A multi-dimensional SRBM: Geometric views of its product form stationary distribution

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Abstract

We present a geometric interpretation of a product form stationary distribution for a \(d\)-dimensional semimartingale reflecting Brownian motion (SRBM) that lives in the nonnegative orthant. The \(d\)-dimensional SRBM data can be equivalently specified by \(d+1\) geometric objects: an ellipse and \(d\) rays. Using these geometric objects, we establish necessary and sufficient conditions for characterizing product form stationary distribution. The key idea in the characterization is that we decompose the \(d\)-dimensional problem to \(\frac{1}{2}d(d-1)\) two-dimensional SRBMs, each of which is determined by an ellipse and two rays. This characterization contrasts with the algebraic condition of Harrison and Williams [14]. A \(d\)-station tandem queue example is presented to illustrate how the product form can be obtained using our characterization. Drawing the two-dimensional results in [1, 7], we discuss potential optimal paths for a variational problem associated with the three-station tandem queue. Except Appendix D, the rest of this paper is almost identical to the QUESTA paper with the same title.

1 Introduction

A multidimensional semimartingale reflecting Brownian motion (SRBM) has been extensively studied in the past as it serves as the diffusion approximation of a multiclass queueing network and even a more general stochastic network; see, e.g., [12, 13]. In this paper, we focus on a \(d\)-dimensional SRBM \(Z = \{Z(t); t \geq 0\}\) that lives on the nonnegative orthant \(\mathbb{R}^d_+\). Its data consists of a (nondegenerate) \(d \times d\) covariance matrix \(\Sigma\), a drift vector \(\mu \in \mathbb{R}^d\)
and a \( d \times d \) reflection matrix \( R \). An SRBM \( Z \) associated with data \((\Sigma, \mu, R)\) is defined as a (weak) solution of the following equations:

\[
Z(t) = Z(0) + X(t) + RY(t) \in \mathbb{R}_+^d, \quad t \geq 0, \\
X = \{X(t), t \geq 0\} \text{ is a } (\Sigma, \mu)\text{-Brownian motion,} \\
Y(0) = 0, Y(\cdot) \text{ is nondecreasing,} \\
\int_0^\infty Z_i(t)dY_i(t) = 0 \text{ for } i = 1, \ldots, d.
\] (1.1) (1.2) (1.3) (1.4)

see, e.g., Definition 1 of [4] for a precise definition. Thus, in the interior of the orthant, \( Z \) behaves as an ordinary Brownian motion with drift vector \( \mu \) and covariance matrix \( \Sigma \), and \( Z \) is pushed in direction \( R^{(j)} \) whenever \( Z \) hits the boundary surface \( \{z \in \mathbb{R}_+^d : z_j = 0\} \), where \( R^{(j)} \) is the \( j \)th column of \( R \), for \( j = 1, \ldots, d \).

A square matrix \( A \) is said to be an \( S \)-matrix if there exists a vector \( w \geq 0 \) such that \( Aw > 0 \). (Hereafter, we use inequalities for vectors as componentwise inequalities.) It is known that \( Z \) exists and is unique in law for each initial distribution of \( Z(0) \) if and only if \( R \) is a completely-\( S \) matrix, that is, if every principal submatrix of \( R \) is an \( S \) matrix (see, e.g., [9, 23]). We refer to the solution \( Z \) as \((\Sigma, \mu, R)\)-SRBM if the data \( \Sigma, \mu \) and \( R \) need to be specified.

In this paper, we are also concerned with \( R \) being a \( P \) matrix, which is a square matrix whose principal minors are positive, that is, each principal sub-matrix has a positive determinant. A \( P \)-matrix is within a subclass of completely-\( S \) matrices; the still more restrictive class of \( M \)-matrices is defined as in Chapter 6 of [2].

It is also known that the existence of a stationary distribution for \( Z \) requires

\[
R \text{ is nonsingular, and } R^{-1} \mu < 0,
\] (1.5)

but this condition is generally not sufficient (see, e.g.,[3]).

For applications of the \( d \)-dimensional SRBM, it is important to obtain the stationary distribution in a tractable form. However, this is a very hard problem even for \( d = 2 \). Harrison and Williams [14] show that the \( d \)-dimensional SRBM has a product form stationary distribution if and only if the following skew symmetry condition

\[
2\Sigma = R\text{diag}(R)^{-1}\text{diag}(\Sigma) + \text{diag}(\Sigma)\text{diag}(R)^{-1}R^T
\] (1.6)

is satisfied. Here, for a matrix \( A \), \( \text{diag}(A) \) denotes the diagonal matrix whose entries are diagonals of \( A \), and \( A^T \) denotes the transpose of \( A \). Although many SRBMs arising from queueing networks do not have product form stationary distributions, approximations based on product form have been developed to assess the performance of queueing networks; see, [16] for an example in the setting of SRBMs and [18] for an example in the setting of reflecting random walks.

This paper develops an alternative characterization for a \( d \)-dimensional SRBM to have a product form stationary distribution. This new characterization is based on the geometric
objects associated with the SRBM data \((\Sigma, \mu, R)\). More specifically, specifying the SRBM data is equivalent to specifying \(d + 1\) geometric objects: an ellipse \(E\) that is specified by \((\Sigma, \mu)\) and \(d\) rays that are specified through \(R\); the \(i\)th ray is the unique one that is orthogonal to \(R^{(j)}\) for each \(j \neq i\). Ray \(i\) intersects the ellipse at a unique point \(\theta^{(i,r)} \neq 0\). For each pair \(i \neq j\), \(\theta^{(i,r)}\) and \(\theta^{(j,r)}\) span a two-dimensional hyperplane \(\Gamma_{\{i,j\}}\) in \(\mathbb{R}^d\) (see (3.5) for its definition). We draw a line on this hyperplane which goes through the point \(\theta^{(i,r)} \in E\) and keeps constant \(\theta^{(i,r)}\) in its \(i\)-th coordinate. This line either is tangent to the ellipse \(E\) at \(\theta^{(i,r)} \in E\) or intersects the ellipse \(E\) at another point. We denote this point by \(\theta^{ij(i,r)}\) which is identical with \(\theta^{(i,r)}\) if the line is tangent to \(E\), and refer to it as a symmetry point of \(\theta^{(i,r)}\). Similarly, one defines \(\theta^{ij(j,r)}\) to be the symmetry of \(\theta^{(j,r)}\) on the hyperplane \(\Gamma_{\{i,j\}}\).

We prove in Theorem 1 that the SRBM has a product form stationary distribution if and only if \(R\) is a \(P\)-matrix and, for every pair \(i \neq j\),

\[
\theta^{ij(i,r)} = \theta^{ij(j,r)}. 
\] (1.7)

Figure 1 gives an example illustrating points \(\theta^{(1,r)}\) and \(\theta^{(2,r)}\) on the ellipse and their symmetry points \(\theta^{12(1,r)}\) and \(\theta^{12(2,r)}\) when \(d = 2\). Theorem 1 generalizes the two-dimensional result which is proved in [7]: assume \(R\) is a \(P\)-matrix; a two-dimensional SRBM has a product form stationary distribution if and only if

\[
\theta^{12(1,r)} = \theta^{12(2,r)}. 
\] (1.8)

We show that those geometric objects on the hyperplane \(\Gamma_{\{i,j\}}\) correspond to a two-dimensional SRBM, and we can characterize the the product form condition of the \(d\)-dimensional SRBM through the two-dimensional SRBMs. Interestingly, this simultaneously shows that, if \(R\) is a \(P\)-matrix, then \(d\) random variables having the stationary distribution of the SRBM are independent if and only if each pair of them are independent (see Corollary 2).
In recent years, there has been an increasing interest in developing explicit expressions for the tail asymptotic delay rate of the stationary distribution. However, results are limited for \( d = 2 \) (e.g., see [6, 7]). There are some studies for \( d \geq 3 \), but partial results are only available under very restrictive conditions (e.g., see [19]). We hope the present geometric interpretations of the product form will make a new step for studying the stationary distribution of a higher dimensional SRBM. We discuss two topics related to this.

The first topic is about approximation for the stationary distribution. Characterization (1.7) has a potential to allow one to develop new product form based approximations for the performance analysis of a general \( d \)-dimensional SRBM. See [18] for an example of incorporating tail asymptotics into product form approximations.

The second topic is about a variational problem (VP) associated with the SRBM. VP is an important, difficult class of problems that are closely related to the large deviations theory of SRBMs. See, for example, [20, 21, 22] for the connection between large deviations and VPs associated with SRBMs. Except for papers [17, 19], there has been not much progress in solving VPs in \( d \geq 3 \) dimensions. When \( d = 2 \), [1] shows that the entrance velocities \( \tilde{a}(1) \) and \( \tilde{a}(2) \) from the first and second boundary, respectively, play a key role in obtaining the optimal paths of a VP; see also [11]. In [7], the authors show that

\[
\tilde{a}(2) = \mu + \Sigma \theta^{12(1,r)}.
\]

Namely, the entrance velocity \( \tilde{a}(2) \) from the second boundary (the \( x \) axis) is equal to the outward normal direction of the ellipse \( E \) at the symmetry point \( \theta^{12(1,r)} \). An analogous formula holds for \( \tilde{a}(1) \).

For a \( d \)-dimensional SRBM with a product form stationary distribution, we have the set of the two dimensional SRBMs which are used to characterize the product form. These two dimensional SRBMs may be useful to find the optimal path for the VP because we can apply the results in [1]. However, we also need to consider higher dimensional versions of the entrance velocities. This topic will be discussed using an example, and we conjecture the optimal path for a three-dimensional product form SRBM.

This paper consists of five sections. In Section 2, we introduce the basic geometric objects and derive the basic adjoint relationship (BAR) using the moment generating functions. We also derive a BAR in quadratic form that characterizes the existence of a product form stationary distribution. This characterization is the foundation of our analysis. In Section D.1, we introduce the projection idea from the \( d \)-dimensional problem to two-dimensional ones and present our main theorem, Theorem 1. In Section D.2, we give a detailed proof of the Theorem 1. In Section 5, we discuss SRBMs arising from tandem queues and the optimal path for some multi-dimensional VPs.

We will use the following notation unless otherwise stated.
\( J \) \{1, 2, \ldots, d\}

\( T^{(i)} \) the \( i \)-th column of a square matrix \( T \)

\( T^{ij} \) 2-dimensional principal matrix composed of the \( i \)-th and \( j \)-th rows of \( T \)

\( x^{ij} \) \((x_i, x_j)^T \in \mathbb{R}^2 \) for \( x \in \mathbb{R}^d \)

\( x_A \) for \( A \subset J \) the \( d \)-dimensional vector whose \( i \)-th entry is \( x_i \) for \( i \in A \) and the others zero

\( x^A \) for \( A \subset J \) \(|A|\)-dimensional vector \((x_i : i \in A)\)

\( (x, y) \) \(\sum_{i=1}^d x_i y_j \) for \( x, y \in \mathbb{R}^d \)

Table 1: A summary of basic notation

2 The stationary distribution and its product form characterization

We assume that \( \Sigma \) is positive definite and \( R \) is completely-\( S \) so that \( Z \) exists. They, together with the drift \( \mu \), constitute the primitive data of the SRBM. We first describe them in terms of \( d \)-dimensional polynomials, which are defined as

\[
\gamma(\theta) = -\frac{1}{2} \langle \theta, \Sigma \theta \rangle - \langle \mu, \theta \rangle, \quad \theta \in \mathbb{R}^d,
\]

\[
\gamma_i(\theta) = \langle R^{(i)} \theta \rangle, \quad \theta \in \mathbb{R}^d, \quad i \in J,
\]

where \( R^{(i)} \) is the \( i \)-th column of the reflection matrix \( R \). Obviously, those polynomials uniquely determine the primitive data, \( \Sigma \), \( \mu \) and \( R \). Thus, we can use those polynomials to discuss everything about the SRBM instead of the primitive data themselves.

Assume the SRBM has a stationary distribution. The stationary distribution must be unique. Our first tool is the stationary equation that characterizes the stationary distribution. For this, we first introduce the boundary measures for a distribution \( \pi \) on \((\mathbb{R}_+^d, \mathcal{B}(\mathbb{R}_+^d))\), where \( \mathcal{B}(\mathbb{R}_+^d) \) is the Borel \( \sigma \)-field on \( \mathbb{R}_+^d \). They are defined as

\[
\nu_i(B) = \mathbb{E}_\pi \left[ \int_0^1 1\{Z(t) \in B\} dY_i(t) \right], \quad B \in \mathcal{B}(\mathbb{R}_+^d), \quad i \in J.
\]

The stationary equation is in terms of moment generating functions, which are defined as

\[
\varphi(\theta) = \mathbb{E}_\pi[e^{\langle \theta, Z(\cdot) \rangle}], \quad \varphi_i(\theta) = \mathbb{E}_\pi \left[ \int_0^1 e^{\langle \theta, Z(t) \rangle} dY_i(t) \right], \quad i \in J,
\]

where \( \mathbb{E}_\pi \) is the expectation operator when \( Z(0) \) is subject to the distribution \( \pi \).

Because for each \( i \in J \), \( Y_i(t) \) increases only when \( Z_i(t) = 0 \), one has \( \varphi_i(\theta) \) depends on \( \theta_{J \setminus \{i\}} \) only, where \( \theta_A \) for \( A \subset J \) is the \( d \)-dimensional vector whose \( i \)-th entry is identical with that of \( \theta \) for \( i \in A \) and the entry is zero for \( i \in J \setminus A \). Therefore,

\[
\varphi_i(\theta) = \varphi_i(\theta_{J \setminus \{i\}}).
\]
The following lemma is identical to Lemma 1 in [8]. We state it here for easy reference.

**Lemma 1.** (a) Assume $\pi$ is the stationary distribution of a $(\Sigma, \mu, R)$-SRBM. For $\theta \in \mathbb{R}^d$, $\varphi(\theta) < \infty$ implies $\varphi_i(\theta) < \infty$ for $i \in J$. Furthermore,

$$
\gamma(\theta) \varphi(\theta) = \sum_{i=1}^{d} \gamma_i(\theta) \varphi_i(\theta),
$$

(2.1)

holds for $\theta \in \mathbb{R}^d$ such that $\varphi(\theta) < \infty$. (b) Assume that $\pi$ is a probability measure on $\mathbb{R}^d_+$ and that $\nu_i$ is a positive finite measure whose support is contained in $\{x \in \mathbb{R}^d_+ : x_i = 0\}$ for $i \in J$. Let $\varphi$ and $\varphi_i$ be the moment generating functions of $\pi$ and $\nu_i$, respectively. If $\varphi$, $\varphi_1$, $\ldots$, $\varphi_d$ satisfy (2.1) for each $\theta \in \mathbb{R}^d$ with $\theta \leq 0$, then $\pi$ is the stationary distribution and $\nu_i$ is the boundary measure of the associated SRBM on $\{x \in \mathbb{R}^d_+ : x_i = 0\}$.

Equation (2.1) is the moment generating function version of the standard basic adjoint relationship (BAR) that was first derived in [13]; for the standard BAR, see also equation (7) of [4]. Part (a) is now standard, following Proposition 3 of [4] and Lemma 4.1 of [6]. For part (b), one can follow a standard procedure (see Proposition 1 in Appendix D) to argue that Equation (2.1) is equivalent to the standard BAR. The rest of part (b) is implied by [5].

From now on, we always assume that $\pi$ is the stationary distribution of the SRBM unless otherwise is stated. It follows from [14] that the stationary distribution of SRBM, when exists, has a density. We use $\zeta(y)$ to denote the stationary density of $d$-dimensional SRBM. Thus, the stationary distribution has product form if and only if

$$
\zeta(y) = \prod_{i=1}^{d} \zeta_i(y_i),
$$

(2.2)

where $\zeta_i$’s are the marginal densities of $\zeta$. It follows from the first Theorem in Section 9 of [13] on page 107 that when the stationary density is of product form in (2.2), each $\zeta_i$ must be exponential. Thus, $d$-dimensional SRBM has a product form if and only if there exists a $d$-dimensional vector $\alpha > 0$ such that

$$
\zeta(y) = \prod_{i=1}^{d} \alpha_i e^{-\alpha_i y_i}.
$$

(2.3)

It is shown in [14] that, under the skew symmetry condition (1.6), the SRBM has a product-form stationary density in (2.3) and $\alpha$ is given by

$$
\alpha = -2\text{diag}(\Sigma)^{-1}\text{diag}(R)R^{-1}\mu.
$$

(2.4)

In this paper, we provide alternative characterizations for the product form in terms of a set of two-dimensional SRBMs. For each two-dimensional SRBM, a geometric interpretation for the product form condition is derived in [7], and therefore the necessary and sufficient condition of this paper has also geometric interpretation.
The following is a key lemma to characterize the product form of SRBM which will be used repeatedly in this paper.

**Lemma 2.** Assume $R$ is completely-$S$ and condition (1.5) is satisfied. The $d$-dimensional SRBM has a product form stationary distribution with its density in (2.3) for some $\alpha = (\alpha_1, \ldots, \alpha_d)^T > 0$ if and only if for some positive constants $C_1, \ldots, C_d$

$$\gamma(\theta) = \sum_{i=1}^{d} C_i \gamma_i(\theta)(\alpha_i - \theta_i) \quad \text{for } \theta \in \mathbb{R}^d. \quad (2.5)$$

Furthermore, if (2.5) holds, then $C_i = \Sigma_{ii}/(2R_{ii})$, $\alpha$ is given in (2.4), and $\gamma(\alpha) = 0$.

**Remark 1.** The above lemma can also be used to show that (1.6) holds if and only if the stationary distribution of SRBM has a product form. See Appendix A.

**Proof.** Assume that SRBM has a product form stationary density as in (2.3) for some $\alpha = (\alpha_1, \ldots, \alpha_d)^T > 0$. Then following from [13], we know that the boundary measure $\nu_i$ has density:

$$\zeta_i(y) = \frac{\Sigma_{ii}}{2R_{ii}} \alpha_i \prod_{k \neq i} e^{-\alpha_k y_k} \quad \text{for } i \in J \text{ and } y \in \mathbb{R}_+^d. \quad (2.6)$$

Following the above equations, we have:

$$\varphi(\theta) = \prod_{i=1}^{d} \frac{\alpha_i}{\alpha_i - \theta_i} \quad \text{for } \theta < \alpha. \quad (2.7)$$

$$\varphi_i(\theta) = \frac{\Sigma_{ii}}{2R_{ii}} \alpha_i \prod_{k \neq i} \frac{\alpha_k}{\alpha_k - \theta_k} \quad \text{for } \theta < \alpha. \quad (2.8)$$

Substituting these $\varphi(\theta)$ and $\varphi_i(\theta)$ into (2.1) of Lemma 1, we have (2.5) for any $\theta < \alpha$. In particular, (2.5) holds for infinitely many $\theta$’s. Since both sides of (2.5) are quadratic in $\theta$, (2.5) holds for all $\theta \in \mathbb{R}^d$. Conversely, if there exists an $\alpha > 0$ such that (2.5) holds, one can define $\varphi(\theta)$ and $\varphi_i(\theta)$ as in (2.7) and (2.8). They satisfy (2.1). Then the moment generating functions of the stationary density (2.3) and boundary densities (2.6) satisfy (2.1) for $\theta \leq 0$. So according to part (b) of Lemma 1, the SRBM must have (2.3) as its stationary density.

Assume (2.5) holds. Comparing the coefficients of $\theta_i^2$ on both sides of (2.5), we obtain $C_i = \Sigma_{ii}/2R_{ii}$ for $1 \leq i \leq d$. By comparing the coefficients of $\theta_i$ for $i = 1, \ldots, d$, we have (2.4). The fact that $\gamma(\alpha) = 0$ in the last statement is easily verified by substituting $\theta = \alpha$ into (2.5). □
3 Geometric objects and main results

We consider a $d$-dimensional SRBM having data $(\Sigma, \mu, R)$. We assume that $\Sigma$ is positive definite, $R$ is completely-$S$, and condition (1.5) holds. From BAR (2.1), one can imagine that the tail decay rate of the stationary distribution would be related to the $\theta$ at which the coefficients of $\varphi(\theta)$ and $\varphi_i(\theta_{J\setminus\{i\}})$ in (2.1) becomes zero. Thus, we introduce the following geometric objects:

$$E = \{\theta \in \mathbb{R}^d; \gamma(\theta) = 0\}, \quad \Gamma_{\{i\}} = \cap_{k \in J \setminus \{i\}} \{\theta \in \mathbb{R}^d; \gamma_k(\theta) = 0\}, \quad i \in J.$$ 

These geometric objects are well defined even when the SRBM does not have a stationary distribution. The object $E$ is an ellipse in $\mathbb{R}^d$. Since $R$ is invertible and $\theta \in \Gamma_{\{i\}}$ implies that $\langle \theta, R^{(k)} \rangle = 0$ for $k \neq i$, $\Gamma_{\{i\}}$ must be a line going through the origin. Clearly, for each $i$, $\Gamma_{\{i\}}$ intersects the ellipse $E$ at most two points, one of which is the origin. We denote its non-zero intersection by $\theta^{(i,r)}$ if it exists. Otherwise, let $\theta^{(i,r)} = 0$. The following lemma shows that the latter is impossible by giving an explicit formula for $\theta^{(i,r)}$. For that let $B = (R^{-1})^T$ and $B^{(i)}$ be the $i$th column of $B$. Equivalently, the transpose of $B^{(i)}$ is the $i$th row of $R^{-1}$.

**Lemma 3.** For each $i \in J$,

$$\theta^{(i,x)} = \Delta_i B^{(i)},$$

where

$$\Delta_i = -\frac{2\langle \mu, B^{(i)} \rangle}{\langle B^{(i)}, \Sigma B^{(i)} \rangle} > 0.$$ (3.2)

**Proof.** Because $R^{-1}R = I$, we have $B^{(i)} \neq 0$ and $B^{(i)} \in \Gamma(i)$. Therefore, (3.1) holds for some $\Delta_i \in \mathbb{R}$. Since $\theta^{(i,x)} \in E$, we have $\gamma(\Delta_i B^{(i)}) = 0$, from which we have the equality in (3.2). If (1.5) holds, we have $\langle \mu, B^{(i)} \rangle < 0$, from which the inequality in (3.2) holds.

Let

$$A = (\theta^{(1,x)}, \ldots, \theta^{(d,x)})$$

and

$$A^{ij} = \begin{pmatrix} \theta^{(i,x)}_i & \theta^{(j,x)}_i \\ \theta^{(i,x)}_j & \theta^{(j,x)}_j \end{pmatrix} \quad \text{for } 1 \leq i < j \leq d.$$ (3.3)

By Lemma 3, $A = B\Delta$, where $\Delta = \text{diag}(\Delta_1, \ldots, \Delta_d)$. Clearly, $A^{ij}$ is a $2 \times 2$ principal sub-matrix of $A$. We let

$$c_{ij} = \text{det}(A^{ij}).$$ (3.4)

For each pair $(i, j)$ with $i, j \in J$ and $i < j$, we define the two-dimensional hyperplane in $\mathbb{R}^d$:

$$\Gamma_{\{i,j\}} = \cap_{k \in J \setminus \{i,j\}} \{\theta \in \mathbb{R}^d; \gamma_k(\theta) = 0\}.$$ (3.5)

We then define a mapping $f^{ij}$ from $\mathbb{R}^2$ to $\Gamma_{\{i,j\}} \subset \mathbb{R}^d$ such that $f^{ij}(\theta_i, \theta_j) = \theta$ for any $\theta \in \Gamma_{\{i,j\}}$. Sometimes, we write $f^{ij}(\theta_i, \theta_j)$ as $f^{ij}(\theta^{ij})$ where $\theta^{ij} = (\theta_i, \theta_j)^T$. The following lemma confirms that $f^{ij}$ is well defined if $c_{ij} \neq 0$. Its proof is given in Appendix B.
Lemma 4. For each $i \neq j \in J$, if $c_{ij} = \det(A^i_j) \neq 0$, then the mapping $f^{ij}$ defined above uniquely exists.

Since both points $\theta^{(i,x)}$ and $\theta^{(j,x)}$ are on $\Gamma_{i,j}$ and they are linearly independent, for $z^{ij} \equiv (z_i, z_j)^T \in \mathbb{R}^2$, $f^{ij}(z^{ij})$ is a linear combination of $\theta^{(i,x)}$ and $\theta^{(j,x)}$. Indeed, one can check that

$$f^{ij}(z^{ij}) = \frac{1}{c_{ij}} \left( (\theta^{(i,x)}_i z_i - \theta^{(j,x)}_i z_j) \theta^{(i,x)} + (-\theta^{(i,x)}_j z_i + \theta^{(j,x)}_j z_j) \theta^{(j,x)} \right). \quad (3.6)$$

We remark that $c_{ij}$ in (3.4) can be zero even if $R$ is completely-$\mathcal{S}$ and condition (1.5) is satisfied; see Example 1 in Appendix C.

The intersection $E \cap \Gamma_{i,j}$ of the ellipse and the hyperplane is an ellipse on hyperplane $\Gamma_{i,j}$. Both $\theta^{(i,x)}$ and $\theta^{(j,x)}$ are on $E \cap \Gamma_{i,j}$. Now we define two points $\theta^{ij(i,x)}$ and $\theta^{ij(j,x)}$ that are symmetries of $\theta^{(i,x)}$ and $\theta^{(j,x)}$ on $E \cap \Gamma_{i,j}$, respectively. If $\theta^{ij(i,x)} = \text{argmax}\{\theta_i : \theta \in E \cap \Gamma_{i,j}\}$, define $\theta^{ij(j,x)} = \theta^{(i,x)}$. Otherwise, define $\theta^{ij(j,x)}$ to be the unique $\theta \in \mathbb{R}^d$ that satisfies

$$\theta \in E \cap \Gamma_{i,j}, \quad \theta_i = \theta^{ij(i,x)}, \quad \text{and} \quad \theta \neq \theta^{(i,x)}. \quad (3.8)$$

The point $\theta^{ij(j,x)}$ is well defined because of the following lemma, which will be proved in Appendix B.

Lemma 5. If (3.7) holds, the quadratic equation

$$\gamma \left( \frac{1}{c_{ij}} \left( (\theta^{ij(i,x)}_i \theta^{ij(i,x)}_i - \theta^{ij(j,x)}_i z_j) \theta^{ij(i,x)} + (-\theta^{ij(i,x)}_j \theta^{ij(i,x)}_i + \theta^{ij(j,x)}_j z_j) \theta^{ij(j,x)} \right) \right) = 0 \quad (3.9)$$

has a unique (double) solution $z_j = \theta^{ij(j,x)}$. Otherwise, (3.9) has two solutions $z_j' = \theta^{ij(j,x)}$ and $z_j \neq \theta^{ij(j,x)}$.

Let $z_j$ be the solution in Lemma 5. Then the symmetry of $\theta^{(i,x)}$ is equal to

$$\theta^{ij(i,x)} = f^{ij}(\theta^{ij(i,x)}, z_j).$$

Similarly we define $\theta^{ij(j,x)} = \theta^{(j,x)}$ if $\theta^{ij(j,x)} = \text{argmax}\{\theta_j : \theta \in E \cap \Gamma_{i,j}\}$. Otherwise, it is defined to the unique $\theta \in \mathbb{R}^d$ that satisfies $\theta \in E \cap \Gamma_{i,j}$, $\theta_j = \theta^{ij(j,x)}$, and $\theta \neq \theta^{(j,x)}$.

We need one more lemma, which will be proved in Appendix B.

Lemma 6. (a) $R^{-1}$ is a $\mathcal{P}$-matrix if $R$ is a $\mathcal{P}$-matrix.
(b) Assume that $R^{-1} \mu < 0$, $c_{ij}$ in (3.4) is positive if $R$ is a $\mathcal{P}$-matrix.
(c) If $d$-dimensional SRBM has a product form stationary distribution, then $R$ is a $\mathcal{P}$-matrix, and therefore $c_{ij} > 0.$
Theorem 1. Assume that $\Sigma$ is a $d \times d$ positive definite matrix and $\mu$ is a $d$-dimensional vector. Assume that $R$ is a $d \times d$ completely-S matrix, and $(R, \mu)$ satisfies (1.5). (a) The $(\Sigma, \mu, R)$-SRBM has a product form stationary distribution if and only if $R$ is a $P$-matrix and
\[
\theta^{ij(i,x)} = \theta^{ij(j,x)} \quad \text{for each } 1 \leq i < j \leq d. \tag{3.10}
\]
(b) If for each $1 \leq i < j \leq d$, $A^{ij}$ in (3.3) is a $P$-matrix and (3.10) is satisfied, then the $(\Sigma, \mu, R)$-SRBM has a product form stationary distribution.

The proof of Theorem 1 will be given in Section D.2. For that, we define
\[
\Sigma^* = A^T \Sigma A \quad \text{and} \quad \mu^* = A^T \mu. \tag{3.11}
\]
The main idea in the proof is to prove that the $d$-dimensional SRBM has a product form stationary distribution if and only if for each $1 \leq i < j \leq d$, the two-dimensional $(\tilde{\Sigma}^{ij}, \tilde{\mu}^{ij}, \tilde{R}^{ij})$-SRBM is well defined and has a product form stationary distribution, where
\[
\tilde{\Sigma}^{ij} = ((A^{ij})^T)^{-1}(\Sigma^*)^{ij}(A^{ij})^{-1},
\]
\[
\tilde{\mu}^{ij} = ((A^{ij})^T)^{-1}(\mu^*)^{ij}, \quad \text{and} \quad \tilde{R}^{ij} = ((A^{ij})^T)^{-1}\text{diag}(\Delta_i, \Delta_j). \tag{3.13}
\]
In the following corollary, we set
\[
\tau = (\theta^{i,x}_1, \ldots, \theta^{i,x}_d)T, \quad \text{and} \quad \tau^{ij} = (\tau_i, \tau_j)^T. \tag{3.14}
\]

Corollary 1. Under the assumptions of Theorem 1, the $(\Sigma, \mu, R)$-SRBM has a product form stationary distribution if and only if and for each $i, j \in J$ with $i < j$, $A^{ij}$ is a $P$-matrix and
\[
\gamma(f^{ij}(\tau^{ij})) = 0, \tag{3.15}
\]
\[
\theta^{i,x}_j \neq \tau_j \quad \text{if } \theta^{i,x}_i \neq \text{argmax}\{\theta_i : \theta \in E \cap \Gamma_{\{i,j\}}\}, \quad \text{and} \tag{3.16}
\]
\[
\theta^{j,x}_i \neq \tau_i \quad \text{if } \theta^{j,x}_j \neq \text{argmax}\{\theta_j : \theta \in E \cap \Gamma_{\{i,j\}}\}. \tag{3.17}
\]
Corollary 1 will be proved shortly below. One may wonder how the two-dimensional $(\tilde{\Sigma}^{ij}, \tilde{\mu}^{ij}, \tilde{R}^{ij})$-SRBM is related to the two-dimensional marginal process $\{(Z_i(t), Z_j(t)), t \geq 0\}$. The next corollary answers this question. The proof of this corollary will be given at the end of Section D.2. To state the corollary, let $Z(\infty)$ be a random vector that has the distribution to the stationary distribution of the $d$-dimensional SRBM $Z = \{Z(t), t \geq 0\}$. $Z_i(\infty)$ and $Z_j(\infty)$ are the $i$th and $j$th components of $Z(\infty)$ for each $i \neq j \in J$.

Corollary 2. Under the assumptions of Theorem 1, we have the following facts. (a) The $d$-dimensional SRBM $Z = \{Z(t), t \geq 0\}$ has a product form stationary distribution if and only if, for each $i \neq j \in J$, $\tilde{R}^{ij}$ is a $P$-matrix and the two-dimensional $(\tilde{\Sigma}^{ij}, \tilde{\mu}^{ij}, \tilde{R}^{ij})$-SRBM has a product form stationary distribution, which is identical to the distribution of $(Z_i(\infty), Z_j(\infty))$. (b) If $R$ is a $P$-matrix, then $Z_1(\infty), Z_2(\infty), \ldots, Z_d(\infty)$ are independent if and only if, for each $i \neq j \in J$, $Z_i(\infty)$ and $Z_j(\infty)$ are independent.
Proof of Corollary 1. We first prove the necessity. Assume $i < j$. Since $R$ is a $P$-matrix, $R^{-1}$ is a $P$-matrix by Lemma 6. This implies that $A$ is a $P$-matrix, and therefore $A^{ij}$ is a $P$-matrix.

Recall that $\theta^{(i,x)}_i = \theta^{(i,x)}_i = \tau_i$ and $\theta^{(j,x)}_j = \theta^{(j,x)}_j = \tau_j$. Thus, (3.10) is equivalent to

$$\theta^{(i,x)}_i = f^{ij}(\tau^{ij}), \quad (3.18)$$

$$\theta^{(j,x)}_j = f^{ij}(\tau^{ij}). \quad (3.19)$$

Assume the product form stationary distribution. Then, by part (a) of Theorem 1, (3.10) holds. As a consequence, both (3.18) and (3.19) hold. Since $\theta^{(i,x)}_i$ is on the ellipse, (3.18) implies (3.15). We now prove (3.16) must hold. Assume that $\theta^{(i,x)}_i \neq \arg\max\{\theta_i : \theta \in E \cap \Gamma_{\{i,j\}}\}. \quad (3.20)$

Suppose on the contrary that $\theta^{(i,x)}_i = \tau_i$. This implies that $\theta^{(i,x)}_j = \theta^{(i,x)}_j$, which contradicts the condition $\theta^{(i,x)}_j \neq \theta^{(i,x)}_j$ in the definition of (3.8). Similarly, we can prove (3.17) holds. This proves the necessity.

Now we prove the sufficiency. Let $i, j \in J$ with $i < j$. Assume $A^{ij}$ is a $P$-matrix and (3.15)-(3.17) hold. Then (3.15) and (3.16) imply (3.18), and (3.15) and (3.17) imply (3.19). Thus, (3.10) holds. It follows from part (b) of Theorem 1 that the SRBM has a product form stationary distribution.

Remark 2. In the two-dimensional case, when $\tau_1 \neq \theta^{(2,x)}_1$ and $\tau_2 \neq \theta^{(1,x)}_2$, the SRBM has a product form stationary distribution if and only if the point $\tau$ is on the ellipse, i.e., $\gamma(\tau) = 0$. Example 2 in Appendix C shows that when $d \geq 3$, the condition $\gamma(\tau) = 0$ is not sufficient for a product form stationary distribution.

We end this section by stating a lemma that will be needed in the proof of Theorem 1, and proved in Appendix B. To state the following lemma, for $z^{ij} = (z_i, z_j)^T \in \mathbb{R}^2$, let

$$\tilde{\gamma}^{ij}(z^{ij}) = -\frac{1}{2}(z^{ij}, \tilde{\Sigma}^{ij}z^{ij}) - \langle \tilde{\mu}^{ij}, z^{ij} \rangle. \quad (3.21)$$

Lemma 7. For $z^{ij} = (z_i, z_j)^T \in \mathbb{R}^2$, $\tilde{\gamma}^{ij}(z^{ij}) = \gamma(f^{ij}(z^{ij})).$

4 Proof of Theorem 1

We first prove the necessity in (a) of Theorem 1. Assume that the $(\Sigma, \mu, R)$-SRBM has a product form stationary distribution. Therefore, for some $\alpha \in \mathbb{R}^d$ with $\alpha > 0$, (2.5) holds for every $\theta \in \mathbb{R}^d$.

By part (c) of Lemma 6, $R$ is a $P$-matrix. We now prove (3.10). By Lemma 4, it suffices to prove that for any $1 \leq i < j \leq d$,

$$\theta^{(i,x)}_i = \theta^{(j,x)}_i = \alpha_i, \quad \text{and} \quad \theta^{(i,x)}_j = \theta^{(j,x)}_j = \alpha_j, \quad (4.1)$$


where the condition $c_{ij} \neq 0$ is satisfied by Lemma 6.

To prove (4.1), we follow the derivation from (5.17) to (5.21) of [7]. Observe that
\[ \gamma(\theta^{(i,r)}) = 0, \quad \gamma_k(\theta^{(i,r)}) = 0 \quad \text{for} \quad k \in J \setminus \{i\}, \quad \text{and} \quad \gamma_i(\theta^{(i,r)}) \neq 0; \] the latter holds because $R$ is
assumed to be invertible in (1.5). Plugging $\theta = \theta^{(i,r)}$ into (2.5), we have $\alpha_i = \theta^{(i,r)}_i$. Thus,
by the definition of $\theta^{ij(i,r)}$, we have $\theta^{ij(i,r)}_i = \theta_i^{(i,r)} = \alpha_i$. We now show that
\[ \theta^{ij(i,r)}_j = \alpha_j. \] (4.2)

To see this, by the definition of $\theta^{ij(i,r)}$, we have $\gamma(\theta^{ij(i,r)}) = 0$, $\gamma_k(\theta^{ij(i,r)}) = 0$ for $k \in J \setminus \{i, j\}$. If $\theta^{ij(i,r)} \neq \theta^{(i,r)}$, we have $\gamma_j(\theta^{ij(i,r)}) \neq 0$. Plugging $\theta = \theta^{ij(i,r)}$ into (2.5), we have
\[ \gamma_j(\theta^{ij(i,r)})(\alpha_j - \theta^{ij(i,r)}_j) = 0, \]
from which we conclude that (4.2) holds. If $\theta^{ij(i,r)} = \theta^{(i,r)}$, then (3.7) holds. According to
Lemma 7, $\theta^{(i,r)} = \arg \max \{ z_i : z_{ij} = (z_i, z_j)^T \in \mathbb{R}^2, \hat{\gamma}^{ij}(z^{ij}) = 0 \}$. At the same time, if (2.5)
holds, then plugging the definition of $f^{ij}(z^{ij})$ in (3.6) into (2.5), we have
\[ \hat{\gamma}^{ij}(z^{ij}) = \gamma(f^{ij}(z^{ij})) = C_i \Delta_j c^{-1}_{ij}(\theta^{(i,r)}_j z_i - \theta^{(j,r)}_i z_j)(\alpha_i - z_i) + C_j \Delta_j c^{-1}_{ij}(-\theta^{(i,r)}_i z_i + \theta^{(i,r)}_j z_j)(\alpha_j - z_j). \] (4.3)

Taking derivative in the both sides of (4.3) with respect to $z_j$, and plugging $(\theta^{ij(i,r)}_i, \theta^{ij(i,r)}_j)^T$
into the new equation, we again conclude that (4.2) holds as
\[ \frac{\partial \hat{\gamma}^{ij}(z^{ij})}{\partial z_j} \bigg|_{z^{ij} = (\theta^{ij(i,r)}_i, \theta^{ij(i,r)}_j)^T} = C_j \Delta_j c^{-1}_{ij}(\theta^{(i,r)}_i - \theta^{(j,r)}_j)(\alpha_j - \theta^{ij(i,r)}_j) = 0. \]

Similarly, we can show that
\[ \theta^{ij(j,x)}_i = \alpha_i, \quad \text{and} \quad \theta^{ij(j,x)}_j = \alpha_j, \]
thus proving (4.1). This concludes the necessity proof.

We note that the sufficiency of (a) in Theorem 1 is immediate from (b) and Lemma 3
because $R^{-1}$ is $\mathcal{P}$-matrix by (a) of Lemma 6. Thus, it remains only to prove (b). For
any fixed pair $1 \leq i < j \leq d$, assume that $A^{ij}$ is a $\mathcal{P}$-matrix and (3.10) holds. Using
Theorem 5.1 of [7], we would like to conclude that two-dimensional $(\tilde{\Sigma}^{ij}, \tilde{\mu}^{ij}, \tilde{R}^{ij})$-SRBM
has a product form stationary distribution.

Now we prove that (3.10) implies condition (5.2) in Theorem 5.1 of [7]. For this,
we define the geometric objects associated with the two-dimensional $(\tilde{\Sigma}^{ij}, \tilde{\mu}^{ij}, \tilde{R}^{ij})$-SRBM.
Recall the definition of $\tilde{\gamma}^{ij}(z^{ij})$ in (3.21). Then, $\tilde{\gamma}^{ij}(z^{ij}) = 0$ defines an ellipse in $\mathbb{R}^2$. Let
\[ \tilde{\gamma}^{ij}_k(z^{ij}) = \langle z^{ij}, (\tilde{R}^{ij})^{(k)} \rangle, \quad k = i, j, \] (4.4)
where \((\tilde{R}^{ij})^{(k)}\) is the kth column of \(\tilde{R}^{ij}\). Then \(\tilde{\gamma}_k^{ij}(z^{ij}) = 0\) defines a line in \(\mathbb{R}^2\) for \(k = 1, 2\). We next find the non-zero intersection points of the ellipse \(\tilde{\gamma}(z^{ij}) = 0\) and the lines \(\tilde{\gamma}_i^{ij}(z^{ij}) = 0\) and \(\tilde{\gamma}_j^{ij}(z^{ij}) = 0\), respectively, on \(\mathbb{R}^2\). By (3.6), we have

\[
f^{ij}(\theta^{(i,x)}_i, \theta^{(i,x)}_j) = \theta^{(i,x)}_i, \quad \text{and} \quad f^{ij}(\theta^{(j,x)}_i, \theta^{(j,x)}_j) = \theta^{(j,x)}_j.
\]

Therefore, we can use Lemma 7 and expressions

\[
\tilde{\gamma}_i^{ij}(z^{ij}) = c_{ij}^{-1} \Delta_i(z_i\theta^{(j,x)}_j - z_j\theta^{(i,x)}_i) \quad \text{and} \quad \tilde{\gamma}_j^{ij}(z^{ij}) = c_{ij}^{-1} \Delta_j(-z_i\theta^{(i,x)}_j + z_j\theta^{(i,x)}_i)
\]

to verify that these intersection points are given by

\[
(\theta^{(i,x)}_i, \theta^{(i,x)}_j)^T, \quad \text{and} \quad (\theta^{(j,x)}_i, \theta^{(j,x)}_j)^T.
\]

Define

\[
\tilde{\theta}^{ij(i,x)}_i = (\theta^{ij(i,x)}_i, \theta^{ij(i,x)}_j)^T, \quad \text{and} \quad \tilde{\theta}^{xij(j,x)} = (\theta^{xij(j,x)}_i, \theta^{xij(j,x)}_j)^T.
\]

Then \(f^{ij}(\tilde{\theta}^{ij(i,x)}_i) = \theta^{ij(j,x)}_i\). By Lemma 7, we have \(\tilde{\gamma}^{ij}(\tilde{\theta}^{ij(i,x)}_i) = \gamma(\tilde{\theta}^{ij(i,x)}_i) = 0\), where the latter equality follows from the definition of \(\theta^{ij(i,x)}_i\). It follows from Lemma 7 that (3.7) holds if and only if

\[
\theta^{(i,x)}_i = \arg\max\{z_i : z^{ij} = (z_i, z_j)^T \in \mathbb{R}^2, \tilde{\gamma}^{ij}(z^{ij}) = 0\}. \quad (4.5)
\]

Therefore, we have \(\tilde{\theta}^{ij(i,x)}_j = \theta^{ij(i,x)}_j\) if and only if (4.5) holds. Thus, we have proved that \(\tilde{\theta}^{ij(i,x)}_j\) is the symmetric point of \((\theta^{(i,x)}_i, \theta^{(i,x)}_j)^T\) on \(\tilde{\gamma}^{ij}(z^{ij}) = 0\). Similarly, we can verify that \(\tilde{\theta}^{ij(j,x)}_i\) is the symmetric point of \((\theta^{(j,x)}_i, \theta^{(j,x)}_j)^T\) on \(\tilde{\gamma}^{ij}(z^{ij}) = 0\). Condition (3.10) implies that

\[
\tilde{\theta}^{ij(i,x)}_j = \tilde{\theta}^{ij(j,x)}_i.
\]

Thus, Condition (5.2) of [7] is satisfied for the \((\tilde{\Sigma}^{ij}, \tilde{\mu}^{ij}, \tilde{R}^{ij})\)-SRBM.

It follows that the two-dimensional \((\tilde{\Sigma}^{ij}, \tilde{\mu}^{ij}, \tilde{R}^{ij})\)-SRBM has a product form stationary distribution. Furthermore, it follows from (5.28) in the proof of Theorem 5.1 in [7], there exist two constants \(d^{ij}_i > 0\) and \(d^{ij}_j > 0\) such that for any \(z^{ij} \in \mathbb{R}^2\)

\[
\tilde{\gamma}^{ij}(z_i, z_j) = d^{ij}_i \Delta_i c_{ij}^{-1}(\theta^{(j,x)}_j z_i - \theta^{(i,x)}_j z_j)(\alpha_i - z_i) + d^{ij}_j \Delta_j c_{ij}^{-1}(-\theta^{(i,x)}_i z_i + \theta^{(i,x)}_j z_j)(\alpha_j - z_j),
\]

where \(\alpha_k = \theta^{(k,x)}_k > 0\) for \(k = i, j\). For \(y^{ij} = (y_i, y_j)^T \in \mathbb{R}^2\), by letting \(z^{ij} = A^{ij} y^{ij}\), we get

\[
\frac{1}{2} \langle y^{ij}, \Sigma^{ij} y^{ij} \rangle - \langle \mu^{ij}, y^{ij} \rangle = \sum_{k=i,j} d^{ij}_k \Delta_k y_k \left(\alpha_k - (\theta^{(i,x)}_k y_i + \theta^{(j,x)}_k y_j)\right). \quad (4.6)
\]
Comparing the coefficients of $y_i$ on the both sides of (4.6), we have

$$d_{ij}^{(ij)}\Delta_i\alpha_i = -\mu_i^* = -\mu_i^{*ij},$$

from which we conclude that $d_{ij}^{(ij)}$ is independent of $j$, and we denote it by $d_i$. Then (4.6) becomes

$$-\frac{1}{2}\langle y, \Sigma^* y \rangle - \langle \mu^*, y \rangle = \sum_{k=1,} d_k \Delta_k y_k \left( \alpha_k - \theta_{i}^{(i,x)} y_i + \theta_{j}^{(j,x)} y_j \right),$$

(4.7)

from which we have

$$\Sigma_{ij}^* = \frac{1}{2}\left( d_i \Delta_i \theta_{i}^{(j,x)} + d_j \Delta_j \theta_{j}^{(i,x)} \right) \quad \text{and} \quad \mu_i^* = -d_i \Delta_i \quad \text{for any } i, j \in J.$$

(4.8)

It follows from (4.8) that

$$-\frac{1}{2}\langle y, \Sigma^* \rangle - \langle \mu^*, y \rangle = \sum_{i=1}^{d} d_i \Delta_i y_i \left( \alpha_i - \sum_{k=1}^{d} \theta_{i}^{(k,x)} y_k \right) \quad \text{for any } y \in \mathbb{R}^d.$$

(4.9)

Setting $\theta = Ay$, we have

$$\gamma(\theta) = \sum_{i=1}^{d} d_i \gamma_i(\theta) (\alpha_i - \theta_i) \quad \text{for any } \theta \in \mathbb{R}^d.$$

Thus, (2.5) holds. It follows from Lemma 2 that the $d$-dimensional $(\Sigma, \mu, R)$-SRBM has a product form stationary distribution. This completes the proof of Theorem 1.

**Proof of Corollary 2.** (a) The if and only if part is immediate from Corollary 1 because we can see in the proof of Theorem 1 that the conditions (3.15), (3.16) and (3.17) are equivalent for the two dimensional $(\tilde{\Sigma}^{ij}, \tilde{\mu}^{ij}, \tilde{R}^{ij})$-SRBM to have a product form stationary distribution. Thus, we only need to prove that, if the $d$-dimensional $(\Sigma, \mu, R)$-SRBM has a product form stationary distribution $Z(\infty)$, then the density function of $(Z_i(\infty), Z_j(\infty))$ is equal to

$$\alpha_i \alpha_j e^{-(\alpha_i y_i + \alpha_j y_j)}.$$

In the proof of Theorem 1, we know that when (2.5) holds for every $\theta \in \mathbb{R}^d$, then (4.3) holds. Equation (4.3) is precisely the two-dimensional analog of (2.5) for the two-dimensional $(\tilde{\Sigma}^{ij}, \tilde{\mu}^{ij}, \tilde{R}^{ij})$-SRBM. Therefore, by invoking Lemma 2 again, this time in two dimensions, we conclude that the stationary distribution of the two-dimensional $(\tilde{\Sigma}^{ij}, \tilde{\mu}^{ij}, \tilde{R}^{ij})$-SRBM is of product form with density $\alpha_i \alpha_j e^{-(\alpha_i y_i + \alpha_j y_j)}$. This proves (a). Then (b) is immediate from (a) and Lemma 6 because $R$ is assumed to be a $\mathcal{P}$-matrix.
5 Tandem queues and variational problems

In this section, we focus on SRBMs that arise from tandem queueing networks. For such an SRBM, we characterize its product form stationary distribution through its basic network parameters. We will also discuss a variational problem (VP) associated with the SRBM.

We assume that the reflection matrix $R$, the covariance matrix $\Sigma$, and the drift vector $\mu$ are given by

$$R_{i,i-1} = -1 \quad \text{and} \quad \Sigma_{i,i-1} = \Sigma_{i-1,i} = -c_{i-1} \quad \text{for } i = 2, \ldots, d,$$  
$$R_{i,i} = 1 \quad \text{and} \quad \Sigma_{i,i} = c_{i-1} + c_i \quad \text{for } i = 1, \ldots, d,$$  
$$\mu_i = \beta_{i-1} - \beta_i \quad \text{for } i = 1, \ldots, d.$$  

with all other entries being zero. An example, when $d = 3$, is given by

$$R = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} c_0 + c_1 & -c_1 & 0 \\ -c_1 & c_1 + c_2 & -c_2 \\ 0 & -c_2 & c_2 + c_3 \end{pmatrix}, \quad \mu = \begin{pmatrix} \beta_0 - \beta_1 \\ \beta_1 - \beta_2 \\ \beta_2 - \beta_3 \end{pmatrix}.$$

Such an SRBM arises from a $d$-station generalized Jackson network in series, also known as a tandem queue. In the tandem queue, the interarrival times are assumed to be iid with mean $1/\beta_0$ and squared coefficient of variation (SCV) $c_0$. The service times at station $i$ are assumed to be iid with mean $1/\beta_i$ and SCV $c_i$, $i \in J$. We assume that $\Sigma$ is nonsingular and condition (1.5) is satisfied. It follows from [13] that the SRBM $Z$ has a unique stationary distribution $\pi$. By using Theorem 1, we first check that the stationary distribution $\pi$ has a product form if and only if

$$c_0 = c_i \quad \text{for } i = 1, \ldots, d - 1.$$  

To see this, set

$$b = -R^{-1} \mu > 0 \quad \text{and} \quad \tau_i = \frac{2b_i}{c_0 + c_i} \quad \text{for } i = 1, \ldots, d.$$  

We can easily compute that

$$\theta_j(i,x) = \tau_i \quad \text{for } j = 1, \ldots, i \quad \text{and} \quad \theta_j(i,x) = 0 \quad \text{for } j = i + 1, \ldots, d.$$  

Recall the definition of $\Sigma^*$ and $\mu^*$ in (3.11). An easy computation leads to

$$\Sigma^*_{i,j} = c_0 \tau_i \tau_j, \quad \Sigma^*_{i,i} = \tau_i^2 (c_0 + c_i), \quad \mu_i^* = \tau_i (\beta_0 - \beta_i), \quad 1 \leq i \neq j \leq d.$$  

Thus, for any $1 \leq i < j \leq d$, we have

$$\tilde{\Sigma}^{ij} = \begin{pmatrix} c_0 + c_i & -c_i \\ -c_i & c_i + c_j \end{pmatrix}, \quad \tilde{\mu}^{ij} = \begin{pmatrix} \beta_0 - \beta_i \\ \beta_i - \beta_j \end{pmatrix}, \quad \tilde{R}^{ij} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$
where, in the derivation, we have used the following formula for \( A_{ij} \) defined in (3.3)

\[
A_{ij} = \begin{pmatrix} \tau_i & \tau_j \\ 0 & \tau_j \end{pmatrix}.
\]

Because \( \tilde{R}_{ij} \) is an \( M \)-matrix, a \((\tilde{\Sigma}_{ij}, \tilde{\mu}_{ij}, \tilde{R}_{ij})\)-SRBM is well defined and this two-dimensional SRBM corresponds to a two-station tandem queue consisting of station \( i \) and station \( j \) from the original \( d \)-station network. So Corollary 2 concludes that the SRBM from a \( d \)-station tandem queue has the product form stationary distribution if and only if each of the \( \frac{1}{2}d(d - 1) \) two-dimensional SRBMs from the two-station queues have product form stationary distributions.

By solving equation (3.9), we have the symmetry points of \( \theta^{(i,r)} \) and \( \theta^{(j,r)} \):

\[
\theta^{(i,r)} = f_{ij}^{(i,r)} \left( \tau_i, \frac{2c_i \tau_i + 2b_j - 2b_i}{c_i + c_j} \right) \quad \text{and} \quad \theta^{(j,r)} = f_{ij}^{(j,r)} \left( \frac{2b_i + c_i \tau_j - c_0 \tau_j}{c_0 + c_i}, \tau_j \right), \tag{5.8}
\]

where \( f_{ij} \) is again the mapping defined in (3.6). Using (5.8), condition (3.10) is equivalent to

\[
\tau_i = \frac{2b_i + c_i \tau_j - c_0 \tau_j}{c_0 + c_i} \quad \text{and} \quad \tau_j = \frac{2c_i \tau_i + 2b_j - 2b_i}{c_i + c_j},
\]

which is further equivalent to (5.5). Thus, we have used Theorem 1 to prove that the \( d \)-dimensional stationary distribution has a product form if and only if

\[
c_0 = c_i \quad \text{for} \quad i = 1, 2, \ldots, d - 1. \tag{5.9}
\]

This fact is well known and can be verified by using skew symmetry condition (1.6) developed in [14].

Recall the variational problem (VP) defined in Definition 2.3 of [1]. The VP is proved to be related to large deviations rate function of the corresponding SRBM; see, for example, [20]. In the two-dimensional case, the VP is solved completely in [1], whose optimal solutions are interpreted geometrically in [7]. In particular, for the two-dimensional \((\tilde{\Sigma}_{ij}, \tilde{\mu}_{ij}, \tilde{R}_{ij})\)-SRBM, the “entrance” velocities in (3.4) of [1] are given by

\[
\tilde{a}^{(i,r)} = \tilde{\Sigma}_{ij} \tilde{\theta}^{(i,r)} + \tilde{\mu}_{ij} \quad \text{and} \quad \tilde{a}^{(j,r)} = \tilde{\Sigma}_{ij} \tilde{\theta}^{(j,r)} + \tilde{\mu}_{ij}, \tag{5.10}
\]

where \( \tilde{\theta}^{(i,r)} \) is the two-dimensional vector whose components are the \( i \)th and \( j \)th component of \( \tilde{\theta}^{(i,r)} \), and \( \tilde{\theta}^{(j,r)} \) is defined similarly. These velocities indicate influence of the boundary faces on an optimal path, that is, a sample path for the optimal solution of the VP. See Section 4 of [7].

Assume the product form condition (5.5). Under condition (5.5), \( \tau_i \) has the simplified expression:

\[
\tau_i = \frac{1}{c_0} (\beta_i - \beta_0), \quad i = 1, 2, \ldots, d - 1, \quad \text{and} \quad \tau_d = \frac{2}{c_0 + c_d} (\beta_d - \beta_0) \tag{5.11}
\]
and the symmetry points are given by
\[ \tilde{\theta}^{ij(x)} = (\tau_i, \tau_j)^T. \quad (5.12) \]
Therefore, it follows from \((5.10)\) that, for \(1 \leq i < j \leq d - 1\),
\[ \tilde{a}^{ij(x)} = \tilde{a}^{ij(y)} = c_0 \left( \frac{\tau_i - \tau_j}{\tau_j} \right) = \left( \frac{\beta_i - \beta_j}{\beta_j - \beta_0} \right), \quad (5.13) \]
and, for \(1 \leq i \leq d - 1\),
\[ \tilde{a}^{id(x)} = \tilde{a}^{id(y)} = \left( \frac{c_0(\tau_i - \tau_d)}{2(c_0 + c_d)\tau_d} \right) = \left( \frac{\beta_i - 2c_0\beta_d + (c_d - c_0)\beta_0}{c_0 + c_d} - \frac{\beta_d - \beta_0}{\beta_4 - \beta_0} \right). \quad (5.14) \]

We now consider the optimal path for the VP for the product form network. To make arguments simplified, we consider the case for \(d = 3\) and assume that \(c_0 = c_3\) in addition to the product form condition \((5.9)\). Then, we have \((5.13)\) for \(1 \leq i < j \leq d = 3\).

Analogously to the two dimensional case in \([7]\), let us consider a normal vector at a point \(\theta\) on the ellipse \(E\). Denote this normal vector by \(n^J(\theta)\) which is denoted by \(n^\Gamma(\theta)\) in \((3.16)\) of \([7]\). We conjecture \(n^J(\tau)\) is the “entrance velocity” for the last segment of an optimal path from origin to a point \(z \in S\). This conjecture is consistent with the result in the two dimensional case; see Figure 3 of \([1]\). Let \(\tilde{a}^J = n^J(\tau)\). Then we have
\[ \tilde{a}^J = \left( \begin{array}{c} \beta_1 - \beta_2 \\ \beta_2 - \beta_3 \\ \beta_3 - \beta_0 \end{array} \right). \]
Combining this \(\tilde{a}^J\) with the two dimensional velocities \(\tilde{a}^{ij}\), we can guess the optimal path for the three-dimensional VP.

To see this, let us consider the case that
\[ \beta_1 < \beta_2 < \beta_3. \quad (5.15) \]
In this case, the first two components of \(\tilde{a}^J\) are negative, and the third component is positive. This suggests that the final segment of the optimal path to a point \(z \in S \equiv \mathbb{R}_+^3\) with \(z_3 > 0\) is parallel to \(\tilde{a}^J\), and is a straight-line from a point \(y\) in the interior of the boundary face \(F_3 = \{ x \in \mathbb{R}_+^3 ; x_3 = 0 \}\). The optimal path from origin to \(y\) should remain on face \(F_3\) and is obtained using the velocity \(\tilde{a}^{1,2}\) as argued in \([1]\). By \((5.13)\) and assumption \((5.15)\), the first component of \(\tilde{a}^{1,2}\) is negative, and the second component is positive.
Hence, the optimal path in $F_3$ has two segments such that the first segment is from the origin to a point on the first coordinate and the second segment is from that point to $y$ by a straight-line.

Thus, we conjecture that the optimal path from origin to $z$ is composed of three segments whose first segment is on the first coordinate axis, the second segment is from the end of the first segment to $y \in F_3$, then the final segment is from $y$ to $z \in S$. The optimality of this path is intuitively appealing because the first queue is a bottleneck among the three queues and the second queue is a bottleneck among the latter two queues under assumption (5.15).

Appendix

A The skew symmetric condition

We will use Lemma 2 to show that SRBM has a product form stationary distribution of the form in (2.3) if and only if (1.6) holds.

For that, we show that (2.5) holds with $C_i = \frac{\Sigma_{ii}}{2R_{ii}}$ for $1 \leq i \leq d$ and $\alpha$ given by (2.4) is equivalent to (1.6). As both sides of (2.5) are quadratic functions of $\theta \in \mathbb{R}^d$, (2.5) holds if and only if coefficients of $\theta_i \theta_j$ and coefficients of $\theta_i$ on the both sides are equal for all $i, j \in J$. Letting the coefficient of $\theta_i \theta_j$ of two sides equal, we arrive at

$$-\frac{1}{2}(\Sigma_{ij} + \Sigma_{ji}) = -C_i R_{ji} - C_j R_{ij}. \quad (A.1)$$

for $1 \leq i, j \leq d$. Let $j = i$ in (A.1), we can get $C_i = \frac{\Sigma_{ii}}{2R_{ii}}$. Let $i = j$ in (A.1), we can get $C_j = \frac{\Sigma_{jj}}{2R_{jj}}$. Rewrite (A.1) into the matrix form with $C_i = \frac{\Sigma_{ii}}{2R_{ii}}$ and $C_j = \frac{\Sigma_{jj}}{2R_{jj}}$, we have (1.6). Letting the coefficient of $\theta_i$ of two sides equal, we have

$$-\mu_i = \sum_{k=1}^{d} \frac{\Sigma_{kk}}{2R_{kk}} R_{ik} \alpha_k.$$ 

We can also rewrite it into matrix form

$$-\mu = 2\text{diag}(\Sigma)\text{diag}(R)^{-1}R\alpha.$$ 

Solving $\alpha$ from it, we can arrive at (2.4).

So if (2.5) holds, then $C_i = \frac{\Sigma_{ii}}{2R_{ii}}$ for $1 \leq i \leq d$ and $\alpha$ must be given by (2.4). And (1.6) holds. Conversely, if (1.6) holds, $C_i = \frac{\Sigma_{ii}}{2R_{ii}}$ for $1 \leq i \leq d$ and $\alpha$ given by (2.4), we have (2.5) holds. Through (2.5), we have now reproduced a result in [14].
B Proofs of lemmas

Proof of Lemma 4. In the proof, \( R^{ij} \) is the \((d - 2) \times (d - 2)\) principal submatrix of \( R \) obtained by deleting rows \( i \) and \( j \) and columns \( i \) and \( j \).

We first prove the existence of a map. Let function \( f^{ij}(z^{ij}) \) be given in (3.6). Clearly, \( f^{ij}(z^{ij}) \in \Gamma_{i,j} \) as it is a linear combination of \( \theta^{(i,x)} \) and \( \theta^{(j,x)} \). We first show that for any \( \theta \in \Gamma_{i,j} \), we have \( f^{ij}(\theta_i, \theta_j) = \theta \). To see this, let \( \theta = a\theta^{(i,x)} + b\theta^{(j,x)} \) for some \( a, b \in \mathbb{R} \). Then,

\[
\theta_i = a\theta^{(i,x)} + b\theta^{(j,x)},
\]

\[
\theta_j = a\theta^{(i,x)} + b\theta^{(j,x)}.
\]

Because \( c_{ij} = \det(A^{ij}) \neq 0 \), we have

\[
\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{c_{ij}} \begin{pmatrix} \theta^{(i,x)} \theta_i - \theta^{(j,x)} \theta_j \\ -\theta^{(j,x)} \theta_i + \theta^{(j,x)} \theta_j \end{pmatrix},
\]

Using the definition of \( f^{ij} \), it is clear that \( f^{ij}(\theta_i, \theta_j) = a\theta^{(i,x)} + b\theta^{(j,x)} = \theta \).

To see the uniqueness of map \( f^{ij} \), let \( \theta^1 \) and \( \theta^2 \) be two points on \( \Gamma_{i,j} \). Assume that \( \theta^1 = \theta^2 = z_i \), \( \theta^1 = \theta^2 = z_j \) for some \( z_i \) and \( z_j \). We now show that \( \theta^1 = \theta^2 \). To see this, let \( \theta = \theta^1 - \theta^2 \). Then \( \langle R^{(k)}, \theta \rangle = 0 \) for \( k \in J \setminus \{i, j\} \) and \( \theta_i = 0 \) and \( \theta_j = 0 \). It follows that

\[
(R^{ij})^T \theta^{ij} = 0,
\]

where \( R^{ij} \) is the \((d - 2) \times (d - 2)\) principal sub-matrix of \( R \) obtained by deleting rows \( i \) and \( j \) and columns \( i \) and \( j \) from \( R \), and \( \theta^{ij} \) is the \((d - 2)\)-dimensional sub-vector of \( \theta \) by deleting components \( i \) and \( j \) from \( \theta \). Later on, we will prove \( R^{ij} \) is non-singular. Hence, (B.1) implies \( \theta^{ij} = 0 \), which, together with \( \theta_i = 0 \) and \( \theta_j = 0 \), implies \( \theta = 0 \). Thus, we have proved the claim.

To see the non-singularity of \( R^{ij} \), if not, then there exists \( \beta^{ij} \neq 0 \) such that \( R^{ij} \beta^{ij} = 0 \). Now let \( w = \sum_{l \neq i,j} \beta^{ij}_l R^{(l)} \), we see \( w \neq 0 \) as \( \beta^{ij} \neq 0 \) and \( R^{(l)} \) are linearly independent for \( l \neq i, j \). On the other hand, we obtained \( w_l = 0 \) for \( l \neq i, j \) due to \( R^{ij} \beta^{ij} = 0 \). Thus, \( (w_i, w_j)^T \neq 0 \). Now considering \( \langle w, \theta^{(i,x)} \rangle \), we get \( \langle w, \theta^{(i,x)} \rangle = 0 \) as \( \langle R^{(l)}, \theta^{(i,x)} \rangle = 0 \) for \( l \neq i, j \). Then we obtain \( w_i \theta^{(i,x)} + w_j \theta^{(j,x)} = 0 \). Similarly, we obtain \( w_i \theta^{(j,x)} + w_j \theta^{(j,x)} = 0 \). So \( A^{ij} \) is singular \( (c_{ij} = 0) \) as \( (w_i, w_j)^T \neq 0 \), a contradiction. Thus, we have proved \( R^{ij} \) is non-singular.

Proof of Lemma 6. (a) First we quote an equivalent definition of \( \mathcal{P} \)-matrix in Section 2.5 of [15]: \( A \) is a \( \mathcal{P} \)-matrix, if and only if, for each nonzero \( x \in \mathbb{R}^d \), there is some \( k \in \{1, 2, \cdots, d\} \) such that \( x_k (Ax)_k > 0 \).
For each nonzero $x \in \mathbb{R}^d$, $R^{-1}x$ is also nonzero. Since $R$ is a $P$-matrix, then there is some $k \in \{1, 2, \cdots, d\}$ such that $(RR^{-1}x)_k(R^{-1}x)_k > 0$. Now we have

$$x_k(R^{-1}x)_k = (RR^{-1}x)_k(R^{-1}x)_k > 0.$$  

So $R^{-1}$ is also a $P$-matrix.

(b) Assume that $R$ is a $P$-matrix, then $R^{-1}$ is a $P$-matrix following (a). Thus, for $i \neq j$, $\det((R^{-1})^{ij}) > 0$. On the other hand, using the fact that $R^{-1} = \Delta^{-1}A^T$ we have $\det((R^{-1})^{ij}) = \frac{c_{ij}}{\Delta_x}$, where $\Delta_x$ defined in (3.2) is positive because of (1.5). Therefore, we have proved that $c_{ij} > 0$.

(c) We next assume that the SRBM has a product form stationary distribution, and prove that $R$ is a $P$-matrix. For that, we quote another equivalent definition of $P$-matrix in [15]: $A$ is a $P$-matrix, if and only if for each nonzero $x \in \mathbb{R}^d$, there is some positive diagonal matrix $D(x) \in M_d(\mathbb{R})$ such that $x^T(D(x)A)x > 0$.

By Lemma 2, (2.5) holds. Therefore, comparing the coefficients of $\theta_i\theta_j$, we have for any nonzero $\theta \in \mathbb{R}^d$,

$$\sum_{i=1}^{d} \sum_{j=1}^{d} C_i R_{ji} \theta_j \theta_i = \frac{1}{2} \langle \theta, \Sigma \theta \rangle > 0,$$

(B.2)

where $C_i$’s are constants in (2.5) and the inequality holds because $\Sigma$ is positive definite. It follows that

$$x^T \text{diag}(C_1, \ldots, C_d)^{-1}Rx > 0$$

for any nonzero $x \in \mathbb{R}^d$, proving that $R$ is a $P$-matrix.

**Proof of Lemma 7.** One can check that

$$\bar{\gamma}^{ij}(z^{ij}) = -\frac{1}{2} \langle (A^{ij})^{-1}z^{ij}, \hat{\Sigma}^{ij}(A^{ij})^{-1}z^{ij} \rangle - \langle \hat{\mu}^{ij}, (A^{ij})^{-1}z^{ij} \rangle$$

$$= -\frac{1}{2} \langle y, A^T \Sigma Ay \rangle - \langle A^T \mu, y \rangle$$

$$= -\frac{1}{2} \langle Ay, \Sigma(Ay) \rangle - \langle \mu, Ay \rangle$$

$$= -\frac{1}{2} \langle f^{ij}(z^{ij}), \Sigma f^{ij}(z^{ij}) \rangle - \langle \mu, f^{ij}(z^{ij}) \rangle$$

$$= \gamma(f^{ij}(z^{ij})), \quad \text{(B.3)}$$

where $y \in \mathbb{R}^d$ is the unique vector whose components $i$ and $j$ are given by $(A^{ij})^{-1}(z_i, z_j)^T$ and other components are zero, and in the second last equality we have used the fact that $Ay = f^{ij}(z^{ij})$.

**Proof of lemma 5.** According to Lemma 7, (3.9) is equivalent to $\bar{\gamma}(\theta^{i,j}, z) = 0$. Also due to Lemma 7, (3.7) is equivalent to (4.5).
If (4.5) holds, then for any solution $(\theta_{i}^{(i,x)}, z_j)$ satisfying $\tilde{\gamma}(\theta_{i}^{(i,x)}, z_j) = 0$, we have 
\[
\frac{\partial \tilde{\gamma}(\theta_{i}^{(i,x)}, z_j) \big|_{z_j = (\theta_{i}^{(i,x)}, z_j)^T}}{\partial z_j} = -\tilde{\Sigma}_{jj} z_j - \tilde{\Sigma}_{ij} \theta_{i}^{(i,x)} - \tilde{\mu}_j = 0.
\]
So $\tilde{\gamma}(\theta_{i}^{(i,x)}, z_j) = 0$ has a unique solution $z_j = \theta_{j}^{(i,x)}$. Otherwise, 
\[
\frac{\partial \tilde{\gamma}(\theta_{i}^{(i,x)}, z_j) \big|_{z_j = (\theta_{i}^{(i,x)}, \theta_{i}^{(i,x)})^T}}{\partial z_j} = -\tilde{\Sigma}_{jj} \theta_{j}^{(i,x)} - \tilde{\Sigma}_{ij} \theta_{i}^{(i,x)} - \tilde{\mu}_j \neq 0,
\]
so $\theta_{j}^{(i,x)} \neq -\frac{\tilde{\Sigma}_{ij} \theta_{i}^{(i,x)} + \tilde{\mu}_j}{\tilde{\Sigma}_{jj}}$. From quadratic equation $\tilde{\gamma}(\theta_{i}^{(i,x)}, z_j) = 0$, the other solution $z_j \neq \theta_{j}^{(i,x)}$ satisfies $z_j + \theta_{j}^{(i,x)} = -2 \frac{\tilde{\Sigma}_{ij} \theta_{i}^{(i,x)} + \tilde{\mu}_j}{\tilde{\Sigma}_{jj}}$. So $z_j \neq \theta_{j}^{(i,x)}$.

\[\Box\]

C Examples

Our first example complements Lemma 6.

**Example 1.** Let
\[
R = \begin{pmatrix}
1 & 1/2 & 1 & 0 \\
2 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{pmatrix},
\]
(C.1)
Since $R$ is a nonnegative matrix, it is easy to check that $R$ is a complete-$S$ matrix. The matrix $R$ in invertible with inverse
\[
R^{-1} = \begin{pmatrix}
0 & 1/2 & 0 & -1/2 \\
2 & 0 & -2 & 0 \\
0 & -1/2 & 1 & 1/2 \\
-2 & 0 & 2 & 1
\end{pmatrix}.
\]
Then condition (1.5) is satisfied with $\mu = -(1.1, 1.1, 1, 1)^T$. However, $c_{34}$ in (3.4) equals zero, demonstrating that Lemma 6 cannot be generalized to completely-$S$ matrix satisfying (1.5).

The next examples shows that, unlike the case when $d = 2$, the condition that the point $\tau$, defined in (3.14), is on the ellipse is not sufficient for a product form stationary distribution.

**Example 2.** Consider the 3-dimensional SRBM with
\[
R = \begin{pmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
0 & -1 & 1
\end{pmatrix}, \quad \Sigma = \begin{pmatrix}
1 & -1 & 0 \\
-1 & 3 & -2 \\
0 & -2 & 3
\end{pmatrix}, \quad \mu = \begin{pmatrix}
-1/2 \\
-2 \\
-1/2
\end{pmatrix}.
\]
(C.2)
Since $R$ is an $\mathcal{M}$-matrix, $R$ is completely-$S$. One can verify that
\[
R^{-1} \mu = \begin{pmatrix}
-1/2 \\
-2 \\
-1/2
\end{pmatrix} < 0.
\]

21
This SRBM arises from a three station tandem queue (see Section 5 for details of this model). A simple computation leads to \(\theta^{(1,r)} = (1, 0, 0)^T\), \(\theta^{(2,r)} = (2, 2, 0)^T\) and \(\theta^{(3,r)} = (1, 1, 1)^T\). Thus, \(\tau = (1, 2, 1)^T\), where \(\tau_i = \theta_i^{(i,r)}\) for \(i = 1, 2, 3\) following the definition in (3.14). One can check that \(\gamma(\tau) = 0\). Now we use Corollary 1 to verify that the SRBM does not have a product form stationary distribution. For that, we have

\[\theta^{(12)} = f^{12}(\tau^{12}) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.\]

Because \(\gamma(\theta^{(12)}) = -1 \neq 0\), by Corollary 1, this SRBM does not have a product form stationary distribution.

D Equivalence of two versions of basic adjoint relationship

This section is devoted to the proof for part (b) of Lemma 1. The key is to establish the equivalence of two versions of basic adjoint relationship (BAR). This equivalence is stated in Proposition 1 below. Since this proposition may be of independent interest, we keep this appendix as self-contained as possible. This means that some of the terminology and notation are reintroduced here in this appendix.

D.1 The main result

We focus on a \(d\)-dimensional semimartingale reflecting Brownian motion (SRBM) that lives on the nonnegative orthant \(\mathbb{R}_+^d\). The SRBM data consists of a \(d \times d\) positive definite matrix \(\Sigma\), a vector \(\mu \in \mathbb{R}^d\) and a \(d \times d\) reflection matrix \(R\). The matrix \(\Sigma\) is known as the covariance matrix, \(\mu\) the drift vector, and \(R\) the reflection matrix. Assume that the SRBM has a stationary distribution. It is known that the stationary distribution is unique and is characterized by a basic adjoint relationship (BAR) ([5]). In this appendix, we show that a moment generating function version of the BAR is equivalent to the standard BAR in [4] and [5]. The equivalence argument is standard. We present details here for easy reference.

Given the primitive data \((\Sigma, \mu, R)\) of an SRBM, we define the following \(d\)-dimensional polynomials

\[\gamma(\theta) = -\frac{1}{2} \langle \theta, \Sigma \theta \rangle - \langle \mu, \theta \rangle, \quad \theta \in \mathbb{R}^d,\]

\[\gamma_i(\theta) = \langle R^{(i)}, \theta \rangle, \quad \theta \in \mathbb{R}^d, \quad i \in J = \{1, 2, \ldots, d\},\]

where, for \(x, y \in \mathbb{R}^d\), \(\langle x, y \rangle\) denotes the standard inner product of \(x\) and \(y\), and \(R^{(i)}\) denotes the \(i\)th column of \(R\). For a finite measure \(\tau\) on \((\mathbb{R}_+^d, \mathcal{B}(\mathbb{R}_+^d))\) with \(\mathcal{B}(\mathbb{R}_+^d)\) being the Borel \(\sigma\)-field on \(\mathbb{R}_+^d\), we define the corresponding moment generating function

\[\varphi_\tau(\theta) = \int_{\mathbb{R}_+^d} e^{\langle \theta, x \rangle} \tau(dx) \quad \text{for } \theta \in \mathbb{R}^d \text{ with } \theta \leq 0.\]
Hereafter, vector inequalities are interpreted componentwise. Because \( \tau \) is a finite measure, \( \varphi_\tau(\theta) \) is well defined for each \( \theta \leq 0 \). When the measure \( \tau \) is clear from the context, we sometimes drop the subscript \( \tau \) from \( \varphi \). For an open set \( U \subset \mathbb{R}^m \) for some \( m \geq 1 \), a function \( f : U \to \mathbb{R} \) is said to be in \( C^k(U) \) if \( f \) and its derivatives up to \( k \)th order are continuous in \( U \). A function \( f : \mathbb{R}^d_+ \to \mathbb{R} \) is said to be in \( C^2_b(\mathbb{R}^d_+) \) if (a) for each \( x \in \mathbb{R}^d_+ \), \( f \) is well defined in a neighborhood \( U \) of \( x \) in \( \mathbb{R}^d \) such that \( f \in C^2(U) \), and (b) 

\[
\|f\|_{\mathbb{R}^d_+} = \max_{i,j \in J} \left| \frac{\partial^2 f}{\partial x_i \partial x_j} \right| + \max_{i \in J} \left| \frac{\partial f}{\partial x_i} \right| + \sup_{x \in \mathbb{R}^d_+} |f(x)| \tag{D.1}
\]

is finite.

**Proposition 1.** Let \((\Sigma, \mu, R)\) be the data of an SRBM. Assume that \( \pi \) is a probability measure on \( \mathbb{R}^d_+ \) and that \( \nu_i \) is a positive finite measure whose support is contained in \( \{x \in \mathbb{R}^d_+ : x_i = 0\} \) for \( i \in J \). Let \( \varphi \) and \( \varphi_i \) be the moment generating functions of \( \pi \) and \( \nu_i \), respectively. Then \( \varphi, \varphi_1, \ldots, \varphi_d \) satisfy

\[
\gamma(\theta)\varphi(\theta) = \sum_{i=1}^d \gamma_i(\theta)\varphi_i(\theta) \quad \text{for each } \theta \in \mathbb{R}^d \text{ with } \theta \leq 0 \tag{D.2}
\]

if and only if

\[
\int_{\mathbb{R}^d_+} Lf(x)\pi(dx) + \sum_{i=1}^d \int_{\mathbb{R}^d_+} D_i f(x)\nu_i(dx) = 0 \quad \text{for each } f \in C^2_b(\mathbb{R}^d_+), \tag{D.3}
\]

where

\[
Lf(x) = \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \Sigma_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^d \mu_i \frac{\partial f}{\partial x_i}(x),
\]

\[
D_if(x) = \sum_{j=1}^d R_{ji} \frac{\partial f}{\partial x_j}(x) \quad \text{for } i \in J.
\]

**Remark 3.** Theorem 1.2 of [5] says if (D.3) holds for each \( f \in C^2_b(\mathbb{R}^d_+) \), then \( \pi \) is the stationary distribution of the SRBM and \( \nu_1, \ldots, \nu_d \) are the corresponding boundary measures associated with the SRBM. Combining Theorem 1.2 of [5] with Proposition 1, we have proved Lemma 1.

### D.2 Proof of Proposition 1

**Proof.** We first argue that (D.3) implies (2.1). Assume that (D.3) holds for every \( f \in C^2_b(\mathbb{R}^d_+) \). For a given \( \theta \in \mathbb{R}^d \) with \( \theta \leq 0 \), let

\[
f(x) = e^{\langle \theta, x \rangle} \text{ for } x \in \mathbb{R}^d. \tag{D.4}
\]
One can verify that \( f \in C^2_b(\mathbb{R}^d_+) \), \( Lf(x) = \gamma(\theta)e^{i\theta \cdot x} \), and \( D_i f(x) = \gamma_i(\theta)e^{i\theta \cdot x} \). Since (D.3) holds for this \( f \), (2.1) holds for this \( \theta \).

Now we argue that (2.1) implies (D.3). Assume that (2.1) holds. We would like to prove that (D.3) holds for each \( f \in C^2_b(\mathbb{R}^d_+) \). In this section, we prove this fact in four steps. Before we present full details of these four steps, we first provide an outline of these steps.

Note that (2.1) implies (D.3) for all functions \( f \) of the form in (D.4) with \( \theta \leq 0 \). In step 1, we argue that (D.3) continues to hold for functions \( f \) of the form in (D.4) when \( \theta \) is replaced by \( (z_1, \ldots, z_d)^T \), where each \( z_j \) is a complex variable with \( \Re z_j \leq 0 \) and the superscript \( T \) represents transpose. In step 2, applying the inverse Fourier theorem for any \( f \in C^\infty(\mathbb{R}^d) \), the space of \( C^\infty \) functions on \( \mathbb{R}^d \) with compact support, we argue that (D.3) holds for each \( f \in C^\infty_c(\mathbb{R}^d) \). In step 3, we prove (D.3) holds for all \( C^2_b(\mathbb{R}^d) \) functions. In step 4, we prove (D.3) holds for all \( C^2_b(\mathbb{R}^d_+) \) functions.

Before we carry out the details of these four steps, we state a standard result from complex analysis in the following lemma. The lemma is used in step 1 below; for its proof, see, for example, Theorem 1.1 on page 73 of [24].

**Lemma 8.** Let \( \Omega \subset \mathbb{C} \) be some connected open subset of the complex plane \( \mathbb{C} \) and let \( f \) be an analytic function defined on \( \Omega \). Suppose that \( f(z_0) = 0 \) for some \( z_0 \in \Omega \). Then, either \( f(z) = 0 \) for all \( z \in \Omega \) or there exists a neighborhood \( U \subset \Omega \) of \( z_0 \) such that \( f(z) \neq 0 \) for all \( z \in U \setminus \{z_0\} \).

**Step 1.** Let \( z = (z_1, \ldots, z_d)^T \) and each \( z_j \) be a complex variable with \( \Re z_j \leq 0 \). Let \( f(x) = e^{i \langle z, x \rangle} \). Define

\[
    h(z) = \int_{\mathbb{R}^d_+} Lf(x) \pi(dx) + \sum_{i=1}^d \int_{\mathbb{R}^d_+} D_i f(x) \nu_i(dx).
\]

Since \( Lf(x) = -\gamma(z)f(x) \), \( D_i f(x) = \gamma_i(z)f(x) \), and \( \pi \) and \( \nu_i \) are finite measures, one can check that \( h(z) \) is well defined and it satisfies

\[
    h(z) = -\gamma(z) \varphi(z) + \sum_{i=1}^d \gamma_i(z) \varphi_i(z). \tag{D.5}
\]

First, we would like to prove \( h(z) = 0 \), where \( z = (z_1, \ldots, z_d)^T \) and each \( z_j \) is a complex variable with \( \Re z_j < 0 \). To see this, fix \( \theta_j < 0 \) for \( 2 \leq j \leq d \). Let \( U = \{ z_1 \in \mathbb{C} : \Re z_1 < 0 \} \).

For any \( z_1 \in U \), define

\[
    g_1(z_1) = h(z_1, \theta_2, \cdots, \theta_d),
\]

which, by (D.5), is equal to

\[
    -\gamma(z_1, \theta_2, \cdots, \theta_d) \varphi(z_1, \theta_2, \cdots, \theta_d) + \sum_{i=1}^d \gamma_i(z_1, \theta_2, \cdots, \theta_d) \varphi_i(z_1, \theta_2, \cdots, \theta_d),
\]

24
where the argument inside functions such as \( h(\cdot) \) should have been the column vector \((z_1, \theta_2, \cdots, \theta_d)^T\); for notational simplicity, we drop the transpose and write \( h(z_1, \theta_2, \cdots, \theta_d) \) in the rest of this document. Clearly, \( \gamma(z_1, \theta_2, \cdots, \theta_d) \) and \( \gamma_i(z_1, \theta_2, \cdots, \theta_d) \) are analytical functions of \( z_1 \) in the entire complex plane \( \mathbb{C} \). Also, one can check that \( \varphi(z_1, \theta_2, \cdots, \theta_d) \) and \( \varphi_i(z_1, \theta_2, \cdots, \theta_d) \) are analytical functions of \( z_1 \) on \( U \). From (2.1), we know that \( h(\theta) = 0 \) for \( \theta \in \mathbb{R}^d \) with \( \theta < 0 \). Therefore, \( g_1(\theta_1) = 0 \) for \( \theta_1 \in (-\infty, 0) \subset U \). Applying Lemma 8, we have \( g_1(z_1) = 0 \) for \( z_1 \in U \). Similarly, by fixing \( z_1 \in U, \theta_3 < 0, \ldots, \theta_d < 0 \), we can prove \( g_2(z_2) = h(z_1, z_2, \theta_3, \ldots, \theta_d) \) is an analytic function on \( U \) and \( g_2(\theta_2) = 0 \) for \( \theta_2 \in (-\infty, 0) \subset U \). Therefore, again by Lemma 8, \( g_2(z_2) = 0 \) for \( z_2 \in U \). Thus, we have proved that for any \( z_i \in \mathbb{C} \) with \( \Re z_i < 0 \) for \( i = 1, 2 \) and \( \theta_i \in (-\infty, 0) \) for \( i = 3, \ldots, d \), \( h(z_1, z_2, \theta_3, \ldots, \theta_d) = 0 \). By an induction argument, one can prove that \( h(z_1, \ldots, z_d) = 0 \) for \( z_i \in \mathbb{C} \) with \( \Re z_i < 0 \) for \( i \in J \).

Next, we would like to prove \( \lim h(z) = 0 \), where \( z = (z_1, \ldots, z_d)^T \) and each \( z_j \) is a complex variable with \( \Re z_j = 0 \). We use an induction argument to prove this. Suppose that \( h(z_1, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_d) = 0 \) for \( z_j \in \mathbb{C} \) with \( \Re z_j = 0 \) for \( j = 1, \ldots, i-1 \) and \( z_j \in \mathbb{C} \) with \( \Re z_j < 0 \) for \( j = i, \ldots, d \). Fix \( z = (z_1, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_d)^T \), where \( z_j \in \mathbb{C} \) for \( j \in J \) with \( \Re z_j = 0 \) for \( j = 1, \ldots, i \) and \( \Re z_j < 0 \) for \( j = i+1, \ldots, d \). For each positive integer \( k \), let \( z_i^k = z_i^{-1/k} \). Then \( \Re z_i^k < 0 \) for each \( k \geq 1 \) and \( \lim_{k \to \infty} z_i^k = z_i \).

Let \( z^k = (z_1, \ldots, z_{i-1}, z_i^k, z_{i+1}, \ldots, z_d)^T \) for \( k \geq 1 \). Then \( \lim_{k \to \infty} z^k = z \). Clearly,

\[
\lim_{k \to \infty} \gamma(z^k) = \gamma(z) \quad \text{and} \quad \lim_{k \to \infty} \gamma_j(z^k) = \gamma_j(z) \quad \text{for} \quad j \in J.
\]

Note that

\[
\varphi(z^k) = \int_{\mathbb{R}^d_+} f_k(x) \pi(dx),
\]

where \( f_k(x) = e^{(z^k \cdot x)} \). Since \( \Re z_j^k \leq 0 \) for each \( j \in J \), one can check that \( |f_k(x)| \leq 1 \) for each \( x \in \mathbb{R}^d_+ \). By the dominated convergence theorem, we have

\[
\lim_{k \to \infty} \int_{\mathbb{R}^d_+} f_k(x) \pi(dx) = \int_{\mathbb{R}^d_+} f(x) \pi(dx),
\]

where \( f(x) \) is given in (D.4). Therefore, we have proved that

\[
\lim_{k \to \infty} \varphi(z^k) = \varphi(z).
\]
Similarly, we can prove
\[
\lim_{k \to \infty} \varphi_j(z^k) = \varphi_j(z) \quad \text{for each } j \in J.
\]
By (D.5),
\[
\lim_{k \to \infty} h(z^k) = \lim_{k \to \infty} \left( -\gamma(z^k)\varphi(z^k) + \sum_{j=1}^d \gamma_j(z^k)\varphi_j(z^k) \right)
= -\gamma(z)\varphi(z) + \sum_{j=1}^d \gamma_j(z)\varphi_j(z) = h(z).
\]
By the induction assumption, \( h(z^k) = 0 \) for \( k \geq 1 \). Therefore, we have \( h(z) = 0 \). Thus, we have proved that (D.3) holds for functions \( f(x) = e^{\langle z, x \rangle} \), where \( z = (z_1, \ldots, z_d)^T \) and each \( z_i \) is a complex variable with \( \Re z_i = 0 \).

**Step 2.** In this step, we prove (D.3) holds for any function \( f \in C^\infty_K(\mathbb{R}^d) \), the space of \( C^\infty \) functions on \( \mathbb{R}^d \) with compact support. Such an \( f \) belongs to the so called Schwartz space on \( \mathbb{R}^d \) (page 236 in [10]). For a function \( f \) in the Schwartz space, its Fourier transform \( \hat{f}(\zeta) = \int_{\mathbb{R}^d} e^{-2\pi i \langle y, \zeta \rangle} f(y) dy \) is well defined for each \( \zeta \in \mathbb{R}^d \). By the Fourier inversion theorem for the functions in Schwartz space (Corollary 8.23 and Theorem 8.26 in [10]), one can recover a function \( f \) in Schwartz space through its Fourier transform \( f(x) = \int_{\mathbb{R}^d} e^{2\pi i \langle x, \zeta \rangle} \hat{f}(\zeta) d\zeta \) for \( x \in \mathbb{R}^d \).

For any function \( f \) belonging to Schwartz space, its Fourier transform and itself are both absolutely integrable (Corollary 8.23 in [10]). Consequently, we have following expressions for \( Lf(x) \) and \( D_i f(x) \) for \( i \in J \)
\[
Lf(x) = \int_{\mathbb{R}^d} Lg(x, \zeta) \hat{f}(\zeta) d\zeta \quad \text{and} \quad D_i f(x) = \int_{\mathbb{R}^d} D_i g(x, \zeta) \hat{f}(\zeta) d\zeta \quad \text{for } x \in \mathbb{R}^d,
\]
where \( g(x, \zeta) = e^{2\pi i \langle x, \zeta \rangle} \). Then one can check,
\[
\int_{\mathbb{R}^d} Lf(x) \pi(dx) + \sum_{i=1}^d \int_{\mathbb{R}^d} D_i f(x) v_i(dx)
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} Lg(x, \zeta) \hat{f}(\zeta) d\zeta \pi(dx) + \sum_{i=1}^d \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} D_i g(x, \zeta) \hat{f}(\zeta) d\zeta v_i(dx)
= \int_{\mathbb{R}^d} \left\{ \int_{\mathbb{R}^d} Lg(x, \zeta) \pi(dx) + \sum_{i=1}^d \int_{\mathbb{R}^d} D_i g(x, \zeta) v_i(dx) \right\} \hat{f}(\zeta) d\zeta
= 0.
\]
The second equality is due to Fubini’s Theorem. Fubini’s Theorem holds because \( \hat{f}(\zeta) \) is absolutely integrable over \( \mathbb{R}^d \). The last equality holds because

\[
\int_{\mathbb{R}^d_+} Lg(x, \zeta) \pi(dx) + \sum_{i=1}^d \int_{\mathbb{R}^d_+} D_i g(x, \zeta) v_i(dx) = 0
\]

for all \( \zeta \in \mathbb{R}^d \) and the result in Step 2. Therefore we have prove that (D.3) holds for C∞ functions with compact support.

**Step 3.** In this step, we prove (D.3) holds for all \( C^2_K(\mathbb{R}^d) \) functions. Fix an \( f(x) \in C^2_K(\mathbb{R}^d) \). We now construct a sequence of functions \( g^n(x) \in C^\infty_K(\mathbb{R}^d) \) that converges to \( f \) in a proper sense. The construction is standard and is adapted from Proposition 8 on page 29 of [25]. Let \( g^n(x) = \eta^n * f(x) = \int_{\mathbb{R}^d} \eta^n(y) f(x-y)dy \), where \( \eta^n(x) = n^d \eta(nx) \),

\[
\eta(x) = \begin{cases} \frac{c \exp (- (1 - |x|^2)^{-1})}{(1 - |x|^2)^{1/2}} & \text{for } |x| < 1, \\ 0 & \text{otherwise,} \end{cases}
\]

and \( c \) is a constant such that \( \int_{\mathbb{R}^d} \eta(x)dx = 1 \). It is known that \( \eta^n(x) \in C^\infty_K(\mathbb{R}^d) \) and \( g^n(x) \in C^\infty_K(\mathbb{R}^d) \). By the result from Step 2, we have

\[
\int_{\mathbb{R}^d_+} Lg^n(x) \pi(dx) + \sum_{i=1}^d \int_{\mathbb{R}^d_+} D_i g^n(x) v_i(dx) = 0 \quad \text{for each } n \geq 1. \tag{D.7}
\]

Because \( f \in C^2_K(\mathbb{R}^d) \), we have for each \( x \in \mathbb{R}^d \)

\[
Lg^n(x) = \eta^n * Lf(x) \quad \text{and} \quad D_i g^n(x) = \eta^n * D_i f(x), \tag{D.8}
\]

and

\[
\lim_{n \to \infty} Lg^n(x) = Lf(x) \quad \text{and} \quad \lim_{n \to \infty} D_i g^n(x) = D_i f(x), \quad i \in J.
\]

Also, by (D.8), one has for each \( n \geq 1 \)

\[
\sup_{x \in \mathbb{R}^d} |Lg^n(x)| \leq \sup_{x \in \mathbb{R}^d} |Lf(x)| \quad \text{and} \quad \sup_{x \in \mathbb{R}^d} |D_i g^n(x)| \leq \sup_{x \in \mathbb{R}^d} |D_i f(x)|.
\]

Taking \( n \to \infty \) on both sides of (D.7), by the bounded convergence theorem, we have

\[
\int_{\mathbb{R}^d_+} Lf(x) \pi(dx) + \sum_{i=1}^d \int_{\mathbb{R}^d_+} D_i f(x) v_i(dx) = 0. \tag{D.9}
\]

**Step 4.** In this step, we prove that (D.3) holds for \( f \in C^2_0(\mathbb{R}^d_+) \). Fix an \( f(x) \in C^2_0(\mathbb{R}^d_+) \). Then \( \|f\|_{\mathbb{R}^d_+} < \infty \), where \( \|f\|_{\mathbb{R}^d_+} \) is defined in (D.1). For any \( \epsilon > 0 \), choose a constant \( B > 0 \) such that

\[
\pi(|x| \geq B) + \sum_{i=1}^d \nu_i(\{|x| \geq B\}) < \epsilon. \tag{D.10}
\]
Since \( f \in C^2(\mathbb{R}^d_+) \), there exists \( \delta > 0 \) such that \( f \), its first order derivatives, and second order derivatives are well defined and continuous on

\[
\{ x \in \mathbb{R}^d : x_i > -4\delta \ for \ i \in J \} \cap \{ |x| < B + 2 \}.
\]

Let

\[
h_1(y) = 1 - \int_0^y \eta(u - (B + 1))du \quad \text{for } y \in \mathbb{R},
\]

\[
h_2(y) = \int_{-(y + 2\delta)/\delta}^{(y + 2\delta)/\delta} \eta(u)du \quad \text{for } y \in \mathbb{R},
\]

where \( \eta : \mathbb{R} \to \mathbb{R} \) is the one variable version defined in (D.6). One can check that \( h_k \in C^2(\mathbb{R}) \), \( k = 1, 2 \).

Define

\[
g(x) = \begin{cases} 
  f(x) h_1(|x|) \prod_{i=1}^d h_2(x_i) & \text{for } x \in \{ x \in \mathbb{R}^d : x_i > -4\delta, i \in J \}, \\
  0 & \text{otherwise},
\end{cases}
\]

where \( |x| = \sqrt{\langle x, x \rangle} \). Since \( |x| \) has derivatives in all orders for \( |x| > B \), one can verify that \( g \in C^2_R(\mathbb{R}^d) \). It follows from Step 3 that

\[
\int_{\mathbb{R}^d_+} Lg(x)\pi(dx) + \sum_{i=1}^d \int_{\mathbb{R}^d_+} Di g(x) v_i(dx) = 0. \tag{D.11}
\]

Because

\[
\sup_{y \in \mathbb{R}} |h_1(y)| \leq 1, \quad \sup_{y \in \mathbb{R}} |h_1'(y)| \leq 1, \quad \text{and} \quad \sup_{y \in \mathbb{R}} |h_1''(y)| \leq 3/2,
\]

there exists a constant \( C > 0 \), independent of \( B \) and \( \delta \), such that

\[
\sup_{x \in \mathbb{R}^d_+} (|Lf(x)| + |Lg(x)|) \leq C\|f\|_{\mathbb{R}^d_+}, \quad \sup_{x \in \mathbb{R}^d_+} (|Di f(x)| + |Di g(x)|) \leq C\|f\|_{\mathbb{R}^d_+}. \tag{D.12}
\]

Note that for \( x \in \mathbb{R}^d_+ \cap \{ |x| < B \}, Lf(x) = Lg(x) \) and \( Di f(x) = Di g(x) \). Therefore,
we have
\[
\left| \int_{\mathbb{R}_+^d} Lf(x)\pi(dx) + \sum_{i=1}^{d} \int_{\mathbb{R}_+^d} D_i f(x) v_i(dx) \right|
\]
\[
= \left| \int_{\mathbb{R}_+^d \cap \{|x|<B\}} Lg(x)\pi(dx) + \sum_{i=1}^{d} \int_{\mathbb{R}_+^d \cap \{|x|<B\}} D_i g(x) v_i(dx) \right|
\]
\[
+ \left| \int_{\mathbb{R}_+^d \cap \{|x|\geq B\}} Lf(x)\pi(dx) + \sum_{i=1}^{d} \int_{\mathbb{R}_+^d \cap \{|x|\geq B\}} D_i f(x) v_i(dx) \right|
\]
\[
\leq \left| \int_{\mathbb{R}_+^d} Lg(x)\pi(dx) + \sum_{i=1}^{d} \int_{\mathbb{R}_+^d} D_i g(x) v_i(dx) \right|
\]
\[
+ \left| \int_{\mathbb{R}_+^d \cap \{|x|\geq B\}} Lg(x)\pi(dx) + \sum_{i=1}^{d} \int_{\mathbb{R}_+^d \cap \{|x|\geq B\}} D_i g(x) v_i(dx) \right|
\]
\[
+ \left| \int_{\mathbb{R}_+^d \cap \{|x|\geq B\}} Lf(x)\pi(dx) + \sum_{i=1}^{d} \int_{\mathbb{R}_+^d \cap \{|x|\geq B\}} D_i f(x) v_i(dx) \right|
\]
\[
\leq C\|f\|_{\mathbb{R}_+^d} \epsilon,
\]
where the last inequality follows from (D.11), (D.12), and (D.10). Since $\epsilon > 0$ can be arbitrarily small, we have that (D.3) holds for $f \in C_b^2(\mathbb{R}_+^d)$.

\[\square\]

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29
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