Large Kernel Polar Codes with efficient Window Decoding

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Abstract—In this paper, we modify polar codes constructed with some $2^t \times 2^t$ polarization kernels to reduce the time complexity of the window decoding. This modification is based on the permutation of the columns of the kernels. This method is applied to some of the kernels constructed in the literature of size 16 and 32, with different error exponents and scaling exponents such as eNBCH kernel. It is shown that this method reduces the complexity of the window decoding significantly without affecting the performance.

I. INTRODUCTION

Polar codes, introduced by Arikan [1], are the first family of capacity-achieving codes with low complexity successive cancellation (SC) decoder. However, the performance of the polar codes at finite block lengths is not comparable with the state of art codes, due to (i) sub-optimality of the SC decoder and (ii) imperfectly polarized bit-channels resulted from large binary kernels. Window decoder exploits the relationship between arbitrary kernels and the Arikan’s kernel. However, the complexity of the window decoder for any arbitrarily large kernel (e.g. eNBCH kernels) is too high for practical implementation.

A heuristic construction was proposed in [12] for binary kernels of dimension 16, which minimizes the complexity of the window decoder and achieves the required rate of polarization. The authors achieved these goals by applying some elementary row operations on Arikan’s kernel. However, a systematic design of large kernels with the required polarization properties, which admit low-complexity decoder, is still an open problem.

In this paper, we modify some $2^t \times 2^t$ large kernels with good polarization rates, to reduce the computational complexity of the window decoder. This modification is based on the search through $(2^t)!$ column permutations of the kernel which do not affect its polarization properties. Since exploring all possible permutations is not practical, we propose a suboptimal algorithm to reduce the search space significantly and find good column permutations independently of the structure of the original kernel. Then, we apply our algorithm to the kernels of size 16 and 32 constructed in the literature with error exponents 0.51828 and 0.537, respectively. The complexity of the window decoding of these modified kernels is significantly lower as compared to the window decoding of the original kernels. Recently, in an independent work, authors in [14] also suggested a suboptimal search algorithm of column permutations to reduce the complexity of the Viterbi algorithm for eNBCH kernels. This algorithm specifically takes advantage of eNBCH structure for reducing the search space. However, the complexity of eNBCH kernels with Viterbi algorithm is still high for practical applications.

II. BACKGROUND

Consider a binary input discrete memoryless channel (B-DMC) $W : \mathcal{X} \to \mathcal{Y}$ with input alphabet $\mathcal{X} = \{0, 1\}$, output alphabet $\mathcal{Y}$, and transition probabilities $W(y|x)$, where $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and $W(y|x)$ is the conditional probability, the channel output $y$ given the transmitted input $x$. An $(N = l^n, k)$ polar code based on the $l \times l$ polarization kernel $K$ is a linear block code generated by $k$ rows of $G_n = K^{\otimes n}$, and $\otimes n$ is $n$-times Kronecker product of matrix with itself. In order to polarize, none of the column permutations of the kernel $K$ should result in an upper triangular matrix. Note that Arikan’s polarization kernel, $F_2 = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right)$, is a special case of the polarization kernel $K$.

By encoding the binary input vector $u_0^{n-1}$ as $c_0^{n-1} = u_0^{n-1} G_n$, it was shown in [3] that the transformation $G_n$ splits the B-DMC channel $W(y|x)$ into $N = l^n$ subchannels

$$W_{n,K}(u_i^t|y_0^{N-1}) = \frac{W_{n,K}(y_0^{N-1}, u_0^{i-1}|u_i)}{2W(y_0^{N-1})} = \sum_{u_{i+1}^N} \prod_{i=0}^{N-1} W((u_0^{N-1} G_n)_i|y_i)$$

with capacities converging to 0 or 1 as $N \to \infty$. Let’s denote by $F$ the set of the indices of subchannels with the lowest reliabilities. Then, setting $|F| = N - k$ entries of the input vector $u_0^{N-1}$ to zero (frozen bits) and using the remaining entries to the information bits payload, will provide almost error-free communication.
At the decoder side, the successive cancellation (SC) decoder first computes the $i$-th LLR in each step $i$, according to the following formula:

$$S^{(i)}_{n}(u_0^{i-1}, y_0^{N-1}) \triangleq \ln \frac{W_{n,K}^{(i)}(u_0^{i-1}, u_4 = 0 | y_0^{N-1})}{W_{n,K}^{(i)}(u_0^{i-1}, u_4 = 1 | y_0^{N-1})}, \quad (2)$$

and then it sets the estimated bit $\hat{u}_i$ to the most likely value according to the following rule:

$$\hat{u}_i(y_0^{N-1}, u_0^{i-1}) = \begin{cases} u_i, & \text{if } i \in F \\ 0, & \text{if } i \in FC \& LLR^{(i)}_N \geq 0 \\ 1, & \text{if } i \in FC \& LLR^{(i)}_N < 0 \end{cases} \quad (3)$$

The complexity of this decoder for a polarization kernel $K$ of dimension $l \times l$ is $O(2^l N \log_2 N)$. Methods for marginally reducing this complexity were proposed in [10] and [11], however, even for small $l$ this is still not practical.

**Window Decoding**

This method, introduced in [7], reduces the complexity of the SC decoding by exploiting the relationship between the given kernel $K$ and Arikan’s kernel ($K_A = F_2^{l/2}$), without any performance loss.

If we write the $l \times l$ $(n = 1)$ polarization kernel $K$ with $l = 2^i$ as a product of the Arikan’s kernel with another matrix $T$, $K = TK_A$, then encoding is given by $u_0^{i-1} = v_0^{i-1}K_A$ and we have $v_0^{i-1} = u_0^{i-1}T^{-1}$. Now, it is possible to reconstruct $u_0^{i}$ from $v_0^{i}$, where $\tau_l$ is the position of the last non-zero bit in the $i$-th row of $T^{-1}$. The relation between the vectors $v_0^{i-1}$, $u_0^{i-1}$ can be written as

$$v_0^{i-1} = u_0^{i-1}K, \quad \text{where } u_0^{i-1} = \text{transposed matrix of } u_0^{i-1} \text{ and } \theta' = (S|I) \text{ and } S \text{ is the } l \times l \text{ matrix obtained by transposing matrix } T \text{ and reversing the order of the columns. Applying row operations can transform matrix } \theta' \text{ into a minimum-span form } \theta, \text{ such that the } i \text{-th row starts in the } i \text{-th column, and ends in column } z_i, \text{ where all } z_s \text{ are distinct. If we denote } j_i = z_i - i - l \text{ and } h_i = \max_{0 \leq j \leq l} j_i, \text{ one can express } u_i \text{ as }$$

$$u_i = \sum_{s=0}^{i-1} u_s \theta_{i-1-i, s-1} + \sum_{t=0}^{j_i} v_t \theta_{i-1-i, t+i}, \quad (4)$$

and, as a result, the $i$-th bit channel of kernel $K$ in terms of the $h_i$-th bit channel of kernel $K_A$ as

$$W_K^{(i)}(u_0^{i} | y_0^{i-1}) = \sum_{v_D \in \{0, 1\}^{|D_i|}} W_K^{(h_i)}(v_0^{h_i} | y_0^{i-1}) = \sum_{v_D \in \{0, 1\}^{|D_i|}} \sum_{v_{D_j} \in \{0, 1\}^{v_D_{j+1}}} W_K^{(j+1)}(v_0^{j} | y_0^{i-1}) \quad \text{(5)}$$

where, $D_i = \{0, 1, ..., h_i\} \setminus \{j_0, j_1, ..., j_i\}$ is the decoding window, where $|D_i| = h_i - i$. Note that the set of the vectors $v_0^{h_i}$ in (5) satisfies (4). Since $n = 1$, we omit the first subscript of $W$.

In particular, (4) and (5) imply that given the previous decoded bits $u_0^{i-1}$, one can decode $u_i$ by using the SC decoder and calculating the transition probabilities $W_K^{(h_i)}(v_0^{h_i} | y_0^{i-1})$ for all $2^{|D_i|}$ values of $v_D$, corresponding to the input bits in $v_0^{h_i}$ that are not decoded yet. Then, for the evaluation of (2), one needs $2^{|D_i|+1}$ operations for both $W_K^{(i)}(u_0^{i-1}, u_i | y_0^{N-1})$ for $i$-th bit channel of the kernel $K$. Thus, the $|D_i|$’s determine the window decoding complexity.

More precisely, the estimated complexity of the implementation of the window decoder for the kernel $K$ of size $l$ is

$$\psi(K) = \sum_{i=0}^{l-1} \phi(i) \text{ where }$$

$$\phi(i) = \begin{cases} 0, & \text{if } h_i = 0 \\ 2^{|D_i|} - 1 + \Lambda(i), & \text{if } h_i > 0 \text{ and } |D_i| > 0 \\ 2^i C_i + 1, & \text{if } h_i = 0 \\ 2^i C_0, & \text{if } h_i = 0 \text{ and } |D_i| = 0 \end{cases} \quad (6)$$

with $\Lambda(i) = \sum_{j=0}^{h_i} j 2^{j+b(h+1)+i - 1}$. The computational cost of a bit channel of $K_A$ for the $i$-th bit channel, let $s$ be the largest integer such that $2^s$ divides $i$, then $C_i = 2^{s+1} - 1$ and $C_0 = 2^l - 1$.

**III. PROPOSED ALGORITHM**

As it is stated in Section [2] the size of the window, $|D_i| = h_i - i$, determines the complexity of the window decoder, when $h_i > h_{i-1}$. Tables [12] and [13] show $|D_i|$’s for each of the kernels $K_{\text{ENICH}}, K_A$ and $K_F$ constructed in [8], [6] and [5], respectively. These tables show that the size of window for some of the bit channels is very large. One solution to this problem is to modify the kernel $K$ to reduce the size of the window, without altering the polarization properties of the kernel. Column permutation of the kernel $K$ doesn’t affect its polarization properties. However, finding the best permutation by exploring all $l!$ cases is not practical and a method for reducing the size of the search space is needed. On the other hand, there are many permuted kernels that have the same $h_i$, resulting in the same complexity. However, reducing the search space to the permutations that result in the unique $h_i$’s is not trivial. One suboptimal method is to limit our search to the permuted kernels that result in matrix $T$ with as many rows with only one non-zero element. This implies that in the new column permuted kernel $K'$, we want to have as many rows from $K_A$ as possible. A threshold, $M_{\text{thr}}$, for satisfying this rule will reduce the search space significantly. After finding the permutations which satisfy this rule, we choose the ones.
which have the minimum complexities based on (6). This solution is not the optimal one, but it reduces the search space significantly and finds a good permutation that results in reduced complexity.

Algorithm 1: Finding good column permutation

| input: Kernel K of size l and threshold $M_{thr}$ |
| output: Good column permutation $\pi$ and permuted kernel $K'$ |

1. Define Lists: $C$, $R$, $M$, $TmpC$, $TmpR$, $TmpM$
2. $(C, R, M) \leftarrow (\{\} , \{1, 2, ..., l\} , \{\})$; // Init.
3. $(\text{TmpC, TmpR, TmpM} ) \leftarrow (\{\} , \{\} , \{\})$; // Init.
4. while $(C = \{\})$ do
   1. for $i \leftarrow 1$ to l do
      1.1. $\text{foreach} (\kappa, \iota, \mu) \in (C, R, M)$ do
      1.2. $\text{CandCol} \leftarrow \{1, 2, ..., l\} \setminus \kappa$; // Cand. for i-th col.
      1.3. for $m \leftarrow 0$ to length(CandCol) − 1 do
          1.3.1. $\text{[Cel, Rel, Metric]} \leftarrow \text{CalculateMetric}(i, \text{CandCol}, \kappa, m, \iota, \mu)$
          1.3.2. if Metric $\geq M_{thr}$ then
              1.3.2.1. $\text{PUSH} ((\text{Cel, Rel, Metric}), \text{(TmpC, TmpR, TmpM)});$ end
          1.3.3. end
      1.4. end
      1.5. if $(\text{TmpC, TmpR, TmpM} ) = (\{\} , \{\} , \{\})$ then
          1.5.1. $(C, R, M) \leftarrow (\{\} , \{1, 2, ..., l\} , \{\})$;
      1.6. else
          1.6.1. $(C, R, M) \leftarrow (\text{TmpC, TmpR, TmpM} );$
          1.6.2. $\text{(TmpC, TmpR, TmpM} ) \leftarrow (\{\} , \{\} , \{\})$;
      1.7. end
   1.8. end
   1.9. $M_{thr} \leftarrow M_{thr} - 1;$
1.10. end
1.11. $\pi = \arg \min_{\kappa \in C} \Psi(\kappa(K))$
1.12. return $K' = \pi(K), \pi$

subroutine $\text{CalculateMetric}(i, \text{CandCol}, \kappa, m, \iota, \mu)$:

2.1. $\text{Cel} \leftarrow (\kappa, \text{CandCol}[m])$; // append CandCol[m] to $\kappa$
2.2. $SK \leftarrow \{K[i] | \text{Cel}\}$; // List of rows of the subkernel $K[i] | \text{Cel}$ (with repetitions)
2.3. $SK_A \leftarrow \{K_A[i] | \text{Cel}\}$; // Set of rows of the subkernel $K_A[i] | \text{Cel}$ (no repetitions)
2.4. Metric $\leftarrow \text{Calculate Metric with (7) for } SK_A \text{ and } SK$
2.5. cnt $\leftarrow 0$; // counter
2.6. for $j \leftarrow 0$ to $\mu - 1$ do
2.7. if $SK[j] \in SK_A$ then
        2.7.1. $\text{cnt} \leftarrow \text{cnt} + 1$
        2.7.2. $\text{Rel[cnt]} \leftarrow i[j]$
    2.8. end
2.9. end
2.10. return Cel, Rel, Metric;

Algorithm 1 shows the process of finding good column permutations for an arbitrary kernel $K$. The inputs to this algorithm are the kernel $K$ of size $l$ and a predefined threshold $M_{thr}$. The outputs are good permutations $\pi$ and the resulting permuted kernel $K' = \pi(K)$. In each step, the best candidates for the first $(i - 1)$ columns of the partially permuted kernel are used to examine all possible candidates for the $i$-th column to determine the best ones based on the following metric,

$$\text{Metric} = \sum_{j \in SK} \mathbb{I}_{SK_A}(j),$$

where $SK$ and $SK_A$ are the sets of some rows of the first $i$ columns of the partially permuted kernel and of $K_A$, respectively, and $\mathbb{I}$ is the indicator function.

Let’s define $C$ as the list of the best candidates for the first $(i - 1)$ columns, $R$ as the list of the indexes of the rows of the first $(i - 1)$ columns of the partially permuted kernel, which belong to the set of the rows of the first $(i - 1)$ columns of the kernel $K_A$. Let $M$ be the list of the Metrics of the best candidates for the $(i - 1)$ columns. Note that for $i = 1$, we assume that $C = \{\}, R = \{1, 2, ..., l\}$ and $M = \{\}$.

The proposed algorithm finds the best candidates for the $i$-th column of the partially permuted kernel with the following steps:

1. For each $(\kappa, \iota, \mu) \in (C, R, M)$, it determines the non yet selected columns as the candidates CandCol for the $i$-th column (line 8). Then, for each of these candidates, it follows the four steps:
   - Appends a candidate from CandCol to $\kappa$ to obtain Cel, the list of the first $i$ possible columns.
   - Picks each row of the sub-kernel $K[i] | \text{Cel}$ and puts it in the list $SK$ accounting for any repetitions and picks each row of the sub-kernel $K_A[i] | \text{Cel}$ and puts it in the set $SK_A$ (without repetitions).
   - Counts the elements of $SK$ belonging to the set $SK_A$ and stores this number in Metric. Stores the corresponding indexes of the rows belonging to set $SK_A$ in Rel.
   - Compares Metric with the threshold $M_{thr}$. If Metric $\geq M_{thr}$, it pushes Cel, Rel and Metric, to the temporary lists TmpC, TmpR and TmpM, respectively.

2. If there is at least one candidate from CandCol with Metric $\geq M_{thr}$, it copies all the collected parameters of these candidates to $(C, R, M)$ to use them for the next column selection; otherwise, it reduces the threshold $M_{thr}$ and repeats the process from the beginning, with $i = 1$.

Finally, the algorithm continues this process until it finds the best candidates for all $l$ columns. Then, it outputs the good column permutations, among the candidates in $C$, which minimize the approximate complexity in (6).

The algorithm significantly reduces the $l!$ search space to the candidates in the list $C$ by using the threshold $M_{thr}$ and the Metric (7). Then, it finds the best ones among them.

To find the initial value of the threshold $M_{thr}$, we define two multisets HWK and HWA as the containing the Hamming weights of the rows of $K$ and $K_A$, respectively. Then, $M_{thr}$ which is the maximum possible threshold will be $M_{thr} = |HWK \cap HWA|$.

Here, we apply our algorithm to two kernels $K_F$ and $K_L$ of sizes 16 constructed in (8) and (6), respectively, and to the kernels $K_{N \times B \times C}$ of size 16 and 32, (8). The error exponents (EE), scaling exponents (SE) and good permutations resulting from our algorithm for these kernels are given in TABLE II.
Due to space limitations, for $K_{NBCH}$ of size 32, we wrote only one of the good permutation.

Although the proposed algorithm is sub-optimal, we conjecture that the obtained permutations for $K_{NBCH}$ may be optimal.

| Size | Kernel | Ef. | Size | Permutation |
|------|--------|-----|------|-------------|
| 16   | $K_L$  | 0.51828 | 3.3627 | 1, 16, 3, 7, 6, 12, 10, 9, 11, 13, 14, 15 |
| 32   | $K_{NBCH}$ | 0.537 | 3.1221 | 1, 2, 3, 20, 4, 7, 21, 13, 5, 31, 8, 29, 22, 10, 14, 26, 6, 12, 32 |

Table I: Good permutations found by Algorithm I for different kernels.

IV. Analysis and Simulation Results

In this section, we first analyze the computational complexity of the kernels and compare the complexity of them before and after applying the permutations. Then, we compare their performances and also their real-time complexities with each other and also with the Arikan’s kernel under SC and SCL decoders.

Tables II and III show $h_i$ and $|D_i|$ for each $i \in \{0, 1, \ldots, l-1\}$ of the different kernels of sizes 16 and 32 before and after applying the column permutations. The corresponding approximate computational complexities (AC) using the expression (6) as well as the computational complexities (CC) using CSE algorithm proposed in [9] are also given in these tables. Note that we use arithmetic complexity as the number of summation and comparison operations for calculating the LLR (13) in [9]. It can be observed that the good column permutations obtained from our algorithm can reduce the maximum size of the window from 12 to 4 for $K_{NBCH}$ of size 16, from 7 to 4 for $K_F$, from 12 to 5 for $K_L$ and from 28 to 17 for $K_{NBCH}$ of size 32. As a result, the computational complexity CC of the window decoder after applying the column permutations is reduced from 38089 to 465 for $K_{NBCH}$ of size 16, from 1851 to 517 for $K_F$ and from 38089 to 728 for $K_L$ without any performance loss. Also, for $K_{NBCH}$ of size 32 the good column permutation results in complexity reduction by factor of 1192 compared to the original kernel.

Note that after applying column permutation proposed in [14] on $K_{NBCH}$ of size 16, Viterbi decoder needs 5019 operations, while using our best permutations for window decoding requires only 446. Indeed, using window decoder for the kernels constructed in [14] requires 3000 and 6714318 operations approximately, while applying viterbi decoder needs 5019 and 299235 operations, respectively.

Fig. 2 illustrates the performance of the (4096, 2048) polar (sub)codes constructed with kernels $K_{NBCH}$ and $K_F$ of size 16 under SC and SCL decoders over the AWGN channel with BPSK modulation. The construction is based on Monte-Carlo simulations. Note that the column permutation doesn’t alter the polarization behaviours of the kernel, so the performance of the kernel $K$ is the same as the performance of the kernel $K'$. It can be seen that polar codes based on kernels $K'_F$, $K'_{NBCH}$ provide significant performance gain compared to polar codes with Arikan’s kernel. Indeed, the kernel $K'_F$ with lower scaling exponent provides better performance as compared to the $K'_{NBCH}$ with higher scaling exponent. It can observed that polar subcodes [13] with $K'_{NBCH}$ under SCL with list size $L = 8$ provides approximately the same performance as polar subcodes with Arikan’s kernel under SCL with $L = 32$.

Fig. 3(a) shows the performance of the (4096, 2048) polar subcodes constructed with $K'_{NBCH}$ and $K'_F$ of size 16 under SCL with different list sizes $L$ at $Eb/No = 1.25$ dB. It can be observed that these kernels need lower list size to
TABLE II: Comparison of the approximate complexity (AC) and complexity with CSE (CC) of the different kernels of size 16 with window decoder before and after applying the permutations.

| $K_{NBCH}$ | $K_{NBCH}'$ | $K_{FH}$ | $K_{FH}'$ | $K_{L}$ | $K_{L}'$ |
|-----------|-------------|---------|---------|--------|--------|
| $h_l$ | $h_l$ | $h_l$ | $h_l$ | $h_l$ | $h_l$ |
| $C_{AC}$ | $C_{AC}$ | $C_{AC}$ | $C_{AC}$ | $C_{AC}$ | $C_{AC}$ |
| $C_{CC}$ | $C_{CC}$ | $C_{CC}$ | $C_{CC}$ | $C_{CC}$ | $C_{CC}$ |

TABLE III: Comparison of the approximate cost (AC) of the $K_{NBCH}$ of size 32 with window decoder before and after applying the permutations.

| $K_{NBCH}$ | $K_{NBCH}'$ | $K_{FH}$ | $K_{FH}'$ | $K_{L}$ | $K_{L}'$ |
|-----------|-------------|---------|---------|--------|--------|
| $h_l$ | $h_l$ | $h_l$ | $h_l$ | $h_l$ | $h_l$ |
| $C_{AC}$ | $C_{AC}$ | $C_{AC}$ | $C_{AC}$ | $C_{AC}$ | $C_{AC}$ |

The kernel, was applied to some kernels constructed in the literature, e.g. eNBCH, and showed that the complexity of the window decoder for these modified kernels is substantially lower as compared to original ones.

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