Testing a Quantum Inequality

with a Meta-analysis of Data for Squeezed Light

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Abstract

In quantum field theory, coherent states can be created that have negative energy density, meaning it is below that of empty space, the free quantum vacuum. If no restrictions existed regarding the concentration and permanence of negative energy regions, it might, for example, be possible to produce exotic phenomena such as Lorentzian transversable wormholes, warp drives, time machines, violations of the second law of thermodynamics, and naked singularities. Quantum Inequalities (QIs) have been proposed that restrict the size and duration of the regions of negative quantum vacuum energy that can be accessed by observers. However, QIs generally are derived for situations in cosmology and are very difficult to test. Direct measurement of vacuum energy is difficult and to date no QI has been tested experimentally. We test a proposed QI for squeezed light by a meta-analysis of published data obtained from experiments with optical parametric amplifiers (OPA) and balanced homodyne detection. Over that last three decades, researchers in quantum optics have been trying to maximize the squeezing of light and the quantum vacuum and have succeeded in reducing the variance in the quantum vacuum fluctuations to -15 dB. To apply the QI, a time sampling function is required. In our meta-analysis different time sampling functions for the QI were examined, but in all physically reasonable cases the QI is violated by much or all of the measured data. This brings into question the basis for QI. Possible explanations are given for this surprising result.

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I. INTRODUCTION

In quantum field theory, the vacuum expectation value of the normally ordered or renormalized energy density \( \langle T_{\text{vac}} \rangle \) need not be positive. For example, a superposition of a vacuum state \( (n=0) \) and a two photon state \( (n=2) \), can have negative renormalized energy density with the proper choice of coefficients. Squeezed light can have a negative energy density. From theory and experiment, we know that static negative energy densities associated with vacuum states are concentrated in narrow spatial regions, e.g., inside a parallel plate Casimir cavity with small plate separation or in the region near the Schwarzschild radius in the Boulware vacuum where the energy density is everywhere negative as seen by static observers. There is no known way to directly measure vacuum energy density. On the other hand, the total energy of a system is believed to always be positive or zero. For example, the sum of the mass energy of the plates plus the negative vacuum energy inside the cavity is positive [1][2].

The classical energy conditions imply that an inertial observer who initially encounters some negative energy density must encounter compensating positive energy density at some arbitrary time in the future. Quantum Inequalities (QIs) have been derived for the free vacuum quantum field, with no sources or boundaries, which constrain the magnitude and duration of negative energy densities relative to the energy density of an underlying reference vacuum state. The QI places bounds on quantum violations of the classical energy conditions [3][4]. The QI is formulated as a mathematical bound on the average of the quantum expectation value of a free field’s energy-momentum tensor in the vacuum state, where the average is taken along an observer’s timelike or null worldline using time sampling functions. Contrary to the classical energy conditions, the QI dictates that the more negative the energy density is in some time interval \( T \), the shorter the duration \( T \) of the interval, so that an inertial observer cannot encounter arbitrarily large negative energy densities that last for arbitrarily long time intervals. An inertial observer must encounter compensating positive energy density no later than after a time \( T \), which is inversely proportional to the magnitude of the initial negative energy density.

In QI, restrictions are placed on the integral of the vacuum expectation value of the energy density multiplied by a sampling function. For the electromagnetic field in flat space-time,
with a normalized time sampling function of \( f(t) = (t_o/\pi)(1/(t^2 + t_o^2)) \), Ford and Roman have shown \[5\]

\[\hat{\rho} \equiv \frac{t_o}{\pi} \int_{-\infty}^{+\infty} \frac{\langle T_{oo} \rangle}{t^2 + t_o^2} dt \geq -\frac{3}{16\pi^2} \frac{\hbar c}{(ct_o)^4} \] (1)

To give a frame of reference, this can be compared to the vacuum energy density within a parallel plate Casimir cavity of separation \( a \):

\[\langle T_{oo} \rangle_{Cas} = -\frac{\pi^2 \hbar c}{720 a^4} \] (2)

The ratio of the numerical factors for the free field to the Casimir cavity is 1.4, so a negative energy density \( \hat{\rho} \) equal to that in a perfectly conducting parallel plate cavity of spacing \( a \) can exist no longer than for a time \( t_o \sim a/c \), about \( 3 \times 10^{-16} \) seconds for a typical experiment. As the sampling time \( t_o \) increases, \( \hat{\rho} \) rapidly goes to zero. (Note however, that as derived, the QIs do not apply directly to the Casimir cavity since it has boundaries.) Also, to test this QI Eq. 1 experimentally, one must make an absolute measurement of the vacuum energy density, an experimental challenge for which no solution has yet been found. Some progress is due to Reik et al who were able to directly probe the spectrum of vacuum fluctuations of the electric field in the multi-THz range using femtosecond laser pulses \[6\].

If the laws of quantum field theory placed no restrictions on negative energy, then it might be possible to produce surprising macroscopic effects such as violations of the second law of thermodynamics, traversable wormholes, warp drives, and possibly time machines \[5\]. QI appear to restrict these violations of the second law \[7\].

A quantum inequality has been derived for squeezed light by Marecki \[8\]. Squeezed light has a nonclassical distribution of the quadrature components, which may be considered as the canonical momentum and position components of an equivalent harmonic oscillator corresponding to the frequency of the electromagnetic radiation being considered. Squeezed states are routinely made in quantum optics experiments in the process of parametric down conversion, in which an incident photon is converted in a non-linear crystal to two entangled photons of the same frequency, which is one half of that of the incident photon. The fluctuations of the electric field in the squeezed light are locally lower than the vacuum fluctuations, the so-called shot-noise level. There appears to be a limit to the amount of squeezing relative to the free vacuum, which has been measured to be from -0.5 dB to the most recent value of -15 dB \[9\]. Marecki’s QI predicts the maximum degree of squeezing
in dB that is possible in terms of the fraction of the cycle during which the variance in the electric field is less than that of the free vacuum limit.

To date no QI has been tested experimentally. One of the reasons for this was noted by Marecki: 

"As far as we know quantum-field theoreticians do not know that their inequalities may influence real experiments nor are quantum opticians aware of the existence of such inequalities.” Most of the quantum inequalities have been developed by quantum cosmologists, who are unaware of measurements of vacuum energy done by quantum opticians. This paper is the first attempt to bridge this gap and test a quantum inequality with data. It appears easiest to test the QI for squeezed light because experimentally one measures the squeezing relative to the free vacuum, which corresponds to the theoretical quantity described in the corresponding quantum inequality. However, there may be some subtleties in the comparison because of differences in the measurement protocols. Indeed, we find that the QI as given is violated by most of the experimental data, yet all experimental data are consistent with a theoretical model of the Optical Parametric Amplifier (OPA) used to generate squeezed light.

II. QUANTUM INEQUALITY FOR SQUEEZED LIGHT

A simplified version of Marecki’s derivation is given in the Appendix. His key result is

\[ <\Delta>_A \geq -\frac{2}{(2\pi)^2} \int_0^{\infty} d\omega \int d^3p \mu_p^2 \omega_p (f^{1/2})_{FT}(\omega + \omega_p)^2 \]  

The minimum value of the variance \(<\Delta>_A\) for a state A is determined by the time window function \(f(t)\), specifically by the Fourier transform of the square root of the window function \(f(t)\). In order to insure convergence of the integral, a spectral function \(\mu_p = \mu(\omega - \omega_p)\) that reflects the frequency response of the apparatus measuring the variance must be included. This result is similar in spirit to that of other researchers in that it involves the Fourier transform of the time window\[11\]. There is no proof that Eq. 3 represents the greatest lower bound for \(<\Delta>_A\).

In other formulations of Quantum Inequalities, other features of the time window, such as the second derivative, determine the minimum average energy over the time sampled\[5\]. In all formulations of Quantum Inequalities to date, the window function determines the minimum energy values. In all cases, these formulations assume a free electromagnetic field
with plane waves, that is without sources, or boundaries. We have found, like others, that
the specific properties of the window function are very important [4]. Only in Marecki’s
calculations of the QI does a spectral function $\mu_p$ appear. This may be a problem, since
the Fourier transform of the time window $f(t)$ implies a certain frequency response of the
apparatus, and this may conflict with the independent requirements for the function $\mu_p$.

The quantity that is generally measured in experiments is the log of the variance for some
state $A$ relative to the variance of the free vacuum: $\Delta A$. The squeezing in dB must exceed in
numeric value $R$ where

$$R = 10\log_{10} \left( \frac{\langle \Delta A \rangle + \langle E^2 \rangle_{\text{vac}}}{\langle E^2 \rangle_{\text{vac}}} \right)$$

(A. Evaluation of $R$ for Specific Time Sampling Functions)

For a Gaussian

$$f(t) = \frac{1}{t_0 \sqrt{2\pi}} e^{-t^2/2t_0^2}$$

we obtain

$$R = 10\log_{10} \left[ -\frac{\int d^3 p \mu_p^2 \omega_p E_r f(\sqrt{2}t_0\omega_p)}{\int d^3 p \mu_p^2 \omega_p} \right]$$

If $\mu_p$ is a sharply peaked function of $\omega_p - \omega_0$, and $\delta\omega \ll \omega_0$, then to a good approximation

$$R(\omega, t_0) = 10\log_{10}[E_r f(\sqrt{2}\omega_0 t_0)]$$

(Marecki has an additional factor of 2 in the Erf function, which we do not get). As a check
on the role of the frequency windows $\mu_p$, we can do all the integrations in $R$ for a Gaussian
frequency function, and we get an additional factor of $\omega_0^3 \delta\omega$ for the $\omega_p$ integration in the
numerator and in the denominator. These factors cancel, giving to lowest order in $\delta\omega$, the
result quoted above. On the other hand, if we do not introduce a frequency function, we
find that $R = 10\log_{10}[1] = 0$, indicating that no squeezing is possible. In other words,
a frequency function is required to get reasonable results, however, as we have noted the
frequency function may not be consistent with the time window.
We can also compute $R$ for a squared Lorentzian time sampling function (an ordinary Lorentzian does not give well-behaved integrals):

$$f(t) = \frac{2}{\pi} \frac{t^3}{(t^2 + t_o^2)^2}$$  \hspace{1cm} (9)

We find

$$R(\omega_o t_o) = 10 \log_{10}(1 - e^{-2\omega_o t_o})$$  \hspace{1cm} (10)

assuming $\mu_p$ is strongly peaked at $\omega_o$. The equation behaves similarly to the one for the Gaussian time function.

We can compute $R$ for a square window function $f(t)$ of width $\Delta T$, with perfectly sharp corners, and using a frequency function. We find that we always get perfect squeezing, $R = 10 \log_{10} 0$, with no dependence on $\Delta T$. Although, a perfectly sharp window is not physically possible, and is mathematically unstable, one still wonders about the meaning of this result. A sharp window allows one to do a perfect measurement (at least in principle) in which only regions of perfect squeezing are measured, and one can avoid the regions with partial or antisqueezing.

We have also evaluated the variance for a symmetric trapezoidal window, with a center region $T_S$ long, and sloping sides that are each $nT_s$ long, normalized to 1.

### III. PRODUCTION OF SQUEEZED LIGHT USING OPTICAL PARAMETRIC AMPLIFICATION

A model for an optical parametric amplifier predicts the relative variance $S$ in the quadrature components of the vacuum electromagnetic field for a state $A$:

$$S = \frac{\langle E^2 \rangle_A - \langle E^2 \rangle_{\text{vac}}}{\langle E^2 \rangle_{\text{vac}}}$$  \hspace{1cm} (11)

The model ([12], [13], [14]) predicts that

$$S(\theta, x, \omega) = 1 + 4\beta x \left[ \frac{\cos^2 \theta}{(1 - x)^2 + (\omega/\gamma)^2} - \frac{\sin^2 \theta}{(1 + x)^2 + (\omega/\gamma)^2} \right]$$  \hspace{1cm} (12)

where $x = P/P_{th}$ is the ratio of the laser power to the power at threshold ($0 < x < 1$); $\beta$ is the optical efficiency, $\theta$ is the phase difference between the local oscillator field and the vacuum field, $\omega$ is the sideband angular frequency of measurement by a spectrum analyzer, $\gamma$ is the
halfwidth or cavity decay rate \( \gamma = c(T + L)/l \) where \( c \) = speed of light, \( T \) = transmissivity of coupling mirror, \( L \) = round trip loss, \( l \) = round trip length). The model has been parameterized so the squeezing is a maximum at \( \omega = 0 \). Generally, the squeezing in given in terms of dB:

\[
R = 10 \log_{10} S(\theta, x, \omega) \tag{13}
\]

To clarify the physical basis of the model and derive equations relating it to the QI, we briefly review the OPA model and experimental results. In recent experiments, values for the full width \( 2\gamma/2\pi \) ranges from 9 MHz to 84 MHz. Measurement frequencies \( \omega \) are typically about 1 MHz to, at most, 8 MHz, \( 0.9 < \beta < 0.99 \), and laser wavelengths vary from about 795 nm to 1064 nm. In the measurement range, \( (\omega/\gamma) \) varies from about 0 to at most 1. Figure 1 shows a recent experimental arrangement[9]. The OPA is operated below threshold and is composed of a cavity with a nonlinear crystal that is fully reflective at one end and a partially reflective mirror at the other end. The OPA non-linear crystal is driven by the output of a frequency doubled laser (SHG). The crystal has a small probability of producing two photons of the same wavelength (twice the driving wavelength) by degenerate parametric down conversion. Detection is by balanced homodyne detection, in which the difference in photodetector current PD1-PD2 is measured for components of the squeezed vacuum SQZ and the laser (local oscillator) LO, that have interfered at a 50-50 beam splitter. The difference current is analyzed by a spectrum analyzer, typically with a measurement bandwidth of about 100kHz to 500kHz.

Data on a squeezed vacuum taken from the apparatus illustrated are shown in Figures 2[9]. Fits based on Eq.12 are shown in dashed lines. In Figure 3, the vacuum squeezing is presented as a function of the phase difference between the LO and the squeezed vacuum SQZ[9]. In this particular experiment the mirror was vibrated periodically and the abscissa given in time rather than angle, but these methods are equivalent. The first minimum would correspond to the antisqueezed quadrature \( \theta = \pi/2 \) radians, the second to \( \pi/2 + \pi \) radians[18]. The fraction of the period for which the squeezing is negative equals \( F_T \), which is an indicator of the squeezing. From measuring the graph (using curves d and a) , one finds that \( F_T \) is about 0.14.

If \( S(\theta, x, \omega) \) is less than one, then the variance or noise of this quadrature component is less than that of the free vacuum, and is squeezed. This implies that the other quadrature component \( R(\theta + \pi/2, x, \omega) \) > 1 and is antisqueezed. The minimum value of S for squeezing
FIG. 1. Schematic of experimental setup. Squeezed vacuum states of light SQZ at a wavelength of 1064 nm were generated in a double resonant, type I optical parametric amplifier (OPA) operated below threshold. SHG: second-harmonic generator; PBS: polarizing beam splitter; DBS: dichroic beam splitter; LO: local oscillator; PD: photodiode; MC1064: three mirror ring cavity for spatiotemporal mode cleaning; EOM: electro-optical modulator; FI: Faraday isolator. The phase shifter for the relative phase $\theta$ between SQZ and LO was a piezoelectric actuated mirror $^{[9]}$.

FIG. 2. Squeezing in $\text{dB} = 10 \log R$, as a function of Power $= P_{\text{th}} x$, $P_{\text{th}} = 16.2 \text{ mW}$, and frequency $\omega$. Theoretical curves are shown as the narrow dashed lines, with $2\gamma/2\pi = 84 \text{ MHz}$, and $\beta = 0.975$. The decrease in noise with increase in frequency is due to the term $(\omega/\gamma)^2$ in Eq $^{[12]}$ $^{[9]}$

occurs for $\theta = \pi/2$ and is

$$S_\omega(x, \omega) = 1 - \frac{4\beta x}{(1+x)^2 + (\omega/\gamma)^2}$$

(14)
FIG. 3. Squeezing dB as a function of phase difference $\theta$, measured here by the time to move a mirror. Curve a: noise level with all inputs blanked; Curve b: phase is locked to the squeezed quadrature ($\theta = \pi/2$); Curve c: phase is locked to the antisqueezed quadrature ($\theta = 0$); Curve d: the phase is scanned. The fraction of the period that the squeezing is below zero equals $F_T$. [9]

FIG. 4. The maximum squeezing $R_\omega = S_\omega(x, \omega)$ as a function of $x$. When $S_\omega(x, \omega)$ is less than 1, there is squeezing below the normal quantum limit. We assume $\beta = 1$ and $\omega/\gamma$ is negligible.

and the maximum antisqueezing occurs for $\theta = 0$ or $\pi$ is

$$S_\omega(x, \omega) = 1 + \frac{4\beta x}{(1 - x)^2 + (\omega/\gamma)^2} \quad (15)$$

The maximum possible squeezing $S_\omega(x, 0)$ is shown as a function of $x$ in Figure [4].

The frequency spectrum of squeezing $1 - S_\omega(x, f = \omega/\gamma)$ is Lorentzian, with a halfwidth of $\gamma$. The product of the maximum and minimum variances is

$$S_\omega(x, \omega) * S_\omega(x, \omega) = 1 - \frac{16\beta(1 - \beta)x^2}{[(1 + x)^2 + (\omega/\gamma)^2][(1 - x)^2 + (\omega/\gamma)^2]} \quad (16)$$
For an ideal optical system with no losses $\beta = 1$ and the product is 1, as it must be according to the Heisenberg Uncertainty Principle.

For comparison to the quantum inequality, we need to know the angular interval $\Delta \theta$ over which the light is squeezed. The light is squeezed if the term in brackets in Eq. 12 is negative which implies

$$\frac{S_+(x, \omega) - 1}{1 - S_-(x, \omega)} < \tan^2 \theta$$

(17)

It follows that $F_T = \Delta \theta / \pi$, which is the fraction of the period during which the light is squeezed, is given by

$$F_T(x, \omega) = 1 - \frac{2}{\pi} \tan^{-1} \sqrt{\frac{S_+(x, \omega) - 1}{1 - S_-(x, \omega)}}$$

(18)

For the special case on an ideal OPA, we can substitute $S_+ = 1/S_-$ to get

$$F_T(x, \omega) = 1 - \frac{2}{\pi} \tan^{-1} \sqrt{\frac{1}{S_-(x, \omega)}}$$

(19)

which can be solved for $S_-$ to obtain

$$S_-(x, \omega) = \tan^2[F_T(x, \omega) \frac{\pi}{2}]$$

(20)

which is valid for $0 < F_T < 0.5$. A plot of $R = 10\log S_-(x, \omega)$ as a function of $F_T(x, \omega)$ for an ideal OPA is shown in Figure 5. We would not expect experimental points to display squeezing greater than the amount allowed for the ideal OPA. Since most OPA measurements were not ideal, we used the general formula Eq. 18 for $F_T(x, \omega)$ to reduce data.

The maximum fraction of time in a period during which squeezing can occur is $F_T = 1/2$, and this only occurs when the $x$ approaches zero, so the amount of squeezing is slight.
IV. ANALYSIS OF DATA FROM OPA

We analyzed data from 12 experiments conducted over the last thirty years [15] - [25]. We obtained values of $F_T$ and squeezing/antisqueezing from plots or from the text, and estimated errors as much as possible. The most recent data is shown in Figure 2 [9]. They fit their squeezing data ($S_-$ and $S_+$) to the model, actually including a small correction for the phase uncertainty, with excellent agreement. To get $F_T$ from their data, we used the equation from the OPA model in terms of the arc tangent Eq 18. On the other hand, for the data in Figure 3, we could do calculations of $F_T$ graphically from $S_-$ and $S_+$ and from $\Delta \theta$, from their plots. Thus we can compare the two methods. (Some papers plot the squeezing vs time shown in Figure 2 as squeezing versus phase difference. We treat both types of plots in the same manner, with an assumed equivalence between time and phase change which corresponds to the rate at which a mirror is moved in degrees/second.) In about half the papers, we could compare the two methods, and found they agreed within about +/-8% rms. When we could use both methods to compute $F_T$, we used the average in our plots. For Vahlbruch we took points at three power levels ($x = 0.8, 0.3, 0.1$) [9]. For all other publications, we had only one power level.

V. INTERPRETATION OF SQUEEZING AND OBSERVATION TIME

To compare the results of the QI and the OPA data requires the assumption that the squeezing in the OPA analysis is equivalent to the squeezing in the QI analysis, as discussed by Marecki [8]. In the OPA case, the squeezing depends on the phase difference $\theta$ between the local oscillator LO and the light SQZ, while in the QI analysis the squeezing depends on the phase change $\omega_o t_o$ occurring during the observation time. In the equation for $R$, which expresses the Quantum Inequality, $t_o$ is the width of the Gaussian time sampling function, and $\omega_o$ is the center frequency in radians/sec of the frequency sampling function. One would naively think this frequency $\omega_o$ would correspond to the frequency of the laser being used in the OPA, which is roughly $2\pi 10^{14}$ radians/sec. On the other hand, the frequency being measured by the spectrum analyzer is typically about $2\pi 10^6$, eight orders of magnitude different! The product $\omega_o t_o$ corresponds to the phase change in radians during the observation time $t_o$. If we observe for a time $M$, which equals the period of the
squeezing $S(\theta, x, \omega)$, then the phase change is $\pi = M\omega_o$. Therefore $\omega_o t = \pi(t/M)$, where $t$ is the observation time. Defining the fractional observation time $F_T = t/M$ we conclude $\omega_o t = \pi F_T$. Thus for a Gaussian time sampling function we have

$$R(F_T) = 10\log_{10}[\text{Erf}(\sqrt{2}\pi F_T)]$$

(21)

On the other hand, Marecki just identified $\omega_o t = \tau$ as the fraction of the period $F_T$ in which squeezing occurred, omitted the factor of $\pi$, and also had an additional factor of 2, thus obtaining $R(F_T) = 10\log_{10}[\text{Erf}(2\sqrt{2}F_T)]$.

For the squared Lorentzian time function we obtained

$$R(F_T) = 10\log_{10}(1 - e^{-2\pi F_T})$$

(22)

whereas Marecki did not have the factor $\pi$ in the exponent.

Note that in the derivation of the Quantum Inequality, $\omega_o t_o$ is the phase change during an observation of the variance of a quadrature component. Nothing is said in the derivation about whether the field is squeezed or not. The QI appears to place a bound on the variance for this phase change for any quadrature component, squeezed or not, during the observation time. Assuming one physically can observe the field only when it is squeezed, then we should obtain the value for $R(F_T)$ as restricted by the QI.

VI. COMPARISON OF QI PREDICTIONS AND OPA DATA

If we assume that we are observing the variance during half of the period, then $F_T = 0.5$, and the quantum inequality gives a value of $R(F_T)$ whose absolute value could not be exceeded with the maximum possible squeezing during the half period. Similarly, if we are observing for the entire period, then $F_T = 1$. By our understanding, the longer we observe, the more likely we will have regions of variance that are above the vacuum level, and the bigger $R$ will be. For the shortest times, we can have the most squeezing.

In Figure 6 for a Gaussian time function, we have plotted our result for $R(F_T)$ (Eq. (21), top dotted curve), and Marecki’s result $R(F_T) = 10\log_{10}[\text{Erf}(2\sqrt{2}F_T)]$ (middle dotted curve), and $R(F_T)$ (Eq. (19), bottom solid curve) for an ideal OPA. The quantum inequality is very restrictive for both equations. The degree of squeezing obtained in the experiments is greater than that allowed by the QI for all but one experimental point. All data are consistent with the ideal OPA model.
FIG. 6. Squeezing $R$ in dB versus $F_T$, the fraction of observation period that is squeezed, for a Gaussian time function. The top dotted curve is derived as $10 \log_{10} \left[ \text{Erf} \left( \sqrt{\frac{2}{\pi}} F_T \right) \right]$. The middle dotted curve is calculated directly from equations of Reference 8. The solid bottom curve is from the ideal OPA model. Experimental points are shown with error bars.

FIG. 7. $R$ dB versus $F_T$ for a Lorentzian squared time function. Solid line, closest to the x-axis, is our result, which includes a factor of $\pi$; middle dashed curve is Marecki’s result with no $\pi$; and the thick dashed line is for an ideal OPA.

The results for the squared Lorentzian time function were better than for the Gaussian, as shown in Figure 7. Almost all points violated Eq. 22 but only about half the points violate Marecki’s version of the QI with no $\pi$ (middle dashed curve).

One phenomenological approach to understanding the disagreement between data and
FIG. 8. $R$ dB versus $F_T$, showing the best fit for the Lorentzian (solid curve) and the Gaussian (dashed curve) time functions. For the Lorentzian, the arguments are $(1/3\pi)$ (or $1/3$ for Marecki) of the theoretical values. For the Gaussian, the arguments are $(1/4\pi)$ (or $1/4$ for Marecki).

the QI is to try reducing the argument in the equations to improve the agreement of the QI prediction with the data. As the arguments in the Erfc function and the exponential decrease, the agreement does improve. Fitting the functional forms to the data give the plots as shown in Figure 8. No points violate these best fits, but the significance of them is not clear. Certainly, for $F_T$ above about 0.3, they do not appear to not be sufficiently restrictive. They are not as restrictive as the ideal OPA curve.

We evaluated the variance for a symmetric trapezoidal window with a center region $T_S$ long, and sloping sides that are each $nT_S$ long, normalized to 1. The results (dashed curves) are displayed in Figure 9 for a range of values of $n$ from 0.001 (most negative black curve, and $f(t)$ is almost a square window) to 5.0 (dashed curve nearest the origin) which corresponds to a nearly triangular window. The solid curves are for the same $n$ values, but a factor of $\pi$ has been omitted in the argument as Marecki did. The ideal OPA bound is the solid curve crossing all other curves. As the window becomes more triangular, the curves are less restrictive on squeezing and do not agree with the data. Only the curve for the nearly square window ($n = 0.001$) without the $\pi$ is not inconsistent with all the data. Yet this curve clearly fails to be sufficiently restrictive for values of $F_T$ greater than about 0.3 and predicts squeezing exceeding that allowed by an ideal OPA. Mathematically this nearly rectangular window is on the edge of instability especially for low values of the power, and as $n$ decreases further, this window becomes a square window for which the limit is $R = 10\log_{10} 0$. 
FIG. 9. $R\ dB = S\ dB$ versus $F_T$ for a symmetric trapezoidal time function with a center region $T_S$ long and sides $nT_S$ long. The dashed curves are for $n = 0.001$ (most negative $R$), $0.2, 0.5, 1.0, 3.0, 5.0$ (nearest the origin). The solid curves are for Marecki’s result, which omits the $\pi$. The solid curve crossing the other curves is for the ideal OPA.

Clearly the form of the time window is very important, yet for all forms examined, the resultant Quantum Inequalities do not appear to have the right functional form or the numerical values one might expect for a Quantum Inequality applicable to this data.

VII. DISCUSSION AND CONCLUSION

The mathematics of the derivation of the Quantum Inequality appear sound, and the model for the OPA data has been experimentally validated, and it is consistent with all the data we examined. Yet the QI and OPA model do not appear to be consistent with each other. The QI was violated by all the data points for the Gaussian time window and about half the points for the Lorentzian squared time window. Although other windows may show improved results, these inconsistencies suggest a deeper problem.

The model for the ideal OPA gave the best results: no data exceeded its maximum squeezing, yet the most recent data came close. It also predicted that the maximum duration of squeezed light does not exceed $1/2$ the cycle, which agreed with the data, yet was not predicted by the QI. Nevertheless, the ideal OPA is a model that is an approximation with limitations.

One of the issues mentioned was the potential conflict between specifying a time function
function $f(t)$ and an independent frequency function $\mu(\omega)$. To explore this effect, we did a calculation assuming the Gaussian time function and a frequency function $\mu(\omega)$ which was given by the Fourier transform of the time function. Explicit calculation showed that the resulting expression for the Quantum Inequality was similar to that obtained without explicitly giving the precise form of frequency window. Although this conflict is real, it does not appear to be responsible for the systemic disagreement seen between the QI and the OPA data.

Another possible issue might be the frequency dependence of the measured squeezing for the OPA data as predicted by Eq. [12] The output beam of a OPA has the Lorentzian squeezing spectrum with center frequency and halfwidth $\gamma$ that depends on the properties of the resonant cavity. Data are typically taken with a phase $\theta$ and frequency which give the maximum squeezing. Since the Quantum Inequality correlates the fraction of time the signal is squeezed with the dB of squeezing, we may need to account for the change in $F_T$ for frequencies away from the sideband used for the measurement. We can compute an "effective" duration of squeezing $F_{TE}(x)$ which is weighted by integrating the frequency over the variance of the squeezed vacuum. The behavior of $F_{TE}(x)$ will depend on the range over which we integrate and the halfwidth. This integration will increase the effective size of the $F_{TE}(x)$, essentially moving all data points to the right, making the disagreement between data and the QI worse, so this is not the explanation.

Another critical issue concerns the nature of the assumed measurement in the derivation of the Quantum Inequality. The assumption is that a measurement of the energy will be made that lasts a fraction of a cycle of oscillation of the electromagnetic field of the laser being employed. To make an accurate measurement requires observation for a number of cycles. Thus, it does not appear possible to make a good measurement of the squeezing when the observation is for a fraction of a cycle. The measurement of the energy in the OPA method is actually done over many cycles. For a fixed phase difference between the LO and the vacuum signal, the balanced homodyne detection automatically selects the corresponding energy output which is measured continuously over as many cycles of the laser light as desired, ensuring significant accuracy. On the other hand, no such mechanism is available for the measurement assumed to occur in the derivation of the Quantum Inequality. Thus, there appears to be a fundamental inconsistency between the measurement assumed in the derivation of the QI and the measurement method of the OPA. However, it is not clear how to prove that this inconsistency is responsible for the disagreement between the
QI and the data.

The choice of window function is probably the most significant factor when applying the QI to real data. Mathematically, the choice of a window function is simple. However, when comparing theory to data, it is not clear what window function is actually appropriate for the experiment being done, even though the choice dominates the restrictions due to the QI. In addition, Heisenberg and Bohr maintained that measured fields were averages over space-time volumes, whereas Marecki (Eq. A-6) and Ford (Eq. 1) only have a time average.

This work represents the only comparison to date of data to the theory of a quantum inequality. Hopefully, the conundrum of the disagreement between the QI and the OPA measurements will be resolved more fully in the future with interdisciplinary collaborations and more experiments and more detailed theoretical derivations. Our results highlight the subtleties that can be implicit in theoretical derivations of Quantum Inequalities, particularly in the proposed measurement process. Ideally, an unambiguous experimental procedure could be associated with the theoretical derivations. These issues may also affect the applicability of the Quantum Inequalities that have been proposed for other situations.

**Appendix A: Derivation of Quantum Inequality**

Marecki[8] derives a Quantum Inequality for squeezed light and squeezed vacuum following the general approach of Fewster and Teo[10] and Pfenning [11]. We briefly describe his derivation to clarify the comparisons to the OPA data. He defines the operator variance of the normally ordered electric field $\Delta E^2(x, t)$:

$$\Delta E^2(x, t) = E^2(x, t) - \langle E^2(x, t) \rangle_{\text{vac}}$$  \hspace{1cm} (A-1)

and considers a time sampling of the field squared

$$\Delta = \int_{-\infty}^{+\infty} dt f(t) \Delta E^2(x, t)$$  \hspace{1cm} (A-2)

where

$$1 = \int_{-\infty}^{+\infty} dt f(t)$$  \hspace{1cm} (A-3)

He also mentions the possibility of including a frequency sampling function $\mu_p = \mu(\omega - \omega_p)$ that reflects the frequency response of the apparatus measuring the variance. Since it is necessary to use a frequency sampling function to get finite results for $\Delta$, we will include it.
in our derivations. However, we note that there is a potential consistency issue using an independently selected frequency sampling function since the time sampling function \( f(t) \) implies a frequency selection determined by its Fourier transform. Using the Coulomb gauge, the vector potential is

\[
A_i(x,t) = \frac{1}{\sqrt{(2\pi)^3}} \int \frac{d^3 k}{\sqrt{2\omega_k}} \sum_{\alpha=1,2} e_\alpha^i(k) \{ a_\alpha^\dagger(k) e^{ikx} + a_\alpha(k) e^{-ikx} \} \quad (A-4)
\]

where \( \omega_k = |k| \) and \( \alpha \) denotes the two polarization states which are normalized and orthogonal to \( k \). In the exponentials, \( kx = -k \cdot x + \omega t \) represents the scalar product. The electric field operator is

\[
E_i = -\frac{\partial A_i}{\partial t} \quad (A-5)
\]

The expectation value of the time sampled free vacuum field squared is

\[
< E^2 >_{\text{vac}} = \int_{-\infty}^{+\infty} dt f(t) < E^2(x,t) >_{\text{vac}}
\]

\[
= \frac{1}{2(2\pi)^2} \int \mu_k d^3 k \mu_p d^3 p \sqrt{\omega_k \omega_p} 2\pi f_{FT}(\omega_p - \omega_k) \times \sum_{\alpha,\beta=1,2} e_\alpha^i(k) e_\beta^i(p) \delta_{\alpha\beta} \delta(p - k) \quad (A-7)
\]

where we have used the commutator \( [a_\alpha(p), a_\beta^\dagger(k)] = \delta_{\alpha\beta} \delta(p - k) \) and included the frequency function. The Fourier transform is defined as

\[
f_{FT}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt f(t) e^{-i\omega t} \quad (A-8)
\]

Integrating Eq. 7 over \( k \), using the unity normalization of the polarization vectors \( \sum_i e_i^\alpha(k) e_i^\beta(p) = 1 \), and that \( f_{FT}(0) = 1/2\pi \) because of the \( f(t) \) normalization, gives

\[
< E^2 >_{\text{vac}} = \frac{1}{(2\pi)^3} \int (\mu_p^2 \omega_p) d^3 p \quad (A-9)
\]

Substituting this result into the expression for the variance \( \Delta \) gives, after integration over time,

\[
\Delta = \frac{1}{2(2\pi)^2} \int d^3 k d^3 p \mu_k \mu_p \sqrt{\omega_k \omega_p} \times \sum_{\alpha,\beta=1,2} e_\alpha^\beta(k) e_\beta^\beta(p) \{ a_\alpha^\dagger(k) a_\beta(p) e^{i(-k+p)x} f_{FT}(\omega_p - \omega_k) \\
- a_\alpha(k) a_\beta^\dagger(p) e^{i(k+p)x} f_{FT}(\omega_p + \omega_k) + HC \} \quad (A-10)
\]
where HC is the Hermitian conjugate. To derive a quantum inequality, Marecki defines a vector operator \( B_i(\omega) \) and he computes the norm of \( B \) which has to be positive

\[
\int_0^\infty d\omega B_i^\dagger(\omega)B_i(\omega) > 0 \quad \text{(A-11)}
\]

and he takes the expectation value with respect to a state \( A \). By choosing

\[
B_i(\omega) = \frac{1}{\sqrt{2\pi^2}} \int d^3p \sqrt{\omega_p} \sum_{\alpha,\beta=1,2} e^{i(p\cdot x)} \left\{ a_{\alpha}(p)f^{1/2}_{FT}(\omega - \omega_p)e^{ipx} - a_{\alpha}^\dagger(p)f^{1/2}_{FT}(\omega + \omega_p)e^{-ipx} \right\}
\]

(A-12)

and using Eq. 10 above we obtain the result the quoted in the text, Eq. 3.

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