TAKIFF ALGEBRAS WITH POLYNOMIAL RINGS OF SYMMETRIC IN INVARI ANTS

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INTRODUCTION

The ground field \( \mathbb{k} \) is algebraically closed and of characteristic 0. Let \( Q \) be a connected algebraic group with \( q = \text{Lie} \ Q \) and \( S(q) = \mathbb{k}[q^*] \) the symmetric algebra of \( q \). The subalgebra of \( Q \)-invariants in \( \mathbb{k}[q^*] \) is denoted by \( \mathbb{k}[q^*]^Q \) or \( \mathbb{k}[q^*]^\mathfrak{g} \). The elements of \( \mathbb{k}[q^*]^Q \) are called symmetric invariants of \( q \). Interesting classes of non-reductive groups \( Q \) such that \( \mathbb{k}[q^*]^Q \) is a polynomial ring have recently been found, see e.g. [J07, P07, PP, PY12, PY13, CM16, Y17]. A quest for this type of groups continues. Let \( q\langle m \rangle := q \otimes \mathbb{k}[T]/(T^m+1) \) be the \( m \)-th Takiff algebra (= truncated current algebra) of \( q \). Since \( q\langle 0 \rangle \simeq q \), we may assume that \( m \geq 1 \).

Our main result is that under a mild restriction, the passage from \( q \) to \( q\langle m \rangle \) preserves the polynomiality of symmetric invariants. We also (1) discover a new phenomenon that a certain ideal of \( q\langle m \rangle \) has a polynomial ring of invariants in \( \mathbb{k}[q\langle m \rangle]^* \), and (2) show that the property of \( q \) that \( \mathbb{k}[q^*] \) is a free \( \mathbb{k}[q^*]^Q \)-module does not always extend to \( q\langle 1 \rangle \).

The story began in 1971, when Takiff proved that if \( \mathfrak{g} \) is semisimple, then \( \mathfrak{g}\langle 1 \rangle = \mathfrak{g} \times \mathfrak{g}^{ab} \) has a polynomial ring of symmetric invariants whose Krull dimension equals \( 2 \cdot \text{rk} \mathfrak{g} \) [Ta71]. Then Raïs and Tauvel proved a similar result for \( \mathfrak{g}\langle m \rangle \) with arbitrary \( m \in \mathbb{N} \) [RT92]. This is the classical analogue of the description of the Feigin-Frenkel centre \( Z(\mathfrak{g}) \subset \mathcal{U}(t^{-1} \mathfrak{g}[t^{-1}]) \), see [FF92]. Recently, Macedo and Savage came up with a multi-parameter generalisation of the Raïs-Tauvel result. Namely, let

\[
(0.1) \quad \hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{k}[T_1, \ldots, T_r]/(T_1^{m_1+1}, \ldots, T_r^{m_r+1}) =: \mathfrak{g}\langle m_1, \ldots, m_r \rangle
\]

be a truncated multi-current algebra of a semisimple \( \mathfrak{g} \). Then \( \mathbb{k}[\hat{\mathfrak{g}}^*] \) is a polynomial ring of Krull dimension \( (m_1 + 1) \ldots (m_r + 1) \cdot \text{rk} \mathfrak{g} \), see [MS16]. The proofs heavily use the fact that \( \mathfrak{g} \) is semisimple, when many structure results are available. For instance, both [RT92] and [MS16] exploit Kostant’s section for the set of the regular elements of \( \mathfrak{g} \). On the other hand, if \( \mathfrak{g} \) is simple and \( q = \mathfrak{g}_e \) is the centraliser of a nilpotent element \( e \in \mathfrak{g} \) such that \( \mathfrak{g}_e \) has the “codim-2 property” and \( e \) admits a “good generating system” in \( \mathbb{k}[\mathfrak{g}]^G \), then \( \mathbb{k}[\mathfrak{g}_e \langle m \rangle^*]^{\mathfrak{g}_e(m)} \) is a polynomial ring for all \( m \in \mathbb{N} \), see [AP17, Theorem 3.1]. In all these cases, the free

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generators of the ring of symmetric invariants of \( \hat{g} \) or \( g_e\langle m \rangle \) are explicitly described via those of \( g \) or \( g_e \), respectively. This goes back to a general construction of [RT92].

Our main theorem provides a substantial generalisation of all these partial results. To state it, we need some notation. The index of \( q \), \( \text{ind} \ q \), is the minimal codimension of the \( Q \)-orbits in \( q^* \), hence \( \text{ind} \ q = \text{rk} \ q \) if \( q \) is reductive. Let \( df \) be the differential of \( f \in \mathbb{k}[q^*] \). We regard \( df \) as a polynomial mapping from \( q^* \) to \( q \) and write \( (df)_\xi \) for its value at \( \xi \in q^* \). If \( f \in \mathbb{k}[q^*]^Q \), then \( df \) is \( Q \)-equivariant. The image of \( q \otimes T + \cdots + q \otimes T^m \) in \( q\langle m \rangle \) is an ideal of codimension \( \dim q \), which is denoted by \( q\langle m \rangle^u \). An open subset of an irreducible variety is called big, if its complement does not contain divisors. Then a brief version of our result is

**Theorem 0.1.** Let \( q \) be an algebraic Lie algebra such that \( \mathbb{k}[q^*]^q = \mathbb{k}[f_1, \ldots, f_l] \) is a graded polynomial ring, where \( l = \text{ind} \ q \). Set \( \Omega_{q^*} = \{ \xi \in q^* \mid (df_1)_\xi \land \cdots \land (df_l)_\xi \neq 0 \} \), and assume that \( \Omega_{q^*} \) is big (in \( q^* \)). For any \( m \geq 1 \), we then have

(i) \( \mathbb{k}[\langle m \rangle^*]^{q\langle m \rangle^u} \) is a graded polynomial ring of Krull dimension \( \dim q + ml \).

(ii) the Takiff algebra \( q\langle m \rangle \) has the same properties as \( q \), i.e., \( \mathbb{k}[\langle m \rangle^*]^{q\langle m \rangle} \) is a graded polynomial ring of Krull dimension \( (m + 1)l = \text{ind} q\langle m \rangle \) and the similarly defined subset \( \Omega_{q\langle m \rangle^*} \subset q\langle m \rangle^* \) is also big.

(See also Theorem 2.2 for a description of free generators and \( \Omega_{q\langle m \rangle^*} \).) As is well-known, a semisimple Lie algebra \( q \) satisfies the assumptions of Theorem 0.1. (This goes back to Chevalley and Kostant.) Therefore, Theorem 0.1 yields another proof and a generalisation of [MS16, Theorem 5.4], see Corollary 2.6. A notable difference between our Theorem 0.1 and results of [AP17] is that we do not impose a constraint on \( \sum_i \deg f_i \), which is a part of the definition of a “good generating system”, and do not require the codim--2 property for \( q \) (see Section 1 for the definition). A weaker assumption that \( \Omega_{q^*} \) is big appears to be sufficient. That is, our result applies to a larger supply of non-reductive Lie algebras, see examples in Sections 3 and 4. For instance, the canonical truncation, \( \tilde{q} \), of a Frobenius Lie algebra \( q \) satisfies the hypotheses of Theorem 0.1, see Section 3.2.

If \( g \) is semisimple, then \( \mathbb{k}[g\langle m \rangle^*] \) is a free \( \mathbb{k}[g\langle m \rangle^*]^{g\langle m \rangle} \)-module for any \( m \) [M01, Appendix]. In Section 5, we prove that this property does not generalise to the truncated multi-current algebras of \( g \) or the truncated current algebras \( q\langle m \rangle \) for arbitrary \( q \) such that \( \mathbb{k}[q\langle m \rangle^*] \) is a free \( \mathbb{k}[q\langle m \rangle^*]^{q\langle m \rangle} \)-module. Namely, \( \mathbb{k}[g\langle 1, 1, 1 \rangle] \) is not a free \( \mathbb{k}[g\langle 1, 1, 1 \rangle]^{g\langle 1, 1, 1 \rangle} \)-module (Theorem 5.5). This can also be interpreted as follows. Since the passage \( g \sim g\langle 1 \rangle \) preserve freeness of the module [G94, M01], in the chain of Takiff extensions

\[
g \sim g\langle 1 \rangle \sim g\langle 1 \rangle \langle 1 \rangle \simeq g\langle 1, 1 \rangle \sim g\langle 1, 1 \rangle \langle 1 \rangle \simeq g\langle 1, 1, 1 \rangle,
\]

...
we lose the freeness of the module at the second or third step (conjecturally, at the third step!). This also implies that, for $g \langle 1, 1, \ldots, 1 \rangle = g(1^n)$ and every $r \geq 3$, $\mathbb{k}[g(1^n)]$ is not a free module over the ring of symmetric invariants.

**Notation.** Let $Q$ act on an irreducible affine variety $X$. Then $\mathbb{k}[X]^Q$ is the algebra of $Q$-invariant regular functions on $X$ and $\mathbb{k}(X)^Q$ is the field of $Q$-invariant rational functions. If $\mathbb{k}[X]^Q$ is finitely generated, then $X//Q := \text{Spec} \mathbb{k}[X]^Q$, and the quotient morphism $\pi_Q : X \to X//Q$ is induced by the inclusion $\mathbb{k}[X]^Q \hookrightarrow \mathbb{k}[X]$. If $\mathbb{k}[X]^Q$ is a graded polynomial ring, then the elements of any set of algebraically independent homogeneous generators are called basic invariants. If $V$ is a $Q$-module and $v \in V$, then $q_v = \{ \zeta \in q \mid \zeta \cdot v = 0 \}$ is the stabiliser of $v$ in $q$ and $Q_v = \{ s \in Q \mid s \cdot v = v \}$ is the isotropy group of $v$ in $Q$; $H^o$ is the identity component of an algebraic group $H$.

1. Preliminaries on the coadjoint representation

Let $Q$ be a connected affine algebraic group with Lie algebra $q$. The symmetric algebra $S(q)$ over $\mathbb{k}$ is identified with the graded algebra of polynomial functions on $q^*$ and we also write $\mathbb{k}[q^*]$ for it.

The index of $q$, ind $q$, is the minimal codimension of $Q$-orbits in $q^*$. Equivalently, ind $q = \min_{\xi \in q^*} \dim q_{\xi}$. By Rosenlicht’s theorem [VP89, 2.3], one also has ind $q = \text{tr.deg} \mathbb{k}(q^*)^Q$. The “magic number” associated with $q$ is $b(q) = (\dim q + \text{ind} q)/2$. Since the coadjoint orbits are even-dimensional, the magic number is an integer. If $q$ is reductive, then ind $q = rk q$ and $b(q)$ equals the dimension of a Borel subalgebra. The Poisson bracket $\{ , \}$ in $\mathbb{k}[q^*]$ is defined on the elements of degree 1 (i.e., on $q$) by $\{ x, y \} := [x, y]$. The centre of the Poisson algebra $S(q)$ is $S(q)^a = \{ H \in S(q) \mid \{ H, x \} = 0 \ \forall x \in q \}$. Since $Q$ is connected, we also have $S(q)^a = S(q)^Q = \mathbb{k}[q^*]^Q$.

The set of $Q$-regular elements of $q^*$ is $q^*_{\text{reg}} = \{ \eta \in q^* \mid \dim Q \cdot \eta \geq \dim Q \cdot \eta' \text{ for all } \eta' \in q^* \}$. We say that $q$ has the codim-$n$ property if $\text{codim} (q^* \backslash q^*_{\text{reg}}) \geq n$. The following useful result appears in [P07’, Theorem 1.2]:

**Theorem 1.1.** Suppose that $q$ has the codim–2 property and there are homogeneous algebraically independent $f_1, \ldots, f_l \in \mathbb{k}[q^*]^Q$ such that $l = \text{ind} q$ and $\sum_{i=1}^l \deg f_i = b(q)$. Then

(i) $\mathbb{k}[q^*]^Q = \mathbb{k}[f_1, \ldots, f_l]$ and

(ii) $(df_1)_\xi, \ldots, (df_l)_\xi$ are linearly independent if and only if $\xi \in q^*_{\text{reg}}$.

Furthermore, if $q$ has the codim–2 property, then for any collection of algebraically independent homogeneous $f_1, \ldots, f_l \in \mathbb{k}[q]^Q$ with $l = \text{ind} q$, one has $\sum_{i=1}^l \deg f_i \geq b(q)$.

**Definition 1** (cf. [P08]). An algebraic Lie algebra $q$ is said to be $n$-wonderful, if

(i) $q$ has the codim–$n$ property.
(ii) \( \mathbb{k}[q^*]^Q \) is a polynomial algebra of Krull dimension \( l = \text{ind} q \);

(iii) If \( f_1, \ldots, f_l \) are basic invariants in \( \mathbb{k}[q^*]^Q \), then \( \sum_{i=1}^l \deg f_i = b(q) \).

For instance, any semisimple Lie algebra is 3-wonderful.

It follows from Theorem 1.1 that if \( q \) is 2-wonderful, then \( \Omega_{q^*} = q_{\text{reg}} \) is big. Therefore, Theorem 0.1 applies to all 2-wonderful Lie algebras. (A more precise statement is given in Corollary 2.5 below.) For instance, it applies to all centralisers of nilpotent elements in types \( A_n \) or \( C_n \), see [PPY, Theorems 4.2 & 4.4] and [AP17, Section 3].

Suppose that \( \mathbb{k}[q^*]^Q \) is a polynomial ring, but nothing is known about the \( \text{codim} \)-2 property. Theorem 0.1 suggests that one needs some tools to decide whether \( \Omega_{q^*} \) is big. In many cases, the following assertion is helpful.

**Proposition 1.2** (see [JS10, Prop. 5.2]). If \( \mathbb{k}[q^*]^Q \) is a polynomial ring and \( Q \) has no proper semi-invariants in \( \mathbb{k}[q^*] \), then \( \Omega_{q^*} \) is big.

**Remark 1.3.** Using some ideas of Knop (see [Kn86, Satz 2]), we can prove a more general assertion, which we do not need here. Namely,

Let an algebraic group \( Q \) act on an irreducible affine factorial variety \( X \). Suppose that \( X//Q \) exists (i.e., \( \mathbb{k}[X]^Q \) is finitely generated) and \( \mathbb{k}[X] \) contains no proper \( Q \)-semi-invariants. Let \( X_{\text{sm}} \) denote the smooth locus of \( X \) and \( \pi_Q : X \to Y := X//Q \) the quotient morphism. Set \( \Omega_X = \{ x \in X_{\text{sm}} | \pi_Q(x) \in Y_{\text{sm}} \& (d\pi_Q)_x \text{ is onto} \} \). Then \( \Omega_X \) is big.

## 2. TAKIFF ALGEBRAS AND THEIR SYMMETRIC INVARIANTS

By definition, the \( m \)-th Takiff algebra of \( q \) is \( q^{(m)} := q \otimes \mathbb{k}[T]/(T^{m+1}) \). In particular, \( q^{(1)} = q \ltimes q^{ab} \) is the semi-direct product, where the second factor is an abelian ideal. For \( j \leq m \), the image of \( q \otimes T^j \) in \( q^{(m)} \) is denoted by \( q^{(j)} \). A typical element of \( q^{(m)} \) can be written as \( x = (x_0, x_1, \ldots, x_m) \), where \( x_j \in q^{(j)} \). Likewise, we have \( q^{(m)^*} \simeq \bigoplus_{j=0}^m (q^{(j)})^* \) as vector space, and \( \xi = (\xi_0, \xi_1, \ldots, \xi_m) \) is an element of \( q^{(m)^*} \), where \( \xi_j \in q^{(j)^*} \). Then the pairing of \( q^{(m)} \) and \( q^{(m)^*} \) is given by \( \langle x, \xi \rangle_{q^{(m)}} = \sum_{i=0}^m \langle x_i, \xi_i \rangle_q \). It is sometimes convenient to regard the elements of \( q^{(m)} \) and \( q^{(m)^*} \) as “polynomials” in \( \epsilon \), where \( \epsilon^{m+1} = 0 \). Namely,

\[
\begin{align*}
\sum_{i=0}^m x_i \epsilon^i & \quad \text{and} \quad \sum_{j=0}^m \xi_j \epsilon^{m-j}.
\end{align*}
\]

Using this notation, the Lie bracket in \( q^{(m)} \) is

\[
[x, y]_\epsilon = \sum_{0 \leq i + j \leq m} [x_i, y_j] \epsilon^{i+j}
\]

and the coadjoint representation \( \text{ad}^*_{q^{(m)}} \) of \( q^{(m)} \) is given by

\[
(\text{ad}^*_{q^{(m)}} x) \xi = \sum_{0 \leq j - i \leq m} (\text{ad}^*_{q^i}(x_i) \xi_j) \epsilon^{m-j+i}.
\]
Then \( q(m)^u = \bigoplus_{j=1}^{m} q[j] \) is an ad-nilpotent ideal of \( q(m) \) and the corresponding connected algebraic group is

\[
Q(m) \simeq Q \ltimes \exp(q(m)^u) = Q \ltimes Q(m)^u.
\]

(If \( Q \) is reductive, then \( Q(m)^u \) is the unipotent radical of \( Q(m) \).) For a non-Abelian \( Q \), the unipotent group \( Q(m)^u \) is commutative if and only if \( m = 1 \).

By [RT92, 2.8], one has \( \text{ind } q(m) = (m + 1) \cdot \text{ind } q \). Hence also \( b(q(m)) = (m + 1) \cdot b(q) \).

Moreover,

\[
(2.1) \quad \xi \in q(m)^*_\text{reg} \text{ if and only if } \xi_m \in q^*_\text{reg}.
\]

Therefore, the presence of \( \text{codim } n \) property for \( q \) implies that for \( q(m) \).

A general method for constructing symmetric invariants of \( q(m) \) is presented in [RT92]. Suppose that \( f \in \mathbb{k}[q^*] \) is homogeneous. Recall that \( df \in \text{Mor}(q^*, q) \) is the differential of \( f \). Consider \( \xi_e \) as an element of \( q^* \otimes \mathbb{k}[\epsilon] \) with \( \epsilon^{m+1} = 0 \), and expand \( f(\xi_e) \) as a polynomial in \( \epsilon \):

\[
f(\xi_m + \epsilon \xi_{m-1} + \cdots + \epsilon^{m-1} \xi_1 + \epsilon^m \xi_0) = \sum_{j=0}^{m} F^j(\xi) \epsilon^j.
\]

It is readily seen that \( F^0(\xi) = f(\xi_m) \) and \( F^1(\xi) = \langle (df)_{\xi_m}, \xi_{m-1} \rangle_q \). More generally, the following assertion is true.

**Proposition 2.1** (see [RT92, Section III]). For any \( j \in \{0, 1, \ldots, m\} \), we have

\[
(i) \quad F^j(\xi) = \langle (df)_{\xi_m}, \xi_{m-j+1} \rangle_q + H_j(\xi_m, \ldots, \xi_{m-j+1}) \text{ for some } H_j \in \mathbb{k}[q(m)^*];
(ii) \quad \text{If } f \in \mathbb{k}[q^*]Q, \text{ then every } F^j \text{ is a symmetric invariant of } q(m), \text{ i.e., } F^j \in \mathbb{k}[q(m)^*]Q(m).
\]

Let \( f_1, \ldots, f_l \) be a set of basic invariants in \( \mathbb{k}[q^*]Q \), where \( l = \text{ind } q \). Using the above construction of [RT92], we associate to each \( f_i \) the set of \( Q(m) \)-invariants \( F^0_i, \ldots, F^m_i \). Now, we are ready to state precisely our main result.

**Theorem 2.2.** Let \( Q \) be a connected algebraic group such that \( \mathbb{k}[q^*]Q = \mathbb{k}[f_1, \ldots, f_l] \) is a graded polynomial ring, where \( l = \text{ind } q \). Set \( \Omega_{q^*} = \{ \xi \in q^* \mid (df_1)_\xi \land \cdots \land (df_l)_\xi \neq 0 \} \), and assume that \( \Omega_{q^*} \) is big. For any \( m \geq 1 \), we then have

\[
(i) \quad \mathbb{k}[q(m)^*]Q(m)^u \text{ is a graded polynomial ring of Krull dimension } \dim q + ml, \text{ which is freely generated by the coordinate functions on } q^*_m \text{ and the } \{ F^j_i \}'s \text{ with } i = 1, \ldots, l \text{ and } j = 1, \ldots, m.
(ii) \quad \text{the Takiff algebra } q(m) \text{ has the same properties as } q, \text{ i.e.,}
\]

- \( \mathbb{k}[q(m)^*]Q(m) \) is a graded polynomial ring of Krull dimension \( \text{ind } q(m) = (m + 1)l \).
- \( \Omega_{q(m)^*} = \bigoplus_{j=0}^{m-1} q^*_j \ltimes \Omega_{q^*} \text{ is big, where } \Omega_{q^*} \subset q^*_m \simeq q^*. \)
Proof. (i) Recall that \( q^m \simeq q \times q^m \), where \( q = q[0] \) and \( q^m = \bigoplus_{j=1}^m q[j] \), and \( Q^m = Q \times Q^m \). Here \( Q^m \) is a unipotent normal subgroup of \( Q^m \).

Note that the subspace \( q|m| \subset q^m \) regarded as a subset of \( k[q^m]^* \) belongs to the subalgebra of \( Q^m \)-invariants, and \( F_j^0 = f_i \in S[q|m] \). Let \( \mathcal{A} \) denote the subalgebra of \( k[q^m]^*Q^m \) generated by \( q|m \) and \( \{ F_j^0 \} \) with \( j = 1, \ldots, m \) and \( i = 1, \ldots, l \). (Note that we do not include \( F_j^0 \) in the generating set for \( A \! \) !)

For \( x = (x_0, \ldots, x_m) \) with \( x_i \in q[i] \), we say that \( x_j \neq 0 \) is the lowest component of \( x \), if \( x_0 = \cdots = x_{j-1} = 0 \). Now, \( (dF_j^0) \xi \in q^m \) and using Proposition 2.1(i), one readily verifies that its lowest component is \( (dF_j^0) \xi \mid_{m-j} = (df_i) \xi_m \in q|m-j| \), where \( j = 0, 1, \ldots, m-1 \).

Clearly, these lowest components are linearly independent if and only if \( \xi_m \in \Omega_q^* \). If \( v_1, \ldots, v_{\dim q} \) is a basis for \( q|m| \), then \( (dF_i) \xi = v_i \in q|m| \). Since all these differentials have a block-triangular form w.r.t. the decomposition \( q^m = \bigoplus_{i=1}^m q[i] \) (cf. Table 1), it follows that the differentials per se are linearly independent at \( \xi \) if and only if \( \xi_m \in \Omega_q^* \). Therefore, the polynomials

\[
v_1, \ldots, v_{\dim q}, \text{ and } \{ F_j^0 \} \text{ with } j = 1, \ldots, m, \ i = 1, \ldots, l
\]

are algebraically independent and generate \( \mathcal{A} \). As the differentials of this family are linearly independent on the big open subset \( \bigoplus_{j=0}^{m-1} q[j] \times \Omega_q^* \) of \( q^m \), Theorem 1.1 in [PPY] guarantee us that \( \mathcal{A} \) is an algebraically closed subalgebra in \( k[q^m]^* \), of Krull dimension \( \dim q + m \).

On the other hand, if \( \xi = (0, \ldots, 0, \xi_m) \) and \( \xi_m \in q^m_{\reg} \), then \( \dim Q^m \cdot \xi = m(\dim q - l) \). Hence \( \text{tr.deg} k[q^m]^*Q^m \leq \dim q^m - \dim Q^m \cdot \xi = \dim q + m \). Therefore \( \mathcal{A} = k[q^m]^*Q^m \) is an algebraic extension, which implies that \( \mathcal{A} = k[q^m]^*Q^m \). In other words, \( k[q^m]^*Q^m \) is \( \{ F_j^0 \} = k[q|m]|F_j^0, \ i = 1, \ldots, l; \ j = 1, \ldots, m \} \).

(ii) Since \( Q^m \simeq Q \times Q^m \) and the \( F_j^0 \)'s are already \( Q^m \)-invariant (Prop. 2.1(ii)), it follows from part (i) that

\[
k[q^m]^*Q^m = (k[q^m]^*Q^m)^Q = k[q^m]^Q[\{ F_i^0 \}, 1 \leq i \leq l; 1 \leq j \leq m] = k[F_i^0, 1 \leq i \leq l; 0 \leq j \leq m].
\]

Furthermore, the differentials of the total set of generators \( \{ F_i^0 \} \), with the value \( j = 0 \) included, are also linearly independent if and only if \( \xi_m \in \Omega_q^* \subset q^m_{\reg} \), see [RT92, Lemma 3.3] and Table 1. Therefore, \( \Omega_{q^m}^* = \bigoplus_{j=0}^{m-1} q[j] \times \Omega_q^* \) is big. \( \square \)

For future use, we record a by-product of the proof:

**Corollary 2.3.** \( \xi \in \Omega_{q^m}^* \iff \xi_m \in \Omega_q^* \).

**Remark 2.4.** It appears that Theorem 2.2 is fully analogous to [P07, Theorem 11.1], where the polynomiality of invariants for the adjoint representation of \( q^m \) is studied.
Table 1. Components of the differentials of basic invariants

|             | \( q_{[m]} \) | \( q_{[m-1]} \) | \( q_{[m-2]} \) | \ldots | \( q_{[0]} \) |
|-------------|---------------|----------------|----------------|-------|-------------|
| \((dF^0)\xi\) | \((df_1)\xi_m\) | 0               | \ldots          | \ldots | 0           |
| \vdots      | \vdots        | \vdots          | \ldots          | \ldots | \vdots      |
| \((dF^0)\xi\) | \((df_i)\xi_m\) | 0               | \ldots          | \ldots | 0           |
| \((dF^1)\xi\) | *             | \((df_1)\xi_m\) | 0               | \ldots | 0           |
| \vdots      | \vdots        | \vdots          | \ldots          | \ldots | \vdots      |
| \((dF^1)\xi\) | *             | \((df_i)\xi_m\) | 0               | \ldots | 0           |
| \((dF^2)\xi\) | *             | *               | \((df_1)\xi_m\) | \ldots | 0           |
| \vdots      | \vdots        | \vdots          | \ldots          | \ldots | \vdots      |
| \((dF^2)\xi\) | *             | *               | \((df_i)\xi_m\) | \ldots | 0           |

**Corollary 2.5.** If \( q \) is an \( n \)-wonderful algebra for \( n \geq 2 \), then so is \( q^{(m)} \) for any \( m \in \mathbb{N} \).

**Proof.** Let us check that the properties of Definition 1 carry over from \( q \) to \( q^{(m)} \).

- As noted above, the presence of \( \text{codim} – n \) property for \( q \) implies that for \( q^{(m)} \). We also have \( \dim q^{(m)} = (m+1) \cdot \dim q \) and \( \text{ind} q^{(m)} = (m+1) \cdot \text{ind} q \).

- If \( q \) is 2-wonderful, then \((df_1)\xi, \ldots, (df_i)\xi\) are linearly independent if and only if \( \xi \in q_{\text{reg}}^* \) (Theorem 1.1). Hence \( \Omega^*_q = q_{\text{reg}}^* \) and its complement does not contain divisors. Therefore, \( \mathbb{k}[q^{(m)}]^*Q^{(m)} \) is polynomial ring of Krull dimension \((m+1)l = (m+1) \cdot \text{ind} q \), freely generated by the \( F_{ij} \)'s.

- Clearly, \( \deg F_{ij} = \deg f_i \) for all \( i \) and \( j \). Therefore

\[
\sum_{i=1}^{l} \sum_{j=0}^{m} \deg F_{ij} = (m+1) \sum_{i=1}^{l} \deg f_i = (m+1) b(q) = b(q^{(m)}). \quad \square
\]

**Corollary 2.6** (cf. [MS16, Thm. 5.4]). For any \( r \)-tuple \( m_1, \ldots, m_r \), the truncated multi-current algebra \( q^{(m_1, \ldots, m_r)} \) has a polynomial ring of symmetric invariants.

**Proof.** A truncated multi-current algebra of any \( q \) is obtained as an iteration of various Takiff algebras. That is,

\[
(2.2) \quad q := q^{(m_1, \ldots, m_r)} \simeq ( \cdots (q^{(m_1)})^{(m_2)}) \cdots)^{(m_r)}. \]

Therefore, if \( q \) satisfies the hypotheses of Theorem 2.2, then so is \( q \). In particular, \( \mathbb{k}[q^*]^Q \) is a polynomial ring. \quad \square
Note that if \( q = g \) is semisimple, then one can use results of [RT92] only for the first iteration \( g \sim g(m_1) \), because afterwards the algebra in question becomes non-reductive.

**Remark 2.7.** An essential point in our proof of Theorem 2.2 is the use of Theorem 1.1 in [PPY]. This ensures that the subalgebra \( \mathcal{A} \) is algebraically closed in \( k[q(m)]^* \) and hence \( \mathcal{A} = k[q(m)]^* Q(m)^u \) for the dimension reason. However, one can use instead an invariant-theoretic (geometric) argument related to Igusa’s lemma (see e.g. [VP89, Theorem 4.12]) or [P07, Lemma 6.1]). Namely, consider the morphism

\[
\tau : q(m)^* \to q[m] \times \mathbb{A}^m_{\mathbb{C}} =: Y
\]

given by \( \tau(\xi) = (\xi_m, F^1_l(\xi), \ldots, F^m_l(\xi), \ldots, F^m_l(\xi)) \). From the assumption on \( \Omega_q^* \) and a “triangular” form of \( \{ F^i_l \} \) (see Prop. 2.1(i)), one derives that

1. \( \text{Im} \tau \supset \Omega_q^* \times \mathbb{A}^m_{\mathbb{C}} \), where the RHS is a big open subset of \( Y \);

2. for any \( y \in \Omega_q^* \times \mathbb{A}^m_{\mathbb{C}} \), the fibre \( \tau^{-1}(y) \) is a sole \( Q(m)^u \)-orbit.

Then Igusa’s lemma asserts that \( k[Y] \simeq k[q(m)]^*[Q(m)^u] \), i.e., \( Y \simeq q(m)^*/Q(m)^u \) and \( \tau = \pi_{Q(m)^u} \). (Cf. the similar use of Igusa’s lemma in [P07, Theorems 6.2 & 11.1] and [P07′, Theorem 5.2].)

3. **Prehomogeneous vector spaces and rings of semi-invariants**

Here we show that some old results of Sato–Kimura [SK77] on prehomogeneous vector spaces allow us to construct Lie algebras satisfying the hypotheses of Theorem 2.2.

**3.1. Prehomogeneous vector spaces.** Let \( H \subset GL(V) \) be a representation of a connected group \( H \) having an open orbit in \( V \), i.e., \( V \) is a prehomogeneous vector space w.r.t. \( H \). By [SK77, § 4], the algebra of \( H \)-semi-invariants in \( k[V] \), denoted \( k[V]^{(H)} \), is polynomial. More precisely, let \( O \subset V \) be the open \( H \)-orbit and \( D_1, \ldots, D_l \) all simple divisors in \( V \setminus O \) (we do not need the irreducible components of codimension \( \geq 2 \) in \( V \)). If \( D_l = \{ f_l = 0 \} \), then \( f_l \in k[V]^{(H)}, f_1, \ldots, f_l \) are algebraically independent, and \( k[V]^{(H)} = k[f_1, \ldots, f_l] \). Moreover, let \( \lambda_i : H \to \mathbb{C}^\times \) be the \( H \)-character corresponding to \( f_i \), i.e., \( h \cdot f_i = \lambda_i(h) f_i \) for all \( h \in H \). Then the differentials of \( \lambda_i \)'s are linearly independent and \( \tilde{H} := \{ h \in H | \lambda_i(h) = 1 \ \forall i \}^o \) is of codimension \( l \) in \( H \). Then \( [H,H] \subset \tilde{H} \subset H \) and \( k[V]^{[H,H]} = k[V]^{\tilde{H}} = k[V]^{(H)} \) is a polynomial ring.

**3.2. Frobenius Lie algebras.** Suppose that \( \text{ind} \ h = 0 \), i.e., \( h \) is Frobenius. Then \( H \) has an open orbit in \( \mathfrak{h}^* \) and the above results apply to \( V = \mathfrak{h}^* \). Then \( k[\mathfrak{h}^*]^{(H)} = k[\mathfrak{h}^*]^{\tilde{H}} \) is a polynomial ring of Krull dimension \( \dim H - \dim \tilde{H} = \text{ind} \ h \). Note that

\[
k[k][\mathfrak{h}^*] = S(\tilde{h}) \subset S(h) = k[\mathfrak{h}^*],
\]

and an important additional feature of the “coadjoint” situation is that \( k[\mathfrak{h}^*]^{\tilde{H}} \subset k[\mathfrak{h}^*] \), see [BGR, Kap. II, § 6]. Hence \( k[\mathfrak{h}^*]^{\tilde{H}} = k[\mathfrak{h}^*]^{\tilde{H}}, \) i.e., \( h \) has a polynomial ring of symmetric
invariants whose Krull dimension equals \( \text{ind} \, \tilde{h} \). By the very construction, \( \tilde{H} \) has no proper semi-invariants in \( \mathbb{k}[h^*] \) and hence in \( \mathbb{k}[\tilde{h}^*] \). It then follows from Proposition 1.2 that \( \Omega_{\tilde{h}} \) is big. Thus, Theorem 2.2 applies to \( \tilde{h} \), and hence \( \tilde{h}(m) \) has a polynomial ring of symmetric invariants for any \( m \geq 1 \).

**Remark 3.1.** More generally, for any Lie algebra \( h \), the ring of symmetric semi-invariants \( \mathbb{k}[h^*]^{(H)} \) (i.e., the Poisson semi-centre of \( S(h) = \mathbb{k}[h^*] \)) is isomorphic to the ring of symmetric invariants of a canonically defined subalgebra \( \tilde{h} \subset h \) [BGR, Kap. II, §6], see also [OVdB, Sect. 3]. The subalgebra \( \tilde{h} \) is called the *canonical truncation* of \( h \). It has the property that \( \dim h - \dim \tilde{h} = \text{ind} \, \tilde{h} - \text{ind} h \) [OVdB, Lemma 3.7], hence \( b(h) = b(\tilde{h}) \). Furthermore, since \( \tilde{H} \) has no proper semi-invariants in \( \mathbb{k}[\tilde{h}^*] \), \( \mathbb{k}(\tilde{h}^*)^{\tilde{H}} \) is the field of fractions of \( \mathbb{k}[\tilde{h}^*]^{\tilde{H}} \) and the Krull dimension of \( \mathbb{k}[\tilde{h}^*]^{\tilde{H}} \) equals \( \text{ind} \, \tilde{h} \). Therefore, if \( \mathbb{k}[h^*]^{(H)} = \mathbb{k}[\tilde{h}^*]^{\tilde{H}} \) is a polynomial ring, then Proposition 1.2 and Theorem 2.2 apply to \( \tilde{h} \), and hence \( \tilde{h}(m) \) has a polynomial ring of symmetric invariants for all \( m \geq 1 \). In the special case, where \( h \) is Frobenius, this is already explained in the previous paragraph.

Let us illustrate this theory in both Frobenius and non-Frobenius cases.

**Example 3.2.** Let \( G \) be a simple algebraic group with \( \text{Lie}(G) = g \), \( b \) a Borel subalgebra of \( g \), and \( [b, b] = u \). The corresponding connected subgroups of \( G \) are \( B \) and \( U \). Here we are interested in the symmetric invariants of \( b \) and \( u \), and the canonical truncation of \( b \).

Most of these results are due to Kostant [K12] and Joseph [J77]. (Actually, many Kostant’s results are rather old and had been cited in [J77].) Our idea is to demonstrate utility of the Sato–Kimura theory in this context.

(\( \diamond_1 \)) If \( \text{ind} \, b = 0 \), then \( \text{ind} \, u = \text{rk} \, g \) and \( \tilde{b} = [b, b] = u \). Hence \( S(b)^U = S(u)^U \) is a polynomial ring of Krull dimension \( \text{rk} \, g \). As explained above, Theorem 2.2 applies to \( u = \tilde{b} \). It is well known that \( \text{ind} \, b = 0 \) *if and only if* \( g \in \{ B_n, C_n, D_{2n}, E_7, E_8, F_4, G_2 \} \).

Let \( f_1, \ldots, f_{\text{rk} \, g} \) be the basic invariants in \( S(u)^U \). Their weights and degrees are pointed out in [J77, Tables I,II], with some corrections in [FJ05, Annexe A]. It follows from those data that \( \sum_{i=1}^{\text{rk} \, g} \text{deg} \, f_i < b(u) = \frac{1}{2} \text{dim} \, b \) unless \( g = C_n \). This means that, for all but one case, the *codim–2* property does not hold for \( u \) (use Theorem 1.1!).

(\( \diamond_2 \)) If \( \text{ind} \, b > 0 \), then \( \text{ind} \, u < \text{rk} \, g \) and \( S(u)^U \) is a proper subalgebra of \( S(b)^U \). (Actually, one always has \( \text{ind} \, u + \text{ind} \, b = \text{rk} \, g \).) There are two possibilities to construct a suitable subalgebra of \( b \): one is related to the Sato–Kimura approach, and the other exploits the canonical truncation.

\( (-1-) \) Since \( B \) has a dense orbit in \( u^* \) [K12], one applies Sato–Kimura results to \( V = u^*, H = B \), and \( U = [B, B] \). This shows that \( S(u)^U \) is still a polynomial ring. Moreover, \( \Omega_{u^*} \) is big for the same reason as above. For all these cases (i.e., \( g \in \{ A_n, D_{2n+1}, E_6 \} \)), we have \( \sum_i \text{deg} \, f_i < b(u) \). Hence there is no *codim–2* property for \( u \), but Theorem 2.2 applies to \( u \).
Now, the canonical truncation of \( b \) is a subalgebra that properly contains \( u \). Namely, the toral part of \( \tilde{b} \) has dimension \( \text{ind} \ b \). If \( b = t \oplus u \) and \( \Delta^+ \) is the set of positive roots (= roots of \( u \)), then one canonically constructs the cascade \( K \) of strongly orthogonal roots in \( \Delta^+ \) (Kostant’s cascade), see [J77, Section 2]. If \( K = \{ \gamma_1, \ldots, \gamma_t \} \), then \( \text{ind} \ b = \dim t - t \) and \( \tilde{b} = t \oplus u \), where \( t = \{ \gamma_1, \ldots, \gamma_t \}^\perp \). Thus, we obtain that

\[
\text{k}[b^*]^U = \text{k}[b^*]^{(B)} = \text{k}[\tilde{b}^*]^\tilde{B}.
\]

By [J77, 4.16], \( S(b)^U \) is a polynomial ring of Krull dimension \( \text{rk} \, g \). Hence Theorem 2.2 applies to \( \tilde{b} \).

The output of this example is that, for any simple Lie algebra \( g \), our main theorem applies to both \( b \) (the canonical truncation of \( b \)) and \( u = [b, b] \). These two subalgebras of \( b \) coincide if and only if \( b \) is Frobenius.

4. More examples

We provide other applications of Theorem 2.2 to Lie algebras with or without the codim–2 property.

**Example 4.1.** Let \( G \subset SL(V) \) be a representation of a connected semisimple algebraic group. Consider the semi-direct product \( q = g \ltimes V^{ab} \). The corresponding connected group \( Q = G \ltimes \exp(V) \) has no non-trivial characters, hence \( \text{k}[q^*] \) does not contain proper \( Q \)-semi-invariants. Therefore, if (we know that) \( \text{k}[q^*]Q \) is a polynomial ring, then \( \Omega_q^\ast \) is big (use Proposition 1.2) and Theorem 2.2 applies to \( q \). The classification of representation \((G : V)\) of simple algebraic groups \( G \) such that \( \text{k}[q^*]Q \) is a polynomial ring is the subject of an ongoing project initiated by the second author. First non-trivial results for \( G = SL_n \) are found in [Y17], and the representations of the exceptional groups are considered in [PY17]. The representations of \( SO_n \) and \( Sp_{2n} \) will be handled in our forthcoming publication. (However, it is not always easy to decide whether the codim–2 property holds for such \( q \).)

Consider a concrete elementary example, where everything can be verified by hand. For an \( n \)-dimensional vector space \( V \) with \( n \geq 2 \), take the semi-direct product \( q = sl(V) \times nV = sl_n \rtimes n\text{k}^n \). The elements of \( nV \) (resp. \( nV^\ast \)) are regarded as \( n \times n \) matrices, where \( sl_n \) acts via left (resp. right) multiplications. Since \( \text{k}[nV^\ast]^{SL(V)} = \text{k}[\det] \) and generic stabilisers for the action \((SL(V) : nV)\) are trivial, we have

\[
\text{k}[q^*]^Q = \text{k}[nV^\ast]^{SL(V)} = \text{k}[\det].
\]

(The first equality here stems from [P07, Theorem 6.4].) Hence \( \text{ind} \, q = 1 \) and \( b(q) = n^2 \). For an \( n \times n \) matrix \( \eta \), one has \( d(\det) \eta = 0 \iff \text{rk} \, \eta < n - 1 \). Therefore, \( d(\det) \) vanishes on the determinantal variety of matrices of rank \( \leq n - 2 \), which is of codimension 4 in \( nV^\ast \). Thus, \( \Omega_q^\ast \) is big.
On the other hand, \( q^*_{\text{reg}} = \mathfrak{sl}_n^* \times (nV)^*_{\det} \) is a principal open subset, i.e., \( q^* \setminus q^*_{\text{reg}} = \mathfrak{sl}_n^* \times \{ \det = 0 \} \) is a divisor. Hence the codim–2 property does not hold here. This also follows from the fact that \( n = \deg(\det) < b(q) = n^2 \).

**Example 4.2.** Let \( \mathfrak{Hei}_n \) be the Heisenberg Lie algebra of dimension \( 2n + 1 \). It has a basis \( x_1, \ldots, x_n, y_1, \ldots, y_n, z \) such that the only nonzero brackets are \([x_i, y_i] = z, i = 1, \ldots, n\). Then \( \text{ind} (\mathfrak{Hei}_n) = 1 \) and \( \mathbb{k}[\mathfrak{Hei}_n^*]^{\mathfrak{Hei}_n} = \mathbb{k}[z] \). Therefore, \( \Omega_{\mathfrak{Hei}_n} = \mathfrak{Hei}_n^* \) and Theorem 2.2 applies here. It is easily seen that the hyperplane \( \{ \xi \in \mathfrak{Hei}_n^* \mid \langle \xi, z \rangle = 0 \} \) consists of the fixed points of the Heisenberg group. Hence, \( \mathfrak{Hei}_n \) does not have the codim–2 property.

This has the following application to centralisers of nilpotent elements:

*Example 4.3.* Let \( G \) be a simple group of type \( G_2 \). If \( G \cdot e \subset \mathfrak{g} \) is the subregular nilpotent orbit, then \( \dim \mathfrak{g}_e = 4 \) and \( \mathfrak{g}_e \cong \mathfrak{Hei}_1 \oplus \mathfrak{k}e \).

**Example 4.4.** Associated with any parabolic subalgebra \( \mathfrak{p} \) of \( \mathfrak{g} \), there is an interesting contraction of \( \mathfrak{g} \), which is called a parabolic contraction, see [PY13]. If \( \mathfrak{p} = \mathfrak{b} \), then such a contraction has much better properties [PY12]. Let \( \mathfrak{b}^- \) be an opposite Borel and \( \mathfrak{u}^- = [\mathfrak{b}^-, \mathfrak{b}^-] \). Then \( \mathfrak{g} = \mathfrak{b} \oplus \mathfrak{u}^- \) is a vector space sum. The contraction in question is \( \mathfrak{q} := \mathfrak{b} \ltimes (\mathfrak{u}^-)^{\text{ab}} \), where \( (\mathfrak{u}^-)^{\text{ab}} \) is an abelian ideal of \( \mathfrak{q} \) and \( (\mathfrak{u}^-)^{\text{ab}} \) is regarded as \( \mathfrak{b} \)-module via isomorphism \( \mathfrak{g}/\mathfrak{b} \cong \mathfrak{u}^- \). Note that \( \mathfrak{q} \) is solvable.

By [PY12, Section 3], we have

1. \( \text{ind} \mathfrak{q} = \text{rk} \mathfrak{g} \),
2. \( \mathbb{k}[\mathfrak{q}^*]^{\mathfrak{q}} \) is a polynomial ring, and
3. the degrees of basic invariants are the same as those for \( \mathfrak{g} \).

In particular, \( b(\mathfrak{q}) = b(\mathfrak{g}) \) and if \( f_1, \ldots, f_l \) are the basic invariants in \( \mathbb{k}[\mathfrak{q}^*]^{\mathfrak{q}} \), then \( \sum_{i=1}^l \deg f_i = b(\mathfrak{q}) \).

However, \( \mathfrak{q} \) does not have the codim–2 property unless \( \mathfrak{g} \) is of type \( A_l \) [PY12, Theorem 4.2]. Furthermore, \( \Omega_{\mathfrak{q}^*} \) is not big, if \( \mathfrak{g} \neq A_l \) [Y14, Remark 5.3]. Therefore, Theorem 2.2 does not apply to \( \mathfrak{b} \ltimes (\mathfrak{u}^-)^{\text{ab}} \), if \( \mathfrak{g} \neq A_l \). But one can look at the canonical truncation of \( \mathfrak{q} \), where the situation improves considerably. Following [Y14, Sect. 5], consider

\[
\tilde{\mathfrak{q}} = \mathfrak{u} \ltimes (\mathfrak{u}^-)^{\text{ab}} \subset \mathfrak{b} \ltimes (\mathfrak{u}^-)^{\text{ab}} = \mathfrak{q}.
\]

Here one has \( \tilde{\mathfrak{q}} = [\mathfrak{q}, \mathfrak{q}] \), \( \text{ind} \tilde{\mathfrak{q}} = \text{ind} \mathfrak{q} + (\text{dim } \mathfrak{q} - \text{dim } \tilde{\mathfrak{q}}) = 2\text{rk } \mathfrak{g} \), and hence \( b(\tilde{\mathfrak{q}}) = b(\mathfrak{q}) = b(\mathfrak{g}) \).

By [Y14, Theorem 5.9], \( S(\tilde{\mathfrak{q}}) \) is a polynomial ring of Krull dimension \( 2\text{rk } \mathfrak{g} \). The situation with the codim–2 property for \( \tilde{\mathfrak{q}} \) remains the “same” as for \( \mathfrak{q} \), but \( \Omega_{\tilde{\mathfrak{q}}^*} \) is already a big open subset of \( \tilde{\mathfrak{q}}^* \) (see the proof of Theorem 5.9 in [Y14]). Thus, Theorem 2.2 applies to \( \tilde{\mathfrak{q}} \) for all simple \( \mathfrak{g} \).
Example 4.5. Let \( g = g_0 \oplus g_1 \) be a \( \mathbb{Z}_2 \)-grading of a simple Lie algebra \( g \) and \( q = g_0 \ltimes g_1^{ab} \) the related \( \mathbb{Z}_2 \)-contraction. Then \( \text{ind} \ q = rk \ g \) and the \( \text{codim} \ = 2 \) property is always satisfied here (see [P07]). Here \( g_0 \) is reductive but not necessarily semisimple, and \( \mathbb{k}[q^*]^Q \) is a polynomial ring (in \( rk \ g \) variables) if and only if the restriction homomorphism \( \mathbb{k}[g]^G \to \mathbb{k}[g_1]^{G_0} \) is onto [Y17, Sect. 6]. This excludes only four \( \mathbb{Z}_2 \)-gradings related to the algebras of type \( E_n \).

Example 4.6. Let \( p \) and \( p' \) be two parabolic subalgebras of \( g \) such that \( p + p' = g \). Then \( s = p \cap p' \) is called a seaweed (or biparabolic) subalgebra of \( g \) [P01]. By work of Joseph and his collaborators, it is known in many cases that \( \mathbb{k}[s^*]^{(S)} \) is a polynomial ring. In particular, this is true for any \( s \), if \( g \) is of type \( A_n \) or \( C_n \) [J07]. (See also a summary of known results and other good cases in [FP].) Therefore, in all such good cases, the canonical truncation of \( s \) (= truncated biparabolic in Joseph’s terminology) is a good example for Theorem 2.2.

5. ON THE EQUIDIMENSIONALITY

Whenever a connected algebraic group \( Q \) has the property that \( \mathbb{k}[q^*]^Q \) is a polynomial ring, it is natural to inquire whether it is true that \( \mathbb{k}[q^*] \) is a free \( \mathbb{k}[q^*]^Q \)-module. The latter is equivalent to that the enveloping algebra \( U(q) \) is a free module over its centre \( Z(q) \simeq \mathbb{k}[q^*]^Q \). Assuming that \( \mathbb{k}[q^*]^Q \) is a polynomial ring, i.e., \( q^* \vert / Q \) is an affine space, the well-known geometric answer to this inquiry is that

\[ \mathbb{k}[q^*] \] is a free \( \mathbb{k}[q^*]^Q \)-module if and only if \( \pi_Q : q^* \to q^* \vert / Q \) is equidimensional,

i.e., equivalently, the zero-fibre of \( \pi_Q \), \( \pi_Q^{-1}(\pi_Q(0)) \), has the ‘right’ dimension \( \dim q - \dim q^* \vert / Q \). In the setting of Takiff algebras, one can raise the following:

Question 1. Suppose that the hypotheses of Theorem 2.2 hold for \( q \) and \( \pi_Q \) is equidimensional. Is it true that \( \pi_{Q(m)} : q^*(m) \to q^*(m) \vert / Q(m) \) is equidimensional, too?

As we shall see below, the general answer to this question is “no”. The celebrated positive result is that if \( g \) is semisimple, then the zero-fibre of \( \pi_{G(m)} \) is irreducible and \( \pi_{G(m)} \) is equidimensional for any \( m \in \mathbb{N} \) [M01, Appendix]. The reason is that the usual nilpotent cone \( N \subset g \simeq g^* \) is an irreducible complete intersection, and it has rational singularities. Here \( N(m) := \pi_{G(m)}^{-1}(\pi_{G(m)}(0)) \) is a jet scheme of \( N \).

For \( m = 1 \), these results are obtained in [G94] via a case-by-case argument. (See also another approach and a generalisation in [P07, Theorem 10.2].)

In this section, we prove that the equidimensionality does not carry over to the multi-current setting, even for semisimple \( g \). Let \( \hat{q} = q(m_1, \ldots, m_r) \) be a truncated multi-current algebra of \( q \), cf. (0.1). As in Section 2, we can write \( \hat{q} = \bigoplus_{i_1, \ldots, i_r} q[i_1, \ldots, i_r] \) and likewise for \( \hat{q}^* \), where \( 0 \leq i_j \leq m_j \), \( j = 1, \ldots, r \). It then follows from (2.1) and the iteration process (2.2)
that
\[ \xi = (\xi_{[i_1, \ldots, i_r]}) \in \hat{q}_{\text{reg}}^* \iff \xi_{[m_1, \ldots, m_r]} \in q_{\text{reg}}^* \]
(see also Prop. 4.1(b) in [MS16]). Assume that \( q \) satisfies all the assumptions of Theorem 2.2 and set \( \mathcal{N} = \pi_Q^{-1}(\pi_Q(0)) \subset q^* \). Then \( \mathcal{N} \langle m_1, \ldots, m_r \rangle \subset q^* \) stands for the zero-fibre of \( \pi_Q : \hat{q}^* \rightarrow \hat{q}^*/\hat{Q} \). We work below with the case in which all \( m_i = 1 \). Then \( \hat{q} \) is obtained as iteration of semi-direct products, the first step being \( q \sim q \ltimes q_{ab} = q(1) \). Let us investigate the relation between \( \mathcal{N} \) and \( N(1) \). This will also apply below to the passage from \( \mathcal{N}(1) \) to \( N(1,1) \).

Recall that \( \xi = (\xi_0, \xi_1) \) is an element of \( q(1)^* \). If \( k[q^*]^Q = k[f_1, \ldots, f_l] \) with \( l = \text{ind} q \), then \( k[q(1)^*]^Q(1) \) is freely generated by \( F_1^0, \ldots, F_l^0, F_1^1, \ldots, F_l^1 \), where \( F_i^0 \) depends only on \( \xi_1 \) and \( F_i^1(\xi_0, \xi_1) = \langle (df_i)_{\xi_1}, \xi_0 \rangle_q \). Therefore
\[
\mathcal{N}(1) = \{ (\xi_0, \xi_1) \mid \xi_1 \in \mathcal{N} \& \langle (df_i)_{\xi_1}, \xi_0 \rangle_q = 0 \ \forall i \}.
\]

Since \( df_i \) is a \( Q \)-equivariant morphism from \( q^* \) to \( q \), we have \( (df_i)_{\xi} \in q_\xi \). Moreover, if \( \xi \in q_{\text{reg}}^* \cap \Omega_q \), then \( \{ (df_i)_{\xi} \} \) is a basis for \( q_\xi \). Consider the stratification of \( \mathcal{N} \) determined by the basic invariants \( f_1, \ldots, f_l \). Set
\[ X_{i,\mathcal{N}} = \{ \xi \in \mathcal{N} \mid \text{dim span} \{ (df_1)_{\xi}, \ldots, (df_l)_{\xi} \} \leq i \} \]
Then \( \{ 0 \} = X_{0,\mathcal{N}} \subset X_{1,\mathcal{N}} \subset \cdots \subset X_{l,\mathcal{N}} = \mathcal{N} \). If \( \mathcal{N} = \bigcup_j \mathcal{N}_j \) is the irreducible decomposition, then \( X_{i,\mathcal{N}_j} \) is similarly defined for any \( j \). Set \( X_{i,\mathcal{N}_j}^0 = X_{i,\mathcal{N}_j} \setminus X_{i-1,\mathcal{N}_j} \) for \( i > 0 \) and \( X_{0,\mathcal{N}_j}^0 = \{ 0 \} \). Clearly, each \( X_{i,\mathcal{N}_j}^0 \) is irreducible and open in \( X_{i,\mathcal{N}_j} \). However, \( X_{i,\mathcal{N}_j}^0 \) can be empty for some \( i, j \). It follows from (5.1) that \( p : \mathcal{N}(1) \rightarrow \mathcal{N}, (\xi_0, \xi_1) \mapsto \xi_1 \), is a surjective projection and
\[ \text{dim} p^{-1}(X_{i,\mathcal{N}_j}^0) = \text{dim} X_{i,\mathcal{N}_j}^0 + \text{dim} q - i. \]
Since \( q(1) \) has a polynomial ring of symmetric invariants, with \( 2l \) basic invariants \( F_1^0, \ldots, F_l^0, F_1^1, \ldots, F_l^1 \), one can consider the corresponding stratification of \( \mathcal{N}(1) \):
\[ \{ 0 \} = X_{0,\mathcal{N}(1)} \subset X_{1,\mathcal{N}(1)} \subset \cdots \subset X_{2l,\mathcal{N}(1)} = \mathcal{N}(1). \]

**Lemma 5.1.** We have \( p^{-1}(X_{i,\mathcal{N}}^0) \subset \bigcup_{j=2i}^{l+i} X_{j,\mathcal{N}(1)}^0 \).

*Proof.* By definition, \( \text{dim span} \{ (df_1)_{\xi}, \ldots, (df_l)_{\xi} \} = i \) for \( \xi \in X_{i,\mathcal{N}}^0 \). This clearly implies that, for \( \xi = (\xi_0, \xi_1) \in p^{-1}(\xi) \), we have \( \text{dim span} \{ (df_1^0)_{\xi}, \ldots, (df_l^0)_{\xi} \} = i \) and \( \text{dim span} \{ (df_1^1)_{\xi}, \ldots, (df_l^1)_{\xi} \} \geq i \) (cf. Table 1 with \( m = 1 \)). Furthermore, the lowest components of \( (df_j^0)_{\xi} \) and \( (df_j^1)_{\xi} \) belong to different graded pieces of \( q(1) \).

By Lemma 5.1, the closures of \( p^{-1}(X_{i,\mathcal{N}}^0) \) for all \( j \) are the only subvarieties of \( \mathcal{N}(1) \) that meet \( \Omega_{q(1)}^* \). Therefore, if \( X_{i,\mathcal{N}_j}^0 \neq \emptyset \), then \( p^{-1}(X_{i,\mathcal{N}_j}^0) \) is an irreducible component of \( \mathcal{N}(1) \) of dimension \( \text{dim} \mathcal{N}_j + \text{dim} q - l \). Since \( \text{dim} \mathcal{N}_j \geq \text{dim} q - l \) for all \( j \), one readily obtains
Proposition 5.2. If \( q \) satisfies all the assumptions of Theorem 2.2, then the following two conditions are equivalent:

1. \( \pi_{Q(1)} \) is equidimensional, i.e., \( \dim N(1) = \dim q(1) - \text{ind} q(1) = 2(\dim q - l) \);
2. (i) \( \pi_Q \) is equidimensional, i.e., \( \dim N_j = \dim q - l \) for all \( j \);
   (ii) \( X_{l,N_j}^o \neq \emptyset \) for all \( j \) (i.e., \( N_j \cap \Omega_{q^*} \neq \emptyset \));
   (iii) \( \text{codim}_{N_j}(X_{l,N_j}^o) \geq l - i \) for \( i < l \).

This yields a sufficient condition for the absence of equidimensionality of \( \pi_{Q(1)} \):

Corollary 5.3. If there is an irreducible component \( N_j \) of \( N \) such that \( N_j \cap \Omega_{q^*} = \emptyset \), then \( \dim p^{-1}(N_j) > 2(\dim q - l) \). Hence \( \pi_{Q(1)} \) is not equidimensional.

We say that such \( N_j \) is a bad irreducible component of \( N \).

Remark 5.4. If \( N \) is irreducible and \( \dim N = \dim q - l \), then a similar analysis shows that \( N(1) \) is irreducible if and only if conditions (i), (ii), and (iii)' hold, where (i), (ii) are as above, with \( N \) in place of \( N_j \), and the last one is a bit stronger than (iii):

(iii)' \( \text{codim}_{N}(X_{l,N}^o) > l - i \) for \( i < l \).

For, the closure of \( p^{-1}(X_{l,N}^o) \) is always an irreducible component of \( N(1) \) of the ‘right’ dimension \( 2(\dim q - l) \), and we need the condition that \( p^{-1}(X_{l,N}^o) \) does not yield another component, i.e., \( \dim p^{-1}(X_{l,N}^o) < 2(\dim q - l) \) for \( i < l \).

From now on, we assume that \( q = g \) is a simple Lie algebra of rank \( l \). Let us recall some properties of the nilpotent cone \( N \subset g^* \cong g \):

- \( N \) is irreducible and contains finitely many \( G \)-orbits;
- \( X_{l,N}^o \) is the principal (or, regular) nilpotent orbit;
- \( X_{l-1,N}^o \) is irreducible of dimension \( \dim N - 2 \) and \( X_{l-1,N}^o \neq \emptyset \) (it contains the sub-regular nilpotent orbit as a dense open subset). Moreover, if \( \deg f_1 \leq \ldots \leq \deg f_l \), then \( \deg f_{l-1} < \deg f_l \) and \( (df_l)_{\xi} = 0 \) for all \( \xi \in X_{l-1,N}^o \).

Then \( N(1) \) is also irreducible, and for the projection \( p : N(1) \to N \), we have:

- \( p^{-1}(X_{l,N}^o) \) is the open dense \( G(1) \)-orbit in \( N(1) \), of dimension \( 2(\dim g - l) \);
- \( \dim p^{-1}(X_{l-1,N}^o) = 2(\dim g - l) - 1 \). Hence the closure of \( p^{-1}(X_{l-1,N}^o) \) is a simple divisor, say \( D \), in \( N(1) \). By Lemma 5.1, \( p^{-1}(X_{l-1,N}^o) \subset X_{2l-2,N(1)}^o \cup X_{2l-2,N(1)}^{ab} \).

The next iteration replaces \( g(1) \) with \( g(1) \cong g(1) \rtimes g(1)^{ab} \) and provides the surjective projection \( p_1 : N(1,1) \cong N(1)\langle 1 \rangle \to N(1) \). Here we are interested in \( p_1^{-1}(D) \). There is a dichotomy: either (1) \( D \cap X_{2l-2,N(1)}^o \neq \emptyset \) or (2) \( D \subset X_{2l-2,N(1)}^{ab} \).

- In the first case, \( \dim p_1^{-1}(D) = 4(\dim g - l) \) and it is an irreducible component of \( N(1,1) \) that is different from the closure of \( p_1^{-1}(X_{2l,N(1)}^o) \). In other words, \( p_1^{-1}(D) \) is a bad irreducible component of \( N(1,1) \). Hence \( \pi_{g(1,1,1)} \) is not equidimensional by Corollary 5.3.
• In the second case, \( \dim p_1^{-1}(D) = 4(\dim g - l) + 1 \). Hence \( \pi_{g(1,1)} \) is already not equidimensional and then \( \pi_{g(1,1,1)} \) is not equidimensional, too.

Thus, we have proved

**Theorem 5.5.** Let \( g \) be a simple Lie algebra. Then

(i) \( N(1, 1) \subset g(1, 1)^* \) is reducible;
(ii) \( N(1, 1, 1) \subset g(1, 1)^* \) is not pure, i.e., \( \pi_{g(1,1,1)} \) is not equidimensional.

**Remark 5.6.** If \( g = g_1 \oplus \cdots \oplus g_s \) is semisimple, where each \( g_i \) is simple and \( s \geq 2 \), then \( g\langle m_1, \ldots, m_r \rangle \simeq \bigoplus_{i=1}^{s} g_i\langle m_1, \ldots, m_r \rangle \). Therefore, Theorem 5.5 readily extends to the semisimple case.

Although \( N(1, 1) \) is reducible, it still might be true that \( \pi_{g(1,1)} \) is equidimensional; in particular, it is likely that the second case above does not materialise. In fact, we hope (conjecture) that \( \pi_{g(1,1)} \) is always equidimensional.

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