Maximally entangled states and Bell’s inequality in relativistic regime

Shahpoor Moradi¹, *

December 21, 2013

¹ Department of Physics, Razi University, Kermanshah, IRAN

Abstract

In this Letter we show that in relativistic regime maximally entangled state of two spin-$\frac{1}{2}$ particles not only gives maximal violation of the Bell-CHSH inequality but also gives the largest violation attainable for any pairs of four spin observables that are noncommuting for both systems. Also we extend our results to three spin-$\frac{1}{2}$ particles. We obtain the largest eigenvalue of Bell operator and show that this value is equal to expectation value of Bell operator on GHZ state.

1 Introduction

Relativistic effects on quantum nonlocality is investigated by many authors [1-6]. M. Czachor [1], investigated Einstein-Podolsky-Rosen experiment with relativistic massive spin-$\frac{1}{2}$ particles. The degree of violation of the Bell’s inequality is shown to depend on the velocity of the pair of particles with respect to the laboratory. He considered the spin singlet of two spin-$\frac{1}{2}$ massive particles moving in the same direction. He introduced the concept of a relativistic spin observable using the relativistic center-of-mass operator. For two observers in the lab frame measuring the spin component of each particle in the same direction, the expectation value of the joint spin measurement, i.e., the expectation value of the tensor product of the relativistic spin observable of each constituent particle, depends on the boost velocity.

Kar [7] has shown that a maximally entangled state of two spin-$\frac{1}{2}$ particles gives a maximum violation of the Bell-CHSH inequality. To prove this, Kar made use of a technique based on the determination of the eigenvectors and eigenvalues of the associated Bell operator. In this Letter we would like to extend these results to the relativistic case.

The paper is organized as follows: In section 2 we obtain the eigenvalue of Bell operator for two qubit system. After that we calculate the expectation value of Bell operator on a maximally entangled state. In section 3 we do the same for three particles case. Finally we conclude with a discussion in section 4.

*e-mail: shahpoor.moradi@gmail.com
### 2 Two qubit system

For two spin $\frac{1}{2}$-particles, the most commonly discussed Bell’s inequality is the CHSH inequality

$$-2 \leq \langle B \rangle \leq 2,$$

where $\langle B \rangle$ denotes the expectation value of the Bell-CHSH operator

$$\mathcal{B} = \vec{a} \cdot \vec{\sigma} \otimes (\vec{b} + \vec{b}’) + \vec{a}’ \cdot \vec{\sigma} \otimes (\vec{b} - \vec{b}’).$$

Here $\vec{a}, \vec{a}’, \vec{b}$ and $\vec{b}’$ are real three-dimensional vectors of unit length and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the Pauli spin operator. For each measurement, one of two possible alternative measurement is performed: $\vec{a}$ or $\vec{a}’$ for particle 1, $\vec{b}$ or $\vec{b}’$ for particle 2. The square of Bell operator is given by

$$\mathcal{B}^2 = 4 \hat{I} \otimes \hat{I} - [\vec{a} \cdot \vec{\sigma}, \vec{a}’ \cdot \vec{\sigma}] \otimes [\vec{b} \cdot \vec{\sigma}, \vec{b}’ \cdot \vec{\sigma}].$$

With the help of the identities

$$[\vec{a} \cdot \vec{\sigma}, \vec{a}’ \cdot \vec{\sigma}] = 2i\vec{\sigma}.(\vec{a} \times \vec{a}’) = 2i\vec{\sigma}.\vec{c},$$

$$[\vec{b} \cdot \vec{\sigma}, \vec{b}’ \cdot \vec{\sigma}] = 2i\vec{\sigma}.(\vec{b} \times \vec{b}’) = 2i\vec{\sigma}.\vec{d},$$

relation (3) reduces to [7]

$$\mathcal{B}^2 = 4 \left[ \hat{I} \otimes \hat{I} + \sin\theta \sin\theta' \sigma_c \otimes \sigma_d \right],$$

where $\theta, \theta’$ is the angle between the vectors $\vec{a}$ and $\vec{a}’$, $\sigma_c$ is the spin observable corresponding to a spin measurement along the unit vector $\vec{c}$ and so on. A straightforward computation shows that $\mathcal{B}^2 \leq 8$. Accordingly, the largest eigenvalue of $\mathcal{B}$ is $2\sqrt{2}$ and the Bell-CHSH inequality can be violated by quantum state by a maximal factor of $\sqrt{2}$.

Now we obtain the relativistic version of (6). The normalized relativistic spin observable $\hat{a}$ is given by [1]

$$\hat{a} = \frac{(\sqrt{1 - \beta^2} a_\perp + a_y)}{1 + \beta^2|\vec{c} \cdot \vec{a}|^2 - 1},$$

where the subscripts $\perp$ and $\parallel$ denote the components which are perpendicular and parallel to the boost direction $\vec{b} = \beta \vec{c}$. Operator $\hat{a}$ is related to the Pauli-Lubanski pseudo vector which is relativistic invariant operator corresponding to spin. Without loss of generality we assume measurements are in $xy$-plane and boost in $x$-direction. In this case square of Bell operator takes the form

$$\mathcal{B}^2 = 4 \left[ \hat{I} \otimes \hat{I} + \frac{(1 - \beta^2) \sin(\phi_a - \phi_{a'}) \sin(\phi_b - \phi_{b'}) \sigma_{1z} \otimes \sigma_{2z}}{\sqrt{(1 + \beta^2(a_{1x}^2 - 1))(1 + \beta^2(a_{2x}^2 - 1))(1 + \beta^2(b_{1x}^2 - 1))(1 + \beta^2(b_{2x}^2 - 1))}} \right],$$

where we labelled the angles from the $x$-axis. The eigenstates are products of eigenstates of $\sigma_{1z}$ and $\sigma_{2z}$, which denotes by $|0\rangle$ and $|1\rangle$. Here 0 and 1 represent spins polarized up and down along the $z$ axis. The largest eigenvalues of $\mathcal{B}^2$ is given by

$$\zeta = 4 \left[ 1 + \frac{(1 - \beta^2)|\sin(\phi_a - \phi_{a'}) \sin(\phi_b - \phi_{b'})|}{\sqrt{(1 + \beta^2(a_{1x}^2 - 1))(1 + \beta^2(a_{2x}^2 - 1))(1 + \beta^2(b_{1x}^2 - 1))(1 + \beta^2(b_{2x}^2 - 1))}} \right].$$
The corresponding degenerate eigenstates are |00⟩ and |11⟩ for \( \sin(\phi_a - \phi_{a'}) \) and \( \sin(\phi_b - \phi_{b'}) \) having the same sign or |01⟩ and |10⟩ for \( \sin(\phi_a - \phi_{a'}) \) and \( \sin(\phi_b - \phi_{b'}) \) of opposite sign. As every eigenvalue for \( \mathcal{B}_H^2 \) must lie in the interval [0, 8] it follows that the eigenvalues for \( \mathcal{B}' \) are necessarily restricted to lie in the interval \([-2\sqrt{2}, 2\sqrt{2}]\). It’s obvious that in ultrarelativistic limit as \( \beta \to 1 \) Bell’s inequality is not violated. For the following set vector
\[
\vec{a} = \frac{1}{\sqrt{2}}(1, -1), \quad \vec{b} = (0, 1), \\
\vec{a}' = \frac{1}{\sqrt{2}}(-1, -1), \quad \vec{b}' = (1, 0),
\] (10)
the square of Bell operator takes the form
\[
\mathcal{B}'^2 = 4 \left[ 1 + \frac{2\sqrt{1 - \beta^2}}{(2 - \beta^2)} \right] I \otimes I,
\] (11)
then the largest eigenvalue of \( \mathcal{B}_H' \) to be
\[
\varepsilon_2 = \zeta^{1/2} = \frac{2}{\sqrt{2 - \beta^2}}(1 + \sqrt{1 - \beta^2}).
\] (12)
In ultrarelativistic limit \( \beta \to 1 \) the amount of violation is 2, which indicates Bell’s inequality is not violated. In non relativistic limit \( \beta \to 0 \) we have the maximum value \( 2\sqrt{2} \) for \( \varepsilon_2 \). Here we obtain identity (12) using the expectation value of Bell operator on a eigenstate. We assume eigenstate is
\[
|\psi⟩ = \frac{1}{\sqrt{2}}(|00⟩ + |11⟩).
\] (13)
A straightforward calculation leads to
\[
⟨\psi|\hat{a} \otimes \hat{b} |\psi⟩ = \frac{a_x b_x + a_y b_y - (1 - \beta^2)a_y b_y}{\sqrt{(1 + \beta^2(a_x^2 - 1))(1 + \beta^2(b_x^2 - 1))}}
\] (14)
For the set vector (10) the expectation value of relativistic Bell observable is exactly (12). The result (12) is obtained by Ahn, et al [3]. They calculate the Bell observables for entangled states in the rest frame seen by the observer moving in the \( x \) direction and show that the entangled states satisfy the Bells inequality when the boost speed approaches the speed of light. The calculated average of the Bell observable for the Lorentz transformed entangled states is (12).

3 Three qubit system

Here we consider to three particle case. For three spin-\( \frac{1}{2} \) particles the Bell operator is
\[
\mathcal{B}_3 = \hat{a} \otimes \hat{b} \otimes \hat{c} + \hat{a}' \otimes \hat{b} \otimes \hat{c}' + \hat{a}' \otimes \hat{b}' \otimes \hat{c} - \hat{a} \otimes \hat{b} \otimes \hat{c},
\] (15)
where \( \hat{a}, \hat{a}' \) denote spin observable on the first qubit, \( \hat{b}, \hat{b}' \) on the second, and \( \hat{c}, \hat{c}' \) on the third. Bell’s inequality for three qubits is given by inequality (1). The square of Bell operator (13) is given by
\[
\mathcal{B}_3^2 = 4I - [\hat{a}, \hat{a}'][\hat{b}, \hat{b}'] - [\hat{a}, \hat{a}'][\hat{c}, \hat{c}'] - [\hat{b}, \hat{b}'][\hat{c}, \hat{c}'].
\] (16)
We assume that three particles move with the same momentums in x-direction. After some algebra we arrive at

\[ B_3^2 = 4 \left[ \hat{I} \otimes \hat{I} + \frac{(1 - \beta^2) \sin(\phi_a - \phi_{a'}) \sin(\phi_b - \phi_{b'}) \sigma_{1z} \otimes \sigma_{2z}}{\sqrt{(1 + \beta^2(a_x^2 - 1))(1 + \beta^2(a_x'^2 - 1))(1 + \beta^2(b_x^2 - 1))(1 + \beta^2(b_x'^2 - 1))}} \right. \\
\left. + \frac{(1 - \beta^2) \sin(\phi_a - \phi_{a'}) \sin(\phi_c - \phi_{c'}) \sigma_{2z} \otimes \sigma_{3z}}{\sqrt{(1 + \beta^2(a_x^2 - 1))(1 + \beta^2(a_x'^2 - 1))(1 + \beta^2(c_x^2 - 1))(1 + \beta^2(c_x'^2 - 1))}} \right. \\
\left. + \frac{(1 - \beta^2) \sin(\phi_b - \phi_{b'}) \sin(\phi_c - \phi_{c'}) \sigma_{1z} \otimes \sigma_{3z}}{\sqrt{(1 + \beta^2(b_x^2 - 1))(1 + \beta^2(b_x'^2 - 1))(1 + \beta^2(c_x^2 - 1))(1 + \beta^2(c_x'^2 - 1))}} \right], \]

we can see that the largest eigenvalue for \( B_3^2 \) is

\[ \lambda_3 = 4 \left[ 1 + \frac{(1 - \beta^2) \sin(\phi_a - \phi_{a'}) \sin(\phi_b - \phi_{b'})}{\sqrt{(1 + \beta^2(a_x^2 - 1))(1 + \beta^2(a_x'^2 - 1))(1 + \beta^2(b_x^2 - 1))(1 + \beta^2(b_x'^2 - 1))}} \right. \\
\left. + \frac{(1 - \beta^2) \sin(\phi_a - \phi_{a'}) \sin(\phi_c - \phi_{c'})}{\sqrt{(1 + \beta^2(a_x^2 - 1))(1 + \beta^2(a_x'^2 - 1))(1 + \beta^2(c_x^2 - 1))(1 + \beta^2(c_x'^2 - 1))}} \right. \\
\left. + \frac{(1 - \beta^2) \sin(\phi_b - \phi_{b'}) \sin(\phi_c - \phi_{c'})}{\sqrt{(1 + \beta^2(b_x^2 - 1))(1 + \beta^2(b_x'^2 - 1))(1 + \beta^2(c_x^2 - 1))(1 + \beta^2(c_x'^2 - 1))}} \right], \]

which attains maximum value 16 with the following suitably chosen measurement settings,

\[ \hat{a} = \hat{b} = \hat{c} = \hat{y}, \]
\[ \hat{a}' = \hat{b}' = \hat{c}' = \hat{x}, \] (17)

On the other hand it can be easily seen that the minimum possible eigenvalue for \( B_3^2 \) is zero. For example \(|001\rangle\) is an eigenvector of \( B_3^2 \) with zero eigenvalue whenever \( \phi_a - \phi_{a'} = \phi_b - \phi_{b'} = \phi_c - \phi_{c'} = \pi/2 \). It is interesting that for three qubits the largest eigenvalue of \( B_3 \) is not depends on boost velocity which is not same as two qubit case. Investigations show that exist a family of pure entangled \( N > 2 \) qubit states that do not violate any Bell’s inequality for N-particle correlations for the case of a standard Bell experiment on N qubits [9]. For \( N = 3 \), one class is Greenberger-Horne-Zeilinger (GHZ) state given by \(|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)\). In three qubits case three experimentalists, Alice, Bob, and Charlotte, can measure the spin component in arbitrary direction. In nonrelativistic domain for a GHZ state, Bell’s inequality is maximally violated if, for example, measurements are made in the xy-plane along some appropriate directions. For example with set vectors (17) and using the algebra of Pauli matrices it is easily verifiable for GHZ state Bell’s inequality is maximally violated with value 4. In relativistic regime again we use the relativistic spin operator (7) and we assume that three particles move with the same momentums in x-direction, then the expectation values of \( \hat{a} \otimes \hat{b} \otimes \hat{c} \) on three qubit sates are

\[ \langle 000|\hat{a} \otimes \hat{b} \otimes \hat{c}|000\rangle = -\langle 111|\hat{a} \otimes \hat{b} \otimes \hat{c}|111\rangle = \]
\[
\left(1 - \beta^2\right)^{3/2}a_x b_x c_x \sqrt{\left[1 + \beta^2(a_x^2 - 1)\right]\left[1 + \beta^2(b_x^2 - 1)\right]\left[1 + \beta^2(c_x^2 - 1)\right]} \tag{18}
\]

\[
(111|\hat{a} \otimes \hat{b} \otimes \hat{c}|000) = \langle 000|\hat{a} \otimes \hat{b} \otimes \hat{c}|111\rangle^* = \frac{(a_x + i\sqrt{1 - \beta^2}a_y)(b_x + i\sqrt{1 - \beta^2}b_y)(c_x + i\sqrt{1 - \beta^2}c_y)}{\sqrt{\left[1 + \beta^2(a_x^2 - 1)\right]\left[1 + \beta^2(b_x^2 - 1)\right]\left[1 + \beta^2(c_x^2 - 1)\right]}} \tag{19}
\]

Then the expectation value on GHZ state is

\[
\langle GHZ|\hat{a} \otimes \hat{b} \otimes \hat{c}|GHZ\rangle = \frac{a_x b_x c_x - (1 - \beta^2)(a_y b_x c_y + a_y b_y c_x + a_x b_y c_y)}{\sqrt{\left[1 + \beta^2(a_x^2 - 1)\right]\left[1 + \beta^2(b_x^2 - 1)\right]\left[1 + \beta^2(c_x^2 - 1)\right]}} \tag{20}
\]

So the expectation value of Bell observable (15) on GHZ state is 4, then Bell’s inequality is maximally violated like nonrelativistic case.

Here we assume particles are emitted in a plane in a configuration in which the three momenta lie at angles of $2\pi/3$ to each other. In this situation particles are in the center of mass frame with the following unit vector boosts

\[
e_1 = -\hat{x}, \tag{21}
\]

\[
e_2 = \frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}, \tag{22}
\]

\[
e_3 = \frac{1}{2}\hat{x} - \frac{\sqrt{3}}{2}\hat{y}, \tag{23}
\]

Then the largest eigenvalue of $B_3$ to be

\[
\lambda = 4 \left(1 + \frac{8\sqrt{1 - \beta^2}}{\sqrt{(4 - \beta^2)(4 - 3\beta^2)}} + \frac{16(1 - \beta^2)}{(4 - \beta^2)(4 - 3\beta^2)}\right). \tag{24}
\]

therefor we have

\[
\varepsilon_3 = \lambda^{1/2} = 2 \left(1 + \frac{4\sqrt{1 - \beta^2}}{\sqrt{(4 - \beta^2)(4 - 3\beta^2)}}\right). \tag{25}
\]

In ultrarelativistic limit as $\beta \to 1$ approaches to 2 and in non relativistic limit $\beta \to 0$ to be 4. which is the maximum value of Bell operator $B_3$. Then similar to two qubit case only in ultrarelativistic limit the Bell’s inequality is satisfied. For set vectors (17) the relativistic spin operators take the forms

\[
\hat{a} = \hat{y}, \quad \hat{a}' = \hat{x}, \tag{26}
\]

\[
\hat{b} = \frac{(3 + \sqrt{1 - \beta^2})\hat{y} + \sqrt{3}(1 - \sqrt{1 - \beta^2})\hat{x}}{2\sqrt{4 - \beta^2}}, \tag{27}
\]

\[
\hat{c} = \frac{(3 + \sqrt{1 - \beta^2})\hat{y} - \sqrt{3}(1 - \sqrt{1 - \beta^2})\hat{x}}{2\sqrt{4 - \beta^2}}, \tag{28}
\]
\[ \dot{b} = \frac{\sqrt{3}(1 - \sqrt{1 - \beta^2})\dot{y} + (3 + \sqrt{1 - \beta^2})\dot{x}}{2\sqrt{4 - 3\beta^2}}, \]  
\[ \dot{c} = \frac{-\sqrt{3}(1 - \sqrt{1 - \beta^2})\dot{y} + (3 + \sqrt{1 - \beta^2})\dot{x}}{2\sqrt{4 - 3\beta^2}}, \]

One can easily show that for the above spin operators the expectation value of Bell operator on GHZ state is same as (25).

4 Conclusions

We show that the maximally entangled state gives the largest possible violation for Bell-CHSH inequality in relativistic formalism also largest eigenvalue of relativistic Bell operator. Furthermore, we have shown that the maximal violation of Bell’s inequality predicted by quantum mechanics decreases in relativistic case. Bell’s inequality in relativistic case is not always violated, because the degree of violation of Bell’s inequality depends on the velocity of the particles. In non relativistic case the spin degrees of freedom and momentum degrees of freedom are independent. But in relativistic regime Lorentz transformation of spin of particle depends on its momentum.

There are some differences between two and three qubit systems. In two particle systems when particles move with the same speed, using the set vector which in non relativistic case yields the maximum value of Bell operator, we see the relativistic Bell operator depends on speed of particles. For three qubit case and the same conditions like two qubit case the amount of violation is independent of speed. On the other hand when three particles move in the center of mass frame amount of violation depends on speed of particles.

References

[1] M. Czachor Phys. Rev. A 55, 72 (1997)
[2] H. Terashima and M. Ueda Quant. Inf. Comput. 3 224 (2003); H. Terashima and M. Ueda Int. J. Quant. Inf. 1, 93 (2003)
[3] D. Ahn, H-J Lee, Y. H. Moon and S. W. Hwang Phys. Rev. A 67, 012103 (2003); D. Ahn, H-J Lee, S.W. Hwang and M. S. Kim Preprint quant-ph/0304119
[4] W. T. Kim and E. J. Son Phys. Rev. A 71, 014102 (2005)
[5] D. Lee and E. Chang-Young New J. Phys 6, 67 (2004)
[6] S. Moradi Phys. Rev. A 77, 024101 (2008)
[7] G. Kar, Phys. Lett. A 204, 99 (1995)
[8] L.J. Landau, Phys. Lett. A 120 4 (1987)
[9] V. Scarani and N. Gisin, J. Phys. A 34, 6043 (2001).