

Thus, in general, we have five independent IRMR constants. $\bar{Z}_1(\epsilon)$ and $\bar{Z}_2(\epsilon)$ determine the renormalization of the ghost-gluon proper vertex and ghost propagator, respectively, i.e., $G_{\mu} = \bar{Z}_1(\epsilon)\bar{G}_{\mu}$ and $G = \bar{Z}_2(\epsilon)\bar{G}$. For simplicity, here and below we omit the dependence on the corresponding momenta. We know that the quark wave function IRMR constant $Z_2(\epsilon)$ cannot be singular, while the ghost self-energy IRMR constant $\bar{Z}(\epsilon) = \bar{Z}_2^{-1}(\epsilon)$, by definition, is either singular or constant as $\epsilon \to 0^+$. It is evident that these very conditions...
and similar ones below govern the concrete \( \epsilon \)-dependence of the IRMR constants which in general remain arbitrary.

So we have three conditions for the above-mentioned five independent IRMR constants. Obviously, this system always has a nontrivial solution determining three of them in terms of two any chosen independent IRMR constants. It is convenient to choose \( \tilde{Z}(\epsilon) \) and \( Z_2(\epsilon) \) as the two independent IRMR constants since we know their possible behavior with respect to \( \epsilon \) as it goes to zero. Then a general solution to the system (2.1) can be written down as follows:

\[
X(\epsilon) = \epsilon Z_2^{-1}(\epsilon)\tilde{Z}(\epsilon), \quad Z_1(\epsilon) = \tilde{Z}_1(\epsilon) = Z_2(\epsilon)\tilde{Z}(\epsilon).
\]

Thus in the quark-ghost sector the self-consistent IRMR program really exists. Moreover, it has room for additional specifications. The most interesting case is the quark propagator which is IR finite from the very, i.e., when the quark wave function IRMR constant \( Z_2(\epsilon) = \text{const} \). The second interesting case is when \( \tilde{Z}(\epsilon) = KZ_2^{-1}(\epsilon) \), where \( K \) is an arbitrary but finite constant (see section IV). However, in Ref. [3] it has been shown explicitly that by redefining all the IRMR constants, all finite but arbitrary constants like \( Z_2 \) and \( K \) can be put to unity without losing generality.

### III. IR FINITE ST IDENTITIES FOR PURE GLUON VERTICES

In order to determine the IR finite bound-state problem within the Bethe-Salpeter (BS) formalism, it is necessary to know the IRMR constants of the three- and four-gluon proper vertex functions which satisfy the corresponding ST identities [1,9-13]. This information is also necessary to investigate the IR properties of all other SD equations in 2D QCD. It is convenient to start from the ST identity for the three-gluon vertex which is [9,10]

\[
[1 + b(k^2)]k_\lambda T_{\lambda\mu\nu}(k, q, r) = d^{-1}(q^2)G_{\lambda\nu}(q, k)(g_{\lambda\mu}q^2 - q_\lambda q_\mu) + d^{-1}(r^2)G_{\lambda\mu}(r, k)(g_{\lambda\nu}r^2 - r_\lambda r_\nu), \tag{3.1}
\]

where \( k + q + r = 0 \) is understood and \( d^{-1} \) is the inverse of the exact gluon form factor, while \( G \)'s are the corresponding ghost-gluon vertices, and \( b(k^2) \) is the ghost self-energy. Let us now introduce the IR renormalized triple gauge field vertex as follows: \( T_{\lambda\mu\nu}(k, q, r) = Z_3(\epsilon)\tilde{T}_{\lambda\mu\nu}(k, q, r) \), where \( \tilde{T}_{\lambda\mu\nu}(k, q, r) \) exists as \( \epsilon \) goes to zero. Passing to the IR renormalized quantities, one obtains

\[
[\tilde{Z}^{-1}(\epsilon) + \tilde{b}(k^2)]k_\lambda \tilde{T}_{\lambda\mu\nu}(k, q, r) = \tilde{G}_{\lambda\nu}(q, k)d^{-1}(q^2)(g_{\lambda\mu}q^2 - q_\lambda q_\mu) + \tilde{G}_{\lambda\mu}(r, k)d^{-1}(r^2)(g_{\lambda\nu}r^2 - r_\lambda r_\nu), \tag{3.2}
\]

so that the following IR convergence relation holds

\[
Z_3(\epsilon) = \tilde{Z}^{-1}(\epsilon)\tilde{Z}_1(\epsilon). \tag{3.3}
\]
In the derivation of the relation (3.3) (as well as those similar below) we use \( b = \tilde{Z}(\epsilon)\tilde{b} \) and \( G_{\lambda\nu} = \tilde{Z}_1(\epsilon)\tilde{G}_{\lambda\nu} \) [3]. Here and below we are considering the inverse of the free gluon propagator as IR finite from the very beginning, i.e., \( d^{-1} \equiv \bar{d}^{-1} = 1 \). It is not a singularity at all and therefore it should not be treated as a distribution [4] (there is no integration over its momentum).

The corresponding ST identity for the quartic gauge field vertex is [9,10]

\[
[1 + b(p^2)]\frac{1}{p\lambda} T_{\lambda\mu\nu\delta}(p, q, r, s) = d^{-1}(q^2)(g_{\lambda\mu}q^2 - q_{\lambda}q_{\mu})B^g_{\lambda\nu\delta}(q, p; r, s) + d^{-1}(r^2)(g_{\lambda\nu}r^2 - r_{\lambda}r_{\nu})B^g_{\mu\lambda\delta}(r, p; q, s) + d^{-1}(s^2)(g_{\lambda\delta}s^2 - s_{\lambda}s_{\delta})B^g_{\lambda\mu\nu}(s, p; q, r) - T_{\mu\lambda\delta}(q, s, -q, -s)G_{\lambda\nu}(q + s, p, r) - T_{\mu\nu\lambda}(q, r, -q, -r)G_{\lambda\delta}(q + r, p, s) - T_{\nu\delta\lambda}(r, s, -r, -s)G_{\lambda\mu}(r + s, p, q),
\]

(3.4)

where \( p + q + r + s = 0 \) is understood. Here T’s and G’s are the corresponding three- and ghost-gluon vertices, respectively. The quantity \( B^g \) with three Dirac indices is the corresponding ghost-gluon scattering kernel which is shown in Fig. 1 (see also Refs. [1,11]).

Let us introduce now its IR renormalized counterpart as follows: \( B^g_{\lambda\nu\delta}(q, p; r, s) = \tilde{Z}_g(\epsilon)\tilde{B}^g_{\lambda\nu\delta}(q, p; r, s) \), where \( \tilde{B}^g_{\lambda\nu\delta}(q, p; r, s) \) exists as \( \epsilon \) goes to zero. From the decomposition of the ghost-gluon proper vertex shown in Fig. 2, it follows that \( \tilde{Z}_1(\epsilon) = (1/\epsilon^2)X(\epsilon)\tilde{Z}_2(\epsilon)\tilde{Z}_g(\epsilon) \), so that

\[
\tilde{Z}_g(\epsilon) = \epsilon X^{-1}(\epsilon)\tilde{Z}(\epsilon)\tilde{Z}_1(\epsilon).
\]

(3.5)

Let us remind that a factor \( \sqrt{X(\epsilon)} \) should be additionally assigned to each ghost-gluon vertex, while to the scattering kernel \( B^g \) with two gluon legs a factor \( X(\epsilon) \) should be additionally assigned.
FIG. 2. The decomposition of the ghost-gluon proper vertex. Here and in all figures $D \rightarrow D^0$ is understood.

Let us now introduce the IR renormalized four-gluon gauge field vertex as follows: $T_{\lambda\mu\nu\delta}(p, q, r, s) = Z_4(\epsilon)\bar{T}_{\lambda\mu\nu\delta}(p, q, r, s)$, where $\bar{T}_{\lambda\mu\nu\delta}(p, q, r, s)$ exists as $\epsilon$ goes to zero. Passing again to the IR renormalized quantities, one obtains

\[
(\tilde{Z}^{-1}(\epsilon) + \bar{b}(p^2))p_\lambda T_{\lambda\mu\nu\delta}(p, q, r, s) = d^{-1}(q^2)(g_{\lambda\mu}q^2 - q_{\lambda}q_{\mu})B_{\lambda\mu\delta}^g(q, p; r, s) + d^{-1}(r^2)(g_{\lambda\nu}r^2 - r_{\lambda}r_{\nu})\bar{B}_{\lambda\mu\delta}^g(r, p; q, s) + d^{-1}(s^2)(g_{\lambda\delta}s^2 - s_{\lambda}s_{\delta})\bar{B}_{\lambda\mu\delta}^g(s, p; q, r) - \tilde{T}_{\mu\lambda\delta}(q, s, -q, -s)G_{\lambda\mu}(q + s, p, r) - \tilde{T}_{\mu\nu\lambda}(q, r, -q, -r)\bar{G}_{\lambda\delta}(q + r, p, s) - \tilde{T}_{\nu\delta\lambda}(r, s, -r, -s)\bar{G}_{\lambda\mu}(r + s, p, q),
\]

iff

\[
Z_4(\epsilon) = Z_3^2(\epsilon) = \tilde{Z}^{-2}(\epsilon)\tilde{Z}_1^2(\epsilon).
\]

Evidently, in the derivation of this expression the general solution (2.2) has been used as well as Eqs. (3.3) and (3.5). Thus we have determined the IRMR constants of the triple and quartic gauge field vertices in Eqs. (3.3) and (3.7), respectively.

IV. IR FINITE BOUND-STATE PROBLEM

Apart from quark confinement and DBCS, the bound-state problem is one of the most important NP problems in QCD. The general formalism for considering it in quantum field theory is the BS equation (see Ref. [14] and references therein). For the color-singlet, flavor-nonsinglet bound-state amplitudes for mesons it is shown in Figs. 3 and 4. Flavor-singlet mesons require a special treatment, since pairs, etc. of gluons in color-singlet states can contribute to the direct-channel processes. The exact BS equation for the bound-state meson amplitude $B(p, p')$ can be written analytically as follows:
\[ S_{q}^{-1}(p)B(p, p')S_{\bar{q}}^{-1}(p') = \int d^n l K(p, p'; l) B(p, p'; l), \quad (4.1) \]

(for simplicity all numerical factors are suppressed), where \( S_{q}^{-1}(p) \) and \( S_{\bar{q}}^{-1}(p') \) are inverse quark and antiquark propagators, respectively, and \( K(p, p'; l) \) is the two-particle irreducible (2PI) BS scattering kernel (its skeleton expansion is shown in Fig. 4) which precisely defines the BS equation itself. The BS equation is a homogeneous linear integral equation for the \( B(p, p') \) amplitude. For this reason the meson bound-state amplitude should always be considered as IR finite from the very beginning, i.e., \( B(p, p') \equiv \tilde{B}(p, p') \). Passing as usual [3] to the IR renormalized quantities in this equation, one obtains

\[ \tilde{S}_{q}^{-1}(p)B(p, p')\tilde{S}_{\bar{q}}^{-1}(p') = \int d^n l \tilde{K}(p, p'; l) B(p, p'; l), \quad (4.2) \]

iff

\[ Z_{2}^{-2}(\epsilon) = Z_{K}(\epsilon), \quad \epsilon \to 0^{+}, \quad (4.3) \]

where we introduce the IRMR constant \( Z_{K}(\epsilon) \) of the BS scattering kernel as follows: \( K(p, p'; l) = Z_{K}(\epsilon)\tilde{K}(p, p'; l) \) and \( \tilde{K}(p, p'; l) \) exists as \( \epsilon \to 0^{+} \). This is the exact BS equation IR convergence condition.

In general, the nth skeleton diagram of the BS equation skeleton expansion contains \( n \) independent loop integrations over the gluon momentum which, as we already know, generate a factor \( 1/\epsilon \) each, \( n_1 \) quark-gluon vertex functions and \( n_2 \) quark propagators. Also it contains \( n_3 \) and \( n_4 \) three and four-gluon vertices, respectively. It is worth reminding that to each

\[ \]
quark-gluon vertex and three-gluon vertex a factor $\sqrt{X(\epsilon)}$ should be additionally assigned, while to the four-gluon vertex a factor $X(\epsilon)$ should be assigned. Thus the corresponding IRMR constant is equal to

$$Z_K^{(n)}(\epsilon) = \epsilon^{-n} \left[ Z_1^{-1}(\epsilon) \right]^{n_1} \left[ Z_2(\epsilon) \right]^{n_2} \left[ Z_3(\epsilon) \right]^{n_3} \left[ Z_4(\epsilon) \right]^{n_4} \left[ X(\epsilon) \right]^{n_4+(n_3+n_1)/2}, \quad \epsilon \to 0^+. \quad (4.4)$$

On the other hand, it is easy to see that for each skeleton diagram the following relations hold $n_2 = n_1 - 2, \quad 2n = n_1 + n_3 + 2n_4$. Substituting these relations into the previous expression as well as using the general solution (2.2) and taking into account results of the previous section, one finally obtains $Z_K^{(n)}(\epsilon) = Z_2^{-2}(\epsilon) \left[ Z_2(\epsilon) \tilde{Z}(\epsilon) \right]^{n-n_3}_1$, so that from Eq. (4.3) it follows that $\left[ Z_2(\epsilon) \tilde{Z}(\epsilon) \right]^{n-n_3}_1 = A_{(n)}$, where $A_{(n)}$ is an arbitrary but finite constant different, in principle, for each skeleton diagram. Evidently, its solution is $\tilde{Z}(\epsilon) = \left[ A_{(n)} \right]^{-1\cdot(n-1)} Z_2^{-1}(\epsilon)$. Let us emphasize now that the relation between these (and all other) IRMR constants cannot depend on which skeleton diagram is considered. This means that the above-mentioned arbitrary but finite constant can only be a common factor for all skeleton diagrams, i.e., $\left[ A_{(n)} \right]^{-1\cdot(n-1)} = K$, where $K$ can be put to unity not losing generality as we already know. Thus the solution becomes

$$\tilde{Z}(\epsilon) = Z_2^{-1}(\epsilon). \quad (4.5)$$

Summarizing, the general system of the IR convergence conditions for removing at this stage all the severe IR singularities on a general ground and in a self-consistent way from the theory is

$$X(\epsilon) = \epsilon Z_2^{-2}(\epsilon), \quad \tilde{Z}_1^{-1}(\epsilon) = \tilde{Z}_2(\epsilon) = Z_3(\epsilon) = \tilde{Z}_4(\epsilon) = Z_4(\epsilon), \quad \tilde{Z}_1(\epsilon) = Z_1^{-1}(\epsilon) = 1, \quad Z_4(\epsilon) = Z_3(\epsilon) = Z_2(\epsilon), \quad (4.6)$$

and the limit $\epsilon \to 0^+$ is always assumed. This system provides the cancellation of all the severe IR singularities in 2D QCD at this stage, and what is most important this system provides the IR finite bound-state problem within our approach. All the IRMR constants are expressed in terms of the quark wave function IRMR constant $Z_2(\epsilon)$ except the quark-gluon and ghost-gluon proper vertices IRMR constants. They have been fixed to be unity though we were unable to investigate the corresponding ST identity for the latter vertex [3], so we avoided this difficulty.

**V. IR FINITE SD EQUATION FOR THE GLUON PROPAGATOR**

Let us now investigate the IR properties of the SD equation for the gluon propagator which is shown diagrammatically in Fig. 5 (see also Refs. [15,16] and references therein). Analytically it can be written down as follows:

$$D^{-1}(q) = D_0^{-1}(q) - \frac{1}{2} T_1(q) - \frac{1}{2} T_1(q) - \frac{1}{2} T_2(q) - \frac{1}{2} T_2(q) + \frac{1}{6} T_3(q) + T_{gh}(q) + T_q(q), \quad (5.1)$$

where numerical factors are due to combinatorics and, for simplicity, Dirac indices determining the tensor structure are omitted. $T_i$ (the so-called tadpole term) and $T_1$ describe
one-loop contributions, while \( T_2 \) and \( T'_2 \) describe two-loop contributions containing three- and four-gluon proper vertices, respectively. Evidently, \( T_{gh}, T_q \) describe ghost- and quark-loop contributions.

Equating \( D = D^0 \) now and passing as usual to the IR renormalized quantities, one obtains

\[
\frac{1}{\epsilon} X(\epsilon) \frac{1}{2} \bar{T}_t(q) + \frac{1}{\epsilon} X(\epsilon) Z_3(\epsilon) \frac{1}{4} \bar{T}_1(q) + \frac{1}{\epsilon^2} X^2(\epsilon) Z_3^2(\epsilon) \frac{1}{2} \bar{T}_2(q) + \frac{1}{\epsilon^2} X^2(\epsilon) Z_4(\epsilon) \frac{1}{6} \bar{T}_2'(q) \\
- \frac{1}{\epsilon} X(\epsilon) \tilde{Z}_2^2 \tilde{Z}_1(\epsilon) \bar{T}_{gh}(q) - X(\epsilon) Z_2^2(\epsilon) Z_1^{-1}(\epsilon) \bar{T}_q(q) = 0,
\]

(5.2)

where the quantities with bar are, by definition, IR renormalized, i.e., they exist as \( \epsilon \to 0^+ \).

Let us also remind that each independent loop integration over the gluon and ghost momenta generates a factor \( 1/\epsilon \), while it is easy to show that there are no additional IR singularities with respect to \( \epsilon \) in the quark loop (since we have found regular at zero solutions for the quark propagator [3]).

Using now the general solution (4.6), one further obtains

\[
\frac{1}{2} \bar{T}_t(q) + Z_2(\epsilon) \frac{1}{2} \bar{T}_1(q) + \frac{1}{2} \bar{T}_2(q) + \frac{1}{6} \bar{T}_2'(q) - Z_2^2(\epsilon) \bar{T}_{gh}(q) - \epsilon Z_2^2(\epsilon) \bar{T}_q(q) = 0.
\]

(5.3)

Since the quark wave function IRMR constant \( Z_2(\epsilon) \) can be only either unity or vanishing as \( \epsilon \) goes to zero, the contribution from the quark loop is always suppressed in the \( \epsilon \to 0^+ \) limit, and we are left with the pure Yang-Mills (YM) SD equation for the gluon propagator. For the quark propagator which is IR renormalized the very beginning (i.e., \( Z_2(\epsilon) = Z_2 = 1 \), so that it is IR finite), the SD equation (5.3) becomes

\[
\frac{1}{2} \bar{T}_t(q) + \frac{1}{2} \bar{T}_1(q) + \frac{1}{2} \bar{T}_2(q) + \frac{1}{6} \bar{T}_2'(q) = \bar{T}_{gh}(q),
\]

(5.4)
while for the IR vanishing type of the quark propagator \((Z_2(\epsilon) \rightarrow 0 \text{ as } \epsilon \rightarrow 0^+)\) the SD equation (5.3) becomes

\[
\bar{T}_1(q) + \bar{T}_2(q) + \frac{1}{3} \bar{T}_2'(q) = 0.
\] 

(5.5)

The tensor structure of the YM SD equations for the gluon propagator is not important here. However, it may simplify substantially the corresponding IR renormalized YM SD equations (5.4) and (5.5). What matters here is that within our approach the self-consistent equations for the gluon propagator free from the severe IR singularities exist in the YM sector. In other words, the general solution (4.6) eliminates all the severe IR singularities from Eq. (5.1), indeed.

VI. IR FINITE SD EQUATION FOR THE THREE-GLUON PROPER VERTEX

It is instructive to investigate the IR properties of the SD equation for the triple gauge field proper vertex since it provides a golden opportunity to fix \(Z_2(\epsilon)\). This equation is shown in Fig. 6. For simplicity, the skeleton expansions of the corresponding kernels \(M, \bar{M}'\) and \(B^g\) are not shown. Let us note that the ghost-gluon scattering kernel \(B^g\) (for which we have already established its IRMR constant from the decomposition of the ghost-gluon proper vertex shown in Fig. 2) is denoted as \(G'\) in Ref. [1].

![Fig. 6. The SD equation for the triple gauge field vertex.](image)

Obviously, there is no need to investigate separately the IR properties of the SD equations for the quark-gluon vertex and for pure gluon vertices since the information about their IRMR constants has been uniquely extracted from the corresponding ST identities (see the general solution (4.6)). Moreover, the IRMR constants of different types of the scattering kernels which enter the above-mentioned SD equations (see, for example, Fig. 6) are to be precisely determined by the general system (4.6). In principle, each skeleton diagram of the above-mentioned expansions can be investigated in the same way as the BS scattering kernel was investigated in section IV.
The above-mentioned golden opportunity is provided by the third and fourth terms of this SD equation. The interesting feature of these terms is that they do not contain unknown scattering kernels, so their IR properties can be investigated directly by using only the known IRMR constants. On the other hand, these terms are nothing but the corresponding independent decompositions of the triple gauge field proper vertex with the IRMR constant is equal to $Z_3(\epsilon) = Z_2(\epsilon)$, so one has

$$Z_2(\epsilon) = \frac{1}{\epsilon} X(\epsilon) Z_2(\epsilon) = Z_2^{-1}(\epsilon) = 1,$$

which, obviously, has only a unique solution given by the last equality. Thus we have fixed finally the quark wave function IRMR constant to be unity. Adopting the same method, it is easy to show that all other IRMR constants for the corresponding scattering kernels are:

$$\tilde{Z}_g(\epsilon) = Z_{M'}(\epsilon) = Z_{M''}(\epsilon) = 1.$$

We are now ready to investigate the IR properties of the SD equation for the triple gauge field proper vertex shown in Fig. 6 without referring to the skeleton expansions of the corresponding scattering kernels. Using the previous results, the IR renormalized version of this equation is

$$\tilde{T}_3 = T_3^{(0)} + \frac{1}{2} \tilde{T}_1 + \frac{1}{2} \tilde{T}_1' + \frac{1}{2} \tilde{T}_2' + \frac{1}{6} \tilde{T}_2 - \tilde{T}_g,$$

where, for simplicity, we omit the dependence on momenta and suppress the Dirac indices (i.e., tensor structure). As usual, the quantities with bar are, by definition, IR renormalized, i.e., they exist as $\epsilon \to 0^+$. The SD equations for all other Green’s functions can be investigated in the same way, in particular for the quark-gluon proper vertex shown for example in Ref. [1]. The general solution (4.6), taking into account the fundamental relation (6.1), provides their IR convergence, i.e., they exist in the $\epsilon \to 0^+$ limit and, hence, similar to the SD equations, explicitly considered here, they are free of the severe IR divergences with respect to $\epsilon$. In particular, let us note that Eq. (5.5) should be ruled out as a possible SD equation for the gluon propagator and the SD equation (5.4) which includes the ghost loop is the only possible one.

VII. DISCUSSION AND CONCLUSIONS

In summary, the main observation is that 2D QCD is an inevitably IR divergent theory, and therefore a few points are worth reemphasizing.

We have shown explicitly how the NP IRMR program should be done to remove all the severe IR singularities from the theory on a general ground and in a self-consistent way. The general system of the IR convergence conditions (4.6), on account of the fundamental relation (6.1), simply becomes

$$X(\epsilon) = \epsilon, \quad \epsilon \to 0^+,$$

while all other independent quantities (Green’s functions) are IR finite from the very beginning, i.e., their IRMR constants are simply unity. Evidently, only the nontrivial IR
renormalization of the coupling constant squared is needed to render the theory IR finite, i.e., to make it free from all the severe IR divergences. Only the condition (7.1) provides a cancellation of all the severe IR singularities in 2D QCD. This completes the proof of the NP IR multiplicative renormalizability of 2D QCD within our approach.

Our proof implies that quark propagator should be IR finite from the very beginning, i.e., $Z_2(\epsilon) = 1$ which means $S(p) = \bar{S}(p)$. In the 't Hooft model [5], the quark propagator is IR vanishing, i.e., $Z_2(\epsilon)$ goes to zero as $\epsilon \to 0^+$. However, there is no contradiction with the above-mentioned since in his model it is implicitly assumed that neither $g^2$ nor $N_c$ depend on $\epsilon$ (i.e., they are IR finite from the very beginning) though both parameters $g^2$ and $N_c$ (since it is free one in the large $N_c$ limit approach) should, in principle, depend on it in the presence of such a severe IR singularity in the theory. From our general solution (4.6) then it follows that $Z_2(\epsilon) = \sqrt{\epsilon}$, indeed, since in this case one has to put $X(\epsilon) = 1$, i.e., the coupling constant squared is IR finite from the very beginning ($g^2 = \bar{g}^2$).

A few remarks are in order. In principle, no regularization scheme (how to introduce the IR regularization parameter in order to parameterize the severe IR divergences) should be introduced "by hand". First of all it should be well defined. Secondly, it should be compatible with the DT [4]. The DR scheme [8] is well defined; in Ref. [7] we have shown how it should be introduced into the DT (complemented by the number of subtractions, if necessary). In principle, other regularizations schemes are also available, such as, e.g., analytical regularization used in Ref. [17] or the so-called Speer's regularization [18]. However, they should be compatible with the DT as was emphasized above. Anyway, not the regularization is important but the DT itself.

Whether the theory is IR multiplicative renormalizable or not depends on neither the regularization nor the gauge. Due to the chosen regularization scheme or the gauge only the details of the corresponding IRMR program can be simplified. For example, in the light-cone gauge at any chosen regularization scheme (the 't Hooft model with different prescriptions how to deal with the severe IR singularities [2] (and references therein)) to prove the IR multiplicative renormalizability of 2D QCD is almost trivial. This is mainly due to the fact that in this case only two sectors survive in QCD, namely quark and BS sectors. In other words, if theory is proven to be IR or UV renormalizable in one gauge, it is IR or UV renormalizable in any other gauge. This is true for the regularization schemes as well. As it follows from the present investigation, to prove the IR multiplicative renormalizability of 2D QCD in the covariant gauge was not so simple. However, it was necessary to get firstly the IR finite bound-state problem (which is important for physical applications), and secondly to generalize our approach on 4D QCD which is real theory of strong interactions. 2D QCD in the light-cone gauge is not appropriate theory for this purpose since its confinement mechanism looks more like that of the Schwinger model [1] of 2D electrodynamics, than it may happen in real QCD, where we believe it is much more complicated.

The structure of the severe IR singularities in Euclidean space is much simpler than in Minkowski space, where kinematical (unphysical) singularities due to light cone also exist. In this case it is rather difficult to correctly untangle them from the dynamical singularities, only ones which are important for the calculation of any physical observable. Also the consideration is much more complicated in configuration space [4]. That is why we always prefer to work in momentum space (where propagators do not depend explicitly on the number of dimensions) with Euclidean signature. We also prefer to work in the covariant
gauges in order to avoid peculiarities of the noncovariant gauges [19], for example how to untangle the gauge pole from the dynamical one. The IR structure of 2D QCD in the lightcone gauge by evaluating different physical quantities has been investigated in more detail in Refs. [2,20-23] (and references therein).

The only dynamical mechanism which can be thought of in 2D QCD is the direct interaction of massless gluons (without explicitly involving some extra degrees of freedom). It is well known that in the deep UV limit this interaction brings to birth asymptotic freedom (AF) [1] in QCD as well. It becomes strongly singular in the deep IR domain and can be effectively correctly absorbed (accumulated) into the gluon propagator. Let us remind that formally the $D = D^0$ solution always exists in the system of the SD equations due to its construction by expansion around the free field vacuum. Otherwise it would be impossible to introduce interaction step by step though the final equations make no reference to perturbation theory [1]. It is either trivial (coupling is zero) or nontrivial, then some additional condition(s) (constraint(s)), involving other Green’s functions, is to be derived. Eq. (5.4) is just this exact constraint. The only question to be asked is whether this solution (precisely to its severe IR structure) is justified to use in order to explain some physical phenomena, such as quark confinement, DBCS, etc. or not. By proving the NP IR multiplicative renormalizability of 2D QCD, we conclude that this is so, indeed. At the same time, let us emphasize that this solution is not justified to use for the above-mentioned purposes in 4D QCD since its IR singularity is not severe (i.e., NP) there as was mentioned in the Introduction.

One can also conclude that in some sense it is easier to prove the IR multiplicative renormalizability of 2D QCD than to prove its UV renormalizability. The reason is, of course, that we know the mathematical theory which has to be used - the theory of distributions [4]. This is due to its fundamental result [4,7] which requires that any NP (severe) singularity with respect to the momentum in the deep IR domain in terms of $\epsilon$ should be always $1/\epsilon$ and this does not depend on how the IR regularization parameter $\epsilon$ has been introduced in the way compatible with the DT itself. On the other hand, the above-mentioned fundamental result relates the IR regularization to the number of space-time dimensions [4,7] (compactification). It is clear that otherwise none of the IRMR programs would be possible. In other words, the DT provides the basis for the adequate mathematical investigation of the global character of the severe IR divergences (each independent skeleton loop diagram diverges as $1/\epsilon$), while the UV divergences have a local character, and thus should be investigated term by term in powers of the coupling constant.

In this connection let us make a few remarks. The full dynamical content of 2D (4D) QCD is contained in its system of the SD equations of motion. To solve 2D (4D) QCD means to solve this system and vice versa. In particular, to prove the IR renormalizability of 2D (4D) QCD means to formulate the IRMR program for removing all the NP IR singularities from this system on a general ground and in a self-consistent way. As was mentioned above, the fortunate feature which makes this possible is the global character of the IR singularities in 2D QCD. Each skeleton diagram is the sum of infinite series of terms, however, the DT shows how their severe IR singularities can be summed up. Moreover, it shows how the severe IR singularities of different scattering kernels (which by themselves are infinite series of the skeleton diagrams) can also be summed up.

The next important step is to impose a number of independent conditions to cancel all the NP IR singularities which inevitably appear in the theory after the above-mentioned
summations have been done with the help of the entire chain of strongly coupled SD and BS equations. They should also be complemented by the corresponding ST identities which are consequences of the exact gauge invariance, and therefore are exact constraints on any solution to QCD \[1\]. The only problem now is to find self-consistent solutions to the system of the IR convergence conditions. If such solutions exist, so everything is O.K. If not, the theory is not IR renormalizable. It is worth reemphasizing that we have found a self-consistent solution to this system (Eq. (7.1)).

Concluding, we would like to make a few things perfectly clear. In Ref. \[3\] we made sharp conclusions that 2D covariant gauge QCD implies quark confinement and dynamical breakdown of chiral symmetry. However, these conclusions would be groundless if in the present investigation we were not be able to prove that 2D QCD is an IR multiplicative renormalizable theory. This is our first conclusion here, and it is based on the compelling mathematical ground provided by the DT itself.

The second one is that we fixed finally the type of the quark propagator. The NP IRMR program implies it to be IR finite from the very beginning.

The third conclusion (the most important for physical applications) is that the bound-state problem becomes tractable within our approach. To any order in the skeleton expansion of the BS scattering kernel shown in Fig. 7, the corresponding BS equation (4.2) can be reduced finally to an algebraic problem. Its derivation is absolutely similar to the derivation of the IR renormalized quark SD equation which has been carried out in Ref. \[3\] (see also Ref. \[24\], where one can find a more detail discussion, conclusions and the comparison with the ’t Hooft model as well).

It was widely believed that the severe IR singularities could not be put under control. However, we show explicitly here and in Ref. \[3\] that the above-mentioned common belief is not justified. They can be controlled in all sectors of QCD of any dimensions by using correctly the DT \[4,7\]. This can be considered also as one of our important results from a mathematical point of view.

This work has been carried out under BPCL-17 Contract and we thank L.P. Csernai for support. One of the authors (V.G.) is grateful to A.V. Kouziouchine for help and support.
REFERENCES

[1] W. Marciano, H. Pagels, Phys. Rep. C 36 (1978) 137.
[2] E. Abdalla, M. C. B. Abdalla, Phys. Rep. 265 (1996) 253.
[3] V. Gogohia, Phys. Lett. B 531 (2002) 321.
[4] I. M. Gel’fand, G. E. Shilov, Generalized Functions, Vol.1, Academic Press, New York, 1968.
[5] G. ’t Hooft, Nucl. Phys. B 75 (1974) 461.
[6] G. ’t Hooft, Nucl. Phys. B 72 (1974) 461;
S. Coleman, in: Proc. Inter. School of Subnuclear Physics, ed. A. Zichichi (Plenum Press, New York and London, 1979).
[7] V. Gogohia, hep-ph/0102261.
[8] G. ’t Hooft, M. Veltman, Nucl. Phys. B 44 (1972) 189.
[9] M. Baker, C. Lee, Phys. Rev. D 15 (1977) 2201.
[10] S.K. Kim, M. Baker, Nucl. Phys. B 164 (1980) 152.
[11] U. Bar-Gadda, Nucl. Phys. B 163 (1980) 312.
[12] B.W. Lee, Phys. Rev. D 9 (1974) 933.
[13] S.-H.H. Tye, E. Tomboulis, E.C. Poggio, Phys. Rev. D 11 (1975) 2839.
[14] Behavior of the Solutions to the Bethe-Salpeter Equation (Ed. N. Nakanishi), Prog.
Theor. Phys. Suppl. 95 (1988) 1.
[15] N. Brown, M.R. Pennington, Phys. Rev. D 39 (1989) 2723.
[16] C.D. Roberts, S.M. Schmidt, Prog. Part. Nucl. Phys. 45 (2000) 2nd issue.
L.v. Smekal, A. Hauck, R. Alcofer, Ann. Phys. (N.Y.) 267 (1998) 1.
[17] H. Pagels, Phys. Rev. D 15 (1977) 2991.
[18] E.R. Speer, Jour. Math. Phys. 15 (1974) 1.
[19] A. Bassetto, G. Nardeli and R. Soldati, Yang-Mills Theories in Algebraic Non Covariant
Gauges (WS, Singapore, 1991);
G. Leibbrandt, Noncovariant Gauges (WS, Singapore, 1994).
[20] A. Bassetto, R. Begliuomini, G. Nardeli, Phys. Rev. D 59 (1999) 125005;
M. Staudacher, W. Krauth, Phys. Rev. D 57 (1998) 2456.
[21] M.B. Einhorn, Phys. Rev. D 14 (1976) 3451.
[22] C.G. Callan, Jr., N. Coote, D.J. Gross, Phys. Rev. D 13 (1976) 1649.
[23] T.T. Wu, Phys. Lett. B 71 (1977) 142;
K.G. Klimenko, G. L. Rcheulishvili, Theor. Math. Phys. 62 (1985) 247.
[24] V. Gogokhia, Gy. Kluge, hep-ph/0104290.