The superpotential method for chiral cosmological models connected with the modified gravity

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We consider the Chiral Cosmological Models (CCM) and modified gravity theories associated with them. Generalization of the superpotential method for a general CCM with several scalar fields is performed and the method of construction of CCM admitting exact solutions is developed. New classes of exact solutions in the two-component CCM connected with a \( f(R) \) gravity model with an additional scalar field have been constructed. We construct new cosmological solutions for a diagonal metric of the target space, including modified power-law solutions. In particular, we proposed the reconstruction procedure based on the superpotential method and presented examples of kinetic part reconstruction for periodic and hyperbolic Hubble parameters. We also focus on a cyclic type of Universe dubbed Quasi-Steady State (QSS) model with the aim to construct single- and double-field potentials for one and the same behaviour of the Hubble parameter using developed superpotential method for CCM. The realization of this task includes a new set of solutions for CCM with scale factor characterized the QSS theory. We also propose the method of reducing the two-field CCM to the single scalar field model.

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I. INTRODUCTION

Scalar fields play an important role in model building for early Universe and late time cosmic evolution. The observations [1–3] show that the Universe evolution can be described by the spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime as background and cosmological perturbations. Models with scalar fields are well suited to describe such an evolution. As for the modified theories of gravity, in general, they can be thought as Einstein theory of General Relativity (GR) plus extra degrees of freedom; for instance, \( f(R) \) gravity models correspond to the General Relativity models with a single self-interacting scalar field.

The scalar fields are important in inflationary scenarios [4–8], including the Starobinsky \( R^2 \) model [8] and the Higgs-driven inflation [10]. Models with a single scalar field non-minimally coupled to gravity as well as \( f(R) \) gravity models can be always transformed to models with minimally coupled scalar field with the canonical kinetic term by the metric and scalar field transformations. On the other hand, models with a few fields non-minimally coupled with gravity in general do not admit such a transformation [11]. After the metric transformation, one obtains the Chiral Cosmological Models (CCM) in the Einstein frame [12]. The analogues models can be obtained from \( f(R) \) gravity models with additional scalar fields [13, 14]. The CCM allows to describe not only the inflationary epoch of the Universe evolution, but also the present accelerated expansion of the Universe [15, 16].

In the case of single scalar cosmology with a generic potential, the integration procedure is reduced to solving Ivanov–Salopek–Bond (ISB) equation [7, 17]. There are many methods of obtaining physically relevant solutions including that with the Higgs potential [17–20]. The reduction of CCM to the single scalar field model has been proposed in [21]. New approach for study CCM when gravitational field equations are represented in a linear form.

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under the point transformation has been developed in [22]. For a specific geometry of the target space and special form of the potential it was found the way of obtaining the solutions to the gravitational field equation¹. In the case of several scalar fields with kinetic interaction the progress in obtaining the exact solutions have been achieved for later universe evolution [23], Emergent universe [24], Einstein-Gauss-Bonnet cosmology [25] and tensor-multi-scalar model [26] as well.

The goal of this paper is to propose the way to get a particular solution of the CCM in the analytic form. We do not seek the solutions for a given potential but construct the potential of the scalar field such that the resulting model has exact solutions with important physical properties. Such a method is similar to the Hamilton-Jacobi method (also known as the superpotential method or the first-order formalism) and is applied to cosmological models with minimally [7, 17, 20, 27–36] and non-minimally [37] coupled scalar fields. This method has been used, in particular, to find the exact solutions in the single scalar field inflationary models² [40–49]. Note that a similar method is used for the reconstruction procedure in brane [31, 51–53] and in holographic models [54, 55].

The key point of the superpotential method is that the Hubble parameter is considered as a function of the scalar fields \( \phi^A(t) \). Note that there is an important difference between one-field and multi-field models. In the case of one-field models, the above-mentioned procedure is straightforward because only one superpotential (up to a constant) corresponds to the given scalar field \( \phi(t) \). In the case of two or more fields, the knowledge of a particular solution \( \phi^A(t) \) does not fix the potential. An explicit example of essentially different potentials of two-field models with the same particular solution \( \phi^A(t) \) is given in [27]. In this paper, we generalize the superpotential method on the CCM and constructing new models with exact solutions.

The term a **multifield model** is similar to a **chiral cosmological model** which is defined as the self-gravitating nonlinear sigma model with the potential of (self)interactions employed in cosmology. Let us mention that the term ”multi-scalar field cosmology” was first introduced as the collection of scalar fields with the sum of kinetic (canonical) parts and with the potential depending on all fields. The model with the kinetic interaction between scalar fields, represented in some articles (for example, in the recent work [22]) where the term ”multi-field” has been introduced first time in the work by V. de Alfaro et al [56], with the aim to obtain instanton and meron solutions in 4D model. They introduced geometrical restriction: all fields take value in the n-dimensional sphere. The potential term was not presented in the model which called as ”Four-dimensional sigma model coupled to the metric tensor field”. A.M. Perelomov in 1981 [57] introduced terminology with exchanging the term ”group invariant sigma model” for ”chiral model” and he also introduces the metric of a chiral model and extends the model from 2D for N-dimensional models, so-called, ”the chiral models of general type”. In the paper [57], it was no connection with gravity. G.G. Ivanov [58] independently of work [57] came to ”non-linear sigma model coupled to gravity” considering the Lorentz’s signature metric of spacetime and the scalar (chiral) fields as the source of gravity, besides the kinetic interaction have been introduced as the metric of ”chiral” space.

The potential of the interaction of chiral fields has been introduced by S. Chervon in 1994 [59]. Such a model in [59] was called as ”Self-gravitating nonlinear sigma model with the potential”. Then in further publications, using terminology introduced by Perelomov, the model was referred to as ”Chiral inflationary model” and then ”Chiral cosmological model”. Thus, the term ”Chiral Cosmological Model” reflects the geometrical interactions of fields via the metric of the target (chiral) space which includes the kinetic interactions.

In this paper, we generalize this reconstruction procedure on models with an arbitrary finite number of scalar fields minimally coupled to gravity. The structure of the paper is as follows. In section II we connect the CCM with modified gravity. In section III the superpotential method develops on CCMs with an arbitrary number of scalar fields. In sections IV, V, VI we consider two-component CCMs. In section IV the considering CCMs correspond to \( f(R) \) gravity models with an additional scalar field. In Sections V and VI we have found models with Ruzmaikins solutions, solutions that correspond to the intermediate inflation and modified power-law solutions for the models with the given kinetic terms of the actions due to the choice of the potential. The procedure of construction of CCMs with trigonometric and hyperbolic Hubble functions due to a suitable choice of the kinetic term has been proposed in Section VI In Section VII we construct models with the Hubble parameter that describes a cyclic type of Universe dubbed quasi-steady state. The method of reducing the two-field CCMs to single scalar field models has been proposed in Section IX. Our results are summarized in section X.

¹ It is difficult to consider such solutions as exact ones without involving the dynamic equations of the chiral fields.
² Other methods allowing to find exact solutions in inflationary models were presented in [12, 13, 21, 34–42] (for recent review, see [43]).
II. THE CONNECTION BETWEEN CHIRAL COSMOLOGICAL MODELS AND THE MODIFIED GRAVITY

Chiral cosmological models with $K$ scalar fields $\phi^A (\bar{\phi} = \phi^1, \phi^2, ..., \phi^K)$ are described by the following action

$$S = \int d^4x \sqrt{-g} \left[ \frac{M_{Pl}^2}{2} R - \frac{1}{2} h_{AB}(\bar{\phi}) \partial_{\mu} \phi^A \partial^B g^{\mu\nu} - V(\bar{\phi}) \right], \quad (1)$$

where the functions $h_{AB}(\bar{\phi})$ and the potential $V(\bar{\phi})$ are differentiable functions, $M_{Pl}$ denotes the reduced Planck mass: $M_{Pl} \equiv 1/\sqrt{8\pi G}$. We assume that $h_{AB} = h_{BA}$ and the determinant of this matrix is not equal to zero, so, this matrix can be considered as the field-space metric.

Varying action (2), we get the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{M_{Pl}^2} T_{\mu\nu}, \quad (2)$$

where the energy-momentum tensor is

$$T_{\mu\nu} = h_{AB}(\bar{\phi}) \partial_{\mu} \phi^A \partial_{\nu} \phi^B - g_{\mu\nu} \left[ \frac{1}{2} h_{AB}(\bar{\phi}) \partial_{\rho} \phi^A \partial_{\beta} \phi^B g^{\rho\beta} + V(\bar{\phi}) \right]. \quad (3)$$

Variation the action (1) on the chiral field $\phi^C$ leads to the field equation

$$\frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu\nu} h_{CB} \phi^B_{,\nu} \right) - \frac{1}{2} g_{\mu\nu} h_{AB,C} \phi^A_{,\mu} \phi^B_{,\nu} - V_{,C}(\bar{\phi}) = 0, \quad (4)$$

where $h_{AB,C} \equiv \partial h_{AB}/\partial \phi^C$.

In the spatially flat Friedmann–Lemaitre–Robertson–Walker (FLRW) metric with the interval

$$ds^2 = -dt^2 + a^2(t) \left( dx_1^2 + dx_2^2 + dx_3^2 \right), \quad (5)$$

the Einstein equations (2) have the following form:

$$3H^2 = \frac{\varrho}{M_{Pl}^2}, \quad (6)$$

$$2\dot{H} + 3H^2 = -\frac{p}{M_{Pl}^2}, \quad (7)$$

where

$$\varrho = -T^0_0 = \frac{1}{2} h_{AB}(\bar{\phi}) \dot{\phi}^A \dot{\phi}^B + V(\bar{\phi}), \quad p = T^1_1 = \frac{1}{2} h_{AB}(\bar{\phi}) \dot{\phi}^A \dot{\phi}^B - V(\bar{\phi}), \quad (8)$$

the dots denote the time derivative and the Hubble parameter $H(t)$ is the logarithmic derivative of the scale factor: $H = \dot{a}/a$.

In the FLRW metric (5), Eq. (4) is transformed to:

$$-h_{CB} \left( \dot{\phi}^B + 3H \dot{\phi}^B \right) - h_{CB,D} \dot{\phi}^D \dot{\phi}^B + \frac{1}{2} h_{DB,C} \dot{\phi}^D \dot{\phi}^B - V_C(\bar{\phi}) = 0. \quad (9)$$

Contracting this equation with $h^{AC}$, we obtain

$$\ddot{\phi}^A + 3H \dot{\phi}^A + \Gamma^A_{DB} \dot{\phi}^D \dot{\phi}^B + h^{AC} V_C = 0, \quad (10)$$

where $\Gamma^A_{DB}$ are the Christoffel symbols for the field-space manifold, defined by the metric $h_{AB}$.

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3 Note that upper index $A$ could not be move down with the chiral metric $h_{AB}$.
Let us introduce a new variable

\[ X = h_{AB} \dot{\phi}^A \dot{\phi}^B. \]  

(11)

From Eqs. (6) and (7), we get

\[ \dot{H} = -\frac{X}{2M_{Pl}}. \]  

(12)

Multiplying Eqs. (9) on \( \dot{\phi}^C \), summing them and using

\[ \dot{X} = 2h_{AB} \ddot{\phi}^A \dot{\phi}^B + h_{AB,C} \dot{\phi}^A \dot{\phi}^B \dot{\phi}^C, \]

we get the following equation

\[ \frac{1}{2} \dot{X} + 3HX + \dot{V} = 0. \]  

(13)

Note that Eq. (13) is a consequence of Eqs. (6) and (12).

Many modified gravity models are connected with chiral cosmological models. In particular, let us consider models with non-minimally coupled scalar fields, that are described by the following action:

\[ S_J = \int d^4x \sqrt{-g} \left[ f(\phi, \bar{\phi}) g^{\mu\nu} \partial_\mu \phi^A \partial_\nu \phi^B - \bar{V}(\phi) \right]. \]  

(14)

By the conformal transformation of the metric

\[ g_{\mu\nu} = \frac{2}{M_{Pl}} f(\phi) \tilde{g}_{\mu\nu}, \]  

(15)

one gets the following action in the Einstein frame [6]:

\[ S_E = \int d^4x \sqrt{-g} \left[ \frac{M_{Pl}^2}{2} R - \frac{1}{2} h_{AB}(\bar{\phi}) \tilde{g}^{\mu\nu} \partial_\mu \phi^A \partial_\nu \phi^B - V_E \right], \]  

(16)

where

\[ h_{AB}(\bar{\phi}) = \frac{M_{Pl}^2}{2f(\phi)} \left[ \tilde{G}_{AB} + \frac{3f_A f_B}{f(\phi)} \right], \quad V_E = \frac{M_{Pl}^4}{4f^2} \bar{V}. \]

(17)

For the class of \( f(R) \) gravity models with scalar fields, described by

\[ S_R = \int d^4x \sqrt{-g} \left[ f(\phi, \bar{\phi}, R) - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{G}_{AB} \partial_\mu \phi^A \partial_\nu \phi^B - \bar{V}(\phi) \right], \]  

(19)

one can introduce an additional scalar field without kinetic term \( \phi^{K+1} \) and rewrite \( S_R \) as follows [60]:

\[ \tilde{S}_J = \int d^4x \sqrt{-g} \left[ \frac{df(\phi, \phi^{K+1})}{d\phi^{K+1}} (\bar{R} - \phi^{K+1}) + f(\phi, \phi^{K+1}) - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{G}_{AB} \partial_\mu \phi^A \partial_\nu \phi^B - \bar{V}(\phi) \right]. \]  

(18)

Therefore, we get the model with \( K + 1 \) scalar fields, describing by the action [24] and can transform it to the chiral cosmological model with the action [10].

III. THE SUPERPOTENTIAL METHOD FOR THE CCM

Generalizing the superpotential method for multifield models with the standard kinetic term [7] and for two-fields models with a constant kinetic term [27], we describe the superpotential for the general CCM.

We assume that functions \( \phi^A \) are solutions of the following system of \( K \) ordinary differential equations

\[ \frac{d\phi^A}{dt} \equiv \dot{\phi}^A = F^A(\bar{\phi}). \]  

(19)
In this case, the Hubble parameter $H(t)$ is a function of all scalar fields:

$$H(t) = W(\phi) + C_W,$$

where superpotential $W$ is a differentiable function and $C_W$ is a constant part of the Hubble function that plays a special role (see, for example, [20]). It is convenient to write $C_W$ separately. Thus, Eqs. (6) and (7) take the following form:

$$3M_{Pl}^2[W(\phi) + C_W]^2 = \frac{1}{2}h_{AB}F^AF_B + V(\phi), \quad (20)$$

$$M_{Pl}^2(2W,AF_A + 3[W(\phi) + C_W]^2) = -\frac{1}{2}h_{AB}F^AF_B + V(\phi). \quad (21)$$

Subtracting Eq. (20) from Eq. (21), we get

$$\left(W,A + \frac{1}{2M_{Pl}^2}h_{AB}F_B^A\right)F^A = 0. \quad (22)$$

One can see that the sufficient condition to satisfy Eq. (22) is

$$W,A = -\frac{h_{AB}F_B^A}{2M_{Pl}^2}, \quad (23)$$

for all $A$. This is rather tight restriction which is equivalent to the decomposition method used in the works [20,23,26].

Using $\ddot{\phi}^B = \ddot{F}^B = F^D_DF^D$, we rewrite the field equations (9) as follow

$$h_{CB}F^B_DF^D + h_{CB,D}F^DF^B - \frac{1}{2}h_{DB,C}F^DF^B + 3[W(\phi) + C_W]h_{CB}F^B + V,C = 0. \quad (24)$$

From (20) it is follows:

$$V,C = 6M_{Pl}^2[W(\phi) + C_W]W,C - \frac{1}{2}h_{DB,C}F^DF^B - h_{DB}F^DF^B. \quad (25)$$

Substituting this expression of $V,C$ into Eq. (24), we get:

$$3[W(\phi) + C_W]\left\{2M_{Pl}^2W,C + h_{DC}F^D\right\} = (h_{DB}F^D_C - h_{DC}F^D_B)F^B + (h_{DB,C} - h_{CB,D})F^DF^B. \quad (26)$$

If condition (23) is satisfied, then

$$[h_{DB}F^D_C - h_{DC}F^D_B]F^B + (h_{DB,C} - h_{CB,D})F^DF^B = 0. \quad (27)$$

Also, from the obvious equality $W,CB = W,BC$ we get

$$(h_{BD,C} - h_{CD,B})F^D = h_{CD}F^D_B - h_{BD}F^D_C. \quad (28)$$

Condition (27) is a consequence of (23). The matrix $h_{AB}$ is symmetric and the conditions (28) are trivial at $B < C$ only.

Thus the task of solving the dynamic equations of the model is reduced to the Eqs. (20), (24) and (28). The last equation gives guaranty that the solution of the chiral fields equation (24) is true.

Further we will use the expression for the potential $V(\phi)$ in terms of the superpotential $W(\phi)$ which follows from

$$V(\phi) = 3M_{Pl}^2[W(\phi) + C_W]^2 + M_{Pl}^2W,AF^A. \quad (29)$$

Applying Eq. (23) once more the physical potential can be presented in the form

$$V(\phi) = 3M_{Pl}^2[W(\phi) + C_W]^2 - 2M_{Pl}^4h^{AB}W,AW,B. \quad (30)$$

From Eq. (10), we obtain

$$\ddot{F}^E + \Gamma_D^E F^DF^D + 3W^E + h^{CE}V,C = 0. \quad (31)$$
To demonstrate how one can get exact solutions due to the superpotential method we consider a such diagonal matrix $h_{AB}$ that each $h_{BB}$ depends only $\phi^A$, with $A \leq B$ and has the following form:

$$h_{11} = s_1(\phi^1), \quad h_{BB} = u_B(\phi^1, \ldots, \phi^{B-1})s_B(\phi^B),$$

(32)

for all $B = 2, \ldots, K$.

Let us prove that for any $h_{BB}$ given by (32) we can construct the CCM with exact solutions obtained either in analytic form or in quadratures. We choose

$$H(t) = \sum_{A=1}^{K} W_K(\phi^K) + C_W,$$

(33)

where $W_K$ are differentiable functions.

The function $W$ fixes the potential $V$ by Eq. (29). For any $A$ the function $F_A$ is defined as follows

$$F_A = -2M_{Pl}^2 \frac{W_A}{h_{AA}}.$$

(34)

Let us check the conditions (28). For the matrix $h_{AB}$, defined by (32) and $B < C$ we get

$$\frac{d}{d\phi^B} (h_{CC} F^C) = 0,$$

(35)

without summing on $C$. It is easy to see that the functions $F_A$, defined (for each index $A$) by (34) satisfy these conditions. Therefore, we obtain all functions $F_A$ and system (19) takes the following form:

$$\dot{\phi}^1 = -2M_{Pl}^2 \frac{W_{1,1}(\phi^1)}{h_{11}(\phi^1)},$$

$$\dot{\phi}^2 = -2M_{Pl}^2 \frac{W_{2,2}(\phi^2)}{h_{22}(\phi^1, \phi^2)},$$

$$\ldots$$

$$\dot{\phi}^K = -2M_{Pl}^2 \frac{W_{K,K}(\phi^K)}{h_{KK}(\phi^1, \phi^2, \ldots, \phi^K)}.$$

(36)

This system can be solved at least in quadratures. Indeed, the first equation of this system as the first order autonomous differential equation can be solved in quadratures. Let us assume that all $\phi^A$ are known for $A < B$ and consider the equation for $\phi^B$. Using $h_{BB} = u_B(\phi^1, \ldots, \phi^{B-1})s_B(\phi^B)$, we get

$$\frac{s_B(\phi^B)}{W_{B,B}(\phi^B)} d\phi^B = -\frac{2M_{Pl}^2}{u_B(\phi^1(t), \ldots, \phi^{B-1}(t))} dt.$$

(37)

This equation is integrable for any $u_B$, given as a function of $t$. So, the solution is found and the statement is proven by induction.

Note that the superpotential method allows us to construct such a one-parametric set of the CCM that the corresponding Hubble parameters differ on an arbitrary constant $C_W$. In the next sections, we restrict ourselves to two-dimensional $h_{AB}$, in particular, we show that two-dimensional $h_{AB}$ in the form (32) naturally arise from $f(R)$ gravity models with one scalar field.

IV. EXACT SOLUTIONS FOR AN $f(R)$ GRAVITY MODEL WITH AN ADDITIONAL SCALAR FIELD

In this and following sections, we consider two-component CCM and denote

$$\phi^1 = \psi, \quad \phi^2 = \chi, \quad \dot{\phi}^1 = F^1 = U(\psi, \chi), \quad \dot{\phi}^2 = F^2 = S(\psi, \chi).$$

It has been shown in [13, 14] that under the metric transformation

$$g_{\mu\nu} = e^{\sqrt{2} \frac{\chi}{M_{Pl}}} \tilde{g}_{\mu\nu},$$

with
the \( f(\tilde{R}) \) gravity model with a scalar field \( \chi \), described by the action

\[
S_J = \int d^4\tilde{x} \sqrt{-\tilde{g}} \left[ f(\chi, \tilde{R}) - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi \right],
\]  

(38)

transforms to the chiral cosmological model, described by the action \( H \) with

\[
h_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & K(\psi) \end{pmatrix}
\]

(39)

where

\[
\psi = \sqrt{\frac{3 M_{Pl}^2}{2}} \ln \left( \frac{2}{M_{Pl}^2} \left| \frac{\partial f}{\partial \tilde{R}} \right| \right), \quad K(\psi) = e^{C \psi},
\]

(40)

and a constant \( C = -\sqrt{\frac{2}{3 M_{Pl}^2}} \).

Let us consider this CCM to obtain exact solutions due to the superpotential method, described in the previous section.

From Eq. (23), we get

\[
W_1 = -\frac{1}{2 M_{Pl}^2} U(\psi, \chi), \quad W_2 = -\frac{K(\psi)}{2 M_{Pl}^2} S(\psi, \chi).
\]

(41)

To get \( W \) in the form (33) we choose

\[
U(\psi, \chi) = U(\psi), \quad S(\psi, \chi) = \frac{Q(\chi)}{K(\psi)}.
\]

(42)

where \( Q(\chi) \) is an arbitrary function.

The superpotential \( W \) is defined as follows:

\[
W = -\frac{1}{2 M_{Pl}^2} \left( \int U d\psi + \int Q d\chi \right).
\]

(43)

To get an exact particular solution we assume the explicit form of the functions \( U(\psi) \) and \( Q(\chi) \) and solve the corresponding system of the first order differential equations:

\[
\dot{\psi} = U(\psi), \quad \dot{\chi} = \frac{Q(\chi)}{K(\psi)}.
\]

(44)

We choose

\[
U(\psi) = F_0 e^{-\Lambda_1 \psi}, \quad \Lambda_1 = \text{const}.
\]

(45)

Therefore,

\[
\psi(t) = \frac{1}{\Lambda_1} \ln(\Lambda_1 F_0 (t - t_0))
\]

(46)

and

\[
K(\psi) = e^{C \psi(t)} = (\Lambda_1 F_0 (t - t_0))^{C/\Lambda_1}.
\]

(47)

Substituting the obtained \( \psi(t) \), we get \( \chi(t) \). For example, for

\[
Q(\chi) = F_0 e^{-\Lambda_2 \chi}, \quad \Lambda_2 = \text{const},
\]

(48)

we obtain

\[
\chi(t) = -\frac{1}{\Lambda_2} \ln \left( \frac{C - \Lambda_1}{\Lambda_2 (C - (\Lambda_1 F_0 (t - t_0))^{1-C/\Lambda_1})} \right),
\]

(49)
where \( C_2 \) is an integration constant.

The Hubble parameter

\[
H(t) = W_1(\psi) + W_2(\chi) + C_W, \tag{50}
\]

where

\[
W_1(\psi) = \frac{F_0}{2\Lambda_1 M_{Pl}^2} e^{-\Lambda_1 \psi}, \quad W_2(\chi) = \frac{F_0}{2\Lambda_2 M_{Pl}^2} e^{-\Lambda_2 \chi}. \tag{51}
\]

The corresponding potential is

\[
V = \frac{3F_0^2}{4M_{Pl}^2 \Lambda_1^2 \Lambda_2^2} \left( \Lambda_2 e^{-\Lambda_1 \psi} + \Lambda_1 e^{-\Lambda_2 \chi} + \frac{2M_{Pl}^2 \Lambda_1 \Lambda_2}{F_0} C_W \right)^2 - \frac{F_0^2}{2} \left( e^{-C\psi} - 2\Lambda_2 \chi + e^{-2\Lambda_1 \psi} \right). \tag{52}
\]

We get a one-parametric set of models with exact solutions. Let us check that for some values of parameters we get slow-roll regime that maybe suitable for inflation.

Let us assume that \( \Lambda_1 > 0 \) and \( \Lambda_2 > 0 \). In this case, the potential has a finite non-negative limit

\[
V \to 3M_{Pl}^2 C_W^2
\]

at the scalar fields tend to plus infinity such that \( 2\Lambda_2 \chi > -C\psi \).

We consider large initial values of scalar field and assume that the scalar fields monotonic decrease during inflation, choosing \( F_0 < 0 \). For large positive values of scalar fields we have quasi de Sitter solutions with \( H \approx C_W \). So, we get a model that looks as suitable to describe inflation. Full analysis of the possible inflationary scenarios with calculations of the inflationary parameters will be a subject of the further investigations.

V. DIAGONAL CONSTANT METRIC OF THE TARGET SPACE

Studying a canonical scalar field equations we set the metric coefficient \( h_{11} = 1 \). Therefore, it will be of interest to study the diagonal metric with constant chiral metric component \( h_{22} = 1 \).

The equation (22) takes the form

\[
\frac{\partial W}{\partial \psi} U(\psi, \chi) + \frac{\partial W}{\partial \chi} S(\psi, \chi) = -\frac{1}{2} U^2 - \frac{1}{2} S^2. \tag{53}
\]

Let us insert the metric components of the target space into fields’ equation (26). After simple algebra we obtain

\[
3W(2W,\psi + U(\psi, \chi)) = S,\psi S - U,\chi S, \tag{54}
\]

\[
3W(2W,\chi + S(\psi, \chi)) = U,\chi U - S,\psi U. \tag{55}
\]

If we suggest that \( U = U(\psi) \) and \( S = S(\chi) \), then the equations (54) are reduced to

\[
2W,\psi + U(\psi) = 0, \tag{56}
\]

\[
2W,\chi + S(\chi) = 0. \tag{57}
\]

Let us note that consistency relation \( W,\psi\chi = W,\chi\psi \) is satisfied.

Such representation gives us possibility to perform integration and find the superpotential \( W(\psi, \chi) \):

\[
W = -\frac{1}{2M_{Pl}^2} \left( \int U(\psi)d\psi + \int S(\chi)d\chi \right). \tag{58}
\]

Let us choose the linear dependence of the chiral fields derivatives

\[
U(\psi) = \mu_1 \psi + c_1, \quad S(\chi) = \mu_2 \chi + c_2 \tag{59}
\]
From here one can find the chiral field evolution

\[
\psi = \frac{1}{\mu_1} e^{\mu_1 t} - \frac{c_1}{\mu_1} \quad \psi = \frac{1}{\mu_2} e^{\mu_2 t} - \frac{c_2}{\mu_2}
\]  

(60)  

(61)

Then the superpotential can be obtained by the integration of (23) and reads

\[
W(\psi, \chi) = -\frac{1}{2} \left( \frac{\mu_1}{2} \psi^2 + c_1 \psi + \frac{\mu_2}{2} \chi^2 + c_2 \chi \right)
\]  

(62)

Further inserting the chiral fields in (62) we can obtain the Hubble function \( H = H(t) \)

\[
H(t) = -\frac{1}{4} \left( \frac{e^{2\mu_1 t}}{\mu_1} - \frac{c_1^2}{\mu_1} + \frac{e^{2\mu_2 t}}{\mu_2} - \frac{c_2^2}{\mu_2} \right)
\]  

(63)

Thus, we obtain double exponent solution for the scalar factor.

Let us choose the following dependence

\[
U(\psi) = \mu_1 \psi^k_1, \quad S(\chi) = \mu_2 \chi^k_2.
\]  

(64)

Exceptional situation \( k_1 = k_2 = 0 \) leads to very interesting solution.

\[
W(\psi, \chi) = -\frac{1}{2} (\mu_1^2 + \mu_2^2) t.
\]  

(65)

Remind that \( H(t) = W(\psi, \chi) + C_W \), we find

\[
a(t) = a_* \exp \left[ -\frac{1}{4} (\mu_1^2 + \mu_2^2) t^2 + C_W t \right]
\]  

(66)

This scale factor corresponds to Ruzmaikin solutions [61, 62].

When \( k_1, k_2 \neq 0,1 \) we have solutions for fields

\[
\psi(t) = [(1 - k_1)(\mu_1 t + c_1)]^{1/k_1}, \\
\chi(t) = [(1 - k_2)(\mu_2 t + c_2)]^{1/k_2}.
\]  

(67)  

(68)

The Hubble function is

\[
H(t) = -\frac{1}{2} \left[ \frac{\mu_1}{k_1 + 1} [(1 - k_1)(\mu_1 t + c_1)]^{k_1+1} + \frac{\mu_2}{k_2 + 1} [(1 - k_2)(\mu_2 t + c_2)]^{k_2+1} + C_W \right].
\]  

(69)

The scale factor is

\[
a(t) = a_* \exp \left[ \frac{\mu_1}{k_1 + 1} [(1 - k_1)(\mu_1 t + c_1)]^{2/k_1} + \frac{\mu_2}{k_2 + 1} [(1 - k_2)(\mu_2 t + c_2)]^{2/k_2} + C_W t \right].
\]  

(70)

Thus, the obtained solution corresponds to the intermediate inflation.

VI. MODIFIED POWER-LAW SOLUTIONS

As known solutions with the power-law Hubble parameter corresponds to the radiation and the matter dominated epochs. It is interesting to get exact solutions with the Hubble parameter \( H = C_0 + C_1/t \), where constants \( C_i \) can be chosen in such a way that the solution has both dark matter and dark energy parts.

We consider the CCM with the metric of the target space

\[
h_{AB} = \begin{pmatrix}
\frac{C_1 M^2_{Pl}}{\psi^2} & 0 \\
0 & \frac{C_2 M^2_{Pl}}{\chi^2}
\end{pmatrix},
\]  

(71)
and assume the following form of the superpotential

$$W = Y_0 - Y_1 \psi^{m_1} - Y_2 \chi^{m_2},$$

(72)

where $Y_i$ and $m_i$ are constants.

Using expression (29), we get the potential

$$V = 3M^2 P_i Y_0^2 - 6M^2 P_i Y_0 Y_1 \psi^{m_1} - 6M^2 P_i Y_0 Y_2 \chi^{m_2} + 6M^2 P_i Y_1 \psi^{m_1} Y_2 \chi^{m_2}$$

$$+ \frac{M^2 P_i Y_1^2 (3C_1 - 2m_1^2)}{C_1} \psi^{2m_1} + \frac{M^2 P_i Y_2^2 (3C_2 - 2m_2^2)}{C_2} \chi^{2m_2}.$$  

(73)

Equations (28) are:

$$\dot{\psi} = 2 \frac{Y_1 m_1}{C_1} \psi^{m_1+1}, \quad \dot{\chi} = 2 \frac{Y_2 m_2}{C_2} \chi^{m_2+1}.$$  

(74)

This system has the following solution:

$$\psi = \left( \frac{-C_1}{2Y_1 m_1^2 (t - t_0)} \right)^{1/m_1}, \quad \chi = \left( \frac{-C_2}{2Y_2 m_2^2 (t - t_0)} \right)^{1/m_2},$$

(75)

where $t_0$ and $\tilde{t}_0$ are integration constants. Substituting the obtained solution to the superpotential, we get:

$$W = Y_0 + \frac{C_1}{2m_1^2 (t - t_0)} + \frac{C_2}{2m_2^2 (t - t_0)}.$$  

(76)

Choosing $t_0 = 0$ and $\tilde{t}_0 = 0$, we get

$$W = Y_0 + \frac{1}{2} \left( \frac{C_1}{m_1^2} + \frac{C_2}{m_2^2} \right).$$  

(77)

So, the Hubble parameter is a sum of a constant that can correspond to the dark energy dominant epoch and the power-law function that corresponds to the radiation dominant epoch at $\frac{C_1}{m_1^2} + \frac{C_2}{m_2^2} = 1$ or to the matter dominant epoch at $\frac{C_1}{m_1^2} + \frac{C_2}{m_2^2} = \frac{4}{3}$. Choosing $t_0 \neq t_0$ we get more complicated solutions.

**VII. TARGET SPACE RECONSTRUCTION FROM THE SUPERPOWERTIAL**

**A. The search of the CCM with the given $H(t)$**

From given superpotential (Hubble function) it is possible to reconstruct kinetic part to get the exact solution of the model.

Considering single scalar fields cosmology we may find the different formulation of the problem because we have two independent differential equations with three functions. Therefore we must choose what function we have to consider as a given one. To fix the potential energy (or, simply, potential) is preferable because it may be collected from the high energy physics. If we study a canonical scalar field we suggest the unit multiplier in the kinetic energy term. Extending such approach to CCM we assume that the metric of a chiral space should be fixed and the analogue to the canonical field will be the unit diagonal metric. In the models with symmetry (for example, SO(3) - invariant CCM) chiral metric is fixed also.

Another approach can be proposed when we use the superpotential method. If we are looking for the CCM which obeys the given superpotential (or, equivalently, Hubble function) we can use so-called 'deformation of a chiral space' method. That is we define such metric components which the match to given data. Such an approach is similar, in some sense, to 'the fine tuning of the potential' method for a single scalar field where given Hubble function allows to define the potential and kinetic energies. Analyzing Eq. (23) which takes for the first field $\psi$ the following form

$$2M^2 P_i W_{\psi} = - h_{11}(\psi),$$

(78)

one can come to conclusion that functional part of the lhs may be included to the chiral metric component $h_{11}(\psi)$ and $\psi$ one can set equal to unity: $\psi = 1$. This gives us possibility (performing the same procedure for the second field $\chi$) to choose the linear dependence the fields on time

$$\psi = t + \psi_*, \quad \chi = t + \chi_*.$$  

(79)
B. Examples of periodic Hubble functions

To demonstrate such approach let us study an example of the following periodic Hubble function

$$H(t) = H_0 \sin(\lambda t) + C_W. \quad (80)$$

We may represent the superpotential in the following form

$$W(\psi, \chi) = H_0 \left[ (1 - \lambda_0) \sin(\lambda\psi) + \lambda_0 \sin(\lambda\chi) \right], \quad (81)$$

where $\lambda$ and $\lambda_0$ are constants.

The field equations (23) are nothing but the definition of the chiral metric component

$$h_{11}(\psi) = -2H_0M_{Pl}^2(1 - \lambda_0)\lambda \cos(\lambda\psi),$$
$$h_{22}(\chi) = -2H_0M_{Pl}\lambda_0 \lambda \cos(\lambda\chi). \quad (82)$$

The potential can be defined by (30):

$$V(\psi, \chi) = 3H_0^2M_{Pl}^2 \left[ (1 - \lambda_0) \sin(\lambda\psi) + \lambda_0 \sin(\lambda\chi) \right] + H_0^2M_{Pl}^2\lambda ((1 - \lambda_0) \cos(\lambda\psi) + \lambda_0 \cos(\lambda\chi)) \quad (83)$$

It is interesting to note that the solution (82) belongs to the two-field case in (36) with $u = 1$. Putting $\psi^* = \chi^* = 0$, we obtain the Hubble parameter (80). If $\psi^* = \chi^* = -\pi/(2\lambda)$, then we get

$$H(t) = H_0 \cos(\lambda t) + C_W. \quad (84)$$

The model constructed has two-parametric set of exact solutions with

$$H(t) = H_0 \left[ (1 - \lambda_0) \sin(\lambda(t + \psi^*)) + \lambda_0 \sin(\lambda(t + \chi^*)) \right] + C_W. \quad (85)$$

Note that the case

$$H(t) = H_0 \sin^2(\lambda t) + C_W \quad (86)$$

is reduced to the previous one due to substitution $\sin^2(\lambda t) = 1 - 2 \cos(2\lambda t)$.

For

$$H(t) = H_0 \exp(-\alpha t \sin t) \quad (87)$$

the same approach gives us the solution

$$H_0^{-1}W(\psi, \chi) = (1 - \lambda_0) \exp(-\alpha \psi \sin \psi) + \lambda_0 \exp(-\alpha \chi \sin \chi), \quad (88)$$

$$h_{11}(\psi) = 2H_0M_{Pl}^2\alpha (1 - \lambda_0) \left[ \sin(\psi) + \psi \cos(\psi) \right] \exp(-\alpha \psi \sin(\psi)), \quad (89)$$

$$h_{22}(\chi) = 2H_0M_{Pl}^2\alpha \lambda_0 \left[ \sin(\chi) + \chi \cos(\chi) \right] \exp(-\alpha \chi \sin(\chi)). \quad (90)$$

where $\psi = t + \psi^*$, $\chi = t + \chi^*$.

The potential $V(\psi, \chi)$ is

$$V(\psi, \chi) = 3H_0^2M_{Pl}^2 \left[ (1 - \lambda_0) \exp(-\alpha \psi \sin \psi) + \lambda_0 \exp(-\alpha \chi \sin \chi) \right]^2 - H_0M_{Pl}^2\alpha \left[ (1 - \lambda_0)(\sin(\psi + \psi \cos(\psi)) \exp(-\alpha \psi \sin(\psi)) + \lambda_0(\sin(\chi + \chi \cos(\chi)) \exp(-\alpha \chi \sin(\chi)) \quad (91)$$

C. Example of an hyperbolic Hubble function

Another example is connected with the scale factor

$$a(t) = a_0[k_\star \sinh(\lambda t)]^{2/3},$$
that corresponds to ΛCDM model. The Hubble function is

\[ H = H_0 \coth(\lambda t), \quad H_0 = \frac{2}{3} \lambda. \]

The corresponding superpotential is

\[ W(\psi, \chi) = H_0 \left[ (1 - \lambda_0) \coth(\lambda \psi) + \lambda_0 \coth(\lambda \chi) \right] \]

The solution is

\[ \psi = t + \psi*, \quad \chi = t + \chi* \quad (92) \]
\[ h_{11}(\psi) = 2M_P^2 H_0 (1 - \lambda_0) \lambda \sinh^{-2}(\lambda \psi) \quad (93) \]
\[ h_{22}(\chi) = 2M_P^2 H_0 \lambda_0 \lambda \sinh^{-2}(\lambda \chi) \quad (94) \]

Here and further we assume \( \psi \to (\psi - \psi*) \) and \( \chi \to (\chi - \chi*) \). The physical potential is

\[ V(\psi, \chi) = 3M_P^2 H_0^2 \left[ (1 - \lambda_0) \coth(\lambda \psi) + \lambda_0 \coth(\lambda \chi) \right]^2 - M_P^2 H_0 \lambda \left[ (1 - \lambda_0) \sinh^{-2}(\lambda \psi) + \lambda_0 \sinh^{-2}(\lambda \chi) \right] \quad (96) \]

Using the superpotential method, it is possible to get different models for the given time dependence of the Hubble parameter, that we demonstrate in the next section.

**VIII. THE CYCLIC UNIVERSE**

In this section, we focus on a cyclic type of Universe dubbed Quasi-Steady State (QSS) introduced to address the outstanding problems of the hot big bang, for instance, the singularity problem, in particular, see Refs. [63–65] and references therein. In this case, the Hubble parameter might increase at late times to account the well-known tension between Planck and local observations. The proposed early Universe modifications, namely, the interaction between known matter components or their interaction with dark energy, does not seem to account for the discrepancy [66].

One might attribute the latter to late time physics, for instance, to the emergence of phantom behaviour at late times. Quasi-steady state model includes such a feature. In what follows we construct single (double) field potentials corresponding to QSS.

**A. One-field models**

Let us construct CCM with the following form of scale factor that characterizes the quasi-steady state theory,

\[ a(t) = a_0 e^{C_W t (1 + \alpha \cos(\mu t))}, \quad (97) \]

where \( a_0, \alpha \) and \( \mu \) are constants. This type of the dynamics corresponds to the quasi-steady state model [63,64]. We assume that \( a(t) > 0 \) for any values of \( t \), so \( |\alpha| < 1 \). The shift of time \( t \to t + \pi/\mu \) is equivalent to the change of the sign of \( \alpha \), so we can assume that \( 0 \leq \alpha < 1 \) without the loss of generality\(^4\). By the same reason we can put \( \mu > 0 \).

The corresponding Hubble parameter is

\[ H = C_W - \frac{\mu \alpha \sin(\mu t)}{1 + \alpha \cos(\mu t)}, \quad (98) \]

and has the following time derivative:

\[ \dot{H} = -\mu^2 \alpha \frac{\alpha + \cos(\mu t)}{(1 + \alpha \cos(\mu t))^2}. \quad (99) \]

---

\(^4\) The case \( \alpha = 0 \) corresponds to a de Sitter solution.
Such behavior of the Hubble parameter can be reproduced in one-field models. Reconstructing the chiral metric component as in previous section we choose $\psi = \mu t$ and get

$$h_{11} = \frac{2\alpha (\cos \phi + \alpha)}{(1 + \alpha \cos \phi)^2}$$

(100)

and

$$V(\phi) = \frac{C_W^2 + \alpha^2(C_W^2 - \mu^2) \cos^2 \phi + \alpha \cos \phi (2C_W^2 - \mu^2) - 2\alpha C_W \mu \sin \phi (1 - \alpha \cos \phi)}{(1 + \alpha \cos \phi)^2}.$$  

(101)

The alternative variant is to consider $\psi(t)$ that tends to finite limits at $t \to \pm \infty$:

$$\psi(t) = \frac{2}{\mu \sqrt{1 - \alpha^2}} \arctan \left( \frac{\sqrt{1 - \alpha^2}}{1 + \alpha} \tan \left( \frac{\mu t}{2} \right) \right).$$

(102)

It is easy to check that

$$\cos^2 \left( \frac{\mu t}{2} \right) = \frac{(1 - \alpha) \cos^2 \left( \frac{\mu \sqrt{1 - \alpha^2} \psi}{2} \right)}{1 + \alpha - 2\alpha \cos^2 \left( \frac{\mu \sqrt{1 - \alpha^2} \psi}{2} \right)}, \quad \Rightarrow \quad \cos (\mu t) = \frac{\alpha - \cos (\mu \sqrt{1 - \alpha^2} \psi)}{\alpha \cos (\mu \sqrt{1 - \alpha^2} \psi) - 1}.$$  

(103)

Thus, the function $\psi(t)$ is a solution of the following equation:

$$\dot{\psi} = \frac{1}{1 + \alpha \cos (\mu t)} = U(\psi) = \frac{1 - \alpha \cos (\mu \sqrt{1 - \alpha^2} \psi)}{1 - \alpha^2}.$$  

(104)

In the case of a one scalar field model:

$$\dot{H} = -\frac{1}{2M^2_{Pl}} h_{11}(\psi) \dot{\psi}^2,$$  

(105)

therefore

$$h_{11} = 2M^2_{Pl} \mu^2 \left[ (1 - \alpha^2) \cos (\mu \sqrt{1 - \alpha^2} \psi) \right] = 2M^2_{Pl} \mu^2 \alpha \left[ \frac{(1 - \alpha^2) \cos (\mu \sqrt{1 - \alpha^2} \psi)}{1 - \alpha \cos (\mu \sqrt{1 - \alpha^2} \psi)} \right].$$

(106)

Using Eq. (23), we obtain

$$W'_{\psi} = \mu^2 \alpha \cos \left( \mu \sqrt{1 - \alpha^2} \psi \right) \quad \Rightarrow \quad W = \frac{\mu \alpha}{\sqrt{1 - \alpha^2}} \sin \left( \mu \sqrt{1 - \alpha^2} \psi \right).$$

(107)

The potential of the model constructed is defined by (20) and has the following form:

$$V = \frac{M^2_{Pl} \mu^2 \alpha}{1 - \alpha^2} \left[ 3\alpha - \cos (\mu \sqrt{1 - \alpha^2} \psi) - 2\alpha \cos^2 \left( \mu \sqrt{1 - \alpha^2} \psi \right) \right]$$

$$- \frac{6\alpha \mu C_W M^2_{Pl}}{1 - \alpha^2} \sin \left( \mu \sqrt{1 - \alpha^2} \psi \right) + 3M^2_{Pl} C_W^2.$$  

Note that $h_{11}$ changes its sign during the scalar field evolution, so $\psi$ is neither an ordinary scalar field, nor a phantom scalar field [67]. Assuming that the considering one-field models describe the dark energy, we get that the obtained exact solutions have the state parameters crossing the cosmological constant barrier. It has been shown in the paper [68], such transitions are physically implausible in one-field models because they are either realized by a discrete set of trajectories in the phase space or are unstable with respect to the cosmological perturbations. To describe such type of the dark energy one can use quintom models [27, 28, 69, 73].
B. The CCM quasi-steady state models

To construct two-field CCM with the Hubble parameter given by (98) we use relations \( W, \psi = -\frac{1}{2}h_{11}(\psi)\dot{\psi}, W, \chi = -\frac{1}{2}h_{22}(\chi)\dot{\chi}, \) and

\[
\frac{dW}{dt} = -\frac{1}{2} h_{11}(\psi)\dot{\psi}^2 - \frac{1}{2} h_{22}(\chi)\dot{\chi}^2. \tag{108}
\]

First of all we can easily make reconstruction of the chiral metric component as in previous section. We choose the superpotential in the form

\[
W(\psi, \chi) = -(1 - \lambda_0)\frac{\alpha \mu \cos \mu t(\psi)}{(1 + \alpha \cos \mu t(\psi))^2} - \lambda_0 \frac{\alpha \mu \cos \mu t(\chi)}{(1 + \alpha \cos \mu t(\chi))^2} \tag{109}
\]

Then we choose the dependence on \( t \) as

\[
\psi = t + \psi_*, \; \chi = t + \chi_* \tag{110}
\]

The chiral metric components will be

\[
h_{11}(\psi) = \frac{2\alpha \mu^2 \cos \mu \psi}{(1 + \alpha \cos \mu t)^2} \tag{111}
\]

\[
h_{22}(\chi) = \frac{2\alpha \mu^2}{(1 + \alpha \cos \mu \chi)^2} \tag{112}
\]

The physical potential can be easily derived by Eq. (29). (It can be wrote here but it is rather large.)

The superpotential evidently defined by (109) with the substitution (110).

There are others possible solutions connecting with the choice of chiral metric components. Let us study a few of them.

From (109) we find

\[
\frac{dW}{dt} = - \frac{\alpha \mu^2}{(1 + \alpha \cos \mu t)^2} (\cos(\mu t) + \alpha). \tag{113}
\]

Further we can decompose the expression above for the following two:

\[
\frac{\partial W(\psi, \chi)}{\partial \psi} = - \frac{\alpha \mu^2 \cos \mu t}{(1 + \alpha \cos \mu t)^2} = -\frac{1}{2} h_{11}(\psi)\dot{\psi}^2 \tag{114}
\]

\[
\frac{\partial W(\psi, \chi)}{\partial \chi} = - \frac{\alpha^2 \mu^2}{(1 + \alpha \cos \mu t)^2} = -\frac{1}{2} h_{22}(\chi)\dot{\chi}^2 \tag{115}
\]

Using the relations (110) we can perform integration of the Eqs. (114) and (115) and obtain two parts of the superpotential \( W_1(\psi) \) and \( W_2(\chi) \) in terms of elementary functions but in rather complicated form. Therefore, our task is to make combination which allows to integrate the expressions for the fields and for the superpotential with more suitable result.

To this end we can choose

\[
\dot{\psi}^2 = \frac{\mu^2}{(1 + \alpha \cos \mu t)^2}, \; h_{11} = 2\alpha \cos \mu t \tag{116}
\]

\[
\dot{\chi}^2 = \frac{\mu^2}{(1 + \alpha \cos \mu t)^2}, \; h_{22} = 2\alpha^2 \tag{117}
\]

As the result we obtain the following solution for the chiral fields

\[
\psi(t) = \frac{2}{\sqrt{1 - \alpha^2}} \arctan \left( \frac{1 - \alpha}{\sqrt{1 - \alpha^2}} \tan \left( \frac{\mu t}{2} \right) \right). \tag{118}
\]
\[ \chi(t) = \frac{2}{\sqrt{1-\alpha^2}} \arctan \left( \frac{1-\alpha}{\sqrt{1-\alpha^2}} \tan \left( \frac{\mu t}{2} \right) \right). \] (119)

From solution (118) one can obtain the chiral metric component \( h_{11}(\psi) \) in the form
\[ h_{11}(\psi) = 2\alpha \frac{(1-\alpha) - (1+\alpha) \tan \left( \frac{\sqrt{1-\alpha^2} \psi}{2} \right)}{(1-\alpha) + (1+\alpha) \tan \left( \frac{\sqrt{1-\alpha^2} \psi}{2} \right)}. \] (120)

Once again such presentation does not give us suitable form for the superpotential and for physical potential. For the same representation (116) one can choose another appearance for \( \chi \) and \( h_{22} \). For example,
\[ \dot{\chi}^2 = 2\alpha^2, \quad h_{22} = \frac{\mu^2}{(1+\alpha \cos \mu t)^2}. \] (121)

Then the solution for \( \chi \) is
\[ \chi = \sqrt{2\alpha t} + \chi_*, \]
and for \( h_{22}(\chi) \)
\[ h_{22}(\chi) = \frac{\mu^2}{(1+\alpha \cos \left( \frac{\mu t}{\sqrt{2\alpha}} \chi \right))^2}. \]

Thus we can state that the chiral metric, fields and physical potential may be different with respect to the given Hubble function. To stress this fact and find the suitable form of the superpotential we can generalize procedure of solutions generating by introducing two arbitrary functions \( f(\mu t) \) and \( y(\mu t) \) by the following way
\[ \alpha \mu^2 \frac{f(\mu t)^2 \cos \mu t}{f(\mu t)^2 (1+\alpha \cos \mu t)^2} = \frac{1}{2} h_{11}(\psi) \psi^2 \] (122)
\[ \alpha^2 \mu^2 \frac{y(\mu t)^2}{y(\mu t)^2 (1+\alpha \cos \mu t)^2} = \frac{1}{2} h_{22}(\chi) \chi^2 \] (123)

The functions \( f(\mu t) \) and \( y(\mu t) \) can be selected in such a way that integration is performed while finding fields. For example, if we are finding the field \( \psi \) we have to perform the integral
\[ \psi = \int \frac{d(\mu t)}{f(\mu t)(1+\alpha \cos \mu t)} \] (124)
To make the integral with elementary function as a solution one can choose as the example \( f(\mu t) = \frac{1}{\sin \mu t} \). Then the solution is
\[ \psi = -\frac{1}{\alpha} \ln(1+ \alpha \cos \mu t), \quad \text{where} \quad \alpha < 1. \] (125)

Under suggestion (124) one can find the chiral metric component \( h_{11} \) by the following way
\[ h_{11}(\psi) = 2\alpha f(\mu t)^2 \cos \mu t. \] (126)

In our case \( h_{11} = 2\alpha \frac{\cos \mu t}{\sin^2 \mu t} \). Finding dependence \( t \) on \( \psi \) from (125) one can obtain dependence \( h_{11} \) on \( \psi \) as following
\[ h_{11}(\psi) = 2\alpha^2 \frac{e^{-\alpha \psi} - 1}{\alpha^2 - (e^{-\alpha \psi} - 1)^2} \] (127)

Further we consider what types of the superpotential and physical potential will correspond to the solution for field \( \psi \) (125) and chiral metric component \( h_{11}(\psi) \) (127) using the freedom of possible choice a function \( y(\chi) \).

Let us start from the solution for \( \psi \)-part in the form (\( |\alpha| < 1 \))
\[ \psi = -\frac{1}{\alpha} \ln(1+ \alpha \cos \mu t), \quad f(\mu t) = \frac{1}{\sin \mu t} \] (128)
Note that
\[ \dot{\psi}(\psi) = \frac{\mu}{\alpha} e^{\alpha \psi} \sqrt{\alpha^2 - (e^{-\alpha \psi} - 1)^2}. \]

To find the dependence \( W \) on \( \psi \) we have to integrate the relation
\[ W_{,\psi} = -\frac{1}{2} h_{11}(\psi) \dot{\psi}, \]
which can be transformed to the following
\[ \frac{\partial W}{\partial \psi} = -\mu \alpha e^{-\alpha \psi} \frac{e^{-\alpha \psi} - 1}{\sqrt{\alpha^2 - (e^{-\alpha \psi} - 1)^2}}. \]

The solution is
\[ W_1(\psi) = -\frac{\mu}{\alpha^2} F(z) + \frac{\mu \alpha^2}{\alpha^3} \arctan \left( \frac{a^2 z - 1}{a F(z)} \right). \]

where
\[ z = \exp(\alpha \psi), \quad F(z) = \sqrt{-a^2 z^2 + 2z - 1}, \quad a^2 = 1 - \alpha^2 > 0. \]

Further we have various possibilities to define \( \chi \) and \( h_{22}(\chi) \). Let us select nontrivial choice \( y(\mu t) = 1 - \alpha \cos \mu t \).

Then we have the solution for the field \( \chi \)
\[ \chi = \frac{1}{\sqrt{1 - \alpha^2}} \arctan \left( \frac{\tan \mu t}{\sqrt{1 - \alpha^2}} \right), \quad \alpha^2 < 1 \]

The chiral metric component \( h_{22} \) is
\[ h_{22}(\chi) = 2 \alpha^2 \left( 1 - \alpha \left( 1 + (1 - \alpha^2) \tan^2(\sqrt{1 - \alpha^2} \chi)^{-1/2} \right) \right)^2 \]

Thus, we get
\[ \frac{dW(\chi)}{d\chi} = -\frac{\alpha^2 \mu}{(1 - \alpha^2)^{3/2}} \left( \sqrt{1 + (1 - \alpha^2) v^2} - \alpha \right)^2 (1 + v^2)^{-2}, \quad v = \tan(\sqrt{1 - \alpha^2} \chi). \]

The solution for the \( \chi \) part of the superpotential is
\[
W_2(\chi) = -\frac{\alpha \mu}{(1 - \alpha^2)^{2/3}} \left\{ 2(\alpha^2 - 1) \arctan(v/P(v)) - (2\alpha^2 - 1) \arctan(\alpha v/P(v)) + \frac{\alpha v(\alpha + 1)P(v)}{1 + v^2} - \frac{\sqrt{1 - \alpha^2}}{2} \chi - \frac{\alpha^2 v}{v^2 + 1} \right\}, \quad P(v) = \sqrt{v^2(1 - \alpha^2) + 1}. \]

The superpotential is equal to
\[ W(\psi, \chi) = W_1(\psi) + W_2(\chi). \]

The next step is to calculate the potential \( V(\psi, \chi) \) by Eq. (29). This is possible but the answer will be too long.

It is possible to get the same time evolution of the scalar factor using a quintom model. The time derivative of the Hubble parameter \( H \) can be presented in the following form:
\[ \dot{H} = -\frac{\mu^2 \alpha (\alpha + \cos(\mu t))}{[1 + \alpha \cos(\mu t)]^2} = \frac{-(\alpha^2 + \alpha) \mu^2}{[1 + \alpha \cos(\mu t)]^2} + \frac{2 \mu^2 \alpha \sin^2 \left( \frac{\mu t}{2} \right)}{[1 - \alpha + 2 \alpha \cos^2 \left( \frac{\mu t}{2} \right)]^2}. \]
Let us introduce two scalar fields with the same time behavior:

\[
\psi(t) = \frac{2}{\mu \sqrt{1-\alpha^2}} \arctan\left(\frac{\sqrt{1-\alpha^2}}{1+\alpha} \tan\left(\frac{\mu t}{2}\right)\right),
\]

\[
\chi(t) = \frac{2}{\mu \sqrt{1-\alpha^2}} \arctan\left(\frac{\sqrt{1-\alpha^2}}{1+\alpha} \tan\left(\frac{\mu t}{2}\right)\right).
\]

Using Eq. 133, the time derivative of \(H\) can be rewritten as follows:

\[
\dot{H} = -(1+\alpha)\alpha \mu^2 \dot{\psi}^2 + 2\alpha \mu^2 \left[1 - \frac{(1-\alpha)\cos^2\left(\frac{\mu \sqrt{1-\alpha^2} \chi}{2}\right)}{1 + \alpha - 2\alpha \cos^2\left(\frac{\mu \sqrt{1-\alpha^2} \chi}{2}\right)}\right] \dot{\chi}^2.
\]

So, we get the the matrix \(h_{AB}\) as diagonal and

\[
h_{11} = 2(1+\alpha)\alpha M_{Pl}^2 \mu^2, \quad h_{22} = \frac{-2\alpha(1+\alpha)\mu^2 M_{Pl}^2 \left[1 - \cos(\mu \sqrt{1-\alpha^2} \chi)\right]}{1 - \alpha \cos(\mu \sqrt{1-\alpha^2} \chi)}.
\]

The field \(\psi(t)\) is an ordinary scalar field, whereas \(\chi\) is a phantom scalar field.

We assume that the superpotential has the form (33):

\[
W = W_1(\psi) + W_2(\chi).
\]

Using Eq. 23, we get

\[
W_1 = \frac{\mu^2 \alpha}{\alpha - 1} \psi - \frac{\mu \alpha^2}{\sqrt{1-\alpha^2}(\alpha - 1)} \sin\left(\mu \sqrt{1-\alpha^2} \psi\right),
\]

\[
W_2 = \frac{\mu^2 \alpha}{\alpha - 1} \chi - \frac{\mu \alpha}{\sqrt{1-\alpha^2}(\alpha - 1)} \sin\left(\mu \sqrt{1-\alpha^2} \chi\right).
\]

Using Eq. 29 we get the potential of the obtained two-field model in the following form:

\[
V = \frac{3 M_{Pl}^2}{(\alpha - 1)^3 (\alpha + 1)} \left[\alpha^2 \mu^4 (\alpha^2 - 1) \left(\psi^2 + \chi^2\right) + 2\mu^4 \alpha^2 (\alpha^2 - 1) \psi \chi
\right.

\left. + 2\mu^2 \alpha C_W (\alpha + 1)(\alpha - 1)^2 \psi + 2\mu^2 \alpha C_W (\alpha + 1)(\alpha - 1)^2 \chi + (\alpha + 1)(\alpha - 1)^3 C_W^2
\right. 

\left. - \alpha^2 (\alpha^2 + 1) \mu^2 - 2\alpha^3 \mu^2 \sin\left(\mu \sqrt{1-\alpha^2} \psi\right) \sin\left(\mu \sqrt{1-\alpha^2} \chi\right)
\right.

\left. + 2\alpha \mu \sqrt{1-\alpha^2} \left[\alpha \mu^2 (\chi + \psi) + (\alpha - 1) C_W\right] \sin\left(\mu \sqrt{1-\alpha^2} \chi\right)
\right.

\left. + 2\alpha^2 \mu \sqrt{1-\alpha^2} \left[\alpha \mu^2 (\chi + \psi) + (\alpha - 1) C_W\right] \sin\left(\mu \sqrt{1-\alpha^2} \psi\right)
\right.

\left. + \frac{1}{3} \mu^2 \alpha^2 \left[(\alpha + 2) \cos\left(\mu \sqrt{1-\alpha^2} \chi\right) + (2 \alpha^2 + \alpha) \cos\left(\mu \sqrt{1-\alpha^2} \psi\right)^2\right]
\right.

\left. + \frac{1}{3} \mu^2 \alpha^2 \left(\alpha (1-\alpha^2) \cos\left(\mu \sqrt{1-\alpha^2} \chi\right) + 2\alpha^2 (\alpha - 1) \cos\left(\mu \sqrt{1-\alpha^2} \psi\right)\right)\right].
\]

**IX. THE REDUCING TWO-FIELD DYNAMIC EQUATIONS TO THE SINGLE FIELD ONES**

For simplification of the generation of the exact solutions for chiral cosmological models (CCM) we consider the possibility of the reducing the dynamic equations in such type of models to a single field case.

For this aim, we write the dynamic equations in terms of the effective field \(\varphi\), which is connected with CCM-fields by the following relation [21]

\[
\varphi^2 = h_{AB} \varphi^A \varphi^B,
\]

(142)
where \( \varphi^A \) are the fields in CCM and \( h_{AB} \) is the metric tensor of a target space.

The double kinetic energy \( X = \dot{\varphi}^2 \) of the effective field \( \varphi \) is considered earlier as \( X \)-field (11). For positive \( X = \dot{\varphi}^2 > 0 \) one has the canonical effective scalar field \( \varphi \) and \( X < 0 \) corresponds to phantom one. The dynamic equations in CCM (6)–(10) can be noted as

\[
3H^2 M_{Pl}^2 = \frac{1}{2} \dot{\varphi}^2 + V(\varphi),
\]

(143)

\[
\dot{\varphi}^2 = -2HM_{Pl}^2,
\]

(144)

\[
D_\nu \dot{\varphi}^A + 3H \dot{\varphi}^A + h^{AB} V_B = 0,
\]

(145)

where

\[
D_{\nu} \dot{\varphi}^A = \frac{d\dot{\varphi}^A}{dt} + \Gamma^A_{BC} \dot{\varphi}^B \dot{\varphi}^C,
\]

(146)

is a covariant derivative in a target space.

Also, one can rewrite the first dynamic equation (143) on the basis of the equation (144) in the following form

\[
V(\varphi) = M_{Pl}^2 (3H^2 + \dot{\varphi}).
\]

(147)

Therefore, in the general case, the connection between CCMs and one field models is defined on the basis of the equation (145) as follows

\[
\ddot{\varphi}^A + \Gamma^A_{BC} \dot{\varphi}^B \dot{\varphi}^C + 3H \dot{\varphi}^A + h^{AB} V_B = \ddot{\varphi} + 3H \dot{\varphi} + V_\varphi = 0.
\]

(148)

Further, we consider the fulfillment of this condition for the case of CCM with two identical scalar fields \( \phi = \psi \) by the specific connections between components of the tensor \( h_{AB} \).

Now, we consider the partial case of CCM with two identical scalar fields \( \phi = \psi \) and the specific connections between components of the tensor \( h_{AB} \).

Firstly, we write the dynamic equations (143)–(145) for CCM with two fields

\[
3H^2 M_{Pl}^2 = \frac{1}{2} h_{11} \dot{\phi}^2 + h_{12} \dot{\phi} \dot{\psi} + \frac{1}{2} h_{22} \dot{\psi}^2 + V(\phi, \psi),
\]

(149)

\[
- \dot{H} M_{Pl}^2 = \frac{1}{2} h_{11} \dot{\phi}^2 + h_{12} \dot{\phi} \dot{\psi} + \frac{1}{2} h_{22} \dot{\psi}^2,
\]

(150)

\[
3H \left( h_{11} \dot{\phi} + h_{12} \dot{\psi} \right) + \frac{\partial}{\partial \phi} \left( h_{11} \phi + h_{12} \dot{\psi} \right) - \frac{1}{2} \frac{\partial h_{11}}{\partial \phi} \dot{\phi}^2 - \frac{\partial h_{12}}{\partial \phi} \dot{\psi} - \frac{1}{2} \frac{\partial h_{22}}{\partial \phi} \dot{\psi}^2 + \frac{\partial V}{\partial \phi} = 0,
\]

(151)

\[
3H \left( h_{12} \dot{\phi} + h_{22} \dot{\psi} \right) + \frac{\partial}{\partial \psi} \left( h_{12} \phi + h_{22} \dot{\psi} \right) - \frac{1}{2} \frac{\partial h_{11}}{\partial \psi} \dot{\phi}^2 - \frac{\partial h_{12}}{\partial \psi} \dot{\phi} - \frac{1}{2} \frac{\partial h_{22}}{\partial \psi} \dot{\psi}^2 + \frac{\partial V}{\partial \psi} = 0,
\]

(152)

Secondly, for identical scalar fields \( \phi = \psi \) with the following metric tensor of targets space

\[
h_{AB} = \begin{pmatrix}
-h_{12} + \frac{n}{2} & \frac{h_{12}}{2} \\
\frac{h_{21}}{2} & -h_{12} + \frac{n}{2}
\end{pmatrix},
\]

where \( h_{11} = h_{22} = \frac{n}{2} - h_{12}, h_{21} = h_{12} \) and \( n \) is an arbitrary constant, we have

\[
h_{11} \dot{\phi} + h_{12} \dot{\psi} = h_{12} \dot{\phi} + h_{22} \dot{\psi} = \frac{n}{2} \dot{\phi} + \frac{n}{2} \dot{\psi},
\]

(153)
\[
\frac{1}{2} \frac{\partial h_{11}}{\partial \phi} \dot{\phi}^2 + \frac{\partial h_{12}}{\partial \phi} \dot{\phi} \dot{\psi} + \frac{1}{2} \frac{\partial h_{22}}{\partial \psi} \dot{\psi}^2 = \frac{1}{2} \frac{\partial h_{11}}{\partial \psi} \dot{\psi}^2 + \frac{\partial h_{12}}{\partial \psi} \dot{\phi} \dot{\psi} + \frac{1}{2} \frac{\partial h_{22}}{\partial \phi} \dot{\phi}^2 = 0, 
\]

(154)

\[
\frac{1}{2} h_{11} \ddot{\phi}^2 + h_{12} \dot{\phi} \dot{\psi} + \frac{1}{2} h_{22} \dot{\psi}^2 = \frac{n}{2} \dot{\phi}^2 = \frac{n}{2} \dot{\psi}^2. 
\]

(155)

Thus, from the equations (149)–(152) we obtain

\[
V(\phi, \psi) = M_{Pl}^2 (3H^2 + \dot{H}) = V(\varphi), 
\]

(156)

\[
- \dot{H} M_{Pl}^2 = \frac{n}{2} \dot{\phi}^2 = \frac{n}{2} \dot{\psi}^2 = \frac{1}{2} \dot{\varphi}^2, 
\]

(157)

\[
\ddot{\varphi} + 3H \dot{\varphi} + \frac{2}{n} \frac{\partial V}{\partial \varphi} = 0. 
\]

(158)

Finally, we note that from (156)–(157) one can obtain (158) and the equation

\[
\ddot{\varphi} + 3H \dot{\varphi} + V,\varphi = 0, 
\]

(159)

is a differential consequence of (156)–(157) as well.

Therefore, in this case, we have the fulfillment of the condition (148).

Further, we consider the following superpotential of the effective field

\[
W(\varphi) \equiv H(t), 
\]

(160)

and write the dynamic equations (156)–(158) as

\[
V(\varphi) = M_{Pl}^2 \left[ 3W^2(\varphi) - 2M_{Pl} \left( \frac{dW(\varphi)}{d\varphi} \right)^2 \right], 
\]

(161)

\[
\ddot{\varphi} = -2M_{Pl}^2 \left( \frac{dW(\varphi)}{d\varphi} \right), 
\]

(162)

for the effective field

\[
\varphi = \pm \sqrt{\frac{n}{2}} (\phi + \psi), 
\]

(163)

where the constant parameter \( n \) defines the character of an effective field \( \varphi \), namely this field can be canonical or phantom for the different signs of \( n \).

Also, we define the metric tensor of target space as

\[
h_{AB} = \left( \begin{array}{cc} -f(\phi) + \frac{n}{2} & \frac{1}{2} (f(\phi) + f(\psi)) \\ \frac{1}{2} (f(\phi) + f(\psi)) & -f(\psi) + \frac{n}{2} \end{array} \right), 
\]

where \( f(\phi) \equiv f(\psi) \) are an arbitrary functions that define the interaction between scalar fields \( \phi \) and \( \psi \).

For example, we obtain the other exact solutions for two identical scalar fields with linear dependence form cosmic time than ones in Sec. VII for (79) and Sec. VIII for the same fields (110) on the basis of proposed approach.

For the following superpotential

\[
W(\varphi) = - \frac{\alpha}{2M_{Pl}^2} \varphi, 
\]

(164)

from equation (162) we have

\[
\varphi(t) = \alpha t - \beta, 
\]

(165)

where \( \beta \) is a constant of integration.
The Hubble parameter and the scale factor are

\[ H(t) = -\frac{\alpha}{2M_P^2}(\alpha t - \beta), \]  
(166)

\[ a(t) = a_0 \exp \left[ \frac{\alpha}{4M_P} t (2\beta - \alpha t) \right], \]  
(167)

corresponding to Ruzmaikins solutions \[61, 62\].

From the equation (161) we obtain the following potential of the effective field

\[ V(\varphi) = \left( \frac{\alpha}{2M_P} \right)^2 \left[ 3\varphi^2 - 2M_P^2 \right]. \]  
(168)

After substituting (163) into the solutions for \( \varphi \) we obtain

\[ \phi(t) = \psi(t) = \pm \frac{1}{\sqrt{n}} (\alpha t - \beta), \]  
(169)

\[ V(\varphi, \psi) = \left( \frac{\alpha}{2M_P} \right)^2 \left[ \frac{3n}{4} (\varphi + \psi)^2 - 2M_P^2 \right], \]  
(170)

the potential and evolution of the CCM-fields corresponding to the same dynamics (166)–(167) of the early Universe.

As the other example, we consider the exact solutions defined by the following superpotential

\[ W(\varphi) = A^8 M_P^2 \varphi^2 + \lambda, \]  
(171)

where \( A \) and \( \lambda \) are an arbitrary constant.

From equations (161)–(162) one has

\[ H(t) = B \exp(-At) + \lambda, \]  
(172)

\[ a(t) = a_0 \exp \left( \lambda t - \frac{B}{A} e^{-At} \right), \]  
(173)

\[ \varphi(t) = \sqrt{\frac{8B}{A}} \exp \left( -\frac{At}{2} \right), \]  
(174)

\[ V(\varphi) = 3 \left( \frac{A}{8M_P} \right)^2 \varphi^4 + \frac{A}{4} \left( 3\lambda - \frac{A}{2} \right) \varphi^2 + 3\lambda^2, \]  
(175)

which correspond to the Higgs potential.

As the special case for \( \lambda = A/6 \) one has the potential for chaotic inflation

\[ V(\varphi) = 3 \left( \frac{A}{8M_P} \right)^2 \varphi^4 + 3\lambda^2. \]  
(176)

After replacing the effective field \( \varphi \) on CCM-fields

\[ \phi(t) = \psi(t) = \pm \sqrt{\frac{8B}{An}} \exp \left( -\frac{At}{2} \right), \]  
(177)

we have

\[ V(\phi, \psi) = \frac{3}{16} \left( \frac{An}{8M_P} \right)^2 (\phi + \psi)^4 + \frac{nA}{16} \left( 3\lambda - \frac{A}{2} \right) (\phi + \psi)^2 + 3\lambda^2, \]  
(178)

or, for \( \lambda = A/6 \) the potential of CCM-fields is

\[ V(\phi, \psi) = \frac{3}{16} \left( \frac{An}{8M_P} \right)^2 (\phi + \psi)^4 + 3\lambda^2. \]  
(179)

Thus, we have obtained these solutions for an arbitrary function \( f(\phi) = f(\psi) \) in the metric tensor of the target space. Under conditions \( f(\phi) = f(\psi) = 0 \) one has the same solutions for the trivial case of constant diagonal tensor \( h_{AB} \).

Similarly, one can generalize any exact solutions in single field models (see, for example, \[20, 74\]) on this special class of chiral cosmological models with two components.
X. CONCLUSIONS

In this paper, we develop the superpotential technique for the chiral cosmological models. The key point in this method is that the Hubble parameter is considered as a function of the scalar fields, and this allows one to reconstruct the scalar field potential. The CCM models are actively used in cosmology and can be connected with modified gravity models due to the conformal transformation of the metric. So, the proposed method allows constructing modified gravity models with exact solutions. In particular, the CCMs that correspond to $f(R)$ gravity models with one scalar field have been considered in Section [V]. The corresponding two-field CCMs with asymptotic de Sitter solutions have been constructed. In future, we shall explore the possibility of application of our results to inflation.

To demonstrate that the proposed reconstruction procedure is powerful, we constructed the CCM with different behaviour of the Hubble parameter that actively used in cosmology. In particular, in Sections [V] and [VI] we have found models with Ruzmaikins solutions that correspond to the intermediate inflation and modified power-law solutions, for which the Hubble parameter is a sum of a constant and a function inverse proportional to the cosmic time. These solutions have been found for the models with the given kinetic terms of the actions. In our case, it is possible to choose both the potential and the function that defines the kinetic term. The construction of trigonometric and hyperbolic Hubble functions due to a suitable choice of the kinetic term has been proposed in Section [VII]. In Section [VIII], we constructed one- and two-field models that correspond to the Hubble parameter that describes a cyclic type of Universe dubbed quasi-steady state. We demonstrated that the superpotential method allows to construct different models with one and the same Hubble parameter.

In Section [IX], we shown how an exact particular solution for the CCM can be obtained from the one-field models. The correspondence between one- and multifield models is actively used for multifield inflationary models in the method of cosmological attractors [75, 76]. Let us note that compared to the method of cosmological attractors, the proposed algorithm, allows to obtain the exact solutions.

The superpotential method is suitable for construction of inflationary scenarios in one-field models [46–49]. In future, we plan to generalize this method to the chiral cosmological inflationary models with many scalar fields.

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