COTANGENT BUNDLES OF PARTIAL FLAG VARIETIES AND CONORMAL VARIETIES OF THEIR SCHUBERT DIVISORS

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Abstract. Let $P$ be a parabolic subgroup in $G = SL_n(k)$, for $k$ an algebraically closed field. We show that there is a $G$-stable closed subvariety of an affine Schubert variety in an affine partial flag variety which is a natural compactification of the cotangent bundle $T^*G/P$. Restricting this identification to the conormal variety $N^*X(w)$ of a Schubert divisor $X(w)$ in $G/P$, we show that there is a compactification of $N^*X(w)$ as an affine Schubert variety. It follows that $N^*X(w)$ is normal, Cohen-Macaulay, and Frobenius split.

1. Introduction

Let the base field $k$ be algebraically closed. Consider a cyclic quiver with $h$ vertices and dimension vector $\underline{d} = (d_1, \cdots, d_h)$. Let

$$\text{Rep}(\underline{d}, \hat{A}_h) = \text{Hom}(V_1, V_2) \times \cdots \times \text{Hom}(V_h, V_1), \quad GL_{\underline{d}} = \prod_{1 \leq i \leq h} GL(V_i)$$

We have a natural action of $GL_{\underline{d}}$ on $\text{Rep}(\underline{d}, \hat{A}_h)$: for $g = (g_1, \cdots, g_h) \in GL_{\underline{d}}$ and $f = (f_1, \cdots, f_h) \in \text{Rep}(\underline{d}, \hat{A}_h)$,

$$g \cdot f = (g_2 f_1 g_1^{-1}, g_3 f_2 g_2^{-1}, \cdots, g_1 f_h g_h^{-1})$$

Let

$$Z = \{(f_1, \cdots, f_h) \in Z | f_h \circ f_{h-1} \circ \cdots \circ f_1 : V_1 \to V_1 \text{ is nilpotent}\}$$

Clearly $Z$ is $GL_{\underline{d}}$-stable. Set $n = \sum d_i$, and let $\hat{SL}_n$ be the affine type Kac-Moody group with Dynkin diagram $\hat{A}_{n-1}$. Lusztig (cf. [Lu2]) has shown that an orbit closure in $Z$ is canonically isomorphic to an open subset of a Schubert variety in $\hat{SL}_n/Q$, where $Q$ is the parabolic subgroup of $\hat{SL}_n$ corresponding to omitting the simple roots $\alpha_0, \alpha_{d_1}, \alpha_{d_1+d_2}, \cdots, \alpha_{d_1+\cdots+d_{n-1}}$.

Let now $h = 2$ and consider the subvariety $Z_0$ of $Z$ given by

$$Z_0 = \{(f_1, f_2) \in \text{Hom}(V_1, V_2) \times \text{Hom}(V_2, V_1) | f_2 \circ f_1 = f_1 \circ f_2 = 0\}$$

with the dimension vector $(d_1, d_2)$. Strickland (cf. [S]) has shown that the irreducible components of $Z_0$ give the conormal varieties of the determinantal varieties in $\text{Hom}(V_1, V_2) = \text{Mat}_{d_2 \times d_1}(k)$, the set of $d_2 \times d_1$ matrices with entries in $k$. A determinantal variety in $\text{Hom}(V_1, V_2)$ being canonically isomorphic to an open subset in a certain Schubert variety in $Gr_{d_1, d_1+d_2}$ (the Grassmannian variety of $d_1$-dimensional subspaces of $k^{d_1+d_2}$) (cf. [LS]), the above two results of Lusztig and Strickland suggest a connection between conormal varieties to Schubert varieties in...
the (finite-dimensional) flag variety and affine Schubert varieties.

Let \( G = SL_n(k) \). We view the loop group \( LG = SL_n(k[t,t^{-1}]) \) as a Kac-Moody group of type \( \widehat{A}_{n-1} \). Let \( P \) be the parabolic subgroup of \( G \) corresponding to omitting the simple root \( \alpha_d \) for some \( 1 \leq d \leq n-1 \), and \( P \) the parabolic subgroup of \( LG \) corresponding to omitting the simple roots \( \alpha_0, \alpha_d \). Lakshmibai (La) has shown that the cotangent bundle \( T^*G/P \) is an open subset of a Schubert variety in \( LG/P \). Let \( w_0 \) be the longest element in the Weyl group \( W \) of \( G \). In [LS], we have shown that the conormal variety of the Schubert variety \( X_P(w) \) is an open subset of a Schubert variety in \( LG/P \) if and only if the Schubert variety \( X_P(w_0w) \) is smooth.

The approach adopted in [La, LS] seems to be quite successful in relating cotangent bundles and conormal varieties of classical Schubert varieties to affine Schubert varieties. In this paper, for any parabolic subgroup \( P \) of \( G \), we first construct an embedding of the cotangent bundle \( T^*G/P \) inside an affine Schubert variety \( X(\kappa) \) (Theorem 4.11). We then show that the conormal variety of a Schubert divisor in \( G/P \) is an open subset of some affine Schubert subvariety of \( X(\kappa) \) (Proposition 5.4).

In particular, we obtain that the conormal variety of a Schubert divisor is normal, Cohen-Macaulay, and Frobenius split (Corollary 5.5).

Let \( L^+G = G(k[t]) \) and let \( B \) be a Borel subgroup of \( LG \) contained in \( L^+G \). The action of \( G \) on \( V = k^n \) defines an action of \( LG \) on \( V[t,t^{-1}] \). The affine Grassmannian \( LG/L^+G \) is an ind-variety whose points are the lattices with virtual dimension 0 (see Section 3.6 for the definition of a lattice and its virtual dimension). The affine flag variety \( LG/B \) is an ind-variety whose points are affine flags \( L_0 \subset L_1 \subset \ldots \subset L_{n-1} \) satisfying the incidence relation \( tl_{n-1} \subset L_0 \).

Let \( N \) be the variety of \( n \times n \) nilpotent matrices on which \( G \) acts by conjugation. In [La1], Lusztig constructs a \( G \)-equivariant embedding \( \psi : N \hookrightarrow LG/L^+G \) which takes \( G \)-orbit closures in \( N \) to \( G \)-stable Schubert varieties in \( LG/L^+G \). Let \( \theta \) be the Springer resolution \( T^*G/B \to N \) given by \( (g,X) \mapsto gXg^{-1} \). In [LSS] Lakshmibai et al. construct a lift of \( \psi \), namely a map \( \eta \) embedding \( T^*G/B \) into the affine flag variety \( LG/B \), leading to the following commutative diagram:

\[
\begin{array}{ccc}
T^*G/B & \xrightarrow{\eta} & LG/B \\
\downarrow & & \downarrow pr \\
N & \xrightarrow{\psi} & LG/L^+G
\end{array}
\]

where \( pr \) is the projection map. Under the identifications of the previous paragraph, it takes the affine flag \( L_0 \subset \ldots \subset L_{n-1} \) to the lattice \( L_0 \). In this paper, for any parabolic subgroup \( P \) of \( G \), we define a generalization \( \phi_P \), as described below, of the map \( \eta \), and then study the conormal variety of a Schubert divisor by identifying its image under \( \phi_P \).

Let \( P \) be a parabolic subgroup of \( G \) corresponding to some subset \( S_P \subset S_0 \). We denote by \( P \) he parabolic subgroup of \( LG \) corresponding to \( S_P \subset S = S_0 \cup \{ \alpha_0 \} \). Let \( U \) be the Lie algebra of the unipotent radical of \( P \). Using the identification \( T^*G/P = G \times_P U \) (see Section 3.6), we define in Section 3.11 the map
\( \phi_P : T^*G/P \to G/P \) by \( \phi_P(g,X) = g(1-t^{-1}X) \mod P \), which sits in the following commutative diagram (see Section 3.13):

\[
\begin{array}{ccc}
T^*G/P & \xleftarrow{\phi_P} & LG/P \\
\downarrow{\theta_P} & & \downarrow{\text{pr}} \\
N_\kappa & \xrightarrow{\psi} & LG/L^+G \\
\end{array}
\]

The variety \( N_\kappa \) is the closure of a \( G \)-orbit in \( N \). The map \( \theta_P : T^*G/P \to N_\kappa \), given by \( (g,X) \mapsto gXg^{-1} \), is a resolution of \( N_\kappa \) (cf. [FM]), and pr is the quotient map. The closure of \( \text{Im}(\phi_P) \) in \( X_P(\kappa) \) is a \( G \)-stable compactification of \( T^*G/P \). We also identify the \( \kappa \in \widehat{W} \) (see Section 4.5 and Theorem 4.11) for which \( \text{Im} \phi_P \subset X_P(\kappa) \), and further, \( \kappa \) is minimal for this property. Finally, given a Schubert divisor \( X_P(w) \subset G/P \), we identify a Schubert variety \( X_P(v) \) which is a compactification of the conormal variety \( N^*X_P(w) \) (see Proposition 5.4).

When \( P \) is maximal, \( \phi_P \) is the same as the map in [La], and gives a dense embedding of \( T^*G/P \), i.e., \( X_P(\kappa) \) is a compactification of \( T^*G/P \). Mirković and Vybornov [MV] have constructed another lift of Lusztig’s embedding along the Springer resolution, which we briefly discuss in Section 3.14. Their lift is in general different, but agrees with \( \phi_P \) exactly when \( P \) is maximal. The reader is cautioned that Mirković and Vybornov work with a different choice of Lusztig’s map given by \( X \mapsto (1-t^{-1}X)^{-1}(\mod L^+G) \). This difference is simply a matter of convention.

The paper is organized as follows. In \( \S 2 \), we discuss some generalities on loop groups, and the associated root system and Weyl group. In \( \S 3 \), we define the embedding \( \phi_P : T^*G/P \to G/P \), relate it to Lusztig’s embedding \( \psi : N \to G/L^+G \) via the Springer resolution, and discuss the relationship between \( \phi_P \) and the map \( \psi \) of [MV]. In \( \S 4 \), we identify the minimal \( \kappa \in \widehat{W} \) for which \( \phi_P(T^*G/P) \subset X_P(\kappa) \). In particular, \( \phi_P \) identifies a \( G \)-stable subvariety of \( X_P(\kappa) \) as a compactification of \( T^*G/P \). We also compute the dimension of \( X_P(\kappa) \). This allows us to recover the result of [La] and further show that \( X_P(\kappa) \) is a compactification of \( T^*G/P \) if and only if \( P \) is a maximal parabolic subgroup. Finally, in \( \S 6 \), we apply \( \phi_P \) to the conormal variety \( N^*X_P(w) \) of a Schubert divisor \( X_P(w) \) in \( G/P \) to show that \( N^*X_P(w) \) can be embedded as an open subset of an affine Schubert variety.

2. The Loop Group

Let \( k \) be an algebraically closed field. In this section, we discuss some standard results on the root system, Weyl groups, and the Bruhat decomposition of the loop group \( LG = SL_n(k[t,t^{-1}]) \). For further details, the reader may refer to [Kai] [Ku] [R].

2.1. The Loop Group and its Dynkin Diagram. Let \( G \) be the special linear group \( SL_n(k) \). The subgroup \( B \) (resp. \( B^- \)) of upper (resp. lower) triangular matrices in \( G \) is a Borel subgroup of \( G \), and the subgroup \( T \) of diagonal matrices is a maximal torus in \( G \). The root system of \( G \) with respect to \((B,T)\) has Dynkin diagram \( A_{n-1} \). We use the standard labeling of simple roots \( S_0 \)

\[
\alpha_1 \quad \alpha_2 \quad \ldots \quad \alpha_{n-2} \quad \alpha_{n-1}
\]
Let $\Delta_0$ be the set of roots and $\Delta_+^+$ the set of positive roots. Let $L^+G = SL_n \left( k \left[ t^{\pm 1} \right] \right)$ and $\pi_\pm : L^+G \to G$ the surjective map given by $t^{\pm 1} \mapsto 0$. Then $B^\pm := \pi_i^{-1}(B)$ are opposite Borel subgroups of $LG$. The loop group $LG$ is a (minimal) Kac-Moody group (cf. [R]) with torus $T$ and Dynkin diagram $\hat{A}_{n-1}$.

\[ \begin{array}{c}
\alpha_0 \\
\alpha_1 \\
\vdots \\
\alpha_{n-2} \\
\alpha_{n-1}
\end{array} \]

We denote the Borel subgroup $B^+$ as just $B$. Parabolic subgroups containing $B$ (resp. $B$) in $G$ (resp. $LG$) correspond to subsets of $S_0$ (resp. $S$). For $P$ the parabolic subgroup in $G$ corresponding to $S_P \subset S_0$, the parabolic subgroup $P \subset LG$ corresponding to $S_P \subset S$ is given by $P = \pi^{-1}(P)$. In particular, the subgroup $L^+G$ is the parabolic subgroup corresponding to $S_0 \subset S$.

2.2. The Root system of $A_{n-1}$. Consider the vector space of $n \times n$ diagonal matrices, with basis $\{E_{i,i} | 1 \leq i \leq n\}$. Writing $\{\epsilon_i | 1 \leq i \leq n\}$ for the basis dual to $\{E_{i,i} | 1 \leq i \leq n\}$, we can identify the set of roots $\Delta_0$ of $G$ as follows:

$$\Delta_0 = \{ (\epsilon_i - \epsilon_j) | 1 \leq i \neq j \leq n \}$$

The positive root $\alpha_i + \ldots + \alpha_{j-1}$, where $i < j$, corresponds to $\epsilon_i - \epsilon_j$, and the negative root $-(\alpha_i + \ldots + \alpha_{j-1})$ corresponds to $\epsilon_j - \epsilon_i$. For convenience, we shall denote the root $\epsilon_i - \epsilon_j$ by $(i,j)$, where $1 \leq i \neq j \leq n$. The action of $W(\cong S_n)$ on $\Delta_0$ is given by $w(i,j) = (w(i), w(j))$.

2.3. The Root system of $\hat{A}_{n-1}$. Let $\Delta$ be the root system associated to the Dynkin diagram $\hat{A}_{n-1}$. We have

$$\Delta = \{n\delta + \alpha | n \in \mathbb{Z}, \alpha \in \Delta_0\} \bigcup \{ n\delta | n \in \mathbb{Z}, n \neq 0 \}$$

where $\delta = \alpha_0 + \theta$ is the basic imaginary root in $\Delta$, and $\theta = \alpha_1 + \ldots + \alpha_{n-1}$ is the highest root in $\Delta_0^+$. The set of positive roots $\Delta^+$ has the following description:

$$\Delta^+ = \{n\delta + \alpha | n > 0, \alpha \in \Delta_0\} \bigcup \Delta_0^+ \bigcup \{n\delta | n > 0 \}$$

Let $\sim$ be the equivalence relation on $\mathbb{Z} \times \mathbb{Z}$ given by

$$(i,j) \sim (i + kn, j + kn) \quad k \in \mathbb{Z}$$

We identify $\Delta$ with $\mathbb{Z} \times \mathbb{Z}/\sim$ as follows:

$$k\delta \mapsto (0,k)$$

$$(i,j) + k\delta \mapsto (i,j + kn) \quad \text{for } (i,j) \in \Delta_0$$

Under this identification, $(i,j) \in \Delta^+$ if and only if $i < j$.

2.5. The Weyl group. Let $N$ be the normalizer of $T$ in $G$. We identify the Weyl group $\hat{W}$ associated to $\hat{A}_{n-1}$ with $N(k[t,t^{-1}])/T$. Let $E_{i,j}$ be the $n \times n$ matrix with 1 in the $(i,j)$ position, and 0 elsewhere. The elements of $N(k[t,t^{-1}])$ are matrices of the form $\sum_{1 \leq i \leq n} \sum_{1 \leq \ell \leq n} t_{i,\ell} E_{\sigma(i),i}$, where

- $(t_{i,\ell})$ is a collection of non-zero monomials in $t$ and $t^{-1}$.
- $\sigma$ is a permutation of $\{1 \ldots n\}$.
Consider the homomorphism $N(\mathbf{k}[t,t^{-1}]) \rightarrow GL_n(\mathbf{k}[t,t^{-1}])$ given by

$$
\sum_{i=1}^{n} t_i E_{\sigma(i),i} \mapsto \sum_{i=1}^{n} t^{|\delta_i|} E_{\sigma(i),i}
$$

The kernel of this map is $T$, and so we can identify $\hat{W}$ with the group of $n \times n$ permutation matrices $M$, with each non-zero entry a power of $t$, and $\text{ord}(\det M) = 0$. For $w \in \hat{W}$, we call the matrix corresponding to $w$ the affine permutation matrix of $w$.

2.7. Generators for $\hat{W}$. We shall work with the set of generators for $\hat{W}$ given by $\{s_0,s_1,\ldots,s_{n-1}\}$, where $s_i, 0 \leq i \leq n-1$ are the reflections with respect to $\alpha_i, 0 \leq i \leq n-1$. Note that $\{\alpha_i | 1 \leq i \leq n-1\}$ being the set of simple roots of $G$, the Weyl group $W$ of $G$ is the subgroup of $\hat{W}$ generated by $s_1, \ldots, s_{n-1}$. The affine permutation matrix of $w \in W$ is $\sum E_{w(i),i}$. The affine permutation matrix of $s_0$ is given by

$$
\begin{pmatrix}
0 & 0 & \cdots & t^{-1} \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & 0 \\
t & 0 & 0 & 0
\end{pmatrix}
$$

For $1 \leq a < b \leq n$, the reflection with respect to the positive root $(a, b)$ is given by the affine permutation matrix

$$s_{(a,b)} = E_{a,b} + E_{b,a} + \sum_{1 \leq i \leq n, i \neq a,b} E_{i,i}
$$

2.8. Decomposition of $\hat{W}$ as a semi-direct product. Consider the element $s_\theta \in W$, reflection with respect to the root $\theta$. There exists (cf. [Ku], §13.1.6) a group isomorphism $\hat{W} \rightarrow W \ltimes Q$ given by

$$s_i \mapsto (s_i,0) \quad \text{for } 1 \leq i \leq n-1$$

$$s_0 \mapsto (s_\theta, -\theta^\vee)$$

where $Q$ is the coroot lattice and $\theta^\vee = \alpha_1^\vee + \cdots + \alpha_{n-1}^\vee$. The simple coroot $\alpha_i^\vee \in \mathfrak{h}, 1 \leq i \leq n-1$, is given by the matrix $E_{i,i} - E_{i+1,i+1}$. As shown in [LSS], Section 2.4, a lift to $N(\mathbf{k}[t,t^{-1}])$ of $(\text{id}, \alpha_i^\vee) \in \hat{W}$, for $1 \leq i \leq n-1$ is given by

$$\tau_{\alpha_i^\vee} = \sum_{k \neq i,i+1} E_{k,k} + t^{-1} E_{i,i} + t E_{i,i}
$$

For $q \in Q$, we will write $\tau_q$ for the image of $(\text{id}, q) \in W \ltimes Q$ in $\hat{W}$. Observe that for $\alpha \in \Delta_0, q \in Q$ such that $\alpha = (a,b)$ and $\tau_q = \sum t_i E_{ii}$, we have

$$\alpha(q) = \text{ord}(t_b) - \text{ord}(t_a)
$$

The action of $\tau_q$ on $\Delta$ is given by $\tau_q(\delta) = \delta$ and

$$\tau_q(\alpha) = \alpha - \alpha(q)\delta \quad \text{for } \alpha \in \Delta_0
$$

In particular, for $\alpha \in \Delta_0^+$, $\tau_q(\alpha) > 0$ if and only if $\alpha(q) \leq 0$. 

\[\bullet \text{ det } \left( \sum_{i=1}^{n} t_i E_{\sigma(i),i} \right) = 1.\]
Corollary 2.11. For \( \alpha \in \Delta^+ \), \( \tau_q s_\alpha > \tau_q \) if and only if \( \alpha(q) \leq 0 \) and \( s_\alpha \tau_q > \tau_q \) if and only if \( \alpha(q) \geq 0 \).

Proof. Follows from the equivalences (cf. [Ku]) \( ws_\alpha > w \iff w(\alpha) > 0 \) and \( s_\alpha w > w \iff w^{-1}(\alpha) > 0 \) applied to \( w = \tau_q \). \( \square \)

2.12. The Coxeter Length. Let \( w \in \hat{W} \) be given by the affine permutation matrix \( w = t^c E_{\sigma(i),i} \), i.e. \( w(i) = \sigma(i) - c_i n \). We have the formula:

\[
(2.13) \quad l(w) = \sum_{1 \leq i < j \leq n} |c_i - c_j - f_\sigma(i,j)|
\]

where \( f_\sigma(i,j) = \begin{cases} 0 & \text{if } \sigma(i) < \sigma(j) \\ 1 & \text{otherwise} \end{cases} \).

Proof. We use Equation (2.4) to write \( \Delta^+ = \{(i,j) | 1 \leq i \leq n, i < j \} \). We know from [Ku] that \( l(w) = \# \{ \alpha \in \Delta^+ \mid w(\alpha) < 0 \} \)

\[
= \# \{(i,j) \mid 1 \leq i \leq n, i < j, w(i) > w(j)\}
\]

\[
= \sum_{1 \leq i < n} \sum_{1 \leq j < n} \# \{(i,j') \mid j' = j \text{ mod } n, i < j', w(i) > w(j')\}
\]

\[
= \sum_{1 \leq i < j \leq n} \# \{(i,j + kn) \mid k \geq 0, w(i) > w(j + kn)\} + \# \{(j,i + kn) \mid k \geq 1, w(j) > w(i + kn)\}
\]

\[
= \sum_{1 \leq i < j \leq n} \# \left\{ k \mid 0 \leq k < \frac{\sigma(i) - \sigma(j)}{n} + c_j - c_i \right\} + \# \left\{ k \mid 1 \leq k < \frac{\sigma(j) - \sigma(i)}{n} - c_j + c_i \right\}
\]

\[
= \sum_{1 \leq i < j \leq n} |c_i - c_j - f_\sigma(i,j)|
\]

\( \square \)

2.14. The Bruhat Decomposition. The Bruhat decomposition of \( LG \) is given by \( LG = \bigsqcup_{w \in \hat{W}} Bw \mathcal{B} \). For \( w \in \hat{W} \), let \( X_B(w) \subset LG/\mathcal{B} \) be the affine Schubert variety:

\[
X_B(w) = \overline{Bw \mathcal{B}}(\text{mod } \mathcal{B}) = \bigsqcup_{v \leq w} Bv \mathcal{B}(\text{mod } \mathcal{B})
\]

The Bruhat order \( \leq \) on \( \hat{W} \) reflects inclusion of Schubert varieties, i.e., \( v \leq w \) if and only if \( X_B(v) \subseteq X_B(w) \).

Let \( \hat{W}^P \subset \hat{W} \) be the set of minimal (in the Bruhat order) representatives of \( \hat{W}/\hat{W}_P \). The Bruhat decomposition with respect to \( \mathcal{P} \) is given by

\[
LG = \bigsqcup_{w \in \hat{W}^P} Bw \mathcal{P}
\]

For \( w \in \hat{W}^P \), the affine Schubert variety \( X_P(w) \subset LG/\mathcal{P} \) is given by

\[
X_P(w) = \overline{Bw \mathcal{P}}(\text{mod } \mathcal{P}) = \bigsqcup_{v \leq w} Bv \mathcal{P}(\text{mod } \mathcal{P})
\]

For \( w \in \hat{W}^P \), \( X_P(w) \) is a normal (cf. [Fa]), projective variety of dimension \( l(w) \), the length of \( w \).
Suppose \( w \in \hat{W} \) is given by the affine permutation matrix

\[
\sum t_i E_{\sigma(i),i} = \sum E_{\sigma(i),i} \sum t_i E_{i,i} = \sigma q
\]

with \( \sigma \in S_n \) and \( \tau_q = \sum t_i E_{i,i} \in Q \). Let \( 1 \leq a < b \leq n \) and set \( s_r = s_{(a,b)} \), \( s_t = \sigma s_r \sigma^{-1} = s_{(\sigma(a),\sigma(b))} \).

**Case 1:** Suppose \( \text{ord}(t_a) = \text{ord}(t_b) \). Then \( s_t w = ws_r \). The set \( \{w, s_t w\} \) has a unique minimal element \( u \), given by

\[
\begin{cases}
  w & \text{if } \sigma(a) < \sigma(b) \\
  s_t w & \text{if } \sigma(a) > \sigma(b)
\end{cases}
\]

**Case 2:** Suppose \( \text{ord}(t_a) \neq \text{ord}(t_b) \). Then \( s_t w \neq ws_r \), and the set \( \{w, s_t w, ws_r, s_t ws_r\} \) has a unique minimal element \( u \), given by

\[
\begin{cases}
  s_t ws_r & \text{if } \text{ord}(t_a) < \text{ord}(t_b) \\
  w & \text{if } \text{ord}(t_a) > \text{ord}(t_b)
\end{cases}
\]

Further, we have \( u < s_t u < s_t us_r \) and \( u < us_r < s_t us_r \).

**Proof.** Recall from Equation (2.9) that \( q(a,b) = \text{ord}(t_b) - \text{ord}(t_a) \). Suppose first that \( \text{ord}(t_a) = \text{ord}(t_b) \), which from Equation (2.10) is equivalent to \( \tau_q(a,b) = (a,b) \).

It follows that

\[
ws_r = \sigma \tau_q s_{(a,b)} = \sigma s_{(a,b)} \tau_q = s_{(\sigma(a),\sigma(b))} \tau_q = s_t w
\]

Recall that for \( \alpha \in \Delta_0^+ \), we have \( w s_\alpha > w \) if and only if \( w(a) > 0 \). Now, \( w(a,b) = \sigma \tau_q(a,b) = \sigma(a,b) = (\sigma(a),\sigma(b)) \) being positive if and only if \( \sigma(a) < \sigma(b) \), it follows that \( w < ws_r \) if and only if \( \sigma(a) < \sigma(b) \).

Suppose now that \( \text{ord}(t_a) \neq \text{ord}(t_b) \), and set \( k = \text{ord}(t_a) - \text{ord}(t_b) \). Then

\[
w(a,b) = \tau_q(a,b) = (\sigma(a),\sigma(b)) + k \delta
\]

\[
(w^{-1}(a,b),\sigma(b)) = \tau_q^{-1}(a,b) = (a,b) + k \delta
\]

It follows that \( w(a,b) \) and \( w^{-1}(a,b),\sigma(b) \) are positive if \( k > 0 \) and negative if \( k < 0 \). Accordingly, we have \( w < ws_r, w < s_t w \) if \( \text{ord}(t_a) > \text{ord}(t_b) \), and \( w > ws_r, w > s_t w \) if \( \text{ord}(t_a) < \text{ord}(t_b) \). Similarly, it follows from

\[
s_t w(a,b) = s_t \tau_q(a,b) = s_t(\sigma(a),\sigma(b)) + k \delta = (\sigma(b),\sigma(a)) + k \delta
\]

that \( s_t w < s_t ws_r \) if \( \text{ord}(t_a) > \text{ord}(t_b) \) and \( s_t w > s_t ws_r \) if \( \text{ord}(t_a) < \text{ord}(t_b) \). Finally,

\[
(ws_r)^{-1}(a,b) = s_r \tau_q^{-1}(a,b) = s_r \tau_q^{-1}(\sigma(a),\sigma(b)) = s_r \tau_q^{-1}(\sigma(a),\sigma(b)) = s_r(a,b) - k \delta = (b,a) - k \delta
\]

implies \( s_t ws_r < ws_r \) if \( \text{ord}(t_a) < \text{ord}(t_b) \) and \( s_t ws_r > ws_r \) if \( \text{ord}(t_a) > \text{ord}(t_b) \). \( \square \)

### 3. Lusztig’s Embedding and the Springer Resolution

In this section, we define the embedding \( \phi_P : T^* G/P \to LG/P \) and relate it to Lusztig’s embedding \( \psi : N \to LG/L^+ G \) via the Springer resolution. We also discuss Mirković and Vybornov’s \( (MV) \) compactification of \( T^* G/P \) and show how it relates to the map \( \phi_P \).

Let \( V \) be the vector space \( \mathbb{k}^n \) with standard basis \( \{e_i \mid 1 \leq i \leq n\} \), and let the group \( G = SL_n(\mathbb{k}) \) act on \( V \) in the usual way. Fix a sequence \( 0 = d_0 < d_1 < \ldots < d_{r-1} < d_r = n \), and let \( P \supset B \) be the parabolic subgroup corresponding to the set of roots \( S_P = S_0 \setminus \{\alpha_{d_i} \mid 1 \leq i < r\} \) in \( S_0 \). Let \( \lambda_i = d_i - d_{i-1} \) and \( \Lambda = (\lambda_1, \ldots, \lambda_r) \).

We denote by \( P \) the parabolic subgroup \( B \subset P \subset LG \) corresponding to \( S_P \subset S \).
3.1. **Partitions.** A partition \( \mu \) of \( n \) is a decreasing sequence \((\mu_1, \mu_2, \ldots, \mu_r)\) of positive integers such that \( \sum \mu_i = n \). Given \( \mu = (\mu_1, \ldots, \mu_r) \), we sometimes view \( \mu \) as an infinite non-negative sequence by defining \( \mu_i = 0 \) for \( i > r \). Let \( \text{Par} \) denote the set of partitions of \( n \). The dominance order \( \preceq \) on \( \text{Par} \) is given by

\[
\mu \preceq \nu \iff \sum_{j \leq i} \mu_j \leq \sum_{j \leq i} \nu_j \quad \forall i \in \mathbb{Z}
\]

3.2. **The Nilpotent Cone.** Let \( \mathcal{N} \) be the variety of nilpotent \( n \times n \) matrices, and let \( G \) act on \( \mathcal{N} \) by conjugation. The variety \( \mathcal{N} \) has the \( G \)-orbit decomposition \( \mathcal{N} = \bigsqcup_{\nu \in \text{Par}} \mathcal{N}_\nu^0 \), where \( \mathcal{N}_\nu^0 \) is the set of nilpotent matrices of Jordan type \( \nu \). We write \( \mathcal{N}_\nu \) for the closure (in \( \mathcal{N} \)) of \( \mathcal{N}_\nu^0 \). Then \( \mathcal{N}_\mu \subset \mathcal{N}_\nu \) if and only if \( \mu \preceq \nu \).

3.3. **Lusztig’s Embedding.** Let \( \nu = (\nu_1, \ldots, \nu_s) \) be the partition of \( n \) which is conjugate to the partition of \( n \) obtained from \( \lambda \) by rearranging the \( \lambda_i \) in decreasing order. In particular, \( r = \nu_1 \) and \( s = \max \{ \lambda_i | 1 \leq i \leq r \} \). Consider the element \( \tau_\nu \in \tilde{W} \) given by the affine permutation matrix

\[
\tau_\nu = \sum_{i=1}^s \nu_i^{-1} E_{i,i} + \sum_{i=s+1}^n t^{-1} E_{i,i}
\]

There exists a \( G \)-equivariant injective map \( \psi : \mathcal{N}_\nu \hookrightarrow X_{L^+G}(\tau_\nu) \) given by

\[
\psi(X) = (1 - t^{-1}X) \pmod{L^+G} \quad \text{for } X \in \mathcal{N}_\nu
\]

The map \( \psi|_{\mathcal{N}_\nu} \) is an open immersion onto the opposite Bruhat cell

\[
Y_{L^+G}(\tau_\nu) := \mathcal{B}^-/L^+G \cap X_{L^+G}(\tau_\nu)
\]

where \( \mathcal{B}^-/L^+G \) denotes the image of \( \mathcal{B}^- \) under the map \( LG \rightarrow LG/L^+G \). This statement is well-known. A proof of can be found in §4.1 of [AH]. A variant of the map \( \psi \) was first introduced in [LmI].

3.6. **The Cotangent Bundle.** We identify the points of the variety \( G/P \) with the partial flags in \( V(= k^n) \) of shape \( \lambda \):

\[
G/P \cong \{ (F_0 \subset F_1 \subset \ldots \subset F_{r-1} \subset F_r) | \dim F_i/F_{i-1} = \lambda_i \}
\]

Using the Killing form on \( \mathfrak{sl}_n \), we can identify the cotangent space at identity \( T^*G/P \) with the Lie algebra \( \mathfrak{u} \) of the unipotent radical \( U_P \) of \( P \). Then \( T^*G/P \) is the fiber bundle over \( G/P \) associated to the principal \( P \)-bundle \( G \to G/P \), for the adjoint action of \( P \) on \( \mathfrak{u} \):

\[
T^*G/P = G \times^P \mathfrak{u} = G \times \mathfrak{u}/\sim
\]

The equivalence relation \( \sim \) is given by \( (g,Y) \sim (gp,p^{-1}Yp) \), where \( g \in G, Y \in \mathfrak{u}, p \in P \). We also have the identification

\[
T^*G/P = \{ (X,F_0 \subset \ldots \subset F_k) | X \in \mathcal{N}, \dim F_i/F_{i-1} = \lambda_i, X(F_i) \subset F_{i-1} \}
\]

The two identifications are related via the isomorphism

\[
(g,X) \mapsto (gXg^{-1}, gV_0 \subset \ldots \subset gV_k)
\]

where \( (V_0 \subset \ldots \subset V_r) \) is the flag of shape \( \lambda \) fixed by \( P \).
3.8. The Affine Grassmannian. A lattice in \( L \subset V[t, t^{-1}] \) is a \( k[t] \) module satisfying \( k[t, t^{-1}] \otimes L = V[t, t^{-1}] \). The virtual dimension of \( L \) is defined as
\[
\text{vdim}(L) := \dim_k(L/L \cap E) - \dim_k(E/L \cap E)
\]
where \( E \) is the standard lattice, namely the \( k[t] \) span of \( V \). The quotient \( LG/L^+G \) is an ind-variety whose points are identified with the lattices of virtual dimension 0 (cf. [Fa]).
\[
LG/L^+G = \{ L \text{ a lattice} \mid \text{vdim}(L) = 0 \}
\]

3.9. Affine Flag Varieties. A partial affine flag of shape \( \lambda \) is a sequence of lattices \( L_0 \subset L_1 \subset \ldots \subset L_r \) satisfying \( tL_r = L_0 \) and \( \dim L_i/L_{i-1} = \lambda_i \) for \( 1 \leq i \leq r \). For \( P \) and \( \mathcal{P} \) as above, we can identify the points of the ind-variety \( LG/ \mathcal{P} \) with partial affine flags of shape \( \lambda \) satisfying the additional condition \( \text{vdim}(L_0) = 0 \) (cf. [Fa]).
\[
(3.10) \quad LG/ \mathcal{P} \cong \{ (L_0 \subset \ldots \subset L_r) \mid \dim L_i/L_{i-1} = \lambda_i, tL_r = L_0, \text{vdim}(L_0) = 0 \}
\]
The ind-variety \( LG/ \mathcal{P} \) is called the affine flag variety associated to \( \mathcal{P} \).

3.11. The map \( \phi_P \): Let \( \phi_P : G \times \mathcal{P} \rightarrow LG/ \mathcal{P} \) be defined by
\[
\phi_P(g, X) = g(1 - t^{-1}X)(\mod \mathcal{P}) \quad g \in G, \ X \in \mathcal{P}
\]
For \( g \in G, p \in P \) and \( X \in \mathcal{P} \), we have
\[
\phi_P(gp, p^{-1}Xp) = gp(1 - t^{-1}p^{-1}Xp)(\mod \mathcal{P})
\]
\[
= g(p - t^{-1}Xp)(\mod \mathcal{P})
\]
\[
= g(1 - t^{-1}X)(\mod \mathcal{P})
\]
\[
= g(1, X)
\]
It follows that \( \phi_P \) is well-defined and \( G \)-equivariant. Under the identifications of Equations (3.7) and (3.10), we have
\[
\phi_P(X, F_0 \subset F_1 \subset \ldots \subset F_r) = L_0 \subset L_1 \subset \ldots \subset L_r
\]
where \( L_i = (1 - t^{-1}X)(V[t] \oplus t^{-1}F_i) \).

Lemma 3.12. The map \( \phi_P \) is injective.

Proof. Suppose \( \phi_P(g, Y) = \phi_P(g_1, Y_1) \), i.e.,
\[
g(1 - t^{-1}Y) = g_1(1 - t^{-1}Y_1)(\mod \mathcal{P})
\]
\[
\Rightarrow g(1 - t^{-1}Y) = g_1(1 - t^{-1}Y_1) \ x
\]
for some \( x \in \mathcal{P} \). Denoting \( h = g_1^{-1}g \) and \( Y' = hYh^{-1} \), we have
\[
h(1 - t^{-1}Y) = (1 - t^{-1}Y_1) \ x
\]
\[
\Rightarrow x = (1 - t^{-1}Y_1)^{-1}h(1 - t^{-1}Y)
\]
\[
= (1 - t^{-1}Y_1)^{-1}(1 - t^{-1}Y')h
\]
\[
\Rightarrow xh^{-1} = (1 + t^{-1}Y_1 + t^{-2}Y_2 + \cdots)(1 - t^{-1}Y')
\]
Now since \( x \in \mathcal{P} \), \( h \in G \), the left hand side is integral, i.e., does not involve negative powers of \( t \). Hence both sides must equal identity. It follows \( x = h \in P = \mathcal{P} \cap G \) and \( Y_1 = Y' = hYh^{-1} \). In particular, \( (g_1, Y_1) = (gh^{-1}, hYh^{-1}) \sim (g, Y) \) as required. \( \square \)
3.13. The Springer Resolution. Let \( \nu \) be the partition of \( n \) which is conjugate to the partition of \( n \) obtained from \( \lambda \) by rearranging the \( \lambda_i \) in decreasing order. The Springer map \( \theta : T^*G/P \rightarrow N \) given by \( \theta(g,X) = gXg^{-1} \) where \( g \in G, X \in \mathfrak{n} \) is a resolution of singularities for the \( G \)-orbit \( \mathcal{N}_\nu \subset \mathcal{N} \). The maps \( \phi_P \) and \( \psi \) from Section 3.11 and eq. (3.5) sit in the following commutative diagram:

\[
\begin{array}{ccc}
T^*G/P & \xleftarrow{\phi_P} & LG/P \\
\downarrow{\theta_P} & & \downarrow{pr} \\
\mathcal{N}_\nu & \xrightarrow{\psi} & LG/L^+G
\end{array}
\]

where \( pr : LG/P \rightarrow LG/L^+G \) is the natural projection. We present a more precise version of this statement in Section 4.5.

3.14. The Mirković-Vybornov Compactification. Consider the convolution Grassmannian \( \tilde{G}^{\lambda} \) whose points are identified with certain lattice flags:

\[
\tilde{G}^{\lambda} = \{ L_0 \subset L_1 \subset \ldots \subset L_r \mid L_r = t^{-1}V[t], \dim L_i/L_{i-1} = \lambda_i, tL_i \subset L_{i-1} \}
\]

In [MV], Mirković and Vybornov construct an embedding \( \tilde{\psi} : T^*G/P \hookrightarrow \tilde{G}^{\lambda} \) given by

\[
\tilde{\psi}(X,F_0 \subset \ldots \subset F_i) = L_0 \subset \ldots \subset L_r
\]

where \( L_i = (1-t^{-1}X)V[t] \oplus t^{-1}F_i \). Once again, we have a commutative diagram

\[
\begin{array}{ccc}
T^*G/P & \xleftarrow{\tilde{\psi}} & \tilde{G}^{\lambda} \\
\downarrow{\theta_P} & & \downarrow{pr} \\
\mathcal{N}_\nu & \xrightarrow{\psi} & \tilde{G}^{\lambda}_0(n)
\end{array}
\]

where the map \( pr : \tilde{G}^{\lambda} \rightarrow G^{\lambda} \) is given by \( (L_0,L_1,\ldots,L_r) \mapsto L_0 \). As we can see, the incidence relations \( tL_i \subset L_{i-1} \) are in general different from the incidence relations \( tL_r = L_0 \) of the partial affine flag variety; accordingly, \( \tilde{G}^{\lambda} \) is different from \( LG/P \). However, when \( P \) is maximal, i.e., \( \lambda = (d,n-d) \) for some \( d \), we have an isomorphism \( \beta : \tilde{G}^{\lambda} \simto LG/P \) given by

\[
(L_0 \subset L_1 \subset t^{-1}V[t]) \mapsto (L_0 \subset L_1 \subset L_2)
\]

where \( L_2 = t^{-1}L_0 \). In this case, one can verify that \( \beta \circ \tilde{\psi} = \phi_P \).

4. The element \( \kappa \)

Let \( \lambda = (\lambda_1,\ldots,\lambda_r), P, \) and \( \mathcal{P} \) be as in the previous section. Further, let \( \nu = (\nu_1,\ldots,\nu_s) \) be the partition of \( n \) which is conjugate to the partition of \( n \) obtained from \( \lambda \) by rearranging the \( \lambda_i \) in decreasing order.

In this section, we describe the element \( \kappa \in \widehat{W}^\mathcal{P} \) for which \( \phi_P(T^*G/P) \subset X_{\mathcal{P}}(\kappa) \); further, \( \kappa \) is minimal for this property. We compute \( \dim X_{\mathcal{P}}(\kappa) = l(\kappa) \) and show that when \( P \) is maximal, \( X_{\mathcal{P}}(\kappa) \) is a compactification of \( T^*G/P \).
4.1. **Tableaux.** We draw a left-aligned tableau with \( r \) rows, with the \( i^{th} \) row from top having \( \lambda_i \) boxes. Fill the boxes of the tableau as follows: the entries of the \( i^{th} \) row are the integers \( k \) satisfying \( d_i < k \leq d_{i+1} \), written in increasing order. We denote by \( \text{Row}(i) \) the set of entries in the \( i^{th} \) row of the tableau. Observe that the number of boxes in the \( i^{th} \) column from the left is \( \nu_i \). The Weyl group \( W_P \) is the set of elements in \( S_n \) that preserve the partition \( \{1, \ldots, n\} = \bigcup_i \text{Row}(i) \).

We define a co-ordinate system \( \chi(\bullet, \bullet) \) on \( \{1, \ldots, n\} \) as follows: For \( 1 \leq i \leq r \), \( 1 \leq j \leq \nu_i \), let \( \chi(j, i) \) denote the \( j^{th} \) entry (from the top) of the \( i^{th} \) column. Note that \( \chi(j, i) \) need not be in \( \text{Row}(j) \).

Finally, let \( F_{j,k}^i \) denote the elementary matrix \( E_{\chi(j,i), \chi(k,i)} \). Observe that \( \chi(b, i) = \chi(c, j) \) if and only if \( i = j \) and \( b = c \). In particular,

\[
F_{a,b}^i F_{c,d}^j = \delta_{ij} \delta_{bc} F_{a,d}^i
\]

4.3. **Red and Blue.** We split the set \( \{1, \ldots, n\} \) into disjoint subsets \( \text{Red} \) and \( \text{Blue} \), depending on their positions in the tableau. Let \( \mathcal{S}_i = \{\chi(1, i) | 1 \leq i \leq s\} \) be the set of entries which are topmost in their column. We write \( \mathcal{S}_1(i) = \mathcal{S}_1 \cap \text{Row}(i) \). For convenience, we also define \( \mathcal{S}_2(i) = \text{Row}(i) \setminus \mathcal{S}_1(i) \) and \( \mathcal{S}_2 = \bigcup_i \mathcal{S}_2(i) \).

The set \( \text{Red}(i) \) is the collection of the \#\( \mathcal{S}_1(i) \) smallest entries in \( \text{Row}(i) \):

\[
\text{Red}(i) := \{j | d_{i-1} < j \leq d_i - \max \{\lambda_j | j < i\}\}
\]

We set \( \text{Blue}(i) = \text{Row}(i) \setminus \text{Red}(i) \), \( \text{Red} = \bigcup_i \text{Red}(i) \), and \( \text{Blue} = \bigcup_i \text{Blue}(i) \). The elements of \( \text{Red}(i) \) are smaller than the elements of \( \text{Blue}(i) \).

The elements of \( \text{Red} \), arranged in increasing order are written \( l(1), \ldots, l(s) \). We enumerate the elements of \( \text{Blue} \), written row by row from bottom to top, each row written left to right, as \( m(1), \ldots, m(n-s) \).

**Example 4.4.** Let \( n = 17 \) and the sequence \( (d_i) \) be \((1, 5, 9, 11, 17)\). The corresponding tableau is

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 \\
6 & 7 & 8 & 9 \\
10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\
\end{array}
\]

- \( r = 5 \), \( s = 6 \).
- The sequence \( (\lambda_i) \) is \((1, 4, 4, 2, 6)\).
- The sequence \( (\nu_i) \) is \((5, 4, 3, 3, 1, 1)\).
- \( \text{Row}(3) = \{6, 7, 8, 9\} \).
- \( \chi(4, 1) = 10 \), \( \chi(3, 4) = 15 \), \( \chi(1, 6) = 17 \) etc.
- \( E_{2,10}^3 = E_{2,10} \), \( E_{3,15}^3 = E_{3,15} \) etc.
- \( \mathcal{S}_1 = \{1, 3, 4, 5, 16, 17\} \).
- The sequence \( l(i) \) is \((1, 2, 3, 4, 12, 13)\).
- The sequence \( m(i) \) is \((14, 15, 16, 17, 10, 11, 6, 7, 8, 9, 5)\).
4.5. **The element** \( \kappa \). Let \( \kappa \in \hat{W} \) be given by the affine permutation matrix

\[
\sum_{i=1}^{s} t^{\nu_i} E_{i,l(i)} + \sum_{i=1}^{n-s} t^{-1} E_{i+s,m(i)}
\]

The commutative diagram of Section 3.13 can be refined to the following:

\[
\begin{array}{ccc}
T^* G/P & \xrightarrow{\phi_P} & X_P(\kappa) \\
\downarrow{\phi_P} & & \downarrow{\psi} \\
N_\nu & \xrightarrow{\psi} & X_{L+G}(\tau_q)
\end{array}
\]

The only additional part is the claim \( \phi_P(T^* G/P) \subset X_P(\kappa) \). This is the content of Theorem 4.11.

4.6. **The Matrix** \( Z \). Let \( Z \) be the \( n \times n \) matrix given by

\[
Z = \sum_{i=1}^{s} \nu_i \sum_{j=1}^{n} F_{i,j+1}
\]

Then

\[
Ze_{\chi(j,i)} = \begin{cases} e_{\chi(j-1,i)} & \text{if } j > 1 \\ 0 & \text{if } j = 1 \end{cases}
\]

It follows that \( \{ e_{\chi(\nu_i,i)} \mid 1 \leq i \leq s \} \) is a minimal generating set for \( V(= k^n) \) as a module over \( k[Z] \) (the \( k \)-algebra generated by \( Z \)). In particular, \( Z(V_i) \subset V_{i-1} \) for all \( i \). It follows that \( Z \in u \), where \( u \) is as in Section 3.6.

An element in \( x \in u \) is called a **Richardson element** of \( u \) if the \( P \)-orbit of \( x \) is dense in \( u \), or equivalently, the \( G \)-orbit of \( (1,Z) \) is dense in \( T^* G/P = G \times P u \). A comprehensive study of Richardson elements in \( u \) can be found in [H]. In particular, we will need the following:

**Lemma 4.7** (Theorem 3.3 of [H]). The matrix \( Z \) is a Richardson element in \( u \).

**Proposition 4.8.** Let \( \tilde{BwB} \) be the Bruhat cell containing \( 1 - t^{-1} Z \). A lift of \( \bar{w} \) to \( N(k[t,t^{-1}]) \) is given by

\[
\bar{\omega} = \sum_{i=1}^{s} \left( t^{\nu_i} F_{i,1} - \sum_{j=2}^{\nu_i} t^{-1} F_{j-1,j} \right)
\]

**Proof.** For \( 1 \leq i \leq s \), let

\[
\begin{align*}
& b_i := \sum_{j=1}^{\nu_i} \sum_{k=j}^{\nu_i} t^{k-j} F_{i,j} = \sum_{j=1}^{\nu_i} \sum_{k=0}^{\nu_i-j} t^{k} F_{i,j} + c_i \\
& c_i := \sum_{j=1}^{\nu_i} F_{j,j} + \sum_{j=2}^{\nu_i} t^{j-1} F_{j-1,1} \\
& Z_i := \sum_{j=1}^{\nu_i} F_{i,j} - \sum_{j=1}^{\nu_i-1} t F_{i,j+1}
\end{align*}
\]
We compute

\[ b_i Z_i c_i = \left( \sum_{j=1}^{\nu_i} \sum_{k=0}^{\nu_i-j} t^k F_{j+k,j}^i \right) \left( \sum_{j=1}^{\nu_i} F_{j,j}^i - t^{j-1} \sum_{j=1}^{\nu_i} F_{j,j+1}^i \right) c_i \]

\[ = \left( \sum_{j=1}^{\nu_i} \sum_{k=0}^{\nu_i-j} t^k F_{j+k,j}^i - \sum_{j=1}^{\nu_i} \sum_{k=0}^{\nu_i-j} t^{j-1} F_{j,k,j+1}^i \right) c_i \]

\[ = \left( \sum_{j=1}^{\nu_i} \sum_{k=0}^{\nu_i-j} t^k F_{j+k,j}^i - \sum_{j=2}^{\nu_i} \sum_{k=-1}^{\nu_i-j} t^k F_{j,j}^i + \sum_{j=2}^{\nu_i} t^{j-1} F_{j-j,j}^i \right) \]

\[ = \sum_{j=0}^{\nu_i-1} t^k F_{k+1,j}^i - \sum_{j=2}^{\nu_i} t^{j-1} F_{j-j,j}^i - \sum_{j=2}^{\nu_i} t^{j-2} F_{j-j,j}^i \]

\[ = t^{\nu_i-1} F_{\nu_i-1,j}^i - \sum_{j=2}^{\nu_i} t^{j-1} F_{j-j,j}^i \]

Observe that \(1 - t^{-1} Z = \sum_{1 \leq i \leq s} Z_i\). It follows from Equation (4.2) that for \(i \neq j\), \(b_i Z_j = 0\) and \(Z_j c_i = 0\). Writing \(b = \sum_i b_i\) and \(c = \sum_i c_i\), we see

\[ b(1 - t^{-1} Z) c = \sum_i b_i Z_i c_i = \omega \]

The result now follows from the observation \(b, c \in \mathcal{B}\). \(\square\)

**Lemma 4.9.** There exist \(w_g \in \hat{W}_{L+O}(= W)\), \(w_p \in \hat{W}_P(= W_P)\) such that \(\omega = w_g w_p\). In particular, \(X_P(\omega) \subset L^{+} G \mathbb{P}(\text{mod} P)\).

**Proof.** Recall the disjoint subsets \(S_1, S_2\) of \(\{1, \ldots, n\}\) from Section 4.3. Consider the bijection \(\iota : S_2 \to \{\chi(j,i) \mid 1 \leq j \leq \nu_i - 1\}\) given by \(\iota(\chi(j,i)) = \chi(j-1,i)\). We reformulate Proposition 4.8 as

\[ \tilde{\omega} = \sum_{i=1}^{s} t^{\nu_i-1} E_{\chi(\nu_i,i),\chi(1,i)} - \sum_{i \in S_2} E_{i(i),i} \]

It follows from Equation (2.6) that the affine permutation matrix of \(\omega\) is given by

\[ \omega = \sum_{i=1}^{s} t^{\nu_i-1} E_{\chi(\nu_i,i),\chi(1,i)} + \sum_{i \in S_2} E_{i(i),i} \]

Observe that \(#\text{Red}(k) = #S_1(k)\) and \(#\text{Blue}(k) = #S_2(k)\). Since both \((l(i))_{1 \leq i \leq s}\) and \((\chi(1,i))_{1 \leq i \leq s}\) are increasing sequences, \(\chi(1,i)\) and \(l(i)\) are in the same row for each \(i\). Furthermore, there exists an enumeration \(t(1), \ldots, t(n-s)\) of \(S_2\) such that \(t(i)\) is in the same row as \(m(i)\) for all \(i\). We define \(w_g \in W\) and \(w_p \in W_P\) via their

\[ \text{...} \]
affine permutation matrices:

\[ w_g = \sum_{i=1}^{s} E_{i,\chi(v_i,i)} + \sum_{i=1}^{n-s} E_{i+s,t(i)} \]

\[ w_p = \sum_{i=1}^{s} E_{\chi(1),l(i)} + \sum_{i=1}^{n-s} E_{l(i),m(i)} \]

A simple calculation shows \( \varpi = w_g \kappa w_p \).

**Proposition 4.10.** The Schubert variety \( X_P(\kappa) \) is stable under left multiplication by \( L^+G \), i.e., \( X_P(\kappa) = L^+G \kappa P(\text{mod} \ P) \).

**Proof.** Consider the affine permutation matrix of \( \kappa \). We will show, for \( 1 \leq i < n \), either \( s_i \kappa = \kappa(\text{mod} \ P) \) or \( s_i \kappa < \kappa \). We split the proof into several cases, and use Proposition 2.15.

1. \( i < s \) and \( \nu_i = \nu_{i+1} \): We deduce from \( \nu_i = \nu_{i+1} \) that the entries \( l(i) \) and \( l(i+1) \) appear in the same row of the tableau. In particular, \( l(i) \in S_P \). The non-zero entries of the \( i^{th} \) and \( (i+1)^{th} \) row are \( \tau_i^{-1} E_{i,l(i)} \) and \( \tau_i^{s_i-1} E_{i+1,l(i+1)} \) respectively. We see \( s_i \kappa = \kappa s_i(l(i)) = \kappa(\text{mod} \ W_P) \).

2. \( i < s \) and \( \nu_i > \nu_{i+1} \): The non-zero entries of the \( i^{th} \) and \( (i+1)^{th} \) row are \( \tau_i^{-1} E_{i,l(i)} \) and \( \tau_i^{s_i+1} E_{i+1,l(i+1)} \) respectively. Case 2 of Proposition 2.15 applies with \( a = l(i), b = l(i+1) \), and we have \( s_i \kappa < \kappa \).

3. \( i = s \): The non-zero entries of the \( i^{th} \) and \( (i+1)^{th} \) row are \( \tau_i^{-1} E_{s,l(s)} \) and \( \tau_i E_{s+1,m(1)} \) respectively. Since \( m(1) \in \text{Blue}(r) \), it follows from Section 4.3 that \( l(s) < m(1) \). Case 2 of Proposition 2.15 applied with \( a = l(s), b = m(1) \) tells us \( s_i \kappa < \kappa \).

4. \( i > s \) and \( m(i-s) \in S_P \): The non-zero entries of the \( i \) and \( (i+1)^{th} \) row are \( \tau_i^{-1} E_{i,m(i-s)} \) and \( \tau_i^{-1} E_{i+1,m(i+1-s)} \) respectively. It follows \( s_i \kappa = \kappa s_i(m(i-s)) = \kappa(\text{mod} \ W_P) \).

5. \( i > s \) and \( m(i-s) \notin S_P \): It follows from \( m(i-s) \notin S_P \) that if \( m(i-s) \in \text{Row}(j) \) then \( m(i+1-s) \in \text{Row}(j-1) \). In particular, \( m(i+1-s) < m(i-s) \). The non-zero entries of the \( i \) and \( (i+1)^{th} \) row are \( \tau_i^{-1} E_{i,m(i-s)} \) and \( \tau_i^{-1} E_{i+1,m(i+1-s)} \) respectively. Case 1 of Proposition 2.15 applied with \( a = m(i+1-s), b = m(i-s) \) tells us \( s_i \kappa < \kappa \).

**Theorem 4.11.** Let \( \phi_P \) be as in Section 3.11 Then \( \text{Im}(\phi_P) \subset X_P(\kappa) \). In particular, the map \( \phi_P \) gives a compactification of \( T^*G/P \). Further \( \kappa \in W^P \) is minimal for the property \( \text{Im}(\phi_P) \subset X_P(\kappa) \).

**Proof.** Let \( \mathcal{O} \) denote the \( G \)-orbit of \((1, Z) \in G \times P \). It follows from Lemma 4.7 that \( T^*G/P = \overline{\mathcal{O}} \), and from Proposition 4.8 that \( \phi_P(1, Z) \in B \overline{\mathcal{O}} P/P \). Since \( \phi_P \) is \( G \)-equivariant, it follows that \( \phi_P(\mathcal{O}) \subset GB \overline{\mathcal{O}} P/P \subset L^+G \overline{\mathcal{O}} P/P \), and so

\[ \phi_P(T^*G/P) = \phi_P(\overline{\mathcal{O}}) \subset \phi_P(\mathcal{O}) \subset L^+G \overline{\mathcal{O}} P/P = X_P(\kappa) \]

where the last equality follows from Lemma 4.9 and Proposition 4.10. Further, \( \phi_P(T^*G/P) \), being a closed subvariety of \( X_P(\kappa) \), is compact.

To prove the minimality of \( \kappa \), we show that there exists a \( a \in G \) such that
Proposition 4.12. The dimension of $X_P(\kappa)$ is $l(\kappa)$, the length of $\kappa$.

Proof. We need to show that $\kappa \in \tilde{W}^P$. For $\alpha_i \in S_P$, we show $\kappa \alpha_i > \kappa$. Recall the partitioning of $\{1, \ldots, n\}$ from Section 4.3 into Red and Blue. Note that $\alpha_i \in S_P$, implies $i$ and $i + 1$ appear in the same row of the tableau. In particular, if $i \in \mathcal{B}$, then $i + 1 \in \mathcal{B}$.

(1) Suppose $i \in \text{Blue}$. Then $i = m(k)$ and $i + 1 = m(k + 1)$ for some $k$. The non-zero entries in the $i^{th}$ and $(i + 1)^{th}$ columns are $t^{-1}E_{k, i}$ and $t^{-1}E_{k+1, i+1}$. We apply Case 1 of Proposition 2.15 with $a = i$, $b = i + 1$.

(2) Suppose $i \in \text{Red}$ and $i + 1 \in \text{Blue}$. The non-zero entries in the $i^{th}$ and $(i + 1)^{th}$ columns are $t^{\nu_k - 1}E_{k, i}$ and $t^{\nu_k + 1}E_{k+1, i+1}$. Since $k \leq s < j$, we can apply Case 2 of Proposition 2.15 with $a = i$, $b = i + 1$, $\nu_k - 1 = \text{ord}(t_a) > \text{ord}(t_b) = -1$ and $i = \sigma(a) < \sigma(b) = i + 1$ to get $\kappa < \kappa s_i$.

(3) Suppose, $i, i + 1 \in \text{Red}$. Since $i$ and $i + 1$ are in the same row of the tableau, we have $i = l(k)$ and $i + 1 = l(k + 1)$ for some $k$. The non-zero entries in the $i^{th}$ and $(i + 1)^{th}$ columns are $t^{\nu_k - 1}E_{k, i}$ and $t^{\nu_k + 1}E_{k+1, i+1}$. If $\nu_k = \nu_{k+1}$, Case 1 of Proposition 2.15 applies with $a = i$, $b = i + 1$ to give $\kappa < \kappa s_i$. If $\nu_k > \nu_{k+1}$, Case 2 of Proposition 2.15 applies with $a = i$, $b = i + 1$, $\nu_k - 1 = \text{ord}(t_a) > \text{ord}(t_b) = \nu_{k+1} - 1$ to give $\kappa < \kappa s_i$.

Lemma 4.13. The length of $\kappa$ is given by the formula

$$l(\kappa) = 2 \dim G/P + \sum_{k' < k} \#\text{Row}(k) \#\text{Blue}(k')$$

Proof. Note that $\kappa = \tau_n \sigma$, where $\tau_n$ is given by Equation (3.4), and $\sigma \in W$ is given by the permutation matrix

$$\sigma = \sum_{i=1}^{s} E_{i, l(i)} + \sum_{i=1}^{n-s} E_{i+s, m(i)}$$

(4.14)

Viewing $\sigma$ as an element of $S_n$, we have $\sigma^{-1}(i) = \begin{cases} l(i) & i \leq s \\ m(i - s) & i > s \end{cases}$. In particular,

$$l(\sigma) = \# \{(i, j) \mid 1 \leq i < j \leq n, \sigma^{-1}(i) > \sigma^{-1}(j)\}$$

$$= \# \{(i, j) \mid i < j \leq s, l(i) > l(j)\}$$

$$+ \# \{(i, j) \mid i \leq s < j, l(i) > m(j - s)\}$$

$$+ \# \{(i, j) \mid s < i < j, m(i - s) > m(j - s)\}$$
Recall that \( l(i) \) is an increasing sequence, i.e., \( i < j < s \implies l(i) < l(j) \), and so

\[
l(\sigma) = \# \{(i, j) | i \leq s, j \leq n - s, l(i) > m(j)\} + \# \{(i, j) | i < j \leq n - s, m(i) > m(j)\}
\]

\[
= \# \{(i, j) | i \in \text{Red}, j \in \text{Blue}, i > j\} + \# \{(i, j) | i < j \leq n - s, m(i) > m(j)\}
\]

\[
= \sum_{k' < k} \# \{(i, j) | i \in \text{Red}(k), j \in \text{Blue}(k')\}
\]

\[
+ \sum_{k' < k} \# \{(i, j) | i \in \text{Blue}(k), j \in \text{Blue}(k')\}
\]

\[
= \sum_{k' < k} \# \{(i, j) | i \in \text{Row}(k), j \in \text{Blue}(k')\}
\]

\[
= \sum_{k' < k} \# \text{Row}(k) \# \text{Blue}(k')
\]

Now, it follows from Equations [2.9] and [2.10] that \( \tau_q \in \widehat{W}^{L+G} \). In particular,

\[
l(\tau_q) = \dim X_{L+G}(\tau_q) = \dim \mathcal{N}_\nu = 2 \dim G/P
\]

Further, \( l(\kappa) = l(\tau_q) + l(\sigma) = 2 \dim G/P + \sum_{k' < k} \# \text{Row}(k) \# \text{Blue}(k') \) as claimed. \( \square \)

**Corollary 4.15.** The Schubert variety \( X_P(\kappa) \) is a compactification of \( T^*G/P \) if and only if \( P \) is a maximal parabolic subgroup.

**Proof.** The parabolic subgroup \( P \) is maximal if and only if \( S_P = S_0 \setminus \{\alpha_d\} \) for some \( d \), equivalently, the corresponding tableau has exactly 2 rows. In this case \( \text{Blue} \subset \text{Row}(2) \), (recall from Section 4.3 that \( \text{Blue} = \bigsqcup_i \text{Blue}(i) \)). It follows that the second term in Lemma 4.13 is an empty sum, which implies \( l(\kappa) = 2 \dim G/P \). Suppose now that the tableau has \( r \geq 3 \) rows. In this case, both \( \text{Blue}(2) \) and \( \text{Row}(r) \) are non-empty, and so the second term in Lemma 4.13 is strictly greater than 0. \( \square \)

5. **Conormal variety of the Schubert Divisor**

Let \( \lambda, P, \) and \( \mathcal{P} \) be as in the previous section. In this section, we show that for \( X_P(w) \) a Schubert divisor in \( G/P \), the conormal variety \( N^*X_P(w) \) is an open subset of a Schubert variety in \( LG/\mathcal{P} \). In particular, \( N^*X_P(w) \) is normal, Cohen-Macaulay, and Frobenius split.

We write \( \mathfrak{h}, \mathfrak{b}, \) and \( \mathfrak{g} \) for the Lie algebras of \( T, B, \) and \( G \) respectively. For \( \alpha \in \Delta_0 \), we denote by \( \mathfrak{g}^\alpha \) the root space corresponding to \( \alpha \).

5.1. **The Conormal Variety.** Let \( X \) be a closed subvariety in \( G/P \), and write \( X_{\text{sm}} \) for the smooth locus of \( X \). For \( x \in X_{\text{sm}} \), the conormal fibre \( N_x^* \) is the annihilator of \( T_xX \) in \( T_x^*G/P \). The conormal variety \( N^*X \) of \( X \hookrightarrow G/P \) is then defined to be the closure in \( T^*G/P \) of the conormal bundle \( N^*X_{\text{sm}} \).
Proposition 5.2. Let \( \Delta_P \) be the subset of \( \Delta_0 \) generated by \( S_P \). The conormal variety \( N^*X_P(w) \) is the closure in \( T^*G/P \) of
\[
\left\{ (bw, X) \in G \times^P u \mid b \in B, X \in \bigoplus_{\alpha \in R} g^\alpha \right\}
\]
where \( R = \left\{ \alpha \in \Delta_0^+ \mid \alpha \notin \Delta_P, w(\alpha) > 0 \right\} \).

Proof. The tangent space of \( G/P \) at identity is \( g/p \). Consider the action of \( P \) on \( g/p \) induced from the adjoint action of \( P \) on \( g \). The tangent bundle \( TG/P \) is the fiber bundle over \( G/P \) associated to the principal \( P \)-bundle \( G \to G/P \), for the aforementioned action of \( P \) on \( g/p \), i.e., \( TG/P = G \times^P g/p \).

Let \( R' = \left\{ \alpha \in \Delta_0^+ \mid w(\alpha) > 0 \right\} \), so that \( \Delta_0^+ = \Delta_P^+ \cup R \cup -R' \). Further, let \( U_w = \langle U_\alpha \mid \alpha \in R' \rangle \). For any point \( b \in B \), we have (see, for example [St]):
\[
BwP(\text{mod } P) = bBwP(\text{mod } P) = b(wU_ww^{-1})wP(\text{mod } P) = bwU_wP(\text{mod } P)
\]
It follows that the tangent subspace at \( bw \) of the big cell \( BwP(\text{mod } P) \) is given by
\[
T_{bw}BwP(\text{mod } P) = \left\{ (bw, X) \in G \times^P g/p \mid X \in \bigoplus_{\alpha \in R} g^\alpha/p \right\}
\]
where \( g^\alpha/p \) denotes the image of a root space \( g^\alpha \) under the map \( g \to g/p \). Recall that the Killing form identifies the dual of a root space \( g^\alpha \) with the root space \( g^{-\alpha} \). Consequently, a root space \( g^\alpha \subset u \) annihilates \( T_{bw}BwP(\text{mod } P) \) if and only if \( \alpha \in \Delta_0^+ \setminus \Delta_P^+ \) and \( -\alpha \notin R' \), or equivalently, \( \alpha \notin R \). The result now follows from the observation that \( BwP(\text{mod } P) \) is a dense open subset of \( X_P(w) \), and is contained in the smooth locus of \( X_P(w) \). \( \square \)

5.3. Schubert divisors. A Schubert divisor in \( G/P \) is a Schubert variety of codimension 1. Let \( w_0^P \) be the longest element in \( W^P \). The affine permutation matrix for \( w_0^P \) is given by
\[
w_0^P = \begin{pmatrix}
0 & 0 & I(\lambda_r) \\
0 & \ddots & 0 \\
I(\lambda_1) & 0 & 0
\end{pmatrix}
\]
where, for \( k > 0 \), \( I(k) \) denotes the \( k \times k \) identity matrix. Codimension one Schubert varieties \( X_P(w) \) in \( G/P \) correspond to \( w = s_kw_0^P \), where \( k = n - d_i \) for some \( 1 \leq i < r \). For \( 1 \leq k < n \), let \( v_k \in \hat{W} \) be given by the affine permutation matrix
\[
v_k = \sum_{i=1}^{n} a_i E_{i,n+1-i}
\]
where \( a_i = \begin{cases} t^{-1} & \text{if } i = k \\ t & \text{if } i = k + 1 \\ 1 & \text{otherwise} \end{cases} \)
We denote by \( v_k^P \) the minimal representative of \( v_k \) with respect to \( \hat{W}_P \).

Proposition 5.4. Let \( w = s_kw_0 \), where \( k = n - d_i \) for some \( 1 \leq i < r \). Then \( X_P(v_k^P) \) is a compactification of \( N^*X_P(w) \) via \( \phi_P \).
Consider $b N$
In particular, we can write a generic point of $a$
further assume $\phi$
show that $\phi$
right square sub-matrix of $w$
Since $G/B$
dim
Let $L$ be the bottom left square sub-matrix of $w_0^P$ of size $d_{k-1}$ and $L'$ the top right square sub-matrix of $w_0^P$ of size $(n - d_{k+1})$. We fix a lift $\tilde{w}$ of $w$ to the normalizer of $T$:
$$\tilde{w} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & L' \\ 0 & 0 & 0 & I(\lambda_{k+1} - 1) & 0 & 0 \\ 0 & e & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & I(\lambda_k - 1) & 0 & 0 & 0 \\ L & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
where $I(k)$ denotes the $k \times k$ identity matrix, and $e = \pm 1$ is determined by the equation $\det \tilde{w} = 1$. Observe that the $e$ is in the $(n - d_k, d_{k-1} + 1)$ position and the $1$ in the $(n - d_k + 1, d_{k+1})$ position. Consider the root
$$\gamma := \sum_{d_{i-1} < j < d_{i+1}} \alpha_j$$
Under the identification of Section 2.2, we have $\gamma = (d_{i-1} + 1, d_{i+1})$. We check that
$$\{ \alpha \in \Delta_0^+ \mid \alpha \notin \Delta_P, w(\alpha) > 0 \} = \{ \gamma \}$$
In particular, we can write a generic point of $N^*X_P(w)$ as $(bw_0^P, aE_\gamma)$. We may further assume $a \neq 0$. It is now sufficient to show that
$$\phi_P(bw_0^P, aE_\gamma) = bw_0^P (1 - at^{-1}E_\gamma) \in X_P(v_k^P)$$
Consider $b_1, b_2, b_3 \in B$ given by
$$b_2 = I(n) + \frac{et}{a} E_{k+1,k} \quad \quad b_3 = I(n) + \frac{t}{a} E_{d_{i+1},d_{i-1}+1}$$
$$b_1 = \sum_{1 \leq i \leq n} c_i E_{ii} \quad \quad c_i = \begin{cases} \frac{e}{a} & \text{for } i = k \\ ea & \text{for } i = k + 1 \\ 1 & \text{otherwise} \end{cases}$$
It is easily verified that $(b_1 b_2 b^{-1})bw_0^P (1 - t^{-1}aE_\gamma)b_3$ is an affine permutation matrix corresponding to $v_k^P \in \tilde{W}$. 

**Corollary 5.5.** Let $X_P(w)$ be a Schubert divisor in $G/P$. The conormal variety $N^*X_P(w)$ is normal, Cohen-Macaulay, and Frobenius split.

**Proof.** Schubert varieties in $LG/P$ are normal, Cohen-Macaulay, and Frobenius split (cf. [Fa, MR]). Therefore, the same is true for $N^*X_P(w)$, since it is an open subset of $X_P(v_k^P)$. 

\text{\hfill $\square$}
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