Tomographic map within the framework of star-product quantization

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Tomograms introduced for the description of quantum states in terms of probability distributions are shown to be related to a standard star-product quantization with appropriate kernels. Examples of symplectic tomograms and spin tomograms are presented.

1 Introduction

Star-product quantization provides us the formulation of quantum mechanics where quantum observables are presented by functions on the phase space instead of operators acting on some Hilbert space. The product of functions is not the point-wise product. This induced product is nonlocal and it is defined by some kernel. Recently the tomographic description of quantum states has been introduced to associate states with standard probability distributions instead of density operators. The tomographic description was introduced also for spin. Properties of tomographic maps and their relation with Heisenberg–Weyl and SU(2) groups were studied. The relation of tomographic representation to the star-product quantization was established. The aim of this contribution is to review this approach and to study properties of the star-product of symbols of quantum observables. We show that the tomographic map can be discussed within the framework of the standard star-product quantization procedure.

2 Star-product

In quantum mechanics, observables are described by operators acting on the Hilbert space of states. In order to consider observables as functions on a phase space, we review first a general construction and provide general relations and properties of a map from operators onto functions without a concrete realization of the map. Given a Hilbert space $H$ and an operator $A$ acting
on this space, let us suppose that we have a set of operators $\hat{U}(x)$ and a $n$-dimensional vector $x = (x_1, x_2, \ldots, x_n)$. We construct the $c$-number function $f_\hat{A}(x)$ (we call it the symbol of operator $\hat{A}$)

$$f_\hat{A}(x) = \text{Tr} \left[ \hat{A} \hat{U}(x) \right].$$

Let us suppose that the relation has an inverse, i.e., there exists a set of operators $\hat{D}(x)$ such that

$$\hat{A} = \int f_\hat{A}(x) \hat{D}(x) \, dx.$$

In fact, we could consider maps of the form

$$\hat{A} \rightarrow f_\hat{A}(x) \quad \text{and} \quad f_\hat{A}(x) \rightarrow \hat{A},$$

and require them to be one the inverse of the other.

The most important property is the existence of associative product (star-product) of the symbols induced by the product of operators. We introduce the product (star-product) of two functions $f_\hat{A}(x)$ and $f_\hat{B}(x)$ corresponding to two operators $\hat{A}$ and $\hat{B}$ by the relations

$$f_\hat{A} \hat{B}(x) = f_\hat{A}(x) * f_\hat{B}(x) := \text{Tr} \left[ \hat{A} \hat{B} \hat{U}(x) \right].$$

Since the standard product of operators on a Hilbert space is an associative product, i.e. $\hat{A}(\hat{B}\hat{C}) = (\hat{A}\hat{B})\hat{C}$, it is obvious that the formula defines an associative product for the functions $f_\hat{A}(x)$, namely,

$$f_\hat{A}(x) * \left( f_\hat{B}(x) * f_\hat{C}(x) \right) = \left( f_\hat{A}(x) * f_\hat{B}(x) \right) * f_\hat{C}(x).$$

The commutator of two operators

$$\hat{C} = [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

is mapped onto the quantum Poisson bracket of two symbols $f_\hat{A}(x)$ and $f_\hat{B}(x)$

$$f_\hat{C}(x) = \left\{ f_\hat{A}(x), f_\hat{B}(x) \right\}_* = \text{Tr} \left[ [\hat{A}, \hat{B}] \hat{U}(x) \right].$$

Since the Jacobi identity is fulfilled for the commutator of the operators, the Jacobi identity is also fulfilled for the Poisson bracket of the functions $f_\hat{A}(x)$.
and $f_B(x)$. Since for operators one has the derivation property with respect to the associative product, the Poisson brackets reproduce this property
\[
\{ f_A(x), f_B(x) * f_C(x) \}_* = \{ f_A(x), f_B(x) \}_* * f_C(x) + f_B(x) * \{ f_A(x), f_C(x) \}_* 
\]
this property qualifies it as a quantum Poisson bracket.

The evolution in the space of observables can be described by the Heisenberg equations of motion
\[
\dot{\hat{A}} = i[\hat{H}, \hat{A}] \quad (\hbar = 1),
\]
where $\hat{H}$ is the Hamiltonian of the system and $\hat{A}$ is a generic observable. This equation can be rewritten in terms of the functions $f_A(x)$ and $f_H(x)$, where
\[
f_H(x) = \text{Tr} \left[ \hat{H} \hat{U}(x) \right]
\]
corresponds to the Hamiltonian, in the form
\[
f_A(x, t) = i \{ f_H(x, t), f_A(x, t) \}_*
\]
via the quantum Poisson bracket we have defined.

Let us suppose that there exist other maps, we choose two different ones. One map is described by a vector $x = (x_1, x_2, \ldots, x_n)$ and operators $\hat{U}(x)$ and $\hat{D}(x)$. Another map is described by a vector $y = (y_1, y_2, \ldots, y_m)$ and operators $\hat{U}_1(y)$ and $\hat{D}_1(y)$. For given operator $\hat{A}$, one has the function
\[
\phi_A(y) = \text{Tr} \left[ \hat{A} \hat{U}_1(y) \right]
\]
and the inverse relation
\[
\hat{A} = \int \phi_A(y) \hat{D}_1(y) \, dy.
\]
One can obtain a relation of the function $f_A(x)$ to the function $\phi_A(y)$ in the form
\[
\phi_A(y) = \int f_A(x) \text{Tr} \left[ \hat{D}(x) \hat{U}_1(y) \right] \, dx
\]
and the inverse relation
\[
f_A(x) = \int \phi_A(y) \text{Tr} \left[ \hat{D}_1(y) \hat{U}(x) \right] \, dy.
\]
We see that functions $f_A(x)$ and $\phi_A(y)$ corresponding to different maps are connected by means of the invertible integral transform. These transforms are determined by means of intertwining kernels

$$K_1(x,y) = \text{Tr} \left[ \hat{D}(x)\hat{U}_1(y) \right] \quad \text{and} \quad K_2(x,y) = \text{Tr} \left[ \hat{D}_1(y)\hat{U}(x) \right].$$

One can write down a composition rule for two symbols $f_A(x)$ and $f_B(x)$, which determines the star-product of these symbols. The composition rule is described by the formula

$$f_A(x) \ast f_B(x) = \int f_A(x'') f_B(x') K(x'', x', x) \, dx' \, dx''.$$

The kernel under the integral is determined by the trace of product of the basic operators, which we use to construct the map

$$K(x'', x', x) = \text{Tr} \left[ \hat{D}(x'')\hat{D}(x')\hat{U}(x) \right].$$

### 3 Symplectic tomograms

According to the general scheme one can introduce for the operator $\hat{A}$ the function $f_A(x)$, where $x = (x_1, x_2, x_3) \equiv (X, \mu, \nu)$, which we denote here as $w_A(X, \mu, \nu)$ depending on the position $X$ and the parameters $\mu$ and $\nu$ of the reference frame

$$w_A(X, \mu, \nu) = \text{Tr} \left[ \hat{A}\hat{U}(x) \right].$$

We call the function $w_A(X, \mu, \nu)$ the tomographic symbol of the operator $\hat{A}$. The operator $\hat{U}(x)$ is given by

$$\hat{U}(x) \equiv \hat{U}(X, \mu, \nu) = \delta (X - \mu\hat{q} - \nu\hat{p}),$$

where $\hat{q}$ and $\hat{p}$ are position and momentum operators.

In the case under consideration, the inverse transform will be of the form

$$\hat{A} = \int w_A(X, \mu, \nu)\hat{D}(X, \mu, \nu) \, dX \, d\mu \, d\nu,$$

where

$$\hat{D}(x) \equiv \hat{D}(X, \mu, \nu) = \frac{1}{2\pi} \exp (iX - i\nu\hat{p} - i\mu\hat{q}).$$
If one takes two operators $\hat{A}_1$ and $\hat{A}_2$, which are expressed through the corresponding functions by the formulae

$$\hat{A}_1 = \int w_{\hat{A}_1}(X', \mu', \nu') \hat{D}(X', \mu', \nu') dX' d\mu' d\nu',$$

$$\hat{A}_2 = \int w_{\hat{A}_2}(X'', \mu'', \nu'') \hat{D}(X'', \mu'', \nu'') dX'' d\mu'' d\nu'',$$

and $\hat{A}$ denotes the product of $\hat{A}_1$ and $\hat{A}_2$, then the function $w_{\hat{A}}(X, \mu, \nu)$, which corresponds to $\hat{A}$, is the star-product of functions $w_{\hat{A}_1}(X, \mu, \nu)$ and $w_{\hat{A}_2}(X, \mu, \nu)$, i.e.,

$$w_{\hat{A}}(X, \mu, \nu) = w_{\hat{A}_1}(X, \mu, \nu) \ast w_{\hat{A}_2}(X, \mu, \nu)$$

reads

$$w_{\hat{A}}(X, \mu, \nu) = \int w_{\hat{A}_1}(X', \mu', \nu') K(x', x') dx' dx',$$

with the kernel given by

$$K(x'', x', x) = \text{Tr} \left[ \hat{D}(X'', \mu'', \nu'') \hat{D}(X', \mu', \nu') \hat{U}(X, \mu, \nu) \right].$$

The explicit form of the kernel reads

$$K(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2, X, \mu, \nu) = \frac{\delta(\nu_1 + \nu_2 - \nu(\mu_1 + \mu_2))}{4\pi^2}$$

$$\times \exp \left[ \frac{i}{2} (\nu_1 \mu_2 - \nu_2 \mu_1) + 2X_1 + 2X_2 - 2 \frac{(\nu_1 + \nu_2)}{\nu} X \right].$$

By using the oscillator’s ground state

$$w_0 = \left[ \pi (\mu^2 + \nu^2) \right]^{-1/2} \exp \left( -\frac{X^2}{\mu^2 + \nu^2} \right),$$

one can check that this kernel satisfies the obvious relation

$$w_0 \ast w_0 = w_0.$$
Following we will derive the expression for an arbitrary operator acting on spin states in terms of measurable mean values of the operator for the spin projection on a given direction, considered in a rotated reference frame. For arbitrary values of spin, let the operator \( \hat{A}^{(j)} \) be represented by the matrix

\[
A_{mm'}^{(j)} = \langle jm | \hat{A}^{(j)} | jm' \rangle , \quad m = -j, -j + 1, \ldots, j - 1, j ,
\]

where

\[
\hat{\mathbf{j}}_3 | jm \rangle = m | jm \rangle ; \quad \hat{\mathbf{j}}^2 | jm \rangle = j(j + 1) | jm \rangle ,
\]

and

\[
\hat{A}^{(j)} = \sum_{m=-j}^{j} \sum_{m'=-j}^{j} A_{mm'}^{(j)} | jm \rangle \langle jm' | ; \quad A_{mm'}^{(j)} = w_0(m) .
\]

Let us consider the diagonal elements of the operator in another reference frame

\[
A_{m_1 m_1}^{(j)} = (UAU^\dagger)_{m_1 m_1} .
\]

The unitary rotation transform \( U \) depends on the Euler angles \( \alpha, \beta, \gamma \).

Below we introduce new notation for the diagonal matrix elements of the operator and rewrite the above equality in the form

\[
\tilde{w}(m_1, \alpha, \beta) = \sum_{m'_1=-j}^{j} \sum_{m''_1=-j}^{j} D_{m_1 m'_1}^{(j)}(\alpha, \beta, \gamma) A_{m'_1 m''_1}^{(j)}(\alpha, \beta, \gamma) .
\]

The defined function is called the tomographic symbol of the operator. In view of the structure of the above formula, the tomogram depends only on two Euler angles. Here the matrix elements \( D_{m_1 m'_1}^{(j)}(\alpha, \beta, \gamma) \) are the matrix elements of the rotation transform \( U \)

\[
D_{m'm}^{(j)}(\alpha, \beta, \gamma) = e^{im'\gamma} d_{m'm}^{(j)}(\beta) e^{im\alpha} ,
\]

where

\[
d_{m'm}^{(j)}(\beta) = \left[ \frac{(j + m')!(j - m)!}{(j + m)!(j - m)!} \right]^{1/2} \left( \cos \frac{\beta}{2} \right)^{m' + m} \left( \sin \frac{\beta}{2} \right)^{m' - m} \times P_n^{(m' - m, m' + m)}(\cos \beta)
\]

and \( P_n^{(a,b)}(x) \) is the Jacobi polynomial.

The inverse relation reads

\[
\sum_{j_3=0}^{2j} \sum_{m_3 = -j_3}^{j_3} (2j_3 + 1)^2 \sum_{m_1 = -j}^{j} (-1)^{m_1} \tilde{w}(m_1, \alpha, \beta) D_{0 m_3}^{(j_3)}(\alpha, \beta, \gamma)
\]
\( \otimes \begin{pmatrix} j & j & j_3 \\ m_1 & -m_1 & 0 \end{pmatrix} \otimes \begin{pmatrix} j & j & j_3 \\ m'_1 & -m'_2 & m_3 \end{pmatrix} \frac{d\omega}{8\pi^2} = (-1)^{m'_2} A^{(j)}_{m'_1 m'_2}, \)

where \( d\omega \) is the volume element in terms of Euler angles. One can obtain an invariant form of this relation. To do this, let us introduce the function on the sphere

\[ \Phi^{(j_3)}_{jm'_1 m'_2}(\alpha, \beta) = (-1)^{m'_2} \sum_{m_3 = -j_3}^{j_3} D^{(j_3)}_{0 m_3}(\alpha, \beta, \gamma) \begin{pmatrix} j & j & j_3 \\ m_1 & -m_2 & m_3 \end{pmatrix} \]

and the operator on the sphere

\[ \tilde{A}^{(j)}(\alpha, \beta) = (2j_3 + 1)^2 \sum_{m'_1 = -j}^{j} \sum_{m'_2 = -j}^{j} |jm'_1\rangle \Phi^{(j_3)}_{jm'_1 m'_2}(\alpha, \beta) \langle jm'_2|. \]

Then using another operator \( x = (m_1, \alpha, \beta) \)

\[ \tilde{B}^{(j)}(x) \equiv \tilde{B}^{(j)}_{m_1}(\alpha, \beta) = (-1)^{m_1} \sum_{j_3 = 0}^{2j} \begin{pmatrix} j & j & j_3 \\ m_1 & -m_1 & 0 \end{pmatrix} \tilde{A}^{(j)}_{j_3}(\alpha, \beta), \]

one can write for the operator under study the invariant expression:

\[ \tilde{A}^{(j)} = \sum_{m_1 = -j}^{j} \int \frac{d\omega}{8\pi^2} \tilde{w}(m_1, \alpha, \beta) \tilde{B}^{(j)}_{m_1}(\alpha, \beta). \]

Let us consider the tomographic star-product of two functions \( W(m_1, \alpha, \beta) \) and \( W'(m_1, \alpha, \beta) \). One has, by definition,

\[ (W \ast W')(m_1, \alpha, \beta) = \sum_{x', x''} W(x', \alpha', \beta') W'(x'', \alpha'', \beta'') K(x, x', x''), \]

where the kernel has the form

\[ K(x, x', x'') = \frac{1}{(8\pi^2)^2} \text{Tr} \left[ \langle jm_1 | jm_1 | U \tilde{B}^{(j)}(x') \tilde{B}^{(j)}(x'') U^\dagger \rangle \right]. \]

The kernel can be given in explicit form. The tomographic star-product defined by means of the above kernel defines an associative product. So we have an invertible map between the operator \( \tilde{A}^{(j)} \) and its symbol \( \tilde{w}(x) \):

\[ \tilde{A}^{(j)} = \sum_x \tilde{w}(x) \tilde{D}(x), \quad \tilde{D}(x) = \frac{1}{8\pi^2} \tilde{B}^{(j)}_{m_1}(x), \]

\[ \tilde{w}(x) = \text{Tr} \left[ \tilde{A}^{(j)} \tilde{U}(x) \right], \quad \tilde{U}(x) = U^\dagger \langle jm_1 | jm_1 | U \rangle, \]

where sum over \( x \) means the integral over Euler angles and sum over angular momentum projection.
5 Conclusions

To conclude, we have embedded the tomographic map into the general scheme of symbols and operators. When the kernel of star-product is known, Heisenberg equations of motion for observables written in the form of equations for their symbols can be presented, in some cases, in the form of partial differential equations. For example, in the tomographic representation, the evolution equation for density matrix takes the form of Fokker–Planck-type equation. One can also obtain such differential form of the evolution equation for spin tomograms.

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