On a subject of diverse improvisations: 
The uncertainty relations on a circle

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Abstract. The disputed question of uncertainty relations (UR) on a circle is regarded as a particular element of a more general problem which refers to the quantum description of angular observables $L_z$ and $\phi$. The improvised $L_z - \phi$ UR are found to be affected by unsourmontable shortcomings. Also in contradiction with a largely accepted belief it is proved that the usual procedures of quantum mechanics are accurately applicable for the $L_z - \phi$ pair. The applicability regards both the known circular motions and the less known non-circular rotational motions. The presented facts contribute as arguments to the following indubitable conclusions: (i) the traditional interpretation of UR must be denied as an incorrect doctrine, (ii) for a natural physical consideration of the the $L_z - \phi$ pair the results from the usual quantum mechanics are sufficient while the improvised $L_z - \phi$ UR must be rejected as senseless formulas and (iii) the descriptions of quantum measurements have to be done in a framework which is distinct and additional in respect with the usual quantum mechanics.

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1. Introduction

A recent work [1] reminds the fact that, in spite of its importance and long history, the problem of uncertainty relations (UR) on a circle still remains open. Then, by using an ingenious labour, it is improvised a new such UR which, in its essence regards the angular variables $L_z$ (or $J$) and $\varphi$ ($z$-component of angular momentum and azimuthal angle). So the known class [2–17] of improvised $L_z - \varphi$ UR is enlarged. The relations from the mentioned class are introduced through esoteric and dissonant considerations. The respective considerations seem to be motivated exclusively by the preoccupation of obtaining an accordance with the traditional interpretation of UR (TIUR). They are also associated with the belief that the usual quantum mechanics (QM) procedures (i.e. operator-based and Fourier transform approaches) are not applicable for the $L_z - \varphi$ pair.

In this paper, in Sec.II, we point out the fact that the class of improvised $L_z - \varphi$ UR are affected by shortcomings (dissimilarities and deficiencies). The respective shortcomings are unsurmountable because they cannot be avoided through valid arguments (of physical and/or mathematical nature).

Contrary to the above-alluded belief, in Sec.III, we reveal that the usual QM procedures are applicable directly, on just ways, to the $L_z - \varphi$ pair. We show that the respective procedures work naturally (i.e. without any esoteric considerations) for all the rotational motions which the improvised $L_z - \varphi$ UR refer to. Moreover we find that the same procedures are also valid for the cases of non-circular rotations, never refered by the mentioned impovisations. Those cases regard the quantum torsion pendulum (QTP) and degenerate spatial rotations.

For QM problems the findings from Sec.III are things of technical nature - i.e. they report bare mathematical results. But as it is known QM is also involved in controversial questions connected with TIUR. The significance of the mentioned findings for TIUR and related questions are dicused in Sec.IV. The respective dicussions are focused on the known fact that the disputes regarding the $L_z - \varphi$ pair originate from TIUR and from the associated belief about the quantum measurements. Such a focusing dicloses a lot of incontestable arguments which, taken together with other doubtless facts [18–26], entail the conclusion that TIUR must be denied as an incorrect doctrine. Associated with the mentioned conclusion we find that the usual QM procedures are sufficient for a natural and correct description of the observables $L_z$ and $\varphi$. Consequently the improvised $L_z - \varphi$ UR have to be rejected as formulas without any physical significance. In the same context we opine that the descriptions of the quantum measurements necesitate a distinct framework, additional to the usual QM.

2. Improvised $L_z - \varphi$ relations and related shortcomings

The largely known searches and disputes about the $L_z - \varphi$ UR refer exclusively to the restricted class of circular motions. As concrete examples (usually not listed explicitely in the literature) the mentioned class includes: the motion of a particle on a circle, the
plane rotations of a rotator with fixed axis and the completely indexed spatial rotations. The alluded kind of spatial rotations refer to: (i) a particle on a sphere, (ii) a rotator with mobile axis and (iii) an electron in a hydrogen-like atom. By completely indexed we denote a situation for which the corresponding wave function has unique values for all the implied quantum numbers (i.e. for \( m \) and \( l \) in the cases (i) and (ii) respectively for \( m, l \) and \( n \) in the case (iii) - \( m, l \) and \( n \) being the magnetic, orbital and principal quantum numbers).

For all circular motions the part of the wave function regarding the \( L_z - \varphi \) pair has the form

\[
\Psi_m(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi}, \quad \varphi \in [0, 2\pi]
\]

with a single value for the integer number \( m \). On the other hand, according to the usual QM procedures, the observables \( L_z \) and \( \varphi \) are described by the operators

\[
\hat{L}_z = -i\hbar \frac{\partial}{\partial \varphi}, \quad \hat{\varphi} = \varphi.
\]

respectively by the commutation relation

\[
[\hat{L}_z, \hat{\varphi}] = -i\hbar
\]

For the observables \( L_z \) and \( \varphi \) in the situations described by (1) the corresponding standard deviations \( \Delta L_z \) and \( \Delta \varphi \) (defined in the usual sense - see also the next section) have the expressions

\[
\Delta L_z = 0, \quad \Delta \varphi = \frac{\pi}{\sqrt{3}}
\]

But such expressions are incompatible with the formula

\[
\Delta L_z \cdot \Delta \varphi \geq \frac{\hbar}{2}
\]

required by TIUR for two observables described by conjugated operators as the ones given by (2) and (3).

In order to avoid the mentioned incompatibility many scientists sustained the belief that for the \( L_z - \varphi \) pair the usual QM procedures do not work correctly. Consequently it was accredited the idea that formula (5) must be prohibited and replaced with other \( L_z - \varphi \) UR. So, along the years, instead of (5) there were promoted a lot of improvised \( L_z - \varphi \) UR such are

\[
\frac{\Delta_x L_z \cdot \Delta \varphi}{1 - 3(\Delta \varphi/\pi)^2} \geq 0.16\hbar
\]

\[
\frac{(\Delta L_z)^2 \cdot (\Delta \varphi)^2}{1 - (\Delta \varphi)^2} \geq \frac{\hbar^2}{4}
\]

\[
(\Delta L_z)^2 + \left( \frac{\hbar}{2} \alpha \right)^2 \cdot (\Delta \varphi)^2 \geq \frac{\hbar^2}{2} \left[ \left( \frac{9}{\pi^2} + \alpha^2 \right)^{1/2} - \frac{3}{\pi^2} \right]
\]

\[
\frac{\Delta_x L_z \cdot \Delta \varphi}{1 - 3(\Delta \varphi/\pi)^2} \geq \hbar \frac{2}{3} \left( \frac{V_{\min}}{V_{\max}} \right)
\]
\[(\Delta L_z)^2 \cdot (\Delta \sin \varphi)^2 \geq \frac{\hbar^2}{4}(\cos^2 \varphi)\] (10)

\[(\Delta L_z)^2 \cdot (\Delta \cos \varphi)^2 \geq \frac{\hbar^2}{4}(\sin^2 \varphi)\] (11)

\[\Delta L_z \cdot \Delta \chi \geq \frac{\hbar}{2}\] (12)

\[\Delta L_z \cdot \Delta \varphi \geq \frac{\hbar}{2} |\langle \varepsilon(\varphi) \rangle|\] (13)

\[(\Delta L_z)^2 + \hbar^2(\Delta \varphi)^2 \geq \hbar^2\] (14)

\[\Delta L_z \cdot \Delta \varphi \geq \frac{\hbar}{2} \left| 1 - 2\pi |\Psi(2\pi)|^2 \right|\] (15)

In (8) \(\alpha\) is a real parameter. In (9) \(V_{\min}\) and \(V_{\max}\) represent the minimum respectively the maximum values of \(V(\beta) = \int_{-\pi}^{\pi} \beta |\Psi(\alpha + \beta)|^2 d\varphi\) where \(\beta \in [-\pi, \pi]\) and \(\Psi\) denotes the wave function. In (12) \(\chi = \varphi + 2\pi N\), \(\Delta \chi = \left[2\pi^2 \left(\frac{1}{12} + N^2 - N_1^2 + N - N_1\right)\right]^{1/2}\), while \(N\) and \(N_1\) with \(N \neq N_1\), denote two arbitrary integer numbers. In (13) \(\varepsilon(\varphi)\) is a complicated expression of \(\varphi\). Relation (14) is written in original version [1] with \(\hbar = 1\) and \(L_z = J\).

A minute examination of the facts shows that, in its essence, the set of improvised relations (6)-(15) is affected by the following shortcomings (Shc):

- **Shc.1**: None of the relations (6)-(15) is unanimously accepted as a correct version for theoretical \(L_z - \varphi\) UR.
- **Shc.2**: From a mathematical perspective the relations (6)-(13) are not mutually equivalent.
- **Shc.3**: The relations (6)-(14) do not have correct supports in the usual formalism of QM (that however works very well in a huge number of applications).
- **Shc.4**: In fact the considerations implied in the promotion of the relations (6)-(14) do not have real physical motivations (argumentations).

**Observation** We do not associate the formula (15) with Shc.3 and Shc.4 because it proves itself to be a relation derivable from the usual QM procedures (see the relation (12) in Sec.III).

Relations (6)-(15) refer to the circular motions in which \(L_z\) and \(\varphi\) are in postures of basic observables. But \(L_z\) and \(\varphi\) are also in similar postures in the cases of non-circular rotations (NCR). Within the class of NCR we include: motion of quantum torsion pendulum (QTP) and degenerated (or incompletely indexed) spatial rotations. The alluded degenerate rotations refer to: (i) a particle on a sphere, (ii) a rotator with mobile axis and (iii) an electron in a hydrogen-like atom. By degenerate motions we refer to the situations when the energy of sistem is well precised while the non-energetic quantum numbers take all the permitted values. Such numbers are \(m\) in the cases (i) and (ii) respectively \(l\) and \(m\) in the case (iii).
From the class of NCR let us firstly refer to the motions of a QTP, which is a quantum harmonic oscillator described by the Hamiltonian

\[ \hat{H} = \frac{1}{2I} \hat{\mathbf{L}}^2 + \frac{I\omega^2}{2} \varphi^2 = -\frac{\hbar^2}{2I} \partial^2 + \frac{I\omega^2}{2} \varphi^2. \]  

with \( I = \) moment of inertia and \( \omega = \) angular frequency. Consequently the states of QTP are described by the wave functions

\[ \Psi_n(\varphi) = \Psi_n(\xi) \sim \exp \left\{-\frac{\xi^2}{2}\right\} \mathcal{H}_n(\xi), \quad \xi = \varphi\sqrt{\frac{I\omega}{\hbar}} \in (-\infty, \infty) \]  

and energies

\[ E_n = \hbar\omega \left(n + \frac{1}{2}\right) \]  

In (17) and (18) \( n = 0, 1, 2, \ldots \) = oscillation quantum number and \( \mathcal{H}_n(\xi) \) denote the Hermite polynomials. Also in the case of QTP the observables \( L_z \) and \( \varphi \) are described by the operators presented in (2) and (3). Then, by using (17), similarly with the case of recti-linear oscillator for the standard deviations \( \Delta L_z \) and \( \Delta \varphi \) one obtains the expressions

\[ \Delta L_z = \sqrt{\hbar I\omega \left(n + \frac{1}{2}\right)}, \quad \Delta \varphi = \sqrt{\frac{\hbar}{I\omega \left(n + \frac{1}{2}\right)}} \]  

With these expressions one finds that for QTP the \( L_z - \varphi \) pair satisfies the “prohibited” formula (5).

From the same class of NCR now let us discuss the cases of degenerate motions such one finds for a particle on a sphere as well as for a rotator with mobile axis (the problem of an atomic electron can be discussed in a completely similar way). In both the mentioned cases a degenerate state corresponds to an energy

\[ E_l = \frac{\hbar^2}{2I} l(l + 1) \]  

where \( I \) is the moment of inertia and \( l \) denotes the orbital quantum number, which has a well-precised value. Such a state is degenerate in respect with the magnetic quantum number \( m \) that has the permitted values \( m = 0, \pm 1, \pm 2, \ldots, \pm l \). Consequently the wave function of the respective state is of the form

\[ \Psi_l(\theta, \varphi) = \sum_{m=-l}^{l} c_m Y_{lm}(\theta, \varphi), \quad \varphi \in [0, 2\pi] \]  

where \( Y_{lm}(\theta, \varphi) \) are the spherical functions and \( c_m \) denote complex coefficients, which satisfy the condition

\[ \sum_{m=-l}^{l} |c_m|^2 = 1 \]
By using the wave functions $\Psi_l(\theta, \phi)$ given by (21) respectively the operators $\hat{L}_z$ and $\hat{\phi}$ as defined in (2)-(3) one finds

\[
(\Delta L'_z)^2 = \sum_{m=-l}^{l} |c_m|^2 \hbar^2 m^2 - \left[ \sum_{m=-l}^{l} |c_m|^2 \hbar m \right]^2
\]  \hspace{1cm} (23)

\[
(\Delta \phi)^2 = \sum_{m=-l}^{l} \sum_{m'=-l}^{l} c_m^* c_{m'} (Y_{lm}, \phi^2 Y_{lm'}) - \left[ \sum_{m=-l}^{l} \sum_{m'=-l}^{l} c_m^* c_{m'} (Y_{lm}, \phi Y_{lm'}) \right]^2
\]  \hspace{1cm} (24)

Where $(f, g)$ denote the scalar product of the functions $f$ and $g$ (for notations see also the next section). With (23) and (24) in the cases described by (21), one finds that it is possible for the “prohibited” relation to be verified. The respective possibility is conditioned by the concrete values of the coefficients $c_m$.

The above presented facts in connection with NCR, regarded together with the mentioned discussions about the improvised $L_z - \phi$ UR (6)-(15), induce the following questions (Q):

- **Q.1**: What is the significance of the respective facts for TIUR?
- **Q.2**: Are the usual QM procedures really inapplicable for the $L_z - \phi$ pair?
- **Q.3**: Must the set of improvised relations (6)-(14) be accepted as a natural and justified thing within the healthful framework of physics?

Related to **Q.1** we opine that the mentioned facts are deprecatory elements for TIUR because they increase the deadlook disclosed by the shortcomings Shc.1-4. The gravity of the respective deadlook and its consequences will be discussed in Sec.IV.

As regards **Q.2** the answer is negative. In the next section we prove effectively that the usual QM procedures are accurately applicable for $L_z - \phi$ pair in respect to all the physical situations. Based on the respective proof in Sec.IV we find a credible verdict of non-acceptance for relations mentioned in **Q.3**.

### 3. The usual QM procedures for $L_z - \phi$ pair

The goal of this section is to prove that, in contradiction with a largely accredited belief, the usual QM procedures are accurately applicable for the $L_z - \phi$ pair. The procedures in question regard the operator-based respectively the Fourier transforms approaches. Details about the main particularities of the alluded approaches for the case of $L_z - \phi$ pair are presented below in the subsections 3.1 respectively 3.2.

#### 3.1. The operator-based approach

In order to present this approach for the problems of $L_z - \phi$ pair let us consider a quantum system in a rotational motion irrespective of circular or NCR type. The state of the system is regarded as described by the wave function $\Psi(q, \phi)$, indifferently of the fact that $\phi \in [0, 2\pi]$ (as in (1) and (21)) or $\phi \in (-\infty, \infty)$ (as in (17)). In $\Psi(q, \phi)$ by $q$
we denote the orbital coordinates other than \( \varphi \) and specific for the considered system (e.g. \( q \) is: (i) absent in the cases of a motion on a circle or of a rotator with fixed axis, (ii) the polar angle \( \theta \) in the cases of a particle on a sphere or of a rotator with a mobile axis, respectively, (iii) the ensemble of both polar angle \( \theta \) and radial distance \( r \) in the case of an atomic electron). In the functions space to which belong \( \Psi(q, \varphi) \) the scalar product is defined as

\[
(\Psi_1, \Psi_2) = \int \Psi_1^*(q, \varphi) \Psi_2(q, \varphi) d\Omega_q d\varphi \tag{25}
\]

where \( d\Omega_q \) denotes the infinitesimal “volume” associated with the variables \( q \) (i.e. \( d\Omega_q = \sin \theta d\theta \) or \( d\Omega_q = r^2 \sin \theta dr d\theta \) in the above mentioned cases (ii) respectively (iii)).

For the considered system \( L_z \) and \( \varphi \) are basic observables described by the operators \( \hat{L}_z \) and \( \hat{\varphi} \) mentioned in (2) and (3). Associated with the respective observables we use the following quantities:

\[
\langle A \rangle = (\Psi, \hat{A}\Psi) \tag{26}
\]

\[
C(A, B) = (\delta \hat{A}\Psi, \delta \hat{B}\Psi), \quad \delta \hat{A} = \hat{A} - \langle A \rangle \tag{27}
\]

\[
\Delta A = \sqrt{C(A, A)} = (\delta \hat{A}\Psi, \delta \hat{B}\Psi)^{1/2} \tag{28}
\]

which denote: \( \langle A \rangle \) = the mean (or expected) value of the observable \( A \), \( C(A, B) \) = the correlation of \( A \) and \( B \) respectively \( \Delta A \) = standard deviation of \( A \). Note that in terms of usual QM the quantities (26)-(28) are probabilistc parameters (characteristics): of first order \( \langle A \rangle \), respectively of second order - \( C(A, B) \) and \( \Delta A \).

In terms of the above-presented notations the following Schwartz relation is always satisfied:

\[
(\delta \hat{L}_z \Psi, \delta \hat{L}_z \Psi) \cdot (\delta \hat{\varphi} \Psi, \delta \hat{\varphi} \Psi) \geq |(\delta \hat{L}_z \Psi, \delta \hat{\varphi} \Psi)|^2 \tag{29}
\]

which gives directly

\[
\Delta L_z \cdot \Delta \varphi \geq |(\delta \hat{L}_z \Psi, \delta \hat{\varphi} \Psi)| \tag{30}
\]

This is a general \( L_z - \varphi \) relation, valid for all types of rotational motions.

A particular but restrictive \( L_z - \varphi \) relation can be obtained from (31) as follows. If in respect to the operators \( \hat{A}_j = \hat{L}_z \) and \( \hat{A}_2 \) the wave function \( \Psi \) of the system satisfy the conditions

\[
(\hat{A}_j \Psi, \hat{A}_k \Psi) = (\Psi, \hat{A}_j \hat{A}_k \Psi), \quad j, k = 1, 2 \tag{31}
\]

one can write

\[
(\delta \hat{L}_z \Psi, \delta \hat{\varphi} \Psi) = \frac{1}{2} \left\langle \left[ \delta \hat{L}_z, \delta \hat{\varphi} \right]_+ \right\rangle + \frac{1}{2} \left\langle [\hat{L}_z, \hat{\varphi}] \right\rangle + \frac{i}{2} \frac{\hbar}{2} \tag{32}
\]
On ... the uncertainty relations on circle

where \( \langle [\delta \hat{L}_z, \delta \hat{\varphi}]_+ \rangle = \langle \delta \hat{L}_z \delta \hat{\varphi} + \delta \hat{\varphi} \delta \hat{L}_z \rangle \) is a real quantity. Then by using (32) from (30) one obtains the restricted formula

\[
\Delta L_z \cdot \Delta \varphi \geq \frac{\hbar}{2} \tag{33}
\]

This is just the disputed and “prohibited” relation (5).

The above results ensure a veridical base for a natural resolution of the known disputes regarding the relation (B3)/(F) respectively the applicability of the usual QM procedures for the observables \( L_z \) and \( \varphi \). The mentioned base has to incorporate obligatorily the whole ensamble of the following evident mathematical findings (MF):

- **MF.1**: The general relation (B0) as well as its restricted form (B3)/(F) are obtainable through rigurous and precisely specified mathematical ways.

- **MF.2**: In the cases described by (1) one finds

\[
(\hat{L}_z \Psi, \hat{\varphi} \Psi) - (\Psi, \hat{L}_z \hat{\varphi} \Psi) = i\hbar \tag{34}
\]

and, consequently, the conditions (B1) are not satisfied. This means that for such cases the relation (B3)/(F) is not mathematically applicable. However even in the respective cases the general relation (B0) remains valid, degenerating into the trivial equality 0 = 0.

- **MF.3**: The conditions (B1) are always satisfied in the case of QTP described by (17). Consequently for the respective case the relation (B3)/(F) is mathematically applicable. But note that for the same case the general relation (B0) remains also valid.

- **MF.4**: For the cases described by (21) one obtains:

\[
(\hat{L}_z \Psi_l, \hat{\varphi} \Psi_l) - (\Psi_l, \hat{L}_z \hat{\varphi} \Psi_l) =
\]

\[
i\hbar \left\{ 1 + 2 \text{Im} \left[ \sum_{m=-l}^{l} \sum_{m'=-l}^{l} c_m^* c_{m'} \langle Y_{lm}, \hat{\varphi} Y_{lm'} \rangle \right] \right\} \tag{35}
\]

(with \( \text{Im} F = \text{imaginary part of} \ F \)). Then it results that for such cases the conditions (B1) are satisfied or violated in the situations when the expression after the equality sign in (B5) is null respectively non-null. Corespondingly one finds situations in which the relation (B3)/(F) is true respectively wrong. But note that in both types of situations the general relation (B0) remains valid.

Now, in order to facilitate a discussion in Sec.IV, let us refer to the pair \( \theta - \varphi \) (polar and azimuthal angles) considered for the situation described by (21). For such a situation, similarly with (B0), it is satisfied the relation

\[
\Delta \theta \cdot \Delta \varphi \geq \left| \langle \delta \hat{\theta} \Psi, \delta \hat{\varphi} \Psi \rangle \right| \tag{36}
\]

In this relation we take \( \hat{\theta} = \theta \cdot \) and the usual notations introduced above. Additionally we specify that for the right hand side term from (B7) we have

\[
\langle \delta \hat{\theta} \Psi, \delta \hat{\varphi} \Psi \rangle = \langle \theta \varphi \rangle - \langle \theta \rangle \cdot \langle \varphi \rangle \tag{37}
\]
with the detailed notation
\[
\langle A \rangle = (\Psi, \hat{A}\Psi) = \sum_{m=-l}^{l} \sum_{m'=-l}^{l} c_m c_{m'} (Y_{lm}, \hat{A}Y_{lm'})
\]  
(38)

From (37) and (38) one observes that, depending on the concrete values of the coefficients \(c_m\), one can find situations when the quantity \(\langle \delta \hat{\theta} \Psi, \delta \hat{\varphi} \Psi \rangle\) have a non-null value. In such situations the relation (36) is satisfied as a formula with a non-null term in its right hand side.

3.2. The Fourier transforms approach

The particularities of \(L_z - \varphi\) pair in respect with various quantum rotational motions can also be approached in terms of Fourier transform as follows.

Firstly let us refer to the case of QTP described by the wave functions \(\Psi = \Psi(\varphi)\) given by (17) and defined for \(\varphi \in (-\infty, \infty)\). For the respective case the alluded approach was not discussed in literature but it can be managed by analogy with the known treatment [28] of the Cartesian coordinate \(x\) and momentum \(p\) for \(x \in (-\infty, \infty)\). Then with \(\Psi(\varphi)\) we associate the Fourier transform \(\tilde{\Psi}(k), k \in (-\infty, \infty)\), defined by
\[
\tilde{\Psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(\varphi) e^{-ik\varphi} \, d\varphi
\]  
(39)

Due to the Parseval theorem as well as to the fact that \(\Psi(\varphi)\) is normalised we can write
\[
\int_{-\infty}^{\infty} |\Psi(\varphi)|^2 \, d\varphi = \int_{-\infty}^{\infty} |\tilde{\Psi}(k)|^2 \, dk = 1
\]  
(40)

This relation shows that both \(|\Psi(\varphi)|^2\) and \(|\tilde{\Psi}(k)|^2\) can be regarded as probability densities for the random variables \(\varphi\) respectively \(k\). The mean (or expected) values of the quantities \(A = A(\varphi)\) and \(B = B(k)\) depending on the respective variables are defined as
\[
\langle A \rangle = \int_{-\infty}^{\infty} A(\varphi)|\Psi(\varphi)|^2 \, d\varphi
\]  
(41)
\[
\langle B \rangle = \int_{-\infty}^{\infty} B(k)|\tilde{\Psi}(k)|^2 \, dk
\]  
(42)

The following relation is evidently true
\[
\int_{-\infty}^{\infty} \left| \lambda(\varphi - \langle \varphi \rangle) \Psi(\varphi) + \left( \frac{d}{d\varphi} - i\langle k \rangle \right) \Psi(\varphi) \right|^2 \, d\varphi \geq 0
\]  
(43)
if \(\lambda\) is an arbitrary real parameter. By means of some simple calculations from (43) one obtains
\[
\langle (k - \langle k \rangle)^2 \rangle \cdot \langle (\varphi - \langle \varphi \rangle)^2 \rangle \geq \frac{1}{4}
\]  
(44)

Now by using the notation \(\Delta A = \langle (A - \langle A \rangle)^2 \rangle^{1/2}\) and taking \(\hbar k = L_z\) from (44) one finds directly
\[
\Delta L_z \cdot \Delta \varphi \geq \frac{\hbar}{2}
\]  
(45)
i.e. just the relation \( \frac{33}{5} \) which is true in the considered case of QTP.

In some publications one finds attempts of approaching with Fourier transforms the “periodic situations” in which \( \varphi \in [0, 2\pi] \) and \( \Psi(0) = \Psi(2\pi) \neq 0 \). The respective attempts can be resumed as follows. For the corresponding wave functions \( \Psi(\varphi) \) are defined the Fourier coefficients

\[
b_m = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \Psi(\varphi) e^{-im\varphi} d\varphi \quad (46)
\]

with \( m = 0, \pm 1, \pm 2, \ldots \). Based on the fact that \( \Psi(\varphi) \) is normalised on the range \([0, 2\pi]\) as well as on the Parseval theorem one writes

\[
\int_0^{2\pi} |\Psi(\varphi)|^2 d\varphi = \sum_m |b_m|^2 = 1 \quad (47)
\]

Then the quantities \( |\Psi(\varphi)|^2 \) and \( |b_m|^2 \) are regarded as probability density respectively probabilities for the continuous respectively discrete random variables \( \varphi \) and \( m \). For quantities \( A = A(\varphi) \) and \( B = B(m) \) depending on the variables \( \varphi \) and \( m \) the mean (or expected) values are

\[
\langle A \rangle = \int_0^{2\pi} A(\varphi) |\Psi(\varphi)|^2 d\varphi \quad (48)
\]

\[
\langle B \rangle = \sum_m B(m) |b_m|^2 \quad (49)
\]

With \( \lambda \) as an arbitrary real parameter the evident relation

\[
\int_0^{2\pi} \left| \lambda(\varphi - \langle \varphi \rangle) \Psi(\varphi) + \left( \frac{d}{d\varphi} - i\langle m \rangle \right) \Psi(\varphi) \right|^2 d\varphi \geq 0 \quad (50)
\]

together with the condition \( \Psi(0) = \Psi(2\pi) \neq 0 \) give

\[
\langle (m - \langle m \rangle)^2 \rangle \cdot \langle (\varphi - \langle \varphi \rangle)^2 \rangle \geq \frac{1}{4} \left( 1 - 2\pi |\Psi(2\pi)|^2 \right) \quad (51)
\]

Taking \( m\hbar = L_z \) and \( \Delta A = \langle (A - \langle A \rangle)^2 \rangle^{1/2} \) from (51) results directly

\[
\Delta L_z \cdot \Delta \varphi \geq \frac{\hbar}{2} \left| 1 - 2\pi |\Psi(2\pi)|^2 \right| \quad (52)
\]

So one finds in fact the relation (52), which sometimes in publications is mentioned as \( L_z - \varphi \) UR.

Now let us inspect the applicability of the relation (52) to the possible “periodic situations”, when \( \Psi(q, 0) = \Psi(q, 2\pi) \). Note that the category of such situations includes the cases of circular motions described by (1) as well as the NCR refered in (21). It is to easy see that in the first cases (52) is applicable under the form of trivial equality \( 0 = 0 \).

The cases refered to they in (21), even if they belong to the “periodic situations” as defined above, they remain outside the field of applicability for (52). This because for the respective cases the sequence of relations (46)-(52) as well as the implied quantities must
be adequately modified. So for $\Psi_l(\theta, \varphi)$ given by (21) instead of constant coefficients $b_m$ from (46) it is necessary to operate with the $\theta$-dependent coefficients

$$b_m(\theta) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \Psi_l(\theta, \varphi) e^{-im\varphi} d\varphi = c_m \Theta_{lm}(\theta)$$  \hspace{1cm} (53)$$

In (53) the coefficients $c_m$ are the same as in (21) while $\Theta_{lm}(\theta)$ denote the $\theta$-dependent part of the spherical function $Y_{lm}(\theta, \varphi)$ (which can be written as $Y_{lm}(\theta, \varphi) = \Theta_{lm}(\theta) (2\pi)^{-1/2} e^{im\varphi}$). For the here discussed cases instead of (50) we have

$$\int_0^{2\pi} \sin \theta d\theta \int_0^{2\pi} |\Psi_l(\theta, \varphi)|^2 d\varphi = \sum_{m=-l}^{l} |c_m|^2 = 1$$  \hspace{1cm} (54)$$

Also for the mean values instead of (48)-(49) we must take the expressions

$$\langle A \rangle = \int_0^{2\pi} \sin \theta d\theta \int_0^{2\pi} A(\varphi)|\Psi_l(\theta, \varphi)|^2 d\varphi$$  \hspace{1cm} (55)$$

$$\langle B \rangle = \sum_{m=-l}^{l} \int_0^{\pi} \sin \theta B(m) |b_m(\theta)|^2 d\theta = \sum_{m=-l}^{l} B(m)|c_m|^2$$  \hspace{1cm} (56)$$

For the NCR described by (21) instead of (50) we have to operate with the relation

$$\int_0^{2\pi} \sin \theta d\theta \int_0^{2\pi} \left| \lambda(\varphi - \langle \varphi \rangle) \Psi_l(\theta, \varphi) + \left( \frac{d}{d\varphi} - i\langle m \rangle \right) \Psi_{lm}(\theta, \varphi) \right|^2 d\varphi \geq 0$$  \hspace{1cm} (57)$$

From this relation, by taking $\hbar m = L_z$ and through some simple calculations, one finds the formula

$$\Delta L_z \cdot \Delta \varphi \geq \frac{\hbar}{2} \left| 1 - \sum_{m=-l}^{l} \sum_{m'=-l}^{l} c_m c_{m'} \gamma_{mm'} \right|$$  \hspace{1cm} (58)$$

where

$$\gamma_{mm'} = \int_0^{2\pi} \Theta_{lm}(\theta) \Theta_{lm'}(\theta) \sin \theta d\theta$$  \hspace{1cm} (59)$$

One can conclude that, from the Fourier transforms perspective, for the NCR described by (21) the true $L_z - \varphi$ relation is given by (58). In the respective relation the right hand term can be a null or non-null quantity depending on the concrete values of the coefficients $c_m$. Note here the fact that, even for $m \neq m'$ the quantities $\gamma_{mm'}$ (defined by (59)), can have non-null values (e.g. when $l = 2$ for $m = 1$ and $m' = -1$, respectively for $m = 2$ and $m' = -2$ one finds $\gamma_{mm'} = 1$.

In the end of this section we note that the above presented considerations offer the complete set of elements required for a correct placement of the $L_z - \varphi$ pair within the mathematical framework of usual QM procedures.
4. Concluding remarks

Now let us discuss the significance of the above findings for the $L_z - \varphi$ problems. As it was shown the respective problems regard the situation of observables $L_z$ and $\varphi$ within QM and related questions of interpretation. The alluded problems constitute the subject of disputes generated by the unsuccessful searches to adjust the $L_z - \varphi$ pair to the TIUR assertions. It is interesting to evaluate the significance of the previous findings for the mentioned disputes. The respective evaluation requires firstly an adequate presentation of TIUR. We try to do such a presentation here below.

The doctrine of TIUR germinates from the preoccupation for giving a unique and generic interpretation for Heisenberg’s relations

$$\Delta A \cdot \Delta B \geq \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle|$$  \hspace{1cm} (60)

$$\Delta_{TE} A \cdot \Delta_{TE} B \geq \hbar$$  \hspace{1cm} (61)

Note that, on the one side, the relations (60) are QM formulas written in terms of usual notations specified in Sec.III. They were introduced originally [29] by Heisenberg for the observables $x$ and $p$ (Cartesian coordinate and momentum). The modern form (60), that regard arbitrary observables $A$ and $B$, were introduced afterwards by other scientists (see [30–35]). On the other side the relations (61) are of “thought-experimental” (TE) or “mental” nature. They were introduced by means of some “thought” or “mental” experiments. In relations (61) the observables $A$ and $B$ are considered as being canonically conjugated. Note that the respective relations were introduced firstly [29] for pairs of observables $x - p$ respectively $E - t$ (energy-time). Later they were adjusted [36] for the $L_z - \varphi$ pair by using only some “mental” considerations (i.e. without any real experimental justification).

Starting from (60) and (61) TIUR was promoted as a new doctrine (for a bibliography of the significant publications in the field see [17, 30–35]. The UR (60)-(61) have a large popularity, being frequently regarded as crucial formulas of physics or [35] even as expression of “the most important principle of the twentieth century physics”. But as, as a strange aspect, in the partisan literature TIUR is often presented so fragmentary and esoteric that it seems to be rather a dim conception than a well-delimited scientific doctrine. However, in spite of such an aspect, from the mentioned literature one can infer that in fact TIUR is reducible to the following set of main assertions (Ass.):

- **Ass.1**: The quantities $\Delta A$ and $\Delta_{TE} A$ from (60) and (61) have similar significances of measurement uncertainty for the QM observable $A$.

- **Ass.2**: With (60) and (61) the same generic interpretation of uncertainty relations (UR) for simultaneous measurements regarding the observables $A$ and $B$ it is associated.

- **Ass.3**: For a solitary QM observable $A$ the quantity $\Delta A$ can be indefinitely small (even null).
• **Ass.4:** In the case of two “compatible” observables $A$ and $B$ (when $[\hat{A}, \hat{B}] = 0$), considered simultaneously, the corresponding quantities $\Delta A$ and $\Delta B$ are mutually independent, each of them being allowed to take an indefinitely small (even null) value.

• **Ass.5:** For two “incompatible” observables $A$ and $B$ (when $[\hat{A}, \hat{B}] \neq 0$), considered simultaneously, the quantities $\Delta A$ and $\Delta B$ are mutually interdependent. Their product $\Delta A \cdot \Delta B$ is lower limited by a non-null quantity dependent on the Planck’s constant $\hbar$.

• **Ass.6:** The relations (60) and (61) are typically QM formulas and they, as well as the Planck’s constant $\hbar$, do not have analogues in classical (non-quantum) physics.

The above assertions constitute the true reference points for the announced evaluation of the significance of the findings from Sec.II and Sec.III. Connected with the respective points the mentioned evaluation is expressible in terms of the following remarks (**Rem.**):

• **Rem.1:** First of all we note that the “mental” improvisations (61) must not be taken into account in evaluation of $L_z - \varphi$ problems or other questions of interest for physics. This because the relations (61) have only a provisional character while any respectable doctrine (as TIUR claims to be) must operate with permanent concepts and relations. The alluded character results from the fact that relations (61) were founded on old classical limitative criteria (introduced by Abbe and Rayleigh - see [37]). But in modern experimental physics are known [38– 40] some super-resolution techniques that overstep the respective criteria. This means that instead of the quantities $\Delta_{TE} A$ and $\Delta_{TE} B$ from (61) are imaginable some super-resolution TE (SRTE) uncertainties $\Delta_{SRTE} A$ and $\Delta_{SRTE} B$. Then it is possible to replace (61) with a SRTE relation of the form

$$\Delta_{SRTE} A \cdot \Delta_{SRTE} B < \hbar \tag{62}$$

Such a possibility clearly evidences the provisional and fictitious character of the relations (61). The respective evidence incriminates TIUR in connection with one of its points of origin, expressed by **Ass.1** and **Ass.2**. Consequently one can conclude that in in the discussions of $L_z - \varphi$ pair the results from the approaches above notified in Sec.III must be regarded as rigorous and indubitable findings. At the actual stage of scientific progress, it is senselessly to contest the
On ... the uncertainty relations on circle

respective results. Then the natural conclusion is that, in the $L_z - \varphi$ case, the debates have to accept the bare (mathematical) results of usual QM procedures and to reconsider the (physical or even philosophical) interpretation of the respective results.

• Rem.3: In the line with the conclusion from Rem.2 let us refer to the term “uncertainty” used by TIUR. The term regards the quantities like $\Delta A$ from (60) or, similarly the quantities $\Delta L_z$ and $\Delta \varphi$ from the relations presented in Sec.III. We think that the respective term is groundless because of the following facts. Being defined in the mathematical QM framework $\Delta A$ signifies a probabilistic parameter (standard deviation) of the observable $A$ regarded as a random variable. The mentioned framework deals with theoretical concepts and models regarding the intrinsic (inner) properties of the considered systems but not with elements referring to the (possible) measurements performable on the respective systems. Consequently, for a physical system, $\Delta A$ refers to the intrinsic characteristics, reflected in the fluctuations (deviations from the mean value) of the observable $A$. So considered, in spite of the assertions Ass.1 and Ass.2 the quantity $\Delta A$ has no connection with the performances (or “uncertainties”) of the possible measurements regarding the observable $A$. Moreover, for a system in a given state, $\Delta A$ has a unique and well defined value (connected with the corresponding wave function) but not conjectural and changeable evaluations (as it is asserted in Ass.3, Ass.4 and Ass.5).

• Rem.4: The above alluded conjectural evaluations can be associated with the measurements errors (due to the possible modifications in the performances of the measuring devices and techniques). But, as a general rule, they regard all the characteristics (i.e. the mean values and fluctuations) for every physical observable. Moreover such evaluations refer both to the quantum and to the classical physical observables, without any essential differences. Also, on a well-principled base, one can state that the description of measurements must not pertain to QM or to other chapters of theoretical physics. Probably that such a statement is not agreed by many of the TIUR partisans. However, we opine that the respective statement is consonant with the thinking [K]: “the word ‘measurement’ has been so abused in quantum mechanics that it would be good to avoid it altogether”. In the spirit of the mentioned statement the QM, as well as the whole theoretical physics, must be concerned only with the (conceptual and mathematical) models for the description of the intrinsic properties (characteristics) of physical systems.

• Rem.5: The quantities $\Delta L_z$ and $\Delta \varphi$ are rigorously implied in diverse relations, corresponding to various rotational motions (see Sec.III). The respective relations are not always consonant with the formula (61) agreed by TIUR. But such a dissonance clearly incriminates the TIUR assertion Ass.5.

The above remarks, directly connected with the $L_z - \varphi$ pair, disclose irregularities of the TIUR doctrine. The same irregularities as well as additional ones are revealed
by some other facts that do not have direct connections with the $L_z - \varphi$ pair. We try to present to present such facts in the following remarks.

- **Rem.6:** In disputes regarding the applicability of TIUR it is also known [4] the case of observables $N - \Phi$ (number-phase). It is easy to see [23] that, from the perspective of usual QM procedures, the case of $N - \Phi$ pair can be approached on a way completely analogue with the one used above in subsection 3.1 for $L_z - \varphi$ pair. This means that for TIUR the $N - \Phi$ pair entails an incrimination similar to the one mentioned in **Rem.5**.

- **Rem.7:** In Sec.III it was shown that it is possible to exist situations in which the angular observables $\theta$ and $\varphi$ satisfy the relation (36) with a non-null value for the term from the right hand side. But as $\theta$ and $\varphi$ are commutable ($[\hat{\theta}, \hat{\varphi}] = 0$) the mentioned situations are in posture to incriminate clearly the assertion Ass.4 of TIUR. Note that a similar situation can also be evidenced [23] for the case of other commutable observables (such is the case for the Cartesian coordinates $x$ and $y$ regarding a particle in a bi-dimensional potential well having inclined walls in respect with $x - y$ axes).

- **Rem.8:** As it is known TIUR promoted the idea that two observables $A$ and $B$ are denotable with the terms “compatible” respectively “incompatible” subsequently of the fact that their operators $\hat{A}$ and $\hat{B}$ are commutable ($[\hat{A}, \hat{B}] = 0$) or no ($[A, B] \neq 0$). The mentioned terms are directly connected with the fact that the product $\Delta A \cdot \Delta B$ does not have respectively has a non-null lower limit. Now one can see directly that the facts presented in the remarks **Rem.2,5,6** and **7** proves the desuetude of the respective TIUR idea. Particularly the alluded TIUR idea becomes self-contradictory in the $L_z - \varphi$ case when the lower limit of the product $\Delta L_z \cdot \Delta \varphi$ can be both null and non-null. Then, strangely, the same quantities $L_z$ and $\varphi$ appears as both “compatible” and “incompatible” observables.

- **Rem.9:** The quantities $\Delta L_z$ and $\Delta \varphi$ from (30) (and similarly $\Delta A$ and $\Delta B$ from (30)) are second order probabilistic parameters. Consequently (30) is a simple second order probabilistic formula. But (30) is generalisable in form of some extended relations referring also to the second order probabilistic parameters. So one obtains [16, 21, 23, 26]: (i) bi-temporal relations, (ii) many-observable relations and (iii) macroscopic quantum statistical relations. For the mentioned extended relations TIUR has to give an interpretation concordant with its own essence, if it is a well-grounded conception. But to find such an interpretation on natural ways (i.e. without esoteric and extra-physical considerations) seems to be a difficult (even impossible) task. In this sense it is significant to remind the lack of success connected with the above noted macroscopic relations. In order to adjust the respective relations to the TIUR assertions, among other things, it was promoted the idea of an appeal to the so-called “macroscopic operators” (see [14] and references). But in fact [23, 26] the mentioned appeal does not ensure for TIUR the avoidance of the involved shortcomings. Moreover the respective “macroscopic operators” appear
as being only fictitious concepts without any real applicability in physics. It is also interesting to observe that, in the last years the problem of the macroscopic relations and operators is eschewed in the partisan literature of TIUR, even if the respective problem still remains unclarified from the TIUR perspective.

**Rem.10:** In classical physics of probabilistic structure a nontrivial interest can also present the higher order parameters (correlations) (see [45, 46]). This fact suggests that in the case of QM, additionally to the second order quantities like $\Delta A = (\delta \hat{A} \Psi, \delta \hat{A} \Psi)^{1/2}$ or $(\delta \hat{A} \Psi, \delta \hat{B} \Psi)$ from (60) or (30), to use also higher order correlations such as $((\delta \hat{A})^r \Psi, (\delta \hat{B})^s \Psi)$ with $r + s \geq 3$. Then, naturally, for the respective correlations TIUR has to give an interpretation concordant with his own doctrine. But, in our opinion, it is less probable (or even excluded) that such an interpretation will be promoted by the TIUR partisans.

**Rem.11:** In contradiction with the assertion Ass.6 of TIUR in classical (non-quantum) physics [23, 47, 48] there are really some formulas that are completely similar to the QM relations (60) and (30). The alluded formulas can be written in the form

$$\Delta_{CF} A \cdot \Delta_{CF} B \geq |\langle \delta A \cdot \delta B \rangle_{CF}| \quad (63)$$

where the standard deviations $\Delta_{CF} A$ and $\Delta_{CF} B$ respectively the correlation $\langle \delta A \cdot \delta B \rangle_{CF}$ are referring to the classical fluctuations (CF) of the macroscopic observables $A$ and $B$. One can see that in fact both classical formulas (63) and QM relations (60) and (30) imply only second order probabilistic parameters (standard deviations and correlations). The respective parameters describe the fluctuations of the corresponding observables considered as random variables. Note that the fluctuations regard the intrinsic (own) properties of the physical systems but not the aspects (uncertainties) of the measurements performed on such systems.

**Rem.12:** The concrete expressions of the fluctuations parameters implied in the quantum-classical similarity mentioned in Rem.11 evidence the fact [48, 49] that the Planck’s constant $\hbar$ has also a classical similar, namely the Boltzmann’s constant $k$. It was shown [49] that both $\hbar$ and $k$ play similar roles of generic indicators of stochasticity (randomness) in the cases of quantum respectively classical observables. The mentioned roles are directly connected with the probabilistic parameters implied in the classical formulas (63) respectively in the QM relations (60) and (30). Such parameters are expressible [49] in terms of products between $k$ respectively $\hbar$ and quantities which do not contain $k$ respectively $\hbar$. But the alluded parameters disclose the level of stochasticity (randomness) of the referred observables and systems. Then it results that $k$ and $\hbar$ have the attributes of stochasticity (randomness) indicators. The noted attributes are generic i.e. they are specific for all the observables of a system respectively for all systems from the same class (classical or quantum). Note that the above mentioned similarity between $\hbar$ and $k$ clearly contradicts the assertion Ass.6 of TIUR.
• **Rem.13** The discussions included in the above remarks Rem.1-12 collect new supporting elements for the opinion that [50] the uncertainty relations “are probably the most controverted formulae in the whole of the theoretical physics”. Also through the respective discussions we update and consolidate the convincingness of the observation [51]: “the idea that there are defects in the foundations of orthodox quantum theory is unquestionable present in the conscience of many physicists”.

Now it is evident that in their whole ensemble the remarks Rem.1-13 indubitably argue for the following concluding remark:

• **Rem.14**: From the perspective of physics TIUR must be denied as an incorrect and useless doctrine which actually generates only senseless and unproductive disputes. Consequently the relations (61) are rejected as fictitious formulas while the relations (60) remain as simple formulas, deprived of any capital (or extraordinary) significance for physics. Then the QM observables appear as (generalised) random variables endowed with fluctuations characterised by some ordinary parameters as the standard deviations and correlations implied in relations of (60) or (61) type. Moreover in the respective relations the commutatibility of the corresponding operators do not play any capital (or extraordinary) role. This fact associated with Rem.8 shows that in connection with QM observables the terms “compatible” respectively “incompatible” are completely obsolete and useless.

Then, in respect with the $L_z - \varphi$ problems that motivate the present work, we are justified to note the next two remarks:

• **Rem.15**: For a correct physical description of the observables $L_z$ and $\varphi$ the usual QM procedures (as presented in Sec.III) are sufficient and they have not to be adjusted with any other esoteric considerations or improvised relations.

• **Rem.16**: The improvised $L_z - \varphi$ UR (6)-(14) must be rejected as formulas without any authentic physical significance. Their associate shortcomings Shc.1-4, noted in Sec.II, are completely unsurmountable because they cannot be avoided by means of some credible physical arguments.

In the context of the above noted remarks it is the place to add some observations about another question often mentioned in connection with TIUR. The question regards the problem of the pair $E-t$ (energy and time). We include the announced observations in the next remark.

• **Rem.17**: The $E-t$ problem was largely disputed during the history of QM (see [17] and references). In the main the alluded disputes were focused on the subordination of the QM description for the $E-t$ pair to the relations (60) and (71). As the respective subordination happened to be neither evident nor a direct one there was promoted diverse adjustments (some of them more or less exotic). Here we do not intend to examine in details the mentioned historical facts or the beliefs germinated from them. Our present goal is that, in connection with the E-t problem, to note
some opinions consonant with the above argued views and confluent discussions. The respective opinions are:

(i) As in fact the relations (60) and (61) are not capital or extraordinary scientific elements the QM description of the $E - t$ pair needn’t be subordinated to the respective relations.

(ii) Due to the things presented in Rem.1, similarly to the $L_z - \phi$ case, in the discussions about the $E - t$ pair the relation (61) must be also completely ignored.

(iii) Connected with the possible significance of the relation (60) it is important to note the following aspects. In many known texts for the QM description of the $E - t$ pair one uses the operators $\hat{E} = i\hbar \frac{\partial}{\partial t}$ and $\hat{t} = t$. But then it must be noted the truth that related to the respective usance the relation (60) is inadequate. The inadequacy is due to the infringement of the conditions of (31)-type. Indeed the alluded operators are associated with the result

$$(\hat{E}\Psi, \hat{t}\Psi) - (\Psi, \hat{E}\hat{t}\Psi) = -i\hbar$$

valid for every wave function $\Psi$. Here the notations are the ones from usual QM reminded in Sec.III. The result (64) shows that the conditions of (31) type are not satisfied and, consequently, it proves that (60) is not applicable. However, as it was noted in Sec.III, independently of a result like (64), the relations of (30)-type are true. This means that with the mentioned operators one obtains the relation

$$\Delta E \cdot \Delta t \geq |(\delta \hat{E}\Psi, \delta \hat{t}\Psi)|$$

which reduces itself to the trivial equality $0 = 0$ because $\langle t \rangle = t$ and $\delta \hat{t} = 0$. Such an equality has as sole physical significance the fact that in the framework of usual QM the time $t$ is a deterministic (dispersion free) variable but not a probabilistic (random) quantity. Note that in the mentioned framework, essentially implied in (64) and (65), the probabilistic load is carried by other variables (of orbital or spin nature) but not by the time.

(iv) We opine that naturally in QM framework the time $t$ must be regarded as a deterministic (non-random) variable, which has no fluctuation characteristics (describable in terms of standard deviations and correlations). Consequently in the respective framework the time $t$ needn’t be associated with an operator. The so-called “time operators” introduced in some publications can be regarded rather as pieces of pure mathematical reasonings. On the other hand we consider that, in the same framework, an operator must describe the energy. In a natural conception the respective operator is identifiable with the QM Hamiltonian.

(v) In the disputes about the $E - t$ pair, beside the relation (60), many publications put forward the “spreading” formula

$$\nabla E \cdot \nabla t \geq \frac{\hbar}{2}$$

specific for the QM wave packets. In association it is promoted the belief that, likewise with (60), the formula (66) is a capital indicator for the QM peculiarity (i.e. for the distinction in comparasion with classical physics). We think that the
respective formula is not an indicator of the mentioned kind. In fact it is completely analogous with the classical “spreading” formula 
\[ \omega \cdot t \geq \frac{1}{2} \] 
(67)
specific for signal pulses (packets) \((\omega = \text{angular frequency and } t = \text{time})\). Mathematically both \((66)\) and \((67)\) can be introduced similarly through relations like \((44)\) from Fourier transforms approach. As significance in both cases \((66)\) and \((67)\) \(\omega \cdot t\) denotes the “duration” of the packet (pulse). In classical context \([52]\) \(\omega\) signifies the spectral width (or “spreading”) of the pulses. Similarly for QM wave packets \(\omega \cdot t\) must be interpreted also as spectral (energetic) width. As regards the likeness between the relations \((66)\) and \((67)\) it is clear only of same nature. This because the respective relations are introduced within different mathematical and conceptual contexts.

Our above discussions are focused on the remarks about the physical significance of some relations from the same family with \((60)\). But, besides the mentioned significance, the respective relations seem to present a mathematical attraction (see \([16, 53, 54]\) and references). Connected with such a fact we note the next remark.

- **Rem.18**: Even if the relation \((60)\) must be regarded as a formula without any capital or extraordinary physical significance from a mathematical view point it appears in posture of an interesting source of inspiration. Such a posture explains the appreciable number of mathematical formulas which extend or generalize the relation \((60)\) (for a relevant bibliography in this sense see \([16, 53, 54]\) and references). In the next years probably the alluded number will increase as a result of some pure mathematical approaches. The physical significance (most probably of non-capital importance) of some of the mentioned mathematical formulas seems to be actually in the attention of scientists. On the other hand, in connection with the respective formulas it is of nontrivial interest to take also into account the idea according to which an increased number of mathematical formulas and reasonings does not provide without fail some significant results for physics.

Now let us note a few observations on the question of quantum measurements, mainly generated by TIUR history. According to its assertions **Ass.1** and **Ass.2** TIUR promoted the idea that the description of the respective measurements must be associated with the relations \((60)\). But, as we have shown above the respective relations have to be deprived of the traditionally assumed significance and TIUR must be denied. Then it results that the alluded question of quantum measurements remains an unelucidated problem that requires some additional considerations. We think that, in a first approximation, the respective considerations can be stated with the following remark.

- **Rem.19**: The descriptions of quantum measurements have to be done in specific approaches and frameworks that must be distinct and additional in respect with the usual QM.
A possible approach of the mentioned kind was formulated in our recent work \cite{55}. The basic idea promoted by us is that, for a quantum microparticle in an orbital motion, the description of measurements can be done in terms of linear transforms for both probability density and probability current. One of the main advantages of our approach is that it avoids any considerations connected with the strange idea about the collapse (reduction) of wave function.

Acknowledgments

Along the years, in the investigations of here approached problems, I have studied a large number of publications. Some of them were obtained directly from their authors to whom I express my deep gratitude. I wish to express my profound thanks to the World Scientific Company, Singapore, for putting at my disposal a copy of the monumental book \cite{17}.

Because of various reasons in the present work I confined myself to a restricted number of references. Doing so I cannot give (and I did not intend to do) an exhaustive presentation and appreciation of the huge number of publications in the field. But, I think, that the mentioned references are works that can offer good and credible guide marks for the discussions approached here.

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Appendix: A reply addendum

Our first opinions about the $L_z - \varphi$ pair were presented in earlier works \cite{18,19}. The presentations were more modest and less complete - e.g. we did not use all the arguments resulting from the above discussed cases which are described by the wave functions \cite{17} and \cite{21}. Newertheles, we think that the alluded opinions were correct in their essence. However in a review \cite{56} of Prof.F.E.Schroeck the respective our opinions were judged as being erroneous. In this addendum we wish to reply to the mentioned judgements by using arguments based on the considerations included in the present work.

The main error reproached to us in \cite{56} is: “most of the results stated concerning angular momentum and angle operators (including the canonical commutation relations) are false, this being a consequence of not using Rieman-Stieljes integration theory which is necessitated since the angle function has a jump discontinuity”.

In order to give an answer to the respective reproach we appeal to the following arguments:

(i) One can see that, mainly, the reproach is founded on the supposition that the variable $\varphi$ has jump (of magnitude $2\pi$ at $\varphi = 0$ or, equivalently, at $\varphi = 2\pi$). Such a supposition implies directly the idea that for the operators $\hat{L}_z$ and $\hat{\varphi}$ the commutation relation is not the one given in \cite{3} but $[\hat{L}_z, \hat{\varphi}] = -i\hbar + i\hbar 2\pi \delta$ (where $\delta$ denotes the
Dirac function at the boundary $\varphi = 0$ or $\varphi = 2\pi$). We inform that the respective idea was confessed to us by Prof. F.E. Schroeck in two letters (dated September 16, 1981 respectively April 2, 1982). Note also the fact that, in a visible or masked manner, the same idea is also promoted in a number of publications dealing with the $L_z - \varphi$ problem.

(ii) The reproach could refer only to the situations described by the wave functions (14) when the real physical range of $\varphi$ is the interval $[0, 2\pi]$ (at the respective date our works [21, 22, 23] regarding QTP had not appeared yet). But then the mentioned supposition and idea reveal themselves as strange things because they imply the hypothesis that the range of $\varphi$ is larger than the interval $[0, 2\pi]$, eventually even the infinite interval $(-\infty, \infty)$. But it is evident that such a hypothesis is wrong and that for the situations under discussions the correct range for $\varphi$ is interval $[0, 2\pi]$ It is also important that in the mentioned situations the normalization of the wave functions is done in fact on the interval $[0, 2\pi]$ but not on other larger domains. Therefore related to the respective situations it is senseless to consider that “the angle function has a jump discontinuity”. Consequently in the alluded context to “using Riemann-Stieljes integration theory” is not necessitated at all.

(iii) In the case of QTP described by (17) the relation $[\hat{L}_z, \hat{\varphi}] = -i\hbar$, given in (8) is indubitably applicable. Then, in the spirit of the mentioned reproach, for the same pair of observables $L_z$ and $\varphi$, one has to tolerate two completely dissimilar and irreconcilable commutation relations $[\hat{L}_z, \hat{\varphi}] = -i\hbar$ and $[\hat{L}_z, \hat{\varphi}] = -i\hbar + i\hbar 2\pi \delta$. But such a tolerance is evidently a senseless thing, without any real (physical) support.

(iv) From the considerations presented above in Sec.III it results directly the irrefutable conclusion that, from a mathematical perspective, we have the unique commutation relation $[\hat{L}_z, \hat{\varphi}] = -i\hbar$ for all the rotational motions (described by any of the wave functions (14), (17) or (22)).

(v) The mathematical findings MF.1-4 from Sec.III reveal the fact that, in the descriptions of various rotational motions, the differences are evidenced not by the commutation relation for $\hat{L}_z$ and $\hat{\varphi}$ but by the resulting formula for $\Delta L_z$ and $\Delta \varphi$. The respective formula is always valid in the general version (30) but, depending on the fulfilment of the conditions (31), it can take the particular and restricted form (33)/(34).

The ensemble of the above noted arguments (i)-(v) proves as unfounded the reproaches [56] of Prof. F.E. Schroeck regarding our opinions about the $L_z - \varphi$ pair.

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List of abbreviations

Ass. = assertion
CF = classical fluctuations
MF = mathematical finding
NCR = non-circular rotations
Q = question
QM = quantum mechanics
Rem = remark
Shc = shortcoming
TIUR = traditional interpretation of uncertainty relations
UR = uncertainty relations