Sehgal Type Contractions on Dislocated Spaces

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Abstract: In this paper, we investigate the contractive type inequalities for the iteration of the mapping at a given point in the setting of dislocated metric space. We consider an example to illustrate the validity of the given result. Further, as an application, we propose a solution for a boundary value problem of the second order differential equation.

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1. Introduction and Preliminaries

In 1968, Bryant relaxed the assumption of Banach contraction mapping principle by using an iteration of the mapping.

Theorem 1 ([1]). Let T be a self mapping on the complete metric space \((M,d)\), and m a positive integer. Suppose that there exists q ∈ [0, 1) such that T satisfies the inequality

\[ d(T^m v, T^m w) \leq q d(v, w), \]  

for all v, w ∈ M, where T^m denotes the mth iterate of T. Then, there exists exactly one fixed point of T.

After then a number of authors deepen the research by considering an iteration of the mapping, see e.g., [2–7]. We recollect some significant results in this direction. One of the pioneer report in this way was given Seghal [4].

Theorem 2 ([4]). Let \((M,d)\) be a complete metric space, T a continuous self-mapping of M which satisfies the condition that there exists a real number q, 0 < q < 1 such that, for each v ∈ M there exists a positive integer m(v) such that, for each w ∈ M,

\[ d(T^{m(v)} v, T^{m(v)} w) \leq q d(v, w). \]

Then T has a unique fixed point in M.

Guseman [2] extended this result by removing the condition of continuity of T and later, other extensions for a single mapping were discussed in several papers, see e.g., Iseki [8], Matkowski [3], Singh [5] and the reference therein. One of the most interesting results for mappings which satisfy a general contractive conditions were announced by Singh.
Theorem 3 ([5]). Let \( (\mathcal{M}, d) \) be a complete metric space and \( T : \mathcal{M} \to \mathcal{M} \) be a mapping such that for all \( v, w \in \mathcal{M} \) we can find a positive integer \( m(v) \) such that

\[
\begin{align*}
d(T^{m(v)}v, T^{m(v)}w) & \leq q(v, w)d(v, w) + r(v, w)d(v, T^{m(v)}v) + s(v, w)(w, T^{m(v)}w) \\
& \quad + l(v, w)d(w, T^{m(v)}v) + p(v, w)d(v, T^{m(v)}w),
\end{align*}
\]

(3)

where \( q(v, w), r(v, w), s(v, w), l(v, w), p(v, w) \) are nonnegative functions such that

\[
\sup \{2l(v, w) + q(v, w) + r(v, w) + s(v, w) + p(v, w)\} = \lambda < 1.
\]

Then \( T \) has a unique fixed point \( v^* \).

In this paper, we consider more general contractive condition in the setting of dislocated metric space. For sake of completeness, we shall recollect some basic notions and fundamental results.

Definition 1. For a nonempty set \( \mathcal{M} \) a dislocated metric is a function \( \mathcal{D} : \mathcal{M} \times \mathcal{M} \to [0, \infty) \) such that for all \( v, w, u \in \mathcal{M} \):

1. \( \mathcal{D}(v, w) = 0 \Rightarrow v = w, \)
2. \( \mathcal{D}(v, w) = \mathcal{D}(w, v), \)
3. \( \mathcal{D}(v, w) \leq \mathcal{D}(v, u) + \mathcal{D}(u, w). \)

The space \( (\mathcal{M}, \mathcal{D}) \) is said to be a dislocated metric space (DMS).

Example 1. Let \( \mathcal{M} = \mathbb{R}^+_0 \) and \( \mathcal{D} : \mathcal{M} \times \mathcal{M} \to [0, \infty) \) defined by \( \mathcal{D}(v, w) = \max \{v, w\} \). The pair \( (\mathcal{M}, \mathcal{D}) \) forms a dislocated metric space.

It is obvious that any metric space is a dislocated metric space, but conversely this is not true.

Definition 2. Let \( (\mathcal{M}, \mathcal{D}) \) be a DMS. A sequence \( \{v_n\} \) in \( \mathcal{M} \) is called:

a) convergent to a point \( v \in \mathcal{M} \) if the following limit exists and is finite

\[
\lim_{n \to \infty} \mathcal{D}(v_n, v) = \mathcal{D}(v, v); \quad (4)
\]

b) Cauchy if the following limit exists and is finite

\[
\lim_{n \to \infty} \mathcal{D}(v_n, v_m)
\]

Moreover, if \( \lim_{n \to \infty} \mathcal{D}(v_n, v_m) = 0 \), then is said that \( \{v_n\} \) is a 0-Cauchy sequence.

Definition 3. The DMS \( (\mathcal{M}, \mathcal{D}) \) is complete if for each Cauchy sequence \( \{v_n\} \) in \( \mathcal{M} \), there is some \( v \in \mathcal{M} \) such that

\[
l = \lim_{n \to \infty} \mathcal{D}(v_n, v) = \mathcal{D}(v, v) = \lim_{n,m \to \infty} \mathcal{D}(v_n, v_m). \quad (5)
\]

Particularly, if each 0-Cauchy sequence \( \{v_n\} \) converges to a point \( v \in \mathcal{M} \) the pair \( (\mathcal{M}, \mathcal{D}) \) is said to be 0-complete-DMS.

Definition 4. Let \( (\mathcal{M}, \mathcal{D}) \) be a DMS. A mapping \( T : \mathcal{M} \to \mathcal{M} \) is continuous if for any sequence \( \{v_n\} \) in \( \mathcal{M} \) converging to \( v \in \mathcal{M} \), we have \( \{Tv_n\} \) converges to \( Tv \).
Proposition 1 ([9]). Let \((M, D)\) be a DMS. For any \(v, w \in M\) we have the following

(i) If \(D(v, w) = 0\) then \(D(v, v) = D(w, w) = 0\).
(ii) If \(v \neq w\) then \(D(v, w) > 0\).
(iii) If \(\{v_n\}\) is a sequence in \(M\) such that \(\lim_{n \to \infty} D(v_n, v_{n+1}) = 0\) then
\[
\lim_{n \to \infty} D(v_n, v_n) = \lim_{n \to \infty} D(v_{n+1}, v_{n+1}) = 0.
\]

Definition 5. By a comparison function we mean a function \(\varphi : [0, \infty) \to [0, \infty)\) with the following properties:

\(cf_1\) \(\varphi\) is increasing;
\(cf_2\) \(\lim_{n \to \infty} \varphi^n(x) = 0\), for \(x \in [0, \infty)\).

We denote by \(\Phi\) the class of the comparison function \(\varphi : [0, \infty) \to [0, \infty)\).

Next we list some basic properties of the comparison functions.

Proposition 2 ([10,11]). If \(\varphi\) is a comparison function then:

\(cf_1\) each \(\varphi^k\) is a comparison function, for all \(k \in \mathbb{N}\);
\(cf_{ii}\) \(\varphi\) is continuous at 0;
\(cf_{iii}\) \(\varphi(x) < x\) for all \(x > 0\).

Definition 6 ([10]). A function \(\varphi_c : [0, \infty) \to [0, \infty)\) is called a \(c\)-comparison function if:

\(ccf_1\) \(\varphi_c\) is monotone increasing;
\(ccf_2\) \(\sum_{n=0}^{\infty} \varphi_c^n(x) < \infty\), for all \(x \in (0, \infty)\).

We denote by \(\Phi_c\) the family of \(c\)-comparison functions.

It can be shown that every \(c\)-comparison function is a comparison function.

Throughout this paper we denote by \(\Psi\) the collection of all \(c\)-comparison functions \(\varphi : [0, \infty) \to [0, \infty)\) that satisfy the following condition
\(ccf_3\) \(\lim_{x \to \infty} (x - \varphi(x)) = \infty\).

In the following we recall the concept of \(\alpha\)-admissible mappings. A function \(T : M \to M\) is said to be \(\alpha\)-admissible if
\(A\) \(\alpha(v, w) \geq 1 \Rightarrow \alpha(Tv, Tw) \geq 1\),
for all \(v, w \in M\) where \(\alpha : M \times M \to [0, \infty)\) is a given function. An \(\alpha\)-admissible map \(T : M \to M\) which satisfies the condition
\(TA\) \(\alpha(v, w) \geq 1\) and \(\alpha(w, u) \geq 1\) implies that \(\alpha(v, u) \geq 1\), \(v, w, u \in M\)
is said to be triangular \(\alpha\)-admissible.

Later, the notion of \(\alpha\)-admissible mapping and triangular \(\alpha\)-admissible mappings are refined by Popescu [12], as follows:

Definition 7 ([12]). Let \(T : M \to M\) and \(\alpha : M \times M \to [0, \infty)\). We say that \(T\) is an \(\alpha\)-orbital admissible mapping if for all \(v \in M\) we have
\(O\) \(\alpha(v, Tv) \geq 1 \Rightarrow \alpha(Tv, T^2v) \geq 1\).

Every \(\alpha\)-admissible mapping is an \(\alpha\)-orbital admissible mapping, for more details on admissible mapping, see e.g., [13–24].
**Definition 8** ([12]). Let \( \alpha : \mathcal{M} \times \mathcal{M} \to [0, \infty) \). An \( \alpha \)-orbital admissible function \( T : \mathcal{M} \to \mathcal{M} \) is said to be triangular \( \alpha \)-orbital admissible if it satisfies

\[
(TO) \quad \alpha(v, w) \geq 1 \text{ and } \alpha(w, Tw) \geq 1 \text{ implies that } \alpha(v, Tw) \geq 1, \text{ for all } v, w \in \mathcal{M}.
\]

At the end of this section, we present two further concepts that will be essential in our next considerations.

A set \( \mathcal{M} \) is regular with respect to mapping \( \alpha : \mathcal{M} \times \mathcal{M} \to [0, \infty) \) if the following condition is satisfied:

\[
(R) \quad \text{for any sequence } \{v_n\} \text{ in } \mathcal{M} \text{ such that } \alpha(v_n, v_{n+1}) \geq 1 \text{ for all } n \text{ and } v_n \to v \in \mathcal{M} \text{ as } n \to \infty \text{ we have } \alpha(v, v_n) \geq 1, \text{ for all } n.
\]

A map \( \alpha : \mathcal{M} \times \mathcal{M} \to [0, \infty) \) is said to satisfy the condition (U) if

\[
(U) \quad \text{for any fixed point } v \text{ of } T^{m(v)} \text{ we have } \alpha(v, w) \geq 1 \text{ for any } w \in \mathcal{M}, \text{ where } m(v) \text{ is a positive integer.}
\]

**2. Main Results**

We are now prepared to establish the main result of this paper.

**Theorem 4.** Let \( (\mathcal{M}, D) \) be a complete DMS, a function \( T : \mathcal{M} \to \mathcal{M}, \psi \in \Psi \) and \( \alpha : \mathcal{M} \times \mathcal{M} \to [0, \infty) \). Suppose that for all \( v \in \mathcal{M} \) we can find a positive integer \( m(v) \) such that for any \( w \in \mathcal{M} \)

\[
\alpha(v, w)D(T^{m(v)}v, T^{m(v)}w) \leq \psi \left( \max \left\{ D(v, w), \frac{D(v, T^{m(v)}v)}{2}, \frac{D(w, T^{m(v)}w)}{2}, \frac{D(v, T^{m(v)}v) + D(v, T^{m(v)}w)}{3} \right\} \right). \quad (6)
\]

Suppose also that:

(i) \( T \) is triangular \( \alpha \)-orbital admissible;
(ii) there exists \( v_0 \) in \( \mathcal{M} \) such that \( \alpha(v_0, T v_0) \geq 1 \);
(iii) either \( T \) is continuous, or
(iv) the \( \mathcal{M} \) space is regular and \( \alpha \) satisfies the condition (U).

Then the function \( T \) has exactly one fixed point.

**Proof.** Consider the initial value \( v_0 \in \mathcal{M} \) and define a sequence \( \{v_n\} \) as follows:

\[
v_1 = T^{m(v_0)}v_0, \quad v_2 = T^{m(v_1)}v_1, \ldots, \quad v_{k+1} = T^{m(v_k)}v_k, \ldots \quad (7)
\]

If we denote \( m_k = m(v_k) \) for any \( k \in \mathbb{N} \), then we can write \( v_{k+1} = T^{m_k}v_k \).

Now, \( T \) is \( \alpha \)-orbital admissible and \( \alpha(v_0, T v_0) \geq 1 \). Thus, from condition (O), we have \( \alpha(T v_0, T^2 v_0) \geq 1 \) and so forth

\[
\alpha(T^{n-1}v_0, T^n v_0) \geq 1, \text{ for all } n \in \mathbb{N}. \quad (8)
\]

Taking into account (TO) and (8) we easily infer that

\[
\alpha(v_k, T v_k) \geq 1 \text{ and } \alpha(T v_k, T^2 v_k) \geq 1 \text{ imply that } \alpha(v_k, T^2 v_k) \geq 1.
\]

Recursively, we can conclude that

\[
\alpha(v_k, T^m v_k) \geq 1, \quad (9)
\]

for all \( m \in \{1, 2, \ldots\} \).
In the initial inequality (6) letting \( v = v_{k-1} \), \( w = T^{m_k}v_{k-1} \) and using (9) we can find a positive integer, \( m_{k-1} \) such that

\[
D(v_k, v_{k+1}) = D(v_k, T^{m_k}v_k) = D(T^{m_{k-1}}v_{k-1}, T^{m_k}(T^{m_{k-1}}v_{k-1}))
\]

\[
= D(T^{m_{k-1}}v_{k-1}, T^{m_{k-1}}(T^{m_k}v_{k-1}))
\]

\[
\leq a(v_{k-1}, T^{m_k}v_{k-1})D(T^{m_{k-1}}v_{k-1}, T^{m_{k-1}}(T^{m_k}v_{k-1}))
\]

\[
\leq \psi \left( \max \left\{ \frac{D(v_{k-1}, T^{m_k}v_{k-1}), D(v_{k-1}, T^{m_k}v_{k-1})}{2}, \frac{D(T^{m_k}v_{k-1}, T^{m_k}v_{k-1})}{2} \right\} \right). \tag{10}
\]

Since \( \psi \in \Psi \), the condition (cf1) is satisfied and applying (D3) we obtain

\[
D(v_k, v_{k+1}) = D(v_k, T^{m_k}v_k) < \max \left\{ \frac{D(v_{k-1}, T^{m_k}v_{k-1}), D(v_{k-1}, T^{m_k}v_{k-1})}{2}, \frac{D(T^{m_k}v_{k-1}, T^{m_k}v_{k-1})}{2} \right\}.
\tag{11}
\]

Let \( p_1 \in \{ m_{k-1}, m_k, m_k + m_{k-1} \} \) such that

\[
\max \left\{ D(v_{k-1}, T^{m_k}v_{k-1}), D(v_{k-1}, T^{m_k}v_{k-1}), D(v_{k-1}, T^{m_k+1}v_{k-1}) \right\} = D(v_{k-1}, T^{p_1}v_{k-1}).
\]

Then from (11) together with (cfii) we get that

\[
D(v_k, T^{m_k}v_k) \leq \psi (D(v_{k-1}, T^{p_1}v_{k-1})) < D(v_{k-1}, T^{p_1}v_{k-1}). \tag{12}
\]

Using the same arguments, we can find a positive integer \( m_{k-2} \) such that

\[
D(v_{k-1}, T^{p_1}v_{k-1}) = D(T^{m_{k-2}}v_{k-2}, T^{p_1}(T^{m_{k-2}}v_{k-2}))
\]

\[
\leq a(v_{k-2}, T^{p_1}v_{k-2})D(T^{m_{k-2}}v_{k-2}, T^{m_{k-2}}(T^{p_1}v_{k-2}))
\]

\[
\leq \psi \left( \max \left\{ \frac{D(v_{k-2}, T^{p_1}v_{k-2}), D(v_{k-2}, T^{m_{k-2}}v_{k-2})}{2}, \frac{D(T^{m_{k-2}}v_{k-2}, T^{m_{k-2}}(T^{p_1}v_{k-2}))}{2} \right\} \right). \tag{13}
\]

where \( p_2 \in \{ m_{k-2}, p_1, m_{k-2} + p_1 \} \) is chosen such that

\[
D(v_{k-2}, T^{p_2}v_{k-2}) = \max \left\{ D(v_{k-2}, T^{p_1}v_{k-2}), D(v_{k-2}, T^{m_{k-2}}v_{k-2}), D(v_{k-2}, T^{m_{k-2}+p_1}v_{k-2}) \right\}.
\]

Very easily we can see from (12), (13) and taking into account (cf1), that

\[
D(v_k, T^{m_k}v_k) \leq \psi (D(v_{k-1}, T^{p_1}v_{k-1})) < \psi^2 (D(v_{k-2}, T^{p_2}v_{k-2})). \tag{14}
\]

Since \( \psi \) is monotone increasing, by continuing this process, we find that

\[
D(v_k, v_{k+1}) = D(v_k, T^{m_k}v_k) \leq \psi^k (D(v_0, T^{p_k}v_0)), \tag{15}
\]
for $p_1, p_2, ..., p_k \in \mathbb{N}$. On one hand the inequality (15) shows us, taking into account (cf2) from Definition 5 that

$$D(v_k, v_{k+1}) = D(v_k, T^{m_0}v_k) \to 0. \quad (16)$$

On the other hand using triangle inequality, for $i \in \mathbb{N}$, we have

$$D(v_k, v_{k+i}) \leq D(v_k, v_{k+1}) + D(v_{k+1}, v_{k+2}) + \ldots + D(v_{k+i-1}, v_{k+i})$$

$$\leq \psi^i(D(v_0, T^{s_0}v_0)) = \psi^i(D(v_0, T^{s_0}v_0)) + \ldots + \psi^i(D(v_0, T^{s_0}v_0)) + \ldots + \psi^i(D(v_0, T^{s_0}v_0)). \quad (17)$$

We should focus our attention on the set \( \{D(v_0, T^iv_0), i \in \mathbb{N}\} \). More precisely, we will show that this set is bounded. In order to prove that, we mention first that by hypothesis there exists a positive integer $m_0 = m(v_0)$ such that (6) holds. Let $x_i = D(v_0, T^{m_0+i}v_0)$, where $s \in \{1, 2, ..., m_0\}$ and $l = \max \{x_0, D(v_0, T^{m_0}v_0)\}$. According to (ccf3), we can find $a > l > 0$ such that

$$t - \psi(t) > l \text{ for any } t \in [a, \infty). \quad (18)$$

It is clear then that $x_0 \leq l < a$ and we will show that $x_i < a$ for all $i \in \mathbb{N}$. We suppose the contrary, that there exists $k \in \mathbb{N}$ such that $x_k < a \leq x_{k+1}$. Note that (according to (6), (9) and triangle inequality)

$$x_k = D(v_0, T^{m_0+s}v_0) \leq D(v_0, T^{m_0}v_0) + D(T^{m_0}v_0, T^{m_0+s}v_0)$$

$$\leq D(v_0, T^{m_0}v_0) + D(T^{m_0}v_0, T^{m_0}(T^{k-1}m_0+s)0)$$

$$\leq l + \psi \left( \max \left\{ \frac{D(v_0, T^{k-1}m_0+s)0, D(T^{k-1}m_0+s)0, 3D(v_0, T^{m_0}v_0)}{2}, \frac{D(v_0, T^{m_0}v_0)}{2} \right\} \right). \quad (19)$$

But,

$$\max \left\{ \frac{D(v_0, T^{k-1}m_0+s)0, D(T^{k-1}m_0+s)0, 3D(v_0, T^{m_0}v_0)}{2}, \frac{D(v_0, T^{m_0}v_0)}{2} \right\} \leq \max \left\{ D(v_0, T^{k-1}m_0+s)0, D(T^{k-1}m_0+s)0, 3D(v_0, T^{m_0}v_0) \right\}$$

$$= \max \left\{ x_{k-1}, \frac{x_k - x_{k-1}}{2}, \frac{x_k - x_{k-1} + D(v_0, T^{m_0}v_0)}{2} \right\} \leq \max \left\{ x_{k-1}, \frac{x_k - x_{k-1}}{2}, \frac{x_k - x_{k-1} + x_0}{2} \right\}.$$

Since $\psi$ is increasing, from (19) we get

$$x_k \leq l + \psi(x_k) \quad (20)$$

which contradicts (18). This contradiction shows that our assumption was false. Thus, for all $i \in \mathbb{N}$

$$x_i = D(v_0, T^{im_0}v_0) < a.$$

We have thus demonstrated that the set \( \{D(v_0, T^{im_0}v_0) : i \in \mathbb{N}\} \) is bounded, and also, varying $s \in \{0, 1, 2, ..., m_0\}$, the set \( \{D(v_0, T^iv_0), i \in \mathbb{N}\} \) is bounded. Hence,

$$r(v_0) = \sup_i \left\{D(v_0, T^iv_0)\right\} < \infty. \quad (21)$$
With this observation, we return to (17) and we get

$$D(v_k, v_{k+1}) \leq D(v_k, v_{k+1}) + D(v_{k+1}, v_{k+2}) + \ldots + D(v_{k+l-1}, v_{k+l})$$

$$\leq \psi^k(r(v_0)) + \psi^{k+1}(r(v_0)) + \ldots + \psi^{k+l-1}(r(v_0))$$

$$= \sum_{j=k}^{k+l-1} \psi^j(r(v_0)).$$

(22)

The series $\sum_{j=0}^{\infty} \psi^j(r(v_0))$ is convergent due to $(cc_2)$ and its sequence of partial sums, denoted by $\{S_n\}$, is convergent at $S$. Then

$$D(v_k, v_{k+1}) \leq S_{k+i-1} - S_k \to 0$$

(23)

as $k \to \infty$, and, therefore $\{v_k\}$ is a 0-Cauchy sequence. By completeness of $(M, D)$, there is some point $v^* \in M$ such that

$$\lim_{n \to \infty} D(v_n, v^*) = 0 = \lim_{n,m \to \infty} D(v_n, v_m).$$

(24)

From the continuity of $T$ it easily follows that

$$\lim_{n \to \infty} D(T^n v_k, T^n v^*) = \lim_{n \to \infty} D(T^n v_k, T^n v^*) = \lim_{n,m \to \infty} D(T^n v_k, T^n v_m) = \lim_{n,m \to \infty} D(v_n, v_{m+1}) = 0$$

and by the uniqueness of the limit, we get $T^n v^* = v^*$.

We claim now that $v^*$ is a fixed point of $T$ under the hypothesis (iv). The first step in our proof is to show that $T^m(v^*) v^* = v^*$ which means that $v^*$ is a fixed point of $T^m(v^*)$. Firstly we claim that

$$\lim_{k \to \infty} D(T^m(v^*) v_k, v_k) = 0.$$ 

Taking $v = T^m(v^*) v_k$ and $w = v_k$ in (6), there exists $m_k$ such that for any $k \geq 1$ we have

$$D(T^m(v^*) v_k, v_k) = D(T^m(v^*) (T^{m_k-1} v_k), T^{m_k-1} v_k)$$

$$\leq a(T^m(v^*) v_k, v_k) D(T^{m_k-1} (T^m(v^*) v_k), T^{m_k-1} v_k)$$

$$\leq \psi \left( \max \left\{ \frac{D(T^m(v^*) v_k, v_k)}{D(T^m(v^*) v_k, T^{m_k-1} v_k)}, \frac{D(T^m(v^*) v_k, T^{m_k-1} v_k)}{D(T^m(v^*), T^{m_k-1} v_k)} \right\} \right)$$

(25)

Let $q_1 \in \{m(v^*), m_k - 1, m(v^*) + m_k - 1\}$ such that

$$\max \left\{ D(T^m(v^*) v_k, v_k), D(T^{m_k-1} v_k, v_k), D(T^{m(v^*) + m_k - 1} v_k, v_k) \right\} = D(T^{q_1} v_k, v_k).$$

Using triangle inequality, we have

$$\frac{D(T^{m(v^*) + m_k - 1} v_k, v_k)}{2} \leq \frac{D(T^{m(v^*)} v_k, v_k) + D(T^{m_k - 1 + w(v^*)} v_k, v_k)}{2}$$

$$\leq \frac{D(T^{q_1} v_k, v_k)}{2}$$

and

$$\frac{D(T^{m(v^*)} v_k, v_k) + D(T^{m_k - 1 + w(v^*)} v_k, v_k)}{3} \leq \frac{D(T^{m(v^*)} v_k, v_k) + D(T^{m_k - 1} v_k, v_k) + D(v_k, T^{m_k - 1 + w(v^*)} v_k)}{3}$$

$$\leq \frac{D(T^{q_1} v_k, v_k)}{3}.$$
Repeating this process and keeping in mind the properties \((c\ell_1), (c\ell_2)\) we find that

\[
D(T^{m(v^*)}v_k, v_k) \leq \psi^k(D(T^{m(v_0)}, v_0)) \to 0. \tag{27}
\]

Suppose now that \(T^{m(v^*)}v^* \neq v^*\). Then \(D(T^{m(v^*)}v^*, v^*) > 0\). Since \(v_k \to v^*\) as \(k \to \infty\) and the space \(M\) is regular, by triangle inequality we have

\[
0 < D(T^{m(v^*)}v^*, v^*) \leq D(T^{m(v^*)}v^*, T^{m(v^*)}v_k) + D(T^{m(v^*)}v_k, v^*) + D(v_k, v^*)
\]

\[
\leq \alpha(v^*, v_k)D(T^{m(v^*)}v^*, T^{m(v^*)}v_k) + D(T^{m(v^*)}v_k, v^*) + D(v_k, v^*)
\]

\[
\leq \psi \left( \max \left\{ D(v^*, v_k), \frac{D(v^*, T^{m(v^*)}v^*)}{2}, \frac{D(v_k, T^{m(v^*)}v_k)}{2}, \frac{D(v_k, T^{m(v^*)}v^*) + D(v^*, T^{m(v^*)}v^*)}{2} \right\} \right)
\]

\[
+ D(T^{m(v^*)}v_k, v_k) + D(v_k, v^*)
\]

\[
< \max \left\{ D(v^*, v_k), D(v^*, T^{m(v^*)}v^*), \frac{D(v_k, T^{m(v^*)}v_k)}{2}, \frac{D(v_k, T^{m(v^*)}v^*) + D(v^*, T^{m(v^*)}v^*)}{2} \right\}
\]

\[
+ D(T^{m(v^*)}v_k, v_k) + D(v_k, v^*).
\]

Letting \(k \to \infty\) in the above inequality, and taking (24) respectively (27) into account, we find that

\[
0 < D(T^{m(v^*)}v^*, v^*) < D(v^*, T^{m(v^*)}v^*)
\tag{28}
\]

which implies that \(D(v^*, T^{m(v^*)}v^*) = 0\). Hence, \(T^{m(v^*)}v^* = v^*\).

Let \(w^* \in M\) another point such that \(T^{m(v^*)}w^* = w^*\) and \(v^* \neq w^*\). Since \(T\) satisfies (6) and the function \(\alpha\) satisfies the condition (U) we get

\[
0 < D(v^*, w^*) = D(T^{m(v^*)}v^*, T^{m(v^*)}w^*) \leq \alpha(v^*, w^*)D(T^{m(v^*)}v^*, T^{m(v^*)}w^*)
\]

\[
\leq \psi \left( \max \left\{ D(v^*, w^*), \frac{D(v^*, T^{m(v^*)}v^*)}{2}, \frac{D(v^*, T^{m(v^*)}w^*)}{2}, \frac{D(w^*, T^{m(v^*)}v^*) + D(v^*, T^{m(v^*)}w^*)}{2} \right\} \right).
\]

Since

\[
\frac{D(w^*, T^{m(v^*)}w^*)}{2} \leq \frac{D(w^*, v^*) + D(v^*, T^{m(v^*)}w^*)}{2} = D(v^*, w^*)
\]

we obtain

\[
0 < \psi(D(v^*, w^*)) < D(v^*, w^*).
\]

But the above inequality is possible only if \(D(v^*, w^*) = 0\), that is \(v^* = w^*\). This is a contradiction.

From the uniqueness of the fixed point we can conclude that \(v^*\) is a fixed point for \(T\). Indeed,

\[
Tv^* = T(T^{m(v^*)}v^*) = T^{m(v^*)}(Tv^*)
\tag{29}
\]

shows that \(Tv^*\) is also fixed point of \(T^{m(v^*)}\). But, \(T^{m(v^*)}\) has a unique fixed point \(v^*\). Hence, \(Tv^* = v^*\). \(\Box\)

**Remark 1.** Denoting by \(S(x) = \sum_{n=0}^{\infty} \psi^n(x)\), we have from (22)

\[
D(v_k, v_{k+1}) \leq \sum_{j=k}^{k+l-1} \psi^j(r(v_0)) = \sum_{j=0}^{l-1} \psi^j(r(v_0)) < \sum_{j=0}^{l-1} \psi^j(r(v_0)).
\tag{30}
\]
Letting \( l \to \infty \) in the above inequality we obtain

\[
\mathcal{D}(v_k, v^*) < S(\psi^h(r(v_0))).
\]

**Example 2.** Let \( \mathcal{M} = \{a, b, c, d\} \) and \( \mathcal{D} : \mathcal{M} \times \mathcal{M} \to [0, \infty) \) with \( \mathcal{D}(v, w) = \mathcal{D}(w, v) \) for any \( v, w \in \mathcal{M} \), defined as follows

\[
\begin{align*}
\mathcal{D}(a, a) &= \mathcal{D}(b, b) = \mathcal{D}(c, c) = 0, \quad \mathcal{D}(d, d) = 1, \\
\mathcal{D}(a, b) &= \mathcal{D}(a, c) = \mathcal{D}(b, c) = 1, \\
\mathcal{D}(a, d) &= \mathcal{D}(b, d) = \mathcal{D}(c, d) = 2.
\end{align*}
\]

(It is easy to see that the pair \((\mathcal{M}, \mathcal{D})\) is a DMS but not a metric space.) Let \( T : \mathcal{M} \to \mathcal{M} \) be defined by

\[
Ta = Tc = a, \quad Tb = c, \quadTd = b.
\]

For \( x = b \) and \( y = c \) we have

\[
\mathcal{D}(Tb, Tc) = \mathcal{D}(c, a) = 1 \geq q\mathcal{D}(b, c) = 1
\]

for any \( q \in (0, 1) \). Thus, \( T \) does not satisfy the Banach contraction condition. We show that the function \( T \) satisfies all the assumptions of Theorem 4 with \( \psi(x) = \frac{2x}{3} \) for any \( x \geq 0 \) and \( a : \mathcal{M} \times \mathcal{M} \to [0, \infty) \), \( a(v, w) = 1 \). Since \( a(v, w) = 1 \) for all \( v, w \in \mathcal{M} \) the assumptions (i), (ii) and (iv) are fulfilled. We discuss the following possible cases:

1. If \( v, w \in \{a, b, c\} \) and \( v = w \) then for \( m(v) = 1 \) we have \( \mathcal{D}(Tv, Tv) = 0 \) and inequality (6) holds.
2. If \( v = w = d \) then \( \mathcal{D}(T^2d, T^2d) = \mathcal{D}(c, c) = 0 \) and also (6) holds.
3. If \( v = b, w = c \) then \( T^2b = T^2c = a \). Then, for \( m(b) = 2 \) we have \( \mathcal{D}(T^2b, T^2c)\mathcal{D}(a, a) = 0 \). Thus, the condition (6) is satisfied.
4. If \( v = b, w = d \) then \( T^2b = a \) and \( T^2d = c \). Then, for \( m(b) = 2 \) we have \( \mathcal{D}(T^2b, T^2d) = \mathcal{D}(a, c) = 1 \) and \( \mathcal{D}(b, d) = 2 \). Thus,

\[
a(b,d)\mathcal{D}(T^2b,T^2d) = \mathcal{D}(a,c) = 1 \leq \frac{2 \cdot 2}{3} = \psi(\mathcal{D}(b,d)).
\]

5. If \( v = c, w = d \) then \( T^2c = a \) and \( T^2d = c \). Then, for \( m(c) = 2 \) we have \( \mathcal{D}(T^2c, T^2d) = \mathcal{D}(a, c) = 1 \) and \( \mathcal{D}(c, d) = 2 \). Thus,

\[
a(c,d)\mathcal{D}(T^2b,T^2d) = \mathcal{D}(a,c) = 1 \leq \frac{2 \cdot 2}{3} = \psi(\mathcal{D}(c,d)).
\]

On the other hand, we can note that:

\[
T^3a = T^3b = T^3c = T^3d = a.
\]

For this reason, there exists \( m(a) = 3 \) such that for any \( w \in \mathcal{M} \) the condition (6) is satisfied (since \( \mathcal{D}(a, a) = 0 \)).

The conclusion is that \( T \) satisfies all the assumptions of Theorem 4. Therefore \( T \) has exactly one fixed point, \( v = a \).

Taking, in Theorem 4, \( a(v, v) = 1 \) we get the following result:

**Corollary 1.** Let \((\mathcal{M}, \mathcal{D})\) be a complete DMS, a function \( T : \mathcal{M} \to \mathcal{M} \) and \( \psi \in \Psi \). Suppose that for all \( v, w \in \mathcal{M} \) we can find a positive integer \( m(v) \) such that

\[
\begin{align*}
\mathcal{D}(T^{m(v)}v, T^{m(v)}w) &\leq \psi \left( \max \left\{ \frac{\mathcal{D}(v, w)}{2}, \frac{\mathcal{D}(v, T^{m(v)}v) + \mathcal{D}(w, T^{m(v)}w)}{2}, \frac{\mathcal{D}(w, T^{m(v)}v) + \mathcal{D}(v, T^{m(v)}w)}{3} \right\} \right).
\end{align*}
\]
Corollary 2. Let \((\mathcal{M}, D)\) be a DMS, a function \(T : \mathcal{M} \to \mathcal{M}\) and \(\psi \in \Psi\). Suppose that for all \(v, w \in \mathcal{M}\) we can find a positive integer \(m(v)\) such that
\[
D(T^{m(v)} v, T^{m(v)} w) \leq \psi(D(v, w)). \tag{32}
\]
Then the function \(T\) has exactly one fixed point.

If we take \(\psi(x) = qx, q \in [0, 1)\) in Corollary 1 respectively in Corollary 2 we find the following consequences:

Corollary 3. Let \((\mathcal{M}, D)\) be a complete DMS, a function \(T : \mathcal{M} \to \mathcal{M}\). Suppose that for all \(v, w \in \mathcal{M}\) we can find a positive integer \(m(v)\) such that
\[
D(T^{m(v)} v, T^{m(v)} w) \leq q \left[ \max \left\{ D(v, w), \frac{D(v, T^{m(v)} v)}{2}, \frac{D(w, T^{m(v)} w)}{2}, \frac{D(w, T^{m(v)} v) + D(v, T^{m(v)} w)}{3} \right\} \right]. \tag{33}
\]
Then the function \(T\) has exactly one fixed point.

Corollary 4. Let \((\mathcal{M}, D)\) be a complete DMS, a function \(T : \mathcal{M} \to \mathcal{M}\). Suppose that for all \(v, w \in \mathcal{M}\) we can find a positive integer \(m(v)\) such that
\[
D(T^{m(v)} v, T^{m(v)} w) \leq q D(v, w). \tag{34}
\]
Then the function \(T\) has exactly one fixed point.

Corollary 5. Let \((\mathcal{M}, D)\) be a complete DMS, a function \(T : \mathcal{M} \to \mathcal{M}\). Suppose that for all \(v, w \in \mathcal{M}\) we can find a positive integer \(m(v)\) such that
\[
D(T^{m(v)} v, T^{m(v)} w) \leq a_1 D(v, w) + a_2 \left[ D(v, T^{m(v)} v) + D(w, T^{m(v)} w) \right] + a_3 \left[ D(w, T^{m(v)} v) + D(v, T^{m(v)} w) \right], \tag{35}
\]
where \(a_1, a_2, a_3 \in \mathbb{R}_0\) and \(a_1 + 4a_2 + 3a_3 < 1\). Then the function \(T\) has exactly one fixed point.

Proof. Since
\[
D(T^{m(v)} v, T^{m(v)} w) \leq a_1 D(v, w) + 2a_2 \left[ \frac{D(v, T^{m(v)} v)}{2} + \frac{D(w, T^{m(v)} w)}{2} \right] + 3a_3 \left[ \frac{D(w, T^{m(v)} v) + D(v, T^{m(v)} w)}{3} \right]
\leq a_1 D(v, w) + 4a_2 \max \left\{ \frac{D(v, T^{m(v)} v)}{2}, \frac{D(w, T^{m(v)} w)}{2} \right\} + 3a_3 \frac{D(w, T^{m(v)} v) + D(v, T^{m(v)} w)}{3}
\leq (a_1 + 4a_2 + 3a_3) \cdot \max \left\{ D(v, w), \frac{D(v, T^{m(v)} v)}{2}, \frac{D(w, T^{m(v)} w)}{2}, \frac{D(w, T^{m(v)} v) + D(v, T^{m(v)} w)}{3} \right\}.
\]

Letting \(\psi(x) = (a_1 + 4a_2 + 3a_3)x\) the result follows from Theorem 4.

If we take \(m(v) = 1\) in Theorem 4 we get:

Corollary 6. Let \((\mathcal{M}, D)\) be a complete DMS, a function \(T : \mathcal{M} \to \mathcal{M}\) and \(\psi \in \Psi\). Suppose that for all \(v, w \in \mathcal{M}\)
\[
a(v, w) D(Tv, Tw) \leq \psi \left( \max \left\{ D(v, w), \frac{D(v, Tv)}{2}, \frac{D(w, Tw)}{2}, \frac{D(w, Tv) + D(v, Tw)}{3} \right\} \right). \tag{36}
\]
Suppose also that:
We must discuss the next cases:

2. If \( v \) is triangular \( T \) is \( \alpha \)-orbital admissible; 

Then the function \( T \) has exactly one fixed point.

**Example 3.** Let a dislocated metric space \((M, \mathcal{D})\), where \( X = [0,1] \) and \( \mathcal{D}(v, w) = \max \{v, w\} \) for any \( v, w \in M \). Let a self mapping \( T \) on \( M \) be defined as follows:

\[
T(v) = \begin{cases} 
\frac{v}{3} & \text{for } v \in [0, \frac{1}{2}] \cup \{1\} \\
1 & \text{for } v = \frac{1}{2} \\
\frac{v^2}{4} + \frac{1}{2} & \text{for } v \in (\frac{1}{2}, 1)
\end{cases}
\]

Let the functions \( \psi(x) = \frac{x}{27}, x \geq 0 \) and \( \alpha : M \times M \rightarrow [0, \infty) \),

\[
\alpha(v, w) = \begin{cases} 
2 & \text{for } (v, w) \in A \times A \\
1 & \text{for } (v, w) = \left(\frac{1}{4}, \frac{1}{2}\right) \\
v + 1 & \text{for } (v, w) \in \left\{\left(v, \frac{1}{2}\right), \left(\frac{1}{2}, v\right) : v \in A\right\} \\
0 & \text{otherwise}
\end{cases}
\]

Let us first notice that \( v_0 = T^n v = \frac{v}{27} \rightarrow 0 \) for any \( v \in A \) and \( T^n 1 = \frac{1}{27^n} \rightarrow 0 \). Since \( \alpha(v_0, 0) = 2 \) we get that assumption (iv) of Theorem 4 is satisfied. Also, since \( \alpha(0, 0) = 2 \geq 1 \) by simple calculation we can conclude that the assumptions (ii) and (iv) are satisfied. We remark that if \( v = \frac{1}{3} \) and \( w = \frac{1}{2} \) then \( T_\frac{1}{3} = \frac{1}{2}, T_\frac{1}{2} = 1 \). Hence,

\[
\mathcal{D}\left(\frac{T_\frac{1}{3}, T_\frac{1}{2}}{2}\right) = \max \left\{\frac{1}{9}, 1\right\} = 1 > q \frac{1}{2} = \max \left\{\frac{1}{3}, \frac{1}{2}\right\} = \mathcal{D}\left(\frac{1}{3}, \frac{1}{2}\right)
\]

and

\[
\alpha\left(\frac{1}{3}, \frac{1}{2}\right) \mathcal{D}\left(\frac{T_\frac{1}{3}, T_\frac{1}{2}}{2}\right) = \left(\frac{1}{3} + 1\right) \max \left\{\frac{1}{9}, 1\right\} = \frac{4}{3} \geq \frac{2}{3} \cdot \frac{1}{2} = \psi\left(\max\left\{\mathcal{D}\left(\frac{1}{3}, \frac{1}{2}\right), \frac{\mathcal{D}\left(\frac{1}{3}, \frac{1}{2}\right)}{2}, \frac{\mathcal{D}\left(\frac{1}{3}, \frac{1}{2}\right)}{3}\right\}\right),
\]

which shows us that \( T \) does not satisfy the contraction condition of Banach, neither condition (36) of Corollary 6.

We must discuss the next cases:

1. If \( v, w \in A \) then for \( m(v) = 3 \) we have \( T^m v = \frac{v}{27} \) and \( T^m w = \frac{w}{27} \). Thus,

\[
\alpha(v, w) \mathcal{D}\left(T^m(v), T^m(w)\right) = 2 \cdot \max\left\{\frac{v, w}{27}\right\} \leq 2 \cdot \frac{3}{3} \cdot \max\{v, w\} = \psi(\mathcal{D}(v, w)) \leq \psi(M(v, w)),
\]

where

\[
M(v, w) = \max\left\{\mathcal{D}(v, w), \frac{\mathcal{D}(v, T^m(v))}{2}, \frac{\mathcal{D}(w, T^m(w))}{2}, \frac{\mathcal{D}(w, T^m(v)) + \mathcal{D}(v, T^m(w))}{3}\right\}
\]

2. If \( v = w = \frac{1}{2} \) we can choose \( m(v) = 4 \). Then, \( T^\frac{1}{2} = \frac{1}{2} \) and

\[
\alpha(\frac{1}{2}, \frac{1}{2}) \mathcal{D}\left(T^\frac{1}{2}, T^\frac{1}{2}\right) = 1 \cdot \frac{1}{27} \leq \frac{2}{3} \cdot \frac{1}{2} = \psi(\mathcal{D}(\frac{1}{2}, \frac{1}{2})) \leq \psi(M(\frac{1}{2}, \frac{1}{2})).
\]
3. If \( v = \frac{1}{2} \) and \( w \in [0, \frac{1}{2}) \) then for \( m(v) = 4 \) we have \( T^4 \frac{1}{2} = \frac{1}{27} \) and \( T^4w = \frac{w}{3^4} \)

\[
a \left( \frac{1}{2}, w \right)D \left( T^4 \frac{1}{2}, T^4w \right) = (w + 1) \cdot \max \left\{ \frac{1}{27}, \frac{w}{3^4} \right\} = \frac{w + 1}{27} \leq \frac{2}{3} \cdot \max \left\{ \frac{1}{2}, w \right\}
\]

The other cases are not interesting since \( a(v, w) = 0 \). Therefore \( v = 0 \) is the unique fixed point for \( T \).

Inspired by Proposition 3 from [7] we will establish a new fixed point result for a \( T \) function on a DMS, not necessarily complete.

**Corollary 7.** Let \((\mathcal{M}, D)\) be a DMS and a function \( T: \mathcal{M} \to \mathcal{M} \). Suppose that for a given \( v \in \mathcal{M} \) such that \( D(v, Tv) > 0 \) we can find a positive integer \( m(v) \) such that the following two conditions hold:

\[
D(v, T^{m(v)}v) < D(v, Tv),
\]

and

\[
D(T^{m(v)}v, T^{m(v)+1}v) \leq a_1 D(v, Tv) + a_2 \left[ D(v, T^{m(v)}v) + D(Tv, T^{m(v)+1}v) \right] + a_3 \left[ D(Tv, T^{m(v)}v) + D(v, T^{m(v)+1}v) \right]
\]

for any \( a_1, a_2, a_3 \in \mathbb{R}_0 \) and \( a_1 + 4a_2 + 4a_3 < 1 \).

Suppose also that there exists a point \( v^* \in \mathcal{M} \) such that \( \sigma(v^*) = \inf \{ \sigma(v) : v \in \mathcal{M} \} \), where \( \sigma(v) = D(v, Tv) \). Then \( v^* \) is fixed point of \( T \).

**Proof.** Suppose that \( D(v^*, T^{m(v^*)}v^*) > 0 \). Then we can find a positive integer \( m(v^*) \) such that

\[
D(v^*, T^{m(v^*)}v^*) < D(v^*, Tv^*).
\]

Replacing \( v = v^* \) in (38), using triangle inequality and keeping in mind (39) we have

\[
D(T^{m(v^*)}v^*, T^{m(v^*)+1}v^*) \leq a_1 D(v^*, Tv^*) + a_2 \left[ D(v^*, T^{m(v^*)}v^*) + D(Tv^*, T^{m(v^*)}v^*) + D(T^{m(v^*)}v^*, T^{m(v^*)+1}v^*) \right] + a_3 \left[ D(Tv^*, v^*) + D(v^*, T^{m(v^*)}v^*) + D(v^*, T^{m(v^*)}v^*) + D(T^{m(v^*)}v^*, T^{m(v^*)+1}v^*) \right] < (a_1 + 3a_2 + 3a_3) D(v^*, Tv^*) + (a_2 + a_3) D(T^{m(v^*)}v^*, T^{m(v^*)+1}v^*)
\]

or, since \( a_1 + 4a_2 + 4a_3 < 1 \) we get

\[
\sigma(T^{m(v^*)}v^*) = D(T^{m(v^*)}v^*, T^{m(v^*)+1}v^*) < \frac{a_1 + 3a_2 + 3a_3}{1 - a_2 - a_3} D(v^*, Tv^*) < D(v^*, Tv^*) = \sigma(v^*).
\]

This is a contradiction. Hence \( D(v^*, Tv^*) = 0 \) and \( v^* \) is fixed point of \( T \).

3. **Ulam-Stability**

**Definition 9.** Let \((\mathcal{M}, D)\) be a DMS and a function \( T: \mathcal{M} \to \mathcal{M} \). We say that the fixed point equation

\[
v = Tv, \ v \in \mathcal{M}
\]
is generalized Ulam-stable if for each \( \varepsilon > 0 \) and \( w \in \mathcal{M} \) there exists \( m(w) \in \{1, 2, \ldots\} \) such that for any \( w^* \in \mathcal{M} \) satisfying the inequality

\[
q(T^m(w^*)w^*) \leq \varepsilon
\]  

(42)

there exists an increasing function \( \eta : [0, \infty) \to [0, \infty) \) continuous at 0, with \( \eta(0) = 0 \) and \( v^* \in \mathcal{M} \) a solution of Equation (41) such that

\[
D(v^*, w^*) \leq \eta(\varepsilon).
\]  

(43)

Remark 2. If \( \eta(x) = ax \) for all \( x \geq 0 \), where \( a > 0 \), the fixed point Equation (41) is said to be Ulam-stable.

Theorem 5. Let the function \( \eta : [0, \infty) \to [0, \infty) \), defined by \( \eta(x) := x - \psi(x) \), with \( \psi \in \Psi \). Under the hypothesis of Corollary 2 the fixed point Equation (41) is generalized Ulam-stable.

Proof. There exists exactly one point \( v^* \in \mathcal{M} \) such that \( Tv^* = v^* \), which means that \( v^* \) is a unique solution of fixed point Equation (41). Let \( w^* \in \mathcal{M} \). There exists \( m(w^*) \in \{1, 2, \ldots\} \) such that (42) holds. Keeping in mind the properties of function \( \psi \), the condition imposed on the alpha function and using the triangle inequality we obtain

\[
D(v^*, w^*) \leq D(v^*, T^m(w^*)w^*) + D(T^m(w^*)w^*, w^*)
= D(T^m(w^*)v^*, T^m(w^*)w^*) + D(T^m(w^*)w^*, w^*)
\leq a(v^*, w^*)D(T^m(w^*)v^*, T^m(w^*)w^*) + D(T^m(w^*)w^*, w^*)
\leq \psi(D(v^*, w^*)) + \varepsilon.
\]  

(44)

Taking into account the definition of the function \( \beta \) we have

\[
D(v^*, w^*) - \psi(D(v^*, w^*)) = \eta(D(v^*, w^*)) \leq \varepsilon
\]

which is equivalent with

\[
D(v^*, w^*) \leq \eta^{-1}(D(v^*, w^*))
\]

From the assumption, \( \eta \) is continuous and strictly increasing. Thus, \( \eta^{-1} \) is also continuous and increasing, with \( \eta^{-1}(0) = 0 \) Therefore, the Equation (41) is generalized Ulam-stable.

\( \square \)

4. Application to Boundary Value Problem

Here we consider the following two point boundary value problems for the second order differential equation.

\[
\begin{aligned}
\frac{d^2 \vartheta}{dt^2} &= \ell(t, \vartheta(t)); \quad t \in [0, 1] \\
\vartheta(0) &= \vartheta(1) = 0
\end{aligned}
\]  

(45)

where \( \ell : [0, 1] \times \mathbb{R} \to \mathbb{R} \) is a continuous function. Recall that the Green’s function associated to (45) is

\[
\Lambda(t, s) = \begin{cases} 
\ell(1-s), & \text{if } 0 \leq t \leq s \leq 1 \\
\ell(t-1), & \text{if } 0 \leq s \leq t \leq 1
\end{cases}
\]  

(46)

Let \( \mathcal{M} = C([0, 1]) \) be the space of all continuous functions defined on \( I = [0, 1] \). We consider on \( \mathcal{M} \), the dislocated metric \( D \) given by \( D(\vartheta, \omega) = ||\vartheta - \omega|| + ||\vartheta|| + ||\omega|| \) for all \( \vartheta, \omega \in \mathcal{M} \), where \( ||\vartheta|| = \max_{t \in I} |\vartheta(t)| \) for each \( \vartheta \in \mathcal{M} \).

Clearly, \( (\mathcal{M}, D) \) is a complete DMS.
It is well known that $\vartheta \in C^2(I)$ is a solution of (45) is equivalent to that $\vartheta \in M = C(I)$ is a solution of the integral equation.

$$\vartheta(t) = \int_0^1 \Lambda(t,s)\ell(s,\vartheta(s))ds, \ \forall t \in I$$ (47)

**Theorem 6.** Let $(M,D)$ be a complete DMS as defined above. Further, we will assume the following conditions hold:

1. there exists a continuous function $\varrho : I \rightarrow \mathbb{R}^+$ such that
   $$|\ell(s,x_1) - \ell(s,x_2)| \leq 8\varrho(s)|x_1 - x_2|$$
   for each $s \in I$ and $x_1, x_2 \in \mathbb{R}$;
2. there exists a continuous function $\Upsilon : I \rightarrow \mathbb{R}^+_0$ such that
   $$|\ell(s,x)| \leq 8\Upsilon(s)|x|$$
   for each $s \in I$ and $|x| \in \mathbb{R}$;
3. $\sup_{s \in I} \varrho(s) = z_1 < \frac{1}{3}$;
4. $\sup_{s \in I} \Upsilon(s) = z_2 < \frac{1}{3}$.

The problem (45) has a solution $\vartheta \in M$.

**Proof.** Define the mapping $T : C(I) \rightarrow C(I)$ by

$$T^m(\vartheta)(t) = \int_0^1 \Lambda(t,s)\ell(s,\vartheta(s))ds$$

for all $\vartheta \in M, s, t \in I$ and $m(\vartheta)$ be a positive integer. Then the Equation (47) is equivalent to finding $\vartheta \in M$ that is a fixed point of $T$.

Now let $\vartheta, \omega \in M = (C[0,1])$. We have,

$$|T^m(\vartheta)(t) - T^m(\omega)(t)| = |\int_0^1 \Lambda(t,s)\ell(s,\vartheta(s))ds - \int_0^1 \Lambda(t,s)\ell(s,\omega(s))ds|$$

$$\leq \int_0^1 \Lambda(t,s)|\ell(s,\vartheta(s)) - \ell(s,\omega(s))|ds$$

$$\leq 8 \int_0^1 \Lambda(t,s)|\varrho(s)||\vartheta(s) - \omega(s)||ds$$

$$\leq 8z_1||\vartheta - \omega|| \sup_{t \in I} \int_0^1 \Lambda(t,s)ds$$ (48)

for each $t \in I$. On the other hand,

$$\int_0^1 \Lambda(t,s)ds = \frac{t}{2} - \frac{t^2}{2}$$ and so $\sup_{t \in I} \int_0^1 \Lambda(t,s)ds = \frac{1}{8}$.

From (48), we get,

$$|T^m(\vartheta)(t) - T^m(\omega)(t)| \leq z_1||\vartheta - \omega||$$ (49)
\[ |T^{m(\theta)} \theta(t)| = | \int_0^1 \Lambda(t,s) \ell(s, \theta(s)) ds | \]
\[ \leq \int_0^1 \Lambda(t,s) |\ell(s, \theta(s))| ds \]
\[ \leq 8 \int_0^1 \Lambda(t,s) Y(s) |\theta(s)| ds \]
\[ \leq 8z_2 |\theta| \sup_{t \in I} \int_0^1 \Lambda(t,s) ds \]
\[ = z_2 |\theta|. \]

Thus \(|T^{m(\theta)} \theta(t)| \leq z_2 |\theta|\). Similarly, we derive that
\[ ||T^{m(\theta)} \omega(t)|| \leq z_2 ||\omega||. \] (51)

Take \(z = z_1 + 2z_2\). Thus \(z < 1\). By using (49)–(51), we get,
\[ D(T^{m(\theta)} \theta(t) - T^{m(\theta)} \omega(t)) = ||T^{m(\theta)} \theta(t) - T^{m(\theta)} \omega(t)|| + ||T^{m(\theta)} \theta(t)|| + ||T^{m(\theta)} \omega(t)|| \]
\[ \leq z_1 ||\theta - \omega|| + z_2 ||\theta|| + z_2 ||\omega|| \]
\[ \leq (z_1 + 2z_2)(||\theta - \omega|| + ||\theta|| + ||\omega||) \]
\[ = zD(\theta, \omega). \] (52)

Hence \(D(T^{m(\theta)} \theta(t), T^{m(\theta)} \omega(t)) \leq zD(\theta(t), \omega(t))\). Thus all the conditions of Corollary 4 are satisfied. Hence \(T\) has exactly one fixed point \(\theta \in M\), i.e., the problem (45) has a solution \(\theta \in C^2(I)\). □

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