A representation of solutions to a scalar conservation law in several dimensions

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Abstract

We find a representation of smooth solutions to the Cauchy problem for a scalar multidimensional conservation law as small diffusion limit of a stochastic perturbation along characteristics. It helps, in particular, to study the process of singularities formation. Further, we introduce an associated system of balance laws that can be interpreted as describing the motion of a continuum with some specific pressure term. This term arises only after the instant when the solution to the initial Cauchy problem looses its smoothness. Before this instant the system coincides partly with the one known as pressure free gas dynamics.

Key words: scalar conservation law, the Cauchy problem, representation of solution, associated conservation laws, stochastic perturbation

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Introduction

We consider the initial value problem

\[ u_t + \sum_{i=1}^n a_i(t, x, u) u_x^i = 0, \quad u(0, x) = u_0(x), \quad u_0(x) \in C^1_b(\mathbb{R}^n; \mathbb{R}), \]  

where \( t \in \mathbb{R}_+, \quad x \in \mathbb{R}^n, \quad a_i(t, x, u), \quad i = 1, \ldots, n, \) is a real-valued \( C^1 \) function defined on some open subset of \( (\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}) \). For a technical reason the functions \( a_i(t, x, u) \) are assumed to grow at infinity not quicker than a linear function in \( u \).

A particular important special case is given by the scalar conservation law in the form

\[ u_t + \text{div} F(t, u) = 0, \]

where \( F(t, \cdot) = (F_1(t, \cdot), \ldots, F_n(t, \cdot)) \) is a \( C^2 \) vector-function defined on some open subset of \( \mathbb{R} \), for any \( t \in \mathbb{R}_+ \), and \( a_i(t, u) = \frac{\partial F_i(t, u)}{\partial u}, \quad i = 1, \ldots, n. \)

The main aim of this paper is to obtain an asymptotic formula for the solution of the Cauchy problem \((\text{1})\) for the case of a scalar conservation law. The formula is obtained by the limit for vanishing perturbation of the corresponding stochastically modified equation (small diffusion limit).

Nevertheless, let us first consider the general case.

Let us write the associated characteristic ODE:

\[ \frac{dx_i}{dt} = a_i(t, x, u), \quad \frac{du}{dt} = 0, \quad i = 1, \ldots, n. \]

Its stochastic analog is

\[ dX_i(t) = a_i(t, X(t), U(t))dt + \sigma_1 d\langle W^1_i \rangle_t, \quad\quad dU(t) = \sigma_2 d\langle W^2 \rangle_t, \]

\[ X_i(0) = x_i, \quad U(0) = u, \quad t > 0, \]

\( i = 1, \ldots, n \), \( X(t) \) and \( U(t) \) are considered as random variables with given initial distributions, \((X(t), U(t))\) runs in the phase space \( \mathbb{R}^n \times \mathbb{R}^1 \), \( \sigma_1 \) and \( \sigma_2 \) are nonnegative constants such that \( |\sigma| \neq 0 \) \((\sigma = (\sigma_1, \sigma_2))\) and \((\langle W^1 \rangle_t, \langle W^2 \rangle_t) = (W^1_1, \ldots, W^1_n, W^2_1)\) is an \( n+1 \) - dimensional Brownian motion, i.e. the \( W^1_i, W^2_i, \quad i = 1, \ldots, n, \) are independent one-dimensional standard Brownian motions.

Let \( P(t, dx, du), \quad t \in \mathbb{R}_+, \quad x \in \mathbb{R}^n, \) be the probability of the joint distribution of the random variables \((X, U)\), subject to the initial data

\[ P_0(dx, du) = \delta_u(u_0(x)) \rho_0(x) dx, \]
where \( \rho_0 \) is a bounded nonnegative function from \( C(\mathbb{R}^n) \) and \( dx \) is Lebesgue measure on \( \mathbb{R}^n \), \( \delta_u \) is Dirac measure concentrated on \( u \). \( P(t, dx, du) \) has the form \( P(t, x, du) dx \), where \( P(t, x, du) \) is a positive measure with respect to \( u \) and a function with respect to \( x \) (density function of \( P(t, dx, du) \) with respect to Lebesgue measure).

We look at \( P = P(t, dx, du) \) as a generalized function (distribution) with respect to the variable \( u \). It satisfies the Fokker-Planck equation

\[
\frac{\partial P}{\partial t} = \left[ -\sum_{k=1}^{n} \frac{\partial}{\partial x_k} a_k(t, x, u) + \sum_{k=1}^{n} \frac{1}{2} \sigma_{1k}^2 \frac{\partial^2}{\partial x_k^2} + \sum_{k=1}^{n} \frac{1}{2} \sigma_{2k}^2 \frac{\partial^2}{\partial u_k^2} \right] P,
\]

subject to the initial data \( (3) \).

There is a standard procedure for finding the fundamental solution for \( (4) \) (see, e.g. [9]). This procedure consists in a reduction of the equation to a Fredholm integral equation, the solution of which can be found in the form of series. We are going to show that for \( a(t, x, u) = a(t, u) \) one can also find an explicit solution to the Cauchy problem \( (4), (3) \).

Let us introduce, still in the general case, the functions, for \( t \in \mathbb{R}_+ \), \( x \in \mathbb{R}^n \), depending on \( \sigma = (\sigma_1, \sigma_2) \):

\[
\rho_{\sigma}(t, x) = \int_{\mathbb{R}} P(t, x, du),
\]

\[
u_{\sigma}(t, x) = \frac{\int_{\mathbb{R}} uP(t, x, du)}{\int_{\mathbb{R}} P(t, x, du)},
\]

\[
a_{\sigma}(t, x) = \frac{\int_{\mathbb{R}} a(t, x, u)P(t, x, du)}{\int_{\mathbb{R}} P(t, x, du)},
\]

the integrals in the numerator being assumed to exist in the Lebesgue sense.

It will readily be observed that \( u_{\sigma}(0, x) = u_0(x) \) and \( a_{\sigma}(0, x) = a(0, x, u_0(x)) \).

We denote

\[
\bar{\rho}(t, x) = \lim_{\sigma \to 0} \rho_{\sigma}(t, x), \quad \bar{u}(t, x) = \lim_{\sigma \to 0} u_{\sigma}(t, x), \quad \bar{a}(t, x) = \lim_{\sigma \to 0} a_{\sigma}(t, x),
\]

provided these limits exist.
1 Case of a conservation law

Now we dwell on the simpler case of a conservation law, where \( a = a(t, u) \). Here the equation (4) can be solved explicitly. Moreover, for the sake of simplicity we set \( \sigma_2 = 0 \) and denote \( \sigma_1 = \sigma \).

**Proposition 1** If \( a = a(t, u) \), then problem (4), (3) has the following solution:

\[
P(t, x, du) = \frac{1}{(\sqrt{2\pi t} \sigma)^n} \int_{\mathbb{R}^n} \delta_u(u_0(y)) \rho_0(y) e^{-\sum_{i=1}^n \left| \int_0^t a_i(r, u_0(y)) dr + y_i - x_i \right|^2 / 2\sigma^2 t} dy, \quad (8)
\]

for all \( t \geq 0, x \in \mathbb{R}^n \), or, in other words,

\[
\int_{\mathbb{R}} \phi(u) P(t, x, du) = \frac{1}{(\sqrt{2\pi t} \sigma)^n} \int_{\mathbb{R}^n} \phi(u_0(y)) \rho_0(y) e^{-\sum_{i=1}^n \left| \int_0^t a_i(r, u_0(y)) dr + y_i - x_i \right|^2 / 2\sigma^2 t} dy, \quad (9)
\]

**Proof.** We act as in [3], [4]. Namely, we apply the Fourier transform to \( P(t, x, du) \) in (4), (3) with respect to the variable \( x \) and obtain the Cauchy problem for the Fourier transform \( \tilde{P} = \tilde{P}(t, \lambda, du) \) of \( P(t, x, du) \):

\[
\frac{\partial \tilde{P}}{\partial t} = -\left( \frac{1}{2} \sigma^2 |\lambda|^2 + i(\lambda, a(t, u)) \right) \tilde{P}, \quad (10)
\]

\[
\tilde{P}(0, \lambda, du) = \int_{\mathbb{R}^n} e^{-i(\lambda, y)} \delta_u(u_0(y)) \rho_0(y) dy, \quad \lambda \in \mathbb{R}^n. \quad (11)
\]

Equation (11) can easily be integrated and we obtain the solution given by the following formula:

\[
\tilde{P}(t, \lambda, du) = \tilde{P}(0, \lambda, du) e^{-\frac{1}{2} \sigma^2 |\lambda|^2 t + i \int_0^t (\lambda, a(\tau, u)) d\tau}. \quad (12)
\]

The inverse Fourier transform (in the distributional sense) allows to find the density function \( P(t, x, du), t > 0 \):

\[
P(t, x, du) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(\lambda, x)} \tilde{P}(t, \lambda, du) d\lambda =
\]

\[
= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n} e^{i(\lambda, x)} \left( \int_{\mathbb{R}^n} e^{-i(\lambda, y)} e^{-\int_0^t (\lambda, a(\tau, u)) d\tau} \delta_u(u_0(y)) \rho_0(y) dy \right) e^{-\frac{1}{2} \sigma^2 |\lambda|^2 t} d\lambda =
\]
\[ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \delta_u(u_0(y)) \rho_0(y) \int_{\mathbb{R}^n} e^{-\frac{1}{2\sigma^2t} \left( \lambda - \int_0^t a(\tau, u_0(y)) \right)^2 - \frac{1}{2\sigma^2t} \int_0^t |a(\tau, u_0(y))| y_i - x_i |^2^2} d\lambda dy = \]

\[ = \frac{1}{(\sqrt{2\pi}t\sigma)^n} \int_{\mathbb{R}^n} \delta_u(u_0(y)) \rho_0(y) e^{-\frac{1}{2\sigma^2t} \int_0^t |a(\tau, u_0(y))| y_i - x_i |^2^2} dy, \quad t \geq 0, x \in \mathbb{R}^n. \]

The third equality is satisfied by Fubini’s theorem, which can be applied by the absolute integrability and the bound on the function involved. Thus, the proposition is proved.

**Remark 1** In the general case \( \sigma_2 \neq 0 \) an analogous formula can be obtained in a similar way.

**Corollary 1** The functions \( \rho_{\sigma}, u_{\sigma} \) and \( a_{\sigma} \) defined in (5) – (7) can be represented by the following formulae:

\[ \rho_{\sigma}(t, x) = \int_{\mathbb{R}^n} \rho_0(y) e^{-\frac{\sum_{i=1}^n \int_0^t |a_i(t, u_0(y))| y_i - x_i |^2}{2\sigma^2t}} dy, \quad \text{(13)} \]

\[ u_{\sigma}(t, x) = \int_{\mathbb{R}^n} u_0(y) \rho_0(y) e^{-\frac{\sum_{i=1}^n \int_0^t |a_i(t, u_0(y))| y_i - x_i |^2}{2\sigma^2t}} dy \]

\[ = \int_{\mathbb{R}^n} \rho_0(y) e^{-\frac{\sum_{i=1}^n \int_0^t |a_i(t, u_0(y))| y_i - x_i |^2}{2\sigma^2t}} dy \]

\[ a_{\sigma}(t, x) = \int_{\mathbb{R}^n} a(t, u_0(y)) \rho_0(y) e^{-\frac{\sum_{i=1}^n \int_0^t |a_i(t, u_0(y))| y_i - x_i |^2}{2\sigma^2t}} ds \]

\[ = \int_{\mathbb{R}^n} \rho_0(y) e^{-\frac{\sum_{i=1}^n \int_0^t |a_i(t, u_0(y))| y_i - x_i |^2}{2\sigma^2t}} ds \]

**Proof.** The result is obtained by substitution of \( P(t, x, du) \) as given by (8) in (5), (6) and (7).
1.1 Asymptotic formula for smooth solutions

Let us define the following subset $\Lambda$ of $\mathbb{R}$:

$$
    t \in \Lambda \quad \text{if} \quad \inf_{y \in \mathbb{R}^n} \int_0^t \sum_{i=1}^n (a_i)_{u(\tau, u_0(y))}(u_0(y))_y \, d\tau > -1,
$$

(16)

where $u_0 \in C^1_b(\mathbb{R}^n)$. It is not difficult to show the $\Lambda$ is an open set. Denote $t^*_*(u_0) = \sup \Lambda$.

The following theorem holds:

**Theorem 1** Let $u(t, x)$ be a solution to the Cauchy problem

$$
    u_t + \sum_{i=1}^n a_i(t, u) u_{x_i} = 0, \quad u(0, x) = u_0(x),
$$

(17)

where $a_i$, $i = 1, ..., n$, are $C^1$-functions defined on some open subset of $(\mathbb{R}_+ \times \mathbb{R})$ and $u_0 \in C^1_b(\mathbb{R}^n)$. Assume $t^*_*(u_0) = \sup \Lambda > 0$, $\Lambda$ being defined by (16). Then for $t \in [0, t^*_*(u_0))$,

$$
    u(t, x) = \bar{u}(t, x) = \lim_{\sigma \to 0} u_\sigma(t, x),
$$

where $u_\sigma(t, x)$ is given by (6) and the limit exists pointwise.

**Proof.** The proof is similar to the one given in [4] for a related problem. According to the classical theory (see, e.g. [6]), Theorem 5.1.1), the solution $u$ of (17) exists on some maximal interval $[0, T)$, $T \leq \infty$ and is a $C^1$-smooth function. Since $u$ is constant along characteristics, its value at any point $(t, x)$, with $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$, satisfies the implicit relation

$$
    u(x, t) = u_0(x - \int_0^t \int_{\mathbb{R}^n} a(\tau, u)d\tau).
$$

(18)

In particular, the range of $u$ coincides with the range of $u_0$.

Differentiating (18) yields

$$
    \partial_x u(t, x) = \frac{\partial_y u_0(y)}{1 + \int_0^t \int_{\mathbb{R}^n} (a_i)_{u(\tau, u_0(y))}(u_0(y))_y \, d\tau}, \quad y = x - \int_0^t \int_{\mathbb{R}^n} a(\tau, u)d\tau.
$$

(19)

This imply $T = t^*_*(u_0)$. If $0 < t^*_*(u_0) < +\infty$, then the solution to the Cauchy problem blows up at the instant $t^*_*(u_0)$. Otherwise, the solution keeps its smoothness for all $t > 0$. 
The formula (6) implies, using the weak convergence of measures and the fact that \(\rho_0\) and \(u_0\) are continuous and bounded and independent of \(\sigma\)

\[
\lim_{\sigma \to 0} u_\sigma(t, x) = \frac{\int_{\mathbb{R}^n} u_0(y) \rho_0(y) \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi \sigma^n}} e^{-\frac{1}{2} \frac{1}{\sigma^2} |t \int_0^t a(\tau, u_0(y)) d\tau + y - x|^2} dy}{\int_{\mathbb{R}^n} \rho_0(y) \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi \sigma^n}} e^{-\frac{1}{2} \frac{1}{\sigma^2} |t \int_0^t a(\tau, u_0(y)) d\tau + y - x|^2} dy}
\]

with

\[
p(t, x, y) := \int_0^t a(\tau, u_0(y)) d\tau + y - x,
\]

where \(\delta_p\) is the Dirac measure at \(p \in \mathbb{R}^n\). We can use locally the implicit function theorem and find \(y = y_{t,x}(p)\) from \(p(t, x, y)\). The condition for existence of this function is the invertibility of the matrix

\[
C_{ij}(t, y) = \frac{\partial p_i(t, x, y)}{\partial y_j}, \quad i, j = 1, \ldots, n.
\]

This matrix fails to be invertible for \(t = t_*(u_0)\). For \(t < t_*(u_0)\)

\[
\tilde{u}(t, x) = \lim_{\sigma \to 0} u_\sigma(t, x) = \frac{\int_{\mathbb{R}^n} u_0(y) \rho_0(y) \delta_{p(t,x,y)} dy}{\int_{\mathbb{R}^n} \rho_0(y) \delta_{p(t,x,y)} dy} = u_0(y_{t,x}(0)).
\]

Let us introduce the new notation \(y_0(t, x) \equiv y_{t,x}(0)\). Then (20) implies the following vectorial equation:

\[
\int_0^t a(\tau, u_0(y_0(\tau, x))) d\tau + y_0(t, x) - x = 0, \quad t \geq 0, \quad x \in \mathbb{R}^n.
\]

(21)

Let us show that \(u(t, x) = u_0(y_0(t, x))\) satisfies equation (1), that is

\[
\sum_{j=1}^n \partial_j u_0(y_0, x) + \sum_{j,k=1}^n a_j(t, u_0) \partial_k u_0(y_0, x) x_j = 0.
\]

and \(u_0(y_0(0, x)) = u_0(x)\). Here we denote by \(y_{0,i}\) the \(i\) - th components of the vector \(y_0\).
For \( t < t^* \) we can differentiate (21) with respect to \( t \) and \( x_j \) to get the matrix equations:

\[
\sum_{j=1}^{n} C_{ij} (y_0,j)_t + u_0,i = 0, \quad i = 1, \ldots, n,
\]

and

\[
\sum_{k=1}^{n} C_{ik} (y_0,k)_x + \delta_{ij} = 0, \quad i, j = 1, \ldots, n,
\]

where \( \delta_{ij} \) is the Kronecker symbol. The equations imply

\[
(y_0,j)_t = - \sum_{i=1}^{n} (C^{-1})_{ij} u_0,i, \quad (y_0,k)_x = -(C^{-1})_{jk}.
\]

(23)

It remains now only to substitute (23) into (22) to see that \( \bar{u}(t, x) \) satisfies the first equation in (17).

Further, (21) implies \( u_0(y_0(0, x)) = u_0(x) \), thus Theorem 1 is proved.

**Remark 2** For \( a_i = a_i(u) \) (i.e. \( a_i \) is independent of variable \( t \)) we have the Conway’ criterium [5]:

\[
t^*_*(u_0) = \sup_{y \in \mathbb{R}^n} \left( - \frac{1}{\sum_{i=1}^{n} (a_i)_u(u_0(y))(u_0(y))_y} \right).
\]

Note that if the denominator vanishes, then \( t^*_*(u_0) = \infty \) and the solution does not blow up. If \( t^*_*(u_0) < 0 \), then the solution is globally smooth for \( t \geq 0 \), as well.

**Proposition 2** Under the assumptions of Theorem 7 the vector

\[
\bar{a}(t, x) = \lim_{\sigma \to 0} a_\sigma(t, x),
\]

where \( a_\sigma(t, x) \) is given by (7), solves the multidimensional Burgers equation

\[
(\bar{a}(t, u))_t + (\bar{a}(t, u), \nabla)\bar{a}(t, u) = 0
\]

with initial data \( \bar{a}(0, u(0, x)) = a(0, u_0(x)) \).

**Proof.** This fact follows directly from Proposition 2.1 of [4].

**Remark 3** The introduction of a small perturbation of deterministic equation to study then the original equation in the limit of vanishing noise has appeared in several contexts, particularly for equations of the reaction-diffusion type, see, e.g. [1], [7] and references therein.
1.2 Associated system of balance laws

Now we consider the following question: what system of equations do the triple \((\rho_\sigma, u_\sigma, a_\sigma)\) and its limit \((\bar{\rho}, \bar{u}, \bar{a})\) satisfy before and after the blow up time \(t_*(u_0)\)?

The following proposition holds:

**Proposition 3** The functions \(\rho_\sigma, u_\sigma\) and \(a_\sigma\), given by \((5)\), \((6)\) and \((7)\), satisfy for \(t \geq 0\) the following PDE system:

\[
\frac{\partial \rho_\sigma}{\partial t} + \text{div}_x(\rho_\sigma a_\sigma) = \frac{1}{2} \sigma^2 \sum_{k=1}^{n} \frac{\partial^2 \rho_\sigma}{\partial x_k^2}, \tag{24}
\]

\[
\frac{\partial (\rho_\sigma u_\sigma)}{\partial t} + \text{div}_x(\rho_\sigma u_\sigma a_\sigma) = \frac{1}{2} \sigma^2 \sum_{k=1}^{n} \frac{\partial^2 (\rho_\sigma u_\sigma)}{\partial x_k^2} - I_u^u, \tag{25}
\]

where

\[
I_u^u = \int_{\mathbb{R}^n} (u - u_\sigma(t, x))((a(t, u) - a_\sigma(t, x)), \nabla_x P(t, x, du));
\]

\[
\frac{\partial (\rho_\sigma a_{\sigma,i})}{\partial t} + \text{div}_x(\rho_\sigma a_{\sigma,i} a_\sigma) = \frac{1}{2} \sigma^2 \sum_{k=1}^{n} \frac{\partial^2 (\rho_\sigma a_{\sigma,i})}{\partial x_k^2} - I_{\sigma,i}^a, \tag{26}
\]

\(i = 1, .., n\), where

\[
I_{\sigma,i}^a = \int_{\mathbb{R}^n} (a_i(t, u) - a_{\sigma,i}(t, x))((a(t, u) - a_\sigma(t, x)), \nabla_x P(t, x, du)) + \int_{\mathbb{R}^n} a_i(t, u)) t P(t, x, du).
\]

**Proof.** The equation \((24)\) follows from the Fokker-Planck equation \((4)\) directly.

Let us prove \((26)\) (the derivation of \((25)\) is analogous). We note that the definitions of \(a_\sigma(t, x)\) and \(\rho_\sigma(t, x)\) imply

\[
\frac{\partial (\rho_\sigma a_\sigma)}{\partial t} = \frac{\partial}{\partial t} \int_{\mathbb{R}^n} a(t, u) P(t, x, du) = \int_{\mathbb{R}^n} a(t, u) P_t(t, x, du) = \\
- \int_{\mathbb{R}^n} a(t, u)(a(t, u), \nabla_x P(t, x, du)) + \frac{1}{2} \sigma^2 \sum_{k=1}^{n} \frac{\partial^2 a_\sigma \rho_\sigma}{\partial x_k^2}, \tag{27}
\]

where \(P_t \equiv \frac{\partial}{\partial t} P\).
Further, we have
\[ \frac{\partial}{\partial x_k} \left( \rho \sigma \frac{a_{\sigma,k}}{a_{\sigma,i}} \right) = a_{\sigma,i} \left( t, x \right) \frac{\partial}{\partial x_k} \left( \int_{\mathbb{R}^n} a_k(t, u) P(t, x, du) \right) + \]
\[ \int_{\mathbb{R}^n} a_k(t, u) P(t, x, du) \frac{\partial}{\partial x_k} \left( \int_{\mathbb{R}^n} a_i(t, u) P(t, x, du) \right) = \int_{\mathbb{R}^n} a_{\sigma,i} \left( t, x \right) a_k(t, u) P_{x_k}(t, x, du) + \]
\[ \int_{\mathbb{R}^n} a_k(t, u) P(t, x, du) \frac{\partial}{\partial x_k} \left( \int_{\mathbb{R}^n} a_i(t, u) P_{x_k}(t, x, du) \right) \int_{\mathbb{R}^n} P(t, x, du) = \]
\[ \int_{\mathbb{R}^n} \left( a_k(t, u) a_{\sigma,i} \left( t, x \right) + a_i(t, u) a_{\sigma,k} \left( t, x \right) - a_{\sigma,k} \left( t, x \right) a_{\sigma,i} \left( t, x \right) \right) P_{x_k}(t, x, du), \]
i, k = 1, ..., n, with \( P_{x_k} \equiv \frac{\partial}{\partial x_k} P. \)

Equation (26) follows immediately from (27) and (28). Thus, Proposition 3 is proved.

**Corollary 2** Before the instant \( t^* \left( u_0 \right) \), the blow up time of the solution to the Cauchy problem (17), the triple \( \left( \bar{\rho}, \bar{u}, \bar{a} \right) \), which constitutes the limit as \( |\sigma| \to 0 \) of the triple \( \left( \rho_{\sigma}, u_{\sigma}, a_{\sigma} \right) \), solves the following system:
\[ \partial_t \bar{\rho} + \text{div}_x (\bar{\rho} \bar{a}) = 0, \]  
(28)
\[ \partial_t (\bar{\rho} \bar{u}) + \nabla_x (\bar{\rho} \bar{u} \bar{a}) = 0, \]  
(29)
\[ \partial_t (\bar{\rho} \bar{a}) + \nabla_x (\bar{\rho} \otimes \bar{a}) = 0. \]  
(30)

**Proof.** Equation (28) follows from the properties of parabolic differential equations with a small parameter in front of the derivatives of second order ([8], Theorem 3.1), since until the instance \( t^* \left( u_0 \right) \) the coefficients of equation (24) are differentiable. Equation (29) follows from (28) and Theorem 1. Proposition 2 implies (30).

**Remark 4** System (28) and (30) constitutes the so called pressureless gas dynamics system, the simplest model introduced to describe the formation of large structures in the Universe, see, e.g. [10].

**Remark 5** As it has been shown in [7] on an example, for discontinuous solutions to (17) the limits as \( \sigma \to 0 \) of the terms \( I_\sigma^a \) and \( I_\sigma^u \) do not vanish as \( \sigma \to 0 \) and yield some specific pressure.
Remark 6  The method of special stochastic perturbations, applied here, was used in [2], [3] for studying other deterministic problems.

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