A Gevrey class semigroup, exponential decay and Lack of analyticity for a system formed by a Kirchhoff-Love plate equation and the equation of a membrane-like electric network with indirect fractional damping

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Abstract

The emphasis in this paper is on the Coupled System of a Kirchhoff-Love Plate Equation with the Equation of a Membrane-like Electrical Network, where the coupling is of higher order given by the Laplacian of the displacement velocity $\gamma \Delta u_t$ and the Laplacian of the electric potential field $\gamma \Delta v_t$, here only one of the equations is conservative and the other has dissipative properties. The dissipative mechanism is given by an intermediate damping $(-\Delta)^\theta v_t$ between the electrical damping potential for $\theta = 0$ and the Laplacian of the electric potential for $\theta = 1$. We show that $S(t) = e^{Bt}$ is not analytic for $\theta \in [0, 1)$ and analytic for $\theta = 1$, however $S(t) = e^{Bt}$ decays exponentially for $0 \leq \theta \leq 1$ and $S(t)$ is of Gevrey class $s > \frac{2+\theta}{\theta}$ when the parameter $\theta$ lies in the interval $(0, 1)$.

Key words and phrases: Electric Network Equation, Kirchhoff-Love Plates, Lack of Analyticity, Exponential Decay, Gevrey Class Semigroup.

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1 Introduction

In the literature there are several mathematical models that describe a single electrical network connecting piezoelectric actuators and/or transducers, see for example [5], [17] or [31]. In particular in [17], equations (2b) and (2c), we have, for example, the equations of a second order electric transmission line with zero order or second order dissipation:

(S.Z) and (S,S)-network: second-order network with zeroth-order dissipation and second-order dissipation

\[
\begin{align*}
 v_{tt} - \beta_2 \Delta v + \delta_0 v_t &= 0 \quad \text{and} \quad v_{tt} - \beta_2 \Delta v - \delta_2 \Delta v_t &= 0. \\
(1)
\end{align*}
\]

Where \( v(x, t) \) denotes the time-integral of the electric potential difference between the nodes and the ground.

The motivation for this research was born from the coupled system of the Kirchhoff-Love Plates and Membrane-Like Electric Network deduced in [31] as follows:

\[
\begin{align*}
u_{tt} + \alpha \Delta^2 u - \gamma \Delta v_t &= 0, \quad x \in \Omega, \quad t > 0, \\
v_t - \beta \Delta v + \gamma \Delta u_t + \delta v_t + \delta \gamma \Delta u &= 0, \quad x \in \Omega, \quad t > 0, \\
(2) & \\
(3) & \\
\end{align*}
\]

satisfying the boundary conditions

\[
u = \Delta u = 0, \quad v = 0, \quad x \in \partial \Omega, \quad t > 0, \\
(4)
\]

and prescribed initial data

\[
u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \quad x \in \Omega. \\
(5)
\]

Here we denote by \( u(x, t) \) the transversal displacements of the plates and \( v(x, t) \) is time-integral of the electric potential difference between the nodes and the ground, and \( \Omega \subset \mathbb{R}^n \) the domain with smooth boundary \( \partial \Omega \). The coefficients \( \alpha, \beta, \delta \) are positive and \( \gamma \) is non-zero, for details of the physical meaning and as determined each of the coefficients consult the deduction of the Physical-Mathematical model on pages 441 and 442 of reference [31]. For more details on modeling, the reference [31] can also be consulted.

Our purpose in this work is to study a more general system, to this end we will consider in the equation of the electrical network the fractional dissipation \(( -\Delta )^\theta v_t\) for \( 0 \leq \theta \leq 1 \), keep in mind that for the particular cases \( \theta = 0 \) and \( \theta = 1 \) the mathematical models are given by equations in (1) of [31] respectively.

We will write the system under study in its abstract form. For this purpose, we introduce some helpful notations beforehand. Let \( \Omega \) a bounded set in \( \mathbb{R}^n \) with smooth boundary and given the operator:

\[
A = -\Delta, \quad D(A) \subset L^2(\Omega) \rightarrow L^2(\Omega), \quad A : D(A) \subset L^2(\Omega) \rightarrow L^2(\Omega), \quad (6)
\]

It is known that this operator given in (6) is selfadjoint, positive, has compact inverse and it has compact resolvent. Using this operator, the system (2)–(5), can
be written in an abstract way as follows:

\begin{align*}
  u_{tt} + \alpha A^2 u + \gamma A v_t &= 0, \quad x \in \Omega, \quad t > 0, \\
  v_{tt} + \beta A v - \gamma A u_t + \delta A^2 v_t &= 0, \quad x \in \Omega, \quad t > 0,
\end{align*}

and contemplates the boundary conditions (4).

In the last decades, many researchers have focused their efforts in the study of the asymptotic stability of several coupled systems with indirect damping (Terminology initially used by Russell in his work [23]). Systems of two coupled equations as wave-wave, plate-plate or plate-wave equations with indirect damping inside of their domains, or on their boundaries, were studied by several authors. We are going briefly mention some of these work:

Alabau et al. in [2]. They considered an abstract evolution equations given by:

\begin{align*}
  u_{tt} + A^2 u + \alpha v &= 0, \quad x \in \Omega, \quad t > 0, \\
  v_{tt} + A_1 v + \beta B v_t + \alpha u &= 0, \quad x \in \Omega, \quad t > 0,
\end{align*}

in which \(\Omega\) be a bounded open set of \(\mathbb{R}^n\) with smooth boundary \(\partial \Omega\) and \(A_1, A_2\) are self-adjoint positive linear operators in Hilbert space and \(B\) is a bounded operator.

When \(A_1 = -\Delta = A, A_2 = \Delta^2\) and \(B\) is the identity operator, we have wave-Petrowsky system, where \(\beta > 0\), with partial frictional damping \(\beta u_t\). For this case, they showed that, if \(0 < |\alpha| < C_{\Omega}^{3/2}\) and

\begin{align*}
  v_0 \in H^3(\Omega) \cap H^2_0(\Omega), \\
  u_0 \in H^6(\Omega) \cap H^3_0(\Omega), \\
  v_1 \in H^2(\Omega) \cap H^1_0(\Omega), \\
  u_1 \in H^4(\Omega) \cap H^2_0(\Omega).
\end{align*}

Then the energy of the solution satisfies, for every \(t > 0\), the estimate

\begin{align*}
  \int_{\Omega} (|\partial_t v|^2 + |\nabla v|^2 + |\partial_t u|^2 + |\Delta u|^2) dx \\
  \leq \frac{C}{t}(\|v_0\|_{3,\Omega}^2 + \|u_0\|_{6,\Omega} + \|v_1\|_{2,\Omega}^2 + \|u_1\|_{4,\Omega}^2) .
\end{align*}

In this direction other results can be found in [4, 6, 11, 13, 21].

Alabau et al. [4] (see also [1, 2, 3]) considered an abstract system of two coupled evolution equations with applications to several hyperbolic systems satisfying hybrid boundary conditions. They have shown the polynomial decay of their solutions using energy method and multiplicative techniques. Tebou [27] considered a weakly coupled system of plate-wave equations with indirect frictional damping mechanisms. He showed this system is not exponentially stable when the damping acts either in the plate equation or in the wave equation and a polynomial decay of the semigroup was showed using a frequency domain approach combined with multiplier techniques, and a recent Borichev and Tomilov [7] result in the characterization of polynomial decay of bounded semigroups. Recently, Guglielmi [11] considered two classes of systems of weakly coupled hyperbolic equations wave-wave equation and to a wave-Petrovsky system. When the wave equation is frictionally damped, he proved that this system is not exponentially stable and a polynomial decay was
obtained. No result about the optimal decay rate was provided. Many other papers were published in this direction, some of them can be viewed in [19, 22, 27, 29].

Now we will mention some concrete problems that motivated the work in of this paper:

Han and Liu in [12] have recently studied the regularity and asymptotic behavior of two-plate system solutions where only one of them is dissipative and indirect system dissipation occurs through the higher order coupling term $\gamma \Delta w_t$ and $-\gamma \Delta u_t$. The damping mechanism considered in this work was the structural or the Kelvin-Voigt damping. More precisely, the system studied in [12] is:

\[
\begin{align*}
    u_{tt} + \Delta^2 u + \gamma \Delta w_t &= 0, \quad x \in \Omega, \quad t > 0, \\
    w_{tt} + \Delta^2 w - \gamma \Delta u_t - d_{sl} \Delta w_t + d_{sv} \Delta^2 w_t &= 0, \quad x \in \Omega, \quad t > 0,
\end{align*}
\]

satisfying the boundary conditions

\[
    u = \frac{\partial u}{\partial \nu} = 0, \quad w = \frac{\partial w}{\partial \nu} = 0, \quad t > 0, \quad x \in \partial \Omega,
\]

where $u(x, t), w(x, t)$ denote the transversal displacements of the plates at time $t$ in the domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial \Omega$, $\gamma \neq 0$ is the coupling coefficient. They showed that if $d_{sl} > 0$ and $d_{sv} = 0$, the semigroup associated with the system is analytic and for $d_{sl} = 0$ and $d_{sv} > 0$, they showed that $S(t)$ is exponential but not analytic.

In 2013, Dell’Oro et al. in [8]. They considered the abstract system with fractional partial damping:

\[
\begin{align*}
    \rho_1 u_{tt} + \gamma_1 A u_{tt} + A^2 u - A^\sigma \phi &= 0, \quad x \in \Omega, \quad t > 0, \\
    \phi_t + A \phi + A^\sigma u_t &= 0, \quad x \in \Omega, \quad t > 0,
\end{align*}
\]

where $\Omega$ be a bounded open set of $\mathbb{R}^n$ with smooth boundary $\partial \Omega$ and $A = -\Delta$ as in (6) this system models a thermoelastic plate, where the parameter $\gamma \geq 0$ is responsible for the rotational inertia, which is proportional to the plate thickness, $\gamma = 0$, corresponding to the case of a thin plate. They showed that if $a_{sl} > 0$ and $a_{sv} = 0$, the semigroup decays polynomially to zero as $t^{-1/(4-4\sigma)}$ for initial data in the domain of the semigroup generator, and such a decay rate is optimal. Moreover, when $1/2 \leq \sigma < 1$, they proved that the semigroup decays polynomially with the optimal rate $t^{-1/(1-2\sigma)}$. Other results in this direction can be found in [6, 24, 26, 28].

A more recent result involving fractional dissipation was published in 2019 by Oquendo-Suárez [18], they studied the following abstract system:

\[
\begin{align*}
    \rho_1 u_{tt} + \gamma_1 A u_{tt} + \beta_1 A^2 u + \alpha v &= 0, \quad x \in \Omega, \quad t > 0, \\
    \rho_2 v_{tt} + \gamma_2 A v_{tt} + \beta_2 A^2 v + \alpha u + \kappa A^\sigma v_t &= 0, \quad x \in \Omega, \quad t > 0,
\end{align*}
\]

where $\Omega$ be a bounded open set of $\mathbb{R}^n$ with smooth boundary $\partial \Omega$ and one of these equations is conservative and the other has fractional dissipative properties given
by $A^\theta v_t$, where $0 \leq \theta \leq 1$ and $A = -\Delta$ as in (6) and where the coupling terms are $\alpha u$ and $\alpha v$. They showed that the semigroup decays polynomially with a rate that depends on $\theta$ and some relations between the structural coefficients of the system. Have also shown that the rates obtained are optimal using a spectral characterization theorem of semigroup polynomial stability due to Borichev and Tomilov [7].

Recently published works explore the regularity of solutions using the Gevrey classes introduced in 1989 in the thesis of Taylor [25]. Among these works, we can mention Hao-Liu-Yong [13] and, more recently, the work of Keyanto-Ute-Warma [30] to be published. In this last work, the authors studied the thermelastic plate model with a fractional Laplacian between the Euler-Bernoulli and Kirchhoff model with two types of boundary conditions, in addition to studying the asymptotic and analytical behavior, the authors show that the underlying semigroups is of Gevrey class $s > \frac{2-\theta}{2}$ for both the clamped and hinged boundary conditions when the parameter $\theta$ lies in the interval $(0, 1/2)$.

This paper is organized as follows. In section 2, we study the well-posedness of the system (7)-(8) through the semigroup theory. We left our main results for the last two sections. In Section 3, we prove the exponential decay of the semigroup $S(t) = e^{Bt}$, for $0 \leq \theta \leq 1$. In section 4 deals with the lack of analiticity of the semigroup $S(t) = e^{Bt}$ for $\theta \in [0, 1)$ and analiticity de $S(t)$ for $\theta = 0$; in particular, we address the case $0 \leq \theta < 1$ in Subsection 4.1, while the case $\theta = 1$ is discussed in Subsection 4.2. Finally in section 5 we show that $S(t) = e^{Bt}$ is of Gevrey class $s > \frac{2+\theta}{\theta}$ when the parameter $\theta$ lies in the interval $(0, 1)$.

2 Well-Posedness of the System

We will use a semigroup approach to show existence uniqueness of strong solutions for the abstrac system (7)-(8). It is important recalling that $A$ defined in (6) is a positive self-adjoint operator with compact inverse on a complex Hilbert space $D(A^0) = L^2(\Omega)$. Therefore, the operator $A^\theta$ is self-adjoint positive for $\theta \in \mathbb{R}$ and the embedding

$$D(A^{\theta_1}) \hookrightarrow D(A^{\theta_2}),$$

is continuous for $\theta_1 > \theta_2$. Here, the norm in $D(A^\theta)$ is given by $\|u\|_{D(A^\theta)} := \|A^\theta u\|$, $u \in D(A^\theta)$, where $\| \cdot \|_{\mathcal{H}}$ denotes the norm in the Hilbert space $\mathcal{H}$. Some of these spaces are: $D(A^{1/2}) = H^1_0(\Omega)$, $D(A^0) = L^2(\Omega)$ and $D(A^{-1/2}) = H^{-1}(\Omega)$.

Now, we will use a semigroup approach to study the well-posedness of the system (7)-(8). Taking $w = u_t$, $v_t = z$ and considering $U = (u, v, w, z)$ and $U_0 = (u_0, v_0, u_1, v_1)$, the system (7)-(8), can be written in the following abstract framework

$$\frac{d}{dt}U(t) = B U(t), \quad U(0) = U_0,$$

where the operator $B$ is given by

$$BU := \left( w, \ z, \ -\alpha A^2 u - \gamma Az, \ -\beta Av + \gamma Aw - \delta A^\theta z \right),$$

(10)
for $U = (u, v, w, z)$. This operator will be defined in a suitable subspace of the phase space 
\[ \mathcal{H} := D(A) \times D(A^{\frac{1}{2}}) \times D(A^0) \times D(A^0). \]
It’s a Hilbert space with the inner product 
\[ \langle U_1, U_2 \rangle := \alpha \langle Au_1, Au_2 \rangle + \beta \langle A^{\frac{1}{2}}v_1, A^{\frac{1}{2}}v_2 \rangle + \langle w_1, w_2 \rangle + \langle z_1, z_2 \rangle, \]
for $U_i = (u_i, v_i, w_i, z_i) \in \mathcal{H}$, $i = 1, 2$. In these conditions, we define the domain of $\mathcal{B}$ as 
\[ D(\mathcal{B}) := \{ U \in \mathcal{H} : (w, z) \in D(A) \times D(A^{\frac{1}{2}}), (-\alpha Au - \gamma z, -\beta v - \delta A^{\theta - 1} z) \in \mathcal{D}(\mathcal{B}) \}. \]
To show that the operator $\mathcal{B}$ is the generator of a $C_0$-semigroup we invoke a result from Liu-Zheng’ book.

**Theorem 1 (see Theorem 1.2.4 in [14])** Let $\mathcal{B}$ be a linear operator with domain $D(\mathcal{B})$ dense in a Hilbert space $\mathcal{H}$. If $\mathcal{B}$ is dissipative and $0 \in \rho(\mathcal{B})$, the resolvent set of $\mathcal{B}$, then $\mathcal{B}$ is the generator of a $C_0$-semigroup of contractions on $\mathcal{H}$.

Let us see that the operator $\mathcal{B}$ in (10) satisfies the conditions of this theorem. Clearly, we see that $D(\mathcal{B})$ is dense in $\mathcal{H}$. Effecting the internal product of $\mathcal{B}U$ with $U$, we have 
\[ \text{Re}\langle \mathcal{B}U, U \rangle = -\delta \| A^{\theta/2} z \|^2 \, dx, \quad \forall \ U \in D(\mathcal{B}), \] (11)
that is, the operator $\mathcal{B}$ is dissipative.

To complete the conditions of the above theorem, it remains to show that $0 \in \rho(\mathcal{B})$. Let $F = (f_1, f_2, f_3, f_4) \in \mathcal{H}$, let us see that the stationary problem $\mathcal{B}U = F$ has a solution $U = (u, v, w, z)$. From the definition of the operator $\mathcal{B}$ given in (10), this system can be written as 
\[ \begin{align*}
  w &= f_1, & \alpha A^2 u &= -[\gamma Af_2 + f_3], \\
  z &= f_2, & \beta Av &= \gamma Af_1 - \delta A^\theta f_2 - f_4. 
\end{align*} \] (12) (13)
This problem can be placed in a variational formulation: to find $t = (u, v)$ such that 
\[ b(t, z) = h(z) := \langle h, z \rangle, \quad \forall \ z = (z_1, z_2) \in D(A) \times D(A^{\frac{1}{2}}), \] (14)
where 
\[ h = (-[\gamma Af_2 + f_3], \gamma Af_1 - \delta A^\theta f_2 - f_4) \in D(A^0) \times D(A^0) \] and 
\[ b(u, v; z_1, z_2) := \alpha \langle Au, Az \rangle + \beta \langle A^{\frac{1}{2}}v, A^{\frac{1}{2}}z \rangle. \]
Consequently 
\[ b(t, t) = \alpha \| Au \|^2 + \beta \| A^{\frac{1}{2}}v \|^2. \] (15)
Of (15) the proof of the coercivity of this sesquiline form $b$ in Hilbert space $D(A) \times D(A^{\frac{1}{2}})$ is immediate, now, applying the Lax-Milgram Theorem and taking into account the first equations of (12)-(13) we have a unique solution $U \in \mathcal{H}$. As
this solution satisfies the system (12)-(13) in a weak sense, from these equations we can conclude that $U \in D(B)$.

Again, from (15) and the second equations of (12)-(13), applying Cauchy-Schwarz and Young inequalities to the second member of this inequality, for $\varepsilon > 0$ there exists $K_{\varepsilon} > 0$, such that
\[
\alpha \|Au\|^2 + \beta \|A^{\frac{1}{2}}v\|^2 \leq C_{\varepsilon} \|F\|^2.
\]
This inequality and the first equations of (12)-(13) imply that $\|U\|_{\mathcal{H}} \leq C \|F\|_{\mathcal{H}}$, then $0$ belongs to the resolvent set $\rho(B)$. Consequently, from Theorem 1 we have $B$ is the generator of a contractions semigroup.

As $B$ is the generator of a $C_0$-semigroup the solution of the abstract system (9) is given by $U(t) = e^{tB}U_0$, $t \geq 0$. Thus, we have shown the following well-posedness theorem:

**Theorem 2 (see [20])** Let us take initial data $U_0$ in $\mathcal{H}$ then there exists only one solution to the problem (9) satisfying
\[
U \in C([0, \infty[; \mathcal{H}).
\]
Moreover, if $U_0 \in D(B)$ then the solution satisfies
\[
U \in C([0, \infty[; D(B)) \cap C^1([0, \infty); \mathcal{H}).
\]

### 3 Stability Results

In this section, we will study the asymptotic behavior of the semigroup of the system (7)-(8). First we will use the following spectral characterization of exponential stability of semigroups due to Gearhart [10] (Theorem 1.3.2 book of Liu-Zheng) and to study analyticity we will use a characterization of the book of Liu-Zheng (Theorem 1.3.3).

**Theorem 3 (see [14])** Let $S(t) = e^{Bt}$ be a $C_0$-semigroup of contractions on a Hilbert space $\mathcal{H}$. Then $S(t)$ is exponentially stable if and only if
\[
\rho(B) \supseteq \{i\lambda/\lambda \in \mathbb{R}\} \equiv i\mathbb{R}
\]
and
\[
\limsup_{|\lambda| \to \infty} \|(i\lambda I - B)^{-1}\|_{L(\mathcal{H})} < \infty
\]
holds.

**Theorem 4 (see [14])** Let $S(t) = e^{Bt}$ be $C_0$-semigroup of contractions on a Hilbert space $\mathcal{H}$. Suppose that
\[
\rho(B) \supseteq \{i\lambda/\lambda \in \mathbb{R}\} \equiv i\mathbb{R}
\]
Then $S(t)$ is analytic if and only if
\[
\limsup_{|\lambda| \to \infty} \|\lambda(i\lambda I - B)^{-1}\|_{L(\mathcal{H})} < \infty
\]
holds.
And using the following spectral characterization of stability of semigroups due to Borichev and Tomilov[7]:

**Theorem 5 (see [7])** Let $B$ be the generator of a $C_0$-semigroup of bounded operators on a Hilbert space such that $i\mathbb{R} \subset \rho(B)$. Then, we have
\[
\|e^{tB}U_0\| \leq Ct^{-1/\eta}\|U_0\|_{D(B)}, \quad t > 0,
\]
if and only if
\[
\limsup_{|\lambda| \to \infty} \frac{1}{|\lambda|^\eta}\|(i\lambda I - B)^{-1}\| < \infty.
\]

In what follows: $C$, $C_\delta$ and $K_\varepsilon$ will denote positive constants that assume different values in different places and the coupling coefficient $\gamma$ will be assumed positive (the results remain valid when this coefficient is negative).

First, note that if $\lambda \in \mathbb{R}$ and $F = (f_1, f_2, f_3, f_4) \in \mathcal{H}$ then the solution $U = (u, v, w, z) \in D(B)$ of the stationary system $(i\lambda I - B)U = F$ can be written in the form
\[
i\lambda u - w = f_1, \quad (19)
i\lambda v - z = f_2, \quad (20)i\lambda w + \alpha A^2 u + \gamma Az = f_3, \quad (21)i\lambda z + \beta Av - \gamma Aw + \delta A^\theta z = f_4. \quad (22)
\]

We have
\[
\delta \|A^\theta z\|^2 = \text{Re}\langle (i\lambda - B)U, U \rangle = \text{Re}\langle F, U \rangle \leq \|F\|_\mathcal{H}\|U\|_\mathcal{H}. \quad (23)
\]
From equations (20) and (23), we have
\[
\lambda^2 \|A^\theta v\|^2 \leq C \left\{ \|F\|_\mathcal{H}\|U\|_\mathcal{H} + \|F\|_\mathcal{H}^2 \right\}. \quad (24)
\]
As $\theta - 2 \leq 0 \leq \theta$, taking into account the continuous embedding $D(A^\theta_2) \hookrightarrow D(A^\theta_1)$, $\theta_2 > \theta_1$ and (23), we obtain
\[
\|A^{\theta_2} z\|^2 \leq C \left\{ \|F\|_\mathcal{H}\|U\|_\mathcal{H} + \|F\|_\mathcal{H}^2 \right\}. \quad (25)
\]
\[
\|z\|^2 \leq C \left\{ \|F\|_\mathcal{H}\|U\|_\mathcal{H} + \|F\|_\mathcal{H}^2 \right\}. \quad (26)
\]

### 3.1 Exponential Decay of $S(t)$ for $0 \leq \theta \leq 1$

In this subsection we show the exponential decay using Theorem (18), let us first check condition (17).

Now, notice that:
\[
\langle A^2 v, A^\sigma w \rangle = \langle A^2 v, A^\sigma (i\lambda u - f_1) \rangle = -i\lambda \langle A^\sigma v, A^2 u \rangle - \langle A^{1+\sigma} v, Af_1 \rangle
\]
\[
\langle A^2 u, A^\sigma z \rangle = \langle A^2 u, A^\sigma (i\lambda v - f_2) \rangle = -i\lambda \langle A^2 u, A^\sigma v \rangle - \langle A^{1+\sigma} u, Af_2 \rangle.
\]

Summing up, both equations and taking the real part, we have
\[
\text{Re}\{\langle A^2 v, A^\sigma w \rangle + \langle A^2 u, A^\sigma z \rangle\} = -\text{Re}\{\langle A^{1+\sigma} u, Af_2 \rangle + \langle A^{1+\sigma} v, Af_1 \rangle\} \quad (27)
\]
To get our first results, we should first demonstrate some lemmas.
Lemma 6 Let \(0 \leq \theta \leq 1\) and \(\sigma \leq -1\). The solutions of equations (19)-(22) satisfy the following equality

\[
\frac{\gamma \alpha}{\beta} \| A^{\frac{\sigma+1}{2}} w \|^2 = \gamma \| A^{\frac{\sigma+1}{2}} z \|^2 - \alpha \Re \{ \langle A^{1+\sigma} u, A f_2 \rangle + \langle A^{1+\sigma} v, A f_1 \rangle \} \\
+ \frac{\delta \alpha}{\beta} \Re \{ A^{\frac{\sigma+1}{2}} z, A^{\frac{\sigma+2}{2}} w \} - \frac{\alpha}{\beta} \Re \{ f_4, A^{\sigma+1} w \} - \Re \{ f_3, A^\sigma z \} \\
- \frac{\lambda \alpha}{\beta} \Im \{ A^{\frac{\sigma+2}{2}} z, A^{\frac{\sigma+2}{2}} w \} - \lambda \Im \{ A^{\frac{\sigma+3}{2}} z, A^{\frac{\sigma+3}{2}} w \}.
\]

Proof: Applying the product duality to equation (21) with \(A^\sigma z\) and recalling that the operator \(A\) is self-adjoint, we have

\[
\gamma \| A^{\frac{\sigma+1}{2}} z \|^2 = -\alpha \langle A^2 u, A^\sigma z \rangle - i \lambda \langle w, A^\sigma z \rangle - \langle f_3, A^\sigma z \rangle.
\]

Similarly, applying the product duality to equation (22) with \(\frac{\alpha}{\beta} A^{\sigma+1} w\) and using the equation (19) we obtain

\[
\frac{\gamma \alpha}{\beta} \| A^{\frac{\sigma+2}{2}} w \|^2 = \alpha \langle A^2 v, A^\sigma w \rangle + \frac{i \lambda \alpha}{\beta} \langle A^{\frac{\sigma+3}{2}} z, A^{\frac{\sigma+2}{2}} w \rangle + \frac{\delta \alpha}{\beta} \langle A^{\frac{\sigma+4}{2}} z, A^{\frac{\sigma+3}{2}} w \rangle \\
- \frac{\alpha}{\beta} \langle f_4, A^{\sigma+1} w \rangle.
\]

Now, to get the conclusion of this Lemma it is sufficient to perform the subtraction of these last two equations, take the real part and use the identity (27).

Taking \(\sigma = -2\), in Lemma (6), we have

\[
\frac{\gamma \alpha}{\beta} \| w \|^2 = \gamma \| A^{\frac{1}{2}} z \|^2 - \alpha \Re \{ \langle A^{-1} u, A f_2 \rangle + \langle A^{-1} v, A f_1 \rangle \} \\
+ \frac{\delta \alpha}{\beta} \Re \{ A^{\theta-1} z, w \} - \frac{\alpha}{\beta} \Re \{ f_4, A^{-1} w \} - \Re \{ f_3, A^{-2} z \} \\
- \frac{\alpha}{\beta} \Im \{ z, A^{-1} \lambda w \} - \lambda \Im \{ w, A^{-2} z \},
\]

From equation (21), we have \(A^{-1} \lambda w = i \alpha A u + i \gamma z - i A^{-1} f_3\), therefore

\[
- \frac{\alpha}{\beta} \Im \{ z, A^{-1} \lambda w \} = - \frac{\alpha}{\beta} \Im \{ z, i \alpha A u + i \gamma z - i A^{-1} f_3 \} \\
= \frac{\alpha^2}{\beta} \Re \{ A^\theta z, A^{\frac{\theta-\sigma}{2}} u \} + \frac{\alpha \gamma}{\beta} \| z \|^2 - \frac{\alpha}{\beta} \Re \{ z, A^{-1} f_3 \} \\
\leq \frac{\alpha^2}{\beta} \Re \{ A^\theta z, A^{\frac{\theta-\sigma}{2}} u \} - \frac{\alpha}{\beta} \Re \{ z, A^{-1} f_3 \} + C \| F \|_H \| F \|_H.
\]

Substituting (29) into (28) and from \(-\frac{1}{2} < \frac{\theta}{2}\), using (23), we have
\[ \frac{\gamma}{\beta} \| u \|^2 \leq C \| F \|_{\mathcal{H}} \| U \|_{\mathcal{H}} - \alpha \text{Re}\{ (A^{-1}u, A f_2) + (A^{-1}v, A f_1) \} + \frac{\delta}{\beta} \text{Re}(A^{\theta-1}z, w) - \frac{\alpha}{\beta} \text{Re}(f_4, A^{-1}w) - \text{Re}(f_3, A^{-2}z) \]

\[ + \frac{\alpha^2}{\beta} \text{Re}(A^{\theta}z, A^{2-\theta}u) - \frac{\alpha}{\beta} \text{Re}(z, A^{-1}f_3) - \text{Im}(A^{-2}\lambda w, z). \]  

(30)

On the other hand of the equation (21), we have \( A^{-2}\lambda w = i\alpha u + i\gamma A^{-1}z - iA^{-2}f_3 \), therefore

\[ \text{Im}(A^{-2}\lambda w, z) = \text{Im}(i\alpha u + i\gamma A^{-1}z - iA^{-2}f_3, z) \]

\[ = \alpha \text{Re}(A^{\frac{\theta}{2}}u, A^{\frac{\theta}{2}}z) + \gamma \| A^{\frac{\theta}{2}}z \|^2 - \text{Re}(A^{-2}f_3, z). \]

(31)

Now, substituting (31) into (30), we have

\[ \frac{\gamma}{\beta} \| u \|^2 \leq C \| F \|_{\mathcal{H}} \| U \|_{\mathcal{H}} - \alpha \text{Re}\{ (A^{-1}u, A f_2) + (A^{-1}v, A f_1) \} + \frac{\delta}{\beta} \text{Re}(A^{\theta-1}z, w) - \frac{\alpha}{\beta} \text{Re}(f_4, A^{-1}w) - \text{Re}(f_3, A^{-2}z) \]

\[ + \frac{\alpha^2}{\beta} \text{Re}(A^{\theta}z, A^{2-\theta}u) - \frac{\alpha}{\beta} \text{Re}(z, A^{-1}f_3) - \text{Re}(A^{-2}f_3, z). \]  

(32)

Applying Cauchy-Schwarz and Young inequalities, taking into account the continuous embedding \( D(A^{\theta_2}) \hookrightarrow D(A^{\theta_1}) \), \( \theta_2 > \theta_1 \), \( \theta - 1 \leq \frac{\theta}{2} \), and using estmative (23), we have, for \( \varepsilon > 0 \), there existe \( k_\varepsilon > 0 \), such that

\[ \| w \|^2 \leq C\{ \| F \|_{\mathcal{H}} \| U \|_{\mathcal{H}} \} + \varepsilon \| w \|^2 + \varepsilon \| A^{\frac{\theta}{2}}u \|^2 + \varepsilon \| A^{\frac{\theta}{2}}u \|^2. \]

(33)

On the other hand, by effecting the product duality of (21) by \( A^{-\theta}u \), we have

\[ \alpha \| A^{\frac{\theta}{2}}u \|^2 = \langle w, A^{-\theta}(i\lambda u) \rangle - \gamma \langle A^{\frac{\theta}{2}}z, A^{\frac{\theta}{2}}u \rangle + \langle f_3, A^{-\theta}u \rangle \]

\[ = \| A^{\frac{\theta}{2}}w \|^2 + \langle w, A^{-\theta}f_1 \rangle - \gamma \langle A^{\frac{\theta}{2}}z, A^{\frac{\theta}{2}}u \rangle + \langle f_3, A^{-\theta}u \rangle. \]

Taking real part and applying Cauchy-Schwarz and Young inequalities, taking into account the continuous embedding, \( -\frac{\theta}{2} \leq \frac{\theta}{2} \), we have

\[ \| A^{\frac{\theta}{2}}u \|^2 \leq C\{ \| F \|_{\mathcal{H}} \| U \|_{\mathcal{H}} \} + \| A^{\frac{\theta}{2}}w \|^2. \]

(34)

Substituting (34) into (33) and taking into account the continuous embedding, \( -\frac{\theta}{2} \leq \frac{\theta}{2} \) and \( \frac{\theta}{2} \leq 0 \), we have

\[ \| w \|^2 \leq C\{ \| F \|_{\mathcal{H}} \| U \|_{\mathcal{H}} \} \quad \text{for} \quad 0 \leq \theta \leq 1. \]

(35)
Taking the duality product between equation (21) and \( u \) and using the equation (19), we obtain
\[
\alpha \| Au \|^2 = -\gamma \langle z, Au \rangle + \| w \|^2 + \langle w, f_1 \rangle + \langle f_3, u \rangle.
\] (36)

Applying Cauchy-Schwarz and Young inequalities, taking into account the continuous embedding \( D(A^{\theta_1}) \to D(A^{\theta_2}) \) with \( \theta_2 > \theta_1, \frac{1}{\theta_2} < \frac{2}{\delta}, 0 \leq \frac{2}{\delta} \) and using estimatives (23) and (35) we have, for \( \varepsilon > 0 \), there exist \( k_\varepsilon > 0 \), such that
\[
\alpha \| Au \|^2 \leq C \{ \| F \|_H \| U \|_H \} \quad \text{for} \quad 0 \leq \theta \leq 1.
\] (37)

Similarly, applying the duality product to equation (22) with \( v \) and using the equation (20), we have
\[
\beta \| A^{1/2} v \|^2 = \gamma \langle Av, v \rangle + \| z \|^2 - \delta \langle A^{1/2} z, A^{1/2} v \rangle + \langle z, f_2 \rangle + \langle f_4, v \rangle.
\] (38)

Subtracting (38) from (36) and taking the real part, we have
\[
\beta \| A^{1/2} v \|^2 = \alpha \| Au \|^2 + \gamma \text{Re} \{ i\lambda Av - Af_2, u \} + \langle i\lambda Au - Af_1, v \rangle - \| w \|^2
\]
\[
-\delta \text{Re} \{ A^{1/2} z, A^{1/2} v \} + \langle z, f_2 \rangle + \text{Re} \{ f_4, v \} - \text{Re} \{ w, f_1 \} - \text{Re} \{ f_3, u \}
\]
\[
\leq \alpha \| Au \|^2 + \gamma \lambda \text{Im} \{ \langle Av, u \rangle + \langle u, Av \rangle \} - \delta \text{Re} \{ f_2, Au \} + \langle f_1, v \rangle
\]
\[
-\delta \text{Re} \{ A^{1/2} z, A^{1/2} v \} + \langle z, f_2 \rangle + \text{Re} \{ f_4, v \} - \text{Re} \{ w, f_1 \} - \text{Re} \{ f_3, u \}
\]

Now, as \( \text{Im} \{ \langle Av, u \rangle + \langle u, Av \rangle \} = 0 \) and \( \frac{2}{\delta} \leq \frac{1}{\theta_2} \), using the estimative (37) and applying Cauchy-Schwarz inequality and Young inequality and continuous embedding we have the inequality
\[
\beta \| A^{1/2} v \|^2 \leq C \{ \| F \|_H \| U \|_H \} \quad \text{for} \quad 0 \leq \theta \leq 1.
\] (39)

Therefore, estimates (20), (35), (37) and (39), condition (17) the Theorem (3) is verified for \( 0 \leq \theta \leq 1 \).

Now let’s show condition (16) the Theorem (3). It’s prove that \( i\mathbb{R} \subset \rho(\mathbb{B}) \) by contradiction, then we suppose that \( i\mathbb{R} \not\subset \rho(\mathbb{B}) \). As \( 0 \in \rho(\mathbb{B}) \) and \( \rho(\mathbb{B}) \) is open, we consider the highest positive number \( \lambda_0 \) such that the interval \( ] -i\lambda_0, i\lambda_0[ \subset \rho(\mathbb{B}) \) then \( i\lambda_0 \) or \( -i\lambda_0 \) is an element of the spectrum \( \sigma(\mathbb{B}) \). We suppose \( i\lambda_0 \in \sigma(\mathbb{B}) \) (if \( -i\lambda_0 \in \sigma(\mathbb{B}) \) the proceeding is similar). Then, for \( 0 < \delta < \lambda_0 \) there exist a sequence of real numbers \( \{ \lambda_n \} \), with \( \delta \leq \lambda_n < \lambda_0 \), \( \lambda_n \to \lambda_0 \), and a vector sequence \( U_n = (u_n, v_n, w_n, z_n) \in D(\mathbb{B}) \) with unitary norms, such that
\[
\| (i\lambda_n - \mathbb{B}) U_n \|_H = \| F_n \|_H \to 0,
\]
as \( n \to \infty \). From (37) and (39) for \( 0 \leq \theta \leq 1 \), we have
\[
\alpha \| Au_n \|^2 \leq C \{ \| F_n \|_H \| U_n \|_H + \| F_n \|_H^2 \},
\]
\[
\beta \| A^{1/2} v_n \|^2 \leq C \{ \| F_n \|_H \| U_n \|_H + \| F_n \|_H^2 \}.
\]
In addition to the estimates and (26) and (35) for $0 \leq \theta \leq 1$, we have
\[ \|w_n\|^2 + \|z_n\|^2 \to 0. \]

Consequently,
\[ \alpha \|Au_n\|^2 + \beta \|A^{1/2}v_n\|^2 + \|w_n\|^2 + \|z_n\|^2 \to 0. \]

Therefore, we have $\|U_n\|_\mathcal{H} \to 0$ but this is absurd, since $\|U_n\|_\mathcal{H} = 1$ for all $n \in \mathbb{N}$. Thus, $i\mathbb{R} \subset \rho(B)$.

This completes the proof of condition (13) of the Theorem(3).

4 $S(t) = e^{Bt}$ is not analytic for $\theta \in [0,1)$ and it is analytical for $\theta = 1$

This section is divided into two subsections: In the first subsection (4.1) we show the lack of analyticity for $0 \leq \theta < 1$ and in subsection (4.2) we test the analyticity of $S(t)$ for $\theta = 1$.

4.1 Lack of analyticity of $S(t)$ for $\theta \in [0,1)$

In the proof of this subsections, we will use the (18) equivalence of the theorem (1), note that non-verification of the right-hand inequality of identity (18) implies the lack of analyticity of the associated semigroup $S(t) = e^{Bt}$ for $\theta \in [0,1)$.

**Theorem 7** The semigroup associated to system (7)-(8), is not analytical for $\theta \in [0,1)$.

**Proof:** The spectrum of operator $A = -\Delta$ defined in (6) is constituted by positive eigenvalues $(\sigma_n)$ such that $\sigma_n \to \infty$ as $n \to \infty$. For $n \in \mathbb{N}$ we denote with $e_n$ an unitary $L^2$-norm eigenvector associated to the eigenvalue $\sigma_n$, that is:
\[ A e_n = \sigma_n e_n, \quad A^\theta e_n = \sigma_n^\theta e_n, \quad \|e_n\|_{L^2(\Omega)} = 1, \quad \text{for} \quad 0 \leq \theta < 1, \quad n \in \mathbb{N} \quad (40) \]

Let's show that the right side of inequality (18) for $\theta \in [0,1)$ is not verified. Consider the eigenvalues and eigenvectors of the operator $A$ as in (6) and (40) respectively.

Let $F_n = (0,0,-e_n,0) \in \mathcal{H}$. The solution $U = (u,v,w,z)$ of the system $(i\lambda I - B)U = F_n$ satisfies $w = i\lambda u$, $z = i\lambda v$ and the following equations
\[
\begin{align*}
\lambda^2 u - \alpha A^2 u - i\lambda \gamma A v &= e_n, \\
\lambda^2 v - \beta A v + i\gamma \lambda A u - i\lambda \delta A^\theta v &= 0.
\end{align*}
\]

Let us see whether this system admits solutions of the form
\[ u = \mu e_n, \quad v = \nu e_n, \]

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for some complex numbers $\mu$ and $\nu$. Then, the numbers $\mu$, $\nu$ should satisfy the algebraic system
\begin{align*}
\{\lambda^2 - \alpha\sigma_n^2\}\mu - i\lambda\gamma\sigma_n\nu & = 1, \\
\lambda\gamma\sigma_n\mu + \{\lambda^2 - \beta\sigma_n - i\delta\sigma_n^\theta\}\nu & = 0.
\end{align*}
(41) (42)

On the other hand solving the system (41)-(42), we find that
\begin{equation}
\mu = \frac{\{p_{2,n}(\lambda^2) - i\delta\sigma_n^\theta\mu\}}{p_{1,n}(\lambda^2)p_{2,n}(\lambda^2) - \gamma^2\lambda^2\sigma_n^2 - i\delta\sigma_n^\theta\lambda p_{1,n}(\lambda^2)},
\end{equation}
where
\begin{equation}
p_{1,n}(\lambda^2) := \lambda^2 - \alpha\sigma_n^2 \quad \text{and} \quad p_{2,n}(\lambda^2) = \lambda_n^2 - \beta\sigma_n.
\end{equation}
(43) (44)

Taking $s_n = \lambda^2 = \lambda_n^2$ and considering the polynomial
\begin{equation}
q_n(s_n) := p_{1,n}(s_n)p_{2,n}(s) - \gamma^2\sigma_n^2 s_n \\
= s_n^2 - [(\alpha + \gamma^2)\sigma_n^2 + \beta\sigma_n]s_n + \alpha\beta\sigma_n^3.
\end{equation}
Now, taking $q_n(s_n) = 0$, we have the roots of the polynomial $q_n$ are given by
\begin{equation}
s_n^\pm = \frac{[\alpha + \gamma^2]\sigma_n^2 + \beta\sigma_n \pm \sigma_n \sqrt{(\alpha + \gamma^2)\sigma_n^2 + 2\beta(\gamma^2 - \alpha)\sigma_n + \beta^2}}{2}.
\end{equation}
(45)

Thus, if we introduce the notation $x_n \approx y_n$ meaning that $\lim_{n \to \infty} \frac{|x_n|}{|y_n|}$ is a positive real number.

Taking $s_n = s_n^+$ from equation (45), we have
\begin{equation}
s_n \approx \sigma_n^2 \quad \text{and} \quad \lambda_n \approx \sigma_n.
\end{equation}
(46)

Then
\begin{equation}
p_{2,n}(s_n) = s_n - \beta\sigma_n \approx \sigma_n^2.
\end{equation}
(47)

From $q_n(s_n) = 0$ in (43), we have
\begin{equation}
\mu_n = \frac{\{p_{2,n}(\lambda_n^2) - i\delta\lambda_n\sigma_n^\theta\}}{-i\delta\sigma_n^\theta\lambda_n p_{1,n}(\lambda^2)} = \frac{p_{2,n}(\lambda_n^2)}{\gamma^2\lambda_n^2\sigma_n^2} + i\frac{p_{2,n}(\lambda_n^2)}{\delta\gamma^2\lambda_n^2\sigma_n^2 + \theta}.
\end{equation}
(48)

Therefore
\begin{equation}
|\mu_n| \approx |\lambda_n|^{-1 - \theta}.
\end{equation}
(49)

Finally, of (10) for $C > 0$, the solution $U_n$ of the system $(i\lambda_n - \Box)U = F_n$, satisfies
\begin{equation}
\|U_n\|_H \geq C\|w_n\| = C|\lambda_n|\|u_n\| = C|\lambda_n|\|\mu_n\|\|e_n\| = C|\lambda_n|\|\mu_n\| \quad \text{for} \quad 0 \leq \theta < 1.
\end{equation}
(50)

Then, using estimates (49) in (50), for $\delta > 0$ and $|\lambda| > 1$, we have
\begin{equation}
\|U_n\|_H \geq \delta \begin{cases}
|\lambda|^{-\theta} > \delta|\lambda|^{-1} & \text{for} \quad 0 < \theta < 1, \\
|\lambda|^{\theta} > |\lambda|^{-1} & \text{for} \quad \theta = 0.
\end{cases}
\end{equation}
(51)
Finally, if we suppose that the semigroup decays with the rate $t\rightarrow \frac{1}{t}$ for some $\theta > -\frac{1}{2}$ for $0 < \theta < 1$, then from Theorem [5] we have that $|\lambda_n|^{-1/(1-\theta)}\|U_n\|_H$ is bounded. On the other hand, the above inequality implies that

$$|\lambda_n|^{-1/(1-\theta)}\|U_n\|_H \geq \delta |\lambda_n|^{-\theta+1} \rightarrow \infty \quad \text{for} \quad 0 < \theta < 1,$$

which is absurd. Therefore decay rate of $S(t)$ is $t^{\frac{1}{\theta}}$ for $0 < \theta < 1$ and since $S(t)$ is exponentially stable particularly for $\theta = 0$, from second equation of \([51]\), we have the optimal rate from $S(t)$ to $\theta = 0$ is exponential. So we can conclude that: $S(t) = e^{\theta t}$ is not analytic for $\theta \in [0, 1)$. This compleat the proof. \[\square\]

### 4.2 Analyticity of $S(t)$ for $\theta = 1$

In this subsection we show the analyticity the $S(t)$ for $\theta = 1$ using Theorem[11], specifically checking to condition \([13]\) $|\lambda| |(i\lambda I - B)^{-1}F|_H \leq C|F|_H\|U\|_H$.)

**Remark 8** Let $\delta > 0$. Exist $C_\delta > 0$ such that, for $0 \leq \theta \leq 1$, we have $\frac{\theta-1}{2} \leq 0$.

Applying continuous immersions and inequality \([35]\), we have

$$\|A^{\theta-\frac{1}{2}}w\|^2 \leq C_\delta \{F|_H\|U\|_H\} \quad \text{for} \quad 0 \leq \theta \leq 1.$$

**Lemma 9** Let $\delta > 0$. Exist $C_\delta > 0$ such that the solutions of equations \([14]\)-\([23]\) for $|\lambda| \geq \delta$, satisfy

$$\|A^{\theta}w\|^2 \leq C_\delta \{F|_H\|U\|_H\} \quad \text{for} \quad 0 \leq \theta \leq 1.$$

**Proof:** From $0 \leq \theta \leq 1$, then $\sigma = \theta - 2 \leq -1$. Therefore taking $\sigma = \theta - 2$ in the Lemma[6], we have

$$\frac{\gamma\alpha}{\beta} \|A^{\theta}w\|^2 = \gamma \|A^{\theta-\frac{1}{2}}z\|^2 - \alpha \text{Re}\{\lambda^\theta u, A^{\theta-1}f_2\} + \alpha \text{Re}\{\hat{A}^{\theta-1}v, A\hat{f}_1\}$$

$$+ \frac{\delta\alpha}{\beta} \text{Re}\{A^{\theta-\frac{2}{2}}z, A^{\theta}w\} - \frac{\alpha}{\beta} \text{Re}\{f_4, A^{\theta-1}w\} - \text{Re}\{f_3, A^{\theta-2}z\} \quad (52)$$

$$- \frac{\lambda\alpha}{\beta} \text{Im}\{A^{\theta-\frac{2}{2}}z, A^{\theta}w\} - \lambda \text{Im}\{A^{\theta}w, A^{\theta-\frac{2}{2}}z\}.$$

From equation \([21]\), we have $A^{\theta-1}\lambda w = i\alpha A^{\theta+1}u + i\gamma A^{\theta}z - iA^{\theta-1}f_3$, therefore

$$- \frac{\lambda\alpha}{\beta} \text{Im}\{z, A^{\theta-1}w\} = - \frac{\alpha}{\beta} \text{Im}\{A^{\theta}z, iA^{\theta}\hat{A}^{\theta-1}\hat{u}\} + \frac{\alpha\gamma}{\beta} \|A^{\theta}z\|^2 - \frac{\alpha}{\beta} \text{Re}\{z, A^{\theta-1}f_3\} \quad (53)$$

Applying Cauchy-Schwarz and Young inequalities, estivimative \([23]\) and for $1 > \epsilon > 0$, exist $K_\epsilon > 0$, we get

$$- \frac{\lambda\alpha}{\beta} \text{Im}\{z, A^{\theta-1}w\} \leq K_\epsilon \|F\|_H\|U\|_H + \epsilon \|A^{\theta+1}u\|^2 \quad (54).$$
On the outer hand, applying the product duality to equation (21) with $A^\theta u$ and recalling that the operator $A$ is self-adjoint, we obtain

$$\alpha \|A^{2+\theta} u\|^2 = \langle w, A^\theta (i\lambda u) \rangle - \gamma \langle A^{\frac{\theta}{2}} z, A^{2+\theta} u \rangle + \langle f_3, A^\theta u \rangle$$

$$= \|A^{\frac{\theta}{2}} w\|^2 + \langle w, A^\theta f_1 \rangle - \gamma A^{\frac{\theta}{2}} z A^{2+\theta} u + \langle f_3, A^\theta \rangle,$$

now applying Cauchy-Schwarz and Young inequalities for every $\varepsilon > 0$, there exists a positive constant $K_\varepsilon$, independent of $\lambda$, such that

$$\|A^{2+\theta} u\|^2 \leq C \{\|F\|_H \|U\|_H \} + \|A^{\frac{\theta}{2}} w\|^2. \quad (55)$$

Using (55) in (53), we obtain

$$-\lambda \alpha \beta \text{Im} \langle z, A^{\theta-1} w \rangle \leq \varepsilon \|A^{\frac{\theta}{2}} w\|^2 + C \{\|F\|_H \|U\|_H \}. \quad (56)$$

Similarly from equation (21), we have

$$A^{\theta-2} \lambda w = i \alpha A^\theta u + i \gamma A^{\theta-1} z - i A^{\theta-2} f_3,$$

therefore

$$-\lambda \text{Im} \langle A^{\theta-2} w, z \rangle = -i \alpha \langle A^\theta u, z \rangle - i \gamma \|A^{\frac{\theta-1}{2}} z\|^2 + i \langle A^{\theta-2} f_3, z \rangle. \quad (57)$$

Applying Cauchy-Schwarz and Young inequalities, estimator (23) and for $1 > \varepsilon > 0$, exist $K_\varepsilon > 0$, we get

$$-\text{Im} \langle A^{\theta-2} w, z \rangle \leq K_\varepsilon \|A^{\frac{\theta}{2}} z\|^2 + \varepsilon \|A^{\frac{\theta}{2}} u\|^2 + C \|A^{\frac{\theta-1}{2}} z\|^2 + C \|F\|_H \|U\|_H. \quad (58)$$

From $\frac{\theta-1}{2} < \frac{\theta}{2} \leq \frac{2+\theta}{2}$ using continuous embedding and estimmatives (23) and (55), we obtain

$$-\text{Im} \langle A^{\theta-2} w, z \rangle \leq \varepsilon \|A^{\frac{\theta}{2}} w\|^2 + C \{\|F\|_H \|U\|_H \}. \quad (59)$$

Applying Cauchy-Schwarz and Young inequalities in equation (52), for $1 > \varepsilon > 0$, exist $K_\varepsilon > 0$ and estimmatives (56) and (59) and from $\frac{\theta}{2} < \frac{\theta-1}{2} < \frac{\theta}{2}$ using continuous embedding for every $\varepsilon > 0$, there exists a positive constant $K_\varepsilon$, independent of $\lambda$, such that

$$\|A^{\frac{\theta}{2}} w\|^2 \leq C \|A^{\frac{\theta}{2}} z\|^2 + C \{\|F\|_H \|U\|_H \} + \varepsilon \|A^{\frac{\theta}{2}} w\|^2. \quad (60)$$

Finally from inequality (23) in the inequality (60) finish to proof.

\[\square\]

**Remark 10** Using Lemma (9) in the inequality (55), we have

$$\|A^{2+\theta} u\|^2 \leq C_\delta \{\|F\|_H \|U\|_H + \|F\|^2 \}. \quad (61)$$

And taking $\theta = 1$ in Lemma (9), we have

$$\|A^{\frac{1}{2}} w\| \leq C_\delta \{\|F\|_H \|U\|_H \}^{\frac{1}{2}}. \quad (62)$$
Lemma 11 Let $0 \leq \theta \leq 1$ and $\delta > 0$. Exist $C_\delta > 0$ such that the solutions of equations (19)-(22) for $|\lambda| \geq \delta$, satisfy:

$$\|A^{\frac{\delta+1}{2}}w\|^2 \leq C_\delta \lambda^2 \{\|F\|_\mathcal{H} \|U\|_\mathcal{H}\}.$$  

Proof: Applying the product duality to equation (21) with $A^{\theta-1}u$ and recalling that the operator $A$ is self-adjoint, we obtain

$$\alpha \|A^{\frac{\theta+1}{2}}u\|^2 = -i\lambda \langle w, A^{\theta-1}u \rangle + \gamma \langle A^\theta z, A^\frac{\theta}{2}u \rangle + \langle f_3, A^{\theta-1}u \rangle,$$

$$= \|A^{\frac{\theta+1}{2}}w\|^2 + \langle w, A^{\theta-1}f_1 \rangle + \gamma \langle A^\frac{\theta}{2}z, A^\frac{\theta}{2}u \rangle + \langle f_3, A^{\theta-1}u \rangle.$$

From Remark (8), Young inequalities, and taking into account that, $\frac{\theta}{2} < \frac{\theta+1}{2}$ and $|\lambda| \geq \delta$, by applying continuous immersions and inequality (23). And applying the operator $A^{\frac{\theta+1}{2}}$ in the equation (19) and applying Holder inequality finish to proof.  

$$\blacksquare$$

Remark 12 Taking $\theta = 1$ in inequality (61) to Remark (10), we have

$$\|A^{\frac{3}{2}}u\| \leq C_\delta \{\|F\|_\mathcal{H} \|U\|_\mathcal{H}\}^{\frac{1}{2}}. \quad (63)$$

Lemma 13 Let $\theta = 1$ and $\delta > 0$. Exist $C_\delta > 0$ such that the solutions of equations (19)-(22) for $|\lambda| \geq \delta$, satisfy:

$$\lambda \|z\|^2 \leq C_\delta \|F\|_\mathcal{H} \|U\|_\mathcal{H}.$$

Proof: Applying the product duality to equation (22) with $z$ and recalling that the operator $A$ is self-adjoint, we have

$$i\lambda \|z\|^2 = -\beta \langle A^{\frac{1}{2}} \left( \frac{-iz}{\lambda} - i\frac{f_2}{\lambda} \right), A^{\frac{1}{2}}z \rangle + \gamma \langle A^{\frac{1}{2}}w, A^{\frac{1}{2}}z \rangle - \delta \|A^\frac{\theta}{2}z\|^2 + \langle f_4, z \rangle.$$

Taking imaginary part and using Cauchy-Schwarz and Young inequalities and for $|\lambda| > 1$, we obtain

$$|\lambda| \|z\| \leq C_\delta \{\|A^{\frac{1}{2}}z\|^2 + \|A^\frac{\theta}{2}w\|^2\} + \|f_4\| \|z\|.$$  

From estimatives (23), (24) and (62) finish to proof.  

$$\blacksquare$$

Lemma 14 Let $\theta = 1$ and $\delta > 0$. The solutions of equations (19)-(22) satisfy the following equality:

$$|\lambda| \|w\|^2 \leq C_\delta \|F\|_\mathcal{H} \|U\|_\mathcal{H}.$$
**Proof:** Applying the product duality to equation (21) with $w$ and recalling that the operator $A$ is self-adjoint, we have

$$i\lambda \|w\|^2 + \alpha \langle Au, Aw \rangle + \gamma \langle Az, w \rangle = \langle f_3, w \rangle.$$  

(65)

Taking imaginary part, we obtain

$$\lambda \|w\|^2 = \text{Im}\{-\alpha \langle A^{1/2}u, A^{1/2}w \rangle - \gamma \langle A^{1/2}z, A^{1/2}w \rangle + \langle f_3, w \rangle\}.$$  

(66)

Finally using the Remark(11), Remark(12) and inequality (23) for $\theta = 1$ in (66). We finished the proof of this lemma.

Now. Applying the operator $A$ in the equation (19), we have

$$Aw = i\lambda Au - Af_1,$$

using this identity in (65) and taking imaginary part, we have

$$\lambda \alpha \|Au\|^2 = \lambda \|w\|^2 - \text{Im}\left[\alpha \langle Au, Af_1 \rangle - \gamma \langle A^{1/2}z, A^{1/2}w \rangle + \langle f_3, w \rangle\right].$$  

(67)

Therefore

$$|\lambda| \|Au\|^2 \leq |\lambda| \|z\|^2 + \text{Im}\left[\alpha \langle Au, Af_1 \rangle + \gamma \langle A^{1/2}z, A^{1/2}w \rangle + \langle f_3, w \rangle\right].$$  

(68)

Applying Cauchy-Schwarz and Young inequalities, estimative Lemma(9) and Lemma(13) in (68), we obtain

$$\alpha |\lambda| \|Au\|^2 \leq C_\delta \|F\|_\mathcal{H} \|U\|_\mathcal{H}.$$  

(69)

For $\theta = 1$, summing estimatives (24), (62), Lemma(13) and (69), we have

$$|\lambda| \|U\|_\mathcal{H} \leq C_\delta \|F\|_\mathcal{H} \|U\|_\mathcal{H} \iff \|\lambda(i\lambda I - B)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C_\delta.$$  

(70)

Finally for $\theta = 1$, the condition (18) of the Theorem(4) is verified.

**5** $S(t) = e^{Bt}$ is of Gevrey class $s > \frac{2+\theta}{\theta}$ when the parameter $\theta$ lies in the interval $(0, 1)$.

Before exposing our results, it is useful to recall the next definition and result presented in [28] (adapted from [25], Theorem 4, p. 153).

**Definition 15** Let $t_0 \geq 0$ be a real number. A strongly continuous semigroup $S(t)$, defined on a Banach space $\mathcal{H}$, is of Gevrey class $s > 1$ for $t > t_0$, if $S(t)$ is infinitely differentiable for $t > t_0$, and for every compact set $K \subset (t_0, \infty)$ and each $\mu > 0$, there exists a constant $C = C(\mu, K) > 0$ such that

$$\|S^{(n)}(t)\|_{\mathcal{L}(\mathcal{H})} \leq C\mu^n (n!)^s, \text{ for all } t \in K, n = 0, 1, 2...$$  

(71)
Theorem 16 (25) Let $S(t)$ be a strongly continuous and bounded semigroup on a Hilbert space $\mathcal{H}$. Suppose that the infinitesimal generator $B$ of the semigroup $S(t)$ satisfies the following estimate, for some $0 < \tau < 1$:

$$\lim_{|\lambda| \to \infty} \sup |\lambda|^\tau \|(i\lambda I - B)^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty. \quad (72)$$

Then $S(t)$ is of Gevrey class $s$ for $t > 0$, for every $\delta > \frac{1}{\tau}$.

Our main result in this section is as follows:

Theorem 17 Let $S(t) = e^{Bt}$ strongly continuos-semigroups of contractions on the Hilbert space $\mathcal{H}$, the semigroup $S(t)$ is of Gevrey class $s$ for every $s > \frac{1}{\tau}$ for $\tau \in (0, 1)$, as there exists a positive constant $C$ such that we have the resolvent estimative:

$$|\lambda|^\tau \|(i\lambda I - B)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C, \quad \lambda \in \mathbb{R}. \quad (73)$$

Proof:

On the other hand. Henceforth, we assume $\lambda \in \mathbb{R}$ with $|\lambda| > 1$, we shall borrow some ideas from [15]. Set $z = z_1 + z_2$, where $z_1 \in D(A^{\frac{1}{2}})$ and $z_2 \in D(A^0)$, with

$$i\lambda z_1 + A z_1 = f_4, \quad i\lambda z_2 = -\beta v - \delta A^{\theta} z + A z_1. \quad (74)$$

Firstly, applying the product duality the first equation in (74) by $z_1$, we have

$$i\lambda \|z_1\|^2 + \|A^{\frac{1}{2}} z_1\|^2 = \langle f_4, z_1 \rangle. \quad (75)$$

Taking first the imaginary part of (75) and in the sequence the real part and applying Cauchy-Schwarz inequality, we have

$$|\lambda| \|z_1\|^2 = |\text{Im}(f_4, z_1)| \leq C\|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}, \quad \|A^{\frac{1}{2}} z_1\|^2 \leq C\|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}. \quad (76)$$

Equivalently

$$\|z_1\| \leq C\left\{\frac{\|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2}{|\lambda|^{\frac{1}{2}}} \right\}^{\frac{1}{2}} \quad \|A^{\frac{1}{2}} z_1\| \leq C\left\{\|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2\right\}^{\frac{1}{2}}. \quad (77)$$

In follows from the second equation in (74) that

$$i\lambda A^{-1} z_2 = -\beta v - \delta A^{\theta-1} z + \gamma w + z_1$$

then

$$|\lambda| \|A^{-1} z_2\| \leq C\left\{\|v\| + \|w\| + \|A^{\theta-1} z\| + \|z_1\|\right\},$$

applying Cauchy-Schwarz and Young inequalities continuous embedding and using first inequality of (76) and estimatives (23), (24) and (35), we obtain

$$|\lambda| \|A^{-1} z_2\| \leq C\left\{\|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2\right\}^{\frac{1}{2}} \left[\frac{|\lambda|^{\frac{1}{2}} + 1}{|\lambda|^{\frac{1}{2}}}\right].$$
then, for $|\lambda|^\frac{1}{2} \geq 1$, we find

$$
\|A^{-1}z_2\| \leq C|\lambda|^{-\frac{1}{2}}\{\|F\|_H\|U\|_H + \|F\|_H^2\}^{\frac{1}{2}}.
$$

(77)

On the other hand, from $z_2 = z - z_1$, (23) and first inequality of (76), we have

$$
\|A^\frac{\theta}{2}z_2\| \leq \|A^\frac{\theta}{2}z_1\| + \|A^\frac{\theta}{2}z_1\|
\leq C\{\|F\|_H\|U\|_H + \|F\|_H^2\}^{\frac{\theta}{2}}.
$$

(78)

Now, by Lions’ interpolations inequality, we derive

$$
|\lambda|^{\frac{1}{2}}\|z_2\| \leq C|\lambda|^{\frac{1}{2}}\|A^{-1}z_2\|^{\frac{\theta}{2+\theta}}\|A^\frac{\theta}{2}z_2\|^{\frac{2}{2+\theta}}.
$$

(79)

From (77) and $|\lambda| \geq 1$, we have

$$
\|A^{-1}z_2\|^{\frac{\theta}{2+\theta}} \leq C|\lambda|^{-\frac{1}{2}}\{\|F\|_H\|U\|_H + \|F\|_H^2\}^{\frac{\theta}{2+\theta}},
$$

(80)

and from (78), we have

$$
\|A^\frac{\theta}{2}z_2\|^{\frac{2}{2+\theta}} \leq C\{\|F\|_H\|U\|_H + \|F\|_H^2\}^{\frac{1}{2+\theta}}.
$$

(81)

Then, using (80) and (81) in (79), for $|\lambda| > 1$, we derive

$$
|\lambda|^{\frac{1}{2}}\|z_2\| \leq C|\lambda|^{\frac{2}{2+\theta}}\{\|F\|_H\|U\|_H + \|F\|_H^2\}^{\frac{1}{2+\theta}}.
$$

(82)

Therefore, from first inequality of (76) and (82), we have

$$
|\lambda|\|z\|^2 \leq C|\lambda|^{\frac{2}{2+\theta}}\{\|F\|_H\|U\|_H + \|F\|_H^2\} \quad \text{for} \quad 0 < \theta < 1.
$$

(83)

On the other hand. Henceforth, we assume $\lambda \in \mathbb{R}$ with $|\lambda| > 1$, we shall borrow some ideas from [15]. Set $w = w_1 + w_2$, where $w_1 \in D(A^\frac{\theta}{2})$ and $w_2 \in D(A^0)$, with

$$
i\lambda w_1 + Aw_1 = f_3, \quad i\lambda w_2 = -\alpha A^2u - \gamma Az + Aw_1.
$$

(84)

Firstly, applying the product duality the first equation in (84) by $w_1$, we have

$$
i\lambda\|w_1\|^2 + \|A^\frac{1}{2}w_1\|^2 = \langle f_3, w_1 \rangle.
$$

(85)

Taking first the imaginary part of (85) and in the sequence the real part and applying Cauchy-Schwarz inequality, we have

$$
|\lambda|\|w_1\|^2 = |\text{Im}(f_3, w_1)| \leq C\|F\|_H\|U\|_H, \quad \|A^\frac{\theta}{2}w_1\|^2 \leq C\|F\|_H\|U\|_H.
$$

Equivalently

$$
\|w_1\| \leq C\{\|F\|_H\|U\|_H + \|F\|_H^2\}^{\frac{1}{2}} \quad \text{and} \quad \|A^\frac{1}{2}w_1\| \leq C\{\|F\|_H\|U\|_H\}^{\frac{1}{2}}.
$$

(86)

In follows from the second equation in (84) that

$$
i\lambda A^{-1}w_2 = -\alpha Au - \gamma z + w_1.
then, we find

\[ |\lambda| \|A^{-1}w_2\| \leq C\{\alpha\|Au\| + \|z\|\} + \|w_1\|, \]

applying Cauchy-Schwarz and Young inequalities and using first inequality of (86) and estimatives (23) and (35) for \(|\lambda|^{\frac{1}{2}} > 1\), we obtain

\[ |\lambda| \|A^{-1}w_2\| \leq C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\hat{H}}^2 \left(\frac{|\lambda|^{\frac{1}{2}} + 1}{|\lambda|^{\frac{1}{2}}}\right)^2 \]

\[ \|A^{-1}w_2\| \leq C|\lambda|^{-1}\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\hat{H}}^2\}^{\frac{1}{2}}. \]  
(87)

On the other hand, from \(w_2 = w - w_1\), we have

\[ \|A^\theta w_2\| \leq \|A^\theta w\| + \|A^\theta w_1\|. \]  
(88)

From estimatives Lemma(9) and second inequality (86), we have

\[ \|A^\theta w_2\| \leq C\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\hat{H}}^2\}^{\frac{1}{2}} \quad \text{for} \quad 0 \leq \theta \leq 1. \]  
(89)

Now, by Lions’ interpolations inequality, we derive

\[ |\lambda|^\frac{1}{2}\|w_2\| \leq C|\lambda|^\frac{1}{2}\|A^{-1}w_2\|^{\frac{\theta}{2}}\|A^\theta w_2\|^{\frac{1}{2}}. \]  
(90)

From (87), we have

\[ \|A^{-1}w_2\|^{\frac{2}{1+\theta}} \leq C|\lambda|^{-\frac{\theta}{1+\theta}}\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\hat{H}}^2\}^{\frac{\theta}{1+\theta}}. \]  
(91)

and from (89), we have

\[ \|A^\theta w_2\|^{\frac{2}{1+\theta}} \leq C\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\hat{H}}^2\}^{\frac{\theta}{1+\theta}}. \]  
(92)

Then, using (91) and (92) in (90), for \(|\lambda|^{\frac{1}{2}} > 1\), we derive

\[ |\lambda|\|w_2\|^2 \leq C|\lambda|^\frac{\theta}{2+\theta}\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\hat{H}}^2\}. \]  
(93)

Therefore, as \(|\lambda|\|w\|^2 \leq |\lambda|\{\|w_1\|^2 + \|w_2\|^2\}\), from first inequality of (86) and esti- mative (93), we have

\[ |\lambda|\|w\|^2 \leq C|\lambda|^\frac{2+\theta}{2+\theta}\{\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\hat{H}}^2\} \quad \text{for} \quad 0 \leq \theta \leq 1. \]  
(94)

Now we will estimate the term \(|\lambda|\|Au\|^2\). Making the duality product between equation (21) and \(\lambda u\) and using the equation (19), we have

\[ \alpha_1\lambda^2\|Au\|^2 = \lambda\langle w, i\lambda u \rangle - \gamma\langle \frac{\lambda}{|\lambda|^{\frac{1}{2}}}z, |\lambda|^\frac{1}{2}Au \rangle + \langle f_3, \lambda u \rangle \]

\[ = \lambda\|w\|^2 + \langle \alpha_1\lambda^2u + i\gamma Az - if_3, f_1 \rangle \]

\[ -\gamma\langle \frac{\lambda}{|\lambda|^{\frac{1}{2}}}z, |\lambda|^\frac{1}{2}Au \rangle + \langle f_3, -iw - if_1 \rangle. \]
Applying Cauchy-Schwarz and Young inequalities, for $\varepsilon > 0$, there exists a positive constant $K_\varepsilon$, independent of $\lambda$, such that:

$$|\lambda||Au|^2 \leq C|\lambda||w|^2 + C\left\{\langle Au, Af_1\rangle + |\langle z, Af_1\rangle| + |\langle f_3, f_1\rangle|\right\} + K_\varepsilon|\lambda||z|^2 + \varepsilon|\lambda||Au|^2 + C|\langle f_3, w\rangle|.$$  \hfill (95)

Now, taking the duality product between equation (22) and $\varepsilon > 0$, we have

$$|\lambda|\alpha_1||Au|^2 \leq C|\lambda|\bar{\theta}\left\{|F||U|| + ||F|^2\right\} \text{ for } 0 \leq \theta \leq 1. \hfill (96)$$

Finally, we'll get the estmative for $|\lambda||Av|^2$, taking the duality product between equation (21) and $w$ and using the equation (19), we have

$$i\lambda||w|^2 - i\lambda\alpha_1||Au|^2 = -\gamma\langle A\frac{1}{2}z, A\frac{1}{2}w\rangle + \alpha_1\langle Au, Af_1\rangle + \langle f_3, w\rangle. \hfill (97)$$

Now, taking the duality product between equation (22) and $z$ and using the equation (20), we have

$$i\lambda||z|^2 + \delta||A\frac{1}{2}z|^2 = i\lambda\alpha_2||A\frac{1}{2}v|^2 + \alpha_2\langle A\frac{1}{2}v, A\frac{1}{2}f_2\rangle + \gamma\langle A\frac{1}{2}w, A\frac{1}{2}z\rangle + \langle f_4, z\rangle. \hfill (98)$$

Subtracting the equations (97) and (98) and taking the imaginary part and noting that $\text{Im}\{\langle A\frac{1}{2}z, A\frac{1}{2}w\rangle + \langle A\frac{1}{2}w, A\frac{1}{2}z\rangle\} = 0$, we obtain

$$\gamma\lambda\alpha_2||A\frac{1}{2}v|^2 = \gamma\text{Im}\{\alpha_1\langle Au, Af_1\rangle + \langle f_3, w\rangle - \alpha_2\langle A\frac{1}{2}v, A\frac{1}{2}f_2\rangle - \langle f_4, z\rangle\} + \gamma\lambda\alpha_1||Au|^2 + \gamma\lambda||z|^2 - ||w|^2\} \hfill (99)$$

On the other hand, now applying Cauchy-Schwarz and Young inequalities in (99), using estimatives (83), (94) and (96), we find

$$|\lambda|\alpha_2||A\frac{1}{2}v|^2 \leq C|\lambda|\bar{\theta}\left\{|F||U|| + ||F|^2\right\} \text{ for } 0 \leq \theta \leq 1. \hfill (100)$$

Finally, adding the estimates (83), (94), (96) and (100), we find:

$$|\lambda||U|^2_{H} \leq C|\lambda|\bar{\theta}\left\{|F||U|| + ||F|^2\right\} \text{ for } 0 \leq \theta \leq 1. \hfill (101)$$

Then, for every $\varepsilon > 0$, there exists positive constant $K_\varepsilon$, independent of $\lambda$ such that:

$$|\lambda||U|^2_{H} \leq C|\lambda|\bar{\theta}\left\{|F||U|| + ||F|^2\right\} \iff \frac{|\lambda||\left((i\lambda - B)^{-1}F\right)||U|}{||F||_{H}} \leq C \hfill (102)$$

where $\tau = \frac{\theta}{2 + \theta} > 0$ for $0 < \theta < 1$. Therefore

$$|\lambda|^\tau\left|\left((i\lambda - B)^{-1}\right)||F||_{H}\leq C. \hfill (102)$$
So, applying $\limsup$ when $|\lambda| \to \infty$ in (102) of Theorem (16), $S(t)$ is of the class Gevrey $s$, for every $s > \frac{1}{\tau}$.

Finally, of the inequalities (102) and Theorem (16), the inequality (71) is verified and $S(t)$ is the Gevrey class $s > \frac{2+\theta}{\theta}$. Therefore, from the definition (15), the semigroups $S(t) = e^{Bt}$ is infinitely differentiable in $B$ for all $t > 0$ and $\theta \in (0, 1)$.

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