ESSENTIAL OPEN BOOK FOLIATION AND FRACTIONAL DEHN TWIST COEFFICIENT

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Abstract. We introduce an essential open book foliation, a refinement of the open book foliation, and develop technical estimates of the fractional Dehn twist coefficient (FDTC) of monodromies and the FDTC for closed braids, which we introduce as well.

As applications, we quantitatively study the ‘gap’ of overtwisted contact structures and a non-right-veering monodromies. We give sufficient conditions for a 3-manifold to be irreducible and atoroidal. We also show that the geometries of a 3-manifold and the complement of a closed braid are determined by the Nielsen-Thurston types of the monodromies of their open book decompositions.

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1. Introduction

We have introduced the open book foliation in [25]. Let $S = S_{g,d}$ be a compact oriented genus $g$ surface with $d \neq 0$ boundary components. Let $\text{Aut}(S, \partial S)$ be the group...
of isotopy classes of diffeomorphisms of \( S \) fixing the boundary \( \partial S \) pointwise. Abusing notation we will often regard \( \phi \in \text{Aut}(S, \partial S) \) as a diffeomorphism representing \( \phi \).

Let \((S, \phi)\) be an open book decomposition of a closed oriented 3-manifold \( M \), where \( \phi \in \text{Aut}(S, \partial S) \). Consider a compact oriented surface \( F \) in \( M \) possibly with boundary. Generic intersection of \( F \) and the pages of the open book yields a singular foliation on \( F \). If it satisfies certain conditions we call it open book foliation and denote it by \( \mathcal{F}_{\text{ob}}(F) \). The open book foliation has its origin in Bennequin’s work \cite{bennequin} and Birman and Menasco’s braid foliation \cite{birman-menasco1, birman-menasco2, birman-menasco3, birman-menasco4, birman-menasco5, birman-menasco6, birman-menasco7, birman-menasco8}, where the underlying manifold \( M \cong S^3 \).

Let us recall Giroux’s seminal result \cite{giroux}:

For a closed oriented 3-manifold \( M \) there exists a one-to-one correspondence between the open book decompositions of \( M \) up to positive stabilization and the contact structures on \( M \) up to contact isotopy. Due to the Giroux-correspondence the open book foliation has extensive application to contact geometry. For instance, in \cite{ito-kawamura} we have reproved Honda, Kazez and Matić’s theorem \cite{honda-kazez-matic}:

The contact structure \((M, \xi)\) is tight if and only if every open book supporting \((M, \xi)\) is right-veering.

Now the notion “right-veering” is a key to tight contact structures. For every boundary component \( C \subset \partial S \) and every essential arc \( \gamma \subset S \) starting on \( C \) if \( \phi(\gamma) \) lies on the right of \( \gamma \) near the starting point, we say that \( \phi \) is right-veering. As a highly effective tool to detect right-veering-ness Honda, Kazez and Matić \cite{honda-kazez-matic} introduced the fractional Dehn twist coefficient. It measures how much \( \phi \in \text{Aut}(S, \partial S) \) contributes twisting along a boundary component \( C \) and is denoted by \( c(\phi, C) \in \mathbb{Q} \). They show that positivity of \( c(\phi, C) \) is almost equivalent to the right-veering-ness of \( \phi \).

In the present paper we introduce the (strongly) essential open book foliation and establish relationship between the fractional Dehn twist coefficient and the essential open book foliation. As applications we obtain results in contact 3-manifolds and topology and geometry of 3-manifolds.

In the previous paper \cite{ito-kawamura} we start from an open book \((S, \phi)\) with certain properties (eg. non-right-veeringness) and construct surfaces \( F \) in \( M_{(S, \phi)} \) (eg. transverse over-twisted discs and a Seifert surface of a closed braid). The combinatorial nature of the open book foliation \( \mathcal{F}_{\text{ob}}(F) \) enables us to extract various properties of the underlying contact manifold \((M_{(S, \phi)}, \xi_{(S, \phi)})\) or the closed braid \( \partial F \), such as overtwistedness or the self-linking number.

In this paper we explore the converse: We start from a surface admitting an open book foliation \( \mathcal{F}_{\text{ob}}(F) \) and read properties of the monodromy \( \phi \) like the fractional Dehn twist coefficient \( c(\phi, C) \). For this purpose, a general open book foliation often contains excessive information. We define the (strongly) essential open book foliation as an optimal foliation for sharper estimates of \( c(\phi, C) \). It is similar to use incompressible surfaces rather than generic ones in order to study topology/geometry of the ambient 3-manifolds. Our procedure breaks into three steps:

1. Replace a given open book foliation \( \mathcal{F}_{\text{ob}} \) with another one \( \mathcal{F}'_{\text{ob}} \) that reflects more properties of the monodromy \( \phi \in \text{Aut}(S, \partial S) \).
2. Read properties of \( \phi \) from \( \mathcal{F}'_{\text{ob}} \).
3. Find applications to contact geometry and topology/geometry of 3-manifolds.

In this paper we mainly treat Steps (2) and (3). We discuss Step (2) in Section 5 and Step (3) in Sections 6–8.
As for Step (1), typically, \( F_{ob} \) is a generic open book foliation and \( F_{ob}' \) is an essential one. In fact, in some cases even an essential open book foliation is insufficient for our purposes. To deal with the insufficiency we further introduce notions of strongly essential b-arcs and elliptic points. In the subsequent paper \([26]\), we will study various techniques (some foliation moves and braid moves) to convert a given \( F_{ob} \) to a better \( F_{ob}' \).

The paper is organized as follows:

In Section 2 we review basic notions and facts about the open book foliation. In Section 3 we introduce the essential open book foliation, and show that every incompressible surface can be isotoped so that it admits an essential open book foliation (Theorem 3.2). We introduce a notion of strongly essential b-arc which plays a crucial role in large part of the paper.

In Section 4 we develop basics of the fractional Dehn twist coefficient \( c(\phi, C) \). We give a practical and efficient method to compute \( c(\phi, C) \) which does not require Nielsen-Thurston classification (Theorem 4.10), and interpret \( c(\phi, C) \) as a translation number of certain natural dynamics of \( \text{Aut}(S, \partial S) \) (Theorem 4.12). We also introduce a new notion, the fractional Dehn twist coefficient \( c(\phi_L, C) \) for a closed braid \( L \) in \((S, \phi)\).

Section 5 is the core of the paper. In Theorem 5.4 we obtain lower and upper bounds of \( c(\phi, C) \) by counting the number of singularities of a strongly essential open book foliation. We observe similar estimates for \( c(\phi_L, C) \).

In Section 6 we characterize non-right-veering monodromy: We show that \( \phi \) is non-right-veering if and only if there exists a special “simplest” transverse overtwisted disc (Theorem 6.2). This highlights the difference between non-right-veering monodromy and overtwistedness:

**Corollary 6.5.** There exists an invariant \( n(S, \phi) \in \mathbb{Z}_{\geq 0} \) such that

1. \( n(S, \phi) = 0 \) if and only if \( \xi(S, \phi) \) is tight (and hence \( \phi \) is right veering).
2. \( n(S, \phi) = 1 \) if and only if \( \xi(S, \phi) \) is overtwisted and \( \phi \) is not right veering.
3. \( n(S, \phi) \geq 2 \) if and only if \( \xi(S, \phi) \) is overtwisted and \( \phi \) is right veering.

In Section 7 we study topology of 3-manifolds. We obtain bounds of \( |c(\phi, C)| \) from data on a closed incompressible surface embedded in \( M(S, \phi) \) (Theorems 7.1 and 7.2). As a consequence, we find sufficient conditions on the monodromy \( \phi \) that \( M(S, \phi) \) is irreducible and/or atoroidal (Corollaries 7.3 and 7.4). Our atoroidal criterion can be refined if we assume \( \xi(S, \phi) \) is tight (Theorem 7.8).

We also apply our technique to study topology of null-homologous braids in open books. We relate the genus of a braid \( L \) and \( c(\phi_L, C) \) (Corollary 7.13), which plays an essential role to prove our main result Theorem 8.3.

In Section 8 we prove the following result parallel to Thurston’s classification of geometry of mapping tori \([32]\). The same result holds for closed braid complements (Theorem 8.4).

**Theorem 8.3.** Let \((S, \phi)\) be an open book decomposition of 3-manifold \( M \). Assume that

- \( \partial S \) is connected and \( |c(\phi, \partial S)| > 1 \), or
- \( |c(\phi, C)| > 4 \) for every boundary component \( C \) of \( S \).

Then,

1. \( M \) is toroidal if and only if \( \phi \) is reducible.
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(2) $M$ is hyperbolic if and only if $\phi$ is pseudo-Anosov.
(3) $M$ is Seifert fibered if and only if $\phi$ is periodic.

The theory of essential lamination and taut foliation has been successful in studying topology/geometry of 3-manifolds. Here, let us compare lamination/foliation theory and open book foliation method. Interestingly, they tend to play complementary roles in the following sense:

The existence of taut foliation, essential lamination or genuine lamination in $M(S, \phi)$ implies that the fundamental group of $M(S, \phi)$ has various nice properties strongly reflecting the geometric structure (cf. [17], [18]). Constructing such ‘good’ foliation/lamination in $M(S, \phi)$ can often be done when $\phi$ is pseudo-Anosov. However, many toroidal manifolds are also equipped with ‘good’ foliations/laminations.

On the other hand, the open book foliation can be used to determine $M(S, \phi)$ is toroidal or atoroidal as shown in Corollary 7.4, Theorem 7.8 and Remark 7.11. While, at this writing, the open book foliation tells few about the fundamental group of $M(S, \phi)$.

We use Gabai and Oertel’s work on essential laminations [18] and open book foliation method to prove Theorems 8.3 and 8.4. One may consider these theorems are fine examples of the above point of view.

The open book foliation gives a direct and elementary way to analyze $M(S, \phi)$ including reducible and toroidal ones, with or without essential laminations, even without knowing the Nielsen-Thurston type of $\phi$. As the open book foliation encodes not only the topology of $M(S, \phi)$ but also its supporting contact structure, in some sense it is a unified method to analyze the relationships among topology, geometry, and contact structures.

2. Quick review of open book foliation

In this section we review basic definitions and techniques of the open book foliation. For details see [25, §2-4].

An open book $(S, \phi)$ is a pair of oriented compact surface $S$ with non-empty boundary $\partial S$ and (the isotopy class of) a diffeomorphism $\phi \in \text{Aut}(S, \partial S)$ fixing the boundary point-wise. By abuse of notation we will often use $\phi$ for its diffeomorphism representative. Given an open book $(S, \phi)$ we define a closed oriented 3-manifold $M(S, \phi)$ by

$$M(S, \phi) = M_{\phi} \bigcup \left( \prod_{[\partial S]} D^2 \times S^1 \right)$$

where $M_{\phi}$ denotes the mapping torus $S \times [0, 1] / (x, 1) \sim (\phi(x), 0)$, and the solid tori are attached so that for each point $p \in \partial S$ the circle $\{p\} \times S^1 \subset \partial M_{\phi}$ bounds a meridian disc of $D^2 \times S^1$. We say that $(S, \phi)$ is an open book decomposition of the 3-manifold $M = M(S, \phi)$. We view the cores of the attached solid tori as an oriented fibered link in $M$ and call it the binding $B$ of the open book. Let $\pi : M \setminus B \to S^1 = \mathbb{R}/\mathbb{Z}$ denote the fibration. The fibers $\pi^{-1}(t) = S_t$ where $t \in [0, 1)$ are called the pages of the open book.

Let $\xi = \xi(S, \phi) = \ker \alpha$ be the contact structure on $M$ supported by $(S, \phi)$ through the Giroux-correspondence: That is, $\alpha > 0$ on the binding $B$ and $d\alpha$ is a positive area form on each page $S_t$. See [31] for a construction of such contact structure.
We say that an oriented link \( L \) in \( M \) is in \textit{braid position} with respect to \((S, \phi)\) if \( L \) is disjoint from \( B \) and positively transverse to each page \( S_t \). The algebraic intersection number of \( L \) and the page \( S_0 \) is called the \textit{braid index} of \( L \). Thanks to Bennequin \cite{Bennequin} and Pavelescu \cite{Pavelescu}, any transverse link in a contact 3-manifold \((M, \xi)\) \((S, \phi)\) can be transversely isotoped to a closed braid in \((S, \phi)\). Conversely, a closed braid in \((S, \phi)\) is naturally regarded as a transverse link in \((M, \xi)\). Hence from now on, we always assume that every (transverse) link is in braid position.

Fix a closed (possibly empty) braid \( L \). Let \( F \) be an oriented connected compact surface embedded in \( M \) such that

- \( F \) is a closed surface that lies in \( M \setminus L \), or
- \( F \) is a Seifert surface of \( L \), i.e., \( \partial F = L \).

Consider the singular foliation \( \mathcal{F} = \mathcal{F}(F) \) of \( F \) induced by the intersections of the pages \( \{S_t\} \) and \( F \). We call each connected component of \( F \cap S_t \) a leaf.

**Definition 2.1.** We say that the above \( F \) is an \textit{open book foliation}, denoted by \( \mathcal{F}_{ob}(F) \), if the following four conditions are satisfied.

\( \text{(OF i)} \): The binding \( B \) pierces the surface \( F \) transversely in finitely many points. Each point \( p \in B \cap F \) is an \textit{elliptic} singularity of \( F \), namely there exists a disc neighborhood \( N_p \subset \text{Int}(F) \) of \( p \) on which the foliation \( \mathcal{F}(N_p) \) is radial with the node \( p \) (see top sketches of Figure 1). The converse also holds: any elliptic singularity of \( F \) is a transverse intersection point of \( B \) and \( \text{Int}(F) \).

\( \text{(OF ii)} \): There exists a tubular neighborhood \( N(L) \subset M \) of \( L \) such that each leaf of the foliation \( \mathcal{F}(F \cap N(L)) \) transversely intersects \( L \).

\( \text{(OF iii)} \): All but finitely many fibers \( S_t \) intersect \( F \) transversely. Each exceptional fiber is tangent to \( \text{Int}(F) \) at a single point. In particular, saddle-saddle connections do not exist.

\( \text{(OF iv)} \): All the tangencies of \( F \) with fibers are saddles, and each saddle corresponds to a \textit{hyperbolic} singularity of \( \mathcal{F}(F) \) (see bottom sketches of Figure 1).

**Definition 2.2.** We say that a page \( S_t \) is \textit{regular} if \( S_t \) intersects \( F \) transversely and is \textit{singular} otherwise. Similarly, we say a leaf \( l \) of \( F \) is \textit{regular} if \( l \) does not contain a tangency point and is \textit{singular} otherwise. The regular leaves are classified into the following three types:

- \textit{a-arc} : An arc where one of its endpoints lies on \( B \) and the other lies on \( L \).
- \textit{b-arc} : An arc whose endpoints both lie on \( B \).
- \textit{c-circle} : A simple closed curve.

**Theorem 2.3.** \cite[Theorem 3.5]{Theorem} By isotopy that fixes the transverse link type of the boundary \( \partial F \) (if \( \partial F \) exists), every surface \( F \) admits an open book foliation \( \mathcal{F}_{ob}(F) \). Moreover, we may also assume that \( \mathcal{F}_{ob}(F) \) has no \textit{c-circle} leaves.

**Definition 2.4.** We say that an elliptic point \( p \) is \textit{positive} (resp. \textit{negative}) if the binding \( B \) is positively (resp. negatively) transverse to \( F \) at \( p \). The sign of the hyperbolic point \( q \) is \textit{positive} (resp. \textit{negative}) if the positive normal direction of \( F \) at \( q \) agrees (resp. disagrees) with the direction of \( t \). See Figure 1 where we describe an elliptic point by a hollowed circle with its sign inside, a hyperbolic point by a dot with the sign nearby, and positive normals to \( F \), \( \vec{n}_F \), by dashed arrows.
We see a hyperbolic singularity as a process of switching the configuration of leaves. As $t$ increases, two regular leaves $l_1$ and $l_2$ approach along an arc $\gamma$ (the dashed arc in Figure 2) connecting $l_1$ and $l_2$. At a critical moment $l_1$ and $l_2$ form a hyperbolic singularity, then the configuration is changed. See the passage in Figure 2. The hyperbolic singularity is determined by the isotopy class of $\gamma$. We call $\gamma$ a description arc of the hyperbolic singularity and use a dashed arc. We will denote the sign of a singular point $x$ of $\mathcal{F}_{ob}$ by $\text{sgn}(x)$.

**Definition 2.5.** We denote the number of positive (resp. negative) elliptic points of $\mathcal{F}_{ob}(F)$ by $e_+ = e_+(\mathcal{F}_{ob}(F))$ (resp. $e_- = e_-(\mathcal{F}_{ob}(F))$). Similarly, the number of positive (resp. negative) hyperbolic points is denoted by $h_+ = h_+(\mathcal{F}_{ob}(F))$ (resp. $h_- = h_-(\mathcal{F}_{ob}(F))$).

Recall that the characteristic foliation $\mathcal{F}_\xi(F)$ of an embedded surface $F \subset (M, \xi)$ is a singular foliation obtained by integrating the vector field $\xi \cap TF$ on $F$. The characteristic
foliation and the convex surfaces play important roles in contact geometry. The next theorem shows a close relation between $F_{ob}(F)$ and $F_{\xi}(F)$. In particular, an open book foliation without c-circles is regarded as a characteristic foliation. For more comparisons of $F_{ob}(F)$ and $F_{\xi}(F)$, see [25, Remark 4.2].

**Theorem 2.6** (Structural stability [25, Theorem 4.1]). Assume that a surface $\Sigma$ in $M_{(S,\phi)}$ admits an open book foliation $F_{ob}(\Sigma)$. With a $C^\infty$-small perturbation of $\Sigma$ fixing the boundary $\partial \Sigma$ we have $e_{\pm}(F_{ob}(\Sigma)) = e_{\pm}(F_{\xi}(\Sigma))$ and $h_{\pm}(F_{ob}(\Sigma)) = h_{\pm}(F_{\xi}(\Sigma))$.

Moreover, if $F_{ob}(\Sigma)$ contains no c-circles, then $F_{ob}(\Sigma)$ and $F_{\xi}(\Sigma)$ are topologically conjugate, namely there exists a homeomorphism of $\Sigma$ that takes $F_{ob}(\Sigma)$ to $F_{\xi}(\Sigma)$. In particular [15, Lemma 2.1] implies that $\Sigma$ is a convex surface.

The above result yields the following:

**Proposition 2.7.** [25, Propositions 5.1, 3.9] Suppose that $F \subset M_{(S,\phi)}$ is a surface admitting an open book foliation.

1. If $\partial F$ is non-empty, the self linking number
   \[ sl(\partial F, [F]) = -\langle e(\xi), [F] \rangle = -(e_+ - e_-) + (h_+ - h_-). \]

2. The Euler characteristic $\chi(F) = (e_+ + e_-) - (h_+ + h_-)$.

Hyperbolic singularities in $F_{ob}(F)$ are classified into six types, according to the types of nearby regular leaves: Type $aa$, $ab$, $bb$, $ac$, $bc$, and $cc$ as depicted in Figure 3. Such a

![Figure 3](image_url)

**Figure 3.** Six types of the region neighborhoods for hyperbolic singularities.

model neighborhood is called a region neighborhood. We denote by $\text{sgn}(R)$ the sign of the hyperbolic point that region $R$ contains. If $R$ is of type $aa$, $ac$, $bc$, or $cc$, some parts of $\partial R$ may be identified. In such case we say that $R$ is degenerated.

**Proposition 2.8** (Region decomposition [25, Proposition 3.11]). If $F_{ob}(F)$ contains a hyperbolic point, the surface $F$ is decomposed into a union of regions so that whose
interiors are disjoint. For each \( \mathcal{F}_{ob}(F) \) the decomposition is unique up to three types of regions foliated only by regular \( a \)-arcs, \( b \)-arcs and \( c \)-circles, cf. [25, Figure 9].

We call the decomposition in Proposition 2.8 region decomposition of \( F \).

**Definition 2.9.** The negativity graph \( G_{--} \) is a graph properly embedded in \( F \). The edges of \( G_{--} \) are the unstable separatrices for negative hyperbolic points in \( aa \)-, \( ab \)- and \( bb \)-tiles. See Figure 4. We regard the negative hyperbolic points as part of the edges, i.e., not vertices. The vertices of \( G_{--} \) are the negative elliptic points in \( ab \)- and \( bb \)-tiles and the end points of the edges of \( G_{--} \) that lie on \( \partial F \), called the fake vertices. Similarly we can define the positivity graph \( G_{++} \).

![Figure 4. The negativity graph \( G_{--} \).](image)

Finally, we recall the transverse overtwisted disc.

**Definition 2.10.** Let \( D \subset M(S, \phi) \) be an embedded oriented disc whose boundary is a positive unknot braid in \( (S, \phi) \). If the following are satisfied \( D \) is called a transverse overtwisted disc:

1. \( G_{--} \) is a tree with no fake vertices,
2. \( \mathcal{F}_{ob}(D) \) contains no c-circles,
3. \( G_{++} \) is homeomorphic to \( S^1 \).

**Theorem 2.11.** [25, Proposition 6.2 and Corollary 6.5] The contact structure \( \xi(S, \phi) \) is overtwisted if and only if there exists a transverse overtwisted disc.

**Remark.** Definition 2.10 implies that the region decomposition of a transverse overtwisted disc consists of \( ab \)- and \( bb \)-tiles with \( \text{sgn}(ab \text{-tile}) = +1 \) and \( \text{sgn}(bb \text{-tile}) = -1 \). Hence \( G_{++} \) lives in the \( ab \)-tiles and \( G_{--} \) lives in the \( bb \)-tiles.

### 3. Essential open book foliation

In this section we introduce an essential open book foliation. Take a closed braid \( L \) and a surface \( F \) as in Section 2 so that \( F \) admits an open book foliation.

**Definition 3.1.** We say that a \( b \)-arc \( l \) in a fiber \( S_l \) is essential if \( l \) is an essential arc in \( S_l \setminus (S_l \cap L) \). Similarly, a \( c \)-circle \( l \) in \( S_l \) is called essential if \( l \) is an essential simple closed curve in \( S_l \setminus (S_l \cap L) \). The open book foliation \( \mathcal{F}_{ob}(F) \) is called essential if all the \( b \)-arcs are essential (\( c \)-circles need not be essential). An elliptic point \( v \in \mathcal{F}_{ob}(F) \) is called essential if every \( b \)-arc that ends at \( v \) is essential.
Our first result is an improvement of Theorem 2.3 for incompressible surfaces.

**Theorem 3.2.** Assume that a surface $F$ is incompressible. Then applying isotopy that fixes $L$, $F$ admits an essential open book foliation.

Theorem 3.2 essentially follows from Roussarie-Thurston’s theorem [30, 33, 13] for an incompressible surface in a taut foliation. A codimension one foliation $\mathcal{F}$ on a 3-manifold is called *taut* if there exists an embedded circle which transversely intersects all the leaves of $\mathcal{F}$. Roussarie-Thurston’s theorem implies that an incompressible surface in a taut foliation can be put in good position similar to the setting of the open book foliation.

**Theorem 3.3** (Roussarie-Thurston [30, 33, 13]). Let $\mathcal{F}$ be a transversely oriented taut foliation on a 3-manifold $M$ other than the product foliation of $S^2 \times S^1$. Let $F$ be an incompressible surface in $M$ such that $\partial F$ is transverse to $\mathcal{F}$ and that $F$ is not isotopic to the leaves of $\mathcal{F}$. Then by isotopy that fixes $\partial F$, $F$ can be transverse to $\mathcal{F}$ except for finitely many isolated saddle tangencies.

Now we prove Theorem 3.2 by using Roussarie-Thurston theorem.

**Proof of Theorem 3.2.** We put the surface $F$ in a general position so that it admits a singular foliation $\mathcal{F}$ satisfying the properties (OF i), (OF ii), (OF iii) in Definition 2.1 and (OF iv′): The type of each tangency in (OF iii) is either saddle or local extremum. Note that (OF iv′) is weaker than (OF iv).

We remove all the inessential b-arcs by the following way (cf. [2, Lemma 1.2]). Let $l$ be an innermost inessential b-arc of $F$ in a regular page $S_t$ that cobounds with a sub-arc of the binding $B = \partial S_t$, a disc $\Delta \subset S_t$, and $\Delta \cap L$ is empty. Since $l$ is innermost $\Delta$ contains no b-arcs. If $\Delta$ contains c-circles $c_1, \ldots, c_k$, let $A_i \subset F$ be a small annular neighborhood of $c_i$ with no singularities. We push each annulus $A_i$ out of $\Delta$ across the binding $B$ as shown in Figure 5. This does not create new b-arcs but new local extrema appear. Now $l$ is boundary parallel in $S_t \setminus (S_t \cap F)$. We push the surface $F$ along $\Delta$ as shown in Figure 5. As a consequence the foliation $\mathcal{F}$ changes. Namely, two elliptic points that are the end points of $l$ disappear and new hyperbolic points and local extremal points appear. Since the number of elliptic points in the original $\mathcal{F}$ is finite, applying the operation finitely many times $\mathcal{F}$ no longer has inessential b-arcs.

Let $N(B)$ be a standard tubular neighborhood of the binding $B$, and $n$ be the braid index of $L$. Since the complement $M \setminus (N(B) \cup L)$ is an $(S_{g,d+n})$-bundle over $S^1$ it admits
a taut foliation, $\mathcal{F}$. By assumption, $F' = F \setminus (F \cap N(B))$ is not isotopic to the leaves of $\mathcal{F}$. By Theorem 3.3 we can isotope $F'$ fixing $\partial F'$ and $L$ so that $F'$ is transverse to $\mathcal{F}$ except for finitely many isolated saddle tangencies and has no local extrema. Hence the surface $F' \cup (F \cap N(B))$ admits an open book foliation with no inessential b-arcs. □

**Remark.** The Roussarie-Thurston theorem cannot apply to compressible surfaces in general, hence a compressible surface may not admit essential open book foliation. The most trivial example is a compressible sphere in the complement of the binding.

In contrast to Theorem 2.3 there may exist a surface $F$ all of whose essential open book foliations contain $c$-circles.

The converse of Theorem 3.2 is false: a compressible surface might admit an essential open book foliation. For example, The boundary torus of a regular neighborhood of a closed braid admits an essential open book foliation.

Next we introduce “strong essentiality” a key concept for estimates of the fractional Dehn twist coefficient.

**Definition 3.4.** A b-arc in a fiber $S_t$ is called **strongly essential** if it is essential in $S_t$. (Hence if a b-arc is strongly essential then it is essential.) An elliptic point $v \in \mathcal{F}_{ob}(F)$ is called **strongly essential** if every b-arc that ends at $v$ is strongly essential.

In braid foliation theory no b-arcs are strongly essential because $S_t \cong D^2$. The existence of strongly essential b-arcs is a unique feature of the open book foliation.

### 4. Fractional Dehn Twist Coefficients

In this section we review the notion of right-veering diffeomorphisms and the fractional Dehn twist coefficient defined by Honda, Kazez and Matić [22] and present its basic properties. We show that the fractional Dehn twist coefficient is effectively computable and give an alternative description which does not require the Nielsen-Thurston classification.

**4.1. Definitions of** $c(\phi, C)$ **and** $c(\phi_L, C)$. 

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**Figure 6.** Removing an inessential b-arc.
Definition 4.1. [22] Let $C$ be a boundary component of $S$, and let $\gamma, \gamma'$ be isotopy classes (rel. to the endpoints) of oriented properly embedded arcs in $S$ which start at the same base point $* \in C \subset \partial S$. We say that $\gamma'$ lies \textit{strictly on the right side} of $\gamma$ if there exist curves representing $\gamma$ and $\gamma'$ realizing the minimal geometric intersection number and $\gamma'$ strictly lies on the right side of $\gamma$ near $*$. In such case, we denote $\gamma > \gamma'$.

Definition 4.2. [22] Definition 2.1] Let $C$ be a boundary component of $S$. Let $\text{Aut}(S, C)$ denote the group of isotopy classes of diffeomorphisms of $S$ fixing $C$ point-wise. We say that $\phi \in \text{Aut}(S, C)$ is \textit{right-veering} (resp. \textit{strictly right-veering}) with respect to $C$ if $\gamma \geq \phi(\gamma)$ (resp. $\gamma > \phi(\gamma)$) for any isotopy classes $\gamma$ of essential arcs in $S$ starting on $C$. For $\phi \in \text{Aut}(S, \partial S)$, we say that $\phi$ is (\textit{strictly}) \textit{right-veering} if $\phi$ is (strictly) right-veering with respect to all the boundary components of $S$. In particular, the identity map is right-veering.

Convention 4.3. Assume that $\chi(S) < 0$, i.e., $S$ admits a complete hyperbolic metric with finite area. By the Nielsen-Thurston classification [12], any $\phi \in \text{Aut}(S, \partial S)$ is freely isotopic to a homeomorphism of $S$ of type either periodic, reducible or pseudo-Anosov. For each case, we say that $\phi \in \text{Aut}(S, \partial S)$ is \textit{periodic}, \textit{reducible}, or \textit{pseudo-Anosov}, respectively.

If $S$ is an annulus or a disc, we will regard every element $\phi \in \text{Aut}(S, \partial S)$ periodic.

The \textit{fractional Dehn twist coefficient} $c(\phi, C)$ measures how much $\phi$ twists the surface $S$ along the boundary component $C$. This number is closely related to the right-veeringness. The origin of $c(\phi, C)$ is Gabai and Oertel’s \textit{degeneracy slope}, which is used to study taut foliations and essential laminations [13]. Here we review the definition due to Honda, Kazez and Matić.

Definition 4.4. [22] p.423]

1) Assume that $\phi \in \text{Aut}(S, \partial S)$ is periodic. Let $C_1, \ldots, C_d$ be the boundary components of $S$. Let $T_C$ denote the right-handed Dehn twist along (a curve parallel to) $C$. There exist numbers $N \in \mathbb{N}$ and $M_1, \ldots, M_d \in \mathbb{Z}$ such that $\phi^N = T_{C_1}^{M_1} \cdots T_{C_d}^{M_d}$, which is freely isotopic to the identity. We define the fractional Dehn twist coefficient $c(\phi, C_i) = \frac{M_i}{N}$.

2) Assume that $\phi \in \text{Aut}(S, \partial S)$ is pseudo-Anosov and $\phi$ is freely isotopic to a pseudo-Anosov homeomorphism, $\Phi$. Fix a boundary component $C$. Let $L$ be the stable (or unstable) geodesic measured lamination for $\Phi$ and $W$ be the connected component of $S \setminus L$ containing $C$. It is known that $W$ is a \textit{crown} [12] Lemma 4.4]: a complete hyperbolic surface of finite area and homeomorphic to $(S^1 \times [0,1]) \setminus A$ where $C$ is identified with $S^1 \times \{0\}$ and $A = \{p_0, \ldots, p_m\}$ a finite set of points in $S^1 \times \{1\}$. Take $m$ semi-infinite geodesics $\lambda_i \subset W$ which start at a point $q_i \in C$ and approach to $p_i$ (see Figure 7 left). There exists $j \in \{0, \ldots, m - 1\}$ such that $\Phi(\lambda_i) = \lambda_{i+j}$. Let $H : C \times [0,1] \to C$ be the free isotopy from $\phi$ to $\Phi$ restricted to $C$. We have $H(q_i, 0) = \phi(q_i) = q_i$ and $H(q_i, 1) = \Phi(q_i) = q_{i+j}$. There exists $n \in \mathbb{Z}$ such that the arc $\alpha_t = H(q_0, t)$ starts from $q_0$ and winds around $C$ in the positive direction for $n + \frac{1}{m}$ times and then stops at $q_j$.

We define $c(\phi, C) = n + \frac{1}{m}$. For example, $\phi$ depicted in Figure 7 has $c(\phi, C) = 1 + \frac{1}{4} = \frac{5}{4}$.

3) Suppose that $\phi \in \text{Aut}(S, \partial S)$ is reducible. For each boundary component $C$ of $S$ there exists subsurface $S'$ of $S$ containing $C$ such that $\phi(S') = S'$ and $\phi|_{S'}$ is either periodic or pseudo-Anosov. We define $c(\phi, C) = c(\phi|_{S'}, C)$. 
Next we introduce a new notion, the fractional Dehn twist coefficients for closed braids.

**Definition 4.5.** Let \( L \) be a closed \( n \)-braid in an open book \((S, \phi)\). Let \( B \) denote the set of bindings of \((S, \phi)\). The manifold \( M_{\phi, L} := M_{(S, \phi)} \setminus (L \cup B) \) is a fibered 3-manifold over \( S' := S - \{n \text{ points}\} \). The boundary of \( S' \) is \( \partial S' = \partial S \cup \{n \text{ points}\} \). Let \( \phi_L \) denote the monodromy of the fibration \( M_{\phi, L} \rightarrow S' \). Since \( \phi_L \) permutes the \( n \) points there exists \( m \in \mathbb{Z} \) such that \( \phi_L^m \) fixes them pointwise, i.e., \( \phi_L^m \in \text{Aut}(S', \partial S') \).

Let \( c(\phi_L, C) := \frac{1}{m}c(\phi_L^m, C) \).

- The definition of \( c(\phi_L, C) \) is independent of the choice of \( \phi_L \). If braids \( L \) and \( L' \) in \((S, \phi)\) are isotopic in \( M \setminus B \) then \( c(\phi_L, C) = c(\phi_{L'}, C) \).
- The difference between \( c(\phi_L, C) \) and \( c(\phi, C) \) can be arbitrary large. For example, the open book decomposition \((S, \phi) = (D^2, \text{id})\) of \( S^3 \) has \( c(\phi, \partial S) = 0 \). If \( L \) is a closed 2-braid representing the \((2, k)\)-torus knot then \( c(\phi_L, \partial S) = \frac{k}{2} \).

4.2. **Properties of** \( c(\phi, C) \). We develop computational techniques for \( c(\phi, C) \), including the key lemma (Lemma 4.9). In later sections we will obtain more estimates by using the open book foliation. We start by listing basic properties of \( c(\phi, C) \) straight forward from the definition.

**Proposition 4.6.** Let \( C \) be a boundary component of \( S \) and \( \phi \in \text{Aut}(S, \partial S) \).

1. \( c(\phi^N, C) = Nc(\phi, C) \) for \( N \in \mathbb{Z} \).
2. \( c(T_C, C) = 1 \) and \( c(T_C \circ \phi, C) = c(\phi \circ T_C, C) = 1 + c(\phi, C) \).
3. \( c(\phi, C) = c(\psi \circ \phi \circ \psi^{-1}, C) \) for any \( \psi \in \text{Aut}(S, \partial S) \).

**Proposition 4.7** (Honda, Kazez and Matić [22]). If \( \phi \) is periodic then \( \phi \) is right-veering if and only if \( c(\phi, C) \geq 0 \) for all \( C \). If \( \phi \) is pseudo-Anosov then \( \phi \) is right-veering with respect to \( C \) if and only if \( c(\phi, C) > 0 \).

The following shows that topology of \( S \) governs \( c(\phi, C) \).

**Proposition 4.8.** Let \( S = S_{g,d} \) be an oriented genus \( g \) surface with \( d > 0 \) boundaries. If \( \phi \in \text{Aut}(S, \partial S) \) is periodic then

\[
(4.1) \quad c(\phi, C) \in \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \in \{1, 2, \ldots, 4g + 2\} \right\}.
\]
If \( \phi \) is pseudo-Anosov then

\[
(4.2) \quad c(\phi, C) \in \left\{ \frac{p}{q} \mid p \in \mathbb{Z}, q \in \{1, 2, \ldots, 4g + d - 3\} \right\}.
\]

**Proof.** If \( S \) is an annulus (or a disc), then \( c \in \mathbb{Z} \) (or \( c = 0 \)) so the statement holds. Hence in the rest of the proof we assume \( \chi(S) < 0 \). By fixing a hyperbolic metric on \( S \), we regard \( S \) as a complete hyperbolic surface with geodesic boundary and finite area.

Assume that \( \phi \) is periodic of period \( M \). Let \( \hat{S} \) be the genus \( g \) surface obtained by capping off the \( d \) boundary circles. Extend \( \phi \) to \( \hat{\phi} \in \text{Aut}(\hat{S}) \) by setting \( \hat{\phi} = \text{id} \) on \( \hat{S} - S \).

Clearly \( \hat{\phi} \) has the period \( M \). The “4g + 2 theorem” \([10] \) Theorem 7.5] implies that \( M \leq 4g + 2 \). Since \( Mc(\hat{\phi}, C) \in \mathbb{Z} \) we get \([4.1] \).

Next assume that \( \phi \) is pseudo-Anosov. Let \( L, W \) and \( m \) be as in Definition \([14] \). By Definition \([14] \) we have:

\[
c(\phi, C) \in \left\{ \frac{p}{m} \mid p \in \mathbb{Z} \right\}.
\]

Since Area\((S-L) = \text{Area}(S)\) by \([12] \) Theorem 4.9], Area\((W) = m\pi\), and Area\((S-W) \geq (d-1)\pi\) we have an inequality:

\[
m\pi + (d-1)\pi \leq \text{Area}(W) + \text{Area}(S-W) = \text{Area}(S) = (4g - 4 + 2d)\pi,
\]

i.e., \( m \leq 4g - 3 + d \) and we obtain \([4.2] \). \( \square \)

The following is a key estimate of \( c(\phi, C) \) that we will use repeatedly in this paper. One can also find a similar result in \([27] \) Corollary 2.6].

**Lemma 4.9 (Key lemma).** Let \( C \) be a boundary component of \( S \) and \( \phi \in \text{Aut}(S, \partial S) \). If there exists an essential arc \( \gamma \subset S \) that starts on \( C \) and satisfies \( T^m_C(\gamma) \geq \phi(\gamma) \geq T^M_C(\gamma) \) for some \( m, M \in \mathbb{Z} \), then \( m \leq c(\phi, C) \leq M \).

**Proof.** Assume contrary that \( M < c(\phi, C) \) then \( c(T^{-M}_C \phi, C) > 0 \). Propositions 3.1 and 3.2 of \([22] \) imply that \( T^{-M}_C \phi \) is strictly right-veering with respect to \( C \). Hence for any immersed geodesic arc \( \alpha \) which begins on \( C \) we have \( \alpha > T^{-M}_C \phi(\alpha) \), hence \( T^M_C(\alpha) > \phi(\alpha) \). This contradicts the assumption. The proof of \( m \leq c(\phi, C) \) is similar. \( \square \)

Practically, in order to compute \( c(\phi, C) \) one may need to know the Nielsen-Thurston normal form for \( \phi \) and its invariant measured lamination. Proposition \([13] \) and Lemma \([19] \) provide effective methods to compute \( c(\phi, C) \) without using Nielsen-Thurston theory. Recall that in \([28] \) Mosher proves that the mapping class group of \( S \) is automatic hence each element of \( \text{Aut}(S, \partial S) \) admits a normal form called Mosher’s normal form.

**Theorem 4.10.** Let \( S = S_{g,d} \) and \( D(S) = \max\{4g + 2, 4g + d - 3\} \). Fix an integer \( N > D(S)(D(S) - 1) \). Suppose that there exists a geodesic arc \( \gamma \subset S \) that starts on \( C \) and an integer \( M \) satisfying

\[
T^M_C(\gamma) \geq \phi^N(\gamma) > T^{M+1}_C(\gamma).
\]

Then the fractional Dehn twist coefficient has

\[
c(\phi, C) = \left\lfloor \frac{M}{N} \cdot \frac{M + 1}{N} \right\rfloor \cap \left\{ \frac{p}{q} \mid p \in \mathbb{Z}, q \in \{1, 2, \ldots, D(S)\} \right\}.
\]
Moreover, \( c(\phi, C) \) can be computed in polynomial time with respect to the length of Mosher’s normal form of \( \phi \).

**Proof.** By Lemma 4.9 and Proposition 4.6, \( M \leq c(\phi^N, C) = Nc(\phi, C) \leq M + 1 \) so \( c(\phi, C) \in \left[ \frac{M}{N}, \frac{M + 1}{N} \right] \). On the other hand, by Proposition 4.8, \( c(\phi, C) \in \left\{ \frac{p}{q} \mid p \in \mathbb{Z}, q = 1, 2, \ldots, D(S) \right\} \). Since we choose \( N \) with \( N > D(S)(D(S) - 1) \), the intersection

\[
\left[ \frac{M}{N}, \frac{M + 1}{N} \right] \cap \left\{ \frac{p}{q} \mid p \in \mathbb{Z}, q \in \left\{ 1, 2, \ldots, D(S) \right\} \right\}
\]

consists of one rational number, which must be \( c(\phi, C) \).

Next we show that \( c(\phi, C) \) is computable in polynomial time. We define a partial ordering \( <_{\gamma} \) on \( \text{Aut}(S, \partial S) \) by \( \phi <_{\gamma} \psi \) if \( \phi(\gamma) \geq \psi(\gamma) \). As shown in [34, Theorem 2.1], this partial ordering is determined in linear time with respect to the length \( l(\phi) \) of Mosher’s automatic normal form of \( \phi \) (see [28] for the definition) by using Mosher’s automatic structure of \( \text{Aut}(S, \partial S) \). By definition of Mosher’s normal form, each generator \( x \) of Mosher’s normal form satisfies \( c(\phi, C) = 0 \) so we have an a priori estimate \( |c(\phi, C)| \leq l(\phi) \). This implies that the above integer \( M \) can be computable in polynomial time with respect to \( l(\phi) \), hence so is \( c(\phi, C) \).

**Corollary 4.11.** If there exists a (possibly immersed) geodesic arc \( \gamma \subset S \) that starts on \( C \subset \partial S \) with \( T^{\phi_n}_{\gamma}(\gamma) = \phi^N(\gamma) \) for some \( m, N \in \mathbb{Z} \) \((N \neq 0)\), then \( c(\phi, C) = \frac{N}{m} \).

4.3. **Alternative description of \( c(\phi, C) \).** We give an alternative description of the fractional Dehn twist coefficient which appears to be natural from theoretical point of view and does not require Nielsen-Thurston classification. Let \( \pi : \tilde{S} \to S \) be the universal covering of \( S \). Fix a base point \( * \in C \subset \partial S \) and its lift \( \tilde{*} \in \pi^{-1}(C) \subset \pi^{-1}(\partial S) \). Let \( \tilde{C} \) be the connected component of \( \pi^{-1}(C) \) that contains \( \tilde{*} \). Since \( S \) admits a hyperbolic metric there is an isometric embedding of \( \tilde{S} \) to the Poincaré disc \( \mathbb{H}^2 \). By attaching points at infinity to \( \tilde{S} \) we obtain a compact disc \( \widehat{S} \subset \mathbb{H}^2 \).

For a homeomorphism \( f : S \to S \) fixing the boundary pointwise we take the lift \( \tilde{f} : \tilde{S} \to \tilde{S} \) with \( \tilde{f}(\tilde{*}) = \tilde{*} \). It uniquely extends to a homeomorphism \( \overline{f} : \overline{S} \to \overline{S} \). The restriction \( \overline{f}|_{\overline{S}} \) is an invariant of the mapping class \( [f] \in \text{MCG}(S) \). Since \( f = id \) on \( \partial S \) and \( \tilde{f}(\tilde{*}) = \tilde{*} \) the map \( \overline{f} \) fixes \( \tilde{C} \) pointwise. We identify \( \overline{S} \setminus \tilde{C} \) with \( \mathbb{R} \) so that \( T^{\overline{f}}_{\tilde{C}}(x) = x + 1 \) for all \( x \in \mathbb{R} \). Let \( \text{Homeo}^+(S^1) \) be the group of orientation-preserving homeomorphisms of \( S^1 \) that are lifts of orientation-preserving homeomorphisms of \( S^1 \).

In other words, \( \text{Homeo}^+(S^1) \) consists of elements of \( \text{Homeo}^+(\mathbb{R}) \) that commute with the translation \( x \mapsto x + 1 \). Since \( f \circ T^C = T^C \circ f \) we can define a homomorphism \( \Theta_C : \text{MCG}(S) \to \text{Homeo}^+(S^1) \) by

\[
\Theta_C([f]) = \overline{f}|_{\overline{S} \setminus \tilde{C}}.
\]

The map \( \Theta_C \) is called the **Nielsen-Thurston homomorphism** and is intensively studied in [35] to describe a total left-invariant ordering of \( \text{MCG}(S) \). It is known that \( \Theta_C \) is injective [35]. We note that \( \Theta_C \) depends on various choices such as hyperbolic metrics on \( S \) and identifications of \( \partial S \) with \( \mathbb{R} \).
Let \( \tau : \text{Homeo}^+(S^1) \rightarrow \mathbb{R} \) be the translation number defined by
\[
\tau(h) = \lim_{N \to \infty} \frac{h^N(x) - x}{N} \quad (x \in \mathbb{R}).
\]

It is well-known that the above limit exists and is independent of the choice of \( x \in \mathbb{R} \).

The fractional Dehn twist coefficient is related to the Nielsen-Thurston map as follows.

**Theorem 4.12.** (cf. [14, p.3]) For \( \phi \in \text{Aut}(S, \partial S) \) we have \( c(\phi, C) = \tau(\Theta_C(\phi)) \).

**Proof.** Let us take a geodesic \( \tilde{\gamma} \) in \( \tilde{S} \subset \mathbb{H}^2 \) which joins \( \ast \) and \( x \in \partial \tilde{S} \setminus \tilde{C} = \mathbb{R} \). Denote \( \gamma = \pi(\tilde{\gamma}) \). For \( N > 0 \) there exists an integer \( M(N) \) such that
\[
T^{M(N)}_C(\gamma) \geq \phi^N(\gamma) \geq T^{M(N)+1}_C(\gamma).
\]
This is equivalent to
\[
\Theta_C(T^{M(N)}_C(x)) \leq \Theta_C(\phi^N(x)) \leq \Theta_C(T^{M(N)+1}_C(x)).
\]
Recall that \( \Theta(T_C) \) translates \( x \mapsto x + 1 \), hence
\[
x + M(N) \leq \Theta_C(\phi^N)(x) \leq x + M(N) + 1,
\]
i.e.,
\[
\frac{M(N)}{N} \leq \frac{\Theta_C(\phi^N)(x) - x}{N} \leq \frac{M(N) + 1}{N}.
\]
By Theorem 4.10 as \( N \to \infty \) both \( \frac{M(N)}{N} \) and \( \frac{M(N)+1}{N} \) converge to \( c(\phi, C) \) and the middle term converges to \( \tau(\Theta_C(\phi)) \), so we obtain \( c(\phi, C) = \tau(\Theta_C(\phi)) \). \( \square \)

Since the translation number \( \tau : \text{Homeo}^+(S^1) \rightarrow \mathbb{R} \) is a homogeneous quasi-morphism of defect 1, we get the following.

**Corollary 4.13.** The fractional Dehn twist coefficient w.r.t. \( C \) defines a homogeneous quasi-morphism of defect 1. That is,
\[
|c(\phi\psi, C) - c(\phi, C) - c(\psi, C)| \leq 1
\]
and
\[
c(\phi^N, C) = Nc(\phi, C)
\]
hold for all \( \phi, \psi \in \text{Aut}(S) \) and \( N \in \mathbb{Z} \).

5. **Estimates of fractional Dehn twist coefficient from open book foliation**

Section 5.1 is devoted to estimates of the fractional Dehn twist coefficient of a monodromy. In Section 5.2 we extend the results to the fractional Dehn twist coefficient for braids.
5.1. Estimates of $c(\phi, C)$. To estimate the fractional Dehn twist coefficient of a monodromy, the notion of strong essentiality (Definition 3.4) plays an important role. Lemma 5.1, a special case of Theorem 5.4, gives a simple but still useful estimate. The proof shows a basic idea how the fractional Dehn twist coefficient and the open book foliation are related to each other.

**Lemma 5.1.** Let $v$ be an elliptic point of $\mathcal{F}_{ob}(F)$ lying on a binding component $C \subset \partial S$. Assume that $v$ is strongly essential and there are no a-arcs around $v$. Let $p$ (resp. $n$) be the number of positive (resp. negative) hyperbolic points that are joined with $v$ by a singular leaf.

1. If $\text{sgn}(v) = +1$ then $-n \leq c(\phi, C) \leq p$.
2. If $\text{sgn}(v) = -1$ then $-p \leq c(\phi, C) \leq n$.

**Proof.** We prove the case $\text{sgn}(v) = -1$. (Similar arguments hold for the positive case.)

Note that when $\text{sgn}(v) = -1$ any regular leaf that ends at $v$ is a b-arc, so the assumption that around $v$ there are no a-arcs is automatically satisfied.

Let $h_1, \ldots, h_{n+p}$ be the hyperbolic points around $v$, and $S_t$ be the singular fiber that contains $h_1$. With no loss of generality we may assume $0 < t_1 < \cdots < t_{n+p} < 1$. For $t \neq t_1, \ldots, t_{n+p}$ let $b_t$ denote the b-arc in $S_t$ that ends at $v$. Since $v$ is strongly essential, all the $b_t$ are strongly essential.

Suppose that $\text{sgn}(h_1) = +1$. Let $\gamma \subset S_0 \setminus (S_0 \cap F)$ be the describing arc for $h_1$. At least one of the endpoints of $\gamma$ lies on $b_0$, which we call $v'$ (if both the endpoints lie on $b_0$, pick the one closer to $v$). We isotope $\gamma$ in $S_0 \setminus (S_0 \cap F)$ by sliding $v'$ along $b_0$ until it reaches $v$. Since $\text{sgn}(v) = -1$, near $v$ the arc $\gamma$ lies strictly on the right of $b_0$ in $S_0$. Hence $b_0 > \gamma$. See Figure 8. On the other hand, since the interiors of $\gamma$ and $b_0$ are disjoint and $b_0$ is strongly essential (i.e., $b_0 \neq T_C(b_0)$), $\gamma$ lies strictly on the left side of $T_C(b_0)$. We have $\gamma > T_C(b_0)$. After passing the critical time $t_1$ we may identify $b_{t_1+\varepsilon}$ with $\gamma$ for sufficiently small $\varepsilon > 0$. Hence we get

$$b_0 > b_{t_1+\varepsilon} > T_C(b_0).$$

Similarly, if $\text{sgn}(h_1) = -1$ we obtain

$$T_C^{-1}(b_0) > b_{t_1+\varepsilon} > b_0.$$

Fix $1 \leq k \leq p+n$. Suppose that $\alpha$ of the hyperbolic points $h_1, \ldots, h_k$ are negative and $\beta = k - \alpha$ of them are positive. If $T_C^{-\alpha}(b_0) > b_{t_k+\varepsilon} > T_C^{\beta}(b_0)$ then the above argument
implies that $T^{-\alpha}(b_0) > b_{t_{k+1}+\epsilon} > b_{t_{k+1}-\epsilon} > T_C(b_{t_{k+1}-\epsilon}) = T_C(b_{t_{k+1}+\epsilon}) > T_C^{\beta+1}(b_0)$ when $\text{sgn}(h_{k+1}) = +1$. Hence

$$T^{-\alpha}(b_0) > b_{t_{k+1}+\epsilon} > T_C^{\beta+1}(b_0) \quad \text{if} \quad \text{sgn}(h_{k+1}) = +1,$$

$$T_C^{-(\alpha+1)}(b_0) > b_{t_{k+1}+\epsilon} > T_C^{\beta}(b_0) \quad \text{if} \quad \text{sgn}(h_{k+1}) = -1.$$

By induction on $k$ we conclude

$$T^{-n}(b_0) > b_1 = \phi^{-1}(b_0) > T_C^{p}(b_0).$$

Since the elliptic point $v$ is strongly essential, the b-arc $b_0$ is strongly essential, i.e., $b_0$ is an essential arc in $S$. Lemma 4.9 implies $-n \leq c(\phi^{-1},C) \leq p$. With Proposition 4.6-(1) we obtain the desired estimate. □

**Remark 5.2.** In [23, 24] the first-named author used the similar arguments to relate the valence of a vertex in the braid foliation (which corresponds to the number of hyperbolic singular points around a elliptic singular points in open book foliation) and the Dehornoy floor, an integer-valued complexity of braids defined by the Dehornoy ordering of the braid groups which roughly corresponds to the absolute value of the fractional Dehn twist coefficient. Lemma 5.1 can be seen as a generalization of the arguments in [23] and [24].

Below is an immediate consequence of Lemma 5.1.

**Corollary 5.3.** Let $v_1, \ldots, v_n$ be the strongly essential elliptic points on a binding component $C$ of the open book $(S, \phi)$ such that all the regular leaves ending on $v_i$ are b-arcs. Let $p_i$ (resp. $n_i$) be the number of positive (resp. negative) hyperbolic points connected to $v_i$ by a singular leaf. Define the upper bound function

$$U(v_i) = \begin{cases} 
  p_i & \text{if} \quad \text{sgn}(v_i) = +1, \\
  n_i & \text{if} \quad \text{sgn}(v_i) = -1,
\end{cases}$$

and the lower bound function

$$L(v_i) = \begin{cases} 
  -n_i & \text{if} \quad \text{sgn}(v_i) = +1, \\
  -p_i & \text{if} \quad \text{sgn}(v_i) = -1.
\end{cases}$$

Then the fractional Dehn twist coefficient has

$$\max_{i=1,\ldots,n} L(v_i) \leq c(\phi,C) \leq \min_{i=1,\ldots,n} U(v_i). \quad (5.1)$$

Lemma 5.1 and Corollary 5.3 above require only local information of the open book foliation, namely, the number of hyperbolic points around a single strongly essential elliptic point. In Theorem 5.4 below we examine more than one strongly essential elliptic points at the same time, and get a sharper estimate of $c(\phi,C)$.

Let $|x| \in \mathbb{Z}$ be the ceiling of $x \in \mathbb{R}$, the smallest integer greater than or equal to $x$.

**Theorem 5.4.** Let $v_1, \ldots, v_n \in \mathcal{O}_b(F)$ be strongly essential elliptic points. Assume that all of them lie on the same component $C$ of $\partial S$ and that all the regular leaves ending on $v_i$ are b-arcs. Let $N$ (resp. $P$) be the total number of negative (resp. positive) hyperbolic
points that are connected to at least one of \( v_1, \ldots, v_n \) by a singular leaf. Let \( f_\pm : \mathbb{N} \to \mathbb{Q} \) be a map defined by

\[
f_(m) = \begin{cases} 
\frac{1}{m} \left[ \frac{Nm}{n} - \frac{(n-1)^2}{4m^2} \right] & (n : \text{odd}) \\
\frac{1}{m} \left[ \frac{Nm}{n} - \frac{n-2}{4m^2} \right] & (n : \text{even})
\end{cases}
\]

and

\[
f_+(m) = \begin{cases} 
\frac{1}{m} \left[ \frac{Nm}{n} - \frac{(n-1)^2}{4m^2} \right] & (n : \text{odd}) \\
\frac{1}{m} \left[ \frac{Nm}{n} - \frac{n-2}{4m^2} \right] & (n : \text{even})
\end{cases}
\]

(1) If \( \text{sgn}(v_1) = \text{sgn}(v_2) = \cdots = \text{sgn}(v_n) = -1 \), then

\[
\inf_{m \in \mathbb{N}} f_+(m) \leq c(\phi, C) \leq \inf_{m \in \mathbb{N}} f_-(m).
\]

(2) If \( \text{sgn}(v_1) = \text{sgn}(v_2) = \cdots = \text{sgn}(v_n) = +1 \) and all the regular leaves from \( v_i \) are \( b \)-arcs, then

\[
\inf_{m \in \mathbb{N}} f_-(m) \leq c(\phi, C) \leq \inf_{m \in \mathbb{N}} f_+(m).
\]

**Remark 5.5.** Lemma 5.1 is a corollary of Theorem 5.4 for the case \( n = 1 \). If \( n \geq 2 \) it depends which estimate of (5.1), (5.2) or (5.3) is the sharpest.

**Proof.** We show the upper bound of \( c(\phi, C) \) for Case (1). The rest of the bounds can be obtained similarly.

Let \( A = F \cap S_0 \) be the multi-curve on \( S = S_0 \). We cut \( F \subset M \) along \( A \) to get a properly embedded oriented surface \( \Sigma \) in \( M \setminus S_0 \simeq S \times [0, 1] \). Let \( \sim_\partial \) be an equivalence relation \( (x, t) \sim_\partial (x, 0) \) for \( x \in \partial S \) and \( t \in [0, 1] \). We orient \( A \) so that

\[
\Sigma \cap S_0 = -A, \quad \Sigma \cap S_1 = \phi^{-1}(A).
\]

Fix an integer \( m \geq 1 \). For \( i = 0, \ldots, m - 1 \), let

\[
\Phi_i : S \times [0, 1] \sim_\partial \to S \times [i, i + 1] \sim_\partial
\]

be a map defined by

\[
\Phi_i(x, t) = (\phi^{-i}(x), t + i).
\]

Let

\[
\Sigma_m = \Sigma \cup \Phi_1(\Sigma) \cup \cdots \cup \Phi_{m-1}(\Sigma) \subset S \times [0, m] \sim_\partial,
\]

a properly embedded surface. Consider a natural quotient map

\[
\pi_m : S \times [0, m] \sim_\partial \to M_{(S,\phi^m)}
\]

identifying \((x, m)\) with \((\phi^m(x), 0)\) for \( x \in \text{Int}(S) \). Note that

\[
\Sigma_m \cap S_0 = -A \quad \text{and} \quad \Sigma_m \cap S_m = \phi^{-m}(A).
\]

Hence we obtain a surface \( F_m := \pi_m(\Sigma_m) \subset M_{(S,\phi^m)} \).

The manifold \( M_{(S,\phi^m)} \) is the cyclic \( m \)-fold branched cover of \( M_{(S,\phi)} \) with branch locus the binding of the open book. Likewise the surface \( F_m \) is the cyclic \( m \)-fold branched cover of \( F \). By abuse of notation let \( v_1, \ldots, v_n \in F_m \) denote the lifts of the branch points \( v_1, \ldots, v_n \in F \). By construction, the lifts \( v_1, \ldots, v_n \) are also strongly essential negative
elliptic points in $F_{ob}(F_m)$ and connected to $Nm$ negative hyperbolic points, $h_1,\ldots,h_{Nm}$, by a singular leaf. We assume that $h_k$ lies on the singular fiber $S_{tk}$ where

$$0 < t_1 < t_2 < \cdots < t_N < 1 \quad \text{and} \quad t_{lN+j} = l + t_j.$$ 

For $i = 1,\ldots,n$ and $t \neq t_k$, we denote the b-arc in $S_t$ that ends at $v_i$ by $b_i^t$. As in the proof of Lemma 5.1, we compute the upper bound of $c(\phi^m, C)$ by comparing $b_i^0$ and $b_m^i = \phi^{-m}(b_i^0)$. To this end we introduce twisting of $b_i^t$ and total twisting on $S_t$.

Consider a natural projection $P : M_{(S, \phi^m)} \to S$ defined by $P(x, t) = x$. Below, for sake of simplicity we freely identify an arc in a page $S_t$ and its image under $P$. If $b_i^t > b_i^0$ near $v_0$, there exists an integer $l_{i,j} = l_{i,j}(t) \in \mathbb{Z}_{\leq 0}$ for each $j = 1,\ldots,n$ such that the geometric intersection number

$$i(b_i^l, T_C^{-1}(b_i^0)) := i(P(b_i^l), P(T_C^{-1}(b_i^0)))$$

is minimized when $l = l_{i,j}$. We define $tw(b_i^t)$, the twisting of $b_i^t$, by

$$tw(b_i^t) := \begin{cases} 0 & \text{if } b_i^0 \geq b_i^t \text{ near } v_i, \\ 1 + \sum_{j=1}^n \left(i(b_i^t, b_j^0) - i(b_i^l, T_C^{-1}(b_i^0))\right) & \text{if } b_i^l > b_i^t \text{ near } v_i. \end{cases}$$

By definition, $tw(b_i^t) \geq 0$. For each $i$, $tw(b_i^t)$ is a step function of $t$ and the value possibly increases at $t = t_1, t_2, \ldots, t_N$, because $\sgn(h_1) = \cdots = \sgn(h_{Nm}) = -1$ and $\sgn(v_i) = -1$. Morally, $tw(b_i^t)$ measures how much the diffeomorphism $\phi^{-m}$ turns $b_i^0$ to the left near $v_i$ on the page $S_t$.

Here is an alternative explanation of $tw(b_i^t)$. Assume that $b_i^t$ and the multi-curve $b_0^i \cup \cdots \cup b_0^n$ attain the minimal geometric intersection number. We consider the subset

$$I_i^t \subset \text{int}(b_i^t) \cap (b_0^i \cup \cdots \cup b_0^n)$$

that represents the $T_C^{-1}$-factor contributing to $\phi^{-m}$. We have

$$tw(b_i^t) = |I_i^t| + 1,$$

where “+1” corresponds to the end point $v_i$. Since $I_i^t$ locates near $C$ we can take an annular collar neighborhood $N(C)$ of $C$ containing $I_1^t \cup \cdots \cup I_n^t$ such that the term

$$\sum_{j=1}^n \left(i(b_i^t, b_j^0) - i(b_i^l, T_C^{-1}(b_i^0))\right) = |I_i^t| = |\text{int}(b_i^t) \cap (b_0^i \cup \cdots \cup b_0^n) \cap N(C)|$$

counts the number of intersection points lying in $N(C)$. See Figure 3 where the gray area represents $N(C)$ and the dashed lines denote the multi-curve $(b_0^i \cup \cdots \cup b_0^n) \cap N(C)$. When $t = m$, from the definition of $tw(b_i^m)$ for every $i = 1,\ldots,n$ we have:

$$T_C^{-\nu}(b_0^i) \geq b_m^i \quad \text{where } \nu = \left[\frac{tw(b_i^m)}{n}\right] \geq 0$$

i.e.,

$$\phi^m(b_m^i) = b_i^m \geq T_C^{-\nu}(b_m^i).$$

Hence Lemma 4.9 implies that

$$c(\phi^m, C) \leq \left[\frac{tw(b_i^m)}{n}\right]$$

for all $i = 1,\ldots,n$. 

(5.4)
The total twisting $TW_t^C$ on the fiber $S_t$ along the boundary $C$ is defined by:

\[
TW_t^C := \sum_{i=1}^{n} tw(b_i^t)
\]

See the right picture in Figure 9. As we have seen in the proof of Lemma 5.1 positive hyperbolic points do not contribute to the total twisting, so in the next two observations we concentrate on the effect of negative hyperbolic points $h_1, \ldots, h_mN$:

\textbf{Observation 5.6.} For each $j = 1, \ldots, mN$ the hyperbolic point $h_j$ increases the total twisting by at most $n$, namely:

\[
0 \leq TW_{t_j+\varepsilon}^C - TW_{t_j-\varepsilon}^C \leq n.
\]

\textit{Proof.} Since $\text{sgn}(h_j) = -1$ and $\text{sgn}(v_i) = -1$, we have $0 \leq TW_{t_j+\varepsilon}^C - TW_{t_j-\varepsilon}^C$.

Let $\gamma_j$ denote the describing arc for $h_j$. We may assume that at least one of the endpoints of $\gamma_j$ stays in the region $N(C)$. We have

\[
TW_{t_j+\varepsilon}^C - TW_{t_j-\varepsilon}^C = |\gamma_j \cap (b_1^t \cup \cdots \cup b_n^t) \cap N(C)| \leq n.
\]

\hfill \Box

In general, the equality $TW_{t_j+\varepsilon}^C - TW_{t_j-\varepsilon}^C = n$ may hold when the b-arcs $b_{t_j-\varepsilon}^1, \ldots, b_{t_j-\varepsilon}^n$ get out of $N(C)$ through (1) a single region between consecutive $b_{t_j-\varepsilon}^0$ and $b_{t_j-\varepsilon}^{0+1}$, or (2) two regions as depicted in Figure 10.

\textbf{Observation 5.7.} For each $j = 1, \ldots, \lfloor \frac{n}{2} \rfloor$ the $j$-th negative hyperbolic point $h_j$ increases the total twisting by at most $2j$, that is,

\[
0 \leq TW_{t_j+\varepsilon}^C - TW_{t_j-\varepsilon}^C \leq 2j.
\]

\textit{Proof.} Suppose that the describing arc $\gamma_1$ for $h_1$ joins $b^i_1$ and $b^i_2$ for some $i_0, i_1 \in \{1, \ldots, n\}$. $\gamma_j$ joins $b^{i_j+j-1}$ and $b^{i_j+j-1}$ for all $j = 1, \ldots, \lfloor \frac{n}{2} \rfloor$ if and only if the the equality $TW_{t_j+\varepsilon}^C - TW_{t_j-\varepsilon}^C = 2j$ holds. See Figure 11 where the cases $j = 1, 2$ are depicted. \hfill \Box
From Observations 5.6 and 5.7 letting $k = \lfloor \frac{n}{2} \rfloor$ we have:
\[
TW_m^C \leq 2 + 4 + \cdots + 2k + (Nm - k)n
\]
\[
= \begin{cases} 
Nm - \frac{(n-1)^2}{4} & n: \text{odd} \\
Nm - \frac{n(n-2)}{4} & n: \text{even}
\end{cases}
\]
Thus by (5.5) we have:
\[
\min_{i=1,\ldots,n} \{tw(b_m^i)\} \leq \begin{cases} 
Nm - \frac{(n-1)^2}{4n} & n: \text{odd} \\
Nm - \frac{(n-2)}{4} & n: \text{even}
\end{cases}
\]
By (5.4) we have for all $m \in \mathbb{N}$:
\[
m \cdot c(\phi, C) = c(\phi^m, C) \leq \begin{cases} 
\left\lceil \frac{Nm}{n} - \frac{(1-n)^2}{4n^2} \right\rceil & n: \text{odd} \\
\left\lceil \frac{Nm}{n} - \frac{n-2}{4n} \right\rceil & n: \text{even}
\end{cases}
\]
5.2. **Estimates of** $c(\phi_L, C)$. Now we apply the results in Section 5.1 to obtain estimates of the fractional Dehn twist coefficients for closed braid, $c(\phi_L, C)$.

One remarkable distinction between the estimates of $c(\phi, C)$ and those of $c(\phi_L, C)$ is that the former requires $\mathcal{F}_{ob}(F)$ to be strongly essential, while the latter requires just essentiality by the following reason:

Let $L$ be an $n$-stranded closed braid in an open book $(S, \phi)$. Let $F$ be either a Seifert surface of $L$ or a closed surface in the complement of $L$. Suppose that $\mathcal{F}_{ob}(F)$ is essential. Recall that $M_{(S,\phi)} \setminus (L \cup B)$ is fibered over $S'$ with the fiber $S' = S \setminus \{n \text{ points}\}$ and the monodromy $\phi_L \in \text{Aut}(S', \partial S)$. Observe that every $a$-leaf of $\mathcal{F}_{ob}(F)$ is an essential arc in $S'$ and if $l$ is an essential $b$-arc of $\mathcal{F}_{ob}(F)$ then $l$ is an essential arc in $S'$. Therefore each non-singular leaf of $\mathcal{F}_{ob}(F)$ is naturally regarded as an essential arc in $S'$.

The following are variations of Lemma 5.1 and Theorem 5.4.

**Lemma 5.8.** Let $v$ be an elliptic point of $\mathcal{F}_{ob}(F)$ lying on a binding component $C \subset \partial S$. Assume that $v$ is essential. Let $p$ (resp. $n$) be the number of positive (resp. negative) hyperbolic points that are joined with $v$ by a singular leaf.

1. If $\text{sgn}(v) = +1$ then $-n \leq c(\phi_L, C) \leq p$.
2. If $\text{sgn}(v) = -1$ then $-p \leq c(\phi_L, C) \leq n$.

**Theorem 5.9.** Let $v_1, \ldots, v_n \in \mathcal{F}_{ob}(F)$ be essential elliptic points lying on the same component $C$ of $\partial S$. Let $N$ (resp. $P$) be the total number of negative (resp. positive) hyperbolic points that are connected to at least one of $v_1, \ldots, v_n$ by a singular leaf. Let $f_{\pm} : N \to \mathbb{Q}$ be a map defined by

$$f_{-}(m) = \begin{cases} \frac{1}{m} \left\lfloor \frac{Nm}{n} - \frac{(n-1)^2}{4n^2} \right\rfloor & \text{(n odd)} \\
\frac{1}{m} \left\lfloor \frac{Nm}{n} - \frac{n-2}{4n} \right\rfloor & \text{(n even)} \end{cases}$$

and

$$f_{+}(m) = \begin{cases} \frac{1}{m} \left\lfloor \frac{Pm}{n} - \frac{(n-1)^2}{4n^2} \right\rfloor & \text{(n odd)} \\
\frac{1}{m} \left\lfloor \frac{Pm}{n} - \frac{n-2}{4n} \right\rfloor & \text{(n even)} \end{cases}$$

1. If $\text{sgn}(v_1) = \text{sgn}(v_2) = \cdots = \text{sgn}(v_n) = -1$, then

$$\inf_{m \in \mathbb{N}} f_{+}(m) \leq c(\phi_L, C) \leq \inf_{m \in \mathbb{N}} f_{-}(m). \tag{5.6}$$

2. If $\text{sgn}(v_1) = \text{sgn}(v_2) = \cdots = \text{sgn}(v_n) = +1$, then

$$\inf_{m \in \mathbb{N}} f_{-}(m) \leq c(\phi_L, C) \leq \inf_{m \in \mathbb{N}} f_{+}(m). \tag{5.7}$$

6. **Non-right-veeringness and open book foliation**

The notion of right-veeringness is closely related to tight/overtwisted-ness of the contact structure supported by the open book: Recall a theorem of Honda-Kazez-Matić [22, Theorem 1.1], (cf. [25, Theorem 6.7] for an alternative proof based on open book foliation).

**Theorem 6.1.** If $\phi \in \text{Aut}(S, \partial S)$ is not right-veering then $(S, \phi)$ supports an overtwisted contact structure.
In this section we characterize non-right-veering monodromy via open book foliation, especially by a special transverse overtwisted disc. Our results highlight the fact that the converse of Theorem 6.1 does not hold in general. Namely, right-veeringness does not imply tightness of the compatible contact structure.

**Theorem 6.2.** A diffeomorphism \( \phi \in \text{Aut}(S, \partial S) \) is not right-veering if and only if there exists a transverse overtwisted disc \( D \) in \((S, \phi)\) whose open book foliation has exactly one negative elliptic point. See Figure 12.

**Proof.** (\(\Rightarrow\)) If \( \phi \) is non-right-veering then there exists an essential properly embedded arc \( \gamma \) in \( S \) such that \( \phi(\gamma) > \gamma \). As shown in the proof of [25, Theorem 6.7] we can construct a transverse overtwisted disc with exactly one negative elliptic point.

(\(\Leftarrow\)) Let \( v \) be the negative elliptic point in \( \mathcal{F}_{ob}(D) \), and \( 0 < t_1 < \cdots < t_k < 1 \) be the numbers such that the page \( S_{t_i} \) contains a positive hyperbolic point \( h_i \) of \( \mathcal{F}_{ob}(D) \). Let \( b_t \subset S_{t} (t \neq t_i) \) be the b-arc emanating from \( v \).

**Claim 6.3.** \( v \) is strongly essential.

**Proof.** Assume contrary that \( v \) is not strongly essential. We may suppose that the b-arc \( b_0 \) is boundary parallel in \( S_0 \). Let \( w \) be the positive elliptic point that is the other end point of \( b_0 \) so that the arc \( \overline{vw} \subset C \) cobounds a disc \( \Delta_0 \subset S_0 \) with \( b_0 \). We may choose \( b_0 \) so that the disc \( \Delta_0 \) is minimal in the sense that if any b-arc \( b_t \) cobounds a disc \( \Delta_t \) with \( C \) then \( \Delta_t \not\subset \Delta_0 \) under the natural projection \( P : S_t \rightarrow S \). In the following for simplicity of notation we denote the image of arcs and discs under \( P \) by the same symbol. There are two cases to consider:

(i): The disc \( \Delta_0 \) lies on the right side of \( b_0 \) as we walk from \( v \) to \( w \).

(ii): The disc \( \Delta_0 \) lies on the left side of \( b_0 \) as we walk from \( v \) to \( w \).

Let us consider the case (i). Since \( \text{sgn}(h_1) = +1 \) and \( \text{sgn}(v) = -1 \), the describing arc for \( h_1 \) (hence \( b_{t_1+\epsilon} \)) lies on the interior of \( \Delta_0 \). That is, the disc \( \Delta_{t_1+\epsilon} \) co-bounded by \( b_{t_1+\epsilon} \) sits inside \( \Delta_0 \), which contradicts the minimality of \( \Delta_0 \).

A similar argument works for case (ii). \( \square \)

Since all the hyperbolic points have \( \text{sgn} = +1 \), using the same argument as in the proof of Lemma 5.1 we get

\[ \phi(b_1) = b_0 > b_{t_1+\epsilon} > b_{t_2+\epsilon} > \cdots > b_{t_k+\epsilon} = b_1 \] near \( v \).
Claim 6.3 guarantees that $b_1$ is an essential arc in $S_1$. Thus $\phi$ is not right-veering with respect to $C$. This concludes Theorem 6.2. □

Definition 6.4. Let

$$n(S, \phi) = \min \left\{ e_-(F_{ob}(D)) \mid D \subset M_{(S, \phi)} : \text{transverse overtwisted disk} \right\}.$$ 

If $\xi_{(S, \phi)}$ is tight let $n(S, \phi) = 0$. It is an invariant of open books and we call it the overtwisted complexity.

Corollary 6.5. As a consequence of Theorem 6.2 we have:

- $n(S, \phi) = 0$ if and only if $\xi_{(S, \phi)}$ is tight (and hence $\phi$ is right veering).
- $n(S, \phi) = 1$ if and only if $\xi_{(S, \phi)}$ is overtwisted and $\phi$ is non right veering.
- $n(S, \phi) \geq 2$ if and only if $\xi_{(S, \phi)}$ is overtwisted and $\phi$ is right veering.

Example 6.6. Infinitely many examples of the case $n(S, \phi) \geq 2$ can be found in [25, Theorem 6.10].

The open book foliation provides more sufficient conditions for non-right-veering-ness. Recall that a bc-annulus is degenerated if the two b-arc boundaries are identified.

Proposition 6.7. If there exists a (possibly closed) surface $F$ in $M_{(S, \phi)}$ whose open book foliation contains a degenerated bc-annulus $R$ with essential c-circles (cf. Figure 13) then $\phi$ is not right-veering.

Figure 13. Degenerated bc-annulus $R$.

Proof. We may assume that $R$ is as in the right sketch of Figure 13. Let $S_{t_0}$ be the fiber on which the unique hyperbolic point $h \in F_{ob}(R)$ lies. Let $v^+$ and $v^-$ be the positive and the negative elliptic points of $F_{ob}(R)$.

Claim 6.8. $v^\pm$ are strongly essential.

Proof. Assume contrary that $v^\pm$ are not strongly essential, i.e., for any $t \neq t_0$ the b-arc $b_t \subset R \cap S_t$ cobounds a disc $\Delta_t \subset S_t$ with a binding. For $t \in (t_0 - 2\varepsilon, t_0)$ let $c_t \subset R \cap S_t$ denote the c-circle. There are two cases to consider:

1. If $c_{t_0 - \varepsilon} \subset \Delta_{t_0 - \varepsilon}$ then $c_{t_0 - \varepsilon}$ bounds a disc $X \subset \Delta_{t_0 - \varepsilon} \subset S_{t_0 - \varepsilon}$.
2. If $c_{t_0 - \varepsilon} \subset (S_{t_0 - \varepsilon} \setminus \Delta_{t_0 - \varepsilon})$ then $c_{t_0 - \varepsilon} \# b_{t_0 - \varepsilon} \simeq b_{t_0 + \varepsilon}$, so $c_{t_0 - \varepsilon}$ bounds a disc $X \subset S_{t_0 - \varepsilon} \setminus \Delta_{t_0 - \varepsilon}$.

Suppose that $m = |X \cap \partial F|$ and $n = |\Delta_{t_0 - \varepsilon} \cap \partial F|$. Because of the hyperbolic point $h$ we have

$$|\Delta_{t_0 + \varepsilon} \cap \partial F| = \begin{cases} n - m & \text{for case (1),} \\ n + m & \text{for case (2).} \end{cases}$$
Since $\Delta_t$ has no interaction with rest of the world when $t \neq t_0$, we have

$$|\Delta_{t_0-\varepsilon} \cap \partial F| = |\Delta_{t_0+\varepsilon} \cap \partial F|.$$  

Hence $m = 0$, i.e., $c_t$ is inessential, which contradicts our assumption. \hfill \Box

Since the $c$-circles $c_t$ are essential, if $\text{sgn}(R) = \pm 1$ we have

$$\phi(b_1) = b_0 = b_{t_0-\varepsilon} > b_{t_0+\varepsilon} = b_1 \text{ near } v^\mp.$$  

Let $C^\pm \subset \partial S$ be the binding component of the open book that contains $v^\pm$ respectively. By Claim 6.8 the $b$-arc $b_1$ is strongly essential, hence $b_1$ is an essential arc in $S$. Applying Lemma 4.9 we obtain $c(\phi, C^\mp) < 0$ when $\text{sgn}(R) = \pm 1$, respectively, i.e., $\phi$ is not right veering. \hfill \Box

**Corollary 6.9.** Let $D \subset (S, \phi)$ be a disc with an open book foliation $F_{ob}(D)$ satisfying

1. $\text{sl}(\partial D, [D]) = 1$,
2. $F_{ob}(D)$ contains $c$-circles, and
3. $e_-(F_{ob}(D)) = 1$.

Then $\phi \in \text{Aut}(S, \partial S)$ is non-right-veering.

**Proof.** By conditions (2) and (3) the region decomposition of $D$ contains one degenerated bc-annulus, $R$, and $D \setminus R$ contains one (possibly degenerated) ac-annulus and aa-tiles.

**Claim 6.10.** The $c$-circles in $F_{ob}(D)$ are essential.

**Proof.** We assume that $F_{ob}(R)$ is as shown in the right sketch of Figure 13 (the same arguments apply to the left sketch case). A possibly degenerated ac-annulus surrounds $R$ as shown in Figure 14. We may assume that the hyperbolic point of the ac-annulus

\[ (\text{resp. } R) \text{ lives on the page } S_{t_0} \text{ (resp. } S_{h_1}) \text{ for some } 0 < t_0 < t_1 < 1 \text{ and we name it } h_{t_0} \text{ (resp. } h_{h_1}).\]

Proposition 2.7 and the assumptions (1) and (3) imply that $e_+(F_{ob}(D)) = h_+(F_{ob}(D))$ and $h_-(F_{ob}(D)) = 0$. Therefore $\text{sgn}(h_{t_0}) = \text{sgn}(h_{h_1}) = +1$.

Suppose contrary that $c$-circles $c_t \subset (D \cap S_{t})$ for $t_0 < t < t_1$ are inessential. For a very small $\varepsilon > 0$ since $\text{sgn}(h_{t_0}) = +1$ the circle $c_{t_0+\varepsilon}$ bounds a disc in $S_{t_0+\varepsilon}$ on its left side with respect to the orientation of $c_{t_0+\varepsilon}$. On the other hand, since $\text{sgn}(h_{f_1}) = +1$ the circle $c_{t_1-\varepsilon}$ bounds a disc in $S_{t_1-\varepsilon}$ on the right side of $c_{t_1-\varepsilon}$. Since $\{c_t \mid t_0 < t < t_1\}$
is a continuous family of oriented circles they can bound discs only on the same side, hence we get a contradiction.

Claim 6.10 and Proposition 6.7 imply that \( \phi \) is not right-veering.

7. Topology of open book manifolds

In this section we apply technique of the open book foliation to obtain results in topology of the 3-manifold \( M = M(S, \phi) \).

7.1. Incompressible surfaces and \( c(\phi, C) \). In the following two theorems we establish estimates of \( c(\phi, C) \) from topological data of incompressible closed surfaces in \( M \).

**Theorem 7.1** (General case). Suppose that there exists a closed, oriented, incompressible, genus \( g \) surface \( F \) in \( M \) which admits essential open book foliation and intersects the binding in \( 2n \) \((n > 0)\) points.

1. If \( g = 0 \) then \( |c(\phi, C)| \leq 3 \) for some boundary component \( C \).
2. If \( g \geq 1 \) then \( |c(\phi, C)| \leq 4 + \lfloor 4g - 4n \rfloor \) for some boundary component \( C \).

If \( S \) has connected boundary we get even sharper estimates:

**Theorem 7.2** (Connected binding case). Under the same setting of Theorem 7.1, assume further that \( \partial S \) is connected. Then

\[
|c(\phi, \partial S)| \leq \inf_{m \in \mathbb{N}} G(m)
\]

where \( G : \mathbb{N} \rightarrow \mathbb{Q} \) is a map defined by:

\[
G(m) = \begin{cases} 
\frac{1}{m} \left[ \frac{(g-1+n)m}{n} - \frac{(n-1)^2}{4n^2} \right] & (n: \text{odd}) \\
\frac{1}{m} \left[ \frac{(g-1+n)m}{n} - \frac{n-2}{4n} \right] & (n: \text{even})
\end{cases}
\]

In particular:

1. If \( g = 0 \) we have \( |c(\phi, \partial S)| \leq 1 \).
2. If \( g \geq 1 \) we have \( |c(\phi, \partial S)| \leq g \).

Once we have Theorems 7.1 and 7.2 in hand, we give a criterion for \( M(S, \phi) \) to be irreducible or atoroidal in terms of the fractional Dehn twist coefficients.

**Corollary 7.3.** Assume that:

1. \( |c(\phi, C)| > 3 \) for every boundary component \( C \) of \( S \), or
2. \( \partial S \) is connected and \( |c(\phi, \partial S)| > 1 \).

Then the 3-manifold \( M \) is irreducible.

**Proof.** Assume that \( M \) is reducible. There exists a sphere \( S \) which does not bound a 3-ball in \( M \). In particular, \( S \) is incompressible. By Theorem 3.2 \( S \) can admit an essential open book foliation. Since \( (e_+ + e_-) - (h_+ + h_-) = \chi(S) = 2 \) we know that \( F_{ob}(S) \) has elliptic points, i.e., \( S \) intersects the binding. Now Theorem 7.1 (1) together with Theorem 7.2 (\( g = 0 \) case) yields the contrapositive of the statement of Corollary 7.3.

**Corollary 7.4** (First atoroidal criterion). Assume that \( \phi \in \text{Aut}(S, \partial S) \) is of irreducible type and that;
Then the 3-manifold $M$ is irreducible and atoroidal.

Remark. See Theorem 7.8 below for a related result.

Proof. By Corollary 7.3 we know that $M$ is irreducible so it remains to show that $M$ is atoroidal. Assume contrary that $M$ contains an incompressible torus $T$. We isotope $T$ so that it admits an essential open book foliation. Theorems 7.1-(2) and 7.2 ($g = 1$ case) guarantee that $T$ does not intersect the binding, i.e., $\mathcal{F}_{ob}(T)$ contains no elliptic points. Since $(e_+ + e_-) - (h_+ + h_-) = \chi(T) = 0$, $\mathcal{F}_{ob}(T)$ contains no hyperbolic points, that is, all the leaves of $\mathcal{F}_{ob}(T)$ are c-circles.

Hence the sets of c-circles $T \cap S_1$ and $T \cap S_0$ are isotopic in $S$. But the definition of open book decomposition imposes $\phi(T \cap S_1) = T \cap S_0$. Therefore $\phi(T \cap S_1) = T \cap S_1$ which contradicts the assumption that $\phi$ is irreducible. □

Remark. Corollary 7.3 is valid regardless of the type of $\phi$. However if we assume that $\phi$ is pseudo-Anosov then [18, Theorem 5.3, Corollary 5.4] of Gabai and Oertel yield the following stronger result. They view $M_{(S, \phi)}$ as a Dehn filling of the mapping torus of $\phi$ and use essential lamination theory.

Theorem 7.5 (Gabai, Oertel [18]). Let $\phi$ be of pseudo Anosov. For each boundary component $C_i \subset \partial S$ put $c(\phi, C_i) = \frac{p_i}{q_i}$, where $q_i$ is the number of prongs of the crown $W$ in Definition 4.4. Note that $p_i$ and $q_i$ may not be coprime. If $|p_i| > 1$ for all the boundary components $C_i$ then the suspension of the (un)stable lamination of $\phi$ gives an essential lamination in $M_{(S, \phi)}$, hence $M_{(S, \phi)}$ is an irreducible manifold with infinite fundamental group.

Before proving Theorems 7.1 and 7.2 we list some observations:

Observation 7.6. When $F$ is an incompressible closed surface in $M$, there are various convenient properties.

1. Since $F$ has no boundary, every essential b-arc is automatically strongly essential, hence there is no distinction between essentiality and strong essentiality for b-arcs. In particular, any b-arc in an essential open book foliation is strongly essential.

2. $\mathcal{F}_{ob}(F)$ does not have a-arcs. Thus, the region decomposition of $\mathcal{F}_{ob}(F)$ consists only of three types; bb-tile, bc-annulus, and cc-pants.

3. Since the binding $B = \partial S$ is null-homologous in $M$, the algebraic intersection of $F$ with $B$ is zero. Hence the numbers of positive and negative elliptic points of $\mathcal{F}_{ob}(F)$ are equal; $e_+(\mathcal{F}_{ob}) = e_-(\mathcal{F}_{ob})$.

The first two properties imply that if $\mathcal{F}_{ob}(F)$ is essential then the hypotheses of Theorem 5.4 are always satisfied. Thus, in order to estimate $c(\phi, C)$ we only need to count the number of singularities of $\mathcal{F}_{ob}(F)$.

Now we prove Theorem 7.1. The strategy is strongly motivated by braid foliation theory and is similar to [23, Theorem 1.2]. In fact, Theorem 7.1 can be seen as a generalization of [23, Theorem 1.2]. By analyzing the Euler characteristic and the region decomposition of a surface we find an elliptic point $v$ such that the number of hyperbolic
points around $v$ is small. Then we apply Lemma 5.1 to get the desired estimate of $c(\phi, C)$.

**Proof of Theorem 7.1.** We construct a cellular decomposition of the surface $F$ by modifying the region decomposition of $\mathcal{F}_{ob}(F)$. To this end, we construct a singular foliation (but not an open book foliation), $\mathcal{F}'$, on $F$ by replacing each cc-pants of $\mathcal{F}_{ob}(F)$ with three bc-annuli as shown in Figure 15. Though $\mathcal{F}'$ is not derived from the intersection of the surface $F$ with the pages, by abuse of notations we keep using the terminologies of the open book foliation, such as region decomposition, be-tile, bc-singular points, etc. We call the newly-inserted elliptic points and hyperbolic points fake elliptic points and fake hyperbolic points, respectively. The sign of a fake elliptic point is canonically determined. The sign of the hyperbolic points (both fake and non-fake) are not important in the following arguments so we omit it from now on.

The region decomposition of $F$ induced by $\mathcal{F}'$ consists only of bb-tiles and bc-annuli. Using Birman and Menasco’s idea [9] we construct a cellular decomposition of $F$ as follows: A bc-annulus always exists in pair because the c-circle boundary is identified with the c-circle boundary of another bc-annulus. Let $W$ be the annulus obtained by gluing two bc-annuli along their c-circle boundaries (note: $W$ is a disc if one of the bc-annuli is degenerated, also ab-annulus does not exist because $F$ is closed). Each component of $\partial W$ has two elliptic points. Choose two disjoint essential arcs each of which connects elliptic points of opposite sign as shown in Figure 16. Call such arcs...
e-edges. We cut $W$ along the e-edges to obtain two 2-cells, which we name be-tiles. We may choose e-edges so that each be-tile contains exactly one hyperbolic point.

Now we view the surface $F$ as a union of bb-tiles and be-tiles. This defines a cellular decomposition of $F$ such that 0-cells (vertices) are the elliptic points, 1-cells are the b-arcs and e-edges on the boundary of the bb- and be-tiles, and 2-cells are the bb-tiles and be-tiles.

Let $v$ be a 0-cell and $\text{Val}(v)$ denote the valence of $v$ in the 1-skeleton graph. We define $\text{Hyp}(v)$ to be the number of non-fake hyperbolic points in $\mathcal{F}_{ob}(F)$ that are connected to $v$ by a singular leaf.

**Claim 7.7.** Let $v$ be a 0-cell of the cellular decomposition of $F$.

1. If $v$ is a fake elliptic point of $F'$, then $\text{Val}(v) = 6$.
2. If $v$ is not a fake elliptic point of $F'$, then $\text{Hyp}(v) \leq \text{Val}(v)$.

**Proof of claim** 7.7. (1) If $v$ is a fake elliptic point then $v$ sits on some cc-pants $P$ in the region decomposition. In $F'$, $P$ decomposes into three bc-annuli (Figure 15). After converting bc-annuli to be-tiles, at $v$ three b-arc 1-cells and three e-edge 1-cells meet, see Figure 17. Totally $\text{Val}(v) = 6$.

![Figure 17. A fake elliptic point has valence six.](image)

(2) We first note that a vertex $v$ is a non-fake elliptic point in $F'$ if and only if it is an elliptic point of the original foliation $F_{ob}(F)$. If $v$ lies in the interior of a degenerated bc-annulus in $F_{ob}(F)$ then $\text{Hyp}(v) = 1 < 2 = \text{Val}(v)$.

Next assume that $x$ bb-tiles and $y$ bc-annuli in $F_{ob}(F)$ meet at $v$. Then $\text{Hyp}(v) = x + y \leq x + 2y \leq \text{Val}(v)$ because each bb-tile contains one non-fake hyperbolic point and each bc-annulus contains one e-edge ending at $v$. \hfill \Box

Let us define

$$s = \min \{ \text{Hyp}(v) \mid v \text{ is a 0-cell and non-fake elliptic point} \} .$$

Fix a vertex $v$ realizing $\text{Hyp}(v) = s$. Suppose that $v$ lies on the binding component $C \subset \partial S$. Observation 7.6(1) guarantees that $v$ is strongly essential hence by Lemma 5.1

$$-s \leq c(\phi, C) \leq s .$$

Our goal is to show that $s \leq 3$ if $g = 0$ and $s \leq 4 + \left\lfloor \frac{4g - 4}{n} \right\rfloor$ if $g > 0$.

Consider the cellular decomposition of $F$. For $i \geq 1$ let $V(i)$ be the number of 0-cells of valence $i$, and let $E$ be the number of 1-cells, and $R$ be the number of 2-cells, or the
total number of bb-tiles and be-tiles. Since each 1-cell is a common boundary of distinct
two 2-cells and each 2-cell has distinct four 1-cells on its boundary, we have:
\[(7.1)\]
\[E = 2R.\]
Since the end points of a 1-cell are distinct two 0-cells we have:
\[(7.2)\]
\[\sum_{i \geq 1} IV(i) = 2E.\]
The Euler characteristic of \(F\) is:
\[(7.3)\]
\[\sum_{i \geq 1} V(i) - E + R = \chi(F)\]
From (7.1), (7.2) and (7.3), we get the Euler characteristic equality:
\[(7.4)\]
\[\sum_{i \geq 1} (4 - i)V(i) = 4\chi(F).\]
(1) First we assume that \(F\) is a sphere. The equality (7.4) implies:
\[3V(1) + 2V(2) + V(3) = 8 + \sum_{i \geq 4} (i - 4)V(i)\]
The right hand side (hence the left hand side) is positive. So there exists a vertex \(v\) with
\[\text{Val}(v) \leq 3\] and Claim 7.7-(1) implies that \(v\) is not a fake vertex. By Claim 7.7-(2) we
obtain
\[|c(\phi, C)| \leq s \leq \text{Hyp}(v) \leq \text{Val}(v) \leq 3.\]
(2) Next we assume that \(F\) has genus \(g > 0\) so \(\chi(F) = 2 - 2g < 0\). The Euler
characteristic equality (7.4) is that:
\[0 \leq 3V(1) + 2V(2) + V(3) + 8g - 8 = \sum_{i \geq 4} (i - 4)V(i).\]
If at least one of \(V(1), V(2)\) and \(V(3)\) is positive then by Claim 7.7 there exists a non-
fake elliptic point \(v\) such that \(s \leq \text{Hyp}(v) \leq \text{Val}(v) \leq 3\). Suppose that \(V(1) = V(2) = V(3) = 0\). By Observation 7.6-(3) the original open book foliation \(\mathcal{F}_{ob}(F)\) contains an
even number \((= 2n)\) of elliptic points. Therefore;
\[8g - 8 = \sum_{i \geq 4} (i - 4)V(i) \geq (s - 4)2n,\]
i.e., \(s \leq 4 + \frac{4g - 4}{n}\). In either case since \(s\) is an integer \(s \leq 4 + \lfloor \frac{4g - 4}{n} \rfloor\). \(\square\)
Finally we prove Theorem 7.2 by using Theorem 5.4 and Observation 7.6.

**Proof of Theorem 7.2.** We see in Observation 7.6-(3) that \(e_-(\mathcal{F}_{ob}(F)) = e_+(\mathcal{F}_{ob}(F)) = n\). Let \(v_1^+, \ldots, v_n^+\) be the positive elliptic points and \(v_1^-, \ldots, v_n^-\) be the negative elliptic points of \(\mathcal{F}_{ob}(F)\). By Proposition 2.7-(2),
\[(7.5)\]
\[h_+ + h_- = -\chi(F) + e_+ + e_- = 2g - 2 + 2n.\]
Since \(\partial S\) is connected, all the elliptic points lie on \(\partial S\). So by Theorem 5.4 we have:
\[|c(\phi, \partial S)| \leq \min \left\{ \inf_{m \in \mathbb{N}} \left[ f_+(m) \right], \inf_{m \in \mathbb{N}} \left[ f_-(m) \right] \right\}\]
where the functions $f_{\pm}(m) : \mathbb{N} \to \mathbb{Q}$ satisfy:

$$f_{\pm}(m) \leq \begin{cases} 
\frac{h_{\pm}m}{n} - \frac{(n-1)^2}{4n^2} & (n : \text{odd}) \\
\frac{h_{\pm}m}{n} - \frac{n-2}{4n} & (n : \text{even})
\end{cases}$$

Hence by the equation (7.6) we get:

$$|c(\phi, \partial S)| \leq \begin{cases} 
\inf_{m \in \mathbb{N}} \left( \frac{1}{m} \left[ \frac{(g-1+n)m}{n} - \frac{(n-1)^2}{4n^2} \right] \right) & (n : \text{odd}) \\
\inf_{m \in \mathbb{N}} \left( \frac{1}{m} \left[ \frac{(g-1+n)m}{n} - \frac{n-2}{4n} \right] \right) & (n : \text{even})
\end{cases}$$

7.2. Tight contact structure and atoroidal manifold. With tightness assumption on the underlying contact manifold we refine the atoroidal criterion in Corollary 7.4.

**Theorem 7.8 (Second atoroidal criterion).** Let $(S, \phi)$ be an open book supporting a tight contact structure. If $\phi$ is of irreducible type and $c(\phi, C) > 2$ for every boundary component $C \subset \partial S$ then $M(S, \phi)$ is atoroidal.

**Proof.** Let $T$ be an incompressible torus in $M(S, \phi)$. Put $T$ in position admitting an essential open book foliation $\mathcal{F}_{ob}(T)$. If $T$ does not intersect the binding, the same argument as in the proof of Corollary 7.4 implies that $\phi$ is reducible. So we may assume that $\mathcal{F}_{ob}(T)$ contains at least one elliptic point. Since $e_+ = e_-$, we may actually assume that $\mathcal{F}_{ob}(T)$ contains at least two elliptic points.

As in the proof of Theorem 7.1, let us consider the cellular decomposition of $T$ induced from the region decomposition of $\mathcal{T}$. The 2-cells are bb-tiles and be-tiles. The region decomposition contains no cc-pants by Euler characteristic argument, so the cellular decomposition has no fake elliptic points.

For a negative elliptic point $v \in \mathcal{F}_{ob}(T)$, let $N(v)$ denote the number of 2-cells around $v$ that contain a negative hyperbolic point. Let $\text{Hyp}^-(v)$ be the number of negative hyperbolic points that are connected to $v$ by a singular leaf. By Lemma 5.1 and Observation 7.6-(1), if $v$ lies on the binding component $C \subset \partial S$ then

$$c(\phi, C) \leq \text{Hyp}^-(v) \leq N(v).$$

(7.6)

For simplicity let $e_{\pm} = e_{\pm}(\mathcal{F}_{ob}(\mathcal{T}))$ and $h_{\pm} = h_{\pm}(\mathcal{F}_{ob}(\mathcal{T}))$ the numbers of $(\pm)$ elliptic/hyperbolic points in $\mathcal{F}_{ob}(\mathcal{T})$. For $i \geq 0$ let $w_i$ be the number of negative elliptic points $v$ with $N(v) = i$. Then

$$\sum_{i \geq 0} w_i = e_-.$$ 

(7.7)

Since each 2-cell contains exactly two negative elliptic points

$$\sum_{i \geq 0} iw_i = 2h_-.$$ 

(7.8)

By Poincaré-Hopf formula,

$$0 = \chi(\mathcal{T}) = (e_+ + e_-) - (h_+ + h_-).$$
Let \( e(\xi) \) be the Euler class of the tight contact structure \( \xi = \xi(S, \phi) \) supported by \((S, \phi)\). By Bennequin-Eliashberg inequality, we have
\[
|e(\xi) - (h_+ - h_-)| = |\langle [T], e(\xi) \rangle| \leq -\chi(T) = 0.
\]
Since \( e_+ = e_- > 0 \) (cf. Observation 7.6-(3)) we conclude that
\[
(7.9) \quad e_+ = e_- = h_+ = h_- > 0.
\]
By (7.7), (7.8) and (7.9), we have
\[
2w_0 + w_1 = \sum_{i>2} (i-2)w_i > 0,
\]
or \( w_2 > 0 \) and \( w_i = 0 \) for \( i \neq 2 \). This shows that either \( w_0, w_1 \) or \( w_2 \) is positive. Hence there exists a negative elliptic point \( v \) with \( N(v) \leq 2 \), so by (7.6) we get \( c(\phi, C) \leq 2 \) for the binding component \( C \) where \( v \) lies.
\( \square \)

If the contact structure \( \xi(S, \phi) \) is tight one may refine the estimates in Section 7.1 by the same technique as in the above proof, namely combination of Bennequin-Eliashberg inequality, counts of vertices and Euler characteristic argument.

7.3. **Incompressible surfaces and \( c(\phi_L, C) \).** By the same proofs of Theorems 7.1 and 7.2 we obtain the following two results.

**Proposition 7.9.** Let \( L \) be a closed \( n \)-braid in an open book \((S, \phi)\). Suppose that there exists a closed, oriented, incompressible, genus \( g \) surface \( F \) in \( M - L \) which admits an essential open book foliation and intersects the binding in \( 2k \) \((k > 0)\) points.

1. If \( g = 0 \) then \( |c(\phi_L, C)| \leq 3 \) for some boundary component \( C \) of \( S \).
2. If \( g > 0 \) then \( |c(\phi_L, C)| \leq 4 + \left\lfloor \frac{4g-4}{k} \right\rfloor \) for some boundary component \( C \) of \( S \).

**Proposition 7.10** (Connected binding case). Under the same setting of Proposition 7.9, assume further that \( \partial S \) is connected. Then
\[
|c(\phi_L, \partial S)| \leq \inf_{m \in \mathbb{N}} G(m)
\]
where \( G : \mathbb{N} \to \mathbb{Q} \) is a map defined by:
\[
G(m) = \begin{cases}
\frac{1}{m} \left[ \frac{(g-1+k)m}{k} - \frac{(k-1)^2}{4k^2} \right] & (k: \text{odd}) \\
\frac{1}{m} \left[ \frac{(g-1+k)m}{k} - \frac{k-2}{4k} \right] & (k: \text{even})
\end{cases}
\]

**Remark 7.11.** The same statements as Corollaries 7.3 and 7.4 (atoroidal criterion), where \( M \) is replaced by \( M - L \) and \( c(\phi, C) \) is replaced by \( c(\phi_L, C) \), hold.

Now we relate \( c(\phi_L, C) \) and an incompressible Seifert surface for the closed braid \( L \). The following theorem or its corollary (Corollary 7.13) plays an essential role in the proof of our main result Theorem 8.3.

**Theorem 7.12.** Let \( L \) be a null-homologous, closed \( n \)-braid in an open book \((S, \phi)\). Let \( F \) be a Seifert surface of \( L \) attaining the maximal Euler characteristic, \( \chi(F) \).

1. If \( \chi(F) > 0 \) then \( |c(\phi_L, C)| \leq 3 \) for some boundary component \( C \subset \partial S \).
(1b) If $\chi(F) < 0$ and $F$ intersects the bindings in $k(> 0)$ points, then there exists a binding component $C \subset \partial S$ such that

$$|c(\phi_L, C)| \leq \min \left\{ -\frac{4}{k} \chi(F), +, -\chi(F) + k \right\}.$$

(2) Moreover, if $\partial S$ is connected and $\chi(F) \leq 0$ then

$$|c(\phi_L, \partial S)| \leq \frac{n - \chi(F)}{n}.$$

Proof. The idea of the proof is similar to that of Theorem 7.1, but we need extra arguments because $F$ has non-empty boundary in this setting. We may assume that $F$ is incompressible, hence by Theorem 3.2 the open book foliation $\mathcal{F}_{ob}(F)$ is essential.

Consider the closed surface $\hat{F}$ obtained by identifying each boundary component of $F$ with a point. As in the proof of Theorem 7.1, we get a cellular decomposition of $\hat{F}$ from the region decomposition of $\mathcal{F}_{ob}(F)$. Since $F$ is not a closed surface we need the following operation in addition to the one described in Figure 16: If there exists an ac-annulus it is paired up with another bc-annulus (or ac-annulus) as in Figure 18. We introduce d-edge(s) to cut the region into two abde-tiles (or two ad-tiles). We may assume that each tile contains one hyperbolic point.

**Figure 18.** From a pair of ac- and bc-annuli to two abde-tiles.

In the cell decomposition of $\hat{F}$ a 0-cell is either an elliptic point, a fake elliptic point or a newly attached point to a boundary component of $F$, which we call a boundary 0-cell. Let us call fake and non-fake elliptic points interior 0-cell. Also call a 1-cell that ends (resp. does not end) on a boundary 0-cell boundary 1-cell (resp. interior 1-cell). Note a boundary 1-cell is an a-arc or a d-edge, and an interior 1-cell is a b-arc or an e-edge. If $\mathcal{F}_{ob}(F)$ contains no c-circles then the set of boundary 1-cells of $\hat{F}$ is identified with the set of a-arc 1-cells in the cell decomposition of $F$.

We say that an interior 0-cell $w$ is of type $(i, j)$ if $w$ has valence $(i + j)$ and is a common endpoint of $i$ boundary 1-cells and $j$ interior 1-cells. Let $V(i, j)$ be the number of interior 0-cells of type $(i, j)$. Let $E_\partial$ be the number of boundary 1-cells, $E$ be the number of interior 1-cells, and $R$ be the number of 2-cells. Since each 1-cell is a common boundary of two 2-cells (degenerated 2-cells are counted with multiplicity = 2) and each 2-cell has four 1-cells (degenerated 1-cells are counted with multiplicity = 2) we have:

$$E + E_\partial = 2R.$$
Since each boundary 1-cell contains one interior 0-cell,

\[(7.11) \quad \sum_{n=1}^{\infty} \sum_{i=0}^{n} iV(i, n-i) = E_{\partial}.\]

Since both the endpoints of an interior 1-cell are two distinct interior 0-cells, counting the number of interior 1-cells we get:

\[(7.12) \quad \sum_{n=1}^{\infty} \sum_{i=0}^{n} (n-i)V(i, n-i) = 2E.\]

Let \(d\) be the number of boundary components of \(F\). The Euler characteristic satisfies:

\[(7.13) \quad \chi(\hat{F}) = d + \chi(F) = \left( d + \sum_{n=1}^{\infty} \sum_{i=0}^{n} V(i, n-i) \right) - (E + E_{\partial}) + R.\]

From (7.10), (7.11), (7.12) and (7.13), we get the Euler characteristic equality:

\[(7.14) \quad 4\chi(F) = \sum_{n=1}^{\infty} \sum_{i=0}^{n} (4-n-i)V(i, n-i).\]

(1a) Suppose that \(\chi(F) > 0\). We view (7.14) as

\[3V(0,1) + 2V(1,0) + 2V(0,2) + V(1,1) + V(0,3) = 4\chi(F) + V(2,1) + 2V(3,0) + \sum_{n=4}^{\infty} \sum_{i=0}^{n} (n+i-4)V(i, n-i) > 0.\]

By Claim 7.7 there exists a non-fake elliptic point of valence at most three, and Lemma 5.8 shows that \(|c(\phi_{L}, C)| \leq 3\) for some \(C \subset \partial S\).

(1b) Suppose that \(\chi(F) < 0\).

First we show \(|c(\phi_{L}, C)| \leq 4\chi(F) + 4\). As in the proof of Theorem 7.1 let \(\text{Hyp}(v)\) denote the number of non-fake hyperbolic points in that are connected to a 0-cell \(v\) by a singular leaf of \(\mathcal{F}_{ob}(F)\). Let us define

\[s = \min\{\text{Hyp}(v) \mid v \text{ is a 0-cell and non-fake elliptic point}\}.\]

Let \(C \subset \partial S\) be the binding component that contains an elliptic point \(v\) which attains \(s\). We rewrite (7.14) as

\[0 < 3V(0,1) + 2V(1,0) + 2V(0,2) + V(1,1) + V(0,3) - 4\chi(F)\]

\[(7.15) \quad = V(2,1) + 2V(3,0) + \sum_{n=4}^{\infty} \sum_{i=0}^{n} (n+i-4)V(i, n-i).\]

If at least one of \(V(0,1), V(1,0), V(0,2), V(1,1), V(0,3), V(2,1),\) and \(V(3,0)\) is positive then \(|c(\phi, C)| \leq s \leq i + j \leq 3\).

\[|c(\phi, C)| \leq s \leq i + j \leq 3 \leq \left\lfloor \frac{-4}{k} \chi(F) \right\rfloor + 4.\]
If $V(i, j) = 0$ whenever $i + j \leq 3$ then (7.15) and a similar argument as in the last paragraph of the proof of Theorem 7.1 show that

$$|c(\phi_L, C)| \leq s \leq \left\lfloor \frac{-4}{h} \chi(F) \right\rfloor + 4.$$ 

Next we show $|c(\phi_L, C)| \leq -\chi(F) + k$. Let $h$ be the number of hyperbolic points of $\mathcal{F}_{ob}(F)$. Since $k$, the number of the intersection of $F$ and the binding, is equal to the number of elliptic points of $\mathcal{F}_{ob}(F)$, by Poincaré-Hopf formula we have $h = -\chi(F) + k$. Hence for any binding component $C \subset \partial S$ that intersects $F$ we have

$$|c(\phi_L, C)| \leq s \leq h = -\chi(F) + k.$$ 

(2) Assume that $\partial S$ is connected. All the elliptic points of $\mathcal{F}_{ob}(F)$ lie on the binding $\partial S$ and the algebraic intersection number of $F$ and $\partial S$ satisfies $n = F \cdot \partial S = e_+ - e_-\,$, where $n$ is the braid index of $L$. For $\varepsilon, \delta \in \{\pm\}$ let us define $f_{\varepsilon, \delta}(m) : N \to \mathbb{Q}$ by

$$f_{\varepsilon, \delta}(m) = \frac{1}{m} \left\lfloor \frac{mh_{\varepsilon}}{e_\delta} \right\rfloor.$$ 

If $e_- = 0$, by Theorem 5.9 (2) we have:

$$|c(\phi_L, \partial S)| \leq \max \left\{ \inf_{m \in \mathbb{N}} f_{++}(m), \inf_{m \in \mathbb{N}} f_{--}(m) \right\}.$$ 

Since $h_+ + h_- = e_+ + e_- - \chi(F) = n - \chi(F)$ (Proposition 2.7 (2)) and $e_\pm, h_\pm \geq 0$ we obtain

$$|c(\phi_L, \partial S)| \leq \inf_{m \in \mathbb{N}} \frac{1}{m} \left\lfloor \frac{m(n - \chi(F))}{n} \right\rfloor \leq \frac{1}{n} \left\lfloor \frac{n(n - \chi(F))}{n} \right\rfloor = \frac{n - \chi(F)}{n}.$$ 

Next assume that $e_- > 0$ i.e., $e_+ = e_- + n$. By Theorem 5.9 we have:

$$-\min \left\{ \inf_{m \in \mathbb{N}} f_{++}(m), \inf_{m \in \mathbb{N}} f_{--}(m) \right\} \leq c(\phi_L, \partial S) \leq \min \left\{ \inf_{m \in \mathbb{N}} f_{++}(m), \inf_{m \in \mathbb{N}} f_{--}(m) \right\}$$ 

hence

$$|c(\phi_L, \partial S)| \leq \max \left\{ \min \left\{ \inf_{m \in \mathbb{N}} f_{++}(m), \inf_{m \in \mathbb{N}} f_{--}(m) \right\}, \min \left\{ \inf_{m \in \mathbb{N}} f_{--}(m), \inf_{m \in \mathbb{N}} f_{++}(m) \right\} \right\}.$$ 

Since $e_+ = n + e_- > e_-$ and $h_+ + h_- = e_+ + e_- - \chi(F) = n + 2e_- - \chi(F)$ we get

$$|c(\phi_L, \partial S)| \leq \inf_{m \in \mathbb{N}} \frac{1}{m} \left\lfloor \frac{m(h_+ + h_-)/2}{e_+} \right\rfloor$$ 

(7.16)

$$\leq \frac{n + 2e_- - \chi(F)}{2(n + e_-)}$$

$$< 1 - \frac{\chi(F)}{2n} \quad \text{(since } \chi(F) \leq 0)$$

$$\leq \frac{n - \chi(F)}{n}. \quad \square$$

Theorem (7.12) gives simple estimates for the genera of null-homologous closed braids.

**Corollary 7.13.** Assume that $L$ is a knot of genus $g(L)$.
(1) We have
\[ g(L) \geq \frac{1}{2} \left( \min_{C \subset \partial S} \{ |c(\phi_L, C)| \} - 3 \right). \]

(2) Assume that \( \partial S \) is connected.
(a) If \( g(L) = 0 \) (i.e., \( L \) is an unknot) then \( |c(\phi_L, \partial S)| < 1 \),
(b) if \( g(L) > 0 \) then \( |c(\phi_L, \partial S)| \leq 2g(L) \).

Proof. (1) If \( g(L) = 0 \) then by (1a) of Theorem 7.12 we have \( |c(\phi_L, C)| \leq 2 \) for some \( C \subset \partial S \). If \( g(L) \geq 1 \) then by (1b) of Theorem 7.12 we have for some \( C \subset \partial S 
\begin{align*}
|c(\phi_L, C)| &\leq \min \left\{ \left\lfloor \frac{-4 \chi(F)}{k} \right\rfloor + 4, \chi(F) + k \right\} \\
&\leq -\chi(F) + 4 \\
&\leq 2g(L) + 3
\end{align*}
(2a) If \( g(L) = 0 \), plugging \( \chi(F) = 1 \) into (7.16) we get
\[ |c(\phi_L, \partial S)| \leq \frac{n + 2e_\pm - 1}{2(n + e_\pm)} < 1. \]
(2b) is a direct consequence of Theorem 7.12 (2).

Remark. In the proof of Theorem 7.12 we use weaker form of the estimates in Theorem 5.4. By using the original form of Theorem 5.4 we may sharpen the estimates in Theorem 7.12 and Corollary 7.13.

8. Geometric structures of open book manifolds and braid complements

We apply results in Section 7 to study geometric structures of open book manifolds. First we observe that periodic monodromy implies Seifert-fibered in most cases.

Proposition 8.1. Assume that \( \phi \in \text{Aut}(S, \partial S) \) is periodic and \( c(\phi, C) \neq 0 \) for every boundary component \( C \) of \( S \). Then the 3-manifold \( M = M(S, \phi) \) is Seifert fibered.

Proof. Let us put \( c(\phi, C_i) = \frac{p_i}{q_i} \) where \((p_i, q_i)\) are coprime integers and \( p_i > 0 \). Let
\[ A_i = C_i \times [1, 2] \cong \{ z \in \mathbb{C} \mid 1 \leq |z| \leq 2 \} \]
be an annular neighborhood of \( C_i \), where \( C_i \) is identified with \( C_i \times \{1\} \). Since \( c(\phi, C_i) = \frac{p_i}{q_i} \), we may choose \( A_i \) so that:
- \( \phi(A_i) = A_i \).
- \( \phi(z) = z \exp(-2\pi \sqrt{-1}(|z| - 1)\frac{p_i}{q_i}) \) for \( z \in A_i \) = \{ \( z \in \mathbb{C} \mid 1 \leq |z| \leq 2 \} \).

Let \( B_i \) be the binding component corresponding to \( C_i \). Take a regular neighborhood \( N_i \) of \( B_i \) so that \( N_i \cap S_i \cong A_i \) for all \( t \in [0, 1] \). The complement of the binding \( M \setminus \bigcup \{ N_i \} \cong M_{\phi} \), the mapping torus of \( \phi \), is Seifert-fibered as \( \phi \) is periodic. The fibers on \( \partial N_i \subset \partial M_{\phi} \) are regular and each represents the homology class \( p_i[C_i] + q_i[S^1] \in H_1(\partial N_i; \mathbb{Z}) \) where \([S^1]\) corresponds to a meridian disc of \( N_i \).

The assumption \( c(\phi, C_i) = \frac{p_i}{q_i} \neq 0 \) implies that the Seifert fibration of \( M_{\phi} \) extends to \( N_i \), by adding the binding \( B_i \) as an exceptional fiber of the Seifert invariant \((\alpha_i, \beta_i)\) where \( 0 \leq \beta_i < \alpha_i = p_i \) and \( \beta_i \equiv q_i \) (mod \( p_i \)).
Remark 8.2. It is necessary to assume \( c(\phi, C_1) \neq 0 \). For example, if \( S = S_{g,1} \) is a genus \( g > 0 \) surface with one boundary then \( M_{(S, \text{id})} = \#_{2g}(S^1 \times S^2) \), which admits no Seifert fibered structure as \( \mathbb{RP}^3 \# \mathbb{RP}^3 \) is the only Seifert fibered manifold that is not prime (cf. [21]).

There is tight relationship among Nielsen-Thurston classification, fractional Dehn twist coefficients, and geometric structures. Thurston [32] proved that the mapping torus \( M_\phi \) of \( \phi \in \text{Aut}(S) \) is Seifert-fibered (toroidal, hyperbolic) if and only if \( \phi \) is periodic (reducible, pseudo-Anosov). In [23, Theorem 1.3] the first named author generalized this to the complements of closed braids in \( S^3 = M_{(D^2, \text{id})} \) by using braid foliation. We prove parallel results for \( M = M_{(S, \phi)} \) and \( M - L \) the braid complement.

Theorem 8.3. Let \((S, \phi)\) be an open book decomposition of 3-manifold \( M \). Assume one of the following:

- \( \partial S \) is connected and \( |c(\phi, \partial S)| > 1 \).
- \( |c(\phi, C)| > 4 \) for every boundary component \( C \) of \( S \).

Then we have the following:

1. \( M \) is toroidal if and only if \( \phi \) is reducible.
2. \( M \) is hyperbolic if and only if \( \phi \) is pseudo-Anosov.
3. \( M \) is Seifert fibered if and only if \( \phi \) is periodic.

Theorem 8.4. Let \((S, \phi)\) be an open book decomposition of 3-manifold \( M \) and \( L \) be a closed braid in \((S, \phi)\). Assume one of the following:

- \( \partial S \) is connected and \( |c(\phi_L, \partial S)| > 1 \).
- \( |c(\phi_L, C)| > 4 \) for every boundary component \( C \) of \( S \).

Then we have the following:

1. The complement \( M - N(L) \) is toroidal if and only if \( \phi_L \) is reducible.
2. The complement \( M - N(L) \) is hyperbolic if and only if \( \phi_L \) is pseudo-Anosov.
3. The complement \( M - N(L) \) is Seifert fibered if and only if \( \phi_L \) is periodic.

The proofs of Theorems 8.3 and 8.4 are almost the same, so we prove Theorem 8.3.

Proof of Theorem 8.3. The crucial point of the proof is the equivalence (1). Once we prove (1) the other equivalences follow by the geometrization theorem.

\((\Rightarrow)\) of (1) is proved in Corollary 7.3.

\((\Leftarrow)\) of (1): Assume that \( \phi \) is reducible. There exists an essential simple closed curve \( c \) in \( S \) such that \( \phi^n(c) = c \) for some \( n \in \mathbb{N} \). Let \( c_i = \phi^{i-1}(c) \) where \( i = 1, \ldots, n \). We may assume that \( c_i \)'s are mutually disjoint. Let \( \mathcal{C} = c_1 \cup \cdots \cup c_n \). Then \( \mathcal{C} \times [0, 1] \subset S \times [0, 1] \) gives rise to an embedded torus \( T = T_C \) in \( M \). Our goal is to prove \( T \) is incompressible.

Assume contrary that \( T \) is compressible. Compressing \( T \) yields an embedded sphere in \( M \), which bounds a 3-ball in \( M \) as \( M \) is irreducible by Corollary 7.3. Hence \( T \) separates \( M \) into two pieces \( X \) and \( M \setminus X \). We may suppose that \( X \) contains a compressing disc, \( D \), for \( T \). Hence \( X \) is homeomorphic to a solid torus or the complement of a knot in \( S^3 \).

Claim 8.5. Let \( S_0 \) be the page \( S \times \{0\} \) of \((S, \phi)\).

- \( \partial S_0 \cap (X \cap S_0) \neq \emptyset \).
- \( X \cap S_0 \) is connected.
Proof. (i): If $\partial S_0 \cap (X \cap S_0) = \emptyset$ then $\partial(X \cap S_0) = T \cap S_0 = C$. Thus, $X$ is a surface bundle over $S^1$ where the fiber is a connected component $S'$ of $X \cap S_0$. Since every $c_i$ is essential, $S'$ is not a disc. So $\partial X$ is not compressible in $X$, which is a contradiction.

(ii): Since $\phi|_{\partial S} = id$ the component of $X \cap S_0$ intersecting $\partial S_0$ is mapped to itself under $\phi$. Thus if $X \cap S_0$ is not connected then $X$ is not connected, a contradiction. □

Now we have $\partial(X \cap S_0) \supset T \cap S_0 = c_1 \cup \cdots \cup c_n$. Cap off $S' := X \cap S_0$ along $c_1, \ldots, c_n$ by discs and call the resulting surface $\tilde{S}'$. Let $\tilde{\phi}$ be the homeomorphism of $\tilde{S}'$ naturally extending $\phi|_{S'}$. Consider the open book $(\tilde{S}', \tilde{\phi})$ and denote $\tilde{M} = \tilde{M}(\tilde{S}', \tilde{\phi})$. The centers of the attached discs give rise to a closed $n$-braid, $L$, in the open book $(\tilde{S}', \tilde{\phi})$. Note that the boundary of the compressing disc $\partial D \subset T = \partial N(L)$ is a cable of $L$.

By the definitions of the fractional Dehn twist coefficient for closed braids and for reducible mapping classes, we have for each component $C$ of $\partial S' \cap \partial S$

$$c(\tilde{\phi}_0 D, C) = c(\tilde{\phi}_L, C) = c(\phi|_{S'}, C) = c(\phi, C).$$

Our hypotheses and Corollary 7.13-(1) show that neither $\partial D$ nor $L$ bounds a disc in $\tilde{M}$, which is a contradiction.

$(\Rightarrow)$ of (2): Gabai and Oertel’s Theorem 7.5 implies that $M$ contains an essential lamination, hence $M$ has infinite fundamental group. Further, Corollary 7.4 shows that $M$ is atoroidal and irreducible. Hence the hyperbolization theorem implies that $M$ is hyperbolic.

$(\Leftarrow)$ of (2): If $M$ is hyperbolic then $\phi$ is irreducible by (1) and pseudo-Anosov by Proposition 8.1.

$(\Leftarrow)$ of (3) is proved by Proposition 8.1.

$(\Rightarrow)$ of (3): If $M$ is atoroidal, (1) and (2) imply that $\phi$ is periodic.

In the following, we assume that $M$ is a toroidal Seifert-fibered manifold. By (2), $\phi$ is reducible so $\phi$ preserves a multi-curve $C$ in $S$. We may choose $C = c_1 \cup \cdots \cup c_k$ so that for any component $X$ of $S \setminus C$ the restriction $\phi|_X$ is irreducible. Let $T \subset M$ be the suspension torus (or tori) of $C$.

Assume that $\phi|_X$ is pseudo-Anosov for some $X$, and let $M_X$ be the component of $M \setminus T$ containing $X$. If $X \cap \partial S = \emptyset$ then $M_X$ is the mapping torus of a pseudo-Anosov map so it is hyperbolic. This is a contradiction. Suppose that $X \cap \partial S \neq \emptyset$ and let $C$ be a component of $X \cap \partial S$. By the above argument for $(\Leftarrow)$ of (1), we may regard $M_X$ as the complement of a closed braid $L$. Since $\phi$ is reducible we have $c(\tilde{\phi}_L, C) = c(\phi|_X, C) = c(\phi, C)$. Now (2) of Theorem 8.4 implies that $M_X$ is hyperbolic, which is also a contradiction.

Therefore, $\phi|_X$ is periodic for any component $X$ of $S \setminus C$. This means there exist integers $n > 0, m_1, \ldots, m_k$ such that $\phi^n$ is freely isotopic to $T_{c_1}^{m_1} \circ \cdots \circ T_{c_k}^{m_k}$. If some of $m_i$’s are non-zero then $M$ is a graph manifold but not Seifert-fibered. Hence all of $m_i$’s are zero and $\phi$ is periodic. □

Corollary 8.6. Theorem 8.3 implies that if $M(S, \phi)$ admits sol-geometry then $c(\phi, C) \leq 4$ for some boundary component $C$ of $S$.

By stabilizing an open book sufficiently many times one can always make $\partial S$ connected while preserving the topological type of the underlying 3-manifold $M(S, \phi)$. However, as shown in the next proposition (see also [27, Theorem 2.16]), stabilized open books have
“small” fractional Dehn twist coefficients. This partially explains why we need an open book with “large” \( c(\phi, C) \) to extract properties of \( M(S, \phi) \).

**Proposition 8.7.** If an open book \((S, \phi)\) is a stabilization of an open book \((S', \phi')\), then there exists a component \( C \) of \( \partial S \) such that \( |c(\phi, C)| \leq 1 \). Moreover, if \( \partial S \) is connected then \( |c(\phi, \partial S)| \leq \frac{1}{2} \).

**Proof.** By definition of stabilization, \( \phi = \phi' \circ T^\pm_\gamma \), where \( \gamma \) denote the core of the plumbed Hopf band. We may regard \( S' \subset S \) and \( \phi' \in \text{Aut}(S, \partial S) \). In the following we suppose that \( \phi = \phi' \circ T^+_\gamma \) (similar arguments hold when \( \phi = \phi' \circ T^-_\gamma \)).

Let \( \delta \) be the co-core of the plumbed Hopf band, i.e., an essential arc in \( S \), and let \( C \) be a component of \( \partial S \) that contains (at least) one of the endpoints of \( \gamma \). Then \( T^{-1}_C(T^{-1}_\gamma \delta) \geq \phi(T^{-1}_\gamma \delta) = \phi'(\delta) = \delta > T^{-1}_C(T^{-1}_\gamma \delta) \) hence by Lemma 4.9 we have \( |c(\phi, C)| \leq 1 \).

Next we assume that \( \partial S \) is connected, which is possible only if \( \partial S' \) has exactly two components, say \( C_1 \) and \( C_2 \). Take an integer \( N \geq \max\{|c(\phi', C_1)|, |c(\phi', C_2)|\} \). Viewing \( C_1 \) and \( C_2 \) as simple closed curves embedded in \( S \), for any essential arc \( l \subset S \) we have

\[
T^{-1}_{\partial S}(l) \geq T^{-N}_{C_1}T^{-N}_{C_2}(l) \geq \phi'(l) \geq T^N_{C_1}T^N_{C_2}(l) \geq T_S(l).
\]

Observe that

\[
T^{-1}_{\partial S}(T^{-1}_\gamma \delta) \geq T^{-1}_{\partial S}(T^{-1}_\gamma \delta) \geq \phi'(T^{-1}_\gamma \delta) = \phi^2(T^{-1}_\gamma \delta)
\]

and

\[
\phi^2(T^{-1}_\gamma \delta) = \phi'(T^{-1}_\gamma \delta) \geq T^N_{C_1}T^N_{C_2}(T^{-1}_\gamma \delta) \geq T_S(T^{-1}_\gamma \delta).
\]

Figure 19 justifies the last inequality \( T^N_{C_1}T^N_{C_2}(T^{-1}_\gamma \delta) \geq T_S(T^{-1}_\gamma \delta) \). By Lemma 4.9 and Proposition 4.6-(1) we conclude \( |c(\phi, \partial S)| \leq \frac{1}{2} \). □
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References

[1] D. Bennequin, Entrelacements et équations de Pfaff, Astérisque, 107-108, (1983) 87-161.
[2] J. Birman, E. Finkelstein, Studying surfaces via closed braids, J. Knot Theory Ramifications, 7, No.3 (1998), 267-334.
[3] J. Birman, W. Menasco, Studying links via closed braids. IV. Composite links and split links. Invent. Math. 102 (1990), no. 1, 115-139.
[4] J. Birman, W. Menasco, Studying links via closed braids. II. On a theorem of Bennequin. Topology Appl. 40 (1991), no. 1, 71-82.
[5] J. Birman, W. Menasco, Studying links via closed braids. V. The unlink. Trans. Amer. Math. Soc. 329 (1992), no. 2, 585-606.
[6] J. Birman, W. Menasco, Studying links via closed braids. I. A finiteness theorem. Pacific J. Math. 154 (1992), no. 1, 17-36.
[7] J. Birman, W. Menasco, Studying links via closed braids. VI. A nonfiniteness theorem. Pacific J. Math. 156 (1992), no. 2, 265-285.
[8] J. Birman, W. Menasco, Studying links via closed braids. III. Classifying links which are closed 3-braids. Pacific J. Math. 161 (1993), no. 1, 25-113.
[9] J. Birman, W. Menasco, Special positions for essential tori in link complements. Topology. 33 (1994), no.3, 525-556.
[10] J. Birman, W. Menasco, Stabilization in the braid groups. I. MTWS. Geom. Topol. 10 (2006), 413-540.
[11] J. Birman, W. Menasco, Stabilization in the braid groups. II. Transversal simplicity of knots. Geom. Topol. 10 (2006), 1425-1452.
[12] A. Casson and S. Bleiler, Automorphisms of surfaces after Nielsen and Thurston, London Mathematical society Student Texts(9), Cambridge,1988.
[13] J. Cantwell, L. Conlon, General position in tautly foliated sutured manifolds, unpublished manuscript, Available at http://euler.slu.edu/~cantwell/KF.pdf
[14] V. Colin and K. Honda, Reeb vector fields and open book decompositions, arXiv:0809.5088.
[15] J. Etnyre, J. Van Horn-Morris, Fibered transverse knots and the Bennequin bound, Int. Math. Res. Not. (2011) no. 7, 1483-1509.
[16] B. Farb and D. Margalit, A Primer on Mapping Class Groups, Princeton University Press. 2011.
[17] D. Gabai and W. Kazez, Group negative curvature for 3manifolds with genuine laminations Geom. Topol. 2, (1998), 65–77.
[18] D. Gabai and U. Oertel, Essential laminations in 3-manifolds, Ann. Math., 130 (1989) 41–73.
[19] H. Geiges, An introduction to contact topology, Cambridge Studies in Advanced Mathematics, 109. Cambridge University Press, Cambridge, 2008.
[20] E. Giroux, Géométrie de contact: de la dimension trois vers les dimensions supérieures, Proceedings of the International Congress of Mathematicians, vol. II (Beijing, 2002), 405-414.
[21] A. Hatcher, The Classification of 3-Manifolds - A Brief Overview, available at http://www.math.cornell.edu/~hatcher/Papers/3ManifoldSurvey.pdf.
[22] K. Honda, W. Kazez, and G. Matić, Right-veering diffeomorphisms of compact surfaces with boundary, Invent. math. 169, No.2 (2007), 427-449.
[23] T. Ito, Braid ordering and the geometry of closed braid, Geom. Topol. 15, (2011), 473–498.
[24] T. Ito, Braid ordering and knot genus, J. Knot Theory Ramification, 20, (2011), 1311–1323.
[25] T. Ito and K. Kawamuro, Open book foliation, Available at arXiv:1112.5874v1
[26] T. Ito and K. Kawamuro, Open book foliation III, in preparation.
[27] W. Kazez and R. Roberts, *Fractional Dehn twists in knot theory and contact topology*, Available at [http://arxiv.org/pdf/1201.5290v4.pdf](http://arxiv.org/pdf/1201.5290v4.pdf).

[28] L. Mosher, *Mapping class groups are automatic*, Ann. of Math. 142, (1995), 303–384.

[29] E. Pavelescu, *Braids and Open Book Decompositions*. Ph.D. thesis, University of Pennsylvania (2008). Available at [http://www.math.upenn.edu/grad/dissertations/ElenaPavelescuThesis.pdf](http://www.math.upenn.edu/grad/dissertations/ElenaPavelescuThesis.pdf).

[30] R. Roussarie, *Plongements dans les variétés feuilletées et classification de feuilletages holonomie*, I.H.E.S. Sci. Publ. Math. 43 (1973), 101-142.

[31] W. Thurston and H. Winkelnkemper, *On the existence of contact forms*. Proc. Amer. Math. Soc. 52 (1975), 345-347.

[32] W. Thurston, *Hyperbolic Structures on 3-manifolds, II: Surface groups and 3-manifolds which fiber over the circle*. Available at [http://arxiv.org/pdf/math/9801045.pdf](http://arxiv.org/pdf/math/9801045.pdf).

[33] W. Thurston, *A norm on the homology of three-manifolds*, Mem. Amer. Math. Soc. 59 (1986), 99-130.

[34] C. Rourke and B. Wiest, *Order automatic mapping class groups*, Pacific J. Math, 194, No.1 (2000), 209-227.

[35] H. Short and B. Wiest, *Ordering of mapping class groups after Thurston*, Enseign. Math. 46 (2000), 279-312.

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