DOMINATION OF SAMPLE MAXIMA AND RELATED EXTREMAL DEPENDENCE MEASURES

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Abstract: For a given $d$-dimensional distribution function (df) $H$ we introduce the class of dependence measures $\mu(H, Q) = -\mathbb{E}\{\ln H(Z_1, \ldots, Z_d)\}$, where the random vector $(Z_1, \ldots, Z_d)$ has df $Q$ which has the same marginal df's as $H$. If both $H$ and $Q$ are max-stable df's, we show that for a df $F$ in the max-domain of attraction of $H$, this dependence measure explains the extremal dependence exhibited by $F$. Moreover we prove that $\mu(H, Q)$ is the limit of the probability that the maxima of a random sample from $F$ is marginally dominated by some random vector with df in the max-domain of attraction of $Q$. We show a similar result for the complete domination of the sample maxima which leads to another measure of dependence denoted by $\lambda(Q, H)$. It turns out that both $\mu(H, Q)$ and $\lambda(Q, H)$ are closely related. If $H$ is max-stable we derive useful representations for both $\mu(H, Q)$ and $\lambda(Q, H)$. Our applications include equivalent conditions for $H$ to be a product df and $F$ to have asymptotically independent components.

Key Words: Max-stable distributions; domination of sample maxima; extremal dependence; inf-argmax formula; de Haan representation; records; multiple maxima; concomitants of order statistics; concurrent probabilities.

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1. Introduction

Let $H$ be a $d$-dimensional distribution function (df) with unit Fréchet marginal df’s $\Phi(x) = e^{-1/x}, x > 0$. We shall assume in the sequel that $H$ is a max-stable df, which in our setup is equivalent with the homogeneity property

$$H^t(x_1, \ldots, x_d) = H(tx_1, \ldots, tx_d)$$

for any $t > 0, x_i \in (0, \infty), 1 \leq i \leq d$, see e.g., [1–3]. The class of max-stable df’s is very large with two extreme instances

$$H_0(x_1, \ldots, x_d) = \Pi_{i=1}^d \Phi(x_i), \quad H_\infty(x_1, \ldots, x_d) = \min_{1 \leq i \leq d} \Phi(x_i)$$

the product df $H_0$ and the upper df $H_\infty$, respectively. Hereafter $G = 1 - G$ stands for the survival function of some univariate df $G$. It follows easily by the lower Fréchet -Hoeding bound that

$$(H(nx_1, \ldots, nx_d))^n \geq \left(\max_{1 \leq i \leq d} \Phi(nx_i)\right)^n \geq e^{\lim_{n \to \infty} n \ln(1 - \frac{1}{d} \sum_{i=1}^d \Phi(nx_i))}$$

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From multivariate extreme value theory, see e.g., [1–4] we know that (1.3)
$$H_0(x_1, \ldots, x_d) \leq H(x_1, \ldots, x_d) \leq H_\infty(x_1, \ldots, x_d), \ x_i \in (0, \infty), 1 \leq i \leq d.$$ (1.3)
Indeed, (1.2) is well-known and follows for instance using the Pickands representation of $H$, see e.g., [3][Eq. (4.3.1)] or the inf-argmax formula as shown in Section 4. Consequently, any max-stable df $H$ lies between $H_0$ and $H_\infty$, i.e.,
$$F_{i} = H_0(x_1, \ldots, x_d), \ x_i \in (0, \infty), \ i \leq d.$$
From multivariate extreme value theory, see e.g., [1–4] we know that $d$-dimensional max-stable df’s $H$ are limiting df’s of the component-wise maxima of $d$-dimensional independent and identically distributed (iid) random vectors with some df $F$. In that case, $F$ is said to be in the max-domain of attraction (MDA) of $H$ (abbreviated $F \in MDA(H)$). For simplicity we shall assume throughout in the following that $F$ is a df on $[0, \infty)^d$ with marginal df’s $F_i \in MDA(\Phi), i \leq d$ that have norming constants $a_n = n, n \in \mathbb{N}$, and thus we have
$$\lim_{n \to \infty} F^n_i(nx) = \Phi(x), \ x \in \mathbb{R}$$ (1.4)
for all $i \leq d$, where we set $\Phi(x) = 0$ if $x \leq 0$. Consequently, $F$ is in the MDA of some max-stable df $H$ if further
$$\sup_{n \to \infty, x_i \in \mathbb{R}, 1 \leq i \leq d} \left| F^n_i(nx_1, \ldots, nx_d) - H(x_1, \ldots, x_d) \right| = 0.$$ (1.5)
In the special case that $F$ has asymptotically independent marginal df’s, meaning that for $(X_1, \ldots, X_d)$ with df $F$
$$\lim_{n \to \infty} n \mathbb{P}\{X_i > nx_i, X_j > nx_j\} = 0, \ x_i, x_j \in (0, \infty), \ \forall i \neq j \leq d,$$ (1.6)
then $F \in MDA(H_0)$ if simply $F_i \in MDA(\Phi), i \leq d$.
In various applications it is important to be able to determine if some max-stable df $H$ resulting from the approximation in (1.5) is equal to $H_0$, which in the light of multivariate extreme value theory means that the component-wise maxima $M_n := (\max_{1 \leq i \leq n} X_{i1}, \ldots, \max_{1 \leq i \leq n} X_{id}), n \geq 1$ of a $d$-dimensional random sample $(X_{i1}, \ldots, X_{id}), i = 1, \ldots, n$ of size $n$ from $F$ has asymptotically independent components.
The strength of dependence of the components of $M_n$, or in other words the extremal dependence manifested in $F$, in view of the approximation (1.5) can be measured by calculating some appropriate dependence measure for $H$ (when the limiting df $H$ is known).
For any random vector $Z = (Z_1, \ldots, Z_d)$ with df $Q$ which has the same marginal df’s as $H$ we introduce a class of dependence measure for $H$ indexed by $Q$ given by
$$\mu(H, Q) = -\mathbb{E}\{\ln H(Z_1, \ldots, Z_d)\}.$$ (1.7)
In view of (1.3), since $-\ln H_i(Z_i)$ is a unit exponential random variable, we have
$$1 = \max_{1 \leq i \leq d} \mathbb{E}\{-\ln H_i(Z_i)\} \leq -\mathbb{E}\{\ln \min_{1 \leq i \leq d} H_i(Z_i)\} \leq \mu(H, Q) \leq -\mathbb{E}\{\ln \prod_{i=1}^d H_i(Z_i)\} = d$$ (1.8)
and in particular
$$\mu(H_0, Q) = d, \ \mu(H_\infty, H_\infty) = 1.$$ (1.9)
Clearly, $\mu(H, Q)$ can be defined for any df $H$ and it does not depend on the choice of the marginal df’s of $H$. In this contribution we shall show that $\mu(H, Q)$ is particularly interesting for $H$ being max-stable.
Next, consider the case that both $H$ and $Q$ are max-stable. It follows that (see Theorem 2.3) for $F$ satisfying (1.5) and $G \in MDA(Q)$

\[(1.10) \quad \mu(Q, H) = \lim_{n \to \infty} \mu_n(G, F^n), \quad \mu_n(G, F^n) = n \int_{\mathbb{R}^d} [1 - G(x_1, \ldots, x_d)] dF^n(x_1, \ldots, x_d),\]

provided that both $F$ and $G$ are continuous. In view of (1.10), we see that $\mu(H, Q)$ relates to $F$ under (1.5).

Let in the following $W$ denote a random vector with df $G$ being independent of $M_n$. We say that $W$ marginally dominates $M_n$, if there exists some $i \leq d$ such that $W_i > M_{ni}$. Consequently, assuming further that $W$ is independent of $M_n$ we have

\[\frac{\mu_n(G, F^n)}{n} = P\{W \text{ marginally dominates } M_n\} =: \pi_n.\]

Re-writing (1.10) we have $\lim_{n \to \infty} n \pi_n = \mu(H, Q)$ and thus $\mu(H, Q)$ appears naturally in the context of marginal dominance of sample maxima.

Our motivation for introducing $\mu(H, Q)$ comes from results and ideas of A. Gnedin, see [5–7] where multiple maxima of random samples is investigated. In the turn, the probability of observing a multiple maximum is closely related to the complete domination of sample maxima as we shall explain below.

We say that $W$ completely dominates $M_n$ if $W_i > M_{ni}$ for any $i \leq d$. Assuming that $F$ and $G$ are continuous, we have

\[\lambda_n(F^n, G) := n \int_{\mathbb{R}^d} F^n(x_1, \ldots, x_d) dG(x_1, \ldots, x_d) = nP\{W \text{ completely dominates } M_n\} =: n\pi_n.\]

If further $F \in MDA(H), G \in MDA(Q)$ we show in Theorem 2.3 below that

\[(1.11) \quad \lim_{n \to \infty} \lambda_n(G^n, F) = \lambda(Q, H) = \int_{(0, \infty)^d} Q(x_1, \ldots, x_d) dv(x_1, \ldots, x_d),\]

where $v$ denotes the exponent measure of $H$ defined on $E = [0, \infty]^d \setminus (0, \ldots, 0)$, see [1, 3] for more details on the exponent measure. Note in passing that by symmetry $\lim_{n \to \infty} \lambda_n(F^n, G) = \lambda(H, Q)$ follows.

Our notation and definitions of $\pi_n$ and $\pi_n$ agree with those in [8] for the particular case that $F = G$. Therein the complete and simple records are discussed. If $F$ is continuous and $F = G$ we have that $(n + 1)\pi_n$ equals

\[P\left\{\max_{1 \leq i \leq n+1} X_{ij} = X_{1j}, \quad j = 1, \ldots, d\right\},\]

which is the probability of observing a multiple maximum, see [6, 7, 9–12]. There are only few contributions that discuss the asymptotics of $\lambda_n(G^n, F)$ for $F \neq G$, see [13–15].

Since the exponent measure can be defined also for max-id. df $H$, i.e., if $H^t$ is a df for any $t > 0$, then as above $\lambda(Q, H)$ can also be defined for any such df $H$ and any given $d$-dimensional df $Q$. We shall show that $\mu(H, Q)$ and $\lambda(Q, H)$ are closely related. In particular, for $d = 2$ we have $\mu(H, Q) = 2 - \lambda(Q, H)$, provided that $H$ is a max-id. df. In particular, we show how to define $\lambda(Q, H)$ for any $H$ and $Q$.

For $H$ being a max-id. df we also show how to calculate $\mu(H, Q)$ and $\lambda(Q, H)$ by a limiting procedure, which relates to domination of $d$-dimensional random vectors, see Theorem 2.1 below.

It turns out that both dependence measures $\mu(H, Q)$ and $\lambda(Q, H)$ are very tractable if $H$ is max-stable (note that such $H$ is also max-id. df). In particular, we show that $\mu(H, Q)$ is the extremal coefficient of some $d$-dimensional max-stable df $H^*$, i.e., $\mu(H, Q) = -\ln H^*(1, \ldots, 1)$. Moreover, we derive in Theorem 2.5 tractable expressions
for $\mu(H, Q)$ and $\lambda(Q, H)$, which are useful for simulations of these dependence measures if the de Haan spectral representation of $H$ is known.

It is of particular interest for multivariate extreme value theory to derive tractable criteria that identify if a max-stable df $H$ is equal to $H_0$. In our first application we show several equivalent conditions to $H = H_0$.

In view of (1.10) and (1.11) we see that both measures of extremal dependence $\mu(H, Q)$ and $\lambda(Q, H)$ capture the extremal properties of $F \in MDA(H)$. Motivated by the relation between $\mu(H, Q)$ and $\lambda(Q, H)$ we derive in our second application several conditions equivalent to (1.6).

Both $\mu(H, Q)$ and $\lambda(Q, H)$ can be defined for any $d$-dimensional df $H$ and $Q$. When $H$ is max-stable, these are dependence measures for $H$, since independent of the choice of $Q$, we can determine if $H = H_0$, see Proposition 3.1, statement ii). A simple choice for $Q$ is taking $Q = H$. Alternatively, one can take $Q = H_0$ or $Q = H_\infty$. Independent of the choice of $Q$ we show in Proposition 3.1 that $\mu(H, Q) = 2$ is equivalent with $H = H_0$. In particular, this result shows that $\mu(H, Q)$ is a measure of dependence of $H$ (and not for $Q$).

Brief organisation of the rest of the paper: In Section 2 we derive the basic properties of both measures of $\mu(H, Q)$ and $\lambda(Q, H)$ if $H$ is a max-id. df. More tractable formulas are then derived for $H$ being a max-stable df. Section 3 is dedicated to applications. We present some auxiliary results in Section 4 followed by the proofs of the main results in Section 5.

2. Main Results

In the following $H$ and $Q$ are $d$-dimensional df’s with unit Fréchet marginals df’s and $Z$ is a random vector with df $Q$. The second dependence measure $\lambda(Q, H)$ defined in (1.11) is determined in terms of the exponent measure $\nu$ of $H$, under the max-stability assumption on $H$.

A larger class of multivariate df’s is that of max-id. df’s. Recall that $H$ is max-id., if $H^t$ is a df for an $t > 0$. For such df’s the corresponding exponent measure can be constructed, see e.g., [1], and therefore we can define $\lambda(Q, H)$ as in the Introduction for any $H$ a max-id. df and any given df $Q$. Note that any max-stable df is a max-id. df, therefore in the following we shall consider first the general case that $H$ is a max-id. df, and then focus on the more tractable case that $H$ is a max-stable df.

2.1. Max-id. df $H$. Our analysis shows that $\mu(H, Q)$ and $\lambda(Q, H)$ are closely related. Specifically, if $d = 2$, then $\mu(H, Q) = 2 - \lambda(Q, H)$, provided that $H$ is a max-id. df. Such a relationship does not hold for the case $d > 2$.

However as we show below it is possible to calculate $\mu(H, Q)$ if we know $\lambda(Q_K, H_K)$ for any non-empty index set $K \subset \{1, \ldots, d\}$. A similar result is shown for $\lambda(Q, H)$. In our notation $Q_K$ denotes the marginal df of $Q$ with respect to $K$ and $|K|$ stands for the number of the elements of the index set $K$. Below $\mu_n$ and $\lambda_n$ are as defined in the Introduction.

**Theorem 2.1.** If $H$ is a max-id. df, then we have

\begin{equation}
\mu(H, Q) = \lim_{n \to \infty} \mu_n(H^{1/n}, Q), \quad \lambda(Q, H) = \lim_{n \to \infty} \lambda_n(Q, H^{1/n}).
\end{equation}

Moreover,

\begin{equation}
\mu(H, Q) = d + \sum_{2 \leq i \leq d} (-1)^{i+1} \sum_{K \subset \{1, \ldots, d\}, |K| = i} \lambda(Q_K, H_K)
\end{equation}
and

\begin{equation}
\lambda(Q,H) = d + \sum_{2 \leq i \leq d} (-1)^{i+1} \sum_{K \subset \{1, \ldots, d\}, |K| = i} \mu(H_K, Q_K).
\end{equation}

**Remark 2.2.** i) For \(H\) a max-stable df and \(Q = H\) the claim in (2.3) is shown in [12][Theorem 2.2, Eq. (13)].

ii) A direct consequence of (2.3) is that we can define \(\lambda(Q,H)\) even if \(H\) is not a max-id. df by simply using the definition of \(\mu(H_K, Q_K)\).

iii) It is clear that \(\mu(H,Q) \geq \mu(H_K, Q_K)\) for any non-empty index set \(K \subset \{1, \ldots, d\}\). Note that (2.1) shows that exactly the opposite relation holds for \(\lambda(Q,H)\) when \(H\) is a max-id. df, namely

\[\lambda(Q,H) \leq \lambda(Q_K,H_K)\].

In fact, (2.3) shows that we can calculate both \(\mu(H,Q)\) and \(\lambda(Q,H)\) by a limit procedure if we assume that \(H\) is a max-id. df, see for more details (5.1). Although such a limit procedure shows how to interpret these dependence measures in terms of domination of random vectors, it does not give a precise relation with extremal properties of random samples. Therefore in the following we shall restrict our attention to the tractable case that \(H\) is a max-stable df.

2.2. Max-stable df \(H\). We show next the relation of \(\mu(H,Q)\) and \(\lambda(Q,H)\) with the marginal and complete domination of sample maxima mentioned in the Introduction. Recall that in our notation \(\bar{F}_i, G_i, i \leq d\) stand for the marginal survival functions of \(F\) and \(G\), respectively.

**Theorem 2.3.** If \(H, Q\) are max-stable df’s with unit Fréchet marginals and \(F,G\) are two \(d\)-dimensional continuous df’s such that \(\lim_{x \to \infty} \bar{F}_i(x)/\bar{G}_i(x) = c_i \in (0, \infty)\) for \(i \leq d\) and further \(F \in MDA(H), G \in MDA(Q)\), then (1.10) and (1.11) hold.

**Remark 2.4.** The relation \(\lim_{n \to \infty} \lambda_n(F^n, F) = \lambda(H,H)\) for \(F \in MDA(H)\) is known from works of A. Gnedin, see e.g., [6, 7]. Explicit formulas are given in [16] for \(d = 2\). See also the recent contributions [8, 12].

In view of [4] (recall \(H\) has unit Fréchet marginal df’s) the assumption that \(H\) is max-stable implies the following de Haan representation (see e.g., [17, 18])

\begin{equation}
-\ln H(x_1, \ldots, x_d) = E\left\{ \max_{1 \leq i \leq d} \frac{Y_i}{x_i} \right\}, \quad (x_1, \ldots, x_d) \in (0, \infty)^d,
\end{equation}

where \(Y_i\)'s are non-negative with \(E\{Y_i\} = 1, 1 \leq i \leq d\). As shown in [19], see also [20, 21] we have the alternative formula

\begin{equation}
-\ln H(x_1, \ldots, x_d) = \sum_{i=1}^{d} \frac{1}{x_i} \Psi_i(x_1, \ldots, x_d), \quad (x_1, \ldots, x_d) \in (0, \infty)^d,
\end{equation}

where \(\Psi_i\)'s are non-negative zero-homogeneous, i.e., \(\Psi_i(cx_1, \ldots, cx_d) = \Psi_i(x_1, \ldots, x_d)\) for any \(c > 0, x_i \in (0, \infty), i \leq d\). Moreover, \(\Psi_i\)'s are bounded by 1, which immediately implies the validity of the lower bound in (1.2).

In the literature \(- \ln H(1, \ldots, 1)\) is also referred to as the extremal coefficient of \(H\), denoted by \(\theta(H)\), see e.g., [12].

Our next result gives alternative formulas for \(\mu(H,Q)\) and shows that it is the extremal coefficient of the max-stable
df \( H^* \) defined by

\[ -\ln H^*(x_1, \ldots, x_d) = \mathbb{E}\left\{ \max_{1 \leq i \leq d} \frac{Y_i}{x_iZ_i} \right\}, \quad (x_1, \ldots, x_d) \in (0, \infty)^d, \tag{2.6} \]

with \( Z \) being independent of \( Y = (Y_1, \ldots, Y_d) \). Note that since

\[ \mathbb{E}\{Y_i\} = \mathbb{E}\{1/Z_i\} = 1, \quad i \leq d \]

and \( Y_i/Z_i \)'s are non-negative, then \( H^* \) has unit Fréchet marginal df’s and moreover also \( \tilde{H} \) defined by

\[ -\ln \tilde{H}(x_1, \ldots, x_d) = \mathbb{E}\left\{ \max_{1 \leq i \leq d} Y_i \right\}, \quad (x_1, \ldots, x_d) \in (0, \infty)^d \tag{2.7} \]

is a max-stable df with unit Fréchet marginal df’s.

Theorem 2.5. If \( H \) is a max-stable df with unit Fréchet marginal df’s and de Haan representation (2.6) with \( Y \) being independent of \( Z \) with df \( Q \) which has unit Fréchet marginal df’s, then we have

\[ \mu(H, Q) = \mathbb{E}\left\{ \max_{1 \leq i \leq d} \frac{Y_i}{Z_i} \right\} = \sum_{i=1}^{d} \mathbb{E}\left\{ \frac{1}{Z_i} \Psi_i(Z_1, \ldots, Z_d) \right\}, \quad (x_1, \ldots, x_d) \in (0, \infty)^d \tag{2.8} \]

and

\[ \lambda(Q, H) = \mathbb{E}\left\{ \min_{1 \leq i \leq d} \frac{Y_i}{Z_i} \right\}. \tag{2.9} \]

Moreover, with \( H^* \) defined in (2.6)

\[ \mu(H, Q) = \theta(H^*) \geq \max\left( \theta(H), \theta(\tilde{H}) \right) \geq 1 \tag{2.10} \]

and

\[ \lambda(Q, H) \leq \min\left( \mathbb{E}\left\{ \min_{1 \leq i \leq d} Y_i \right\}, \mathbb{E}\left\{ \min_{1 \leq i \leq d} \frac{1}{Z_i} \right\} \right) \leq 1. \tag{2.11} \]

Remark 2.6. i) If \( Z_1 = \cdots = Z_d = Z \) with \( Z \) a unit Fréchet random variable, then the zero-homogeneity of \( \Psi_i \)'s, (2.5) and (2.8) imply that

\[ \mu(H, Q) = \sum_{i=1}^{d} \Psi_i(1, \ldots, 1) \mathbb{E}\left\{ \frac{1}{Z} \right\} = \sum_{i=1}^{d} \Psi_i(1, \ldots, 1) = \mathbb{E}\left\{ \max_{1 \leq i \leq d} Y_i \right\} = -\ln H(1, \ldots, 1) \geq 1. \tag{2.12} \]

Further, by (2.9) we have \( \lambda(Q, H) = \mathbb{E}\{\min_{1 \leq i \leq d} Y_i\} \).

ii) In view of [12][Theorem 2.2] (see also [16][Eq. (6.9)]) for \( H \) with de Haan representation (2.6)

\[ \lambda(H, H) = -\mathbb{E}\left\{ \frac{1}{\ln H(Y_1, \ldots, Y_d)} \right\} \]

holds, which together with (2.10) implies that

\[ \mu(H_\infty, H_\infty) = \lambda(H_\infty, H_\infty) = 1 \]

and thus the lower bound in (1.8) is sharp. We note in passing that there are numerous papers where \( \lambda_n(F^n, F) \) and \( \lambda(H, H) \) appear, see e.g., [8, 16, 22–24] and the references therein.

iii) For common max-stable df’s \( H \) the spectral random vector \( Y \) that defines (2.4) is explicitly known. Consequently, for any given random vector \( Z \), using the first expression in (2.8) and (2.9), we can easily evaluate \( \mu(H, Q) \) and \( \lambda(Q, H) \) by Monte Carlo simulations, respectively.
3. Applications

In multivariate extreme value theory it is important to have conditions that show if a given max-stable df $H$ is equal to $H_0$. In case $d = 2$ it is well-known that $H = H_0$ if and only if $\lambda(H, H) = 0$, see [12][Proposition 2.2] or [6][Theorem 2]. Consequently, when $d > 2$, in view of [3][Theorem 4.3.3] we have that $H = H_0$ if and only if

\[ \lambda(H_K, H_K) = 0 \]

for any index set $K \subset \{1, \ldots, d\}$ with two elements. Therefore, in the sequel we consider for simplicity the case $d = 2$ discussing some tractable conditions that are equivalent with $H = H_0$ and (1.6).

As in Balkema and Resnick [25], for a given bivariate df $H$ with unit Fréchet margins define $\xi_H : (0, \infty)^2 \to [0, 1]$ by (set $A = H(x_1, x_2), B = H(x_1 + h, x_2 + h)$)

\[ \xi_H(x_1, x_2) = \lim_{h \to 0} \frac{[B - H((x_1, x_2) + (h, -h))][B - H((x_1, x_2) + (-h, h))]}{A[A + B - H((x_1, x_2) + (h, -h)) - H((x_1, x_2) + (-h, h))]} \]

If $H$ is a continuous max-id. df, then in view of [25] the function $\xi_H$ is non-negative, measurable and bounded by 1, almost everywhere with respect to $dH$.

**Proposition 3.1.** Let $H$ and $Q$ be two bivariate df’s with unit Fréchet marginals. If $H$ is a max-id. df, then we have

\[ \lambda(Q, H) = \int_{(0, \infty)^2} [1 - \xi_H(x_1, x_2)] \frac{Q(x_1, x_2)}{H(x_1, x_2)} dH(x_1, x_2). \]

Moreover, if $H$ is a max-stable df, then the following conditions are equivalent:

i) $H = H_0$;

ii) $\theta(H) = -\ln H(1, 1) = 2$;

iii) $\mu(H, Q) = 2 - \lambda(Q, H)$;

iv) $\xi_H$ equals 1 almost everywhere $dH$;

v) $\frac{dH}{dH'} = \frac{\xi_H}{\lambda^2}$ almost everywhere $dH$ for any $t > 0$.

**Remark 3.2.** i) By [6][Theorem 2] we have that $\lambda(H, H) = 0$ is equivalent with $H = H_0$ and $\lambda(H, H) = 1$ is equivalent with $H = H_\infty$.

ii) Statement iii) above holds for any df $Q$ with continuous marginal df’s, and thus $\mu(H, Q)$ and $\lambda(Q, H)$ are both dependence measures for $H$.

We conclude this section with equivalent conditions to (1.6).

**Proposition 3.3.** Let $F, G$ be two continuous bivariate df’s with marginal df’s $F_i, G_i, i = 1, 2$ satisfying $\lim_{t \to \infty} \frac{F_i(t)}{G_i(t)} = 1$. If further $F_1, F_2$ satisfy (1.4) and $(X_1, X_2)$ has df $F$, then the following are equivalent:

i) $F$ has asymptotically independent components;

ii) $\lim_{n \to \infty} n \mathbb{P}\{X_1 > n, X_2 > n\} = 0$;

iii) $\lim_{n \to \infty} \lambda_n(G^n, F) = 0$;

iv) $\lim_{n \to \infty} \mu_n(F, G^n) = 2$;
v) $\lim_{n \to \infty} n P\{G(X_1, X_2) > 1 - 1/n\} = 0.$

**Remark 3.4.** i) The equivalence of i) and ii) in Proposition 3.3 is well-known and relates to Takahashi theorem, i.e., it is enough to know that the limiting max-stable df $H$ is a product df at one point, say $(1,1)$. See for more details in the $d$-dimensional setup [3][p. 452].

ii) Recall that the assumption $F_i \in MDA(\Phi)$ means that $\lim_{n \to \infty} F_i^n(a_n x) = \Phi(x), x \in \mathbb{R}$ for some norming constants $a_n > 0, n \in \mathbb{N}$. For notational simplicity, in this paper we assume that $a_n$'s equal $n$. If this is not the case, then we need to re-formulate statement ii) in Proposition 3.3 as $n \lim_{n \to \infty} n P\{X_1 > a_{n1}, X_2 > a_{n2}\} = 0$. Note that if $F \in MDA(H)$ with $H$ a max-stable df, then

$$\lim_{n \to \infty} n P\{X_1 > a_{n1}, X_2 > a_{n2}\} = 2 + \ln H(1,1) = 2 - \theta(H) =: \lambda_F.$$  

In the literature, $\lambda_F$ is commonly referred to as the coefficient of upper tail dependence of $F$, see [3] for more details.

4. Auxiliary Results

**Lemma 4.1.** Let $(V_1, \ldots, V_d)$ be a random vector with continuous marginal df's $H_i, i \leq d$. If further $G$ is a $d$-dimensional df with $G(x_1, \ldots, x_d) < 1$ for any $(x_1, \ldots, x_d) \in (0, \infty)^d$ and the upper endpoint of $H_i, 1 \leq i \leq d$ equals $\infty$, then we have

$$\lim_{n \to \infty} n E\{G^{-1}(V_1, \ldots, V_d)\} = \lim_{n \to \infty} n P\{G(V_1, \ldots, V_d) > 1 - 1/n\} = \kappa \in [0, \infty),$$

if either of the limits exists. Further if

$$G(x_1, \ldots, x_d) \leq \min_{1 \leq i \leq d} H_i(x_i), \quad (x_1, \ldots, x_d) \in (0, \infty)^d,$$

then $\kappa \in [0, 1]$.

**Proof of Lemma 4.1** The proof of (4.1) follows from [26][Lemma 2.4], see also [6][Proposition 4]. Assuming (4.2), if $H$ denotes the df of $(V_1, \ldots, V_d)$, then we have

$$0 \leq n E\{G^{-1}(V_1, \ldots, V_d)\} \leq n \int_{(0, \infty)^d} \min_{1 \leq i \leq d} H_i^{-1}(x_i) dH(x_1, \ldots, x_d) \leq n \int_0^\infty H_i^{-1}(x_i) dH_i(x_i) = 1$$

establishing the proof. \hfill \square

**Proposition 4.2.** Let $F_n, G_n, n \geq 1$ be two continuous df's on $[0, \infty)^d$ satisfying

$$\lim_{n \to \infty} F_n(x_1, \ldots, x_d) = H(x_1, \ldots, x_d), \quad \lim_{n \to \infty} G_n(x_1, \ldots, x_d) = Q(x_1, \ldots, x_d), \quad (x_1, \ldots, x_d) \in [0, \infty)^d,$$

with $H, Q$ two max-id. df's with unit Fréchet marginal df's $\Phi$. If for all $n$ large and some $C_1 > 0$

$$G_n(x_1, \ldots, x_d) \leq C_1 \sum_{1 \leq i \leq d} F_{ni}(x_i), \quad (x_1, \ldots, x_d) \in (0, \infty)^d,$$

where $F_{ni}$ is the $i$th marginal df of $F_n$, then

$$\lim_{n \to \infty} n \int_{[0, \infty)^d} G_n(x_1, \ldots, x_d) dF_n(x_1, \ldots, x_d) = \int_{(0, \infty)^d} Q(x_1, \ldots, x_d) d\nu(x_1, \ldots, x_d),$$
where \( v(\cdot) \) is the exponent measure pertaining to \( H \) defined on \( E := [0, \infty]^d \setminus \{(0, \ldots, 0)\} \). If further for all \( n \) large and any \( x_1, \ldots, x_d \) positive

\[
1 - G_n(x_1, \ldots, x_d) \leq C_2 \sum_{1 \leq i \leq d} \tilde{F}_n(x_i),
\]

then we have

\[
\lim_{n \to \infty} n \int_{[0, \infty)^d} [1 - G_n(x_1, \ldots, x_d)] dF_n(x_1, \ldots, x_d) = - \int_{(0, \infty)^d} \ln Q(x_1, \ldots, x_d) \, dH(x_1, \ldots, x_d).
\]

**Proof of Proposition 4.2** For notational simplicity we consider below only the case \( d = 2 \). From the assumptions

\[
\lim_{n \to \infty} F_n^n(x_{n1}, x_{n2}) = H(x_1, x_2), \quad \lim_{n \to \infty} G_n^n(x_{n1}, x_{n2}) = Q(x_1, x_2)
\]

for every sequence \((x_{n1}, x_{n2}) \to (x_1, x_2) \in (0, \infty)^2 \) as \( n \to \infty \).

Let \( v \) be the exponent measure of \( H \) defined on \( E \), see [1] for details. For any \( x_0, y_0 \) positive, since by our assumptions

\[
\lim_{n \to \infty} n[1 - F_n(x_1, x_2)] = - \ln H(x_1, x_2)
\]

holds locally uniformly for \((x_1, x_2) \in (0, \infty)^2 \), using further (4.8) and [19][Lemma 9.3] we obtain

\[
\lim_{n \to \infty} \int_{(x_0, \infty) \times [y_0, \infty)} G_n^n(x_1, x_2) \, d(nF_n(x_1, x_2)) = \int_{(x_0, \infty) \times [y_0, \infty)} Q(x_1, x_2) \, dv(x_1, x_2) =: I(x_0, y_0).
\]

Moreover, by (4.4)

\[
n \int_{(0, \infty)^2} G_n^n(x_1, x_2) \, dF_n(x_1, x_2)
\]

\[
\leq nC_1 \left( \int_{[0, x_0]} F_n^{n-1}(x) \, dF_n(x) + \int_{[0, y_0]} F_n^{n-2}(x) \, dF_n(x) \right)
\]

\[
+ \int_{[x_0, \infty) \times [y_0, \infty)} G_n^n(x_1, x_2) \, d(nF_n(x_1, x_2))
\]

\[
= C_1(F_n^n(x_0) + F_n^n(y_0)) + \int_{[x_0, \infty) \times [y_0, \infty)} G_n^n(x_1, x_2) \, d(nF_n(x_1, x_2))
\]

\[
\to C_1(e^{-1/x_0} + e^{-1/y_0}) + \int_{[x_0, \infty) \times [y_0, \infty)} Q(x_1, x_2) \, dv(x_1, x_2), \quad n \to \infty
\]

\[
\to \int_{(0, \infty)^2} Q(x_1, x_2) \, dv(x_1, x_2), \quad x_0 \downarrow 0, y_0 \downarrow 0,
\]

where the equality above is a consequence of the assumption that \( F_n, G_n \) have continuous marginal df's. Hence (4.5) follows and we show next (4.7). Similarly, for \( x_0, y_0 \) as above

\[
\limsup_{n \to \infty} \int_{(0, \infty)^2} n[1 - G_n(x_1, x_2)] \, dF_n^n(x_1, x_2)
\]

\[
= \limsup_{n \to \infty} \left[ \int_{(x_0, \infty) \times [y_0, \infty)} n[1 - G_n(x_1, x_2)] \, dF_n^n(x_1, x_2) \right]
\]

\[
+ \int_{[x_0, \infty) \times [y_0, \infty)} n[1 - G_n(x_1, x_2)] \, dF_n^n(x_1, x_2)
\]

\[
\leq C_2 \limsup_{n \to \infty} \int_{(x_0, \infty) \times [y_0, \infty)} n[\tilde{F}_n(x_1) + \tilde{F}_n(x_2)] \, dF_n^n(x_1, x_2)
\]

\[
+ \limsup_{n \to \infty} \int_{(x_0, \infty) \times [y_0, \infty)} n[1 - G_n(x_1, x_2)] \, dF_n^n(x_1, x_2)
\]
\[
\begin{align*}
&\leq C_2 \limsup_{n \to \infty} (F_{n1}(x_0) + F_{n2}(y_0)) \left[ nF_{n1}(x_0) + nF_{n2}(y_0) \right] \\
&\quad - \int_{[x_0,\infty) \times [y_0,\infty)} \ln Q(x_1, x_2) \, dH(x_1, x_2) \\
&= C_2 \left[ e^{-1/x_0} + e^{-1/y_0} \right] \left[ \frac{1}{x_0} + \frac{1}{y_0} \right] - \int_{[x_0,\infty) \times [y_0,\infty)} \ln Q(x_1, x_2) \, dH(x_1, x_2) \\
&\to - \int_{(0,\infty)^2} \ln Q(x_1, x_2) \, dH(x_1, x_2), \quad x_0 \downarrow 0, y_0 \downarrow 0,
\end{align*}
\]

hence the proof follows. \(\square\)

**Remark 4.3.** The validity of (4.4) has been shown under the assumption that \(G_n\) is a continuous df. From the proof above it is easy to see that (4.4) still holds if we assume instead that \(G_n\) is continuous and positive such that \(G_n^*\) is a df. Similarly, for the validity of (4.7) it is enough to assume that \(F_n^*\) is a continuous df.

**Corollary 4.4.** If \(H\) is a bivariate max-stable df with unit Fréchet marginal df’s \(H_1\) and \(H_2\), then for \(u, t\) positive (4.9)
\[
\int_{(0,\infty)^2} \min(H_1^{1/u}(x_1), H_2^{1/t}(x_2)) \, dv(x_1, x_2) = u + t + \ln H(1/u, 1/t).
\]

**Proof of Corollary 4.4** The proof follows using Fubini Theorem and the homogeneity property of the exponent measure inherited by (1.1). We give below an alternative proof. Let \((V_1, V_2)\) have df \(H\) and set \(U_i = H_i(V_i), i = 1, 2\). By the assumptions since the df \(H\) is continuous, applying Theorem 2.3 and (4.1) with \(u, t > 0\) we obtain
\[
\int_{(0,\infty)^2} \min(H_1^{1/u}(x_1), H_2^{1/t}(x_2)) \, dv(x_1, x_2)
= \lim_{n \to \infty} n \int_{(0,\infty)^2} \min(H_1^{1/u}(x_1), H_2^{1/t}(x_2)) \, dH(x_1, x_2)
= \lim_{n \to \infty} n \mathbb{P} \left\{ \min(H_1^{1/u}(V_1), H_2^{1/t}(V_2)) > 1 - \frac{1}{n} \right\}
= \lim_{n \to \infty} n \mathbb{P} \left\{ U_1 > 1 - \frac{u}{n}, U_2 > 1 - \frac{t}{n} \right\} = u + t + \ln H(1/u, 1/t)
\]
establishing the proof. \(\square\)

5. **Proofs**

**Proof of Theorem 2.1** For \(n > 0\) set \(A_n = Q^{1/n}\) and \(B_n = H^{1/n}\). Since \(H\) is a max-id. df, then \(B_n\) is a df for any \(n > 0\). Furthermore, since \(H_i = Q_i, i \leq d\) (recall \(H_i, Q_i\) are the marginal df’s of \(H\) and \(Q\), respectively), it can be easily checked that we can apply Proposition 4.2, which together with Remark 4.3 imply
\[
\lim_{n \to \infty} n \int_{\mathbb{R}^d} \left| 1 - H^{1/n}(x_1, \ldots, x_d) \right| \, dQ(x_1, \ldots, x_d)
= \lim_{n \to \infty} n \int_{\mathbb{R}^d} \left| 1 - B_n(x_1, \ldots, x_d) \right| \, dA_n(x_1, \ldots, x_d)
\]
\[
= - \int_{\mathbb{R}^d} \ln H(x_1, \ldots, x_d) \, dQ(x_1, \ldots, x_d) = \mu(H, Q).
\]
The second claim in (2.1) follows with similar arguments and therefore we omit its proof.

Next, for any non-empty subset \(K\) of \(\{1, \ldots, d\}\) with \(m = |K|\) elements by (2.1)
\[
\mu(H_K, Q_K) = \lim_{n \to \infty} n \int_{\mathbb{R}^m} \left| 1 - F_{n,K}(x_1, \ldots, x_d) \right| \, dQ_K(x_1, \ldots, x_d)
\]
and

$$\lambda(Q_K, H_K) = \lim_{n \to \infty} n \int_{\mathbb{R}^m} Q_K(x_1, \ldots, x_d) dF_{nK}(x_1, \ldots, x_d),$$

where $F_{nK}, Q_K$ are the marginals of $F_n$ and $Q$ with respect to $K$. Note that for notational simplicity we write the marginal df’s with respect to $K$ as functions of $x_1, \ldots, x_d$ and not as functions of $x_{j1}, \ldots, x_{jm}$ where $K = \{j_1, \ldots, j_m\}$ has $m = |K|$ elements. By Fubini Theorem

$$\int_{\mathbb{R}^m} Q_K(x_1, \ldots, x_d) dF_{nK}(x_1, \ldots, x_d) = \int_{\mathbb{R}^m} F_{nK}(x_1, \ldots, x_d) dQ_K(x_1, \ldots, x_d),$$

where $F_{nK}$ stands for the joint survival function of $F_{nK}$. In the light of the inclusion-exclusion formula

$$1 - F_n(x_1, \ldots, x_d) = \sum_{1 \leq i \leq d} (-1)^{i+1} \sum_{K \subset \{1, \ldots, d\}, |K| = i} F_{nK}(x_1, \ldots, x_d), \quad (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

Using further the fact that $H$ and $Q$ have the same marginal df’s, for any index set $K$ with only one element we have

$$\lim_{n \to \infty} n \int_{\mathbb{R}^d} F_{nK}(x_1, \ldots, x_d) dQ(x_1, \ldots, x_d) = \lim_{n \to \infty} n \int_0^1 (1 - t^{1/n}) dt = 1,$$

hence

$$\mu(H, Q)$$

$$= \lim_{n \to \infty} n \int_{\mathbb{R}^{|K|}} [1 - F_n(x_1, \ldots, x_d)] dQ(x_1, \ldots, x_d)$$

$$= d + \lim_{n \to \infty} n \int_{\mathbb{R}^{|K|}} \sum_{2 \leq i \leq d} (-1)^{i+1} \sum_{K \subset \{1, \ldots, d\}, |K| = i} F_{nK}(x_1, \ldots, x_d) dQ(x_1, \ldots, x_d)$$

$$= d + \sum_{2 \leq i \leq d} (-1)^{i+1} \lim_{n \to \infty} n \int_{\mathbb{R}^{|K|}} \sum_{K \subset \{1, \ldots, d\}, |K| = i} F_{nK}(x_1, \ldots, x_d) dQ(x_1, \ldots, x_d)$$

$$= d + \sum_{2 \leq i \leq d} (-1)^{i+1} \sum_{K \subset \{1, \ldots, d\}, |K| = i} \lambda(Q_K, H_K)$$

and thus (2.2) follows. Since by the inclusion-exclusion formula we have further

$$F_n(x_1, \ldots, x_d) = \sum_{1 \leq i \leq d} (-1)^{i+1} \sum_{K \subset \{1, \ldots, d\}, |K| = i} [1 - F_{nK}(x_1, \ldots, x_d)], \quad (x_1, \ldots, x_d) \in \mathbb{R}^d$$

the claim in (2.3) follows with similar arguments as above.

**Proof of Theorem 2.5** The claim in (2.8) follows by the de Haan and inf-argmax representation of $H$. Since by the independence of $Y_i$’s and $Z_i$’s and the fact that $\mathbb{E}\{Y_i\} = \mathbb{E}\{1/Z_i\} = 1$ we have that

(5.2) \hspace{1cm} \mathbb{E}\{Y_i/Z_i\} = \mathbb{E}\{Y_i\} \mathbb{E}\{1/Z_i\} = 1

is valid for any $i \leq d$. Consequently, by (2.3), (2.8) and the fact that for given constants $c_1, \ldots, c_d$

$$\min_{1 \leq i \leq d} c_i = \sum_{i=1}^d (-1)^{i+1} \sum_{K \subset \{1, \ldots, d\}, |K| = i} \max_{j \in K} c_j,$$

then we have

$$\lambda(Q, H) = \sum_{i=1}^d \mathbb{E}\left\{\frac{Y_i}{Z_i}\right\} + \sum_{2 \leq i \leq d} (-1)^{i+1} \sum_{K \subset \{1, \ldots, d\}, |K| = i} \mathbb{E}\left\{\max_{j \in K} \frac{Y_j}{Z_j}\right\}$$

$$= \mathbb{E}\left\{\min_{1 \leq j \leq d} \frac{Y_j}{Z_j}\right\}.$$
establishing (2.9).

Further, since (5.2) holds, then by de Haan representation of max-stable df’s we have that the df’s $H^*, \tilde{H}$ defined in (2.6) and (2.7), respectively are max-stable with unit Fréchet marginal df’s. Hence (2.8) implies that $\mu(H, Q) = \theta(H^*)$. Note in passing that for $Q = H$ this follows also from [12][Proposition 2.2].

Using again that $Y_i$’s are independent of $Z_i$’s and $E\{Y_i\} = 1, i \leq d$ we obtain (recall $Y_i$’s and $Z_i$’s are non-negative random variables)

$$
\mu(H, Q) = E\left\{ E\left\{ \max_{1 \leq i \leq d} \frac{Y_i}{Z_i} \big| (Z_1, \ldots, Z_d) \right\} \right\} \\
\geq E\left\{ \max_{1 \leq i \leq d} \frac{E\{Y_i\}}{Z_i} \right\} \\
\geq E\left\{ \max_{1 \leq i \leq d} \frac{1}{Z_i} \right\} = \theta(\tilde{H}) \\
\geq \max_{1 \leq i \leq d} E\left\{ \frac{1}{Z_i} \right\} = 1.
$$

With the same arguments using now that $E\{1/Z_i\} = 1, i \leq d$ we have

$$
\mu(H, Q) = E\left\{ E\left\{ \max_{1 \leq i \leq d} \frac{Y_i}{Z_i} \big| (Y_1, \ldots, Y_d) \right\} \right\} \\
\geq E\left\{ \max_{1 \leq i \leq d} Y_i \right\} = -\ln H(1, \ldots, 1) = \theta(H).
$$

The lower bound in (2.11) follows with similar arguments, hence the proof is complete. □

**Proof of Theorem 2.3** Suppose without loss of generality that $F$ satisfies (1.5). If $F_i = G_i, i = 1, 2$, then the claim follows from Lemma 4.1 and Proposition 4.2. We consider next the general case that $F_i$’s are tail equivalent to $G_i$’s and suppose for simplicity that $d = 2$. In view of [26][Lemma 2.4] we have

$$
\lim_{n \to \infty} n \int_{[0, \infty)} G_i^n(x) dF_i(x) = c_i \in [0, \infty), \quad i = 1, 2
$$

if and only if $\lim_{n \to \infty} n \mathbb{P}\{G_i(X_i) > 1 - 1/n\} = c_i$ or equivalently

$$
\lim_{x \to \infty} \frac{\tilde{F}_i(x)}{G_i(x)} = c_i.
$$

By the assumption $c_i \in (0, \infty)$ for $i = 1, 2$. Consequently, for all $x > 0$ there exist $a_1, a_2$ positive such that

$$
a_1 \tilde{F}_i(x) \leq \tilde{G}_i(x) \leq a_2 \tilde{F}_i(x).
$$

Assume for simplicity that $c_i = 1, i = 1, 2$. By the assumptions

$$
n\tilde{F}_i(nx) \to 1/x, \quad n\tilde{G}_i(nx) \to 1/x, \quad n \to \infty
$$

uniformly for $x$ in $[t, \infty), t > 0$. Further, for $i = 1, 2$ we have

$$
\lim_{t \downarrow 0} \lim_{n \to \infty} n \int_{[0, t]} G_i^n(nx) dF_i(nx) = \lim_{t \downarrow 0} \lim_{n \to \infty} n \int_{[0, t]} \tilde{G}_i(nx) d\tilde{F}_i^n(nx) = 0,
$$

which implies

$$
\lim_{t \downarrow 0} \lim_{n \to \infty} n \int_{[0, t]^2} G^n(nx, ny) dF(nx, ny) = \lim_{t \downarrow 0} \lim_{n \to \infty} n \int_{[0, t]^2} [1 - G(nx, ny)] d\tilde{F}^n(nx, ny) = 0.
$$
As in the proof of Proposition 4.2, using that \( F \) and \( G \) are in the MDA of \( H \) and \( Q \), respectively, it follows that for any integer \( k \)
\[
\lim_{n \to \infty} n \int_{[0,\infty)^2} G^{n-k}(x_1, x_2) dF(x_1, x_2) = \int_{(0,\infty)^2} Q(x_1, x_2) dv(x_1, x_2) = \lambda(Q, H)
\]
and further
\[
\lim_{n \to \infty} n \int_{[0,\infty)^2} [1 - F(x_1, x_2)] dG^{n-k}(x_1, x_2) = - \int_{(0,\infty)^2} \ln F(x_1, x_2) dQ(x_1, x_2) = \mu(H, Q)
\]
establishing the proof.

**Proof of Proposition 3.1** In view of Theorem 2.1, since \( H \) being a max-id. df implies that \( H^{1/n} \) is a df for any \( n \geq 1 \) we have with \( F_n = Q^{1/n} \)
\[
\int_{(0,\infty)^2} Q(x_1, x_2) dv(x_1, x_2) = \lim_{n \to \infty} n \int_{(0,\infty)^2} Q(x_1, x_2) dH^{1/n}(x_1, x_2)
\]
\[
= 2 - \lim_{n \to \infty} n \int_{[0,\infty)^2} [1 - H^{1/n}(x_1, x_2)] dF_n(x_1, x_2)
\]
\[
= \int_{(0,\infty)^2} [2 + \ln H(x_1, x_2)] dQ(x_1, x_2).
\]
Since further by [25][Theorem 7] the restriction of \( v \) on \( (0,\infty)^2 \) denoted by \( v_0 \) satisfies
\[
\frac{dv_0}{dH} = \frac{1 - \xi_H}{H}
\]
and \( \xi_H(x_1, x_2) \in [0, 1] \) almost everywhere \( dH \), then the first claim follows.

The equivalence of i) and ii) is known as Takahashi Theorem, see [3][Theorem 4.3.2]. Since \( \xi_H \in [0, 1] \) almost everywhere \( dH \), the equivalence of ii) and iii) is a direct consequence of (3.3) and the fact that \( \lambda(Q, H) = 2 - \mu(H, Q) \), see (2.3). Clearly, by (3.3) we have thus \( \xi_H = 1 \) almost everywhere \( dH \) is equivalent with \( H = H_0 \), whereas iv) is equivalent with v) is consequence of [25][Theorem 7].

**Proof of Proposition 3.3** If \( F (1.6) \) holds, then clearly ii) is satisfied and thus i) implies ii). If ii) holds, then
\[
\limsup_{n \to \infty} F^n(nx_1, nx_2) = \exp \left( \limsup_{n \to \infty} n \ln (1 - \lfloor 1 - F(nx_1, nx_2) \rfloor) \right)
\]
\[
= \exp \left( - \limsup_{n \to \infty} n \lfloor 1 - F(nx_1, nx_2) \rfloor \right)
\]
\[
\leq \exp \left( - \limsup_{n \to \infty} n \left[ F_1(nx_1) + nF_2(nx_2) - n\mathbb{P}\{X_1 > n\min(x_1, x_2), X_2 > n\min(x_1, x_2)\} \right] \right)
\]
\[
= \exp \left( - 1/x_1 - 1/x_2 \right), \quad x_1, x_2 > 0.
\]
As for the derivation of (1.2) we obtain further
\[
\liminf_{n \to \infty} F^n(nx_1, nx_2) \geq \exp \left( -1/x_1 - 1/x_2 \right), \quad x_1, x_2 > 0
\]
implicating that \( F \in MDA(H_0) \), hence i) follows.

Assuming iii) and since the marginal df’s of \( G \) are in the MDA of \( \Phi \), with the same calculations as in (5.3) for the df \( G \) we obtain
\[
0 = \lim_{n \to \infty} n \int_{(0,\infty)^2} G^n(x_1, x_2) dF(x_1, x_2) \geq \lim_{n \to \infty} n \mathbb{P}\{X_1 > n, X_2 > n\} G^n(n, n)
\]
\[
\geq c \lim_{n \to \infty} n \mathbb{P}\{X_1 > n, X_2 > n\}
\]
for some $c \in (0, e^{-2})$, hence ii) follows.

Next, assume that ii) holds. We have that $G(x_1, x_2) \leq G_1(x_1)G_2(x_2) =: K(x_1, x_2)$ and by the assumption that $G_i$'s are in the MDA of $\Phi$ it follows that $K$ is in the MDA of $H_\infty$. Further ii) implies that $F \in MDA(H_0)$ and $-\ln H(1, 1) = 2$. Consequently, Theorem 2.3 yields

$$\lim_{n \to \infty} \lambda_n(K^n, F) = \lambda(H_\infty, H_0).$$

But from Corollary 4.4 we have that $\lambda(H_\infty, H_0) = 0$, hence ii) implies iii).

Let $\overline{G}$ be the joint survival function of the bivariate df $G$. For any positive integer $n$, we have that $F^n$ is a bivariate df. Hence by Fubini theorem and the fact that $F_i = G_i, i = 1, 2$ are continuous df's, for any positive integer $n$ we obtain

$$\int_{\mathbb{R}^2} F^n(x_1, x_2) dG(x_1, x_2) = \int_{\mathbb{R}^2} \overline{G}(x_1, x_2) dF^n(x_1, x_2) = \frac{2n}{n+1} - \int_{\mathbb{R}^2} [1 - G(x_1, x_2)] dF^n(x_1, x_2)$$

and thus the equivalence of iii) and iv) follows. The equivalence iv) and v) follows from Lemma 4.1 and thus the proof is complete. \[\square\]

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