EXISTENCE AND MULTIPLICITY RESULTS FOR SECOND-ORDER DISCONTINUOUS PROBLEMS VIA NON-ORDERED LOWER AND UPPER SOLUTIONS

Rubén Figueroa, Rodrigo López Pouso and Jorge Rodríguez–López*

Departamento de Estatística, Análise Matemática e Optimizació
Instituto de Matemáticas
Universidade de Santiago de Compostela
15782, Facultade de Matemáticas, Campus Vida, Santiago, Spain

Abstract. We present existence and multiplicity principles for second–order discontinuous problems with nonlinear functional conditions. They are based on the method of lower and upper solutions and a recent extension of the Leray–Schauder topological degree to a class of discontinuous operators.

1. Introduction and preliminaries. Consider the second–order problem with nonlinear functional boundary conditions

\[ \begin{align*}
    x'' &= f(t, x, x'), \quad t \in I = [a, b], \\
    0 &= L_1(x(a), x(b), x'(a), x'(b), x), \\
    0 &= L_2(x(a), x(b)),
\end{align*} \tag{1} \]

where \( L_1 : \mathbb{R}^4 \times C(I) \to \mathbb{R} \) is continuous and it is nonincreasing in the third and in the fifth variables, and nondecreasing in the fourth one; and \( L_2 : \mathbb{R}^2 \to \mathbb{R} \) is a continuous function and it is nondecreasing with respect to its first argument. The typical example with this type of boundary conditions is the periodic problem.

We prove existence of Carathéodory solutions for (1), i.e. solutions in \( W^{2,1}(I) \). We assume that we have lower and upper solutions, either well–ordered or not, and the nonlinear part of the differential equation can be discontinuous in the second argument. The method employed involves a new degree theory for a class of discontinuous operators (see [6]).

This work was inspired by the papers [14] devoted to the existence of solutions for \( \phi \)-Laplacian and [11, 12, 13] where the second–order periodic problem with impulses was studied. The main novelty here is that we allow \( f \) to be discontinuous in the second variable over some curves which we call admissible discontinuity curves and they were presented for the first time for second–order equations in the paper of Pouso [9].

2010 Mathematics Subject Classification. Primary: 34A36, 34B15; Secondary: 47H11.

Key words and phrases. Discontinuous differential equations, upper and lower solutions, degree theory.

Rodrigo López Pouso was partially supported by Ministerio de Economía y Competitividad, Spain, and FEDER, Project MTM2016-75140-P, and Xunta de Galicia ED341D R2016/022 and GRC2015/004. Jorge Rodríguez-López was financially supported by Xunta de Galicia Scholarship ED481A-2017/178.

* Corresponding author.
In addition, following the ideas of [5, 16] we achieve multiplicity results for problem (1) in presence of more than a pair of lower and upper solutions.

Now we present some preliminaries concerning the new degree theory for a class of discontinuous operators that we need for our results (for further details see [6]).

**Definition 1.1.** Let \( X \) be a normed space and let \( \Omega \subset X \) be nonempty and open.

The closed-convex envelope of an operator \( T : \overline{\Omega} \to X \) is the multivalued mapping \( T : \overline{\Omega} \to 2^X \) given by

\[
T x = \bigcap_{\varepsilon > 0} \overline{B}_\varepsilon(x) \cap \overline{\Omega} \quad \text{for every} \ x \in \overline{\Omega},
\]

where \( \overline{B}_\varepsilon(x) \) denotes the closed ball centered at \( x \) and radius \( \varepsilon \), and \( \overline{\Omega} \) means closed convex hull.

**Remark 1.** The closed-convex envelope (cc-envelope, for short) is similar to the Krasovskij regularization (see [8]), but here we are ‘convexifying’ operators instead of nonlinear parts of differential equations.

**Definition 1.2.** Let \( \Omega \) be an open subset of a Banach space \( X \) and \( T : \overline{\Omega} \to X \) be such that \( T \overline{\Omega} \) is relatively compact, \( Tx \neq x \) for every \( x \in \partial \Omega \), and

\[
\{x\} \cap Tx \subset \{Tx\} \quad \text{for every} \ x \in \overline{\Omega} \cap \overline{T \Omega},
\]

where \( \overline{\Omega} \) is the cc-envelope of \( T \).

We define the degree of \( I - T \) on \( \Omega \) as follows:

\[
\deg(I - T, \Omega) = \deg(I - T, \Omega),
\]

where the degree in the right-hand side is that of usc multivalued operators (see, e.g. [3, 17]).

**Remark 2.** Notice that condition (3) is nothing but \( \text{Fix}(T) \subset \text{Fix}(T) \), where \( \text{Fix}(S) \) denotes the set of fixed points of operator \( S \).

**Proposition 1.** Let \( T : \overline{\Omega} \to X \) be a mapping in the conditions of Definition 1.2. Then the degree \( \deg(I - T, \Omega) \) satisfies the following properties:

1. **(Additivity)** Let \( \Omega_1, \Omega_2 \subset \Omega \) be open and such that \( \Omega_1 \cup \Omega_2 = \Omega \) and \( \Omega_1 \cap \Omega_2 = \emptyset \).

   If \( 0 \notin (I - T)(\overline{\Omega \setminus (\Omega_1 \cup \Omega_2)}) \), then we have

   \[
   \deg(I - T, \Omega) = \deg(I - T, \Omega_1) + \deg(I - T, \Omega_2).
   \]

2. **(Excision)** Let \( A \subset \Omega \) be a closed set such that \( 0 \notin (I - T)(\partial \Omega) \cup (I - T)(A) \).

   Then

   \[
   \deg(I - T, \Omega) = \deg(I - T, \Omega \setminus A).
   \]

3. **(Existence)** If \( \deg(I - T, \Omega) \neq 0 \) then there exists \( x \in \Omega \) such that \( Tx = x \).

4. **(Normalization)** \( \deg(I, \Omega) = 1 \) if and only if \( 0 \in \Omega \).

**Theorem 1.3.** Let \( H : \overline{\Omega} \times [0, 1] \to X \) be a map satisfying the following conditions:

(a) for each \((x, t) \in \overline{\Omega} \times [0, 1] \) and all \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon, x, t) > 0 \) such that

\[
\text{for } s \in [0, 1], \ |t - s| < \delta \implies \|H(z, t) - H(z, s)\| < \varepsilon \quad \forall z \in \overline{B}_\delta(x) \cap \overline{\Omega};
\]

(b) \( H(\overline{\Omega} \times [0, 1]) \) is relatively compact;

(c) \( \{x\} \cap \overline{H_i(x)} \subset \{H_i(x)\} \) is satisfied for all \( x \in \overline{\Omega} \cap \overline{\Omega} \) when \( t = 0 \) and \( t = 1 \).

If \( x \notin \overline{H}(x, t) \) for all \((x, t) \in \partial \Omega \times [0, 1] \), then

\[
\deg(I - H_0, \Omega) = \deg(I - H_1, \Omega).
\]
2. Well–ordered lower and upper solutions. Following [2] we define lower and upper solutions for problem (1), which we will combine with degree theory methods, in the spirit of [14].

Definition 2.1. We say that a function \( \alpha \in W^{2,1}(I) \) is a lower solution of problem (1) if
\[
\alpha''(t) \geq f(t, \alpha(t), \alpha'(t)) \quad \text{for a.e. } t \in I, \\
L_1(\alpha(a), \alpha(b), \alpha'(a), \alpha'(b), \alpha) \leq 0, \\
L_2(\alpha(a), \alpha(b)) = 0, \quad L_2(\alpha(\cdot), \cdot) \text{ is injective.}
\]

Similarly \( \beta \in W^{2,1}(I) \) is an upper solution for (1) if it satisfies the inequalities in the reverse order.

Now we present a Nagumo condition which provides an a priori bound on the first derivative of all possible solutions between the lower and upper solutions. We omit the standard proof, see [4].

Proposition 2. Let \( \bar{\alpha}, \bar{\beta} \in C(I) \) be such that \( \bar{\alpha} \leq \bar{\beta} \) and define
\[
r = \max \{ \bar{\beta}(b) - \bar{\alpha}(a), \bar{\beta}(a) - \bar{\alpha}(b) \} / (b - a).
\]

Assume there exist a continuous function \( \bar{N} : [0, \infty) \rightarrow (0, \infty) \), \( \bar{M} \in L^1(I) \) and \( R > r \) such that
\[
\int_r^R \frac{1}{\bar{N}(s)} ds > \| \bar{M} \|_{L^1}.
\]

Define \( E := \{ (t, x, y) \in I \times \mathbb{R}^2 : \bar{\alpha}(t) \leq x \leq \bar{\beta}(t) \} \). Then, for every function \( f : E \rightarrow \mathbb{R} \) such that for every \( x, y \in \mathbb{R} \), the mapping \( t \in I \mapsto f(t, x, y) \) is measurable and for a.e. \( t \in I \) and all \( (x, y) \in \mathbb{R}^2 \) with \( (t, x, y) \in E \),
\[
|f(t, x, y)| \leq \bar{M}(t) \bar{N}(|y|),
\]
and for every solution \( x \) of \( x'' = f(t, x, x') \) such that \( \bar{\alpha} \leq x \leq \bar{\beta} \) on \( I \), we have
\[
\|x'\|_{\infty} < R.
\]

The function \( f(t, x, x') \) in the ODE of (1) could be discontinuous in the second argument over admissible discontinuity curves, according to the following definition.

Definition 2.2. An admissible discontinuity curve for the differential equation \( x''(t) = f(t, x, x') \) is a \( W^{2,1} \) function \( \gamma : [c, d] \subset I \rightarrow \mathbb{R} \) satisfying one of the following conditions:

either \( \gamma''(t) = f(t, \gamma(t), \gamma'(t)) \) for a.a. \( t \in [c, d] \) (and we then say that \( \gamma \) is viable for the differential equation),

or there exist \( \varepsilon > 0 \) and \( \psi \in L^1(c, d) \), \( \psi(t) > 0 \) for a.a. \( t \in [c, d] \), such that either
\[
\gamma''(t) + \psi(t) < f(t, y, z) \quad \text{for a.a. } t \in [c, d], \text{ all } y \in [\gamma(t) - \varepsilon, \gamma(t) + \varepsilon], \quad \text{and all } z \in [\gamma'(t) - \varepsilon, \gamma'(t) + \varepsilon], \quad \text{or}
\]
\[
\gamma''(t) - \psi(t) > f(t, y, z) \quad \text{for a.a. } t \in [c, d], \text{ all } y \in [\gamma(t) - \varepsilon, \gamma(t) + \varepsilon], \quad \text{and all } z \in [\gamma'(t) - \varepsilon, \gamma'(t) + \varepsilon].
\]

We say that the admissible discontinuity curve \( \gamma \) is inviable for the differential equation if it satisfies (4) or (5).
Consider the Banach space $X = C^1(I)$ endowed with its usual norm
\[
\|x\|_{C^1} = \|x\|_\infty + \|x'\|_\infty = \max_{t \in I} |x(t)| + \max_{t \in I} |x'(t)|.
\]

Assume that there exist $\alpha, \beta \in W^{2,1}(I)$ lower and upper solutions to (1) well-ordered, i.e. $\alpha < \beta$ on $I$, and that for $f : I \times \mathbb{R}^2 \to \mathbb{R}$ the following conditions hold:

(C1) Compositions $t \mapsto f(t, x(t), y(t))$ are measurable whenever $x(t)$ and $y(t)$ are measurable;

(C2) There exist a continuous function $M : [0, \infty) \to (0, \infty)$, $M \in L^1(I)$ and $R > r$ such that
\[
 \int_r^R \frac{1}{N(s)} ds > \|M\|_{L^1},
\]
and for a.a. $t \in I$, all $x \in [\alpha(t), \beta(t)]$ and all $y \in \mathbb{R}$, we have
\[
|f(t, x, y)| \leq M(t)N(|y|);
\]

(C3) There exist admissible discontinuity curves $\gamma_n : I_n = [a_n, b_n] \to \mathbb{R}$ $(n \in \mathbb{N})$ such that $\alpha \leq \gamma_n \leq \beta$ on $I_n$ and their derivatives are uniformly bounded, and for a.a. $t \in I$ the function $(x, y) \mapsto f(t, x, y)$ is continuous on
\[
\left(\{\alpha(t), \beta(t)\} \setminus \bigcup_{n \in I_n} \{\gamma_n(t)\}\right) \times \mathbb{R}.
\]

The idea is to transform the problem (1) with nonlinear boundary conditions into a Dirichlet BVP in the line of [2], or [14] where the periodic problem was studied. Hence we consider the following equivalent BVP

\[
\begin{aligned}
x''(t) &= f(t, x(t), x'(t)), \quad t \in I, \\
x(a) &= x(a) - L_1(x(a), x(b), x'(a), x'(b), x), \\
x(b) &= x(b) + L_2(x(a), x(b)),
\end{aligned}
\]

whose associated fixed point operator, $T : C^1(I) \to C^1(I)$, is given by
\[
T x(t) = x(a) - L_1(x) \\
+ \frac{t-a}{b-a} \left[ x(b) - x(a) + L_1(x) + L_2(x) - \int_a^b \int_a^s f(r, x(r), x'(r)) \, dr \, ds \right] \\
+ \int_a^t \int_a^s f(r, x(r), x'(r)) \, dr \, ds,
\]

where $L_1(x)$ and $L_2(x)$ denote, respectively, the functions $L_1(x(a), x(b), x'(a), x'(b), x)$ and $L_2(x(a), x(b))$.

For each $l > 0$ we define the open set
\[
\Omega_l = \{x \in C^1(I) : \alpha < x < \beta \text{ on } I \text{ and } ||x'|| < l\}.
\]

**Theorem 2.3.** Assume that there exist $\alpha, \beta \in W^{2,1}(I)$ lower and upper solutions to (1) well-ordered, i.e. $\alpha < \beta$ on $I$, and that for $f : I \times \mathbb{R}^2 \to \mathbb{R}$ conditions (C1)–(C3) hold.

Let $R$ be as in (C2) and such that $R \geq \max\{\|\alpha\|_\infty, \|\beta\|_\infty, \|\gamma_n\|_\infty\}$ for all $n \in \mathbb{N}$. 

Then for all values of $l \geq R$ we have
\[ \deg (I - T, \Omega_t) = 1 \] provided that $Tx \neq x$ for $x \in \partial \Omega_t$.

In particular, problem (1) has at least one solution $x$ such that $\alpha \leq x \leq \beta$.

**Proof.** Let $l \geq R$, define $\Omega := \Omega_t$, and assume that
\[ Tx \neq x \quad \text{for every } x \in \partial \Omega. \] (8)

Now consider the modified problem
\[
\begin{cases}
  x'' = f(t, \varphi(t, x(t)), \delta((\varphi(t, x(t))))'), & \text{for a.a. } t \in I, \\
  x(a) = \varphi(a, x(a) - L_1(x)), \\
  x(b) = \varphi(b, x(b) + L_2(x)),
\end{cases}
\] (9)

where
\[ \varphi(t, x) = \max \{ \min \{ x, \beta(t) \}, \alpha(t) \} \quad \text{for } (t, x) \in I \times \mathbb{R}, \] (10)

and
\[ \delta(y) = \max \{ \min \{ y, l \}, -l \} \quad \text{for all } y \in \mathbb{R}. \] (11)

Denote by $\tilde{T}$ the integral operator associated to the modified problem (9). For all $x \in C^1(I)$ and a.a. $t \in I$, we have
\[ |f(t, \varphi(t, x(t)), \delta((\varphi(t, x(t))))')| \leq \tilde{M}(t) := \max_{s \in [0,1]} \{ N(s) \} M(t), \]

so there exists $R_0 > 0$ such that $\Omega \subset B_{R_0/2}(0)$ and $\tilde{T}x \in B_{R_0/2}(0)$ for all $x \in C^1(I)$. In particular, $\|x\| < R_0/2$ for every $\lambda \in [0,1]$ and $x \in C^1(I)$ such that $x = \lambda \tilde{T}x$. Hence $x \notin \lambda \tilde{T}x$ if $\|x\| = R_0$.

Now we define the homotopy $H : \overline{B}_{R_0}(0) \times [0,1] \to \overline{B}_{R_0}(0)$ given by $H(x, \lambda) = \lambda \tilde{T}x$. By virtue of Theorem 1.3 we have that
\[ \deg \left( I - \tilde{T}, B_{R_0}(0) \right) = \deg (I, B_{R_0}(0)) = 1, \] (12)

provided that $\{x\} \cap \overline{B}_x \subset \left\{ \tilde{T}x \right\}$ for all $x \in \overline{B}_{R_0}(0) \cap \overline{T\overline{B}}_{R_0}(0)$.

Observe that $\overline{B}_{R_0}(0) \cap \overline{T\overline{B}}_{R_0}(0) \subset K$, where
\[ K = \left\{ x \in X : \quad \begin{array}{l}
  \alpha(a) \leq x(a) \leq \beta(a), \quad \alpha(b) \leq x(b) \leq \beta(b), \\
  |x'(t) - x'(s)| \leq \int_s^t \tilde{M}(r) \, dr \quad (a \leq s \leq t \leq b)
\end{array} \right\}, \]

and for every function $x \in K$ we can prove that $\{x\} \cap \overline{T}x \subset \left\{ \tilde{T}x \right\}$ just by following the steps of the proof of [9, Theorem 4.4], and it is exactly the same proof we did in [7, Theorem 2.8]. For the convenience of the reader we recall its main ideas. Thus, we fix $x \in K$ and consider the following three cases.

**Case 1.** $m(\{t \in I_n : x(t) = \gamma_n(t)\}) = 0$ for all $n \in \mathbb{N}$. Then $T$ is continuous at $x$.

The assumption implies that for a.a. $t \in I$ the mapping $f(t, \cdot, \cdot)$ is continuous at $(\varphi(t, x(t)), (\varphi(t, x(t)))')$. Hence if $x_k \to x$ in $K$, then
\[ f(t, \varphi(t, x_k(t)), \delta((\varphi(t, x_k(t))))') \to f(t, \varphi(t, x(t)), \delta((\varphi(t, x(t))))') \quad \text{for a.a. } t \in I. \]

Moreover,
\[ |f(t, \varphi(t, x(t)), \delta((\varphi(t, x(t))))')| \leq \tilde{M}(t) \] (13)
for a.a. \( t \in I \), hence \( T x_k \to Tx \) in \( C^1(I) \).

**Case 2.** \( m(\{ t \in I_n : x(t) = \gamma_n(t) \}) > 0 \) for some \( n \in \mathbb{N} \) such that \( \gamma_n \) is inviable. In this case we can prove that \( x \notin \mathbb{T}x \).

First, we fix some notation. Let us assume that for some \( n \in \mathbb{N} \) we have \( m(\{ t \in I_n : x(t) = \gamma_n(t) \}) > 0 \) and there exist \( \varepsilon > 0 \) and \( \psi \in L^1(I_n) \), \( \psi(t) > 0 \) for a.a. \( t \in I_n \), such that (5) holds with \( \gamma \) replaced by \( \gamma_n \). (The proof is similar if we assume (4) instead of (5), so we omit it.)

We denote \( J = \{ t \in I_n : x(t) = \gamma_n(t) \} \), and we observe that \( m(\{ t \in J : \gamma_n(t) = \beta(t) \}) = 0 \). Indeed, if \( m(\{ t \in J : \gamma_n(t) = \beta(t) \}) > 0 \), then from (5) it follows that \( \beta''(t) - \psi(t) > f(t, \beta(t), \beta'(t)) \) on a set of positive measure, which is a contradiction with the definition of upper solution.

Now we distinguish between two sub-cases.

**Case 2.1.** \( m(\{ t \in J : x(t) = \gamma_n(t) = \alpha(t) \}) > 0 \). Since \( m(\{ t \in J : \gamma_n(t) = \beta(t) \}) = 0 \), we deduce that \( m(\{ t \in J : x(t) = \alpha(t) \neq \beta(t) \}) > 0 \), so there exists \( n_0 \in \mathbb{N} \) such that

\[
m\left( \left\{ t \in J : x(t) = \alpha(t), \ x(t) < \beta(t) - \frac{1}{n_0} \right\} \right) > 0.
\]

We denote \( A = \{ t \in J : x(t) = \alpha(t), \ x(t) < \beta(t) - 1/n_0 \} \) and technical results on measure theory (see, [9, Lemma 4.1 and Corollary 4.2]) ensure that there exist subsets \( J_1 \subset J_0 \subset A \) such that for a point \( \tau_0 \in J_1 \) fixed there exist \( t_- < \tau_0 \) and \( t_+ > \tau_0 \), \( t_\pm \) sufficiently close to \( \tau_0 \) so that the following inequalities are satisfied:

\[
2 \int_{[\tau_0, t_+] \cap A} M(s) \, ds < \frac{1}{4} \int_{\tau_0}^{t_+} \psi(s) \, ds,
\]

(14)

\[
\int_{[\tau_0, t_+] \cap A} \psi(s) \, ds \geq \int_{[\tau_0, t_+] \cap J_0} \psi(s) \, ds > \frac{1}{2} \int_{\tau_0}^{t_+} \psi(s) \, ds,
\]

(15)

\[
2 \int_{[t_-, \tau_0] \cap A} M(s) \, ds < \frac{1}{4} \int_{t_-}^{\tau_0} \psi(s) \, ds,
\]

(16)

\[
\int_{[t_-, \tau_0] \cap A} \psi(s) \, ds > \frac{1}{2} \int_{t_-}^{\tau_0} \psi(s) \, ds.
\]

(17)

Finally, we define a positive number

\[
\rho = \min \left\{ \frac{1}{4} \int_{t_-}^{\tau_0} \psi(s) \, ds, \frac{1}{4} \int_{\tau_0}^{t_+} \psi(s) \, ds \right\},
\]

(18)

and we are now in a position to prove that \( x \notin \mathbb{T}x \). It is sufficient to prove the following claim:

**Claim.** Let \( \varepsilon > 0 \) be defined as \( \varepsilon = \min\{\varepsilon, 1/n_0\} \), where \( \varepsilon \) is given by our assumptions over \( \gamma_n \) and \( 1/n_0 \) by the definition of the set \( A \), and let \( \rho \) be as in (18). For every finite family \( x_i \in \mathcal{B}(x) \cap K \) and \( \lambda_i \in [0, 1] \) \( (i = 1, 2, \ldots, m) \), with \( \sum \lambda_i = 1 \), we have \( \|x - \sum \lambda_i T x_i\|_{C^1} \geq \rho \).

Let \( x_i \) and \( \lambda_i \) be as in the Claim and, for simplicity, denote \( y = \sum \lambda_i T x_i \). For a.a. \( t \in J = \{ t \in I_n : x(t) = \gamma_n(t) \} \) we have

\[
y''(t) = \sum_{i=1}^{m} \lambda_i (T x_i)''(t) = \sum_{i=1}^{m} \lambda_i f(t, \varphi(t, x_i(t)), \delta((\varphi(t, x_i(t)))')).
\]

(19)
On the other hand, for every $i \in \{1, 2, \ldots, m\}$ and for a.a. $t \in J$ we have

$$|x_i(t) - \gamma_n(t)| + |x_i'(t) - \gamma_n'(t)| = |x_i(t) - x(t)| + |x_i'(t) - x'(t)| < \varepsilon. \tag{20}$$

Since $\gamma_n(t) \in [\alpha(t), \beta(t)]$, for a.a. $t \in A$ we have $|\varphi(t, x(t)) - \gamma_n(t)| \leq |x(t) - \gamma_n(t)|$, and $|((\varphi(t, x(t)))') - \gamma_n'(t)| \leq |x_i'(t) - \gamma_n'(t)|$ because if $x_i(t) < \alpha(t)$, then $(\varphi(t, x(t)))' = \alpha'(t) = \gamma_n'(t)$.

Hence, from (5) it follows that

$$\gamma_n'(t) - \psi(t) > f(t, \varphi(t, x_i(t)), (\varphi(t, x_i(t)))')$$

for a.a. $t \in A$ and for all $x_i(t)$ satisfying (20).

Moreover, since for a.a. $t \in A$ we have $|\gamma_n'(t)| < R$ and $|x_i'(t) - \gamma_n'(t)| < \varepsilon$, without loss of generality we can suppose $|(\varphi(t, x_i(t)))'| \leq R$ and thus

$$\gamma_n''(t) - \psi(t) > f(t, \varphi(t, x_i(t)), \delta((\varphi(t, x_i(t)))')) \tag{21}$$

for a.a. $t \in A$.

Therefore the assumptions on $\gamma_n$ ensure that for a.a. $t \in A$ we have

$$y''(t) = \sum_{i=1}^{m} \lambda_i f(t, \varphi(t, x_i(t)), \delta((\varphi(t, x_i(t)))')) < \sum_{i=1}^{m} \lambda_i (\gamma_n''(t) - \psi(t)) = x''(t) - \psi(t). \tag{22}$$

Now we compute

$$y'(t_0) - y'(t_-) = \int_{t_-}^{t_0} y''(s) \, ds = \int_{[t_- , t_0] \cap A} y''(s) \, ds + \int_{[t_- , t_0] \setminus A} y''(s) \, ds$$

$$< \int_{[t_- , t_0] \cap A} x''(s) \, ds - \int_{[t_- , t_0] \setminus A} \psi(s) \, ds$$

$$+ \int_{[t_- , t_0] \setminus A} \tilde{M}(s) \, ds \quad \text{(by (22), (19) and (13))}$$

$$= x'(t_0) - x'(t_-) - \int_{[t_- , t_0] \cap A} x''(s) \, ds - \int_{[t_- , t_0] \setminus A} \psi(s) \, ds$$

$$+ \int_{[t_- , t_0] \setminus A} \tilde{M}(s) \, ds$$

$$\leq x'(t_0) - x'(t_-) - \int_{[t_- , t_0] \cap A} \psi(s) \, ds + 2 \int_{[t_- , t_0] \setminus A} \tilde{M}(s) \, ds$$

$$< x'(t_0) - x'(t_-) - \frac{1}{4} \int_{t_-}^{t_0} \psi(s) \, ds \quad \text{(by (16) and (17))},$$

hence $\|x - y\|_{C^1} \geq y'(t_0) - x'(t_-) \leq \rho$ provided that $y'(t_0) \geq x'(t_0)$.

Similar computations with $t_+$ instead of $t_-$ show that if $y'(t_0) \leq x'(t_0)$ then we also have $\|x - y\|_{C^1} \geq \rho$. The claim is proven.

**Case 2.2.** $m\{t \in J : \gamma_n(t) \in (\alpha(t), \beta(t))\} > 0$. It follows the ideas of the previous one, but working on one set $A_{n_0} = \{t \in J : \alpha(t) + 1/n_0 < x(t) < \beta(t) - 1/n_0\}$ for some $n_0 \in \mathbb{N}$ such that $m(A_{n_0}) > 0$.

**Case 3.** $m\{t \in I_n : x(t) = \gamma_n(t)\} > 0$ only for some of those $n \in \mathbb{N}$ such that $\gamma_n$ is viable. In this case the relation $x \in TX$ implies $x = TX$.

Let us consider the subsequence of all viable admissible discontinuity curves in the conditions of Case 3, which we denote again by $\{\gamma_n\}_{n \in \mathbb{N}}$ to avoid overloading.
notation. We have $m(J_n) > 0$ for all $n \in \mathbb{N}$, where

$$J_n = \{ t \in I_n : x(t) = \gamma_n(t) \}.$$ 

For each $n \in \mathbb{N}$ and for a.a. $t \in J_n$ we have $\gamma_n''(t) = f(t, \gamma_n(t), \gamma_n'(t))$ and from $\alpha \leq \gamma_n \leq \beta$ and $|\gamma_n'(t)| < R$ it follows that $\gamma_n''(t) = f(t, \varphi(t, \gamma_n(t)), \delta((\varphi(t, \gamma_n(t)))'))$.

Then for a.a. $t \in J_n$ we have

$$x''(t) = \gamma_n''(t) = f(t, \varphi(t, \gamma_n(t)), \delta((\varphi(t, \gamma_n(t)))')) = f(t, \varphi(t, x(t)), \delta((\varphi(t, x(t)))')),$$

and therefore

$$x''(t) = f(t, \varphi(t, x(t)), \delta((\varphi(t, x(t)))')) \quad \text{a.e. in } J = \bigcup_{n \in \mathbb{N}} J_n. \quad (23)$$

On the other hand, $x \in T \Omega$ implies that $x''(t) = f(t, \varphi(t, x(t)), \delta((\varphi(t, x(t)))'))$ a.e. in $I \setminus J$, by the continuity of $f$ at $x(t)$, thus showing that $x = T \Omega$.

So far, we have shown that the operator $T$ satisfies condition (3) for all $x \in K$. Thus (12) is justified.

Let

$$\tilde{\Omega} = \{ x \in \Omega : \alpha(a) < x(a) - L_1(x) < \beta(a), \alpha(b) < x(b) + L_2(x) < \beta(b) \}.$$ 

We shall see that $\tilde{T}x = x$ implies that $x \in \tilde{\Omega}$.

If $x$ is a fixed point of $T$, then we have that:

(a) $\alpha(t) \leq x(t) \leq \beta(t)$ for all $t \in I$;
(b) $\|x'\|_\infty < R \leq L$, as a consequence of Proposition 2;
(c) $\alpha(a) \leq x(a) - L_1(x) \leq \beta(a)$ and $\alpha(b) \leq x(b) + L_2(x) \leq \beta(b)$;

the details of the previous claims can be seen in [2, Lemmas 3.3–3.5] or [7, proof of Theorem 2.8].

Therefore $x \in \Omega$. Notice that $\tilde{T} = T$ on $\Omega$, so from (8) it follows that $x \in \tilde{\Omega}$.

Finally by the excision property of the degree (see Proposition 1) we can conclude that

$$\deg(I - T, \Omega) = \deg(I - T, \tilde{\Omega}) = \deg(I - \tilde{T}, \tilde{\Omega}) = \deg(I - \tilde{T}, B_{R_0}(0)) = 1,$$

and the proof is over. \hfill \Box

**Remark 3.** The existence of a solution for (1), even in the non-strict case $\alpha \leq \beta$, can be proven by applying an extension of Schauder’s fixed point theorem in the line of [9, Theorem 3.1] (see [7]). Nevertheless, the fact that the degree is equal to 1 gives more information and it will be crucial in the next sections.

**Remark 4.** Notice that for the existence of solutions of (1) condition (C3) can be relaxed as follows: for a.a. $t \in I$, the mapping $(x, y) \mapsto f(t, x, y)$ is continuous on $\left(\left[\alpha(t), \beta(t)\right] \setminus \bigcup_{n \in I_n} \{ \gamma_n(t) \}\right) \times \left[-M, M\right]$, where $M = \max_{s \in [0, R]} \| N(s) \|_1$. Moreover, it is possible to define some admissible discontinuity curves for the derivative, see [7].

Observe that the derivatives of the admissible discontinuity curves need not be uniformly bounded as in condition (C3) if a global $L^1$ bound is considered for the function $f$. This is due to that in this case the truncation $\delta$ is not necessary when the modified problem is constructed what simplifies the proof of condition (3), see Case 2 in the proof of Theorem 2.3. The following result is a direct consequence of this fact.
Theorem 2.4. Assume that there exist $\alpha, \beta \in W^{2,1}(I)$ lower and upper solutions to (1) well-ordered and that $f : I \times \mathbb{R}^2 \to \mathbb{R}$ satisfies conditions (C1) and

(C2) There exists $M \in L^1(I)$ such that for a.a. $t \in I$, all $x \in [\alpha(t), \beta(t)]$ and $y \in \mathbb{R}$, we have $|f(t, x, y)| \leq M(t)$;

(C3) There exist admissible discontinuity curves $\gamma_n : I_n = [a_n, b_n] \to \mathbb{R}$ ($n \in \mathbb{N}$) such that $\alpha \leq \gamma_n \leq \beta$ on $I_n$, and for a.a. $t \in I$ the function $(x, y) \mapsto f(t, x, y)$ is continuous on

$$\left([\alpha(t), \beta(t)] \setminus \bigcup_{n:t \in I_n} \{\gamma_n(t)\}\right) \times \mathbb{R}.$$

Let $R$ be as given by Proposition 2 and such that $R \geq \max \{\|\alpha'\|_{\infty}, \|\beta'\|_{\infty}\}$. Then for all values of $l \geq R$ we have

$$\deg(I - T, \Omega_I) = 1 \quad \text{provided that } Tx \neq x \text{ for } x \in \partial \Omega_I.$$  

In particular, problem (1) has at least one solution $x$ such that $\alpha \leq x \leq \beta$.

3. Non-ordered lower and upper solutions. In this section we establish the main result of this paper when the lower and upper solutions are not well-ordered. To do so, we follow the ideas of [14], where the authors consider the particular case of (1) with a Carathéodory function $f$ and periodic boundary conditions.

Here and henceforth, we shall be concerned with the following problem:

$$x'' = f(t, x, x'), \quad t \in I = [a, b],$$

$$0 = L_1(x(a), x(b), x'(a), x'(b), x);$$

$$x(a) = x(b),$$

where $L_1$ is as in the previous section. Notice that $L_2(x(a), x(b)) = x(a) - x(b)$ satisfies the conditions in the introduction and, as a result, (24) is a particular case of (1).

Let us start with a technical result.

Lemma 3.1. Let $\overline{\alpha}, \overline{\beta} \in \mathcal{C}(I)$ and $x \in \mathcal{C}(I)$ such that

$$x(t_x) < \overline{\alpha}(t_x) \text{ and } x(s_x) > \overline{\beta}(s_x) \quad \text{for some } t_x, s_x \in I.$$  

Then there exist $\tau_x \in I$ such that

$$\min \left\{\overline{\alpha}(\tau_x), \overline{\beta}(\tau_x)\right\} \leq x(\tau_x) \leq \max \left\{\overline{\alpha}(\tau_x), \overline{\beta}(\tau_x)\right\}. $$

Proof. See [14, Lemma 3.1].

We are ready for our main result.

Theorem 3.2. Assume that there exist $\alpha, \beta \in W^{2,1}(I)$ lower and upper solutions to (24), respectively, such that

$$\alpha(\tau) > \beta(\tau) \quad \text{for some } \tau \in I,$$

and that $f$ satisfies the following conditions:

(C1*) Compositions $t \in I \mapsto f(t, x(t), y(t))$ are measurable whenever $x(t)$ and $y(t)$ are measurable;

(C2*) There exists $M \in L^1(I)$ such that for a.a. $t \in I$ and all $x, y \in \mathbb{R}$, we have $|f(t, x, y)| \leq M(t)$;
(C3*) There exist admissible discontinuity curves \( \gamma_n : I_n = [a_n, b_n] \rightarrow \mathbb{R} \) (\( n \in \mathbb{N} \)) such that they are uniformly bounded, and for a.a. \( t \in I \) the function \( (x, y) \mapsto f(t, x, y) \) is continuous on \( \left( \mathbb{R} \setminus \bigcup_{\{t : t \in I_n\}} \{\gamma_n(t)\} \right) \times \mathbb{R} \).

Then problem (24) has a solution \( x \) such that for some \( \tau_x \in I \)
\[
\min \{\alpha(\tau_x), \beta(\tau_x)\} \leq x(\tau_x) \leq \max \{\alpha(\tau_x), \beta(\tau_x)\}.
\]

Proof. Define \( \tilde{M}(t) := 2M(t) + 1 \) and take \( R > 0 \) such that \( \|x'\|_\infty < R \) for all \( x \in X \) satisfying
\[
|\tilde{\gamma}'(t)| \leq \tilde{M}(t), \quad x(a) = x(b).
\]

Now fix some value \( r \geq \|\alpha\|_\infty + \|\beta\|_\infty + (b - a)R \) and \( r \geq \|\gamma\|_\infty \) for all \( n \in \mathbb{N} \), and define
\[
\tilde{f}(t, x, y) = \begin{cases} 
 f(t, x, y) - M(t) - 1 & \text{if } x \leq -(r + 1), \\
 f(t, x, y) + (x + r)(M(t) + 1) & \text{if } -(r + 1) < x < -r, \\
 f(t, x, y) & \text{if } -r \leq x \leq r, \\
 f(t, x, y) + (x - r)(M(t) + 1) & \text{if } r < x < r + 1, \\
 f(t, x, y) + M(t) + 1 & \text{if } x \geq r + 1,
\end{cases}
\]

and
\[
\tilde{L}_1(x, y, z, w, \xi) = \begin{cases} 
 w - z & \text{if } r + 1 \leq -x, \\
 (r + 1 + x)L_1(x, y, z, w, \xi) - (x + r)(w - z) & \text{if } r - x < r + 1, \\
 L_1(x, y, z, w, \xi) & \text{if } -r \leq x \leq r, \\
 (r + 1 - x)L_1(x, y, z, w, \xi) + (x - r)(w - z) & \text{if } r < x < r + 1, \\
 w - z & \text{if } x \geq r + 1.
\end{cases}
\]

Consider the auxiliary problem
\[
\begin{aligned}
\begin{cases}
x'' = \tilde{f}(t, x, x') & t \in I, \\
0 = \tilde{L}_1(x(a), x(b), x'(a), x'(b), x) , \\
x(a) = x(b).
\end{cases}
\end{aligned}
\]

Notice that \( \alpha \) and \( \beta \) are, respectively, lower and upper solutions of (30).

In addition
\[
|\tilde{f}(t, x, y)| \leq \tilde{M}(t) := 2M(t) + 1 \quad \text{for a.a. } t \in I \text{ and all } x, y \in \mathbb{R},
\]
and
\[
\tilde{f}(t, x, y) < 0 \quad \text{for a.a. } t \in I \text{ and all } x \in (-\infty, -r - 1], y \in \mathbb{R}, \\
\tilde{f}(t, x, y) > 0 \quad \text{for a.a. } t \in I \text{ and all } x \in [r + 1, \infty), y \in \mathbb{R}.
\]

In particular, \( \tilde{\alpha}(t) \equiv -\rho \) and \( \tilde{\beta}(t) \equiv \rho \) (where \( \rho = r + 2 \)) are, respectively, lower and upper solutions for the auxiliary problem (30).

Denote
\[
\Omega_0 = \left\{ x \in C^1(I) : \tilde{\alpha} < x < \tilde{\beta} \text{ on } I, \|x'\|_\infty < R \right\},
\]
\[
\Omega_1 = \left\{ x \in \Omega_0 : \tilde{\alpha} < x < \beta \text{ on } I \right\},
\]
\[
\Omega_2 = \left\{ x \in \Omega_0 : \alpha < x < \tilde{\beta} \text{ on } I \right\}
\]
and
\[
\Omega = \Omega_0 \setminus (\Omega_1 \cup \Omega_2),
\]
which is the set of points \( x \in \Omega_0 \) such that (25) holds with \( \overline{\sigma} = \alpha \) and \( \overline{\beta} = \beta \).
Problem (30) is equivalent to the equation $\bar{T}x = x$ where $\bar{T} : C^1(I) \to C^1(I)$ is given as in (7) but replacing $f$ by $\bar{f}$ and $L_1(x)$ by $\bar{L}_1(x)$. Observe that $\bar{T}$ maps bounded sets into relatively compact sets.

Now we shall show that if $\bar{T}x = x$ and $x \in \overline{\Omega}_0$, then $x \in \Omega_0$. Let $x \in \partial \Omega_0$ be such that $\bar{T}x = x$, then $\|x\|_{\infty} < R$ by (27) and (31). This can only occur if

$$x(\sigma_1) = \max_{t \in I} x(t) = \rho \quad \text{or} \quad x(\sigma_1) = \min_{t \in I} x(t) = -\rho$$

for some $\sigma_1 \in [a, b]$. In the first case, if $\sigma_1 \in (a, b)$ we have $x'(\sigma_1) = 0$ and $x(t) > r + 1$ on $[\sigma_1, \sigma_2]$ for some $\sigma_2 \in (\sigma_1, b]$. By (32), $x''(t) = \bar{f}(t, x(t), x'(t)) > 0$ for a.a. $t \in [\sigma_1, \sigma_2]$, which implies $x'(t) > 0$ on $(\sigma_1, \sigma_2]$, a contradiction with the fact that there is a maximum for $x$ at $\sigma_1$. If $\sigma_1 = a$, since $x'(a) \leq 0$, $x'(b) \geq 0$ and

$$0 = \bar{L}_1(x(a), x(b), x'(a), x'(b), x) = x'(b) - x'(a),$$

then $x'(a) = 0$ and we can argue as before. Similarly the other case is impossible.

Moreover if $\bar{T}x = x$ with $x \in \overline{\Omega}$, then $\|x\|_{\infty} < r$, because by (27) and (31) we obtain $\|x''\|_{\infty} < R$ and by Lemma 3.1 we have

$$\|x\|_{\infty} < \|\alpha\|_{\infty} + \|\beta\|_{\infty} + (b-a)R = r.$$

Then there are two possible cases:

(i) $\bar{T}x = x$ for some $x \in \partial \Omega_0 \cup \partial \Omega_1 \cup \partial \Omega_2$. Then $\|x\|_{\infty} < r$, so $Tx = \bar{T}x = x$ and $x$ is a solution for (24).

(ii) $\bar{T}x \neq x$ on $\partial \Omega_0 \cup \partial \Omega_1 \cup \partial \Omega_2$. By Theorem 2.4

$$\deg \left( I - \bar{T}, \Omega_0 \right) = \deg \left( I - \bar{T}, \Omega_1 \right) = \deg \left( I - \bar{T}, \Omega_2 \right) = 1.$$

Since $\alpha(\tau) > \beta(\tau)$ for some $\tau \in I$, $\Omega_1 \cap \Omega_2 = \emptyset$. Therefore, by the additivity property of the degree

$$\deg \left( I - \bar{T}, \Omega \right) = \deg \left( I - \bar{T}, \Omega_0 \right) - \deg \left( I - \bar{T}, \Omega_1 \right) - \deg \left( I - \bar{T}, \Omega_2 \right) = -1.$$

Hence there exists $x \in \Omega$ such that $\bar{T}x = x$. Then $\|x\|_{\infty} < r$ which implies $\bar{f} = f$ and $\bar{L}_1(x) = L_1(x)$, so $Tx = \bar{T}x = x$, that is, $x$ is a solution for problem (24).

To finish we have to show that

$$\{x\} \cap \overline{\bar{T}x} \subset \{\bar{T}x\} \quad \text{for every} \quad x \in \overline{\Omega_0} \cap \overline{\Omega_0}$$

in order to guarantee that the degree is well–defined. The sets where the function $(x, y) \mapsto \bar{f}(t, x, y)$ can be discontinuous are the graphs of the curves $\gamma_n$ and these curves satisfy that $-r \leq \gamma_n \leq r$, so at these points $\bar{f} = f$ and condition (34) can be proven as in Theorem 2.3.

Remark 5. Observe that Theorem 3.2 allows to obtain solutions via non-ordered lower and upper solutions to BVP with functional nonlinear boundary conditions, such as multipoint or maximum conditions, which fall outside the scope of the papers [10, 11, 12, 13, 14] where the periodic conditions were considered. In this sense, as far as the authors are aware, Theorem 3.2 gives new existence results even in the case of Carathéodory nonlinearities.

The following example illustrates the existence of solutions for (24) with non-ordered lower and upper solutions.
Example. Consider the problem (1) along with the following functional boundary conditions

\[ 0 = L_1(x(0), x(1), x'(0), x'(1), x) = -x(1/2) - \max_{t \in [0,1]} x(t), \]
\[ 0 = L_2(x(0), x(1)) = x(0) - x(1), \]

and

\[ f(t, x, y) = \lfloor 1/(t + |x|) \rfloor^{1/2} \cos(y) + 1/2, \]

for all \( x \in \mathbb{R}, t \in [0,1], t > 0 \) and where \( \lfloor x \rfloor \) denotes the integer part of \( x \).

Observe that the integrable function \( M(t) = t^{-1/2} + 1/2 \) satisfies condition \((C2^*)\).

We take the functions \( \alpha(t) = \pi(t - 1/2)^2 \) and \( \beta(t) = 0 \) for \( t \in [0,1] \) which are lower and upper solutions for our problem, respectively. Indeed,

\[ f(t, \alpha(t), \alpha'(t)) = \lfloor 1/(t + \pi(t - 1/2)^2) \rfloor^{1/2} \cos(2\pi(t - 1/2)) + 1/2 \]
\[ \leq \lfloor 4\pi/(2\pi - 1) \rfloor + 1/2 \]
\[ = \sqrt{2} + 1/2 < 2\pi = \alpha''(t), \]

and

\[ L_1(\alpha(0), \alpha(1), \alpha'(0), \alpha'(1), \alpha) = -\alpha(1/2) - \max_{t \in [0,1]} \alpha(t) = 0 - \pi/4 \leq 0, \]
\[ L_2(\alpha(0), \alpha(1)) = \alpha(0) - \alpha(1) = \pi/4 - \pi/4 = 0. \]

Notice that \( \alpha \) and \( \beta \) are not well–ordered.

For a.a. \( t \in [0,1] \), the function \( f(t, \cdot, \cdot) \) is continuous on

\[ \left[ \alpha(t), \beta(t) \right] \setminus \bigcup_{\{n:t \in I^*_n, i=1,2\}} \{\gamma_{n,i}(t)\} \times \mathbb{R}, \]

where for each \( n \in \mathbb{N}, \)

\[ \gamma_{n,1}(t) = -t + n^{-1} \quad \text{for all } t \in I^*_n = [0, n^{-1}], \]

and

\[ \gamma_{n,2}(t) = t - n^{-1} \quad \text{for all } t \in I^*_n = [n^{-1}, 1]. \]

These curves are inviable for the differential equation (see Definition 2.2). Indeed, there exists \( \varepsilon > 0 \) small enough such that \( \cos(1 + \varepsilon) \geq 0 \) and thus for all \( n \in \mathbb{N} \) and \( i = 1, 2, \)

\[ f(t, x, y) \geq 1/2 > 1/4 + \gamma''_{n,i}(t) \]

for a.a. \( t \in I^*_n \), all \( x \in [\gamma_{n,i}(t) - \varepsilon, \gamma_{n,i}(t) + \varepsilon] \) and all \( y \in [(-1)^i - \varepsilon, (-1)^i + \varepsilon] \).

Hence Theorem 3.2 guarantees the existence of a solution \( x \in W^{2,1}(0,1) \) such that

\[ \min \{\alpha(\tau_x), \beta(\tau_x)\} \leq x(\tau_x) \leq \max \{\alpha(\tau_x), \beta(\tau_x)\} \]

for some \( \tau_x \in [0,1] \).

4. Multiplicity results. In [5, 10, 16, 18] conditions which guarantee the existence of several solutions for second-order problems were given. They are based on degree theory and lower and upper solutions technique. Here by combining the results we got in the two previous sections we obtain multiplicity results.

Definition 4.1. We say that \( \alpha \in W^{2,1}(I) \) is a strict lower solution for the differential problem (1) if it satisfies the following conditions:
(i) For any $t_0 \in (a, b)$ there exist an open interval $I_0$ and $\varepsilon_0 > 0$ such that $t_0 \in I_0$ and for a.a. $t \in I_0$, all $u \in [\alpha(t), \alpha(t) + \varepsilon_0]$ and all $v \in [\alpha'(t) - \varepsilon_0, \alpha'(t) + \varepsilon_0]$, 
\[ \alpha''(t) \geq f(t, u, v). \]

(ii) $L_1(\alpha(a), \alpha(b), \alpha'(a), \alpha'(b), \alpha) < 0$.

(iii) $L_2(\alpha(a), \alpha(b)) = 0$, $L_2(\alpha(a), \cdot)$ is injective.

Similarly $\beta \in W^{2,1}(I)$ is a strict upper solution for (1) if it satisfies:

(i) For any $t_0 \in (a, b)$ there exist an open interval $I_0$ and $\varepsilon_0 > 0$ such that $t_0 \in I_0$ and for a.a. $t \in I_0$, all $u \in [\beta(t) - \varepsilon_0, \beta(t)]$ and all $v \in [\beta'(t) - \varepsilon_0, \beta'(t) + \varepsilon_0]$, 
\[ \beta''(t) \leq f(t, u, v). \]

Lemma 4.2. Let $\alpha, \beta$ be strict lower and upper solutions and let $x$ be a solution for problem (1). Assume that $L_2(x, \cdot)$ is injective for all $x \in \mathbb{R}$. Then $\alpha \leq x$ implies $\alpha < x$ and $x \leq \beta$ implies $x < \beta$.

Proof. Let $\alpha \leq x$ and $0 = x(t_0) - \alpha(t_0)$ at $t_0 \in (a, b)$. Assume that $x(t) > \alpha(t)$ for all $t \in (t_0, b)$. As $x - \alpha$ attains its minimum at $t_0$, we have that $x'(t_0) = \alpha'(t_0)$ so, by the definition of strict lower solution, there exist an open interval $I_0$ and $\varepsilon_0 > 0$ such that $t_0 \in I_0$ and for a.a. $t \in I_0$, all $u \in [\alpha(t), \alpha(t) + \varepsilon_0]$ and all $v \in [\alpha'(t) - \varepsilon_0, \alpha'(t) + \varepsilon_0]$, 
\[ \alpha''(t) \geq f(t, u, v). \]

On the other hand, 
\[ \forall r > 0 \exists t_r \in (t_0, t_0 + r) \text{ such that } \alpha'(t_r) < x'(t_r). \]

Hence, we can choose $t_r \in I_0$, $t_r > t_0$ such that $\alpha'(t_r) < x'(t_r)$ and for every $t \in (t_0, t_r)$,
\[ x(t) \leq \alpha(t) + \varepsilon_0, \quad |x'(t) - \alpha'(t)| < \varepsilon_0. \]
Then for a.a. $t \in (t_0, t_r)$ we have that
\[ \alpha''(t) \geq f(t, x(t), x'(t)), \]
and thus
\[ x'(t_r) - \alpha'(t_r) = \int_{t_0}^{t_r} (x''(s) - \alpha''(s)) \, ds = \int_{t_0}^{t_r} (f(s, x(s), x'(s)) - \alpha''(s)) \, ds \leq 0, \]
a contradiction.

If $x(a) = \alpha(a)$, since $L_2(x(a), x(b)) = 0 = L_2(\alpha(a), \alpha(b))$ and $L_2(\alpha(a), \cdot)$ is injective then $x(b) = \alpha(b)$, so $x - \alpha$ attains its minimum at $a$ and $b$, so $x'(a) \geq \alpha'(a)$ and $x'(b) \leq \alpha'(b)$. Then, by using the monotonicity conditions of $L_1$, we obtain the contradiction
\[ 0 = L_1(x(a), x(b), x'(a), x'(b), x) \leq L_1(\alpha(a), \alpha(b), \alpha'(a), \alpha'(b), \alpha) < 0. \]

Note that condition $L_2(x, \cdot)$ injective is equivalent to $L_2(x, \cdot)$ strictly increasing. Hence, if $x(b) = \alpha(b)$ and $x(a) > \alpha(a)$, then
\[ 0 = L_2(x(a), x(b)) = L_2(x(a), \alpha(b)) > L_2(\alpha(a), \alpha(b)) = 0, \]
a contradiction. \qed
Remark 6. Observe that $L_2(x,\cdot)$ is injective for all $x \in \mathbb{R}$, for example, when $L_2(x,y) = x - y$.

Remark 7. Notice that the proof of Lemma 4.2 remains valid for the following statement:

Let $\alpha$ be a strict lower solution for problem (1) and be $\beta$ an upper solution. Assume that $L_2(x,\cdot)$ is injective for all $x \in \mathbb{R}$. Then $\alpha \leq \beta$ implies $\alpha < \beta$.

Theorem 4.3. Assume there exist $\alpha_1, \alpha_2 \in W^{2,1}(I)$ lower solutions and $\beta \in W^{2,1}(I)$ an upper solution for (24) such that $\alpha_1 \leq \beta$, $\alpha_2(\tau) > \beta(\tau)$ for some $\tau \in I$ and $\alpha_2, \beta$ are strict lower and upper solutions, respectively.

Suppose that the function $f$ satisfies the hypotheses (C1$^*$)-(C3$^*$).

Then the problem (24) has at least two solutions $x_1, x_2 \in W^{2,1}(I)$ such that $\alpha_1 \leq x_1 \leq \beta$ and there exist $t_1, t_2 \in I$ such that $x_2(t_1) < \alpha_2(t_1)$ and $x_2(t_2) > \beta(t_2)$.

Proof. By Theorem 2.4 there exists a solution $x_1 \in W^{2,1}(I)$ for problem (1) such that $\alpha_1 \leq x_1 \leq \beta$. On the other hand, the fact that $\alpha_2, \beta$ be strict lower and upper solutions implies that the case (i) in the proof of Theorem 3.2 is not possible (due to Lemma 4.2), so (24) has a solution $x_2 \in \Omega$ where $\Omega$ is defined as in (33) taking $\alpha = \alpha_2$. Observe that $\Omega$ is the set of functions satisfying (25) with $\alpha = \alpha_2$. 

Remark 8. A similar result can be obtained by interchanging the role of lower and upper solutions.

Example. Consider the problem (1) along with the following nonlinear boundary conditions

$$0 = L_1(x(0), x(1), x'(0), x'(1), x) = -x(0)^2 - x(1/4),$$
$$0 = L_2(x(0), x(1)) = x(0) - x(1),$$

and

$$f(t, x, y) = [1/(t + |x|)]^{1/2} \cos(y) + 1/2,$$

for all $x \in \mathbb{R}$, $t \in [0,1]$, $t > 0$.

Observe that the discontinuity curves of $f$ are inviable and that it is bounded by an integrable function.

One can easily verify that the functions $\alpha_1(t) = \pi(t - 1/2)^2 - \pi$ and $\alpha_2(t) = \pi(t - 1/2)^2$ are strict lower solutions for the previous problem and $\beta(t) = -(t - 1/2)^2/2$ is a strict upper solution. Moreover, they satisfy the following order conditions: $\alpha_1 \leq \beta \leq \alpha_2$. Hence, Theorem 4.3 ensures that the considered problem has at least two different solutions.

Following the ideas of Amann [1], in [5] a three solution theorem is given in presence of two pairs of lower and upper solutions with order relations. Here an analogous result is possible for our problem (24) as an immediate consequence of Theorems 2.4 and 3.2.

Theorem 4.4. Assume there exist $\alpha_1, \alpha_2 \in W^{2,1}(I)$ lower solutions and $\beta_1, \beta_2 \in W^{2,1}(I)$ upper solutions for (24) such that $\alpha_1 \leq \beta_1$, $\alpha_2 \leq \beta_2$, $\alpha_2(\tau) > \beta_1(\tau)$ for some $\tau \in I$ and $\alpha_2, \beta_1$ are strict lower and upper solutions, respectively.

Suppose that the function $f$ satisfies the hypotheses (C1$^*$)-(C3$^*$).
Then the problem (24) has at least three solutions \( x_1, x_2, x_3 \in W^{2,1}(I) \) such that \( \alpha_1 \leq x_1 \leq \beta_1, \alpha_2 \leq x_2 \leq \beta_2 \) and there exist \( t_1, t_2 \in I \) such that \( x_3(t_1) < \alpha_2(t_1) \) and \( x_3(t_2) > \beta_1(t_2) \).

If the lower and upper solutions are strict, by means of the topological degree, it is possible to obtain a three-solution result (see [18]) where \( f \) may satisfy a weaker boundary condition (of Nagumo type) than in the previous one and, besides, the more general boundary conditions of problem (1) can be considered.

**Theorem 4.5.** Assume there exist \( \alpha_1, \alpha_2 \in W^{2,1}(I) \) strict lower solutions and \( \beta_1, \beta_2 \in W^{2,1}(I) \) strict upper solutions for (1) with the following order relations:

\[\alpha_1 \leq \beta_1 \leq \beta_2, \alpha_2 \leq \beta_2, \alpha_2(\tau) \geq \beta_1(\tau) \text{ for some } \tau \in I.\]

Suppose that \( L_2(x, \cdot) \) is injective for all \( x \in \mathbb{R} \) and the function \( f \) satisfies the hypotheses (C1)-(C3) taking \( \alpha = \alpha_1 \) and \( \beta = \beta_2. \)

Then the problem (1) has at least three solutions \( x_1, x_2, x_3 \in W^{2,1}(I) \) such that \( \alpha_1 \leq x_1 \leq \beta_1, \alpha_2 \leq x_2 \leq \beta_2 \) and there exist \( t_1, t_2 \in I \) such that \( x_3(t_1) < \alpha_2(t_1) \) and \( x_3(t_2) > \beta_1(t_2). \)

**Proof.** Consider the sets

\[
\begin{align*}
\Omega_0 &= \{ x \in C^1(I) : \alpha_1 < x < \beta_2 \text{ on } I, \; \|x'\|_{\infty} < R \}, \\
\Omega_1 &= \{ x \in \Omega_0 : \alpha_1 < x < \beta_1 \text{ on } I \}, \\
\Omega_2 &= \{ x \in \Omega_0 : \alpha_2 < x < \beta_2 \text{ on } I \},
\end{align*}
\]

where \( R \) is given by condition (C2). Note that since the lower and upper solutions are strict then the operator \( T \) has no fixed points on the boundary of \( \Omega_i \) (\( i = 0, 1, 2 \)). Hence by Theorem 2.3

\[\deg(I - T, \Omega_0) = \deg(I - T, \Omega_1) = \deg(I - T, \Omega_2) = 1.\]

Then \( T \) has fixed points \( x_1 \in \Omega_1 \) and \( x_2 \in \Omega_2 \) which are different fixed points because \( \Omega_1 \cap \Omega_2 = \emptyset \) due to the order hypotheses about the lower and upper solutions. Now the properties of the degree ensure that

\[\deg(I - T, \Omega_0 \setminus (\overline{\Omega_1} \cup \overline{\Omega_2}) ) = \deg(I - T, \Omega_0) - \deg(I - T, \Omega_1) - \deg(I - T, \Omega_2) = -1,\]

so \( T \) has a fixed point \( x_3 \in \Omega_0 \setminus (\overline{\Omega_1} \cup \overline{\Omega_2}) \). Since \( x_3 \not\in \overline{\Omega_1} \cup \overline{\Omega_2} \), there exist \( t_1, t_2 \in I \) such that \( x_3(t_1) < \alpha_2(t_1) \) and \( x_3(t_2) > \beta_1(t_2) \). \( \Box \)

To finish we illustrate our multiplicity results with an example.

**Example.** Consider the problem (1) along with the following nonlinear boundary conditions

\[\begin{align*}
0 &= L_1(x(0), x(3\pi), x'(0), x'(3\pi), x) = \cos(x(0)), \\
0 &= L_2(x(0), x(3\pi)) = x(0) - x(3\pi) - \sin(x(3\pi)),
\end{align*}\]

and the function

\[f(t, x, y) = \phi(x - t^2) \cos(x) - |y|,\]

for all \( x, y \in \mathbb{R} \) and \( t \in [0, 3\pi] \), where the function \( \phi \) is defined below. Consider a bijection between the rational numbers and the positive integers and denote as \( \{q_n\}_{n \in \mathbb{N}} \) the sequence of rational numbers. The function \( \phi : \mathbb{R} \to \mathbb{R} \) is given by

\[\phi(u) = \sum_{n : q_n < u} 2^{-n}.\]
Notice that $\phi$ is continuous at the irrational points and discontinuous at the rational numbers, see [15, Prop. 2, p. 108-109]. Moreover, $\phi(u) \in (0, 1)$ for each $u \in \mathbb{R}$. Indeed, for every $u \in \mathbb{R}$, $\phi(u) < \sum_{n=1}^{\infty} 2^{-n} = 1$.

Observe that $\alpha_1 \equiv -\pi$ and $\alpha_2 \equiv \pi$ are strict lower solutions for the problem and $\beta_1 \equiv 0$ and $\beta_2 \equiv 2\pi$ are strict upper solutions.

On the other hand, for a.a. $t \in [0, 3\pi]$ the function $(x, y) \mapsto f(t, x, y)$ is continuous on $\left([-\pi, 2\pi] \setminus \bigcup_{\{n : t \in I_n\}} \{\gamma_n(t)\}\right) \times \mathbb{R}$, where for each $n \in \mathbb{N}$,

$$\gamma_n(t) = t^2 + q_n \quad \text{for all } t \in I_n = [0, 3\pi].$$

In addition, the curves $\gamma_n$ are inviable admissible discontinuity curves. Indeed, for every $n \in \mathbb{N}$ and for all $x, y \in \mathbb{R}$ and $t \in [0, 3\pi]$, we have

$$f(t, x, y) = \phi(x - t^2) \cos(x) - |y| \leq 1 < 2 = \gamma''_n(t).$$

Therefore, Theorem 4.5 guarantees the existence of at least three solutions for the considered problem satisfying some suitable localization conditions.

REFERENCES

[1] H. Amann, On the number of solutions of nonlinear equations in ordered Banach spaces, J. Funct. Anal., 11 (1972), 346–384.
[2] A. Cabada and R. L. Pouso, Extremal solutions of strongly nonlinear discontinuous second-order equations with nonlinear functional boundary conditions, Nonlinear Analysis, 42 (2000), 1377–1396.
[3] A. Cellina and A. Lasota, A new approach to the definition of topological degree for multivalued mappings, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Natur., 47 (1969), 434–440.
[4] C. De Coster and P. Habets, Two-Point Boundary Value Problems: Lower and Upper Solutions, Mathematics in Science and Engineering, 205. Elsevier B. V., Amsterdam, 2006.
[5] C. De Coster and S. Nicaise, Lower and upper solutions for elliptic problems in nonsmooth domains, J. Differential Equations, 244 (2008), 599–629.
[6] R. Figueroa, R. L. Pouso and J. Rodríguez-López, Degree theory for discontinuous operators, Fixed Point Theory, accepted.
[7] R. Figueroa, R. L. Pouso and J. Rodríguez-López, Extremal solutions for second-order fully discontinuous problems with nonlinear functional boundary conditions, Electron. J. Qual. Theory Differ. Equ., (2018), 14 pp.
[8] A. F. Filippov, Differential Equations with Discontinuous Righthand Sides, Kluwer Academic, Dordrecht, 1988.
[9] R. López Pouso, Schauder’s fixed-point theorem: New applications and a new version for discontinuous operators, Bound. Value Probl., (2012), Art. ID 2012:92, 14 pp.
[10] I. Rachůnková, Upper and lower solutions and multiplicity results, J. Math. Anal. Appl., 246 (2000), 446–464.
[11] I. Rachůnková and M. Tvrdý, Existence results for impulsive second order periodic problems, Nonlinear Anal., 59 (2004), 133–146.
[12] I. Rachůnková and M. Tvrdý, Impulsive periodic boundary value problem and topological degree, Functional Differential Equations, Israel Seminar, 9 (2002), 471–498.
[13] I. Rachůnková and M. Tvrdý, Non-ordered lower and upper functions in second order impulsive periodic problems, Dyn. Contin. Discrete Impuls. Syst., 12 (2005), 397–415.
[14] I. Rachůnková and M. Tvrdý, Periodic problems with $\phi$-Laplacian involving non-ordered lower and upper functions, Fixed Point Theory, 6 (2005), 99–112.
[15] H. L. Royden and P. M. Fitzpatrick, Real Analysis, 4th Ed., Boston, Prentice Hall, 2010.
[16] B. Rudolf, An existence and multiplicity result for a periodic boundary value problem, Math. Bohem., 133 (2008), 41–61.
[17] J. R. L. Webb, On degree theory for multivalued mappings and applications, *Bolletino Un. Mat. Ital.*, 9 (1974), 137–158.

[18] X. Xian, D. O’Regan and R. P. Agarwal, Multiplicity results via topological degree for impulsive boundary value problems under non-well-ordered upper and lower solution conditions, *Bound. Value Probl.*, (2008), Art. ID 197205, 21 pp.

Received February 2019; revised May 2019.

E-mail address: ruben.figueroa.sestelo@gmail.com
E-mail address: rodrigo.lopez@usc.es
E-mail address: jorgerodriguez.lopez@usc.es