On removable singular sets for solutions of higher order differential inequalities

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Abstract
We obtain conditions guaranteeing that weak solutions of higher order differential inequalities have a removable singularity on a compact set. These conditions depend on the fractal dimension of the singular set. For solutions of the nonlinear Poisson equation in the case of an isolated singularity, i.e. in the case where the fractal dimension of the singular set is equal to zero, they coincide with the well-known Brezis-Véron condition.

Keywords Removable singular sets · Nonlinear differential inequalities of higher order · Fractal dimension

Mathematics Subject Classification 35G20 · 35D30 · 28A80 · 35B33 · 35J30

1 Introduction

We study solutions of the inequality

\[ \sum_{|\alpha|=m} \partial^\alpha a_\alpha(x, u) \geq f(x)g(|u|) \quad \text{in } \Omega \setminus S, \] (1.1)

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where $\Omega$ is a bounded domain in $\mathbb{R}^n$, $m, n \geq 1$ are integers, $S$ is a compact subset of $\Omega$, and $a_\alpha$ are Caratheodory functions such that

$$|a_\alpha(x, \xi)| \leq A|\xi|, \quad |\alpha| = m,$$

with some constant $A > 0$ for almost all $x \in \Omega$ and for all $\xi \in \mathbb{R}$.

Throughout this paper, it is assumed that $f : \Omega \to [0, \infty)$ is a measurable function and $g : [0, \infty) \to [0, \infty)$ is a twice continuously differentiable function such that $g(0) = 0$ and, in addition, $g(\xi) > 0, g'(\xi) > 0, \text{ and } g''(\xi) > 0$ for all $\xi \in (0, \infty)$. As is customary, by $\alpha = (\alpha_1, \ldots, \alpha_n)$ we mean a multi-index with $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $\partial^\alpha = \partial^{\alpha_1}/(\partial^{\alpha_1} \ldots \partial^{\alpha_n})$, where $x = (x_1, \ldots, x_n)$.

Let us denote by $B_r^x$ the open ball of radius $r > 0$ centered at $x \in \mathbb{R}^n$. In the case of $x = 0$, we write $B_r$ instead of $B_r^0$. For an arbitrary set $\omega \subset \mathbb{R}^n$ and a real number $r > 0$ by $\omega_r$ we mean the set of the points $x \in \mathbb{R}^n$ the distance from which to $\omega$ is less than $r$.

A function $u$ is called a weak solution of (1.1) if $u \in L_1(\Omega \setminus S)$ and $f(x)g(|u|) \in L_1(\Omega \setminus S)$ for any $r > 0$ and, moreover,

$$\int_\Omega \sum_{|\alpha| = m} (-1)^m a_\alpha(x, u) \partial^\alpha \varphi \, dx \geq \int_\Omega f(x)g(|u|)\varphi \, dx \quad (1.2)$$

for any non-negative function $\varphi \in C^\infty_0(\Omega \setminus S)$. In so doing, we say that the singular set $S$ is removable if $u \in L_1(\Omega), f(x)g(|u|) \in L_1(\Omega)$, and inequality (1.2) is valid for all non-negative functions $\varphi \in C^\infty_0(\Omega)$.

The problem of removability of singularities for solutions of differential equations and inequalities is a traditional area of interest for many mathematicians [1–11]. Most of the papers in this area are devoted to the case of second-order differential operators [1–7]. For higher order differential inequalities (1.1), sufficient conditions for the removability of a singularity are known only in the case of the power nonlinearity $g(t) = t^\lambda$ [8, 9] or in the case of an isolated singularity [10].

Singular sets for solutions of linear higher order elliptic equations were studied in [11]. For these equations, in order to remove a singularity, some growth conditions have to be imposed on the solutions when approaching a singular set.

In the case of an arbitrary nonlinearity, conditions for the removability of a singular set remained unknown. The paper presented to your attention provides an answer to this question. It turned out unexpected that the fractal dimension of the singular set is responsible for the removability of the singularity in this case. In so doing, for solutions of the nonlinear Poisson equation with an isolated singularity, our results coincide with the well-known Brezis–Véron condition (see Example 3).

Note that we do not impose any requirements on the growth of solutions near a singularity. We also do not impose any ellipticity conditions on the coefficients $a_\alpha$ of the differential operator. Thus, our results can be applied to a wide class of differential inequalities.
Consider the Legendre transformation

\[ g^*(\xi) = \begin{cases} \int_{g'(0)}^{\xi} (g')^{-1}(\xi) d\xi, & \xi > g'(0), \\ 0, & \xi \leq g'(0), \end{cases} \]

of the function \( g \), where \((g')^{-1}\) is the inverse function to \( g' \). From the Fenchel-Young inequality, it obviously follows that

\[ ab \leq g(a) + g^*(b) \]

for all real numbers \( a \geq 0 \) and \( b \geq 0 \). In the case of \( g(t) = t^\lambda/\lambda, \lambda > 1 \), this inequality takes the form

\[ ab \leq \frac{1}{\lambda} a^\lambda + \frac{\lambda - 1}{\lambda} b^{\lambda/(\lambda-1)} \]

for all real numbers \( a \geq 0 \) and \( b \geq 0 \).

We put

\[ \gamma(\xi) = \frac{g^*(\xi)}{\xi} \]

and

\[ \mu(\xi) = \frac{g(\xi)}{\xi}. \quad (1.3) \]

Let \( N_r(\omega) \) be the minimal number of sets of diameter at most \( r \) which can cover a bounded set \( \omega \). The lower and upper fractal dimensions of \( \omega \) respectively are defined by

\[ \underline{dim}_F \omega = \lim \inf_{r \to +0} \frac{\log N_r(\omega)}{\log 1/r} \]

and

\[ \overline{dim}_F \omega = \lim \sup_{r \to +0} \frac{\log N_r(\omega)}{\log 1/r} \]

It does not present any particular problem to verify that

\[ \underline{dim}_F \omega = n - \lim \sup_{r \to +0} \frac{\log \text{mes} \omega_r}{\log r} \]
and

\[ \overline{\dim} F \omega = n - \lim_{r \to +0} \inf \frac{\log \text{mes } \omega_r}{\log r}. \]

Therefore, the last two expressions can be used as another definition of the lower and upper fractal dimensions.

If \( \overline{\dim} F \omega = \overline{\dim} F \omega \), then we say that \( \omega \) has the fractal dimension

\[ \dim F \omega = \lim_{r \to +0} \frac{\log N_r(\omega)}{\log 1/r}. \]

In the literature, the fractal dimensions are sometimes called Minkowski dimensions or box-counting dimensions [12], [13]. The upper fractal dimension is also known as Kolmogorov’s entropy or Kolmogorov’s capacity [12].

We say [11] that \( \omega \) has finite \( k \)-dimension if

\[ \limsup_{r \to +0} \frac{\text{mes } \omega_r}{r^{n-k}} < \infty. \]

In so doing, \( \omega \) is a \( k \)-dimensional set if \( \text{mes } \omega_r/r^{n-k} \) tends to a positive finite limit as \( r \to +0 \) [12]. If \( \omega \) is a \( k \)-dimensional set, then it obviously has both the fractal dimension \( k \) and finite \( k \)-dimension.

2 Main results

Theorem 1 Suppose that

\[ \int_1^\infty g^{-1/m}(\zeta)\zeta^{1/m-1}d\zeta < \infty \quad (2.1) \]

and

\[ \int_{\Omega \setminus S} \gamma \left( \frac{1}{f(x)} \right) dx < \infty. \quad (2.2) \]

Also let there exist a real number \( \delta > 0 \) such that

\[ \lim_{r \to +0} r^{n-k-m-\delta} G^{-1} \left( \text{Cress inf } f^{1/m} \right) = 0 \quad (2.3) \]

for all real numbers \( C > 0 \), where

\[ k = \overline{\dim} F \partial S \quad (2.4) \]
and $G^{-1}$ is the inverse function to

$$G(t) = \int_t^\infty g^{-1/m}(\zeta)\zeta^{1/m-1} d\zeta.$$  \hfill (2.5)

Then the singular set $S$ is removable for any weak solution of (1.1).

**Theorem 2** Suppose that conditions (2.1) and (2.2) are valid and, moreover, $\partial S$ has finite $k$-dimension. If

$$\limsup_{r \to +0} r^{n-k-m} G^{-1}\left(\text{Cress inf}_{S_R \setminus S} f^{1/m}\right) < \infty$$ \hfill (2.6)

and

$$\liminf_{r \to +0} r^m \mu(Cr^{m-n+k})\text{ess inf}_{S_R \setminus S} f > 0$$ \hfill (2.7)

for all real numbers $C > 0$, where $G^{-1}$ is the inverse function to (2.5), then the singular set $S$ is removable for any weak solution of (1.1).

**Remark 1** Condition (2.1) generalizes the well-known Keller–Osserman condition [14], [15]. It also arises in the study of the blow-up phenomenon for entire solutions of higher order differential inequalities [16].

Theorems 1 and 2 are proved in Section 3. Now we dwell on some results that follow from these theorems. A special case of (1.1) is inequalities of the form

$$\sum_{|\alpha| = m} \partial^\alpha a_\alpha(x, u) \geq f(x)|u|^\lambda$$ \hfill (2.8)

in $\Omega \setminus S$, where $\lambda$ is a real number.

**Corollary 1** Suppose that $\lambda > 1$,

$$\int_{\Omega \setminus S} f^{-1/(\lambda-1)}(x) \, dx < \infty,$$ \hfill (2.9)

and

$$\lim_{r \to +0} r^{n-k-\lambda m/\lambda-1-\delta} \text{ess sup}_{S_R \setminus S} f^{-1/(\lambda-1)} = 0$$

for some real number $\delta > 0$, where $k$ is given by (2.4). Then the singular set $S$ is removable for any weak solution of (2.8).
Corollary 2. Suppose that $\lambda > 1$, condition \((2.9)\) is valid and, moreover, $\partial S$ has finite $k$-dimension. If

$$
\limsup_{r \to +0} r^{n-k-\lambda m/(\lambda-1)} \operatorname{ess sup}_{S \setminus S} f^{-1/(\lambda-1)} < \infty
$$

and

$$
\liminf_{r \to +0} r^{n-k+\lambda(m-n+k)} \operatorname{ess sup}_{S \setminus S} f > 0,
$$

then the singular set $S$ is removable for any weak solution of \((2.8)\).

Proof of Corollaries 1 and 2 follows directly from Theorems 1 and 2.

Example 1. In \((2.8)\), let

$$
f(x) = \begin{cases} 
\operatorname{dist}^\sigma(x, S), & x \in \Omega \setminus S, \\
0, & x \in S,
\end{cases}
$$

where $\sigma$ is a real number. In so doing, let $\Omega$ be a bounded plane domain and $S$ be the von Koch snowflake that can be constructed by the following procedure. At first step, we divide the sides of an equilateral triangle into three equal parts. Then, on each side, we replace the middle segment with two segments equal to it so that the protrusion faces outside the triangle. Further, this procedure is repeated with each of the segments obtained at the previous step (see Figure 1).

It is well-known [13] that

$$
\dim_F \partial S = \log_3 4.
$$

By Corollary 1, if

$$
\lambda > 1
$$

and

$$
\lambda(2 - \log_3 4 - m) - 2 + \log_3 4 - \sigma > 0,
$$

then the singular set $S$ is removable for any weak solution of \((2.8)\).
Now, we consider the critical exponent $\lambda = 1$ in condition (2.11). In (1.1), let the function $f$ satisfy (2.10) and

$$g(t) = t \log^\nu (e + t).$$

(2.12)

As above, we assume that $\Omega$ is a bounded plane domain and $S$ is the von Koch snowflake. By Theorem 1, if

$$\nu > m$$

(2.13)

and

$$\sigma < -m,$$

(2.14)

then the singular set $S$ is removable for any weak solution of (1.1). This result can be generalized to the case of

$$g(t) = t \log^m (t_0 + t) \cdots \log^m \log \cdots \log (t_l + t) \log^\nu \log \cdots \log (t_{l+1} + t)$$

(2.15)

where $t_0 = e$ and $t_{l+1} = e^{t_l}$, $i = 0, 1, \ldots, l$. In this case, in accordance with Theorem 1 conditions (2.13) and (2.14) also guarantee that the singular set $S$ is removable for any weak solution of (1.1).

**Example 2** In (2.8), let the function $f$ satisfy (2.10) and

$$S = C \times \cdots \times C$$

be the Cantor dust, where $C$ is the usual middle thirds Cantor set constructed as follows. At the first step, we remove the central third $(1/3, 2/3)$ from the interval $[0, 1]$. After this procedure, the two intervals $[0, 1/3]$ and $[2/3, 1]$ remain of the interval $[0, 1]$. Further, at each of the subsequent steps, we remove the central third from each interval remaining at the previous step.

For $3^{-i-1} \leq r < 3^{-i}$, the set $\partial S$ can be obviously covered with $2^{ni}$ open cubes with side lengths $3^{-i+1}$ such that the union of these cubes contains $S_r$. As the $n$-dimensional Lebesgue measure of the union of these cubes does not exceed $2^{ni} 3^{(-i+1)n}$, the set $\partial S$ has finite $k$-dimension, where

$$k = n \log_3 2.$$

By Corollary 2, if (2.11) is valid and

$$\lambda(n - n \log_3 2 - m) - n + n \log_3 2 - \sigma \geq 0,$$

then the singular set $S$ is removable for any weak solution of (2.8).
Now, let us examine the case of the critical exponent $\lambda = 1$ in (2.11). Consider inequality (1.1), where $f$ and $g$ are given by (2.10) and (2.12), respectively. If both conditions (2.13) and (2.14) are valid, then Theorem 2 implies that the singular set $S$ is removable for any weak solution of (1.1). In its turn, if $\sigma = -m$, then in accordance with Theorem 2 the removability of the singular set $S$ for any weak solution of (1.1) is guaranteed by (2.13) and the inequality

$$n - n \log_2 2 - m \geq 0.$$ 

This statement is also true for the function $g$ given by (2.15).

**Example 3** In (2.8), let the function $f$ satisfy (2.10) and

$$S = \{x = (x_1, \ldots, x_n) : x_1^2 + \ldots + x_k^2 \leq 1, \ x_{k+1} = \ldots = x_n = 0\},$$

where $0 \leq k < n$ is an integer. In the partial case of $k = 0$, we have $S = \{0\}$.

It can easily be seen that $\partial S$ has finite $k$-dimension. Moreover, $\partial S$ is a $k$-dimensional set. Thus, if (2.11) is valid and

$$\lambda(n - k - m) - n + k - \sigma \geq 0,$$

then in accordance with Corollary 2 the singular set $S$ is removable for any weak solution of (2.8).

We note that, in the case of the inequality

$$\Delta u \geq |u|^\lambda \text{ in } \Omega \setminus \{0\},$$

where $\Omega$ is a bounded domain of dimension $n \geq 3$ containing zero, condition (2.16) coincides with the well-known Brezis–Véron condition

$$\lambda \geq \frac{n}{n - 2}$$

obtained in [1].

Now, consider inequality (1.1), where the functions $f$ and $g$ are defined by (2.10) and (2.12). By Theorem 2, the singular set $S$ is removable for any weak solution of (1.1) if (2.13) and (2.14) are satisfied. In so doing, for the critical exponent $\sigma = -m$, the removability of the singular set $S$ for any weak solution of (1.1) is guaranteed by (2.13) and the condition

$$n - k - m \geq 0.$$ 

This result can be generalized to the case of the function $g$ defined by (2.15).

**Example 4** In (2.8), let $S$ be the closure of the unit ball $B_1$. As before, we assume that the function $f$ satisfies relation (2.10).
It is clear that \( \partial S \) has finite \((n-1)\)-dimension. Thus, by Corollary 2, if (2.11) is valid and
\[
\lambda(1-m) - 1 - \sigma \geq 0,
\]
then the singular set \( S \) is removable for any weak solution of (2.8).

For weak solutions of (1.1), where \( f \) and \( g \) are defined by (2.10) and (2.12), in accordance with Theorem 2 the removability of the singular set \( S \) is guaranteed by conditions (2.13) and (2.14). This statement is also true for \( g \) given by (2.15). In so doing, for the critical exponent \( \sigma = -m \), the singular set \( S \) is removable if \( m = 1 \) and \( \nu > 1 \).

**Theorem 3** Suppose that conditions (2.1) and (2.2) are valid. Also let there exist a real number \( \delta > 0 \) such that (2.3) holds for all real numbers \( C > 0 \), where
\[
k = n - \lim inf_{r \to +0} \frac{\log mes S_r \setminus S}{\log r} \tag{2.17}
\]
and \( G^{-1} \) is the inverse function to (2.5). Then the singular set \( S \) is removable for any weak solution of (1.1).

**Theorem 4** Suppose that conditions (2.1) and (2.2) are valid. Also let \( k \) be a real number such that
\[
\lim sup_{r \to +0} \frac{mes S_r \setminus S}{r^{n-k}} < \infty \tag{2.18}
\]
and, moreover, (2.6) and (2.7) hold for all real numbers \( C > 0 \), where \( G^{-1} \) is the inverse function to (2.5). Then the singular set \( S \) is removable for any weak solution of (1.1).

Theorems 3 and 4 are proved in Section 3.

**Remark 2** In Theorems 1–4 and Corollaries 1 and 2, the condition \( g(0) = 0 \) can be dropped if \( f = 0 \) almost everywhere on \( S \). In this case, one should put
\[
\mu(\xi) = \inf_{t \geq \xi} \frac{g(t)}{t}
\]
instead of (1.3). We also note that, in the case of \( g(0) = 0 \), the function \( g(t)/t \) increases on the interval \((0, \infty)\) since \( g' \) and \( g'' \) are positive; therefore,
\[
\inf_{t \geq \xi} \frac{g(t)}{t} = \frac{g(\xi)}{\xi}
\]
for all \( \xi \in (0, \infty) \). Indeed, we have
\[
g(t) = \int_0^t g'(\xi) \, d\xi = -\int_0^t (t - \xi)' g'(\xi) \, d\xi = tg'(0) + \int_0^t (t - \xi)g''(\xi) \, d\xi.
\]
Thus,
\[ \frac{g(t)}{t} = g'(0) + \int_0^t \left( 1 - \frac{\xi}{t} \right) g''(\xi) \, d\xi, \]
whence it obviously follows that
\[ \left( \frac{g(t)}{t} \right)' = \frac{1}{t^2} \int_0^t \xi g''(\xi) \, d\xi > 0 \]
for all \( t \in (0, \infty) \).

3 Proof of Theorems 1–4

In this section, we assume that \( u \) is a weak solution of (1.1). Take a real number \( \varepsilon \in (0, 1) \) such that the closure of \( S_\varepsilon \) in \( \mathbb{R}^n \) belongs to \( \Omega \). Since \( S \) is a compact subset of \( \Omega \), such a real number obviously exists.

We need the following simple lemma.

Lemma 1 For any set \( \omega \subset \mathbb{R}^n \) and real number \( r > 0 \) there exists a non-negative function \( \psi \in C^\infty (\mathbb{R}^n) \) such that
\[
\psi |_{\omega} = 1, \quad \psi |_{\mathbb{R}^n \setminus \omega} = 0, \quad \text{and} \quad \| \psi \|_{C^m (\mathbb{R}^n)} \leq Cr^{-m},
\]
where the constant \( C > 0 \) depends only on \( n \) and \( m \).

Proof We construct a sequence (countable or finite) of points \( x_i \in \omega, i = 1, 2, \ldots \), as follows. Let \( x_1 \) be an arbitrary point belonging to \( \omega \). Assume further that \( x_j \) is already constructed for all \( 1 \leq j \leq i \). If
\[
\omega \subset \bigcup_{j=1}^i B_{r/2}^{x_j},
\]
then we stop; otherwise we take
\[
x_{i+1} \in \omega \setminus \bigcup_{j=1}^i B_{r/2}^{x_j}
\]
such that
\[
|x_{i+1}| < \inf \left\{ |x| : x \in \omega \setminus \bigcup_{j=1}^i B_{r/2}^{x_j} \right\} + 1.
\]
Since \( B_{r/4}^x \cap B_{r/4}^y = \emptyset \) if \( i \neq j \), we obviously obtain
\[
\omega \subset \bigcup_i B_{r/2}^x.
\]

In so doing, for any point \( x \in \mathbb{R}^n \) the number of the balls \( B_{r/2}^x \) containing \( x \) does not exceed a certain value depending only on \( n \). Really, if \( x \in B_{r/2}^x \), then \( B_{r/4}^x \subset B_{5r/4} \). Therefore, the number of the balls \( B_{r/2}^x \) containing \( x \) can not exceed \( \text{mes} B_{5r/4} / \text{mes} B_{r/4} = 5^n \).

Consider a non-negative function \( \varphi \in C^\infty_0 (B_1) \) equal to one on the ball \( B_{1/2} \). Putting
\[
\varphi_i (x) = \varphi \left( \frac{x - x_i}{r} \right),
\]
we obtain
\[
\sum_i \varphi_i (x) \geq 1
\]
for all \( x \in \omega \) and
\[
\sum_i \varphi_i (x) = 0
\]
for all \( x \in \mathbb{R}^n \setminus \omega_r \). Thus, to complete the proof, it remains to take
\[
\psi (x) = \eta \left( \sum_i \varphi_i (x) \right),
\]
where \( \eta \in C^\infty (\mathbb{R}) \) is a non-negative function such that
\[
\eta|_{(-\infty,0]} = 0 \quad \text{and} \quad \eta|_{[1,\infty)} = 1.
\]

The next two lemmas generalize Lemmas 3.1 and 3.2 of paper [10], where the case of \( \Omega = B_1 \) and \( S = \{0\} \) was considered.

**Lemma 2** Let \( 0 < \rho < r \leq \varepsilon/2 \) be real numbers. Then
\[
\int_{S_r \setminus S_{r/2}} |u| \, dx + \frac{1}{(r - \rho)^m} \int_{S_r \setminus S_{\rho}} |u| \, dx \geq C \int_{S_{r/2} \setminus S_{\rho}} f(x) g(|u|) \, dx, \tag{3.1}
\]
where the constant \( C > 0 \) depends only on \( A, n, m, \) and \( \varepsilon \).
Proof We agree to denote by \( C \) various positive constants which can depend only on \( A, n, m, \) and \( \varepsilon \). In accordance with Lemma 1 there is a non-negative function \( \tau \in C^\infty(\mathbb{R}^n) \) with the norm \( \| \tau \|_{C^m(\mathbb{R}^n)}\) depending only on \( n, m, \) and \( \varepsilon \) such that

\[
\tau|_{S_{\varepsilon/2}} = 1 \quad \text{and} \quad \tau|_{\Omega \setminus S_\varepsilon} = 0.
\]

Analogously, there exists a non-negative function \( \psi \in C^\infty(\mathbb{R}^n) \) satisfying the conditions

\[
\psi|_{S_{\rho}} = 0, \quad \psi|_{\Omega \setminus S_\varepsilon} = 1, \quad \text{and} \quad \| \psi \|_{C^m(\mathbb{R}^n)} \leq C(r - \rho)^{-m}. \quad (3.2)
\]

Taking \( \phi(x) = \psi(x)\tau(x) \) as a test function in (1.2), we obtain

\[
\int_{\Omega} \sum_{|\alpha| = m} (-1)^m a_{\alpha}(x, u) \psi \partial^\alpha \tau \, dx
\]

\[
+ \int_{\Omega} \sum_{|\alpha'| + |\alpha''| = m, \ |\alpha'| > 1} a_{\alpha'\alpha''}(x, u) \partial^{\alpha'} \psi \partial^{\alpha''} \tau \, dx
\]

\[
\geq \int_{\Omega} f(x) g(|u|) \psi \tau \, dx, \quad (3.3)
\]

where \( a_{\alpha'\alpha''} \) are some Carathéodory functions such that

\[
|a_{\alpha'\alpha''}(x, \zeta)| \leq C|\zeta|, \quad |\alpha'| + |\alpha''| = m, \ |\alpha'| > 1, \quad (3.4)
\]

for almost all \( x \in \Omega \) and for all \( \zeta \in \mathbb{R} \). In view of (3.2), we have

\[
\left| \int_{\Omega} \sum_{|\alpha'| + |\alpha''| = m, \ |\alpha'| > 1} a_{\alpha'\alpha''}(x, u) \partial^{\alpha'} \psi \partial^{\alpha''} \tau \, dx \right| \leq \frac{C}{(r - \rho)^m} \int_{S_r \setminus S_\rho} |u| \, dx.
\]

It can also be seen that

\[
\left| \int_{\Omega} \sum_{|\alpha| = m} (-1)^m a_{\alpha}(x, u) \psi \partial^\alpha \tau \, dx \right| \leq C \int_{S_r \setminus S_{r/2}} |u| \, dx
\]

and

\[
\int_{\Omega} f(x) g(|u|) \psi \tau \, dx \geq \int_{S_{r/2} \setminus S_r} f(x) g(|u|) \, dx.
\]

Thus, (3.3) implies (3.1). \( \square \)
Lemma 3 Suppose that
\[
\frac{1}{(r_i - r_{i+1})^m} \int_{S_{r_i} \setminus S_{r_{i+1}}} |u| \, dx \to 0 \quad \text{as } i \to \infty
\]
for some decreasing sequence of real numbers \( r_i \in (0, \varepsilon/2), \ i = 1, 2, \ldots, \) tending to zero as \( i \to \infty. \) Then the singular set \( S \) is removable.

Proof As in the proof of Lemma 2, by \( C \) we denote various positive constants which can depend only on \( A, n, m, \) and \( \varepsilon. \) From the Fenchel-Young inequality, it follows that
\[
\int_{S_{\varepsilon/2} \setminus S} |u| \, dx \leq \int_{S_{\varepsilon/2} \setminus S} f(x) g(|u|) \, dx + \int_{S_{\varepsilon/2} \setminus S} f(x) g^\star \left( \frac{1}{f(x)} \right) \, dx.
\]
Putting \( r = r_i \) and \( \rho = r_{i+1} \) in Lemma 2 and passing to the limit as \( i \to \infty, \) we obtain
\[
\int_{S_{\varepsilon/2} \setminus S} f(x) g(|u|) \, dx < \infty. \tag{3.5}
\]
At the same time,
\[
\int_{S_{\varepsilon/2} \setminus S} f(x) g^\star \left( \frac{1}{f(x)} \right) \, dx = \int_{S_{\varepsilon/2} \setminus S} \gamma \left( \frac{1}{f(x)} \right) \, dx < \infty
\]
by condition (2.2). Therefore, we have
\[
\int_{S_{\varepsilon/2} \setminus S} |u| \, dx < \infty.
\]
By Lemma 1, there are non-negative functions \( \psi_i \in C^\infty(\mathbb{R}^n) \) such that
\[
\psi_i |_{S_{r_{i+1}}} = 0, \quad \psi_i |_{\mathbb{R}^n \setminus S_{r_i}} = 1, \quad \text{and} \quad \|\psi_i\|_{C^m(\mathbb{R}^n)} \leq C (r_i - r_{i+1})^{-m}
\]
for all \( i = 1, 2, \ldots. \)
Let \( \varphi \in C_0^\infty(\Omega) \) be an arbitrary non-negative function. Taking
\[
\varphi_i(x) = \psi_i(x) \varphi(x)
\]
as a test function in (1.2), we obtain
\[
\int_{\Omega} \sum_{|\alpha| = m} (-1)^m a_\alpha(x, u) \psi_i \partial^\alpha \varphi \, dx
\]
\[
+ \int_{\Omega} \sum_{|\alpha'| + |\alpha''| = m, |\alpha'| > 1} a_{\alpha'\alpha''}(x, u) \partial^{\alpha'} \psi_i \partial^{\alpha''} \varphi \, dx
\]
\[
\geq \int_{\Omega} f(x) g(|u|) \psi_i \varphi \, dx, \tag{3.6}
\]
where \( a_{\alpha'\alpha''} \) are Carathéodory functions satisfying condition (3.4).

It can easily be seen that
\[
\left| \int_{\Omega} \sum_{|\alpha'|=m, |\alpha''|>1} a_{\alpha'\alpha''}(x,u) \partial^{\alpha'} \psi_i \partial^{\alpha''} \varphi \, dx \right| \leq \frac{C \| \varphi \|_{C^m(\Omega)}}{(r_i - r_{i+1})^m} \int_{S_{r_i} \setminus S_{r_{i+1}}} |u| \, dx \to 0 \quad \text{as} \quad i \to \infty.
\]

At the same time, by Lebesgue’s bounded convergence theorem, we obviously have
\[
\int_{\Omega} \sum_{|\alpha|=m} (-1)^m a_{\alpha}(x,u) \partial^{\alpha} \varphi \, dx \to \int_{\Omega \setminus S} \sum_{|\alpha|=m} (-1)^m a_{\alpha}(x,u) \partial^{\alpha} \varphi \, dx
\]
and
\[
\int_{\Omega} f(x) g(|u|) \psi_i \varphi \, dx \to \int_{\Omega \setminus S} f(x) g(|u|) \varphi \, dx
\]
as \( i \to \infty \). Thus, (3.6) implies (1.2), where \( u \) is extended by zero to \( S \). \( \square \)

According to Lemma 3, if the singular set \( S \) is not removable, then there exist \( \varepsilon_0 \in (0, \infty) \) and \( r_0 \in (0, \varepsilon/2) \) such that
\[
\frac{1}{(r - \rho)^m} \int_{S_{r} \setminus S_{\rho}} |u| \, dx > \varepsilon_0 \quad (3.7)
\]
for all real numbers \( 0 < \rho < r < r_0 \).

**Lemma 4** Let (2.1) and (3.7) be valid. Then
\[
\frac{1}{\text{mes} S_{\rho_2} \setminus S_{\rho_1}} \int_{S_{\rho_2} \setminus S_{\rho_1}} |u| \, dx \leq \frac{2}{\beta} G^{-1} \left( C B^{1/m}(\rho_1 - \rho_0) \text{ess inf}_{S_{\rho_2} \setminus S_{\rho_0}} f^{1/m} \right) \quad (3.8)
\]
for all real numbers \( 0 < \rho_0 < \rho_1 < \rho_2 < r_0 \), where
\[
\beta = \frac{\text{mes} S_{\rho_2} \setminus S_{\rho_1}}{\text{mes} S_{\rho_2} \setminus S_{\rho_0}},
\]
\( G^{-1} \) is the function inverse to (2.5), and \( C > 0 \) is a constant depending only on \( A, n, m, \varepsilon, \varepsilon_0 \), and the first summand in the left-hand side of (3.1).

**Proof** Consider the function
\[
J(\rho) = \frac{1}{\text{mes} S_{\rho_2} \setminus S_{\rho_1}} \int_{S_{\rho_2} \setminus S_{\rho_1}} |u| \, dx, \quad \rho_0 \leq \rho \leq \rho_1.
\]
If $J(\rho_1) = 0$, then (3.8) is obvious; therefore, one can assume that $J(\rho_1) > 0$.

We construct a finite sequence of real numbers $\{\zeta_i\}_{i=1}^l$ as follows. Let $\zeta_1 = \rho_1$. Assume further that $\zeta_i$ is already known for some integer $i \geq 1$. If $\zeta_i \leq (\rho_0 + \rho_1)/2$, then we put $l = i$ and stop; otherwise we take

$$\zeta_{i+1} = \inf \{ \zeta \in (\rho_0, \zeta_i) : J(\zeta) \leq 2J(\zeta_i) \}.$$

Since $J$ is a non-increasing positive continuous function on the closed interval $[\rho_0, \rho_1]$, this procedure must complete in a finite number of steps.

We denote by $C$ various positive constants which can depend only on $A, n, m, \varepsilon, \varepsilon_0$, and the first summand in the left-hand side of (3.1). From inequalities (3.1) and (3.7), where $\rho = \zeta_{i+1}$ and $r = \zeta_i$, it follows that

$$\frac{1}{(\zeta_i - \zeta_{i+1})^m} \int_{S_{\zeta_i} \setminus S_{\zeta_{i+1}}} |u| \, dx \geq C \int_{S_{\zeta_i/2} \setminus S_{\zeta_i}} f(x) g(|u|) \, dx$$

for all $1 \leq i \leq l - 1$. This obviously yields

$$\int_{S_{\zeta_i} \setminus S_{\zeta_{i+1}}} |u| \, dx \geq C (\zeta_i - \zeta_{i+1})^m \text{ess inf}_{S_{\zeta_i}} f \int_{S_{\zeta_i/2} \setminus S_{\zeta_i}} g(|u|) \, dx$$

or, in other words,

$$J(\zeta_{i+1}) - J(\zeta_i) \geq C (\zeta_i - \zeta_{i+1})^m \text{ess inf}_{S_{\zeta_i}} f \frac{1}{\text{mes} S_{\zeta_i} \setminus S_{\zeta_i}} \int_{S_{\zeta_i/2} \setminus S_{\zeta_i}} g(|u|) \, dx \quad (3.9)$$

for all $1 \leq i \leq l - 1$. Since $g$ is a convex function, we have

$$\frac{1}{\text{mes} S_{\zeta_i} \setminus S_{\zeta_i}} \int_{S_{\zeta_i/2} \setminus S_{\zeta_i}} g(|u|) \, dx \geq g \left( \frac{1}{\text{mes} S_{\zeta_i} \setminus S_{\zeta_i}} \int_{S_{\zeta_i/2} \setminus S_{\zeta_i}} |u| \, dx \right)$$

for all $1 \leq i \leq l - 1$. Consequently, (3.9) implies the inequalities

$$J(\zeta_{i+1}) - J(\zeta_i) \geq C (\zeta_i - \zeta_{i+1})^m g(\beta J(\zeta_i)) \text{ess inf}_{S_{\zeta_i}} f, \quad i = 1, 2, \ldots, l - 1,$$

whence it follows that

$$\frac{J^{1/m}(\zeta_i)}{g^{1/m}(\beta J(\zeta_i))} \geq C (\zeta_i - \zeta_{i+1}) \text{ess inf}_{S_{\zeta_i}} f^{1/m}, \quad i = 1, 2, \ldots, l - 1. \quad (3.10)$$

If $\zeta_i > \rho_0$, then $J(\zeta_{i+1}) = 2J(\zeta_i)$ for all $1 \leq i \leq l - 1$; therefore,

$$\int_{J(\zeta_i)}^{J(\zeta_{i+1})} g^{-1/m}(\beta \xi/2) \xi^{1/m-1} \, d\xi \geq \frac{2^{1/m-1} J^{1/m}(\zeta_i)}{g^{1/m}(\beta J(\zeta_i))}, \quad i = 1, 2, \ldots, l - 1.$$
Combining this with (3.10), we obtain
\[
\int_{J(\zeta_i)} J^{\ell/m}(\beta \zeta / 2) \zeta^{1/m-1} \, d\zeta \geq C(\xi_i - \xi_{i+1}) \text{ess inf}_{S_{\rho_2} \setminus S} f^{1/m}, \quad i = 1, 2, \ldots, l - 1.
\]

Summing the last expression over all \(1 \leq i \leq l - 1\), one can conclude that
\[
\int_{J(\rho_1)} g^{-1/m}(\beta \zeta / 2) \xi^{1/m-1} \, d\zeta \geq C(\rho_1 - \rho_0) \text{ess inf}_{S_{\rho_2} \setminus S} f^{1/m}.
\]

(3.11)

Now, let \(\xi_l = \rho_0\). In this case, \(\xi_{l-1} - \xi_l \geq (\rho_1 - \rho_0)/2\) and (3.10) implies the estimate
\[
\frac{\int_{J(\xi_l)} 1/m(\xi_{l-1})}{g^{1/m}(\beta J(\xi_{l-1}))} \geq C(\rho_1 - \rho_0) \text{ess inf}_{S_{\rho_2} \setminus S} f^{1/m},
\]
combining which with the inequality
\[
\int_{J(\xi_{l-1})} g^{-1/m}(\beta / 2) \xi^{1/m-1} \, d\zeta \geq \frac{2^{1/m-1} \int_{J(\xi_{l-1})} 1/m(\xi_{l-1})}{g^{1/m}(\beta J(\xi_{l-1}))},
\]
we again arrive at (3.11). In its turn, by the change of variable \(t = \beta \zeta / 2\), formula (3.11) can be transformed into
\[
\left(\frac{2}{\beta}\right)^{1/m} \int_{J(\rho_1)/2} g^{-1/m}(t) t^{1/m-1} \, dt \geq C(\rho_1 - \rho_0) \text{ess inf}_{S_{\rho_2} \setminus S} f^{1/m},
\]
whence (3.8) follows at once. \(\square\)

**Proof of Theorem 3** Assume the converse. Let the singular set \(S\) is not removable. In this case, in accordance with Lemma 3 condition (3.7) is valid. As in the proof of Lemma 4, we denote by \(C\) various positive constants which can depend only on \(A, n, m, \varepsilon, \varepsilon_0\), and the first summand in the left-hand side of (3.1).

Consider a subsequence of the sequence \(r_i = 2^{-i} r_0, i = 1, 2, \ldots\), such that
\[
\text{mes } S_{r_{ij}} \setminus S_{r_{ij+1}} \geq \text{mes } S_{r_{ij+1}} \setminus S_{r_{ij+2}} \quad (3.12)
\]
for all \(j = 1, 2, \ldots\). Such a subsequence obviously exists; otherwise we have
\[
\text{mes } S_{r_{i+1}} \setminus S_{r_{i+2}} > \text{mes } S_{r_i} \setminus S_{r_{i+1}}
\]
for all \(i \geq l\), where \(l\) is some positive integer. Hence,
\[
\text{mes } S_{r_{i+1}} \setminus S_{r_{i+2}} > \text{mes } S_{r_i} \setminus S_{r_{i+1}} > 0
\]
for all \( i \geq l \). This contradicts the fact that

\[
\sum_{i=l}^{\infty} \mes S_{r_i} \setminus S_{r_{i+1}} = \mes S_{r_i} \setminus S < \infty.
\]

Taking into account (2.17), we obtain

\[
n - k - \delta_j \leq \frac{\log \mes S_{r_{i_j}} \setminus S}{\log r_{i_j}}, \quad j = 1, 2, \ldots,
\]

where \( \delta_j \to 0 \) as \( j \to \infty \), whence it follows that

\[
\mes S_{r_{i_j}} \setminus S \leq r_{i_j}^{n-k-\delta_j}, \quad j = 1, 2, \ldots
\]

(3.13)

Lemma 4 with \( \rho_0 = r_{i_j+2}, \rho_1 = r_{i_j+1}, \) and \( \rho_2 = r_{i_j} \), yields

\[
\frac{1}{\mes S_{r_{i_j}} \setminus S_{r_{i_j+1}}} \int_{S_{r_{i_j}} \setminus S_{r_{i_j+1}}} |u| \, dx \leq 2 \beta_j G^{-1} \left( C \rho_{i_j}^{1/m} r_{i_j} \essinf_{S_{r_{i_j}} \setminus S} f^{1/m} \right),
\]

(3.14)

where

\[
\beta_j = \frac{\mes S_{r_{i_j}} \setminus S_{r_{i_j+1}}}{\mes S_{r_{i_j}} \setminus S_{r_{i_j+2}}}, \quad j = 1, 2, \ldots
\]

In view of (3.13) and the inequality

\[
\frac{1}{2} \leq \beta_j \leq 1
\]

(3.15)

which follows from (3.12), formula (3.14) leads to the estimate

\[
\frac{1}{(r_{i_j} - r_{i_j+1})^m} \int_{S_{r_{i_j}} \setminus S_{r_{i_j+1}}} |u| \, dx \leq 2^{m+2} r_{i_j}^{n-k-m-\delta_j} G^{-1} \left( C r_{i_j} \essinf_{S_{r_{i_j}} \setminus S} f^{1/m} \right)
\]

for all \( j = 1, 2, \ldots \). According to (2.3), the right-hand side of the last expression tends to zero as \( i \to \infty \). This contradicts (3.7).

\[\boxdot\]

**Proof of Theorem 1** We have

\[
S_r \setminus S \subset (\partial S)_r
\]

(3.16)
for all real numbers $r > 0$, where $(\partial S)_r$ is the set of the points $x \in \mathbb{R}^n$ the distance from which to \( \partial S \) is less than \( r \). Consequently,

$$\liminf_{r \to +0} \frac{\log \text{mes}(\partial S)_r}{\log r} \geq n - \liminf_{r \to +0} \frac{\log \text{mes} S_r \setminus S}{\log r}.$$ 

Thus, to complete the proof, it remains to use Theorem 3. \( \Box \)

**Proof of Theorem 4** Assume by contradiction that the singular set \( S \) is not removable. Then in accordance with Lemma 3 condition (3.7) is valid. We agree to denote by \( C \) various positive constants which can depend only on \( A, n, m, k, \varepsilon, \varepsilon_0 \), the limit in (2.18), and the first summand in the left-hand side of (3.1).

Let us put \( r_i = 2^{-i} r_0 \), \( i = 1, 2, \ldots \). By (2.18), we have

$$\text{mes} S_{r_i} \setminus S \leq C r_i^{n-k}$$

for all sufficiently large \( i \). As in the proof of Theorem 3, consider a subsequence of the sequence \( \{r_i\}_{i=1}^{\infty} \) satisfying condition (3.12). Taking \( \rho_0 = r_{i+2} \), \( \rho_1 = r_{i+1} \), and \( \rho_2 = r_i \) in Lemma 4, we obviously obtain (3.14), whence in accordance with (3.15) and (3.17) it follows that

$$\frac{1}{(r_{ij} - r_{ij+1})^m} \int_{S_{r_{ij}} \setminus S_{r_{ij+1}}} |u| \, dx \leq 2^{m+2} r_{ij}^{n-k-m} G^{-1} \left( C r_{ij} \text{ess inf} \int_{S_{r_{ij}} \setminus S} f^{1/m} \right)$$

for all \( j = 1, 2, \ldots \). In view of (2.6), this yields

$$\limsup_{j \to \infty} \frac{1}{(r_{ij} - r_{ij+1})^m} \int_{S_{r_{ij}} \setminus S_{r_{ij+1}}} |u| \, dx < \infty.$$ 

Hence, putting \( r = r_i \) and \( \rho = r_{i+1} \) in Lemma 2 and passing to the limit as \( j \to \infty \), we obtain (3.5), whence it follows that

$$\lim_{i \to \infty} \int_{S_{r_i} \setminus S_{r_{i+1}}} f(x) g(|u|) \, dx = 0. \quad (3.18)$$

Since \( g \) is a convex function, we have

$$\frac{1}{\text{mes} S_{r_i} \setminus S_{r_{i+1}}} \int_{S_{r_i} \setminus S_{r_{i+1}}} g(|u|) \, dx \geq g \left( \frac{1}{\text{mes} S_{r_i} \setminus S_{r_{i+1}}} \int_{S_{r_i} \setminus S_{r_{i+1}}} |u| \, dx \right)$$

for all \( i = 1, 2, \ldots \). At the same time, by (3.7), Lemma 2 implies the estimate

$$\int_{S_{r_i} \setminus S_{r_{i+1}}} |u| \, dx \geq C r_i^m \int_{S_{r/2} \setminus S_{r}} f(x) g(|u|) \, dx, \quad i = 1, 2, \ldots .$$
Hence,
\[
\frac{1}{\text{mes } S_{r_i} \setminus S_{r_{i+1}}} \int_{S_{r_i} \setminus S_{r_{i+1}}} g(|u|) \, dx 
\geq g \left( \frac{C r_i^m}{\text{mes } S_{r_i} \setminus S_{r_{i+1}}} \int_{S_{r_i/2} \setminus S_{r_i}} f(x) g(|u|) \, dx \right)
\]
for all \( i = 1, 2, \ldots \). Combining this with the inequalities
\[
\int_{S_{r_i} \setminus S_{r_{i+1}}} f(x) g(|u|) \, dx \geq \text{ess inf } f \int_{S_{r_i} \setminus S_{r_{i+1}}} g(|u|) \, dx, \quad i = 1, 2, \ldots,
\]
we obtain
\[
\int_{S_{r_i} \setminus S_{r_{i+1}}} f(x) g(|u|) \, dx \geq \text{ess inf } f \text{ mes } S_{r_i} \setminus S_{r_{i+1}} \times g \left( \frac{C r_i^m}{\text{mes } S_{r_i} \setminus S_{r_{i+1}}} \int_{S_{r_i/2} \setminus S_{r_i}} f(x) g(|u|) \, dx \right)
\]
for all \( i = 1, 2, \ldots \). This can obviously be written in the form
\[
\int_{S_{r_i} \setminus S_{r_{i+1}}} f(x) g(|u|) \, dx \geq \text{Cess inf } f r_i^m \frac{g(t_i)}{t_i} \int_{S_{r_i/2} \setminus S_{r_i}} f(x) g(|u|) \, dx, \quad (3.19)
\]
where
\[t_i = \frac{C r_i^m}{\text{mes } S_{r_i} \setminus S_{r_{i+1}}} \int_{S_{r_i/2} \setminus S_{r_i}} f(x) g(|u|) \, dx, \quad i = 1, 2, \ldots.\]

By (2.18), we have
\[t_i \geq C r_i^{m-n+k} \int_{S_{r_i/2} \setminus S_{r_i}} f(x) g(|u|) \, dx\]
for all sufficiently large \( i \). In view of (2.7), \( f \) is a positive function in a neighborhood of the set \( S \); therefore,
\[
\lim_{i \to \infty} \int_{S_{r_i/2} \setminus S_{r_i}} f(x) g(|u|) \, dx = \int_{S_{r_i/2} \setminus S} f(x) g(|u|) \, dx > 0.
\]
Really, if the last inequality is not valid, then \( g(u) \) and, consequently, \( u \) are identically equal to zero almost everywhere in \( S_{r_i/2} \setminus S \). This obviously means that the singular set \( S \) is removable, while we have assumed that this is not the case.

Thus, in accordance with (2.7) for all sufficiently large \( i \) the right-hand side of (3.19) is bounded from below by a positive constant independent of \( i \). We arrive at a contradiction with (3.18). \( \square \)
**Proof of Theorem 2** By (3.16), if \( \partial S \) has finite \( k \)-dimension, then (2.18) is valid. Thus, Theorem 2 follows immediately from Theorem 4. \( \square \)

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**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest

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