An intertwining between conformal dualities and ordinary dualities

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We discuss and reinterpret a 4d conformal triality recently discovered in the literature in terms of ordinary Seiberg duality. We observe that a non-abelian global symmetry is explicitly realized by only two out of the three phases. We corroborate the result by matching the superconformal index in terms of an expansion on the fugacities.

I. INTRODUCTION

The space spanned by the exactly marginal deformations, denoted as conformal manifold, is a powerful tool to explore the strongly coupled dynamics of a SCFT \[1-5\]. Recently a classification program of 4d \(\mathcal{N} = 1\) gauge theory admitting a conformal manifold passing through weak coupling has been developed in \[6,7\]. This is a starting point for finding conformal dualities among non-necessarily weakly coupled, SCFTs.

Many lagrangian descriptions of otherwise strongly coupled SCFT have been indeed obtained \[7,8\]. Furthermore, new dualities between SCFTs admitting both a lagrangian description have been obtained \[9-11\]. Such generalizations.

Motivated by this last open problem here we prove our interpretation by calculating the superconformal index and studying its behavior between the phases with the enhanced global symmetry.

We find some slices of the conformal manifold connecting pairs of weakly coupled cusps along which some discrete symmetries are preserved and we check this by charging the superconformal index with respect to these symmetries. These discrete symmetries become the Weyl subgroups of the non-abelian global symmetries in the loci where we have symmetry enhancement, for example at the weakly coupled cusps. We briefly comment on this behavior and make similar considerations about \(\mathcal{N} = 4\) SYM with Leigh-Strassler \(\gamma\)-deformation \[1\].

II. THE CONFORMAL TRIALITY

In this section we review the conformal triality found in \[1\]. This triality connects three frames with a conformal manifold passing through weak coupling. By keeping the notation of \[1\] these three frames are realized by the following field theories.

Frame A. An \(USp(6)\) gauge theory with six totally antisymmetric tensors \(A_{i=2,...,7}\) and superpotential

\[
W = \sum_{2\leq i<j<k\leq 7} \mathrm{Tr} A_i A_j A_k + \sum_{i=2}^{7} \text{Pf} A_i \tag{1}\]

Frame B. An \(\prod_{i=1}^{7} SU(2)^7\) quiver gauge theories with all the possible chiral fields \(X_{i,j}\) with \(i \neq j\) connecting \(SU(2)_i\) to \(SU(2)_j\) and superpotential

\[
W = \sum_{1\leq i<j<k\leq 7} X_{ij} X_{jk} X_{ki} \tag{2}\]

Frame C. An \(SU(4)_1 \times SU(2)_3 \times SU(2)_6\) quiver with six antisymmetric tensors and three pairs of bifundamentals of \(SU(4)_1 \times SU(2)_3\) and three antibifundamentals of \(SU(4)_1 \times SU(2)_6\) and superpotential

\[
W = A_6(t_{16}^2 + w_{16}v_{16}) + A_5(w_{16}^2 + t_{16}v_{16}) + A_7(v_{16}^2 + t_{16}w_{16}) + (A_2 + A_3 + A_4)(w_{16}v_{16} + t_{16}v_{16} + t_{16}w_{16}) + A_3(t_{31}^2 + w_{31}v_{31}) + A_4(v_{31}^2 + w_{31}t_{31}) + A_2(w_{31}^2 + v_{31}t_{31}) + (A_6 + A_5 + A_7)(w_{31}v_{31} + v_{31}t_{31} + w_{31}t_{31}) \tag{3}\]
Observe that for the ease of the reader above we have fixed the labels of the gauge groups compatibly with the ones obtained after the chains of dualities that we will study below.

The three models have the same central charges $a$ and $c$ and they only have an $U(1)_R$ global symmetry, that implies that there are no other 't Hooft anomalies to match. The conformal manifold has dimension $\dim \mathcal{M}_c = 21$ in all the three phases and the superconformal index has been shown to match at very large order in the fugacities.

Throughout this paper the color indices of all the fields are omitted and the contractions between them are understood.

### III. CONFORMAL TRIALITY FROM SEIBERG DUALITY

In this section we show that the three models introduced above can be mapped through chains of Seiberg and Intriligator-Pouliot dualities. This idea is similar to the one used to show the enhancement of $A_2$ to $E_7$ for $USp(2)$ SQCD with eight fundamentals \cite{22,23}. Indeed one can consider either a gauge group as an $SU(2)$ or as $USp(2)$, and dualize such node with a different interpretation for the flavor symmetry. Indeed if the gauge group is considered as $SU(2)$ one needs to distinguish fundamental and antifundamental representations, and it reflects in the interpretation of the mesonic and baryonic deformations. On the other hand if a gauge group is considered as $USp(2)$ only mesons are available. Such a different interpretation becomes relevant after the duality, because if the number of $USp(2)$ fundamentals is higher than eight then the rank of the gauge group is $SU(n > 2)$ or $USp(2n > 2)$. Furthermore the superpotential deformations have a different role in the dual phase because, as anticipated, they can involve either mesons or baryons. Another consequence of our derivations is that the matching of the integrals representing the superconformal index and the operator mapping on the conformal manifold follows automatically from the duality.

#### A. From Frame B to Frame A

In this case we start by by treating the $SU(2)_i$ gauge node as $USp(2)$. The node has $2N_f = 12$ fundamental and it is dual to an $USp(6)$ gauge group. The mesons associated to this gauge group are of two types. There are mesons in the form $M_{ij} \equiv X_{ij} X_{ji}$, in the bifundamental of $SU(2)_i \times SU(2)_j$ with $i \neq j$ and mesons $S_i \equiv X_i^2$, that correspond to singlets of $SU(2)_i$. The final superpotential is

$$W = \sum_{2 \leq i < j \leq 7} M_{ij} X_{ij} + \sum_{i<j} M_{ij} q_{i1} q_{1j} + \sum_{2 \leq i < j < k \leq 7} X_{ij} X_{jk} X_{ki} \tag{4}$$

where $q_{i1}$ are the dual bifundamentals charged under $USp(6)_1 \times SU(2)_i$. By integrating out the massive fields one is left with

$$W = \sum_{i=2}^7 S_i q_{i1} q_{1i} + \sum_{2 \leq i < j < k \leq 7} (q_{i1} q_{1j} q_{1k})^2 \tag{5}$$

Next we can dualize each $SU(2)_i = 2,...,7$ by treating them as $USp(2)$ nodes, each one with six fundamentals. Each $USp(2)$ factor gives rise to 15 singlets, corresponding to six antisymmetric $A_2$,...,$7$ tensors of $USp(6)$. The superpotential is

$$W = \sum_{i=2}^7 S_i A_i^{(0)} + \sum_{2 \leq i < j < k \leq 7} \text{Tr} A_i A_j A_k + \sum_{i=2}^7 \text{Pf} A_i \tag{6}$$

The F-terms of $S_i$ set $A_i^{(0)}$ to zero, and we are left with six totally antisymmetric tensors $A_i$ (i.e. of dimension 14) interacting through the superpotential $[I]$.

#### B. From Frame B to Frame C

In the second case we consider $SU(2)_1$ as unitary, i.e. we apply the ordinary Seiberg duality. This implies that we have to make a choice on the $SU(2)$ representations, dividing them into fundamentals and antifundamentals. The dual gauge group is then $SU(4)$ and we must pay attention to the structure of the dual superpotentials, because of the presence of interactions involving the baryonic deformations that have to be threaten with some care. We proceed by splitting the index $I = 1,...,7$ as $I = \{1, i = 2, 3, 4, \alpha = 5, 6, 7\}$ and the fields are then $X_{1i}, X_{a1}, X_{ij} = X_{ji}$ and $X_{a\beta} = X_{\beta a}$. The superpotential is

$$W = X_{23} X_{34} X_{42} + X_{56} X_{67} X_{75} + \sum_{2 \leq i < j \leq 4} \sum_{\alpha = 3}^7 X_{ij} X_{ja} X_{\alpha j}$$

$$+ \sum_{5 \leq \alpha < \beta \leq 7} \sum_{i=2}^4 X_{\alpha i} X_{\beta i} X_{\beta i} + \sum_{i=2}^7 \sum_{\alpha = 5}^7 X_{1i} X_{\alpha a} X_{a1}$$

$$+ \sum_{2 \leq i < j \leq 4} X_{1i} X_{1j} X_{ij} + \sum_{5 \leq \alpha < \beta \leq 7} X_{a1} X_{\beta 1} X_{a\beta} \tag{7}$$

The first four terms are spectator of the duality, while the others involve the mesonic deformations (first term) and the baryonic ones (last two terms). The dual superpo-
One type of term corresponds to the contribution of the fundamental and anti-fundamentals (see (SU) + N) + (SU) + p = 6

FIG. 1. Quiver obtained after Seiberg duality on SU(2)1 and Intriligator-Poulit on SU(2)4 and SU(2)6. The bifundamentals between the nodes 2,4 and 5,7 are massive and are integrated out.

\[ W = X_{23}X_{34}X_{42} + \sum_{i=2}^{4} \sum_{5\leq\alpha<\beta\leq7} q_{i\alpha}^2 q_{\alpha\beta} X_{i\beta} \]
\[ + \prod_{i=2}^{4} q_{i1} \left( \sum_{i\neq j\neq k} q_{i1} X_{jk} \right) \]
\[ + X_{56}X_{67}X_{75} + \sum_{\alpha=5}^{7} \sum_{\beta=1}^{4} q_{\alpha1}^2 q_{\alpha\beta} X_{i\beta} \]
\[ + \prod_{\alpha=5}^{7} q_{\alpha1} \left( \sum_{\alpha\neq \beta\neq \gamma} q_{\alpha1} X_{\beta\gamma} \right) \]

We then proceed by applying a Intriligator-Poulit duality on SU(2)3 = USp(2)3 and SU(2)6 = USp(2)6. The quiver becomes the one in the Figure 1 where we highlighted in red the mesons of the dualities. The dual superpotential, after integrating out the massive fields, becomes

\[ W = p_{32}p_{34}(A_3 q_{21} q_{41} + A_6 q_{21} q_{41} + q_{21} q_{41} q_{15} + q_{17} q_{21} q_{41}) \]
\[ + p_{56}p_{76}(A_3 q_{15} q_{17} + A_6 q_{15} q_{17} + q_{15} q_{17} q_{21} + q_{15} q_{17} q_{21}) \]
\[ + s_2 p_{32} + s_4 p_{34} + s_5 p_{56} + s_7 p_{76} + A_3 p_{13} + A_6 p_{16} \]
\[ + N_{21} p_{13} + N_{41} p_{13} p_{34} + N_{15} p_{56} p_{61} + N_{17} p_{76} p_{61} \]
\[ + A_6 (N_{21} q_{21} + N_{41} q_{41} + A_3 (N_{15} q_{15} + N_{17} q_{17})) \]
\[ + N_{15} (q_{15} q_{17} + q_{15} q_{17} + q_{15} q_{17}) \]
\[ + N_{41} (q_{41} q_{41} + q_{17} q_{41} + q_{21} q_{41}) \]
\[ + N_{17} (q_{17} q_{17} + q_{17} q_{17} + q_{17} q_{17}) \]
\[ + N_{21} (q_{21} q_{21} + q_{21} q_{21} + q_{21} q_{21}) \]

The next step consists of acting with two Seiberg dualities on the nodes SU(2)3 and SU(2)6. It requires a choice on the fundamentals and anti-fundamentals (see Figure 2).

The final dualities that we need to perform are Intriligator-Poulit dualities on nodes 2,4,5,7. These four gauge nodes are s-confining and we are left with the confined degrees of freedom consisting of the following fields

\[ M_2 = \begin{pmatrix} q_{21}^2 & t_{23} q_{21}^2 & t_{23} q_{21} \end{pmatrix} \begin{pmatrix} w_{31} & w_{31} & r_2 \end{pmatrix} \]
\[ M_4 = \begin{pmatrix} q_{41}^2 & t_{43} q_{41} \end{pmatrix} \begin{pmatrix} r_4 \end{pmatrix} \]
\[ M_5 = \begin{pmatrix} q_{15}^2 & t_{65} q_{15} \end{pmatrix} \begin{pmatrix} w_{16} & r_5 \end{pmatrix} \]

In this case we are left with an SU(4)1 x SU(2)3 x SU(2)6 gauge theory. The fields A2,...,7 are in the antisymmetric of USp(4), the fields t16, v16 and w16 are in the bifundamental of SU(4)1 x SU(2)6, and the fields t31, v31 and w21 are in the anti-bifundamental of SU(2)3 x SU(4)1. The fields r3,..,6 and s3,..,6 are gauge singlets. There are two types of superpotential terms after the confinement. One type of term corresponds to the contribution of the

Figure 2. After the duality we have represented the dual quiver by highlighting the mesons of this step in green. The dual superpotential, after integrating out the massive fields, becomes

\[ W = s_2 q_{21}^2 + s_4 q_{41}^2 + s_5 q_{56}^2 + s_7 q_{76}^2 + A_3 q_{31}^2 + A_6 q_{61}^2 \]
\[ + t_{23} t_{43} (A_3 q_{21} q_{41} + A_6 q_{21} q_{41} + q_{21} q_{41} q_{15} + q_{17} q_{21} q_{41}) \]
\[ + t_{65} t_{65} (A_3 q_{15} q_{17} + A_6 q_{15} q_{17} + q_{15} q_{17} q_{21} + q_{15} q_{17} q_{21}) \]
\[ + t_{16} q_{16} (q_{16} q_{16} + q_{16} q_{16} + q_{16} q_{16} + A_3 q_{16}) \]
\[ + t_{43} q_{43} (q_{43} q_{43} + q_{43} q_{43} + q_{43} q_{43} + A_6 q_{43}) \]
\[ + t_{65} q_{65} (q_{65} q_{65} + q_{65} q_{65} + q_{65} q_{65} + A_6 q_{65}) \]
\[ + t_{23} q_{23} (q_{23} q_{23} + q_{23} q_{23} + q_{23} q_{23} + A_2 q_{23}) \]
Pfaffians $\text{Pf}M_{2,4,5,7}$, while the other terms are deformations obtained by applying the duality map \[10\]–\[13\] to $\text{USp}$. After integrating out the massive fields we are left with \[3\].

### IV. ONE STEP FURTHER: TURNING OFF $W$

In this section we go beyond the results obtained above and we turn off some of the superpotential couplings in the $SU(2)^7$ model. We then follow the chain of dualities studied above. We distinguish two cases, in the first case we keep only the superpotential term

$$W_B = \sum_{2 \leq i < j < k \leq 7} X_{ij}X_{jk}X_{ki}$$

(14)

as non vanishing, while we turn off the terms $X_{ij}X_{jk}X_{ki}$ with $2 \leq i < j < k \leq 7$. In the second case keep only

$$W_B = X_{23}X_{34}X_{42} + X_{56}X_{67}X_{75} + \sum_{i=2}^{7} \sum_{\alpha=5}^{4} X_{i\alpha}X_{\alpha i}X_{\alpha i}$$

(15)

Repeating the derivation above we start dualizing in the first case as done in Subsection III A and in the second case as done in Subsection III B. In the first case we obtain the $USp(6)$ gauge theory with six totally antisymmetric tensors and

$$W_A = \sum_{i=2}^{7} \text{Pf} \hat{A}_i$$

(16)

while in the second case we obtain the $SU(2)^2 \times SU(4)$ quiver, but this time with superpotential

$$W_C = A_1^2 + A_2^2 + A_4^2 + A_5^2 + A_3v_{16}^2 + A_2w_{31}^2 + A_4v_{31}^2$$

(17)

Some comments are in order. First, turning off the superpotential deformations in the $SU(2)^7$ model does not rescue any continuous symmetry. This can be observed also by acting with Intriligator-Pouliot duality on $SU(2)_1$: the $SU(2)^6 \times USp(6)$ dual quiver has an anomalous axial symmetry associated to each bifundamental. In the $USp(6)$ dual theory this symmetry breaking pattern is more subtle, because it is broken by the non-perturbative contribution to the superpotential \[16\].

On the other hand a global symmetry enhancement is observed in the dual phase. Indeed by keeping only the superpotential \[15\] the final model has an emergent non-abelian $SU(3)^2$ global symmetry.

This symmetry enhancement can be understood from the duality structure that we have performed. Indeed in the second step we have made the choice of dualizing the nodes $SU(2)_3$ and $SU(2)_6$. There are other symmetric choices, we could have either dualized one group between $SU(2)_2$ and $SU(2)_4$ or one group between $SU(2)_5$ and $SU(2)_7$. For each choice the final $SU(4)^2 \times SU(2)$ model would have been completely analogous, reflecting a self duality. This self duality is actually the Weyl group of the $SU(3)^2$ global symmetry that emerges in the model.

We have then computed the superconformal index of the model in Frame C, by charging it with respect to the $SU(6) \times SU(3)^2$ non abelian global symmetry at the free point \[14\] \[15\]. By expanding in the fugacities we obtain

$$I_C(pq, a, b, c) = 1 + (pq)^{2/3} 21_6 + (pq) (-35_6 - 8_6 + 6_6(6_6 + 6_6)) + \ldots$$

(18)

where $a, b, c$ are the fugacities for the $SU(6)$ global symmetry and $b, c$ are the fugacities for the $SU(3)^2$ global symmetries. We use conventions where the character of the fundamental representation of $SU(N)$ is $N_d = d_1 + \cdots + d_N$ and the fugacities $t_i$ satisfy $\prod_{i=1}^{N} t_i = 1$. In the slice of the conformal manifold where only a diagonal $SU(3)^3$ subgroup of the global symmetry is preserved the superconformal index is obtained form the one above by identifying $a_i = b_i$ and $a_{i+3} = c_i$ for $i = 1, 2, 3$.

A further step consists in looking for the presence of such a symmetry in Frame A. At the free point of this frame the global symmetry is $SU(6)$. The index charged under this symmetry is

$$I_A(pq, t) = 1 + (pq)^{2/3} 21_6 + (pq) (-35_6 + 56_t) + \ldots$$

(19)

where $t_1, \ldots, 6$ are the fugacities associated with $SU(6)$. In the slice of the conformal manifold where an $SU(3)^2$ global symmetry is preserved the superconformal index is obtained by imposing $t_3 = (t_1t_2)^{-1}$ and $t_6 = (t_1t_5)^{-1}$. We observed that the two expression \[19\] and \[18\] match if we identify $t_i = a_i$ for $i = 1, \ldots, 6$.

We can interpret this result as an identification of a sub-manifold of the conformal manifold preserving the $SU(3)^2$ symmetry. This corroborates the idea that also the theory in Frame B has such an enhancement when the superpotential \[15\] is turned on, even if such a symmetry is not emergent at lagrangian level. We regard this as an example of an accidental non-abelian symmetry. As we argued above the Weyl group of the symmetry is visible as permutations of the gauge nodes 2, 3, 4 and 5, 6, 7, while the Cartan subalgebra emerges in the IR.

Finally we consider the superconformal index for Frame A with superpotential \[16\]. The Weyl subgroup of the $SU(6)$ global symmetry is preserved while the Cartan is broken to $U(1)^5 \to \langle Z_3 \rangle^5 / Z_3$. We can implement this in the superconformal index \[24\] by imposing $t_i = 1$. By looking for example at order $pq$ we find:

$$-35_6 + 56_t = -5 - \sum_{j > i}^{6} t_i t_j + \sum_{i=1}^{6} t_i^6 + \sum_{i > j}^{6} t_i^2 t_j$$

(20)

where with an abuse of notation we organize the contributions to the superconformal index in characters of
SU(6) in order to have a more compact notation. We stress that the fugacities are associated to the discrete symmetry of the model and the index is meaningful in every point of the conformal manifold where this symmetry is preserved, not only at the free point where we actually have an SU(6) global symmetry. The full index reads:

$$\mathcal{I}_A(pq, t) = 1 + (pq)^{2/3} 21_t + (pq)(1 + 20_t) + \ldots$$

(21)

This theory is dual to Frame B with superpotential [14], which is consistent with a discrete symmetry $S_6$ exchanging the nodes 2, . . . , 7 and an $(Z_3)^6/Z_3$ acting as:

$$(Z_3)_i : X_{i1} \rightarrow e^{2\pi i/3} X_{i1}$$

$$X_{ij} \rightarrow e^{-\pi i/3} X_{ij}, \quad j \neq 1$$

(22)

We can charge the superconformal index with respect to the discrete global symmetry along the lines of [12] where the discrete $Z_{2N}$ symmetry of $SO(N)$ SQCD was studied. We charge the index under the $U(1)_i$ symmetries under which $X_{i1}$ has charge $-\frac{1}{2}$ and $X_{ij}$ with $j \neq 1$ has charge $\frac{1}{2}$. These symmetries are anomalous and are broken to the $(Z_3)^6/Z_3$ discrete symmetry by instanton effects [17] (see also [16]), therefore we impose $w_1^3 = 1$ on the corresponding fugacities. We obtain:

$$\mathcal{I}_B(pq, w) = 1 + (pq)^{2/3} 21_w + (pq)(1 + 20_w) + \ldots$$

(24)

The two indices match under the identification $w_i = t_i$. We checked that this is true up to order $(pq)^{3/2}$. We interpret this as a matching between the discrete global symmetries. As we already pointed out, in Frame A the $S_6$ discrete symmetry is the Weyl of the SU(6) global symmetry at the free point, broken by the superpotential [16]. The full SU(6) symmetry can be recovered by turning off the superpotential [16], which corresponds to a marginal deformation parameterized by an operator unchanged under the $S_6$ discrete symmetry. From the superconformal index [21] we see that there is only one such operator, therefore we claim that Frame B with the superpotential [14] and the (unique) additional exactly marginal deformation unchanged under $S_6$ is dual to Frame A at the free point, for a suitable value of the additional deformation. In this point the global symmetry enhances to SU(6). The Weyl of this symmetry is visible in the UV as permutation of the nodes 2, . . . , 7 while the Cartan is accidental.

Similarly one can analyze the behavior of the discrete symmetries between Frame B with superpotential [15] and Frame C, the procedure is analogous to the one carried out in this section and we will not describe it in this paper.

V. DISCUSSION

In this note we have interpreted the conformal triality found in [21] in terms of ordinary Seiberg and Intriligator-Pouliot duality. We have first considered the SU(2)\(^7\) quiver gauge theory with the whole set of superpotential marginal deformation turned on. This corresponds to Frame B. This deformation explicitly breaks the global symmetry to $U(1)_R$. We have then dualized one of the SU(2) gauge node by treating it as USp(2). The resulting quiver, corresponding to $USp(6) \times USp(2)^6$ with bifundamentals connecting $USp(6)$ with each SU(2) factor, corresponds to an intermediate phase, where the various USp(2) gauge nodes are actually confining gauge theories. In the IR such gauge groups can be traded with sets of antisymmetric mesons. Such antisymmetric mesons correspond to the antisymmetric tensors of the USp(6) gauge group, that is treated as a spectator in this case. By carefully tracing the superpotential deformations only the totally antisymmetric part of these fields survives in the IR. This final model corresponds to Frame A. Analogously, starting from the model denoted as Frame B we have constructed the one denoted as Frame C, by iterating a series of Seiberg and Intriligator-Pouliot dualities.

As a further step we have shown that such dualities between the models persist if we turn off opportune superpotential deformations in the SU(2)\(^7\) quiver. Actually in this case we have found an interesting picture when computing the superconformal index. Indeed the visible global symmetries in this case is still $U(1)_R$ in Frame B and Frame A, while it enhances to $SU(3)^2 \times U(1)_R$ in Frame C.

This symmetry breaking structure in Frame A resembles the discussion of the mismatch of global symmetries in the Berkooz deconfinement of gauge charged two-index tensors [13][20]. Actually in the case of Berkooz deconfinement the situation is different, indeed in that case the claim is that the presence of non perturbative effects breaks the extra global symmetry in the deconfined phase. Here instead the non-perturbative effect break the non abelian global symmetry in the confined phase identified by Frame A. On the other hand we observed a symmetry enhancement in Frame C, because of a self duality emergent from a permutational $S^2_4$ symmetry.

This is an important issue that needs a clarification. On one hand the integral identities used to fully match the superconformal indices between Frame B and Frame C holds only when $U(1)_R$ is the only global symmetry. This is because the constraints (usually referred as balancing conditions) necessary to match the superconformal index requires that any other global symmetry is absent. Anyway we are still free to charge the superconformal index with respect to $SU(3)^2$ in Frame C, because this global symmetry is realized by the action.

By a direct computation we have further shown the model in Frame A can be deformed to reach a point in the conformal manifold where an SU(6) global sym-
metry is realized. We have shown that this index can be matched with the one obtained from Frame C once it is specialized on $SU(3)^2$ fugacity. This is compatible with the claim that the theories are conformally dual and signals the fact that along these symmetries one can move along weakly coupled directions in the conformal manifold preserving the global symmetry structure. One can also use the operator map inherited from the underlining Seiberg and Intriligator-Pouliot dualities in order to have an explicit parametrization of such submanifold. On the other hand, for more general enhancements and for a matching with the model in Frame B strongly coupled directions have to be taken into account and the lagrangian description is not sufficient to have the whole picture.

It should be interesting to further investigate on such phenomenon, looking for other conformally dual models, of the type studied in $[9][8]$. Finding examples of such dualities connected through ordinary Seiberg-like dualities with possibly some gauge factor that locally confines. We have not found so far examples of this type in the literature. Nevertheless an useful starting point consists in finding models that are conformally dual to the ones discussed in $[7]$, because in such cases the conformal manifold passes through weak coupling.

Another interesting question that arises in this discussion is the possibility of observing the pattern of global symmetry enhancement from the superconformal index along the lines of $[21][25]$. We have shown that, through a sequence of controlled motions on the conformal manifold and Seiberg-like dualities, some discrete subgroup of the global symmetry can be mapped between two weakly coupled frames. For example we find a one-dimensional slice connecting the free points of Frame B and A along which a $S_6 \ltimes (Z_3^6)/Z_3$ symmetry is preserved. This can be seen from the superconformal index by changing it with respect to the discrete symmetry. At the free point of Frame A the symmetry enhances to $SU(6)$ and $S_6$ becomes the Weyl group. It would be interesting to investigate this symmetry enhancement directly from Frame B. Indeed, even at the level of the index, the enhancement to $SU(6)$ is only visible thanks to the dual weakly coupled description provided by Frame A. Generally we do not expect that such a dual description is available in a point of the conformal manifold with enhanced symmetry, therefore an alternative approach to the study of this phenomenon is desirable.

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Appendix A: Deconfining $\mathcal{N} = 4$ $SO(6)$ SYM

In this appendix we reproduce some of the manipulations exploited throughout the main body of the paper, such as matching of discrete symmetries and symmetry enhancement on the conformal manifold, in the more familiar context of $\mathcal{N} = 4$ SYM. Consider a $\mathcal{N} = 1$ theory with one $SO(6)$ gauge group, three $USp(2)$ gauge groups and one bifundamental field $X_i$ between $SO(6)$ and each of the symplectic groups. The superpotential is:

$$W = X_i^2 \{X_2^3, X_3^3\}$$  \hspace{1cm} (A1)

The symplectic gauge groups confine and upon dualising them we obtain an $\mathcal{N} = 1$ $SO(6)$ gauge theory with three adjoints $A_i$ and superpotential:

$$W = W^{\mathcal{N}=4} + \sum_{i=1}^{3} Pf(A_i) = W^{\mathcal{N}=4} + \sum_{i=1}^{3} Tr(A_i^3)$$  \hspace{1cm} (A2)

This is nothing but $\gamma$-deformed $\mathcal{N} = 4$ SYM $[1]$. The $\gamma$ deformation is exactly marginal and we can turn it off, moving on the conformal manifold, and recover $\mathcal{N} = 4$ supersymmetry. In particular, from the $\mathcal{N} = 1$ perspective, we recover the $SU(3)_R$ global symmetry. This $SU(3)_R$ is broken by the $\gamma$ deformation to $S_3 \ltimes (Z_3^6)/Z_3$ where $S_3$ permutes the three adjoints and $Z_3^6$ rotates $A_i$ by a phase $\frac{2\pi}{3} i$.

The quiver theory has the same discrete symmetry, indeed there is a classical $S_3$ permuting the bifundamentals $X_i$ while $(Z_3^6)/Z_3$ comes from the anomalous “axial” symmetries as follows. Consider the $U(1)_i$ symmetries under which $X_i$ has charge $\frac{1}{3}$ and the other fields are not charged. These have an anomaly with one of the $USp(2)$ gauge groups and are broken to $Z_3$, which are consistent with the superpotential. A $Z_2$ subgroup of these symmetries is already contained in the $SO(6)$ gauge group, therefore they are broken to $Z_3$. Anomaly cancellation with respect to $SO(6)$ further break them to $(Z_3^6)/Z_3$.

We can check the map between the global discrete symmetry by charging the superconformal index with $Z_3$-valued fugacities. For $\mathcal{N} = 4$ we have:

$$I^{\mathcal{N}=4} = 1 + (pq)^{\frac{1}{3}}(p + q)3_t + (pq)^{\frac{2}{3}}6_s + pq(1 + 10_t - 8_s) + \ldots = 1 + (pq)^{\frac{1}{3}}(p + q)3_t + (pq)^{\frac{2}{3}}6_s + (pq)^33 + \ldots$$  \hspace{1cm} (A3)

where $t_i$, $i = 1, 2, 3$ parametrize $SU(3)_R$ and the second equality is true after imposing $t_2^3 = 1$, consistently with the $\gamma$-deformation. Similarly the superconformal index for the quiver theory is:

$$I^{\mathcal{N}=1} = 1 + (pq)^{\frac{1}{3}}(p + q)3_w + (pq)^{\frac{2}{3}}6_w + (pq)^33 + \ldots$$  \hspace{1cm} (A4)
where \( w_i \) are \( \mathbb{Z}_3 \)-valued fugacities associated with \( \langle \mathbb{Z}_3^2 \rangle / \mathbb{Z}_3 \). The two indices match with \( w_i = t_i \). We see that we are able to match the discrete \( \langle \mathbb{Z}_3^2 \rangle / \mathbb{Z}_3 \) symmetry that is preserved along the direction of the conformal manifold parametrized by the \( \gamma \)-deformation. As a byproduct the \( S_3 \) symmetry is visible at the level of the index as well, and we can track it as we turn off the \( \gamma \)-deformation until we reach the point in the conformal manifold where \( N = 4 \) is recovered. Here \( S_3 \) becomes the Weyl of the enhanced global symmetry \( SU(3)_R \).

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