DIRICHLET FORMS AND STOCHASTIC COMPLETENESS OF GRAPHS AND SUBGRAPHS

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Abstract. We characterize stochastic completeness for regular Dirichlet forms on discrete sets. We then study how stochastic completeness of a subgraph is related to stochastic completeness of the whole graph. We show that any graph is a subgraph of a stochastically complete graph and that stochastic incompleteness of a suitably modified subgraph implies stochastic incompleteness of the whole graph. Along our way we give a sufficient condition for essential selfadjointness of generators of Dirichlet forms on discrete sets and explicitly determine the generators on all $\ell^p$, $1 \leq p < \infty$, in this case.

Introduction

There is a long history to the study of the heat equation and spectral theory on graphs (see e.g. the monographs \cite{4, 5} and references therein). A substantial part of this literature is devoted to graphs giving operators, which are bounded on $\ell^2$ or even on $\ell^\infty$. Recently, certain questions concerning unbounded operators have received attention. This is the starting point for our paper.

More precisely, there are recent investigations of essential selfadjointness by Jorgensen \cite{12} and of essential selfadjointness and stochastic completeness of graphs by Wojciechowski \cite{20} (see \cite{21} as well) and Weber \cite{19}. These investigations deal with locally finite graphs and the associated operators. They do not assume a uniform bound on the vertex degree or a modification of the measure and, accordingly, the resulting operators are not necessarily bounded. It turns out that the operators in question are special instances of generators of regular Dirichlet forms on discrete sets. In fact, there is a one-to-one correspondence between the regular Dirichlet forms on a discrete set and graphs over this set with weights satisfying a certain summability condition. This naturally raises the question to which extend similar results to the ones in \cite{12, 20, 19} also hold for arbitrary regular Dirichlet forms on discrete sets. Our first aim is to answer this question. In particular, we

- characterize stochastic completeness for all regular Dirichlet forms on discrete sets.

This generalizes a main result of \cite{20} (see \cite{19} as well for related results and a sufficient condition for stochastic completeness), which in turn is inspired by Grigor’yan’s corresponding result for manifolds \cite{11}.

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Of course, in terms of methods our considerations concerning stochastic completeness heavily draw on existing literature as e.g. Sturms corresponding considerations for strongly local Dirichlet forms [16] and Grigor’yan’s results [11]. A crucial difference, however, is that our Dirichlet forms are not local. In this sense our results can be understood as providing some non-local counterpart to [16, 11].

As far as methods are concerned we note that our basic line of argument is based on resolvents while [20, 19] consider the heat equation. Let us emphasize that our treatment requires intrinsically more effort than the considerations of [20, 19] as in our setting the generators of the Dirichlet forms are known much less explicit. In fact, in general not even the functions with compact support will be in their domain of definition. If these functions belong to the domain of definition and a certain geometric condition – called \((A)\) below – is satisfied, we can

- prove essential selfadjointness of generators on the continuous functions with compact support.

This result extends the corresponding result of [12, 20, 19] to all regular Dirichlet forms on discrete sets. Note that this includes the presence of an arbitrary killing term. We also give examples in which essential selfadjointness fails (as does \((A)\)).

After these considerations our second aim is to study how completeness of a subgraph is related to completeness of the whole graph. There, we obtain two results:

- Any graph is a subgraph of a stochastically complete graph.
- Stochastic incompleteness of a suitably modified subgraph implies stochastic incompleteness of the whole graph.

Along our way, we can also

- determine the generators for the corresponding semigroups on all \(\ell^p\), \(p \in [1, \infty)\) for all regular Dirichlet forms on graphs satisfying \((A)\). These generators turn out to be the “maximal” ones.

These results seem to be new even in the situations considered in [12, 20, 19].

We have tried to make this paper as accessible and self-contained as possible for both people with a background in Dirichlet forms and people with a background in geometry. For this reason some arguments are given, which are certainly well known.

The paper is organized as follows. In Section 1 we present the notation and our main results. A study of basic properties of Dirichlet forms on graphs can be found in Section 2. In Section 3 we consider Dirichlet forms on graphs satisfying the condition \((A)\) mentioned above. For these forms we calculate the generators of the \(\ell^p\) semigroups for \(p \in [1, \infty)\) and we show essential selfadjointness of the generators on \(\ell^2\) (whenever the functions with compact support are in the domain of definition). In Section 4 we give examples where essential selfadjointness fails as well as examples of non-regular Dirichlet forms on graphs. A short discussion of the heat equation in our framework is given in Section 5. Section 6 deals with extending the semigroup and resolvent in question to a somewhat larger space of functions.
In Section 7 we can then prove our first main result characterisation of stochastic completeness for arbitrary Dirichlet forms on graphs. Section 8 contains a proof that any graph is a subgraph of a stochastically complete graph. Section 9 contains an incompleteness criteria.

1. Framework and results

Throughout \( V \) will be a discrete countable set. Let \( m \) be a measure on \( V \) with full support (i.e. \( m \) is a map \( m : V \rightarrow (0, \infty) \)). Then, \( (V, m) \) is a measure space. We will deal exclusively with real valued functions. Thus, \( \ell^p(V, m) \), \( 0 < p < \infty \) is defined by

\[
\{ u : V \rightarrow \mathbb{R} : \sum_{x \in V} m(x)|u(x)|^p < \infty \}.
\]

Obviously, \( \ell^2(V, m) \) is a Hilbert space with inner product given by

\[
\langle u, v \rangle := \sum_{x \in V} m(x)u(x)v(x) \quad \text{and norm } \|u\| := \langle u, u \rangle^{\frac{1}{2}}.
\]

Moreover we denote by \( \ell^\infty(V) \) the space of bounded functions on \( V \). Note that this space does not depend on the choice of \( m \). It is equipped with the supremum norm \( \| \cdot \|_\infty \).

A symmetric non-negative form on \( (V, m) \) is given by a dense subspace \( D \) of \( \ell^2(V, m) \) called the domain of the form and a map

\[
Q : D \times D \rightarrow \mathbb{R}
\]

with \( Q(u, v) = Q(v, u) \) and \( Q(u, u) \geq 0 \) for all \( u, v \in D \). Such a map is already determined by its values on the diagonal. For \( u \in \ell^2(V, m) \) we then define \( Q(u) \) by \( Q(u) := Q(u, u) \) if \( u \in D \) and \( Q(u) = \infty \) otherwise. If \( \ell^2(V, m) \rightarrow [0, \infty) \), \( u \mapsto Q(u) \) is lower semicontinuous \( Q \) is called closed. If \( Q \) has a closed extension it is called closable and the smallest closed extension is called the closure of \( Q \). A map \( C : \mathbb{R} \rightarrow \mathbb{R} \) with \( C(0) = 0 \) and \( |C(x) - C(y)| \leq |x - y| \) is called a normal contraction. If \( Q \) is both closed and satisfies \( Q(Cu) \leq Q(u) \) for all \( u \in \ell^2(V, m) \) it is called a Dirichlet form on \( (V, m) \) (see [3, 6, 10, 14] for background on Dirichlet forms).

Let \( C_c(V) \) be the space of finitely supported functions on \( V \). A Dirichlet form on \( (V, m) \) is called regular if \( D(Q) \cap C_c(V) \) is dense in both \( C_c(V) \) and \( D(Q) \). As discussed below, for such a regular form the set \( C_c(V) \) is necessarily contained in the form domain. Thus, a Dirichlet form \( Q \) is regular if it is the closure of its restriction to the subspace \( C_c(V) \). Regular Dirichlet forms on \( (V, m) \) are given by graphs on \( V \), as we discuss next (see Section 2 for details). A symmetric weighted graph over \( V \) or a symmetric Markov chain on \( V \) is a pair \((b, c)\) consisting of a map \( b : V \times V \rightarrow [0, \infty) \) with \( b(x, x) = 0 \) for all \( x \in V \) and a map \( c : V \rightarrow [0, \infty) \) satisfying the following two properties:

(b1) \( b(x, y) = b(y, x) \) for all \( x, y \in V \).
(b2) \( \sum_{y \in V} b(x, y) < \infty \) for all \( x \in V \).

We can then think of \((b, c)\) or rather the triple \((V, b, c)\) as a weighted graph with vertex set \( V \) in the following way: An \( x \in V \) with \( c(x) \neq 0 \) is then thought to be connected to the point \( \infty \) by an edge with weight \( c(x) \).
Moreover, \( x, y \in V \) with \( b(x, y) > 0 \) are thought to be connected by an edge with weight \( b(x, y) \). The map \( b \) is called the edge weight. The map \( c \) is called killing term. Vertices \( x, y \in V \) with \( b(x, y) > 0 \) are called neighbors. More generally, \( x, y \in V \) are called connected if there exist \( x_0, x_1, \ldots, x_n, x_{n+1} \in V \) with \( b(x_i, x_{i+1}) > 0, \ i = 0, \ldots, n \) and \( x_0 = x, x_n = y \). This allows us to define connected components of \( V \) in the obvious way.

To \( (V, b, c) \) we associate the form \( Q^{\text{comp}} = Q^{\text{comp}}_{b,c} \) on \( C_c(V) \) with diagonal \( Q^{\text{comp}} : C_c(V) \rightarrow [0, \infty] \) given by

\[
Q^{\text{comp}}(u) = \frac{1}{2} \sum_{x,y \in V} b(x, y)(u(x) - u(y))^2 + \sum_x c(x)u(x)^2.
\]

Obviously, \( Q^{\text{comp}} \) is a restriction of the form \( Q^{\text{max}} = Q^{\text{max}}_{b,c,m} \) defined on \( \ell^2(V, m) \) with diagonal given by

\[
Q^{\text{max}}(u) = \frac{1}{2} \sum_{x,y \in V} b(x, y)(u(x) - u(y))^2 + \sum_x c(x)u(x)^2.
\]

Here, the value \( \infty \) is allowed. It is not hard to see that \( Q^{\text{max}} \) is closed and hence \( Q^{\text{comp}} \) is closable on \( \ell^2(V, m) \) (see Section 2) and the closure will be denoted by \( Q = Q_{b,c,m} \) and its domain by \( D(Q) \). Thus, there exists a unique selfadjoint operator \( L = L_{b,c,m} \) on \( \ell^2(V, m) \) such that

\[
D(Q) := \{ u \in \ell^2(V, m) : Q(u) < \infty \} = \text{Domain of definition of } L^{1/2}
\]
and

\[
Q(u) = \langle L^{1/2}u, L^{1/2}u \rangle
\]
for \( u \in D(Q) \) (see e.g. Theorem 1.2.1 in [6]). As \( Q \) is non-negative so is \( L \). Moreover, it is not hard to see that \( Q^{\text{max}}(Cu) \leq Q^{\text{max}}(u) \) for all \( u \in \ell^2(V, m) \) (and in fact any function \( u \)) and every normal contraction \( C \).

Theorem 1.3.1 of [10] then implies that \( Q \) also satisfies \( Q(Cu) \leq Q(u) \) for all \( u \in \ell^2(V, m) \) and hence is a Dirichlet form. By construction it is regular. In fact, every regular Dirichlet form on \( (V, m) \) is of the form \( Q = Q_{b,c,m} \) (see Theorem 6 in Section 2).

**Remark.** Our setting generalizes the setting of [12] [19] [20] to Dirichlet forms on countable sets. In our notation, the situation of [19] [20] can be described by the assumptions \( m \equiv 1, c \equiv 0 \), and \( b(x, y) \in \{0, 1\} \) for all \( x, y \in V \) with \( x \neq y \) and the setting of [12] can be described by \( m \equiv 1, c \equiv 0 \) and \( b(x, y) = 0 \) for all but finitely many \( y \) for each \( x \in V \). In particular, unlike [12] [19, 20] we do not assume finiteness of the sets \( \{ y \in V : b(x, y) > 0 \}, \ (x \in V) \).

Let now a measure \( m \) on \( V \) with full support and a graph \( (b, c) \) over \( V \) be given. Let \( Q \) be the associated form and \( L \) its generator. Then, by standard theory [7] [10] [14], the operators of the associated semigroup \( e^{-tL}, \ t \geq 0, \) and the associated resolvent \( \alpha(L + \alpha)^{-1}, \ \alpha > 0 \) are positivity preserving and even markovian. Positivity preserving means that they map non-negative functions to non-negative functions. In fact, if \( (V, b, c) \) is connected they are even positivity improving, i.e. map non-negative nontrivial functions to
positive functions (see Section 2). Markovian means that they map non-
negative functions bounded by one to non-negative functions bounded by
one. This can be used to show that semigroup and resolvent extend to all
$\ell^p(V, m), 1 \leq p \leq \infty$. These extensions are consistent i.e. two of them agree
on their common domain [6]. The corresponding generators are denoted
by $L_p$, in particular $L = L_2$. We can describe the action of the operator
$L_p$ explicitly in Section 2) as follows (see Theorem 8): Define the formal
Laplacian $\tilde{L} = \tilde{L}_{b,c,m}$ on the vectorspace
\[ \tilde{F} := \{ u : V \rightarrow \mathbb{R} : \sum_y |b(x,y)u(y)| < \infty \text{ for all } x \in V \} \]
by
\[ \tilde{L}u(x) := \frac{1}{m(x)} \sum_y b(x,y)(u(x) - u(y)) + \frac{c(x)}{m(x)} u(x), \]
where, for each $x \in V$, the sum exists by assumption on $u$. Then, $L_p$ is a
restriction of $\tilde{L}$ for any $p \in [1, \infty]$.

After having discussed the fact that these are different semigroups on
different $\ell^p$ spaces, we will now follow the custom and write
$e^{-tL}$ for all of them.

The preceding considerations show that
\[ 0 \leq e^{-tL}1(x) \leq 1 \]
for all $t \geq 0$ and $x \in V$. The question, whether the second inequality is actu-
ally an equality has received quite some attention. In the case of vanishing
killing term, this is discussed under the name of stochastic completeness or
conservativeness. In fact, for $c \equiv 0$ and $b(x,y) \in \{0,1\}$ for all $x,y \in V$,
there is a characterization of stochastic completeness of Wojciechowski [20]
(see [19] for related results as well). This characterization is an analogue to
corresponding results on manifolds of Grigor’yan [11] and results of Sturm
for general strongly local Dirichlet forms [16].

Our first main result concerns a version of this result for arbitrary regular
Dirichlet forms on graphs (see Section 7 for details and proofs concerning
the subsequent discussion): In this case, we have to replace $e^{-tL}1$ by the
function
\[ M_t(x) := e^{-tL}1(x) + \int_0^t e^{-sL}c(x)ds, x \in V. \]
This is well defined, satisfies $0 \leq M \leq 1$ and for each $x \in V$, the function
t $\mapsto M_t(x)$ is continuous and even differentiable. Note that for $c \equiv 0$, we
obtain $M = e^{-tL}1$ whereas for $c \neq 0$ we obtain $M_t > e^{-tL}1$ on any connected
component of $V$ on which $c$ does not vanish identically (as the semigroup is
positivity improving). The term $e^{-tL}1$ can be interpreted as the amount of
heat contained in the graph at time $t$ and the integral can be interpreted as
the amount of heat killed within the graph up to the time $t$. Thus, $1 - M_t$
is the amount of heat transported to the boundary of the graph by the time
$t$ and $M_t$ can be interpreted as the amount of heat, which has not been
transported to the boundary of the graph at time $t$. Our question then
becomes whether the quantity
\[ 1 - M_t \]
vanishes identically or not. Our result then reads.

**Theorem 1.** (Characterization of heat transfer to the boundary) Let \((V, b, c)\) be a weighted graph and \(m\) a measure on \(V\) of full support. Then, for any \(\alpha > 0\), the function 
\[
w := \int_0^\infty \alpha e^{-t\alpha}(1 - M_t)dt
\]
satisfies \(0 \leq w \leq 1\), solves \((\tilde{L} + \alpha)w = 0\), and is the largest non-negative \(l \leq 1\) with \((\tilde{L} + \alpha)l \leq 0\). In particular, the following assertions are equivalent:

(i) For any \(\alpha > 0\) there exists a nontrivial, non-negative bounded \(l\) with \((\tilde{L} + \alpha)l \leq 0\).

(ii) For any \(\alpha > 0\) there exists a nontrivial bounded \(l\) with \((\tilde{L} + \alpha)l = 0\).

(iii) For any \(\alpha > 0\) there exists a nontrivial, non-negative bounded \(l\) with \((\tilde{L} + \alpha)l = 0\).

(iv) The function \(w\) is nontrivial

(v) \(M_t(x) < 1\) for some \(x \in V\) and some \(t > 0\).

(vi) There exists a nontrivial bounded non-negative \(N : V \times [0, \infty) \rightarrow [0, \infty)\) satisfying \(\tilde{L}N + \frac{d}{dt}N = 0\) and \(N_0 \equiv 0\).

**Remark.** Conditions (ii) and (iii) deal with eigenvalues of \(\tilde{L}\) considered as an operator on \(\ell^\infty(V)\). In particular, (ii) must fail (for sufficiently large \(\alpha\)) whenever \(\tilde{L}\) gives rise to a bounded operator on \(\ell^\infty(V)\).

This theorem suggests the following definition for stochastic completeness and stochastic incompleteness in the general case of Dirichlet forms on graphs.

**Definition 1.1.** The symmetric weighted graph \((V, b, c)\) with the measure \(m\) of full support is said to satisfy (SI) if for all \(\alpha > 0\), there exists a positive bounded solution \(v\) to \((\tilde{L} + \alpha)v \leq 0\). The symmetric weighted graph \((V, b, c)\) is said to satisfy (SC) if it does not satisfy (SI).

**Remark.** Note that validity of (SI) depends on both \((V, b, c)\) and \(m\). In fact, for given \((V, b, c)\) it is always possible to choose \(m\) in such a way that \(\tilde{L}\) becomes a bounded operator on \(\ell^\infty(V, m)\). Then, (SC) holds (by the previous remark).

A subgraph \((W, b_W, c_W)\) of a weighted graph \((V, b, c)\) is given by a subset \(W\) of \(V\) and the restriction \(b_W\) of \(b\) to \(W \times W\) and the restriction \(c_W\) of \(c\) to \(W\). The graph \((V, b, c)\) is then called a supergraph to \((W, b_W, c_W)\). Given a measure \(m\) on \(V\) we denote its restriction to \(W\) by \(m_W\). The subgraph \((W, b_W, c_W)\) then gives rise to a form on \(\ell^2(W, m_W)\) with associated operator \(L_{b_W, c_W, m_W}\).

The following result seems to be new even in the setting considered in \([19, 20]\).

**Theorem 2.** Any weighted graph is the subgraph of a graph satisfying (SC). This supergraph can be chosen to have vanishing killing term if the original graph has vanishing killing term.
Remark. Note that (in the common definitions) the volume growth of balls in a graph dominates the volume growth of balls in any of its subgraph. Thus, the theorem has the consequence that failure of (SC) cannot be inferred from lower bounds on the growth of volumes of ball.

While subgraphs do not force incompleteness according to the previous theorem, suitably adjusted subgraphs do force incompleteness on the whole graph. In order to be more precise, we need some more notation.

Let \((V, b, c)\) be a symmetric graph with measure \(m\) of full support and \(W\) a subset of \(V\). Let \(m_W\) be the restriction of \(m\) to \(W\). Let \(i_W: \ell^2(W, m_W) \rightarrow \ell^2(V, m)\) be the canonical embedding, i.e. \(i_W(u)\) is the extension of \(u\) to \(V\) by setting \(i_W(u)\) identically zero outside of \(W\). Let \(p_W: \ell^2(V, m) \rightarrow \ell^2(W, m_W)\) be the canonical projection i.e. the adjoint of \(i_W\). Then, \(W\) gives rise to the form \(Q_W^{(D)}\) on \(\ell^2(W, m_W)\) defined by

\[
Q_W^{(D)}(u) = Q(i_W u) = Q_{b_W, c_W}(u) + \sum_{x \in W} d_W(x)u(x)^2.
\]

Here, \(d_W(x) = \sum_{y \in V \setminus W} b(x, y)\) describes the edge deficiency of vertices in \(W\) compared to the same vertex in \(V\). Thus, \(Q_W^{(D)}\) is in fact the form to the weighted graph \((W, b_W^{(D)}, c_W^{(D)})\) with

\[
b_W^{(D)} = b_W \quad \text{and} \quad c_W^{(D)} = c_W + d_W.
\]

In particular, \(Q_W^{(D)}\) is a Dirichlet form. The associated self adjoint operator will be denoted by \(L_W^{(D)}\). It is given by

\[
L_W^{(D)} = p_W L i_W.
\]

This operator is sometimes thought of as a restriction of the original operator to \(W\) with Dirichlet boundary condition. For this reason we include the superscript \(D\) in the notation. Another interpretation (suggested by the above expression for the form) is to think about the graph which arises from the subgraph \(W\) by adding one way edges to a vertex at infinity according to the mentioned edge deficiency. Again, it is not hard to express the action of \(L_W^{(D)}\) explicitely. In fact, the above considerations applied to \((W, b_W^{(D)}, c_W^{(D)}, m_W)\) show that

\[
\tilde{L}_W^{(D)} u = L_W^{(D)} u
\]

for any \(u \in D(L_W^{(D)})\). Here, the formal Dirichlet Laplacian \(\tilde{L}_W^{(D)}\) on \(W\) is given by

\[
\tilde{L}_W^{(D)} u(x) = \frac{1}{m(x)} \left( \sum_{y \in V} b(x, y)u(x) - \sum_{y \in W} b(x, y)u(y) + c(x)u(x) \right)
\]

for \(x \in W\). These considerations give that for a function \(u\) on \(W\) (which is extended by 0 to \(V\)) the equality

\[
\tilde{L}_W^{(D)} u(x) = \tilde{L} u(x)
\]

(2)
holds for any \( x \in W \). This will be used repeatedly in the sequel. Note also that for \( W = V \) we recover the operator on the whole graph i.e. \( \tilde{L}_V^{(D)} = \tilde{L} \) and \( L_V^{(D)} = L \).

The following result seems to be new even in the setting considered in [19, 20].

**Theorem 3.** Let \((b, c)\) be a graph over \( V \) and \( m \) a measure on \( V \) of full support. Then (SI) holds, whenever there exists \( W \subseteq V \) such that the graph \((b_W^{(D)}, c_W^{(D)})\) over the measure space \((W, m_W)\) satisfies (SI).

So far, we have not discussed the precise domains of definition for our operators. In fact, the actual domains have been quite irrelevant for our considerations.

To determine the domains we need a geometric condition saying that any infinite path has infinite measure. More precisely, we define condition \((A)\) as follows:

\[(A)\text{ For any sequence } (x_n) \text{ of pairwise different elements of } V \text{ such that } b(x_n, x_{n+1}) > 0 \text{ for all } n \in \mathbb{N}, \text{ the equality } m(\{x_n : n \in \mathbb{N}\}) = \infty \text{ holds.}\]

Note that \((A)\) is a condition on \((V, m)\) and \((b, c)\) together. If
\[
\inf_{x \in V} m_x > 0
\]
holds, then \((A)\) is satisfied for all graphs \((b, c)\) over \( V \).

Our result reads as follows. We are not aware of an earlier result of this form in this context.

**Theorem 4.** Let \((V, b, c)\) be a weighted graph and \( m \) a measure on \( V \) of full support such that \((A)\) holds. Then, for any \( p \in [1, \infty) \) the operator \( L_p \) is the restriction of \( \tilde{L} \) to
\[
D(L_p) = \{ u \in \ell^p(V, m) : \tilde{L}u \in \ell^p(V, m) \}.
\]

**Remark.** The theory of Jacobi matrices already provides examples showing that without \((A)\) the statement becomes false for \( p = 2 \). This is discussed in Section 4.

The condition \((A)\) does not imply that \( \tilde{L}f \) belongs to \( \ell^2(V, m) \) for all \( f \in C_c(V) \). However, if this is the case, then \((A)\) does imply essential selfadjointness. In this case, \( Q \) is the “maximal” form associated to the graph \((b, c)\). More precisely, the following holds.

**Theorem 5.** Let \( V \) be a set, \( m \) a measure on \( V \) with full support, \((b, c)\) a graph over \( V \) and \( Q \) the associated regular Dirichlet form. Assume \( L C_c(V) \subseteq \ell^2(V, m) \). Then, \( D(L) \) contains \( C_c(V) \). If furthermore \((A)\) holds, then the restriction of \( L \) to \( C_c(V) \) is essentially selfadjoint and the domain of \( L \) is given by
\[
D(L) = \{ u \in \ell^2(V, m) : \tilde{L}u \in \ell^2(V, m) \}
\]
and the associated form \( Q \) satisfies \( Q = Q^{\text{max}} \) i.e.
\[
Q(u) = \frac{1}{2} \sum_{x, y \in V} b(x, y)(u(x) - u(y))^2 + \sum_x c(x)u(x)^2
\]
for all \( u \in \ell^2(V,m) \).

**Remark.** (a) If \( \inf m_x > 0 \) then both (A) and \( \bar{L}C_0(V) \subseteq \ell^2(V,m) \) hold for any graph \((b,c)\) on \(V\). In this case, we recover the corresponding results of [9, 20, 19] on essential selfadjointness, as these works assume \( m \equiv 1 \). (They also have additional restrictions on \( b \) but this is not relevant here).

(b) The statement on the form being the maximal one seems to be new even in the context of [9, 20, 19].

(c) Essential selfadjointness fails if (A) does not hold as can be seen by examples (see Section 4 and previous remark).

### 2. Dirichlet forms on graphs - basic facts

In this section we consider a countable set \( V \) together with a measure \( m \) of full support.

**Lemma 2.1.** Let \( Q \) be a regular Dirichlet form on \((V,m)\). Then, \( C_c(V) \) is contained in \( D(Q) \).

**Proof.** Let \( x \in V \) be arbitrary. Choose \( \varphi \in C_c(V) \) with \( \varphi(x) = 2 \) and \( \varphi(y) = 0 \) for all \( y \neq x \). As \( C_c(V) \cap D(Q) \) is dense in \( C_c(V) \) with respect to the supremum norm, there exists \( \psi \in D(Q) \) with \( \psi(x) > 1 \) and \( |\psi(y)| < 1 \) for all \( y \neq x \). As \( Q \) is a Dirichlet form, \( D(Q) \) is invariant under taking modulus and we can assume that \( \psi \) is non-negative. As \( Q \) is a Dirichlet form, also \( \tilde{\psi} := \psi \wedge 1 \) belongs to \( D(Q) \). (Here, \( \wedge \) denotes the minimum.) As \( D(Q) \) is a vector space it contains \( \psi - \tilde{\psi} \) and this is a (nonzero) multiple of \( \varphi \) by construction. As \( x \in V \) was arbitrary, the statement follows. \( \square \)

**Lemma 2.2.** Let \( Q \) be a regular Dirichlet form on \((V,m)\). Then, there exists a weighted graph \((b,c)\) over \( V \) such that the restriction of \( Q \) to \( C_c(V) \) equals \( Q_{b,c}^{\text{comp}} \).

**Proof.** By the previous lemma, \( C_c(V) \) is contained in \( D(Q) \). Then, for any finite \( K \subseteq V \), the restriction \( Q_K \) of \( Q \) to \( C_c(K) \) is a Dirichlet form as well. By standard results (see e.g. [1]), there exists then \( b_K, c_K \) with \( Q_K = Q_{b_K,c_K}^{\text{comp}} \). For \( K \subseteq K' \) and \( x,y \in K \) it is not hard to see that \( b_K(x,y) = b_{K'}(x,y) \) and \( c_K(x) \geq c_{K'}(x) \). Thus, a simple limiting procedure gives the result. \( \square \)

**Lemma 2.3.** Let \( m \) be a measure on \( V \) of full support. Let \((b,c)\) be a weighted graph over \( V \). Then, \( Q_{b,c,m}^{\text{max}} \) is closed and \( Q_{b,c}^{\text{comp}} \) is closable and its closure \( Q_{b,c} \) is a restriction of \( Q_{b,c,m}^{\text{max}} \).

**Proof.** It suffices to show that \( Q_{b,c,m}^{\text{max}} \) is closed. Thus, it suffices to show lower semicontinuity of \( u \mapsto Q_{b,c,m}^{\text{max}}(u,u) \). This follows easily from Fatous lemma. \( \square \)

**Theorem 6.** The regular Dirichlet forms on \((V,m)\) are exactly given by the forms \( Q_{b,c,m} \) with weighted graphs \((b,c)\) over \( V \).

**Proof.** By the previous lemma and the discussion in Section 1, any \( Q_{b,c,m} \) is a regular Dirichlet form. The converse follows from the previous lemmas. \( \square \)
We now discuss two results on solutions of the associated difference equation. These results will be rather useful for our further considerations. We start with a version of a minimum principle.

**Theorem 7.** (Minimum principle) Let \((V, b, c)\) be a weighted graph and \(m\) a measure on \(V\) of full support. Let \(U \subseteq V\) be given. Assume that the function \(u\) on \(V\) satisfies

- \((\overline{L} + \alpha)u \geq 0\) on \(U\) for some \(\alpha > 0\),
- The negative part \(u^{-}_{U} := u|_{U} \wedge 0\) of the restriction of \(u\) to \(U\) attains its minimum,
- \(u \geq 0\) on \(U^{c}\).

Then, \(u \equiv 0\) or \(u > 0\) on each connected component of \(U\). In particular, \(u \geq 0\).

**Proof.** Without loss of generality we can assume \(U\) is connected. If \(u > 0\) there is nothing left to show. It remains to consider the case that there exists \(x \in U\) with \(u(x) \leq 0\). As the negative part of \(u\) on \(U\) attains its minimum, there exists then \(x_{m} \in U\) with \(u(x_{m}) \leq u(y)\) for all \(y \in U\). As \(u(y) \geq 0\) for \(y \in U^{c}\), we obtain \(u(x_{m}) - u(y) \leq 0\) for all \(y \in V\). By the supersolution assumption we find

\[
0 \leq \sum b(x_{m}, y)(u(x_{m}) - u(y)) + c(x_{m})u(x_{m}) + \alpha u(x_{m}) \leq 0.
\]

As \(b\) and \(c\) are non-negative and \(\alpha > 0\), we find \(0 = u(x_{m})\) and \(u(y) = u(x_{m}) = 0\) for all \(y\) with \(y \sim x_{m}\). As \(U\) is connected, iteration of this argument shows \(u \equiv 0\) on \(U\). \(\square\)

The following lemma will be a key tool in our investigations. Note that its proof is rather simple due to the discreteness of the underlying space.

**Lemma 2.4.** (Monotone convergence of solutions) Let \(\alpha \in \mathbb{R}, f : V \rightarrow \mathbb{R}\) and \(u : V \rightarrow \mathbb{R}\) be given. Let \((u_{n})\) be a sequence of bounded non-negative functions on \(V\) with \(u_{n} \leq u_{n+1}\) for all \(n \in \mathbb{N}\), and \(u_{n} \to u\) pointwise and \((\overline{L} + \alpha)u_{n}(x) \to f(x)\) for any \(x \in V\). Then, \(u\) belongs to the set \(\overline{F}\) given in (1) on which \(\overline{L}\) is defined and the equation \((\overline{L} + \alpha)u = f\) holds.

**Proof.** Without loss of generality we assume \(m \equiv 1\). By assumption

\[
(\overline{L} + \alpha)u_{n}(x) = \sum_{y \in V} b(x, y)(u_{n}(x) - u_{n}(y)) + (c(x) + \alpha)u_{n}(x)
\]

converges to \(f(x)\) for any \(x \in V\). As \(\sum_{y \in V} b(x, y)u_{n}(x)\) converges increasingly to \(u(x)\sum_{y \in V} b(x, y) < \infty\), the assumptions on \(u_{n}\) show that \(\sum_{y \in V} b(x, y)u_{n}(y)\) must converge as well and we easily obtain the statement. \(\square\)

We next discuss some fundamental properties of regular Dirichlet forms. These properties do not depend on the graph setting. They are true for general Dirichlet forms and can, for example, be found in the works [17], [18]. For the convenience of the reader we include short proofs based on the previous minimum principle.
Proposition 2.5. (Domain monotonicity) Let \((V,b,c)\) be a symmetric graph. Let \(K_1 \subseteq V\) be finite and \(K_2 \subset V\) with \(K_1 \subseteq K_2\) be given. Then, for any \(x \in K_1\)

\[(L_{K_1}^{(D)} + \alpha)^{-1} f(x) \leq (L_{K_2}^{(D)} + \alpha)^{-1} f(x)\]

for all \(f \in \ell^2(V,m)\) with \(f \geq 0\) and \(\text{supp } f \subseteq K_1\).

Proof. Consider \(f \in \ell^2(V,m)\) with \(f \geq 0\) and \(\text{supp } f \subset K_1\) and define \(u_i := (L_{K_i}^{(D)} + \alpha)^{-1} f, i = 1, 2\). Extending \(u_i\) by zero we can assume that \(u_i\) are defined on the whole of \(V\). Then,

\[(\tilde{L} + \alpha)u_i = f \text{ on } K_i\]

for \(i = 1, 2\). Therefore, \(w := u_2 - u_1\) satisfies

- \(w = u_2 \geq 0\) on \(K_1^c\).
- The negative part of \(w\) attains its minimum on \(K_1\) (as \(K_1\) is finite).
- \((\tilde{L} + \alpha)w = f - f = 0\) on \(K_1\).

The minimum principle yields \(w \geq 0\) on \(V\). \(\square\)

Regularity is crucial for the proof of the following result.

Proposition 2.6. (Convergence of resolvents/semigroups) Let \((V,b,c)\) be a symmetric graph, \(m\) a measure on \(V\) with full support and \(Q\) the associated regular Dirichlet form. Let \((K_n)\) be an increasing sequence of finite subsets of \(V\) with \(V = \bigcup K_n\). Then, \((L_{K_n}^{(D)} + \alpha)^{-1} f \to (L + \alpha)^{-1} f, n \to \infty\) for any \(f \in \ell^2(K_1,m_{K_1})\). (Here, \((L_{K_n}^{(D)} + \alpha)^{-1} f\) is extended by zero to all of \(V\).) The corresponding statement also holds for the semigroups.

Proof. By general principles it suffices to consider the resolvents. After decomposing \(f\) in positive and negative part, we can restrict attention to \(f \geq 0\). Define \(u_n := (L_{K_n}^{(D)} + \alpha)^{-1} f\). Then, \(u_n \geq 0\). Now, by standard characterization of resolvents (see e.g. Section 1.4 in \([10]\)), \(u_n\) is the unique minimizer of

\[Q_{K_n}(u) + \alpha\|u - \frac{1}{\alpha}f\|^2.\]

By domain monotonicity, the sequence \((u_n(x))\) is monotonously increasing for any \(x \in V\). Moreover, by standard results on Dirichlet forms we have \(u_n \leq \frac{1}{\alpha}\|f\|_\infty\) and \(\|u_n\| \leq \frac{1}{\alpha}\|f\|\). Thus, the sequence \(u_n\) converges pointwise and in \(\ell^2(V,m)\) towards a function \(u \in \ell^2(V,m)\). Let now \(w \in C_c(V)\) be arbitrary. Assume without loss of generality that the support of \(w\) is contained in \(K_1\). Then, \(Q(w) = Q_{K_n}(w)\) for all \(n \in \mathbb{N}\). Closedness of \(Q\),
convergence of the \((u_n)\) and the minimizing property of each \(u_n\) then give
\[
Q(u) + \alpha\|u - \frac{1}{\alpha}f\|^2 \leq \liminf_{n \to \infty} Q(u_n) + \alpha\|u - \frac{1}{\alpha}f\|^2
\]
\[
= \liminf_{n \to \infty} \left( Q(u_n) + \alpha\|u_n - \frac{1}{\alpha}f\|^2 \right)
\]
\[
= \liminf_{n \to \infty} \left( Q_K(u_n) + \alpha\|u_n - \frac{1}{\alpha}f\|^2 \right)
\]
\[
\leq \liminf_{n \to \infty} \left( Q_K(w) + \alpha\|w - \frac{1}{\alpha}f\|^2 \right)
\]
\[
= Q(w) + \alpha\|w - \frac{1}{\alpha}f\|^2.
\]
As \(w \in C_c(V)\) is arbitrary and \(Q\) is regular (!), this implies
\[
Q(u) + \alpha\|u - \frac{1}{\alpha}f\|^2 \leq Q(v) + \alpha\|u - \frac{1}{\alpha}f\|^2
\]
for any \(v \in D(Q)\). Thus, \(u\) is a minimizer of
\[
Q(u) + \alpha\|u - \frac{1}{\alpha}f\|^2.
\]
By characterization of resolvents again, \(u\) must then be equal to \((L + \alpha)^{-1}f\).
\[\square\]

We can use the previous result to connect the operators \(L\) to the formal operator \(\bar{L}\). To do so we need two further results.

**Proposition 2.7.** Let \((V, m)\) be given and \((b, c)\) a graph over \(V\). Let \(K \subset V\) be finite. Then, \(L_K^{(D)}\) is a bounded operator with
\[
L_K^{(D)} f(x) = \frac{1}{m(x)} \left( \sum_{y \in K} b(x, y)(f(x) - f(y)) + \left( \sum_{y \in V \setminus K} b(x, y) + c(x) \right) f(x) \right).
\]
In particular, \(\tilde{L}_K f(x) = L_K^{(D)} f(x)\) for all \(x \in K\), where \(i_K : \ell^2(K, m_K) \to \ell^2(V, m)\) is the canonical embedding by extension by zero.

**Proof.** Every linear operator on the finite dimensional \(\ell^2(K, m_K)\) is bounded. Thus, we can directly read off the operator \(L_K^{(D)}\) from the form \(Q_K^{(D)}\) given by \(Q_K^{(D)}(u) := Q(i_K u)\). This gives the first claim. The last statement follows easily. \[\square\]

**Lemma 2.8.** Let \((V, m)\) be given and \((b, c)\) a graph over \(V\). Let \(p \in [1, \infty)\) be given. For any \(g \in \ell^p(V, m)\), the function \(u := (L_p + \alpha)^{-1}g\) belongs to the set \(\tilde{F}\) given in (\(\Pi\)) on which \(\bar{L}\) is defined and solves \((\bar{L} + \alpha)u = g\).

**Proof.** We first consider the case \(p = 2\). If suffices to consider the case \(f \geq 0\). Choose an increasing sequence \((K_n)\) of finite subsets of \(V\) with \(\bigcup K K_n = V\) and let \(g_n\) be the restriction of \(g\) to \(K_n\). Then, \((g_n)\) converges monotonously increasing to \(g\) in \(\ell^2(V, m)\) and consequently \((L + \alpha)^{-1}g_n\) converges monotonously increasing to \(u\). Thus, by monotone convergence of solutions, we can assume without loss of generality that \(g\) has compact support contained in \(K_1\). By convergence of resolvents, \(u_n := (L_{K_n}^{(D)} + \alpha)^{-1}g\)
then converges increasingly to $u := (L + \alpha)^{-1} g$. Moreover, by the previous proposition $u_n$ satisfies $(\tilde{L} + \alpha) u_n = g$ on $K_n$. Thus, the statement follows, again by monotone convergence of solutions.

We now turn to general $p \in [1, \infty]$. Again, it suffices to consider the case $g \geq 0$. Choose an increasing sequence $(K_n)$ of finite subsets of $V$ with $\bigcup K_n = V$ and let $g_n$ be the restriction of $g$ to $K_n$. Then, $u_n := (L_p + \alpha)^{-1} g_n$ converges to $u$. Moreover, as $g_n$ belongs to $\ell^2(V, m)$ consistency of the resolvents gives $u_n = (L + \alpha)^{-1} g_n$. Now, on the $\ell^2(V, m)$ level we can apply the considerations for $p = 2$ to obtain

$$(\tilde{L} + \alpha) u_n = (\tilde{L} + \alpha)(L + \alpha)^{-1} g_n = g_n.$$ 

Taking monotone limits now yields the statement. □

We can now give the desired information on the generators.

**Theorem 8.** Let $(V, b, c)$ be a weighted graph and $m$ a measure on $V$ of full support. Let $p \in [1, \infty]$ be given. Then, $L_p f = \tilde{L} f$ for any $f \in D(L_p)$.

**Proof.** Let $f \in D(L_p)$ be given. Then, $g := (L_p + \alpha) f$ exists and belongs to $\ell^p(V, m)$. By the previous lemma then $f = (L_p + \alpha)^{-1} g$ solves

$$(\tilde{L} + \alpha) f = g = (L_p + \alpha) f$$

and we infer the statement. □

We also note the following by product of our investigation (see [20, 19, 7] for this result for locally finite graphs).

**Corollary 2.9.** (Positivity improving) Let $(V, b, c)$ be a connected graph and $L$ the associated operator. Then, both the semigroup $e^{-tL}$, $t > 0$, and the resolvent $(L + \alpha)^{-1}$, $\alpha > 0$, are positivity improving (i.e. they map non-negative nontrivial $\ell^2$-functions to strictly positive functions).

**Proof.** By general principles it suffices to consider the resolvent. Let $f \in \ell^2(V, m)$, $f \geq 0$ be given and consider $u := (L + \alpha)^{-1} f$. Then $u \geq 0$ as the resolvent of a Dirichlet form is positivity preserving. If $u$ is not strictly positive, there exists an $x$ with $u(x) = 0$. As $u$ is non-negative, $u$ attains its minimum in $x$. By Lemma 2.8 $u$ satisfies $(\tilde{L} + \alpha) u = f \geq 0$. We can therefore apply the minimum principle (with $U = V$) to obtain that $u \equiv 0$. This implies $f \equiv 0$. □

3. Generators of the semigroups on $\ell^p$ and essential selfadjointness on $\ell^2$

In this section we will consider a symmetric weighted graph $(V, b, c)$ and a measure $m$ on $V$ of full support. We will be concerned with explicitly describing the generators of the semigroups on $\ell^p$ and studying essential selfadjointness of the generator on $\ell^2$. Both issues will be tackled by proving uniqueness of solutions on the corresponding $\ell^p$ spaces. The results of this section are not needed to deal with stochastic completeness.

Recall the geometric assumption introduced in the first section:
(A) For any sequence \((x_n)\) of pairwise different elements of \(V\) such that \(b(x_n, x_{n+1}) > 0\) for all \(n \in \mathbb{N}\), the set \(\{x_n : n \in \mathbb{N}\}\) has infinite \(m\)-measure.

The relevance of (A) comes from the following variant of the minimum principle:

**Proposition 3.1.** Assume (A). Let \(\alpha > 0\), \(p \in [1, \infty)\) and \(u \in \ell^p(V, m)\) with \((\tilde{L} + \alpha)u \geq 0\) be given. Then, \(u \geq 0\).

**Proof.** Assume the contrary. Then, there exists an \(x_0 \in V\) with \(u(x_0) < 0\). By

\[
0 \leq (\tilde{L} + \alpha)u(x_0) = \frac{1}{m(x_0)} \sum_y b(x_0, y)(u(x_0) - u(y)) + \frac{c(x_0)}{m(x_0)}u(x_0) + \alpha u(x_0)
\]

there must exist an \(x_1\) connected to \(x_0\) with \(u(x_1) < u(x_0)\). Continuing in this way we obtain a sequence \((x_n)\) of connected points with \(u(x_n) < u(x_0) < 0\). Combining this with (A) we obtain a contradiction to \(u \in \ell^p(V, m)\).

Let us note the following consequence of the previous minimum principle.

**Lemma 3.2.** (Uniqueness of solutions on \(\ell^p\)) Assume (A). Let \(\alpha > 0\), \(p \in [1, \infty)\) and \(u \in \ell^p(V, m)\) with \((\tilde{L} + \alpha)u = 0\) be given. Then, \(u \equiv 0\).

**Proof.** Both \(u\) and \(-u\) satisfy the assumptions of the previous proposition. Thus, \(u \equiv 0\).

**Remark.** The situation for \(p = \infty\) is substantially more complicated as can be seen by (part (ii) of) our first theorem.

This lemma allows us the describe the generators.

**Proof of Theorem 4.** Define

\[
\tilde{D}_p := \{u \in \ell^p(V, m) : \tilde{L}u \in \ell^p(V, m)\}
\]

By Theorem 8 we already know \(L_p f = \tilde{L} f\) for any \(f \in D(L_p)\). It remains to show \(\tilde{D}_p \subseteq D(L_p)\): Let \(f \in \tilde{D}_p\) be given. Then, \(g := (\tilde{L} + \alpha)f\) belongs to \(\ell^p(V, m)\). Thus, \(u := (L_p + \alpha)^{-1}g\) belongs to \(D(L_p)\). Now, as shown above \(u\) solves \((\tilde{L} + \alpha)u = g\). Moreover, \(f\) also solves this equation. Thus, by the uniqueness of solutions given in Lemma 3.2 we infer \(f = u\) and \(f\) belongs to \(D(L_p)\). This finishes the proof.

We now turn to a study of essential selfadjointness on \(C_c(V)\). Clearly, the question of essential selfadjointness on \(C_c(V)\) only makes sense if \(\tilde{L}C_c(V) \subseteq \ell^2(V, m)\). In this context, we have the following result:

**Proposition 3.3.** Let \((V, m)\) be given and \((b, c)\) a graph over \(V\). Then, the following assertions are equivalent:

(i) \(\tilde{L}C_c(V) \subseteq \ell^2(V, m)\).

(ii) For any \(x \in V\), the function \(V \longrightarrow [0, \infty), y \mapsto b(x, y)/m(y)\) belongs to \(\ell^2(V, m)\).
In this case any \( u \in \ell^2(V, m) \) belongs to the set \( \tilde{F} \) of \( \mathbb{1} \) on which \( \tilde{L} \) is defined and three sums

\[
\sum_{x \in V} u(x)\tilde{L}v(x)m(x), \quad \sum_{x \in V} \tilde{L}u(x)v(x)m(x)
\]

and

\[
\frac{1}{2} \sum_{x, y \in V} b(x, y)u(x) - u(y)(v(x) - v(y)) + \sum_{x \in V} c(x)u(x)v(x)
\]

converge absolutely and agree. for all \( u \in \ell^2(V, m) \) and \( v \in C_c(V) \).

Proof. Without loss of generality we assume \( c \equiv 0 \). For any \( x \in V \) define \( \delta_x : V \rightarrow \mathbb{R} \) by \( \delta_x(y) = 1 \) if \( x = y \) and \( \delta_x(y) = 0 \) if \( x \neq y \).

Obviously, (i) is equivalent to \( \tilde{L}\delta_x \in \ell^2(V, m) \) for all \( x \in V \). This latter condition can easily be seen to be equivalent to (ii). This shows the stated equivalence.

Let \( u \in \ell^2(V, m) \) be given. Then, for any \( x \in V \), Cauchy-Schwarz inequality and (ii) give

\[
(*) \quad \sum_{y \in V} |b(x, y)u(y)| \leq \left( \sum_{y \in V} \frac{b^2(x, y)}{m(y)} \right)^{1/2} \left( \sum_{y \in V} u^2(y)m(y) \right)^{1/2} < \infty.
\]

Thus, \( u \) belongs to \( \tilde{F} \). To show the statement on the sums, it suffices to consider \( u \in \ell^2(V, m) \) and \( v = \delta_z \), for \( z \in V \) arbitrary. In this case, the desired statements can easily be reduced to the question of absolute convergence of

\[
\sum_{x, y \in V} b(x, y)u(x)\delta_z(x) \quad \text{and} \quad \sum_{x, y \in V} b(x, y)u(x)\delta_z(y).
\]

This absolute convergence in turn is shown in \((*)\). \( \square \)

Proof of Theorem 5. Without loss of generality let \( c \equiv 0 \). As \( \tilde{L}C_c(V) \subseteq \ell^2(V, m) \) we can define the minimal operator \( L_{\text{min}} \) to be the restriction of \( \tilde{L} \) to

\[
D(L_{\text{min}}) := C_c(V)
\]

and the maximal operator \( L_{\text{max}} \) to be the restriction of \( \tilde{L} \) to

\[
D(L_{\text{max}}) := \{ u \in \ell^2(V, m) : \tilde{L}u \in \ell^2(V, m) \}.
\]

The previous proposition gives

\[
\langle u, L_{\text{min}}v \rangle = Q_{b,c}^{\text{comp}}(u, v)
\]

for all \( u, v \in C_c(V) \). This extends to give

\[
\langle u, L_{\text{min}}v \rangle = Q_{b,c,m}(u, v)
\]

for all \( u \in D(Q) \) and \( v \in C_c(V) \). Thus, \( L_{\text{min}} \) is a restriction of \( L \) in this case.

Moreover, the previous proposition gives also

\[
\langle u, L_{\text{min}}v \rangle = \sum_{x \in V} \tilde{L}u(x)v(x)m(x)
\]
for all $v \in C_c(V)$ and $u \in \ell^2(V, m)$. Thus,

$$L^*_{\text{min}} = L_{\text{max}}.$$ 

Thus, essential selfadjointness of $L_{\text{min}}$ is equivalent to selfadjointness of $L_{\text{max}}$. This in turn is equivalent to $L = L_{\text{max}}$ (as we have $L \subseteq L_{\text{max}}$ by Theorem 4). As $(A)$ and Theorem 4 yield $D(L) = \{u \in \ell^2(V, m) : Lu \in \ell^2(V, m)\}$, we infer $L = L_{\text{max}}$ and essential selfadjointness of the restriction of $L$ to $C_c(V)$ ($= L_{\text{min}}$) follows.

It remains to show the statement on the form. Let $Q_{\text{max}}$ be the maximal form i.e. $Q_{\text{max}}(u) := \frac{1}{2} \sum_{x, y \in V} b(x, y)(u(x) - u(y))^2 + \sum_{x \in V} c(x)u(x)^2$ for all $u \in \ell^2(V, m)$ and $L_{Q_{\text{max}}}$ the associated operator. Then, another application of the previous proposition shows

$$\sum_{x \in V} \tilde{L}u(x)v(x)m(x)$$

$$= \frac{1}{2} \sum_{x, y \in V} b(x, y)(u(x) - u(y))(v(x) - v(y)) + \sum_{x \in V} c(x)u(x)v(x)$$

$$= (Q_{\text{max}}(u), v) = \langle L_{Q_{\text{max}}}u, v \rangle$$

for all $u \in D(L_{Q_{\text{max}}})$ and $v \in C_c(V)$. This gives that the self-adjoint operator $L_{Q_{\text{max}}}$ associated to $Q_{\text{max}}$ satisfies

$$L_{Q_{\text{max}}}u = \tilde{L}u = Lu$$

for all $u \in D(L_{Q_{\text{max}}})$. Thus,

$$L_{Q_{\text{max}}} \subseteq L.$$ 

As $L$ is selfadjoint, we infer $L_{Q_{\text{max}}} = L$ and the statement on the form follows. \qed

4. SOME COUNTEREXAMPLES

In this section, we first discuss an example showing that without condition $(A)$ Theorem 4 and Theorem 5 fail in general. We then present an example of a non-regular Dirichlet form on a graph.

**Example for failure of Theorem 4 and 5 without assumption (A).**

Let $V \equiv \mathbb{Z}$. Let (at first) every point of $\mathbb{Z}$ have measure 1. Consider the bounded operator

$$\Delta : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z}), \quad (\Delta \varphi)(x) = -\varphi(x - 1) + 2\varphi(x) - \varphi(x + 1).$$ 

It corresponds to the Dirichlet form $Q_{b,0,1}$ with $b(x, y) = 1$ whenever $|x - y| = 1$. A direct calculation shows that the function

$$u : V \rightarrow \mathbb{R}, \quad u(x) := e^{\lambda x}$$
is a positive solution to the equation \((\tilde{\Delta} + \alpha)u = 0\) for \(\alpha = e^{-\lambda} + e^{\lambda} - 2\). Obviously, we have \(\alpha > 0\) for \(\lambda\) large enough. Now let \(w \in \ell^1(\mathbb{Z})\), \(w > 0\) and define the measure

\[
m : V \to (0, \infty), \quad m(x) = \min\{1, \frac{w(x)}{u^2(x)}\}
\]

and the killing term

\[
c : V \to [0, \infty), \quad c(x) := \max\{0, \frac{u^2(x)}{w(x)} - 1\} \alpha m(x).
\]

By construction, then, \(u\) belongs to \(\ell^2(\mathbb{Z}, m)\) and

\[
-\frac{\alpha}{m} + \frac{c}{m} + \alpha \equiv 0.
\]

Let \(\tilde{L}\) be defined by

\[
\tilde{L}v(x) := \frac{1}{m(x)} \sum_{y \in V} b(x, y)(v(x) - v(y)) + \frac{c(x)}{m(x)} v(x)
\]

i.e. in a formal sense \(\tilde{L} = \frac{1}{m}(\tilde{\Delta} + c)\). Then, the restriction \(L_{\text{max}}\) of \(\tilde{L}\) to

\[
\{v \in \ell^2(\mathbb{Z}, m) : \tilde{L}v \in \ell^2(\mathbb{Z}, m)\}
\]

has the eigenvalue \(-\alpha < 0\) since

\[
(L_{\text{max}} + \alpha)u(x) = \left(-\frac{\alpha}{m} + \frac{c}{m} + \alpha\right) u(x) = 0.
\]

Consider now the operator \(L\) associated with the Dirichlet form \(Q_{b,c,m}\) on \(\ell^p(\mathbb{Z}, m)\). Of course, \(L\) is a positive operator and therefore can not have a negative eigenvalue. Moreover, by the results of the previous section, this operator is a restriction of \(\tilde{L}\). This implies that \(u\) can not belong to \(D(L)\) and therefore \(D(L) \neq D(L_{\text{max}})\). Thus, the domain of definition \(D(L)\) is not given by Theorem 4. In this case the restriction of \(\tilde{L}\) to \(C_c(V)\) is not essentially self-adjoint (as the proof of Theorem 5 showed that otherwise \(L = L_{\text{max}}\)).

**Example of a non-regular Dirichlet form on \(V\)**

We consider connected graphs \((V, b, c)\) with \(c \equiv 0\) and \(b(x, y) \in \{0, 1\}\) for all \(x, y \in V\). As discussed by Dodziuk-Kendall [9] (see [8, 13] as well) in the context of isoperimetric inequalities, any such graph with positive Cheeger constant \(\alpha > 0\) has the property that

\[
\frac{1}{2} \sum_{x,y} b(x, y)(\varphi(x) - \varphi(y))^2 \geq \frac{\alpha^2}{2} \sum d(x) \varphi(x)^2
\]

for all \(\varphi \in C_c(V)\), where \(d(x) = \sum_{y \in V} b(x, y)\). Let now such a graph be given. Fix an arbitrary \(x_0 \in V\). Choose a measure \(m\) with support \(V\) and \(m(V) = 1\). Thus, the constant function 1 belongs to \(\ell^2(V, m)\). Define the form \(Q\) by

\[
Q(u) := \frac{1}{2} \sum_{x,y} b(x, y)(u(x) - u(y))^2
\]

for all \(u \in \ell^2(V, m)\) for which \(Q(u)\) is finite. Obviously, \(Q\) is a Dirichlet form and the constant function 1 satisfies \(Q(1) = 0\). Let now \(\varphi_n\) be any
sequence in $C_c(V)$ converging to 1 in $\ell^2(V, m)$. Then, $\varphi_n(x_0)$ converges to 1. In particular,
$$Q(\varphi_n) \geq \frac{\alpha^2}{2}d(x_0)\varphi_n(x_0)^2 \to \frac{\alpha^2}{2}d(x_0) > 0, \ n \to \infty.$$ Thus, $Q(\varphi_n)$ does not converge to 0 = $Q(1)$. Hence, $Q$ is not regular.

5. THE HEAT EQUATION ON $\ell^\infty$

In this section we consider a symmetric graph $(b, c)$ over the measure space $(V, m)$ with associated formal operator $\tilde{L}$.

A function $N : [0, \infty) \times V \to \mathbb{R}$ is called a solution of the heat equation if for each $x \in V$ the function $t \mapsto N_t(x)$ is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$ and for each $t > 0$ the function $N_t$ belongs to the domain of $\tilde{L}$ and the equality
$$\frac{d}{dt}N_t(x) = -\tilde{L}N_t(x)$$ holds for all $t > 0$ and $x \in V$. For a bounded solution $N$ validity of this equation can easily be seen to automatically extend to $t = 0$ i.e. $t \mapsto N_t(x)$ is differentiable on $[0, \infty)$ and $\frac{d}{dt}N_t(x) = -\tilde{L}N_t(x)$ holds for any $t \geq 0$.

The following theorem is essentially a standard result in the theory of semigroups. In the situation of special graphs it has been shown in [20, 19]. For completeness we give a proof in our situation as well.

**Theorem 9.** Let $L$ be a self-adjoint restriction of $\tilde{L}$, which is the generator of a Dirichlet form on $\ell^2(V, m)$. Let $v$ be a bounded function on $V$ and define $N : [0, \infty) \times V \to \mathbb{R}$ by $N_t(x) := e^{-tL}v(x)$. Then, the function $N(x) : [0, \infty) \to \mathbb{R}$, $t \mapsto N_t(x)$, is differentiable and satisfies
$$\frac{d}{dt}N_t(x) = -\tilde{L}N_t(x)$$ for all $x \in V$ and $t \geq 0$.

**Proof.** Without loss of generality we can assume $m \equiv 1$. As $v$ is bounded, continuity of $t \mapsto N_t(x)$ follows from general principles on weak $\ell^1, \ell^\infty$ continuity of the semigroup on $\ell^\infty(V, m)$, see e.g. [6]. It remains to show differentiability and the validity of the equation.

As discussed already, it suffices to consider $t > 0$. After decomposing $v$ into positive and negative part, we can assume without loss of generality that $v$ is non-negative.

Let $(K_n)$ be sequence of finite increasing subsets of $V$ with $\bigcup K_n = V$. Let $v_n$ be the function on $V$ which agrees with $v$ on $K_n$ and equals zero elsewhere. Thus, $v_n \in \ell^2(V, m)$ and we can consider $e^{-tL}v_n$ for any $n \in \mathbb{N}$. For each fixed $x \in V$ the function $t \mapsto e^{-tL}v_n(x)$ converges monotonously to $t \mapsto N_t(x)$ (by definition of the semigroup on $\ell^\infty$). As $t \mapsto N_t(x)$ is continuous, this convergence is even uniformly on compact subintervals of $(0, \infty)$. Moreover, standard $\ell^2$ theory shows that $N_n = e^{-tL}v_n$ satisfies
$$\frac{d}{dt}N_n(x) = -\tilde{L}N_n(x)$$ for all $t > 0$ and $x \in V$. By assumption $L$ is a restriction of $\tilde{L}$ and we infer
$$\frac{d}{dt}N_n(x) = \sum_{y \in V} b(x, y)(N_n(x) - N_n(y)) + c(x)N_n(x)$$
for all $x \in V$ and $t > 0$. This equality together with the uniform convergence on compact intervals in $(0, \infty)$ and the summability of the $b(x, y)$ in $y$ gives uniform convergence of the $\frac{d}{dt} N_t(x)$ on compact intervals. Hence, $t \to N_t(x)$ is differentiable on $(0, \infty)$ and satisfies the desired equation. □

**Lemma 5.1.** Let $N$ be a bounded solution of $\frac{d}{dt} N + \tilde{L} N = 0$, $N_0 \equiv 0$. Then, $v := \int_0^\infty e^{-\alpha t} N_t dt$ solves $(\tilde{L} + \alpha) v = 0$.

**Proof.** This follows by a short calculation: By boundedness of $N$ and $\sum_y b(x, y) < \infty$, we can interchange two limits to obtain

$$
\tilde{L} v(x) = \lim_{T \to \infty} \int_0^T e^{-\alpha t} \tilde{L} N_t(x) dt.
$$

Using that $N$ solves the heat equation and partial integration we find

$$
\tilde{L} v(x) = \lim_{T \to \infty} \int_0^T e^{-\alpha t} (-\frac{d}{dt} N_t(x)) dt
= \lim_{T \to \infty} \left( -e^{-\alpha t} N_t(x) \bigg|_0^T - \int_0^T \alpha e^{-\alpha t} N_t(x) dt \right)
= -\alpha \int_0^\infty e^{-\alpha t} N_t(x) dt
= -\alpha v(x).
$$

Here, we used boundedness of $N$ and $N_0 = 0$ to get rid of the boundary terms after the partial integration. □

**6. Extended semigroups and resolvents**

We are now going to extend the resolvents/semigroups to a larger class of functions. To do so we note that for a function $f$ on $V$ with $f \geq 0$ the functions $g \in C_c(V)$ with $0 \leq g \leq f$ form a net with respect to the natural ordering $g \prec h$ whenever $g \leq h$. Limits along this net will be denoted by $\lim_{g \prec f}$. As the resolvents and semigroups are positivity preserving, for $f \geq 0$, $\alpha > 0$, $t > 0$, we can define the functions $(L + \alpha)^{-1} f : V \to [0, \infty]$ and $e^{-tL} f : V \to [0, \infty]$ by

$$(L + \alpha)^{-1} f(x) := \lim_{g \prec f} (L + \alpha)^{-1} g(x)$$

and

$$e^{-tL} f(x) := \lim_{g \prec f} e^{-tL} g(x).$$

In fact, as we are in a discrete setting, the operators have kernels, i.e. for any $t \geq 0$ there exists a unique function

$$e^{-tL} : V \times V \to [0, \infty) \text{ with } e^{-tL} f(x) = \sum_{y \in V} e^{-tL} (x, y) f(y)$$

for any $f \geq 0$ (and similarly for the resolvent). It is not hard to see that for functions in $\ell^\infty(V)$, these definitions are consistent with our earlier definitions.

**Theorem 10.** (Properties of extended resolvents and semigroups) Let $\alpha > 0$ be given. Let $f$ be a non-negative function on $V$. 

(a) Let $K_n$ be an increasing sequence of finite subsets of $V$ with $\bigcup K_n = V$. Let $f_n$ be the restriction of $f$ to $K_n$, and $u_n := (L_{K_n}^{(D)} + \alpha)^{-1}f_n$. Then, $u_n$ converges pointwise monotonously to $(L + \alpha)^{-1}f$.

(b) The following statement are equivalent:

(i) There exists a non-negative $l : V \rightarrow [0, \infty)$ with $(\bar{L} + \alpha)l \geq f$.
(ii) $(L + \alpha)^{-1}f(x)$ is finite for any $x \in V$.
In this case $u := (L + \alpha)^{-1}f$ is the smallest non-negative function $l$ with $(\bar{L} + \alpha)l \geq f$ and it satisfies $(\bar{L} + \alpha)u = f$.

(c) The equation $(L + \alpha)^{-1}f(x) = \int_0^\infty e^{-t\alpha}e^{-tL}f(x)dt$ holds for any $x \in V$.

Remark. Note that the functions in (a) and (c) are allowed to take the value $\infty$.

Proof. Throughout the proof we let $u$ denote the function $(L + \alpha)^{-1}f$.

(a) Let $x \in V$ be given. By domain monotonicity $u_n(x) = (L_{K_n}^{(D)} + \alpha)^{-1}f_n(x)$ is increasing. Moreover, again by domain monotonicity and $f_n \leq f$ we have

$$u_n(x) = (L_{K_n}^{(D)} + \alpha)^{-1}f_n(x) \leq (L + \alpha)^{-1}f_n(x) \leq (L + \alpha)^{-1}f(x) = u(x)$$

for all $n$. It remains to show the ‘converse’ inequality. We consider two cases.

Case 1. $u(x) < \infty$. Let $\varepsilon > 0$ be given. By definition of the exented resolvents there exists then $g \in C_c(V)$ with $0 \leq g \leq f$ and

$$u(x) - \varepsilon \leq (L + \alpha)^{-1}g.$$

As $g$ has compact support we can assume without loss of generality that the support of $g$ is contained in $K_n$ for all $n$. By convergence of resolvents, we conclude

$$(L + \alpha)^{-1}g(x) - \varepsilon \leq (L_{K_n}^{(D)} + \alpha)^{-1}g(x)$$

for all sufficiently large $n$. Thus, for such $n$ we find

$$u(x) - 2\varepsilon \leq (L_{K_n}^{(D)} + \alpha)^{-1}g(x).$$

By $g \leq f$ and $\text{supp} \ g \subseteq K_1$, we have $g \leq f_n$ for all $n$. Thus, the last inequality gives

$$u(x) - 2\varepsilon \leq (L_{K_n}^{(D)} + \alpha)^{-1}f_n(x) = u_n(x).$$

This finishes the considerations for this case.

Case 2. $u(x) = \infty$. Let $\kappa > 0$ be arbitrary. By definition of the extended resolvents there exists then $g \in C_c(V)$ with $0 \leq g \leq f$ and

$$\kappa \leq (L + \alpha)^{-1}g.$$

Now, we can continue as in Case 1 to obtain

$$\kappa - 2\varepsilon \leq (L_{K_n}^{(D)} + \alpha)^{-1}f_n(x) = u_n(x)$$

for all sufficiently large $n$. As $\kappa > 0$ is arbitrary the statement follows.

(b) We first show (ii) $\Rightarrow$ (i): Recall that $u = (L + \alpha)^{-1}f$ and consider $g \in C_c(V)$ with $0 \leq g \leq f$. Then, $u_g := (L + \alpha)^{-1}g$ solves

$$(\bar{L} + \alpha)u_g = g.$$
Taking monotone limits on both sides and using the finiteness assumption (ii) we obtain 
\[(\bar{L} + \alpha)u = f.\]

This shows (i) (with \(l = u\)).

We next show (i) \(\implies\) (ii): Let \(l \geq 0\) satisfy \((\bar{L} + \alpha)l \geq f\). Let \((K_n)\) be an increasing sequence of finite subsets of \(V\) as in (a) and let \(f_n\) be the restriction of \(f\) to \(K_n\). Extend \(u_n := (L^{(D)}_{K_n} + \alpha)^{-1}f_n\) by zero to all of \(V\).

Then, \(w_n := l - u_n\) satisfies:

- \(w_n = l \geq 0\) on \(K_n^c\).
- The negative part of \(w\) attains its minimum on \(K_n\) (as \(K_n\) is finite).
- \((\bar{L} + \alpha)w_n = (\bar{L} + \alpha)l - (\bar{L} + \alpha)u_n \geq f - f = 0\) on \(K_n\).

The minimum principle then gives \(w_n = l - u_n \geq 0\).

As \(n\) is arbitrary and \(u_n\) converges to \(u\) by part (a), we find that \(u \leq l\) is finite. This finishes the proof of the equivalence statement of (b). The last statement of (b) has already been shown in the proof of (i) \(\implies\) (ii).

(c) For \(g \in C_c(V)\) with \(0 \leq g \leq f\) the equation
\[(L + \alpha)^{-1}g = \int_0^\infty e^{-t\alpha}e^{-tL}g(x)dt\]
holds by standard theory on semigroups. Now, (c) follows by taking monotone limits on both sides. \(\square\)

There is a special function \(v\) to which our considerations can be applied:

**Proposition 6.1.** For any \(\alpha > 0\) the estimate
\[0 \leq (L + \alpha)^{-1}(\alpha 1 + c) \leq 1\]
holds.

**Remark.** Let us stress that \(c\) is not assumed to be bounded.

**Proof.** As \(\alpha 1 + c \geq 0\), we have \(0 \leq (L + \alpha)^{-1}(\alpha 1 + c)\). Moreover, we obviously have
\[(\bar{L} + \alpha)1 = \alpha 1 + c.\]

Thus, (b) of the previous theorem shows \((L + \alpha)^{-1}(\alpha 1 + c) \leq 1.\) \(\square\)

We will also need the following consequence of the proposition.

**Proposition 6.2.** Let \((V, b, c)\) be a symmetric graph and define \(S : V \rightarrow [0, \infty]\) by
\[S(x) := \int_0^\infty e^{-sL}c(x)ds.\]

Then, \(S\) satisfies \(0 \leq S \leq 1\) and \(\bar{L}S = c.\)

**Proof.** For \(g \in C_c(V)\) with \(0 \leq g \leq c\) and \(\alpha > 0\) we define \(S_{g,\alpha}\) by \(S_{g,\alpha} = \int_0^\infty e^{-\alpha s}e^{-sL}g(x)ds\) and \(S_g\) by \(S_g = \lim_{\alpha \to 0} S_{g,\alpha}\). Then,
\[S_{g,\alpha} = (L + \alpha)^{-1}g\ \text{i.e.}\ (L + \alpha)S_{g,\alpha} = g.\]
By \( g \leq \alpha 1 + c \) for any \( \alpha > 0 \) and Proposition 6.1 we have

\[
S_{g,\alpha} = (L + \alpha)^{-1}g(x) \leq (L + \alpha)^{-1}(\alpha 1 + c)(x) \leq 1.
\]

As \( S_g \) is the monotone limit of the \( S_{g,\alpha} \) this shows that \( S_g \) is bounded by 1. Moreover, using the uniform bound on the \( S_{g,\alpha} \) and taking the limit \( \alpha \to 0 \) in

\[
(\tilde{L} + \alpha)S_{g,\alpha} = g
\]

we find

\[
\tilde{L}S_g = g \geq 0.
\]

As \( S = \lim_{\alpha \to 0} S_{g,\alpha} \) and the \( S_{g,\alpha} \) are uniformly bounded, we obtain the statement. \( \Box \)

**Lemma 6.3.** Let \( u \geq 0 \) be given. Then, the following assertions are equivalent:

(i) \( e^{-tL}u \leq u \) for all \( t > 0 \).

(ii) \( (L + \alpha)^{-1}u \leq \frac{1}{\alpha}u \) for all \( \alpha > 0 \).

Any \( u \geq 0 \) with \( \tilde{L}u \geq 0 \) satisfies these equivalent conditions.

**Proof.** The implication (i) \( \implies \) (ii) follows easily from \( (L + \alpha)^{-1} = \int e^{-t\alpha}e^{-tL}dt \).

Similarly, the implication (ii) \( \implies \) (i) follows from the standard

\[
e^{-tL}u = \lim_{n \to \infty} \left( \frac{t}{n} \left( L + \frac{n}{t} \right) \right)^{-n} u.
\]

As for the last statement we note that \( \tilde{L}u \geq 0 \) implies

\[
\frac{1}{\alpha}(\tilde{L} + \alpha)u \geq u.
\]

By (b) of Theorem 10 the desired statement (ii) follows. \( \Box \)

### 7. Characterisation of stochastic completeness

In this section, we can finally characterize stochastic completeness. We begin by introducing the crucial quantity in our studies.

**Lemma 7.1.** Let \( (V, b, c) \) be a symmetric graph and \( m \) a measure on \( V \) with full support. Then, the function \( M : [0, \infty) \times V \to [0, \infty] \) defined by

\[
M_t(x) := e^{-tL}1(x) + \int_0^t e^{-sL}c(x)ds
\]

satisfies \( 0 \leq M_s \leq M_t \leq 1 \) for all \( s \geq t \geq 0 \) and, for each \( x \in V \), the map \( t \mapsto M_t(x) \) is differentiable and satisfies \( \frac{d}{dt}M_t(x) + \tilde{L}M_t = c(x) \).

**Remark.** We can give an interpretation of \( M \) in terms of a diffusion process on \( V \) as follows. For \( x \in V \), let \( \delta_x \) be the characteristic function of \( \{x\} \). A diffusion on \( V \) starting in \( x \) with normalized measure is then given by \( \frac{1}{m(x)}\delta_x \) at time \( t = 0 \). It will yield to the amount of heat

\[
\langle e^{-tL} \frac{\delta_x}{m(x)}, 1 \rangle = \langle \frac{\delta_x}{m(x)}, e^{-tL}1 \rangle = \sum_{y \in V} e^{-tL}(x, y)
\]

within \( V \) at the time \( t \). Moreover, at each time \( s \) the rate of heat killed at \( y \) by the killing term \( c \) is given by \( c(y) e^{-sL}(x, y) \). The total amount of heat
killed at $y$ till the time $t$ is then given by $\int_0^t c(y)e^{-sL}(x,y)ds$. The total amount of heat killed at all vertices by $c$ till the time $t$ is accordingly given by

$$\sum_{y \in V} \int_0^t c(y)e^{-sL}(x,y)ds = \int_0^t \sum_{y \in V} e^{-sL}(x,y)c(y)ds = \int_0^t (e^{-sL}c)(x)ds.$$  

This means that $M$ measures the amount of heat at time $t$ which has not been transferred to the boundary of $V$.

**Proof.** By definition we have $M \geq 0$. By Proposition 6.2 $S(x) = \int_0^\infty e^{-sL}c(x)ds$ is finite and we can therefore calculate

$$\int_0^t e^{-sL}c(x)ds = S(x) - \int_t^\infty e^{-sL}c(x)ds = S(x) - e^{-tL}S(x),$$

where the last statement follows by taking monotone limits along the net of $g \in C_c(V)$ with $0 \leq g \leq c$. Thus,

$$M_t = e^{-tL}1 + S - e^{-tL}S = S + e^{-tL}(1 - S).$$

From this equality the desired statements follow easily: By Proposition 6.2 we have $1 - S \geq 0$ and $\tilde{L}(1 - S) = \tilde{L}1 - \tilde{L}S = c - c = 0$. Lemma 6.3 then yields

$$e^{-sL}(1 - S) \leq e^{-tL}(1 - S) \leq 1 - S$$

for all $s \geq t \geq 0$. Plugging this into the formula for $M_t$ gives

$$0 \leq M_s \leq M_t \leq 1$$

for all $0 \leq t \leq s$.

Moreover, as $S$ and the constant function 1 are bounded we can apply Theorem 9 to $M_t = e^{-tL}(1 - S) + S$ to infer that $t \mapsto M_t(x)$ is differentiable with

$$\frac{d}{dt}M_t(x) = -\tilde{L}e^{-tL}1(x) + \tilde{L}e^{-tL}S(x) = -\tilde{L}M_t(x) + \tilde{L}S(x) = -\tilde{L}M_t(x) + c(x),$$

where we used $\tilde{L}S = c$.  \hfill $\square$

We now show that integration over $M$ yields a resolvent.

**Lemma 7.2.** $(L + \alpha)^{-1}(\alpha 1 + c)(x) = \int_0^\infty \alpha e^{-\alpha t}M_t(x)dt$.

**Proof.** As shown in (c) of Theorem 10 we have

$$(L + \alpha)^{-1}(\alpha 1 + c)(x) = \int_0^\infty \alpha e^{-\alpha t}e^{-tL}(1 + \frac{c}{\alpha})(x)dt.$$  

Thus, it suffices to show that

$$\int_0^\infty e^{-\alpha t}e^{-tL}c(x)dt = \int_0^\infty \alpha e^{-\alpha t}\left(\int_0^t e^{-sL}c(x)ds\right)dt.$$  

This follows by partial integration applied to each (non-negative) term of the sum

$$e^{-tL}c(x) = \sum_y e^{-tL}(x,y)c(y).$$  

This finishes the proof. \hfill $\square$
Remark. Let us stress that the care taken with monotone convergence in the above arguments is quite necessary. For example one might think that 1 = (L + α)⁻¹(L + α) 1. Combined with the previous lemma, this would lead to 1 = (L + α)⁻¹(α1 + c) = \int_0^\infty \alpha e^{-t \alpha} Mdt. However, the phenomenon we describe is exactly that the integral can be strictly smaller than 1!

After these preparations we can now prove our first main result.

Proof of Theorem[2]. As \int_0^\infty \alpha e^{-t \alpha} dt = 1, the previous lemma gives w = 1 − (L + α)⁻¹(α1 + c). Thus, w solves (\tilde{L} + α)w = 0. Moreover, the minimality properties of the extended resolvent yield the maximality property of w. More precisely, let l be any nonnegative function bounded by 1 with (\tilde{L} + α)l ≥ 0. Then, 1 − l is nonnegative and satisfies

(\tilde{L} + α)(1 − l) = c + α1 − (\tilde{L} + α)l ≥ c + α1.

The minimality property of 1 − w = (L + α)⁻¹(α1 + c) then gives 1 − w ≤ 1 − l, and the desired inequality l ≤ w follows.

It remains to show the equivalence statements.

(v)⇒(iv): This is clear as 0 ≤ M_l ≤ 1 and M is continuous.

(iv)⇒(iii): This is clear.

(iii)⇒(ii): This is clear.

(ii)⇒(i): Let l⁺ be the positive part of l, i.e. l⁺(x) = l(x) if l(x) > 0 and l⁺(x) = 0 otherwise. If l⁺ is trivial, the function −l is a nontrivial solution and (i) follows. Otherwise a direct calculation shows that l⁺ is a nontrivial subsolution. Obviously, l⁺ is non-negative. By the maximality property of w, we infer nontriviality of w.

(i)⇒(v): If there exists a nontrivial non-negative subsolution, then w as the largest subsolution must be nontrivial. Hence, there must exist t > 0 and x ∈ V with M_l(x) < 1.

(v)⇒(vi): Lemma [7,14] gives that N := 1 − M satisfies N_0 = 0 and \frac{d}{dt}N = LN = 0. This gives the desired implication.

(vi)⇒(i): This is a direct consequence of Lemma [5,11].

8. Stochastically complete graphs with incomplete subgraphs

In this section we prove Theorem[2] The basic idea is to attach stochastic complete graphs to each vertex of a possibly stochastically incomplete graph such that the resulting graph will be stochastically complete.

The graphs we attach will be the following. Let (N, b_N, 0) the symmetric weighted graph with vertex set N = \{0, 1, 2, \ldots\}, b_N(x, y) = 1 if |x − y| = 1 and b_N(x, y) = 0 otherwise and c = 0. Moreover let the measure m on N be constant m ≡ 1. The next lemma shows that when u solves (\tilde{L}_N + α)u(x) = 0 for some α and all x ∈ N \ {0} then it is only bounded if it is exponentially decreasing.

Lemma 8.1. For (N, b_N, 0), m ≡ 1, let u be a positive solution to the equation (\tilde{L}_N + α)u(x) = 0 for α > 0 and x ≥ 1. If for some x ≥ 1

u(x) ≥ \frac{2}{2 + α}u(x − 1)
then $u$ increases exponentially.

**Proof.** Let $u$ be a positive solution. If $(1 + \frac{\alpha}{2})u(x) \geq u(x-1)$ for some $x \geq 1$ we get by the equation $(\bar{L} + \alpha)u(x) = 0$

\[
0 = (1 + \frac{\alpha}{2})u(x) - u(x+1) + (1 + \frac{\alpha}{2})u(x) - u(x-1)
\]
\[
\geq (1 + \frac{\alpha}{2})u(x) - u(x+1).
\]

This implies $u(x+1) \geq (1 + \frac{\alpha}{2})u(x)$. By induction we get $u(y) \geq (1 + \frac{\alpha}{2})u(y-1)$ for $y \geq x$ and thus

\[
u(y) \geq (1 + \frac{\alpha}{2})^{y-x}u(x)
\]

which gives the statement. \hfill \square

**Proof of Theorem 2** Let $(W, b_W, c_W)$ be a symmetric weighted graph and $m$ a measure of full support on $W$. If (SC) holds we are done, so assume the contrary. We will construct a stochastically complete graph $(V, c)$ such that $W \subseteq V$ and $b|_{W \times W} = b_W$. Define

\[
\deg_{b_W}(x) = \frac{1}{m(x)} \sum_{y \in W} b(x, y).
\]

Let $d : W \to (0, \infty)$ be a function which satisfies $d(x) \deg_{b_W}(x)m(x) \in \mathbb{N}$ and

\[
\sum_{j=1}^{\infty} d(x_j) = \infty
\]

for any sequence $(x_j)$ in $V$. (For example we can set $d(x) = \lceil \deg_{b_W}(x)m(x) \rceil$, where $\lceil x \rceil$ denotes the smallest integer not exceeding $x$.)

To each vertex $x \in W$ we associate $d(x) \deg_{b_W}(x)m(x)$ copies of the graph $(N, b_N)$ defined in the beginning of the section. We do this by identifying $x \in W$ with the vertices 0 in the associated copies of $N$. We denote the resulting graph by $V$ and define $b$ on $V \times V$ by letting

\[
b(x, y) = \begin{cases} 
b_W(x, y) & x, y \in W, \\
b_N(x, y) & x, y \text{ in the same copy of } N, \\
0 & \text{otherwise.}
\end{cases}
\]

Moreover we extend $c$ and $m$ to $V$ by letting $c = 0$ and $m = 1$ on $V \setminus W$ and denote $\bar{L} = \bar{L}_V$. We will show now that for all $\alpha > 0$ every positive function $u$ on $V$ which satisfies $(\bar{L} + \alpha)u = 0$ is unbounded. Let $u$ be such a positive solution of $(\bar{L} + \alpha)u = 0$ and assume it is bounded.

Fix an arbitrary $x_0 \in W$ and an arbitrary sequence $(\rho_r)$ in $\mathbb{R}$ with $(2 + \alpha)/2 > \rho_r > 1$ und $\sum (\rho_r - 1) < \infty$. By induction we can now define for each $r \in \mathbb{N}$ an $x_r \in V$ such that $b(x_r, x_{r-1}) > 0$ and $\rho_r u(x_{r+1}) \geq \sup_{y \in V, b(x_r, y) > 0} u(y)$. Since we assumed $u$ bounded, Lemma 8.1 gives $u(y) < 2/(2 + \alpha)u(x_r)$ for each vertex $y$ in a copy of $N$ which is adjacent to $x_r$. 


Thus $x_r$ belongs to $W$. The equation $(\tilde{\mathcal{L}} + \alpha)u(x_r) = 0$ now gives

$$0 = \frac{1}{m(x_r)} \sum_{y \in V} b(x_r, y)(u(x_r) - u(y)) + \frac{c(x_r)}{m(x_r)}u(x_r) + \alpha u(x_r)$$

$$\geq \deg_{b_W}(x)u(x_r) + \frac{1}{m(x_r)} \left( \sum_{y \in V \setminus W} b(x_r, y)(u(x_r) - u(y)) - \sum_{y \in W} b(x_r, y)u(y) \right)$$

$$\geq \left( 1 + \frac{\alpha d(x_r)}{2 + \alpha} \right) \deg_{b_W}(x)u(x_r) - \rho r \deg_{b_W}(x)u(x_{r+1}).$$

In the second inequality we used $\alpha, c(x_r), u(x_r) \geq 0$. In the third inequality we estimated the sum over $y \in V \setminus W$ by the inequality of Lemma 8.1 and the sum over $y \in W$ by our choice of $x_{r+1}$. We get by direct calculation and iteration

$$u(x_{r+1}) \geq \frac{1}{\rho r} \left( \frac{\alpha d(x_r)}{2 + \alpha} + 1 \right) u(x_r) \geq \left( \prod_{j=1}^{r} \frac{1}{\rho_j} \right) \left( \prod_{j=1}^{r} \left( \frac{\alpha d(x_j)}{2 + \alpha} + 1 \right) \right) u(x_0).$$

Letting $r$ tend to infinity the product on the right hand side diverges if and only if $d$ is chosen such that $\sum_{j=1}^{\infty} d(x_j)$ is not converging. (Notice that the infinite product over $(1/\rho_j)$ is greater than zero since we assumed $(\rho_j - 1)$ summable). Thus by our choice of $d$ we arrive at the contradiction that $u$ is unbounded. This construction shows that for every graph $(W, b_W, c_W)$ there is a graph $(V, b, c)$ which is stochastically complete and $(W, b_W, c_W)$ is a subgraph of $(V, b, c)$.

\[\square\]

**Remark.** An alternative construction is to add single vertices instead of copies of $N$. For the resulting graph and a function $u$ satisfying $(\tilde{\mathcal{L}} + \alpha)u = 0$ the value of $u$ on an added vertex $y$ adjacent to the vertex $x$ in the original graph is then determined by $(1 + \alpha)u(y) = u(x)$. The rest of the proof can now be carried out in a similar manner. We chose to do the construction above to avoid the impression that the stochastic completeness is the result of adding some type of boundary to the graph.

9. **An incompleteness criterion**

In this section we prove Theorem 3 which is the counterpart to Theorem 2. We have seen there it is not sufficient for a graph to contain a stochastically incomplete subgraph to be stochastically incomplete. Theorem 3 shows under which additional condition on a subgraph its stochastic incompleteness implies stochastic incompleteness of the whole graph. This condition is about how heavy the incomplete subgraph is connected with the rest of the graph. Not having control over the amount of connections leads possibly to stochastic completeness as we have seen in Theorem 3.

For a subset $W$ of a symmetric weighted graph $(V, b, c)$ we define the outer boundary $\partial W$ of $W$ in $V$ by

$$\partial W = \{ x \in V \setminus W : \exists y \in W, b(x, y) > 0 \}.$$ 

Note that the outer boundary of $W$ is a subset of $V \setminus W$. We will be concerned with decompositions of the whole set $V$ into two sets $W$ and
Let \( W' := V \setminus W \). In this case there are two outer boundaries. Our intention is to extend positive bounded functions \( u \) on \( W \) with \((\tilde{L}_W^{(D)} + \alpha)u \leq 0\) to positive bounded functions \( v \) on the whole space satisfying \((\tilde{L} + \alpha)v \leq 0\). To do so we will have to take particular care on what happens on the two boundaries.

**Lemma 9.1.** Let \((V, b, c)\) be a connected weighted symmetric graph. Let \( W \subseteq V \) be non-empty. Then, any connected component of \( W' = V \setminus W \) contains a point \( x \in \partial W \).

**Proof.** Choose \( x \in W \) be arbitrary. By assumption, any \( y \in W' \) is connected to \( x \) by a path in \( V \) i.e. there exist \( x_0, x_1, \ldots, x_n \in V \) with \( b(x_i, x_{i+1}) > 0 \) and \( x_0 = x, \ x_n = y \). Let \( m \in \{0, \ldots, n\} \) be the largest number with \( x_m \in W \). Then, \( x_{m+1} \) belongs to both the boundary of \( W \) and the connected component of \( y \).

**Lemma 9.2.** Let \((V, b, c)\) be a symmetric weighted graph and \( m \) a measure of full support. Let \( U \subseteq V \) be given. Let \( \varphi \) be a non-negative function in \( \ell^2(U, m) \). Then, \((L_U^{(D)} + \alpha)^{-1}\varphi \) is non-negative on \( U \) and positive on the connected components of any \( x \in U \) with \( \varphi(x) > 0 \).

**Proof.** The operator \( L_U^{(D)} \) is associated to the weighted graph \((U, b_U^{(D)}, c_U^{(D)})\). Hence Corollary 2.2 gives the statement.

**Lemma 9.3.** Let \((V, b, c)\) be a weighted symmetric graph and \( m \) a measure of full support. Let \( U \subseteq V \) be arbitrary. Let \( u \) be a function in the formal domain \( \tilde{L} \) of \( \tilde{L} \) and denote the restriction of \( u \) to \( U \) by \( v \) and the restriction of \( u \) to \( V \setminus U \) by \( v' \). Then, for any \( x \in U \)

\[
(\tilde{L} + \alpha)u(x) = (\tilde{L}_U^{(D)} + \alpha)v(x) - \frac{1}{m(x)} \sum_{y \in V \setminus U} b(x, y) v'(y).
\]

**Proof.** This follows by direct calculation.

**Proof. of Theorem 3** Let \( W \subseteq V \) be given such that for every \( \alpha > 0 \) there is a bounded positive function \( u \) on \( W \) satisfying

\[(\tilde{L}_W^{(D)} + \alpha)u \leq 0.\]

It suffices to show that any such \( u \) can be extended to a positive and bounded function \( v \) on \( V \) such that

\[(\tilde{L} + \alpha)v \leq 0.\]

To do so we proceed as follows: Set \( W' = V \setminus W \). Define

\[\psi : W' \to \mathbb{R}, \quad \psi(x) = \frac{1}{m(x)} \sum_{y \in W} b(x, y) u(y).\]

Thus, \( \psi \) vanishes on \( W' \setminus \partial W \) and is positive on \( \partial W \). Now, choose \( \varphi \in \ell^2(W') \) with \( 0 \leq \varphi \leq \psi \) and \( \varphi(x) \neq 0 \) whenever \( \psi(x) \neq 0 \). Thus,

\[\varphi > 0 \text{ on } \partial W \text{ and } \varphi \equiv 0 \text{ on } W' \setminus \partial W.\]

Define \( u' \) on \( W' \) by

\[u' := (L_{W'}^{(D)} + \alpha)^{-1}\varphi.\]
As $\varphi$ is positive on $\partial W$, combining Lemma 9.1 and Lemma 9.2 shows that $u'$ is positive (on $W'$). Now, define $v$ on $V$ by setting $v$ equal to $u$ on $W$ and setting $v$ equal to $u'$ on $W'$. We now investigate the value of

$$(\tilde{L} + \alpha)v(x).$$

We consider four cases.

Case 1: $x \in W \setminus \partial W'$. Then, $$(\tilde{L} + \alpha)v(x) = (\tilde{L}_W^{(D)} + \alpha)u(x) \leq 0$$ by assumption on $u$.

Case 2: $x \in W' \setminus \partial W$. Then, $$(\tilde{L} + \alpha)v(x) = (\tilde{L}_W^{(D)} + \alpha)u'(x) = \varphi(x) = 0$$ by construction of $u'$.

Case 3: $x \in \partial W'$. Lemma 9.3 with $U = W$ gives

$$(\tilde{L} + \alpha)v(x) = (L_W^{(D)} + \alpha)u(x) - \sum_{y \in W'} b(x, y)u'(y) \leq 0.$$ 

Here, the last inequality follows as $(L_W^{(D)} + \alpha)u(x) \leq 0$ by assumption on $u$ and $u'$ is positive.

Case 4: $x \in \partial W$. Lemma 9.3 with $U = W'$ gives

$$(\tilde{L} + \alpha)v(x) = (L_W^{(D)} + \alpha)u'(x) - \sum_{y \in W} b(x, y)u(y) = \varphi(x) - \psi(x) \leq 0.$$ 

This finishes the proof. $\Box$

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