The homogenization of one-dimensional acoustic or elastic structures of finite extent is considered. A new homogenization method based on transfer matrices is derived. The new homogenization method may account for variable cross sectional area and for Willis coupling, which couples the stress-strain and momentum-velocity constitutive relations. The homogenization method is then demonstrated by considering acoustic waves normally incident upon a rigidly-backed double-layered wall and plane waves propagating in a duct with a section of exponentially-growing cross-sectional area.
I. INTRODUCTION

The study of acoustic metamaterials hinges on the ability to determine the effective material properties of a system, also known as homogenization. Homogenization of one-dimensional systems has been extensively studied in the static case. Analytical methods of homogenization can be especially useful as efficient design tools due to the fact that they provide explicit results. Examples of previous analytical homogenization methods in one dimension include averaging equations for quasi-static deformations, analyzing periodic systems of layered media, and collective modes in the systems of lumped elements.

As an example important to the present work, Kutsenko, et al. used a $4 \times 4$ transfer matrix to describe propagation in an infinite, one-dimensional, periodic, layered piezoelectric medium. They were able to homogenize the system both in the quasi-static limit and for finite frequencies by analyzing the dispersion relation of the propagated waves. While their approach is quite general, except for in the quasi-static limit it assumes an infinitely-periodic system and cannot account for finite sizes of materials. Finite-sized systems materials can be important for designing inclusions for multiscale homogenization methods in periodic and non-periodic media and for analyzing the behavior of composite plates. In addition, it assumes that all layers are of infinite lateral extent and therefore cannot account for one-dimensional ducts with variable cross-sectional area. The purpose of this paper is to present a related but alternative homogenization method to that of Kutsenko, et al. that accurately homogenizes one-dimensional systems that may include finite sizes and variable cross-sectional areas, though not piezoelectric properties.
The outline of the paper is as follows. In Sec. II the alternative transfer matrix homogenization method is presented in both discrete and continuous representations. Section III provides examples of the homogenization method. Finally, Sec. IV summarizes the conclusions.

II. TRANSFER MATRIX HOMOGENIZATION

The Willis constitutive equations in one dimension may be written as\(^1\)

\[-p = \kappa \varepsilon + \psi^{(1)} \dot{v}, \quad \mu = \rho v + \psi^{(2)} \dot{\varepsilon},\]  

where \(p\) is the acoustic pressure, \(\varepsilon\) is the volume strain, \(\mu\) is the momentum density, \(v\) is the particle velocity, and over-dots denote time derivatives. The material properties are the bulk modulus \(\kappa\), the mass density is \(\rho\), and the Willis coupling is represented by \(\psi^{(1)}\) and \(\psi^{(2)}\). For passive and causal systems the Willis coupling coefficients are equal,\(^2\) i.e., \(\psi^{(1)} = \psi^{(2)}\). The constitutive equations supplement the dynamic equation and the definition of the strain rate:

\[\dot{\mu} = -p', \quad \dot{\varepsilon} = v',\]  

where the primes denote spatial derivatives. Combining these equations together leads to the standard wave equation with the wave speed \(c = \sqrt{\kappa/\rho}\).\(^3\) Assuming time-harmonic motion \((e^{-i\omega t} \text{ time convention})\) leads to the conclusion that the wavenumber \(k = \omega/c\).

While the analysis presented here and below assumes all materials are fluids, it is worthwhile to note that in isotropic solids the longitudinal and shear waves are independent of each other, and in one dimension there is no mathematical distinction between these elastic
One-dimensional homogenization

waves and fluid waves. Thus if $G$ is the shear modulus, replacing the bulk modulus $\kappa$ with the plane wave modulus $\kappa + 4G/3$ yields the same results for longitudinal elastic waves and replacing $\kappa$ with $G$ yields the same results for shear elastic waves. Note that this correspondence is only valid for one-dimensional propagation, as the interface conditions become coupled for oblique incidence.

Given an inhomogeneous domain $\Omega = (a, b)$ where $k(b - a) \equiv kL \ll 1$, these constitutive equations may be used to define the effective material properties of the domain. These effective material properties may be written as

$$
\begin{align*}
\kappa_{\text{eff}} &\equiv i\omega \left. \langle \frac{p}{v'} \rangle \right|_{(v')=0}, \\
\rho_{\text{eff}} &\equiv \frac{1}{i\omega} \left. \langle \frac{p'}{v} \rangle \right|_{(v)=0}, \\
\psi_{\text{eff}}^{(1)} &\equiv \frac{1}{i\omega} \left. \langle \frac{p}{v} \rangle \right|_{(v')=0}, \\
\psi_{\text{eff}}^{(2)} &\equiv \frac{1}{i\omega} \left. \langle \frac{p'}{v'} \rangle \right|_{(v)=0}.
\end{align*}
$$

These averages may be written in terms of the field quantities at the edges of the domain. If the domain is $\Omega = (a, b)$, where $b - a = L > 0$, then the average fields may be written as

$$
\begin{align*}
\langle p \rangle &\approx \frac{p(a) + p(b)}{2}, & \langle v \rangle &\approx \frac{v(a) + v(b)}{2}, \\
\langle p' \rangle &\approx \frac{p(b) - p(a)}{L}, & \langle v' \rangle &\approx \frac{v(b) - v(a)}{L}.
\end{align*}
$$

The fields at the edges of the domain $\Omega$ are generally related by an ABCD transmission matrix

$$
\begin{bmatrix}
p(a) \\
v(a)
\end{bmatrix} =
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
p(b) \\
v(b)
\end{bmatrix}.
$$

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\end{bmatrix} =
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
p(b) \\
v(b)
\end{bmatrix}.
$$
Using Eqs. (3)–(5) the effective material properties may then be written as

\[
\kappa_{\text{eff}} = -i\omega L \frac{A + D + 1 + (AD - BC)}{4C}, \quad (6a)
\]
\[
\rho_{\text{eff}} = \frac{1}{-i\omega L} \frac{A + D - 1 - (AD - BC)}{C}, \quad (6b)
\]
\[
\psi^{(1)}_{\text{eff}} = \frac{1}{-i\omega} \frac{D - A - 1 + (AD - BC)}{2C}, \quad (6c)
\]
\[
\psi^{(2)}_{\text{eff}} = \frac{1}{-i\omega} \frac{D - A + 1 - (AD - BC)}{2C}. \quad (6d)
\]

Thus the effective material properties of a one-dimensional system may be obtained with knowledge of the systems ABCD transmission matrix.

If the structure is passive and reciprocal, then the determinant of the ABCD matrix is \(AD - BC = 1\). In this case the effective material properties simplify to the expressions

\[
\kappa_{\text{eff}} = -i\omega L \frac{A + D + 2}{4C}, \quad (7a)
\]
\[
\rho_{\text{eff}} = \frac{1}{-i\omega L} \frac{A + D - 2}{C}, \quad (7b)
\]
\[
\psi^{(1)}_{\text{eff}} = \frac{1}{-i\omega} \frac{D - A}{2C} = \psi^{(2)}_{\text{eff}} \equiv \psi_{\text{eff}}. \quad (7c)
\]
Consider a one-dimensional layered material of length $L$ as shown in Fig. 1. The system consists of $N$ layers and the $n^{\text{th}}$ layer has length $L_n$, has mass density $\rho_n$, bulk modulus $\kappa_n$, and (for ducts) cross-sectional area $S_n$. The acoustic pressure and the volume velocity on the left-hand side of the $n^{\text{th}}$ layer, $p_{n-1}$ and $q_{n-1}$ respectively, may be related to the acoustic pressure and volume velocity on the right-hand side, $p_n$ and $v_n$, by a standard $ABCD$ matrix:

$$
\begin{bmatrix}
    p_{n-1} \\
    q_{n-1}
\end{bmatrix} =
\begin{bmatrix}
    \cos(k_n L_n) & -i Z_n \sin(k_n L_n) \\
    -\frac{i}{Z_n} \sin(k_n L_n) & \cos(k_n L_n)
\end{bmatrix}
\begin{bmatrix}
    p_n \\
    q_n
\end{bmatrix},
$$

(8)

where $k_n = \omega \sqrt{\rho_n / \kappa_n}$ and $Z_n = \sqrt{\rho_n \kappa_n / S_n}$ are the wavenumber and acoustic impedance of the $n^{\text{th}}$ layer. These expressions may be combined to relate the fields at the left-hand side of the entire structure to the fields at the right-hand side as

$$
\begin{bmatrix}
    p_0 \\
    q_0
\end{bmatrix} =
\begin{bmatrix}
    \prod_{n=1}^{N}
    \begin{bmatrix}
        \cos(k_n L_n) & -i Z_n \sin(k_n L_n) \\
        -\frac{i}{Z_n} \sin(k_n L_n) & \cos(k_n L_n)
    \end{bmatrix}
    \end{bmatrix}
\begin{bmatrix}
    p_N \\
    q_N
\end{bmatrix} \equiv
\begin{bmatrix}
    A' & B' \\
    C' & D'
\end{bmatrix}
\begin{bmatrix}
    p_N \\
    q_N
\end{bmatrix}.
$$

(9)

While this analysis does indeed yield an $ABCD$ matrix, it is written for the volume velocity rather than the particle velocity. Defining $S_{\text{ref}}$ as a reference or effective cross-sectional area it is straightforward to find that the elements of the $ABCD$ matrix in terms of the particle velocity may be written as $A = A'$, $B = B'S_{\text{ref}}$, $C = C'/S_{\text{ref}}$, and $D = D'$.

For $\omega$ small enough such that $k_n L_n \ll \pi/2$ the $N \times 2$ matrices

$$
A_n \equiv
\begin{bmatrix}
    \cos(k_n L_n) & -i Z_n \sin(k_n L_n) \\
    -\frac{i}{Z_n} \sin(k_n L_n) & \cos(k_n L_n)
\end{bmatrix},
$$

(10)
One-dimensional homogenization may be expanded in a matrix series as

$$A_n = I - ik_n L_n D_n - \frac{(k_n L_n)^2}{2} I + O([k_n L_n]^3),$$  \hfill (11)

where $I$ is the identity $2 \times 2$ matrix and

$$D_n \equiv \begin{bmatrix} 0 & Z_n \\ \frac{1}{Z_n} & 0 \end{bmatrix}. \hfill (12)$$

Define $\varepsilon$ as the largest value of $k_n L_n \equiv \theta_n$ such that $k_n L_n$ is of order $\varepsilon$ for all $n$. Then, using the results from Appendix A we find that

$$\prod_{n=1}^{N} A_n \approx I - i \sum_{n=1}^{N} k_n L_n D_n - \frac{1}{2} \sum_{n=1}^{N} (k_n L_n)^2 I - \sum_{n=1}^{N-1} \sum_{m=n+1}^{N} k_n L_n k_m L_m D_n D_m. \hfill (13)$$

The elements of the composite $ABCD$ may then be approximated as

$$A' \approx 1 - \frac{1}{2} \sum_{n=1}^{N} (k_n L_n)^2 - \sum_{n=1}^{N-1} \sum_{m=n+1}^{N} k_n L_n k_m L_m \frac{Z_n}{Z_m}, \hfill (14a)$$

$$B' \approx -i \sum_{n=1}^{N} k_n L_n Z_n, \hfill (14b)$$

$$C' \approx -i \sum_{n=1}^{N} \frac{k_n L_n}{Z_n}, \hfill (14c)$$

$$D' \approx 1 - \frac{1}{2} \sum_{n=1}^{N} (k_n L_n)^2 - \sum_{n=1}^{N-1} \sum_{m=n+1}^{N} k_n L_n k_m L_m \frac{Z_m}{Z_n}. \hfill (14d)$$

Since $k_n Z_n = \omega \rho_n / S_n$ and $k_n / Z_n = \omega S_n / \kappa_n$, the effective material properties may then be written to lowest order as

$$\frac{1}{\kappa_{\text{eff}}} = \frac{1}{S_{\text{ref}}} \left\langle \frac{S_n}{\kappa_n} \right\rangle_n, \hfill (15a)$$

$$\rho_{\text{eff}} = S_{\text{ref}} \left\langle \frac{\rho_n}{S_n} \right\rangle_n, \hfill (15b)$$

$$\psi_{\text{eff}} = \frac{\kappa_{\text{eff}}}{2} \left\langle \sum_{m=n+1}^{N} L_m \left( \frac{\rho_m S_n}{\kappa_n S_m} - \frac{\rho_n S_m}{\kappa_m S_n} \right) \right\rangle_n. \hfill (15c)$$
where

$$\langle \cdot \rangle_n \equiv \frac{1}{L} \sum_{n=1}^{N} L_n [ \cdot ]$$  \hspace{1cm} (16)$$

is the spatial average operator.

There are multiple interesting features of the predicted effective material properties in Eqs. (15). First every term depends on the stiffness, meaning that simple averages of the mass density and Willis coupling coefficient are inaccurate. Another point of interest is that the stiffness always appears in summations as its inverse, the compressibility. Thus, layers with very low stiffness tend to dominate the overall response of the system. The Willis coupling coefficient approaches a real constant, even in the zero-frequency limit. Since the summand of the Willis coupling coefficient is odd with respect to \( m \) and \( n \), symmetric systems will not display any Willis coupling. Additionally, two layers, \( m \) and \( n \), do not contribute to the Willis coupling if \( \rho_m S_n / \kappa_n S_m = \rho_n S_m / \kappa_m S_n \), which reduces to equality of the acoustic impedances squared, \( (Z_m / S_m)^2 = (Z_n / S_n)^2 \).

A one-dimensional system with continuously varying properties may be treated with the above framework by letting \( P_n \to P(x) \), where \( P \in \{ \rho, \kappa, S \} \), and \( L_n \to dx \). In this case the effective material properties become

$$\frac{1}{\kappa_{\text{eff}}} = \frac{1}{S_{\text{ref}}} \int_0^L dx \frac{S(x)}{\kappa(x)} \equiv \frac{1}{S_{\text{ref}}} \frac{\left\langle S(x) \right\rangle}{\kappa(x)}$$  \hspace{1cm} (17a)$$

$$\rho_{\text{eff}} = \frac{S_{\text{ref}}}{L} \int_0^L dx \frac{\rho(x)}{S(x)} \equiv S_{\text{ref}} \left\langle \frac{\rho(x)}{S(x)} \right\rangle$$  \hspace{1cm} (17b)$$

$$\psi_{\text{eff}} = \frac{\kappa_{\text{eff}}}{2} \left\langle \int_x^L dy \left( \frac{\rho(y)S(x)}{\kappa(x)S(y)} - \frac{\rho(x)S(y)}{\kappa(y)S(x)} \right) \right\rangle$$  \hspace{1cm} (17c)$$
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FIG. 2. Schematic of a plane wave normally incident upon a bi-layer wall with a rigid backing.

III. EXAMPLES

A. Finite System With Discrete Layers

Consider the reflection problem described schematically in Fig. 2. The background material has mass density and bulk modulus of \( \rho_0 \) and \( \kappa_0 \), and the \( j^{\text{th}} \) layer has the properties \( \rho_j \) and \( \kappa_j \) and is of width \( L_j \). The acoustic pressure field in the background medium for a normally incident plane wave may then be written as

\[
p_0 = A_0 \left[ e^{ik_0 z} + R e^{-ik_0 z} \right],
\]

(18)

where \( A_0 \) is the amplitude of the incoming wave, \( R \) is the reflection coefficient, and \( k_0 = \frac{\omega}{\sqrt{\rho_0/\kappa_0}} \) is the incident wavenumber. It is then straightforward to apply continuity of particle velocity and acoustic pressure at the interfaces and show that the reflection coefficient may be written \( R = (1 - \zeta)/(1 + \zeta) \) where \( \zeta \) is a normalized input impedance given by

\[
\zeta = -i \frac{Z_0}{Z_1} \frac{\tan(k_1 L_1) + \tan(k_2 L_2) \frac{Z_1}{Z_2}}{1 - \tan(k_1 L_1) \tan(k_2 L_2) \frac{Z_1}{Z_2}},
\]

(19)
\( Z_i = \sqrt{\rho_i \kappa_i} \), and \( k_i = \omega \sqrt{\rho_i / \kappa_i} \). For low frequencies the normalized input impedance may be approximated as

\[
\zeta \approx -i \omega Z_0 \left[ \left( \frac{L_1}{\kappa_1} + \frac{L_2}{\kappa_2} \right) + \frac{\omega^2}{3} \left( \frac{\rho_1 \kappa_2^3}{\kappa_1^2} L_1^3 + 3 \frac{\rho_1}{\kappa_1 \kappa_2} L_1^2 L_2 + 3 \frac{\rho_1}{\kappa_2^2} L_1 L_2^2 + \frac{\rho_2}{\kappa_2^3} L_2^3 \right) \right].
\] (20)

The bi-layer wall may be approximated at low frequencies by a single layer of width \( L = L_1 + L_2 \) with effective material properties as prescribed by Eqs. (15). Since all cross sectional areas are equal we thus obtain

\[
\kappa_{\text{eff}} = \frac{L_1 + L_2}{\frac{L_1}{\kappa_1} + \frac{L_2}{\kappa_2}},
\] (21a)

\[
\rho_{\text{eff}} = \frac{L_1 \rho_1 + L_2 \rho_2}{L_1 + L_2},
\] (21b)

\[
\psi_{\text{eff}} = \frac{L_1 L_2}{2} \frac{\rho_2}{\kappa_1} - \frac{\rho_1}{\kappa_2}.
\] (21c)

In line with the above comments \( \psi_{\text{eff}} = 0 \) if the layers have equal impedance. As noted above the acoustic pressure and particle velocity in the Willis layer may be described by the wave equation with the standard wave speed. Then, the acoustic fields may be written in terms of trigonometric functions as

\[
p_{\text{eff}} = A_0 \left[ A_1 \cos(k_{\text{eff}}(L - z)) + B_1 \sin(k_{\text{eff}}(L - z)) \right],
\] (22a)

\[
v_{\text{eff}} = \frac{A_0}{i Z_{\text{eff}}} \left[ (W_{\text{eff}} A_1 - B_1) \cos(k_{\text{eff}}(L - z)) + (A_1 + W_{\text{eff}} B_1) \sin(k_{\text{eff}}(L - z)) \right],
\] (22b)

where \( Z_{\text{eff}} = \sqrt{\rho_{\text{eff}} \kappa_{\text{eff}}} \) is the effective characteristic impedance and \( W_{\text{eff}} = \omega \psi_{\text{eff}} / Z_{\text{eff}} \) is the effective asymmetry factor (a non-dimensional measure of the importance of Willis coupling to total impedance\(^\text{15} \)). Requiring the backing to be rigid leads to the requirement \( B_1 = \)
One-dimensional homogenization

$W_{\text{eff}A_1}$. Then matching the pressure and particle velocity at $z = 0$ leads to the equations

\[ 1 + R = A_1 \left[ \cos(k_{\text{eff}}L) + W_{\text{eff}} \sin(k_{\text{eff}}L) \right], \quad (23a) \]

\[ \frac{1}{Z_0} \left[ 1 - R \right] = \frac{A_1}{iZ_{\text{eff}}} \sin(k_{\text{eff}}L), \quad (23b) \]

which combine to yield $R = (1 - \zeta_{\text{eff}})/(1 + \zeta_{\text{eff}})$, where the effective normalized input impedance is given by

\[ \zeta_{\text{eff}} = -i \frac{Z_0}{Z_{\text{eff}}} \frac{\tan(k_{\text{eff}}L)}{1 + W_{\text{eff}} \tan(k_{\text{eff}}L)}. \quad (24) \]

For very small frequency we may then approximate

\[ \frac{i \zeta_{\text{eff}}}{\omega Z_0} \approx \frac{L}{\kappa_{\text{eff}}} + \frac{\omega^2}{3} \left( \frac{\rho_{\text{eff}}}{\kappa_{\text{eff}}^2} L^3 - 3 \frac{\psi_{\text{eff}}}{\kappa_{\text{eff}}^2} L^2 \right) \]

\[ = \left( \frac{L_1}{\kappa_1} + \frac{L_2}{\kappa_2} \right) + \frac{\omega^2}{3} \left( \frac{\rho_1}{\kappa_1^2} L_1^3 + \frac{7 \rho_1}{2 \kappa_1 \kappa_2} - \frac{1 \rho_2}{2 \kappa_2^2} \right) L_2^2 L_2 + \left[ \frac{5 \rho_1}{2 \kappa_2} + \frac{1 \rho_2}{2 \kappa_1 \kappa_2} \right] L_1 L_2^2 + \frac{\rho_2}{\kappa_2^2} L_2^3 \right). \quad (25) \]

The difference between the low-frequency approximations of $\zeta$ and $\zeta_{\text{eff}}$ is

\[ \zeta - \zeta_{\text{eff}} = i Z_0 \frac{\omega^3}{6} \frac{Z_1^2 - Z_2^2}{\kappa_1 \kappa_2} L_1 L_2 \left( \frac{L_1}{\kappa_1} - \frac{L_2}{\kappa_2} \right). \quad (26) \]

This residual may be explicitly made zero in the case that $L_1/\kappa_1 = L_2/\kappa_2$. Thus, the effective material yields the same normalized input impedance as the full case to $O(\omega^3)$. Note that if $\psi_{\text{eff}}$ were neglected then the difference would yield the error

\[ \zeta - \zeta_{\text{eff}}|_{\psi_{\text{eff}}=0} = i Z_0 \frac{\omega^3}{6} \frac{Z_1^2 - Z_2^2}{\kappa_1 \kappa_2} L_1 L_2 \left( -2 \frac{L_1}{\kappa_1} - 4 \frac{L_2}{\kappa_2} \right), \quad (27) \]

which is still $O(\omega^3)$, but is greater error magnitude than the case where $\psi_{\text{eff}}$ is included.

Since $L_1/\kappa_1$ and $L_2/\kappa_2$ are both strictly positive, it becomes apparent that there is no way to reduce the $O(\omega^3)$ error to zero given $Z_1 \neq Z_2$ without accounting for Willis coupling. An
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![Schematic of a uniform circular duct with cross-sectional area $S_0$ of length $L$ with a small section of exponentially growing cross-sectional area of length $d$ embedded in the center.](image)

FIG. 3. Schematic of a uniform circular duct with cross-sectional area $S_0$ of length $L$ with a small section of exponentially growing cross-sectional area of length $d$ embedded in the center.

analysis of the $O(\omega^5)$ error (not presented here) exhibits a similar behavior. Thus, while neglecting Willis coupling in the effective layer provides an accurate reflection coefficient in the quasi-static limit, as the frequency increases Willis coupling becomes more important.

**B. Finite Duct With an Embedded Exponential Horn**

Consider a uniform circular duct with cross-sectional area $S_0 = S_{\text{ref}}$ of length $L$ with a small section of exponentially growing cross-sectional area of length $d$ centered in the duct, as shown in Fig. 3. In this case the cross-sectional area may be written as

$$S(x) = S_0 \begin{cases} e^{m(x+d/2)} & -d/2 < x < d/2 \\ 1 & \text{else} \end{cases},$$

(28)
The mass density and bulk modulus inside the duct are $\rho_0$ and $\kappa_0$. Then, using Eqs. (17), we obtain

$$\kappa_{\text{eff}} = \kappa_0 \left[ 1 - \phi + \frac{\phi}{md} (e^{md} - 1) \right]^{-1},$$  \hspace{1cm} (29a)$$

$$\rho_{\text{eff}} = \rho_0 \left[ 1 - \phi + \frac{\phi}{md} (1 - e^{-md}) \right],$$  \hspace{1cm} (29b)$$

$$\psi_{\text{eff}} = \phi \rho_0 \frac{1 - \sinh(md)}{m 1 - \phi + \frac{\phi}{md} (e^{md} - 1)},$$  \hspace{1cm} (29c)$$

where $\phi = d/L$. For $\phi = 1$, that is for $d = L$ and the entire duct consists of the exponentially varying portion, the effective material properties reduce to the forms

$$\kappa_{\text{eff}} = \kappa_0 \frac{mL}{e^{mL} - 1},$$  \hspace{1cm} (30a)$$

$$\rho_{\text{eff}} = \rho_0 \frac{1 - e^{-mL}}{mL},$$  \hspace{1cm} (30b)$$

$$\psi_{\text{eff}} = \frac{\rho_0}{m} \frac{mL - \sinh(mL)}{e^{mL} - 1}.$$  \hspace{1cm} (30c)$$

**IV. CONCLUSIONS**

This paper has developed and demonstrated a one-dimensional homogenization method based on transmission line theory. Effective material properties, including the mass density, bulk modulus (or other one-dimensional measures of stiffness), and Willis coupling, may be readily evaluated in the long-wavelength limit. The homogenization method has been formulated for both discrete systems and systems that vary smoothly in space. The discrete homogenization method was demonstrated by considering the reflection of a plane acoustic pressure wave from a rigidly-backed bi-layer wall, and the reflection from an effective single-layer wall. The true and effective reflection coefficients were shown to be equal at lowest
order in frequency, and by including Willis coupling the effective reflection coefficient was shown to better approximate the true reflection coefficient at higher frequencies. Finally, the continuous homogenization formulation was demonstrated by considering an exponentially growing horn embedded in an otherwise-uniform duct.

**APPENDIX A: PRODUCT OF NEAR-IDENTITY MATRICES**

Consider two matrices, $A$ and $B$, that are given by

$$A = I + A_1 + A_2, \quad (A1a)$$

$$B = I + B_1 + B_2, \quad (A1b)$$

where $A_1$ and $B_1$ are $O(\varepsilon)$ and $A_2$ and $B_2$ are $O(\varepsilon^2)$ for some $\varepsilon \ll 1$. The product of these two matrices may then be written as

$$AB = I + [A_1 + B_1] + [A_2 + B_2 + A_1B_1] + O(\varepsilon^3). \quad (A2)$$

Multiplying a third matrix with similar form $C = I + C_1 + C_2$ yields

$$ABC = I + [A_1 + B_1 + C_1] + [A_2 + B_2 + C_2 + A_1B_1 + A_1C_1 + B_1C_1] + O(\varepsilon^3). \quad (A3)$$

Inductively, we conclude that for the product

$$\Pi = \prod_{n=1}^{N} A^{(n)} = \prod_{n=1}^{N} \left( I + A_1^{(n)} + A_2^{(n)} \right) \quad (A4)$$

where $A_1^{(n)} = O(\varepsilon)$ and $A_2^{(n)} = O(\varepsilon^2)$ we may write

$$\Pi = I + \Pi_1 + \Pi_2 + O(\varepsilon^3) \quad (A5)$$
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where

\[ \Pi_1 = \sum_{n=1}^{N} A_1^{(n)} = O(\varepsilon), \]  
(A6a)

\[ \Pi_2 = \sum_{n=1}^{N} A_2^{(n)} + \sum_{n=1}^{N-1} \sum_{m=n+1}^{N} A_1^{(n)} A_1^{(m)} = O(\varepsilon^2). \]  
(A6b)

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