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Simplicity of vacuum modules and associated varieties
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SIMPPLICITY OF VACUUM MODULES AND ASSOCIATED VARIETIES

by Tomoyuki Arakawa, Cuipo Jiang & Anne Moreau

Abstract. — In this note, we prove that the universal affine vertex algebra associated with a simple Lie algebra $g$ is simple if and only if the associated variety of its unique simple quotient is equal to $g^\ast$. We also derive an analogous result for the quantized Drinfeld-Sokolov reduction applied to the universal affine vertex algebra.

Résumé (Simplicité des algèbres vertex affines et variétés associées). — Dans cet article, nous démontrons que l’algèbre vertex affine universelle associée à une algèbre de Lie simple $g$ est simple si et seulement si la variété associée à son unique quotient simple est égale à $g^\ast$. Nous en déduisons un résultat analogue pour la réduction quantique de Drinfeld-Sokolov appliquée à l’algèbre vertex affine universelle.

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1. Introduction

Let $V$ be a vertex algebra, and let

$$V \to (\text{End} V)[z,z^{-1}], \quad a \mapsto a(z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1},$$

be the state-field correspondence. The Zhu $C_2$-algebra [Zhu96] of $V$ is by definition the quotient space $R_V = V/C_2(V)$, where $C_2(V) = \text{span}_C \{a_{(-2)}b \mid a,b \in V\}$, equipped with the Poisson algebra structure given by

$$\{a,b\} = a_{(0)}b,$$

$$\overline{a.b} = a_{(-1)}b,$$
for $a, b \in V$ with $\pi := a + C_2(V)$. The associated variety $X_V$ of $V$ is the reduced scheme $X_V = \text{Specm}(R_V)$ corresponding to $R_V$. It is a fundamental invariant of $V$ that captures important properties of the vertex algebra $V$ itself (see, for example, [BFM, Zhu96, ABD04, Miy04, Ara12a, Ara15a, Ara15b, AM18a, AM17, AK18]). Moreover, the associated variety $X_V$ conjecturally [BR18] coincides with the Higgs branch of a 4D $\mathcal{N} = 2$ superconformal field theory $\mathcal{T}$, if $V$ corresponds to a theory $\mathcal{T}$ by the 4D/2D duality discovered in [BLL+15]. Note that the Higgs branch of a 4D $\mathcal{N} = 2$ superconformal field theory is a hyperkähler cone, possibly singular.

In the case where $V$ is the universal affine vertex algebra $V^k(\mathfrak{g})$ at level $k \in \mathbb{C}$ associated with a complex finite-dimensional simple Lie algebra $\mathfrak{g}$, the variety $X_V$ is just the affine space $\mathfrak{g}^*$ with Kirillov-Kostant Poisson structure. In the case where $V$ is the unique simple graded quotient $L_k(\mathfrak{g})$ of $V^k(\mathfrak{g})$, the variety $X_V$ is a Poisson subscheme of $\mathfrak{g}^*$ which is $G$-invariant and conic, where $G$ is the adjoint group of $\mathfrak{g}$.

Note that if the level $k$ is irrational, then $L_k(\mathfrak{g}) = V^k(\mathfrak{g})$, and hence $X_{L_k(\mathfrak{g})} = \mathfrak{g}^*$. More generally, if $L_k(\mathfrak{g}) = V^k(\mathfrak{g})$, that is, $V^k(\mathfrak{g})$ is simple, then obviously $X_{L_k(\mathfrak{g})} = \mathfrak{g}^*$.

In this article, we prove that the converse is true.

**Theorem 1.1.** The equality $L_k(\mathfrak{g}) = V^k(\mathfrak{g})$ holds, that is, $V^k(\mathfrak{g})$ is simple, if and only if $X_{L_k(\mathfrak{g})} = \mathfrak{g}^*$.

It is known by Gorelik and Kac [GK07] that $V^k(\mathfrak{g})$ is not simple if and only if

$$r^\vee(k + h^\vee) \in \mathbb{Q}_{\geq 0} \setminus \{1/m \mid m \in \mathbb{Z}_{\geq 1}\},$$

where $h^\vee$ is the dual Coxeter number and $r^\vee$ is the lacing number of $\mathfrak{g}$. Therefore, Theorem 1.1 can be rephrased as

$$X_{L_k(\mathfrak{g})} \subseteq \mathfrak{g}^* \iff (1.1) \text{ holds.}$$

Let us mention the cases when the variety $X_{L_k(\mathfrak{g})}$ is known for $k$ satisfying (1.1).

First, it is known [Zhu96, DM06] that $X_{L_k(\mathfrak{g})} = \{0\}$ if and only if $L_k(\mathfrak{g})$ is integrable, that is, $k$ is a nonnegative integer. Next, it is known that if $L_k(\mathfrak{g})$ is admissible [KW89], or equivalently, if

$$k + h^\vee = \frac{p}{q}, \quad p, q \in \mathbb{Z}_{\geq 1}, \ (p, q) = 1, \ p \geq \begin{cases} h^\vee & \text{if } (r^\vee, q) = 1, \\ h & \text{if } (r^\vee, q) \neq 1, \end{cases}$$

where $h$ is the Coxeter number of $\mathfrak{g}$, then $X_{L_k(\mathfrak{g})}$ is the closure of some nilpotent orbit in $\mathfrak{g}$ ([Ara15a]). Further, it was observed in [AM18a, AM18b] that there are cases when $L_k(\mathfrak{g})$ is non-admissible and $X_{L_k(\mathfrak{g})}$ is the closure of some nilpotent orbit. In fact, it was recently conjectured in physics [XY19] that, in view of the 4D/2D duality, there should be a large list of non-admissible simple affine vertex algebras whose associated varieties are the closures of some nilpotent orbits. Finally, there are also cases [AM17] where $X_{L_k(\mathfrak{g})}$ is neither $\mathfrak{g}^*$ nor contained in the nilpotent cone $\mathcal{N}(\mathfrak{g})$ of $\mathfrak{g}$.

In general, the problem of determining the variety $X_{L_k(\mathfrak{g})}$ is wide open.
Now let us explain the outline of the proof of Theorem 1.1. First, Theorem 1.1 is known for the critical level $k = -h^\vee$ ([FF92, FG04]). Therefore, since $R_{V^\ast}(g)$ is a polynomial ring $\mathbb{C}[g^\ast]$, Theorem 1.1 follows from the following fact.

**Theorem 1.2.** — Suppose that the level is non-critical, that is, $k \neq -h^\vee$. The image of any nonzero singular vector $v$ of $V^k(g)$ in the Zhu $C_2$-algebra $R_{V^\ast}(g)$ is nonzero.

The symbol $\sigma(w)$ of a singular vector $w$ in $V^k(g)$ is a singular vector in the corresponding vertex Poisson algebra $\text{gr}V^k(g) \cong S(t^{-1}g[t^{-1}]) \cong \mathbb{C}[J_\infty g^\ast]$, where $J_\infty g^\ast$ is the arc space of $g^\ast$. Theorem 1.2 states that the image of $\sigma(w)$ of a non-trivial singular vector $w$ under the projection

$$\mathbb{C}[J_\infty g^\ast] \longrightarrow \mathbb{C}[g^\ast] = R_{V^\ast}(g)$$

is nonzero, provided that $k$ is non-critical. Here the projection (1.3) is defined by identifying $\mathbb{C}[g^\ast]$ with the Zhu $C_2$-algebra of the commutative vertex algebra $\mathbb{C}[J_\infty g^\ast]$. Hence, Theorem 1.2 would follow if the image of any nontrivial singular vector in $\mathbb{C}[J_\infty g^\ast]$ under the projection (1.3) is nonzero. However, this is false as there are singular vectors in $\mathbb{C}[J_\infty g^\ast]$ that do not come from singular vectors of $V^k(g)$ and that belong to the kernel of (1.3) (see Section 3.4). Therefore, we do need to make use of the fact that $\sigma(w)$ is the symbol of a singular vector $w$ in $V^k(g)$. We also note that the statement of Theorem 1.2 is not true if $k$ is critical (see Section 3.4).

For this reason the proof of Theorem 1.2 is divided roughly into two parts. First, we work in the commutative setting to deduce a first important reduction (Lemma 3.1). Next, we use the Sugawara construction — which is available only at non-critical levels — in the non-commutative setting in order to complete the proof.

Now, let us consider the $W$-algebra $\mathcal{W}^k(g,f)$ associated with a nilpotent element $f$ of $g$ at the level $k$ defined by the generalized quantized Drinfeld-Sokolov reduction [FF90, KRW03]:

$$\mathcal{W}^k(g,f) = H^0_{DS,f}(V^k(g)).$$

Here, $H^0_{DS,f}(M)$ denotes the BRST cohomology of the generalized quantized Drinfeld-Sokolov reduction associated with $f \in N(g)$ with coefficients in a $V^k(g)$-module $M$.

By the Jacobson-Morosov theorem, $f$ embeds into an $\mathfrak{sl}_2$-triple $(e, h, f)$. The Slodowy slice $\mathcal{J}_f$ at $f$ is the affine space $\mathcal{J}_f = f + g^\circ$, where $g^\circ$ is the centralizer of $e$ in $g$. It has a natural Poisson structure induced from that of $g^\ast$ (see [GG02]), and we have [DSK06, Ara15a] a natural isomorphism $R_{\mathcal{W}^k(g,f)} \cong \mathbb{C}[\mathcal{J}_f]$ of Poisson algebras, so that

$$X_{\mathcal{W}^k(g,f)} = \mathcal{J}_f.$$

The natural surjection $V^k(g) \twoheadrightarrow L_k(g)$ induces a surjection $\mathcal{W}^k(g,f) \twoheadrightarrow H^0_{DS,f}(L_k(g))$ of vertex algebras ([Ara15a]). Hence the variety $X_{H^0_{DS,f}(L_k(g))}$ is a $C^\ast$-invariant Poisson subvarieties of the Slodowy slice $\mathcal{J}_f$.

Conjecturally [KRW03, KW08], the vertex algebra $H^0_{DS,f}(L_k(g))$ coincides the unique simple (graded) quotient $\mathcal{W}^k(g,f)$ of $\mathcal{W}^k(g,f)$ provided that $H^0_{DS,f}(L_k(g)) \neq 0$. (This conjecture has been verified in many cases [Ara05, Ara07, Ara11, AvE19].)
As a consequence of Theorem 1.1, we obtain the following result.

**Theorem 1.3.** — Let \( f \) be any nilpotent element of \( \mathfrak{g} \). The following assertions are equivalent:

1. \( V^k(\mathfrak{g}) \) is simple,
2. \( \mathcal{W}^k(\mathfrak{g}, f) = H^0_{DS,f}(L_k(\mathfrak{g})) \),
3. \( X_{H^0_{DS,f}(L_k(\mathfrak{g}))} = \mathcal{F}_f \).

Note that Theorem 1.3 implies that \( V^k(\mathfrak{g}) \) is simple if \( \mathcal{W}^k(\mathfrak{g}, f) = \mathcal{F}_f \) and \( H^0_{DS,f}(L_k(\mathfrak{g})) \neq 0 \) since \( \mathcal{W}^k(\mathfrak{g}, f) \subseteq X_{H^0_{DS,f}(L_k(\mathfrak{g}))} \).

The remainder of the paper is structured as follows. In Section 2 we set up notation in the case of affine vertex algebras that will be the framework of this note. Section 3 is devoted to the proof of Theorem 1.1. In Section 4, we have compiled some known facts on Slodowy slices, \( W \)-algebras and their associated varieties. Theorem 1.3 is proved in this section.

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### 2. Universal affine vertex algebras and associated graded vertex Poisson algebras

Let \( \hat{\mathfrak{g}} \) be the affine Kac-Moody algebra associated with \( \mathfrak{g} \), that is,

\[
\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K,
\]

where the commutation relations are given by

\[
[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m(x|y)\delta_{m+n,0}K, \quad [K, \hat{\mathfrak{g}}] = 0,
\]

for \( x, y \in \mathfrak{g} \) and \( m, n \in \mathbb{Z} \). Here,

\[
(\ | \ ) = \frac{1}{2h} \times \text{Killing form of } \mathfrak{g}
\]

is the usual normalized inner product. For \( x \in \mathfrak{g} \) and \( m \in \mathbb{Z} \), we shall write \( x(m) \) for \( x \otimes t^m \).

#### 2.1. Universal affine vertex algebras. — For \( k \in \mathbb{C} \), set

\[
V^k(\mathfrak{g}) = U(\hat{\mathfrak{g}}) \otimes U(\mathfrak{g}[t] \oplus \mathbb{C}K) \mathbb{C}_k,
\]

where \( \mathbb{C}_k \) is the one-dimensional representation of \( \mathfrak{g}[t] \oplus \mathbb{C}K \) on which \( K \) acts as multiplication by \( k \) and \( \mathfrak{g} \oplus \mathbb{C}[t] \) acts trivially.

By the Poincaré-Birkhoff-Witt Theorem, the direct sum decomposition, we have

(2.1) \[
V^k(\mathfrak{g}) \cong U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) = U(t^{-1}\mathfrak{g}[t^{-1}]).
\]

The space \( V^k(\mathfrak{g}) \) is naturally graded,

\[
V^k(\mathfrak{g}) = \bigoplus_{\Delta \in \mathbb{Z}_{\geq 0}} V^k(\mathfrak{g})_\Delta,
\]
where the grading is defined by
\[ \text{deg}(x^1(−n_1) \cdots x^r(−n_r)1) = ∑_{i=1}^r n_i, \quad r \geq 0, \ x^{ij} ∈ \mathfrak{g}, \]
with 1 the image of 1 ⊗ 1 in \( V^k(\mathfrak{g}) \). We have \( V^k(\mathfrak{g})_0 = \mathbb{C}1 \), and we identify \( \mathfrak{g} \) with \( V^k(\mathfrak{g})_1 \) via the linear isomorphism defined by \( x ⇔ x(−1)1 \).

It is well-known that \( V^k(\mathfrak{g}) \) has a unique vertex algebra structure such that 1 is the vacuum vector,

\[ x(z) := Y(x ⊗ t^{-1}, z) = ∑_{n ∈ \mathbb{Z}} x(n)z^{-n-1}, \]

and

\[ [T, x(z)] = \partial_z x(z) \]

for \( x ∈ \mathfrak{g} \), where \( T \) is the translation operator. Here, \( x(n) \) acts on \( V^k(\mathfrak{g}) \) by left multiplication, and so, one can view \( x(n) \) as an endomorphism of \( V^k(\mathfrak{g}) \). The vertex algebra \( V^k(\mathfrak{g}) \) is called the universal affine vertex algebra associated with \( \mathfrak{g} \) at level \( k \) [FZ92, Zhu96, LL04].

The vertex algebra \( V^k(\mathfrak{g}) \) is a vertex operator algebra, provided that \( k + h^\vee \neq 0 \), by the Sugawara construction. More specifically, set

\[ S = \frac{1}{2} ∑_{i=1}^d x_i(-1)x_i(-1)1, \]

where \( \{x_i \mid i = 1, \ldots, d\} \) is the dual basis of a basis \( \{x^i \mid i = 1, \ldots, \dim \mathfrak{g}\} \) of \( \mathfrak{g} \) with respect to the bilinear form \( ( \quad | \quad ) \), with \( d = \dim \mathfrak{g} \). Then for \( k ≠ -h^\vee \), the vector \( \omega = S/(k + h^\vee) \) is a conformal vector of \( V^k(\mathfrak{g}) \) with central charge

\[ c(k) = \frac{k \dim \mathfrak{g}}{k + h^\vee}. \]

Note that, writing \( \omega(z) = ∑_{n ∈ \mathbb{Z}} L_n z^{-n-2} \), we have

\[ L_0 = \frac{1}{2(k + h^\vee)} \left( ∑_{i=1}^d x_i(0)x_i(0) + ∑_{i=1}^∞ ∑_{n=1}^∞ (x_i(-n)x^i(n) + x^i(-n)x_i(n)) \right), \]

\[ L_n = \frac{1}{2(k + h^\vee)} \left( ∑_{i=1}^∞ ∑_{m=1}^d x_i(-m)x_i(m+n) + ∑_{m=0}^∞ ∑_{i=1}^d x^i(-m+n)x_i(m) \right), \quad \text{if } n ≠ 0. \]

**Lemma 2.1 ([Kac90]).** — We have

\[ [L_n, x(m)] = -nx(m+n), \quad \text{for } x ∈ \mathfrak{g}, \ m, n ∈ \mathbb{Z}, \]

and \( L_n 1 = 0 \) for \( n ≥ -1 \).

We have \( V^k(\mathfrak{g})_Δ = \{v ∈ V^k(\mathfrak{g}) | L_0v = Δv\} \) and \( T = L_{-1} \) on \( V^k(\mathfrak{g}) \), provided that \( k + h^\vee ≠ 0 \).

Any graded quotient of \( V^k(\mathfrak{g}) \) as \( \mathfrak{g} \)-module has the structure of a quotient vertex algebra. In particular, the unique simple graded quotient \( L_k(\mathfrak{g}) \) is a vertex algebra, and is called the simple affine vertex algebra associated with \( \mathfrak{g} \) at level \( k \).
2.2. Associate graded vertex Poisson algebras of affine vertex algebras

It is known by Li [Li05] that any vertex algebra $V$ admits a canonical filtration $F^*V$, called the Li filtration of $V$. For a quotient $V$ of $V^k(\mathfrak{g})$, $F^*V$ is described as follows. The subspace $F^pV$ is spanned by the elements

$$y_1(-n_1 - 1)\cdots y_r(-n_r - 1)1$$

with $y_i \in \mathfrak{g}$, $n_i \in \mathbb{Z}_{\geq 0}$, $n_1 + \cdots + n_r \geq p$. We have

$$V = F^0V \supset F^1V \supset \cdots, \quad \bigcap_p F^pV = 0,$$

(2.2)

$$TF^pV \subset F^{p+1}V,$$

$$a(q)F^pV \subset F^{p+q-n-1}V \text{ for } a \in F^pV, \quad n \in \mathbb{Z},$$

$$a(q)F^V \subset F^{p+q-n}V \text{ for } a \in F^pV, \quad n \geq 0.$$ Here we have set $F^pV = V$ for $p < 0$.

Let $\text{gr}^FV = \bigoplus_p F^pV/F^{p+1}V$ be the associated graded vector space. The space $\text{gr}^FV$ is a vertex Poisson algebra by

$$\sigma_{p}(a)\sigma_{q}(b) = \sigma_{p+q}(a_{(-1)}b),$$

$$T\sigma_{p}(a) = \sigma_{p+1}(Ta),$$

$$\sigma_{p}(a)\sigma_{q}(b) = \sigma_{p+q-n}(a_{n}b)$$

for $a, b \in V$, $n \geq 0$, where $\sigma_p: F^p(V) \to F^pV/F^{p+1}V$ is the principal symbol map. In particular, $\text{gr}^FV$ is a $\mathfrak{g}[t]$-module by the correspondence

$$\mathfrak{g}[t] \ni x(t) \mapsto \sigma_0(x(t)) \in \text{End}(\text{gr}^FV)$$

for $x \in \mathfrak{g}$, $n \geq 0$.

The filtration $F^*V$ is compatible with the grading: $F^pV = \bigoplus_{\Delta \in \mathbb{Z}_{\geq 0}} F^pV\Delta$, where $F^pV\Delta := V\Delta \cap F^pV$.

Let $U_*\,(t^{-1}\mathfrak{g}[t^{-1}])$ be the PBW filtration of $U(t^{-1}\mathfrak{g}[t^{-1}])$, that is, $U_p\,(t^{-1}\mathfrak{g}[t^{-1}])$ is the subspace of $U(t^{-1}\mathfrak{g}[t^{-1}])$ spanned by monomials $y_1y_2\cdots y_r$ with $y_i \in \mathfrak{g}$, $r \leq p$. Define

$$G_pV = U_p\,(t^{-1}\mathfrak{g}[t^{-1}])1.$$ Then $G_*V$ defines an increasing filtration of $V$. We have

(2.4)

$$F^pV\Delta = G_{\Delta-p}G\Delta,$$

where $G_pV\Delta := G_pV \cap V\Delta$, see [Ara12a, Prop. 2.6.1]. Therefore, the graded space $\text{gr}^GV = \bigoplus_{p \in \mathbb{Z}_{\geq 0}} G_pV/G_{p-1}V$ is isomorphic to $\text{gr}^FV$. In particular, we have

$$\text{gr}^k(\mathfrak{g}) \cong \text{gr}U_*\,(t^{-1}\mathfrak{g}[t^{-1}]) \cong S(t^{-1}\mathfrak{g}[t^{-1}]).$$ The action of $\mathfrak{g}[t]$ on $\text{gr}^k(\mathfrak{g}) = S(t^{-1}\mathfrak{g}[t^{-1}])$ coincides with the one induced from the action of $\mathfrak{g}[t]$ on $\mathfrak{g}[t, t^{-1}]/\mathfrak{g}[t] \cong t^{-1}\mathfrak{g}[t^{-1}]$. More precisely, the element $x(m)$, for $x \in \mathfrak{g}$
and \( m \in \mathbb{Z}_{\geq 0} \), acts on \( S(t^{-1}g[t^{-1}]) \) as follows:

\[
x(m) \cdot 1 = 0,
\]

(2.5) \[
x(m) \cdot v = \sum_{j=1}^{r} \sum_{j-m \geq 0} y_1(-n_1) \cdots [x, y_j](m - n_j) \cdots y_r(-n_r),
\]

if \( v = y_1(-n_1) \cdots y_r(-n_r) \) with \( y_i \in g, n_1, \ldots, n_r \in \mathbb{Z}_{> 0} \).

2.3. Zhu’s \( C_2 \)-algebras and associated varieties of affine vertex algebras

We have \([Li05, Lem. 2.9]\) \( F_p^p V = \text{span}_\mathbb{C}\{a_{(-i-1)} b \mid a \in V, i \geq 1, b \in F_p^{-1}V\} \) for all \( p \geq 1 \). In particular,

\[
F^1 V = C_2(V),
\]

where \( C_2(V) = \text{span}_\mathbb{C}\{a_{(-2)} b \mid a, b \in V\} \). Set

\[
R_V = V/C_2(V) = F^0 V/F^1 V \subset \text{gr} F V.
\]

It is known by Zhu \([Zhu96]\) that \( R_V \) is a Poisson algebra. The Poisson algebra structure can be understood as the restriction of the vertex Poisson structure of \( \text{gr} F V \). It is given by

\[
\pi \cdot \overline{b} = a_{(-1)} b, \quad \{\pi, \overline{b}\} = a_{(0)} b,
\]

for \( a, b \in V \), where \( \pi = a + C_2(V) \).

By definition \([Ara12a]\), the associated variety of \( V \) is the reduced scheme

\[
X_V := \text{Specm}(R_V).
\]

It is easily seen that

\[
F^1 V^k(g) = C_2(V^k(g)) = t^{-2}g[t^{-1}]V^k(g).
\]

The following map defines an isomorphism of Poisson algebras

\[
\mathbb{C}[g^*] \cong S(g) \rightarrow R_{V^k(g)}
\]

\( g \ni x \mapsto x(-1)1 + t^{-2}g[t^{-1}]V^k(g) \).

Therefore, \( R_{V^k(g)} \cong \mathbb{C}[g^*] \) and so, \( X_{V^k(g)} \cong g^* \).

More generally, if \( V \) is a quotient of \( V^k(g) \) by some ideal \( N \), then we have

(2.6) \[
R_V \cong \mathbb{C}[g^*]/I_N
\]

as Poisson algebras, where \( I_N \) is the image of \( N \) in \( R_{V^k(g)} = \mathbb{C}[g^*] \). Then \( X_V \) is just the zero locus of \( I_N \) in \( g^* \). It is a closed \( G \)-invariant conic subset of \( g^* \).

Identifying \( g^* \) with \( g \) through the bilinear form \((\ ,\ )\), one may view \( X_V \) as a subvariety of \( g \).

**J.É.P. — M., 2021, tome 8**
2.4. PBW basis. — Let \( \Delta_+ = \{ \beta_1, \ldots, \beta_q \} \) be the set of positive roots for \( g \) with respect to a triangular decomposition \( g = n_- \oplus h \oplus n_+ \), where \( q = (d-\ell)/2 \) and \( \ell = \text{rk}(g) \).

Form now on, we fix a basis

\[
\{ u^i, e_{\beta_j}, f_{\beta_j} \mid i = 1, \ldots, \ell, j = 1, \ldots, q \}
\]

of \( g \) such that \( \{ u^i \mid i = 1, \ldots, \ell \} \) is an orthonormal basis of \( h \) with respect to \( ( \mid ) \) and \( (e_{\beta_i}, f_{\beta_j}) = 1 \) for \( i = 1, 2, \ldots, q \). In particular, \( [e_{\beta_i}, f_{\beta_j}] = \beta_i \) for \( i = 1, \ldots, q \) (see, for example, [Hum72, Prop. 8.3]), where \( h^* \) and \( h \) are identified through \( ( \mid ) \). One may also assume that \( \text{ht}(\beta_i) \leq \text{ht}(\beta_j) \) for \( i < j \), where \( \text{ht}(\beta_i) \) stands for the height of the positive root \( \beta_i \).

We define the structure constants \( c_{\alpha, \beta} \) by

\[
[e_{\alpha}, e_{\beta}] = c_{\alpha, \beta} e_{\alpha+\beta},
\]

provided that \( \alpha, \beta \) and \( \alpha + \beta \) are in \( \Delta \). Our convention is that \( e_{-\alpha} \) stands for \( f_{\alpha} \) if \( \alpha \in \Delta_+ \). If \( \alpha, \beta \) and \( \alpha + \beta \) are in \( \Delta_+ \), then from the equalities,

\[
c_{-\alpha, \alpha+\beta} = (f_{\beta} | [f_{\alpha}, e_{\alpha+\beta}]) = -(f_{\beta} | [e_{\alpha+\beta}, f_{\alpha}]) = -((f_{\beta}, e_{\alpha+\beta}) | f_{\alpha}) = -c_{-\beta, \alpha+\beta},
\]

we get that

\[
c_{-\alpha, \alpha+\beta} = -c_{-\beta, \alpha+\beta}.
\]

By (2.1), the above basis of \( g \) induces a basis of \( V^k(g) \) consisted of \( 1 \) and the elements of the form

\[
z = z^{(+)} z^{(-)} \cdot 1,
\]

with

\[
z^{(+)} := e_{\beta_1}(-1)^{a_{1,1}} \cdots e_{\beta_1}(-r_1) a_{1,1} \cdots e_{\beta_q}(-1)^{a_{q,1}} \cdots e_{\beta_q}(-r_q) a_{q,1}.
\]

\[
z^{(-)} := f_{\beta_1}(-1)^{b_{1,1}} \cdots f_{\beta_1}(-s_1) b_{1,1} \cdots f_{\beta_q}(-1)^{b_{q,1}} \cdots f_{\beta_q}(-s_q) b_{q,1}.
\]

\[
z^{(0)} := u^1(-1)^{c_{1,1}} \cdots u^1(-t_1) c_{1,1} \cdots u^\ell(-1)^{c_{\ell,1}} \cdots u^\ell(-t_\ell) c_{\ell,1}.
\]

where \( r_1, \ldots, r_q, s_1, \ldots, s_q, t_1, \ldots, t_\ell \) are positive integers, and \( a_{i,m}, b_{i,n}, c_{i,j} \), for \( l = 1, \ldots, q \), \( m = 1, \ldots, r_l \), \( n = 1, \ldots, s_l \), \( i = 1, \ldots, \ell \), \( j = 1, \ldots, t_l \) are nonnegative integers such that at least one of them is nonzero.

**Definition 2.2.** — Each element \( x \) of \( V^k(g) \) is a linear combination of elements in the above PBW basis, each of them will be called a *PBW monomial* of \( x \).

**Definition 2.3.** — For a PBW monomial \( v \) as in (2.8), we call the integer

\[
\text{depth}(v) = \sum_{i=1}^q \left( \sum_{j=1}^{r_i} a_{i,j}(j-1) + \sum_{j=1}^{s_i} b_{i,j}(j-1) \right) + \sum_{i=1}^\ell \sum_{j=1}^{t_i} c_{i,j}(j-1)
\]

the depth of \( v \). In other words, a PBW monomial \( v \) has depth \( p \) means that \( v \in F^p V^k(g) \) and \( v \not\in F^{p+1} V^k(g) \). By convention, \( \text{depth}(1) = 0 \).
For a PBW monomial \( v \) as in (2.8), we call degree of \( v \) the integer
\[
\deg(v) = \sum_{i=1}^{g} \left( \sum_{j=1}^{r_i} a_{i,j} + \sum_{j=1}^{b_i} \right) + \sum_{i=1}^{\ell} \sum_{j=1}^{c_i,j},
\]
In other words, \( v \) has degree \( p \) means that \( v \in G^k_p V_k(\mathfrak{g}) \) and \( v \notin G^k_{p-1} V_k(\mathfrak{g}) \) since the PBW filtration of \( V_k(\mathfrak{g}) \) coincides with the standard filtration \( G^k V_k(\mathfrak{g}) \). By convention, \( \deg(1) = 0 \).

Recall that a singular vector of a \( \mathfrak{g}[[t]] \)-representation \( M \) is a vector \( m \in M \) such that \( e_\alpha(0) \cdot m = 0 \), for all \( \alpha \in \Delta_+ \), and \( f_\theta(1) \cdot m = 0 \), where \( \theta \) is the highest positive root of \( \mathfrak{g} \). From the identity
\[
L_{-1} = \frac{1}{k + h^\vee} \left( \sum_{i=1}^{\ell} \sum_{m=0}^{\infty} u^i(-1-m)u^i(m) + \sum_{\alpha \in \Delta_+} \sum_{m=0}^{\infty} (e_\alpha(-1-m)f_\alpha(m) + f_\alpha(-1-m)e_\alpha(m)) \right),
\]
we deduce the following easy observation, which will be useful in the proof of the main result.

**Lemma 2.4.** If \( w \) is a singular vector of \( V_k(\mathfrak{g}) \), then
\[
L_{-1}w = \frac{1}{k + h^\vee} \left( \sum_{i=1}^{\ell} u^i(-1)u^i(0) + \sum_{\alpha \in \Delta_+} e_\alpha(-1)f_\alpha(0) \right)w.
\]

**2.5. Basis of associated graded vertex Poisson algebras.** – Note that \( \mathrm{gr} V_k(\mathfrak{g}) = S(t^{-1} \mathfrak{g}[t^{-1}]) \) has a basis consisting of \( 1 \) and elements of the form (2.8). Similarly to Definition 2.2, we have the following definition.

**Definition 2.5.** Each element \( x \) of \( S(t^{-1} \mathfrak{g}[t^{-1}]) \) is a linear combination of elements in the above basis, each of them will be called a monomial of \( x \).

As in the case of \( V_k(\mathfrak{g}) \), the space \( S(t^{-1} \mathfrak{g}[t^{-1}]) \) has two natural gradations. The first one is induced from the degree of elements as polynomials. We shall write \( \deg(v) \) for the degree of a homogeneous element \( v \in S(t^{-1} \mathfrak{g}[t^{-1}]) \) with respect to this gradation.

The second one is induced from the Li filtration via the isomorphism \( S(t^{-1} \mathfrak{g}[t^{-1}]) \cong \mathrm{gr}^t V_k(\mathfrak{g}) \). The degree of a homogeneous element \( v \in S(t^{-1} \mathfrak{g}[t^{-1}]) \) with respect to the gradation induced by Li filtration will be called the depth of \( v \), and will be denoted by \( \text{depth}(v) \).

Notice that any element \( v \) of the form (2.8) is homogeneous for both gradations. By convention, \( \deg(1) = \text{depth}(1) = 0 \).

As a consequence of (2.5), we get that
\[
\deg(x(m) \cdot v) = \deg(v) \quad \text{and} \quad \text{depth}(x(m) \cdot v) = \text{depth}(v) - m,
\]
for \( m \geq 0 \), \( x \in \mathfrak{g} \), and any homogeneous element \( v \in S(t^{-1} \mathfrak{g}[t^{-1}]) \) with respect to both gradations.
In the sequel, we will also use the following notation, for $v$ of the form (2.8), viewed either as an element of $V^k(g)$ or of $S(t^{-1}g[t^{-1}])$:

\[(2.10) \quad \deg^{(0)}(v) := \sum_{j=1}^{\ell} c_{j,1},\]

which corresponds to the degree of the element obtained from $v^{(0)}$ by keeping only the terms of depth 0, that is, the terms $w^i(-1)$, $i = 1, \ldots, \ell$.

Notice that a nonzero depth-homogeneous element of $S(t^{-1}g[t^{-1}])$ has depth 0 if and only if its image in 

\[R_{V^k(g)} = V^k(g) \cap t^{-2}g[t^{-1}]V^k(g)\]

is nonzero.

3. Proof of the main result

This section is devoted to the proof of Theorem 1.1.

3.1. Strategy. — Let $N_k$ be the maximal graded submodule of $V^k(g)$, so that $L_k(g) = V^k(g)/N_k$. Our aim is to show that if $V^k(g)$ is not simple, that is, $N_k \neq \{0\}$, then $X_{L_k(g)}$ is strictly contained in $\mathfrak{g}^* \cong g$, that is, the image $I_k := I_{N_k}$ of $N_k$ in $R_{V^k(g)} = \mathbb{C}[\mathfrak{g}^*]$ is nonzero.

For $k = -h^\vee$, it follows from [FG04] that $I_k$ is the defining ideal of the nilpotent cone $N(g)$ of $g$, and so $X_{L_k(g)} = N(g)$ (see [Ara12b] or Section 3.4 below). Hence, there is no loss of generality in assuming that $k + h^\vee \neq 0$.

Henceforth, we suppose that $k + h^\vee \neq 0$ and that $V^k(g)$ is not simple, that is, $N_k \neq \{0\}$. Then there exists at least one non-trivial (that is, nonzero and different from 1) singular vector $w$ in $V^k(g)$. Theorem 1.2 states that the image of $w$ in $I_k$ is nonzero, and this proves Theorem 1.1. The rest of this section is devoted to the proof of Theorem 1.2.

Let $w$ be a nontrivial singular vector of $V^k(g)$. One can assume that $w \in F^p V^k(g) \setminus F^{p+1}V^k(g)$ for some $p \in \mathbb{Z}_{\geq 0}$.

The image 

\[w := \sigma(w)\]

of this singular vector in $S(t^{-1}g[t^{-1}]) \cong \text{gr}^F V^k(g)$ is a nontrivial singular vector of $S(t^{-1}g[t^{-1}])$. Here $\sigma: V^k(g) \to \text{gr}^F V^k(g)$ stands for the principal symbol map. It follows from (2.9) that one can assume that $w$ is homogeneous with respect to both gradations on $S(t^{-1}g[t^{-1}])$. In particular $w$ has depth $p$. It is enough to show that $p = 0$, that is, $w$ has depth zero. Write

\[w = \sum_{j \in J} \lambda_j w_j,\]

where $J$ is a finite index set, $\lambda_j$ are nonzero scalar for all $j \in J$, and $w_j$ are pairwise distinct PBW monomials of the form (2.8). Let $I \subset J$ be the subset of $i \in J$ such that
depth $\overline{w}^i = p = \text{depth} \overline{w}$. Since $w \in F^p V^k(\mathfrak{g}) \times F^{p+1} V^k(\mathfrak{g})$, the set $I$ is nonempty. Here, $\overline{w}^i$ stands for the image of $w^i$ in $\text{gr} F V^k(\mathfrak{g}) \cong S(t^{-1} \mathfrak{g}[t^{-1}])$.

More specifically, for any $j \in I$, write

$$w^j = (w^j)^{(+))(w^j)^{(-)}(w^j)^{(0)} 1,$$

with

$$(w^j)^{(+)} := e_{\beta_1} (-1)^{a(i)} \cdots e_{\beta_t} (-r_i) b(i) \cdots e_{\beta_n} (-1)^{a(i)} u_{\beta_q} (-r_q) b(i)$$

$$(w^j)^{(-)} := f_{\beta_1} (-1)^{a(i)} \cdots f_{\beta_t} (-s_i) b(i) \cdots f_{\beta_n} (-1)^{a(i)} u_{\beta_q} (-s_q) b(i)$$

$$(w^j)^{(0)} := u^1 (-1)^{c(i)} \cdots u^l (-t_1)^{c(i)} \cdots u^\ell (-t_\ell)^{c(i)},$$

where $r_1, \ldots, r_q, s_1, \ldots, s_q, t_1, \ldots, t_\ell$ are nonnegative integers, and $a(i), b(i), c(i)$, for $l = 1, \ldots, q$, $m = 1, \ldots, r_t$, $n = 1, \ldots, s_t$, $i = 1, \ldots, \ell$, $p = 1, \ldots, t_i$, are nonnegative integers such that at least one of them is nonzero.

The integers $r_l$’s, for $l = 1, \ldots, q$, are chosen so that at least one of the $u_{\beta_l}$’s is nonzero for $j$ running through $J$ if for some $j \in J$, $(w^j)^{(+) \neq 1$. Otherwise, we just set $(w^j)^{(+) := 1$. Similarly define the integers $s_l$’s and $m$’s, for $l = 1, \ldots, q$. By our assumption, note that for all $i \in I$,

$$\sum_{n=1}^q \left( \sum_{l=1}^r a(n_{l,i}) + \sum_{l=1}^s b(n_{l,i}) \right) + \sum_{n=1}^t \sum_{l=1}^\ell c(n_{l,i}) = \text{deg}(\overline{w})$$

$$\sum_{n=1}^q \left( \sum_{l=1}^r a(n_{l,i}) (l-1) + \sum_{l=1}^s b(n_{l,i}) (l-1) \right) + \sum_{n=1}^t \sum_{l=1}^\ell c(n_{l,i}) (l-1) = \text{depth}(\overline{w}) = p.$$

3.2. A technical lemma. — In this paragraph we remain in the commutative setting, and we only deal with $\overline{w} \in S(t^{-1} \mathfrak{g}[t^{-1}])$ and its monomials $\overline{w}^i$’s, for $i \in I$.

Recall from (2.10) that,

$$\text{deg}_{(0)}(w^i) = \sum_{j=1}^\ell c(n_{l,i})$$

for $i \in I$. Set

$$d_{(0)}^{(0)}(I) := \max \{ \text{deg}_{(0)}(w^i) \mid i \in I \},$$

and

$$I_{(0)} := \{ i \in I \mid \text{deg}_{(0)}(w^i) = d_{(0)}^{(0)}(I) \}.$$ If $(w^i)^{(0)} = 1$ for all $i \in I$, then $(\overline{w})^{(%) = 1$. In other words, for $i \in I_{(0)}$, we have $\overline{w}^i = (\overline{w})^{(0)}(\overline{w})^{(+)1}$.  

Proof. — Suppose the assertion is false. Then for some positive roots $\beta_1, \ldots, \beta_n \in \Delta_+$, one can write for any $i \in I_{(0)}$,

$$(\overline{w})^{(-)} = f_{\beta_1} (-1)^{a_1^{(0)} \beta_1} \cdots f_{\beta_t} (-s_1) b_1^{(0)} \cdots f_{\beta_n} (-1)^{a_1^{(0)} \beta_n} \cdots f_{\beta_i} (-s_1) b_i^{(0)} \cdots f_{\beta_i} (-s_1) b_i^{(0)} \cdots f_{\beta_i} (-s_1) b_i^{(0)}.$$

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so that for any \( l \in \{1, \ldots, t\} \),
\[
\{ b_{j_l,s_j}^{(l)} \mid i \in I_{-1}^{(l)} \} \neq \{0\}.
\]

Set
\[
K_{-1}^{(0)} = \{ i \in I_{-1}^{(0)} \mid b_{j_i,s_{j_i}}^{(i)} > 0 \}.
\]

Since \( \overline{w} \) is a singular vector of \( S(t^{-1} \mathfrak{g}[t^{-1}]) \) and \( s_{j_i} - 1 \in \mathbb{Z}_{\geq 0} \), we have
\[
eq (s_{j_i} - 1) \cdot \overline{w} = 0.
\]

On the other hand, using the action of \( g[t] \) on \( S(t^{-1} \mathfrak{g}[t^{-1}]) \) as described by (2.5), we see that
\[
0 = e_{\beta_{j_i}}(s_{j_i} - 1) \cdot \overline{w} = \sum_{i \in K_{-1}^{(0)}} \lambda_i b_{j_i,s_{j_i}}^{(i)} v^i + v,
\]
where for \( i \in K_{-1}^{(0)} \),
\[
v^i := (\overline{w})^{(i)} b_{j_i}^{(i)} (-1) f_{\beta_{j_i}} (-1)^{j_i} \cdots f_{\beta_{j_i}} (-s_{j_i}) b_{j_i,s_{j_i}}^{(i)} - 1 \cdots f_{\beta_{j_i}} (-1)^{j_i} \cdots f_{\beta_{j_i}} (-s_{j_i}) b_{j_i,s_{j_i}}^{(i)} (w^i)(+) 1,
\]
and \( v \) is a linear combination of monomials \( x \) such that
\[
\text{deg}_{-1}^{(0)}(x) \leq d_{-1}^{(0)}(I).
\]

Indeed, for \( i \in K_{-1}^{(0)} \), it is clear that
\[
eq e_{\beta_{j_i}}(s_{j_i} - 1) \cdot w^i = b_{j_i,s_{j_i}}^{(i)} v^i + y^i,
\]
where \( y^i \) is a linear combination of monomials \( y \) such that \( \text{deg}_{-1}^{(0)}(y) \leq d_{-1}^{(0)}(I) \) because \( \text{ht}(\beta_{j_i}) \leq \text{ht}(\beta_{j_i}) \) for all \( l \in \{1, \ldots, t\} \). Next, for \( i \in I_{-1}^{(0)} \backslash K_{-1}^{(0)} \), \( e_{\beta_{j_i}}(s_{j_i} - 1) \cdot \overline{w} \) is a linear combination of monomials \( z \) such that \( \text{deg}_{-1}^{(0)}(z) \leq d_{-1}^{(0)}(I) \) because \( b_{j_i,s_{j_i}}^{(i)} = 0 \). Finally, for \( i \in I \backslash I_{-1}^{(0)} \), we have \( \text{deg}_{-1}^{(0)}(\overline{w}) < d_{-1}^{(0)}(I) \) and, hence, \( e_{\beta_{j_i}}(s_{j_i} - 1) \cdot \overline{w} \) is a linear combination of monomials \( z \) such that \( \text{deg}_{-1}^{(0)}(z) \leq d_{-1}^{(0)}(I) \) as well.

Now, note that for each \( i \in K_{-1}^{(0)} \),
\[
\text{deg}_{-1}^{(0)}(v^i) = \text{deg}_{-1}^{(0)}(\overline{w}) + 1 = d_{-1}^{(0)}(I) + 1.
\]

Hence by (3.3) we get a contradiction because all monomials \( v^i \), for \( i \) running through \( K_{-1}^{(0)} \), are linearly independent while \( \lambda_i b_{j_i,s_{j_i}}^{(i)} \neq 0 \), for \( i \in K_{-1}^{(0)} \). This concludes the proof of the lemma. \( \square \)

3.3. Use of Sugawara operators. — Recall that \( w = \sum_{j \in J} \lambda_j w^j \). Let \( J_1 \subseteq J \) be such that for \( i \in J_1 \), \( (w^i)(-) = 1 \). Then by Lemma 3.1,
\[
\emptyset \neq I_{-1}^{(0)} \subseteq J_1.
\]

So \( J_1 \neq \emptyset \). Set
\[
d_{-1}^{(0)} := d_{-1}^{(0)}(J_1) = \max\{ \text{deg}_{-1}^{(0)}(w^i) \mid i \in J_1 \},
\]
and
\[ J^{(0)}_{-1} := \{ i \in J_1 \mid \deg^{(0)}_{-1}(w^i) = d^{(0)}_{-1} \} . \]
Then \( d^{(0)}_{-1}(I) \leq d^{(0)}_{-1} \). Set
\[ d^+ := \max\{ \deg(w^i)^{(+) \mid i \in J^{(0)}_{-1}} \} \]
and let
\[ J^+ = \{ i \in J^{(0)}_{-1} \mid \deg(w^i)^{(+) = d^+} \} \subseteq J^{(0)}_{-1} . \]
Our next aim is to show that for \( i \in J^+ \), \( w^i \) has depth zero, whence \( p = 0 \) since \( p \) is by definition the smallest depth of the \( w^i \)'s, and so the image of \( w \) in \( R_{V^k(\mathfrak{g})} = F^0V^k(\mathfrak{g})/F^1V^k(\mathfrak{g}) \) is nonzero.

This will be achieved in this paragraph through the use of the Sugawara construction.

Recall that by Lemma 2.4,
\[ L_{-1}w = \tilde{L}_{-1}w \]
since \( w \) is a singular vector of \( V^k(\mathfrak{g}) \), where
\[ \tilde{L}_{-1} := \frac{1}{k + h} \left( \sum_{i=1}^t u^i(-1)u^i(0) + \sum_{\alpha \in \Delta_+} e_\alpha(-1)f_\alpha(0) \right) . \]

**Lemma 3.2.** — Let \( z \) be a PBW monomial of the form (2.8). Then \( \tilde{L}_{-1}z \) is a linear combination of PBW monomials \( x \) satisfying all the following conditions:

(a) \( \deg(x)^{(+) \leq \deg(z)^{(+) + 1} \text{ and } \deg(x^{(0)} \leq \deg(z^{(0)}) + 1} \),
(b) if \( z^{(-)} \neq 1 \), then \( x^{(-)} \neq 1 \),
(c) if \( x^{(-)} = z^{(-)} \), then either \( \deg(x^{(0)} = \deg(z^{(0)}) + 1 \), or \( x^{(0)} = z^{(0)} \),
(d) if \( \deg(x^{(0)} = \deg(z^{(0)}) + 1 \), then \( x^{(-)} = z^{(-)} \text{ and } \deg(x^{(+) \leq \deg(z^{(+))} \),

**Proof.** — Parts (a)–(c) are easy to see. We only prove (d). Assume that \( \deg(x^{(0)} = \deg(z^{(0)}) + 1 \). Either \( x \) comes from the term \( \sum_{i=1}^t u^i(-1)u^i(0)z \), or it comes from a term \( e_\alpha(-1)f_\alpha(0)z \) for some \( \alpha \in \Delta_+ \).

If \( x \) comes from the term \( \sum_{i=1}^t u^i(-1)u^i(0)z \), then it is obvious that \( x^{(-)} = z^{(-)} \) and \( x^{(+)} = z^{(+)} \).

Assume that \( x \) comes from \( e_\alpha(-1)f_\alpha(0)z \) for some \( \alpha \in \Delta_+ \). We have
\[ e_\alpha(-1)f_\alpha(0)z = e_\alpha(-1)f_\alpha(0), z^{(+)}z^{(0)}1 + e_\alpha(-1)z^{(+)}f_\alpha(0), z^{(-)}(0)1 + e_\alpha(-1)z^{(+)z^{(-)}f_\alpha(0), z^{(0)}1} . \]
Clearly, any PBW monomials \( x \) from
\[ e_\alpha(-1)z^{(+)}f_\alpha(0), z^{(-)}(0)1 \quad \text{or} \quad e_\alpha(-1)z^{(+)z^{(-)}f_\alpha(0), z^{(0)}1} \]
satisfies that \( \deg(x^{(0)} \leq \deg(x^{(0)}) \). Then it is enough to consider PBW monomials in \( e_\alpha(-1)f_\alpha(0), z^{(+)z^{(-)}f_\alpha(0), z^{(0)}1} \).

The only possibility for a PBW monomial \( x \) in \( e_\alpha(-1)f_\alpha(0), z^{(+)z^{(-)}f_\alpha(0), z^{(0)}1} \) to satisfy \( \deg(x^{(0)} = \deg(x^{(0)}) + 1 \) is that it comes from a term \( [f_\alpha(0), e_\alpha(-n)] = -\alpha(-n) \) for
some $n \in \mathbb{Z}_{>0}$, where $e_\alpha(-n)$ is a term in $z^{(+)}$. But then, for PBW monomials $x$ in $e_\alpha(-1)[f_\alpha(0), z^{(+)}]z^{(0)}1$ such that $\deg(x^{(0)}) = \deg(z^{(0)}) + 1$, we have $x^{(-)} = z^{(-)}$ and $\deg(x^{(+)}) \leq \deg(z^{(+)}). \blacksquare$

We now consider the action of $\widetilde{L}_{-1}$ on particular PBW monomials.

**Lemma 3.3.** Let $z$ be a PBW monomial of the form (2.8) such that $z^{(-)} = 1$ and $\text{depth}(z^{(+)}) = 0$, that is, either $z^{(+)} = 1$, or for some $j_1, \ldots, j_q \in \{1, \ldots, q\}$ (with possible repetitions),

$$z = e_{\beta_1}(-1)e_{\beta_2}(-1)\cdots e_{\beta_j}(-1)z^{(0)}1.$$

Then $\widetilde{L}_{-1}z$ is a linear combination of PBW monomials $y$ satisfying one of the following conditions:

1. $y^{(-)} = 1$, $\text{depth}(y^{(+)}) \geq 1$, $\deg(y^{(+)}) \leq \deg(z^{(+)})$, $y^{(0)} = z^{(0)}$.
2. $y^{(-)} = 1$, $\text{depth}(y^{(+)}) = 0$, $\deg(y^{(+)}) \leq \deg(z^{(+)}) - 1$, and $\deg(y^{(0)}) < \deg(z^{(0)})$.
3. $y^{(-)} = 1$, $\text{depth}(y^{(+)}) \geq 1$, $\deg(y^{(+)}) < \deg(z^{(+)}) - 1$, and $\deg(y^{(0)}) = \deg(z^{(0)}) + 1$.
4. $y^{(-)} \neq 1$.

**Proof.** First, we have

$$\sum_{i=1}^\ell u^{(-)}(-1)u^i(0)z = \sum_{r=1}^\ell e_{\beta_{j_1}}(-1)\cdots \sum_{i=1}^\ell u^{(-)}(-1)u^i(0), e_{\beta_{j_r}}(-1) \cdots e_{\beta_{j_r}}(-1)z^{(0)}1,$$

and

$$\sum_{i=1}^\ell u^{(-)}(-1)u^i(0), e_{\beta_{jr}}(-1) = \sum_{i=1}^\ell (u^{(-)}(-1)[u^i(0), e_{\beta_{jr}}(-1)] + [u^{(-)}(-1), e_{\beta_{jr}}(-1)]u^i(0))$$

$$= \beta_{jr}(-1)e_{\beta_{jr}}(-1) + e_{\beta_{jr}}(-2)\beta_{jr}(0).$$

So

(3.4) \[ \sum_{i=1}^\ell u^{(-)}(-1)u^i(0)z \]

$$= \sum_{i=1}^\ell e_{\beta_{j_1}}(-1)\cdots (\beta_{jr}(-1)e_{\beta_{jr}}(-1) + e_{\beta_{jr}}(-2)\beta_{jr}(0)) \cdots e_{\beta_{jr}}(-1)z^{(0)}1.$$ Second, we have

$$\sum_{\alpha \in \Delta_+} e_\alpha(-1)f_\alpha(0)z = \sum_{\alpha \in \Delta_+} \sum_{r=1}^\ell e_\alpha(-1)e_{\beta_{j_1}}(-1)\cdots [f_\alpha(0), e_{\beta_{jr}}(-1)]\cdots e_{\beta_{j_r}}(-1)z^{(0)}1$$

$$+ \sum_{\alpha \in \Delta_+} e_\alpha(-1)e_{\beta_{j_1}}(-1)e_{\beta_{j_2}}(-1)\cdots e_{\beta_{j_r}}(-1)[f_\alpha(0), z^{(0)}]1.$$

It is clear that any PBW monomial $y$ in

$$\sum_{\alpha \in \Delta_+} e_\alpha(-1)e_{\beta_{j_1}}(-1)\cdots e_{\beta_{j_r}}(-1)[f_\alpha(0), z^{(0)}]1$$

satisfies

(3.5) \[ y^{(-)} \neq 1. \]
We now consider
\[ u_r := \sum_{\alpha \in \Delta_+} c_\alpha(-1)e_{\beta_1}(-1) \cdots [f_{\alpha}(0), e_{\beta_r}(-1)] \cdots e_{\beta_t}(-1)z^{(0)} \mathbf{1}, \text{ for } 1 \leq r \leq t. \]

- If \( \beta_{j_r} = \alpha + \beta \) for some \( \alpha, \beta \in \Delta_+ \), then there is a partial sum of two terms in \( u_r \):

\[
\begin{align*}
&c_{-\alpha, \alpha + \beta} c_\alpha(-1)e_{\beta_1}(-1) \cdots e_\beta(-1) \cdots e_{\beta_{j_r}}(-1) z^{(0)} \mathbf{1} \\
&\quad + c_{-\beta, \alpha + \beta} e_{\beta}(-1)e_{\beta_1}(-1) \cdots e_\alpha(-1) \cdots e_{\beta_{j_r}}(-1) z^{(0)} \mathbf{1}.
\end{align*}
\]

Rewriting the above sum to a linear combination of PBW monomials, and noticing that

\[
\begin{align*}
c_{-\alpha, \alpha + \beta} e_{\beta}(-1) + c_{-\beta, \alpha + \beta} e_\beta(-1) e_\alpha(-1) &= c_{-\alpha, \alpha + \beta} c_{\alpha + \beta} e_\alpha(-1) = c_{-\alpha, \alpha + \beta} e_\alpha(-1) = -c_{-\alpha, \alpha + \beta} e_\alpha(-1) = -c_{-\alpha, \alpha + \beta} c_{\alpha + \beta} e_\alpha(-1),
\end{align*}
\]

due to (2.7), we deduce that it is a linear combination of PBW monomials \( y \) such that

\[
y(-) = z(-) = 1, \quad y^{(0)} = z^{(0)}, \quad \text{depth}(y^{(+)}) \geq 1, \quad \deg(y^{(+)}) \leq \deg(z^{(+)})
\]

where \( c_{-\alpha, \alpha + \beta}, c_{-\beta, \alpha + \beta}, c_{\alpha + \beta} \in \mathbb{R}^* \).

- If \( \alpha = -\beta_{j_r} \in \Delta_+ \) for some \( \alpha \in \Delta_+ \), then there is a term in \( u_r \):

\[
\begin{align*}
c_{-\alpha, \beta_{j_r}} e_\alpha(-1)e_{\beta_1}(-1) \cdots e_{\beta_{j_r-1}}(-1)f_{\alpha-\beta_{j_r}}(-1)e_{\beta_{j_r+1}}(-1) \cdots e_{\beta_t}(-1) z^{(0)} \mathbf{1}.
\end{align*}
\]

It is easy to see that (3.7) is a linear combination of PBW monomials \( y \) such that \( y \)

satisfies one of the following:

\[
y^{(-)} = 1, \quad \text{depth}(y^{(+)}) \geq 1, \quad \deg(y^{(+)}) \leq \deg(z^{(+)})\]

\[
y^{(-)} = 1, \quad \text{depth}(y^{(+)}) = 0, \quad \deg(y^{(+)}) \leq \deg(z^{(+)}) - 1,
\]

\[
deg(y^{(0)}) > \deg(z^{(0)}), \quad \deg_{-1}(y) = \deg_{-1}(z),
\]

\[
y^{(-)} \neq 1.
\]

Notice also that with \( \alpha = \beta_{j_r} \), there is a term in \( u_r \):

\[-e_{\beta_{j_r}}(-1)e_{\beta_1}(-1) \cdots e_{\beta_{j_r-1}}(-1)\beta_{j_r}(-1)e_{\beta_{j_r+1}}(-1) \cdots e_{\beta_t}(-1) z^{(0)} \mathbf{1}.
\]

Together with (3.4), we see that

\[
\sum_{i=1}^t u_i(-1)u_i^{(0)}z + \sum_{r=1}^t e_{\beta_1}(-1)e_{\beta_1}(-1) \cdots [f_{\beta_{j_r}}(0), e_{\beta_{j_r}}(-1)] \cdots e_{\beta_t}(-1) z^{(0)} \mathbf{1}
\]

\[
= \sum_{r=1}^t e_{\beta_1}(-1) \cdots (\beta_{j_r}(-1)e_{\beta_{j_r}}(-1) + e_{\beta_{j_r}}(-2)\beta_{j_r}(0)) \cdots e_{\beta_t}(-1) z^{(0)} \mathbf{1}
\]

\[-\sum_{r=1}^t \sum_{s=1}^{r-1} e_{\beta_1}(-1) \cdots [e_{\beta_{j_r}}(-1), e_{\beta_{j_r}}(-1)]
\]

\[
\cdots e_{\beta_{j_r-1}}(-1)\beta_{j_r}(-1)e_{\beta_{j_r+1}}(-1) \cdots e_{\beta_t}(-1) z^{(0)} \mathbf{1}
\]

\[-\sum_{r=1}^t e_{\beta_1}(-1) \cdots e_{\beta_{j_r-1}}(-1)e_{\beta_{j_r}}(-1)\beta_{j_r}(-1)e_{\beta_{j_r+1}}(-1) \cdots e_{\beta_t}(-1) z^{(0)} \mathbf{1}
\]
is a linear combination of PBW monomials $y$ satisfying one of the following:

\begin{align}
(3.11) & \quad y^{(-)} = 1, \quad \text{depth}(y^{(+)}) \geq 1, \quad \text{deg}(y^{(+)}) \leq \text{deg}(z^{(+)}), \quad y^{(0)} = z^{(0)}, \\
(3.12) & \quad y^{(-)} = 1, \quad \text{depth}(y^{(+)}) \geq 1, \quad \text{deg}(y^{(+)}) \leq \text{deg}(z^{(+)}) - 1, \quad \text{deg}_{-1}(y) = \text{deg}_{-1}(z) + 1.
\end{align}

Then the lemma follows from (3.5), (3.6), (3.8)–(3.12). \hfill \square

**Lemma 3.4.** — Let $z$ be a PBW monomial of the form (2.8) such that $z^{(-)} = 1$. Then

\[ \tilde{L}_{-1}z = cz^{(+)}(\gamma - \sum_{j=1}^{q} a_{j,1} \beta_j)(-1)z^{(0)} + y', \]

where $c$ is a nonzero constant, $\gamma = \sum_{j=1}^{q} \sum_{s=1}^{r} a_{j,s} \beta_j$, and $y'$ is a linear combination of PBW monomials $y$ such that

\[ \text{deg}_{-1}(y) = \text{deg}_{-1}(z) + 1, \quad \text{deg}(y^{(+)}) \leq \text{deg}(z^{(+)}) - 1, \]

or

\[ \text{deg}_{-1}(y) \leq \text{deg}_{-1}(z). \]

**Proof.** — Since the proof is similar to that of Lemma 3.3, we left the verification to the reader. \hfill \square

**Lemma 3.5.** — For $i \in J^+$, we have that $\text{depth}((w^i)^{(+)}) = 0$.

**Proof.** — First we have

\[ w = \sum_{j \in J^+} \lambda_j w^j + \sum_{j \in J^+ \setminus J_1^+} \lambda_j w^j + \sum_{j \in J_1^+ \setminus J_1^+} \lambda_j w^j + \sum_{j \in J_1^+} \lambda_j w^j. \]

Then by Lemma 3.2(b) and Lemma 3.4, we have

\[ (k + h^+)\tilde{L}_{-1}w = \sum_{i \in J^+} (w^i)^{(+)} \left( \gamma_i - \sum_{j=1}^{q} a_{j,1} \beta_j \right)(-1)(w^i)^{(0)} + \sum_{i \in J_1^+} (w^i)^{(0)} \left( \gamma_i - \sum_{j=1}^{q} a_{j,1} \beta_j \right)(-1)(w^i)^{(0)} + y', \]

where $\gamma_i = \sum_{j=1}^{q} \sum_{s=1}^{r} a_{j,s} \beta_j$, for $i \in J_1$, and $y'$ is a linear combination of PBW monomials $y$ satisfying one of the following conditions:

\[ \text{deg}_{-1}(y) = d_{-1}^{(0)} + 1, \quad \text{deg}(y^{(+)}) \leq d^{(+)} - 1, \]

\[ \text{deg}_{-1}(y) \leq d_{-1}^{(0)}, \]

\[ y^{(-)} \neq 1. \]

On the other hand, by Lemma 2.4

\[ L_{-1}w = \tilde{L}_{-1}w. \]
By Lemma 2.1, there is no PBW monomial $y$ in $L_{-1}w$ such that \( \deg(y^{(+)}) = d^+ \), $y^{(-)} = 1$, and $\deg_{-1}^{(0)}(y) = -d_{-1}^{(0)} + 1$. Then we deduce that

$$
\sum_{i \in J^+} (w^i)^{(+)\left(\gamma_i - \sum_{j=1}^{q} a_{j,1} \beta_j\right)}(-1)(w^i)^{(0)} = 0,
$$

which means that $\left(\gamma_i - \sum_{j=1}^{q} a_{j,1} \beta_j\right) = 0$, for $i \in J^+$, that is, depth$((w^i)^{(+)}) = 0$. □

As explained at the beginning of §3.3, Theorem 1.1 will be a consequence of the following lemma.

**Lemma 3.6.** For each $i \in J^+$, we have depth$(w^i) = 0$.

**Proof.** By definition, for $i \in J^+$, $(w^i)^{(0)} = 1$. Moreover, by Lemma 3.5, depth$((w^i)^{(+)}) = 0$. Hence it suffices to prove that for $i \in J^+$,

$$
(w^i)^{(0)} = u^1(-1)^{a_{1,1}} \cdots u^\ell(-1)^{a_{\ell,1}}.
$$

Suppose the contrary. Then there exists $i \in J^+$ such that

$$
w^i = e_{\beta_1}(-1)^{\ell,1} \cdots e_{\beta_\ell}(-1)^{\ell,m_\ell} u^1(-1)^{a_{1,1}} \cdots u^\ell(-1)^{a_{\ell,1}} u^1(-m_{1})^{c_{1,m_1}} \cdots u^\ell(-m_{\ell})^{c_{\ell,m_\ell}} 1,
$$

with at least one of the $m_j$’s, for $j = 1, \ldots, \ell$, strictly greater than 1 and $c_{j,m_j}^{(i)} \neq 0$ for such a $j$. Without loss of generality, one may assume that $1 \in J^+$, that

$$
m_1 = \max\{m_j | j = 1, \ldots, \ell\} \quad \text{and} \quad 0 \neq c_{1,m_1}^{(i)} \geq c_{1,m_1}^{(i)} \quad \text{for } i \in J^+.
$$

Writing $L_{-1}w$ as

$$
L_{-1}w = \sum_{i \in J^+} L_{-1}w^i + \sum_{i \in J_0^1 \setminus J^+} L_{-1}w^i + \sum_{i \in J_1 \setminus J_0^1} L_{-1}w^i + \sum_{i \in J_0 \setminus J_1} L_{-1}w^i,
$$

we see by Lemma 2.1 that

$$
L_{-1}w = \lambda_1 m_1 c_{1,m_1}^{(i)} v^1 + \sum_{i \in J^+ \setminus \setminus} \lambda_i m_i c_{i,m_i}^{(i)} v^i + v + v',
$$

where for $i \in J^+$, $v^i$ is the PBW monomial defined by:

$$
\begin{align*}
(v^i)^{(−)} &= (w^i)^{(−)} = 1, \\
(v^i)^{(+)} &= (w^i)^{(+) +} = e_{\beta_1}(-1)^{a_{1,1}} \cdots e_{\beta_\ell}(-1)^{a_{\ell,1}}, \\
(v^i)^{(0)} &= u^1(-1)^{c_{1,1}^{(i)}} \cdots u^\ell(-m_{1})^{c_{1,m_1}^{(i)}} \cdots u^1(-m_{\ell})^{c_{\ell,m_\ell}^{(i)}} - 1 \cdots u^\ell(-m_{\ell})^{c_{\ell,m_\ell}^{(i)}}, \\
\deg_{-1}^{(0)}(v^i) &= d_{-1}^{(0)},
\end{align*}
$$

$v$ is a linear combination of PBW monomials $x$ such that

$$
x^{(0)} = u^1(-1)^{c_{1,1}^{(x)}} \cdots u^\ell(-1)^{c_{\ell,1}^{(x)}} \cdots u^1(-1)^{c_{1,m_1}^{(x)}} \cdots u^\ell(-1)^{c_{\ell,m_\ell}^{(x)}}
$$

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Lemma 3.2(c), either

\[ n_1^{(x)} \leq m_1, \quad \text{or} \quad \deg(x^{(+)}) \leq d^+-1, \quad \text{or} \quad \deg^{(0)}(x) \leq d^{(0)}-1, \]

and \( v' \) is a linear combination of PBW monomials \( x \) such that \( x^{(-)} \neq 1 \). Note that the assumption that \( m_1 \geq 2 \) makes sure that (3.17) holds, and that \( \text{depth}(v') = \text{depth}(w^i) + 1 \) for all \( i \in J^+ \).

On the other hand, by Lemma 2.4,

\[ L_{-1}w = \tilde{L}_{-1}w, \]

since \( w \) is a singular vector of \( V^k(\mathfrak{g}) \). Hence \( v^l \) must be a PBW monomial of \( \tilde{L}_{-1}w \).

Our strategy to obtain the expected contradiction is to show that there is no PBW monomial \( v^l \) in \( \tilde{L}_{-1}w \) for each \( i \in J \).

– Assume that \( i \in J^+ \), and suppose that \( v^l \) is a PBW monomial in \( \tilde{L}_{-1}w^i \). First of all, \( \deg((w^i)^{(+)} = d^+ \) because \( i \in J^+ \). Moreover, by the definition of \( J_1 \) and Lemma 3.5, we have \( (w^i)^{(-)} = 1 \) and \( \text{depth}((w^i)^{(+)} = 0 \). Hence by Lemma 3.3(2),

\[ \deg((v^l)^{(+)} < \deg((w^i)^{(+)} = d^+ \]

because \( (v^l)^{(-)} = 1 \) and \( \text{depth}((v^l)^{(+)} = 0 \) by (3.14) and (3.15). But \( d^+ = \deg((v^l)^{(+)} \) by (3.15), whence a contradiction.

– Assume that \( i \in J^+_0 \setminus J^+. \) By the definition of \( J^+_0 \) and (3.15),

\[ \deg((w^i)^{(+)} < d^+ = \deg((v^l)^{(+}) \]

Suppose that \( v^l \) is a PBW monomial in \( \tilde{L}_{-1}w^i \). Then

\[ (w^i)^{(-)} = 1 = (v^l)^{(-)} \]

by Lemma 3.1 since \( i \in J^+_0 \). The last equality follows from (3.14). Then by Lemma 3.2(c), either \( \deg((v^l)^{(-)} = \deg((w^i)^{(-)} + 1 \), or \( (v^l)^{(0)} = (w^i)^{(0)} \). But it is impossible that \( \deg((v^l)^{(0)} = \deg((w^i)^{(0)} + 1 \), by (d) of Lemma 3.2 because \( \deg((v^l)^{(+)} \geq \deg((w^i)^{(+)} \). Therefore,

\[ (v^l)^{(0)} = (w^i)^{(0)} \]

Computing \( \tilde{L}_{-1}w^i \), we deduce from

\[ (v^l)^{(+)} = e_{\beta_1}(-1)^{a_{i,1}^{(1)}} \cdots e_{\beta_4}(-1)^{a_{i,1}^{(4)}}, \]

that

\[ (w^i)^{(+)} = e_{\beta_1}(-1)^{a_{i,1}^{(1)}} \cdots e_{\beta_4}(-1)^{a_{i,1}^{(4)}}, \]

Since \( (v^l)^{(-)} = (w^i)^{(-)} = 1 \), it results from Lemma 3.3 that \( \deg((v^l)^{(+)} \leq \deg((w^i)^{(+)} \), which contradicts (3.18).

– Assume that \( i \in J_1 \setminus J^+_0 \). Then

\[ \deg^{(0)}(w^i) < d^{(0)}_{-1} = \deg^{(0)}_{-1}(v^l) \]

by (3.17). Suppose that \( v^l \) is a PBW monomial in \( \tilde{L}_{-1}w^i \). By Lemma 3.2(b) and (c),

\[ (w^i)^{(-)} = 1, \quad \deg_{-1}(v^l) = \deg_{-1}(w^i) + 1, \]
It is known \cite{RT92, BD, EF01} that
\[ f \in \mathfrak{S} \] yields a one-parameter subgroup \( \mathfrak{S} \rightarrow \mathfrak{SL} \), which is always zero. Indeed, the \( \mathfrak{S} \)-module \( \mathfrak{S}[\mathfrak{g}]_{t^{-1}} \) is generated by \( \mathfrak{S}[\mathfrak{g}]_{t^{-1}} \mathfrak{g} \) (where \( \mathfrak{S}[\mathfrak{g}]_{t^{-1}} \mathfrak{g} \) is the arc space of \( \mathfrak{g} \)). For \( k = -h \), the maximal module \( \mathcal{N}_k \) of \( V^k(\mathfrak{g}) \) is generated by Feigin-Frenkel center \( (\mathfrak{g}^n_0) \). Hence \( \mathfrak{S}[\mathfrak{g}]_{t^{-1}} \mathfrak{g} \) is exactly the argumentation ideal of \( \mathfrak{S}[\mathfrak{g}]_{t^{-1}} \mathfrak{g} \). Therefore, the above argument shows that the statement of Theorem 1.2 is false at the critical level.

4. \( W \)-algebras and proof of Theorem 1.3

Let \( f \) be a nilpotent element of \( \mathfrak{g} \). By the Jacobson-Morosov theorem, it embeds into an \( \mathfrak{sl}_2 \)-triple \( (e, h, f) \) of \( \mathfrak{g} \). Recall that the Slodowy slice \( \mathcal{S}_f \) is the affine space \( f + \mathfrak{g}^e \), where \( \mathfrak{g}^e \) is the centralizer of \( e \) in \( \mathfrak{g} \). It has a natural Poisson structure induced from that of \( \mathfrak{g}^* \) \cite{GG02}.

The embedding \( \text{span}_C \{e, h, f\} \cong \mathfrak{sl}_2 \hookrightarrow \mathfrak{g} \) exponentiates to a homomorphism \( \mathfrak{SL}_2 \rightarrow G \). By restriction to the one-dimensional torus consisting of diagonal matrices, we obtain a one-parameter subgroup \( \rho : \mathbb{C}^* \rightarrow G \). For \( t \in \mathbb{C}^* \) and \( x \in \mathfrak{g} \), set
\[ \tilde{\rho}(t)x := t^2 \rho(t)(x) \]
We have \( \tilde{\rho}(t)f = f \), and the \( \mathbb{C}^* \)-action of \( \tilde{\rho} \) stabilizes \( \mathcal{J}_f \). Moreover, it is contracting to \( f \) on \( \mathcal{J}_f \), that is, for all \( x \in \mathfrak{g}^c \),

\[
\lim_{t \to 0} \tilde{\rho}(t)(f + x) = f.
\]

The following proposition is well-known. Since its proof is short, we give below the argument for the convenience of the reader.

**Proposition 4.1 ([Slo80, Pre02, CM16]).** — The morphism

\[
\theta_f : G \times \mathcal{J}_f \to \mathfrak{g}, \quad (g, x) \mapsto g \cdot x
\]

is smooth onto a dense open subset of \( \mathfrak{g}^* \).

**Proof.** — Since \( \mathfrak{g} = \mathfrak{g}^c + [f, \mathfrak{g}] \), the map \( \theta_f \) is a submersion at \((1, f)\). Therefore, \( \theta_f \) is a submersion at all points of \( G \times (f + \mathfrak{g}^c) \) because it is \( G \)-equivariant for the left multiplication in \( G \), and

\[
\lim_{t \to \infty} \rho(t) \cdot x = f
\]

for all \( x \in f + \mathfrak{g}^c \). So, by [Har77, Ch.III, Prop.10.4], the map \( \theta_f \) is a smooth morphism onto a dense open subset of \( \mathfrak{g}^* \). □

As in the introduction, let \( \mathcal{W}^k(\mathfrak{g}, f) \) be the affine \( W \)-algebra associated with a nilpotent element \( f \) of \( \mathfrak{g} \) defined by the generalized quantized Drinfeld-Sokolov reduction:

\[
\mathcal{W}^k(\mathfrak{g}, f) = H^0_{\text{DS}, f}(V^k(\mathfrak{g})).
\]

Here, \( H^0_{\text{DS}, f}(M) \) denotes the BRST cohomology of the generalized quantized Drinfeld-Sokolov reduction associated with \( f \in N(\mathfrak{g}) \) with coefficients in a \( V^k(\mathfrak{g}) \)-module \( M \). Recall that we have [DSK06, Ara15a] a natural isomorphism \( R_{\mathcal{W}^k(\mathfrak{g}, f)} \cong \mathbb{C}[\mathcal{J}_f] \) of Poisson algebras, so that

\[
X_{\mathcal{W}^k(\mathfrak{g}, f)} = \mathcal{J}_f.
\]

We write \( \mathcal{W}^k(\mathfrak{g}, f) \) for the unique simple (graded) quotient of \( \mathcal{W}^k(\mathfrak{g}, f) \). Then \( X_{\mathcal{W}^k(\mathfrak{g}, f)} \) is a \( \mathbb{C}^* \)-invariant Poisson subvariety of the Slodowy slice \( \mathcal{J}_f \).

Let \( \mathcal{O}_k \) be the category \( \mathcal{O} \) of \( \mathfrak{g} \) at level \( k \). We have a functor

\[
\mathcal{O}_k \to \mathcal{W}^k(\mathfrak{g}, f)\text{-Mod}, \quad M \mapsto H^0_{\text{DS}, f}(M),
\]

where \( \mathcal{W}^k(\mathfrak{g}, f)\text{-Mod} \) denotes the category of \( \mathcal{W}^k(\mathfrak{g}, f) \)-modules.

The full subcategory of \( \mathcal{O}_k \) consisting of objects \( M \) on which \( \mathfrak{g} \) acts locally finitely will be denoted by \( \mathcal{K}_L_k \). Note that both \( V^k(\mathfrak{g}) \) and \( L_k(\mathfrak{g}) \) are objects of \( \mathcal{K}_L_k \).

**Theorem 4.2 ([Ara15a])**

1. \( H^i_{\text{DS}, f}(M) = 0 \) for all \( i \neq 0 \), \( M \in \mathcal{K}_L_k \). In particular, the functor

\[
\mathcal{K}_L_k \to \mathcal{W}^k(\mathfrak{g}, f)\text{-Mod}, \quad M \mapsto H^0_{\text{DS}, f}(M),
\]

is exact.
(2) For any quotient $V$ of $V^k(g)$, 

$$X_{H^0_{DS,f}}(V) = X_V \cap \mathcal{J}_f.$$ 

In particular $H^0_{DS,f}(V) \neq 0$ if and only if $G \cdot f \subset X_V$.

By Theorem 4.2(1), $H^0_{DS,f}(L_k(g))$ is a quotient vertex algebra of $\mathcal{W}^k(g,f)$ if it is nonzero. Conjecturally [KRW03, KW08], we have

$$\mathcal{W}_k(g,f) \cong H^0_{DS,f}(L_k(g))$$ 

provided that $H^0_{DS,f}(L_k(g)) \neq 0$.

(This conjecture has been verified in many cases [Ara05, Ara07, Ara11, AvE19].)

**Proof of Theorem 1.3.** — The directions $(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$ are obvious. Let us show that $(3)$ implies $(1)$. So suppose that $X_{H^0_{DS,f}}(L_k(g)) = \mathcal{J}_f$. By Theorem 1.1, it is enough to show that $X_{L_k(g)} = g^*$. Assume the contrary. Then $X_{L_k(g)}$ is contained in a proper $G$-invariant closed subset of $g$. On the other hand, by Theorem 4.2 and our hypothesis, we have

$$\mathcal{J}_f = X_{H^0_{DS,f}}(L_k(g)) = X_{L_k(g)} \cap \mathcal{J}_f.$$ 

Hence, $\mathcal{J}_f$ must be contained in a proper $G$-invariant closed subset of $g$. But this contradicts Proposition 4.1. The proof of the theorem is completed. \hfill \Box

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