ON ROTATING SOLITONS IN GENERAL RELATIVITY

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Abstract

We review a number of results recently obtained in the area of constructing rotating solitons in a four dimensional asymptotically flat spacetime. Various models are examined, special attention being paid to the monopole-antimonopole and gauged skyrmion configurations, which have a nonvanishing total angular momentum. For all known examples of rotating solitons, the angular momentum is fixed by some conserved charge of the matter fields.

1 Introduction

Rotation is a universal phenomenon, which seems to be shared by all objects, at all possible scales. However, as shown first by Lichnerowicz [1], the vacuum Einstein equations admit no particle-like solutions, in particular no rotating regular configurations. For a gravitating Maxwell field, the Kerr-Newman black hole solutions represent the only physically reasonable, asymptotically flat configurations with nonzero angular momentum. However, again no regular rotating solution is found in the limit of zero event horizon radius.

The inclusion of more general (nonlinear-) matter sources in the theory leads to the possibility of finding localized, globally regular, particle-like objects with finite energy – so-called solitons. Most of these solutions correspond to gravitating generalizations of the flat spacetime solitons, e.g. the monopoles [2], dyons [3], sphalerons [4] and Skyrmions [5]. There are also examples of solitons which have no Minkowski space counterparts, the Bartnik-McKinnon (BK) solutions [6] in Einstein-Yang-Mills (EYM) theory and the boson stars [7, 8] being the best known cases.

However, most of particle-like solutions discussed in the literature are spherically symmetric. Then it is natural to wonder whether one can find axially symmetric solitons with a nonzero angular momentum. This problem enjoyed recently some interest, the issue of rotating soliton solutions being systematically considered for various models, with some surprising results. In this work we review the situation for several different cases, presenting the basic features of rotating solutions. For simplicity, we’ll restrict to asymptotically flat solutions and four spacetime dimensions.

We consider a generic action principle describing the Einstein gravity coupled with some matter fields with a lagrangean \( L_m \)

\[
S = \int d^4x \sqrt{-g} \left( \frac{1}{16\pi G} R + L_m \right),
\]

which implies the Einstein equations

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad \text{where} \quad T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta L_m}{\delta g^{\mu\nu}}.
\]

All known rotating regular solutions have been found within the usual Lewis-Papapetrou ansatz [9] for a stationary, axially symmetric spacetime with two Killing vector fields \( \partial/\partial \varphi \) and \( \partial/\partial t \). A suitable parametrization of the metric line element in terms of the spherical coordinates \( r, \theta \) and \( \varphi \), used in most studies on this subject, is

\[
ds^2 = -f \, dt^2 + \frac{m}{f} \left( dr^2 + r^2 d\varphi^2 \right) + \frac{l}{f} r^2 \sin^2 \theta \left( d\varphi - \frac{\omega}{r} dt \right)^2,
\]

where \( f, m, l \) and \( \omega \) are only functions of \( r \) and \( \theta \).

1Here we do not ask the solitons to be stable; also the spacetime is supposed to possess a \( R^4 \) topology.
The asymptotic flatness imply that the metric function are given at infinity by $f = m = l = 1, \omega = 0$. For solutions with a regular origin one has $\partial_r f = \partial_r m = \partial_r l = \omega = 0$ as $r \to 0$. Also, all known examples of rotating solitons possess a parity symmetry with respect the $\theta = \pi/2$ plane. Therefore, it is enough to solve the field equations for $0 \leq \theta \leq \pi/2$; the derivatives $\partial_\theta f, \partial_\theta l, \partial_\theta m, \partial_\theta \omega$, have to vanish for both $\theta = 0$ and $\theta = \pi/2$.

The mass $M$ and the angular momentum $J$ of the soliton solutions can be read from the metric function expansion as $r \to \infty$

$$f \sim 1 - \frac{2GM}{r}, \quad \omega \sim \frac{2GJ}{r^2},$$

(4)

or equivalently from (9)

$$M = - \int (2T_t^t - T_\mu^\mu) \sqrt{-g} dr d\theta d\varphi, \quad J = \int T_\varphi^\varphi \sqrt{-g} dr d\theta d\varphi,$$

(5)

the angular momentum being the charge associated with the Killing vector $\partial/\partial \varphi$.

The existence and the properties of the rotating solitons depends on the matter fields choice. To find the features of these types of configuration, we have to solve extremely complicated partial differential equations, no closed form rotating soliton solution being known in the literature.

2 Nonabelian rotating solitons

We consider first the physically interesting case of a spontaneously broken gauge theory, described by a matter lagrangean

$$-L_m = Tr\{\frac{1}{2} F_{\mu\nu} F^{\mu\nu}\} + Tr\{\frac{1}{2} D_\mu \Phi D^\mu \Phi\} + V(\Phi).$$

(6)

with the Higgs field in the adjoint representation, $V(\Phi) = \frac{\lambda}{8} Tr(\Phi^2 - \eta^2)^2$ being the usual scalar potential. The nonabelian field strength tensor is (here we restrict to a SU(2) gauge group)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu],$$

and the covariant derivative $D_\mu = \partial_\mu + i[A_\mu, ]$. Varying the action with respect to $A_\mu$ and $\Phi$ one finds the Yang-Mills-Higgs field equations

$$\frac{1}{\sqrt{-g}} D_\mu (\sqrt{-g} F^{\mu\nu}) = \frac{1}{4} i[\Phi, D^\nu \Phi],$$

(7)

$$\frac{1}{\sqrt{-g}} D_\mu (\sqrt{-g} D^\mu \Phi) + \lambda (\Phi^2 - \eta^2) \Phi = 0.$$  

(8)

The variation of the lagrangian with respect to the metric $g_{\mu\nu}$ yields the energy-momentum tensor which enters the Einstein equations

$$T_{\mu\nu} = 2Tr\{F_{\mu\alpha}F_{\nu\beta}g^{\alpha\beta} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\} + Tr\{\frac{1}{2} D_\mu \Phi D_\nu \Phi - \frac{1}{4} g_{\mu\nu} (D_\alpha \Phi D^\alpha \Phi)\} + V(\Phi) g_{\mu\nu}.$$  

We mention also the expression for the electric and magnetic charges derived by using the ’t Hooft field strength tensor (with $\hat{\Phi} = \Phi/|\Phi|$)

$$Q_e = \frac{1}{4\pi} \oint dS_\mu Tr\{\hat{\Phi} F_{\mu\nu}\}, \quad Q_m = \frac{1}{4\pi} \oint dS_\mu \frac{1}{2} \epsilon_{\mu\nu\alpha} Tr\{\hat{\Phi} F_{\nu\alpha}\}.$$  

(10)

In the purely magnetic case (no electric potential $A_t = 0$), the Einstein-Yang-Mills-Higgs (EYMH) field equations admits a variety of solutions, the gravitating version of the spherically symmetric ’t Hooft-Polyakov monopoles being the best known example. Of interest here are also the composite axially symmetric solutions
containing magnetic charges of both signs. The simplest case consists in two opposite charges located on the z-axis and forming a monopole-antimonopole pair, i.e. a magnetic dipole \([10]\).

The issue of rotating solutions in EYMH model was first addressed within a perturbative approach in the nongravitating limit. However, in the absence of gravity, it has been shown that ’t Hooft Polyakov monopoles and Julia-Zee dyons do not admit slowly rotating excitations \([11]\).

Further progress in this area was made possible by the discovery that the angular momentum \(J\) in \([9]\) admits a simple expression as a surface integral in terms of YM fields only \([12]\). One starts by noticing that, at the level of the matter ansatz, a rotation around the z-axis can be compensated by a gauge rotation

\[
\mathcal{L}_\varphi A = D\Psi,
\]

and therefore \(F_{\mu\varphi} = D_\mu W, \quad D_\varphi \Phi = i[W, \Phi]\), where \(W = A_\varphi - \Psi\).

Therefore one may write the following expression for the \(T^t_\varphi\)-component of the energy-momentum tensor associated with rotation

\[
T^t_\varphi = 2Tr\left\{\frac{1}{\sqrt{-g}} D_\tau (WF^{t\tau}) + \frac{1}{\sqrt{-g}} D_\theta (WF^{\theta t}) - W\left(\frac{1}{\sqrt{-g}} D_\tau (\sqrt{-g} F^{t\tau}) + \frac{1}{\sqrt{-g}} D_\theta (\sqrt{-g} F^{\theta t})\right) + i\frac{1}{4}[W, \Phi] D^t \Phi\right\}.
\]

As a consequence of the YM equations \([7]\) and making use of the fact that the trace of a commutator vanishes we obtain

\[
T^t_\varphi = 2Tr\left\{\frac{1}{\sqrt{-g}} \partial_\mu (WF^{\mu t}) \right\}.
\]

Thus, for globally regular solutions, the total angular momentum can be expressed as an integral over the two-sphere at spacelike infinity \([12, 13]\)

\[
J = \oint_{\infty} 2Tr\{WF^{\mu t}\} dS_\mu = 2\pi \lim_{r \to \infty} \int_0^\pi d\theta \sin \theta \ r^2 [W^{(r)} F^{\mu t(r)} + W^{(\theta)} F^{r t(\theta)} + W^{(\varphi)} F^{\varphi t(\varphi)}].
\]

This generic relation is evaluated for a specific axially symmetric ansatz, within a set of boundary conditions consistent with finite energy and regularity assumptions.

As proven in \([12, 13]\), the angular momentum of a gravitating monopole-antimonopole solution is nonzero and equals the electric charge \(Q_e\). In fact, it appears to exist a very general (and still poorly understood) connection between the angular momentum and the topological charge of an axially symmetric, electrically charged solution in EYMH theory. The total angular momentum of any solution with a nonvanishing global magnetic charge is zero, although the configurations rotates locally, \(g_{\varphi \varphi} \neq 0\). The solutions without a global topological charge (which presents, however, a nonvanishing magnetic charge density) have a nonzero angular momentum proportional to the electric charge,

\[
J = nQ_e,
\]

where \(n\) is an integer - the winding number of solutions.

The only numerical solutions exhibited so far in literature correspond to the charged monopole \([14]\) and monopole-antimonopole \([15]\) cases. They were found within a suitable parametrization of the axially symmetric ansatz derived by Rebbi and Rossi \([16]\), with a SU(2) gauge connection

\[
A_\mu dx^\mu = A_\varphi d\varphi + A_4 dt = \frac{1}{2er} \left[\tau_3 (H_1 dr + (1 - H_2) r d\theta) - (\tau_r H_3 + \tau_\theta (1 - H_4)) r \sin \theta d\varphi + (\tau_r H_5 + \tau_\theta H_6) dt\right],
\]

and a Higgs field of the form

\[
\Phi = (\phi_1 \tau_r + \phi_2 \tau_\theta).
\]

The SU(2) matrices \((\tau_r, \tau_\theta, \tau_\varphi)\) are defined in terms of the Pauli matrices \(\tau = (\tau_r, \tau_\theta, \tau_\varphi)\) by \(\tau_r = \tau_\varphi \cdot (\sin \theta \cos n\varphi, \sin \theta \sin n\varphi, \cos \theta), \quad \tau_\theta = \tau_\varphi \cdot (\cos \theta \cos n\varphi, \cos \theta \sin n\varphi, -\sin \theta), \quad \tau_\varphi = \tau_\varphi \cdot (-\sin n\varphi, \cos n\varphi, 0)\).
Figure 1. The mass-energy density $\epsilon = -T^t_t$ and the angular momentum density $T^i_{\phi}$ of a typical rotating dyon solution in EYMH theory are plotted as a function of the coordinates $\rho = r \sin \theta$ and $z = r \cos \theta$.

The six gauge field functions $H_i$ and the two Higgs field function $\phi_i$ depend only on the coordinates $r$ and $\theta$.

The boundary conditions they satisfy depend on the presence or not of a global magnetic charge and are essentially fixed by the asymptotic behaviour of the Higgs field $|\Phi| \to \eta$

$$\lim_{r \to \infty} \phi_1 = \eta \cos m \theta, \quad \lim_{r \to \infty} \phi_2 = \eta \sin m \theta, \quad (15)$$

with $m = 0, 1, \ldots$ plus regularity requirements. The solutions with even $m$ have a nonzero magnetic charge and $J = 0$; odd $m$ solutions have a different picture, with $Q_m = 0$ and a nonzero angular momentum.

Solving the field equations reveals a complicated branch structure. For both dyons and electrically charged monopole-antimonopoles, a branch of rotating solutions emerges from the flat spacetime configurations.
Figure 2. The same as Figure 1 for a typical rotating monopole-antimonopole solution.

Apart from this fundamental branch, there are also excited solutions related to axially symmetric solutions of EYM theory (for more details, see [14, 15]).

In Figures 1, 2 we show the energy density $\epsilon = -T^t_t$ and the angular momentum density $T^\phi_\varphi$ for typical gravitating dyon and monopole-antimonopole solutions as function of the coordinates $z = r \cos \theta$ and $\rho = r \sin \theta$. As seen from these Figures, the distributions of the dyon mass-energy density shows a peak along the $\rho$-axis and decreases monotonically along the $z$-axis. Equal density contours reveal a torus-like shape of the solutions. The picture is different for the angular momentum density which vanishes on the $\rho$-axis and changes the sign as $z \to -z$. Thus, although it will rotate locally, the total angular momentum of the dyon solutions is zero (although $g_{\rho \varphi} \neq 0$), and the spacetime consists in two regions rotating in opposite directions.
The energy density of a monopole-antimonopole configuration possesses two maxima symmetrically located on the $z-$axis and a saddle point at the origin, describing a composite configuration. A different picture is found for the angular momentum density. As seen in Figure 2, the magnetic dipole system rotates as a single object and the $T^3_z$-component of the energy momentum tensor associated with rotation presents a maximum in the $z = 0$ plane and no local extrema at the locations of the monopole and the antimonopole.

However, it is interesting that no rotating solutions are found in the limit of vanishing Higgs field. Spinning generalizations of the BK solitons in EYM theory, although predicted perturbatively \[17\], appears do not exist within a nonperturbative approach \[13\]. For such configurations, the $A_t$-components of the gauge field act like an isotriplet Higgs field with negative metric, and by themselves cause the magnetic components of the nonabelian potential to oscillate rather than decrease exponentially as $r \to \infty$, which would give an infinite mass. Therefore we are forced to take $\lim_{r \to \infty} A_t = 0$. However, this implies a vanishing electric field, i.e. a zero Poynting vector and a static configuration $g_{st} = 0$.

Here we should remark that another example of a four dimensional rotating soliton was found recently in a closed related model presenting a $U(1)$ gauge field and a dilaton interacting in a nontrivial way with the Yang-Mills and Higgs fields \[19\]. This model originates in Kaluza-Klein reduction of the five dimensional EYM theory. The $d = 4$ rotating solitons are found by reducing along the $x^5$-direction the boosted five dimensional static axisymmetric nonabelian vortices. As expected, the resulting rotating configurations share many features with the EYM dyons and monopole-antimonopole solutions discussed above.

3 Spinning $U(1)$ gauged Skyrmions

To the best of our knowledge, the only example of rotating solution residing in the one-soliton sector of the theory which has a topologically stable limit was found in $U(1)$ gauged Skyrme theory.

The Skyrme model has been proposed a long time ago \[5\] as an effective theory for nucleons in the large $N$ limit of QCD at low energy \[20\], the baryon number being identified with the topological charge. The classical as well as the quantum properties are in relatively good agreement with the observed features of small nuclei. The $U(1)$ gauged Skyrme model was originally proposed by Callan and Witten to study the decay of the nucleons in the vicinity of a monopole \[21\].

In terms of the $O(4)$ sigma model field $\phi^a = (\phi^a, \phi^A)$, $a = 1, 2; \ A = 3, 4$, satisfying the constraint $|\phi^a|^2 + |\phi^A|^2 = 1$, the Lagrangean of the Maxwell gauged Skyrme model is (up to an overall factor which we set equal to one)

\[-L_m = \frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu \phi^a D^\mu \phi^a + \frac{\kappa^2}{8} |D_\mu \phi^a D_\nu \phi^a|^2\]  
(16)

in terms of the Maxwell field strength $F_{\mu\nu}$, and the covariant derivatives defined by the gauging prescription \[22\]

\[D_\mu \phi^a = \partial_\mu \phi^a + A_\mu (\varepsilon \phi)^a, \ \ \ D_\mu \phi^A = \partial_\mu \phi^A.\]  
(17)

The energy-momentum tensor which follows from \[16\] is

\[T_{\mu\nu} = 2 \left( F_{\mu\lambda} F^{\lambda}_\nu - \frac{1}{4} g_{\mu\nu} F_{\tau\lambda} F^{\tau\lambda} \right) + \left( D_\mu \phi^a D_\nu \phi^a - \frac{1}{2} g_{\mu\nu} D_\lambda \phi^a D^\lambda \phi^a \right) + 2 \frac{\kappa^2}{4} \left[ (D_\mu \phi^a D_\lambda \phi^b) (D_\nu \phi^a D^\lambda \phi^b) - \frac{1}{4} g_{\mu\nu} (D_\tau \phi^a D_\lambda \phi^b) (D^\tau \phi^a D^\lambda \phi^b) \right].\]  
(18)

The rotating gauged Skyrmions are found within the following matter ansatz \[3\]

\[A_\mu dx^\mu = (a(r, \theta) - n)d\phi + b(r, \theta) d\theta,\]  
(19)

\[\phi^a = \sin F(r, \theta) \sin G(r, \theta) n^a, \ \ \ \ \phi^3 = \sin F(r, \theta) \cos G(r, \theta), \ \ \ \ \phi^4 = \cos F(r, \theta),\]  
\[\text{Rotating black hole solutions in EYM model are known to exist} \ [18]. \ \text{However, these configurations are found within a set of boundary conditions with a vanishing electric potential at infinity and are sustained by the existence of an event horizon.}\]

\[\text{A nongravitating axially symmetric, spinning soliton of the ungauged Skyrme model, has been recently constructed in} \ [23]. \ \text{However, this is a Q-ball type of solution featuring time-dependent fields.}\]
Figure 3. The mass-energy density $\epsilon = -T_t^t$ and the angular momentum density $T_{\phi t}$ of a typical rotating gauged Skyrmeon solution.

$a$ and $b$ corresponding to the magnetic and electric components of the abelian potential, with $n$ a positive integer - the Baryon number.

By using the field equations, the volume integral of $T_{\phi t}^t$ can be converted into a surface integral at infinity in terms of Maxwell potentials

$$J = 4\pi \lim_{r \to \infty} \int_0^\pi d\theta \sin \theta r^2 a \ b_r.$$  \hspace{1cm} (20)

The field equations imply the asymptotic behaviour of the electric potential, $b \sim V - Q/(2r) + O(1/r^2)$, the
parameter $Q$ corresponding to the electric charge of the solutions. Therefore the following relation holds
\[ J = nQ. \] (21)

Note that the solutions discussed here possess also a magnetic dipole moment which can be read from the asymptotics of the $U(1)$ magnetic potential, $A_{\varphi} \sim \mu \sin \theta/r^2$, while the magnetic charge is zero.

This system has been discussed in the flat spacetime limit in Ref. 24, where the basic properties of the configurations are exposed. Gravitating generalizations of these solutions can easily be constructed 25. For a given Baryon number, the solutions depend on two continuous parameters, the values $V$ of the electric potential at infinity and the Skyrme coupling constant $\kappa$. The solutions with $V = 0$ have $b = 0$ and correspond to static dipoles. A nonvanishing $V$ leads to rotating regular configurations, with nontrivial functions $f, l, m, \omega$ and $F, G, a, b$.

In Figure 3 we plot the energy density and the angular momentum density of a typical gravitating $n = 1$ Skyrmion solution as a function of the coordinates $\rho, z$. We notice that the energy density does not exhibit any distinctly localised individual components, a surface of constant energy density being topologically a sphere. Also, the electrically charged $U(1)$ gauged Skyrmion rotates as a single object and the $T^t_\varphi$-component of the energy-momentum tensor associated with rotation presents a maximum in the $z = 0$ plane and no local extrema.

4 Rotating boson stars

The first example of rotating soliton in general relativity was found in a theory containing a complex scalar field with an harmonic time dependence.

Spherically symmetric, gravitationally bound states of scalar field were first obtained by Kaup 7 and Ruffini and Bonazzola 8. These boson stars are macroscopic quantum states and are only prevented from collapsing gravitationally by the Heisenberg uncertainty principle (see 26 for a recent review of this type of soliton solutions).

The lagrangian density of a complex self-gravitating scalar field $\Phi$ reads
\[ -L_m = \sqrt{-g} \left( g^{ij} \Phi^*_{,i} \Phi_{,j} + V(\Phi) \right), \] (22)
where the asterisc denotes complex conjugate. Here we consider only the case $V(\Phi) = \mu^2 \Phi^* \Phi$, where $\mu$ is the scalar field mass 4.

The Lagrangian density 22 is invariant under a global phase rotation $\Phi \rightarrow \Phi e^{-i\alpha}$; that implies the existence of a conserved current
\[ J^k = ig^{kl} (\Phi^*_j \Phi_{,l} - \Phi_{,l} \Phi^*_j), \] (23)
and an associated conserved charge, namely, the number of scalar particles
\[ N = \int d^3x \sqrt{-g} J^t. \] (24)

The energy momentum tensor is given by
\[ T_{ij} = \Phi^*_i \Phi_{,j} + \Phi^*_{,j} \Phi_i - g_{ij} (g^{km} \Phi^*_{,k} \Phi_{,m} + \mu^2 |\Phi|^2). \] (25)

Rotating boson star solutions are found for a scalar field ansatz
\[ \Phi = \phi(r, \theta) e^{i(m\varphi - \omega t)}. \] (26)

Single-valuedness of the scalar field requires $\Phi(\varphi) = \Phi(\varphi + 2\pi)$. Thus the constant $m$ must be an integer $m = 0, \pm 1, \ldots$.

\[ A \text{ potential on the form } V(\Phi) = \lambda |\Phi|^6 - a|\Phi|^4 + b|\Phi|^2 \text{ leads to nontopological soliton solutions (Q-balls) which exist even in the absence of gravity. Rotating solutions of this model are discussed in 27}. \]
Rotating boson stars solutions have been considered by various authors [28], [29], [30], these being the best understood examples of spinning configurations. One can easily see that the angular momentum of these solutions is quantized

\[ J = mN. \]  

(27)

The results in the literature indicate that rotating boson stars exist only for a limited frequency range \( \omega_{\text{min}} < \omega < \omega_{\text{max}} \). The mass \( M \) and particle number \( N \) tends to zero when the maximal frequency is approached. For each rotational quantum number \( m \), there are even and odd parity solutions, although only even parity rotating solutions have been exhibited in the literature.

A boson star rotates as a single object, without being possible to distinct localised individual components. The typical profiles of the energy density and angular momentum density have a similar shape with those presented in Figure 3.

5 Further remarks

Recently, some progress was achieved in understanding the nature of rotating soliton solutions in general relativity. Several different models have been examined in literature, with a number of surprising results.

A general feature of all known rotating solitons is that the angular momentum is fixed by some conserved charge of the matter fields (the electric charge in the presence of a gauge field or the particle number for a complex scalar field with an harmonic time dependence).

Concerning the case of a spontaneously broken gauge theory, a deep connection appears to exist between the global magnetic charge and angular momentum. The rotating solitons of this theory have a vanishing magnetic charge and are unstable. Also, we expect the EYMH theory to present a whole sequence of rotating solutions generalizing for a nonzero electric potential the magnetic chains and rings found in [31]. These configurations will satisfy the generic relation between the nonabelian charges and angular momentum.

We should also remark that all known rotating solitons in general relativity are curved space generalizations of flat space rotating configurations. No example of rotating soliton sustained by gravity is known in the literature. In particular, although predicted perturbatively, no rotating generalisations of the BK solution seem to exist. However, recently it has been realized that, apart from BK solution, the EYM equations admit a general set of static axially symmetric configurations. Although we expect that no rotating generalizations will be found in this case also, this issue may deserve a careful study.

Considering the case of a double Higgs field, the authors of [33] concluded that the well known SU(2) sphalerons do not admit spinning generalizations within the stationary, axially symmetric ansatz. However, similar to the case of a Higgs field in the adjoint representation, this argument does not exclude the existence of more complex, composed configurations which possible may rotate.

It would be interesting to look for rotating solitons in other models admitting particle-like solutions. Particularly interesting are various supersymmetric theories, which may lead to new qualitative features. Also, very little is known on the question of rotating solitons with higher gauge groups.

The question of rotating solutions may still hold further surprises. For example, all know solutions have been found within the Lewis-Papapetrou ansatz. However, the inclusion of nonabelian matter fields leaves open the possibility of existence of more general rotating solutions, exciting other extradiagonal metric components as well.

We close by remarking that the issue of rotating solitons crucially depends on asymptotic structure of spacetime. For example, the Melvin magnetic geon is known to possess rotating generalizations. Also, the EYM-SU(2) system presents rotating soliton solutions for anti-de Sitter asymptotics.

Acknowledgement

This work was carried out in the framework of Enterprise–Ireland Basic Science Research Project SC/2003/390.

\footnote{The boson star solutions are not a genuine counterexample given the existence of flat space Q-ball solutions with many similar properties.}
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