We study the contact terms that appear in the correlation functions of exactly marginal operators using the AdS/CFT correspondence. It is known that CFT with an exactly marginal deformation requires the existence of the contact terms is crucial for a consistency of with their coefficients having a geometrical interpretation in the context of conformal manifolds. We show that the AdS/CFT correspondence captures properly the mathematical structure of the correlation functions that is expected from the CFT analysis. For this purpose, we employ holographic RG to formulate a most general setup in the bulk for describing an exactly marginal deformation. The resultant bulk equations of motion are nonlinear and solved perturbatively to obtain the on-shell action. We compute three- and four-point functions of the exactly marginal operators using the GKP-Witten prescription, and show that they match with the expected results precisely. It is pointed out that The cut-off surface prescription in the bulk provides us with a regularization scheme for performing a conformal perturbation. serves as a regularization scheme for conformal perturbation theory in the boundary CFT. around a fixed point is regularized by putting a cut-off surface in the bulk. As an application, we examine a double OPE limit of the four-point functions. The anomalous dimensions of double trace operators are written in terms of the geometrical data of a conformal manifold.

Subject Index

B21 AdS/CFT correspondence, B32 Renormalization and renormalization group equation
1. Introduction

Marginal operators of CFT are defined as a scalar operator that has the conformal dimension $\Delta = d$ at a conformal fixed point with $d$ being the spacetime dimension. Perturbing the CFT with marginal operator may spoil conformal symmetry. In fact, the renormalization group (RG) $\beta$-function for a coupling constant $\lambda^I$ can be computed using conformal perturbation theory as

$$\beta^I(\lambda) = \frac{d}{d\log \Lambda} \lambda^I = (\Delta - d)\lambda^I + c_{IJK} \lambda^J \lambda^K + \mathcal{O}(\lambda^3) . \tag{1}$$

Here, $\Delta$ is the conformal dimension of a scalar operator $O_I$ that is added to the CFT as a perturbation. $c_{IJK}$ is the OPE coefficients that are proportional to the three-point function of the scalar operators. The $\beta$-function (1) shows that the marginal couplings with $\Delta = d$ break conformal symmetry if $c_{IJK} \neq 0$. The exactly marginal couplings are defined as the subset of marginal couplings with the vanishing $\beta$-functions at all order in $\lambda$. CFT with an exactly marginal deformation is then characterized by a fixed surface rather than a fixed point in the parameter space. The fixed surface is referred to as a conformal manifold. The conformal manifold admits a natural metric that is defined by the two-point function of the exactly marginal operators $[1]$

$$g_{IJ}(\lambda) = \langle O_I(\infty)O_J(0) \rangle_{\lambda} .$$

Here, the suffix $\lambda$ implies that the two-point function is evaluated with the perturbed CFT action.

It is discussed in $[2]$ that the three-point function of the exactly marginal couplings has a nontrivial contact term

$$\langle O_I(x)O_J(y)O_K(z) \rangle_{\lambda} = \frac{1}{(x-z)^{2d}} \Gamma_{KIJ}(\lambda) \delta^d(x-y) + \text{(cyclic permutations)} , \tag{2}$$

with $\Gamma_{KIJ}$ being the Christoffel symbol defined from $g_{IJ}$. One way for understanding it is to set $\lambda^I = \lambda_0^I + \delta \lambda^I$ with $\delta \lambda^I$ regarded as a perturbation about a reference coupling $\lambda_0^I$. Then, $g_{IJ}(\lambda) = g_{IJ}(\lambda_0 + \delta \lambda)$ is required to be computed by the conformal perturbation in $\delta \lambda^I$. More precisely,

$$g_{IJ}(\lambda_0 + \delta \lambda) = \left. \left. \langle O_I(1)O_J(0) \exp \left( - \int d^d x \delta \lambda^K O_K(x) \right) \right\rangle_{\lambda_0} . \right. \tag{3}$$

The contact terms in the higher-point functions of the exactly marginal operators are analyzed in detail in $[3]$.

The purpose of this paper is to revisit these structures using the AdS/CFT correspondence (for a review, see $[4]$). We first formulate a bulk gravity model in $d+1$ dimensions on the basis of holographic RG $[5]$ (for a review, see also $[6]$). It is argued that the gravity dual of the exactly marginal operator is given by a massless scalar in AdS$_{d+1}$ with a trivial potential term. This guarantees that the RG $\beta$-function for the marginal couplings vanishes.$^1$ This type of models has been studied extensively so far. A typical example is given in $[12, 13]$, which discuss a dilaton-axion system in AdS$_5$ that is dual to the gauge coupling and the

$^1$The existence conditions for an exactly marginal deformation have been discussed extensively so far. For recent works from the viewpoint of conformal perturbation theory, see $[7–11]$.}
θ-angle in $\mathcal{N} = 4$ super YM in four dimensions. For a review, see [14]. For an analysis of exactly marginal deformation within the context of the AdS/CFT correspondence, see also [15–18] and [9].

Starting with this model, we work out the on-shell action by solving the equations of motion of the bulk scalars. These are given by a non-linear equation and can be solved perturbatively in the gravitational coupling. Then, the on-shell action is evaluated in the form of a power series of the boundary values that specify the Dirichlet boundary condition of the bulk scalars at a cut-off surface. It is emphasized that this procedure is interpreted as a conformal perturbation theory by identifying the boundary value with perturbations to a marginal coupling constant. For an early work on conformal perturbation theory in the context of the AdS/CFT correspondence, see [19]. Conformal perturbations become singular when an integrated operator collides with another. It is standard to regularize these divergences by limiting the integral region to avoid the collision of the operators. It is found that the GKP-Witten prescription [20, 21], where a cut-off surface is set near the AdS boundary, plays a role of a regularization scheme for the holographic conformal perturbation theory. From the on-shell action, we derive the three- and four-point functions of the exactly marginal operators. It is seen that they reproduce exactly the contact terms that are expected from the CFT analysis. We also discuss the four-point function in a double OPE limit. We read off the anomalous dimensions of double trace operators composed of the exactly marginal operators as a function of the geometric data of the conformal manifold.

In this paper, we fix the bulk metric to be an AdS$_{d+1}$ background metric with $d$ even, and treat only the bulk scalars as a dynamical field. This implies that the exchange diagrams due to the stress tensor are missed from the results in the four-point function. This simplification is valid for most part of the computations made in this paper, except for an incomplete analysis of the anomalous dimensions of the double trace operators. We leave it as a future work to incorporate the effect from the dynamical, bulk graviton.

This paper is organized as follows. In section 2, we formulate the bulk model of an exactly marginal deformation on the basis of holographic RG. Section 3 is devoted to computing the correlation functions of exactly marginal operators by analyzing this model. The appendix summarizes some useful formulae for bulk-to-bulk and bulk-to-boundary propagators of a scalar field in AdS$_{d+1}$.

## 2. Setup

One of the most efficient ways for describing RG flows in the context of the AdS/CFT correspondence is to utilize the Hamilton-Jacobi (HJ) formulation of a bulk gravity theory [5]. This formalism is demonstrated in more detail in [6].

We start with a bulk action in $M_{d+1}$, a $(d+1)$-dimensional bulk spacetime

$$S \left[ \hat{\gamma}_{\mu\nu}(x, \tau), \hat{\phi}^I(x, \tau) \right] = \int_{M_{d+1}} d^{d+1}X \sqrt{\hat{\gamma}} \left\{ V(\hat{\phi}) - \hat{R}_{(d+1)} + \frac{1}{2} \hat{L}_{IJJ}(\phi) \hat{\gamma}^{\mu\nu} \partial_{\mu} \hat{\phi}^I \partial_{\nu} \hat{\phi}^J \right\}. \quad (4)$$

Here $\hat{\gamma}_{\mu\nu}$ denotes the bulk metric in $M_{d+1}$, and is regarded as a dynamical field only in this section. Using ADM decomposition, it becomes

$$ds^2 = \hat{\gamma}_{\mu\nu} dX^\mu dX^\nu = \hat{N}^2(x, \tau) d\tau^2 + \hat{h}_{ij}(x, \tau)(dx^i + \hat{\lambda}^i(x, \tau) d\tau)(dx^j + \hat{\lambda}^j(x, \tau) d\tau), \quad (5)$$
with \( \tau \geq \tau_0 \). \( h_{ij} \) is an induced metric on a \( d \)-dimensional hypersurface at a fixed \( \tau \). The hatted quantities mean off-shell without the equations of motion imposed. \( M_{d+1} \) has a \( d \)-dimensional boundary \( \Sigma_d \) at \( \tau = \tau_0 \). We omit writing the boundary terms to be added to the action (10) for simplicity.

We solve the bulk equations of motion under the boundary condition

\[
\hat{h}_{ij}(x, \tau = \tau_0) = h_{ij}(x), \quad \hat{\phi}^I(x, \tau = \tau_0) = \phi^I(x).
\]

By inserting these classical solutions into the action, we obtain the on-shell action \( S \), which is the functional of the boundary values. In the dictionary of the AdS/CFT correspondence, \( S \) is identified with the generating functional of CFT\(_d\). Use of the HJ formalism in the bulk gravity allows one to derive an RG equation for the generating functional. To see this, we note that \( S \) obeys the HJ equation by construction, which follows from the Hamiltonian constraint. We divide \( S \) into the local and non-local parts

\[
\frac{1}{2\kappa_{d+1}^2} S[h(x), \phi(x)] = \frac{1}{2\kappa_{d+1}^2} S_{\text{loc}}[h(x), \phi(x)] - \Gamma[h(x), \phi(x)], \quad (6)
\]

with

\[
S_{\text{loc}}[h(x), \phi(x)] = \int d^d x \sqrt{\gamma} \left[ W(\phi) + (\text{derivative terms}) \right] \quad (7)
\]

Inserting this into the HJ equation and employing a derivative expansion, we find that \( W(\phi) \) is related with the scalar potential \( V(\phi) \) as

\[
V(\phi) = -\frac{d}{4(d-1)} W^2(\phi) + \frac{1}{2} L^{IJ}(\phi) \partial_I W(\phi) \partial_J W(\phi) \quad (8)
\]

Here, \( \partial_I = \partial / \partial \phi^I \) and \( L^{IJ} = L^{-1}_{IJ} \). Furthermore, it is found that the non-local part \( \Gamma \) obeys the local RG equation [22] with the RG \( \beta \)-function for the coupling function \( \phi^I(x) \) given by

\[
\beta^I(\phi) = -\frac{2(d-1)}{W(\phi)} L^{IJ}(\phi) \partial_J W(\phi) \quad (9)
\]

This shows that the \( \beta \)-function vanishes iff \( W(\phi) \) is independent of \( \phi^I \). It then follows from (8) that the scalar potential is a constant, being equal to the bulk cosmological constant of AdS\(_{d+1}\).

To summarize, the bulk setup for studying the exactly marginal deformation of CFT\(_d\) is given by the massless scalar fields propagating in AdS\(_{d+1}\) with a trivial scalar potential

\[
S_{\text{bulk}} = \frac{1}{4\kappa_{d+1}^2} \int d^{d+1} X \sqrt{\gamma} G_{IJ}(\phi) \gamma^{\mu\nu} \partial_\mu \phi^I \partial_\nu \phi^J. \quad (10)
\]

Hereafter, the unhatted field \( \phi^I \) denotes an off-shell quantity. The cosmological constant term is neglected because this is irrelevant to the rest of the analysis. This model is studied recently in [9] to argue that it provides us with the holographic description of an exactly marginal deformation. The bulk scalar fields have self-couplings due to \( G_{IJ}(\phi) \), and also are coupled with the bulk metric \( \gamma_{\mu\nu} \). As mentioned before, this is taken to be a non-dynamical, AdS\(_{d+1}\) background metric

\[
\gamma_{\mu\nu} dX^\mu dX^\nu = \frac{dz^2 + d\vec{x}^2}{z^2},
\]

with \( \vec{x} \in \mathbb{R}^d \).
3. Evaluation of on-shell action

The equation of motion of the bulk scalar field $\phi^I$ reads

$$\Box \phi^I + \Gamma^I_{IJ}(\phi) \partial_\mu \phi^J \partial^\mu \phi^K = 0.$$  

Here, $\Box$ is the Laplacian in AdS$_{d+1}$, and $\Gamma^I_{JK}(\phi)$ is the Christoffel symbol for $G_{IJ}(\phi)$. It is impossible to find the exact solutions of this. In this paper, we attempt to solve it perturbatively in $\alpha \equiv (2\kappa_{d+1}^2)^{1/2}$. Expand $\phi^I$ around a constant $\varphi^I$ as

$$\phi^I(X) = \varphi^I + \sum_{n \geq 1} \alpha^n \eta^I_n(X).$$

Then, the EOM decomposes as

$$\Box \eta^I_{(1)} = 0,$$  

$$\Box \eta^I_{(2)} = -\Gamma^I_{JK} \partial_\mu \eta^J_{(1)} \partial^\mu \eta^K_{(1)},$$  

$$\Box \eta^I_{(3)} = -2\Gamma^I_{JK} \partial_\mu \eta^J_{(2)} \partial^\mu \eta^K_{(1)} - \partial_L \Gamma^I_{JK} \eta^L_{(1)} \partial_\mu \eta^K_{(1)} \partial^\mu \eta^L_{(1)},$$  

... 

We set $G_{IJ} = G_{IJ}(\phi = \varphi)$ and $\Gamma^I_{JK} = \Gamma^I_{JK}(\phi = \varphi)$ from now on. These equations are solved by imposing the boundary conditions at the cut-off surface $z = \epsilon > 0$

$$\eta^I_{(1)}(z = \epsilon, \vec{x}) = J^I(\vec{x}), \quad \eta^I_{(n)}(z = \epsilon, \vec{x}) = 0. \ (n \geq 2)$$  

The boundary value $J^I(\vec{x})$ is identified with an external field coupled with the CFT operator that is dual to $\phi^I$. When interpreting our results along the line of conformal perturbation theory, $\varphi^I$ is identified with the reference coupling $\lambda^I_0$ and $J^I$ with a perturbation about it.

(11) is solved by using the modified bulk-to-boundary operator $K^\Delta$ as

$$\eta^I_{(1)}(X) = \int d^d y K^\Delta(X, \vec{y}) J^I(\vec{y}),$$

with $\Delta = d$. The modified propagators are defined to obey an appropriate boundary condition at the cut-off surface $z = \epsilon$ rather than $z = 0$. A full account of them is given in [23]. See also the appendix A for a review. (12) can be solved by using the modified bulk-to-bulk propagator as

$$\eta^I_{(2)}(X) = \Gamma^I_{JK} \int d^{d+1} Y \sqrt{\gamma(Y)} G^\Delta(X, Y) \partial_\mu \eta^I_{(1)} \partial^\mu \eta^K_{(1)}.$$  

Note that this obeys the boundary condition (13) because $G^\Delta(X, Y)$ vanishes at the cut-off surface by definition. The higher-order terms $\eta^I_{(n)}(X)$ with $n \geq 3$ can be computed recursively. By inserting the solutions into $S_{\text{bulk}}$, we obtain the on-shell action $I$ as a functional of $J^I(\vec{x})$ order-by-order in $\alpha$

$$I[J] = \sum_{n \geq 2} \alpha^{n-2} I_n[J].$$

$I_n[J]$ is given by a functional of $J$ of power $n$. 

5/15
As an exercise, \( I_2 \) is evaluated as

\[
I_2 = -\frac{1}{2} G_{1J} \int_{z=\epsilon} d^d x \frac{1}{\epsilon^{d-1}} \eta_{(1)}^I \partial_z \eta_{(1)}^J = -\frac{1}{2 \epsilon^{d-1}} G_{1J} \int d^d x \ d^d y \ J^I(\vec{x}) J^J(\vec{y}) K^e(|\vec{x} - \vec{y}|) .
\]

(17)

Here,

\[
K^e(|\vec{x} - \vec{y}|) \equiv \partial_z K^e_{\Delta=d}(X,\vec{y})|_{z=\epsilon} = \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k} \cdot (\vec{x} - \vec{y})} k \partial_z \log \left[ z^{d/2} K_{d/2}(z) \right] \mid_{z=\epsilon} .
\]

(18)

For the purpose of obtaining the two-point function, we need to expand the integrand in \( \epsilon \) and extract a log divergent term in \( \epsilon \), which is a non-analytic function of \( k^2 \). A standard power expansion of the Bessel function \( K_{\nu}(z) \) in \( z \) is ill-defined for \( \nu = d/2 \in \mathbb{Z}_{\geq 1} \). This problem is resolved by defining \( K_{d/2}(z) \) as \( \lim_{\nu \to d/2} K_{\nu}(z) \). It is verified that

\[
K_{d/2}(z) = \frac{1}{2} \Gamma(d/2) \left( \frac{z}{2} \right)^{-d/2} \left[ \sum_{m=0}^{d/2-1} \frac{(-)^m}{m!} \frac{\Gamma(d/2 - m)}{\Gamma(d/2)} \left( \frac{z}{2} \right)^{2m} \right.
\]

\[
+ \sum_{m=0}^{\infty} \frac{(-)^{d/2-1}}{m!(d/2-1)!(m+d/2)!} \left( \frac{z}{2} \right)^{2m+d} \left\{ 2\gamma_E + 2 \log \left( \frac{z}{2} \right) - H_m - H_{m+d/2} \right\} \right] .
\]

(19)

Here, \( \gamma_E \) is the Euler gamma constant, and \( H_m \) is the harmonic number. Then, we find

\[
I_2 = -\frac{1}{2} \frac{(-)^{d/2-1}}{2^{d/2-2} \Gamma(d/2)^2} G_{1J} \int d^d x \ d^d y \ J^I(\vec{x}) J^J(\vec{y}) \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k} \cdot (\vec{x} - \vec{y})} k^d \log(\epsilon k) + \cdots
\]

\[
= -\frac{1}{2} C_d G_{1J} \int d^d x \ d^d y \ J^I(\vec{x}) J^J(\vec{y}) \frac{1}{(\vec{x} - \vec{y})^{2d}} + \cdots ,
\]

(19)

with

\[
C_d = d \frac{\Gamma(d)}{\pi^{d/2} \Gamma(d/2)} .
\]

Here, the power divergent terms in \( \epsilon \) are neglected because these lead to analytic functions in \( k^2 \), which are removed by local counter terms. It follows that the Zamolodchikov metric is given by

\[
g_{1J} = C_d G_{1J} .
\]

(20)

### 3.1. Cubic terms in \( J \)

The cubic term reads

\[
I_3[J] = \frac{1}{2} G_{1J} \int d^{d+1} X \sqrt{\gamma} \left( \partial_{\mu} \eta_{(1)}^I \partial^{\mu} \eta_{(2)}^J + \partial_{\mu} \eta_{(1)}^J \partial^{\mu} \eta_{(2)}^I \right) + \frac{1}{2} \partial K G_{1J} \int d^{d+1} X \sqrt{\gamma} \eta_{(1)}^K \partial_{\mu} \eta_{(1)}^I \partial^{\mu} \eta_{(1)}^J .
\]

The first term vanishes because of (11) and the boundary condition for \( \eta_{(2)}^I \) in (13). The second term can be rewritten as a boundary term at \( z = \epsilon \). To see this, we use integration
by parts:

\[ I_3 = -\frac{1}{2} \partial_K G_{IJ} \int_{z=1} d^d x \frac{1}{e^{d-1}} \eta^K_{(1)} \eta^I_{(1)} \partial_z \eta^J_{(1)} - \frac{1}{4} \partial_K G_{IJ} \int d^{d+1} x \sqrt{\gamma} \partial^\mu \eta^K_{(1)} \left\{ \eta^I_{(1)} \partial_\mu \eta^J_{(1)} + \eta^J_{(1)} \partial_\mu \eta^I_{(1)} \right\}. \]

The integrand in the second term is rewritten as a total derivative because of (11). As a consequence, we obtain

\[
I_3 = -\frac{1}{2} \partial_K G_{IJ} \int_{z=1} d^d x \frac{1}{e^{d-1}} \eta^K_{(1)} \eta^I_{(1)} \partial_z \eta^J_{(1)} + \frac{1}{4} \partial_K G_{IJ} \int_{z=1} d^d x \frac{1}{e^{d-1}} \partial_z \eta^K_{(1)} \eta^I_{(1)} \eta^J_{(1)}
\]

\[= -\frac{1}{2e^{d-1}} \Gamma_{K,IJ} \int d^d x d^d y J^I(\vec{x}) J^J(\vec{x}) J^K(\vec{y}) K^L(\vec{y}) \delta(\vec{x} - \vec{y}). \]

Our focus is on the log divergent term in \(\epsilon\). The computation of \(I_3\) proceeds exactly in the same manner as of \(I_2\). We find that

\[ I_3 = -\frac{1}{2} C_d \Gamma_{K,IJ} \int d^d x d^d y \delta \frac{1}{(\vec{x} - \vec{y})^2} \cdots. \tag{21} \]

Here, the ultra contact terms that come from power divergent terms in \(\epsilon\) are neglected.

As expected, this result implies that the OPE coefficient among the CFT operators dual to \(\phi^I\) vanishes. Furthermore, we find that the functional derivative \(-\delta^2 I_3 / \delta J^I(\vec{x}) \delta J^J(\vec{y}) |_{J=\delta \lambda / \alpha}\) reproduces the first-order term of \(\delta \lambda\) in (3).

### 3.2. Quartic terms in \(J\)

It is found that

\[ I_4[J] = \frac{1}{2} G_{IJ} \int d^{d+1} x \sqrt{\gamma} \left( 2 \partial_\mu \eta^I_{(1)} \partial^\mu \eta^J_{(2)} + \partial_\mu \eta^I_{(2)} \partial^\mu \eta^J_{(1)} \right)
\]

\[+ \frac{1}{2} \partial_K G_{IJ} \int d^{d+1} x \sqrt{\gamma} \left( \eta^K_{(2)} \partial_\mu \eta^I_{(1)} \partial^\mu \eta^J_{(1)} + 2 \eta^K_{(1)} \partial_\mu \eta^I_{(2)} \partial^\mu \eta^J_{(1)} \right)
\]

\[+ \frac{1}{4} \partial_K G_{IJ} \int d^{d+1} x \sqrt{\gamma} \left( \eta^K_{(1)} \partial_\mu \eta^I_{(1)} \partial^\mu \eta^J_{(1)} \right)
\]

\[= -\frac{1}{2} G_{IJ} \int d^{d+1} x \sqrt{\gamma} \left( \eta^K_{(2)} \partial_\mu \eta^I_{(1)} \partial^\mu \eta^J_{(1)} \right)
\]

\[+ \frac{1}{4} \partial_K G_{IJ} \int d^{d+1} x \sqrt{\gamma} \left( \eta^K_{(1)} \partial_\mu \eta^I_{(1)} \partial^\mu \eta^J_{(1)} \right). \tag{22} \]

Here, (11) and (13) are used. Inserting the solution (15) gives

\[ I_4[J] = I_{\text{ex}}[J] + I_{\text{cont}}[J], \]

with

\[ I_{\text{ex}} = -\frac{1}{2} \Gamma_{M,IJ} \Gamma_{KL} \int d^{d+1} x \int d^{d+1} Y \sqrt{\gamma(X) \sqrt{\gamma(Y)}} G^\alpha_{\Delta}(X,Y) \partial_\mu \eta^I_{(1)}(X) \partial^\mu \eta^J_{(1)}(X) \partial_\nu \eta^K_{(1)}(Y) \partial^\nu \eta^L_{(1)}(Y) \]

\[= -\frac{1}{2} \Gamma_{M,IJ} \Gamma_{KL} \int d^{d+1} x \int d^{d+1} Y \sqrt{\gamma(X) \sqrt{\gamma(Y)}} G^\alpha_{\Delta}(X,Y) \partial_\mu \eta^I_{(1)}(X) \partial^\mu \eta^J_{(1)}(X) \partial_\nu \eta^K_{(1)}(Y) \partial^\nu \eta^L_{(1)}(Y). \tag{23} \]

\[ I_{\text{cont}} = \frac{1}{4} \partial_K G_{IJ} \int d^{d+1} x \sqrt{\gamma} \left( \eta^K_{(1)} \eta^I_{(1)} \partial_\mu \eta^J_{(1)} \right). \tag{24} \]

\(I_{\text{ex}}\) and \(I_{\text{cont}}\) are interpreted as the contributions from an exchange and contact diagram, respectively.

It is verified that the exchange diagram can be rewritten into the form of the contact diagram (24) up to a contact term. This result was first obtained in [12] in an analysis of
the dilaton-axion system, although no attention is paid to the contact terms. To see this, we first perform integration by parts with respect to $\partial / \partial X^\mu$ twice. Then,

$$I_{\text{ex}} = \frac{1}{4} \Gamma_{M,II} \Gamma_{KL}^M \int d^{d+1}X d^{d+1}Y \sqrt{\gamma} \frac{\gamma(Y)}{\gamma(X)} \partial^\mu \eta^L_{(1)}(Y) \cdot \partial^\mu \eta^I_{(1)}(X) \partial^\mu \eta^I_{(1)}(Y)$$

$$- \frac{1}{4} \Gamma_{M,II} \Gamma_{KL}^M \int d^{d+1}X d^d y \sqrt{\gamma} \partial^\mu \eta^I_{(1)}(X) \partial^\mu \eta^I_{(1)}(Y) J^I(y) J^I(\bar{y})$$

$$+ \frac{1}{4} \Gamma_{M,II} \Gamma_{KL}^M \int d^{d+1}X \sqrt{\gamma} \partial^\mu \eta^I_{(1)} \partial^\mu \eta^L_{(1)} \eta^I_{(1)} \eta^I_{(1)} .$$

Here, (A10) and

$$\Box_X G^\Delta(X, Y) = - \frac{1}{\sqrt{\gamma}} \delta^{d+1}(X - Y) ,$$

are used. The first term in (25) can be further rewritten using integration by parts again and (A12) to show that this is given by a contact term. The second term in (25) takes exactly the same form of the contact diagram (24) up to constant prefactors, as promised. We obtain

$$I_{\text{ex}} = \frac{1}{4} \Gamma_{M,II} \Gamma_{KL}^M \int d^{d+1}X \sqrt{\gamma} \partial^\mu \eta^I_{(1)} \partial^\mu \eta^I_{(1)}$$

$$+ \frac{1}{4 \epsilon^{d-1}} \Gamma_{M,II} \Gamma_{KL}^M \int d^{d+1}X d^d y \epsilon^d K^\epsilon(\bar{x} - \bar{y}) J^I(\bar{x}) J^J(\bar{y}) J^K(\bar{y}) \left( J^L(\bar{x}) - \frac{1}{2} J^L(\bar{y}) \right) .$$

In order to examine if $I_4$ reproduces the contact terms that are consistent with exactly marginal deformation, we study

$$- \alpha \frac{\delta^3 I_4}{\delta J^I(\bar{x}_1) \delta J^J(\bar{x}_2) \delta J^K(\bar{x}_3)} \bigg|_{J = \delta \lambda / \alpha} .$$

This should be interpreted as the linear correction in $\delta \lambda$ to the three-point function that results from $I_3$. We start by analyzing the contributions from the contact diagram $I_{\text{cont}}$. It is useful to rewrite it as

$$I_{\text{cont}} = \frac{1}{4} \partial_I \partial_J G_{KL} \int d^d y_1 \cdots d^d y_4 J^I(\bar{y}_1) J^J(\bar{y}_2) J^K(\bar{y}_3) J^L(\bar{y}_4) \mathcal{F}(\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4) ,$$

with

$$\mathcal{F}(\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4) = \int_0^\infty \frac{dz}{z^{d-1}} \int d^d x K^\Delta(X, \bar{y}_1) K^\Delta(X, \bar{y}_2) \partial^\mu K^\Delta(X, \bar{y}_3) \partial^\mu K^\Delta(X, \bar{y}_4) .$$

It is found that

$$\frac{\delta^3 I_{\text{cont}}}{\delta J^I(\bar{x}_1) \delta J^J(\bar{x}_2) \delta J^K(\bar{x}_3)} \bigg|_{J = \text{const.}} = J^L \partial_I \left[ \partial_I G_{KL} f(\bar{x}_1, \bar{x}_2, \bar{x}_3) + \partial_J G_{KL} f(\bar{x}_2, \bar{x}_3, \bar{x}_1) + \partial_K G_{IJ} f(\bar{x}_3, \bar{x}_1, \bar{x}_2) \right] .$$

Here, we define

$$f(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \int \frac{d^d y}{\sqrt{\gamma}} \mathcal{F}(\bar{x}_1, \bar{y}, \bar{x}_2, \bar{x}_3),$$

and used the relation

$$\int \frac{d^d y}{\sqrt{\gamma}} \mathcal{F}(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{y}) = 0 ,$$

8/15
which follows from (A15). It is not difficult to see that \( f(\vec{x}_1, \vec{x}_2, \vec{x}_3) \) is given by a contact term of the form

\[
\frac{1}{2}\epsilon^{d-1} \left[ \delta^d(\vec{x}_{12}) K^d(x_{13}) + \delta^d(\vec{x}_{13}) K^d(x_{12}) - \delta^d(\vec{x}_{23}) K^d(x_{12}) \right].
\]

Here, \( \vec{x}_{ij} = \vec{x}_i - \vec{x}_j \). It follows from these results that

\[
-\alpha \frac{\delta^3 I_{\text{cont}}}{\delta J^I(\vec{x}_1) \delta J^J(\vec{x}_2) \delta J^K(\vec{x}_3)} \bigg|_{J=\delta \lambda/\alpha} = \frac{1}{\epsilon^{d-1}} \delta \lambda^L \partial_L \left[ \delta^d(\vec{x}_{12}) K^d(x_{13}) \Gamma_{K,IJ} + \delta^d(\vec{x}_{23}) K^d(x_{31}) \Gamma_{I,JK} + \delta^d(\vec{x}_{31}) K^d(x_{12}) \Gamma_{J,IK} \right].
\]

(30)

The functional derivative of \( I_{\text{ex}} \) is computed in the same manner. It can be proved that

\[
\frac{\delta^3 I_{\text{ex}}}{\delta J^I(\vec{x}_1) \delta J^J(\vec{x}_2) \delta J^K(\vec{x}_3)} \bigg|_{J=\text{const.}} = 0,
\]

due to cancellation of the functional derivative of the bulk term in (26) with that of the boundary term in (26). Therefore, we find

\[
-\alpha \frac{\delta^3 I_4}{\delta J^I(\vec{x}_1) \delta J^J(\vec{x}_2) \delta J^K(\vec{x}_3)} \bigg|_{J=\delta \lambda/\alpha} = \frac{1}{\epsilon^{d-1}} \delta \lambda^L \partial_L \left[ \delta^d(\vec{x}_{12}) K^d(x_{23}) \Gamma_{K,IJ} + \delta^d(\vec{x}_{23}) K^d(x_{31}) \Gamma_{I,JK} + \delta^d(\vec{x}_{31}) K^d(x_{12}) \Gamma_{J,IK} \right].
\]

(31)

As expected, this coincides with the linear correction to the three-point function of the exactly marginal operators that comes from \( I_3 \).

### 3.3. Double OPE limit and double trace operators

In this subsection, we examine the double OPE limit of the four-point functions of the exactly marginal operators with an emphasis on the structure of double trace operators. The results given below are a simple extension of the papers \([13, 24]\).

The leading contribution to the four-point functions in \( \alpha \) expansions is given by disconnected diagrams. Including them, the four-point function reads

\[
\langle O_I(\vec{x}_1) O_J(\vec{x}_2) O_K(\vec{x}_3) O_L(\vec{x}_4) \rangle = \frac{g_{IJKL}}{x_{12}^{2\Delta} x_{34}^{2\Delta}} + \frac{g_{IJKL}}{x_{13}^{2\Delta} x_{24}^{2\Delta}} + \frac{g_{IJKL}}{x_{14}^{2\Delta} x_{23}^{2\Delta}} - \alpha^2 \frac{\delta^4 I_4[J]}{\delta J^I(\vec{x}_1) \delta J^J(\vec{x}_2) \delta J^K(\vec{x}_3) \delta J^L(\vec{x}_4)}.
\]

(32)

Here, no contact term in the four-point function comes out because the positions of the operators are taken to be different from each other. This implies that we are allowed to neglect the boundary terms at \( z = \epsilon \) and set \( \epsilon = 0 \) in \( I_4 \) before performing the \( z \) integral. Then, we have

\[
I_4 = \frac{1}{4} A_{IJKL} \int d^{d+1} X \sqrt{\gamma} \eta^I_1 \eta^J_1 \partial_\mu \eta^I_1 \partial^\mu \eta^J_1 \cdot \Gamma(d) \left( \frac{\pi^{d/2}}{\Gamma(d/2)} \right)^4 \int d^d y_1 \cdots d^d y_4 J^I(y_1) J^J(y_2) J^K(y_3) J^L(y_4) \mathcal{I}(y_1, y_2, y_3, y_4),
\]

(33)
with
\[ A_{IJKL} = \partial_I \partial_J G_{KL} + \Gamma_{M, IJK} \Gamma_{KL}^M, \] (34)
and
\[ \mathcal{I}(\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4) = \int_0^\infty \frac{dz}{z^{d-1}} \int d^d x \left( \frac{z}{z^2 + (\bar{x} - \bar{y}_1)^2} \right)^d \left( \frac{z}{z^2 + (\bar{x} - \bar{y}_2)^2} \right)^d \times \delta^{\mu \nu} \frac{\partial}{\partial X^\mu} \left( \frac{z}{z^2 + (\bar{x} - \bar{y}_3)^2} \right)^d \frac{\partial}{\partial X^\nu} \left( \frac{z}{z^2 + (\bar{x} - \bar{y}_4)^2} \right)^d. \] (35)

Here, (A13) is used. It then follows that
\[ - \frac{\delta^4 I_4[J]}{\delta J^I(\bar{x}_1) \delta J^J(\bar{x}_2) \delta J^K(\bar{x}_3) \delta J^L(\bar{x}_4)} = - \left( \frac{\Gamma(d)}{\pi^{d/2} \Gamma(d/2)} \right)^4 \left( A_{IKJL} \mathcal{I}(\bar{x}_1, \bar{x}_3, \bar{x}_2, \bar{x}_4) + A_{ILJK} \mathcal{I}(\bar{x}_2, \bar{x}_1, \bar{x}_3, \bar{x}_4) \right. \]
\[ + A_{IJKL} \mathcal{I}(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4) + A_{KLIJ} \mathcal{I}(\bar{x}_3, \bar{x}_4, \bar{x}_1, \bar{x}_2) \]
\[ + A_{ILJK} \mathcal{I}(\bar{x}_1, \bar{x}_4, \bar{x}_2, \bar{x}_3) + A_{JKIL} \mathcal{I}(\bar{x}_2, \bar{x}_3, \bar{x}_1, \bar{x}_4) \right). \] (36)

It is useful to rewrite \( \mathcal{I} \) by employing the \( D \)-function defined in [13]
\[ D_{\Delta_1, \Delta_2, \Delta_3, \Delta_4}(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4) = \int_0^\infty \frac{dz}{z^{d+1}} \int d^d x \prod_{i=1}^4 \left( \frac{z}{z^2 + (\bar{x} - \bar{x}_i)^2} \right)^{\Delta_i}, \] (37)

Using the formulae summarized in the appendix gives
\[ \mathcal{I}(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4) = d^2 \left( D_{dd;dd;dd;dd}(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4) - 2x_{34}^2 D_{dd;dd;dd;dd}(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4) \right) \]
\[ = d^2 \left( 1 + \frac{3}{d} x_{34}^2 \frac{\partial}{\partial x_{34}^2} \right) D_{dd;dd;dd;dd}(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4). \] (38)

The \( D \)-function depends on two conformal invariants defined as
\[ s = \frac{1}{2} \frac{x_{13}^2 x_{24}^2}{x_{12}^4 x_{34}^2 + x_{14}^2 x_{23}^2}, \quad t = \frac{x_{12}^2 x_{14}^2 - x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2 + x_{14}^2 x_{23}^2}. \]

A double OPE limit is defined as
\[ |x_{13}| \ll |x_{12}|, \quad |x_{24}| \ll |x_{12}|. \]

In this limit, the conformal invariants become
\[ s \sim \frac{1}{4} \frac{x_{13}^2 x_{24}^2}{x_{12}^4} \rightarrow 0, \quad t \sim - \frac{1}{x_{12}^2} \left[ \bar{x}_{13} \cdot \bar{x}_{24} - 2 \frac{(\bar{x}_{12} \cdot \bar{x}_{13})(\bar{x}_{12} \cdot \bar{x}_{24})}{x_{12}^2} \right] \rightarrow 0, \]
with \( s/t^2 \) kept finite. As shown in [13], this limit gives rise to finite and log divergent terms in the \( D \)-function:
\[ D_{dd;dd;dd;dd}(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4) \sim - \frac{\pi^{d/2} \Gamma(3d/2)}{2 \Gamma(d+2)} \frac{1}{x_{12}^4} \left( 3H_{d-1} - H_{d-1/2} + \log s \right), \]
Then, the four-point function in the double OPE limit becomes
\[ \langle O_I(\vec{x}_1) O_J(\vec{x}_2) O_K(\vec{x}_3) O_L(\vec{x}_4) \rangle \sim \frac{g_{IK} g_{JL}}{x_{12}^{2\Delta} x_{24}^{2\Delta}} + \frac{g_{IJ} g_{KL} + g_{IL} g_{JK}}{(x_{12}^{2d})^2} \]
\[ + \alpha^2 \frac{d^2}{2 \pi^{3d/2}} \frac{\Gamma(3d/2)}{\Gamma(2d)} \left( \frac{\Gamma(d)}{\Gamma(d/2)} \right)^4 \frac{1}{(x_{12}^{2d})^2} \left[ (3H_d - H_{d-1/2}) A_{IKJL} - 5 (3H_{d-1} - H_{d-1/2}) B_{IKJL} \right. \]
\[ \left. + (A_{IKJL} - 5 B_{IKJL}) \log s \right], \quad (39) \]
with
\[ A_{IKJL} = A_{IKJL} + A_{JLIK} , \quad B_{IKJL} = A_{IJKL} + A_{KLIJ} + A_{ILJK} + A_{JKIL} . \]

We interpret this result in terms of the OPE of the exactly marginal operators
\[ O_I(x_1) O_K(x_3) \sim \frac{g_{IK}}{x_{13}^{2\Delta}} + \sum_a \frac{C_{IK}^a}{x_{13}^{2\Delta-a}} O_a(x_1) . \quad (40) \]

Some comments about this are in order. We neglect the terms in the RHS that depends on the stress tensor for simplicity, because these are of no interest in this paper. As shown in section 3.1, the OPE coefficients among the exactly marginal operators vanish. \{O_a\} denotes the whole set of the double trace operators that are defined by the product of the exactly marginal operators. In this paper, we choose the basis of the double trace operators in such a way that the two-point functions are orthonormal:
\[ \langle O_a(x_1) O_b(x_2) \rangle = \frac{\delta_{ab}}{x_{12}^{2\Delta}} . \]

\( \Delta_a \) is the conformal dimensions of \( O_a \), and related to \( \Delta \) as
\[ \Delta_a = 2\Delta + \gamma_a , \]
with \( \gamma_a \) equal to the anomalous dimension of \( O_a \). \( C_{IK}^a \) is the OPE coefficient among the exactly marginal operators and the double trace operators. Using the OPE (40), the four-point function becomes
\[ \langle O_I(\vec{x}_1) O_J(\vec{x}_2) O_K(\vec{x}_3) O_L(\vec{x}_4) \rangle \sim \frac{g_{IK} g_{JL}}{x_{13}^{2\Delta} x_{24}^{2\Delta}} + \frac{1}{(x_{12}^{2d})^{2\Delta}} \sum_a C_{IK}^a C_{JL}^a (4s)^{\gamma_a/2} . \quad (41) \]

Now we equate (39) with (41) to relate the OPE coefficients and the anomalous dimensions of the double trace operators with the bulk data given in (39). This is possible if the conformal dimensions of the single trace operators \( O_I \) and \( T_{ij} \) receive no \( \alpha \) correction. The stress tensor is not renormalized because it is conserved. We assume that \( O_I \) remains to be an exactly marginal operator beyond the planar limit. See [9] for a discussion about it from the holographic viewpoint. We furthermore assume that \( \gamma_a = O(\alpha^2) \) and the OPE coefficients...
$C_{IJ}^a$ receive an $O(\alpha^2)$ correction

$$C_{IJ}^a = C_{IJ}^{(0)} + \alpha^2 C_{IJ}^{(1)}.$$  

For instance, the bulk AdS$_5$ has $\alpha \propto 1/N$ so that the anomalous dimensions are of $O(1/N^2)$. Then, by expanding the R.H.S. of (41) in $\alpha$, we find

$$\langle O_I(x_1) O_J(x_2) O_K(x_3) O_L(x_4) \rangle \sim \frac{g_{IK} g_{JL}}{x_{13} x_{24}}$$

$$+ \frac{1}{(x_{12}^2)^{2\Delta}} \sum_a \left[ C_{IK}^{(0)} C_{JL}^{(0)} + \alpha^2 \left( C_{IK}^{(0)} C_{JL}^{(1)} + C_{IK}^{(1)} C_{JL}^{(0)} \right) + C_{IK}^{(0)} \gamma_a C_{JL}^{(0)} \log 2 \right]$$

$$+ \frac{1}{2} \log s \sum_a C_{IK}^{(0)} \gamma_a C_{JL}^{(0)} + O(\alpha^4).$$  

(42)

Comparing (39) with (42) yields

$$\sum_a C_{IK}^{(0)} C_{JL}^{(0)} = g_{IK} g_{JL} + g_{IL} g_{JK},$$  

(43)

$$\sum_a C_{IK}^{(0)} \gamma_a C_{JL}^{(0)} = 2 \alpha^2 \frac{d^2}{2\pi^3 d/2} \frac{\Gamma(3d/2)}{\Gamma(2d)} \left( \frac{\Gamma(d)}{\Gamma(d/2)} \right)^4 \left( A_{IKJL} - 5 B_{IKJL} \right),$$  

(44)

$$\sum_a \left( C_{IK}^{(0)} C_{JL}^{(1)} + C_{IK}^{(1)} C_{JL}^{(0)} \right) = \frac{d^2}{2\pi^3 d/2} \frac{\Gamma(3d/2)}{\Gamma(2d)} \left( \frac{\Gamma(d)}{\Gamma(d/2)} \right)^4$$

$$\times \left[ \left( 3H_d - H_{d-1/2} - \log 4 \right) A_{IKJL} - 5 \left( 3H_{d-1} - H_{d-1/2} - \log 4 \right) B_{IKJL} \right].$$  

(45)

In order to obtain a full set of the equations that relate the OPE data with the geometric data in the bulk, we have to work out the double OPE limit of the four-point functions

$$\langle O_I(x_1) T_{ij}(x_2) O_K(x_3) T^{ij}(x_4) \rangle, \quad \langle T_{ij}(x_1) T_{kl}(x_2) T^{ij}(x_3) T^{kl}(x_4) \rangle.$$  

These are computed by promoting the bulk metric to a dynamical field, which amounts to incorporating the bulk graviton exchange into (32). We leave it for a future work.

(43) shows that $C_{IK}^{(0)}$ is regarded as an orthogonal matrix that relates the two operator bases $IK$ and $a$. (44) implies that this diagonalizes the symmetric matrix $A_{IKJL} - 5 B_{IKJL}$ with the eigenvalues proportional to the anomalous dimensions $\gamma_a$. $C_{IK}^{(1)}$ is obtained by solving the linear equation (45).

Acknowledgments

We would like to thank Ken Kikuchi and Naoki Watamura for discussions.

A. Modified propagators

Here, we make a brief review of how to derive the modified Green functions used in this paper.
The Green function of a bulk scalar field of mass \(m\) in \(\text{AdS}_{d+1}\) is defined as

\[
(\Box_X - m^2)G_\Delta(X, Y) = -\frac{1}{\sqrt{\gamma}} \delta^{d+1}(X - Y) .
\]  

(A1)

Fourier-tranforming \(G_\Delta\) as

\[
G_\Delta(X, Y) = \int \frac{d^dk}{(2\pi)^d} e^{i\vec{k} \cdot (\vec{x} - \vec{y})} \tilde{G}_\Delta(z, w, k) ,
\]  

(A2)

\(
\tilde{G}_\Delta\) is found to be solved as

\[
\tilde{G}_\Delta(z, w, k) = z^{d/2} \tilde{H}_\Delta(z, w, k)
\]  

(A3)

with \(\tilde{H}_\Delta\) defined by

\[
(D_z - k^2) \tilde{H}_\Delta(z, w, k) = -z^{d/2-1} \delta(z - w)
\]  

(A4)

Here, \(D_z\) is the differential operator given by

\[
D_z = \partial_z^2 + \frac{1}{z} \partial_z + \nu^2 - \frac{1}{z^2}
\]  

(A5)

and \(\nu = \sqrt{m^2 + d^2/4}\). It is not difficult to show that the complete set for \(D_z\) is formed by the Bessel functions \(\{J_\nu(\lambda z)\}_{\lambda \geq 0}\) with the orthonormal condition given by

\[
\int_0^\infty dz \, J_\nu(\lambda z) J_\nu(\lambda' z) = \frac{1}{\lambda} \delta(\lambda - \lambda')
\]  

(A6)

Then, \(\tilde{H}_\Delta(z, w, k)\) can be solved as a linear combination of the Bessel function

\[
\tilde{H}_\Delta(z, w, k) = w^{d/2} \int_0^\infty d\lambda \, \frac{\lambda}{\lambda^2 + k^2} J_\nu(\lambda z) J_\nu(\lambda w) = \begin{cases} 
  w^{d/2} K_\nu(kz) I_\nu(kw) , & (z \geq w > 0) \\
  w^{d/2} K_\nu(kw) I_\nu(kz) , & (w > z > 0)
\end{cases}
\]  

(A7)

Here, \(I_\nu\) is the modified Bessel function.

Now we define the modified bulk-to-bulk propagator

\[
G^\epsilon_\Delta(X, Y) \equiv G_\Delta(X, Y) - (zw)^{d/2} \int \frac{d^dk}{(2\pi)^d} e^{i\vec{k} \cdot (\vec{x} - \vec{y})} K_\nu(kz) K_\nu(kw) \frac{I_\nu(k\epsilon)}{K_\nu(k\epsilon)} .
\]  

(A8)

This solves the equation (A1) because

\[
(D_z - k^2) K_\nu(kz) = 0.
\]  

(A9)

Then, it is easy to verify that \(G^\epsilon_\Delta\) vanishes at the cut-off surface \(z = \epsilon\).
In order to define the modified bulk-to-boundary propagator $K_\Delta^\epsilon(\bar{x}, \bar{y})$, consider

$$\frac{\partial}{\partial w} G^\Delta(X, Y) \big|_{z = \epsilon, w = \epsilon} = (z\epsilon)^{d/2} \int \frac{d^dk}{(2\pi)^d} e^{ik \cdot (\bar{x} - \bar{y})} k \left[ K_\nu(\epsilon k) I'(\epsilon k) - K_\nu(\epsilon k) K'_\nu(\epsilon k) \frac{I_\nu(\epsilon k)}{I_\nu(\epsilon k)} \right]$$

$$= \epsilon^{d-1} \int \frac{d^dk}{(2\pi)^d} e^{ik \cdot (\bar{x} - \bar{y})} \frac{(z\epsilon)^{d/2} K_\nu(\epsilon k)}{(\epsilon k)^{d/2} K_\nu(\epsilon k)}$$

$$\equiv \epsilon^{d-1} \cdot \epsilon^{-\Delta} K_\Delta^\epsilon(X, \bar{y}),$$

(A10)

with $\Delta = \nu + d/2$. Here, we used the formula

$$I_\nu(z) K'_\nu(z) - I'_\nu(z) K_\nu(z) = -\frac{1}{z}.$$  

(A11)

It is found that

$$K_\Delta^\epsilon(X, \bar{y}) \big|_{z = \epsilon} = \epsilon^{d-\Delta} \delta^d(\bar{x} - \bar{y}).$$

(A12)

As $\epsilon \to 0$, $K_\Delta^\epsilon(X, \bar{y})$ reduces to

$$K_\Delta^\epsilon(X, \bar{y}) \to \frac{1}{\pi^{d/2}} \frac{\Gamma(\Delta)}{\Gamma(\Delta - d/2)} \left( \frac{z}{\epsilon^{d/2}} + (\bar{x} - \bar{y})^2 \right)^{\Delta}.$$  

(A13)

The following formulae are useful:

$$\frac{\partial}{\partial z} K_\Delta^\epsilon(X, \bar{y}) \big|_{z = \epsilon} = \epsilon^{d-\Delta} \int \frac{d^dk}{(2\pi)^d} e^{ik \cdot (\bar{x} - \bar{y})} k \partial_z \log(\epsilon^{d/2} K_\nu(z)) \big|_{z = \epsilon},$$

(A14)

$$\int d^d y K_\Delta^\epsilon(X, \bar{y}) = \epsilon^{d-\Delta}.$$  

(A15)

References

[1] A. B. Zamolodchikov, “Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory,” JETP Lett. 43, 730 (1986) [Pisma Zh. Eksp. Teor. Fiz. 43, 565 (1986)].

[2] N. Seiberg, “Observations on the Moduli Space of Superconformal Field Theories,” Nucl. Phys. B 303, 286 (1988). doi:10.1016/0550-3213(88)90183-6

[3] D. Kutasov, “Geometry on the Space of Conformal Field Theories and Contact Terms,” Phys. Lett. B 220, 153 (1989). doi:10.1016/0370-2693(89)90028-2

[4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large N field theories, string theory and gravity,” Phys. Rept. 323, 183 (2000) [hep-th/9905111].

[5] J. de Boer, E. P. Verlinde and H. L. Verlinde, “On the holographic renormalization group,” JHEP 0008, 003 (2000) doi:10.1088/1126-6708/2000/08/003 [hep-th/9912012].

[6] M. Fukuma, S. Matsuura and T. Sakai, “Holographic renormalization group,” Prog. Theor. Phys. 109, 489 (2003) doi:10.1143/PTP.109.489 [hep-th/0212314].

[7] M. R. Gaberdiel, A. Konchney and C. Schmidt-Colinet, “Conformal perturbation theory beyond the leading order,” J. Phys. A 42, 105402 (2009) doi:10.1088/1751-8113/42/10/105402 [arXiv:0811.3149 [hep-th]].

[8] Z. Komargodski and D. Simmons-Duffin, “The Random-Bond Ising Model in 2.01 and 3 Dimensions,” J. Phys. A 50, no. 15, 154001 (2017) doi:10.1088/1751-8121/aa6087 [arXiv:1603.04444 [hep-th]].

[9] V. Bashmakov, M. Bertolini and H. Raj, “On non-supersymmetric conformal manifolds: field theory and holography,” JHEP 1711, 167 (2017) doi:10.1007/JHEP11(2017)167 [arXiv:1709.01749 [hep-th]].

[10] C. Behan, “Conformal manifolds: ODEs from OPEs,” JHEP 1803, 127 (2018) doi:10.1007/JHEP03(2018)127 [arXiv:1709.03967 [hep-th]].

[11] K. Sen and Y. Tachikawa, “First-order conformal perturbation theory by marginal operators,” arXiv:1711.05947 [hep-th].

[12] H. Liu and A. A. Tseytlin, “On four point functions in the CFT / AdS correspondence,” Phys. Rev. D 59, 086002 (1999) doi:10.1103/PhysRevD.59.086002 [hep-th/9807097].

[13] E. D’Hoker, D. Z. Freedman, S. D. Mathur, A. Matusis and L. Rastelli, “Graviton exchange and complete four point functions in the AdS / CFT correspondence,” Nucl. Phys. B 562, 353 (1999) doi:10.1016/S0550-3213(99)00525-8 [hep-th/9903196].
[14] E. D'Hoker and D. Z. Freedman, “Supersymmetric gauge theories and the AdS / CFT correspondence,” hep-th/0201253.

[15] Y. Tachikawa, “Five-dimensional supergravity dual of a-maximization,” Nucl. Phys. B 733, 188 (2006) doi:10.1016/j.nuclphysb.2005.11.010 [hep-th/0507057].

[16] J. Louis, H. Triendi and M. Zagermann, “$\mathcal{N} = 4$ supersymmetric AdS$_5$ vacua and their moduli spaces,” JHEP 1510, 083 (2015) doi:10.1007/JHEP10(2015)083 [arXiv:1507.01623 [hep-th]].

[17] A. Ashmore, M. Gabella, M. Graña, M. Petrini and D. Waldram, “Exactly marginal deformations from exceptional generalised geometry,” JHEP 1701, 124 (2017) doi:10.1007/JHEP01(2017)124 [arXiv:1605.05730 [hep-th]].

[18] S. Lust, P. Ruter and J. Louis, “Maximally Supersymmetric AdS Solutions and their Moduli Spaces,” JHEP 1803, 019 (2018) doi:10.1007/JHEP03(2018)019 [arXiv:1711.06180 [hep-th]].

[19] D. Berenstein and A. Miller, “Conformal perturbation theory, dimensional regularization, and AdS/CFT correspondence,” Phys. Rev. D 90, no. 8, 086011 (2014) doi:10.1103/PhysRevD.90.086011 [arXiv:1406.4142 [hep-th]].

[20] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from noncritical string theory,” Phys. Lett. B 428, 105 (1998) doi:10.1016/S0370-2693(98)00377-3 [hep-th/9802109].

[21] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998) doi:10.4310/ATMP.1998.v2.n2.a2 [hep-th/9802150].

[22] H. Osborn, “Weyl consistency conditions and a local renormalization group equation for general renormalizable field theories,” Nucl. Phys. B 363, 486 (1991).

[23] W. Mueck and K. S. Viswanathan, “Conformal field theory correlators from classical scalar field theory on AdS(d+1),” Phys. Rev. D 58, 041901 (1998) doi:10.1103/PhysRevD.58.041901 [hep-th/9804035].

[24] E. D’Hoker, S. D. Mathur, A. Matusis and L. Rastelli, “The Operator product expansion of N=4 SYM and the 4 point functions of supergravity,” Nucl. Phys. B 589, 38 (2000) doi:10.1016/S0550-3213(00)00523-X [hep-th/9911222].