Global higher integrability for minimisers of convex obstacle problems with \((p,q)\)-growth

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Abstract
We prove global \(W^{1,q}(\Omega, \mathbb{R}^N)\)-regularity for minimisers of \(\mathcal{F}(u) = \int_{\Omega} F(x, Du) \, dx\) satisfying \(u \geq \psi\) for a given Sobolev obstacle \(\psi\). \(W^{1,q}(\Omega, \mathbb{R}^N)\) regularity is also proven for minimisers of the associated relaxed functional. Our main assumptions on \(F(x,z)\) are a uniform \(\alpha\)-Hölder continuity assumption in \(x\) and natural \((p,q)\)-growth conditions in \(z\) with \(q < \frac{(n+\alpha)p}{n-1}\). In the autonomous case \(F \equiv F(z)\) we can improve the gap to \(q < \min\left(\frac{np}{n-1}, p+1\right)\), a result new even in the unconstrained case.

Mathematics Subject Classification 35J60 · 35J70

1 Introduction and results
We are interested in the following vectorial obstacle problem: given a domain \(\Omega \subset \mathbb{R}^n\), boundary datum \(g\) and an obstacle \(\psi\), consider the obstacle problem

\[
\min_{u \in K_g^\psi(\Omega)} \mathcal{F}(u) \quad \text{where} \quad \mathcal{F}(u) = \int_{\Omega} F(x, Du) \, dx.
\]

(P)

Here we write

\[
K_g^\psi(\Omega) = \{ u \in W^{1,p}_g(\Omega, \mathbb{R}^N) : u \geq \psi \text{ a.e. in } \Omega\}.
\]

Here and throughout we understand vector-valued inequalities such as \(u \geq \psi\) to be applied row-wise, that is if \(u = (u_1, \ldots, u_N)^T\), \(\psi = (\psi_1, \ldots, \psi_N)^T\), then \(u_i \geq \psi_i\) for \(i = 1, \ldots, N\).

In order to ensure that \(K_g^\psi(\Omega)\) is non-empty we assume throughout that \(g, \psi\) are Sobolev functions satisfying \(g \geq \psi\) on \(\partial \Omega\) in the sense of traces. We refer to Sect. 2 for unexplained notation.

The integrand \(F \equiv F(x,z) : \Omega \times \mathbb{R}^{N} \times \mathbb{R}^n \to \mathbb{R}\) is convex and satisfies \((p,q)\)-growth in \(z\) as well as a natural uniform \(\alpha\)-Hölder condition in \(x\). We stress that we work in the multi-
dimensional vectorial case $n > 1$, $N \geq 1$. We prove global higher integrability properties of the minimiser as well as global higher integrability properties of minimisers of a relaxed functional related to (P). We also study the autonomous case $F \equiv F(z)$ where we obtain stronger results which are in fact new even in the unconstrained case where no obstacle is present.

In order to state our results precisely and to compare them with the literature on (constrained) functionals with $(p, q)$-growth we list our assumptions.

We suppose throughout that $F \equiv F(x, z) : \Omega \times \mathbb{R}^{N \times n} \to \mathbb{R}$ is measurable in $x$ and continuously differentiable in $z$. We further introduce the following assumptions, assumed to hold for all $z \in \mathbb{R}^{N \times n}$ and almost every $x, y \in \Omega$, which we invoke as required:

(H1) $F(x, z) - \lambda (\mu^2 + |z|^2)^{\frac{p}{2}}$ is convex in $z$

(H2) $F(x, z) \leq \Lambda (1 + |z|^q)$

(H3) $|F(x, z) - F(y, z)| \leq \Lambda |x - y|^\alpha (1 + |z|^2)^{\frac{q}{2}}$.

Here $\mu \geq 0$, $\Lambda, \lambda > 0$ and $\alpha \in (0, 1]$. Moreover, we assume that $2 \leq p < q < \infty$. If $p < n$, we further suppose $q \leq \frac{np}{n-p}$.

We remark that (H1), (H2) are usually referred to as natural growth conditions. Under these assumptions $F$ is convex in $z$ and (after adding a constant to $F$ if necessary) the following bounds apply for almost every $x \in \Omega$ and every $z, w \in \mathbb{R}^{N \times n}$:

$$\left(\mu^2 + |z|^2 + |w|^2\right)^{\frac{p}{2}} \lesssim \frac{F(x, z) - F(x, w) - \langle \partial_z F(x, w), z - w \rangle}{|z - w|^2}$$

(H4)

$$F(x, z) \gtrsim |z|^p - 1.$$  

(H5)

Let us also give a precise definition of the notions of minimisers we are interested in.

**Definition 1** Given $\psi, g \in W^{1,q}(\Omega) = W^{1,q}(\Omega, \mathbb{R}^N)$, we say that $u \in K_{g}^\psi(\Omega)$ is a (pointwise) minimiser of (P) if it holds that $F(x, Du) \in L^1(\Omega)$ and

$$\int_\Omega F(x, Du) \, dx \leq \int_\Omega F(x, D\phi) \, dx$$

for all $\phi \in K_{g}^\psi(\Omega)$.

Further $u \in K_{g}^\psi(\Omega)$ is a relaxed minimiser of (P) if $u$ minimises the relaxed functional

$$\overline{\mathcal{F}}(u) = \inf \left\{ \liminf_{j \to \infty} \int_\Omega F(x, Du_j) \, dx : u_j \in K_{g}^{*,\psi}(\Omega), u_j \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega) \right\}$$

where

$$K_{g}^{*,\psi}(\Omega) = \{ u \in W^{1,q}_g(\Omega) : u \geq \psi \text{ a.e. in } \Omega \},$$

that is

$$\overline{\mathcal{F}}(u) \leq \overline{\mathcal{F}}(v) \text{ for all } v \in K_{g}^\psi(\Omega).$$
We remark immediately that \( K_{g,\psi}^*(\Omega) \) is dense in \( K_{g,\psi}^*(\Omega) \) (see Corollary 7) and that by weak lower semicontinuity of \( \mathcal{F}(\cdot) \) in \( W^{1,q}(\Omega) \), for \( u \in K_{g,\psi}^*(\Omega) \), \( \mathcal{F}(u) = \mathcal{F}(u) \). Further using the direct method it is not difficult to establish the existence of both types of minimisers.

When no obstacle is present, the study of elliptic systems and functionals when \( p = q \) is well established with a long list of important results. For an introduction and references, we refer to [42, 43].

In the unconstrained case, the systematic study of regularity theory for minimisers when \( p < q \) started with the seminal papers [55, 56]. We refer to [57, 58] for an overview of the theory and further references. We only list the, to our knowledge, best available \( W_{loc}^{1,q} \)-regularity results for general autonomous convex functionals with \((p,q)\)-growth when \( n \geq 2 \). The results vary depending on the precise assumptions made on the integrand.

The results vary depending on the precise assumptions made on the integrand and \( F \). Commonly imposed growth conditions, aside from the natural growth condition we use in this paper, are regularity results for general autonomous convex functionals with \( \lambda \)-growth. Conditions, the gap may be widened to \( q < \frac{np}{n-1} \) (if \( n = 2 \) it suffices to take \( q < \infty \)) [20]. In all three cases, higher integrability goes hand in hand with a higher differentiability result. In the autonomous case our main result is new even in the unconstrained setting and gives a global analogue of the result in [15]. In the non-autonomous case the author in [52] extended the work of [28, 34] giving global versions of local \( W^{1,q} \)-regularity results obtained in [28, 34] for relaxed minimisers of functionals \( F(x,z) \), convex and with \((p,q)\)-growth in \( z \), while satisfying a uniform \( \alpha \)-Hölder condition in \( x \) under the sharp assumption \( q < \frac{(n+\alpha)p}{n} \). An exception are functionals modelled on the double-phase functional [3]. See also [23].

Recently, constrained problems with \((p,q)\)-growth have gained interest. Obstacle problems have a long history in their own right and we refer to [48] for an introduction and further references with regards to the theory in the case of \( p \)-growth. When considering linear elliptic obstacle problems, while solutions are in general not \( C^2 \), in lower regularity regimes, e.g. \( W^{1,\infty} \) or \( C^{1,\lambda} \), the regularity of the solution agrees with the regularity of the obstacle [7, 14, 51]. In the non-linear setting, this is not the case and, usually, more regularity has to be assumed on the obstacle to overcome the effect of the non-linearity. One reason for the recent
interest in obstacle problems with non-standard growth is that such problems have appeared in
the construction of comparison problems for the study of fine properties of the solutions of certain non-linear PDEs with non-standard growth [16, 38, 48, 50].

We remark that at the moment the theory in the vectorial setting and in the case of
\((p, q)\)-growth in particular is considerably less developed than that in the scalar setting, in
particular when comparing to scalar quadratic growth. In the case of the scalar Laplacian,
\(F(x, z) = |z|^2\), the regularity of the free boundary \(\partial \{ u > \psi \}\) has been studied in great detail
[12, 13, 31, 32]. However already when considering the scalar \(p\)-Laplacian \(F(x, z) = |z|^p\),
considerably less is known [2, 30]. Finally we note that our constraint \(u \geq \psi\) is a convex
constraint. The case of non-convex constraints is harder, but nevertheless of interest, see e.g.
the study of functionals with \((p, q)\)-growth constrained to lie on a manifold [17, 18, 21].

Returning to vectorial functionals with \((p, q)\)-growth, until recently most of the focus has
been on results concerning functionals with additional structure assumptions. These include
improved integrability and differentiability results in the case of variable exponent functionals
\(\int_{\Omega} |Du|^{p(z)} \, dx\) [24, 25, 35, 38], as well as the double phase functional \(\int_{\Omega} |Du|^{p} + a(x)|Du|^{q} \, dx\)
[11, 16, 67]. Another direction of research has been Caldéron–Zygmund estimates for both
double phase [10] and variable exponent [54] obstacle problems. We mention that improved
integrability results are also available in the setting of almost linear growth [39, 59] as well
as certain parabolic settings, see for example [6, 26] for results and further references. As
is usually observed in problems with non-standard growth, for radial integrands of the form
\(F(x, z) \equiv F(x, |z|)\), minimisers are better behaved and understood. In this setting, sharp
assumptions giving Lipschitz regularity of minimisers can be found in [22] and integrands
with generalised Orlicz-growth are studied in [49]. Regularity results in the scale of Besov
spaces in this setting were obtained in [45].

Obstacle problems for integrands satisfying \((p, q)\)-growth, but no further structural
assumptions, were first studied in [19, 40, 41] where local improved integrability and differ-
entiability as well as local higher regularity properties of minimisers were established. The
assumptions on the integrand in [19] are similar to the ones that we assume in this paper. The
results here, particularly regarding the non-autonomous case, may be regarded as global
versions of those in [19]. We remark, however, that our assumptions on the obstacle \(\psi\) are
stronger than those in [19]. In particular, there it suffices to assume \(\psi \in W^{1+\alpha,q}(\Omega)\) in the
non-autonomous case whereas we require \(\psi \in W^{2,\infty}(\Omega)\). In fact, under the assumption
that \(\psi \in W^{2,\infty}(\Omega)\), local Hölder-continuity of the minimiser is derived in [19]. We do not
prove an analogous global result here. We remark that we require the stronger assumption
\(\psi \in W^{2,\infty}(\Omega)\) only for the \(L^1\)-penalisation part of the argument, see below for an overview
of the proof, and we point out that the sharpness of either assumption is not known. Neverthe-
less, our assumption on the gap \(q < \frac{(n+\alpha)p}{n}\) appears already in the unconstrained case and is
sharp [28]. We remark that a similar set-up has been studied (locally) in the non-autonomous
case but with (H3) replaced by a Sobolev-dependence, c.f. [5].

The author believes the results in this paper to be the first global results on obstacle prob-
lems for vectorial functionals with \((p, q)\)-growth but without further structure assumptions. Our
main theorem is the following:

**Theorem 1** Suppose that \(\Omega\) is a Lipschitz domain and the \(C^2\) integrand \(F \equiv F(z)\) satisfies
(H1) and (H2) with \(2 \leq p \leq q < \min\left(\frac{np}{n-1}, \ p + 1\right)\). Let \(\psi \in W^{2,\infty}(\Omega)\) and \(g \in W^{2,q}(\Omega)\).
Then \(u \in W^{1,q}(\Omega)\) where \(u\) is the relaxed minimiser of (P).

Suppose \(F \equiv F(x, z)\) is \(C^2\) in \(z\) and satisfies (H1), (H2) and (H3) with \(2 < p \leq q < \frac{(n+\alpha)p}{n}\). Suppose \(\psi \in W^{2,\infty}(\Omega)\) and \(g \in W^{1+\alpha,q}(\Omega)\). Then \(u \in W^{1,q}(\Omega)\) where \(u\) is the
relaxed minimiser of (P).
To the best of our knowledge this is the first global $W^{1,q}$-regularity result valid for a large class of general convex $(p, q)$-growth functionals in the constrained case. The autonomous result is new even in the unconstrained case, and is the global equivalent of results in [15]. The proof of Theorem 1 also gives the following improved differentiability result:

**Corollary 1** Suppose the assumptions of Theorem 1 are satisfied and use the notation of that theorem. Then it holds that $u \in B^{1 + \frac{1}{p}, p}_\infty(\Omega)$ in the autonomous case and $u \in B^{1 + \frac{\alpha}{p}, p}_\infty(\Omega)$ in the non-autonomous case.

The proof of Theorem 1 is based on the difference quotient method. To apply the difference quotient method globally, we rely on an argument developed in [61], which the author also used in the unconstrained case in [52]. However, the constraint causes additional difficulties, since all competitors need to satisfy the constraint. To overcome this issue we rely on an observation that is well known in the numerics literature. We refer to [62, 64] for the necessary results in our set-up. For sufficiently large parameter $\kappa$, the $L^1$-penalisation of obstacle problems is exact, that is, minimisers $u$ of

$$W_g^{1, p}(\Omega) \ni v \to \int_\Omega F(x, Dv) + \kappa(\psi - v)_+ \, dx$$

satisfy $u \geq \psi$ for $\kappa > \kappa_0(\|\psi\|_{W^{2, \infty}(\Omega)})$. The improvement in the autonomous case is due to a trick first used in [4] and used in the context of vector-valued $(p, q)$-growth in [63]. Optimising the choice of cut-off function in dependence on the minimiser we can employ a Sobolev embedding at a crucial step of the proof on spheres instead of balls. Due to the dimensional dependence of the constants in the embedding this gives a stronger result, allowing to widen the gap from $q < \left(\frac{n+1}{n}\right)p$ to $q < \frac{np}{n-1}$.

We remark that Theorem 1 is phrased for relaxed minimisers. This is forced by the possible occurrence of the Lavrentiev phenomenon, which describes the possibility that

$$\inf_{u \in K_\psi^g(\Omega)} \mathcal{F}(u) < \inf_{u \in K_\psi^{*, g}(\Omega)} \mathcal{F}(u).$$

This phenomenon occurs in the unconstrained case. A first example of such behaviour was given in [53]. In the context of $(p, q)$-growth functionals, the theory was further developed in [68–70]. The Lavrentiev phenomenon is closely related to properties of the relaxed functional. We adopt the viewpoint and terminology of [9], and consider a topological space $X$ of weakly differentiable functions with a dense subspace $Y \subset X$. We introduce the following sequentially lower semi-continuous (slsc) envelopes:

$$\overline{\mathcal{F}}_X = \sup\{ G : X \to [0, \infty] : G \text{ slsc, } G \leq \mathcal{F} \text{ on } X \}$$

$$\overline{\mathcal{F}}_Y = \sup\{ G : X \to [0, \infty] : G \text{ slsc, } G \leq \mathcal{F} \text{ on } Y \}.$$

Then we define the Lavrentiev gap functional for $u \in X$ as

$$\mathcal{L}(u, X, Y) = \begin{cases} 
\overline{\mathcal{F}}_Y(u) - \overline{\mathcal{F}}_X(u) & \text{if } \overline{\mathcal{F}}_X(u) < \infty \\
0 & \text{else.}
\end{cases}$$

We note that the gap functional is non-negative.

There is extensive literature on the Lavrentiev phenomenon and gap functional, an overview of which can be found in [8, 36], as well as further references. The phenomenon is also of interest in non-linear elasticity [37]. Considering the common choice $X = W^{1,p}(\Omega)$ endowed with the weak topology and $Y = W^{1,q}_{\text{loc}}(\Omega) \cap W^{1,p}(\Omega)$, a question related to the
Lavrentiev phenomenon is to study measure representations of $\overline{\mathcal{F}}(\cdot)$. We refer to [1, 33] for results and further references in this direction.

In this paper, we always consider the choice $X = K^\psi_g(\Omega)$ endowed with the weak topology inherited from $W^{1,p}(\Omega)$ and $Y = K^{*,\psi}_g(\Omega)$. Since $F(x, z)$ is convex, standard methods show that $\overline{\mathcal{F}}(X) = \mathcal{F}(\cdot)$ [43, Chapter 4]. Furthermore, $\overline{\mathcal{F}}(Y) = \mathcal{F}(\cdot)$. We also note that if $\mathcal{L}(u, X, Y) = 0$ for all $u \in X$, then the Lavrentiev phenomenon cannot occur. Non-occurrence of the Lavrentiev phenomenon allows us to transfer the estimates obtained in Theorem 1 to pointwise minimisers and thus to establish $W^{1,q}$-regularity.

In general, it is necessary to assume that $\mathcal{L}(u, K^\psi_g(\Omega), K^{*,\psi}_g(\Omega)) = 0$ for minimisers of (P) in order to replace relaxed minimisers in Theorem 1 with pointwise minimisers. Nevertheless, combining the arguments used in [47, 52], we are able to prove the following result:

**Proposition 1** Let $\alpha \in (0, 1)$. Suppose that $\Omega$ is a $C^{1,\alpha}$-domain and the assumptions of Theorem 1 hold for this choice of $\alpha$. In the non-autonomous case, assume additionally that (H6) holds and $q < p + 1$. Then $u \in W^{1,q}(\Omega)$, where $u$ is the pointwise minimiser of (P).

The structure of the paper is as follows. In Sect. 2, we collect some background results. We present the proof of exactness of $L^1$-penalisation in Sect. 3, before proving an apriori estimate for a regularised version of the $L^1$-penalised functional in Sect. 4. This allow us to prove our in Sect. 5 our main theorem Theorem 1, as well as Proposition 1.

## 2 Preliminaries

### 2.1 Notation

In this section we introduce our notation. The set $\Omega$ always denotes a open, bounded domain in $\mathbb{R}^n$. Given a set $\omega \subset \mathbb{R}^n$, $\overline{\omega}$ denotes its closure. We write $B_r(x)$ for the usual open Euclidean ball of radius $r$ in $\mathbb{R}^n$ and $S^{n-1}$ for the unit sphere in $\mathbb{R}^n$. We denote the cone of height $\rho$, aperture $\theta$ and axis in direction $n$ by $C_\rho(\theta, n)$, that is, the set

$$C_\rho(\theta, n) = \{ h \in \mathbb{R}^n : |h| \leq \rho, h \cdot n \geq |h| \cos(\theta) \}.$$

Here $|\cdot|$ denotes the Euclidean norm of a vector in $\mathbb{R}^n$ and likewise the Euclidean norm of a matrix $A \in \mathbb{R}^{n \times n}$. The identity matrix in $\mathbb{R}^{n \times n}$ is denoted $\text{Id}$. Given an open set $\Omega$ we denote $\Omega_\lambda = \{ x \in \Omega : d(x, \partial \Omega) > \lambda \}$ and $\lambda \Omega = \{ \lambda x : x \in \Omega \}$, where $d(x, \partial \Omega) = \inf_{y \in \partial \Omega} |x - y|$ denotes the distance of $x$ from the boundary of $\Omega$.

If $p \in [1, \infty]$ denote by $p' = \frac{p}{p-1}$ its Hölder conjugate. The symbols $a \sim b$ and $a \lesssim b$ mean that there exists some constant $C > 0$, depending only on $n, N, p, \Omega, \mu, \Lambda$, and independent of $a$ and $b$ such that $C^{-1} a \leq b \leq C a$ and $a \leq C b$, respectively.

Write $V_{\mu,t}(z) = (\mu^2 + |z|^2)^{\frac{t-2}{2}}z$. We recall the useful well-known inequality:

**Lemma 2** For every $s > -1$, $\mu \in [0, 1]$, $z_1, z_2 \in \mathbb{R}^N$, with $\mu + |z_1| + |z_2| > 0$, we have

$$\int_0^1 (\mu^2 + |z_1 + \lambda(z_2 - z_1)|^2)^\frac{t-2}{2} |\lambda| d\lambda \sim (\mu^2 + |z_1|^2 + |z_2|^2)^\frac{t-2}{2},$$

with the implicit constants only depending on $s$. Furthermore,

$$|V_{\mu,t}(z_1) - V_{\mu,t}(z_2)| \sim (\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{t-2}{2}} |z_1 - z_2|^2.$$

The implicit constants depend on $s$ and $N$ only.
We often find it useful to write for a function \( v \) defined on \( \mathbb{R}^n \) and a vector \( h \in \mathbb{R}^n \),
\[
v(x + h).
\]

We fix a family \( \{ \phi_e \} \) of radially symmetric, non-negative mollifiers of unitary mass. We denote convolution with \( \phi_e \) as
\[
u \ast \phi_e (x) = \int_{\mathbb{R}^n} u(y) \phi_e (x - y) \, dy.
\]

### 2.2 Function spaces

We recall some basic properties of Sobolev and Besov spaces following the exposition in [61]. The theory can also be found in [65].

For \( 0 \leq \alpha \leq 1 \) and \( k \in \mathbb{N} \), \( C^k(\Omega) \) and \( C^{k,\alpha}(\Omega) \) denote the spaces of functions that are \( k \)-times continuously differentiable in \( \Omega \) and \( k \)-times \( \alpha \)-Hölder differentiable in \( \Omega \), respectively.

For \( 1 \leq p \leq \infty \), \( k \in \mathbb{N} \), we let \( L^p(\Omega) = L^p(\Omega, \mathbb{R}^N) \) and \( W^{k,p}(\Omega) = W^{k,p}(\Omega, \mathbb{R}^N) \) denote the usual Lebesgue and Sobolev spaces, respectively. We write \( W^{k,p}_0(\Omega) \) for the closure of \( C^\infty(\Omega) \)-functions with respect to the \( W^{k,p} \)-norm. For \( g \in W^{k,p}(\Omega) \), we write \( W^{k,p}_g(\Omega) = g + W^{k,p}_0(\Omega) \).

We freely identify \( W^{k,p} \)-functions with their precise representatives.

We denote by \([\cdot, \cdot]_{s,q}\) the real interpolation functor. Let \( s \in (0, 1) \) and \( p, q \in [1, \infty] \). We define
\[
B^{s,p}_q(\Omega) = B^{s,p}_q(\Omega, \mathbb{R}^N) = [W^{1,p}(\Omega, \mathbb{R}^N), L^p(\Omega, \mathbb{R}^N)]_{s,q}
\]
\[
B^{1+s,p}_q(\Omega) = [W^{2,p}(\Omega), W^{1,p}(\Omega)]_{s,q} = \{ v \in W^{1,p}(\Omega) : Dv \in B^{s,p}_q(\Omega) \}
\]

Further, we recall that \( W^{1+s,p}_0(\Omega) = B^{1+s,p}_p(\Omega) \) and that for \( 1 \leq q < \infty \), \( B^{s,p}_q(\Omega) \) embeds continuously in \( B^{\infty,p}_\infty(\Omega) \). We use a characterisation of these spaces in terms of difference quotients as follows: let \( D \) be a set generating \( \mathbb{R}^n \), star-shaped with respect to 0. For \( s \in (0, 1) \), \( p \in [1, \infty] \), consider
\[
[v]_{s,p,\Omega}^p := \sup_{h \in D \setminus \{0\}} \int_{\Omega_h} \left| \frac{v_h(x) - v(x)}{h} \right|^p \, dx.
\]

This semi-norm characterises \( B^{s,p}_\infty(\Omega) \) in the sense that
\[
v \in B^{s,p}_\infty(\Omega) \iff v \in L^p(\Omega) \text{ and } [v]_{s,p,\Omega}^p < \infty.
\]

Moreover there are positive constants \( C_1, C_2 > 0 \) depending only on \( s, p, D, \Omega \) such that
\[
C_1 \| v \|_{B^{s,p}_\infty(\Omega)} \leq \| v \|_{L^p(\Omega)} + [v]_{s,p,\Omega} \leq C_2 \| v \|_{B^{s,p}_\infty(\Omega)}.
\]  

(2.1)

If \( \Omega = B_r(x_0) \), then the constants \( C_1, C_2 \) are unchanged by replacing \( D \) with \( QD \), where \( Q \) is an orthonormal matrix. In particular, when \( D = C_p(\theta, n) \) is a cone, they are independent of the choice of \( n \).

Finally, we recall that \( B^{s,p}_q(\Omega) \) may be localised. If \( \{ U_i \}_{i \leq M} \) is a finite collection of balls covering \( \Omega \), then \( v \in B^{s,p}_q(\Omega) \) if and only if \( v_{|\Omega \cap U_i} \in B^{s,p}_q(\Omega \cap U_i) \) for \( i = 1, \ldots, M \). Moreover, there are constants \( C_3, C_4 \) such that
\[ C_3 \| v \|_{B^{s,p}_q(\Omega)} \leq \sum_{i=1}^M \| v \|_{B^{s,p}_q(\Omega \cap U_i)} \leq C_4 \| v \|_{B^{s,p}_q(\Omega)}. \] (2.2)

We recall the following well-known embedding theorem; see, for example, [66].

**Theorem 2** Suppose that \( \Omega \) is a Lipschitz domain. Let \( 0 < s \leq 1 \) and \( p, p_1 \in [1, \infty] \). Assume that \( s - \frac{n}{p} = -\frac{n}{p_1} \) and \( v \in B^{s,p}_q(\Omega) \). Then, for any \( \varepsilon \in (0, 1 - p_1] \), we have
\[ \| v \|_{L^{p_1-\varepsilon}(\Omega)} \lesssim \| v \|_{B^{s,p}_q(\Omega)}. \]

We have a trace theorem in the following form; see, for example, [29].

**Lemma 3** Let \( \Omega \) be a Lipschitz domain and let \( 1 < p < \infty \). There is a bounded linear operator \( \text{Tr}: W^{1+\frac{1}{p},p}(\Omega) \to W^{1,p}(\partial \Omega) \). Moreover, \( \text{Tr}(u) \) may be defined to be the values of the precise representative of \( u \) on \( \partial \Omega \).

Finally, we recall the following well-known result, which will justify extending \( u \in W^{1,p}_g(\Omega) \) by extensions of \( g \). We refer to [29] for the ingredients of the proof.

**Lemma 4** Let \( p \in [1, \infty] \) and \( V \ni \Omega \) an open, bounded set. Suppose that \( u \in W^{1,p}(\Omega) \) and \( v \in W^{1,p}_g(\Omega) \cap W^{1,p}(V) \). Then the map
\[ w = \begin{cases} u & \text{in } \Omega \\ v & \text{in } V \setminus \Omega \end{cases} \]
is an element of \( W^{1,p}(V) \).

We require a Fubini-type theorem on spheres, that has independent interest. The statement is likely known to the expert, but we have been unable to find a reference in the literature. In the context of fractional Sobolev spaces such a result has been obtained in [44]. We remark that, while the proof in [44] relied on a geometric construction and direct calculation, our argument is based on interpolation.

**Lemma 5** Let \( \sigma > \rho \geq 0 \) and denote \( B^s_\sigma = B^s(B_\sigma,0) \), \( B^s_\rho = B^s(B_\rho,0) \). Let \( s \in (0, 1) \) and \( 1 \leq \tau \leq q \leq p \). If \( v \in B^{s,p}_q(B^s_\sigma \setminus B^s_\rho) \), then we have
\[ \int_\sigma^\rho \| v \|_{B^{s,p}_q(\partial B_r)}^\tau \, dr \lesssim \| v \|_{B^{s,p}_q(\partial B_\sigma \setminus B_\rho)}^\tau, \]
where the implicit constant depends only on \( s, \tau, q, p, \sigma \) and \( \rho \).

**Proof** Let \( v \in B^{s,p}_q(B^s_\sigma \setminus B^s_\rho) \). Since \( B^{s,p}_q(\partial B_r) = [L^p(\partial B_r), W^{1,p}(\partial B_r)]_{s,q} \), we have
\[ \| v \|_{B^{s,p}_q(\partial B_r)} = \left( \int_0^\infty \left( t^{-s} K(v, t, r) \right)^q \frac{dr}{t} \right)^{1/q}. \]
where
\[ K(v, t, r) = \inf \left\{ \| v_1 \|_{L^p(\partial B_r)} + t \| v_2 \|_{W^{1,p}(\partial B_r)} : v = v_1 + v_2, (v_1, v_2) \in X \right\} \]
for \( X = L^p(\partial B_r) \times W^{1,p}(\partial B_r) \). Using Jensen’s inequality and Fubini’s theorem, we find...
where
\[ K'(v, t, r) = \inf \{ \| v_1 \|_{L^p(\partial B_r)} + t \| v_2 \|_{W^{1,p}(\partial B_r)} : v = v_1 + v_2 \text{ in } B_\sigma \setminus B_\rho, (v_1, v_2) \in Y \} \]
for \( Y = L^p(B_\sigma \setminus B_\rho) \times W^{1,p}(B_\sigma \setminus B_\rho) \). Using that \( q \leq p \), we have
\[
I \lesssim \left( \int_0^\infty t^{-sq} \inf \left( \| v_1 \|_{L^p(B_\sigma \setminus B_r)} + t \| v_2 \|_{W^{1,p}(B_\sigma \setminus B_r)} \right)^q \, \frac{dr}{t^q} \right)^{\frac{1}{q}}.
\]
where the infimum is taken over
\[ \{(v_1, v_2) \in Y : v = v_1 + v_2 \text{ in } B_\sigma \setminus B_\rho \}. \]

\[ \square \]

We use the following interpolation inequality, which is a direct consequence of Hölder’s inequality. Given \( p_1, p_2 \in [1, \infty] \) and \( 0 < \theta < 1 \), define \( p_\theta \) by the identity \( \frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \). Then, for \( u \in L^{p_1}(\Omega) \cap L^{p_2}(\Omega) \), we have
\[ \| u \|_{L^{p_\theta}(\Omega)} \leq \| u \|_{L^{p_1}(\Omega)}^{\theta} \| u \|_{L^{p_2}(\Omega)}^{1-\theta}. \tag{2.3} \]

### 2.3 Some properties of Lipschitz and \( C^{1,\alpha} \) domains

In this section we recall some properties of Lipschitz and \( C^{1,\alpha} \)-domains. For further details we refer to [46]. We say \( \Omega \subset \mathbb{R}^n \) is a Lipschitz \((C^{1,\alpha})\) domain if \( \Omega \) is an open subset of \( \mathbb{R}^n \) and for every \( x \in \partial \Omega \), there exist a neighbourhood \( V \) of \( x \) in \( \mathbb{R}^n \) and orthogonal coordinates \((y_i)_{1 \leq i \leq n}\) such that the following holds:

1. \( V \) is a hypercube in the new coordinates:
\[ V = \{(y_1, \ldots, y_n) : -a_i < y_i < a_i, \ 1 \leq i \leq n - 1 \}. \]
2. there exists a Lipschitz \((C^{1,\alpha})\) function \(\phi\) defined in
\[
V' = \{ (y_1, \ldots, y_{n-1}) : -a_i < y_i < a_i, \ 1 \leq i \leq n-1 \}
\]
such that
\[
|\phi(y')| \leq a_n/2 \text{ for every } y' = (y_1, \ldots, y_{n-1}) \in V',
\]
\[
\Omega \cap V = \{ y = (y', y_n) \in V : y_n < \phi(y') \},
\]
\[
\partial \Omega \cap V = \{ y = (y', y_n) \in V : y_n = \phi(y') \}.
\]

Let \(\Omega\) be a Lipschitz domain. Then \(\Omega\) satisfies a uniform exterior cone condition [46,Section 1.2.2]. Indeed, there exist \(\rho_0, \theta_0 > 0\) and a map \(n : \mathbb{R}^n \to S^{n-1}\) such that for every \(x \in \mathbb{R}^n\)
\[
C_{\rho_0}(\theta_0, n(x)) \subset O_{\rho_0}(x) = \{ h \in \mathbb{R}^n : |h| \leq \rho_0, \ (\Omega \setminus B_{3\rho_0}(x)) + h \subset \mathbb{R}^n \setminus \Omega \}. \quad (2.4)
\]

Moreover there is a smooth vector field transversal to \(\partial \Omega\), that is, there exist \(\kappa > 0\) and \(X \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)\) such that
\[
X \cdot v \geq \kappa \quad \text{a.e. on } \partial \Omega
\]
where \(v\) is the exterior unit normal to \(\partial \Omega\) [46,Lemma 1.5.1.9.].

We recall the following Lemma from [52] which enables us to to stretch a small neighbourhood of the boundary in a controlled manner. This is crucial in constructing sequences with improved integrability but unchanged boundary behaviour.

**Lemma 6** (Lemma 2.6 in [52]) Suppose \(\Omega\) is a \(C^{1,\alpha}\)-domain. Then there is a family of domains \(\Omega^s \supseteq \Omega\) and a family of \(C^{1,\alpha}\)-diffeomorphisms \(\Psi_s : \Omega^s \to \Omega\) such that

1. \(|D\Psi_s - \text{Id}| \to 0\) uniformly in \(\Omega^s\) as \(s \searrow 1\), so that \(|\det D\Psi_s \to 1|\). Equivalently, \(|D\Psi_s^{-1} - \text{Id}| \to 0\) uniformly in \(\Omega\) as \(s \searrow 1\) and \(|\det D\Psi_s^{-1}| \to 1\).

2. If \(g \in W^{1+\frac{1}{q},q}(\Omega)\) there is an extension \(\hat{g}\) of \(g\) to \(\Omega^s\) such that \(\hat{g} \in W^{1,q}(\Omega^s)\) and \(\hat{g} \circ \Psi_s^{-1} \in W^{1,\frac{1}{q}}(\Omega)\).

We conclude this section by noting a number of extensions, that we may carry out if \(\Omega\) is a Lipschitz domain. Let \(\Omega \Subset B(0, R)\). From [60], if \(g \in W^{s,p}(\Omega)\), there is an extension \(\hat{g} \in W^{s,p}(\mathbb{R}^n, \mathbb{R}^N)\) of \(g\) with
\[
\|\hat{g}\|_{W^{s,p}(\mathbb{R}^n)} \lesssim \|g\|_{W^{s,p}(\Omega)}. \quad (2.5)
\]

Furthermore, we can extend \(F(x, z)\) to a function on \(B(0, R) \times \mathbb{R}^N\), still denoted \(F(x, z)\), such that it satisfies
\[
|F(x, z) - F(y, z)| \lesssim \Lambda |x - y|^{\alpha} (1 + |z|^2)^{\frac{q}{2}}
\]
\[
|F(x, z)| \leq \Lambda (1 + |z|^2)^{\frac{q}{2}}.
\]

We do this by setting, for \(x \in B(0, R) \setminus \Omega\),
\[
F(x, z) = \inf_{y \in \Omega} \left( F(y, z) + \Lambda \left(1 + |z|^2\right)^{\frac{q}{2}} |x - y| \right).
\]
2.4 Relaxed minimisers and Lavrentiev

We begin by showing that \( K_{g,\psi}^{*,\psi}(\Omega) \) is dense in \( K_{g,\psi}^{\psi}(\Omega) \) with respect to the \( W^{1,p}(\Omega) \) norm. Note that this is a requirement in order for us to make sense of the relaxed functional \( \overline{F}(\cdot) \).

**Lemma 7** Suppose \( g, \psi \in W^{1+1/q, q}(\Omega) \). Then \( K_{g,\psi}^{*,\psi}(\Omega) \) is dense in \( K_{g,\psi}^{\psi}(\Omega) \) with respect to the \( W^{1,p}(\Omega) \) norm.

**Proof** Let \( u \in K_{g,\psi}^{\psi}(\Omega) \). We first show that without loss of generality we may assume that \( u \in W^{1,q}(U) \) for some open neighbourhood \( U \) of \( \partial \Omega \). Using the notation of Lemma 6, let \( s \in (1/2, 1) \) and extend \( u \) by \( \hat{g} \) and \( \psi \) by \( \hat{\psi} \) to \( \Omega^s \). Define

\[
    u^s(x) = u(\psi_s^{-1}(x)), \quad \psi^s(x) = \psi(\psi_s^{-1}(x))
\]

and consider \( v^s = u^s - \psi^s + \psi \). Then we have \( v^s \in K_{g,\psi}^{\psi}(\Omega) \) and clearly \( v^s \to u \) in \( W^{1,p}(\Omega) \) as \( s \to 1 \). Further \( v \in W^{1,q}(U) \) for an open neighbourhood \( U \) of \( \partial \Omega \).

Assuming now that \( u \in W^{1,q}(U) \) for an open neighbourhood \( U \) of \( \partial \Omega \), let \( \eta \in C_c^\infty(\Omega) \) be a smooth cut-off function with \( \eta = 1 \) in \( \Omega \setminus U \). We then consider

\[
    u_\varepsilon = \eta u + \psi^s + \psi + (1 - \eta) u.
\]

We note, because \( u \geq \psi \), we have \( u^s + \psi^s + \psi \geq \psi^s + \psi \). Thus \( u_\varepsilon \in K_{g,\psi}^{*,\psi}(\Omega) \). Furthermore, \( u_\varepsilon \to u \) in \( W^{1,p}(\Omega) \) as \( \varepsilon \to 0 \). Hence the proof is complete. \(\Box\)

Given \( F(x, z) \) satisfying \((p, q)\)-growth, we consider the regularised functional

\[
    \overline{F}_\varepsilon(u) = \int_\Omega F(x, Du) + \varepsilon|Du|^q \, dx.
\]

We wish to relate minimisers of \( \overline{F}(\cdot) \) to minimisers of \( \overline{F}_\varepsilon(\cdot) \).

**Lemma 8** (c.f. Lemma 6.4. in [55]) Let \( g, \psi \in W^{1+1/q, q}(\Omega) \). Suppose that \( F(x, z) \) satisfies \((H1)\) and \((H2)\). We take \( u, \) a relaxed minimiser of \( \overline{F}(\cdot) \) in the class \( K_{g,\psi}^{\psi}(\Omega) \), and \( u_\varepsilon, \) the pointwise minimiser of \( \overline{F}_\varepsilon(\cdot) \) in the class \( K_{g,\psi}^{*,\psi}(\Omega) \). Then \( \overline{F}_\varepsilon(u_\varepsilon) \to \overline{F}(u) \) as \( \varepsilon \to 0 \). Moreover, up to a subsequence, we have \( u_\varepsilon \to u \) in \( W^{1,p}(\Omega) \).

**Proof** Existence and uniqueness of \( u_\varepsilon \) follows from the direct method and strict convexity, respectively. We further note that

\[
    \overline{F}(u) \leq \liminf_{\varepsilon \to 0} \overline{F}(u_\varepsilon) \leq \liminf_{\varepsilon \to 0} \overline{F}_\varepsilon(u_\varepsilon).
\]

To prove the reverse implication, we note that, for any \( v \in K_{g,\psi}^{*,\psi}(\Omega) \),

\[
    \limsup_{\varepsilon \to 0} \overline{F}_\varepsilon(u_\varepsilon) \leq \lim_{\varepsilon \to 0} \overline{F}_\varepsilon(v) = \overline{F}(v) = \overline{F}(v).
\]

By definition of \( \overline{F}(\cdot) \) the inequality above extends to all \( v \in K_{g,\psi}^{\psi}(\Omega) \). In particular, it holds with the choice \( v = u \). Thus \( \overline{F}_\varepsilon(u_\varepsilon) \to \overline{F}(u) \) as \( \varepsilon \to 0 \).

Using \((H4)\), we may extract a (non-relabelled) subsequence of \( u_\varepsilon \) such that \( u_\varepsilon \to v \) weakly in \( W^{1,p}(\Omega) \) for some \( v \in W^{1,p}(\Omega) \). Using our calculations above we see that \( v \) is a relaxed minimiser of \( \overline{F}(\cdot) \) in the class \( K_{g,\psi}^{\psi}(\Omega) \). Using \((H4)\), it is easy to deduce that for every \( w_1, w_2 \in K_{g,\psi}^{*,\psi} \),

\[
    \overline{F}(\frac{w_1 + w_2}{2}) + \frac{v}{p} \| Dw_1 - Dw_2 \|_{L^p(\Omega)}^p \leq \frac{1}{2} (\overline{F}(w_1) + \overline{F}(w_2)). \tag{2.6}
\]
Using the definition of $\mathcal{F} (\cdot)$ and weak lower semicontinuity of norms, this estimate extends to $w_1, w_2 \in K^\psi_g$. In particular, $\mathcal{F} (\cdot)$ is convex and so $u = v$. Moreover the choice $w_1 = u$, $w_2 = u_\varepsilon$ in the estimate shows that $u_\varepsilon \rightarrow u$ in $W^{1, p}(\Omega)$. $\Box$

We comment that in the autonomous case, relaxed minimisers agree with pointwise minimiser if $g \in W^{1+1/q, q}(\Omega)$ (or alternatively $g \in W^{1, q}(\partial \Omega)$ and $\psi \in W^{2, \infty}(\Omega)$. This is a direct consequence of the following observation. Some of the calculations are based on arguments in [47].

**Lemma 9** Suppose that $1 < p \leq q < p + 1$ and $F \equiv F (z)$ satisfies (H1) and (H2). Let $\psi \in W^{2, \infty}(\Omega)$ and $g \in W^{1+1/q, q}(\Omega)$. Then, given $u \in K^\psi_g(\Omega)$, there exists a sequence of functions $u_k \in K^\psi_g(\Omega)$ such that $u_k \rightarrow u$ in $W^{1, p}(\Omega)$ and $\mathcal{F} (u_k) \rightarrow \mathcal{F} (u)$.

**Proof** We utilise the notation and construction of Lemma 7. We find

$$
\int_{\Omega} F(Dv^s) \, dx \leq \int_{\Omega} F(Du^s) \, dx + \int_{\Omega} |D(\psi^s - \psi)| (1 + |D\psi| + |D\psi^s| + |Du^s|)^{q-1} \, dx
$$

$$
= A_1 + A_2.
$$

We have

$$
A_1 = \int_{\Omega} |\det \Psi_x| F(Du) \, dx + \int_{\Omega \setminus \Omega^s} |\det \Psi_x| F(D\hat{g}) \, dx \rightarrow \int_{\Omega} F(Du) \, dx < \infty
$$

as $s \rightarrow 1$. For the other term, as $q \leq p + 1$ and using (H2) and the mean value theorem, we have

$$
A_2 \lesssim \|\psi\|_{W^{2, \infty}(\Omega)} (1 + \|u^s\|_{W^{1, p}(\Omega)} + \|\psi^s - \psi\|_{W^{1, p}(\Omega)})^p
$$

$$
\rightarrow \|\psi\|_{W^{2, \infty}(\Omega)} (1 + \|u\|_{W^{1, p}(\Omega)})^p
$$

Thus, by a version of dominated convergence, $\mathcal{F} (v^s) \rightarrow \mathcal{F} (u)$ as $s \rightarrow 1$. Hence, by a diagonal subsequence argument, as in Lemma 7 we may assume that $u \in W^{1, q}(U)$ for some open neighbourhood $U$ of $\partial \Omega$.

Considering $u_\varepsilon$ defined as in Lemma 7 we write

$$
\int_{\Omega} F(Du_{\varepsilon}) \, dx = \int_{\{\eta = 0\}} F(Du_{\varepsilon}) \, dx + \int_{\{\eta = 1\}} F(Du_{\varepsilon}) \, dx + \int_{0 < \eta < 1} F(Du_{\varepsilon}) \, dx.
$$

In the region where $\eta = 0$, we have $u_{\varepsilon} = u$. When $\eta = 1$, using (H2), the mean value theorem and Jensen’s inequality, we see that

$$
\int_{\{\eta = 1\}} F(Du_{\varepsilon}) \, dx \leq \int_{\{\eta = 1\}} F(Du \ast \phi_{\varepsilon}) \, dx
$$

$$
+ c \int_{\{\eta = 1\}} |D(\psi \ast \phi_{\varepsilon} - \psi)| (1 + |Du \ast \phi_{\varepsilon}| + |D(\psi - \psi \ast \phi_{\varepsilon})|)^{q-1} \, dx
$$

$$
\lesssim \int_{\{\eta = 1\}} F(Du) \ast \phi_{\varepsilon} \, dx + \|\psi\|_{W^{2, \infty}(\Omega)} (1 + \|u \ast \phi_{\varepsilon}\|_{W^{1, p}(\Omega)} + \|\psi - \psi \ast \phi_{\varepsilon}\|_{W^{1, p}(\Omega)})^p
$$

$$
\rightarrow \int_{\{\eta = 1\}} F(Du) \, dx + \|\psi\|_{W^{2, \infty}(\Omega)} (1 + \|u\|_{W^{1, p}(\Omega)})^p.
$$

as $\varepsilon \rightarrow 0$. By a version of dominated convergence, we deduce that

$$
\int_{\{\eta = 1\}} F(Du_{\varepsilon}) \, dx \rightarrow \int_{\{\eta = 1\}} F(Du) \, dx.
$$
as $\varepsilon \to 0$. It remains to deal with the region where $\eta \in (0, 1)$. This region is contained in $U$ and so the only thing we need to check is that we are able to apply dominated convergence. Using (H2), we estimate

$$\int_{\{0 < \eta < 1\}} F(Du_\varepsilon) \, dx$$

$$\lesssim \int_{U} 1 + |D(u_\varepsilon \ast \phi_\varepsilon - \psi \ast \phi_\varepsilon + \psi)|^q + |D\eta|^q |u_\varepsilon \ast \phi_\varepsilon - \psi \ast \phi_\varepsilon + \psi| \, dx$$

$$\leq \int_{U} 1 + |D(u_\varepsilon \ast \phi_\varepsilon - \psi \ast \phi_\varepsilon + \psi)|^q + \|D\eta\|_{L^\infty(U)}^q |u_\varepsilon \ast \phi_\varepsilon - \psi \ast \phi_\varepsilon + \psi| \, dx$$

$$\to \int_{U} 1 + |Du|^q + \|D\eta\|_{L^\infty(U)} |u|^q \, dx < \infty.$$  

Thus the proof is complete. \qed

Now we turn to the non-autonomous case, where we prove a similar statement to Lemma 9, as long as (H6) is satisfied. We begin by recalling the following key lemma from [27], which we use to prove Lemma 11.

**Lemma 10** Assume that $1 < p \leq q \leq \frac{(n+\alpha)p}{\eta \, n}$. Let $\Omega$ be a domain and $u \in W^{1,p}(\Omega)$. Suppose that $F(x, \cdot)$ satisfies (H1), (H2), (H3) and (H6). Then, for $x \in \Omega$ and with $\varepsilon \leq \min(\varepsilon_0, d(x, \partial \Omega))$,

$$F(x, Du(-) \ast \phi_\varepsilon(x)) \lesssim 1 + (F(-, Du(-)) \ast \phi_\varepsilon)(x).$$

**Lemma 11** Let $\alpha \in (0, 1)$. Suppose that $F(x, z)$ satisfies (H1), (H2), (H3) and (H6) and that $1 < p \leq q < \min(p + 1, \frac{(n+\alpha)p}{\eta \, n})$. Let $\psi \in W^{2,\infty}(\Omega)$ and $g \in W^{1+1/q,q}(\Omega)$. Given $u \in K_{g,\psi}^\psi(\Omega)$ there exists a sequence of functions $u_k \in K_{g,\psi}^\psi$ such that $u_k \rightharpoonup u$ in $W^{1,p}(\Omega)$ and a sequence of integrands $F_k(x, z)$ satisfying (H1), (H2), (H3) and (H6), with bounds independent of $k$, such that $\int_\Omega F_k(x, Du_k) \, dx \to \mathcal{F}(u)$ as $k \to \infty$.

**Proof** We proceed along the same lines as in Lemma 9. Let $s \in (1/2, 1)$ and extend $u$ by $\hat{g}$ and $\psi$ by $\psi_\Omega$ to $\Omega^s$. Using the notation of Lemma 6, we then define

$$u^s(x) = u(\Psi^s_1(x)), \quad \psi^s(x) = \psi(\Psi^s_1(x))$$

$$F^s(x, z) = F(\Psi^s_1(x), z D\Psi^s_1(\Psi^s_1(x)))$$

and denote $v^s = u^s - \psi^s + \psi$. We have $v^s \in K_{g,\psi}^\psi(\Omega)$ and clearly $v^s \to v$ in $W^{1,p}(\Omega)$ as $s \to 1$. It is straightforward to check that $F^s(x, z)$ satisfies the required assumptions and we can argue exactly as in Lemma 9 to see that it suffices to find $(u_j) \subset K_{g,\psi}^\psi(\Omega)$ such that $\int_\Omega F^s(x, Du_j) \, dx \to \int_\Omega F^s(x, Du) \, dx$ as $j \to \infty$.  

Let $\eta \in C^{\infty}_c(\Omega)$ be a smooth cut-off function with $\eta = 1$ in $\Omega \setminus U$. We then consider

$$u_\varepsilon = \eta(u \ast \phi_\varepsilon - \psi \ast \phi_\varepsilon + \psi) + (1 - \eta)u.$$
The estimate proceeds exactly as in Lemma 9, the only difference being the estimate in the region where \( \eta = 1 \). Here, we argue using \((\text{H2})\), the mean value theorem, Jensen’s inequality and Lemma 10, to see that

\[
\int_{\{\eta = 1\}} F^s(x, Du_\varepsilon) \, dx \leq \int_{\{\eta = 1\}} F^s(x, Du_\phi_\varepsilon) \, dx
\]

\[
+ c \int_{\{\eta = 1\}} |D(\psi_\phi_\varepsilon - \psi)| (1 + |Du_\phi_\varepsilon| + |D(\psi - \psi_\phi_\varepsilon)|)^{q-1} \, dx
\]

\[
\lesssim \int_{\{\eta = 1\}} 1 + F(\cdot, Du(\cdot)) \phi_\varepsilon(x) \, dx
\]

\[
+ \|\psi\|_{W^{2,\infty}(\Omega)} (1 + \|u\|_{W^{1,\infty}(\Omega)} + \|\psi - \psi_\phi_\varepsilon\|_{W^{1,\infty}(\Omega)})^p
\]

\[
\to \int_{\{\eta = 1\}} 1 + F^s(x, Du) \, dx + \|\psi\|_{W^{2,\infty}(\Omega)} (1 + \|u\|_{W^{1,\infty}(\Omega)})^p.
\]

as \( \varepsilon \to 0 \). By a version of dominated convergence, we deduce that

\[
\int_{\Omega} F^s(x, Du_\varepsilon) \, dx \to \int_{\Omega} F^s(x, Du) \, dx,
\]

as \( \varepsilon \to 0 \). □

3 Exactness of \( L^1 \)-penalisation

In this section we prove that \( L^1 \)-penalisation of the obstacle problem \((P)\) is exact. A version of the result can be found in [62] but we present a full argument for completeness. We consider the functional

\[
\tilde{\mathcal{F}}(v) = \int_{\Omega} F(Dv) + \kappa |(\psi - v)_+| \, dx.
\]

(3.1)

and prove that for sufficiently large \( \kappa > 0 \), pointwise minimisers \( \tilde{u} \) of \( \tilde{\mathcal{F}}(\cdot) \) satisfy \( \tilde{u} \geq \psi \) a.e. in \( \Omega \). Here and throughout this section, we denote \( (\psi - \tilde{u})_+ = ((\psi - \tilde{u})_-(\tilde{u})_+) \). In particular, pointwise minimisers of \((P)\) and (3.1) agree. Consequently, we will prefer to work with the unconstrained functional \( \tilde{\mathcal{F}}(\cdot) \) rather than the original functional \( \mathcal{F}(\cdot) \).

Proposition 2 (c.f. [62]) Suppose that \( \Omega \) is a Lipschitz domain. Let \( \psi \in W^{2,\infty}(\Omega) \) and \( g \in W^{1,p}(\Omega) \) be given such that \( \psi \geq g \) on \( \partial \Omega \) in the sense of traces. Suppose \( F \equiv F(x, z): \Omega \times \mathbb{R}^{N \times n} \to \mathbb{R} \) is measurable in \( x \) and \( C^2 \) in \( z \), \((\text{H1})\) holds and there exists \( u_0 \in W^{1,p}_g(\Omega) \) such that \( \int_{\Omega} F(x, Du_0) \, dx < \infty \). Then there exists \( \kappa_0 = \kappa_0(\|\psi\|_{W^{2,\infty}(\Omega)}) \) such that, for \( \kappa > \kappa_0 \), the minimiser \( \tilde{u} \) of the functional \( \tilde{\mathcal{F}}(\cdot) \) in the class \( W^{1,p}_g(\Omega) \) satisfies \( u \geq \psi \) a.e. in \( \Omega \).

Proof Fix \( \kappa > 0 \), the value of which is yet to be determined. By the direct method, a minimiser \( \tilde{u} \) of \( \tilde{F}(v) \) in the class \( W^{1,p}_g(\Omega) \) exists and is unique.

Introduce

\[
w = \tilde{u} + (\psi - \tilde{u})_+.
\]

Note that \( w \in W^{1,p}_g(\Omega) \) as \( \tilde{u} \geq \psi \) on \( \partial \Omega \) in the sense of traces. Thus, using \((\text{H4})\), we see that...
\[ \mathcal{F}(w) = \int_{\Omega} F(x, Dw) \, dx \leq \int_{\Omega} F(x, D\tilde{u}) - \partial_z F(x, Dw) \cdot D(x, \tilde{u} - w) \, dx \]
\[ = \int_{\Omega} F(x, D\tilde{u}) + \partial_z F(x, Dw) \cdot D((\psi - \tilde{u})_+) \, dx = A \]

However, \( D(\psi - \tilde{u})_+ = 0 \) on \( \{ \psi \leq \tilde{u} \} \) and \( D(\psi - \tilde{u})_+ = D\psi - D\tilde{u} \) on \( \{ \psi \geq \tilde{u} \} \). Furthermore, \( w = \psi \) on \( \{ \psi \geq \tilde{u} \} \). Hence, using integration by parts, we find that
\[
A = \int_{\Omega} F(x, D\tilde{u}) + \partial_z F(x, D\psi) \cdot D((\psi - \tilde{u})_+) \, dx
\]
\[
= \int_{\Omega} F(x, D\tilde{u}) - \text{div}(\partial_z F(x, D\psi))(\psi - \tilde{u})_+ \, dx
\]
\[
\leq \int_{\Omega} F(x, D\tilde{u}) + \|\text{div}(\partial_z F(x, D\psi))\|_{L^\infty(\Omega)} \int_{\Omega} |(\psi - \tilde{u})_+| \, dx.
\]

Thus if \( \kappa > \kappa_0 = \|\text{div}(\partial_z F_e(D\psi))\|_{L^\infty(\Omega)} \) the claim follows. \( \square \)

**Corollary 2** Suppose that the hypothesis of Proposition 2 hold. Then, if \( \kappa > \kappa_0 \), we have
\[
\inf_{u \in K^\delta_k(\Omega)} \mathcal{F}(u) = \inf_{u \in W^{1,p}_k(\Omega)} \mathcal{F}(u).
\]

Further minimisers exist and agree.

### 4 An apriori estimate for a regularised functional

A drawback of working with (3.1) is that the integrand is not differentiable. This causes issues in applying the Euler-Lagrange equation and monotonicity inequalities, such as (H5). This motivates us to introduce the following regularisation.

Fix \( H_\delta \in C^\infty(\mathbb{R}) \) such that \( H_\delta(x) \to \max(0, x) \) uniformly in \( \mathbb{R} \), and \( H_\delta(\cdot) \) is non-negative, convex, non-decreasing, with \( \|H_\delta\|_{L^\infty(\mathbb{R})} \leq 2 \) for \( \delta \leq 1 \). Given \( \mathcal{G}(\cdot) \) satisfying (H1) and (H2) with \( p = q \), we define for \( \delta \in (0, 1) \),
\[
\mathcal{G}_\delta(w) = \int_{\Omega} G(x, Dw) + \kappa H_\delta((H_\delta(\psi_i - w_i))) \, dx.
\] (4.1)

By the direct method, minimisers \( \tilde{u}_\delta \) of \( \mathcal{G}_\delta(\cdot) \) in the class \( W^{1,p}_k(\Omega) \) exist and are unique. Let \( \tilde{u} \) be the minimiser of the functional \( \mathcal{G}_\delta(\cdot) \) defined through (3.1) posed for \( \mathcal{G}(\cdot) \). Then \( \tilde{u} \) is a valid competitor in the problem (4.1). Hence if \( \tilde{u}_\delta \) minimises (4.1), we have
\[
\int_{\Omega} G(x, D\tilde{u}_\delta) + \kappa H_\delta((H_\delta(\psi_i - \tilde{u}_\delta,i))) \, dx
\]
\[
\leq \int_{\Omega} G(x, D\tilde{u}) + \kappa H_\delta((H_\delta(\psi_i - \tilde{u}_i))) \, dx
\]
\[
\to \int_{\Omega} G(x, D\tilde{u}) + \kappa (\psi - \tilde{u})_+ \, dx < \infty,
\]
as \( \delta \to 0 \). In particular, using (H5), we extract a subsequence, not relabelled, such that \( \tilde{u}_\delta \rightharpoonup v \) weakly to some \( v \in W^{1,p}_k(\Omega) \). Necessarily, \( v \) is a minimiser of \( \mathcal{G}(\cdot) \). In fact, for any \( \delta > 0 \), we have
\[
\int_{\Omega} G(x, D\tilde{u}_\delta) + \kappa H_\delta \left( (H_\delta(\psi_i - \tilde{u}_{\delta,i})) \right) \, dx \\
\leq \int_{\Omega} G(x, D\tilde{u}) + \kappa H_\delta \left( (H_\delta(\psi_i - \tilde{u}_i)) \right) \, dx.
\]

Taking limits as \( \delta \to 0 \) on both sides the claim follows. Since \( \tilde{G}(\cdot) \) is convex, we deduce that \( v = u \). Moreover, due to (H1) and the convexity of \( H_\delta(\cdot) \), we have
\[
\int_{\Omega} G(x, D\tilde{u}_\delta) + \kappa H_\delta \left( H_\delta \left( \psi_i - \frac{\tilde{u}_{\delta,i} + \tilde{u}_i}{2} \right) \right) \, dx + c \|\tilde{u} - \tilde{u}_\delta\|_{W^{1,p}(\Omega)} \\
\leq \frac{1}{2} \left( \int_{\Omega} G(x, D\tilde{u}_\delta) + \kappa H_\delta \left( (H_\delta(\psi_i - \tilde{u}_{\delta,i})) \right) + G(x, D\tilde{u}) \right. \\
\left. + \kappa H_\delta \left( (H_\delta(\psi_i - \tilde{u}_i)) \right) \, dx \right).
\]

Letting \( \delta \to 0 \) and combining that \( \tilde{u}_\delta \to u \) strongly in \( L^p(\Omega) \) with the lower semi-continuity of \( G(\cdot) \) in \( W^{1,p}(\Omega) \), we deduce that \( \tilde{u}_\delta \to u \) strongly in \( W^{1,p}(\Omega) \).

Recalling Lemma 8 and taking a diagonal subsequence, the key tool we require in order to prove our main theorem is an apriori bound for minimisers of the functional
\[
\mathcal{F}_{\varepsilon,\delta}(u) = \int_{\Omega} F(x, Du) + \varepsilon |Du|^q + \kappa H_\delta \left( (H_\delta(\psi_i - u_i)) \right) \, dx,
\]
where the bound is independent of \( \delta \) and \( \varepsilon \).

### 4.1 The autonomous case

In this section we consider minimisers of \( \mathcal{F}_{\varepsilon,\delta}(\cdot) \) in the autonomous case, that is, when \( F(x, z) = F(z) \), and derive an apriori estimate that is independent of \( \varepsilon \) and \( \delta \).

Our goal is to prove the following.

**Proposition 3** Suppose \( 2 \leq p \leq q < \min \left( p + 1, \frac{np}{n-1} \right) \). Let the data \( g \in W^{2,q}(\Omega) \) and \( \psi \in W^{2,\infty}(\Omega) \) be given. Suppose that \( F \equiv F(z) \) satisfies (H1) and (H2). Then, for minimisers \( u_{\varepsilon,\delta} \) of \( \mathcal{F}_{\varepsilon,\delta}(\cdot) \), for any \( \alpha \in (0, 1/2) \) and some \( \beta > 0 \), we have that
\[
\|Du_{\varepsilon,\delta}\|_{L^p(\Omega)}^q \lesssim \|V_{\mu,p}(Du_{\varepsilon,\delta})\|_{B^{\alpha,2}_{\infty}(\Omega)}^2 \\
\lesssim \left( 1 + \|V_{\mu,p}(Du_{\varepsilon,\delta})\|_{L^2(\Omega)}^2 + \|g\|_{W^{2,q}(\Omega)}^2 \right)^{\frac{\beta}{1-p}} \left( 1 + \int_{\Omega} F_\varepsilon(Du_{\varepsilon,\delta}) \, dx \right)^{\frac{1}{1-p}},
\]
where the implicit constant is independent of \( \varepsilon, \delta \).

The key tool in the proof is the following Lemma. It is a version of the key lemma in [63] but adapted to our purposes.

**Lemma 12** Fix \( n \geq 2 \) and let \( t > 1 \). For given \( 0 < \rho < \sigma < \infty \) with \( \sigma - \rho < 1 \) and \( w \in L^1(B_\sigma) \), consider
\[
J(\rho, \sigma, w) = \inf \left\{ \int_{B_\sigma} |w|(|D\phi| + |D\phi|^t) \, dx : \phi \in C_0^1(B_\sigma), \phi \geq 0, \phi = 1 \text{ in } B_\rho \right\}.
\]

Then, for every \( \delta \in (0, 1) \), we have
\[
J(\rho, \sigma, w) \leq (\sigma - \rho)^{t-1/\delta} \left( \int_{\rho}^{\sigma} \left( \int_{\delta B_r} |w| \, d\sigma \right)^{\delta} \, dr \right)^{\frac{1}{\delta}}.
\]
Moreover, given $\varepsilon > 0$, if $|w| \geq 1$ in $B_\sigma$, there exists a radial symmetric $\tilde{\phi} \in C^1_0(B_\sigma)$ with $\tilde{\phi} \geq 0$ and $\tilde{\phi} = 1$ in $B_\sigma$ such that

$$\int_{B_\sigma} |w|(|D\tilde{\phi}| + |D\tilde{\phi}|^t) \, dx \leq (\sigma - \rho)^{-t-1/\delta} \left( \int_\rho^\sigma \left( \int_{\partial B_r} |w| \, d\sigma \right)^\delta \, dr \right)^{1/\delta} + \varepsilon,$$

and, for almost every $\rho \leq r_1 < r_2 \leq \sigma$,

$$\frac{\tilde{\phi}(r_2) - \tilde{\phi}(r_1)}{(r_2 - r_1)} \leq (\sigma - \rho)^{-t-1/\delta} \left( \int_\rho^\sigma \left( \int_{\partial B_r} |w| \, d\sigma \right)^\delta \, dr \right)^{1/\delta} + \varepsilon.$$

**Proof** The estimate follows by considering appropriate radially symmetric cut-off functions. Indeed, for any $\varepsilon \geq 0$,

$$J(\rho, \sigma, w) \leq \inf \left\{ \int_\rho^\sigma (|\phi'| + |\phi'|^t) \left( \int_{\partial B_\rho} |w| + \varepsilon \, d\sigma \right) \, dr : \phi \in C^1([\rho, \sigma]), \phi(\rho) = 1, \phi(\sigma) = 0 \right\} = J_{1d,\varepsilon}.$$

Since $w \in L^1(B_\sigma)$, by employing a standard approximation argument we can replace $\phi \in C^1([\rho, \sigma])$ with $\tilde{\phi} \in W^{1,\infty}$. Thus, we have that

$$J_{1d,\varepsilon} \leq \int_\rho^\sigma (|\tilde{\phi}'| + |\tilde{\phi}'|^t) b(r) \, dr = \frac{\sigma - \rho}{\int_\rho^\sigma b(r)^{-1} \, dr} + \int_\rho^\sigma b(r)^{1-t} \, dr = I + II.$$

By Hölder’s inequality, for any $s > 1$, we have

$$(\sigma - \rho) = \int_\rho^\sigma \left( \frac{b(r)}{b(r)^{s-1}} \right)^{s-1} \leq \left( \int_\rho^\sigma b(r)^{s-1} \, dr \right)^{\frac{1}{s}} \left( \int_\rho^\sigma b(r)^{-1} \, dr \right)^{\frac{s-1}{s}}.$$

Thus, for any $\delta > 0$,

$$I \leq (\sigma - \rho)^{-(1+\delta)/\delta} \left( \int_\rho^\sigma \left( \int_{\partial B_r} |w| \, d\mathcal{H}^{n-1} + \varepsilon \, d\sigma \right)^\delta \right)^{1/\delta}.$$

We now focus on the second term $II$. Using Jensen’s inequality, we see that

$$\int_\rho^\sigma b(r)^{1-t} \, dr \leq (\sigma - \rho)^{2-t} \left( \int_\rho^\sigma b(r)^{-1} \, dr \right)^{t-1}.$$

Hence, for any $\delta > 0$, using the estimate for the term $I$, we obtain
\[ II \leq (\sigma - \rho)^{2-t} \left( \int_{\sigma}^{\rho} b(r)^{-1} \, dr \right)^{-1} \]
\[ \leq (\sigma - \rho)^{1-t-(1+\delta)/\delta} \left( \int_{\rho}^{\sigma} b(r)^{\delta} \, dr \right)^{1/\delta} \]

Collecting estimates and letting \( \varepsilon \to 0 \) the result follows.

For the latter statement in the lemma, since \(|w| \geq 1\), we note that
\[
\frac{\left| \tilde{\phi}(r_1) - \tilde{\phi}(r_2) \right|}{r_2 - r_1} = \frac{\int_{r_1}^{r_2} b(r)^{-1} \, dr}{(r_2 - r_1) \int_{\rho}^{\sigma} b(r)^{-1} \, dr} \leq \frac{1}{\int_{\rho}^{\sigma} b(r)^{-1} \, dr}.
\]
The claim follows, estimating as for the term \( I \).

We note the following consequence of the previous Lemma. We set \( \xi = \frac{n-1}{n-2} \) if \( n > 2 \) and \( \xi = \infty \) if \( n = 2 \). Suppose that \( w \in L^q(B_\sigma) \), \( t > 1 \) and \( q < \frac{np}{n-1} \). Then, for any \( \delta \in (0, 1] \) and \( \rho < \sigma \) with \( |\rho - \sigma| < 1 \),
\[ J(\sigma, \rho, |w|^q) \leq (\sigma - \rho)^{-t-1/\delta} \left( \int_{\rho}^{\sigma} \|w\|^{q \delta}_{L^\theta(\partial B_r)} \, dr \right)^{\frac{1}{\delta}}. \]

By Hölder’s inequality, we find that
\[
\left( \int_{\rho}^{\sigma} \|w\|^{q \delta}_{L^\theta(\partial B_r)} \, dr \right)^{\frac{1}{\delta}} \leq \left( \int_{\rho}^{\sigma} \|w\|^{q \delta}_{L^p(\partial B_r)} \|w\|^{(1-\theta)q \delta}_{L^{\frac{np}{n-1}}(\partial B_r)} \, dr \right)^{\frac{1}{\delta}}
\leq \left( \int_{\rho}^{\sigma} \|w\|^{q \delta}_{L^p(\partial B_r)} \, dr \right)^{\frac{1-s}{1-\delta}} \left( \int_{\rho}^{\sigma} \|w\|^{(1-\theta)q \delta}_{L^{\frac{np}{n-1}}(\partial B_r)} \, dr \right)^{\frac{1}{\delta}}
\]
where \( \theta \in (0, 1) \) is such that
\[
\frac{\theta}{p} + \frac{1-\theta}{q} = 1, \quad s > 1.
\]
We make the admissible choice
\[ \delta = \frac{p}{q}, \quad s = \frac{1}{1-\theta}. \]

It follows that
\[
J(\sigma, \rho, |w|^q) \leq (\sigma - \rho)^{-t-q/p} \left( \int_{B_\rho \setminus B_\sigma} |w|^p \, dr \right)^{\frac{\xi}{\epsilon - t} \left( 1 - \frac{q}{p} \right)} \left( \int_{\rho}^{\sigma} \|w\|^p_{L^{\frac{np}{n-1}}(\partial B_r)} \, dr \right)^{\frac{\xi}{\epsilon - t} \left( \frac{q}{p} - 1 \right)}.
\]

Here we understand that \( \frac{\infty}{\infty - 1} = 1 \).

By the same argument, if \(|w| \geq 1 \) in \( B_\sigma \) and \( \varepsilon > 0 \), we can pick a radial test function \( \tilde{\phi} \) such that, for almost every \( \rho \leq r_1 < r_2 \leq \sigma \),
\[
\int_{B_\sigma} |w|^q (|D\tilde{\phi}| + |D^2\tilde{\phi}|^p) \, dx + \frac{|\tilde{\phi}(r_2) - \tilde{\phi}(r_1)|}{|r_2 - r_1|}
\leq (\sigma - \rho)^{-p-q/p} \left( \int_{B_\rho \setminus B_\sigma} |w|^p \, dr \right)^{\frac{\xi}{\epsilon - t} \left( 1 - \frac{q}{p} \right)}
\times \left( \int_{\rho}^{\sigma} \|w\|^p_{L^{\frac{np}{n-1}}(\partial B_r)} \, dr \right)^{\frac{\xi}{\epsilon - t} \left( \frac{q}{p} - 1 \right)} + \varepsilon.
\]
This statement is exactly the result we need and so we can proceed to prove the main estimate of this section.

**Proof of Proposition 3** By the direct method and strict convexity deriving from (H1), we obtain the existence of minimisers \( v_{\varepsilon, \delta} \in W^{1,q}_{0}(\Omega) \) that solve

\[
\min_{v \in W^{1,q}_{0}(\Omega)} \mathcal{F}_{\varepsilon, \delta}(v + g)
\]

Moreover, \( v_{\varepsilon, \delta} \) satisfies the Euler–Lagrange equation

\[
\int_{\Omega} \partial_{v} F_{\varepsilon}(Dv_{\varepsilon, \delta} + Dg) \cdot D\phi + \partial_{y} \bar{H}_{\delta}(v_{\varepsilon, \delta} + g) \cdot \phi \, dx = 0 \quad \forall \phi \in W^{1,q}_{0}(\Omega). \tag{4.4}
\]

Here \( \bar{H}_{\delta}(u) = H_{\delta}(\langle H_{\delta}(\psi_{i} - u_{i}) \rangle) \). For notational simplicity we suppress the dependence on \( \varepsilon \) and \( \delta \) for the time being and write \( v = v_{\varepsilon, \delta} \).

Let \( \rho_{0} > 0 \) and \( n : \mathbb{R}^{n} \to \mathbb{S}^{n-1} \) be such that the uniform cone property (2.4) holds. Without loss of generality, we may assume that \( \rho_{0} < 1 \). Let \( \rho_{0} \geq \sigma \geq \rho \geq \rho_{0}/2 \). Fix \( x_{0} \in \Omega \) and take \( \phi \in C^{1}_{c}(\mathbb{R}^{n}) \), a radially symmetric, monotonic decreasing function with \( \phi = 1 \) in \( B_{\rho}(x_{0}) \), \( \supp \phi \subset B_{\sigma}(x_{0}) \). We denote the extension of \( v \) by 0 to \( \mathbb{R}^{n} \) by \( \tilde{v} \) and write \( T_{h}D\tilde{v} = \phi D\tilde{v}_{h} + (1 - \phi)D\tilde{v} \) and \( T_{h}v = \phi \tilde{v}_{h} + (1 - \phi)\tilde{v} \). Using (H1), the convexity of \( \bar{H}_{\delta}(\cdot) \) and (4.4) for \( h \in C_{\rho_{0}/4}(\theta_{0}, n(x_{0})) \), we see that

\[
\int_{\Omega} F_{\varepsilon}(D T_{h} \tilde{v} + Dg) + \bar{H}_{\delta}(T_{h} \tilde{v} + g) - F_{\varepsilon}(D \tilde{v} + Dg) - \bar{H}_{\delta}(\tilde{v} + g) \, dx
\]

\[
\geq \int_{\Omega} (\mu^{2} + |DT_{h} \tilde{v}|^{2} + |D\tilde{v}|^{2} + |Dg|^{2})^{p/2} \phi^{2} |D\tilde{v}_{h} - D\tilde{v}|^{2}
\]

\[
+ \partial_{v} F_{\varepsilon}(D \tilde{v} + Dg) \cdot D(\phi(\tilde{v}_{h} - \tilde{v})) + \partial_{y} \bar{H}_{\delta}(\tilde{v} + g) \cdot ((\phi(\tilde{v}_{h} - \tilde{v})) \, dx
\]

\[
\geq \int_{B_{\rho}(x_{0})} |V_{\mu, p}(D\tilde{v}_{h}) - V_{\mu, p}(D\tilde{v})|^{2} \, dx. \tag{4.5}
\]

To obtain the last line we use Lemma 2 and note that by our choice of \( h \), we have \( \phi \tilde{v}_{h} \in W^{1,q}_{0}(\Omega) \).

We continue by estimating

\[
|h|^{-1} \int_{\Omega} \bar{H}_{\delta}(T_{h} \tilde{v} + g) - \bar{H}_{\delta}(\tilde{v} + g) \, dx \lesssim |h|^{-1} \| H'_{\delta} \|_{L^{\infty}(\mathbb{R})}^{2} \int_{\Omega} |\tilde{v}_{h} - \tilde{v}| \, dx
\]

\[
\lesssim \| H'_{\delta} \|_{L^{\infty}(\mathbb{R})}^{2} \| v \|_{W^{1, p}(\Omega)}
\]

\[
\lesssim \mathcal{F}_{\varepsilon}(v).
\]

We further estimate, for \( h \in C_{\rho_{0}/4}(\theta_{0}, n(x_{0})) \), that

\[
|h|^{-1} \int_{\Omega} F_{\varepsilon}(DT_{h} \tilde{v} + Dg) - F_{\varepsilon}(D \tilde{v} + Dg) \, dx
\]

\[
= |h|^{-1} \int_{\Omega} F_{\varepsilon}(T_{h}D\tilde{v} + D(\tilde{v}_{h} - v) + Dg) - F_{\varepsilon}(T_{h}D\tilde{v} + Dg) \, dx
\]

\[
+ |h|^{-1} \int_{\Omega} F_{\varepsilon}(T_{h}D\tilde{v} + Dg) - F_{\varepsilon}(D \tilde{v} + Dg) \, dx
\]

\[
=: A_{1} + A_{2}.
\]
First using (H2) and Young’s inequality, we find that

\[
|A_1| \lesssim |h|^{-1} \int_{\Omega} |D\phi| |\tilde{v}_h - \tilde{v}| (1 + (|T_h D\tilde{v}| + |D\phi \tau_h \tilde{v}| + |Dg|)^{q-1})\, dx
\]

\[
\leq \varepsilon_1 \int_{B_\sigma(x_0)} \frac{|\tilde{v}_h - \tilde{v}|^q}{|h|^q} \, dx + C(\varepsilon_1) \int_{\Omega} (|D\phi| + |D\phi|^q) \frac{q^q}{q-1} \, w_1 \, dx
\]

\[
\lesssim \varepsilon_1 \int_{B_{\sigma + \rho_0/4}(x_0)} |D\tilde{v}|^q \, dx + C(\varepsilon_1) \int_{\Omega} \left( |D\phi| + \frac{|D\phi|^q}{q-1} \right) \, w_1 \, dx,
\]

where \( w_1 = 1 + |Dg|^q + |D| \tilde{v}|^q + |D\tilde{v}_h|^q + |\tilde{v}|^q + |\tilde{v}_h|^q. \)

Next we turn to \( A_2. \) By convexity of \( F_\varepsilon(\cdot), \) we have

\[
F_\varepsilon(T_h D\tilde{v} + Dg) - F_\varepsilon(D\tilde{v} + Dg)
\]

\[
\leq (1 - \phi) F_\varepsilon(D\tilde{v} + Dg) + \phi F_\varepsilon(D\tilde{v}_h + Dg) - F_\varepsilon(D\tilde{v} + Dg)
\]

\[
= \phi(F_\varepsilon(D\tilde{v}_h + Dg) - F_\varepsilon(D\tilde{v} + Dg)). \tag{4.6}
\]

Thus, we deduce that

\[
|A_2| \leq |h|^{-1} \int_{B_\sigma(x_0)} \phi(F_\varepsilon(D\tilde{v}_h + D\tilde{g}_h) - F_\varepsilon(D\tilde{v} + Dg)) \, dx
\]

\[
+ |h|^{-1} \int_{B_\sigma(x_0)} \phi(F_\varepsilon(D\tilde{v}_h + Dg) - F_\varepsilon(D\tilde{v} + D\tilde{g}h)) \, dx
\]

\[
=: B_1 + B_2.
\]

Here \( \tilde{g}_h \) is a \( W^{2,q}(\mathbb{R}^n) \) extension of \( g \) to \( \mathbb{R}^n \) with \( \|\tilde{g}\|_{W^{2,q}(\mathbb{R}^n)} \lesssim \|g\|_{W^{2,q}(\Omega)}, \) the existence of which is guaranteed by (2.5).

Using a change of coordinates and the fact that \( D\tilde{v}_h = D\tilde{v} = 0 \) in \( B_{\rho_0} \setminus \Omega, \) we see that

\[
|B_1| \leq |h|^{-1} \int_{B_\sigma(x_0)} \phi(F_\varepsilon(D\tilde{v}_h + D\tilde{g}_h) - F_\varepsilon(D\tilde{v} + D\tilde{g})) \, dx
\]

\[
= |h|^{-1} \int_{B_\sigma(x_0) + \varepsilon} \phi(x - h) F_\varepsilon(D\tilde{v} + D\tilde{g}) \, dx - |h|^{-1} \int_{B_\sigma(x_0)} \phi(x) F_\varepsilon(D\tilde{v} + D\tilde{g}) \, dx
\]

\[
\leq |h|^{-1} \int_{B_{\sigma + \rho_0/4}(x_0) \setminus B_{\sigma/4}(x_0)} |\phi(x - h) - \phi(x)| F_\varepsilon(D\tilde{v} + D\tilde{g}) \, dx
\]

\[
\leq |h|^{-1} \int_{B_{\sigma + \rho_0/4}(x_0) \setminus B_{\sigma/4}(x_0)} |\hat{\phi}(|x - h|) - \hat{\phi}(|x|)| F_\varepsilon(D\tilde{v} + D\tilde{g}) \, dx
\]

Here \( \hat{\phi} \) is defined through the relation \( \phi(x) = \phi^\prime(|x|) \) for \( x \in \mathbb{R}^n. \)

Using (H2) and Hölder, we estimate \( B_2 \) as follows:

\[
|B_2| \leq |h|^{-1} \int_{B_\sigma(x_0)} |D\tilde{g} - D\tilde{g}_h| (1 + |D\tilde{v}_h| + |D\tilde{v}| + |Dg| + |D\tilde{g}_h|)^{q-1} \, dx
\]

\[
\lesssim |h|^{-1} \|D\tilde{g} - D\tilde{g}_h\|_{L^q(B_{\sigma + \rho_0}(x_0))} (1 + \|D\tilde{v}\|_{L^q(B_{\sigma}(x_0))}^{q-1} + \|D\tilde{v}_h\|_{L^q(B_{\sigma}(x_0))}^{q-1})
\]

\[
\lesssim \|g\|_{W^{2,q}(B_{\sigma}(x_0))} (1 + \|D\tilde{v}\|_{L^q(B_{\sigma}(x_0))}^{q-1} + \|D\tilde{v}_h\|_{L^q(B_{\sigma}(x_0))}^{q-1})
\]

\[
+ \|D\tilde{g}\|_{L^q(B_{\sigma}(x_0))}^{q-1} + \|D\tilde{g}_h\|_{L^q(B_{\sigma}(x_0))}^{q-1}) = D_1,
\]
where the last line follows from using the difference quotient characterisation of Sobolev spaces.

Collecting terms, we conclude that

\[
[V_{\mu,p}(D\tilde{v})]^2_{\frac{3}{2},\ddot{B}_p} \lesssim \varepsilon_1 \int_{B_{\sigma+\rho/4}(x_0)} |D\tilde{v}|^q dx + C(\varepsilon_1) \int_{B_{\sigma+\rho/4}(x_0)} (|D\phi| + |D\phi|^{\frac{q^2}{q-1}}) w_1 dx
\]

\[
+ |h|^{-1} \int_{B_{\sigma+\rho/4}(x_0) \setminus B_{\rho-\rho_0}(x_0)} |\phi(x-h) - \phi(x)| F_\varepsilon(D\tilde{v}) dx + \mathcal{F}_\varepsilon(v)
\]

\[
\lesssim \varepsilon_1 \int_{B_{\sigma+\rho/4}(x_0)} |D\tilde{v}|^q dx + C(\varepsilon_1) \int_{B_{\sigma+\rho/4}(x_0)} (|D\phi| + |D\phi|^{\frac{q^2}{q-1}}) w_2 dx
\]

\[
+ |h|^{-1} \int_{B_{\sigma+\rho/4}(x_0) \setminus B_{3\rho/4}(x_0)} |\phi(x-h) - \phi(x)| F_\varepsilon(D\tilde{v}) dx + D_1 + \mathcal{F}_\varepsilon(v), \quad (4.7)
\]

where \( w_2 = 1 + |V_{\mu,p}(D\tilde{v})|^2 + |V_{\mu,p}(\tilde{v}_{\varepsilon,h})|^2 + |V_{\mu,p}(\tilde{v}_\varepsilon)|^2 + |V_{\mu,p}(\tilde{v}_{\varepsilon,h})|^2 \).

We immediately note that with a choice of \( \phi \in C^1_\varepsilon(B_\sigma) \) such that \( |D\phi| \leq 2/2(\sigma - \rho) \), we can conclude that

\[
[V_{\mu,p}(D\tilde{v})]^2_{\frac{3}{2},\ddot{B}_p} \lesssim \frac{1}{\varepsilon(\sigma - \rho)^{\frac{q^2}{q-1}} (1 + \int \Omega F_\varepsilon(D\tilde{v}) dx + \|g\|_{W^{2,q}((\Omega))}) + \mathcal{F}_\varepsilon(v). \quad (4.8)
\]

Choosing \( \rho = \rho_0/2, \sigma = \rho \) and applying a standard covering argument, we deduce that \( V_{\mu,p}(D\tilde{v}) \in B_{\frac{3}{2},\ddot{B}_p}((\mathbb{R}^n) \). This observation ensures that our calculations below are valid.

Combining (4.7) and (4.3) with \( w = w_2^\frac{1}{2} \), for any \( \varepsilon_2 > 0 \), we see that

\[
\int_\Omega 1 + |V_{\mu,p}(D\tilde{v})|^2 dx + [V_{\mu,p}(D\tilde{v})]^2_{\frac{3}{2},\ddot{B}_p(x_0))}
\]

\[
\lesssim \frac{C(\varepsilon_1)}{(\sigma + \rho_0/2 - \rho)^{\frac{q^2}{2(q-1)+q/p}}}
\]

\[
\times \left( \int_{B_{\sigma+\rho_0/2}(x_0) \setminus B_{\rho-\rho_0/4}(x_0)} 1 + |V_{\mu,p}(D\tilde{v})|^2 + |V_{\mu,p}(D\tilde{v})|^2 dx \right) \frac{\varepsilon_1}{(\sigma - \rho)^{-1}(\frac{q}{p} - 1)}
\]

\[
\times \left( \int_{\rho - \rho_0/4}^{\sigma + \rho_0/2} \left[ \left\| V_{\mu,p}(D\tilde{v}) \right\|_{L^{\frac{2q}{q-1}}(\partial B_{\varepsilon}(x_0))} + \left\| V_{\mu,p}(D\tilde{v}) \right\|_{L^{\frac{2q}{q-1}}(\partial B_{\varepsilon}(x_0))} \right] dr \right) \frac{\varepsilon}{(\frac{q}{p} - 1)}
\]

\[
\times \left( 1 + \int \Omega F_\varepsilon(D\tilde{v}) dx \right) + \varepsilon_1 \int_\Omega |D\tilde{v}|^q dx + \varepsilon_2 + D_1 + \mathcal{F}_\varepsilon(v),
\]

using the Poincaré inequality, the bound \( \int_\Omega 1 + |V_{\mu,p}(D\tilde{v})|^2 dx \geq |\Omega| \) and the fact that

\[
\frac{\varepsilon}{(\sigma - \rho)^{-1}(\frac{q}{p} - 1)} \left( 1 - \frac{q}{\xi p} + \frac{q}{p} - 1 \right) = \frac{q}{p} \geq 1.
\]

Let \( 0 < \alpha < 1/2 \) and choose \( \rho = \rho_0/2, \sigma = \rho_0 \). Using Theorem 2, followed by Lemma 5, we conclude that
\[ \int_{\Omega} 1 + |V_{\mu,p}(D\tilde{v})|^2 \, dx + [V_{\mu,p}(D\tilde{v})]_{B^1_{\infty,2}(B_{\rho}(x_0) \cap \Omega)}^2 \leq C(\varepsilon_1) \rho_0^{-\frac{q^2}{(q-1)^2} - \frac{q}{p}} \left( \int_{\Omega} 1 + |V_{\mu,p}(D\tilde{v})|^2 \, dx \right) \frac{\varepsilon_1}{\tilde{\varepsilon}} \left( 1 - \frac{q}{\tilde{p}} \right) \]

\times \left( 1 + ||V_{\mu,p}(D\tilde{v})||_{L^2(\Omega)}^2 + [V_{\mu,p}(D\tilde{v})]_{B^1_{\infty,2}(\Omega)}^2 \right) \frac{\varepsilon_1}{\tilde{\varepsilon}} \left( \frac{q}{\tilde{p}} - 1 \right) \]

\times \left( 1 + \int_{\Omega} F_\varepsilon(D\tilde{v}) \, dx \right) + \varepsilon_1 \int_{\Omega} |D\tilde{v}|^q \, dx + \varepsilon_2 + D_1 + \mathcal{F}_\varepsilon(v). \quad (4.9) \]

By a standard covering argument and (2.2), we deduce that

\[ \int_{\Omega} 1 + |V_{\mu,p}(D\tilde{v})|^2 \, dx + [V_{\mu,p}(D\tilde{v})]_{B^1_{\infty,2}(\Omega)}^2 \]

\[ \leq C(\varepsilon_1) \rho_0^{-\frac{q^2}{(q-1)^2} - \frac{q}{p}} \left( \int_{\Omega} 1 + |V_{\mu,p}(D\tilde{v})|^2 \, dx \right) \frac{\varepsilon_1}{\tilde{\varepsilon}} \left( 1 - \frac{q}{\tilde{p}} \right) \]

\times \left( 1 + ||V_{\mu,p}(D\tilde{v})||_{L^2(\Omega)}^2 + [V_{\mu,p}(D\tilde{v})]_{B^1_{\infty,2}(\Omega)}^2 \right) \frac{\varepsilon_1}{\tilde{\varepsilon}} \left( \frac{q}{\tilde{p}} - 1 \right) \]

\times \left( 1 + \int_{\Omega} F_\varepsilon(D\tilde{v}) \, dx \right) + \varepsilon_1 \int_{\Omega} |D\tilde{v}|^q \, dx + \varepsilon_2 + \|g\|_{W^{2,q}(\Omega)} \left( 1 + ||Dv||_{L^q(\Omega)}^{\frac{q-1}{2}} + ||Dg||_{L^q(\Omega)}^{\frac{q-1}{2}} \right) + \mathcal{F}_\varepsilon(v). \]

Next fix $0 < \beta < 1$. We note that, by interpolation between $L^p(\Omega)$ and $L^{\frac{np}{n-\beta}}(\Omega)$,

\[ ||Dv||_{W^{2,q}(\Omega)} \leq ||Dv||_{L^p(\Omega)}^{(1-\theta)(q-1)} ||Dv||_{L^{\frac{np}{n-\beta}}(\Omega)}^{\theta(q-1)} \]

\[ \leq ||V_{\mu,p}(Dv)||_{L^2(\Omega)}^{2(1-\theta)(q-1)/p} ||V_{\mu,p}(Dv)||_{L^{\frac{2n}{n-\beta}}(\Omega)}^{2\theta(q-1)/p} \]

with $\theta = \frac{n}{\beta} \left( 1 - \frac{p}{q} \right)$. Since $q < p + 1$, it is straightforward to check that $\frac{2\theta(q-1)}{p} < 2$ for $\beta$ chosen sufficiently close to 1.

Since $\frac{q}{p} < 1 + \frac{1}{n-1}$, using the definition of $\xi$ we find that

\[ \frac{\xi}{\xi - 1} \cdot \frac{q - p}{p} < 1. \]

Choosing $\varepsilon_1$ sufficiently small and $\varepsilon_2 = 1$, after re-arranging and using the embedding of $B^1_{\infty,2}(\Omega)$ into $W^{1,2n/(n-\beta)}(\Omega)$ (which holds for a sufficiently large choice of $\alpha$), we deduce the estimate

\[ [V_{\mu,p}(D\tilde{v})]_{B^1_{\infty,2}(\Omega)}^2 \]

\[ \lesssim \left( 1 + ||V_{\mu,p}(D\tilde{v})||_{L^2(\Omega)}^2 + ||g||_{W^{2,q}(\Omega)}^q \right) \frac{\tilde{\beta}}{1-\beta} \left( 1 + \int_{\Omega} F_\varepsilon(D\tilde{v}) \, dx \right)^{\frac{1}{1-\beta}} \quad (4.10) \]

where $\tilde{\beta} = \max \left( \frac{\xi}{\xi - 1} \cdot \frac{2n(q-1)(q-p)}{p} \right)$. As $||V_{\mu,p}(D\tilde{v})||_{L^2(\Omega)}^2 \lesssim 1 + \mathcal{F}_{\varepsilon,\beta}(D\tilde{v}) + ||g||_{W^{2,q}(\Omega)}$, using Lemma 8 we conclude that the bound is independent of $\varepsilon$. 
Finally, by Sobolev embedding, for any $\beta < 1$, we can find $\alpha \in (0, 1/2)$ such that

$$\|Dv\|_{L^{\frac{np}{n-\beta}}(\Omega)} \lesssim \|Dv\|_{B^{2\alpha/p,p}_{\infty}(\Omega)} \lesssim \|V_{\mu,p}(Dv)\|_{B^{q,2}_{\infty}(\Omega)}.$$

Considering the regularity of $g$, the result now follows. \qed

**Remark 13** We have chosen to present the argument leading to a global apriori estimate. However, it is straightforward to adapt our argument in order to obtain a local version of the estimate. This is achieved by considering (4.9). Instead of a covering argument, the estimates follow arguing along similar lines as before but requires an iteration lemma. See, for example, [19] for similar arguments.

**Remark 14** Our results also imply an improved differentiability result. For this we return to (4.10). Due to the $W^{1,q}$-regularity of $u_{\epsilon,\delta}$, the right-hand side remains bounded in the limit as $\epsilon, \delta \to 0$. Thus, using also that $\|Dv\|_{B^{1+\frac{1}{p},p}_{\infty}(\Omega)} \lesssim \|V_{\mu,p}(Dv)\|_{B^{\frac{1}{2},p}_{\infty}(\Omega)}$, we obtain a $1+\frac{1}{p},p(\Omega)$-estimate for $u_{\epsilon,\delta}$, independent of $\epsilon, \delta$.

### 4.2 The non-autonomous case

For the non-autonomous case we prove the following result.

**Proposition 4** Let $\alpha \in (0, 1]$ and $2 \leq p \leq q < \frac{(n+\alpha)p}{n}$. Suppose that $g \in W^{1+\alpha,q}(\Omega)$ and $\psi \in W^{2,\infty}(\Omega)$. Let $F \equiv F(x,z)$ satisfy (H1),(H2) and (H3). Then, for minimisers $u_{\epsilon,\delta}$ of $\mathcal{F}_{\epsilon,\delta}(\cdot)$, for any $\tau \in (0, \alpha)$ and some $\beta > 0$, we have that

$$\|Du_{\epsilon,\delta}\|_{{L^q}(\Omega)} \lesssim [V_{\mu,p}(Du_{\epsilon,\delta})]^{\frac{2}{B_{\infty}^{r,2}(\Omega)}} \lesssim \left(1 + \|V_{\mu,p}(Du_{\epsilon,\delta})\|_{{L^r}(\Omega)}^2 + \|g\|_{W^{2,q}(\Omega)}\right)^{\beta},$$

where the implicit constant is independent of $\epsilon, \delta$.

**Proof** We employ the notation of the proof of Proposition 3. The proof follows the same outline as the autonomous case. However the trick of applying the Sobolev embedding on spheres cannot be applied and hence the proof is simpler. We state only the key steps.

We argue exactly as before up to (4.6). Here, using the convexity of $F_{\epsilon}(\cdot)$ as before, we find that

$$|A_2| \leq \int_{B_{3\gamma_0}(x_0)} \tau_{-h} \phi F_{\epsilon}(x - h , D\tilde{v} + D\tilde{g} - h) + \phi(x)(F_{\epsilon}(x - h , D(\tilde{g} - h + \tilde{v})) - F_{\epsilon}(x - h, D(\tilde{g} - \tilde{v}))) \, dx$$

$$+ \int_{B_{2\gamma_0}(x_0)} \phi(x)(F_{\epsilon}(x - h , D\tilde{v} + D\tilde{g}) - F_{\epsilon}(x, D\tilde{v} + D\tilde{g})) \, dx.$$

Using (H2) and (H3), the regularity of $\phi$ and $g$ and (2.5), we estimate each term in turn to conclude that

$$|A_2| \lesssim |h|^\alpha \left(1 + \|Dv\|_{{L^q}(\Omega)}^q + \|g\|_{W^{1+\alpha,q}(\Omega)}^q\right).$$

Combining this with the estimates as obtained in the autonomous case, we find that, $x_0 \in \mathbb{R}^n$, 

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Recalling the definition of \( T_h \), using the triangle inequality and regularity of \( \tilde{g} \), we see that
\[
\sup_{h \in C^0_{\rho_0}(\theta_0, \mathbf{x}(x_0))} |h|^{-\alpha} \| V_{\mu, p}(Dv) - V_{\mu, p}(DT_h \tilde{v}_h) \|_{L^2(B_{\rho_0}(x_0))}^2 \\
\lesssim 1 + \| Dv \|_{L^q(\Omega)}^q + \| g \|_{W^{1, q}(\Omega)}^q.
\]
(4.11)

Using the characterisation of Besov spaces (2.1), we conclude that for every \( x_0 \in \mathbb{R}^n \),
\[
[Dv]_{p, \rho, B_{\rho_0}(x_0)}^p \lesssim (1 + \| Dv \|_{L^q(\Omega)}^q + \| g \|_{W^{1, q}(\Omega)}^q).
\]
(4.12)

Covering \( \Omega \) by a finite number of balls of radius \( \rho_0 \), using (2.2) we conclude that
\[
\| v \|_{B_{\infty}^{1 + \frac{\alpha}{p}, p} (\Omega)}^{p} \lesssim \left( 1 + \| v \|_{W^{1, q}(\Omega)}^{q} + \| g \|_{W^{1, q}(\Omega)}^{q} \right).
\]
(4.13)

We recall that \( B_{\infty}^{1 + \frac{\alpha}{p}, p} (\Omega) \) embeds continuously into \( W^{1, \frac{np}{n-\beta}} (\Omega) \) for any \( \beta < \alpha \) by Theorem 2. Hence we choose \( \beta \) such that \( q < \frac{p(\beta+n)}{n} \) and use (2.3) with \( \theta = \frac{np}{\beta} \left( \frac{1}{p} - \frac{1}{q} \right) \) to see that
\[
\| Dv \|_{L^q(\Omega)}^{1-\theta} \leq \| Dv \|_{L^p(\Omega)}^{\frac{np}{\beta}} \| Dv \|_{L^\infty(\Omega)}^{\frac{np}{n-\beta}}.
\]
(4.14)

As \( q < \frac{(n+\beta)p}{n} \), it follows that \( q\theta < p \). Using (4.14) in (4.13), we find after using Young’s inequality that
\[
\| v \|_{W^{1, \frac{np}{n-\beta}}(\Omega)}^p \lesssim 1 + \frac{1}{2} \| v \|_{W^{1, \frac{np}{n-\beta}}(\Omega)}^{\frac{np}{\beta}} + C(\theta) \| v \|_{W^{1, \frac{np}{n-\beta}}(\Omega)}^{\frac{np}{n-\beta}} + \| g \|_{W^{1, q}(\Omega)}^q.
\]

Rearranging, as well as recalling (H5) and the regularity of \( g \), we obtain the desired result. \( \square \)

**Remark 15** The results of this section can easily be extended to the case \( 1 < p \leq 2 \). We refer to [52] for the details.

**Remark 16** Our results also imply an improved differentiability result. For this we return to (4.13) and take limits as \( \varepsilon \to 0 \). Due to the \( W^{1, q} \)-regularity of \( u_{\varepsilon, \delta} \), the right-hand side remains bounded. Thus we obtain \( B_{\infty}^{1 + \frac{\alpha}{p}, p} (\Omega) \)-estimates for \( u_{\varepsilon, \delta} \), uniform in \( \varepsilon, \delta \).

## 5 Proof of Theorem 1

We are now in a position to prove Theorem 1 and Proposition 2.

**Proof of Theorem 1** For minimisers \( u_{\varepsilon, \delta} \) of \( F_{\varepsilon, \delta} (\cdot) \), due to minimiality as well as Proposition 3 and Proposition 4, we have the bound
\[
\| u_{\varepsilon, \delta} \|_{W^{1, q}(\Omega)} \leq C < \infty
\]
(5.1)

where \( C \) is independent of \( \varepsilon, \delta \). Next, we recall that, as \( \delta \to 0 \), we have \( u_{\varepsilon, \delta} \to u_\varepsilon \) in \( W^{1, q}(\Omega) \) where \( u_\varepsilon \) is a minimiser of
\[
\int_{\Omega} F(x, Dv) + \varepsilon |Dv|^q + \kappa (\psi - v)_+ \, dx
\]
in the class $W^{1,q}_g(\Omega)$. Using Proposition 2, we see that if $\kappa$ is chosen sufficiently large, independently of $\epsilon, u_\epsilon$ minimises
\[ \int_\Omega F(x, Dv) + \epsilon |Dv|^q \, dx \]
in the class $K_{g,\psi}^*(\Omega)$. Due to Lemma 8, $u_\epsilon \to u$ in $W^{1,p}(\Omega)$, where $u$ is the relaxed minimiser of (P). In particular, we can choose a suitable diagonal subsequence of $u_{\epsilon,\delta}$, denoted $u_k$, with $u_k \to u$ as $k \to \infty$. Passing to the limit in (5.1) we deduce that $u \in W^{1,q}(\Omega)$. □

**Proof of Proposition 3** Due to Lemmas 9 and 11, given a pointwise minimiser $u \in K_{g,\psi}^*(\Omega)$ of (P), we can find $(u_j) \subset K_{g,\psi}^*(\Omega)$ and $F_j \equiv F_j(x, z) : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfying the same assumptions as $F$, with constants in the various bounds that do not depend on $j$ such that
\[ u_j \to u \quad \text{in} \quad W^{1,p}(\Omega) \quad \text{and} \quad \int_\Omega F_j(x, Du_j) \, dx \to \mathcal{F}(u) \quad \text{as} \quad j \to \infty. \] (5.2)

Since $F_j(x, z)$ satisfies the same assumptions as $F$, we can apply Propositions 3 and 4 to see that minimisers $u_{\epsilon,\delta,j}$ of
\[ \mathcal{F}_{\epsilon,\delta}^j := \int_\Omega F^j(x, Du) + \epsilon |Du|^q + \kappa H_\delta(\psi - u) \, dx \]
satisfy
\[ \|u_{\epsilon,\delta,j}\|_{W^{1,q}(\Omega)} \leq C < \infty \] (5.3)
independent of $\epsilon, \delta, j$. Thus, repeating the arguments of the proof of Theorem 1 we extract a subsequence $u_k = u_{\epsilon_k,\delta_k,j_k}$ with $\epsilon_k, \delta_k \to 0$ and $j_k \to \infty$ as $k \to \infty$ that converges weakly in $W^{1,q}(\Omega)$ to the minimiser $v$ and such that
\[ \mathcal{F}_{\epsilon_k,\delta_k}^j(u_k^j) \to \mathcal{F}(v) \] as $k \to \infty$. Moreover, by weak lower semicontinuity of norms and (5.3), we have $v \in W^{1,q}(\Omega)$ and so $\mathcal{F}(v) = \mathcal{F}(v)$. By weak semicontinuity and minimaliy of $u_{\epsilon,\delta,j}$, we note that
\[ \mathcal{F}(v) = \lim_{k \to \infty} \mathcal{F}_{\epsilon_k,\delta_k}^j(u_k^j) \leq \lim_{k \to \infty} \mathcal{F}_{\epsilon_k,\delta_k}^j(u_j) = \mathcal{F}(u). \]
For a sufficiently large choice of $\kappa$ we have $v \geq \psi$ by Proposition 2. Using the minimality of $u$ in the class $K_{g,\psi}^*(\Omega)$, we conclude that $\mathcal{F}(v) = \mathcal{F}(u)$. By convexity of $\mathcal{F} (\cdot)$, it follows that $v = u$, concluding the proof. □

We finally prove Corollary 1.

**Proof of Corollary 1** Corollary 1 follows directly from Remarks 14 and 16, lower semicontinuity of norms and the convergence statements in the proof of Theorem 1. □

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**Declarations**

**Conflict of interest** The author declare that he has no conflict of interest.
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