c=1 String Theory as a Topological $\frac{G}{G}$ Model

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ABSTRACT

The physical states on the free field Fock space of the $\frac{SL(2,R)}{SL(2,R)}$ model at any level are computed. Using a similarity transformation on $Q_{BRST}$, the cohomology of the latter is mapped into a direct sum of simpler cohomologies. We show a one to one correspondence between the states of the $k = -1$ model and those of the $c = 1$ string model. A full equivalence between the $\frac{SL(2,R)}{SL(2,R)}$ and $\frac{SL(2,R)}{U(1)}$ models at the level of their Fock space cohomologies is found.

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A deeper understanding of the relation between topological theories, \( N=2 \) theories and string theories is one of the more challenging problems of string theory in recent years. The question “Is the bosonic string topological?” which was raised in ref. [1] should be stated in a broader context, namely, to what extent do string theories admit a topological description. Unravelling this relation may lead not only to a different formulation of string theory but also to better computational tools.

In previous publications\([2,3,4]\) we have put forward the equivalence between twisted topological \( G/H \) models of \( A^{(1)}_{N-1} \) at level \( k = \frac{p}{q} - N \) and the \( (p,q) \) \( W_N \) minimal models coupled to \( W_N \) gravity. We have also demonstrated that the cohomology ring is the same for twisted \( G/H \) topological models with rank \( G = \text{rank} \, H \).

In this paper we investigate the twisted \( \frac{SL(2,R)}{SU(2)} \) theory at level \( k = -1 \), derive the space of physical states and establish its equivalence to that of the \( c = 1 \) string theory coupled to gravity.\(^{[5]}\) First indications towards this equivalence were already given in ref. [2]. In particular we identify the tachyon operators as well as the ground ring generators. We further discuss the generalization of our methods to other \( SL(2,R) \) levels which correspond to \( c < 1 \) non-critical string theories, as well as to the twisted \( \frac{SL(2,R)}{SU(1)} \) case which is related to the 2D black hole.\(^{[6]}\)

The \( \frac{SL(2,R)}{SU(1)} \) topological model at level \( k = -3 \) is studied at length in a recent paper by Mukhi and Vafa.\(^{[7]}\) In their paper the equivalence of this topological model and the \( c = 1 \) string model is analyzed. In the present paper we show that the Fock space cohomology of all these models at different levels are essentially the same. However, unlike the \( c = 1 \) case, for the \( c < 1 \) models one has to employ a further reduction.\(^{[8]}\)

The paper is organized as follows: In section 1 the basic building blocks are presented namely the free field bosonization, OPEs, and the BRST operator. A special map is invoked to translate the cohomology into a sum of simpler cohomologies. The complete BRST cohomology is then extracted. In section 2 a comparison between the Fock space states of non-critical string models and that of the various
models is made. In particular the ground ring as well as the tachyonic branches of the $c = 1$ model are identified in the $\mathcal{G}$ picture. The full equivalence is achieved only after allowing arbitrary powers of certain current components. Section 3 is devoted to a brief description of the application of the procedure developed for the $\mathrm{SL}(2,R)$ models to the $\mathrm{SL}(2,R)/\mathrm{U}(1)$ models. We then summarize, briefly compare to some recent works and discuss some open questions. An appendix is devoted to the derivation of the transformation of the BRST charge and its corresponding cohomology.

1. BRST Cohomology

The $\mathcal{G}$ topological model$^{[16,17]}$ is constructed by gauging the anomaly free diagonal $G$ group of the WZW model for the group $G$. The quantum action of the model was shown to be composed of three decoupled parts:$^{[9,10,11]}$ $S_k(g)$—a WZW model of level $k$ with $g \in G$, $S_{-(k+2C_G)}(h)$—a WZW model of level $-(k+2C_G)$ with $h \in G$, and a dimension $(1,0)$ system of anticommuting ghosts $\rho$ and $\chi$ in the adjoint representation of the group. The action, thus, reads$^{[18]}$

$$S_k(g,h,\rho,\chi) = S_k(g) + S_{-(k+2C_G)}(h) - i \int d^2z \text{Tr}[\bar{\rho} \partial \bar{\chi} + \rho \bar{\partial} \chi], \quad (1)$$

where $C_G$ is the second Casimir of the adjoint representation.

Invariance of each of the three terms under holomorphic $G$ transformations implies that there are three Kac-Moody currents $J(z) = g^{-1}\partial g$, $I(z) = h^{-1}\partial h$ and $J^{(gh)\alpha} = f^{\alpha}_{\beta\gamma} \bar{\chi}^{\beta} \rho^{\gamma}$ of levels $k$, $-(k+2C_G)$ and $2C_G$ respectively. The twisted theory for $G = \mathrm{SL}(2,R)$ is obtained by replacing the energy momentum tensor $T$ of this theory with $\tilde{T} = T + \partial J^{(\text{tot})0}$, where $J^{(\text{tot})}$ is the sum of the holomorphic currents from all sectors (and, therefore, is at level $k = 0$).

The space of physical states of the twisted $\mathcal{G}$ models which correspond to minimal matter models coupled to gravity was extracted$^{[2,3,4]}$ using a spectral sequence decomposition approach$^{[12]}$. In that formulation we had to use a particular
Wakimoto realization of the matter \((J)\) and “gauge” \((I)\) sectors\[^2\]. There are two possible bosonizations of the \(SL(2,R)\) current algebra, which are related by the automorphism \(J^+ \leftrightarrow J^-, J^0 \leftrightarrow -J^0\). Let us denote the bosonization given in eqn. (2) as the \(+\) bosonization, and the one related to it by the automorphism as the \((-\) bosonization. In \[^2\] we used a \(+\) bosonization in the \(J\) sector and a \((-\) bosonization in the \(I\) sector. We shall call this the \((+, -)\) bosonization of the theory. The physical states were associated with the cohomology ring on the space of Kac-Moody irreducible representations via a projection due to Bernard and Felder\[^8\]. Note that the \((+, -)\) bosonization lacks an \(SL(2,R)\) invariant vacuum. For the 2D gravity coupled to minimal matter this was not a problem, since after the Bernard-Felder cohomology is performed both bosonizations give an irreducible representation of the Kac-Moody algebra, (The \(SL(2,R)\) invariant vacuum is restored, since the only problem arises from \(L^-|vacuum, v| \neq 0\). However, this is a descendant of a null in any case and therefore is zero after the Bernard-Felder reduction.) We now introduce the following bosonization\[^{20}\] (with normal ordering assumed everywhere):

\[
\begin{align*}
J^+ &= \beta_J, \quad I^+ = \beta_I, \\
J^0 &= \beta_J \gamma_J + i \sqrt{\frac{t}{2}} \partial \phi_J, \quad I^0 = \beta_I \gamma_I + \sqrt{\frac{t}{2}} \partial \phi_I, \\
J^- &= -\beta_J \gamma_J^2 - i \sqrt{2 t} \gamma_J \partial \phi_J - (t - 2) \partial \gamma_J, \quad I^- = -\beta_I \gamma_I^2 - \sqrt{2 t} \gamma_I \partial \phi_I + (t + 2) \partial \gamma_I.
\end{align*}
\]

(2)

where \(\beta_J, \gamma_J\) and \(\beta_I, \gamma_I\) form bosonic \((1,0)\) systems, \(\phi_J\) and \(\phi_I\) are free scalar fields with background charges of \(-\frac{i}{\sqrt{2t}}\) and \(\frac{1}{\sqrt{2t}}\) respectively, and \(t = k + 2\). It will turn out to be convenient to use the following linear combinations of the free fields of the \(J\) and \(I\) sector:

\[
\begin{align*}
\beta^+ &= \beta_J + \beta_I, \quad \gamma^+ = \frac{1}{2} (\gamma_J + \gamma_I), \quad \phi^+ = \frac{1}{\sqrt{2t}} (\phi_J + i \phi_I), \\
\beta^- &= \beta_J - \beta_I, \quad \gamma^- = \frac{1}{2} (\gamma_J - \gamma_I), \quad \phi^- = \sqrt{\frac{t}{2}} (\phi_J - i \phi_I).
\end{align*}
\]

(3)
They obey the following OPEs:

\[
\gamma^+(z)\beta^+(\omega) = \gamma^-(z)\beta^-(\omega) = \frac{1}{z - \omega} + \ldots
\]

\[
\partial\phi^+(z)\phi^-(\omega) = \partial\phi^-(z)\phi^+(\omega) = \frac{-1}{z - \omega} + \ldots
\]

\[
\chi^+(z)\rho^-(\omega) = \chi^-(z)\rho^+(\omega) = 2\chi^0(z)\rho^0(w) = \frac{1}{z - \omega} + \ldots
\]

(4)

In terms of this free field realization \( J^{(\text{tot})0} = J^0 + I^0 + J^{(gh)}0 \) and the energy momentum tensor now read

\[
J^{0(\text{total})} = \beta^+\gamma^+ + \beta^-\gamma^- + i\partial\phi^- + \chi^+\rho^- - \chi^-\rho^+.
\]

(5)

\[
T^{(\text{total})} = -\partial\phi^+\partial\phi^- - i\partial^2\phi^+ - \beta^+\partial\gamma^+ - \beta^-\partial\gamma^- - \rho^+\partial\chi^- - \rho^-\partial\chi^+ - 2\rho^0\partial\chi^0.
\]

(6)

The twisted energy momentum tensor with which we define our theory is therefore given by:

\[
\tilde{T} = (-\partial\phi^+\partial\phi^- - i\partial^2\phi^+ + i\partial^2\phi^- - 2\rho^-\partial\chi^+ + \chi^+\partial\rho^-) + \gamma^+\partial\beta^+ + \gamma^-\partial\beta^- - \chi^-\partial\rho^+ - 2\rho^0\partial\chi^0.
\]

(7)

As noted in [2], if we identify \( \phi_J \) with \( X \) of the \( c \leq 1 \) Liouville models, \( \phi_I \) with the Liouville field, \( \chi^+ \) with \( c \) and \( \rho^- \) with \( b \), the first part (in parenthesis) of the twisted energy momentum tensor, as depicted, in (7) equals exactly the energy tensor of the Liouville theory at \( c = \frac{-6t^2 + 13t - 6}{t} \). The rest of \( \tilde{T} \) is composed of two pairs of \((1,0)\) bosons and \((1,0)\) fermions which could be treated as additional "topological sectors". Note that both \( t \) and \( \frac{1}{t} \) give an energy momentum tensor which can be identified with the Liouville theory at the same \( c \). Moreover, we can also identify the Liouville theory at \( c(t) \) with the \( \frac{SL(2,R)}{SL(2,R)} \) theory at \(-t\) if we change...
our identifications so that $\phi_J$ is identified with the Liouville field and $\phi_I$ with the matter field. The BRST charge takes the form

$$Q_{BRST} = \oint \frac{dz}{2\pi i} (\chi^- (J^+ + I^+) + 2\chi^0 (J^0 + I^0) + \chi^+ (J^- + I^-)$$

$$+ 2(\chi^+ \chi^- \rho^0 + \chi^- \chi^0 \rho^+ + \chi^0 \chi^+ \rho^-)).$$

(8)

Both $\tilde{T}$ (with which we will work from here on) and $J^{(tot)}_a$ are $Q$ exact: $\tilde{T} = \{Q, G\}$ for $G = \rho^-(J^+ - I^+) + 2\rho^0 (J^0 - I^0) + \rho^+(J^- - I^-) + \partial \rho^0$ and $J^{(tot)}_a = \{Q, \rho^a\}$. Since both $J^{(tot)}_n$ and $L_n$ are $Q$ exact it follows that

$$L_0 |\text{phys} >= 0 \quad J^{(tot)}_0 |\text{phys} >= 0.$$ 

(9)

Let us select a subspace of the space of physical states on which we further impose $\rho_0^0 |\text{phys} >= 0$. On this subspace $Q_{BRST}$ is reduced to $Q_{BRST}^{(red)}$ which does not include $\rho_0^0$ and $\chi_0^0$ as follows from the decomposition

$$Q_{BRST} = \chi_0^0 J^0_{0}^{(total)} + \rho_0^0 M + Q_{BRST}^{(red)}$$

(10)

where $M = 2 \sum_n \chi^+_n \chi^-_n$. Once we compute the $Q_{BRST}^{(red)}$ cohomology, it will be easy to obtain the full $Q_{BRST}$ cohomology, since the representatives (modulo $Q_{BRST}$ exact states) of the physical states would be $^{[12]}$

$$N |\Psi > + \chi_0^0 |\Psi > + |\Psi' >$$

(11)

where $N = \sum_n \chi^+_n \chi^-_n$, and $|\Psi >$ and $|\Psi' >$ are representatives of the $Q_{BRST}^{(red)}$ cohomology. A direct computation of the cohomology of $Q_{BRST}^{(red)}$ in the bosonization of eqn. (2) is not an easy task due to the appearance of cubic and quartic terms. We thus follow a different route where the latter is mapped $^{[14]}$ into a nilpotent operator which is a sum of anti-commuting terms acting on different sectors.
of the theory. Therefore, the corresponding cohomology is a direct sum of simpler cohomologies. Let us define the dimension \((0,0)\) operators of zero ghost number

\[
R = \oint \frac{dz}{2\pi i} (\chi^+ \rho^- \gamma^- + 2\chi^+ \rho^0 \gamma^+ - \chi^+ \rho^+ \gamma^+) \\
P = -\oint \frac{dz'}{2\pi i} (i\phi^+ (\beta^+ \gamma^+ + \beta^- \gamma^- + \chi^+ \rho^- - \chi^- \rho^+))',
\]

(12)

where \(\oint \frac{dz}{2\pi i}\) means that the zero modes of \(\phi^+\) were excluded. We then use these operators to transform \(Q_{BRST}^{(red)}\) to the desired form (for details see the appendix) in the following way

\[
e^{-P} e^R Q_{BRST}^{(red)} e^{-R} e^P = Q_{\text{tr}}^{(red)}
\]

(13)

with

\[
Q_{\text{tr}}^{(red)} = \oint \frac{dz}{2\pi i} [\chi^- \beta^+ + 2i\chi^0 \partial \phi^- - 2t\chi^+ \partial \gamma^- - 2t\phi^+_0 \chi^+ \gamma^-] \\
= 2 \sum_{n \neq 0} \chi^0_n \phi^-_n + \sum_n (\chi^-_n \beta^+_n - 2t(\phi^+_0 - n - 1))\chi^+_n \gamma^-_n).
\]

(14)

The mode expansions are relative to the vacuum of the twisted theory (i.e. \(\gamma(z) = \sum_n \gamma_n z^{-(n+1)}\)). From (13) it follows that the cohomologies of \(Q_{BRST}^{(red)}\) and of \(Q_{\text{tr}}^{(red)}\) are isomorphic, namely, for every state \(|\Phi_0\rangle\) in the cohomology of \(Q_{\text{tr}}^{(red)}\), the state \(|\Psi\rangle = e^{-R} e^P |\Phi_0\rangle\) is in the cohomology of \(Q_{BRST}^{(red)}\) and vice versa.

On the following direct sum of Fock spaces

\[
\bigoplus_{n \neq 0} F(\chi^0_{-n}, \rho^0_{-n}, \phi^-_{-n}, \phi^+_{-n}) \bigoplus_n F(\chi^-_{-n}, \rho^+_n, \gamma^+_n, \beta^+_n) \bigoplus_n F(\chi^+_{-n}, \rho^-_{-n}, \beta^-_{-n}, \gamma^-_{-n})
\]

(15)

the first term is subjected to the action of the first term in eqn. (14), and similarly for the second and third terms. It is thus apparent that \(Q_{\text{tr}}^{(red)}\) indeed decomposes into a sum of anti-commuting terms which act on separate Fock spaces and, therefore, that the cohomology ring is a direct sum of smaller ones. In the first
and second parts of the Fock space, the cohomology ring includes only a single state which is the corresponding vacuum state. In the third part one finds states, in addition to the vacuum, for $\phi_0^+ = n + 1$ when $n$ is an integer. The vacuum $|0\rangle_{phys}$ corresponds to the twisted energy-momentum tensor $\tilde{T} = T + \partial J^{(tot)}_0$.

It is related to the $SL(2, C)$ invariant vacuum $|0\rangle_{SL(2, C)}$, and it is annihilated by the following zero modes: $\chi_0^- |0\rangle_{phys} = \gamma_0^+ |0\rangle_{phys} = \rho_0^- |0\rangle_{phys} = 0$. The only zero modes acting on the vacuum are, therefore, $\chi_0^+, \rho_0^+$ and $\beta_0^\pm$. This is a lowest weight vacuum for both $SL(2, R)$ algebras.

In the case $\phi_0^+ = -n + 1$, $n > 0$, the cohomology of $Q^{(red)}_{tr}$ is in the span of the states

$$\rho_{-n}^s \gamma_{-n}^r |\phi_0^-, \phi_0^+ >$$

where $s = 0, 1$ and $r = 0, 1, \ldots$ and $|\phi_0^-, \phi_0^+ >$ is the vacuum state with momenta $\phi_0^-, \phi_0^+$. The condition of vanishing $J^{(total)}_0$ on the physical states, namely,

$$J^{(total)}_0 \rho_{-n}^s \gamma_{-n}^r |\phi_0^-, \phi_0^+ >= (1 - s - r + \phi_0^-) \rho_{-n}^s \gamma_{-n}^r |\phi_0^-, \phi_0^+ >= 0$$

(16)

gives the relation

$$s + r = \phi_0^+ + 1$$

(17)

which also agrees with the $L^{(total)}_0 = 0$ condition, i.e. $(\phi_0^- + 1)(\phi_0^+ - 1) = -n(s + r)$.

For positive $\phi^+$ momentum, $\phi_0^+ = n + 1$, $n \geq 0$, the cohomology is in the span of the states $\chi_{-n}^s \beta_{-n}^r |\phi_0^-, \phi_0^+ >$ where $s = 0, 1$ and $r = 0, 1, \ldots$. The $J^{(total)}_0 = 0$ constraint now implies that

$$s + r = -(\phi_0^- + 1)$$

(18)

which again agrees with the $L^{(total)}_0 = 0$ condition, i.e.

$$(\phi_0^- + 1)(\phi_0^+ - 1) = -n(s + r).$$

When $\phi_0^+$ is not an integer we have a one dimensional cohomology spanned by
the vacuum $|\phi_0^+, \phi_0^- = -1 >$.

The nontrivial $Q_{tr}^{(red)}$-cohomology states are therefore spanned by

$$|\Phi_0 > = \rho_{-n,\gamma_{-n}}^{r-} |\phi_0^+ = -n + 1, n > 0, \phi_0^- = r >$$

$$|\Phi_0 > = \gamma_{-n}^r |\phi_0^+ = -n + 1, n > 0, \phi_0^- = r - 1 >$$

$$|\Phi_0 > = \chi_{-n,\beta_{-n}}^{+} |\phi_0^+ = n + 1 >, n > 0, \phi_0^- = -r - 2 >$$

$$|\Phi_0 > = \beta_{-n}^r |\phi_0^+ = n + 1 >, n > 0, \phi_0^- = -r - 1 >$$

for $r = 0, 1, 2, \ldots$, and

$$|\Phi_0 > = |any\phi_0^+, \phi_0^- = -1 > .$$

We can now insert $|\Phi_0 >$ into the expressions for the states in the cohomology of $Q_{BRST}^{(red)}$ as follows

$$|\Psi > = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} R^n \sum_{m=0}^{\infty} \frac{P_0^m}{m!} |\Phi_0 > = e^{-R} e^{P_0} |\Phi_0 >$$

where

$$P_0 = - \sum_{n \neq 0} \frac{1}{n} \phi_n^+ (\beta_m \gamma_{-m-n}^- + \chi_m \rho_{-m-n}^-)$$

($P_0$ is obtained from $P$ by omitting the operators that are zero on $|\Phi_0 >$), and

$$R = \sum \chi_n^+ \rho_l^+ \gamma_m \gamma_{-n-l-m}^- - \chi_n^+ \rho_l^+ \chi_m^+ \gamma_{-n-l-m}^+ + 2 \sum \chi_n^+ \rho^0_n \gamma_{-n-m}^+.$$

This way we obtain explicitly the cohomology of $Q_{BRST}^{(red)}$. 

9
2. Physical states of the $SL(2,R)/SL(2,R)$ models versus those of the $c \leq 1$ models.

Let us now compare the structure of the cohomology ring of our twisted $SL(2,R)/SL(2,R)$ model to that of the $c \leq 1$ Liouville model. At ghost number $N_G = -1$, we expect that the discrete states found above would correspond to elements of the ground ring \cite{15}(recall the shift in the ghost number when moving from states to operators because $|0 >_{phys} = |\chi_1^+ > |0 >_{SL(2,C)}$). The lowest level state is simply $\rho_{-1}^- |\phi_0^+ = \phi_0^- = 0$ which corresponds to the identity operator. The next two states of the cohomology of $Q^{(red)}_{tr}$ which are at level 2 translate into operators in the cohomology of $Q_{BRST}^{(red)}$ as follows:

\[ \rho_{-2}^- |\phi_0^+ = -1, \phi_0^- = 0 \rightarrow \tilde{y} = [-i\partial \phi^+ + \chi^+(\rho^- + 2\rho^0 \gamma^+ + \rho^+ [(\gamma^-)^2 - (\gamma^+)^2])] e^{-i\phi^-} \]

These states are (with the appropriate identification) at the same momenta as those of the ground ring generators in the $c \leq 1$ models. In fact $\tilde{y}$ is equal to $y$ of ref. \cite{15} with some additions from the “topological sectors”. One can also change the form of $\tilde{x}$ so it resembles that of the ground ring $x$ by adding a $Q_{BRST}^{(red)}$ exact term as follows

\[ \tilde{x} = \{Q_{BRST}^{(red)}, \frac{1}{2} \rho^0 (\beta^-)^{-1} e^{i\phi^+} \} + (\beta^-)^{-1} (\chi^+ \rho^- + i\partial \phi^- + \beta^+ \gamma^+ - \chi^- \rho^+) e^{i\phi^+} \]

The ground ring cohomology is now generated by $\tilde{x}^n \tilde{y}^m$. As in the ground ring of ref. \cite{15}, it is easy to realize that area preserving diffeomorphisms leave the ground ring invariant. These $W_\infty$ transformations are generated by currents constructed by acting on the $N_G = 1$ cohomology operators with $G_{-1}$. Recall that $G = \rho^- (J^+ - I^+) + 2\rho^0 (J^0 - I^0) + \rho^+ (J^- - I^-) + \partial \rho^0$. For instance the generators $\partial \tilde{x}$ and $\partial \tilde{y}$ take the following form

\[ \partial \tilde{x} = G_{-1} (\chi^+ e^{-i\phi^+}) = \beta^- e^{-i\phi^+} \]
\[ \partial \tilde{y} = G_{-1} (\chi^+ (\beta^-)^{-1} e^{i\phi^-}) = e^{i\phi^-} \]
It is easy to check that indeed, as is hinted by the notations,

\[ \partial_{\tilde{x}} x = \partial_{\tilde{y}} y = 1 \quad \partial_{\tilde{x}} y = \partial_{\tilde{y}} x = 0. \tag{22} \]

One may wonder about the operator \((\beta^{-})^{-1}\) which does not seem to be an appropriate operator to use since \(\beta^{-} = J^+ - I^+\). Without the inclusion of arbitrary powers of \(\beta^{-}\) the space of physical states of the \(\frac{SL(2,R)}{SL(2,R)}\) model does not recover that of the \(c \leq 1\) models. As will be clarified soon a similar situation is facing us also in the tachyonic sector. In the summary we raise another possible prescription for regaining a full equivalence in the states. Here we implement an idea of ref. \[7\] where a further bosonization is invoked for the \((\beta^- , \gamma^-)\) system as follows \(\beta^- \equiv e^{u} - iv\) and \(\gamma^- \equiv -i\partial ve^{-u+iv}\), where \(u, v\) are free bosons with a background charge of \(-\frac{1}{2}\) and \(\frac{1}{2}\) respectively. In terms of the latter bosons, one is entitled to take any arbitrary power of \(\beta^-\) and hence we complete the missing states in the comparison with the gravitational models.

One branch of the tachyons of the \(c \leq 1\) model can be easily identified with a sector of the cohomology of the \(\frac{SL(2,R)}{SL(2,R)}\) model: this is the vacuum of the latter, \(|\phi^+_0 = \text{p}^+, \phi^-_0 = -1\rangle\) which corresponds to the operator \(\chi^+ e^{ip^+\phi^- - i\phi^+}\). If one identifies \(\phi_J\) with the matter field \(X\), \(\phi_I\) with the Liouville field \(\phi\) and \(\chi^+\) with \(c\), the tachyonic states of one branch are indeed found. However, the other branch \(\chi^+ e^{ip^-\phi^+ + i\phi^-}\), is missing in the cohomology of \(Q_{BRST}^{\text{red}}\). There are, however, additional states with no excitations at \(N_G = 0\). These are the states \((\beta^{-}_0)^r |\phi^+_0 = 1, \phi^-_0 = -r - 1\rangle\) corresponding to the operators \(\chi^+(\beta^-)^r e^{i\phi_0} \phi^+ + i\phi^-\). Apart from the appearance of the operator \(\beta^-\) these states are identical to a discrete series of the other branch of the tachyons. If again we bosonize \(\beta^-\) then \(r\) can take any real number and thus one finds states which correspond to the full missing branch. For \(k = -1\), restricting the values of \(r\) to the integers would correspond to the \(c = 1\) model at the self-dual radius.

The states of other ghost number are also in one to one correspondence with those of the \(c \leq 1\) Fock space relative cohomology. The only exception is that our
second branch of the tachyon appears in both $N_G = 1$ and $N_G = 2$ whereas in the
Liouville model it appears only in the former. A similar situation was revealed in the
$\frac{SL(2,R)}{U(1)}$ analysis of ref. [7].

3. The Twisted $\frac{SL(2,R)}{U(1)}$ Case

The twisted $G$ model[17] is a twisted $N = 2$ supersymmetric $G$-WZW model of
level $k$ coupled to gauge fields in the algebra of $H \in G$. It is thus the usual $G$ model
with an additional set of $(1,0)$ anti-commuting ghosts which take their values in the (negative, positive) roots of $G$ respectively. The derivation of the quantum
action and the analysis of the algebraic structure of these models were presented
in refs. [23,4]. In particular a Kac-Moody algebra at level zero associated with the
group $H$ was identified and used to further twist the model in a similar manner
to the $G$ case, namely $T \rightarrow \tilde{T} = T + \partial \sum_{i \in CSA} (J^{(tot)})^i$, where the summation
is over the Cartan sub-algebra. Using the $(+,−)$ bosonization the physical states
in the free Fock space, as well as in the space of irreducible representations, were
found in ref. [4] by computing the cohomology of $Q$, the sum of the BRST gauge-fixing charge and the twisted supersymmetry charge. An elaborate analysis of the
$\frac{SL(2,R)}{U(1)}$ case at level $k = −3$ was recently given in ref. [7]. Here we briefly describe
the application of the method used above for the $G$ case to that of $\frac{SL(2,R)}{U(1)}$. We
continue to parametrize the $J$ sector as in eqn.(2), whereas from the $I$ sector only
$I^0 = \sqrt{\frac{L}{2}} \partial \phi_I$ is left over. This implies that now we have $\beta^- = \beta^+ = \beta_J$, and
$\gamma^- = \gamma^+ = \frac{1}{2} \gamma_J$. The other differences relative to the content of the $\frac{SL(2,R)}{SL(2,R)}$ model
are the absence of the $\chi^-, \rho^+$ pair and the change of the pre-factor in front of the
$\rho^0, \chi^0$, for instance in eqns. (4),(5), from 2 to 1. Introducing all these alterations
to the derivation of $Q^{(red)}_{tr}$ one finally finds an expression which differs from eqn.
(14) by the omission of the $\chi^- \beta^+$ term. Obviously, this also implies the absence
of the second term in the sum of Fock spaces in eqn.(15). Since this term was
anyhow irrelevant in the determination of the non-trivial cohomology, it follows
that the space of physical states of the $\frac{SL(2,R)}{U(1)}$ model is identical to that of the
$\frac{SL(2,R)}{SL(2,R)}$ model. In particular the $c = 1$ Liouville model, that was shown to have
the same physical states as the $k = -1 \frac{SL(2,R)}{SL(2,R)}$ model, can now be associated with either a $k = -3 \frac{SL(2,R)}{U(1)}$ model if the $J$ sector corresponds to the gravity sector, or a $k = -1 \frac{SL(2,R)}{U(1)}$ model if the $J$ sector is related to the matter sector. Notice that in the former case $\beta^- = \beta_J$ is related to the Liouville sector of the model and thus assigning it to $\sqrt{\mu}$ ($\mu$ here is the cosmological constant), as was suggested in ref. [7], seems more natural than in the $\frac{SL(2,R)}{SL(2,R)}$ model. The equivalence of the cohomologies of the $\frac{SL(2,R)}{SL(2,R)}$ and $\frac{SL(2,R)}{U(1)}$ was revealed also in the $(+, -)$ bosonization scheme,[4] and recently in the appendix of ref. [7].

4. Summary and Discussion

In this note we have constructed the cohomology of the BRST charge of the twisted $\frac{SL(2,R)}{SL(2,R)}$ models at an arbitrary level $k$. The space of physical states was expressed on a Fock space of free fields. The corresponding operators were also written down. As a by product we have identified the cohomology of the twisted $\frac{SL(2,R)}{U(1)}$, as well. At level $k = -1$ we have demonstrated that all the states of the $c = 1$ Liouville theory were recovered. In the general case of $k = \frac{p}{q} - 2$ if one further reduces the cohomology to a Kac-Moody irreducible representation,[8] a correspondence to $(p, q)$ minimal models coupled to 2D gravity is revealed.[2] It is important to note that on the level of the Fock space, all the models of different $k$ are essentially the same, as is depicted by the fact that $t = k + 2$ could in fact be absorbed into the definitions of the free fields $\phi^+$ and $\phi^-$. Moreover, the twisted $\frac{SL(2,R)}{U(1)}$ model also produces the same cohomology on the Fock space. An area preserving diffeomorphism associated with a $W_\infty$ algebra could be identified (as long as we stay on the Fock space). This is the situation for the $c = 1$ case where no further Felder reduction is needed. For $c < 1$ the screening charges do not commute in general with the $W_\infty$ currents. A similar analysis of general $\frac{G}{G}$ and $\frac{G}{H}$ models is under current investigation.

In order to account for all the states of the $c = 1$ gravitational model it was not sufficient to work in the $\beta^{\pm}, \gamma^{\pm}, \phi^{\pm}$ Fock space associated with the Wakimoto
representation of the Kac-Moody currents. Following ref. [7] we had to consider states/operators involving also \((\beta^-)^{-\tau}\). Negative (and, in fact, arbitrary real) powers of \(\beta^-\) could make sense if we further bosonize the \(\beta, \gamma\) system. Had we limited ourselves to the \(\beta^\pm, \gamma^\pm, \phi^\pm\) fields using the \((+,+)\) bosonization, we would have recovered the ground ring generators \(x, y\) (and all other discrete states) as well as one tachyonic branch \(T^+\) and discrete states from the other branch. If instead we use the \((+, -)\) bosonization, (as was used in ref. [2,3,4]), we obtain only the other tachyonic branch \(T^-\). The union of these two Wakimoto\(^{[20]}\) representations produces all the physical states. However, we do not have any good reason why both representations should be considered simultaneously. We also note that the transformation \(\phi^+ \to -\phi^-\), which amounts to \(t \to \frac{1}{t}\) and \(\phi_J \to -\phi_J\), leads to a cohomology composed of the discrete states and \(T^-\). Recall that the analogous transformation in \(c(t)\) matter theory coupled to gravity, \(t \to \frac{1}{t}\) and \(X \to -X\), is a symmetry of the energy momentum tensor.

To establish the full isomorphism between the string theory and the corresponding topological model we have to show that the correlation functions match as well. It is quite important to establish this relation also from a practical point of view. It may allow us to develop and use topological tools for calculations of string correlation functions. Such calculations were done for the \(\frac{SL(2,R)}{U(1)}\) case in ref. [25]. A step towards establishing such a correspondence was made very recently in ref. [7] relating the string theory and the topological \(\frac{SL(2,R)}{U(1)}\) model. It was argued, and explicitly demonstrated for the 4-tachyon amplitudes, that the latter model yields the correlators of the \(c = 1\) string theory at non-zero cosmological constant. It is important to perform calculations at higher genus to establish whether one needs to further couple the topological \(\frac{SL(2,R)}{U(1)}\) model to topological gravity.\(^{[25]}\) The cosmological constant was identified in ref. [7] with \(\beta\). Note that this identification makes the appearance of negative powers of \(\beta^-\) more acceptable. Similar arguments can be put forward for the \(\frac{SL(2,R)}{SL(2,R)}\) model. Note that for this case \(\beta_I\) has to be identified with \(-\sqrt{\mu}\) while \(\beta_J\) should be set to zero due to the matter momentum conservation, namely, there is no need for a screening charge in this
sector. This is in accordance with the argument of ref. [22] that the topological models and the standard continuum description of $W$-gravity yield string theories at different values of the cosmological constant. The real challenging question is whether the duality between string theories and topological theories always holds, and in particular whether (super) string theories admit a TFT description.

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**APPENDIX**

Explicit Derivation of the Transformed Cohomology

We start by splitting $Q_{BRST}^{(red)}$ into two terms which are distinguished by their $(\chi^+, \rho^-)$-ghost number as follows:

$$Q_{BRST}^{(red)} = Q_1 + Q_2,$$

(A.1)

where

$$Q_1 = \oint \frac{dz}{2\pi i} [\chi^- \beta^+ + 2i\chi^0 \partial \phi^- + 2\chi^0 (\beta^+ \gamma^+ + \beta^- \gamma^- + \chi^+ \rho^- - \chi^- \rho^+)]$$

$$Q_2 = \oint \frac{dz}{2\pi i} [\chi^+ (4\partial \gamma^+ - 2t \partial \gamma^- - \beta^+ \gamma^+ \gamma^- - 2\beta^- \gamma^- \gamma^+ - \beta^+ \gamma^- \gamma^- - 2i\gamma^+ \partial \phi^- - 2it \gamma^- \partial \phi^+ + 2\chi^- \rho^0)].$$

(A.2)

The zero modes $\rho^0_0$ and $\chi^0_0$ are understood to be omitted in the following mode expansions. We start by noting that $Q_2 = [Q_1, R] + \tilde{Q}_2$ with

$$R = \oint \frac{dz}{2\pi i} (\chi^+ \rho^+ \gamma^- \gamma^- + 2\chi^+ \rho^0 \gamma^+ - \chi^+ \rho^+ \gamma^+ \gamma^+)$$

$$\tilde{Q}_2 = \oint \frac{dz}{2\pi i} (-2t \chi^+ \partial \gamma^- - 2it \chi^+ \gamma^- \partial \phi^+).$$
It can be checked that: $[\widetilde{Q}_2, R] = 0$ and $[[Q_1, R], R] = 0$. Thus,

$$e^{-R}Q_1e^R = Q_1 + [Q_1, R] + \frac{1}{2}[[Q_1, R], R] + \ldots = Q_1 + Q_2 - \widetilde{Q}_2 = Q_{BRST} - \widetilde{Q}_2$$

and so

$$e^RQ_{BRST}e^{-R} = Q_1 + \widetilde{Q}_2.$$

We will now further decompose $\widetilde{Q}_2 = \widetilde{Q}_2' + \widetilde{Q}_2''$ where

$$\widetilde{Q}_2' = -2t \oint \frac{dz}{2\pi i} (\chi^+ \partial \gamma^- + \phi^+_0 \chi^+ \gamma^-) \quad \widetilde{Q}_2'' = -2it \oint \frac{dz}{2\pi i} (\chi^+ \gamma^- \partial \phi^+)$$

(the rhs of the second expression does not include the zero mode $\phi^+_0$), and,

$$Q_1 = Q_1' + Q_1''$$

$$Q_1' = \oint \frac{dz}{2\pi i} (\chi^- \beta^+ + 2i\chi^0 \partial \phi^-)$$

$$Q_1'' = \oint \frac{dz}{2\pi i} (2\chi^0 (\beta^+ \gamma^- + \beta^- \gamma^- + \chi^+ \rho^- - \chi^- \rho^+)).$$

We now define the zero dimension zero ghost number operator $P$ by

$$P = -\oint \frac{dz}{2\pi i} (i\phi^+ (\beta^+ \gamma^- + \beta^- \gamma^- + \chi^+ \rho^- - \chi^- \rho^+)) \quad (A.3)$$

where it should be understood that in $\phi^+$ we do not include the terms $\widetilde{\phi}^+ + \phi^+_0 \log(z)$ ($\widetilde{\phi}^+$ being canonically conjugate to $\phi^-_0$). In other words in mode expansion $P$ takes
the form

\[
P = -\sum_{n \neq 0} \frac{1}{n} \phi_n^+ (\beta_n^+ \gamma_{-m-n}^+ + \beta_n^- \gamma_{-m-n}^- + \chi_n^+ \rho_{-m-n}^- - \chi_n^- \rho_{-m-n}^+).
\]

Now we can calculate:

\[
Q_1'' = -[Q_1', P] \quad [Q_1'', P] = -[[Q_1', P], P] = 0
\]

\[
[\tilde{Q}_2', P] = -\tilde{Q}_2'' \quad [\tilde{Q}_2'', P] = 0
\]

So,

\[
e^P \tilde{Q}_2' e^{-P} = \tilde{Q}_2' + \tilde{Q}_2'' = \tilde{Q}_2
\]

\[
e^P Q_1' e^{-P} = Q_1' + Q_1'' = Q_1
\]

and finally,

\[
e^{-P} e^R Q_{BRST} e^{-R} e^P = Q_1' + \tilde{Q}_2' = Q_{tr}
\]

The cohomology of \( Q_{BRST} \) is, therefore, isomorphic to that of \( Q_{tr} \) (defined by the above equation) which is much easier to compute.
REFERENCES

1. R. Dijkgraaf, E. Verlinde, and H. Verlinde, *Nucl. Phys.* B352 (1991) 59; “Notes On Topological String Theory And 2D Quantum Gravity,” Princeton preprint PUPT-1217 (1990).

2. O. Aharony, O. Ganor N. Sochen J. Sonnenschein and S. Yankielowicz, “Physical states in the $G$ models and two dimensional gravity”, TAUP- 1947-92 April 1992 to appear in *Nucl. Phys.* B.

3. O. Aharony, J. Sonnenschein and S. Yankielowicz, “$G$ models and $W_N$ strings”, *Phys. Lett.* B289B (1992) 309.

4. O. Aharony, O. Ganor J. Sonnenschein and S. Yankielowicz, “On the twisted $G/H$ topological models”, TAUP- 1990-92 August 1992 to appear in *Nucl. Phys.* B.

5. F. David *Mod. Phys. Lett.* A3 (1988) 1651; J. Distler and H. Kawai *Nucl. Phys.* B321 (1989) 509.

6. E. Witten *Phys. Rev.* D44 (1991) 314.

7. S. Mukhi and C. Vafa “Two Dimensional Black hole as a Topological Coset model of $c = 1$ String Theory”, HUTP-93/A002, TIFR/TH/93-01.

8. D. Bernard and G. Felder *Comm. Math. Phys.* 127 (1991) 145.

9. K. Bardacki, E. Rabinovici, and B. Serin *Nucl. Phys.* B299 (1988) 151.

10. K. Gawedzki and A. Kupianen, *Phys. Lett.* 215B (1988) 119, *Nucl. Phys.* B320 (1989) 649.

11. D. Karabali and H. J. Schnitzer, *Nucl. Phys.* B329 (1990) 625.

12. P. Bouwknegt, J. McCarthy and K. Pilch Cern Preprint TH-6162/91

13. P. Bouwknegt, J. McCarthy and K. Pilch *Phys. Lett.* B234B (1990) 297, *Comm. Math. Phys.* 131 (1990) 125.
14. H. Ishikawa and M. Kato “Equivalence of BRST cohomology for 2-d black hole and c=1 Liouville theory” UT-Komaba/92-11 Nov. 1992.

15. E. Witten *Nucl. Phys. B*377 (1992) 55.

16. M. Spiegelglas and S. Yankielowicz “$\hat{G}/G$ Topological Field Theories by Cosetting $G_k$” TAUP-1934 ; “Fusion Rules As Amplitudes in $G/G$ Theories,” Technion PH- 35-90 to appear in *Nucl. Phys. B*

17. E. Witten, *Comm. Math. Phys.* 144 (1992) 189.

18. For a comment about a possible difficulty in case of non-compact groups see ref. [3]

19. Y. Kazama and H. Suzuki, *Nucl. Phys. B*321 (1989).

20. M. Wakimoto, *Comm. Math. Phys.* 104 (1989) 605.

21. B. Feigin and E. Frenkel *Phys. Lett.* 246B (1990) 75.

22. M. Bershadsky, W. Lerche, D. Nemeschansky and N. P. Warner “A BRST operator for non-critical W strings” CERN-TH 6582/92.

23. T. Nakatsu and Y. Sugawara “Topological gauged WZW models and 2 D gravity” Tokyo preprint UT-598 “BRST fixed points and Topological Conformal Symmetry” UT-599.

24. L. H. Hu and M. Yu, “On BRST cohomology of SL(2)/SL(2) gauged WZWN models” Academia Sinica preprint AS-ITP-92-32.

25. E. Witten, *Nucl. Phys. B*371 (1992) 191.