Research Article

Some Remarks on Fixed Point Theorems for Interpolative Kannan Contraction

Youssef Errai, El Miloudi Marhrani, and Mohamed Aamri

Laboratory of Algebra, Analysis and Applications (L3A), Faculty of Sciences Ben M’Siik, Hassan II University of Casablanca, B.P 7955, Sidi Othmane, Casablanca, Morocco

Correspondence should be addressed to El Miloudi Marhrani; marhrani@gmail.com

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In this paper, we use interpolation to obtain fixed point and common fixed point results for a new type of Kannan contraction mappings in complete metric and $b$-metric spaces. Our results extend and improve some results on fixed point theory in the literature. We also give some examples to illustrate the given results.

1. Introduction and Preliminaries

It is well known that fixed point theory played a central role in various scientific fields. The well-known result in this area is undoubtedly the famous Banach contraction principle (see [1]) which motivated researchers to find other forms of contractions. In this line, we cite the well-known Kannan contraction that does not require continuous mapping.

**Definition 1** (see [2]). Let $E$ be a metric space. A self-mapping on $E$ is said to be a Kannan contraction if there exists $\lambda \in [0, 1/2]$ such that

$$d(Tx, Ty) \leq \lambda (d(x, Tx) + d(y, Ty)),$$

for all $x, y \in E$.

Kannan obtained the following theorem.

**Theorem 2** (see [2]). If $(E, d)$ is a complete metric space, then every Kannan contraction on $E$ has a unique fixed point.

In 2018, Karapinar published a new type of contraction obtained from the definition of the Kannan contraction by interpolation as follows.

**Definition 3** (see [3]). Let $(E, d)$ be a metric space. A self-mapping $T : E \to E$ is said to be an interpolative Kannan-type contraction if there are two constants $\lambda \in [0, 1]$ and $\alpha \in [0, 1]$ such that

$$d(Tx, Ty) \leq \lambda (d(x, Tx))^{\alpha} (d(y, Ty))^{1-\alpha},$$

for all $x, y \in E$ with $x \neq Tx$ and $y \neq Ty$.

Karapinar obtained the following result.

**Theorem 4** (see [3]). Let $(E, d)$ be a complete metric space and $T : E \to E$ be an interpolative Kannan-type contraction mapping. Then, $T$ has a fixed point.

This theorem has been generalized by Noorwali [4] who proposed a common fixed point for two maps, and in 2019, Gabba and Karapinar defined the $(\lambda, \alpha, \beta)$-interpolative Kannan contraction as follows.

**Definition 5** (see [5]). A self-mapping $T$ on a metric space $(E, d)$ is called $(\lambda, \alpha, \beta)$-interpolative Kannan contraction if $\lambda \in [0, 1], \alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$ and such that

$$d(Tx, Ty) \leq \lambda (d(x, Tx))^{\alpha} (d(y, Ty))^{\beta},$$

for all $x, y \in E$ with $x \neq Tx$ and $y \neq Ty$. 
And they gave the following theorem.

**Theorem 6** (see [5]). Let \((X, d)\) be a complete metric space and let \(T : X \to X\) be an \((\lambda, \alpha, \beta)\)-interpolative Kannan contraction. Then, \(T\) has a fixed point in \(X\).

The interpolative method has been used by several researchers to obtain generalizations of other forms of contractions (see [6, 7]).

In this paper, we discuss the results of [4], and we give some generalizations for existence of fixed points for \((\lambda, \alpha, \beta)\)-interpolative Kannan contraction on complete metric spaces and complete \(b\)-metric spaces.

Let us recall some basic results of \(b\)-metric spaces:

**Definition 7** (see [8, 9]). Let \(X\) be a nonempty set and \(s \geq 1\) be a given real number. A function \(d : X \times X \to \mathbb{R}^+\) is a \(b\)-metric if for all \(x, y, z \in X\), the following conditions are satisfied:

\[
\begin{align*}
(b_1) & \quad d(x, y) = 0 \text{ if and only if } x = y; \\
(b_2) & \quad d(x, y) = d(y, x); \\
(b_3) & \quad d(x, z) \leq s(d(x, y) + d(y, z)).
\end{align*}
\]

The \((X, d, s)\) is called a \(b\)-metric space.

**Definition 8** (see [10]). Let \((X, d)\) be a \(b\)-metric space

- (a) \(\{x_n\}\) is \(b\)-convergent in \((X, d)\) if there exists \(x \in X\) such that \(d(x_n, x) \to 0\) as \(n \to \infty\). In this case, we write \(\lim_{n \to \infty} x_n = x\).
- (b) \(\{x_n\}\) is a \(b\)-Cauchy sequence in \((X, d)\) if \(d(x_n, x_m) \to 0\) as \(n, m \to \infty\).
- (c) \((X, d)\) is \(b\)-complete if every \(b\)-Cauchy sequence in \((X, d)\) \(b\)-converges.

**Definition 9** (see [11]). Let \((X, d)\) be a \(b\)-metric space and \(g, h : X \to X\). An element \(x\) in \(X\) is called a coincidence point of \(g\) and \(h\) if \(gx = hx\).

**Lemma 10** (see [12]). Let \(\{x_n\}\) be a sequence in a \(b\)-metric space \((X, d, s)\) with \(s \geq 1\) such that

\[
\frac{1}{s^2} d(x, s) \leq \liminf_{n \to \infty} d(x_n, y_n) - \limsup_{n \to \infty} d(x_n, y_n) \leq s^2 d(x, y) .
\]

Moreover, for each \(z \in X\), we have

\[
\frac{1}{s^2} d(x, z) \leq \liminf_{n \to \infty} d(x_n, z) - \limsup_{n \to \infty} d(x_n, z) \leq s d(x, z) .
\]

**2. Discussion and Results**

Using the \((\lambda, \alpha, \beta)\)-interpolative Kannan contraction defined as above, we give our first main result.

**Theorem 12.** Let \((X, d)\) be a complete metric space and \(T\) is a self-mapping on \(X\) such that

\[
d(Tx, Ty) \leq \lambda(d(x, Tx))^\alpha(d(y, Ty))^\beta,
\]

for all \(x, y \in X\) with \(x \neq Tx\) and \(y \neq Ty\), and where \(\lambda \in [0, 1]\) and \(\alpha, \beta \in [0, 1]\) such that \(\alpha + \beta \geq 1\). If there exists \(x \in X\) such that \(d(x, Tx) \leq 1\), then \(T\) has a fixed point in \(X\).

**Proof.** For the case \(\alpha + \beta = 1\), see Karapinar et al. [3]. Assume \(\alpha + \beta > 1\) and define a sequence \(\{x_n\}\) by \(x_0 = x\) and \(x_{n+1} = T x_n\), for all integer \(n\) and assume that \(x_n \neq Tx_n\) for all \(n\).

We have

\[
d(x_n, x_{n+1}) \geq \lambda d(x_n, x_{n-1})^\alpha d(x_n, x_{n+1})^\beta,
\]

for all \(n \geq 1\).

(8)

Since \(d(x_0, x_1) \leq 1\), we obtain by (8) \(d(x_1, x_2) \leq \lambda\). Assume that there exists a real \(\gamma(n)\) such that \(d(x_n, x_{n+1}) \leq \lambda^\gamma(n)\); we obtain

\[
d(x_{n+1}, x_{n+2}) \leq \lambda(d(x_n, x_{n+1}))^\alpha(d(x_{n+1}, x_{n+2}))^\beta,
\]

which gives

\[
(d(x_{n+1}, x_{n+2}))^{1-\beta} \leq \lambda(d(x_n, x_{n+1}))^\alpha \leq \lambda^\alpha \gamma(n).
\]

(10)

It follows that

\[
d(x_{n+1}, x_{n+2}) \leq \lambda^\gamma(n+1),
\]

(11)

where \(\gamma(n+1) = 1/(1-\beta)(1+\alpha \gamma(n))\), for all \(n \geq 1\) with \(\gamma(0) = 0\) and \(\gamma(1) = 1\).

Since \(\alpha/(1-\beta) > 1\), we have \(\lim_{n \to \infty} \gamma(n) = +\infty\). It follows that

\[
\sum_{n=0}^{+\infty} d(x_n, x_{n+1}) \leq \sum_{n=0}^{+\infty} \lambda^\gamma(n),
\]

(12)

which is convergent, and consequently, \(\{x_n\}\) is a Cauchy sequence in \((X, d)\). Thus, \(\{x_n\}\) converges to some \(\bar{x} \in X\). Assume that \(\bar{x} \neq Tx\), we obtain, by (1):

\[
d(x_{n+1}, Tx) \leq \lambda(d(x_n, Tx_n))^\alpha(d(\bar{x}, Tx))^\beta,
\]

for all \(n\).

If \(n \to +\infty\), we obtain \(d(\bar{x}, Tx) = 0\), which is a contradiction. Then, \(Tx = \bar{x}\).
Example 13. Let $X = \{x, y, z, w\}$ be set endowed with the metric $d$ defined by
\[
d(x, x) = d(y, y) = d(z, z) = d(w, w) = 0,
\]
\[
d(x, y) = 3, d(x, z) = 4, d(x, w) = \frac{5}{2}, d(y, y) = 3, d(y, w) = 2, d(z, w) = \frac{3}{2},
\]
(14)

and define the self-mapping $T$ on $X$ by
\[
Tx = x, \, Ty = w, \, Tz = x, \, Tw = y.
\]
(15)

For $\lambda = 99/100, \alpha = 5/8$, and $\beta = 1/2$, we have
\[
d(Tu, Tv) \leq \lambda (d(u, Tu))^\alpha (d(v, Tv))^\beta,
\]
(16)

for all $u, v \in X - \{x\}$. Moreover, $T$ has a fixed point in $X$.

In [4], Noorwali gave the following result.

Theorem 14. Let $(\mathcal{M}, d)$ be a complete metric space and $S, T : \mathcal{M} \to \mathcal{M}$ be self-mappings. Assume that there are two constants $\lambda \in [0, 1], \alpha \in [0, 1]$ such that the condition
\[
d(Tp, Sq) \leq \lambda (d(p, Tp))^\alpha (d(q, Sq))^\beta,
\]
is satisfied for all $p, q \in \mathcal{M}$ such that $Tp \neq p$ and $Sq \neq q$. Then, $S$ and $T$ have a unique common fixed point.

The uniqueness is not true in the case $S = T$; moreover, in the proof, the author considers the sequence defined by $p_0 \in \mathcal{M}, p_{2n+1} = Tp_{2n}$, and $p_{2n+2} = Sp_{2n+1}$. In the case where there is no three consecutive identical terms in the sequence $\{p_n\}$, the author uses the inequality
\[
d(p_{2n+1}, p_{2n+2}) \leq \lambda (d(p_{2n+1}, p_{2n+1}))^\alpha (d(p_{2n+1}, p_{2n+2}))^\beta.
\]
(18)

We note that the above inequality is not valuable if $p_{2n} = p_{2n+1}$ or $p_{2n+1} = p_{2n+2}$. Moreover, the following example shows that the theorem is not true in this form.

Example 15. If $S$ is the identity map on $M$, it is clear that the result is not valid for any mapping $T$ without a fixed point.

Example 16. Let $M = \{p, q, r, s\}$ be endowed with the metric defined by the following Table 1 of values:

|   | p | q | r | s |
|---|---|---|---|---|
| p | 3 | 4 | 5 | 2 |
| q | 3 | 0 | 4 | 2 |
| r | 4 | 3 | 0 | 3 |
| s | 5 | 2 | 3 | 0 |

Table 1: Table of metric $d$.

for all $x, y \in M$ with $Sx \neq x$ and $Ty \neq y$, but $S$ and $T$ have no common fixed point in $M$.

As an alternative of this theorem, we give the following result.

Theorem 17. Let $(M, d)$ be a complete metric space, $S, T : M \to M$ be two self-mappings on $M$. Assume that there are some $\lambda \in [0, 1], \alpha \in [0, 1]$ such that the followings conditions hold:

(i) $d(Sp, Tq) \leq \lambda (d(p, Sp))^\alpha (d(q, Tq))^\beta$, for all $p, q \in M$ with $Sp \neq p$ and $Tq \neq q$

(ii) $d(Sp, Sq) \leq \lambda (d(p, Sp))^\alpha (d(q, Sq))^\beta$, for all $p, q \in M$ with $Sp \neq p$ and $Sq \neq q$

(iii) $d(Tp, Tq) \leq \lambda (d(p, Tp))^\alpha (d(q, Tq))^\beta$, for all $p, q \in M$ with $Tp \neq p$ and $Tq \neq q$

Then, $S$ and $T$ have a common fixed point.

Proof. Let $x \in M$; we define by induction a sequence $\{x_n\}$ by
\[
x_0 = xx_{2n+1} = Sx_{2n}, \text{ and } x_{2n+2} = Tx_{2n+1}, \text{ for all } n.
\]
We shall discuss the following cases:

(a) $d(x_n, x_{n+1}) > 0$, for all $n$. Then, by the same arguments as in the proof of Theorem 2.1 in [4], we can prove that $\{x_n\}$ is a Cauchy sequence in $M$ which converges to a common fixed point of $S$ and $T$. Note that this common fixed point is not necessarily unique.

(b) There exists $n_0$ such that $x_{n_0} = x_{n_0+1} = x_{n_0+2}$. Then, $x_{n_0}$ is a common fixed point of $S$ and $T$.

(c) Assume that $\{x_n\}$ is without three consecutive identical terms, but it contains a subsequence $\{x_{n_k}\}$ such that $x_{n_k} = x_{n_k+1}$, for all $k \in \mathbb{N}$. We have the following situations

(c1) Let $p \in \mathbb{N}$ such that $x_p = x_{p+1}$; if $p = 2n$ for some integer $n$, we have
\[
Tx_{2n-1} = x_{2n} \neq x_{2n-1} \text{ and } Tx_{2n+1} = x_{2n+2} \neq x_{2n+1}.
\]
(21)
It follows that
\[
d(x_{2n+1}, x_{2n+2}) = d(Tx_{2n-1}, Tx_{2n+1}) \\
\leq \lambda d(x_{2n-1}, x_{2n})^\alpha (d(x_{2n+1}, x_{2n+2}))^{1-\alpha},
\]
(22)
and then
\[
d(x_{2n+1}, x_{2n+2}) \leq \lambda^{1/\alpha} d(x_{2n-1}, x_{2n}) \leq \lambda d(x_{2n-1}, x_{2n}).
\]
(23)
And if $p = 2n - 1$ for some integer $n$, we have
\[
Sx_{2n-2} = x_{2n-1} \neq x_{2n-2}, Sx_{2n} = x_{2n+1} \neq x_{2n}.
\]
(24)
By (ii), we obtain
\[
d(x_{2n+1}, x_{2n+2}) = d(x_{2n-1}, x_{2n+1}) \\
= d(Sx_{2n-2}, Sx_{2n}) \\
\leq \lambda d(x_{2n-2}, x_{2n-1})^\alpha (d(x_{2n+1}, x_{2n+2}))^{1-\alpha}.
\]
(25)
It follows that
\[
d(x_{2n+1}, x_{2n+2}) \leq \lambda^{1/\alpha} d(x_{2n-2}, x_{2n-1}).
\]
(26)
From (23) and (26) and since $\lambda^{1/\alpha} \leq \lambda$, we obtain
\[
x_p = x_{p+1} \Rightarrow d(x_{p+1}, x_{p+2}) \leq \lambda d(x_{p-1}, x_p).
\]
(27)
(c2) Assume that $x_{p-1} \neq x_p$ and $x_p \neq x_{p+1}$; if $p = 2n$, for some $n \in \mathbb{N}$, we obtain
\[
T_{x_p} \neq x_{p-1}, Sx_p \neq x_p.
\]
(i) \(\Rightarrow\) \(d(x_p, x_{p+1}) = d(Tx_{p-1}, Sx_p) \leq \lambda (d(x_p, x_{p+1}))^\alpha (d(x_{p-1}, x_p))^{1-\alpha},\)
(28)
which leads to
\[
d(x_p, x_{p+1}) \leq \lambda^{\alpha(1-\alpha)} d(x_{p-1}, x_p) \leq \lambda d(x_{p-1}, x_p).
\]
(29)
And if $p = 2n + 1$, for some $n \in \mathbb{N}$, we obtain
\[
T_{x_p} = x_{p+1} \neq x_p, Sx_{p-1} \neq x_{p-1}.
\]
(30)
By (i), we obtain
\[
d(x_p, x_{p+1}) = d(Sx_{p-1}, Tx_p) \leq \lambda (d(x_{p-1}, x_p))^\alpha (d(x_p, x_{p+1}))^{1-\alpha},
\]
(31)
which leads to
\[
d(x_p, x_{p+1}) \leq \lambda d(x_{p-1}, x_p).
\]
(32)
The inequalities (29) and (32) imply that
\[
d(x_p, x_{p+1}) \leq \lambda d(x_{p-1}, x_p),
\]
(33)
where $p$ is such that $x_{p-1} \neq x_p$ and $x_p \neq x_{p+1}$.
In view of (27) and (33), we obtain
\[
d(x_p, x_{p+1}) \leq \lambda \max \left( d(x_{p-2}, x_{p-1}), d(x_{p-1}, x_p) \right),
\]
(34)
for all $p \in \mathbb{N}$.
Let $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ defined by $\varphi(p) = p$ if $x_p \neq x_{p+1}$ and $\varphi(p) = p + 1$ if $x_p = x_{p+1}$. It is clear that $\varphi$ is monotone nondecreasing and the sequences \(\{x_p\}\) and \(\{x_{\varphi(p)}\}\) have the same set of values. Moreover, $x_{\varphi(p)} \neq x_{\varphi(p+1)}$, for all $p \in \mathbb{N}^*$, and we have
\[
d(x_{\varphi(p)}, x_{\varphi(p+1)}) \leq \lambda d(x_{\varphi(p-1)}, x_{\varphi(p)}),
\]
(35)
for all $p \in \mathbb{N}$. It follows that \(\{x_{\varphi(n)}\}\) is a Cauchy sequence, and then \(\{x_n\}\) is also a Cauchy sequence. And since $M$ is complete, there exists $u \in M$ such that $\lim_{n \to \infty} d(x_n, u) = 0$.
(c3) Assume that $Su \neq u$; we have the following cases: First case: there exists $n_0$ such that $x_{2n_0} = x_{2n_0+1}$, for all $n \geq n_0$. Thus, $Tx_{2n-1} \neq x_{2n-1}$, for all $n \geq n_0$. And we obtain
\[
d(Su, Tx_{2n-1}) \leq \lambda (d(u, Su))^\alpha (d(x_{2n-1}, x_{2n}))^{1-\alpha}.
\]
(36)
It follows that $\lim_{n \to \infty} d(Su, x_{2n}) = d(Su, u) = 0$, which is a contradiction.
Second case: there exists a subsequence \(\{x_{2n_k}\}\) with $x_{2n_k} \neq x_{2n_{k+1}}$, for all $k$; we have $Sx_{2n_k} \neq x_{2n_{k+1}}$, for all $k$. Thus,
\[
d(Su, Sx_{2n_k}) \leq \lambda (d(u, Su))^\alpha (d(x_{2n_k}, x_{2n_{k+1}}))^{1-\alpha}.
\]
(37)
It follows that $\lim_{k \to \infty} d(Su, x_{2n_{k+1}}) = d(Su, u) = 0$, which is a contradiction. Then, $Su = u$.
By identical arguments, we can prove that $Tu = u$, which ends the proof.

Remark 18. If $S = T$, the conditions (i), (ii), and (iii) are identical, and we obtain Theorem 2.2 of [3].
Example 19. Let $X = [0, +\infty]$; we define on $X$ a metric $d$ and two self-mappings $S$ and $T$ by
\[
d(x, y) = \begin{cases} x + y, & \text{if } x \neq y, \\ 0, & \text{if } x = y, \end{cases}
\]
\[
Sx = \begin{cases} 1, & \text{if } x \in [0, 2], \\ e^{x^2}, & \text{if } x \in [2, +\infty], \end{cases}
\]
\[
Tx = \begin{cases} 1, & \text{if } x \in [0, 2], \\ 1/y, & \text{if } x \in [2, +\infty]. \end{cases}
\]
For $\lambda = 24/25$ and $\alpha = 1/2$, we have

For the first case: $x, y \in [0, 2]$; the inequality (39) is obvious.

Second case: $x \in [0, 2]$ and $y \in [2, +\infty]$ with $x \neq 1$; we have
\[
d(Sx, Ty) = 1 + \frac{1}{y} \leq \frac{3}{2},
\]
\[
\lambda(d(Sx, Sx))^\alpha(d(y, Ty))^{-1/\alpha} = \frac{24}{25} \left(1 + \frac{1}{y}\right)^{1/2} \geq \frac{3}{2}.
\]

Third case: $x \in [2, +\infty]$ and $y \in [0, 2]$ with $y \neq 1$; we have
\[
d(Sx, Ty) = e^{x^2} + 1 \leq e^{-2} + 1,
\]
\[
\lambda(d(Sx, Sx))^\alpha(d(y, Ty))^{-1/\alpha} = \frac{24}{25} \left(1 + \frac{1}{y}\right)^{1/2} \geq \frac{3}{2}.
\]

Fourth case: $x, y \in [2, +\infty]$; we have
\[
d(Sx, Ty) = e^{x^2} + 1 \leq e^{-2} + 1,
\]
\[
\lambda(d(Sx, Sx))^\alpha(d(y, Ty))^{-1/\alpha} = \frac{24}{25} \left(1 + \frac{1}{y}\right)^{1/2} \geq \frac{3}{2}.
\]

Then, equation (39) is satisfied.

(ii) To prove
\[
d(Sx, Sy) \leq \lambda(d(Sx, Sx))^\alpha(d(y, Ty))^{-1/\alpha},
\]
we consider the following cases:

First case: $x, y \in [0, 2]$; the inequality (43) is obvious.

Second case: $x \in [0, 2]$ and $y \in [2, +\infty]$ with $x \neq 1$; we have
\[
d(Sx, Sy) = 1 + e^{-y} \leq 1 + e^{-2},
\]
\[
\lambda(d(Sx, Sx))^\alpha(d(y, Ty))^{-1/\alpha} = \frac{24}{25} \left(1 + e^{-y}\right)^{1/2} \geq \frac{3}{2}.
\]
assume that $Tu \neq u$; we have
\[
\lambda d(x_{n+1}, Tu) \leq \lambda (d(x_n, u) + d(x_n, x_{n+1}))^{1-a}(d(u, Tu))^{1-a},
\]
for all $n$. Then, by Lemma 11, we have
\[
\frac{1}{s} d(u, Tu) \leq \limsup_{n} (d(x_n, x_{n+1}))^{1-a}(d(u, Tu))^{1-a} = 0,
\]
which is a contradiction, and then $u$ is a fixed point of $T$.

**Definition 21.** Let $(X, d, s)$ be a $b$-metric space and $T, g : X \to X$ be self-mappings on $X$. We say that the mapping $T$ is an $g$-interpolative Kannan-type contraction if there exist two constants $\lambda \in [0, 1)$ and $\alpha \in (0, 1]$ such that
\[
d(Tx, Ty) \leq \lambda [d(gx, Tx)]^{1-a}[d(gy, Ty)]^{1-a},
\]
for all $x, y \in X$ with $Tx \neq gx$ and $Ty \neq gy$.

**Theorem 22.** Let $(X, d, s)$ be a $b$-complete a $b$-metric space and $T$ a $g$-interpolative Kannan-type contraction. Assume that $TX \subseteq gX$ and $gX$ is closed. If there exist $\lambda, \alpha \in [0, 1]$ such that (53) holds. Then, $T$ and $g$ have a coincidence point in $X$.

**Proof.** Let $x \in X$; since $TX \subseteq gX$, we can define inductively a sequence $\{x_n\}$ such that $x_0 = x$, and $gx_{n+1} = Tx_n$ for all integer $n$.

If there exists $n \in \{0, 1, 2, \cdots\}$ such that $gx_n = Tx_n$, then $x_n$ is a coincidence point of $g$ and $T$. Assume that $gx_k \neq T x_k$ for all $n$. By (53), we obtain
\[
d(Tx_{n+1}, Tx_n) \leq \lambda [d(gx_n, Tx_n)]^{1-a}[d(gx_n, Tx_n)]^{1-a} = \lambda [d(Tx_n, Tx_{n+1})]^{1-a}[d(Tx_{n-1}, Tx_n)]^{1-a}.
\]
Thus,
\[
d(Tx_{n+1}, Tx_n) \leq \lambda^{1/(1-a)} d(Tx_n, Tx_{n-1}) \leq \lambda d(Tx_n, Tx_{n-1}).
\]

By Lemma 11, we obtain
\[
\frac{1}{s} d(z, Tu) \leq \lambda \limsup_{n} (d(gx_n, Tx_n))^{1-a}[d(gu, Tu)]^{1-a} = 0,
\]
which is a contradiction. Then, $z = gu = Tu$.

**Example 23.** Let $X = [0, +\infty]$ and $d$ the $b$-metric defined on $X$ by
\[
d(x, y) = \begin{cases} (x + y)^2, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}
\]
$(X, d)$ is a complete $b$-metric space.

Define two self-mappings $T$ and $g$ on $X$ by $g(x) = x^2$, for all $x \in X$ and
\[
T_x = \begin{cases} 1, & \text{if } x \in [0, 2], \\ \frac{1}{x^2}, & \text{if } x \in [2, +\infty]. \end{cases}
\]
$T$ is a $g$-interpolative Kannan contraction for $\lambda = 3/4$ and $\alpha = 2/5$. For this, we discuss the following cases:

(i) If $x, y \in [0, 2]$, then $d(Tx, Ty) = 0 \leq \lambda [d(gx, Tx)]^{1-a}[d(gy, Ty)]^{1-a} = \lambda (x^2 + \frac{1}{y^2})^{2/(1-a)} \geq \frac{3}{4} (\frac{9}{2})^{2/(1-a)} \geq d(Tx, Ty)$.

(ii) If $x, y \in [2, +\infty]$ with $x \neq y$, we have
\[
d(Tx, Ty) = \left(1 + \frac{1}{y^2}\right)^2 = 1.
\]
\[
\lambda [d(gx, Tx)]^{1-a}[d(gy, Ty)]^{1-a} = \lambda (x^2 + \frac{1}{y^2})^{2/(1-a)} \geq \frac{3}{4} (\frac{9}{2})^{2/(1-a)} \geq d(Tx, Ty).
\]

(iii) If $x \in [0, 2]$ and $y \in [2, +\infty]$ with $x \neq 1$, we have
\[
d(Tx, Ty) = d\left(1 + \frac{1}{y}\right)^2 = \left(1 + \frac{1}{y}\right)^2 \leq \left(\frac{3}{2}\right)^2,
\]
\[
\lambda [d(gx, Tx)]^{1-a}[d(gy, Ty)]^{1-a} = \lambda (x^2 + \frac{1}{y^2})^{2/(1-a)} \geq \frac{3}{4} (\frac{9}{2})^{6/5} \geq d(Tx, Ty).
\]

(iv) If $x \in [2, +\infty]$ and $y \in [0, 2]$ with $y \neq 1$, we obtain
\[ d(Tx, Ty) = d\left( \frac{1}{x}, 1 \right) = \left( 1 + \frac{1}{x} \right)^2 \leq \left( \frac{3}{2} \right)^2, \]

\[
\lambda(d(gx, Tx))^\alpha(d(gy, Ty))^{1-\alpha} = \lambda\left( x^2 + \frac{1}{x} \right)^{2\alpha} (y^2 + 1)^{2(1-\alpha)} \\
\geq \frac{3}{4} \left( \frac{9}{2} \right)^{4\alpha} \geq d(Tx, Ty). \tag{63}
\]

We conclude that \( d(Tx, Ty) \leq \lambda(d(gx, Tx))^\alpha(d(gy, Ty))^{1-\alpha}, \) for all \( x, y \in X \) such that \( Tx \neq x \) and \( Ty \neq y. \) And we remark that \( 1 \) is a coincidence point of \( g \) and \( T. \)

**Remark 24.** In Theorem 22, \( S \) and \( g \) need not have a common fixed point.

**Example 25.** Let \( X = \{0, 1, 2, 3\} \) be endowed with the metric \( d(u, v) = |u - v| \). We define \( S \) and \( g \) by their matrix of values as follows:

\[
S : \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad g : \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \end{pmatrix}. \tag{64}
\]

For \( \lambda = 1/2 \) and \( \alpha = 1/2 \), we have

\[ d(Su, Sv) \leq \lambda(d(0u, Su))^\alpha(d(0v, Sv))^{1-\alpha}. \tag{65} \]

0 and 3 are coincidence points of \( g \) and \( S \), but \( g \) and \( S \) have no common fixed point.

**Remark 26.** If we define \( X, d, T, \) and \( \alpha \) as in Example 23 and \( \lambda(t) = t/(1 + t) \), we obtain

\[ d(Tx, Ty) \leq \lambda(d(x, y))(d(x, Tx))^\alpha(d(y, Ty))^{1-\alpha}, \tag{66} \]

for all \( x, y \in X \) such that \( Tx \neq x \) and \( Ty \neq y. \) And \( T \) has a fixed point.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest regarding this article.

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