Mild solutions and spacetime integral bounds for Stokes and Navier-Stokes flows in Wiener amalgam spaces

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Abstract

We first prove decay estimates and spacetime integral bounds for Stokes flows in amalgam spaces $E^r_q$ which connect the classical Lebesgue spaces to the spaces of uniformly locally $r$-integrable functions. Using these estimates, we construct mild solutions of the Navier-Stokes equations in the amalgam spaces satisfying the corresponding spacetime integral bounds. Time-global solutions are constructed for small data in $E^3_q$, $1 \leq q \leq 3$. Our results provide new bounds for the strong solutions classically constructed by Kato and the more recent solutions in uniformly local spaces constructed by Maekawa and Terasawa. As an application we obtain a result on the stability of suitability for weak solutions to the perturbed Navier-Stokes equation where the drift velocity solves the Navier-Stokes equations and has small data in a local $L^3$ class. Extending an earlier result, we also construct global-in-time local energy weak solutions in $E^2_q$, $1 \leq q < 2$.

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1 Introduction

This paper considers mild solutions with decay estimates and spacetime integral bounds of the nonstationary Stokes system and the Navier-Stokes equations in $\mathbb{R}^d$, $d = 3$, for initial data in uniformly local $L^q$ spaces and Wiener amalgam spaces. The nonstationary Stokes system in $\mathbb{R}^d$ reads

$$u_t - \Delta u + \nabla \pi = f \quad \text{in} \quad \mathbb{R}^d \times (0, \infty),$$

with initial condition

$$u(\cdot, 0) = u_0, \quad \text{div} \, u_0 = 0.$$

Here $u = (u_1, \ldots, u_d)$ is the velocity, $\pi$ is the pressure, and $f = (f_1, \ldots, f_d)$ is the external force. The Navier-Stokes equations reads

$$u_t - \Delta u + \nabla \pi = -u \cdot \nabla u \quad \text{div} \, u = 0 \quad \text{in} \quad \mathbb{R}^d \times (0, \infty),$$

with initial condition (1.2).

A classical approach to studying (1.3) is to first establish estimates for the heat equation and (1.1) and then extend these to the nonlinear system (1.3) using a Picard iteration. This was done in $L^p$ spaces for $d < p < \infty$ by Fabes, Jones and Riviere [15] and $L^d$ by Kato [30]. The endpoint case $p = \infty$ has been examined by Giga, Inui and Matsui [20] as well as Kukavica [33]. The solutions generated by these arguments satisfy an integral formula and are referred to as mild solutions. In the present context, they are smooth and unique. The cases $L^d$ and $L^p$ for $p > d$ are qualitatively distinct in that, for $L^d$, global existence is known but only for small data and local existence is known for large data but the time-scale is not related to the size of the data. On the other hand, for $L^p$ where $p > d$, local existence is known for arbitrarily large data and the time-scale is dependent on the size of the data, but global existence is unknown.

To develop a better understanding of this subject, note that large and small scales play different roles in the regularity of solutions to the Navier-Stokes equations, which represents the fact that, for $p_1 > p_2$, $L^{p_1}$ has better decay at small scales and worse decay at large scales than $L^{p_2}$. Additionally, for parabolic equations, small physical scales in the initial data act primarily on short time scales in the solution while the same is true for large physical scales and large time scales. Hence, if the data is sufficiently regular at small scales, then it should be smooth for small enough times. Decay of the data at large scales should lead to regularity at large times. For $d = 3$, this theme holds true for the Navier-Stokes equations in $L^p$ spaces where the endpoint space for both scenarios is $L^3$. Indeed, for data in $L^2$, the time-global weak solutions of Leray exhibit eventual regularity. For data in $L^p$ with $p > 3$, the time-local mild solutions of Fabes, Jones and Riviere [15] as well as those of Giga, Inui and Matsui [20] are smooth.

The small and large scale decay assumptions on the data can be refined. For example, Maekawa and Terasawa [38] extended the $L^\infty$ theory to a scale of spaces $L^{r}_{uloc}$ defined by the norm

$$\|a\|_{L^r_{uloc}} := \sup_{x_0 \in \mathbb{R}^d} \|a\|_{L^r(B_1(x_0))}.$$

In the case of $L^d_{uloc}$, we are considering a space with local $L^d$ integrability properties and globally worse than $L^d$ decay properties. Hence we only expect short time existence of a smooth solution for small data and this is what is proven in [38]—the long time result of [30] is out of reach due to a lack of decay. When $d < r < \infty$, Maekawa and Terasawa [38] prove local existence of strong mild
solutions in analogy with [15]. Note that space-time integral estimates of Kato [30] (due to Giga) aren’t included in [38].

The classical Lebesgue spaces and the uniformly local spaces can be connected in a sense by the Wiener amalgam spaces. These spaces have a rich literature [7, 12, 14, 19, 26, 31, 36] and have been applied to the analysis of fluids in, e.g., [31, 3, 10]. The Wiener amalgam spaces are denoted $E^p_q$ and defined by the norm

$$
\|a\|_{E^p_q} := \left\| \left( \int_{B_1(k)} |a(x)|^p \, dx \right)^{1/p} \right\|_{\ell^q(k \in \mathbb{Z}^d)} < \infty.
$$

We identify $E^0_q$ with $L^p_{uloc}$. So, $E^p_q$ encodes local $L^p$ integrability and global $L^q$ decay. We have the embeddings

$$
E^p_q \subset E^m_q \text{ for } m > q \text{ and } E^p_q \subset E^r_q \text{ for } r < p.
$$

For the classical Lebesgue spaces, $L^p = E^p_p$, we therefore have the embeddings

$$
L^p \subset E^p_q \text{ and } L^q \subset E^p_q \text{ if } q > p,
$$
as well as

$$
L^p \supset E^p_q \text{ and } L^q \supset E^p_q \text{ if } p > q.
$$

We will use the Hölder inequality for $p, p_1, p_2, q, q_1, q_2 \in [1, \infty]$

$$
\|fg\|_{E^p_q} \leq \|f\|_{E^{p_1}_{q_1}} \|g\|_{E^{p_2}_{q_2}}, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.
$$

It implies (1.4) by taking $g = 1$.

The main goal of this paper is to develop a theory of strong solutions in the Wiener amalgam spaces $E^p_q$ where $d = 3 \leq p \leq \infty$ and $1 \leq q \leq \infty$. Our theory encompasses the following classical results on strong solutions in $L^p$-type spaces in $\mathbb{R}^d$ when $d = 3$, in chronological order (this is an incomplete list, of course; it does not include solutions in Sobolev or Besov spaces, nor weak solutions):

1. Fabes, Jones and Riviere [15] show local existence for any $u_0 \in L^r(\mathbb{R}^d)$, $r > d \geq 3$. Their Theorem (4.3) gives global existence for small $u_0 \in L^{r_1} \cap L^{r_2}(\mathbb{R}^d)$, $r_1 < d < r_2$. Their solutions belong to $L^s(0, \infty; L^p)$, $\frac{d}{p} + \frac{2}{s} \leq 1$, for each finite $T$.

2. Giga-Miyakawa [22] (submitted before [30] although published later) includes global solutions for small $L^d$ initial data in bounded domains in $\mathbb{R}^d$—see Theorem 2.6. It estimates the nonlinear term by means of suitable fractional powers of the Stokes operator. This approach is extended to exterior domains (see, e.g., [23, 24]) and, although it has not been explored in the literature, it seems likely that it also extends to $\mathbb{R}^d$ ($d \geq 3$) where the Stokes operator and Helmholtz projector have explicit forms.

3. Kato [30] constructs global strong solutions for small initial data in $L^d(\mathbb{R}^d)$ and establishes the spacetime integral bound $u \in L^s(0, \infty; L^p(\mathbb{R}^d))$ using Giga’s estimate (1.11); see [21]. This spacetime integral bound is not necessary for existence, but is used in the proof of Theorem 2’ which asserts that $\|u(t)\|_{L^d} \to 0$ in time average as $t \to \infty$; see (1.6) and [30, (2.15)].

---

1Let $E^p$ be the closure of the test functions under the $L^p_{uloc}$ norm, a notation used in [37]. Our convention in this paper is $E^p \subsetneq E^p_{uloc} = L^p_{uloc}$. We used $E^p_{uloc} = E^p$ in [10] since $E^p$ preserves a decay property at spatial infinity which is also enjoyed by $E^p_q$ where $q < \infty$. 

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4. Giga [21] shows (1.11) and develops the $L^r$-theory for general domains and equations.

5. Giga, Inui and Matsui [20] and Kukavica [33] prove local existence of strong solutions in $L^\infty(\mathbb{R}^d)$. The analyticity of the Stokes semigroup is established by Abe and Giga [2] for bounded domains and by Hieber and Maremonti [25] for an exterior domains in $\mathbb{R}^d$ for $d \geq 3$. More results on the Navier-Stokes equations are given by Abe [1] in the $L^\infty$ framework for a class of domains including bounded and exterior domains.

6. Maekawa and Terasawa [38] establish local existence of strong solutions in $L^r_{uloc}(\mathbb{R}^d)$, $r \geq d$.

Global spacetime integral bounds are shown only in [30, 21].

Note that because $E^r_q \subset L^r_{uloc}$ where a strong solution theory exists, the novel contribution of our work concerns the persistence of norms in the Wiener amalgam spaces, in addition to new spacetime estimates. Moreover, our second exponent $q$ in $E^r_q$ (measuring spatial decay), is allowed to go down to $q = 1$, which will lead to solutions which are more localized in space. The lower bound $r \geq 3 = d$ is expected since it is the borderline for local regularity theory.

Some classical estimates in Wiener amalgam spaces can be found in [12]. The theme of “filling in” spaces between a classical Lebesgue spaces and a uniformly locally Lebesgue space using amalgam spaces is explored for weak solutions in spaces $E^2_q$ between $L^2$ and $L^2_{uloc}$ by the first and last author in [10]. Hence this paper can be considered a companion of [10]. The exponent $q$ in [10] for weak solutions is limited to $2 \leq q < \infty$. We will sketch necessary changes to extend the weak solution theory to $E^q$, $1 \leq q < 2$, in Section 4.1.

### 1.1 Spacetime integrals

Spacetime integral bounds are useful in analyzing solutions of the Navier-Stokes equations because they provide more information about spatial-temporal decay. In particular, the borderline integrability inferred from spacetime integral bounds at $t = 0^+$ and $t = T_{\max}$ can be used to study the regularity of the solutions. Moreover, spacetime integral bounds are used by Kato [30, (2.15)] to show the convergence of time average of the $L^d$ norm

$$\frac{1}{T} \int_0^T \|u(t)\|_{L^d(\mathbb{R}^d)} \to 0 \quad \text{as} \quad T \to \infty,$$

for all $d \geq 2$, and $\|u(t)\|_{L^2} \to 0$ for $d = 2$.

We will consider two kinds of spacetime integrals: For $0 < T \leq \infty$, $x \in \mathbb{R}^d$, and $1 \leq s, p, q \leq \infty$, define the norms $L^s_T E^p_q$ and $E^{s,p}_{T,q}$ as follows:

$$\|u\|_{L^s_T E^p_q} := \|u\|_{L^s(0,T;E^p_q(\mathbb{R}^d))},$$

and

$$\|u\|_{E^{s,p}_{T,q}} := \left\| \int_0^T \|u\|_{L^s(B_1(k))} \mathrm{d}t \right\|_{\ell^q(k \in \mathbb{Z}^d)}.$$  

These norms are different from each other when $s \neq q$. By Minkowski’s integral inequality,

$$\|u\|_{L^s_T E^p_q} \leq \|u\|_{E^{s,p}_{T,q}}, \quad \text{if} \quad q \leq s,$$

and

$$\|u\|_{E^{s,p}_{T,q}} \leq \|u\|_{L^s_T E^p_q}, \quad \text{if} \quad q \geq s.$$  

The reversed inequalities are typically wrong. See Example 5.2.
The class $E_{r,q}^{s,p}$ seems more appropriate to parabolic equations than the class $L_{r,q}^s$. Indeed, for the heat semigroup in $\mathbb{R}^d$, when $s < \infty$ it follows from Giga [21] that

$$
\|e^{t\Delta}a\|_{L^s(0,\infty;L^p(\mathbb{R}^d))} \lesssim \|a\|_{L^r(\mathbb{R}^d)},
$$

if

$$
1 < r \leq s \leq \infty, \quad r \leq p < \infty, \quad \frac{2}{s} + \frac{d}{p} = \frac{d}{r}.
$$

(1.12)

The case $s = \infty$ is classical. A special case of our Lemma 2.4 extends the above to

$$
\mathbb{1}_{m \leq s} \|e^{t\Delta}a\|_{L_{T=\infty}^s E^p_m} \lesssim \|e^{t\Delta}a\|_{E_{T=\infty,m}^{s,p}} \lesssim \|a\|_{E^p_q},
$$

if

$$
1 < r \leq s \leq \infty, \quad r \leq p < \infty, \quad \frac{2}{s} + \frac{d}{p} = \frac{d}{r}, \quad 1 < q < m < \infty, \quad \frac{2}{s} + \frac{d}{m} \leq \frac{d}{q}.
$$

(1.13)

Lemma 2.4 also has finite time estimates which allow $1 \leq q \leq m \leq \infty$ without the restriction $\frac{2}{s} + \frac{d}{m} \leq \frac{d}{q}$.

In (1.11), the case $r < s$ is proved in [21, Lemma, p. 196] using the Marcinkiewicz interpolation theorem. It is mentioned in [21, Acknowledgments] that the case $r = s$ in (1.11) is also valid if one appeals to the generalized Marcinkiewicz theorem. We will give details for $r = s$ in the Appendix (§5.1). The case $r > s$ was unclear. We will show in Example 2.6 that (1.11) is actually false when $r > s$ even for $T < \infty$. Moreover, Example 2.6 shows that if $s < q \leq \infty$, then there is $a \in E^p_q$ with $e^{t\Delta}a \notin L_{T=1}^s E^p_m$. Thus the factor $\mathbb{1}_{m \leq s}$ in (1.13) is necessary. In particular, when $a \in L^r_{uloc}$, $e^{t\Delta}a$ is bounded in $E_{T=1,\infty}^{s,p}$, but not in $L_{T=1}^s E^p_q$.

### 1.2 Mild solutions in amalgam spaces

We now present three theorems on mild solutions to (1.3) in $\mathbb{R}^3$. Recall that a mild solution is a solution $u$ to (1.3) satisfying

$$
u(x,t) = e^{t\Delta}u_0 - B(u,u)(t); \quad B(f,g)(t) = \int_0^t e^{(t-s)\Delta}P\nabla \cdot (f \otimes g) \, ds,$$

where $P$ is the Helmholtz projection operator. When the spatial domain is $\mathbb{R}^3$, we can express $B$ by the Oseen tensor, see (3.2).

The following theorem concerns data which is locally subcritical, i.e., $u_0 \in E^q_q$ with $r > 3$. When $q = \infty$ it includes [38, Theorem 1.1 (i)] and when $r = q < \infty$ it includes the results from [15].

**Theorem 1.1** (Subcritical data). Let $r \in (3,\infty]$ and $q \in [1,\infty]$. If $u_0 \in E^q_q$ is divergence free, then, for any positive time $T = T(\|u_0\|_{E^q_q})$ chosen so that

$$
T^{1/2-3/(2r)} + T^{1/2} \lesssim \|u_0\|_{E^q_q}^{-1},
$$

(1.14)

there exists a unique mild solution $u \in L^\infty(0,T;E^r_q) \cap C((0,T);E^r_q)$ to (1.3). Moreover, $u$ satisfies

$$
\sup_{0 \leq t \leq T} \|u(t)\|_{E^r_q} \leq C\|u_0\|_{E^q_q}.
$$

(1.15)

If $q,r < \infty$, then $u \in C([0,T];E^r_q)$. If $q = \infty$ or $r = \infty$, then we still have $\|e^{t\Delta}u_0 - u(t)\|_{E^q_q} \to 0$ as $t \to 0^+$.
Furthermore, if \( r < \infty \), then for any \( s \in [r, \infty) \) and \( p \in [r, 3r] \) with \( \frac{2}{s} + \frac{3}{p} = \frac{3}{r} \),
\[
\|u\|_{E^{s,p}_{T,m \geq q}} \leq C \|u_0\|_{E^q_q}
\]
provided \((1 + T^{\frac{1}{r} + \epsilon})(T^{\frac{1}{r} - \frac{2}{s}} + T^{1 - \frac{1}{s}}) \lesssim \|u_0\|_{E^q_q}^{-1} \) for all \( \epsilon > 0 \).

Comments on Theorem 1.1:

1. The apparent dimensional mismatch between terms in (1.14) is corrected by suppressed dimensional constants of magnitude \( \sim 1 \) and this comment holds throughout the paper.

2. Our uniqueness class does not require the bound (1.15). The uniqueness class can be weakened to the uniqueness class of Maekawa and Terasawa [38] by the embedding \( E^r_q \subset L^r_{uloc} \).

3. Also note that, when \( q = \infty \) or \( r = \infty \), we do not have \( \|e^{t\Delta}u_0 - u_0\|_{E^q} \to 0 \) as \( t \to 0^+ \). See Lemma 2.3. For \( q = \infty \), we have by [38, Theorem 1.2] that, for any ball \( B \),
\[
\lim_{t \to 0^+} \|u - u_0\|_{L^r(B)} = 0.
\]

4. When \( q \leq 3 < r \), we expect global solutions for small data in \( E^r_q \) in
\[
u \in C([0, \infty); E^r_q).
\]

It should be the same solution considered in \( u \in C([0, \infty); E^3_q) \) in Theorem 1.3. The estimate (1.16) does not give better spatial-temporal decay than Theorem 1.3. Since we expect regularity for \( t > 1 \), the main gain of (1.16) is extra integrability for \( t \sim 0^+ \), which may alternatively be obtained from local smoothing estimates in [27, 6, 34]. Hence we do not pursue it.

We now turn to the case of data \( u_0 \in E^3_q \), which we refer to as critical.

**Theorem 1.2** (Critical data I). Let \( q \in [1, \infty) \). Fix \( T > 0 \). There exists \( \varepsilon = \varepsilon(T) > 0 \) such that for all divergence-free \( u_0 \in E^3_q \) with \( \|u_0\|_{E^3_q} \leq \varepsilon \), there exists a mild solution \( u \) to (1.3) with
\[
u \in L^\infty(0, T; E^3_q) \quad \text{and} \quad t^{\frac{1}{r}} u \in L^\infty(0, T; E^\infty_q).
\]

The solution is unique in the class
\[
\sup_{0 < t < T} t^{\frac{1}{r}} \|u\|_{E^q_q} \leq 2 \sup_{0 < t < T} t^{\frac{1}{r}} \|e^{t\Delta}u_0\|_{E^q_q}.
\]

Furthermore, \( \|u\|_{L^\infty_q E^3_q} + \|t^{1/2}u\|_{L^\infty_q E^\infty_q} \lesssim \|u_0\|_{E^3_q} \). We have \( u \in C([0, T); E^q_q) \) for \( q = \infty \) and \( u \in C([0, T); E^3_q) \) for \( q < \infty \). If \( q = \infty \), then we have for any ball \( B \) and \( \delta \in (0, 2) \) that
\[
\lim_{t \to 0^+} \|u(t) - u_0\|_{L^{3-\delta}(B)} = 0.
\]

For any \( s \in [3, \infty) \) and \( p \in [3, 9] \) given by \( \frac{2}{s} + \frac{3}{p} = 1 \), by taking \( \varepsilon \leq \varepsilon_0(T, s) \) sufficiently small, this solution further satisfies
\[
\|u\|_{E^{s,p}_{T,m \geq q}} + \1_{q \leq s} \|u\|_{L^p_q E^p_{m \geq q}} \leq C \|u_0\|_{E^3_q}, \quad \forall m \in [q, \infty].
\]
Comments on Theorem 1.2:

1. When $1 \leq q \leq 3$, this theorem and Theorem 1.3 extend Kato [30], see also Giga [21]. Note that, if $1 \leq q < 3$, then $E^3_q \subset (L^3 \cap L^q)$, considered in [30, Theorems 3&4]. The only spacetime integral estimate in [30, 21] is $L^s L^p$.

When $q = \infty$, this theorem extends Maekawa-Terasawa [38, Theorem 1.1 (iii)]. The spacetime integral bound in $E^{s,p}_{T,m=\infty}$ for $s < \infty$ is not considered in [38].

2. The uniqueness part of Theorem 1.2, unlike Theorem 1.1, assumes smallness given by the condition (1.17). Such a smallness condition for uniqueness is also implicitly assumed in [38, Theorem 1.1 (iii)] when $q = \infty$.

3. We refer to the convergence in (1.18) as convergence in $L^{3-\delta}_{loc}$. For sub-critical data $u_0$ in $L^p_{uloc}$, $p > 3$, it is shown in [38] that $u \rightarrow u_0$ in $L^p_{loc}$. When $p = 3$, it is shown in [38] that $u \rightarrow u_0$ in $L^3_{loc}$ provided $u_0$ is in the $L^3_{uloc}$ closure of bounded uniformly continuous functions.

This complements our result (1.18) which does not require $u_0$ to be in this closure.

Theorem 1.3 (Critical data II). Let $1 \leq q \leq 3$. For all divergence-free $u_0 \in E^3_q$, there exist $T = T(u_0) > 0$ and a unique mild solution $u$ to (1.3) satisfying

$$u \in BC([0,T); E^3_q) \quad \text{and} \quad \frac{1}{2}u \in L^\infty(0,T; E^\infty_{q_2}),$$

with $1/q_2 = 1/q - 1/3$, $q_2 \in [\frac{3}{2}, \infty]$. For any $s \in [3, \infty)$, $\frac{2}{s} + \frac{3}{q} = 1$, and $m \in [q, \infty]$, there is $T_1 \in (0,T]$ such that

$$u \in E^{s,p}_{T_1,m}. \quad (1.20)$$

Furthermore, there is $\varepsilon(q) > 0$ such that $T = \infty$ if $\|u_0\|_{E^3_q} \leq \varepsilon(q)$. If we assume further

$$m > p' = \frac{p}{p-1}, \quad \text{and} \quad m \geq m_1, \quad \frac{2}{s} + \frac{3}{m_1} = \frac{3}{q}, \quad (1.21)$$

with $m > m_1(s,q)$ when $q = 1$, then there exists $\varepsilon_1(s,q,m) > 0$ such that $T_1 = \infty$ if $\|u_0\|_{E^3_q} \leq \varepsilon_1(s,q,m)$. Instead of (1.21), if we assume

$$m \geq \max(p', m_1), \quad \text{and} \quad \begin{cases} m > m_1 & \text{if} \quad q = 1, \\ m \geq p & \text{if} \quad 3s < 5q, \end{cases} \quad (1.22)$$

then there exists $\varepsilon_2(s,q,m) > 0$ such that $u \in L^{s}_{T=\infty} E^p_{m}$ if $\|u_0\|_{E^3_q} \leq \varepsilon_2(s,q,m)$.

Comments on Theorem 1.3:

1. Theorem 1.3 is limited to $q \in [1,3]$ and improves Theorem 1.2 in two aspects: First, it does not require smallness of the norm $\|u_0\|_{E^3_q}$ for local existence. Second, it gives global existence for small data. For global existence and for either $\frac{2}{s} + \frac{3}{m} < \frac{3}{q}$ or $\frac{2}{s} + \frac{3}{m} = \frac{3}{q}$, we need $q < m$. Unlike Theorem 1.2, we require $s < \infty$ in (1.20).

2. The inclusion $\frac{1}{2}u \in L^\infty(0,T; E^\infty_{q_2})$ is worse than $\frac{1}{2}u \in L^\infty(0,T; E^\infty_{q_2})$ in Theorem 1.2 since $q < q_2$. This relaxed choice of space, possible only when $q \leq 3$, allows us to remove the smallness assumption on the initial data for local existence.
3. For global space-time integral estimates, the exponent $m$ is always no less than $\max(p', m_1) \in (q, p]$. In particular, $q < m$. The rectangle $[\frac{1}{s}, \frac{1}{q}] \in (0, \frac{1}{3}] \times [\frac{1}{3}, 1]$ is divided into 3 regions as shown in Figure 1:

They are bordered by two line segments:

$$L_1 : \frac{1}{q} = \frac{2}{3} + \frac{4}{3s}, \quad L_2 : \frac{1}{q} = \frac{5}{3s}.$$ 

For $E_{T=\infty}^{s,p}$ estimates, the lower bound of $m$ is $p'$ in region I, and $m_1$ in both regions II and III. Equality is disallowed on region I, line $L_1$ and the line $L_3 : \frac{1}{q} \leq \frac{1}{3} \leq \frac{1}{3}, q = 1$.

For $L^s E_m^{p}$ estimates, the lower bound of $m$ is $p'$ in region I, $m_1$ in region II, and $p$ in region III. There is a jump across the line $L_2$. Equality is disallowed on line $L_3$ only.

4. The lower bound of $m$ for global $L^s E_m^{p}$ estimates has a jump across the line $L_2$. It is because our linear $L^s E_m^{p}$ estimates in Lemma 2.4 are based on $E_{T=\infty}^{s,p}$ estimates and not optimal. We may hope to decrease the lower bound of $m$ in region III if we could prove Marcinkiewicz interpolation theorem for subadditive maps on the Wiener amalgam spaces $E_q^p$. See Remark 2.5 (iii).

5. For the classical case $q = 3$, small $u_0 \in E_3^3 = L^3(\mathbb{R}^3)$, and $T_1 = \infty$, we have $m_1(s, 3) = p$ for $\frac{2}{s} + \frac{3}{p} = 1$ and Theorem 1.3 gives $u \in E_{s,m}^{\infty} \cap L_s^s E_m^p$ if $3 < p \leq 9$ and $m \geq p$, recovering the classical result of [30, 21].

6. As soon as $q < 3$, we can choose $m = \max(p', m_1) < p$ for the $E_{\infty,m}^{s,p}$ estimate. To choose $m < p$ for the $L_s^s E_{m}^{p}$ estimate, we also need $\frac{1}{q} \geq \frac{5}{3s}$.

We will prove estimates for the solutions of the Stokes equations (1.1) in Section 2, and construct mild solutions of Navier-Stokes equations (1.3) in Section 3, proving Theorems 1.1–1.3.

### 1.3 Weak solutions in amalgam spaces

The existence of global local energy weak solutions in Wiener amalgam spaces $E_q^p$, $2 \leq q < \infty$, in $\mathbb{R}^3$ was considered by the first and last author in [10], as a way to bridge (or interpolate) the
classical theories in $L^2$ and $L^2_{uloc}$. It turns out that the theory can be extended to $E^2_q$ for $1 \leq q < 2$ (extrapolation). We will study such local energy solutions, in the sense of [10, Definition 1.1], in Section 4.1.

Smaller $q$ means that the solutions are more spatially localized. An advantage is that we have a sequence of a priori bounds whose time spans go to infinity. Such property is shown in [10] for $2 \leq q < 6$. As a result, when we construct global solutions, there is no need to consider perturbed Navier-Stokes equations as in [10]. On the other hand, as Young’s convolution inequality for sequence

$$\|a * b\|_{\ell^r} \leq C \|a\|_{\ell^1} \cdot \|b\|_{\ell^r}$$

is valid only for $r \geq 1$, we need to adjust many estimates of the pressure from far field.

As in [10], we will first show eventual regularity for local energy weak solutions with $u_0 \in E^2_q$, $1 \leq q < 2$, in Theorem 4.2.

Moreover, if such a solution has finite $LE_{q}(0,T)$-norm

$$\|u\|_{LE_{q}(0,T)} := \|u\|_{E^{\infty,2}_{q,q}} + \|\nabla u\|_{E^{2,2}_{q,q}},$$

which we refer to as $\ell^q$ local energy, then it satisfied the a priori bounds in Lemma 4.3 for all scales up to time $T$.

Finally, we will prove the following existence theorem.

**Theorem 1.4** (Existence in $E^2_q$). Assume $u_0 \in E^2_q$ where $1 \leq q < 2$ and is divergence free. Then, there exists a time-global local energy solution $u$ and associated pressure $\pi$ to the Navier-Stokes equations (1.3) in $\mathbb{R}^3$ with initial data $u_0$ so that, for any $0 < T < \infty$,

$$\|u\|_{LE_{q}(0,T)} < \infty. \quad (1.24)$$

In particular, $u \in L^\infty(0,T; E^2_q)$ and satisfies the a priori bounds in Lemma 4.3.

**Comments on Theorem 1.4:**

1. Theorem 1.4 extends the range of $q \in [2, \infty)$ in [10, Theorem 1.5] to $q \in (1, 2)$. It can also be viewed as a $\ell^q$-version of [9, Theorem 1.5] except the decay condition [9, (1.12)] on initial data is not needed.

2. The main point of Theorem 1.4 is the finiteness of estimate (1.24). Since $u_0 \in E^2_q \subset L^2$, Leray’s original global weak solution is already a local energy solution, but it may not satisfy (1.24).

In Section 4.2, we discuss the local existence of a local energy solution $u$ of the perturbed Navier-Stokes equations (4.13) when the perturbation $v$ is small in $L^\infty(0,T; L^p_{uloc}); \; p = 3$. Such a solution $u$ is needed in the construction of time-global local energy solutions of (1.3) with initial data in $E^2_q$, $2 \leq q < \infty$ in [10]. The choice of $p = 3$ is natural, but the bound $v \in L^\infty(0,T; L^3_{uloc})$ was insufficient in [10], due to lack of compactness. It turns out the spacetime integral bound (1.19) in Theorem 1.2 for small mild solution $v \in L^\infty L^3_{uloc}$ can be used to show the local energy inequality of $u$. This was actually a motivation for the present paper, and will be shown in Proposition 4.8.

**Organization:** Section 2 contains estimates for the solutions of the Stokes equation while Section 3 contains the proofs of our main results. Section 4 considers weak solutions. In the appendix Section 5, we give details of the end point case of Giga’s estimate (1.11), and gives examples showing the strict inclusions of various functional spaces.
2 Linear estimates in Wiener amalgam spaces

In this section we consider linear equations in $\mathbb{R}^d$ for general space dimension $d \in \mathbb{N}$. We first prove decay estimates of $e^{t\Delta}$, its gradient, and $\mathbb{P} e^{t\Delta} \nabla \cdot$ and limits of $e^{t\Delta}$ in amalgam spaces in Lemmas 2.1–2.3. We then show spacetime integral estimates for $e^{t\Delta}$ and the Duhamel term in Lemmas 2.4 and 2.7–2.8 ($d = 3$), and provide examples in Example 2.6. At the end of this section, we derive $\ell^n$ local energy estimates of $e^{t\Delta}$ and $\mathbb{P} e^{t\Delta} \nabla \cdot$ for $d = 3$ in Lemma 2.9.

Let $S_{ij}(x, t)$ denote the Oseen tensor in $\mathbb{R}^d$, $d \geq 2$, the fundamental solution of the Stokes system (1.1) in $\mathbb{R}^d$, found by Oseen [39] for $d = 3$. We have the following pointwise estimate by Solonnikov [40],

$$| \partial_t^m \nabla_x^k S(x, t) | \leq \frac{C_{k,m}}{(|x| + \sqrt{t})^{d+k+2m}}. \quad (2.1)$$

2.1 Decay estimates and continuity of heat semigroup

Lemma 2.1. Let $d \in \mathbb{N}$, $1 \leq \tilde{p} \leq p \leq \infty$, $1 \leq \tilde{q} \leq q \leq \infty$. Then for any $f \in E^\tilde{p}_q(\mathbb{R}^d)$, we have for $h = 0, 1$

$$\left\| \nabla^h e^{t\Delta} f \right\|_{E^\tilde{p}_q} \lesssim \left( \frac{1}{t^{d/2}} \left( \frac{1}{p} - \frac{1}{q} \right) + \frac{1}{t} \right) \left( \frac{1}{t^{d/2}} \left( \frac{1}{q} - \frac{1}{\tilde{q}} \right) + \frac{1}{t} \right) \left\| f \right\|_{E^\tilde{p}_q}. \quad (2.2)$$

For $d \geq 2$ and $F \in (E^\tilde{p}_q)^{d \times d}$, we have

$$\left\| \int_{\mathbb{R}^d} \partial_x S_{ij}(x-y, t) F_{ij}(y) \, dy \right\|_{E^\tilde{p}_q} \lesssim \left( \frac{1}{t^{d/2}} \left( \frac{1}{p} - \frac{1}{q} \right) + \frac{1}{t} \right) \left\| F \right\|_{E^\tilde{p}_q}. \quad (2.3)$$

When $\frac{1}{p} - \frac{1}{\tilde{p}} \leq \frac{1}{q} - \frac{1}{\tilde{q}}$, (2.2) and (2.3) reduce to

$$\left\| e^{t\Delta} f \right\|_{E^\tilde{p}_q} \lesssim \frac{1}{t^{d/2}} \left( \frac{1}{p} - \frac{1}{q} \right) \left\| f \right\|_{E^\tilde{p}_q}, \quad (2.4)$$

and

$$\left\| \int_{\mathbb{R}^d} \partial_x S_{ij}(x-y, t) F_{ij}(y) \, dy \right\|_{E^\tilde{p}_q} \lesssim \frac{1}{t^{d/2}} \left( \frac{1}{p} - \frac{1}{q} \right) \left\| F \right\|_{E^\tilde{p}_q}. \quad (2.5)$$

The above lemma recovers the known cases of $\tilde{L}^p-L^p$ estimates [15] when $q = p$ and $\tilde{q} = \tilde{p}$ as well as $L^p_{uloc}-L^p_{uloc}$ estimates when $q = \tilde{q} = \infty$ ((2.2) by [4] and (2.2)-(2.3) by [38]). Estimates (2.4) and (2.5) are similar to the usual $L^p-L^p$ estimates, and are convenient for our applications in Section 3. Note that $\tilde{q} \leq q$ since we cannot improve decay, and $\tilde{p} \leq p$ means we can improve regularity. We can allow $\tilde{p} > p$ by the embedding $E^\tilde{p}_q \subset E^p_{\tilde{q}}$. In that case we replace $\left( \frac{1}{p} - \frac{1}{\tilde{p}} \right)$ by $\left( \frac{1}{p} - \frac{1}{\tilde{p}} \right)$. We will give two proofs of Lemma 2.1. We do not use the case $h = 1$ in (2.2) or later in (2.8), but include it because its proofs are carried out simultaneously to those for $h = 0$.

First proof of Lemma 2.1. For $x \in B_1(k)$, $k \in \mathbb{Z}^d$, decompose $\nabla^h e^{t\Delta} f = f_k^1 + f_k^2$ where $f_k^1 = \nabla^h e^{t\Delta}(f \chi_{B_4(k)})$ and $f_k^2 = \nabla^h e^{t\Delta}(f(1 - \chi_{B_4(k)}))$. By Minkowski’s inequality

$$\left\| \nabla^h e^{t\Delta} f \right\|_{E^\tilde{p}_q} \lesssim \left\| f_k^1 \right\|_{L^p(B_1(k))} \left\| f \right\|_{L^q(B_1(k))} + \left\| f_k^2 \right\|_{L^p(B_1(k))} \left\| f \right\|_{L^q(B_1(k))}.$$

We have by standard $L^\tilde{p}-L^p$ estimates that

$$\left\| f_k^1 \right\|_{L^p(B_1(k))} \left\| f \right\|_{L^q(B_1(k))} \lesssim t^{-\frac{d}{2} - \frac{d}{q} - \frac{d}{\tilde{p}} - \frac{d}{\tilde{q}}} \left\| f \right\|_{L^q(B_4(k))} \lesssim t^{-\frac{d}{2} - \frac{d}{q} - \frac{d}{\tilde{p}} - \frac{d}{\tilde{q}}} \left\| f \right\|_{E^\tilde{p}_q} \lesssim t^{-\frac{d}{2} - \frac{d}{p} - \frac{d}{q} - \frac{d}{\tilde{p}} - \frac{d}{\tilde{q}}} \left\| f \right\|_{E^\tilde{p}_q},$$
using the embedding $\ell^q \subset \ell^q$. On the other hand, for $x \in B_1(k)$ we have

$$|f^k_2(x, t)| \lesssim \sum_{|k'| \geq 1} \frac{1}{t^\frac{d}{2} + h} |k'|^h e^{-|k'|^2/(4t)} |B_1| \frac{1}{\lambda^q} \|f\|_{L^p(B_1(k' - k))},$$

and, consequently,

$$\|f^k_2\|_{L^p(B_1(k))}(t) \lesssim |B_1| \frac{1}{\lambda^q} \sum_{|k'| \geq 1} \frac{1}{t^\frac{d}{2} + h} |k'|^h e^{-|k'|^2/(4t)} \|f\|_{L^p(B_1(k' - k))}.$$ 

For small $t \leq 1$, applying the $\ell^q$ norm and using Young’s inequality on the discrete convolution leads to

$$\||f^k_2\|_{L^p(B_1(k))}\|_{\ell^q}(t) \lesssim |B_1| \frac{1}{\lambda^q} \|t^{-\frac{d}{2} - h} |k'|^h e^{-|k'|^2/(4t)} \chi_{k' \neq 0}\|_{\ell^1} \|f\|_{E^q_0} \lesssim t^{-\frac{d}{q} + \frac{1}{p}} \|f\|_{E^q_0},$$

where we’ve used the embedding $\ell^q \subset \ell^q$. For large $t > 1$, we get

$$\||f^k_2\|_{L^p(B_1(k))}\|_{\ell^q}(t) \lesssim |B_1| \frac{1}{\lambda^q} \|t^{-\frac{d}{2} - h} |k'|^h e^{-|k'|^2/(4t)} \chi_{k' \neq 0}\|_{\ell^1} \|f\|_{E^q_0},$$

$$\lesssim t^{-\frac{d}{q} + \frac{q}{p} \left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_{E^q_0}.$$

This proves (2.2).

The proof of (2.3) is logically similar except we set $F^k_1 = \int_{\mathbb{R}^d} \partial_t S_{ij}(x - y, t)(F_{ij} \chi_{B_1(k)})((y) dy$ and $F^k_2 = \int_{\mathbb{R}^d} \partial_t S_{ij}(x - y, t)(F_{ij}(1 - \chi_{B_1(k)}))(y) dy$. Using Young’s convolution inequality,

$$\|F^k_1\|_{L^p(B_1(k))} \lesssim \|F^k_1\|_{L^p(\mathbb{R}^d)} \lesssim \|\partial_t S_{ij}\|_{L^r(\mathbb{R}^d)} \|F \chi_{B_1(k)}\|_{L^p},$$

Then, by Oseen tensor estimate (2.1), we have

$$\|\partial_t S_{ij}\|_{L^r(\mathbb{R}^d)} \lesssim t^{-\frac{d}{2} + \frac{q}{p} \left(\frac{1}{p} - \frac{1}{q}\right)}.$$

So

$$\|F^k_1\|_{L^p(B_1(k))} \lesssim t^{-\frac{d}{2} + \frac{q}{p} \left(\frac{1}{p} - \frac{1}{q}\right)} \|F\|_{L^p(B_1(k))}.$$ 

Taking $\ell^q$-norm over $k \in \mathbb{Z}^d$ on both sides and using the embedding $\ell^q \subset \ell^q$ for $q \leq q$, we derive

$$\||F^k_1\|_{L^p(B_1(k))}\|_{\ell^q}(t) \lesssim t^{-\frac{d}{2} + \frac{q}{p} \left(\frac{1}{p} - \frac{1}{q}\right)} \|F\|_{E^q_0} \lesssim t^{-\frac{d}{2} + \frac{q}{p} \left(\frac{1}{p} - \frac{1}{q}\right)} \|F\|_{E^q_0}.$$

For $F^k_2$, we carry out the same estimates as for $f^k_2$ but replace $t^{-d/2} e^{-|k'|^2/(4t)} \chi_{k' \neq 0}$ by $(|k'| + \sqrt{t})^{-(d+1)} \chi_{k' \neq 0}$. For small $t$, since

$$\|(|k'| + \sqrt{t})^{-(d+1)} \chi_{k' \neq 0}\|_{\ell^1} \lesssim t^{-1/2},$$

we obtain

$$\||F^k_2\|_{L^p(B_1(k))}\|_{\ell^q}(t) \lesssim t^{-\frac{d}{2} \left(\frac{1}{p} - \frac{1}{q}\right)} \|F\|_{E^q_0} \lesssim t^{-\frac{d}{2} \left(\frac{1}{p} - \frac{1}{q}\right)} \|F\|_{E^q_0}.$$
For large $t$, we instead use
\[
\|(|k'| + \sqrt{t})^{-(d+1)} \chi_{k' \neq 0} \|_{r'} \leq t^{\frac{d}{2p} - \frac{q}{2} - \frac{1}{2}}, \quad \frac{1}{q} + 1 = \frac{1}{r'} + \frac{1}{q},
\]
and obtain
\[
\| | F^h_2 \|_{L^p(B_1(k))} \|_{E_q}(t) \leq t^{-\frac{1}{2} - \frac{d}{2} \left( \frac{1}{q} - \frac{1}{r'} \right)},
\]
Combining these estimates for $F^h_1$ and $F^h_2$ leads to the desired bound (2.3).

It is worth noting that Lemma 2.1 can also be proved by using a Young-type convolution inequality for amalgam spaces from Busby and Smith [12]. We include the details to paint a complete picture of the available tools in the amalgam space.

Lemma 2.2. Let $d \in \mathbb{N}$ and $p, q \in [1, \infty]$. For the heat kernel $\Gamma(x, t) = (4\pi t)^{-d/2} e^{-x^2/4t}$ in $\mathbb{R}^d$, we have for $h = 0, 1$
\[
\| \nabla^h \Gamma(\cdot, t) \|_{E_q} \leq C t^{-\frac{d}{2}} \left( t^{-\frac{d}{2p} + \frac{d}{2q}} + 1_{t > 1} t^{-\frac{d}{2} + \frac{d}{2q}} \right), \quad (0 < t < \infty). \tag{2.8}
\]
For $\Phi(x, t) = (|x| + \sqrt{t})^{-d-1}$ defined for $(x, t) \in \mathbb{R}^d \times \mathbb{R}_+$, we have
\[
\| \Phi(\cdot, t) \|_{E_q} \leq C t^{-\frac{d}{2}} \left( t^{-\frac{d}{2p} + \frac{d}{2q}} + 1_{t > 1} t^{-\frac{d}{2} + \frac{d}{2q}} \right), \quad (0 < t < \infty). \tag{2.9}
\]

Proof. First, we have
\[
\| \nabla^h \Gamma(\cdot, t) \|_{E_q} \lesssim \sum_{k \in \mathbb{Z}^d} \left( \int_{B_1(k)} \left| t^{\frac{d}{2} - h} |x|^{h} e^{-\frac{|x|^2}{2t}} \right|^p \, dx \right)^{1/p}.
\]
Note that if $|k| \geq 3$ and $x \in B_1(k)$, then $|x| \in [\frac{2}{3} |k|, \frac{4}{3} |k|]$. Hence,
\[
\| \nabla^h \Gamma(\cdot, t) \|_{E_q} \lesssim \left( \int_{B_1(0)} |\nabla^h \Gamma(x, t)|^p \, dx \right)^{1/p} + \sum_{|k| \geq 3} \left( \int_{B_1(k)} \left| t^{\frac{d}{2} - h} |k|^{h} e^{-\frac{|k|^2}{20t}} \right|^p \, dx \right)^{1/p}
\]
\[
\lesssim \left( \int_{\mathbb{R}^d} |\nabla^h \Gamma(x, t)|^p \, dx \right)^{1/p} + \sum_{|k| \geq 3} t^{-\frac{d}{2} - h} |k|^{h} e^{-\frac{|k|^2}{20t}}
\]
\[
\lesssim t^{-\frac{d}{2} - \frac{d}{2p} + \frac{d}{2q}} + t^{-\frac{d}{2} + \frac{d}{2h}} \int_{\mathbb{R}^d} |y|^{h} e^{-\frac{|y|^2}{20t}} \, dy \lesssim t^{-\frac{d}{2} - \frac{d}{2p} + \frac{d}{2q}} + t^{-\frac{d}{2}}.
\]
The embedding $\ell^1 \subset \ell^q$ implies
\[
\| \nabla^h \Gamma(\cdot, t) \|_{E_q} \leq C \left( t^{-\frac{d}{2} - \frac{d}{2p} + \frac{d}{2q}} + t^{-\frac{d}{2}} \right).
\]
On the other hand,

\[
\left\| \nabla^h \Gamma (\cdot, t) \right\|_{E^p_{\Gamma}} \lesssim \left[ \left( \int_{B_4(0)} |\nabla^h \Gamma(x, t)|^p \, dx \right)^{q/p} + \sum_{|k| \geq 3} \left( \int_{B_1(k)} \left| t^{-\frac{d}{2} - h} |k|^h e^{-\frac{|k|^2}{20t^q}} \right|^p \, dx \right)^{q/p} \right]^{1/q} \\
\lesssim \left[ \left( \int_{\mathbb{R}^d} |\nabla^h \Gamma(x, t)|^p \, dx \right)^{q/p} + \sum_{|k| \geq 3} t^{-\frac{d}{2} + h + \frac{d}{2q} + \frac{d}{2}} \int_{\mathbb{R}^d} |y| |h|^q e^{-\frac{|y|^2}{20t^q}} \, dy \right]^{1/q} \\
\lesssim \left[ t^{-\frac{d}{2} + h + \frac{d}{2q} + \frac{d}{2}} + \int_{\mathbb{R}^d} \Phi(x, t) \, dx \right]^{1/q} \sim t^{-\frac{d}{2} + h + \frac{d}{2q} + \frac{d}{2}}.
\]

This proves (2.8). Similarly,

\[
\left\| \Phi(\cdot, t) \right\|_{E^p_\Gamma} \lesssim \left( \int_{B_4(0)} \Phi(x, t)^p \, dx \right)^{1/p} + \sum_{|k| \geq 3} \left( \int_{B_1(k)} \Phi(k, t)^p \, dx \right)^{1/p} \\
\lesssim \left( \int_{\mathbb{R}^d} \Phi(x, t)^p \, dx \right)^{1/p} + \sum_{|k| \geq 3} \Phi(k, t) \\
\lesssim t^{-\frac{d}{2} + \frac{d}{2p} - \frac{1}{2}} + \int_{\mathbb{R}^d} \Phi(x, t) \, dx \lesssim t^{-\frac{d}{2} + \frac{d}{2p} - \frac{1}{2}} + t^{-\frac{1}{2}}.
\]

Also,

\[
\left\| \Phi(\cdot, t) \right\|_{E^q_\Gamma} \lesssim \left[ \left( \int_{B_4(0)} \Phi(x, t)^p \, dx \right)^{q/p} + \sum_{|k| \geq 3} \left( \int_{B_1(k)} \Phi(x, t)^p \, dx \right)^{q/p} \right]^{1/q} \\
\lesssim \left[ \left( \int_{\mathbb{R}^d} \Phi(x, t)^p \, dx \right)^{q/p} + \sum_{|k| \geq 3} \Phi(k, t)^q \right]^{1/q} \\
\lesssim \left[ t^{-\frac{d}{2} + \frac{d}{2p} - \frac{1}{2}} + \int_{\mathbb{R}^d} \Phi(x, t)^q \, dx \right]^{1/q} \lesssim t^{-\frac{d}{2} + \frac{d}{2p} - \frac{1}{2}} + t^{-\frac{d}{2} + \frac{d}{2q} - \frac{1}{2}}.
\]

This proves (2.9) and completes the proof of Lemma 2.2.

\[\square\]

Remark. Estimate (2.8) for \( h = 1 \) also follows from (2.9) since \(|\nabla \Gamma| \lesssim \Phi|.

Second proof of Lemma 2.1. Define \( p_0 \) and \( q_0 \) by

\[
\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p} - 1 \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_0} + \frac{1}{q} - 1,
\]

which implies \( \frac{1}{q_0} + \frac{1}{q} \geq 1 \). We have

\[
\left\| \nabla^h e^{tA} f \right\|_{E^p_{\Gamma}} \lesssim \left\| \nabla^h \Gamma \right\|_{E^{p_0}_{\Gamma}} \| f \|_{E^p_\Gamma} \lesssim \frac{1}{t^{\frac{d}{2}} \left( \frac{1}{p_0} \left( \frac{1}{p} \left( \frac{1}{p_0} \frac{1}{q_0} + \frac{1}{q} > 1 \right) \right) \right)} \| f \|_{E^p_\Gamma},
\]

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where we first use Busby and Smith’s inequality (2.7), and then (2.8) of Lemma 2.2. This proves (2.2). To prove (2.3), we replace \( \|\Gamma\|_{E_{q_0}^p} \) above by \( \|\nabla S\|_{E_{q_0}^p} \), use the upper bound on first spatial derivatives of the Oseen tensor (2.1), and use (2.9) of Lemma 2.2.

When \( p, q < \infty \), the spaces \( E_q^p \) preserve the good convergence properties of \( e^{t\Delta}u_0 \) to \( u_0 \) as \( t \to 0 \), as the next lemma shows. If \( p = \infty \) or \( q = \infty \) the lemma fails. For \( q = \infty \) this failure is detailed in [38].

**Lemma 2.3** (Continuity and vanishing of \( e^{t\Delta}f \) in \( E_q^p \)). Assume \( f \in E_q^p = E_q^p(\mathbb{R}^d) \), \( d \in \mathbb{N} \), \( 1 \leq p < \infty \) and \( 1 \leq q < \infty \). Then

\[
\lim_{|\tau| \to 0} \| f(\cdot + \tau) - f(\cdot) \|_{E_q^p} = 0, \tag{2.10}
\]

and

\[
\lim_{t \to 0^+} \| e^{t\Delta} f - f \|_{E_q^p} = 0. \tag{2.11}
\]

Moreover, for \( t > 0 \) and \( p, q \in [1, \infty] \), (including \( q = \infty \) or \( p = \infty \)), we have

\[
\lim_{h \to 0} \| e^{(t+h)\Delta} f - e^{t\Delta} f \|_{E_q^p} = 0, \tag{2.12}
\]

with no restriction on the sign of \( h \).

Moreover, if \( 1 \leq \tilde{p} < p \leq \infty \) (with strict inequality) and \( 1 \leq \tilde{q} \leq q \leq \infty \), then for any \( f \in E_{\tilde{q}}^{\tilde{p}}(\mathbb{R}^d) \), and \( f \in E_{\tilde{q}}^{\tilde{p}} \) if \( \tilde{q} = q = \infty \), we have

\[
\lim_{t \to 0^+} t^{\frac{d}{\tilde{p}}(\frac{1}{\tilde{p}} - \frac{1}{p})} \| e^{t\Delta} f \|_{E_{\tilde{q}}^{\tilde{p}}} = 0. \tag{2.13}
\]

When \( \tilde{q} = q = \infty \), (2.13) is false if we only assume \( f \in E_{\infty}^{\tilde{p}} = L_{uloc}^{\tilde{p}} \); see Example 2.6. However, it is true for \( f \in E_{\tilde{p}}^{\tilde{p}} \), i.e., the closure of \( C_c^\infty \) in \( L_{uloc}^{\tilde{p}} \), as the following proof using a density argument still works.

**Proof.** For (2.10), note that for \( \tau \in \mathbb{R}^d \) we have the uniform bound

\[
\| f(\cdot + \tau) - f(\cdot) \|_{E_q^p} \leq \| f(\cdot + \tau) \|_{E_q^p} + \| f \|_{E_q^p} \leq \| f \|_{E_q^p} < \infty.
\]

For \( |\tau| < 1 \), we can make \( \| f(\cdot + \tau) - f(\cdot) \|_{L_p(B_1(\cdot))} \) arbitrarily small by taking \( m \) sufficiently large. Once \( m \) is fixed, \( \| f(\cdot + \tau) - f(\cdot) \|_{L_p(B_1(\cdot))} \) \( |\tau| \to 0 \) as \( |\tau| \to 0 \) by properties of \( L^p \)-spaces. These show (2.10).

For (2.11), note that

\[
e^{t\Delta} f - f)(x) = \int_{\mathbb{R}^d} e^{-|z|^2/4} g(x, z, t) \, dz, \quad g(x, z, t) = f(x - \sqrt{t}z) - f(x).
\]

By Minkowski’s integral inequality in \( x \in B_1(k) \),

\[
\| e^{t\Delta} f - f \|_{L_p^p(B_1(k))} \leq \int_{\mathbb{R}^d} e^{-|z|^2/4} \| g(\cdot, z, t) \|_{L_p^p(B_1(k))} \, dz,
\]
By Minkowski’s integral inequality again in $k \in \mathbb{Z}^d$,
\[
\|e^{t \Delta} f - f\|_{E_q^p} = \left\|\left\|e^{t \Delta} f - f\right\|_{L^p_q(B_1(k))}\right\|_{\ell^q}
\leq \int_{\mathbb{R}^d} e^{-|z|^2/4} \left\|g(\cdot, z, t)\right\|_{L^p_q(B_1(k))} \, dz
= \int_{\mathbb{R}^d} e^{-|z|^2/4} \|f(\cdot - \sqrt{t}z) - f(\cdot)\|_{E_q^p} \, dz.
\]

The last integral vanishes as $t \to 0_+$ by the dominated convergence theorem and (2.10). This shows (2.11).

We now show (2.12), continuity at $t > 0$. For any $p, q \in [1, \infty]$, by the Busby-Smith convolution inequality (2.7), we have
\[
\left\|e^{(t+h) \Delta} f - e^{t \Delta} f\right\|_{E_q^p} \lesssim \|\Gamma_{t+h} - \Gamma_t\|_{L^1} \|f\|_{E_q^p} \to 0,
\]
as $h \to 0$ from either the left or right, by the dominated convergence theorem with $|\Gamma_{t+h} - \Gamma_t| \leq Ct^{-d/2}e^{-|x|^2/(6t)}$ when $|h| < t/2$.

We now show (2.13). Since we will send $t \to 0_+$, we assume $t \leq 1$. Denote $\sigma = \frac{d}{2} \left( \frac{1}{p} - \frac{1}{q} \right) > 0$. For any $\varepsilon > 0$, we can choose $b \in C^\infty_c$ with $\|f - b\|_{E_q^p} \leq \varepsilon/4C$ where $C$ is the constant in (2.2). Then by (2.2),
\[
t^\sigma \left\|e^{t \Delta}(f - b)\right\|_{E_q^p} \leq 2C \|f - b\|_{E_q^p} \leq \varepsilon/2.
\]
By (2.2) again,
\[
t^\sigma \left\|e^{t \Delta} b\right\|_{E_q^p} \leq \tilde{C} t^{\sigma} \|b\|_{E_q^p} \to 0 \quad \text{as} \quad t \to 0_+.
\]
These show (2.13). \(\square\)

In the following we give a direct proof of (2.11) which does not use (2.10).

**Second proof of (2.11).** Fix an arbitrarily small $\varepsilon > 0$. We may write
\[
e^{t \Delta} f = e^{t \Delta}(f \chi_{B_R}) + e^{t \Delta}(f(1 - \chi_{B_R})).
\]
For large enough $R$ we have by (2.2) that
\[
\left\|e^{t \Delta}(f(1 - \chi_{B_R})) - f(1 - \chi_{B_R})\right\|_{E_q^p} \lesssim \|f(1 - \chi_{B_R})\|_{E_q^p} < \varepsilon/2,
\]
by summability in $\ell^q$. We now show for sufficiently large $R$ and small $t$ that
\[
\|e^{t \Delta}(f \chi_{B_R}) - f \chi_{B_R}\|_{E_q^p} < \varepsilon/2.
\]
We treat this term in two steps using the decomposition
\[
e^{t \Delta}(f \chi_{B_R}) - f \chi_{B_R} = \chi_{B_2R}(e^{t \Delta}(f \chi_{B_R}) - f \chi_{B_R}) + (1 - \chi_{B_2R})e^{t \Delta}(f \chi_{B_R}). \quad (2.14)
\]
Focusing on the second term, we have for $|k| \geq 2R$ that
\[
\left\|e^{t \Delta}(f \chi_{B_R})\right\|_{L^p(B_1(k))} \lesssim \frac{1}{2^d/2} e^{-c|k|^2/\ell} R^{d-d/q} \|f\|_{E_q^p}.
\]
Hence, noting $\ell^1 \subset \ell^q$,
\[
\|(1 - \chi_{B2R})e^{t\Delta}(f\chi_{B_R})\|_{E^p_q} \lesssim \left\| t^{-d/2}e^{-c|k|^2/t}\| \chi_{\{ 1 \leq |k| \leq 2R \}} \|_{L^4} d^{-d/q} \right\| \| f \|_{E^p_q} \\
\lesssim \left\| t^{-d/2}e^{-c|k|^2/t}\| \chi_{\{ 1 \leq |k| \leq 2R \}} \|_{L^4} d^{-d/q} \right\| \| f \|_{E^p_q} \\
\to 0 \text{ as } R \to \infty,
\]
provided $t \leq 1$. Hence we choose $R$ large enough that the left hand side above is bounded by $\varepsilon/4$. Then, the first term of (2.14) is bounded by
\[
\|\chi_{B2R}(e^{t\Delta}(f\chi_{B_R}) - f\chi_{B_R})\|_{E^p_q} \lesssim R(\| e^{t\Delta}(f\chi_{B_R}) - f\chi_{B_R} \|_{L^p} \to 0 \text{ as } t \to 0).
\]
Since $\varepsilon > 0$ is arbitrarily small, the above shows (2.11).

\[\square\]

**2.2 Space time integral estimates**

We now develop spacetime integral bounds for the heat equation with data in $E^r_q$. Recall that the spacetime integrals $L^p_{B2}\, E^p_q$ and $E^{p,q}_{T,q}$ are defined in (1.7) and (1.8).

**Lemma 2.4.** Let $d \in \mathbb{N}$, $1 < r \leq s \leq \infty$, $r \leq p < \infty$, $\frac{2}{s} + \frac{4}{d} = \frac{4}{d}$, and $1 \leq q \leq m \leq \infty$. Suppose $f \in E^r_q(\mathbb{R}^d)$.

(a) For $0 < T < \infty$, we have
\[
\| e^{t\Delta}f \|_{L^p_{T,T} E^p_{0,T}} \lesssim (1 + T^\beta) \| f \|_{E^p_{0,q}},
\]
for any $\beta, \beta' \in [0, \infty)$ and $\beta > \alpha, \beta' > \alpha'$ with
\[
\alpha = \frac{d}{2m} - \frac{d}{2q} + \frac{1}{s}, \quad \beta' = \frac{d}{2s} - \frac{d}{2q} + \frac{1}{s}.
\]
We can take $\beta = \alpha$ if $\alpha \geq 0$ and $1 < q < m < \infty$, and we can take $\beta' = \alpha'$ if $\alpha' \geq 0$ and $1 < q < s$.

(b) For $T = \infty$, we have
\[
\| 1_A \| e^{t\Delta}f \|_{E^p_{\infty,m}} \lesssim \| f \|_{E^p_{q}},
\]
\[
(1\{m \leq s\} \cap A + 1E) \| e^{t\Delta}f \|_{L^p_{\infty,m} E^p_{m,m}} \lesssim \| f \|_{E^p_{q}},
\]
where $A$ is the subset of parameters that further satisfy
\[
A = \{ \alpha \leq 0; \ 1 < q < m < \infty \text{ if } \alpha = 0 \},
\]
and $E = E_1 \cup E_2$ is the subset of parameters of $\{ q \leq s < m \}$, unrelated to $A$,
\[
E_1 = \{ q \leq r \leq s \leq p \leq m, s \neq m, r \neq p \},
\]
\[
E_2 = \left\{ s < m \text{ and } \frac{ds}{d+2} \text{ with } q > 1 \text{ if } q = \frac{ds}{d+2} \right\}.
\]

**Remark 2.5.** (i) For finite $T < \infty$, we can allow $\frac{2}{s} + \frac{d}{p} > \frac{4}{d}$ for estimates (2.16)–(2.17). Since such estimates are not optimal and (2.17) follows directly from Lemma 2.1, they are avoided for simplicity.
(ii) This lemma extends Giga [21, Lemma, p. 196]. Indeed, if we take $1 < q = r < m = p < \infty$ (so that $\alpha = 0$) in (2.19), then we get

$$\|e^{t\Delta} f\|_{L^s_T \rightarrow L^p} \lesssim \|f\|_{L^r}, \quad 1 < r \leq s \leq \infty, \quad r \leq p < \infty, \quad \frac{2}{s} + \frac{d}{p} = \frac{d}{r},$$

which recovers (1.11)–(1.12).

(iii) Giga’s estimate (1.11) is proved in [21] using the Marcinkiewicz interpolation theorem (MIT) for subadditive maps on $L^p$ ([11, Appendix]). As our proof below uses (1.11), it uses MIT implicitly. It is natural to ask if we can prove Lemma (MIT) for subadditive maps on $E^s_p$ which recovers (2.19). Its complement set \{q \leq s < m\} \setminus E^s_p$ is valid on $E^s_p$. If yes, then we may try to prove (2.19) directly and possibly enlarge the set of admissible parameters.

(iv) Recall $q \leq m$. When $m \leq s$, we have estimates of $\|e^{t\Delta} f\|_{L^s_T \rightarrow E^m_p}$ in (2.17) and (2.19). When $s < q$, $\|e^{t\Delta} f\|_{L^s_T \rightarrow E^m_p}$ cannot be controlled by $\|f\|_{E^q_p}$, as to be shown by Example 2.6. For the remaining case $q \leq s < m$, we have finite time estimate in (2.17), but no infinite time estimate (2.19) unless the parameters belong to the set $E$. Its complement set \{q \leq s < m\} \setminus E$ is unclear to us.

Proof. We first prove the $E^{s,p}_{T,m}$ estimates. For every $k \in \mathbb{Z}^d$ and $x \in B_1(k)$, we decompose

$$e^{t\Delta} f(x, t) = e^{t\Delta} (f \chi_{B_4(k)}) + e^{t\Delta} (f (1 - \chi_{B_4(k)})) =: f^k_1(x, t) + f^k_2(x, t).$$

Then

$$\left\| e^{t\Delta} f \right\|_{E^{s,p}_{T,m}} = \left\| e^{t\Delta} f \right\|_{L^p(B_1(k))} \left\| e^{t\Delta} f \right\|_{L^s(0,T)} \left\| e^{t\Delta} f \right\|_{\ell^m(k \in \mathbb{Z}^d)}$$

$$\leq \left\| f^k_1(\cdot, t) \right\|_{L^p(B_1(k))} \left\| f^k_1(\cdot, t) \right\|_{L^s(0,T)} \left\| f^k_1(\cdot, t) \right\|_{\ell^m(k \in \mathbb{Z}^d)} + \left\| f^k_2(\cdot, t) \right\|_{L^p(B_1(k))} \left\| f^k_2(\cdot, t) \right\|_{L^s(0,T)} \left\| f^k_2(\cdot, t) \right\|_{\ell^m(k \in \mathbb{Z}^d)}$$

$$=: A_1 + A_2.$$ 

For $A_1$, we use Giga’s integral estimate (1.11) to get

$$\left\| f^k_1(\cdot, t) \right\|_{L^p(B_1(k))} \left\| f^k_1(\cdot, t) \right\|_{L^s(0,T)} \lesssim \left\| e^{t\Delta} f \chi_{B_4(k)} \right\|_{L^s(0,\infty; L^p(\mathbb{R}^d))} \lesssim \left\| f \right\|_{L^r(B_4(k))}.$$

Hence

$$A_1 \lesssim \left\| f \right\|_{L^r(B_4(k))} \left\| f \right\|_{\ell^m(k \in \mathbb{Z}^d)} \approx \left\| f \right\|_{E^m_q} \lesssim \left\| f \right\|_{E^q_p},$$

(2.20)

since $q \leq m$ so $\ell^q$ embeds continuously in $\ell^m$.

For $A_2$, as

$$|f^k_2(x, t)| \lesssim \sum_{k': |k' - k| > 4} \int_{B_1(k)} t^{-d/2} e^{-\frac{|x-y|^2}{4t}} |f(y)| dy,$$

we have

$$\left\| f^k_2(\cdot, t) \right\|_{L^p(B_1(k))} \lesssim t^{-d/2} \sum_{|k'| \geq 1} e^{-|k'|^2/(4t)} \left\| f \right\|_{L^r(B_1(k' - k))}.$$
Taking $L^p_T$ on both sides and applying Minkowski’s integral inequality, we get
\[
\left\| f_k^s \right\|_{L^p_T(L_p^r(B_1(k)))} \lesssim \left[ \int_0^T \left( t^{-d/2} \sum_{|k'| \geq 1} e^{-|k'|^2/(4t)} \| f \|_{L^r(T_k(k'-k))} \right)^s dt \right]^{1/s}
\]
\[
\lesssim \sum_{|k'| \geq 1} \left( \int_0^T | t^{-d/2} e^{-|k'|^2/(4t)} \| f \|_{L^r(T_k(k'-k))} |^s dt \right)^{1/s}
\]
\[
= \sum_{|k'| \geq 1} \| f \|_{L^r(T_k(k'-k))} \left( \int_0^T t^{-\frac{d}{s}} e^{-\frac{|k'|^2}{4t}} dt \right)^{1/s}, \quad t = |k'|^2 \tau
\]
\[
= \sum_{|k'| \geq 1} \| f \|_{L^r(T_k(k'-k))} |k'|^{-d+\frac{2}{s}} \left( \int_0^T \tau^{-\frac{d}{2}} e^{-\frac{\tau}{4\tau}} d\tau \right)^{1/s}.
\]

Since $\frac{d}{\tau} = \frac{2}{s} + \frac{d}{p} > \frac{2}{s}$, we have $ds > 2$ and
\[
\int_0^\infty \tau^{-\frac{d}{2}} e^{-\frac{\tau}{4\tau}} d\tau < \infty.
\]
On the other hand, if $S < 1$,
\[
\int_0^S \tau^{-\frac{d}{2}} e^{-\frac{\tau}{4\tau}} d\tau \lesssim \int_0^S e^{-\frac{\tau}{4\tau}} d\tau \lesssim e^{-\frac{\tau}{4\tau}}.
\]
This estimate also holds for $S > 1$. We conclude
\[
\left\| f_k^s \right\|_{L^p_T(L_p^r(B_1(k)))} \lesssim \sum_{|k'| \geq 1} \| f \|_{L^r(T_k(k'-k))} |k'|^{-d+\frac{2}{s}} e^{-\frac{|k'|^2}{8T}}.
\]
\[
(2.21)
\]

For fixed $0 < T < \infty$, we can bound for any $\beta \in [0, \infty)$
\[
|k'|^{-d+\frac{2}{s}} e^{-\frac{|k'|^2}{8T}} \lesssim |k'|^{-d+\frac{2}{s}} \left( \frac{|k'|^2}{8T} \right)^{-\beta} \lesssim |k'|^{-d+\frac{2}{s} - 2\beta T^\beta}.
\]

By the Young’s convolution inequality,
\[
A_2 \lesssim \left\| \sum_{|k'| \geq 1} \| f \|_{L^r(T_k(k'-k))} |k'|^{-d+\frac{2}{s} - 2\beta T^\beta} \right\|_{\ell^m(k \in \mathbb{Z}^d)} \lesssim \| f \|_{E_q^\gamma} \left\| |k|^{-d+\frac{2}{s} - 2\beta T^\beta} \right\|_{\ell^m(k \neq 0 \in \mathbb{Z}^d)},
\]

where $\frac{1}{m} + 1 = \frac{1}{q} + \frac{1}{n}$, and $n \geq 1$ thanks to $m \geq q$. For the last norm to be finite, we need
\[
d - \frac{2}{s} + 2\beta > \frac{d}{n} = \frac{d}{m} + d - \frac{d}{q},
\]
that is, $2\beta > \frac{d}{m} - \frac{d}{q} + \frac{2}{s}$. Thus if $\frac{d}{m} - \frac{d}{q} + \frac{2}{s} < 0$, we can take $\beta = 0$. If $\frac{d}{m} - \frac{d}{q} + \frac{2}{s} \geq 0$, we take any $\beta > \alpha$, with
\[
\alpha = \frac{d}{2m} - \frac{d}{2q} + \frac{1}{s}.
\]
In all cases, we obtain
\[
A_2 \lesssim \| f \|_{E_q^\gamma} T^\beta.
\]
\[
(2.22)
\]
If $\alpha \geq 0$ and $1 < q < m < \infty$, we can choose $\beta = \alpha$ and use the discrete version of the Hardy-Littlewood-Sobolev inequality in [43, Proposition (a)] to derive (2.22), noting that $d - \frac{2}{s} + 2\alpha = \frac{d}{m} + d - \frac{4}{q} \in (0, d)$.

Eq. (2.20) and (2.22) show the $E_{T,m}^{s,p}$ estimate in (2.16).

For $L_T^p E_m^p$ estimate in (2.17), note that it follows from the $E_{T,m}^{s,p}$ estimates when $m \leq s$ using (1.9) due to Minkowski’s integral inequality.

If $s < m$, then

$$
\|e^{t\Delta} f\|_{L_T^p E_m^p} \lesssim \|e^{t\Delta} f\|_{L_T^p E_s^s} = \|e^{t\Delta} f\|_{E_{T,s}^{s,p}} \lesssim (1 + T^{\beta'}) \|f\|_{E_q^q}
$$

(2.23)

by $E_{T,m}^{s,p}$ estimate in (2.16) (with $m$ replaced by $s$) for any $\beta' \in [0, \infty)$ and $\beta' > \alpha'$ with

$$
\alpha' = \frac{d}{2s} - \frac{d}{2q} + \frac{1}{s}.
$$

The condition to use (2.16) is $1 \leq q \leq s \leq \infty$. Hence we need to further assume $q \leq s$, and we have (2.23) if $q < s < m$. We can take $\beta' = \alpha'$ if $\alpha' \geq 0$ and $1 < q < s$.

Alternatively, if $p \leq m$ and $q \leq r$, (then $\alpha = \frac{d}{2m} - \frac{d}{2q} + \frac{1}{s} \leq \frac{d}{2p} - \frac{d}{2r} + \frac{1}{s} = 0$), we have

$$
\|e^{t\Delta} f\|_{L_T^p E_m^p} = \|e^{t\Delta} f\|_{E_m^p} \lesssim \|e^{t\Delta} f\|_{L_T^p} \lesssim \|f\|_{L_r^r} \quad \text{(since } p \leq m) \\
\lesssim \|f\|_{E_q^q} \quad \text{(since } q \leq r).
$$

In this case we can take $\beta' = 0$. This and (2.23) prove the $L_T^p E_m^p$ estimate in (2.17).

Part (b) is a consequence of part (a), when the constants do not depend on $T$ and we can send $T \to \infty$. The constant in (2.16) is independent of $T$ if $\alpha \leq 0$ (and $\alpha = 0$ requires $1 < q < m < \infty$). Hence we have (2.18). For (2.19), when $m \leq s$, we have the estimate if the parameters are in the set $A$. When $q \leq s < m$, we need one of the following 2 cases:

1. $q \leq r < p \leq m$ or 2. $q \leq s < m$ and $\alpha' \leq 0$ ($\alpha' = 0$ requires $1 < q < s$).

Note that $\alpha' \leq 0$ if and only if $q \leq \frac{ds}{d+2}$. This implies that $q = \frac{ds}{d+2} < s$ when $\alpha' = 0$. So we only need to require $q > 1$ for $\alpha' = 0$. Case 2 corresponds to the set $E_2$. There are two subcases for Case 1:

1.1. $q \leq r$, $s < p \leq m$, or 1.2. $q \leq r$, $p \leq s < m$.

Since $p \leq s$ if and only if $r \leq \frac{ds}{d+2}$, Subcase 1.2 is included in Case 2. (The endpoint case $1 = q = r = \frac{ds}{d+2}$ does not occur since $r > 1$ by assumption.) Subcase 1.1 corresponds to the set $E_1$. This completes the proof of the lemma.

We digress to discuss the condition $q \leq s$ in the factor $\mathbf{1}_{\{q \leq s\}}$ in (2.17). In particular, it prevents spacetime integral estimates in $L^s L^p_{uloc}$—namely for the Maekawa-Terasawa solutions—of the form

$$
\|u\|_{L_T^s L^p_{uloc}} \lesssim \|u\|_{L^s_{uloc}}, \quad \frac{2}{s} + \frac{3}{p} \leq 1,
$$

(2.24)

when $d = r = 3$. This means the spaces $E_{T,q}^{s,p}$ are better suited for the analysis of the Navier-Stokes problem. A similar situation is evident for Besov spaces [5, Section 2.6.3] where the spacetime integral is applied to individual Littlewood-Paley blocks and then summed. If we estimate
\[ \|u\|_{L^q_t L^p_{x,t}} \text{ naive using the linear estimate of Maekawa-Terasawa [38, Corollary 3.1], i.e., (2.2)} \]

with \( q = \tilde{q} = \infty \), then we obtain for \( d = r = 3 \)

\[ \|e^{t \Delta} a\|_{L^q_t} \lesssim \|a\|_{L^p_{x,t}} 1 \leq \|a\|_{L^p_{x,t}}, \]

which diverges if \( 2/s + 3/p \leq 1 \). It converges if we decrease \( s \) so that \( \frac{2}{s} + \frac{3}{p} > 1 \), but the integrability at time 0 is then not as good as \( E_{T;q}^{5/3} \) without decreasing \( s \).

In fact, our next example shows that (2.24) is wrong. Thus, for the existence theorem, we cannot replace the space \( E_{T,q}^{5/3} \) by the strict subspace \( L^5_t L^5_{x,t} \).

**Example 2.6.** Let the exponents \( p, q, r, s, m \) be as in Lemma 2.4. The following example shows that \( e^{t \Delta} a \) is not necessarily in \( L^q_t E^p_m \) for \( a \in E^p_q \) if \( s < q \). In particular, the factor \( \mathbb{1}_{q \leq s} \) in (2.17) is necessary.

First note that for \( \delta > 0, x_0 \in \mathbb{R}^d, x \in B_\delta(x_0) \) and \( t \in (\delta^2, 2\delta^2) \),

\[ \int_{B_\delta(x_0)} \Gamma(x - y, t) dy = \int_{B_\delta(x_0)} C t^{-d/2} e^{-|x - y|^2/4t} dy \geq C, \quad (2.25) \]

where \( C \) is independent of \( \delta \). For \( \beta = \frac{d}{r} \) and \( \{c_k\}_{k \in \mathbb{N}} \in \ell^\beta, c_k \geq 0 \), define

\[ a(x) = \sum_{k \in \mathbb{N}} c_k a_k(x), \quad a_k(x) = 2^{\frac{k\beta}{r}} \cdot \chi_{B_k}(x), \quad B_k = B(x_k, 2^{-k/2}). \]

We may take \( x_k = 2k e_1 \). Note that \( \|a_k\|_{L^r} \) is constant in \( k \), and hence \( \|a\|_{E^p_q} \approx \|c_k\|_{\ell^\beta} \).

Let \( u(x, t) = \int_{\mathbb{R}^d} \Gamma(x - y, t)a(y) dy \), i.e., \( u(\cdot, t) = e^{t \Delta} a \). By (2.25),

\[ u(x, t) \geq c_k 2^{\frac{k\beta}{r}}, \quad \text{if } x \in B_k \text{ and } t \in (2^{-k}, 2^{-k+1}]. \]

Hence for \( t \in (2^{-k}, 2^{-k+1}] \) and \( 1 \leq m \leq \infty \),

\[ \|u(t)\|_{E^m_T} \gtrsim \|u(t)\|_{L^p_{x,t}} \gtrsim \|u(t)\|_{L^p_{x,B_1(x_0)}} \gtrsim c_k (2^{\frac{k\beta}{r}})^{\frac{1}{r} - \frac{d}{r}} \gtrsim c_k t^{-\frac{1}{2}(\beta - \frac{d}{r})} = c_k t^{-\frac{1}{2}}, \quad (2.26) \]

and

\[ \int_0^1 \|u(t)\|_{L^p_{x,t}}^s dt = \sum_{k=1}^\infty \int_{2^{-k}}^{2^{-k+1}} \|u(t)\|_{L^p_{x,t}}^s dt \gtrsim \sum_{k=1}^\infty \int_{2^{-k}}^{2^{-k+1}} (c_k t^{-\frac{1}{2}})^s dt \gtrsim \ln 2 \sum_{k=1}^\infty c_k s. \quad (2.27) \]

Thus, if \( s < q \leq \infty \), we can choose \( \{c_k\}_{k \in \mathbb{N}} \in \ell^\beta \) so that \( \{c_k\}_{k \in \mathbb{N}} \not\in \ell^s \). Then \( u \not\in L^5_t E^5_{x,t} \) and hence \( u \not\in L^5_t E^5_{x,c} \).

Note that, when \( q = \infty \), we can take \( c_k \to 0 \) very slowly, so that \( a \) is in \( E^r_\infty \subset E^r_\infty \), (i.e., \( a \) is in the closure of \( C_c^1 \) in \( E^r_{x,c} \)), but not in \( E^p_{x,c} \) for any \( q < \infty \). For example, \( c_k = (\ln(2 + k))^{-1} \).

The above construction has an interesting application: In the special case \( q = r = 1 \) although \( a \in E^r_q \).

Note that, when \( q = \infty \), we can take \( c_k \to 0 \) very slowly, so that \( a \) is in \( E^r_\infty \subset E^r_\infty \), (i.e., \( a \) is in the closure of \( C_c^1 \) in \( E^r_{x,c} \)), but not in \( E^p_{x,c} \) for any \( q < \infty \). For example, \( c_k = (\ln(2 + k))^{-1} \).

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We next obtain space-time integral estimates for the Duhamel term for solutions to the Stokes equations. For simplicity, we limit ourselves to \( d = 3 \). On the right side we use \( \|F\|_{E^{1,\tilde{p}}_{T,\tilde{m}}} \) in Lemma 2.7, and \( \|F\|_{L^1_T E^p_m} \) in Lemma 2.8.

**Lemma 2.7.** For \( F : \mathbb{R}^3 \times (0, T) \to \mathbb{R}^{3 \times 3} \) let

\[
L(F)_i(x,t) = \int_0^t \int_{\mathbb{R}^3} \partial_t S_{ij}(x-y,t-\tau) F_{ij}(y,\tau) \, dy \, d\tau. \tag{2.28}
\]

Let \( 1 \leq \tilde{p} \leq p \leq \infty \), \( 1 \leq \tilde{m} \leq m \leq \infty \) and \( 1 \leq \tilde{s} \leq s \leq \infty \). Further assume

\[
\sigma := \frac{1}{2} - \frac{3}{2} \left( \frac{1}{\tilde{p}} - \frac{1}{p} \right) - \left( \frac{1}{s} - \frac{1}{\tilde{s}} \right) \geq 0 \tag{2.29}
\]

with \( 1 < \tilde{s} < s < \infty \) in the case of equality \( \sigma = 0 \).

(a) For \( 0 < T < \infty \),

\[
1_{m \leq s} \|L(F)\|_{L^1_T E^p_m} \lesssim \|L(F)\|_{E^{1,\tilde{p}}_{T,\tilde{m}}} \lesssim (T^\sigma + T^3) \|F\|_{E^{1,\tilde{p}}_{T,\tilde{m}}}, \tag{2.30}
\]

where \( \beta \in (0, 1 - \frac{1}{s} + \frac{1}{s}] \) and \( \beta > \alpha \), with

\[
\alpha = \frac{1}{2} - \frac{3}{2} \left( \frac{1}{\tilde{m}} - \frac{1}{m} \right) - \left( \frac{1}{s} - \frac{1}{\tilde{s}} \right).
\]

We can take \( \beta = \alpha \) if \( \alpha \geq 0 \) and \( 1 < \tilde{m} < m < \infty \).

(b) Suppose \( \sigma = 0 \) and \( \alpha \leq 0 \). Assume further \( 1 < \tilde{m} < m < \infty \) if \( \alpha = 0 \). Then

\[
1_{m \leq s} \|L(F)\|_{L^1_T E^p_m} \lesssim \|L(F)\|_{E^{1,\tilde{p}}_{T,\tilde{m}}} \lesssim \|F\|_{E^{1,\tilde{p}}_{T,\tilde{m}}}, \tag{2.31}
\]

For our application to the Navier-Stokes equations (1.3), we will take \( F = u \otimes u \),

\[
\tilde{p} = \frac{p}{2}, \quad \tilde{s} = \frac{s}{2}, \quad \tilde{m} = \max(1, \frac{m}{2}).
\]

**Proof.** Note that the first inequalities in (2.30) and (2.31) follow from (1.9). For every \( k \in \mathbb{Z}^3 \) we may decompose for \( x \in B_1(k) \),

\[
L(F)_i(x,t) = \int_0^t \int_{\mathbb{R}^3} \partial_t S_{ij}(x-y,t-\tau) [F_{ij}\chi_{B_4(k)} + F_{ij}(1-\chi_{B_4(k)})](y,\tau) \, dy \, d\tau =: F_1^k + F_2^k.
\]

By Oseen tensor estimates (2.1) and Young’s convolution inequality, for \( 1 \leq \tilde{p} \leq p \leq \infty \),

\[
\left\| F^k_1 (\cdot, t) \right\|_{L^p(B_1(k))} \leq \int_0^t \left\| \partial_t S_{ij} (\cdot, t-\tau) \ast (F(\cdot, \tau)\chi_{B_4(k)}) \right\|_{L^p(\mathbb{R}^3)} d\tau
\]

\[
\leq \int_0^t \left\| \partial_t S_{ij} (\cdot, t-\tau) \right\|_{L^r(\mathbb{R}^3)} \left\| F (\cdot, \tau)\chi_{B_4(k)} \right\|_{L^\tilde{p}(\mathbb{R}^3)} d\tau
\]

\[
\leq \int_0^t (t-\tau)^{-\frac{1}{2} \left( \frac{3}{\tilde{p}} - \frac{1}{p} \right) - \frac{1}{2}} \left\| F (\cdot, \tau) \right\|_{L^{\tilde{p}}(B_4(k))} d\tau,
\]

where \( \frac{1}{r} = \frac{1}{p} + \frac{1}{\tilde{p}} \), and \( r \geq 1 \) due to \( \tilde{p} \leq p \). Note that the exponent \( \frac{3}{2} \left( \frac{1}{\tilde{p}} - \frac{1}{p} \right) + \frac{1}{2} = -\sigma + 1 - \left( \frac{1}{s} - \frac{1}{\tilde{s}} \right) \in (0, 1) \) due to \( \sigma \geq 0 \) with \( \tilde{s} < s \) in the case of equality in (2.29).
If \( \sigma = \frac{1}{2} - \frac{3}{2} \left( \frac{1}{p} - \frac{1}{p} \right) - \left( \frac{1}{s} - \frac{1}{s} \right) > 0 \), we use the Young's convolution inequality for the right hand side of (2.32) for \( t \in [0, T] \) to get
\[
\left\| F^k_1 \right\|_{L^\sigma(0,T;L^p(B_1(k)))} \leq CT^\sigma \left\| F \right\|_{L^\sigma(0,T;L^p(B_1(k)))}.
\] (2.33)
If \( \sigma = 0 \) and \( 1 < \tilde{s} < s < \infty \), we use the Hardy–Littlewood–Sobolev inequality for the right hand side of (2.32) to get the same estimate (2.33) with \( \sigma = 0 \). The constant \( C \) in (2.33) is independent of \( T \). It follows that, for \( m \geq \tilde{m} \) and \( 0 < T < \infty \),
\[
\left\| F^k_1 \right\|_{L^\sigma(0,T;L^p(B_1(k)))} \mid_{\ell^m(k \in \mathbb{Z}^d)} \leq CT^\sigma \left\| F \right\|_{L^\sigma(0,T;L^p(B_1(k)))} \mid_{\ell^m(k \in \mathbb{Z}^d)} \leq CT^\sigma \left\| F \right\|_{E^{\tilde{p},\tilde{p}}_{T,\tilde{m}}}.
\] (2.34)
If \( \sigma = 0 \), the above estimate is uniform for all \( T \) and we can take \( T = \infty \).

The estimate for \( F^k_2 \) is more intricate but follows themes already explored in this paper and [10]. Note that for \( x \in B_1(k) \) and \( t \in (0, T) \),
\[
|F^k_2(x,t)| \lesssim \int_0^t \int_{|y|>2} \frac{1}{(|y|+\sqrt{\tau})^4} |F(x-y,t-\tau)| \, dy \, d\tau
\]
\[
\lesssim \sum_{|k'| \geq 1} \int_0^t \frac{1}{(|k'|+\sqrt{\tau})^4} \int_{B_1(k')} |F(x-y,t-\tau)| \, dy \, d\tau
\]
\[
\lesssim \sum_{|k'| \geq 1} \int_0^t \frac{1}{(|k'|+\sqrt{\tau})^4} \int_{B_2(k-k')} |F(z,t-\tau)| \, dz \, d\tau
\]
\[
\lesssim \sum_{|k'| \geq 1} \int_0^t \frac{1}{(|k'|+\sqrt{\tau})^4} \left\| F(t-\tau) \right\|_{L^\sigma(B_2(k-k'))} \, d\tau.
\] (2.35)

Using the above \( L^\infty \) estimate to bound \( \left\| F^k_2(t) \right\|_{L^p(B_1(k))} \) and then Young’s convolution inequality for \( t \in (0, T) \), we get
\[
a_k := \left\| F^k_2 \right\|_{L^\sigma(0,T;L^p(B_1(k)))} \lesssim \sum_{|k'| \geq 1} b_{k'} f_{k-k'},
\] (2.36)
where
\[
b_{k'} = \left\| \frac{1}{(|k'|+\sqrt{\tau})^4} \right\|_{L^\sigma(0,T)}, \quad f_k = \left\| F \right\|_{L^\sigma(0,T;L^p(B_2(k))}, \quad 1 + \frac{1}{s} = 1 + \frac{1}{s^*} = 1 + \frac{1}{s}.\]

Note \( 1/s^* \in [1/2, 1] \) since \( \sigma \geq 0 \). We may set \( b_0 = 0 \). For \( k \neq 0 \), we have
\[
b_k = \left( \int_0^T \frac{d\tau}{(|k|+\sqrt{\tau})^{4s^*}} \right)^{-\frac{1}{s^*}} = |k|^{-4+\frac{2}{s^*}} \left( \int_0^{T/|k|^2} \frac{dt}{(1+\sqrt{t})^{4s^*}} \right)^{\frac{1}{s^*}}.
\]
The integral \( \int_0^x \frac{dt}{(1+\sqrt{t})^{4s^*}} \) is bounded by 1 for \( x \geq 1 \) and by \( x \) for \( x \in (0, 1) \). Hence it is bounded by \( x/(1+x) \) for \( 0 < x < \infty \), and we get
\[
b_k \lesssim |k|^{-4+\frac{2}{s^*}} \left( \frac{T}{T+|k|^2} \right)^{\frac{1}{s^*}} = |k|^{-2-(\frac{2}{s^*}-\frac{2}{s})} \left( \frac{T}{T+|k|^2} \right)^{1-(\frac{1}{m}+\frac{1}{m})}. \] (2.37)

Let \( \gamma \in [1, \infty] \) be defined by
\[
1 + \frac{1}{m} = \frac{1}{\gamma} + \frac{1}{m}.
\] 22
Note $b_k \leq CT^{\frac{1}{s}}|k|^{-\frac{2}{s}+\frac{3}{m}-\frac{3}{m}}$ hence $b_k \in \ell^\gamma$. Applying Young’s convolution inequality to (2.36), we get $\|a\|_{\ell^m} \lesssim \|b\|_{\ell^\gamma} \|f\|_{\ell^\alpha}$ and hence
\[
\|a\|_{\ell^m} \lesssim T^\beta \|f\|_{\ell^\alpha}
\] (2.38)
with $\beta = \frac{1}{s}$. By going back to (2.37), it is clear that, for certain parameters, the choice of $\beta$ in (2.38) can be improved. We pursue this presently.

**Case 1:** $\frac{2}{s} - \frac{2}{s} + \frac{3}{m} - \frac{3}{m} > 1$. In this case we have
\[
2 + \left(\frac{2}{s} - \frac{2}{s}\right) > \frac{3}{\gamma} = 3 - \left(\frac{3}{m} - \frac{3}{m}\right),
\] (2.39)
and, noting the second factor in (2.37) is bounded by 1,
\[
\|b\|_{\ell^\gamma} \lesssim \| |k|^{-\left(\frac{2}{s} - \frac{2}{s}\right)} \|_{\ell^\gamma} \lesssim 1.
\]
We get (2.38) with $\beta = 0$.

**Case 2:** $\frac{2}{s} - \frac{2}{s} + \frac{3}{m} - \frac{3}{m} = 1$. In this case we have equality in (2.39), and hence
\[
b_k \lesssim |k|^{-\left(\frac{2}{s} - \frac{2}{s}\right)} = |k|^{-3/\gamma}.
\]
Instead of Young’s inequality, we use the discrete version of the Hardy–Littlewood–Sobolev inequality stated in [43, Proposition (a)] to obtain the same estimate (2.38) with $\beta = 0$. For [43, Proposition (a)] we need to avoid end points, hence we assume $1 < \tilde{m} < m < \infty$. That $3/\gamma < 3$ is given by $\tilde{m} < m$.

**Case 3:** $\frac{2}{s} - \frac{2}{s} + \frac{3}{m} - \frac{3}{m} < 1$. In this case we have inequality “<” in (2.39), and cannot avoid $T$ dependence. If $1 < \tilde{m} < m < \infty$, we choose
\[
\alpha = \frac{1}{2} - \frac{1}{s} + \frac{1}{s} - \frac{3}{2m} + \frac{3}{2m}, \quad 0 < \alpha < 1 - \frac{1}{s} + \frac{1}{s}.
\]
Here, $1 - \frac{1}{s} + \frac{1}{s} > 0$. In fact, the case $(\tilde{s}, s) = (1, \infty)$ is neither in Case 2 nor in Case 3 because $\frac{2}{s} - \frac{2}{s} + \frac{3}{m} - \frac{3}{m} \geq 2$ when $\tilde{s} = 1, s = \infty$. We have
\[
b_k \lesssim |k|^{-\left(\frac{2}{s} - \frac{2}{s}\right)} \left(\frac{T}{T + |k|^2}\right)^\alpha \lesssim |k|^{-3/\gamma T^\alpha}.
\]
By the discrete Hardy–Littlewood–Sobolev inequality in [43] again, we get (2.38) with $\beta = \alpha$.

For all cases including the end points $1 = \tilde{m}$, $\tilde{m} = m$, or $m = \infty$, we can choose $\beta \in (\alpha, 1 - \frac{1}{s} + \frac{1}{s})$,
\[
b_k \lesssim |k|^{-\left(\frac{2}{s} - \frac{2}{s}\right)} \left(\frac{T}{T + |k|^2}\right)^\beta \lesssim |k|^{-3/\gamma - 2(\beta - \alpha)T^{\beta}},
\] (2.40)
and use Young’s inequality.

Combining (2.34) and (2.38) for all 3 cases of $\alpha$, we get part (a). Part (b) is a consequence of part (a), when the constants do not depend on $T$ and we can send $T \to \infty$. This completes the proof of the lemma.

We finally give bilinear estimates in $L^\alpha_T E^\beta_m$. On the right side we use $\|F\|_{L^\alpha_T E^\beta_m}$, not $\|F\|_{E^\beta_{T,m}}$ in Lemma 2.7.
Lemma 2.8. Let $0 < T \leq \infty$. For $F : \mathbb{R}^3 \times (0, T) \to \mathbb{R}^3$ let $L(F)$ be defined by (2.28). For $1 \leq \bar{p} \leq p \leq \infty$, $1 \leq \bar{m} \leq m \leq \infty$ and $1 < \bar{s} < s < \infty$, if $\frac{1}{2} - \frac{3}{2} \left( \frac{1}{\bar{m}} - \frac{1}{m} \right) - \left( \frac{1}{s} - \frac{1}{\bar{s}} \right) = 0$ and $\frac{1}{2} - \frac{3}{2} \left( \frac{1}{\bar{m}} - \frac{1}{m} \right) = \left( \frac{1}{s} - \frac{1}{\bar{s}} \right) = 0$, then for a $T$-independent constant $C$,

$$\|LF\|_{L^p_tE^p_m} \leq C \|F\|_{L^p_tE^p_m}.$$

Proof. Denote $\mu = \frac{1}{2} + \frac{3}{2} \left( \frac{1}{\bar{p}} - \frac{1}{p} \right) = \frac{1}{2} + \frac{3}{2} \left( \frac{1}{\bar{m}} - \frac{1}{m} \right) = 1 - \frac{1}{s} + \frac{1}{\bar{s}}$, $0 < \mu < 1$ since $1 < \bar{s} < s < \infty$. By Lemma 2.1,

$$\|LF\|_{L^p_tE^p_m} = \left\| \int_0^t \left( \int_{\mathbb{R}^3} \partial_t S_{ij}(x - y, t - \tau) F_{ij}(y, \tau) \, dy \right) \, d\tau \right\|_{L^p_tE^p_m},$$

which is bounded by $\|F\|_{L^p_tE^p_m}$ by the Hardy-Littlewood-Sobolev inequality. \hfill \Box

2.3 Energy estimates

In this subsection, we prove the energy estimates in Wiener amalgam spaces, which can be viewed as the $E^2_q$-version of [35, Lemma 2.4]. The energy estimates are used in Section 4.1 to construct global-in-time weak solutions in $E^2_q$ for $1 \leq q < 2$. Recall the $\ell^q$ local energy space $LE_q(0, T)$ is defined by the norm (1.23).

Lemma 2.9. For any $T > 0$ and $q \geq 1$, if $f \in E^2_q$ and $F \in E^{2,2}_{T,q}$, defined for $x \in \mathbb{R}^3$, then we have for any $\beta > 0$,

$$\|e^{t\Delta} f\|_{LE_q(0,T)} \lesssim (1 + T^\beta) \|f\|_{E^2_q}, \quad (2.41)$$

and

$$\left\| \int_0^t e^{(t-\tau)\Delta} \nabla \cdot F(\tau) \, d\tau \right\|_{LE_q(0,T)} \lesssim (1 + T^\beta) \|F\|_{E^{2,2}_{T,q}}, \quad (2.42)$$

Proof. We first prove (2.41). By (2.16) in Lemma 2.4, $\|f\|_{E^{2,2}_{T,q}} \lesssim (1 + T^\beta) \|f\|_{E^2_q}$ for any $\beta > 0$. It remains to estimate $\|\nabla e^{t\Delta} f\|_{E^{2,2}_{T,q}}$. The proof is almost identical to that of Lemma 2.4 except that we use the energy estimate instead of Giga’s estimate for the localized part. For every $k \in \mathbb{Z}^3$ and $x \in B_1(k)$, we decompose

$$\nabla e^{t\Delta} f(x,t) = \nabla e^{t\Delta} (f \chi_{B_4(k)}) + \nabla e^{t\Delta} (f(1 - \chi_{B_4(k)}) =: f_1^k(x,t) + f_2^k(x,t).$$

Then

$$\|\nabla e^{t\Delta} f\|_{E^{2,2}_{T,q}} = \left\| \left\| \nabla e^{t\Delta} f \right\|_{L^2(B_1(k))} \right\|_{L^2(0,T)} \|_{\ell^q(k \in \mathbb{Z}^3)}$$

$$\leq \left\| \left\| f_1^k(\cdot,t) \right\|_{L^2(B_1(k))} \right\|_{L^2(0,T)} \|_{\ell^q(k \in \mathbb{Z}^3)} + \left\| \left\| f_2^k(\cdot,t) \right\|_{L^2(B_1(k))} \right\|_{L^2(0,T)} \|_{\ell^q(k \in \mathbb{Z}^3)}$$

$$=: A_1 + A_2.$$
For $A_1$, we use the usual energy estimate for the heat equation to get

$$\left\| \int_0^T f_1^k(\cdot,t) \right\|_{L^2(B_1(k))} \leq \left\| \nabla e^{t\Delta} (f \chi B_4(k)) \right\|_{L^2(0,\infty;L^2(\mathbb{R}^3))} \lesssim \|f\|_{L^2(B_4(k))}.$$  

Hence

$$A_1 \lesssim \left\| \int_0^T f(\cdot,t) \right\|_{L^2(B_4(k))} \approx \left\| f \right\|_{E^2_t}.$$ (2.43)

For $A_2$, as

$$|f^k_2(x,t)| \lesssim \sum_{k'} \int_{B_1(k)} t^{-2} e^{-\frac{|x-y|^2}{at}} |f(y)| dy,$$

we have

$$\left\| f^k_2(\cdot,t) \right\|_{L^2(B_1(k))} \lesssim t^{-2} \sum_{|k'| \geq 1} e^{-|k'|^2/(4t)} \|f\|_{L^2(B_1(k'-k))}.$$  

Taking $L_t^2$ on both sides and applying Minkowski’s integral inequality, we get

$$\left\| f^k_2 \right\|_{L_t^2 L^2(B_1(k))} \lesssim \left[ \int_0^T \left( t^{-2} \sum_{|k'| \geq 1} e^{-|k'|^2/(8t)} \|f\|_{L^2(B_1(k'-k))} \right)^2 \right]^{1/2} dt$$

$$\leq \sum_{|k'| \geq 1} \left( \int_0^T t^{-4} e^{-|k'|^2/(4t)} \|f\|_{L^2(B_1(k'-k))}^2 \right)^{1/2} dt$$

$$= \sum_{|k'| \geq 1} \|f\|_{L^2(B_1(k'-k))} \left( \int_0^T t^{-4} e^{-|k'|^2/(4t)} dt \right)^{1/2}, \quad t = |k'|^2 \tau$$

$$= \sum_{|k'| \geq 1} \|f\|_{L^2(B_1(k'-k))} |k'|^{-3} \left( \int_0^\tau t^{-\frac{1}{4}} e^{-\frac{|k'|^2}{4t}} dt \right)^{1/2}.$$  

We have $\int_0^\infty \tau^{-4} e^{-\frac{|k'|^2}{4t}} d\tau < \infty$ and, if $S < 1$,

$$\int_0^S \tau^{-4} e^{-\frac{|k'|^2}{4t}} d\tau \lesssim \int_0^S e^{-\frac{|k'|^2}{4t}} d\tau \lesssim e^{-\frac{|k'|^2}{4t}}.$$  

This estimate is also true if $S > 1$. We conclude

$$\left\| f^k_2 \right\|_{L_t^2 L^2(B_1(k))} \lesssim \sum_{|k'| \geq 1} \|f\|_{L^2(B_1(k'-k))} |k'|^{-3} e^{-\frac{|k'|^2}{8t}}.$$  

For fixed $0 < T < \infty$, we can bound for any $\beta \in [0,\infty)$

$$|k'|^{-3} e^{-\frac{|k'|^2}{8t}} \lesssim |k'|^{-3} \left( \frac{|k'|^2}{8T} \right)^{-\beta} = |k'|^{-3-2\beta} T^\beta.$$  

By Young’s convolution inequality,

$$A_2 \lesssim \left\| \sum_{|k'| \geq 1} \|f\|_{L^2(B_1(k'-k))} |k'|^{-3-2\beta} T^\beta \right\|_{L_t^2} \lesssim \left\| f \right\|_{E^2_t} \left\| \sum_{|k'| \geq 1} |k'|^{-3-2\beta} T^\beta \right\|_{L_t^2(k \neq 0 \in \mathbb{Z}^3)},$$ (2.44)
where the last norm is finite since $\beta > 0$. The estimate (2.41) follows from (2.43) and (2.44).

Next, we prove (2.42). By (2.30) in Lemma 2.7, $\left\| \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot F(\tau) \, d\tau \right\|_{E_{T,q}^{\infty,2}} \lesssim (1+T^\beta) \left\| F \right\|_{E_{T,q}^{2,2}}^2$ for any $\beta > 0$. It remains to estimate $\left\| \nabla \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot F(\tau) \, d\tau \right\|_{E_{T,q}^{2,2}}$. The proof is almost identical to that of Lemma 2.7 except that we use the energy estimate instead of the Oseen’s tensor estimate for the localized part. For every $k \in \mathbb{Z}^3$ we may decompose for $x \in B_1(k)$,

$$\nabla \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot F(\tau) \, d\tau = \nabla \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (F(\tau) \chi_{B_4(k)}) \, d\tau + \nabla \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot \left[ F(\tau) (1 - \chi_{B_4(k)}) \right] \, d\tau =: F_1^k + F_2^k.$$

For $F_1^k$, by the usual energy estimate for the Stokes system we get

$$\left\| F_1^k(.,t) \right\|_{L^2(B_1(k))} \leq \left\| \nabla \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (F(\tau) \chi_{B_4(k)}) \, d\tau \right\|_{L^2_{t} L^2(\mathbb{R}^3)} \lesssim \left\| F \right\|_{L^2_{t} L^2(B_4(k))}.$$

Hence

$$\left\| F_1^k(.,t) \right\|_{L^2(B_1(k))} \left\| F_2^k(.,t) \right\|_{L^2(B_1(k))} \lesssim \left\| F \right\|_{E_{T,q}^{2,2}}.$$

For $F_2^k$, note that for $x \in B_1(k)$ and $t \in (0,T)$,

$$|F_2^k(x,t)| \lesssim \int_0^t \int_{|y|>2} \frac{1}{(|y| + \sqrt{\tau})^5} |F(x-y, t-\tau)| \, dy \, d\tau \lesssim \sum_{|k'|\geq 1} \int_0^t \int_{B_1(k')} \frac{1}{(|k'| + \sqrt{\tau})^5} |F(x-y, t-\tau)| \, dy \, d\tau$$

$$\lesssim \sum_{|k'|\geq 1} \int_0^t \int_{B_2(k-k')} \frac{1}{(|k'| + \sqrt{\tau})^5} |F(z, t-\tau)| \, dz \, d\tau \lesssim \sum_{|k'|\geq 1} \int_0^t \left\| F(t-\tau) \right\|_{L^2(B_2(k-k'))} \, d\tau.$$

(2.46)

Using the above $L^\infty$ estimate to bound $\left\| F_2^k(t) \right\|_{L^2(B_1(k))}$ and then Young’s convolution inequality for $t \in (0,T)$, we get

$$a_k := \left\| F_2^k \right\|_{L^2(0,T;L^2(B_1(k)))} \lesssim \sum_{|k'|\geq 1} b_{k'} \phi_{k-k'},$$

(2.47)

where

$$b_{k'} = \left\| \frac{1}{(|k'| + \sqrt{\tau})^5} \right\|_{L^1(0,T)} \quad \phi_k = \left\| F \right\|_{L^2(0,T;L^2(B_2(k)))}.$$

We may set $b_0 = 0$. For $k \neq 0$, we have

$$b_k = \int_0^T \frac{d\tau}{(|k| + \sqrt{\tau})^5} = |k|^{-3} \int_0^{T/|k|} \frac{dt}{(1 + \sqrt{\tau})^5}.$$
The integral \( \int_0^x \frac{dt}{(1 + \sqrt{x^2 t^2})} \) is bounded by 1 for \( x \geq 1 \) and by \( x \) for \( x \in (0, 1) \). Hence it is bounded by \( x/(1 + x) \) for \( 0 < x < \infty \), and we get

\[
\frac{1}{T + |k|^2} \lesssim |k|^{-3} \left( \frac{T}{|k|^2} \right)^{\min(1, \beta)}. 
\]

Hence \( \|b_k\|_{L^1_t} \lesssim 1 + T^\beta \) for any \( \beta > 0 \). Applying Young’s convolution inequality to (2.47) we get \( \|a\|_{L_t^0} \lesssim \|b\|_{L_t^0} \|\phi\|_{L_t^0} \) and hence

\[
\left\| \left\| \cdot F_k^+ \right\|_{L^2_t(B_1(k))} \right\|_{L^2_t(0, T)} \left\| \right\|_{L^2_t(k \in \mathbb{Z}^d)} = \|a\|_{L_t^q} \lesssim (1 + T^\beta) \|\phi\|_{L_t^q}. 
\]

Combining (2.45) and (2.48), (2.42) follows. \( \square \)

## 3 The Navier-Stokes equations in Wiener amalgam spaces

In this section we consider the nonlinear Navier-Stokes equations in space dimension \( d = 3 \).

Recall the Picard contraction principle which states: If \( E \) is a Banach space and \( B : E \times E \to E \) is a bounded bilinear transform satisfying

\[
\|B(e, f)\|_E \leq C_B \|e\|_E \|f\|_E,
\]

and if \( \|e_0\|_E \leq \delta \leq (4C_B)^{-1} \), then the equation \( e = e_0 - B(e, e) \) has a solution with \( \|e\|_E \leq 2\delta \) and this solution is unique in \( B(0, 2\delta) \).

For our application to Navier-Stokes equations, \( e_0 = e^{t\Delta} u_0 \) and the bilinear operator is

\[
B(f, g)_t(x, t) = \int_0^t \int \partial_t \mathcal{S}_{ij}(x - y, t - \tau) f_i g_j(y, \tau) \, dy \, d\tau,
\]

which is just the vector components of \( B(f, g) \) as defined in Section 1.2, in terms of the Oseen tensor \( \mathcal{S}_{ij} \).

### 3.1 Mild solutions in subcritical spaces

We first prove Theorem 1.1 for subcritical data \( u_0 \in E_r^q \), \( 3 < r \leq \infty \).

**Proof of Theorem 1.1.** Let

\[
\|f\|_{E_T} = \sup_{0 \leq t \leq T} \|f(t)\|_{E_r^q}.
\]

By Lemma 2.1 we have

\[
\|e^{t\Delta} u_0\|_{E_T^q} \lesssim \|u_0\|_{E_r^q}.
\]

So

\[
\|e^{t\Delta} u_0\|_{E_T^q} \leq C_1 \|u_0\|_{E_r^q}. 
\]

For bilinear estimate, we have by Lemma 2.1

\[
\|B(f, g)(t)\|_{E_T^q} \lesssim \int_0^t \left( \frac{1}{(t - \tau)^{1/2}} + \frac{1}{(t - \tau)^{1/2}} \right) \|f \otimes g(\tau)\|_{E_r^{q/2}} \, d\tau 
\lesssim (t^{1/2 - 3/(2r)} + t^{1/2}) \sup_{0 < \tau < t} \|f(\tau)\|_{E_r^q} \sup_{0 < \tau < t} \|g(\tau)\|_{E_r^q} 
\lesssim (t^{1/2 - 3/(2r)} + t^{1/2}) \|f\|_{E_T^q} \|g\|_{E_T^q}.
\]
where we’ve used the inclusion $E^r_q \subset E^\infty_q$ in the second to last inequality. Therefore, we have

$$\|B(f, g)\|_{\mathcal{E}_T} \leq C_2(T^{1/2-3/(2r)} + T^{1/2})\|f\|_{\mathcal{E}_T}\|g\|_{\mathcal{E}_T}.$$  \hspace{1cm} (3.4)

We seek a solution of the form

$$u = e^{t\Delta}u_0 - B(u, u).$$

Assume $T$ is chosen small enough that $\|u_0\|_{E^r_q} < (8C_1C_2(T^{1/2-3/(2r)} + T^{1/2}))^{-1}$. Then the Picard contraction principle implies there exists a unique strong mild solution satisfying

$$\|u\|_{\mathcal{E}_T} \leq 2C_1 \|u_0\|_{E^r_q}.$$  \hspace{1cm} (3.5)

We now prove the various statements in the theorem concerning continuity. Assume $r, q < \infty$. Then,

$$\|u(t) - u_0\|_{E^r_q} \leq \|B(u, u)(t)\|_{E^r_q} + \|e^{t\Delta}u_0 - u_0\|_{E^r_q}$$

$$\lesssim (t^{1/2-3/(2r)} + t^{1/2}) \sup_{0 < \tau < t} \|u(\tau)\|_{E^r_q} \sup_{\tau < t} \|u(\tau)\|_{E^\infty_q} + \|e^{t\Delta}u_0 - u_0\|_{E^r_q} \hspace{1cm} (3.6)$$

Since the powers on $t$ are positive in the first term and since the latter term vanishes by Lemma 2.3, we have

$$\|u(t) - u_0\|_{E^r_q} \to 0 \text{ as } t \to 0^+.$$ 

The continuity at $t \in (0, T)$ can be shown as usual, see e.g., [44, lines 3-8, page 86], including $r = \infty$ or $q = \infty$. Compare the proof of Theorem 1.2.

If either $r = \infty$ or $q = \infty$ we no longer have $\|e^{t\Delta}u_0 - u_0\|_{E^r_q} \to 0$ but we do still have

$$(t^{1/2-3/(2r)} + t^{1/2}) \sup_{0 < \tau < t} \|u(\tau)\|_{E^r_q} \sup_{\tau < t} \|u(\tau)\|_{E^r_q} \to 0.$$

Hence, we still have

$$\|u(t) - e^{t\Delta}u_0\|_{E^r_q} \to 0 \text{ as } t \to 0^+,$$

as asserted in the theorem.

Next we prove the uniqueness of the mild solution in $L^\infty(0, T; E^r_q) \cap C((0, T); E^r_q)$ without the bound (3.5). Let $u_1, u_2 \in L^\infty(0, T; E^r_q) \cap C((0, T); E^r_q)$ be two mild solutions with initial data $u_0 \in E^r_q$. Then for $0 < t < T' \leq T$

$$\|(u_1 - u_2)(t)\|_{E^r_q} \leq \|B(u_1 - u_2, u_2)(t)\|_{E^r_q} + \|B(u_2, u_1 - u_2)(t)\|_{E^r_q}$$

$$\lesssim (t^{1/2-3/(2r)} + t^{1/2}) \left(\|u_1\|_{\mathcal{E}_T} + \|u_2\|_{\mathcal{E}_T}\right) \sup_{0 < t < T'} \|(u_1 - u_2)(t)\|_{E^r_q},$$

so that we have

$$\sup_{0 < t < T'} \|(u_1 - u_2)(t)\|_{E^r_q} \lesssim (T'^{1/2 - \frac{n}{2r}} + T'^{1/2}) \left(\|u_1\|_{\mathcal{E}_T} + \|u_2\|_{\mathcal{E}_T}\right) \sup_{0 < t < T'} \|(u_1 - u_2)(t)\|_{E^r_q}.$$ 

Thus, for sufficiently small $T' > 0$ it follows that $u_1 = u_2$ on $(0, T')$. Repeating this argument, we see that $u_1 = u_2$ on $(0, T)$. 

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To obtain the spacetime integral bound we assume $3 < r \leq s \leq \infty$, $r \leq p < \infty$, $\frac{2}{s} + \frac{3}{p} = \frac{3}{r}$ and $1 \leq q = m \leq \infty$, and perform the Picard iteration in the Banach space

$$X_T = \mathcal{E}_T \cap E^{s,p}_{T,q}.$$  

We may assume $m = q$ since $\|u\|_{E^{s,p}_{T,m}} \leq \|u\|_{E^{s,p}_{T,q}}$ for $m \geq q$. From $\frac{3}{p} = \frac{3}{r} - \frac{2}{s} \geq \frac{3}{r} - \frac{2}{r}$ we get $p \leq 3r < \infty$. For the linear term, by (3.3) and Lemma 2.4 (which needs $r < \infty$ and $r \leq s$), we have for a fixed $\epsilon > 0$

$$\|e^{t\Delta} u_0\|_{X_T} = \|e^{t\Delta} u_0\|_{\mathcal{E}_T} + \|e^{t\Delta} u_0\|_{E^{s,p}_{T,q}} \leq C_3(1 + T^{1/\epsilon + \epsilon}) \|u_0\|_{E^r_T}.$$  

For the bilinear term, by (3.4) and Lemma 2.7 with $\tilde{p} = p/2$ and $\tilde{s} = s/2$ so that $\sigma = \frac{1}{2} - \frac{3}{2r} > 0$ due to $r > 3$, (allowing $s = \infty$),

$$\|B(f, g)\|_{X_T} = \|B(f, g)\|_{\mathcal{E}_T} + \|B(f, g)\|_{E^{s,p}_{T,q}} \leq C_4 \left[ T^{\frac{1}{2} - \frac{3}{2r} + T^{\frac{1}{2}}} \|f\|_{\mathcal{E}_T} \|g\|_{\mathcal{E}_T} + \left(T^{\frac{1}{2} - \frac{3}{2r} + T^{\frac{1}{2}}}ight) \|f \otimes g\|_{E^{s,p}_{T,q}} \right].$$

Since

$$\|f \otimes g\|_{E^{s,p}_{T,q}} \leq \|f\|_{E^{s,p}_{T,q}} \|g\|_{E^{s,p}_{T,q}} \leq \|f\|_{E^{s,p}_{T,q}} \|g\|_{E^{s,p}_{T,q}},$$

we derive

$$\|B(f, g)\|_{X_T} \leq 2C_4 \left(T^{\frac{1}{2} - \frac{3}{2r} + T^{\frac{1}{2}}}\right) \|f\|_{X_T} \|g\|_{X_T}.$$  

If $T > 0$ is chosen small enough so that $\|u_0\|_{E^r_T} < \left[8C_3 C_4(1 + T^{\frac{1}{2}\epsilon})(T^{\frac{1}{2} - \frac{3}{2r}} + T^{\frac{1}{2}})^{-1}\right]^{-1}$, then the Picard contraction principle implies there exists a unique strong mild solution $\tilde{u} \in X_T$. Since $X_T \subset L^\infty T_E^q$, we have $\tilde{u} = u$ on $(0, T)$ by the uniqueness in $L^\infty(0, T; E^r_T)$. Thus, $u$ satisfies the spacetime integral bound. \(\square\)

### 3.2 Mild solutions in critical spaces with small data

We next prove Theorem 1.2 for small critical data $u_0 \in E^3_q$, $1 \leq q \leq \infty$. For simplicity of presentation, we first consider finite time exponent $s < \infty$ in the spacetime integral estimates (1.19) in $E^{s,p}_{T,m}$, and postpone the case $s = \infty$ (hence $p = 3$) after we have proved Lemmas 3.1 and 3.2.

**Proof of Theorem 1.2 when $s < \infty$.** The overall logic of the proof follows that in [38] which, in turn, draws on [30]. Let

$$\|f\|_{\tilde{F}^r_T} := \sup_{0 < t < T} \|f(\cdot, t)\|_{E^3_q} + \sup_{0 < t < T} t^{\frac{1}{2}} \|f(\cdot, t)\|_{E^\infty_q}$$

and

$$\|f\|_{\tilde{F}^r_T} := \sup_{0 < t < T} t^{\frac{1}{2}} \|f(\cdot, t)\|_{E^3_q}.$$  

It is obvious that $\tilde{E}_T \subset \tilde{F}_T$. Indeed,

$$t^{\frac{1}{2}} \|f\|_{E^3_q} = t^{\frac{1}{2}} \left\|\left(f(\cdot, t)\right)_{L^3(B_1(k))}\right\|_{L^6(k \in \mathbb{Z}^3)} 
\leq t^{\frac{1}{2}} \left\|\left(f(\cdot, t)\right)_{L^3(B_1(k))}\right\|_{L^6(k \in \mathbb{Z}^3)} 
\leq t^{\frac{1}{2}} \left\|\left(f(\cdot, t)\right)_{L^3(B_1(k))}\right\|_{L^6(k \in \mathbb{Z}^3)} 
\leq (t^{\frac{1}{2}} \|f(\cdot, t)\|_{E^\infty_q})^{\frac{1}{2}} \|f(\cdot, t)\|_{E^3_q}.$$

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Now, we define
\[ \mathcal{E}_T := \tilde{\mathcal{E}}_T \cap E^{s,p}_{T,m} \quad \text{and} \quad \mathcal{F}_T := \tilde{\mathcal{F}}_T \cap E^{s,p}_{T,m}, \]
with norms
\[ \| f \|_{\mathcal{E}_T} := \| f \|_{\tilde{\mathcal{E}}_T} + \| f \|_{E^{s,p}_{T,m}} \quad \text{and} \quad \| f \|_{\mathcal{F}_T} := \| f \|_{\tilde{\mathcal{F}}_T} + \| f \|_{E^{s,p}_{T,m}}, \]
respectively. We also have \( \mathcal{E}_T \subset \mathcal{F}_T \) by intersecting \( E^{s,p}_{T,m} \) with both sides of the inclusion.

By (2.16) in Lemma 2.4 we have for a fixed \( \epsilon > 0 \) chosen so that \( \epsilon > \frac{3}{2m} - \frac{3}{2q} \) from the definition of \( \beta \) in Lemma 2.4,
\[ \| e^{t\Delta} u_0 \|_{E^{s,p}_{T,m}} \lesssim (1 + T^{\frac{1}{s} + \epsilon}) \| u_0 \|_{E^{q}_q}. \]
Also, by Lemma 2.1,
\[ t^{\frac{1}{s}} \| e^{t\Delta} u_0 \|_{E^{q}_q} \lesssim (1 + T^{\frac{1}{s} + \epsilon}) \| u_0 \|_{E^{q}_q}. \]
Hence
\[ \| e^{t\Delta} u_0 \|_{\mathcal{F}_T} \leq C_1 (1 + T^{\frac{1}{s} + \epsilon} + T^{\frac{1}{s}}) \| u_0 \|_{E^{q}_q}. \]

For the bilinear term, using \( s < \infty \) and Lemma 2.7 with \( \tilde{s} = s/2, \tilde{p} = p/2, \) and \( \tilde{m} = m \) (so that \( \sigma = 0 \)), we obtain
\[ \| B(f, g) \|_{E^{s,p}_{T,m}} \leq C(1 + T^{1 - \frac{1}{2}}) \| f \otimes g \|_{E^{s,p}_{T,m}} \leq C(1 + T^{1 - \frac{1}{2}}) \| f \|_{E^{s,p}_{T,m}} \| g \|_{E^{s,p}_{T,m}} \]
\[ \leq C(1 + T^{1 - \frac{1}{2}}) \| f \|_{E^{s,p}_{T,m}} \| g \|_{E^{s,p}_{T,m}}, \]
by Hölder’s inequality and we have used the inclusion \( E^{s,p}_{T,m} \subset E^{s,p}_{T,\infty} \) in the last inequality. Also, by Lemma 2.1 and Hölder inequality (1.5)
\[ \| B(f, g) \|_{E^{q}_q}(t) \lesssim \int_0^t \left( \frac{1}{(t-\tau)^{\frac{1}{s}}} + \frac{1}{(t-\tau)^{\frac{1}{s}}(t-\tau)^{\frac{1}{2}}} \right) \| (f \otimes g)(\tau) \|_{E^{q}_q} \ d\tau \]
\[ \leq \int_0^t \left( \frac{1}{(t-\tau)^{\frac{1}{s}}} + \frac{1}{(t-\tau)^{\frac{1}{2}}} \right) \| f(\tau) \|_{E^{q}_q} \| g(\tau) \|_{E^{q}_q} \ d\tau \]
\[ \lesssim (1 + t^{-1/4}) \| f \|_{\tilde{\mathcal{F}}_T} \| g \|_{\tilde{\mathcal{F}}_T}. \]
Hence
\[ \| B(f, g) \|_{\mathcal{F}_T} \leq C_2 (1 + T^{\frac{1}{4}} + T^{1 - \frac{1}{2}}) \| f \|_{\mathcal{F}_T} \| g \|_{\mathcal{F}_T}. \]
By (3.10), taking \( \| u_0 \|_{E^{q}_q} \) small it is possible to ensure
\[ \| e^{t\Delta} u_0 \|_{\mathcal{F}_T} \leq [4C_2 (1 + T^{1/4} + T^{1 - \frac{1}{2}})]^{-1}. \]

The Picard contraction theorem then guarantees the existence of a mild solution \( u \) to (1.3) so that
\[ \| u \|_{\mathcal{F}_T} \leq 2 \| e^{t\Delta} u_0 \|_{\mathcal{F}_T}. \]
This solution is unique among all mild solutions \( v \) with data \( u_0 \) satisfying \( \| v \|_{\mathcal{F}_T} \leq 2 \| e^{t\Delta} u_0 \|_{\mathcal{F}_T}. \) In fact, since we can also apply the Picard contraction to \( \tilde{\mathcal{F}}_T \), the solution is also unique in the class \( \| v \|_{\tilde{\mathcal{F}}_T} \leq 2 \| e^{t\Delta} u_0 \|_{\tilde{\mathcal{F}}_T}. \)
Next, we show that a solution $u \in \mathcal{F}_T$ with small enough initial data $u_0 \in E^3_T$ also belongs to $\mathcal{E}_T$. Let $\{u^{(n)}\}_{n \geq 1}$ be the Picard iteration sequence in $\mathcal{F}_T$. By construction,

$$\left\| u^{(n)} \right\|_{\mathcal{F}_T} \leq 2C_1(1 + T^{\frac{1}{4} + \epsilon} + T^{\frac{1}{4}}) \left\| u_0 \right\|_{E^3_T} < 1.$$ 

Note that

$$\left\| u^{(n)} \right\|_{\mathcal{E}_T} \leq \left\| e^{t\Delta} u_0 \right\|_{\mathcal{E}_T} + \left\| B(u^{(n-1)}, u^{(n-1)}) \right\|_{\mathcal{E}_T}.$$ 

By Lemma 2.1,

$$\left\| e^{t\Delta} u_0 \right\|_{\mathcal{E}_T} \leq C(1 + T^{1/2}) \left\| u_0 \right\|_{E^3_T}.$$ 

As is usual in arguments like this, we now seek estimates for $B(f, g)$ in $\mathcal{E}_T$ in terms of measurements of $f$ and $g$ in $\mathcal{F}_T$ and $\mathcal{E}_T$. We have by Lemma 2.1 and Hölder inequality (1.5),

$$\left\| B(f, g) \right\|_{E^3_T} \lesssim \int_0^t \left( \frac{1}{(t-\tau)^{\frac{1}{2}}} + \frac{1}{(t-\tau)^{\frac{1}{4}}} \right) \left\| (f \otimes g)(\tau) \right\|_{E^3_T} d\tau \leq \int_0^t \left( \frac{1}{(t-\tau)^{\frac{1}{2}}} + \frac{1}{(t-\tau)^{\frac{1}{4}}} \right) \left\| f(\tau) \right\|_{E^3_T} \left\| g(\tau) \right\|_{E^3_T} d\tau \lesssim (1 + T^{1/4}) \left\| f \right\|_{\mathcal{E}_T} \left\| g \right\|_{\mathcal{E}_T}.$$ 

and

$$t^{\frac{1}{2}} \left\| B(f, g) \right\|_{E^3_T} \lesssim t^{\frac{1}{2}} \int_0^t \left( \frac{1}{(t-\tau)^{\frac{1}{2}}} + \frac{1}{(t-\tau)^{\frac{1}{4}}} \right) \left\| (f \otimes g)(\tau) \right\|_{E^3_T} d\tau \leq t^{\frac{1}{2}} \int_0^t \left( \frac{1}{(t-\tau)^{\frac{1}{2}}} + \frac{1}{(t-\tau)^{\frac{1}{4}}} \right) \left\| f(\tau) \right\|_{E^3_T} \left\| g(\tau) \right\|_{E^3_T} d\tau \lesssim (1 + T^{1/4}) \left\| f \right\|_{\mathcal{E}_T} \left\| g \right\|_{\mathcal{E}_T}.$$

By switching $f, g$ in the estimates,

$$\left\| B(f, g) \right\|_{\mathcal{E}_T} \lesssim (1 + T^{1/4}) \min \left( \left\| f \right\|_{\mathcal{E}_T} \left\| g \right\|_{\mathcal{E}_T}, \left\| g \right\|_{\mathcal{E}_T} \left\| f \right\|_{\mathcal{E}_T} \right).$$

Returning to our main goal, we have now that

$$\left\| u^{(n)} \right\|_{\mathcal{E}_T} \leq \left\| e^{t\Delta} u_0 \right\|_{\mathcal{E}_T} + \left\| B(u^{(n-1)}, u^{(n-1)}) \right\|_{\mathcal{E}_T} \leq C_T \left\| u_0 \right\|_{E^3_T} + C'_T \left\| u^{(n-1)} \right\|_{\mathcal{E}_T} \left\| u^{(n-1)} \right\|_{\mathcal{E}_T} \leq C_T \left\| u_0 \right\|_{E^3_T} + C'_T C''_T \left\| u_0 \right\|_{E^3_T} \left\| u^{(n-1)} \right\|_{\mathcal{E}_T}.$$ 

Thus, if $\left\| u_0 \right\|_{E^3_T} \leq (2C'_T C''_T)^{-1}$, then $\left\| u^{(n)} \right\|_{\mathcal{E}_T}$ is uniformly bounded by $2C_T \left\| u_0 \right\|_{E^3_T}$. We now obtain the inclusion $u \in \mathcal{E}_T$ as follows:

$$\left\| u^{(n+1)} - u^{(n)} \right\|_{\mathcal{E}_T} = \left\| B(u^{(n)}, u^{(n)}) - B(u^{(n-1)}, u^{(n-1)}) \right\|_{\mathcal{E}_T} \leq \left\| B(u^{(n)} - u^{(n-1)}, u^{(n)}) \right\|_{\mathcal{E}_T} + \left\| B(u^{(n-1)}, u^{(n)} - u^{(n-1)}) \right\|_{\mathcal{E}_T} \lesssim \left\| u^{(n)} - u^{(n-1)} \right\|_{\mathcal{E}_T} \left( \left\| u^{(n)} \right\|_{\mathcal{E}_T} + \left\| u^{(n-1)} \right\|_{\mathcal{E}_T} \right) \lesssim \left\| u^{(n)} - u^{(n-1)} \right\|_{\mathcal{E}_T} \left( \left\| u^{(n)} \right\|_{\mathcal{E}_T} + \left\| u^{(n-1)} \right\|_{\mathcal{E}_T} \right).$$
In particular, the above implies \( \{u^{(n+1)} - u^{(n)}\} \) is Cauchy in \( \tilde{E}_T \) and hence the limit of \( \{u^{(n)}\} \) is in \( E_T \) where we’re noting that convergence in the spacetime norm in the definition of \( E_T \) is already implied by convergence in \( F_T \).

When \( q \leq s \), note that

\[
\|u\|_{L^s_TE^p_m} \leq \|u\|_{L^s_TE^p_q} \leq \|u\|_{E^p_{s,q}},
\]

using (1.9) and \( q \leq s \) for the second inequality. This shows the \( L^s_TE^p_m \)-estimate in (1.19).

We now address convergence to the initial data when \( q < \infty \). By Lemma 2.3 we have

\[
\lim_{T' \to 0^+} \sup_{0 < t < T'} t^{\frac{1}{q}} \|e^{t\Delta}u_0\|_{E^q_1} = \lim_{T' \to 0^+} \|e^{t\Delta}u_0\|_{\tilde{F}_{T'}} = 0,
\]

whenever \( u_0 \in E^3_q \). We extend this property to all Picard iterates via induction. Our base case is (3.15) where we note that \( u^{(0)} \) can be viewed as \( e^{t\Delta}u_0 \). Our inductive hypothesis is that

\[
\lim_{T' \to 0^+} \|u^{(n-1)}\|_{\tilde{F}_{T'}} = 0.
\]

By our above estimates in the class \( \tilde{F}_{T'} \), where we are taking \( T' \leq T \), we have

\[
\|u^{(n)}\|_{\tilde{F}_{T'}} \leq \|e^{t\Delta}u_0\|_{\tilde{F}_{T'}} + \|B(u^{(n-1)}, u^{(n-1)})\|_{\tilde{F}_{T'}} \leq \|e^{t\Delta}u_0\|_{\tilde{F}_{T'}} + \|u^{(n-1)}\|_{\tilde{F}_{T'}}^2.
\]

Hence

\[
\lim_{T' \to 0^+} \|u^{(n)}\|_{\tilde{F}_{T'}} = 0,
\]

by (3.15) and (3.16). This completes the induction.

The limit (3.15), convergence of the Picard iterates in \( \tilde{F}_{T} \) and (3.17) imply that, by taking \( T' \) small, we can make \( \sup_{0 < t < T'} t^{\frac{1}{q}} \|u(t)\|_{E^q_1} \) small. To elaborate, we have

\[
\|u\|_{\tilde{F}_{T'}} \leq \|u - u^{(n)}\|_{\tilde{F}_{T'}} + \|u^{(n)}\|_{\tilde{F}_{T'}}.
\]

We may choose \( n \) large so that the first term is small and then make the second term small by taking \( T' \) small. Hence,

\[
\lim_{T' \to 0^+} \|u\|_{\tilde{F}_{T'}} = 0.
\]

Using (3.13) and (3.19), we have

\[
\lim_{T' \to 0^+} \sup_{0 < t < T'} \|B(u, u)\|_{E^3_q}(t) = 0.
\]

This and Lemma 2.3 imply

\[
\lim_{t \to 0} \|u - u_0\|_{E^3_q} = 0.
\]

If \( q = \infty \) then we have a weaker mode of convergence. Fix a ball \( B \). Take \( R > 0 \) large so that \( B \subset B_R(0) \). We re-write the bilinear form as

\[
B(u, u) = B(u, u\chi_{B_R(0)}) + B(u, u(1 - \chi_{B_R(0)})).
\]

If \( 1 < \omega < 3 \) then we have

\[
\|B(u, u\chi_{B_R(0)})\|_{L^\omega} \leq \left( \int_0^t \frac{1}{(t - \tau)^\frac{\omega}{2} + \frac{3}{4} \left( \frac{3}{2} - \frac{1}{\omega} \right)}\|u^2 \chi_{B_R(0)}\|_{L^{3/2}}(\tau) \, d\tau \right)^{\frac{1}{2}} \lesssim_s t^{\frac{1}{2} - \frac{3}{4} \left( \frac{3}{2} - \frac{1}{\omega} \right)} \|u\|_{L^\omega}^2 L^3_{abc}.
\]
For any $R > 0$, the above vanishes as $t \to 0$ provided $\omega < 3$.

By taking $R = 2 \max_{x \in B} |x|$, we can ensure that for all $x \in B$ and $|y| \geq R$ we have $\frac{1}{2} |y| \leq |x - y| \leq \frac{3}{2} |y|$. Hence, for $x \in B$,

$$|B(u, u\chi_{B(0)})(x, t)| \leq \int_0^t \int_{y \in B(t, 0)} \left( \frac{1}{2} |y| + \sqrt{t - \tau} \right)^4 |u|^2(y, \tau) \, dy \, d\tau$$

$$\leq t \sup_{0 < \tau < t} \sum_{k=1}^{\infty} \frac{1}{2^{4k-4}R^4} \int_{R2k-1 \leq |y| < R2k} |u|^2(y, \tau) \, dy$$

$$\leq t \sup_{0 < \tau < t} \sum_{k=1}^{\infty} \frac{R^{2k}}{R^{4}2^{4k-4}} \left( \int_{|y| < R2k} |u|^3(y, \tau) \, dy \right)^\frac{2}{3}$$

$$\leq t \sup_{0 < \tau < t} \frac{R^3}{R^42^{4k-4}} \sup_{0 < \tau < t} \|u(\tau)\|_{L^3_{\text{loc}}}^2$$

$$\leq \frac{t}{R} \sup_{0 < \tau < t} \|u(\tau)\|_{L^3_{\text{loc}}}^2.$$

Now,

$$\|B(u, u)(t)\|_{L^\infty(B)} \leq R \|B(u, u\chi_{B(0)})(t)\|_{L^\infty} + \|B(u, u\chi_{B(0)})(t)\|_{L^\infty(B)}$$

$$\leq R t^{\frac{1}{2} - \frac{3}{3} - \frac{1}{3}} \|u\|^2_{L^\infty(B)} + t \sup_{0 < \tau < t} \|u(\tau)\|^2_{L^3_{\text{loc}}}.$$

Hence

$$\lim_{t \to 0^+} \|B(u, u)(t)\|_{L^\infty(B)} = 0.$$

Referring to [38, p. 394], we have

$$\lim_{t \to 0} \|e^{t\Delta} u_0 - u_0\|_{L^\infty(B)} = 0.$$

It follows that

$$\lim_{t \to 0} \|u(\cdot, t) - u_0(\cdot)\|_{L^\infty(B)} = 0.$$

We now prove continuity at positive times. Let $t_1 > 0$ be fixed. We will send $t \to t_1$. Note that by Lemma 2.3 we have $e^{t_1 \Delta} u_0 - e^{t_1 \Delta} u_0 \to 0$ in $E^3_q$ as $t \to t_1$. We therefore only need to show $B(u, u)(t) \to B(u, u)(t_1)$. Following [44, p. 86], we take $\rho$ slightly less than 1 so that $\rho t_1 < t$ and write

$$B(u, u)(t) = B(u, u)(t) = \int_{t_1}^t e^{(t-\tau)\Delta} \mathbb{F} \nabla \cdot F \, d\tau - \int_{t_1}^{t_1} e^{(t_1-\tau)\Delta} \mathbb{F} \nabla \cdot F \, d\tau$$

$$+ \int_0^t e^{(t-\rho t_1)\Delta} - e^{(t_1-\rho t_1)\Delta} \mathbb{F} \nabla \cdot F \, d\tau$$

where $F = u \otimes u(\tau)$. For the first and second terms, by (2.3) with $p = \bar{p} = 3$, $q = q$ and using the embedding $E^\infty_q \subset E^\infty$, we have

$$\int_{\rho t_1}^t \|e^{(t-\tau)\Delta} \mathbb{F} \nabla \cdot F\|_{E^3_q} \, d\tau \leq \int_{\rho t_1}^t \frac{1}{(t - \tau)^\frac{1}{2} \tau^\frac{1}{2}} \|u\|_{E^3_q(\tau)} \tau^\frac{1}{2} \|u\|_{E^\infty_q} \, d\tau \leq \frac{(t - \rho t_1)^{\frac{1}{2}}}{(\rho t_1)^{\frac{1}{2}}} \|u\|^2_{L^s}.$$
and
\[
\int_{\rho t_1}^{t_1} \|e^{(t_1-\tau)\Delta} \mathbb{P} \nabla \cdot F\|_{E_3^p} d\tau \lesssim \int_{\rho t_1}^{t_1} \frac{1}{(t_1-\tau)^{\frac{3}{2}}} \|u\|_{E_3^q(\tau)} \|u\|_{E_3^\infty} d\tau \lesssim \frac{(t_1-\rho t_1)}{(\rho t_1)^{\frac{3}{2}}} \|u\|^2_{E_T},
\]
both of which can be made arbitrarily small by taking \(\rho t_1\) close to \(t_1\) and \(t\) close to \(t_1\).

For the third term we note that by Lemma 2.3, for each \(0 < \tau < \rho t_1\), we have
\[
\|e^{(t-\rho t_1)\Delta} - e^{(t_1-\rho t_1)\Delta} \|e^{(\rho t_1-\tau)\Delta} \mathbb{P} \nabla \cdot F(\tau)\|_{E_3^q} \to 0 \text{ as } t \to t_1,
\]
which follows (even if \(q = \infty\)) the fact that \(e^{(\rho t_1-\tau)\Delta} \mathbb{P} \nabla \cdot F(\tau)\in E_3^3\), which is a consequence of Lemma 2.1. Additionally,
\[
\|e^{(t-\rho t_1)\Delta} - e^{(t_1-\rho t_1)\Delta}\|e^{(\rho t_1-\tau)\Delta} \mathbb{P} \nabla \cdot F(\tau)\|_{E_3^q} \lesssim \left(\frac{1}{(t-\tau)^{\frac{3}{2}}} + \frac{1}{(t_1-\tau)^{\frac{3}{2}}}\right) \|u\|^2_{E_T} \in L^1(0, \rho t_1),
\]
where integration in \(L^1(0, \rho t_1)\) is with respect to \(\tau\). So, by Lebesgue’s dominated convergence theorem,
\[
\int_0^{\rho t_1} \|e^{(t-\rho t_1)\Delta} - e^{(t_1-\rho t_1)\Delta}\|e^{(\rho t_1-\tau)\Delta} \mathbb{P} \nabla \cdot F(\tau)\|_{E_3^q} d\tau \to 0 \text{ as } t \to t_1.
\]
The above show the continuity of \(u(t)\) at positive times.

In the following we will prove spacetime integral bound (1.19) for \(p = 3\), \(s = \infty\), i.e. in \(E_{T,q}^{3,\infty}\). For this purpose we will need time-weighted space-time integral estimates for both linear and nonlinear terms in the following two lemmas. Because the constants depend on \(T\), these estimates cannot be extended to \(T = \infty\).

**Lemma 3.1.** Let \(d = 3\) and \(0 < T < \infty\). For \(3 < p \leq \infty\) and \(a = \frac{p-3}{2p} \in (0, 1/2)\), we have
\[
\|t^a e^{\Delta} u_0\|_{E_{T,q}^{\infty,p}} \lesssim (1 + T^a) \|u_0\|_{E_3^q} \tag{3.20}
\]
For \(p = 3\) and \(a = 0\), we also have
\[
\|e^{\Delta} u_0\|_{E_{T,q}^{3,\infty}} \lesssim \ln(2 + T) \|u_0\|_{E_3^q} \tag{3.21}
\]

**Proof.** For \(x \in B_1(k)\) we write
\[
t^a |e^{\Delta} u_0(x)| \lesssim \int_{\mathbb{R}^3} \frac{e^{-\frac{|x-y|^2}{4t}}}{(|x-y| + \sqrt{t})^{3-2a}} |u_0(y)| \, dy =: U_1^k + U_2^k,
\]
where \(U_1^k = \int_{\mathbb{R}^3} \frac{e^{-\frac{|x-y|^2}{4t}}}{(|x-y| + \sqrt{t})^{3-2a}} (|u_0| \chi_{B_4(k)}(y)) \, dy\) and \(U_2^k = \int_{\mathbb{R}^3} \frac{e^{-\frac{|x-y|^2}{4t}}}{(|x-y| + \sqrt{t})^{3-2a}} (|u_0| (1 - \chi_{B_4(k)})) \, dy\). Using Young’s convolution inequality,
\[
\|U_1^k\|_{L^p(B_1(k))} \lesssim \|U_1^k\|_{L^p(\mathbb{R}^3)} \lesssim \left\| \frac{e^{-\frac{|x|^2}{4t}}}{(|x| + \sqrt{t})^{3-2a}} \right\|_{L^{\frac{3p}{3p-2a}}(\mathbb{R}^3)} \|u_0\|_{L^3(B_4(k))} \lesssim \|u_0\|_{L^3(B_4(k))}.
\]

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Thus,
\[
\left\| \sup_{0 < t < T} \left\| U_1^k \right\|_{L^p(B_1(k))} \right\|_{\ell^1(k \in \mathbb{Z}^3)} \lesssim \left\| u_0 \right\|_{E^3_q}.
\]
On the other hand,
\[
\left\| U_2^k \right\|_{L^p(B_1(k))} \lesssim \sum_{|k'| \geq 1} \frac{e^{-\frac{|k'|^2}{4\epsilon}}}{(|k'| + \sqrt{T})^{3-2a}} \left\| u_0 \right\|_{L^3(B_1(k'-k))}.
\]
So
\[
\left\| \sup_{0 < t < T} \left\| U_2^k \right\|_{L^p(B_1(k))} \right\|_{\ell^1(k \in \mathbb{Z}^3)} \lesssim \left\| \sup_{0 < t < T} \sum_{|k'| \geq 1} \frac{e^{-\frac{|k'|^2}{4\epsilon}}}{(|k'| + \sqrt{T})^{3-2a}} \left\| u_0 \right\|_{L^3(B_1(k'-k))} \right\|_{\ell^1(k \in \mathbb{Z}^3)}
\]
\[
\leq \sum_{|k'| \geq 1} \frac{e^{-\frac{|k'|^2}{4\epsilon}}}{|k'|^{3-2a}} \left\| u_0 \right\|_{L^3(B_1(k'-k))} \lesssim \left\| u_0 \right\|_{E^3_q} \lesssim T^a \left\| u_0 \right\|_{E^3_q}.
\]
In the last inequality we have used \( a > 0 \). Therefore,
\[
\left\| t e^{t\Delta} u_0 \right\|_{E^{\infty, p}_{T,q}} \lesssim \left\| \sup_{0 < t < T} \left\| U_1^k \right\|_{L^p(B_1(k))} \right\|_{\ell^1(k \in \mathbb{Z}^3)} + \left\| \sup_{0 < t < T} \left\| U_2^k \right\|_{L^p(B_1(k))} \right\|_{\ell^1(k \in \mathbb{Z}^3)} \lesssim (1 + T^a) \left\| u_0 \right\|_{E^3_q}.
\]
This shows (3.20). When \( p = 3 \) and \( a = 0 \), since
\[
\left\| e^{-\frac{|k|^2}{4\epsilon}} \right\|_{\ell^1(k \in \mathbb{Z}^3 \setminus \{0\})} \lesssim \ln(2 + T),
\]
the same argument gives (3.21).

**Lemma 3.2.** Let \( d = 3 \) and \( 0 < T < \infty \). For nonnegative \( a, b \) with \( a + b < 1 \) and \( p \in [1, \infty] \), \( \tilde{p} \in (3, \infty) \) with \( \frac{1}{\tilde{p}} - \frac{3}{2p} - b \geq 0 \),
\[
\left\| t^a B(f, g) \right\|_{E^{\infty, \tilde{p}}_{T,q}} \lesssim \left( T^{\frac{1}{2} - \frac{3}{2p} - b} + T^{1-b} \right) \min \left( \left\| t^a f \right\|_{E^{\infty, p}_{T,q}}, \left\| t^b g \right\|_{E^{\infty, p}_{T,q}}, \left\| t^b f \right\|_{E^{\infty, \tilde{p}}_{T,q}} \right).
\]

**Proof.** Decompose in \( B_1(k) \) for \( k \in \mathbb{Z}^3 \),
\[
B_1(f, g)(x, t) = \int_0^t \left( F_1^k(x, t, \tau) + F_2^k(x, t, \tau) \right) d\tau,
\]
with \( F_1^k(x, t, \tau) = \int_{\mathbb{R}^3} \partial_t S_{ij}(x - y, t - \tau)(f_t g_{ij}(X_B(k))(y, \tau)) dy \) and \( F_2^k(x, t, \tau) = \int_{\mathbb{R}^3} \partial_t S_{ij}(x - y, t - \tau)(f_t g_{ij}(1 - X_B(k))(y, \tau)) dy \). By the computation in (2.32) without time integration,
\[
\left\| F_1^k(\cdot, t, \tau) \right\|_{L^p(B_1(k))} \lesssim (t - \tau)^{-\frac{3}{2} \left( \frac{1}{p} - \frac{1}{\tilde{p}} \right) - \frac{1}{2}} \left\| f \otimes g \right\|_{L^r(B_4(k))}, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{\tilde{p}},
\]
\[
\lesssim (t - \tau)^{-\frac{3}{2} \left( \frac{1}{p} - \frac{1}{\tilde{p}} \right) - \frac{1}{2}} \left\| f \right\|_{L^p(B_4(k))} \left\| g \right\|_{L^p(B_4(k))} \left( \tau \right).
\]

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The lemma follows by summing (3.24) without time integration yields

\[
\| F_2^k (\cdot, t) \|_{L^p(B_1(k))} \lesssim \sum_{|k'| \geq 1} \frac{1}{|k'| + \sqrt{t - \tau}} \| (f \otimes g)(\tau) \|_{L^r(B_2(k-k'))}, \quad r \geq 1.
\]

Choosing \( r \) so that \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \), one has

\[
\left\| \sup_{0 < t < T} t^a \int_0^t \| F_2^k (\cdot, t) \|_{L^p(B_1(k))} \ d\tau \right\|_{\ell^q(k \in \mathbb{Z}^3)} \lesssim \sum_{|k'| \geq 1} \frac{1}{|k'| + \sqrt{t - \tau}} \| F_2^k (\cdot, t) \|_{L^p(B_1(k))} d\tau
\]

\[
\lesssim \sup_{0 < t < T} \sum_{|k'| \geq 1} \int_0^t \frac{t^a}{|k'| + \sqrt{t - \tau}} \| f(\cdot, \tau) \|_{L^p(B_2(k-k'))} \| g(\cdot, \tau) \|_{L^q(B_2(k-k'))} d\tau
\]

\[
\lesssim \sum_{|k'| \geq 1} \frac{1}{|k'|} \| t^a f \|_{L^p(B_2(k-k'))} \| t^b g \|_{L^q(B_2(k-k'))}
\]

Using Young’s convolution inequality for the \( \ell^q \) norm above,

\[
\left\| \sup_{0 < t < T} t^a \int_0^t \| F_2^k (\cdot, t) \|_{L^p(B_1(k))} \ d\tau \right\|_{\ell^q(k \in \mathbb{Z}^3)} \lesssim T^{1-b} \left\| \sum_{|k'| \geq 1} \frac{1}{|k'|} \| t^a f \|_{L^p(B_2(k-k'))} \| t^b g \|_{L^q(B_2(k-k'))} \right\|_{\ell^q(k \in \mathbb{Z}^3)}
\]

\[
\lesssim T^{1-b} \left\| \sup_{k \neq 0} \frac{1}{|k|} \| t^a f \|_{L^p(B_2(k))} \| t^b g \|_{L^q(B_2(k))} \right\|_{\ell^q(k \in \mathbb{Z}^3)}
\]

\[
\lesssim T^{1-b} \left\| \| t^a f \|_{L^p(B_2(k))} \| t^b g \|_{L^q(B_2(k))} \right\|_{\ell^q(k \in \mathbb{Z}^3)}
\]

\[
\lesssim T^{1-b} \left\| \| t^a f \|_{E_{T,q}^{\infty,p}} \| t^b g \|_{E_{T,q}^{\infty,p}} \right\|_{\ell^q(k \in \mathbb{Z}^3)}
\]

The lemma follows by summing (3.23) and (3.24), and by switching \( f, g \) in the estimates. \( \square \)
Remark 3.3. It is possible to prove time-weighted space-time integral estimates for the linear and nonlinear terms, with finite time exponent $s < \infty$. For example,

$$\|t^a e^{t \Delta} u_0\|_{E_{T,q}^{s,p}} \lesssim_T \|u_0\|_{E_0^q}, \quad a(s,p) = \frac{1}{2} - \frac{3}{2p} - \frac{1}{s} > 0. \quad (3.25)$$

It extends Lemma 2.4. We limit ourselves here to $s = \infty$.

Proof of spacetime integral estimates in $E_{T,q}^{s,p}$ in Theorem 1.2.

To prove the spacetime integral bound (1.19) for $p = 3, s = \infty$, we work in the spaces with the norms

$$\|f\|_{E_{T,q}^3} = \|f\|_{E_{T,q}^{3,3}} + \|t^{\frac{3}{2}} f\|_{E_{T,q}^{\infty,\infty}}, \quad \|f\|_{F_{T,q}^3} = \|t^{\frac{1}{2}} f\|_{E_{T,q}^{\infty,6}}.$$

We also have $E_{T,q}^3 \subset F_{T,q}^3$ since

$$\|t^{\frac{1}{2}} f\|_{E_{T,q}^{\infty,6}} = \|t^{\frac{1}{2}} \|f\|_{L_1^0(B_t(1))}\|_{L_1^\infty} \|f\|_{L_1^\infty(B_t(1))}\|_{L_1^\infty} \leq \|t^{\frac{1}{2}} \|f\|_{L_1^\infty(B_t(1))}\|_{L_1^\infty} \leq \|t^{\frac{1}{2}} \|f\|_{L_1^\infty(B_t(1))}\|_{L_1^\infty} \leq \|t^{\frac{1}{2}} \|f\|_{L_1^\infty(B_t(1))}\|_{L_1^\infty}.$$

For the linear estimate in $F_{T,q}^3$, taking $p = 6$ so that $a = 1/4$ in Lemma 3.1, we have

$$\|e^{t \Delta} u_0\|_{F_{T,q}^3} = \|t^{\frac{1}{2}} e^{t \Delta} u_0\|_{E_{T,q}^{\infty,6}} \leq C_1^a (1 + T^\frac{3}{2}) \|u_0\|_{E_0^3}. \quad (3.26)$$

For bilinear estimate, taking $a = b = 1/4, p = \tilde{p} = 6$ in Lemma 3.2,

$$\|B(f,g)\|_{F_{T,q}^3} = \|t^{\frac{3}{2}} B(f,g)\|_{E_{T,q}^{\infty,6}} \leq C_2^a (1 + T^\frac{3}{2}) \|t^{\frac{3}{2}} f\|_{E_{T,q}^{\infty,6}} \|t^{\frac{3}{2}} g\|_{E_{T,q}^{\infty,6}} = C_2^a (1 + T^\frac{3}{2}) \|f\|_{F_{T,q}^3} \|g\|_{F_{T,q}^3}.$$

By choosing $\|u_0\|_{E_0^3}$ small enough so that $\|u_0\|_{E_0^3} < \frac{1}{4C_1 C_2^a (1 + T^\frac{3}{2}) (1 + T^\frac{3}{2})}$, Picard iteration yields a unique mild solution satisfying

$$\|u\|_{F_{T,q}^3} \leq 2C_1^a (1 + T^\frac{3}{2}) \|u_0\|_{E_0^3}.$$

Now, we claim that a solution $u \in F_{T,q}^3$ with small enough $u_0 \in E_0^3$ also belongs to $E_{T,q}^3$. By (3.21) of Lemma 3.1 (or by (2.16) with $d = p = r = 3, s = \infty$, and constant $1 + T^\frac{1}{2}$),

$$\|e^{t \Delta} u_0\|_{E_{T,q}^{3,3}} \lesssim \ln(2 + T) \|u_0\|_{E_0^3}.$$

Taking $p = \infty$ so that $a = 1/2$ in Lemma 3.1,

$$\|t^{\frac{3}{2}} u_0\|_{E_{T,q}^{\infty,6}} \lesssim (1 + T^\frac{3}{2}) \|u_0\|_{E_0^3}.$$

Therefore, one has

$$\|e^{t \Delta} u_0\|_{E_{T,q}^3} \lesssim (1 + T^\frac{3}{2}) \|u_0\|_{E_0^3}. \quad (3.27)$$
We next show
\[ \|B(f, g)\|_{E^q_T} \lesssim T \min \left( \|f\|_{E^q_T}, \|g\|_{E^q_T}, \|f\|_{F^{q_2}_q}, \|g\|_{F^{q_2}_q} \right). \]  
(3.28)
Indeed, by choosing \((a, b, p, \tilde{p}) = (0, 1/4, 3, 6)\) and \((a, b, p, \tilde{p}) = (1/2, 1/4, \infty, 6)\) in Lemma 3.2, we get
\[ \|B(f, g)\|_{E^q_{T,q}} \lesssim (1 + T^{\frac{q}{6}}) \min \left( \|f\|_{E^q_{T,q}}, \|t^{\frac{q}{6}}g\|_{E^q_{T,q}}, \|g\|_{E^{q_2}_{T,q}}, \|t^{\frac{q}{6}}f\|_{E^{q_2}_{T,q}} \right) \]
and
\[ \|t^{\frac{q}{6}}B(f, g)\|_{E^{q_2}_{T,q}} \lesssim (1 + T^{\frac{q}{6}}) \min \left( \|t^{\frac{q}{6}}f\|_{E^{q_2}_{T,q}}, \|t^{\frac{q}{6}}g\|_{E^{q_2}_{T,q}}, \|t^{\frac{q}{6}}g\|_{E^{q_2}_{T,q}}, \|t^{\frac{q}{6}}f\|_{E^{q_2}_{T,q}} \right), \]
respectively. By the same argument as before, we can show \(u \in E^s_T\), by taking possibly a smaller \(T > 0\). In particular, \(u \in E^{q_2}_{T,q}\) and its norm is bounded \(\|u\|_{K^q_T}\). The case \(s = \infty\) of Theorem 1.2 then follows from the embeddings \(\|u\|_{L^\infty E^q_T} \leq \|u\|_{E^{q_2}_{T,q}}\) and \(\|t^{\frac{q}{6}}u\|_{L^\infty E^q_T} \leq \|t^{\frac{q}{6}}u\|_{E^{q_2}_{T,q}}\).

3.3 Mild solutions in critical spaces with enough decay

We finally prove Theorem 1.3 for critical data \(u_0 \in E^3_q\) with enough decay, \(1 \leq q \leq 3\). For local wellposedness, we can allow the data to be large thanks to the decay. We will assume smallness for global wellposedness.

**Proof of Theorem 1.3.** Although this proof is very similar to the proof of Theorem 1.2, there are technical modifications throughout which necessitate we revisit the details. By Lemma 2.1, we have for \(1 \leq q \leq 3\),
\[ \sup_{0 < t < \infty} \left( \|e^{t\Delta}u_0\|_{E^3_q} + t^{\frac{q}{6}}\|e^{t\Delta}u_0\|_{E^{q_2}_q} + t^{\frac{q}{6}}\|e^{t\Delta}u_0\|_{E^{q_1}_q} \right) \leq C \|u_0\|_{E^3_q}, \]
where
\[ \frac{1}{q_1} = \frac{1}{q} - \frac{1}{6}, \quad \frac{1}{q_2} = \frac{1}{q} - \frac{1}{3}, \quad q < q_1 < q_2 \leq \infty. \]
(3.30)
Motivated by (3.29), for \(0 < T \leq \infty\) and \(1 \leq q \leq 3\), let \(E_T, F_T\) be Banach spaces defined as
\[ E_T := \left\{ f \in L^\infty(0, T; E^3_q) : t^{\frac{q}{6}}f(\cdot, t) \in L^\infty(0, T; E^{q_2}_q) \right\}, \]
and
\[ F_T := \left\{ f : t^{\frac{q}{6}}f(\cdot, t) \in L^\infty(0, T; E^{q_1}_q) \right\}, \]
(3.32)
with norms
\[ \|f\|_{E_T} := \sup_{0 < t < T} \|f(\cdot, t)\|_{E^3_q} + \sup_{0 < t < T} t^{\frac{1}{q}}\|f(\cdot, t)\|_{E^{q_2}_q} \quad \text{and} \quad \|f\|_{F_T} := \sup_{0 < t < T} t^{\frac{1}{q}}\|f(\cdot, t)\|_{E^{q_1}_q}, \]
respectively. We claim that \(E_T \subset F_T\). Indeed, using \(1/q_1 = 1/(2q_2) + 1/(2q)\),
\[ t^{\frac{q}{6}}\|f(\cdot, t)\|_{E^{q_1}_q} = t^{\frac{q}{6}}\left\|f(\cdot, t)\right\|_{L^6(B_1(k))} \leq t^{\frac{1}{q}}\left\|f(\cdot, t)\right\|_{L^6(B_1(k))} \leq t^{\frac{1}{q}}\left\|f(\cdot, t)\right\|_{L^6(B_1(k))} \leq t^{\frac{1}{q}}\left\|f(\cdot, t)\right\|_{L^6(B_1(k))} \leq \|f(\cdot, t)\|_{E^{q_2}_q} \cdot \|f(\cdot, t)\|_{E^{q_2}_q} \cdot \|f(\cdot, t)\|_{E^{q_2}_q} = (t^{\frac{1}{q}}\|f(\cdot, t)\|_{E^{q_2}_q})^{1/2} \cdot \|f(\cdot, t)\|_{E^{q_2}_q}^{1/2}. \]
The above is true even if \( q = 3 \) and \( q_2 = \infty \).

By Lemma 2.1 again and Hölder inequality (1.5) using \( 2q \geq q_1 \) due to \( q \leq 3 \),

\[
\|B(f, g)\|_{E_q^q}(t) \leq \int_0^t \frac{1}{(t - \tau)^{\frac{2}{3}}} \|f \otimes g(\tau)\|_{E_q^q} d\tau \\
\leq \int_0^t \frac{1}{(t - \tau)^{\frac{2}{3}}} \|f(\tau)\|_{E_{q_1}^q} \|g(\tau)\|_{E_{q_1}^q} d\tau \\
\leq \int_0^t \frac{1}{(t - \tau)^{\frac{2}{3}}} \tau^{-1/4} \|f\|_{F_T} \tau^{-1/4} \|g\|_{F_T} d\tau \\
\leq t^{-1/4} \|f\|_{F_T} \|g\|_{F_T}.
\]

Hence,

\[
\|B(f, g)\|_{F_T} \leq c_* \|f\|_{F_T} \|g\|_{F_T},
\]

where \( c_* \) is a universal constant.

Concerning the caloric extension of \( u_0 \), we have for \( \|u_0\|_{E_q^q} \) of any size that

\[
\lim_{T \to 0} \|e^{t\Delta} u_0\|_{F_T} = 0,
\]

by (2.13) of Lemma 2.3. Hence, there exists \( T = T(u_0) \) so that

\[
\|e^{t\Delta} u_0\|_{F_T} \leq c_*^{-1}.
\] (3.34)

If, on the other hand, \( \|u_0\|_{E_q^q} \leq c_*^{-1} \), then by (3.29), we have (3.34) for \( T = \infty \). The Picard contraction theorem then guarantees the existence of a mild solution \( u \) to (1.3) so that

\[
\|u\|_{F_T} \leq 2 \|e^{t\Delta} u_0\|_{F_T}.
\]

This solution is unique among all mild solutions \( v \) with data \( u_0 \) satisfying \( \|v\|_{F_T} \leq 2 \|e^{t\Delta} u_0\|_{F_T} \).

Next, we show that a solution \( u \in F_T \) with initial data \( u_0 \in E_q^q \) also belongs to \( E_T \). Let \( \{u^{(n)}\}_{n \geq 1} \) be the Picard iteration sequence in \( F_T \). By construction,

\[
\|u^{(n)}\|_{F_T} \leq 2 \|e^{t\Delta} u_0\|_{F_T}.
\] (3.35)

Note that

\[
\|u^{(n)}\|_{E_T} \leq \|e^{t\Delta} u_0\|_{E_T} + \|B(u^{(n-1)}, u^{(n-1)})\|_{E_T}.
\]

We now bound \( B(f, g) \) in \( E_T \) in terms of \( f \) and \( g \) in \( F_T \) and \( E_T \). We have by Lemma 2.1 and Hölder inequality (1.5) using \( q_1 \geq 6 \),

\[
\|B(f, g)\|_{E_q^q}(t) \leq \int_0^t \frac{1}{(t - \tau)^{\frac{2}{3}}} \|f \otimes g(\tau)\|_{E_q^q} d\tau \\
\leq \int_0^t \frac{1}{(t - \tau)^{\frac{2}{3}}} \|f(\tau)\|_{E_{q_1}^q} \|g(\tau)\|_{E_{q_1}^q} d\tau \\
\leq \int_0^t \frac{1}{(t - \tau)^{\frac{2}{3}}} \tau^{-1/4} \|f\|_{F_T} \tau^{-1/4} \|g\|_{F_T} d\tau \\
\leq \|f\|_{F_T} \|g\|_{F_T}.
\]
By $E_T \subset \mathcal{F}_T$, we have

$$\|B(f, g)\|_{E_T^3}(t) \lesssim \|f\|_{E_T^3} \|g\|_{\mathcal{F}_T} \wedge \|g\|_{E_T^3} \|f\|_{\mathcal{F}_T}.$$ 

Also by Lemma 2.1 and Hölder inequality (1.5),

$$\|B(f, g)\|_{E_T^3}(t) \lesssim \int_0^t \frac{1}{(t - \tau)^{4\over 3}} \|f \otimes g\|_{E_T^3}(\tau) \, d\tau$$

$$\lesssim \int_0^t \frac{1}{(t - \tau)^{4\over 3}} (\tau^{1/2} \|f\|_{E_T^3}^{1/4} \|g\|_{E_T^3}^{1/4} \wedge \tau^{1/2} \|g\|_{E_T^3}^{1/4} \|f\|_{E_T^3}) \, d\tau$$

$$\lesssim t^{-2}(\|f\|_{E_T^3} \|g\|_{\mathcal{F}_T} \wedge \|g\|_{E_T^3} \|f\|_{\mathcal{F}_T}).$$

(3.37)

Based on the above estimates we conclude

$$\|B(f, g)\|_{E_T^3} \lesssim \|f\|_{E_T^3} \|g\|_{\mathcal{F}_T} \wedge \|g\|_{E_T^3} \|f\|_{\mathcal{F}_T}.$$ 

(3.38)

We can now conclude that $\{u^{(n)}\}$ is Cauchy in $E_T$ by the calculation preceding and including (3.14). However, the smallness of the constant is now provided by (3.34)-(3.35), not by $\|u_0\|_{E_T^3}$.

We now show continuity. For small data, we can try to inherit continuity from Theorem 1.2. But we will provide a proof valid for general data. We first address convergence to the initial data. By Lemma 2.3 we have

$$\lim_{T' \to 0^+} \sup_{0 < t < T'} t^{4\over 3} \|e^{t\Delta} u_0\|_{E_T^3} = \lim_{T' \to 0^+} \|e^{t\Delta} u_0\|_{\mathcal{F}_{T'}} = 0,$$ 

(3.39)

whenever $u_0 \in E_T^3$. By our estimates in the class $\mathcal{F}_{T'}$ where we are taking $T' \leq T$, we have

$$\|u^{(n)}\|_{\mathcal{F}_{T'}} \leq \|e^{t\Delta} u_0\|_{\mathcal{F}_{T'}} + \|B(u^{(n-1)}, u^{(n-1)})\|_{\mathcal{F}_{T'}} \lesssim \|e^{t\Delta} u_0\|_{\mathcal{F}_{T'}} + \|u^{(n-1)}\|_{\mathcal{F}_{T'}}^2.$$ 

From this and by induction, for any $n$ we have

$$\lim_{T' \to 0^+} \|u^{(n)}\|_{\mathcal{F}_{T'}} = 0.$$

The limit (3.39), convergence of the Picard iterates in $\mathcal{F}_T$ and the above inequality imply that, by taking $T'$ small, we can make sup$_{0 < t < T'} t^{4\over 3} \|u(t)\|_{E_T^3}$ small. To elaborate, we have

$$\|u\|_{\mathcal{F}_{T'}} \leq \|u - u^{(n)}\|_{\mathcal{F}_{T'}} + \|u^{(n)}\|_{\mathcal{F}_{T'}}.$$ 

(3.40)

We may choose $n$ large so that the first term is small and then make the second term small by taking $T'$ small. Hence,

$$\lim_{T' \to 0^+} \|u\|_{\mathcal{F}_{T'}} = 0.$$ 

(3.41)

Using (3.36), this implies

$$\lim_{T' \to 0^+} \sup_{0 < t < T'} \|B(u, u)\|_{E_T^3}(t) = 0.$$

This and Lemma 2.3 imply

$$\lim_{t \to 0} \|u - u_0\|_{E_T^3} = 0.$$ 

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We now prove continuity at positive times using the same argument as in the proof of Theorem 1.2. Let $t_1 > 0$ be fixed. We will send $t \to t_1$. Note that by Lemma 2.3 we have $e^{t \Delta} u_0 - e^{t_1 \Delta} u_0 \to 0$ in $E^3_q$ as $t \to t_1$. We therefore only need to show $B(u, u)(t) \to B(u, u)(t_1)$. Take $\rho$ slightly less than 1 so that $\rho t_1 < t$ and write

$$B(u, u)(t) - B(u, u)(t_1) = \int_{\rho t_1}^t e^{(t-\tau)\Delta} \nabla F d\tau + \int_{\rho t_1}^t e^{(t_1-\tau)\Delta} \nabla F d\tau$$

where $F = u \otimes u(\tau)$. For the first and second terms we have by the sequence of inequalities in (3.36) that,

$$\int_{\rho t_1}^t \|e^{(t-\tau)\Delta} \nabla F\|_{E^3_q} d\tau \lesssim \frac{1}{t-\tau} \frac{1}{t} \frac{1}{2} \frac{1}{2} \|u\|_{F_T}^2 \lesssim \frac{(t-\rho t_1)^{\frac{1}{2}}}{(\rho t_1)^{\frac{1}{2}}} \|u\|_{F_T}^2,$$

and

$$\int_{\rho t_1}^{t_1} \|e^{(t_1-\tau)\Delta} \nabla F\|_{E^3_q} d\tau \lesssim \frac{1}{t_1-\tau} \frac{1}{t_1} \frac{1}{2} \frac{1}{2} \|u\|_{F_T}^2 \lesssim \frac{(t_1-\rho t_1)^{\frac{1}{2}}}{(\rho t_1)^{\frac{1}{2}}} \|u\|_{F_T}^2,$$

both of which can be made arbitrarily small by taking $\rho t_1$ close to $t_1$ and $t$ close to $t_1$.

For the third term we note that by Lemma 2.3, for each $0 < \tau < \rho t_1$, we have

$$\|(e^{(t-\rho t_1)\Delta} - e^{(t_1-\rho t_1)\Delta}) e^{(\rho t_1 - \tau)\Delta} \nabla F(\tau)\|_{E^3_q} \to 0 \text{ as } t \to t_1,$$

which follows from the fact that $e^{(\rho t_1 - \tau)\Delta} \nabla F(\tau) \in E^3_q$, which is a consequence of Lemma 2.1. Additionally,

$$\|(e^{(t-\rho t_1)\Delta} - e^{(t_1-\rho t_1)\Delta}) e^{(\rho t_1 - \tau)\Delta} \nabla F(\tau)\|_{E^3_q} \lesssim \left( \frac{1}{(t-\tau)} + \frac{1}{(t_1-\tau)} \right) \|u\|_{F_T}^2 \in L^1(0, \rho t_1),$$

where integration in $L^1(0, \rho t_1)$ is with respect to $s$. So, by Lebesgue’s dominated convergence theorem,

$$\int_{0}^{\rho t_1} \|(e^{(t-\rho t_1)\Delta} - e^{(t_1-\rho t_1)\Delta}) e^{(\rho t_1 - \tau)\Delta} \nabla F(\tau)\|_{E^3_q} d\tau \to 0 \text{ as } t \to t_1. \quad (3.42)$$

The above show the continuity of $u(t)$ at positive times.

We now prove the spacetime integral bound (1.20) for $p \in (3, 9]$ and $\frac{2}{s} + \frac{3}{p} = 1$. Note that we exclude $p = 3$, i.e., $s = \infty$. By imbedding (1.4), we may assume $m < \infty$. (We do not take $m = q$ since we need $q < m$ for global existence). Denote the Banach space

$$X_T = E_T \cap E^{s,p}_{T,m}.$$

For the linear term, by Lemmas 2.1 and 2.4,

$$\|e^{t \Delta} u_0\|_{X_T} = \sup_{0 < t < T} \|e^{t \Delta} u_0\|_{E^3_q} + \sup_{0 < t < T} t^\frac{1}{2} \|e^{t \Delta} u_0\|_{E^3_{q_2}} + \|e^{t \Delta} u_0\|_{E^{s,p}_{T,m}}$$

$$\leq C_3(1 + T^{\beta_0}) \|u_0\|_{E^3_q}, \quad (3.43)$$
for any \( \beta_0 \in [0, \infty) \) and \( \beta_0 > \alpha_0 = \frac{3}{2m} - \frac{3}{2q} + \frac{1}{s} \). Note that
\[
\lim_{T \to 0^+} \| e^{t\Delta} u_0 \|_{E^{s,p}_{T,m}} = 0. \tag{3.44}
\]

Here is a proof using \( m < \infty \): Denote
\[
a_k(t) = \| e^{t\Delta} u_0 \|_{L^s(0,t;L^p(B_I(k)))}, \quad k \in \mathbb{Z}^3.
\]

For any \( \varepsilon > 0 \), since \( \{ a_k(T) \}_k \in \ell^m \), there is \( N > 0 \) such that \( \sum_{|k| > N} a_k(T)^m \leq \varepsilon^m \). Then, since \( s < \infty \), there is \( t \in (0, T) \) so that
\[
\sup_{|k| \leq N} a_k(t) \leq \frac{\varepsilon}{N^3/m}.
\]

For any \( \tau \in (0, t] \), we have
\[
\sum_{k \in \mathbb{Z}} a_k(\tau)^m \leq \sum_{|k| \leq N} a_k(t)^m + \sum_{|k| > N} a_k(T)^m \leq CN^3 \left( \frac{\varepsilon}{N^3/m} \right)^m + \varepsilon^m = C\varepsilon^m.
\]

Hence \( \| a_k(\tau) \|_{\ell^m(k \in \mathbb{Z}^3)} \leq C\varepsilon \) for all \( \tau \leq t \). This shows (3.44).

For the bilinear term, by Lemma 2.7 with \( s = s/2, \beta = p/2 \), and \( m = \max(1, m/2) \), so that
\[
\sigma = 0, \quad \alpha = \begin{cases} \frac{1}{2} - \frac{3}{2m} - \frac{1}{s}, & \text{if } 2 \leq m \leq \infty, \\ -1 + \frac{3}{2m} - \frac{1}{s}, & \text{if } 1 < m < 2, \end{cases} \tag{3.45}
\]
we have
\[
\| B(f, g) \|_{E^{s,p}_{T,m}} \leq C_4(1 + T^{\beta}) \| f \otimes g \|_{E^{s,p}_{T,m}}
\]
for any \( \beta \in [0, 1 - \frac{1}{s}] \) and \( \beta > \alpha \). Note
\[
\| f \otimes g \|_{E^{s,p}_{T,m}} \leq \| f \|_{E^{s,p}_{T,2m}} \| g \|_{E^{s,p}_{T,2m}} \leq \| f \|_{E^{s,p}_{T,2m}} \| g \|_{E^{s,p}_{T,m}}
\]
no matter \( m \geq 2 \) or \( 1 < m < 2 \). We conclude, also using (3.38),
\[
\| B(f, g) \|_{E^{s,p}_{T,m}} \leq C_4(1 + T^{\beta}) \| f \|_{E^{s,p}_{T,m}} \| g \|_{E^{s,p}_{T,m}}, \tag{3.46}
\]
\[
\| B(f, g) \|_{X_T} \leq 2C_4(1 + T^{\beta}) \| f \|_{X_T} \| g \|_{X_T}.
\]

By (3.44), we can find \( T_1 \in (0, T) \) so that
\[
\| e^{t\Delta} u_0 \|_{E^{s,p}_{T_1,m}} \leq \delta = \left[ 4C_4(1 + T^{\beta}) \right]^{-1}.
\]

Then the Picard sequence \( u^{(k)} \) satisfies \( \| u^{(k)} \|_{E^{s,p}_{T_1,m}} \leq 2\delta \) for all \( k \in \mathbb{N} \), and we get \( \| u \|_{E^{s,p}_{T_1,m}} \leq 2\delta \). Thus, \( u \) satisfies the spacetime integral bound (1.20).

We next show the global \( E^{s,p}_{T,m} \)-estimates when \( u_0 \) is sufficiently small in \( E^{3}_{q} \). We need to avoid \( T \)-dependence in the constants. In other words, we want to choose \( \beta_0 = 0 \) in (3.33) and \( \beta = 0 \) in (3.46).

To choose \( \beta_0 = 0 \) in (3.33), we first require \( \frac{2}{s} + \frac{3}{m} \leq \frac{3}{q} \) which makes \( \alpha_0 = \frac{3}{2m} - \frac{3}{2q} + \frac{1}{s} \leq 0 \) in Lemma 2.4. If \( \frac{2}{s} + \frac{3}{m} = \frac{3}{q} \) then \( \alpha_0 = 0 \) and we require additionally that \( 1 < q < m < \infty \). Then we can take \( \beta_0 = 0 \).
We now consider the conditions for $\beta = 0$ in (3.46). When $m \geq 2$, we assume further that $\frac{2}{s} + \frac{3}{m} \geq 1$ so that $\alpha \leq 0$. We can take $\beta = 0$ when $\alpha = 0$ since the condition $1 < \tilde{m} < m < \infty$ in Lemma 2.7 is met. Note that all conditions when $m \geq 2$, in particular the upper bound $\frac{2}{s} + \frac{3}{m} \geq 1$, are satisfied for $m = p$. Once we have shown $u \in E^{*, p}_{T = \infty, m}$ for one $m$, we have $u \in E^{*, p}_{T = \infty, \tilde{m}}$ for all $\tilde{m} \in [m, \infty]$. Hence the condition $\frac{2}{s} + \frac{3}{m} \geq 1$ can be removed.

We finally consider the $L^p_T E^p_m$ estimate of $u$. For simplicity, we only consider $T = \infty$. Fix $s \in [3, \infty)$ and $q \in [1, 3]$, and let $p$ be given by $\frac{2}{p} + \frac{3}{s} = 1$. By Lemma 2.8 with $\tilde{p} = p/2$, $\tilde{s} = s/2$ and $\tilde{m} \geq 1$ such that $\frac{1}{m} - \frac{1}{\tilde{m}} = \frac{1}{p} - \frac{1}{\tilde{p}} = \frac{1}{2}$, and by Hölder inequality, we get

$$
\|B(f, g)\|_{L^p_T E^p_m} \lesssim \|f \otimes g\|_{L^{\frac{2}{s}}_T E^p_m} \lesssim \|f\|_{L^{\frac{s}{p}}_T E^p_m} \|g\|_{L^{\frac{s}{p}}_T E^p_m},
$$

if $p \geq m$. The condition $\tilde{m} \geq 1$ is the same as $m \geq p' = \frac{p}{p-1}$. Hence we have

$$
\|B(f, g)\|_{L^p_T E^p_m} \lesssim \|f\|_{L^{\frac{s}{p}}_T E^p_m} \|g\|_{L^{\frac{s}{p}}_T E^p_m}, \quad \text{if } p' \leq m \leq p. \quad (3.47)
$$

Denote the set of acceptable $m$ that we can prove $u \in L^s E^p_m$ as $\mathcal{M}(s, q)$. Since $L^s E^p_m \subset L^s E^p_{m_2}$ if $m < m_2$, , the region $\mathcal{M}(s, q)$ of acceptable $m$, if non-empty, is an interval of the form

$$
m < m \leq \infty, \quad \text{or} \quad \tilde{m} \leq m \leq \infty,
$$

with $\tilde{m} = m(s, q) \in [1, \infty]$. We expect $\tilde{m} \geq q$ due to Example 2.6 for the heat equation.

Define $m_1$ by

$$
\frac{1}{m_1} = \frac{1}{q} - \frac{2}{3s}.
$$

As $s < \infty$, $q \leq 3$ and $\frac{1}{p} = \frac{1}{q} - \frac{2}{3s}$, we have

$$
q < m_1 \leq p. \quad (3.49)
$$

By Lemma 2.4 with $d = r = 3$, we have

$$
\|e^{t\Delta} f\|_{L^3_T E^p_m} \lesssim \|f\|_{E^q_3}, \quad (3.50)
$$

with $T$-independent constants, if one of the following holds:

A. $m_1 \leq m \leq s$, (and $m_1 < m \leq s$ if $q = 1$),

E1. $s < p = m$,

E2. $s < m$ and $\frac{1}{q} \geq \frac{5}{3s}$.

If the linear estimate (3.50) holds, and if $p' \leq m \leq p$ so that the bilinear estimate (3.47) holds, then we can prove that the Picard sequence $u^{(n)}$ converges in $L^s E^p_m$, if $\|u_0\|_{E^q}$ is sufficiently small.

Case A has nontrivial $m$ as soon as $m_1 \leq s$, i.e., $\frac{1}{q} \geq \frac{5}{3s}$. Note that we have strict inequality $m_1 < s$ when $q = 1$. By (3.49) and $p' < 2 < s$, the number

$$
m^*(s, q) = \max(p', m_1)
$$

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is in both \([p', p]\) and \([m_1, s]\). Hence the Picard sequence \(u^{(n)}\) converges in \(L^s E_m^p\) for \(m = m^*\), or for \(m\) slightly larger than \(m^*\) when \(q = 1\) and \(3 \leq s \leq 4\) (When \(q = 1\) and \(4 < s < \infty\), we are in region I of Figure 1, and \(m^* = p' > m_1\).) By imbedding, \(u \in L^s E_m^p\) for all \(m > m^*\).

Case \(E_1\) applies when \(3 \leq s < 5\) and we can take \(m = p\). In particular, it covers region III: \(1 \leq q \leq 3 \leq s < 5\) and \(\frac{1}{q} < \frac{s}{5s}\). In region III we define \(m^*(s, q) = p\).

Case \(E_2\) requires \(\frac{1}{q} \geq \frac{s}{3s}\) and does not give smaller \(m\) than Case A.

This completes the proof of \(L^s E_m^p\)-estimates, and concludes the proof of Theorem 1.3.

\[\Box\]

4 Related remarks on weak solutions

4.1 Global weak solutions in \(E_q^2\) for \(1 \leq q < 2\)

In [10] weak solutions are constructed by the first and third authors for data in \(E_q^2\) where \(2 \leq q < \infty\). These solutions are viewed as bridging the gap between Leray weak solutions where the data is in \(L^2 = E_q^2\) and the time-global weak solutions of Lemarié-Rieusset [37] where the data is in \(E^2\). The motivation for [10] was to identify scalings at which properties for Leray solutions, such as eventual regularity, and Lemarié-Rieusset solutions break down. It appears that the construction in [10] can be extended to amalgam spaces \(E_q^2\) where \(1 \leq q < 2\). In this subsection, we extend the range of the exponent \(q\) in [10, Theorem 1.3 (Eventual regularity), Theorem 1.4 (Explicit growth rate), Theorem 1.5 (Global existence)] down to \(q = 1\). The structure of the proofs is similar, but parts of the old proof break down for small \(q\)—for example, in the last display of the proof of [10, Lemma 4.1] (see [10, p.2013]), the first term \(\|K(k) \ast (a^2)\|_{q/2}\) is infinite for \(q\) close to 1. To address this and to simplify the overall argument, we use the fact that the time-scale in Lemma 4.3 can be pushed to large times by passing to large scales. This avoids the iterative extension of a time-local solution to a time-global solution in [10], which involves solving a perturbed system. This idea only works for \(q < 6\) (see Lemma 4.3 for \(1 \leq q < 2\) and [10, Lemma 3.1] for \(2 \leq q < 6\) and so cannot be used in place of the construction in [10] for \(6 \leq q < \infty\).

4.1.1 Eventual regularity

We recall the notation \(u \in \mathcal{N}(u_0)\) in [10, Definition 1.1] meaning that \(u\) is a local energy solution to the Navier-Stokes equations (1.3) in \(\mathbb{R}^3\), and define as in [10, §2] the quantities

\[N^0_{q,R}(u_0) = \frac{1}{R} \left( \sum_{k \in \mathbb{Z}^3} \left( \int_{B_R(kR)} |u_0|^2 \, dx \right)^{q/2} \right)^{2/q}, \quad 1 \leq q < \infty,\]

\[N^0_{\infty,R} = N^0_{R} := \sup_{x_0 \in \mathbb{R}^3} \frac{1}{R} \int_{B_R(x_0)} |u_0|^2 \, dx.\]

By the concavity of \(f(t) = t^p, \; 0 < p < 1\),

\[f(a + b) \leq f(a) + f(b), \quad f(\sum_{i=1}^n a_i) \leq \sum_{i=1}^n f(a_i), \quad \text{if } a_i \geq 0. \tag{4.1}\]

We prove the following lemma corresponding to [10, Lemma 2.2].

**Lemma 4.1.** Let \(1 \leq q < 2\). If \(u_0 \in E_q^2\), then

\[\lim_{R \to \infty} N^0_{R}(u_0) = 0.\]
Proof. Denote \( a_i = \|u_0\|_{L^2(B_{\ell}(i))} \), \( i \in \mathbb{Z}^3 \). Fix \( k = (k_1,k_2,k_3) \), \( k' = (k'_1,k'_2,k'_3) \in \mathbb{Z}^3 \) so that \(|k-k'| \geq 2\). Without loss of generality \(|k_1-k'_1| \geq 2\). Suppose \( i \) satisfies \(|i-kR| < R\), then, because we have \(|k_1R-k'_1R| \geq 2R\), which implies \( R \leq 2R-|i-kR| \leq |i-kR|\), it follows that \(|i-k'\ell| \geq R\). So, sets of the form \( S_k = \{i : |i-kR| < R\} \) form a cover of \( \mathbb{Z}^3 \) in which \( S_k \) only overlaps those \( S_{k'} \) where \(|k'-k| < 2\). Hence, there are 27 sets that can overlap \( S_k \). Using this fact and concavity (4.1) we have for \( R \geq 1\),

\[
N_0^{q,R}(u_0) \leq \frac{C}{R} \left( \sum_{k \in \mathbb{Z}^3} \left( \sum_{|i-kR| < R} a_i^2 \right)^{q/2} \right)^{2/q} \leq \frac{C}{R} \left( \sum_{k \in \mathbb{Z}^3} \sum_{|i-kR| < R} a_i^2 \right)^{2/q} \leq \frac{C}{R} \|u_0\|^2_{E^2_q}.
\]

The lemma follows as \( N_0^{q,R}(u_0) \leq N_0^{q,R}(u_0) \). \( \square \)

Using Lemma 4.1, the following eventual regularity result, which corresponds to [10, Theorem 1.3], is a direct consequence of [9, Theorem 1.2 (1)] (also [10, Theorem 2.1]).

**Theorem 4.2** (Eventual regularity in \( E^2_q \)). Assume \( u_0 \in E^2_q \) where \( 1 \leq q < 2\), is divergence free and \( u \in \mathcal{N}(u_0) \). Then \( u \) has eventual regularity, i.e., there is \( t_1 < \infty \) such that \( u \) is regular at \((x,t)\) whenever \( t \geq t_1 \), and

\[
\|u(\cdot,t)\|_{L^\infty} \lesssim t^{1/2},
\]

for sufficiently large \( t \).

In fact, Theorem 4.2 also follows from [10, Theorem 1.3] without using Lemma 4.1 since \( u_0 \in E^2_q \subset L^2 \) for \( q < 2\). However, Lemma 4.1 will also be useful in the proof of Theorem 4.4 and Lemma 4.7.

### 4.1.2 A priori bounds and explicit growth rate

Now we extend the main tool involved in proving the global existence, an \textit{a priori} estimate [10, Lemma 3.1] for \( 2 \leq q < \infty \), to the range \( 1 \leq q < 2\). Note that, aside from changing the range of \( q \), the only difference here compared to [10] is that we have restricted the permissible values of \( \sigma \) in (4.5) and changed the definitions of \( \lambda_0 \) and \( \lambda_R \). Recall the \( \ell^q \) local energy space \( LE_q(0,T) \) is defined by the norm (1.23).

**Lemma 4.3.** Assume \( u_0 \in E^2_q \) for some \( 1 \leq q < 2 \) is divergence free and that \( u \in \mathcal{N}(u_0) \) satisfies, for some \( T_2 > 0 \),

\[
\|u\|_{LE_q(0,T_1)} < \infty, \quad \text{for all } T_1 \in (0,T_2).
\]

Then there are positive constants \( C_1 \) and \( \lambda_0 < 1 \), both independent of \( q \) and \( R \), such that, for all \( R > 0 \) with \( \lambda_R R^2 \leq T_2 \),

\[
\esssup_{0 \leq t \leq \lambda_R R^2} \left( \int_{B_R(x_0)} \frac{|u|^2}{2} \, dx + \int_0^{\lambda_R R^2} \int_{B_R(x_0 R)} |\nabla u|^2 \, dx \, dt \right)_{L^{q/2}(x_0 \in \mathbb{Z}^3)} \leq C_1 A_{0,q}^0(R),
\]

where

\[
A_{0,q}^0(R) = RN_0^{q,R} = \left( \int_{B_R(x_0)} |u_0(x)|^2 \, dx \right)_{L^{q/2}(x_0 \in \mathbb{Z}^3)}.
\]
and
\[ \lambda_R = \min \left\{ \lambda_0, \lambda_0 R^2, \frac{\lambda_0 R^2}{A_0,q(R)^2} \right\}. \]

Furthermore, for all \( R > 0 \) and \( \sigma \geq \frac{3}{5} \),
\[
\left\| \int_0^{\lambda_R R^2} \int_{B_R(x_0 R)} \left( |u|^\frac{10}{3} + |\pi - c_{Rz_0,R}(t)|^\frac{5}{2} \right) dx dt \right\|_{\ell^\sigma(x_0 \in \mathbb{Z}^3)} \leq C A_{0,q}(R)^\frac{5}{2},
\]
where \( c_{Rz_0,R} \) is a function of time given in the definition of local energy solutions in [10].

The bound (4.4) is stronger than the a priori bound for Leray’s weak solutions, as
\[
\sup_{0 \leq t \leq \lambda_R R^2} \int_{\mathbb{R}^3} \frac{|u|^2}{2} dx + \int_0^{\lambda_R R^2} |u|^2 dx dt \leq \left( \int_0^{\lambda_R R^2} |\nabla u|^2 dx dt \right) \left( \int_0^{\lambda_R R^2} |u|^2 dx dt \right)^{\frac{q}{2}}.
\]

**Proof of Lemma 4.3.** Let \( \phi_0 \in C_0^\infty(\mathbb{R}^3) \) be radial, non-increasing, identically 1 on \( B_1(0) \), supported on \( B_2(0) \), and satisfy \( |\nabla \phi_0(x)| \leq 1 \) and \( |\nabla \phi_0^\frac{1}{2}(x)| \leq 1 \). Let \( R > 0 \) be as in the statement of the lemma. Let \( \phi(x) = \phi_0(x/R) \). Let \( 0 < \lambda \leq 1 \).

We decompose the pressure according to [10, Definition 1.1], namely for a fixed ball \( B_{2R}(\kappa) \) where \( \kappa \in \mathbb{R}^3 \), we write
\[ \pi(x,t) = - \Delta^{-1} \text{div div}[(u \otimes u)\chi_{4R}(x - \kappa)] \]
\[ - \int_{\mathbb{R}^3} (K(x - y) - K(\kappa - y))(u \otimes u)(y,t)(1 - \chi_{4R}(y - \kappa)) dy + c_{\kappa,R}(t) \]
\[ = \pi_1(x,t) + \pi_2(x,t) + c_{\kappa,R}(t). \]

For \( \kappa \in \mathbb{R}^3 \), let
\[ e_{R,\lambda}(\kappa) := \sup_{0 \leq t \leq \lambda R^2} \int |u(t)|^2 \phi(x - \kappa) dx + \int_0^{\lambda R^2} |\nabla u|^2 \phi(x - \kappa) dx dt, \]
and
\[ E_{R,q,\lambda} := \left\| \sup_{0 \leq t \leq \lambda R^2} \int |u(t)|^2 \phi(x - Rk) dx + \int_0^{\lambda R^2} |\nabla u|^2 \phi(x - Rk) dx ds \right\|_{\ell^{q/2}(k \in \mathbb{Z}^3)}. \]

The idea of [10, Proof of Lemma 3.1] is to first bound each \( e_{R,\lambda}(\kappa) \) using the local energy inequality and then bound \( E_{R,q,\lambda} \) by summing the bounds for \( e_{R,\lambda}(\kappa) \). Following [10, Proof of Lemma 3.1] exactly, we obtain
\[
e_{R,\lambda}(\kappa) \leq \int |u_0|^2 \phi(x - \kappa) dx + C\lambda \sum_{\kappa' \in \mathbb{R}^3: |\kappa' - \kappa| \leq 2R} e_{R,\lambda}(\kappa')
\]
\[ + C \lambda^{1/4} \int_{\kappa' \in \mathbb{R}^3: |\kappa' - \kappa| \leq 10R} (e_{R,\lambda}(\kappa'))^{3/2} + \int_0^{\lambda R^2} 2\pi_2 u \cdot \nabla \phi(x - \kappa) dx ds, \]
\[ (4.6) \]
provided $\lambda \leq 1$. Recall $\pi_2$ in $B_{2R}(\kappa)$ is bounded by $R^{-3}\mathbf{K} \ast e_{R,\lambda}(\kappa)$, and the convolution $\mathbf{K} \ast e_{R,\lambda}$ is understood over $R\mathbb{Z}^3$, and, for $x \in R\mathbb{Z}^3$,

$$\mathbf{K}(x) = \frac{1}{|x/R|^4}, \quad \text{if } |x| > 4R; \quad \mathbf{K}(x) = 0 \quad \text{otherwise.}$$

Using the fact that $\pi_2$ in $B_R(k)$ is bounded by $R^{-3}\mathbf{K} \ast e_{R,\lambda}(k)$, the last integral in (4.6) is bounded as

$$\int_0^{\lambda R^2} \int 2\pi_2 u \cdot \nabla \phi(x - \kappa) \, dx \, ds \lesssim \frac{\lambda R^2}{|u|_{L^\infty(0,\lambda R^2; L^2(B_{2R}(\kappa)))}} |\mathbf{K} \ast e_{R,\lambda}(\kappa)|$$

$$\lesssim \frac{\lambda}{R^{1/2}} \left( |u|_{L^\infty(0,\lambda R^2; L^2(B_{2R}(\kappa)))} + |\mathbf{K} \ast e_{R,\lambda}(\kappa)|^2 \right) \quad (4.7)$$

In the above estimates (4.6) and (4.7), the constants do not depend on $q$. Note that the exponent of $|\mathbf{K} \ast e_{R,\lambda}(\kappa)|^2$ is 2, instead of 3/2 in (10, (3.12)), so that when we take its $\ell^q/2$-norm for $1 \leq q$ we can still apply Young’s convolution inequality.

Next, raise both sides of the inequality (4.6) to the power $q/2$ and sum over $\kappa \in R\mathbb{Z}^3$. The left hand side becomes $E_{R,q,\lambda}$. For the non-convolution terms on the right hand side of (4.6) where the last term is replaced by the bound in (4.7), we have

$$\sum_{\kappa \in R\mathbb{Z}^3} \left( \int |u_0|^2 \phi(x - \kappa) \, dx \right)^{q/2} \leq C^q A_{0,q}(R)^{q/2},$$

$$\sum_{\kappa \in R\mathbb{Z}^3} \left( C + \frac{\lambda}{R^{1/2}} \sum_{\kappa' \in R\mathbb{Z}^3; |\kappa' - \kappa| \leq 2R} e_{R,\lambda}(\kappa') \right)^{q/2} \leq C^q \left( \lambda + \frac{\lambda}{R^{1/2}} \right)^{q/2} E_{R,q,\lambda},$$

and

$$\sum_{\kappa \in R\mathbb{Z}^3} \left( C \frac{\lambda^{1/4}}{R^{1/2}} \sum_{\kappa \in R\mathbb{Z}^3; |\kappa' - \kappa| \leq 10R} (e_{R,\lambda}(\kappa'))^{3/2} \right)^{q/2}$$

$$\leq C^q \left( \frac{\lambda}{R^2} \right)^{3/8} \sum_{\kappa \in R\mathbb{Z}^3} \sum_{\kappa \in R\mathbb{Z}^3; |\kappa' - \kappa| \leq 10R} e_{R,\lambda}(\kappa')^{3q/4} \leq C^q \left( \frac{\lambda}{R^2} \right)^{3/8} \sum_{\kappa \in R\mathbb{Z}^3} e_{R,\lambda}(\kappa)^{3q/4}.$$ 

Above we have used $(\sum_{i=1}^n a_i)^p \leq \sum_{i=1}^n a_i^p$ for $a_i \geq 0$ since $0 < q/2 < 1$, see (4.1). Also note,

$$\sum_{\kappa \in R\mathbb{Z}^3} e_{R,\lambda}(\kappa)^{3q/4} = \|e_{R,\lambda}\|_{\ell^{3q/4}}^q \leq \|e_{R,\lambda}\|_{\ell^q} \leq \|e_{R,\lambda}\|_{\ell^{q/2}} = E_{R,q,\lambda}^{3/4}.$$

For the convolution term we use Young’s convolution inequality to obtain

$$\sum_{\kappa \in R\mathbb{Z}^3} \left( \frac{C \lambda}{R^{1/2}} \right)^{q/2} \left( \mathbf{K} \ast e_{R,\lambda}(\kappa) \right)^{q/2} = \left( \frac{C \lambda}{R^{1/2}} \right)^{q/2} \sum_{\kappa \in R\mathbb{Z}^3} |\mathbf{K} \ast e_{R,\lambda}(\kappa)|^q$$

$$\leq \left( \frac{C \lambda}{R^{1/2}} \right)^{q/2} \|e_{R,\lambda}\|_{\ell^q} \|e_{R,\lambda}\|_{\ell^{q/2}} \leq \left( \frac{C \lambda}{R^{1/2}} \right)^{q/2} E_{R,q,\lambda}.$$
where we used the fact that \( \|R\|_{L^1(\mathbb{R}^3)} \) is bounded independently of \( R \).

We conclude that, for some constant \( C_2 \geq 1 \) independent of \( q, R \), we have

\[
E_{R,q,\lambda} \leq C_2^q A_{0,q}(R)^{q/2} + C_2^q \left( \lambda + \frac{\lambda}{R^{1/2}} \right)^{q/2} E_{R,q,\lambda} + C_2^q \left( \frac{\lambda}{R^{1/2}} \right)^{q/2} E_{R,q,\lambda} \tag{4.8}
\]

This is achieved if

\[
\lambda \leq \lambda_R := \min \left\{ \lambda_0, \frac{\lambda_0 R^2}{A_{0,q}(R)^2} \right\}, \quad \lambda_0 = cC_2^{-12}
\]

for some positive \( c < 1 \). This shows the first estimate (4.4) of Lemma 4.3 with \( C_1 = CC_2^2 \). Note that the constants \( C_2, \lambda_0 \) and \( C_1 \) do not depend on \( q \) and \( R \).

We now show (4.5). Denote \( N = \sup_{0 \leq t \leq \lambda R^2} \int_{B_R} |u(t)|^2 \, dx + 2 \int_0^{\lambda R^2} \int_{B_R} |\nabla u|^2 \, dx \, dt \) with \( \lambda = \lambda_R \).

We have by the Gagliardo-Nirenberg inequality

\[
\int_0^{\lambda R^2} \int_{B_R} |u(t)|^{10} \, dx \, dt \leq N^{2/3} \int_0^{\lambda R^2} \left( \int_{B_R} |\nabla u|^2 \right) \, dt + R^{-2} N^{5/3} \lambda R^2 \\
\leq N^{5/3} + \lambda N^{5/3} \lesssim N^{5/3},
\]

using \( \lambda \leq 1 \). For \( k \in R^3 \) and \( Q(k) = B_R(k) \times (0, \lambda R^2) \), by the preceding estimate with \( B_R \) replaced by \( B_R(k) \), we have \( N(B_R(k)) \leq e_{R,\lambda}(k) \) and hence, for \( \sigma > 0 \) satisfying \( \frac{5\sigma}{2} \geq \frac{q}{2} \) (which is implied if \( \sigma \geq 3/5 \)), we have

\[
\sum_{k \in R^3} \left( \int_{Q(k)} |u|^{10} \, dx \, dt \right)^{\sigma} \leq C \sum_{k \in R^3} e_{R,\lambda}(k)^{\frac{5\sigma}{2}} \leq C \left( \sum_{k \in R^3} e_{R,\lambda}(k) \right)^{\frac{2}{3} \frac{5\sigma}{2}} \leq C E_0^q \frac{5\sigma}{3}.
\]

By Calderon-Zygmund estimates,

\[
\sum_{k \in R^3} \left( \int_{Q(k)} |\pi_1|^{q} \, dx \, dt \right)^{\sigma} \leq C \sum_{k \in R^3} \sum_{k' \in R^3 : |k-k'| < 10R} \left( \int_{Q(k')} |u|^{10} \, dx \, dt \right)^{\sigma} \leq C E_0^q \frac{2\frac{5\sigma}{q}}{3}.
\]

We now address the second component of the pressure. Recall \( \pi_2 \) in \( B_R(k) \) is bounded by \( R^{-3}R \ast e_{R,\lambda}(k) \). Hence,

\[
\int_{Q(k)} |\pi_2|^{\frac{q}{2}} \, dx \, dt \leq C \lambda (R \ast e_{R,\lambda}(k))^\frac{q}{2}.
\]
Thus, if \( \frac{5q}{3} \geq 1 \), which is implied by our assumptions, then we have

\[
\sum_{k \in \mathbb{R}^3} \left( \int_{Q(k)} |\pi_2|^\frac{2}{3} \, dx \, dt \right)^{\sigma} \leq C \lambda^{\sigma} \sum_{k \in \mathbb{R}^3} (K * e_{R,\lambda}(k))^{\frac{5q}{3}} \\
\leq C \lambda^{\sigma} \|K\|_{L^2}^{\frac{5q}{3}} \sum_{k \in \mathbb{R}^3} (e_{R,\lambda}(k))^{\frac{5q}{3}} \leq C E_0^{\frac{2}{3}}. 
\]

We conclude that

\[
\left\| \int_{Q(k)} |u|^\frac{5q}{3} + |\pi_1 + \pi_2|^\frac{2}{3} \, dx \, dt \right\|_{L^{\sigma}(k \in \mathbb{R}^3)} \leq C E_0^{\frac{2}{3}} = CA_{0,q}(R) \frac{\sigma}{3}.
\]

This shows (4.5).

By (4.2), we have \( A_{0,q}(R) \lesssim \|u_0\|_{E_q^2}^2 \). By Lemma 4.3, we immediately have the following theorem of explicit growth rate that corresponds to [10, Theorem 1.4].

**Theorem 4.4** (Explicit growth rate in \( E_q^2 \)). Assume \( u_0 \in E_q^2 \) where \( 1 \leq q < 2 \), is divergence free and \( u \in \mathcal{N}(u_0) \) satisfies, for some \( T_2 > 0 \),

\[
\|u\|_{\text{LE}_{q}(0,T_1)} < \infty, \quad \forall T_1 \in (0,T_2).
\]

Then, for any \( R \geq 1 \), with \( T = \min \left( \lambda_1 (1 + \|u_0\|_{E_q^2})^{-4} R^2, T_2 \right) \), we have

\[
\left\| \text{ess sup}_{0 \leq t \leq T} \int_{B_R(Rk)} |u|^2 \, dx + \int_0^T \int_{B_R(Rk)} |\nabla u|^2 \, dx \, dt \right\|_{L^{\sigma/2}(k \in \mathbb{R}^3)} \leq C \|u_0\|_{E_q^2},
\]

for positive constants \( \lambda_1 \) and \( C \) independent of \( u_0 \) and \( R \). In particular, if \( T_2 = \infty \) then \( T \to \infty \) as \( R \to \infty \).

### 4.1.3 Global existence

To prove global existence we follow the approach of [9, Theorem 1.5], (modified from [35, §3]), via the localized and regularized Navier-Stokes equations

\[
\begin{align*}
\partial_t u^\epsilon - \Delta u^\epsilon + (\mathcal{J}_\epsilon(u^\epsilon) \cdot \nabla)(u^\epsilon \Phi^\epsilon) + \nabla \pi^\epsilon &= 0, \\
\text{div} \, u^\epsilon &= 0,
\end{align*}
\]

where \( \mathcal{J}_\epsilon(f) = \eta_\epsilon * f \) for a spatial mollifier \( \eta_\epsilon(x) = e^{-3} \eta(x/\epsilon) \) and \( \Phi_\epsilon(x) = \Phi(\epsilon x) \) for a fixed radially decreasing cutoff function \( \Phi \) satisfying \( \Phi = 1 \) on \( B_1(0) \) and \( \text{supp}(\Phi) \subset B_{3/2}(0) \).

Next we construct a mild solution in \( \text{LE}_{q}(0,T) \), defined in (1.23), of the regularized Navier-Stokes equations (4.10). The following lemma corresponds to [35, Lemma 3.3].

**Lemma 4.5.** Let \( q \geq 1 \). For each \( 0 < \epsilon < 1 \) and \( u_0 \) with \( \text{div} \, u_0 = 0 \) and \( \|u_0\|_{E_q^2} \leq B \), if \( 0 < T < \min(1, c^3 B^{-2}) \), we can find a unique solution \( u = u^\epsilon \) to the integral form of (4.10)

\[
u(t) = e^{t \Delta} u_0 - \int_0^t e^{(t-\tau) \Delta} \mathbb{P} \nabla \cdot (\mathcal{J}_\epsilon(u) \otimes u \Phi_\epsilon)(\tau) \, d\tau
\]

satisfying

\[
\|u\|_{\text{LE}_{q}(0,T)} \leq 2C_0 B,
\]

where \( c > 0 \) and \( C_0 > 1 \) are absolute constants.
Proof. Let $\Psi(u)$ be the map defined by the right side of (4.11) for $u \in \mathbf{L}E_q(0, T)$. By Lemma 2.9 and $T \leq 1$,

$$
\|\Psi(u)\|_{\mathbf{L}E_q(0, T)} \lesssim \|u_0\|_{E^q_2} + \|\mathcal{J}_\epsilon(u) \otimes u\Phi_\epsilon\|_{E^2_{T,q}}
$$

$$
\lesssim \|u_0\|_{E^q_2} + \|\mathcal{J}_\epsilon(u)\|_{L^\infty(0, T; L^\infty(\mathbb{R}^3))} \|u\|_{E^2_{T,q}}
$$

$$
\lesssim \|u_0\|_{E^q_2} + \epsilon^{-\frac{3}{2}} \sqrt{T} \|u\|^2_{E^\infty_{T,q}}.
$$

Thus

$$
\|\Psi(u)\|_{\mathbf{L}E_q(0, T)} \leq C_0 \|u_0\|_{E^q_2} + C_1 \epsilon^{-\frac{3}{2}} \sqrt{T} \|u\|^2_{\mathbf{L}E_q(0, T)},
$$

for some constants $C_0, C_1 > 0$. Similarly, for $u, v \in \mathbf{L}E_q(0, T)$,

$$
\|\Psi(u) - \Psi(v)\|_{\mathbf{L}E_q(0, T)} \leq C_1 \epsilon^{-\frac{3}{2}} \sqrt{T} \left( \|u\|_{\mathbf{L}E_q(0, T)} + \|v\|_{\mathbf{L}E_q(0, T)} \right) \|u - v\|_{\mathbf{L}E_q(0, T)}.
$$

By the Picard contraction theorem, if $T$ satisfies

$$
T < \frac{\epsilon^3}{64(C_0C_1)^2} = \epsilon^3 B^{-2},
$$

then we can find a unique fixed point $u \in \mathbf{L}E_q(0, T)$ of $u = \Psi(u)$, i.e., (4.11), satisfying $\|u\|_{\mathbf{L}E_q(0, T)} \leq 2C_0B$.

The following lemma corresponds to [35, Lemma 3.4].

**Lemma 4.6.** Let $u_0 \in E^q_2$, $q \geq 1$, with div $u_0 = 0$. For each $\epsilon \in (0, 1)$, we can find $u^\epsilon$ in $\mathbf{L}E_q(0, T)$ and $\pi^\epsilon$ in $L^\infty(0, T; L^2(\mathbb{R}^3))$ for some positive $T = T(\epsilon, \|u_0\|_{E^q_2})$ which solve the regularized Navier-Stokes equations (4.10) in the sense of distributions, and $u^\epsilon(t) \to u_0$ in $L^2(E)$ as $t \to 0^+$ for any compact subset $E$ of $\mathbb{R}^3$.

**Proof.** The proof is nearly identical to [35, Proof of Lemma 3.4] except that Lemma 4.5 ($E^2_q$-version of [35, Lemma 3.3]) is applied and that the first displayed estimate in [35, Proof of Lemma 3.4] is adjusted as

$$
\left\|u^\epsilon - e^{\tau \Delta}u_0\right\|_{E^2_{T,q}} = \left\|\int_0^t e^{(t-\tau)\Delta} \mathbf{P} \nabla \cdot \left(\mathcal{J}_\epsilon(u) \otimes u\Phi_\epsilon\right)(\tau) \, d\tau\right\|_{E^2_{T,q}}
$$

$$
\lesssim \|\mathcal{J}_\epsilon(u) \otimes u\Phi_\epsilon\|_{E^2_{T,q}} \lesssim \epsilon^{-\frac{3}{2}} \sqrt{T} \|u\|^2_{E^\infty_{T,q}},
$$

where we have used Lemma 2.9 and are assuming $t' \leq T \leq 1$.

We next show global existence for the regularized system (4.10). The following lemma corresponds to [9, Lemma 3.3]. Thanks to Lemma 4.1, the decay condition [9, (1.12)] on initial data is implied by $u_0 \in E^q_2$, $q < 2$.

**Lemma 4.7.** Assume $u_0 \in E^q_2$, $1 \leq q < 2$, is divergence free, and fix $\epsilon \in (0, 1)$. Then, there exists a global solution $(u^\epsilon, \pi^\epsilon)$ to the regularized Navier-Stokes equations (4.10) such that $u^\epsilon \in \mathbf{L}E_q(0, T)$ for any $T < \infty$, and $(u^\epsilon, \pi^\epsilon)$ satisfies the a priori bounds (4.4)-(4.5) in Lemma 4.3 for all $R = n \in \mathbb{N}$ up to time $T_n$ defined in (4.12).
Proof. We will take radius \( R = n \in \mathbb{N} \). By (4.2) of Lemma 4.1, \( A_{0,q}(n) \leq c_3 \|u_0\|_{E^q_2}^2 \) for all \( n \). For \( n \in \mathbb{N} \), let

\[
T_n = \lambda_0 n^2 \min \left\{ 1, n^2 (c_3 \|u_0\|_{E^q_2}^2)^{-2} \right\} \leq \lambda_n n^2,
\]

(4.12)

where \( \lambda_0 \) and \( \lambda_n \) are defined in Lemma 4.3. The sequence \( T_n \) is increasing and \( T_n \to \infty \). By the same proof of Lemma 4.3, if a solution \( u^\varepsilon, \pi^\varepsilon \) of (4.10) satisfies \( u^\varepsilon \in \mathbf{LE}_q(0,T) \), then it satisfies the a priori bounds (4.4) and (4.5) for \( R = n \) up to time \( \min(T,T_n) \).

As the system (4.10) can be considered as a Stokes system with localized and regularized source, it has a global unique solution \( (u^\varepsilon, \pi^\varepsilon) \). Its uniqueness makes it agree with the \( \mathbf{LE}_q \)-solution of Lemma 4.6, hence \( u^\varepsilon \in \mathbf{LE}_q(0,T) \) for some \( T = \tau(\varepsilon, \|u_0\|_{E^q_2}) > 0 \). Fix \( n \in \mathbb{N} \). By (4.4) with \( R = n \), \( \|u^\varepsilon(\tau)\|_{E^q_2} \leq C(n) \|u_0\|_{E^q_2} \). By Lemma 4.6, there is an \( \mathbf{LE}_q \)-solution on \( (\tau, \tau + \tau_1) \) for some \( \tau_1 = \tau_1(\varepsilon, C(n) \|u_0\|_{E^q_2}) > 0 \). By uniqueness, it agrees with \( u^\varepsilon \) and we have \( u^\varepsilon \in \mathbf{LE}_q(0,\tau + \tau_1) \) with the a priori bound (4.4) up to \( \tau + \tau_1 \). We can repeat this extension to show \( u^\varepsilon \in \mathbf{LE}_q(0,\tau + k\tau_1) \), \( k \in \mathbb{N} \), until \( \tau + k\tau_1 \geq T_n \). We conclude, for each \( n \in \mathbb{N} \), \( u^\varepsilon \in \mathbf{LE}_q(0,T_n) \) and satisfies the a priori bound (4.4) for \( R = n \) up to time \( T_n \).

As \( T_n \to \infty \), the lemma is proved.

With the above lemmas, we are ready to consider the Navier-Stokes equations with \( E^2_q \) data, \( 1 \leq q < 2 \), and construct a global-in-time local energy solution in \( \mathbf{LE}_q(0,T) \).

Proof of Theorem 1.4. For \( k \in \mathbb{N} \), let \( u^k \) be the solution of the regularized system (4.10) with \( \varepsilon = 1/k \), given in Lemma 4.7. They share the same a priori bound (4.4) for \( R = n \) up to time \( T_n \), thus

\[
\sup_{k \in \mathbb{N}} \|u^k\|_{\mathbf{LE}_q(0,T_n)} < \infty, \quad \forall n \in \mathbb{N}.
\]

Using this a priori bound, we can construct the desired global solutions as the limit of \( u^k \) defined in \( (0,T_k) \), \( T_k \to \infty \), in the same manner as [9, Proof of Theorem 1.5]. The only difference is that all the supremums \( \sup_{x_0 \in \mathbb{R}^3} \) are replaced by \( \|\cdot\|_{\ell^p(x_0 \in \mathbb{R}^3)} \). For example, [9, (3.7)] is replaced by the \( \ell^q \)-version, (4.9). To avoid redundancy, we omit further details.

### 4.2 Application to the stability of suitability for the perturbed Navier-Stokes equations

In [10], the first and third authors constructed time-global local energy solutions with initial data in the \( L^2 \)-based Wiener amalgam classes \( E^2_q \), \( 2 \leq q < \infty \), which is related to work in [37, 32, 35, 9, 8, 16]. As a technical step, a suitable weak solution was needed solving the perturbed Navier-Stokes equations

\[
\partial_t u - \Delta u + u \cdot \nabla u + v \cdot \nabla u + u \cdot \nabla v + \nabla \pi = 0; \quad \nabla \cdot u = 0, \quad \text{in} \quad \mathbb{R}^3 \times (0,\infty),
\]

(4.13)

and \( v \in L^\infty L^p_{uloc} \) where \( p > 3 \) is itself a solution to (1.3) in \( \mathbb{R}^3 \). It is natural to ask if an analogous statement holds when \( v \) is a small \( L^3_{uloc} \) solution in the sense of Maekawa and Terasawa [38] and our Theorem 1.2. The new spacetime estimates in Theorem 1.2 allow us to confirm this, a claim which is explored presently.

Note that in the global existent proof for \( E^2_q \) where \( 1 \leq q < 2 \) the perturbed system is not needed; so, this subsection is independent of the work done above.

The following is an exact copy of [10, Lemma 4.4] except the condition

\[
\text{ess sup}_{0 < t \leq T_0} \|v(t)\|_{L^4_{uloc}} < \infty,
\]
has been replaced by scaling invariant space-time integral which is finite for the solutions constructed in Theorem 1.2.

**Proposition 4.8.** Let \( c_0 \) and \( \lambda_0 \) be the constants in [10, Lemma 4.4] Assume \( u_0 \in E_q^2 \) \( 2 \leq q < \infty \) is divergence free, and \( v: \mathbb{R}^3 \times [0, T_0] \rightarrow \mathbb{R}^3 \) satisfies \( \text{div} \, v = 0 \) and

\[
\text{ess sup}_{0 < t \leq T_0} \|v(t)\|_{L_{\text{loc}}^3} < \delta \leq c_0; \quad \|v\|_{E^{5,5}_{q,0,\infty}} < \infty.
\]

Let \( T = \min(T_0, \lambda_0, \lambda_0 A_{0,q}^{-2}) \). Then, there exists a weak solution \( u \) and pressure \( \pi \) satisfying (4.13) in the distributional sense on \( \mathbb{R}^3 \times [0, T] \). Furthermore, \( u \) and \( \pi \) are a local energy solution to the perturbed Navier-Stokes equations (4.13) satisfying

\[
\|u\|_{\text{LE}_q(0,T)} \leq C\|u_0\|_{E_q^2},
\]

for a constant \( C \).

**Proof of Proposition 4.8.** We adopt all of the notation and setup of [10, Proof of Lemma 4.4]. The only difference in the present proof compared to [10, Proof of Lemma 4.4] involves the local energy inequality, which should hold for non-negative test functions \( \phi \in C_c^\infty(\mathbb{R}^3 \times [0, T]) \). The steps in [10, pp. 2017-2018] apply here except for those using the assumption that \( \text{ess sup}_{0 < t \leq T_0} \|v(t)\|_{L^4_{\text{loc}}} < \infty \), namely the estimation of \( I_{1,n} \) and \( I_{3,n} \). Note for any ball \( B \subset \mathbb{R}^3 \) and taking \( \{u_n\} \) to be the sequence defined on [10, p. 2017], \( u_n \rightarrow u \) weakly in \( L^2(0, T_0; L^5(B)) \) and strongly in \( L^2(0, T_0; L^4(B)) \), \( \frac{4}{3} + \frac{4}{q} = \frac{3}{2} \), \( q < 6 \). Letting \( B \) be a ball containing the spatial support of \( \phi(t) \) for all \( 0 \leq t \leq T_0 \), we have,

\[
I_{1,n} = \left| \int_0^{T_0} \int ((u_n - u) \cdot \nabla u_n) \cdot (v \phi) \, dx \, dt \right| 
\leq \phi \|u - u_n\|_{L^{10/3}(B \times (0,T_0))} \|v\|_{E^{5,5}_{q,0,\infty}} \|\nabla u_n\|_{L^2(B \times (0,T_0))} \rightarrow 0.
\]

For the remaining term we have that

\[
I_{3,n} = \left| \int_0^{T_0} \int (u_n \cdot \nabla u_n) \cdot (\eta \ast v - v)\phi \, dx \, dt \right| 
\leq \|u_n\|_{L^{10/3}(B \times (0,T_0))} \|\eta \ast v - v\|_{L^5(0,T_0; L^5(B))} \|\nabla u_n\|_{L^2(B \times (0,T_0))}.
\]

Recall from [10] that \( \eta \ast v \) is a spatial mollifier applied to \( v \). So, by properties of spatial mollifiers and the dominated convergence theorem, the middle term above vanishes. This and the work in [10, pp. 2017-2018] establishes the local energy inequality for \( \phi \in C_c^\infty(\mathbb{R}^3 \times [0, T]) \).

Note that without the spacetime integral estimate we would only have the inclusion \( v \in L^\infty(0, T_0; L^3_{\text{loc}}) \) for our drift velocity. The best this gives is when bounding \( I_{1,n} \) is

\[
\|\nabla u_n\|_{L^2(B \times (0,T_0))} \|u_n v - uv\|_{L^2(B \times (0,T_0))} 
\leq \|\nabla u_n\|_{L^2(B \times (0,T_0))} \|u_n - u\|_{L^2(0,T_0; L^6)} \|v\|_{L^\infty(0,T_0; L^3_{\text{loc}})},
\]

which is not implied to vanish since we only have weak convergence in \( L^2 \tilde{H}^1_{\text{loc}} \). This is why the assumption \( v \in L^\infty L^4_{\text{loc}} \) is included in [10, Lemma 4.4]. On the other hand, the bound for \( I_{3,n} \) requires a space-time Lebesgue norm is finite with finite spatial integrability exponent in order to apply the dominated convergence theorem.

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Remark 4.9. In [10, p. 2018], it is asserted that \((\eta_\varepsilon \ast v)\phi \to v\phi\) in \(L^\infty(0,T;L^3_{uloc})\). This appears to be false. In the reference, and adopting its notation, this can be fixed using the following estimate

\[
\int_0^t \int (u_n \cdot \nabla u_n) \cdot (\eta_\varepsilon \ast v - v)\phi \, dx \, ds 
\leq \frac{1}{4} \|u_n\|_{L^\infty(0,T;L^2_{uloc})}^{1/4} \|\eta_\varepsilon \ast v - v\|_{L^4(B)}^{7/4} \int_0^t \|u_n\|_{L^4(B)}^{7/4} \|\eta_\varepsilon \ast v - v\|_{L^8(0,T;L^4(B))} \, ds
\]

The last quantity above vanishes as \(\varepsilon \to 0\) by the dominated convergence theorem and the fact that \(v \in L^\infty(0,T;L^4_{uloc})\). Since the other terms are uniformly bounded in \(n\), the left hand side vanishes as well.

5 Appendix

In this appendix, we first give details of the endpoint case of Giga’s estimates, and then give examples showing the strict inclusions between \(L^p, E^p_q, E^{p,p}_{r,q}\) and \(L^r_q, E^p_q\).

5.1 Endpoint case of Giga’s estimates

It is mentioned in [21, Acknowledgments] that the case \(r = s > 1\) in (1.11) is also valid if one appeals to the generalized Marcinkiewicz theorem. Here we give the details. Let \(\Omega\) be a bounded smooth open set in \(\mathbb{R}^d\) or \(\mathbb{R}^d\) itself. Let \(A = -\Delta\) or the Stokes operator. For fixed \(p \in (1,\infty)\), the subadditive map \(U(a) = \|e^{-tA}a\|_p\) for \(a \in L^r(\Omega)\) or \(a \in L^r_s(\Omega)\),

\[U : L^r(\Omega) \to L^s_{wk}(0,\infty)\]

is of weak type \((r, s)\) if \(2/s = d/r - d/p\), by the \(L^r-L^p\) estimate of \(e^{-tA}\). By Marcinkiewicz’s theorem in Lorentz spaces, Theorem V.3.15 of Stein-Weiss [42, p.197],

\[\|U(a)\|_{L^{r,m}(0,\infty)} \lesssim \|a\|_{L^{r,m}(\Omega)}, \quad \forall m \in [1,\infty], \quad \forall a \in L^r \cap L^{r,m},\]

when \(r, s \in (1,\infty)\) satisfy \(2/s = d/r - d/p\). In particular, when \(r = m \leq s\),

\[\|U(a)\|_{L^s(0,\infty)} = \|U(a)\|_{L^{s,r}(0,\infty)} \leq \|U(a)\|_{L^{s,r}(0,\infty)} \lesssim \|a\|_{L^{r,r}(\Omega)} = \|a\|_{L^r(\Omega)}\]

Marcinkiewicz’s theorem for subadditive maps in Lorentz spaces for full range of exponents was first shown by Hunt [28] and Calderón [13]. Indeed, the theorem of Hunt is for quasi-linear maps: \(|T(f + g)| \leq K(|Tf| + |Tg|)\). A subadditive map is when \(K = 1\). There is a proof for linear maps in Lorentz spaces in Bergh-Löfström [7, Theorem 5.3.2]. See [28, 13, 29], [7, p.20] and [42, p.216] for more references.

The case \(s = \infty\) in (1.11) is a direct consequence of the \(L^p\) estimate of Stokes semigroup:

\[\|e^{-tA}a\|_p \lesssim \|a\|_p.\]

5.2 Strict inclusions between functional spaces

Lemma 5.1. Consider \(L^p(\mathbb{R}^d)\) and \(E^p_q(\mathbb{R}^d)\) for \(p, q \in [1,\infty]\). When \(p > q\),

\[E^p_q \subset (L^p \cap L^q)\].

(5.1)
In contrast, when \( p < q \), then
\[
L^p + L^q \subset E^p_q. \tag{5.2}
\]
Both inclusions are strict.

**Proof.** Both inclusions are straight-forward to verify. We now show they are strict. Let
\[
a(x) = \sum_{k \in \mathbb{Z}^d} c_k \phi \left( \frac{x - k}{\delta_k} \right),
\]
for \( \phi = \chi_{B_{1/2}} \) and \( c_k, \delta_k \in (0, 1] \). We have
\[
\|a\|_{E^p_q} = c_{p,q} \left( \sum_{k \in \mathbb{Z}^d} c_k^q \delta_k^{dq/p} \right)^{1/q}. \tag{5.3}
\]

When \( p > q \), using (5.3), we have \( a \in (L^p \cap L^q) \setminus E^p_q \) if
\[
\sum_{k \in \mathbb{Z}^d} c_k^p \delta_k^d < \infty, \quad \sum_{k \in \mathbb{Z}^d} c_k^q \delta_k^{dq/p} = \infty.
\]
We may take for example
\[
c_k = 1, \quad \delta_k = (1 + |k|)^{-\frac{q}{p}}.
\]
This shows that the inclusion (5.1) is strict.

When \( p < q \), using (5.3), we have \( a \in E^p_q \), \( a \not\in L^p \) and \( a \not\in L^q \) if
\[
\sum_{k \in \mathbb{Z}^d} c_k^p \delta_k^d = \infty, \quad \sum_{k \in \mathbb{Z}^d} c_k^q \delta_k^{dq/p} < \infty.
\]
We may take for example
\[
c_k = 1, \quad \delta_k = (1 + |k|)^{-1}.
\]
For this choice of \( c_k \) and \( \delta_k \), we claim \( a \in E^p_q \setminus (L^p + L^q) \). Suppose \( a = f + g \) with \( f \in L^p \) and \( g \in L^q \). We may assume \( f \geq 0 \) by replacing \( f, g \) by
\[
\tilde{f} = f \chi_{\{f > 0\}}, \quad \tilde{g} = a - f \chi_{\{f > 0\}} = f \chi_{\{f < 0\}} + g.
\]
Note \( 0 \leq \tilde{f} \leq f \in L^p \) and, where \( f < 0, 0 \leq a = \tilde{g} \leq g \in L^q(f < 0) \), and where \( f \geq 0, \tilde{g} = g \in L^q(f \geq 0) \). Similarly we may assume \( g \geq 0 \). For \( k \in \mathbb{Z}^d \), let \( \varepsilon_k = 1 \) if the set \( \{ f > \frac{1}{3} \} \cap B_{\delta_k/2}(k) \) has measure at least \( \frac{1}{3} \) of \( B_{\delta_k/2} \), and \( \varepsilon_k = 0 \) otherwise. Also let \( \mu_k = 1 \) if the set \( \{ g > \frac{1}{3} \} \cap B_{\delta_k/2}(k) \) has measure at least \( \frac{1}{3} \) of \( B_{\delta_k/2} \), and \( \mu_k = 0 \) otherwise. Since \( a = 1 \) on \( B_{\delta_k/2}(k) \), we have
\[
\varepsilon_k + \mu_k \geq 1, \quad \forall k \in \mathbb{Z}^d. \tag{5.4}
\]
Let
\[
F(x) = \sum_{k \in \mathbb{Z}^d} c_k \varepsilon_k \phi \left( \frac{x - k}{\delta_k} \right), \quad G(x) = \sum_{k \in \mathbb{Z}^d} c_k \mu_k \phi \left( \frac{x - k}{\delta_k} \right).
\]
By definition of $\varepsilon_k$, $\|f\|_{L^p(B_t(k))} \geq C \|F\|_{L^p(B_t(k))}$ for all $k$, hence $\|f\|_{L^p(\mathbb{R}^d)} \geq C \|F\|_{L^p(\mathbb{R}^d)}$. Similarly, $\|g\|_{L^q(\mathbb{R}^d)} \geq C \|G\|_{L^q(\mathbb{R}^d)}$. However, by (5.3), $c_k = 1$, and (5.4),

$$
\int_{\mathbb{R}^d} |f|^p + |g|^q \geq C \|F\|_{L^p(\mathbb{R}^d)}^p + C \|G\|_{L^q(\mathbb{R}^d)}^q
= C \sum_{k \in \mathbb{Z}^d} (c_k \varepsilon_k)^p \delta_k^d + C \sum_{k \in \mathbb{Z}^d} (c_k \mu_k)^q \delta_k^d
= C \sum_{k \in \mathbb{Z}^d} \varepsilon_k \delta_k^d + C \sum_{k \in \mathbb{Z}^d} \mu_k \delta_k^d
\geq C \sum_{k \in \mathbb{Z}^d} \delta_k^d = \infty,
$$

which is a contradiction to $f \in L^p$ and $g \in L^q$. This shows that the inclusion (5.2) is strict. 

**Example 5.2.** Here we show that (1.9) and (1.10) may fail if the order of $q$ and $s$ are switched. For (1.9), take for example $s = p = 1$, $q = \infty$, and $T = 1$. We have $\|u\|_{L^1_{T} \cap L^1_{u_{\text{loc}}}} \leq \|u\|_{L^1_{T} \cap L^1_{u_{\text{loc}}}}$ by (1.10). Let

$$
u(x, t) = \sum_{k \in \mathbb{N}} \frac{2^k}{|B_1|} \cdot 1_{B_1(2ke_1)}(x) \cdot 1_{(2^{-k}, 2^{-k+1})}(t). \quad (5.5)
$$

then $\|u(t)\|_{L^1_{u_{\text{loc}}}} = 2^k$ for $t \in (2^{-k}, 2^{-k+1})$. Hence

$$\|u\|_{L^1_{T = 1, u_{\text{loc}}}}^1 = 1, \quad \|u\|_{L^1_{T = 1, u_{\text{loc}}}}^1 = 1 = \infty, \quad \|u\|_{L^1_{T = 1, u_{\text{loc}}}}^1 \geq C \|u\|_{L^1_{T = 1, u_{\text{loc}}}}^1.
$$

This shows the failure of (1.9) if $q > s$. As another example, take $p = q = 1$ and $s = \infty$. We have $\|u\|_{L^\infty_{T} E_1^1} \leq \|u\|_{L^\infty_{T_{u_{\text{loc}}}}}$ by (1.9). Let $x_0(t) = (\cot t, 0, \ldots, 0)$ and

$$u(x, t) = 1 \text{ if } |x - x_0(t)| < 1; \quad u(x, t) = 0 \text{ otherwise.} \quad (5.6)
$$

Then $\|u\|_{L^\infty_{T_{u_{\text{loc}}}} E_1^1} < \infty$ while $\|u\|_{L^\infty_{T_{u_{\text{loc}}}}} = \infty$. This shows the failure of (1.10) if $q < s$.

**Acknowledgments**

Bradshaw was supported in part by the Simons Foundation (635438). The research of both Lai and Tsai was partially supported by the NSERC grant RGPIN-2018-04137. Lai acknowledges support by the Simons Foundation Math + X Investigator Award #376319 (Michael I. Weinstein). A preliminary version of this paper was presented in a RIMS (Research Institute for Mathematical Sciences, Kyoto University) workshop on December 7, 2021, and received fruitful questions. We thank Professor Yoshikazu Giga for sharing two insightful remarks and referring us to several related papers.

**Disclosure statement**

The authors report there are no competing interests to declare.
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