BLM REALIZATION FOR $\mathcal{U}_\mathbb{Z}(\widehat{\mathfrak{gl}}_n)$

QIANG FU

Abstract. In 1990, Beilinson–Lusztig–MacPherson (BLM) discovered a realization [1, 5.7] for quantum $\mathfrak{gl}_n$ via a geometric setting of quantum Schur algebras. We will generalize their result to the classical affine case. More precisely, we first use Ringel–Hall algebras to construct an integral form $\mathcal{U}_\mathbb{Z}(\widehat{\mathfrak{gl}}_n)$ of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$, where $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ is the universal enveloping algebra of the loop algebra $\widehat{\mathfrak{gl}}_n := \mathfrak{gl}_n(\mathbb{Q}) \otimes \mathbb{Q}[t, t^{-1}]$. We then establish the stabilization property of multiplication for the classical affine Schur algebras. This stabilization property leads to the BLM realization of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ and $\mathcal{U}_\mathbb{Z}(\widehat{\mathfrak{gl}}_n)$. In particular, we conclude that $\mathcal{U}_\mathbb{Z}(\widehat{\mathfrak{gl}}_n)$ is a $\mathbb{Z}$-Hopf subalgebra of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$. As a bonus, this method leads to an explicit $\mathbb{Z}$-basis for $\mathcal{U}_\mathbb{Z}(\widehat{\mathfrak{gl}}_n)$, and it yields explicit multiplication formulas between generators and basis elements for $\mathcal{U}_\mathbb{Z}(\widehat{\mathfrak{gl}}_n)$. As an application, we will prove that the natural algebra homomorphism from $\mathcal{U}_\mathbb{Z}(\widehat{\mathfrak{gl}}_n)$ to the affine Schur algebra over $\mathbb{Z}$ is surjective.

1. Introduction

The positive part of the integral form of quantum enveloping algebras of finite type was realized by Ringel, see [22, 23]. This is an important breakthrough for the structure of quantum groups. Almost at the same time, A.A. Beilinson, G. Lusztig and R. MacPherson [1, 5.7] realized the entire quantum $\mathfrak{gl}_n$ over the rational function field $\mathbb{Q}(v)$ (with $v$ being an indeterminate) via quantum Schur algebras. It is natural to ask how to realize the integral form of the entire quantum $\mathfrak{gl}_n$. If this can be achieved, then one can realize quantum $\mathfrak{gl}_n$ over any field. The remarkable BLM’s work has important applications to the investigation of quantum Schur-Weyl reciprocity. The classical Schur-Weyl reciprocity relates representations of the general linear and symmetric groups over $\mathbb{C}$ (cf. [27]). This reciprocity is also true over any field (cf. [2, 3, 7]). The quantum Schur-Weyl reciprocity at nonroots of unity was first formulated in [16]. Using BLM’s work, the integral quantum Schur-Weyl reciprocity was established in [8, 9].

The BLM realization problem of quantum affine $\mathfrak{gl}_n$ was investigated in [10, 5]. In particular, it was proven that the natural algebra homomorphism from quantum affine $\mathfrak{gl}_n$ to affine quantum Schur algebras over $\mathbb{Q}(v)$ is surjective in [5] (cf. [12, 20]). Furthermore, the universal enveloping algebra $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ of $\widehat{\mathfrak{gl}}_n$ was realized in [5] using a modified BLM approach. However, in the affine case, there are still many important unsolved problems. For example, the stabilization

Supported by the National Natural Science Foundation of China, the Program NCET, Fok Ying Tung Education Foundation and the Fundamental Research Funds for the Central Universities.
property of multiplication for quantum Schur algebras given in \([1] 4.2\) is the key to the BLM realization of quantum \(\mathfrak{gl}_n\). Furthermore, explicit multiplication formulas between generators and basis elements for the quantum enveloping algebra of \(\mathfrak{gl}_n\) were obtained in \([1] 5.3\). But it seems hard to generalize these results to the quantum affine case. In addition, it is difficult to construct a suitable integral form for quantum affine \(\mathfrak{gl}_n\) such that the integral quantum affine Schur reciprocity holds (cf. \([5] 3.8.6\)).

In this paper, we will solve the above problems in the classical case. First, we will use Ringel–Hall algebras to construct a free \(\mathbb{Z}\)-submodule \(U_\mathbb{Z}(\hat{\mathfrak{gl}}_n)\) of the universal enveloping algebra \(U(\hat{\mathfrak{gl}}_n)\) of the loop algebra \(\hat{\mathfrak{gl}}_n\) in \(\S 3\). We then prove in \([6,1] 6.1\) a stabilization property for the structure constants of an affine Schur algebra, which is the affine analogue of \([1] 4.2\). This property allows us to construct an algebra \(K_\mathbb{Z}(n)\) without unity. Then we consider the completion algebra \(\hat{K}_\mathbb{Q}(n)\) of \(K_\mathbb{Z}(n)\) and construct a \(\mathbb{Z}\)-submodule \(V_\mathbb{Z}(n)\) of \(\hat{K}_\mathbb{Q}(n)\). We will prove in \([7,3] 7.3\) and \([8,5] 8.5\) that \(V_\mathbb{Z}(n)\) is a \(\mathbb{Z}\)-subalgebra of \(\hat{K}_\mathbb{Q}(n)\) with nice multiplication formulas, which is the affine analogue of \([1] 5.3\) and \([5, 5.7\]). Finally, we will prove in \([9,2] 9.2(1)\) that \(V_\mathbb{Q}(n) := V_\mathbb{Z}(n) \otimes \mathbb{Q}\) is isomorphic to \(U(\hat{\mathfrak{gl}}_n)\), which is the affine analogue of \([1] 5.7\). Furthermore, we will prove in \([9,2] 9.2(2)\) that \(U_\mathbb{Z}(\hat{\mathfrak{gl}}_n)\) is a \(\mathbb{Z}\)-subalgebra of \(U(\hat{\mathfrak{gl}}_n)\) and \(V_\mathbb{Z}(n)\) is the realization of \(U_\mathbb{Z}(\hat{\mathfrak{gl}}_n)\). As a result, we derive an explicit \(\mathbb{Z}\)-basis for \(U_\mathbb{Z}(\hat{\mathfrak{gl}}_n)\) together with explicit multiplication formulas between generators and arbitrary basis elements for \(U_\mathbb{Z}(\hat{\mathfrak{gl}}_n)\) (see \([7,3] 7.3\), \([8,1] 8.1\) and \([9,2] 9.2\)). As a byproduct, we will establish affine Schur-Weyl reciprocity at the integral level in \([9,5] 9.5\).

We organize this paper as follows. We recall the definition of Ringel–Hall algebras and extended Ringel–Hall algebras in \(\S 2\). Using Ringel–Hall algebras, we will construct a \(\mathbb{Z}\)-submodule \(U_\mathbb{Z}(\hat{\mathfrak{gl}}_n)\) of \(U(\hat{\mathfrak{gl}}_n)\) in \(\S 3\). We review in \(\S 4\) the definition of affine quantum Schur algebras and generalize \([1] 3.9\) to the affine case. Certain useful multiplication formulas for affine Schur algebras \(S_\mathbb{Z}(n, r)\) will be established in \(\S 5\). These formulas will be used to establish the stabilization property for affine Schur algebras in \([6,1] 6.1\). Then we use this property to construct an algebra \(K_\mathbb{Z}(n)\) without unity and derive some important multiplication formulas for the completion algebra \(\hat{K}_\mathbb{Q}(n)\) of \(K_\mathbb{Z}(n)\) in \([7,3] 7.3\). In \([8,5] 8.5\) we will use these formulas to prove that the \(\mathbb{Z}\)-submodule \(V_\mathbb{Z}(n)\) of \(\hat{K}_\mathbb{Q}(n)\) constructed in \(\S 8\) is a \(\mathbb{Z}\)-subalgebra of \(\hat{K}_\mathbb{Q}(n)\). Finally, we will prove that \(U(\hat{\mathfrak{gl}}_n) \cong V_\mathbb{Q}(n)\) and \(U_\mathbb{Z}(\hat{\mathfrak{gl}}_n) \cong V_\mathbb{Z}(n)\) in \([9,2] 9.2\). Furthermore, we will prove that \(U_\mathbb{Z}(\hat{\mathfrak{gl}}_n)\) is a Hopf algebra over \(\mathbb{Z}\) in \([9,3] 9.3\). Using this realization for \(U_\mathbb{Z}(\hat{\mathfrak{gl}}_n)\), we will prove in \([9,5] 9.5\) that the natural algebra homomorphism from \(U_\mathbb{Z}(\hat{\mathfrak{gl}}_n)\) to \(S_\mathbb{Z}(n, r)\) is surjective.

**Notation 1.1.** For a positive integer \(n\), let \(M_{\Delta,n}(\mathbb{Q})\) be the set of all matrices \(A = (a_{i,j})_{i,j \in \mathbb{Z}}\) with \(a_{i,j} \in \mathbb{Q}\) such that

(a) \(a_{i,j} = a_{i+n,j+n}\) for \(i,j \in \mathbb{Z}\);

(b) for every \(i \in \mathbb{Z}\), both sets \(\{j \in \mathbb{Z} \mid a_{i,j} \neq 0\}\) and \(\{j \in \mathbb{Z} \mid a_{j,i} \neq 0\}\) are finite.
Let $M_{\triangle n}(\mathbb{Z})$ be the subset of $M_{\triangle n}(\mathbb{Q})$ consisting of matrices with integer entries. Let

$$(1.1.1) \quad \tilde{\Theta}_\triangle(n) := \{A \in M_{\triangle n}(\mathbb{Z}) \mid a_{i,j} \geq 0, \forall i \neq j\}, \quad \Theta_\triangle(n) := \{A \in M_{\triangle n}(\mathbb{Z}) \mid a_{i,j} \in \mathbb{N}, \forall i,j\}$$

Let $\mathbb{Z}_\triangle^n = \{(\lambda_i)_{i \in \mathbb{Z}} \mid \lambda_i \in \mathbb{Z}, \lambda_i = \lambda_{i-n} \text{ for } i \in \mathbb{Z}\}$ and $\mathbb{N}_\triangle^n = \{(\lambda_i)_{i \in \mathbb{Z}} \in \mathbb{Z}_\triangle^n \mid \lambda_i \geq 0 \text{ for } i \in \mathbb{Z}\}$. We will identify $\mathbb{Z}_\triangle^n$ with $\mathbb{Z}^n$ via the following bijection

$$(1.1.2) \quad b : \mathbb{Z}_\triangle^n \longrightarrow \mathbb{Z}^n, \quad j \mapsto b(j) = (j_1, \cdots, j_n).$$

There is a natural order relation $\leq$ on $\mathbb{Z}_\triangle^n$ defined by

$$(1.1.3) \quad \lambda \leq \mu \iff \lambda_i \leq \mu_i \text{ for all } 1 \leq i \leq n.$$ 

We say that $\lambda < \mu$ if $\lambda \leq \mu$ and $\lambda \neq \mu$.

Let $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$, where $v$ is an indeterminate, and let $\mathbb{Q}(v)$ be the fraction field of $\mathcal{Z}$. Specializing $v$ to 1, $\mathbb{Q}$ and $\mathcal{Z}$ will be viewed as $\mathcal{Z}$-modules.

2. Ringel–Hall algebras and extended Ringel–Hall algebras

Let $\triangle(n)$ ($n \geq 2$) be the cyclic quiver

with vertex set $I = \mathbb{Z}/n\mathbb{Z} = \{1, 2, \ldots, n\}$ and arrow set $\{i \to i + 1 \mid i \in I\}$. Let $\mathbb{F}$ be a field. A representation $V = (V_i, f_i)_{i \in I}$ of $\triangle(n)$ over $\mathbb{F}$ is called nilpotent if $f_n \cdots f_2 f_1 : V_1 \to V_1$ is nilpotent. We will denote by $\text{Rep}_\mathbb{F}^0\triangle(n) = \text{Rep}_\mathbb{F}^0\triangle(n)$ the category of finite-dimensional nilpotent representations of $\triangle(n)$ over $\mathbb{F}$.

For $i \in I$, let $S_i$ denote the one-dimensional representation in $\text{Rep}_\mathbb{F}^0\triangle(n)$ with $(S_i)_i = \mathbb{F}$ and $(S_i)_j = 0$ for $i \neq j$. Let

$$\Theta^+_\triangle(n) = \{A \in \Theta_\triangle(n) \mid a_{i,j} = 0 \text{ for } i \geq j\}.$$ 

For any $A = (a_{i,j}) \in \Theta^+_\triangle(n)$, let

$$M(A) = M\mathcal{F}(A) = \bigoplus_{1 \leq i < n, i < j} a_{i,j} M^{i,j},$$

where $M^{i,j}$ is the unique indecomposable representation of length $j - i$ with top $S_i$. For $A \in \Theta^+_\triangle(n)$, let $d(A) \in NI = \mathbb{N}^n$ be the dimension vector of $M(A)$. We will sometimes identify $NI$ with $\mathbb{N}_\triangle^n$ under (1.1.2).

Given modules $M, N_1, \cdots, N_m$ in $\text{Rep}_\mathbb{F}^0\triangle(n)$, let $F_{N_1 \cdots N_m}^M$ be the number of the filtrations

$$0 = M_m \subseteq M_{m-1} \subseteq \cdots M_1 \subseteq M_0 = M.$$
Thus we obtain a map \( \mathbb{P} \) Let \( \tilde{\mathbb{P}} \) where \( \mathfrak{M} \) element defined by
\[
\mathfrak{M}(A) = A(v^2) \in \mathbb{Z}[v^2]
\]
\[
\mathfrak{M}(A) = A(v^2) = |\text{Aut}(M(F))|.
\]

For \( a = (a_i) \in \mathbb{Z}_n^1 \) and \( b = (b_i) \in \mathbb{Z}_n^1 \), the Euler form associated with the cyclic quiver \( \Delta(n) \) is the bilinear form \( \langle \cdot, \cdot \rangle : \mathbb{Z}_n^1 \times \mathbb{Z}_n^1 \to \mathbb{Z} \) defined by
\[
\langle a, b \rangle = \sum_{i \in I} a_i b_i - \sum_{i \in I} a_i b_{i+1}.
\]

Let \( \mathfrak{H}_n \) be the (generic) Ringel–Hall algebra of the cyclic quiver \( \Delta(n) \), which is by definition the free module over \( \mathbb{Z}[v, v^{-1}] \) with basis \( \{ u_A = u_{[M(A)]} \mid A \in \Theta_n^+(n) \} \). The multiplication is given by
\[
u_A u_B = v^{(d(A) + d(B))} \sum_{C \in \Theta_n^+(n)} \mathfrak{M}(C)(v^2) u_C
\]
for \( A, B \in \Theta_n^+(n) \). For \( A \in \Theta_n^+(n) \), let
\[
\tilde{u}_A = v^{\dim \text{End}(M(A)) - \dim M(A)} u_A.
\]

Now let us recall the triangular relation given in [6, (9.2)] for the Ringel–Hall algebra \( \mathfrak{H}_n \). For \( M, N \in \text{Rep}^0\Delta(n) \), there exists a unique extension \( G \) (up to isomorphism) of \( M \) by \( N \) with minimal \( \dim \text{End}(G) \), which will be denoted by \( M \rtimes N \) in the sequel (see [4, 21]). Let \( \mathcal{M} \) be the set of isoclasses of nilpotent representations of \( \Delta(n) \) and define a multiplication \( \ast \) on \( \mathcal{M} \) by
\[
[M] \ast [N] = [M \rtimes N]
\]
for any \( [M], [N] \in \mathcal{M} \). Then by [4, §3] \( \mathcal{M} \) is a monoid with identity \( 1 = [0] \).

For \( \lambda \in \mathbb{N}_n^1 \) let
\[
S_\lambda = \bigoplus_{i=1}^n \lambda_i S_i
\]
be the semisimple representation in \( \text{Rep}^0\Delta(n) \). A semisimple representation \( S_\lambda \) is called sincere if \( \lambda \) is sincere, namely, all \( \lambda_i \) are positive. For \( 1 \leq i \leq n \) let \( e_i^\lambda \in \mathbb{N}_n^1 \) be the element satisfying
\[
b(e_i^\lambda) = e_i = (0, \cdots, 0, 1, 0, \cdots, 0),
\]
where \( b \) is defined in [1, 1.2]. Let
\[
\tilde{I} = \{ e_1^\lambda, e_2^\lambda, \cdots, e_n^\lambda \} \cup \{ \text{all sincere vectors in } \mathbb{N}_n^1 \}.
\]

Let \( \Sigma \) be the set of words on the set \( \tilde{I} \). For \( w = a_1 a_2 \cdots a_m \in \Sigma \), let \( \varphi^+(w) \in \Theta_n^+(n) \) be the element defined by
\[
[S_{a_1}] \ast \cdots \ast [S_{a_m}] = [M(\varphi^+(w))].
\]

Thus we obtain a map \( \varphi^+ : \Sigma \to \Theta_n^+(n) \). Let
\[
\Theta_n^-(n) = \{ A \in \Theta_n(n) \mid a_{i,j} = 0 \text{ for } i \leq j \}.
\]
For $w = a_1 a_2 \cdots a_m \in \tilde{\Sigma}$ we let
\[ t_w = a_m \cdots a_2 a_1. \]
Let $\varphi^-$ be the map from $\tilde{\Sigma}$ to $\Theta^+_\Delta(n)$ defined by $\varphi^-(w) = t(\varphi^+(t_w))$, where $t(\varphi^+(t_w))$ is the transpose of $\varphi^+(t_w)$. By [6, 3.3] the maps $\varphi^+$ and $\varphi^-$ are all surjective.

Following [1, 3.5] and [10] we may define the order relation $\leq$ on $M_{\Delta n}(\mathbb{Z})$ as follows. For $A \in M_{\Delta n}(\mathbb{Z})$ and $i \neq j \in \mathbb{Z}$, let
\[ \sigma_{i,j}(A) = \sum_{s \leq i, t \geq j} a_{s,t} \text{ if } i < j, \text{ and } \sigma_{i,j}(A) = \sum_{s \geq i, t \leq j} a_{s,t} \text{ if } i > j. \]
For $A, B \in M_{\Delta n}(\mathbb{Z})$, define $B \leq A$ if $\sigma_{i,j}(B) \leq \sigma_{i,j}(A)$ for all $i \neq j$. Put $B < A$ if $B \leq A$ and, for some pair $(i,j)$ with $i \neq j$, $\sigma_{i,j}(B) < \sigma_{i,j}(A)$.

For $\lambda \in \mathbb{N}^n$ let $u_{\lambda} = u_{(S_{\lambda})}$ and let
\[ \tilde{u}_{\lambda} = v^{\dim \text{End}(S_{\lambda}) - \dim S_{\lambda}} u_{\lambda}. \]
Any word $w = a_1 a_2 \cdots a_m$ in $\tilde{\Sigma}$ can be uniquely expressed in the tight form $w = b_1^{x_1} b_2^{x_2} \cdots b_t^{x_t}$ where $x_i = 1$ if $b_i$ is sincere, and $x_i$ is the number of consecutive occurrences of $b_i$ if $b_i \in \{ e_1^\underline{0}, e_2^\underline{1}, \ldots, e_n^\underline{\lambda} \}$. For $w = a_1 a_2 \cdots a_m \in \tilde{\Sigma}$ with the tight form $b_1^{x_1} b_2^{x_2} \cdots b_t^{x_t}$ let
\[ \tilde{u}_{(w)} = \tilde{u}_{x_1 b_1} \tilde{u}_{x_2 b_2} \cdots \tilde{u}_{x_t b_t} \in \mathcal{F}_H(n). \]
By [6] (9.2) and [10] 6.2, we have the following triangular relation in $\mathcal{F}_H(n)$.

**Proposition 2.1.** For $A \in \Theta^+_{\Delta}(n)$, there exist $w_A \in \tilde{\Sigma}$ such that $\varphi^+(w_A) = A$ and
\[ \tilde{u}_{(w_A)} = \sum_{B \in \Theta^+_{\Delta}(n) \mid \varphi^+(A) = \varphi^+(B)} f_{B,A} \tilde{u}_{(B)} \]
where $f_{B,A} \in \mathbb{Z}$ and $f_{A,A} = 1$. In particular, $\mathcal{F}_H(n)$ is generated by $u_{\lambda}$ for $\lambda \in \mathbb{N}^n$.

Let $\mathcal{F}_H(n) = \mathcal{F}_H(n) \otimes \mathbb{Q}(v)$. The algebra $\mathcal{F}_H(n)$ does not have a Hopf algebra structure. However, if we add the torus algebra to $\mathcal{F}_H(n)$, we may get a Hopf algebra $\mathcal{F}_H(n)^{\geq 0}$, called the extended Ringel–Hall algebra. Let $\mathcal{F}_H(n)^{\geq 0}$ be a $\mathbb{Q}(v)$-space with basis $\{ u_A^+ K_\alpha \mid \alpha \in ZI, A \in \Theta^+_{\Delta}(n) \}$. Let $\Theta^+_{\Delta}(n)_1 := \Theta^+_{\Delta}(n) \setminus \{0\}$.

**Proposition 2.2.** The $\mathbb{Q}(v)$-space $\mathcal{F}_H(n)^{\geq 0}$ with basis $\{ u_A^+ K_\alpha \mid \alpha \in ZI, A \in \Theta^+_{\Delta}(n) \}$ becomes a Hopf algebra with the following algebra, coalgebra and antipode structures.

(a) Multiplication and unit:
\[ u_A^+ u_B^+ = \sum_{C \in \Theta^+_{\Delta}(n)} v^{\langle d(A), d(B) \rangle} \varphi^+_{A,B} u_C^+, \text{ for all } A, B \in \Theta^+_{\Delta}(n), \]
\[ K_\alpha u_A^+ = v^{\langle d(A), \alpha \rangle} u_A^+ K_\alpha, \text{ for all } \alpha \in ZI, A \in \Theta^+_{\Delta}(n), \]
\[ K_\alpha K_\beta = K_{\alpha + \beta}, \text{ for all } \alpha, \beta \in ZI. \]
with unit $1 = u_0^+ = K_0$.

(b) **Comultiplication and counit (Green [14]):**

$$\Delta(u_C^+) = \sum_{A,B \in \Theta_0^+(n)} u^{(A,B)} a_{AB} \varphi_C a_{C,B} u_B^+ \otimes u_A^+ K_d(B),$$

$$\Delta(K_\alpha) = K_\alpha \otimes K_\alpha, \text{ where } C \in \Theta_0^+(n) \text{ and } \alpha \in \mathbb{Z}.$$  

with counit $\varepsilon$ satisfying $\varepsilon(u_C^+) = 0$ for all $C \in \Theta_0^+(n)$ and $\varepsilon(K_\alpha) = 1$ for all $\alpha \in \mathbb{Z}$. Here, for each $\alpha = (a_i) \in \mathbb{Z}I$, $\tilde{K}_\alpha$ denotes $(\tilde{K}_1)^{a_1} \cdots (\tilde{K}_n)^{a_n}$ with $\tilde{K}_i = K_i K_{i+1}^{-1}$.

(c) **Antipode (Xiao [28]):**

$$\sigma(u_C^+) = \varepsilon_{C;0} + \sum_{m \geq 1} (-1)^m \sum_{D \in \Theta_0^+(n)} \frac{ac_1 \cdots ac_m \varphi_{C_1} \cdots \varphi_m \varphi_{D;C_1} \cdots \varphi_{C_m;D} u_D^+ K_{d(C)}}{ac},$$

for all $C \in \Theta_0^+(n)$, and $\sigma(K_\alpha) = K_{-\alpha}$, for all $\alpha \in \mathbb{Z}I$.

We conclude this section by introducing the integral form $\mathcal{H}_0(n)^{\geq 0}$ for $\mathcal{H}_0(n)^{\geq 0}$. For $c, t \in \mathbb{Z}$ with $t \geq 1$, let

$$\left[K_i; c \right] = \prod_{s=1}^t \frac{|K_{i+c-s+1} - K_{i-c+s-1}|_{v^s-v^{-s}}}{v^s-v^{-s}} \quad \text{and} \quad \left[K_i; 0 \right] = 1.$$  

Let $\mathcal{H}_0(n)^{\geq 0}$ be the $\mathbb{Z}$-submodule of $\mathcal{H}_0(n)^{\geq 0}$ spanned by all $u_A^+ \prod_{1 \leq i \leq n} \left[ K_i; \lambda_i \right] K_i^\delta_i$, where $A \in \Theta_0^+(n)$, $\delta_i \in \{0, 1\}$ and $\lambda \in \mathbb{N}_0^n$.

**Lemma 2.3.** $\mathcal{H}_0(n)^{\geq 0}$ is a $\mathbb{Z}$-Hopf subalgebra of $\mathcal{H}_0(n)^{\geq 0}$.

**Proof.** Clearly, for $A \in \Theta_0^+(n)$, $1 \leq i \leq n$ and $t \in \mathbb{N}$,

$$\left[ K_i; 0 \right] u_A^+ = u_A^+ \left[ K_i; (d(A), e_i^t) \right].$$  

This together with [19, 2.14] implies that $\mathcal{H}_0(n)^{\geq 0}$ is a $\mathbb{Z}$-subalgebra of $\mathcal{H}_0(n)^{\geq 0}$.

For $m \in \mathbb{N}$, let $[m]^t = [1][2] \cdots [m]$ where $[i] = \frac{v^i - v^{-i}}{v - v^{-1}}$. Clearly, for $\lambda, \lambda^{(1)}, \cdots, \lambda^{(m)} \in \mathbb{N}_0^n$ with $\lambda = \sum_{1 \leq i \leq m} \lambda^{(i)}$,

$$\varphi_{\lambda^{(1)} \cdots \lambda^{(m)}} = \prod_{1 \leq l \leq m} \left[ \lambda_l^{(1)} \cdots \lambda_l^{(m)} \right], \quad a_{\lambda} = \prod_{1 \leq i \leq n} (v^{2\lambda_i} - v^{2s}),$$

where $\left[ \lambda_l^{(1)} \cdots \lambda_l^{(m)} \right] = \frac{[\lambda_l^{(1)}]}{[\lambda_l^{(1)}] \cdots [\lambda_l^{(m)}]}$. We conclude that

$$\Delta(\bar{u}_{\lambda^2}) = \sum_{\lambda = \lambda^{(1)} + \lambda^{(2)}} v^{(\lambda_2, \lambda^{(1)})} \bar{u}_{\lambda^{(1)}}^+ \otimes u_{\lambda^{(2)}}^+ K_{\lambda^{(1)}} \in \mathcal{H}_0(n)^{\geq 0} \otimes \mathcal{H}_0(n)^{\geq 0}$$

(2.3.2)
and
\[
\sigma(u^+_\lambda) = \delta_{\lambda,0} + \sum_{m \geq 1} (-1)^m \sum_{D \in \Theta^+_\lambda(n)} v^{-2\sum_{1 \leq i \leq n} \sum_{1 \leq k \leq m} \lambda(i)^\lambda(k)} \varphi^{(i)}_{\lambda(m)} \cdots \varphi^{(1)}_{\lambda(1)} u^+_D K_{-\lambda} \\
\in \mathfrak{h}_\lambda(n)^{>0}.
\]
Furthermore, by \cite[(2.1),(2.2)]{25}, we have
\[
\Delta \left( \begin{bmatrix} K_i; 0 \\ t \end{bmatrix} \right) = \sum_{0 \leq j < t} K_i^{-t+j} \begin{bmatrix} K_i; 0 \\ j \end{bmatrix} \otimes K_i^j \begin{bmatrix} K_i; 0 \\ t-j \end{bmatrix} \in \mathfrak{h}_\lambda(n)^{>0} \otimes \mathfrak{h}_\lambda(n)^{>0}
\]
and
\[
\sigma \left( \begin{bmatrix} K_i; 0 \\ t \end{bmatrix} \right) = (-1)^t \begin{bmatrix} K_i; -1 + t \\ t \end{bmatrix} \in \mathfrak{h}_\lambda(n)^{>0}.
\]
Consequently, \( \mathfrak{h}_\lambda(n)^{>0} \) is a Hopf algebra over \( \mathbb{Z} \), since \( \mathfrak{h}_\lambda(n)^{>0} \) is generated by \( u^+_\lambda, K_i^{\pm 1} \) and \( \begin{bmatrix} K_i; 0 \\ t \end{bmatrix} \) for all \( \lambda \in \mathbb{N}_0^n, 1 \leq i \leq n \) and \( t \in \mathbb{N} \).

3. THE INTEGRAL FORM \( \mathcal{U}_\mathbb{Z}^-(\hat{\mathfrak{g}}_l_n) \) OF \( \mathcal{U}(\hat{\mathfrak{g}}_l_n) \)

Let \( \mathfrak{gl}_n(\mathbb{Q}) \) be the general linear Lie algebra over \( \mathbb{Q} \). Let
\[
\hat{\mathfrak{g}}_l_n := \mathfrak{gl}_n(\mathbb{Q}) \otimes \mathbb{Q}[t, t^{-1}]
\]
be the loop algebra associated to \( \mathfrak{gl}_n(\mathbb{Q}) \). For \( i, j \in \mathbb{Z} \), let \( E_{i,j}^\Delta \in M_{\Delta,n}(\mathbb{Q}) \) be the matrix \( (e_{k,l}^{i,j})_{k,l \in \mathbb{Z}} \) defined by
\[
e_{k,l}^{i,j} = \begin{cases} 
1 & \text{if } k = i + sn, l = j + sn \text{ for some } s \in \mathbb{Z}; \\
0 & \text{otherwise},
\end{cases}
\]
Recall the set \( M_{\Delta,n}(\mathbb{Q}) \) defined in \cite{11}. Clearly, the map
\[
M_{\Delta,n}(\mathbb{Q}) \rightarrow \hat{\mathfrak{g}}_l_n, \ E_{i,j}^\Delta \rightarrow E_{i,j} \otimes t^l, 1 \leq i, j \leq n, l \in \mathbb{Z}
\]
is a Lie algebra isomorphism. We will identify the loop algebra \( \hat{\mathfrak{g}}_l_n \) with \( M_{\Delta,n}(\mathbb{Q}) \) in the sequel.

Let \( \mathcal{U}(\hat{\mathfrak{g}}_l_n) \) be the universal enveloping algebra of the loop algebra \( \hat{\mathfrak{g}}_l_n \). Let \( \mathcal{U}^+(\hat{\mathfrak{g}}_l_n) \) (resp., \( \mathcal{U}^-(\hat{\mathfrak{g}}_l_n) \)) be the subalgebra of \( \mathcal{U}(\hat{\mathfrak{g}}_l_n) \) generated by \( E_{i,j}^\Delta \) (resp., \( E_{i,j}^{-\Delta} \)) for \( i < j \). Let \( \mathcal{U}^0(\hat{\mathfrak{g}}_l_n) \) be the subalgebra of \( \mathcal{U}(\hat{\mathfrak{g}}_l_n) \) generated by \( E_{i,i}^\Delta \) for \( 1 \leq i \leq n \).

We may interpret the \( \pm \)-part of \( \mathcal{U}(\hat{\mathfrak{g}}_l_n) \) as the specialization of Hall algebras. Let \( \mathfrak{h}_\lambda(n)_\mathbb{Q} = \mathfrak{h}_\lambda(n) \otimes \mathbb{Q} \), where \( \mathbb{Q} \) is regarded as a \( \mathbb{Z} \)-module by specializing \( v \) to 1. We shall denote \( u_A \otimes 1 \) by \( u_{A,1} \) for \( A \in \Theta^+_\lambda(n) \).
Lemma 3.1 ([5. 6.1.2]). (1) The set
\[ \left\{ \prod_{1 \leq i < j \leq n} u_{E_{i,j}}^{a_{i,j}} \left| A = (a_{i,j}) \in \Theta_+^0(n) \right. \right\} \]
forms a \( \mathbb{Q} \)-basis of \( \mathfrak{H}_n \), where the products are taken with respect to any fixed total order on \( \{(i, j) \mid 1 \leq i \leq n, i < j, j \in \mathbb{Z}\} \).

(2) There is a unique injective algebra homomorphism \( \iota^+ : \mathfrak{H}_n \rightarrow \mathcal{U}(\widehat{\mathfrak{g}}_n) \) (resp., \( \iota^- : \mathfrak{H}_n^{op} \rightarrow \mathcal{U}(\widehat{\mathfrak{g}}_n) \)) taking \( u_{E_{i,j}} \mapsto E_{i,j}^\Delta \) for all \( i < j \) such that \( \iota^+(\mathfrak{H}_n) = \mathcal{U}^+(\widehat{\mathfrak{g}}_n) \) and \( \iota^-(\mathfrak{H}_n^{op}) = \mathcal{U}^-(\widehat{\mathfrak{g}}_n) \).

Let \( \mathcal{U}^0(\widehat{\mathfrak{g}}_n) = \mathcal{U}^+(\widehat{\mathfrak{g}}_n) \mathcal{U}^0(\widehat{\mathfrak{g}}_n) \mathcal{U}^0(\widehat{\mathfrak{g}}_n) \) be the the Borel subalgebra \( \mathcal{U}^0(\widehat{\mathfrak{g}}_n) \) of \( \mathcal{U}(\widehat{\mathfrak{g}}_n) \). We now show that the algebra isomorphism \( \iota^+ : \mathfrak{H}_n \rightarrow \mathcal{U}(\widehat{\mathfrak{g}}_n) \) can be extended to a Hopf algebra isomorphism between the specialization \( \mathfrak{H}_n^{\mathbb{Q}} \) of \( \mathfrak{H}_n \) at \( v = 1 \) for all \( K_i = 1 \) and \( \mathcal{U}^0(\widehat{\mathfrak{g}}_n) \).

Let \( \mathfrak{H}_n^{\mathbb{Q}} = \mathfrak{H}_n \otimes \mathbb{Q} \). We shall denote \( u_{A_{i,1}}^+ = u_A^+ \otimes 1, K_{i,1} = K_i \otimes 1 \) and \( [K_i;0]_1 = [K_i;0] \otimes 1 \) for \( A \in \Theta_+^0(n), 1 \leq i \leq n \) and \( t \in \mathbb{N} \). Let
\[ \mathfrak{H}_n^{\mathbb{Q}} = \mathfrak{H}_n^{\mathbb{Q}}/(K_{i,1} - 1; 1 \leq i \leq n). \]
We will use the same notation for elements in \( \mathfrak{H}_n^{\mathbb{Q}} \) and \( \mathfrak{H}_n^{\mathbb{Q}}/(K_{i,1} - 1) \).

Lemma 3.2. There is a Hopf algebra isomorphism
\[ \iota^0 : \mathfrak{H}_n^{\mathbb{Q}} \rightarrow \mathcal{U}^0(\widehat{\mathfrak{g}}_n) \]
defined by sending \( u_{E_{i,j}}^+ \) to \( E_{i,j}^\Delta \), and \( [K_i;0]_1 \) to \( E_{i,i}^\Delta \) for \( i < j \).

Proof. By [19 2.3(g9),(g10)] we have in \( \mathfrak{H}_n^{\mathbb{Q}}, [K_i;\pm1]_1 = [K_i;0]_1 \pm 1 \). Thus by (2.3.1) we have in \( \mathfrak{H}_n^{\mathbb{Q}} \),
\[ [K_i;0]_1 u_{E_{i,j}}^+ = u_{E_{k,l}}^+ \left[ K_i;0 \right]_1 - \left[ K_i;0 \right]_1 \delta_{i,k} - \delta_{i,l} \]
for \( 1 \leq i \leq n \) and \( k < l \). This together with (3.1.2) implies that there is an algebra homomorphism \( f : \mathcal{U}^0(\widehat{\mathfrak{g}}_n) \rightarrow \mathfrak{H}_n^{\mathbb{Q}} \) such that \( f(E_{i,j}^\Delta) = u_{E_{i,j}}^+ \) and \( f(E_{i,i}^\Delta) = [K_i;0]_1 \) for \( i < j \). Since, by [18 4.1(d)], \( [K_i;0]_1 - [K_{i-1};0]_1 = j \) for \( 1 \leq i \leq n \) and \( j \in \mathbb{Z} \), it follows from [18 4.1(f)] that, we have in \( \mathfrak{H}_n^{\mathbb{Q}} \),
\[ [K_i;0]_1 = \frac{1}{t!} \prod_{0 \leq j \leq t-1} [K_i;j]_1 = f \left( \left( E_{i,i}^\Delta \right)_{t} \right), \]
where

\[
\left( E_{i,j}^\lambda \right)_{t} = \frac{E_{i,j}^\lambda (E_{i,j}^\lambda - 1) \cdots (E_{i,j}^\lambda - t + 1)}{t!}.
\]

Thus by [19, 2.14] and [3.1(1)] \( f \) sends the PBW-basis

\[
\left\{ \prod_{1 \leq i < j \leq \mathbb{N}^n} (E_{i,j}^\lambda)_{a_{i,j}} \prod_{1 \leq i < j \leq \mathbb{N}^n} (E_{i,j}^\lambda)_{t} \bigg| A = (a_{i,j}) \in \Theta_\lambda^+(\mathbb{N}), \, \lambda \in \mathbb{N}^n \right\}
\]

for \( \mathcal{U}^\lambda_{\mathbb{Q}}(\hat{\mathfrak{g}}_n) \) to a \( \mathbb{Q} \)-basis for \( \hat{\mathfrak{h}}(\mathbb{N})_{\mathbb{Q}}^\lambda \). Consequently, \( f \) is an algebra isomorphism. Finally, since

\[
\Delta(u_{E_{i,j},1}^+) = u_{E_{i,j},1}^+ \otimes 1 + 1 \otimes u_{E_{i,j},1}^+ \quad \Delta \left( \left[ K_i; 0 \right]_1 \right) = \left[ K_i; 0 \right]_1 \otimes 1 + 1 \otimes \left[ K_i; 0 \right]_1
\]

and

\[
\sigma(u_{E_{i,j},1}^+) = -u_{E_{i,j},1}^+, \quad \sigma \left( \left[ K_i; 0 \right]_1 \right) = - \left[ K_i; 0 \right]_1
\]

for \( i < j \), we conclude that \( f \) is a Hopf algebra isomorphism. \( \square \)

We now use Ringel–Hall algebras to define the integral form \( \mathcal{U}_Z(\hat{\mathfrak{g}}_n) \) of \( \mathcal{U}(\hat{\mathfrak{g}}_n) \). Let \( \mathcal{U}^+_Z(\hat{\mathfrak{g}}_n) = \iota^+ (\hat{\mathfrak{h}}(\mathbb{N})_{\mathbb{Z}}^\lambda) \) and \( \mathcal{U}^-_Z(\hat{\mathfrak{g}}_n) = \iota^- (\hat{\mathfrak{h}}(\mathbb{N})_{\mathbb{Z}}^{op}) \), where \( \hat{\mathfrak{h}}(\mathbb{N})_{\mathbb{Z}} = \hat{\mathfrak{h}}(\mathbb{N}) \otimes_{\mathbb{Z}} \mathbb{Z} \) and \( \mathbb{Z} \) is regarded as a \( \mathbb{Z} \)-module by specializing \( v \) to 1. Let \( \mathcal{U}^0_Z(\hat{\mathfrak{g}}_n) \) be the \( \mathbb{Z} \)-submodule of \( \mathcal{U}^+_Z(\hat{\mathfrak{g}}_n) \) spanned by \( \prod_{1 \leq i < n} (E_{i,i}^\lambda)_{a_{i,i}} \), for \( \lambda \in \mathbb{N}^n \). Let

\[
\mathcal{U}_Z(\hat{\mathfrak{g}}_n) = \mathcal{U}^+_Z(\hat{\mathfrak{g}}_n) \mathcal{U}^0_Z(\hat{\mathfrak{g}}_n) \mathcal{U}^-_Z(\hat{\mathfrak{g}}_n).
\]

We will prove that \( \mathcal{U}_Z(\hat{\mathfrak{g}}_n) \) is a \( \mathbb{Z} \)-subalgebra of \( \mathcal{U}^+_Z(\hat{\mathfrak{g}}_n) \) and give a BLM realization of \( \mathcal{U}_Z(\hat{\mathfrak{g}}_n) \) in 9.2. Furthermore, we will use 3.2 to show that \( \mathcal{U}_Z(\hat{\mathfrak{g}}_n) \) is a \( \mathbb{Z} \)-Hopf subalgebra of \( \mathcal{U}^+_Z(\hat{\mathfrak{g}}_n) \) in 9.3

4. AFFINE QUANTUM SCHUR ALGEBRAS

Let \( \mathfrak{S}_\Delta \) be the group consisting of all permutations \( w : \mathbb{Z} \to \mathbb{Z} \) such that \( w(i + r) = w(i) + r \) for \( i \in \mathbb{Z} \). Let \( W \) be the subgroup of \( \mathfrak{S}_\Delta \) consisting of \( w \in \mathfrak{S}_\Delta \) with \( \sum_{i=1}^r w(i) = \sum_{i=1}^r i \). By [17], \( W \) is the Weyl group of affine type \( A \) with generators \( s_i \) \((1 \leq i \leq \mathbb{N}) \) defined by setting \( s_i(j) = j \) for \( j \neq i, i + 1 \) \( \mod \mathbb{N} \), \( s_i(j) = j - 1 \) for \( j \equiv i + 1 \) \( \mod \mathbb{N} \) and \( s_i(j) = j + 1 \) for \( j \equiv i \) \( \mod \mathbb{N} \). The subgroup of \( \mathfrak{S}_\Delta \) generated by \( s_1, \ldots, s_{r-1} \) is isomorphic to the symmetric group \( \mathfrak{S}_r \). Let \( \rho \) be the permutation of \( \mathbb{Z} \) sending \( j \) to \( j + 1 \) for all \( j \in \mathbb{Z} \). Then \( \mathfrak{S}_\Delta = \langle \rho \rangle \rtimes W \). We extend the length function \( \ell \) on \( W \) to \( \mathfrak{S}_\Delta \) by setting \( \ell(\rho^m w) = \ell(w) \) for all \( m \in \mathbb{Z}, w \in W \).

The extended affine Hecke algebra \( \mathcal{H}_\sigma(r) \) over \( \mathbb{Z} \) associated to \( \mathfrak{S}_\lambda \) is the (unital) \( \mathbb{Z} \)-algebra with basis \( \{ T_w \}_{w \in \mathfrak{S}_\lambda} \), and multiplication defined by

\[
\begin{cases}
T_{s_i}^2 = (v^2 - 1)T_{s_i} + v^2, & \text{for } 1 \leq i \leq n \\
T_w T_{w'} = T_{ww'}, & \text{if } \ell(ww') = \ell(w) + \ell(w').
\end{cases}
\]
For \( \lambda = (\lambda_i)_{i \in \mathbb{Z}} \in \mathbb{Z}^n_0 \) let \( \sigma(\lambda) = \sum_{1 \leq i \leq n} \lambda_i \). For \( r \geq 0 \) we set

\[
\Lambda_\circ(n, r) = \{ \lambda \in \mathbb{N}^n_0 | \sigma(\lambda) = r \}.
\]

For \( \lambda \in \Lambda_\circ(n, r) \), let \( \mathfrak{S}_\lambda := \mathfrak{S}_{(\lambda_1, \ldots, \lambda_n)} \) be the corresponding standard Young subgroup of \( \mathfrak{S}_r \). For each \( \lambda \in \Lambda_\circ(n, r) \), let \( x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w \in \mathcal{H}_\circ(r) \). The endomorphism algebras

\[
\mathcal{S}_\circ(n, r) := \text{End}_{\mathcal{H}_\circ(r)} \left( \bigoplus_{\lambda \in \Lambda_\circ(n, r)} x_\lambda \mathcal{H}_\circ(r) \right).
\]

are called affine quantum Schur algebras (cf. [12, 13, 20]). Following [13], we will introduce a \( \mathcal{Z} \)-basis of \( \mathcal{S}_\circ(n, r) \) as follows. For \( \lambda \in \Lambda_\circ(n, r) \), let

\[
\mathcal{R}_\lambda = \{ d | d \in \mathfrak{S}_{\lambda, r}, \ell(wd) = \ell(w) + \ell(d) \text{ for } w \in \mathfrak{S}_\lambda \}.
\]

Note that we have

\[
(4.0.2) \quad d^{-1} \in \mathcal{R}_\lambda \iff d(\lambda_{0, i-1} + 1) < d(\lambda_{0, i-1} + 2) < \cdots < d(\lambda_{0, i-1} + \lambda_i), \quad \forall 1 \leq i \leq n,
\]

where \( \lambda_{0, i-1} = \sum_{1 \leq t \leq i-1} \lambda_t \). Let \( \mathcal{R}_{\lambda, \mu} = \mathcal{R}_\lambda \cap \mathcal{R}_{\mu}^{-1} \). For \( \lambda, \mu \in \Lambda_\circ(n, r) \) and \( d \in \mathcal{R}_{\lambda, \mu} \), define \( \phi^d_{\lambda, \mu} \in \mathfrak{S}_\circ(n, r) \) as follows:

\[
(4.0.3) \quad \phi^d_{\lambda, \mu}(x_v h) = \delta_{v\mu} \sum_{w \in \mathfrak{S}_\lambda d \mathfrak{S}_\mu} T_w h
\]

where \( v \in \Lambda_\circ(n, r) \) and \( h \in \mathcal{H}_\circ(r) \). Then by [13] the set \( \{ \phi^d_{\lambda, \mu} | \lambda, \mu \in \Lambda_\circ(n, r), d \in \mathcal{R}_{\lambda, \mu} \} \) forms a basis for \( \mathcal{S}_\circ(n, r) \).

Recall the sets \( \tilde{\Theta}_\circ(n) \) and \( \Theta_\circ(n) \) defined in [11.1]. For \( A \in \tilde{\Theta}_\circ(n) \), let \( \sigma(A) = \sum_{1 \leq i \leq n, j \in \mathbb{Z}} a_{i,j} \). For \( r \geq 0 \), let

\[
\Theta_\circ(n, r) = \{ A \in \tilde{\Theta}_\circ(n) | \sigma(A) = r \}.
\]

The basis for \( \mathcal{S}_\circ(n, r) \) is indexed by triples \( \{ (\lambda, d, \mu) | \lambda, \mu \in \Lambda_\circ(n, r), d \in \mathcal{R}_{\lambda, \mu} \} \). In what follows next, it will be convenient to reindex these basis elements by the set \( \Theta_\circ(n, r) \). For \( 1 \leq i \leq n, k \in \mathbb{Z} \) and \( \lambda \in \Lambda_\circ(n, r) \) let

\[
R^\lambda_{i+kn} = \{ \lambda_{k,i-1} + 1, \lambda_{k,i-1} + 2, \ldots, \lambda_{k,i-1} + \lambda_i = \lambda_{k,i} \},
\]

where \( \lambda_{k,i-1} = kr + \sum_{1 \leq t \leq i-1} \lambda_t \). By [26, 7.4] (see also [10, 9.2]), there is a bijective map

\[
\iota_\circ : \{ (\lambda, d, \mu) | d \in \mathcal{R}_{\lambda, \mu}, \lambda, \mu \in \Lambda_\circ(n, r) \} \rightarrow \Theta_\circ(n, r)
\]

sending \( (\lambda, w, \mu) \) to \( A = (a_{k,l}) \), where \( a_{k,l} = |R^\lambda_k \cap wR^\mu_l| \) for all \( k, l \in \mathbb{Z} \). If \( \lambda, \mu \in \Lambda_\circ(n, r) \) and \( d \in \mathcal{R}_{\lambda, \mu} \) are such that \( A = \iota_\circ(\lambda, d, \mu) \), we will denote \( \phi^d_{\lambda, \mu} \) by \( e_A \) and let

\[
[A] = v^{-d_A} e_A, \quad \text{where} \quad d_A = \sum_{1 \leq i \leq n \atop i \neq k, j \neq l} a_{i,j} a_{k,l}.
\]
By [20, 1.11], the $\mathbb{Z}$-linear map

\begin{equation}
\tau_r : S_\Delta(n, r) \rightarrow S_\Delta(n, r), \quad [A] \mapsto [^t A]
\end{equation}

is an algebra anti-involution, where $[^t A]$ is the transpose of $A$.

Let

$$\Theta^\pm_\Delta(n) = \{ A \in \Theta_\Delta(n) \mid a_{i,i} = 0 \text{ for all } i \}.$$ 

For $A \in \Theta^\pm_\Delta(n)$ and $\mathbf{j} \in \mathbb{Z}_n$, define $A(\mathbf{j}, r) \in S_\Delta(n, r)$ by

$$A(\mathbf{j}, r) = \sum_{\lambda \in \Lambda_0(n-r - \sigma(A))} v^{\sum_{1 \leq i \leq n} \lambda_{i,i}} [A + \text{diag}(\lambda)].$$

The affine quantum Schur algebra $S_\Delta(n, r)$ and the Ring-Hall algebra $S_\Delta(n)$ can be related by the following algebra homomorphism defined in [26, 7.6].

**Proposition 4.1.** (1) There is a $\mathbb{Z}$-algebra homomorphism

$$\eta^-_r : S_\Delta(n) \rightarrow S_\Delta(n, r), \quad \tilde{u}_A \mapsto ([^t A](0, r) \text{ for all } A \in \Theta^+_\Delta(n).$$

(2) Dually, there is a $\mathbb{Z}$-algebra homomorphism

$$\eta^+_r : S_\Delta(n) \rightarrow S_\Delta(n, r), \quad \tilde{u}_A \mapsto A(0, r) \text{ for all } A \in \Theta^+_\Delta(n).$$

We end this section by generalizing [11, 3.9] to the affine case. This is the first key result in proving stabilization property of multiplication for affine Schur algebras.

First we will use the triangular relation for Ringel–Hall algebras to get similar relations for affine quantum Schur algebras. For $a \in \mathbb{N}_0^n$ let

\begin{equation}
A_a = \sum_{1 \leq i \leq n} a_i E^\Delta_{i, i+1} \in \Theta_\Delta(n), \quad B_a = [^t A_a] = \sum_{1 \leq i \leq n} a_i E^\Delta_{i, i+1} \in \Theta^-_\Delta(n).
\end{equation}

For $w = a_1 a_2 \cdots a_m \in \tilde{\Sigma}$ with the tight form $b_1^t b_2^t \cdots b_t^t$, let

\begin{align*}
m^+_r(w), r &= \eta^+_r(\tilde{u}_e(w)) = A_{x_1 b_1(0, r)} A_{x_2 b_2(0, r)} \cdots A_{x_t b_t(0, r)} \in S_\Delta(n, r), \\
m^-_r(w), r &= B_{x_1 b_1(0, r)} B_{x_2 b_2(0, r)} \cdots B_{x_t b_t(0, r)} \in S_\Delta(n, r).
\end{align*}

For $A \in \tilde{\Theta}_\Delta(n)$, we write

\begin{equation}
A = A^+ + A^0 + A^-
\end{equation}

where $A^+ \in \Theta^+_\Delta(n)$, $A^- \in \Theta^-_\Delta(n)$ and $A^0$ is a diagonal matrix.

**Lemma 4.2.** For any $A \in \Theta^\pm_\Delta(n)$, there exist $w_{A^+}, w_{A^-} \in \tilde{\Sigma}$ such that $\varphi^+(w_{A^+}) = A^+$, $\varphi^-(w_{A^-}) = A^-$ and

\begin{align*}
m^+_r(w_{A^+}), r &= \sum_{B \in \Theta^+_\Delta(n), B \leq A^+, \lambda(A^+) = \lambda(B)} f_{B, A^+} B(0, r), \\
m^-_r(w_{A^-}), r &= \sum_{B \in \Theta^-_\Delta(n), B \leq A^-, \lambda(A^-) = \lambda(B)} g_{B, A^-} B(0, r)
\end{align*}

for any $r \geq 0$, where $f_{B, A^+}, g_{B, A^-} \in \mathbb{Z}$ is independent of $r$ and $f_{A^+, A^+} = g_{A^-, A^-} = 1.$
Proof. The first equation follows from 2.1 and 4.1. Now we assume \( A \in \Theta_\alpha^+(n) \). Then \( \varphi^+(w) = tA \) and

\[
\sum_{B \in \Theta_\alpha^+(n), B \leq tA} f_{B,tA} B(0,r).
\]

Since \( X \preceq Y \) if and only if \( tX \preceq tY \) for \( X, Y \in \Theta_\alpha(n) \), applying the antiautomorphism \( \tau_r \) defined in (4.1.3) to the above equation yields

\[
\sum_{B \in \Theta_\alpha^+(n), B \leq tA} f_{B,tA} B(0,r) = \sum_{C \in \Theta_\alpha^+(n), C \leq A} g_{C,A} C(0,r),
\]

where \( w_A = t w \) and \( g_{C,A} = f_{C,tA}. \)

For \( A \in \tilde{\Theta}_\alpha(n) \) let

\[
\text{ro}(A) = \left( \sum_{j \in \mathbb{Z}} a_{i,j} \right)_{i \in \mathbb{Z}}, \quad \text{co}(A) = \left( \sum_{i \in \mathbb{Z}} a_{i,j} \right)_{j \in \mathbb{Z}} \in \mathbb{Z}_+^n.
\]

For \( A \in \tilde{\Theta}_\alpha(n) \) with \( \sigma(A) = r \), we denote \( [A] = 0 \in S_\alpha(n,r) \) if \( a_{i,i} < 0 \) for some \( i \in \mathbb{Z} \). For \( A \in \Theta_\alpha(n,r) \), let

\[
\sigma(A) = (\sigma_i(A))_{i \in \mathbb{Z}} \in \Lambda_\alpha(n,r)
\]

where \( \sigma_i(A) = a_{i,i} + \sum_{j < i} (a_{i,j} + a_{j,i}) \). In the following discussion we shall denote by \([A] + \text{lower terms}\) an element of \( S_\alpha(n,r) \) which is equal to \([A] \) plus a \( \mathbb{Z} \)-linear combination of elements \([A']\) with \( A' \in \Theta_\alpha(n,r) \), \( A' < A \), \( \text{co}(A') = \text{co}(A) \) and \( \text{ro}(A') = \text{ro}(A) \). The following triangular relation for affine quantum Schur algebras is given in [5, 3.7.7].

**Proposition 4.3.** For \( A \in \Theta_\alpha^+\alpha(n) \) and \( \lambda \in \Lambda_\alpha(n,r) \), we have

\[
A^+(0,r)[\text{diag}(\lambda)]A^-(0,r) = [A + \text{diag}(\lambda - \sigma(A))] + \text{lower terms}.
\]

In particular, the set

\[
\{A^+(0,r)|\text{diag}(\lambda)]A^-(0,r) \mid A \in \Theta_\alpha^+(n), \lambda \in \Lambda_\alpha(n,r), \lambda \preceq \sigma(A)\}
\]

forms a \( \mathbb{Z} \)-basis for \( S_\alpha(n,r) \), where the order relation \( \preceq \) is defined in (1.1.3).

**Lemma 4.4.** Let \( A_i, B_j \in \Theta_\alpha^+(n) \) \((1 \leq i \leq s, 1 \leq j \leq t)\) and \( \lambda \in \Lambda_\alpha(n,r) \). Then we have

\[
A_1(0,r) \cdots A_s(0,r)[\text{diag}(\lambda)]B_1(0,r) \cdots B_t(0,r) = [A_1 + \text{diag}(\lambda^{(1)})] \cdots [A_s + \text{diag}(\lambda^{(s)})][B_1 + \text{diag}(\mu^{(1)})] \cdots [B_t + \text{diag}(\mu^{(t)})]
\]

where \( \lambda^{(i)} = \lambda - \text{co}(A_i) + \sum_{i+1 \leq k \leq s} (\text{ro}(A_k) - \text{co}(A_k)) \) and \( \mu^{(j)} = \lambda - \text{ro}(B_j) + \sum_{1 \leq k \leq j-1} (\text{co}(B_k) - \text{ro}(B_k)) \) for \( 1 \leq i \leq s \) and \( 1 \leq j \leq t \).
Proof. Clearly, for any $A \in \Theta^+(n)$ and $\lambda \in \Lambda_\delta(n, r)$, we have $[\text{diag}(\lambda)] = [\text{diag}(\lambda)]^2$ and

\[(4.4.1) \quad A(0, r)[\text{diag}(\lambda)] = [\text{diag}(\lambda - \text{co}(A) + \text{ro}(A))]A(0, r).\]

Repeatedly using (4.4.1), we conclude the assertion. \hfill \Box

Now we can prove the affine version of [1, 3.9].

**Proposition 4.5.** Let $A \in \Theta_\delta(n, b)$. We choose $w_{A^+}, w_{A^-} \in \tilde{\Sigma}$ such that (4.2.1) hold. We assume $w_{A^+}$ and $w_{A^-}$ have tight form $w_{A^+} = a_1^{r_1} \cdots a_s^{r_s}$, $w_{A^-} = b_1^{r_1} \cdots b_t^{r_t}$ and let $A_i = A_{x_i}a_i$, $B_j = B_{y_j}b_j$ for $1 \leq i \leq s$ and $1 \leq j \leq t$. Then the following identity holds in $S_\delta(n, an + b)$

\[\begin{align*}
[a(A_1 + \text{diag}(\lambda(1)))] & \cdots [a(A_s + \text{diag}(\lambda(s)))][a(B_1 + \text{diag}(\mu(1)))] \cdots [a(B_t + \text{diag}(\mu(t)))] \\
& = [aA] + \text{lower terms}
\end{align*}\]

for any $a \geq 0$, where $\lambda(i) = \sigma(A) - \text{co}(A_i) + \sum_{i+1 \leq k \leq s} \text{ro}(A_k) - \text{co}(A_k)$ and $\mu(j) = \sigma(A) - \text{ro}(B_j) + \sum_{1 \leq k \leq j-1} \text{co}(B_k) - \text{co}(B_k)$ for $1 \leq i \leq s$ and $1 \leq j \leq t$.

**Proof.** By (4.2.1) and 4.3, for any $r \geq 0$ and $\lambda \in \Lambda_\delta(n, r)$, we have

\[m^+_{(w_{A^+})^r}[\text{diag}(\lambda)]m^-_{(w_{A^-})^r} = [A^\pm + \text{diag}(\lambda - \sigma(A^\pm))] + \text{lower terms}\]

Now the assertion follows from 4.4. \hfill \Box

5. **The fundamental multiplication formulas for affine Schur algebras**

We derive certain useful multiplication formulas for affine Schur algebras in 5.3. These formulas are the second key result for the proof of the stabilization property of multiplication for affine Schur algebras.

We need some preparation before proving 5.3. For a finite subset $X \subseteq S_\delta$, let $X = \sum_{x \in X} x \in \mathcal{Q}S_\delta$, It is clear that for $\lambda, \mu \in \Lambda_\delta(n, r)$ and $w \in S_\delta$,

\[(5.0.1) \quad \overline{S_\lambda w \overline{S_\mu}} = [w^{-1}S_\lambda w \cap \overline{S_\mu}]\overline{S_\lambda w \overline{S_\mu}}.
\]

**Lemma 5.1 ([5, 3.2.3]).** Let $\lambda, \mu \in \Lambda_\delta(n, r)$ and $d \in \mathcal{D}_{\lambda, \mu}$. Assume $A = \gamma(\lambda, d, \mu)$. Then $d^{-1} \overline{S_\lambda d} \cap \overline{S_\mu} = \overline{S_\nu}$, where $\nu = (\nu^{(1)}, \ldots, \nu(n))$ and $\nu^{(i)} = (a_k)_{k \in \mathbb{Z}} = (\ldots, a_{1i}, \ldots, a_{ni}, \ldots)$.

**Lemma 5.2.** Assume $\mu \in \Lambda_\delta(n, r)$, $\beta \in \mathbb{N}^n$ is such that $\mu \geq \beta$.

1. Let $\alpha = \sum_{1 \leq i \leq n} (\mu_i - \beta_i)e^\rho_{\mu_i} - \delta$, $\delta = (\alpha_n, \beta_1, \alpha_1, \beta_2, \ldots, \alpha_{n-1}, \beta_n)$ and $\gamma = \{(Y_0, Y_1, \ldots, Y_{n-1}) \mid Y_i \subseteq R_{i+1}^\mu, \ |Y_i| = \alpha_i, \ for \ 0 \leq i \leq n-1\}$.

Then there is a bijective map

\[\vartheta : D^\mu \cap \overline{S_\mu} \rightarrow \gamma
\]

defined by sending $w$ to $(w^{-1}X_0, w^{-1}X_1, \cdots, w^{-1}X_{n-1})$ where $X_i = \{\mu_0, i + 1, \mu_0, i + 2, \cdots, \mu_0, i + \alpha_i\}$, with $\mu_0, i = \sum_{1 \leq s \leq i} \mu_s$ and $\mu_0, 0 = 0$. 


(2) Let $\gamma = \mu - \beta$, $\theta = (\beta_1, \gamma_1, \beta_2, \gamma_2, \cdots, \beta_n, \gamma_n)$ and
$$
\gamma' = \{(Y'_1, Y'_2, \cdots, Y'_n) \mid Y'_i \subseteq R_1^\mu, |Y'_i| = \gamma_i, \text{ for } 1 \leq i \leq n\}.
$$

Then there is a bijective map
$$
\vartheta' : \mathcal{D}_0 \cap \mathcal{S}_\mu \rightarrow \gamma'
$$
defined by sending $w$ to $(w^{-1}X'_1, w^{-1}X'_2, \cdots, w^{-1}X'_n)$ where $X'_i = \{\mu_0, i - 1 + \beta_i + 1, \mu_0, i - 1 + \beta_i + 2, \cdots, \mu_0, i\}$.

**Proof.** We only prove (1). The proof of (2) is similar.

We assume $w_1, w_2 \in \mathcal{D}_0 \cap \mathcal{S}_\mu$ is such that $\vartheta(w_1) = \vartheta(w_2)$. Since $w_1, w_2 \in \mathcal{D}_0$, by (4.0.2), for $1 \leq i \leq 2$ we have

$$(5.2.1) \quad w_i^{-1}(j) < w_i^{-1}(j + 1)$$

if either $\{i, j + 1\} \subseteq X_k$ or $\{j, j + 1\} \subseteq R_{k+1}^\mu \setminus X_k$ for some $0 \leq k \leq n - 1$. Furthermore, we have $w_i^{-1}(R_{k+1}^\mu \setminus X_i) = w_i^{-1}(R_{k+1}^\mu \setminus X_i)$ for $0 \leq i \leq n - 1$, since $w_i^{-1}X_i = w_i^{-1}X_i$ and $w_1, w_2 \in \mathcal{S}_\mu$.

Thus, $w_i^{-1}(j) = w_2^{-1}(j)$ for $j \in \bigcup_{0 \leq k \leq n - 1} (X_k \cup R_{k+1}^\mu \setminus X_k) = \bigcup_{0 \leq k \leq n - 1} R_{k+1}^\mu = \{1, 2, \cdots, r\}$. Consequently, $w_1 = w_2$.

Let $(Y_0, \cdots, Y_{n-1}) \in \gamma'$. We write $Y_i = \{k_{i,1}, \cdots, k_{i,\alpha_i}\}$ and $R_{i+1}^\mu \setminus Y_i = \{k_{i,\alpha_i+1}, \cdots, k_{i,\mu_{i+1}}\}$ where $k_{i,1} < k_{i,2} \cdots < k_{i,\alpha_i}$ and $k_{i,\alpha_i+1} < k_{i,\alpha_i+2} < \cdots < k_{i,\mu_{i+1}}$ for all $i$. Define $w \in \mathcal{S}_\mu$ by letting $w^{-1}(\mu_0, i + s) = k_{i,s}$ for $0 \leq i \leq n - 1$ and $1 \leq s \leq \mu_{i+1}$. Then $w \in \mathcal{D}_0 \cap \mathcal{S}_\mu$ and $\vartheta(w) = (Y_0, Y_1, \cdots, Y_{n-1})$. This finishes the proof. \hfill $\square$

Let $S_0(n, r)_\mathbb{Z} = S_0(n, r) \otimes \mathbb{Z}$ and $S_0(n, r)_\mathbb{Q} = S_0(n, r) \otimes \mathbb{Q}$, where $\mathbb{Z}$ and $\mathbb{Q}$ are regarded as $\mathbb{Z}$-modules by specializing $v$ to 1. We will identify $S_0(n, r)_\mathbb{Z}$ as a subalgebra of $S_0(n, r)_\mathbb{Q}$. For $A \in \Theta_0(n, r)$ we will denote $[A] \otimes 1$ by $[A]_1$.

There is a natural map

$$(5.2.2) \quad \tilde{\cdot} : \Theta_0(n) \rightarrow \Theta_0(n) \quad A = (a_{i,j}) \mapsto \tilde{A} = (\tilde{a}_{i,j}),$$

where $\tilde{a}_{i,j} = a_{i,1-j}$ for all $i, j \in \mathbb{Z}$. We now give some multiplication formulas in the affine Schur algebra $S_0(n, r)_\mathbb{Z}$ over $\mathbb{Z}$, which are the affine version of [1] 3.1.

**Proposition 5.3.** Let $A \in \Theta_0(n, r)$ and $\mu = \text{ro}(A)$. Assume $\beta \in \mathbb{N}_0^n$ is such that $\beta \leq \mu$. Let $\alpha = \sum_{1 \leq i \leq n} (\mu_i - \beta_i)e_{i-1}^\Lambda$ and $\gamma = \mu - \beta$. Then in $S_0(n, r)_\mathbb{Z}$

$$
(1) \quad \left[ \sum_{1 \leq i \leq n} \alpha_i E_{i,i+1}^\Lambda + \text{diag}(\beta) \right]_1 [A]_1 = \sum_{T \in \Theta_0(n), \text{ro}(T) = \alpha} \prod_{1 \leq i \leq n} \prod_{1 \leq i \leq n} (a_{i,j} - t_{i-1,j} + t_{i,j}) [A + T - \tilde{T}]_1;
$$
(2) \[ \sum_{1 \leq i \leq n} \gamma_i E_{i+1,i}^\Delta + \text{diag}(\beta) \] \[ A \]_1 = \sum_{T \in \Theta(n), \text{ro}(T) = \gamma} \prod_{a_{i,j} + t_{i-1,j} - t_{i,j} \in \mathbb{Z}} (a_{i,j} + t_{i-1,j} - t_{i,j}) [A - T + \tilde{T}]_1.

Proof. We only prove (1). The proof for (2) is entirely similar.

Let \( B = \sum_{1 \leq i \leq n} \alpha_i E_{i+1,i}^\Delta + \text{diag}(\beta) \), \( \lambda = \text{ro}(B) \), \( \nu = \text{co}(A) \). Assume \( d_1 \in \mathcal{R}_\lambda,\mu \) and \( d_2 \in \mathcal{R}_\mu,\nu \) are such that \( \delta(\lambda, d_1, \mu) = B \) and \( \delta(\mu, d_2, \nu) = A \). Clearly, by (4.0.2), we have

\[ d_1(i) = -\alpha_n + i \text{ for } 1 \leq i \leq r. \]

By 5.1 and 5.0.1,
\[ [B]_1[A]_1(\mathbb{S}_\nu) = \mathbb{S}_\lambda d_1 \mathbb{S}_\mu \cdot d_2 \cdot \mathcal{R}_\delta \cap \mathbb{S}_\nu \]
\[ = \frac{1}{|\mathbb{S}_\mu|} \mathbb{S}_\lambda d_1 \mathbb{S}_\mu \cdot \mathbb{S}_\nu d_2 \mathbb{S}_\nu \]
\[ = \frac{1}{|\mathbb{S}_\mu|} \prod_{1 \leq i \leq n} \frac{1}{a_{i,j}!} \mathbb{S}_\lambda d_1 \mathbb{S}_\mu \cdot \mathbb{S}_\nu d_2 \cdot \mathbb{S}_\nu \]
\[ = \prod_{1 \leq i \leq n} \frac{1}{a_{i,j}!} \mathbb{S}_\lambda \cdot d_1 \cdot \mathcal{R}_\delta \cap \mathbb{S}_\mu \cdot d_2 \cdot \mathbb{S}_\nu \]
\[ = \sum_{w \in \mathcal{R}_\delta \cap \mathbb{S}_\mu} \prod_{1 \leq i \leq n} \frac{c^{(w)}_{i,j}}{a_{i,j}!} [C^{(w)}]_1(\mathbb{S}_\nu) \]

where \( \mathbb{S}_\omega = d_2^{-1} \mathbb{S}_\mu d_2 \cap \mathbb{S}_\nu \), \( \mathbb{S}_\delta = d_1^{-1} \mathbb{S}_\lambda d_1 \cap \mathbb{S}_\mu \) with \( \delta = (\alpha_n, \beta_1, \alpha_1, \beta_2, \ldots, \alpha_{n-1}, \beta_n) \) and \( C^{(w)} = (c^{(w)}_{i,j}) \) with \( c^{(w)}_{i,j} = |R^\lambda_i \cap d_1 wd_2 R^\nu_j| \). Thus we have

\[ [B]_1[A]_1 = \sum_{w \in \mathcal{R}_\delta \cap \mathbb{S}_\mu} \prod_{1 \leq i \leq n} \frac{c^{(w)}_{i,j}}{a_{i,j}!} [C^{(w)}]_1 \]

Let us compute \( C^{(w)} \) for \( w \in \mathcal{R}_\delta \cap \mathbb{S}_\mu \) as follows. Since, by (5.3.1), \( d_1^{-1}(j) = \alpha_n + j \) for \( 1 \leq j \leq r \), we have

\[ d_1^{-1}R^\lambda_i = \alpha_n + R^\lambda_i = \{\mu_{0,i-1} + \alpha_{i-1} + 1, \mu_{0,i-1} + \alpha_{i-1} + 2, \ldots, \mu_{0,i} + \alpha_i\} = (R^\mu_i \setminus X_{i-1}) \cup X_i \]
for \( 1 \leq i \leq n \), where \( X_i = \{\mu_{0,i} + 1, \mu_{0,i} + 2, \ldots, \mu_{0,i} + \alpha_i\} \). Thus, since \( \delta(\mu, d_2, \nu) = A \), we have

\[ c^{(w)}_{i,j} = |R^\lambda_i \cap d_1 wd_2 R^\nu_j| = |w^{-1}d_1^{-1}R^\lambda_i \cap d_2 R^\nu_j| = a_{i,j} - t^{(w)}_{i-1,j} + t^{(w)}_{i,j} \]
for \( w \in \mathcal{P}_0 \cap \mathfrak{S}_\mu, 1 \leq i \leq n \) and \( j \in \mathbb{Z} \), where \( t_{i,j}^{(w)} = |w^{-1}X_i \cap d_2 R_j^w| \). Note that, for \( w \in \mathcal{P}_0 \cap \mathfrak{S}_\mu \), the numbers \( t_{i,j}^{(w)} \) with \( 1 \leq i \leq n \) and \( j \in \mathbb{Z} \) determine an unique matrix \( T^{(w)} = (t_{i,j}^{(w)})_{i,j \in \mathbb{Z}} \) in \( \Theta_0(n) \) by letting \( t_{i,kn+j,n}^{(w)} = t_{i,j}^{(w)}. \) Consequently,

\[
C^{(w)} = A + T^{(w)} - \overline{T}^{(w)}
\]

for \( w \in \mathcal{P}_0 \cap \mathfrak{S}_\mu \).

Now, by [5.3.2] and noting \( \text{ro}(T^{(w)}) = \alpha \) for \( w \in \mathcal{P}_0 \cap \mathfrak{S}_\mu \),

\[
[B]_1[A]_1 = \sum_{w \in \mathcal{P}_0 \cap \mathfrak{S}_\mu} \prod_{1 \leq i \leq n, j \in \mathbb{Z}} \frac{(a_{i,j} - t_{i-1,j}^{(w)} + t_{i,j}^{(w)})!}{a_{i,j}!} [A + T^{(w)} - \overline{T}^{(w)}]_1
\]

(5.3.4)

\[
= \sum_{T \in \Theta_0(n), \text{ro}(T) = \alpha} |X(T)| \prod_{1 \leq i \leq n, j \in \mathbb{Z}} \frac{(a_{i,j} - t_{i-1,j}^{(w)} + t_{i,j}^{(w)})!}{a_{i,j}!} [A + T - \overline{T}]_1
\]

where \( X(T) = \{ w \in \mathcal{P}_0 \cap \mathfrak{S}_\mu \mid T^{(w)} = T \} \). By restriction, for each \( T \), the bijective map \( \vartheta \) defined in [5.2] induces a bijective map \( \vartheta_T : X(T) \rightarrow \gamma(T) \), where

\[
\gamma(T) = \{(Y_0, \cdots, Y_{n-1}) \in \gamma \mid |Y_i \cap d_2 R_j^{\mu}| = t_{i,j}, \text{ for } 0 \leq i \leq n - 1, j \in \mathbb{Z} \}.
\]

Furthermore, for each \( T \) there is a natural bijective map \( \kappa : \gamma(T) \rightarrow Z(T) \) defined by sending \( (Y_0, \cdots, Y_{n-1}) \) to \( (Y_i \cap d_2 R_j^{\mu})_{0 \leq i \leq n-1, j \in \mathbb{Z}} \), where

\[
Z(T) = \{(Z_{i,j})_{0 \leq i \leq n-1, j \in \mathbb{Z}} \mid |Z_{i,j}| = t_{i,j}, Z_{i,j} \subseteq R_{i+1}^{\mu} \cap d_2 R_j^{\mu}, \text{ for } 0 \leq i \leq n - 1, j \in \mathbb{Z} \}.
\]

Consequently,

\[
|X(T)| = |\gamma(T)| = |Z(T)| = \prod_{0 \leq i \leq n-1, j \in \mathbb{Z}} \left( \frac{a_{i+1,j}}{t_{i,j}} \right) = \prod_{1 \leq i \leq n, j \in \mathbb{Z}} \left( \frac{a_{i,j}}{t_{i-1,j}} \right).
\]

Thus, by [5.3.3] and noting \( \prod_{1 \leq i \leq n, j \in \mathbb{Z}} t_{i,j}! = \prod_{1 \leq i \leq n, j \in \mathbb{Z}} t_{i-1,j}! \),

\[
[B]_1[A]_1 = \sum_{T \in \Theta_0(n), \text{ro}(T) = \alpha} \prod_{1 \leq i \leq n, j \in \mathbb{Z}} \frac{(a_{i,j} - t_{i-1,j}^{(w)} + t_{i,j}^{(w)})!}{t_{i-1,j}!} [A + T - \overline{T}]_1
\]

\[
= \sum_{T \in \Theta_0(n), \text{ro}(T) = \alpha} \prod_{1 \leq i \leq n, j \in \mathbb{Z}} \left( \frac{a_{i,j} - t_{i-1,j}^{(w)} + t_{i,j}^{(w)}}{t_{i,j}} \right) [A + T - \overline{T}]_1,
\]

proving (1). \( \square \)

For \( A \in \Theta_0(n) \) and \( a \in \mathbb{Z} \) we set

\[
aA = A + aI
\]

where \( I \in \Theta_0(n) \) is the identity matrix. Note that if \( a \) is large enough, we have \( aA \in \Theta_0(n) \).

Let \( x \) be an indeterminate. We denote by \( \mathcal{Z}_1 \) the subring of \( \mathbb{Q}[x] \) generated by 1 and \( \left( \frac{a+x}{t} \right) \) for \( a \in \mathbb{Z} \) and \( t \in \mathbb{N} \).
Corollary 5.4. From 5.3, we immediately have the following result.

Let $\alpha$ is diagonal for some $S$. We have in $K$ all algebras, which is the affine analogue of [1, 4.2]. This property allow us to construct an algebra in the affine Schur algebra $\mathcal{S}(n, r)_Z$.

For $T \in \Theta_\delta(n)$ and $A \in \tilde{\Theta}_\delta(n)$ let

$$P_{T,A}(x) = \prod_{1 \leq i \leq n, j \neq i} \left( a_{i,j} - t_{i-1,j} + t_{i,j} \right) \prod_{1 \leq i \leq n} \left( a_{i,i} - t_{i-1,i} + t_{i,i} + x \right) \in \mathbb{Z}_1$$

and

$$Q_{T,A}(x) = \prod_{1 \leq i \leq n, j \neq i} \left( a_{i,j} + t_{i-1,j} - t_{i,j} \right) \prod_{1 \leq i \leq n} \left( a_{i,i} + t_{i-1,i} - t_{i,i} + x \right) \in \mathbb{Z}_1.$$ 

From 5.3 we immediately have the following result.

**Corollary 5.4.** Let $A, B \in \tilde{\Theta}_\delta(n)$ is such that $\text{co}(B) = \text{ro}(A)$ and let $b = \sigma(A) = \sigma(B)$.

1. If $B - \sum_{1 \leq i \leq n} \alpha_i E_{i,i+1}^\Delta$ is diagonal for some $\alpha \in \mathbb{N}_\delta^\alpha$. Then for large $a$ and $r = an + b$, we have in $\mathcal{S}(n, r)_Z$,

$$[aB]_1[aA]_1 = \sum_{T \in \Theta_\delta(n), \text{ro}(T) = \alpha} \sum_{a_{i,j} - t_{i-1,j} + t_{i,j} \geq 0, \forall i \neq j} P_{T,A}(a)(A + T - \tilde{T})_1.$$ 

2. If $B - \sum_{1 \leq i \leq n} \alpha_i E_{i+1,i}^\Delta$ is diagonal for some $\alpha \in \mathbb{N}_\delta^\alpha$. Then for large $a$ and $r = an + b$, we have in $\mathcal{S}(n, r)_Z$,

$$[aB]_1[aA]_1 = \sum_{T \in \Theta_\delta(n), \text{ro}(T) = \alpha} \sum_{a_{i,j} + t_{i-1,j} - t_{i,j} \geq 0, \forall i \neq j} Q_{T,A}(a)(A - T + \tilde{T})_1.$$ 

6. THE ALGEBRA $K_Z(n)$

We now use 5.3 and 5.4 to derive the stabilization property of multiplication for affine Schur algebras, which is the affine analogue of [1 4.2]. This property allow us to construct an algebra $K_Z(n)$ without unity.

For $A \in \Theta_\delta(n)$, define (cf. [1])

$$[A] = \sum_{1 \leq i \leq n, j \neq i} \frac{(j - i)(j - i + 1)}{2} a_{i,j} + \sum_{1 \leq i \leq n, i > j} \frac{(i - j)(i - j + 1)}{2} a_{i,j}.$$ 

Let $\tilde{\Theta}_\delta(n)^{ss}$ be the set of $X \in \tilde{\Theta}_\delta(n)$ such that either $X - \sum_{1 \leq i \leq n} \alpha_i E_{i,i+1}^\Delta$ or $X - \sum_{1 \leq i \leq n} \alpha_i E_{i+1,i}^\Delta$ is diagonal for some $\alpha \in \mathbb{N}_\delta^\alpha$ and let $\Theta_\delta(n)^{ss} = \tilde{\Theta}_\delta(n)^{ss} \cap \Theta_\delta(n)$.

**Proposition 6.1.** Let $A, B \in \tilde{\Theta}_\delta(n)$ such that $\text{co}(B) = \text{ro}(A)$. Then there exist unique $X_1, \cdots, X_m \in \tilde{\Theta}_\delta(n)$, unique $P_i(x), \cdots, P_m(x) \in \mathbb{Z}_1$ and an integer $a_0 \geq 0$ such that, for all $a \geq a_0$,

$$[aB]_1[aA]_1 = \sum_{1 \leq i \leq m} P_i(a)[aX_i]_1$$

in the affine Schur algebra $\mathcal{S}(n, an + \sigma(A))_Z$. 


Proof. If $B \in \widetilde{\Theta}_\alpha(n)^{ss}$ then the assertion follows from [5, 3.7.6]. So, by induction, if $B_1, \ldots, B_l \in \widetilde{\Theta}_\alpha(n)^{ss}$ are such that $\text{co}(B_i) = \text{ro}(B_{i+1})$ and $\text{co}(B_l) = \text{ro}(A)$, then there exist $Y_j \in \widetilde{\Theta}_\alpha(n)$, $Q_j(x) \in Z_1$ ($1 \leq j \leq m$), and $a_0 \in \mathbb{N}$ such that

$$\sum_{1 \leq j \leq m} Q_j(a)[aY_j]_1$$

for all $a \geq a_0$.

In general, we apply induction on $\|B\|$. If $\|B\| = 0$ then $B$ is diagonal and $[aB]_1[aA]_1 = [aA]_1$ for all large enough $a$. Assume $\|B\| > 1$ and the result is true for those $\|B_1\|$ with $\|B_1\| < \|B\|$. Choose $b \in \mathbb{N}$ such that $bB \in \Theta_\alpha(n)$ and apply 4.5 to $bB$. Thus, there exist $B_1, \ldots, B_N \in \Theta_\alpha(n)^{ss}$, such that $\text{co}(B_i) = \text{ro}(B_{i+1})$ and

$$[aB_1]_1 \cdots [aB_N]_1 = [a+bB]_1 + \text{lower terms}$$

for $a \in \mathbb{N}$. Let $A_i = B_i - bI \in \widetilde{\Theta}_\alpha(n)^{ss}$ for $1 \leq i \leq N$. Then we have

$$[cA_1]_1 \cdots [cA_N]_1 = [c-B_1]_1 \cdots [c-B_N]_1 = [cB]_1 + \text{lower terms}$$

for $c \geq b$. By (6.1.2), there exist $Z_i, Z'_j \in \widetilde{\Theta}_\alpha(n)$ and $Q_i(x), Q'_j(x) \in Z_1$ ($1 \leq i \leq m$, $1 \leq j \leq m'$) such that

$$[cA_1]_1 \cdots [cA_N]_1 = \sum_{1 \leq i \leq m} Q_i(c)[cZ_i]_1,$$

and

$$[cA_1]_1 \cdots [cA_N]_1[cA]_1 = \sum_{1 \leq j \leq m'} Q'_j(c)[cZ'_j]_1$$

for all large enough $c$. Comparing (6.1.3) with (6.1.4), we see that we may assume that $Z_1 = B$, $Q_1(x) = 1$ and $Z_i \prec B$ for $2 \leq i \leq m$. Thus by (6.1.4) and (6.1.5), for large $c$,

$$[cB]_1[cA]_1 = [cA_1]_1 \cdots [cA_N]_1[cA]_1 - \sum_{2 \leq i \leq m} Q_i(c)[cZ_i]_1[cA]_1$$

for $i > 1$ and large $c$. Consequently, $[cB]_1[cA]_1$ is of the required form. \hfill \Box

We now use 6.1 to construct the $\mathbb{Z}$-algebra $\mathbb{K}_Z(n)$ as follows. Let $\mathbb{K}(n)$ be the free $Z_1$-module with basis $\{A \mid A \in \widetilde{\Theta}_\alpha(n)\}$. There is a unique structure of associative $Z_1$-algebra (without unit) on this module in which $B \cdot A = \sum_{1 \leq i \leq m} P_i(x)X_i$ (notation of 6.1) if $\text{co}(B) = \text{ro}(A)$ and $B \cdot A = 0$, otherwise. Consider the specialization $Z_1 \rightarrow \mathbb{Z}$ obtained by sending $x$ to 0 and let

$$\mathbb{K}_Z(n) = \mathbb{K}(n) \otimes_{Z_1} \mathbb{Z}.$$
Then $\mathcal{K}_Z(n)$ is an associative algebra over $\mathbb{Z}$ with basis $\{A \otimes 1 \mid A \in \Theta_\alpha(n)\}$. We will denote the element $A \otimes 1$ by $[A]_1$ in the sequel. By \[5.4\] and \[4.5\] the following multiplication formulas hold in the algebra $\mathcal{K}_Z(n)$.

**Proposition 6.2.** Let $A, B \in \Theta_\alpha(n)$ be such that $\text{co}(B) = \text{ro}(A)$. In the algebra $\mathcal{K}_Z(n)$, the following statements hold.

1. If $B - \sum_{1 \leq i \leq n} \alpha_i E^\delta_{i,i+1}$ is diagonal for some $\alpha \in \mathbb{N}_\alpha$, then

   $[B]_1[A]_1 = \sum_{T \in \Theta_\alpha(n), \text{ro}(T) = \alpha} \prod_{1 \leq i \leq n, j \in \mathbb{Z}} \left( a_{i,j} - t_{i-1,j} + t_{i,j} \right) [A + T - \bar{T}]_1;$

2. If $B - \sum_{1 \leq i \leq n} \alpha_i E^\delta_{i+1,i}$ is diagonal for some $\alpha \in \mathbb{N}_\alpha$, then

   $[B]_1[A]_1 = \sum_{T \in \Theta_\alpha(n), \text{ro}(T) = \alpha} \prod_{1 \leq i \leq n, j \in \mathbb{Z}} \left( a_{i,j} + t_{i-1,j} - t_{i,j} \right) [A - T + \bar{T}]_1.$

3. There exist upper triangular matrix $A_i$ and lower triangular matrix $B_j$ in $\Theta_\alpha(n)^ss$ ($1 \leq i \leq s, 1 \leq j \leq t$) such that

   $[A_1]_1 \cdots [A_s]_1 [B_1]_1 \cdots [B_t]_1 = [A]_1 + \text{lower terms},$

where “lower terms” stands for a $\mathbb{Z}$-linear combination of terms $[A']_1$ with $A' \in \Theta_\alpha(n)$, $A' < A$, $\text{co}(A') = \text{co}(A)$ and $\text{ro}(A') = \text{ro}(A)$.

The algebra $\mathcal{K}_Z(n)$ and $S_\alpha(n,r)_\mathbb{Z}$ are related by the following algebra homomorphism.

**Proposition 6.3.** The linear map $\hat{\zeta}_r : \mathcal{K}_Z(n) \rightarrow S_\alpha(n,r)_\mathbb{Z}$ defined by

$$\hat{\zeta}_r([A]_1) = \begin{cases} [A]_1 & \text{if } A \in \Theta_\alpha(n,r); \\ 0 & \text{otherwise} \end{cases}$$

is a surjective algebra homomorphism.

**Proof.** Since, by \[5.2\] 3, the algebra $\mathcal{K}_Z(n)$ is generated by $[A]$ with $A \in \Theta_\alpha(n)^{ss}$, it is enough to prove that

\begin{equation}
\hat{\zeta}_r([B]_1[A]_1) = \hat{\zeta}_r([B]_1)[\hat{\zeta}_r([A]_1)]
\end{equation}

for $B \in \Theta_\alpha(n)^{ss}$ and $A \in \Theta_\alpha(n)$ with co$(B) = \text{ro}(A)$. If $\sigma(A) \neq r$, then $\hat{\zeta}_r([B]_1[A]_1) = 0 = \hat{\zeta}_r([B]_1)\hat{\zeta}_r([A]_1)$.

Now we assume $\sigma(A) = r$, co$(B) = \text{ro}(A)$, $B = \sum_{1 \leq i \leq n} \alpha_i E^\delta_{i,i+1} + \text{diag}(\beta) \in \Theta_\alpha(n)^{ss}$ for some $\alpha \in \mathbb{N}_\alpha$ and $\beta \in \mathbb{Z}_\alpha^n$. Let us prove \[6.3.1\] in three cases.

Case 1 If $A, B \in \Theta_\alpha(n,r)$, then the assertion follows from \[5.3\] and \[6.2\] 1.
Case 2 Suppose $a_{i_0,i_0} < 0$ for some $1 \leq i_0 \leq n$. If $T \in \Theta_\delta(n)$ is such that $\text{ro}(T) = \alpha$ and $A + T - \tilde{T} \in \Theta_\delta(n)$, then

$$\zeta_r \left( \left( a_{i_0,i_0} + t_{i_0,i_0} - t_{i_0-1,i_0} \right) [A + T - \tilde{T}]_1 \right) = 0$$

since $a_{i_0,i_0} + t_{i_0,i_0} - t_{i_0-1,i_0} < t_{i_0,i_0}$. It follows from (6.2.1) that $\zeta_r([B]_1[A]_1) = 0 = \zeta_r([B]_1) \zeta_r([A]_1)$.

Case 3 Suppose $\beta_{i_0} < 0$ for some $1 \leq i_0 \leq n$. Let $T \in \Theta_\delta(n)$ be such that $\text{ro}(T) = \alpha$ and $A + T - \tilde{T} \in \Theta_\delta(n)$. Since $\beta + \sum_{1 \leq i \leq n} \alpha_i e_{i+1} = \text{co}(B) = \text{ro}(A)$ and $\text{ro}(T) = \alpha$, we have

$$\sum_{s \in \mathbb{Z}} (a_{i_0,s} - t_{i_0-1,s}) = \sum_{s \in \mathbb{Z}} a_{i_0,s} - \alpha_{i_0-1} = \beta_{i_0} < 0,$$

and hence $a_{i_0,k} - t_{i_0-1,k} + t_{i_0,k} < t_{i_0,k}$ for some $k \in \mathbb{Z}$. Thus,

$$\zeta_r \left( \left( a_{i_0,k} + t_{i_0,k} - t_{i_0-1,k} \right) [A + T - \tilde{T}]_1 \right) = 0.$$

This together with (6.2.1) implies that $\zeta_r([B]_1[A]_1) = 0 = \zeta_r([B]_1) \zeta_r([A]_1)$.

Similarly, if $B - \sum_{1 \leq i \leq n} \alpha_i E_{i+1,i}$ is diagonal for some $\alpha \in \Theta_\delta(n)$, $\zeta_r([B]_1[A]_1) = 0 = \zeta_r([B]_1) \zeta_r([A]_1)$. The proof is completed. \qed

7. The completion algebra $\hat{K}_n(n)$ of $\mathcal{K}_n(n)$ and multiplication formulas

Let $\mathcal{K}_n(n) = \mathcal{K}_Z(n) \otimes_\mathbb{Z} \mathbb{Q}$. As in [1, 5.1], let $\hat{K}_n(n)$ be the vector space of all formal (possibly infinite) $\mathbb{Q}$-linear combinations $\sum_{A \in \Theta_\delta(n)} \beta_A[A]$ such that for any $x \in \mathbb{Z}^n$, the sets $\{ A \in \Theta_\delta(n) \mid \beta_A \neq 0, \text{ro}(A) = x \}$ and $\{ A \in \Theta_\delta(n) \mid \beta_A \neq 0, \text{co}(A) = x \}$ are finite. We shall regard $\mathcal{K}_n(n)$ naturally as a subset of $\hat{K}_n(n)$. We can define the product of two elements $\sum_{A \in \mathbb{Z}} \beta_A[A]_1, \sum_{B \in \mathbb{Z}} \gamma_B[B]_1$ in $\hat{K}_n(n)$ to be $\sum_{A,B} \beta_A \gamma_B[A]_1 \cdot [B]_1$ where $[A]_1 \cdot [B]_1$ is the product in $\mathcal{K}_n(n)$. This defines an associative algebra structure on $\hat{K}_n(n)$. This algebra has a unit element: $\sum_{\lambda \in \mathbb{Z}_n^\circ} [\text{diag}(\lambda)]_1$.

We now establish some important multiplication formulas in $\hat{K}_n(n)$ and $S_n(n,r)\mathbb{Q}$, which will be used to realize $U_\mathbb{Z}(\mathfrak{G}_n)$ as a $\mathbb{Z}$-subalgebra of $\hat{K}_n(n)$. These formulas are the affine analogue of [1, 5.3].

For $m, k_1, k_2, \ldots, k_l \in \mathbb{N}$ with $\sum_{1 \leq i \leq l} k_i = m$, let $(k_1, k_2, \ldots, k_l) = \frac{m!}{k_1! k_2! \cdots k_l!}$. For $\lambda, \mu^{(1)}, \ldots, \mu^{(s)} \in \mathbb{N}_0$ with $\lambda = \sum_{1 \leq j \leq s} \mu^{(j)}$, let

$$\left( \mu^{(1)}, \ldots, \mu^{(s)} \right) = \prod_{1 \leq i \leq n} \left( \frac{\lambda_i}{\mu^{(1)}_i, \ldots, \mu^{(s)}_i} \right).$$

Recall the order relation $\leq$ defined in (1.1.3). We need the following well known combinatorial formulas.
Lemma 7.1. For \( m, n \in \mathbb{Z} \), \( a, b \in \mathbb{N} \) we have

1. \( \binom{n}{a} = \sum_{0 \leq j \leq a} \binom{m}{j} \binom{n-m}{a-j} \);
2. \( \binom{m}{a} \binom{b}{a} = \sum_{0 \leq c \leq \min\{a, b\}} \binom{a+b-c}{c} \binom{a-b-c}{a} \binom{m}{a+b-c} \).

Corollary 7.2. For \( \lambda, \mu \in \mathbb{N}_0^n \) and \( \alpha, \beta \in \mathbb{Z}_0^n \) we have

1. \( \binom{\alpha+\beta}{\lambda} = \sum_{\mu \in \mathbb{N}_0^n, \mu \leq \lambda} \binom{\alpha}{\mu} \binom{\beta}{\lambda-\mu} \);
2. \( \binom{\alpha}{\lambda} = \sum_{\gamma \in \mathbb{N}_0^n, \gamma \leq \mu} \binom{\lambda+\mu-\gamma}{\gamma} \binom{\lambda+\mu-\gamma}{\lambda+\mu-\gamma} \).

For \( A \in \Theta_\Delta^+(n) \), \( \lambda \in \mathbb{N}_0^n \) let

\[
A\{\lambda\} = \sum_{\mu \in \mathbb{Z}_0^n} \binom{\mu}{\lambda} [A + \text{diag}(\mu)]_{1} \in \hat{K}_Q(n)
\]

\[
A\{\lambda, r\} = \sum_{\mu \in \Lambda_\alpha(n, r - \sigma(A))} \binom{\mu}{\lambda} [A + \text{diag}(\mu)]_{1} \in S_\lambda(n, r)_{\mathbb{Z}}.
\]

Also, for \( A \in M_{\alpha,n}(\mathbb{Z}) \), define

\[ A\{\lambda\} = 0 \text{ and } A\{\lambda, r\} = 0 \text{ if } a_{ij} < 0 \text{ for some } i \neq j. \]

For \( A \in \tilde{\Theta}_\Delta(n) \) let \( \lambda(A) \) be the element in \( \mathbb{Z}_0^n \) such that

\[ \text{diag}(\lambda(A)) = A^0, \]

where \( A^0 \) is defined in \((4.1.2)\). Recall the map \( \tilde{\lambda} : \Theta_\Delta(n) \to \Theta_\Delta(n) \) defined in \((5.2.2)\).

Proposition 7.3. Let \( A \in \Theta_\Delta^+(n) \), \( B = \sum_{1 \leq i \leq n} \alpha_i E_{i,i+1} \) and \( C = \sum_{1 \leq i \leq n} \alpha_i E_{i+1,i} \) with \( \alpha \in \mathbb{N}_0^n \).

Let \( \lambda, \mu \in \mathbb{N}_0^n \). The following identities holds in \( \hat{K}_Q(n) \):

1. \( \text{0}\{\mu\} A\{\lambda\} = \sum_{\delta \in \mathbb{N}_0^n} \sum_{\beta \in \mathbb{N}_0^n, \beta \leq \lambda} \binom{\beta}{\mu} \binom{\text{ro}(A)}{(\delta + \lambda)} \binom{\delta + \lambda}{\mu - \beta - \delta} A\{\lambda + \delta\} \)

2. \( B\{0\} A\{\lambda\} = \sum_{T \in \Theta_\Delta(n), \delta \in \mathbb{N}_0^n} a(T, \delta) \prod_{1 \leq i \leq n, j \in \mathbb{Z}, j \neq i} (a_{i,j} - t_{i-1,j} + t_{i,j}) (A + T^{-} + \tilde{T}^{-}) \{\lambda(T) + \delta\} \)

where

\[ a(T, \delta) = \sum_{\beta \in \mathbb{N}_0^n, \beta \leq \lambda(T)} \binom{\lambda(T) - \lambda(T)}{(\lambda(T) + \delta)} \binom{\lambda(T) + \delta}{\beta, \delta, \lambda(T) - \beta} \in \mathbb{Z}; \]

3. \( C\{0\} A\{\lambda\} = \sum_{T \in \Theta_\Delta(n), \delta \in \mathbb{N}_0^n} b(T, \delta) \prod_{1 \leq i \leq n, j \in \mathbb{Z}, j \neq i} (a_{i,j} + t_{i-1,j} - t_{i,j}) (A - T^{-} + \tilde{T}^{-}) \{\lambda(T) + \delta\} \)
where
\[
b(T, \delta) = \sum_{\beta \in \mathbb{N}_+^n, \beta \leq \lambda(T), \beta \leq \lambda - \delta} \left( \lambda(T) - \lambda(T) \right) \left( \lambda(T) + \delta \beta, \delta, \lambda(T) - \beta \right) \in \mathbb{Z}.
\]

The same formulas hold in \(S_0(n, r) \mathbb{Z} \) with \(A\{\lambda\} \) replaced by \(A\{\lambda, r\}\).

Proof. First, by \[7.2\]
\[
0\{\mu\}A\{\lambda\} = \sum_{\alpha \in \mathbb{Z}_n} \left( \frac{\mu}{\lambda} \right) \left( A + \text{diag}(\alpha) \right) A \{\lambda + \alpha\}
\]
\[
= \sum_{\alpha \in \mathbb{Z}_n} \left( \frac{\mu}{\lambda} \right) \left( A + \text{diag}(\alpha) \right) A \{\lambda + \alpha\}
\]
\[
= \sum_{\alpha \in \mathbb{Z}_n} \left( \frac{\mu}{\lambda} \right) \left( \frac{\lambda + \alpha}{\lambda} \right) A \{\lambda + \alpha\}
\]
\[
= \sum_{\delta \in \mathbb{Z}_n} \left( \sum_{\beta \in \mathbb{Z}_n} \left( \frac{\mu + \delta}{\lambda} \right) \left( \frac{\lambda + \delta}{\lambda} \right) \right) A \{\lambda + \delta\},
\]
proving (1). To prove (2), we conclude from \[6.2\] that
\[
B\{0\}A\{\lambda\} = \sum_{\gamma \in \mathbb{Z}_n} \left( \frac{\gamma}{\lambda} \right) \left[ B + \text{diag} \left( \frac{\gamma + \text{ro}(A) - \sum_{1 \leq i \leq n} \alpha_i e_{i+1}}{\lambda} \right) \right] \left( A + \text{diag}(\gamma) \right)
\]
\[
= \sum_{T \in \Theta_0(n) \text{ with } T \neq \alpha} \prod_{1 \leq i \leq n} \left( a_{i,j} - t_{i-1,j} + t_{i,j} \right) x_T
\]
where
\[
x_T = \sum_{\gamma \in \mathbb{Z}_n} \left( \frac{\gamma}{\lambda} \right) \left( \frac{\gamma - \lambda(T) + \lambda(T)}{\lambda(T)} \right) \left( A + T^+ - \widetilde{T}^+ + \text{diag}(\gamma - \lambda(T) + \lambda(T)) \right).
\]
Furthermore, by \[7.2\] we have
\[
x_T = \sum_{\nu \in \mathbb{Z}_n} \left( \nu - \lambda(T) + \lambda(T) \right) \left( \nu \lambda(T) \right) \left( A + T^+ - \widetilde{T}^+ + \text{diag}(\nu) \right)
\]
\[
= \sum_{j \in \mathbb{N}_+, \nu \in \mathbb{Z}_n} \left( \lambda(T) - \lambda(T) \right) \left( \nu \lambda(T) \right) \left( A + T^+ - \widetilde{T}^+ + \text{diag}(\nu) \right)
\[ \begin{align*}
&= \sum_{J, \lambda \in N_n^\alpha} \begin{pmatrix} \lambda(T) - \lambda(T) \\ \lambda - j \end{pmatrix} \begin{pmatrix} j + \lambda(T) - \beta \\ \beta, j - \beta, \lambda(T) - \beta \end{pmatrix} \\
&\quad \times \sum_{\nu \in Z_n^n} \begin{pmatrix} \nu \\ j + \lambda(T) - \beta \end{pmatrix} [A + T^\pm - \tilde{T}^\pm + \text{diag}(\nu)]_1 \\
&= \sum_{J, \lambda \in N_n^\alpha} \begin{pmatrix} \lambda(T) - \lambda(T) \\ \lambda - j \end{pmatrix} \begin{pmatrix} j - \beta + \lambda(T) \\ \beta, j - \beta, \lambda(T) - \beta \end{pmatrix} (A + T^\pm - \tilde{T}^\pm) \{j - \beta + \lambda(T)\} \\
&= \sum_{\delta \in N_n^\alpha, \delta \leq \lambda} a(T, \delta)(A + T^\pm - \tilde{T}^\pm) \{\delta + \lambda(T)\}.
\end{align*} \]

Therefore, (2) holds. Formula (3) is proved similarly. \(\square\)

8. The algebra \(V_Z(n)\)

We shall denote by \(V_Z(n)\) the \(Z\)-submodule of \(\hat{K}_Q(n)\) spanned by

\[ \mathcal{B} := \{A\{\lambda\} \mid A \in \Theta^\alpha_{\lambda}(n), \lambda \in N_n^\alpha \}. \]

We will prove that \(V_Z(n)\) is actually a \(Z\)-subalgebra of \(\hat{K}_Q(n)\) in \(8.5\).

Lemma 8.1. The set \(\mathcal{B}\) forms a \(Z\)-basis for \(V_Z(n)\).

Proof. It is enough to prove the linear independence of \(\mathcal{B}\). Suppose

\[ \sum_{A \in \Theta^\alpha_{\lambda}(n) \atop \lambda \in N_n^\alpha} k_{A, \lambda} A\{\lambda\} = 0, \]

for some \(k_{A, \lambda} \in Z\). Then

\[ 0 = \sum_{A \in \Theta^\alpha_{\lambda}(n) \atop \lambda \in N_n^\alpha} k_{A, \lambda} A\{\lambda\} = \sum_{A \in \Theta^\alpha_{\lambda}(n) \atop \lambda \in N_n^\alpha} \left( \sum_{\mu \in N_n^\alpha} k_{A, \lambda} \left( \begin{array}{c} \mu \\ \lambda \end{array} \right) \right) [A + \text{diag}(\mu)] \]

for some \(k_{A, \lambda} \in Z\). Thus, \(\sum_{\lambda \in N_n^\alpha} k_{A, \lambda} \left( \begin{array}{c} \mu \\ \lambda \end{array} \right) = 0\), for any \(A \in \Theta^\alpha_{\lambda}(n)\), \(\mu \in Z_n^\alpha\). We want to show that \(k_{A, \lambda} = 0\) for all \(A, \lambda\). If this is not the case, then there exist \(B \in \Theta^\alpha_{\lambda}(n)\) such that \(X_B := \{\lambda \in N_n^\alpha \mid k_{B, \lambda} \neq 0\} \neq \emptyset\). Let \(\nu\) be the minimal element in \(X_B\) with respect to the lexicographic order. Then

\[ k_{B, \nu} = \sum_{\lambda \in X_B, \nu \neq \lambda} k_{B, \lambda} \left( \begin{array}{c} \nu \\ \lambda \end{array} \right) = \sum_{\lambda \in N_n^\alpha} k_{B, \lambda} \left( \begin{array}{c} \nu \\ \lambda \end{array} \right) = 0 \]

since \(\nu\) is minimal. This is a contradiction. \(\square\)
Let \( \mathcal{V}_Z^+(n) = \text{span}_Z\{A\{0\} \mid A \in \Theta_+^+(n)\} \), \( \mathcal{V}_Z^-(n) = \text{span}_Z\{A\{0\} \mid A \in \Theta_-^+(n)\} \) and \( \mathcal{V}_2^0(n) = \text{span}_Z\{0\{\lambda\} \mid \lambda \in \mathbb{N}_0^n\} \). By \( \text{6.3(1)} \), \( \mathcal{V}_2^0(n) \) is a \( \mathbb{Z} \)-subalgebra of \( \widehat{\mathcal{K}}_Q(n) \). Also, we will see in \( \text{8.2} \) that \( \mathcal{V}_Z^+(n) \) and \( \mathcal{V}_Z^-(n) \) are \( \mathbb{Z} \)-subalgebras of \( \widehat{\mathcal{K}}_Q(n) \).

The maps \( \tau_r \) defined in \( \text{4.0.4} \) induce an algebra anti-automorphism

\[
\tau : \mathcal{K}_Z(n) \to \mathcal{K}_Z(n) \quad ([A]_1 \to [A]_1).
\]

Consequently, we get an algebra anti-automorphism

\[
(8.1.1) \quad \hat{\tau} : \widehat{\mathcal{K}}_Q(n) \to \widehat{\mathcal{K}}_Q(n)
\]
defined by sending \( \sum_A \beta_A[A]_1 \) to \( \sum_A \beta_A[A]_1 \). Clearly, \( \hat{\tau}(A\{\lambda\}) = (\hat{\lambda})\{A\} \) for \( A \in \Theta_+^+(n) \) and \( \lambda \in \mathbb{N}_0^n \). Thus,

\[
(8.1.2) \quad \mathcal{V}_Z^-(n) = \hat{\tau}(\mathcal{V}_Z^+(n)).
\]

**Lemma 8.2.** (1) \( \mathcal{V}_Z^+(n) \) (resp., \( \mathcal{V}_Z^-(n) \)) is a \( \mathbb{Z} \)-subalgebra of \( \widehat{\mathcal{K}}_Q(n) \) and the linear map \( \theta^+ : \mathcal{S}_h(n)_{\mathbb{Z}} \to \mathcal{V}_Z^+(n) \) (resp., \( \theta^- : \mathcal{S}_h(n)_{\mathbb{Z}} \to \mathcal{V}_Z^-(n) \)) taking \( u_{A,1} \to A\{0\} \) (resp., \( u_{A,1} \to (\hat{\lambda})\{A\} \)) for \( A \in \Theta_+^+(n) \) is an algebra isomorphism.

(2) \( \mathcal{V}_Z^+(n) \) (resp., \( \mathcal{V}_Z^-(n) \)) is generated by \( \sum_{\alpha \in \mathbb{N}_0^n} \alpha_i E_{i,i+1}^\Delta \{0\} \) (resp., \( \sum_{\alpha \in \mathbb{N}_0^n} \alpha_i E_{i+1,i}^\Delta \{0\} \)) for \( \alpha \in \mathbb{N}_0^n \) as a \( \mathbb{Z} \)-algebra.

**Proof.** Statement (2) follows from (1) and \( \text{2.1} \). We now prove (1). Let \( \mathcal{V}_Z^+(n) \) be the \( \mathbb{Z} \)-submodule of \( \prod_{r \geq 0} \mathcal{S}_h(n, r)_Q \) spanned by the elements \( \{A\{0, r\}\}_{r \geq 0} \) for \( A \in \Theta_+^+(n) \). Since the elements \( \{A\{0, r\}\}_{r \geq 0} \) are linearly independent, the map \( \eta_r^+ \) defined in \( \text{4.1} \) induce an injective algebra homomorphism

\[
\eta^+ = \prod_{r \geq 0} \eta_r^+ : \mathcal{S}_h(n)_{\mathbb{Z}} \to \prod_{r \geq 0} \mathcal{S}_h(n, r)_Q.
\]

Thus \( \mathcal{V}_Z^+(n) = \eta^+ (\mathcal{S}_h(n)_{\mathbb{Z}}) \) is a \( \mathbb{Z} \)-subalgebra of \( \prod_{r \geq 0} \mathcal{S}_h(n, r)_Q \) and the restriction of \( \eta^+ \) to \( \mathcal{S}_h(n)_{\mathbb{Z}} \) induces a \( \mathbb{Z} \)-algebra isomorphism

\[
(8.2.1) \quad \eta^+ : \mathcal{S}_h(n)_{\mathbb{Z}} \xrightarrow{\sim} \mathcal{V}_Z^+(n).
\]

On the other hand, the map \( \zeta_r \) defined in \( \text{6.3} \) induces a surjective algebra homomorphism

\[
(8.2.2) \quad \zeta_r : \widehat{\mathcal{K}}_Q(n) \to \mathcal{S}_h(n, r)_Q
\]
sending \( \sum_{A \in \mathcal{S}_h(n)} \beta_A[A]_1 \) to \( \sum_{A \in \mathcal{S}_h(n)} \beta_A \zeta_r ([A]_1) \). Consequently, we get an algebra homomorphism

\[
(8.2.3) \quad \zeta = \prod_{r \geq 0} \zeta_r : \widehat{\mathcal{K}}_Q(n) \to \prod_{r \geq 0} \mathcal{S}_h(n, r)_Q.
\]
Since $\hat{\zeta}(A\{0\}) = (A\{0, r\})_{r \geq 0}$ for $A \in \Theta_{\Delta}^+(n)$ and the elements $(A\{0, r\})_{r \geq 0}$ ($A \in \Theta_{\Delta}^+(n)$) are linearly independent, the restriction of $\hat{\zeta}$ to $V_{Z}^+(n)$ is injective and hence we get a bijective map

$$\hat{\zeta} : V_{Z}^+(n) \rightarrow \tilde{V}_{Z}^+(n).$$

This, together with (8.2.1), implies that $V_{Z}^+(n)$ is a subalgebra of $\hat{\mathcal{K}}_Q(n)$ and $\theta^+ = \hat{\zeta}^{-1} \circ \eta^+$ is an algebra isomorphism. Finally, using (8.1.2), we get the similar result for $V_{Z}^-(n)$. □

Recall the notation $A_a$ and $B_a$ introduced in (4.1.1). For $w = a_1a_2 \cdots a_m \in \tilde{\Sigma}$ with the tight form $b_1^{x_1}b_2^{x_2} \cdots b_t^{x_t}$ we let

$$n^+_w = A_{x_1b_1}\{0\}A_{x_2b_2}\{0\} \cdots A_{x_tb_t}\{0\} \in \hat{\mathcal{K}}_Q(n),$$

$$n^-_w = B_{x_1b_1}\{0\}B_{x_2b_2}\{0\} \cdots B_{x_tb_t}\{0\} \in \hat{\mathcal{K}}_Q(n).$$

The triangular relation for affine Schur algebras can be lifted to the level of $\hat{\mathcal{K}}_Q(n)$ as follows.

**Lemma 8.3.** Let $A \in \Theta_{\Delta}^\pm(n)$ and $\lambda \in \mathbb{N}_{\Delta}^n$.

1. We have

$$A^+\{0\}0\{\lambda\}A^-\{0\} = A\{\lambda\} + \sum_{j \in \mathbb{N}_{\Delta}^n \atop j < \lambda} \binom{\sigma(A)}{\lambda - j} A\{j\} + f$$

where $f$ is a $\mathbb{Z}$-linear combination of $B\{\nu\}$ such that $B \in \Theta_{\Delta}^\pm(n)$, $\lambda < A$ and $\nu \in \mathbb{N}_{\Delta}^n$.

2. There exist $w_{A^+}, w_{A^-} \in \tilde{\Sigma}$ such that $\varphi(w_{A^+}) = A^+$, $\varphi(w_{A^-}) = A^-$ and

$$n^+_{(w_{A^+})}\{\lambda\}n^-_{(w_{A^-})} = A\{\lambda\} + \sum_{j \in \mathbb{N}_{\Delta}^n \atop j < \lambda} \binom{\sigma(A)}{\lambda - j} A\{j\} + g$$

where $g$ is a $\mathbb{Z}$-linear combination of $B\{\nu\}$ such that $B \in \Theta_{\Delta}^\pm(n)$, $\lambda < A$ and $\nu \in \mathbb{N}_{\Delta}^n$.

**Proof.** By 4.3 and 6.1 for any $\mu \in \mathbb{Z}_{\Delta}^n$ we have

$$A^+\{0\}[\text{diag}(\mu)]A^-\{0\} = [A + \text{diag}(\mu - \sigma(A))] + f_\mu$$
where $f_\mu$ is a $\mathbb{Z}$-linear combination of $|B|$ such that $B < A$ and $\text{co}(B) = \text{co}(A) + \mu - \sigma(A)$ and $\text{ro}(B) = \text{ro}(A) + \mu - \sigma(A)$. This equality together with (7.2) implies that

$$A^+\{0\}0\{\lambda\}A^-\{0\} = \sum_{\mu \in \mathbb{Z}_n} \binom{\mu}{\lambda} \left( [A + \text{diag}(\mu - \sigma(A))] + f_\mu \right)$$

$$= \sum_{\nu \in \mathbb{Z}_n} \binom{\nu + \sigma(A)}{\lambda} [A + \text{diag}(\nu)] + f$$

$$= \sum_{\nu \in \mathbb{Z}_n} \sum_{j \in \mathbb{N}_\lambda} \binom{\nu}{j} \binom{\sigma(A)}{\lambda - j} [A + \text{diag}(\nu)] + f$$

$$= \sum_{j \in \mathbb{N}_\lambda} \binom{\sigma(A)}{\lambda - j} A\{j\} + f$$

where $f = \sum_{\mu \in \mathbb{Z}_n} \binom{\mu}{\lambda} f_\mu$. By (7.3) and (8.2), $f$ must be a $\mathbb{Z}$-linear combination of $B\{\nu\}$ for various $B \in \Theta^+(n)$ such that $B < A$ and various $\nu \in \mathbb{N}_\lambda$. This proves (1). The assertion (2) follows from (1), 2.1 and 8.2(1). \qed

**Corollary 8.4.** We have $\mathcal{V}_Z(n) = \mathcal{V}_Z^+(n)\mathcal{V}_Z^0(n)\mathcal{V}_Z^-(n) \cong \mathcal{V}_Z^+(n) \otimes \mathcal{V}_Z^0(n) \otimes \mathcal{V}_Z^-(n)$. 

**Proof.** Clearly, (8.3) implies that $\mathcal{V}_Z(n) = \mathcal{V}_Z^+(n)\mathcal{V}_Z^0(n)\mathcal{V}_Z^-(n)$. Furthermore, by 8.1 and 8.3, the set $\{A^+\{0\}0\{\lambda\}A^-\{0\} | A \in \Theta^+(n), \lambda \in \mathbb{N}_\lambda\}$ is linearly independent. The proof is completed. \qed

Now we can prove the main result of this section, which is the affine analogue of [1, 5.5].

**Proposition 8.5.** (1) $\mathcal{V}_Z(n)$ is a $\mathbb{Z}$-subalgebra of $\widehat{\mathcal{K}}_\mathbb{Q}(n)$. 

(2) The elements $\sum_{1 \leq i \leq n} \alpha_i E^\lambda_{i,i+1}\{0\}$, $\sum_{1 \leq i \leq n} \alpha_i E^\lambda_{i+1,i}\{0\}$, $0\{\lambda_i e_i^\lambda\}$ (for $\alpha \in \mathbb{N}_\lambda$, $\lambda_i \in \mathbb{N}$, $1 \leq i \leq n$) generate $\mathcal{V}_Z(n)$ as a $\mathbb{Z}$-algebra.

**Proof.** Let $\mathcal{V}_Z(n)_1$ be the $\mathbb{Z}$-subalgebra of $\widehat{\mathcal{K}}_\mathbb{Q}(n)$ generated by the elements indicated in (2). From (7.3) we see that $\mathcal{V}_Z(n)_1 \subseteq \mathcal{V}_Z(n)_1 \mathcal{V}_Z(n) \subseteq \mathcal{V}_Z(n)$. So it is enough to prove that $A\{\lambda\} \in \mathcal{V}_Z(n)_1$ for all $A \in \Theta^+(n)$ and $\lambda \in \mathbb{N}_\lambda$. We shall prove this by induction on $|A|$. If $|A| = 0$, then $A = 0$ and $0\{\lambda\} = 0\{\lambda_1 e_1^\lambda\} \cdots 0\{\lambda_n e_n^\lambda\} \in V_Z(n)_1$.

Now we assume that $|A| > 0$ and our statement is true for $A'$ with $|A'| < |A|$. By (8.3), there exist $w_{A'}$, $w_{A^-} \in \widehat{\mathcal{S}}$ such that

$$\mathfrak{n}^+_{(w_{A'})} \mathfrak{n}^-_{(w_{A^-})} = A\{0\} + g$$

where $g$ is a $\mathbb{Z}$-linear combination of $B\{\nu\}$ with $B \in \Theta^+(n)$, $B < A$ and $\nu \in \mathbb{N}_\lambda$. Since, by [5, 3.7.6], $B < A$ implies that $\|B\| < \|A\|$, we have by the induction hypothesis $g \in \mathcal{V}_Z(n)_1$. 


Consequently, $A\{0\} \in \mathcal{V}_{\mathbb{Z}}(n)_1$. Furthermore, by \[\ref{thm:indep}(1),
\]
\[(8.5.1) \quad 0\{\lambda\}A\{0\} = A\{\lambda\} + \sum_{\delta < \lambda} \left( \ro(A) \right) \lambda - \delta A\{\delta\} = A\{\lambda\} + \sum_{\delta < \sigma(\lambda)} \left( \ro(A) \right) \lambda - \delta A\{\delta\}.\]

Thus, using induction on $\sigma(\lambda)$, we see that $A\{\lambda\} \in \mathcal{V}_{\mathbb{Z}}(n)_1$ for $\lambda \in \mathbb{N}_0^n$. This finishes the proof. \hfill $\square$

Let $\mathcal{V}_Q(n) = \text{span}_Q \mathfrak{B}$. Then, by \[\ref{subalg}(1), \] $\mathcal{V}_Q(n)$ is a $Q$-subalgebra of $\hat{\mathcal{K}}_Q(n)$. We will prove that $\mathcal{V}_Q(n)$ is isomorphic to $\mathcal{U}((\hat{\mathfrak{g}}_n))$. By \[\ref{thm:indep} \] the set $\mathfrak{B}$ forms a $Q$-basis for $\mathcal{V}_Q(n)$. We end this section with the construction of another $Q$-basis for $\mathcal{V}_Q(n)$. For $A \in \Theta_\alpha^+(n)$ and $j \in \mathbb{N}_0^n$, define (cf. \[\ref{subalg} \])
\[\begin{align*}
A[j] &= \sum_{\lambda \in \mathbb{N}_0^n} \lambda^j[A + \text{diag}(\lambda)]_1 \in \hat{\mathcal{K}}_Q(n) \\
A[j, r] &= \sum_{\lambda \in \Lambda_\Theta(n, r - \sigma(A))} \lambda^j[A + \text{diag}(\lambda)]_1 \in \mathcal{S}_\Theta(n, r)_\mathbb{Z},
\end{align*}\]
where $\lambda^j = \prod_{1 \leq i \leq n} \lambda_i^j$. Note that, by definition, $A[0] = A\{0\} = 0$ and $0[e_i^\lambda] = 0$ for $A \in \Theta_\Theta^+(n)$ and $i \in \mathbb{Z}$. Clearly, the following multiplication formula follows immediately from the definition.

**Lemma 8.6.** For $A \in \Theta_\alpha^+(n)$ and $j, j' \in \mathbb{N}_0^n$, we have
\[0[j']A[j] = \sum_{\alpha \in \mathbb{N}_0^n, \alpha \subseteq j'} \binom{j'}{\alpha} (\ro(A))^{j' - \alpha} A[\alpha + j].\]

In particular, we have $0[j']0[j] = 0[j + j']$.

**Lemma 8.7.** Let $\mathcal{V}_Q^0(n)$ be the $Q$-subspace of $\mathcal{V}_Q(n)$ spanned by the elements $0\{\lambda\}$ for $\lambda \in \mathbb{N}_0^n$. Then the set $\{0[j] \mid j \in \mathbb{N}_0^n\}$ forms a $Q$-basis for $\mathcal{V}_Q^0(n)$.

**Proof.** Let $\mathcal{V}_Q^0(n)_1 = \text{span}_Q \{0[j] \mid j \in \mathbb{N}_0^n\}$. By \[\ref{subalg}(1), \] it is enough to prove that $\mathcal{V}_Q^0(n) = \mathcal{V}_Q^0(n)_1$. Since, by \[\ref{thm:indep}(1), \] $\mathcal{V}_Q^0(n)$ is a $Q$-subalgebra of $\hat{\mathcal{K}}_Q(n)$, we have $0[j] = 0[e_1^\lambda]^{j_1} \cdots 0[e_n^\lambda]^{j_n} = 0\{e_1^{\lambda_1} \cdots e_n^{\lambda_n}\}^{j_1} \cdots {j_n} \in \mathcal{V}_Q^0(n)$ for $j \in \mathbb{N}_0^n$. Furthermore, by \[\ref{subalg} \] $\mathcal{V}_Q^0(n)_1$ is a $Q$-subalgebra of $\hat{\mathcal{K}}_Q(n)$. This implies that
\[0\{j\} = \prod_{1 \leq i \leq n} 0\{e_i^{\lambda_i}\} = \prod_{1 \leq i \leq n} \frac{0[e_i^{\lambda_i}][0[e_i^{\lambda_i}] - 1] \cdots [0[e_i^{\lambda_i}] - j_i + 1]}{j_i!} \in \mathcal{V}_Q^0(n)_1.\]
Consequently, $\mathcal{V}_Q^0(n) = \mathcal{V}_Q^0(n)_1$. \hfill $\square$

**Proposition 8.8.** The set $\mathcal{C} := \{A[j] \mid A \in \Theta_\alpha^+(n), j \in \mathbb{Z}_0^n\}$ forms a $Q$-basis for $\mathcal{V}_Q(n)$.

**Proof.** For $A \in \Theta_\alpha^+(n)$, let $\mathcal{V}_A = \text{span}_Q \{A[\lambda] \mid \lambda \in \mathbb{N}_0^n\}$ and $\mathcal{W}_A = \text{span}_Q \{A[\lambda] \mid \lambda \in \mathbb{N}_0^n\}$. Since, by \[\ref{subalg}(1), \] the set $\mathcal{C}$ is linearly independent, it is enough to prove that $\mathcal{V}_A = \mathcal{W}_A$ for each $A$. Fix $A \in \Theta_\alpha^+(n)$. We now prove $A[\lambda] \in \mathcal{W}_A$ and $A[\lambda] \in \mathcal{V}_A$ by induction on $\sigma(\lambda)$. If
\(\sigma(\lambda) = 0\) then \(A\{0\} = A[0] = 0\) for all \(\lambda\). Now we assume \(\sigma(\lambda) > 1\) and \(A\{\mu\} \in \mathcal{W}_A, A[\mu] \in \mathcal{V}_A\) for \(\mu \in \mathbb{N}^n_\lambda\) with \(\sigma(\mu) < \sigma(\lambda)\). By \(8.6\) and \(8.7\) we have \(0\{\lambda\}A\{0\} = 0\{\lambda\}A[0] \in \mathcal{W}_A\). This, together with \(8.5.1\) and the induction hypothesis, implies that \(A\{\lambda\} \in \mathcal{W}_A\). On the other hand, by \(8.6\) we have

\[
A[\lambda] = 0[\lambda]A[0] - \sum_{\alpha \in \Lambda^\vee_{\lambda} : \alpha \in \lambda} \left(\frac{\lambda}{\alpha}\right) \text{ro}(A)^{\lambda - \alpha} A[\alpha].
\]

Furthermore, by \(7.3(1)\) and \(8.7\) we have \(0[\lambda]A[0] = 0[\lambda]A\{0\} \in \mathcal{V}_A\). Thus, by the induction hypothesis, \(A[\lambda] \in \mathcal{V}_A\). Consequently, \(\mathcal{V}_A = \mathcal{W}_A\) for all \(A \in \Theta^\pm_\lambda(n)\). This finishes the proof. □

9. Realization of \(U_\mathbb{Z}(\widehat{\mathfrak{g}}_n)\) and affine Schur-Weyl duality

In this section, we will prove that \(\mathcal{V}_\mathbb{Z}(n)\) is the realization of \(U_\mathbb{Z}(\widehat{\mathfrak{g}}_n)\) and use it to prove that the natural surjective algebra homomorphism \(\xi_r : U(\widehat{\mathfrak{g}}_n) \to S_n(n, r)_\mathbb{Q}\) remains surjective at the integral level.

By \([29]\) and \([5, 6.1.3]\), there is a unique surjective algebra homomorphism

\[
(9.0.1) \quad \xi_r : U(\widehat{\mathfrak{g}}_n) \to S_n(n, r)_{\mathbb{Q}}
\]
such that \(\xi_r(E_{i,j}^\lambda) = E_{i,j}^\lambda[0, r]\) and \(\xi_r(E_{i,i}^\lambda) = 0[e_i^\lambda, r]\) for \(i \neq j\). We will see that the maps \(\xi_r\) induce an algebra isomorphism \(\xi\) from \(U(\widehat{\mathfrak{g}}_n)\) to \(\mathcal{V}_\mathbb{Z}(n)\) such that \(\xi(U(\widehat{\mathfrak{g}}_n)) = \mathcal{V}_\mathbb{Z}(n)\).

Lemma 9.1. There is a unique algebra homomorphism

\[
\xi : U(\widehat{\mathfrak{g}}_n) \to \mathcal{V}_\mathbb{Q}(n)
\]
such that \(\xi(E_{i,j}^\lambda) = E_{i,j}^\lambda[0]\) and \(\xi(E_{i,i}^\lambda) = 0[e_i^\lambda]\) for \(i \neq j\).

Proof. Note that \(U(\widehat{\mathfrak{g}}_n)\) has a presentation with generators \(E_{i,j}^\lambda\) (1 ≤ \(i \leq n, j \in \mathbb{Z}\)), subject to the following relations:

(a) \([E_{i,i}^\lambda, E_{k,l}^\lambda] = (\delta_{i,k} - \delta_{i,l})E_{k,l}^\lambda\).

(b) \([E_{i,j}^\lambda, E_{k,l}^\lambda] = \delta_{j,k}E_{i+l,j-k}^\lambda - \delta_{k,i}E_{k+j+l-i}^\lambda\) for \(i \neq j\) and \(k \neq l\).

Thus it is enough to prove that

(R1) \(0[e_i^\lambda][0][e_k^\lambda] = 0[e_k^\lambda][0][e_i^\lambda]\) for all \(i, k\);

(R2) \(0[e_i^\lambda][E_{k,l}^\lambda] = E_{k,l}^\lambda[0][0][e_i^\lambda] = (\delta_{i,k} - \delta_{i,l})E_{k,l}^\lambda[0]\) for \(k \neq l\);

(R3) \(E_{i,j}^\lambda[0][E_{k,l}^\lambda] - E_{k,l}^\lambda[0][E_{i,j}^\lambda] = \delta_{j,k}E_{i+l,j-k}^\lambda[0] - \delta_{j,l}E_{k+j+l-i}^\lambda[0]\) for \(i \neq j\) and \(k \neq l\).

For \(i, k \in \mathbb{Z}\), we have

\(0[e_i^\lambda][0][e_k^\lambda] = 0[0][e_i^\lambda + e_k^\lambda] = 0[e_k^\lambda][0][e_i^\lambda]\).
proving (R1). By definition, for \(i \in \mathbb{Z} \) and \(k \neq l \in \mathbb{Z} \), we have
\[
0[e_i^\lambda]E_{k,l}^\lambda[0] - E_{k,l}^\lambda[0]0[e_i^\lambda] = \sum_{\mu \in \mathbb{Z}^n} (\mu_i + \delta_{i,k})[E_{k,l}^\lambda + \text{diag}(\mu)] - \sum_{\mu \in \mathbb{Z}^n} (\mu_i + \delta_{i,l})[E_{k,l}^\lambda + \text{diag}(\mu)]
\]
\[
= (\delta_{i,k} - \delta_{i,l})E_{k,l}^\lambda[0],
\]
proving (R2).

It remains to prove (R3). Assume \(i \neq j \) and \(k \neq l \). Applying \(\xi_r \) to (b) yields
\[
E_{i,j}^\lambda[0, r]E_{k,l}^\lambda[0, r] - E_{k,l}^\lambda[0, r]E_{i,j}^\lambda[0, r] = \delta_{j,k}E_{i,j}^\lambda[0, r] - \delta_{i,l}E_{k,j}^\lambda[0, r].
\]
Multiplying on both sides by \([\text{diag}(\lambda)]_1 \) (\(\lambda \in \Lambda_\mathbb{Z}(n, r) \) and \(\lambda \geq e_1^\lambda + e_k^\lambda \)) gives the following formula in \(S_\mathbb{Z}(n, r)Q\):
\[
[E_{i,j}^\lambda + \text{diag}(\lambda^{(1)})]_1[E_{k,l}^\lambda + \text{diag}(\lambda^{(2)})]_1 - [E_{k,l}^\lambda + \text{diag}(\lambda^{(3)})]_1[E_{i,j}^\lambda + \text{diag}(\lambda^{(4)})]_1
\]
\[
= \delta_{j,k}[E_{i,j}^\lambda + \text{diag}(\lambda^{(1)})]_1 - \delta_{i,l}[E_{k,l}^\lambda + \text{diag}(\lambda^{(3)})]_1,
\]
where \(\lambda^{(1)} = \lambda - e_i^\lambda, \lambda^{(2)} = \lambda - e_j^\lambda - e_k^\lambda, \lambda^{(3)} = \lambda - e_k^\lambda, \lambda^{(4)} = \lambda - e_i^\lambda + e_k^\lambda - e_j^\lambda. \) Thus by \(\xi_r \) and the definition of \(K_\mathbb{Z}(n)\), for any \(\lambda \in \mathbb{Z}^n\), we have in \(K_\mathbb{Q}(n)\),
\[
[\text{diag}(\lambda)]_1(E_{i,j}^\lambda[0]E_{k,l}^\lambda[0] - E_{k,l}^\lambda[0]E_{i,j}^\lambda[0]) = [E_{i,j}^\lambda + \text{diag}(\lambda^{(1)})]_1[E_{k,l}^\lambda + \text{diag}(\lambda^{(2)})]_1
\]
\[
- [E_{k,l}^\lambda + \text{diag}(\lambda^{(3)})]_1[E_{i,j}^\lambda + \text{diag}(\lambda^{(4)})]_1
\]
\[
= \delta_{j,k}[E_{i,j}^\lambda + \text{diag}(\lambda^{(1)})]_1 - \delta_{i,l}[E_{k,l}^\lambda + \text{diag}(\lambda^{(3)})]_1
\]
\[
= [\text{diag}(\lambda)]_1(\delta_{j,k}E_{i,j}^\lambda[0] - \delta_{i,l}E_{k,j}^\lambda[0]).
\]
This implies that,
\[
E_{i,j}^\lambda[0]E_{k,l}^\lambda[0] - E_{k,l}^\lambda[0]E_{i,j}^\lambda[0] = \sum_{\lambda \in \mathbb{Z}^n} [\text{diag}(\lambda)]_1(E_{i,j}^\lambda[0]E_{k,l}^\lambda[0] - E_{k,l}^\lambda[0]E_{i,j}^\lambda[0])
\]
\[
= \sum_{\lambda \in \mathbb{Z}^n} [\text{diag}(\lambda)]_1(\delta_{j,k}E_{i,j}^\lambda[0] - \delta_{i,l}E_{k,j}^\lambda[0])
\]
\[
= \delta_{j,k}E_{i,j}^\lambda[0] - \delta_{i,l}E_{k,j}^\lambda[0],
\]
proving (R3). 

We can now prove that \(\mathcal{V}_\mathbb{Z}(n)\) gives a BLM realization of \(\mathcal{U}_\mathbb{Z}(\widehat{\mathfrak{gl}}_n)\), which is the affine version of \([1] \ 5.7\).

**Theorem 9.2.** (1) The algebra homomorphism \(\xi : \mathcal{U}(\widehat{\mathfrak{gl}}_n) \to \mathcal{V}_\mathbb{Q}(n)\) defined in \([9.7\) is an algebra isomorphism.

(2) \(\mathcal{U}_\mathbb{Z}(\widehat{\mathfrak{gl}}_n)\) is a \(\mathbb{Z}\)-subalgebra of \(\mathcal{U}(\widehat{\mathfrak{gl}}_n)\) and the restriction of \(\xi\) to \(\mathcal{U}_\mathbb{Z}(\widehat{\mathfrak{gl}}_n)\) gives a \(\mathbb{Z}\)-algebra isomorphism \(\xi : \mathcal{U}_\mathbb{Z}(\widehat{\mathfrak{gl}}_n) \to \mathcal{V}_\mathbb{Z}(n)\).
Proof. We first prove (1). Let \( \mathcal{L}^+ = \{(i, j) \mid 1 \leq i \leq n, j \in \mathbb{Z}, i < j \} \) and \( \mathcal{L}^- = \{(i, j) \mid 1 \leq i \leq n, j \in \mathbb{Z}, i > j \} \). By 3.1(1), 8.2(1), and 8.1.2, the set
\[
\left\{ \prod_{(i, j) \in \mathcal{L}^+} (E_{i,j}^\delta [0])^{a_{i,j}} \mid A \in \Theta_\delta^+(n) \right\} \left( \text{resp.,} \left\{ \prod_{(i, j) \in \mathcal{L}^-} (E_{i,j}^\delta [0])^{a_{i,j}} \mid A \in \Theta_\delta^-(n) \right\} \right)
\]
forms a \( \mathbb{Q} \)-basis for \( \mathcal{V}_Q^+(n) \) (resp., \( \mathcal{V}_Q^-(n) \)), where the products are taken with respect to a fixed total order on \( \mathcal{L}^+ \) (resp., \( \mathcal{L}^- \)). This, together with 3.3 and 8.7, implies that the set
\[
\left\{ \prod_{(i, j) \in \mathcal{L}^+} (E_{i,j}^\delta [0])^{a_{i,j}} 0[j] \prod_{(i, j) \in \mathcal{L}^-} (E_{i,j}^\delta [0])^{a_{i,j}} \mid A \in \Theta_\delta^+(n), j \in \mathbb{N}_n \right\}
\]
forms a \( \mathbb{Q} \)-basis for \( \mathcal{V}_Q(n) \). Thus \( \xi \) sends a PBW-basis of \( \mathcal{U}(\widehat{\mathfrak{gl}}_n) \) to a basis of \( \mathcal{V}_Q(n) \). Consequently, \( \xi \) is an algebra isomorphism.

To see (2), by (1) and 3.3(1), it is enough to prove \( \xi(\mathcal{U}_Z(\widehat{\mathfrak{gl}}_n)) = \mathcal{V}_Z(n) \). Recall the algebra homomorphism \( \iota^+ : \mathfrak{so}_\lambda(n)_\mathbb{Q} \to \mathcal{U}(\widehat{\mathfrak{gl}}_n) \) defined in 3.1 and the algebra isomorphism \( \theta^+ : \mathfrak{so}_\lambda(n)_\mathbb{Z} \to \mathcal{V}_Z^+(n) \) described in 8.2. The map \( \theta^+ \) induces an injective algebra isomorphism \( \mathfrak{so}_\lambda(n)_\mathbb{Q} \to \mathcal{V}_Q(n) \), which is also denoted by \( \theta^+ \). Since, by 3.1 \( \mathfrak{so}_\lambda(n)_\mathbb{Q} \) is generated by \( E_{i,j}^\delta \) for \( i < j \), and since
\[
\xi \circ \iota^+(u_{E_{i,j}^\delta}) = \xi(E_{i,j}^\delta) = E_{i,j}^\delta [0] = \theta^+(u_{E_{i,j}^\delta,1})
\]
for \( i < j \), we conclude that \( \xi \circ \iota^+ = \theta^+ \). So \( \mathcal{V}_Z^+(n) = \theta^+(\mathfrak{so}_\lambda(n)_\mathbb{Z}) = \xi \circ \iota^+(\mathfrak{so}_\lambda(n)_\mathbb{Z}) = \xi(\mathcal{U}_Z^+(\widehat{\mathfrak{gl}}_n)) \).

Similarly we have \( \mathcal{V}_Z^-(n) = \xi(\mathcal{U}_Z^-(\widehat{\mathfrak{gl}}_n)) \). Furthermore, since
\[
\xi \left( \prod_{1 \leq i \leq n} \left( \frac{E_{i,i}^\Delta}{\lambda_i} \right) \right) = 0\{\lambda\},
\]
we have \( \mathcal{V}_Z^0(n) = \xi(\mathcal{U}_Z^0(\widehat{\mathfrak{gl}}_n)) \). Thus, by 8.4 we conclude that
\[
\mathcal{V}_Z(n) = \mathcal{V}_Z^+(n) \mathcal{V}_Z^0(n) \mathcal{V}_Z^-(n) = \xi(\mathcal{U}_Z^+(\widehat{\mathfrak{gl}}_n) \mathcal{U}_Z^0(\widehat{\mathfrak{gl}}_n) \mathcal{U}_Z^-(\widehat{\mathfrak{gl}}_n)) = \xi(\mathcal{U}_Z(\widehat{\mathfrak{gl}}_n)),
\]
proving (2). \( \square \)

Corollary 9.3. \( \mathcal{U}_Z(\widehat{\mathfrak{gl}}_n) \) is a \( \mathbb{Z} \)-Hopf subalgebra of \( \mathcal{U}(\widehat{\mathfrak{gl}}_n) \) with comultiplication given by
\[
\Delta(\iota^+(u_{\lambda,1})) = \sum_{\lambda = \lambda(1)+\lambda(2)} \iota^+(u_{\lambda(1)}) \otimes \iota^+(u_{\lambda(2)})
\]
\[
\Delta(\iota^- (u_{\lambda,1})) = \sum_{\lambda = \lambda(1)+\lambda(2)} \iota^- (u_{\lambda(1)}) \otimes \iota^- (u_{\lambda(2)})
\]
\[
\Delta \left( \left( \frac{E_{i,i}^\delta}{t} \right) \right) = \sum_{0 \leq j \leq t} \left( \frac{E_{i,j}^\delta}{j} \right) \otimes \left( \frac{E_{i,i}^\delta}{t-j} \right)
\]
for \( \lambda \in \mathbb{N}_n^n, 1 \leq i \leq n \) and \( t \in \mathbb{N} \), where \( u_{\lambda,1} = u_{[S\lambda]} \otimes 1 \).
Proof. By \(2.3.4\), \(2.3.2\), \(2.3.3\), \(3.2.1\) and \(3.2\), \(U_\mathbb{Z}^+(\hat{\mathfrak{gl}}_n)\) and \(U_\mathbb{Z}^0(\hat{\mathfrak{gl}}_n)\) are \(\mathbb{Z}\)-Hopf subalgebras of \(U(\hat{\mathfrak{gl}}_n)\). Clearly, there is a natural algebra anti-isomorphism
\[
\Phi : U(\hat{\mathfrak{gl}}_n) \rightarrow U(\hat{\mathfrak{gl}}_n)(E_{i,j}^\lambda \mapsto E_{j,i}^\lambda \forall i, j).
\]
Since \(\Phi(U_\mathbb{Z}^+(\hat{\mathfrak{gl}}_n)) = U_\mathbb{Z}^+(\hat{\mathfrak{gl}}_n)\), and \(\Phi\) preserves comultiplication and antipode, \(U_\mathbb{Z}^-(\hat{\mathfrak{gl}}_n)\) is also a \(\mathbb{Z}\)-Hopf subalgebra of \(U(\hat{\mathfrak{gl}}_n)\). The proof is completed. \(\square\)

Finally, we will prove that \(\xi_r : U_\mathbb{Z}(\hat{\mathfrak{gl}}_n) \rightarrow S_\mathbb{Z}(n, r)_\mathbb{Z}\) is surjective. By restriction, the algebra homomorphism \(\zeta_r : \hat{\mathcal{K}}_Q(n) \rightarrow S_\mathbb{Q}(n, r)_\mathbb{Q}\) defined in \(\ref{9.2.2}\) induces an algebra homomorphism
\[
(9.3.1)\quad \zeta_r := \zeta_r|_{\mathcal{V}_Q(n)} : \mathcal{V}_Q(n) \rightarrow S_\mathbb{Q}(n, r)_\mathbb{Q}.
\]
Clearly, \(\zeta_r(A\{\lambda\}) = A\{\lambda, r\}\) for \(A \in \Theta_\mathbb{Q}^+(n)\) and \(\lambda \in \mathbb{N}_0^n\).

**Lemma 9.4.** The algebra homomorphism
\[
\zeta_r : \mathcal{V}_Q(n) \rightarrow S_\mathbb{Q}(n, r)_\mathbb{Q}
\]
is surjective and we have \(\zeta_r(\mathcal{V}_\mathbb{Z}(n)) = S_\mathbb{Z}(n, r)_\mathbb{Z}\).

**Proof.** Since \(0\{\lambda, r\} = [\text{diag}(\lambda)]\) for \(\lambda \in \Lambda_\mathbb{Z}(n, r)\), we have
\[
\zeta_r(\mathcal{V}_\mathbb{Z}(n)) = \text{span}_\mathbb{Z}\{0\{\lambda, r\} | \lambda \in \mathbb{N}_0^n\} = \text{span}_\mathbb{Z}\{[\text{diag}(\lambda)] | \lambda \in \Lambda_\mathbb{Z}(n, r)\}.
\]
Thus, by \(\ref{8.4}\) and \(\ref{4.3}\)
\[
\zeta_r(\mathcal{V}_\mathbb{Z}(n)) = \text{span}_\mathbb{Z}\{A^+\{0, r\}0\{\lambda, r\}A^-\{0, r\} | A \in \Theta_\mathbb{Z}^+(n), \lambda \in \mathbb{N}_0^n\}
\]
\[= \text{span}_\mathbb{Z}\{A^+\{0, r\}[\text{diag}(\lambda)]A^-\{0, r\} | A \in \Theta_\mathbb{Z}^+(n), \lambda \in \Lambda_\mathbb{Z}(n, r)\}
\]
\[= S_\mathbb{Z}(n, r)_\mathbb{Z},
\]
proving the assertion. \(\square\)

**Theorem 9.5.** The restriction of \(\xi_r\) to \(U_\mathbb{Z}(\hat{\mathfrak{gl}}_n)\) gives a surjective \(\mathbb{Z}\)-algebra homomorphism
\[
\xi_r : U_\mathbb{Z}(\hat{\mathfrak{gl}}_n) \rightarrow S_\mathbb{Z}(n, r)_\mathbb{Z}.
\]

**Proof.** Clearly, \(\zeta_r \circ \xi(E_{i,j}^\lambda) = \xi_r(E_{i,j}^\lambda|0) = E_{j,i}^\lambda|0, r\) for any \(i \neq j \in \mathbb{Z}\) and \(\zeta_r \circ \xi(E_{i,i}^\lambda) = \xi_r(0|e_i^\lambda) = 0|e_i^\lambda, r\) = \(\xi_r(E_{i,i}^\lambda)\). Thus we have the following commutative diagram:

\[
\begin{array}{ccc}
U(\hat{\mathfrak{gl}}_n) & \xrightarrow{\xi} & \mathcal{V}_Q(n) \\
\downarrow{\xi_r} & & \downarrow{\zeta_r} \\
S_\mathbb{Z}(n, r)_\mathbb{Q} & & \\
\end{array}
\]

It follows from \(\ref{9.2}\) and \(\ref{9.4}\) that \(\xi_r(U_\mathbb{Z}(\hat{\mathfrak{gl}}_n)) = \zeta_r \circ \xi(U_\mathbb{Z}(\hat{\mathfrak{gl}}_n)) = \zeta_r(\mathcal{V}_\mathbb{Z}(n)) = S_\mathbb{Z}(n, r)_\mathbb{Z} .\square\)

Let \(k\) be a field. We denote \(U_k(\hat{\mathfrak{gl}}_n) = U_\mathbb{Z}(\hat{\mathfrak{gl}}_n) \otimes k\) and \(S_\mathbb{Z}(n, r)_k = S_\mathbb{Z}(n, r)_\mathbb{Z} \otimes k .\)
Corollary 9.6. For any field $k$, the algebra homomorphism

$$\xi_r \otimes \text{id} : \mathcal{U}_k(\widehat{\mathfrak{gl}_n}) \to \mathcal{S}_\delta(n,r)_k \cong \text{End}_k \mathcal{S}_{\delta,r} \left( \bigoplus_{\lambda \in \Lambda_\delta(n,r)} k \mathcal{S}_\lambda \mathcal{S}_{\lambda,r} \right)$$

is surjective.

REFERENCES

[1] A. A. Beilinson, G. Lusztig and R. MacPherson, A geometric setting for the quantum deformation of $GL_n$, Duke Math. J. 61 (1990), 655–677.
[2] R. Carter and G. Lusztig, On the modular representations of the general linear and symmetric groups, Math. Zeit. 136 (1974), 193–242.
[3] C. de Concini and C. Procesi, A characteristic free approach to invariant theory, Adv. Math. 21 (1976), 330–354.
[4] B. Deng and J. Du Monomial bases for quantum affine $\mathfrak{sl}_n$, Adv. Math. 191 (2005), 276–304.
[5] B.B. Deng, J. Du and Q. Fu, A double Hall algebra approach to affine quantum Schur–Weyl theory, preprint, ArXiv: 1010.4619.
[6] B. Deng, J. Du and J. Xiao, Generic extensions and canonical bases for cyclic quivers, Can. J. Math. 59 (2007), 1260–1283.
[7] S. Donkin, Invariants of several matrices, Invent. Math. 110 (1992), 389–401.
[8] J. Du, A note on the quantized Weyl reciprocity at roots of unity, Alg. Colloq. 2 (1995), 363–372.
[9] J. Du, B. Parshall and L. Scott, Quantum Weyl reciprocity and tilting modules, Commun. Math. Phys. 195 (1998), 321–352.
[10] J. Du and Q. Fu, A modified BLM approach to quantum affine $\mathfrak{sl}_n$, Math. Z. 266 (2010), 747–781.
[11] Q. Fu On Schur algebras and little Schur algebras, J. Algebra 322 (2009), 1637–1652.
[12] V. Ginzburg and E. Vasserot, Langlands reciprocity for affine quantum groups of type $A_n$, Internat. Math. Res. Notices 1993, 67–85.
[13] R. M. Green, The affine $q$-Schur algebra, J. Algebra 215 (1999), 379–411.
[14] J. A. Green, Hall algebras, hereditary algebras and quantum groups, Invent. Math. 120 (1995), 361–377.
[15] J. Y. Guo, The Hall polynomials of a cyclic serial algebra, Comm. Algebra 23 (1995), 743–751.
[16] M. Jimbo, A $q$-analogue of $U(\mathfrak{gl}(N + 1))$, Hecke algebras, and the Yang–Baxter equation, Lett. Math. Phys. 11(1986), 247–252.
[17] G. Lusztig Some examples of square integrable representations of semisimple $p$-adic groups Trans. Amer. Math. Soc. 277 (1983), 623–653.
[18] G. Lusztig, Modular representations and quantum groups, Contemp. Math. 82 (1989), 59–77.
[19] G. Lusztig, Finite dimensional Hopf algebras arising from quantized universal enveloping algebras, J. Amer. Math. Soc. 3 (1990), 257–296.
[20] G. Lusztig, Aperiodicity in quantum affine $\mathfrak{gl}_n$, Asian J. Math. 3 (1999), 147–177.
[21] M. Reineke, Generic extensions and multiplicative bases of quantum groups at $q = 0$, Represent. Theory 5 (2001), 147–163.
[22] C. M. Ringel, Hall algebras and quantum groups, Invent. Math. 101 (1990), 583–592.
[23] C. M. Ringel, Hall algebras revisited, Israel Mathematical Conference Proceedings, Vol. 7 (1993), 171–176.
[24] C. M. Ringel, The composition algebra of a cyclic quiver, Proc. London Math. Soc. 66 (1993), 507–537.
[25] M. Takeuchi, Some topics on $GL_q(n)$, J. Algebra 147 (1992), 379–410.
[26] M. Varagnolo and E. Vasserot, On the decomposition matrices of the quantized Schur algebra, Duke Math. J. 100 (1999), 267–297.
[27] H. Weyl, The classical groups, Princeton U. Press, Princeton, 1946.
[28] J. Xiao, Drinfeld double and Ringel–Green theory of Hall algebras, J. Algebra 190 (1997), 100–144.
[29] D. Yang, On the affine Schur algebra of type A, Comm. Algebra 37 (2009), 1389–1419.

Department of Mathematics, Tongji University, Shanghai, 200092, China.

E-mail address: q_fu@hotmail.com