Communication, Distortion, and Randomness in Metric Voting

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Abstract

In distortion-based analysis of social choice rules over metric spaces, one assumes that all voters and candidates are jointly embedded in a common metric space. Voters rank candidates by non-decreasing distance. The mechanism, receiving only this ordinal (comparison) information, aims to nonetheless select a candidate approximately minimizing the sum of distances from all voters to the chosen candidate. It is known that while the Copeland rule and related rules guarantee distortion at most 5, many other standard voting rules, such as Plurality, Veto, or \( k \)-approval, have distortion growing unboundedly in the number \( n \) of candidates.

An advantage of Plurality, Veto, or \( k \)-approval with small \( k \) is that they require less communication from the voters; all deterministic social choice rules known to achieve constant distortion require voters to transmit their complete ranking of all candidates. This motivates our study of the tradeoff between the distortion and the amount of communication in deterministic social choice rules.

We show that any one-round deterministic voting mechanism in which each voter communicates only the candidates she ranks in a given set of \( k \) positions must have distortion at least \( \frac{2n-k}{k} \); we give a mechanism achieving an upper bound of \( O\left(\frac{n}{k}\right) \), which matches the lower bound up to a constant. For more general communication-bounded voting mechanisms, in which each voter communicates \( b \) bits of information about her ranking, we show a slightly weaker lower bound of \( \Omega\left(\frac{n}{b}\right) \) on the distortion.

For randomized mechanisms, the situation looks much brighter: it is known that Random Dictatorship achieves expected distortion strictly smaller than 3, almost matching a lower bound of \( 3 - \frac{2}{n} \) for any randomized mechanism that only receives each voter’s top choice. We close this gap, by giving a simple randomized social choice rule which only uses each voter’s first choice, and achieves expected distortion \( 3 - \frac{2}{n} \).

1 Introduction

In voting or social choice, there is a set of \( n \) alternatives (such as political candidates or courses of action) from which a group (such as a country or an organization) wants to select a winner. Each voter submits a ranking (or preference order) of the candidates, and the mechanism (or social choice rule) chooses a winner based on these submitted rankings.

Many different social choice rules have been proposed, and it is an important question how to compare them. One fruitful and long line of work, dating back at least to the correspondence of Borda and Condorcet [25, 26], formulates axioms that a social choice rule “should” satisfy; one can then compare social choice rules by which or how many of these axioms they satisfy [19]. Unfortunately, many results in this area are impossibility results, most notably Arrow’s result.

\footnote{A large and important part of the literature studies the goal of choosing a complete consensus ranking of all candidates; we will not study this alternative goal here, and therefore identify social choice with the selection of a single winner.}
for producing a consensus ranking and the Gibbard-Satterthwaite Theorem ruling out truthful voting rules with minimal additional properties.

An alternative to the axiomatic approach is to consider social choice as an optimization problem with the goal of selecting the “best” candidate for the population. A natural way to express the notion of “best” is to assume that each voter has a utility (or cost) for each candidate; the mechanism’s goal is to optimize the aggregate (e.g., average or median) utility or cost of all voters. However, as articulated in, the social choice rule has to optimize with crucial information missing: a voter can only communicate her ranking according to the utility/cost. In other words, the mechanism receives only ordinal information — which candidate is preferred over which other candidate — even though it needs to optimize a cardinal objective function. From an optimization perspective, this means that the mechanism should simultaneously optimize over all utility/cost functions that are consistent with the reported rankings, in that they would give rise to the observed rankings. The worst-case ratio (over all cost/utility functions) between the mechanism’s cost/utility and that of the optimum candidate for the specific function is called the mechanism’s distortion. (Formal definitions of all concepts and terms are given in Section 2.)

In applying this general framework, an important question is what class of cost/utility functions to consider. A natural approach was suggested in (see also the expanded/improved journal version and general overview): all candidates and voters are jointly embedded in a metric space, and the cost of voter $v$ for candidate $x$ is their metric distance $d(v, x)$. The assumption that voters rank candidates by non-decreasing distance in a latent space dates back to earlier work on so-called single-peaked preferences, though much of the earlier work focuses on the special case when the metric is the line. Using the framework of distortion and metric costs, show a remarkable separation. While many commonly used voting rules (such as Plurality, Veto, $k$-approval, Borda count) have either unbounded distortion or distortion linear in the number of candidates, and indeed all score-based rules have distortion $\omega(1)$ (in terms of the number of candidates), uncovered-set rules have distortion at most 5. To describe uncovered-set rules, consider a tournament graph $G$ on the $n$ candidates which contains the directed edge $(x, y)$ iff at least as many voters prefer $x$ to $y$ as vice versa. The uncovered set of $G$ is the set of all candidates with paths of length at most 2 to all other candidates; an example of such a candidates is the candidate $x$ with maximum outdegree, which is selected by the Copeland rule. show that any candidate in the uncovered set of $G$ has distortion at most 5, and also show a lower bound of 3 on the distortion of every deterministic voting mechanism.

One advantage of some of the mechanisms with large distortion — such as Plurality, Veto, or $k$-approval with small $k$ — is that they require little communication from the voters. Instead of having to transmit her entire ranking, a voter under Plurality only needs to share her first choice; similarly a voter under Veto only needs to share her last choice. This observation raises the question of whether high distortion is inherently a consequence of limited communication between voters and the mechanism.

The answer to the preceding question is clearly “No:” there are simple randomized mechanisms achieving constant distortion. Perhaps the simplest is Random Dictatorship: “Return the first choice of a uniformly random voter.” This mechanism is known to have distortion strictly smaller than 3, a smaller distortion than any deterministic mechanism can achieve. However, despite the frequent mathematical appeal and elegance of randomized algorithms and mechanisms, most organizations are leery of using randomization for making important decisions; hence, we consider

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2For consistency and clarity, we will always refer to voters using female and candidates using male pronouns.

3A reader taking issue with this statement may want to think about his/her own computer science, mathematics, economics, or operations research department. Even though these are likely among the most savvy organizations in terms of understanding randomization, decision making procedures practically never involve randomization, except
Determinism a very desirable property in the design of voting mechanisms. Considering the following three properties: (1) low distortion, (2) low communication, (3) determinism, it is known that any two can be achieved simultaneously:

- Random Dictatorship satisfies (1), (2).
- Uncovered-set mechanisms satisfy (1), (3).
- Plurality and many other mechanisms satisfy (2), (3).

The big-picture question we investigate in this article is the tradeoff between all three of these desirable properties.

1.1 Our Models and Results

We only consider the goal of minimizing the average (or total) metric distance of all voters from the winning candidate. Our main result, proved in Section 4, is essentially a negative answer to the question of whether any voting mechanisms can simultaneously have all three desirable properties. We consider a model in which each voter communicates $b$ bits of information about her ranking to the mechanism, in a single round. Associated with each $b$-bit string $\mu$ is a subset $\Pi_\mu$ of rankings. The $\Pi_\mu$ must form a disjoint cover of all possible rankings. If they did not form a cover, some voters might not have any message to send, making the mechanism ill-defined. And if the $\Pi_\mu$ were not disjoint, then it is not clear how a voter with multiple possible messages $\mu$ would make the (non-deterministic) choice which one to send; in particular, this choice could depend on the actual metric distances, and it might require much more subtle definitions to place meaningful restrictions on a mechanism to not exploit such information. Each voter communicates the (unique) $\mu$ such that her permutation is in $\Pi_\mu$. We require that the same set $\Pi_\mu$ is associated with the string $\mu$, regardless of the identity of the voter sending the string. Under this model, in Section 4, we prove the following lower bound:

**Theorem 1.1** Every one-round deterministic voting mechanism in which each voter sends only a $b$-bit string to the mechanism has distortion at least $2^n - \frac{4}{b} - 1$.

Most mechanisms with limited communication are of a fairly specific form: voters can communicate only their choices in a (small) set $K$ of their ranking, typically at the top or bottom of their ballots. (Either giving the candidate for each such position, or specifying them as a set, as in $k$-approval.) For such restricted mechanisms, a simpler proof (in Section 3) gives a lower bound that is stronger by a factor $\Theta(\log n)$:

**Theorem 1.2** Any deterministic one-round social choice rule which receives, from each voter, no information about candidates outside positions $K$ in her ranking, has distortion at least $\frac{2n - |K|}{|K|}$.

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4Recall that [4] and several follow-up articles studied both the average and median distance.

5Analyzing the distortion of multi-round deterministic mechanisms with limited communication is a very interesting direction for future work.

6Our results require this assumption. While studying the power of mechanisms that allow different voters to use different encodings of their preferences would be interesting theoretically, voting mechanisms which treat votes differently a priori tend to not be accepted in practice.
The proof of Theorem 1.2 is significantly easier and cleaner than the proof of Theorem 1.1, while still containing some of the key ideas. Therefore, we present the proof of Theorem 1.2 before that of Theorem 1.1.

Theorem 1.2 provides a generalization of Theorem 1 of the recent work [31], which proves linear distortion for the special case when $K$ consists of the top $k$ positions, for constant $k$. In fact, [31] shows these lower bounds on the expected squared distortion of randomized mechanisms; this directly implies the same bounds for deterministic mechanisms.

The fact that the lower bound of Theorem 1.2 is stronger than that of Theorem 1.1 by a factor of $\Theta(\log n)$ is discussed in more detail in Section 4. To see it most immediately, consider the case $|K| = k = \omega(n / \log n), k = o(n)$. Because $k = o(n)$, Theorem 1.2 provides a super-constant lower bound on the distortion. On the other hand, communicating the positions of $k$ candidates requires $b = \omega(n)$ bits, so the lower bound of Theorem 1.1 is vacuous. Closing this $\Theta(\log n)$ gap is an interesting direction for future work, discussed in Section 7.

The reason we consider Theorem 1.1 our main contribution is that it helps us pinpoint the source of high distortion. Several recent works have shown lower bounds on the distortion of different specific classes of social choice rules, such as score-based rules [3] or the above-mentioned top-$k$ ballots [31]. Our result implies that regardless of the intricacy of the mechanism, low communication (within the context studied here) and determinism are enough to force high distortion. Communication as a measure of complexity is fairly natural, as evidenced by the mechanisms typically used in practice for large numbers of alternatives. Communication can also be regarded as a proxy for cognitive effort imposed on the voters, although admittedly, the computation of a message $\mu$ in a general $b$-bit bounded mechanism may still require the voter to first determine her full ranking of all candidates.

The results of Theorems 1.1 and 1.2 are lower bounds, raising the question of how small one can make a mechanism's distortion when communication is limited. In Section 5 we address this question, proving the following theorem.

**Theorem 1.3** There is a one-round deterministic social choice rule which, given only each voter's top $k$ candidates (in order), selects a candidate with distortion at most $\frac{79n}{k}$.

The deterministic social choice rule of Theorem 1.3 is a generalization of the Copeland rule to such top-$k$ ballots. Up to constant factors, the bounds of Theorems 1.2 and 1.3 match. Closing the gap between the upper and lower bound is likely difficult, as even for $k = n$, the best-known lower bound of 3 does not match the best current upper bound of $2 + \sqrt{5} \approx 4.23$ due to [45]; whether there is a deterministic mechanism with metric distortion 3 is a well-known open question. Notice also that Theorem 1.3 implies that knowing each voter’s ranking for a constant fraction of candidates is sufficient to achieve constant distortion, a fact that may not be a priori obvious.

As we discussed earlier, the main focus in this article is on deterministic mechanisms: as discussed earlier, the Random Dictatorship mechanism has distortion strictly smaller than 3, achieving small distortion and low communication simultaneously. However, even for $k = 1$, this leaves a gap between the upper bound of essentially 3 for Random Dictatorship and the lower bound of $3 - \frac{2}{n}$. Recently, [31] shrunk this gap: they proved that the Random Oligarchy mechanism — which

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7 An application of Corollary 5.3 of [38] gives an upper bound of $\frac{12n}{k}$, which, however, is still far from matching the lower bound.

8 The amount by which it is smaller is of order $1/|V|$; here, $|V|$ is the number of voters, which we consider “large.”
samples three voters and outputs a majority of first-place votes if it exists, and otherwise the choice of a random voter among the three — achieves expected distortion close to $3 - \frac{2}{n}$, though there still remains a small gap between the upper and lower bounds. As an additional result, in Section 6 we close this remaining gap:

**Theorem 1.4** There is a simple randomized social choice rule in which each voter only communicates her first-choice candidate, and which achieves distortion at most $3 - \frac{2}{n}$.

**Nature of Latent Distances**

The optimization objective of the mechanism is expressed in terms of latent utilities, or more specifically, distances. A subtle question is whether voters “know” their utilities for (or distances to) candidates, or — perhaps more philosophically — whether these utilities/distances are “real.” In general, one attractive feature of the distortion framework is that it completely obviates the need to address this question: when a mechanism achieves low distortion, it optimizes robustly over all possible utility/distance functions consistent with the rankings, and the question of whether voters could actually quantify the utilities in a meaningful way is irrelevant.

However, when we focus on the design of mechanisms with low communication, the question should be addressed explicitly, as the answer has a strong impact on the design space for mechanisms. When the mechanism designer has control not only over the aggregation of ballots, but also over the type of information about voter preferences that is elicited, this opens the door to designing mechanisms in which agents explicitly communicate numerical estimates of their utilities for some candidates; in turn, having such information may allow a mechanism to achieve lower distortion (as we will see in related work below). If agents themselves cannot quantify their utilities, then not only is communication of a ranking imposed by the class of typically used mechanisms, but it is inherently the only information about the utilities that agents themselves may have access to.

Which of these two assumptions (or something between the two along a more fine-grained spectrum) is more realistic likely depends on the envisioned application. For example, if software agents vote on a preferred alternative in a mostly economically motivated setting, then it is very reasonable to assume that the agents can compute (good approximations of) their utilities. On the other hand, when human voters choose between political candidates, assuming an ability to quantify a metric distance in some abstract space of political positions is much less realistic. Thus, we believe that for both assumptions, there are important and natural settings in which they are justified, motivating studies of communication-distortion tradeoffs in both types of scenarios.

**1.2 Related Work**

Communication complexity [39] generally studies the required communication between multiple parties wishing to jointly compute an outcome. Several recent works have studied the communication required specifically for jointly computing particular economic outcomes, or — conversely — to bound the effects of limited communication on such economic outcomes. These include work on auctions and allocations [1, 8, 16, 15, 28], persuasion [30], and general mechanism design [42]. While the high-level concerns are similar across different domains, the specific approaches and techniques do not appear to carry over.

The impact of communication more specifically on social choice rules has been explored before; see, for instance, [18] for an overview. However, most of the focus in past work has been on the number of bits that need to be communicated in order to compute the outcome of a particular social choice rule, rather than on proving lower bounds arising due to limited communication
when the social choice rule is not pre-specified. A classic paper in this context is by Conitzer and Sandholm [24]: they study vote elicitation rules, i.e., protocols by which a mechanism can interact with voters to determine the winner under a particular voting rule while not eliciting the full ranking information. This raises algorithmic questions about whether the information obtained so far uniquely determines a winner as well as incentive issues, among others, and a large amount of follow-up literature (e.g., [27]) has studied these issues. Relatedly, Conitzer [23] studies how many comparisons need to be elicited from voters to be able to reconstruct their complete ranking, and shows that the number is linear (as opposed to quadratic) when preferences are single-peaked (on the line).

Several very recent papers have explicitly considered the tradeoff between communication and distortion in social choice, both in deterministic and randomized settings. Perhaps most immediately related is recent work by Fain et al. [31]. Their focus is on mechanisms with extremely low communication which achieve low expected squared distortion, a measure somewhere between expected distortion and deterministic distortion. They prove that the Random Referee mechanism, which asks two randomly chosen voters for their top choices, and asks a third voter to choose between these two choices, achieves constant expected squared distortion. Notice that this mechanism elicits different information from different voters. Theorem 1 of [31] shows that this is unavoidable, in that any mechanism that only obtains top-$k$ lists (for constant $k$), even from all voters, must have linear expected squared distortion, implying the same result for the distortion of deterministic mechanisms. Our Theorem 1.2 generalizes this result for deterministic mechanisms to non-constant $k$ and sets other than the top $k$ positions.

Another very related piece of work is due to Mandal et al. [40], studying the communication-distortion tradeoff in a setting where the voters have utilities (instead of costs) for the candidates, and these utilities are only assumed to be non-negative and normalized, but do not need to satisfy any other properties (such as being derived from a metric). The other major modeling difference between our work and [40] is that they assume that agents compute their message $\mu$ to the mechanism directly from their utility vector, rather than the ranking. In particular, the mechanism can be designed to allow voters to express the strength of their preferences, albeit in possibly coarse form. This allows for a choice of deterministic/randomized algorithms in two places: (1) the voters’ computation of their message, and (2) the mechanism’s aggregation of the messages into a winner. [40] give upper and lower bounds for deterministic and randomized voting rules in this setting.

The positive/algorithmic results in [40] are obtained primarily by generalizing an approach of Benadè et al. [11], asking voters to communicate their top few candidates as well as a suitably rounded version of their utility for those nominated candidates. The bounds are improved in some parameter regimes by having the mechanism randomly select a subset of candidates and restricting voters to choose from this subset.

While the results of [40] are clearly directly related to our work, they are not immediately comparable. Because the utilities are not derived from metrics, the mechanisms need to deal with much broader classes of inputs, resulting in (generally) weaker upper bounds and stronger lower bounds. On the other hand, the assumption that voters can explicitly quantify their utilities — and hence have them elicited by a mechanism — gives a mechanism more power than in our setting.

Another related recent piece of work is on approval-based voting, due to Pierczyński and Skowron [46]. While much of this work focuses on a different notion of distortion — analyzing the fraction of voters who approve of the winning candidate in the sense of being “close enough” — [46] also analyzes the (traditional) distortion of approval-based voting. Under the type of mechanism that they consider, rather than approving a given number of voters (as in $k$-approval), voters approve all candidates within a given distance of themselves, i.e., within a ball of given radius around themselves. This approval radius can be voter-specific or uniform across voters. In this
context, the main result of [46] is to show specific constant distortion whenever a uniform approval radius ensures that a constant fraction of voters, bounded away from 0 and 1, have the optimum candidate within their approval radius. It is of course not clear how a mechanism (or the voters) could determine such a radius. Also note that this type of approval-based mechanism does require voters to quantify their distances, rather than just interact with their individual ordinal rankings.

Note that Theorem 1.3 can be considered as somewhat related to this result. It shows that whenever voters communicate their top \( k \) candidates, where \( k \) is a constant fraction of the number of candidates, there is a mechanism with constant distortion. However, in contrast to the result of [46], not just the identity, but also the ranking of these top \( k \) candidates must be communicated; on the other hand, the theorem makes no assumptions about whether the optimum candidate appears in any of these top-\( k \) rankings.

Low communication complexity of voter preferences is also the focus of a recent preprint by Bentert and Skowron [12]. They study the more “traditional” goal of implementing given voting rules with low communication [18], but are interested in approximate implementation of these rules. To make approximation meaningful, they focus on score-based rules, which naturally assign each candidate a score (such as Borda Count, Plurality, or MiniMax). Then, the quality of approximation is the ratio between the score of the winner under full information vs. the score of the winner under limited communication. They focus on mechanisms in which each voter is asked to rank a small subset of candidates; this subset is either the voter’s top \( k \) candidates (a deterministic mechanism) or a random subset of \( k \) candidates (a randomized mechanism).

Given that the goal in [12] is the approximate implementation of specific scoring-based voting rules rather than achieving low distortion, the results are not directly comparable. However, the techniques in Section 3.2 of [12] readily yield a randomized mechanism with distortion \( 5 + O(\epsilon) \) and very low communication complexity per voter when the number of voters is sufficiently large. By asking each voter to compare a uniformly random pair of candidates (see also [37]), and using the majority of returned votes, with high probability (by Chernoff and Union Bounds), one obtains a tournament graph in which each directed edge \((x, y)\) corresponds to at least a \( \frac{1}{2} - \epsilon \) fraction of voters preferring \( x \) over \( y \). Then, a straightforward modification of the analysis of the distortion of uncovered set rules in [3] (or a simple application of Corollary 5.3 in [38]) gives a distortion of \( 5 + O(\epsilon) \). This rule only requires each voter to compute 1 bit in total. However, different voters are asked to answer different questions, which is often considered undesirable. Furthermore, the total communication complexity is \( n \) bits, whereas the Random Dictator mechanism only needs to elicit \( \log_2 n \) bits from one voter.

The recent work of Bentert and Skowron is somewhat related to earlier work of Filmus and Oren [33]: they are also interested in the question of when top-\( k \) ballots from voters are sufficient to obtain the correct candidate. However, [33] study this question under probabilistic models for the ballots, significantly changing the nature of the results.

The metric-based distortion view of social choice has proved to be a very fruitful analysis framework. In fact, it has been extended beyond social choice to other optimization problems in which it is natural to assume that a mechanism only receives ordinal information; see, e.g., [6, 2].

Several modeling assumptions have been proposed that yield lower distortion than the worst-case bounds of [3]. One such assumption is termed decisiveness [5, 36]: it posits that for every voter, there is a sufficiently clear first choice among candidates. When the metric space is sufficiently decisive, significantly stronger upper bounds on the distortion can be proved. An alternative approach was proposed in [21, 22]. The authors assumed that the candidates were “representative,” in that they themselves were drawn i.i.d. uniformly from the set of voters. Under this assumption,

\[\text{In particular, when that fraction is between } \frac{1}{4} \text{ and } \frac{1}{2}, \text{ the distortion is at most 3.}\]
the authors obtained improved expected distortion bounds for the case of two candidates \[21\], and constant expected distortion for Borda count and several other position-based scoring rules \[22\].

As mentioned above, the gap between the upper bound of 5 (achieved, e.g., by the Copeland rule) and the lower bound of 3 has posed an interesting open question for several years now. One initial conjecture of \[4\] was that the Ranked Pairs mechanism might achieve a distortion of 3. This conjecture was disproved by \[35\], who showed a lower bound of 5 on the distortion of Ranked Pairs. Very recently, Munagala and Wang \[45\] have presented a (deterministic) social choice rule with distortion at most \(2 + \sqrt{5} \approx 4.23\), which is the first piece of progress towards closing the gap.

In our and much of the preceding work on metric voting, the focus is on distortion, while ignoring incentive compatibility. (Recall the strong impossibility result of \[34\], \[49\].) The connection between strategy proofness and distortion in this type of setting was studied in \[32\].

2 Preliminaries

2.1 Voters, Candidates, and Social Choice Rules

There are \(n\) candidates, which we always denote by lowercase letters at the end of the alphabet. Sets of candidates are denoted by uppercase letters, and \(X\) is the set of all candidates. The preference order (or ranking) of voter \(v\) over the candidates is a bijection \(\pi_v : \{1, \ldots, n\} \rightarrow X\), mapping positions \(i\) to the candidate \(x = \pi_v(i)\) which voter \(v\) ranks in position \(i\). We say that \(v\) (strictly) prefers \(x\) to \(y\) iff \(\pi_v^{-1}(x) < \pi_v^{-1}(y)\). When only the ranking, but not the identity, of a voter is relevant, we will omit the subscript \(v\) for legibility. The set of all voters is denoted by \(V\). We write \(S_n\) for the set of all possible rankings \(\pi : \{1, \ldots, n\} \rightarrow X\), and \(P = (\pi_v(i))_{v \in V; i \in \{1, \ldots, n\}}\) for the rankings of all voters, which we call the vote profile.

In the traditional full-information view, a social choice rule (we use the terms mechanism or voting mechanism interchangeably) \(f : S_n^V \rightarrow X\) is given the rankings of all voters, i.e., \(P\), and produces as output one winning candidate \(w = f(P)\). For most of this article, we are interested only in deterministic social choice rules \(f\).

2.2 Communication-bounded mechanisms

Our main contribution is to consider communication-bounded social choice rules. As in the standard model described above, we still only consider deterministic single-round mechanisms, i.e., each voter can only send a single message to the mechanism. However, this message is now also restricted to be at most \(b\) bits long.

This induces \(M = 2^b\) sets \(\Pi_1, \Pi_2, \ldots, \Pi_M\) of rankings; when the mechanism receives a message \(\mu\) from voter \(v\), all it learns is that \(\pi_v \in \Pi_\mu\). As discussed in the introduction, we assume that the \(\Pi_\mu\) form a disjoint partition of \(S_n\), i.e., they are pairwise disjoint and cover all rankings: \(\bigcup_{\mu=1}^M \Pi_\mu = S_n\). The fact that \(M\) is a power of 2 is not relevant anywhere in our proofs, so we also consider mechanisms with arbitrary numbers \(M\) of sets.

**Definition 2.1** (\(M\)-communication bounded social choice rule) An \(M\)-communication bounded social choice rule consists of pairwise disjoint sets \(\Pi_1, \Pi_2, \ldots, \Pi_M \subseteq S_n\) with \(\bigcup_{\mu=1}^M \Pi_\mu = S_n\), and a deterministic mapping \(f : \{1, \ldots, M\}^V \rightarrow X\).

\(^{10}\)We will not need to reference the number of voters explicitly. In general, we treat the number of voters as “much larger” than the number of candidates, and are only interested in bounds in terms of the number of candidates. 

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Communication-bounded social choice rules that are used in practice, such as Plurality, Veto, k-approval, and combinations thereof, are of a specific form: there is a set $K$ of $k$ positions, and voters can communicate the set of candidates they have in positions in $K$, possibly with an ordering, but cannot communicate any additional information about their ranking of candidates in positions outside $K$. For such mechanisms, we will be able to prove stronger lower bounds on the distortion, and with a significantly simpler proof. We define them formally as follows:

**Definition 2.2** A $k$-entry social choice rule is an $M$-communication bounded social choice rule with the following additional restriction on the sets $\Pi_1, \Pi_2, \ldots, \Pi_M$: there exists a set $K \subseteq \{1, \ldots, n\}$ of at most $k$ positions such that if $\pi, \pi'$ agree for all positions in $K$, i.e., $\pi(i) = \pi'(i)$ for all $i \in K$, then $\pi \in \Pi_\mu$ if and only if $\pi' \in \Pi_\mu$.

### 2.3 Metric Space and Distortion

The key modeling contribution of the metric-based distortion \[4\] objective is to assume that all voters and candidates are embedded in a pseudo-metric space $d$. $d(v, x)$ denotes the distance between voter $v$ and candidate $x$. Being a pseudo-metric, it satisfies non-negativity and the triangle inequality $d(v, x) \leq d(v, y) + d(v', y) + d(v', x)$ for all voters $v, v'$ and candidates $x, y$. Given our choice of defining the metric only for pairs consisting of a voter and a candidate, symmetry is not directly relevant. One can naturally extend the pseudo-metric to pairs of candidates or pairs of voters, but those distances will never appear in our mechanisms or proofs. For our upper bounds, we explicitly allow the distance between candidates and voters (and thus also between pairs of candidates or pairs of voters) to be 0; however, for improved flow, we will still refer to $d$ as a metric. In our lower-bound constructions, all distances will be strictly positive; that is, we do not exploit the increased generality for negative results.

We say that a vote profile $P$ is consistent with the metric $d$, and write $d \sim P$, if $\pi_v(x) < \pi_v(y)$ whenever $d(v, x) < d(v, y)$. That is, $P$ is consistent with $d$ iff all voters rank candidates by non-decreasing distance from themselves. Notice that in case of ties among distances, i.e., $d(v, x) = d(v, y)$, several vote profiles are consistent with $d$. None of our results depend on any tie breaking assumptions.

The cost of candidate $x$ is $C(x) = \sum_v d(v, x)$, i.e., the sum of distances of $x$ to all voters. An optimum candidate is any candidate $x_\ast^d \in \arg\min_{x \in \mathcal{X}} C(x)$; in our analysis, it will not matter which candidate is considered “the” optimum candidate in case of ties.

The social choice rule is handicapped by not knowing the metric $d$, instead only observing the consistent vote profile $P$ (or some limited information about it, when communication is restricted). Due to this handicap, and possibly other suboptimal choices, it will typically choose candidates with higher cost than $C(x^\ast)$. The distortion of $f$ is the worst-case ratio between the cost of the candidate chosen by $f$, and the optimal candidate $x_\ast^d$ (determined with knowledge of the actual distances $d$). Formally,

$$\rho(f) = \max_{P \text{ }d \sim P} \sup_{d \sim P} \frac{C(f(P))}{C(x_\ast^d)}.$$ 

We can think of the distortion in terms of a game between the social choice rule and an adversary. First, the adversary chooses the vote profile $P$. Then, the social choice rule, knowing only $P$ (or part of that information, in case of communication restrictions), chooses a winning candidate $w = f(P)$. Then, the adversary chooses a metric $d$ consistent with $P$ that maximizes the ratio between the cost of the candidate chosen by $f$ and the optimum candidate for $d$.

\[4\] also consider the median distance as an optimization objective; here, we only focus on the sum/average objective.
The goal now is to define a social choice rule $f$— under suitable constraints — that achieves small distortion $\rho(f)$, and to prove lower bounds on all social choice rules under the given constraints.

3  A Lower Bound for $k$-Entry Social Choice Rules

In this section, we establish the lower bound of Theorem 1.2, restated here formally.

**Theorem 3.1** Every one-round deterministic $k$-entry social choice rule has distortion at least $\frac{2n-k}{k}$.

**Proof.** Let $K = \{\kappa_1 < \kappa_2 < \cdots < \kappa_k\}$. Because every deterministic social choice rule has distortion at least 3 [4], we only need to consider the case where $2n-k > 3k$, i.e., $k < n/2$. We will prove the theorem by induction on $n$, with the base case $n = 2$ holding because the only such case with $k < n/2$ is $k = 0$, where the mechanism receives no information about any voter’s preferences, and hence has unbounded distortion.

First, we consider the case when $n \in K$. We designate one candidate $\hat{x}$ who is “infinitely” far from all other candidates and voters, and thus ranked last by all voters. The mechanism clearly cannot choose $\hat{x}$ as a winner. This reduces the problem to one of $n-1$ candidates, and a set $K' = K \setminus \{n\}$ of $k-1$ positions at which voters specify their ranking. By induction hypothesis, applied to this instance, the distortion is lower-bounded by $\frac{2(n-1)-(k-1)}{k-1} = \frac{2n-k-1}{k-1} > \frac{2n-k}{k}$; the inequality holds because $k < n$.

For the remainder of the proof, we can assume that $n \notin K$, i.e., voters do not specify their least favorite candidate. In this case, we will not need to use the induction hypothesis for $n-1$. For each subset $S \subseteq X, |S| = k$ of $k$ candidates, and each ordering $\sigma : \{1, \ldots, k\} \rightarrow S$, we say that a voter $v$ has type $(S, \sigma)$ if she puts the candidates from $S$ in the positions $K$, in the order given by $\sigma$. That is, $v$ has type $(S, \sigma)$ iff $\pi_v(\kappa_i) = \sigma(i)$ for $i = 1, \ldots, k$. There are $t = \binom{n}{k} \cdot k!$ types of voters. We define a vote profile which has exactly a $1/t$ fraction of voters of type $(S, \sigma)$, for each type. Throughout, we will talk about fractions, rather than numbers, of voters, so that the total adds up to 1.

Each subset of candidates and each order among those candidates is equally frequent, and in aggregate, the vote profile expresses no preference by the voters for any candidate over any other. Let $w$ be the candidate chosen by the social choice rule for this input. $w$ is well-defined as a function of all voters’ types, because (1) for each voter $v$, the message sent by $v$ is uniquely determined by her ranking of candidates in positions in $K$, and (2) the mechanism’s output is a deterministic function of only the messages sent by the voters.

We now define a metric space. Let $\epsilon$ be a very small constant (we will let $\epsilon \rightarrow 0$), and $0 < \epsilon_1 < \epsilon_2 < \cdots < \epsilon_n < \epsilon$. Consider a voter $v$ of type $(S, \sigma)$. We distinguish two cases:

1. In the first case, $w \notin S$. Let $\pi_v$ be any ordering that puts the candidates in $S$ in positions $K$ in the order $\sigma$, and which additionally has $\pi_v(n) = w$, i.e., candidate $w$ is in the last position in $v$’s ranking. Apart from this, $\pi_v$ is arbitrary. By construction, a voter $v$ with ranking $\pi_v$ has type $(S, \sigma)$. We now set the distance between $v$ and the candidate $w$ to 1, and the distance from $v$ to every candidate $\pi_v(i)$ (for $i < n$) to $\epsilon + \epsilon_i$. These distances are consistent with the ranking $\pi_v$.

2. In the second case, $w \in S$. Again, let $\pi_v$ be any permutation that puts the candidates in $S$ in positions $K$ in the order $\sigma$ (ensuring that $\pi_v$ is consistent with $v$ having type $(S, \sigma)$). This time, the position of $w$ in $\pi$ is prescribed by $S, \sigma$, and we let the remaining positions of $\pi_v$
be arbitrary. Voter $v$ has distance exactly $\frac{1}{2} + \epsilon + \epsilon_i$ from each candidate $\pi_v(i)$, including the case when $\pi_v(w) = i$. Again, $v$ ranks the candidates in the order given by $\pi_v$.

We now verify that these distances satisfy the triangle inequality. Consider voters $v, v'$ and candidates $x, y$. We will show that $d(v, y) \leq d(v, x) + d(v', x) + d(v', y)$, by distinguishing two cases for $y$:

1. In the first case, $y = w$. Then, $\frac{1}{2} + \epsilon \leq d(v, y) \leq 1$. Either the distance $d(v', y) = 1$, in which case the triangle inequality holds obviously, or $d(v', y) \geq \frac{1}{2} + \epsilon$, in which case our definition ensures that $d(v', x) \geq \frac{1}{2} + \epsilon$ as well. In either case, the triangle inequality holds.

2. In the second case, $y \neq w$, so either $\epsilon < d(v, y) < 2\epsilon$ or $\frac{1}{2} + \epsilon < d(v, y) < \frac{1}{2} + 2\epsilon$, depending on the case of the definition. Because all distances are lower-bounded by $\epsilon$, the triangle inequality clearly holds if $d(v, y) < 2\epsilon$. In the other case $\frac{1}{2} + \epsilon < d(v, y)$, we have that $\frac{1}{2} + \epsilon < d(v, x)$, which together with $\epsilon < d(v', x)$ again ensures that the triangle inequality holds.

Recall that $w$ is selected by the social choice rule under the given rankings. Each voter of type $(S, \sigma)$ with $w \notin S$ has cost 1 for candidate $w$, and cost at most $2\epsilon$ for any candidate $x \neq w$. Each voter of type $(S, \sigma)$ with $w \in S$ has cost at least $\frac{1}{2}$ for candidate $w$, and cost at most $\frac{1}{2} + 2\epsilon$ for each candidate $x \neq w$.

Of the $t$ types $(S, \sigma)$, exactly $(\frac{n-1}{k-1}) \cdot k!$ have $w \in S$. Thus, the cost of candidate $w$ is at least $\frac{1}{2} \cdot (\frac{n-1}{k-1}) \cdot k! + 1 \cdot (t - (\frac{n-1}{k-1}) \cdot k!)$, while the cost of any other candidate is at most $\frac{1}{2} \cdot (2\epsilon + \frac{1}{2} \cdot (\frac{n-1}{k-1}) \cdot k!)$. Letting $\epsilon \to 0$, the distortion approaches

$$1 + 2\left(\frac{n!}{(n-k)!} - \frac{k \cdot (n-1)!}{k \cdot (n-k)!}\right) = 1 + \frac{2(n-k)}{k} = \frac{2n-k}{k}.$$

4 The General Lower Bound

In this section, we prove the more general lower bound of Theorem 1.1. The bound applies to all $M$-communication bounded social choice rules, but is slightly weaker than that of Theorem 3.1. To gain some insight into general communication-bounded social choice rules, we begin with an easy proposition, independently obtained as Lemma 4.1 in [40]. We include a proof here for completeness, and because it illustrates some of the type of reasoning required for the proof of Theorem 1.1.

**Proposition 4.1** Assume that there exists a set $\Pi_\mu$ containing two rankings $\pi, \pi'$ with $\pi(1) \neq \pi'(1)$, i.e., there is a $\mu$ which does not uniquely specify the voter’s top-ranked candidate. Then, the corresponding social choice rule has unbounded distortion.

**Proof.** Let $x = \pi(1), y = \pi'(1)$. Consider a vote profile in which all voters communicate the message $\mu$ to the mechanism, i.e., state that their ranking is in $\Pi_\mu$. If the mechanism chooses $x$ as the winner, then the metric will be such that all voters have distance 0 from $y$, and distance 1 from all other candidates, including $x$. Then, the cost of $y$ is 0, while the cost of $x$ is 1, giving infinite cost ratio, i.e., distortion. Similarly, if the mechanism does not choose $x$ as the winner, then all voters will be at distance 0 from $x$ and at distance 1 from all other candidates, including $y$. Again, the cost ratio between the optimum candidate $x$ and the winner will be infinite.

---

\[\text{At the cost of small } \epsilon_i, \text{ which we could then let go to } 0, \text{ we could avoid ties here; in the limit, we would obtain exactly the same result. See the proof of Theorem 3.1 for spelled-out details.}\]
**Theorem 4.2** Let $f$ be any one-round $M$-communication bounded social choice rule on $n$ candidates. Then, $f$ must have distortion at least $\frac{2n-1}{nM} - 1$.

**Proof.** The high-level idea of the proof is to use induction on the number of candidates, to show that when communication is “sufficiently bounded,” any social choice rule must have high distortion. After completing the proof by induction, we would like to apply the result to $n$ candidates, and “sufficiently bounded” must then include $M$-communication bounded. Therefore, the relationship between the number of candidates in the induction proof and the bound on communication depends on $n, M$, and to avoid notational ambiguity, we will use different variable names for the induction. Specifically, we use $\nu$ for the number of candidates within the induction proof, and $M_\nu$ for the upper bound on communication.

Let $\gamma = 1 - M^{-1/(n-2)}$. We will prove by induction on $\nu$ that every $M_\nu$-communication bounded social choice rule on $\nu$ candidates with $M_\nu \leq \frac{1}{(1-\gamma)^\nu}$ has distortion at least $\frac{2}{\gamma} - 1$.

The base case $\nu = 2$ is easy: the communication bound is $M_2 \leq \frac{1}{(1-\gamma)^2} = 1$, so the voters cannot communicate any preference. By Proposition 4.1 the social choice rule has unbounded distortion. For the induction step, we distinguish two cases:

1. In the first case, we assume that for each candidate $x$, at least a $1 - \gamma$ fraction of all sets $\Pi_\mu$ contain a ranking $\pi_\mu \in \Pi_\mu$ that ranks $x$ last, i.e., $\pi_\mu(\nu) = x$. Then, we consider a vote profile with $M_\nu$ voters in which for each $\mu = 1, \ldots, M_\nu$, exactly one voter submits $\mu$.

   Let $w$ be the candidate chosen by $f$. Consider the following metric space: For every voter $\nu$ who submitted $\mu$ such that there is a ranking $\pi_\mu \in \Pi_\mu$ ranking $w$ last, we define the distance between $v$ and $w$ to be 1, and the distance from all other candidates to $w$ to be 0. For all other voters, the distance to all candidates is $\frac{1}{2}$. Said differently, all candidates $x \neq w$ are at distance 0 from each other, and at distance 1 from $w$. All voters who could possibly rank $w$ last are in the same location as the candidates different from $w$, while all other voters are halfway between $w$ and the other candidates.

   Then, the cost of $w$ is at least $\gamma \cdot \frac{1}{2} + (1 - \gamma) \cdot 1 = 1 - \frac{\gamma}{2}$, while the cost of each other candidate is at most $\gamma \cdot \frac{1}{2} + (1 - \gamma) \cdot 0 = \frac{\gamma}{2}$. Thus, the distortion of the mechanism is at least $\frac{2}{\gamma} - 1$, completing the proof directly.

2. Otherwise, let $x$ be a candidate such that at most a $1 - \gamma$ fraction of all sets $\Pi_\mu$ contain a ranking $\pi_\mu \in \Pi_\mu$ that ranks $x$ last. Define $M_{\nu-1}$ to be the number of such sets, and assume w.l.o.g. (by renumbering) that $\Pi_1, \Pi_2, \ldots, \Pi_{M_{\nu-1}}$ are all the sets which contain at least one ranking with $x$ in the last position. By the assumption in this part of the proof, we have that $M_{\nu-1} \leq (1 - \gamma) \cdot M_\nu$. We will only construct instances in which all voters rank $x$ last; thus, no voter communicates any message $\mu > M_{\nu-1}$.

   No mechanism with finite distortion can select $x$ as a winner, by the same argument as in the preceding case. (That is, the metric puts $x$ at distance 1 from all voters, and all other candidates at distance 0 from all voters.) As a result, we obtain an instance with $\nu - 1$ candidates, only $(\nu - 1)!$ remaining possible rankings, and — crucially — only $M_{\nu-1} \leq (1 - \gamma) \cdot M_\nu$ remaining sets of rankings. We can therefore apply the induction hypothesis for $\nu - 1$, and conclude that the mechanism’s distortion is at least $\frac{2}{\gamma} - 1$.

To show that we can apply the inductive claim with $\nu = n$ in the end, observe that $M_n = M = \frac{n}{(1-\gamma)^n}$. \footnote{Again, ties could be broken by using small $\epsilon_i \to 0$ without affecting the final result.}
It remains to show that \( \frac{2}{\gamma} - 1 \geq \frac{2n-4}{\ln M} - 1 \). To do so, we rewrite \( \gamma \) by using the Taylor expansion of \( t^{1/(n-2)} \) around \( t = 1 \), then apply straightforward bounds:

\[
\gamma = 1 - M^{-1/(n-2)} \\
= \frac{1}{n-2} \sum_{k=1}^{\infty} \frac{1}{k} \cdot (1 - 1/M)^k \cdot \prod_{j=1}^{k-1} \left( 1 - \frac{1}{j \cdot (n-2)} \right) \\
\leq \frac{1}{n-2} \sum_{k=1}^{\infty} \frac{1}{k} \cdot (1 - 1/M)^k \\
= \frac{1}{n-2} \cdot \ln M.
\]

Substituting this bound for \( \gamma \) into the distortion completes the proof.

To compare the bound of Theorem 4.2 with that of Theorem 3.1, observe that when voters get to specify the candidates in each of \( k \) (given) positions in a ranking, this generates a partition of \( S_n \) into \( M = \binom{n}{k} \cdot k! = \frac{n!}{(n-k)!} \) sets: one for each subset and order within that subset. These sets of rankings do in fact form a disjoint cover. For the “interesting” range \( k \leq n/2 \), we can simply bound \( (n/2)^k \leq M \leq n^k \), so we get that \( \ln M \approx k \ln n \). This shows that the lower bound of Theorem 4.2 is weaker than that of Theorem 3.1 by a factor of \( \Theta(\log n) \). Closing this gap is an interesting direction for future work, briefly discussed in Section 7.

5 A Near-Matching Upper Bound

While the results of Theorems 3.1 and 4.2 are negative, there are parameter ranges, such as \( k = o(n) \), \( k = \omega(1) \), in which they leave room for non-trivial positive results, in particular, sublinear distortion. In this section, we investigate how well one-round mechanisms can do with limited communication.

Our main result is a \( k \)-entry social choice rule which — up to constants — matches the lower bound of Theorem 3.1. This shows that the lower bound of Theorem 3.1 is essentially tight. Not surprisingly, the mechanism is a variation on uncovered set mechanisms, which are the only type of mechanism known to achieve constant distortion even with access to the full vote profile.

In our mechanism, each voter communicates her top \( k \) choices. We say that voter \( v \) prefers \( x \) over \( y \) if either: (1) Both \( x \) and \( y \) are among her top choices, and she ranks \( x \) higher than \( y \), or (2) \( x \) is among her top choices, and \( y \) is not. Obviously, the mechanism does not know which of two candidates she prefers if neither candidate is among her top \( k \) candidates.

As in uncovered set mechanisms like Copeland, we construct a comparison graph \( G \) among the \( n \) candidates. Define \( \alpha = \frac{k}{2n} \). For each ordered pair \( x, y \), the graph \( G \) contains a directed edge \((x, y)\) if and only if at least an \( \alpha \) fraction of all voters prefer \( x \) over \( y \). Notice that because \( \alpha \leq \frac{1}{2} \), it is possible that \( G \) contains both \((x, y)\) and \((y, x)\). Similarly, it is possible that for a pair \( \{x, y\} \), \( G \) contains neither \((x, y)\) nor \((y, x)\); for instance, this will happen if no voter ranks either \( x \) or \( y \) among her top \( k \) candidates.

Let \( S_2 \) be the set of candidates \( x \) such that at least a \( 2\alpha \) fraction of voters rank \( x \) among their top \( k \) candidates. (We will show in the proof of Lemma 5.2 that \( S_2 \) is not empty.) The winner \( w \) returned by \( M \) is a candidate in the induced graph \( G[S_2] \) with largest outdegree; notice that edges leaving \( S_2 \) are not counted.

**Theorem 5.1** \( M \) has distortion at most \( \frac{79n}{k} \). 

13
We begin with a lemma showing the key structural property of the winning candidate $w$.

**Lemma 5.2** In $G$, for every candidate $x$, there is a directed path of length at most 3 from $w$ to $x$.

**Proof.** Similar to the definition of $S_2$, let $S_3$ be the set of candidates $x$ such that at least a $3\alpha$ fraction of the voters ranks $x$ somewhere among their top $k$ candidates. By the Pigeon Hole Principle, because each voter ranks a $\frac{k}{n}$ fraction of candidates in her top $k$, and $\alpha = \frac{k}{3n}$, at least one candidate occurs in a $3\alpha$ fraction of top-$k$ lists. In particular, $S_3$ (and thus $S_2$) is non-empty.

Each candidate $x \in S_3$ has a directed edge to each candidate $y \notin S_2$. This is because $x$ appears in at least a $3\alpha$ fraction of top-$k$ lists, while $y$ appears in at most a $2\alpha$ fraction. In particular, at least an $\alpha$ fraction of voters rank $y$, but not $x$, in their top-$k$ lists, and thus prefer $x$ to $y$.

Now consider the induced graph $G[S_2]$. For each pair $x, y \in S_2$, at least one of the edges $(x, y)$ or $(y, x)$ is in $G[S_2]$. This is because of the (at least) $2\alpha$ fraction of voters with $x$ in their lists, at least an $\alpha$ fraction rank $y$ higher in their lists, or at least an $\alpha$ fraction rank $y$ lower (or not in their lists). Hence, $G[S_2]$ is a supergraph of a tournament graph.

Because $w$ has maximum degree in $G[S_2]$, it also has maximum degree in at least one tournament subgraph of $G[S_2]$. It is well known (see, e.g., [44, 4]) that the maximum-degree node in a tournament graph is in the uncovered set, i.e., it has a directed path of length at most 2 to every other node. This of course still holds in the supergraphs $G[S_2]$ and $G$. Thus, $w$ has a directed path of length 2 in $G$ to every candidate $x \in S_2$.

Let $y \in S_3$ be arbitrary. By the preceding two paragraphs, $y$ has a directed edge to each $x \notin S_2$, and $w$ has a directed path of length at most 2 to $y$. In summary, $w$ has a directed path of length at most 3 to each candidate $x$.

Next, we show a lemma upper-bounding the cost ratio of two candidates $x, y$ when $x$ has a directed path of length at most 3 to $y$.

**Lemma 5.3** Let $w, z$ be two candidates such that there is a directed path of length at most $\ell$ edges from $w$ to $z$ in $G$. Then, $C(w) \leq (1 + \frac{3\ell-1}{\alpha}) \cdot C(z)$.

**Remark 5.4** Lemma 5.3 can be considered a (somewhat weaker) generalization of a result proved in the proof of Theorem 7 in [4] (see also the discussion in the subsequent remark in [4]). By Lemma 6 of [4], if an $\alpha$ fraction of voters prefer $x$ over $y$, then $C(x) \leq 1 + \frac{2(1-\alpha)}{\alpha} \cdot C(y)$. In particular, for $\alpha = \frac{1}{2}$, this implies an upper bound of 3 on the cost ratio. If $x$ has a directed path of length $\ell$ to $y$, then this bound implies that $C(x) \leq 3\ell \cdot C(y)$. However, since we are interested in a regime where $\alpha = o(1)$, the exponential dependence on $\ell$ (recall that we have $\ell = 3$) would result in bounds that do not match our lower bounds asymptotically. The point of Lemma 5.3 is to improve upon this exponential dependence.

The exponential dependence on $\ell$ is an artifact of our relatively simple proof. Applying Corollary 5.3 from [38] instead would yield an improved bound of $\frac{\ell}{\alpha} + 1$ or $\frac{\ell+1}{\alpha} - 1$, depending on whether $\ell$ is even or odd.

**Proof.** Let $(w, y_1, y_2, \ldots, y_{\ell-1}, y_{\ell} := z)$ be a directed path of $\ell$ edges from $w$ to $z$. We distinguish two cases, based on the relative lengths of the distances $d(w, y_i)$ and $d(y_i, z)$, compared to $d(w, z)$.

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14 The opposite edge may be in $G$ as well; this is irrelevant.

15 Theorem 7 of [4] uses a more intricate proof to improve the upper bound for length-2 paths from this immediate 9 to 5.
1. If there exists a candidate \( y_i \) (with \( i < \ell \)) such that \( \ell(d(y_i, y_{i+1}) \geq \frac{2^{3^\ell-1}}{3^{\ell-1}} \cdot d(w, z) \), then let \( i \) be maximal with this property.

All the voters who prefer \( y_i \) over \( y_{i+1} \), which comprise at least an \( \alpha \) fraction of all voters, are at distance at least \( \frac{d(y_i, y_{i+1})}{2} \geq \frac{2^{3^\ell-1}}{3^{\ell-1}} \cdot d(w, z) \) from \( y_{i+1} \).

By maximality of \( i \), all candidates \( y_j \) with \( j > i \) have \( d(y_j, y_{j+1}) < \frac{2^{3^j-1}}{3^{j-1}} \cdot d(w, z) \). Using the triangle inequality and summing this inequality for all \( j > i \) gives us that \( d(y_{i+1}, z) < \frac{2^{3^\ell-1}}{3^{\ell-1}} \cdot d(w, z) \cdot \sum_{j=i+1}^{\ell-1} 3^{-j} = \frac{2^{3^\ell-1}}{3^{\ell-1}} \cdot d(w, z) \cdot \frac{3^{\ell-1}-3^{\ell-(\ell-1)}}{2} = \frac{2^{3^\ell-1} \cdot 3^{\ell-1}}{3^{\ell-1}} \cdot d(w, z) \).

Again by triangle inequality, the voters who prefer \( y_i \) over \( y_{i+1} \) are at distance at least \( d(y_i, z) \geq \frac{d(y_i, y_{i+1})}{2} - d(y_{i+1}, z) > \frac{2^{3^\ell-1}}{3^{\ell-1}} \cdot d(w, z) - \frac{2^{3^\ell-1} \cdot 3^{\ell-1}}{3^{\ell-1}} \cdot d(w, z) = \frac{2^{3^\ell-1}}{3^{\ell-1}} \cdot d(w, z) \).

In both cases, we have thus shown that at least an \( \alpha \) fraction of voters are at distance at least \( \frac{1}{3^{\ell-1}} \cdot d(w, z) \) from \( z \). Thus, the cost of \( z \) is at least \( \frac{\alpha}{3^{\ell-1}} \cdot d(w, z) \). By the triangle inequality,

\[
C(w) \leq C(z) + d(w, z) \leq (1 + \frac{3^\ell - 1}{\alpha}) \cdot C(z).
\]

This completes the proof of the lemma.

**Proof of Theorem 5.1.** By Lemma 5.2 \( w \) has a path of length at most 3 in \( G \) to every candidate \( x \); in particular, to the optimum candidate \( x = x^* \). Thus, by Lemma 5.3 with \( \ell = 3 \), \( C(w) \leq (1 + \frac{26}{\alpha}) \cdot C(x^*) \). Substituting \( \alpha = \frac{k}{3n} \) and bounding \( 1 \leq \frac{n}{k} \) now completes the proof.

6 A Tight Upper Bound for Randomized Algorithms

We have seen that limited communication is a serious handicap for deterministic social choice rules, in that all communication-bounded deterministic social choice rules must have essentially linear distortion. It is well known [5, 36] that this lower bound disappears for randomized social choice rules: for example, the Random Dictatorship mechanism, which elects the first choice of a uniformly random voter, has distortion slightly smaller than 3, even though each voter only communicates her first choice.

When each voter can only communicate her first choice, [34] proved a lower bound of \( 3 - \frac{2}{n} \) on the distortion of every randomized mechanism. Fain et al. [31] showed that the Random Oligarchy mechanism has an upper bound on the distortion almost matching the \( 3 - \frac{2}{n} \) bound. Here, we give a simple randomized mechanism which achieves an expected distortion of exactly \( 3 - \frac{2}{n} \), thereby closing the remaining gap. The mechanism \( M \) is as follows:
• With probability \( \frac{1}{n-1} \), select a candidate using the \textit{Proportional to Squares} mechanism. That is, for each candidate \( x \), let \( \nu_x \) be the fraction of voters who rank \( x \) first. Select candidate \( x \) with probability \( \frac{\nu_x^2}{\sum_x \nu_x^2} \).

• With the remaining probability \( \frac{n-2}{n-1} \), select a candidate using the \textit{Random Dictatorship} mechanism. That is, choose a voter uniformly at random, and return her first choice. Notice that this mechanism selects candidate \( x \) with probability exactly \( \nu_x \).

We prove the following theorem:

\textbf{Theorem 6.1} The expected distortion of \( M \) is at most \( 3 - \frac{2}{n} \).

The proof is straightforward: it consists of a bit of arithmetic and using Lemma 3 of [36], restated here in our notation.

\textbf{Lemma 6.2} (Lemma 3 of [36]) Let \( \nu = (\nu_x)_x \) be the vector of the fractions of voters ranking candidate \( x \) first, for all \( x \). Suppose that for every such first-place vote vector \( \nu \) and every candidate \( x \), the probability of electing \( x \) under \( M \) is at most \( q_x(\nu) \). Then, the distortion of \( M \) is at most

\[ 1 + 2 \max_{\nu,x} (q_x(\nu) \cdot \frac{1-\nu_x}{\nu_x}) \]

The main technical lemma, proved momentarily, is the following:

\textbf{Lemma 6.3} For all \( t \in [0,1] \), we have that

\[ (1 - \frac{1}{n-1}) \cdot (1 - t) + \frac{t(1-t)}{(n-1)t^2+(1-t)^2} \leq 1 - \frac{1}{n} \]

\textbf{Proof of Theorem 6.1} Let candidate \( x \) be the first choice of a fraction \( \nu := \nu_x \) of voters. The probability that \( x \) is chosen under \( M \) is

\[ (1 - \frac{1}{n-1}) \cdot \nu + \frac{1}{n-1} \cdot \frac{\nu^2}{\sum_y \nu_y^2} \leq (1 - \frac{1}{n-1}) \cdot \nu + \frac{1}{n-1} \cdot \frac{\nu^2}{\nu + (n-1) \left( \frac{1-\nu}{n-1} \right)^2} \]

\[ = (1 - \frac{1}{n-1}) \cdot \nu + \frac{\nu \cdot (1-\nu)}{(n-1) \cdot \nu^2 + (1-\nu)^2} \]

Multiplying with the term \( \frac{1-\nu}{\nu} \), we now have

\[ (1 - \frac{1}{n-1}) \cdot (1 - \nu) + \frac{\nu \cdot (1-\nu)}{(n-1) \cdot \nu^2 + (1-\nu)^2} \]

By Lemma 6.3 this quantity is bounded by \( 1 - \frac{1}{n} \). Since this bound holds for all \( x \) and all \( \nu_x \), we can substitute it into Lemma 6.2 and obtain a bound of \( 3 - \frac{2}{n} \) on the distortion, as claimed.

\textbf{Proof of Lemma 6.3} We want to upper-bound \( f(t) = (1 - \frac{1}{n-1}) \cdot (1 - t) + \frac{t(1-t)}{(n-1)t^2+(1-t)^2} \). First, we have that \( f(0) = 1 - \frac{1}{n-1} \), and \( f(1) = 0 \), so the inequality holds at the extreme points.

We lower-bound the denominator \( g(t) = (n-1)t^2 + (1-t)^2 \) of the second term. By setting the derivative \( g'(t) = 0 \), we get that the only local extremum is a minimum at \( t = \frac{1}{n} \), where \( g(1/n) = \frac{n-1}{n} \), whereas \( g(0) = 1 \) and \( g(1) = n-1 \). Thus, \( g(t) \geq \frac{n-1}{n} \). Substituting the lower bound on \( g(t) \), we can bound

\[ f(t) \leq (1 - \frac{1}{n-1}) \cdot (1 - t) + \frac{n \cdot t \cdot (1-t)}{n-1} \]

A derivative test shows that this expression has a local maximum at \( t = \frac{1}{n} \), where its value is \( 1 - \frac{1}{n} \). Thus, we have shown that \( f(t) \leq 1 - \frac{1}{n} \) for all \( t \in [0,1] \).
7 Conclusions

As we already discussed in the introduction and Section 4, there is a gap of $\Theta(\log n)$ in the lower bound on distortion we achieve for $k$-entry social choice rules and more general $M$-communication bounded social choice rules. It does not appear that our techniques from Section 4 can be directly generalized to produce bounds matching the ones of Theorem 3.1. Thus, if the stronger bound holds more generally, a proof will likely require a deeper understanding of the combinatorial structure of partitions of $S_n$. An intriguing alternative is that there may be a mechanism in which voters communicate only $\Theta(1)$ bits of information per candidate, but which nonetheless achieves constant distortion. An obstacle to designing such mechanisms is that it is very unclear how a mechanism would make use of information in which it cannot distinguish between several very different rankings.

Throughout this article, we assumed that all voters use the same “encoding” in communicating with the mechanism. For both $k$-entry social choice rules and $M$-communication bounded rules, one could consider relaxing this uniformity, although voting mechanisms which treat voters differently a priori are typically not widely accepted. For $k$-entry social choice rules, our lower-bound proof can be directly adapted to give the same lower bound so long as no voter (or almost no voter) gets to specify which candidate she ranks last. However, the proof does not carry over directly when some, but not all, voters can specify their bottom-ranked candidate, since our technique of “sacrificing” a candidate may come at a higher cost to the adversary. For $M$-communication bounded rules, it is much less clear how to deal with arbitrarily differing encodings.

A further generalization would be to let voters choose which encoding to use, or which subset of positions to fill in. Mechanisms allowing such a choice by the voters would have to be considered as “non-deterministic,” because there is not a unique message any more for each ranking. This raises the issue of how a voter would determine which of many possible messages to send. In particular, the specific choice of message may encode additional (e.g., cardinal) information about the voter’s ranking. It would require some subtlety to define a model to rule out the revelation of a lot of cardinal information, while still allowing voters non-trivial choices.

Here, we only considered single-round mechanisms. It is well-known that in many settings, including in the implementation of social choice rules [18, 50], multiple rounds of communication can lead to significantly (including exponentially) lower overall communication. Indeed, [36, 31] studied randomized multi-round voting mechanisms with the explicit goal of reducing the required communication, while achieving low distortion. In the case of randomized mechanisms, receiving $\log_2 n$ bits of information from each voter is enough to achieve distortion $3 - \frac{2}{n}$ (as we showed in Section 6— it was known previously how to achieve distortion 3), so the room for improving the required communication with multiple rounds is limited. However, for deterministic mechanisms, there is potential for significant improvement, and a natural question is whether one might even achieve constant distortion with only $O(\log n)$ (or $O(\text{polylog}(n))$) communication from each voter.

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