A Framework for the Evaluation of the Hierarchical Complexity of Network Topology in EEG Functional Connectivity

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Abstract

Background: Understanding the complex hierarchical topology of functional brain networks is a key aspect of functional connectivity research. Such topics are obscured by the widespread use of sparse binary network models which are fundamentally different to the complete weighted networks derived from functional connectivity.

New Methods: We offer an alternative framework for functional brain networks designed specifically for CWN analysis, generalising widely used binary network models to CWN form and analysing the main topological features of brain networks- integration, regularity and modularity- over the full range of densities. Importantly, we introduce two techniques to probe the hierarchical complexity of topologies which complement and enhance this framework. Firstly, a new metric to measure hierarchical complexity, secondly, a Weighted Complex Hierarchy (WCH) null model for EEG functional connectivity.

Results: By controlling the parameters of the model, the highest complexity is found to arises between a random topology and a strict ‘class-based’ topology. Further, the model has equivalent complexity to EEG phase-lag networks at peak performance.

Comparison to existing methods: Hierarchical complexity attains greater magnitude and range of differences between different networks than the previous commonly used complexity metric and our WCH model offers a much broader range of network topology than the standard scale-free and small-world models at a full range of densities.
Conclusions: Our metric and model provide a full characterisation of hierarchical complexity. Importantly, our framework shows a scale of complexity arising between random and regular topologies at one extreme and star and strict 'class-based' topologies at the other.

1 Introduction

Graph theory is an important tool in functional connectivity research for understanding the interdependent activity occurring over multivariate brain signals [1][2][3]. In this setting, Complete Weighted Networks (CWNs) are produced from all common recording platforms including the Electroencephalogram (EEG), the Magnetoencephalogram (MEG) and functional Magnetic Resonance Imaging (fMRI), where every pair of nodes in the network share a connection whose weight is the output of some connectivity measure. However, the current field has largely lacked any concerted effort to build a framework specifically targeted at CWNs, preferring instead to manipulate the functional connectivity CWNs into sparse binary form (e.g. [4][5][6] as well as wide-spread use of the Watts-Strogatz [7] and Albert-Barabasi [8] models) and using the pre-existing framework built around other research areas which have different aims and strategies in mind [9]. The current understanding of functional CWN topology is thus limited because of this oversight [10]. Here we address this issue by offering a novel framework for understanding CWN topology and implement it on EEG networks to gain important insights on the broader structures of functional brain networks and how they relate at all levels of connection density. Particularly, we offer novel methods to probe the complex hierarchical structures found in real networks [11], including brain networks [1].

The framework is a mixture of entirely new concepts and novel generalisations of existing concepts to CWN form. It is constituted of the following elements:

1. Four metrics (one novel- \( R \)) characterising four important and distinct topological features:

   (a) Hierarchical complexity, \( R \), for complexity,

   (b) Clustering coefficient, \( C \), for integration (or lack thereof),
(c) Degree variance, $V$, for irregularity,

(d) Modularity, $Q$, for modularity,

2. Five CWN archetypal models (all novelly formulated, but based on sparse archetypes) which, combined, classify minimal and maximal values of the above topological features:

(a) The Random CWN: maximal for integration,

(b) The Star CWN: maximal for irregularity, minimal for complexity and modularity,

(c) The Regular Lattice CWN: minimal for complexity and irregularity,

(d) The Fractal Modular CWN: maximal for modularity, minimal for integration,

(e) The Grid Lattice CWN: neither minimal nor maximal but important for structural comparisons for functional connectivity,

3. The Weighted Complex Hierarchy (WCH) model: A new CWN null model for EEG functional connectivity which, importantly, simulates functional brain network topologies before binarisation and other network processing steps.

The method we choose to analyse CWNs is to subject the network weights over a range of thresholds to obtain binary networks whose differences are found in topologies resulting from the existence and non-existence of connections \[12][13]. This then generalises strongest weighted connections, providing snapshots of the network of connectivity over the full spectrum of connectivity strengths. Here we use this technique to obtain metric curves plotted against connection density which provides a detailed analysis of the CWN topology. Other methods exist to analyse CWNs such as weighted metrics \[14] or density integrated metrics \[15], but these metrics still give
only singular values for a given network which belies little of topological behaviour at different scales of connectivity strength within the network.

In [17][16] an 'architecture' of network topology is proposed involving the three most widely studied properties of brain networks—integration (and segregation) [2][7][18], 'scale-freeness' [8][19] and modularity [20][21]. For our framework, we choose and fully justify a straightforward metric for each of these topological factors.

In addition to these, we introduce a new metric—hierarchical complexity—which offers a measure of the complex hierarchical patterns in networks. Complexity is understood neither to mean regularity, where obvious patterns and repetition are evident, nor randomness, where no pattern or repetition can be established, but attributed to systems in which patterns are irregular and unpredictable such as in many real world phenomena [22]. Tononi et al. [23] introduced an information theory metric called neural complexity which was explained to be maximal for middle ranges of integration, indicating that high complexity arises from a moderately integrated system.

For graph topology network entropy has been introduced [16] for sparse networks to target similar characteristics based on measuring the 'entropy' of the degree distribution. Hierarchical complexity, on the other hand, targets the structural consistency at each hierarchical level of network topology, targeting the complex relationships of hierarchical systems. We compare our metric with network entropy and find that we can offer a greater magnitude and density range for establishing differences in complexity of different graph topologies.

We apply these metrics to a wide range of CWN types, novelty extended from sparse network concepts and relevant to at least one of the aforementioned topological features of functional brain networks. We also explain methods for their creation which are of a general nature and thus can be used as a basis from which to form other CWNs. Due to the intrinsic properties of these graph types we find minimal and maximal topologies which can help to shed light on a wide variety of topological forms and their possible limitations [16].

Finally, we introduce the WCH model which simulates the hierarchical structures of weighted networks. This model works by modifying uniform random weights by addition of multiples of a constant, which is essentially a weighted preferential selection method with a highly unpredictable component provided by the original random weights. We show that it follows very similar topological characteristics of networks formed from EEG phase-lag connectivity. Intrinsic to our model is a strict control of weight ranges for hierarchical levels which offers unprecedented ease, flexibility and rigour for
topological comparisons in applied settings and for simulations in technical exploration for brain network analysis. This also provides an unconvoluted alternative to methods which randomise connections [7][6] or weights [14] of the original network.

This framework offers a different perspective of network topology, revealing new insights into EEG functional connectivity networks and their formation and providing greater comparative abilities for future clinical and technical research.

2 Network Science

We adopt the notation in [24] so that a graph, \( G(V, A) \), is a set of \( n \) nodes, \( V \), connected according to an \( n \times n \) weighted adjacency matrix, \( A \). Entry \( A_{ij} \) of \( A \) corresponds to the weight of the connection from node \( i \) to node \( j \) and can be zero. An unweighted graph is one in which connections are distinguished only by their existence or non-existence, so that, without loss of generality, all existing connections have weight 1 and non-existent connections have weight 0. The graph is undirected if connections are symmetric, which gives symmetric \( A \). A simple graph is unweighted, undirected, with no connections from a node to itself and with no more than one connection between any pair of nodes. This corresponds to a graph with a symmetric binary adjacency matrix with zero diagonal. Such graphs are easy to study and measure[9]. The degree, \( k_i \), of node \( i \) is defined as the number of its adjacent connections, which is the number of non zero entries of the \( i \)th column of \( A \). Then, for a simple graph, \( k_i = \sum_{j=1}^{n} A_{ij} \). For a graph with \( 2m \) edges, the connection density, \( P \), of a graph is \( P = 2m/n(n-1) \). A path from node \( i \) to node \( j \) consists of a sequence of existing connections, \( A_{ik}, A_{kl}, \ldots, A_{pq}, A_{qj} \). Intuitively we see this as following the connections from node \( i \) to node \( j \) sequentially, as if walking along a path.

A CWN is represented by a symmetric adjacency matrix with zero diagonal (no self-loops) and weights, \( w_{ij} \in [0,1] \), elsewhere. To analyse CWNs it is beneficial to convert it to simple form by binarising the adjacency matrix using a threshold, where a percentage of strongest connections are set to 1 and the remaining values set to 0. This stays true to the network activity[10] whilst reducing computational complexity and weight issues found with weighted metrics[2]. Hereafter, all mathematics will refer to simple graphs.

In the coming section we will outline the four different topological factors we include in our framework- complexity, integration, regularity and
modularity. We will assign and justify a metric for their evaluation, and where necessary we will comment on where CWNs differ from sparse networks and where conceptual clarification is required. We will also outline relevant types of CWNs which will be formulated in section 3.

2.1 Metrics

2.1.1 Complexity

The ideas of order and complexity are well known in the discussion of networks. Indeed, real world networks are often called complex networks \[1][3][25]. In mathematics, the graphs studied derive from some theoretical principles. These can involve set patterns, without random fluctuations of connections, such as regular networks, fractal networks, star networks and grid networks. On the other hand much interest is shown in more randomly generated topologies, such as random graphs and other graphs involving random processes, as these express something of the more erratic and irregular quality of connections in networks constructed from real world phenomena \[7][26]. However, real world phenomena differ from random processes in that there is a clear organisational behaviour apparent throughout the hierarchical structure, both within hierarchical levels and between hierarchical levels \[11][21]. This structure however, is difficult to analyse and impossible to retrace because its formation inevitably involves many unknown generative processes.

Hierarchies in networks are generally determined by degrees of nodes, where highly connected nodes create a rich club \[25] on the top hierarchical level and nodes of lowest connections exist on a peripheral bottom level. Further, it is seen that a node’s relationship within the context of the network is greatly determined by the other nodes to which it is connected \[27]. Thus, to understand the complexity of a network we propose to study the behaviour of nodes of a given degree by looking at the degrees of nodes in their neighbourhoods. We define \(D\) as the set of degrees of a graph, \(G\). Similar to the idea of node degree sequences \[28], we can construct neighbourhood degree sequences, specific to each node in the graph. That is, for a node, \(i\), of degree \(k \in D\) we have a sequence

\[
s_i = \{d_{i,1}, d_{i,2}, \ldots, d_{i,k}\} \text{ s.t. } d_{i,1} \leq d_{i,2} \leq \cdots \leq d_{i,k} \in D,
\]

where \(d_{i,j}\) is the degree of the \(j\)th node connected to node \(i\) (see Fig. 1). For all nodes of a given degree, \(k\), the corresponding neighbourhood degree sequences have equal length, \(k\).
Figure 1: Example of a node degree neighbourhood. Here is shown a part of a network relating to the neighbourhood of the blue node. The blue node has neighbourhood degree sequence \( \{1, 2, 3, 4, 4\} \), i.e. the ordered degrees of the orange nodes. Grey connections indicate all the additional connections of the orange nodes in the network.

Figure 2: Example for graph complexity. Here is shown a 20 node network with varying 'orderedness' at different degree levels.

Something which exhibits high order can be expressed as an object displaying regularly spaced and repeating motifs. For example, the line graph, the complete graph, the star and quasi-star graphs, regular graphs and fractal graphs should all be characterised as highly ordered, low complexity topologies. Randomly generated graphs should also be characterised by low levels of complexity and networks reflecting interactions of real world systems should be characterised by high complexity.

We define the complexity, \( R \), of a network as the average variance of the \( k \)-degree neighbourhood degree sequences and can be expressed as:

\[
R = \sum_{k=1}^{k_{\text{max}}} \frac{1}{k} \frac{1}{k_{\text{max}}} \left( \sum_{s \in D_k} \left( \sum_{j=1}^{k} (s_{k,i}(j) - \mu_j)^2 \right) \right),
\]

where \( k_{\text{max}} \) is the maximum degree of the graph, \( D_k \) is the set of nodes of degree \( k \), \( s_{k,i}(j) \) is the \( j \)th element of the \( i \)th \( k \)-length sequence \( s_{k,i} \), \( \mu_j \) is the
mean value of element $j$ over all $k$-length sequences and $r_k$ is the number of nodes of degree $k$ which is squared to normalise the value between 0 and 1.

Organisation of the graph at the level of $k$-degree nodes can be seen by comparing the $j$th elements of their neighbourhood sequences. If all of the $j$th elements of all the sequences are equal, that is $s_i = s_j$ for all $s_i, s_j$ of length $k$, then there is a high degree of order present in the $k$-degree nodes of the graph. If these sequences differ widely however, then it can be said that the $k$-degree nodes are either disorganised or more complexly organised. For example, in Fig.2 the two degree nodes all have the same degree sequences- \{3, 4\}- whereas the three degree nodes are split into two different degree sequences- \{1, 2, 2\} and \{1, 1, 4\}- and finally the neighbourhood degree sequences of the four degree nodes are all different- \{1, 1, 1, 4\}, \{1, 2, 2, 4\} and \{2, 3, 3, 3\}. So the complexity of just the two degree nodes is 0, the complexity of just the three degree nodes is $((2 - 1.5)^2 + (1 - 1.5)^2)/2 + ((2 - 3)^2 + (4 - 3)^2)/2)/4/3 = 5/48$ and the complexity of just the four degree nodes is $((2(1 - 4/3)^2 + (2 - 4/3)^2)/3 + 2((1 - 2)^2 + (3 - 2)^2)/3 + (2(4 - 11/3)^2 + (3 - 11/3)^2)/3)/3/4 = 4/27$, the complexity over all three levels being the average- $0.0841$.

This measure is minimal for graphs in which, for each $k$ and $k'$, every $k$-degree node is connected to exactly the same number of $k'$-degree nodes. This property, for example, is seen in ring lattices, and quasi-star graphs and is close to minimal in the line graph, fractal graphs and grid lattices. Furthermore, the degrees of random networks are known to have a fairly small spread which is a factor penalised by our complexity value. Thus random networks should obtain low values of our complexity measure. We expect, then, that real-world networks and their simulations will, almost exclusively, obtain high values of $R$.

2.1.2 Integration

The concept of integration in brain networks is closely tied in to the small world phenomenon [29], where real world networks are found to have an efficient ‘trade off’ between integrative and segregative behaviours [30]. The most widely used topological metrics in network science- The characteristic path length, $L$, and the global clustering coefficient, $C$- are commonly noted as measures of these quantities, respectively. Here, $L$ is defined as the average of the shortest paths between each pair of nodes and $C$ is defined as the probability that a path of length 2, or triple, in the graph has a shortest
path of length 1. That is,

$$C = \frac{\text{closed triples}}{\text{triples}},$$

where a closed triples is such that, for triple \( \{A_{ik}, A_{kj}\} \), \( A_{ij} = 1 \), for \( i, j, k \) distinct.

The Watts-Strogatz small world model [7] is based on randomly rewiring a small number of connections in a sparse binary construction called a regular ring lattice (see Fig. 4A). This results in similar measurements of high \( C \) and low \( L \) to small world networks. The regular ring lattice itself has high \( C \) and \( L \) and the fully randomised network, also known as an Erdős-Rényi random graph [26], has low \( C \) and \( L \). The small world property is then \( \sigma = C/L \) which is normalised by random graph values to \( \sigma_{SW} = (C/C_{\text{ran}})/(L/L_{\text{ran}}) \) [31].

In fact, since integration implies a non-discriminative behaviour in choice, we argue that the random graph ensemble [26], defined by its equal probability of existent connections between all pairs of nodes, is the most exemplary model of an integrated network. Anything which deviates from equal probability is a discriminative factor which favours certain connections or nodes over others, likely leading to more segregated activity. Further, it is clear that integration and segregation are opposite ends of the same spectrum—something which is not integrated must be segregated and vice versa. Having one metric to inform on where a network lies on that spectrum is therefore sufficient. Contrarily, choosing \( C \) and \( L \), two correlated metrics [4], to measure them separately is difficult to justify.

Thus, here we propose that \( C \) is sufficient as a topological measure to evaluate the levels of integration (and so segregation) of a given network. Firstly, we note that values of \( C \) for random graphs and small-world graphs are often much more distinguishable than those of \( L \) [7] and it is certainly assumed that these graphs have very different levels of integration. Secondly, since the random network is optimally integrated and \( E\{C_{\text{ran}}\} = E\{P_{\text{ran}}\} \) [9], where \( P_{\text{ran}} \) is the connection density of the random network, then the larger the deviation from 1 of the value \( \gamma = C/E\{C_{\text{ran}}\} = C/E\{P_{\text{ran}}\} = C/P \), the more segregated is the network. We will include both \( L \) and \( C \) in our analysis in order to provide evidence to back the above proposal.

2.1.3 Regularity

Another topological factor of small world networks is noted as a scale-free nature characterised by a power law degree distribution [32]. To under-
stand this aspect of network topology another factor of network behaviour is formulated distinguishing between 'line' like and 'star' like graphs [2] [33].

Here, we show that characterisation of scale-freeness can be reduced largely to the regularity of the network. Regular graphs have been studied for over a century [34]. They are defined as graphs for which every node has the same degree. An almost regular graph is a graph for which the highest and lowest degree differs by only 1. Thus a highly irregular graph can be thought of as any graph whose vertices have a high variability. Such behaviour can be captured simply by the variance of the degrees present in the graph, that is

\[ V = \text{var}(D), \]

where \( D = \{k_i\}_{i \in V} \) is the set of node degrees on a given graph [35].

For regular graphs \( V = 0 \) by definition, but more probing is necessary to distinguish high \( V \) topology. For a graph with degrees \( k = \{k_1, k_2, \ldots, k_n\} \), and \( \sum_{i=1}^{n} k_i = 2m \), on multiplying out the brackets \( V \) simplifies to

\[
V = \frac{1}{(n-1)} \sum_{i=1}^{n} \left( \frac{2m}{n} - k_i \right)^2 \\
= \frac{\|k\|_2^2}{(n-1)} - 2mP,
\]

where \( P = 2m/n(n-1) \) is the connection density and \( \|k\|_2^2 = \sum_{i=1}^{n} k_i^2 \), is the squared \( \ell_2 \) norm of \( k \). This tells us that \( V \) is proportional to the sum of the squares of the degrees of the graph, \( \|k\|_2^2 \), and, for fixed number of connections, \( m \), \( V \) in fact depends only on \( \|k\|_2^2 \). Now, it is known that \( \|k\|_2^2 \) is maximal in quasi-star graphs and quasi-complete graphs [36]. Essentially, the quasi-star graph has a maximal number of maximum degree nodes in the graph for the given connection density (see Fig. 2.B for a quasi-star graph) and the quasi-complete graph has a maximal number of isolated, or zero-degree, nodes in the graph. This tells us that, for low \( P \), high \( V \) denotes the presence of a few high degree nodes and a majority of relatively low degree nodes, i.e. scale-free-like graphs. Thus the irregularity of degrees, which can be measured straightforwardly by \( V \), is a strong indicator of scale-freeness.

### 2.1.4 Modularity

A third commonly noted quality of real networks is modularity. This is where networks have interconnected modules of nodes. High modularity indicates that a network is comprised of a number of modules such that
there is a high density (or average weight) of connections within the module and a relatively low density of connections connecting the module to the rest of the network. This is a distinct issue to that of integration [17]. In order to quantify this behaviour the metric $Q$ for modularity was proposed [20]:

$$Q = \frac{1}{2m} \sum_{i,j} \left( A_{ij} - \frac{k_i k_j}{2m} \right) \delta(c_i, c_j),$$

where $c_i$ is the module containing node $i$ and $\delta(\cdot)$ is the Kronecker delta function. This metric implies that the modules are already known, but the detection of modules within networks is not simple since it is rare to have a clear distinction between the 'boundaries' of one module and the next. For this task, highly efficient algorithms have been created [38][37] aiming to maximise the value of $Q$ for a given network. To compute the modularity of our networks, we use the undirected modularity function [38] in the Brain Connectivity Toolbox [39].

### 2.2 Network models

We will now introduce the formulations of different CWNs, including our null model. Before introducing the WCH model, it is necessary to first explain the Erdős-Rényi Random CWN on which it is based. After these, we will go into detail of different ordered CWNs relating to ideal network types found in the sparse binary framework.

For the archetypal CWNs we require that there are obvious higher density versions of lower density forms which can be arranged in adjacency matrix form such that each non-zero entry, $A_{ij}$, of the lower density adjacency matrix exists as a non-zero entry, $\hat{A}_{ij}$ in the higher density adjacency matrix. This is indeed the case for the Star, Regular Ring Lattice, Grid Lattice and Fractal Modular CWNs which we will demonstrate below. We explain these higher and lower density forms of the binarised CWN in terms of weight categories where, if we choose an appropriate threshold, $T$, we can recover all edges in the same and all higher weight categories and none of the edges existing in all lower categories.

#### 2.2.1 Erdős-Rényi Random

The most general random network is the Erdős-Rényi (E-R) random network [26] which is formed by assigning a probability, $p$, to the question of the existence or non-existence of connections on a network with $n$ nodes. Such a construct is, in fact, an ensemble of graphs denoted $G(n, p)$. A sample of
this ensemble is obtained by generating a random value for every possible connection and applying the probability value \( p \) as a threshold to see whether or not that connection should exist in our sample. The expected connection density of the graph obtained is \( p \).

The CWN random model is just the basis from which a sample of \( G(n, p) \) is formed. We simply generate a symmetric matrix with zero diagonal and randomly generated values \( A_{ij} \in [0, 1] \) elsewhere. This is then the adjacency matrix for a complete network with randomly weighted connections. Note, if we threshold the CWN at weight \( T = p \), we get an Erdős-Rényi random graph from the random graph ensemble \( G(n, p) \).

2.2.2 Weighted Complex Hierarchy Model

We now detail the algorithm for the generation of the WCH model. Starting from an Erdős-Rényi CWN we randomly distribute the nodes into hierarchy levels based on some discrete cumulative distribution function, \( p \), by generating a random number, \( r \), between 0 and 1 for each node and putting the node in the level for which \( r - p \) is first less than 0. We then distribute \( ls \) additional weight to all connections of adjacent nodes in the \( l \)th level, for some suitably chosen \( s \). The parameters of this model are then \((n, s, l, p)\) where \( n \) is the number of nodes in the network; \( s \) is the strength parameter, which is constant since the random generation of the initial weights is enough to contribute to weight randomness; \( l \) is the number of levels of the hierarchy, with a default setting of a random integer between 2 and 5, and \( p \) is the cumulative probability distribution vector denoting the probabilities that a given node will belong to a given level where the default, which we use here, is a geometric distribution with \( p = 0.6 \) in hierarchical levels \((0, 1, 2, \ldots, l)\) where the nodes with highest connectivity (top hierarchical level) are at the tail end of the distribution. Fig. 3 plots an example of the geometric distribution for a three level hierarchy. The text inside the box plots, above, indicates the additional weights given to connections adjacent to nodes inside the given level. The graphic below explains the additional weights provided by the strength parameter of connections between nodes in different levels as well as in the same level. For example, a connection between Level 1 and Level 2 has additional strength \( 3s \) which consists of one \( s \) provided by the node in Level 1 and \( 2s \) provided by the node in Level 2. At \( s = 0 \), we have the E-R random network and at \( s = 1 \) the weights of the network are linearly separable by the hierarchical structure producing a strict 'class-based' topology. Between these values a weak class-based topology emerges.
Figure 3: This diagram explains the construction of the WCH model. Above is the probability distribution function for a geometric distribution with $p = 0.6$ for a three level hierarchy. Below is a graphic displaying the additional weight added between nodes in given hierarchy levels.

### 2.2.3 Star

A star graph is the archetypal scale-free graph with one node sharing connections to every other node and no other connections. Thus it has one node of degree $n - 1$ and $n - 1$ nodes of degree 1 (Fig. 4.C). We can construct a complete weighted generalisation of the star graph by taking the classic star as the first weight category and the subsequent weight categories associated with the increasingly higher density quasi-star graphs. Thus the second weight category is constructed by connecting any one of the 1 degree nodes to all other nodes in the network, creating a network of two $n - 1$ degree nodes and $n - 2$ nodes of degree 2 with two weight categories. Adding a third category creates a network of three $n - 1$ degree nodes and $n - 3$ nodes of degree 3 and so on. Eventually we have a CWN with $n - 1$ categories consisting of $n - 1$, $n - 2$, $\ldots$, 2, 1 connections, respectively. Fig. 4.B, shows
the connections corresponding to the first four weight categories in a star CWN.

### 2.2.4 Regular Ring Lattice

A regular ring lattice network is a network which we can illustrate by evenly spacing nodes in a circle and connecting each node to its $k$ closest neighbours, giving a regular graph of degree $k$ (Fig. 4A). Note that $k$ must be an even number since equal spacing on a circle means that closest nodes come in pairs. The exception to this is when $n - 1$ is odd and $k = n - 1$ forms the complete graph. The regular ring lattice is then defined by the parameters $(n, k)$. Some special examples are the closed triple with $(3, 2)$ and the regular ring lattice with parameters $(n, 4)$, which was presented by Watts and Strogatz to represent regular networks for comparison with small world networks [7].

We propose to form a complete weighted network for the ring lattice with weight categories associated with increasing value of $k$. Thus for $n - 1$ even, we have decreasing weight categories for increasing $k = 2, 4, \ldots, n - 1$. For $n - 1$ odd we have decreasing weight categories for increasing $k = 2, 4, \ldots, n - 4, n - 2, n - 1$. That is, the $k/2$th weight category belongs to all of the connections in graph $(n, k)$ which are not present in graph $(n, k - 2)$ (or, for $k = n - 1$ odd, the $(k + 1)/2$th weight category belongs to all of the connections in graph $(n, k)$ not present in graph $(n, k - 1)$).

### 2.2.5 Grid Lattice

Another common lattice graph is the grid lattice where nodes are placed at the intersection of lines in a grid. This graph is a strictly 'class-based' graph by which we mean nodes can be separated into clearly distinct classes based on degree: corner nodes ($k = 2$), side nodes ($k = 3$) and central nodes ($k = 4$). We use this as an archetype for a distance based graph where closer nodes are more strongly connected. In order to construct a complete weighted graph following from the grid lattice topology we propose to categorise the connections as shown in Fig. 4C. This graph is similar to the regular lattice in that the nodes have strongest connections to nodes they are close to. However weight categorisation by closeness is instead best represented by placing square ‘catchment’ areas around the nodes (Fig. 4C). Each category then consists of the connections within the corresponding catchment area placed around every node minus all the connections in the previous category. This results in a highly inhomogeneous number of con-
Figure 4: A. A 12 node ring lattice of degree 6, comprising the three strongest weight categories of the ring lattice CWN. B. The quasi-star with 4 nodes of degree $n - 1$ and $n - 4$ nodes of degree 4, also comprising the first four categories of the star CWN. C. The grid lattice weight categorisation of a 30 node network. Colours of connections denote category: black, blue, green, orange and red are categories 1, 2, 3, 4 & 5, respectively. The increasingly light grey boundaries are the 'catchment' areas around a node for the categories. D. Fractal modular CWN weight categorisation on 30 nodes. Connections shown are 1st weight category edges. Increasingly light grey background represents consecutive weight categories within which all nodes become connected.

Connections at each node in a given category, creating a hierarchy of nodes based on degree, contrasting with the regular ring lattice where all nodes have equal degree by definition.

2.2.6 Fractal Modular

In order to obtain an ordered graph with a highly modular topology, we define here methods for constructing fractal modular graphs from some number of $K3$s and $K4$s, the complete graphs on 3 and 4 nodes, respectively. Here, we simply connect these $K3$ and $K4$ subgraphs in a ring as shown in Fig. 4D. All integers above 5 can be expressed as a sum of 3s and 4s so this
method can be used to construct a graph with any $n > 5$. These networks are fractal because at each step the smaller modules merge into larger modules until we eventually have a complete graph, these steps are shown in Fig. 4D by the increasingly lighter grey backgrounds where nodes within the shaded area indicate that connections exist at that category level. To select a 3, 4-summation of $n$ as well as the ordering of module forming at each step we can simply use our discretion for graphs with a fairly low number of nodes. Here, for 30 nodes we choose six $K4$s and two $K3$s connected in a ring to construct the first weight category and progressive module forming as depicted in Fig. 4D. For 64 nodes we choose an initial weight category consisting of sixteen $K4$s connected in a ring with similar progressive module forming. Generally, the higher the power of 2 which is a factor of the initial combined number of $K3$ and $K4$ modules, the better the 3, 4-summation it is for the fractal composition of the network.

3 Methods

Here we apply methods to graphs of 64 nodes, typical of medium density EEG. For analysis we employ connection density thresholds at integer percentages of strongest weighted connections, rounded to the nearest whole number of connections. We then implement metric algorithms on these binary networks and plot the obtained values on a curve against connection density. For random and WCH CWNs we use sample sizes of 100 for each network and for the EEG functional connectivity CWNs we have a sample size of 109. On the metric curves for these we plot the median with the interquartile range shaded in. For ordered networks there is only one network per type by definition.

3.1 Comparison for Hierarchical Complexity

We compare our hierarchical complexity metric with a commonly used metric for analysing the entropy of the network degrees [16]. This is defined using the normalised degree distribution $q(k) = kp_k/\sum_j j p_j$. Then the entropy of graph $G$ is

$$H(G) = -\sum_k q(k) \log(q(k)).$$

(5)
3.2 Comparisons for the WCH model

We implement comparisons with the Watts-Strogatz small-world model [7] which randomly rewires a set proportion of edges starting from a regular lattice. We use the full range of parameters for initial degree specification (2 up to 62) and random rewiring parameters from 0.05 in steps of 0.05 up to 0.95. For each combination of parameter, 100 realisations of the model were computed and C, V, Q, and R were measured. We further compare with Albert-Barabasi’s scale-free model [8] which begins with a graph consisting of core of highly connected nodes to which the rest of the nodes are added one by one with a set degree but paired by edges to randomly selected nodes. We use an initial number of nodes of 15 and the additional node’s degree from 3 up to 14 in order to reach larger densities.

3.3 EEG networks

We use an eyes open, resting EEG data set with 64 nodes. We report on networks created from the beta (12.5-32Hz) band using coherence (COH) and the debiased Weighted Phase-Lag Index (dWPLI) in order to account for different possible types of EEG networks while reducing redundancy of similar topological forms found between the frequency bands (see supplementary material).

The dataset, recorded using the BCI2000 instrumentation system [45], was freely acquired from Physionet [46]. The signals were recorded from 64 electrodes placed in the main in accordance with the international 10-10 system. We took the eyes open resting state condition data, consisting of 1 minute of continuously streamed data which were partitioned into 1s epochs and averaged for each of 109 volunteers.

FieldTrip [42] was used for pre-processing, frequency analysis and connectivity analysis to obtain the adjacency matrices of complete weighted networks. The 64 channels were re-referenced using an average reference, the multi-taper method was implemented from 0 seconds onwards using Slepian sequences and 2Hz spectral smoothing. A 0.5Hz resolution was obtained using one second of zero padding. We chose to analyse the matrices obtained from both the coherence and the debiased Weighted Phase-Lag Index (dWPLI) [43] to look for differences between network topologies of zero and non-zero phase lag dependencies in the channels [44]. We treat the data of all tasks as a single dataset to allow for the variability of the EEG network topologies since we are not interested here in the tasks themselves but on the behaviour of general EEG networks obtained from dWPLI and
coherence.

3.4 Statistical analysis

Due to the polynomial formulation of the complexity measure, producing a non-normal distribution, we compare metric distributions using the Wilcoxon rank sum test. The z-score is used to ascertain the magnitude and direction of the relationship of the distributions.

4 Results

4.1 Metric Comparisons

Fig. 5 shows the metric curves for \( C, V, Q, R, L \) and \( H \) for all archetypes as well as for the EEG dWPLI and coherence networks. From these plots we see experimental evidence of maximal and minimal topologies for the given topological characteristics. Fractal Modular networks are maximal for both \( C \) and \( Q \) (Fig. 3, rows 1 & 3, respectively). This is to be expected since the modules are complete subnetworks and there are very few connections between modules (maximising \( Q \)) and also minimising the number of open triples in the graph (maximising \( C \)). The star CWN acts as a maximal topology for \( V \), as expected from the theory. Regular graphs, such as the ring lattice network, give 0 degree variance and hierarchical complexity, thus are minimal topologies of these features. The star CWN is shown to be a minimal topology for \( L \) (row 5). The results of fig. 5 for 30 node networks follow the same relationships and can be found in the supplementary material which also includes details of the power-law properties of the degree distributions of EEG networks and the WCH model.

Comparing the plots of \( C \) with \( L \) and \( R \) with \( H \) it is immediately clear that \( L \) and \( H \) show extreme behaviour at low densities while remaining consistent at higher densities. This exemplifies how these metrics are aimed at analysis of sparse networks, where it appears that values can take a much greater range than for higher density networks.

To explore these comparisons further we perform statistical analysis with Wilcoxon rank sum tests on the differences of distributions of metric values of EEG dWPLI and E-R random networks as well as of EEG dWPLI and EEG coherence networks (Fig 6). The results show that \( C \) and \( R \) attain a greater range over edge density, \( P \), of significant differences. Particularly, \( R \) distinguishes differences from 1% up to 44% densities in the EEG dWPLI and coherence comparison, whilst entropy only can distinguish differences.
from 1% up to 27%. Further, the z-scores indicate that in the range 1-27%, the differences found in $R$ are greater than those found using $H$. Comparing the EEG dWPLI networks with E-R random networks, both metrics find differences at all levels, but the magnitude of difference found by $R$ is consistently greater than those found by $H$. Thus, our metric outperforms entropy in both magnitude and range of differences found. Similarly, $C$ finds a greater range and magnitude of differences than $L$. In fact, $C$ discerns differences at all connection densities for the two comparisons, while $L$ fails to find differences after 62% in comparing dWPLI and coherence networks and after 73% in comparing dWPLI and random networks. Furthermore, $L$ displays inverse differences at low densities (1-12%) compared to higher densities in the dWPLI vs random comparison. This inconsistency is undesirable for translatability of integrative behaviour of network types from sparse networks to more dense networks. $C$ does not suffer from such behaviour, displaying a constant relationship of metric values through the full range of densities.

Given these results, for the rest of our analysis, we will drop $L$ and $H$ and focus on the four proposed metric, $C$, $V$, $Q$ and $R$.

4.2 Weighted Complex Hierarchy Null Model

Fig. 4 shows the mean results of $C$, $V$, $Q$ and $R$ over 100 realisations of each of the WCH models with strength parameter $s = 0.05, 0.1, \ldots, 0.75$, after which the parameter begins to saturate as the weights of the hierarchy levels tend to linear separability. We see that WCH networks exhibit curve behaviour similar to the EEG networks and E-R random graphs. The scale-free model also exhibits a similar behaviour, however in stark contrast, the small-world model exhibits very different behaviours than those of the EEG or WCH networks, exhibiting a strong unsuitability for comparisons with EEG networks with much higher modularity and highly skewed $V$ curve towards high densities as well as a similar skew in $R$ which is opposite to the skew found for other network types. Although the scale-free model exhibits similar tendencies in topological metrics to the WCH and EEG networks, its range of values and densities is clearly very limited.

By increasing the strength parameter of the WCH model we change the topology in a smooth fashion with decreasing integration, regularity and modularity. Interestingly, $R$ rises with increasing strength parameter from $s = 0.05$ up to $s = 0.3$ where it takes its maximum values at densities ranging from 1-30% before falling again from $s = 0.35$ until $s = 0.75$. Further, above $s = 0.3$, the curves begin to deviate significantly from those of the EEG.
networks, exhibiting greater plateaus of high complexity.

Interestingly, the complexity of the EEG dWPLI networks appears to attain maximal values of $\mathcal{R}$ of all the networks studied here. The only model which comes close is the WCH model. To clarify this observation we perform Wilcoxon rank sum tests on $\mathcal{R}$ values of the EEG dWPLI networks against that of the WCH model with strength parameters ranging from $s = 0.2$ up to $s = 0.4$, i.e. two steps before and after the maximal complexity setting of $s = 0.3$. The results are displayed in Fig. 8. In the vast majority of instances of strength parameter and density, the EEG dWPLI networks do indeed exhibit greater complexity than the WCH model. The strong exception to this is an inability to distinguish significant differences between the maximal complexity $s = 0.3$ WCH model and dWPLI networks within 7-23% densities. Also, as the weight parameter increases, the high plateaus previously mentioned begin to take effect as in the medium ranges of density the $\mathcal{R}$ values of the dWPLI networks and WCH model becomes more indistinguishable, with greater complexity found in the range 55-57% in the WCH model with $s = 0.4$.

4.3 Null model approaching EEG phase-lag networks

Fig. 9 shows the values of the four topological features - complexity, integration, regularity and modularity for EEG dWPLI networks and the WCH network with strength parameter 0.2. We see clearly that these networks behave very similarly with respect to the given metrics. The most obvious difference is that the modularity of dWPLI EEG networks is higher. Also, as previously discussed, the dWPLI network complexity is greater than the WCH model, but it is still by far the most comparable model for complexity of those presented here.

5 Discussion

5.1 Complexity as revealed by Weighted Complex Hierarchy model

The behaviour demonstrated by the WCH model with respect to $\mathcal{R}$ indicates that high complexity arises from a hierarchical structure in which a greater degree of variability is present in the rankings of weights with respect to hierarchy level. Too little difference between levels and the hierarchy is too weak to maintain complex interactions, too much difference between levels and the complexity of the hierarchy is dampened by a more ordered structure. Thus,
we show that topological complexity does not arise as a middle ground between regular and random systems as previously conjectured \cite{7,23} (see Fig. 8), but, in the middle ground between weak hierarchical topology, such as anarchic (random) or egalitarian (regular) systems, and strong hierarchical topology, such as authoritarian (star) or strict class-based (grid lattice) systems.

5.2 Weighted Complex Hierarchy as null model

There are two clear reasons why the WCH model is a good fit for functional connectivity networks from EEG recordings. Not only does it create several hub like nodes giving a high degree variability, but furthermore it simulates the rich club phenomena found in complex brain networks \cite{25,47}, as the higher the hierarchy levels of two nodes, the stronger the weight of the connection will be between them, see Fig. 3.

One of the greatest benefits of this model over others is that it simulates brain networks previous to network processing steps because it creates CWNs rather than sparse networks. This means that any and all techniques one wants to use on the brain networks can be applied elegantly and in parallel with this single null model free from any complications. Particularly, methods which create sparse binary networks directly, whether these models are built independently from the brain networks \cite{8,6} or are constructed by the randomisation of connections of the networks being compared \cite{7}, run into problems with density specification (in the case of independent models) and reproducibility (in both types of model). With the WCH model, we can simply create a bank of simulated CWNs which can be used throughout the study in exactly the same way as we use the functional connectivity CWNs.

As an example of the power and elegance of the proposed model, say we want to find maximum spanning trees \cite{48} of our brain networks and compare with a null model, then we simply take the maximum spanning trees of our null model. In contrast, in \cite{5} they use a convoluted reverse engineering process by assigning random weightings to the connections of Watts-Strogatz small world networks (which are themselves of limited comparability to brain networks) and computing the MST from these resulting sparse weighted networks.

Further, as seen in Fig. 5 for technical studies which rely on network simulations, the WCH model is built on parameters which can be altered to subtly change the resulting topology. This allows for sensitive analysis of a new techniques ability to distinguish subtle topological differences. Such paradigms are evident in clinical studies where, for example, one may try
to distinguish between healthy and ill patients [13][33] or between different cognitive tasks [11], so that this null model offers simulations which are directly relatable to clinical settings.

5.3 EEG coherence and WPLI networks

We see there is a large difference in the integration, modularity and complexity of the EEG coherence and dWPLI networks. The EEG coherence networks behave similarly to the ring and grid lattice networks, which agrees with the volume conduction effects believed to dominate zero-lag dependency measures [41], i.e. the closer the nodes are the stronger are the edge weights. The dWPLI networks on the other hand have a more integrated and less modular nature, where the structure of the brain plays less of a role.

The very high complexity of the dWPLI (and very possibly phase-lag measures in general [49]) provides evidence to support that it does indeed largely overcome the volume conduction effect and maintain a large amount of information with respect to the complex interactions of the brain. Thus we recommend it as a useful measure of true functional connectivity in the brain.

5.4 Degree Variance curves

A striking feature seen is in the degree variance curves where, unobviously, a highly symmetric parabolic curve is noted with a central maximum value for random graphs, WCH networks and EEG networks. This tells us that the curve of $V$ for these CWNs must be very similar to the $V$ of the complement of the network. Perhaps this is most striking for EEG networks, where it has been understood that the low weights are spurious due to noise and general absence of true connectivity [12]. The symmetry tells us, though, that low weighted connections do not behave particularly differently to the high weighted connections in terms of regularity. If, for example, the low weighted connections had a more random distribution, the $V$ curve would surely reflect this at high densities by convergence to the $V$ curve of the E-R random CWNs. Furthermore, this feature reveals to us a 'scale-free' paradigm at all density levels and not just the classic sparse network scale-free at low densities. As the density of the network increases one obtains more even distributions of high and low density nodes, indicated by the high values of $V$, and, eventually, towards high densities the symmetry of $V$ values with low densities tells us that the scale-free network is characterised
by a small number of low degree nodes and a majority of high degree nodes, i.e. the inverse (or complement) of the low density behaviour. In network science the rich-club of highly connected nodes are generally regarded as the most important for the study. However, an absence of connectivity does not mean an absence of informative behaviour- such connectivity 'cold spots' may also play a vital role in the topological structure of real-world networks.

5.5 Topological Randomness

It is very apparent that random networks have a restricted topology at all density levels, where the interquartile range is much smaller in comparison with that of the EEG networks and the proposed null model. This provides evidence that E-R random networks are not in fact topologically random in the precise sense that one cannot expect a large variability of topological metric values of random networks. Instead, we see that uniformly distributed random weights of edges results in a very particular optimally integrated, moderately regular, lowly modular and low complexity topology at all densities. Based on this evidence and previous discussion of random networks in the methods section, we suggest that E-R random networks be re-understood as optimally integrated networks.

Following from this the randomisation of connections used widely in null models is not a topologically randomising process but, more accurately, a topologically integrative process. Such a feature is then not necessarily typical of network topology and thus one must be cautious to use this as a null model unless one wants to specifically target integrative behaviour. Further, the practice of normalisation of graph values by E-R random graph values must also be used with due caution. The basis of such a normalisation is to contrast a networks values with those of the 'average' network topology, rather than contrasting with a highly specific topology which behaves very differently to real world networks.

6 Conclusion

We proposed a novel framework for CWNs for brain functional connectivity to replace the framework for sparse networks adopted from other network science research areas. This included the synthesis of concepts from the literature in a succinct manner- proposing single metrics as indicators of separate topological factors and extending the most important archetypal sparse network models to CWN form. Particularly, we introduced a metric for measuring the hierarchical complexity of a network providing insight of
what distinguishes real-world networks from both ordered and spontaneous forms as generally the most complex kind of topology. We also introduced a highly flexible, reproducible and elegant WCH model which negates the use of spurious connection randomisation processes and convoluted methodologies. These insights help towards a comprehensive understanding of the framework within which functional connectivity networks are set and thus provide invaluable information and tools for future clinical and technical research which may also extend far beyond neuroscience into the many domains in which network science is utilised. The supplementary material attached to this article shows results for a 30 node network size and a comparison of different frequency bands. Matlab codes for all synthesis and analysis of the networks as introduced in this paper will be made publicly available upon publication.

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Figure 5: Topological metric values for integration ($C$), regularity ($V$), modularity ($Q$), hierarchical complexity ($R$), characteristic path length ($L$) and network entropy ($H$) against number of strongest connections kept. Curves relate to network models as indicated in the legend (bottom right).
Figure 6: Positive (negative) values indicate the contrasted distributions exhibit the relationship provided in the legend (or its opposite). Zero indicates p-value insignificant at 5% level. a) The hierarchical complexity, $R$ (blue), compared with network entropy, $H$ (yellow). b) The clustering coefficient, $C$ (blue), compared with characteristic path length, $L$ (yellow).
Figure 7: These plots demonstrate the topological characterisation of network models by clustering coefficient, $C$, degree variance, $V$, modularity, $Q$ and hierarchical complexity, $R$. Grey lines indicate mean values of the weighted complex hierarchy model with increasing light grey indicating increasing strength parameter from $s = 0$ (E-R random) in steps of 0.05 up to $s = 0.75$. Red errorbars indicate values of the Albert Barabasi scale-free model. Blue errorbars indicate values of the Watts Strogatz small-world model with increasingly light blue indicating increasing proportion of edges being randomly rewired.
Figure 8: The z-statistics of distributions with significant differences from a Wilcoxon rank sum test. Positive (negative) values indicate the contrasted distributions exhibit the relationship provided in the legend (or its opposite). Zero indicates p-value insignificant at 5% level.
Comparison of the Randomly Hierarchical null model with strength parameter 0.2 (yellow) and EEG dWPLI networks (blue)

Figure 9: Clustering coefficient, $C$, degree variance, $V$, modularity, $Q$, and complexity, $R$, against density of binarised weighted networks.