EXISTENCE AND LOCAL UNIQUENESS OF NORMALIZED PEAK SOLUTIONS FOR A SCHRÖDINGER-NEWTON SYSTEM

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Abstract. In this paper, we investigate the existence and local uniqueness of normalized peak solutions for a Schrödinger-Newton system under the assumption that the trapping potential is degenerate and has non-isolated critical points.

First we investigate the existence and local uniqueness of normalized single-peak solutions for the Schrödinger-Newton system. Precisely, we give the precise description of the chemical potential \( \mu \) and the attractive interaction \( a \). Then we apply the finite dimensional reduction method to obtain the existence of single-peak solutions. Furthermore, using various local Pohozaev identities, blow-up analysis and the maximum principle, we prove the local uniqueness of single-peak solutions by precise analysis of the concentrated points and the Lagrange multiplier. Finally, we also prove the nonexistence of multi-peak solutions for the Schrödinger-Newton system, which is markedly different from the corresponding Schrödinger equation. The nonlocal term results in this difference.

The main difficulties come from the estimates on Lagrange multiplier, the different degenerate rates along different directions at the critical point of \( P(x) \) and some complicated estimates involved by the nonlocal term. To our best knowledge, it may be the first time to study the existence and local uniqueness of solutions with prescribed \( L^2 \)-norm for the Schrödinger-Newton system.

Keywords: Normalized solutions; the Schrödinger-Newton system; Degenerated trapping potential.

AMS Subject Classifications: 35A01 · 35B25 · 35J20 · 35J60

1. Introduction and our main results

In this paper, we investigate the following nonlinear Schrödinger-Newton system

\[
\begin{align*}
-\Delta u + P(x)u &= a\psi u + \mu u, \quad x \in \mathbb{R}^3, \\
-\Delta \psi &= \frac{u^2}{2}, \quad x \in \mathbb{R}^3,
\end{align*}
\]

under the mass constraint \( \int_{\mathbb{R}^3} u^2(x)dx = 1 \), where \( P(x) \) is a degenerate trapping potential with non-isolated critical points, \( \mu \in \mathbb{R} \) and \( a \geq 0 \) denote the chemical potential and the attractive interaction respectively.

The Schrödinger-Newton problem arises from describing the quantum mechanics of a polaron at rest, see [32]. Also it was used to describe an electron trapped in its own hole in a certain approximating to Hartree-Fock theory of one component plasma in [22]. In addition, Penrose in [32] applied it as a model of self-gravitating matter, where quantum

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state reduction is understood as a gravitational phenomenon. Specifically, if \( m \) is the mass of the point, the interaction leads to the system in \( \mathbb{R}^3 \)

\[
\begin{cases}
-\frac{\epsilon^2}{2m} \Delta u + P(x)u = \psi u, & x \in \mathbb{R}^3, \\
-\Delta \psi = 4\pi \tau u^2, & x \in \mathbb{R}^3,
\end{cases}
\]

where \( u \) is the wave function, \( \psi \) is the gravitational potential energy, \( P(x) \) is a given Schrödinger potential, \( \epsilon \) is the Planck constant, \( \tau = Gm^2 \) and \( G \) is the Newton’s constant of gravitation.

Set

\[
u(x) \mapsto \frac{u}{4\epsilon \sqrt{\pi \tau m}}, \quad P(x) \mapsto \frac{1}{2m} P(x), \quad \psi(x) \mapsto \frac{1}{2m} \psi(x).
\]

Then system (1.2) can be written, maintaining the original notations, as

\[
\begin{cases}
-\epsilon^2 \Delta u + P(x)u = \psi u, & x \in \mathbb{R}^3, \\
-\epsilon^2 \Delta \psi = \frac{u^2}{2}, & x \in \mathbb{R}^3.
\end{cases}
\]

From the second equation of (1.3), we know

\[
\psi(x) = \frac{1}{8\pi \epsilon^2} \left( \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy \right).
\]

Then the system (1.3) turns into the following single nonlocal equation

\[
-\epsilon^2 \Delta u + P(x)u = \frac{1}{8\pi \epsilon^2} \left( \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy \right) u, \quad x \in \mathbb{R}^3,
\]

which also appears in the study of standing waves for the following nonlinear Hartree equations

\[
i\epsilon \frac{\partial \varphi}{\partial t} = -\epsilon^2 \Delta_x \varphi + (P(x) + E)\varphi - \frac{1}{8\pi \epsilon^2} \left( \int_{\mathbb{R}^3} \frac{\varphi^2(y)}{|x-y|} dy \right) \varphi, \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R}^+,
\]

with the form \( \varphi(x, t) = e^{-iEt/\epsilon} u(x) \), where \( i \) is the imaginary unit and \( \epsilon \) is the Planck constant.

In recent decades, problem (1.4) has been extensively investigated. When \( \epsilon = 1 \) and \( P(x) = 1 \), (1.4) becomes

\[
-\Delta u + u = \frac{1}{8\pi} \left( \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy \right) u, \quad x \in \mathbb{R}^3.
\]

The existence and uniqueness of ground states for (1.5) was obtained with variational methods by Lieb [22], Lions [24] and Menzala [28]. The nondegeneracy of the ground states for (1.5) was proved by Tod-Moroz [35] and Wei-Winter [38].

When \( \epsilon \) is small and \( P(x) \) is not a constant, in [25] Lions proved the existence of solutions with ground states for (1.4) under some conditions on \( P(x) \) since problem (1.4) has a variational structure. Moreover, the solution with ground states concentrates at certain point. Applying the finite dimensional reduction method, in [38] Wei and Winter proved that (1.4) has a solution concentrating at \( k \) points which are the local minimum points of \( P(x) \). This also means the existence of multiple solutions. Very recently, in [26] Luo, Peng and Wang proved the uniqueness of the concentrated solutions of (1.4) obtained in [38] by
some local Pohozaev identities, blow-up analysis and the maximum principle. For more other results of the existence of solutions with concentration, one can refer to \cite{6, 10, 34, 36} and the references therein. For a very similar nonlocal problem i.e. Schrödinger-Possion equations, one can refer to \cite{17, 18, 37}, where some existence of peak solutions are obtained by the finite dimensional reduction method under various assumptions of the potential function.

Actually, the Schrödinger-Newton problem \eqref{eq1.4} is a special type of following Choquard equation:

\begin{equation}
\label{eq1.6}
-\epsilon^2 \Delta u + P(x)u = \frac{a}{\epsilon^2} (I_\alpha * F(u)) f(u) + \mu u, \quad N \geq 2, \quad p > 1, \quad x \in \mathbb{R}^N,
\end{equation}

where $F \in C^1(\mathbb{R}, \mathbb{R})$, $F'(s) = f(s)$, $\mu \in \mathbb{R}$ denotes the chemical potential, $a \geq 0$ denotes the attractive interaction and the Riesz potential $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is defined(cf. Ref \cite{33}) as

\begin{equation}
\label{eq1.7}
I_\alpha := \frac{\Gamma(N-\alpha)}{\Gamma(\frac{N}{2}) \pi^{\frac{N}{2}} 2^\alpha} \frac{1}{|x|^{N-\alpha}}, \quad x \in \mathbb{R}^N \setminus \{0\}, \quad \alpha \in (0, N).
\end{equation}

Especially, when $N = 3$ and $\alpha = 2$ in \eqref{eq1.7}, then $I_\alpha = \frac{1}{4\pi|x|}$, $x \in \mathbb{R}^3 \setminus \{0\}$.

There are many results about the case that $f(s) = |s|^{p-2}s$. When $\epsilon = 1$, $N = 3$, $\alpha = 2$, $P(x) = constant > 0$, $\mu = 0$ and $p \to 2$, in \cite{39} Xiang proved the uniqueness and non-degeneracy of ground states to \eqref{eq1.6}. When $\epsilon = 1$, $N = 3, 4, 5$, $P(x) = 1$, $\alpha = 2$, $p = 2$ and $\mu = 0$, in \cite{4} Chen proved the non-degeneracy of ground states of \eqref{eq1.6}. As an application of the non-degeneracy result he obtained, he then used a Lyapunov-Schmidt reduction argument to construct multiple semi-classical solutions to \eqref{eq1.1} with an external potential. When $N = 3, \alpha = 2$ and $\mu = 0$ in \eqref{eq1.6}, in \cite{29, 30} Moroz and Van Schaftingen obtained some results about the existence and concentration of positive solutions to the Choquard equation. For ground states of \eqref{eq1.6} with $\epsilon = 1$, one can refer to \cite{8, 14, 19-21, 40} under various assumptions of trapping potential $P(x)$, which can be described equivalently by positive $L^2$ minimizers of the following Hartree-type energy functional(cf. \cite{14})

\[ E_\alpha(u) = \int_{\mathbb{R}^N} (|\nabla u|^2 + P(x)u^2)dx - \frac{a}{p} \int_{\mathbb{R}^N} (I_\alpha * |u(x)|^p)|u(x)|^pdx. \]

Very recently, taking $a = 1$, in \cite{7} Cingolani and Tanaka developed a new variational approach and proved the existence of a family of solutions concentrating to a local minimum of $P(x)$ as $\epsilon \to 0$ under general conditions on $F(s)$. In \cite{3}, assuming that the nonlinear term is subcritical and satisfies almost optimal assumptions, Cingolani1, Gallo and Tanaka proved the existence of a spherically symmetric solution to the following fractional Schrödinger equation with a nonlocal nonlinearity of Choquard type

\[ \begin{cases}
(-\Delta)^s u + \mu u = (I_\alpha * F(u)) f(u), & x \in \mathbb{R}^N, \\
\int_{\mathbb{R}^N} u^2 = c, & (\mu, u) \in (0, +\infty) \times H^s_0(\mathbb{R}^N),
\end{cases} \]

where $s \in (0, 1)$, $N \geq 2$, $\alpha \in (0, N)$, $F \in C^1(\mathbb{R}, \mathbb{R})$, $f(s) = F'(s)$ and $c > 0$. Also they proved the solution obtained is a ground state. In \cite{16}, Jeanjean and Le studied the
existence of solutions to the Schrödinger-Poisson-Slater equation

\begin{equation}
-\Delta u + \mu u - \gamma(|x|^{-1} \ast |u|^2)u - \beta |u|^{p-2}u, \quad x \in \mathbb{R}^3, \int_{\mathbb{R}^3} u^2 = c, \quad u \in H^1(\mathbb{R}^3),
\end{equation}

where $c > 0$, $\gamma \in \mathbb{R}$, $p \in (\frac{10}{3}, 6]$ and $\beta \in \mathbb{R}$. When $\gamma > 0$ and $\beta > 0$ and $p \in (\frac{10}{3}, 6]$, they proved that there exists a $c_1 > 0$ such that, for any $c \in (0, c_1)$, (1.8) admits two solutions $u_+ c$ and $u_- c$ which can be characterized respectively as a local minima and as a mountain pass critical point of the associated Energy functional restricted to the norm constraint.

In the case $\gamma > 0$ and $\beta < 0$, they proved that, for any $p \in (\frac{10}{3}, 6]$ and any $c > 0$, (1.8) admits a solution which is a global minimizer. When $\gamma < 0$, $\beta > 0$ and $p = 6$, they proved that (1.8) does not admit positive solutions.

Note that all the references about the existence of normalized solutions above are studied by the variational methods. Very recently, Luo etc in [27] applied the finite-dimensional reduction method to study the existence of normalized solutions for the Bose-Einstein condensates. Also in [31], Pellacci etc studied normalized solutions for a nonlinear elliptic system by the Lyapunov-Schmidt reduction.

To our best knowledge, up to now there are no results about the existence and the local uniqueness of peak solutions to (1.9), that is,

\begin{equation}
-\Delta u + P(x)u = \frac{a}{8\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy u + \mu u, \quad \text{in } \mathbb{R}^3,
\end{equation}

where $P(x)$ denotes a class of degenerate trapping potential which has non-isolated critical points. So in this paper, we are aimed to investigate these problems. Precisely, we study the existence and the local uniqueness of peak solutions for (1.1) with the following $L^2$ constraint

\begin{equation}
\int_{\mathbb{R}^3} u^2 = 1.
\end{equation}

We first recall the existence result for the ground state. Denote by $U(x)$ the unique positive solution of

\begin{equation}
-\Delta u + u = \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy u(x), \quad u \in H^1(\mathbb{R}^3).
\end{equation}

Form [23, 38], we know that $U(0) = \max_{x \in \mathbb{R}^3} U(x)$ and the solution $U(x)$ is strictly decreasing and

\[
\lim_{|x| \to \infty} U(0) e^{c|x|} = \lambda_0 > 0, \quad \lim_{|x| \to \infty} \frac{U'(x)}{U(x)} = -1,
\]

for some constant $\lambda_0 > 0$. Moreover, if $\phi(x) \in H^1(\mathbb{R}^3)$ solves the linearized equation

\[
-\Delta \phi(x) + \phi(x) = \frac{1}{8\pi} \left( \int_{\mathbb{R}^3} \frac{U^2(y)}{|x-y|} dy \right) \phi(x) + \frac{1}{4\pi} \left( \int_{\mathbb{R}^3} \frac{U(y) \phi(y)}{|x-y|} dy \right) U(x),
\]

then $\phi(x)$ is a linear combination of $\frac{\partial U}{\partial x_j}$, $j = 1, 2, 3$. 
In this paper, we let \( a_* = \int_{\mathbb{R}^3} U^2 \). First of all, we study the following problem without constraint:

\[
\begin{cases}
-\Delta w + (\lambda + P(x))w = \int_{\mathbb{R}^3} \frac{w^2(y)}{|x-y|} dy w, & \text{in } \mathbb{R}^3; \\
w \in H^1(\mathbb{R}^3),
\end{cases}
\]

where \( \lambda > 0 \) is a large parameter. It is well known that for large \( \lambda > 0 \), we can construct various positive solutions concentrating at some stable critical points of \( P(x) \). Particularly, we can construct positive single-peak solutions for (1.12) in the sense that \( w_{\lambda}(x) = \lambda \left( U(\sqrt{\lambda}(x - x_{\lambda})) + \varphi_{\lambda}(x) \right) \),

with \( \int_{\mathbb{R}^3} \left[ \frac{1}{\lambda} |\nabla \varphi_{\lambda}|^2 + \varphi_{\lambda}^2 \right] = o(\lambda^{-\frac{3}{2}}) \). Let \( u_{\lambda} = \frac{w_{\lambda}}{\left( \int_{\mathbb{R}^3} w_{\lambda}^2 \right)^{1/2}} \). Then \( \int_{\mathbb{R}^3} u_{\lambda}^2 = 1 \), and

\[
\begin{cases}
-\Delta u_{\lambda} + (\lambda + P(x))u_{\lambda} = a_{\lambda} \int_{\mathbb{R}^3} \frac{u_{\lambda}^2(y)}{|x-y|} dy u_{\lambda}(x), & \text{in } \mathbb{R}^3; \\
u_{\lambda} \in H^1(\mathbb{R}^3),
\end{cases}
\]

with

\[
a_{\lambda} = \int_{\mathbb{R}^3} w_{\lambda}^2 = \lambda^{2-\frac{2}{p}} \left( \int_{\mathbb{R}^3} U^2 + o(1) \right) = \lambda^{\frac{4}{p}} (a_* + o(1)).
\]

Note that \( a_{\lambda} > 0 \), and as \( \lambda \to +\infty \), so \( a_{\lambda} \to +\infty \). Therefore, we obtain a concentrated solution with single peak for (1.9)–(1.10) with \( \mu = -\lambda \) and suitable \( a_{\lambda} \). Now the crucial question is for any \( a > 0 \) large, whether we can choose a suitable large \( \lambda_a > 0 \), such that (1.9)–(1.10) holds with

\[
\mu = -\lambda_a, \quad u_a = \frac{w_{\lambda_a}}{\left( \int_{\mathbb{R}^3} w_{\lambda_a}^2 \right)^{1/2}}.
\]

The above discussions show that the existence of concentrated solutions for (1.9)–(1.10) is closely related to the existence of peaked solutions for the nonlinear Schrödinger equation (1.12). In this paper, we will mainly investigate concentrated solutions \( u_a \) of (1.9)–(1.10) in the sense that

\[
\max_{x \in B_\vartheta(b_0)} u_a(x) \to +\infty, \text{ while } u_a(x) \to 0 \text{ uniformly in } \mathbb{R}^3 \setminus B_\vartheta(b_0), \text{ for any } \vartheta > 0,
\]
as \( a \to +\infty \), where \( b_0 \) is a point in \( \mathbb{R}^3 \). Here, we are concerned with the following three aspects: possible values for \( \mu_a \); the exact location of the concentrated points of the concentrated solutions for (1.9)–(1.10); the existence and the local uniqueness of the concentrated solutions for (1.9)–(1.10).

The first result of our paper is as follows.

**Theorem 1.1.** Suppose that \( u_a \) is a solution of (1.9)–(1.10) concentrated at some points as \( a \to +\infty \). Then it holds

\[
\mu_a \to -\infty, \text{ as } a \to +\infty.
\]
Moreover, \( u_a \) satisfies
\[
(1.13) \quad u_a(x) = \frac{-\mu_a}{\sqrt{a}} \left( U\left( \sqrt{-\mu_a}(x - x_a) \right) + \varpi_a(x) \right),
\]
with
\[
\int_{\mathbb{R}^3} \left[ -\frac{1}{\mu_a} |\nabla \varpi_a|^2 + \varpi_a^2 \right] = o\left( \frac{1}{(-\mu_a)^{\frac{3}{2}}} \right).
\]

Throughout this paper, we call \( u_a \) a single-peak solution of (1.9)–(1.10) if \( u_a \) satisfies (1.13). To the best knowledge of us, if the critical point of \( P(x) \) is not isolated, not much is known for the exact location of the concentrated point, nor for the local uniqueness of the solutions. In the paper, we assume that \( P(x) \) obtains its local minimum or local maximum at \( \Gamma_i \) \((i = 1, \cdots, m)\) and \( \Gamma_i \) is a closed 2 dimensional hyper-surface satisfying \( \Gamma_i \cap \Gamma_j = \emptyset \) for \( i \neq j \). More precisely, we assume that the following conditions hold.

\( (P) \). There exist \( \delta > 0 \) and some \( C^2 \) compact hypersurfaces \( \Gamma_i \) \((i = 1, \cdots, m)\) without boundary, satisfying
\[
P(x) = P_i, \quad \frac{\partial P(x)}{\partial v_i} = 0, \quad \frac{\partial^2 P(x)}{\partial v_i^2} \neq 0, \quad \text{for any } x \in \Gamma_i \text{ and } i = 1, \cdots, m,
\]
where \( P_i \in \mathbb{R} \), \( v_i \) is the unit outward normal of \( \Gamma_i \) at \( x \in \Gamma_i \). Moreover, \( P(x) \in C^4 \left( \bigcup_{i=1}^{m} W_{\delta,i} \right) \) and \( P(x) = O(e^{\alpha|x|}) \) for some \( \alpha \in (0, 2) \). Here we denote \( W_{\delta,i} := \{ x \in \mathbb{R}^3, \text{dist}(x, \Gamma_i) < \delta \} \).

We would like to point out that the assumption \( (P) \) was first introduced in [27] by Luo etc. A specific example of \( P(x) \) was also given by [27]. Observe that the assumption \( (P) \) implies that \( P(x) \) obtains its local minimum or local maximum on the hypersurface \( \Gamma_i \) for \( i = 1, \cdots, m \). It is also easy to see that if \( \delta > 0 \) is small, the set \( \Gamma_{t,i} = \{ x : P(x) = t \} \bigcap W_{\delta,i} \) consists of two compact hypersurfaces in \( \mathbb{R}^3 \) without boundary for \( t \in [P_i - \theta, P_i] \) \( \left( \text{or } t \in [P_i - \theta, P_i] + \theta \right) \) provided \( \theta > 0 \) is small. Moreover, the outward unit normal vector \( v_{t,i}(x) \) and the \( j \)-th principal tangential unit vector \( \tau_{t,i,j}(x) (j = 1, 2) \) of \( \Gamma_{t,i} \) at \( x \) are Lip-continuous in \( W_{\delta,i} \).

**Remark 1.2.** When \( m = 1 \), for simplicity of notations we denote \( P_0 := P_i \) and omit all the subscript \( i \).

Applying the local Pohozaev identities, we can easily prove that a single peak solution of (1.9)–(1.10) must concentrate at a critical point of \( P(x) \), for which we can also refer to [12]. If the assumption \( (P) \) holds and the concentrated points belong to \( \Gamma \), we ask further where the concentrating points locate on \( \Gamma \). The following result gives our answer for this question.

**Theorem 1.3.** Assume that the assumption \( (P) \) holds. If \( u_a \) is a single-peak solution of (1.9)–(1.10), concentrating at \( \{ b_0 \} \) with \( b_0 \in \Gamma \) as \( a \) goes to \( \infty \), then
\[
(1.14) \quad (D_{\tau_j} \Delta V)(b_0) = 0, \quad \text{with } j = 1, 2.
\]
where \( \tau_j \) is the \( j \)-th principal tangential unit vector of \( \Gamma \) at \( b_0 \).
Theorem 1.3 implies that not every $\{b_0\}$ with $b_0 \in \Gamma$ can generate a single-peak solution for (1.9)–(1.10). In order to investigate the converse of Theorem 1.3, we have to add the following non-degenerate assumption on the critical point of $P(x)$. We say that $x_0 \in \Gamma$ is non-degenerate on $\Gamma$ if it satisfies:

$$\frac{\partial^2 P(x_0)}{\partial \nu^2} \neq 0 \text{ and } \det \left( \frac{\partial^2 P(x_0)}{\partial \tau_i \partial \tau_j} \right)_{1 \leq i, j \leq 2} \neq 0.$$ 

**Theorem 1.4.** Under the assumption $(P)$, if $b_0 \in \Gamma$ is a non-degenerate critical point of $P(x)$ on $\Gamma$ satisfying (1.14), then there exists a large constant $a_0 > 0$, such that (1.9)–(1.10) has a single-peak solution $u_a$ concentrating at $b_0$ as $a \in [a_0, +\infty)$.

The existence result in Theorem 1.4 is new even without the $L^2$-norm constraint since our potential includes the degenerate case. Also if $P(x)$ does not achieve its global minimum on $\Gamma$, any solution concentrating at a point on $\Gamma$ is not a ground state. So our existence result can not be obtained by the method in [8, 14, 20, 21].

To state the local uniqueness result of single-peak solutions for (1.9)–(1.10), we give another assumption $(\tilde{P})$ of $P(x)$: if $b_0$ is non-degenerate and

$$\left( \frac{\partial^2 P(b_0)}{\partial \tau_i \partial \tau_j} \right)_{1 \leq i, j \leq 2} + \frac{\partial^2 P(b_0)}{\partial \nu} \text{diag}(\kappa_1, \kappa_2)$$

is non-singular, where $\kappa_j$ is the $j$-th principal curvature of $\Gamma$ at $b_0$ for $j = 1, 2$.

Our second main result is the following.

**Theorem 1.5.** Suppose that the assumptions $(P)$ and $(\tilde{P})$ hold. Let $u_a^{(1)}(x)$ and $u_a^{(2)}(x)$ be two single-peak solutions of (1.9)–(1.10) concentrating at $b_0$ with $b_0 \in \Gamma$. Then there exists a large positive number $a_0$, such that $u_a^{(1)}(x) \equiv u_a^{(2)}(x)$ for all $a$ with $a_0 \leq a < +\infty$.

**Remark 1.6.** Our method can also be used to study the following Choquard equation with the dimensions $N = 4, 5$

\[
-\epsilon^2 \Delta u + P(x)u = \frac{a}{8\pi\epsilon^2} \left( \int_{\mathbb{R}^N} \frac{u^2(y)}{|x-y|^{N-2}} dy \right) u + \mu u, \quad x \in \mathbb{R}^N. 
\]

We would like to point out that in this case $a$ goes to

$$a^* = \int_{\mathbb{R}^4} U^2(x) dx, \quad N = 4,$$

$$0, \quad N = 5.$$ 

The main difference in the discussion of the local uniqueness for (1.9) and [26] is that the Lagrange multiplier $\mu_a$ in (1.9) also depends on the solution $u_a$. Hence its corresponding linearized operator has changed. For the details, see Lemma B.1. Fortunately, such change does not bring much difficulties.

To our knowledge, except [26], there seems to be no other local uniqueness results for peak (or bubbling) solutions of a Schrödinger-Newton system. Even for Schrödinger equations, there are very few results about the local uniqueness of peak solutions. The classical moving plane method is not still effective to study the uniqueness of concentrating solutions. There are two main tools to investigate the uniqueness of concentration solutions, i.e., the topological degree method and Pohozaev identities. Before the local uniqueness result of
multi-bump solutions in \cite{27}, other local uniqueness results for peak (or bubbling) solutions of Schrödinger equations are available only for the case when the solutions blow up at $x_0$, an isolated critical point of the potential $P(x)$. When $x_0$ is a non-degenerate critical point of $P(x)$, that is, $(D^2P)$ is non-singular at $x_0$, one can prove the local uniqueness of the peak solution concentrating at $x_0$ either by counting the local degree of the corresponding reduced finite dimensional problem as in \cite{1, 3, 11}, or by using Pohozaev type identities as in \cite{2, 9, 12, 13, 15}.

Compared with the topological degree method, it can deal with the degenerate potential to use the Pohozaev identities to prove the local uniqueness, one can refer to \cite{2, 9, 15}. We want to point out that in \cite{2, 9, 15}, though the critical point $x_0$ is degenerate, the rate of degeneracy along each direction is the same. Moreover, in \cite{12} Grossi gave an example which shows that local uniqueness may not be true at a degenerate critical point $x_0$ of $P(x)$. When peak solutions concentrates at a degenerate critical point, it is more subtle to study the local uniqueness of it. One can refer to \cite{27}. Under our assumption $(P)$, the potential $P(x)$ is non-degenerate along the normal direction $\nu$ of $\Gamma$. But along each tangential direction of $\Gamma$, $P(x)$ is degenerate. Such non-uniform degeneracy causes the estimates more complicated. Moreover, there are two terms involving volume integral in the corresponding local Pohozaev identity, which brings us some difficulties.

Finally, we give a non-existence of multi-peak solutions for (1.9)–(1.10).

\textbf{Theorem 1.7.} Under the assumption $(P)$, assume that $b_i \in \Gamma_i (i = 1, \ldots, m)$ are non-degenerate critical points of $P(x)$ on $\Gamma_i$ satisfying 
\[(D_{\tau_{i,j}} \Delta P)(b_i) = 0, \text{ with } i = 1, \cdots, m \text{ and } j = 1, 2,\]

where $\tau_{i,j}$ is the $j$-th principal tangential unit vector of $\Gamma$ at $b_i$. Then there exists a large constant $a_0 > 0$, such that problem (1.9)–(1.10) has no $m$-peaks solutions ($m \geq 2$) of this form
\begin{equation}
(1.16) \quad u_a(x) = \frac{-\mu_a}{\sqrt{a}} \sum_{i=1}^{m} \left( U\left( \sqrt{-\mu_a}(x - x_{a,i}) \right) + \omega_a(x) \right),
\end{equation}

with \(\int_{\mathbb{R}^3} \left[ -\frac{1}{\mu_a} |\nabla \omega_a|^2 + \omega_a^2 \right] = o\left( \frac{1}{(-\mu_a)^2} \right)\) and some points $x_{a,i} \in \mathbb{R}^3 (i = 1, \cdots, m)$ satisfying $x_{a,i} \to b_i, i = 1, \ldots, m$ as $a \to +\infty$ and $b_i \neq b_j$ for $i \neq j$.

\textbf{Remark 1.8.} The nonexistence result of multi-peak solutions for (1.9)–(1.10) is very different from the Schrödinger problem studied in \cite{27}, which is mainly caused by the nonlocal term.

To prove Theorem 1.7 we mainly use some local Pohozaev identities and some contradiction argument. We have to obtain some accurate estimates involved by the nonlocal term as \cite{26}.

This paper is organized as follows. We prove Theorem 1.1 in section 2. In section 3 we estimate the Lagrange multiplier $\mu_a$ in terms of $a$. In section 4 we prove the results for the location of the peaks and for the existence of single-peak solutions. We study the local
uniqueness of single-peak solutions in section 5. In section 6 we prove the nonexistence result of multi-peak solutions. We put some known results and some basic and technical estimates in Appendices A to C.

For simplicity, we use $|u|^q (2 \leq q \leq 6)$ to denote $\left( \int_{\mathbb{R}^3} |u(x)|^q dx \right)^{\frac{1}{q}}$ and $\|u\|$ the usual $H^1(\mathbb{R}^3)$ norm. In this paper, we always assume that $b_0 \in \Gamma$.

2. The proof of Theorem 1.1

First, we study the following problem:

$$\begin{align}
-\Delta u &= P_1(x)u, \quad u > 0, \quad \text{in } \mathbb{R}^3, \tag{2.1}
\end{align}$$

where the function $P_1(x)$ satisfies $P_1 > 1$ in $B_R(0) \setminus B_t(0)$ for some fixed $t > 0$ and large $R > 0$.

In order to prove Theorem 1.1, first we give the following result.

**Proposition 2.1.** (Proposition 2.1, [27]) Problem (2.1) has no solution.

With Proposition 2.1 at hand, now we are in a position to prove Theorem 1.1.

**Proof of Theorem 1.1** First, we prove that $\mu_a \to -\infty$. We argue by an indirect method. Suppose that $|\mu_a| \leq M$. Since $\int_{\mathbb{R}^3} u_a^2 = 1$, by the Moser iteration, we can prove that $u_a$ is uniformly bounded. That is, $u_a$ does not blow up.

Suppose that $\mu_a \to +\infty$. Set $P_1(x) = \mu_a - P(x) + \frac{a}{8\pi} \int_{\mathbb{R}^3} \frac{u_a^2(y)}{|x-y|} dy$. Noting that $u_a$ concentrates at some points, we have for $x \in \mathbb{R}^3 \setminus B_t(0)$ and $t > 0$

$$\begin{align}
\int_{\mathbb{R}^3} \frac{u_a^2(y)}{|x-y|} dy &= \int_{|x-y| \leq \frac{t}{2}} \frac{u_a^2(y)}{|x-y|} dy + \int_{|x-y| \geq \frac{t}{2}} \frac{u_a^2(y)}{|x-y|} dy \\
&= o(1) \int_{|x-y| \leq \frac{t}{2}} \frac{1}{|x-y|} dy + \left( \int_{|x-y| \geq \frac{t}{2}} |u_a(y)|^{20} dy \right)^{\frac{1}{20}} \left( \frac{1}{|x-y|^{\frac{20}{3}}} dy \right)^{\frac{3}{20}} \\
&= o(1)t^2 + O\left( \frac{1}{t^{10}} \right),
\end{align}$$

where $o(1)$ goes to zero as $a \to \infty$. Hence from (2.2) we may suppose that $\frac{a}{8\pi} \int_{\mathbb{R}^3} u_a^2(y) dy \geq -1$ in $\mathbb{R}^3 \setminus B_t(0)$ for some $t > 0$. Hence for any fixed $R > 0$, we usually have

$$P_1(x) = \mu_a - P(x) + \frac{a}{8\pi} \int_{\mathbb{R}^3} \frac{u_a^2(y)}{|x-y|} dy > 1, \quad x \in B_R(0) \setminus B_t(0).$$

From Proposition 2.1, we obtain a contradiction. So we have proved that $\mu_a \to -\infty$. Let $\lambda_a = -\mu_a$. Let $x_a$ be the maximum point of $u_a$. By equation (1.9), we find

$$\frac{a}{8\pi} \int_{\mathbb{R}^3} \frac{u_a^2(y)}{|x-y|} dy u_a(x) \geq (\lambda_a + P(x_a)) u_a(x_a) > 0,$$
which implies that \( a > 0 \).

Denote \( \bar{u}_a(x) = \frac{1}{\lambda_a} u_a\left(\frac{x}{\sqrt{\lambda_a}}\right) \). Then

\[
(2.3) \quad -\Delta \bar{u}_a + \left(1 + \frac{1}{\lambda_a} P\left(\frac{x}{\sqrt{\lambda_a}}\right)\right) \bar{u}_a = \frac{a}{8\pi} \int_{\mathbb{R}^3} \frac{\bar{u}_a^2(y)}{|x-y|} dy \bar{u}_a(x), \quad \text{in } \mathbb{R}^3,
\]

and

\[
(2.4) \quad \int_{\mathbb{R}^3} \bar{u}_a^2 = \frac{1}{\lambda_a^2}.
\]

By (2.3) and (2.4), applying Moser iteration, we can show that \(|u_a| \leq C\) for some constant independent of \( a \). Let \( \bar{x}_a \) be a maximum point of \( \bar{u}_a \). Then

\[
\frac{a}{8\pi} \int_{\mathbb{R}^3} \frac{\bar{u}_a^2(y)}{|\bar{x}_a - y|} dy \bar{u}_a(\bar{x}_a) \geq \left(1 + \frac{1}{\lambda_a} P\left(\frac{\bar{x}_a}{\sqrt{\lambda_a}}\right)\right) \bar{u}_a(\bar{x}_a),
\]

which gives \( a \geq 8\pi \left(\int_{\mathbb{R}^3} \frac{\bar{u}_a^2(y)}{|\bar{x}_a - y|} dy\right)^{-1} \geq c_0 > 0 \). Applying the standard blow-up argument, by (2.4) we can check that there holds

\[
(2.5) \quad \bar{u}_a = U_a(x - \bar{x}_a) + \varphi_a(x),
\]

for some \( \bar{x}_a \in \mathbb{R}^3 \) with

\[
\int_{\mathbb{R}^3} \left[|\nabla \varphi_a|^2 + (\varphi_a)^2\right] = o(1),
\]

and \( U_a \) is the unique positive solution of

\[
-\Delta u + u = \frac{a}{8\pi} \int_{\mathbb{R}^3} \frac{u^2(x)}{|x-y|} dy u(x) , \quad u \in H^1(\mathbb{R}^3), \quad u(0) = \max_{x \in \mathbb{R}^3} u(x).
\]

Observing that \( U_a = \frac{1}{\sqrt{\lambda_a}} U \), it follows from (2.4) and (2.5) that \( a \to a^* \lambda_a^\frac{1}{2} \to +\infty \). \( \square \)

3. Some estimates for general potentials

In this section, we shall estimate \( \mu_a \) respect to \( a \).

Let \( \epsilon = \frac{1}{\sqrt{-\mu_a}} \) and \( u(x) \mapsto \frac{\mu_a^{\epsilon}}{\sqrt{a}} u(x) \). Then (1.9) can be rewritten as

\[
(3.1) \quad -\epsilon^2 \Delta u + \left(1 + \epsilon^2 P(x)\right) u = \frac{1}{8\pi \epsilon^2} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy u(x) , \quad u \in H^1(\mathbb{R}^3).
\]

For any \( a \in \mathbb{R}^+ \), we define \( \|u\|_a := \int_{\mathbb{R}^3} \left(\epsilon^2 |\nabla u|^2 + u^2\right)^{\frac{1}{2}} \).

By (1.13), we know that a single-peak solution of (3.1) has the following form

\[
\bar{u}_a(x) = U_{\epsilon,x_a}(x) + \varphi_a(x), \quad \text{with } \|\varphi_a\|_a = o(\epsilon^\frac{3}{2}),
\]
where $U_{\epsilon,x_a}(x) := (1 + \epsilon^2 P_0)U\left(\frac{\sqrt{1 + \epsilon^2 P_0(x-x_a)}}{\epsilon}\right)$. Then, there holds

$$
-\epsilon^2 \Delta \varphi_a + \left((1 + \epsilon^2 P(x))\varphi_a - \frac{1}{4\pi\epsilon^2} \int_{\mathbb{R}^3} \frac{U_{\epsilon,x_a}(y)\varphi_a(y)}{|x-y|} dy U_{\epsilon,x_a}(x) \right)
$$

$$
- \frac{1}{8\pi\epsilon^2} \int_{\mathbb{R}^3} \frac{(U_{\epsilon,x_a}(y))^2}{|x-y|} dy \varphi_a(x) = \mathcal{R}_{a,\epsilon}(\varphi_a) + \mathcal{L}_a(x),
$$

where

$$
\mathcal{R}_{a,\epsilon}(\varphi_a) = \left(\frac{1}{8\pi\epsilon^2} \int_{\mathbb{R}^3} \frac{(U_{\epsilon,x_a} + \varphi_a)^2(y)}{|x-y|} dy (U_{\epsilon,x_a} + \varphi_a)(x) - \frac{1}{8\pi\epsilon^2} \int_{\mathbb{R}^3} \frac{(U_{\epsilon,x_a}(y))^2}{|x-y|} dy U_{\epsilon,x_a}(x) \right)
$$

$$
- \frac{1}{4\pi\epsilon^2} \int_{\mathbb{R}^3} \frac{(U_{\epsilon,x_a}\varphi_a)(y)}{|x-y|} dy U_{\epsilon,x_a}(x) - \frac{1}{8\pi\epsilon^2} \int_{\mathbb{R}^3} \frac{(U_{\epsilon,x_a}(y))^2}{|x-y|} dy \varphi_a(x)
$$

$$
= \frac{1}{4\pi\epsilon^2} \int_{\mathbb{R}^3} \frac{(U_{\epsilon,x_a}\varphi_a)(y)}{|x-y|} dy \varphi_a(x) + \frac{1}{8\pi\epsilon^2} \int_{\mathbb{R}^3} \frac{(\varphi_a)^2(y)}{|x-y|} dy U_{\epsilon,x_a}(x)
$$

$$
+ \frac{1}{8\pi\epsilon^2} \int_{\mathbb{R}^3} \frac{(\varphi_a)^2(y)}{|x-y|} dy \varphi_a(x),
$$

and

$$
\mathcal{L}_a(x) = -\epsilon^2 (P(x) - P_0) U_{\epsilon,x_a}(x).
$$

We can move $x_a$ a bit (still denoted by $x_a$), so that the error term $\varphi_a \in E_{a,x_a}$, where

$$
E_{a,x_a} := \left\{ u(x) \in H^1(\mathbb{R}^3) : \left\langle u, \frac{\partial U_{\epsilon,x_a}(x)}{\partial x_j} \right\rangle_a = 0, \ j = 1,2,3 \right\}.
$$

Let $L_a$ be the bounded linear operator from $H^1(\mathbb{R}^3)$ to itself, defined by

$$
\langle L_a u, v \rangle_a = \int_{\mathbb{R}^3} \left(\epsilon^2 \nabla u \nabla v + (1 + \epsilon^2 P(x)) u v \right) - \frac{3}{8\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)(uv)(y)}{|x-y|} dxdy.
$$

Then, it is standard to prove the following lemma.

**Lemma 3.1.** There exist constants $\varrho > 0$ and large $a_0 > 0$ such that for all $a$ with $a_0 \leq a < +\infty$, it holds

$$
\|L_a u\|_a \geq \varrho \|u\|_a, \text{ for all } u \in E_{a,x_a}.
$$

**Lemma 3.2.** There exists a constant $C > 0$ independent of $a$ such that

$$
\|\mathcal{R}_{a,\epsilon}(\varphi_a)\|_a = O\left(\epsilon^{-3}\|\varphi_a\|_a^3 + \epsilon^{-\frac{3}{2}}\|\varphi_a\|_a^2\right).
$$
Proof. From (3.3), for any \( v \in H^1(\mathbb{R}^3) \) we have

\[
(\mathcal{R}_{a,d}(\varphi_a), v) = \frac{1}{4\pi \epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\varphi_a(x)v(x)(U_{\epsilon,x_a}\varphi_a)(y)}{|x-y|} \, dx \, dy + \frac{1}{8\pi \epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U_{\epsilon,x_a}(x)v(x)(\varphi_a)^2(y)}{|x-y|} \, dx \, dy
\]

\[
= A_1 + \frac{1}{8\pi \epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\varphi_a(x)v(x)(\varphi_a)^2(y)}{|x-y|} \, dx \, dy.
\]

By Lemma A.2, we have

\[
A_1 \leq C \epsilon^{-1} \||\varphi_a||_a^\delta ||v||_a ||U_{\epsilon,x_a}||_a \leq C \epsilon^{\frac{1}{2}} ||\varphi_a||_a^2 ||v||_a, \quad A_2 \leq C \epsilon^{\frac{1}{2}} ||\varphi_a||_a^2 ||v||_a,
\]

and

\[
A_3 \leq C \epsilon^{-1} ||\varphi_a||_a^3 ||v||_a.
\]

It follows from (3.6) and (3.7) that (3.5) holds. \( \square \)

**Lemma 3.3.** There holds

\[
(\mathcal{L}_a, v) = O\left(\left(\left|P(x_a) - P_0\right)|\epsilon^{7/2} + \left|\nabla P(x_a)\right|\epsilon^{9/2} + \epsilon^{11/2}\right)\right).
\]

Proof. For any \( v \in H^1(\mathbb{R}^3) \), we have

\[
(\mathcal{L}_a(x), v) = -\epsilon^2 \int_{\mathbb{R}^3} (P(x) - P_0)U_{\epsilon,x_a}(x)v(x)dx
\]

\[
= -\epsilon^2 \int_{\mathbb{R}^3} (P(x) - P(x_a))U_{\epsilon,x_a}(x)v(x)dx - \epsilon^2 \int_{\mathbb{R}^3} (P(x_a) - P_0)U_{\epsilon,x_a}(x)v(x)dx.
\]

By Hölder inequality and the exponential of \( U(x) \) at infinity, we can estimate

\[
B_1 = \int_{B_\delta(x_a)} (P(x) - P(x_a))U_{\epsilon,x_a}(x)v(x)dx + \int_{B_\delta^c(x_a)} (P(x) - P(x_a))U_{\epsilon,x_a}(x)v(x)dx
\]

\[
= \int_{B_\delta(x_a)} (\nabla P(x_a) \cdot (x - x_a) + O(|x - x_a|)U_{\epsilon,x_a}(x)v(x)dx + C\left(\int_{B_\delta^c(x_a)} U_{\epsilon,x_a}^2(x)dx\right)^{\frac{1}{2}}|v|_2
\]

\[
\leq |\nabla P(x_a)|\left(\int_{B_\delta(x_a)} |x - x_a|U_{\epsilon,x_a}^2(x)dx\right)^{\frac{1}{2}}|v|_2 + C\left(\int_{B_\delta(x_a)} |x - x_a|U_{\epsilon,x_a}^2(x)dx\right)^{\frac{1}{2}}|v|_2
\]

\[
+ O(\epsilon^{\frac{6}{\epsilon}})||v||_a
\]

\[
= O(|\nabla P(x_a)|\epsilon^{\frac{2}{3}}||v||_a) + O(\epsilon^{\frac{2}{3}}||v||_a),
\]

where \( \delta \) is a constant such that \( \epsilon \delta \leq 1 \).
\[ B_2 = \int_{\mathbb{R}^3} \left( P(x_a) - P_0 \right) U_{\epsilon, x_a}(x) v(x) \, dx \]
\[ \leq |P(x_a) - P_0| \left( \int_{\mathbb{R}^3} U_{\epsilon, x_a}(x) \, dx \right)^{\frac{1}{2}} |v|_2 = O(|P(x_a) - P_0| \epsilon^2) \|v\|_a. \]

Combining all the estimates above, we know that (3.8) holds. \( \Box \)

With Lemmas 3.1 to 3.3 at hand, we can prove

**Lemma 3.4.** A single-peak solution \( \tilde{u}_a \) for (3.1) concentrating at \( b_0 \) has the following form

\[ \tilde{u}_a(x) = U_{\epsilon, x_a}(x) + \varphi_a(x), \]
with \( \varphi_a \in E_{a, x_a} \) and

\[ \| \varphi_a \|_a = O \left( |P(x_a) - P_0| \epsilon^2 + \| \nabla P(x_a) \| \epsilon^2 + \epsilon^{\frac{11}{2}} \right). \]

**Proof.** Then from (3.2), (3.4), (3.5), (3.8) and by applying the contraction mapping theorem, we can get (3.9) and (3.10) using the standard argument. \( \Box \)

Let \( \tilde{\varphi}_a(x) = \varphi_a(\epsilon x + x_a) \). Then, \( \tilde{\varphi}_a \) satisfies \( \| \tilde{\varphi}_a \|_a = O(\epsilon^2) \). Using the Moser iteration, we can prove \( \| \tilde{\varphi}_a \|_{L^\infty(\mathbb{R}^3)} = O(1) \). From this and the comparison theorem, similar to Proposition 2.2 in [26], we can prove the following estimates for \( \tilde{u}_a(x) \) away from the concentrated point \( b_0 \).

**Proposition 3.5.** Suppose that \( \tilde{u}_a(x) \) is a single-peak solution of (3.1) concentrating at \( b_0 \). Then for any fixed \( R \gg 1 \), there exist some \( \theta > 0 \) and \( C > 0 \), such that

\[ |\tilde{u}_a(x)| + |\nabla \tilde{u}_a(x)| \leq Ce^{-\theta|x-x_a|/\epsilon}, \quad \text{for } x \in \mathbb{R}^3 \setminus B_{R_0}(x_a). \]

**Lemma 3.6.** There holds

\[ \int_{\mathbb{R}^3} U^2 = \frac{3}{32 \pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U^2(x)U^2(y)}{|x-y|} \, dx \, dy. \]

**Proof.** It follows directly from the following two identities:

\[ \int_{\mathbb{R}^3} (|\nabla U|^2 + U^2) = \frac{1}{8 \pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U^2(x)U^2(y)}{|x-y|} \, dx \, dy \]

and

\[ \int_{\mathbb{R}^3} (|\nabla U|^2 + 3U^2) = \frac{5}{16 \pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U^2(x)U^2(y)}{|x-y|} \, dx \, dy, \]

where the last equality can be deduced by multiplying \( \langle x, \nabla U \rangle \) on both sides of (1.11) and integrating on \( \mathbb{R}^3 \). \( \Box \)

**Proposition 3.7.** Letting \( a \to +\infty \), there holds

\[ \frac{a}{\sqrt{-\mu_a}} = a_* - \frac{P_0}{2\mu_a} a_* + O \left( |P(x_a) - P_0| \left( \frac{1}{(-\mu_a)} + \| \nabla P(x_a) \| \left( \frac{1}{\sqrt{-\mu_a}} \right)^2 + \frac{1}{\mu_a} \right) \right). \]
Proof. From (1.9) and (1.10), we have

\[ 1 = \int_{\mathbb{R}^3} (u_a(x))^2 = \int_{\mathbb{R}^3} \left( \bar{U}_{a,x_a}(x) + \frac{\mu_0}{\sqrt{a}} \varphi_a(x) \right)^2, \]

where \( \bar{U}_{a,x_a}(x) = \frac{\mu_0 + P_0}{\sqrt{a}} \mathbf{U}(\sqrt{-\mu_a} + P_0(x - x_a)) \). By direct computation, we can obtain

\[ a = a_\ast \sqrt{-\mu_a} + P_0 + O \left( \left| \mathbf{P}(x_a) - P_0 \right| \frac{1}{\sqrt{-\mu_a}} + \left| \nabla \mathbf{P}(x_a) \right| \frac{1}{(\mu_a)^{1/2}} + \frac{1}{(\sqrt{-\mu_a})^{3/2}} \right), \]

which implies that (3.13) is true. \( \square \)

4. Locating the peak and the existence of single-peak solutions

First, we locate the peak for a single-peak solution. Let \( \tilde{u}_a \) be a single-peak solution of (3.1). Then for any small fixed \( \rho > 0 \), from (B.4) and (C.1) we find

\[ \begin{align*}
\epsilon^2 \int_{B_\rho(x_a)} & \left( \frac{\partial P(x)}{\partial x_j} \right)^2 \tilde{u}_a^2 dx \\
&= -2\epsilon^2 \int_{\partial B_\rho(x_a)} \frac{\partial \tilde{u}_a}{\partial \nu} \frac{\partial \tilde{u}_a}{\partial x_j} d\sigma + \epsilon^2 \int_{\partial B_\rho(x_a)} |\nabla \tilde{u}_a|^2 \tilde{v}_j(x) d\sigma \\
&\quad + \int_{\partial B_\rho(x_a)} \left( 1 + \epsilon^2 P(x) \right) \tilde{u}_a^2 \tilde{v}_j(x) d\sigma - \frac{1}{8\pi \epsilon^2} \int_{\partial B_\rho(x_a)} \int_{\mathbb{R}^3} \frac{\tilde{u}_a^2}{|x - y|^2} dy d\tilde{v}_j(x) d\sigma \\
&\quad - \frac{1}{8\pi \epsilon^2} \int_{B_\rho(x_a)} \int_{\mathbb{R}^3} \frac{(\tilde{u}_a(y))^2 (\tilde{u}_a(x))^2 (x_j - y_j)}{|x - y|^3} dy dx \\
&= O(e^{-\theta}), \text{ with some } \theta > 0,
\end{align*} \]

where \( j = 1, 2, 3 \) and \( \tilde{v}(x) = (\tilde{v}_1(x), \tilde{v}_2(x), \tilde{v}_3(x)) \) is the outward unit normal of \( \partial B_\rho(x_a) \). And then (1.11) implies the first necessary condition for the concentrated point \( b_0: \nabla \mathbf{P}(b_0) = 0 \).

Now we are in a position to prove Theorem 1.3.

**Proof of Theorem 1.3** Since \( x_a \to b_0 \in \Gamma \), we find that there is a \( t_a \in [P_0, P_0 + \sigma] \) if \( \Gamma \) is a local minimum set of \( \mathbf{P}(x) \), or \( t_a \in [P_0 - \sigma, P_0] \) if \( \Gamma \) is a local maximum set of \( \mathbf{P}(x) \), such that \( x_a \in \Gamma_{t_a} \). Let \( \tau_a \) be the unit tangential vector of \( \Gamma_{t_a} \) at \( x_a \). Then

\[ G(x_a) = 0, \text{ where } G(x) = \langle \nabla \mathbf{P}(x), \tau_a \rangle. \]

We have the following expansion:

\[ \begin{align*}
G(x) &= \langle \nabla G(x_a), x - x_a \rangle + \frac{1}{2} \langle \nabla^2 G(x_a), x - x_a \rangle, x - x_a \rangle + o(|x - x_a|^2), \text{ for } x \in B_\rho(x_a).
\end{align*} \]
Then it follows from (3.10), (4.1) and the above expansion that
\[
\int_{\mathbb{R}^3} G(x)U^2_{\epsilon,x_a}(x) = \int_{B_\rho(x_a)} G(x)U^2_{\epsilon,x_a}(x) + O(e^{-\frac{\rho}{\epsilon}})
\]
(4.2)
\[
= -2 \int_{B_\rho(x_a)} G(x)U_{\epsilon,x_a}(x)\varphi_a - \int_{B_\rho(x_a)} G(x)\varphi_a^2 + O(e^{-\frac{\rho}{\epsilon}})
\]
\[
= O\left(\xi^2 |\nabla G(x_a)| + \xi^2 \|\varphi_a\|_a + \xi|\nabla G(x_a)| \cdot \|\varphi_a\|^2_a + O(e^{-\frac{\rho}{\epsilon}})\right)
\]
\[
= O(\epsilon^6).
\]
On the other hand, noting that $G(x_a) = 0$, it is easy to check
\[
\int_{\mathbb{R}^3} G(x)U^2_{\epsilon,x_a}(x) = \frac{1}{6}\epsilon^5 \Delta G(x_a) \int_{\mathbb{R}^3} |\xi|^2 U^2 + O(\epsilon^7).
\]
(4.3)
Then it follows from (4.2) and (4.3) that $(\Delta G)(x_a) = O(\epsilon)$. Thus by the assumption $(P)$, we get (4.4).

Now, we study the existence of single-peak solutions for (1.12) with $\lambda > 0$ a large parameter. Letting $\eta = \frac{1}{\sqrt{x}}$ and $w(x) \mapsto \lambda w(x)$, then (1.12) can be changed to the following problem:

\[
-\eta^2 \Delta w + (1 + \eta^2 P(x))w = \frac{1}{8\pi \eta^2} \int_{\mathbb{R}^3} \frac{w^2}{|x-y|} dy w, \; w \in H^1(\mathbb{R}^3).
\]
(4.4)

In the sequel, we denote $\langle u, v \rangle_\eta = \int_{\mathbb{R}^3} (\eta^2 \nabla u \nabla v + uv)$ and $\|u\|_\eta = \langle u, u \rangle_\eta^{\frac{1}{2}}$. Now for $\eta > 0$ small, we construct a single-peak solution $u_\eta$ of (1.12) concentrating at $b_0$. Here we can prove the following result in a standard way.

**Proposition 4.1.** There is an $\eta_0 > 0$, such that for any $\eta \in (0, \eta_0]$, and $z_0$ close to $b_0$, there exists $u_{\eta,z_0} \in F_{\eta,z_0}$ such that

\[
\int_{\mathbb{R}^3} (\eta^2 \nabla w_{\eta} \nabla \psi + (1 + \eta^2 P(x)) w_{\eta} \psi) = \frac{1}{8\pi \eta^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w_{\eta}^2(y)}{|x-y|} w_{\eta}(x) \psi(x) dy dx, \; \text{for all } \psi \in F_{\eta,z_0},
\]
where
\[
w_{\eta}(x) = U_{\eta,z_0}(x) + \varphi_{\eta,z_0}(x)
\]
and
\[
F_{\eta,z_0} = \left\{ u(x) \in H^1(\mathbb{R}^3) : \langle u, \frac{\partial U_{\eta,z_0}(x)}{\partial x_j} \rangle_\eta = 0, \; j = 1, 2, 3 \right\}.
\]
Moreover, it holds
\[
\|\varphi_{\eta,z_0}\|_\eta = O\left(\|P(z_0) - P_0\|_{\eta^2} + \|\nabla P(z_0)\|_{\eta^2} + \eta^2 \right).
\]
(4.5)

To obtain a true solution for (4.4), we have to choose $z_0$ such that

\[
\int_{B_\rho(x_a)} (-\eta^2 \Delta w_{\eta} \frac{\partial w_{\eta}}{\partial x_j} + (1 + \eta^2 P(x)) w_{\eta} \frac{\partial w_{\eta}}{\partial x_j}) = \frac{1}{8\pi \eta^2} \int_{\mathbb{R}^3} \frac{w_{\eta}^2(y)}{|x-y|} dy w_{\eta} \frac{\partial w_{\eta}}{\partial x_j}) = 0, \; j = 1, 2, 3.
\]
Similar to (4.1), it is easy to check that the above identities are equivalent to

\[
(4.7) \quad \int_{B_\rho(x_0)} \frac{\partial P(x)}{\partial x_j} u^2_\eta = O\left(e^{-\frac{\eta}{5}}\right), \quad \forall \, j = 1, 2, 3.
\]

For \( z_0 \) close to \( b_0 \), and \( z_0 \in \Gamma_t \) for some \( t \) close to \( P_0 \), now we use \( \nu \) to denote the unit normal vector of \( \Gamma_t \) at \( z_0 \), while we use \( \tau_j \) (\( j = 1, 2 \)) to denote the principal directions of \( \Gamma_t \) at \( x_0 \). Then, at \( z_0 \), it holds

\[
D_{\tau_j} P(z_0) = 0, \quad \text{for } j = 1, 2, \text{ and } |\nabla P(z_0)| = |D_{\nu} P(z_0)|.
\]

We first prove the following result.

**Lemma 4.2.** Under the assumption (P), \( \int_{B_\nu(x_0)} D_{\nu} P(x) u^2_\eta = O\left(e^{-\frac{\eta}{5}}\right) \) is equivalent to

\[
(4.8) \quad D_{\nu} P(z_0) = O\left(\eta^2\right).
\]

**Proof.** First, from (4.5) we have

\[
\int_{\mathbb{R}^3} D_{\nu} P(x) U_{\eta,z_0}^2(x) = -2 \int_{\mathbb{R}^3} D_{\nu} P(x) U_{\eta,z_0}(x) \varphi_{\eta,z_0} - \int_{\mathbb{R}^3} D_{\nu} P(x) \varphi_{\eta,z_0}^2 + O\left(e^{-\frac{\eta}{5}}\right),
\]

\[
= O\left(|D_{\nu} P(z_0)| \eta^2 \cdot \| \varphi_{\eta,z_0} \|_\eta + \eta^2 \| \varphi_{\eta,z_0} \|_\eta + \| \varphi_{\eta,z_0} \|_\eta^2 \right) = O\left(\eta^5\right).
\]

On the other hand, we have

\[
(4.10) \quad \int_{\mathbb{R}^3} D_{\nu} P(x) U_{\eta,z_0}^2(x) = a_s \eta^3 D_{\nu} P(z_0) + O\left(\eta^5\right).
\]

Then it follows from (4.9) and (4.10) that (4.8) holds.

**Lemma 4.3.** Under the assumption (P), \( \int_{B_\nu(x_0)} D_{\tau} P(x) u^2_\eta = O\left(e^{-\frac{\eta}{5}}\right) \) is equivalent to

\[
(4.11) \quad (D_{\tau} \Delta P)(z_0) = O\left(\|P(z_0) - P_0\|_\eta + \eta^2\right).
\]

**Proof.** Let \( G(x) = \langle \nabla P(x), \tau \rangle \). Then, similar to the estimate (4.2), by (4.6) we have

\[
(4.12) \quad \int_{\mathbb{R}^3} G(x) U_{\eta,z_0}^2(x) = -2 \int_{\mathbb{R}^3} G(x) U_{\eta,z_0}(x) \varphi_{\eta,z_0} - \int_{\mathbb{R}^3} G(x) \varphi_{\eta,z_0}^2 + O\left(e^{-\frac{\eta}{5}}\right)
\]

\[
= O\left(\|P(z_0) - P_0\|_5 + \eta^5\right).
\]

On the other hand, in view of \( G(z_0) = 0 \), it is easy to show

\[
(4.13) \quad \int_{\mathbb{R}^3} G(x) U_{\eta,z_0}^2(x) = \frac{1}{2} \eta^5 \Delta G(z_0) B + O\left(\eta^7\right),
\]

where

\[
(4.14) \quad B = \frac{1}{3} \int_{\mathbb{R}^3} |x|^2 U^2.
\]
Thus, from (1.12) and (1.13) we can obtain (1.11). □

**Theorem 4.4.** For \( \lambda > 0 \) large, (1.12) has a solution \( u_\lambda \) satisfying

\[
u(x) = \lambda \left( U \left( \sqrt{\lambda} (x - x_\lambda) \right) + \varphi_\lambda \right),
\]

where \( x_\lambda \to b_0 \) and \( \int_{\mathbb{R}^3} \left( |\nabla \varphi_\lambda|^2 + \varphi_\lambda^2 \right) \to 0 \) as \( \lambda \to +\infty \).

**Proof.** As pointed out earlier, we need to solve (1.7). By Lemmas 4.2 and 4.3, the equation (4.7) is equivalent to

\[
D_\nu P(z_0) = O(\eta^2), \quad (D_\nu \Delta P)(z_0) = O\left( |P(z_0) - P_0| \eta + \eta^2 \right).
\]

Let \( \tilde{z}_0 \in \Gamma \) be the point such that \( z_0 - \tilde{z}_0 = \alpha_0 \nu \) for some \( \alpha_0 \in \mathbb{R} \). Then, we have \( D_\nu P(\tilde{z}_0) = 0 \). As a result,

\[
D_\nu P(z_0) = D_\nu P(z_0) - D_\nu P(\tilde{z}_0) = D_\nu^2 P(z_0)(z_0 - \tilde{z}_0, \nu) + O(|z_0 - \tilde{z}_0|^2).
\]

By the non-degenerate assumption, we find that \( D_\nu P(z_0) = O(\eta^2) \) is equivalent to \( (z_0 - \tilde{z}_0, \nu) = O(\eta^2 + |z_0 - \tilde{z}_0|^2) \). This means that \( D_\nu P(z_0) = O(\eta^2) \) can be written as

(4.15) \[ |z_0 - \tilde{z}_0| = O(\eta^2). \]

Let \( \tilde{\tau}_j \) be the \( j \)-th tangential unit vector of \( \Gamma \) at \( \tilde{z}_0 \). Now by the assumption \( (P) \), we have

\[
(D_\tau_\nu \Delta P)(z_0) = (D_\tau_\nu \Delta P)(\tilde{z}_0) + O(|z_0 - \tilde{z}_0|) = (D_\tau_\nu \Delta P)(\tilde{z}_0) + O(\eta^2),
\]

and

\[
(D_\tau_\nu \Delta P)(\tilde{z}_0) = (D_\tau_\nu \Delta P)(\tilde{z}_0) - (D_{\tau_j,0} \Delta P)(b_0) = \langle (\nabla_T D_{\tau_j,0} \Delta P)(b_0), \tilde{z}_0 - b_0 \rangle + O|\tilde{z}_0 - b_0|^2,
\]

where \( \nabla_T \) is the tangential gradient on \( \Gamma \) at \( b_0 \in \Gamma \), and \( \tau_{j,0} \) is the \( j \)-th tangential unit vector of \( \Gamma \) at \( b_0 \). Therefore, \( (D_\tau_\nu \Delta P)(z_0) = O\left( |P(z_0) - P_0| \eta + \eta^2 \right) \) can be rewritten as

(4.16) \[ \langle (\nabla_T D_{\tau_j,0} \Delta P)(b_0), \tilde{z}_0 - b_0 \rangle = O(\eta^2 + |\tilde{z}_0 - b_0|^2). \]

So we can solve (4.15) and (4.16) to obtain \( z_0 = x_{n,0} \) with \( x_{n,0} \to b_0 \) as \( \eta \to 0 \). □

**Proof of Theorem 1.4** Let \( w_\lambda \) be a single-peak solution as in Theorem 4.4 and we define

\[
u_\lambda = \frac{w_\lambda}{\left( \int_{\mathbb{R}^3} w_\lambda^2 \right)^{1/2}}.
\]

Then \( \int_{\mathbb{R}^3} u_\lambda^2 = 1 \), and

\[
-\Delta u_\lambda + P(x)u_\lambda = \frac{a_\lambda}{8\pi} \int_{\mathbb{R}^3} \frac{u_\lambda^2(y)}{|x - y|} dy u_\lambda(x) - \lambda u_\lambda, \quad \text{in } \mathbb{R}^3,
\]

with \( a_\lambda = \int_{\mathbb{R}^3} w_\lambda^2 \).
Similar to (3.13), we can prove
\[
\frac{1}{\sqrt{\lambda}} \int_{\mathbb{R}^3} w_\lambda^2 = a_* + o(1), \quad \text{as } a \to +\infty.
\]
Take \( \lambda_0 > 0 \) large and let \( a_0 = \int_{\mathbb{R}^3} w_{\lambda_0}^2 \). For any \( a > 0 \), let \( f(\lambda) = \int_{\mathbb{R}^3} w_\lambda^2 - a \). Then for \( a \geq a_0 \), we have
\[
f(\lambda_0) = a_0 - a \leq 0 \quad \text{and} \quad \lim_{\lambda \to +\infty} f(\lambda) = \lim_{\lambda \to +\infty} \left( \sqrt{\lambda}(a_* + o(1)) - a \right) = +\infty.
\]
Hence by the continuity of the function \( f(\lambda) \), for any \( a \geq a_0 \), there exists \( \lambda = \lambda_a > 0 \) large such that \( f(\lambda_a) = 0 \), i.e. \( \int_{\mathbb{R}^3} w_\lambda^2 = a \), which yields that there exists \( \lambda = \lambda_a > 0 \) large such that the solution \( u_a \) of (1.12) with \( \lambda = \lambda_a \) satisfies \( \int_{\mathbb{R}^3} w_\lambda^2 = a \). Thus, for such \( a \), we obtain a single-peak solution for (1.9), where \( \mu_a = -\lambda_a \).

\[\square\]

5. Local uniqueness of single peak solutions

From Lemma 3.4, a single-peak solution \( \tilde{u}_a \) to (3.1) can be written as
\[
\tilde{u}_a(x) = U_{\epsilon,x_a} + \varphi_a(x),
\]
with \( |x_a - b_0| = o(1) \), \( \epsilon = \frac{1}{\sqrt{-\mu_a}} \), \( \varphi_a \in E_{a,x_a} \) and
\[
\|\varphi_a\|_a = O\left(\|P(x_a) - P_0\|^{\frac{3}{2}} + \|\nabla P(x_a)\|^{\frac{3}{2}} + \epsilon^{\frac{3}{2}} + \epsilon^{\frac{11}{2}}\right).
\]
Also we know \( x_a \in \Gamma_{t_a} \) for some \( t_a \to P_0 \). Similar to the last section, we use \( \nu_a \) to denote the unit normal vector of \( \Gamma_{t_a} \) at \( x_a \), while we use \( \tau_{a,j} \) to denote the principal direction of \( \Gamma_{t_a} \) at \( x_a \). Then, at \( x_a \), it holds
\[
D_{\tau_{a,j}} P(x_a) = 0, \quad |\nabla P(x_a)| = |D_{\nu_a} P(x_a)|.
\]

We first prove the following result.

**Lemma 5.1.** Under the assumption \((P)\), we have
\[
D_{\nu_a} P(x_a) = O(\epsilon^2).
\]

**Proof.** We use (4.1) to obtain
\[
\int_{B_\rho(x_a)} D_{\nu_a} P(x) \tilde{u}_a^2 = O(\epsilon^{-\frac{4}{7}}).
\]
Proof. It follows from (5.6) and (5.7) that
\[
\int_{B_\rho(x_0)} D_{\nu_a} P(x) U^2_{\epsilon,x_0} \quad (5.10)
\]
then (5.2) and (5.10) imply
\[
\tau - \phi_a = O\left( |\varphi_a| + \|\varphi_a\|^2 \right) + O\left( e^{-\frac{\rho}{\epsilon}} \right) \quad (5.9)
\]
On the other hand, by Taylor’s expansion, we have
\[
\int_{B_\rho(x_0)} D_{\nu_a} P(x) U^2_{\epsilon,x_0} = \epsilon^3 \left[ a_s D_{\nu_a} P(x_0) + \frac{B\epsilon^2}{2} \Delta D_{\nu_a} P(x_0) + O(\epsilon^4) \right] \quad (5.7)
\]
where \( B \) is the constant in (4.14). And then (5.4) follows from (5.6) and (5.7). \( \square \)

Let \( \bar{x}_a \in \Gamma \) be the point such that \( x_a - \bar{x}_a = \beta_a \nu_a \) for some \( \beta_a \in \mathbb{R} \). Then we can prove

**Lemma 5.2.** If the assumption (P) holds, then we have

\[
\begin{cases}
\bar{x}_a - b_0 = Le^2 + O(\epsilon^4), \\
x_a - \bar{x}_a = -\frac{B}{2a_s} \frac{\partial \Delta P(b_0)}{\partial \nu} \left( \frac{\partial^2 P(b_0)}{\partial \nu^2} \right)^{-1} \epsilon^2 + O(\epsilon^4),
\end{cases}
\]

where \( B \) is the constant in (4.14) and \( L \) is a vector depending on \( b_0 \).

**Proof.** It follows from (5.6) and (5.7) that

\[
(a_s + O(\epsilon^2)) D_{\nu_a} P(x_0) + \frac{B\epsilon^2}{2} \Delta D_{\nu_a} P(x_0) = O(\epsilon^4 + \epsilon^2|P(x_0) - P_0|) = O(\epsilon^4 + \epsilon^2|x_a - \bar{x}_a|^2).
\]

Since \( \frac{\partial^2 P(b_0)}{\partial \nu^2} \neq 0 \), the outward unit normal vector \( \nu_a(x) \) and the tangential unit vector \( \tau_a(x) \) of \( \Gamma_{\nu_a} \) at \( x_a \) are Lip-continuous in \( W_\delta \), from (5.9), we find

\[
x_a - \bar{x}_a = -\frac{B}{2a_s} \left( \Delta D_{\nu} P(b_0) \right) \left( \frac{\partial^2 P(b_0)}{\partial \nu^2} \right)^{-1} \epsilon^2 + O(\epsilon^4 + \epsilon^2|\bar{x}_a - b_0|^2).
\]

Then (5.2) and (5.10) implies

\[
\|\varphi_a\|_a = O\left( |x_a - \bar{x}_a|^2 e^\frac{\rho}{\epsilon} + e^\frac{\delta}{\epsilon} \right) = O\left( e^\frac{\delta}{\epsilon} \right).
\]
Recall that $G(x) = \langle \nabla P(x),\tau_a \rangle$. Then $G(x_a) = 0$. Similar to (4.12) and (5.6), we have
\begin{equation}
\int_{B_\rho(x_a)} G(x)U^2_{\epsilon,x_a} = -2 \int_{B_\rho(x_a)} G(x)U_{\epsilon,x_a}\varphi_a - \int_{B_\rho(x_a)} G(x)\varphi_a^2 + O(e^{-\frac{\rho}{\epsilon}}) \\
= -2 \int_{B_\rho(x_a)} G(x)U_{\epsilon,x_a}\varphi_a + O(\|\varphi_a\|_a^2) + O(e^{-\frac{\rho}{\epsilon}}) \\
= -2 \int_{B_\rho(x_a)} \left\langle \nabla G(x_a), x - x_a \right\rangle U_{\epsilon,x_a}\varphi_a + O(\epsilon^9).
\end{equation}
(5.12)
On the other hand, in view of $\nabla P(x) = 0$, $x \in \Gamma$, we find
\begin{equation}
\int_{B_\rho(x_a)} \left\langle \nabla G(x_a), x - x_a \right\rangle U_{\epsilon,x_a}\varphi_a = O(\epsilon^{10} \|\nabla G(x_a)\|_a \|\varphi_a\|_a) = O(\|x_a - \bar{x}_a\|) = O(\epsilon^{10}).
\end{equation}
(5.13)
Then by (5.12) and (5.14), we find
\begin{equation}
\int_{B_\rho(x_a)} G(x)U^2_{\epsilon,x_a} = O(\epsilon^9).
\end{equation}
(5.15)
On the other hand, by the Taylor’s expansion, we can prove
\begin{equation}
\int_{B_d(x_a)} G(x)U^2_{\epsilon,x_a} = \frac{B_{d5}}{2}(1 + P_0\epsilon^2)^{\frac{1}{2}}(D_{\tau_a}\Delta P)(x_a) + \frac{H_{r}\epsilon^7}{24} + O(\epsilon^9),
\end{equation}
(5.16)
where
\[H_{r_i} = \sum_{l=1}^{2} \sum_{m=1}^{2} \frac{\partial^4 G(b_0)}{\partial x_l^2 \partial x_m^2} \int_{\mathbb{R}^N} x_l^2 x_m^2 U^2.\]
So (5.15) and (5.16) give
\begin{equation}
(D_{\tau_a}\Delta P)(x_a) = -\frac{H_r \epsilon^2}{12B} + O(\epsilon^4).
\end{equation}
(5.17)
We denote by $\bar{\tau}_a$ the tangential vector of $\Gamma$ at $\bar{x}_a$. Then by (5.10), we get
\begin{align*}
(D_{\tau_a}\Delta P)(x_a) &= (D_{\tau_a}\Delta P)(\bar{x}_a) + \langle A_{\tau}, x_a - \bar{x}_a \rangle + O(|x_a - \bar{x}_a|^2) \\
&= (D_{\tau_a}\Delta P)(\bar{x}_a) + B_r \epsilon^2 + O(\epsilon^4),
\end{align*}
where $A_{\tau}$ is a vector depending on $b_0$ and $B_r$ is a constant depending on $b_0$. Moreover,
\begin{equation}
(D_{\tau_a}\Delta P)(\bar{x}_a) = \left( D_{\tau}^2(\Delta P)(b_0) \right)(\bar{x}_a - b_0) + O(|\bar{x}_a - b_0|^2).
\end{equation}
(5.18)
Therefore, from (5.17)–(5.18), we find
\begin{equation}
D_{\tau_a}^2(\Delta P)(b_0)(\bar{x}_a - b_0) = -\left( \frac{H_r}{12B} + B_r \right) \epsilon^2 + O(\epsilon^4) + O(|\bar{x}_a - b_0|^2).
\end{equation}
(5.19)
Since $D^2_t(\Delta P)(b_0)$ is non-singular, we can complete the proofs of (5.8) from (5.10) and (5.19).

Let

$$\delta_a := \frac{a_x}{a}.$$

**Proposition 5.3.** Under the assumption (P), there holds

$$-\mu_0 \delta_a^2 = 1 + \gamma_1 \delta_a^2 + O(\delta_a^4),$$

and

$$x_a - b_0 = \tilde{L} \delta_a^2 + O(\delta_a^4),$$

where $\gamma_1$ and the vector $\tilde{L}$ are constants.

**Proof.** First, (3.13) shows that (5.20) holds. Then we can find (5.21) by (5.8) and (5.20).

Let $u(x) \mapsto a^{-\frac{1}{2}} \delta_a^2 u(x)$. Then the problem (1.9)–(1.10) can be changed into the following problem

$$-\delta_a^2 \Delta u + (-\mu_0 \delta_a^2 + \delta_a^2 P(x)) u = \frac{1}{8 \pi \delta_a^2} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy u(x), \ u \in H^1(\mathbb{R}^3),$$

and

$$\int_{\mathbb{R}^3} u^2 = a \delta_a^4.$$

Then similar to Lemma 3.4 the single-peak solution of (5.22)–(5.23) concentrating at $b_0$ can be written as $\tilde{U}_{\delta_a,x_a} \varphi_a(x)$, with $|x_a - b_0| = o(1)$, $\|\varphi_a\|_{\delta_a} = o(\delta_a^3)$, and

$$\varphi_a \in E_{a,x_a} := \left\{ \varphi \in H^1(\mathbb{R}^3) : \left< v, \frac{\partial \tilde{U}_{\delta_a,x_a}}{\partial x_j} \right>_{\delta_a} = 0, \ j = 1, 2, 3 \right\},$$

where $\tilde{U}_{\delta_a,x_a} := (1 + (\gamma_1 + P_0) \delta_a^2) U \left( \frac{\sqrt{1 + (\gamma_1 + P_0) \delta_a^2 (x-x_a)}}{\delta_a} \right)$, $\|\varphi\|_{\delta_a}^2 := \int_{\mathbb{R}^3} (\delta_a^2 |\nabla \varphi|^2 + \varphi^2)$ and $\gamma_1$ is the constant in (5.20). Then we can write the equation (5.22) as follows:

$$\tilde{L}_a(\tilde{\varphi}_a) = \mathcal{R}_{a,\delta_a}(\tilde{\varphi}_a) + \tilde{L}_a(x),$$

where $\mathcal{R}_{a,\delta_a}$ is defined by (3.3),

$$\tilde{L}_a(\tilde{\varphi}_a) := -\delta_a^2 \Delta \tilde{\varphi}_a + \left( (-\mu_0 \delta_a^2 + \delta_a^2 P(x)) \tilde{\varphi}_a - \frac{1}{4 \pi \delta_a^2} \int_{\mathbb{R}^3} \frac{\tilde{U}_{\delta_a,x_a}(y) \tilde{\varphi}_a(y)}{|x-y|} dy \tilde{U}_{\delta_a,x_a}(x) \right)$$

$$- \frac{1}{8 \pi \delta_a^2} \int_{\mathbb{R}^3} \frac{(\tilde{U}_{\delta_a,x_a}(y))^2}{|x-y|} dy \tilde{\varphi}_a(x)$$

and

$$\tilde{\mathcal{L}}_a = - \sum_{i=1}^{m} \left( -\mu_0 \delta_a^2 - (1 + \gamma_1 \delta_a^2) + (P(x) - P_0) \delta_a^2 \right) \tilde{U}_{\delta_a,x_a}.$$
Lemma 5.4. There holds
\begin{equation}
\|\tilde{\varphi}_a\|_{\delta_a} = O\left(\delta_a^\frac{11}{10}\right).
\end{equation}

Proof. The proofs are similar to that of Lemma 3.4, the difference is
\begin{equation}
\|\tilde{L}_a\|_{\delta_a} = O\left(\|P(x_a) - P_0\|\delta_a^\frac{9}{10} + |\nabla P(x_a)| \delta_a^\frac{9}{10} + \delta_a^{\frac{11}{10}}\right) = O\left(\delta_a^\frac{11}{10}\right).
\end{equation}

Similar to Lemma 3.1, we can also check that \(\tilde{L}_a\) is invertible in \(\tilde{E}_{a,x_a}\). Finally, (5.25) and the contradiction mapping theorem imply (5.24).

For simplicity of notations, hereafter we denote \(p_0 := 1 + (\gamma_1 + P_0)\delta_a^3\). Hence
\[\tilde{U}_{\delta_a,x_a} := p_0 U\left(\sqrt{p_0}(x - x_a)\right)\]

Let \(u_a^{(1)}\) and \(u_a^{(2)}\) be two single-peak solutions of (5.22)–(5.23) concentrating at some point \(b_0\), which can be written as
\[u_a^{(l)} = \tilde{U}_{\delta_a,x_a} + \tilde{\varphi}_a^{(l)}(x), \quad l = 1, 2, \quad \text{and} \quad \tilde{\varphi}_a^{(l)} \in \tilde{E}_{a,x_a}\).

Now we set \(\xi_a(x) = \frac{u_a^{(1)}(x) - u_a^{(2)}(x)}{\|u_a^{(1)}(x) - u_a^{(2)}(x)\|_{L_\infty(R^3)}}\). Then \(\xi_a(x)\) satisfies \(\|\xi_a\|_{L_\infty(R^3)} = 1\). And from (5.22), we find that \(\xi_a\) satisfies
\[-\delta_a^2 \Delta \xi_a(x) + C_a(x)\xi_a(x) - D_a(x)\xi_a(x) - E_a(x) = g_a(x),\]
where
\[C_a(x) = \delta_a^2 P(x) - \delta_a^2 \mu_a^{(1)}, \quad D_a(x) = \frac{1}{8\pi\delta_a^2} \int_{R^3} \frac{(u_a^{(1)}(y))^2}{|x - y|} dy, \]
\[E_a(x) = \frac{u_a^{(2)}(x)}{8\pi\delta_a^2} \int_{R^3} \frac{(u_a^{(1)}(y) + u_a^{(2)}(y))\xi_a(y)}{|x - y|} dy, \quad g_a(x) = \frac{\delta_a^2 (\mu_a^{(1)} - \mu_a^{(2)})}{\|u_a^{(1)}(x) - u_a^{(2)}(x)\|_{L_\infty(R^3)}} u_a^{(2)}(x)\]

Also, similar to (3.11), for any fixed \(R \gg 1\), there exist some \(\theta > 0\) and \(C > 0\), such that
\begin{equation}
|u_a^{(l)}(x)| + |\nabla u_a^{(l)}(x)| \leq C e^{-\theta|x - x_a|/\delta_a}, \quad \text{for} \ l = 1, 2, \ x \in R^3 \setminus B_{R\delta_a}(x_a).
\end{equation}

Now let \(\tilde{\xi}_a(x) = \xi_a\left(\frac{\delta_a}{\sqrt{p_0}} x + x_a^{(1)}\right)\), we have
\begin{equation}
-\Delta \tilde{\xi}_a(x) + \frac{C_a\left(\frac{\delta_a}{\sqrt{p_0}} x + x_a^{(1)}\right)}{p_0}\tilde{\xi}_a(x) - \frac{D_a(x)}{p_0}\tilde{\xi}_a(x) - \frac{E_a(x)}{p_0} = \frac{g_a\left(\frac{\delta_a}{\sqrt{p_0}} x + x_a^{(1)}\right)}{p_0},
\end{equation}
where
\[\tilde{D}_a(x) = \frac{1}{8\pi p_0} \int_{R^3} \frac{(u_a^{(1)}(y))^2 \left(\frac{\delta_a}{\sqrt{p_0}} y + x_a^{(1)}\right)}{|x - y|} dy, \]
\[\tilde{E}_a(x) = \frac{u_a^{(2)}\left(\frac{\delta_a}{\sqrt{p_0}} x + x_a^{(1)}\right)}{8\pi p_0} \int_{R^3} \frac{(u_a^{(1)}(y) + u_a^{(2)}(y))\left(\frac{\delta_a}{\sqrt{p_0}} y + x_a^{(1)}\right)\tilde{\xi}_a(y)}{|x - y|} dy.\]
Lemma 5.5. For $x \in B_{r_0}(0)$, it holds
\[
\frac{\bar{D}_a(x)}{\rho_0^2} = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{U^2(y)}{|x-y|} dy + O(\delta_a^4),
\]
and
\[
\frac{\bar{E}_a(x)}{\rho_0^2} = \frac{U(x)}{4\pi} \int_{\mathbb{R}^3} \frac{U(y)\bar{\xi}_a(y)}{|x-y|} dy + O\left(\delta_a + \bar{\varphi}_a^2 \left(\frac{\delta_a x_0}{\rho_0} + x_a^{(1)}\right)\right).
\]

Proof. By direct computations, from (C.4) we have
\[
\frac{\bar{D}_a(x)}{\rho_0^2} = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{(u_a^{(1)})^2 \left(\frac{\delta_a x}{\rho_0} + x_a^{(1)}\right)}{|x-y|} dy
\]
\[
= \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{U^2(y)}{|x-y|} dy + \frac{1}{4\pi\rho_0^2} \int_{\mathbb{R}^3} \frac{U(y)\varphi_a^{(1)} \left(\frac{\delta_a y}{\rho_0} + x_a^{(1)}\right)}{|x-y|} dy
\]
\[
= \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{U^2(x)}{|x-y|} dy + O(\delta_a^4).
\]

Also, it follows from (C.7) that
\[
\frac{\bar{E}_a(x)}{\rho_0^2} = \frac{u_a^{(2)} \left(\frac{\delta_a x}{\rho_0} + x_a^{(1)}\right)}{8\pi\rho_0^2} \int_{\mathbb{R}^3} \frac{(u_a^{(1)} + u_a^{(2)}) \left(\frac{\delta_a y}{\rho_0} + x_a^{(1)}\right)\bar{\xi}_a(y)}{|x-y|} dy
\]
\[
= \frac{U(x)}{4\pi} \int_{\mathbb{R}^3} \frac{U(y)\bar{\xi}_a(y)}{|x-y|} dy
\]
\[
+ \frac{1}{4\pi\rho_0^2} \left[u_a^{(2)} \left(\frac{\delta_a x}{\rho_0} + x_a^{(1)}\right) - \rho_0 U(x)\right] \int_{\mathbb{R}^3} \frac{U(y)\bar{\xi}_a(y)}{|x-y|} dy
\]
\[
= \frac{U(x)}{4\pi} \int_{\mathbb{R}^3} \frac{U(y)\bar{\xi}_a(y)}{|x-y|} dy + O\left(\delta_a + \bar{\varphi}_a^2 \left(\frac{\delta_a x_0}{\rho_0} + x_a^{(1)}\right)\right).
\]
\[
\square
Lemma 5.6. For $x \in B_{r \sqrt{\rho_0}}^{-1}(0)$, it holds

\begin{equation}
C_a \left( \frac{\delta x + x^{(1)}_a}{\rho_0} \right) = 1 + O \left( \delta^4_a + \sum_{l=1}^{2} \bar{\varphi}^{(l)}_a \left( \frac{\delta x + x^{(1)}_a}{\rho_0} \right) \right),
\end{equation}

and

\begin{equation}
g_a \left( \frac{\delta x + x^{(1)}_a}{\rho_0} \right) = - \frac{1}{4\pi a_0} U(x) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U^2(x)U(y)\bar{\xi}_a(y)}{|x-y|} \, dx \, dy + O \left( \delta_a + \sum_{l=1}^{2} \bar{\varphi}^{(l)}_a \left( \frac{\delta x + x^{(1)}_a}{\rho_0} \right) \right).
\end{equation}

Proof. First, (5.28) can be deduced by (5.20) and (5.21) directly. Now we prove (5.29).

From (5.22) and (5.23), for $l = 1, 2$, we find

\begin{equation}
a\mu^{(l)}_a \delta^6_a = \delta^2_a \int_{\mathbb{R}^3} \left( \nabla u^{(l)}_a \right)^2 + P(x)(u^{(l)}_a)^2 - \frac{1}{8\pi \delta^2_a} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u^{(l)}_a)^2(x)(u^{(l)}_a)^2(y)}{|x-y|} \, dx \, dy,
\end{equation}

which gives

\begin{equation}
\frac{a\delta^6_a(\mu^{(1)}_a - \mu^{(2)}_a)}{\|u^{(1)}_a - u^{(2)}_a\|_{L^\infty(\mathbb{R}^3)}} = - \mu^{(2)}_a \delta^2_a \int_{\mathbb{R}^3} (u^{(1)}_a + u^{(2)}_a) \xi_a + \delta^2_a \int_{\mathbb{R}^3} \left( \nabla (u^{(1)}_a + u^{(2)}_a) \cdot \nabla \xi_a + P(x)(u^{(1)}_a + u^{(2)}_a) \xi_a \right)
\end{equation}

\begin{equation}
- \frac{1}{8\pi \delta^2_a} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u^{(1)}_a)^2(x)(u^{(1)}_a + u^{(2)}_a) \xi_a(y) + (u^{(1)}_a + u^{(2)}_a) \xi_a(x)(u^{(2)}_a)^2(y)}{|x-y|} \, dx \, dy
\end{equation}

\begin{equation}
= - (\mu^{(2)}_a - \mu^{(1)}_a) \delta^2_a \int_{\mathbb{R}^3} u^{(1)}_a \xi_a - \frac{1}{8\pi \delta^2_a} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u^{(1)}_a)^2(x)(u^{(2)}_a)^2(y) + (u^{(2)}_a)^2(x)(u^{(1)}_a(y)) \xi_a(y)}{|x-y|} \, dx \, dy,
\end{equation}

here we use the following identity:

\begin{equation}
\int_{\mathbb{R}^3} (u^{(1)}_a + u^{(2)}_a) \xi_a = \frac{1}{\|u^{(1)}_a - u^{(2)}_a\|_{L^\infty(\mathbb{R}^3)}} \left( \int_{\mathbb{R}^3} (u^{(1)}_a)^2 - \int_{\mathbb{R}^3} (u^{(2)}_a)^2 \right) = 0.
\end{equation}

Then from (5.20), (5.21), (5.24) and (5.30), we know

\begin{equation}
\frac{\delta^2_a(\mu^{(1)}_a - \mu^{(2)}_a)}{\|u^{(1)}_a - u^{(2)}_a\|_{L^\infty(\mathbb{R}^3)}} = \frac{1}{a_0 \delta^3_a (\mu^{(2)}_a - \mu^{(1)}_a) \delta^2_a} \int_{\mathbb{R}^3} u^{(1)}_a \xi_a
\end{equation}

\begin{equation}
- \frac{1}{8\pi a_0 \delta^5_a} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u^{(1)}_a)^2(x)(u^{(2)}_a)^2(y) + (u^{(2)}_a)^2(x)(u^{(1)}_a(y)) \xi_a(y)}{|x-y|} \, dx \, dy.
\end{equation}
By (5.20), H̄older inequality and (5.24), we have
\[ \frac{1}{a_0} \left( \mu_a^{(2)} - \mu_a^{(1)} \right) \delta_a^2 \int_{\mathbb{R}^3} u_a^{(1)} \xi_a \]
\[ = -\frac{1}{a_0} \left( \mu_a^{(2)} - \mu_a^{(1)} \right) \delta_a^2 \left( \int_{\mathbb{R}^3} p_0 U \left( \frac{\sqrt{p_0}}{\delta_a} (x - x_a^{(1)}) \xi_a + \int_{\mathbb{R}^3} \varphi_a^{(1)} \xi_a \right) \right) \]
\[ = \frac{1}{a_0} O(\delta_a^4) \left( C \delta_a^2 + C \delta_a^{11/2} \right) = O(\delta_a^2). \]

From (C.12), we can check that
\[ \frac{1}{8\pi a_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{[(u_a^{(1)})^2(x)u_a^{(2)}(y) + (u_a^{(2)})^2(x)]u_a^{(1)}(y)]\xi_a(y)}{|x-y|} \ dx \ dy \]
\[ = \frac{1}{8\pi a_0} \left( 2\hat{a}^5 p_0 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U^2(x)U(y)\xi_a(y)}{|x-y|} \ dx \ dy + O(\delta_a^6) \right) \]
\[ = \frac{1}{4\pi a_0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U^2(x)U(y)\xi_a(y)}{|x-y|} \ dx \ dy + O(\delta_a^6). \]

Noting that in \( x \in B_{\rho_0, p_0, -1}(0) \), by the mean value theorem and (5.21) we have
\[ \frac{\delta_a x}{\sqrt{p_0}} + x_a^{(1)} \]
\[ = p_0 U(x) + p_0 \left( U(x) + \frac{\sqrt{p_0}}{\delta_a} (x_a^{(1)} - x_a^{(2)}) - U(x) \right) + \varphi_a^{(2)} \left( \frac{\delta_a x}{\sqrt{p_0}} + x_a^{(1)} \right) \]
\[ = p_0 U(x) + p_0 \nabla U \left( \xi x + (1 - \zeta) \frac{\sqrt{p_0}}{\delta_a} (x_a^{(1)} - x_a^{(2)}) \right) \cdot \frac{\sqrt{p_0}}{\delta_a} (x_a^{(1)} - x_a^{(2)}) + \varphi_a^{(2)} \left( \frac{\delta_a x}{\sqrt{p_0}} + x_a^{(1)} \right) \]
\[ = p_0 U(x) + O(\delta_a) + \varphi_a^{(2)} \left( \frac{\delta_a x}{\sqrt{p_0}} + x_a^{(1)} \right), \]

which combining all the estimates above implies that (5.29) holds. \( \square \)

Then from Lemmas 5.5 and 5.6 we have the following result.

**Lemma 5.7.** From \( |\xi_a| \leq 1 \), we suppose that \( \xi_a(x) \rightarrow \xi(x) \) in \( C^1_{\text{loc}}(\mathbb{R}^3) \). Then \( \xi(x) \) satisfies following system:
\[ -\Delta \xi(x) + \xi(x) - \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{U^2(y)}{|x-y|} \ dy \xi(x) - \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{U(y)\xi(y)}{|x-y|} \ dy \ U(x) \]
\[ = \frac{1}{4\pi a_0} U(x) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U^2(x)U(y)\xi(y)}{|x-y|} \ dx \ dy. \]

To prove \( \xi = 0 \), we write
\[ (5.31) \quad \tilde{\xi}_a(x) = \sum_{j=0}^3 \beta_{a,j} \psi_j + \tilde{\xi}_a(x), \text{ in } H^1(\mathbb{R}^3), \]
where \( \psi_j(j = 0, 1, 2, 3) \) are the functions in (5.33) and \( \tilde{\xi}_a(x) \in \tilde{E} \) with
\[
\tilde{E} = \{ u \in H^1(\mathbb{R}^3), \langle u, \psi_j \rangle = 0, \text{ for } j = 0, 1, 2, 3 \}.
\]
It is standard to prove the following result.

**Lemma 5.8.** For any \( u \in \tilde{E} \), there exists \( \tilde{\gamma} > 0 \) such that
\[
\| \tilde{L}(u) \| \geq \tilde{\gamma} \| u \|,
\]
where \( \tilde{L} \) is defined by
\[
\tilde{L}(u) := -\Delta u(x) + u(x) - \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{U^2(y)}{|x-y|} dy u(x) - \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{U(y)u(y)}{|x-y|} dy U(x)
\]
\[
+ \frac{1}{4\pi a_*} U(x) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U^2(x)U(y)u(y)}{|x-y|} dx dy.
\]

**Proposition 5.9.** Let \( \tilde{\xi}_a(x) \) be as in (5.31). Then
\[
\| \tilde{\xi}_a \| = O(\delta_a).
\]

**Proof.** First, Lemma 5.8 gives
\[
\| \tilde{\xi}_a \| \leq C \| \tilde{L}(\tilde{\xi}_a) \|.
\]
On the other hand, from (5.27)–(5.31), we can prove
\[
\tilde{L}(\tilde{\xi}_a) = \left(1 - \frac{C_a(\frac{\delta_a}{\sqrt{p_0}} x + x_a^{(1)})}{p_0}\right) \tilde{\xi}_a(x) + \left(- \frac{D_a(x)}{p_0} \tilde{\xi}_a(x) - \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{U^2(y)}{|x-y|} dy \tilde{\xi}_a(x)\right)
\]
\[+ \left(- \frac{E_a(x)}{p_0} \tilde{\xi}_a(x) - \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{U(y)\tilde{\xi}_a(y)}{|x-y|} dy U(x)\right)
\]
\[+ \left( \frac{g_a(\frac{\delta_a}{\sqrt{p_0}} x + x_a^{(1)})}{p_0} \tilde{\xi}_a(x) + \frac{1}{4\pi a_*} U(x) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U^2(x)U(y)\tilde{\xi}_a(y)}{|x-y|} dx dy\right).
\]

So from (5.27), (5.31), Lemma 5.5 and Lemma 5.6, we have for any \( v(x) \in H^1(\mathbb{R}^3) \)
\[
\langle \tilde{L}(\tilde{\xi}_a), v \rangle = O(\delta_a) \| v \| + O\left( \int_{\mathbb{R}^3} \sum_{l=1}^{2} \varphi_a^{(l)}(\delta_a x + x_a^{(1)} ) |v| \right)
\]
\[\leq O(\delta_a) \| v \| + C\delta_a^{-\frac{3}{2}} \| \varphi_a^{(l)} \| \delta_a \| v \|
\]
\[\leq O(\delta_a) \| v \|,
\]
which implies (5.32).

Letting \( \beta_{a,j} \) be as in (5.31) and using \( |\tilde{\xi}_a| \leq 1 \), we find
\[
\beta_{a,j} = \frac{\langle \tilde{\xi}_a, \psi_j \rangle}{\| \psi_j \|^2} = O(\| \tilde{\xi}_a \|) = O(1), \ j = 0, 1, 2, 3.
\]
Lemma 5.10. There holds

(5.35) \[ \beta_{a,0} = o(1). \]

Proof. On one hand, from (5.31), (5.32) and (5.21) we get

\[ \int_{B_{\rho}(x^{(1)}_a)} (u^{(1)}_a + u^{(2)}_a) \xi_a \]
\[ = \frac{\delta^3}{\rho^7} \int_{B_{\frac{\rho}{\delta_a}}(0)} \left( 2p_0 U(x) + \nabla U \left( \zeta x + (1 - \zeta) \frac{\nu P_0}{\delta_a} (x^{(1)}_a - x^{(2)}_a) \right) \cdot \frac{\nu P_0}{\delta_a} (x^{(1)}_a - x^{(2)}_a) \right) \]
\[ + \sum_{l=1}^2 \phi^{(l)} \left( \frac{\nu P_0}{\delta_a} x + x^{(1)}_a \right) \xi_a(x) dx \]
\[ = 2 \frac{\delta^3}{\sqrt{P_0}} \int_{\mathbb{R}^3} U(x) \xi_a(x) - \int_{\mathbb{R}^3 \setminus B_{\frac{\rho}{\delta_a}}(0)} U(x) \xi_a(x) \]
\[ + O\left( \delta^4 |x^{(1)}_a - x^{(2)}_a| \right) \left| \nabla U \left( \zeta x + (1 - \zeta) \frac{\nu P_0}{\delta_a} (x^{(1)}_a - x^{(2)}_a) \right) \right| \]
\[ + \frac{\delta^3}{\rho^7} \sum_{l=1}^2 \left| \phi^{(l)} \left( \frac{\nu P_0}{\delta_a} x + x^{(1)}_a \right) \right| \xi_a(x) \]
\[ = 2 \frac{\delta^3}{\sqrt{P_0}} \left( \int_{\mathbb{R}^3} U(x) \beta_{a,0} (2U(x) + x \cdot \nabla U(x)) + \int_{\mathbb{R}^3} U(x) \xi_a(x) \right) + O(e^{-\frac{\rho}{\delta_a}}) \]
\[ + O\left( |x^{(1)}_a - x^{(2)}_a| \delta^2 + \delta^2 \left( \sum_{l=1}^2 \| \phi^{(l)} \|_{\delta_a} \right) \right) \]
\[ = \beta_{a,0} \frac{\delta^3}{\sqrt{P_0}} \int_{\mathbb{R}^3} U^2(x) + O(\delta^4) + O\left( |x^{(1)}_a - x^{(2)}_a| \delta^2 + O(\delta^4) + \delta^2 \| \phi^{(2)} \|_{\delta_a} \right) \]
\[ = a_x \beta_{a,0} \delta^2 + O(\delta^4). \]

On the other hand, noting that from (5.26)

\[ \int_{\mathbb{R}^3 \setminus B_{\rho}(x^{(1)}_a)} (u^{(1)}_a + u^{(2)}_a) \xi_a = O\left( \int_{\mathbb{R}^3 \setminus B_{\rho}(x^{(1)}_a)} |u^{(1)}_a + u^{(2)}_a| \right) \]
\[ = O\left( \int_{\mathbb{R}^3 \setminus B_{\rho}(x^{(1)}_a)} |u^{(1)}_a| \right) + O\left( \int_{\mathbb{R}^3 \setminus B_{\rho}(x^{(2)}_a)} |u^{(2)}_a| \right) \]
\[ = O(e^{-\frac{\rho}{\delta_a}}), \]
then we can have

$$\int_{B_\rho(x_a^{(1)})} (u_a^{(1)}(x) + u_a^{(2)}(x)) \xi_a = \int_{\mathbb{R}^3} (u_a^{(1)} + u_a^{(2)}) \xi_a + O(e^{-\frac{\theta}{\delta_a}})$$

(5.37)

$$= \frac{1}{\|u_a^{(1)} - u_a^{(2)}\|_{L^\infty(\mathbb{R}^3)}} \int_{\mathbb{R}^3} \left[ (u_a^{(1)})^2 - (u_a^{(2)})^2 \right] + O(e^{-\frac{\theta}{\delta_a}})$$

$$= O(e^{-\frac{\theta}{\delta_a}}),$$

since

$$\int_{\mathbb{R}^3} (u_a^{(1)})^2 = \int_{\mathbb{R}^3} (u_a^{(2)})^2 = 1.$$

It follows from (5.36) and (5.37) that (5.35) holds. □

Remark 5.11. We would like to point out that since here we consider the single-peak case and observing this fact $$\hat{R}_3(\sum_{l=1}^2 u_a^{(l)}(x) + x \cdot \nabla U_a(x))dx = 1$$, the proof of Proposition 5.10 is much simple.

Proposition 5.12. It holds

(5.38) $$\beta_{a,j} = o(1), \ j = 1, 2, 3.$$  

Proof. Step 1: To prove $$\beta_{a,3} = O(\delta_a).$$

Using (4.11), we deduce

(5.39) $$\int_{B_\rho(x_a^{(1)})} \frac{\partial P(x)}{\partial \nu_a} B_a(x) \xi_a = O(e^{-\frac{\theta}{\delta_a}}),$$

where $$\nu_a$$ is the outward unit vector of $$\partial B_a(x_a^{(1)})$$ at $$x$$, $$B_a(x) = \sum_{l=1}^2 u_a^{(l)}(x).$$

On the other hand, by (5.21), we have

(5.40) $$B_a(x) = \left(2 + O(\delta_a^2)\right) \tilde{U}_{\delta_a, x_a^{(1)}}(x) + O\left(\sum_{l=1}^2 |\tilde{\varphi}_a^{(l)}(x)|\right), \ x \in B_\rho(x_a^{(1)}).$$

Also, from (5.8), we find

$$\frac{\partial P(x_a^{(1)})}{\partial \nu_a} = \frac{\partial P(x_a^{(1)})}{\partial \nu_a} - \frac{\partial P(\tilde{x}_a^{(1)})}{\partial \nu_a} = O\left(|\tilde{x}_a^{(1)}|\right) = O(\delta_a^2),$$

and

$$\frac{\partial^2 P(x_a^{(1)})}{\partial \nu_a \partial \tau_{a,j}} = \frac{\partial^2 P(x_a^{(1)})}{\partial \nu_a \partial \tau_{a,j}} - \frac{\partial^2 P(\tilde{x}_a^{(1)})}{\partial \nu_a \partial \tau_{a,j}} = O\left(|\tilde{x}_a^{(1)}|\right) = O(\delta_a^2), \ for \ j = 1, 2.$$
From (3.11), (4.1) and (5.40), we get

\[
\int_{B_{\rho}(x_a^{(1)})} \frac{\partial P(x)}{\partial \nu_a} B_a(x) \xi_a
\]
\[
= \int_{\mathbb{R}^3} \frac{\partial P(x_a^{(1)})}{\partial \nu_a} B_a(x) \xi_a + \int_{\mathbb{R}^3} \left( \nabla \frac{\partial P(x_a^{(1)})}{\partial \nu_a}, x - x_a^{(1)} \right) B_a(x) \xi_a + O(\delta^5_a)
\]
\[
= - \frac{\partial^2 P(x_a^{(1)})}{\partial \nu_a^2} a_s \beta_{a,3} \delta^4_a + O(\delta^5_a).
\]

Then (5.39) and (5.41) imply \( \beta_{a,3} = O(\delta_a) \).

**Step 2:** To prove \( \beta_{a,j} = o(1) \) for \( j = 1, 2 \).

Similar to (5.39), we have

\[
\int_{B_{\rho}(x_a^{(1)})} \frac{\partial P(y)}{\partial \tau_{a,j}} B_a(y) \xi_a = O(e^{-\frac{\rho}{\xi_a}}), \text{ for } j = 1, 2.
\]

Using suitable rotation, we assume that \( \tau_{a,1} = (1, 0, 0), \tau_{a,2} = (0, 1, 0) \) and \( \nu_a = (0, 0, 1) \). Under the assumption (P), we obtain

\[
\frac{\partial P(\frac{\delta_a}{\sqrt{\rho}} y + x_a^{(1)})}{\partial \tau_{a,j}}
\]
\[
= \delta_a \sum_{0}^{3} \frac{3}{2} \frac{\partial^2 P(x_a^{(1)})}{\partial y_l \partial \tau_{a,j}} y_l + \frac{\delta_a^2}{2 \rho} \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{\partial^3 P(x_a^{(1)})}{\partial y_k \partial y_l \partial \tau_{a,j}} y_k y_l
\]
\[
+ \delta_a^3 \sum_{s=1}^{3} \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{\partial^4 P(x_a^{(1)})}{\partial y_s \partial y_l \partial y_k \partial \tau_{a,j}} y_s y_k y_l + O(\delta^3_a |y|^3), \text{ in } B_{\frac{\rho}{\xi_a} \sqrt{\rho_0}(0)}.
\]

By (1.7), (5.8), (5.24), (5.31), (5.40) and the symmetry of \( \psi_j(x) \), we find, for \( j = 1, 2 \),

\[
\sum_{k=1}^{3} \sum_{l=1}^{3} \frac{\partial^2 P(x_a^{(1)})}{\partial y_k \partial y_l \partial \tau_{a,j}} \int_{B_{\frac{\rho}{\xi_a} \sqrt{\rho_0}(0)}} B_a(\frac{\delta_a}{\sqrt{\rho_0}} y + x_a^{(1)}) \tilde{\xi}_a y_l y_l
\]
\[
= 2 \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{\partial^2 P(x_a^{(1)})}{\partial y_k \partial y_l \partial \tau_{a,j}} \int_{B_{\frac{\rho}{\xi_a} \sqrt{\rho_0}(0)}} U(y) \tilde{\xi}_a y_l y_l + O(\delta^2_a)
\]
\[
= B \beta_{a,0} \frac{\partial \Delta P(x_a^{(1)})}{\partial \tau_{a,j}} + O(\delta^2_a) = O(|x_a^{(1)} - b_0|) + O(\delta^2_a) = O(\delta^2_a).
\]
Also from (5.38) and (5.40), we get

\[ \sum_{s=1}^{3} \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{\partial^4 P(x_a^{(1)})}{\partial y_s \partial y_l \partial y_k \partial \tau_{a,j}} \int_{B_{\sqrt{\frac{a}{r}}}} B_a \left( \frac{\delta_a}{\sqrt{P_0}} y + x_a^{(1)} \right)\xi_a y_s y_l y_m \]

\[ = 2 \sum_{s=1}^{2} \sum_{k=1}^{2} \sum_{l=1}^{2} \frac{\partial^4 P(x_a^{(1)})}{\partial y_s \partial y_l \partial y_k \partial \tau_{a,j}} \int_{B_{\sqrt{\frac{a}{r}}}} U(y) \left( \sum_{h=1}^{2} \beta_{a,h} \psi_h(y) y_s y_l y_k + o(1) \right) \]

\[ = 2 \sum_{h=1}^{2} \beta_{a,j} \int_{B_{\sqrt{\frac{a}{r}}}} U(y) \psi_h(y) y_h \left[ \frac{\partial^4 P(x_a^{(1)})}{\partial y_i^3 \partial \tau_{a,j}} y_i^2 + 3 \sum_{l=1, l \neq h}^{3} \frac{\partial^4 P(x_a^{(1)})}{\partial y_l^2 \partial \tau_{a,j}} y_l^3 \right] + o(1) \]

\[ = -3B \left( \sum_{h=1}^{2} \frac{\partial^2 \Delta P(x_a^{(1)})}{\partial \tau_{a,h} \partial \tau_{a,j}} \beta_{a,h} \right) + o(1) = -3B \left( \sum_{h=1}^{2} \frac{\partial^2 \Delta P(b_0)}{\partial \tau_{a,h} \partial \tau_{a,j}} \beta_{a,h} \right) + o(1). \]

By (A.3), we estimate

\[ \frac{\partial^2 P(x_a^{(1)})}{\partial y_l \partial \tau_{a,j}} = - \frac{\partial P(x_a^{(1)})}{\partial \nu} \kappa_l(x_a^{(1)}) \delta_{ij}, \quad l, j = 1, 2. \]

Since \( \frac{\partial P(x_a^{(1)})}{\partial \nu} = 0 \), from (5.8), we find

\[ \frac{\partial^2 P(x_a^{(1)})}{\partial y_l \partial \tau_{a,j}} = - \left( \frac{\partial P(x_a^{(1)})}{\partial \nu} - \frac{\partial P(x_a^{(1)})}{\partial \nu} \right) \kappa_l(x_a^{(1)}) \delta_{ij} = - \frac{\partial^2 P(x_a^{(1)})}{\partial y_l^2} (x_a^{(1)} - \bar{x}_a^{(1)}) \cdot \nu_a \kappa_l(x_a^{(1)}) \delta_{ij} + o(\delta_a^2) \]

\[ = - \frac{\partial^2 P(b_0)}{\partial y^2} (x_a^{(1)} - \bar{x}_a^{(1)}) \cdot \nu_a \kappa_l(b_0) \delta_{ij} + o(\delta_a^2) \]

\[ = B \frac{\partial \Delta P(b_0)}{\partial \nu} \delta_a^2 \kappa_l(b_0) \delta_{ij} + o(\delta_a^2). \]

Therefore from (5.21), (5.31), (5.40) and (5.44), we get

\[ \sum_{l=1}^{3} \frac{\partial^2 P(x_a^{(1)})}{\partial y_l \partial \tau_{a,j}} \int_{B_{\sqrt{\frac{a}{r}}}} B_a \left( \frac{\delta_a}{\sqrt{P_0}} y + x_a^{(1)} \right)\xi_a y_l \]

\[ = \frac{B}{2a_s} \frac{\partial \Delta P(b_0)}{\partial \nu} \delta_a^2 \kappa_j(b_0) \beta_{a,j} \int_{B_{\sqrt{\frac{a}{r}}}} B_a \left( \frac{\delta_a}{\sqrt{P_0}} y + x_a^{(1)} \right)\xi_a y_j + o(\delta_a^2) \]

\[ = \frac{B}{a_s} \frac{\partial \Delta P(b_0)}{\partial \nu} \delta_a^2 \kappa_j(b_0) \beta_{a,j} \int_{\mathbb{R}^3} U(y) \frac{\partial P(b_0)}{\partial y_j} y_j + o(\delta_a^2) \]

\[ = - \frac{B}{2} \frac{\partial \Delta P(b_0)}{\partial \nu} \delta_a^2 \kappa_j(b_0) \beta_{a,j} + o(\delta_a^2). \]
Combining (5.43) to (5.45), we obtain

\[
\int_{B_{\hat{R}}(x^{(1)}_a)} \frac{\partial P(y)}{\partial \tau_{a,j}} B_a(y) \xi_a = - \frac{B}{2} \frac{\partial \Delta P(b_0)}{\partial \nu} \delta(a, j) \beta_a, j - \frac{B}{2} \left( \sum_{l=1}^{2} \frac{\partial^2 \Delta P(b_0)}{\partial \tau_l \partial \tau_j} \beta_a, l \right) \delta_a + o(\delta_a).
\]  

(5.46)

From (5.42) and (5.46), we find

\[
\frac{\partial \Delta P(b_0)}{\partial \nu_i} \kappa_j(b_0) \beta_a, j + \left( \sum_{l=1}^{2} \frac{\partial^2 \Delta P(b_0)}{\partial \tau_l \partial \tau_j} \beta_a, l \right) = o(1),
\]

which together with the assumption (\(\hat{P}\)) implies \(\beta_{a,j} = o(1)\) for \(j = 1, 2\).

\[\square\]

Finally, we prove Theorem 1.5.

**Proof of Theorem 1.5:** First, for large fixed \(R\), (5.20), (5.26), (A.1) and (A.2) give

\[
C_a(x) - D_a(x) \geq \frac{1}{2}, \quad |g_a(x) + E_a(x)| \leq Ce^{-\theta R}, \quad x \in \mathbb{R}^3 \setminus B_{R\hat{R}}(x^{(1)}_a),
\]

for some \(\theta > 0\).

Using the comparison principle, we get

\[
\xi_a(x) = o(1), \quad \text{in} \quad \mathbb{R}^3 \setminus \bigcup B_{R\hat{R}}(x^{(1)}_a).
\]

On the other hand, it follows from (5.32), (5.35) and (5.38) that

\[
\xi_a(x) = o(1), \quad \text{in} \quad B_{R\hat{R}}(x^{(1)}_a).
\]

This is in contradiction with \(\|\xi_a\|_{L^\infty(\mathbb{R}^3)} = 1\). So \(u^{(1)}_a(x) \equiv u^{(2)}_a(x)\) as \(a\) goes to \(+\infty\).

\[\square\]

6. The non-existence of multi-peak solutions

From (3.1), we know that in order to prove Theorem 1.7 it suffices to prove the following result.

**Proposition 6.1.** Under the assumption \((P)\), there exists a small constant \(\epsilon_0 > 0\) such that problem (3.1) has no \(m\)-peak solutions \((m \geq 2)\) of the form

\[
\tilde{u}_a(x) = \sum_{i=1}^{m} U_{\epsilon, x_{a,i}}(x) + \varphi_a(x)
\]

with \(\|\varphi_a\|_a = O(\epsilon^\gamma), \quad x_{a,i} \to b_i\) as \(0 < \epsilon \leq \epsilon_0\) for each \(i = 1, \ldots, m\), and \(b_i \neq b_j\) for \(i \neq j\), where \(U_{\epsilon, x_{a,i}}(x) := (1 + \epsilon^2 P_i U \left( \frac{\sqrt{1 + \epsilon^2 P_i(x - x_{a,i})}}{\epsilon} \right) \).
Let $\tilde{u}_a$ be a $m$-peak solution of (3.1). Then for any small fixed $\rho > 0$, from (B.4) we find
\[
\epsilon^2 \int_{B_\rho(x_{a,i})} \frac{\partial P(x)}{\partial x_j}(\tilde{u}_a)^2 dx = -2\epsilon^2 \int_{\partial B_\rho(x_{a,i})} \frac{\partial \tilde{u}_a}{\partial \nu} \frac{\partial \tilde{u}_a}{\partial x_j} d\sigma + \epsilon^2 \int_{\partial B_\rho(x_{a,i})} |\nabla \tilde{u}_a|^2 \hat{v}_j(x) d\sigma
\]
\[
\begin{align*}
&= \hat{c}_1 + \int_{\partial B_\rho(x_{a,i})} (1 + \epsilon^2 P(x)) (\tilde{u}_a)^2 \hat{v}_j(x) d\sigma - \frac{1}{8\pi\epsilon^2} \int_{\partial B_\rho(x_{a,i})} \int_{\mathbb{R}^3} \frac{(\tilde{u}_a)^2}{|x-y|^3} dy d\nu_j(x) d\sigma \\
&\quad - \frac{1}{8\pi\epsilon^2} \int_{B_\rho(x_{a,i})} \int_{\mathbb{R}^3} \frac{(\tilde{u}_a(y))^2(\tilde{u}_a(x))^2(x_j - y_j)}{|x-y|^3} dy dx,
\end{align*}
\]
where $j = 1, 2, 3$ and $\hat{v}_j(x) = (\hat{v}_1(x), \hat{v}_2(x), \hat{v}_3(x))$ is the outward unit normal of $\partial B_\rho(x_{a,i})$.

Let $\tilde{\varphi}_a(x) = \varphi_a(\epsilon x + x_{a,i})$. Then, $\tilde{\varphi}_a$ satisfies $\|\tilde{\varphi}_a\|_a = O(\epsilon^2)$. Using the Moser iteration, we can prove $\|\tilde{\varphi}_a\|_{L^\infty(\mathbb{R}^3)} = o(1)$. From this fact and the comparison theorem, similar to Proposition 2.2 in [26], we can prove the following estimates for $\tilde{u}_a(x)$ away from the concentrated points $b_1, \ldots, b_m$.

**Proposition 6.2.** Suppose that $\tilde{u}_a(x)$ is a $m$-peak solution of (3.1) concentrated at $b_1, \ldots, b_m$. Then for any fixed $R \gg 1$, there exist some $\theta > 0$ and $C > 0$, such that
\[
(6.2) \quad |\tilde{u}_a(x)| + |\nabla \tilde{u}_a(x)| \leq C \sum_{i=1}^m e^{-\theta|x-x_{a,i}|/\epsilon}, \text{ for } x \in \mathbb{R}^3 \setminus \bigcup_{i=1}^m B_{R\epsilon}(x_{a,i}).
\]

From the proof of Proposition 4.2 in [26], we have the following result.

**Lemma 6.3.** For the small fixed constant $\hat{\rho} > 0$ and any $\rho \in (\hat{\rho}, 2\hat{\rho})$, then there exist $i_0 \in \{1, \ldots, m\}$, $j_0 \in \{1, 2, 3\}$ and $C^* = C(i_0, j_0) \neq 0$ such that
\[
(6.3) \quad \hat{c}_5 = C^* \epsilon^4 + o(\epsilon^4).
\]

Now we are ready to prove Proposition 6.1.

**Proof of Proposition 6.1.** Here we prove it by contradiction. Assume that (3.1) has a $m$-peak solution $\tilde{u}_a(x)$. Then taking $i = i_0$ and $j = j_0$ as in Lemma 6.3, it follows from (6.1) and (6.3) that
\[
\epsilon^2 \int_{B_\rho(x_{a,i})} \frac{\partial P(x)}{\partial x_j}(\tilde{u}_a)^2 dx = C^* \epsilon^4 + o(\epsilon^4) + O(e^{-\frac{\epsilon}{2}}),
\]
since similar to (C.2) and by (6.2) we can prove
\[
\sum_{i=1}^4 \hat{c}_i \leq C \int_{\partial B_\rho(x_{a,i})} (\epsilon^2 |\nabla \tilde{u}_a|^2 + \tilde{u}_a^2(x)) d\sigma \leq C \int_{\partial B_\rho(x_{a,i})} \sum_{j=1}^m e^{-\frac{\epsilon}{2}} d\sigma = O(e^{-\frac{\epsilon}{2}}),
\]
where we use the fact that \( \{ x : \partial B_{b_i}(x_{a,i}) \} \subset \{ x : \mathbb{R}^3 \setminus \bigcup_{j=1}^{m} B_{b_j}(x_{a,j}) \} \). Since \( x_{a,i} \to b_i \in \Gamma_i \), we find that there is a \( t_a \in [P_i, P_i + \sigma] \) if \( \Gamma \) is a local minimum set of \( P(x) \), or \( t_a \in [P_i - \sigma, P_i] \) if \( \Gamma_i \) is a local maximum set of \( P(x) \), such that \( x_{a,i} \in \Gamma_{t_a,i} \). Let \( \tau_{a,i} \) be the unit tangential vector of \( \Gamma_{t_a,i} \) at \( x_{a,i} \). Then
\[
G(x_{a,i}) = 0, \text{ where } G(x) = \langle \nabla P(x), \tau_{a,i} \rangle.
\]
We have the following expansion:
\[
G(x) = \langle \nabla G(x_{a,i}), x - x_{a,i} \rangle + \frac{1}{2} \langle \nabla^2 G(x_{a,i}), (x - x_{a,i}) \rangle \ + o(|x - x_{a,i}|^2), \text{ for } x \in B_{b_i}(x_{a,i}).
\]
Then it follows from (6.4) and (6.5) that
\[
\begin{align*}
\int_{\mathbb{R}^3} G(x)U^2_{\epsilon,x_{a,i}}(x) &= \int_{B_{b_i}(x_{a,i})} G(x)U^2_{\epsilon,x_{a,i}}(x) + O(\epsilon^{-\hat{\theta}}) \\
&= -2 \int_{B_{b_i}(x_{a,i})} G(x)U_{\epsilon,x_{a,i}}(x) \varphi_a - \int_{B_{b_i}(x_{a,i})} G(x)\varphi_a^2 + C^* \epsilon^2 + o(\epsilon^2) \\
&= O\left( [\hat{\theta}^2 |\nabla G(x_{a,i})| + \epsilon^2] \| \varphi_a \| + \epsilon |\nabla G(x_{a,i})| \cdot \| \varphi_a \|^2 + C^* \epsilon^2 + o(\epsilon^2) \right)
\end{align*}
\]
where \( 0 < \hat{\theta} < \theta \) small.

On the other hand, noting that \( G(x_{a,i}) = 0 \), it is easy to check
\[
\int_{\mathbb{R}^3} G(x)U^2_{\epsilon,x_{a,i}}(x) = \frac{1}{6} \epsilon^5 \Delta G(x_{a,i}) \int_{\mathbb{R}^3} |x|^2 U^2 + O(\epsilon^7) = O(\epsilon^5).
\]
Then it follows from (6.4) and (6.5) that
\[
C^* = o(1),
\]
which contradicts with \( C^* \neq 0 \). Therefore \( (3.1) \) has no \( m \)-peak solution \( \tilde{u}_a(x) \). \( \square \)

With Proposition 6.1 at hand, we can prove Theorem 1.7 at once.

**Proof of Theorem 1.7.** Theorem 1.7 follows directly from Proposition 6.1 and the relation between the solutions of equation (1.9) and equation (3.1). \( \square \)

**Appendix A. Some known results**

In this section, we give some known results which are used before. Denote
\[
H_{\epsilon} = \left\{ u \in H^1(\mathbb{R}^3), \int_{\mathbb{R}^3} (\epsilon^2|\nabla u|^2 + P(x)u^2(x))dx < \infty \right\}
\]
and the corresponding norm
\[
\| u \|_{\epsilon} = (u(x),u(x))_{\epsilon}^{\frac{1}{2}} = \left( \int_{\mathbb{R}^3} (\epsilon^2|\nabla u|^2 + P(x)u^2(x))dx \right)^{\frac{1}{2}}.
\]
Lemma A.1. (Lemma 2.1, [10]) For each $u \in L^q(\mathbb{R}^3)(2 \leq q \leq 6)$, we have

$$|u|_q \leq C \delta_a^{3(\frac{1}{q} - \frac{1}{2})} \|u\|_{\delta_a}.$$

Lemma A.2. (Lemma 2.2, [10], Lemma A.5, [26]) For any fixed $u \in H^1(\mathbb{R}^3)$, then

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_1(x)u_2(x)u_3(y)u_4(y)}{|x - y|} dx dy \leq C \epsilon^{-1} \|u_1\|_\epsilon \|u_2\|_\epsilon \|u_3\|_\epsilon \|u_4\|_\epsilon.$$

Lemma A.3. (Lemma A.6, [26]) For any $u_1, u_2, u_3, u_4 \in H^1(\mathbb{R}^3)$, then

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_1(x)u_2(x)u_3(y)u_4(y)}{|x - y|} dx dy \leq C \|u_1\|_{H^1(\mathbb{R}^3)} \|u_2\|_{H^1(\mathbb{R}^3)} \|u_3\|_{H^1(\mathbb{R}^3)} \|u_4\|_{H^1(\mathbb{R}^3)}.$$

Lemma A.4. (Lemma B.2, [26]) For any fixed $R > 0$, it holds

$$(A.1) \quad D_\alpha(x) = o(1) R + O\left(\frac{1}{R}\right), \quad \text{for } x \in \mathbb{R}^3 \setminus B_{R\delta_a}(x_0),$$

and

$$(A.2) \quad E_\alpha(x) = O(e^{-\bar{\theta} R}), \quad \text{for } x \in \mathbb{R}^3 \setminus B_{R\delta_a}(x_0), \quad \text{and some } \bar{\theta} > 0.$$

Now let $\tilde{\Gamma} \subset C^2$ be a closed hypersurface in $\mathbb{R}^3$. For $y \in \tilde{\Gamma}$, let $\nu(y)$ and $T(y)$ denote respectively the outward unit normal to $\tilde{\Gamma}$ at $y$ and the tangent hyperplane to $\tilde{\Gamma}$ at $y$. The curvatures of $\Gamma$ at a fixed point $y_0 \in \tilde{\Gamma}$ are determined as follows. By a rotation of coordinates, we can assume that $y_0 = 0$ and $\nu(0)$ is the $x_3$-direction, and $x_j$-direction is the $j$-th principal direction.

In some neighborhood $N = N(0)$ of 0, we have

$$\tilde{\Gamma} = \{ x : x_3 = \phi(x') \},$$

where $x' = (x_1, x_2), \quad \phi(x') = \frac{1}{2} \sum_{j=1}^2 \kappa_j x_j^2 + O(|x'|^3),$$

where $\kappa_j$ is the $j$-th principal curvature of $\tilde{\Gamma}$ at 0. The Hessian matrix $[D^2 \phi(0)]$ is given by

$$[D^2 \phi(0)] = \text{diag} [\kappa_1, \kappa_2].$$

Suppose that $W$ is a smooth function, such that $W(x) = \text{constant}$ for all $x \in \tilde{\Gamma}$. It follows from [27] that

Lemma A.5. (Lemma B.1, [27]) We have

$$\left. \frac{\partial W(x)}{\partial x_l} \right|_{x=0} = 0, \quad l = 1, 2,$$

$$(A.3) \quad \left. \frac{\partial^2 W(x)}{\partial x_m \partial x_l} \right|_{x=0} = - \left. \frac{\partial W(x)}{\partial x_3} \right|_{x=0} \kappa_m \delta_{ml}, \quad \text{for } m, l = 1, 2,$$

where $\kappa_1, \kappa_2$, are the principal curvatures of $\tilde{\Gamma}$ at 0.
Appendix B. Linearization and A Pohozaev identity

Lemma B.1. Let \( \xi_0 \) be the solution of following system:

\[
- \Delta \xi(x) + \xi(x) - \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{U^2(y)}{|x-y|} \, dy \xi(x) - \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{U(y)\xi(y)}{|x-y|} \, dy \, U(x) = 0,
\]

(B.1)

\[
= - \frac{1}{4\pi a_*} U(x) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U^2(x)U(y)\xi(y)}{|x-y|} \, dx \, dy.
\]

Then it holds

\[
\xi(x) = \sum_{j=0}^{3} \gamma_j \psi_j,
\]

where \( \gamma_j \) are some constants,

(B.2)

\[
\psi_0 = 2U + x \cdot \nabla U, \ \psi_j = \frac{\partial U}{\partial x_j}, \text{ for } j = 1, 2, 3.
\]

Proof. We set \( \tilde{L}(u) := -\Delta u(x) + u(x) - \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{U^2(y)}{|x-y|} \, dy \, u(x) - \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{U(y)u(y)}{|x-y|} \, dy \, U(x) \). It is obvious that \( \tilde{L}(\psi_j) = 0 \) and \( \psi_j \) is the solution of (B.1), for \( j = 1, 2, 3 \). Also, then using (3.12), we find that \( \psi_0 \) is also the solution of (B.1). And we know that \( \psi_0, \psi_1, \psi_2, \psi_3 \) are linearly independent. Then we get (B.2).

\[\square\]

Proposition B.2. Let \( \bar{u}_a(x) \) be the solution of equation (3.1). Then we have following local Pohozaev identity:

(B.3)

\[
\epsilon^2 \int_{\Omega} \frac{\partial P(x)}{\partial x_j} \bar{u}_a^2(x) \, dx = -2\epsilon^2 \int_{\partial \Omega} \frac{\partial \bar{u}_a(x)}{\partial \nu} \frac{\partial \bar{u}_a(x)}{\partial x_j} \sigma \partial \sigma + \epsilon^2 \int_{\partial \Omega} |\nabla \bar{u}_a(x)|^2 \nu_j(x) \, d\sigma
\]

\[
+ \int_{\partial \Omega} (1 + \epsilon^2 P(x)) \bar{u}_a^2(x) \nu_j(x) \, d\sigma
\]

\[
- \frac{1}{8\pi \epsilon^2} \int_{\partial \Omega} \int_{\mathbb{R}^3} \frac{\bar{u}_a^2(y)\bar{u}_a^2(x)}{|x-y|} \nu_j(x) \, dy \, d\sigma - \frac{1}{8\pi \epsilon^2} \int_{\Omega} \int_{\mathbb{R}^3} \frac{\bar{u}_a^2(y)\bar{u}_a^2(x)}{|x-y|^3} \frac{x_j - y_j}{|x-y|^3} \, dy \, dx,
\]

where \( \Omega \) is a bounded open domain of \( \mathbb{R}^3 \), \( j = 1, 2, 3 \), \( \nu(x) = (\nu_1(x), \nu_2(x), \nu_3(x)) \) is the outward unit normal of \( \partial \Omega \) and \( x_j, y_j \) are the \( j \)-th components of \( x, y \).

Proof. Since (B.3) is obtained just by multiplying \( \frac{\partial \bar{u}_a(x)}{\partial x_j} \) on both sides of (1.4) and integrating on \( \Omega \), here we omit the details.

\[\square\]

Appendix C. Some basic and useful estimates

In this section, we mainly give some basic and useful estimates which are used before.

Lemma C.1. There holds

(C.1)

\[
C_1 + C_2 + C_3 + C_4 + C_5 = O(e^{-\frac{a}{2}}).
\]
Proof. From (A.1) and (3.11), we have

\[(C.2) \quad C_1 + C_2 + C_3 + C_4 \leq C \int_{\partial B_\rho(x_a)} (e^2 |\nabla \tilde{u}_a|^2 + \tilde{u}_a^2(x))d\sigma = O(e^{-\frac{\rho}{\tau}}).\]

By symmetry and (3.11), we have

\[(C.3) \quad C_5 = \frac{1}{8\pi \rho^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(\tilde{u}_a(y))^2(\tilde{u}_a(x))^2(x_j - y_j)}{|x - y|^3} dy dx\]

\[= \frac{1}{8\pi \rho^2} \int_{B_\rho^c(x_a)} \int_{\mathbb{R}^3} \frac{(\tilde{u}_a(y))^2(\tilde{u}_a(x))^2(x_j - y_j)}{|x - y|^3} dy dx = O(e^{-\frac{\rho}{\tau}}).\]

By (C.2) and (C.3), (C.1) is true. \(\square\)

Lemma C.2. For any small and fixed \(\rho > 0\), if \(x \in B_{\rho \delta_a^{-1}} \sqrt{p_0}(0)\), then we have

\[(C.4) \quad F_1 + F_2 = O(\delta_a^4).\]

Proof. From (5.24) and Lemma A.1, by Hölder inequality we have

\[(C.5) \quad F_1 = \frac{1}{4\pi p_0} \int_{\mathbb{R}^3} U(y) \tilde{\varphi}_a^{(1)}(\frac{\delta_a y}{\sqrt{p_0}} + x_a^{(1)}) dy = \frac{1}{4\pi p_0} \int_{\mathbb{R}^3} U(\frac{\sqrt{p_0}}{\delta_a}(y - x_a^{(1)})) \tilde{\varphi}_a^{(1)}(y) dy\]

\[\leq \frac{1}{4\pi p_0} \left| U\left(\frac{\sqrt{p_0}}{\delta_a}(y - x_a^{(1)})\right)\right| \left| \tilde{\varphi}_a^{(1)}(y)\right| \left( \int_{|y - x_a^{(1)}| - \frac{\delta_a}{\sqrt{p_0}} x_a^{(1)}| < c_0 \delta_a} \frac{1}{|y - x_a^{(1)} - \frac{\delta_a}{\sqrt{p_0}} x_a^{(1)}|^{2}} \right)\]

\[+ \frac{1}{4\pi p_0} \left| \int_{\mathbb{R}^3} U(\frac{\sqrt{p_0}}{\delta_a}(y - x_a^{(1)})) \tilde{\varphi}_a^{(1)}(y) dy \right| \delta_a\]

\[\leq C \delta_a^{-2\frac{3}{2}} \delta_a^{-1} \| \tilde{\varphi}_a^{(1)}(y) \| \delta_a \delta_a^3 \delta_a^{\frac{1}{11}} = O(\delta_a^4),\]

and

\[(C.6) \quad F_2 = \frac{1}{8\pi p_0} \int_{\mathbb{R}^3} \left( \tilde{\varphi}_a^{(1)}(\frac{\delta_a y}{\sqrt{p_0}} + x_a^{(1)}) \right)^2 dy = \frac{1}{8\pi p_0} \int_{\mathbb{R}^3} \left( \frac{\sqrt{p_0}}{\delta_a}(y - x_a^{(1)})) \right)^2 dy\]

\[\leq \frac{1}{8\pi p_0} \left| \tilde{\varphi}_a^{(1)}(y)\right|^2 \left( \int_{|y - x_a^{(1)}| - \frac{\delta_a}{\sqrt{p_0}} x_a^{(1)}| < c_0 \delta_a} \frac{1}{|y - x_a^{(1)} - \frac{\delta_a}{\sqrt{p_0}} x_a^{(1)}|^{2}} \right)^\frac{1}{2}\]

\[+ \frac{1}{4\pi p_0} \left| \int_{\mathbb{R}^3} U(\frac{\sqrt{p_0}}{\delta_a}(y - x_a^{(1)})) \tilde{\varphi}_a^{(1)}(y) dy \right| \delta_a\]

\[\leq C \delta_a^{-2} \delta_a^{-\frac{3}{2}} \| \tilde{\varphi}_a^{(1)}(y) \| \delta_a \delta_a^{\frac{1}{2}} + C \delta_a^{-3} \delta_a^{\frac{11}{2}} = O(\delta_a^8).\]

Hence, it follows from (C.5) to (C.6) that (C.4) holds. \(\square\)
Lemma C.3. For any small and fixed \( \rho > 0 \), if \( x \in B_{\rho \delta}^{-1} \sqrt{p_0}(0) \), there holds

\[
G_1 + G_2 = O \left( \delta_a + \hat{\varphi}_a^{(2)} \left( \frac{\delta_a}{\sqrt{p_0}} x + x_a^{(1)} \right) \right).
\]

Proof. By the mean value theorem and (5.21), we have

\[
G_1 = \frac{1}{4\pi p_0} \left( u_a^{(2)} \left( \frac{\delta_a}{\sqrt{p_0}} x + x_a^{(1)} \right) - p_0 U(x) \right) \int_{\mathbb{R}^3} \frac{U(y) \xi_a(y)}{|x - y|} dy
\]

\[
= \frac{1}{4\pi p_0} \left( \nabla U(\zeta x + (1 - \zeta) \frac{\sqrt{p_0}}{\delta_a} (x_a^{(1)} - x_a^{(2)})) \cdot \frac{\sqrt{p_0}}{\delta_a} (x_a^{(1)} - x_a^{(2)}) + \hat{\varphi}_a^{(2)} \left( \frac{\delta_a}{\sqrt{p_0}} x + x_a^{(1)} \right) \right) \int_{\mathbb{R}^3} \frac{U(y) \xi_a(y)}{|x - y|} dy
\]

\[
\leq C \left( \delta_a + \hat{\varphi}_a^{(2)} \left( \frac{\delta_a}{\sqrt{p_0}} x + x_a^{(1)} \right) \right) \int_{\mathbb{R}^3} \frac{U(y) \xi_a(y)}{|x - y|} dy
\]

\[
= O \left( \delta_a + \hat{\varphi}_a^{(2)} \left( \frac{\delta_a}{\sqrt{p_0}} x + x_a^{(1)} \right) \right), \quad \text{for } \zeta \in (0, 1),
\]

since

\[
\int_{\mathbb{R}^3} \frac{U(y) \xi_a(y)}{|x - y|} dy \leq \int_{\mathbb{R}^3} \frac{U(y)}{|x - y|} dy
\]

\[
\leq |U(y)|_2 \left( \int_{|x - y| \leq 2R} \frac{1}{|x - y|^2} dy \right)^{\frac{1}{2}} + \frac{1}{2R} \int_{\mathbb{R}^3} U(y) dy
\]

\[
= O \left( R^{-\frac{1}{2}} + \frac{1}{R} \right) = O(1).
\]

We can also check

\[
G_2 = \frac{u_a^{(2)} \left( \frac{\delta_a}{\sqrt{p_0}} x + x_a^{(1)} \right)}{8\pi p_0^2} \left( \left( u_a^{(1)} + u_a^{(2)} \right) \left( \frac{\delta_a}{\sqrt{p_0}} x + x_a^{(1)} \right) - 2p_0 U(y) \right) \xi_a(y) \int_{\mathbb{R}^3} \frac{d}{|x - y|}
\]

\[
\leq C \left| x_a^{(1)} - x_a^{(2)} \right| \delta_a \int_{\mathbb{R}^3} \frac{\nabla U(\zeta y + (1 - \zeta) \frac{\sqrt{p_0}}{\delta_a} (x_a^{(1)} - x_a^{(2)}) \cdot \xi_a(y))}{|x - y|} dy
\]

\[
+ C \sum_{l=1}^{2} \int_{\mathbb{R}^3} \frac{\hat{\varphi}_l^{(1)} \left( \frac{\delta_a}{\sqrt{p_0}} x + x_a^{(1)} \right) \cdot \xi_a(y)}{|x - y|} dy
\]

\[
= O(\delta_a),
\]

since similar to (C.9), we have

\[
\int_{\mathbb{R}^3} \frac{\nabla U(\zeta y + (1 - \zeta) \frac{\sqrt{p_0}}{\delta_a} (x_a^{(1)} - x_a^{(2)}) \cdot \xi_a(y)}{|x - y|} dy = O(1),
\]
Lemma C.4. Then (C.7) follows from (C.8) and (C.10). \[ \square \]

**Lemma C.7**. There holds

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\left( (u_a^{(1)})^2(x)u_a^{(2)}(y) + (u_a^{(2)})^2(x)u_a^{(1)}(y) \right) \xi_a(y)}{|x-y|} dxdy = 2\delta_a^5 \rho \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U^2(x)U(y)\bar{\xi}_a(y)}{|x-y|} dxdy + O(\delta_a^6). \tag{C.11}
\]

**Proof.** By direct computations, from (3.12) we have

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_a^{(1)})^2(x)u_a^{(2)}(y)\xi_a(y)}{|x-y|} dxdy = \delta_a^5 \rho \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U^2(x)U(y)\bar{\xi}_a(y)}{|x-y|} dxdy + \frac{\delta_a^5}{\rho} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U^2(x)(u_a^{(1)}(y) \delta_a \bar{\rho}_0 y + x_a^{(1)}) - \rho_0 U(y)\bar{\xi}_a(y)}{|x-y|} dxdy := H_1
\]

\[
+ \delta_a^5 \rho \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\bar{\xi}_a(y) \left( (u_a^{(1)}) (u_a^{(2)} \delta_a \bar{\rho}_0 y + x_a^{(1)}) + \rho_0 U(x) u_a^{(2)} (\delta_a \bar{\rho}_0 y + x_a^{(1)}) \right) \bar{\xi}_a(y)}{|x-y|} dxdy := H_2
\]

\[
= \delta_a^5 \rho \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U^2(x)U(y)\bar{\xi}_a(y)}{|x-y|} dxdy + O(\delta_a^6),
\]

and

\[
\int_{\mathbb{R}^3} \frac{\varphi_a^{(1)}(\delta x \bar{\rho}_0 + x_a^{(1)})\bar{\xi}_a(y)}{|x-y|} dy \leq \int_{|y| \leq 2 \sqrt{\frac{\rho_0}{\delta a}}} \frac{\varphi_a^{(1)}(\delta x \bar{\rho}_0 + x_a^{(1)})\bar{\xi}_a(y)}{|x-y|} dy + \int_{|y| \geq 2 \sqrt{\frac{\rho_0}{\delta a}}} \frac{\varphi_a^{(1)}(\delta x \bar{\rho}_0 + x_a^{(1)})\bar{\xi}_a(y)}{|x-y|} dy \leq C \varphi_a^{(1)}(\frac{\delta a y}{\sqrt{\rho_0}} + x_a^{(1)}) \int_{|x-y| \leq 3 \sqrt{\frac{\rho_0}{\delta a}}} \frac{1}{|x-y|^2} dxdy + C \delta^a \varphi_a^{(1)}(\frac{\delta a y}{\sqrt{\rho_0}} + x_a^{(1)}) \int_{|x-y| \geq 3 \sqrt{\frac{\rho_0}{\delta a}}} \frac{1}{|x-y|^2} dxdy \leq C \delta^a \hat{\varphi}_a^{(1)}(\frac{\delta a y}{\sqrt{\rho_0}} + x_a^{(1)}) + C \delta^a \hat{\varphi}_a^{(1)}(\frac{\delta a y}{\sqrt{\rho_0}} + x_a^{(1)}) = O(\delta^a).
\]
since by Lemmas A.2, A.3 and (5.24), we have

\[
H_1 = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U^2(x)P_0(U(y + \sqrt{\beta} (x_0^{(1)} - x_0^{(2)}))-U(y)) \xi_a(y)}{|x-y|} dxdy
+ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U^2(x) \tilde{\varphi}_a^{(2)}(\frac{\delta_a}{\sqrt{P_0}} y + x_a^{(1)}) \tilde{\xi}_a(y)}{|x-y|} dxdy
= O(\delta_a) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U^2(x)\nabla U(\zeta y + (1 - \zeta) \sqrt{\beta} (x_0^{(1)} - x_0^{(2)}))||\xi_a(y)||}{|x-y|} dxdy
+ C\delta_a^{-1}||U||_2^2 \left\| \tilde{\varphi}_a^{(2)}(\frac{\delta_a}{\sqrt{P_0}} y + x_a^{(1)}) \right\|_\delta_a \left\| \xi_a \right\|_\delta_a
= O(\delta_a)||U||_2^2 \left\| \nabla U(\zeta y + (1 - \zeta) \sqrt{\beta} (x_0^{(1)} - x_0^{(2)})\right\| \left\| \xi_a \right\| + O(\delta_a^3)
= O(\delta_a), \text{ for some } \zeta \in (0, 1),
\]

and

\[
H_2 = 2P_0^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U(x) \tilde{\varphi}_a^{(1)}(\frac{\delta_a}{\sqrt{P_0}} x + x_a^{(1)}) U(y + \sqrt{\beta} (x_0^{(1)} - x_0^{(2)})) \tilde{\xi}_a(y)}{|x-y|} dxdy
+ 2P_0 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U(x) \tilde{\varphi}_a^{(1)}(\frac{\delta_a}{\sqrt{P_0}} x + x_a^{(1)}) \tilde{\varphi}_a^{(2)}(\frac{\delta_a}{\sqrt{P_0}} y + x_a^{(1)}) \tilde{\xi}_a(y)}{|x-y|} dxdy
+ P_0 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(\tilde{\varphi}_a^{(1)}(\frac{\delta_a}{\sqrt{P_0}} x + x_a^{(1)})^2 \tilde{\varphi}_a^{(2)}(\frac{\delta_a}{\sqrt{P_0}} y + x_a^{(1)}) \tilde{\xi}_a(y))}{|x-y|} dxdy
\leq C\delta_a^{-1}||U||_2^2 \left\| \tilde{\varphi}_a^{(1)}(\frac{\delta_a}{\sqrt{P_0}} x + x_a^{(1)}) \right\|_\delta_a \left\| \xi_a(y) \right\|_\delta_a
+ C\delta_a^{-1}||U||_\delta_a \left\| \tilde{\varphi}_a^{(1)}(\frac{\delta_a}{\sqrt{P_0}} x + x_a^{(1)}) \right\|_\delta_a \left\| \tilde{\varphi}_a^{(2)}(\frac{\delta_a}{\sqrt{P_0}} x + x_a^{(1)}) \right\|_\delta_a \left\| \xi_a(y) \right\|_\delta_a
+ C\delta_a^{-1}||\tilde{\varphi}_a^{(1)}(\frac{\delta_a}{\sqrt{P_0}} x + x_a^{(1)})||_\delta_a^2 \left\| U \right\|_\delta_a \left\| \xi_a(y) \right\|_\delta_a
+ C\delta_a^{-1}||\tilde{\varphi}_a^{(1)}(\frac{\delta_a}{\sqrt{P_0}} x + x_a^{(1)})||_\delta_a^2 \left\| \tilde{\varphi}_a^{(2)}(\frac{\delta_a}{\sqrt{P_0}} x + x_a^{(1)}) \right\|_\delta_a \left\| \xi_a(y) \right\|_\delta_a
= O(\delta_a^3) + O(\delta_a^7) + O(\delta_a^{11}) = O(\delta_a^3).
\]

Just by the same argument as that of (C.12), we can also check that

\[
(C.13) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_a^{(2)})^2(x)u_a^{(1)}(y)\xi_a(y)}{|x-y|} dxdy = \delta_a^{\frac{1}{2}} P_0 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U^2(x)U(y)\xi_a(y)}{|x-y|} dxdy + O(\delta_a^6).
\]
It follows from (C.12) and (C.13) that (C.11) holds. □

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