DEFORMATIONS IN $G_2$ MANIFOLDS

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Abstract. Here we study the deformations of associative submanifolds inside a $G_2$ manifold $M^7$ with a calibration 3-form $\varphi$. A choice of 2-plane field $\Lambda$ on $M$ (which always exits) splits the tangent bundle of $M$ as a direct sum of a 3-dimensional associate bundle and a complex 4-plane bundle $TM = E \oplus V$, and this helps us to relate the deformations to Seiberg-Witten type equations. Here all the surveyed results as well as the new ones about $G_2$ manifolds are proved by using only the cross product operation (equivalently $\varphi$). We feel that mixing various different local identifications of the rich $G_2$ geometry (e.g. cross product, representation theory and the algebra of octonions) makes the study of $G_2$ manifolds looks harder then it is (e.g. the proof of McLean’s theorem [M]). We believe the approach here makes things easier and keeps the presentation elementary. This paper is essentially self contained.

1. $G_2$ manifolds

We first review the basic results about $G_2$ manifolds, along the way we give a self contained proof of the McLean’s theorem and its generalization [M], [AS1]. A $G_2$ manifold $(M, \varphi, \Lambda)$ with an oriented 2-plane field gives various complex structures on some of subbundles of $T(M)$. This imposes interesting structures on the deformation theory of its associative submanifolds. By using this we relate them to the Seiberg-Witten type equations.

Let us recall some basic definitions (c.f. [B1], [B2], [HL]): Octonions give an 8 dimensional division algebra $\mathbb{O} = \mathbb{H} \oplus \mathbb{H} = \mathbb{R}^8$ generated by $\langle 1, i, j, k, l, li, lj, lk \rangle$. The imaginary octonions $im\mathbb{O} = \mathbb{R}^7$ is equipped with the cross product operation $\times : \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$ defined by $u \times v = im(\bar{v}u)$. The exceptional Lie group $G_2$ is the linear automorphisms of $im\mathbb{O}$ preserving this cross product. It can also be defined in terms of the orthogonal 3-frames:

\begin{equation}
G_2 = \{ (u_1, u_2, u_3) \in (im\mathbb{O})^3 \mid \langle u_i, u_j \rangle = \delta_{ij}, \langle u_1 \times u_2, u_3 \rangle = 0 \}.
\end{equation}

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Alternatively, $G_2$ is the subgroup of $GL(7, \mathbb{R})$ which fixes a particular 3-form $\varphi_0 \in \Omega^3(\mathbb{R}^7)$, [B1]. Denote $e^{ijk} = dx^i \wedge dx^j \wedge dx^k \in \Omega^3(\mathbb{R}^7)$, then
\[
G_2 = \{ A \in GL(7, \mathbb{R}) \mid A^* \varphi_0 = \varphi_0 \}.
\]
(2)

**Definition 1.** A smooth 7-manifold $M^7$ has a $G_2$ structure if its tangent frame bundle reduces to a $G_2$ bundle. Equivalently, $M^7$ has a $G_2$ structure if there is a 3-form $\varphi \in \Omega^3(M)$ such that at each $x \in M$ the pair $(T_x(M), \varphi(x))$ is isomorphic to $(T_0(\mathbb{R}^7), \varphi_0)$ (pointwise condition). We call $(M, \varphi)$ a manifold with a $G_2$ structure.

A $G_2$ structure $\varphi$ on $M^7$ gives an orientation $\mu_\varphi = \mu \in \Omega^7(M)$ on $M$, and $\mu$ determines a metric $g = g_\varphi = \langle \ , \ \rangle$ on $M$, and a cross product operation $TM \times TM \mapsto TM$: $(u, v) \mapsto u \times v = u \times_\varphi v$ defined as follows: Let $i_v = v_\perp$ be the interior product with a vector $v$, then
\[
\langle u, v \rangle = [(u_\perp \varphi) \wedge (v_\perp \varphi) \wedge \varphi]/6\mu.
\]
(3)

$\varphi(u, v, w) = (v_\perp u_\perp \varphi)(w) = \langle u \times v, w \rangle$.

**Definition 2.** A manifold with $G_2$ structure $(M, \varphi)$ is called a $G_2$ manifold if the holonomy group of the Levi-Civita connection (of the metric $g_\varphi$) lies inside $G_2$. In this case $\varphi$ is called integrable. Equivalently $(M, \varphi)$ is a $G_2$ manifold if $\varphi$ is parallel with respect to the metric $g_\varphi$, that is $\nabla_{g_\varphi}(\varphi) = 0$; which is in turn equivalent to $d\varphi = 0$, $d(*_{g_\varphi} \varphi) = 0$ (i.e. $\varphi$ harmonic). Also equivalently, at each point $x_0 \in M$ there is a chart $(U, x_0) \rightarrow (\mathbb{R}^7, 0)$ on which $\varphi$ equals to $\varphi_0$ up to second order term, i.e. on the image of $U$, $\varphi(x) = \varphi_0 + O(|x|^2)$.

**Remark 1.** One important class of $G_2$ manifolds are the ones obtained from Calabi-Yau manifolds. Let $(X, \omega, \Omega)$ be a complex 3-dimensional Calabi-Yau manifold with Kähler form $\omega$ and a nowhere vanishing holomorphic 3-form $\Omega$, then $X^6 \times S^1$ has holonomy group $SU(3) \subset G_2$, hence is a $G_2$ manifold. In this case $\varphi = Re \Omega + \omega \wedge dt$. Similarly, $X^6 \times \mathbb{R}$ gives a noncompact $G_2$ manifold.

**Definition 3.** Let $(M, \varphi)$ be a $G_2$ manifold. A 4-dimensional submanifold $X \subset M$ is called coassociative if $\varphi|_X = 0$. A 3-dimensional submanifold $Y \subset M$ is called associative if $\varphi|_Y \equiv vol(Y)$; this condition is equivalent to $\chi|_Y \equiv 0$, where $\chi \in \Omega^2(M, TM)$ is the tangent bundle valued 3-form given by:
\[
\langle \chi(u, v, w), z \rangle = \ast \varphi(u, v, w, z).
\]
(4)
Equivalence of these conditions follows from the ‘associator equality’ of [HL]
\[ \varphi(u, v, w)^2 + |\chi(u, v, w)|^2/4 = |u \wedge v \wedge w|^2. \]

Sometimes \( \chi \) is also called the triple cross product operation and denoted by \( \chi(u, v, w) = u \times v \times w \). By imitating the definition of \( \chi \), we can view the usual cross product as a tangent bundle 2-form \( \psi \in \Omega^2(M, TM) \) defined by
\[ \langle \psi(u, v), w \rangle = \varphi(u, v, w). \]

As in the case of \( \varphi \), \( \chi \) can be expressed in terms of in cross product and metric
\[ \chi(u, v, w) = -u \times (v \times w) - \langle u, v \rangle w + \langle u, w \rangle v \]
(c.f. [H], [HL], [K]). From (6) and the identity \( u \times v = u.v + \langle u, v \rangle \), the reader can easily check that \( 2\chi(u, v, w) = (u.v).w - u.(v.w) \), which shows that the associative submanifolds of \( (M, \varphi) \) are the manifolds where the octonion multiplication of the tangent vectors is “associative”.

We call a 3-plane \( E \subset TM \) associative plane if \( \varphi|_E = vol(E) \), so associate submanifolds \( Y^3 \) are submanifolds whose tangent planes are associative. From (2) and (3) we see that an associative 3-plane \( E \subset TM \) is a plane generated by three orthonormal vectors in the form \( \langle u, v, u \times v \rangle \); and also if \( V = E^\perp \) is its orthogonal complement (coassociative), the cross product induces maps:

\[ E \times V \rightarrow V, \text{ and } V \times V \rightarrow E, \text{ and } E \times E \rightarrow E. \]

Note that (4) implies that the 3-form \( \chi \) assigns a normal vector to every oriented 3-plane in \( T(M) \), [AS1], which is zero on the associative planes. Therefore, we can view \( \chi \) as a section of the 4-plane bundle \( V = \mathbb{E}^\perp \rightarrow G_3(M) \) over the Grassmannian bundle of orientable 3-planes in \( T(M) \), where \( V \) is orthogonal bundle to the canonical bundle \( \mathbb{E} \rightarrow G_3(M) \). In particular, \( \chi \) gives a normal vector field on all oriented 3-dimensional submanifolds \( f : Y^3 \hookrightarrow (M, \varphi) \), which is zero if the submanifold is associative. This gives an interesting first order flow \( \partial f/\partial t = \chi(f, vol(Y)) \) (which is called \( \chi \)-flow in [AS2]), which appears to push \( f(Y) \) towards associative submanifolds.

Finally, a useful fact which will be used later is the following: The \( SO(3) \)-bundle \( \mathbb{E} \) is the reduction of the \( SO(4) \)-bundle \( \mathbb{V} \) by the projection to the first factor \( SO(4) = (SU(2) \times SU(2))/\mathbb{Z}_2 \rightarrow SU(2)/\mathbb{Z}_2 = SO(3) \), i.e. \( \mathbb{E} = \Lambda^3_+ \mathbb{V} \).
2. 2-FRAME FIELDS OF $G_2$ MANIFOLDS

By a theorem of Emery Thomas, all orientable 7-manifolds admit non-vanishing 2-frame fields $[\mathbb{T}]$, in particular they admit non-vanishing oriented 2-plane fields. Using this, we get a useful additional structure on the tangent bundle of $G_2$ manifolds.

**Lemma 1.** A non-vanishing oriented 2-plane field $\Lambda$ on a manifold with $G_2$-structure $(M, \varphi)$ induces a splitting of $T(M) = E \oplus V$, where $E$ is a bundle of associative 3-planes, and $V = E^\perp$ is a bundle of coassociative 4-planes. The unit sections $\xi$ of the bundle $E \to M$ give complex structures $J_{\xi}$ on $V$.

**Proof.** Let $\Lambda = \langle u, v \rangle$ be the 2-plane spanned by the basis vectors of an orthonormal 2-frame $\{u, v\}$ in $M$. Then we define $E = \langle u, v, u \times v \rangle$, and $V = E^\perp$. We can define the complex structure on $V$ by $J_{\xi}(x) = x \times \xi$.

Similar complex structures were studied in [HL]. The complex structure $J_{\Lambda}(z) = \chi(u, v, z)$ of [AS1] turns out to coincide with $J_{v \times u}$ because by (6):

\[
\chi(u, v, z) = \chi(z, u, v) = -z \times (u \times v) - \langle z, u \rangle v + \langle z, v \rangle u = J_{v \times u}(z).
\]

$J_{\xi}$ also defines a complex structure on the bigger bundle $\xi^\perp \subset TM$. So it is natural to study manifolds $(M, \varphi, \Lambda)$, with a $G_2$ structure $\varphi$, and a nonvanishing oriented 2-plane field $\Lambda$ inducing the splitting $T(M) = E \oplus V$, and $J = J_{v \times u}$. Note that each of these terms depend on $\varphi$ and $\Lambda$.

**Definition 4.** We call $Y^3 \subset (M, \Lambda)$ a $\Lambda$-spin submanifold if $\Lambda|_Y \subset TY$, and call $Y^3 \subset (M, \varphi, \Lambda)$ a $\Lambda$-associative submanifold if $E|_Y = TY$.

Clearly $\Lambda$-associative submanifolds $Y \subset (M, \varphi, \Lambda)$ are $\Lambda$-spin. Also since $Y$ has a natural metric induced from the metric of $(M, \varphi)$, we can identify the set of $Spin^c$ structures $Spin^c(Y) \cong H^2(Y, \mathbb{Z})$ on $Y$ by the homotopy classes of 2-plane fields on $Y$ (as well as the homotopy classes of vector fields on $Y$). So, any $\Lambda$-spin submanifold $Y$ inherits a natural $Spin^c$ structure $s = s(\Lambda)$ from $\Lambda$.

How abundant are the $\Lambda$-associative (or $\Lambda$-spin) submanifolds? Some answers:

**Lemma 2.** Let $M^7$ be an orientable 7-manifold, then every $Spin^c$ submanifold $(Y^3, s) \subset M^7$ is $\Lambda$-spin for some $\Lambda$ with $s = s(\Lambda)$, and every associative $Y \subset (M, \varphi)$ is $\Lambda$-associative for some $\Lambda$.

**Proof.** Let $s = \langle u', v' \rangle$ be the $Spin^c$ structure generated by an orthonormal frame field on $TY$. By using $[\mathbb{T}]$ we choose a nonvanishing orthonormal 2-frame field $\{u, v\}$ on $M$. Let $V_2(\mathbb{R}^7) \to V_2(M) \to M$ be the Steifel bundle of 2-frames in $T(M)$. Now the restriction of this bundle to $Y$ has two sections $\{u', v'\}$
and \{u, v\} which are homotopic, since the fiber \(V_2(\mathbb{R}^7)\) is 4-connected. By the homotopy extension property \{u', v'\} extends to orthonormal 2-frame field \{u'', v''\} to \(M\), then we let \(\Lambda = \langle u'', v''\rangle\). Furthermore when \(Y\) is associative, we can start with an orthonormal 3-frame of \(TY\) of the form \{u', v', u' \times v'\}, then get the corresponding \(E_\Lambda = \langle u'', v'', u'' \times v''\rangle\), which makes \(Y\) \(\Lambda\)-associative. \(\square\)

More generally, for any manifold with a \(G_2\) structure \((M, \varphi)\) we can study the bundle of oriented 2-planes \(G_2(\mathbb{M}) \to \mathbb{M}\) on \(\mathbb{M}\), and construct the corresponding universal bundles \(E \to G_2(\mathbb{M})\) and \(V \to G_2(\mathbb{M})\), and a complex structure \(J\) on \(V\), where \(J = J_\Lambda\) on the fiber over \(\Lambda = \langle u, v\rangle\). Then each \((M, \varphi, \Lambda)\) is a section of \(G_2(M) \to \mathbb{M}\), inducing \(E, V, J\). We can do the same construction on the bundle of oriented 2-frames \(V_2(\mathbb{M}) \to \mathbb{M}\) and get the same quantities, in this case we get a hyper-complex structure on \(V\), i.e. we get three complex structures \(J = J_1, J_2, J_3\) on \(V\) corresponding to \(J_{u \times v}, J_u, J_v\), over each fiber \{u, v\}, and they anti-commute and cyclically commute e.g. \(J_1 J_2 = J_3\). Notice also that \(J_1\) depends only on the oriented 2-plane field, whereas \(J_2, J_3\) depend on the 2-frame field.

By using \(J_1\) (or one of the other \(J_p, p = 2, 3\)) we can split \(V_C = W \oplus \bar{W}\), as a pair of conjugate \(C^2\)-bundles (\(\pm i\) eigenspaces of \(J_1\)). This gives a complex line bundle \(K = \Lambda^2(W)\) which corresponds to the 2-plane field \(\Lambda\). Corresponding to \(K\) we get a canonical Spin\(^c\) structure on \(V\). More specifically, recall that \(U(2) = (S^1 \times S^3)/\mathbb{Z}_2, SO(4) = (S^3 \times S^3)/\mathbb{Z}_2, Spin^c(4) = (S^3 \times S^3 \times S^1)/\mathbb{Z}_2,\)

\[
\begin{align*}
Spin^c(4) & \overset{U(2)}\rightarrow SO(4) \times S^1
\end{align*}
\]

(9)

where the horizontal map \([\lambda, A] \mapsto ([\lambda, A], \lambda^2)\) canonically lifts to the map \([\lambda, A] \mapsto (\lambda, A, \lambda)\), where the transition functions \(\lambda^2\) corresponds to \(K\) (see for example \([\lambda]\)). This means that there are pair of complex \(C^2\)-bundles, \(W^\pm \to V_2(\mathbb{M})\) with \(V_C = W^+ \otimes \bar{W}^\pm\). This fact can be checked directly by taking \(W^+ = K^{-1} + \mathbb{C}\) and \(\bar{W}^- = \bar{W}\) (note \(\Lambda^2(W) \otimes \bar{W} \cong \Lambda^2(W) \otimes W^* \cong W\)). This gives an action \(E = \Lambda^2(V) : W^+ \to W^+;\) in our case this action will come from cross product structure, Lemma 3 will do this by identifying \(W^+\) with \(S\).

Note also that from (6) an (7) the cross product operation \(\rho(a)(w) = a \times w\) induces a Clifford representation by \(\rho(u \times v) = -J_1, \rho(u) = -J_2, \rho(v) = -J_3\)

\[
\rho : E \to End(V).
\]

(10)
2.1. $G_2$ frame fields on $G_2$ manifolds.

In the case of a manifold with $G_2$ structure $(M, \varphi)$, Thomas’s theorem can be strengthened to the conclusion that $M$ admits a 2-frame field $\Lambda$, with the property that on the tubular neighborhood of the 3-skeleton of $M$, $\Lambda$ is the restriction of a $G_2$ frame field. To see this, we start with an orthonormal 2-frame field $\{u_1, u_2\}$, then let $\Lambda = \langle u_1, u_2, u_1 \times u_2 \rangle$ and $\mathbb{V} \to M$ be the corresponding universal 4-plane bundle as in the last section. Then we pick a unit section $u_3$ of $\mathbb{V} \to M$ over the 3-skeleton $M^{(3)}$; there is no obstruction doing this since we are sectioning an $S^3$-bundle over the 3-skeleton of $M$. Now, from the definition of $G_2$ in (1) we see that $\{u_1, u_2, u_3\}$ is a $G_2$ frame on $M^{(3)}$.

**Definition 5.** We call $(M, \varphi, \Lambda)$ a framed $G_2$ manifold if $\Lambda$ is the restriction of a $G_2$ frame field on $M$.

The above discussion says that every $(M, \varphi)$ admits a 2-frame field $\Lambda$ such that $(M^{(3)}, \varphi, \Lambda)$ is a $G_2$-framed manifold. From now on, the notation $(M, \varphi, \Lambda)$ will refer to a manifold with a $G_2$ structure and a 2-frame field $\Lambda$, such that on $M^{(3)}$, $\Lambda$ is the restriction of a $G_2$ frame as above. From the above discussion, the last condition is equivalent to picking a nonvanishing section of $\mathbb{V} \to M^{(3)}$ (called $u_3$ above). This will be useful when studying local deformations of associative submanifolds $Y^3 \subset M$ (they live near $M^{(3)}$). Using the same notations of the last section we state:

**Lemma 3.** Let $(M, \varphi, \Lambda)$ be a framed $G_2$ manifold. Then we can decompose $\mathbb{V}_C = S \oplus \overline{S}$ as a pair of bundles, each of which is isomorphic to $W^+ = K^{-1} + \mathbb{C}$, and the cross product $\rho$ induces a representation $\rho_C : \mathbb{E}_C \to \text{End}(S)$ given by:

$$u \times v \mapsto \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad u \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad v \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

**Proof.** We choose a local orthonormal frame $\{e_1, \ldots, e_7\}$ which $\varphi$ is in the form (2) with $\{u \times v, u, v\} = \{e_1, e_2, e_3\}$ (because of the canonical metric we will not distinguish the notations of local frames and coframes). From (2) and (3) we compute the cross product operation, $\mathbb{J}_1, \mathbb{J}_2, \mathbb{J}_3$, and $\mathbb{W}$ from the tables below.

| $\times$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---------|-------|-------|-------|-------|
| $e_1$   | $e_5$ | $-e_4$| $e_7$ | $-e_6$|
| $e_2$   | $e_6$ | $-e_7$| $-e_4$| $e_5$ |
| $e_3$   | $-e_7$| $-e_6$| $e_5$ | $e_4$ |
This is the quadratic map which appears in Seiberg-Witten theory, after identifying \( E \) with the Lie algebra \( \mathfrak{su}(2) \) (skew adjoint endomorphisms of \( \mathbb{C}^2 \) with the inner product given by the Killing form) we get

\[
\sigma(z, w) = \left( \frac{|z|^2 - |w|^2}{2} \right) u \times v + \text{Re}(z\bar{w})u + \text{Im}(w\bar{z})v.
\]

This is the quadratic map which appears in Seiberg-Witten theory, after identifying \( E \) with the Lie algebra \( \mathfrak{su}(2) \) (skew adjoint endomorphisms of \( \mathbb{C}^2 \) with the inner product given by the Killing form) we get

\[
\sigma(x) = \sigma(z, w) = \left( \frac{|z|^2 - |w|^2}{2} \right) u \times v + \frac{z\bar{w}}{|w|^2 - |z|^2}.
\]

\[
\langle \sigma(x), x \rangle = 2|\sigma(x)|^2 = \frac{1}{2}|x|^4.
\]

These identifications are standard tools used Seiberg-Witten theory (c.f [A]).
2.2. Deforming $G_2$ structures.

For a 7-manifold with a $G_2$ structure $(M, \varphi)$, the space of all $G_2$ structures on $M$ is identified with an open subset of 3-forms $\Omega^3_+(M) \subset \Omega^3(M)$, which is the orbit of $\varphi$ by the gauge transformations of $T(M)$. The orbit is open by the dimension reason (recall that the action of $GL(7, \mathbb{R})$ on $\Omega^3(X)$ has $G_2$ as the stabilizer). The structure of $\Omega^3_+(M)$ is nicely explained in [B2] as follows: By definition, $\Omega^3_+(M)$ is the space of sections of a bundle over $M$ with fiber $GL(7, \mathbb{R})/G_2$ (which is homotopy equivalent to $\mathbb{R}P^7$). Furthermore, the subspace of the $G_2$ structures inducing the same metric can be parametrized with the space of sections of the bundle $\mathbb{R}P^7 \to P(T^*M \oplus \mathbb{R}) \to M$ with fibers $SO(7)/G_2 = \mathbb{R}P^7$, where $P(T^*M \oplus \mathbb{R})$ is the projectivization of $T^*(M) \oplus \mathbb{R}$. That is, if $\lambda = [a, \alpha]$ with $a^2 + \alpha^2 = 1$, then the corresponding $\varphi_\lambda \in \Omega^3_+(M)$ is

$$\varphi_\lambda = \varphi - 2a^\# \perp [a(\ast \varphi) + \alpha \wedge \varphi]$$

where $a^\#$ is the metric dual of $a$. This is given in [B2], written slightly differently. Therefore, if we start with an integrable $G_2$ structure with harmonic $\varphi$, the space of integrable $G_2$ structures inducing the same metric are parametrized by the sections $\lambda = [a, \alpha]$, such that $d\theta = d(\ast \theta) = 0$, where $\theta = a^\# \perp [a(\ast \varphi) + \alpha \wedge \varphi]$ and $\ast \theta = \alpha \wedge [a(\varphi) - (a^\# \ast \varphi)]$. It is a natural question whether a submanifold $Y^3 \subset (M, \varphi)$ is associative. The following says that any $Y$ can be made associative in $(M, \varphi_\lambda)$, after deforming $\varphi$ to $\varphi_\lambda$.

**Proposition 4.** Let $(M^7, \varphi)$ be a manifold with a $G_2$ structure, then any Spin$^c$ submanifold $(Y^3, s) \subset M^7$ is a $\Lambda$-associative submanifold of $(M, \varphi_\lambda, \Lambda)$ for some choice of $\lambda = [a, \alpha]$ and a plane field $\Lambda$.

**Proof.** By Lemma 2, we can assume $Y$ is $\Lambda$-spin for some $\Lambda = \langle u, v \rangle$. Hence this gives an orthogonal splitting $T(M) = E \oplus V$, with $E = \langle u, v, u \times v \rangle$. Choose a unit vector field $w$ in $Y$ orthogonal to $\langle u, v \rangle|_Y$, then extend $w$ to $M$. Now we want to choose $\lambda = [a, \alpha]$ so that if $(u \times v)_\lambda$ is the cross product corresponding to the $G_2$ structure $\varphi_\lambda$, then $(u \times v)_\lambda|_Y = w$.

By (13), and the rules $(u \times v)^\# = v \perp u \perp \varphi$ and $(u \times v)_{\lambda}^\# = v \perp u \perp \varphi_\lambda$ we get

$$(u \times v)_{\lambda}/2 = (u \times v)/2 - |\alpha|^2(u \times v) - a\chi(u, v, a^\#) + a\chi(v, a^\#) - a\chi(v, \alpha^#) + a\chi(u, v, \alpha^#).$$

This formula holds for any $u, v \in TM$. In our case $\{u, v\}$ are orthonormal generators of $\Lambda$, so by (8) the third term on the right is $aJ(\alpha^#)$ where $J = J_{v \times u}$ is the complex structure of $V$ given by Lemma 1 (and remarks following it).
Now if we call \( w_0 = \frac{1}{2}[(u \times v) - w] \), and choose \( \alpha \) among 1-forms whose \( E \) component zero (i.e. section of \( V \)) with \( |\alpha^\#| < 1 \) (hence \( a \neq 0 \)), the equation \((u \times v)_{\lambda|Y} = w\) gives \( w_0 = |\alpha|^2(u \times v) + aJ(\alpha^\#)\). By taking inner products of both sides with basis elements of \( E \), we see that \( w_0^\dagger = a J(\alpha^\#) \) where \( w_0^\dagger \) is the \( V \)-component of \( w_0 \). We can apply \( J \) to both sides and solve \( \alpha^\# = -\frac{1}{a}J(w_0^\dagger) \).

2.3. Deforming associative submanifolds.

Let \( G(3,7) \cong SO(7)/SO(3) \times SO(4) \) be the Grassmannian manifold of oriented 3-planes in \( \mathbb{R}^7 \), and \( G^{\varphi_0}(3,7) = \{ L \in G(3,7) \mid \varphi_0|_L = vol(L) \} \) be the submanifold of associative 3-planes. Recall that \( G_2 \) acts on \( G^{\varphi_0}(3,7) \) with stabilizer \( SO(4) \) giving the identification \( G^{\varphi_0}(3,7) \cong G_2/\text{SO}(4) \) \([HL]\). Recall also that if \( E \rightarrow G(3,7) \) and \( V = E^\perp \rightarrow G(3,4) \) are the canonical 3-plane bundle and the complementary 4-plane bundle, then we can identify the tangent bundle by \( TG(3,7) = E^* \otimes V \). How does the tangent bundle of \( G^{\varphi_0}(3,7) \) sit inside of this? The answer is given by the following Lemma. By \((7)\) the cross product operation maps \( E \times V \rightarrow V \), and the metric gives an identification \( E^* \cong E \), now if \( L = \langle e_1, e_2, e_3 \rangle \in G^{\varphi_0}(3,7) \) with \( \{ e_1, e_2, e_3 = e_1 \times e_2 \} \) orthonormal, then

**Lemma 5.** \( T_L G^{\varphi_0}(3,7) = \{ \sum_{j=1}^3 e^j \otimes v_j \in E^* \otimes V \mid \sum e_j \times v_j = 0 \} \).

**Proof.** A tangent vector of \( G(3,7) \) at \( L \) is a path of planes generated by three orthonormal vectors \( L(t) = \langle e_1(t), e_2(t), e_3(t) \rangle \), such that \( L(0) = L \), in other words \( \dot{L} = \sum e_j \otimes \dot{e}_j \). Clearly this tangent vector lies in \( G^{\varphi_0}(3,7) \) if \( e_3(t) = e_1(t) \times e_2(t) \). So \( \dot{e}_3 = \dot{e}_1 \times e_2 + e_1 \times \dot{e}_2 \). By taking cross product of both sides with \( e_3 \) and then using the identity \((6)\) we get

\[
\chi(\dot{e}_1, e_2, e_3) + \chi(e_1, \dot{e}_2, e_3) + \chi(e_1, e_2, \dot{e}_3) = 0.
\]

Now by using \((8)\) and the fact that the cross product of two of the vectors in \( \{ e_1, e_2, e_3 \} \) is equal to the third (in cyclic ordering), we get the result. \( \square \)

It is easy to see that the normal bundle of \( G^{\varphi_0}(3,7) \) in \( G(3,7) \) is isomorphic to \( V \) giving the exact sequence of the bundles over \( G^{\varphi_0}(3,7) \):

\[
0 \rightarrow TG^{\varphi_0}(3,7) \rightarrow TG(3,7) \xrightarrow{\chi} V \rightarrow 0
\]

From \((7)\) we know that, if \( Y^3 \subset (M, \varphi) \) associative and \( \nu \) is its normal bundle, then the cross product operation maps: \( TY \times \nu \rightarrow \nu \), \( \nu \times \nu \rightarrow TY \), and \( TY \times TY \rightarrow TY \). Let \( \{ e_1, e_2, e_3 \} \) and \( \{ e^1, e^2, e^3 \} \) be local frames and the dual coframes on \( TY \) and \( A_0 \) be the background Levi-Civita connection on \( \nu \) induced from the metric on \( M \) (there is also the identification \( TY \cong T^*Y \) by
induced metric). Then we can define a Dirac operator \( \mathcal{D}_{A_0} : \Omega^0(\nu) \to \Omega^0(\nu) \) as the covariant derivative \( \nabla_{A_0} = \sum e^j \otimes \nabla_{e_j} \) followed by the cross product:

\[
(14) \quad \mathcal{D}_{A_0} = \sum e^j \times \nabla_{e_j}.
\]

So the cross product plays the role of the Clifford multiplication in defining the Dirac operator in the normal bundle. We can extend this multiplication to 2-forms: \((a \wedge b) \times x = \frac{1}{2}[a \times (b \times x) - b \times (a \times x)]\) then by using (6) we get:

\[
(a \wedge b) \times x = \frac{1}{2}[x \times (a \wedge b)] - \chi(a, b, x).
\]

In particular, when \(a, b \in TY\) and \(x \in \nu\) then \((a \wedge b) \times x = -\chi(a, b, x)\). As usual we can twist this Dirac operator by connections on \(\nu\), by replacing \(A_0\) with \(A_0 + a\), where \(a \in \Omega^1(Y, \text{ad}\nu)\) is an endomorphism of \(\nu\) valued 1-form. The following from [AS1], is a generalized version of McLean’s theorem [M].

**Theorem 6.** The tangent space to associative submanifolds of a manifold with a \(G_2\) structure \((M, \varphi)\) at an associative submanifold \(Y\) is given by the kernel of the the twisted Dirac operator \(\mathcal{D}_A : \Omega^0(\nu) \to \Omega^0(\nu)\), where \(A = A_0 + a\) for some \(a \in \Omega^1(Y, \text{ad}(\nu))\). The term \(a = 0\) when \(\varphi\) is integrable.

**Proof.** Recall the notations of Lemma 5. Let \(L = \langle e_1, e_2, e_3 \rangle\) be a tangent plane to \(Y \subset M\). Any normal vector field \(\nu\) to \(Y\) moves \(L\) by one parameter group of diffeomorphisms giving a path of 3-planes in \(M\), hence it gives a vertical tangent vector \(\hat{L} = \sum e_j \otimes L_\nu(e_j) \in T_LG_3(M)\) of the Grassmannian bundle of 3-planes \(G_3(M) \to M\) where \(L_\nu\) is the Lie derivative along \(\nu\). By Lemma 5 this path of planes remain associative if \(\sum e_j \times L_\nu(e_j) = 0\). Since \(L_\nu(e_j) = \nabla_{e_j}(\nu) - \nabla_\nu(e_j)\), where \(\nabla\) is the (torsion free) metric connection of \(M\); then the result follows by letting \(a(\nu) = \sum e^j \times \nabla_\nu(e_j)\) where \(\nabla\) is the normal component of \(\nabla\). If \(\varphi\) is integrable, then on a local chart it coincides with \(\varphi_0\) up to quadratic terms, so \(0 = \nabla_\nu(\varphi)|_Y = \nabla_\nu(e^1 \wedge e^2 \wedge e^3)\), which implies \(a = 0\). Also, by using the fact that the cross product operation preserves the tangent space of the associative manifold \(Y\), it is easy to check that the expression of \(a\) is independent of the choice of the orthonormal basis of \(L\).

Notice that at any point by choosing normal coordinates we can make \(a = 0\). This reflects the fact that \(\varphi\) coincides only pointwise with \(\varphi_0\), not on a chart. To make the Dirac operator onto, we can twist it by 1-forms \(a \in \Omega^1(Y)\), i.e.

**Lemma 7.** For associative \(Y \subset (M, \varphi)\) the map \(\Omega^1 \times \Omega^0(\nu) \to \Omega^0(\nu)\) defined by \((x, a) \mapsto D_A(x) + a \times x\) is onto, (by using appropriate Sobolev norms)
Proof. It suffices to show that the orthogonal complement of the image of this map is zero: Assume \( \langle D_A(x), y \rangle + \langle a \times x, y \rangle = 0 \) for all \( x \) and \( a \), then by taking \( a = 0 \) and using the self adjointness of the Dirac operator we get \( D_A(y) = 0 \). Hence \( \langle a \times x, y \rangle = 0 \), then the fact that the map \( (x, a) \mapsto a \times x \) is surjective gives the result. Note that by (6) \( a \times (a \times x) = -|a|^2 x \).

So for a generic choice of \( a \) this twisted Dirac operator is onto, but what does this mean in terms of the deformation space of the associative submanifolds? The next Proposition \([\text{AS1}]\) gives an answer. It says that if we perturb the deformation space with the gauge group (i.e. allowing a slight rotation of \( TY \) by the gauge group of \( TM \) during deformation) then it becomes smooth.

Note that Theorem 6 may be explained by a Gromov-Witten set-up: Let \( G_3^\epsilon(M) \subset G_3(M) \to M \) be the subbundle of associative 3-planes with fiber \( G_3^\epsilon \cong G_2/\text{SO}(4) \). We can form a bundle \( G_3(Y, M) \to \text{Im}(Y^3, M) \) over the space of imbeddings, whose fiber over \( f : Y \hookrightarrow M \) are the liftings \( F \):

\[
G_3(M) \supset G_3^\epsilon(M) \\
F \nearrow \\
Y \quad \xrightarrow{f} \\
M
\]

The Gauss map \( f \mapsto \sigma(f) \) gives a natural section to this bundle, and \( Y \) is associative if and only if this section maps into \( G_3^\epsilon(M) \). Theorem 6 gives the condition that the derivative of \( \sigma \) maps into the tangent space of \( G_3^\epsilon(M) \), which is the subbundle of \( G_3(Y, M) \) consisting of \( F \)'s mapping into \( G_3^\epsilon(M) \). Recall that if \( P \to M \) denotes the tangent frame bundle of \( M \), then the gauge group \( \mathcal{G}(M) \) of \( M \) is defined to be the sections of the \( \text{SO}(7) \)-bundle \( \text{Ad}(P) \to M \), where \( \text{Ad}(P) = P \times \text{SO}(7)/(p, h) \sim (pg, g^{-1}hg) \). By perturbing the Gauss map with the gauge group (i.e. by composing \( \sigma \) with the gauge group action \( G_3(M) \to G_3(M) \)) we can make it transversal to \( G_3^\epsilon(Y, M) \).

**Proposition 8.** The map \( \tilde{\sigma} : \mathcal{G}(M) \times \text{Im}(Y, M) \to G_3(Y, M) \) is transversal to \( G_3^\epsilon(Y, M) \), where \( \tilde{\sigma}(s, f) = s(f)\sigma(f) \)

**Proof.** We start with the local calculation of the proof of Theorem 6, except in this setting we need to take \( \tilde{L} = \sum e_j \otimes \mathcal{L}_v(se_j) \), where \( s \in \text{SO}(7) \) is the gauge group in the chart. Then the resulting equation is \( \mathcal{D}_A(v) + \sum e_j \times v(s)e_j = 0 \), where \( v(s)e_j \) denotes the normal component of \( v(s)e_j \) (here we are doing the calculation in normal coordinates where \( \nabla_v(e_k) = 0 \) pointwise). Then the argument as in the proof of Lemma 7 (by showing the second term is surjective) gives the proof. 

\[\square\]
The kernel of this operator gives the deformations of \textit{pseudo associative submanifolds} of $Y \subset (M, \varphi)$, defined in [AS1] as the manifolds where the perturbed Gauss map $\tilde{\sigma}$ maps $Y$ into $G^3_3(M)$. We can either choose a generic $a$, or constraint the new variable $a$ by a natural second equation, which results equations resembling the Seiberg-Witten equations as follows:

Let $Y \subset (M, \varphi, \Lambda)$ be a $\Lambda$-associative submanifold, we deform $Y$ in the complex bundle $S_\sim = K^{-1} + \mathbb{C} = W^+$ defined in Lemma 3. From projections (9), the background $SO(4)$ metric connection on the normal bundle $\nu$ along with a choice of a connection $A$ on the line bundle $K \to Y$ gives a connection of the $\text{Spin}^c$ bundle, which in turn induces a connection on the associated $U(2)$ bundle $W^+ \to Y$. By using the Clifford multiplication $TY_\mathbb{C} \otimes W^+ \to W^+$ coming from the cross product (Lemma 3) we can form the Dirac operator $D_A : \Omega^0(W^+) \to \Omega^0(W^+)$, whose kernel identifies locally the deformations of $Y$ in the bundle $W^+$. Then if we constraint the new variable $A$ by (11) we obtain deformations resembling to the Seiberg-Witten equations.

\begin{equation}
D_A(x) = 0 \\
*F_A = \sigma(x).
\end{equation}

where $*$ is the star operator of $Y$ induced from the background metric of $M$, and $(x, A) \in \Omega^0(W^+) \times \mathcal{A}(K)$, and $\mathcal{A}(K)$ is the space of connections on $K$. From Weitzenböck formula and (12), the above equations give compactness to this type of local deformation space, hence allow us to assign Seiberg-Witten invariant to $Y$. Now the natural question is how easy to produce $\Lambda$-associative submanifolds $Y^3 \subset (M, \varphi, \Lambda)$? One answer is that any zero set $Y^3$ of a transverse section of $V \to M$ gives a $\Lambda$-associative submanifold. This is because the transversality gives a canonical identification $TY \cong E|_Y$. Then the natural question is: Are there natural sections of $V$? We can obtain such things from the other $G_2$ structures as follows. Recall that $\Lambda = \langle u, v \rangle$ gives the section $s = \langle u, v, u \times v \rangle$ of the bundle $G_3(M) \to M$. For any other $G_2$ structure $\psi$ on $(M, \varphi)$ defines a section $\chi_\psi$ of $V \to G_3(M)$. Then by pulling $\chi_\psi$ with $s$ over $M$ produces a natural section $s(\varphi, \psi)$ of $V \to M$. To these sections we can associate an integer valued invariant, i.e. the Seiberg-Witten invariant of their zero set (15) (the nontransverse sections we can associate zero). Consequences of this will be explored in a future paper.

Note that in the usual Seiberg-Witten equations on $Y$, we use an action of $T^*Y : W^+ \to W^-$ coming from $\text{Spin}^c$ structure, which then extends an action of $\Lambda^2(Y)$ to $W^+ \to W^+$. Here $T^*Y$ acts as $W^+ \to W^+$ by Lemma 3. On the other hand by the background metric we have the identification $T^*Y \cong \Lambda^2(Y)$.
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