ZEROS OF RANDOM POLYNOMIALS ON $\mathbb{C}^m$

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Abstract. For a regular compact set $K$ in $\mathbb{C}^m$ and a measure $\mu$ on $K$ satisfying the Bernstein-Markov inequality, we consider the ensemble $\mathcal{P}_N$ of polynomials of degree $N$, endowed with the Gaussian probability measure induced by $L^2(\mu)$. We show that for large $N$, the simultaneous zeros of $m$ polynomials in $\mathcal{P}_N$ tend to concentrate around the Silov boundary of $K$; more precisely, their expected distribution is asymptotic to $N^m \mu_{eq}$, where $\mu_{eq}$ is the equilibrium measure of $K$. For the case where $K$ is the unit ball, we give scaling asymptotics for the expected distribution of zeros as $N \to \infty$.

1. Introduction

A classical result due to Hammersley [Ha] (see also [SV]), loosely stated, is that the zeros of a random complex polynomial

$$f(z) = \sum_{j=0}^{N} c_j z^j$$

mostly tend towards the unit circle $|z| = 1$ as the degree $N \to \infty$, when the coefficients $c_j$ are independent complex Gaussian random variables of mean zero and variance one. In this paper, we will prove a multivariable result (Theorem 3.1), a special case (Example 3.5) of which shows, loosely stated, that the common zeros of $m$ random complex polynomials in $\mathbb{C}^m$,

$$f_k(z) = \sum_{|J| \leq N} c^k_J z_1^{j_1} \cdots z_m^{j_m} \quad \text{for } k = 1, \ldots, m,$$

tend to concentrate on the product of the unit circles $|z_j| = 1$ ($j = 1, \ldots, m$) as $N \to \infty$, when the coefficients $c^k_J$ are i.i.d. complex Gaussian random variables.

The following is our basic setting: We let $K$ be a compact set in $\mathbb{C}^m$ and let $\mu$ be a Borel probability measure on $K$. We assume that $K$ is non-pluripolar and we let $V_K$ be its pluricomplex Green function. We also assume that $K$ is regular (i.e., $V_K = V^*_K$) and that $\mu$ satisfies the Bernstein-Markov inequality (see [2]). We give the space $\mathcal{P}_N$ of holomorphic polynomials of degree $\leq N$ on $\mathbb{C}^m$ the Gaussian probability measure $\gamma_N$ induced by the Hermitian inner product

$$(f, g) = \int_K f \overline{g} \, d\mu.$$  

The Gaussian measure $\gamma_N$ can be described as follows: We write $f = \sum_{j=1}^{d(N)} c_j p_j$, where $\{p_j\}$ is an orthonormal basis of $\mathcal{P}_N$ with respect to (3) and $d(N) = \dim \mathcal{P}_N = \binom{N+m}{m}$. Identifying

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f ∈ \mathcal{P}_N with c = (c_1, \ldots, c_{d(N)}) ∈ \mathbb{C}^{d(N)}, we have
\[ d\gamma_N(s) = \frac{1}{\pi^{d(N)}} e^{-|c|^2} dc. \]
(The measure \(\gamma_N\) is independent of the choice of orthonormal basis \(\{p_j\}\).) In other words, a random polynomial in the ensemble \((\mathcal{P}_N, \gamma_N)\) is a polynomial \(f = \sum_j c_j p_j\), where the \(c_j\) are independent complex Gaussian random variables with mean 0 and variance 1.

Our main result, Theorem 3.1, gives asymptotics for the expected zero current of \(k\) i.i.d. random polynomials (\(1 \leq k \leq m\)). In particular, the expected distribution \(E(Z_{f_1}, \ldots, f_m)\) of simultaneous zeros of \(m\) independent random polynomials in \((\mathcal{P}_N, \gamma_N)\) has the asymptotics
\[ \frac{1}{N^m} E(Z_{f_1}, \ldots, f_m) \rightarrow \mu_{eq} \text{ weak}^*, \quad (4) \]
where \(\mu_{eq} = (i\partial \bar{\partial} V_K)^m\) is the equilibrium measure of \(K\). Here, \(E(X)\) denotes the expected value of a random variable \(X\).

The reader may notice from (4) that the distributions of zeros for the measures on \(\mathcal{P}_N\) considered here are quite different from those of the SU\((m+1)\) ensembles studied, for example, in [SZ1, SZ4, BSZ1, BSZ2, DS]. The Gaussian measure on the SU\((m+1)\) polynomials is based on the inner product
\[ \langle f, g \rangle_N = \int_{S^{2m+1}} F_N G_N, \]
where \(F_N, G_N ∈ \mathbb{C}[z_0, z_1, \ldots, z_m]\) denote the degree \(N\) homogenizations of \(f\) and \(g\) respectively. It follows easily from the SU\((m+1)\)-invariance of the inner product that the expected distribution of simultaneous zeros equals \(\frac{N^m}{m!} \omega(m)\) (exactly), where \(\omega\) is the Fubini-Study Kähler form (on \(\mathbb{C}^m \subset \mathbb{CP}^m\)).

In this paper, we also give scaling limits for the expected zero density in the case of the unit ball in \(\mathbb{C}^m\) (Theorem 4.1). The problem of finding scaling limits for more general sets in \(\mathbb{C}^m\) remains open. Another open problem is to establish the multivariable version of the following one variable result: For a regular subset \(K ⊂ \mathbb{C}\), it is known (see [SZ1]) that with probability one, a sequence \(\{f_N\}_{N=1,2,\ldots}\) of random polynomials of increasing degree satisfies:
\[ \lim_{N→∞} \frac{1}{N} Z_{f_N} = \mu_{eq} \text{ weak}^*. \]

2. Background

We let \(\mathcal{L}\) denote the Lelong class of plurisubharmonic (PSH) functions on \(\mathbb{C}^m\) of at most logarithmic growth at \(∞\). That is
\[ \mathcal{L} := \{ u ∈ \text{PSH}(\mathbb{C}^m) \mid u(z) ≤ \log^+ \|z\| + O(1) \}\]
For \(K\) a compact subset of \(\mathbb{C}^m\), we define its pluricomplex Green function \(V_K(z)\) via
\[ V_K(z) = \sup \{ u(z) \mid u ∈ \mathcal{L}, u ≤ 0 \text{ on } K \}. \]
We will assume \(K\) is regular, that is by definition, \(V_K\) is continuous on \(\mathbb{C}^m\) (and so \(V_K = V^*_K\), its uppersemicontinuous regularization). The function \(V_K\) is a locally bounded PSH function.
on \( \mathbb{C}^m \) and, in fact
\[
V_K - \log^+ \| z \| = O(1) .
\] (7)

By a basic result of Bedford and Taylor [BT1] (see [K]), the complex Monge-Ampère operator \((dd^c)^m = (2i\partial\bar{\partial})^m\) is defined on any locally bounded PSH function \( \mathbb{C}^m \) and in particular on \( V_K \). The equilibrium measure of \( K \) is defined by (see [Kl, Cor. 5.5.3])
\[
\mu_{eq}(K) := \left( \frac{i}{\pi} \partial\bar{\partial} V_K \right)^m
\] (8)

Since \( V_K \) satisfies (7), it is a positive Borel measure, here normalized to have mass 1. The support of the measure \( \mu_{eq}(K) \) is the Silov boundary of \( K \) for the algebra of entire analytic functions [BT2]. In one variable, i.e. \( K \subset \mathbb{C} \), \( V_K \) is the Green function of the unbounded component of \( \mathbb{C} \setminus K \) with a logarithmic pole at \( \infty \), and \( \mu_{eq}(K) = \frac{1}{2\pi} \Delta V_K \), where \( \Delta \) is the Laplacian [Ra].

Let \( \mu \) be a finite positive Borel measure on \( K \). The measure \( \mu \) is said to satisfy a Bernstein-Markov (BM) inequality, if, for each \( \varepsilon > 0 \) there is a constant \( C = C(\varepsilon) > 0 \) such that
\[
\| p \|_{L^2(\mu)} \leq C e^{\varepsilon \deg(p)} \| p \|_{L^2(\mu)}
\] (9)
for all holomorphic polynomials \( p \). Essentially, the BM inequality says that the \( L^2 \) norms and the sup norms of a sequence of holomorphic polynomials of increasing degrees are “asymptotically equivalent”.

The question arises as to which measures actually satisfy the BM inequality. It is a result of Nguyen-Zeriahi [NZ] combined with [Kl, Cor. 5.6.7] that for \( K \) regular, \( \mu_{eq}(K) \) satisfies BM. This fact is used in Examples 3.5–3.6. In [Bl1, Theorem 2.2], a “mass-density” condition for a measure to satisfy BM was given. (See also [BL].)

Our proof uses the probabilistic Poincaré-Lelong formula for the zeros of random functions (Proposition 2.1 below). Considering a slightly more general situation, we let \( g_1, \ldots, g_d \) be holomorphic functions with no common zeros on a domain \( U \subset \mathbb{C}^m \). (We are interested in the case where \( U = \mathbb{C}^m \) and \( \{g_j\} \) is an orthonormal basis of \( \mathcal{P}_N \) with respect to the inner product [3], as discussed above.) We let \( \mathcal{F} \) denote the ensemble of random holomorphic functions of the form \( f = \sum c_j g_j \), where the \( c_j \) are independent complex Gaussian random variables with mean 0 and variance 1. We consider the Szegő kernel
\[
S_\mathcal{F}(z, w) = \sum_{j=1}^d g_j(z) \overline{g_j(w)} .
\]
For the case where the \( g_j \) are orthonormal functions with respect to an inner product on \( \mathcal{O}(U) \), \( S_\mathcal{F}(z, w) \) is the kernel for the orthogonal projection onto the span of the \( g_j \).

Under the assumption that the \( g_j \) have no common zeros, it is easily shown using Sard’s theorem (or a variation of Bertini’s theorem) that for almost all \( f_1, \ldots, f_k \in \mathcal{F} \), the differentials \( df_1, \ldots, df_k \) are linearly independent at all points of the zero set
\[
\text{loc}(f_1, \ldots, f_k) := \{ z \in U : f_1(z) = \cdots = f_k(z) = 0 \} .
\]
This condition implies that the complex hypersurfaces \( \text{loc}(f_j) \) are smooth and intersect transversely, and hence \( \text{loc}(f_1, \ldots, f_k) \) is a codimension \( k \) complex submanifold of \( U \). We
then let $Z_{f_1,\ldots,f_k} \in \mathcal{D}^{k,k}(U)$ denote the current of integration over $\text{loc}(f_1,\ldots,f_k)$:

\[
(Z_{f_1,\ldots,f_k}, \varphi) = \int_{\text{loc}(f_1,\ldots,f_k)} \varphi, \quad \varphi \in \mathcal{D}^{m-k,m-k}(U).
\]

We shall use the following Poincaré-Lelong formula from [SZ3, SZ4]:

**Proposition 2.1.** The expected zero current of $k$ independent random functions $f_1,\ldots,f_k \in \mathcal{F}$ is given by

\[
\mathbf{E}(Z_{f_1,\ldots,f_k}) = \left(\frac{i}{2\pi} \partial \bar{\partial} \log S_{\mathcal{F}}(z,z)\right)^k.
\]

The proof follows by a verbatim repetition of the proof of Proposition 5.1 in [SZ3] (which gives the case where the $g_j$ are normalized monomials with exponents in a Newton polytope). The codimension $k = 1$ case was given in [SZ1] (for sections of holomorphic line bundles), and in dimension 1 by Edelman-Kostlan [EK]. (The formula also holds for infinite-dimensional ensembles; see [So, SZ4].) The general case follows from the codimension 1 case together with the fact that

\[
\mathbf{E}(Z_{f_1,\ldots,f_k}) = \mathbf{E}(Z_{f_1}) \wedge \cdots \wedge \mathbf{E}(Z_{f_k}) = \mathbf{E}(Z_f)^k,
\]

which is a consequence of the independence of the $f_j$. The wedge product of currents is not always defined, but $Z_{f_1} \wedge \cdots \wedge Z_{f_k}$ is almost always defined (and equals $Z_{f_1,\ldots,f_k}$ whenever the hypersurfaces $\text{loc}(f_j)$ are smooth and intersect transversely), and a short argument given in [SZ3] yields (10). (In fact, the left equality of (10) holds for independent non-identically-distributed $f_j$, as proven in [SZ3].) We note that the expectations in (10) are smooth forms.

### 3. Random polynomials on polynomially convex sets

**Theorem 3.1.** Let $\mu$ be a Borel probability measure on a regular compact set $K \subset \mathbb{C}^m$, and suppose that $(K, \mu)$ satisfies the Bernstein-Markov inequality. Let $1 \leq k \leq m$, and let $(\mathcal{P}_N^k, \gamma_N^k)$ denote the ensemble of $k$-tuples of i.i.d. Gaussian random polynomials of degree $\leq N$ with the Gaussian measure $d\gamma_N$ induced by $L^2(\mu)$. Then

\[
\frac{1}{N^k} \mathbf{E}_{\gamma_N^k}(Z_{f_1,\ldots,f_k}) \to \left(\frac{i}{\pi} \partial \bar{\partial} V_K\right)^k \text{ weak*}, \quad \text{as } N \to \infty,
\]

where $V_K$ is the pluricomplex Green function of $K$ with pole at infinity.

To prove Theorem 3.1 we consider the Szegö kernels

\[
S_N(z, w) := S_{(\mathcal{P}_N, \gamma_N)}(z, w) = \sum_{j=1}^{d(N)} p_j(z)\overline{p_j(w)},
\]

where $\{p_j\}$ is an $L^2(\mu)$-orthonormal basis for $\mathcal{P}_N$. Our proof is based on approximating the extremal function $V_K$ by the (normalized) logarithms of the Szegö kernels $S_N(z, z)$ (Lemma 3.4).

We begin by considering the polynomial suprema

\[
\Phi_N^k(z) = \sup\{ |f(z)| : f \in \mathcal{P}_N, \|f\|_K \leq 1\}.
\]

Since $1/N \log f \in \mathcal{L}$, for $f \in \mathcal{P}_N$, it is clear that $1/N \log \Phi_N^k \leq V_K$, for all $N$. Pioneering work of Zaharjuta [Za] and Siciak [Si1, Si2] established the convergence of $1/N \log \Phi_N^k$ to $V_K$. The
uniform convergence when \( K \) is regular seems not to have been explicitly stated and we give the proof below.

**Lemma 3.2.** Let \( K \) be a regular compact set in \( \mathbb{C}^m \). Then

\[
\frac{1}{N} \log \Phi^K_N(z) \to V_K(z)
\]

uniformly on compact subsets of \( \mathbb{C}^m \).

**Proof.** We first note that \( 1 \leq \Phi_j \leq \Phi_j \Phi_k \leq \Phi_{j+k} \), for \( j, k \geq 0 \). By a result of Siciak [Si1] and Zaharjuta [Za] (see [Kl, Theorem 5.1.7]),

\[
V_K(z) = \lim_{N \to \infty} \frac{1}{N} \log \Phi^K_N(z) = \sup_N \frac{1}{N} \log \Phi^K_N(z), \tag{12}
\]

for all \( z \in \mathbb{C}^m \).

We use the regularity of \( K \) to show that the convergence is uniform: let \( \psi_N = \frac{1}{N} \log \Phi^K_N \geq 0 \).

Thus for \( N, k \geq 1, j \geq 0 \), we have

\[ Nk \psi_{Nk} + j \psi_j \leq (Nk + j) \psi_{Nk+j} . \]

Since \( \psi_N \leq \psi_{Nk} \), we then obtain the inequality

\[ \psi_{Nk+j} \geq \frac{Nk}{Nk+j} \psi_N + \frac{j}{Nk+j} \psi_j \geq \frac{Nk}{Nk+j} \psi_N . \tag{13} \]

Fix \( \varepsilon > 0 \). For each \( a \in \mathbb{C}^m \), we choose \( N_a \in \mathbb{Z}^+ \) such that

\[ V_K(a) - \psi_{N_a}(a) < \varepsilon \quad \text{and} \quad \frac{V_K(a)}{N_a} < \varepsilon , \]

and then choose a neighborhood \( U_a \) of \( a \) such that

\[
|V_K(z) - V_K(a)| < \varepsilon, \quad \psi_{N_a}(z) \geq \psi_{N_a}(a) - \varepsilon, \quad \frac{V_K(z)}{N_a} < \varepsilon, \quad \text{for } z \in U_a .
\]

Now let \( N \geq N_a^2 \), and write \( N = N_a k + j \), where \( k \geq N_a, 0 \leq j < N_a \). By (12)–(13), we have

\[
0 \leq V_K - \psi_N \leq V_K - \frac{N_a k}{N_a k + j} \psi_{N_a} \leq V_K - \frac{N_a}{N_a + 1} \psi_{N_a} \leq V_K - \psi_{N_a} + \frac{1}{N_a + 1} V_K . \tag{14}
\]

Hence, for all \( N \geq N_a^2 \) and for all \( z \in U_a \), we have

\[
0 \leq V_K(z) - \psi_N(z) < V_K(z) - \psi_{N_a}(z) + \varepsilon
= [V_K(a) - \psi_{N_a}(a)] + [V_K(z) - V_K(a)] + [\psi_{N_a}(a) - \psi_{N_a}(z)] + \varepsilon
< 4\varepsilon . \tag{15}
\]

Hence for each compact \( A \subset \mathbb{C}^m \), we can cover \( A \) with finitely many \( U_{a_i} \), so that we have by (15),

\[ \| V_K - \psi_N \|_A \leq 4\varepsilon \quad \forall \ N \geq \max_i N_{a_i}^2 . \]
Lemma 3.3. For all \( \varepsilon > 0 \), there exists \( C = C_\varepsilon > 0 \) such that \[
\frac{1}{d(N)} \leq \frac{S_N(z, z)}{\Phi_N(z)^2} \leq C e^{\varepsilon N} d(N).
\]

Proof. Let \( f \in \mathcal{P}_N \) with \( \|f\|_K \leq 1 \). Then \[
|f(z)| = \left| \int_K S_N(z, w) f(w) \, d\mu(w) \right| \leq \int_K |S_N(z, w)| \, d\mu(w) 
\leq \int_K S_N(z, z)^{\frac{3}{2}} S_N(w, w)^{\frac{1}{2}} \, d\mu(w) = S_N(z, z)^{\frac{1}{2}} \|S_N(w, w)^{\frac{1}{2}}\|_{L^1(\mu)} 
\leq S_N(z, z)^{\frac{1}{2}} \|1\|_{L^2(\mu)} \|S_N(w, w)^{\frac{1}{2}}\|_{L^2(\mu)} = S_N(z, z)^{\frac{1}{2}} d(N)^{\frac{1}{2}}.
\]

Taking the supremum over \( f \in \mathcal{P}_N \) with \( \|f\|_K \leq 1 \), we obtain the left inequality of the lemma.

To verify the right inequality, we let \( \{p_j\} \) be a sequence of \( L^2(\mu) \)-orthonormal polynomials, obtained by applying Gram-Schmidt to a sequence of monomials of non-decreasing degree, so that \( \{p_1, \ldots, p_{d(N)}\} \) is an orthonormal basis of \( \mathcal{P}_N \) (for each \( N \in \mathbb{Z}^+ \)). By the Bernstein-Markov inequality \([\text{BMT}]\), we have \( \|p_j\|_K \leq C e^{\varepsilon \deg p_j} \) and hence \[
|p_j(z)| \leq \|p_j\|_K \Phi^K_{\deg p_j}(z) \leq C e^{\varepsilon \deg p_j} \Phi^K_{\deg p_j}(z) \leq C e^{\varepsilon N} \Phi^K_N(z), \quad \text{for } j \leq d(N).
\]

Therefore, \[
S_N(z, z) = \sum_{j=1}^{d(N)} |p_j(z)|^2 \leq d(N) C^2 e^{2\varepsilon N} \Phi^K_N(z)^2.
\]

Lemma 3.4. Under the hypotheses of Theorem 3.1, we have \[
\frac{1}{2N} \log S_N(z, z) \rightarrow V_K(z)
\]
uniformly on compact subsets of \( \mathbb{C}^m \).

Proof. Let \( \varepsilon > 0 \) be arbitrary. Recalling that \( d(N) = \binom{N+m}{m} \), we have by Lemma 3.3 \[
-\frac{m}{N} \log(N + m) \leq \frac{1}{N} \log \left( \frac{S_N(z, z)}{\Phi_N(z)^2} \right) \leq \frac{\log C}{N} + \varepsilon + \frac{m}{N} \log(N + m).
\]

Since \( \varepsilon > 0 \) is arbitrary, we then have \[
\frac{1}{N} \log \left( \frac{S_N(z, z)}{\Phi_N(z)^2} \right) \rightarrow 0.
\]

The conclusion follows from Lemma 3.2 and (16). \( \square \)

Proof of Theorem 3.1: It follows from Lemma 3.4 and the fact that the complex Monge-Ampere operator is continuous under uniform limits \([\text{BTT}]\), \[
\left( \frac{i}{2\pi N} \partial \bar{\partial} \log S_N(z, z) \right)^k \rightarrow \left( \frac{i}{\pi} \partial \bar{\partial} V_K(z) \right)^k \quad \text{weak*}.
\]
The conclusion then follows from Proposition 2.3.

Example 3.5. Let \( K \) be the unit polydisk in \( \mathbb{C}^m \). Then \( V_K = \max_{j=1}^m \log^+ |z_j| \), the Silov boundary of \( K \) is the product of the circles \( |z_j| = 1 \) (\( j = 1, \ldots, m \)), and \( d\mu_{eq} = \left( \frac{1}{2\pi} \right)^m d\theta_1 \cdots d\theta_m \) where \( d\theta_j \) is the angular measure on the circle \( |z_j| = 1 \).

The monomials \( z^J := z_1^{j_1} \cdots z_m^{j_m} \), for \( |J| \leq N \), form an orthonormal basis for \( \mathcal{P}_N \). A random polynomial in the ensemble is of the form

\[
f(z) = \sum_{|J| \leq N} c_J z^J
\]

where the \( c_J \) are independent complex Gaussian random variables of mean zero and variance one. By Theorem 3.3, \( \mathbb{E} \gamma_K^N (Z_{f_1, \ldots, f_m}) \rightarrow \left( \frac{1}{2\pi} \right)^m d\theta_1 \cdots d\theta_m \) weak*, as \( N \to \infty \). In particular, the common zeros of \( m \) random polynomials tend to the product of the unit circles \( |z_j| = 1 \) for \( j = 1, \ldots, m \).

Example 3.6. Let \( K \) be the unit ball \( \{ \|z\| \leq 1 \} \) in \( \mathbb{C}^m \). Then the Silov boundary of \( K \) is its topological boundary \( \{ \|z\| = 1 \} \), \( V_K(z) = \log^+ \|z\| \), and \( \mu_{eq} \) is the invariant measure on the unit sphere \( S^{2m-1} \). A more general compact set \( K \subset \mathbb{C} \) with an analytic boundary, scaling limits are found in [SZ2].

In this section, we consider the case where \( K = \{ z \in \mathbb{C}^m : \|z\| \leq 1 \} \) is the unit ball and \( \mu \) is its equilibrium measure, i.e. invariant measure on the unit sphere \( S^{2m-1} \). We have the following scaling asymptotics for the expected distribution of zeros of \( m \) random polynomials orthonormalized on the sphere:

**Theorem 4.1.** Let \( (\mathcal{P}_N^m, \gamma_N^m) \) denote the ensemble of \( m \)-tuples of i.i.d. Gaussian random polynomials of degree \( \leq N \) with the Gaussian measure \( d\gamma_N \) induced by \( L^2(S^{2m-1}, \mu) \), where \( \mu \) is the invariant measure on the unit sphere \( S^{2m-1} \subset \mathbb{C}^m \). Then

\[
\mathbb{E} \gamma_N^m (Z_{f_1, \ldots, f_m}) = D_N \left( \log \|z\|^2 \right) \left( \frac{i}{2} \partial \bar{\partial} \|z\|^2 \right)^m,
\]

where

\[
\frac{1}{N^{m+1}} D_N \left( \frac{u}{N} \right) = \frac{1}{\pi^m} F_m''(u) F_m'(u)^{m-1} + O \left( \frac{1}{N} \right), \quad F_m(u) = \log \left[ \frac{d^{m-1}}{d\mu^{m-1}} \left( \frac{e^u - 1}{u} \right) \right].
\]

**Proof.** We write

\[
z^J = z_1^{j_1} \cdots z_m^{j_m}, \quad z = (z_1, \ldots, z_m), \quad J = (j_1, \ldots, j_m).
\]

An easy computation yields

\[
\int_{S^{2m-1}} |z^J|^2 d\mu(z) = \frac{(m-1)! j_1! \cdots j_m!}{(|J| + m - 1)!} = \frac{1}{(m-1)!} \binom{|J| + m - 1}{|J|} \binom{|J|}{|J|}, \tag{17}
\]

where

\[
|J| = j_1 + \cdots + j_m, \quad \binom{|J|}{J} = \binom{|J|}{j_1 \cdots j_m}.
\]
Thus an orthonormal basis for $P_N$ on $S^{2m-1}$ is:

$$\varphi_J(z) = \left(\frac{|J| + m - 1}{m - 1}\right)^{\frac{1}{2}} \left(\frac{|J|}{J}\right)^{\frac{1}{2}} z^J, \quad |J| \leq N.$$  \hfill (18)

We have

$$S_N(z, z) = \sum_{|J| \leq N} |\varphi_J(z)|^2 = \sum_{k=0}^N \left(\frac{k + m - 1}{m - 1}\right) \sum_{|J| = k} \left(\frac{k}{J}\right) |z_1|^{2j_1} \cdots |z_m|^{2j_m}$$

$$= \sum_{k=0}^N \left(\frac{k + m - 1}{m - 1}\right) \|z\|^{2k}.$$  

Hence

$$S_N(z, z) = g_N(\|z\|^2), \quad \text{where} \quad g_N(x) = \sum_{k=0}^N \left(\frac{k + m - 1}{m - 1}\right) x^k.$$  \hfill (19)

We note that

$$g_N = \frac{1}{(m - 1)!} G_N^{(m-1)}, \quad \text{where} \quad G_N(x) = \frac{1 - x^{N+m}}{1 - x}. $$

We denote by $O(\frac{1}{N})$ any function $\lambda(N, u) = \lambda_N(u) : \mathbb{Z}^+ \times \mathbb{R} \to \mathbb{R}$ satisfying:

$$\forall R > 0, \forall j \in \mathbb{N}, \exists C_R \in \mathbb{R}^+ \text{ such that } \sup_{|u| < R} |\lambda^{(j)}_N(u)| < \frac{C_R}{N}.$$  \hfill (20)

We note that

$$N \log \left(1 + \frac{u}{N}\right) = u + u^2 O \left(\frac{1}{N}\right) \quad \text{(for } |u| < N),$$

and hence

$$\left(1 + \frac{u}{N}\right)^N = e^u + u^2 O \left(\frac{1}{N}\right).$$

Thus we have

$$\frac{1}{N} G_N \left(1 + \frac{u}{N}\right) = \frac{e^u - 1}{u} + O \left(\frac{1}{N}\right).$$  \hfill (21)

Hence

$$\frac{1}{N} g_N \left(1 + \frac{u}{N}\right) = \frac{1}{(m - 1)!} \frac{d^{m-1}}{du^{m-1}} \left(\frac{e^u - 1}{u}\right) + O \left(\frac{1}{N}\right).$$  \hfill (22)

Therefore

$$\log \left[\frac{(m - 1)!}{N^m} g_N \left(1 + \frac{u}{N}\right)\right] = F_m(u) + O \left(\frac{1}{N}\right),$$  \hfill (23)

where $F_m$ is given in the statement of the theorem.

Since the zero distribution is invariant under the $\text{SO}(2m)$-action on $\mathbb{C}^m$, we can write

$$E_{\gamma_N}(Z_{f_1, \ldots, f_m}) = D_N \left(\log \|z\|^2\right) \left(\frac{i}{2} \partial \bar{\partial} \|z\|^2\right)^m.$$  \hfill (24)
Then $D_N(\frac{u}{N})$ is the density at the point 

$$z^N := \left(\frac{1}{\sqrt{m}} e^{u/2N}, \ldots, \frac{1}{\sqrt{m}} e^{u/2N}\right) \in \mathbb{C}^m, \quad \|z^N\|^2 = e^{u/N}.$$ 

We shall compute using the local coordinates $\zeta_j = \rho_j + i\theta_j = \log z_j$. Let 

$$\Omega = \left(\frac{i}{2} \partial \bar{\partial} \sum |\zeta_j|^2 \right)^m.$$ 

By Proposition 2.4 and (19), we have 

$$E_{\gamma_m}(Z_{f_1,\ldots,f_m}) = \left(\frac{1}{2\pi}\right)^m \det \left(\frac{1}{2} \frac{\partial^2}{\partial \rho_j \partial \rho_k} \log g_N \left(\sum e^{2\rho_j}\right)\right) \Omega. \quad (25)$$

We note that 

$$\Omega = m^m \left[1 + O\left(\frac{1}{N}\right)\right] \left(\frac{i}{2} \partial \bar{\partial} \|z\|^2\right)^m \text{ at the point } z^N. \quad (26)$$

We let $1$ denote the $m \times m$ matrix all of whose entries are equal to 1 (and we let $I$ denote the $m \times m$ identity matrix). By (23) and (25)–(26), we have 

$$D_N\left(\frac{u}{N}\right) = \left(\frac{m}{2\pi}\right)^m \left[1 + O\left(\frac{1}{N}\right)\right] \times \det \left(2m^{-2} e^{2u/N} (\log g_N)''(e^{u/N}) \mathbf{1} + 2m^{-1} e^{u/N} (\log g_N)'(e^{u/N}) I\right) = \frac{1}{\pi^m} \left[1 + O\left(\frac{1}{N}\right)\right] \det \left(m^{-1} N^2 F''_m(u) \mathbf{1} + N F'_m(u) I\right).$$

Therefore,

$$\frac{1}{N^{m+1}} D_N\left(\frac{u}{N}\right) = \frac{1}{N^{m+1} \pi^m} \left[1 + O\left(\frac{1}{N}\right)\right] \times \left\{[N F'_m(u)]^m + m \left[m^{-1} N^2 F''_m(u)\right] [N F'_m(u)]^{m-1}\right\} = \frac{1}{\pi^m} F''_m(u) F'_m(u)^{m-1} + O\left(\frac{1}{N}\right).$$

□

**Remark:** There is a similarity between the scaling asymptotics of Theorem 4.1 and that of the one-dimensional SU(1, 1) ensembles in [BR] with the norms $\|z^j\| = \left(L^{-1+j}\right)^{-1/2}$, for $L \in \mathbb{Z}^+$. Then the expected distribution of zeros of random SU(1, 1) polynomials of degree $N$ has the asymptotics [BR, Th. 2.1]:

$$E_N(Z_f) = \tilde{D}_N \left(\log |z|^2\right) \frac{i}{2} dz \wedge d\bar{z},$$

where (in our notation)

$$\frac{1}{N^2} \tilde{D}_N\left(\frac{u}{N}\right) = \frac{1}{\pi} F''_{L-1}(u) + O\left(\frac{1}{N}\right).$$
REFERENCES

[BT1] E. Bedford and B. A. Taylor, A new capacity for plurisubharmonic functions, Acta Math. 149 (1982), 1–40.

[BT2] E. Bedford and B. A. Taylor, Fine topology, Silov boundary and $(dd^c)^n$, J. Funct. Anal. 72 (1987), 225–251.

[BR] P. Bleher and R. Ridzal, SU(1, 1) random polynomials, J. Statist. Phys. 106 (2002), 147–171.

[BSZ1] P. Bleher, B. Shiffman and S. Zelditch, Poincaré-Lelong approach to universality and scaling of correlations between zeros, Comm. Math. Phys. 208 (2000), 771–785.

[BSZ2] P. Bleher, B. Shiffman and S. Zelditch, Universality and scaling of correlations between zeros on complex manifolds, Invent. Math. 142 (2000), 351–395.

[Bi1] T. Bloom, Orthogonal polynomials in $\mathbb{C}^n$, Indiana Univ. Math. J. 46 (1997), 427–452.

[Bi2] T. Bloom, Random polynomials and Green functions, Int. Math. Res. Not. 2005 (2005), 1689–1708.

[BL] T. Bloom and N. Levenberg, Capacity convergence results and applications to a Bernstein-Markov inequality, Trans. Amer. Math. Soc. 351 (1999), 4753–4767.

[DS] T.-C. Dinh and N. Sibony, Distribution des valeurs de transformations méromorphes et applications, Comment. Math. Helv. 81 (2006), 221–258.

[EK] A. Edelman and E. Kostlan, How many zeros of a random polynomial are real? Bull. Amer. Math. Soc. 32 (1995), 1–37.

[Ha] J. H. Hammersley, The zeros of a random polynomial. Proceedings of the Third Berkeley symposium on Mathematical Statistics and Probability, 1954-55, vol II, University of California Press, California 1956, pp 89–111.

[IZ] I. Ibragimov and O. Zeitouni, On roots of random polynomials, Trans. Amer. Math. Soc. 349 (1997), 2427–2441.

[Kl] M. Klimek, Pluripotential Theory, London Math. Soc. Monographs, New Series 6, Oxford University Press, New York, 1991.

[NZ] T. V. Nguyen and A. Zériahia, Famille de polynômes presque partout bornées, Bull. Sci. Math., Paris, 107 (1983), 81–91.

[Ra] T. Ransford, Potential Theory in the Complex Plane, London Mathematical Society Student Texts 28, Cambridge University Press, 1995.

[SV] L. A. Shepp and R. J. Vanderbei, The Complex zeros of random polynomials, Trans. Amer. Math. Soc. 347 (1995), 4365–4384.

[SZ1] B. Shiffman and S. Zelditch, Distribution of zeros of random and quantum chaotic sections of positive line bundles, Comm. Math. Phys. 200 (1999), 661–683.

[SZ2] B. Shiffman and S. Zelditch, Equilibrium distribution of zeros of random polynomials, Int. Math. Res. Not. 2003 (2003), 25–49.

[SZ3] B. Shiffman and S. Zelditch, Random polynomials with prescribed Newton polytope, J. Amer. Math. Soc. 17 (2004), 49–108.

[SZ4] B. Shiffman and S. Zelditch, Number variance of random zeros, preprint 2005, math.CV/0512652.

[Si1] J. Siciak, Extremal plurisubharmonic functions in $\mathbb{C}^n$, Ann. Polon. Math. 39 (1981), 175–211.

[Si2] J. Siciak, Extremal Plurisubharmonic Functions and Capacities in $\mathbb{C}^n$, Sophia Kokyuroku in Mathematics, No. 14, Sophia Univ., Tokyo, 1982.

[So] M. Sodin, Zeros of Gaussian analytic functions, Math. Res. Lett. 7 (2000), 371–381.

[Za] V. P. Zaharjuta, Extremal plurisubharmonic functions, orthogonal polynomials, and the Bernstein-Walsh theorem for functions of several complex variables (Russian), Ann. Polon. Math. 33 (1976/77), 137–148.

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