The Hasimoto Transformation for a Finite Length Vortex Filament and its Application

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Abstract

We consider two nonlinear equations, the Localized Induction Equation and the cubic nonlinear Schrödinger Equation, and prove that the solvability of certain initial-boundary value problems for each equation is equivalent through the generalized Hasimoto transformation. As an application, we prove the orbital stability of plane wave solutions of the nonlinear Schrödinger equation based on stability estimates obtained for the Localized Induction Equation by the author in a paper in preparation. As far as the author knows, this is the first time that the analysis of the Localized Induction Equation, along with the Hasimoto transformation, provided new insight for the nonlinear Schrödinger equation.

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1 Introduction and Problem Setting

A vortex filament is a space curve on which the vorticity of the fluid is concentrated. Vortex filaments are used to model very thin vortex structures such as vortices that trail off airplane wings or propellers. The model equation we consider in this paper is the Localized Induction Equation (LIE) given by

$$\mathbf{x}_t = \mathbf{x}_s \times \mathbf{x}_{ss}$$

where $\mathbf{x}(s,t) = (x_1(s,t), x_2(s,t), x_3(s,t))$ is the position vector of the vortex filament parametrized by its arc length $s$ at time $t$, $\times$ is the exterior product in the three-dimensional Euclidean space, and subscripts $s$ and $t$ are differentiations with the respective variables.

The LIE, which is derived by applying the localized induction approximation to the Biot–Savart integral, was first derived by Da Rios [13] in 1906 and was re-derived twice independently by Murakami et al. [30] in 1937 and by Arms and Hama [4] in 1965. Since
then, many researches have been done on the LIE and many results have been obtained. Nishiyama and Tani \[32, 33\] proved the unique solvability of initial and initial-boundary value problems in Sobolev spaces. The author \[1\] and the author and Iguchi \[3\] proved the unique solvability of initial-boundary value problems in Sobolev spaces with different boundary conditions. In the above papers, the equation for the tangent vector, \( \mathbf{v} := \mathbf{x}_s \), given by

\[
\mathbf{v}_t = \mathbf{v} \times \mathbf{v}_{ss} \tag{1.1}
\]

is introduced in the analysis and plays an important role. Equation (1.1) is sometimes called the Vortex Filament Equation (VFE).

Koiso \[29\] considered a geometrically generalized setting in which he rigorously proved the equivalence of the solvability of the initial value problem for the VFE and the cubic nonlinear Schrödinger equation. This equivalence was first shown by Hasimoto \[23\] in which he studied the formation of solitons on a vortex filament. He defined a transformation of variable known as the Hasimoto transformation to transform the VFE into a nonlinear Schrödinger equation. The Hasimoto transformation is a change of variable given by

\[
q(s, t) = \kappa(s, t) \exp \left( i \int_0^s \tau(r, t) \, dr \right),
\]

where \( i \) is the imaginary unit, \( \kappa \) is the curvature, and \( \tau \) is the torsion of the filament. Defined as such, it is well known that \( q \) satisfies the nonlinear Schrödinger equation given by

\[
iq_t = q_{ss} + \frac{1}{2} |q|^2 q. \tag{1.2}
\]

The original transformation proposed by Hasimoto uses the torsion of the filament in its definition, which means that the transformation is undefined at points where the curvature of the filament is zero. Koiso \[29\] constructed a transformation, sometimes referred to as the generalized Hasimoto transformation, and gave a mathematically rigorous proof of the equivalence of the VFE and (1.2). More precisely, Koiso proved that the solvability of initial value problems for the VFE and (1.2) are equivalent. More recently, Banica and Vega \[5, 6, 7\] and Gutiérrez, Rivas, and Vega \[22\] utilized the generalized Hasimoto transformation to construct and analyze a family of self-similar solutions of the LIE which forms a corner in finite time. In Chang, Shatah, and Uhlenbeck \[12\] and Nahmod, Shatah, Vega, and Zeng \[28\], they considered the Schrödinger maps and employed a Hasimoto type transformation to prove the correspondence between the solution of the Schrödinger maps and the solution of a nonlinear Schrödinger type equation. The Schrödinger maps are a generalization of the Heisenberg model for a ferromagnetic spin system given by

\[
\mathbf{m}_t = \mathbf{m} \times \Delta \mathbf{m},
\]

where \( \Delta \) is the Laplacian in \( \mathbb{R}^n \) and the unknown variable \( \mathbf{m} \) takes values in \( S^2 \). The Heisenberg model can be seen as a multi-dimensional version of the VFE.

All of the results mentioned above which utilizes the Hasimoto transformation consider either an infinitely long filament defined on the whole line or a closed filament defined on
the torus. As far as the author knows, the rigorous justification and application of the Hasimoto transformation for problems describing the motion of filaments with end-points have not been done. In this paper, we justify the Hasimoto transformation and prove the equivalence of the solvability of the following initial-boundary value problems.

\begin{align}
\begin{cases}
    v_t = v \times v_{ss}, & s \in I_L, \ t > 0, \\
    v(s,0) = v_0, & s \in I_L, \ t > 0, \\
    v(0,t) = e_1, \ v(L,t) = b, & t > 0,
\end{cases}
\end{align}

(1.3)

\begin{align}
\begin{cases}
    iq_t = q_{ss} + \frac{1}{2} |q|^2 q, & s \in I_L, \ t > 0, \\
    q(s,0) = q_0(s), & s \in I_L, \\
    q_s(0,t) = q_s(L,t) = 0, & t > 0.
\end{cases}
\end{align}

(1.4)

Problem (1.3) is an initial-boundary value problem for the VFE which describes the motion of a vortex filament on a slanted plane, considered in a previous paper by the author [1]. Here, $L > 0$ is the length of the initial filament, $I_L \subset \mathbb{R}$ is the interval $(0, L)$, $e_1 = (1, 0, 0)$, and $b \in \mathbb{R}^3$ is an arbitrary constant vector with unit length. The boundary datum at $s = 0$ was chosen as $e_1$ without loss of generality, because the VFE is invariant under rotation. Problem (1.4) is an initial-boundary value problem for the focusing cubic nonlinear Schrödinger equation, where $q = q(s, t)$ is a complex-valued function.

As an application of this equivalence, we will prove the orbital stability of the plane wave solution $q_R$ of (1.4) given in the form

$$q_R(t) = -\frac{1}{R} \exp \left\{-\frac{it}{2R^2}\right\}$$

for $R > L/\pi$ in the Sobolev space $H^2(I_L)$. This will be done by considering problem (1.3) with appropriate initial and boundary data, utilizing energy estimates for the solution $v$ of (1.3) obtained by the author in [1, 2], and transferring the estimates for $v$ into estimates for solutions of (1.4).

As far as the author knows, the results of this paper is the first time the generalized Hasimoto transformation is utilized to give new insight on the nonlinear Schrödinger equation from known facts about the VFE. All of the preceding works utilizing the generalized Hasimoto transformation did so to analyze the solution of the VFE utilizing known facts about the nonlinear Schrödinger equation.

The initial value problem for equation (1.2) on the torus, explicitly given by

\begin{align}
\begin{cases}
    iu_t = u_{ss} + \frac{1}{2} |u|^2 u, & s \in T, \ t > 0, \\
    u(s,0) = u_0(s), & s \in T,
\end{cases}
\end{align}

(1.5)

where $T = \mathbb{R}/[-L, L]$, is closely related to problem (1.4) since the solvability of problem (1.4) can be reduced to the solvability of problem (1.3) by reflection and periodic extension.
It is known by Zakharov and Shabat [38] that equation (1.2) is completely integrable, and the solution to problem (1.5) possesses infinitely many conserved quantities. This in turn implies that solutions in the Sobolev space $H^m$ for $m \in \mathbb{N}$ is bounded in $H^m$ for all time. The solvability of problem (1.5) in Lebesgue or Sobolev spaces are known, for example, by Bourgain [9]. Hence, it is natural to ask if particular types of solutions are stable in Sobolev spaces. Namely, the stability of plane wave solutions of equation (1.2) has been studied by many researchers.

Zhidkov [39] gives a detailed analysis of the plane wave solutions for the initial value problem on the whole space $\mathbb{R}$. The stability of plane wave solutions and periodic wave solutions for problem (1.5) was investigated by Rowlands [35], Gallay and Hărăguş [18, 19], Faou, Gauckler, and Lubich [14], and Wilson [37].

In particular, Gallay and Hărăguş [18, 19] considered problem (1.5) and proved the orbital stability of periodic wave solutions in $H^1_{per}$, i.e., the orbital stability in $H^1$ with the perturbation restricted to periodic perturbations with the same period as the periodic wave solution. By definition, periodic wave solutions include plane wave solutions and hence, the stability results in [18, 19] are valid for plane wave solutions as well.

Faou, Gauckler, and Lubich [14] considered the initial value problem on the torus with general dimension $d \geq 1$, and proved the long-time orbital stability of plane wave solutions in $H^r$. By long-time they mean stability up to time of order $O(\varepsilon N)$ where $\varepsilon > 0$ is the size of the perturbation in $H^r$ and $N \in \mathbb{N}$. The index $r > 0$ must be chosen sufficiently large, depending on $N$ and the $L^2$ norm of the perturbation.

Wilson [37] considered the stability problem of plane wave solutions for the nonlinear Schrödinger equation with general power nonlinearities, and obtained similar results as [14].

Other results on the initial–boundary value problems for the nonlinear Schrödinger equation have been obtained by Holmer [24], Fokas and Its [15], Fokas, Its, and Sung [17], Lenells and Fokas [25, 26], Bona, Sun, and Zhang [8], and Fokas, Himonas, and Mantzavinos [16]. These results prove the well-posedness of initial-boundary value problems under various boundary conditions as well as obtain representation formulas for boundary values which represent unknown boundary values of a solution by known boundary values, but do not address the problem of stability of specific solutions.

In summary, we see that up until this paper, the stability of plane wave solutions in higher order Sobolev spaces is only partially known. Specifically, the regularity for which the stability is proved in [14, 37] is not given explicitly, and the time-span for which the stability holds is not global. On the other hand, the results in this paper give the time-global orbital stability of plane wave solutions in Sobolev space $H^2$. This is possible greatly due to the fact that the method of the proof given in this paper is vastly different from the preceding works.

The methods utilized to prove the stability of solutions in [39, 35, 18, 19, 14, 37] can be broadly categorized into two types. Variational methods and methods utilizing the Hamiltonian structure of the equation. The variational approach was utilized in Cazenave and Lions [11] to prove orbital stability of standing waves for the nonlinear Schrödinger equation. The approach utilizing the Hamiltonian structure of the equation was introduced by Grillakis, Shatah, and Strauss [20, 21] for a broad range of equations having a Hamiltonian structure. These methods are widely adopted to approach stability.
problems for a wide variety of dispersive equations. In contrast, the method employed in this paper is tailor-made specifically for problem (1.3) and (1.4). Hence, we are able to obtain more information for our specific problem, but the method is not as widely applicable to other problems compared to traditional methods.

The contents of the rest of the paper are as follows. In Section 2, we introduce basic notations and define compatibility conditions related to problems (1.3) and (1.4). Then, we give a brief explanation of the Hasimoto transformation and state our main theorems. In Section 3, we prove our main theorems. We first prove that the solvability of problems (1.3) and (1.4) is equivalent through the generalized Hasimoto transformation. We further prove that plane wave solutions of problem (1.4) correspond to a particular type of solution, which we call arc-shaped solutions, of problem (1.3).

Then, we prove stability estimates for arc-shaped solutions of problem (1.3) and also prove that these stability estimates can be transferred to stability estimates for plane wave solutions of problem (1.4) through the generalized Hasimoto transformation. The stability estimates for arc-shaped solutions are essentially derived from standard energy estimates for the perturbation, which is much more simple than traditional methods.

### 2 Function Spaces, Notations, and Main Theorem

We introduce some function spaces that will be used throughout this paper, and notations associated with the spaces. For a non-negative integer $m$ and $1 \leq p \leq \infty$, $W^{m,p}(I_L)$ is the Sobolev space containing all real-valued functions that have derivatives in the sense of distribution up to order $m$ belonging to $L^p(I_L)$. We set $H^m(I_L) := W^{m,2}(I_L)$ as the Sobolev space equipped with the usual inner product, and set $H^1_0(I_L)$ as the closure, with respect to the $H^1$-norm, of the set of smooth functions with compact support. The norm in $H^m(I_L)$ is denoted by $\| \cdot \|_m$ and we simply write $\| \cdot \|$ for $\| \cdot \|_0$. Otherwise, for a Banach space $X$, the norm in $X$ is written as $\| \cdot \|_X$. The inner product in $L^2(I_L)$ is denoted by $(\cdot, \cdot)$.

For $0 < T \leq \infty$ and a Banach space $X$, $C^m([0,T];X)$ ($C^m([0,\infty);X)$ when $T = \infty$), denotes the space of functions that are $m$ times continuously differentiable in $t$ with respect to the norm of $X$. The space $L^\infty(0,\infty;X)$ denotes the space of functions that are essentially bounded in $t$ with respect to the norm of $X$.

For any function space described above, we say that a vector valued function belongs to the function space if each of its components does, and the same for complex-valued functions if both the real and imaginary parts do.

Finally, vectors $e_j \in \mathbb{R}^3$ for $j = 1, 2, 3$ denote the standard basis of $\mathbb{R}^3$. In other words, $e_1 = i(1,0,0)$, $e_2 = i(0,1,0)$, and $e_3 = i(0,0,1)$. Additionally, we denote the unit sphere in $\mathbb{R}^3$ by $S^2$.

We next introduce some definitions in order to state the main theorems of this paper. First we define the compatibility conditions for both (1.3) and (1.4).

### 2.1 Compatibility Conditions for (1.3) and (1.4)

First we define the compatibility conditions needed in this paper for problem (1.3).
Definition 2.1. For $m = 0$ or $1$, we say that $v_0 \in H^{2m+1}(I_L)$ and $b \in S^2$ satisfy the $m$-th order compatibility condition for (1.3) if
\[ v_0(0) = e_1, \quad v_0(L) = b, \]
when $m = 0$, and
\[ v_0(0) \times v_0s(0) = v_0(L) \times v_0s(L) = 0 \]
when $m = 1$. We also say that $v_0$ and $b$ satisfy the compatibility conditions for (1.3) up to order 1 if $v_0$ and $b$ satisfy both the 0-th order and the 1-st order compatibility condition for (1.3).

Next we define the corresponding compatibility condition for (1.4).

Definition 2.2. For $q_0 \in H^2(I_L)$, we say that $q_0$ satisfies the 0-th order compatibility condition for (1.4) if
\[ q_0s(0) = q_0s(L) = 0 \]
are satisfied.

2.2 The Hasimoto Transformation and the Main Theorems

To state our main theorem, we give a brief explanation of the Hasimoto transformation. The Hasimoto transformation is a map that relates the solution of equation (1.1) to the solution of equation (1.2) proposed by Hasimoto [23]. The original transformation proposed by Hasimoto isn’t always well-defined because the transformation is defined using the torsion of the filament, which is not defined at points where the curvature of the filament is zero. Later, Koiso [29] proved that the solvability of the initial value problem on the torus for (1.1) and (1.2) is equivalent. Koiso did so by constructing a modified transformation, to which we refer to as the generalized Hasimoto transformation, which maps solutions of equation (1.1) to solutions of equation (1.2). This transformation is invertible, and hence, the solvability is equivalent.

One of the aims of this paper is to prove that the generalized Hasimoto transformation given by Koiso [29] can be further modified to prove the equivalence of the solvability of problem (1.3) and problem (1.4). More precisely, we prove the following.

Theorem 2.3. For $q_0 \in H^2(I_L)$ satisfying the 0-th order compatibility condition for (1.4), there exists $v_0 \in H^3(I_L)$ and $b \in S^2$ satisfying $|v_0| \equiv 1$ and the compatibility conditions for (1.3) up to order 1 such that the following holds. The solution $v \in C\left(\left[0, \infty\right); H^3(I_L)\right) \cap C^1\left(\left[0, \infty\right); H^1(I_L)\right)$ of problem (1.3) with initial datum $v_0$ and boundary datum $b$ corresponds to the solution $q \in C\left(\left[0, \infty\right); H^2(I_L)\right) \cap C^1\left(\left[0, \infty\right); L^2(I_L)\right)$ of problem (1.4) with initial datum $q_0$ through the generalized Hasimoto transformation.

Theorem 2.4. For $v_0 \in H^3(I_L)$ and $b \in S^2$ satisfying $|v_0| \equiv 1$ and the compatibility conditions for (1.3) up to order 1, there exists $q_0 \in H^2(I_L)$ satisfying the 0-th order compatibility condition for (1.4) such that the following holds. The solution
Theorem 2.5. The plane wave solution \( q \in C([0, \infty); H^2(I_L)) \cap C^1([0, \infty); L^2(I_L)) \) of problem (1.4) with initial datum \( q_0 \) corresponds to the solution \( v \in C([0, \infty); H^3(I_L)) \cap C^1([0, \infty); H^1(I_L)) \) of problem (1.3) with initial datum \( v_0 \) and boundary datum \( b \) through the inverse generalized Hasimoto transformation.

Theorem 2.3 and 2.4 together states that the solvability of problem (1.3) and problem (1.4) are equivalent in suitable Sobolev spaces. The two theorems also imply that the compatibility conditions for (1.3) and (1.4) correspond to each other through the generalized Hasimoto transformation. Recall from the introduction that the solvability of problem (1.3) and problem (1.4) are already known. Hence, the solvability of either problem in itself is not new, but the fact that the solvability of the two problems are equivalent is new.

As an application of the above two theorems, we prove the following theorem.

**Theorem 2.5.** The plane wave solution \( q_R \) of problem (1.4) given by

\[
q_R(t) = -\frac{1}{R} \exp \left\{ -\frac{it}{2R^2} \right\}
\]

with \( R > \frac{L}{\pi} \) is orbitally stable in \( H^2(I_L) \). More specifically, for any \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that for any \( \phi_0 \in H^2(I_L) \) satisfying the 0-th order compatibility condition for (1.4) and \( \| \phi_0 \|_2 \leq \delta \), the solution \( q \in C([0, \infty); H^2(I_L)) \cap C^1([0, \infty); L^2(I_L)) \) of problem (1.4) with initial datum \( q_0 = q_R(0) + \phi_0 \) satisfies

\[
\sup_{t \geq 0} \inf_{\theta \in \mathbb{R}} \| \exp(i\theta)q(t) - q_R(t) \|_2 < \varepsilon.
\]

From here on, we will refer to the maps \( q_0 \mapsto v_0 \) and \( v \mapsto q \) implied in Theorem 2.3 as the generalized Hasimoto transformation, and the maps \( v_0 \mapsto q_0 \) and \( q \mapsto v \) implied in Theorem 2.4 as the inverse generalized Hasimoto transformation.

## 3 Proof of Main Theorems

In this section, we prove Theorem 2.3, 2.4, and 2.5. The transformation utilized in the proof of theorem 2.3 and 2.4 is mostly due to Koiso [29] with some modifications to accommodate the presence of boundary conditions.

Although it is implied in Theorem 2.3 and 2.4, we explicitly state the solvability of problems (1.3) and (1.4) in the form that we will take for granted throughout this paper.

**Theorem 3.1.** Let \( v_0 \in H^3(I_L) \) and \( b \in S^2 \) satisfy \( |v_0| \equiv 1 \) and the compatibility condition for (1.3) up to order 1. Then, there exists a unique solution \( v \in C([0, \infty); H^3(I_L)) \cap C^1([0, \infty); H^1(I_L)) \) of problem (1.3). Furthermore, \( |v(s, t)| = 1 \) for all \( s \in I_L \) and \( t > 0 \). Additionally, there exists \( c_* > 0 \) such that \( v \) satisfies

\[
(3.1) \quad \sup_{t > 0} \| v(t) \|_3 \leq c_*,
\]

where \( c_* > 0 \) depends on \( L \) and \( \| v_0 \|_3 \) and is non-decreasing with respect to \( \| v_0 \|_3 \).
Theorem 3.2. Let $q_0 \in H^2(I_L)$ satisfy the 0-th order compatibility condition for (1.4). Then, there exists a unique solution $q \in C([0, \infty); H^2(I_L)) \cap C^1([0, \infty); L^2(I_L))$ of problem (1.4).

Theorem 3.1 is proved by the author in [1] and Theorem 3.2 is essentially due to Bourgain [9, 10] since problem (1.4) can be reduced to the initial value problem on the torus by reflection and periodic extension.

3.1 Proof of Theorem 2.3

Let $q_0 \in H^2(I_L)$ satisfy the 0-th order compatibility condition for problem (1.4) and define $q_0^0$ and $q_0^2$ by

$$q_0(s) = q_0^0(s) + iq_0^2(s).$$

Furthermore, we define $v_0, e^0, w^0$ as the solution of the following system of ordinary differential equations.

$$\begin{cases} v_{0s} = q_0^0 e^0 + q_0^2 w^0, & s \in I_L, \\ e_s^0 = -q_1^0 v_0, & s \in I_L, \\ w_s^0 = -q_2^0 v_0, & s \in I_L, \\ (v_0, e^0, w^0)(0) = (e_1, -e_2, e_3). \end{cases}$$

By direct calculations, we see that the triplet $\{v_0, e^0, w^0\}$ is an orthonormal basis of $\mathbb{R}^3$ for all $s \in I_L$. Set $b := v_0(L)$, and we see that by definition, $v_0$ and $b$ satisfy the 0-th order compatibility condition of problem (1.3). We further calculate and see that

$$v_{0ss} = (-q_1^2)(q_2^2)v_0 + q_{1s}^0 e^0 + q_{2s}^0 w^0,$$

and hence,

$$v_0 \times v_{0ss} = q_{1s}^0 v_0 \times e^0 + q_{2s}^0 v_0 \times w^0 = -q_{1s}^0 w^0 + q_{2s}^0 e^0.$$

Here, we have used the fact that $\{v_0, e^0, w^0\}$ is an orthonormal basis of $\mathbb{R}^3$ and the orientation given at $s = 0$ yields $v_0 \times e^0 = -w^0$ and $v_0 \times w^0 = e^0$. Since $q_0$ satisfies the 0-th order compatibility condition for (1.4), we see that

$$v_0(0) \times v_{0ss}(0) = v_0(L) \times v_{0ss}(L) = 0$$

and $v_0$ and $b$ satisfies the 1-st order compatibility condition for (1.3).

Let, $v \in C([0, \infty); H^3(I_L)) \cap C^1([0, \infty); H^1(I_L))$ be the solution of problem (1.3) with initial datum $v_0$ and boundary datum $b$ just obtained. From Theorem 3.1, we know that $|v(s, t)| = 1$ for all $s \in I_L$ and $t > 0$. We now define $e$ as the solution of

$$\begin{cases} e_s = -(v_s \cdot e)v, & s \in I_L, \\ e(0) = e_2, \end{cases}$$

from which we see that $e$ satisfies the 1-st order compatibility condition for (1.3).
and set $w = v \times e$. Here, $\cdot$ is the standard inner product of $\mathbb{R}^3$. Note that $e$ and $w$ depend on $t > 0$ as a parameter through $v$. Again, we see that \( \{v, e, w\} \) is an orthonormal basis of $\mathbb{R}^3$ for all $s \in I_L$ and $t > 0$. Since $v \cdot v_s \equiv 0$, we have the decomposition

\begin{equation}
(3.2) \quad v_s = \psi_1 e + \psi_2 w
\end{equation}

for some $\psi_1(s, t)$ and $\psi_2(s, t)$. Then we also see that

\begin{align}
(3.3) & \quad e_s = -\psi_1 v, \\
(3.4) & \quad w_s = -\psi_2 v.
\end{align}

Taking the inner product of $e$ and $w$ with equation (3.2) we have

$$
\psi_1 = e \cdot v_s,
$$

$$
\psi_2 = w \cdot v_s,
$$

from which we deduce that $\psi_1, \psi_2 \in C([0, \infty); H^2(I_L))$ since $v \in C([0, \infty); H^3(I_L))$. From equation (3.2), we see that

$$
|\psi_1|^2 + |\psi_2|^2 = |v_s|^2,
$$

which implies

$$
\|\psi_1(t)\|^2 + \|\psi_2(t)\|^2 = \|v_s(t)\|^2.
$$

Furthermore, we have

$$
\psi_{1s} e + \psi_{2s} w = v_{ss} + (\psi_1)^2 v + (\psi_2)^2 v,
$$

$$
\psi_{1ss} e + \psi_{2ss} w = v_{sss} + 3\psi_1\psi_{1s} v + 3\psi_2\psi_{2s} v + \psi_1((\psi_1)^2 + (\psi_2)^2) e
$$

$$
+ \psi_2((\psi_1)^2 + (\psi_2)^2) w,
$$

and after taking the inner product of the above equations with $e$ and $w$, we derive

$$
\|\psi_{1s}(t)\| \leq \|v_{ss}(t)\|,
$$

$$
\|\psi_{2s}(t)\| \leq \|v_{ss}(t)\|,
$$

$$
\|\psi_{1ss}(t)\| \leq \|v_{sss}(t)\| + C\|v_s(t)\|^2\|v_s(t)\|,
$$

$$
\|\psi_{2ss}(t)\| \leq \|v_{sss}(t)\| + C\|v_s(t)\|^2\|v_s(t)\|,
$$

where $C > 0$ is determined from the embedding $H^1(I_L) \hookrightarrow L^\infty(I_L)$. Hence, we have

\begin{equation}
(3.5) \quad \|\psi_1(t)\|_2 + \|\psi_2(t)\|_2 \leq C\left(\|v(t)\|_3 + \|v(t)\|_3^2\right),
\end{equation}

for all $t > 0$, where $C > 0$ is independent of $t$ and $v$. 
Since $\mathbf{v} \cdot \mathbf{v}_t \equiv 0$, a decomposition of the form $\mathbf{v}_t = p_1 \mathbf{e} + p_2 \mathbf{w}$ holds. From the equation $(\mathbf{v}_t)_s = (\mathbf{v}_s)_t$ and $(\mathbf{e}_t)_s = (\mathbf{e}_s)_t$, we deduce that $p_1 = -\psi_{2s}$ and $p_2 = \psi_{1s}$. Furthermore,

\begin{equation}
\mathbf{v}_t = -\psi_{2s} \mathbf{e} + \psi_{1s} \mathbf{w},
\end{equation}

\begin{equation}
\mathbf{e}_t = \psi_{2s} \mathbf{v} + \left\{ -\frac{1}{2} ((\psi_1)^2 + (\psi_2)^2) + \frac{1}{2} ((\psi_1)^2 + (\psi_2)^2) \right\} \mathbf{w},
\end{equation}

\begin{equation}
\mathbf{w}_t = -\psi_{1s} \mathbf{v} - \left\{ -\frac{1}{2} ((\psi_1)^2 + (\psi_2)^2) + \frac{1}{2} ((\psi_1)^2 + (\psi_2)^2) \right\} \mathbf{e},
\end{equation}

holds. Here, $|_{s=0}$ denotes the trace at $s = 0$. The above three equations along with the fact that $\mathbf{v} \in C^1([0, \infty); H^2(I_L))$, $\psi_1 = \mathbf{e} \cdot \mathbf{v}_s$, and $\psi_2 = \mathbf{w} \cdot \mathbf{v}_s$ implies that $\psi_1, \psi_2 \in C^1([0, \infty); L^2(I_L))$. This in turn allows us to calculate as follows.

\begin{align*}
\psi_{1t} &= -\psi_{2ss} + \left\{ -\frac{1}{2} ((\psi_1)^2 + (\psi_2)^2) + \frac{1}{2} ((\psi_1)^2 + (\psi_2)^2) \right\} \psi_2, \\
\psi_{2t} &= \psi_{1ss} - \left\{ -\frac{1}{2} ((\psi_1)^2 + (\psi_2)^2) + \frac{1}{2} ((\psi_1)^2 + (\psi_2)^2) \right\} \psi_1.
\end{align*}

Setting $\psi = \psi_1 + i\psi_2$, we see that

\[ i\psi_t - \psi_{ss} = \frac{1}{2} |\psi|^2 \psi - \frac{1}{2} |\psi(0, t)|^2 \psi, \]

and after a gauge transform given by

\begin{equation}
q(s, t) = \psi(s, t) \exp \left\{ \frac{i}{2} \int_0^t |\psi(0, \tau)|^2 d\tau \right\},
\end{equation}

we see that $q$ satisfies

\[ iq_t = q_{ss} + \frac{1}{2} |q|^2 q. \]

As $t$ tends to zero in equation (3.7), we see from the uniqueness of the solution of ordinary differential equations that

\[ q(s, 0) = q_0(s). \]

Finally, we see that by taking the derivative of the boundary condition in problem (1.3) with respect to $t$, we see that $\mathbf{v}_t(0, t) = \mathbf{v}_t(L, t) = 0$, and from equation (3.6), we see that

\[ q_s(0, t) = q_s(L, t) = 0. \]

Also note that $q \in C([0, \infty); H^2(I_L)) \cap C^1([0, \infty); L^2(I_L))$ from the definition of $q$ along with the fact that $\psi_1, \psi_2 \in C([0, \infty); H^2(I_L)) \cap C^1([0, \infty); L^2(I_L))$. Additionally, from equations (3.5) and (3.7) we have

\begin{equation}
||q(t)||_2 \leq C(||v(t)||_\infty + ||v(t)||_3^3),
\end{equation}

for all $t \geq 0$, where $C > 0$ is independent of $t$ and $\mathbf{v}$. Estimate (3.8) will be utilized later. This finishes the proof of Theorem 2.3.

\[ \square \]
3.2 Proof of Theorem [2.4]

Let \( v_0 \in H^3(I_L) \) satisfying \(|v_0| \equiv 1\) and \( b \in S^2 \) be initial and boundary datum for problem \((1.3)\) satisfying the compatibility conditions up to order 1. Define \( e^0 \) as the solution of

\[
\begin{align*}
    e^0_s &= -(v_{0s} \cdot e^0)v_0, \quad s \in I_L, \\
    e^0(0) &= -e_2,
\end{align*}
\]

and set \( w^0 = e^0 \times v_0 \). Similarly to Section 3.1, the triplet \( \{v_0, e^0, w^0\} \) is an orthonormal basis of \( \mathbb{R}^3 \) and the decomposition

\[
v_{0s} = q_1^0e^0 + q_2^0w^0
\]

holds and direct calculation yields

\[
v_0 \times v_{0ss} = q_2^0e^0 - q_{1s}w^0.
\]

Hence, setting \( q_0 = q_1^0 + iq_2^0 \) we see that \( q_0 \) satisfies \( q_{0s}(0, t) = q_{0s}(L, t) = 0 \) since \( v_0 \) and \( b \) satisfy the 1-st order compatibility condition for \((1.3)\). In other words, \( q_0 \) satisfies the 0-th order compatibility condition for \((1.4)\).

Let \( q \in C([0, \infty); H^2(I_L)) \cap C^1([0, \infty); L^2(I_L)) \) be the solution of problem \((1.4)\) with initial datum \( q_0 \) just obtained and set \( q(s, t) = q_1(s, t) + iq_2(s, t) \). We define \( v, e, \) and \( w \) as the solution of the following system of ordinary differential equations with respect to \( s \).

\[
\begin{align*}
    v_s &= q_1e + q_2w, \quad s \in I_L, \\
    e_s &= -q_1v, \quad s \in I_L, \\
    w_s &= -q_2v, \quad s \in I_L, \\
    (v, e, w)(0) &= (e_1, -e_2, e_3).
\end{align*}
\]

(3.9)

Again, \( \{v, e, w\} \) depends on \( t > 0 \) as a parameter and \( \{v, e, w\} \) is an orthonormal basis of \( \mathbb{R}^3 \) for all \( s \in I_L \) and \( t > 0 \). From (3.9), we deduce that \( v \in C([0, \infty); H^3(I_L)) \cap C^1([0, \infty); H^1(I_L)) \) and

\[
\begin{align*}
    \|v_s(t)\|^2 &= \|q_1(t)\|^2 + \|q_2(t)\|^2, \\
    \|v_{ss}(t)\| &\leq C(\|q_{1s}(t)\| + \|q_{2s}(t)\| + \|q_1(t)\|^2 + \|q_2(t)\|^2), \\
    \|v_{sss}(t)\| &\leq C(\|q_{1ss}(t)\| + \|q_{2ss}(t)\| + \|q_1(t)\|^3 + \|q_2(t)\|^3),
\end{align*}
\]

where \( C > 0 \) is independent of \( t \) and \( q \). Hence we have

\[
\begin{align*}
    |v(s, t)| &= 1 \text{ for all } s \in I_L \text{ and } t \geq 0, \\
    \|v_s(t)\|_2 &\leq C(\|q(t)\|_2 + \|q(t)\|^2).
\end{align*}
\]
Note that the definition of $v_0$, $e_0$, and $w_0$ coincides with the limit as $t$ tends to zero in equations (3.9).

Next, we see that

$$v_t = q_{2s}e - q_{1s}w,$$

$$e_t = -q_{2s}v + \left(\frac{1}{2}|q|^2 - \frac{1}{2}|q(0,t)|^2\right)w,$$

$$w_t = q_{1s}v - \left(\frac{1}{2}|q|^2 - \frac{1}{2}|q(0,t)|^2\right)e.$$  

From equations (3.9), we see that

$$v \times v_{ss} = q_{2s}e - q_{1s}w,$$

and hence, along with (3.10) we have

$$v_t = v \times v_{ss}.$$  

The first equation in (3.9) with $t = 0$ together with the uniqueness of the solution of ordinary differential equations show that

$$v(s,0) = v_0(s).$$

Finally, taking the trace at $s = 0$ and $s = L$ in equation (3.10) we see that $v_t(0,t) = v_t(L,t) = 0$ which in turn yields

$$v(0,t) = v_0(0) = e_1,$$

$$v(L,t) = v_0(L) = b,$$

and hence, $v$ satisfies the boundary condition of problem (1.3). This shows that $v$ is the desired solution of problem (1.3). This finishes the proof of Theorem 2.4. \[\square\]

### 3.3 Proof of Theorem 2.5

We divide the proof of Theorem 2.5 into five steps.

We first establish that perturbations to the initial datum of problem (1.4) correspond to perturbations to the initial datum of problem (1.3). Furthermore, we derive estimates that show in what norms the perturbations translate.

Secondly, we show that $q_R(0)$ corresponds to a particular type of initial datum of problem (1.3). Solutions of problem (1.3) with this type of initial datum will be referred to as an arc-shaped solution of problem (1.3).

Then, we prove stability estimates for arc-shaped solutions of problem (1.3), which is related to the stability problem for plane wave solutions of problem (1.4). The stability estimates for arc-shaped solutions are already utilized by the author in [2], but we reiterate it here for completeness.

Finally, we combine the results of the previous steps to prove that the plane wave solution $q_R$ is orbitally stable. More precisely, we show that the stability estimates for problem (1.3) yield stability estimates for problem (1.4) through the generalized Hasimoto transformation.
3.3.1 The Generalized Hasimoto Transformation of Perturbations

In this section, we show that perturbations to the initial datum of problem (1.4) translate to perturbations to the initial datum of problem (1.3) through the generalized Hasimoto transformation.

Let \( q_0 \in H^2(I_L) \) and \( \varphi_0 \in H^2(I_L) \) be such that \( q_0 \) and \( \varphi_0 \) satisfy the 0-th order compatibility condition for problem (1.4). Note that \( q_0 + \varphi_0 \) also satisfy the same compatibility condition. Setting \( \tilde{q}_0 = q_0 + \varphi_0 \) and

\[
q_0 = q_1^0 + iq_2^0 \quad \text{and} \quad \tilde{q}_0 = \tilde{q}_1^0 + i\tilde{q}_2^0,
\]

define \( \{v_0, e^0, w^0\} \) and \( \{\tilde{v}_0, \tilde{e}^0, \tilde{w}^0\} \) as the solution of

\[
\begin{align*}
    v_{0s} &= q_1^0 e^0 + q_2^0 w^0, & s &\in I_L, \\
    e_s^0 &= -q_0^0 v_0, & s &\in I_L, \\
    w_s^0 &= -q_2^0 v_0, & s &\in I_L, \\
    (v_0, e^0, w^0) &= (e_1, -e_2, e_3),
\end{align*}
\]

\[
\begin{align*}
    \tilde{v}_{0s} &= q_1^0 \tilde{e}^0 + q_2^0 \tilde{w}^0, & s &\in I_L, \\
    \tilde{e}_s^0 &= -\tilde{q}_0^0 \tilde{v}_0, & s &\in I_L, \\
    \tilde{w}_s^0 &= -\tilde{q}_2^0 \tilde{v}_0, & s &\in I_L, \\
    (\tilde{v}_0, \tilde{e}^0, \tilde{w}^0) &= (e_1, -e_2, e_3),
\end{align*}
\]

respectively. Setting \( V := v_0 - \tilde{v}_0, \ E := e^0 - \tilde{e}^0, \) and \( W := w^0 - \tilde{w}^0, \) we see that

\[
\begin{align*}
    V_s &= (q_1^0 - \tilde{q}_1^0) e^0 + \tilde{q}_1^0 E + (q_2^0 - \tilde{q}_2^0) w^0 + \tilde{q}_2^0 W, & s &\in I_L, \\
    E_s &= (q_1^0 - \tilde{q}_1^0) v_0 - \tilde{q}_1^0 V, & s &\in I_L, \\
    W_s &= (q_2^0 - \tilde{q}_2^0) v_0 - \tilde{q}_2^0 V, & s &\in I_L, \\
    (V, E, W)(0) &= (0, 0, 0),
\end{align*}
\]

(3.11)

holds. Standard energy estimates yield

\[
\frac{1}{2} \frac{d}{ds} \left\{ |V|^2 + |E|^2 + |W|^2 \right\} \leq C \left( 1 + \|q_1^0\|^2_{L^\infty(I_L)} + \|\tilde{q}_1^0\|^2_{L^\infty(I_L)} \right) \left( |V|^2 + |E|^2 + |W|^2 \right)
\]

\[
+ \left( \|q_1^0 - \tilde{q}_1^0\|^2_{L^\infty(I_L)} + \|\tilde{q}_1^0\|^2_{L^\infty(I_L)} \right),
\]

and from Gronwall’s inequality, we have

\[
|V|^2 + |E|^2 + |W|^2 \leq C e^{CL} \left\{ \|q_1^0 - \tilde{q}_1^0\|^2_{L^\infty(I_L)} + \|\tilde{q}_2^0 - q_2^0\|^2_{L^\infty(I_L)} \right\},
\]

(3.12)
where $C > 0$ depends on $\|q_0\|_{L^\infty(I_L)}$ and $\|\tilde{q}_0\|_{L^\infty(I_L)}$. Hence, estimate (3.12) implies that if $\tilde{q}_0 - q_0$ is controlled in $L^\infty(I_L)$, then we have a control on $V, E,$ and $W$ in $L^\infty(I_L)$, and as a consequence, in $L^2(I_L)$. Equations (3.11) together with inequality (3.12) show that

$$
\|V_s\|^2 + \|E_s\|^2 + \|W_s\|^2 \leq C(\|\tilde{q}_1^0 - q_1^0\|^2_{L^\infty(I_L)} + \|\tilde{q}_2^0 - q_2^0\|^2_{L^\infty(I_L)}),
$$

$$
\|V_s\|_{L^\infty(I_L)} + \|E_s\|_{L^\infty(I_L)} + \|W_s\|_{L^\infty(I_L)}
\leq C(\|\tilde{q}_1^0 - q_1^0\|_{L^\infty(I_L)} + \|\tilde{q}_2^0 - q_2^0\|_{L^\infty(I_L)}),
$$

hold, where $C > 0$ depends on $\|q_0\|_{L^\infty(I_L)}$ and $\|\tilde{q}_0\|_{L^\infty(I_L)}$.

Taking the derivative with respect to $s$ of equations (3.11), we have

$$
V_{ss} = (q_1^0 - \tilde{q}_1^0)_s e^0 - q_1^0(q_1^0 - \tilde{q}_1^0)v_0 + \tilde{q}_1^0(q_1^0 - \tilde{q}_1^0) - q_1^0(q_1^0 - \tilde{q}_1^0) + (\tilde{q}_2^0 - q_2^0)_s w^0 + \tilde{q}_2^0(q_2^0 - \tilde{q}_2^0)v_0 + \tilde{q}_2^0(q_2^0 - \tilde{q}_2^0) + (\tilde{q}_2^0 - q_2^0)_s w^0
\leq -((q_1^0 - \tilde{q}_1^0)^2 + (q_2^0 - \tilde{q}_2^0)^2)v_0 + (q_1^0 - \tilde{q}_1^0)_s e^0 + (q_2^0 - \tilde{q}_2^0)_s w^0
\leq q_1^0 e + \tilde{q}_2^0 w - ((\tilde{q}_1^0)^2 + (\tilde{q}_2^0)^2) + q_2^0_0 + \tilde{q}_2^0_0 + \tilde{q}_2^0_0 + \tilde{q}_2^0_0 + \tilde{q}_2^0_0,
$$

from which we have the estimate

$$
\|V_{ss}\| \leq C(\|q_1^0 - \tilde{q}_1^0\|_{L^\infty(I_L)} + \|q_2^0 - \tilde{q}_2^0\|_{L^\infty(I_L)} + \|\tilde{q}_1^0 - q_1^0\|_{L^\infty(I_L)} + \|\tilde{q}_2^0 - q_2^0\|_{L^\infty(I_L)}).
$$

In a similar fashion, we can derive the estimate

$$
\|V_{sss}\| \leq C(\|q_1^0 - \tilde{q}_1^0\|_{L^\infty(I_L)} + \|q_2^0 - \tilde{q}_2^0\|_{L^\infty(I_L)} + \|\tilde{q}_1^0 - q_1^0\|^2_{L^\infty(I_L)} + \|\tilde{q}_2^0 - q_2^0\|^2_{L^\infty(I_L)}).
$$

In summary, we have the estimate

$$
(3.13) \quad \|\tilde{v}_0 - v_0\|_3 \leq C(\|\tilde{q}_0 - q_0\|_2 + \|\tilde{q}_0 - q_0\|_2^2),
$$

where $C > 0$ depends on $\|\tilde{q}_0\|_2 + \|q_0\|_2$. We also see that $C > 0$ is non-decreasing with respect to $\|\tilde{q}_0\|_2 + \|q_0\|_2$. Inequality (3.13) shows that $H^2$-perturbations of $q_0$ correspond to $H^3$-perturbations of $v_0$.

We summarize the conclusion of this section in the following proposition.

**Proposition 3.3.** For $q_0 \in H^2(I_L)$ and $\varphi_0 \in H^2(I_L)$ which satisfy the $0$-th order compatibility condition for (1.4), let $v_0$ and $\tilde{v}_0$ be initial data constructed by the generalized Hasimoto transformation from $q_0$ and $q_0 + \varphi_0$, respectively. Then, $v_0, \tilde{v}_0 \in H^3(I_L)$ and satisfy

$$
\|\tilde{v}_0 - v_0\|_3 \leq C(\|\tilde{q}_0 - q_0\|_2 + \|\tilde{q}_0 - q_0\|_2^2),
$$

where $C > 0$ depends on $\|q_0\|_2 + \|\varphi_0\|_2$, and is non-decreasing with respect to $\|q_0\|_2 + \|\varphi_0\|_2$. 

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3.3.2 The Inverse Generalized Hasimoto Transformation of the Plane Wave Solution

We investigate the image of the plane wave solution by the inverse generalized Hasimoto transformation. Recall that for $R > 0$,

$$q_R(t) = -\frac{1}{R} \exp \left\{ -\frac{it}{2R^2} \right\},$$

and $q_R(0) = -\frac{1}{R}$. Following the arguments in Section 3.1, the generalized Hasimoto transformation applied to the initial datum $-\frac{1}{R}$ corresponds to defining $v_0$ and $e^0$ as the solution to

$$\begin{align*}
\left\{ \begin{array}{l}
v_{0s} = -\frac{1}{R}e^0, \quad s \in I_L, \\
e^0_s = \frac{1}{R}v_0, \quad s \in I_L,
\end{array} \right.

(3.14)

\end{align*}$$

and setting $w^0 \equiv e_3$. System (3.14) can be solved explicitly to obtain

$$v_0(s) = t(\cos(s/R), -\sin(s/R), 0).$$

The above $v_0$ corresponds to an arc-shaped filament, which is an explicit solution of problem (1.3). Namely,

$$v_0 \times v_{0ss} \equiv 0$$

and hence, $v_0$ is a stationary solution of problem (1.3) with $b = t(\cos(L/R), -\sin(L/R), 0)$. We set

$$v^R(s, t) = v^R(s) := t(\cos(s/R), -\sin(s/R), 0)$$

for $s \in I_L$ and $t > 0$ and refer to $v^R$ as the arc-shaped solution of problem (1.3). This shows that $q_R$ corresponds to $v^R$ through the inverse generalized Hasimoto transformation.

3.3.3 Stability of Arc-Shaped Solutions of Problem (1.3)

We derive stability estimates for arc-shaped solutions of problem (1.3). These estimates are already derived by the author in [2], but we reiterate the statements and proofs for completeness.

For $R > 0$, $v^R$ is the solution of problem (1.3) with

$$v_0(s) = v^R(s) = t(\cos(s/R), -\sin(s/R), 0),$$

$$b = t(\cos(L/R), -\sin(L/R), 0).$$

Recall that $v^R$ is a stationary solution. We consider perturbations of the arc-shaped solution $v^R$. Let, $\varphi_0 \in H^3(I_L)$ be the initial perturbation satisfying the following.
Assumption 3.4. For the initial perturbation \( \varphi_0 \), we assume the following.

(A1) \(|v_0(s) + \varphi_0(s)| = 1\) for all \( s \in I_L \).

(A2) \( v_0 + \varphi_0 \) and \( b \) satisfy the compatibility conditions for problem (1.3) up to order 1.

Note that because \( v_0 \) and \( b \) satisfy the 0-th order compatibility condition, assumption (A2) implies \( \varphi_0|_{s=0} = \varphi_0|_{s=L} = 0 \). Let \( v \in C([0, \infty); H^3(I_L)) \cap C^1([0, \infty); H^1(I_L)) \) be the solution of problem (1.3) with initial datum \( v_0 + \varphi_0 \) and boundary datum \( b \). For convenience, we refer to this solution \( v \) as the perturbed arc-shaped solution. Set \( \varphi = v - v^R \). Then, \( \varphi \) satisfies

\[
\begin{align*}
\varphi_t &= \varphi \times \varphi_{ss} + \varphi \times v_{ss}^R + v^R \times \varphi_{ss}, \quad s \in I_L, t > 0, \\
\varphi(s, 0) &= \varphi_0(s), \quad s \in I_L, \\
\varphi(0, t) &= \varphi(L, t) = 0.
\end{align*}
\]

(3.15)

Note that \( \varphi \in C([0, \infty); H^3(I_L)) \cap C^1([0, \infty); H^1(I_L)) \). We first prove the following lemma.

Lemma 3.5. For \( \varphi \in C([0, \infty); H^3(I_L)) \cap C^1([0, \infty); H^1(I_L)) \), set

\[
E(\varphi(t)) := \|\varphi_s(t)\|^2 - \frac{1}{R^2}\|\varphi(t)\|^2.
\]

Then, if \( \varphi \) satisfies (3.15), \( E(\varphi(t)) = E(\varphi_0) \) for all \( t > 0 \). In other words, \( E \) is a conserved quantity.

Proof. From the first equation in (3.15), we have

\[
\begin{align*}
\frac{d}{dt}\|\varphi\|^2 &= 2(\varphi, \varphi_t) = 2(\varphi, v^R \times \varphi_{ss}) = -2(\varphi, v_{ss}^R \times \varphi_s),
\end{align*}
\]

where integration by parts was used. Furthermore,

\[
\begin{align*}
\frac{d}{dt}\|\varphi_s\|^2 &= 2(\varphi_s, \varphi_{st}) = -2(\varphi_{ss}, \varphi_t) = -2(\varphi_{ss}, \varphi \times v_{ss}^R) \\
&= 2(\varphi_s, \varphi \times v_{sss}^R) \\
&= -\frac{2}{R^2}(\varphi, v_s^R \times \varphi_s),
\end{align*}
\]

where we substituted \( v_{sss}^R = -\frac{1}{R^2} v_s^R \). Combining the two equalities show that \( \frac{d}{dt}E(\varphi(t)) = 0 \), and this proves the lemma.

From Lemma 3.5, the following Corollary immediately follows.

Corollary 3.6. For \( \varphi \in C([0, \infty); H^3(I_L)) \cap C^1([0, \infty); H^1(I_L)) \) satisfying (3.15), if \( R > L/\pi \), there exists \( C > 0 \) such that

\[
\|\varphi(t)\|_1 \leq C\|\varphi_0\|_1
\]

holds for all \( t > 0 \), where \( C > 0 \) depends on \( L \) and \( R \).
Proof. Since $\varphi|_{s=0} = \varphi|_{s=L} = 0$, the Poincaré inequality
\[ \|\varphi(t)\| \leq \frac{L}{\pi} \|\varphi_s(t)\|, \]
is applicable and hence
\[ E(\varphi(t)) \geq \|\varphi_s(t)\|^2 - \frac{1}{R^2} \left( \frac{L}{\pi} \right)^2 \|\varphi_s(t)\|^2 = c_0 \|\varphi_s(t)\|^2 \]
holds for all $t > 0$, where $c_0 = 1 - \frac{L^2}{R^2 \pi^2} \in (0, 1)$. This with Lemma 3.5 yields
\[ \|\varphi_t(t)\| \leq C \|\varphi_0\| \leq C \|\varphi_0\|_1 \]
for all $t > 0$.

Next, we show higher-order estimates.

**Lemma 3.7.** For $\varphi \in C((0, \infty); H^3(I_L)) \cap C^1([0, \infty); H^1(I_L))$ satisfying (3.15), there exists $C > 0$ depending only on $L$ and $R$ such that if $R > L/\pi$, $\varphi$ satisfies
\[ \|\varphi_{ss}(t)\|_1 \leq C(\|\varphi_0\|_3 + \|\varphi_0\|^3_3) \]
for all $t > 0$.

**Proof.** We make use of conserved quantities for problem (1.3), which was also utilized in [1]. Properties of solutions of problem (1.3) that we make use of here are either well known, or proved in [1] and hence, we will use them without proof. The perturbed arc-shaped solution $\psi$ satisfies $|\psi(s, t)| = 1$ for all $s \in I_L$ and $t > 0$. Hence,
\[ 1 = |\psi(s, t)|^2 = |\psi^R(s) + \varphi(s, t)|^2 \]
for all $s \in I_L$ and $t > 0$. Since $|\psi^R(s)| = 1$, we have
\[ 2\psi^R(s) \cdot \varphi(s, t) = -|\varphi(s, t)|^2 \]
for all $s \in I_L$ and $t > 0$. We also have the following conserved quantities.
\[ \frac{d}{dt} \left\{ \|\psi_{ss}\|^2 - \frac{5}{4} \|\psi_s\|^2 \right\} = 0, \]
\[ \frac{d}{dt} \left\{ \|\psi_{sss}\|^2 - \frac{7}{2} \|\psi_s\| \|\psi_{ss}\|^2 - 14 \|\psi_s \cdot \psi_{ss}\|^2 + \frac{21}{8} \|\psi_s|^3 \|^2 \right\} = 0. \]
These quantities are well known, and the above equality can be verified by direct calculation.
First we set

$$E_1(v(t)) := \|v_{ss}(t)\|^2 - \frac{5}{4}||v_s(t)||^2.$$  

For $t \geq 0$ we substitute $v = v^R + \varphi$ and decompose as follows.

$$E_1(v(t)) = E_1(v^R) + E_1(\varphi(t)) + R_1(v^R, \varphi(t)),$$

where $R_1(v^R, \varphi(t))$ is given by

$$R_1(v^R, \varphi(t)) = 2(v^R_{ss}, \varphi_{ss}) - 5(|v^R_s|^2v^R_s, \varphi^R_s) - 5((v^R_s \cdot \varphi_s)v^R_s, \varphi_s)$$

$$- \frac{5}{2}(|v^R_s|^2 \varphi_s, \varphi_s) - 5(|\varphi_s|^2 v^R_s, \varphi_s).$$

On the other hand, since $E_1$ is conserved, we have

$$E_1(v(t)) = E_1(v_0) = E_1(v^R + \varphi_0)$$

$$= E_1(v^R) + E_1(\varphi_0) + R_1(v^R, \varphi_0),$$

and combining the two equalities, we have

$$E_1(\varphi(t)) = E_1(\varphi_0) - R_1(v^R, \varphi) + R_1(v^R, \varphi_0).$$

We estimate $R_1(v^R, \varphi)$ in detail and omit the details for $R_1(v^R, \varphi_0)$ since they are the same. The terms that are linear with respect to $\varphi$ can be estimated as follows. First we have

$$2(v^R_{ss}, \varphi_{ss}) = -\frac{2}{R^2}(v^R_{ss}, \varphi_{ss}) = \frac{2}{R^2}(v^R_{ss}, \varphi_s) - \frac{2}{R^2}[v^R \cdot \varphi_s]_{s=0}^L$$

$$= -\frac{2}{R^2}(v^R_{ss}, \varphi) - \frac{2}{R^2}[v^R \cdot \varphi_s]_{s=0}^L$$

$$= \frac{2}{R^4}(v^R, \varphi) - \frac{2}{R^2}[v^R \cdot \varphi_s]_{s=0}^L.$$

Integrating equation (3.16) with respect to $s$ shows that $(v^R, \varphi) = -\frac{1}{2}||\varphi||^2$. Furthermore, differentiating equation (3.16) with respect to $s$ and taking the trace $s = 0, L$ yields $v^R_s \cdot \varphi_{s=0,L} = 0$. Hence we have

$$2(v^R_{ss}, \varphi_{ss}) = -\frac{1}{R^4}||\varphi||^2.$$  

Similarly, we have

$$-5(|v^R_s|^2v^R_s, \varphi_s) = -\frac{5}{R^2}(v^R_s, \varphi_s) = \frac{5}{2R^4}||\varphi_s||^2,$$

where $|v^R_s| \equiv \frac{1}{R}$ was substituted. Taking into account the above two equalities, we have

$$|2(v^R_{ss}, \varphi_{ss}) - 5(|v^R_s|^2v^R_s, \varphi_s)| \leq C||\varphi||^2_1 \leq C||\varphi_0||^2_1.$$  

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where Corollary 3.6 was applied in the last inequality. For the other terms, we have

\[
| - 5((v_s^R \cdot \varphi_s)v_s^R, \varphi_s) - \frac{5}{2}(|v_s^R|^2 \varphi_s, \varphi_s) - 5(|\varphi_s|^2 v_s^R, \varphi_s)|
\leq C(\|\varphi_s\|^2 + \|\varphi_s\|^2\|\varphi_s\|_1)
\leq C(\|\varphi_0\|^2 + \|\varphi_0\|^3)
\]

where \(C > 0\) depends only on \(R\). Corollary 3.6 and the interpolation inequality yield

\[
E_1(\varphi(t)) \geq \|\varphi_{ss}\|^2 - C\|\varphi_{ss}\|\|\varphi_s\|^3 - C\|\varphi_s\|^2
\]

\[
\geq c_1\|\varphi_{ss}\|^2 - C\|\varphi_s\|^2 - C\|\varphi_s\|^6
\]

\[
\geq c_1\|\varphi_{ss}\|^2 - C\|\varphi_0\|^2 - C\|\varphi_0\|^6
\]

where \(c_1 > 0\) depends only on \(L\) and \(C > 0\) depends on \(L\) and \(R\). Combining all of the obtained estimates yields

(3.19) \[
\|\varphi_{ss}(t)\|^2 \leq C(\|\varphi_0\|_2^2 + \|\varphi_0\|^6)
\]

for all \(t > 0\). Here, \(C > 0\) depends on \(L\) and \(R\).

Similarly, if we set

\[
E_2(v(t)) = \|v_{sss}(t)\|^2 - \frac{7}{2}\|v_s(t)\|v_{ss}(t)\|^2 - 14\|v_s(t) \cdot v_{ss}(t)\|^2 + \frac{21}{8}\|v_s(t)|^2|^2,
\]

we have

\[
E_2(v^R + \varphi_0) = E_2(v^R) + E_2(\varphi_0) + R_2(v^R, \varphi_0),
\]

\[
E_2(v^R + \varphi(t)) = E_2(v^R) + E_2(\varphi(t)) + R_2(v^R, \varphi(t)),
\]

and the conservation of \(E_2\) implies that the above two are equal. Hence,

\[
E_2(\varphi(t)) = E_2(\varphi_0) + R_2(v^R, \varphi_0) - R_2(v^R, \varphi(t)).
\]

Here,

\[
R_2(v^R, \varphi(t)) = 2(v_{sss}^R, \varphi_{ss}) + 2(|v_s^R|^2 v_{ss}^R, \varphi_{ss}) + (|v_s^R|^2 v_s^R, \varphi_s)
\]

\[
\quad + 8(|v_s^R|^4 v_s^R, \varphi_s) + N(v^R, \varphi(t)),
\]

where \(N(v^R, \varphi(t))\) are terms that are nonlinear with respect to \(\varphi\). We estimate as follows. First we have

\[
2(v_{sss}^R, \varphi_{ss}) = -\frac{2}{R^2}(v_s^R, \varphi_{ss}) = -\frac{2}{R^2}[v_s^R \cdot \varphi_{ss}]_s=0 + \frac{2}{R^2}(v_{ss}^R, \varphi_{ss})
\]
Taking the exterior product of $v^R_s$ from the left with the first equation in problem (3.15), we have
\begin{equation}
(v^R_s \times \varphi_t) = v^R_s \times (\varphi \times \varphi_{ss}) + v^R_s \times (\varphi \times v^R_{ss}) + v^R_s \times (v^R \times \varphi_{ss}).
\end{equation}
Since $\varphi_t|_{s=L} = 0$, taking the trace at $s = 0, L$ in (3.20) yields
\begin{equation}
(v^R_s \cdot \varphi_{ss})|_{s=0,L} = 0.
\end{equation}

$|v^R| \equiv 1$ further implies that $v^R_s \cdot \varphi_{ss}|_{s=0,L} = 0$. Hence, we have
\begin{equation}
2(v^R_{ss}, \varphi_{ss}) = \frac{2}{R^2}(v^R_s, \varphi_s) = -\frac{1}{R^6} \| \varphi \|^2.
\end{equation}

Next we estimate
\begin{equation}
2(|v^R_s|^2 v^R_s, \varphi_{ss}) = \frac{2}{R^2}(v^R_s, \varphi_s) = -\frac{1}{R^6} \| \varphi \|^2;
\end{equation}
\begin{equation}
2(|v^R_{ss}|^2 v^R_s, \varphi_s) + 8(|v^R_s|^4 v^R_s, \varphi_s) = \frac{9}{R^2}(v^R_s, \varphi_s) = -\frac{9}{2R^5} \| \varphi_s \|^2;
\end{equation}
and finally,
\begin{equation}
|N(v^R, \varphi(t))| \leq C(\| \varphi \|_2^2 + \| \varphi \|_2^5).
\end{equation}

Additionally,
\begin{equation}
E_2(\varphi(t)) \geq \| \varphi_{sss}(t) \|^2 - C(\| \varphi(t) \|_2^2 + \| \varphi(t) \|_2^6)
\end{equation}
along with all the estimates previously obtained shows that
\begin{equation}
\| \varphi_{sss}(t) \|^2 \leq C(\| \varphi_0 \|_3^2 + \| \varphi_0 \|_3^5)
\end{equation}
holds for all $t > 0$, where $C > 0$ depends on $L$ and $R$. Estimates (3.19) and (3.21) combined proves Lemma 3.7.

In summary, we have the following.

**Proposition 3.8.** For $R > 1/L$, the arc-shaped solution $v^R$ of problem (1.3) is stable in the following sense. For $\varphi_0 \in H^3(I_L)$ satisfying Assumption 3.4, the perturbed arc-shaped solution $v \in C([0, \infty); H^3(I_L)) \cap C^1([0, \infty); H^1(I_L))$ satisfies
\begin{equation}
\| v(t) - v^R \|_3 \leq C(\| \varphi_0 \|_3 + \| \varphi_0 \|_3^3)
\end{equation}
for all $t > 0$. Here, $C > 0$ depends on $L$ and $R$. 

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3.3.4 Stability Estimates and the generalized Hasimoto transformation

In this section, we show that stability estimates for solutions of problem (1.3) transfer to stability estimates for solutions of problem (1.4) through the generalized Hasimoto transformation.

Let, \( v_1 \) and \( v_2 \) be solutions of problem (1.3) belonging to \( C([0, \infty); H^3(I_L)) \cap C^1([0, \infty); H^1(I_L)) \) with possibly different initial datum, but the same boundary datum. Additionally, suppose that there exists \( C_0 > 0 \) independent of \( v_1 \) and \( v_2 \) and \( M \geq 0 \) such that

\[
\| v_1(t) - v_2(t) \|_3 \leq C_0 M
\]

holds for all \( t > 0 \). Define \( e^1 \) and \( e^2 \) as the solution of

\[
\begin{align*}
e^1_s &= -(v_{1s} \cdot e^1)v_1, & s \in I_L, \\
|e^1|_{s=0} &= e_2,
\end{align*}
\]

and

\[
\begin{align*}
e^2_s &= -(v_{2s} \cdot e^2)v_2, & s \in I_L, \\
|e^2|_{s=0} &= e_2,
\end{align*}
\]

respectively, and define \( w^1 \) and \( w^2 \) by

\[
w^1 = v_1 \times e^1 \quad \text{and} \quad w^2 = v_2 \times e^2.
\]

Again, note that \( e^i \) and \( w^i \) depend on \( t \) through \( v_i \). Similarly to Section 3.3.1 estimating \( e^1 - e^2 \) and \( w^1 - w^2 \) using (3.23), (3.24), and (3.25) yields

\[
\| e^1(t) - e^2(t) \|_3 \leq CM,
\]

\[
\| w^1(t) - w^2(t) \|_3 \leq CM,
\]

for \( t > 0 \). Here, \( C > 0 \) depends on \( C_0 \) and \( \| v_{1s} \|_2 + \| v_{2s} \|_2 \) and is non-decreasing with respect to \( C_0 > 0 \) and \( \| v_{1s} \|_2 + \| v_{2s} \|_2 \).

Since \( |v_i| \equiv 1 \) \((i = 1, 2)\), we have the decomposition

\[
v_{1s} = \psi_1^1 e^1 + \psi_{12}^1 w^1,
\]

\[
v_{2s} = \psi_2^1 e^2 + \psi_{22}^1 w^2,
\]

for some \( \psi_{ij}(s, t) \) \((i, j = 1, 2)\). Furthermore, from equations (3.28) and (3.29), along with (3.23), (3.24), and (3.25), we have

\[
\| v_{1s}(t) \|_2 + \| v_{2s}(t) \|_2 \leq C \left( \sum_{i,j=1}^2 \| \psi_{ij}^1(t) \|_2 + \sum_{i,j=1}^2 \| \psi_{ij}^2(t) \|_2^3 \right),
\]
where $C > 0$ is independent of $\psi_j^i (i,j = 1,2)$. Substituting (3.28) and (3.29) into the first equation of (3.23) and (3.24) yields

$$ (\psi_1^1 - \psi_1^2) v_1 = -(e_1^1 - e_1^2)_s + \psi_1^2 (v_2 - v_1), $$

$$ (\psi_2^1 - \psi_2^2) v_1 = -(w_1^1 - w_1^2)_s + \psi_2^2 (v_2 - v_1), $$

and taking the inner product of the above two equations with $v_1$ yields

$$ (\psi_1^1 - \psi_2^1) = -(e_1^1 - e_1^2)_s \cdot v_1 + \psi_1^2 ((v_2 - v_1) \cdot v_1), $$

$$ (\psi_2^1 - \psi_2^2) = -(w_1^1 - w_1^2)_s \cdot v_1 + \psi_2^2 ((v_2 - v_1) \cdot v_1). $$

We see from direct estimates based on equations (3.31) and (3.32) that

$$ \|\psi_1^1(t) - \psi_1^2(t)\|_2 \leq C_1 M, $$

$$ \|\psi_2^1(t) - \psi_2^2(t)\|_2 \leq C_1 M $$

holds for all $t > 0$, where $C_1 > 0$ depends on $C_0$ and $\sum_{i,j=1}^2 \|\psi_i^j(t)\|_2$ and is non-decreasing with respect to $C_0 > 0$ and $\sum_{i,j=1}^2 \|\psi_i^j(t)\|_2$. Here, (3.26), (3.27), and (3.30) was also utilized.

Estimate (3.1) along with estimate (3.5) implies that there exists $C_* > 0$ such that $\sum_{i,j=1}^2 \|\psi_i^j(t)\|_2 \leq C_*$ holds for all $t > 0$. Here, $C_* = C_*(K)$ depends on $K = \|v_1(0)\|_3 + \|v_2(0)\|_3$ and is non-decreasing with respect to $K > 0$. We briefly remark that estimate (3.1) was utilized in [1] and is derived from the conserved quantities (3.17), (3.18), and

$$ \frac{d}{dt} \|v_s(t)\| = 0, $$

along with the fact that $|v(s,t)| = 1$ for all $s \in I_L$ and $t > 0$. Setting $\psi_1(s,t) := \psi_1^1(s,t) + i\psi_1^2(s,t)$ and $\psi_2(s,t) := \psi_2^1(s,t) + i\psi_2^2(s,t)$, we see that

$$ \|\psi_1(t) - \psi_2(t)\|_2 \leq C_2 M $$

holds for all $t > 0$, where $C_2 > 0$ depends on $C_0$ and $K$ and is non-decreasing with respect to $C_0 > 0$ and $K > 0$. Finally, setting

$$ q_1(s,t) := \psi_1(s,t) \exp \left\{ \frac{i}{2} \int_0^t |\psi_1(0, \tau)|^2 \, d\tau \right\}, $$

$$ q_2(s,t) := \psi_2(s,t) \exp \left\{ \frac{i}{2} \int_0^t |\psi_2(0, \tau)|^2 \, d\tau \right\}, $$

we have

$$ \|\psi_1(t) - \psi_2(t)\|_2 = \left\| \exp \left\{ \frac{i}{2} \int_0^t (|\psi_2(0, \tau)|^2 - |\psi_1(0, \tau)|^2) \, d\tau \right\} q_1(t) - q_2(t) \right\|_2 $$

$$ \geq \inf_{\theta \in \mathbb{R}} \| \exp \{i\theta\} q_1(t) - q_2(t) \|_2 $$

for all $t > 0$. In summary, we have
Proposition 3.9. Let $v_1, v_2 \in C([0, \infty); H^3(I_L)) \cap C^1([0, \infty); H^1(I_L))$ be solutions of problem (1.3) with initial datum $v_{0,1}, v_{0,2} \in H^3(I_L)$, respectively, and a common boundary datum $b \in S^2$. Let $q_1, q_2 \in C([0, \infty); H^3(I_L)) \cap C^1([0, \infty); L^2(I_L))$ be solutions of problem (1.4) corresponding to $v_1$ and $v_2$ respectively, through the generalized Hasimoto transformation. Furthermore, assume that there exists $C_0 > 0$ independent of $v_1$ and $v_2$ and $M \geq 0$ such that

$$\|v_1(t) - v_2(t)\|_3 \leq C_0 M$$

holds for all $t > 0$. Then, there exists $C_{**} > 0$ depending on $C_0$ and $\|v_{0,1}\|_3 + \|v_{0,2}\|_3$ such that

$$\inf_{\theta \in \mathbb{R}} \|\exp\{i\theta\}q_1(t) - q_2(t)\|_2 \leq C_{**} M$$

holds for all $t > 0$. Here, $C_{**} > 0$ is non-decreasing with respect to $C_0 > 0$ and $\|v_{0,1}\|_3 + \|v_{0,2}\|_3$.

Proposition 3.9 with $M \geq 0$ being small implies that Lyapunov stability estimates for problem (1.3) translates to orbital stability estimates for problem (1.4) through the generalized Hasimoto transformation.

3.3.5 Final Step of the Proof of Theorem 2.5

We combine all of the results of the preceding sections to finish the proof of Theorem 2.5. Let $q_R(t) = -\frac{1}{R} \exp\{-\frac{t}{2R^2}\}$ with $R > L/\pi$ be the plane wave solution of problem (1.4) and $\phi_0 \in H^2(I_L)$ satisfy the 0-th order compatibility condition for (1.4). Let $v^R$ be the arc-shaped solution of problem (1.3) which corresponds to $q_R$ through the inverse generalized Hasimoto transformation as shown in Section 3.3.2. Recall that $v^R = v^R(s)$ is a stationary solution. Furthermore, let $v_0 \in H^3(I_L)$ be the initial datum corresponding to $q_R(0) + \phi_0$ and set $\varphi_0 := v_0 - v^R$. Finally, let $v \in C([0, \infty); H^3(I_L)) \cap C^1([0, \infty); H^1(I_L))$ be the solution of problem (1.3) with initial datum $v_0$ and boundary datum $b = v^R(L)$ and set $\varphi := v - v^R$. From the results of Section 3.3.1 we have

$$\|\varphi_0\|_3 = \|v_0 - v^R\|_3 \leq C(\|\phi_0\|_2 + \|\phi_0\|_2^3),$$

where $C > 0$ depends on $\|q_R(0)\|_2 + \|\phi_0\|_2$ and is non-decreasing with respect to $\|q_R(0)\|_2 + \|\phi_0\|_2$. Note that $q_R$ doesn’t depend on $s$, so that $C$ essentially depends on $R$ and $\|\phi_0\|_2$, and is non-decreasing with respect to $\|\phi_0\|_2$. From the above estimate and Proposition 3.8 there exists $C_3 > 0$, which depends on $L$, $R$, and $\|\phi_0\|_2$, such that

$$\|v(t) - v^R\|_3 \leq C_3(\|\phi_0\|_2 + \|\phi_0\|_2^3)$$

holds for all $t > 0$. Proposition 3.9 further shows that there exists $C_4 > 0$ depending on $L$, $R$, $\|\phi_0\|_2$, and $\|v^R\|_3 + \|v_0\|_3$ such that

$$\inf_{\theta \in \mathbb{R}} \|\exp\{i\theta\}q(t) - q_R(t)\|_2 \leq C_4(\|\phi_0\|_2 + \|\phi_0\|_2^3)$$
holds for all $t > 0$, and $C_4 > 0$ is non-decreasing with respect to $\|\phi_0\|_2$ and $\|v^R\|_3 + \|v_0\|_3$. Again, note that $\|v^R\|_3$ is essentially a constant depending on $R$. From inequality (3.33) we see that

$$\|v_0\|_3 \leq C(1 + \|\phi_0\|_2 + \|\phi_0\|_2^3),$$

where $C > 0$ depends on $R$ and $\|\phi_0\|_2$, and is non-decreasing with respect to $\|\phi_0\|_2$. Hence, $C_4 > 0$ can be chosen uniformly with respect to $\phi_0$ satisfying $\|\phi_0\|_2 \leq 1$.

Finally, for an arbitrary $\varepsilon > 0$, choose $0 < \delta < 1$ small enough such that $C_4\left(\delta + \delta^0\right) < \varepsilon$ holds. Then, if $\|\phi_0\|_2 \leq \delta$,

$$\inf_{\theta \in \mathbb{R}} \|\exp\{i\theta\}q(t) - q_R(t)\|_2 < \varepsilon$$

holds for all $t > 0$, and this finishes the proof of Theorem 2.5.

\[\square\]

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