NON-COMMUTATIVE DEFORMATIONS OF PERVERSE COHERENT SHEAVES AND RATIONAL CURVES

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Abstract. We consider non-commutative deformations of sheaves on algebraic varieties. We develop some tools to determine parameter algebras of versal non-commutative deformations for partial simple collections and the structure sheaves of smooth rational curves. We apply them to universal flopping contractions of length 2 and higher. We confirm Donovan-Wemyss conjecture in the case of deformations of Laufer’s flops.

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1. INTRODUCTION

The purpose of this paper to develop tools for calculating versal non-commutative (NC) deformations. It is a continuation of our papers [14], [15] and [16]. We develop two methods; one to determine the versal NC deformation as well as the NC deformation algebra for a partial simple collection of perverse coherent sheaves, and another to determine the degree 2 parts of defining equations of the NC deformation algebra for a smooth rational curve on a smooth variety. Then we apply these methods to examples of flopping contractions of rational curves in the case of length 2 and higher.

We assume that the base field $k$ is an algebraically closed field of characteristic 0. The first method is a generalization of Theorems 6.1 and 6.2 of [15]. We consider a projective morphism $f : Y \to X$ such that $X = \text{Spec}(R)$ for a complete local Noetherian $k$-algebra $R$ whose residue field is $k$. We assume that there is a locally free coherent sheaf $P$ on $Y$
which is a tilting generator of $D^b(\text{coh}(Y))$ as in \cite{15}. We define a category of perverse coherent sheaves $\text{Perv}(Y/X)$ to be the one corresponding to the category of modules $\text{mod-}A$ under Bondal-Rickard derived equivalence $D^b(\text{coh}(Y)) \cong D^b(\text{mod-}A)$, where $A = f_*\mathcal{E}nd(P)$ is a coherent sheaf of associative $\mathcal{O}_X$-algebras. Let $\{s_j\}_{j=1}^m$ be the set of all simple objects in $\text{Perv}(Y/X)$, and take arbitrary non-empty subset $J \subset \{1, \ldots, m\}$. Then we determine explicitly the versal multi-pointed NC deformation of the simple collection $\{s_j\}_{j \in J}$ as well as the NC deformation algebra, the parameter associative algebra of the versal deformation (Theorem 2.1). (15) Theorem 6.1 treated the case when $J = \{1, \ldots, m\}$, and Theorem 6.2 when $f$ has at most 1-dimensional fibers and $J$ is the complement of an element which corresponds to the structure sheaf). This is applied in §4 for determining NC deformations of a non-reduced fiber of a flopping contraction.

The second method concerns formal NC deformations of a smooth rational curve $C$ on a smooth variety $X$. In this case, the NC deformation algebra is a quotient ring of a NC power series ring $k\langle\langle\text{Ext}^1(\mathcal{O}_C, \mathcal{O}_C)\rangle\rangle$ divided by an ideal $I$ coming from an obstruction space $\text{Ext}^2(\mathcal{O}_C, \mathcal{O}_C)$ (cf. \cite{16}). An example by Donovan-Wemyss (6) Example 1.3) shows that a certain flopping curve on a smooth 3-fold has a deformation algebra $k\langle\langle a, b \rangle\rangle/(ab + ba, a^2 - b^3)$. The composition map

$$\text{Ext}^1(\mathcal{O}_C, \mathcal{O}_C) \otimes \text{Ext}^1(\mathcal{O}_C, \mathcal{O}_C) \to \text{Ext}^2(\mathcal{O}_C, \mathcal{O}_C)$$

determines the second order terms of the generators of $I$. We calculate the behavior of this map in terms of the positivity and the negativity of the normal bundle $N_{C/X}$ (Theorem 3.1 and Proposition 3.2). The reason why anti-symmetric relations like $ab + ba = 0$ appear in the NC deformation algebra is revealed to be the mixture of the positivity and the negativity of the normal bundle.

We apply these two results to investigate NC deformations of fibers for some flopping contractions. We consider projective birational morphisms $f : Y \to X$ from smooth varieties with at most 1-dimensional fibers and such that the canonical divisors $K_Y$ are relatively trivial.

The first case is a universal flopping contraction of length 2 constructed by Curto-Morrison (15). We have $\dim X = 6$, and the scheme theoretic central fiber $f^{-1}(0)$ has multiplicity 2 in this case. We determine NC deformation rings of the reduced central fiber and that of the scheme theoretic non-reduced fiber (Theorems 4.5 and 4.7). They correspond to the two irreducible components of the singular locus $\text{Sing}(X)$ of the base space $X$. When we consider only commutative deformations, then there are no obstructions in both cases. But their NC deformations look very different; one has more NC deformations over an irreducible component of $\text{Sing}(X)$ which is again singular, and the other has only commutative deformations over another irreducible component of $\text{Sing}(X)$ which is non-singular.
Next we consider deformations of Laufer’s flopping contraction. We consider a family of flopping contractions of 3-dimensional varieties over 2n-dimensional affine space for a positive integer n. This family of flops of length 2 was also considered independently by Van Garderen [25] in the context of Donaldson-Thomas invariants. We calculate the NC deformation algebra (Theorem 5.5) using Theorem 4.5. We prove that the base affine space has a stratification such that there are only 2n + 1 isomorphism types in this deformation family (Theorem 5.2), and then prove that the isomorphism types of the NC deformation algebras correspond bijectively to those of flopping contractions (Proposition 5.9 and Theorem 5.10). This is an affirmative answer to a conjecture of Donovan-Wemyss ([6] Conjecture 1.4) in this case.

Finally we determine NC deformation algebras for higher length universal flopping contractions (Theorem 6.2) using a classification result of Karmazyn ([10]). There has been a problem to understand all flopping rational curves on smooth 3-folds since 1980’s. It looks a simple problem, but is actually quite complicated probably because of its non-commutative nature, which is indicated by the difference of the proofs of Proposition 5.9 and Theorem 5.10.

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2. Versal deformations of partial simple collections

Let k be an algebraically closed field of characteristic 0, and let f : Y → X be a projective morphism of Noetherian k-schemes such that X = Spec(R) for a complete local algebra R whose residue field is k. A locally free coherent sheaf P on Y is called a tilting generator if the following conditions are satisfied: (1) all higher direct images of End(P) for f vanishes, (2) P generates the derived category of quasi-coherent sheaves D(Qcoh(Y)). Then the derived Morita equivalence theorem of Bondal and Rickard ([2], [21]) tells us that there is an equivalence of triangulated categories

\[ \Phi : D^b(\text{coh}(Y)) \cong D^b(\text{mod-A}) \]

given by \( \Phi(\bullet) = R\text{Hom}(P, \bullet) \), where A = f_* End(P) is a coherent sheaf of associative O_X-algebras. The category of perverse coherent sheaves Perv(Y/X) is defined to be the abelian subcategory of \( D^b(\text{coh}(Y)) \) corresponding to the category of finitely generated right A-modules (mod-A) under this equivalence ([15]). We note that P becomes a projective object in Perv(Y/X), because \( \Phi(P) \cong A \). For example, the category of perverse coherent sheaves \( p\text{Perv}(Y/X) \) by Bridgeland [3] and Van den Bergh [23] is a special case where f has at most 1-dimensional fibers.

The following is a generalization of [6] Definition 2.9, Definition 3.8, Lemma 3.9, and [15] Theorems 6.1 and 6.2. (For a flopping contraction
of a smooth 3-fold, the contraction algebra and the NC deformation algebra are defined in Definitions 2.9 and 3.8 of [6] as the factor algebra and the endomorphism algebra as in the theorem below, and they are shown to coincide in Lemma 3.9 of [6]. Theorem 6.1 treated the case when $J = \{1, \ldots, m\}$, and Theorem 6.2 when $f$ has at most 1-dimensional fibers and $J$ is the complement of an element which corresponds to the structure sheaf).

**Theorem 2.1.** Let $\{s_j\}_{j=1}^m$ be the set of all simple objects in $\text{Perv}(Y/X)$ above the closed point $x_0 \in X$, and let $P = \bigoplus_{i=1}^m P_i$ be the direct sum of all indecomposable projective objects in the category of perverse coherent sheaves $\text{Perv}(Y/X)$ such that $\dim \text{Hom}(P_i, s_j) = \delta_{ij}$. Let $J \subset \{1, \ldots, m\}$ be any non-empty subset, and let $J^c = \{1, \ldots, m\} \setminus J$ be the complement. Denote $P_J = \bigoplus_{i \in J} P_i$ and $P_{J^c} = \bigoplus_{i \in J^c} P_i$. Let $A = \text{End}(P)$ be an associative algebra of endomorphisms, and let $I \subset A$ be the two-sided ideal generated by $g \in A$ which factorizes in the form $P \to P_{J^c} \to P$ as $O_Y$-homomorphisms.

(1) Let $Q$ be defined by the following distinguished triangle in $D^b(\text{coh}(Y))$:

$$Q[-1] \to \text{Hom}(P_{J^c}, P) \otimes_{\text{End}(P_{J^c})} P_{J^c} \to P \to Q$$

where the tensor product is defined in $\text{Perv}(Y/X)$ (see the proof) and the morphism is a natural one. Then $Q \in \text{Perv}(Y/X)$.

(2) $Q$ is a versal NC deformation of a simple collection $\bigoplus_{j \in J} s_j$.

(3) The parameter algebra $A_{\text{def}} := \text{End}(Q)$ of the versal NC deformation is given by $A/I$.

**Proof.** (1) We denote $B = \text{End}(P_{J^c})$, and let

$$F_1 \to F_0 \to \text{Hom}(P_{J^c}, P) \to 0$$

be a resolution by free right $B$-modules. Then we define a tensor product $\text{Hom}(P_{J^c}, P) \otimes_B P_{J^c}$ in $\text{Perv}(Y/X)$ by an exact sequence in $\text{Perv}(Y/X)$:

$$F_1 \otimes_B P_{J^c} \to F_0 \otimes_B P_{J^c} \to \text{Hom}(P_{J^c}, P) \otimes_B P_{J^c} \to 0.$$ 

In order to prove that $Q \in \text{Perv}(Y/X)$, it is sufficient to show that the morphism $\text{Hom}(P_{J^c}, P) \otimes_B P_{J^c} \to P$ is injective in $\text{Perv}(Y/X)$. It is in turn sufficient to prove that the homomorphism which is obtained by applying $\text{Hom}(P, \bullet)$:

$$\text{Hom}(P_{J^c}, P) \otimes_B \text{Hom}(P, P_{J^c}) \to \text{Hom}(P, P)$$

is injective, because $P$ is a projective generator of $\text{Perv}(Y/X)$. We have an injective homomorphism of a direct summand

$$\text{Hom}(P_{J^c}, P) \otimes_k \text{Hom}(P, P_{J^c}) \to \text{Hom}(P, P) \otimes_k \text{Hom}(P, P).$$

The right (resp. left) action of $A$ on $\text{Hom}(P, P)$ induces the right (resp. left) action of $B$ on a direct summand $\text{Hom}(P_{J^c}, P)$ (resp. $\text{Hom}(P, P_{J^c})$). Therefore the homomorphism

$$\text{Hom}(P_{J^c}, P) \otimes_B \text{Hom}(P, P_{J^c}) \to \text{Hom}(P, P) \otimes_A \text{Hom}(P, P)$$

is injective, hence our claim, because $\text{Hom}(P, P) \otimes_A \text{Hom}(P, P) \cong \text{Hom}(P, P)$. 


(2) Since $\text{Hom}(P_{J^c}, \bullet)$ is an exact functor on $\text{Perv}(Y/X)$, we have an exact sequence

$$F_1 \to F_0 \to \text{Hom}(P_{J^c}, \text{Hom}(P_{J^c}, P) \otimes_B P_{J^c}) \to 0.$$ 

Hence

$$\text{Hom}(P_{J^c}, P) \cong \text{Hom}(P_{J^c}, \text{Hom}(P_{J^c}, P) \otimes_B P_{J^c}).$$

Then we have $R\text{Hom}(P_{J^c}, Q) = 0$.

We will prove that $Q$ is an inverse limit of iterated extensions of the $s_j$ for $j \in J$ by a similar argument to the proof of [15] Theorem 6.2. It is sufficient to prove that the quotient $Q/m^nQ$ for any $n$ is an iterated extension of the $s_j$ for only $j \in J^c$, where $m$ is the maximal ideal of $R$, because we have $Q = \varprojlim Q/m^nQ$. There is a filtration of $Q/m^nQ$ whose quotients are isomorphic to some $s_j$. We only need to prove that there appear no $s_j$ for $j \in J^c$. For this purpose, we use $\text{Hom}(P_{J^c}, s_j) = k$ for $j \in J^c$ and $\text{Hom}(P_{J^c}, s_j[1]) = 0$ for all $j$ together with $\text{Hom}(P_{J^c}, Q) = 0$. If there is a quotient which is isomorphic to some $s_j$ with $j \in J^c$, then there is a morphism from $P_{J^c}$ to this quotient, which extends successively to $Q/m^nQ$ for all $n$, and we obtain a contradiction with $\text{Hom}(P_{J^c}, Q) = 0$.

Now in order to prove the versality of $Q$, it is sufficient to prove that $\text{Hom}(Q, s_j) = k$ and $\text{Hom}(Q, s_j[1]) = 0$ for $j \in J$ by the argument of [15] Theorem 6.1. (The first assertion implies that the successive extensions of the $s_j$ for $j \in J$ towards $Q$ are all non-trivial, while the second implies that the final extension $Q$ is maximal.) Any non-zero homomorphism $\text{Hom}(P_{J^c}, P) \otimes_B P_{J^c} \to s_j$ is surjective in $\text{Perv}(Y/X)$ since $s_j$ is simple. Then the composed morphism $F_0 \otimes P_{J^c} \to s_j$ is also surjective. Therefore $\text{Hom}(\text{Hom}(P_{J^c}, P) \otimes_B P_{J^c}, s_j) = 0$ for $j \in J$. Then it follows that $\text{Hom}(Q, s_j) \cong \text{Hom}(P, s_j) \cong k$ and $\text{Hom}(Q, s_j[1]) = 0$ for $j \in J$.

(3) We know that the parameter algebra of the versal deformation is given as $A_{\text{def}} := \text{End}(Q)$ by [13]. $Q \oplus \bigoplus_{j \in J^c} s_j$ is an NC deformation of a simple collection $\bigoplus_{j=1}^m s_j$. Since $P$ is a versal NC deformation of the simple collection $\bigoplus_{j=1}^m s_j$, there is a homomorphism $h : P \to Q \oplus \bigoplus_{j \in J^c} s_j$ of NC deformations. We have an associated ring homomorphism of the parameter algebras $h_* : A \to A_{\text{def}}$.

Let $M, M_{\text{def}}$ be respectively the two-sided ideals of $A, A_{\text{def}}$ consisting of elements which induce 0 maps on the central fiber $\bigoplus_{j=1}^m s_j$. We have $A/M \cong A_{\text{def}}/M_{\text{def}} \cong k^m$. Then the ring homomorphism $h_*$ induces a homomorphism between the Zariski cotangent spaces $M/M^2 \to M_{\text{def}}/M_{\text{def}}^2$, which is the same as the projection

$$\bigoplus_{i,j=1}^m \text{Ext}^1(s_i, s_j)^* \to \bigoplus_{i,j \in J} \text{Ext}^1(s_i, s_j)^*$$

(cf. [16]). Since this is surjective, so is the ring homomorphism $h_*$, because $A = \varprojlim A/M^n$ and $A_{\text{def}} = \varprojlim A_{\text{def}}/M_{\text{def}}^n$. It follows that $h$ is also surjective.
Let \( g \in \text{End}(P) \) be an endomorphism which factors through \( \bigoplus_{i \in J^c} P_i \):

\[
P \longrightarrow \bigoplus_{i \in J^c} P_i \longrightarrow P
\]

\[
\downarrow h_*(g) \downarrow
\]

\[
Q \longrightarrow Q
\]

Since \( \text{Hom}(P_i, s_j) = 0 \) for \( i \in J^c \) and \( j \in J \), we have \( \text{Hom}(\bigoplus_{j \in J^c} P_i, Q) = 0 \).

Hence \( h_*g = 0 \).

Conversely, assume that \( g \in \text{End}(P) \) satisfies \( h_*g = 0 \). We have a commutative diagram of exact sequences in \( \text{Perv}(Y/X) \):

\[
0 \longrightarrow \text{Hom}(P_{J^c}, P) \otimes_B P_{J^c} \longrightarrow P \longrightarrow Q \longrightarrow 0
\]

\[
\downarrow \quad g \quad \downarrow 0
\]

\[
0 \longrightarrow \text{Hom}(P_{J^c}, P) \otimes_B P_{J^c} \longrightarrow P \longrightarrow Q \longrightarrow 0.
\]

By the diagram chasing, there is a homomorphism \( \tilde{g} : P \to \text{Hom}(P_{J^c}, P) \otimes_B P_{J^c} \) which lifts \( g \). Since \( F_0 \otimes_B P_{J^c} \to \text{Hom}(P_{J^c}, P) \otimes_B P_{J^c} \) is a surjection in \( \text{Perv}(Y/X) \) and \( P \) is a projective object, \( \tilde{g} \) is lifted to \( \tilde{g}_0 : P \to F_0 \otimes_B P_{J^c} \).

Therefore \( g \) belongs to the ideal \( I \). \( \square \)

### 3. Second infinitesimal neighborhood of a smooth rational curve

Let \( F \) be a coherent sheaf on an algebraic variety \( X \) with proper support. We can describe the NC deformation ring \( A_{\text{def}} \) of \( F \), the parameter algebra of the versal NC deformation, by using \( A^\infty \) algebra multiplications \([10]\).

The tangent space of the deformation ring is given by \( \text{Ext}^1(F, F) \), and the relations by \( \text{Ext}^2(F, F) \) in the following way.

There are \( A^\infty \)-multiplications \( m_n : (\text{Ext}^1(F, F))^\otimes n \to \text{Ext}^2(F, F) \) for \( n \geq 2 \). \( m_2 \) coincides with the usual composition. Let

\[
m = \sum m_n : \hat{T}(\text{Ext}^1(F, F)) \to \text{Ext}^2(F, F)
\]

be their formal sum, where \( \hat{T}(\text{Ext}^1(F, F)) = \prod_{n=0}^\infty (\text{Ext}^1(F, F))^\otimes n \) is the completed tensor algebra and we put \( m_0 = m_1 = 0 \). Then the NC deformation ring is given by

\[
A_{\text{def}} = \hat{T}(\text{Ext}^1(F, F)^*)/m^*(\text{Ext}^2(F, F)^*)
\]

where

\[
m^* : \text{Ext}^2(F, F)^* \to \hat{T}(\text{Ext}^1(F, F)^*)
\]

is the dual map. In particular, it is a quotient algebra of a non-commutative power series ring of dimension \( \dim \text{Ext}^1(F, F) \) divided by a two-sided ideal generated by \( \dim \text{Ext}^2(F, F) \) elements.
Now we apply the above description to the case where $C \subset X$ is a smooth rational curve on a smooth variety. We consider NC deformations of a sheaf $O_C$.

The quadratic terms of the relations are determined by the composition $$m_2 : \text{Ext}^1(O_C, O_C) \otimes \text{Ext}^1(O_C, O_C) \to \text{Ext}^2(O_C, O_C).$$ If $m_2$ is skew-symmetric (resp. symmetric), then the quadratic term is a commutative (resp. anti-commutative) relation by the following reason.

The quadratic terms of the relations are the image of the homomorphism $$m_2^* : \text{Ext}^2(O_C, O_C)^* \to \text{Ext}^1(O_C, O_C)^* \otimes \text{Ext}^1(O_C, O_C)^*$$ which is the dual of $m_2$. If $m_2(a_i, a_j) = \sum c_{ijk} b_k$, then we have $m_2^* b_k^* = \sum c_{ijk} a_i^* \otimes a_j^*$. Therefore, if $m_2$ is symmetric (resp. skew-symmetric), i.e., $m_2(a, b) = m_2(b, a)$ (resp. $m_2(a, b) + m_2(b, a) = 0$), then the image of $m_2^*$ is generated by elements of the form $a^* b^* + b^* a^*$ (resp. $a^* b^* - b^* a^*$).

If the normal bundle of $C$ has a direct sum decomposition with mixed signs, then the composition is symmetric:

**Theorem 3.1.** Let $X$ be a smooth 3-dimensional algebraic variety and let $C$ be a subvariety which is isomorphic to $\mathbb{P}^1$. Assume that the normal bundle $N_{C/X} \cong O_C(1) \oplus O_C(-a)$ for some $a > 0$. Then a natural bilinear form

$$\text{Ext}^1(O_C, O_C) \otimes \text{Ext}^1(O_C, O_C) \to \text{Ext}^2(O_C, O_C)$$

is symmetric.

**Proof.** Since $C$ is a locally complete intersection, we have $\text{Ext}^p(O_C, O_C) \cong O_C, N_{C/X}, \Lambda^p N_{C/X}$ for $p = 0, 1, 2$. We have a spectral sequence

$$E_2^{p,q} = H^p(\text{Ext}^q(O_C, O_C)) \Rightarrow \text{Ext}^{p+q}(O_C, O_C).$$

Since $H^p(O_C) = 0$ for $p = 1, 2$, we have $\text{Ext}^1(O_C, O_C) \cong H^0(N_{C/X}) \cong H^0(O_C(1))$.

Let $I_C$ be the ideal sheaf of $C$, i.e., $O_C = O_X/I_C$, and let $O_{2C} = O_X/I_C^2$. We have an exact sequence

$$0 \to N_{C/X}^* \to O_{2C} \to O_C \to 0$$

and a long exact sequence

$$0 \to \text{Hom}(O_C, O_C) \to \text{Hom}(O_{2C}, O_C) \to \text{Hom}(N_{C/X}^*, O_C) \to \text{Ext}^1(O_C, O_C).$$

Since the first two terms are isomorphic to $k$ and the next two terms have the same dimension, the connecting homomorphism $\text{Hom}(N_{C/X}^*, O_C) \to \text{Ext}^1(O_C, O_C)$ is a bijection, which is given by the extension class $e \in \text{Ext}^1(O_C, N_{C/X}^*)$ of (3.2) and a composition map

$$\text{Ext}^1(O_C, N_{C/X}^*) \times \text{Hom}(N_{C/X}^*, O_C) \to \text{Ext}^1(O_C, O_C)$$
Indeed we have an exact sequence \( L \otimes O \) for any open subset \( U \) to prove that \( G \) obtained by the following commutative diagram is obtained by the following commutative diagram:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & N^*_{C/X} & \longrightarrow & O_{2C} & \longrightarrow & O_C & \longrightarrow & 0 \\
   &   & \downarrow p &   & \downarrow &   & \downarrow &   & \\
0 & \longrightarrow & O_C(-1) & \longrightarrow & O_{C_2} & \longrightarrow & O_C & \longrightarrow & 0.
\end{array}
\]

Let \( g, g' \in \Ext^1(O_C, O_C) \). Then our bilinear form is given by \( (g, g') \mapsto g[1]g' \in \Hom(O_C, O_C[2]) = \Ext^2(O_C, O_C) \). We write \( g = h_1[1]p[1]e \) and \( g' = h'_1[1]p[1]e \). In order to prove that \( gg' = g'g \), it is sufficient to prove that \( h_1[1]p[1]eh'_1 = h'_1[1]p[1]eh_1 \in \Hom(O_C(-1), O_C[1]) \).

Let \( G, G' \) be the extensions of \( O_C(-1) \) by \( O_C \) corresponding to \( h_1[1]p[1]eh'_1 \) and \( h'_1[1]p[1]eh_1 \). We will prove that they are isomorphic as extensions. \( G \) is obtained by the following commutative diagram:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & O_C(-1) & \longrightarrow & O_{C_2} & \longrightarrow & O_C & \longrightarrow & 0 \\
   &   & \downarrow = &   & \downarrow h'_1 &   & \downarrow h_1 &   & \\
0 & \longrightarrow & O_C(-1) & \longrightarrow & F & \longrightarrow & O_C(-1) & \longrightarrow & 0 \\
   &   & \downarrow h_1 &   & \downarrow = &   &   &   & \\
0 & \longrightarrow & O_C & \longrightarrow & G & \longrightarrow & O_C(-1) & \longrightarrow & 0.
\end{array}
\]

\( G' \) is similarly defined by interchanging \( h_1 \) and \( h'_1 \).

We need to prove that \( G' \cong G \) as extensions of \( O_C(-1) \) by \( O_C \). We note that the \( O_X \)-submodule \( O_C \subset G \) is characterized by

\[
\Gamma(U, O_C) = \{ s \in \Gamma(U, G) \mid I_C s = 0 \}
\]

for any open subset \( U \subset C \), and similarly for \( G' \). Therefore it is sufficient to prove that \( G' \cong G \) as \( O_X \)-modules.

Let \( P, P' \) be the supports of the cokernels of the injective homomorphisms \( h_1, h'_1 \), respectively. Let \( L \) be any invertible sheaf on \( C_2 \subset 2C \) such that \( L \otimes O_C = O_C(P' - P) \cong O_C \). We have \( L \cong O_{C_2} \) because \( H^1(O_C(-1)) = 0 \). Indeed we have an exact sequence

\[
H^1(O_C(-1)) \rightarrow H^1(O_{C_2}') \rightarrow H^1(O_C') \rightarrow 0.
\]
Locally around $P$ and $P'$, we take analytic local coordinates $(x, y, z)$ and $(x', y', z')$ respectively, which satisfy the following:

\[ \mathcal{O}_{C, P} = k\{ x, y, z \}/(y, z) \cong k\{ x \}, \quad \mathcal{O}_{C, P'} = k\{ x', y', z' \}/(y', z') \cong k\{ x' \}, \]
\[ \mathcal{O}_{C_{2}, P} = k\{ x, y, z \}/(y^2, z) \cong k\{ x, y \}/(y^2), \]
\[ \mathcal{O}_{C_{2}, P'} = k\{ x', y', z' \}/(y'^2, z') \cong k\{ x', y' \}/(y'^2), \]
\[ h_1 \mathcal{O}_{C, P}(-1) = x\{ k\{ x, y, z \}/(y, z) \} \cong xk\{ x \}, \]
\[ h'_1 \mathcal{O}_{C, P'}(-1) = x'(k\{ x', y', z' \}/(y', z')) \cong x'k\{ x' \}. \]

Then
\[ G_{P'} = (x' + p'(y'), y')k\{ x', y' \}/(y'^2) = (x', y')k\{ x', y' \}/(y'^2) \]
for some function $p'(y') \in y'k\{ x', y' \}$, and
\[ G_{P'} = (1, x^{-1}y)k\{ x, y \}/(x^{-1}y^2) \]
where $z, z'$ act trivially on these modules. On the other hand, we have $G_{P} = (x, y)k\{ x, y \}/(y^2)$ and $G'_{P'} = (1, x'^{-1}y')k\{ x', y' \}/(x'^{-1}y'^2)$. Therefore we have $G' \cong G \otimes L \cong G$. 

On the other hand, if the normal bundle is positive, then the composition is skew-symmetric:

**Proposition 3.2.** Let $X$ be a smooth algebraic variety and $C \cong \mathbb{P}^1$ a subvariety. Assume that $H^1(N_{C/X}) = 0$. Then the bilinear form (3.1) is skew-symmetric.

**Proof.** We have Ext$^1(\mathcal{O}_C, \mathcal{O}_C) = H^0(N_{C/X})$, Ext$^2(\mathcal{O}_C, \mathcal{O}_C) = H^0(\wedge^2 N_{C/X})$, and the bilinear form (3.1) comes from the wedge product. Therefore it is skew-symmetric.

4. **Example: universal flopping contraction of length 2**

A projective birational morphism $f : Y \to X$ from a variety with only terminal (or canonical, or more) singularities to a normal variety is called a **flopping contraction** if the exceptional locus has codimension at least 2 and the canonical divisor $K_Y$ is numerically trivial along the exceptional curves. We consider only the case where $Y$ is smooth in this paper.

Let us consider the case where $\dim Y = 3$, the exceptional curve is isomorphic to $\mathbb{P}^1$, and $X$ is a germ of a singularity. The analytic type of a generic hyperplane section of $X$ through its singularity is classified by Katz and Morrison [12] using a result of [20] (an easy alternative proof is found in [13]) in the case where the exceptional locus is irreducible:

**Theorem 4.1.** Let $f : Y \to X$ be a flopping contraction of a smooth 3-fold such that the exceptional locus $C$ of $f$ is a smooth rational curve. Let $P = f(C)$ be the singular point of $X$, let $H$ be a general hyperplane section of $X$ through $P$, and let $L = f^{-1}(H)$ be its inverse image. Let $l$ be the length of the scheme theoretic fiber $f^{-1}(P)$ at the generic point of $C$. Then the length
$l$ takes value in a set $\{1, 2, 3, 4, 5, 6\}$, and the singularity of $H$ together with the partial resolution $f_H : L \to H$ is determined by $l$. More precisely, $H$ has a rational double point of type $A_1, D_4, E_6, E_7, E_8$, or $E_8$, if $l = 1, 2, 3, 4, 5$ or $6$, respectively, and the exceptional divisor of $f_H$ is the rational curve which appears in the minimal resolution of $H$ and uniquely determined by the condition that it has multiplicity $l$ in the fundamental cycle.

A universal flopping contraction morphism $\tilde{f} : \tilde{Y} \to \tilde{X}$ of length $l$ is a versal deformation of the contraction morphism $f_H : L \to H$ of surfaces with the given length as a neighborhood of the exceptional curve $C$ described in the above theorem. We have $\dim \tilde{X} = 3, 6, 8, 9, 10$ if $l = 1, 2, 3, 4, 5, 6$ (cf. [10]).

As a corollary of the above theorem, we deduce that a universal flopping contraction morphism is universal in the following sense: for any flopping contraction $f : Y \to X$ of smooth 3-fold with irreducible exceptional locus and length $l$, there is a morphism $p : X \to \tilde{X}$ such that $f$ is isomorphic to the pull back of $\tilde{f}$ by $p$, so that $Y \cong \tilde{Y} \times \tilde{X}$ and $f$ corresponds to the second projection.

Curto-Morrison [5] constructed a universal flopping contraction morphism of length $l = 2$ explicitly:

**Theorem 4.2.** Let $\tilde{X}$ is a hypersurface given by the following equation in $k^7$:

$$F = x^2 + uy^2 + 2vyz + wz^2 + (uw - v^2)t^2 = 0.$$  

Then there exists a maximal Cohen-Macaulay sheaf $f_*M$ of rank 2 on $\tilde{X}$ such that a universal flopping contraction morphism of length $l = 2$ is given as a Grassmann blowup $\tilde{f} : \tilde{Y} \to \tilde{X}$, a universal projective birational morphism such that the inverse image modulo torsion $M$ of $f_*M$ become locally free.

Let $S = k[x, y, z, t, u, v, w]$ be a polynomial ring and let $R = S/(F) = O_{\tilde{X}}$. The sheaf $f_*M$ has a matrix factorization (7) as follows (5): it has a resolution by free $S$-modules

$$\ldots \xrightarrow{\Psi} S^4 \xrightarrow{\Phi} S^4 \xrightarrow{\Psi} S^4 \xrightarrow{\Phi} S^4 \xrightarrow{\Phi} f_*M \longrightarrow 0$$

where $\Phi = xI - \Xi$ and $\Psi = xI + \Xi$ with

$$\Xi = \begin{pmatrix} -vt & y & z & t \\ -uy - 2vt & vt & -ut & z \\ -wz & wt & -vt & -y \\ -uvt & -wz & uy + 2vt & vt \end{pmatrix}$$

such that

$$\Phi \Psi = \Psi \Phi = FI_4.$$  

$f_*M$ and $R = O_{\tilde{X}}$ are the indecomposable maximal Cohen-Macaulay sheaves on $\tilde{X}$. According to Van den Bergh (23), $M \oplus O_{\tilde{X}}$ become a tilting generator of $D(Qcoh(\tilde{Y}))$, and the category of the perverse coherent sheaves
$^{-1}\text{Perv}(\tilde{Y}/\tilde{X})$ (denoted $\text{Perv}(\tilde{Y}/\tilde{X})$ in §2) is defined as the subcategory of $D^b(\text{coh}(\tilde{Y}))$ which corresponds to the category of modules (mod-$A$) under the Bondal-Rickard equivalence

$$D^b(\text{coh}(\tilde{Y})) \cong D^b(\text{mod-}A)$$

where $A = \text{End}(M \oplus O_{\tilde{Y}})$ is a sheaf of associative algebras on $\tilde{X}$.

The algebra $A$ is determined by Aspinwall-Morrison [1]. There are generators: $a, b : f_*M \to f_*M$, $c : f_*M \to R$, $d : R \to f_*M$ expressed by using the matrix factorization: we have

$$a = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -u & 0 & 0 & 0 \\ -2v & 0 & 0 & 1 \\ 0 & 2v & -u & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -w & 0 & 0 & 0 \\ 0 & w & 0 & 0 \end{pmatrix},$$

$$c = (x - vt \ y \ z \ t), \quad d = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For example, we have a commutative diagram

$$\begin{array}{cccccc}
\ldots & \xrightarrow{\Psi} & S^4 & \xrightarrow{\Phi} & S^4 & \xrightarrow{f_*M} \to 0 \\
& a & \downarrow & a & \downarrow & a \\
\ldots & \xrightarrow{\Psi} & S^4 & \xrightarrow{\Phi} & S^4 & \xrightarrow{f_*M} \to 0
\end{array}$$

where we used the same symbol $a$ for an element of $A$ and its lift given by a matrix.

**Theorem 4.3.** The normal bundle of the reduced fiber $C = \tilde{f}^{-1}(0)_{\text{red}}$ is given by

$$N_{C/\tilde{X}} \cong \mathcal{O}_C(1) \oplus \mathcal{O}_C \oplus \mathcal{O}_C(-1)^3.$$

**Proof.** By [5], a neighborhood of $C \subset \tilde{Y}$ is covered by two open subsets $U_1, U_2$. $U_1$ has coordinates $(z, t, u, v, \alpha_{12}, \alpha_{22})$ such that the coordinates $x, y, w$ of $X$ are given by ([5] formulas (44), (45), (48)):

$$y + \alpha_{12}z + \alpha_{22}t = 0,$$

$$w + \alpha_{22}^2 + \alpha_{12}^2u - 2\alpha_{12}v = 0,$$

$$x + vt - \alpha_{12}ut + \alpha_{22}z = 0,$$

while $U_2$ has coordinates $(y, t, v, w, \beta_{12}, \beta_{22})$ such that ([5] formula (57)):

$$\beta_{12}y + z + \beta_{22}t = 0,$$

$$\beta_{12}^2w + \beta_{22}^2 + u - 2\beta_{12}v = 0,$$

$$x - vt + \beta_{12}ut - \beta_{22}y = 0.$$
These formulas are equivalent under the transformation:
\[ \alpha_{12} \beta_{12} = 1, \quad \alpha_{22} \beta_{12} = \beta_{22}. \]

The conormal bundle of the reduced central fiber \( C \) is generated by \((z, t, u, v, \alpha_{22})\) in \( U_1 \) and \((y, t, v, w, \beta_{22})\) in \( U_2 \). Since we have
\[ y \equiv -\alpha_{12} z \mod I_C^2 \]
there is a subbundle \( L_1 \) of degree 1 of \( N_{C/\tilde{Y}}^* \) generated by \( z \) in \( U_1 \) and \( y \) in \( U_2 \). We have also
\[ w \equiv -\alpha_{12}^2 u + 2\alpha_{12} v = \alpha_{12}(-\alpha_{12} u + 2v) \mod I_C^2 \]
and there is a subbundle \( L_2 \) of degree 1 generated by \(-\alpha_{12} u + 2v\) in \( U_1 \) and by \( w \) in \( U_2 \). A relation
\[ 2v \equiv \alpha_{12} u \mod L_2 \]
gives a subbundle \( L_3 \) of degree 1 generated by \( u \) in \( U_1 \) and by \( 2v \) in \( U_2 \).

\[ t \] generates a subbundle \( L_4 \) of degree 0. There is a subbundle \( L_5 \) of degree \(-1\) generated by \( \alpha_{22} \) in \( U_1 \) and by \( \beta_{22} \) in \( U_2 \). Therefore we have our claim. \( \square \)

**Remark 4.4.** We have \( H^1(N_{C/\tilde{Y}}^*) = 0 \). Hence the commutative deformations of \( C \) in \( \tilde{Y} \) have no obstruction.

We have \( \dim \text{Ext}^1(O_C, O_C) = \dim H^0(N_{C/X}^*) = 3 \), hence the tangent spaces of commutative and non-commutative deformations are the same. We have a basis \( \{a, b, t\} \) of the cotangent space \( H^0(N_{C/X}^*)^* = \text{Ext}^1(O_C, O_C)^* \).

On the other hand, we have
\[ \bigwedge^2 N_{C/X} \cong \mathcal{O}_C(1) \oplus \mathcal{O}_C^3 \oplus \mathcal{O}_C(-1)^3 \oplus \mathcal{O}_C(-2)^3. \]

Hence \( \dim \text{Ext}^2(O_C, O_C) = \dim H^0(\bigwedge^2 N_{C/X}) = 5 \), and there are 5 relations among the generators \( a, b, t \) in NC deformations.

First we determine the NC deformation algebra for the reduced fiber \( C = \tilde{f}^{-1}(0)_{\text{red}} \):

**Theorem 4.5.** The NC deformation algebra of \( O_C \) on \( \tilde{Y} \) is given by
\[ A_{\text{def}}^1 = k[\langle a, b, t \rangle]/(at - ta, bt - tb, tab - tba, ab^2 - b^2a, a^2b - ba^2) \]
\[ = k[[t]][\langle a, b \rangle]/(tab - tba, ab^2 - b^2a, a^2b - ba^2). \]

**Proof.** The set of simple objects in \( \text{Perv}(\tilde{Y}/\tilde{X}) \) corresponding to the set of indecomposable projective objects \( \{M, O_Y\} \) is \( \{O_C(-1)[1], O_C\} \), where \( \tilde{C} \) is the scheme theoretic fiber \( \tilde{f}^{-1}(0) \) which has length 2. In other words, we have
\[ \text{Hom}(M, O_C(-1)[1]) \cong k, \quad \text{Hom}(M, O_C) \cong 0 \]
\[ \text{Hom}(O_Y, O_C(-1)[1]) \cong 0, \quad \text{Hom}(O_Y, O_C) \cong k. \]
Therefore we have $A_{\text{def}}^1 = A/I^1$ by Theorem 2.1 where $I^1$ is a two-sided ideal generated by endomorphisms which are compositions of 2 homomorphisms of the form $M \oplus \mathcal{O}_Y \to \mathcal{O}_Y \to M \oplus \mathcal{O}_Y$. Therefore $A_{\text{def}}^1$ is generated by $a, b$ over $R$.

We can check the following equations by examining the matrices as endomorphisms of $S^4$:

\[
\begin{align*}
a^2 &= -u \\
b^2 &= -w \\
ab + ba &= -2v.
\end{align*}
\]

We have

\[
y - tb + bdc + dcb \\
= \begin{pmatrix}
y & 0 & -t & 0 \\
0 & y & 0 & t \\
w & 0 & y & 0 \\
0 & -w & 0 & y
\end{pmatrix}
+ \begin{pmatrix}
-x + vt & -y & -z & -t \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-w & w & x - vt & -y
\end{pmatrix}
\equiv 0 \mod \text{Im}(\Phi)
\]

as endomorphisms of $S^4$, so that we have

\[
y - tb + bdc + dcb = 0
\]

as endomorphisms of $f_*M = \text{Coker}(\Phi)$, hence

\[
(4.3) \quad y = tb
\]
in \( A_{\text{def}}^1 \). We also have
\[
z + ta - adc - dca
\]
\[
= \begin{pmatrix} z & t & 0 & 0 \\ -ut & z & 0 & 0 \\ -2vt & 0 & z & t \\ 0 & 2vt & -ut & z \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ x - vt & y & z & t \\ 0 & 0 & 0 & 0 \\ -uy - 2vz & x + vt & -ut & z \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -uy - 2vz & x + vt & -ut & z \end{pmatrix}
\]
\[
= \begin{pmatrix} z & t & 0 & 0 \\ -ut & z & 0 & 0 \\ -x - vt & -y & 0 & 0 \\ uy + 2vz & -x + vt & 0 & 0 \end{pmatrix} \equiv 0 \mod \text{Im(}\Phi\text{)}
\]
\[
x + vt + tba - dcab + badc
\]
\[
= \begin{pmatrix} x - vt & 0 & 0 & t \\ 0 & x - vt & ut & 0 \\ 0 & -wt & x + vt & 0 \\ -uwt & 0 & 0 & x + vt \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ uwt & wz & -uy - 2vz & -x - vt \end{pmatrix}
\]
\[
- \begin{pmatrix} x - vt & y & z & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -y & -z & 0 \\ 0 & x - vt & ut & 0 \\ 0 & -wt & x + vt & 0 \\ 0 & wz & -uy - 2vz & 0 \end{pmatrix} \equiv 0 \mod \text{Im(}\Phi\text{)}.
\]

Hence
\[
z = -ta \\
x = -tba - vt
\]
in \( A_{\text{def}}^1 \). Therefore \( A_{\text{def}}^1 \) is generated by \( a, b, t \) over \( k \).

We determine quadratic relations. Since \( H^1(N_{C/X}) = 0 \), they are skew-symmetric bilinear form (Proposition 3.2). Since \( t \in R \) is in the center, we have \( ta - at = 0 \) and \( tb - bt = 0 \), and there are no more quadratic terms. Since there are 5 generators of the relation ideal, there are 3 more relations. Since \( ay = ya \), we have \( tab - ath = tba \), hence \( tab - tba = 0 \). Since \( a^2, b^2 \) are in the center, we have \( ab^2 - b^2a = a^2b - ba^2 = 0 \). These 3 relations are order 3 and linearly independent. Therefore there are no more relations. □

The commutative deformations of \( \mathcal{O}_C \) on \( \bar{Y} \) is given by the following. We denote by \( (A_{\text{def}}^1)_{ab} \) the abelianization of the NC deformation algebra \( A_{\text{def}}^1 \), which is the parameter algebra of commutative deformations. (This is because the natural homomorphism \( A_{\text{def}}^1 \to (A_{\text{def}}^1)_{ab} \) is universal among local homomorphisms \( A_{\text{def}}^1 \to B \) to Artin local algebras).

**Corollary 4.6.** \( (A_{\text{def}}^1)_{ab} = k[[a, b, t]] \).

Next we investigate NC deformations of the scheme theoretic fiber \( \bar{C} = \bar{f}^{-1}(0) \).
Theorem 4.7. The NC deformation algebra of $\mathcal{O}_\tilde{C}$ on $\tilde{Y}$ is given by

$$A^2_{\text{def}} = k[[u, v, w]].$$

Proof. We have $A^2_{\text{def}} = A/I^2$ by Theorem 2.1, where $I^2$ is a two-sided ideal generated by endomorphisms which are compositions of 2 homomorphisms of the form $M \oplus \mathcal{O}_\tilde{Y} \to M \to M \oplus \mathcal{O}_\tilde{Y}$. Therefore $A^2_{\text{def}}$ is a quotient ring of $R$. In particular, there are only commutative deformations.

We have

$$cd = t, \quad cad = z, \quad cbd = -y, \quad cbad = x - vt.$$ 

Therefore $A^2_{\text{def}}$ is a quotient ring of $k[[u, v, w]]$.

There is an affine space $D$ of dimension 3 with coordinates $(u, v, w)$ defined by $x = y = z = t = 0$ contained in $\tilde{X}$. If we pull it back by $\tilde{f}$, we obtain a 4-dimensional smooth subspace $E$ of $\tilde{Y}$ with coordinates $(u, v, \alpha_{12}, \alpha_{22})$ on $U_1$ and $(v, w, \beta_{12}, \beta_{22})$ on $U_2$, where we used the notation of the proof of Theorem 4.3. The morphism $\tilde{f}_E : E \to D$ is given by

$$w = -\alpha_{22}^2 - \alpha_{12}^2 u + 2 \alpha_{12} v$$
$$u = -\beta_{22}^2 - \beta_{12}^2 w + 2 \beta_{12} v.$$ 

Thus $\tilde{f}_E$ is a flat morphism and gives a commutative deformation of the fiber $\tilde{C}$ with the whole space $D$ as a parameter space. By taking the completion at the origin, we obtain the formal power series ring as stated. □

We determine the singular locus of $\tilde{X}$ as well as the singularities of along it:

Lemma 4.8. (1) The singular locus of $\tilde{X}$ consists of two irreducible components $\text{Sing}(\tilde{X}) = S_1 \cup S_2$ with

$$S_1 = \{ x = uw - v^2 = uy + vz = y^2 + wt^2 = z^2 + ut^2 = 0 \},$$

$$S_2 = \{ x = y = z = t = 0 \}.$$

(2) $\tilde{X}$ has $A_1$ singularities along generic points of the singular locus $\text{Sing}(\tilde{X}) = S_1 \cup S_2$.

Proof. (1) The partial derivations of $F$ yield equations of the singular locus

$$x = uy + vz = vy + wz = t(uw - v^2)$$
$$y^2 + wt^2 = yz - vt^2 = z^2 + ut^2 = 0.$$ 

If $t = 0$, then $x = y = z = t = 0$, thus we have $S_2$. If $t \neq 0$ and $u = 0$, then $x = z = u = v = y^2 + wt^2 = 0$. This locus is denoted by $S_3$. If $t \neq 0$ and $w = 0$, then $x = y = v = w = z^2 + ut^2 = 0$. This locus is denoted by $S_4$.

If $uw \neq 0$, then $vyz \neq 0$, and $y = -vz/u = -wz/v$. Hence $x = 0$, $uw = v^2$, $uy + vz = 0$, $y^2 + wt^2 = 0$, $z^2 + ut^2 = 0$, thus we obtain $S_1$ which contains $S_3, S_4$. 15
(2) Let \( A = uw - v^2, B = uy + vz, C = z^2 + ut^2 \). Then we have
\[ uF = ux^2 + B^2 + AC. \]
Thus we have a family of \( A_1 \) along generic points of \( S_1 \).
Since \( F \) is a quadratic form on \( x, y, z, t, X \) has also a family of \( A_1 \) along generic points of \( S_2 \).

**Proposition 4.9.** The versal commutative deformation of \( O_C \) (resp. \( O_{\bar{C}} \)) is along the component \( S_1 \) (resp. \( S_2 \)).

**Proof.** We have the following equalities in the abelianization \((A^1_{\mathrm{der}})^{ab}\), the parameter algebra of the versal commutative deformation:
\[
\begin{align*}
x &= -tba + \frac{1}{2} t(ab + ba) = \frac{1}{2} t(ab - ba) = 0, \\
 uw - v^2 &= a^2 b^2 - \frac{1}{4} (ab + ba)^2 = -\frac{1}{4} (ab - ba)^2 = 0, \\
 uy + vz &= -ta^2 b + \frac{1}{2} t(aba + baa) = \frac{1}{2} t(ab - ba^2) = 0, \\
 z^2 + ut^2 &= t^2 a^2 - t^2 a^2 = 0.
\end{align*}
\]
Since \( \dim S_1 = 3 \), the deformation of \( O_C \) covers the whole \( S_1 \).
The statement for the versal deformation of \( O_{\bar{C}} \) is already proved in Theorem 4.7.

**Remark 4.10.** The genus zero Gopakumar-Vafa invariants \( n_j \) for \( 1 \leq j \leq l \) defined by Katz [11] counts the number of rational curves on deformations of a flopping contraction of a 3-fold. In our case of the universal flopping contraction, these numbers come from the rational curves above the components \( S_j \). Toda [22] Theorem 1.1 proved a formula connecting the intersection multiplicities of \( S_j \) and the dimensions of the NC deformation algebras.

5. Example: deformations of Laufer’s flopping contraction

We consider a family of hypersurfaces \( X_\lambda \subset k^4 \) defined by equations:
\[
F_\lambda = x^2 + y^3 + \sum_{i=1}^{2n} \lambda_i y^2 (-w)^i + z^2 w + yw^{2n+1} - \sum_{i=1}^{2n} \lambda_i (-w)^{i+2n+1} = 0
\]
where \( n \) is a positive integer, \( \lambda = (\lambda_i) \), and \( \lambda_i \in k = \mathbb{C} \). They are obtained from the universal flopping contraction of Curto-Morrison [11] by the substitution:
\[
(5.1) \quad t = (-w)^n, \quad u = y + \sum_{i=1}^{2n} \lambda_i (-w)^i, \quad v = 0.
\]
The example of Morrison-Pinkham [19] is the case where \( n = 1 \). Laufer’s example [17] of a flopping contraction of length 2 is constructed from \( X_0 \),
and $F_0$ is weighted homogeneous with weights

$$wt(x, y, z, w) = (6n + 3, 4n + 2, 6n + 1, 4).$$

We assume that the parameters $\lambda_i$ are small in the sense that $\lambda \in U \subset k^{2n}$ for a neighborhood $U$ of 0, but Theorem 5.2 below implies that $\lambda \in k^{2n}$ can be arbitrary indeed.

**Lemma 5.1.** Let $\tilde{f} : \tilde{Y} \to \tilde{X}$ be the Grassmann blowup in Theorem 4.2. Then the morphism $f_\lambda : Y_\lambda \to X_\lambda$ obtained by the pull back by a morphism (5.1) is a flopping contraction of length 2.

**Proof.** By construction, $\tilde{f}$ is a birational morphism which is an isomorphism above the smooth locus of $\tilde{X}$, the fibers of $\tilde{f}$ above the singular locus are 1-dimensional, and the canonical divisor $K_{\tilde{Y}}$ is relatively numerically trivial.

When $\lambda = 0$, $Y_0 \to X_0$ is Laufer’s flopping contraction. Thus the image of $X_0$ on $\tilde{X}$ intersects the singular locus only at 0. Therefore the only singularity of $X_\lambda$ is isolated at 0 and $Y_\lambda$ is smooth because they are small deformations of $Y_0 \to X_0$. $K_{Y_\lambda}/X_\lambda$ is relatively numerically trivial because it is the pull-back of $K_{\tilde{Y}}/\tilde{X}$. The scheme theoretic fibers coincide $f_\lambda^{-1}(0) = f_0^{-1}(0) = \tilde{f}^{-1}(0)$ as schemes, hence the length is 2. □

We note that the exceptional locus of the contraction morphism $Y_\lambda \to X_\lambda$ is always the same curve $C$ inside $\tilde{Y}$.

The deformations of the hypersurface $X_0$ is parametrized by a quotient ring $U_0 = k[x, y, z, w]/J_0$ where $J_0$ is an ideal generated by the partial derivatives:

$$x, zw, 3y^2 + w^{2n+1}, z^2 + (2n+1)yw^{2n}$$

because $F_0$ is weighted homogeneous. Then we have mod $J_0$:

$$x \equiv zw \equiv 0, \quad z^3 \equiv -(2n+1)yzw^{2n} \equiv 0,$$

$$yw^{2n+1} \equiv -\frac{1}{2n+1}z^2w \equiv 0, \quad y^3 \equiv -\frac{1}{3}yw^{2n+1} \equiv 0,$$

$$w^{4n+2} \equiv 9y^4 \equiv 0, \quad y^2z \equiv -\frac{1}{3}z^2w^{2n+1} \equiv 0,$$

$$yz^2 \equiv -(2n+1)y^2w^{2n} \equiv \frac{2n+1}{3}w^{4n+1}, \quad y^2 \equiv -\frac{1}{3}w^{2n+1}.$$}

Thus the deformation space $U_0$ for $f_0$ is generated by the following monomials:

$$1, \quad w, \quad \ldots, \quad w^{4n+1}, \quad z, \quad z^2$$

$$y, \quad yw, \quad \ldots, \quad yw^{2n}, \quad yz.$$}

If we discard monomials of degree $\leq 2$ and those smaller than a monomial $yw^{2n+1}$, then the remaining monomials are $w^{2n+2}, \ldots, w^{4n+1}$, which are equivalent to $y^2w, \ldots, y^2w^{2n}$ mod $J_0$. In this way we obtain our $2n$-dimensional deformation family.
**Theorem 5.2.** Define a stratification \( \{ \Sigma_i \}_{i=0}^{2n} \) of the affine \( \lambda \)-space \( k^{2n} \) by \( \Sigma_0 = \{ 0 \} \) and \( \Sigma_i = \{ \lambda \mid \min \{ j \mid \lambda_j \neq 0 \} = i \} \) for \( i > 0 \). Then \( X_\lambda \cong X_{\lambda'} \) if \( \lambda, \lambda' \in \Sigma_i \) for fixed \( i \).

**Proof.** We take a \( \lambda \in \Sigma_i \). The versal deformations of the hypersurface \( X_\lambda \) is parametrized by a quotient ring \( U_\lambda = k[x, y, z, w]/J_\lambda \), where \( J_\lambda \) is an ideal generated by \( F_\lambda \) and the partial derivatives of \( F_\lambda \):

\[
x, \quad zw, \quad 3y^2 + \sum_{j=1}^{2n} 2\lambda_j y(-w)^j + w^{2n+1},
\]

\[
z^2 + (2n+1)yw^{2n} - \sum_{j=1}^{2n} \lambda_j y^j (-w)^{j-1} + \sum_{j=1}^{2n} \lambda_j (-w)^{j+2n}.
\]

We have

\[
(6n+3)x\partial F_\lambda/\partial x + (4n+2)y\partial F_\lambda/\partial y + (6n+1)z\partial F_\lambda/\partial z + 4w\partial F_\lambda/\partial w = (12n+6)F_\lambda + \sum_{j=1}^{2n} (4j - 4n - 2)\lambda_j (y^2(-w)^j - (-w)^{j+2n+1}).
\]

Therefore \( \sum_{j=i}^{2n} (4j - 4n - 2)\lambda_j (y^2(-w)^j - (-w)^{j+2n+1}) \in J_\lambda \) with \( \lambda_i \neq 0 \). Then it follows that \( y^2(-w)^j - (-w)^{j+2n+1} \in J_\lambda \) for all \( j \geq i \). Indeed, since \( X_\lambda \) has an isolated singularity, the ring \( U_\lambda \) is an Artin local ring. Then \( \sum_{j=i}^{2n} (4j - 4n - 2)\lambda_j (-w)^{j-i} \) becomes invertible in \( U_\lambda \).

The restricted deformation family \( X_{\lambda'} \) for \( \lambda' \in \Sigma_i \) induces a Kodaira-Spencer map \( \kappa_\lambda : T_{\Sigma_i, \lambda} \rightarrow U_\lambda \). Since \( T_{\Sigma_i, \lambda} \) is a vector space which has a basis corresponding to monomials \( y^2(-w)^j - (-w)^{j+2n+1} \) for \( j \geq i \), we deduce that \( \kappa_\lambda = 0 \). Since this is true for any \( \lambda \in \Sigma_i \), we conclude that the restricted deformation family is locally trivial. \( \square \)

**Remark 5.3.** We have an alternative proof of the above theorem which uses Theorem 5.8 below. Indeed the above proof shows that the commutative deformation algebras of hypersurfaces for \( \lambda \in \Sigma_i \) are independent of the coefficients \( \lambda_j \) for \( j > i \).

**Corollary 5.4.** There are only \( 2n + 1 \) isomorphism classes in the deformation family \( X_\lambda \). They are isomorphic to \( X_0 \) defined by \( F_0 = 0 \) or \( X_i \) defined by

\[
F_i = x^2 + y^3 + y^2(-w)^i + z^2w + yw^{2n+1} - (-w)^{i+2n+1} = 0
\]

for \( 1 \leq i \leq 2n \).

We calculate the NC deformation algebra of \( O_C \) on \( Y_\lambda \):

**Theorem 5.5.** The NC deformation algebra of \( O_C \) on \( Y_\lambda \) is given by

\[
k\langle \langle a, b \rangle \rangle/(ab + ba, a^2 + b^{2n+1} + \sum_{i=1}^{2n} \lambda_i b^{2i}).
\]
Proof. We add relations (5.1) with the help of (4.2) and (4.3). Then we have
\[ t = b^{2n}, \quad -a^2 = t b + \sum_{i=1}^{2n} \lambda_i b^{2i}, \quad ab + ba = 0 \]
hence
\[ a^2 + b^{2n+1} + \sum_{i=1}^{2n} \lambda_i b^{2i} = 0. \]
We check that the relations of Theorem 4.5 follow from these equations.
\[ ab^{2n+1} = -b^{2n+1} a, \]
\[ a(b^{2n+1} + \sum_{i=1}^{2n} \lambda_i b^{2i}) = (b^{2n+1} + \sum_{i=1}^{2n} \lambda_i b^{2i}) a, \]
hence
\[ (5.3) 0 = ab^{2n+1} = tab = tba. \]

Remark 5.6. (1) The normal bundle of \( C \) is given by \( N_{C/Y} \cong \mathcal{O}_C(1) \oplus \mathcal{O}_C(-3) \). Hence \( \dim \text{Ext}^1(\mathcal{O}_C, \mathcal{O}_C) = \dim H^0(N_{C/Y}) = 2 \) and \( \dim \text{Ext}^2(\mathcal{O}_C, \mathcal{O}_C) = \dim H^1(N_{C/Y}) = 2 \). Thus the cotangent space of the NC deformation algebra is generated by \( a, b \) and there are 2 relations. The quadratic terms of the relations are anti-symmetric: \( ab + ba, a^2 \).

(2) When \( \lambda_i = 0 \) for all \( i \), then the equation is weighted homogeneous with weights \( ut(a, b) = (2n + 1, 2) \). Thus the isomorphism type of the NC deformation algebra does not change under the substitution \( \lambda_i \mapsto \alpha^{2i-2n+1} \lambda_i \) for \( \alpha \in k \).

(3) Since \( \dim Y = 3 \) and \( (K_Y, C) = 0 \), there is a superpotential \( W \) (24). It is expressed by using non-commutative variables \( a, b, c, d, w \) which are generators of \( \text{End}(R \oplus M) \), where \( a, b \in \text{Hom}(M, M), c \in \text{Hom}(M, R), d \in \text{Hom}(R, M) \) and \( w \in \text{Hom}(R, R) \). It seems to be given by
\[ W = \frac{1}{2}dcdc + b^2 dc + a^2 b + dwc - \frac{(w)^{n+1}}{n+1} + \frac{b^{2n+2}}{2n+2} + \sum_{i=1}^{2n} \lambda_i b^{2i+1} \]
By cyclically differentiating \( W \), we then obtain the relations among the variables:
\[ ab + ba = 0, \]
\[ a^2 + bdc + dcb + b^{2n+1} + \sum \lambda_i b^{2i} = 0, \]
\[ c(b^2 + dc) = (b^2 + dc)d = 0, \]
\[ cd + (-w)^n = 0. \]
By putting \( c = d = w = 0 \), we obtain the relations of the NC deformation algebra.
(4) As already remarked in [3], non-commutative associative algebras may be non-isomorphic even if their abelianizations are isomorphic. It may even happen that one is finite dimensional and the other is infinite dimensional.

We know that $k[[a,b]]/(a^2 + b^2 + b^3) \cong k[[a,b]]/(a^2 + b^2)$, because $1 + b$ is invertible. But $k\langle\langle a,b\rangle\rangle/(ab + ba, a^2 + b^2 + b^3)$ is finite dimensional (9-dimensional), and $k\langle\langle a,b\rangle\rangle/(ab + ba, a^2 + b^2)$ is infinite dimensional. Indeed we have an injective homomorphism

$$k[[a^2]] = k[[a^2, b^2]]/(a^2 + b^2) = k\langle\langle a^2, b^2\rangle\rangle/(a^2b^2 - b^2a^2, a^2 + b^2)$$

$$\rightarrow k\langle\langle a,b\rangle\rangle/(ab + ba, a^2 + b^2).$$

We consider the following conjecture of Donovan and Wemyss:

**Conjecture 5.7** ([6] Conjecture 1.4). Let $f_i : Y_i \to X_i$ ($i = 1, 2$) be flopping contractions of smooth 3-folds whose exceptional loci are irreducible smooth rational curves $C_i$, and let $A_i$ be the NC deformation algebras of $\mathcal{O}_{C_i}$ on $Y_i$. Then the completions of $X_i$ at the singular points $f_i(C_i)$ are isomorphic if and only if $A_i$ are isomorphic.

Conjecture 5.7 has some partial positive answers in [9] and [8]. Conjecture 5.7 seems more reasonable than it appears because it can be regarded as a non-commutative generalization of the following theorem:

**Theorem 5.8** ([18]). Let $(X, 0), (X', 0) \subset \mathbb{C}^{n+1}$ be germs of hypersurfaces with isolated singularities at the origin defined by equations $f, f'$. Then they are isomorphic (i.e., biholomorphically equivalent) if and only if their (commutative) deformation algebras are isomorphic:

$$\mathcal{O}_{\mathbb{C}^{n+1}, 0}/(f, \frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}) \cong \mathcal{O}_{\mathbb{C}^{n+1}, 0}/(f', \frac{\partial f'}{\partial z_0}, \ldots, \frac{\partial f'}{\partial z_n})$$

as $\mathbb{C}$-algebras, where $z_0, \ldots, z_n$ are local coordinates on $\mathbb{C}^{n+1}$ at the origin.

We define

$$A_0 = k\langle\langle a, b\rangle\rangle/(ab + ba, a^2 + b^{2n+1})$$

$$A_i = k\langle\langle a, b\rangle\rangle/(ab + ba, a^2 + b^{2n+1} + b^{2i})$$

for $1 \leq i \leq 2n$. We denote by $A_i^{ab} = A_i/([A_i, A_i])$ their abelianizations.

We confirm Conjecture 5.7 for deformations of Laufer’s flops in Proposition 5.9 and Theorem 5.10. This is a generalization of [3] Theorem 4.7.

**Proposition 5.9.** $A_0^{ab}, A_1^{ab}, \ldots, A_n^{ab}$ are not isomorphic to each other, while $A_0^{ab}, A_{n+1}^{ab}, \ldots, A_{2n}^{ab}$ are isomorphic.

**Proof.** We have

$$A_i^{ab} = \begin{cases} 
  k[[a, b]]/(ab, a^2 + b^{2n+1}), & i = 0, \\
  k[[a, b]]/(ab, a^2 + b^{2i}), & 1 \leq i \leq n, \\
  k[[a, b]]/(ab, a^2 + b^{2n+1}), & n + 1 \leq i \leq 2n.
\end{cases}$$

Thus $\dim A_0^{ab} = 2 + 2n + 1$ and $\dim A_i^{ab} = 2 + 2i$ for $1 \leq i \leq n$. □
Theorem 5.10. $A_0, A_{n+1}, \ldots, A_{2n}$ have the same dimension $6n + 3$ as $k$-linear spaces, and are non-isomorphic to each other as associative $k$-algebras.

Proof. We consider $A_0$ and $A_{n+i}$ for $1 \leq i \leq n$. We have $ab^2 = b^2a$.

We claim that $a^3 = ab^{2n+1} = b^{4n+2} = 0$ in $A_{n+i}$. Indeed

$$a^3 + ab^{2n+2i} = -ab^{2n+1} = b^{2n+1}a = -a^3 - ab^{2n+2i},$$

hence $ab^{2n+1} = 0$. Then $a^3 = -ab^{2n+1} - ab^{2n+2i} = 0$. We have

$$b^{4n+2} = -a^2b^{2n+1} - b^{4n+2i+1} = -b^{4n+2i+1} = b^{3n+4i} = \ldots.$$ Since $\bigcap_{m=1}^{\infty}(b^m) = 0$, we obtain $b^{4n+2} = 0$. We have $a^3 = ab^{2n+1} = b^{4n+2} = 0$ also in $A_0$.

Since $a^2 + b^{2n+1}$ is weighted homogeneous with $wt(a, b) = (2n + 1, 2)$, $A_0$ is a graded algebra. $A_0$ has a $k$-linear basis $a^ib^t$ for $0 \leq s \leq 2$ and $0 \leq t \leq 2n$.

Indeed since they have all different weights:

$$wt(1, b, \ldots, b^n, a, b^{n+1}, ab, b^{n+2}, ab^2, \ldots, b^{2n},$$

$$a^{bn}, a^2b^{n+1}, a^{2b}b, a^{2b^2}, ab^{n+2}, a^2b^{n+1}, a^2b^{2n})$$

$$= (0, 2, \ldots, 2n, 2n + 1, 2n + 2, 2n + 3, 2n + 4, 2n + 5, \ldots, 4n, 4n + 1, 4n + 2, 4n + 3, 4n + 4, \ldots, 6n + 1, 6n + 2, 6n + 4, \ldots, 8n + 2)$$

they are linearly independent.

Assuming that there is an isomorphism $f : A_{n+i} \to A_0$ for some $i$, we will derive a contradiction.

Since $a^3 = 0$ in $A_{n+i}$, we have $f(a)^3 = 0$. Since $wt(b) = 2$ is minimal, $(b + \ldots)^3 \neq 0$, hence $b \not\in f(a)$, i.e., the monomial $b$ does not appear in $f(a)$.

Since $f(m) = m$ for maximal ideals $m$, we have $a \in f(a)$.

We claim that $b^k \not\in f(a)$ for $1 \leq k \leq 2n$. We proceed by induction on $k$. Assume that $b^k \not\in f(a)$ for $k < k_0$ for some $k_0 \leq 2n$. If $k_0 \leq n$, then $wt(b^{k_0}) < wt(a)$, and we have $b^{3k_0} \in (b^{k_0} + a + \ldots)^3$, i.e., $b^{3k_0}$ does not cancel since its weight attains the minimum, where we note that $3k_0 \leq 4n + 1$. Here we omitted coefficients as long as they are non-zero, because we are dealing with only monomials. Since $f(a)^3 = 0$, we deduce that $b^{k_0} \not\in f(a)$. Similarly, if $k_0 > n$, then $a^2b^{k_0} \in (a + b^{k_0} + \ldots)^3$, because $wt(a) < wt(b^{k_0})$. Therefore $b^{k_0} \not\in f(a)$.

We have $f(a)f(b) + f(b)f(a) = 0$. We claim that $a \notin f(b)$. Indeed if $a \in f(b)$, then since $a \in f(a)$ and $b^k \not\in f(a)$ for $1 \leq k \leq 2n$, $a^2 \in f(a)f(b) + f(b)f(a)$ is not cancelled, a contradiction. Thus $a \notin f(b)$. Since $f(m) = m$, we have $b \in f(b)$.

We claim that $b^{2k} \not\in f(b)$ for all $1 \leq k \leq n$. Otherwise, $a^2b^{2k} \in f(a)f(b) + f(b)f(a)$ does not cancel for some $k$.

We use $f(a)^2 + f(b)^2f(a) + f(b)f(a) = 0$, and we look at a monomial $a^2b^{2i-1} = -b^{2i+1}$. We have $a^2b^{2i-1} \in f(b)^{2n+2i}$, because $f(b) = b + \ldots$. On the other hand, we claim that $a^2b^{2i-1} \not\in f(a)^2$. Indeed, since $b^k \not\in f(a)$ for $1 \leq k \leq 2n$, we need to look at $(a + ab^{2i-1})^2$. But

$$(a + ab^{2i-1})^2 = a^2 + a^2b^{2i-1} - a^2b^{2i-1} - a^2b^{4i-2} = a^2 - a^2b^{4i-2},$$
and $a^2b^{2i-1}$ disappears.

We claim that $b^{2n+2i} \not\in f(b)^{2n+1}$. The form $(b + ab + \ldots)^{2n+1}$, where $a$ does not appear inside the parentheses, contains $a^2b^m$ with $m \geq 2n + 1$, and $a^2b^{2i-1}$ does not appear. The form $(b + b^{m_1} + \ldots + b^{m_s} + \ldots)^{2n+1}$, where the $m_k$ are odd, does not contain $b^{2n+2i}$, because an odd sum of odd numbers is odd and cannot be $2n + 2i$. Therefore $b^{2n+2i} \not\in f(b)^{2n+1}$. But this is a contradiction with $f(a)^2 + f(b)^{2n+1} + f(b)^{2n+2i} = 0$. Thus we conclude that there is no isomorphism $f : A_{n+i} \to A_0$.

Next we generalize the above argument to prove that there is no isomorphism $f : A_{n+i} \to A_{n+j}$ for $1 \leq i < j \leq n$. We claim first that the monomials $a^s b^t$ for $0 \leq s \leq 2$ and $0 \leq t \leq 2n$ are $k$-linear basis of $A_{n+j}$. Indeed, by using the relation $ba = -ab$, any monomials are written as $a^s b^t$ for some integers $s, t$. Since $a^3 = ab^2 + b^4 = 0$, all possible non-zero monomials are $b^t$ for $0 \leq t \leq 4n + 1$, $ab^t$ for $0 \leq t \leq 2n$ and $a^2b^t$ for $0 \leq t \leq 2n$. The only relations among them are given by $a^2b^t + b^{2n+1+t} + b^{2n+2t} = 0$, where the last terms for $t \geq 2n - 2j + 2$ vanish. Thus $b^{2n+1+t}$ for $0 \leq t$ are expressed by other monomials while $b^t$ for $0 \leq t \leq 2n$ are not. The multiplication of $A_{n+j}$ is also determined by this rule.

We have still $f(a)^3 = 0$. Then we have again $b \not\in f(a)$, hence $a \in f(a)$.

We prove again that $b_k \not\in f(a)$ for $1 \leq k \leq 2n$ by induction on $k$. Assume that $b_k \not\in f(a)$ for $k < k_0$ for some $k_0 \leq 2n$. If $k_0 \leq n$, then $b^{3k_0} \in (b^{k_0} + a + \ldots)^3$ as before, and obtain a contradiction if $3k_0 \leq 2n$. If $3k_0 > 2n$, then we rewrite

$$b^{3k_0} = a^2b^{3k_0-2n-1} - b^{3k_0+2j-1}$$

and obtain a contradiction. Hence $b^{3k_0} \not\in f(a)$. Similarly, if $k > n$, then we consider $a^2b^{k_0} \in (a + b^{k_0} + \ldots)^3$. Therefore $b^{k_0} \not\in f(a)$.

We use again $f(a)f(b) + f(b)f(a) = 0$. We claim that $a \not\in f(b)$. Indeed, if $a \in f(b)$, then $a^2 \in f(a)f(b) + f(b)f(a)$, since we have $a \in f(a)$ and $b \not\in f(a)$ for $1 \leq k \leq 2n$, a contradiction. Thus $a \not\in f(b)$, hence $b \in f(b)$.

We have also that $b_k \not\in f(b)$ for $1 \leq k \leq n$, otherwise $ab^{2k} \in f(a)f(b) + f(b)f(a)$.

We use $f(a)^2 + f(b)^{2n+1} + f(b)^{2n+2i} = 0$ again. We look at a monomial $b^{2n+2i}$. We have

$$b^{2n+2i} = a^2b^{2i-1} - b^{2n+2i+2j-1} = -a^2b^{2i-1} + a^2b^{2i+2j-2} - \ldots.$$ 

Hence $a^2b^{2i-1} \in f(b)^{2n+2i}$. On the other hand, $a^2b^{2i-1} \not\in f(a)^2$, because

$$(a + ab^{2i-1})^2 = a^2 + a^2b^{2i-1} - a^2b^{2i-1} - a^2b^{4i-2} = a^2 - a^2b^{4i-2}.$$ 

We claim that $a^2b^{2i-1} \not\in f(b)^{2n+1}$. The form $(b + ab)^{2n+1}$ contains $a^2b^m$ with $m \geq 2n + 1$, but does not contain $a^2b^{2i-1}$. The form $(b + b^{m_1} + \ldots + b^{m_s})^{2n+1}$ with odd $m_k$ does not contain $b^{2n+2i}$, because an odd sum of odd numbers is odd and cannot be $2n + 2i$. We note also that $j > i$. Therefore
we obtain a contradiction with \( f(a)^2 + f(b)^{2n+1} + f(b)^{2n+2i} = 0 \) again, and we conclude that there is no isomorphism \( f : A_{n+i} \rightarrow A_{n+j} \). \hfill \Box

6. Example: Universal Flopping Contraction of Higher Length

By using Karmazyn [10], we can describe NC deformation algebras of the reduced fiber \( \mathcal{O}_C \) for the universal flopping contractions in the case of higher length \( l \geq 3 \). We recall a description of the endomorphism algebras of the tilting bundles:

**Theorem 6.1** ([10] Theorem 1.3). Let \( \tilde{f} : \tilde{Y} \rightarrow \tilde{X} = \text{Spec}(R) \) be a universal flopping contraction of length \( l \) for \( l = 1, 2, 3, 4, 5, 6 \), and let \( A = \text{End}(\mathcal{O}\tilde{Y} \oplus M) \) be the endomorphism algebra of a tilting generator \( \mathcal{O}\tilde{Y} \oplus M \) of \( D^b(\text{coh}(\tilde{Y})) \) over \( \tilde{X} \). Then \( A \) is a quiver algebra \( A = H[Q]/I \) over a polynomial algebra \( H \) with relations \( I \) as follows, where there are two vertices \( v_0, v_1 \) of the quiver \( Q \) with the corresponding idempotents \( e_0, e_1 \) and edges \( a_0, a_1, a^* \in \text{Hom}(v_0, v_1) \), \( a_0^*, a_1^* \in \text{Hom}(v_1, v_0) \) and \( b, c, d \in \text{Hom}(v_1, v_1) \).

1. \( l = 1 \), \( H = k[t] \), and \( I \) is generated by
   \[
   a_0a_0 - a_1a_1^* = te_0, \quad a_1^*a_1 - a_0a_0^* = -te_1.
   \]

2. \( l = 2 \), \( H = k[t, u_1, u_2, u_3] \), and \( I \) is generated by
   \[
   a^*a = te_0, \quad b^2 = u_1e_1, \quad c^2 = u_2e_1, \quad d^2 = u_3e_1,
   \]
   \[
   aa^* + b + c + d = \frac{1}{2}te_1.
   \]

3. \( l = 3 \), \( H = k[t, u_1, u_2, u_3, u_4, u_5] \), and \( I \) is generated by
   \[
   a^*d = ta^*, \quad da = ta, \quad a^*a = (t^2 - u_5)e_0, \quad aa^* = d^2 - u_5e_1,
   \]
   \[
   b^3 = u_2b + u_1e_1, \quad c^3 = u_4c + u_3e_1, \quad b + c + d = \frac{1}{3}te_1.
   \]

4. \( l = 4 \), \( H = k[t, u_1, u_2, u_3, u_4, u_5, u_6] \), and \( I \) is generated by
   \[
   a^*d = ta^*, \quad da = ta, \quad a^*a = (t^3 - u_6t - u_5)e_0,
   \]
   \[
   aa^* = d^3 - u_6d - u_5e_1, \quad b^2 = u_1e_1, \quad c^4 = u_4c^2 + u_3c + u_2e_1,
   \]
   \[
   b + c + d = \frac{1}{4}te_1.
   \]

5. \( l = 5 \), \( H = k[t, u_1, u_2, u_3, u_4, u_5, u_6, u_7] \), and \( I \) is generated by
   \[
   a^*d = ta^*, \quad da = ta, \quad a^*a = (t^4 - u_7t^2 - u_2t - u_1)e_0,
   \]
   \[
   aa^* = d^4 - u_3d^2 - u_2d - u_1e_1,
   \]
   \[
   cbc + bc^2 + b^3c = -u_7bc - u_6c - u_4e_1,
   \]
   \[
   (c + b^2)^2 + bcb = -u_7(c + b^2) - u_6b - u_5e_1, \quad d - b = \frac{1}{5}te_1.
   \]
(6) \( l = 6, H = k[t, u_1, u_2, u_3, u_4, u_5, u_6, u_7], \) and \( I \) is generated by

\[
a^*d = ta^*, \quad da = ta, \quad a^*a = (t^5 - u_7t^3 - u_6t^2 - u_5t - u_4)e_0, \\
aa^* = d^5 - u_7d^3 - u_6d^2 - u_5d - u_4e_1, \quad b^2 = u_1e_1, \\
c^3 = u_3c + u_2e_1, \quad b + c + d = \frac{1}{6}te_1.
\]

We calculate the NC deformation algebra of the reduced central fiber by using the above theorem:

**Theorem 6.2.** Let \( \tilde{f} : \tilde{Y} \to \tilde{X} = \text{Spec}(R) \) be as in Theorem 6.1. Then the NC deformation algebra \( A_{\text{def}} \) of the reduced central fiber \( O_C \) of \( \tilde{f} \) is the completion of a quotient algebra \( A/Ae_0A \), and is given by the following:

1. \( l = 1: \) \( A_{\text{def}} = k. \)
2. \( l = 2: \) \( A_{\text{def}} = k[[t]]\langle\langle (b, c)\rangle\rangle/(tbc - tcb, \ bc^2 - c^2b, \ b^2c - cb^2). \)
3. \( l = 3: \)

\[
A_{\text{def}} = k[[t, u_2, u_4]]\langle\langle (b, c)\rangle\rangle/(u_2(bc - cb) - (b^3c - cb^3), \ u_4(bc - cb) - (bc^3 - cb^3), \ 3(bc^2 - c^2b) + 3(b^2c - cb^2) - 2(tbc - tcb)).
\]

4. \( l = 4: \)

\[
A_{\text{def}} = k[[t, u_3, u_4, u_6]]\langle\langle (b, c)\rangle\rangle/(b^2c - cb^2, u_3(bc - cb) + u_4(bc^2 - c^2b) - (bc^4 - c^4b), (16u_6 - 3t^2)(bc - cb) + 12t(bc^2 - c^2b) - 16(b^3c - cb^3 + b^2c^2 - c^2b^2 + bc - cb - c^3b - cb^3)).
\]

5. \( l = 5: \)

\[
A_{\text{def}} = k[[t, u_2, u_3, u_6, u_7]]\langle\langle (b, c)\rangle\rangle/(u_7(bc - cb) + (bc^2 - c^2b) + (b^3c - cb^3), u_6(bc - cb) + u_7(b^2c - cb) + (bc - cb) + (b^2c^2 - bc - cb) + (b^4c - b^3cb), u_5(bc - cb) + u_3(b^2c - cb^2 + (2/5)t(bc - cb)) - (4/125)t^3(bc - cb) - (6/25)t^2(b^2c - cb^2) - (4/5)t(b^3c - b^3) - (b^4c - cb^4)).
\]
(6) \( l = 6 \):

\[
A_{\text{def}} = k[[t, u_3, u_5, u_6, u_7]]/(b^2 c - cb^2, \ u_3(bc - cb) - bc^3 - c^3b, \\
(-5/6^3)t^4 + (1/12)t^2u_7 + (1/3)tu_6 + u_5)(bc - cb) \\
+ ((10/6^3)t^3 - (1/2)t - u_6)(b^2 c - cb^2 + bc^2 - c^2b) \\
+ (-10/6^2)t^2 + (1/6)u_7) \\
(b^3 c - cb^3 + b^2c^2 - ca^2 + bc - cb + bc^3 - c^3b) \\
+ (5/6)t(c^4b - bc^4 + c^3b^2 - b^2c^3 + c^2bc - bc^2 + cbc^2b - bc^2bc \\
+ c^2b^3 - b^3c^2 + cb^2c - bc^2c - a^2bc + 2a^2c - c^4b) \\
- (b^3c - c^3b) \\
+ (b^4c - c^2b^4 + b^3c^2c - c^2bc + b^2c^3c - c^2bc + b^2c^2b - b^2bc^2b) \\
+ (b^3c - c^3b^2 + b^2bc^2 - c^2bc^2 + b^2c^2b - c^2bc^2 + c^2b^2c - c^2b^2c) \\
- bc^2b^2c - c^2b^2c + c^2bc^2b) \\
+ (b^2c^4 - c^4b^2 + bc^3c - bc^3b + bc^2bc^2 - bc^2b^2 + bc^3bc - c^3bc) \\
+ (bc^3 - c^3b).
\]

Proof. We note that the two-sided ideal \( A e_0 A \) is generated by those endomorphisms of \( \mathcal{O}_\tilde{Y} \oplus M \) which factor through \( \mathcal{O}_\tilde{Y} \). Therefore its completion is the NC deformation algebra by Theorem 2.1.

We deduce our result by changing variables from \( A/Ae_0 A \). (1) is clear because \( C \) is a \((-1, -1)\)-curve, and is rigid inside \( \tilde{Y} \).

(2) This is already proved in Theorem 4.5 but we give an alternative proof. We obtain

\[
A/Ae_0 A = H(b, c, d)/\langle b^2 - u_1, \ c^2 - u_2, \ d^2 - u_3, \ b + c + d - t/2 \rangle.
\]

We remove \( d \) by using \( d = t/2 - b - c \). Since \( b^2, c^2 \) and \( d^2 = t^2/4 -(tb+tc) + b^2 + c^2 + (bc+cb) \) are in the center of \( A/Ae_0 A \), so is \( (bc+cb) - (tb+tc) \). Then \( (bc+cb - bc - cb) - t(bc - cb) = 0 \). Therefore we have \( tbc - tcb = 0 \).

We have \( \dim \tilde{Y} = 6 \) and \( (K_{\tilde{Y}}, C) = 0 \). Since \( C \subseteq \tilde{Y} \) is a deformation of a \((1, -3)\)-curve, there is an exact sequence

\[
0 \rightarrow \mathcal{O}_C^3 \rightarrow N_{\tilde{Y}/Y} \rightarrow \mathcal{O}_C(1) \oplus \mathcal{O}_C(-3) \rightarrow 0
\]

for a normal bundle \( N_{\tilde{Y}/Y} \). Since the pair \( C \subseteq \tilde{Y} \) has no more deformation, we deduce that \( H^1(T_{\tilde{Y}}|C) = 0 \) for the tangent bundle \( T_{\tilde{Y}} \). Indeed if \( H^1(T_{\tilde{Y}}|C) \neq 0 \), then \( C \subseteq \tilde{Y} \) has a non-trivial deformation. Thus there is no factor of degree \( \leq -2 \) in \( T_{\tilde{Y}}|C \), and we obtain

\[
N_{\tilde{Y}/Y} \cong \mathcal{O}_C(1) \oplus \mathcal{O}_C \oplus \mathcal{O}_C(-1)^3.
\]
Then
\[ 2 N_{C/Y} \cong \mathcal{O}_C(1) \oplus \mathcal{O}_C^3 \oplus \mathcal{O}_C(-1)^3 \oplus \mathcal{O}_C(-2)^3. \]
Therefore
\[
\dim H^0(N_{C/Y}) = 3, \quad \dim H^1(N_{C/Y}) = 0, \quad \dim H^0(\bigwedge^2 N_{C/Y}) = 5.
\]
We have 3 generators \( t, b, c \) of the maximal ideal of \( A_{\text{def}} \), and they satisfy 5 relations. Since
\[
\dim(\text{Im}((\bigwedge^2 H^0(N_{C/Y}) \to H^0(\bigwedge^2 N_{C/Y}))) = 2,
\]
there are only 2 relations of order 2.

Since \( t, a^2, b^2 \) are in the center, we have relations \( at - ta = bt - tb = tab - tba = a^2b - b^2a = 0 \). We have thus 3 linearly independent relations of order 3 besides 2 relations of order 2. It follows that there are no more relations. Therefore we have
\[
A_{\text{def}} = k\langle t, b, c \rangle/(at - ta, bt - tb, tab - tba, a^2b - b^2a, a^2b - ba^2).
\]

(3) We obtain
\[ A/Ae_0A = H(b, c, d)/(b^3 - u_2b - u_1, \ c^3 - u_4c - u_3, \ d^2 - u_5, \ b + c + d - t/3). \]
We remove \( d \) by using \( d = t/3 - b - c \).

Since \( b^3 - u_2b \) and \( c^3 - u_4c \) are in the center, we have
\[
0 = (b^3 - u_2b)c - c(b^3 - u_2b) = -u_2(bc - cb) + (b^3c - cb^3),
\]
\[
0 = (c^3 - u_4c)b - b(c^3 - u_4c) = u_4(bc - cb) - (bc^3 - c^3b).
\]
Since \( d^2 = t^2/9 - \frac{2}{3}(tb + tc) + b^2 + c^2 + (bc + cb) \) is in the center, so is \( b^2 + c^2 + (bc + cb) - \frac{2}{3}(tb + tc). \) Hence
\[
(bc^2 - c^2b) + (bbc + bcb - bcb - cbb) - \frac{2}{3}t(bc - cb) = 0.
\]
Thus \( 3(bc^2 - c^2b) + 3(b^2c - cb^2) - 2(tbc - tcb) = 0 \).

We have \( N_{C/Y} \cong \mathcal{O}_C(1) \oplus \mathcal{O}_C^3 \oplus \mathcal{O}_C(-1)^3 \) as before. Since \( \bigwedge^2 N_{C/Y} \cong \mathcal{O}_C(1)^3 \oplus \mathcal{O}_C^3 \oplus \mathcal{O}_C(-1)^9 \oplus \mathcal{O}_C(-2)^3 \), we have
\[
\dim H^0(N_{C/Y}) = 5, \quad \dim H^1(N_{C/Y}) = 0, \quad \dim H^0(\bigwedge^2 N_{C/Y}) = 12.
\]
Thus we have 5 generators \( t, u_2, u_4, b, c \) and 12 relations. Since
\[
\dim(\text{Im}((\bigwedge^2 H^0(N_{C/Y}) \to H^0(\bigwedge^2 N_{C/Y}))) = 9,
\]
there are 9 relations of order 2, which are commutations with central variables. We already have 3 relations of order 3, and there are no more relations.
(4) We obtain
\[ A/Ae_0A = H(b, c, d)/(b^2 - u_1, e^4 - u_4c^2 - u_3c - u_2, d^3 - u_6d - u_5, b + c + d - t/4). \]

We remove \( d \) by using \( d = t/4 - b - c \).

We have \( N_{C/Y} \cong \mathcal{O}_C(1) \oplus \mathcal{O}_C^4 \oplus \mathcal{O}_C(-1)^3 \), \( \bigwedge^2 N_{C/Y} \cong \mathcal{O}_C(1)^4 \oplus \mathcal{O}_C^6 \oplus \mathcal{O}_C(-1)^{12} \oplus \mathcal{O}_C(-2)^3 \). Hence

\[ \dim H^0(N_{C/Y}) = 6, \quad \dim H^1(N_{C/Y}) = 0, \quad \dim H^0(\bigwedge^2 N_{C/Y}) = 17. \]

Thus we have 6 generators \( t, u_3, u_4, u_6, b, c \) and 17 relations. Since

\[ \dim(\text{Im}(\bigwedge^2 H^0(N_{C/Y}) \to H^0(\bigwedge^2 N_{C/Y}))) = 14, \]

there are 14 relations of order 2, which are commutations with central variables. We need 3 more relations of order \( \geq 3 \).

Since \( b^2 \) and \( e^4 - u_4c^2 - u_3c \) are in the center, we have

\[ 0 = b^2c - cb^2, \]
\[ 0 = b(e^4 - u_4c^2 - u_3c) - (e^4 - u_4c^2 - u_3c)b \]
\[ = -u_3(bc - cb) - u_4(bc^2 - c^2b) + (bc^4 - c^4b). \]

Since

\[ d^3 - u_6d = (t/4)^3 - (3/16)t^2(b + c) + (3/4)t(b + c)^2 - (b + c)^3 - u_6(t/4 - b - c) \]

is in the center, so is

\[ (3/4)t(c^2 + bc + cb) - (b^3 + c^3 + bcc + bcb + cbc + ccb + bcc + bcc) + (u_6 - (3/16)t^2)(b + c). \]

Thus

\[ 0 = 16((3/4)t(bc^2 - c^2b) - (b^3c - cb^3) + b^2c^2 - c^2b^2 + bcb - cbcb - c^3b + bc^3) \]
\[ + (u_6 - (3/16)t^2)(bc - cb)) \]
\[ = (16u_6 - 3t^2)(bc - cb) + 12t(bc^2 - c^2b) \]
\[ - 16(b^3c - c^3b + b^2c^2 - c^2b^2 + bcb - cbcb - c^3b + bc^3). \]

Therefore we have our claim.

(5) We obtain
\[ A/Ae_0A = H(b, c, d)/(b + c) + b + u_7bc + u_6c + u_4, \]
\[ (c + b)^2 + bcb + u_7(c + b^2) + u_6b + u_5, d^4 - u_3d^2 - u_2d - u_1, -b + d - t/5). \]

We remove \( d \) by using \( d = t/5 + b \).

We have \( N_{C/Y} \cong \mathcal{O}_C(1) \oplus \mathcal{O}_C^5 \oplus \mathcal{O}_C(-1)^3 \), \( \bigwedge^2 N_{C/Y} \cong \mathcal{O}_C(1)^5 \oplus \mathcal{O}_C^{10} \oplus \mathcal{O}_C(-1)^{15} \oplus \mathcal{O}_C(-2)^3 \). Hence

\[ \dim H^0(N_{C/Y}) = 7, \quad \dim H^1(N_{C/Y}) = 0, \quad \dim H^0(\bigwedge^2 N_{C/Y}) = 23. \]
Thus we have 7 generators $t, u_2, u_3, u_6, u_7, b, c$ and 23 relations. Since $$\dim(\text{Im} (\bigwedge^2 H^0(N_{C/Y}) \to H^0(\bigwedge^2 N_{C/Y}))) = 20,$$
there are 20 relations of order 2, which are commutations with central variables. We need 3 more relations of order $\geq 3$.

We have

\begin{align*}
0 &= b(cbc + bc^2 + b^3c + u_7bc + u_6c) - (cbc + bc^2 + b^3c + u_7bc + u_6c)b \\
&= u_6(bc - cb) + u_7(b^2c - bc) + (bcbc - cbcb) + (b^2c^2 - bc^2b) + (b^4c - b^3cb), \\
0 &= b((c + b^2) + bcb + u_7(c + b^2) + u_6b) - ((c + b^2)^2 + bcb + u_7(c + b^2) + u_6b)b \\
&= (bc^2 - c^2b) + (bc^2 + b^3c - cb^3 - b^2cb) + (b^2cb - bc^2b) + u_7(bc - cb) \\
&= u_7(bc - cb) + (bc^2 - c^2b) + (b^3c - cb^3).
\end{align*}

We have $d^4 - u_3d^2 - u_2d = (t/5 + b^4 - u_3(t/5 + b)^2 - u_2(t/5 + b)$. Hence $b^4 + (4/5)tb^3 + (6/25)t^2b^2 + (4/125)t^3b - u_3(b^2 + (2/5)t) - u_2b$ is in the center, and we obtain

\begin{align*}
0 &= c(b^4 + (4/5)tb^3 + (6/25)t^2b^2 + (4/125)t^3b - u_3(b^2 + (2/5)t) - u_2b) \\
&- (b^4 + (4/5)tb^3 + (6/25)t^2b^2 + (4/125)t^3b - u_3(b^2 + (2/5)t) - u_2b)c \\
&= u_2(bc - cb) + u_3(b^2c - cb^2 + (2/5)t(bc - cb)) - (4/125)t^3(bc - cb) - (6/25)t^3(b^2c - cb^2) \\
&- (4/5)t(b^2c^3) - (b^2c^3 - cb^3)
\end{align*}

and our result.

(6) We obtain

\begin{align*}
A/Ae_0 A &= H(b, c, d)/b^2 - u_1, \ c^3 - u_3c - u_2, \\
d^5 - u_7d^5 - u_6d^2 - u_5d - u_4, \ b + c + d - t/6).
\end{align*}

We remove $d$ by using $d = t/6 - b - c$.

We have $N_{C/Y} \cong O_C(1) \oplus O_C^3 \oplus O_C(-1)^3$, $\bigwedge^2 N_{C/Y} \cong O_C(1)^5 \oplus O_C^{10+3} \oplus O_C(-1)^{15} \oplus O_C(-2)^3$. Hence

$$\dim H^0(N_{C/Y}) = 7, \quad \dim H^1(N_{C/Y}) = 0, \quad \dim H^0(\bigwedge^2 N_{C/Y}) = 23.$$ 

Thus we have 7 generators $t, u_3, u_5, u_6, u_7, b, c$ and 23 relations. Since

$$\dim(\text{Im}(\bigwedge^2 H^0(N_{C/Y}) \to H^0(\bigwedge^2 N_{C/Y}))) = 20,$$

there are 20 relations of order 2, which are commutations with central variables. We need 3 more relations of order $\geq 3$.

We have

\begin{align*}
0 &= b(c^3 - u_3c) - (c^3 - u_3c)b = -u_3(bc - cb) + bc^3 - c^3b.
\end{align*}
Remark 6.3. Hence the result. Since \(a, b\) have \(\dim \text{Hom}(l, b - c) = 0\) as in [23], we have \(\dim \text{Hom}(M, O_C) = \dim \text{Ext}^1(L, O_C) = 0\), hence \(\dim \text{Hom}(M, O_C) = r - 1\). Since \(\dim \text{Hom}(M, O_C(-1)) = 0\) and \(\dim \text{Ext}^1(M, O_C(-1)) = 1\), we have \(\dim \text{Hom}(M, O_{C_i}) = \dim \text{Hom}(M, O_{C_i-1}) - 1\). Since \(\dim \text{Hom}(M, O_{C_i}) = 0\), we have \(l = r\).

(3) We would like to ask whether the global NC deformation algebra of \(O_C\) is given in the form \(H'(b, c)/J\) over a polynomial algebra \(H'\) with variables \(t\) and some of the \(u_i\), so that our (formal) deformation algebra is its completion.

(4) The abelianizations \(A_{ab}^{\text{def}}\) are (commutative) formal power series rings for all \(l\), because \(H^1(N_{C/Y}) = 0\). That is why the generators of the defining ideals of the NC deformation rings are commutators.
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