Gauge Invariance of QED$_{2+1}$

I. V. Tyutin† and Vad. Yu. Zeitlin‡

I. E. Tamm Theory Department, P. N. Lebedev Physical Institute
Russia, 117924, Moscow, Leninsky prospect, 53

The problem of gauge invariance of the physical sector of (2+1)-dimensional Maxwell-Chern-Simons quantum electrodynamics (QED$_{2+1}$) is studied. It is shown that using Proca mass term for the infrared regularization one obtains gauge-invariant fermion mass and the physical mass shell of QED$_{2+1}$ is well-defined in all orders of the perturbation theory. We are demonstrating also a class of gauges in the framework of QED$_{2+1}$, including transversal and Feynman-like ones, where the physical sector is well defined and independent of the gauge parameter.

† E-mail address: tyutin@lpi.ac.ru
‡ E-mail address: zeitlin@lpi.ac.ru
1 Introduction

Recent years have witnessed a great interest to field theory models in the space-time of dimension (2+1) (see e.g. Refs. [1–7] and references to [1]). These models arise as a temperature reduction of four-dimensional models [8, 9] or may be used for describing low-dimensional models like films and wires [4, 5, 10]. Besides, field theories in low-dimensional space-time may be of considerable interest as being simplified (toy) models of more realistic (and more complicated) four-dimensional models.

Many unusual properties of 3-dimensional models are due to the gauge-invariant mass of the gauge field, which is not induced by spontaneous symmetry breaking. The mass may be introduced by adding the Chern-Simons term to the bare action [1, 12–14] (topologically massive gauge theories) and/or be generated dynamically via interaction with fermions [1].

Low-dimensional field theories have severe infrared singularities. In Refs. [1, 14] where (2+1)-dimensional quantum electrodynamics (QED_{2+1}) was first studied it was shown that the singularities lead to dependence of the one-loop physical mass of the fermion on the gauge parameter in an arbitrary \( \alpha \)-gauge, and the higher order perturbative diagrams are apparently divergent. It was assumed in the above papers that one should use transversal (Landau) gauge in the perturbative calculations, since the photon propagator is less infrared singular in this case and infrared divergencies may be absent from the Green functions.

Nevertheless, it is not clear whether gauge variance of the physical mass of the fermion indicates an anomaly (of Adler’s anomaly type) even in the transversal gauge. The latter problem may be raised in other way: whether the theory is unitary in the subspace of states of fermions and massive photons? As far as we know [1, 15], the physical unitarity of QED_{2+1} in the transversal gauge has been stated by comparing it to the Coulomb gauge where the nonphysical sector is absent. The equivalence of the two gauges has been shown by modifying longitudinal terms in the gauge field propagator. This does not affect the Green function on the physical mass shell, but transforms the Coulomb photon propagator into that in the transversal gauge. Unfortunately, the photon propagator is ill-defined in the Coulomb gauge because of \( 1/p^2 \) infrared singularities, thus the latter gauge requires an additional definition.

In this paper we present a detailed analysis of unitarity, both by introducing appropriate infrared regularization – by adding the Proca mass term to the bare action (this modification of QED_{2+1} we shall call the Proca model) and in the framework of QED_{2+1} in a class of gauges including transversal and Feynman-like ones. In the Proca model all the photon modes are massive and infrared divergencies are absent (however, just two modes are physical). We shall show that the observables are independent of the gauge parameter and after the infrared regularization is removed (by letting the Proca mass tend to zero) the scattering amplitudes involving one of the photons (that with vanishing mass) die off, and the resulting theory is just QED_{2+1} in the physical subspace of the transversal gauge. Furthermore, the unitarity in the physical sector is demonstrated directly in the framework of QED_{2+1}.

The paper is organized as follows. In Sec. 2 we are showing the reasons why the fermion mass in QED_{2+1} is gauge dependent despite the Ward identities are explicitly satisfied. In Sec. 3 we are demonstrating by analyzing the corresponding Ward identities that addition of the Proca mass term to QED_{2+1} Lagrangian removes dependence of the physical sector on the gauge parameter. Gauge-variant mode of the electromagnetic field splits off and does not contribute to any scattering amplitude, while vertex functions are gauge-invariant on the fermion mass
shell. The limit of zero Proca mass is studied and it is shown that the regularization may be taken off. In Sec. 4 the unitarity and gauge independence of the physical sector in a class of gauges is shown in the framework of QED$_{2+1}$. In the Appendix the normalization conditions in the theory with the Chern-Simons term are obtained.

2 Gauge variance of QED$_{2+1}$

First, let us consider the reasons for gauge dependence of the fermion mass (generally speaking, any observable) in QED$_{2+1}$.

The Lagrangian of QED$_{2+1}$ is the following:

$$\mathcal{L}_{(0,\alpha)} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\theta}{4} \varepsilon_{\mu\nu\sigma} F^{\mu\nu} A^{\sigma} - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 + \bar{\psi}(i\hat{\partial} + e\hat{A} - M)\psi \ .$$  (1)

The first term in the left-hand side of Eq. (1) is the Maxwell term ($F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$), the second one is so-called Chern-Simons term, the third one is the gauge fixing term, and the last one describes fermions and interaction. We shall call the above model QED$_{2+1}$ in $\alpha$-gauge.

In a standard way one may obtain the Ward identities for generating functional of vertex functions $\Gamma$:

$$\partial_\mu \frac{\delta \Gamma}{\delta A_\mu} = ie\bar{\psi} \frac{\delta \Gamma}{\delta \psi} - ie\bar{\psi} \frac{\delta \Gamma}{\delta \bar{\psi}} \ ,$$  (2)

where

$$e^{iW(I,\eta,\bar{\eta})} = \int DAD\psi D\bar{\psi} e^{\int dx (\mathcal{L} + IA + \eta\psi + \bar{\eta}\bar{\psi})} \ ,$$  (3)

$$\Gamma = W - IA - \eta\psi - \bar{\eta}\bar{\psi} \equiv \bar{\Gamma} - \frac{1}{2\alpha} (\partial A)^2 \ ,$$  (4)

$$A_\mu = \frac{\delta W}{\delta I^\mu} \ , \ \psi = \frac{\delta W}{\delta \eta} \ , \ \bar{\psi} = \frac{\delta W}{\delta \bar{\eta}} \ .$$  (5)

Taking into account Eq. (3) one has the following expression for the variation of the $\bar{\Gamma}$ with respect to the gauge parameter:

$$\frac{\delta \bar{\Gamma}}{\delta \alpha} = \frac{e}{2\alpha} \int dx dy \left[ \partial_\mu \frac{\delta^2 W}{\delta I^\mu(x)\delta \eta(y)} D_{(0)}(x-y) \frac{\delta \bar{\Gamma}}{\delta \psi(y)} - \partial_\mu \frac{\delta^2 W}{\delta I^\mu(x)\delta \bar{\eta}(y)} D_{(0)}(x-y) \frac{\delta \bar{\Gamma}}{\delta \bar{\psi}(y)} \right] \ ,$$  (6)

$$D_{(m^2)}(x-y) = \int \frac{dp}{(2\pi)^3} \frac{e^{-ip(x-y)}}{p^2 - m^2} \ .$$  (7)

If only vertex-function-type diagrams (i.e. one-particle-irreducible diagrams) were nonsingular when the external momenta are on the mass shell, then by using the standard arguments employed to prove the theorem of the equivalency and gauge invariance Ref. [16], one could show that the transversality of the vertex functions (except two-point function, the inverse photon propagator) in photon momenta with fermion momenta on the mass shell follows from
Eq. (2), and independence of the vertex functions (except two-point photon vertex) on the
gauge parameter $\alpha$ appears from Eq. (6).

Unfortunately, in QED$_{2+1}$ in $\alpha$-gauge due to the ill infrared behavior of the longitudinal
part of the gauge field propagator $D_{\mu\nu}^{(0,\alpha)}(p)$,

$$D_{\mu\nu}^{(0,\alpha)}(p) = \frac{-i}{p^2 - \theta^2 + i\epsilon} \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) + \theta \varepsilon_{\mu\nu\alpha} \frac{p^\alpha}{p^2} - i\alpha \frac{p_\mu p_\nu}{p^4}, \quad (8)$$

one-particle irreducible diagrams may be singular on the mass shell, thus the gauge invariance
cannot be proven despite validity of Eqs. (2) and (6). Consider fermion mass operator as an
illustration. It follows from Eq. (6) that

$$\frac{\partial \Sigma(x - y)}{\partial \alpha} = -\frac{i}{2} \int dz \left[ \int dz_1 dz_2 D_\mu(z - z_1) \Gamma^\mu(z_1, x, z_2) S(z_2 - z) \right] S^{-1}(z - y) -$$

$$-\frac{i}{2} \int dz S^{-1}(x - z) \left[ \int dz_1 dz_2 D_\mu(z - z_2) S(z - z_1) \Gamma^\mu(z_1, z_2, y) \right], \quad (9)$$

$$\Gamma^\mu(x, y, z) = \frac{\delta}{\delta A_\mu(x)} \frac{\delta}{\delta \psi(y)} \frac{\delta}{\delta \bar{\psi}(z)} \Gamma \bigg|_{\mu=\psi=\bar{\psi}=0}, \quad (10)$$

$S$ is an exact fermion propagator (vacuum expectation value of the $T$-product) and the mass
operator $\Sigma$ is defined as follows:

$$S^{-1} = S_0^{-1} + i\Sigma, \quad (11)$$

$S_0$ is the bare fermion propagator and the function $D_\mu$ is equal to

$$D_\mu(x) = \int \frac{dp}{(2\pi)^3} e^{-ipx} \frac{p_\mu}{p^4}. \quad (12)$$

Consider Eq. (11) in the one-loop approximation (two terms provide the same contribution):

$$\frac{\partial \Sigma_1(x - y)}{\partial \alpha} = -ie^2 \int dz S_0^{-1}(x - z) [D_\mu(x - y) S_0(x - y) \gamma^\mu] \equiv ie^2 \int dz S_0^{-1}(x - z) \Sigma'(z - y). \quad (13)$$

In the momentum space the latter is:

$$\frac{\partial \Sigma_1(p)}{\partial \alpha} = ie^2 S_0^{-1}(p) \Sigma'_1(p) = e^2 (\hat{p} - M) \Sigma'_1(p), \quad (14)$$

$$\Sigma'_1(p) = -i \int \frac{dk}{(2\pi)^3} \frac{(\hat{p} - \hat{k} + M)\hat{k}}{k^4((\hat{p} - \hat{k})^2 - M^2)} = \frac{1}{8\pi} \frac{1}{\hat{p} - M} + \ldots, \quad (15)$$

where (…) denotes terms which are logarithmically singular and finite on the mass shell. The
validity of Eqs. (14), (15) may be easily checked by direct calculating of $\Sigma_1$. However, the
latter equation does not manifest gauge invariance of $\Sigma_1$ on the mass shell (at $\hat{p} = M$) despite
the multiplier vanishing at \( \hat{p} = M \), i.e. the physical mass of the fermion is not gauge invariant. This is a consequence of the singularity of \( \Sigma' \) on the mass shell (see Eq. (15)), which is provided by the infrared behavior of the photon propagator in \( \alpha \)-gauge.

### 3 Proca term and gauge invariance

We shall demonstrate in this section that the gauge invariance in QED\(_{2+1}\) may be restored. As a regularization parameter we are adding the Proca mass\(^1\) to the bare Lagrangian Eq. (1). The Lagrangian of the Proca model is the following:

\[
\mathcal{L}_{(m,\alpha)} = -\frac{1}{4} V_{\mu\nu} V^{\mu\nu} + \frac{\theta}{4} \varepsilon_{\mu\nu\alpha} V^{\mu\nu} V^\alpha + \frac{m^2}{2} V_\mu V^\mu - \frac{1}{2} (\partial_\mu V^\mu) \frac{1}{\alpha} (\partial_\nu V^\nu) + \bar{\psi} (i\hat{\partial} + e\hat{V} - M) \psi \, ,
\]

\[
V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu \, .
\]

We have written the gauge-fixing term in that form, since we shall use below an arbitrary function of d’Alambertian as a gauge parameter, \( \alpha = \alpha (-\Box) \).

The solutions to the free equation of motion for the field \( V_\mu \) are the following (correct procedure of normalization of the polarization vectors is discussed in the Appendix):

\[
V_\mu(x) = \sum_{i=1}^{3} \int \frac{d\mathbf{p}}{2\pi \sqrt{2\omega_i}} (e^{-ip^\mu x} u_\mu^{(i)} a_i(\mathbf{p}) + \text{h.c.}) \, , \quad p_\mu^{(i)} = (\omega_i, \mathbf{p}) \, .
\]

\[
\begin{align*}
u_\mu^{(1,2)} &= \frac{\omega_{1,2}}{|\mathbf{p}| \sqrt{\omega_{1,2}^2 - \mathbf{p}^2 + m^2}} (p_\mu^{(1,2)} - g_{\mu \alpha} \frac{\omega_{1,2}^2 - \mathbf{p}^2}{\omega_{1,2}} + i \frac{\omega_{1,2}^2 - \mathbf{p}^2 - m^2}{\theta \omega_{1,2}} \varepsilon_{\mu\alpha\nu} p^{(1,2)\nu}) \, , \\
\omega_{1,2} &= \sqrt{\mathbf{p}^2 + m_{1,2}^2} \, , \quad m_1^2 = m^2 + \frac{\theta^2}{2} + \theta \sqrt{m^2 + \frac{\theta^2}{4}} \, , \quad m_2^2 = \frac{m^4}{m^2 + \frac{\theta^2}{2} + \theta \sqrt{m^2 + \frac{\theta^2}{4}}} \, ,
\end{align*}
\]

\[
\begin{align*}
u_\mu^{(3)} &= \frac{1}{m} p_\mu^{(3)} \, , \quad \omega_3 = \sqrt{\mathbf{p}^2 + m_3^2} \, , \quad m_3^2 = \alpha m^2 \, .
\end{align*}
\]

There are three excitations in the gauge sector of the model. Excitations with polarization vectors \( u_\mu^{(1)} \), \( u_\mu^{(2)} \) are massive physical modes, while the excitation with the polarization vector \( u_\mu^{(3)} \) is nonphysical with its mass depending on the gauge parameter (as we shall show below the latter mode is free and does not contribute to scattering amplitudes).

The free vector field \( V_\mu \) propagator is the following

\[
D_{\mu\nu}^{(m,\alpha)}(p) = -i \frac{p^2 - m^2}{(p^2 - m^2)^2 - \theta^2 p^2} \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) + \theta \varepsilon_{\mu\nu\alpha} \frac{p^\alpha}{p^2 - m^2} - i \frac{p_\mu p_\nu}{p^2} \frac{\alpha}{p^2 - \alpha m^2} \, .
\]

\(^1\)\((2+1)\)-dimensional models with the Proca term have been considered in Refs. [17, 18] in a different context.
It is nonsingular when \( p \to 0 \) for any \( \alpha \). Therefore, the Proca model is free of infrared problems. For comparison, the similar quantities in QED\(_{2+1}^\text{independence of } \bar{\Gamma} \text{ on modes and the coefficients are chosen to satisfy commutation relations } [a_{\mu h}(p), a_{\mu h}^\dagger(q)] = [a_2(p), a_3^\dagger(q)] = [a_3(p), a_2^\dagger(q)] = \delta(p - q), \text{ other commutators are vanishing.} \)

\[
A_\mu(x) = \int \frac{d\mathbf{p}}{2\pi \sqrt{2\omega}} (e^{-i(p\cdot x)} e^{(1)\mu} a_{\mu h}(\mathbf{p}) + \text{h.c.}) + \int \frac{d\mathbf{p}}{2\pi \sqrt{2\omega}} (e^{-i(p\cdot x)} (e^{(2)\mu} a_2(\mathbf{p}) + e^{(3)\mu} a_3(\mathbf{p})) + \text{h.c.}) ,
\]

(23)
with the polarization vectors\(^2\):

\[
e^{(1)\mu} = \frac{\omega_1}{\theta|\mathbf{p}|} (p^{(1)\mu} - \delta_{\mu 0} \frac{\theta^2}{\omega_1} + i \frac{\theta}{\omega_1} \epsilon_{\mu\alpha 0} p^{(1)\alpha}) , \quad \omega_1 = p^{(1)}_0 = \sqrt{\theta^2 + \mathbf{p}^2} . \quad (24)
\]

\[
e^{(2)\mu} = \frac{1}{\sqrt{\theta|\mathbf{p}|}} p^{(2)\mu} , \quad p^{(2)}_0 = |\mathbf{p}| \quad (25)
\]

\[
e^{(3)\mu} = -\frac{1}{2} \sqrt{\frac{\theta}{|\mathbf{p}|}} \left( \alpha g_{\mu 0} + \frac{2i}{\theta} \epsilon_{\mu\beta 0} p^{(2)\beta} + \left( \frac{|\mathbf{p}|}{\theta^2} - \frac{\alpha}{2|\mathbf{p}|} - i \alpha x_0 \right) p^{(2)\mu} \right) , \quad (26)
\]
the propagator is presented in Eq. \([5]\). There is only one physical excitation in the gauge-field sector of QED\(_{2+1}^\text{infrared limit} \), the propagators in the Proca model are nonsingular in the infrared limit, the arguments underlying the equivalency theorem are valid in the theory, thus Eq. \([28]\) implies the independence of \( \bar{\Gamma} \) on \( \alpha \) with the external fermion momenta on the mass shell, and the transversality in the photon momenta of the vertex functions with more than two external lines does follow from Eq. \([27]\) (see also the next section).

\[
\delta \Gamma = \frac{e}{2} \int dx dy \left[ \frac{\partial^2 W}{\partial I^\mu(x) \partial \eta(y)} \frac{\delta \alpha}{\alpha} D_{(am^2)}(x - y) \frac{\delta \bar{\Gamma}}{\delta \psi(x)} - \partial_\mu \frac{\partial^2 W}{\partial I^\mu(x) \partial \bar{\eta}(y)} \frac{\delta \alpha}{\alpha} D_{(am^2)}(x - y) \frac{\delta \bar{\Gamma}}{\delta \bar{\psi}(y)} \right] , \quad (28)
\]

\[
\Gamma = \bar{\Gamma} - \frac{1}{2} (\partial_\mu V^\mu) \frac{1}{\alpha} (\partial_\nu V^\nu) + \frac{m^2}{2} V_\mu V^\mu . \quad (29)
\]

\(^2\)Free equation of motion has two linearly independent massless (nonphysical) solutions, longitudinal mode \( e^{(2)\mu} = c p_\mu \) and gauge-dependent mode \( e^{(3)\mu} = e [|\mathbf{p}| g_{\mu 0} + \frac{2i|\mathbf{p}|}{\alpha} \epsilon_{\mu\alpha 0} p^\alpha - i |\mathbf{p}| x_0 g_{\mu 0}] \) (in an arbitrary relativistic gauge one cannot present all modes as plane waves, see e.g. Ref. \([19]\)). Polarizations Eq. \([26]\) are superpositions of these modes and the coefficients are chosen to satisfy commutation relations \([a_{\mu h}(p), a_{\mu h}^\dagger(q)] = [a_2(p), a_3^\dagger(q)] = [a_3(p), a_2^\dagger(q)] = \delta(p - q)\), other commutators are vanishing.
One of the corollaries of the gauge invariance of the vertex function is the fact that the physical masses of the fermion and of the two photon modes with polarizations \( u^{(3)}_\mu \) and \( u^{(2)}_\mu \) do not depend on \( \alpha \). Consider one-loop corrections to the fermion mass as an illustration. Taking into account Eq. (28) one has

\[
\frac{\partial \Sigma'_1(p)}{\partial \alpha} = e^2 (\hat{p} - M) \Sigma'_1(p)
\]

(30)

\[
\Sigma'_1(p) = -\frac{i e^2}{(2\pi)^3} \int \frac{d\hat{k}}{(\hat{k}^2 - \alpha m^2)^2((\hat{p} - \hat{k})^2 - M^2)} (\hat{p} - \hat{k} + M) \hat{k} \left( \frac{k^2 - m^2}{(p - k)^2 - M^2} \right),
\]

(31)

\[
\Sigma'_1(\hat{p} = M) = \frac{1}{8\pi m \sqrt{\alpha}}.
\]

(32)

Since \( \Sigma'_1(p) \) is nonsingular on the fermion mass shell, it follows from Eq. (30) that the fermion physical mass does not depend on \( \alpha \). Explicit calculation of the mass operator confirms this:

\[
\Sigma_1(p) = \Sigma_1^{(0)}(p) + \Sigma_1^{(\alpha)}(p),
\]

(33)

\[
\Sigma_1^{(0)}(p) = -ie^2 \int \frac{d\hat{k}}{(2\pi)^3} \frac{\gamma^\mu(\hat{p} - \hat{k} + M)\gamma^\nu}{((\hat{p} - \hat{k})^2 - M^2)} \left( \frac{k^2 - m^2}{(k^2 - m^2)^2} \right) \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + i\theta \varepsilon_{\mu\nu\alpha} \frac{k^\alpha}{k^2 - m^2},
\]

\[
\Sigma_1^{(\alpha)}(p) = -ie^2 \alpha(\hat{p} - M) \int \frac{d\hat{k}}{(2\pi)^3} \left( \frac{\gamma^\mu}{((p - k)^2 - M^2)} \right) \left( \frac{k_\mu k_\nu}{k^2 - \alpha m^2} \right) \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + i\theta \varepsilon_{\mu\nu\alpha} \frac{k^\alpha}{k^2 - m^2},
\]

(34)

(35)

One can see that at any \( \alpha \) only \( \Sigma_1^{(0)} \) contributes to the mass renormalization. Therefore the mass does not depend on \( \alpha \), and in the \( m \to 0 \) limit it coincides with the mass renormalization in QED\(_{2+1}\) in the transversal gauge \( \alpha = 0 \).

There are two important consequences of the transversality in photon momenta of the vertex functions (with more than two external lines) on the fermion mass shell. Since \( u^{(3)}_\mu \sim p_\mu \), the corresponding photon mode splits off from the physical sector which contains the fermion and two other photon modes.

Then, Eq. (19) for the polarization vector \( u^{(2)}_\mu \) may be rewritten as follows:

\[
u^{(2)}_\mu|_{m \to 0} = \frac{1}{m} p_\mu + ml_\mu(p). \]

(36)

the vector \( l_\mu \) has a finite limit when \( m \to 0 \). Therefore, due to the transversality of vertex functions the elements of the \( S \)-matrix corresponding to radiation of \( n \) photons with the polarization \( u^{(2)}_\mu \) (we shall call these photons "soft") have the factor of \( m^n \). In case the vertex functions after taking the regularization off, \( m \to 0 \), have lower singularities in \( m \), \( S \)-matrix
elements containing soft external photons vanish in this limit, and the physical sector will contain one massive photon and the fermion only. Actually, it will be demonstrated in the next section that all the vertex functions exist in QED\(_{2+1}\) in the transversal gauge \(\alpha = 0\) on the mass shell.

Thus the analysis of the Proca model shows that after the infrared regularization is taken off, the Feynman rules become exactly those of QED\(_{2+1}\) in the transversal gauge thus establishing unitarity of QED\(_{2+1}\) in the transversal gauge, the physical sector of the resulting theory comprising of a fermion and a massive photon.

Note that the changing of the constant \(\alpha\) to an arbitrary function of d’Alambertian, \(\alpha \rightarrow \alpha(-\Box)\), does not affect the expression for the photon propagator and Ward identities Eq. (27) and Eq. (28). Transversality of the vertex functions in photon momenta and the fact that \(\bar{\Gamma}\) does not depend on \(\alpha(-\Box)\) on the fermion mass shell also hold true. Let us choose \(\alpha(-\Box)\) to be

\[
\alpha(-\Box) = \frac{\Box(\Box + m^2)}{(\Box + m^2)(\Box + (1 - \xi)m^2) + \theta^2 \Box} .
\]  

(37)

The corresponding photon propagator is the following

\[
D^{(m,\xi)}_{\mu\nu}(p) = -\frac{t}{(p^2 - m^2)^2 - \theta^2 p^2} \left( g_{\mu\nu} - (1 - \xi) \frac{p_{\mu} p_{\nu}}{p^2} \right) (p^2 - m^2) + i \theta \varepsilon_{\mu\nu\alpha} p^\alpha .
\]

(38)

In the \(m \rightarrow 0\) limit Eq. (38) becomes:

\[
D^{(0,\xi)}_{\mu\nu}(p) = -\frac{t}{p^2 - \theta^2} \left( g_{\mu\nu} - (1 - \xi) \frac{p_{\mu} p_{\nu}}{p^2} + i \theta \varepsilon_{\mu\nu\alpha} p^\alpha \right) .
\]

(39)

Infrared behavior of the propagator \(D^{(0,\xi)}_{\mu\nu}(p)\) is exactly the same as that of the photon propagator in QED\(_{2+1}\) in the transversal gauge, therefore \(D^{(0,\xi)}_{\mu\nu}(p)\) may also be used as the QED\(_{2+1}\) photon propagator. Since the physical spectrum of the Proca model is independent of \(\xi\), it will hold true for QED\(_{2+1}\) as well. On the other hand, as far as the use of \(D^{(0,\xi)}_{\mu\nu}(p)\) for the QED\(_{2+1}\) propagator does not lead to any infrared divergency one can expect the independence of the physical sector on \(\xi\), also the unitarity in this sector may be demonstrated directly in the framework of QED\(_{2+1}\). We shall justify this suggestion in the next section.

\section{Gauge invariance in the \(\xi\)-gauge}

Let us consider the Lagrangian

\[
\mathcal{L}_{(0,\xi)} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\theta}{4} \varepsilon_{\mu\nu\alpha} F^{\mu\nu} A^\alpha - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 + B \partial_\mu A^\mu - \frac{\xi}{2\theta^2} \partial_\mu B \partial^\mu B + \bar{\psi}(i \partial + e A - M) \psi ,
\]

(40)

which we shall refer to as Lagrangian of QED\(_{2+1}\) in \(\xi\)-gauge. The free propagator of the gauge field \(A_\mu\) in this model is the following
\begin{align}
D_{\mu \nu}^{(0, \xi)}(p) &= -\frac{i}{p^2 - \theta^2} \left( g_{\mu \nu} - (1 - \xi) \frac{p_{\mu} p_{\nu}}{p^2} + i \theta \varepsilon_{\mu \nu \alpha} \frac{p^\alpha}{p^2} \right), \quad (41)
\end{align}
and at \( \xi = 0 \) it transforms to the propagator of QED\(_{2+1} \) in the transversal gauge. Obviously, the main infrared singularity of the propagator \( D_{\mu \nu}^{(0, \xi)}(p) \) is independent of \( \xi \). To calculate the spectrum of the gauge field modes we should solve the free equations of motion

\begin{align}
\begin{pmatrix}
A_\mu \\
B
\end{pmatrix} &= \sum_{i=1}^{4} \int \frac{d^4 p}{2\pi \sqrt{2\omega_p}} \left( e^{-ip(x)} \epsilon_i \psi_i p + \text{h.c.} \right), \quad (42)
\end{align}

\begin{align}
p_i^{(i)} &= (\omega_i, p), \quad \omega_{1,2} = \sqrt{p^2 + \theta^2}, \quad \omega_{3,4} = |p|, \quad (43)
\end{align}

\begin{align}
\epsilon_i^{(1)} &= \frac{\omega}{\theta|p|} \left( \begin{array}{c}
p_i^{(1)} - \delta \theta \omega_2 + i \frac{\theta}{\omega_1} \varepsilon_{\mu \nu \alpha} p^{(1)\alpha} \\
0
\end{array} \right), \quad \epsilon_i^{(2)} = \frac{\sqrt{\xi}}{\theta} \left( \begin{array}{c}
p_i^{(2)} \\
-i \frac{\theta}{\omega_1}
\end{array} \right), \quad (44)
\end{align}

\begin{align}
\epsilon_i^{(3)} = \frac{1}{\sqrt{\theta|p|}} \left( \begin{array}{c}
p_i^{(3)} \\
0
\end{array} \right), \quad \epsilon_i^{(4)} = -\frac{1}{\sqrt{\theta|p|}} \left( \begin{array}{c}
p_i^{(4)} + \theta \varepsilon_{\mu \nu \alpha} p^{(4)\alpha} \\
i|p|
\end{array} \right), \quad (45)
\end{align}

where the creation and annihilation operators are subject to the following commutation relations:

\begin{align}
[a_p^{(1)} \psi_q^{(1)\dagger}] = [a_p^{(2)} \psi_q^{(2)\dagger}] = [a_p^{(3)} \psi_q^{(3)\dagger}] = [a_p^{(4)} \psi_q^{(4)\dagger}] = \delta(p - q), \quad (46)
\end{align}

other commutators vanish.

As we shall demonstrate below just one mode of four, that of the polarization \( \psi_i^{(1)} \), is physical. It does not depend on \( \xi \) and coincides with the physical mode in \( \alpha \)-gauge. One can obtain the Ward identities in this model. For the generating functional of Green functions \( W \) they are the following:

\begin{align}
\left( \partial_\mu \frac{\delta}{\delta I^\mu(x)} + \frac{\xi}{\theta^2} \square \frac{\delta}{\delta I^B(x)} \right) W + I_B(x) = 0, \quad (47)
\end{align}

\begin{align}
\left( -\square + \frac{\theta^2}{\xi} \partial_\mu \frac{\delta}{\delta I^\mu(x)} + ie \left( \eta(x) \frac{\delta}{\delta \eta(x)} - \bar{\eta}(x) \frac{\delta}{\delta \bar{\eta}(x)} \right) \right) W - \frac{\theta^2}{\xi} I_B(x) - \partial_\mu I^\mu(x) = 0, \quad (48)
\end{align}

\begin{align}
\frac{\partial W}{\partial \xi} = \frac{1}{2} \int dx \left[ \frac{1}{\xi^2} \left( \partial_\mu \frac{\delta W}{\delta I^\mu(x)} \right)^2 + \frac{1}{\theta^2} \frac{\delta W}{\delta I^B(x)} \square \frac{\delta W}{\delta I^B(x)} \right] + \frac{e}{2\xi} \int dx dy D_0(x - y) \partial_\mu \frac{\delta}{\delta I^\mu(x)} \left( \eta(y) \frac{\delta}{\delta \eta(y)} - \bar{\eta}(y) \frac{\delta}{\delta \bar{\eta}(y)} \right) W, \quad (49)
\end{align}

\((I_B \text{ is the source to the field } B)\).
For the generating functional of vertex functions $\bar{\Gamma}$,

$$\bar{\Gamma} \equiv \Gamma + \frac{1}{2\xi} (\partial A)^2 - B \partial_\mu A^\mu - \frac{\xi}{2\theta^2} B \Box B \; ,$$

(50)

the Ward identities become

$$\frac{\delta \bar{\Gamma}}{\delta B} = 0 \; ,$$

(51)

$$\partial_\mu \frac{\delta \bar{\Gamma}}{\delta A_\mu (x)} = i e \left( \psi (x) \frac{\delta}{\delta \psi (x)} - \bar{\psi} (x) \frac{\delta}{\delta \bar{\psi} (x)} \right) \bar{\Gamma} \; ,$$

(52)

$$\frac{\partial \bar{\Gamma}}{\partial \xi} = \frac{e}{2\xi} \int dx dy D_0 (x - y) \left( \partial_\mu \frac{\delta^2 W}{\delta I_\mu (x) \delta \eta (y)} \partial \bar{\psi} (y) - \partial_\mu \frac{\delta^2 W}{\delta I_\mu (x) \delta \bar{\eta} (y)} \delta \psi (y) \right) \; .$$

(53)

We shall show below that the generating functional of vertex functions $\bar{\Gamma}$ in this model on the mass shell exists for all the modes and the arguments underlying the equivalency theorem are valid here, too. Therefore, the transversality of $\bar{\Gamma}$ in photon momenta on the fermion mass shell (f.m.s) follows from Eqs. (47), (48) or Eqs. (51), (52),

$$\frac{\delta \bar{\Gamma}}{\delta B} \bigg|_{f.m.s.} = 0 \; ,$$

(54)

and independence of $\bar{\Gamma}$ on parameter $\xi$ on the fermion mass shell

$$\frac{\partial \bar{\Gamma}}{\partial \xi} \bigg|_{f.m.s.} = 0 \; ,$$

(55)

appears from Eq. (49) or Eq. (53).

It follows from Eq. (54) that the $S$-matrix depends on two creation and annihilation operators out of four, namely $a^{(1)}$, $a^{(1)\dagger}$ and $a^{(4)}$, $a^{(4)\dagger}$. Due to commutation relations Eq. (16) this implies that the $S$-matrix is unitary in the physical subspace comprising fermion and photon physical mode with the polarization $e^{(1)}_l$. Then, it follows from Eq. (55) that the matrix elements are independent of $\xi$ in the physical subspace. In particular, the physical mass of the fermion and propagating photon mode are $\xi$-invariant. Consider one-loop correction to the fermion mass as an illustration:

$$\Sigma_1 (p) = \Sigma_1^{(1)} (p) + \Sigma_1^{(\xi)} (p)$$

(56)

$$\Sigma_1^{(1)} (p) = -i e^2 \int \frac{dk}{(2\pi)^3} \gamma^\mu \frac{(\hat{p} - \hat{k} + M)}{((p - k)^2 - M^2)(k^2 - \theta^2)} \gamma^\nu \left( g_{\mu\nu} + i \theta \varepsilon_{\mu\nu\alpha} \frac{k^\alpha}{k^2} \right)$$

(57)

$$\Sigma_1^{(\xi)} (p) = -i e^2 (1 - \xi) \int \frac{dk}{(2\pi)^3} \frac{\hat{k} (\hat{p} - \hat{k} + M) \hat{k}}{k^2 ((p - k)^2 - M^2)(k^2 - \theta^2)} =$$

$$-i e^2 (1 - \xi) (\hat{p} - M) \int \frac{dk}{(2\pi)^3} \frac{(\hat{p} - \hat{k} + M) \hat{k}}{k^2 ((p - k)^2 - M^2)(k^2 - \theta^2)}$$

(58)
\[ \Sigma_1^{(\xi)}(p) \text{ vanishes on the fermion mass shell and the only contribution to the mass renormalization comes from } \Sigma_1^{(1)}(p), \text{ which is independent of } \xi \text{ and coincides with the mass renormalization in the transversal gauge.} \]

Let us prove the existence of vertex functions and validity of the equivalency theorem. The most infrared singular term in the photon propagator Eq. (41) at any \( \xi \) is

\[ \frac{\theta}{p^2 - \theta^2} \frac{\varepsilon_{\mu\nu\lambda} p^\lambda}{p^2}, \quad (59) \]

i.e. the propagator in the infrared region is effectively proportional to \( \varepsilon_{\mu\nu\lambda} p^\lambda/p^2 \). Let us suppose that a diagram contains a closed fermion loop as a subdiagram (or it is just a closed fermion loop). We shall denote such a loop with \( n \) external photon lines as \( \Pi_{\mu_1...\mu_n}(p_1, \ldots, p_n) \). Due to the transversality of the diagram in any photon momenta,

\[ p^{\mu_1} \Pi_{\mu_1...\mu_n}(p_1, \ldots, p_n) = 0, \quad (60) \]

\( \Pi_{\mu_1...\mu_n}(p_1, \ldots, p_n) \) is proportional to every photon momentum for \( n \geq 3 \). Therefore, infrared singularities of photon propagators outgoing the fermion loop with more than two external lines are completely suppressed. For \( n = 2 \) (the polarization diagram) it follows from Eq. (60) that

\[ \Pi_{\mu\nu} = (p^2 g_{\mu\nu} - p_\mu p_\nu) A + \varepsilon_{\mu\nu\lambda} p^\lambda B, \quad (61) \]

where \( A \) and \( B \) have finite limit at \( p \to 0 \), thus the effective photon line resulting after insertion of the polarization diagrams is proportional to \( \varepsilon_{\mu\nu\lambda} p^\lambda/p^2 \) again in the infrared limit. Therefore, the (sub)diagrams with closed fermion lines are infrared innocuous and internal photon lines may be considered as effective ones.

Diagrams left after exclusion of fermion loops are non-closed fermion lines with outgoing and incoming internal and external photon lines. One may suppose that each loop momentum integration is carried out over some internal photon momentum. Since the internal photon line has infrared singularity \( 1/k \), infrared divergency of the diagram may take place when the diagram contains two other lines with propagators \( \sim 1/k \). First, suppose the diagram has no external lines. Since the photon line is tied to two fermion lines, the divergence may take place if only terms nonvanishing in \( k \to 0 \) limit in the denominators of the fermion propagators are cancelled due to the mass shell condition. Thus the infrared divergency may arise just in the following type of integrals:

\[ \int dk \frac{\varepsilon_{\mu\nu\lambda}}{((p_1 + k)^2 - M^2)((p_2 - k)^2 - M^2)k^2} T_{\beta_1\beta_2} = \quad (62) \]

\( T \) has no infrared singularities in \( k \). Potentially divergent part of the integral is

\[ \int dk \frac{\gamma^\mu (\hat{p}_1 + M)_{\alpha_1 \beta_1} (\gamma^\nu (\hat{p}_2 + M)_{\alpha_2 \beta_2} \varepsilon_{\mu\nu\lambda} k^\lambda}{((p_1 + k)^2 - M^2)((p_2 - k)^2 - M^2)k^2} T_{\beta_1\beta_2} = \quad (63) \]
\[
\int dk \frac{((\hat{p}_1 - M)\gamma^\mu - 2p_1^\mu)_{\alpha_1\beta_1}((\hat{p}_2 - M)\gamma^\nu - 2p_2^\nu)_{\alpha_2\beta_2}\varepsilon_{\mu\nu\lambda\kappa}k^\lambda}{(k^2 + 2p_1k + \delta_1)(k^2 - 2p_2k + \delta_2)k^2} T_{\beta_1\beta_2} =
\]
\[
\int \frac{dk}{(k^2 + 2p_1k + \delta_1)(k^2 - 2p_2k + \delta_2)k^2} \left( (\hat{p}_1 - M)\gamma^\mu \right)_{\alpha_1\beta_1} (\hat{p}_2 - M)\gamma^\nu \right)_{\alpha_2\beta_2} - 2((\hat{p}_1 - M)\gamma^\mu)_{\alpha_1\beta_1}p_2^\nu \delta_{\alpha_2\beta_2} - 2p_1^\mu \delta_{\alpha_1\beta_2} (\hat{p}_2 - M)\gamma^\nu \right)_{\alpha_2\beta_2} \varepsilon_{\mu\nu\lambda\kappa}k^\lambda T_{\beta_1\beta_2}
\]
\[
\delta_1 = p_1^2 - M^2 , \quad \delta_2 = p_2^2 - M^2 . \quad (64)
\]

Since in the vicinity of the mass shell one has
\[
\hat{p}_1 - M = \frac{\delta_1}{2M} , \quad \hat{p}_2 - M = \frac{\delta_2}{2M} , \quad (65)
\]
the integral behaves as \(\delta \ln \delta\) and it is not singular on the mass shell.

Then, suppose the diagram contains external photon lines. Potentially dangerous are the following diagrams:
\[
\int dk
\]

The denominator of the additional fermion propagator is
\[
(p_1 + k + p)^2 - M^2 = k^2 + 2pk + p^2 + \delta_1 + 2p_1p + 2kp_1 \quad (67)
\]
and even if the massless photon \((p^2 = 0)\) is radiated it has infrared regularization \(2p_1p\). Therefore, infrared singularities of diagrams containing photon radiation are the same as those of diagrams without radiation and all these diagrams are infrared finite.

Thus we have shown that the vertex functions of QED_{2+1} in \(\xi\)-gauge has no singularities at any \(\xi\). In particular, it implies that the vertex functions in the Proca model have finite limit at \(m \to 0\) (we have used this in the previous section).

Let us prove Eq. (54) and Eq. (55) now. We shall consider just Eq. (55), since Eq. (54) may be proven along the same lines (this approach is also valid in proving the similar equalities in the Proca model). It is more convenient to demonstrate that the last term in Eq. (13) is vanishing on the mass shell. That would mean that Eq. (55) is satisfied (we are following the approach elaborated in Ref. [16]).

The last term in the right-hand side of Eq. (13) has the following structure:
\[
\int dp \psi_{in}(p) \tilde{K}(p) \int dk
\]

where the dotted line denotes the propagator

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\[
\frac{k_\mu}{k^2(k^2 - \theta^2)} \quad .
\] (69)

We have used above the Ward identities for the photon propagator valid both for free and exact functions:

\[
k^{\mu}D_{\mu\nu} = -i\frac{\xi k_\nu}{k^2 - \theta^2} \quad .
\] (70)

The operator \( K(p) \) in Eq. (68) is

\[
K(p) = \hat{p} - M \quad ,
\] (71)

\( \psi_{in}(p) \) is a solution of the free equation of motion:

\[
K(p)\psi_{in}(p) = 0 \quad , \quad \psi_{in}(p) \sim \delta(p^2 - M^2) \quad .
\] (72)

As \( K(p) \) acts rightwards on the function with no \( 1/\hat{p} - M \) singularities, Eq. (68) vanishes.

Let the diagram Eq. (68) be one-particle irreducible. In this case \( K(p) \) operates on a function of the form

\[
\int dk \frac{(\hat{p} - \hat{k} + M)k_\mu}{(k^2 - 2pk + \delta)k^2(k^2 - \theta^2)}M_\mu(k) \quad , \quad \delta = p^2 - M^2 \quad ,
\] (73)

which is obviously nonsingular on the mass shell \( \hat{p} = M, \delta = 0 \). If the diagram Eq. (68) is a one-particle reducible one, it has the following structure:

\[
\gamma \rightarrow M' \quad ,
\] (74)

\[
\gamma = \quad p - k \quad .
\] (75)

where \( M' \) is one-particle irreducible diagram, and Eq. (68) is nonvanishing. However, since \( \gamma \) is nonsingular on the mass shell \( \hat{p} = M \), its contribution is exactly cancelled by the fermion wave function renormalization (one may find the details in Ref. [16]). Note that in \( \alpha \)-gauge the propagator (69) in Eq. (68) should be changed to

\[
\frac{k_\mu}{k^4} \quad ,
\] (76)

thus \( K(p) \) in Eq. (68) acts on the function singular on the fermion mass shell \( \hat{p} = M \) and Eq. (68) is nonvanishing (even if vertex functions exist in \( \alpha \)-gauge).

Finally, we have shown that QED\(_{2+1} \) is well-defined in the class of \( \xi \)-gauges and it contains \( \xi \)-independent physical sector where \( S \)-matrix is unitary. In \( \xi \rightarrow 0 \) limit one has exactly the
same expression for the generating functional of vertex functions $\bar{\Gamma}$ and for operators $A_\mu$, $\psi$ and $\bar{\psi}$ as in QED$_{2+1}$ in the transversal gauge, i.e. the transversal gauge belongs to $\xi$-gauges class. At $\xi = 1$ the expression for the free photon propagator is the simplest:

$$D^{(1)}_{\mu\nu} = -\frac{i}{p^2 - \theta^2} \left( g_{\mu\nu} + \psi \frac{\varepsilon_{\mu\alpha\beta}}{p^2} \right) ,$$

and the latter gauge is the analogue of the Feynman gauge.

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**Appendix**

We shall discuss here the normalization condition of the polarization vectors in the Proca model. This question requires special discussion since the antisymmetric structure associated with the Chern-Simons coefficient spoils the naive normalization condition $u^{(i)}_\alpha u^{(j)}_\alpha = \delta_{ij}$. However, correct definition of the normalization coefficients of the polarization modes is crucial for reduction formulae and the study of the massless limit of the Proca model.

Let us rewrite the free action for the gauge field $V_\mu$ as follows:

$$S = \frac{1}{2} \int d^3x V_\mu \Lambda^{\mu\nu} V_\nu ,$$

$$\Lambda^{\mu\nu} = g^{\mu\nu} \partial_\alpha - \frac{1}{\alpha} \partial_\mu \partial_\nu - \frac{1}{2} \theta \varepsilon_{\mu\nu\alpha} \partial_\alpha + m^2 g_{\mu\nu} .$$

Then, one may introduce the scalar product of two functions $\phi_\mu(x)$ and $\psi_\mu(x)$ in a standard way (see, e.g. [20]):

$$(\phi, \psi) = i \int dx J^0(x; \phi, \psi) ,$$

where the current $J^\mu(x; \phi, \psi)$ is:

$$J^\mu(x; \phi, \psi) = \phi^{(\alpha)} (j^\mu_{\alpha\beta} \psi^\beta) - (j^\mu_{\alpha\beta} \phi^\beta) \psi^{(\alpha)} ,$$

$$j^\mu_{\alpha\beta} = g_{\alpha\beta} \partial^\mu - \frac{1}{\alpha} (g^\mu_\alpha \partial_\beta + g^\mu_\beta \partial_\alpha) - \frac{\theta}{2} \varepsilon^\mu_{\alpha\beta} ,$$

and its derivative $\partial_\mu J^\mu(x; \phi, \psi)$ may be written as follows

$$\partial_\mu J^\mu(x; \phi, \psi) = \phi_\mu \left( \Lambda^{\mu\nu} \psi_\nu \right) - \left( \Lambda^{\mu\nu} \phi_\nu \right) \psi_\mu .$$

Therefore, $\partial_\mu J^\mu(x; \phi, \psi)$ vanishes on solutions of the free equation of motion. The zeroth component of the current is equal to
\[ J^0(x; \phi, \psi) = P^\mu_{(\phi)} \psi^\mu - \phi^\mu P^\mu_{(\psi)} \]  

(A7)

with \( P^\mu_{(\phi)} \) being the momentum canonically conjugated to field \( \phi^\mu \) for action \( S \).

The gauge field \( V_\mu(x) \) may be decomposed in the polarization vectors

\[ V_\mu(x) = \sum_{i=1}^{3} \int dP \left[ e^{(i)}_{p,\mu}(x) a^{(i)}_p + h.c. \right] \]  

(A8)

\[ e^{(i)}_{p,\mu}(x) = \frac{1}{2\pi\sqrt{2\omega_i}} e^{-ip(x)^\mu} u^{(i)}_\mu(p) \]  

(A9)

Vectors \( e^{(i)}_{p,\mu}(x) \) are subjects to two conditions. First, they are solutions of the free equation of motion,

\[ \Lambda^{\mu\nu} e^{(i)}_{p,\mu}(x) = 0 \]  

(A10)

(in the Proca model the latter condition defines the polarizations vectors up to normalization coefficients).

Second, polarization vectors satisfy normalization conditions,

\[ (e^{(i)}_p, e^{(j)}_q) = 0 \]  
\[ (e^{(i)*}_p, e^{(j)}_q) = \delta_{ij} \delta(p - q) \]  

(A11)

In this case

\[ a^{(i)}_p = (e^{(i)*}_p, V) \]  
\[ a^{(i)*}_p = (e^{(i)}_p, V) \]  

(A12)

The normalization conditions (A11), (A12) may be derived also by using canonical commutation relations

\[ [P_\mu(x), V_\nu(y)]|_{x_0 = y_0} = -ig_{\mu\nu}\delta(x - y) \]  

(A13)

\[ [a^{(i)}(p), a^{(j)*}_q] = \delta_{ij} \delta(p - q) \]  
\[ i = 1, 2 \quad [a^{(3)}(p), a^{(3)*}_q] = -\delta(p - q) \]  

(A14)

Other commutators vanishes, \( P_\mu = \partial L/\partial \dot{V}_\mu \) are momenta canonically conjugated \( V_\mu \).

Let us emphasize that the thus defined polarization vectors enter the matrix elements of the scattering matrix when these are expressed through the Green functions (reduction formulae):

\[ \langle \ldots | S | p, i; \ldots \rangle = \langle \ldots | S a^{(i)*}_p | \ldots \rangle = \]  
\[ i \int d^3 x e^{(i)}_{p,\mu}(x) \tilde{\Lambda}^{\mu\nu}(0) TV_\nu(x) \ldots | 0 \rangle = \int d^3 x e^{(i)}_{p,\mu}(x) \Gamma^{\mu\nu}(x, \ldots) \]  

(A15)
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