Control and controllability of microswimmers by a shearing flow

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Abstract

With the continuing rapid development of artificial microrobots and active particles, questions of microswimmer guidance and control are becoming ever more relevant and prevalent. In both the applications and theoretical study of such microscale swimmers, control is often mediated by an engineered property of the swimmer, such as in the case of magnetically propelled microrobots. In this work, we will consider a modality of control that is applicable in more generality, effecting guidance via modulation of a background fluid flow. Here, considering a model swimmer in a commonplace flow and simple geometry, we analyse and subsequently establish the efficacy of flow-mediated microswimmer positional control, later touching upon a question of optimal control. Moving beyond idealised notions of controllability and towards considerations of practical utility, we then evaluate the robustness of this control modality to sources of variation that may be present in applications, examining in particular the effects of measurement inaccuracy and rotational noise. This exploration gives rise to a number of cautionary observations, which, overall, demonstrate the need for the careful assessment of both policy and behavioural robustness when designing control schemes for use in practice.

1 Introduction

Synthetic microscale swimmers are becoming increasingly prevalent, with the canonical biological examples of spermatozoa and bacteria informing and motivating the development of modern artificial microrobots and active particles. A natural question that arises, particularly for engineered swimmers with targeted applications, is that of guidance and control. Examples to date have included the navigation of an intricate environment or microdevice [1, 2], the delivery of therapeutic cargo for medical interventions [3, 4, 5], or hydrodynamically disguising the motions of another swimmer [6]. In particular, as continued developments in fabrication and biohybridisation advance the potential utility of microswimmers, the task of directing a microswimmer along a path or towards a desired goal becomes ever more pertinent.

Realising the seemingly simple objectives of guidance and control, however, can be complex. Much of this complexity is determined by the modality of swimmer control or propulsion, which can range from magnetically propelled microrobots to acoustically driven synthetic swimmers [7, 8]. As such, there are many different strategies and modalities for realising swimmer control. Magnetotactic microswimmers, such as the rotating magnetic helices that are the subject of many recent studies [9, 10, 11, 12], naturally admit simple control via modulation of the applied magnetic field, whilst the recent work of Khalil et al. [13] exploits the magnetism-driven rotation of a microrobot to drive fluid flow, in turn directing passive microbeads. Other approaches consider swimmers as internally controlled or ‘smart agents’, capable of effecting changes in their own direction or speed in response to their environment. These self-regulated swimmers, often coupled with adaptive or learned strategies [11, 2, 6], are able to navigate changing multiswimmer environments whilst achieving sophisticated goals, for example, the hydrodynamic cloaking of another swimmer through maintenance of particular relative positioning [6]. However, methods tailored to guiding magnetic or smart sensing swimmers are inherently restricted to these subclasses of microswimmer, with neither of these approaches being suitable for controlling the movement of simpler, non-magnetic, synthetic swimmers or biological cells, such as bacteria and spermatozoa.

A mechanism that avoids these restrictions is that of flow control, in which the guidance of a generic microswimmer is realised via the modulation of a generated fluid flow. Indeed, the field of microfluidics has advanced rapidly over the past two decades, with finely calibrated changes to fluid flows now able to be effected on timescales of milliseconds [15]. Whilst this method of control is naturally restricted to contexts in which flow can be applied, it is otherwise universally applicable to microswimmers, in particular not requiring on-board sensors or magnetic components. Despite this broad applicability, this flow-based approach has, to the best of our knowledge, only briefly been explored [10, 13]. However, removed from the context of control, the
general interactions of swimmers with background flows have been extensively considered, both experimentally and theoretically \cite{17, 18, 19, 20, 21, 22}, with observations including the remarkable upstream rheotaxis of spermatozoa when confined \cite{23, 24, 25}. Such studies build upon classical works of the 20th century for the dynamics of bodies in flow, with perhaps the best known being the study of Jeffery \cite{26} on the intricate orbits of passive ellipsoids. Recently, it has been noted that Jeffery’s results can be applied in more generality, with Jeffery’s orbits accurately representing the time-averaged dynamics of both a flagellated swimmer and bodies with particular symmetries \cite{16, 27}. Thus, the consideration of the motion of an ellipsoid in flow presents itself as a viable first avenue for the exploration of the guidance of more complex swimmers. Therefore, with the governing equations of such an idealised swimmer model being sufficiently simple so as to be amenable to analysis and ready exploration, the first objective of this work will be to formulate the control system describing a flow-guided ellipsoid, from which we will formally evaluate the theoretical controllability properties of the swimmer.

Controllability addresses the theoretical existence of controlled trajectories between given initial and target states of a system. Once this existence is established, one can seek to find the best trajectory with respect to the minimisation of a certain objective function, such as time, using optimal control theory. The first theoretical studies of microswimmer control, as well as many subsequent works, considered swimmers that were set in motion by their own deformation, whether for a general deformable body \cite{28, 29, 30, 31, 32}, or a slender robot comprised of rigid links connected by elastic joints \cite{33, 34, 35, 36}. This latter type of swimmer has also been more recently analysed when driven by an external magnetic field, from the points of view of both controllability \cite{37, 38, 39} and optimal control \cite{40}. Further to laying a formal mathematical foundation that fosters understanding of microswimming at a theoretical level, the control-theory standpoint can provide both guidance for designing microdevices and an optimal way of controlling them. The present analysis will therefore seek to establish the viability of flow control as a means of guidance in the simple but broadly applicable considered scenario, whilst motivating and informing future considerations of complex fluid and swimmer control problems.

Whilst theoretical results of swimmer controllability would serve to address the abstract and idealised problem of flow-mediated control, it will only go so far in evaluating more practical degrees of suitability. In particular, in order for any method of control to be of practical efficacy, a certain level of robustness must be established. For instance, any approach that prescribes a policy of flow control to elicit a particular swimmer behaviour should be robust to inaccuracies in the measured geometry of the swimmer, with the theory of Jeffery’s orbits highlighting the impacts of shape on the response of even simple bodies to simple flows. Another significant factor in many microscale processes is diffusive noise, for example, the rotational diffusion of bacterial swimmers that invariably leads to non-straight swimming paths in a quiescent environment. This robustness is naturally present in feedback-control mechanisms of guidance, such as smart and programmable agents \cite{11, 14, 2, 6}, but is not guaranteed in the broader context of microswimmer control. Hence, as the second objective of this study, we will investigate the robustness of a range of both designed and optimised flow control policies to variations in swimmer shape, as might be particularly pertinent when considering a heterogeneous population, in addition to the interactions between flow and diffusive effects.

Hence, we will proceed by formulating the simple control system that captures the planar dynamics of an idealised swimmer in a background shear flow. Applying well-known tools of control theory, we will analytically interrogate the local controllability around trajectories and minimal-time control of the system, additionally providing numerical exemplars to illustrate the key features of the coupled flow-control system. With these theoretical explorations in hand, we will move to evaluate the impacts of shape on the design of control policies, including the resulting complex range of behaviours exhibited by an immersed swimmer. Further, we will incorporate the effects of rotational diffusion into our swimmer model, exploring by explicit example the relation between applied flow and the measured diffusivity of a swimmer. In doing so, we will seek to evaluate the plausibility of flow-mediated guidance as a mechanism for practical microswimmer control.

2 A simple flow-driven control system

We will consider the planar motion of an active prolate spheroid in a background Newtonian shear flow at zero Reynolds number, with this idealised axisymmetric swimmer being a common model for a range of biological and artificial microswimmers, applicable even to the beat-averaged dynamics of a flagellated swimmer or non-axisymmetric bodies \cite{16, 27}. We parameterise its shape via the aspect ratio \( r \geq 1 \), the dimensionless ratio of its major to minor axes, with a sphere corresponding to \( r = 1 \) and an elongated ellipsoid to \( r > 1 \). In more generality, this may be replaced with an effective aspect ratio where justified, with the analysis that follows otherwise unchanged. As an explicit example, Walker et al. \cite{16} calculated that an effective aspect ratio of \( r \approx 5.49 \) for a model \textit{Leishmania mexicana} in shear flow, a flagellated microswimmer that, despite periodic yawing due to its flagellar motion, can be well represented by the simple spheroid model. Written with respect to the laboratory frame, with the plane containing the swimmer and its major axis being described
Consider a constant shear rate $\bar{\omega}$, that we will call the reference control. Given an initial state $\bar{z}_0 = [\bar{x}_0, \bar{y}_0, \bar{\theta}_0]^T$ for the swimmer, let $\bar{z} = [\bar{x}, \bar{y}, \bar{\theta}]^T$ be the solution of System (2) starting at $\bar{z}_0$ with the constant rate $\bar{u}$, on the time interval $[0, T]$; we will call $\bar{z}$ the reference trajectory. In this section, we address the controllability properties of the flow-driven swimmer in a local sense around the reference trajectory $\bar{z}$, trying to answer the following question, which constitutes an important theoretical feature for a controlled system: do “small” variations of the reference control $\bar{u}$ allow one to target any position and orientation within a neighbourhood of the reference trajectory? A particular situation of interest is the case of $\bar{u} = 0$, in which we seek to control the position and orientation of the swimmer by generating a shear flow of “small” amplitude.
The formal definition of this notion can be stated as follows [46]: System (2) is said to be locally controllable around the trajectory \((\bar{z}, \bar{u})\) if, for all \(\varepsilon > 0\), there exists neighbourhoods \(\mathcal{V}\) and \(\mathcal{W}\) of \(z_0\) and \(\bar{z}(T)\) such that, for all \(z_0\) in \(\mathcal{V}\), and \(z_1\) in \(\mathcal{W}\), there exists a bounded control \(u\) defined on the time interval \([0, T]\), satisfying \(|u(t) - \bar{u}| \leq \varepsilon\) for all \(t \in [0, T]\), and such that the solution of System (2) starting at \(z_0\) satisfies \(z(T) = z_1\).

To study the local controllability of System (2), we consider the linearised control system of System (2) around \((\bar{z}, \bar{u})\), that reads

\[
\dot{z} = A(t)z + B(t)u(t),
\]

with \(A = \frac{\partial f(x,u)}{\partial x}(\bar{z}, \bar{u})\) and \(B = \frac{\partial f(x,u)}{\partial u}(\bar{z}, \bar{u})\). It is well known (see for instance [46, Theorem 3.6]) that the nonlinear system is locally controllable around \((\bar{z}, \bar{u})\) if the linearised system is (globally) controllable. To check if this is the case, we can apply a simple algebraic “Kalman-type” condition for time-varying linear control systems [47, 48]: let \(B_1(t) = B - AB\) and \(B_2(t) = B_1 - AB\); then System (3) is controllable in time \(T\) if and only if there exists \(t \in [0, T]\) such that the matrix \((B(t)|B_1(t)|B_2(t))\) is invertible.

The determinant \(\Delta\) of \((B(t)|B_1(t)|B_2(t))\) can be obtained through straightforward calculations:

\[
\Delta = \frac{1}{32} \bar{u} \left( E \cos 2\bar{\theta} - 1 \right) \left[ 2\bar{u}E^2 \left( \cos 4\bar{\theta} + 4 \cos \bar{\theta} - 1 \right) + (E - 1) \left\{ 4 + 2 \cos \bar{\theta} + E(1 + \cos \bar{\theta}) \right\} \right].
\]

Let us first assume that the reference control is null \((\bar{u} = 0)\). It is worth noticing that, in this case, the reference trajectory explicitly reads \(\bar{z}(t) = [\bar{x}_0 + t \cos \bar{\theta}_0, \bar{y}_0 + t \sin \bar{\theta}_0, \bar{\theta}_0]^T\), i.e. the swimmer goes in a straight line in the direction given by the initial angle \(\bar{\theta}_0\). Moreover, we immediately see that \(\Delta = 0\), hence the linearised system is not controllable. Although the fact that the linearised system is not controllable does not mean that the swimmer is not locally controllable around the straight reference trajectory \(\bar{z}(t)\), it suggests that some directions in the state space are difficult to attain in a neighbourhood of the reference trajectory by using a variable shear rate of small amplitude.

We may, however, seek to control only the position \((x, y)\), rather than both the position and orientation \((x, y, \theta)\). For this purpose, we define partial local controllability as in [49, 50] by restricting controllability requirements to the variables \(x\) and \(y\). A similar analysis on the linearised system can be conducted to evaluate the local partial controllability properties of the system projected onto the \(xy\)-plane [50]. For \(\bar{u} = 0\), the determinant of the partial Kalman matrix, defined in the same manner as above, is equal to

\[
\frac{1}{2} (\bar{y}_0 + t \sin \bar{\theta}_0) \cos \bar{\theta}_0 \left( E \cos 2\bar{\theta}_0 - 1 \right).
\]

We readily conclude that, if \(\bar{u} = 0\), the swimmer is partially locally controllable around the trajectory \(\bar{z}\) for the position variables \(x, y\), unless \(\cos \bar{\theta}_0 = 0\) or both \(\sin \bar{\theta}_0 = 0\) and \(\bar{y}_0 = 0\).

We now turn to the general case with \(\bar{u} \neq 0\), in which the reference trajectories are known to be Jeffery orbits. Given a time interval \([0, T]\), the determinant \(\Delta\) of the full system is therefore an analytic function of time and, thus, cannot be identically equal to zero on \([0, T]\). Thus, the algebraic Kalman condition is satisfied, so we deduce that the linearised system is controllable and that System (2) is locally controllable, for both position and orientation, around any reference trajectory.

Overall, the analysis of the linearised system shows that we have broad local controllability of the swimmer position, except in the two particular cases noted above: namely, if the reference control \(\bar{u}\) is zero and either \(\cos \bar{\theta}_0 = 0\) (swimmer perpendicular to the shear direction) or both \(\sin \bar{\theta}_0 = 0\) and \(\bar{y}_0 = 0\) (swimmer moving along the midline). On the other hand, controllability of the combined positional and rotational dynamics is also possible, provided the slightly more restrictive condition \(\bar{u} \neq 0\) holds. Whilst this result is of a local nature, it suggests that long time global controllability can be achieved in most cases, without requiring elaborate trajectories that avoid unreachable regions.

We illustrate these results by attempting to numerically visualise the reachable space. In order to do so, we plot some trajectories corresponding to a range of controls taken “around” the reference control \(\bar{u}\). More precisely, let \(u\) be a control of small amplitude and \(z\) the trajectory associated to the control \(\bar{u} + u\). If the swimmer is locally controllable, the differences \(z - \bar{z}\) for several different controls \(u\) can be expected to roughly cover a neighbourhood of the origin. On the contrary, when controllability does not hold, some regions will likely appear unattainable.

The results for \(\bar{u} = 0\) are featured in Figure 2. The expression for the controls used for the simulations is given in Appendix C. Panels (a) and (b) show that a neighbourhood of the origin is attained in the \(xy\)-plane, but seemingly not in the full state space \((x, y, \theta)\). Panels (d) and (e) illustrate the lack of partial controllability that occurs in the cases \(\cos \bar{\theta}_0 = 0\) (swimmer perpendicular to the shear direction) or both \(\sin \bar{\theta}_0 = 0\) and \(\bar{y}_0 = 0\) (swimmer parallel to the shear direction and located on the midline).

Figure 2 features the results for \(\bar{u} = 20\). In agreement with the controllability analysis, it suggests that nonzero reference control allows the recovery of local controllability in the full state space. It is worth keeping in mind the local nature of this result. The size of the neighbourhood of the origin where controllability holds
FIG. 2: Illustration of the local controllability properties on swimmer trajectories. In panels (a), (b), (d) and (e), the components of $z - \bar{z}$ are shown for 100 controlled trajectories, with controls taken around $\bar{u} = 0$ as realisations of the randomised control featured in Appendix C. The reference trajectory $\bar{z}$ and the controlled trajectories are shown as red and grey curves, respectively, in the upper right inset in panel (a) and are approximately indistinguishable at the resolution of this figure. (a) A neighbourhood of the origin is covered in the plane $(x, y)$, but seemingly not in the full state space (b), with the initial condition here set to $z(0) = [0, 0, \pi/4]^T$. In panels (d) and (e), the initial conditions are $z(0) = [0, 0, \pi/2]^T$ and $[0, 0, 0]^T$, respectively, with the swimmer appearing to not be partially controllable for $(x, y)$ from these initial conditions. In panels (c) and (f), 100 trajectories are plotted, with the reference control now $\bar{u} = 20$ and the initial condition again set to $z(0) = [0, 0, \pi/4]^T$. In (c), the reference trajectory $\bar{z}$ is shown in red, with controlled trajectories being indistinguishable from the reference trajectory at the resolution of this figure, which underlines the local nature of our controllability results. Plotting the difference $z - \bar{z}$ between the reference trajectory and the controlled trajectories of (c) in (f), we observe that the trajectories cover a neighbourhood of the origin in the state space.
may be rather small compared to the characteristic length of the system. Figure 2 illustrates this fact, showing that the controlled trajectories deviate little from the reference trajectory in our simulations.

Knowing that the swimmer is locally controllable, let us address the more practical situation in which we aim to reach a given target position \((x_f, y_f)\), by solving the minimal time control problem:

\[
\begin{align*}
\min_{u} &\quad T \\
\text{s.t.} &\quad \dot{z} = f(z, u) \text{ on } [0, T], \\
&\quad u \in [u_{\min}, u_{\max}], \\
&\quad z(0) = z_0, \\
&\quad x(T) = x_f, y(T) = y_f,
\end{align*}
\]

(5)

where we assume \(u_{\min} < 0\) and \(u_{\max} > 0\), therefore allowing the flow to be switched off.

Let us define the adjoint state vector \(p = [p_x, p_y, p_\theta]^T\) and write the Hamiltonian associated with (5):

\[
H = p^0 + p \cdot f(z, u) = H_0 + uH_1,
\]

(6)

with \(p^0 \in \mathbb{R}, H_0 = p_x \cos \theta + p_y \sin \theta\) and \(H_1 = p_x y + \frac{1}{2} p_\theta (-1 + E \cos 2\theta)\). By virtue of the Pontryagin maximum principle [51], the optimal control \(u\) must maximize the Hamiltonian at all times. As expected in a minimal time control problem, this shows that \(u\) is made of “bang arcs” for which \(u\) is equal to \(u_{\min}\) or \(u_{\max}\) depending on the sign of \(H_1\), and “singular arcs” when \(H_1\) vanishes on a subinterval of \([0, T]\). The value of \(u\) on potential singular arcs can be determined by computing \(H_1 = 0 = \{\{H_1, H_0\}, H_0\} + u\{\{H_1, H_0\}, H_1\}\), where we used the Poisson bracket notation [52]. For the optimal control problem (5), we serendipitously have \(\{\{H_1, H_0\}, H_0\} = 0\) (and \(\{\{H_1, H_0\}, H_1\}\) is not identically zero), so \(u\) always vanishes on the singular arcs.

Overall, \(u\) should follow the command law

\[
u(t) = \begin{cases} 
    u_{\min} & \text{if } H_1 < 0, \\
    u_{\max} & \text{if } H_1 > 0, \\
    0 & \text{if } H_1 = 0.
\end{cases}
\]

(7)

Thus, the optimal trajectory will be a concatenation of Jeffery orbit portions (when \(u\) is equal to \(u_{\min}\) or \(u_{\max}\)) and line segments when \(u = 0\). This is nicely illustrated by the optimal trajectories numerically computed in Fig. 3. In Appendix D, we additionally prescribe the final orientation of the swimmer, noting that the control is once again given by Eq. (7) and explore the broad range of computed optima as a function of this final orientation.

4 Practical robustness of flow control

4.1 Evaluating robustness necessitates care

With partial controllability results established, evaluating the robustness of swimmer behaviour under a flow policy to error or uncertainty in measurements remains a necessary precursor to designing and using flow control in practice. As a concrete example, consider the task of guiding a swimmer from \((x, y) = (0, 0)\) to \((x, y) = (1, 2)\), where the initial orientation of the swimmer is fixed and a background shear flow may be prescribed as detailed in Section 2. For a given swimmer shape, here encoded in the aspect ratio \(r\), one may construct a policy of flow control that effectively realises the swimmer target position, here minimising travel time in line with the controllability analysis, having taken \(u_{\min} = -1\) and \(u_{\max} = 1\). An example such policy is shown inset in Fig. 3, tailored to a swimmer with aspect ratio \(r = 1.5\). For a range of swimmer shapes under this policy we also show the trajectories in the \(xy\)-plane, when \(r \in [1, 2]\). The trajectory for \(r = 1.5\), for which the policy was designed, can be seen to affect the desired movement of the swimmer. The other curves in Fig. 3 showcase the variability in path and outcome were there to be uncertainty in the measured swimmer shape, which appears to demonstrate that such differences have minimal qualitative and quantitative effects on the outcomes of the policy. From this evaluation alone, one may be tempted to conclude that the guidance of a swimmer to \((1, 2)\) from the origin is robust to measurement error in this setup, with little sensitivity of the trajectory to such error.

However, having fixed a policy and varied the swimmer shape, we now consider the effects of a variety of tailored policies on a fixed swimmer shape. This scenario can be thought of as designing a flow policy for inaccurately measured swimmer geometry, complimentary to the previous analysis, which applied a fixed policy to swimmers of various shapes. In Fig. 3b, we show the paths of a swimmer of fixed aspect ratio \(r = 1.5\) under policies of flow control designed for swimmers with shapes of \(r \in [1, 2]\). These policies, again designed to affect movement from \((0, 0)\) to \((1, 2)\) in their idealised swimmer, are again shown inset, each being qualitatively similar though with the precise timings of flow switching differing between policies. In contrast to Fig. 3a, from which robustness seems apparent, here we see significant qualitative differences in the paths taken by the
FIG. 3: Evaluating robustness of a simple control problem. (a) Fixing a flow-control policy tailored to a swimmer with $r = 1.5$ (shown inset), we show the trajectories of swimmers with shapes $r \in [1, 2]$, each starting from the origin with fixed initial orientation. We observe qualitative similarity in the trajectory endpoints and swimmer paths, suggesting that one can conclude robustness of overall behaviour to uncertainty in swimmer shape. (b) Now tailoring flow-control policies to swimmers of various shapes (shown inset), we compute the trajectories of a swimmer of fixed shape $r = 1.5$ subject to this range of policies. In contrast to (a), now apparent are significant qualitative differences between the paths, with flow policies designed for different swimmers yielding greatly altered swimmer behaviour and, thus, demonstrating a lack of robustness and the presence of marked sensitivity to measurement error in the trajectories under different policies. Hence, the consideration of a single panel in isolation may lead to unreliable assessments of robustness, with thorough investigation that varies both shape and policy therefore warranted.

swimmer under the different policies, naturally accompanied by large quantitative distinctions. In particular, large variation is present between trajectories around $y \approx 1.4$, in stark contrast to those of Fig. 3a. Thus, from consideration of this plot, one is drawn to conclude that, relatively, there is a distinct sensitivity and a lack of robustness in swimmer paths to measurement errors in the swimmer shape.

Thus, whilst both approaches may a priori be thought of as sufficient explorations to establish some practical notion of robustness or sensitivity, we have seen in this simple example that the conclusions drawn from each of them may be at odds with one another, at least when considering the whole of the swimmer trajectory. With each of these assessments therefore being incomplete and potentially misleading in isolation, this highlights that careful and thorough examination of robustness is warranted in order to draw reliable conclusions. Indeed, with this exemplar representing a very simple short-time guidance problem, this further suggests that careful examination of sensitivity from multiple viewpoints is required when considering flow-mediated swimmer control in more generality.

4.2 Non-monotonic responses to variation

The previous example explored a simple control scenario, demonstrating that care is necessitated when attempting to evaluate robustness of control to variations in swimmer shape. However, despite exhibiting large differences in swimmer paths in some cases, the relations between applied policy and swimmer shape in Fig. 3 appeared simplistic, with policies that switched off the background flow at earlier times resulting in shorter swimmer trajectories, for example. One might therefore expect that such simple rules might be applicable in more generality.

We will explore this notion via a more complex example, one in which the background flow is prescribed such that a spherical swimmer, of shape $r = 1$, undergoes a closed and symmetric trajectory, as shown in Fig. 4a in grey. For reference, the initial condition of the swimmer and the accompanying flow policy required to elicit this behaviour are given in Appendix E and are fixed throughout. In order to explore the effect of swimmer shape on swimming path, we examine the behaviours of swimmers with different shapes from this initial example, with the trajectories shown as solid black curves in Fig. 4b for $r \in [1, 2]$, where the motion of each swimmer has been simulated over the interval $t \in [0, 5\pi]$. Evident is a qualitative similarity in the overall shape of the swimmer paths, though the symmetry of the $r = 1$ trajectory appears broken to varying extents.
FIG. 4: Complex, non-monotonic responses of swimmer behaviour to changes in shape. Prescribing an initial condition and fixed flow policy so as to elicit the symmetric closed trajectory in the $xy$-plane shown in (a) for a spherical swimmer ($r = 1$), we simulate the motion of a range of swimmers in this background flow, with $r \in [1, 2]$, with these trajectories shown as black curves in (b). Panels (c) and (d) highlight single trajectories in comparison to the symmetric target path, the latter being shown as a grey curve throughout the figure. (c) For $r = 1.1$, the swimmer overshoots the desired endpoint. (d) Increasing $r$ further to 1.7 yields an undershoot in final position, thereby highlighting the non-monotonic response of a swimmer behaviour to changes in shape.

The black arrowhead shows the direction of traversal of the swimmers around their trajectories, with the initial position shown as a filled grey circle.

degrees for $r > 1$. Isolating a single trajectory, again shown in black, in [Fig. 4] we see that the swimmer with $r = 1.1$ has approximately followed the path of the spherical example, though has overshot the target endpoint by some considerable margin due to increased initial rotation in the background flow compared to the spherical counterpart. Extrapolating from this minimal comparison, reassured by the simple relation between shape and trajectory seen in Fig. 3a, one might expect that further elongating the swimmer may yield successively greater overshoots.

However, consideration of another elongated swimmer in Fig. 4d, with $r = 1.7$, reveals a different narrative, with the trajectory significantly departing from the symmetry of the spherical base case and substantially undershooting the target endpoint. Here, the fast initial rotation of the swimmer has resulted in a path that only briefly enters $y > 0$ in the first half of the motion, whilst in the latter portion of the motion the non-spherical swimmer is oriented approximately $30^\circ$ away from the spherical body (not shown), yielding the large difference in final position. Thus, with this undershoot not compatible with the tentative observations drawn from Fig. 4c, this simple relation between shape and dynamics in flow does not hold. Indeed, the lack of monotonicity in the observed relationship between swimmer geometry and the flow precludes any such simple rule for predicting, in generality, the manifestations of geometric variation or uncertainty in the observed behaviours of flow-guided swimmers. We remark, however, that simple quantifications may exist for certain features or subsets of the dynamics. In particular, a pertinent example of this is the evolution of the swimmer orientation $\theta$, which, if the swimmer is subject to constant shear flow, follows a canonical Jeffery’s orbit with a period that is monotonically increasing with respect to swimmer eccentricity, whilst the corresponding trajectory in the $xy$-plane may nevertheless be intricate and defy similar analysis.

4.3 Flow-amplification of diffusive effects

Thus far we have considered the trajectories of swimmers in the absence of any diffusive effects, though many potential biological swimmers, such as motile bacteria, warrant the inclusion of such rotational noise. We therefore modify the orientation dynamics, replacing the ordinary differential equation of Eq. (2) for $\theta$ with the stochastic differential equation

$$d\theta_t = \frac{1}{2} u(t) (-1 + E \cos 2\theta_t) \, dt + \sqrt{D} \, dW_t,$$

where $W$ denotes a Wiener process and $D$ is a diffusion coefficient that will characterise the contribution of rotational noise. Here, $D$ is constant, therefore there is no distinction between treating Eq. (8) as an Ito or Stratonovich stochastic differential equation, thus solution via the Ito algorithm of the Euler-Maruyama method, as used here, is legitimate even though Stratonovich calculus is often advocated for physical systems.

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In order to investigate the interactions between a guiding flow and rotational diffusion, we consider the
dynamics of a spherical swimmer, taking \( r = 1 \), with and without a background flow that is prescribed so as to
yield a oscillatory trajectory, as shown in Fig. 5a. This flow is given by

\[
u(t) = \begin{cases} 
  +1, & 2n\pi \leq t < (2n + 1)\pi, \\
  -1, & (2n + 1)\pi \leq t < (2n + 2)\pi,
\end{cases} \quad n \in \mathbb{Z},
\]

where we simulate the motion for \( t \in [0, 5\pi] \) from the initial condition \((x(0), y(0), \theta(0)) = (0, 0, \pi/4)\). In
particular such a flow is a pertinent exemplar in the consideration of swimmer behaviour and guidance since
consecutive bang arcs can feature in solutions to minimal time control problems, as highlighted by equation
7.

In Fig. 5a, we observe that the swimmer’s trajectories in the absence of noise are indeed the expected
oscillatory and straight paths, corresponding to the cases with and without the guiding flow, respectively.
Introducing diffusive effects, in Fig. 5b, we note the progressive divergence of the swimmer paths away from
the noise-free case, with these effects clearly evident in Fig. 5c and Fig. 5d. Indeed, evident in these latter panels
is a wider spread of the final swimmer positions when the flow is present compared to motion in the absence
of flow. In order to assess this quantitatively, in the separate cases of motion with and without flow we compute
a measure of the spread of the final positions of the swimmers for a fixed \( D \), defined explicitly as

\[
\sigma_c = \sqrt{\text{Var}[x_c] + \text{Var}[y_c]},
\]

where the subscripts denote evaluation at the end of the path. Simulating 1000 trajectories for a range of
considered \( D \), this measure is plotted in Fig. 5e, from which a clear distinction between the cases of flow and
no flow is evident. In agreement with qualitative assessments of Fig. 5a-d, the effect of the flow is tantamount
to increasing the diffusion coefficient of the swimmer, in this case amplifying diffusive effects by approximately
65%. Of note, though not shown, this conclusion is qualitatively unchanged when considering motion away
from the axis of the shear at \( y = 0 \), as well as when considering a flow with constant shear rate.

Therefore, whilst the inclusion of rotational diffusion might be expected to negatively impact on robustness,
we have observed that the presence of a background guiding flow in this case enhances the effects of this noise,
with robustness therefore necessitating an evaluation that includes the details of the controlling flow and is
unable to be extrapolated from considerations swimming in a quiescent environment.

5 Discussion

In this work we have considered a simply posed problem of guiding a model microswimmer with a modifiable
shear flow, examining various desirable control properties of the associated dynamical system via both theoretical
and numerical explorations. By direct simulation, we have seen that uncertainty in swimmer morphology can
lead to widely varying behaviours under the same imposed flow conditions, with the evaluation of practical
robustness also requiring a thorough analysis to afford confidence in a regime of control. Further compounding
this potential lack of reliability in the presence of uncertainty was the existence of complex non-monotonic
responses to simple variations in swimmer geometry, which was here quantified by a single parameter for an
idealised class of swimmer morphologies. Whilst the extension of this investigation to swimmers with more
intricate geometry represents a pertinent direction for future work, the cautionary notes of this work serve to
exemplify the care that will likely be needed in more complex and general settings, with a perhaps surprising
degree of sensitivity to uncertainty found in the explorations presented in this study.

Despite these practical concerns, which may be significantly mitigated by confidence in swimmer morphology
and flow control, theoretical analysis has demonstrated that flow control represents a plausible method for
realising microswimmer guidance, having established the local controllability of the positional and rotational
dynamics of the swimmer, with a notable exception when the reference flow is zero. In that latter case, numerical
investigation of the full nonlinear system reveals apparently inaccessible regions of the state space. With focus
therefore drawn to controlling swimmer position, which nonetheless could represent a feat of significant utility
in practice, we addressed the minimal time control problem, establishing that the optimally chosen flow rate is
either extremal or zero. This simple result provides helpful guidance for optimal control design in this context
and illustrates the general principle of optimising flow-mediated guidance.

Lastly, we considered the effects of rotational noise, which is of pertinence to helical small-scale swimmers
such as bacteria. In particular, augmenting our simple control system with unbiased rotational noise, we
quantitatively assessed the interactions between flow and diffusive effects on swimmer position following a
prescribed policy of shear control. Computing a lengthscale of effective dispersion, we observed that the
background flow acts to increase the effects of diffusive noise, resulting in a marked increase in the variability
of swimmer location in this case. Therefore, we are led to conclude that assessments of the impacts of noise
on swimming in the absence of flow cannot naively be extrapolated to cases where flow is present. As such,
FIG. 5: Amplification of rotational noise by flow. With the diffusion parameter $D$ fixed in each of the panels (a)–(d), we simulate the motion of a spherical swimmer from a prescribed initial condition for $t \in [0, 5\pi]$, prescribing either no flow or the background flow given in Eq. (9), with trajectories shown as solid and dot-dashed lines in each of these cases, respectively. With $D = 0$ in (a), we recover noise-free trajectories, straight in the absence of flow and oscillatory when the guiding flow is present. Increasing the level of noise, we observe an expected increase in the spread of swimmer trajectories across panels (b)–(d), with the endpoints of trajectories shown as blue and red dots for the no-flow and with-flow cases, respectively. Noting a somewhat wider spread of swimmers in the presence of background flow, particularly evident in (d), we plot a measure of the spread of the final swimmer position, defined in Eq. (10) and denoted by $\sigma_e$, for a range of values of the diffusion parameter in (e). Now quantitatively, we observe a marked increase in the effective diffusion of the swimmer when the background flow is present, with the flow amplifying diffusive effects by approximately 65% in this case. In panels (a)–(d), 100 realisations of the swimmer motion of presented, whilst the quantitative measure $\sigma_e$ of (e) is computed using 1000 realisations in each case.
careful evaluation of diffusive effects is warranted in general, with the potential for complex interplay between multiple aspects of the control system. Though we have considered prescribed controls in this work, addressing the impacts of noise may also warrant the use of control-feedback loops, in which the control is adjusted in real time to account for the measured impacts of noise or other complicating factors [6, 54].

The evaluations and assessments presented in this work have been naturally restricted to a simple framework, though the implications of the findings apply more generally as cautionary examples that highlight a lack of universal robustness that has been without thorough investigation. In practice, the neglect of translational Brownian noise and inertial effects in this work limits the presented analysis to swimmers on the microscale, with, in general, larger swimmers experiencing inertial effects and smaller swimmers being significantly impacted by translational noise. Despite these restrictions, this class of swimmers is broad, including canonically studied examples such as motile bacteria and spermatozoa. This work may be extended to consider more complex systems, for example, multiswimmer environments and confined domains, with the latter already having been associated with effecting long-time changes in behaviour of swimmers in shear [16]. Sophisticated flows and three-dimensional motion additionally represent promising avenues for further exploration, with the local controllability analysis being readily generalisable to wider contexts. Indeed, with established robustness and feasibility being a desirable, if not necessary, precursor to practical applications of swimmer guidance, such an analysis could be extended to other modalities of microswimmer control, with simple acoustically guided swimmers expected to give rise to a similarly tractable control system.

In summary, we have posed a simple control system for flow-mediated swimmer guidance, establishing the theoretical controllability of the planar positional dynamics of a spheroidal swimmer in a shear flow. Numerical explorations of the model dynamics have revealed a surprising sensitivity of swimmer motion in flow to uncertainty, with the effects of rotational diffusion in particular being amplified by the presence of flow. Nevertheless, utilising both practical and theoretical evaluations of control and controllability, this analysis has suggested the potential efficacy of a widely applicable mechanism of swimmer guidance, whilst highlighting the importance of thorough assessments of robustness, control, and controllability.

Data Access

The data used and generated in this paper is freely available from the University of Oxford Research Archive (ORA) via the link http://dx.doi.org/xxx/xxx. For the purposes of review we have generated a temporary link https://cloud.maths.ox.ac.uk/index.php/s/tJZ5HZzA3xoFQ6X, accessible with password: MoreauSubmission

Author’s Contributions

CM carried out the controllability analysis, drafted and critically revised the manuscript, and designed the study. KI and EAG designed the study and critically revised the manuscript. BJW carried out the robustness analysis, drafted and critically revised the manuscript, and designed the study.

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A Controlling Leishmania mexicana and the impacts of flow response times

As a particular example, we consider the control of a model Leishmania mexicana cell, with Walker et al. [16] reporting an effective aspect ratio of $r = 5.49$. We will also make use of typical Leishmania mexicana length and velocity scales, taking the cell body length to be $10 \mu m$ and the propulsive velocity to be $20 \mu m s^{-1}$ [53]. Of particular note, the computation of this motion is performed by simulating the general dimensionless system Eq. (2) with quantities subsequently rescaled to reflect Leishmania mexicana length and velocity scales. In order to illustrate the effects of the response times of flows in practice on swimmer control, we specify three similar flow controls, each of which seeks to transition between specified shear rates, which we constrain to be
FIG. 6: A model flagellated swimmer, *Leishmania mexicana*, and its control by flow. (a) We show a caricatured swimmer with typical body length of 10 µm, which would swim in approximately the direction of the black arrow in the absence of external flow. Shown in colour are the trajectories of the swimmer under various flow controls, with red dots corresponding to the times at which the shear rate of the flow is changed. Black dots highlight the position of the swimmer after 1 s. (b) The controls used to effect the trajectories shown in (a), each corresponding to different flow response times, which are given in the legend. The shear rate begins to switch at the times indicated by the red dots, with the flows responding over times ranging from 1 ms to 50 ms. Irrespective of the response time of the setup, the background flow is able to enact significant change in the swimmer path, with all trajectories ending in qualitatively similar locations after 1 s. All quantities are dimensional in this figure.

at most $10 \text{s}^{-1}$. These controls differ in the time taken for each to achieve the intended shearing rates after it is prescribed, with the sharpest control switching between shear rates within 1 ms and the least responsive control transitioning over a period of 50 ms. These piecewise linear controls are illustrated in Fig. 6b, with flow switching initiated at the points marked with red dots. Shown in green is the least responsive control, which we see indeed transitions most gradually, in contrast to the near-instantaneous switching of the blue and yellow curves, which correspond to response times of 1 ms and 10 ms, respectively.

The corresponding paths, as well as the caricatured morphology of *Leishmania mexicana*, are shown in Fig. 6a, from which we see a significant deviation of the swimmer from its otherwise straight rightward path due to the flow. Perhaps surprisingly, the range of considered flow response times realise approximately the same swimmer location at the end of the motion, suggesting that the impacts of practical flow response times on the feasibility of control are relatively minor, in agreement with the observations of Walker et al. [16]. Of particular note, however, is that flow response times may be readily incorporated into any numerical optimisation or control design problems, so that swimmer control may be reliably modelled and realised even if switching times are not near-instantaneous, unlike those of Section 3 and Section 4.

B Deriving the equations of motion

The linear velocity $U$ and angular velocity $\omega$ of a force and torque-free prolate spheroid with unit major axis in a general flow $u^\infty$ are given by the Faxén laws, which, following Kim and Karrila [43], may be stated as

$$U = \frac{1}{2e} \int_{-e}^{e} \left[ 1 + (e^2 - \xi^2) \frac{1 - e^2}{4e^2} \nabla^2 \right] u^\infty(x) \, d\xi$$

and

$$\omega = \frac{3}{8e^2} \int_{-e}^{e} (e^2 - \xi^2) \nabla \times u^\infty(x) \, d\xi + \frac{3}{4e(2 - e^2)} \int_{-e}^{e} \left( e^2 - \xi^2 \right) \left( 1 + (e^2 - \xi^2) \frac{1 - e^2}{8e^2} \nabla^2 \right) d \cdot (e^\infty(x) \cdot d) \, d\xi.$$  

(12)

Here, the unit vector $d$ points along the long axis of the ellipsoid, making an angle $\theta$ to the $x$ axis of the laboratory frame, $e^\infty = [\nabla u^\infty + (\nabla u^\infty)^T]/2$ is the rate of strain tensor of the background flow, and $e = \sqrt{1 - 1/r^2}$ is the eccentricity of the spheroid. The variable of integration $\xi$ parameterises the major axis of the ellipsoid, with the position $x$ on this axis being given explicitly by

$$x(\xi) = x_c + \xi d, \quad \xi \in [-e, e],$$

(13)

where $x_c$ is the location of the centre of the ellipsoid. In the case of planar motion, we may identify $x_c = xe_x + ye_y$, as in Section 2.
These equations can be applied to general background flows, generating a control system akin to that of Eq. (2) though with complexity naturally inherited from the form of the background flow. For the case of the linear shear flow considered in this work, as given in Eq. (1), these integrals can be readily computed by hand, yielding

\[ U = u(t)(x_e \cdot e_y)e_x = u(t)y e_x, \quad \omega = \frac{1}{2} u(t) \left( \frac{e^2 - 2e^2 \cos 2\theta - 1}{2} \right) e_z, \]  

(14)

having written \( d = \cos \theta e_x + \sin \theta e_y \) and defined the out-of-plane unit vector \( e_z \) to complete the right-handed orthonormal basis \( \{e_x, e_y, e_z\} \). Differentiating this expression for \( d \) with respect to time and applying the standard relation \( d = \omega \wedge d \), we arrive at the angular evolution equation

\[ \dot{\theta} = \frac{1}{2} u(t) \left( \frac{e^2 - 2e^2 \cos 2\theta - 1}{2} \right). \]  

(15)

This reduces to the expression for the angular component of the swimmer motion in Eq. (2), noting that the shape parameter \( E \) may be written in terms of the eccentricity \( e \) via \( E = e^2/(2 - e^2) \). By the linearity of Stokes equations, the velocity of the swimmer due to self propulsion may simply be added to \( U \), yielding precisely the control system of Eq. (2) noting that the direction of self propulsion is given by \( d \).

C Flow controls for Section 3

The controls \( u \) used for the numerical simulations of Fig. 2 are realisations of the control

\[ u(t) = \bar{u} + \frac{1}{30} \left( u_1 \sin(10\omega_1 t + \phi_1) + u_2 \sin(10\omega_2 t + \phi_2) \right), \]

where \( \bar{u} = 0 \) on panels (a), (b), (d) and (e), \( \bar{u} = 20 \) on panels (c) and (f), and \( u_1, u_2, \omega_1, \omega_2, \phi_1, \phi_2 \) are taken uniformly randomly in \([-1, 1]\).

D Controlling rotational dynamics

Despite being unable to establish as broad a local controllability result for the full system given in Eq. (2) in contrast to the reduced \((x, y)\) system, the analogue of the time minimisation problem Eq. (5) with additional restrictions on \( \theta \) may still be explored with sufficiently large bounds on the control \( u \). This augmented problem naturally inherits the potential issues and sensitivities explored in Section 4 and retains the bang-bang form of the optimal controls. Through exemplar computation, shown in Fig. 7 we see highlighted the significance of rotational restrictions on feasible swimmer trajectories, with great qualitative variation exhibited by the computed trajectories as a function of prescribed eventual orientation. Further, the computed controls, shown in Fig. 7b, highlight the larger variation in time taken to reach the endpoint of the trajectory, which here has coordinates \((x, y) = (1, 2)\) Thus, in practice, the goal of controlling the final orientation of the swimmer may need to be carefully prioritised against other objectives, such as minimising the time or energy required to realise an identified control.

E Setup and flow policy for Section 4.2

The initial condition and flow policy corresponding to Section 4.2 are given by

\[ (x(0), y(0), \theta(0)) = (6.0635, -3.0989, 2.5725), \quad u(t) = \begin{cases} +1, & t - t^* < \pi, \\ -1, & \pi \leq t - t^* < 2\pi, \\ +1, & 2\pi \leq t - t^*. \end{cases} \]

(16)

Here \( t^* = 3.5742 \) and \( t \in [0, 3\pi + 2t^*] \), yielding an approximately closed trajectory.

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FIG. 7: Prescribing final orientation in an optimal-time problem. Optimal trajectories (a) and the corresponding controls (b) for minimizing the travel time between the origin and the point (1, 2), each shown as black dots in (a). Here, with an initial swimmer orientation of $\pi/2$, indicated by the black arrow in (a), we explore the effects of prescribing various final orientations $\theta_f$, noting large variation in both the character of the resulting trajectories and their length. Correspondingly, indicated as vertical dashed lines in (b), the minimal times vary significantly with $\theta_f$. Axes in (a) are shown stretched for visual clarity and we have constrained $u(t)$ to lie between $\pm 5 \text{s}^{-1}$.

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