A new generalization of a system of two-sided coupled Sylvester-like quaternion tensor equations

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Abstract: This study establishes consistency conditions and a general solution for a coupled system that consists of five two-sided Sylvester-like tensor equations in ten quaternion variables throughout the Einstein tensor product. Certain specific cases are thus established. In a direct application, we investigate certain necessary and sufficient conditions for the existence of an $\eta$-Hermitian solution to five coupled two-sided Sylvester-like quaternion tensor equations. Finally, we present an algorithm and a numerical example to validate the main result.

Keywords: Tensor, Moore-Penrose inverse, Quaternion, Tensor equation

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1. Introduction

We introduce certain notations and definitions for convenience. Consider $I_1, \ldots, I_M$ to be positive integers for the positive integer $M$. An $M$ order tensor $D$ with entry $D_{i_1 \ldots i_M}$ ($1 \leq i_j \leq I_j$, $i = 1, \ldots, M$) is a multidimensional array with the subscripts $i_1, i_2, \ldots, i_M$ \cite{19, 12, 33, 50, 41}. We utilize the notation that $I(M)$ represents $I_1 \times I_2 \times \ldots \times I_M$. The quaternion concept was investigated by Hamilton in \cite{19}, and quaternion algebra can be considered a non-commutative skew field. Let $\mathbb{R}$ and $\mathbb{C}$ be the fields of real numbers and complex numbers, respectively, and let $\mathbb{H}$ be the quaternion algebra

$$\mathbb{H} = \{d_0 + d_1 i + d_2 j + d_3 k \mid i^2 = j^2 = k^2 = ijk = -1, \ d_0, d_1, d_2, d_3 \in \mathbb{R}\}.$$ 

Let $\mathbb{H}^{I(M)}$ be the set of the order $M$ dimension $I(M)$ tensors over the quaternion algebra $\mathbb{H}$. Tensors are the natural expansions of vectors and matrices. Tensor equations and computations have applications in machine learning, signal processing, mechanics, physics, Markov processes, control theory, numerical analysis, partial differential equations, and engineering problems \cite{5, 6, 7, 8, 9, 10}. Tensor decompositions, tensor eigenvalue, and non-negative tensors \cite{33, 11, 12} have implementations in signal processing, color image processing \cite{28}, quantum mechanics \cite{7}, quaternion tensor computing \cite{13}, Iterative algorithms for solving some tensor equations \cite{14, 15, 30, 32, 40, 41, 50}. Let $\mathcal{A} \in \mathbb{H}^{I(N) \times J(N)}$ and $\mathcal{B} \in \mathbb{H}^{J(N) \times K(M)}$, then the Einstein tensor product \cite{42} of tensors $\mathcal{A}$

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and $\mathcal{B}$ is denoted by $A \ast_{N} \mathcal{B} \in \mathbb{H}^{I(N) \times K(M)}$, where

$$
(A \ast_{N} \mathcal{B})_{i_1\ldots i_Nk_1\ldots k_M} = \sum_{j_1\ldots j_N} a_{i_1\ldots i_Nj_1\ldots j_N} b_{j_1\ldots j_Nk_1\ldots k_M}.
$$

The operation $\ast_{N}$ is associative over the set of all quaternion tensors with qualified order.

Let $\psi$ be a nonstandard involution of the quaternion algebra $\mathbb{H}$ (Definition 3.4.5 [38]). If $D \in \mathbb{H}^{m \times n}$, then $(D)_{\psi}$ is an $n \times m$ matrix over $\mathbb{H}$ obtained by applying $\psi$ entrywise to the transpose of $D$. Let $D$ be an $n \times n$ matrix over $\mathbb{H}$. $D$ is called a $\psi$-Hermitian matrix if $(D)_{\psi} = D$ (Definition 3.6.1 [38]). Took et al. [43] introduced an example of a $\psi$-Hermitian matrix called an $\eta$-Hermitian matrix. For fixed $\eta \in \{1, j, k\}$, a square matrix $A$ is called an $\eta$-Hermitian matrix if $D^{\eta \ast} = D$, where $D^{\eta \ast} = -\eta D^\ast \eta$. An $\eta$-Hermitian matrix has applications in linear modeling and statistical signal processing [43, 46]. He [25] gave a generalization of an $\eta$-Hermitian matrix. A square quaternion tensor $D$ is called an $\eta$-Hermitian tensor if $D = D^{\eta \ast}$, where $D^{\eta \ast} = -\eta D^\ast \eta$. We [25] investigated the consistency conditions and the exact general solution formula for the following two-sided quaternion tensor equations:

$$
A_1 \ast_{N} X_1 \ast_{M} B_1 + A_2 \ast_{N} X_2 \ast_{M} B_2 + A_2 \ast_{N} (C_3 \ast_{N} X_3 \ast_{M} D_3 + C_4 \ast_{N} X_4 \ast_{M} D_4) \ast_{M} B_1 = \mathcal{E}_1.
$$

(1.1)

where $A_i, B_i, C_j, D_j$ ($i = 1, 2, j = 3, 4$), and $\mathcal{E}_i$ are given quaternion tensors. A tensor equation (1.1) has applications in the discretization of higher-dimension linear partial differential equations, even including its generalizations [29]. Recently, Wang et al. [51] gave a proper extension to the quaternion tensor equation (1.1). They established consistency conditions and a general solution to the following coupled two-sided Sylvester-type quaternion system of tensor equations in terms of the Moore–Penrose inverses for certain given tensors:

$$
\begin{align*}
A_1 \ast_{N} X_1 \ast_{M} B_1 + A_2 \ast_{N} W \ast_{M} B_2 &= \mathcal{E}_1 \\
A_3 \ast_{N} Y_1 \ast_{M} B_3 + A_4 \ast_{N} W \ast_{M} B_4 &= \mathcal{E}_2.
\end{align*}
$$

(1.2)

This is based on the various uses of quaternions, rank characterizations of some matrix expressions, matrix decompositions, the coupled Sylvester-like quaternion systems of matrix equations [2, 4, 10, 11, 20, 21, 24, 27, 39, 40, 41, 50, 52, 53, 55, 57], and the theoretical studies surrounding Sylvester-like quaternion tensor equations. This paper investigates the consistency and general solution to the following coupled two-sided Sylvester-like quaternion system of tensor equations:

$$
\begin{align*}
F_4 \ast_{N} Z_1 \ast_{M} G_4 &= \mathcal{E}_4, \\
A_i \ast_{N} X_i \ast_{M} G_i + C_i \ast_{N} Y_i \ast_{M} D_i + C_i \ast_{N} (F_i \ast_{N} Z_i \ast_{M} G_i + H_i \ast_{N} Z_{i+1} \ast_{M} J_i) \ast_{M} B_i &= \mathcal{E}_i, \\
H_4 \ast_{N} Z_4 \ast_{M} J_4 &= \mathcal{E}_5,
\end{align*}
$$

(1.3)

$(i = 1, 3)$, which gives us a proper generalization of both systems, (1.1) and (1.2). As a direct conclusion, we derive certain necessary and sufficient conditions for the consistency of:

$$
\begin{align*}
F_4 \ast_{N} Z_1 \ast_{M} G_4 &= \mathcal{E}_4, \\
F_1 \ast_{N} Z_1 \ast_{M} G_1 + H_1 \ast_{N} Z_2 \ast_{M} J_1 &= \mathcal{E}_1, \\
F_2 \ast_{N} Z_2 \ast_{M} G_2 + H_2 \ast_{N} Z_3 \ast_{M} J_2 &= \mathcal{E}_2, \\
F_3 \ast_{N} Z_3 \ast_{M} G_3 + H_3 \ast_{N} Z_4 \ast_{M} J_3 &= \mathcal{E}_3, \\
H_4 \ast_{N} Z_4 \ast_{M} J_4 &= \mathcal{E}_5.
\end{align*}
$$

(1.4)
As an implementation of [1.4], we obtain the consistency conditions for the existence of an 
\(\eta\)-Hermitian solution to the following two-sided quaternion system of tensor equations:

\[
\begin{align*}
\mathcal{F}_1 \ast_N Z_1 \ast_N F_1^\eta &= \mathcal{E}_1, \\
\mathcal{F}_2 \ast_N Z_2 \ast_N F_2^\eta &= \mathcal{E}_2, \\
\mathcal{F}_3 \ast_N Z_3 \ast_N F_3^\eta &= \mathcal{E}_3, \\
\mathcal{H}_4 \ast_N Z_4 \ast_N \mathcal{H}_4^\eta &= \mathcal{E}_5.
\end{align*}
\]

(1.5)

If we set \(C_i = B_i = I\) in (1.3), we obtain the following Sylvester-like quaternion system of tensor equations:

\[
\begin{align*}
\mathcal{A}_i \ast_N X_i + \mathcal{Y}_i \ast_M D_i + F_i \ast_N Z_i \ast_M \mathcal{G}_i + \mathcal{H}_i \ast_N Z_{i+1} \ast_M \mathcal{J}_i &= \mathcal{E}_i, \\
\mathcal{F}_4 \ast_N Z_1 \ast_M \mathcal{G}_4 &= \mathcal{E}_5, \\
\mathcal{H}_4 \ast_N Z_4 \ast_M \mathcal{H}_4^\eta &= \mathcal{E}_5.
\end{align*}
\]

(1.6)

The remainder of this manuscript is described as follows. The concept of an \(\eta\)-Hermitian quaternion tensor and the Moore—Penrose inverse for a general tensor are reminiscent of Section 2. Section 3 expresses the general solution to the two-sided Sylvester-type quaternion system of tensor equations (1.3) when the solvability conditions are applicable. In Section 4, we provide the necessary and sufficient conditions for the existence of a \(\eta\)-Hermitian solution to a system (1.3) as a system (1.4) application. We briefly summarize the key results in Section 5.

2. Preliminaries

Throughout this paper tensors are considered quaternion tensors. A tensor \(C \in \mathbb{H}^{I(N) \times J(N)}\) is
called an even-order tensor. An even-order tensor \(C \in \mathbb{H}^{I(N) \times I(N)}\) is called an even-order square
tensor. Let \(c \in \mathbb{H}\), then \(\overline{c}\) stands for the conjugate of \(c\). A quaternion tensor \(C^* = (\overline{C}_{i_3, \ldots, i_{M+1} \ldots i_N}) \in \mathbb{H}^{J(M) \times I(N)}\) calls the conjugate transpose of the tensor \(C = (c_{i_1 \ldots i_N}) \in \mathbb{H}^{I(N) \times J(M)}\). If \(C = C^*\), then \(C\) is called Hermitian tensor.

**Definition 2.1.** \(\mathbb{H}\) An even order square tensor \(C = (c_{i_1 \ldots i_M}) \in \mathbb{H}^{I(M) \times I(M)}\) is called a diagonal tensor if \(c_{i_1 \ldots i_M} \neq 0\) and all its entries are zero. A diagonal tensor is said to be a unit tensor if \(c_{i_1 \ldots i_M} = 1\), which denotes by \(I\).

**Definition 2.2.** \(\mathbb{H}\) Let \(C = (c_{i_1 \ldots i_N}) \in \mathbb{H}^{I(N) \times J(M)}\), \(D = (d_{i_1 \ldots i_Nk_1 \ldots k_M}) \in \mathbb{H}^{I(N) \times K(M)}\). The "row block tensor" of \(C\) and \(D\) is denoted by

\[
\begin{pmatrix}
C \\
D
\end{pmatrix}
\in \mathbb{H}^{I(N) \times L(M)},
\]

(2.1)

where \(L_s = J_s + K_s\), \(s = 1, \ldots, M\) define as

\[
\begin{pmatrix}
C \\
D
\end{pmatrix}_{i_1 \ldots i_Ni_1 \ldots i_M} = \begin{cases}
c_{i_1 \ldots i_N i_1 \ldots i_M}, & \text{if } i_1 \ldots i_N \in [I_1] \times \ldots \times [I_N], \ i_1 \ldots i_M \in [J_1] \times \ldots \times [J_M], \\
d_{i_1 \ldots i_N i_1 \ldots i_M}, & \text{if } i_1 \ldots i_N \in [I_1] \times \ldots \times [I_N], \ i_1 \ldots i_M \in \Gamma_1 \times \ldots \times \Gamma_M, \\
0, & \text{otherwise},
\end{cases}
\]

where \(\Gamma_s = \{J_s + 1, \ldots, J_s + K_s\}\), \(s = 1, \ldots, M\). For a given tensors \(A = (a_{j_1 \ldots j_M}) \in \mathbb{H}^{J(M) \times I(N)}\), \(B = (b_{k_1 \ldots k_M}) \in \mathbb{H}^{K(M) \times I(N)}\). The "column block tensor" of \(A\) and \(B\) is
denoted by
\[
\begin{pmatrix}
A \\
B
\end{pmatrix}_{l_1...l_M i_1...i_N} = \begin{cases}
a_{l_1...l_M i_1...i_N}, & \text{if } l_1...l_M \in [J_1] \times ... \times [J_M], \ i_1...i_N \in [I_1] \times ... \times [I_N], \\
b_{l_1...l_M i_1...i_N}, & \text{if } l_1...l_M \in \Gamma_1 \times ... \times \Gamma_M, \ i_1...i_N \in [I_1] \times ... \times [I_N], \\
0, & \text{otherwise},
\end{cases}
\]

where \(\Gamma_s = \{J_s + 1, ..., J_s + K_s\}, \ s = 1, ..., M\).

**Proposition 2.1.** [42] Let \(A \in \mathbb{H}^{I(P) \times K(N)}\) and \(B \in \mathbb{H}^{K(N) \times J(M)}\). Then

1. \((A * N B)^* = B^* * N A^*;\)
2. \(I_N * N B = B, \ B * M I_M = B,\) where \(I_N \in \mathbb{H}^{K(N) \times K(N)}\) and \(I_M \in \mathbb{H}^{J(M) \times J(M)}\) are units.

**Proposition 2.2.** [42] Consider the tensors \((A \ B)\) and \((C \ D)\) given in (2.1) and (2.2). For a given quaternion tensor \(G \in \mathbb{H}^{I(N) \times I(N)}\), we have that

1. \(G * N (A \ B) = (G * N A \ G * N B) \in \mathbb{H}^{I(I) \times L(M)},\)
2. \((C \ D) * N G = \begin{pmatrix}
C * N G \\
D * N G
\end{pmatrix} \in \mathbb{H}^{L(M) \times I(N)},\)
3. \((A \ B) * M (C \ D) = A * M C + B * M D \in \mathbb{H}^{I(I) \times I(I)}.

**Definition 2.3.** [22] For a given quaternion tensor \(D \in \mathbb{H}^{I(I) \times I(I)}\), the Moore-Penrose inverse of \(D\) is the unique quaternion tensor \(X \in \mathbb{H}^{J(I) \times I(I)}\) satisfies the following axioms:

1. \(D \ * N X \ * N D = D,\)
2. \(X \ * N D \ * N X = X,\)
3. \((D \ * N X)^* = D \ * N X,\)
4. \((X \ * N D)^* = X \ * N D.\)

which denotes by \(D^\dagger\). Furthermore, \(R_D\) and \(L_D\) denote the projections along \(D\).

**Proposition 2.3.** [22] Let \(D \in \mathbb{H}^{I(I) \times I(I)}\). Then

1. \(L_D * N D^\dagger = D \ * N L_D = 0, \ R_D * N D = D^\dagger \ * N R_D = 0,\)
2. \((D^\dagger)^\dagger = (D^\dagger), \ (D\eta)^\dagger = (D^\dagger)\eta,\)
3. \((L_D\eta)^\dagger = R_D\eta, \ (R_D)\eta^\dagger = L_D\eta,\)
4. \((D^\dagger * N D)^\dagger = D^\dagger \ * N (D^\dagger), \ (D \ * N D)^\dagger = (D^\dagger)^\dagger \ * N D^\dagger.\)

**Lemma 2.4.** [22] Let \(A_1 \in \mathbb{H}^{I(I) \times J(N)}, \ A_2 \in \mathbb{H}^{I(I) \times G(N)}, \ B_1 \in \mathbb{H}^{K(M) \times L(M)}, \ B_2 \in \mathbb{H}^{H(M) \times L(M)}, \ C_3 \in \mathbb{H}^{G(N) \times Q(N)}, \ C_4 \in \mathbb{H}^{G(N) \times T(N)}, \ D_3 \in \mathbb{H}^{S(M) \times K(M)}, \ D_4 \in \mathbb{H}^{P(M) \times K(M)}\) and \(E_1 \in \mathbb{H}^{I(I) \times L(M)}\)
be given. Set
\[ M_1 = R_{A_1} * N A_2, \quad N_1 = B_2 * M L_{B_1}, \quad S_1 = A_2 * N L_{M_1}, \quad \hat{A}_1 = M_1 * N C_3, \]
\[ \hat{A}_2 = M_1 * N C_4, \quad \hat{B}_1 = D_3 * M B_1 * M L_{B_1}, \quad \hat{B}_2 = D_3 * M B_1 * M L_{B_2}, \]
\[ \hat{M}_1 = R_{\hat{A}_1} * N \hat{A}_2, \quad \hat{N}_1 = B_2 * M L_{\hat{B}_1}, \quad \hat{S}_1 = \hat{A}_2 * N E_{\hat{M}_1}, \quad \hat{E}_1 = R_{A_1} * N E_1 * M L_{B_2}, \]
\[ \hat{E}_1 = E_1 - A_2 * N (C_3 * N X_3 * M D_3 + C_4 * N W * M D_4) * M B_1. \]

Then the following statements are equivalent:

(1) \([1.1]\) is solvable.

(2)
\[ R_{M_1} * N R_{A_1} * N E_1 = 0, \quad E_1 * M L_{B_1} * M L_{N_1} = 0, \quad R_{A_2} * N E_1 * M L_{B_1} = 0, \]
\[ R_{\hat{N}_1} * N R_{\hat{A}_1} * N \hat{E}_1 = 0, \quad \hat{E}_1 * M L_{\hat{B}_1} * M L_{\hat{N}_1} = 0, \]
\[ R_{\hat{A}_1} * N \hat{E}_1 * M L_{\hat{B}_2} = 0, \quad R_{\hat{X}_2} * N \hat{E}_1 * M L_{\hat{B}_2} = 0. \]

In that case, the general solution to \([1.1]\) can be expressed as follows:

\[ X_1 = A_1^{\dagger} * N \hat{E}_1 * M B_1^\dagger - A_1^{\dagger} * N A_2 * N M_1^{\dagger} * N \hat{E}_1 * M B_1^\dagger - A_1^{\dagger} * N S_1 * N A_2^\dagger * N \hat{E}_1 * M N_1^\dagger \]
\[ + L_{A_1} * N U_1 + U_5 * M R_{B_1}, \]
\[ X_2 = M_1^{\dagger} * N \hat{E}_1 * M B_2^\dagger + S_1^{\dagger} * N S_1 * A_2^{\dagger} * N \hat{E}_1 * M N_1^\dagger + L_{M_1} * N L_{S_1}, \]
\[ * N U_1 + L_{M_1} * N U_2 * M R_{N_1} + U_3 * M R_{B_2}, \]
\[ X_3 = A_1^{\dagger} * N \hat{E}_1 * M \hat{B}_1^\dagger - A_1^{\dagger} * N \hat{A}_2 * N \hat{M}_1^{\dagger} * N \hat{E}_1 * M \hat{B}_1^\dagger - A_1^{\dagger} * N \hat{S}_1 * N \hat{A}_2^{\dagger} * N \hat{E}_1 * M \hat{N}_1^\dagger \]
\[ + L_{\hat{A}_1} * N \hat{U}_1 + \hat{U}_5 * M R_{\hat{B}_1}, \]
\[ X_4 = \hat{M}_1^{\dagger} * N \hat{E}_1 * M \hat{B}_2^\dagger + \hat{S}_1^{\dagger} * N \hat{S}_1 * \hat{A}_2^{\dagger} * N \hat{E}_1 * M \hat{N}_1^\dagger + L_{\hat{M}_1} * N L_{\hat{S}_1}, \]
\[ * N \hat{U}_1 + L_{\hat{M}_1} * N \hat{U}_2 * M R_{\hat{N}_1} + \hat{U}_3 * M R_{\hat{B}_2}, \]

where \( U_i, \hat{U}_i \) (\( i = 1, 5 \)) are arbitrary tensors with suitable orders.

In case of \( A_2 = B_1 = I \) and \( A_1 = B_2 = 0 \), we have the following special case of \([1.1]\)
\[ C_3 * N X_3 * M D_3 + C_4 * N X_4 * M D_4 = E_1, \]

which is solvable if and only if
\[ R_{\hat{M}_1} * N R_{C_3} * N E_1 = 0, \quad E_1 * M L_{D_3} * M L_{N_1} = 0, \]
\[ R_{C_4} * N E_1 * M L_{D_4} = 0, \quad R_{C_4} * N E_1 * M L_{D_3} = 0. \]

In that case, the general solution can be expressed as follows:

\[ X_3 = C_3^{\dagger} * N E_1 * M D_3^\dagger - C_3^{\dagger} * N C_4 * N \hat{M}_1^{\dagger} * N E_1 * M D_3^\dagger - C_3^{\dagger} * N \hat{S}_1 * N C_4^\dagger \]
\[ N \hat{E}_1 * M \hat{N}_1^{\dagger} * M D_3^\dagger - C_3^{\dagger} * N \hat{S}_1 * N \hat{U}_2 * M R_{\hat{N}_1} * M D_4 * M D_3^\dagger \]
\[ + L_{C_3} * N \hat{U}_1 + \hat{U}_5 * M R_{D_3}, \]
\[ X_4 = \hat{M}_1^{\dagger} * N \hat{E}_1 * M D_4^\dagger + \hat{S}_1^{\dagger} * N \hat{S}_1 * C_4^{\dagger} * N \hat{E}_1 * M \hat{N}_1^{\dagger} + L_{\hat{M}_1} * N L_{\hat{S}_1}, \]
\[ * N \hat{U}_1 + L_{\hat{M}_1} * N \hat{U}_2 * M R_{\hat{N}_1} + \hat{U}_3 * M R_{D_4}, \]
3. The consistency conditions and the general Solution to (1.4)

In the following Theorem, we provide consistency conditions and general solution of a coupled Two-sided Sylvester-like quaternion system of tensor equations \([13]\).

**Theorem 3.1.** Consider the quaternion system of tensor equations \([13]\), where

\[
\begin{align*}
\mathcal{F}_i &\in \mathbb{H}^{I(N)\times J(N)}, \quad \mathcal{G}_i \in \mathbb{H}^{L(M)\times K(M)}, \quad \mathcal{H}_i \in \mathbb{H}^{I(N)\times Q(N)}, \quad \mathcal{J}_i \in \mathbb{H}^{S(M)\times K(M)}, \\
\mathcal{E}_i &\in \mathbb{H}^{I(N)\times K(M)}, \quad \mathcal{A}_i \in \mathbb{H}^{I(N)\times J(N)}, \quad \mathcal{B}_i \in \mathbb{H}^{F(M)\times K(M)}, \quad \mathcal{C}_i \in \mathbb{H}^{I(N)\times A(N)}, \\
\mathcal{B}_2 &\in \mathbb{H}^{F(M)\times K(M)}, \quad \mathcal{B}_2 \in \mathbb{H}^{G(M)\times K(M)}, \quad \mathcal{B}_3 \in \mathbb{H}^{H(M)\times K(M)}, \\
\mathcal{C}_1 &\in \mathbb{H}^{I(N)\times A(N)}, \quad \mathcal{C}_2 \in \mathbb{H}^{I(N)\times B(N)}, \quad \mathcal{C}_3 \in \mathbb{H}^{I(N)\times C(N)}, \quad \mathcal{D}_1 \in \mathbb{H}^{L(M)\times K(M)}, \\
\mathcal{D}_2 &\in \mathbb{H}^{L(M)\times K(M)}, \quad \mathcal{D}_3 \in \mathbb{H}^{L(M)\times K(M)}, \quad \mathcal{F}_1 \in \mathbb{H}^{A(N)\times J(N)}, \quad \mathcal{F}_2 \in \mathbb{H}^{B(N)\times P(N)}, \\
\mathcal{F}_3 &\in \mathbb{H}^{C(N)\times J(N)}, \quad \mathcal{G}_1 \in \mathbb{H}^{L(M)\times F(M)}, \quad \mathcal{G}_2 \in \mathbb{H}^{Q(M)\times G(M)}, \quad \mathcal{G}_3 \in \mathbb{H}^{L(M)\times H(M)}, \\
\mathcal{H}_1 &\in \mathbb{H}^{A(N)\times P(N)}, \quad \mathcal{H}_2 \in \mathbb{H}^{B(N)\times J(N)}, \quad \mathcal{J}_2 \in \mathbb{H}^{L(M)\times M}, \quad \mathcal{J}_3 \in \mathbb{H}^{S(M)\times M}, \quad \mathcal{E}_i \in \mathbb{H}^{I(N)\times K(M)}, \quad (i = 1, 3).
\end{align*}
\]

are given tensors over \(\mathbb{H}\). Set

\[
\begin{align*}
\hat{E}_i &= E_i - C_i \ast N (F_i \ast_N Z_i \ast_M G_i - H_i \ast_N Z_{i+1} \ast_M J_i) \ast_M B_i, \\
\mathcal{M}_i &= \mathcal{R}_{A_i} \ast_N C_i, \quad \mathcal{N}_i = D_i \ast_M L_{B_i}, \quad S_i = C_i \ast_N L_{M_i}, \quad \hat{A}_i = M_i \ast_N F_i, \\
\hat{C}_i &= M_i \ast_N H_i, \quad \hat{B}_i = G_i \ast_M B_i \ast_M L_{D_i}, \quad \hat{D}_i = J_i \ast_M B_i \ast_M L_{D_i}, \quad \hat{M}_i = \mathcal{R}_{A_i} \ast_N \hat{C}_i, \\
\hat{N}_i &= \hat{D}_i \ast_M L_{B_i}, \quad \hat{S}_i = \hat{C}_i \ast_N L_{\hat{M}_i}, \quad \hat{E}_i = \mathcal{R}_{A_i} \ast_N \hat{E}_i \ast_M L_{D_i}, \quad (i = 1, 3), \\
\hat{A}_{11} &= \left[ L_{F_1} - L_{\hat{A}_1} \right], \quad \hat{D}_{11} = \left[ \begin{array}{c} \mathcal{R}_{\hat{G}_1} \\ -\mathcal{R}_{\hat{B}_1} \end{array} \right], \quad \hat{A}_{11} = \hat{A}_1 \ast_N \hat{S}_1, \quad \hat{B}_{11} = R_{\hat{N}_1} \ast_M \hat{D}_1 \ast_M \hat{B}_1, \\
\hat{E}_{11} &= \hat{A}_1 \ast_N \hat{E}_1 \ast_M \hat{B}_1 - \hat{A}_1 \ast_N \hat{C}_1 \ast_N \hat{M}_1 \ast_N \hat{E}_1 \ast_M \hat{B}_1 - \hat{A}_1 \ast_N \hat{S}_1 \ast_N \hat{C}_1 \ast_N \hat{E}_1 \ast_M \hat{N}_1 \ast_M \hat{D}_1 \ast_M \hat{B}_1 \ast_M \hat{F}_1 \ast_N \hat{E}_4 \ast_M \hat{G}_4, \\
\hat{A}_{22} &= \left[ L_{H_4} - L_{\hat{M}_3} \ast_N L_{\hat{S}_3} \right], \quad \hat{D}_{22} = \left[ \begin{array}{c} \mathcal{R}_{\hat{J}_4} \\ -\mathcal{R}_{\hat{B}_3} \end{array} \right], \quad \hat{A}_{22} = \hat{L}_{\hat{M}_3}, \quad \hat{B}_{22} = R_{\hat{N}_3}, \\
\hat{E}_{22} &= \hat{M}_3 \ast_N \hat{E}_3 \ast_M \hat{D}_3 + \hat{S}_3 \ast_N \hat{S}_3 \ast_N \hat{C}_3 \ast_N \hat{E}_3 \ast_M \hat{N}_3 \ast_N \hat{H}_3 \ast_N \hat{E}_5 \ast_M \hat{J}_4, \\
\hat{A}_{ii} &= \mathcal{R}_{A_i} \ast_N \hat{A}_{ii}, \quad \hat{B}_{ii} = \hat{B}_i \ast_M \hat{L}_{D_i}, \quad \hat{E}_{ii} = \mathcal{R}_{A_i} \ast_N \hat{E}_{ii} \ast_M \hat{L}_{D_i}, \quad (i = 1, 2), \\
\hat{F}_1 &= \left[ -L_{\hat{M}_1} \ast_N L_{\hat{S}_2} \right], \quad \hat{F}_2 = \left[ -L_{\hat{M}_2} \ast_N L_{\hat{S}_3} \right], \quad \hat{F}_1 = \hat{A}_2 \ast_N \hat{S}_2, \quad \hat{F}_2 = \hat{A}_3 \ast_N \hat{S}_3, \\
\hat{E}_1 &= \left[ L_{\hat{M}_1} \ast_N L_{\hat{S}_2} \right], \quad \hat{E}_2 = \left[ L_{\hat{M}_2} \ast_N L_{\hat{S}_3} \right], \quad \hat{E}_1 = \hat{D}_2 \ast_N \hat{B}_2, \quad \hat{E}_2 = \hat{D}_3 \ast_N \hat{B}_3, \\
\hat{E}_1 &= \left[ L_{\hat{M}_1} \ast_N L_{\hat{S}_2} \right], \quad \hat{E}_2 = \left[ L_{\hat{M}_2} \ast_N L_{\hat{S}_3} \right], \quad \hat{E}_1 = \hat{D}_2 \ast_N \hat{B}_2, \quad \hat{E}_2 = \hat{D}_3 \ast_N \hat{B}_3, \\
\hat{E}_1 &= \left[ L_{\hat{M}_1} \ast_N L_{\hat{S}_2} \right], \quad \hat{E}_2 = \left[ L_{\hat{M}_2} \ast_N L_{\hat{S}_3} \right], \quad \hat{E}_1 = \hat{D}_2 \ast_N \hat{B}_2, \quad \hat{E}_2 = \hat{D}_3 \ast_N \hat{B}_3.
\end{align*}
\]
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\begin{align*}
\bar{E}_2 &= -\bar{\mathcal{M}}_2 \ast N \bar{E}_2 \ast M \bar{D}_1 - \bar{S}_2 \ast N \bar{S}_2 \ast N \bar{C}_1 \ast N \bar{\mathcal{E}} \ast M \bar{\mathcal{N}}_2 + \bar{\mathcal{A}}_1 \ast N \bar{\mathcal{E}}_3 \ast M \bar{B}_3 - \bar{\mathcal{A}}_3 \ast N \bar{C}_3 \ast N \bar{\mathcal{M}}_3 \ast N \bar{E}_3 \ast M \bar{\mathcal{D}}_3 \ast M \bar{B}_3, \\
\mathcal{F}_{ii} &= \mathcal{R}_{\mathcal{F}} \ast N \mathcal{F}_i, \quad \mathcal{G}_{ii} = \mathcal{G}_i \ast M \mathcal{L}_{\mathcal{B}}, \quad \mathcal{H}_{ii} = \mathcal{R}_{\mathcal{F}} \ast N \mathcal{H}_i, \quad \mathcal{J}_{ii} = \mathcal{J}_i \ast M \mathcal{L}_{\mathcal{B}}, \\
\mathcal{E}_{ii} &= \mathcal{R}_{\mathcal{E}} \ast N \mathcal{E}_i \ast M \mathcal{L}_{\mathcal{B}}, \quad \mathcal{M}_{ii} = \mathcal{R}_{\mathcal{F}_{ii}} \ast N \mathcal{H}_{ii}, \quad \mathcal{N}_{ii} = \mathcal{J}_{ii} \ast M \mathcal{L}_{\mathcal{G}_{ii}}, \quad \mathcal{S}_{ii} = \mathcal{H}_{ii} \ast N \mathcal{L}_{\mathcal{S}_{ii}}, \\
\mathcal{A}_1 &= \left[ \mathcal{L}_{\mathcal{F}_{11}} - \mathcal{L}_{\mathcal{M}_{22}} \ast N \mathcal{L}_{\mathcal{S}_{22}} \right], \quad \mathcal{B}_1 = \begin{bmatrix} \mathcal{R}_{\mathcal{F}_{11}} \\ -\mathcal{R}_{\mathcal{G}_{11}} \end{bmatrix}, \quad \mathcal{F}_1 = \mathcal{F}_{11} \ast N \mathcal{S}_{11}, \\
\tilde{\mathcal{G}}_1 &= \mathcal{R}_{\mathcal{G}_{11}} \ast M \mathcal{G}_{11}, \quad \tilde{\mathcal{H}}_1 = \mathcal{R}_{\mathcal{H}_{11}} \ast N \mathcal{H}_{11}, \quad \tilde{\mathcal{J}}_1 = \mathcal{J}_{11} \ast M \mathcal{L}_{\mathcal{B}_{11}}, \\
\tilde{\mathcal{E}}_1 &= \mathcal{E}_{11} \ast N \mathcal{E}_{11} \ast M \mathcal{G}_{11} - \mathcal{E}_{11} \ast N \mathcal{M}_{11} \ast N \mathcal{G}_{11} - \mathcal{E}_{11} \ast N \mathcal{S}_{11} \ast N \mathcal{H}_{11} \ast N \mathcal{E}_{11} \ast M \mathcal{N}_{11}, \\
\tilde{\mathcal{E}}_2 &= \mathcal{E}_{22} \ast N \mathcal{E}_{22} \ast M \mathcal{G}_{22} - \mathcal{E}_{22} \ast N \mathcal{M}_{22} \ast N \mathcal{G}_{22} - \mathcal{E}_{22} \ast N \mathcal{S}_{22} \ast N \mathcal{H}_{22} \ast N \mathcal{E}_{22} \ast M \mathcal{N}_{22}, \\
\tilde{\mathcal{F}}_1 &= \left[ \mathcal{L}_{\tilde{\mathcal{C}}_{11}} - \mathcal{L}_{\tilde{\mathcal{M}}_{11}} \right], \quad \tilde{\mathcal{F}}_2 = \left[ \mathcal{L}_{\tilde{\mathcal{M}}_{11}} \ast N \mathcal{L}_{\tilde{\mathcal{S}}_{11}} - \mathcal{L}_{\tilde{\mathcal{C}}_{22}} \right], \quad \tilde{\mathcal{H}}_1 = \mathcal{F}_{11} \ast N \mathcal{S}_{11}, \\
\tilde{\mathcal{J}}_1 &= \mathcal{R}_{\mathcal{J}_{11}} \ast M \mathcal{J}_{11} \ast M \mathcal{G}_{11}, \quad \tilde{\mathcal{G}}_2 = \begin{bmatrix} \mathcal{R}_{\mathcal{J}_{11}} \\ -\mathcal{R}_{\mathcal{G}_{22}} \end{bmatrix}, \quad \tilde{\mathcal{J}}_1 = \mathcal{J}_{11} \ast M \mathcal{L}_{\mathcal{G}_{22}}, \quad \tilde{\mathcal{H}}_2 = \mathcal{H}_{11}, \quad \tilde{\mathcal{J}}_2 = \mathcal{R}_{\mathcal{J}_{22}}, \\
\tilde{\mathcal{E}}_1 &= \mathcal{E}_{11} \ast N \mathcal{E}_{11} \ast M \mathcal{G}_{11} - \mathcal{E}_{11} \ast N \mathcal{M}_{11} \ast N \mathcal{G}_{11} - \mathcal{E}_{11} \ast N \mathcal{S}_{11} \ast N \mathcal{H}_{11} \ast N \mathcal{E}_{11} \ast M \mathcal{N}_{11}, \\
\tilde{\mathcal{E}}_2 &= \mathcal{E}_{22} \ast N \mathcal{E}_{22} \ast M \mathcal{G}_{22} - \mathcal{E}_{22} \ast N \mathcal{M}_{22} \ast N \mathcal{G}_{22} - \mathcal{E}_{22} \ast N \mathcal{S}_{22} \ast N \mathcal{H}_{22} \ast N \mathcal{E}_{22} \ast M \mathcal{N}_{22}, \\
\tilde{\mathcal{H}}_{11} &= \mathcal{R}_{\tilde{\mathcal{H}}_1} \ast N \tilde{\mathcal{H}}_1, \quad \tilde{\mathcal{H}}_{22} = \mathcal{R}_{\tilde{\mathcal{H}}_2} \ast N \tilde{\mathcal{H}}_2, \quad \tilde{\mathcal{J}}_{11} = \mathcal{J}_{11} \ast M \mathcal{L}_{\tilde{\mathcal{G}}_1}, \quad \tilde{\mathcal{J}}_{22} = \mathcal{J}_{22} \ast M \mathcal{L}_{\tilde{\mathcal{G}}_2}, \\
\tilde{\mathcal{E}}_{11} &= \mathcal{R}_{\tilde{\mathcal{E}}_{11}} \ast N \tilde{\mathcal{E}}_{11} \ast M \tilde{\mathcal{G}}_{11}, \quad \tilde{\mathcal{E}}_{22} = \mathcal{R}_{\tilde{\mathcal{E}}_{22}} \ast N \tilde{\mathcal{E}}_{22} \ast M \tilde{\mathcal{G}}_{22}, \quad \tilde{\mathcal{A}} = \left[ \mathcal{L}_{\tilde{\mathcal{H}}_{11}} - \mathcal{L}_{\tilde{\mathcal{H}}_{22}} \right], \\
\mathcal{B} &= \begin{bmatrix} \mathcal{R}_{\tilde{\mathcal{J}}_{11}} \\ -\mathcal{R}_{\tilde{\mathcal{J}}_{22}} \end{bmatrix}, \quad \bar{E} = \mathcal{E}_{11} \ast N \bar{E}_{22} \ast M \mathcal{J}_{22} - \mathcal{H}_{11} \ast N \bar{E}_{11} \ast M \mathcal{J}_{11}.
\end{align*}
Then the system (1.3) is consistent if and only if

\begin{align}
R_Mi \ast N R_Ai \ast N E_i &= 0, \ E_i \ast M L_Bi \ast M L_Ni = 0, \ R_Ci \ast N E_i \ast M L_Bi = 0, \\
R_{\tilde{M}}i \ast N R_{\tilde{A}}i \ast N \tilde{E}_i &= 0, \ \tilde{E}_i \ast M L_{\tilde{B}}i \ast M L_{\tilde{N}}i = 0, \\
(3.3) \\
R_{\tilde{M}}kk \ast N R_{\tilde{F}}kk \ast N \tilde{E}_{kk} &= 0, \ \tilde{E}_{kk} \ast M L_{\tilde{B}}kk = 0, \\
R_{\tilde{F}}kk \ast N \tilde{E}_{kk} \ast M L_{\tilde{J}}kk &= 0, \ R_{\tilde{H}}kk \ast N \tilde{E}_{kk} \ast M L_{\tilde{G}}kk = 0, \\
(3.7) \\
R_{\tilde{F}}11 \ast N \tilde{E}_{11} \ast M L_{\tilde{J}}11 &= 0, \ R_{\tilde{H}}11 \ast N \tilde{E}_{11} \ast M L_{\tilde{G}}11 = 0, \\
(3.11) \\
R_{\tilde{A}} \ast N \tilde{E} \ast M L_{\tilde{G}} &= 0. \\
(3.14)
\end{align}
Under these conditions, the general solution to system (1.3) can be expressed as follows:

\[ X_i = A_i^1 * N \hat{E}_i * M B_i^1 - A_i^1 * N C_i * N M_i^1 * N \hat{E}_i * M B_i^1 - A_i^1 * N S_i * N C_i^1 * N \hat{E}_i \]
\[ * M N_i^1 * M D_i * M B_i^1 - A_i^1 * N S_i * N U_{2i} * M R_{N_i} * M D_i * M B_i^1 + L_{A_i} * N U_{4i} + U_{5i} * M R_{B_i}, \]
\[ Y_i = M_i^1 * N \hat{E}_i * M D_i^1 + S_i^1 * N S_i * N C_i^1 * N \hat{E}_i * M N_i^1 + L_{M_i} * N L_{S_i} * N U_{1i} + L_{M_i} * N U_{2i} * M R_{N_i} + U_{5i} * M R_{D_i}, \]
\[ Z_1 = J_1^1 * N \varepsilon_4 * M G_i^1 + L_{J_1^1} * N W_1 + W_3 * M R_{G_i}, \]
\[ Z_4 = H_1^1 * N \varepsilon_5 * M J_1^1 + L_{H_4} * N W_1 + W_3 * M R_{J_4}, \]
\[ Z_2 = \hat{M}_1 * N \hat{E}_1 * M \hat{D}_1^1 + \hat{S}_1^1 * N \hat{S}_1 * N \hat{C}_1^1 * N \hat{E}_1 * M \hat{N}_1^1 + L_{M_1} * N \hat{L}_{S_1} * N \hat{U}_1 \]
\[ + L_{M_1} * N \hat{U}_2 * M R_{\hat{N}_1} + \hat{U}_3 * M R_{\hat{D}_1}, \]
\[ or \ Z_2 = \hat{A}_1^1 * N \hat{E}_2 * M \hat{D}_2^1 + \hat{A}_2^1 * N \hat{C}_2 * N \hat{M}_2^1 * N \hat{E}_2 * M \hat{D}_2^1 - \hat{A}_2^1 * N \hat{S}_2 * N \hat{C}_2^1 * N \hat{E}_2 \]
\[ * M \hat{N}_2^1 * M \hat{D}_2^1 * M \hat{B}_2^1 - \hat{A}_2^1 * N \hat{S}_2 * N \hat{V}_2 * M R_{N_2} * M \hat{D}_2 * M \hat{B}_2^1 + L_{\hat{A}_2} * N \hat{V}_1 + \hat{V}_5 * M R_{\hat{D}_2}, \]
\[ Z_3 = \hat{M}_2^1 * N \hat{E}_2 * M \hat{D}_2^1 + \hat{S}_2^1 * N \hat{S}_2 * N \hat{C}_2^1 * N \hat{E}_2 * M \hat{N}_2^1 + L_{M_2} * N \hat{L}_{S_2} * N \hat{V}_1 \]
\[ + L_{M_2} * N \hat{V}_2 * M R_{\hat{N}_2} + \hat{V}_3 * M R_{\hat{D}_2}, \]
\[ or \ Z_3 = \hat{A}_2^1 * N \hat{E}_3 * M \hat{D}_3^1 + \hat{A}_3^1 * N \hat{C}_3 * N \hat{M}_3^1 * N \hat{E}_3 * M \hat{D}_3^1 - \hat{A}_3^1 * N \hat{S}_3 * N \hat{C}_3^1 * N \hat{E}_3 \]
\[ * M \hat{N}_3^1 * M \hat{D}_3 * M \hat{B}_3^1 - \hat{A}_3^1 * N \hat{S}_3 * N \hat{R}_2 * M R_{\hat{N}_3} * M \hat{D}_3 * M \hat{B}_3^1 + L_{\hat{A}_3} * N \hat{R}_4 \]
\[ + \hat{R}_5 * M R_{\hat{D}_3}, \]
\[ \text{where} \]
\[ W_1 = \begin{bmatrix} I & 0 \end{bmatrix} * N [A_{11}^1 * N (E_{11} - \hat{A}_{11} * N \hat{U}_{2} * M \hat{B}_{11}) - \varepsilon_{11} * M D_{11} + L_{A_{11}} * N \varepsilon_{22}], \]
\[ \tilde{U}_4 = \begin{bmatrix} 0 \end{bmatrix} * N [A_{11}^1 * N (E_{11} - \hat{A}_{11} * N \hat{U}_{2} * M \hat{B}_{11}) - \varepsilon_{11} * M D_{11} + L_{A_{11}} * N \varepsilon_{22}], \]
\[ W_2 = [R_{A_{11}} * N (E_{11} - \hat{A}_{11} * N \hat{U}_{2} * M \hat{B}_{11}) * M D_{11}^1 + A_{11} * N \varepsilon_{11} \]
\[ + \varepsilon_{33} * M R_{D_{11}} ] * M \begin{bmatrix} I \\ 0 \end{bmatrix}, \]

(3.15) (3.16) (3.17) (3.18) (3.19) (3.20) (3.21) (3.22) (3.23) (3.24) (3.25)
\( \hat{U}_5 = [R_{A_{11}} * N (E_{11} - \hat{A}_{11} * N \hat{U}_2 * M \hat{B}_{11}) * M D_{11}^{\dagger} + A_{11} * N \nu_{11} + \nu_{33} * M R_{D_{11}}] * M \begin{bmatrix} 0 \\ I \end{bmatrix}, \)  

(3.26a)

\( \hat{W}_1 = [I \ 0] * N [A_{22}^{\dagger} * N (E_{22} - \hat{A}_{22} * N \hat{K}_{2} * M \hat{B}_{22}) - \nu_{44} * M D_{22} + \nu_{44} * M \nu_{55}], \)  

(3.26b)

\( \hat{K}_1 = [0 \ I] * N [A_{11}^{\dagger} * N (E_{22} - \hat{A}_{22} * N \hat{K}_{2} * M \hat{B}_{22}) - \nu_{44} * M D_{22} + \nu_{44} * M \nu_{55}], \)  

(3.26c)

\( W_3 = [R_{A_{22}} * N (E_{22} - \hat{A}_{22} * N \hat{K}_{2} * M \hat{B}_{22}) * M D_{22}^{\dagger} + A_{22} * N \nu_{44} + \nu_{66} * M R_{D_{22}}] * M \begin{bmatrix} 0 \\ I \end{bmatrix}, \)  

(3.26d)

\( \hat{K}_3 = [R_{A_{22}} * N (E_{22} - \hat{A}_{22} * N \hat{K}_{2} * M \hat{B}_{22}) * M D_{22}^{\dagger} + A_{22} * N \nu_{44} + \nu_{66} * M R_{D_{22}}] * M \begin{bmatrix} 0 \\ I \end{bmatrix}, \)  

(3.26e)

\( \hat{U}_2 = \hat{A}_{11}^{\dagger} * N \hat{E}_{11} * M \hat{B}_{11}^{\dagger} + L_{A_{11}} * N \nu_{77} + \nu_{88} * M R_{\hat{B}_{11}}, \)  

(3.26f)

\( \hat{K}_2 = \hat{A}_{22} * N \hat{E}_{22} * M \hat{B}_{22}^{\dagger} + L_{A_{22}} * N \nu_{99} + W_{11} * M R_{\hat{B}_{22}}, \)  

(3.26g)

\( \hat{U}_1 = [I \ 0] * N [A_{11}^{\dagger} * N (\hat{F}_1 * N \hat{V}_2 * M \hat{G}_1 + \hat{H}_1 * N \hat{V}_2 * M \hat{J}_1 - \hat{E}_1) + P_{11} * M \hat{B}_1 + \nu_{33} * M R_{\hat{B}_1}] * M \begin{bmatrix} I \\ 0 \end{bmatrix}, \)  

(3.26h)

\( \hat{V}_4 = [0 \ I] * N [A_{11}^{\dagger} * N (\hat{F}_1 * N \hat{V}_2 * M \hat{G}_1 + \hat{H}_1 * N \hat{U}_2 * M \hat{J}_1 - \hat{E}_1) + P_{11} * M \hat{B}_1 + \nu_{33} * M R_{\hat{B}_1}] * M \begin{bmatrix} 0 \\ I \end{bmatrix}, \)  

(3.26i)

\( \hat{U}_3 = [R_{A_{11}} * N (\hat{F}_1 * N \hat{V}_2 * M \hat{G}_1 + \hat{H}_1 * N \hat{V}_2 * M \hat{J}_1 - \hat{E}_1) * M \hat{F}_1^{\dagger} + \hat{A}_1 * N P_{11} + P_{33} * M R_{\hat{B}_1}] * M \begin{bmatrix} I \\ 0 \end{bmatrix}, \)  

(3.26j)

\( \hat{V}_5 = [R_{A_{11}} * N (\hat{F}_1 * N \hat{V}_2 * M \hat{G}_1 + \hat{H}_1 * N \hat{U}_2 * M \hat{J}_1 - \hat{E}_1) * M \hat{F}_1^{\dagger} + \hat{A}_1 * N P_{11} + P_{33} * M R_{\hat{B}_1}] * M \begin{bmatrix} 0 \\ I \end{bmatrix}, \)  

(3.26k)

\( \hat{V}_1 = [I \ 0] * N [A_{22} * N (\hat{F}_2 * N \hat{K}_{2} * M \hat{G}_2 + \hat{H}_2 * N \hat{V}_2 * M \hat{J}_2 - \hat{E}_2) + P_{22} * M \hat{B}_2 + \nu_{33} * M R_{\hat{B}_2}], \)  

(3.26l)

\( \hat{K}_4 = [0 \ I] * N [A_{22} * N (\hat{F}_2 * N \hat{K}_{2} * M \hat{G}_2 + \hat{H}_2 * N \hat{V}_2 * M \hat{J}_2 - \hat{E}_2) + P_{22} * M \hat{B}_2 + \nu_{33} * M R_{\hat{B}_2}], \)  

(3.26m)

\( \hat{V}_3 = [R_{A_{22}} * N (\hat{F}_2 * N \hat{K}_{2} * M \hat{G}_2 + \hat{H}_2 * N \hat{V}_2 * M \hat{J}_2 - \hat{E}_2) * M \hat{F}_2^{\dagger} + \hat{A}_2 * N P_{22} + Q_{33} * M R_{\hat{B}_2}] * M \begin{bmatrix} I \\ 0 \end{bmatrix}, \)  

(3.26n)

\( \hat{K}_5 = [R_{A_{22}} * N (\hat{F}_2 * N \hat{K}_{2} * M \hat{G}_2 + \hat{H}_2 * N \hat{V}_2 * M \hat{J}_2) * M \hat{F}_2^{\dagger} + \hat{A}_2 * N P_{22} + Q_{33} * M R_{\hat{B}_2}] * M \begin{bmatrix} 0 \\ I \end{bmatrix}, \)  

(3.26o)

\( \hat{V}_2 = [\hat{F}_{11} * N \hat{E}_{11} * M \hat{G}_{11}^{\dagger} - \hat{F}_{11} * N \hat{H}_{11} * N \hat{M}_{11}^{\dagger} * N \hat{E}_{11} * M \hat{G}_{11}^{\dagger} - \hat{F}_{11} * N \hat{S}_{11} * N \hat{H}_{11} * N \hat{E}_{11} * M \hat{G}_{11}^{\dagger} + \hat{L}_{11} * N P_{55} + P_{66} * M R_{\hat{B}_{11}}, \)  

(3.26p)
\[\begin{align*}
\text{or } \hat{\mathcal{V}}_2 &= \mathcal{M}_{22} \ast N \mathcal{E}_{22} \ast \mathcal{M} \mathcal{F}_{22} + \mathcal{S}_{22} \ast N \mathcal{E}_{22} \ast \mathcal{N}_{22} \mathcal{E}_{22} \ast \mathcal{M} \mathcal{N}_{22} + \mathcal{L}_{\mathcal{M}_{22}} \\
&= \mathcal{N} \mathcal{L}_{\mathcal{S}_{22}} \ast N \mathcal{Q}_{77} + \mathcal{L}_{\mathcal{M}_{22}} \ast N \mathcal{Q}_{55} \ast \mathcal{M} \mathcal{R}_{\mathcal{M}_{22}} + \mathcal{Q}_{88} \ast \mathcal{M} \mathcal{R}_{\mathcal{F}_{22}}, \\
\mathcal{P}_{55} &= \left[ \mathcal{I} \ast N \mathcal{A}_1 \ast N \left( -\overline{\mathcal{E}}_1 + \mathcal{F}_1 \ast \mathcal{N} P_{44} \ast \mathcal{M} \overline{\mathcal{G}}_1 + \overline{\mathcal{H}}_1 \ast N \mathcal{Q}_{55} \ast \mathcal{M} \overline{\mathcal{F}}_1 \right) \right] \\
&\quad + \mathcal{K}_{11} \ast \mathcal{M} \mathcal{B}_1 + \mathcal{L}_{\mathcal{A}_1} \ast N \mathcal{K}_{22}, \\
\mathcal{Q}_{77} &= \left[ \mathcal{I} \ast N \mathcal{A}_1 \ast N \left( -\overline{\mathcal{E}}_1 + \mathcal{F}_1 \ast \mathcal{N} P_{44} \ast \mathcal{M} \overline{\mathcal{G}}_1 + \overline{\mathcal{H}}_1 \ast N \mathcal{Q}_{55} \ast \mathcal{M} \overline{\mathcal{F}}_1 \right) \right] \\
&\quad + \mathcal{K}_{11} \ast \mathcal{M} \mathcal{B}_1 + \mathcal{L}_{\mathcal{A}_1} \ast N \mathcal{K}_{22}, \\
\mathcal{P}_{66} &= \left[ \mathcal{R}_{\mathcal{A}_1} \ast N \left( -\overline{\mathcal{E}}_1 + \mathcal{F}_1 \ast \mathcal{N} P_{44} \ast \mathcal{M} \overline{\mathcal{G}}_1 + \overline{\mathcal{H}}_1 \ast N \mathcal{Q}_{55} \ast \mathcal{M} \overline{\mathcal{F}}_1 \right) \right] \\
&\quad \ast \mathcal{M} \mathcal{B}_1 + \mathcal{A}_1 \ast N \mathcal{K}_{11} + \mathcal{K}_{33} \ast \mathcal{M} \mathcal{R}_{\mathcal{B}_1} \ast \mathcal{M} \left[ \mathcal{I} \ast N \left( -\overline{\mathcal{E}}_1 + \mathcal{F}_1 \ast \mathcal{N} P_{44} \ast \mathcal{M} \overline{\mathcal{G}}_1 + \overline{\mathcal{H}}_1 \ast N \mathcal{Q}_{55} \ast \mathcal{M} \overline{\mathcal{F}}_1 \right) \right] \\
&\quad + \mathcal{K}_{11} \ast \mathcal{M} \mathcal{B}_1 + \mathcal{L}_{\mathcal{A}_1} \ast N \mathcal{K}_{22}, \\
\mathcal{Q}_{88} &= \left[ \mathcal{R}_{\mathcal{A}_1} \ast N \left( -\overline{\mathcal{E}}_1 + \mathcal{F}_1 \ast \mathcal{N} P_{44} \ast \mathcal{M} \overline{\mathcal{G}}_1 + \overline{\mathcal{H}}_1 \ast N \mathcal{Q}_{55} \ast \mathcal{M} \overline{\mathcal{F}}_1 \right) \right] \\
&\quad \ast \mathcal{M} \mathcal{B}_1 + \mathcal{A}_1 \ast N \mathcal{K}_{11} + \mathcal{K}_{33} \ast \mathcal{M} \mathcal{R}_{\mathcal{B}_1} \ast \mathcal{M} \left[ \mathcal{I} \ast N \left( -\overline{\mathcal{E}}_1 + \mathcal{F}_1 \ast \mathcal{N} P_{44} \ast \mathcal{M} \overline{\mathcal{G}}_1 + \overline{\mathcal{H}}_1 \ast N \mathcal{Q}_{55} \ast \mathcal{M} \overline{\mathcal{F}}_1 \right) \right]
\end{align*}\]
that the Sylvester-like tensor equation (3.28) is consistent if and only if

$$A_{1 * M} B_1 + C_{1 * N} \mathcal{V}_1 * M D_1 + C_{1 * N} (F_{1 * N} Z_1 * M G_1 + H_{1 * N} Z_2 * M J_1) * M B_1 = \mathcal{E}_1,$$

equation (3.29)

$$A_{2 * N} X_2 * M B_2 + C_{2 * N} \mathcal{V}_2 * M D_2 + C_{2 * N} (F_{2 * N} Z_2 * M G_2 + H_{2 * N} Z_3 * M J_2) * M B_2 = \mathcal{E}_2,$$

equation (3.30)

$$A_{3 * N} X_3 * M B_3 + C_{3 * N} \mathcal{V}_3 * M D_3 + C_{3 * N} (F_{3 * N} Z_3 * M G_3 + H_{3 * N} Z_4 * M J_3) * M B_3 = \mathcal{E}_3,$$

equation (3.31)

$$\mathcal{F}_4 * N Z_4 * M G_4 = \mathcal{E}_4,$$

equation (3.32)

The main idea is to implement the conditions of consistency to enable this group to have a solution, and hence, we establish an expression of this solution. Applying Lemma 2.4, we have that the Sylvester-like tensor equation (3.28) is consistent if and only if

$$\mathcal{R}_{A_1} * N \mathcal{R}_{A_1} * N \mathcal{E}_1 = 0, \mathcal{E}_1 * M \mathcal{L}_{B_1} * M \mathcal{L}_{N_1} = 0, \mathcal{R}_{C_1} * N \mathcal{E}_1 * M \mathcal{L}_{B_1} = 0$$

$$\mathcal{R}_{A_1} * N \mathcal{R}_{A_1} * N \mathcal{E}_1 = 0, \mathcal{E}_1 * M \mathcal{L}_{B_1} * M \mathcal{L}_{N_1} = 0, \mathcal{R}_{C_1} * N \mathcal{E}_1 * M \mathcal{L}_{B_1} = 0$$

$$\mathcal{R}_{A_1} * N \mathcal{R}_{A_1} * N \mathcal{E}_1 = 0, \mathcal{E}_1 * M \mathcal{L}_{B_1} * M \mathcal{L}_{N_1} = 0, \mathcal{R}_{C_1} * N \mathcal{E}_1 * M \mathcal{L}_{B_1} = 0.$$
In that case, the general solution can be expressed as

\[
X_1 = A^*_1 * N \hat{X}_1 * M B^\dagger_1 - A^*_1 * N C_1 * N M^*_1 * N \hat{X}_1 * M B^\dagger_1 - A^*_1 * N S_1 * N U_{21} * M R_{N_1} * M D_1 * M B^\dagger_1 + L_{A_1} * N U_{41} \\
+ U_{61} * M R_{B_1},
\]

\[
Y_1 = M^*_1 * N \hat{Y}_1 * M D_1 + S^*_1 * N S_1 * N C^*_1 * N \hat{Y}_1 * M N^*_1 + L_{M_1} * N L_{S_1} * N U_{11} \\
+ L_{M_1} * N U_{21} * M R_{N_1} + U_{31} * M R_{D_1},
\]

\[
Z_1 = \hat{A}^*_1 * N \hat{Z}_1 * M \hat{B}^\dagger_1 - \hat{A}^*_1 * N \hat{C}_1 * N \hat{M}^*_1 * N \hat{Z}_1 * M \hat{B}^\dagger_1 - \hat{A}^*_1 * N \hat{S}_1 * N \hat{C}^*_1 * N \hat{Z}_1 \\
* M \hat{N}^*_1 * M \hat{D}_1 * M \hat{B}^\dagger_1 - \hat{A}^*_1 * N \hat{S}_1 * N \hat{U}_2 * M R_{N_1} * M \hat{D}_1 * M \hat{B}^\dagger_1 + L_{\hat{A}_1} * N \hat{U}_4 \\
+ \hat{U}_5 * M R_{\hat{B}_1},
\]

\[
Z_2 = \hat{M}^*_1 * N \hat{Z}_1 * M \hat{D}_1 + S^*_1 * N \hat{S}_1 * N \hat{C}^*_1 * N \hat{E} * M \hat{N}^*_1 + L_{\hat{M}_1} * N L_{\hat{S}_1} * N \hat{U}_1 \\
+ L_{\hat{M}_1} * N \hat{U}_2 * M R_{\hat{N}_1} + \hat{U}_3 * M R_{\hat{D}_1},
\]

where \(\hat{A}_1, \hat{B}_1, \hat{C}_1, \hat{D}_1, \hat{E}_1, \hat{M}_1, \hat{N}_1\) and \(\hat{S}_1\) given by (3.1b)-(3.1d) whenever \(i = 1\). It can follow the same technique to determine the consistency conditions and the general solution to the Sylvester-like quaternion tensor equation (3.29). So, we have that Eq. (3.29) is solvable if and only if the conditions (3.3a)-(3.3b) satisfy whenever \(i = 2\). In this case, the quaternion tensors \(X_2\) and \(Y_2\) can be given by (3.15)-(3.16) whenever \(i = 2\) and

\[
Z_2 = \hat{A}^*_2 * N \hat{Z}_2 * M \hat{D}_2 - \hat{A}^*_2 * N \hat{C}_2 * N \hat{M}^*_2 * N \hat{Z}_2 * M \hat{B}^\dagger_2 - \hat{A}^*_2 * N \hat{S}_2 * N \hat{C}^*_2 * N \hat{Z}_2 \\
* M \hat{N}^*_2 * M \hat{D}_2 * M \hat{B}^\dagger_2 - \hat{A}^*_2 * N \hat{S}_2 * N \hat{V}_2 * M R_{N_2} * M \hat{D}_2 * M \hat{B}^\dagger_2 + L_{\hat{A}_2} * N \hat{V}_4 \\
+ \hat{V}_5 * M R_{\hat{B}_2},
\]

\[
Z_3 = \hat{M}^*_2 * N \hat{Z}_2 * M \hat{D}_2 + \hat{S}^*_2 * N \hat{S}_2 * N \hat{C}^*_2 * N \hat{Z}_2 * M \hat{N}^*_2 + L_{\hat{M}_2} * N L_{\hat{S}_2} * N \hat{V}_1 \\
+ L_{\hat{M}_2} * N \hat{V}_2 * M R_{\hat{N}_2} + \hat{V}_3 * M R_{\hat{D}_2},
\]

where \(\hat{A}_2, \hat{B}_2, \hat{C}_2, \hat{D}_2, \hat{E}_2, \hat{M}_2, \hat{N}_2\) and \(\hat{S}_2\) given by (3.15)-(3.16) whenever \(i = 2\).

Similarly, we can provide that Eq. (3.31) is solvable if and only if the conditions (3.3a)-(3.3b) are satisfying whenever \(i = 3\). In this case, the quaternion tensors \(X_3\) and \(Y_3\) can be given by (3.15)-(3.16) whenever \(i = 3\) and

\[
Z_3 = \hat{A}^*_3 * N \hat{Z}_3 * M \hat{B}^\dagger_3 - \hat{A}^*_3 * N \hat{C}_3 * N \hat{M}^*_3 * N \hat{Z}_3 * M \hat{B}^\dagger_3 - \hat{A}^*_3 * N \hat{S}_3 * N \hat{C}^*_3 * N \hat{Z}_3 \\
* M \hat{N}^*_3 * M \hat{D}_3 * M \hat{B}^\dagger_3 - \hat{A}^*_3 * N \hat{S}_3 * N \hat{K}_2 * M R_{N_3} * M \hat{D}_3 * M \hat{B}^\dagger_3 + L_{\hat{A}_3} * N \hat{K}_4 \\
+ \hat{K}_5 * M R_{\hat{B}_3},
\]

\[
Z_4 = \hat{M}^*_3 * N \hat{Z}_3 * M \hat{D}_3 + \hat{S}^*_3 * N \hat{S}_3 * N \hat{C}^*_3 * N \hat{Z}_3 * M \hat{N}^*_3 + L_{\hat{M}_3} * N L_{\hat{S}_3} * N \hat{K}_1 \\
+ L_{\hat{M}_3} * N \hat{K}_2 * M R_{\hat{N}_3} + \hat{K}_3 * M R_{\hat{D}_3},
\]

where \(\hat{A}_3, \hat{B}_3, \hat{C}_3, \hat{D}_3, \hat{E}_3, \hat{M}_3, \hat{N}_3\) and \(\hat{S}_3\) given by (3.15)-(3.16), whenever \(i = 3\).

It follows from Lemma 2.3 that the necessary and sufficient conditions for the Sylvester-like quaternion tensor equation (3.31) and (3.32) to be consistent are given by (3.6), respectively. Consequently, the solutions to these two equations are expressed as

\[
Z_1 = F^*_1 * N E_4 * M G^\dagger_1 + L_{F_4} * N W_1 + W_2 * M R_{G_4},
\]

\[
Z_4 = H^*_4 * N E_5 * M J^\dagger_4 + L_{H_4} * N \hat{W}_1 + \hat{W}_2 * M R_{J_4},
\]
Let $Z_1$ in (3.34c) be equal to $Z_1$ in (3.37a), and $Z_4$ in (3.36b) be equal to $Z_4$ in (3.37b). Then we have the following equations:

\[ A_{11} * N \left[ \begin{array}{c} W_1 \\ \hat{U}_1 \end{array} \right] + \left[ W_2 \ \hat{U}_3 \right] * M D_{11} = \mathcal{E}_{11} - \hat{A}_{11} * N \hat{U}_2 * M \hat{B}_{11}, \quad (3.38) \]

\[ A_{22} * N \left[ \begin{array}{c} W_1 \\ \hat{K}_1 \end{array} \right] + \left[ W_3 \ \hat{K}_3 \right] * M D_{22} = \mathcal{E}_{22} - \hat{A}_{22} * N \hat{K}_2 * M \hat{B}_{22}, \quad (3.39) \]

Apply Lemma 2.4 to Eq. (3.38), we have that it is solvable if and only if there exists a quaternion tensor $\hat{U}_2$ satisfies

\[ \hat{A}_{11} * N \hat{U}_2 * M \hat{B}_{11} = \hat{\mathcal{E}}_{11}. \quad (3.40) \]

In that case, the general solution can be express as

\[ \left[ \begin{array}{c} W_1 \\ \hat{U}_1 \end{array} \right] = A_{11}^\dagger * N (\mathcal{E}_{11} - \hat{A}_{11} * N \hat{U}_2 * M \hat{B}_{11}) - \mathcal{V}_{11} * M D_{11} + \mathcal{L}_{A_{11}} * N \mathcal{V}_{22}, \quad (3.41) \]

\[ \left[ \begin{array}{c} W_2 \\ \hat{U}_3 \end{array} \right] = \mathcal{R}_{A_{11}} * N (\mathcal{E}_{11} - \hat{A}_{11} * N \hat{U}_2 * M \hat{B}_{11}) * M D_{11}^\dagger + A_{11} * N \mathcal{V}_{11} + \mathcal{V}_{33} * M \mathcal{R}_{D_{11}}, \quad (3.42) \]

By applying Proposition 2.2 to equations (3.41)-(3.42), we can find expressions for quaternion tensors $W_1, \hat{U}_4, W_2,$ and $\hat{U}_5$ in (3.23)-(3.26a). In the same way, we have that Eq. (3.39) is solvable if and only if there exists a quaternion tensor $\hat{K}_2$ satisfies

\[ \hat{A}_{22} * N \hat{K}_2 * M \hat{B}_{22} = \hat{\mathcal{E}}_{22}. \quad (3.43) \]

In that case, the general solution can be express as

\[ \left[ \begin{array}{c} W_1 \\ \hat{K}_1 \end{array} \right] = A_{22}^\dagger * N (\mathcal{E}_{22} - \hat{A}_{22} * N \hat{K}_2 * M \hat{B}_{22}) - \mathcal{V}_{44} * M D_{22} + \mathcal{L}_{A_{22}} * N \mathcal{V}_{55}, \quad (3.44) \]

\[ \left[ \begin{array}{c} W_3 \\ \hat{K}_3 \end{array} \right] = \mathcal{R}_{A_{22}} * N (\mathcal{E}_{22} - \hat{A}_{22} * N \hat{K}_2 * M \hat{B}_{22}) * M D_{22}^\dagger + A_{22} * N \mathcal{V}_{44} + \mathcal{V}_{66} * M \mathcal{R}_{D_{22}}, \quad (3.45) \]

It can be utilized Proposition 2.2 to equations (3.44)-(3.45), we can get quaternion tensors $W_1, \hat{K}_1, W_2$ and $\hat{K}_3$ in (3.26b)-(3.26b). Meanwhile, the quaternion tensor equations (3.40) and (3.43) are solvable if and only if the conditions (3.7) are satisfying, respectively, for $k = 1, 2$. In that case, the general solution can be written as

\[ \hat{U}_2 = \tilde{A}_{11}^* N \hat{\mathcal{E}}_{11} * M \tilde{B}_{11}^* + \mathcal{L}_{\tilde{A}_{11}} * N \mathcal{V}_{77} + \mathcal{V}_{88} * M \mathcal{R}_{\tilde{B}_{11}}, \quad (3.46a) \]

\[ \hat{K}_2 = \tilde{A}_{22}^* N \hat{\mathcal{E}}_{22} * M \tilde{B}_{22}^* + \mathcal{L}_{\tilde{A}_{22}} * N \mathcal{V}_{99} + \mathcal{V}_{11} * M \mathcal{R}_{\tilde{B}_{22}}. \quad (3.46b) \]

Now, $Z_2$ in (3.34d) should be equal to $Z_2$ in (3.35a) and $Z_3$ in (3.35b) should be equal to $Z_3$ in (3.36a), yields:

\[ \mathcal{A}_{11} * N \left[ \begin{array}{c} \hat{U}_1 \\ \hat{V}_3 \end{array} \right] + \left[ \hat{V}_3 \ \hat{V}_5 \right] * M \mathcal{E}_1 = -\mathcal{F}_1 + \mathcal{F}_1 * N \hat{V}_2 * M \mathcal{G}_1 + \mathcal{F}_4 * N \hat{U}_2 * M \mathcal{J}_1, \quad (3.47) \]

\[ \mathcal{A}_{22} * N \left[ \begin{array}{c} \hat{V}_1 \\ \hat{K}_3 \end{array} \right] + \left[ \hat{V}_3 \ \hat{K}_5 \right] * M \mathcal{E}_2 = -\mathcal{F}_2 + \mathcal{F}_2 * N \hat{K}_2 * M \mathcal{G}_2 + \mathcal{F}_2 * N \hat{V}_2 * M \mathcal{J}_2. \quad (3.48) \]
The system of tensor equations which consists of the two equations (3.47) and (3.48) is consistent if and only if there exists quaternion tensors \( \hat{V}_2, \hat{U}_2, \) and \( \hat{K}_2 \) satisfy the following system:

\[
\begin{align*}
\mathcal{F}_{11} \ast N \hat{V}_2 \ast M \mathcal{G}_{11} + \mathcal{H}_{11} \ast N \hat{U}_2 \ast M \mathcal{J}_{11} &= \mathcal{E}_{11}, \\
\mathcal{F}_{22} \ast N \hat{K}_2 \ast M \mathcal{G}_{22} + \mathcal{H}_{22} \ast N \hat{V}_2 \ast M \mathcal{J}_{22} &= \mathcal{E}_{22},
\end{align*}
\]

(3.49, 3.50)

In utilizing Lemma 2.4, we have that the general solution to quaternion tensor equations (3.47)-(3.48) can be given by (3.26a)-(3.26b). The quaternion system of tensor equations (3.49) - (3.50) is solvable if and only if Eq. (3.49) is solvable, Eq. (3.50) is solvable, and the quaternion tensor \( \hat{V}_2 \) in (3.50) coincide with \( \hat{V}_2 \) in (3.50). So, Eq. (3.49) and Eq. (3.50) are solvable if and only if the conditions (3.8) - (3.9) are satisfying, respectively, for \( k = 1, 2 \). In this case, the general solution can be express as

\[
\begin{align*}
\hat{V}_2 &= \mathcal{F}_{11} \ast N \mathcal{F}_{11} \ast M \mathcal{G}_{11} - \mathcal{F}_{11} \ast N \mathcal{H}_{11} \ast N \mathcal{M}_{11} \ast N \mathcal{E}_{11} \ast M \mathcal{G}_{11} - \mathcal{F}_{11} \ast N \mathcal{S}_{11} \ast N \mathcal{H}_{11} \\
&\quad + \mathcal{L}_{11} \ast N \mathcal{P}_{55} + \mathcal{P}_{66} \ast M \mathcal{R}_{11}, \\
\hat{U}_2 &= \mathcal{M}_{11} \ast N \mathcal{E}_{11} \ast M \mathcal{J}_{11} + \mathcal{S}_{11} \ast N \mathcal{S}_{11} \ast N \mathcal{H}_{11} \ast N \mathcal{E}_{11} \ast M \mathcal{N}_{11} + \mathcal{L}_{11}, \\
&\quad \ast N \mathcal{L}_{11} \ast N \mathcal{Q}_{14} + \mathcal{L}_{11} \ast N \mathcal{P}_{44} \ast M \mathcal{R}_{11} + \mathcal{Q}_{66} \ast M \mathcal{R}_{11},
\end{align*}
\]

(3.51a, 3.51b)

\[
\begin{align*}
\hat{K}_2 &= \mathcal{F}_{22} \ast N \mathcal{F}_{22} \ast M \mathcal{G}_{22} - \mathcal{F}_{22} \ast N \mathcal{H}_{22} \ast N \mathcal{M}_{22} \ast N \mathcal{E}_{22} \ast M \mathcal{G}_{22} - \mathcal{F}_{22} \ast N \mathcal{S}_{22} \ast N \mathcal{H}_{22} \\
&\quad + \mathcal{L}_{22} \ast N \mathcal{P}_{77} + \mathcal{P}_{88} \ast M \mathcal{R}_{22}, \\
\hat{V}_2 &= \mathcal{M}_{22} \ast N \mathcal{E}_{22} \ast M \mathcal{J}_{22} + \mathcal{S}_{22} \ast N \mathcal{S}_{22} \ast N \mathcal{H}_{22} \ast N \mathcal{E}_{22} \ast M \mathcal{N}_{22} + \mathcal{L}_{22}, \\
&\quad \ast N \mathcal{L}_{22} \ast N \mathcal{Q}_{77} + \mathcal{L}_{22} \ast N \mathcal{Q}_{55} \ast M \mathcal{R}_{22} + \mathcal{Q}_{88} \ast M \mathcal{R}_{22},
\end{align*}
\]

(3.51c, 3.51d)

By equating \( \hat{V}_2 \) in (3.51a) with \( \hat{V}_2 \) in (3.51d), we have the following equation:

\[
\mathcal{A}_1 \ast N \left[ \mathcal{P}_{55} \mathcal{P}_{77} \right] + \left[ \mathcal{P}_{66} \mathcal{Q}_{88} \right] \ast N \mathcal{B}_1 = -\mathcal{E}_1 + \mathcal{F}_1 \ast N \mathcal{P}_{44} \ast M \mathcal{G}_1 + \mathcal{H}_1 \ast N \mathcal{Q}_{55} \ast M \mathcal{J}_1.
\]

(3.52)

It follows from Lemma 2.4 that Eq. (3.52) is solvable if and only if there exist quaternion tensors \( \mathcal{P}_{44} \) and \( \mathcal{Q}_{55} \) satisfy

\[
\mathcal{F}_{11} \ast N \mathcal{P}_{44} \ast M \mathcal{G}_{11} + \mathcal{H}_{11} \ast N \mathcal{Q}_{55} \ast M \mathcal{J}_{11} = \mathcal{E}_{11}.
\]

(3.53)

In utilizing Lemma 2.4, we have that the general solution to quaternion tensor equation (3.52) can be given by (3.26c)-(3.26d). On applying Lemma 2.4, we have that Eq. (3.53) is solvable if and only if conditions (3.10)-(3.11) satisfy. In that case, the general solution can be given by

\[
\begin{align*}
\mathcal{P}_{44} &= \mathcal{F}_{11} \ast N \mathcal{F}_{11} \ast M \mathcal{G}_{11} - \mathcal{F}_{11} \ast N \mathcal{H}_{11} \ast N \mathcal{M}_{11} \ast N \mathcal{E}_{11} \ast M \mathcal{G}_{11} - \mathcal{F}_{11} \ast N \mathcal{S}_{11} \\
&\quad + \mathcal{L}_{11} \ast N \mathcal{K}_{44} \ast M \mathcal{R}_{11}, \\
\mathcal{Q}_{55} &= \mathcal{M}_{11} \ast N \mathcal{E}_{11} \ast M \mathcal{J}_{11} + \mathcal{S}_{11} \ast N \mathcal{S}_{11} \ast N \mathcal{H}_{11} \ast N \mathcal{E}_{11} \ast M \mathcal{N}_{11} + \mathcal{L}_{11},
\end{align*}
\]

(3.54, 3.55)
Quaternion tensors $\tilde{U}_2$ in (3.46a) and $\tilde{K}_2$ in (3.46b) should coincide with $\hat{U}_2$ in (3.51b) and $\hat{K}_2$ in (3.51c), respectively. In that case, we have the following equations:

\[
\tilde{A}_1 *_{N} \begin{bmatrix} Q_{14} \\ \psi_{77} \end{bmatrix} + \begin{bmatrix} Q_{66} & V_{88} \end{bmatrix} *_{M} \tilde{B}_1 = \tilde{C}_1 - \tilde{C}_1 *_{N} P_{44} *_{M} \tilde{D}_1, \tag{3.56}
\]

\[
\tilde{A}_2 *_{N} \begin{bmatrix} V_{99} \\ \psi_{77} \end{bmatrix} + \begin{bmatrix} W_{11} & P_{88} \end{bmatrix} *_{M} \tilde{B}_2 = \tilde{C}_2 - \tilde{C}_2 *_{N} Q_{55} *_{M} \tilde{D}_2, \tag{3.57}
\]

Apply Lemma 2.4 to Eq. (3.56) and Eq. (3.57). we have that (3.56) and Eq. (3.57) are solvable if and only if there are quaternion tensors $\tilde{P}$ and $\tilde{Q}$ in (3.60) and (3.61), respectively. In that case, we have the following system of tensor equations:

\[
\tilde{Q}_{44} = \tilde{C}_{44} *_{N} \tilde{E}_{44} *_{M} \tilde{D}_{44} + L_{\tilde{C}_{44}} *_{N} \tilde{W}_{88} + \tilde{W}_{99} *_{M} R_{\tilde{D}_{44}}, \tag{3.60}
\]

\[
\tilde{Q}_{55} = \tilde{C}_{55} *_{N} \tilde{E}_{55} *_{M} \tilde{D}_{55} + L_{\tilde{C}_{55}} *_{N} \tilde{W}_{88} + \tilde{W}_{99} *_{M} R_{\tilde{D}_{55}}. \tag{3.61}
\]

If the quaternion tensor equations (3.58) and (3.59) are solvable and $\tilde{P}$ and $\tilde{Q}$ in (3.60) and (3.61) are satisfying, respectively, for $j = 1, 2$. In that case, the general solution can be given by

\[
\tilde{A}_1 *_{N} \begin{bmatrix} W_{88} \\ K_{55} \end{bmatrix} + \begin{bmatrix} \tilde{W}_{99} & K_{66} \end{bmatrix} *_{M} \tilde{G}_1 = \tilde{C}_1 - \tilde{H}_1 *_{N} K_{44} *_{M} \tilde{J}_1, \tag{3.62}
\]

\[
\tilde{A}_2 *_{N} \begin{bmatrix} K_{77} \\ \tau_{11} \end{bmatrix} + \begin{bmatrix} \tilde{K}_{88} & \tau_{22} \end{bmatrix} *_{M} \tilde{G}_2 = \tilde{C}_2 - \tilde{H}_2 *_{N} K_{44} *_{M} \tilde{J}_2. \tag{3.63}
\]

Apply Lemma 2.4 to Eq. (3.62) and Eq. (3.63). Consequently, we have that

\[
\tilde{H}_{11} *_{N} K_{44} *_{M} \tilde{J}_{11} = \tilde{C}_1, \tag{3.64}
\]

\[
\tilde{H}_{22} *_{N} K_{44} *_{M} \tilde{J}_{22} = \tilde{C}_2. \tag{3.65}
\]

In that case, the general solution to Equations (3.62) and (3.63) can be given by (3.66) and (3.67). Meanwhile, the quaternion system of tensor equations (3.64)-(3.65) is solvable if and only if Eq. (3.56) and Eq. (3.57) are solvable, and $\tilde{K}_{44}$ in (3.64) coincide with $\tilde{K}_{44}$ in (3.65). Eq. (3.64) and Eq. (3.65) are solvable if and only if the conditions (3.12) are satisfying, respectively, for $l = 1, 2$. In that case, the general solution to Equations (3.62) and (3.63) can be given by

\[
\tilde{K}_{44} = \tilde{H}_{11} *_{N} \tilde{E}_{11} *_{M} \tilde{J}_{11} + L_{\tilde{H}_{44}} *_{N} \tilde{W}_2 + \tilde{W}_3 *_{M} R_{\tilde{J}_{11}}, \tag{3.66}
\]

\[
\tilde{K}_{44} = \tilde{H}_{22} *_{N} \tilde{E}_{22} *_{M} \tilde{J}_{22} + L_{\tilde{H}_{44}} *_{N} \tilde{W}_4 + \tilde{W}_5 *_{M} R_{\tilde{J}_{22}}. \tag{3.67}
\]

Ultimately, equating $\tilde{K}_{44}$ in (3.66) by $\tilde{K}_{44}$ in (3.67), yield:

\[
\tilde{A} *_{N} \begin{bmatrix} \tau_{33} \\ \tau_{55} \end{bmatrix} + \begin{bmatrix} \tau_{44} & \tau_{66} \end{bmatrix} *_{M} \tilde{B} = \tilde{C}. \tag{3.68}
\]
Apply Lemma 2.4 to Eq. (3.68), we have that Eq. (3.68) is solvable if and only if condition (3.13) satisfies. In that case the general solution can be given by (3.27p) - (3.27s).

\[ \square \]

**Algorithm 3.2.** The general solution to the system of two-sided four coupled Sylvester-like quaternion tensor equations (1.3) gives by the following:

1. **Input** the system of two-sided four coupled Sylvester-like quaternion tensor equations (1.3) with viable orders over \( \mathbb{H} \).
2. **Compute** all quaternion tensors, which appeared in (3.1a) - (3.2s).
3. **Check** whether the Moore-Penrose inverses conditions in Theorem 3.1 are satisfied or not. If not, return “The system (1.3) is inconsistent”.
4. **Else** compute the quaternion unknowns \( X_i, Y_i, Z_j \), where \( i = 1, 3 \) and \( j = 1, 4 \) by (3.15) - (3.22).
5. **Output** the general solution of the system (1.3) is \( X_i, Y_i, Z_j \).

We give an example to illustrate Theorem 3.1.

**Example 3.3.** Consider the two-sided four coupled Sylvester-like quaternion system of tensor equations (1.3), where

\[
F_4(:, 1, 1) = \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}, \quad F_4(:, 1, 2) = \begin{bmatrix} 1 & 0 \\ 0 & j \end{bmatrix}, \quad F_4(:, 2, 1) = \begin{bmatrix} 0 & 0 \\ 0 & -k \end{bmatrix}, \quad F_4(:, 2, 2) = \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix},
\]

\[
G_4(:, 1, 1) = \begin{bmatrix} 0 & 0 \\ 2 & i \end{bmatrix}, \quad G_4(:, 1, 2) = \begin{bmatrix} 2 & 0 \\ 0 & 2i \end{bmatrix}, \quad G_4(:, 2, 1) = \begin{bmatrix} 0 & 0 \\ 0 & k \end{bmatrix}, \quad G_4(:, 2, 2) = \begin{bmatrix} 0 & 3 - j \\ 0 & 0 \end{bmatrix},
\]

\[
E_4(:, 1, 1) = \begin{bmatrix} -2j & 2i \\ 0 & -2j \end{bmatrix}, \quad E_4(:, 1, 2) = \begin{bmatrix} 0 & -4i \\ 0 & 2i - 6j \end{bmatrix}, \quad E_4(:, 2, 1) = \begin{bmatrix} 0 & 0 \\ 0 & 2 + k \end{bmatrix},
\]

\[
H_4(:, 1, 1) = \begin{bmatrix} i & j \\ 0 & 0 \end{bmatrix}, \quad H_4(:, 1, 2) = \begin{bmatrix} j & k \\ 0 & 0 \end{bmatrix},
\]

\[
J_4(:, 1, 1) = \begin{bmatrix} 1 & 0 \\ 2 & -i \end{bmatrix}, \quad J_4(:, 1, 2) = \begin{bmatrix} 0 & 3 - k \\ 0 & 3 + k \end{bmatrix}, \quad J_5(:, 1, 1) = \begin{bmatrix} i + 2j - k & 1 + i \\ 1 + 2i + 2j - k & 0 \end{bmatrix},
\]

\[
E_5(:, 1, 2) = \begin{bmatrix} 10i + 2j & -2i + 6j + k \\ 3 + 6i & 0 \end{bmatrix}, \quad E_5(:, 2, 1) = \begin{bmatrix} -2 + i + 4j + 5k & 1 - 2i + 2j + k \\ 9j + 3k & 0 \end{bmatrix},
\]

\[
E_5(:, 2, 2) = \begin{bmatrix} -2 + 11i + 7j + 6k & -9i + 9j \\ 1 + 7i + 5j + 5k & 0 \end{bmatrix}, \quad A_1(:, 1, 1) = \begin{bmatrix} i & -i \\ 0 & 0 \end{bmatrix}, \quad A_1(:, 1, 2) = \begin{bmatrix} j & -j \\ 0 & 0 \end{bmatrix},
\]

\[
A_1(:, 2, 1) = \begin{bmatrix} k & -k \\ 0 & 0 \end{bmatrix}, \quad A_1(:, 2, 2) = \begin{bmatrix} 0 & i \\ i & -i \end{bmatrix}, \quad B_1(:, 1, 1) = \begin{bmatrix} 0 & 0 \\ j & -j \end{bmatrix}, \quad B_1(:, 1, 2) = \begin{bmatrix} 0 & 0 \\ k & -k \end{bmatrix},
\]

\[
B_1(:, 2, 1) = \begin{bmatrix} i & i + j \\ 0 & 0 \end{bmatrix}, \quad B_1(:, 2, 2) = \begin{bmatrix} j & j + k \\ 0 & 0 \end{bmatrix}, \quad C_1(:, 1, 1) = \begin{bmatrix} k & i + k \\ 0 & 0 \end{bmatrix}, \quad C_1(:, 2, 1) = \begin{bmatrix} 5i & 1 \\ 0 & 0 \end{bmatrix},
\]

\[
C_1(:, 1, 2) = \begin{bmatrix} 2 - i & 0 \\ 0 & 2k \end{bmatrix}, \quad C_1(:, 2, 2) = \begin{bmatrix} 0 & i \\ k & 0 \end{bmatrix}, \quad D_1(:, 1, 1) = \begin{bmatrix} k & 2 - k \\ j & 0 \end{bmatrix}, \quad D_1(:, 1, 2) = \begin{bmatrix} 0 & 0 \\ 0 & 3i \end{bmatrix},
\]
\[ \mathcal{D}_1(\cdot, 2, 1) = \begin{bmatrix} i & j \\ k & 2 \end{bmatrix}, \quad \mathcal{D}_1(\cdot, 2, 2) = \begin{bmatrix} 2 & i \\ 0 & j \end{bmatrix}, \quad \mathcal{F}_1(\cdot, 1, 1) = \begin{bmatrix} 0 & k \\ 0 & 2i \end{bmatrix}, \quad \mathcal{F}_1(\cdot, 1, 2) = \begin{bmatrix} -1 & -1+i \\ 0 & 0 \end{bmatrix}, \]

\[ \mathcal{F}_1(\cdot, 2, 1) = \begin{bmatrix} 0 & 1 \\ 0 & 1-i \end{bmatrix}, \quad \mathcal{F}_1(\cdot, 2, 2) = \begin{bmatrix} 0 & 0 \\ 2 & 2-i \end{bmatrix}, \quad \mathcal{G}_1(\cdot, 1, 1) = \begin{bmatrix} i & 2-i \\ j & 0 \end{bmatrix}, \quad \mathcal{H}_1(\cdot, 1, 2) = \begin{bmatrix} 0 & i \\ 0 & j \end{bmatrix}, \]

\[ \mathcal{G}_1(\cdot, 1, 1) = \begin{bmatrix} 0 & 0 \\ 0 & 1+j+k \end{bmatrix}, \quad \mathcal{G}_1(\cdot, 2, 1) = \begin{bmatrix} j & 2-j \\ k & 0 \end{bmatrix}, \quad \mathcal{H}_1(\cdot, 2, 2) = \begin{bmatrix} 0 & k \\ i & 0 \end{bmatrix}, \quad \mathcal{J}_1(\cdot, 1, 1) = \begin{bmatrix} 0 & j \\ k & 0 \end{bmatrix}, \]

\[ \mathcal{G}_1(\cdot, 2, 2) = \begin{bmatrix} 0 & 0 \\ 0 & i+k \end{bmatrix}, \quad \mathcal{H}_1(\cdot, 1, 1) = \begin{bmatrix} i-k & 0 \\ 0 & -2k \end{bmatrix}, \quad \mathcal{J}_1(\cdot, 1, 2) = \begin{bmatrix} j & 0 \\ 0 & k \end{bmatrix}, \quad \mathcal{J}_1(\cdot, 2, 1) = \begin{bmatrix} i & 0 \\ 0 & j \end{bmatrix}, \]

\[ \mathcal{H}_1(\cdot, 2, 1) = \begin{bmatrix} i & j-k \\ 0 & 0 \end{bmatrix}, \quad \mathcal{E}_1(\cdot, 1, 1) = \begin{bmatrix} 49+19i+5j+23k \\ 1+3i+2j \end{bmatrix}, \quad \mathcal{E}_1(\cdot, 1, 2) = \begin{bmatrix} 3-10i-3j-14k \\ -12+6i+3j-7k \end{bmatrix}, \]

\[ \mathcal{J}_1(\cdot, 2, 2) = \begin{bmatrix} 1-i & 0 \\ 0 & j \end{bmatrix}, \quad \mathcal{E}_1(\cdot, 2, 1) = \begin{bmatrix} -2-4i-39j+82k \\ 3-15i5j-k \end{bmatrix}, \quad \mathcal{E}_1(\cdot, 2, 2) = \begin{bmatrix} 58-18i-45j+34k \\ -4-10i-8j-12k \end{bmatrix}, \quad \mathcal{A}_2(\cdot, 1, 1) = \begin{bmatrix} 2i \\ 0 \end{bmatrix}, \quad \mathcal{A}_2(\cdot, 1, 2) = \begin{bmatrix} 0 & -5k \\ 0 & k \end{bmatrix}, \]

\[ \mathcal{A}_2(\cdot, 2, 1) = \begin{bmatrix} 3 & 0 \\ -i & 0 \end{bmatrix}, \quad \mathcal{A}_2(\cdot, 2, 2) = \begin{bmatrix} 0 & -6 \\ 0 & -i \end{bmatrix}, \quad \mathcal{B}_2(\cdot, 1, 1) = \begin{bmatrix} 4 & j \\ 0 & 0 \end{bmatrix}, \quad \mathcal{B}_2(\cdot, 1, 2) = \begin{bmatrix} -i & 7 \\ 0 & 0 \end{bmatrix}, \]

\[ \mathcal{B}_2(\cdot, 2, 1) = \begin{bmatrix} 5 & 0 \\ -j & 0 \end{bmatrix}, \quad \mathcal{B}_2(\cdot, 2, 2) = \begin{bmatrix} j & -8 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{C}_2(\cdot, 1, 1) = \begin{bmatrix} 6 & k \\ 0 & -k \end{bmatrix}, \quad \mathcal{C}_2(\cdot, 1, 2) = \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix}, \]

\[ \mathcal{C}_2(\cdot, 2, 1) = \begin{bmatrix} 7 & 0 \\ 0 & 3j \end{bmatrix}, \quad \mathcal{C}_2(\cdot, 2, 2) = \begin{bmatrix} 2k & 0 \\ 8 & 0 \end{bmatrix}, \quad \mathcal{D}_2(\cdot, 1, 1) = \begin{bmatrix} 8 & 0 \\ 0 & 3j \end{bmatrix}, \quad \mathcal{D}_2(\cdot, 1, 2) = \begin{bmatrix} 0 & 0 \\ 7 & 0 \end{bmatrix}, \]

\[ \mathcal{D}_2(\cdot, 2, 1) = \begin{bmatrix} 9 & i-k \\ 0 & 0 \end{bmatrix}, \quad \mathcal{D}_2(\cdot, 2, 2) = \begin{bmatrix} i & 0 \\ 6 & -k \end{bmatrix}, \quad \mathcal{F}_2(\cdot, 1, 1) = \begin{bmatrix} 0 & 0 \\ i & i-k \end{bmatrix}, \]

\[ \mathcal{F}_2(\cdot, 2, 1) = \begin{bmatrix} 0 & 2i \\ k & 0 \end{bmatrix}, \quad \mathcal{F}_2(\cdot, 2, 2) = \begin{bmatrix} 0 & k-j \\ 0 & j \end{bmatrix}, \quad \mathcal{G}_2(\cdot, 1, 1) = \begin{bmatrix} 0 & -k \\ 2k & 0 \end{bmatrix}, \]

\[ \mathcal{G}_2(\cdot, 2, 1) = \begin{bmatrix} 1 & 0 \\ 2i & 0 \end{bmatrix}, \quad \mathcal{G}_2(\cdot, 2, 2) = \begin{bmatrix} 0 & 5 \\ 0 & 3j \end{bmatrix}, \quad \mathcal{H}_2(\cdot, 1, 1) = \begin{bmatrix} 2i & 0 \\ 2j & 0 \end{bmatrix}, \quad \mathcal{H}_2(\cdot, 1, 2) = \begin{bmatrix} 0 & i-k \\ 0 & i+k \end{bmatrix}, \]

\[ \mathcal{H}_2(\cdot, 2, 1) = \begin{bmatrix} -i+j & 0 \\ 0 & i+k \end{bmatrix}, \quad \mathcal{H}_2(\cdot, 2, 2) = \begin{bmatrix} 0 & 3 \\ 3-i & j \end{bmatrix}, \quad \mathcal{J}_2(\cdot, 1, 1) = \begin{bmatrix} i & 0 \\ 0 & 2j \end{bmatrix}, \]

\[ \mathcal{J}_2(\cdot, 2, 1) = \begin{bmatrix} 0 & 0 \\ 0 & 2i \end{bmatrix}, \quad \mathcal{E}_2(\cdot, 1, 1) = \begin{bmatrix} -89+649i-668j+5k \\ -2+88i+25j-20k \end{bmatrix}, \]

\[ \mathcal{J}_2(\cdot, 2, 2) = \begin{bmatrix} 3i & 0 \\ -j & 0 \end{bmatrix}, \quad \mathcal{E}_2(\cdot, 1, 2) = \begin{bmatrix} 146+329i-635j-147k \\ 108-95i-860j+77k \end{bmatrix}, \quad \mathcal{E}_2(\cdot, 2, 1) = \begin{bmatrix} -131+82i1-879j+151k \\ 19-682i-160j-3k \end{bmatrix}, \quad \mathcal{E}_2(\cdot, 2, 2) = \begin{bmatrix} 166+368i-705j+192k \\ 402-114i-975j-15k \end{bmatrix}, \]

\[ \mathcal{A}_3(\cdot, 1, 1) = \begin{bmatrix} 0 & i-k \\ 0 & j \end{bmatrix}, \quad \mathcal{A}_3(\cdot, 1, 2) = \begin{bmatrix} i+j & 0 \\ i-2j & 0 \end{bmatrix}, \quad \mathcal{A}_3(\cdot, 2, 1) = \begin{bmatrix} 0 & i-k \\ 0 & j \end{bmatrix}, \quad \mathcal{A}_3(\cdot, 2, 2) = \begin{bmatrix} i+j & 0 \\ i-2j & 0 \end{bmatrix}, \]
We now look at the system \( \mathbf{13} \). Rendering of direct calculations

\[
\mathcal{A}_3(\,::, 2, 1) = \begin{bmatrix} 0 & 0 \\ j + k & 1 \end{bmatrix}, \quad \mathcal{A}_3(\,::, 2, 2) = \begin{bmatrix} j & -k \\ 0 & 0 \end{bmatrix}, \quad \mathcal{B}_3(\,::, 1, 1) = \begin{bmatrix} i & 0 \\ j & 0 \end{bmatrix}, \quad \mathcal{B}_3(\,::, 1, 2) = \begin{bmatrix} 0 & j \\ k & 0 \end{bmatrix}.
\]

\[
\mathcal{B}_3(\,::, 2, 1) = \begin{bmatrix} k & i \\ 0 & 0 \end{bmatrix}, \quad \mathcal{B}_3(\,::, 2, 2) = \begin{bmatrix} i & 0 \\ j & 0 \end{bmatrix}, \quad \mathcal{C}_3(\,::, 1, 1) = \begin{bmatrix} 0 & j \\ i & 0 \end{bmatrix}, \quad \mathcal{C}_3(\,::, 1, 2) = \begin{bmatrix} 0 & k \\ i & 0 \end{bmatrix}.
\]

\[
\mathcal{D}_3(\,::, 2, 1) = \begin{bmatrix} i + j & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{D}_3(\,::, 2, 2) = \begin{bmatrix} 0 & j \\ i & 0 \end{bmatrix}, \quad \mathcal{D}_3(\,::, 1, 1) = \begin{bmatrix} k & i \\ 0 & 0 \end{bmatrix}, \quad \mathcal{F}_3(\,::, 1, 1) = \begin{bmatrix} 2j & 0 \\ 3k & 0 \end{bmatrix}.
\]

\[
\mathcal{F}_3(\,::, 2, 1) = \begin{bmatrix} 0 & 0 \\ i + k & -k \end{bmatrix}, \quad \mathcal{F}_3(\,::, 2, 2) = \begin{bmatrix} k & 0 \\ 2k & 0 \end{bmatrix}, \quad \mathcal{G}_3(\,::, 1, 1) = \begin{bmatrix} 0 & j \\ 2j & 0 \end{bmatrix}, \quad \mathcal{G}_3(\,::, 1, 2) = \begin{bmatrix} 0 & i \\ 3i & 0 \end{bmatrix}.
\]

\[
\mathcal{G}_3(\,::, 2, 1) = \begin{bmatrix} i - j & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{G}_3(\,::, 2, 2) = \begin{bmatrix} j + k & 0 \\ 0 & -k \end{bmatrix}, \quad \mathcal{H}_3(\,::, 1, 1) = \begin{bmatrix} 0 & i + j \\ 0 & k \end{bmatrix}.
\]

\[
\mathcal{H}_3(\,::, 2, 1) = \begin{bmatrix} i + j & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{H}_3(\,::, 2, 2) = \begin{bmatrix} j + k & 0 \\ j & k \end{bmatrix}, \quad \mathcal{J}_3(\,::, 1, 1) = \begin{bmatrix} i + j & 0 \\ j & k \end{bmatrix}.
\]

\[
\mathcal{J}_3(\,::, 2, 1) = \begin{bmatrix} 0 & 0 \\ j - k & -j \end{bmatrix}, \quad \mathcal{J}_3(\,::, 1, 2) = \begin{bmatrix} 4 - 7i - 2j + 10k & 9 + 2i + 2j - 9k \\ 21 - 4i - 5j - 13k & -1 - 5i + 4j - 4k \end{bmatrix}.
\]

\[
\mathcal{J}_3(\,::, 2, 2) = \begin{bmatrix} i + j & i \\ 0 & 0 \end{bmatrix}, \quad \mathcal{E}_3(\,::, 1, 2) = \begin{bmatrix} -31 - 20i - 18j + 12k & -17 - 9i + 4j + 10k \\ -14 - 13i + 11j & -22 - 22i + 4j + 7k \end{bmatrix}, \quad \mathcal{F}_3(\,::, 1, 2) = \begin{bmatrix} 0 & j \\ 0 & j - k \end{bmatrix}.
\]

\[
\mathcal{E}_3(\,::, 2, 1) = \begin{bmatrix} 24 + 15i + 8j - 6k \\ 23 + 2j + 6k \end{bmatrix}, \quad \mathcal{E}_3(\,::, 1, 2) = \begin{bmatrix} 11 + 11i - 7j - k \\ 27i - 7j + 6k \end{bmatrix}, \quad \mathcal{J}_3(\,::, 1, 2) = \begin{bmatrix} k & -2i \\ 0 & 0 \end{bmatrix}.
\]

\[
\mathcal{F}_3(\,::, 2, 1) = \begin{bmatrix} i - j \\ k \end{bmatrix}, \quad \mathcal{F}_3(\,::, 1, 2) = \begin{bmatrix} i - k \\ 0 \end{bmatrix}, \quad \mathcal{H}_3(\,::, 1, 2) = \begin{bmatrix} 0 & j \\ j & k \end{bmatrix}.
\]

\[
\mathcal{J}_3(\,::, 1, 2) = \begin{bmatrix} 0 & i + k \\ i & k \end{bmatrix}, \quad \mathcal{G}_3(\,::, 1, 2) = \begin{bmatrix} -2 & 0 \\ i - k & 0 \end{bmatrix}.
\]
Remark 3.4. If we set $C_i = B_i = \mathcal{I}$ in (1.3) where $i = \overline{1,3}$, we obtain the Sylvester-like quaternion system of tensor equations (1.6).

Remark 3.5. If we set $A_i = D_i = 0$ in (1.6) where $i = \overline{1,3}$, we derive the Sylvester-like quaternion system of tensor equations (1.4).

Remark 3.6. If we set $G_i = F_i^T = 0$, $J_i = H_i^T = 0$ and $E_i = F_i^T = 0$ in (1.4) where $i = \overline{1,3}$, we investigate $\eta$-Hermitian solution for (1.4).

In the following Section, we establish the consistency conditions and the general solution to (1.4). In a direct implementation, we investigate some necessary and sufficient conditions for the existence of a common $\eta$-Hermitian solution of (1.5).
4. Some implementations of the central system \((1.4)\)

**Theorem 4.1.** Consider the quaternion system of tensor equations \((1.4)\), where

\[
\mathcal{F}_4 \in \mathbb{H}^{(N)\times J(N)}, \quad \mathcal{G}_4 \in \mathbb{H}^{L(M)\times K(M)}, \quad \mathcal{H}_4 \in \mathbb{H}^{(N)\times Q(N)}, \quad \mathcal{J}_4 \in \mathbb{H}^{S(M)\times K(M)}, \\
\mathcal{E}_4 \in \mathbb{H}^{(N)\times K(M)}, \quad \mathcal{E}_5 \in \mathbb{H}^{(N)\times J(M)}, \quad \mathcal{F}_5 \in \mathbb{H}^{A(N)\times J(N)}, \quad \mathcal{G}_5 \in \mathbb{H}^{L(M)\times F(M)}, \\
\mathcal{H}_5 \in \mathbb{H}^{A(N)\times P(N)}, \quad \mathcal{J}_5 \in \mathbb{H}^{S(M)\times F(M)}, \quad \mathcal{E}_5 \in \mathbb{H}^{A(N)\times F(M)} \quad \text{(i = 1, 3)}
\]

are given tensors over \(\mathbb{H}\). Set

\[
\hat{\mathcal{M}}_i = R_{\mathcal{F}_i} * N \mathcal{N}_i, \quad \hat{\mathcal{N}}_i = \mathcal{J}_i * M \mathcal{L}_i, \quad \hat{\mathcal{S}}_i = \mathcal{H}_i * N \mathcal{L}_{\hat{\mathcal{M}}_i}, \quad \text{(i = 1, 3)},
\]

\[
A_{i1} = \begin{bmatrix} \mathcal{L}_{F_4} & -\mathcal{L}_{F_4} \end{bmatrix}, \quad D_{i1} = \begin{bmatrix} R_{G_4} & -R_{G_4} \end{bmatrix}, \quad \hat{A}_{i1} = \mathcal{F}_4^\dagger * N \hat{S}_1, \quad \hat{B}_{i1} = R_{\hat{\mathcal{N}}_1} * M \mathcal{J}_1 * M \mathcal{G}_1^\dagger, \quad \text{(4.1a)}
\]

\[
\mathcal{E}_{i1} = \mathcal{F}_4^\dagger * N \mathcal{E}_1 * M \mathcal{G}_1^\dagger - \mathcal{F}_1^\dagger * N \mathcal{H}_1 * M \hat{\mathcal{M}}_1^\dagger * N \mathcal{E}_1 * M \mathcal{G}_1^\dagger - \mathcal{F}_4^\dagger * N \hat{S}_1 * N \mathcal{H}_1^\dagger * N \mathcal{E}_1 * M \mathcal{G}_1^\dagger, \quad \text{(4.1c)}
\]

\[
A_{i2} = \begin{bmatrix} \mathcal{L}_{H_4} & -\mathcal{L}_{\hat{\mathcal{M}}_3} * N \mathcal{L}_{S_3} \end{bmatrix}, \quad D_{i2} = \begin{bmatrix} R_{J_4} & -R_{J_4} \end{bmatrix}, \quad \hat{A}_{i2} = \mathcal{L}_{\hat{\mathcal{M}}_3}, \quad \hat{B}_{i2} = R_{\hat{\mathcal{N}}_3}, \quad \text{(4.1d)}
\]

\[
\mathcal{E}_{i2} = \hat{\mathcal{M}}_3 * N \mathcal{E}_3 * M \mathcal{J}_3^\dagger + \hat{S}_3 * N \hat{S}_3 * N \mathcal{H}_3^\dagger * N \mathcal{E}_3 * M \hat{\mathcal{M}}_3^\dagger - \mathcal{H}_3^\dagger * N \mathcal{E}_5 * M \mathcal{J}_3^\dagger, \quad \text{(4.1e)}
\]

\[
A_1 = \begin{bmatrix} -R_{J_4} & R_{J_4} \end{bmatrix}, \quad B_2 = \begin{bmatrix} -R_{G_4} \end{bmatrix}, \quad \mathcal{F}_2 = \mathcal{F}_4^\dagger * N \hat{S}_3, \quad \mathcal{G}_1 = \mathcal{J}_2 * N \mathcal{G}_2^\dagger, \quad \mathcal{G}_2 = \mathcal{J}_3 * N \mathcal{G}_3^\dagger, \quad \text{(4.1h)}
\]

\[
\mathcal{H}_1 = \mathcal{L}_{\hat{\mathcal{M}}_1}, \quad \mathcal{J}_1 = R_{\hat{\mathcal{N}}_1}, \quad \mathcal{H}_2 = \mathcal{L}_{\hat{\mathcal{M}}_2}, \quad \mathcal{J}_2 = R_{\hat{\mathcal{N}}_2}, \quad \text{(4.1i)}
\]

\[
\mathcal{E}_1 = -\hat{\mathcal{M}}_1^\dagger * N \mathcal{E}_1 * M \mathcal{J}_1^\dagger - \hat{S}_1^\dagger * N \hat{S}_1 * N \mathcal{H}_1^\dagger * N \mathcal{E}_1 * M \hat{\mathcal{M}}_1^\dagger + \mathcal{F}_2^\dagger * N \mathcal{E}_2 * M \mathcal{G}_2^\dagger - \mathcal{F}_1^\dagger * N \mathcal{H}_2 * M \hat{\mathcal{M}}_2^\dagger * N \mathcal{E}_2 * M \mathcal{G}_2^\dagger - \mathcal{F}_3^\dagger * N \hat{S}_2 * N \mathcal{H}_2^\dagger * N \mathcal{E}_2 * M \hat{\mathcal{M}}_3^\dagger * M \mathcal{J}_3 * M \mathcal{G}_3^\dagger, \quad \text{(4.1j)}
\]

\[
\mathcal{F}_{i1} = R_{\mathcal{N}_i} * N \mathcal{J}_i, \quad \mathcal{G}_{i1} = \mathcal{G}_i * M \mathcal{L}_{B_i}, \quad \mathcal{H}_{i1} = R_{\mathcal{N}_i} * N \mathcal{H}_i, \quad \mathcal{J}_{i1} = \mathcal{J}_i * M \mathcal{L}_{B_i}, \quad \text{(4.1l)}
\]

\[
\mathcal{F}_{i2} = R_{\mathcal{N}_i} * N \mathcal{E}_i * M \mathcal{L}_{B_i}, \quad \mathcal{G}_{i2} = R_{\mathcal{N}_i} * N \mathcal{G}_i, \quad \mathcal{H}_{i2} = \mathcal{H}_i * N \mathcal{L}_{\mathcal{M}_{i1}}, \quad \text{(4.1m)}
\]

\[
\mathcal{F}_{i1} = R_{\mathcal{N}_i} * N \mathcal{F}_{i1}, \quad \mathcal{G}_{i1} = \mathcal{G}_i * M \mathcal{L}_{B_i}, \quad \mathcal{H}_{i1} = R_{\mathcal{N}_i} * N \mathcal{H}_i, \quad \mathcal{J}_{i1} = \mathcal{J}_i * M \mathcal{L}_{B_i}, \quad \text{(4.1q)}
\]
Then the system (4.13) is consistent if and only if

\( R_{\tilde{\chi}_i} * N R_{\tilde{F}_i} * N \tilde{E}_i = 0, \quad \tilde{E}_i * M L_{G_i} * M \tilde{\chi}_i = 0, \quad (4.3) \)

\( R_{\tilde{F}_i} * N \tilde{E}_i * M L_{J_i} = 0, \quad R_{H_i} * N \tilde{E}_i * M L_{G_i} = 0, \quad (i = 1, 3), \quad (4.4) \)

\( R_{\tilde{F}_4} * N \tilde{E}_4 = 0, \quad \tilde{E}_4 * M L_{G_4} = 0, \quad R_{H_4} * N \tilde{E}_5 = 0, \quad \tilde{E}_5 * M L_{J_4} = 0, \quad (4.5) \)

\( R_{\tilde{\chi}_k} * N \tilde{E}_{kk} = 0, \quad \tilde{E}_{kk} * M L_{\tilde{B}_{kk}} = 0, \quad (4.6) \)

\( R_{\tilde{\chi}_{kk}} * N R_{\tilde{F}_{kk}} * N \tilde{E}_{kk} = 0, \quad \tilde{E}_{kk} * M L_{G_{kk}} * M \tilde{\chi}_{kk} = 0, \quad (4.7) \)

\( R_{\tilde{F}_{kk}} * N \tilde{E}_{kk} * M L_{J_{kk}} = 0, \quad R_{H_{kk}} * N \tilde{E}_{kk} * M L_{G_{kk}} = 0, \quad (k = 1, 2), \quad (4.8) \)

\( R_{\tilde{\chi}_{kk}} * N R_{\tilde{F}_{kk}} * N \tilde{E}_{kk} = 0, \quad \tilde{E}_{kk} * M L_{G_{kk}} * M \tilde{\chi}_{kk} = 0, \quad (4.9) \)

\( R_{\tilde{F}_{kk}} * N \tilde{E}_{kk} * M L_{J_{kk}} = 0, \quad R_{H_{kk}} * N \tilde{E}_{kk} * M L_{G_{kk}} = 0, \quad (4.10) \)

\( R_{\tilde{C}_{jj}} * N \tilde{E}_{jj} = 0, \quad \tilde{E}_{jj} * M L_{B_{jj}} = 0, \quad (j = 1, 2), \quad (4.11) \)

\( R_{\tilde{\chi}_{ll}} * N \tilde{E}_{ll} = 0, \quad \tilde{E}_{ll} * M L_{B_{ll}} = 0, \quad \tilde{E}_{ll} * M L_{\tilde{B}}, \quad (l = 1, 2), \quad (4.12) \)

\( R_{\tilde{\chi}_{kk}} * N \tilde{E}_{kk} * M L_{\tilde{B}} = 0. \quad (4.13) \)
Under these conditions, the general solution to system (1.3) can be expressed as follows:

\[ Z_1 = F_4^\dagger * N E_4 * M G_1^\dagger + L_{F_4} * N W_1 + W_2 * M R_{G_4}, \]

\[ Z_4 = H_4^* * N E_5 * M J_4^\dagger + \hat{L}_{H_4} * N \hat{W}_1 + \hat{W}_3 * M \hat{R}_{J_4}, \]

\[ Z_2 = \tilde{M}_1 * M \tilde{S}_1 + \tilde{S}_1 * N \hat{S}_1 * N H_1 * N \tilde{E}_1 * M \tilde{N}_1^\dagger + L_{\tilde{X}_1} * N L_{\tilde{S}_1} * N \hat{U}_1 + L_{\tilde{X}_1} * N \hat{U}_2 * M R_{\tilde{X}_1} + \hat{U}_3 * M R_{J_1}, \]

\[ Z_2 = F_2 * N E_2 * M G_2^\dagger - F_2 * N H_2 * N \tilde{M}_2 * N \tilde{E}_2 * M G_2^\dagger - F_2 * N \tilde{S}_2 * N H_2 * N \tilde{E}_2 * M G_2^\dagger + L_{\tilde{F}_2} * N \hat{V}_2 * M R_{\tilde{X}_2} * M J_2 * M G_2^\dagger + L_{\tilde{F}_2} * N \hat{V}_4 \]

\[ + \hat{V}_5 * M R_{F_2}, \]

\[ Z_3 = \tilde{M}_3 * N E_3 * M J_3^\dagger + \tilde{S}_3 * N \hat{S}_3 * N H_3 * N \tilde{E}_3 * M \tilde{N}_3^\dagger + L_{\tilde{X}_3} * N L_{\tilde{S}_3} * N \hat{V}_1 + L_{\tilde{X}_3} * N \hat{V}_2 * M R_{\tilde{X}_3} + \hat{V}_3 * M R_{J_3}, \]

\[ Z_3 = F_3 * N E_3 * M G_3^\dagger - F_3 * N H_3 * N \tilde{M}_3 * N \tilde{E}_3 * M G_3^\dagger - F_3 * N \tilde{S}_3 * N H_3 * N \tilde{E}_3 * M G_3^\dagger + L_{\tilde{F}_3} * N \hat{V}_3 * M R_{\tilde{X}_3} * M J_3 * M G_3^\dagger + L_{\tilde{F}_3} * N \hat{V}_4 \]

\[ + \hat{V}_5 * M R_{F_3}, \]

Where the arbitrary tensors \( W_j, \hat{V}_i, \hat{U}_j, \hat{K}_k \) and \( W_1 \) \((j = 1, 3, i = 1, 5, k \in \{2, 4, 5\}) \) can be reduced by (3.22a)-(3.26d).

Proof. See Remark (3.1), Remark (3.5).

Corollary 4.2. Consider the quaternion system of tensor equations (1.4), where

\[ F_i \in \mathbb{H}^{(N) \times J(N)}, \quad H_i \in \mathbb{H}^{(N) \times Q(N)}, \quad E_i \in \mathbb{H}^{(N) \times I(N)}, \quad E_i \in \mathbb{H}^{(A(N) \times A(N)} \quad \text{(i = 1, 3)} \]

are given tensors over \( \mathbb{H} \). Set

\[ \tilde{M}_i = R_{F_i} * N H_i, \quad \tilde{N}_i = (\tilde{M}_i)^\dagger, \quad \tilde{S}_i = H_i * N L_{\tilde{X}_i}, \quad (i = 1, 3) \quad A_{11} = [L_{F_4} - L_{F_1}], \]

\[ D_{11} = \begin{bmatrix} R_{F_1} & -R_{F_1} \\ \dagger & \dagger \end{bmatrix}, \quad \tilde{A}_{11} = F_1 * N \tilde{S}_1, \quad \tilde{B}_{11} = R_{\tilde{X}_1} * N H_1^\dagger * N (F_1^\dagger), \]

\[ E_{11} = F_1 * N E_1 * N (F_1^\dagger) - F_1 * N H_1 * N \tilde{M}_1 * N \tilde{E}_1 * N (F_1^\dagger) - F_1 * N \tilde{S}_1 * N H_1^\dagger * N \tilde{E}_1 * N (F_1^\dagger), \]

\[ \tilde{A}_{22} = \begin{bmatrix} L_{H_4} & -L_{\tilde{X}_3} * N L_{\tilde{S}_3} \end{bmatrix}, \quad \tilde{D}_{22} = \begin{bmatrix} R_{H_4}^\dagger & -R_{H_4}^\dagger \end{bmatrix}, \quad \tilde{A}_{22} = L_{\tilde{M}_3}, \quad \tilde{B}_{22} = R_{\tilde{N}_3}, \]

\[ \tilde{E}_{22} = \tilde{M}_3 * N E_3 * N (H_3^\dagger)^\dagger + \tilde{S}_3 * N \hat{S}_3 * N H_3^\dagger * N E_3 * N \tilde{N}_3^\dagger - H_4 * N E_5 * N (J_4^\dagger)^\dagger, \]

\[ \tilde{A}_{ii} = R_{A_{ii}} * N \tilde{A}_{ii}, \quad \tilde{B}_{ii} = \tilde{B}_{ii} * N L_{D_{ii}}, \quad \tilde{E}_{ii} = R_{A_{ii}} * N E_{ii} * N L_{D_{ii}}, \quad (i = 1, 2), \]

\[ \tilde{A}_1 = \begin{bmatrix} -L_{\tilde{X}_1} * N L_{\tilde{S}_1} & L_{F_3} \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} -L_{\tilde{X}_2} * N L_{\tilde{S}_2} & L_{F_3} \end{bmatrix}, \quad \tilde{F}_1 = F_3 * N \tilde{S}_3, \quad \tilde{F}_2 = F_3 * N \tilde{S}_3, \quad \tilde{F}_3 = R_{\tilde{N}_2}, \]

\[ \tilde{G}_1 = \begin{bmatrix} -R_{H_1}^\dagger \\ R_{F_1}^\dagger \end{bmatrix}, \quad \tilde{G}_2 = \begin{bmatrix} -R_{H_2}^\dagger \\ R_{F_2}^\dagger \end{bmatrix}, \quad \tilde{G}_3 = \begin{bmatrix} -R_{H_3}^\dagger \\ R_{F_3}^\dagger \end{bmatrix}, \quad \tilde{F}_2 = F_3 * N \tilde{S}_3, \quad \tilde{G}_1 = F_3 * N (G_3^\dagger), \]

\[ \tilde{G}_2 = J_3 * N (G_3^\dagger), \quad \tilde{G}_1 = L_{\tilde{M}_1}, \quad \tilde{F}_1 = R_{\tilde{X}_1}, \quad \tilde{G}_2 = L_{\tilde{M}_2}, \quad \tilde{F}_2 = R_{\tilde{X}_2}, \]

\[ \tilde{G}_3 = J_3 * N (G_3^\dagger), \quad \tilde{G}_1 = L_{\tilde{M}_1}, \quad \tilde{F}_1 = R_{\tilde{X}_1}, \quad \tilde{G}_2 = L_{\tilde{M}_2}, \quad \tilde{F}_2 = R_{\tilde{X}_2}, \]

\[ \tilde{G}_3 = J_3 * N (G_3^\dagger), \]

\[ \tilde{G}_1 = L_{\tilde{M}_1}, \quad \tilde{F}_1 = R_{\tilde{X}_1}, \quad \tilde{G}_2 = L_{\tilde{M}_2}, \quad \tilde{F}_2 = R_{\tilde{X}_2}, \]

\[ \tilde{G}_3 = J_3 * N (G_3^\dagger), \]

\[ \tilde{G}_1 = L_{\tilde{M}_1}, \quad \tilde{F}_1 = R_{\tilde{X}_1}, \quad \tilde{G}_2 = L_{\tilde{M}_2}, \quad \tilde{F}_2 = R_{\tilde{X}_2}, \]

\[ \tilde{G}_3 = J_3 * N (G_3^\dagger), \]

\[ \tilde{G}_1 = L_{\tilde{M}_1}, \quad \tilde{F}_1 = R_{\tilde{X}_1}, \quad \tilde{G}_2 = L_{\tilde{M}_2}, \quad \tilde{F}_2 = R_{\tilde{X}_2}, \]

\[ \tilde{G}_3 = J_3 * N (G_3^\dagger), \]
\[ \mathcal{F}_1 = -\mathcal{M}_{11}^* \mathcal{E}_1 * M (\mathcal{H}_1^{*})^\dagger - \mathcal{S}_{11}^* \mathcal{S}_1 * N \mathcal{H}_1^i * N \mathcal{E}_1 * N \hat{\mathcal{N}}_1^i + \mathcal{F}_{11}^* \mathcal{E}_2 * N (\mathcal{F}_{11}^i)^\dagger - \mathcal{F}_{11}^i \mathcal{S}_2 * N \hat{\mathcal{H}}_2^i * N \mathcal{E}_2 
\]
\[ \mathcal{E}_2 = -\mathcal{M}_{11}^* \mathcal{E}_2 * N (\mathcal{H}_1^{*})^\dagger - \mathcal{S}_{11}^* \mathcal{S}_1 * N \mathcal{H}_1^i * N \mathcal{E}_2 * N \hat{\mathcal{N}}_1^i + \mathcal{F}_{11}^* \mathcal{E}_2 * N (\mathcal{F}_{11}^i)^\dagger - \mathcal{F}_{11}^i \mathcal{S}_2 * N \hat{\mathcal{H}}_2^i * N \mathcal{E}_2 
\]
\[ \mathcal{F}_{11} = \mathcal{R}_{\mathcal{A}_1}^* \mathcal{E}_1 * N \mathcal{L}_{11}^i, \quad \mathcal{G}_{11} = \mathcal{R}_{\mathcal{A}_1}^* \mathcal{E}_1 * N \mathcal{L}_{11}^i, \quad \mathcal{J}_{11} = \mathcal{R}_{\mathcal{A}_1}^* \mathcal{E}_1 * N \mathcal{L}_{11}^i, \quad \mathcal{S}_{11} = \mathcal{R}_{\mathcal{A}_1}^* \mathcal{E}_1 * N \mathcal{L}_{11}^i, \quad \mathcal{A}_1 = [\mathcal{L}_{11}^i - \mathcal{L}_{11}^i, \mathcal{B}_1 = \left[ \begin{array}{c} \mathcal{R}_{\mathcal{S}_{11}}^{11} \\ \mathcal{R}_{\mathcal{A}_1}^{11} \end{array} \right], \quad \mathcal{F}_{11} = \mathcal{R}_{\mathcal{A}_1}^{11} * N \mathcal{S}_{11}, \quad \mathcal{G}_{11} = \mathcal{R}_{\mathcal{A}_1}^{11} * N \mathcal{S}_{11}, \quad \mathcal{J}_{11} = \mathcal{R}_{\mathcal{A}_1}^{11} * N \mathcal{S}_{11}, \quad \mathcal{S}_{11} = \mathcal{R}_{\mathcal{A}_1}^{11} * N \mathcal{S}_{11}. \]
\[ \mathcal{E}_{11} = \left[ \begin{array}{c} \mathcal{L}_{11}^i * N \mathcal{L}_{11}^i, \mathcal{B}_{11} = \left[ \begin{array}{c} \mathcal{R}_{\mathcal{S}_{11}}^{11} \\ \mathcal{R}_{\mathcal{A}_1}^{11} \end{array} \right], \quad \mathcal{F}_{11} = \mathcal{R}_{\mathcal{A}_1}^{11} * N \mathcal{S}_{11}, \quad \mathcal{G}_{11} = \mathcal{R}_{\mathcal{A}_1}^{11} * N \mathcal{S}_{11}, \quad \mathcal{J}_{11} = \mathcal{R}_{\mathcal{A}_1}^{11} * N \mathcal{S}_{11}, \quad \mathcal{S}_{11} = \mathcal{R}_{\mathcal{A}_1}^{11} * N \mathcal{S}_{11}. \]
\[ \mathcal{F}_1 = \mathcal{R}_{\mathcal{A}_1}^{11} * N \mathcal{F}_{11}, \quad \mathcal{G}_1 = \mathcal{R}_{\mathcal{A}_1}^{11} * N \mathcal{G}_{11}, \quad \mathcal{J}_1 = \mathcal{R}_{\mathcal{A}_1}^{11} * N \mathcal{J}_{11}, \quad \mathcal{S}_1 = \mathcal{R}_{\mathcal{A}_1}^{11} * N \mathcal{S}_{11}, \quad \mathcal{A}_1 = \left[ \begin{array}{c} \mathcal{L}_{11}^i - \mathcal{L}_{11}^i, \mathcal{B}_1 = \left[ \begin{array}{c} \mathcal{R}_{\mathcal{S}_{11}}^{11} \\ \mathcal{R}_{\mathcal{A}_1}^{11} \end{array} \right], \quad \mathcal{F}_1 = \mathcal{R}_{\mathcal{A}_1}^{11} * N \mathcal{S}_{11}, \quad \mathcal{G}_1 = \mathcal{R}_{\mathcal{A}_1}^{11} * N \mathcal{S}_{11}, \quad \mathcal{J}_1 = \mathcal{R}_{\mathcal{A}_1}^{11} * N \mathcal{S}_{11}, \quad \mathcal{S}_1 = \mathcal{R}_{\mathcal{A}_1}^{11} * N \mathcal{S}_{11}. \]
\[ \mathcal{E}_1 = \left[ \begin{array}{c} \mathcal{L}_{11}^i * N \mathcal{L}_{11}^i, \mathcal{B}_1 = \left[ \begin{array}{c} \mathcal{R}_{\mathcal{S}_{11}}^{11} \\ \mathcal{R}_{\mathcal{A}_1}^{11} \end{array} \right], \quad \mathcal{F}_1 = \mathcal{R}_{\mathcal{A}_1}^{11} * N \mathcal{S}_{11}, \quad \mathcal{G}_1 = \mathcal{R}_{\mathcal{A}_1}^{11} * N \mathcal{S}_{11}, \quad \mathcal{J}_1 = \mathcal{R}_{\mathcal{A}_1}^{11} * N \mathcal{S}_{11}, \quad \mathcal{S}_1 = \mathcal{R}_{\mathcal{A}_1}^{11} * N \mathcal{S}_{11}. \]
Then the system (3.3) is consistent if and only if
\[
\begin{align*}
R_{\tilde{\mathcal{M}}_i} \ast N R_{\mathcal{F}_i} \ast N E_i &= 0, \quad R_{\mathcal{F}_i} \ast N E_i \ast M \mathcal{L}_{\mathcal{H}_i^{\ast}} = 0, \quad (i = \overline{1,3}), \quad (4.22) \\
R_{\mathcal{F}_i} \ast N E_4 &= 0, \quad R_{\mathcal{H}_4} \ast N E_5 = 0, \quad R_{\tilde{\mathcal{A}}_{kk}} \ast N \tilde{E}_{kk} = 0, \quad \tilde{E}_{kk} \ast M \mathcal{L}_{\tilde{\mathcal{B}}_{kk}} = 0, \quad (4.23) \\
R_{\tilde{\mathcal{A}}_{kk}} \ast N R_{\mathcal{F}_{kk}} \ast N \tilde{E}_{kk} &= 0, \quad \tilde{E}_{kk} \ast M \mathcal{L}_{\tilde{\mathcal{V}}_{kk}} = 0, \quad (k = 1, 2), \quad (4.24) \\
R_{\tilde{\mathcal{A}}_{11}} \ast N R_{\mathcal{F}_{11}} \ast N \tilde{E}_{11} &= 0, \quad \tilde{E}_{11} \ast M \mathcal{L}_{\tilde{\mathcal{V}}_{11}} = 0, \quad (4.25) \\
R_{\tilde{\mathcal{A}}_{11}} \ast N \tilde{E}_{11} \ast M \mathcal{L}_{\tilde{\mathcal{V}}_{11}} &= 0, \quad (4.26) \\
R_{\tilde{\mathcal{C}}_{jj}} \ast N \tilde{E}_{jj} &= 0, \quad \tilde{E}_{jj} \ast M \mathcal{L}_{\tilde{\mathcal{D}}_{jj}} = 0, \quad (j = 1, 2), \quad R_{\tilde{\mathcal{H}}_{ll}} \ast N \tilde{E}_{ll} = 0, \quad (4.28) \\
\tilde{E}_{ll} \ast M \mathcal{L}_{\tilde{\mathcal{V}}_{ll}} &= 0, \quad R_{\tilde{\mathcal{A}} \ast N \tilde{E} \ast M \mathcal{L}_{\tilde{\mathcal{B}}}}, \quad (l = 1, 2), \quad R_{\tilde{\mathcal{A}} \ast N \tilde{E} \ast M \mathcal{L}_{\tilde{\mathcal{B}}} = 0. \quad (4.29)
\end{align*}
\]
Under these conditions, the general solution to system (3.3) can be expressed as follows:
\[
Z_k = \frac{\tilde{Z}_k + \tilde{Z}_k^{0\ast}}{2}, \quad (k = \overline{1,4}), \quad (4.30)
\]
where
\[
\begin{align*}
\tilde{Z}_1 &= \mathcal{F}_d \ast N E_4 \ast M (\mathcal{F}_d^{0\ast})^\dagger + \mathcal{L}_{\mathcal{F}_d} \ast N \mathcal{W}_1 + \mathcal{W}_2 \ast M R_{\mathcal{F}_d^{0\ast}}, \quad (4.31) \\
\tilde{Z}_2 &= \tilde{\mathcal{A}}_1 \ast N \tilde{E}_1 \ast M (\tilde{\mathcal{H}}_1^{0\ast})^\dagger + \tilde{\mathcal{S}}_1 \ast N \tilde{S}_1 \ast N \tilde{H}_1 \ast N \tilde{E}_1 \ast N \tilde{N}_1^\dagger + \tilde{\mathcal{L}}_{\tilde{\mathcal{M}}_1} \ast N \tilde{S}_1 \ast N \tilde{U}_1 \\
&\quad + \tilde{\mathcal{L}}_{\tilde{\mathcal{M}}_1} \ast N \tilde{U}_2 \ast M R_{\tilde{\mathcal{N}}_1} + \tilde{U}_3 \ast M R_{\tilde{\mathcal{H}}_1^{0\ast}}, \quad (4.33) \\
or \tilde{Z}_2 &= \mathcal{F}_4 \ast N E_2 \ast M (\mathcal{F}_2^{0\ast})^\dagger - \mathcal{F}_2 \ast N \mathcal{H}_2 \ast N \tilde{\mathcal{M}}_2 \ast N E_2 \ast N (\mathcal{F}_2^{0\ast})^\dagger - \mathcal{F}_4 \ast N \tilde{E}_2 \ast N \tilde{H}_2 \ast N \tilde{E}_2 \ast N \tilde{N}_2^\dagger + \mathcal{L}_{\mathcal{F}_2} \ast N \tilde{V}_1 + \tilde{V}_5 \ast M R_{\mathcal{F}_2}, \quad (4.34) \\
\tilde{Z}_3 &= \tilde{\mathcal{M}}_2 \ast N \tilde{E}_2 \ast N (\tilde{\mathcal{H}}_2^{0\ast})^\dagger + \tilde{\mathcal{S}}_2 \ast N \tilde{S}_2 \ast N \tilde{H}_2 \ast N \tilde{E}_2 \ast N \tilde{N}_2^\dagger + \tilde{\mathcal{L}}_{\tilde{\mathcal{M}}_2} \ast N \tilde{S}_2 \ast N \tilde{V}_1 \ast M R_{\tilde{\mathcal{N}}_2} + \tilde{V}_3 \ast M R_{\tilde{\mathcal{H}}_2^{0\ast}}, \quad (4.35) \\
or \tilde{Z}_3 &= \mathcal{F}_3 \ast N E_3 \ast M (\mathcal{F}_3^{0\ast})^\dagger - \mathcal{F}_3 \ast N \mathcal{H}_3 \ast N \tilde{\mathcal{M}}_3 \ast N E_3 \ast N (\mathcal{F}_3^{0\ast})^\dagger - \mathcal{F}_3 \ast N \tilde{E}_3 \ast N \tilde{H}_3 \ast N \tilde{E}_3 \ast N \tilde{N}_3^\dagger \quad (4.36) \\
&\quad + \mathcal{L}_{\mathcal{F}_3} \ast N \tilde{V}_3 + \tilde{V}_5 \ast M R_{\mathcal{F}_3^\ast}, \quad (i = \overline{1,3}).
\end{align*}
\]
Where the arbitrary tensors \(\mathcal{W}_j, \tilde{\mathcal{V}}_1, \tilde{U}_i, \tilde{K}_k \) and \(\tilde{W}_1 \) (\(j = 1, 3, \ i = 1, 5, \ k \in \{2, 4, 5\}\)) can be reduced by \(4.23\) under the definitions \(4.26\) under the definitions \(4.26\).

**Proof.** Consider the following quaternion system of tensor equations:
\[
\begin{cases}
\mathcal{F}_4 \ast N \tilde{Z}_1 \ast N \mathcal{F}_4^{0\ast} = \mathcal{E}_4, \\
\mathcal{F}_4 \ast N \tilde{Z}_4 \ast N \mathcal{F}_4^{0\ast} + \mathcal{H}_4 \ast N \tilde{Z}_4 \ast N \mathcal{H}_4^{0\ast} = \mathcal{E}_3, \\
\mathcal{H}_4 \ast N \tilde{Z}_4 \ast N \mathcal{H}_4^{0\ast} = \mathcal{E}_5,
\end{cases}
\]
where \((i = \overline{1,3})\). Suppose that the system (1.5) is consistent. Claim that \((\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3, \tilde{Z}_4)\) is a solution to the quaternion system of tensor equations (1.5), then it is evident that \((\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3, \tilde{Z}_4)\)
\( (Z_1, Z_2, Z_3, Z_4) \) is a solution to the system \((4.37)\). Conversely, if the system \((4.37)\) has a solution \((\dot{Z}_1, \dot{Z}_2, \dot{Z}_3, \dot{Z}_4)\). It is sufficient to show that

\[
(Z_1, Z_2, Z_3, Z_4) = \left( \frac{\dot{Z}_1 + \dot{Z}_1^*}{2}, \frac{\dot{Z}_2 + \dot{Z}_2^*}{2}, \frac{\dot{Z}_3 + \dot{Z}_3^*}{2}, \frac{\dot{Z}_4 + \dot{Z}_4^*}{2} \right),
\]

is a solution to system \((1.5)\). Clearly, the quaternion tensors \(Z_i, (i = 1, 4)\) are \(\eta\)-Hermitian tensors. By Applying \((4.38)\) on the system \((1.5)\) yields:

\[
\mathcal{F}_4 \ast_N Z_1 \ast_N \mathcal{F}_4 \ast_N = \mathcal{F}_4 \ast_N \left( \frac{\dot{Z}_1 + \dot{Z}_1^*}{2} \right) \ast_N \mathcal{F}_4 \ast_N
\]

\[
= \frac{1}{2} \mathcal{F}_4 \ast_N \dot{Z}_1 \ast_N \mathcal{F}_4 \ast_N + \frac{1}{2} \left( \mathcal{F}_4 \ast_N \dot{Z}_1 \ast_N \mathcal{F}_4 \ast_N \right)^\eta = \mathcal{E}_4.
\]

Similarly, it can be shown that

\[
\mathcal{H}_4 \ast_N Z_4 \ast_N \mathcal{H}_4 \ast_N = \mathcal{E}_5.
\]

Moreover,

\[
\mathcal{F}_i \ast_N Z_i \ast_N \mathcal{F}_i \ast_N + \mathcal{H}_i \ast_N Z_{i+1} \ast_N \mathcal{H}_i \ast_N
\]

\[
= \mathcal{F}_i \ast_N \left( \frac{\dot{Z}_i + \dot{Z}_i^*}{2} \right) \ast_N \mathcal{F}_i \ast_N + \mathcal{H}_i \ast_N \left( \frac{\dot{Z}_{i+1} + \dot{Z}_{i+1}^*}{2} \right) \ast_N \mathcal{H}_i \ast_N
\]

\[
= \frac{1}{2} \left[ \mathcal{F}_i \ast_N \dot{Z}_i \ast_N \mathcal{F}_i \ast_N + \mathcal{H}_i \ast_N \dot{Z}_{i+1} \ast_N \mathcal{H}_i \ast_N \right] + \frac{1}{2} \left[ \mathcal{F}_i \ast_N \dot{Z}_i \ast_N \mathcal{F}_i \ast_N + \mathcal{H}_i \ast_N \dot{Z}_{i+1} \ast_N \mathcal{H}_i \ast_N \right]^\eta = \mathcal{E}_i, (i = 1, 3).
\]

Therefore, \((4.38)\) is a solution to the system \((1.5)\). Consequently, apply Theorem \(4.1\) on the system \((4.37)\), we can establish the solvability conditions and the general solution to the quaternion system \((1.5)\).

\[
\Box
\]

5. Conclusion

Having first established the necessary and sufficient conditions for the presence of a solution to \((1.3)\), we, therefore, manifest an expression of its general solution. If \(A_i = D_i = 0\) in \((1.6)\), where \((i = 1, 3)\), we obtain the Sylvester-like quaternion system of tensor equations \((1.4)\). As an application of system \((1.4)\), we investigate an \(\eta\)-Hermitian solution to system \((1.5)\). We also construct a numerical example to validate the system \((1.3)\). It is notable that the primary conclusions of this study are particularly beneficial for the corresponding systems over the real and complex number fields. These conclusions can also obtain the corresponding matrix equation systems to \((1.3)-(1.5)\).

All results are valid over an arbitrary division ring. As a direct consequence, the corresponding systems of quaternion matrix equations to the systems \((1.3), (1.4), (1.5), \) and \((1.6)\) can be described by rank equalities and Moore-Penrose inverses of matrices whenever \(N = M = 1\).
In further future work, we infer the solvability constraints to the \( n \)-system of the matrix equations

\[
\begin{align*}
A_1X_1B_1 + C_1Y_1D_1 + C_1(G_1Z_1F_1 + H_1Z_2J_1)B_1 &= E_1 \\
A_2X_2B_2 + C_2Y_2D_2 + C_2(G_2Z_2F_2 + H_2Z_3J_2)B_2 &= E_2 \\
&\vdots \\
A_nX_nB_n + C_nY_nD_n + C_n(G_nZ_nF_n + H_nZ_{n+1}J_n)B_n &= E_n
\end{align*}
\]

can be characterized by rank equalities and Moore-Penrose inverses of some known matrices and hence, we can derive a formula of its general solution. Moreover, we intend to study that system over an arbitrary regular ring.

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