A HYBRID PARAMETRIZATION APPROACH FOR A CLASS OF NONLINEAR OPTIMAL CONTROL PROBLEMS

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Abstract. In this paper, a suitable hybrid iterative scheme for solving a class of non-linear optimal control problems (NOCPs) is proposed. The technique is based upon homotopy analysis and parametrization methods. Actually an appropriate parametrization of control is applied and state variables are computed using homotopy analysis method (HAM). Then performance index is transformed by replacing new control and state variables. The results obtained from the given method are compared with the results which are obtained using the spectral homotopy analysis method (SHAM), homotopy perturbation method (HPM), optimal homotopy perturbation method (OHPM), modified variational iteration method (MVIM) and differential transformations. The existence and uniqueness of the solution are presented. The comparison and ability of the given approach is illustrated via two examples.

1. Introduction. Recently theory and applications of optimal control problem have been widely used in different fields such as biomedicine [10], spacecraft [4, 15], robotic [37], and so on. Because of the important role of non-linear optimal control problems (NOCPs) in science and engineering, considerable attentions have been received on this kind of problems. However, most practical problems are too difficult to solve analytically. Therefore, there is a need for new computational methods to overcome these problems. Many methods have been proposed to solve these equations. Stryk and Bulirsch [33] gives a brief list of commonly used direct and indirect efficient methods for the numerical solution of OCPs. In (6, [33]) converts the problem into a non-linear programming by using the discretization or parametrization techniques. Recently solutions based on He’s variational iteration

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method [32], boubaker polynomials expansion scheme [17], the control parameterization method [26], OHPM [12], SHAM [31], differential transform method [9, 29], Galerkin approximation solutions [38], homotopy analysis method [2], optimal homotopy analysis method [14], modified homotopy perturbation method [3], RBF collocation method [28], Legendre approximations [16] have been used for solving NOCPs.

Computational methods for solving more general optimal control problems are also available; see, for example, [5, 13, 34, 18, 24, 25, 20, 27, 30, 35].

In 1992, Liao employed the basic ideas of the homotopy in topology to propose a general analytic method for non-linear problems, namely homotopy analysis method (HAM)[21]. This method has been used effectively to solve various non-linear problems in science and engineering such as Davey-Stewartson equation [11], Kawahara equations [1], and so on. The advantage of the HAM method is the presence of the auxiliary parameter $h$ that provides a way to adjust and control the convergence region and the rate of convergence of solution series.

It is well-known that NOCPs are much more difficult to solve than linear OCP, especially by means of analytic methods. Our goal is to offer a method that can easily solve NOCPs. For this purpose, we combined the methods of parametrization [36] and homotopy analysis method [23] for solving a class of nonlinear optimal control problems, in which control variable is considered as continuous function. In this method, the control variables can be approximated by choosing an appropriate function with finitely many unknown parameters as follows:

$$u(t) = \sum_{j=0}^{k} a_j z_j(t),$$  \hspace{1cm} (1)

where $a_j$ denote unknown parameters, $z_j(t)$ are some polynomial functions and state variables will be computed by HAM. High accuracy and ease of applying this method for OCPs are two important advantages. Comparison between the results obtained by the proposed method with the numerical results obtained by SHAM, HPM, OHPM, MVIM and differential transformations methods demonstrate the efficiency of the given method. The proposed method is a direct method that the advantage to indirect methods, is their broader radius of convergence to an optimal solution. In addition, since the necessary conditions are difficult to obtain, the direct methods can quickly be utilized to solve a number of practical trajectory optimization [7]. Examples show that the proposed method is suitable for its simplicity and small computation costs. This paper is organized as follows: Section 2 is devoted to the basic idea of HAM. State the problem mentioned in section 3. In section 4, talk about the method of solution. The convergence analysis stated in Section 5. In Section 6, we demonstrate the accuracy of the method by considering two test examples. We sum up the section 7 with conclusion.

2. Basic idea of HAM. Consider the following nonlinear equation:

$$N[u(t)] = 0,$$  \hspace{1cm} (2)

where $N$ is a non-linear operator, $t$ denotes independent variable, and $u(t)$ is an unknown function. By means of generalizing the traditional homotopy method, Liao [22] constructed the so-called zero-order deformation equation

$$(1 - q)L[\phi(t; q) - u_0(t)] = qhN[\phi(t; q)],$$  \hspace{1cm} (3)
where \( q \in [0, 1] \) is an embedding parameter, \( h \) is a non-zero auxiliary parameter, \( L \) is an auxiliary linear operator with the property \( L(c_1) = 0 \), where \( c_1 \) is an integral constant, \( u_0(t) \) is an initial guess of \( u(t) \) and \( \phi(t; q) \) is an unknown function. It is important to note that one has great freedom to choose auxiliary objects such as \( h \) and \( L \) in HAM. Obviously, when \( q = 0 \) and \( q = 1 \), both \( \phi(t; 0) = u_0(t) \) and \( \phi(t; 1) = u(t) \), hold. Thus, as \( q \) increases from 0 to 1, the solutions \( \phi(t; q) \) varies from the initial guess \( u_0(t) \) to the solution \( u(t) \). Expanding \( \phi(t; q) \) in Taylor series with respect to \( q \), we have

\[
\phi(t; q) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t)q^m, \tag{4}
\]

where

\[
u_m(t) = \frac{1}{m!} \frac{\partial^m \phi(t; q)}{\partial q^m} \big|_{q=0}; \tag{5}\]

if the auxiliary linear operator, the initial guess, the auxiliary parameter \( h \), and the auxiliary function are properly chosen, then the series equation (4) converges at \( q = 1 \) and have

\[
u(t) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t), \tag{6}\]

which must be one of the solutions of the original non-linear equation, as proved by Liao [22]. With \( h = -1 \) equation (3) becomes

\[
(1 - q)L[\phi(t; q) - u_0(t)] + qN[\phi(t; q)] = 0, \tag{7}\]

which is mostly used in the HPM [8]. As stated in (5), the governing equations can be concluded from the zero-order deformation equations (3). Define the vectors

\[
u_n \rightarrow = [u_0(t), u_1(t), ..., u_n(t)]. \tag{8}\]

Differentiating equation (3), \( m \) times with respect to the embedding parameter \( q \) and then setting \( q = 0 \) and finally dividing them by \( m! \), we have the so-called \( m \)th-order deformation equations

\[
L[u_m(t) - \chi_m u_{m-1}(t)] = hR_m(v_{m-1} \rightarrow), \tag{9}\]

subject to initial condition

\[
u_m(0) = 0, \tag{10}\]

where

\[
R_m(v_{m-1} \rightarrow) = \frac{1}{(m-1)!} \frac{\partial^{m-1}N[\phi(t; q)]}{\partial q^{m-1}} \big|_{q=0}, \tag{11}\]

and

\[
\chi_m = \left\{ \begin{array}{ll} 0, & m \leq 1, \\ 1, & m > 1. \end{array} \right. \tag{12}\]

If we are not able to determine the sum of series in (6) then we can accept the partial sum of this series

\[
u(t) \approx \sum_{i=0}^{m} u_i(t), \tag{13}\]

as the approximate solution of the considered equation.

It should be emphasized that \( u_m(t) \) for \( m \geq 1 \) is governed by linear equation (3) under the linear boundary condition that come from the original problem, which can be easily solved by symbolic computation softwares such as Mathematica and
Maple. For the convergence of the above method we refer the reader to Liao’s work [22]. If Eq. (2) admits unique solution, then this method will produce the unique solution. If Eq. (2) does not possess unique solution, the HAM will give a solution among many other (possible) solutions.

3. **Non-linear quadratic optimal control problem.** Consider the non-linear dynamical system

\[ x'(t) = f(t, x(t)) + g(t, x(t))u(t), \quad t \in [t_0, t_f], \quad (14) \]

\[ x(t_0) = x_0, \quad x(t_f) = x_f, \quad (15) \]

With \( x(t) \in \mathbb{R}^n \) denoting the state variable, \( u(t) \in \mathbb{R}^p \) the control variable and \( x_0 \) and \( x_f \) the given initial and final states at \( t_0 \) and \( t_f \), respectively. Moreover, \( f(t, x(t)) \in \mathbb{R}^n \) and \( g(t, x(t)) \in \mathbb{R}^{n \times p} \) are two continuously differentiable functions in all arguments. Aim is to minimize the convex quadratic objective functional

\[ J[x, u] = \frac{1}{2} \int_{t_0}^{t_f} (x^T(t)Qx(t) + u^T(t)Ru(t))dt, \quad (16) \]

subject to the nonlinear system (14), where \( Q \in \mathbb{R}^{n \times n} \) and \( R \in \mathbb{R}^{p \times p} \) are positive semi-definite and positive definite matrices, respectively.

4. **Method of solution.** First, consider \( z_k(t) \) as a polynomial basis, which is dense in the space of \( C(\Omega) \). The continuous control function \( u(t) \) can be approximated by a finite combination from elements of this basis. Now consider NOCP mentioned in section (3). For solving this problem using the hybrid method, first solve equation (14) with the HAM. Let \( x = (x_1, x_2, ..., x_n) \) and \( u = (u_1, u_2, ..., u_p) \), in the given scheme consider the control function as follows:

\[ u_s(t) = \sum_{j=0}^{k} a_{s,j}z_j(t), \quad s = 1, ..., p, \quad (17) \]

let \( L_r \) be the linear operator defined by

\[ L_r = \frac{\partial \phi_r(t, q)}{\partial t}, \quad r = 1, ..., n, \quad (18) \]

define a non-linear operator as follows:

\[ N_r[\phi_r(t, q)] = \frac{\partial \phi_r(t, q)}{\partial t} - f(t, \phi_r(t, q)) - g(t, \phi_r(t, q))u_s(t), \quad (19) \]

such that

\[ x_r(t) = \phi_r(t, q), \quad u_s(t) = \sum_{j=0}^{k} a_{s,j}z_j(t); \quad (20) \]

the zero order deformation equation is

\[ (1-q)L_r[\phi_r(t, q) - x_r,0(t)] = qh_rN_r[\phi_r(t, q)]. \quad (21) \]

Differentiating equation (21), \( m \) times with respect to the embedding parameter \( q \), then setting \( q = 0 \) and finally dividing them by \( m! \), we have the so-called \( m \)th-order deformation equation for \( (m \geq 1) \)

\[ L_r[x_{r,m}(t, a_{s,0}, ..., a_{s,k}) - \chi_m x_{r,m-1}(t, a_{s,0}, ..., a_{s,k})] = h_rR_{r,m-1}, \quad (22) \]

subject to

\[ \phi_{r,m}(0) = 0, \quad (23) \]
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\[ R_{r,m-1} = \frac{1}{(m-1)!} \frac{\partial^{m-1} N_r[\phi_r(t;q)]}{\partial q^{m-1}} \bigg|_{q=0}. \]  

(24)

Now the solution of the \( m \)th-order \((m \geq 1)\) deformation equation (22) becomes

\[ x_{r,m}(t, a_s, 0, \ldots, a_s, k) = x_{r,0}(t) + \left. \sum_{m=1}^{+\infty} x_{r,m}(t, a_s, 0, \ldots, a_s, k) \right|_{q=0}. \]

(26)

Since we are not able to determine the sum of series in (26), then we truncate the series to approximate \( x(t) \) as follows:

\[ x_r(t, a_s, 0, \ldots, a_s, k) = x_{r,0}(t) + \sum_{m=1}^{l} x_{r,m}(t, a_s, 0, \ldots, a_s, k), \]

(27)

now by substituting (17) and (27) in (16) for \( r = 1, \ldots, n \), and \( s = 1, \ldots, p \), obtain an approximate solution of the OCP for \( J \). Consider

\[ \text{Minimize} \ J_k(a_s, 0, \ldots, a_s, k) + \left( x_{r, t_f, a_s, 0, \ldots, a_s, k} \right)^2. \]

(30)

Equation (30) can be solved by Mathematica optimization Toolbox (FindMinimum) or any other suitable optimization Toolbox. Assuming \( J^*_k \) as the optimal value of (28) in the \( k \)th iteration, a stopping criterion may be considered as follows:

\[ || J^*_k - J^*_{k-1} || < \epsilon, \]

(31)

that \( \epsilon \) is a small positive number and should be chosen according to the accuracy desired.

5. Convergence Analysis. In this section, the convergence of the proposed method is discussed. Define \( U \) as the set of admissible control functions

\[ U = \{ u : \Omega \to W | u(.) \in C(\Omega) \}, \]

in which \( W \subseteq \mathbb{R}^p \) is a compact set.

Definition 5.1. The pair \( (x, u) \) is called an admissible pair, if it satisfies in (14) and (15). Define \( \Xi \) as the set of admissible pairs. Define \( \Xi^m \) and \( \Xi^m_k \) as follows:
\[ \xi^m := \{(x_m(;u), u(\cdot))|u \in U\}, \]
\[ \xi^m_k := \{(x_m(;u_k), u_k(\cdot))|u_k \in P_k \cap U\}, \]
\[(33)\]
where \( P_k \) is the set of all polynomials of degree at most \( k \). Define
\[ \alpha^m_k := \inf_{(x_m, u_k) \in \xi^m_k} J(x_m, u_k), \]
\[ \alpha^m := \inf_{(x_m, u) \in \xi^m} J(x_m, u). \]
\[(34)\]

**Assumption:** Assume \( \alpha^m_k, \alpha^m \) exist for all \( m, k \in \mathbb{N} \).

Now consider the system (14) which can be written as follows:
\[ L[x(t)] + N[x(t)] = \phi(t), \]
\[(35)\]
where \( L \) is a linear operator of (14) and \( N \) is the remaining nonlinear component. Define the nonlinear operator \( N \) as follows:
\[ N[x(t)] = \sum_{k=0}^{\infty} N_k(x_0, x_1, ..., x_k). \]
\[(36)\]
With HAM method we have the following equation, which is referred to as the \( m \)th order (or higher order) deformation equation
\[ L[x_m(t) - \chi_m x_{m-1}(t)] = hR_m(x_{m-1}), \]
\[(37)\]
subject to the initial condition
\[ x_m(0) = 0, \]
\[(38)\]
where
\[ R_m(x_{m-1}) = L[x_{m-1}] + N_{m-1}(x_0, x_1, ..., x_m) - (1 - \chi_m)\phi(t). \]
\[(39)\]
From (37) have
\[ x_1(t) = hR_1(x_0), \]
\[(40)\]
and for \( m \geq 2 \)
\[ x_m(t) = x_{m-1}(t) + hR_m(x_{m-1}), \]
\[(41)\]
and
\[ x(t) = \sum_{m=0}^{\infty} x_m(t). \]
\[(42)\]

**Theorem 5.2.** If the series solution defined in (42) is convergent, then it converges to an exact solution of the non-linear problem (14).

**Proof.** Since, by hypothesis, the series is convergent, it holds:
\[ s(t) = \sum_{m=0}^{\infty} x_m(t). \]
\[(43)\]
So, the necessary condition for the convergence of the series is valid; that is,
\[ \lim_{m \to \infty} x_m(t) = 0. \]
\[(44)\]
Lemma 5.3. The following relations hold:

Using (37), (44) and \(L\) is linear operator, we have

\[
\lim_{n \to \infty} \sum_{m=1}^{n} L[x_m(t) - \chi_m x_{m-1}(t)] = \lim_{n \to \infty} \sum_{m=1}^{n} L[x_m(t)] = 0.
\]

Since \(h \neq 0\), we must have

\[
\sum_{m=1}^{\infty} R_m(x_{m-1}) = 0.
\]

On the other side, we have according to the definition (39)

\[
\sum_{m=1}^{\infty} R_m(x_{m-1}) = \sum_{m=1}^{\infty} \left( L[x_{m-1}] + N_{m-1}(x_0, x_1, ..., x_m) - (1 - \chi_m)\phi(t) \right)
\]

\[
= \sum_{m=1}^{\infty} L[x_{m-1}] + \sum_{m=1}^{\infty} N_{m-1}(x_0, x_1, ..., x_m) - \phi(t).
\]

From (36) and (42) have

\[
L[x(t)] + N[x(t)] - \phi(t) = 0,
\]

this completes the proof. ~

Lemma 5.3. The following relations hold:

1. \(\alpha_1^m \geq \alpha_2^m \geq ... \geq \alpha_k^m \geq \alpha^m\);
2. \(\lim_{k \to \infty} \alpha_k^m = \alpha^m\);
3. \(\lim_{m \to \infty} \lim_{k \to \infty} \alpha_k^m = \alpha\), in which \(\alpha = \inf_{(x,u) \in \xi} J(x,u) \equiv J(x^*, u^*)\).

The proof can be found in [19].

6. Numerical experiments and practical applications. In this section, hybrid homotopy analysis and the parametrization method is applied to obtain approximate solutions of OCPs. To assess the advantages and the accuracy of this method for solving OCPs, consider the following examples.

Example 1.

Consider the optimal maneuvers of a rigid asymmetric spacecraft [15]. The Euler equations for the angular velocities of the spacecraft are given by:

\[
x'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} -\frac{I_2-I_3}{I_3} x_2(t)x_3(t) \\ -\frac{I_1-I_3}{I_3} x_1(t)x_3(t) \\ -\frac{I_1-I_2}{I_2} x_1(t)x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{I_1} & 0 \\ 0 & 0 & \frac{1}{I_2} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix},
\]

where \(x_1, x_2\) and \(x_3\) are angular velocities of the spacecraft, \(u_1, u_2,\) and \(u_3\) are control torques, \(I_1 = 86.24, I_2 = 85.07\) and \(I_3 = 113.59kgm^2\) are the spacecraft principle inertia.

The quadratic cost functional to be minimized is given by:

\[
J[x,u] = \frac{1}{2} \int_0^{100} (x^T(t)Qx(t) + u^T(t)Ru(t))dt,
\]

where \(Q\) and \(R\) are positive definite matrices, and \(x(0) = x_0, x(100) = x_1\).
Table 1. Minimum of performance index value $J_k$ of the proposed method

| Itr | CPU time (sec.) | HAM and parametrization approaches |
|-----|-----------------|-----------------------------------|
| m=4, k=1 | 0.109 | 0.00468778 |
| m=4, k=2 | 0.121 | 0.00468778 |

Table 2. The Max error of the proposed method for $x_1(t)$ that $k = 2$ and $h = -1$ in comparison to SHAM and HPM

| Method | CPU time (sec.) | Max error |
|--------|-----------------|-----------|
| proposed method (m=4, k=2) | 1.55 | $2.93152 \times 10^{-17}$ |
| SHAM (Legendre) (m=6, N=50, h=-1.2) | 0.224 | $1.0589 \times 10^{-9}$ |
| SHAM (Chebyshev) (m=6, N=50, h=-1.2) | 0.224 | $1.0586 \times 10^{-9}$ |
| HPM (m=6) | 46.401 | $3.1420 \times 10^{-8}$ |

Figure 1. Approximate solution of $x_1(t)$ and $u_1(t)$ for $(m=4, k=2)$

where $Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. In addition, the following boundary conditions should be satisfied:

$$
x_1(0) = 0.01 r/s, \quad x_2(0) = 0.005 r/s, \quad x_3(0) = 0.001 r/s, \\
x_1(100) = x_2(100) = x_3(100) = 0 r/s.
$$

Consider initial conditions $x_1(0) = -0.0001t + 0.01, x_2(0) = -0.00005t + 0.005$ and $x_3(0) = -0.000001t + 0.001$. By applying the HAM and parametrization method (proposed method) the computed results for $J_k$ are given in table 1. The maximum absolute error of proposed method, HPM and SHAM [31] are given in table 2. Furthermore, in table 3, minimum of $J$ for our approach is obtained, and shows comparison between proposed method, SHAM, HPM, OHPM [12] and MVIM [27] of $J$. It is noteworthy that the given method improves the maximum absolute error. Also the obtained numerical solution for $m = 4, k = 2$ and $h = -1$ are depicted in Figures 1 to 3. In Figures 4 and 5 the $h$ curves for $m = 4$ are plotted.
Table 3. Minimum of performance index value $J$ of the proposed method and other methods

| Method                     | Cost function | CPU time (sec.) |
|----------------------------|---------------|-----------------|
| Proposed Method (m=4, k=2, h=-1) | 0.0046877837 | 0.141           |
| SHAM Chebyshev (m=6, N=50, h=-1.2) | 0.0046877944625923 | 0.226           |
| SHAM Legendre (m=6, N=50, h=-1.2)  | 0.0046877944625906 | 0.227           |
| HPM (m=3)                  | 0.004687795533 | 10.821          |
| OHPM (m=1)                 | 0.004688009428 | -               |
| MVIM (m=3)                 | 0.004687986656 | -               |

Figure 2. Approximate solution of $x_2(t)$ and $u_2(t)$ for (m=4, k=2)

Figure 3. Approximate solution of $x_3(t)$ and $u_3(t)$ for (m=4, k=2)

Figure 4. $h$-curve at 4-order of approximation of $x_1(t)$ and $x_2(t)$
Example 2.

Consider the following OCP for the Van Der Pol oscillator [9]

$$
\text{Min} J = \int_0^2 \frac{1}{2}(x_1^2 + x_2^2 + u^2) dt,
$$

subject to

$$
x_1' = x_2,  \quad x_2' = -x_1 + x_2(1 - x_1^2) + u,  \quad x_1(0) = 1, \quad x_2(0) = 0,
$$

where $x(t) \in \mathbb{R}^2$ and $u(t) \in \mathbb{R}$.

Consider initial conditions $x_1(0) = 1$ and $x_2(0) = 0$. By applying the proposed method and considering $\epsilon = 0.0002$, the computed results of applying our method for $J_k$ are given in table 4. The maximum absolute errors of HAM and parametrization method (proposed method), DT [9] and SHAM [31] are given in table 5. Furthermore, in table 6, a minimum of $J$ for HAM and parameterization method is obtained, and shows comparison between proposed method and other methods of $J$. Also the obtained numerical solution for $m = 7, k = 3$ and $h = -0.9$ are depicted in Figures 6 and 7. In Figure 8 the $h$ curves for $m = 7$ are plotted.

7. Conclusions. In this paper proposed a method to solve NOCPs problem with efficient hybrid method based on homotopy analysis and parametrization. This method is a direct method to solve this kind of the problem. The suggested method has been compared with other methods, and numerical results affirm the effectiveness of the proposed method. This approach can be applied to any application of OCPs that the control function be continuous. In this method, the control functions can be parametrization based on Legendre and Chebyshev polynomial which may or may not be better. As a future research direction, we can apply this method for
Table 5. The Max error of our method of $x_1(t)$ in comparison to SHAM and HPM

| Itr | Max error |
|-----|-----------|
| Proposed Method (m=7, k=3, h=-0.9) | $3.16673 \times 10^{-5}$ |
| SHAM Chebyshev (m=15, N=50, h=-0.5) | $4.2749 \times 10^{-4}$ |
| SHAM Legendre (m=15, N=50, h=-0.5) | $4.2749 \times 10^{-4}$ |
| DT (m=15) | $4.4380 \times 10^{-4}$ |

Table 6. Minimum of performance index value $J$ of the proposed method and other methods

| Method | Cost function | CPU time (sec.) |
|--------|---------------|-----------------|
| Proposed Method (m=7, k=3, h=-0.9) | 1.01184 | 0.032 |
| SHAM Chebyshev (m=15, N=50, h=-0.5) | 1.0472 | 0.200 |
| SHAM Legendre (m=15, N=50, h=-0.5) | 1.0472 | 0.188 |
| DT (m=15) | 1.0478 | 87.74 |

Figure 6. Approximate solution of $x_1(t)$ and $x_2(t)$ for (m=7, k=3)

Figure 7. Approximate solution of $u(t)$

Table 7. Minimum of performance index value $J_k$ of the proposed method

| Itr | CPU time (sec.) | HAM and parametrization approaches |
|-----|-----------------|-----------------------------------|
| m=7, k=1 | 0.016 | 1.07504 |
| m=7, k=2 | 0.031 | 1.0136 |
| m=7, k=3 | 0.032 | 1.01184 |
uncertainty problem in the field of optimal control that is an emerging real-world necessity. All calculations are done with the Mathematica software.

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