Auto-similarity
in rational base number systems

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Abstract

This work is a contribution to the study of set of the representations of integers in a rational base number system. This prefix-closed subset of the free monoid is naturally represented as a highly non regular tree whose nodes are the integers and whose subtrees are all distinct. With every node of that tree is then associated a minimal infinite word.

The main result is that a sequential transducer which computes for all $n$ the minimal word associated with $n + 1$ from the one associated with $n$, has essentially the same underlying graph as the tree itself.

These infinite words are then interpreted as representations of real numbers; the difference between the numbers represented by these two consecutive minimal words is the called the span of a node of the tree. The preceding construction allows to characterise the topological closure of the set of spans.

1 Introduction

The purpose of this work is a further exploration and a better understanding of the set of words that represent integers in a rational base number systems. These numeration systems have been introduced and studied in [1], leading to some progress in the results around the so-called Mahler’s problem (cf. [4]). We give below a precise definition of rational base number systems and of the representation of numbers in such a system. But one can hint at the results established in this paper by just looking at the figure showing the ‘representation tree’ of the integers – that is, the compact way of describing the words that represent the integers – in a

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rational base number system (Fig. 1(b) for the base $\frac{3}{2}$) and by comparison
with the representation tree in a integer base number system (Fig. 1(a) for
the base 3).

Every subtree in the second tree is the full ternary tree whereas every
subtree in the first one is different from all other subtrees. As a result,
the language of the representations of the integers is not only a non regular
language, but the situation is even worse as this language indeed satisfies no
iteration lemma of any kind ([5]). With the hope of finding some order or
regularity within what seems to be closer to complete randomness (which,
on the other hand, is not established either) we consider the minimal words
originating from every node of the tree.

In the case of an integer base, this is perfectly uninteresting: all these
minimal words are equal to $0^\omega$. In the case of a rational base these words are
on the contrary all distinct, none are even ultimately periodic (as the other
infinite words in the representation tree). In order to find some invariant
of all these distinct words, or at least a relationship between them, we have
studied the function that maps the minimal word $w_n^-$ associated with $n$
onto the one associated with $n + 1$. We tried to describe this function by
a possibly infinite transducer.

![Figure 1: Representation trees in two number systems](image)

The computation of such a transducer in the case the base $\frac{3}{2}$, and more
generally in the case of a base $\frac{q}{p}$ with $p = 2q - 1$, leads to a surprising
and unexpected result. The transducer, denoted by $D_{\frac{q}{p}}$, is obtained by
replacing in the representation tree, denoted by $T_{\frac{q}{p}}$, the label of every edge
by a set of pairs of letters that depends upon this label only. In other
words, the underlying graphs of $T_p^q$ and $D_p^q$ coincide, and $D_p^q$ is obtained from $T_p^q$ by a substitution from the alphabet of digits into the alphabet of pairs of digits, in this special and remarkable case.

The general case is hardly more difficult to describe, once it has been understood. In the special case, the canonical digit alphabet has $p = 2q - 1$ elements; in the general case, we still consider a digit alphabet with $2q - 1$ elements denoted by $B_{p,q}$, either by keeping the larger $2q - 1$ elements of the canonical digit alphabet, when $p$ is is greater than $2q - 1$, or by enlarging the canonical alphabet with enough negative digits, when $p$ is is smaller than $2q - 1$; in both cases, $p - 1$ is the largest digit.

From $T_p^q$ and with the digit alphabet $B_{p,q}$, we then define another ‘representation graph’ denoted by $\hat{T}_p^q$: either by deleting the edges of $T_p^q$ labelled by digits that do not belong to $B_{p,q}$ in the case where $p > 2q - 1$ or, in the case where $p < 2q - 1$ by adding edges labelled with the new negative digits. Then, $D_p^q$ is obtained from $\hat{T}_p^q$ exactly as above, by a substitution from the alphabet of digits into the alphabet of pairs of digits. This construction of $D_p^q$, which we call the derived transducer, and the proof of its correctness are presented in Sect. 3.

In [1], the tree $T_p^q$, which is built from the representations of integers, is used to define the representations of real numbers: the label of an infinite branch of the tree is the development ‘after the decimal point’ of a real number and the drawing of the tree as a fractal object — like in Fig. 1 — is fully justified by this point of view. The same idea leads to the definition of the (renormalized\(^1\)) span of a node $n$ of the representation tree: it is the difference between the real represented respectively by the maximal and the minimal words originating in the node $n$.

Again, this notion is perfectly uninteresting in the case of an integer base $p$: the span of node $n$ is always 1. And again, the notion is far more richer and complex in the case of a rational base $p^q$. The trivial relationship between the minimal word originating at node $n + 1$ and the maximal word originating at node $n$ leads to the connexion between the construction of the derived transducer $D_p^q$ and the description of the set of spans $S_p^q$. Not only the digit-wise difference between maximal and minimal words is written on the alphabet $B_{p,q}$, but all these ‘difference words’ are infinite branches in the tree $\hat{T}_p^q$. This is explained in Sect. 4. From the structure of $\hat{T}_p^q$, it then follows (Theorem 16) that the topological closure of $S_p^q$ is an interval in the case where $p < 2q - 1$, and a set with empty interior in the case where $p > 2q - 1$.

With every node $n$ of the tree structure $T_p^q$ is associated the infinite

\(^1\)The classical definition of span of the node $n$ is, in the fractal drawing, the width of the subtree rooted in $n$. This value is obviously decreasing (exponentially) with the depth of the node $n$, hence the span of two nodes cannot be easily compared. In this article we only consider the renormalized span which is the span multiplied by $(\frac{p}{q})^k$, where $k$ is the depth of the node $n$. 
minimal word $w_n^-$, an irregular infinite word that looks as complex as the whole tree. In conclusion, we have shown that a straightforward computation of $w_{n+1}^-$ from $w_n^-$ require the same structure as $T^p_q$ itself – despite the fact that every minimal word looks as complex as the whole tree – whether it be performed directly on the words, or indirectly via the span of the nodes. It is this phenomenon that we call auto-similarity of the structure $T^p_q$. In this process, the value cases $p = 2q - 1$ appear to mark the boundary between two different behaviour, in a more deeper way than that was described in the first study of rational base number systems [1].

This paper is meant to be self-contained and gives, in particular, all necessary definitions concerning rational base number systems. However, our paper [1] where these systems have been defined and the sets of representations first studied will probably be useful.

2 Preliminaries and notations

2.1 Numbers and words

Given two real numbers $x$ and $y$, we denote by $x/y$ or $\frac{x}{y}$ their division in $\mathbb{R}$ (even if $x$ or $y$ happened to be integers), by $[x, y]$ the corresponding interval of $\mathbb{R}$ and by $\lfloor x \rfloor$ the integer $n$ such that $(n - 1) < x \leq n$. On the other hand, given two positive integers $n$ and $m$, we denote by $n \div m$ and $n \mod m$ respectively the quotient and the remainder of the Euclidean division of $n$ by $m$, that is, $n = (n \div m)m + (n \mod m)$ and $0 \leq (n \mod m) < m$. Additionally, we denote by $[n, m]$ the integer interval $\{n, (n+1), \ldots, m\}$.

An alphabet is a finite set of symbols called letters or digits when they are integers. Given an alphabet $A$, we consider both finite and infinite words over $A$ respectively denoted by $A^*$ and $A^\omega$. As in most cases letters will be digits, we denote the empty word by $\varepsilon$. For every positive integer $p$, we denote by $A_p$ the canonical digit alphabet of the base $p$ number system: $A_p = \{0, 1, \ldots, p-1\}$. For clarity, we as much as possible denote finite words by $u, v$ and infinite words by $w$. The concatenation of two words $u, v$ is either explicitly denoted by a low dot, as in $uv$, or implicitly when there is no ambiguity, as in $uv$. A finite word $u$ is said to be a prefix of a finite word $v$ (resp. an infinite word $w$) if there exists a finite word $v'$ (resp. an infinite word $w'$) such that $v = uv'$ (resp. $w = uw'$). The set of subsets of an alphabet $A$ is denoted by $\mathcal{P}(A)$.

2.2 Automata and transducers

We deal here with a very special class of automata and transducers only: they are infinite, their state set is $\mathbb{N}$, they are deterministic (or letter-to-letter and sequential), the initial state is 0, and all states are final.

As usual, an automaton $X$ over $A$ is denoted by a 5-tuple $X = (\mathbb{N}, A, \delta, 0, \mathbb{N})$, where $\delta : \mathbb{N} \times A \rightarrow \mathbb{N}$ is the transition function. The partial function $\delta$ is
extended to $\mathbb{N} \times A^*$, and $\delta(n, u) = m$ is also denoted by $n \cdot u = m$ or by $n \cdot u \rightarrow m$. Given an integer $n$, every state $n \cdot a$ for some $a \in A$ is called a successor of $n$. A word $u$ in $A^*$ (resp. a word $w$ in $A^*$) is accepted by $\mathcal{X}$ if $0 \cdot u$ exists (resp. if $0 \cdot v$ exists for every finite prefix $v$ of $w$). The language of finite words (resp. of infinite words) accepted by $\mathcal{X}$ is denoted by $L(\mathcal{X})$ (resp. by $L(\mathcal{X}^\omega)$).

For transducers, we essentially use the notation of [2], adapted for the infinite case. A transducer is an automaton whose transitions are labelled by pair of letters, it is formally a tuple $\mathcal{Y} = \langle \mathbb{N}, A \times B, \delta, \eta, 0, \mathbb{N} \rangle$ where $\langle \mathbb{N}, A, \delta, 0, \mathbb{N} \rangle$ is an automaton, called the underlying input automaton of $\mathcal{Y}$. $A$ is called the input alphabet, $B$ is the output alphabet and $\eta: \mathbb{N} \times A \rightarrow B$ is the output function. The transition function $\delta$ is extended as in automata, and $\eta$ is as usual extended to $\mathbb{N} \times A^* \rightarrow B^*$ by $\eta(n, \varepsilon) = \varepsilon$ and $\eta(n, u a) = \eta(n, u). \eta(n \cdot u, a)$, and $\eta(n, u)$ is also denoted by $n \cdot u$ for short.

Moreover, given two finite words $u$ and $v$, we denote by $n \cdot u \rightarrow m$ the combination of $n \cdot u = m$ and $n \cdot u = v$. We say that the image of a finite word $u$ by $\mathcal{Y}$, denoted by $\mathcal{Y}(u)$, is the word $v$, if it exists, such that $0 \cdot u \rightarrow v$ for some $k$. Similarly, the image of the infinite word $w$ is $w'$ if, for every finite prefix $u$ of $w$, $\mathcal{Y}(u)$ is a prefix of $w'$.

### 2.3 Rational base number system

Let $p$ and $q$ be two co-prime integers such that $p > q > 1$. Given a positive integer $N$, let us define $N_0 = N$ and, for all $i > 0$,

$$q N_i = p N_{i+1} + a_i,$$

where $a_i$ is the remainder of the Euclidean division of $q N_i$ by $p$, hence in $A_p = \{0, p - 1\}$. Since $p > q$, the sequence $(N_i)_{i \in \mathbb{N}}$ is strictly decreasing and eventually stops at $N_{k+1} = 0$. Moreover, it holds that

$$N = \sum_{i=0}^{k} \frac{a_i}{q} \left(\frac{p}{q}\right)^i.$$

The evaluation function $\pi$ is derived from this formula. Given a word $a_n a_{n-1} \cdots a_0$ over $A_p$, and indeed over any alphabet of digits, its value is defined by

$$\pi(a_n a_{n-1} \cdots a_0) = \sum_{i=0}^{n} \frac{a_i}{q} \left(\frac{p}{q}\right)^i \quad (1)$$

Conversely, a word $u$ in $A_p^*$ is called a $\frac{p}{q}$-representation of an integer $x$ if $\pi(u) = x$. Since the representation is unique up to leading 0’s (see [1, Theorem 1]), $u$ is denoted by $\langle x \rangle_{\frac{p}{q}}$ (or $\langle x \rangle$ for short) and can be computed with the modified Euclidean division algorithm above. By convention, the
representation of 0 is the empty word \( \varepsilon \). The set of \( \frac{p}{q} \)-representations of integers is denoted by \( L_{\frac{p}{q}} \):

\[
L_{\frac{p}{q}} = \left\{ \langle n \rangle_{\frac{p}{q}} \mid n \in \mathbb{N} \right\}.
\]

It should be noted that a rational base number system is not a \( \beta \)-numeration — where the representation of a number is computed by the (greedy) Rényi algorithm (cf. [3, Chapter 7]) — in the special case where \( \beta \) is a rational number. In such a system, the digit set is \{0, 1, \ldots, \lceil \frac{p}{q} \rceil \} and the weight of the \( i \)-th leftmost digit is \((\frac{p}{q})^i\); whereas in the rational base number system, they are \{0, 1 \ldots (p - 1)\} and \( \frac{1}{q}(\frac{p}{q})^i \) respectively.

It is immediate that \( L_{\frac{p}{q}} \) is prefix-closed (since, in the modified Euclidean division algorithm \( \langle N \rangle = \langle N_1 \rangle . a_0 \) and prolongable (for every representation \( \langle n \rangle \), there exists (at least) an \( a \) in \( A_p \) such that \( q \) divides \((np + a)\) and then \( \langle \frac{np+a}{q} \rangle = \langle n \rangle . a \). As a consequence, \( L_{\frac{p}{q}} \) can be represented as an infinite tree (cf. Figure 2).

It is known that \( L_{\frac{p}{q}} \) is not rational (not even context-free), and the following automaton (in fact accepting the language \( 0^* L_{\frac{p}{q}} \)) is infinite.

**Definition 1.** Let \( \tau_{\frac{p}{q}} : \mathbb{N} \times \mathbb{Z} \to \mathbb{N} \) be the (partial) function defined\(^2\) by:

\[
\forall n \in \mathbb{N}, \forall a \in \mathbb{Z} \quad \tau_{\frac{p}{q}}(n, a) = \left( \frac{np + a}{q} \right) \quad \text{if} \ (np + a) \text{ is divisible by} \ q.
\]

\(^2\)The function \( \tau_{\frac{p}{q}} \) is defined on \( \mathbb{N} \times \mathbb{Z} \) instead of \( \mathbb{N} \times A_p \) in anticipation of future developments.
We denote\(^3\) by \(T_p^q\) the automaton \(T_p^q = \langle \mathbb{N}, A_p, \tau_p^q, 0, \mathbb{N} \rangle\).

In \(T_p^q\), we then have the transitions \(n \xrightarrow{a} \frac{np+q}{a}\) for every \(n \in \mathbb{N}\), and every \(a \in A_p\) such that \((np + q)\) is divisible by \(q\). The tree representation of \(L_p^q\), as in Figure 2 augmented by an additional loop labelled by 0 on the state 0 becomes a representation of \(T_p^q\).

We call minimal alphabet (resp. maximal alphabet) the subalphabet \(A_q = \llbracket 0, (q - 1) \rrbracket\) (resp. the subalphabet \(\llbracket (p - q), (p - 1) \rrbracket\)) of \(A_p\). Any letter of \(A_q\) is then called a minimal letter, maximal letter being defined analogously. The definition of \(\tau_p^q\) implies that every state of \(T_p^q\) has a successor by a unique minimal (resp. maximal) letter.

**Definition 2** (minimal word). A minimal word (in the \(p^q\)-system) is an infinite word in \(A_q^\omega\) labelling an (infinite) path of \(T_p^q\) (not necessarily starting from the initial state 0).

It is immediate that there exists a unique infinite word in \(A_q^\omega\) starting from the state \(n\) of \(T_p^q\). We call this word the minimal word associated with \(n\) and denote it by \(w_n^-\). Additionally, we will use the term minimal outgoing label of \(n\), to designate the first letter of \(w_n^-\) and minimal successor of \(n\) the unique successor of \(n\) by a minimal letter.

We define in a similar way the maximal word \(w_n^+\) associated with \(n\).

### 3 The derived transducer

The purpose of this section is to build an automaton over \(A_q \times A_q\), that is, a letter-to-letter transducer realising the function \(w_n^- \mapsto w_{(n+1)}^-\). We call this transducer the derived transducer and denote it by \(D_p^q\). It will be obtained from \(T_p^q\) by a local\(^4\) transformation and this is the subject of Section 3.1.

#### 3.1 From \(T_p^q\) to \(D_p^q\)

The transformation of \(T_p^q\) into \(D_p^q\) is a two-step process. First, the structure of \(T_p^q\) is changed locally, by changing the alphabet, and a new automaton \(\hat{T}_p^q\) is thus obtained. The second step consists in replacing the labels in \(\hat{T}_p^q\) by a subset of \(A_q \times A_q\) by means of a substitution (meaning that two transitions of \(\hat{T}_p^q\) labelled by the same letter will be replaced by the same set of transitions) and produces \(D_p^q\).

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\(^3\)In [1], \(T_p^q\) is denoted an infinite directed tree. The labels of the (finite) paths starting from the root precisely formed the language \(0^*L_p^q\), as is \(L(T_p^q)\) in our case.

\(^4\)The term local is arguable see Remark 19, in the appendix.
3.1.1 Changing the alphabet

We denote by $B_{p,q}$ the alphabet $[p - (2q - 1), (p - 1)]$. In particular, if $p = (2q - 1)$, $B_{p,q} = A_p$; if $p < (2q - 1)$, $B_{p,q}$ contains negative digits; and if $p > (2q - 1)$, $B_{p,q}$ is an uppermost subset of $A_p$. Note that $B_{p,q}$ is always of cardinal $(2q - 1)$, an odd number, that the digit $(p - q)$ is then the centre of $B_{p,q}$ and that its maximal element $p - 1$ coincides with the one of $A_p$.

The automaton $\widehat{T}_{q}$ is then defined by:

$$\widehat{T}_{q} = \langle \mathbb{N}, B_{p,q}, \tau_{q}, 0, \mathbb{N} \rangle.$$  

This is possible, even if $B_{p,q}$ is larger than $A_p$ because, in Equation 2, $\tau_{q}$ is defined on $\mathbb{N} \times \mathbb{Z}$, hence on $\mathbb{N} \times B_{p,q}$.

Figure A.5, in the appendix, shows an example of the case when $p$ is (strictly) smaller than $(2q - 1)$, i.e. one has to add edges (thicker arrows). In this case, the resulting automaton is a DAG (more complex than a tree with one loop). Figure 3a shows an example of the case when $p$ is (strictly) greater than $(2q - 1)$, i.e. one has to remove edges (dotted arrows). In this case, the resulting automaton is a forest (that is, an infinite union of trees).

As already noted, if $p = (2q - 1)$, $B_{p,q} = A_p$ and $T_{q} = \widehat{T}_{q}$.

It is easy to verify that the process ensures that every state of $\widehat{T}_{q}$ congruent to $-1$ modulo $q$ has a unique successor and that all other states have exactly two successors.

3.1.2 Changing the labels

Every label of $\widehat{T}_{q}$ (which is a letter of $B_{p,q}$) is replaced by a set of pairs of digits in $A_q \times A_q$. The label replacement function $\omega_{q} : B_{p,q} \rightarrow \mathcal{P}(A_q \times A_q)$ (or $\omega$ for short), is more easily defined in two steps, as follows. First, the function $\omega$ computes the distance of the input to the centre of $B_{p,q}$.

\begin{figure}[h]
\centering
\begin{subfigure}[h]{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Transforming $T_{q}$ into $\widehat{T}_{q}$}
\end{subfigure} \hspace{0.5cm} \begin{subfigure}[h]{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{The derived transducer $D_{q}$}
\end{subfigure}
\caption{From $T_{q}$ to $D_{q}$}
\end{figure}
ω(a) = a – (p – q), for every a in B_{p,q}. Then, the image of a by ω is the set of pairs of letters in A_q whose difference is ω(a):

∀a ∈ B_{p,q}, \omega(a) = \{(b|c) ∈ A_q × A_q | c − b = ω(a)\}.

(3)

**Example 3** (The case \(q > \frac{p}{2}\)). The functions \(\omega_{\frac{q}{2}}\) and \(\omega_{\frac{q}{2}}\) are as follows:

\[
\begin{align*}
\omega_{\frac{q}{2}} & : 0 \mapsto \{- \rightarrow \frac{1}{2}\} \\
\omega_{\frac{q}{2}} & : 1 \mapsto \{- \rightarrow \frac{1}{2}\} \\
\omega_{\frac{q}{2}} & : 2 \mapsto \{- \rightarrow \frac{1}{2}\}
\end{align*}
\]

and Fig. 4 shows \(D_{\frac{q}{2}}\).

![Figure 4: The derived transducer \(D_{\frac{q}{2}}\)](image)

Figure 3b shows the transducer \(D_{\frac{p}{2}}\) and \(D_{\frac{q}{2}}\) is represented by Figure A.6 in the appendix.

Formally, the transducer \(D_{\frac{q}{2}} = \langle N, A_q × A_q, \delta, \eta, 0, N \rangle\) is defined implicitly or, more precisely, the transition function \(\delta\) and the output function \(\eta\) are implicit functions defined by the following statement:

∀n ∈ N, \forall a ∈ B_{p,q}, \forall (b,c) ∈ \omega(a)

\(\tau_{\frac{q}{2}}(n, a)\) is defined \(\Rightarrow n \xrightarrow{b,c} \tau_{\frac{q}{2}}(n, a)\) is a transition of \(D_{\frac{q}{2}}\),

that is, \(\delta(n, b) = \tau_{\frac{q}{2}}(n, a)\) and \(\eta(n, b) = c\).

(4)

In other words, the transitions of \(D_{\frac{q}{2}}\) are labelled as follows:

if \(n ≡ -1 [q]\), the state \(n\) has exactly one outgoing transition with labels 0|0, 1|1, …, \(q − 1|q − 1\). Otherwise, the state \(n\) has two outgoings transitions and their labels are 0|\(k\), 1|\(k + 1\), …, (\(q − k − 1|q − 1\) for the upper transition and \(q − k|0\), (\(q − k + 1|1\), …, \(q − 1|k − 1\) for the lower transition, with \(k = a − (p − q)\) and \(a\) being the maximal outgoing label of \(n\) in \(T_{\frac{q}{2}}\).

The transducer constructed in this manner is sequential and input-complete, as stated by the following lemma whose proof is given in the appendix.
Lemma 4. For every state $n$ of $D_{pq}$ and every letter $b$ of $A_q$, there exists a unique state $m$ and a unique letter $c$ such that $n \xrightarrow{b|c} m$.

Corollary 5. For every infinite word $w$ in $A_q^\omega$, $D_{pq}(w)$ exists and is unique.

3.2 Correctness of $D_{pq}$

It remains to establish that $D_{pq}$ has the expected behaviour, as stated in the following.

Theorem 6. For every $n$ in $\mathbb{N}$, $D_{pq}(w_n^-) = w_{(n+1)^-}$.

The proof of this theorem relies on the equivalent (and more explicit) definition of the transition of $D_{pq}$, stated in the following proposition whose proof is given in the appendix.

Proposition 7. If $n \xrightarrow{b|c} m$ is a transition of $D_{pq}$, then

$$c = (b - (n + 1)p) \% q \quad \text{and} \quad m = \left\lceil \frac{(n + 1)p - b}{q} - 1 \right\rceil.$$ 

In the case of finite words, a stronger version can be stated.

Theorem 8. Given a base $\mathbb{Z}$ and two finite words $u, v$ over $A_q$, $u|v$ labels a run of $D_{pq}$ if, and only if there exists an integer $n$ such that $u$ is a prefix of $w_n^-$ and $v$ is a prefix of $w_{n+1}^-$. 

This theorem is purposely stated on finite words, as a similar statement for infinite words would be false: for every infinite word $w$ of $A_q^\omega$, $D_{pq}(w)$ exists, hence there is uncountably many pairs of infinite words $w_n^-|D_{pq}(w)$ accepted by $D_{pq}$ while there is only countably many pairs $w_n^-|w_{n+1}^-$. 

4 Span of a node

In this part, we consider the real value of infinite words. We denote by $\rho : A_p^\omega \to \mathbb{R}$, the real evaluation function, defined as follows:

$$\rho(a_1a_2\cdots a_n\cdots) = \sum_{i>0} a_i \left(\frac{p}{q}\right)^{-i}.$$  

We denote by $W_p^\mathbb{Z}$ the language of infinite words $\mathcal{L}(T_p^\mathbb{Z})$. It is proven in [1, Theorem 2] that $\rho(W_p^\mathbb{Z})$ is the interval $[0, \rho(w_n^\mathbb{Z})]$. By extension, we denote by $W_{pq}^n$ (or, for short, $W_n$) the language of infinite words $\langle n \rangle^{-1}\mathcal{L}(T_p^\mathbb{Z})$. Intuitively, an infinite word $w$ over $A_p$ is in $W_n$ if $n \cdot u$ exists in $T_p^\mathbb{Z}$ for every finite prefix $u$ of $w$. Analogously to $W_p^\mathbb{Z}$, the following holds.
Lemma 9. For every integer \( n \), \( \rho(W_n) \) is the interval \([\rho(w^-_n), \rho(w^+_n)]\).

Definition 10. For every integer \( n \), the span of \( n \), denoted by \( \text{span}(n) \), is the size of \( \rho(W_n) \): \( \text{span}(n) = (\rho(w^+_n) - \rho(w^-_n)) \).

Let \( a \) be a letter from the minimal alphabet \( A_q = \llbracket 0, (q - 1) \rrbracket \) and \( b \) a letter from the maximal alphabet \( \llbracket (p - q), (p - 1) \rrbracket \). The integer \( (b - a) \) is necessarily in \( \llbracket p - (2q - 1), p - 1 \rrbracket = B_{p,q} \). Hence, through this digit-wise subtraction, denoted as ‘\( \ominus \)’, \( (w^+_n \ominus w^-_n) \) is a word over \( B_{p,q} \), and is called the span-word of \( n \). It is routine to check that the following statement is true.

Lemma 11. For all integer \( n \), \( \text{span}(n) = \rho(w^+_n \ominus w^-_n) \).

We denote by \( S_{\frac{p}{q}} \) the set of real numbers \( \{\text{span}(n) \mid n \in \mathbb{N}\} \). In order to establish properties of \( S_{\frac{p}{q}} \) (Theorem 16, below) we first need to consider span-words.

Theorem 12. All span-words are accepted by \( \hat{T}_{\frac{p}{q}} \).

The proof of this theorem is a direct consequence of Proposition 13 below and requires more definitions. The span-words is closely related to the derived transducer. There exists a (trivial) map \( m \) from the minimal alphabet to the maximal alphabet, such that, for all integer \( n \), \( m(w^+_n) = w^+_n \).

\[
m : A_q \rightarrow \llbracket (p - q), (p - 1) \rrbracket \\
a \mapsto \text{maxLetter}(a + p)
\] (6)

where \( \text{maxLetter}(x) \) is the greatest integer congruent to \( x \) modulo \( q \) and strictly smaller than \( p \). By extending \( m \) to \( A_{\infty} \), Theorem 12 is reduced to say that \( \hat{T}_{\frac{p}{q}} \) accepts \( (m(w^+_n) \ominus w^-_n) \) for every \( n \):

Proposition 13. If \( w | w' \) is a pair of infinite words accepted by \( D_{\frac{p}{q}} \) then \( \hat{T}_{\frac{p}{q}} \) accepts the word \( (m(w') \ominus w) \).

Analogously to the case of \( D_{\frac{p}{q}} \), \( \hat{T}_{\frac{p}{q}} \) accepts uncountably many infinite words, therefore words that are not \( (w^+_n \ominus w^-_n) \) for any \( n \). That being said, it seems to be the best result we can hope, as the following two corollaries hold.

Corollary 14. Every finite word accepted by \( \hat{T}_{\frac{p}{q}} \) is the prefix of a span-word.

Corollary 15. The language of infinite words of \( \hat{T}_{\frac{p}{q}} \) is the topological closure of the span-words.
In [1], it was hinted that there might be structural differences between two classes of rational base number systems. Indeed, those where $p \geq 2q - 1$ had an additional property, namely that every $W_n$ contains at least two words (hence infinitely many). It was however never proved that this property was false when $p < 2q - 1$. The next statement provides a first element to differentiate these two classes of rational base number systems.

**Theorem 16.**

(i) If $(p < 2q)$, $S_{\frac{p}{q}}$ is dense in $[0, \rho(w_0^+)]$.

(ii) If $(p > 2q)$, $S_{\frac{p}{q}}$ is nowhere dense.

The proof of this theorem is not difficult but heavily relies on the notions and properties developed in [1]. We give a sketch of it in the appendix.

5 Conclusion

In the search of elucidating the structure of the set of representations of integers in a rational base number system, we have shown that the correspondence between two consecutive minimal words is achieved by a transducer that exhibits essentially the same structure as the one of the set of representations we started with. We have called this property an “*autosimilarity*” of the structure, as we have not shown that the structure is indeed *self-similar*.

Let us note that the infinite transducer we have thus built realises the correspondence for *all minimal words*. It is not a very good omen, but does not contradict the following conjecture.

**Conjecture 17.** For every integer $n$, there exists a finite transducer that transforms $w_n^{-}$ into $w_{n+1}^{-}$.

It is also remarkable that in this construction, the case $p = 2q - 1$ appears as the frontier between two completely different behaviours of the system, in a much stronger way than it was described in our first work on rational base number systems.

References

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Appendix

The section headings and numbers of the paper body are recalled, and prefixed with an A, for an easier navigation.

A.3 The derived transducer

A.3.1 From $T_q^L$ to $D_q^L$

A.3.1.1 Changing the alphabets

Figure A.5: Transforming $T_q^3$ into $\hat{T}_q^3$

A.3.1.2 Changing the labels

**Lemma 4.** For all state $n$ of $D_q^L$ and every letter $b$ of $A_q$, there exists a unique state $m$ and a unique letter $c$ such that $n \xrightarrow{b|c} m$.

Proof. Case where $n$ is congruent to $-1$ modulo $q$: by definition, $n$ has a unique successor associated with the letter $(p-q)$, $n \xrightarrow{\omega(p-q)} \left( \frac{np+a}{q} \right)$. In this case, the lemma’s statement is immediate as $\omega(p-q) = 0$, hence $\omega(p-q)$ is constituted of every pair $b|b$, for every $b$ in $A_q$.

Case where $n$ is not congruent to $-1$ modulo $q$: let $a$ be a maximal letter different than $(p-q)$, and $d$ a minimal letter. It is sufficient to prove that $\omega(a) \cup \omega(a-q)$ contains exactly one pair of the form $d|e$ for some $e$.

Since $\overline{\omega}(a) = (a-p+q)$ and $\overline{\omega}(a-q) = (a-p)$, the difference between the two is $q$, hence at most one integer of $\{ (d+\overline{\omega}(a)), (d+\overline{\omega}(a-q)) \}$ is in $A_q$.

Since $a$ is a maximal letter, $\overline{\omega}(a)$ is contained in $[0,(q-1)]$, as is $d$, by definition of $A_q$. Follows that $d+\overline{\omega}(a)$ is in $[0,2(q-1)]$, hence either $d+\overline{\omega}(a)$ is in $A_q$, or it is in $[q,2(q-1)]$, in which case $(d+\overline{\omega}(a-q)) = (d+\overline{\omega}(a)-q)$ is in $A_q$. $\square$
A.3.2 Correctness of $D_{pq}$

We establish now that $D_{pq}$ has the expected behaviour, that is, we prove the main Theorem 6 as stated in the following.

**Theorem 6.** For every $n$ in $\mathbb{N}$, $D_{pq}(w_n) = w_{(n+1)}$.

After the description of $D_{pq}$ by a transformation of $T_{pq}$, we characterise its transition and output functions by relations that will be used in further demonstrations.

**Proposition 7.** If $n \overset{b}{\rightarrow} c$ is a transition of $D_{pq}$, then

$$c = (b - (n + 1)p) \mod q$$

and

$$m = \lfloor \frac{(n + 1)p - b}{q} - 1 \rfloor.$$

**Proof.** If $n \overset{b}{\rightarrow} c$ is a transition of $D_{pq}$, then by hypothesis there exists a letter $a$ in $B_{pq}$ such that $b|c$ is in $\omega(a)$, in which case: $\overline{\omega}(a) = c - b$, hence $a = (p - q) + (c - b)$.

From Equation 4, we know that $(np + a)$ is congruent to 0 modulo $q$. By replacing $a$ with $((p - q) + (c - b))$, we finally obtain that $c$ is congruent to $(b - (n + 1)p)$ modulo $q$. Since $c$ is in $A_q$, $c = (b - (n + 1)p) \mod q$.

From Equation 4, we know as well that $m = \frac{np + a}{q}$, hence $m = \frac{np + ((p - q) + (c - b))}{q}$, and after simplification, $m = \frac{(n + 1)p - b}{q} - 1$. Since $c$ is in $A_q$, it is strictly smaller than $q$, then $0 \leq \frac{c}{q} < 1$ which concludes the proof.

Theorem 6 is then a corollary of the next proposition which describes the behaviour of $D_{pq}$ starting from all states, not only the initial one.
Proposition 18. Let $u$ and $v$ be two words over $A_q$. If $n \xrightarrow{u} m$ in $T_q^p$ and $i \xrightarrow{u \mid v} j$ in $D_q^p$, then $(n + i + 1) \xrightarrow{u \mid v} (m + j + 1)$ in $T_q^p$.

Proof. Let us first consider the special case where $u$ is a single letter $a$. The first hypothesis implies (from Equation 2) that $m = \frac{np + a}{q}$; the second (from Proposition 7) that $v$ is the single letter $(a - (i + 1)p) \% q$ and $j = \lceil \frac{(i + 1)p - a}{q} - 1 \rceil$.

It is routine to check that $(a - (i + 1)p) \% q$ is indeed an outgoing letter of $(n + i + 1)$. The successor in $T_q^p$ of $(n + i + 1)$ by this letter is

$$
\frac{(n + i + 1)p + (a - (i + 1)p) \% q}{q} = \frac{np + (i + 1)p}{q} + \frac{(a - (i + 1)p) \% q}{q} = m + \frac{(i + 1)p - a}{q} + \frac{(a - (i + 1)p) \% q}{q} = m + \frac{(i + 1)p - a}{q} = m + j + 1
$$

The general case then consists in a simple induction over the length of $u$. \qed

Remark 19 (Locality). At the start of Section 3, it was claimed that the transformation from $T_q^p$ to $D_q^p$ is local. Although it undoubtedly is when $p \geq (2q - 1)$, it is less clear when $p < (2q - 1)$.

Indeed, at some point, one has to add an edge $n \rightarrow (m - 1)$ while having access to the edge $n \rightarrow m$, and must then access the state $(m - 1)$. Considering $T_q^p$ has an undirected graph, the path from $(m - 1)$ to $n$ can be arbitrarily large, which would contradict locality. However, we deemed it reasonable to have access to either a map from $\mathbb{N}$ to the states of $T_q^p$ or simply a ‘decrementer’ operator linking every state $m$ to $(m - 1)$.

A.4 Span of a node

Proposition 13. If $w \mid w'$ is a pair of infinite words accepted by $D_q^p$, then $\hat{T}_q^p$ accepts the word $(m(w') \ominus w)$.

Proof. It is enough to prove that, for every pair $b \mid c$ in $\omega(a)$, $m(c) - b = a$.

With this denotation, by definition of $D_q^p$ (and more particularly $\omega$, from Equation 3), $c = b + \overline{\omega}(a)$ and $0 \leq c < q$, or more precisely

$$
0 \leq c < q \\
0 \leq b + \overline{\omega}(a) < q \\
0 \leq b + a - (p - q) < q \\
(p - q) \leq (b + a) < p
$$
Therefore, \((b + a)\) is a maximal letter, hence \(\text{maxLetter}(b + a) = b + a\), and finally \(m(c) - b = a\) when replacing \(c\) and \(\mathcal{U}\) by their expression.

\[\text{\textbf{Theorem 16.}}\]

(i) If \((p < 2q)\), \(S_{\frac{p}{q}}\) is dense in \([0, \rho(w_0^+))]\).

(ii) If \((p > 2q)\), \(S_{\frac{p}{q}}\) is nowhere dense.

The proof of (i) essentially consists in the next Lemma and its corollary, stating that even though \(\hat{T}_{\frac{p}{q}}\) accepts words that \(T_{\frac{p}{q}}\) doesn’t, their values are redundant.

\[\text{\textbf{Lemma 20.}}\]

If \(p < 2q - 1\), given a finite word \(u\) over \(B\) accepted by \(\hat{T}_{\frac{p}{q}}\), there exists a finite word \(v\) over \(A_p\) such that \(v\) is accepted by \(T_{\frac{p}{q}}\), \(\pi(u) = \pi(v)\) and \(|u| = |v|\).\(^5\)

\[\text{Proof.}\]

Through a simple induction, one can reduce the statement to the special case where \(u\) is part of \(A_p^* B\). We denote by \(n\) the non-negative integer \(\pi(u)\). It is then enough to prove that \(|\langle n \rangle| \leq |u|\), since setting \(v = 0^k \langle n \rangle\) would satisfy both equations.

We denote by \(u'\) (resp. \(v\)) the word in \(A_p^*\) and by \(b\) (resp. \(a\)) the letter in \(B\) (resp. \(A_p\)) such that \(u = u'b\), (resp. \(\langle n \rangle = v'a\)).

Since 1. \(v'\) is the representation of the integer \(\pi(v')\),
2. \(\pi(u')\) is smaller than \(\pi(v')\),
3. \(u'\) is in \(A_p^*\);

\[|v'| = \frac{1}{|\langle \pi(v') \rangle|} \leq \frac{2}{|\langle \pi(u') \rangle|} \leq \frac{3}{|u'|}\]

hence \(|\langle n \rangle| \leq |u|\).

\[\text{\textbf{Corollary 21.}}\]

If \(p < 2q\), \(\rho(\mathcal{L}(\hat{T}_{\frac{p}{q}})) = \rho(\mathcal{L}(T_{\frac{p}{q}}))\).\(^6\)

\[\text{\textit{Proof of Theorem 16.(i).}}\]

Since \(\mathcal{L}(\hat{T}_{\frac{p}{q}})\) is the topological closure of the span words, and that (from Corollary 21) \(\rho(\mathcal{L}(\hat{T}_{\frac{p}{q}})) = \rho(\mathcal{L}(T_{\frac{p}{q}})) = W_0\), the set \(\{\text{span}(n) \mid n \in \mathbb{N}\}\) is dense in \(W_0 = [0, \rho(w_0^+))]\).

The proof of Theorem 16.(ii) requires more notation. For all integer \(n\), we denote by \(W_n^*\) the set of words \(0^* \langle n \rangle W_n\), that is, the (infinite) words of \(\mathcal{L}(T_{\frac{p}{q}})\) whose run passes through the state \(n\). In particular, with this notation, if \(n \xrightarrow{\pi} m\), then \(W_m^* \subseteq W_n^*\), which “basically” reduces Theorem 16.(ii) to the following statement.

\(^5\)The condition on \(p\) and \(q\) can be relaxed, but the case where \((p \geq 2q - 1)\) is trivial and unnecessary in the following.

\(^6\)Here however the condition on \(p\) and \(q\) is mandatory.
**Lemma 22.** If $p > 2q$, for every integer $n$, there exists an integer $m$ such that $m$ is reachable from $n$ in $\mathcal{T}_{p,q}$ but not in $\hat{\mathcal{T}}_{p,q}$.

*Proof.* We denote by $S_i$ the set $\{n' \mid n \xrightarrow{u} n' \text{ and } |u| = i\}$. For all $i$, $S_i$ is an integer interval, and since $p > 2q$, $|S_i|$ increases strictly with $i$. It follows that $S_{p+1}$ contains at least an integer $m$ congruent to 0 modulo $p$ (beware, it is $p$ and not $q$). The state $m$ is reachable in $\mathcal{T}_{p,q}$ by a unique transition labelled by 0, and since 0 is not in $B$ (because $p > 2q$), $k$ is not reachable in $\hat{\mathcal{T}}_{p,q}$.

*Proof of Theorem 16 (ii).* We denote by $S$ the set $\{\text{span}(n) \mid n \in \mathbb{N}\}$, and for all $n$ in $\mathbb{N}$, we denote by $W'_n$ the set of words $\langle n \rangle W_n$.

Let us assume that $S$ is dense in an interval $[x, y]$. There exists a positive integer $n$ such that $\langle n \rangle w^-_n$ and $\langle n \rangle w^+_n$ are both in $[x, y]$, hence $S$ is dense in $\rho(W'_n)$. From Lemma 22, there exists an integer $m$ reachable from $n$ in $\mathcal{T}_{p,q}$ but not in $\hat{\mathcal{T}}_{p,q}$, hence no word of $W'_m$ is accepted by $\hat{\mathcal{T}}_{p,q}$. The real values of these words form a (non-trivial) sub-interval of $\rho(W'_n)$.

From [1, Corollary 38] we know that every real number has either one $\frac{p}{q}$-representation, or two, in which case one is $\langle i + 1 \rangle w^-_{i+1}$ and the other $\langle i \rangle w^+_i$ for some $i$. It implies that every word of the interior of $\rho(W'_n)$ has no $\frac{p}{q}$-representation outside of $W'_m$, hence $\rho(W'_n)$ contains an open set whose intersection with $S$ is empty, a contradiction. \qed