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A stochastic SIQR epidemic model with Lévy jumps and three-time delays

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\textbf{A B S T R A C T}

Isolation and vaccination are the two most effective measures in protecting the public from the spread of illness. The SIQR model with vaccination is widely used to investigate the dynamics of an infectious disease at population level having the compartments: susceptible, infectious, quarantined and recovered. The paper mainly aims to extend the deterministic model to a stochastic SQIR case with Lévy jumps and three-time delays, which is more suitable for modeling complex and unstable environment. The existence and uniqueness of the global positive solution are obtained by using the Lyapunov method. The dynamic properties of stochastic solution are studied around the disease-free and endemic equilibria of the deterministic model. Our results reveal that stochastic perturbation affect the asymptotic properties of the model. Numerical simulation shows the effects of interested parameters of theoretical results, including quarantine, vaccination and jump parameters. Finally, we apply both the stochastic and deterministic models to analyze the outbreak of mutant COVID-19 epidemic in Gansu Province, China.

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1. Introduction

Infectious disease is a strong antagonist, posing a threat to human life and health. Recently, the outbreak of the novel coronavirus (COVID-19) had a huge impact on social life all over the world. The control and eradication of the disease have been an urgent problem and received considerable attention from the researchers. Mathematical modeling is one of the most commonly used methods for analyzing the transmission mechanisms as well as controlling various epidemic diseases. Kermack and McKendrick [1] used a rigorous mathematical approach and for the first time, they formulated a susceptible-infected-recovered (SIR) epidemic model to discuss the spread of contagious diseases. Following that, numerous researchers used ordinary differential equations for modelling different types of infectious diseases, for instance, one can see [2–4]. During the transmission of many diseases, quarantine and vaccination are the two most effective methods to prevent the epidemic from spreading further, such as COVID-19. Hence, more and more compartments and factors that inhibit the spread of diseases were considered in the SIR model. Suppose that the total host population is partitioned into susceptible, infectious, quarantined and recovered individuals, respectively, whose size is $S(t)$, $I(t)$, $Q(t)$, and $R(t)$ at any time $t$. Ma et al. [5] formulated and analyzed an SIQR model assuming the hybrid strategies and the bilinear incidence rate in the following

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form
\[
\begin{align*}
    dS &= (A - \phi S(t))l(t) - (\mu + p)S(t)dt, \\
    dl &= \left(\phi S(t)l(t) - (\mu + \zeta_1 + \pi + \rho + q)l(t)\right)dt, \\
    dQ &= \left(\pi l(t) - (\mu + \zeta_2 + \kappa)Q(t)\right)dt, \\
    dR &= (\rho l(t) + pS(t) + \kappa Q(t) - \mu R(t))dt,
\end{align*}
\]  
(1)

where the parameter \(A\) is the constant recruitment into the susceptible compartment, \(\phi\) is the rate at which the disease is spreading in the population, \(p\) is the vaccination rate of the susceptible individuals, \(q\) represents the elimination rate of the infective class and the natural death rate is denoted by \(\mu\) which is constant for all compartments. The parameter \(\rho\) denotes the recovery rate of individuals that remain infectious for a period of \(1/\rho\), the notations \(\zeta_1\) and \(\zeta_2\) are the respective disease-related death rates in the infected and quarantined populations. The rate at which the infected population becoming quarantined is denoted by \(\pi\), whereas, \(\kappa\) is the rate at which quarantine population is getting recovery. For biological purposes, we assume that all of the parameters used in the model are constants and positive.

In practice, the history related to an epidemic can play a significant role in studying the dynamics of certain infectious diseases. That is because, at any time \(t\), the dynamics of diseases are subject to some previous conditions. Time delay is the most generally used tool to simulate the infectious period in the models which describe dynamics of epidemics. Such conditions occur frequently and extensively in the actual world such as temporary immunity and incubation period. Therefore, it is natural to introduce time delay into the SIQR model describing the dynamics of an infectious disease [6–8]. Lu et al. [9] proposed an SIQR epidemic model with one time-delay to describe the spreading behavior of COVID–19. However, there are two limitations of this model: (i) It fails to consider the effect of COVID-19 vaccination. (ii) There exist other time delays in the susceptible and quarantined periods, similar to the infectious period. Inspired by the works of Ma et al. [5,9], we introduce three-time delays into the SIQR model (1) with vaccination, and establish a novel model as
\[
\begin{align*}
    dS &= (A - \phi S(t))l(t) - (p + \mu)S(t) + pme^{-\mu t_1}S(t - t_1) + \rho e^{-\mu t_2}l(t - t_2) + \kappa e^{-\mu t_3}Q(t - t_3)dt, \\
    dl &= \left(\phi S(t)l(t) - (\mu + \zeta_1 + \rho + \pi)l(t)\right)dt, \\
    dQ &= \left(\pi l(t) - (\mu + \zeta_2 + \kappa)Q(t)\right)dt, \\
    dR &= (\rho l(t) + pS(t) + \kappa Q(t) - \mu R(t))dt,
\end{align*}
\]  
(2)

where \(t_1\) is positive and stands for the length of immunity period of the vaccinated individuals, the term \(e^{-\mu t_1}\) represents the probability that a susceptible individual is vaccinated at time \(t - t_1\) and he/she is still alive at time \(t\). Similarly, the second and third delay terms \(t_2\) and \(t_3\) are positive and these reflect the durations of the immunity period of individuals being recovered from infected and quarantined compartment, respectively. In the same way, the quantities \(e^{-\mu t_2}\) and \(e^{-\mu t_3}\) denote the probabilities of infected and quarantined individuals who are respectively immunized at time \(t - t_2\) and \(t - t_3\), lose their immunity and still alive at time \(t\). The notion \(m\) is the rate of recovered people who forget to re-vaccinate before losing immunity. Since the first three equations in model (2) do not depend on the fourth equation, thus, we have the following simplified form of the model
\[
\begin{align*}
    dS &= (A - \phi S(t))l(t) - (p + \mu)S(t) + pmS(t - t_1)e^{-\mu t_1} + \rho l(t - t_2)e^{-\mu t_2} + \kappa Q(t - t_3)e^{-\mu t_3}dt, \\
    dl &= \left(\phi S(t)l(t) - (\mu + \zeta_1 + \pi + \rho)l(t)\right)dt, \\
    dQ &= \left(\pi l(t) - (\mu + \zeta_2 + \kappa)Q(t)\right)dt.
\end{align*}
\]  
(3)

From model (3), we can easily obtain the basic reproduction number
\[
R_0 = \frac{\phi A}{(\mu + (1 - me^{-\mu t_1})p)(\zeta_1 + \mu + \rho + \pi)},
\]  
(4)

which determines whether the disease is tend to extinct or not. The deterministic model (3) can describe the general trends of epidemic and has a good prediction effect for large-scale infectious diseases. The dynamic properties of this model will be discussed later in this work.

However, epidemics are inevitably affected by the environmental noises, which are the important components to be taken into account by mathematical models. Therefore, it is suitable to use the tools of stochastic modelling to describe such characteristics of infectious diseases, and perform better on local infectious diseases or a small number of infections, comparing to the deterministic model (3). In this regard, various stochastic epidemic models were formulated and analyzed to reveal the influence of environmental noises [10–16]. El Fatini et al. [17] proposed a triple delayed SIQR model with white noises. Further, Pitchaimani [18] studied the triple delayed SICR model with general incidence rate. In these results, the authors only assumed that the stochastic models were driven by the white noises. For some highly infectious diseases, the environmental noises may lead to severe fluctuations in the number of infected people, such as COVID-19 [19,20], Tuberculosis [21], etc. Thus, it is inappropriate and unrealistic to prefer the deterministic over stochastic approach while modeling such epidemics. Based on these facts, numerous researchers take into account the stochastic epidemic modeling driven by Lévy jumps [22–30]. Recently, Koufi et al. [31] proposed a stochastic SIQR epidemic model and discussed the effect of Lévy jumps as well as utilized the incidence rate of Beddington-DeAngelis type. The model of Koufi et al. [31] is indeed a great contribution to stochastic epidemic model from various perspectives. However, the model ignored the effect of time delays
which arise naturally. Besides that, they did not investigate the asymptotic properties of solutions around the equilibrium points.

Motivated by the aforementioned work, in this paper, we employ three white noises and a stationary Poisson point process as the driven jump process into model (3). Let us consider a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ that is right continuous as well as increasing. Also, $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Suppose that $\mathbb{N}$ is a Poisson counting measure with characteristic measure $\nu$ on measurable subset $Z$ of $[0, \infty)$, satisfying $\nu(Z) < \infty$. Define $\eta_i(z) : Z \times \Omega \rightarrow \mathbb{R}$ for $i = 1, 2, 3$, which are used for reflecting the impacts of random jumps. The compensated random measure is denoted by $\tilde{N}$ and is defined by $\tilde{N}(dt, du) = N(dt, du) - \nu(du)dt$. Based on model (3), we propose a stochastic epidemic model with Lévy jumps given by the following system

$$
\begin{align*}
    dS &= (A - \phi S(t))I(t) - (p + \mu)S(t) + pmS(t - \tau_1)e^{-\mu t_2} + \rho I(t - \tau_2)e^{-\mu t_2} + \kappa Q(t - \tau_2)e^{-\alpha t_3}dt + \sigma_1 S(t)dB_1(t) + \int_{\mathbb{R}} \eta_1(z)S(t - z)N(dt, dz), \\
    dl &= (\phi S(t))I(t) - (\mu + \gamma + \rho)I(t)dt + \sigma_2 I(t)dB_2(t) + \int_{\mathbb{R}} \eta_2(z)I(t - z)N(dt, dz), \\
    dQ &= (\tau I(t) - (\mu + \gamma + \kappa)Q(t))dt + \sigma_3 Q(t)dB_3(t) + \int_{\mathbb{R}} \eta_3(z)Q(t - z)N(dt, dz),
\end{align*}
$$

where $S^-(t), I^-(t)$, and $Q^-(t)$ denote the left limits of $S(t), I(t)$ and $Q(t)$, respectively. The notions $B_i(t)(i = 1, 2, 3)$ are the mutually independent standard Brownian motions. Similarly, $\sigma_i^2 > 0 (i = 1, 2, 3)$ are the perturbation volatilities.

To the best of our knowledge, due to the complexity of constructing appropriate Lyapunov functions and related calculations, there are very few studies on the asymptotic properties around the equilibrium points of delayed models. The present work is indeed a great novelty for the stochastic delay approach and for the asymptotic properties around the equilibrium points in stochastic models. The rest of this paper is organized as follows. We review some definitions and notation which are used in the later parts of the study in Section 2. In Section 3, we firstly prove that there is a unique positive global solution of the stochastic SIQR model (5). Then, we investigate the dynamic properties of stochastic solutions around the equilibrium points of the model (3). In Section 4, some numerical solutions are presented to validate the obtained analytical findings. In Section 5, the spread of mutant COVID-19 is investigated through our proposed models in Gansu Province, China. A brief conclusion is given in Section 6.

2. Preliminaries

Consider the following three-dimensional stochastic differential equation with Lévy jumps

$$
\begin{align*}
    dY(t) &= f(Y(t), Y(t - \tau), \mu)dt + g(Y(t), \tau)dB_t \\
    &+ \int_{\mathbb{R}} H(Y(t), \gamma, z)\tilde{N}(dt, dz) \quad \text{for } \tau \geq 0, t \geq -\tau
\end{align*}
$$

with the initial value $Y(t_0) = Y_0 \in \mathbb{R}^3$, where $\mathbb{R}^3_+ = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_j > 0, i = 1, 2, 3\}$. The operator $L$ acting on the equation (6) is defined by

$$
\begin{align*}
    LY(t) &= \frac{\partial f}{\partial t} + \sum_{i=1}^3 \frac{\partial f_i}{\partial y_i} f_i(t, Y(t), Y(t - \tau), \tau) + \frac{1}{2} \sum_{i,j=1}^3 \frac{\partial^2 f_i}{\partial y_i \partial y_j} \left[ g_i^2(t, Y(t), \tau) \right]_{ij} \\
    &+ \int_{\mathbb{R}} \left[ Y(t) - H(Y(t), \gamma, z) - Y(t - \tau) - \frac{\partial f_i}{\partial y_i} H(Y(t), \gamma, z) \right] \nu(dz).
\end{align*}
$$

Let $C([-\tau, 0], \mathbb{R}^3)$ be the Banach space of all continuous mappings which is equipped with the norm $\|\varphi\| = \sup \{ \varphi(\theta) \}$ such that $-\tau \leq \theta \leq 0$. Denote $C^2([0, \infty); \mathbb{R}^3_+)$ as the family of all nonnegative functions $F(Y, \tau) \in C^2([0, \infty); \mathbb{R}^3_+) \times [-\tau, \infty)$ as the family of all nonnegative functions $F(Y, \tau)$ defined on $C([-\tau, 0], \mathbb{R}^3_+)$ such that they are continuously twice differentiable in $Y$ and once in $\tau$. By applying the operator $L$ on a Lyapunov function $F(Y, \tau) \in C^2([-\tau, 0]; \mathbb{R}^3_+) \times [-\tau, \infty); \mathbb{R}_+)$, it follows that

$$
\begin{align*}
    LF(Y(t)) &= F_i(Y(t)) + F_j(Y(t))f_i(Y(t), Y(t - \tau), \pi) + \frac{1}{2} \text{trace} \left[ g_i^2(t, Y(t), \tau) \right]_{ij} \\
    &+ \int_{\mathbb{R}} \left[ F_i(Y(t) + H(Y(t), \gamma, z)) - F_i(Y(t)) - \frac{\partial f_i}{\partial y_i} H(Y(t), \gamma, z) \right] \nu(dz),
\end{align*}
$$

where

$$
\begin{align*}
    F_i &= \frac{\partial F}{\partial x_i}, \\
    F_j &= \left( \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \frac{\partial F}{\partial x_3} \right), \\
    F_{ij} &= \left( \frac{\partial^2 F}{\partial x_i \partial x_j} \right)_{3 \times 3}.
\end{align*}
$$

By utilizing the Itô’s formula, we obtain

$$
\begin{align*}
    dF(Y(t)) &= LF(Y(t))dt + F_j(Y(t))g(t, Y)dB_t \\
    &+ \int_{\mathbb{R}} [F_i(Y(t) + H(Y(t), \gamma, z)) - F_i(Y(t))] \tilde{N}(ds, dz).
\end{align*}
$$

3. Main results

Before studying the dynamical behavior of model (5), it is perhaps the most significant step to show the existence of a global positive solution of the model. Let $\tau = \max \{\tau_1, \tau_2, \tau_3\}$. Denote $\mathbb{R}^3_+ = \{(S(t), I(t), Q(t)) \in \mathbb{R}^3 : S(t) > 0, I(t) > 0, Q(t) > 0\}$ and $C([-\tau, 0], \mathbb{R}^3_+)$. 
**Theorem 1.** For an initial value \((S(0), I(0), Q(0)) \in \mathbb{R}_+^3\), there exists a unique positive solution \((S(t), I(t), Q(t))\) of model (5) for all time \(t \geq -\tau\). The solution will remain in the space \(\mathbb{R}_+^3\) with unit probability.

**Proof.** Obviously, all the coefficients of model (5) are locally Lipschitz continuous. Hence, the existence of a unique local solution \((S(t), I(t), Q(t))\) is sure for all \(t \in [-\tau, \tau_e)\) and an initial value \((S(0), I(0), Q(0))\) that lies within the space \(\mathbb{R}_+^3\). Here, the notion \(\tau_e\) represents the duration of the explosion. To show that this solution is global as well, it is sufficient to prove that \(\tau_e = \infty\) almost surely. To prove this, we consider a sufficiently large positive integer \(\nu\) such that \((S(0), I(0), Q(0))\) belong to the interval \([\frac{1}{\nu}, \nu]\), respectively. Further, for every integer \(\nu\) greater or equal than \(\nu_0\), the stopping time is defined as

\[
\tau_i = \inf \left\{ t \in [0, \tau_e) : S(t) \notin \left( \frac{1}{\nu}, \nu \right) \text{ or } I(t) \notin \left( \frac{1}{\nu}, \nu \right) \text{ or } Q(t) \notin \left( \frac{1}{\nu}, \nu \right) \right\},
\]

where \(\inf \Phi = \infty\) and \(\Phi\) is the null set. Surely, \(\tau_i\) is a monotonically increasing function of \(i\). Set \(\tau_{\infty} = \lim_{\nu \to \infty} \tau_i\), which implies \(\tau_{\infty} \leq \tau_e\) almost surely. If we prove that \(\tau_{\infty} = \infty\), it will reflect that \(\tau_e = \infty\) and ultimately assures that \((S(t), I(t), Q(t))\) lies within the space \(\mathbb{R}_+^3\) a.s. for all \(t \geq -\tau\). If this statement is false, then there exists a constant \(T > 0\) and \(\epsilon\) from the interval \((0,1)\) such that

\[
\epsilon < P(T \geq \tau_{\infty}).
\]

As a result, there must exist an integer \(\nu_1 \geq \nu_0\) such that

\[
\epsilon < P(T \geq \tau_i), \forall \nu_1 \leq i.
\]

To proceed further, a \(C^2\)-function \(V_1\) from the space \(\mathbb{R}_+^3\) to \(\mathbb{R}_+\) is assumed as

\[
V_1(S, I, Q) = \left( S(t) - a + a \ln \frac{S(t)}{a} \right) + (I(t) - 1 - \ln I(t)) + (Q(t) - 1 - \ln Q(t)) + pme^{-\mu \tau_1} \int_{S_1 - \tau_1} S(s) ds + e^{-\mu \tau_1} \rho \int_{S_1 - \tau_1} I(s) ds + e^{-\mu \tau_1} \kappa \int_{S_1 - \tau_1} Q(s) ds, \tag{9}
\]

where \(a\) is a positive constant to be determined later. By using the Itô formula (8), we have

\[
dV_1(S, I, Q) = dV_1 dt + \sigma_1(S(t) - a) dB_1(t) + \sigma_2(I(t) - 1) dB_2(t) + \sigma_3(Q(t) - 1) dB_3(t) + \int_{\frac{1}{\nu}} \frac{1}{\nu} \ln(1 + \eta_1(z)) dN(dt, dz) + \int_{\frac{1}{\nu}} \frac{1}{\nu} \ln(1 + \eta_3(z)) dN(dt, dz), \tag{10}
\]

where

\[
IV_1 = A - \phi S(t) I(t) - (p + \mu) S + pme^{-\mu \tau_1} S(t - \tau_1) + \rho e^{-\mu \tau_1} I(t - \tau_2) + \kappa Q(t - \tau_3) e^{-\mu \tau_3}
\]

\[
+ k + \phi l(t) S(t) - (\mu + \zeta_1 + \rho + \pi) I(t) + \rho l(t) - (\mu + \zeta_2 + \kappa) Q(t) + \frac{A}{S(t)}
\]

\[
- (\mu + p) - \phi l(t) - \frac{a p S(t - \tau_1) e^{-\mu \tau_1}}{S(t)} - \frac{\rho l(t - \tau_2) e^{-\mu \tau_2}}{S(t)} - \frac{\kappa Q(t - \tau_3) e^{-\mu \tau_3}}{S(t)}
\]

\[
+ \phi S(t) - (\mu + \zeta_1 + \pi + \rho) + \frac{\pi I(t)}{Q(t)} - (\mu + \zeta_2 + \kappa) + \frac{a \sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2} + \int_{\frac{1}{\nu}} [a \eta_1(z) - a \ln(1 + \eta_1(z)) + \eta_2(z) - \ln(1 + \eta_2(z)) + \eta_3(z) - \ln(1 + \eta_3(z))] \nu(dz)
\]

\[
+ pme^{-\mu \tau_1} S(t) - pme^{-\mu \tau_1} S(t - \tau_1) + e^{-\mu \tau_1} I(t - \tau_2) + \kappa e^{-\mu \tau_3} C(t)
\]

\[
- \kappa e^{-\mu \tau_3} C(t - \tau_3)
\]

\[
\leq A - (\zeta_1 + \rho + \pi) I + a \phi l + (q + \mu) a + \mu + \zeta_1 + \pi + \rho + \mu + \zeta_2 + \kappa + \frac{a \sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2} + \int_{\frac{1}{\nu}} [a \eta_1(z) - a \ln(1 + \eta_1(z)) + \eta_2(z) - \ln(1 + \eta_2(z)) + \eta_3(z) - \ln(1 + \eta_3(z))] \nu(dz).
\]

Choosing \(a = \frac{\mu + \zeta_1}{\pi}\) and keeping in mind the fact that

\[
0 \leq \int_{\frac{1}{\nu}} [\eta_i(z) - \ln(1 + \eta_i(z))] \nu(dz) \leq K_0, \quad i = 1, 2, 3,
\]

where \(K_0\) is a constant, we have

\[
IV_1 \leq A + \mu (2 + a) + pa + \rho + \zeta_1 + \pi + (a \sigma_1^2 + \sigma_2^2 + \sigma_3^2)/2 + (a + 2)K_0 \triangleq K > 0.
\]
Hence, it yields that
\[
\begin{align*}
  dV_1(S, I, Q) &= Kdt + \sigma_1(S(t) - a)dB_1(t) + \sigma_2(I(t) - 1)dB_2(t) + \sigma_3(Q(t) - 1)dB_3(t) \\
  &+ \int_\mathcal{Z} [\eta_1(z)S(t-\cdot) - a \ln (1 + \eta_1(z))]N(dt, dz) \\
  &+ \int_\mathcal{Z} [\eta_2(z)I(t-\cdot) - \ln (1 + \eta_2(z))]N(dt, dz) \\
  &+ \int_\mathcal{Z} [\eta_3(z)Q(t-\cdot) - \ln (1 + \eta_3(z))]N(dt, dz).
\end{align*}
\]

By integrating both sides of Eq. (11) from 0 to \(\tau, T\) and then assuming the expectation, we have
\[
\begin{align*}
  0 &\leq EV_1(S(\tau, T), I(\tau, T), Q(\tau, T)) \\
  &\leq V_1(S(0), I(0), Q(0)) + KE(\tau, T) \\
  &\leq V_1(S(0), I(0), Q(0)) + KT.
\end{align*}
\]
For any \(\varepsilon \geq \varepsilon_1\), we have \(\Omega = \{\varepsilon \leq \Omega\}\). Consequently, at least one of the classes \(S(\tau, \omega), I(\tau, \omega), Q(\tau, \omega)\) equals either \(\varepsilon\) or \(\frac{1}{\varepsilon}\) for each \(\omega \in \Omega\). Thus
\[
V_1(S(\tau), I(\tau), Q(\tau)) \geq \left(-a + \varepsilon - a \ln \frac{1}{\varepsilon}\right) \wedge (-1 + \varepsilon - \ln \varepsilon) \wedge \left(-a + \frac{1}{\varepsilon} + a \ln (\varepsilon a)\right) \wedge \left(\frac{1}{\varepsilon} - 1 + \ln \varepsilon\right).
\]
From the above, we have
\[
\begin{align*}
  V_1(S(0), I(0), Q(0)) + KT \\
  &\geq \mathbb{E}[1_{\Omega} V(S(\omega), I(\omega), Q(\omega))] \\
  &\geq \mathbb{E}(-a + \varepsilon - a \ln \frac{1}{\varepsilon}) \wedge (-1 + \varepsilon - \ln \varepsilon) \wedge (-a + \frac{1}{\varepsilon} + a \ln (\varepsilon a)\) \wedge \left(\frac{1}{\varepsilon} - 1 + \ln \varepsilon\right).
\end{align*}
\]
where \(1_{\Omega}\) is the indicator function of \(\Omega\). If \(\varepsilon \to \infty\), then
\[
\lim_{\varepsilon \to \infty} V_1(S(0), I(0), Q(0)) + KT = \infty.
\]
Thus, we have \(\tau_\infty = \infty\) a.s. and the theorem is proved. □

Theorem 1 reveals the existence and uniqueness of the stochastic solution of model (5). In the model (3), the disease-free equilibrium is denoted by
\[
E_0 = (S_0, 0, 0) = \left(\frac{A}{\mu + p(1 - me^{-\mu T})}, 0, 0\right).
\]
It is handy to prove that if \(R_0 \leq 1\), the equilibrium \(E_0\) is both locally and globally asymptotically stable. In the case of stochastic models like the system (5), researchers have put a question mark on the existence of \(E_0\). Consequently, one can explain the behavior of the solution of the models around the disease-free equilibrium. Particularly, it is interesting to study what kind of changes will appear around the disease-free equilibrium. The following theorem aims to answer such and related questions. For convenience, denote
\[
\begin{align*}
  l_1 &= \mu + p - \sigma_1^2 + \frac{1}{2}\omega_1 e^{-\mu T} - \omega_2 e^{-\mu T} - \kappa e^{-\mu T} + \int_\mathcal{Z} (\eta_1(z) + \frac{1}{2} \eta_1(z) \eta_2(z) \nu(dz)) \\
  l_2 &= \zeta_1 + \mu + \frac{1}{2}\omega_2 + \int_\mathcal{Z} (\eta_2(z) + \frac{1}{2} \eta_1(z) \eta_2(z) \nu(dz)) \\
  l_3 &= \mu + \zeta_1 + \kappa - \omega_1 e^{-\mu T} - \frac{1}{2} \omega_1 \eta_1^2(z) \nu(dz).
\end{align*}
\]

Theorem 2. Assume that \((S(t), I(t), Q(t))\) is a solution of model (5) with the initial value \((S(0), I(0), Q(0)) \in \mathbb{R}^3_+\). If \(l_1 > 0, l_2 > 0, l_3 > 0\) and \(R_0 < 1\), then
\[
\limsup_{t \to \infty} E \int_0^t \left[ (S(s) - S_0)^2 + I^2(s) + Q^2(s) \right] ds \leq \frac{S_0^2}{M_1} \left[ \frac{1}{2} \sigma_1^2 + \int_\mathcal{Z} (\eta_1^2(z) + \eta_1(z) \eta_2(z) \nu(dz)) \right].
\]
where \(M_1 = \min\{l_1, l_2, l_3\}\).

Proof. Let \(u(t) = S(t) - S_0, v(t) = I(t), w(t) = Q(t)\) in model (5). Thus, model (5) is equivalent to the following equations
\[
\begin{align*}
  du(t) &= -(\phi u(t) + \phi t)S_0 - (p + \mu u(t) + p u(t) - \tau_1) me^{-\mu T} + \rho v(t - \tau_2) e^{-\mu T} \\
  &+ K v(t - \tau_2) e^{-\mu T} dt + \sigma_1 (u(t) + S_0) dB_1(t) + \int_\mathcal{Z} \eta_1(z) (u(t) - S_0) \tilde{N}(dt, dz), \\
  dv(t) &= (\phi u(t) + \phi t)S_0 - (p + \mu + \zeta_1) v(t) dt + \sigma_2 I(t) dB_2(t) + \int_\mathcal{Z} \eta_2(z) v(t - \cdot) \tilde{N}(dt, dz), \\
  dw(t) &= (\pi v(t) - (\mu + \zeta_2 + \kappa) w(t)) dt + \sigma_3 w(t) dB_3(t) + \int_\mathcal{Z} \eta_3(z) w(t - \cdot) \tilde{N}(dt, dz),
\end{align*}
\]
where \(u(t) \in \mathbb{R}\) and \(v(t), w(t)\) are positive functions. Define another \(C^2\)-function of the form
\[
V_2 = \frac{1}{2} (u(t) + v(t))^2 + b_1 v(t) + \frac{1}{2} w^2(t) + \rho m e^{-\mu T} \int_{t-\tau_1}^t u^2(s) ds + \rho e^{-\mu T} \int_{t-\tau_2}^t v^2(s) ds + \kappa e^{-\mu T} \int_{t-\tau_1}^t w^2(s) ds.
\]
where $b_1$ is a positive real number to be calculated later. By utilizing the formula due to Itô (8), we have

$$dV_2 = D_2 dt + (u(t) + v(t)) σ_1(u(t) + S_0) dB_1(t) + b_1 σ_2(v(t) db_2(t) + σ_3(u(t) + S_0) + σ_2(v(t))^2 N(dt, dz)$$

$$+ ∫ f_2(1/2σ_2^2(z)^2 w^2(t))^2 N(dt, dz) + (u(t) + v(t)) f_2(η_1(z)(u(t) + S_0) N(dt, dz)$$

$$+ (b_1 + u(t) + v(t)) f_2(η_2(z)v(t) N(dt, dz) + f_2(η_3(z)^2 w^2(t))^2 N(dt, dz),$$

where the term $V_2$ in equation (13) can be written as

$$V_2 = (u(t) + v(t))(-p + ρ μ)u(t) + pme^{-μs} u(t - τ_1) + pe^{-μs} v(t - τ_2)$$

$$+ pe^{-μs} w(t - τ_3) - (ζ_1 + π + ρ + μ)u(t)) + 1/2 σ_1^2(μτ + S_0)^2 + 1/2 σ_2^2 v^2(t)$$

$$+ 1/2 σ_2^2 w^2(t) + 1/2 b_1(η_1(z) + φ t + φ(t) + φ v(t) - (μ + π + ρ + S_0) + v(t))$$

$$+ 1/2 f_2(η_1(z) + η_2(z)w^2(t) + pme^{-μs}(u^2(t) - u^2(t - τ_1) + pe^{-μs}(w^2(t) - w^2(t - τ_2)$$

Let $V_2 = -[μ - p - 1/2(p + μ)u^2(t) - (κ + π + ρ + μ)u(t)) + 1/2 σ_1^2(μτ + S_0)^2 + 1/2 σ_2^2 v^2(t) + 1/2 σ_2^2 w^2(t)$$

$$+ 1/2 b_1(η_1(z) + φ t + φ(t) + φ v(t) - (μ + π + ρ + S_0) + v(t))$$

$$+ 1/2 f_2(η_1(z) + η_2(z)w^2(t) + pme^{-μs}(u^2(t) - u^2(t - τ_1) + pe^{-μs}(w^2(t) - w^2(t - τ_2)$$

Choose $b_1 = φ/(ζ_1 + π + ρ + 2μ + p)$, that is, $b_1 = φ - (ζ_1 + π + ρ + 2μ + p) = 0$. Thus, for $R_0 < 1$, we have

$$0 ≤ EV_2(u(t), v(t), w(t)) ≤ V_2(u(0), v(0), w(0)) + E ∫_0^t dV_2(u(s), v(s), w(s)) ds.$$ 

By using the inequality (14) in the above relation, we have

$$E ∫_0^t [l_1 u^2(s) + l_2 v^2(s) + l_3 w^2(s)] ds$$

$$≤ V_2(u(0), v(0), w(0)) + S_0(1/2 σ_1^2 + f_2(η_1(z) + η_2(z) + η_3(z)w^2(t))^2 N(dt, dz)$$

Therefore

$$lim sup_{t→∞} 1/t E [∫_0^t ((S(s) - S_0)^2 + P^2(s) + Q^2(s))] ds ≡ M_1 [1/2 σ_1^2 + f_2(η_1(z) + η_2(z))^2 N(dt, dz)],$$

where $M_1 = min \{l_1, l_2, l_3\}$. 

**Remark 1.** In Theorem 2, we observed that for $R_0 < 1$ and $l_i > 0 (i = 1, 2, 3)$, every solution of model (5) will be swinging in the vicinity of disease-free equilibrium of the model (3). Further, the disturbance range is proportional to the values of $σ_1$ and $η_2(z)$ for each $i = 1, 2, 3$. In other words, if one assumes very small white and Lévy noise intensities, then every solution of model (5) will cluster very close to $E_0$. In particular, if $σ_1 = 0, η_1(z) = 0$, and $η_2(z) = 0$, then $E_0 = (S_0, 0, 0)$ is also the disease-free equilibrium of model (5)

According to the proof process in Theorem 2, we know that $IV_2$ is negative definite and $V_2$ is positive definite. It means that the trivial solution of model (5) is stochastically asymptotically stable.

**Corollary 1.** If $l_1 > 0, l_2 > 0, l_3 > 0, σ_1 = 0, η_1(z) = 0, η_2(z) = 0$ and $R_0 < 1$, then the disease-free equilibrium $E_0 = (S_0, 0, 0)$ of model (5) is stochastically asymptotically stable in the large.
Next, we investigate the asymptotic behavior of the stochastic solution \((S(t), I(t), Q(t))\) of model (5) around the endemic equilibrium of the model (3) as

\[
E^* = (S^*, I^*, Q^*) = \left( \frac{\mu + \zeta_1 + \rho + \pi}{\phi}, \frac{A - S^*(p + \mu + mpe^{-\mu t_1})}{\mu + \zeta_1 + \rho + \pi - \rho e^{-\mu t_2} - \frac{\pi}{\mu + \zeta_2 + \kappa}}, \frac{\pi}{\mu + \zeta_2 + \kappa} \right).
\]

Here again, we introduce the following notations for the sake of simplicity

\[
\begin{align*}
l_4 &= \mu + p - 3 p\rho e^{-\mu t_1} - \rho e^{-\mu t_2} - \kappa e^{-\mu t_1} - \sigma_1^2 - \int_2 \eta_2(z)v(dz), \\
l_5 &= \rho + \pi + \mu + \zeta_1 - p\rho e^{-\mu t_1} - 3 \rho e^{-\mu t_2} - \kappa e^{-\mu t_1} - \sigma_2^2 - \frac{1}{2} \pi - \int_2 \eta_2(z)v(dz), \\
l_6 &= \rho + \mu + \kappa - \frac{1}{2} \pi - \sigma_2^2 - 2 \kappa e^{-\mu t_1} - \int_2 \eta_2(z)v(dz), \\
M_2 &= (3 \rho e^{-\mu t_1} + \sigma_1^2 + \int_2 \eta_2(z)v(dz))S^* + (3 \rho e^{-\mu t_1} + \sigma_2^2 + \int_2 \eta_2(z)v(dz))I^* + \left( \frac{1}{2} \rho e^{-\mu t_1} + \frac{1}{2} \kappa e^{-\mu t_1} + \sigma_3^2 + \int_2 \eta_2(z)v(dz) \right)Q^* + \frac{2 \mu + p + \pi + \zeta_1 + \rho}{\phi} \int_2 \{\eta_2(z) - \ln(1 + \eta_2(z))\}v(dz).
\end{align*}
\]

**Theorem 3.** Assume that \((S(t), I(t), Q(t))\) is a solution of model (5) with the initial value \((S(0), I(0), Q(0)) \in \mathbb{R}_+^3\). If \(l_4 > 0, l_5 > 0, l_6 > 0\) and \(R_0 > 1\), then

\[
\lim_{t \to \infty} \sup \frac{1}{t} \mathbb{E} \left[ \int_0^t \{ (S(s) - S^*)^2 + (I(s) - I^*)^2 + (Q(s) - Q^*)^2 \} \, ds \right] \leq \frac{M_2}{k_2},
\]

where \(k_2 = \min \{l_4, l_5, l_6\} \).

**Proof.** Define a non-negative function \(V_3\) of the form

\[
\begin{align*}
V_3 &= \frac{1}{2} (S(t) - l^*)^2 + c_1 (I(t) - l^*)^2 + c_1 (1 + \eta_2(z)) \dot{N}(dt, dz) \\
&\quad + \frac{1}{2} \int_2 \eta_2(z)v(dz) (S(t) - l^*) \dot{N}(dt, dz)
\end{align*}
\]

where \(c_1\) is a positive real number to be calculated later. By the Itô’s formula (8), we have

\[
\begin{align*}
dV_3 &= LV_3 \, dt + (S(t) - l^* + I(t) - S^*)(\sigma_1 S(t)dB_1(t) + \sigma_2 I(t)dB_2(t)) \\
&\quad + \sigma_2 (Q(t) - Q^*)Q(t)dB_3(t) \\
&\quad + \int_2 \left( \frac{1}{2} \eta_1(z)S(t) + \eta_2(z)I(t) \right) \dot{N}(dt, dz) \\
&\quad + \int_2 (\eta_2(z)I(t) - l^* \ln(1 + \eta_2(z))) \dot{N}(dt, dz) \\
&\quad + \int_2 (S(t) - S^* + l^* - l^*)(\eta_1(z)S(t) + \eta_2(z)I(t) - l^*) \dot{N}(dt, dz) \\
&\quad + \int_2 (\eta_2(z)(Q(t) - Q^*)Q(t) - l^* \eta_2(z)Q^2(t)) \dot{N}(dt, dz) \\
&\quad + \int_2 (\eta_1(z)(S(t) - S^*)^2 - \eta_2(z)Q^2(t)) \dot{N}(dt, dz),
\end{align*}
\]

where

\[
\begin{align*}
LV_3 &= (S(t) - S^* + l^* - l^*)(A - p + \mu + \pi + \mu + \zeta_1)I(t) + \frac{\pi^2}{2} \dot{S}^2(t) \\
&\quad + \rho e^{-\mu t_1}S(t - \tau_1) + \rho e^{-\mu t_2}I(t - \tau_2) + \kappa e^{-\mu t_1}Q(t - \tau_3) + \frac{\pi^2}{2} \dot{I}^2(t) \\
&\quad + \frac{1}{2} \int_2 \eta_1(z)S(t) + \eta_2(z)I(t) \dot{N}(dt, dz) + c_1 \eta_2(z)Q^2(t) \dot{N}(dt, dz) \\
&\quad + c_1 \eta_1(z)S(t - \tau_1) + \frac{\pi}{\phi} \dot{I}^2(t) - (\theta + \mu + \kappa)Q(t) + \rho e^{-\mu t_1} \dot{I}^2(t) - \eta_2(z)Q^2(t) \dot{N}(dt, dz) \\
&\quad + \phi \dot{S}^2(t) - \dot{S}^2(t - \tau_1).
\end{align*}
\]

From the model (3), the endemic equilibrium \(E^* = (S^*, I^*, Q^*)\) satisfies the following equations

\[
\begin{align*}
A &= \phi S^* I^* + (p + \mu) S^* - pmS^* e^{-\mu t_1} - \rho I^* e^{-\mu t_2} - \kappa Q^* e^{-\mu t_1}, \\
\phi I^* S^* &= (\mu + \zeta_1 + \rho + \pi) I^*, \\
\pi I^* &= (\mu + \zeta_2 + \kappa) Q^*.
\end{align*}
\]
Therefore, by substituting these relations in Eq. (16), we have
\[
IV_3 = - (\mu + p)(S(t) - S^*)^2 - (\rho + \pi + \mu + \zeta_1)(I(t) - I^*)^2
\]
\[
- (\rho + \pi + \mu + \zeta_1)(S(t) - S^*)(l(t) - I^*) + c_1\phi(S(t) - S^*)(l(t) - I^*)
\]
\[
+ \rho e^{-\mu_s}(l(t - \tau_2) - I^*)(S(t) - S^*) + \kappa e^{-\mu_s}Q(t - \tau_3 - \tau_4 + \rho + \mu + \pi)(I(t) - I^*)
\]
\[
- (\mu + p)(S(t) - S^*)(l(t) - I^*) - (\zeta_1 + \rho + \mu + \pi)(I(t) - I^*)^2
\]
\[
+ \rho e^{-\mu_s}(S(t) - S^*)(l(t) - I^*) + \rho e^{-\mu_s}((l(t - \tau_2) - I^*)(I(t) - I^*)
\]
\[
+ \kappa e^{-\mu_s}Q(t - \tau_3 - \tau_4 + \rho + \mu + \pi)(S(t) - S^*)(l(t) - I^*) - (\mu + p)(S(t) - S^*)(l(t) - I^*) + \frac{\zeta_1^2}{2}S^2
\]
\[
+ \frac{1}{2} \int_0^t \eta_1(z)S(t) + \eta_2(z)I(t)\)^2 \eta\(dz) + \rho e^{-\mu_s}(S(t) - S^*)(l(t) - I^*)
\]
\[
+ c_1\phi \int_0^t \eta_2(z) - \ln(1 + \eta_2(z))\)\(v\(dz) + \pi (Q(t) - Q^*)(I(t) - I^*) + \frac{\zeta_1^2}{2}I^2(t)
\]
\[
+ \frac{\zeta_1^2}{2}I^2(t) + \frac{1}{2} \int_0^t \eta_2^2(z)Q(t)\)\(v\(dz) + \rho e^{-\mu_s}\(I^2(t) - S^2(t) - Q^2(t - \tau_1) + \delta_1^2(t)
\]
\[
+ \rho e^{-\mu_s}(I^2(t) - S^2(t) - Q^2(t - \tau_1)) + \kappa e^{-\mu_s}(Q^2(t - Q^2(t - \tau_1)) - (\theta + \mu + \kappa)(Q(t) - Q^2(t - \tau_1))
\].

Choose \(c_1 = \frac{2(\mu + p + \pi + \zeta_1 + \rho)}{\phi}\) so that \(2\mu + p + \pi + \zeta_1 + \rho - c_1\phi = 0\). Utilizing the inequalities \(\frac{(a+b)^2}{2} \leq a^2 + b^2\) and \(\frac{(a+b+c)^2}{3} \leq a^2 + b^2 + c^2\), we have
\[
IV_3 \leq -l_4(S(t) - S^*)^2 - l_5(I(t) - I^*)^2 - l_6(\rho + \mu + \pi)(I(t) - I^*)^2 + M_2.
\]

Here again, if we integrate both sides of the Eq. (15) over the interval \([0, t]\), and then take the expectations, we obtain the following inequality
\[
0 \leq EV_2[S(t), I(t), Q(t)] \leq V_2[S(0), I(0), Q(0)] + E\int_0^t [-l_4(S(t) - S^*)^2
\]
\[
- l_5(I(t) - I^*)^2 - l_6(\rho + \mu + \pi)(I(t) - I^*)^2 + M_2]dt.
\]

Denote \(k_2 = \min\{l_4, l_5, l_6\}\). Then
\[
\lim_{t \to \infty} \sup \frac{1}{t} \mathbb{E}\left[\int_0^t \left(\left(S(s) - S^*)^2 + (I(s) - I^*)^2 + (Q(s) - Q^*)^2\right) ds\right] \leq \frac{M_2}{k_2}.
\]

\(\square\)

**Remark 2.** Physically, Theorem 3 indicates that if \(R_0 > 1\) and \(l_i > 0 (i = 4, 5, 6)\), all the solutions of model (5) fluctuates around the endemic equilibrium \(E^*\) of the model (3). Here again, the disturbance range of the fluctuations is proportional to the values of \(\eta_i(z)\) and \(\sigma_i (i = 1, 2, 3)\). In other words, if the intensities of Lévy noise and white noise become smaller, each solution of system (5) will remain close in the vicinity of \(E^*\).

4. Numerical simulations

To verify the obtained theoretical results, different settings of parameters together with numerical simulations are proposed in this section. Model (5) consists of three parts: deterministic part with delays, white noises determined by Wiener process, and Lévy jumps. For the general stochastic model with delays, we usually use the Milstein higher-order method and Euler-Maruyama algorithm [32]. Similar to [20], the jump part is the compound Poisson process defined by \(Z_t = \sum_{i=1}^{N_t} Y_i\), where \(N_t\) is a Poisson process with mean \(\lambda t\). The jump size \(Y_i\) is independent and identically distributed random variable with distribution function \(F\). Here, \(Y_i\) follows the standard normal distribution, that is, \(F(y) = \frac{1}{\sqrt{2\pi}} \exp(-y^2/2)\). The Lévy jump measure of \(Z_t\) is given by \(v(A) = \lambda \int_A dF(z)\). In the simulation, we assumed \(\lambda = 1\).

**Example 4.1.** Assume that \(A = 0.4, \mu = 0.2, p = 0.2, \phi = 0.2, \rho = 0.09, \pi = 0.034, k = 0.0859, m = 0.03, \zeta_1 = 0.2, \sigma_2 = 0.53, \eta_1(z) = 0.03, \eta_2(z) = 0.03, \eta_3(z) = 0.01, \sigma_1 = 0.03, \sigma_2 = 0.01, \sigma_3 = 0.1, \tau_0 = 0.1, \tau_2 = 0.2, \tau_1 = 0.3\). The time step size \(\Delta t = 0.1\). The initial value of all the compartments in vectorized form is given as \(S(0), I(0), Q(0)\) which clearly fluctuates around the disease-free equilibrium proposed by the underlying deterministic model (3).

**Example 4.2.** Assume that \(\phi = 0.4\) and other parameters are similar to those of Example 4.1. Through calculation, we have \(l_4 = 0.00072 > 0, l_5 = 0.15 > 0, l_6 = 0.106 > 0\) and \(R_0 = 1.5 > 1\). Hence, the condition of Theorem 3 holds. From Fig. 2, each solution of model (5) fluctuates around the endemic equilibrium of model (3).

**Example 4.3.** Assume that \(A = 0.3, \phi = 0.23, \sigma_2 = 0.2, \sigma_3 = 0.2\) and consider other values of the parameters in Example 4.1. It can be seen from Fig. 3 that when \(R_0 < 1\) and \(l_i > 0 (i = 1, 2, 3)\), then by increasing the intensity of the noise will decrease the numbers of infectious and isolated individuals.
Fig. 1. (a) The trajectories of $S(t)$ in models (3) and (5) with parameters values defined in Example 4.1. Panel (b) and (c) shows the trajectories of $I(t)$ and $Q(t)$, respectively.

Fig. 2. (a) The trajectories of $S(t)$ in models (3) and (5). Panel (b) and (c) shows the trajectories of $I(t)$ and $Q(t)$ with parameters values defined in Example 4.2.

Fig. 3. (a) describes the trajectories of $I(t)$ in stochastic model (5) with or without jumps, which the parameters values are defined in Example 4.3. (b) describes the trajectories of $Q(t)$ in the stochastic model (5) with or without jumps, compared with model (3). Here, model (5) with jumps has Lévy noise and white noise. Model (5) without jumps only has white noise.

Fig. 4. The values of $r(t), i(t), q(t)$ reflect the trajectories of model (3). The values of $S(t), I(t), Q(t)$ correspond the trajectories of model (5). (a) describes the trajectories of the solutions of models (3) and (5) with the isolation rate $\pi = 0.05$ and other parameter values of Example 4.2. Panel (b) and (c) are the trajectories of models (3) and (5) under the cases $\pi = 0.1$ and $\pi = 0.2$, respectively.
5. Mutant COVID - 19 epidemic

Mutant COVID-19 strains are now spreading geographically much faster than at any time in the history. Studying the spread of COVID-19 epidemic transmission can help better control and prevent the epidemic. Recently, China has suffered another hardest-hit by the virus since the beginning of the 2022. In order to study the outbreak, we obtained the number of existing confirmed cases in Gansu Province from March 11 to April 2 from the Municipal Health Commission (http://wsjk.gansu.gov.cn/). According to the National Bureau of Statistics, the total population of Gansu Province has reached 25,019,831 from the seventh national population census. Thus, \(S(0), I(0), Q(0) = (25019519, 156, 156)\) and \(A = 261\).

The unknown parameters are estimated by the least-square method [33] with the initial value \((S(0), I(0), Q(0))\). Table 1 lists the estimated values of parameters of models (3) and (5). Take other parameters \(\eta_1(z) = 0.03, \eta_2(z) = 0.03, \eta_3(z) = 0.01, \sigma_1 = 0.03, \sigma_2 = 0.01, \tau_1 = 0.9, \tau_2 = 0.2, \tau_3 = 0.3\) from Example 4.1. Through calculation, \(R_0 = 0.0856 < 1, I_1 = 0.0013 > 0, I_2 = 0.116 > 0\) and \(I_3 = 0.123 > 0\), satisfying the conditions of Theorem 2. Therefore, the mutant COVID - 19 epidemic of Gansu Province will become extinct over time. Based on all values of parameters, Fig. 6 shows curves of the fitted mutant COVID-19 cases in models (3) and (5), compared with the real data from March 11 to April 2 in Gansu Province. Obviously, it can be seen from Fig. 6 that the stochastic model is considered to be better in fitting the real data.
Fig. 6. The trajectories of the fitted mutant COVID-19 cases in models (3) and (5), compared with the reported existing infections of Gansu Province, China. Model (3) corresponds to a blue dotted line. Model (5) corresponds to a red solid line, and the red asterisks are the real data. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

6. Conclusion

The spread of infectious disease is influenced by various factors including recruitment, transmission, recovery, vaccination, and isolated rates. Time delays and Lévy jumps are also very important for modelling the spread of epidemic. In this paper, we have proposed a stochastic SIQR epidemic model with triple delays and Lévy jumps. Through the theoretical analysis of the stochastic model, the main findings are listed as follows: (i) Using the suitable Lyapunov method with time delays, we obtain the existence and uniqueness of global positive solution of model (5) (see Theorem 1). (ii) If $R_0 < 1$ and $l_i > 0 (i = 1, 2, 3)$, the stochastic solution $(S(t), I(t), I(t))$ of model (5) fluctuates around the disease-free equilibrium $E_0 = (S_0, 0, 0)$ of model (3) (see Theorem 2 and Remark 1). Especially, if $\sigma_1 = 0, \eta_1(z) = 0, \eta_2(z) = 0$, then the solution of deterministic model (3) and stochastic model (5) have the same asymptotic behavior (see Corollary 1). (iii) If $R_0 > 1$ and $l_i > 0 (i = 4, 5, 6)$, then the stochastic solution $(S(t), I(t), I(t))$ varies around the endemic equilibrium $E^* = (S^*, I^*, Q^*)$ of model (3) (see Theorem 3 and Remark 2). Because of the complexity of the stochastic model, all the theoretical results are obtained by constructing Lyapunov function with time delays. Further, they reveal the relationship of models (3) and (5).

The numerical simulations are used to illustrate our analysis results. Under different parameter settings, we discussed and analyzed the effects of interested parameters in models (3) and (5) according to the basic reproductive number $R_0$ and other conditions. We summarize some useful results: (i) The disease becomes extinct if the transmission rate is smaller (Figs. 1 and 2). (ii) The increasing of noises may lead to a decrease the number of infectious and isolated individuals (Fig. 3). (iii) The infected individuals will decrease if the isolation and vaccination rates increase (Figs. 1 and 5). All the results are based on the fixed Lévy jumps and three time delays. Finally, we apply the proposed models to investigate the spread of mutant COVID-19 epidemic in Gansu Province. The result reflects that the mutant COVID-19 epidemic eventually go to extinct.

In our work, there are still some interesting issues which needs further research. Because of the complexity in the proposed models (3) and (5), more of the theoretical properties has not been solved such as the existence of unique ergodic stationary distribution. They will be considered in our further work.

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