On the reachability and observability of path and cycle graphs

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Abstract

In this paper we investigate the reachability and observability properties of a network system, running a Laplacian based average consensus algorithm, when the communication graph is a path or a cycle. More in detail, we provide necessary and sufficient conditions, based on simple algebraic rules from number theory, to characterize all and only the nodes from which the network system is reachable (respectively observable). Interesting immediate corollaries of our results are: (i) a path graph is reachable (observable) from any single node if and only if the number of nodes of the graph is a power of two, \( n = 2^i, i \in \mathbb{N} \), and (ii) a cycle is reachable (observable) from any pair of nodes if and only if \( n \) is a prime number. For any set of control (observation) nodes, we provide a closed form expression for the (unreachable) unobservable eigenvalues and for the eigenvectors of the (unreachable) unobservable subsystem.

I. INTRODUCTION

Distributed computation in network control systems has received great attention in the last years. One of the most studied problems is the consensus problem. Given a network of processors, the task of reaching consensus consists of computing a common desired value by performing local computation and exchanging local information. A variety of distributed algorithms for diverse system dynamics and consensus objectives has been proposed in the literature.

An early short version of this work appeared as [1]: differences between this early short version and the current article include the reachability analysis, a much improved comprehensive and thorough treatment, revised complete proofs for all statements.

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We are interested in two problems that may arise in a network running a consensus algorithm when only a subset of nodes is controlled by an external input or measured by an external processor. Namely, is it possible to reach all the node configurations just controlling a limited number of nodes? Respectively, is it possible to reconstruct the entire network state just knowing the state of a limited number of nodes?

In this paper we will concentrate on a first-order network system running a Laplacian based average consensus algorithm with fixed communication graph topology of path or cycle type. Average consensus has been widely studied in the last years. Several distributed feedback laws have been proposed. A survey on these algorithms and their performance may be found e.g. in [2] and references therein. The dynamical system arising from a consensus network with fixed topology is a linear time-invariant system. The problem of understanding if all the node configurations can be reached by controlling a given subset of nodes is a reachability problem. Respectively, the problem of understanding if the entire network state may be reconstructed is an observability problem.

We organize the relevant literature in two areas. First, the reachability problem in a first-order network arises in network systems where all (or most of) the nodes run a linear average consensus algorithm, while a subset of them can be driven by an exogenous input. These networks are often called leader-follower networks in the sense that the nodes controlled by an external input are leaders that drive the followers to desired configurations. The reachability (controllability) problem for a leader-follower network was introduced in [3] for a single control node. Intensive simulations were provided showing that it is “unlikely” for a Laplacian based consensus network to be completely controllable. In [4] and [5], necessary conditions for controllability, based on suitable properties of the graph, have been provided. The conditions rely on algebraic graph tools based on the notion of equitable partitions of a graph. A more exhaustive analysis of the controllability and other structural system properties for network systems on the basis of graph structural properties can be found in [6]. In [7] preliminary results are given for the controllability of first-order network with switching communication topology applied to the formation control problem. In [8] sufficient conditions for controllability in first and second order multi-agent systems with delay are given. The conditions are based on the eigenstructure of the network system and delay matrices. In [9] necessary conditions and sufficient conditions for the controllability of tree graphs are given, based on the eigenvalues of suitable subgraphs. A first
contribution to the controllability of multi-agent systems with nonlinear dynamics is provided in [10], where the controllability of pairs of identical nonholonomic vehicles maintaining a constant distance is studied.

Second, observability for a network system running an average consensus algorithm has been studied for the first time in [11]. In that paper the authors provide necessary conditions for observability. The conditions are based on equitable partitions of a graph as in the reachability setting investigated in [4] and [5]. A recent reference on observability for network dynamic systems is [12]. Here, the linear dynamical systems of the network are decoupled and the coupling among the systems appears through the output. A parallel research line investigates a slightly different property called structural observability [13]. Here, the objective is to choose the nonzero entries of the consensus matrix (i.e. the state matrix of the resulting network system) in order to obtain observability from a given set of nodes. However, in many contexts the structure of the system matrix is given (e.g. the Laplacian for average consensus).

It is worth noting that, the observability property is an important property in distributed estimation, [14], [15], and intrusion detection problems [16], [17] for steady state analysis. In the literature this property is often assumed or considered as non generic. Finally, preliminary results on the controllability and observability of path and cycle graphs were given in [18], where, using these properties, a formation control strategy was proposed.

The contribution of the paper is twofold. First, we provide necessary and sufficient conditions based on simple algebraic relations from number theory that completely characterize the reachability (observability) of path and cycle graphs. More in detail, on the basis of the node labels and the total number of nodes in the graph we are able to: (i) identify all and only the reachable (observable) nodes of the graph, (ii) say if the graph is reachable (observable) from a given set of nodes and (iii) construct a set of control (observation) nodes from which the graph is reachable (observable). Interesting immediate corollaries of our results are: (i) a path graph is reachable (observable) from any single node if and only if the number of nodes of the graph is a power of two, \( n = 2^i, i \in \mathbb{N} \), (ii) a path graph is reachable (observable) from a single node \( i \) if and only if there is no odd prime factor \( p \) of \( n \) such that \( n - i = (i - 1) + \alpha p \) for some integer \( \alpha \); (iii) a cycle graph is reachable (observable) from any pair of nodes if and only if \( n \) is a prime number, and (iv) a cycle graph is reachable (observable) from two nodes, say \( i_1 \) and \( i_2 \), if and only if \( i_2 - i_1 \) and \( n + i_1 - i_2 \) are coprime. Thus, e.g., any cycle is observable from...
two adjacent nodes.

Second, we provide a closed form expression for the unreachable (unobservable) eigenvalues and eigenvectors, and characterize the orthogonal complement to the reachable subspace (respectively the unobservable subspace) for any unreachable (unobservable) set of nodes. This result is based on the complete characterization of the spectrum of suitable submatrices of the path and cycle Laplacians. As a consequence of these linear algebra results, we also provide a closed form for all the Laplacian eigenvalues of a path graph. At the best of our knowledge both the characterization of the Laplacian eigenvalues and the mathematical tools used to characterize them are new.

The paper is organized as follows. In Section II we introduce some preliminary definitions and properties of undirected graphs, describe the network model used in the paper and set up the reachability and observability problems. In Section III we provide a complete characterization of the eigenvalues and eigenvectors of the Laplacian of a path graph and other matrices used to study the path and cycle reachability (observability). Finally, in Section IV we provide a complete characterization of the path and cycle graph reachability (observability), and provide some useful example explaining the main results.

Notation: We let \( \mathbb{N} \), \( \mathbb{N}_0 \), and \( \mathbb{R}_{\geq 0} \) denote the natural numbers, the non-negative integer numbers, and the non-negative real numbers, respectively. We denote \( 0_d, d \in \mathbb{N} \), the vector of dimension \( d \) with zero components and \( 0_{d_1 \times d_2}, d_1, d_2 \in \mathbb{N} \), the matrix with \( d_1 \) rows and \( d_2 \) columns with zero entries. For \( i \in \mathbb{N} \) we let \( e_i \) be the \( i \)-th element of the canonical basis, e.g. \( e_1 = [1 \ 0 \ \ldots \ 0]^T \). For a matrix \( A \in \mathbb{R}^{d_1 \times d_2} \) we denote \( [A]_{ij} \) the \((i,j)\)th element and \( [A]_i \) the \( i \)th column of \( A \). For a vector \( v \in \mathbb{R}^d \) we denote \( (v)_i \) the \( i \)th component of \( v \) so that \( v = [(v)_1 \ldots (v)_d]^T \). Also, we denote \( \Pi \in \mathbb{R}^{d \times d} \) the permutation matrix reversing all the components of \( v \) so that \( \Pi v = [(v)_d \ldots (v)_1]^T \) (the \( j \)-th column of \( \Pi \) is \( [\Pi]_j = e_{n-j+1} \)). Adapting the usual terminology of number theory, we will say that \( k \) divides a nonzero integer \( m \) (written \( k \mid m \)) if there is an integer \( q \) with the property that \( m = kq \). When this relation holds, \( k \) is said a factor or divisor of \( m \). If two integers \( b \) and \( c \) satisfy for a given \( m \) the relation \( m \mid (b - c) \) then we say that \( b \) is congruent to \( c \) modulo \( m \) (written \( b \equiv c \mod(m) \) or equivalently \( b \equiv c \mod(m) \)). The greatest common divisor of two positive integers \( a \) and \( b \) is the largest divisor common to \( a \) and \( b \), and we will denote it \( GCD(a, b) \). The greatest common divisor can also be defined for three or more positive integers as the largest divisor shared by all of them. Two or more
positive integers that have greatest common divisor 1 are said relatively prime or coprime. A prime number is a positive integer that has no positive integer divisors other than 1 and itself. Every natural number \( n \) admits a prime factorization (Fundamental Theorem of Arithmetic), i.e. we can factorize \( n \) as \( n = 2^{n_0} \prod_{\alpha=1}^{l} p_\alpha \), where \( n_0 \in \mathbb{N} \) and each \( p_\alpha \) is an odd prime number. Notice that in our factorization we allow two or more factors \( p_\alpha \) to be equal.

II. PRELIMINARIES AND PROBLEM SET-UP

In this section we present some preliminary terminology on graph theory, introduce the network model, set up the reachability and observability problems, and provide some standard results on reachability (observability) of linear systems that will be useful to prove the main results of the paper.

A. Preliminaries on graph theory

Let \( G = (I, E) \) be a static undirected graph with set of nodes \( I = \{1, \ldots, n\} \) and set of edges \( E \subset I \times I \). We denote \( \mathcal{N}_i \) the set of neighbors of agent \( i \), that is, \( \mathcal{N}_i = \{ j \in I \mid (i,j) \in E \} \), and \( d_i = \sum_{j \in \mathcal{N}_i} 1 \) the degree of node \( i \). The maximum degree of the graph is defined as \( \Delta = \max_{i \in I} d_i \). The degree matrix \( D \) of the graph \( G \) is the diagonal matrix defined as \([D]_{ii} = d_i\). The adjacency matrix \( A \in \mathbb{R}^{n \times n} \) associated to the graph \( G \) is defined as

\[
[A]_{ij} = \begin{cases} 
1 & \text{if } (i,j) \in E \\
0 & \text{otherwise.}
\end{cases}
\]

The Laplacian \( L \) of \( G \) is defined as \( L = D - A \). The Laplacian is a symmetric positive semidefinite matrix with \( k \) eigenvalues in 0, where \( k \) is the number of connected components of \( G \). If the graph is connected the eigenvector associated to the eigenvalue 0 is the vector \( 1 = [1 \ldots 1]^T \).

We introduce the two undirected graphs that will be of interest in the rest of the paper, namely the path and cycle graphs. A path graph is a graph in which there are only nodes of degree two except for two nodes of degree one. The nodes of degree one are called external nodes, while the other are called internal nodes. From now on, without loss of generality, we will label the external nodes with 1 and \( n \), and the internal nodes so that the edge set is \( E = \{(i, i+1) \mid i \in \{1, \ldots, n-1\}\} \). A cycle graph is a graph in which all the nodes have degree two. From now on, without loss of generality, we will label the nodes so that the edge set is \( E = \{(i, i \mod(n) + 1) \mid i \in \{1, \ldots, n\}\} \).
B. Network of agents running average consensus

We consider a collection of agents labeled by a set of identifiers \( I = \{1, \ldots, n\} \), where \( n \in \mathbb{N} \) is the number of agents. We assume that the agents communicate according to a time-invariant undirected communication graph \( G = (I, E) \), where \( E = \{(i, j) \in I \times I \mid i \text{ and } j \text{ communicate}\} \). That is, we assume that the communication between any two agents is bi-directional. The agents run a consensus algorithm based on a Laplacian control law (see e.g. [2] for a survey). The dynamics of the agents evolve in continuous time \( (t \in \mathbb{R}_\geq 0) \) and are given by

\[
\dot{x}_i(t) = -\sum_{j \in N_i} (x_i(t) - x_j(t)), \quad i \in \{1, \ldots, n\}.
\]

Using a compact notation the dynamics may be written as

\[
\dot{x}(t) = -Lx(t), \quad t \in \mathbb{R}_\geq 0,
\]

where \( x = [ (x)_1 \ldots (x)_n]^T = [x_1 \ldots x_n]^T \) is the vector of the agents’ states and \( L \) is the graph Laplacian.

**Remark 2.1 (Discrete time system):** In discrete time, we can consider the following dynamics

\[
x_i(t + 1) = x_i(t) - \epsilon \sum_{j \in N_i} (x_i(t) - x_j(t)), \quad i \in \{1, \ldots, n\},
\]

where \( \epsilon \in \mathbb{R} \) is a given parameter. A compact expression for the dynamics is

\[
x(t + 1) = (I - \epsilon L)x(t), \quad t \in \mathbb{N}_0.
\]

For \( \epsilon \in (0, 1/\Delta) \) (\( \Delta \) is the maximum degree of the graph), \( P=(I-\epsilon L) \) is a nonnegative, doubly stochastic, stable matrix.

It can be easily shown that the continuous and discrete time systems have the same reachability and observability properties (namely the same unreachable and unobservable eigenvalues and eigenvectors). Therefore, the results shown in the paper also hold in this discrete time set-up. □

C. Network reachability and observability

In this section we describe the mathematical framework that we will use to study the reachability and observability of a network system. We start by describing the scenarios that motivate our work. As regards the reachability problem, we imagine that in a network of agents running average consensus as in Section II-B a subset of nodes can be controlled by an external input.
In the literature these nodes are often called \textit{leader nodes} or \textit{pinned nodes}. The idea is that they are special nodes with higher computation capabilities so that more sophisticated control laws can be designed. Let $I_\ell = \{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$ be the set of control nodes, the dynamical system modeling this scenario is

$$\dot{x}(t) = -Lx(t) + Bu(t),$$

where $u(t) = [u_{i_1}(t), \ldots, u_{i_m}(t)]^T$ is the input and $B = [e_{i_1} \mid \ldots \mid e_{i_m}]$.

As regards the observability problem, we imagine that an external processor (not running the consensus algorithm) collects information from some nodes in the network. We call these nodes \textit{observation nodes}. In particular, we assume that the external processor may read the state of each observation node. Equivalently, we can think of one or more observation nodes, running the consensus algorithm, that have to reconstruct the state of the network by processing only their own state. We can model these two scenarios with the following mathematical framework.

For each observation node $i \in I$, we have the following output

$$y_i(t) = x_i(t).$$

Given the set of observation nodes $I_o = \{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$, the output is $y(t) = [x_{i_1}(t) \ x_{i_2}(t) \ \ldots \ x_{i_m}(t)]^T$. Therefore, the system dynamics is given by

$$\dot{x}(t) = -Lx(t),$$

$$y(t) = Cx(t),$$

where the output matrix is $C = [e_{i_1} \mid \ldots \mid e_{i_m}]^T$.

It is a well known result in linear systems theory that the reachability properties of the pair $(L, B)$ are the same as the observability properties of the pair $(L, C) = (L^T, B^T) = (L, B^T)$.

\textbf{Remark 2.2 (Duality and regulator design):} Due to the symmetry of the state matrix $L$, the network is reachable from a given subset of nodes if and only if it is observable from it. This important property allows, for example, to design a regulator at a subset of (control/observation) nodes that estimates the entire network state and controls it to a desired configuration. This has, for example, certainly an impact on security issues.

\textbf{Remark 2.3 (Equivalence with other problem set-ups):} Straightforward results from linear system theory can be used to prove that the controllability problem studied, e.g., in [4] and [5] and the dual observability problem studied in [11] can be equivalently formulated in our set up.
D. Standard results on reachability and observability of linear systems

The reachability problem consists of looking for those states that can be reached in finite time from the origin. Respectively, the observability problem consists of looking for nonzero values of \( x(0) \) that produce an identically zero output \( y(t) \). Using known results in linear system theory the two problems are equivalent to studying the rank of the reachability matrix,

\[
R_n = \begin{bmatrix} B & L \cdot B & \cdots & L^{n-1} \cdot B \end{bmatrix},
\]

respectively of the observability matrix

\[
O_n = \begin{bmatrix} C \\ C \cdot L \\ \vdots \\ C \cdot L^{n-1} \end{bmatrix}.
\]

The image \( X_r \) of \( R_n \) is the reachable subspace, i.e. the set of states that are reachable from the origin. The kernel \( X_{no} \) of \( O_n \) is the unobservable subspace, i.e. the set of initial states that produce an identically zero output.

Here, we recall an interesting result on the reachability (observability) of time-invariant linear systems known as Popov-Belevich-Hautus (PBH) lemma.

**Lemma 2.4 (PBH lemma):** Let \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m} \) and \( C \in \mathbb{R}^{m \times n}, n, m \in \mathbb{N} \), be the state, input and output matrices of a linear time-invariant system. The pair \((A, B)\) is reachable if and only if

\[
\text{rank} \begin{bmatrix} A - \lambda I & B \end{bmatrix} = n,
\]

for all \( \lambda \in \mathbb{C} \). Respectively the pair \((A, C)\) is observable, if and only if

\[
\text{rank} \begin{bmatrix} C \\ A - \lambda I \end{bmatrix} = n,
\]

for all \( \lambda \in \mathbb{C} \). \( \square \)

Combining the PBH lemma with the fact that the state matrix is symmetric (and therefore diagonalizable) the following corollary may be proven.

**Corollary 2.5 (PBH lemma for symmetric matrices):** Let \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m} \) and \( C \in \mathbb{R}^{m \times n}, n, m \in \mathbb{N} \), be the state, input and output matrices of a linear time-invariant system, where
A is symmetric. Then, the orthogonal complement to the reachable subspace $X_r$, associated to the pair $(A, B)$, is spanned by vectors $v_l$ satisfying

$$
B^T v_l = 0_m
$$

$$
L v_l = \lambda v_l.
$$

Respectively, the unobservable subspace $X_{no}$ associated to the pair $(L, C)$ is spanned by vectors $v_l$ satisfying

$$
C v_l = 0_m
$$

$$
L v_l = \lambda v_l,
$$
for $\lambda \in \mathbb{R}$. □

In the rest of the paper we will denote the eigenvalues and eigenvectors for which (1) is satisfied, unreachable eigenvalues and eigenvectors. Respectively, we will denote unobservable eigenvalues and eigenvectors the ones for which (2) is satisfied.

### III. Spectral Properties of the Laplacian of a Path and Related Submatrices

In this section we provide a closed form expression for the eigenvalues and eigenvectors of suitable submatrices of the Laplacian of path and cycle graphs. This characterization will play a key role in the characterization of the reachability (observability) properties of path and cycle graphs. As a self-contained result of this section, we provide a closed form expression for the Laplacian spectrum of a path graph.

We start motivating the analysis in this section. Recall that, without loss of generality, we assume that nodes of the path are labeled so that the undirected edges are $(i, i + 1)$ for $i \in \{1, \ldots, n - 1\}$. Let $L_n$ denote the Laplacian of a path graph of length $n$ and $B$ ($C$) the input (output) matrix associated to the set of control (observation) nodes $I_o = \{i_1, \ldots, i_m\}$. Using the PBH Lemma in the version of Corollary 2.5, unreachability (unobservability) for the path graph from $I_o$ is equivalent to the existence of a nonzero solution of the linear (algebraic) system $L_n v = \lambda v$, where $v$ satisfies $B^T v = 0$ ($C v = 0$), i.e. $(v)_j = 0$ for each $j \in I_o$. Exploiting the structure of the linear system we can write
where the vertical line on the $i_1$-th column means that it is multiplied by $(v)_{i_1} = 0$. The same holds for each $j$-th column, $j \in I_o$. Now, define the matrices $N_\nu \in \mathbb{R}^{\nu \times \nu}$ and $M_\mu \in \mathbb{R}^{\mu \times \mu}$ as

$$N_\nu = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & \ddots & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad \text{and} \quad M_\mu = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & \ddots & -1 \\ 0 & -1 & 2 \end{bmatrix},$$

where the subindex refers to their dimensions. The Laplacian $L_n$ can be compactly written as

$$L_n = \begin{bmatrix} N_{i_1-1} & \ldots & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & M_{i_2-i_1-1} & 0 \\ 0 & 0 & \ldots & 0 & 2 \\ 0 & 0 & \ldots & 0 & \Pi N_{n-i_m} \Pi \end{bmatrix},$$

where $\Pi = \Pi^T = \Pi^{-1}$ is the (symmetric) permutation matrix reversing all the components of a vector.
Remark 3.1 (Partition of the Laplacian of a cycle): Applying the same procedure to the Laplacian of a cycle, under the agreement of labeling the nodes so that \( i_1 = 1 \), we get a partition of the Laplacian where the submatrices are all matrices of type \( M_\mu, \mu \in \mathbb{N} \).

We are now ready to investigate the spectral properties of these matrices. We begin with a useful lemma.

Lemma 3.2: The eigenvectors of \( N_\mu \), \( M_\mu \) and \( L_\mu \), \( \mu \in \mathbb{N} \), have nonzero first and last components.

Proof: We prove the statement by proving that if an eigenvector has zero first component, then all the other components have to be zero. The same line of proof can be followed to prove that if the last component is zero, then the same must hold for the previous \( \mu - 1 \).

Let \( \lambda \) be an eigenvalue of \( N_\mu \) (respectively \( M_\mu \) and \( L_\mu \)) and \( v = [v_1 \ldots v_\mu] \) the corresponding eigenvector. The following conditions must hold

\[
\begin{align*}
   av_1 - v_2 &= \lambda v_1 \\
   -v_{i-1} + bv_i - v_{i+1} &= \lambda v_i, \quad i \in \{2, \ldots, \mu - 1\} \\
   -v_{\mu-1} + bv_\mu &= \lambda v_\mu,
\end{align*}
\]

for suitable (positive) \( a \) and \( b \). We proceed by induction. We use the inductive assumption that \( v_{j-1} = v_j = 0 \) and prove that \( v_{j+1} = 0 \). The statement is obviously true for \( j = 1 \). Indeed, from the first equation it follows easily that if \( v_1 = 0 \) then \( v_2 = 0 \) (we have considered a fake \( v_0 = 0 \) in the first equation). Then, plugging the inductive assumption in the second equation, we get \( v_{j+1} = 0 \) for \( j \in \{2, \ldots, \mu - 1\} \), while the last equation gives the result for \( j = \mu \).

Remark 3.3 (A path is reachable (observable) from an external node): Combining the previous lemma with Corollary [2,5] it follows easily that a path graph is reachable (observable) from each of the external nodes as shown, e.g., in [4] ([19]).

Proposition 3.4 (Eigenstructure of \( N_\nu \) and \( M_\mu \)): For any two matrices \( N_\nu \in \mathbb{R}^{\nu \times \nu} \) and \( M_\mu \in \mathbb{R}^{\mu \times \mu} \) the following holds:

(i) All the eigenvalues of \( N_\nu \) are eigenvalues of \( M_{2\nu} \) and the corresponding eigenvectors, respectively \( v \in \mathbb{R}^\nu \) and \( w \in \mathbb{R}^{2\nu} \), are related by \( w = \begin{bmatrix} \Pi v \\ v \end{bmatrix} \).
(ii) Eigenvalues and eigenvectors of $N_\nu$ and $M_\mu$ have the following closed form expression:

\[
\begin{align*}
\lambda_{N_\nu} &= 2 - 2 \cos \left[ (2k - 1) \frac{\pi}{2\nu + 1} \right], \\
(v_k)_j &= \sin \left[ \frac{(\nu + j)(2k - 1)\pi}{2\nu + 1} \right], \quad j = 1, \ldots, \nu, \quad k = 1, \ldots, \nu \\
\lambda_{M_\mu} &= 2 - 2 \cos \left( k\frac{\pi}{\mu + 1} \right), \\
(w_k)_j &= \sin \left( \frac{jk\pi}{\mu + 1} \right), \quad j = 1, \ldots, \mu, \quad k = 1, \ldots, \mu.
\end{align*}
\]

(3) (4)

**Proof:** To prove the first statement, consider a vector $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \Pi v$, with $v \in \mathbb{R}^\nu$. We show that $w$ is an eigenvector of $M_{2\nu}$ with eigenvalue $\lambda_N$ if and only if $v$ is an eigenvector of $N_\nu$ with eigenvalue $\lambda_N$. Indeed, by exploiting the structure of $M_{2\nu}$ we get

\[
M_{2\nu} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \Pi N_\nu \Pi + e_\nu e_\nu^T & -e_\nu e_1^T \\ -e_1 e_\nu^T & N_\nu + e_1 e_1^T \end{bmatrix} \begin{bmatrix} \Pi v \\ v \end{bmatrix}
\]

\[
= \begin{bmatrix} \Pi N_\nu \Pi v + e_\nu \left( e_\nu^T \Pi v - e_1^T v \right) \\ -e_1 \left( e_\nu^T \Pi v \right) + N_\nu v + e_1 (v)_1 \end{bmatrix}
\]

\[
= \begin{bmatrix} \Pi N_\nu v + e_\nu [(v)_1 - (v)_1] \\ N_\nu v + e_1 [(v)_1 - (v)_1] \end{bmatrix} = \begin{bmatrix} \Pi N_\nu v \\ N_\nu v \end{bmatrix}
\]

Now, notice that, if $\lambda_N$ is an eigenvalue of $N_\nu$, i.e. $N_\nu v = \lambda_N v$ for some nonzero $v \in \mathbb{R}^\nu$, then $w$ is an eigenvector of $M_{2\nu}$ with the same eigenvalue. Viceversa, if $w = \begin{bmatrix} \Pi v \\ v \end{bmatrix} \neq 0$ satisfies $M_{2\nu}w = \lambda_N w$, then, from the last $\nu$ relations, it follows easily that $N_\nu v = \lambda_N v$, thus concluding the first part of the proof.

To prove the second statement, we observe that $M_\mu = 2I - A_p$, where $A_p$ is the adjacency matrix of the path, reported in Appendix A. Thus, the eigenvalues of $M_\mu$ can be computed by summing 2 to the eigenvalues of $A_p$ and the eigenvectors are the same, so that (4) follows.
now show that the $\nu$ eigenvalues and eigenvectors of $N_\nu$ are the eigenvalues and eigenvectors of $M_{2\nu}$ corresponding to the odd values of $k$ in (4). To see this, notice that the $\nu$ eigenvectors of $M_{2\nu}$ with structure $w = \begin{bmatrix} \Pi v \\ v \end{bmatrix}$ satisfy $(w)_j = (w)_{2\nu-j+1}$, $j = 1, \ldots, \nu$. Referring to the parametrization of $w$ in (4), we look for $k$s satisfying

$$\sin\left(\frac{jk\pi}{2\nu+1}\right) = \sin\left(\frac{(2\nu-j+1)k\pi}{2\nu+1}\right), \forall j \in \{1, \ldots, \nu\}.$$ 

This is easily verified for odd $k$s because the arguments of the sine functions on the two sides sum to $(2l+1)\pi$ for some integer $l$. This concludes the proof.

Next, the eigenvalues of the Laplacian $L_n$ of a path graph of length $n$ are expressed in closed form by relating them to the eigenvalues of the $M_{n-1}$ matrix. The following technical lemma gives the tools to directly compute the eigenvalues of the laplacian matrix of a path graph.

**Lemma 3.5:** The characteristic polynomials of $N_\mu$ and $M_\mu$, $\mu \in \mathbb{N}$, satisfy the following relations,

- \[ \det(sI - N_\mu) = (s - 1) \det(sI - M_{\mu-1}) - \det(sI - M_{\mu-2}) \]
- \[ \det(sI - N_\mu) = (s - 2) \det(sI - N_{\mu-1}) - \det(sI - N_{\mu-2}) \]
- \[ \det(sI - M_\mu) = (s - 2) \det(sI - M_{\mu-1}) - \det(sI - M_{\mu-2}) \]

**Proof:** The result is obtained by applying the Laplace expansion to compute the determinant of the matrices $(sI - N_\mu)$ and $(sI - M_\mu)$. In particular, the first and second relations are obtained by expanding respectively along the first and last row of $(sI - N_\mu)$. The third relation is obtained by expanding along, e.g., the first row of $(sI - M_\mu)$.

**Proposition 3.6 (Eigenvalues of the Laplacian of a path):** The characteristic polynomial of the Laplacian, $L_n$, of a path graph of length $n$ can be written as

$$\det(sI - L_n) = s \det(sI - M_{n-1}).$$

Thus, the eigenvalues of the Laplacian are given by

$$\lambda_{L_n} = 2 - 2 \cos\left((k-1)\frac{\pi}{n}\right), k = 1, \ldots, n.$$ 

**Proof:** Applying the Laplace expansion for the computation of the determinant and using
the results of the previous lemma, the following equalities hold:

\[
\det(sI - L_n) = (s - 1) \det(sI - N_{n-1}) - \det(sI - N_{n-2})
\]

\[
= (s - 1)^2 \det(sI - M_{n-1}) - 2(s - 1) \det(sI - M_{n-2}) - \det(sI - M_{n-3})
\]

\[
= (s^2 - 2s) \det(sI - M_{n-1}) - s \det(sI - M_{n-2})
\]

\[
= (s^2 - 2s) \det(sI - M_{n-1}) - s[(s - 2) \det(sI - M_{n-2}) - \det(sI - M_{n-1})]
\]

\[
= s^2 \det(sI - M_{n-1}).
\]

\[\blacksquare\]

IV. Reachability and Observability of Path and Cycle Graphs

In this section we completely characterize the reachability and observability of path and cycle graphs.

A. Reachability and observability of path graphs

We characterize the reachability (observability) of a path graph by using the PBH lemma in the form expressed in Corollary 2.5. First, as recalled in Remark 3.3, a path graph is always (reachable) observable from nodes 1 or \(n\). Next two lemmas give necessary and sufficient conditions for reachability (observability) from a given subset of nodes in terms of the \(N_\nu\) and \(M_\mu\) submatrices introduced in the previous section.

**Lemma 4.1**: A path graph of length \(n\) is reachable (observable) from a node \(i \in \{2, \ldots, n-1\}\) if and only if the matrices \(N_{i-1}\) and \(N_{n-i}\) do not have any common eigenvalue. The eigenvalues common to the two matrices are all and only the unreachable (unobservable) eigenvalues of the Laplacian \(L_n\) from node \(i\).

**Proof**: Applying Corollary 2.5 we have that \(i\) is not reachable (observable) if and only if \(L_nv = \lambda v\) and \(e_i^Tv = 0\) for some \(\lambda\) and \(v\). Equivalently, \(i\) is not reachable (observable) if and only if there exists an eigenvector \(v\) of \(L_n\) with \(v = [v_1 0 v_2]^T, v_1 \in \mathbb{R}^{i-1}, v_2 \in \mathbb{R}^{n-1}\). Component-wise this is written as

\[
N_{i-1}v_1 = \lambda v_1
\]

\[
(v_1)_{i-1} + (v_2)_1 = 0
\]

\[
(\Pi N_{n-i} \Pi) v_2 = \lambda v_2.
\]
The necessary condition follows easily by the above equations. Indeed, if $N_{i-1}$ and $N_{n-i}$ have at least one common eigenvalue $\lambda_0$ with corresponding eigenvectors respectively $v_{10}$ and $v_{20}$, then the conditions in the above equations are satisfied for $v = [v_{10} \ 0 \ \rho v_{20}]$ and $\lambda = \lambda_0$, where $\rho \in \mathbb{R}$ is just a scaling factor to satisfy $(v_{10})_i - 1 + \rho (v_{20})_1 = 0$.

To prove the converse we proceed by analyzing when these three conditions are satisfied. The first equation in (5) is verified in two cases: i) $v_1 = 0$ and $\lambda$ arbitrary, and ii) $v_1$ and $\lambda$ respectively eigenvector and eigenvalue of $N_{i-1}$. From the first condition it follows easily that $(v_2)_1 = 0$ and, using Lemma 3.2, that $v_2 = 0$. Therefore, the only possibility to have unreachability (unobservability) is ii). Now, using the second equation it follows easily that $v_2 \neq 0$, and, using the third equation, that $\Pi v_2$ must be an eigenvector of $N_{n-i}$ corresponding to the same eigenvalue $\lambda$ of $N_{i-1}$. This concludes the first part of the proof.

The fact that the unreachable (unobservable) eigenvalues of $L_n$ from node $i$ are all and only the eigenvalues common to $N_{i-1}$ and $N_{n-i}$ follows straight from the previous argument. Indeed, by definition the unreachable (unobservable) eigenvalues are all and only the ones satisfying the condition in Corollary 2.5 and, thus, the equations in (5).

**Remark 4.2 (Paths with odd number of nodes):** A straightforward consequence of the previous lemma is that a path graph with an odd number, $n$, of nodes is not reachable (observable) from the central node. Also, $(n - 1)/2$ eigenvalues are unreachable (unobservable) from that node, namely the eigenvalues of $L_n$ that are also eigenvalues of $N_{(n-1)/2}$. The corresponding $(n - 1)/2$ unreachable (unobservable) eigenvectors of $L_n$ are of the form $[v^T_0 \ -v^T_0 \ 0]^T$ with $v$ being the eigenvectors of $N_{(n-1)/2}$ as in (3).

A generalization to the multi input (output) case is given in the following lemma.

**Lemma 4.3:** A path graph of length $n$ is reachable (observable) from the set of control (observation) nodes $I_o = \{i_1, \ldots, i_m\}$ if and only if the matrices $N_{i_1-1}$, $M_{i_2-i_1-1}$, $\ldots$, $M_{i_m-i_{m-1}-1}$ and $N_{n-i_m}$ do not have common eigenvalues. The eigenvalues common to the matrices are all and only the (unreachable) unobservable eigenvalues of $L_n$ from the set $I_o$.

**Proof:** We proceed as in the single input (output) case and apply the PBH Lemma in the version expressed in Corollary 2.5. The path is not reachable (observable) from the set $I_o = \{i_1, \ldots, i_m\}$ if and only if there exists an eigenvector $v$ of $L_n$ with $(v)_i = \ldots = (v)_{i_m} = 0$, so that $v = [v^T_1 \ 0 \ v^T_2 \ \ldots \ 0 \ v^T_m]^T$ for suitable vectors $v_1 \in \mathbb{R}^{i_1-1}, \ldots, v_m \in \mathbb{R}^{i_m-1}$. This is
equivalent to
\[ N_{i_1-1}v_1 = \lambda v_1 \]
\[ (v_1)_{i_1-1} + (v_2)_1 = 0 \]
\[ M_{i_2-i_1-1}v_2 = \lambda v_2 \]
\[ (v_2)_{i_2-1} + (v_3)_1 = 0 \]
\[ \vdots \]
\[ (v_{m-1})_{i_m-1} + (v_m)_1 = 0 \]
\[ (\prod N_{n-i_m}\Pi)v_m = \lambda v_m. \]

The proof follows by using the same arguments in Lemma 4.1.

We are now ready to completely characterize the reachability (observability) of a path by means of simple algebraic rules from number theory. For the sake of clarity, we state the theorem for path graphs of length \( n \), where \( n \) has a prime factorization with distinct odd prime factors. The general case follows straight and is discussed in a remark.

**Theorem 4.4 (Path reachability and observability):** Given a path graph of length \( n \), let \( n = 2^{m_0} \prod_{\nu=1}^{k} p_{\nu} \) be a prime number factorization for some \( k \in \mathbb{N} \) and distinct (odd) prime numbers \( p_1, \ldots, p_k \). The following statements hold:

(i) the path is not completely reachable (observable) from a node \( i \in \{2, \ldots, n-1\} \) if and only if

\[ (n - i) \mod p = (i - 1) \]

for some odd prime \( p \) dividing \( n \);

(ii) the path is not completely reachable (observable) from a set of nodes \( I_o = \{i_1, \ldots, i_m\} \subset \{2, \ldots, n-1\} \) if and only if

\[ 2(i_1 - 1) + 1 \mod p = (i_2 - i_1) \mod p = \ldots = (i_m - i_{m-1}) \mod p = 2(n - i_m) + 1, \]

for some odd prime \( p \) dividing \( n \);

(iii) for each odd prime factor \( p \in \{p_1, \ldots, p_k\} \) of \( n \), the path is not reachable (observable) from each set of nodes \( I^p_o = \{\ell p - \frac{p-1}{2}\} \ell \in \{1, \ldots, \frac{n}{p}\} \) with the following unreachable (unobservable) eigenvalues

\[ \lambda_{\nu} = 2 - 2 \cos \left( \frac{(2\nu - 1)\pi}{p} \right), \quad \nu \in \{1, \ldots, \frac{p-1}{2}\}; \]
and unreachable (unobservable) eigenvectors

\[ V_\nu = \begin{bmatrix} v_\nu^T & 0 & -\nu v_\nu^T & 0 & \cdots & (-1)^{\frac{\nu}{2}} \nu v_\nu^T \end{bmatrix}^T, \tag{8} \]

where \( v_\nu \in \mathbb{R}^{(p-1)/2} \) is the eigenvector of \( N_{(p-1)/2} \) corresponding to the eigenvalue \( \lambda_\nu \) for \( \nu \in \{1, \ldots, (p-1)/2\} \); and

(iv) if node \( i \) belongs to \( I_\nu^0 = \{\ell q_j - q_i-1\}_{\ell \in \{1, \ldots, \frac{n}{q_j}\}^l} \) for \( l \leq k \) distinct prime factors \( q_i \neq \ldots \neq q_l \) of \( n \), then the set of unreachable (unobservable) eigenvalues from node \( i \) is given by

\[ \lambda_\nu = 2 - 2 \cos \left( (2\nu - 1) \frac{\pi}{q_1 \cdot \ldots \cdot q_l} \right), \quad \nu \in \{1, \ldots, (q_1 \cdot \ldots \cdot q_l - 1)/2\}. \]

Also, the orthogonal complement to the reachable subspace, \( (X_r)^\perp \), (respectively the unobservable subspace, \( X_{uo} \)) is spanned by all the corresponding eigenvectors of the form

\[ V_\nu = \begin{bmatrix} v_\nu^T & 0 & -\nu v_\nu^T & 0 & \cdots & (-1)^{\frac{\nu}{2}} \nu v_\nu^T \end{bmatrix}^T, \]

where \( v_\nu \in \mathbb{R}^{(q_1 \cdot \ldots \cdot q_l - 1)/2} \) is the eigenvector of \( N_{((q_1 \cdot \ldots \cdot q_l) - 1)/2} \) corresponding to the eigenvalue \( \lambda_\nu \) for \( \nu \in \{1, \ldots, ((q_1 \cdot \ldots \cdot q_l) - 1)/2\} \).

**Proof:** Using Lemma 4.1 we have that the path graph is not completely reachable (observable) from node \( i \) if and only if \( N_{i-1} \) and \( N_{n-i} \) have at least one common eigenvalue. Therefore, using Proposition 3.4 we have that it must hold \( 2 - 2 \cos \frac{(2j_1-1)\pi}{2(i-1)+1} = 2 - 2 \cos \frac{(2j_2-1)\pi}{2(n-i)+1} \) for some \( j_1 \in \{1, \ldots, i-1\} \) and \( j_2 \in \{1, \ldots, (n-i)\} \). In the admissible range of \( j_1 \) and \( j_2 \) the cosine arguments are less than \( \pi \) so that the cosine is one to one. Thus, the equality holds if and only if \( \frac{(2j_1-1)}{2(i-1)+1} = \frac{(2j_2-1)}{2(n-i)+1} \). Two integers \( j_1 \) and \( j_2 \) satisfying this equation exist in the admissible range if only if \( 2(i-1)+1 \) and \( 2(n-i)+1 \) are not coprime, that is, if and only if \( GCD(2i-1, 2n-2i+1) \) is greater than one. Now, \( GCD(2i-1, 2n-2i+1) \) is odd because \( 2i-1 \) and \( 2n-2i+1 \) are. Therefore, we can write \( 2i-1 = p\alpha_1 \) and \( 2n-2i+1 = p\alpha_2 \) with \( p, \alpha_1 \) and \( \alpha_2 \) odd. This is equivalent to \( n+n-2i+1 = p\alpha_2 \) and, since \( p \) divides \( n \), \( p \) must divide also \( (n-2i+1) \) thus concluding the first part of the proof.

To prove statement (ii), we have by Lemma 4.3 that the path graph is not reachable (observable) from the set \( I_o = \{i_1, \ldots, i_m\} \) if and only if the matrices \( N_{i_1-1}, M_{i_2-i_1-1}, \ldots, M_{i_m-i_{m-1}-1} \) and \( N_{n-i_m} \) do not have common eigenvalues. The proof follows by using again Proposition 3.4 and the same arguments as in the single node case.
To prove statement (iii), we start observing that the set of nodes $I_o^p = \{\ell p - \frac{p-1}{2} \}_{\ell \in \{1, \ldots, \frac{n}{p}\}}$, is the set of all nodes satisfying condition in (i) for a given $p \in \{p_1, \ldots, p_k\}$. Using Lemma 4.3, we have that the unreachable (unobservable) eigenvalues from this set of nodes are the common eigenvalues to $N_{(p-1)/2}$ and $M_{p-1}$. From Proposition 3.4, it follows easily that the common eigenvalues between $N_{(p-1)/2}$ and $M_p$ are all the eigenvalues of $N_{(p-1)/2}$ and have the form in equation (7). As regards the unreachable (unobservable) eigenvectors, using equation (6) for this special set $I_o$, it follows straights that the eigenvectors have zero components as in equation (8). To prove that the nonzero components have that special structure in equation (8), we observe that they must be eigenvectors of respectively $N_{(p-1)/2}$, $M_{p-1}$, $\ldots$, $M_{p-1}$ and $\Pi N_{(p-1)/2} \Pi$. From point (i) of Proposition 3.4, the eigenvectors of $N_{(p-1)/2}$, $M_{p-1}$, $\ldots$, $M_{p-1}$ and $\Pi N_{(p-1)/2} \Pi$ are respectively $\alpha_0 v_{\nu}$, $\alpha_1 [(\Pi v_{\nu})^T v_{\nu}]^T$, $\ldots$, $\alpha_{2^{k-1}} [(\Pi v_{\nu})^T v_{\nu}]^T$ and $\alpha_{2^k} \Pi v_{\nu}$, where $v_{\nu}$ is an eigenvector of $N_{(p-1)/2}$ and $\alpha_\mu \in \mathbb{R}$, $\mu \in \{0, \ldots, \frac{n}{p}\}$. Finally, using again equation (6), $\alpha_\mu = -\alpha_{\mu-1}$, $\mu \in \{1, \ldots, \frac{n}{p}\}$, so that the proof follows by choosing $\alpha_0 = 1$.

The proof of statement (iv) follows from the definition of unreachable (unobservable) eigenvalues and eigenvectors, and arguments as in the previous statements. \hfill \Box

Remark 4.5 (General version of Theorem 4.4): In the general case of a path graph of length $n = 2^{n_0} \prod_{\nu=1}^{k} p_{\nu}$, where $p_1, \ldots, p_k$ are not all distinct, statement (i) and (ii) of Theorem 4.4 continue to hold in the same form. As regards statement (iii), it still holds in the same form, but it can also be strengthen with a slight modification. That is, for each multiple factor $\bar{p}$ with multiplicity $\bar{k}$, the statement continues to hold if $\bar{p}$ is replaced by $\bar{p}^\alpha$ with $\alpha \in \{1, \ldots, \bar{k}\}$. Statement (iv) holds if for each prime factor $\bar{p}$ with multiplicity $\bar{k}$ we check if node $i$ belongs not only to $I_o^p$, but also to each $I_o^{\bar{p}^\alpha}$ with $\alpha \in \{1, \ldots, \bar{k}\}$. Consistently the unreachable (unobservable) eigenvalues and eigenvectors considered in the statement must be constructed by using $\bar{p}^\alpha$ instead of $\bar{p}$, where $\bar{\alpha} = \max_\alpha \{\alpha \in \{1, \ldots, \bar{k}\} | i \in I_o^{\bar{p}^\alpha}\}$. \hfill \Box

The following corollary follows straight from Theorem 4.4 and characterizes all and only the path graphs that are observable from any node.

Corollary 4.6 (Reachable (observable) paths from any node): Given a path graph of length $n = 2^k$ for some $k \in \mathbb{N}$, then the path is reachable (observable) from any node. \hfill \Box

Next, we provide a simple routine giving a graphical interpretation of the results of the theorem. We describe the routine for paths with simple factorization leaving the generalization to a specific example. We proceed by associating a unique symbol to each set of nodes defined at point

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(iii) of Theorem 4.4 for a given $p$. Thus, each group of nodes sharing the same unreachable (unobservable) eigenvalues has the same symbol and nodes of different groups (and so associated to different unreachable (unobservable) eigenvalues) have different symbols. Formally, let $n = 2^{n_0} \prod_{\nu=1}^{k} p_\nu$ for some $n_0 \in \mathbb{N}$ and $p_1, \ldots, p_k$ prime integers. At the beginning of the procedure we initialize all the nodes without any symbol. For any $p_\nu, \nu \in \{1, \ldots, k\}$, we partition the nodes into $n/p_\nu$ groups of $p_\nu$ nodes and assign the same symbol to all the nodes in position $i = j p_\nu - \frac{p_\nu - 1}{2}, j \in \{1, \ldots, \frac{n}{p_\nu}\}$. A set of nodes from which the path is reachable (observable) is obtained by selecting any node without symbols, if there are any, or a set of nodes having no symbols in common.

Three examples for $n = 6$ (even), $n = 15$ (odd) and $n = 9$ (multiple factor) are shown respectively in Figure 1 in Figure 2 and in Figure 3. In Figure 1 nodes with the triangle symbol are unable to reconstruct the state of the network by themselves. Indeed, they share the same unreachable (unobservable) eigenvalue $\lambda = 1$. In view of the previous results, focusing on node $i_1 = 2$, notice that $N_{i-1} = N_1 = [1]$ (whose eigenvalue is 1), $N_{n-i} = N_4$ and its eigenvalues are: $\{0.12, 1, 2.35, 3.53\}$. The common eigenvalue is of course $\lambda = 1$. The unreachability (unobservability) can be checked more easily using the test $(n - i) = 4 \mod 3 = 1 = (i - 1)$.

![Diagram](image.png)

Fig. 1. Reachable (observable) nodes for a path with $n = 6$ nodes.

In Figure 2 nodes with the triangle belong to the set $I_3^{p_1}$, with $p_1 = 3$, and nodes with the square to $I_5^{p_2}$, with $p_2 = 5$. The “triangle nodes” share the same (unreachable) unobservable eigenvalue $\lambda = 1$ ($N_{i-1} = N_1 = [1]$), while two unreachable (unobservable) eigenvalues, 0.3820 and 2.6180, are associated to the “square nodes”. Finally, the central node has both the triangle and square symbols.

In Figure 3 we consider a path of length $n = 9 = 3^2$, with $p = 3$ being a multiple factor. Nodes with the triangle belong to the set $I_3^{p_1}$, with $p_1 = 3$. To the central node is associated both a triangle (since it belongs to $I_3^3$) and a square since it is the unique node in $I_5^{p_2}$, with $p_2^2 = 9$. The “triangle nodes” share the same (unreachable) unobservable eigenvalue $\lambda = 1$ ($N_{i-1} = N_1 = [1]$), while the central node has four unreachable (unobservable) eigenvalues being the eigenvalues of
Fig. 2. Reachable (observable) nodes for a path with \( n = 15 \) nodes.

\( N_4 \). This example suggests how to associate symbols in the general case when \( n \) has multiple factors. That is, for each multiple factor \( \bar{p} \) with multiplicity \( \bar{k} \), we use \( \bar{k} \) different symbols, one for each \( \bar{p}^\alpha, \alpha \in \{1, \ldots, \bar{k}\} \).

Fig. 3. Reachable (observable) nodes for a path with \( n = 9 \) nodes.

**B. Reachability and observability of cycle graphs**

Next, we characterize the reachability (observability) of a cycle graph. We start with a negative result, namely that a cycle graph is not reachable (observable) from a single node. First, we need a well known result in linear systems theory [20].

**Lemma 4.7:** If a state matrix \( A \in \mathbb{R}^{n \times n}, n \in \mathbb{N} \), has an eigenvalue with geometric multiplicity \( \mu \geq 2 \), then for any \( B \in \mathbb{R}^n \) (respectively \( C \in \mathbb{R}^{1 \times n} \)) the pair \( (A, B) \) is unreachable (respectively the pair \( (A, C) \) is unobservable). \( \square \)

As shown in Appendix B, all but at most two eigenvalues of the Laplacian of the cycle have geometric multiplicity two. Thus, applying the previous lemma next proposition follows.

**Proposition 4.8:** A cycle graph is not completely reachable (observable) from a single node for any choice of the control (observation) node. Furthermore, \( \lceil \frac{n-1}{2} \rceil \) eigenvalues are unreachable (unobservable).
**Proof:** The unreachability (unobservability) follows directly by noting that all but at most two eigenvalues of the cycle have geometric multiplicity two and by applying the previous lemma. By using the PBH lemma, it follows straight that the unreachable (unobservable) eigenvalues are all but the zero eigenvalue.

Recall that, without loss of generality, we label the nodes of the cycle so that the undirected edges are \((i, (i \mod (n) + 1))\) for \(i \in \{1, \ldots, n\}\). Following the same line as in the reachability (observability) analysis of path graphs, we can prove the following lemma.

**Lemma 4.9:** A cycle graph of length \(n\) is reachable (observable) from the set of nodes \(I_o = \{i_1, \ldots, i_m\}\) if and only if the matrices \(M_{i_2-i_1-1}, \ldots, M_{i_m-i_{m-1}-1}\) and \(M_{(i_1-i_{m-1}) \mod n}\) do not have common eigenvalues.

**Proof:** Without loss of generality, we set node \(i_1 = 1\), so that the matrices in the statement of the theorem become \(M_{i_2-i_1-1}, \ldots, M_{i_m-i_{m-1}-1}\) and \(M_{n-i_m}\). Following the same lines of the proof of Lemma 4.3, loss of reachability (observability) is equivalent to the existence of a nonzero solution \(v = [0 \ v_1^T \ 0 \ v_2^T \ldots \ 0 \ v_m^T]^T\) to:

\[
(v_1)_1 + (v_m)_{n-i_m} = 0
\]
\[
M_{i_2-i_1-1}v_1 = \lambda v_1
\]
\[
\vdots
\]
\[
(v_{m-1})_{i_{m-1}} + (v_m)_1 = 0
\]
\[
M_{n-i_m}v_m = \lambda v_m.
\]

Now, if \(v_j = 0\) for some \(j \in \{1, \ldots, m\}\), using Lemma 3.2 the only solution of the above system is \(v = 0\). With this condition in hand, it follows straight that \(v\) is a (nonzero) solution if and only if all the \(v_j\) are nonzero eigenvectors of \(M_{i_2-i_1-1}, \ldots, M_{n-i_m}\) with common eigenvalue \(\lambda\), thus concluding the proof.

It is worth noting that, due to the symmetry of the cycle, the reachability (observability) properties are determined by the relative distance between each pair of consecutive control (observation) nodes. The following theorem parallels Theorem 4.4. As for the path, we state the theorem for cycle graphs of length \(n\), where \(n\) has a prime factorization with distinct prime factors. The general case follows straight from similar arguments as in Remark 4.5.

**Theorem 4.10 (Cycle reachability and observability):** Given a cycle graph of length \(n\), let \(n = \prod_{\nu=1}^{k} p_{\nu}\) be a prime number factorization for some \(k \in \mathbb{N}\) and distinct prime numbers...
\[ p_1, \ldots, p_k \] (including the integer 2). The following statements hold:

(i) the cycle graph is reachable (observable) from the set of nodes \( I_o = \{i_1, \ldots, i_m\} \) if and only if
\[
GCD((i_2 - i_1), (i_3 - i_2), \ldots, (n + i_1 - i_m)) = 1;
\]  

(ii) for each prime factor \( p \) of \( n \) and for each fixed \( \kappa \in \{1, \ldots, p\} \), the set of nodes \( I_p^\kappa = \{\kappa + \ell p\}_{\ell \in \{0, \ldots, \frac{n}{p} - 1\}} \), is unreachable (unobservable) with the following unreachable (unobservable) eigenvalues
\[
\lambda_\nu = 2 - 2 \cos \left( \nu \frac{\pi}{p} \right), \quad \nu \in \{1, \ldots, p - 1\}
\]
and, for \( \kappa = 1 \), eigenvectors
\[
V_\nu = \begin{bmatrix} 0 & w^T_\nu & \ldots & 0 & w^T_\nu \end{bmatrix}^T,
\]
where \( w_\nu \in \mathbb{R}^{(p-1)} \) is the eigenvector of \( M_{(p-1)} \) corresponding to the eigenvalue \( \lambda_\nu \).

(iii) if a set of control (observation) nodes \( I_o \) with cardinality greater than 1, satisfies
\[
GCD((i_2 - i_1), (i_3 - i_2), \ldots, (n + i_1 - i_m)) = q_1 \cdot \ldots \cdot q_l
\]
where \( q_1 \neq \ldots \neq q_l \) are \( l \leq k \) distinct prime factors of \( n \), then the set of unreachable (unobservable) eigenvalues from \( I_o \) is given by
\[
\lambda_\nu = 2 - 2 \cos \left( \nu \frac{\pi}{q_1 \cdot \ldots \cdot q_l} \right), \quad \nu \in \{1, \ldots, (q_1 \cdot \ldots \cdot q_l) - 1\}.
\]
Also, without loss of generality, setting \( i_1 = 1 \), the orthogonal complement to the reachable subspace, \( (X_r)^\perp \), (respectively the unobservable subspace, \( X_{no} \)) is spanned by all the corresponding eigenvectors
\[
V_\nu = \begin{bmatrix} 0 & w^T_\nu & \ldots & 0 & w^T_\nu \end{bmatrix}^T,
\]
where \( w_\nu \in \mathbb{R}^{(q_1 \cdot \ldots \cdot q_l) - 1} \) is the eigenvector of \( M_{(q_1 \cdot \ldots \cdot q_l) - 1} \) corresponding to the eigenvalue \( \lambda_\nu \).

**Proof:** We provide just a sketch of the proof since it follows the same line as the proof of Theorem 4.4. Statement (i) is proven by using Lemma 4.9 and the structure of the eigenvalues of the \( M \) matrices given in Proposition 3.4 with the same argument as in Theorem 4.4 (i) and (ii). To prove statement (ii), we start observing that the set of nodes \( I_o^p = \{\kappa + \ell p\}_{\ell \in \{1, \ldots, \frac{n}{p}\}} \), is the set
of all nodes satisfying condition in (i) for a given \( p \in \{p_1, \ldots, p_k\} \). Using Lemma 4.9 we have that the unreachable (unobservable) eigenvalues from this set of nodes are the eigenvalues of \( M_{p-1} \). The proof follows by the same arguments as in Theorem 4.4 (iii). Finally, statement (iii) follows straight.

Next corollaries provide respectively an easy way to choose two control (observation) nodes to get reachability (observability) for any cycle length and the class of cycle graphs (lengths) for which reachability (observability) is guaranteed for any pair of nodes.

**Corollary 4.11:** Any cycle graph is reachable (observable) from two adjacent nodes. □

**Corollary 4.12:** A cycle graph of length \( n \) is reachable (observable) from any pair of nodes if and only if \( n \) is prime. □

As for the path, we provide a simple routine giving a graphical interpretation of the results of the theorem. Again, we give the procedure for the case of simple factors.

We will mark each unreachable (unobservable) node with a different symbol. Let \( n = \prod_{\nu=1}^{k} p_{\nu} \) for some \( k \in \mathbb{N} \) and \( p_1, \ldots, p_k \) distinct prime integers (here we include 2 among the \( p_\nu \) as well).

At the beginning of the procedure all the nodes are initialized without any symbol. For any \( p_{\nu}, \nu \in \{1, \ldots, k\} \) and \( i \in \{1, \ldots, p_\nu\} \), partition the nodes into \( n/p_\nu \) groups of \( p_\nu \) nodes and assign the same symbol to all the nodes in position \( i + k \cdot p_\nu, j \in \{1, \ldots, n/p_\nu - 1\} \). Nodes with the same symbol have the same unreachable (unobservable) eigenvectors, in the sense that controlling (observing from) all the nodes at the same time gives the same unreachable (unobservable) eigenvectors. A set of nodes from which the cycle is reachable (observable) is obtained by selecting any subset of nodes having no symbols in common.

In Figure 4 there are two symbols for each node. This is because \( n = 15 = 3 \cdot 5 \). Symbols closer to the nodes have periodicity 5 and the others have periodicity 3. Notice the ease of design using the above procedure. For example \( \{4, 13\} \) and \( \{8, 14\} \) are unreachable (unobservable) pairs since they share respectively the square and the parallelogram, while \( \{2, 13\} \) and \( \{5, 12\} \) are reachable (observable) pairs. Finally, notice that two neighboring nodes have always different symbols in accordance with the result in Corollary 4.11.

V. Conclusions

In this paper we have characterized the reachability (observability) of path and cycle graphs in terms of simple algebraic rules from number theory. In particular, we have shown what are
all and only the unreachable (unobservable) set of nodes and provided simple routines to select a set of control (observation) nodes that guarantee reachability (observability).

Promising avenues for further research include the extension of the proposed methodologies to more complex graphs having paths and cycles as constitutive graphs (e.g., grid, torus and cylinder graphs).

**APPENDIX**

**A. Adjacency matrix of path graphs**

Eigenvalues and eigenvectors of the adjacency matrix, \( A_p \in \mathbb{R}^{n \times n} \), of a path graph of length \( n \) can be easily computed [21]. Here we briefly summarize some of the steps.

Consider the matrix

\[
A_p = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
1 & 0 & 1 & \ddots & \\
\vdots & \ddots & \ddots & \ddots & 1 \\
0 & \ldots & 1 & 0
\end{bmatrix}
\]

By definition, an eigenvalue \( \lambda \) and a corresponding eigenvector \( v \in \mathbb{R}^n \) of \( A_p \) satisfy \( (v)_{i-1} - \lambda (v)_i + (v)_{i+1} = 0 \) with \( (v)_0 = (v)_{n+1} = 0 \) and at least one \( (v)_i, i \in \{1, \ldots, n\} \), nonzero. Set \( (v)_1 = a \) and build the sequence \( (v)_i \), according to \( (v)_{i+1} = -(v)_{i-1} + 2 \lambda (v)_i \) (e.g. \( (v)_2 = \lambda a \), \( (v)_3 = \lambda^2 a - a \), \( (v)_4 = \lambda^3 a - 2 \lambda a \), \ldots, \( (v)_\ell = p_\ell(\lambda)a = (\lambda \cdot p_{\ell-1}(\lambda) - p_{\ell-2}(\lambda))a \)). Imposing
\( p_{n+1}(\lambda) = 0 \) one finds \( \lambda_k = 2 \cos \left( k \cdot \frac{\pi}{n+1} \right) \) as all possible values giving \( (v)_{n+1} = 0 \) with nonzero \( a \). Now, choose \( a_k = \sin \left( k \cdot \frac{\pi}{n+1} \right) \), the corresponding eigenvector can be expressed componentwise \( (v_k)_i = \sin \left( i \cdot k \cdot \frac{\pi}{n+1} \right) \) according the recursive formula above and simple trigonometric rules.

**B. Circulant matrices and eigenstructure of the Laplacian of a cycle graph**

An \( n \times n \) matrix \( C \) of the form

\[
C = \begin{bmatrix}
c_0 & c_{n-1} & \ldots & c_2 & c_1 \\
c_1 & c_0 & c_{n-1} & \ldots & \vdots \\
\vdots & c_1 & c_0 & \ddots & \vdots \\
c_{n-2} & \ddots & \ddots & \ddots & c_0 \\
c_{n-1} & c_{n-2} & \ldots & c_1 & c_0
\end{bmatrix}
\]

is called a *circulant matrix* \[22\]. A circulant matrix is fully specified by the first column \( c = [c_0, \ldots, c_{n-1}]^T \) of \( C \). The other columns are obtained by a cyclic permutation of the first. The eigenvalues of a circulant matrix can be expressed in terms of the coefficients \( c_0, \ldots, c_{n-1} \) \[22\]:

\[
\lambda_j = \sum_{k=0}^{n-1} \omega^{jk} c_k, \quad \omega = e^{i \frac{2\pi}{n}}, \quad \text{where here } i \text{ represents the imaginary unit.}
\]

The Laplacian matrix of a cycle graph is a special case of this family corresponding to \( c_0 = 2 \), \( c_1 = c_{n-1} = -1 \), \( c_j = 0 \), \( j = 2, \ldots, n-2 \):

\[
\lambda_j = \omega^{j0} 2 - e^{i \frac{2\pi}{n} j(n-1)} - e^{i \frac{2\pi}{n} j} = 2 - 2 \cos \left( \frac{2\pi}{n} j \right),
\]

\( j = 0, 1, \ldots, n-1 \). Notice that the eigenvalues \( \lambda_0 = 0 \) and \( \lambda_{\frac{n}{2}} = 4 \) (only if \( n \) is even) are simple, all the others verify \( \lambda_j = \lambda_{n-j} \).

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