Ashtekar’s New Variables and Positive Energy

James M. Nester†§, Roh-Suan Tung†∥ and Yuan Zhong Zhang†¶

† Department of Physics, National Central University, Chung-Li, Taiwan
‡ Institute of Theoretical Physics, Academia Sinica, Beijing

Abstract. We discuss earlier unsuccessful attempts to formulate a positive gravitational energy proof in terms of the New Variables of Ashtekar. We also point out the difficulties of a Witten spinor type proof. We then use the special orthonormal frame gauge conditions to obtain a locally positive expression for the New Variables Hamiltonian and thereby a “localization” of gravitational energy as well as a positive energy proof.

§ E-mail address: nester@phyast.phy.ncu.edu.tw
∥ E-mail address: m792001@phy.ncu.edu.tw
¶ E-mail address: yuanzhong%bepc2@scs.slac.stanford.edu
1. Introduction

A promising development in gravity research has been the discovery (Ashtekar 1987) of certain complex New Variables which transform the initial value constraints of Einstein’s theory into a rather nice form. There is hope that these variables may truly capture the real physics of the theory. Consequently, considerable effort has been and is being devoted to developing and exploring applications and features of these New Variables (Ashtekar 1988, 1991; Rovelli 1991).

An essential fundamental property of any theory of gravity is that the total energy (for solutions with a suitable asymptotic region) must be positive. It is thus appropriate to examine this issue in the light of the New Variables. Of course, there is no doubt that the New Variables reformulation does satisfy this requirement. Since, when certain reality and non-degeneracy conditions are imposed, the New Variables formulation is essentially equivalent to that in terms of the standard variables; hence we can appeal to any of the existing positive energy proofs for Einstein’s theory.

However, a proof of positive energy directly in terms of the new variables (not just a transcription of a known proof) is very desirable. Indeed, if such were not possible it would detract from the prestige of this formulation. Moreover, for a truly good set of variables for the gravitational field one might hope for a mathematically natural and physically reasonable quasi-localization of gravitational energy.

2. Ashtekar’s New Variables

Our conventions are summarized in the appendix; generally we follow Ashtekar (1988, 1991). The theory can be obtained from the self-dual action

$$S[e_I, A^{JK}] = \int F^{IJ} \wedge \eta_{IJ}$$

where

$$F^{IJ} := dA^{IJ} + A^I_M \wedge A^M_J = \frac{1}{2} (\Omega^{IJ} - \frac{1}{2} i \epsilon^{IJ}_{MN} \Omega^{MN})$$

and

$$A^{IJ} := \frac{1}{2} (\omega^{IJ} - \frac{1}{2} i \epsilon^{IJ}_{MN} \omega^{MN})$$

are the self-dual parts of the curvature and connection forms, and $\Omega^{IJ} := d\omega^{IJ} + \omega^{I}_K \wedge \omega^{K}_J$ is the Riemannian curvature 2-form.
The gravitational Hamiltonian in terms of Ashtekar’s New Variables has the form

\[ H(A, \tilde{E}) = \int d^3 x \tilde{N} C + N^a V_a + A_0^i \mathcal{G}_i + \oint \ldots, \]  

(2.4)

where \( \tilde{N} \) is a density of weight \(-1\), representing the lapse, \( N^a \) is the shift vector and

\[ C := -F_{ab} \tilde{E}^a \tilde{E}^b, \]  

(2.5)

\[ V_a := \frac{1}{2} \epsilon_{ij}^k F_{ab} \tilde{E}^b \tilde{E}^k, \]  

(2.6)

\[ \mathcal{G}_i := D_a \tilde{E}^a_i, \]  

(2.7)

are the Hamiltonian, momentum and Gauss constraint respectively, with \( \tilde{E}^a_i \) the densitized triad and

\[ A_{a}^{ij} := \Gamma_{a}^{ij} - i\epsilon_{0k} \Gamma_{a}^{0k} = \Gamma_{a}^{ij} + i\epsilon_{ijk} K_{ak}, \]  

(2.8)

is the complex connection. It is sometimes more convenient to use

\[ A_{ak} := \frac{1}{2} \epsilon_{ijk} A_{a}^{ij} = \Gamma_{ak} + iK_{ak}, \]  

(2.9)

One advantage of this formulation is that the constraints are simpler, in fact they are low order polynomials in the variables. If we impose suitable reality conditions this formulation is equivalent to the standard formulation as long as the triad is non-degenerate.

3. Positive energy

A fundamental requirement for any gravitation theory is that gravity be attractive. Technically, all solutions (with realistic sources and suitable asymptotic regions) should have positive total energy. The proper definition of total energy at time \( t_0 \) for a solution is just the value of the Hamiltonian at that time. The Hamiltonian is an integral over the spatial hypersurface, \( t = t_0 \), of a suitable Hamiltonian density. Thus the Hamiltonian density can serve as an energy density. Hence for a positive energy proof (and more, a localization of gravitational energy), we can attempt to “diagonalize” the Hamiltonian density, representing it as a quadratic expression and try to suppress the negative definite part.
In the particular case we are concerned with here, Ashtekar’s New Variables formulation, the variables are complex and so is the Hamiltonian. We want to make the real part of the Hamiltonian positive, to achieve this we naturally expect that it will be necessary to use the reality conditions.

Moreover, we will freely use the non-degeneracy conditions. We know of no theoretical principle which would require positivity in the degenerate sector. Hence we were not surprised to learn that there is an explicitly constructed negative total energy solution with a degenerate metric (Varadarajan 1991).

Although we think it might be revealing to do so, in this work we make no attempt to minimize our use of non-degeneracy, the reality conditions and vanishing torsion.

Clearly we can revert to the traditional variables and do proofs and localizations. But can we do proofs and localizations directly in terms of the New Variables? We regard this as one desirable quality for “good variables”.

4. Some earlier “proofs”

Given the importance that was attached to positive energy proofs a decade ago as well as the recent resurgence of interest in the quasi-localization of gravitational energy we are surprised that there has been so little done on this topic in connection with the New Variables. A few attempts have been made but in our estimation none have been successful.

Ashtekar (1987) credits Joohan Lee with the simple form of the Hamiltonian with \( N^k = 0 \)

\[
H(A, \tilde{E}) = \int d^3x \sqrt{g} \left( A^{ab} A_{ba} - \tilde{A} A \right) \tag{4.1}
\]

(We know of no published derivation, in the appendix we derive a more general expression.) It was noted that this would be a positive energy proof if \( A_{ab} \) were symmetric and traceless. However it was soon realized that the antisymmetric part of \( A_{ab} \) must be non-vanishing except for the trivial case for which the total energy vanishes (Ashtekar 1988, 1991).

One quite interesting approach has been developed by Mielke (1990, 1992). He considered the teleparallel equivalent of Einstein’s theory, introducing new complex variables of the Ashtekar type. In one part of this work he noted the results (4.1) and realized that a positive energy proof using the special orthonormal frame (SOF) gauge conditions should be possible. However, in his analysis
it was incorrectly assumed that the SOF gauge (see appendix) guaranteed that certain relations (eqs (12.3) and (12.4) in Mielke 1992) held. Hence his proof, as he appears to have suspected (one of his papers is entitled “Positive-gravitational-energy proof from complex variables?”), is not correct. In section 7 we shall present the correct application of the SOF gauge to this question.

Wallner (1990, 1992) has presented an insightful reexamination of Ashtekar’s New Variables in terms of exterior forms. Among other topics this work claims to include a simple proof of positive total energy. However, the total energy was presumed to be the limiting value of the integral over an increasingly large 2-surface of its extrinsic curvature. Although it is known that there is a relationship between this integral and the total energy, nevertheless, this quantity by itself diverges—even for flat space. After renormalization it is no longer clearly positive. Hence this proof also is not correct.

5. A Witten spinor type proof?

It is natural to try to obtain a positive energy proof of the spinor type (Witten 1981) in terms of New Variables especially in view of the important role that Sen connections (Sen 1982) have played in both of these developments. Indeed, at first we wondered why a recent work (Mason and Frauendiener 1990) covering both of these topics did not include such a proof. The following analysis reveals certain difficulties with such a proof.

We can express the New Variables Hamiltonian for asymptotically flat space in the form:

\[ H(N) = \int N^K F^{IJ} \wedge \eta_{IJK} + \oint \ldots = \int d^3 x 2N^\mu G^\nu_{\mu} + \oint \ldots \]  

(5.1)

where \( G^\nu_{\mu} \) is the complex Einstein tensor obtained from the self-dual curvature \( F^{IJ} \).

It is easy to verify that the spinor-curvature identity (Nester 1984)

\[ \mathcal{H}(\psi) := 2\{D(\bar{\psi}\gamma_5 \gamma) \wedge D\psi - D\bar{\psi} \wedge D(\gamma_5 \gamma \psi)\} \equiv 2N^\mu G^\nu_{\mu} \eta_{\nu} + dB \]  

(5.2)

where \( B := (\bar{\psi}\gamma_5 \gamma) \wedge D\psi - \bar{\psi} \wedge D(\gamma_5 \gamma \psi) + D(\bar{\psi}\gamma_5 \gamma) \psi + D\bar{\psi} \wedge (\gamma_5 \gamma \psi), \gamma = \gamma_{\mu} dx^{\mu}, \) and \( N^\mu = \bar{\psi} \gamma^\mu \psi, \) extends to the covariant differential using the self-dual connection, \( D\psi := d\psi - \frac{1}{4} A^{IJ} \gamma_{IJ} \psi \) (for spinor conventions see appendix). It is also easy to verify that \( \mathcal{H}(\psi) \) is a good expression for the New Variables Hamiltonian, one need merely note that (i) up to an exact differential it is the complex Einstein tensor of the self-dual connection so it has the desired variational derivatives,
and (ii) it is asymptotically of order \( O(r^{-4}) \) and so its variation has an asymptotically vanishing boundary integral.

The Hamiltonian 3-form \( \mathcal{H}(\psi) \) is apparently quadratic; next we aim to make it positive definite following the argument which succeeded for the usual variables in Einstein’s theory. Choosing a spacelike hypersurface we can decompose \( \mathcal{H}(\psi) \) with respect to the normal into

\[
\mathcal{H}(\psi) = 4D\overline{\psi} \wedge \gamma_5 \gamma \wedge D\psi + 2[\overline{\psi} \gamma_5 (D\gamma) \wedge D\psi - D\overline{\psi} \wedge \gamma_5 (D\gamma)\psi] \tag{5.3}
\]

In order to make (5.3) locally positive, we can eliminate an apparently negative term by choosing \( \psi \) to solve the Witten equation \( \gamma^a D_a \psi = 0 \). However there remain some formidable obstacles to a positive energy proof: (1) the connection is now complex so that the first term in (5.3b) is not really positive definite, (2) the factor \( D\gamma = \gamma_I D\vartheta^I \), because of the self dual connection, is not simply the torsion and hence no longer vanishes.

Another approach begins from the spinor form of the New Variables Hamiltonian (5.2) and decomposes the self dual covariant differential

\[
D\psi := d\psi - \frac{1}{4} A^{IJ} \gamma_{IJ} \psi = d\psi - \frac{1}{8}(\omega^{IJ} - \frac{1}{2} i \epsilon^{IJ}_{\; MN} \omega^{MN}) \gamma_{IJ} \psi. \tag{5.4}
\]

Since \( \frac{1}{2} \epsilon^{IJ}_{\; MN} \gamma_{IJ} = -\gamma_{MN} \gamma_5 \), we have

\[
D\psi := d\psi - \frac{1}{4} \omega^{IJ} \gamma_{IJ} \frac{1}{2} (1 + i \gamma_5) \psi = d\psi_- + D\psi_+, \tag{5.5}
\]

where \( \psi_{\pm} := P_{\pm} \psi := \frac{1}{2} (1 \pm i \gamma_5) \psi \) are the chiral projections. Hence a natural approach is to restrict our considerations to chiral spinors, \( \psi = i \gamma_5 \psi = \psi_+ \), then the covariant differential using the self dual connection \( A^{IJ} \) is just the ordinary covariant differential: \( D\psi_+ = D\psi_+ \). Since \( P_{\pm} \gamma_\mu = \gamma_\mu P_{\mp} \) and \( \overline{\psi}_\pm = (\overline{\psi})_\mp \), the Hamiltonian 3-form (5.2) reduces to the simple expression

\[
\mathcal{H}(\psi_+) = 2[D(\overline{\psi_+} \gamma_5 \gamma) \wedge D\psi_+ - d\overline{\psi_+} \wedge d(\gamma_5 \gamma \psi_+)], \tag{5.6}
\]

Vanishing torsion simplifies the first term to the nice real form \( 2D(\overline{\psi_+}) \wedge \gamma_5 \gamma \wedge D\psi_+ \), however the second term, although an exact differential, is quite problematical.
Of course we will succeed in getting locally positive expressions if we carefully decompose everything into the usual variables, however it seems that we cannot get a Witten type proof directly in terms of the Ashtekar variables.

6. A three spinor field proof?

We next attempt a positive energy proof of the recently found 3-spinor type (Nester and Tung 1993). The New Variables Hamiltonian (2.4) with vanishing shift and gauge parameter

\[ H = \int d^3x \mathcal{N} \mathcal{C} + \oint \ldots, \]  

(6.1)
can be replaced (with \( N = \varphi^\dagger \varphi \)) by the spatial integral of the 3-form

\[ \mathcal{H}(\varphi) := 2[D(\varphi^\dagger i\sigma) \wedge D\varphi - D\varphi^\dagger \wedge D(i\sigma\varphi)] = dB - (\varphi^\dagger \varphi)F^{ij} \wedge \zeta_{ij} \]  

(6.2)

where \( B = \varphi^\dagger i\sigma \wedge D\varphi - \varphi^\dagger D(i\sigma\varphi) + D(\varphi^\dagger i\sigma)\varphi + (D\varphi^\dagger) \wedge i\sigma\varphi \) and \( \sigma := \sigma_a dx^a \). On the one hand it is easy to establish that this 3-spinor identity holds also for the connection \( A^{ij} \) as well as for \( \omega^{ij} \), consequently \( \mathcal{H}(\varphi) \) has the correct value \( -\mathcal{N}F_{abij} \bar{E}_{ai} \bar{E}_{bj} d^3x \) up to an exact differential and hence generates the correct equations of motion. On the other hand it is asymptotically of order \( O(r^{-4}) \) and hence the boundary term in its variation vanishes asymptotically so no additional total differential is needed. The Hamiltonian density \( \mathcal{H}(\varphi) \) can be decomposed as follows:

\[
2[D(\varphi^\dagger i\sigma) \wedge D\varphi - D\varphi^\dagger \wedge D(i\sigma\varphi)]
= 4[D\varphi^\dagger \wedge i\sigma \wedge D\varphi] + 2[\varphi^\dagger D(i\sigma) \wedge D\varphi - D\varphi^\dagger \wedge D(i\sigma)\varphi]
= 4[g^{ab}D_a \varphi^\dagger D_b \varphi - D_a \varphi^\dagger \sigma^a \sigma^b D_b \varphi] \zeta + 2[\varphi^\dagger D(i\sigma) \wedge D\varphi - D\varphi^\dagger \wedge D(i\sigma)\varphi].
\]  

(6.3)

We may try to eliminate the apparently negative term by the Witten like equation \( \sigma^a D_a \varphi = 0 \) however we find the same difficulties as mentioned in connection with the 4-spinors: (1) the connection is complex so that the first term in (6.3) is not positive definite, in fact decomposing this term reveals that it contains the kinetic energy term \( K_{ab}K^{ab} \) of the ADM Hamiltonian with an incorrect negative sign! (2) the factor \( D\sigma = \sigma_i D\theta^i \) which, because of the self dual connection, is not simply the torsion and hence does not vanish. Explicitly the covariant differential \( D \) in the self-dual connection \( A \) is

\[
D\varphi := d\varphi + \frac{1}{2}A^{ij} \sigma_i \sigma_j \varphi
= d\varphi + \frac{1}{2}(\omega^{ij} + i\epsilon^{ijk} K_k) \sigma_i \sigma_j \varphi = \nabla \varphi - \frac{1}{2}K_k \sigma^k \varphi,
\]  

(6.4)
and
\[ \mathcal{D} \varphi^\dagger = \nabla \varphi^\dagger + \frac{1}{2} K_k \varphi^\dagger \sigma^k. \]  
(6.5)

Consequently the quadratic in \( \mathcal{D} \varphi \) terms in the Hamiltonian (6.3) further decompose as

\[ 4[g^{ab} \mathcal{D}_a \varphi^\dagger \mathcal{D}_b \varphi - \mathcal{D}_a \varphi^\dagger \sigma^a \sigma^b \mathcal{D}_b \varphi] \]
\[ = [4g^{ab} \nabla_a \varphi^\dagger \nabla_b \varphi - (\varphi^\dagger \varphi) K^{ab} K_{ab} + 2K_{ai}(\varphi^\dagger \sigma^i \nabla^a \varphi - \nabla^a \varphi^\dagger \sigma^i \varphi)] \]
\[ - [4 \nabla_a \varphi^\dagger \sigma^a \sigma^b \nabla_b \varphi - (\varphi^\dagger \varphi) K^2 - 2 \nabla_a \varphi^\dagger \sigma^a \sigma^b K_{bi} \sigma^i \varphi + 2K_{ai} \varphi^\dagger \sigma^i \sigma^a \sigma^b \nabla_b \varphi]. \]  
(6.6)

By imposing a 3-dimensional Dirac equation on \( \varphi \)
\[ \sigma^a \nabla_a \varphi = 0, \]  
(6.7)
we obtain
\[ 4g^{ab} \nabla_a \varphi^\dagger \nabla_b \varphi - (\varphi^\dagger \varphi)(K^{ab} K_{ab} - K^2) + 2K_{ai}(\varphi^\dagger \sigma^i \nabla^a \varphi - \nabla^a \varphi^\dagger \sigma^i \varphi), \]  
(6.8)
which, as claimed above, contains the quadratic extrinsic curvature terms with the opposite sign as appears in the ADM Hamiltonian. This sign is reversed by the quadratic \( K_{ab} \) terms in the expression
\[ +2[\varphi^\dagger \mathcal{D}(i\sigma) \wedge \mathcal{D} \varphi - \mathcal{D}_a \varphi^\dagger \wedge \mathcal{D}(i\sigma) \varphi]. \]  
(6.9)

Thus if we decompose everything into the usual variables, we get
\[ \mathcal{H}(\varphi) = [4g^{ab} \nabla_a \varphi^\dagger \nabla_b \varphi + (\varphi^\dagger \varphi)(K^{ab} K_{ab} - K^2)] \zeta \]  
(6.10)
so this type of argument will work but again it seems that we cannot get a spinor type proof directly in terms of the Ashtekar variables.

7. A special orthonormal frames proof

In this section we present a positive energy proof and gravitational energy localization using Ashtekar’s New Variables and the special orthonormal frame rotational gauge conditions. The proof and localization are similar to the SOF proof and localization (Nester 1989bc, 1991) for the usual formulation.
The New Variables Hamiltonian (2.4) (with vanishing shift vector $N^a$) is given by:

$$H(A, \tilde{E}) = -\int d^3x \, (F_{ab}^{\ ij} \tilde{E}^a_i \tilde{E}^b_j) + 2 \oint dS_a \, A_b^{\ ij} \tilde{E}^a_i \tilde{E}^b_j,$$  

(7.1)

Taking the divergence of the boundary term, using the vanishing torsion and reality conditions (see appendix for explicit calculation) gives

$$H(A, \tilde{E}) = \int d^3x \, 2(\partial_a N) \epsilon^{ijk} A_{bk} \tilde{E}^a_i \tilde{E}^b_j + N g^{1/2}(\overline{A^{ab}} A_{ba} - \overline{A}),$$  

(7.2)

a generalization of (4.1).

Clearly we should consider the symmetric and antisymmetric parts of $A_{ab}$ separately. Introducing the vector $A^i := A_{a}^{\ ij} E^a_j = \epsilon^{ijk} E^a_j A_{ak}$ the Hamiltonian density of (7.2) takes the form

$$\mathcal{H}(A, \tilde{E}, N) = 2 g^{1/2}(\partial_a N) A^a - N \frac{1}{2} g^{1/2} \overline{A^b} A_b + N g^{1/2}(\overline{A^{(ab)}} A_{(ba)} - \overline{A}).$$  

(7.3)

Since torsion vanishing implies that $K_{ij}$ is symmetric we find, from (2.9), that $A^i = \Gamma^i_{aj} E^a_j$. Hence, in the special orthonormal frame rotational gauge (see appendix and Nester 1989a) $A_b = 4 \partial_b \ln \Phi$, consequently the Hamiltonian density (7.3) takes the (already real) form

$$\mathcal{H}(A, \tilde{E}, N) = 8 g^{1/2} g^{ab} \partial_a (N \Phi^{-1}) \partial_b \Phi + N g^{1/2}(\overline{A^{ab}} A_{(ba)} - \overline{A}).$$  

(7.4)

It is worth remarking that this expression is good for compact spatial surfaces as well as for the asymptotically flat spatial surfaces which are of interest to us here as they permit a definition of total energy.

Many choices of the lapse $N$ will yield a locally positive Hamiltonian density and thus a positive energy proof. A particularly good choice for the lapse in eq (7.4) is $N = \Phi$ which leads to the following expression for the gravitational energy density

$$\mathcal{H}(\Phi) = \Phi \sqrt{g} (\overline{A^{(ab)}} A_{(ab)} - \overline{A}).$$  

(7.5)

This very succinct expression for the gravitational energy density is reminiscent of Lee’s expression (4.1). It is locally non-negative if the complex scalar $A = \Gamma + iK$ vanishes, i.e. on maximal ($K = 0$) asymptotically flat ($\Gamma = 0$, see appendix) spatial hypersurfaces thereby providing (in addition to a
positive energy proof) a localization of gravitational energy. The total gravitational energy within a volume $V$ with surface $S$ is given by

$$16\pi GE = \int_V \mathcal{H}(\Phi) \, d^3x = 8 \oint_S g^{1/2} g^{ab} \partial_a \Phi \, dS_b. \tag{7.6}$$

For the Schwarzschild black hole, this expression localizes all of the energy inside. The second equality in eq (7.6) follows from the SOF version of the Hamiltonian initial value constraint, which, with the addition of a source,

$$8g^{1/2} \nabla^2 \Phi = \mathcal{H}(\Phi) + 16\pi G g^{1/2} \Phi \rho, \tag{7.7}$$

generalizes the Poisson equation of Newtonian gravity, reveals the special function $\Phi$ as a generalization of the Newtonian potential, justifies our identification of the gravitational energy density (7.5) and is closely related to the scale equation of the usual approach (see e.g., Choquet-Bruhat and York 1981) to the initial value constraints.

8. Discussion

Admitting a positive energy proof is a desirable property for a good set of variables. We have discussed some earlier unsuccessful attempts at a positive energy proof in terms of Ashtekar’s New Variables. We have also pointed out the difficulties of a Witten spinor type proof (although we have not yet given up trying). We were surprised to find this difficulty and this apparent mismatch between these two applications of Sen connections. To us it suggests that either the Witten spinor or the New Variables formulation does not properly represent an important part of the gravitational physics.

Ashtekar’s New Variables are a triad and a connection. The special orthonormal frame gauge conditions are independent but nicely complimentary: they fix the rotational gauge freedom. In the special orthonormal frame gauge with the special choice of lapse, $N = \Phi$, the New Variables Hamiltonian is locally positive on maximal asymptotically flat spacelike surfaces, thereby providing a positive energy proof and more, a gravitational energy localization. Moreover the SOF gauge links the New variables Hamiltonian constraint to the scale equation of the standard approach to the initial value constraints.
Beyond this the SOF-New Variables formulation also offers prospects for a quasi-localization of energy. The SOF gauge conditions are elliptic equations. The solution depends upon the choice of boundary values. Concerning the solution on the whole spacelike hypersurface $t = t_0$ the natural choice is the Cartesian frame at spatial infinity. This leads to the nice localization of the total energy mentioned above. However, the same Hamiltonian and total energy expressions may also be used for a strictly finite region to give a real quasi-localization if they are supplemented by some satisfactory method of choosing the boundary values on a finite surface.

The success of the special orthonormal frame approach complimenting the New Variables formulation increases our confidence in both the New Variables formulation and the SOF gauge conditions.

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Appendix:

(I) Conventions

Generally we follow the conventions of Ashtekar (1988, 1991); in particular, the metric signature is $(-,+,+,+)$, the labels $IJK...$ are used for spacetime orthonormal frames, $\mu\nu...$ are spacetime coordinate indices, $ijk...$ are spatial orthonormal frames, and $abc...$ are spatial coordinate indices. The respective orthonormal coframes are $\vartheta^I$ and $\theta_i$. The anti-symmetric Levi-Civita tensors are denoted by $\eta_{IJKL}$ and $\epsilon_{ijk}$, with $\eta_{0123} = +1$. We also use the combinations $\eta_I := \ast \vartheta_I$, $\eta_{IJ} := \ast (\vartheta_I \wedge \vartheta_J)$, and $\eta_{IJK} := \ast (\vartheta_I \wedge \vartheta_J \wedge \vartheta_K)$. The 3-dimensional volume 3-form is $\zeta = \vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3 = \sqrt{g} d^3x$ and $\zeta_{ij} := \epsilon_{ijk} \vartheta^k$. The notation $\bar{E}$ and $\bar{N}$ indicates 3-dimensional densities of weight +1 and −1 respectively.

The 4-dimensional Dirac spinor conventions used here are: $\gamma_I \gamma_J + \gamma_J \gamma_I = 2g_{IJ}$, $\gamma_I = \gamma_I \gamma_5$, $\gamma_5 := \gamma^0 \gamma^1 \gamma^2 \gamma^3$, and $\gamma_I \gamma_J + \gamma_J \gamma_I = 2 \gamma_5 \eta_{IJKL} \gamma^L$. For 3-dimensional Pauli spinors we use $\sigma_i \sigma_j = \delta_{ij} + i \epsilon_{ijk} \sigma^k$.

(II) Derivation of New Variables Hamiltonian density
Here we present a derivation of the Hamiltonian expression (7.2). We find it convenient to work in terms of the connection one-form $A_{ij} := A_{aij} dx^a$, the curvature 2-form $F_{ij} := dA_{ij} + A_k^i \wedge A_k^j = \frac{1}{2} F_{ab}^{ij} dx^a \wedge dx^b$ and the basis one forms $\theta^i = E_{ai}^b g_{ab} dx^b$ (we presume a nondegenerate triad). The New Variables Hamiltonian (with vanishing shift vector $N^a$) (7.1) takes the form

$$H(A, \tilde{E}) = -\int NF_{ij} \wedge \epsilon_{ijk} \theta^k + \oint N A_{ij} \wedge \epsilon_{ijk} \omega^k \wedge \theta^l. \quad (A1)$$

Converting the boundary integral into a volume integral yields the the Hamiltonian 3-form:

$$\mathcal{H}(A, \tilde{E}, N) d^3x = -NA^i_k \wedge A^{kj} \wedge \epsilon_{ijkl} \theta^l + dN \wedge A_{ij} \wedge \epsilon_{ijk} \theta^k + NA^i_j \wedge \epsilon_{ijk} \omega^k \wedge \theta^l, \quad (A2)$$

where we have assumed that the torsion $d\theta^k + \omega^k \wedge \theta^l$ vanishes. Hence the New Variables Hamiltonian density is

$$\mathcal{H}(A, \tilde{E}, N) = -Ng^{1/2}(A^{ab} A_{ba} - A^2) + 2N_a g^{1/2} E_{a}^{i} A_{b}^{i} E_{b}^{j} + 2Ng^{1/2}(A^{ab} \Gamma_{ba} - A \Gamma), \quad (A3)$$

where $A_{ab} := A_{ak} E_{b}^{k}$ and $A := A_{ai} E_{ai}^i$, likewise $\Gamma_{ab}$ and $\Gamma$. Using the reality condition $2\Gamma_{ab} = A_{ab} + \overline{A_{ab}}$ leads to eq (7.2).

(III) Special orthonormal frames

For 3-dimensions the rotational state of a frame field is specified by a function and a covector field. In terms of the connection coefficients these quantities are $\Gamma$ and $\Gamma_a := -E_{ai}^a \Gamma_{ai}^{ij} E_{bj}$. The rotational gauge conditions (for existence and uniqueness for this elliptic system see Dimakis and Müller-Hoissen 1989)

$$\Gamma = \text{constant}, \quad \Gamma_a = \text{gradient} \quad (A5)$$

select a special orthonormal frame. For asymptotically flat spaces the frame should be Cartesian at infinity, consequently $\Gamma$ must vanish at spatial infinity and hence everywhere. This leads to the additional bonus that the gauge conditions are conformally invariant for asymptotically flat spaces. The gradient condition on $\Gamma_a$ determines a special function, the choice $\Gamma_a = 4\partial_a \ln \Phi$ is convenient.
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