On the Intrinsic Privacy of Stochastic Gradient Descent

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Abstract—Protecting the privacy of training data is important for the safe deployment of machine learning models. Private learning algorithms have been proposed that ensure strong differential-privacy (DP) guarantees. However, the additional noise required for such protection comes at the cost of reduced model utility. Meanwhile, the stochastic gradient descent (SGD) method — the most common optimization algorithm for neural networks — contains intrinsic randomness which has not been leveraged for privacy. Arguing that SGD guarantees intrinsic privacy, we investigate the extent to which this privacy can be quantified and used to improve the utility of privately learned models. In effect, we ask the question; “If SGD were a differentially-private mechanism, how good would it be?”

In this work, we take the first step towards analysing the intrinsic privacy properties of SGD. Our primary contribution is a large-scale empirical analysis of SGD on both convex and non-convex objectives. To this end, we evaluate the inherent variability due to the stochasticity in SGD on 3 different datasets and calculate the $\epsilon$ values due to the intrinsic noise. First, we show that the variability in model parameters due to the random sampling almost always exceeds that due to changes in the data. We observe that SGD provides intrinsic $\epsilon$ values of 7.8, 6.9, and 2.8 on MNIST, Adult, and Forest Covertype datasets respectively, without addition of external noise. Next, we propose a method to augment the intrinsic noise of SGD with additional noise to achieve the desired $\epsilon$. Our augmented SGD outputs model that outperform existing approaches with the same privacy guarantee, thus closing the gap to noiseless utility between 0.19% and 10.07% on our benchmark datasets. Finally, we show that the existing theoretical bound on the sensitivity of SGD is not tight. By estimating the tightest bound empirically, we achieve near-noiseless performance at $\epsilon \approx 1$, closing the utility gap to the noiseless model between 5.13% and 100%. Our experiments provide concrete evidence that changing the seed in SGD is likely to have a far greater impact on the resulting model than including or excluding any given training example. By properly accounting for this intrinsic randomness, higher utility can be achieved without sacrificing further privacy. With these results, we hope to inspire the research community to further explore and characterise the randomness in SGD, its impact on privacy, and the parallels with generalisation in machine learning.

I. INTRODUCTION

Respecting the privacy of users contributing their data to train machine learning models is important. For example, a HIV patient providing their medical record to train a classification model may not want to reveal their participation. Attacks on deep learning models such as membership or attribute inference have shown to reveal significant information about the underlying dataset [19], [44], [59]. These attacks raise privacy concerns about the deployment of models in sensitive applications such as healthcare [49]. To address these concerns, privacy-enhanced stochastic gradient descent (SGD) have been proposed to train models [10], [20], [41], [47], [51]. Making SGD private is especially attractive due to its position as the commonly used optimization method in deep learning.

Private learning algorithms typically use $\epsilon$-differential-privacy [16]($\epsilon$-DP) due to its strong privacy guarantees. $\epsilon$-DP protects privacy by ensuring that the outcome of a query performed on a dataset does not change ‘much’ when a single record in that dataset is changed. Ensuring $\epsilon$-DP often requires the addition of noise to the query, proportional to its sensitivity to changes in the data. While more noise corresponds to smaller $\epsilon$ (and thus more privacy), for machine learning models this noise can have disastrous effects on the model utility. This apparent trade-off between privacy and utility is therefore the main focus while designing differentially-private learning algorithms [3], [37], [38]. However, learning algorithms are often stochastic, and thus framing them as ‘fixed’ queries on a dataset neglects an important source of intrinsic noise. Accounting for this intrinsic noise would allow the addition of less noise to achieve the same $\epsilon$-DP guarantees while resulting in improved model utility. We focus on stochastic gradient descent (SGD) as a widely-applicable stochastic learning algorithm. Its stochasticity is a by-product of avoiding gradient computation over large training datasets but has enhanced generalisation as compared to its non-stochastic counterpart [27]. The properties of SGD have since become an object of study [6], [29], [45], with particular focus on the generalisation performance of the solutions it finds [14], [22], [25]. The fact that SGD exhibits stochasticity is well known to machine learning practitioners, as choice of initialisation point and random seed can impact model performance [18], [23]. At the same time, the fact that SGD is relatively ‘insensitive’ to variations in its input data has been established [22] in terms of uniform stability. Combining these observations, we speculate that the variability in the model parameters (weights) produced by the intrinsic stochasticity of SGD may exceed its sensitivity to perturbations in the input data. To this end, we ask the question: “If SGD were a differentially-private mechanism, how good would it be?”

Our Approach. To answer our research question, we aim to theoretically and empirically analyse the variability present in the output of SGD. During training, SGD introduces randomness in the output from two main sources — (1) random initialization of the model parameters and (2) random sampling of the input dataset. We highlight that training a model on the same data, even with the same initialisation produces different final weights, purely due to the random order of traversal of the dataset - an example is shown in Figure [1]. We argue that rather than viewing this variability as a pitfall
Each curve shows the evolution model weights $w$ can nonetheless produce different models, due to its intrinsic randomness. Fig. 1: A demonstration of the phenomenon that SGD run on a fixed dataset can nonetheless produce different models, due to its intrinsic randomness. Each curve shows the evolution model weights $w^{26}$ and $w^{36}$ throughout training, using runs of SGD on the same data with identical initialisation, with different random seeds and thus data sampling order. The model is logistic regression on the Forest dataset.

of stochastic optimisation, it can instead be seen as a source of noise that can mask information about participants in the training data. Our theoretical analysis shows that we can bound the variability in weights due to randomness, and that this bound tends to greatly exceed the variability in weights due to changes in the data (the sensitivity). This prompts us to consider SGD as an $\epsilon$-differentially-private mechanism, with some intrinsic $\epsilon_i$ value, referred to as $\epsilon_i$. To calculate $\epsilon_i$, we propose a novel method that characterises SGD as a Gaussian mechanism and estimates the noise variance for a given dataset. To the best of our knowledge, ours is the first work to report the empirical calculation of $\epsilon_i$ values based on the observed distribution. We claim that training models using SGD provides reasonable intrinsic privacy under certain assumptions on the loss function. To obtain enhanced privacy, we propose an algorithm for an augmented differentially-private SGD that takes into account the intrinsic $\epsilon_i$ guarantees and thereby provides better utility. We use the output perturbation techniques as proposed in prior work to demonstrate that accounting for the randomness of SGD ensures the same privacy guarantees with consistently higher utility.

Results. We empirically evaluate the connection of variability in SGD to its privacy guarantees. First, we compute the intrinsic $\epsilon_i$ values using the known theoretical sensitivity bounds of SGD for 3 different datasets. We observe that SGD provides $\epsilon_i$ values of 7.8, 6.9, and 2.8 on MNIST (binary), Adult and Forest Covertype datasets respectively. Next, we add noise to the model parameters trained on each of these datasets to achieve a desired $\epsilon$ value of 1 and compare the utility of these models to prior work. We observe that accounting for the intrinsic noise in SGD consistently results in higher utility, indicating that this $\epsilon_i$ is an important parameter to consider while designing provide learning algorithms.

In addition, we calculate the empirical sensitivity bounds using a public partition of our benchmarks datasets. We observe that the earlier theoretical bounds on the sensitivity aren’t tight, and that improved utility could be attained with tighter bounds, which we estimate empirically. These empirical sensitivity bounds allow us to retain utility values close to the original accuracy of the noiseless model parameters. Our experiments confirm our theoretical claim that the randomness in SGD helps to mask the sensitivity of the algorithm towards changes of a single input sample.

Contributions. We summarize our contributions below:

- Variability in SGD: We theoretically demonstrate that the change in weights due to the randomness in SGD has — with high probability — a strictly higher bound than the dataset sensitivity. We empirically demonstrate that this variability due to randomness almost always exceeds the upper bound on the sensitivity due to changes in the data, and that this effect is magnified when both model initialisation and dataset traversal order are allowed to vary.

- Computing intrinsic $\epsilon_i$ values: We propose a first concrete technique to compute the intrinsic $\epsilon_i$ values from the output distribution of model parameters trained using SGD. In essence, we characterise SGD as a Gaussian mechanism and provide techniques to estimate its parameters. We report that these $\epsilon_i$ values range from 0.9 to 7.9, indicating that SGD can provide intrinsic privacy similar to that achieved on moderately complex tasks with dedicated private learning algorithms.

- Utility Evaluation: We empirically evaluate the utility of models trained on 3 datasets with $(\epsilon, \delta)$-DP guarantees taking into account the inherent noise from SGD with the theoretical sensitivity bounds, with particular focus on $\epsilon = 1$ and $\delta$ as the inverse of the training data size. We observe that this approach closes the gap between the performance of a model trained privately without accounting for intrinsic noise, and a noiseless model, by between 0.19% and 10.07% for our target benchmark datasets. This improvement increases to between 3.13% and 100% when using a tighter sensitivity bound computed empirically, allowing our approach to provide $(1, \delta)$-DP with almost no loss in utility for that dataset.

II. Problem & Background

In this work, we investigate whether the popular optimization algorithm — stochastic gradient descent exhibits inherent privacy properties, and evaluate the extent to which these allow for improving the utility of private models.

A. Attacker Model

We consider a scenario where the training happens using SGD as an optimization algorithm in a trusted environment and only the final model parameters are released to an untrusted party. This is a common application scenario, for example, a DNA analysis service like 23andMe might train their models in-house but upload the model parameters on the cloud to offer inference-as-a-service. Hence, we consider a white-box threat model where the adversary such as a compromised cloud provider has access to the final model parameters. Note
that our white-box model differs from prior work that allows the adversary to observe the gradients and weights after each iteration during training \[33\]. In our setting, the adversary has access to the full trained model, including its architecture, and we assume details of the training procedure are public (e.g. batch size, number of training iterations, learning rate), but not the random seed used to initialise the model parameters and sample inputs from the dataset. This means the adversary can run arbitrary unlimited inference queries on the model and observe its output. With the above knowledge set, the attacker aims to infer private information about the data present in the training dataset. We assume that even leaking whether a single data sample was present or absent in the training dataset poses a serious privacy risk to the user contributing the sample.

**B. Preliminaries**

In our analysis of the privacy guarantees of stochastic gradient descent (SGD) algorithm, we use the well-known notion of \((\epsilon, \delta)\)-differential-privacy. We provide a brief overview of differential privacy mechanism and SGD algorithm below.

**Differential Privacy.** Differential privacy provides one of the strongest notion of privacy guarantees to hide the participation of an individual sample in the dataset \[16\]. To state informally, it ensures that the presence or absence of a single data point in the input dataset does not change the output by much. It is formally defined as follows:

**Definition 1** \((\epsilon, \delta)\)-Differential Privacy. A mechanism \(M\) with domain \(I\) and range \(O\) satisfies \((\epsilon, \delta)\)-differential privacy if for any two neighbouring datasets \(S, S' \in I\) that differ only in one input and for a set \(E \subseteq O\), the following holds:

\[
\Pr(M(S) \in E) \leq e^{\epsilon} \Pr(M(S') \in E) + \delta
\]

For \(\delta = 0\), the mechanism \(M\) is simply \(\epsilon\)-differentially private. Informally, \((\epsilon, \delta)\)-differential privacy ensures that for all adjacent datasets \(S \) and \(S'\), the privacy loss of any individual datapoint is bounded by \(\epsilon\) with probability at least \(1 - \delta\) \[17\]. In our setting, datasets consist of \(N\) training examples, and two datasets are adjacent if they share \(N - 1\) training examples in the same order. A well established method to design \(\epsilon\)-differentially private mechanism is to add noise proportional to the \(\ell_1\) or \(\ell_2\) sensitivity of single input sample either at the input, during the algorithm or at the output. Since our threat model assumes only the final parameters are observable to the adversary, we use the output perturbation approach for designing \(\epsilon\)-DP mechanism, adding noise right before releasing the model parameters. We measure the influence of an input sample using the \(\ell_2\)-sensitivity and define it as follows:

**Definition 2** \(\ell_2\)-Sensitivity (From Def 3.8 in \[17\]). Let \(f\) be a function that maps a dataset to a vector in \(\mathbb{R}^d\). Let \(S, S'\) be two datasets such that they differ in one data point. Then the \(\ell_2\)-sensitivity of a function \(f\) is defined as:

\[
\Delta_2(f) = \max_{S, S'} \|f(S) - f(S')\|_2
\]

From prior work on differential privacy, we have the following theorem that ensures \((\epsilon, \delta)\)-differential privacy by addition of Gaussian noise to the output.

**Theorem 1. From \[17\]** Let \(f\) be a function that maps a dataset to a vector in \(\mathbb{R}^d\). Let \(\epsilon \in (0, 1)\) be arbitrary. For \(c^2 > 2 \ln (1.25/\delta)\), adding Gaussian noise sampled using the parameters \(\sigma \geq c\Delta_2(f)/\epsilon\) guarantees \((\epsilon, \delta)\)-differentially privacy.

Thus, a Gaussian mechanism that adheres to the above theorem is inherently differentially-private. Using this information, we analyse the privacy properties of SGD by characterising it as a Gaussian mechanism and inferring its parameters.

**Stochastic Gradient Descent.** Stochastic gradient descent (SGD) is the most common optimisation method for training machine learning models \[5\]. The traditional gradient descent (GD) method optimizes the model parameters \((w)\) by computing the gradients over all the input-label pairs \((x, y)\) of the dataset in each training step. Using \(\eta\) as the constant step-size or learning rate, \(L(w, (x, y))\) as the loss function, and \(N\) as the number of training samples, the gradient descent algorithm consists of iterative steps:

\[
w_{t+1} = w_t - \eta \sum_{i=1}^{N} \nabla_w \mathcal{L}(w_t, (x_i, y_i))
\]

SGD provides a stochastic approximation of the traditional gradient descent by estimating the gradient of the objective function \(\mathcal{L}\) at a random input instead of computing gradients over the entire dataset at once. At step \(t\), on selecting a random sample \((x_t, y_t)\) the gradient update function \(G\) performs:

\[
w_{t+1} = G(w_t) = w_t - \eta \nabla_w \mathcal{L}(w_t)
\]

Gradient estimation in this way allows for training on very large datasets. In practice, instead of a single sample, the gradient update is performed using a mini-batch of samples. There is a rich line of work exploring the impact of minibatch size \((B)\) and learning rate \((\eta)\) on model performance and generalisation \[22, 31\]. However, these hyperparameters are typically chosen empirically using cross-validation. SGD is guaranteed to converge \[8\], to a unique minimum if \(\mathcal{L}\) is convex in \(w\), and to a local minimum otherwise. There are variants of SGD depending on the point in the algorithm when the shuffling or randomization of the dataset is performed. For our analysis, we consider that the dataset is shuffled at the beginning of each epoch and the mini-batches are sampled without replacement. This approach has shown to provide better convergence properties \[24, 43\], however we expect other variants share the variability we study.

**C. Problem Statement**

We investigate whether the stochasticity in SGD provides appreciable privacy, through theoretical and empirical analysis.
The variability due to random sampling in SGD is a well-known phenomenon in machine learning. It is often overcome by reporting results averaged over different random seeds, or treating the random seed as a hyperparameter and selecting it using cross-validation. We argue that this randomness is actually helpful for ensuring privacy. Assuming the random seed is unknown to the adversary, it adds inherent noise to the output of SGD and protects the privacy of training data points as per the notion of differential privacy. To this end, we formulate our problem with the following concrete questions:

1. **Does the variability in SGD exceed the sensitivity due to change in an individual input sample?**

   To answer this, we use the results from Hardt et al. [22] and Wu et al. [51] that allow us to bound the difference (variability) in the model parameters when trained with SGD using a convex objective function. Specifically, we use the **boundedness** and **expansiveness** properties of the gradient update rule in SGD to establish our results. Our theoretical analysis shows that variability in SGD is bounded by a quantity which dominates the upper bound of sensitivity due to the change in an individual sample. We provide a detailed theoretical analysis in Section III. We validate our theoretical finding with experimental results in Section VI-B.

2. **What is the intrinsic privacy guarantee of SGD?**

   To quantify the intrinsic privacy guarantees of SGD, we aim to calculate the intrinsic $\epsilon_i$ values for any given dataset. As lower $\epsilon_i$ values correspond to better privacy, we use $\epsilon_i$ as our metric to comment on the privacy guarantees of SGD. For this, we propose a novel algorithm to calculate the $\epsilon_i$ values for a given distribution of model parameters trained using SGD (discussed in Section IV). Essentially, we interpret the posterior distribution returned by SGD computed with many random seeds as the output of a Gaussian mechanism and estimate its parameters using the theoretical sensitivity bounds from prior work. Further, we propose a variant that achieves tighter sensitivity bounds using empirical analysis on public data of the same distribution. We empirically compute $\epsilon_i$ values for our target datasets (Section VI-C).

3. **Can the intrinsic privacy of SGD improve utility?**

   Lastly, we question whether the intrinsic privacy can be leveraged while designing privacy-preserving SGD algorithms to achieve superior model utility without damaging privacy guarantees. We propose an augmented DP-SGD algorithm using the output perturbation technique as shown in prior work [4], [10], [47], [51]. Section V gives the details of this algorithm. Our augmented DP-SGD approach entails the addition of strictly less noise to achieve target $\epsilon$ values and thus provides improved utility. Section VI-D evaluates the utility improvement using the intrinsic $\epsilon_i$ values as compared to prior work on the three datasets we consider.

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1. Recall that variability refers to the difference in the model parameters due to the randomness of SGD, while sensitivity is due to the change in the input sample (as in Def. 22), but both refer to a difference in model weights trained and hence are comparable.

### III. Theoretical Analysis of Variability in SGD

We theoretically analyse the variability in SGD which arises due to the randomness in the algorithm, and compare it to the sensitivity due to changes in the data, which has been analysed previously. Both the variability and sensitivity capture the difference in the weights and hence are comparable values. We first state the assumptions under which our results hold.

#### A. Assumptions

We consider that SGD uses a loss function $L: W \mapsto \mathbb{R}$ for optimizing the model parameters and $L$ satisfies the following properties. Let $W \subseteq \mathbb{R}^d$ be the hypothesis space.

- $L$ is convex for any $u, v \in W$; i.e., $L(u) \geq L(v) + \langle \nabla L(v), u - v \rangle$.
- $L$ is $L$-Lipschitz for any $u, v \in W$; i.e., $\|L(u) - L(v)\| \leq L\|u - v\|$.
- $L$ is $\beta$-smooth; i.e., $\|\nabla L(u) - \nabla L(v)\| \leq \beta\|u - v\|$.

#### B. Sensitivity of SGD without Randomness

Recently, Wu et al. provided a theoretical sensitivity bound for SGD due to the change in a single sample [51]. Their results rely on the boundedness and expansiveness properties for the gradient update rule $(G)$ of SGD as proposed by Hardt et al. [22]. We present their results for the sensitivity of the model parameters trained using SGD due to a change in a single input datapoint.

**Property 1. (Boundedness of $G$.)** For a loss function that is $L$-Lipschitz, the gradient update of SGD is $\eta L$ bounded i.e.,

$$\sup_{w \in W} \|G(w) - w\| \leq \eta L$$

**Property 2. (Expansiveness of $G$.)** For a loss function that is $\beta$-smooth, and $\eta \leq 2/\beta$, then the gradient update of SGD is $1$-expansive i.e.,

$$\sup_{w, w'} \frac{\|G(w) - G(w')\|}{\|w - w'\|} \leq 1$$

**Theorem 2. From [51].** Let $A$ denote the SGD algorithm using $r$ as the random seed then the upper bound for sensitivity for $k$-passes of SGD is given by:

$$\hat{\Delta}_S = \max_r \|A(r; S) - A(r; S')\| \leq 2k L \eta$$

Here, $\hat{\Delta}_S$ gives the maximum difference in the model parameters due to the presence or absence of a single input sample. As this is not the main contribution of our paper, we refer interested readers to the original paper for a formal proof [51]. This section provides a brief intuition for achieving the bound. For a single pass of SGD over neighbouring datasets $S$ and $S'$ with a fixed initialization and fixed sampling strategy, the two executions $G$ and $G'$ will differ only at a single step — when the differing sample gets selected. In that case, from the above boundedness property, we have that $G(w) - G'(w') \leq 2L \eta$. For all the other steps, the samples selected are exactly same and hence the $1$-expansiveness...
property applies. Therefore, after $k$-passes of SGD over the dataset, the difference in the model parameters will have an upper bound of $2kL\eta$. In practice, when trained using a batchsize of $B$, the sensitivity bound can be reduced by a factor of $B$ i.e., $\Delta_S \leq 2kL\eta/B$. Henceforth, the theoretical sensitivity always refers to the one with batchsize $B$.

### C. Variability due to the Randomness in SGD

We use the boundedness and expansivity properties of the gradient update rule to calculate the upper bound for variability in SGD. Recall that variability captures the difference in the model parameters due to the stochastic process of selecting the input samples. We use a similar argument as in prior work for calculating the bound at each step of SGD. For a single pass of SGD on dataset $S$ with fixed initialization but different random seeds $r$ and $r'$ for sampling inputs, in the best case, every step encounters different samples. Thus, using the boundedness property, each step will add a $2L\eta$ deviation between the model parameters. Therefore, for a $k$-pass SGD on dataset of size $N$ such that each step selects a differing sample, we get a variability bound of:

$$\hat{\Delta}_V = \max_{r,r'}||A(r;S) - A(r';S')|| \leq 2kLN\eta$$

**Claim 1.** The upper bound of variability due to the randomness in SGD is strictly greater than or equal to the sensitivity of SGD due to the change in a single input sample i.e., $\hat{\Delta}_V > \Delta_S$

The above claim gives a weak guarantee about the inherent noise in SGD as it considers the upper bound of variability. To improve the guarantee, we consider $\Delta_V$ to be a random variable itself and apply the Chebyshev inequality. We need the expected value and variance of $\Delta_V$. The bounds stated previously rely on the fact that every iteration of SGD with mis-matching samples introduce a term of $2L\eta$ to the (maximum) difference in outputs. For the upper bound, we assumed that every sample is mis-matching, that is we compare runs of SGD where one is a perfect derangement of the training-set traversal order of the other. In reality, between two runs with different random seeds, the same example may be encountered at the same time-point; this would constitute a permutation of the training data with a fixed point. If we assume that $X_i$ is the number of fixed points of the training data in epoch $i$ (relative to a fixed reference permutation), the number of mismatches is therefore $N - X_i$, and the bound on the difference of weights is

$$\hat{\Delta}_V = \sum_{i=1}^{k} 2L\eta(N - X_i)$$

The probability distribution of $X_i$ is

$$P(X_i = j) = \frac{D_{N,j}}{N!}$$

where $N$ is the number of training examples, and $D_{N,j}$ is a rencontres number giving the number of permutations of length $N$ with $j$ fixed points. For large $N$, the distribution of rencontres numbers approaches a Poisson distribution with rate parameter $\lambda = 1/2$. and so both the expected value and variance of $X_i$ are 1: This allows us to use standard properties of expectation and variance, and the fact that the permutation (and thus $X_i$) selected at each epoch is independent.

$$E[\hat{\Delta}_V] = \sum_{i=1}^{k} 2L\eta(N - E[X_i]) = 2kLN(N - 1)$$

$$\forall[\hat{\Delta}_V] = \sum_{i=1}^{k} \forall[2L\eta(N - X_i)] = (2L\eta)^2 k$$

We then use the Chebyshev inequality to bound the probability that $\hat{\Delta}_V$ is far from its mean. Doing so is interesting because we can prove that $\hat{\Delta}_V$ is unlikely to be near $\Delta_S$. If we define $t = \lfloor E[\hat{\Delta}_V] - \Delta_S \rceil/2 = kLN(N - 2)$ then by Chebyshev inequality:

**Claim 2.** The bound on the variability of SGD is larger than its sensitivity with high probability.

$$P \left[ |\hat{\Delta}_V - E[\hat{\Delta}_V]| \geq kLN(N - 2) \right] \leq \frac{4}{k(N - 2)^2}$$

Since $N$ is typically large, we see that the probability $\hat{\Delta}_V$ is sufficiently far from its mean and near $\Delta_S$ is very low.

### IV. Estimating $\epsilon$ for SGD

Every run of SGD produces a potentially different result, thanks to the stochasticity arising from the unspecified random seed. We therefore think of SGD as a mechanism for sampling model weights from some distribution, and equally as a randomised mechanism in the sense of differential privacy. To characterise the intrinsic privacy of SGD, we need to understand the parameters of the resulting weight distribution. While theoretically characterising SGD as a sampling mechanism is a subject of ongoing research [32], in this section, we propose an algorithm (see Algorithm 1) for empirically estimating the privacy properties of SGD, under certain assumptions.

**Compute $\epsilon_i$.** In particular, we wish to estimate what we call the ‘intrinsic’ $\epsilon$ of SGD - $\epsilon_i$. We assume that the noise of SGD is normally distributed. This is a common assumption made in the analysis of SGD [32, 46], and we empirically validate it in our evaluation (as shown in Figure 3). If $A$ is the SGD algorithm, and $S$ is a training dataset, we therefore assume

$$A(S) = w = w_S + w_\rho$$

where $w_S$ is deterministic and dataset-dependent and $w_\rho$ is the intrinsic random noise induced by the stochasticity of SGD having a distribution of mean 0 and intrinsic variance $\sigma_i$,

$$w_\rho \sim N(0, \lambda_i^2)$$

Based on Theorem 1 in Section 1.3, we characterize SGD as a Gaussian mechanism with parameters $c^2 > 2 \ln(1.25/\delta)$ and $\sigma_i \geq c\Delta_2(f)/\epsilon_i$. The value of $\delta$ is arbitrary, but following convention [17] we can set $\delta = 1/N$ where $N$ is the size of the training data set passed to SGD. Thus, knowing $\sigma_i$, $\delta$, and $\Delta_2(f)$, we calculate $\epsilon_i$ as:

$$\epsilon_i = \frac{\sqrt{2 \log 1.25/\delta \Delta_2(f)}}{\sigma_i}$$
Computing Sensitivity. For calculating the sensitivity $\Delta_2(f)$, we describe three different alternative approaches. First, we take the (bound on the) theoretical sensitivity of SGD $\Delta_S$ estimated by \cite{13,21} and described in Section III. Next, we further push this bound by taking the tightest possible bound, estimated empirically from public data. Denoting this as $\Delta_S^*$, we obtain it as:

$$\Delta_S^* = \max_{i,j,r} \| A(S_i, r) - A(S_j, r) \|$$  \hspace{1cm} (17)

where $A(S_i, r)$ is the output of SGD run on dataset $S_i$ with random seed $r$. In practice, we consider a subset of all datasets and seeds (so $i,j,$ and $r$ come from a restricted range). Finally, we consider an absolute ‘pairwise’ sensitivity defined between a pair of datasets by dropping the max above:

$$\Delta_S = \| A(S_i, r) - A(S_j, r) \|$$  \hspace{1cm} (18)

Each variant of the sensitivity implies a different interpretation for the resulting $\epsilon_i$. If we consider the pairwise sensitivity, we obtain a distribution of $\epsilon_i$ values which would be obtained by considering subsets of permissible neighbouring datasets. This variant of sensitivity computation is similar to the notion of data-dependent or smooth sensitivity which is emerging as a promising approach for designing better differentially-private mechanisms \cite{22}. We further discuss the implication of these different sensitivity values ($\Delta_S, \Delta_S^*, \Delta_S$) in our evaluation.

Computing Variance. The object of central interest is $\sigma_i$, which is the (diagonal) covariance of $w_p$ - the ‘noise’ added to the final weights by the stochasticity in SGD. We can obtain samples of $w_p$ by running SGD with different random seeds and subtracting the data-dependent mean value $w_S$. Estimating $\sigma_i$ then amounts to computing the standard deviation of the (flattened) estimated $w_i$. In practice, a separate $\sigma_i$ can be estimated for each dataset $S$, but as we assume $\sigma_i$ is independent of $S$ (and empirically find this to be the case, see Section VI-B), this variation need not be considered and we take the median $\sigma_i$ across all datasets. We observe that other choices for estimating an aggregate $\sigma_i$, such as computing a per-weight $\sigma_i^k$ and then averaging, or estimating the variance of the norm of the weights, produce largely consistent results. It is likely that a superior method for estimating $\sigma_i$ exists, which we leave as a question for future work.

Some care needs to be taken in obtaining $\sigma_i$, as estimating it on the private dataset would leak some information. For the purpose of this work, we assume that the user has access to a public dataset whose distribution is sufficiently similar to the private dataset, such that $\sigma_i$ can be estimated. This is a similar assumption to the setting of fine-tuning described in \cite{21}, where the private dataset is used predominantly to fine-tune a model already trained on a similar, public dataset. We summarize the steps for computing $\epsilon_i$ using sensitivity and variance of SGD mechanism in Algorithm 1.

V. AUGMENTED DIFFERENTIALLY PRIVATE SGD

In this section, we show how to extend the method of \cite{21} to account for the intrinsic noise of SGD. The basic premise of output perturbation techniques is to train a model in secret, and then release a noisy version of the final weights. For a desired $\epsilon$, $\delta$, and a known sensitivity value ($\Delta_S, \Delta_S^*, \Delta_S$) the Gaussian mechanism (Theorem 1) gives us the required level of noise that needs to be present in the output, which we call $\sigma_{\text{target}}$. In \cite{21}, this $\sigma_{\text{target}}$ defines the variance of the noise vector sampled and added to the model weights, to produce a $(\epsilon, \delta)$-DP model. Since the sum of two independent Gaussians with variance $\sigma_a^2$ and $\sigma_b^2$ is a Gaussian with variance $\sigma_a^2 + \sigma_b^2$, if the intrinsic noise of SGD is $\sigma_i$, to achieve the desired $\epsilon$ we need to add noise from the variance,

$$\sigma_{\text{augment}} = \sqrt{\sigma_{\text{target}}^2 - \sigma_i^2}$$  \hspace{1cm} (19)

If $\sigma_i$ already exceeds the required $\sigma_{\text{target}}$, no additional noise is required. Given the degradation of model performance with output noise, accounting for $\sigma_i$ can only help utility without compromising privacy in theory. The resulting algorithm for augmented differentially-private SGD is in Algorithm 2.

Algorithm 1 Estimating $\epsilon_i$ empirically

1: Given neighbouring datasets $S = \{S_a\}^{|S|}$, random seeds $R = \{r\}^{|R|}$, SGD algorithm $A$ with batch size $B$, fixed learning rate $\eta$, number of epochs $k$, $\delta$.
2: for all $S_a \in S$ do
3: for all $r \in R$ do
4: $w_{r,a} \leftarrow A(S_a; r)$  \hspace{1cm} > Run SGD on $S_a$ with seed $r$
5: procedure COMPUTE SENSITIVITY
6: for $r \in R$ do
7: for $S_a, S_b \in S$ do
8: $\Delta_{ra} \leftarrow \| w_{r,a} - w_{r,b} \|$  \hspace{1cm} > Pairwise sensitivity
9: $\hat{\Delta} \leftarrow 2k\ln(1/\delta)/B$  \hspace{1cm} > Theoretical bound
10: $\Delta_{S}^* \leftarrow \max_{r,a,b} \Delta_{ra}$  \hspace{1cm} > Empirical bound
11: procedure COMPUTE VARIANCE
12: for all $S_a \in S$ do
13: $w_a \leftarrow \frac{1}{R} \sum_{r} w_{r,a}$
14: $\sigma_a \leftarrow \text{stddev} (\text{flatten}(w_{r,a} - w_a))$
15: $\sigma_i \leftarrow \text{median}_{a} \| \sigma_a \|
16: procedure COMPUTE EPSILON
17: $c \leftarrow \sqrt{2\log(1.25)/\delta + 1 \times 10^{-5}}$
18: $\epsilon_i \leftarrow \epsilon \Delta_S^*/\sigma_i$  \hspace{1cm} > Get values for Sensitivity & Variance
19: $\epsilon_i^* \leftarrow \epsilon \Delta_S^*/\sigma_i$  \hspace{1cm} > Using empirical bound
20: return $\epsilon_i, \epsilon_i^*$

Algorithm 2 Augmented differentially private SGD

1: Given $\tau, \epsilon_{\text{target}}, \delta, \Delta_S(f)$, model weights $w_{\text{private}}$.
2: $c \leftarrow \sqrt{2\log(1.25)/\delta + 1 \times 10^{-5}}$
3: $\sigma_{\text{target}} \leftarrow c \Delta_S(f)/\epsilon_{\text{target}}$
4: if $\sigma_i < \sigma_{\text{target}}$ then
5: $\sigma_{\text{augment}} \leftarrow \sqrt{\sigma_{\text{target}}^2 - \sigma_i^2}$
6: else
7: $\sigma_{\text{augment}} \leftarrow 0$
8: $\rho \sim N(0, \sigma_{\text{augment}})$
9: $w_{\text{public}} \leftarrow w_{\text{private}} + \rho$
10: return $w_{\text{public}}$

Theorem 3. Assuming SGD is a Gaussian mechanism with intrinsic noise $\sigma_i$, Algorithm 2 is $(\epsilon, \delta)$-differentially private.

The proof is a straightforward application of the fact that the sum of Gaussians is a Gaussian, and so the construction
in Algorithm 2 produces the desired value of $\sigma$ to achieve a $(\epsilon, \delta)$-differentially private mechanism as per Theorem 1.

VI. EMPIRICAL STUDY

We perform an extensive empirical study on the sensitivity and variability of SGD and relate it to differential-privacy guarantees. Our broad objective is to understand and quantify the variability in the weights of a trained model, and to relate this variability to the intrinsic privacy of the algorithm.

Our evaluation goals are three-fold:
1) Quantify and compare the variability of SGD arising from changes in data and/or random seed
2) Compute the intrinsic $\epsilon_i$ of SGD, thus quantifying how much privacy SGD provides inherently
3) Evaluate the utility of a private model accounting for this intrinsic noise in SGD

Summary of Results. We summarize our key findings below.

- **Variability in SGD is strictly greater than sensitivity.** Throughout all our experiments, we find that the variability in the model parameters due to the randomness in SGD is strictly greater than the sensitivity (or change in parameters) due to a single input sample.

- **SGD provides reasonable intrinsic privacy.** SGD provides an intrinsic $\epsilon_i$ of 2.8, 6.9 and 7.8 for Forest, Adult and MNIST-binary datasets which are reasonably low values for privacy.

- **Empirical sensitivity bounds are tighter than theoretical bounds.** Existing theoretical bounds are often loose and can be improved. This finding is of particular interest as it motivates researchers to explore data-dependent sensitivity bounds instead of generic theoretical bounds. Using tighter sensitivity bound improves the intrinsic $\epsilon_i$ values to 0.89, 1.37 and 6.36 for Forest, Adult and MNIST-binary datasets.

- **High utility & privacy are possible for SGD.** Our results show that SGD provides acceptable intrinsic privacy $\epsilon_i < 2$ while retaining the original utility of the model. We provide a novel perspective for balancing the privacy vs. utility trade-off.

A. Experimental Setup

We study how the output of SGD depends on:

- Variation in data to notions of algorithmic stability [22] or sensitivity in the sense of differential privacy.
- Variation in random seed corresponding to the ‘intrinsic randomness’ of SGD.

**Variation in Data.** We run a grid of experiments, where either data or (non-exclusively) seed is varied. To vary the data, we consider neighbouring datasets derived from a given data source $D$. A pair of datasets is neighbouring if they differ in exactly one example. We construct a set of $|S|$ neighbouring datasets in $D$ by replacing the $s$-th training example of $D$ with the first example (wlog). We then drop the first example from the training data to avoid unnecessary duplication. In this way, the derived datasets $S_i$ and $S_j$, differ in that $S_i$ is missing $x_i$, and $S_j$ is missing $x_j$ (each contains $x_0$ in the $i$th and $j$th positions respectively). Note that it is important to replace the $i$-th example and not delete it, to ensure that for a fixed seed (and thus order of traversal of the data), the mini-batches sampled from $S_i$ and $S_j$ only differ when they would contain the $i$th or $j$th elements. To make the study more computationally tractable, we consider $|S| < N$, where $N$ is the size of the training data.

**Variation in Random Seed.** To vary the random seed, we simply run each experiment (on a given dataset $S$) multiple times with different seeds provided to the random number generator. The random seed impacts the training procedure by impacting the initialisation of the weights, and subsequently the order of traversal of the dataset. We follow the traditional setting of SGD where a random permutation (determined by the random seed) is applied to the training data at the start of each epoch, and batches of examples are sequentially drawn. We also include the setting where the initialisation is fixed, which would correspond to fine-tuning a ‘public’ (albeit very poor) model, which is considered known. For all datasets, we scale each example $x$ such that $|x| \leq 1$. If we test $R$ random seeds for each derived dataset, the total number of experiments for a data source $D$ is $2R|S|$. The total number of experiments run for each data source are shown in the column $E$ in Table I.

**Benchmark Datasets.** We perform our experiments using three commonly-used data sources.

- **MNIST-binary** [11], [30]. This is a popular dataset of handwritten digits in greyscale. We restrict to digits 3 and 5 only, to convert to a binary classification task. These were chosen randomly but likely constitute a more challenging pairing. We follow [51] and project the digits from (28, 28) to 50-dimensional feature vectors using Gaussian random projection [40].
- **Forest** [12], [15] Forest cover type prediction from cartographic information. Although the original data has seven classes, we convert this to a binary classification task by subsetting to classes 1 and 2, which are the most numerous.
- **Adult** [15] The task is to predict whether an individual’s income exceeds $50k/year based on census data from 1994. We one-hot encode categorical-valued features, dropping the first level.

**Training and Implementation Details.** For each task, we train a logistic regression model with cross-entropy loss, as is

| Dataset     | Training size | Validation size | Test size | $d$ | $E$ |
|-------------|---------------|-----------------|-----------|-----|-----|
| MNIST-binary | 10,397        | 1,155           | 1,902     | 50  | 9,799 |
| MNIST-binary | 378,783       | 42,086          | 74,272    | 49  | 16,116 |
| MNIST-binary | 29,305        | 3,256           | 16,284    | 100 | 25,702 |

TABLE I: Sizes of considered data sets. The dimension of feature vectors is $d$, and the number of experiments performed is $E$ (the number of seeds tested is approximately $\sqrt{E}$). For MNIST-binary, the original examples are size 28 $\times$ 28, but we project them to $d = 50$ with Gaussian random projections [40] following [51].
| dataset     | batch size | learning rate | convergence point (batches) |
|-------------|------------|---------------|-----------------------------|
| MNIST-binary| 32         | 0.1           | 1850                        |
| Forest      | 50         | 1.0           | 8400                        |
| Adult       | 32         | 0.5           | 3400                        |

TABLE II: Hyperparameters for logistic regression. For Forest, we chose the batch size to replicate [21].

...standard. In section [VI-E] we consider non-convex models. The loss function for logistic regression is Lipschitz with constant $L = \sup_{x} |x|$ and smooth with $\beta = \sup_{x} |x|^2$. Since we include a bias term in the model, and scale the data to have norm $\|x\| \leq 1$, we have $L = \sqrt{2}$. We hold out a portion of the training data as a validation set (see Table I), and use this to select the model hyperparameters and convergence point. We did not perform extensive hyperparameter search as our focus is not on finding the best-performing model. The hyperparameters for each setting are shown in Table II. Experiments were implemented in Keras [11] and TensorFlow [2], with Sacred [21] for experiment management. Initialisation of model parameters was done using default initialisation settings proposed by [51]. All results shown use logistic regression for binary classification.

B. Variability in SGD

Variability in the learned model depends on two things: the dataset, and the random seed of SGD. In this section, we empirically compare these sources of variability with the following quantities:

1) Variation only due to dataset;
   \[ \Delta_S := \| A(r; S) - A(r; S') \| \]

2) Variation only due to seed;
   \[ \Delta_V := \| A(r; S) - A(r'; S') \| \]

a) allowing for fixed initialisation ($\Delta^\text{fix}_V$)
   b) and seed-dependent initialisation ($\Delta^\text{vary}_V$)

3) Variation in both; \[ \Delta_{S+V} := \| A(r; S) - A(r'; S') \| \]

where $r$ and $r'$ are two random seeds, $S$ and $S'$ are two neighbouring datasets, and $A$ is the SGD algorithm. Using the experimental design outlined above, we can explore the properties of $\Delta_S$, $\Delta_V$, and $\Delta_{S+V}$. We are interested in understanding how the variability due to seed ($\Delta^\text{fix}_V$, $\Delta^\text{vary}_V$) compares to the data sensitivity $\Delta_S$ and the bound $\Delta_{S+V}$.

Results. Figure 2 (a), (b), (c) give the variability results for MNIST-binary, Forest and Adult dataset respectively. The vertical dashed line represents the theoretical sensitivity bound, $\Delta_S$. We observe that the variability $\Delta^\text{vary}_V$ (shaded) and $\Delta^\text{fix}_S$ (dotted) overlap significantly, demonstrating that the variability due to randomness completely dominates or hides the sensitivity due to change in the input sample. Similar observation holds for the fixed initialization values $\Delta^\text{fix}_V$ (shaded) and $\Delta^\text{fix}_{S+V}$ (dashed). We summarise our key findings are as follows:

- Both the fixed and varying initialization variants of $\Delta_V >> \Delta_S$ almost all the time for all the datasets. This confirms our hypothesis that changing the random seed has a larger impact on the resulting model weights than using a neighbouring dataset.

- Stronger than the previous observation, we see that $\Delta_V >> \Delta_S$ i.e., variability is greater than the theoretical upper bound of sensitivity almost all the time as well. This result validates our theoretical analysis of variability in Section III.

- $\Delta^\text{var}_V >> \Delta^\text{fix}_S$: including the effect of the unknown initialisation of the network further increases the difference in weights from runs with different seeds. Random initialisation of the model can increase the variability by up to a factor of 5. This indicates that, even for a convex objective...
as in logistic regression, the model has not yet ‘forgotten’ its initialisation at the convergence point studied here. For the remaining sections, we omit the fixed initialisation setting, but for completeness include these results in the Appendix.

- The bound $\Delta_S$ is quite loose at times; with the exception of MNIST-binary in our experiments, the bound is much larger than any value of $\Delta_S$ we tested. This suggests that even without accounting for the variability of SGD, existing output perturbation approaches are likely over-estimating the sensitivity of SGD and adding more noise than necessary.

- While the magnitude of variability in the Adult dataset appears largest, this is likely explained by the larger dimensionality (and thus model size). For MNIST-Binary and Forest, the dimensionality is very similar, but much larger variability is observed in Forest. This is likely due to the large discrepancy in the relative sizes of these dataset, as Forest has approximately 30 times the number of examples. While larger data size is good for hiding the influence of single datapoints, we also see that it appears to increase the variability in SGD.

C. What is $\epsilon$ for SGD?

The results in Figure 2 show that the variability due to intrinsic randomness of SGD exceeds the data-dependent sensitivity. Next, we aim to estimate the resulting intrinsic $\epsilon_i$ for SGD. We follow the procedure outlined in Section IV and Algorithm I using the large grid of experiments on the datasets Adult, Forest, MNIST-binary.

Validating Normal Distribution of Noise. In designing Algorithm I we assume that the noise in the weights of SGD follows a normal distribution. We evaluate this assumption for the MNIST-binary dataset. To do this, we estimate the distribution of each of the resulting weights of the model, and compare it to a normal distribution. We perform the comparison by conducting a statistical test of normality (Shapiro-Wilk [42]). Figure 3 shows the distribution of p-values obtained from running tests for each weight, for models trained on many datasets. Small p-values indicate the hypothesis that the distribution is normal can be rejected. The thresholds for rejection are marked by two vertical lines - the line at $p = 0.05$ reflects a standard threshold for such a statistical test, however as we are performing many tests we also indicate the corrected threshold at $p = 0.05/d$ (Bonferroni correction). This correction is applied to avoid spurious rejections of the null hypothesis while performing multiple tests. As we can see, the majority of weights would not be rejected at $p = 0.05$, and none would be rejected at the corrected threshold. This indicates that the distribution of each weight is consistent with a normal distribution.

$\epsilon_i$ of SGD. Table III shows the values of $\sigma_i^{\text{var}}, \Delta_S, \Delta^*_S, \delta,$ and subsequently $\epsilon_i$ values computed using $\Delta_S$. As observed in the previous section, the theoretical bound $\Delta_S$ is generally not tight. We obtain the tightest bound by selecting the maximum $\Delta^*_S$ of the observed sensitivity values and calculate the $\epsilon^*_i$ in this case. The results are shown in the row of Table III Further, we explore the ‘pairwise $\epsilon_i$’ in Figure 4. In this case, we use $\Delta_S = \|A(r; S) - A(r; S')\|$ as the sensitivity to estimate pairwise $\epsilon_i$, and hence not use the bound on sensitivity, but the distribution itself. We provide the results for the fixed initialization setting in Figure 5 in Appendix.

Results. We outline our key findings and observations below.

- Table III indicates that the intrinsic variability of SGD ($\sigma_i$) differs between datasets; characterising the nature of $\sigma_i$ as a function of hyperparameters of SGD is an open question. We observe the $\sigma_i$ values for the categorical dataset Adult and Forest to be close to each other i.e., 0.10 and 0.11 respectively while that for continuous dataset MNIST-binary is very small. Note that, these findings conform with the variability results observed from Figure 2.

- Using the theoretical bound on the sensitivity, the intrinsic $\epsilon_i$ is 2.8, 6.9 and 7.8 for Forest, Adult and MNIST-binary respectively. With the empirical (tighter) bound, $\epsilon^*_i$ is 0.8, 1.3 and 6.3 respectively. While these values of $\epsilon_i$ are in some cases too high (e.g., 7.8) for use without further noise. As we demonstrate in the next section, starting
D. What is the impact on utility?

We have seen that the intrinsic \( \epsilon_i \) of SGD can be quantified, but in many cases is insufficient alone to provide a desirable level of privacy. In this section, we demonstrate that by accounting for \( \epsilon_i \) (via \( \sigma_i \)), model performance can be improved over an existing approach based on output perturbation. We apply the procedure described in Section V, using the empirical results obtained thus far. We examine the model performance (binary accuracy) as a function of \( \epsilon \), comparing four settings:

1) Noiseless (\( \sigma = 0 \))

2) ‘SGD as deterministic’; the setting in [51]. We estimate the required \( \sigma \) using the Gaussian mechanism and the sensitivity \( \Delta_S \) of SGD, and denote this \( \sigma_{\text{target}} \).

3) ‘SGD with unknown seed’; proposing that we think of SGD as a randomised mechanism, we estimate the required \( \sigma \) as \( \sigma_{\text{augment}} = \sqrt{\sigma_{\text{target}}^2 - \sigma_i^2} \)

In the third case, if the intrinsic noise (\( \sigma_i \)) would exceed the target noise, we set \( \sigma_{\text{target}} = 0 \). This scenario occurs when the desired \( \epsilon \) is greater than the intrinsic \( \epsilon_i \) reported in Table III. We also include the setting where the bound is the tightest possible, as described above. In Table IV, we report the utility for \( \epsilon = 1 \). Figure 5 shows how the utility varies with \( \epsilon \) for a randomly-selected experiment.

### Results

Table IV demonstrates that treating SGD as random and augmenting its intrinsic noise produces a model with consistently and significantly superior utility to one which does not take intrinsic randomness into account. At \( \epsilon = 1 \), these improvements range from 0.001 to 0.006, but are consistent across experiments and datasets (which themselves display variability in performance). Figure 5 shows how this performance gap becomes more pronounced as \( \epsilon \) becomes larger, and then gradually closes as the magnitude of noise drops. The setting with fixed model initialisation but variable dataset traversal order is shown in the Appendix in Figure 9. We have the following findings:

- Accounting for the variability in SGD results in improved model performance at the same desired \( \epsilon \).
- The magnitude of the improvement is greater when the intrinsic noise \( \sigma_i \) is larger, as is the case for Forest where we see the most dramatic gains.
- Using the empirical sensitivity further improves the performance, especially when the theoretical bound is loose.
- We confirm that the point where the performance becomes equivalent to the noiseless setting corresponds to

### Table IV: Private performance of models at \( \epsilon = 1 \)

| Data Set | Noiseless (\( \sigma = 0 \)) | SGD as deterministic \( \sigma_{\text{augment}} \) | SGD as random \( \sigma_{\text{augment}} \) |
|----------|-------------------------------|-----------------------------------------------|-----------------------------------------------|
| MNIST-binary | 0.83(2) | 0.0015 | +0.0015 |
| Forest | 0.771(3) | +0.0013 | +0.0029 |
| Adult | 0.823(6) | 1.55% of gap | 10.07% of gap |
| MNIST-binary | 0.927(2) | 0.0015 | +0.0015 |
| Forest | 0.771(3) | +0.0008 | +0.0060 |
| Adult | 0.8341(5) | 3.13% of gap | 56.07% of gap |

Fig. 4: Distribution of pairwise intrinsic \( \epsilon_i \) values reflects the variability in pairwise sensitivity, indicating a relatively small number of dataset pairs contribute a rightward skew to the pairwise \( \epsilon_i \) distribution. In each case, we estimate the overall variability of SGD for the dataset with variable initialisation (\( \sigma^\text{aux} \)), then use the pairwise distance between weights \( \Delta_S \) computed between pairs of datasets to estimate \( \epsilon_i \).

with \( \epsilon_i \) allows for the addition of less noise.

- The distribution of pairwise \( \epsilon_i \) in Figure 4 shows that it may be possible to achieve a lower \( \epsilon_i \) by restricting to a subset of the data space. This reflects the distribution of \( \Delta_S \) seen in Figure 2, some pairs of datasets are more different than others, in the sense of the resulting learned model. This motivates the study for data-dependent differentially-private mechanisms that use better sensitivity bounds [35].

D. What is the impact on utility?

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In the third case, if the intrinsic noise (\( \sigma_i \)) would exceed the target noise, we set \( \sigma_{\text{target}} = 0 \). This scenario occurs when the desired \( \epsilon \) is greater than the intrinsic \( \epsilon_i \) reported in Table III. We also include the setting where the bound is the tightest possible, as described above. In Table IV, we report the utility for \( \epsilon = 1 \). Figure 5 shows how the utility varies with \( \epsilon \) for a randomly-selected experiment.

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- Using the empirical sensitivity further improves the performance, especially when the theoretical bound is loose.
- We confirm that the point where the performance becomes equivalent to the noiseless setting corresponds to
the intrinsic $\epsilon_i$ reported in Table III

E. Non-Convex Objectives

The empirical analyses described in the previous sections can also be applied to as-yet theoretically intractable models. In these cases, we have no theoretical bound on the sensitivity of SGD, however we can use the tightest possible bound by estimating it from the empirical distribution, as before. We study neural networks trained on two of the previous tasks and one new setting. The model architecture we use is a fully-connected feed-forward neural network (FFNN) with one hidden layer. As observed for convex models, the variability due to random seed is larger than the data sensitivity.

1) Adult - binary classification, as before. With a hidden size of 8, the total number of parameters is 817.
2) MNIST-binary - binary classification, as before. The hidden size is 16, resulting in 521 parameters.

The drawback of models more expressive than logistic regression is that they typically have many more parameters, and such models can’t be guaranteed to satisfy the assumptions required for the theoretical results in Section III. As the magnitude of noise added in the Gaussian mechanism grows with $\sqrt{d}$, this results in values of $\epsilon_i$ far exceeding those obtained for logistic regression in previous sections. Nonetheless, we demonstrate here that models of arbitrary complexity are amenable to empirical analysis. Figure 6 demonstrates the distribution of $\Delta_S, \Delta_V, \Delta_{S+V}$ for these non-convex models. We observe the same qualitative results as in the non-convex setting, but the separation between $\Delta_S$ and

Fig. 5: Private model performance as a function of $\epsilon$, for example models trained on each dataset. Dashed lines indicate the sensitivity is computed empirically as $\hat{\Delta}^*_S$. $\sigma_{\text{target}}$ indicates the ‘SGD as deterministic’ setting, where we add the full target noise. $\sigma_{\text{augment}}$ is the ‘SGD as random’ setting where intrinsic noise $\sigma_i$ is accounted for before noise $\sigma_{\text{augment}}$ is added to achieve an overall level of $\sigma_{\text{target}}$. While noiseless (and thus noisy) performance is variable as outlined in Table IV, the $\sigma_{\text{augment}}$ setting is consistently superior to $\sigma_{\text{target}}$ as assessed with a dependent t-test. The setting with fixed model initialisation but variable dataset traversal order is shown in the Appendix in Figure 9.

Fig. 6: Distribution of $\Delta = \|w - w'\|$ across pairs of experiments differing in data ($\Delta_S$), random seed ($\Delta_{\text{fix}V}, \Delta_{\text{vary}V}$), or both ($\Delta_{S+V}$). $\Delta_{\text{fix}V}$ refers to the setting where the random seed is variable, but the initialisation of the model is fixed. All results shown use a fully-connected neural network with one hidden layer. As observed for convex models, the variability due to random seed is larger than the data sensitivity.
$\Delta V$ is less marked. This suggests that high-capacity models may be more susceptible to changes in the training data, even when performing early stopping. However, the tail of $\Delta S$ has very low density, suggesting a small number of dataset pairs are responsible for these large $\Delta S$ values. In Figure 7, we plot the distribution of pairwise $\epsilon_i$ for the non-convex settings. We see that the pairwise $\epsilon_i$ is concentrated around 12 and 18 for MNIST-binary and Adult respectively, but the maximum observed values are 32.3 and 61.9 - essentially, SGD does not provide appreciable intrinsic privacy in these cases. The effect of apparent ‘outlier’ datasets is quite evident in Figure 7b, where the density of pairwise $\epsilon_i$ values drops sharply after 30 or so. Adding noise to such high-dimensional models, even accounting for the tight empirical bound on the sensitivity, and the enhanced variability due to randomness in the initialisation, tends to destroy their utility and so we exclude the analysis of performance in this section.

VII. DISCUSSION

The intrinsic variability of SGD far exceeds its sensitivity to dataset perturbations. Our theoretical result demonstrates that the upper bound on the seed-dependent variability in the weights is with high probability greater than the data-dependent sensitivity. While further work is required to establish the properties of the \textit{lower} bound of this variability, the empirical results indicate that the lower bound also tends to exceed the data-dependent sensitivity. The gap between this expected change in model weights due to changes in the random seed and changes in the training data is pronounced. While earlier work on the stability of SGD \cite{22, 51} established the bound on the sensitivity, the results here empirically demonstrate that this bound tends to be far below the seed-dependent variability. In our analysis and experiments, we have focused on SGD with fixed learning rate, and without modifications such as momentum or variants like Adam or RMSprop \cite{28}, \cite{48}. Decaying the learning rate would aid in convergence to local minima, which would likely reduce the variability in outputs on convex objectives with unique minima. For non-convex losses with potentially many local minima \cite{34} however, it is unclear how this would impact the resulting $\sigma_i$. Furthermore, although we performed mild hyperparameter optimisation to identify models with reasonable performance, this analysis did not explore the relationship between the variability of SGD and these hyperparameters. In particular, the variability likely depends strongly on the choice of batch size in terms of both its magnitude and the characteristics of the resulting noise distribution. As the batch size increases (assuming the training data is larger still), the noise in the gradients tends to Gaussian by the central limit theorem. In the finite batch limit however, the nature of the noise distribution remains an open question \cite{45}, and further relating the noise in the gradients to the variability in the final model parameters requires additional analysis. We have established that the intrinsic noise in SGD can be used for additional privacy, but making practical use of this insight requires further theory to safely state in which circumstances SGD exhibits the desired level of noise.

Accounting for intrinsic noise produces significant improvements in utility. One of the interesting findings of this analysis is that the noise in SGD is not trivial in the sense of utility. Accounting for the variability across seeds produces a statistically significant improvement in the accuracy of the resulting private model. This improvement is more pronounced for intermediate values of $\epsilon$ (from Figure 2b around $\epsilon = 4$ using the theoretical bound), but reflects the non-linear relationship between the magnitude of the perturbation to the weights and the resulting drop in accuracy. With further understanding of the factors most strongly contributing to the variability of SGD, this suggests that learning algorithms could be engineered to promote (non-harmful) variability across runs of SGD, and thus enhance the intrinsic privacy and resulting utility gains.

There is within-dataset variability in sensitivity. Looking deeper into the estimation of the intrinsic $\epsilon_i$ of SGD prompted the analysis of the ‘pairwise’ $\epsilon_i$ using sensitivity computed across pairs of datasets. This analysis, alongside the results in Figure 2, illuminates the perhaps-surprising variability of the data-dependent sensitivity. Even though the datasets in
question are normalised (|x| ≤ 1), the distribution of \( \Delta_S \) (which follows the distribution of pairwise \( \epsilon_i \)) shows that there is a small number of dataset pairs which result in an unusually large distance between weights (potentially also as a function of the random seed which is fixed between the pair of experiments). A pre-processing strategy of identifying and removing "problematic" examples from the training data could potentially remove the long tail on this distribution and provide further privacy improvements. Removing outliers in this way is already common practice when fitting potentially-sensitive models to noisy data. A closely related line of work is that on calculating dataset specific smooth sensitivity that provides stronger privacy guarantees [35].

The theoretical bound on sensitivity is not tight. Our experiments provide empirical evidence that the existing theoretical bounds on the sensitivity of SGD [22, 51] are not tight. Further empirical analyses (not shown) indicate that while the Lipschitz constant of logistic regression is \( L = \sqrt{2} \) in our experiments, this bound is only reasonably tight at the start of training, and subsequent gradients have small norms, as expected for a gradient-based optimisation scheme. By computing an 'empirical bound' (\( \Delta_S^\ast \)), which is certainly optimistic, we can demonstrate the potential dramatic improvement in utility that would be gained by producing a tighter theoretical bound on the sensitivity. We hope this proof of concept inspires further study into this question.

VIII. Related Work

Stability of SGD. Owing to its centrality in machine learning (specifically deep learning), the stochastic gradient method has been an object of much study. The results in this paper build from a line of work examining the algorithmic stability of SGD, a concept closely related to sensitivity as used in differential privacy. [7] first defined the notion of algorithmic stability for a learning algorithm such as SGD and related it to generalisation error. [22] more recently built on and expanded these results, relating them to modern deep learning techniques and hyperparameters of SGD such as the number of training iterations; [29] extend the analysis to data-dependent notions of stability. In these cases, and in much work studying SGD, the focus is on understanding the generalisation properties of the algorithm, however the links between general notions of stability, privacy, and generalisation have already been established [7, 22, 36]. That stochasticity in SGD (via small batches) affords superior generalisation to the large or full-batch counterpart [27] therefore suggests that this same stochasticity could be exploited for privacy, prompting the investigation in this paper.

SGD as a randomised mechanism. Our argument that SGD can be viewed as an instance of the Gaussian mechanism casts SGD as a sampling procedure from some posterior distribution over weights. Recently, Mandt et al. demonstrated that, under certain assumptions, SGD is equivalent to performing approximate Bayesian inference with a particular choice of variational distribution [32]. Following existing work viewing SGD as a stochastic differential equation [46], they characterise the posterior distribution over weights as the solution to the SDE, which happens to be Gaussian. Dieuleveut et al. view SGD with constant step size as a homogeneous Markov chain which converges to a stationary distribution in the convex case, but they do not further characterise this distribution [13].

Differentially-private machine learning. Various methods exist to train machine learning models in a privacy-preserving manner. [10] demonstrated that perturbing the objective in empirical risk minimisation would produce an \( \epsilon \)-differentially private learning algorithm under strong convexity assumptions. Later, [3] proposed a modification to the stochastic gradient descent (SGD) algorithm to provide for differentially-private learning of deep (typically non-convex) models, and [37, 38] use a teacher-student framework on partitioned data for training (potentially deep) classifiers. These methods provide privacy throughout the training procedure by design, by ensuring that every time private information is accessed (to compute a gradient as in [3] or a label as in [37]), it is fed through a differentially-private mechanism. Composition theorems then allow for the overall privacy loss to be estimated in such iterative settings. While providing privacy throughout training may be advantageous in some settings, we consider a scenario where only the final weights are released at the end of training. This is the setting addressed by [51], who treat SGD as a ‘black box’ and inject Laplace or Gaussian noise on the final model weights.

IX. Conclusion

In this work, we aim to answer whether the inherent randomness in stochastic gradient descent (SGD) can be exploited to improve the performance of privately-trained models. Our intuition is that the stochasticity of SGD, as well as providing apparent gains in generalisation performance, can be thought of as providing a degree of ‘plausible deniability’ for participants in the training data. Making concrete this intuition in the framework of differential privacy, we interpret SGD as a randomized mechanism (the Gaussian mechanism) and estimate its intrinsic noise parameter.

We conclude with stating our four concrete observations while answering the above research question. First, the variability in weights due to the stochasticity in SGD is often strictly greater than the sensitivity due to change in a single input sample in the training dataset. This is consistent with our theoretical results indicating that this variability is upper bounded by a quantity typically much larger than the data sensitivity. Second, SGD provides sufficient intrinsic privacy to be considered while designing privacy-preserving systems. Third, the existing known theoretical sensitivity bounds for SGD are often loose and concrete solutions to compute data-dependent sensitivity are essential to improve privacy guarantees of SGD. Lastly, our work highlights a novel angle for balancing the privacy vs. utility trade-off by utilizing the inherent noise due to the randomness in SGD.
X. APPENDIX

A. Distribution of $\epsilon$ with fixed initialization

Figure 8 shows the pairwise $\epsilon$ values for fixed initialization of model parameters for MNIST-binary, Forest and Adult dataset. We observe similar results for $\epsilon$ values with fixed initialization as for the variable initialization setting discussed in Section VI-C.

B. Performance versus $\epsilon$ with fixed initialisation

Figure 9 shows the utility comparison for fixed initialization of model parameters of our augmented SGD approach and prior work [51]. With the fixed initialisation, the improvement in utility is less pronounced, but still consistent.

Fig. 8: Distribution of pairwise intrinsic $\epsilon_i$ values shows that some pairs of datasets produce more different models than others. In each case, we estimate the overall variability of SGD for the dataset where the initialisation of the model is fixed ($\sigma^{fix}_i$), then use the pairwise distance between weights $\Delta S$ computed between pairs of datasets to estimate $\epsilon_i$. 

Fig. 9: Private model performance as a function of $\epsilon$, similar to Figure 5 but including the setting where we assume the initialisation of the model is fixed and variability only arises from the order of sampling the data ($\sigma^{fix}_i$). Dashed lines indicate the sensitivity is computed empirically as $\hat{\Delta}^*_S$. We see that as the variability due to the initialisation is quite large, the utility gains from the fixed initialisation setting are more modest.