Universal $T$-matrix, Representations of $OSp_q(1/2)$ and Little $Q$-Jacobi Polynomials

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Abstract

We obtain a closed form expression of the universal $T$-matrix encapsulating the duality between the quantum superalgebra $U_q[osp(1/2)]$ and the corresponding supergroup $OSp_q(1/2)$. The classical $q \rightarrow 1$ limit of this universal $T$ matrix yields the group element of the undeformed $OSp(1/2)$ supergroup. The finite dimensional representations of the quantum supergroup $OSp_q(1/2)$ are readily constructed employing the said universal $T$-matrix and the known finite dimensional representations of the dually related deformed $U_q[osp(1/2)]$ superalgebra. Proceeding further, we derive the product law, the recurrence relations and the orthogonality of the representations of the quantum supergroup $OSp_q(1/2)$. It is shown that the entries of these representation matrices are expressed in terms of the little $Q$-Jacobi polynomials with $Q = -q$. Two mutually complementary singular maps of the universal $T$-matrix on the universal $R$-matrix are also presented.

PACS numbers: 02.20.Uw, 02.30.Gp
I. INTRODUCTION

The representation theory of quantum groups and algebras have richer structures compared to their classical counterparts. Various non-classical features in the representation theory of the former have been found for specific values of deformation parameters such as roots of unity or crystal base limit. It is known that, for generic values of deformation parameters, each irreducible representation of the classical Lie groups and algebras has its quantum analog (see for example Ref.1). Even for such a generic case, however, there exist representations of the quantum algebra that do not have classical partners.2, 3, 4 Extending our studies to the quantum supergroups, we expect further richness of representations as the nilpotency of Grassmann variables in classical supergroups are, in many cases, lost at quantum level. Grassmann coordinates of quantum superspaces5 and quantum superspheres6, as well as the Grassmann elements of quantum supermatrices5, 6 are instances of lost nilpotency. When representations of such algebraic objects are considered, nonvanishing squares of Grassmann variables cause a drastic shift from the classical cases even for generic values of deformation parameters. Influenced by this observation, we here study the representations of the simplest quantum supergroup $OSp_q(1/2)$. Precise theory of matrix representations of quantum groups has been developed in Ref. 7. Physical motivations of the present work is provided by the investigations on solvable models having quantized $osp(1/2)$ symmetry. For instance, vertex models,8 Gaudin model9 and 2D field theories10 based on $U_q[osp(1/2)]$ symmetry have been proposed and investigated. Fully developed representation theory of $OSp_q(1/2)$ will provide useful tools for analyzing these models and building new ones.

It is known that the representation theories of two quantum algebras $U_q[sl(2)]$ and $U_q[osp(1/2)]$ are quite parallel for a generic $q$. We naturally anticipate that the known results on the quantum group $SL_q(2)$ may be extended to the quantum supergroup $OSp_q(1/2)$. To accomplish the extension, we employ the universal $T$-matrix, which is a generalization of the exponential mapping relating a Lie algebra with its corresponding group. Underlying reasons for this are as follows: (i) The universal $T$ matrix succinctly embodies the representations of the dually related conjugate Hopf structures, $U_q[osp(1/2)]$ and $OSp_q(1/2)$. In particular, contributions of the nonvanishing square of the odd elements of $OSp_q(1/2)$ may be directly read from the expression of the universal $T$-matrix. (ii) Moreover, the universal $T$-matrix allows us to map each irreducible representation of $U_q[osp(1/2)]$ on the corresponding one of $OSp_q(1/2)$. Therefore, various properties of representations follow from the corresponding ones of the universal $T$-matrix, and the role of the lost nilpotency becomes explicit. Specifically, we demonstrate that the nonvanishing contributions of odd elements assume polynomial structures in the representation matrices.

The plan of this article is as follows. After fixing notations and conventions in the next section, the basis set of the Hopf dual to the $U_q[osp(1/2)]$ algebra is explicitly obtained. The finitely generated basis sets of the dually related Hopf algebras are now used to derive a closed form expression of the universal $T$-matrix via the method of Frønsdal and Galindo11. Singular, and, therefore, non-invertible maps of the universal $T$-matrix on the universal $R$-matrix exist11. Two such mutually complementary maps are studied in §IV.

General properties of the finite dimensional representations of the quantum supergroup
OSp_q(1/2), such as the product law, the recurrence relations and the orthogonality of representations, follow, as observed in §V, from the duality encompassed in the universal \(\mathcal{T}\)-matrix. Explicit form of the representation matrices are derived in §VI and its relation to the little \(Q\)-Jacobi polynomials are discussed. It is shown that the entries of representation matrix are expressed in terms of the little \(Q\)-Jacobi polynomials with \(Q = -q\). This provides a new link of the representation theory of quantum supergroups with the hypergeometric functions. Section §VII is devoted to concluding remarks. Corresponding results on \(SL_q(2)\) and other quantum groups are mentioned in each section.

II. \(U_q[osp(1/2)]\) AND ITS REPRESENTATIONS

The quantum superalgebra \(U = U_q[osp(1/2)]\) and the dually related quantum supergroup \(\mathcal{A} = OSp_q(1/2)\), dual to \(U\), have been introduced in Ref.[12]. Structures and representations of \(U\) have been investigated in Refs[8] [12]. For the purpose of fixing our notations and conventions we here list the relations that will be used subsequently.

The algebra \(U\) is generated by three elements \(H\) (parity even) and \(V_\pm\) (parity odd) subject to the relations

\[
[H, V_\pm] = \pm \frac{1}{2} V_\pm, \quad \{V_+, V_-\} = -\frac{q^{2H} - q^{-2H}}{q - q^{-1}} \equiv [-2H]_q.
\] (2.1)

The deformation parameter \(q\) is assumed to be generic throughout this article. The Hopf algebra structures defined via the coproduct (\(\Delta\)), the counit (\(\epsilon\)) and the antipode (\(S\)) maps read as follows:

\[
\Delta(H) = H \otimes 1 + 1 \otimes H, \quad \Delta(V_\pm) = V_\pm \otimes q^{-H} + q^H \otimes V_\pm, \quad \epsilon(H) = \epsilon(V_\pm) = 0, \quad S(H) = -H, \quad S(V_\pm) = -q^{\mp 1/2} V_\pm.
\] (2.2-2.4)

Using the flip operator \(\sigma\): \(\sigma(a \otimes b) = (-1)^{p(a)p(b)} b \otimes a\), where \(p(a)\) denotes the parity of \(a\), we define the transposed coproduct: \(\Delta' = \sigma \circ \Delta\). The universal \(R\)-matrix intertwining \(\Delta\) and \(\Delta'\) is given by\[12\]

\[
R_q = q^{4H \otimes H} \sum_{k \geq 0} \frac{(q - q^{-1})^k q^{-k/2}}{k!} (q^H V_+ \otimes q^{-H} V_-)^k,
\] (2.5)

where

\[
\binom{k}{q} = \frac{1 - (-1)^k q^k}{1 + q}, \quad \binom{k}{q}! = \binom{k}{q} \binom{k - 1}{q} \cdots \binom{0}{q}, \quad \binom{0}{q}! = 1.
\] (2.6)

Two properties of \(R_q\) that will be used later are listed below: (i) It satisfies Yang-Baxter equation

\[
R_{q12}R_{q13}R_{q23} = R_{q23}R_{q13}R_{q12},
\] (2.7)

and (ii) its antipode map reads \((S \otimes \text{id})R_q = R_{q^{-1}} = R_q^{-1}.\)
The finite dimensional irreducible representations of $\mathcal{U}$ is specified by the highest weight $\ell$ which takes any non-negative integral value. The irreducible representation space $V^{(\ell)}$ of highest weight $\ell$ is $2\ell + 1$ dimensional. We denote its basis as $\{ e^\ell_m(\lambda) | m = \ell, \ell - 1, \ldots, -\ell \}$, where the index $\lambda = 0, 1$ specifies the parity of the highest weight vector $e^\ell_0(\lambda)$. The parity of the vector $e^\ell_m(\lambda)$ equals $\ell - m + \lambda$, as it is obtained by the action of $V^{\ell-m}$ on $e^\ell_{\ell}$ for the superalgebras the norm of the representation basis need not be chosen positive definite. In this work, however, we assume the positive definiteness of the basis elements:

$$(e^\ell_m(\lambda), e^{\ell'}_{m'}(\lambda)) = \delta_{\ell\ell'}\delta_{mm'}.$$  

(2.8)

With these settings, the irreducible representation of $\mathcal{U}$ on $V^{(\ell)}$ is given by

$$He^\ell_m(\lambda) = \frac{m}{2} e^\ell_m(\lambda),$$

(2.9)

$$V_+ e^\ell_m(\lambda) = \left(\frac{1}{\{2\}_q} \{\ell - m\}_q \{\ell + m + 1\}_q\right)^{1/2} e^\ell_{m+1}(\lambda),$$

$$V_- e^\ell_m(\lambda) = (-1)^{\ell-m-1} \left(\frac{1}{\{2\}_q} \{\ell + m\}_q \{\ell - m + 1\}_q\right)^{1/2} e^\ell_{m-1}(\lambda),$$

where

$$\{m\}_q = \frac{q^{-m/2} - (-1)^m q^{m/2}}{q^{-1/2} + q^{1/2}}.$$  

(2.10)

Tensor product of two irreducible representations is, in general, reducible and may be decomposed into irreducible ones without multiplicity:

$$V^{(\ell_1)} \otimes V^{(\ell_2)} = V^{(\ell_1 + \ell_2)} \oplus V^{(\ell_1 + \ell_2 - 1)} \oplus \cdots \oplus V^{(|\ell_1 - \ell_2|)}.$$  

The decomposition of the tensored vector space in the irreducible basis is provided by the Clebsch-Gordan coefficients (CGC):

$$e^\ell_{m}(\ell_1, \ell_2, \Lambda) = \sum_{m_1, m_2} C^\ell_{m_1 m_2 m} e^\ell_{m_1}(\lambda) \otimes e^\ell_{m_2}(\lambda),$$  

(2.11)

where $m = m_1 + m_2$, and $\Lambda = \ell_1 + \ell_2 + \ell \pmod{2}$ signify the parity of the highest weight vector $e^\ell_{\Lambda}(\ell_1, \ell_2, \Lambda)$. The CGC for $\mathcal{U}$ is extensively studied in Ref. \[13\]. In spite of our assumption (2.8) regarding the positivity of the basis vectors, the norm of tensored vector space is not always positive definite. Indeed, the basis (2.11) is pseudo orthogonal:

$$(e^{\ell'}_{m'}(\ell_1, \ell_2, \Lambda), e^\ell_m(\ell_1, \ell_2, \Lambda)) = (-1)^{(\ell - m + \lambda)(\ell_1 + \ell_2 + \ell + \lambda)} \delta_{\ell'} \delta_{m'}.$$  

(2.12)

The CGC satisfies two pseudo orthogonality relations

$$\sum_{m_1, m_2} (-1)^{(\ell_1 - m_1 + \lambda)(\ell_2 - m_2 + \lambda)} C^\ell_{m_1 m_2 m} C^\ell_{m_1 m_2 m'} = (-1)^{(\ell - m + \lambda)(\ell_1 + \ell_2 + \ell + \lambda)} \delta_{\ell'} \delta_{m'},$$  

(2.13)

$$\sum_{\ell, m} (-1)^{(\ell - m + \lambda)(\ell_1 + \ell_2 + \ell + \lambda)} C^\ell_{m_1 m_2 m} C^\ell_{m_1 m_2 m'} = (-1)^{(\ell_1 - m_1 + \lambda)(\ell_2 - m_2 + \lambda)} \delta_{m_1} \delta_{m_1'} \delta_{m_2}.$$  

(2.14)
Equation (2.14) immediately provides the inversion of the construction (2.11):
\[
e^{\ell_1}_{m_1}(\lambda) \otimes e^{\ell_2}_{m_2}(\lambda) = (-1)^{(\ell_1-m_1)(\ell_2-m_2)} \sum_{\ell,m} (-1)^{(\ell-m)(\ell_1+\ell_2+\ell)} C^{\ell_1}_{m_1} \ell_2 \ell C^{\ell}_{m} \ell_1 \ell_2 \ell e^{\ell}_{m}(\ell, \ell_2, \Lambda). \tag{2.15}
\]

Before closing this section, we make two remarks: (i) All the CGC are of parity even state. (ii) The explicit realization of CGC for \( \mathcal{U} \) is found in Ref[6] and also in Ref[13]. As we maintain the phase convention for the representation of \( \mathcal{U} \) given in Ref[13], we use the results obtained therein.

### III. UNIVERSAL \( \mathcal{T} \)-MATRIX VIA DUALITY

Two Hopf algebras \( \mathcal{U} \) and \( \mathcal{A} \) are in duality\(^{11}\) if there exists a doubly-nondegenerate bilinear form \( \langle , \rangle : \mathcal{A} \otimes \mathcal{U} \rightarrow \mathbb{C} \) such that, for \((a, b) \in \mathcal{A}, (u, v) \in \mathcal{U},\)
\[
\langle a, uv \rangle = \langle \Delta_{\mathcal{A}}(a), u \otimes v \rangle, \quad \langle ab, u \rangle = \langle a \otimes b, \Delta_{\mathcal{U}}(u) \rangle, \quad \langle a, 1_{\mathcal{U}} \rangle = \epsilon_{\mathcal{A}}(a), \quad \langle 1_{\mathcal{A}}, u \rangle = \epsilon_{\mathcal{U}}(u), \quad \langle a, S_{\mathcal{U}}(u) \rangle = \langle S_{\mathcal{A}}(a), u \rangle. \tag{3.1}
\]

Let the ordered monomials \( E_{k\ell m} = V^k_+ H^\ell V^-_m, \) \((k, \ell, m) \in (0, 1, 2, \cdots)\) be the basis elements of the algebra \( \mathcal{U} \) obeying the multiplication and the induced coproduct rules given by
\[
E_{k\ell m} E_{k'\ell' m'} = \sum_{pqr} f_{k\ell m, k'\ell' m'}^{pqr} E_{pqr}, \quad \Delta(E_{k\ell m}) = \sum_{pqr} g_{k\ell m}^{pqr} p^{q'} r^{r'} E_{pqr} \otimes E_{p'r'q'}. \tag{3.2}
\]

The basis elements \( e^{k\ell m} \) of the dual Hopf algebra \( \mathcal{A} \) follow the relation
\[
\langle e^{k\ell m}, E_{k'\ell' m'} \rangle = \delta_{k, k'}^g \delta_{\ell, \ell'} \delta_{m, m'}. \tag{3.3}
\]

In particular, the generating elements of the algebra \( \mathcal{A} \), defined as \( x = e^{100}, y = e^{001} \) and \( z = e^{010} \), satisfy the following duality structure:
\[
\langle x, V_+ \rangle = 1, \quad \langle z, H \rangle = 1, \quad \langle y, V_- \rangle = 1. \tag{3.4}
\]

Thus, \( x \) and \( y \) are of odd parity, while \( z \) is even. The duality condition (3.1) requires the basis set \( e^{k\ell m} \) to obey the multiplication and coproduct rules given below:
\[
e_{k\ell m}^{pqr} e_{p'q'r'}^{k'\ell' m'} = \sum_{k\ell m} f_{k\ell m}^{pqr} p^{q'} r^{r'} e_{k\ell m}, \quad \Delta(e_{k\ell m}^{pqr}) = \sum_{k\ell m} f_{k\ell m}^{pqr} p^{q'} r^{r'} e_{k\ell m} \otimes e_{k'\ell' m'}. \tag{3.5}
\]

To derive the Hopf properties of the dual algebra \( \mathcal{A} \), we, therefore, need to extract the structure constants defined in (3.2). Towards this end we note that the induced coproduct map of the elements \( E_{k\ell m} \) may be obtained via (2.2):\(^{12}\)
\[
\Delta(E_{k\ell m}) = \Delta(V_+^k \Delta(H)^\ell \Delta(V_-)^m \quad \sum_{a=0}^k \sum_{b=0}^\ell \sum_{c=0}^m \binom{k}{a} \binom{\ell}{b} \binom{m}{c} (-1)^{(m-c)(a+c)} q^{-a(k-a)/2-c(m-c)/2} \\
\times V^k_{+a} q^{(a+c)H} H^{-b} V^m_{-} \otimes V^a_{+} H^{-(k+m-a-c)H} V^c_{-}, \tag{3.6}
\]

\[4\]
where
\[
\binom{k}{a}_q = \frac{k!}{a! (k-a)!}_q.
\]

The second equality in (3.6) can be verified by using the commutation relations
\[
q(V_+ \otimes q^{-H})(q^H \otimes V_+) + (q^H \otimes V_+)(V_+ \otimes q^{-H}) = 0,
\]
\[
q(q^H \otimes V_-)(V_- \otimes q^{-H}) + (V_- \otimes q^{-H})(q^H \otimes V_-) = 0,
\]
and an extension of binomial theorem that is easily proved by induction: Arbitrary operators $A, B$ subject to the commutation properties $qAB + BA = 0$, satisfy the following expansion
\[
(A + B)^n = \sum_{k=0}^{n} \binom{n}{k}_q A^{n-k}B^k. \tag{3.7}
\]

Employing (3.6) we now obtain a set of structure constants:
\[
g_{k \ell m}^{001} = \delta_{k1}\delta_{\ell 0}\delta_{m1}, \quad g_{k \ell m}^{001} = -\delta_{k1}\delta_{\ell 0}\delta_{m1},
\]
\[
g_{k \ell m}^{010} = -\ln q \delta_{k1}\delta_{\ell 0}\delta_{m0} + \delta_{k1}\delta_{\ell 1}\delta_{m0},
\]
\[
g_{k \ell m}^{010} = \ln q \delta_{k1}\delta_{\ell 0}\delta_{m0} + \delta_{k1}\delta_{\ell 1}\delta_{m0},
\]
\[
g_{k \ell m}^{001} = \ln q \delta_{k0}\delta_{\ell 0}\delta_{m1} + \delta_{k0}\delta_{\ell 1}\delta_{m1},
\]
\[
g_{k \ell m}^{001} = -\ln q \delta_{k0}\delta_{\ell 0}\delta_{m1} + \delta_{k0}\delta_{\ell 1}\delta_{m1}.
\]

The above structure constants immediately yield the algebraic relations obeyed by the generators of the algebra $\mathcal{A}$:
\[
\{x, y\} = 0, \quad [z, x] = 2 \ln q \ x, \quad [z, y] = 2 \ln q \ y. \tag{3.8}
\]

Proceeding towards constructing the coproduct maps of the generating elements of the dual algebra $\mathcal{A}$ we notice that the defining properties (3.5) provide the necessary recipe:
\[
\Delta(x) = \sum_{k \ell m} f_{k \ell m}^{100} \epsilon^{k \ell m} \otimes \epsilon^{k' \ell' m'},
\]
\[
\Delta(z) = \sum_{k \ell m} f_{k \ell m}^{010} \epsilon^{k \ell m} \otimes \epsilon^{k' \ell' m'},
\]
\[
\Delta(y) = \sum_{k \ell m} f_{k \ell m}^{001} \epsilon^{k \ell m} \otimes \epsilon^{k' \ell' m'}. \tag{3.9}
\]

The relevant structure constants obtained via (3.2) are listed below:
\[
f_{k \ell m}^{100} = \delta_{k1}\delta_{\ell 0}\delta_{m0}\delta_{k'0}\delta_{\ell'0}\delta_{m'0} + (-1)^m \frac{1}{2^e} \sigma_{m+1} \delta_{k0}\delta_{k'm+1}\delta_{\ell0}\delta_{\ell'm0},
\]
\[
f_{k \ell m}^{010} = \delta_{k0}\delta_{m0}\delta_{k'0}\delta_{\ell0}\delta_{\ell'0}(\delta_{\ell1}\delta_{\ell'0} + \delta_{\ell0}\delta_{\ell1}) + (-1)^m \frac{4 \ln q}{q - q^{-1}} \sigma_m \delta_{k0}\delta_{m0}\delta_{k'0}\delta_{\ell0}\delta_{\ell'0}\delta_{m'0},
\]
\[
f_{k \ell m}^{001} = \delta_{k0}\delta_{m0}\delta_{k'0}\delta_{\ell0}\delta_{\ell'0}\delta_{m1} + (-1)^{k'1} \frac{1}{2^e} \sigma_{k'1} \delta_{k0}\delta_{m0}\delta_{k'm}\delta_{\ell0}\delta_{\ell'm0}, \tag{3.10}
\]
\[
\sigma_1 = 1, \quad \sigma_m = \prod_{k=1}^{m-1} \sum_{\ell=0}^{k-1} (-1)^{\ell} [k - \ell]_q, \quad (m > 1).
\]
The coproduct maps of the generators of $\mathcal{A}$ may now be explicitly obtained à la (3.9) provided the basis elements $e^{k\ell m}$ of the algebra $\mathcal{A}$ are known. We complete this task subsequently.

As the algebra $\mathcal{A}$ is finitely generated, we may start with the generators $(x, y, z)$ and obtain all dual basis elements $e^{k\ell m}$, $(k, \ell, m) \in (0, 1, 2, \cdots)$ by successively applying the multiplication rule given in the first equation in (3.5). The necessary structure constants may be read from the relation (3.2) of the algebra $\mathcal{U}$. In the procedure described below we maintain the operator ordering of the monomials as $x^k z^\ell y^m$, $(k, \ell, m) \in (0, 1, 2, \cdots)$. The product rule

$$e^{100} e^{n00} = \sum_{k\ell m} g^{100 n00}_{k\ell m} e^{k\ell m}$$

(3.11)

and the explicit evaluation of the structure constant

$$g^{100 n00}_{k\ell m} = \{n + 1\}_q \delta_{k n+1} \delta_{\ell 0} \delta_{m0}$$

(3.12)

obtained from (3.6) immediately provide

$$e^{n00} = \frac{x^n}{\{n\}_q!}, \quad \{n\}_q! = \prod_{\ell=1}^{n} \{\ell\}_q, \quad \{0\}_q! = 1.$$  

(3.13)

Employing another product rule

$$e^{nr0} e^{010} = \sum_{k\ell m} g^{nr0 010}_{k\ell m} e^{k\ell m}$$

(3.14)

and the value of the relevant structure constant

$$g^{nr0 010}_{k\ell m} = -n \ln q \delta_{kn0} \delta_{\ell r} \delta_{m0} + (r + 1) \delta_{kn} \delta_{\ell r+1} \delta_{m0},$$

obtained in the aforesaid way we produce the following result:

$$e^{nr0} = \frac{x^n}{\{n\}_q! r!}(z + n \ln q)^r.$$  

(3.15)

Continuing the above process of building of the dual basis set we use the product rule

$$e^{nrs} e^{001} = \sum_{k\ell m} g^{nrs 001}_{k\ell m} e^{k\ell m}$$

(3.16)

and the value of the corresponding structure constant

$$g^{nrs 001}_{k\ell m} = \{s + 1\}_{q-1} r \sum_{j=0}^{r} \frac{1}{j!} (\ln q)^j \delta_{kn} \delta_{\ell r+j} \delta_{m s+1},$$

obtained via (3.6). This finally leads us to the complete construction of the basis element of the algebra $\mathcal{A}$:

$$e^{nrs} = \frac{x^n}{\{n\}_q!} (z + (n - s) \ln q)^r \frac{y^s}{\{s\}_{q-1}!}.$$  

(3.17)
Combining our results in (3.9), (3.10) and (3.17), we now provide the promised coproduct structure of the generators of the algebra $\mathcal{A}$:

$$\Delta(x) = x \otimes 1 + \sum_{m=0}^{\infty} (-1)^m \sigma_{m+1} q^{-m/2} e^{z/2} y^m \{m\}_{q^{-1}}^{1} \otimes x^{m+1} \{m+1\}_q$$

$$\Delta(z) = z \otimes 1 + 1 \otimes z + \frac{4 \ln q}{q - q^{-1}} \sum_{m=1}^{\infty} (-1)^m \sigma_m y^m \{m\}_{q^{-1}}^{1} \otimes x^m$$

$$\Delta(y) = 1 \otimes y + \sum_{k=0}^{\infty} (-1)^k \sigma_{k+1} q^{k/2} \frac{y^{k+1}}{\{k+1\}_{q^{-1}}^{1}} \otimes x^k e^{z/2}.$$  \hspace{1cm} (3.18)

Algebraic simplifications allow us to express the coproduct maps of the above generators more succinctly:

$$\Delta(x) = x \otimes 1 + \sum_{m=0}^{\infty} (-1)^{m(m-1)/2} \left(\frac{1+q^{-1}}{q - q^{-1}}\right)^m y^m \otimes x^{m+1},$$

$$\Delta(z) = z \otimes 1 + 1 \otimes z + \frac{4 \ln q}{q - q^{-1}} \sum_{m=1}^{\infty} (-1)^{m(m+1)/2} \frac{q^{1/2} + q^{-1/2}}{q - q^{-1}} \left(\frac{q^1 + q^{-1}}{q - q^{-1}}\right)^{m-1} y^m \otimes x^m$$

$$\Delta(y) = 1 \otimes y + \sum_{m=0}^{\infty} (-1)^{m(m-1)/2} \frac{q+1}{q - q^{-1}} \left(\frac{q^1 + q^{-1}}{q - q^{-1}}\right)^m y^{m+1} \otimes x^m e^{z/2}.$$ \hspace{1cm} (3.19)

With the aid of the result (3.19) we may explicitly demonstrate that the coproduct map is a homomorphism of the algebra $\mathcal{A}$: namely,

$$\{\Delta(x), \Delta(y)\} = 0, \quad [\Delta(z), \Delta(x)] = 2 \ln q \Delta(x), \quad [\Delta(z), \Delta(y)] = 2 \ln q \Delta(y).$$

The coassociativity constraint

$$(\text{id} \otimes \Delta) \circ \Delta(X) = (\Delta \otimes \text{id}) \circ \Delta(X) \quad \forall X \in \{x, y, z\}$$

may also be established by using the following identity:

$$\exp(\Delta(z)) = (\exp(z) \otimes 1) \prod_{m=1}^{\infty} \mathcal{P}_m \left(1 \otimes \exp(z)\right),$$

$$\mathcal{P}_m = \exp \left((-1)^{m(m+1)/2} \frac{2m q^1}{m \{m\}_{q^{-1}}^{1}} \left(\frac{q^{1/2} + q^{-1/2}}{q - q^{-1}}\right)^m y^m \otimes x^m\right).$$ \hspace{1cm} (3.20)

The counit map of the generators of the algebra $\mathcal{A}$ reads as

$$\epsilon(x) = \epsilon(y) = \epsilon(z) = 0.$$ \hspace{1cm} (3.21)

The antipode map of the dual generators follows from the last equation in (3.1). We quote
the results here:

\[ S(x) = -\sum_{m=0}^{\infty}(-1)^m (m-1/2) q^{-1} \left( \frac{1 + q^{-1}}{q - q^{-1}} \right)^m x^m \exp \left( -\frac{m + 1}{2} y \right) y^m, \]

\[ S(z) = -z + \frac{4 \ln q}{q - q^{-1}} \sum_{m=1}^{\infty} (-1)^{m(m+1)/2} (m-1/2) \left( \frac{q^{1/2} + q^{-1/2}}{q - q^{-1}} \right)^{m-1} x^m e^{-mx/2} y^m, \]

\[ S(y) = -\sum_{m=0}^{\infty} (-1)^m (m+1/2) q \left( \frac{q + 1}{q - q^{-1}} \right)^m x^m \exp \left( -\frac{m + 1}{2} z \right) y^{m+1}. \]

This completes our construction of the Hopf algebra \( A \) dually related to the quantum superalgebra \( U \).

Our explicit listing of the complete set of dual basis elements in (3.17) allows us to obtain à la Frønsdal and Galindo the universal \( T \)-matrix for the supergroup:

\[ T_{e,E} = \sum_{k \ell m} (-1)p(e k \ell m)(p(e k \ell m) - 1)/2 e^{k \ell m} \otimes E_{k \ell m}, \quad (3.23) \]

where the parity of basis elements is same for two Hopf algebras \( U \) and \( A \)

\[ p(e k \ell m) = p(E_{k \ell m}) = k + m. \quad (3.24) \]

The notion of the universal \( T \)-matrix is a key feature capping the Hopf duality structure. Consequently, the duality relations (3.1) may be concisely expressed in terms of the \( T \)-matrix as

\[ T_{e,E} T_{e',E} = T_{e \Delta(E),E}, \quad T_{e,E} T_{e,E'} = T_{e,\Delta(E)}, \]

\[ T_{e,E} T_{e,E} = T_{e,\varepsilon(E)} = 1, \quad T_{S(e),E} = T_{e,S(E)}. \quad (3.25) \]

where \( e \) and \( e' \) (\( E \) and \( E' \)) refer to the two identical copies of algebra \( A (U) \). A general discussion of the universal \( T \)-matrix for supergroups is found in the Appendix.

As both the Hopf algebras in our case are finitely generated, the universal \( T \)-matrix may now be obtained as an operator valued function in a closed form:

\[ T_{e,E} = \sum_{k=0}^{\infty} \frac{(x \otimes V_+ q^H k)}{k! \left( \frac{\sqrt{q}}{q} \right)} \exp(z \otimes H) \left( \sum_{m=0}^{\infty} \frac{(y \otimes q^{-H} V_- m)}{m! \sqrt{q}} \right) \]

\[ \equiv \mathcal{E} \exp_q(x \otimes V_+ q^H) \exp(z \otimes H) \mathcal{E} \exp_{q^{-1}}(y \otimes q^{-H} V_-) \mathcal{E}, \quad (3.26) \]

where we have introduced a deformed exponential that is characteristic of the quantum \( OSp_q(1/2) \) supergroup:

\[ \mathcal{E} \exp_q(x) \equiv \sum_{n=0}^{\infty} \frac{x^n}{n! \sqrt{q}}, \quad (3.27) \]

The operator ordering has been explicitly indicated in (3.26). The closed form of the universal \( T \)-matrix in (3.26) will be used in the computation of representation matrices of
the quantum supergroup $A$. In Ref. 14 using the Gauss decomposition of the fundamental representation a universal $T$-matrix for $U$ is given in terms of the standard $q$-exponential instead of the deformed exponential (3.27) characterizing quantum supergroups.

In the classical limit of $q \to 1$, it is immediately evident that the structure constant (3.12) is truncated at $n = 2$ so that $x$ remains nilpotent. Similarly, $y^2 = 0$ holds in this limit. It is interesting to observe the $q \to 1$ limit of the universal $T$ matrix (3.26). For this purpose we note

$$
\lim_{q \to 1} \begin{cases} 
2n & \to n(1 - q), \\
2n + 1 & \to 1,
\end{cases} \quad n = 0, 1, 2, \cdots.
$$

Assuming the finite limit

$$
\lim_{q \to 1} \frac{x^2}{q - 1} = x, \quad \lim_{q \to 1} \frac{y^2}{q^{-1} - 1} = \eta,
$$

it immediately follows that the universal $T$ matrix (3.26) reduces to the group element of the undeformed $OSp(1/2)$, and by definition constitutes its universal $T$-matrix:

$$
T_{e,E}|_{q=1} = (1 \otimes 1 + x \otimes V_+) \exp(x \otimes V_+^2) \exp(z \otimes H) \exp(\eta \otimes V_+^2) (1 \otimes 1 + y \otimes V_-). \quad (3.30)
$$

The finite limiting elements $(x, \eta)$ are bosonic in nature, and in the context of the classical limit of the function algebra $A$ they are dually related to the squares of the odd generators of the undeformed $osp(1/2)$. The elements $(V_+^2, H)$ of the classical $osp(1/2)$ algebra form a $sl(2)$ subalgebra. The corresponding classical $SL(2)$ subgroup structure is evident from (3.30). In fact, the correct limiting structure (3.30) emphasizes the essential validity of the quantum universal $T$ matrix derived in (3.26). Obviously, there is a striking difference between the quantum and classical universal $T$-matrices caused by the absence of nilpotency of parity odd elements in the former case. The infinite series of operators summarized in the deformed exponential contribute to new polynomial matrix elements in the representations of quantum supergroup $A$. This is considered in detail in §V.

The above construction of the universal $T$-matrix for the algebra $U$ is parallel to the one for the generalized Heisenberg algebra $15$ which is a bosonization $16$ of the superalgebra $U$. Dual basis to the two-parametric deformation of $GL(2)$ is studied in Ref. 17. The universal $T$-matrix for the two-parametric quantum $GL(2)$ is given in Refs.11, 18. The generalization to the quantum $gl(n)$ is found in Ref19 and a supersymmetric extension is initiated in Ref20.

### IV. MAPPING $\mathcal{T}$ ON $\mathcal{R}$

Two singular and mutually complementary maps connecting the universal $T$-matrix in (3.26) and the universal $R$-matrix in (2.5) are discussed in this section. The first map $\Phi : A \to U$ reads

$$
\Phi(x) = 0, \quad \Phi(z) = (4 \ln q)H, \quad \Phi(y) = q^{-1/2}(q - q^{-1})q^H V_+.
$$

It is easily observed to satisfy the following properties: (i) $(\Phi \otimes \text{id})(\mathcal{T}_{e,E}) = \mathcal{R}_q$, and (ii) $\Phi$ is an algebra homomorphism though not a Hopf algebra homomorphism; that is, $\Phi$
respects the commutation relations in \([3.8]\) but does not maintain the Hopf coalgebra maps.

For introducing the second map, we recast the universal \(T\)-matrix in the form given below:

\[
\mathcal{T}_{E,e} = \sum_{k\ell m} (-1)^{(k+m)(k+m-1)/2} E_{k\ell m} \otimes e^{k\ell m}
\]

\[
= \left( \sum_k \frac{1}{k!} (V_q^H \otimes x)^k \right) e^{H \otimes z} \left( \sum_m \frac{1}{m!} (q^{-H} V_- \otimes y)^m \right). \tag{4.2}
\]

The universal \(R\)-matrix is also rewritten as

\[
\mathcal{R}_q = \sum_{k \geq 0} \frac{(q - q^{-1})^k}{k!} (V_q^H \otimes q^H V_-)^k q^{4H \otimes H}. \tag{4.3}
\]

Comparison of above two expressions immediately yields the promised map \(\Psi : \mathcal{A} \to \mathcal{U}\) defined as follows:

\[
\Psi(x) = (q^{-1} - q) q^{-H} V_- , \quad \Psi(z) = (-4 \ln q) H , \quad \Psi(y) = 0. \tag{4.4}
\]

One can immediately verify that \((\text{id} \otimes \Psi)(\mathcal{T}_{E,e}) = \mathcal{R}_q^{-1}\), and that \(\Psi\) is an algebraic homomorphism but not a Hopf algebra homomorphism. It is interesting to observe that in both the maps introduced here one Borel subalgebra of the function algebra \(\mathcal{A}\) is mapped on the corresponding Borel subalgebra of the universal enveloping algebra \(\mathcal{U}\). Therefore, the two conjugate Borel subalgebras of the \(\mathcal{U}\) algebra are acted upon by two distinct, but complementary maps. Being singular in nature, these maps are, however, not invertible.

The maps \(\Phi\) and \(\Psi\) may be utilized to connect the universal \(T\)-matrix and the Yang-Baxter equation. As shown in the Appendix, the universal \(T\)-matrix satisfies \(RTT\)-type relations. Using the tensored operators

\[
\mathcal{T}_{e,E}^{(1)} = \sum_k (-1)^{(k+m)(k+m-1)/2} e^{k\ell m} \otimes E_{k\ell m} \otimes 1 ,
\]

\[
\mathcal{T}_{e,E}^{(2)} = \sum_k (-1)^{(k+m)(k+m-1)/2} 1 \otimes E_{k\ell m} , \tag{4.5}
\]

the following identity may be established:

\[
(1 \otimes \mathcal{R}_q) \mathcal{T}_{e,E}^{(1)} \mathcal{T}_{e,E}^{(2)} = \mathcal{T}_{e,E}^{(2)} \mathcal{T}_{e,E}^{(1)} (1 \otimes \mathcal{R}_q). \tag{4.6}
\]

Mirroring the structure in \([4.5]\) we also define the transposed \(T\)-matrices in the tensored space as

\[
\mathcal{T}_{E,e}^{(1)} = \sum_k (-1)^{(k+m)(k+m-1)/2} E_{k\ell m} \otimes 1 \otimes e^{k\ell m} ,
\]

\[
\mathcal{T}_{E,e}^{(2)} = \sum_k (-1)^{(k+m)(k+m-1)/2} 1 \otimes E_{k\ell m} \otimes e^{k\ell m}. \tag{4.7}
\]
These matrices also obey another $RTT$-type relation:
\[(R_q \otimes 1) \mathcal{T}^{(1)}_{E,e} \mathcal{T}^{(2)}_{E,e} = \mathcal{T}^{(2)}_{E,e} \mathcal{T}^{(1)}_{E,e} (R_q \otimes 1). \quad (4.8)\]

Application of the tensored map $\Phi \otimes \text{id} \otimes \text{id}$ to (4.6) converts the $RTT$-type relation into the Yang-Baxter equation of the form
\[R_q^{23} R_q^{12} R_q^{13} = R_q^{13} R_q^{12} R_q^{23}, \quad (4.9)\]
while similar action of the conjugate map $\text{id} \otimes \text{id} \otimes \Psi$ on (4.7) provides the another form of Yang-Baxter equation:
\[R_q^{12} R_q^{13} R_q^{23} = R_q^{23} R_q^{13} R_q^{12}. \quad (4.10)\]

Mappings from a universal $\mathcal{T}$-matrix to a universal $R$-matrix has been discussed for only a few quantum algebras. Frønsdal\cite{10} considered such mappings for quantum $gl(n)$ and the particular case of two-parametric quantum $gl(2)$ is discussed in Ref\cite{21}. The maps for Alexander-Conway quantum algebra is studied in Ref\cite{22}.

V. REPRESENTATIONS OF $\mathcal{A}$

We do not yet have the explicit formulae of finite dimensional representation matrices of the function algebra $\mathcal{A}$. These expressions will be derived in the next section. But, prior to that, the general properties of such finite dimensional representation matrices may be understood via the duality arguments interrelating $\mathcal{A}$ and $\mathcal{U}$ algebras. We will address to this task in the present section.

To be explicit, for us the representations of the universal $\mathcal{T}$-matrix on $V^{(\ell)}$ defined in §II:
\[T_{m'm}(\lambda) = (e_{m'}^\ell(\lambda), T_{E,e} e_{m}(\lambda)) = \sum_{abc} (-1)^{(a+c)(a+c-1)/2+(a+c)(\ell-m'+\lambda)} e^{abc} (e_{m'}^\ell(\lambda), E_{abc} e_{m}(\lambda)). \quad (5.1)\]

Under the assumption of the completeness of the basis vectors $e_{m}^\ell(\lambda)$, it is not difficult to verify the relations:
\[\Delta(T_{m'm}(\lambda)) = \sum_{k} T_{m'k}(\lambda) \otimes T_{km}(\lambda), \quad \epsilon(T_{m'm}(\lambda)) = \delta_{m'm}. \quad (5.2)\]

The relations in (5.2) imply that the matrix elements (5.1) satisfy the axiom of comodule\cite{11}. We may, therefore, regard $T_{m'm}(\lambda)$ as the $2\ell + 1$ dimensional matrix representation of the algebra $\mathcal{A}$.

We now consider a product of two representations in order to obtain their composition rule. We evaluate the matrix elements of $T_{e,\Delta(E)}$ on the coupled basis vector space in two different ways. The first evaluation is as follows:
\[(e_{m'}^\ell(\ell_1, \ell_2, \Lambda), T_{e,\Delta(E)} e_{m}(\ell_1, \ell_2, \Lambda)) = \sum_{abc} (-1)^{(a+c)(a+c-1)/2+(a+c)(\ell-m'+\lambda)} e^{abc} (e_{m'}^\ell(\ell_1, \ell_2, \Lambda), \Delta(E_{abc}) e_{m}(\ell_1, \ell_2, \Lambda)) = \delta_{\ell'}(\ell_1+\ell_2+\ell'+\lambda) T_{m'm}(\lambda), \quad (5.3)\]
In the last equality, the pseudo orthogonality \((2.12)\) of the coupled basis has been used. An alternate evaluation of the said matrix element explicitly uses the Clebsch-Gordan coupling of the basis vectors. With the aid of the relation

\[
T_{\alpha \Delta(E)} = \sum_{p'q'r'} (-1)^{(p+r+p'+r')(p+r+p'+r'-1)/2} e^{pq'q'r'} \otimes E_{pq'q'r'},
\]

we obtain

\[
(e_{m'}^{\ell'}(\ell_1, \ell_2, \Lambda), T_{\alpha \Delta(E)} e_m^{\ell}(\ell_1, \ell_2, \Lambda)) = \sum_{m_1, m_2} (-1)^{(\ell_1-m_1+\lambda)(\ell_2-m_2'\lambda)} C_{m_1 m_2 m}^{\ell_1 \ell_2 \ell_1'} C_{m_1 m_2 m}^{\ell_1 \ell_2 \ell_1'} T_{m_1 m_2}^{\ell_1 m_1}(\Lambda) T_{m_2 m_2}^{\ell_2 m_2}(\Lambda). \tag{5.4}
\]

Since the results of the two evaluations have to be identical, we obtain the product law for the quantum supergroup \(A\) of the following form:

\[
\delta_{\ell'} T_{m'm}^{\ell}(\Lambda) = (-1)^{(\ell'-m'+\lambda)(\ell_1+\ell_2+\ell'+\lambda)} \times \sum_{m_1, m_2} (-1)^{(\ell_1-m_1+\lambda)(\ell_2-m_2'\lambda)} C_{m_1 m_2 m}^{\ell_1 \ell_2 \ell_1'} C_{m_1 m_2 m}^{\ell_1 \ell_2 \ell_1'} T_{m_1 m_2}^{\ell_1 m_1}(\Lambda) T_{m_2 m_2}^{\ell_2 m_2}(\Lambda). \tag{5.5}
\]

Another derivation of the product law \((5.5)\) is found in Ref.\(^6\)

Two alternate forms of the product law are readily derived by using the pseudo orthogonality of CGC \((2.13)\) and \((2.14)\):

\[
\sum_{m'} C_{n_1 n_2 m'}^{\ell_1 \ell_2 \ell} T_{m'm}^{\ell}(\Lambda) = \sum_{m_1, m_2} (-1)^{(n_1+m_1)(n_2-m_2+\lambda)} C_{m_1 m_2 m}^{\ell_1 \ell_2 \ell_1'} T_{n_1 m_1}^{\ell_1 m_1}(\Lambda) T_{n_2 m_2}(\Lambda), \tag{5.6}
\]

\[
\sum_{m} (-1)^{(n_1+m_2')(\ell_1+\ell_2+\ell'+\lambda)} C_{n_1 n_2 m}^{\ell_1 \ell_2 \ell_1'} T_{m'm}^{\ell}(\Lambda) = \sum_{m_1', m_2'} (-1)^{(n_2+m_2')(\ell_1-n_1+\lambda)} C_{m_1' m_2' m}^{\ell_1 \ell_2 \ell_1'} T_{m_1'n_1}^{\ell_1 m_1}(\Lambda) T_{m_2'n_2}(\Lambda). \tag{5.7}
\]

The product law allows us to derive the orthogonality and the recurrence relations of the representation matrix \(T_{m'm}^{\ell}\). Setting \(\ell_1 = \ell_2, \ell = m = 0\) in \((5.6)\) and using the formula of CGC given in Ref.\(^6\) one can verify the orthogonality relation:

\[
\sum_{m} (-1)^{m_1(m_1+m_2)+m_1(m_1-1)/2} q^{(m_1-m)/2} T_{m_1 m_1}(\Lambda) T_{-m_2 m_2}(\Lambda) = \delta_{m_1 m_2}. \tag{5.8}
\]

Another orthogonality relation is similarly obtained by setting \(\ell_1 = \ell_2, \ell' = m' = 0\) in the product law \((5.7)\):

\[
\sum_{m} (-1)^{m_1(m_1+m_2)+m_1(m_1-1)/2} q^{(m_1-m)/2} T_{m_1 m_1}(\Lambda) T_{-m_2 m_2}(\Lambda) = \delta_{m_1 m_2}. \tag{5.9}
\]
For the choice of \( \ell_2 = 1 \) in (5.6), the recurrence relations for the representation matrices are obtained below. These relations are classified into three sets according to the values of \( \ell \):

- The first set has the value of \( \ell = \ell_1 + 1 \). It comprises of three relations corresponding to all possible values of \( n_2 \). The recurrence relations listed below correspond to \( n_2 = 1, 0 \) and \(-1\), respectively:

\[
\begin{align*}
(-1)^{(n+m)} & q^{-(\ell-n)/2} F^\ell(n, 0, -1) T^\ell_{nm}(\Lambda) = q^{-(\ell-m)/2} F^\ell(m, 0, -1) T^{\ell-1}_{n-1, m-1}(\lambda) a \\
-(-1)^\lambda q^{m/2} G^\ell(m, 0, 0) T^{\ell-1}_{n-1, m}(\lambda) a + q^{(\ell+m)/2} F^\ell(-m, 0, -1) T^{\ell-1}_{n-1, m+1}(\lambda) b, \\
(-1)^{(n+m)(1+\lambda)} q^{n/2} G^\ell(n, 0, 0) T^\ell_{nm}(\Lambda) = -q^{-(\ell-m)/2} F^\ell(m, 0, -1) T^{\ell-1}_{n, m-1}(\lambda) \gamma \\
+ q^{m/2} G^\ell(m, 0, 0) T^{\ell-1}_{nm}(\lambda) e - q^{(\ell+m)/2} F^\ell(-m, 0, -1) T^{\ell-1}_{n, m+1}(\lambda) \beta, \\
(-1)^{(n+m)(1+\lambda)} q^{(\ell-n)/2} F^\ell(-n, 0, -1) T^\ell_{nn}(\Lambda) = q^{-(\ell-m)/2} F^\ell(m, 0, -1) T^{\ell-1}_{n+1, m-1}(\lambda) c \\
+ q^{m/2} G^\ell(m, 0, 0) T^{\ell-1}_{n+1, m}(\lambda) \delta + q^{(\ell+m)/2} F^\ell(-m, 0, -1) T^{\ell-1}_{n+1, m+1}(\lambda) \beta,
\end{align*}
\]

where

\[
F^\ell(m, a, b) = \sqrt{\{\ell + m + a\}_q \{\ell + m + b\}_q},
\]

\[
G^\ell(m, a, b) = \sqrt{\{2\}_q \{\ell + m + a\}_q \{\ell - m + b\}_q},
\]

and the matrix elements for the fundamental representation (\( \ell = 1 \)) are denoted as

\[
\begin{pmatrix}
a & \alpha & b \\
\gamma & e & \beta \\
c & \delta & d
\end{pmatrix} = \begin{pmatrix}
T^1_{11}(\lambda) & T^1_{10}(\lambda) & T^1_{1-1}(\lambda) \\
T^1_{01}(\lambda) & T^1_{00}(\lambda) & T^1_{0-1}(\lambda) \\
T^1_{-11}(\lambda) & T^1_{-10}(\lambda) & T^1_{-1-1}(\lambda)
\end{pmatrix}.
\]

In this set, the highest weight \( \Lambda \) assumes a constant value of 0 (mod 2).

- The second set corresponds to \( \ell = \ell_1 \). It also contains three recurrence relations and, in this instance, we have \( \Lambda = 1 \) (mod 2):

\[
\begin{align*}
(-1)^{\ell-n+\lambda+(n+m+1)\lambda} q^{(n-m)/2} G^\ell(n, 0, 1) T^\ell_{nm}(\Lambda) &= (-1)^{\ell-m} G^\ell(m, 0, 1) T^\ell_{n-1, m-1}(\lambda) a \\
-H^\ell_m T^\ell_{n-1, m}(\lambda) a + \{2\}_q^{-1/2} G^\ell(m, 1, 0) T^\ell_{n-1, m+1}(\lambda) b, \\
(-1)^{(n+m)(1+\lambda)} q^{(n-m)/2} H^\ell_n T^\ell_{nm}(\Lambda) &= (-1)^{\ell-m} G^\ell(m, 0, 1) T^\ell_{n, m-1}(\lambda) \gamma \\
+ H^\ell_m T^\ell_{nm}(\lambda) e + \{2\}_q^{-1/2} G^\ell(m, 1, 0) T^\ell_{n, m+1}(\lambda) \beta, \\
(-1)^{(n+m)\lambda} q^{(n-m)/2} \{2\}_q^{-1/2} G^\ell(n, 0, 1) T^\ell_{nm}(\Lambda) &= (-1)^{\ell-m} G^\ell(m, 0, 1) T^\ell_{n+1, m-1}(\lambda) c \\
-H^\ell_m T^\ell_{n+1, m}(\lambda) \delta + \{2\}_q^{-1/2} G^\ell(m, 1, 0) T^\ell_{n+1, m+1}(\lambda) d,
\end{align*}
\]

where

\[
H^\ell_m = q^{-\ell/2} \{\ell + m + 1\}_q - (-1)^{\ell-m} q^{\ell/2} \{\ell - m + 1\}_q.
\]
Similarly, the third set contains three recurrence relations for $\ell = \ell_1 - 1$. For this case, the highest weight reads $\Lambda = 0 \text{ (mod 2)}$, and the recurrence relations are given by

$$q^{(\ell - m + n + 1)/2} F^{\ell} (-n, 1, 2) T_{nm}^{\ell} (\Lambda) = q^{(\ell + 1)/2} F^{\ell} (-m, 1, 2) T_{n-1,m-1}^{\ell+1} (\Lambda) a$$
$$+ (-1)^{\ell - m + \lambda} G^{\ell} (m, 1, 1) T_{n-1,m}^{\ell+1} (\Lambda) \alpha - q^{-(\ell + 1)/2} F^{\ell} (m, 1, 2) T_{n-1,m}^{\ell+1} (\Lambda) b,$n
$$(-1)^{\ell - n + \lambda} q^{(n - m)/2} G^{\ell} (n, 1, 1) T_{nm}^{\ell} (\Lambda) = q^{(\ell + 1)/2} F^{\ell} (-m, 1, 2) T_{n,m-1}^{\ell+1} (\Lambda) c$$
$$+ (-1)^{\ell - m + \lambda} G^{\ell} (m, 1, 1) T_{n,m}^{\ell+1} (\Lambda) e - q^{-(\ell + 1)/2} F^{\ell} (m, 1, 2) T_{n+1,m+1}^{\ell+1} (\Lambda) \beta,$n
$$q^{-(\ell - n + m + 1)/2} F^{\ell} (n, 1, 2) T_{nm}^{\ell} (\Lambda) = -q^{(\ell + 1)/2} F^{\ell} (-m, 1, 2) T_{n+1,m-1}^{\ell+1} (\Lambda) d$$

The discussion so far is independent of the explicit form of the universal $\mathcal{T}$-matrix. The general properties of the universal $\mathcal{T}$-matrix and the the Clebsch-Gordan decomposition of tensor product representations play a seminal role in the derivation of all properties of the representation of $\mathcal{A}$. Thus one can repeat the same arguments for other quantum deformations of $OSp(1/2)$, namely the Jordanian\cite{23} and the super-Jordanian\cite{24, 25} analogs for deriving their product law, the orthogonality and the recurrence relations. Being triangular algebras, the Jordanian and the super-Jordanian deformations of $OSp(1/2)$ possess the same Clebsch-Gordan decomposition as in the present case.

The representations of $SL_q(2)$ (or $SU_q(2)$) have been discussed by many authors. Among the properties of representation matrices, the product law\cite{26}, recurrence relations\cite{26, 27}, orthogonality and $RTT$-relation\cite{27} and generating functions\cite{25} are found in literature. In Ref\cite{27}, the representation matrices are interpreted as the wave functions of quantum symmetric top in noncommutative space. Representations of the Jordanian quantum group $SL_h(2)$ have been considered in Refs\cite{29, 30}.

VI. REPRESENTATION MATRIX AND LITTLE $Q$-JACOBI POLYNOMIALS

Explicit formulae for the representation matrices of the quantum group $SU_q(2)$ have been obtained by several authors\cite{21, 22, 23}. It is observed that, for the finite dimensional representations, the matrix elements are expressed in terms of the little $q$-Jacobi polynomials. Investigating the Jordanian quantum group $SL_h(2)$ in a similar framework, it has also been noted that the conventional Jacobi polynomials contribute\cite{22} to the representation matrices therein. The corresponding matrix elements for the two-parametric quantum group $GL(2)$ have been computed in Ref\cite{31}. Their relation to orthogonal polynomials, however, is still an open problem.

In this section, we provide the explicit form of the representation matrices (5.1) of $\mathcal{A}$ by direct computation, and study the resulting polynomial structure. Towards this end,
we proceed by noticing the identities obtained by repeated use of (2.9):

\[ V^c e_m^\ell(\lambda) = (-1)^c (\ell - m + c - 3)/2 \left( \frac{1}{\{2\}_q} \frac{\ell + m + c}{\ell - m + c} \right)^{1/2} e_m^{\ell - c}(\lambda), \]

\[ V^m e_m^\ell(\lambda) = \left( \frac{1}{\{2\}_q} \frac{\ell + m + a}{\ell - m - a} \right)^{1/2} e_m^{\ell + a}(\lambda). \]

The explicit listing of the basis elements of \( A \) in (3.17) renders the computation of the matrix elements straightforward. The final result is quoted below:

\[ T^\ell_{m^\prime m}(\lambda) = \left( -1 \right)^c (m^\prime - m)(m^\prime - m - 1)/2 + (m^\prime - m)(\ell - m + \lambda) q^{m(m^\prime - m)/2} \sum \left( \frac{\ell + m + c}{\ell - m - c} \right)^{1/2} \frac{x^{m^\prime - m + c}}{\{2\}_q^{\ell - m + c}} \exp \left( \frac{m - c}{2} \right) \frac{y^c}{\{2\}_q^c}, \]

where the index \( c \) runs over all non-negative integers maintaining the argument of \( \{A\}_q \) non-negative.

The fundamental representation \( (\ell = 1, \lambda = 0) \) may be identified with the quantum supermatrix of Ref[6] à la (5.11). This identification allows us to realize the entries of the quantum supermatrix in terms of the generators of the quantum supergroup \( A \):

\[ a = xy + e^{z/2} + \frac{x^2 e^{-z/2} y^2}{\{2\}_q^2}, \quad \alpha = x - \frac{x^2 e^{-z/2}}{q^{1/2} \{2\}_q}, \quad b = \frac{x^2 e^{-z/2}}{q \{2\}_q}, \]

\[ \gamma = y + \frac{q^{1/2} x e^{-z/2} y^2}{\{2\}_q^2}, \quad e = 1 - x e^{-z/2}, \quad \beta = -q^{-1/2} x e^{-z/2}, \]

\[ c = -\frac{q e^{-z/2} y^2}{\{2\}_q}, \quad \delta = q^{1/2} e^{-z/2}, \quad d = e^{-z/2}. \]

Straightforward computation using the commutation relations (6.3) allows us to infer that the realization (6.3) recovers all the commutation relations of the supermatrix listed in Ref[6]. The realization (6.3), more importantly, implies the following Gaussian decomposition of the quantum supermatrix (6.3):

\[ \begin{pmatrix} 1 & 0 & 0 & 0 \\ -q^{-1/2} x & 1 & 0 & 0 \\ -\frac{1}{q^{1/2} x^2} & 0 & 1 & 0 \\ x & 1 & 0 & e^{z/2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & q^{1/2} y \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta & c & \beta & a \\ e & \gamma & \alpha & b \\ \beta & \gamma & \alpha & d \\ c & \delta & \beta & a \end{pmatrix} C^{-1}, \]

where

\[ C = \begin{pmatrix} a & \alpha & b & c \\ \beta & \gamma & \delta & d \end{pmatrix}. \]
where

\[
C = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}, \quad C^{-1} = C.
\]

We now turn our attention to the polynomial structure built into the general matrix element (6.2) in terms of the variable

\[
\zeta = \frac{q^{-1/2}}{2} xe^{-z/2y}.
\]

(6.5)

To demonstrate this, the product of generators in (6.2) for the case \(m' - m \geq 0\) may be rearranged as follows:

\[
x^{m'-m+c} \exp \left( \frac{m - c}{2} z \right) y^c = (-1)^{c(c-1)/2} q^{-mc} x^{m'-m} e^{mz/2} (xe^{-z/2y})^c.
\]

The matrix element \(T_{m'm'}(\lambda)\) may now be succinctly expressed as a polynomial structure given below:

\[
T_{m'm'}(\lambda) = (-1)^{(m'-m)(m'-m+1)/2 + (m'-m)(\ell-m')/2} q^{m(m'-m)/2} \frac{m' - m \gamma q}{\ell - m - c} \frac{(\ell + m)^{\gamma q}}{(\ell - m)^{\gamma q}} P_{m'm'}(\zeta).
\]

(6.6)

The polynomial \(P_{m'm'}(\zeta)\) in the variable \(\zeta\) is defined by

\[
P_{m'm'}(\zeta) = \sum_c (-1)^{c(\ell-m)+c(c-1)/2} q^{-c(m'+m+1)/2}
\]

\[
\times \frac{m' - m \gamma q}{\ell - m - c} \frac{\gamma q}{\ell + m - c} \frac{\gamma q}{\ell - m} \frac{\gamma q}{\ell + m}. \]

(6.7)

where the index \(c\) runs over all non-negative integers maintaining the arguments of \(\{X\}_q\) non-negative. For the case \(m' - m \leq 0\), we make a replacement of the summation index \(c\) with \(a = m' - m + c\). Rearrangement of the generators now provides the following expression of the general matrix element:

\[
T_{m'm'}(\lambda) = (-1)^{(m'-m)(m'-m+1)/2 - (m'-m)(\lambda-1)} q^{m(m'-m)/2} \frac{m - m' \gamma q}{\ell - m'} \frac{(\ell + m')^{\gamma q}}{(\ell - m)^{\gamma q}} P_{m'm'}(\zeta).
\]

(6.8)
where the polynomial $P_{m'm}(\zeta)$ for $m' - m \leq 0$ is defined by
\[
P_{m'm}(\zeta) = \sum_a (-1)^{a(\ell-m')} a(a-1)/2 q^{a(m'+m-1)/2} \times \frac{\{m-m'\}_q! \{l+m'\}_q! \{\ell-m' + a\}_q! \{m-m'+a\}_q! \{l+m'-a\}_q! \{\ell-m'+a\}_q! \{a\}_q! \zeta^a}{\{m-m'+a\}_q! \{l+m'+a\}_q! \{\ell-m'+a\}_q! \{a\}_q!).
\] (6.9)

It is immediate to note that the polynomials are symmetric with respect to the transposition $m \leftrightarrow m'$: $P_{m'm'}(\zeta) = P_{m'm}(\zeta)$, and that $P_{-\ell m}(\zeta) = P_{m'-\ell}(\zeta) = 1$.

Polynomials obtained above are related to the basic hypergeometric functions. We define the basic hypergeometric function $2\phi_1$ by
\[
2\phi_1(a_1, a_2; b; Q; z) = \sum_{n=0} (a_1; Q)_n (a_2; Q)_n b^n z^n,
\] (6.10)
where the shifted factorial is defined as usual:
\[
(x; Q)_n = \begin{cases} 1, & n = 0 \\ \prod_{k=0}^{n-1} (1 - xQ^k), & n \neq 0 \end{cases}
\] (6.11)

The little $Q$-Jacobi polynomials are defined via $2\phi_1$ as standard theory of orthogonal polynomials
\[
p_m^{(a,b)}(z) = 2\phi_1(Q^{-m}; abQ^{m+1}; aQ; Qz).
\] (6.12)
Setting $a = Q^\alpha, b = Q^\beta$, we have the following form of little $Q$-Jacobi polynomials
\[
p_m^{(\alpha,\beta)}(z) = \sum_n \frac{(Q^{-m}; Q)_n (Q^{\alpha+\beta+m+1}; Q)_n}{(Q^{\alpha+1}; Q)_n (Q; Q)_n} (Qz)^n.
\] (6.13)

Rewriting our polynomials (6.7) and (6.9) in terms of the shifted factorial with $Q = -q$, one can identify our polynomials with the little $Q$-Jacobi polynomials. For the choice $m' - m \geq 0$, the polynomial structure reads
\[
P_{m'm}(\zeta) = \sum_a \frac{(-q)^{-\ell-m}; -q}_a (-q)^{\ell-m+1}; -q)_a (-q\zeta)^a = p_{\ell+m,-m'-m}(\zeta),
\] (6.14)
and for the $m' - m \leq 0$ case its identification is given by
\[
P_{m'm}(\zeta) = \sum_a \frac{(-q)^{-\ell-m'; -q}_a (-q)^{\ell-m'+1}; -q)_a (-q\zeta)^a = p_{\ell+m',-m'-m}(\zeta).
\] (6.15)

It is amazing that $Q = -q$ polynomials appear for the supergroup $\mathcal{A}$ in contrast to the $Q = q$ polynomials being present for the quantum group $SU_q(2)$ [31, 32, 33].
VII. CONCLUDING REMARKS

Starting from the construction of the universal $\mathcal{T}$-matrix, we have investigated the finite dimensional representations of the quantum supergroup $\mathcal{A}$. A qualitative difference between the universal $\mathcal{T}$-matrices for the classical and the quantum $OSp(1/2)$ algebras exists due to the nilpotency of the classical parity odd elements. The absence of the said nilpotency in the quantum case induces a new polynomial structure in the matrix elements of $\mathcal{T}$. We observe that these polynomials are expressed in terms of the little $Q$-Jacobi polynomials. This suggests a new link between orthogonal polynomials and representations of quantum supergroups. It is likely to be a general property that if the Grassmann variables in a classical supergroup lose nilpotency at the quantum level, it may be reflected in the representations of the quantum supergroup so that the entries of representation matrices may have a new quantized polynomial structure. The present work provides an example of this statement. Another likely candidate for the existence of similar polynomial structure is the super-Jordanian $OSp(1/2)$ where the loss of nilpotency has been observed. We believe that the investigation along this line will give a new algebraic background to basic hypergeometric series.

We have also tried to extend the known properties of the representations for $SL_q(2)$ to the quantum supergroup $\mathcal{A}$. An extension of the product law, the orthogonality and the recurrence relations was shown to be possible. However, two known results are not extended in the present work, that is, the generating function of the representation matrices and the Peter-Weyl theorem. In order to discuss the Peter-Weyl theorem, a Haar measure has to be defined on $\mathcal{A}$. Since the loss of nilpotency makes the superspace on $\mathcal{A}$ more complex than the classical case, studying the Peter-Weyl theorem may be interesting from the viewpoint of harmonic analysis on quantum supergroups. We will discuss these issues elsewhere.

ACKNOWLEDGEMENTS

A comment by the Referee motivated us in finding the present form of (3.30). We thank him for this. One of us (N.A.) would like to thank R. Jagannathan for his warm hospitality at The Institute of Mathematical Sciences where a part of this work was done. The work of N.A. is partially supported by the grants-in-aid from JSPS, Japan (Contract No. 15540132). Other authors (R.C., S.S.N.M. and J.S.) are partially supported by the grant DAE/2001/37/12/BRNS, Government of India.

APPENDIX: UNIVERSAL $\mathcal{T}$-MATRIX FOR QUANTUM SUPERGROUPS

This Appendix is devoted to a general discussion of the universal $\mathcal{T}$-matrix for quantum supergroups. In particular, the relations used in §III and §IV are proved in general setting.

Let $\mathcal{U}$ and $\mathcal{A}$ be dually conjugate unital $\mathbb{Z}_2$ graded Hopf algebras. The basis of the algebras $\mathcal{U}$ and $\mathcal{A}$ are denoted by $E_k$, $e^k$, respectively. One may assume that $E_0 = 1_{\mathcal{U}}$, $e^0 =
\[ E_k E_\ell = \sum_m f_{k \ell}^m E_m, \quad \Delta(E_k) = \sum_{pq} g_k^{pq} E_p \otimes E_q, \quad (A.1) \]

\[ e^k e^\ell = \sum_m g_m^{k \ell} e^m, \quad \Delta(e^k) = \sum_{pq} f_p^k q e^p \otimes e^q. \quad (A.2) \]

The universal \( T \)-matrix is defined by
\[ T_{e,E} = \sum_k (-1)^{p(e^k)(p(e^k)-1)/2} e^k \otimes E_k, \quad e, E \in A \otimes \mathcal{U} \quad (A.3) \]

Although the factor \((-1)^{p(e^k)(p(e^k)-1)/2}\) is trivial, it is convenient to keep it in the discussion of universal \( T \)-matrix.

We start with the proof of the relations in (3.25). The proof of the first relation in (3.25) is straightforward
\[ T_{e,E} T_{e',E} = \sum_{k,f,m} (-1)^{(p(e^k)+p(e'))(p(e^k)+p(e')-1)/2} e^k \otimes e' \otimes f_{k \ell}^m E_m \]
\[ = \sum_m (-1)^{p(e^m)(p(e^m)-1)/2} \Delta(e^m) \otimes E_m \]
\[ = T_{\Delta(e),E}. \]

The second equality is due to the fact that the parity of the both sides of the first equation in (A.1) are equal, and that the structure constants are of even parity. The second relation in (3.25) can be proved similarly. The third relation in (3.25) follows from \( \epsilon(e^k) = \delta_0^k, \quad \epsilon(E_k) = \delta_{k0}. \)

The last relation in (3.25) is a consequence of the identities:
\[ T_{e,S(E)} T_{e,E} = T_{e,E} T_{e,S(E)} = 1 \otimes 1, \quad T_{S(e),E} T_{e,E} = T_{e,E} T_{S(e),E} = 1 \otimes 1. \quad (A.4) \]

The above identities can be proved by using the axiom of antipode and \( p(S(e^k)) = p(e^k). \)

We next derive the \( RTT \)-type relations (4.4) and (4.8). Defining
\[ T_{e,E}^{(1)} = \sum_k (-1)^{p(e^k)(p(e^k)-1)/2} e^k \otimes E_k \otimes 1, \quad T_{e,E}^{(2)} = \sum_k (-1)^{p(e^k)(p(e^k)-1)/2} e^k \otimes 1 \otimes E_k, \]
we obtain
\[ T_{e,E}^{(1)} T_{e,E}^{(2)} = \sum_m (-1)^{p(e^m)(p(e^m)-1)/2} e^m \otimes \Delta(E_m). \]

On the other hand the transposed coproduct appears in the reverse-ordered product:
\[ T_{e,E}^{(2)} T_{e,E}^{(1)} = \sum_m (-1)^{p(e^m)(p(e^m)-1)/2} e^m \otimes \Delta'(E_m) \]
\[ = \sum_m (-1)^{p(e^m)(p(e^m)-1)/2} e^m \otimes \mathcal{R}\Delta(E_m)\mathcal{R}^{-1} \]
\[ = (1 \otimes \mathcal{R}) T_{e,E}^{(1)} T_{e,E}^{(2)} (1 \otimes \mathcal{R}^{-1}). \]
In the last equality, the fact that the universal $R$-matrix is of even parity has been used. This completes our proof of the $RTT$-type relation (4.6). The proof of (4.8) may be done similarly with the definitions

$$T_{E,e}^{(1)} = \sum_k (-1)^{p(e^k)p(e^k)-1/2} E_k \otimes 1 \otimes e^k, \quad T_{E,e}^{(2)} = \sum_k (-1)^{p(e^k)p(e^k)-1/2} 1 \otimes E_k \otimes e^k.$$

**Note added:** After submitting the manuscript, we have learnt about the work of Zou, where the representations of $OSp_{q}(1/2)$ and their relation to the basic hypergeometric functions are computed by using other basis states, and adopting a method (similar to Ref. [33]) different from ours. Peter-Weyl theorem has also been established there.

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