Hartree Approximation to the One Loop Quantum Gravitational Correction to the Graviton Mode Function on de Sitter

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ABSTRACT

We use the Hartree approximation to the Einstein equation on de Sitter background to solve for the one loop correction to the graviton mode function. This should give a reasonable approximation to how the ensemble of inflationary gravitons affects a single external graviton. At late times we find that the one loop correction to the plane wave mode function $u(\eta, k)$ goes like $GH^2 \ln(a)/a^2$, where $a$ is the inflationary scale factor. One consequence is that the one loop corrections to the “electric” components of the linearized Weyl tensor grow compared to the tree order result.

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1 Introduction

Primordial cosmological perturbations are believed originate in the scalars and gravitons produced by inflation [1]. These inflationary scalars and gravitons also interact among themselves, and they should affect the dynamics of other particles. One analyzes the latter sort of effect by first computing the contribution of inflationary scalars and/or gravitons to the appropriate 1PI (one-particle-irreducible) 2-point function, then using this 1PI 2-point function to quantum-correct the linearized effective field equation of the particle whose dynamics are being studied.

The past decade has witnessed a number of studies of this type. It is easier to work with massless, minimally coupled (MMC) scalars than with gravitons, so scalar effects were probed first:

- When a MMC scalar is endowed with a quartic self-interaction the scalar mode function behaves as if its mass were growing [2];
- Yukawa-coupled MMC scalars cause fermions to develop a growing mass, even when the scalar has zero potential [3];
- Charged MMC scalars induce so much vacuum polarization that the photon develops mass [4] and there are corresponding corrections to electromagnetic forces [5] and
- Gravitationally coupled MMC scalars do not induce any secular change in the graviton mode function [6].

Three studies have been made of what inflationary gravitons do to other particles:

- There is a slow secular growth in the field strength of massless fermions [7, 8], driven by the spin-spin interaction [9], and a much larger effect driven by a small, nonzero mass [10];
- The absence of any spin-spin coupling prevents inflationary gravitons from having a comparable effect on MMC scalars [11]; and
- Inflationary gravitons induce a slow secular growth in the electric components of the photon field strength but no comparable growth in the magnetic field [12].
The purpose of this paper is to begin the study of what inflationary gravitons do to other gravitons. The first step of an exact analysis would be to compute the graviton self-energy at one loop order in de Sitter background. That has been done \cite{13}, but only using an old formalism for which dimensional regularization cannot be employed, so the result is only valid away from coincidence. The formalism required for a fully dimensionally regulated computation has since then been developed \cite{14}, but its application to the graviton self-energy has not yet been completed. In the meantime we can gain a qualitative understanding of the potential results by employing the Hartree approximation \cite{15}. This has been shown to predict the correct time dependence for the effects of MMC scalars on photons \cite{16,17}, the effects of gravitons on fermions \cite{7,8}, and for the effects of gravitons on photons \cite{12}.

This paper contains five sections, of which the first is this Introduction. In section 2 we describe the de Sitter background geometry, the quantum gravity Lagrangian whose field equations we will solve in the Hartree approximation and our choice of gauge. The Hartree approximation to the linearized effective field equation is derived in section 3 and solved for plane wave gravitons in section 4. Our discussion comprises section 5.

2 Feynman Rules

In this section we give the Feynman rules that we will use in this study. These will derive from the Lagrangian of quantum gravity whose dynamical field corresponds to a conformally rescaled graviton field. We will employ a generalized version of the de Donder gauge fixing term and we will present the associated ghost Lagrangian. All this will be done in the context of conventional perturbation theory around a de Sitter background, so we find it necessary to start this section with a review of de Sitter space.

We are interested in the effects of gravitons produced during primordial inflation. Cosmological observations \cite{18,19} support the idea that de Sitter space can be considered as a paradigm for inflation. However, because we are employing de Sitter as an approximation for the true inflationary background — which is a homogeneous, isotropic and spatially flat geometry — we want to work on the open coordinate submanifold of the full de Sitter geometry. Because we shall be using dimensional regularization we work in $D$-dimensional conformal coordinates $x^\mu = (\eta, x^i)$, where

\[ -\infty < \eta < 0 \ , \ -\infty < x^i < \infty \ . \quad (1) \]
We express the full metric as,
\[
g_{\mu\nu} = a^2[\eta_{\mu\nu} + \kappa h_{\mu\nu}] \equiv a^2 \tilde{g}_{\mu\nu} .
\] (2)

Here \(a(\eta) \equiv -1/H\eta\) is the scale factor, \(H\) is the (constant) Hubble parameter of de Sitter, \(\eta_{\mu\nu}\) is the Lorentz metric with spacelike signature, \(h_{\mu\nu}\) is a perturbation to this background which we identify with the graviton field (whose indices are raised and lowered with \(\eta_{\mu\nu}\)), and \(\kappa^2 \equiv 16\pi G\) is the loop counting parameter of quantum gravity. We note that \(\tilde{g}^{\alpha\beta}\) inverts its covariant counterpart,
\[
\tilde{g}^{\alpha\beta} = \eta^{\alpha\beta} - \kappa h^{\alpha\beta} + \kappa^2 h^{\alpha\rho}h_\rho^\beta - ... \] (3)

The Einstein-Hilbert Lagrangian for quantum gravity is,
\[
L_{\text{inv}} = \kappa^{-2} \sqrt{-g} \left[ R - (D-2)\Lambda \right] ,
\] (4)
where \(R\) is the \(D\)-dimensional Ricci scalar and \(\Lambda \equiv (D-1)H^2\) is the cosmological constant. Expanding this Lagrangian with the metric (2) and extracting a presumably irrelevant surface term (whose expression we will not write here), we obtain \(L_{\text{inv}} - S_{\mu,\mu} = \) \(L_{\text{gt}}\):
\[
L_{\text{inv}} - S_{\mu,\mu} = \left(\frac{D}{2} - 1\right) H a^{D-1} \sqrt{-g} \tilde{g}^{\rho\sigma} g^{\mu\nu} h_{\rho\sigma,\mu} h_{\nu\rho} + a^{D-2} \sqrt{-\tilde{g}} \tilde{g}^{\alpha\beta} \tilde{g}^{\rho\sigma} \tilde{g}^{\mu\nu} \times \left( \frac{1}{2} h_{\alpha\rho,\mu} h_{\nu\sigma,\beta} - \frac{1}{2} h_{\alpha\rho,\rho} h_{\nu\sigma,\mu} + \frac{1}{4} h_{\alpha\beta,\rho} h_{\mu\nu,\sigma} - \frac{1}{4} h_{\alpha\rho,\mu} h_{\beta\sigma,\nu} \right) .
\] (5)

We fix the gauge by adding an analogue of the de Donder term used in flat space \[14\],
\[
L_{\text{gt}} = -\frac{1}{2} a^{D-2} \eta^{\mu\nu} F_\mu F_\nu , \quad F_\mu = \eta^{\sigma\rho} \left[ h_{\mu\rho,\sigma} - \frac{1}{2} h_{\rho\sigma,\mu} + (D-2) H a h_{\mu\rho} \delta_\sigma^0 \right] .
\] (6)

Because space and time components are treated differently it will convenient to define the purely spatial parts of the Minkowski metric and the Kronecker delta function,
\[
\bar{\eta}_{\mu\nu} \equiv \eta_{\mu\nu} + \delta_\mu^0 \delta_\nu^0 , \quad \bar{\delta}_\nu^\mu \equiv \delta^\mu_\nu - \delta_0^\mu \delta_\nu^0 , \quad \bar{\eta}^{\mu\nu} \equiv \eta^{\mu\nu} + \delta^\mu_0 \delta^\nu_0 . \] (7)
The graviton kinetic operator can be found by partially integrating the quadratic part $L_{\text{inv}} + L_{\text{gf}}$ to obtain the form $\frac{1}{2} h_{\mu\nu}^{\rho\sigma} \mathcal{D}_{\mu\nu}^{\rho\sigma}$, where

\[
\mathcal{D}_{\mu\nu}^{\rho\sigma} = \left[ \frac{1}{2} \delta_{\mu}^{\rho} \delta_{\nu}^{\sigma} - \frac{1}{4} \eta_{\mu\nu} \eta^{\rho\sigma} - \frac{1}{2(D-3)} \delta_{\mu}^{0} \delta_{\nu}^{0} \delta_{0}^{\rho} \delta_{0}^{\sigma} \right] \mathcal{D}_{A} + \delta_{(\mu}^{0} \delta_{\nu)}^{\rho} \mathcal{D}_{B} + \frac{1}{2} \left( \frac{D-2}{D-3} \right) \delta_{\mu}^{0} \delta_{\nu}^{0} \delta_{0}^{\rho} \delta_{0}^{\sigma} \mathcal{D}_{C} .
\]  

The three scalar differential operators are,

\[
\mathcal{D}_{A} \equiv \partial_{\mu} \left( a^{D-2} \eta^{\mu\nu} \partial_{\nu} \right) , \quad \mathcal{D}_{B} \equiv \mathcal{D}_{A} - (D-2)a^{D} H^{2} , \quad \mathcal{D}_{C} \equiv \mathcal{D}_{A} - 2(D-3)a^{D} H^{2} .
\]

The associated ghost Lagrangian is,

\[
L_{\text{ghost}} \equiv - a^{D-2} \omega^{\mu} \delta F_{\mu}
\]

\[
= \omega^{\mu} \left( \delta_{\mu}^{\rho} \mathcal{D}_{A} + \delta_{\mu}^{0} \delta_{0}^{\rho} \mathcal{D}_{B} \right) \omega_{\nu} - 2\kappa a^{D-2} \omega^{\mu\nu} \left( h_{\mu}^{\rho} \partial_{\nu} + \frac{1}{2} h^{\mu\nu} \partial_{\nu} - H a h_{\mu\nu} \delta_{0}^{\rho} \right) \omega_{\mu}
\]

\[
+ \kappa \left( a^{D-2} \omega^{\mu} \right)_{\mu} \left( h^{\rho\sigma} \partial_{\sigma} + \frac{1}{2} h_{\sigma}^{\rho\sigma} - H a h \delta_{0}^{\rho} \right) \omega_{\rho} .
\]

The ghost and graviton propagator in this gauge can be written in a simple form as a sum of constant tensor factors times scalar propagators,

\[
i[\mu \Delta_{\nu}](x; x') = \tau_{\mu \nu} i \Delta_{A}(x; x') - \delta_{\mu}^{0} \delta_{\nu}^{0} i \Delta_{B}(x; x') ,
\]

\[
i[\mu \Delta_{\alpha\beta}](x; x') = \sum_{I=A,B,C} [\mu \nu \mathcal{T}^{I}_{\alpha\beta}] i \Delta_{I}(x; x') .
\]

The tensor factors are given by,

\[
[\mu \nu \mathcal{T}^{A}_{\alpha\beta}] = 2 \tau_{\mu \nu} \tau_{\alpha \beta} - \frac{2}{D-3} \tau_{\mu \nu} \tau_{\alpha \beta} , \quad [\mu \nu \mathcal{T}^{B}_{\alpha\beta}] = -4 \delta_{(\mu}^{0} \tau_{\nu) (\alpha \delta_{\beta)}^{0} ,
\]

\[
[\mu \nu \mathcal{T}^{C}_{\alpha\beta}] = \frac{2}{(D-2)(D-3)} \left[ \tau_{\mu \nu} + (D-3) \delta_{\mu}^{0} \delta_{\nu}^{0} \right] \left[ \tau_{\alpha \beta} + (D-3) \delta_{\alpha}^{0} \delta_{\beta}^{0} \right] .
\]

and the three scalar propagators, which we discuss below, obey:

\[
\mathcal{D}_{I} i \Delta_{I}(x; x') = i \delta^{D}(x - x') , \quad I = A, B, C .
\]

It follows that the graviton propagator satisfies the equation,

\[
\mathcal{D}_{\rho}^{\mu\nu} i [\mu \nu \Delta_{\alpha\beta}](x; x') = i \delta_{(\alpha}^{\rho} \delta_{\beta)}^{\sigma} \delta^{D}(x - x') .
\]
To write expressions for the three scalar propagators, we note that our
gauge fixing term (6) will produce a graviton propagator that contains a de Sitter
invariant part as well as a de Sitter symmetry breaking piece. For the
former it will be useful to introduce a function
\[ y(x; x') = 4 \sin^2 \left( \frac{1}{2} H \ell(x; x') \right) \]
of the de Sitter invariant \( \ell(x; x') \) between the points \( x^\mu \) and \( x'^\mu \) defined by,
\[ y(x; x') \equiv a a' H^2 \left[ \| \vec{x} - \vec{x}' \|^2 - (|\eta - \eta'| - i \epsilon)^2 \right]. \] (17)

Because \( y(x; x') \) is a de Sitter invariant, any function of \( y(x; x') \) is also de Sitter invariant. Moreover covariant derivatives of it are de Sitter invariant,
this means that the first three derivatives of \( y(x; x') \) produce a convenient basis of de Sitter invariant bi-tensors [20],
\[ \frac{\partial y(x; x')}{\partial x^\mu} = H a \left( y^0_{\mu} + 2 a' H \Delta x_{\mu} \right), \] (18)
\[ \frac{\partial y(x; x')}{\partial x'^\nu} = H a' \left( y^0_{\nu} - 2 a H \Delta x_{\nu} \right), \] (19)
\[ \frac{\partial^2 y(x; x')}{\partial x^\mu \partial x'^\nu} = H^2 a a' \left( y_{\mu\nu}^0 + 2 a' H \Delta x_{\mu} \delta_{\nu}^0 - 2 a \delta_{\mu}^0 H \Delta x_{\nu} - 2 \eta_{\mu\nu} \right). \] (20)

Here and subsequently \( \Delta x_{\mu} \equiv \eta_{\mu\nu}(x-x')^\nu \).

It turns out that only the \( A \)-type propagator contains a de Sitter breaking part as it corresponds to a massless, minimally coupled scalar for which it is well known that no de Sitter invariant solution exists [21]. Preserving only the symmetries of homogeneity and isotropy this scalar propagator can be written as [22],
\[ i \Delta_A(x; x') = A(y(x; x')) + K \ln(aa'), \] (21)
where the constant \( K \) is,
\[ K \equiv \frac{H^{D-2} \Gamma(D-1)}{(4\pi)^{D/2} \Gamma(D/2)}. \] (22)

The function \( A(y) \) is,
\[ A(y) = \frac{H^{D-2}}{(4\pi)^{D/2}} \left\{ \Gamma \left( \frac{D}{2} - 1 \right) \left( \frac{4}{y} \right)^{D/2 - 1} + \Gamma \left( \frac{D}{2} + 1 \right) \left( \frac{4}{y} \right)^{D/2 - 2} + A_1 \right. \\
- \sum_{n=1}^{\infty} \frac{\Gamma(n + D/2 + 1)}{(n-D/2+2)\Gamma(n+2)} \left( \frac{y}{4} \right)^{n-D/2 + 2} - \frac{\Gamma(n + D - 1)}{n \Gamma(n + D/2)} \left( \frac{y}{4} \right)^n \right\}. \] (23)
Here the constant $A_1$ is defined by,

$$A_1 = \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \left\{ -\psi\left(1 - \frac{D}{2}\right) + \psi\left(\frac{D-1}{2}\right) + \psi(D-1) + \psi(1) \right\}. \quad (24)$$

On the other hand the $B$-type and the $C$-type propagators are de Sitter invariant,

$$i\Delta_B(x; x') \equiv B(y) = \frac{H^{D-2}}{(4\pi)^{D/2}} \Gamma\left(\frac{D}{2} - 1\right) \left(\frac{4}{y}\right)^{\frac{D}{2} - 1}$$

$$- \frac{H^{D-2}}{(4\pi)^{D/2}} \sum_{n=0}^{\infty} \left\{ \frac{\Gamma(n+D-2)}{\Gamma(n+D/2)} \left(\frac{y}{4}\right)^n - \frac{\Gamma(n+D/2)}{\Gamma(n+2)} \left(\frac{y}{4}\right)^{n-D/2+2} \right\}, \quad (25)$$

$$i\Delta_C(x; x') \equiv C(y) = \frac{H^{D-2}}{(4\pi)^{D/2}} \Gamma\left(\frac{D}{2} - 1\right) \left(\frac{4}{y}\right)^{\frac{D}{2} - 1} + \frac{H^{D-2}}{(4\pi)^{D/2}} \sum_{n=0}^{\infty} \left\{ \frac{(n+1)\Gamma(n+D-3)}{\Gamma(n+D/2)} \left(\frac{y}{4}\right)^n - \left(n-D/2+3\right) \frac{\Gamma(n+D/2-1)}{\Gamma(n+2)} \left(\frac{y}{4}\right)^{n-D/2+2} \right\}. \quad (26)$$

Each of the three invariant functions $A(y), B(y)$ and $C(y)$ contains an infinite series in powers of $y$. That may seem a little discouraging at first, but inspection reveals that each of the three series vanishes for $D = 4$. This means that we need to keep only those terms which multiply potentially divergent terms. Another simplification is that our computation will require only the coincidence limits of these functions (and their first derivatives). To take the coincidence limit means to set $x^\mu = x'^\mu$, hence it follows that in this limit $a = a'$, $\Delta x^\mu = 0$, and $y = 0$. This gives,

$$\lim_{x' \to x} \frac{\partial y(x; x')}{\partial x^\mu} = 0, \quad \lim_{x' \to x} \frac{\partial y(x; x')}{\partial x'^\nu} = 0, \quad \lim_{x' \to x} \frac{\partial^2 y(x; x')}{\partial x^\mu \partial x'^\nu} = -2H^2 a^2 \eta_{\mu\nu}. \quad (27)$$

Furthermore we recall that in dimensional regularization, any $D$–dependent power of zero is automatically set equal to zero. We can then summarize the coincidence limits we will need in the following way. For the three types ($I = A, B, C$) of propagators $i\Delta_I(x; x') = I(y) + \delta_I^A K \ln(aa')$ the coincidence limits are,

$$\lim_{x' \to x} i\Delta_I(x; x') = I(0) + \delta_I^A \times 2K \ln(a), \quad (28)$$
\[
\lim_{x' \to x} \partial_\mu i \Delta_I (x; x') = \delta_I^A \times K H a \delta^0_\mu ,
\]
\[
\lim_{x' \to x} \partial'_\nu i \Delta_I (x; x') = \delta_I^A \times K H a \delta^0_\nu ,
\]
\[
\lim_{x' \to x} \partial_\mu \partial'_\nu i \Delta_I (x; x') = \Gamma' (0) \times -2 H^2 a^2 \eta_{\mu \nu}.
\]

From expressions (23)-(26) we obtain,
\[
A(0) = \frac{H^{D-2}}{(4\pi)^{D/2}} A_1 , \quad B(0) = -\frac{H^{D-2}}{(4\pi)^{D/2}} \frac{\Gamma(D-2)}{\Gamma\left(\frac{D}{2}\right)} , \quad C(0) = \frac{H^{D-2}}{(4\pi)^{D/2}} \frac{\Gamma(D-3)}{\Gamma\left(\frac{D}{2}\right)} ,
\]
and
\[
A'(0) = \frac{H^{D-2}}{(4\pi)^{D/2}} \frac{\Gamma(D)}{4 \Gamma\left(\frac{D}{2}+1\right)} , \quad B'(0) = \frac{(D-2)}{2D} B(0) , \quad C'(0) = \frac{(D-3)}{D} C(0).
\]

3 The Hartree Approximation

To calculate the effects inflationary gravitons have on dynamical gravitons we would need to calculate the graviton self-energy and then solve the effective field equations at least to one-loop order. That work is currently in progress. However, we can anticipate the result by employing the Hartree, or mean-field, approximation \[15, 16, 17, 7, 8, 12\]. The idea is that whenever we encounter a product of graviton fields in the equations of motion we can consider one graviton field as being the “external” one and approximate the others in the same product by taking their expectation value in the vacuum. To illustrate this, let us denote an external graviton field by \(E_{\alpha\beta}(x)\). The Hartree approximation then consists of the following replacements,
\[
h_{\mu\nu} \rightarrow E_{\mu\nu} ,
\]
\[
h_{\mu\nu} h_{\rho\sigma} \rightarrow E_{\mu\nu} \langle h_{\rho\sigma} \rangle + E_{\rho\sigma} \langle h_{\mu\nu} \rangle ,
\]
\[
h_{\mu\nu} h_{\rho\sigma} h_{\alpha\beta} \rightarrow E_{\mu\nu} \langle h_{\rho\sigma} h_{\alpha\beta} \rangle + E_{\rho\sigma} \langle h_{\mu\nu} h_{\alpha\beta} \rangle + E_{\alpha\beta} \langle h_{\mu\nu} h_{\rho\sigma} \rangle ,
\]
for one, two and three gravitons respectively. Because it is important to understand what this approximation includes and what it does not, we digress to discuss the technique in the context of a simple quantum mechanical model. We then implement the approximation for quantum gravity.
3.1 Hartree approximation for anharmonic oscillator

The point of this sub-section is to explicate the meaning and validity of the Hartree approximation in the context of a point particle $q(t)$ whose Lagrangian is,

$$L = \frac{1}{2} m \dot{q}^2(t) - \frac{1}{2} m \omega^2 q^2(t) - \frac{1}{3} m \omega^4 g q^3(t) - \frac{1}{4} m \omega^2 g^2 q^4(t). \quad (37)$$

Here $g$ is a parameter with the dimensions of inverse length which quantifies how far the model is from being a simple harmonic oscillator. We first give a precise definition for the effective mode function $u(t)$. We then compute the full result for $u(t)$ at one loop ($g^2$) order and compare this with what the Hartree approximation gives. The sub-section closes with a discussion of previous Hartree computations of the effective mode function in various quantum field theories.

Even though our model (37) is not a harmonic oscillator for $g \neq 0$, we can still form the initial position and velocity into the raising and lowering operators of a harmonic oscillator,

$$a \equiv \sqrt{\frac{m \omega}{2 \hbar}} \left[ q(0) + \frac{i \dot{q}(0)}{\omega} \right] \implies [a, a^\dagger] = 1. \quad (38)$$

Similarly, there is a Heisenberg state $|\Omega\rangle$ which is the normalized ground state of the harmonic oscillator as perceived by the $t = 0$ operators,

$$a |\Omega\rangle = 0 = \langle \Omega | a^\dagger , \quad \langle \Omega | \Omega \rangle = 1. \quad (39)$$

The effective mode function $u(t)$ is the matrix element of $q(t)$ between the ground state and the $(g = 0)$ first excited state,

$$u(t) \equiv \langle \Omega | q(t) a^\dagger | \Omega \rangle = \langle \Omega | [q(t), a^\dagger] | \Omega \rangle. \quad (40)$$

In quantum field theory it would be the matrix element of the field between free vacuum and the free one particle state, at whatever time the system is released. One might also include perturbative corrections to the $t = 0$ states which would alter the initial time dependence of $u(t)$ but not its late time form [23].

To derive an exact expression for $u(t)$ at order $g^2$ we require the initial value solution of the Heisenberg equation of motion,

$$\ddot{q}(t) + \omega^2 q(t) + \omega^2 g q^2(t) + \omega^2 g^2 q^3(t) = 0. \quad (41)$$
We solve this equation by perturbatively expanding the solution in powers of the parameter \( g \),

\[
q(t) = q_0(t) + gq_1(t) + g^2q_2(t) + g^3q_3(t) + \ldots
\]

(42)

Collecting powers of \( g \), the zeroth order equation is,

\[
\ddot{q}_0(t) + \omega^2 q_0(t) = 0.
\]

(43)

We treat the initial value data as zeroth order so \( q_0(t) \) is,

\[
q_0(t) = q(0) \cos(\omega t) + \dot{q}(0) \omega \sin(\omega t) = \sqrt{\frac{\hbar}{2m\omega}} \left( e^{-i\omega t} a + e^{i\omega t} a^\dagger \right).
\]

(44)

Similarly, the first order equation is,

\[
\ddot{q}_1(t) + \omega^2 q_1(t) = -\omega^2 q_2^0(t).
\]

(45)

The solution to this equation is,

\[
q_1(t) = \int_0^t dt' \sin[\omega(t-t')] \left[ q_0^2(t') + q_0(t')q_1(t') + q_1(t')q_0(t') \right].
\]

(46)

In the same way the second order equation is,

\[
\ddot{q}_2(t) + \omega^2 q_2(t) = -\omega^2 q_3^0(t) - \omega^2 \left[ q_0(t)q_1(t) + q_1(t)q_0(t) \right],
\]

(47)

whose solution is found to be,

\[
q_2(t) = -\omega \int_0^t dt' \sin[\omega(t-t')] \left[ q_0^2(t') + q_0(t')q_1(t') + q_1(t')q_0(t') \right].
\]

(48)

The commutator of \( q_0(t) \) with \( a^\dagger \) facilitates our computation,

\[
[q_0(t), a^\dagger] = \sqrt{\frac{\hbar}{2m\omega}} e^{-i\omega t} \equiv u_0(t).
\]

(49)

With our perturbative solutions (46) and (48), this gives,

\[
[q(t), a^\dagger] = u_0(t) - 2g\omega \int_0^t dt' \sin[\omega(t-t')] q_0(t') u_0(t')
\]

\[
- g^2 \omega \int_0^t dt' \sin[\omega(t-t')] \left[ 3q_0^2(t') u_0(t') + 2q_1(t') u_0(t') \right]
\]

\[
- 2\omega \int_0^t dt'' \sin[\omega(t-t'')] \left[ q_0(t'), q_0(t'') \right] u_0(t'') + O(g^3).
\]

(50)
Because the expectation value of any odd number of $q_0$'s vanishes, the perturbative expansion of the effective mode function is,

$$u(t) = u_0(t) - g^2\omega\int_0^t dt' \sin[\omega(t-t')] \left[ 3\langle q_0^2(t') \rangle u_0(t') + 2\langle q_1(t') \rangle u_0(t') \right]$$

$$-2\omega\int_0^t dt' \sin[\omega(t-t')] \left\{ \left\{ q_0(t'), q_0(t') \right\} \right\} u_0(t') + O(g^4) \ . \ (51)$$

Expression \( (51) \) is the full one loop result. We can recognize the Hartree contribution by acting the kinetic operator,

$$\left[ \left( \frac{d}{dt} \right)^2 + \omega^2 \right] u(t) = -g^2\omega^2 \left[ 3\langle q_0^2(t) \rangle + 2\langle q_1(t) \rangle \right] u_0(t)$$

$$+2g^2\omega^3\int_0^t dt' \sin[\omega(t-t')] \left\{ \left\{ q_0(t), q_0(t') \right\} \right\} u_0(t') + O(g^4) \ . \ (52)$$

The one loop Hartree contribution comes from the term $\langle q_0^2(t) \rangle$ on the first line of \( (52) \),

$$u_{\text{Hartree}} = u_0(t) - 3g^2\omega\int_0^t dt' \sin[\omega(t-t')] \langle q_0^2(t') \rangle u_0(t') + O(g^4) \ . \ (53)$$

Of course this is the contribution from the 4-point vertex. The other terms on the right hand side of expression \( (52) \) also have simple interpretations. The factor of $\langle q_1(t) \rangle$ gives the contribution due to an order $g$ shift in the background field, and the integral on the last line represents the nonlocal contribution from two 3-point vertices.

It is worth working out the three one loop contributions to $u(t)$ so that they can be compared in detail,

$$-3g^2\omega\int_0^t dt' \sin[\omega(t-t')] \langle q_0^2(t') \rangle u_0(t') = \frac{g^2\hbar}{m\omega} \left\{ \frac{3}{8} [1 + i2\omega t] u_0(t) + \frac{3}{8} u_0^*(t) \right\} \ . \ (54)$$

$$-2g^2\omega\int_0^t dt' \sin[\omega(t-t')] \langle q_1(t') \rangle u_0(t')$$

$$= \frac{g^2\hbar}{m\omega} \left\{ \frac{1}{6} u_0(2t) + \frac{1}{4} [1 + i2\omega t] u_0(t) - \frac{1}{2} u_0(0) + \frac{1}{12} u_0^*(t) \right\} \ . \ (55)$$

$$2g^2\omega^2\int_0^t dt' \sin[\omega(t-t')] \int_0^{t'} dt'' \sin[\omega(t'-t'')] \left\{ \left\{ q_0(t'), q_0(t'') \right\} \right\} u_0(t'')$$

$$= \frac{g^2\hbar}{m\omega} \left\{ \frac{1}{6} u_0(2t) - \left[ \frac{5}{36} - \frac{1}{6} i\omega t \right] u_0(t) - \frac{1}{12} u_0^*(t) + \frac{1}{18} u_0^*(2t) \right\} \ . \ (56)$$
Recall that the Hartree contribution is (54). The only two things which seem to distinguish it from the vacuum shift (55) and the nonlocal contribution (56) are the absence of the nonoscillatory term $u_0(0)$ and the absence of higher harmonics — $u_0(2t)$ and $u_0^*(2t)$. The form of the dominant late time behavior is $g^2\hbar/m\omega \times i\omega tu_0(t)$, and all three contributions possess it, with coefficients $-\frac{3}{8}$, $+\frac{1}{4}$, and $+\frac{1}{6}$, respectively.

From the preceding discussion we see that there is nothing particularly distinctive about the one loop Hartree contribution (54) to the effective mode function $u(t)$. It does predict the form of the dominant late time behavior — $ig^2\hbar/m\omega \times i\omega tu_0(t)$ — but not the numerical coefficient of this term. That is typical of what has been found in recent computations of quantum corrections to the effective mode function from inflationary scalars and gravitons [2, 3, 4, 7, 8, 9, 12]. In some cases — such as the photon wave function in scalar QED [4] — the Hartree result gives the correct numerical coefficient of the dominant late time behavior. In other cases — such as scalar corrections to the scalar wave function [2] and graviton corrections to the fermion wave function [7, 8, 9] — it predicts the form of the dominant late time behavior but not the correct numerical coefficient. And there are some cases — such as scalar corrections to the fermion wave function in Yukawa theory [3] — in which Hartree contribution vanishes even though there are very significant late time corrections.

What the one loop Hartree approximation to the effective mode function always gives is the contribution from the 4-point vertex. (That is why it happens to vanish for Yukawa.) Our reasons for considering it for graviton corrections to other gravitons are not that it dominates in any particular regime but rather:

- It is vastly easier to compute than the nonlocal contribution from two 3-point vertices; and
- Whatever it gives is additively present in the full result.

It therefore sets a sort of minimum level for what one expects for the dominant late time behavior of the full result. We hope to have the full result to compare in about a year’s time. In the meanwhile, it seems reasonable to explore the minimum late time effect which the Hartree approximation predicts.
3.2 The effective field equation

The linearized, quantum corrected effective field equation for gravitons is,
\[ \mathcal{D}^{\mu\nu\rho\sigma} h_{\rho\sigma}(x) - \int d^4x' [i^{\mu\nu}] (x; x') h_{\rho\sigma}(x') = 0, \tag{57} \]
where \(-i^{\mu\nu}[\Sigma^\rho\sigma](x; x')\) is the graviton self-energy. To obtain the approximate version of this we employ the perturbation theory scheme. Expanding the total Lagrangian \(\mathcal{L}_{\text{inv}} + \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{ghost}}\) in powers of \(\kappa\) to the order we shall require, our equation of motion follows from the functional derivative of the action with respect to \(h_{\mu\nu}(x)\),
\[ \frac{\delta S[h]}{\delta h_{\mu\nu}(x)} = \mathcal{D}^{\mu\nu\rho\sigma} h_{\rho\sigma} + \kappa h^2 + \kappa^2 h^3 + ... = 0, \tag{58} \]
where the last two terms represent collectively all possible terms in the expansion that contain two and three factors of \(h_{\mu\nu}\) respectively. In applying the Hartree approximation \((34)-(36)\) to the equation above we consider the single field in the first term as the external one, and we can neglect all terms in the \(O(h^2)\) group since the expectation value of one field is zero. The interesting effects come form the \(O(h^3)\) group. Here all terms will be of the form \(hhh, h\partial h\) or \(h\partial h\partial h\), which can be seen from the structure of \(\mathcal{L}_{\text{inv}}\) in \([5]\).

Let us give an example of the replacement used in the Hartree approximation applied to a generic term from the second of these cases. The replacement is,
\[ h_{\alpha\beta} h_{\mu\nu} \partial_\lambda h_{\rho\sigma} \rightarrow E_{\rho\sigma,\lambda}(x) \lim_{x' \rightarrow x} i^{[\alpha\beta}[\Delta_{\mu\nu}](x; x') \]
\[ + E_{\mu\nu}(x) \lim_{x' \rightarrow x} \partial_\lambda' i^{[\alpha\beta}[\Delta_{\rho\sigma}](x; x') + E_{\alpha\beta}(x) \lim_{x' \rightarrow x} \partial_\lambda' i^{[\mu\nu}[\Delta_{\rho\sigma}](x; x'). \tag{59} \]

After applying similar substitutions on all terms in the \(O(h^3)\) group, the next step would be to substitute and contract our expression for the graviton propagator \((13)\) and then apply the coincidence limit using equations \((28)-(31)\). More simplifications arise when we impose the conditions of transversality and tracelessness on physical gravitons, namely, \(E_{\mu\nu,\mu} = 0\) and \(E_{\mu\nu} = 0\). We also consider only purely spatial gravitons \(E_{00} = 0 = E_{0i}\).

Once all this has been done we can extract our effective field equation perturbatively. To do this we can similarly expand the graviton field in powers of \(\kappa^2\),
\[ E_{\alpha\beta}(x) = \sum_{n=0}^{\infty} \kappa^{2n} E_{\alpha\beta}^{(n)}(x). \tag{60} \]
Substituting the expression for the kinetic operator \((8)\) in \((58)\) and collecting powers of \(\kappa^2\) give us the zeroth order equation,

\[
\mathcal{D}_A E_{ij}^{(0)}(x) = 0 .
\]  

(61)

Similarly the order \(\kappa^2\) equation is,

\[
\frac{1}{2} \mathcal{D}_A E_{ij}^{(1)}(x) + a^{D-2} \mathcal{D}_\eta E_{ij}^{(0)}(x) = 0 .
\]  

(62)

The operator \(\mathcal{D}_\eta\) has the form,

\[
\mathcal{D}_\eta = (c_1 + \alpha_1 \ln a) \partial^2 + (c_2 + \alpha_2 \ln a) (D-2) Ha \partial_\eta \\
- (c_3 + \alpha_3 \ln a) \partial^\rho \partial_\rho + c_4 H^2 a^2 ,
\]  

(63)

where \(c_i\) and \(\alpha_i\) are constants whose expressions we omit here in favor of writing below only those terms that will contribute the most in the late-time limit. We will give solutions to these equations in the next section.

## 4 The One Loop Mode Function

This section comprises our main result. Here we present and solve the one-loop order graviton mode function equation in the late-time regime. To solve this equation we will consider a spatial plane-wave expansion for the graviton field in terms of its mode function \(u(\eta, k)\) and the same transverse, traceless and purely spatial polarization tensor \(\epsilon_{\alpha\beta}\) as in flat space,

\[
E_{\alpha\beta}(x) = \epsilon_{\alpha\beta} u(\eta, k) e^{i \vec{k} \cdot \vec{x}} .
\]  

(64)

The mode functions \(u(\eta, k)\) have a similar perturbative expansion,

\[
u(\eta, k) = \sum_{n=0}^{\infty} \kappa^{2n} u^{(n)}(\eta, k) .
\]  

(65)

Substituting this expansion in \((64)\) and expanding the operator \(\mathcal{D}_A\) according to its definition \((9)\), the zeroth order equation \((61)\) becomes,

\[
[\partial^2_\eta + (D-2) Ha \partial_\eta + k^2] u^{(0)}(\eta, k) = 0 .
\]  

(66)
The solution to this mode equation is well known in terms of Hankel functions $H_{\nu}^{(1)}(z)$,

$$u^{(0)}(\eta, k) = \sqrt{\frac{\pi}{4H}} a^{-\frac{D-4}{2}} H_{\frac{D-1}{2}}^{(1)} \left( \frac{k}{Ha} \right).$$  \hspace{1cm} (67)

We may as well specialize $u^{(0)}(\eta, k)$ to $D = 4$, and its late time behavior is of crucial importance for us,

$$D = 4 \implies u^{(0)}(\eta, k) = \frac{H}{\sqrt{2k^3}} \left( 1 - \frac{ik}{Ha} \right) \exp \left[ \frac{ik}{Ha} \right],$$  \hspace{1cm} (68)

$$= \frac{H}{\sqrt{2k^3}} \left[ 1 + \frac{1}{2} \left( \frac{k}{Ha} \right)^2 + \frac{i}{3} \left( \frac{k}{Ha} \right)^3 + \ldots \right].$$  \hspace{1cm} (69)

The one-loop order mode equation follows similarly. Collecting terms of order $\kappa^2$ we can express the one loop corrections in terms of a differential operator $\mathcal{D}_{\eta}$ acting on $u^{(0)}(\eta, k)$,

$$\left[ \partial_{\eta}^2 + (D-2)Ha\partial_{\eta} + k^2 \right] u^{(1)}(\eta, k) - 2\mathcal{D}_{\eta} u^{(0)}(\eta, k) = 0.$$  \hspace{1cm} (70)

Because we are interested in the late-time limit of this equation we need consider only the most relevant terms in $\mathcal{D}_{\eta} u^{(0)}$, which are those that exhibit the largest growth with time during this period. From (63), the leading contribution comes from the $c_4 H^2 a^2$ term. However, the fact that this term survives for $k = 0$ means that it must be removed by the same counterterm which completely absorbs the one loop 1-point function in the same gauge [24]. Hence the $c_4 H^2 a^2$ term does not contribute at all after renormalization.

The next leading terms are those proportional to $\ln a$. Keeping only the late-time relevant terms, the operator $\mathcal{D}_{\eta}$ is,

$$\mathcal{D}_{\eta} = \frac{(D^3 - 12D^2 + 31D - 4)}{8(D-3)} \times 2K \ln a \times [\partial_{\eta}^2 + (D-2)Ha\partial_{\eta}].$$  \hspace{1cm} (71)

Here we have neglected terms proportional to $(D - 4)$ and we remind the reader that the constant $K$ was defined in expression [22]. Hence the late time form of our first order mode equation (70) becomes

$$\left[ \partial_{\eta}^2 + (D-2)Ha\partial_{\eta} + k^2 \right] u^{(1)}(\eta, k)$$

$$= \frac{(D^3 - 12D^2 + 31D - 4)}{2(D-3)} K \ln a \times [\partial_{\eta}^2 + (D-2)Ha\partial_{\eta}] u^{(0)}(\eta, k)$$  \hspace{1cm} (72)

$$= -\frac{(D^3 - 12D^2 + 31D - 4)}{2(D-3)} K \ln a \times k^2 u^{(0)}(\eta, k).$$  \hspace{1cm} (73)
The last equality follows upon substitution of the zeroth order mode equation (66).

At this point we note that there are no divergent terms when \( D \to 4 \). As a consequence we can specialize equation (73) to \( D = 4 \) to obtain,

\[
\left[ \partial^2_{\eta} + 2 Ha \partial_{\eta} + k^2 \right] u^{(1)}(\eta, k) = \frac{H^2}{2\pi^2} \times \ln a \times k^2 u^{(0)}(\eta, k) .
\] (74)

To obtain the leading late time behavior of \( u^{(1)}(\eta, k) \) recall from (69) that \( u^{(0)}(\eta, k) \) approaches the constant \( H/\sqrt{2k^3} \) at late times. Hence the right hand side of (74) grows like \( \ln(a) \) at late times. Now consider acting the differential operator on the left hand side on \( \ln(a)/a^2 \),

\[
\left[ \partial^2_{\eta} + 2 Ha \partial_{\eta} + k^2 \right] \left( \frac{\ln a}{a^2} \right) = -2H^2 \ln a - \frac{H^2}{2} + k^2 \ln a.
\] (75)

The last two terms can be neglected when compared to the leading term \( \ln a \). Hence up to first order, the leading late-time limit contribution of the graviton mode function can be written as

\[
u(\eta, k) = u^{(0)}(\eta, k) + \kappa^2 u^{(1)}(\eta, k) = \left( 1 - \frac{4k^2}{\pi H^2} \times \frac{GH^2 \ln a}{a^2} \right) u^{(0)}(\eta, k) .
\] (76)

This result gives us a rough idea about the enhancement inflationary gravitons acquire in the late-time regime from other gravitons. We expect that the same time dependence will be present in the full result, the only discrepancy being a different numerical factor. Only the fully renormalized and dimensionally regulated calculation will tell us how good the approximation really is.

5 Epilogue

We have used perturbative quantum gravity on de Sitter background to calculate the effects of inflationary gravitons on dynamical gravitons at one loop order in the late-time regime. We decomposed the graviton field using a plane wave expansion and we employed the Hartree approximation [15, 16, 17, 7, 8, 12] to obtain the first order graviton mode equation (74). In equation (76) we have found that the time dependence of the graviton mode functions is modified by a factor of \( GH^2 \ln(a)/a^2 \) which decays exponentially with time.
Our result (76) might be thought to demonstrate the irrelevance of quantum loops corrections to gravitons, but that is not so. It does show that there are no significant loop corrections to the tensor power spectrum, if we define $\Delta_2^h(k)$ using the norm-squared of the graviton mode function [25], because the late time limit of the norm-squared of the mode function is identical to its tree order result,

$$\Delta_2^h(k) = \frac{k^3}{2\pi^2} \times 64\pi G \times \left| u(\eta, k) \right|^2_{\eta \to 0} = \frac{16}{\pi} GH^2. \quad (77)$$

However, let us instead compare the curvature of the zeroth order term with the curvature induced by the quantum correction. For transverse-traceless graviton fields $h_{\mu\nu}(\eta, \vec{x})$ the linearized Weyl tensor is,

$$C_{\rho\sigma\mu\nu}^{\text{lin}} = -\frac{\kappa}{2a^2} \left( h_{\rho\mu,\sigma\nu} - h_{\mu\sigma,\nu\rho} + h_{\sigma\nu,\rho\mu} - h_{\nu\rho,\mu\sigma} \right). \quad (78)$$

Now specialize to a spatial plane wave $h_{ij}(\eta, \vec{x}) = \epsilon_{ij} u(\eta, k) e^{i\vec{k} \cdot \vec{x}}$ with purely spatial polarization, and examine the “electric” components,

$$C_{0i0j}^{\text{lin}} = -\frac{\kappa}{2a^2} \left( \partial_\eta^2 h_{ij} \right). \quad (79)$$

Next use relations (69) and (76) to compare the late time limits of second (conformal) time derivatives of the 0th and 1st order mode functions,

$$\partial_\eta^2 u^{(0)}(\eta, k) \to \frac{Hk^2}{\sqrt{2k^3}} \times 1, \quad (80)$$

$$\partial_\eta^2 u^{(1)}(\eta, k) \to \frac{Hk^2}{\sqrt{2k^3}} \times -\frac{4}{\pi} GH^2 \left[ 2 \ln(a) - 3 \right]. \quad (81)$$

By substituting (80),(81) into the electric components (79) we see two things:

- That the magnitude of the one loop corrections to the electric components of the linearized curvature grows (without bound) relative to the tree order result; and
- That the one loop correction tends to cancel the tree order result.

So it seems fair to conclude that quantum corrections make geometrically significant changes to gravitons. This might be important in trying to understand how two loop effects — which include the gravity sourced by these
one loop perturbations — might induce a secular change in the expansion rate \(26\).

It is interesting to compare what we have found about how inflationary gravitons affect other gravitons (in the one loop Hartree approximation) with how they affect other particles. Recall that we found a fractional correction to the mode function of the form \(-GH^2 \ln(a)/a^2\). The effects of gravitons on massless fermions produce a fractional secular growth in the field strength of the form \(+GH^2 \ln(a)\) [7, 8]. By contrast, the fractional change (in the Hartree approximation) on the photon wave function is of the form \(-GH^2 \ln(a)/a\) [12]. Just like what we found for gravitons, the one loop correction to photon mode functions goes to zero, but the effect on the electric components of the relevant field strength grow in magnitude [12]. In each of these three cases the particles being followed have spin, which seems to be why they continue to interact with inflationary gravitons even when their kinetic energies have red-shifted to zero [9].

To conclude this work we would like to comment on the current situation concerning the de Sitter (non-)invariance of the graviton propagator. It is worthwhile to note that the \(-GH^2 \ln(a)/a^2\) enhancement we have found for dynamical gravitons arises from the fact that, as can be seen from eqn. (74), the first order correction to the mode functions is sourced by a factor of \(\ln a\) in the late time limit. This can be traced back to the de Sitter breaking logarithmic term in the \(A\)-type scalar propagator (21). We will summarize the long controversy [27, 28] about the existence of such symmetry breaking terms in the graviton propagator.

Mathematical physicists have for decades believed that the graviton propagator must be de Sitter invariant because they could use analytic continuation techniques to find explicit, de Sitter invariant solutions for it when they add de Sitter invariant gauge fixing terms to the action [29]. Researchers who approach the problem from the perspective of cosmology have been equally convinced that there must be de Sitter breaking because free dynamical gravitons obey the same equation as massless, minimally coupled scalars [30], which possesses no normalizable, de Sitter invariant states [21]. Indeed, the \(A\)-type propagator is precisely that of a massless, minimally cou-

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1 It might be argued that the prefactor of \(1/a^2\) in expression (79) makes \(\ln(a)\) corrections irrelevant as a source of corrections to the de Sitter background. However, one must recall that (79) represents the effect from a single graviton. The actual source comes from adding up the contribution from all super-horizon gravitons, and this sum compensates the factor of \(1/a^2\), to leave the \(\ln(a)\).
pled scalar in the homogeneous and isotropic state required by cosmology [22], and its presence in any valid graviton propagator is required by the scale invariance of the tensor power spectrum [28]. Although our particular graviton propagator was derived in a de Sitter breaking gauge [14], one can show that its de Sitter breaking is physical by adding the compensating gauge transformation [31].

The two views have been converging recently because it has been demonstrated that there is an obstacle to adding invariant gauge fixing terms on any manifold which possesses a linearization instability such as de Sitter [32]. Ignoring the problem in scalar quantum electrodynamics leads to unphysical, on-shell singularities for one loop scalar self-mass-squared [20] and would produce similar problems in quantum gravity. It has also been shown that the analytic continuation techniques employed by mathematical physicists automatically subtract power law infrared divergences to produce formal solutions to the propagator equation which are not true propagators in the sense of being the expectation value, in some normalized state, of the time-ordered product of two field operators [33].

It is still valid to employ de Sitter invariant gauge conditions which are “exact”; that is, the condition is enforced as a strong operator equation. When this was done, without using invalid analytic continuations, the result was a de Sitter breaking propagator [34], whose spin two part agrees with the one we used [35]. The same result persists for the entire 1-parameter family of exact, de Sitter invariant gauges [36].

Mathematical physicists have conceded the point about gauge fixing, but some of them still insist on the validity of analytic continuation because it does produce solutions to the propagator equation [37]. A recent paper by Morrison [38] has identified precisely the two deviations which would convert the cosmological derivation of a de Sitter breaking propagator [34, 36] into the derivation of a de Sitter invariant result. One of these deviations corresponds to regarding the scalar propagator for any $M^2$ as a de Sitter invariant and meromorphic function of $M^2$, even for the tachyonic case of $M^2 < 0$ [39]. The other deviation corresponds to adding a constant to the scalar equation for the spin two structure function, when no such constant can be added for any other slow roll parameter $\epsilon(t) \equiv -\dot{H}/H^2$ [39]. Thus we feel confident in adopting the de Sitter breaking propagator.

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