On idempotents of commutative rings

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Abstract

In the present work, a formula is provided for determining the idempotent elements of a
commutative ring $R$ from those of the quotient ring $R/N$, where $N$ is in most cases a nilpotent
ideal of $R$. As an application of this formula, idempotent elements of certain commutative
rings are described. Several examples are included illustrating the main results.

Keywords Idempotent element, nilpotent ideal, commutative ring, chain ring, group algebra.

1 Introduction

Idempotent elements of an algebra are a topic of considerable research with a variety of applica-
tions including (theoretical) physics and chemistry ([5],[7],[10],[11]), econometrics and various areas
of mathematics such as, representation theory involving the decompositions of modules [3]. Information
theory is not the exception, and idempotent elements are of great importance, particularly in
coding theory with error detecting-correcting linear codes whose alphabet is a finite (commutative)
ring ([2],[4],[6],[9],[12],[14],[15]).

Determining the idempotent elements of an arbitrary ring is not an easy task in general. There
are in the literature cases in which these elements have been obtained. For example, in [8 Propo-
sition 7.14, pag.405] a method of constructing idempotent elements of a ring $R$ from those of a
quotient ring $R/N$ where $N$ is a nil ideal of $R$ is given. A set of primitive idempotent elements are
determined for the group algebra $\mathbb{C}G$ by using the group of characters of the group $G$ ([13], Thm.
5.1.11, pag.185). This method can be interpreted in terms of the Discrete Fourier Transform. There
are other cases in which a set of idempotent elements of a ring have been described. For example, in
[15] the idempotent elements of $\mathbb{F}_q[x]/\langle x^n - 1 \rangle$ are determined by means of cyclotomic classes. In [6]
these elements are given for the group algebra $\mathbb{F}_qG$ by using the lattice of subgroups of the abelian
group $G$ where in both cases $\mathbb{F}_q$ is a finite field with $q$ elements. In [9] (see also [12]) the idempotents

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of the algebra \( \mathbb{Z}_{p^k}[x]/(x^n - 1) \) are obtained by means of the irreducible factors of \( x^n - 1 \), where \( \mathbb{Z}_{p^k} \) is the ring of integers modulo \( p^k \) with \( p \) a prime and \( k \geq 1 \) an integer. In this manuscript, starting from the result [3 Proposition 7.14, pag.405] which lifts an idempotent element from a quotient ring to the ambient ring, under certain general conditions, a formula to determine the lifted idempotent which is simpler than the one provided in the mentioned reference is given (Thm. 3.3). This result has several consequences, for instance the set of idempotent elements are determined in cases which include commutative rings containing a nilpotent ideal; commutative group rings \( RG \), where the ring \( R \) contains a nilpotent ideal; commutative group ring \( RG \) where the ring \( R \) is a chain ring; and the commutative group ring \( \mathbb{Z}_m G \).

The paper is organized as follows. In Section 2, basic facts about idempotent elements and for completeness the proof of [3 Proposition 7.14, pag.405] is included. In Section 3 the main result of this manuscript, Theorem 3.3 is presented. As consequence, the set of idempotent elements of several rings is determined in Section 4. Examples are included to illustrate the main ideas.

## 2 Basic facts

The starting point for the results presented in this manuscript is the following appearing in [3, Proposition 7.14, pag.405] on the construction of idempotent elements on a ring from those of a quotient ring. This result is recalled in the case of a commutative ring and the main steps of its proof are included. The interested reader can see further details in the mentioned reference.

Recall that an element \( e \) of a ring \( R \) is called idempotent if \( e^2 = e \), and two idempotent elements \( e_1 \) and \( e_2 \) of a ring \( R \) are said to be orthogonal if \( e_1 e_2 = 0 \). An idempotent \( e \in R \) is primitive if it can not be written as a sum of two non-trivial orthogonal idempotent elements.

**Proposition 2.1.** Let \( R \) be a ring, \( N \) a nil ideal of \( R \) and \( \bar{f} = f + N \) an idempotent element of the quotient ring \( R/N \). Then there exists an idempotent element \( e \) in \( R \) such that \( \bar{e} = \bar{f} \), where ’’ \( \bar{} \) ’’ denotes the canonical homomorphism from \( R \) to \( R/N \). Furthermore, if \( R \) is commutative, the element \( e \) is unique.

**Proof:** Since \( \bar{f} \) is idempotent, \( f^2 - f \in N \) and because \( N \) is a nil ideal, \( (f^2 - f)^n = 0 \) for some integer \( n > 0 \). Then, if \( g = 1 - f \), \( 0 = (fg)^n = f^n g^n \). From the relation \( f + g = 1 \) it follows that

\[
1 = 1^{2n-1} = (f + g)^{2n-1} = h + e,
\]

where

\[
h = \sum_{i=0}^{n-1} \binom{2n-1}{i} f^i g^{2n-1-i}, \quad e = \sum_{i=n}^{2n-1} \binom{2n-1}{i} f^i g^{2n-1-i}.
\]

Since \( f^n g^n = 0 \) it follows that \( eh = he = 0 \) and since \( e + h = 1, \ e^2 = e \) and \( h^2 = h \). From this it is easy to see that \( f^{2n-1} \equiv e \mod N \) and that \( f \equiv f^2 \equiv \cdots \equiv f^{2n-1} \mod N \) from which it is concluded that \( e \equiv f \mod N \).

To prove the uniqueness consider the idempotent element of the form \( e + z \) with \( z \) nilpotent. The condition \( (e + z)^2 = e + z \) gives \( (1 - 2e)z = z^2 \). Then \( z^3 = (1 - 2e)z^2 = (1 - 2e)^2 z \) and by
induction, \((1 - 2e)^n z = z^{n+1}\). Since \((1 - 2e)^2 = 1 - 4e + 4e = 1\), this implies that \(z = 0\) and hence \(e + z = e\).

Given an idempotent element \(T\) of \(R/N\), if \(R\) is commutative, the unique element \(e \in R\) determined as in Proposition 2.1 will be called the *lifted* idempotent of \(T\).

**Remark 2.2.** The following observations are easy consequences of the previous result.

1. If \(\bar{f}_1\) and \(\bar{f}_2\) are orthogonal idempotent elements of \(R/N\), then the corresponding lifted idempotent \(e_1\) and \(e_2\) of \(R\) are also orthogonal. This follows from the fact that \(e_1 e_2\) is an idempotent element in the nil ideal \(N\).

2. If \(\bar{f} \in R/N\) is a primitive idempotent then the corresponding lifted idempotent \(e \in R\) is also primitive. In fact, if \(e\) is not primitive, orthogonal idempotent elements \(g, h\) exist in \(R\) such that \(e = g + h\). Then \(\bar{e} = \bar{f} = \bar{g} + \bar{h}\) with \(\bar{g}\) and \(\bar{h}\) orthogonal idempotents in \(R/N\), so either \(\bar{g} = 0\) or \(\bar{h} = 0\), i.e., \(g \in N\) or \(h \in N\) and the claim follows from the fact that \(N\) is a nil ideal.

3. From the previous claims it follows that if \(\{\bar{f}_1, \ldots, \bar{f}_r\}\) is a set of primitive orthogonal idempotent elements in \(R/N\), the corresponding set \(\{e_1, \ldots, e_r\}\) of lifted idempotent elements of \(R\) has the same properties.

4. For a finite set \(X\), \(|X|\) will denote its cardinality. For a commutative ring \(R\), let \(E(R)\) denote the set of idempotent elements of \(R\). Assuming the conditions of Proposition 2.1

\[|E(R)| = |E(R/N)|.\]

This follows from the fact that the canonical homomorphism from \(R\) onto \(R/N\) restricted to \(E(R)\) is a bijection onto \(E(R/N)\). The surjectivity follows from the construction of the idempotent element of \(R\) from an idempotent element of \(R/N\), and the injectivity follows from the uniqueness of this construction.

### 3 A formula to compute lifted idempotents

In this section, under certain general conditions, a simple way to compute the lifted idempotent \(e\) of \(\bar{f}\) given in Proposition 2.1 is presented. More precisely, if a collection \(\{N_1, \ldots, N_k\}\) of ideals of a ring \(R\) satisfies the CNC-condition (see Definition 3.2 below), the lifted idempotent \(e\) of the idempotent \(f + N_1 \in R/N_1\) can be computed as a power of \(f\) in the ring \(R\) (see Theorem 3.3 below).

**Proposition 3.1.** Let \(R\) be a commutative ring and \(N\) a nilpotent ideal of index \(t \geq 2\) in \(R\). If \(\bar{f}\) is an idempotent element in the quotient ring \(R/N\) and \(e\) the corresponding lifted idempotent element in \(R\) determined as in Proposition 2.1 then:

1. For any prime number \(p\) such that \(p \geq t\) and for all \(n \in N\), there exists \(r \in R\) such that

\[(e + n)^p = e + pnr.\]
2. If a natural number \( s > 1 \) exists, such that \( sN = 0 \), and all the prime factors of the number \( s \) are greater than or equal to the nilpotency index \( t \) of the ideal \( N \), then the lifted idempotent \( e \) is
\[
e = f^s.
\]

3. In particular, when the nilpotency index of the ideal \( N \) is \( t = 2 \) and \( sN = 0 \) for some \( s \geq 2 \), then \( e = f^s \).

Proof:

1. Since \( n^t = 0 \) and \( e \) is an idempotent element in the ring \( R \),
\[
(e + n)^p = \sum_{j=0}^{p} \binom{p}{j} e^{p-j}n^j = e + \sum_{j=1}^{t-1} \binom{p}{j} en^j.
\]
Since \( p \) is a prime number, \( p \) divides \( \binom{p}{j} \) for all \( 1 \leq j \leq p-1 \). Also, since \( t \leq p \),
\[
(e + n)^p = e + pn(k_1e + k_2en + \cdots + k_{t-1}en^{t-2})
\]
where \( k_i = \binom{p}{i}/p \). Therefore,
\[
(e + n)^p = e + pr
\]
with \( r = k_1e + k_2en + \cdots + k_{t-1}en^{t-2} \in R \).

2. Let \( \{p_1p_2\cdots p_m\} \) be the set of primes in the prime decomposition of the integer \( s \). Since \( \bar{f} = \bar{e} \), \( f = e + n \) for some \( n \in N \), and since \( p_1 \geq t \), from item [1] of Observation 2.2 it follows that there exists \( r_1 \in R \) such that
\[
f^{p_1} = (e + n)^{p_1} = e + p_1nr_1.
\]
Similarly, since \( p_2 \geq t \) and \( p_1nr_1 \in N \), item [1] of Observation 2.2 implies that there exists \( r_2 \in R \) such that
\[
f^{p_1p_2} = (f^{p_1})^{p_2} = (e + p_1nr_1)^{p_2} = e + p_2(p_1nr_1)r_2.
\]
Continuing with this process \( r_3, r_4, \cdots, r_m \in R \) exists such that
\[
f^s = e + sn(r_1r_2\cdots r_m).
\]
In other words,
\[
f^s = e + sh,
\]
where \( h = nr_1r_2\cdots r_s \in N \). Finally, since \( h \in N \), and \( sN = 0 \), it follows that \( e = f^s \).

3. Observe that if the nilpotency index of the ideal \( N \) is \( t = 2 \), it is obvious that all prime factors of \( s \) are greater or equal to \( t = 2 \). Therefore the proof of this claim is an immediate consequence of item [2] of Observation 2.2.
The following definition will play an important role in the rest of the manuscript.

Recall that given an ideal \( N \) of the ring \( R \) and \( k > 1 \), \( N^k \) denotes the ideal generated by all products \( x_1x_2 \cdots x_k \), where each \( x_i \in N \) for \( i = 1, 2, ..., k \).

**Definition 3.2.** We say that a collection \( \{N_1, ..., N_k\} \) of ideals of a commutative ring \( R \) satisfies the CNC-condition if the following properties hold:

1. **Chain condition:** \( \{0\} = N_k \subset N_{k-1} \subset \cdots \subset N_2 \subset N_1 \subset R \).

2. **Nilpotency condition:** for all \( i = 1, 2, 3, ..., k-1 \), there exists an integer \( t_i \geq 2 \) such that \( N_i^{t_i} \subset N_{i+1} \).

3. **Characteristic condition:** for all \( i = 1, 2, 3, ..., k-1 \), there exists an integer \( s_i \geq 1 \) such that \( s_iN_i \subset N_{i+1} \). In addition, the prime factors of \( s_i \) are greater or equal to \( t_i \).

The minimum number \( t_i \) satisfying the nilpotency condition will be called the nilpotency index of the ideal \( N_i \) in the ideal \( N_{i+1} \). Similarly, the minimum number \( s_i \) satisfying the characteristic condition will be called the characteristic of the ideal \( N_i \) in the ideal \( N_{i+1} \).

The nilpotency and characteristic conditions introduced above can be stated as follows:

a. The nilpotency condition is equivalent to the following: for all \( i = 1, 2, ..., k-1 \), the quotient \( N_i/N_{i+1} \) is a nilpotent ideal of index \( t_i \) in the ring \( R/N_{i+1} \). In fact, if condition 2 holds, then for all \( n_1, n_2, ..., n_{t_i} \in N_i \), \( n_1n_2n_3 \cdots n_{t_i} \in N_i^{t_i} \subset N_{i+1} \). Thus,

\[
(n_1 + N_{i+1})(n_2 + N_{i+1}) \cdots (n_{t_i} + N_{i+1}) = n_1n_2 \cdots n_{t_i} + N_{i+1} = N_{i+1},
\]

which implies that \( N_i/N_{i+1} \) is a nilpotent ideal of index \( t_i \) in the ring \( R/N_{i+1} \). Conversely, if \( N_i/N_{i+1} \) is a nilpotent ideal of index \( t_i \) in the ring \( R/N_{i+1} \), then for all \( n_1, n_2, ..., n_{t_i} \in N_i \),

\[
(n_1 + N_{i+1})(n_2 + N_{i+1}) \cdots (n_{t_i} + N_{i+1}) = N_{i+1},
\]

which implies that \( n_1n_2 \cdots n_{t_i} \in N_{i+1} \). Thus any product of \( t_i \) elements in the ideal \( N_i \) are in the ideal \( N_{i+1} \) implying that \( N_i^{t_i} \subset N_{i+1} \). Therefore, condition 2 holds.

b. The characteristic condition is equivalent to the following: for all \( i = 1, 2, ..., k-1 \), there exists a natural number \( s_i \geq 1 \) such that \( s_i(N_i/N_{i+1}) = 0 \) in the ring \( R/N_{i+1} \). In fact, if condition 3 holds, for all \( n \in N_i \), \( s_in \in N_{i+1} \). Then

\[
s_i(n + N_{i+1}) = s_in + N_{i+1} = N_{i+1},
\]

implying that \( s_i(N_i/N_{i+1}) = 0 \) in the ring \( R/N_{i+1} \). Conversely, if we assume that \( s_i(N_i/N_{i+1}) = 0 \) in the ring \( R/N_{i+1} \), then for all \( n \in N_i \),

\[
s_i(n + N_{i+1}) = N_{i+1},
\]
which implies that \( s_i n \in N_{i+1} \), proving that \( s_i N_i \subset N_{i+1} \). Note that the characteristic \( s_i \) of the ideal \( N_i \) in the ideal \( N_{i+1} \) satisfies \( s_i \leq r_i \), where \( r_i \) denotes the characteristic of the ring \( R/N_{i+1} \). In fact, since for all \( x \in R \), \( r_i x + N_{i+1} = N_{i+1}, \), \( r_i x \in N_{i+1} \). Hence, \( r_i R \subset N_{i+1} \), and then \( r_i N_i \subset N_{i+1} \), showing that \( s_i \leq r_i \).

Now, we are in a position to give the following result.

**Theorem 3.3.** Let \( R \) be a commutative ring, \( \{N_1, N_2, \ldots, N_k\} \) be a collection of ideals of \( R \) satisfying the CNC-condition. If \( s_i \) is the characteristic of the ideal \( N_i \) in the ideal \( N_{i+1} \) and \( f + N_1 \) is an idempotent element of the ring \( R/N_1 \), then

\[
fs_1s_2\cdots s_{k-1},
\]

is an idempotent element of the ring \( R \). Furthermore, \( |E(R)| = |E(R/N_{k-1})| = \cdots = |E(R/N_1)| \).

**Proof:** If \( f + N_1 \) is an idempotent element of the ring

\[
R/N_1 \simeq \frac{(R/N_2)}{(N_1/N_2)},
\]

then \((f + N_2) + (N_1/N_2)\) is an idempotent element of the ring \( (R/N_2)/(N_1/N_2) \). Since \( N_1/N_2 \) is a nilpotent ideal of index \( t_1 \) in the ring \( R/N_2 \) and \( s_1 \) satisfies the hypothesis of claim 2 of Proposition 3.1, it follows that \( f s_1 + N_2 \) is an idempotent element of the ring \( R/N_2 \). From the isomorphism

\[
R/N_2 \simeq \frac{(R/N_3)}{(N_2/N_3)},
\]

it follows that \((f s_1 + N_3) + (N_2/N_3)\) is an idempotent element of the ring \( (R/N_3)/(N_2/N_3) \). Since \( N_2/N_3 \) is a nilpotent ideal of index \( t_2 \) in the ring \( R/N_3 \) and \( s_2 \) satisfies the hypothesis of claim 2 of Proposition 3.1, \( f s_1 s_2 + N_3 \) is an idempotent element of the ring \( R/N_3 \). Continuing with this process, since

\[
f s_1 s_2 \cdots s_{i-1} + N_{i+1}\]

is an idempotent element of the ring \( R/N_{i+1} \). Finally, in the last step of the chain of ideals,

\[
f s_1 s_2 \cdots s_{k-1} + N_k = f s_1 s_2 \cdots s_{k-1},
\]

is an idempotent element of the ring \( R/N_k = R \).

From the ring isomorphism given in (3), it follows that

\[
|E(R/N_i)| = \left| E\left( \frac{R/N_{i+1}}{N_i/N_{i+1}} \right) \right|,
\]
for all $i = 1, 2, 3 \ldots, k - 1$. Since $N_i/N_{i+1}$ is a nilpotent ideal of index $t_i$ in $R/N_{i+1}$, and $R$ is a commutative ring by Remark 2.2

$$\left| E\left( \frac{R/N_{i+1}}{N_i/N_{i+1}} \right) \right| = |E(R/N_{i+1})|.$$  

Therefore $|E(R/N_i)| = |E(R/N_{i+1})|$, for all $i = 1, 2, 3 \ldots, k - 1$, and the claim is proved.

Remark 3.4. Observe that if $\{N_1, N_2, \ldots, N_k\}$ is a collection of ideals of the ring $R$ satisfying the CNC-condition, any idempotent element $f + N_1$ of the ring $R/N_1$ is lifted up to the idempotent element $f^{s_1} + N_2$ of the ring $R/N_2$. This new idempotent element is lifted up to the idempotent element $f^{s_1s_2} + N_3$ of the ring $R/N_3$, and so on. At the end of this process, $f^{s_1s_2 \ldots s_k-1}$ is an idempotent element of the ring $R$. The following chain of ring homomorphisms

$$R \xrightarrow{\phi_{k-1}} \frac{R}{N_{k-1}} \xrightarrow{\phi_{k-2}} \ldots \xrightarrow{\phi_3} \frac{R}{N_3} \xrightarrow{\phi_2} \frac{R}{N_2} \xrightarrow{\phi_1} \frac{R}{N_1},$$

appears naturally in the lifting process of the idempotent element $f + N_1 \in R/N_1$. In addition, each homomorphism $\phi_i$ of this chain induces a bijection when restricted to the set of idempotents elements $E(R/N_{i+1})$, i.e.,

$$\phi_i : E(R/N_{i+1}) \to E(R/N_i), \quad x + N_{i+1} \to x + N_i$$

is bijective for $i = 1, 2, \ldots, k - 1$.

Remark 3.5. Under the conditions of Theorem 3.3, the set of idempotent elements $E(R)$ of the commutative ring $R$ is given by

$$E(R) = \{ f^{s_1s_2 \ldots s_k-1} : \bar{f} \in E(R/N_1) \}.$$ 

Hence, in order to determine the idempotent elements of the ring $R$, it is necessary to have a collection $\{N_1, N_2, \ldots, N_k\}$ of ideals of the ring $R$ satisfying the CNC-condition together with all idempotent elements of the quotient ring $R/N_1$.

4 Consequences and applications

In this section, by means of Theorem 3.3 the set of idempotent elements of a variety of cases which include: commutative rings containing a nilpotent ideal; commutative group rings $RG$, where $R$ contains a nilpotent ideal; commutative group rings $RG$ where $R$ is a chain ring; the group ring $\mathbb{Z}_m G$, where $\mathbb{Z}_m$ is the ring of integers modulo $m$. In each of these cases examples are included illustrating the results.
4.1 Commutative rings containing a nilpotent ideal

The following result describes the set of idempotent elements of a commutative ring \( R \) containing a nilpotent ideal \( N \) in terms of the idempotent elements of the quotient ring \( R/N \).

**Proposition 4.1.** Let \( R \) be a commutative ring and \( N \) a nilpotent ideal of nilpotency index \( k \geq 2 \) of \( R \). Let \( s > 1 \) be the characteristic of the quotient ring \( R/N \). If \( f + N \) is an idempotent element of \( R/N \), then

\[
fs^{k-1}
\]

is an idempotent element of the ring \( R \). Moreover, \( |E(R)| = |E(R/N)| \).

**Proof:** The proof of this proposition is a consequence of Theorem 3.3. It will be enough to show that the collection \( B = \{N, N^2, \ldots, N^k\} \) of ideals of the ring \( R \) satisfies the CNC-condition with nilpotency index and characteristic of the ideal \( N_i \) in the ideal \( N_{i+1} \) being \( t_i = 2 \) and \( s_i = s \) for all \( i = 1, 2, 3, \ldots, k - 1 \). Indeed,

1. It is clear that the collection \( B \) satisfies the chain condition.
2. Since \((N^i)^2 = N^{2i}\) and \( i + 1 \leq 2i \) for all \( i = 1, 2, 3, \ldots, k - 1 \), it follows that \((N^i)^2 \subset N^{i+1}\). Hence, the collection \( B \) satisfies the nilpotency condition.
3. Since the ring \( R/N \) has characteristic \( s \), there exists \( n \in N \) such that \( \sum_{i=1}^{k} 1_R = n \). Then

\[
sN^i = (1_R + \cdots + 1_R)N^i = nN^i \subset N^{i+1},
\]

and it follows that \( sN^i \subset N^{i+1} \) for all \( i = 1, 2, 3, \ldots, k - 1 \). In addition, all prime factors of \( s_i = s \) are greater or equal to the nilpotency index \( t_i = 2 \). It proves that the collection \( B \) satisfies the characteristic condition.

Therefore, Theorem 3.3 implies that \( fs^{k-1} \) is an idempotent element of the ring \( R \) and \( |E(R)| = |E(R/N)| \).

An immediate consequence of Proposition 4.1 is the following:

**Corollary 4.2.** Let \( R \) be a commutative ring, \( a \in R \) a nilpotent element of index \( k \), and \( s > 1 \) the characteristic of the quotient ring \( R/\langle a \rangle \). If \( f + \langle a \rangle \) is an idempotent element of \( R/\langle a \rangle \), then

\[
fs^{k-1}
\]

is an idempotent element of the ring \( R \). Moreover, \( |E(R)| = |E(R/\langle a \rangle)| \).

**Proof:** Since \( N = \langle a \rangle \) is a nilpotent ideal of nilpotency index \( k \) in \( R \), the result follows from Proposition 4.1.

The following example illustrates the previous corollary.
Example 4.3. Let
\[ \mathbb{Z}_{p^k}[i] = \{ a + bi : a, b \in \mathbb{Z}_{p^k}, i^2 = -1 \}, \]
where \( p > 2 \) is a prime and \( k > 1 \) a natural number. It is easy to see that \( a = p \) is a nilpotent element of index \( k \) in the ring \( \mathbb{Z}_{p^k}[i] \). Since
\[ \frac{\mathbb{Z}_{p^k}[i]}{(p)} \cong \mathbb{Z}_p[i] \]
and the ring \( \mathbb{Z}_p[i] \) has characteristic \( s = p \), Corollary 4.2 implies that
\[ E(\mathbb{Z}_{p^k}[i]) = \left\{ f^r : r = p^{k-1}, \text{ and } \bar{f} \in E(\mathbb{Z}_p[i]) \right\}, \]
and
\[ |E(\mathbb{Z}_{p^k}[i])| = |E(\mathbb{Z}_p[i])|. \]
The idempotent elements of the ring \( \mathbb{Z}_p[i] \) are determined as follows. Since the idempotent elements of \( \mathbb{Z}_p[i] \) are of the form \( z = a + bi \), where \( a = a^2 - b^2 \) and \( b = 2ab \), if \( b = 0 \) then \( z = 0 \) or \( z = 1 \). If \( b \neq 0 \), \( 2a = 1 \) in \( \mathbb{Z}_p \), hence \( a = (p + 1)/2 \). From the relation \( a = a^2 - b^2 \) it can be seen that \( 4b^2 = (2b)^2 = -1 \) in \( \mathbb{Z}_p \). This last equation has solution if and only if \( p \equiv 1 \mod 4 \), i.e., \(-1\) is a quadratic residue in \( \mathbb{Z}_p \), (Theorem 3.1, pag.132, [10]). It is easy to see (by Wilson’s Theorem) that the solutions are \( 2b = \pm (\frac{p-1}{2})! \). If \( x = (\frac{p-1}{2})! \), the idempotent elements of the ring \( \mathbb{Z}_p[i] \) are
\[ f_1 = 0, \quad f_2 = 1 \]
\[ f_3 = (p+1)/2 + (x/2)i, \quad f_4 = (p+1)/2 - (x/2)i. \]
Hence, the set of idempotents elements of the ring \( \mathbb{Z}_{p^k}[i] \) is
\[ E(\mathbb{Z}_{p^k}[i]) = \{ f_1^r, f_2^r, f_3^r, f_4^r \}, \]
where \( r = p^{k-1} \). Observe that the ring \( \mathbb{Z}_{p^k}[i] \) is isomorphic to the ring \( \mathbb{Z}_{p^k}[x]/(x^2 + 1) \).

4.2 Commutative group rings

In the following the set of idempotent elements for the group ring \( RG \) where \( R \) is a commutative ring containing a collection of ideals satisfying the CNC-condition and \( G \) a finite commutative group is determined.

Proposition 4.4. Let \( R \) be a commutative ring and \( G = \{ g_1, g_2, \ldots, g_m \} \) a finite commutative group. Let \( \{ N_1, N_2, \ldots, N_k \} \) be a collection of ideals of \( R \) satisfying the CNC-condition. Let \( s_i \) be the characteristic of the ideal \( N_i \) in the ideal \( N_{i+1} \). If \( f + N_1G \) is an idempotent element of the group ring \( (R/N_1)G \), then
\[ f^{s_1s_2\cdots s_{k-1}} \]
is an idempotent element of the group ring \( RG \). Furthermore, \( |E(RG)| = |E((R/N_1)G)| \).
Proof: The proof of this proposition is a consequence of Theorem 3.3. First, we prove that the collection \( B = \{N_1 G, N_2 G, \ldots, N_k G\} \) of ideals of the group ring \( RG \) satisfies the CNC-condition with nilpotency index and characteristic of the ideal \( N_i G \) in the ideal \( N_{i+1} G \) exactly the same nilpotency index and characteristic of the ideal \( N_i \) in the ideal \( N_{i+1} \). Indeed,

1. It is clear that the collection \( B \) satisfies the chain condition.

2. If \( t_i \) denotes the nilpotency index of the ideal \( N_i \) in the ideal \( N_{i+1} \), then \( (N_i G)^{t_i} = N_i^{t_i} G \subset N_{i+1} G \), proving that the collection \( B \) satisfies the nilpotency condition.

3. Since \( s_i \) is the characteristic of the ideal \( N_i \) in the ideal \( N_{i+1} \), then \( s_i N_i \subset N_{i+1} \). It is clear that \( s_i (N_i G) = (s_i N_i) G \). Hence,

\[
s_i (N_i G) = (s_i N_i) G \subset N_{i+1} G.
\]

Now, since the collection \( \{N_1, N_2, \ldots, N_k\} \) satisfies the CNC-condition, it is obvious that all prime factors of the characteristic \( s_i \) are greater or equal to the nilpotency index \( t_i \) for all \( i = 1, 2, 3, \ldots, k - 1 \). It proves that the collection \( B \) satisfies the characteristic condition.

From Theorem 3.3 and the isomorphism

\[
\frac{RG}{N_1 G} \cong \left( \frac{R}{N_1} \right) G,
\]

it follows that \( f^{s_1 s_2 \cdots s_{k-1}} \) is an idempotent element of the group ring \( RG \), and \( |E(RG)| = |E((R/N_1)G)| \).

\[ \blacksquare \]

Corollary 4.5. Let \( R \) be a commutative ring, \( N \) a nilpotent ideal of index \( k \) in \( R \), \( G = \{g_1, g_2, \ldots, g_n\} \) a finite commutative group, and \( s \) the characteristic of the quotient ring \( R/N \). If \( f + NG \) is an idempotent element in the group ring \( (R/N)G \), then

\[
f^{s \cdot s \cdots s^{k-1}}
\]

is an idempotent element in the group ring \( RG \). Furthermore, \( |E(RG)| = |E((R/N)G)| \).

Proof: The proof of the Proposition 4.1 shows that the collection \( \{N, N^2, \ldots, N^k\} \) of ideals of the ring \( R \) satisfies the CNC-condition with constant characteristic \( s_i = s \) for all \( i = 1, 2, 3, \ldots, k - 1 \). The result follows from Proposition 4.4. \[ \blacksquare \]
Example 4.6. Let $\mathbb{F}_p$ be the finite field of order $p$, $R = \mathbb{F}_p[x_1, ..., x_m]/\langle x_1^p, ..., x_m^p \rangle$ and $N = \langle x_1, ..., x_m \rangle$ the ideal generated by $x_j = X_j + \langle x_1^p, ..., x_m^p \rangle$. It is not difficult to see that $N$ has nilpotency index equal to $m(p - 1) + 1$ and that $R/N \simeq \mathbb{F}_p$. Then for $G$ a finite abelian group, Corollary 4.5 implies that $E(RG) = \{ f^{p^{m(p-1)}} : \bar{f} \in E(\mathbb{F}_pG) \}$.

Remark 4.7. The following collection of ideals

$$\{ N, N^2, ..., N^{m(p-1)}, N^{m(p-1)+1} = (0) \},$$

is important in the study of Reed-Muller codes over the finite field $\mathbb{F}_p$, $p$ a prime, since for an integer $\nu$ with $0 \leq \nu \leq m(p - 1)$,

$$N^{m(p-1)-\nu} = RM_\nu(m, p),$$

where $RM_\nu(m, p)$ is the Reed-Muller code of order $\nu$ over the alphabet $\mathbb{F}_p$. Details can be found in [1].

Corollary 4.8. Let $R$ be a commutative ring, $a$ be a nilpotent element of index $k$ in $R$, $G = \{g_1, g_2, \ldots, g_n\}$ a finite commutative group and $s$ the characteristic of the quotient ring $R/\langle a \rangle$. If $f + \langle a \rangle G$ is an idempotent element of the group ring $(R/\langle a \rangle)G$, then

$$f^{s^{k-1}}$$

is an idempotent element of the group ring $RG$. Furthermore, $|E(RG)| = |E((R/\langle a \rangle)G)|$.

Proof: It is enough to observe that $N = \langle a \rangle$ is a nilpotent ideal of index $k$ in $R$. The result follows from Corollary 4.5. ■

4.3 Commutative group ring $RG$ with $R$ a chain ring

Let $R$ be a finite commutative chain ring and $G$ a finite commutative group. It is well known that $R$ contains a unique maximal nilpotent ideal $N = \langle a \rangle$ for some $a \in R$. If $k$ denotes the nilpotency index of $a$, and $p$ the characteristic of the (residue) field $\mathbb{F} = R/\langle a \rangle$ from Corollary 4.8 it follows that

$$E(RG) = \{ f^r : r = p^{k-1}, \text{ and } \bar{f} \in E(\mathbb{F}G) \}.$$ (6)

Examples of finite commutative chain rings include the modular ring of integers $R = \mathbb{Z}_{p^k}$, where $p$ is a prime number and $k > 1$ is an integer. In this example, the maximal nilpotent ideal is $N = \langle p \rangle$, $p$ has nilpotency index equal to $k$ in $\mathbb{Z}_{p^k}$, and $\mathbb{F} \cong \mathbb{Z}_p$. Thus,

$$E(\mathbb{Z}_{p^k}G) = \{ f^r : r = p^{k-1}, \text{ and } \bar{f} \in E(\mathbb{Z}_pG) \}.$$ (7)

When the group $G$ is cyclic of order $n$, it is known that $\mathbb{Z}_{p^k}G \cong \mathbb{Z}_{p^k}[x]/\langle x^n - 1 \rangle$, then (7) can be written in the following equivalent form:

$$E(\mathbb{Z}_{p^k}[x]/\langle x^n - 1 \rangle) = \{ f^r : r = p^{k-1}, \text{ and } \bar{f} \in E(\mathbb{Z}_p[x]/\langle x^n - 1 \rangle) \}.$$
Another interesting class of chain rings are Galois rings which are of the form \( R = \mathbb{Z}_p[x]/\langle q(x) \rangle \) where \( p \) is a prime number, \( k > 1 \) and \( q(x) \) is a monic polynomial of degree \( r \) whose image in \( \mathbb{Z}_p[x] \) is irreducible. In this example, the maximal nilpotent ideal is \( N = pR, p \) has nilpotency index equal to \( k \) in \( R \), and \( \mathbb{F} = R/N \cong \mathbb{F}_{p^r} \) is the finite field of order \( p^r \). Thus,

\[
E((\mathbb{Z}_p[x]/\langle q(x) \rangle))G) = \{ f^r : r = p^{k-1}, \text{ and } \bar{f} \in E(\mathbb{F}_{p^r}G) \}.
\]  

(8)

Combining the theory of Ferraz and Polcino in [6] the results discussed above, it will be shown how to compute the idempotent elements in group rings \( \mathbb{Z}_pG \) for certain commutative groups \( G \). This is illustrated in the following examples.

**Example 4.9.** Let \( p = 5 \) and let \( G = C_7 = \langle g \rangle = \{ e, g, g^2, g^3, g^4, g^5, g^6 \} \) be the cyclic group of order 7 generated by \( g \). From [6, Lemma 3] and [6, Corollary 4] it can be seen that the idempotent elements of the group algebra \( \mathbb{Z}_5C_7 \) are:

\[
\bar{f}_1 = 0, \quad \bar{f}_2 = 1,
\]

\[
\bar{f}_3 = \hat{G} = 3e + 3g + 3g^2 + 3g^3 + 3g^4 + 3g^5 + 3g^6,
\]

\[
\bar{f}_4 = 1 - \hat{G} = 3e + 2g + 2g^2 + 2g^3 + 2g^4 + 2g^5 + 2g^6,
\]

where for a subgroup \( H \) of \( G \), the idempotent element \( \hat{H} \) is defined by

\[
\hat{H} = \frac{1}{|H|} \sum_{h \in H} h.
\]

Therefore, from (8), the idempotent elements of the group ring \( \mathbb{Z}_5^3C_7 \) are:

\[
\bar{f}_1^{5^2} = 0, \quad \bar{f}_2^{5^2} = 1,
\]

\[
\bar{f}_3^{5^2} = 18e + 18g + 18g^2 + 18g^3 + 18g^4 + 18g^5 + 18g^6,
\]

\[
\bar{f}_4^{5^2} = 108e + 107g + 107g^2 + 107g^3 + 107g^4 + 107g^5 + 107g^6.
\]

**Example 4.10.** Let \( p = 2 \) and \( G = \langle a, b | a^5 = b^5 = 1, ab = ba \rangle \) be an abelian group of order 25. From [6, Lemma 5] and [6, Theorem 4.1], (see also [14, Section III]), it follows that the primitive orthogonal idempotent elements of the group ring \( \mathbb{Z}_2(C_5 \times C_5) \) are:

\[
f_0 = \hat{G}, \quad f_1 = \langle a \rangle - \hat{G},
\]

\[
f_2 = \langle b \rangle - \hat{G}, \quad f_i = \langle ab^i \rangle - \hat{G}, \quad i = 1, 2, 3, 4,
\]

where for a subgroup \( H \) of \( G \), \( \hat{H} = \frac{1}{|H|} \sum_{h \in H} h \).

Therefore, the set of primitive orthogonal idempotent elements of the group ring \( \mathbb{Z}_2^3(C_5 \times C_5) \) is

\[
E(\mathbb{Z}_2^3(C_5 \times C_5)) = \{ f_0^{2^2}, f_1^{2^2}, f_2^{2^2}, f_3^{2^2} \},
\]

the other idempotents are obtained as a sum of the primitive orthogonal idempotent elements.

In general, for any \( k \in \mathbb{N} \), the set of primitive orthogonal idempotent elements of the group ring \( \mathbb{Z}_2^k(C_5 \times C_5) \) is

\[
E(\mathbb{Z}_2^k(C_5 \times C_5)) = \{ f_0^{2^{k-1}}, f_1^{2^{k-1}}, f_2^{2^{k-1}}, f_3^{2^{k-1}} \}.
\]
4.4 Commutative group rings $\mathbb{Z}_mG$ with $G$ a commutative group

The previously discussed results for $\mathbb{Z}_pG$ can be extended to the group ring $\mathbb{Z}_mG$, where $\mathbb{Z}_m$ is the ring of integers modulo $m > 1$, and $G$ a finite commutative group.

**Theorem 4.11.** Let $m = p_1^{r_1} p_2^{r_2} \cdots p_j^{r_j}$ be the prime factorization of the integer $m \geq 2$. Set $m_i = m/p_i^{r_i}$ and let $s_i$ be a natural number such that $s_im_i = 1 \mod (p_i^{r_i})$ for $i = 1, 2, \cdots, j$. If $\bar{i}$ is an idempotent element of the group ring $\mathbb{Z}_{p_i}G$ for $i = 1, 2, \cdots, j$, by setting $\alpha_i = p_i^{r_i-1}$, the element $e$ given by
\[ e = s_1 m_1 f_1^{\alpha_1} + s_2 m_2 f_2^{\alpha_2} + \cdots + s_j m_j f_j^{\alpha_j}, \]
is an idempotent element of the group ring $\mathbb{Z}_mG$. Moreover,
\[ |E(\mathbb{Z}_mG)| = |E(\mathbb{Z}_{p_1}G)||E(\mathbb{Z}_{p_2}G)| \cdots |E(\mathbb{Z}_{p_j}G)|. \]

**Proof:** From the Chinese Remainder Theorem (CRT), $\mathbb{Z}_m \cong \mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \mathbb{Z}_{p_j^{r_j}}$ and, therefore,
\[ \mathbb{Z}_mG \cong \phi \mathbb{Z}_{p_1^{r_1}}G \times \mathbb{Z}_{p_2^{r_2}}G \times \cdots \times \mathbb{Z}_{p_j^{r_j}}G. \]

If $\bar{i}$ is an idempotent element of $\mathbb{Z}_{p_i}G$, for $i = 1, 2, \cdots, j$, from Corollary 4.8, $f_i^{\alpha_i}$ is an idempotent element of $\mathbb{Z}_{p_i}G$ for $i = 1, 2, 3, \cdots, j$. Thus,
\[ (f_1^{\alpha_1}, f_2^{\alpha_2}, \cdots, f_j^{\alpha_j}), \]
is an idempotent element of the product ring given in (11). Consequently, $\phi^{-1}((f_1^{\alpha_1}, f_2^{\alpha_2}, \cdots, f_j^{\alpha_j})) = e$ is an idempotent element of the group ring $\mathbb{Z}_mG$. Finally, from the CRT, $e$ can be expressed in the form (9). The equality (10) follows from Corollary 4.8 and (11). \[ \blacksquare \]

The following result provides an alternative way to compute the idempotents of $\mathbb{Z}_mG$.

**Theorem 4.12.** Let $m = p_1^{r_1} p_2^{r_2} \cdots p_j^{r_j}$ be the prime factorization of the integer $m \geq 2$. Set $k = \max\{r_1, r_2, \cdots, r_j\}$, $c_i = (p_1 p_2 \cdots p_j)/p_i$, and let $t_i$ be a natural number such that $t_i c_i = 1 \mod (p_i)$ for $i = 1, 2, 3, \cdots, j$. If $\bar{i}$ is an idempotent element of the group ring $\mathbb{Z}_{p_i}G$ for $i = 1, 2, 3, \cdots, j$, then
\[ e = [t_1 c_1 f_1 + t_2 c_2 f_2 + \cdots + t_j c_j f_j]^{(p_1 p_2 \cdots p_j)^k-1} \]
is an idempotent element of the group ring $\mathbb{Z}_mG$. Therefore,
\[ |E(\mathbb{Z}_mG)| = |E(\mathbb{Z}_{p_1 p_2 \cdots p_j}G)|. \]

**Proof:** If $\bar{i}$ is an idempotent element of the group ring $\mathbb{Z}_{p_i}G$ for $i = 1, 2, \cdots, j$, from Theorem 4.11 it follows that
\[ g = t_1 c_1 \bar{i} + t_2 c_2 \bar{i} + \cdots + t_j c_j \bar{i}, \]
is an idempotent element of the group ring $\mathbb{Z}_{p_1 p_2 \cdots p_j}G$. Observe that $a = p_1 p_2 \cdots p_j$ has nilpotency index $k$ in the ring $\mathbb{Z}_m$ and
\[ \frac{\mathbb{Z}_m}{a\mathbb{Z}_m} \cong \mathbb{Z}_a \]
has characteristic $a = p_1p_2 \cdots p_j$. Hence from Corollary 4.8 it follows that,

$$e = g^{(m_1p_2 \cdots p_j)}k^{-1},$$

is an idempotent element of the group ring $\mathbb{Z}_mG$, proving (12). Relation (13) also follows from Corollary 4.8.

It is worth noting that Theorem 4.11 can be extended to the group ring $RG$, where $G$ is a finite commutative group and $R = \mathbb{F}_q[x]/\langle m(x) \rangle$, with $\mathbb{F}_q$ a finite field with $q = p^s$ elements, and $m(x) \in \mathbb{F}_q[x]$. Indeed, let $m(x) = p_1^{\alpha_1}(x)p_2^{\alpha_2}(x) \cdots p_j^{\alpha_j}(x)$ be the factorization of the polynomial $m(x)$ in $\mathbb{F}_q[x]$, $m_i(x) = m(x)/p_i^{\alpha_i}(x)$ and $s_i(x) \in \mathbb{F}_q[x]$ such that $s_i(x)m_i(x) \equiv 1 \mod (p_i^{\alpha_i}(x))$ for $i = 1, 2, \ldots, j$. If $f_i$ is an idempotent element in the group ring $\mathbb{F}_q[x]/\langle p_i(x) \rangle)G$, the element $e$ given by

$$e = s_1(x)m_1(x)f_1^{\alpha_1}(x) + s_2(x)m_2(x)f_2^{\alpha_2}(x) + \cdots + s_j(x)m_j(x)f_j^{\alpha_j}(x), \quad \alpha_i = p^r \cdot 1,$$

is an idempotent element of the group ring $(\mathbb{F}_q[x]/\langle m(x) \rangle)G$. Furthermore,

$$|E(\mathbb{F}_q[x]/\langle m(x) \rangle)G| = |E(\mathbb{F}_q[x]/\langle p_1(x) \rangle)G)| \cdots |E(\mathbb{F}_q[x]/\langle p_j(x) \rangle)G)|.$$

The proof of the former result is similar to the one given for Theorem 4.11. However, in the following lines the main steps of the proof are given. If $R_i = \mathbb{F}_q[x]/\langle p_i(x) \rangle$, from the Chinese Remainder Theorem (CRT), it follows that

$$RG \cong_\phi R_1G \times \cdots \times R_jG.$$

Since $a_i = p_i(x) + \langle p_i(x) \rangle$ is a nilpotent element in the ring $R_i$ of nilpotency index $r_i$ and the field $R_i/\langle a_i \rangle \cong \mathbb{F}_q[x]/\langle p_i(x) \rangle$ has characteristic $p$ for all $i = 1, 2, \ldots, j$, from Corollary 4.8 it follows that, if $f_i$ is an idempotent element in $(R_i/\langle a_i \rangle)G$, for each $i = 1, 2, 3, \ldots, j$, then $f_i^{\alpha_i}$ is an idempotent element in $R_iG$ for each $i = 1, 2, 3, \ldots, j$. Thus, $(f_1^{\alpha_1}, f_2^{\alpha_2}, \cdots, f_j^{\alpha_j})$ is an idempotent element in the ring $R_1G \times \cdots \times R_jG$. Consequently, $\phi^{-1}_1((f_1^{\alpha_1}, f_2^{\alpha_2}, \cdots, f_j^{\alpha_j})) = e$ is an idempotent element in the ring $RG$. Finally, from the CRT, $e$ can be expressed in the form (15).

Next, Theorems 4.11 and 4.12 will be illustrated with examples. In example 4.13, the idempotent elements of the group ring $\mathbb{Z}_{200}C_3 \cong \mathbb{Z}_{200}[x]/\langle x^3 - 1 \rangle$ are determined. In example 4.14, the idempotent elements of the group ring $\mathbb{Z}_{200}(C_5 \times C_5)$ are given. Here, $C_n$ denotes the cyclic group of order $n$.

**Example 4.13.** Let $R = \mathbb{Z}_{200}C_3$, where $C_3 = \langle g \rangle = \{g_0, g_1, g_2 \}$ is the cyclic group of order three. First note that the elements

$$u_1 = g_0 + g_1 + g_2, \quad u_3 = g_0,$$

$$u_2 = g_1 + g_2, \quad u_4 = 0,$$

are the 4 idempotent elements of the group ring $\mathbb{Z}_2C_3$ (the calculation of these idempotent elements can be done by hand). In addition,

$$v_1 = 4g_0 + 3g_1 + 3g_2, \quad v_3 = g_0,$$

$$v_2 = 2g_0 + 2g_1 + 2g_2, \quad v_4 = 0,$$

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are the 4 idempotent elements of the group ring \( \mathbb{Z}_5 \mathbb{C}_3 \). In order to determine the idempotents elements of the group ring \( R \), Theorem 4.11 or Theorem 4.12 are applied.

- **In terms of Theorem 4.11**: since \( m = 200 = 2^3 5^2 \), \( p_1 = 2 \), \( p_2 = 5 \), \( m_1 = 5^2 \) and \( m_2 = 2^3 \), so \( s_1 = 1 \) and \( s_2 = 22 \). Then,
  \[
  E(\mathbb{Z}_{200} \mathbb{C}_3) = \{ 25f^4 + 176g^5 : f \in E(\mathbb{Z}_2 \mathbb{C}_3), \ g \in E(\mathbb{Z}_5 \mathbb{C}_3) \}.
  \] (19)

- **In terms of Theorem 4.12**: since \( m = 200 = 2^3 5^2 \), \( p_1 = 2 \), \( p_2 = 5 \), \( k = 3 \), \( c_1 = 5 \), and \( c_2 = 2 \), so \( t_1 = 1 \) and \( t_2 = 3 \). Then,
  \[
  E(\mathbb{Z}_{200} \mathbb{C}_3) = \{(5f + 6g)^{100} : f \in E(\mathbb{Z}_2 \mathbb{C}_3), \ g \in E(\mathbb{Z}_5 \mathbb{C}_3) \}.
  \] (20)

From relations (17), (18) and by using an algorithm implemented by the authors in the programing language \( C \), one can see that

\[
\begin{align*}
  h_1 &= 184g_0 + 8g_1 + 8g_2, & h_9 &= 126g_0 + 125g_1 + 125g_2, \\
  h_2 &= 192g_0 + 192g_1 + 192g_2, & h_{10} &= 17g_0 + 192g_1 + 192g_2, \\
  h_3 &= 142g_0 + 117g_1 + 117g_2, & h_{11} &= 67g_0 + 67g_1 + 67g_2, \\
  h_4 &= 134g_0 + 133g_1 + 133g_2, & h_{12} &= 59g_0 + 83g_1 + 83g_2, \\
  h_5 &= 51g_0 + 75g_1 + 75g_2, & h_{13} &= 9g_0 + 8g_1 + 8g_2, \\
  h_6 &= 150g_1 + 125g_1 + 125g_2, & h_{14} &= 75g_0 + 75g_1 + 75g_2, \\
  h_7 &= g_0, & h_{15} &= 25g_0, \\
  h_8 &= 0, & h_{16} &= 176g_0.
\end{align*}
\]

are the 16 idempotent elements of \( \mathbb{Z}_{200} \mathbb{C}_3 \).

**Example 4.14.** Let \( \mathbb{Z}_{25} \mathbb{C}_5 \) where \( G = C_5 \times C_5 = \langle a, b | a^5 = b^5 = 1, ab = ba \rangle \) be an abelian group of order 25. First, note that from example 4.10, the elements

\[
\begin{align*}
  f_0 &= \hat{G}, & f_1 &= \hat{a} - \hat{G}, \\
  f_2 &= \langle b \rangle - \hat{G}, & f_i &= \langle ab^i \rangle - \hat{G}, \ i = 1, 2, 3, 4,
\end{align*}
\]

are the primitive orthogonal idempotent elements of the group ring \( \mathbb{Z}_2(C_5 \times C_5) \).

Also, from \([6, \text{Lemma 5}]\), \([6, \text{Theorem 4.1}]\) and \([14, \text{Section III}]\) it follows that the primitive orthogonal idempotent elements of the group ring \( \mathbb{Z}_3(C_5 \times C_5) \) are:

\[
\begin{align*}
  g_0 &= \hat{G}, & g_1 &= \hat{a} - \hat{G}, \\
  g_2 &= \langle b \rangle - \hat{G}, & g_i &= \langle ab^i \rangle - \hat{G}, \ i = 1, 2, 3, 4,
\end{align*}
\]

and the primitive orthogonal idempotent elements of the group ring \( \mathbb{Z}_{15}(C_5 \times C_5) \) are:

\[
\begin{align*}
  h_0 &= \hat{G}, & h_1 &= \hat{a} - \hat{G}, \\
  h_2 &= \langle b \rangle - \hat{G}, & h_i &= \langle ab^i \rangle - \hat{G}, \ i = 1, 2, 3, 4.
\end{align*}
\]
In order to determine the idempotent elements of the group ring \( \mathbb{Z}_{936}G \), Theorem 4.11 can be applied. Since \( m = 936 = 2^2 \cdot 3^2 \cdot 13 \), \( p_1 = 2 \), \( p_2 = 3 \), \( p_3 = 13 \), \( m_1 = 3^2 \cdot 13 \), \( m_2 = 2^3 \cdot 13 \) and \( m_3 = 2^3 \cdot 3^2 \), so \( s_1 = 5 \), \( s_2 = 2 \) and \( s_3 = 2 \). Then,

\[
E(\mathbb{Z}_{936}G) = \{ 585 f^4 + 208 g^3 + 144 h : f \in E(\mathbb{Z}_2G), \ g \in E(\mathbb{Z}_3G), \ h \in E(\mathbb{Z}_{13}G) \}.
\]

is a set of primitive orthogonal idempotent elements of the group ring \( \mathbb{Z}_{936}G \).

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References

[1] H. Andriatahiny, “Generalized Reed-Muller Codes and the Radical Powers of a Modular Algebra.” British J. of Mathematics & Computer Science 18(5): 1-14, (2016).

[2] G.K. Bakshi, S. Gupta and I.B.S. Passi, “Semisimple metacyclic group algebras”, Proc. Indian Acad. Sci. (Math. Sci.), 121, 379396, (2012).

[3] O. Broche and Á. del Río, “Wedderburn decomposition of finite group algebras”, Finite Fields Appl., vol 13, 7179, (2017).

[4] J. de la Cruz and W. Willems, “On group codes with complementary duals”, Des. Codes Cryptogr., 86, 2065-2073 (2018).

[5] A.M. Dobrotvorskii, “New method of calculating idempotent density matrices and the total energy of many-electron systems.”, Theoretical and Experimental Chemistry, vol. 26, No.2, 1-4 (1990).

[6] R.A. Ferraz and C. Polcino Milies, “Idempotents in group algebras and minimal abelian codes”, Finite Fields and Appl., vol 13, 382-393, (2007).

[7] R.U. Haq and L.H. Kaufmann, “Iterants, Idempotents and Clifford Algebra in Quantum Theory”, arXiv: 1705.06600v2 [quant-ph], 2 nov. 2017.

[8] N. Jacobson, Basic II. W. H. Freeman and Company, New York, (1989).

[9] P. Kanwar and S. López-Permouth, “Cyclic Codes over the Integers Modulo \( p^m \)”, Finite Fields and Appl., vol 3, pp. 334-352, (1997).

[10] I. Niven, H.S. Zuckerman and H.L. Montgomery, Introduction to the Theory of Numbers, John Wilwy & Sons, Inc., 5th ed.,1991.
[11] Per-Olov Löwdin (ed.), *Advances in Quantum Chemistry*, vol. 8-1974, Academic Press, N.Y., London, (1974).

[12] V. Pless and Z. Qian, “Cyclic codes and quadratic codes over $\mathbb{Z}_4$”, *IEEE Trans. Inform. Theory*, vol 42, 15941600, 1996.

[13] C. Polcino Milies and S.K. Sehgal, *An Introduction to Group Rings*. Kluwer Academic, Dordrecht, 2002.

[14] C. Polcino Miles and F. D. de Melo, “On Cyclic and Abelian Codes”, *IEEE Trans. Inform. Theory*, vol.59, No.11, 7314-7319, (2013).

[15] M. Pruthi and S.K. Arora, “Minimal codes of prime power length”, *Finite Fields Appl.*, vol 3, 99113, (1997).

[16] Y. Zhang, Sh. Pang, Z. Jiao and W. Zhao, “Group partition and systems of orthogonal idempotents”, *Linear Algebra and its Applications*, 278, 249-262, (1998).