Gravity Probe B data analysis: I. Coordinate frames and analysis models

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Abstract
Gravity Probe B (GP-B) was a cryogenic, space-based experiment testing the geodetic and frame-dragging predictions of Einstein’s theory of general relativity (GR) by means of gyroscopes in Earth orbit. This first of three data analysis papers reviews the GR predictions and details the models that provide the framework for the relativity analysis. In the second paper we describe the flight data and their preprocessing. The third paper covers the algorithms and software tools that fit the preprocessed flight data to the models to give the experimental results published in Everitt et al (2011 Phys. Rev. Lett. 106 221101–4).

Keywords: general relativity, GP-B, data analysis

1. Introduction

In 1960 Schiff [2] showed that the spin axis direction of a free gyroscope orbiting the Earth would exhibit two drift rates due to general relativity (GR): (1) geodetic, due to the gyroscope’s motion through the curved spacetime about the massive Earth, (2) frame-dragging, due to the dragging of spacetime by the rotating Earth. Gravity Probe B (GP-B) was designed to measure these two effects. An overview in this issue by Everitt et al [3] gives the concept and implementation of the experiment, critical on-orbit gyroscope observations that motivate the data analysis, and a summary of results. The early history of the GP-B development can
be found in [4]; a detailed description of on-orbit operations is presented in [5]; the data analysis status at the end of 2008 is summarized in [6].

In this, the first of three data analysis papers, abbreviated DA I, we derive the required theoretical models supporting the experiment. The second part, DA II [7], discusses the experimental data and the preparation of these data for the final analysis. Estimation algorithms and their computational realization, as well as the resulting relativity estimates and associated uncertainties are found in the third part, DA III [8].

To perform the GP-B experiment, a gyroscope, located within a spacecraft, was placed in a circular polar orbit about the Earth. The two GR effects are perpendicular to one another in this orbit allowing them to be independently measured (figure 1). The spacecraft tracked a nearly fixed reference star (guide star, GS) with an on-board telescope, so that the gyroscope drift rate could be referenced to inertial space. A superconducting quantum interference device (SQUID) measured the drift using the gyroscope’s London moment (LM), a magnetic marker that is intrinsically aligned with the spin axis. The spacecraft rolled about the line-of-sight to the star, thereby modulating the drift data at roll frequency; measurement of roll phase allowed demodulation.

This paper has six further sections, as follows:

II. The GR predictions for the orbiting gyroscope are discussed and the reference frames needed for the data analysis are defined, along with the nearly fixed vectors of gyro spin, spacecraft roll, and their difference the spin-to-roll misalignment.

III. We introduce the time-varying influence of the rotor’s trapped magnetic flux on the LM signal. This influence is analyzed by using the SQUID signal after it has been lightly filtered to a \( \sim 1 \) kHz bandwidth; a signal henceforth referred to as the high frequency (HF) signal. The model for its analysis, called trapped flux mapping (TFM), is derived in appendix A. TFM allows the trapped flux (TF) signal to be converted from volts to arc-seconds; the conversion quantity is referred to as the scale factor. TFM, described in detail in DA III, also provides the evolving polhode, critical to analyzing scale factor variations and the torques discussed later in this paper (sections 4 and 5).

IV. The model for the SQUID signal, heavily filtered onboard, is derived in this section. This low frequency (LF) signal is described by the main measurement equation for the experiment. It contains a gyroscope scale factor term comprised of three components: the LM, the \( \sim 1\% \) contribution from TFM and a \( \sim 1\% \) adjustment from SQUID electronics variation.
Other most important submodels involved in the measurement equation are gyro spin motion caused by patch effect (PE) torques (section 5), and the model of telescope pointing (section 6). The measurement equation is mainly used for that part of the orbit when the telescope tracks the guide star (guide star valid or GSV); a generalization of this central equation is found in appendix B for guide star invalid (GSI) periods during which the Earth occults the line-of-sight to the star.

V. Contrary to the pre-launch expectations [9], disturbance torques had a significant impact on gyroscope drift rate. This section gives the exact solution to the equations of motion for the trajectory of the spin axis in the presence of torques due to patch potentials on the rotor and housing. In appendix C we derive these torques and examine implications.

VI. Here we consider telescope pointing and give the model for one phenomenon that is essential to the experiment, the orbital and annual aberration of starlight. This effect causes the telescope to point towards an apparent rather than true guide star position. The apparent position is precisely known so that aberration may be used to accurately calibrate the telescope and LM readouts. Next, we develop the model for this mispointing and give the scale factor matching method that removes its effect upon the relativity result.

VII. This section summarizes all of the GP-B analysis models, which in combination provide the foundation of the estimation algorithms and software that fit the flight data to give the GP-B relativity result.

2. Coordinate frames. Relativistic drift predictions. Spin-to-roll misalignment

Practically all the models we used to analyze the GP-B data are either expressed in terms of certain coordinate frames, or are derived using them. So it is natural to start with the definitions of those frames and the quantities, mostly vectors, that are needed for our analysis. We also discuss here the GR predictions for the gyro drift, to clarify exactly what GP-B was measuring.

The $2 \times 2$ matrix $R(\alpha)$ describes the rotation of the plane by the angle $\alpha$,

$$R(\alpha) = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}.$$ 

The $3 \times 3$ matrices

$$R_1(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{pmatrix}, \quad R_2(\alpha) = \begin{pmatrix} \cos \alpha & 0 & \sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & \cos \alpha \end{pmatrix},$$

$$R_3(\alpha) = \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

each containing the block $R(\alpha)$, describe the rotation by the angle $\alpha$ about the corresponding axis of a Cartesian frame. These notations are used in what follows.

2.1. Inertial frames

Let us write $x = x_1$, $y = x_2$, $z = x_3$ for the Cartesian coordinates of the inertial frame JE2000 [10], with the unit vectors $\hat{x} = \hat{x}_1$, $\hat{y} = \hat{x}_2$, $\hat{z} = \hat{x}_3$ along the corresponding axes. We need a Cartesian inertial frame more appropriate for the GP-B data reduction. It is natural to
set one axis in the direction to the guide star (GS) with unit vector $\hat{e}_{gs}$,

$$\hat{e}_{gs} = \cos \delta \cos \alpha \hat{x} + \cos \delta \sin \alpha \hat{y} + \sin \delta \hat{z};$$  \hspace{1cm} (1)

here $\delta$ is the declination and $\alpha$ is the right ascension of the GS (for IM Pegasi—used as the GS in the GP-B mission—$\alpha = 343.25949461^\circ$ and $\delta = 16.84126106^\circ$ [11]).

Now, the $z$ axis of the JE2000 frame was exactly parallel to the Earth rotation axis on noon GMT, 1 January 2000, and stayed within a few arc-seconds (as; 1 as $= 4.848 \times 10^{-6}$ rad) of it throughout the entire GP-B flight in 2004–2005. The ideal GP-B polar orbit would thus contain both $\hat{e}_{gs}$ and $\hat{z}$, hence a good choice for the second unit vector of the frame under construction is

$$\hat{e}_{we} \equiv \frac{\hat{e}_{gs} \times \hat{z}}{\left| \hat{e}_{gs} \times \hat{z} \right|} = \frac{\hat{e}_{gs} \times \hat{z}}{\sqrt{1 - \left( \hat{e}_{gs} \cdot \hat{z} \right)^2}}.$$  \hspace{1cm} (2)

The index WE stands for the West–East direction perpendicular to the ideal orbit plane, in which the gyroscope drifts due to the relativistic Lense–Thirring effect (see the next section). We define the third axis in the usual way (see figure 2):

$$\hat{e}_{ns} \equiv \hat{e}_{we} \times \hat{e}_{gs}.$$  \hspace{1cm} (3)

The unit vector $\hat{e}_{ns}$ lies in the ideal orbit plane and is orthogonal to $\hat{e}_{gs}$; as shown below, the geodetic relativistic drift goes in this NS direction. From the definitions (2) and (3) using expression (1) we find $\hat{e}_{in}$ and $\hat{e}_{we}$ in the basic frame JE2000:

---

**Figure 2.** Inertial frame for the GP-B data reduction. $\alpha$, $\delta$—right ascension and declination of the guide star; $\beta_0$—ecliptic declination from the equatorial plane.
\[
\begin{align*}
\dot{e}_x &= -\sin \delta \cos \alpha \dot{x} - \sin \delta \sin \alpha \dot{y} + \cos \delta \dot{z} ; \\
\dot{e}_y &= \sin \alpha \dot{y} - \cos \alpha \dot{\alpha} , \\
\dot{e}_z &= \sin \alpha \dot{\alpha} - \cos \alpha \dot{y} . \\
\end{align*}
\]

Inverse formulas to (1), (4) for the basic unit vectors through \{\hat{e}_g, \hat{e}_n, \hat{e}_{we}\} read:
\[
\begin{align*}
\dot{x} &= \sin \alpha \, \dot{e}_{we} + \cos \alpha \left( \cos \delta \, \dot{e}_g - \sin \delta \, \dot{e}_n \right) ; \\
\dot{y} &= -\cos \alpha \, \dot{e}_{we} + \sin \alpha \left( \cos \delta \, \dot{e}_g - \sin \delta \, \dot{e}_n \right) ; \\
\dot{z} &= \sin \delta \, \dot{e}_g + \cos \delta \, \dot{e}_n .
\end{align*}
\]

The GS–NS–WE right-handed Cartesian inertial frame with the unit vectors \{\hat{e}_g, \hat{e}_n, \hat{e}_{we}\} is the most used in the GP-B data analysis.

2.2. GR drift rate predictions and the gyro spin axis in inertial space

2.2.1. Geodetic and frame-dragging effects predicted by GR. In 1960, L I Schiff [2], using the linearized GR theory (weak field approximation, e.g. [12], ch 18) showed that an ideal gyroscope in orbit around the Earth would undergo a relativistic precession according to the equations
\[
\begin{align*}
\frac{d\vec{s}}{dt} &= \vec{\Omega} \times \vec{s} + \text{non-precession terms} ; \\
\vec{\Omega} &= \vec{\Omega}_g + \vec{\Omega}_{fd} , \quad \vec{\Omega}_g = \frac{3GM}{2c^2r^3} \left( \vec{r} \times \vec{v} \right) , \quad \vec{\Omega}_{fd} = \frac{GI}{c^2r^3} \left[ \frac{3F}{r^2} \left( \vec{\omega}_e \cdot \vec{r} \right) - \vec{\omega}_e \right] .
\end{align*}
\]

Here \( M \) is the Earth’s mass, \( I \) is Earth’s moment of inertia about its rotation axis, \( \vec{\omega}_e \) is the Earth angular velocity vector (whose direction was close to \( \vec{z} \) during the mission), \( G \) is the gravitational constant, and \( \vec{r} \) and \( \vec{v} \) are the S/C orbital radius and velocity, \( r = |\vec{r}| \). Notation \( \vec{\Omega}_g \) stands for the geodetic (or gravito-electric, or de Sitter [13]) precession caused by the gyro motion in the spacetime curved by the massive Earth, and \( \vec{\Omega}_{fd} \) is the frame-dragging (or gravito-magnetic, or Lense–Thirring [14]) precession caused by the Earth’s rotation.

The non-precession terms in the first equation (6) average out over a closed free-fall orbit [2, 19], and the change in \( \vec{s} \) is entirely negligible over such short periods of time; thus the orbit averaged dynamics becomes purely precessional:
\[
\frac{d\vec{s}}{dt} = (\vec{\Omega})_{orb} \times \vec{s} .
\]

Since \( \vec{\Omega}_g \) is proportional to the orbital angular momentum, both the instant vector and its orbit average are perpendicular to the orbit plane. The average magnitude of the precession is \( 3GMv/2c^2r^2 \) \((v = |\vec{v}|) \) for a perfectly circular orbit by Schiff’s formula (6). If the orbit is also precisely polar, and the GS is exactly in the orbit plane, then the geodetic precession is totally in the WE direction defined by (2). For a circular polar orbit, with \( \vec{\omega}_e = \omega_e \, \vec{z} \), the \( x \)- and \( y \)-components of the Lense–Thirring precession \( \vec{\Omega}_{fd} \) average to zero, leaving just the \( z \)-component equal to \( GIo_\omega /2c^2r^3 \), so the WE precession due to frame-dragging vanishes for a perfect orbit. Thus the proper polar orbit provides the best spatial separation of the two relativistic effects: they are orthogonal in this case. The geodetic drift is in the NS direction, and the frame-dragging drift in the WE one. In reality the GP-B orbit differed but slightly from the ideal one: the GS was off the orbit plane by \( \sim 2 \times 10^{-4} \) on average, and the Earth’s rotation axis deviated from the \( z \) axis by less than \( 10^{-4} \). So the frame-dragging effect contribution to the NS drift rate \((\lesssim 0.04 \text{ mas/yr})\) and geodetic effect contribution to the WE drift \((\lesssim 1 \text{ mas/yr})\) were both negligible.
Formulas (6) are rigorously valid for the spherically symmetric gravitational field of a rotating mass \( M \), which is not exactly the case with the Earth. The effect of the Earth quadrupole moment, \( J_2 \approx 1.083 \times 10^{-3} \), on the geodetic precession was found independently by D C Wilkins and R F O’Connel in 1968 (see papers [15] and [16]). The correction to the orbit averaged geodetic precession due to the Earth oblateness and small orbit eccentricity was calculated in [15, 17] using rather elaborate analytical techniques; its simple derivation was presented later by J V Breakwell [18].

The general treatment of both the geodetic and frame-dragging precessions in the gravitational field that contains all the multipoles was given in [19]. In particular, it was shown that the desired accuracy \( \sim 1 \) mas/yr for measuring the geodetic effect requires including the Earth quadrupole moment only; all other terms are two to four orders of magnitude smaller [20]. The proper expressions from [19] based on Earth’s gravitational potential (\( r_e \) is the Earth equatorial radius),

\[
\Phi = - \frac{GM}{r} \left[ 1 - J_2 \frac{r_2}{2r^2} \left( \frac{3z^2}{r^2} - 1 \right) \right],
\]

were used to calculate the instantaneous values of \( \hat{\Omega}_g \) and \( \hat{\Omega}_{fd} \) (in fact, the \( J_2 \) correction to the frame-dragging was negligible). These were computed for every orbit from the precise GPS orbit data [21] using each available valid data point. The averages found differed slightly from orbit to orbit due its small classical variation, so \( \langle \Omega_{we} \rangle \) and \( \langle \Omega_{ns} \rangle \) were additionally averaged over all orbits, and relativistic drift rate predictions were determined from the result given by the formulas (11). These resulting numbers for \( r_{we} \) (\( \sim 6, 606 \) mas/yr) and \( r_{ns} \) (\( \sim 39 \) mas/yr) are shown in figure 1 (precise definitions of these quantities are found in section 2.2.3); note that the contribution of the geodetic precession in the WE drift rate, due to the orbit imperfection, was not completely insignificant, \( \sim 1 \) mas/yr. As an important cross-check, the averages of the non-precession terms in the first of the equations (6) were also computed and found to be small as expected: \( 0.1 \) mas/yr at most, and typically much less.

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2.2.2. Gyro spin axis in inertial space. Starting here and throughout all the three papers on data analysis, we use the term ‘(gyro) spin axis’ in place of ‘instant (gyro) rotation axis’, for brevity and out of the GP-B tradition. As explained below, for GP-B this axis practically coincides with the axis of the angular momentum (usually called ‘spin axis’ in physics, and meant in the Schiff’s equations (6)), so we will not make a difference between the two.

After the on-orbit spin-up of all the GP-B gyroscopes [1, 3, 5], their spin axes were torqued to point within less than \( 10^{-2} \) as of the direction to the GS. During the mission the axes remained within less than \( 20 \) as \( \approx 10^{-4} \) of the GS, with a few arc-seconds average over the whole mission. Consequently, we have (writing \( \hat{s} \) for the unit vector of a gyro spin axis looking towards the GS):

\[
\hat{s}(t) = s_{ps} \hat{e}_{ps} + s_{ns}(t) \hat{e}_{ns} + s_{we}(t) \hat{e}_{we}, \quad s_{gs} = \sqrt{1 - \left[ s^2_{ps}(t) + s^2_{we}(t) \right]},
\]

where the transverse components \( |s_{ns}(t)|, |s_{we}(t)| < 10^{-4} \) throughout the whole mission. So, with more than sufficient accuracy (\( 10^{-8} \) or better),
\[
\hat{s}(t) = \hat{e}_{gs} + \hat{s}'(t), \quad \hat{s}'(t) = s_{ns}(t)\hat{e}_{ns} + s_{we}(t)\hat{e}_{we} + O\left(s_{ns}^2 + s_{we}^2\right)\hat{e}_{gs},
\]

\[
s = |\hat{s}| = \sqrt{s_{ns}^2 + s_{we}^2} \left[ 1 + O\left(\sqrt{s_{ns}^2 + s_{we}^2}\right)\right] \sim 10^{-4}, \quad s = O\left(\sqrt{s_{ns}^2 + s_{we}^2}\right)
\]

(note \(\sqrt{s_{ns}^2 + s_{we}^2} \leq s\), so the opposite to the above estimate, \(\sqrt{s_{ns}^2 + s_{we}^2} = O(s)\), is trivially correct).

The vector \(\hat{s}(t)\) to lowest order (l.o.) is two-dimensional and lies in the plane perpendicular to the GS direction. It describes the drift of the gyroscope, i.e., its change carries the information on the relativistic drift rates; thus we can call \(\hat{s}\) the drift vector.

More precisely, GP-B had to trace the change in the vector of angular momentum, rather than in the spin vector; however, the difference proves to be insignificant, as mentioned in the beginning of this section. Indeed, in inertial space the instantaneous rotation axis of a free rotor precesses about its angular momentum vector, and the sine of the angle between them is proportional to the relative difference of the corresponding moments of inertia (e.g. [22]). For the GP-B gyroscopes \(\Delta t / t \sim 10^{-6}\) [3, 23, 24] for any two body axes, so the angle between the spin and angular momentum is of the same order: very tiny. In addition, \(\hat{s}\) averaged over each spin period is exactly aligned with the angular momentum. Since the sampling rates of all the GP-B LF science signals (0.5Hz) are much smaller than the spin speed (~60 or 80 Hz), the data are effectively spin-averaged, so we can use \(\hat{s}(t)\) as the unit vector of angular momentum as well. Gyroscopes #1 and #3 were rotating counterclockwise, and #2 and #4 clockwise around the \(\hat{s}(t)\) direction.

2.2.3. GR drift rate predictions for GP-B. The needed NS and WE projections of the precession equation (13) are (the dot stands for the derivative d/dt; we drop the subscript ‘orb’ in the averaging notation):

\[
\begin{align*}
\dot{s}_{ns} &= \langle \Omega_{we} \rangle s_{gs} - \langle \Omega_{gs} \rangle s_{we}, \\
\dot{s}_{we} &= -\langle \Omega_{ns} \rangle s_{gs} + \langle \Omega_{gs} \rangle s_{ns}; \\
\Omega_{\nu} &= \hat{\Omega} \cdot \hat{e}_{\nu}, \quad \nu = gs, ns, we.
\end{align*}
\]

With the numbers given above it is straightforward to estimate that \(\langle \Omega_{gs} \rangle \approx 0.2\langle \Omega_{we} \rangle \approx 0.001\langle \Omega_{ns} \rangle\). Also \(s_{gs} \approx 1\) with accuracy \(10^{-8}\) or better, and \(s_{ns}, s_{we} \approx 0\) with accuracy of at least \(10^{-4}\) (section 2.1). Therefore we can neglect the second terms in the rhs of the above equations and write them, accurately enough for GP-B, as

\[
\begin{align*}
\dot{s}_{ns} &= \langle \Omega_{we} \rangle, \\
\dot{s}_{we} &= -\langle \Omega_{ns} \rangle;
\end{align*}
\]

introducing

\[
\begin{align*}
r_{ns} &= \langle \Omega_{we} \rangle, \\
r_{we} &= -\langle \Omega_{ns} \rangle,
\end{align*}
\]

we obtain the spin dynamics equations in their final form:

\[
\dot{s}_{ns} = r_{ns}, \quad \dot{s}_{we} = r_{we}.
\]

The estimates of relativistic drift rates \(r_{ns}, r_{we}\) are the main result of the GP-B experiment.

By the equations (12), purely relativistic motion of the gyroscopes is uniform,

\[
\begin{align*}
s_{ns}(t) &= s_{ns}^0 + r_{ns}(t - t_0), \\
s_{we}(t) &= s_{we}^0 + r_{we}(t - t_0),
\end{align*}
\]

with \(s_{ns}^0, s_{we}^0\) being the initial spin axis position at \(t = t_0\). However, if non-negligible classical torques are present, their contribution must be added to the rhs of equations (12), and it appears then in the gyro trajectory \(\hat{s}(t)\) as well. As stated in the introduction, this is exactly what happened with the GP-B gyroscopes because of the PE torques. The corresponding equations of motion and their solution are given in section 5.2. Unlike the expressions (13), it
is nonlinear in time, and also contains relativistic drift rates (11) to be estimated from GP-B data.

2.3. Gyro spin in the frame fixed in the rotor’s body

We write \( I_1, I_2, I_3 \) for the gyro principal moments of inertia,

\[
0 < I_1 < I_2 < I_3. \tag{14}
\]

Although the GP-B flight rotors are the best spheres ever made, with \((I_1 - I_j)/I_k \sim 10^{-6}\), the value of the inertial asymmetry parameter, defined as

\[
0 \leq Q = \frac{I_2 - I_1}{I_3 - I_1} \leq 1, \tag{15}
\]

can be anywhere between zero and one. The case \( Q = 0 \) corresponds to a rotor symmetric about the maximum inertia axis, \( I_1 = I_2 \); \( Q = 1 \) describes a top symmetric about the minimum inertia axis, \( I_1 = I_2 \). Three GP-B gyroscopes were within the range \( Q = 0.12 - 0.20 \), and the fourth had \( Q \approx 0.3 \) ([24], table 3).

The principal axes of inertia with unit vectors \( \hat{I}_1, \hat{I}_2, \hat{I}_3 \) form a Cartesian rotor-fixed frame, in which the direction of the instant spin axis, \( \vec{\omega}_s = \omega_\parallel \hat{s} \), is specified by the two angles, \( \gamma_p = \gamma_p(t) \) and \( \phi_p = \phi_p(t) \) (figure 3) as

\[
\hat{s} = \sin \gamma_p \left( \cos \phi_p \hat{I}_1 + \sin \phi_p \hat{I}_2 \right) + \cos \gamma_p \hat{I}_3. \tag{16}
\]

We call \( \gamma_p \) and \( \phi_p \) the polhode angle and phase; given \( \omega_\parallel \), the free motion of a gyroscope is described by these two quantities (see appendix A). These two angles play a very significant role in the whole data analysis; if \( Q \neq 0 \), both change with the time and depend on the value of the asymmetry parameter.
2.4. Spacecraft roll axis and quasi-inertial frame related to it. Spin-to-roll misalignment

2.4.1. Roll axis and the frame related to it. Another very important axis in the GP-B experiment set-up is that of roll, the axis about which the spacecraft rotated uniformly. Its unit vector \( \hat{\tau} \) is

\[
\hat{\tau}(t) = \tau_{gs}(t) \hat{e}_{gs} + \tau_{ns}(t) \hat{e}_{ns} + \tau_{we}(t) \hat{e}_{we}, \quad \tau_{gs}(t) = \sqrt{1 - \left[ \tau_{ns}^2(t) + \tau_{we}^2(t) \right]},
\]

similar to \( \hat{s} \) from (12). We can also write

\[
\hat{\tau} = \left[ 1 - 0.5 \left( \tau_{ns}^2 + \tau_{we}^2 \right) \right] \hat{e}_{gs} + \tau_{ns} \hat{e}_{ns} + \tau_{we} \hat{e}_{we} + O \left( \left( \tau_{ns}^2 + \tau_{we}^2 \right)^2 \right) \hat{e}_{gs},
\]

since the projections \( \tau_{ns}, \tau_{we} \) are small. Indeed, during GSV the roll axis nearly coincided with the S/C axis, i.e., the boresight axis of the telescope, the pointing sensor. The S/C axis, and hence also the roll axis, deviates from \( \hat{e}_{gs} \) mainly by the amount of the optical aberration of stralight (see section 6.1), within \( \sim 20 \text{ as} \approx 10^{-4} \). During GSI the roll axis deviated from the S/C axis by at most \( \sim 1 \text{ as} \approx 1.25 \times 10^{-4} \), and typically smaller; hence \( |\tau_{ns}|, |\tau_{we}| \sim 10^{-4} \) at any time, similar to \( s_{ns}, s_{we} \).

We use unit vectors \( \hat{x}, \hat{y}, \hat{z} \) to form a right-handed Cartesian basis, \( \{ \hat{x}, \hat{y}, \hat{z} \} \), together with \( \hat{\tau} \). For convenience, we pick the first vector perpendicular to \( \hat{e}_{gs} \),

\[
\hat{x} = \tau_{gs} \hat{e}_{gs} + \tau_{ns} \hat{e}_{ns} + \tau_{we} \hat{e}_{we},
\]

Then from \( \hat{x} \cdot \hat{x} = 0 \) and \( \hat{x} \cdot \hat{e}_{ns} = 1 \) we obtain, to the second order in \( \tau_{ns}, \tau_{we} \):

\[
\hat{\tau} = \hat{e}_{gs}, \quad \hat{\tau} \approx \hat{e}_{gs}, \quad \hat{\tau} \approx -\hat{e}_{ns}, \quad \hat{\tau} \approx -\hat{e}_{we}.
\]

We call \( \{ \hat{x}, \hat{y}, \hat{z} \} \) the roll axis frame; since it wanders in inertial space no more than \( 10^{-4} \), it is called quasi-inertial. This is clearly seen also from the zeroth order expressions of its unit vectors,

\[
\hat{\tau} \approx \hat{e}_{gs}, \quad \hat{\tau} \approx -\hat{e}_{ns}, \quad \hat{\tau} \approx -\hat{e}_{we},
\]

rendering them to the GS inertial frame of section 2.1. The roll axis frame is used to derive the main measurement equation for the SQUID signal (section 4.5 and appendix B). When deriving it, we are interested in exactly those small deviations, to establish the linearity of the GP-B readout. For this reason, we carefully kept quadratic terms in all the formulas (18) and (19).

2.4.2. Spin-to-roll misalignment. If, instead of (18), one writes \( \hat{\tau} \) to the linear order, then it looks very similar to the gyro spin vector \( \hat{s} \) from (9):
Like the vector $\vec{s}$, the vector $\vec{\tau}$ is two-dimensional to l.o.; it is called the pointing vector, and gives the deviation of the roll axis from the true direction to the GS.

We now introduce the vector of the spin-to-roll axis misalignment, $\vec{\mu}$, as the difference between $\hat{\tau}$ and $\hat{s}$ (figure 4). Using formulas (9) and (20), we find

$$\tilde{\mu}(t) = \vec{\tau}(t) - \vec{s}(t) = \vec{\tau}(t) - \vec{s}(t)
= \left[ (\tau_{ns}(t) - s_{ns}) \hat{e}_{ns} + (\tau_{we}(t) - s_{we}) \hat{e}_{we} + O\left( \tau^2 \right) \hat{e}_{gs} \right].$$

(recall that, up to higher order terms, $s_{ns}^2 = \tau_{ns}^2 + \tau_{we}^2$, $\tau_{ns}^2 = \tau_{ns}^2 + \tau_{we}^2$). Since the transverse components are all small, the $\vec{\mu}$ is yet another small vector, $\mu \lesssim 10^{-4}$:

$$\mu^2 = |\vec{\mu}|^2 = (\tau_{ns} - s_{ns})^2 + (\tau_{we} - s_{we})^2 + O\left( \tau^2 + \tau^2 \right)^2$$

$$\lesssim 2\left( s^2 + \tau^2 \right) + O\left( \left( s^2 + \tau^2 \right)^2 \right) = O\left( s^2 + \tau^2 \right) \lesssim 10^{-8}.$$  

We will sometimes use the polar representation of $\vec{\mu}$,

$$\mu_{ns} = \tau_{ns} - s_{ns} = \mu \cos \theta, \quad \mu_{we} = \tau_{we} - s_{we} = \mu \sin \theta,$$

where $\mu$ is taken to l.o., $\mu = \sqrt{\mu_{ns}^2 + \mu_{we}^2}$. The angle $\theta$ called the misalignment phase is counted from the NS axis in the NS–WE plane.
To gain more insight in the meaning of $\vec{\mu}$, let $\psi = \psi(t)$ be the angle between the spin and roll, and $\vec{\psi}$ the unit vector of the corresponding rotation axis,

$$\vec{\psi} = \frac{\dot{\vec{t}} \times \dot{\vec{s}}}{|\dot{\vec{t}} \times \dot{\vec{s}}|}.$$ 

Using $\dot{s}$ and $\dot{t}$ first by equations (8), (17), and then by equations (9) and (20), one calculates:

$$\dot{\vec{t}} \times \dot{\vec{s}} = \left(\tau_{ms} s_{we} - \tau_{we} s_{ms}\right) \hat{e}_{gs} + \left(\tau_{we} s_{gs} - \tau_{gs} s_{we}\right) \hat{e}_{ns} + \left(\tau_{gs} s_{ms} - \tau_{ms} s_{gs}\right) \hat{e}_{ws}$$

$$= \left(\tau_{we} - s_{we}\right) \hat{e}_{ns} + \left(s_{ns} - \tau_{ns}\right) \hat{e}_{we} + 0.5 \left[\mu_{ns} \left(s_{we} + \tau_{we}\right) - \mu_{we} \left(s_{ns} + \tau_{ns}\right)\right] \hat{e}_{gs}$$

$$+ O\left(\mu \left(s^2 + \tau^2\right)\right) \left(\hat{e}_{ns} + \hat{e}_{we}\right)$$

$$= \mu_{we} \hat{e}_{ns} - \mu_{ns} \hat{e}_{we} + O\left(\mu \sqrt{s^2 + \tau^2}\right) = \vec{\mu} \times \hat{e}_{gs} + O\left(\mu \sqrt{s^2 + \tau^2}\right).$$

(24)

The above expressions imply

$$\sin \psi = |\dot{\vec{t}} \times \dot{\vec{s}}| = \sqrt{\mu_{ns}^2 + \mu_{we}^2 + O\left(\mu^2 \left(s^2 + \tau^2\right)\right)} = \mu \left[1 + O\left(\left(s^2 + \tau^2\right)\right)\right].$$

(25)

and, as $\psi \sim \mu \lesssim 10^{-4}$ is also small,

$$\psi = \sin \psi + O\left(\psi^3\right) = \mu \left[1 + O\left(\left(s^2 + \tau^2\right)\right)\right] + O\left(\mu^3\right).$$

(26)

This equality is rather obvious: the arc ($\psi$) between two close vectors of the same length ($|\vec{s}| = |\vec{t}| = 1$) is equal, to l.o., to the length ($\mu$) of the chord connecting the ends of these vectors (see figure 4). This clarifies the meaning of $\mu$: to l.o. it is the angle between the spin and roll axes.

Results (24) and (26) allow us to find the l.o. approximation for the $\vec{\psi}$:

$$\vec{\psi} = \frac{\dot{\vec{t}} \times \dot{\vec{s}}}{|\dot{\vec{t}} \times \dot{\vec{s}}|} = \frac{1}{\mu} \left(\mu_{we} \hat{e}_{ns} - \mu_{ns} \hat{e}_{we}\right) + O\left(\left(s^2 + \tau^2\right)\right)$$

$$= \sin \theta \hat{e}_{ns} - \cos \theta \hat{e}_{we} + O\left(\left(s^2 + \tau^2\right)\right).$$

(27)

The last expression using representation (23) remains finite and meaningful even when $\mu \to 0$; the corresponding limit value of $\theta$ is also well defined, since $\vec{\mu}$ in the limit position is perpendicular to the aligned vectors $\vec{s}$ and $\vec{t}$.

2.5. Frame fixed in the S/C body and the pick-up loop orientation

Here we define the Cartesian frame $\{\hat{\xi}_b, \hat{\eta}_b, \hat{z}_b\}$ attached to the spacecraft body. We choose the unit vector $\hat{z}_b$ along the S/C axis, i.e., the boresight axis of the science telescope. We can take any other two orthogonal body-fixed unit vectors $\hat{\xi}_b$ and $\hat{\eta}_b$ such that the three form a right-handed Cartesian frame. It is convenient to take the vector $\hat{\xi}_b$ in the plane formed by $\hat{z}_b$ and the normal, $\hat{n}_i$, to the SQUID pick-up loop. Since the frame we are building is only an intermediate, it might be different for each GP-B gyroscope; the GP-B instrument set-up allows, however, for one common frame for gyroscopes #1 and #2, and another one for gyroscopes #3 and #4.

Due to unavoidable assembly imperfections, the pick-up loop plane is at a constant angle to the axis $\hat{z}_b$. 

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\[ \alpha = \text{const} < 55 \text{ as } \approx 2.75 \times 10^{-4}. \] (28)

According to the frame choice, the body–fixed pick-up loop normal is thus
\[ \hat{n} = \cos \alpha \hat{x}_b + \sin \alpha \hat{z}_b = \left( 1 - 0.5 \alpha^2 \right) \hat{x}_b + \alpha \hat{z}_b + O(\alpha^3). \] (29)

We want now to convert this to the inertial frame for any GSV period. During GSV, the S/C roll axis coincides with the telescope axis,
\[ \hat{z}_b = \hat{\tau}. \] (30)

The body-fixed vectors \( \hat{x}_b \) and \( \hat{y}_b \) rotate about \( \hat{z} = \hat{\tau} \) of the roll-axis frame in the previous section. Hence they are expressed through the vectors \( \hat{x}^r \), equation (19), and \( \hat{y}^r \), equation (19), by the rotation \( R_3(\phi_r + \delta_r) \) with the satellite roll phase, \( \phi_r \):
\[
\hat{x}_b(t) = \cos(\phi_r + \delta_r) \hat{x}^r + \sin(\phi_r + \delta_r) \hat{y}^r
= \left[ \tau_{ns} \cos(\phi_r + \delta_r) + \tau_{we} \sin(\phi_r + \delta_r) \right] \hat{e}_{gs}
+ \left[ -\left( 1 - 0.5 \tau_{ns}^2 \right) \cos(\phi_r + \delta_r) + \tau_{we} \tau_{ns} \sin(\phi_r + \delta_r) \right] \hat{e}_{ns}
- \left( 1 - 0.5 \tau_{we}^2 \right) \sin(\phi_r + \delta_r) \hat{e}_{we} + O\left( \tau^3 \right); \] (31)
\[
\hat{y}_b(t) = -\sin(\phi_r + \delta_r) \hat{x}^r + \cos(\phi_r + \delta_r) \hat{y}^r
= \left[ -\tau_{ns} \sin(\phi_r + \delta_r) + \tau_{we} \cos(\phi_r + \delta_r) \right] \hat{e}_{gs}
+ \left[ \left( 1 - 0.5 \tau_{ns}^2 \right) \sin(\phi_r + \delta_r) + \tau_{we} \tau_{ns} \cos(\phi_r + \delta_r) \right] \hat{e}_{ns}
- \left( 1 - 0.5 \tau_{we}^2 \right) \cos(\phi_r + \delta_r) \hat{e}_{we} + O\left( \tau^3 \right). \] (32)

These formulas are obtained by substituting expressions \( \hat{x}^r \) and \( \hat{y}^r \) from (19), with quadratic terms in them. We wrote \( \delta_r = \text{const} \) for the roll phase offset, i.e., the angle between \( \hat{x}^r \) and \( \hat{x}_b \) at a moment when \( \phi_r = 0 \) (mod \( 2\pi \)).

We can now introduce expressions (31) and (32) into equation (29) to obtain \( \hat{n} \) in the inertially fixed frame during GSV. After carefully gathering all similar terms, \( \hat{n} \) acquires a relatively compact form:
\[
\hat{n} = \left[ \left( 1 - 0.5 \alpha^2 - 0.5 \tau_{ns}^2 \right) \cos(\phi_r + \delta_r) + \tau_{we} \tau_{ns} \sin(\phi_r + \delta_r) + \alpha \tau_{ns} \right] \hat{e}_{ns}
+ \left[ \tau_{ns} \cos(\phi_r + \delta_r) + \tau_{we} \sin(\phi_r + \delta_r) + \alpha \tau_{we} \right] \hat{e}_{we}
+ \left[ \tau_{ns} \cos(\phi_r + \delta_r) + \tau_{we} \sin(\phi_r + \delta_r) + \alpha \right] \hat{e}_{gs} + \text{cubic terms in } \tau \text{ and } \alpha. \] (33)

This formula is the key to the main measurement equation, that is, the model of the science SQUID signal during GSV (section 4.5). The inertial orientation of the pick-up loop during GSI, which is less important, is derived in appendix B.

**3. HF SQUID signal: changing polhode and TFM**

As we pointed out in the Introduction, the GP-B SQUID signal came through two channels, at LF and HF. The signal in the LF channel, originating mostly in the LM, contained the relativity information and was therefore in the center of the GP-B data analysis. The HF signal, due to magnetic field trapped in the rotor, was originally considered to be auxiliary
data allowing us to determine the gyro spin and polhode frequencies and the spin–down rate. The role of this HF signal for the main science analysis was significantly re-evaluated on-orbit, after finding the changing polhode period, and especially after discovering the anomalously large PE torques whose imprint on the gyro motion strongly depends on polhoding (see section 5). We developed and implemented a special procedure to analyze the HF signal called TFM; it is described in detail in our paper [24] and briefly below. In appendix A we give the complete derivation of all the model equations for TFM for the first time. The ultimate goal of TFM was to estimate accurate polhode parameters and LF SQUID scale factor variations throughout the mission, which were critically important for the science analysis.

3.1. Origin and structure of HF SQUID signal

The dipole LM field is the main but not the only source of the GP-B magnetic readout. When a thin film superconductor is cooled below the critical temperature, residual magnetic field, which existed even under the unique low–field GP-B conditions [3], is trapped inside the rotor. Magnetic quanta $\pm \theta_0$ comprise point sources on the rotor surface (fluxons). The multipole field of these point sources contributes to the total flux through the SQUID pick-up loop, which rolls with the S/C in inertial space; the roll period was $T_r = 77.5\, s$.

The time signature of the TF data emerges from the following argument. Fluxons are frozen in the rotor surface and spin with it; the function that converts a fluxon’s position into its magnetic flux through the pick-up loop is strongly nonlinear (almost a step-function, due to the relatively small gap between the rotor and the loop [25]). Therefore the TF signal consists of multiple harmonics of spin, and, since the pick-up loop is rolling, they are actually the harmonics of $\pm$ spin + roll frequency (recall that two gyroscopes were rotating counterclockwise, and the other two clockwise). Finally, since the spin axis moves in the rotor body (polhoding), the spin harmonics are modulated by the polhode frequency. All this combined leads to the following expression for TF through the pick-up loop (this formula is derived in detail in appendix A resulting in (A.18)–(A.20)):

$$\Phi_{TF}(t) = \sum_{n} H_n(t)e^{in(\phi + \phi_0)}$$

$$= \sum_{|n| \neq \text{odd}} H_n(t)e^{in(\phi + \phi_0)} + \beta(t) \sum_{|n| = \text{even}} h_n(t)e^{in(\phi + \phi_0)}, \quad (34)$$

where $H_n(t) = H_n(\phi(t), \gamma_p(t))$, $h_n(t) = h_n(\phi(t), \gamma_p(t))$ (by section 2.2, $\phi > 0$ for gyroscopes #1 and #3, and $\phi < 0$ for #2 and #4). Here $\beta(t)$ is the angle between the gyro spin axis and the pick–up loop plane shown to be $\lesssim 10^{-4}$ in section 4.3 and appendix B. The even harmonics of spin are multiplied by this small angle for a purely geometric reason: the spin axis resided almost in the pick-up loop plane in the GP-B experiment. Note the $n = 0$ contribution of TF to the LF signal:

$$\Phi_{TF}(t)|_{LF} = C_{TF}(t)\beta(t), \quad C_{TF}(t) \equiv h_0(t) ; \quad (35)$$

we will return to it in section 4.

3.2. Sources of the GP-B HF data

During the GP-B flight there were two telemetry sources for the HF SQUID signal: (1) the first six harmonics of spin obtained by the FFT, and (2) the so called snapshots, which were $\sim 2\, s$ stretches of original SQUID signal sampled at 2200 Hz. Both were received only during
the part of the orbit when the GS was occulted by the Earth, due to the limited capacity of the on-board CPU. When available, a snapshot would come about every 40 s, but gaps between the snapshot arrays might be up to 2 d.

HF FFT harmonics were analyzed during the mission as soon as the data were available. Snapshots, being the most accurate source of HF information, were thoroughly treated after the mission. All 976,478 snapshots from the mission science period were processed after the flight.

3.3. Changing polhode period and path: kinetic energy dissipation

As mentioned, one of the two on–orbit discoveries most strongly affecting GP-B data analysis was the changing polhode period. The measured time history of the polhode period for each of the GP-B gyroscopes is shown in figure 5, with very good agreement of the results obtained using the two HF signals, FFT harmonics (red) and HF snapshots (blue). Notably, the \( T_p(t) \) plot for gyro 1, 2 has a peak where formally \( T_p = \infty \). Nevertheless, throughout the science mission (started 13 September 2004) polhode periods of all the four gyros decreased monotonically tending to specific asymptotic values \( T_{pa} \).

Such behavior contradicts the Euler solution for free gyro motion, which gives constant polhode frequency and period. It can only be explained by energy dissipation that reduces the kinetic energy, \( E \), of the rotor without changing its angular momentum, \( L \). Dissipation moves the spin axis in the body to the only stable position, the maximum inertia axis, at which energy is minimum when \( L = \text{const} \). So dissipation also changes the polhode path controlled by the parameter \( L^2/2E \); in contrast, the spin–down torque changes both quantities \( L \) and \( E \) keeping this ratio constant, and the polhode path unchanged.

The two types of behavior seen in the plots of figure 5 are also explained: by pure chance, gyroscopes 1 and 2 start their evolution with the spin vector precessing about \( \hat{I}_1 \) (see section 2.3 for notations), so \( T_p \) tends to infinity when \( \vec{\omega}_s \) crosses the separatrix [22, 26]. Contrary to this, the evolution of gyroscopes 3 and 4 starts with \( \vec{\omega}_s \) already precessing about \( \hat{I}_3 \), so \( T_p(t) \) just decreases monotonically.

From angular momentum conservation one can readily determine the relative energy loss and spin speed reduction when the spin axis moves in the rotor body all the way from the minimum, \( \hat{I}_1 \), to the maximum, \( \hat{I}_3 \), inertia axis. They both are equal to the same ratio \( (I_3 - I_1)/I_1 < 4 \times 10^{-6} \) for all the GP-B gyroscopes. The gyro kinetic energy is \(~1\ J\), so the total energy loss when \( \omega_s \) moves from \( \hat{I}_1 \) to \( \hat{I}_3 \) is less than \(~4 \mu J\); in one year, the average dissipated power required for this is just \( 10^{-13} \) W!
The physical origin of this energy dissipation is not completely clear. It might be related to inelastic deformations of the rotor, but probably is dominated by dissipative PE torques [27, 28]. Independent of the origin, we have found a general dissipation model in the form of an additional term in the Euler equations unique up to a scalar factor (its brief derivation can be found in [29]). Fitting the polhode period time history obtained from the model to the measurements allowed us to determine the rotor asymmetry parameter, the asymptotic period \( T_{pa} \sim 1 - 2 \) hours, and the characteristic time of dissipation \( \tau_{dis} \sim 1 - 2 \) months, depending on the gyroscope (see table 2 in [24] for details). The gyroscopes started the science period with \( T_p \) ranging between 1.5 and 8 hours; thus the dissipation is slow \( (T_p \ll \tau_{dis}) \), so that the polhode motion of the GP-B gyros is quasi-adiabatic.

3.4. TFM and its products

TFM is a procedure for finding the distribution of trapped magnetic field and characteristics of gyro motion from odd harmonics of the HF SQUID signal by fitting them to their theoretical model. It is based on the solution for the scalar magnetostatic potential outside the spherical rotor with fluxons on its surface in the rotor-fixed frame. The complete derivation of formula (34) and the harmonic amplitudes \( H_n(t) \) and \( h_n(t) \) involved in it, through the coefficients \( A_{lm} \) of the spherical harmonics expansion of body-fixed potential distribution, is given in appendix A. The corresponding formulas (A.18)–(A.20) constitute the main model of TFM, which is presented in our paper [24]. Here we briefly review some points of TFM relevant to science analysis.

To perform TFM, one first of all needs to process the measured HF SQUID signal, \( z_{HF}(t) \propto \Phi_{HF}(t) \), observed in almost one million snapshots, to extract from them the amplitudes of multiple harmonics of spin, \( H_n(t) \), according to the formula (34):

\[
\text{measured } z_{HF}(t) = \sum_n H_n(t)e^{in(\phi + \phi_0)}. \tag{36}
\]

Because of the small factor \( \beta \) involved in the even harmonics, their amplitudes are orders of magnitude smaller than those of the odd ones. So one uses expressions (A.19) for \( H_n(t) \), \( n = \text{odd} \), through the coefficients \( A_{lm} \), polhode phase and angle to determine them by fitting to the measured \( H_n(t) \).

The TFM estimation process is broken up into three parts called levels, in which various groups of parameters are estimated somewhat independently. Level A and B data processing provides estimates of the polhode phase and angle, \( \phi_p(t) \) and \( \gamma_p(t) \), the spin phase, \( \phi_s(t) \), and the asymmetry parameter, \( \Omega \); the estimation at levels A and B is essentially nonlinear. Once the three Euler angles are known at all times along with the rotor asymmetry, coefficients \( A_{lm} \) can be found in level C by a linear fit. The details of processing are given in [24] and [8].

For the entire duration of the mission, TFM produced spin speed to 10 nHz and the spin-down rate to 1 pHz s\(^{-1}\) [30]; spin phases to 0.05 rad; polhode phases to 0.02 rad, and the polhode angles to 0.01 – 0.1 rad [24]. With these results we computed the LF SQUID scale factor variations \( C_{TF}(t) \) for the whole mission using formula (A.22) from appendix A.

The TFM results play a crucial role in the GP-B science analysis, which simply cannot be accomplished without them. First of all, accurate polhode phase and angle values at all times of the mission are required to model both the scale factor variations and PE torques, because all the torque coefficients are modulated by the polhode harmonics in the same way as the LF SQUID scale factor. TFM enables a separate determination of the LM scale factor and the slowly varying D.C. part of the TF scale factor, \( C_{TF}(t) \). The latter is directly used in the LF...
science analysis, as described in DA II, significantly simplifying the nonlinear estimation problem.

3.5. Characteristic frequencies and their values

It is already clear, and will be more evident from the rest of this paper and papers [7, 8], that our data analysis was pretty much about proper handling of fundamental characteristic frequencies involved in the GP-B signals. They were numerous, of both natural and artificial origin, and ranged ten orders of magnitude, from $10^{-8}$ to $10^1$ Hz. There were two groups of frequencies, one not related to gyroscopes, the other gyro-dependent. For convenience, we list all of them here along with their values.

The frequencies of the first group are: (1) annual, $f_a \approx 31$ nHz, with the period of 1 year; (2) orbital, $f_o \approx 170$ µHz, with 97.6 minute period; (3) roll, i.e., the frequency of S/C rotation, $f_r \approx 13$ mHz, with 77.5 second period; (4) the frequency of the calibration signal infused in the SQUID readouts, $f_c \approx 8$ mHz, with the period of 124 second. There were also two dither frequencies infused in the ATC system, to facilitate SQUID-to-telescope signal matching (section 6.3), one per telescope axis. At the start of the mission those were $f_{dx} \approx 33$ mHz and $f_{dy} = 25$ mHz, with the periods of 30 and 40 seconds, respectively. Later, in December 2005, they were switched to $f_{dx} \approx 34$ and $f_{dy} = 29$ mHz (29 and 34 second periods), to move $f_{dy}$ away from the second harmonics of the roll frequency.

Every gyroscope had its own frequency of rotation (spin), $f_s$, and the polhode frequency, $f_p$, growing through the Mission; their values are shown in table 1.

4. LF SQUID signal: model for science readout (main measurement equation)

Within the linear range of the SQUID, the LF gyroscope readout signal is proportional to the total magnetic flux, $\Phi$, through the pick-up loop:

$$z(t) = C_{el} \Phi + b + \text{noise.} \tag{37}$$

Here $b$ is the bias, and $C_{el}$ is the scale factor of the whole chain of the electronic onboard signal processing, starting with the SQUID to pick-up loop scale factor. The output signal is a changing voltage, so the physical dimension of both $z(t)$ and $b$ is volts, and the dimension of $C_{el}$ is volts per magnetic flux (standard unit is $V/\Phi_0$, with $\Phi_0$ being the magnetic flux quantum).

Unfortunately, both $C_{el}$ and $b$ may be not perfectly constant; this issue is addressed later. Meanwhile, to be able to estimate relativistic drift rates, we need first to express the total flux through other variables, in particular, the gyro orientation $s(t)$.

| Gyroscope | Spin | Polhode: start | Polhode: end |
|-----------|------|----------------|--------------|
| 1         | 79.4 Hz | 143 µHz | 320 µHz |
| 2         | 61.8 Hz | 40 µHz | 94 µHz |
| 3         | 82.1 Hz | 168 µHz | 182 µHz |
| 4         | 64.8 Hz | 55 µHz | 67 µHz |

Table 1. Gyro-specific frequencies.
4.1. LM signal

The main contributor to the flux through the pick-up loop is the dipole LM aligned with the gyro spin ([31], section 1). A standard magnetostatics solution provides the flux in the form:

\[ \Phi_{Lm}(t) = C_{Lm} \sin \beta(t), \quad C_{Lm} = \frac{2\pi N_a M_L}{R} = \frac{4\pi^2 N_a}{e} \frac{r m_e c^2}{R^2} f_s. \]  

(38)

The angle \( \beta(t) \) between the gyro spin axis and the pick-up loop plane changes due to the relativistic drift of the gyroscope spin axis (as well as the classical torques, if any), and thus contains the quantities we want to measure. Otherwise \( N_a \approx 4 \) is the effective number of turns of the pick-up loop, \( M_L \) is the LM expressed through physical parameters, \( C_{Lm} \) is the LM scale factor, \( r \) and \( R \) are the radii of the rotor and the pick-up loop, respectively, \( m_e \) and \( e \) are the mass and charge of the electron, and \( f_s \) is the rotor spin frequency. As we explicitly show in section 4.5, \( \beta \lesssim 10^{-4} \), so that

\[ \sin \beta = \beta + O(\beta^3) = \beta, \]  

(39)

with more than sufficient accuracy of \( \sim 10^{-12} \) during GSV. For this reason, we permanently drop the sine in the formula (38) and write the linear LM flux through the pick-up loop as

\[ \Phi_{Lm}(t) = C_{Lm} \beta(t). \]  

(40)

The LM scale factor \( C_{Lm} \), whose dimension is magnetic flux per angle, with the standard unit \( \Phi_0/\text{as} \), depends only on the spin speed, which is extremely stable: the spin–down rate of the four gyros is shown to be within \( 0.3 - 1.3 \mu \text{Hz hr}^{-1} \) [30, 32]. Hence \( C_{Lm} \) can be considered constant for all the short- and mid-term analyses. However, for 10 months of observations the relative change due to the spin–down can be up to \( 5 \times 10^{-5} \), so we write:

\[ C_{Lm} = C_{Lm}^0 \left( 1 - \Delta_{sd}(t) \right); \]

\[ C_{Lm}^0 \equiv \frac{4\pi^2 N_a}{e} \left( \frac{r}{R} \right) \left( \frac{m_e c^2}{e} \right) = \text{const}, \quad \Delta_{sd}(t) = \left[ f_s^0 - f_s(t) \right] / f_s^0, \]  

(41)

where \( f_s^0 \) is the gyro spin frequency at the start of the data stretch, and \( f_s(t) \) is its current value available from TFM throughout the whole Mission accurate to \( 10 \text{nHz} \). Thus the small correction \( \Delta_{sd}(t) \) is known for the whole mission.

4.2. TF addition to LM signal: polhode variations of scale factor

According to equation (35), the LF part of the TF contributes to the LF readout, adding to the LM flux. These two contributions provide the total magnetic flux through the SQUID pick-up loop:

\[ \Phi(t) = \Phi_{Lm}(t) + \Phi_{TF}(t)|_{LF} = \left[ C_{Lm} + C_{TF}(t) \right] \beta(t), \]  

(42)

so the TF adds, in fact, to the SQUID scale factor. At the start of the science mission, the added factor ranges between 0.1 and 5% of \( C_{Lm} \), depending on the gyroscope, and eventually approaches a constant. First \( C_{TF}(t) \) is expressed by the formula (A.24) through the actual polhode phase, \( \phi_p = \phi_p(t, Q) \), and angle, \( \gamma_p = \gamma_p(t, Q) \), of a given gyroscope with its specific asymmetry parameter \( Q > 0 \). As explained in appendix A, this representation is not convenient for the analysis, so it is converted into the final formula (A.29) using the polhode phase and angle \( \phi_0(t) \) and \( \gamma_0(t) \) of the symmetric \((Q = 0)\) rotor with the same polhode period time history:
Thus the polhode motion combined with TF creates polhode variations of the readout scale factor. Expression (43) is good for modeling these variations because the dynamics of two very different scales, polhode period and dissipation time, are well separated in it. Indeed, the coefficients $C_m^{\text{pol}}(\epsilon_0)$ change slowly due to dissipation, and all but $C_m^0(\epsilon_0)$ tend to zero as the spin axis asymptotically approaches its stable position along the maximum axis of inertia. This is seen from the explicit model for these coefficients (formula (A.31), appendix A):

$$C_m^0(\epsilon_0) = \sum_{j=0}^{\infty} C_0^j \epsilon_0^j, \quad C_m^{\text{pol}}(\epsilon_0) = \epsilon_0 \sum_{j=0}^{\infty} C_{mj}^{\text{pol}} \epsilon_0^j, \quad m = 1, 2, \ldots$$

(44)

The constant parameters $C_{mj}^{\text{pol}}$ that are estimated in the signal analysis depend only on the distribution of fluxons on the rotor surface and the asymmetry parameter.

Introducing expressions (41), (43) and (44) to the formula (42), we arrive at the final model of the total SQUID scale factor:

$$\Phi(t) = C_{\text{SQ}}^0(\epsilon_0)^{\beta(t)} = C_{\text{pol}}^0 \left[ 1 - \Delta_{\text{ad}}(t) + \Delta_{\text{pol}}(t) \right]^{\beta(t)};$$

$$\Delta_{\text{pol}}(t) = \sum_{m=0}^{\infty} \left[ \Delta_m^0(\epsilon_0) \cos m\phi_0(t) + \Delta_m^0(\epsilon_0) \sin m\phi_0(t) \right];$$

$$\Delta_0^m(\epsilon_0) = \sum_{j=0}^{\infty} \Delta_{0j}^j \epsilon_0^j, \quad \Delta_m^{\text{pol}}(\epsilon_0) = \epsilon_0 \sum_{j=0}^{\infty} \Delta_{mj}^{\text{pol}} \epsilon_0^j, \quad m = 1, 2, \ldots$$

(45)

Of course, in numerical calculations all the infinite series here and elsewhere are replaced with finite sums with the number of terms providing sufficient accuracy. A universal and unambiguously defined procedure for choosing these cutoffs is discussed in the DA III paper.

### 4.3. Long-term variations of electronics scale factor

Next we study the electronics scale factor, $C_{el}$, from (37), which was supposed to be constant. Accurate investigation of the LF SQUID data has shown that at the level $10^{-3} - 10^{-5}$ some long term variations were in it, with characteristic times of several weeks or months. Most probably they reflect the changing environment (thermal, electromagnetic, material aging, etc) of the onboard circuitry. There is no way to detect, trace, and model all of them, for which reason the model for $C_{el}(t)$ is the only one in the whole data analysis which is not derived from the underlying physics, but is motivated by computational convenience and efficiency; the same model is used also for the misalignment torque coefficient (see section 5.2).

Namely, $C_{el}(t)$, as any function on a given interval of the length $T_{\text{sm}}$ (‘sm’ stands for the science mission, about one year), can be represented by its Fourier series with the basic frequency $1/T_{\text{sm}}$. Thus the analysis model is
\[ C_{ct} (t) = C_{ct}^0 \left[ 1 + \Delta_{ct} (t) \right]; \]
\[ \Delta_{ct} (t) = d_0^c \cos \left( \pi t / T_{sm} \right) + d_0^s \sin \left( \pi t / T_{sm} \right) \]
\[ + \sum_{m=1}^{\infty} \left[ d_m^c \cos \left( 2\pi nt / T_{sm} \right) + d_m^s \sin \left( 2\pi nt / T_{sm} \right) \right], \] (46)

where \( d_m^{c,s} \) are the constant dimensionless parameters to estimate, and a subharmonics \((m = 1/2)\) was added to facilitate secular changes.

4.4. Complete scale factor model

Everything is ready now for the complete model of LF SQUID signal scale factor. Introducing (45) and (46) to the formula (37), we find:

\[ z(t) = C_{ct}^0 \left[ 1 + \Delta_{ct} (t) \right] C_{ctm}^0 \left[ 1 - \Delta_{sd} (t) + \Delta_{pol} (t) \right] \beta (t) + b + \text{noise}. \]

Since the corrections are small, their products are neglected, and the final model is:

\[ z(t) = C_{ct}^0 \beta + b + \text{noise}, \]
\[ C_{ct} (t) = C_{ct}^0 \left[ 1 - \Delta_{sd} (t) + \Delta_{pol} (t) + \Delta_{ct} (t) \right]. \] (47)

Here the parameter \( C_{ct}^0 = C_{ct}^0 C_{ctm}^0 \) [V/\( \text{as} \)] is the constant part of the scale factor, and the expressions for the known, \( \Delta_{sd} (t) \), and unknown \( \Delta_{pol} (t) \), \( \Delta_{ct} (t) \) varying corrections are found in (41), (45), and (46), respectively. When analyzing the real data numerically, the series in the formulas (45), and (46), as well as the series in all other models, are truncated at some finite number of terms, as is discussed in the DA III paper.

4.5. Main measurement equation: SQUID signal in terms of spin-to-roll misalignment

Representation (47) is not helpful for the analysis and thus for the determination of relativistic drift rates, until angle \( \beta (t) \) is expressed in terms of the experiment’s observables. By its definition and the formula (39) it is determined to be

\[ \beta \approx \sin \beta = \hat{s} \cdot \hat{n}, \] (48)

where \( \hat{n} \) is the unit normal to the pick-up loop. Using its GS frame expression (33) for GSV, and expression (9) for the unit vector of gyro spin \( \hat{s} \) in the same inertial frame, one calculates the dot product in (48) to find:

\[ \beta = (\tau_{ns} - s_{ns}) \cos \left( \phi_s + \delta_n \right) + (\tau_{wc} - s_{wc}) \sin \left( \phi_s + \delta_n \right) + \alpha \]
\[ + \text{cubic terms in } \alpha, \tau, \text{ and } s + O \left( \beta^3 \right) \]
\[ = \mu_{ns} \cos \left( \phi_s + \delta_n \right) + \mu_{wc} \sin \left( \phi_s + \delta_n \right) + \alpha + \text{cubic terms in } \alpha, \tau, s, \text{ and } \mu. \] (49)

The last estimate of the remainder here is due to the equality \( \beta = O (\mu + \alpha) \) implied by the leading term. It is amazing that all the quadratic terms on the right of equation (49) cancel out completely, leaving just the leading linear term and its cubic corrections. For any GSV outside the S/C anomalies the cubic corrections should be not larger than about one part in \( 10^{11} \) (worst case); this is definitely negligible as compared to \( \lesssim 10^{-4} \) of the main linear term. In the majority of anomalies where pointing was not lost, the spin–to–roll misalignment was still within \( \sim 500 \) as. Even then the corrections remained \( \sim 10^{-10} \), which is negligible.
After introducing expression (49) in the readout equation (47), we obtain

\[
zt = C_g(t) \beta + b + \text{noise}
\]

\[
= C_g(t) \left[ (r_{ns} - s_{ns}) \cos(\theta_t + \delta_t) + (r_{we} - s_{we}) \sin(\theta_t + \delta_t) + \alpha \right] + b + \text{noise}
\]

\[
= C_g(t) \left[ H_0 \cos(\theta_t + \delta_t) + H_0 \sin(\theta_t + \delta_t) \right] + B(t) + \text{noise},
\]

(50)

where \( B(t) = C_g(t) \alpha + b \) may be treated as the new time-varying bias. This is the main measurement equation of the whole GP-B data analysis. Here, as before, \( z(t) \) is the LF SQUID output (in volts), and \( C_g(t) \) (V as\(^{-1}\)) is the scale factor whose model was given in the previous section; \( b \) is the constant bias, \( \delta_t \) is the constant roll phase offset, and \( \alpha \) is the constant assembly angle. These are estimated in the signal analysis, the latter as a part of the total bias. Next, \( \dot{\phi}_t(t) \) is the phase of the S/C roll measured independently by S/C star trackers, and \( \tilde{\tau}(t) \), entering through its NS and WE projections, is also known for any GSV (see section 6.1). Relativistic drift rates \( r_{ns} \) and \( r_{we} \) are hidden in the gyro orientation described by the projections of the vector \( \tilde{s}(t) \) from equation (9) \( s_{ns}(t) \) and \( s_{we}(t) \). If all the classical torques were negligible, these would be linear functions of time (13) with the slope coefficients \( r_{ns} \) and \( r_{we} \); the real dependences, much more complicated due to PE torques, will be derived in the next section. Note that the SQUID signal model for GSI is derived in appendix B; it was not used for estimating relativistic drift rates, but only for determining the S/C pointing, \( \tilde{\tau}(t) \), during every GSI period. It has just one more term than equation (50).

The bias variability can be modelled by some additional states, \( B_j \),

\[
B(t) = \sum_{j=1}^{N_b} B_j h_j(t),
\]

(51)

with the known time signatures \( h_j(t) \). However, the bias is totally removed before the final relativistic drift estimation. The analyzed signal was free of both the bias and pointing error removed using the telescope signal (see section 6.3).

4.6. Linear readout and nonlinear estimation problem

We have thus proved that the GP-B readout is linear in all the small angles \( \mu, \alpha, \beta \), etc, with only cubic corrections; those are several orders smaller than required by the desired precision. Nevertheless, the problem of estimating the relativistic drift rates and other parameters by the equation (50) is nonlinear. Indeed, the measurement equation of a linear estimation problem has the form

\[
z(t) = \sum_{n=1}^{N} x_n h_n(t) + \text{noise},
\]

(52)

where \( z(t) \) is the known signal, and the rhs is its model. In particular, \( x_n, n = 1, 2, \ldots, N \) are the unknown parameters to estimate (states), and the known time signatures \( h_n(t), n = 1, 2, \ldots, N \) are linearly independent functions of time. The integer \( N \) is the dimension of the problem; its unique solution is proved to exist and can be found by the least square fit (LSQ) method or, equivalently, with the help of the Kalman filter [33]. Algebraic LSQ equations implied by the measurement equation (52) are linear, as the name ‘linear estimation problem’ implies.

At first glance, our equation (50) is far from having the structure or equation (52): for instance, the roll phase offset \( \delta_t \) is involved in equation (50) in what seems to be a very
nonlinear way. Moreover, other states, such as the drift rates \(r_{ns}\), \(r_{we}\) in the components of \(\mathbf{\bar{s}}(t)\), are multiplied by a scale factor that also needs to be estimated—yet another nonlinearity! However, under some natural simplifying conditions the measurement equation (50) can be reduced to the linear form (52) by just a change of variables [34–36]. This is important for fully understanding the difficulties of the GP-B data analysis, so we discuss it in more detail.

Let us assume for a moment that the scale factor variations, bias and classical torques are negligible, so that

\[
C_g(t) = C^0_g = \text{const}; \quad b = \text{const};
\]

\[
s_{ns}(t) = s_{ns}^0 + r_{ns}(t - t_0), \quad s_{we}(t) = s_{we}^0 + r_{we}(t - t_0)
\]

(53)

(see equation (13)). The measurement equation (50) then becomes

\[
z(t) = b + \text{noise} + C^0_g \times \left\{ \left[ r_{ns} - s_{ns}^0 - r_{ns}(t - t_0) \right] \cos (\phi_I + \delta_I) \right.
\]

\[
+ \left[ r_{we} - s_{we}^0 - r_{we}(t - t_0) \right] \sin (\phi_I + \delta_I) + \alpha \right\},
\]

(54)

with just seven parameters to estimate: \(C^0_g, s_{ns}^0, s_{we}^0, r_{ns}, r_{we}, \delta_I\) and \(B = C^0_g \alpha + b\) (recall that \(r_{ns}(t)\), \(r_{we}(t)\) and \(\phi_I(t)\) are considered to be known). The last equation shows no particular resemblance to (52), either. Still, first using addition formulas for trigonometric functions, and then introducing new states according to

\[
x_1 = C_g^0 \cos \delta_I, \quad x_2 = C_g^0 \sin \delta_I; \quad x_7 = B = C_g^0 \alpha + b;
\]

\[
x_3 = -C_g^0 \left( s_{ns}^0 \cos \delta_I + s_{we}^0 \sin \delta_I \right), \quad x_4 = C_g^0 \left( s_{ns}^0 \sin \delta_I - s_{we}^0 \cos \delta_I \right);
\]

\[
x_5 = -C_g^0 \left( r_{ns} \cos \delta_I + r_{we} \sin \delta_I \right), \quad x_6 = C_g^0 \left( r_{ns} \sin \delta_I - r_{we} \cos \delta_I \right);
\]

(55)

one reduces equation (54) to the linear form (52) with the dimension \(N = 7\), the above states (55), and the following time signatures:

\[
h_1(t) = r_{ns}(t) \cos \phi_I(t) + r_{we}(t) \sin \phi_I(t); \quad h_3(t) = \cos \phi_I(t);
\]

\[
h_2(t) = -r_{ns}(t) \sin \phi_I(t) + r_{we}(t) \cos \phi_I(t); \quad h_4(t) = \sin \phi_I(t);
\]

\[
h_5(t) = (t - t_0) \cos \phi_I(t), \quad h_6(t) = (t - t_0) \sin \phi_I(t); \quad h_7(t) \equiv 1.
\]

(56)

Formulas (55) can be inverted, so as soon as all the estimates of \(x_n\) are found, the originally needed parameters \(C_g, \ldots, \delta_I\) can be determined uniquely through them. The covariance matrix of the latter can also be recalculated from that gotten for \(x_n\), giving the full solution to the problem.

Next it occurs that dropping the first of the simplifying assumptions (53), i.e., allowing for the scale factor variation, does not spoil the picture much, because those variations are relatively small and can be treated as perturbations. Indeed, the full model (47) for the scale factor can be written as

\[
C_g(t) = C_g^0 \left[ 1 + \sum_{m=1}^{N} c_m y_m(t) \right],
\]

where the sum represents variations of the scale factor, see formula (47). The time signatures \(y_m(t)\) are assumed to be linearly independent functions normalized to unity, and \(c_m\), the
additional parameters to estimate, are all small, \( |c_n| \ll 1 \), \( c \equiv \sqrt{c_1^2 + c_2^2 + \ldots + c_N^2} \ll 1 \) (typically \( \sim 10^{-3} \) and smaller, for GP-B). Still assuming linear gyro drifts (53), we arrive at the following measurement equation (50) in terms of the ‘linearizing’ variables (55) with time signatures (56):

\[
z(t) = \left[ 1 + \sum_{n=1}^{N} c_n y_n(t) \right] \sum_{n=1}^{N} x_n h_n(t) + \text{noise.} \tag{57}
\]

We have considered the exact LSQ equations for the model (57); they are no longer linear, but contain a cubic nonlinearity involving the states \( c_n \). We proved that for small enough \( c \) their unique solution exists and can be obtained iteratively with an exponential rate of convergence [the actual bound on \( c \) can be calculated from the data, \( z(t) \)]. Iterations can be implemented by the method of ‘frozen coefficients’: first one assumes \( c_n = 0 \), \( n = 1, 2, \ldots, N \) and determines the first estimates \( x_n^{(1)} \) of the states \( x_n \) from the linear model. Then, using the found \( x_n^{(1)} \), one carries out a linear LSQ fit for \( c_n \) and determines their first iteration values \( c_n^{(1)} \). The next iteration starts with the LSQ fit for \( x_n \) using the found values \( c_n^{(1)} \) in equation (57), and thus determines \( x_n^{(2)} \); inserting them in equation (57), one concludes the second iteration by finding \( c_n^{(2)} \), and so on. In this way, only a linear estimation problem needs to be solved at each of the two steps of any iteration.

Our results show that it is only the combination of relatively small but significant variations of the gyro scale factor with a nonlinear time change of the gyro orientation \( \vec{s}(t) \) that makes the main GP-B estimation problem essentially and strongly nonlinear. It can be no longer reduced to iterations of linear estimation problems. Contributions of the PE torques to the gyro motion were not only non-negligible, but could not be even treated as small perturbations of the relativistic drift.

5. LF SQUID signal: gyro dynamics in the presence of PE torques

5.1. PE torques on GP-B rotors

The electrostatic PE [37–39] is a non-uniform distribution of electrical potential on metal surface. It occurs when a non-uniform dipole layer forms on the surface due to impurities or microcrystal structure, resulting in a varying potential and electric field not necessarily perpendicular to the surface. In the idealized case of an isolated conductor, the net force and torque on the body are still equal to zero. But for two metallic surfaces at a finite distance, the net force and torque on each of them do not in general vanish. Naturally, the effect is larger, the closer the surfaces, as first confirmed by the calculation of the PE force for two parallel conducting planes [40].

In GP-B, the two metallic surfaces close to each other are the rotor and its housing. Various manifestations of PE in our experiment were studied in paper [28], mostly by fitting a PE model to experimental data of different physical origin. PE torques acting on GP-B rotors are discussed in [41]: their experimental discovery is described, and some key theoretical results are presented (see also [3, 5]). A theory of PE torques is developed in detail in appendix C. Its results effectively simplify due to two small parameters: (1) the ratio, \( (d/a) \sim 10^{-3} \), of the rotor–to–housing gap, \( d \), to the rotor radius, \( a \), and (2) the spin-to-roll misalignment, \( \mu \lesssim 10^{-4} \) (see section 2.4, in particular equations (21) and (22)). Based on these results, we examine the motion of GP-B gyroscopes with PE torques acting on them and
obtain an explicit formula for \( \vec{s}(t) \). In the data analysis it is a submodel of the main measurement equation (50).

Two distinct types of PE torques were found during and after the flight mission, which were confirmed by the PE torque theoretical model derived in appendix C. They are the misalignment torque and the roll-polhode resonance torque.

The **misalignment torque** was discovered in the post-flight calibration period of the experiment when the S/C was deliberately maneuvered up to 7 degrees away from the GS. The torque increased with the misalignment, and, within a \( \leq 1^\circ \) range, was proportional to it; the direction of the torque was perpendicular to the misalignment vector \( \vec{\mu} \). This is exactly as one of the PE torques behaves, as shown in appendix C.

The **roll-polhode resonance torque** was discovered in the data as sizeable ‘jumps’ in the estimated gyro orientation: from time to time one of the gyroscopes would change its spin direction by as much as 20–100 mas in one day or less. The jumps occurred when some high harmonic of changing polhode frequency became equal to the roll frequency. Before this, one additional PE torque was found theoretically; it had zero order in the misalignment (i.e., was independent of it), thus potentially it could be very large, without the misalignment factor \( \leq 10^{-4} \). But it was proportional to the sine or cosine of roll and thus usually averaged out during each roll period; so it was neglected. However, since those roll harmonics were modulated by the polhode ones, the roll averaging broke down at the times when the resonance condition, \( \omega_r = n \omega_p \), holds. This led to a significant gyro drift in the vicinity of the resonance, when the frequency detuning was small.

### 5.2. Rotor motion equations and their exact solution: model for \( \vec{s}(t) \)

The results of appendix C lead to the following equation for the gyro motion when PE torques are present (see our paper [1]):

\[
\begin{align*}
\dot{s}_n &= r_n + k_n \mu_n + \sum_{n} \left( a_n \cos \Delta \phi_n - b_n \sin \Delta \phi_n \right); \\
\dot{s}_m &= r_m - k_m \mu_m + \sum_{n} \left( a_n \sin \Delta \phi_n + b_n \cos \Delta \phi_n \right). 
\end{align*}
\]

(58)

Here \( r_n \), \( r_m \) are the relativistic drift rates (11), the terms \( k_n \mu_n \), \( k_m \mu_m \) represent the drift rates from the misalignment torque, with \( \mu_n \), \( \mu_m \) being the components of the misalignment vector (21); \( k \) is the time varying torque coefficient, see equations (C.20) and (C.21).

The contributions of the roll-polhode resonance torque are given by the third term on the right of equations (58), with \( \Delta \phi_n(t) = n \phi_0(t) - \phi(t) \), \( \phi \) and \( \phi_0 \) being the known roll and polhode phases (of the corresponding symmetric gyroscope, see below). The resonance occurs when the roll frequency coincides with some harmonics of the polhode frequency, \( \omega_r = n \omega_p \). Note that the value of \( n \) necessary for a resonance is large, from 40 up to about 300, depending on the gyro. Therefore it was difficult to believe that polhode harmonics of such high order could cause a significant effect until it was seen in the data.

Unlike the general formulas (C.26), only the terms coming into resonance during the flight were included in equation (58) when used in the data analysis: the contribution of other terms was entirely negligible due to averaging, as explained. The range of \( n \) and the total number of resonances, \( N_{\text{res}} \), are given by

\[
\begin{align*}
n_{\text{min}} &\leq n \leq n_{\text{max}}, \quad n_{\text{min}} = \omega_r / \omega_p^{\text{min}}, \quad n_{\text{max}} = \omega_r / \omega_p^{\text{max}}, \quad N_{\text{res}} = n_{\text{max}} - n_{\text{min}} + 1; 
\end{align*}
\]

here \( \omega_p^{\text{min}} \) and \( \omega_p^{\text{max}} \) are the values of \( \omega_p \) at the start and end of the mission, see table I, section 3.5. The total resonance number \( N_{\text{res}} \) ranged from \( N_{\text{res}} = 6 \) for gyroscope #3 to
$N_{\text{res}} = 245$ for gyroscope #2. Its value is determined by the span between the start and end values of the polhode frequency, and by how they compare to the roll frequency.

The time distribution of resonances depends also on the rate of the polhode change, the largest in the beginning and monotonically decreasing to the asymptotic zero. Due to this, the resonances are dense at the start and become more and more scarce towards the end of the mission. The time gaps between the neighbor resonances range from a few days for gyroscope #2 to several weeks and even a month for gyroscope #3. The picture of resonances for gyroscope #2 may be seen in figures 13, 14 of paper [7]; only the resonances whose contribution to the drift was visible were included in these plots. Naturally, the number, $n$, of the occurring resonances decreases with the time, since the polhode frequency increases: higher order resonances come earlier.

As shown in appendix C, the misalignment torque coefficient $k = k(t)$ is modulated by the polhode motion exactly like the TF scale factor. The only difference is that the modulation is caused not by the motion of magnetic fluxons on the rotor relative to the pick-up loop, but by the relative motion of voltage patches on the rotor and housing. Therefore the model for $k(t)$ (formula (C.24)) is exactly the same as the model (44) for $C_{\text{TF}}(t)$, namely:

$$k(t) = \sum_{n=0}^{\infty} \left[ k_n^a(e_0) \cos n\phi_0(t) + k_n^b(e_0) \sin n\phi_0(t) \right],$$

$$e_0(t) = \tan \left( \frac{\gamma_0(t)}{2} \right) \to 0, \quad t \to \infty;$$

$$k_n^a(e_0) = \sum_{j=0}^{\infty} k_n^a e_j^a; \quad k_n^b e_j^b(e_0) = e_0 \sum_{j=0}^{\infty} k_n^b e_j^b; \quad n = 1, 2, \ldots.$$

(59)

Recall that $\gamma_0$, $\phi_0$ are the polhode angle and phase of a symmetric rotor with the same polhode period history as of the given one.

Just as with the electronic scale factor $C_{\text{el}}$ (section 4.3), small long–term variations in $k$ were found by signal analysis. They could come from very high spherical harmonics of the patch distribution whose significant presence in the field is known. This translates, in particular, in a very large number of terms needed in the expansion (59) for $k_n^a(e_0)$ to get it accurately, which is difficult to implement.

To capture these long-term variations, a Fourier series in harmonics of the period $T_{\text{sm}}$, the duration of the science mission, was added to the expression (59) for $k(t)$. The addition has exactly the same structure as the variable part $\Delta_d$ of the electronic scale factor in the formula (46). It is important to note that relativistic drift rates and the misalignment torque coefficient were found simultaneously from the science signal. For this reason, any model of the torque coefficient that adequately describes its variation is sufficient to determine the relativistic drift.

The resonance torque contribution to equations (58) already includes modulation by polhode harmonics, so the coefficients $a_n(t)$, $b_n(t)$ change only slowly due to energy dissipation. They are thus the same as $k_n^{a,b}$ in (59), see equations (C.25) and (C.28):

$$\xi_0(e_0) = \sum_{j=0}^{\infty} \xi_j^a e_j^a; \quad \xi_n(e_0) = e_0 \sum_{j=0}^{\infty} \xi_j^{a,b} e_j^b, \quad n = 1, 2, \ldots;$$

(60)

here $\xi$ means either $a$ or $b$. The constant coefficients $k_n^{a,b}$, $a_n$, $b_n$ are completely determined by the patch patterns on the rotor and housing, and the rotor asymmetry parameter $Q$. They
are estimated in our signal analysis. Same as in all other cases, the actual number of terms in the sums (60) was determined by a universal procedure described in DA III. However, the signal variations due to these terms were so small that only the first terms, i.e., the coefficients $a_{ao}, b_{ao}$, allowed for a statistically significant estimates determined predominantly by the corresponding resonances.

Since $\bar{\mu} = \vec{r} - \vec{s}$, equation (58) form a linear first order system of differential equations for $s_{ns}(t), s_{we}(t)$ needed for the analysis. Surprisingly, the exact closed form solution can be found for an arbitrary time–dependent coefficient $k(t)$. Consider first the corresponding homogeneous system

$$\dot{s}_m = -k(t) s_{we}, \quad \dot{s}_{we} = k(t) s_{ns}.$$  \hspace{1cm} (61)

Dividing both parts by $k(t)$ and introducing the new time variable $K$ according to

$$\frac{dK}{dt} = k(t) dt, \quad K = K(t, t_0) = \int_{t_0}^{t} k(t') dt', \quad K(t_0, t_0) = 0,$$

we obtain equations with constant coefficients describing a linear oscillator,

$$\frac{ds_{ns}}{dK} = -s_{we}, \quad \frac{ds_{we}}{dK} = s_{ns}.$$  

The well-known general solution is a rotation of an arbitrary vector $\vec{s} = c_{ns} \hat{e}_{ns} + c_{we} \hat{e}_{we}$ by the angle $[-K(t, t_0)]$. Using the standard Lagrange variation of arbitrary constants $c_{ns}, c_{we}$, one finds a solution of the equations with any right hand side (equation (58), in our case). Its vector form is the most compact, so we first rewrite the system (58) in terms of vectors:

$$\dot{s} = k \hat{e}_{gs} \times \vec{s} + \vec{D}(t);$$  \hspace{1cm} (63)

$$\vec{D} = \vec{D}^{rel} + \vec{D}^{\tau} + \vec{D}^{res};$$

$$\vec{D}^{rel} = r_{ns} \hat{e}_{ns} + r_{we} \hat{e}_{we}; \quad \vec{D}^{\tau} = k \vec{s} \times \hat{e}_{gs};$$

$$\vec{D}^{res} = \sum_n R(-\Delta \phi) \vec{a}_n, \quad \vec{a}_n = a_n \hat{e}_{ns} + b_n \hat{e}_{we};$$  \hspace{1cm} (64)

the two-dimensional rotation matrix $R(\alpha)$ was defined in the beginning of section 2.

The unique solution to the system (63) with the initial condition $\vec{s}(t_0) = \vec{s}^0$ is:

$$\vec{s}(t) = \mathcal{M}(t, t_0) \vec{s}^0 + \int_{t_0}^{t} \mathcal{M}(t, t') \vec{D}(t') dt'.$$  \hspace{1cm} (65)

Here the rotation matrix $\mathcal{M}(t, t_0)$ is defined as

$$\mathcal{M}(t, t_0) \equiv R(-K(t, t_0)) = \begin{bmatrix} \cos K(t, t_0) & -\sin K(t, t_0) \\ \sin K(t, t_0) & \cos K(t, t_0) \end{bmatrix}.$$  \hspace{1cm} (66)

This matrix is, of course, orthogonal, so $\mathcal{M}^{-1} = \mathcal{M}^T$. Moreover, the solution to the Cauchy problem for the homogeneous equations (61) with the same initial value $\vec{s}^0$ is expressed through $\mathcal{M}(t, t_0)$ by the first term on the rhs of formula (65) corresponding to $\vec{D} = 0$. Equations (61) represent an autonomous dynamical system whose resolving flow $\mathcal{M}(t, t_0)$ enjoys the group property: for any $t_0 \leq t' \leq t$,
\[ M(t, t_0) = M(t, \tau) M(\tau, t_0). \]  

(67)

This expresses the uniqueness of a trajectory of the dynamical system (61) specified by any initial point on it, an implication of the arbitrary time shift that does not change the system. In our case the system is linear, which is why it is straightforward to derive the property (67) from the definition (66) by simply multiplying appropriate matrices.

The two properties of \( M \) imply
\[ M(t, \tau) = M(t, t_0) M^{-1}(\tau, t_0) = M(t, t_0) M^T(\tau, t_0). \]  

(68)

Using expression (68) in the solution (65), we write the latter in a different way:
\[ \vec{s}(t) = M(t, t_0) \vec{s}_0 + \int_{t_0}^{t} \int_{t_0}^{\tau} M^T(\tau, t_0) \vec{D}(\tau') \, d\tau' \]
\[ = M(t, t_0) \left[ \vec{s}_0 + \int_{t_0}^{t} M^T(\tau', t_0) \vec{D}(\tau') \, d\tau' \right]. \]  

(69)

Formulas (65) or (69), along with (64), (59), and (60), provide a model for the drift vector \( \vec{s}(t) \), which differs strongly from uniform relativistic motion (12). This explicit model is crucial for the GP-B data analysis: the alternative of numerical integration of the motion equations (58) at each of the billions of estimation steps was computationally unacceptable, and might have had serious numerical errors even if implemented. The form (69) of the solution saves a lot of computational time and resources because, unlike (65), it requires only a sequential computation of the matrix \( M(t, t_0) \) at each data point starting with the beginning of the stretch, \( t_0 \) (the actual time step was 2 s, see DA III).

Note that the relativistic drift rates \( r_{\text{ns}}, r_{\text{re}} \) enter model (69) linearly, although their time signatures are no longer linear functions of time; the same is true for the other group of states need to estimate, namely, \( a_{ni}, b_{ni} \). However, this is not so for the states \( k_{ni} \): in addition to a linear involvement in \( \vec{D} \) as defined in (64), they also populate the model (69) in a very nonlinear way, sitting inside \( K(t, t_0) \) in the matrix \( M(t, t_0) \equiv R(-K(t, t_0)) \). Expression (64) for the pointing part of the misalignment torque contribution requires continuous knowledge of S/C pointing, \( \vec{r}(t) \), at each time point of data.

6. LF signal: model of spacecraft pointing

6.1. Spacecraft pointing, \( \vec{r} \)

A glance at our main measurement equation (50) shows that we have already provided models for all the quantities involved there, except for two components \( r_{\text{ns}}, r_{\text{re}} \) of the pointing vector \( \vec{r} \). The equation is used for relativity estimates only during GSV periods, when the science telescope serves as the S/C attitude reference. Accordingly, we need the telescope pointing during GSV, given by
\[ \vec{r}(t) = \vec{A}(t) + \vec{P}(t) + \vec{B}(t) + \vec{\theta}^{\text{in}}(t). \]  

(70)

The first three terms on the right are optical effects causing the telescope to deviate from the true direction to the GS, namely, the optical aberration (consisting of the annual and orbital motion contributions), the parallax, and the bending of the guide star light by the Sun. The first two are classical (although we use the first special relativity correction in the aberrations, see below), while the third one, the bending of light, is the famous GR effect absent in Newtonian gravity. The last term in equation (70), \( \vec{\theta}^{\text{in}}(t) \), is the telescope inertial pointing
error whose value is provided by the telescope signal as described in the next two sections. The magnitude of the annual and orbital aberration is ≈ 20 as and ≈ 5 as, respectively. The parallax, due to the Earth motion around the Sun, is about three orders of magnitude smaller, ≈ 10 mas, and the bending of light is comparable, ≈ 20 mas. The pointing error during GSV was oscillatory with ≈ 100 mas amplitude, for the majority of orbits.

Let \( \vec{R}_E \) and \( \vec{V} \) be the position and velocity of the Earth on its orbit, so that \( \vec{R}_E \) is the vector from the Sun to the Earth. For any inertial velocity vector \( \vec{w} \) let us introduce a vector operation

\[
\vec{A} = \vec{w} \hat{w} = \vec{w}_n \hat{e}_n + \vec{w}_s \hat{e}_s.
\]  

(71)

Then the annual aberration including the first relativity correction [44] is:

\[
\vec{A}^a = \vec{A} \left( \vec{V} \right); \quad A^a_n = \frac{V_n}{c} \left( 1 - \frac{V_{gs}}{2c} \right), \quad A^a_w = \frac{V_w}{c} \left( 1 - \frac{V_{gs}}{2c} \right).
\]

The orbital aberration differs only by the replacement of \( \vec{V} \) with \( \vec{v} \),

\[
\vec{A}^o = \vec{A} \left( \vec{v} \right); \quad A^o_n = \frac{V_n}{c} \left( 1 - \frac{V_{gs}}{2c} \right), \quad A^o_w = \frac{V_w}{c} \left( 1 - \frac{V_{gs}}{2c} \right).
\]

\( \vec{v} \) being the S/C orbital velocity. However, the total aberration \( \vec{A} \) is not simply the sum of the annual and orbital ones:

\[
\vec{A} = \vec{A} \left( \vec{V} + \vec{v} \right) \neq \vec{A} \left( \vec{V} \right) + \vec{A} \left( \vec{v} \right);
\]

\[
A_n = \frac{V_n + v_n}{c} \left( 1 - \frac{V_{gs} + V_{gs}}{2c} \right), \quad A_w = \frac{V_w + v_w}{c} \left( 1 - \frac{V_{gs} + V_{gs}}{2c} \right).
\]  

(72)

The inequality here is due to the relativistic term making the operation \( \vec{A} \left( \vec{w} \right) \) nonlinear; to l. o., i.e., in the non-relativistic approximation, the total aberration is, of course, just the sum of its two constituents. By the way, the special relativistic correction to the velocity addition is too small to matter, \( vV/c^3 \sim 10^{-13} \) at most.

The time signature of the annual aberration components is a sinusoid with annual frequency and the proper amplitude and phase. The same is true for the parallax,

\[
\vec{p} = \frac{\vec{R}_E \cdot \hat{e}_n}{L} \hat{e}_n + \frac{\vec{R}_E \cdot \hat{e}_w}{L} \hat{e}_w.
\]

(73)

where \( L = 6.25 \times 10^{15} \) km is the distance to the GS [11]. However, not only the amplitude, but also the phase of its sinusoid is different from that of the annual aberration: the parallax depends on the Earth position, rather than its velocity.

The time signature of the orbital aberration projected is also a sinusoid with orbital frequency. For the ideal GP-B orbit \( \nu_{we} = 0 \), and the entire orbital aberration is in the NS direction. In reality, the WE orbital aberration was small, with the amplitude <10^{-3} of the total 5 as.

It remains now to produce the model for the bending of star light. Although the derivation of this effect is found in a large number of books, the form required by our data analysis is hard to find anywhere. So we took the result closest to the one needed from paper [45], and developed it to the proper answer, namely:
Here $R_E = |\vec{R}_E|$ is the distance from the Sun to the Earth, and $m = GM_\odot/c^2 \approx 1.45$ km is the geometrical mass (gravitational radius) of the Sun. This result is valid in the weak field approximation of GR to l.o. in $m/R_E$, also taking into account that $r \ll R_E \ll L$, with $L$ being the distance to the GS, and $r$ being the S/C orbital radius. The time signature of the star light deflection signal is a complicated function containing all harmonics of the annual frequency.

All three effects (72)–(74) were precalculated for the entire science period at the same moments with the 2 s intervals as all other signals (SQUID signal, telescope signal, etc). The Earth’s position and velocity were taken from JPL Ephemerides, and the S/C velocity was provided by the most precise post–processed GPS data. Finally, the pointing error came from the telescope signal, and was eventually removed from the SQUID signal (see the next two sections). In principle, any other optical effect can be added to those included, if known; one such effect is the orbital motion of the guide star IM Pegassi, but its amplitude is way too small, $\sim 1$ mas.

This would be the end of the story, but for the fact that the misalignment torque was acting continuously, so $\vec{\tau}(t)$ was permanently involved in the drift rate, see formulas (64) and (69). For this reason, we needed S/C pointing during GSI as well, even though these periods were not used for reliability estimation: to calculate the misalignment torque contribution at a given moment of time, we had to find it for all previous moments of time, starting with the initial time of the data stretch. During GSI the S/C attitude reference was not the telescope (see paper [42]), so expression (70) was not valid there. In fact, $\vec{\tau}(t)$ during GSI was found using gyro SQUID signals in the following way.

The gyro orientation does not change much during one GSV: even with the PE torques acting, the change is within 10 mas $= 5 \times 10^{-8}$ at most. The SQUID scale factor also does not change much: its relative variation is just $10^{-3}$ or less. So the average GSV values $C_{gs}^m$ and $\vec{s}_{gs}$ are meaningful and accurate enough. Thus the SQUID signal of every science gyroscope was analyzed at each GSV period (more than 5000 throughout the mission) to determine $C_{gs}^m$ and $\vec{s}_{gs}$ using measurement equation (50) without either torque or scale factor variation modeling (the estimation problem is then converted to a linear one using the two–step scheme, see section 4.6). After this, for any GSI one specifies $C_{gs}^*, \vec{s}_{gs}$ as the average of the corresponding values from the two neighboring GSV stretches. Then, by fitting the model (50) to the SQUID in a window of the length of the roll period or larger, one finds the estimates of the weighted components of the spin-to-roll misalignment, $m_{ns} = C_{gs}h_m$ and $m_{we} = C_{gs}h_w$. The approximate pointing direction at the middle of the window is calculated by the definition (21) of the misalignment:

$$\vec{\tau}_{ns} = \mu_{ns} + \vec{s}_{ns} = m_{ns}/C_{gs} + s_{ns}, \quad \vec{\tau}_{we} = \mu_{we} + \vec{s}_{we} = m_{we}/C_{gs} + s_{we}. \quad (75)$$

In practice, we used a moving window of the length twice roll period with a 2 s time step through each GSI, getting the values of pointing at the proper time instants. This procedure was carried out for each of the four gyrosopes, and the final numbers for pointing were found by averaging the four appropriate values. In this way $\vec{\tau}(t)$ was obtained for the whole mission, both at GSV and GSI; the GSI pointing was refined later by a more sophisticated modeling taking into account some features of the GSI spacecraft motion, as described in [42].
6.2. Telescope signal model

The telescope signal model is derived from the telescope system design and from on-orbit analysis of its pointing signal as a function of the pointing measured by the four gyroscopes, as described above. The scheme and construction of star-tracking telescope are given in detail in [3, 5]. The telescope observes the guide star when it is not occulted by the Earth (GSV) serving as the S/C attitude sensor. The telescope pointing signals are produced as follows.

A beam splitter in front of the focal plane reflects half of the light and transmits the rest. This arrangement provides two focal planes. Roof prisms with very sharp and straight edges are placed at the focal positions: one roof prism is oriented to measure the telescope position relative to the S/C x-axis and the other about the y-axis. When the telescope is exactly pointed toward the GS, half of the light is reflected from each side of the roof prism, but when the telescope is pointed slightly away from the GS more light is reflected from one side than the other. The light reflected from each side of the roof prism is directed toward one photodiode of a photodiode pair. The measured photocurrents from the photodetector pair, and , whose values depend thus on the deviation from the GS, are used to derive the telescope pointing signal. For redundancy there is a beam splitter in front of the primary photodetector pair (side A) so that the second photodetector pair (side B) is activated. This photodiode set-up thus generates the following four pairs of signals: ; ; ; .

If the channels of each photodiode pair were identical, then the normalized telescope signal could be taken as the ratio of the currents’ difference to their sum. In reality, these channels are not perfectly symmetric, so a weighting factor compensating for the difference in the light transport from each side of the roof prism, as well as the difference in the photodiode efficiencies, is introduced. Using it, the normalized telescope signal, , for a particular axis (X or Y) and side (A or B) is defined as

\[ z_T = \frac{i^+ - w_i^-}{i^+ + w_i^-} \]  

(here and below we ignore the , , and subscripts). We estimated the weighting factor for each photodetector pair by assuming that the weighted sum of the photocurrents is a constant and by using the natural variation in the pointing resulting from disturbances and the response of the S/C attitude control system.

\[ i^+ + w_i^- = \text{const.} \]

We only used data for which the normalized pointing signal was small, \(|z_T(t)| < 0.2\), to avoid larger pointing angles where light can be lost in optical transport.

To l.o. in the small deviation from the direction to the GS, the normalized telescope signal (76) is just proportional to the pointing error \(\theta\),

\[ \frac{i^+ - w_i^-}{i^+ + w_i^-} = z_T = C_T \theta, \]

where \(C_T\) is the telescope scale factor. However, even for a perfect telescope the normalized telescope signal is inherently a nonlinear measure of the telescope pointing angle, so the question is just how large the nonlinear corrections are. On-orbit measurements of the GP-B telescope pointing demonstrated that the linear approximation is not accurate enough. The correction was represented by a defocus and by a wavefront error using the axially symmetric 4th order Zernike polynomial. In this way we established the conversion from the normalized telescope signal to the telescope pointing angle \(\theta\) for both axes in the form
with \( z_T \) given by the expression (76). The scale factor was found to be \( C_T^{-1} = 3.03 \) as, the cubic correction coefficient was \( \kappa = 16.2 \). With these numbers the nonlinearity correction was accurate to 4% for pointing angles less than 400 mas. The value of the scale factor was refined during data analysis by matching the telescope and gyroscope signals discussed in the following section.

6.3. SQUID to telescope signal matching

As just pointed out, the estimate of the telescope scale factors \( C_{Tx} \) and \( C_{Ty} \), relating the body-fixed pointing error \( \vec{\theta} = \{ \theta_x, \theta_y \} \) to the normalized telescope signal needs to be refined using on-orbit science data. The body-fixed vector \( \vec{\theta} \) and the inertial vector \( \vec{\theta}_\text{in} \), appearing in the formula (70) for the GSV pointing, are connected by a rotation due to S/C roll,

\[
\vec{\theta} = R(\vec{\phi})\vec{\theta}_\text{in}, \quad \vec{\theta}_\text{in} = R(-\vec{\phi})\vec{\theta}.
\]

Using this and denoting \( \vec{\theta} \) the sum of the optical effects in the formula (70),

\[
\vec{r}(t) = \vec{O}(t) + \vec{\theta}_\text{in}(t), \quad \vec{O}(t) = \vec{A}(t) + \vec{P}(t) + \vec{B}(t),
\]

we rewrite the SQUID measurement equation (50) with the body-fixed pointing error entering explicitly:

\[
z_{tx}(t) = C_{Tx} \left[ (O_{tx} - s_{tx}) \cos(\phi_x + \delta_x) + (O_{wy} - s_{wy})\sin(\phi_y + \delta_y) + \nu_x + \text{noise}; \right.

\]

\[
C_{gx} = C_g \cos \delta_x, \quad C_{gy} = C_g \sin \delta_x.
\]

(80)

For brevity, we use the linear telescope signal model (77),

\[
z_{Tx}(t) = C_{Tx} \theta_x(t) + \nu_x, \quad z_{Ty}(t) = C_{Ty} \theta_y(t) + \nu_y,
\]

(81)

with \( \nu_x \) denoting the noise in the telescope signal channel \( \xi \); the nonlinear correction included in the equation (78) was properly treated in the GP-B estimation process. Apparently, pointing errors can be eliminated from equation (80) by using equation (81) if the ratio of the SQUID and telescope scale factors is known. The procedure for finding this ratio and eliminating \( \theta_x, \theta_y \) is called SQUID-to-telescope scale factor matching.

To implement it, a dither signal, a sinusoid at a known stable frequency, was injected into the attitude control system which forced the S/C to slowly oscillate about the line of sight to the GS. The two dither signals in the \( x \)- and \( y \)-channels of the attitude control system were at different frequencies denoted \( \omega_{dx} \) and \( \omega_{dy} \). Cross-coupling due the S/C inertia and the attitude control system response made each of the dither frequencies appear in both channels, so the ‘true’ pointing errors \( \theta_x(t) \) and \( \theta_y(t) \) should be replaced, respectively, with

\[
\theta_x(t) + c_{xx} \cos(\omega_{dx}t) + s_{xx} \sin(\omega_{dx}t) + c_{xy} \cos(\omega_{dy}t) + s_{xy} \sin(\omega_{dy}t),
\]

\[
\theta_y(t) + c_{yx} \cos(\omega_{dx}t) + s_{yx} \sin(\omega_{dx}t) + c_{yy} \cos(\omega_{dy}t) + s_{yy} \sin(\omega_{dy}t),
\]

(82)

where all the coefficients are unknown. Thus both the telescope, (81), and SQUID, (80), signal equations are rewritten as (see the definition (79) of vector \( \vec{O} \)):
\[ z_g(t) = C_f \left[ (O_{ia} - s_{ia}) \cos(\varphi_f + \delta_f) + (O_{ae} - s_{ae}) \sin(\varphi_e + \delta_e) + \alpha \right] \\
+ b + C_{gx} \theta_y(t) + C_{gy} \theta_y(t) \\
+ \left( C_{gx}^2 c_{xx} + C_{gx}^2 c_{xx} \right) \cos(\omega_{dx} t) + \left( C_{gy}^2 s_{xx} + C_{gy}^2 s_{xx} \right) \sin(\omega_{dx} t) \\
+ \left( C_{gx} c_{xy} + C_{gy} c_{xy} \right) \cos(\omega_{dy} t) + \left( C_{gx} s_{xy} + C_{gy} s_{xy} \right) \sin(\omega_{dy} t); \]
\[ z_\tau_X(t) = C_{\tau_X} \left[ \theta_x(t) + c_{xx} \cos(\omega_{dx} t) + s_{xx} \sin(\omega_{dx} t) + c_{xy} \cos(\omega_{dy} t) + s_{xy} \sin(\omega_{dy} t) \right]; \]
\[ z_\tau_Y(t) = C_{\tau_Y} \left[ \theta_y(t) + c_{yx} \cos(\omega_{dx} t) + s_{yx} \sin(\omega_{dx} t) + c_{yy} \cos(\omega_{dy} t) + s_{yy} \sin(\omega_{dy} t) \right], \tag{83} \]

with the noise dropped from all the formulas, to save space.

The subtraction method of matching (see [36]) consists of two stages. At the first stage we analyze the SQUID and telescope signals separately, finding the amplitudes at the cosines and sines of the dither frequencies in each signal. Then, using the dither component as a benchmark for the whole pointing error, we find such a linear combination of the SQUID and telescope signals which is free of it.

To complete the first stage, we use the model (83) to analyze the SQUID signal without modeling pointing errors themselves except for their dither part. We find thus the estimates of all the dither frequency amplitudes,

\[ C_{gx} c_{xx} + C_{gy} c_{xx}, \ C_{gx} s_{xx} + C_{gy} s_{xx}, \ C_{gx} c_{xy} + C_{gy} c_{xy}, \ C_{gx} s_{xy} + C_{gy} s_{xy}. \tag{84} \]

As for the telescope signals \( z_\tau_X(t) \) and \( z_\tau_Y(t) \), we carry out their LSQ fit to the dither components shown in equations (83). This provides the other set of values

\[ C_{\tau_X} c_{xx}, \ C_{\tau_X} s_{xx}, \ C_{\tau_X} c_{xy}, \ C_{\tau_X} s_{xy}, \ C_{\tau_Y} c_{xx}, \ C_{\tau_Y} c_{xy}, \ C_{\tau_Y} s_{xx}, \ C_{\tau_Y} s_{xy}. \tag{85} \]

We then move on to the second stage of the subtraction method by looking for such a linear combination of the three signals,

\[ z_g^{\text{match}}(t) = z_g(t) + Q_1 z_\tau_X(t) + Q_2 z_\tau_Y(t), \tag{86} \]

that does not contain the dither terms, and, presumably, the pointing errors as well. Using equations (83), we require the coefficients at the two cosines and two sines of the dither frequencies to vanish. We thus obtain the following four linear equations for the two unknowns \( Q_{1,2} \) (in fact, they are simply the negatives of the scale factor ratios):

\[ C_{gx} c_{xx} + C_{gy} c_{xx} + Q_1 C_{\tau_X} c_{xx} + Q_2 C_{\tau_Y} c_{xx} = 0; \]
\[ C_{gx} s_{xx} + C_{gy} s_{xx} + Q_1 C_{\tau_X} s_{xx} + Q_2 C_{\tau_Y} s_{xx} = 0; \]
\[ C_{gx} c_{xy} + C_{gy} c_{xy} + Q_1 C_{\tau_X} c_{xy} + Q_2 C_{\tau_Y} c_{xy} = 0; \]
\[ C_{gx} s_{xy} + C_{gy} s_{xy} + Q_1 C_{\tau_X} s_{xy} + Q_2 C_{\tau_Y} s_{xy} = 0. \]

We solve this overdetermined system by means of the LSQ fit and find the values of \( Q_{1,2} \), thus completing the second stage of the procedure. With these estimated coefficients, the signal (86) becomes free of the pointing errors, and it is this signal that was used to estimate relativistic drift rates. The model for its analysis is given by equation (80) with \( \bar{\theta} = 0 \), i.e., it is the GSV SQUID signal model without pointing errors.

Note that the matching method can immediately incorporate the nonlinear model telescope signal (78), instead of the linear model (77): one only needs to replace everywhere \( z_\tau_X \) and \( z_\tau_Y \) with \( z_\tau_X \left( 1 + \kappa_\tau_{\tau_X} \right) \) and \( z_\tau_Y \left( 1 + \kappa_\tau_{\tau_Y} \right) \), respectively.
The described matching procedure was actually used at the preliminary stage of GP-B data analysis carried out by means of the 2-floor filter ([7], section 5.3). In it, the model for finding the combination of signals free of pointing errors was augmented by several other frequencies known to be present in the pointing signal. Those are the harmonics of the roll, roll plus or minus twice orbital frequency, and some others. In this way both the injected and natural dithers were exploited to achieve the most accurate estimate of the scale factor ratios.

The final GP-B results were obtained using the 2 s estimator ([8], section III) where the scale factor matching was not exploited: the algorithm used the ready pointing signal prepared by the recipes of section 6.1 (see also [7], section 5). However, the matching procedure not only worked nicely at preliminary analysis stages, but was also fundamentally important for validating the whole data analysis process.

7. Summary of science signal models

The description of all the models needed in the nonlinear multiparameter estimation problem of the GP-B data analysis is complete; here we summarize.

The reference frames needed for the model construction were defined in section 2, where we also discussed the GR predictions of the orbiting gyroscope precessions, see sections 2.2.1 and 2.2.3. In sections 2.2.2 and 2.4 two-dimensional inertially fixed vectors in the plane perpendicular to the direction to the GS, which play the key role in our modeling, were defined. Those are: \( \mathbf{s} \) (t), representing the gyro drift, \( \mathbf{r} \) (t), giving the S/C deviation from the direction to the GS, and their difference, the spin-to-roll misalignment vector \( \mathbf{\mu} \) (t).

The main measurement equation, i.e., the model of the LF SQUID signal, is given by the formula (50). Its submodels are: (a) SQUID scale factor model, equations (47), (41), (45) and (46), and bias model, equation (51); (b) the model of gyro drift \( \mathbf{s} \) (t) in the presence of the PE torques, equations (69), (66), (62), (64), (59), and (60); (c) the model of spacecraft pointing \( \mathbf{r} \) (t) during GSV, equation (70). The effects of optical aberration, paralax, and bending of GS light by the Sun, constituting \( \mathbf{r} \) (t) during GSV, are calculated by equations (72)–(74), respectively. Pointing errors, being also a part of \( \mathbf{r} \) (t), were removed from the SQUID signal using the normalized telescope signal whose model is given by the equation (78). The removal is possible due to the SQUID-to-telescope scale factor matching described in section 6.3; the resulting SQUID signal model for GSV free of pointing errors is given by equation (80) with \( \theta = 0 \).

Gyro polhode motion history in terms of the polhode angle \( \gamma_0 \) (t) and phase \( \phi_0 \) (t) is crucial for using models (a) and (b). Their accurate values, and other important parameters, were obtained by the analysis of HF SQUID signal called TFM found in section 5. The model for it is given by equations (A.18) and (A.19).

Note that all the models, with two minor exceptions, were carefully derived from the underlying physics, and their complete derivations are given in the proper sections and the three appendices. In just two exceptional cases (see sections 4.3 and 5.2) the model is based on the general principles of approximation theory.

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Appendix A. TF signal and polhode variations of SQUID scale factor.

Derivations

Here we derive the HF SQUID signal model used in TFM (section 3) along with its implications for the LF science analysis, that is, the model for the polhode variations of the gyro scale factor (see section 4.2). Certain versions of the calculations given in the next two sections can be found in theses [32, 43].

A.1. Magnetic field due to fluxons on the rotor surface

Introducing the magnetostatic potential in the usual way, \( \overline{B} = -\nabla \Psi_{\text{mag}} \), we obtain the following boundary value problem:

\[
\Delta \Psi_{\text{mag}} = 0, \quad r > a; \quad \Psi_{\text{mag}} \big|_{r \to \infty} = 0; \quad \frac{\partial \Psi_{\text{mag}}}{\partial r} \bigg|_{r=a} = \Phi_0 \sum_{n=1}^{N_{f}} \left[ \delta(\vec{r} - \vec{r}_n^+) - \delta(\vec{r} - \vec{r}_n^-) \right].
\] (A.1)

Here \( a = r_g \) is the rotor radius, and \( \{r, \theta, \phi\} \) are spherical coordinates in some rotor–fixed frame, in particular, in the principal inertial axis frame introduced in section 2.3. For brevity, we use the notations:

\[
\sum_{l=0}^{\infty} \sum_{m=-l}^{l} = \sum_{|m|=0}^{\infty} \sum_{|m|=0}^{\infty}, \quad \sum_{l=0}^{\infty} \sum_{m=-l}^{l} = \sum_{l=0}^{\infty} \sum_{m=0}^{\max(l,1)} .
\] (A.2)

Moreover, as before, \( \Phi_0 \) is the rotor flux quantum, \( N_{f} \) is the number of fluxon–antifluxon pairs, \( \vec{r}_n^\pm = \{a, \theta_0^\pm, \phi_0^\pm\} \) are the fluxon and antifluxon coordinates, respectively, and \( \delta(\vec{r} - \vec{r}_n^+) \) is the surface delta–function.

Standard separation of variables in the problem (A.1) leads to the solution

\[
\Psi_{\text{mag}} = \frac{\Phi_0}{2a} \sum_{l, m} \frac{1}{l + 1} \left( \frac{a}{r} \right)^{l+1} A_{lm} Y_{lm}(\theta, \phi), \quad r > a,
\] (A.3)

where \( \{Y_{lm}(\theta, \phi)\} \) is the orthonormal set of spherical harmonics [46]:

\[
Y_{lm}(\theta, \phi) = \sqrt{\frac{2l + 1}{4\pi} \frac{(l - m)!}{(l + m)!}} P_{m}^{l}(\cos \theta) e^{im\phi}; \quad Y_{lm}^*(\theta, \phi) = (-1)^m Y_{l-m}^-(\theta, \phi),
\] (A.4)

star(*) means, as usual, complex conjugation, and \( P_{m}^{l}(x) \) is the Legendre function. Due to the above property of spherical harmonics and the fact that \( \Psi_{\text{mag}} \) is real, the expansion coefficients satisfy the relation:

\[
A_{lm} = (-1)^m A_{l-m}^*. \quad \text{(A.5)}
\]

The boundary conditions imply

\[
A_{lm} = \sum_{n=1}^{N_{f}} \left[ Y_{lm}(\theta_n^+, \phi_n^+) - Y_{lm}(\theta_n^-, \phi_n^-) \right];
\] (A.6)

thus the coefficients \( A_{lm} \) are completely determined by the fluxon positions on the rotor surface. Because the latter are not known, coefficients \( A_{lm} \) can only be determined from the HF SQUID signal by TFM. Hence we need an expression for the flux, \( \Phi_{TF} \), through the pickup loop based on the potential \( \Psi_{\text{mag}} \) and the field determined by it. However, it is virtually impossible to integrate the field implied by the potential (A.3) as is, because in the rotor-fixed
frame the pick-up loop moves in a rather complicated way due to the rotor spin and spacecraft roll.

We therefore transform the result (A.3) to the pick-up loop frame where calculating the flux becomes easy. This is done by two coordinate rotations. The first goes from the rotor-fixed frame \( \{x, y, z\} \) of section 2.3 to the frame \( \{x', y', z'\} \) with the \( z' \)-axis along the instant spin vector \( \vec{\omega}_s = a_s \delta \). As seen in figure 3, this is a rotation by the three Euler angles \( \phi_p, \gamma_p, \phi'_p \):

\[
\vec{r}'' = R_3(\phi_p) R_2(\gamma_p) R_3(\phi'_p) \vec{r}; \quad \vec{r} = R_3(-\phi_p) R_2(-\gamma_p) R_3(-\phi'_p) \vec{r}''.
\]

The second equality gives the inverse transformation of the vector radius of an arbitrary point.

Spherical harmonics in both frames transform as in (4.8) from [46]:

\[
Y_{lm}(\theta, \phi) = \sum_{n=-l}^{l} d_{mn}^{l}(-\gamma) e^{i\mu \phi} Y_{ln}(\theta', \phi'),
\]

where \( d_{mn}^{l}(\alpha) \) is the spherical harmonics rotation matrix ([46], (4.13), (4.14); up to a trigonometric factor, it is some hypergeometric polynomial of the argument \( \tan^2(\alpha/2) \). This leads to the expression for the flux in the spin frame:

\[
\Psi_{\text{mag}} = \frac{\Phi_0}{2\alpha} \sum_{l,m} \frac{1}{l+1} \left( \frac{\alpha}{r} \right)^{l+1} A_{ln} e^{i\mu \phi} \sum_{n=-l}^{l} d_{mn}^{l}(-\gamma) e^{i\mu \phi} Y_{ln}(\theta', \phi').
\]

We have to change coordinates one more time, to the housing-fixed frame \( \{x'', y'', z''\} \) whose \( z'' \)-axis is aligned with the normal to the pick-up loop. According to figure A1, we use the following transformation (the last Euler angle is zero since we do not care about the particular directions of \( x'' \) and \( y'' \) axes):

\[
\vec{r}'' = R_2(0.5\pi - \beta) R_3(\phi) \vec{r}'''; \quad \vec{r}' = R_3(-\phi) R_2(-0.5\pi + \beta) \vec{r}''.
\]

Applying relation (A.7) to this rotation, we find

\[
Y_{ln}(\theta', \phi') = \sum_{n=-l}^{l} e^{i\mu \phi} d_{mn}^{l}(-\pi/2 + \beta) Y_{ln}(\theta'', \phi'').
\]
Introducing expression (A.10) to the formula (A.8), we arrive at:

\[
\psi_{mag} = \frac{\Phi_0}{2a} \sum_{l=1}^{\infty} \sum_{m,n,l=-l}^{m,n,l=1} A_{lm} e^{i\text{m} \phi} d_{nm}^l \left( -c_{l} \right) e^{i(n \phi + \phi)} d_{nm}^l \left( -\frac{\pi}{2} + \beta \right) Y_{l}^{m} \left( \theta^*, \phi^* \right). \tag{A.11}
\]

This expression is in the proper frame and is ready for the TF calculation.

A.2. TF through the pick-up loop. HF SQUID signal

Let \( R \) be the pick-up loop radius. To get the total trapped magnetic flux, \( \Phi_{TF} \), through the loop we need to integrate the magnetic field normal to it over the circle \( r^* < R \) in the plane \( z^* = 0 \). Technically it is more convenient to find the equal flux through the upper hemisphere \( \{ S^+ : r^* = R, \ 0 < \theta < (\pi/2) \} \), so that

\[
\Phi_{TF} = \int_{S^+} B_n \, dA = -\int_{0}^{\pi/2} \int_{0}^{2\pi} \frac{\partial \psi_{mag}}{\partial r} R^2 \sin \theta^* \, d\theta d\phi^*.
\]

Using (A.11) and integrating over the azimuthal angle

\[
\int_{0}^{2\pi} d\phi^* Y_{l}^{m} \left( \theta^*, \phi^* \right) = \sqrt{\pi} \delta_{lm} R_{l} \left( \cos \theta \right)
\]

eliminates the \( n' \) \( \neq 0 \) harmonics giving thus

\[
\Phi_{TF} = \frac{\Phi_0}{2} \sum_{l=1}^{\infty} \left( \frac{a}{R} \right)^l \sum_{m,n,l=-l}^{m,n,l=1} A_{lm} e^{i\text{m} \phi} d_{nm}^l \left( -c_{l} \right) e^{i(n \phi + \phi)} d_{nm}^l \left( -\frac{\pi}{2} + \beta \right) I_l. \tag{A.12}
\]

where

\[
I_l = \sqrt{\pi} \int_{0}^{\pi/2} \sin \theta^* R_{l} \left( \cos \theta \right) \left( \frac{(-1)^j (2j - 1)!!}{2^{j+2} (j + 1)! \sqrt{\pi}} \right) \delta_{l, 2j+1}, \quad l = 1, 2, 3, \ldots. \tag{A.13}
\]

There are no even harmonics of fluxon distribution in the result (A.12), since \( I_l = 0 \) for \( l = \) even; i.e., (A.12) contains only the coefficients \( A_{lm} \) with \( l = \) odd. Hence, remarkably, these coefficients, but not the ones with \( l = \) even, can only be found from the HF SQUID signal by TFM.

Finally, we need the flux expanded in spin harmonics (34). We find it by changing the order of summation in (A.12), namely:

\[
\Phi_{TF}(t) = \sum_{n=-\infty}^{\infty} e^{i n (\phi + \phi)} H_n(t) = \sum_{n=-\infty}^{\infty} e^{i n (\phi + \phi)} \times \frac{\Phi_0}{2} \sum_{l=odd, \ |l|, |l|}^{\infty} \left( \frac{a}{R} \right)^l \sum_{m,n,l=-l}^{m,n,l=1} A_{lm} e^{i\text{m} \phi} d_{nm}^l \left( -c_{l} \right) e^{i(n \phi + \phi)} d_{nm}^l \left( -\frac{\pi}{2} + \beta \right) I_l. \tag{A.14}
\]

The values of the complex harmonic amplitudes \( H_n(t) = H_n^*(t) \) are clearly seen from (A.14). However, we need to verify the second representation of the TF signal from (34) for our case of small spin-to-pick-up loop misalignment \( \beta \). To do this, we use formula (4.30) from [46] which gives:

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Using this and 3.4.(20), (23) from [48], for \( l = \text{odd} \) and \( \beta \to 0 \) we obtain:

\[
d_{ln}(-\pi/2 + \beta) = (1)^n \frac{[(l - n)!]}{l + n!} P_l^n (\sin \beta), \quad (A.15)
\]

(A.15)

where \( F(x) \) is the Euler gamma-function. Substituting expressions (A.16) and (A.17) in the formula (A.14) provides exactly the second expression (34) for the TF with explicitly written spin harmonics:

\[
\Phi_{Tl}(t) = \sum_{|n|=\text{odd}} H_n(t) e^{i\nu n} + \beta(t) \sum_{|n|=\text{even}} h_n(t) e^{i\nu n}; \quad (A.18)
\]

\[
H_n\left(\phi, \gamma_p\right) = \frac{\Phi_0}{2} \sum_{i=\text{odd}, \beta|n|}^{\infty} \left( \alpha \frac{1}{R} \right) \left( l \sum_{m=-l}^{l} A_{ln} \sin \theta d_m l \left( -\gamma_p \right) \right), \quad n = \text{odd}; \quad (A.19)
\]

\[
h_n\left(\phi, \gamma_p\right) = \frac{\Phi_0}{2} \sum_{i=\text{odd}, \beta|n|}^{\infty} \left( \alpha \frac{1}{R} \right) \left( l \sum_{m=-l}^{l} A_{ln} \sin \theta d_m l \left( -\gamma_p \right) \right), \quad n = \text{even}; \quad (A.20)
\]

quantities \( I_l, p_{ln}, q_{ln} \) are given in (A.13), (A.16) and (A.17). This is the model for TFM.

A.3. Perturbations: robustness of the TFM model

The above TFM model is actually derived under a number of idealised assumptions about the geometry of the problem. In reality each pick-up loop is protected by a local superconducting shield. The shield has the form of a circular cylinder and, in the perfect case, it is centered on the pick-up loop and aligned with its normal. The shield radius, \( R_s \), is about 1.5 times larger than that of the pick-up loop, \( R_s \sim 1.5 R \), so the shield is by no means a small perturbation. To model it, we have found the solution of the magnetostatics problem for an infinite cylindrical shield in terms of the coefficients \( A_{ln} \), and recalculated the flux through the pick-up loop. The result is the same equations (A.18) – (A.20) with a single replacement

\[
A_{ln} \longrightarrow A_{ln} \left[ 1 + X_l \left( R/R_s \right) \right], \quad (A.21)
\]

where the positive correction \( X_l < 0.4 \) and drops exponentially with \( l \):

\[
0 < X_l < 2^{-1/2} \left( R/R_s \right)^{1/2}, \quad l = 1, 2, \ldots
\]

Thus there is no change in the TFM due to the local shield, except in the interpretation: the obtained estimates were not of \( A_{ln} \), but of the quantities \( A_{ln} \left[ 1 + X_l \left( R/R_s \right) \right] \) close to it. In particular, the TFM results for the polhode angle and phase, and for the contribution (A.22) to the SQUID scale factor remain valid.

The other deviations from the ideal configuration are all small effects, which were examined independently to lowest (linear) order in five pertinent small parameters:
1. Small misalignment of the pick-up loop and the local shield.
2. Small deviation of the pick-up loop shape from the circular one.
3. Small additional in-plane pick-up loop area.
4. Small miscentering of the pick-up loop relative to the local shield.
5. Small miscentering of the rotor relative to the pick-up loop.

We obtained the TF expression for all these cases with the local shield present. In case 1 the odd harmonics (the only ones participating in TFM) are not changed at all to l.o. Case 2 leads to another replacement in (A.21),

\[ A_{lm} \rightarrow A_{lm} \left[ 1 + X_l \left( \frac{R}{R_s} \right) + \lambda Z_l \left( \frac{R}{R_s} \right) \right], \]

where \( \lambda \ll 10^{-3} \) is the maximum relative out-of-roundness of the pick-up loop, and \( Z_l \) is a correction with the same properties as \( X_l \). The TFM results remain unchanged.

Finally, the corrections in the remaining cases 3–5 are small enough to be neglected at the accuracy of TFM. Therefore the model (A.18)–(A.20) for TFM is robust.

A.4. TF addition to LM scale factor

By definition (35) (see also expressions (42) and (A.20)), the TF addition to the LM scale factor is:

\[ C_{TF}(t) = h_0(t) = \frac{\Phi_0}{2} \sum_{|m| \leq l_0} \left( \frac{a}{R} \right)^l \sum_{m=1}^l A_{lm} d_{lm}^l \left[ -\gamma_p \right] e^{i m \phi} q_{l0}. \]  

(A.22)

This formula, however, is inconvenient for estimating the variations (as explained in DA III, it is used to compute an initial approximation for \( C_{TF}(t) \) using the TFM results). For this reason, we convert it to a different form. Writing \( e_p(t) = \tan \left[ \frac{\gamma_p(t)}{2} \right] \rightarrow 0, \ t \rightarrow \infty \) and using the definition of the rotation matrix \( d_{lm}^l (-\gamma_p) \) ([46], (4.14)) we get its power series in the small parameter \( e_p \):

\[ d_{lm}^l (-\gamma_p) = e_p^m \sum_{k=0}^\infty d_{lmk} e_p^{2k} \quad l = \text{odd}; \]

(A.23)

here \( d_{lmk} \) are known numbers. Introducing this in the expression (A.22), changing the order of summation and slightly polishing the result, we obtain:

\[ C_{TF}(t) = \sum_{n=0}^\infty \left[ e_n^l (e_p) \cos n \phi_p(t) + e_n^s (e_p) \sin n \phi_p(t) \right]. \]

(A.24)

\[ e_n^l (e_p) = e_p^n \sum_{k=0}^\infty e_{lk}^l e_p^{2k}, \quad m = 1, 2, \ldots, \]

(A.25)

where the coefficients \( e_{lk}^l \) are the proper linear combinations of \( A_{lm}, l = \text{odd} \), weighted by \( d_{lmk} \), and \( e_p = \tan(\gamma_p/2); \gamma_p = \gamma_p(t, Q), \phi_p = \phi_p(t, Q) \) are the rotor polhode angle and phase dependent on the asymmetry parameter \( Q \) (section 2.3).

The advantage of the estimation model (A.24) and (A.25) is that \( e_p \rightarrow 0 \) with time, as the spin axis approaches its asymptotic position along the maximum inertia axis. There are also significant disadvantages, such as the dependence on \( Q \) in both \( \gamma_p \) and \( \phi_p \), and especially the lack of separation between oscillatory (polhode) and monotonic (dissipative) motion. Indeed, in the dissipative free motion of an asymmetric rotor, the angle \( \gamma_p \) (and hence \( e_p \)) changes with the time in two ways: it oscillates with the period \( T_p/2 \), and also...
slowly goes to zero. Thus \( \epsilon_p \), and with it \( \epsilon_{nk}^p(\epsilon_p) \), are expanded in even harmonics of \( \phi_p \). In turn, the phase \( \phi_p(t) \) differs from a straight line due to the rotor asymmetry within a single polhode period, but the period itself is slowly shrinking. So two very different time scales are present in these variables: \( \gamma_p = \gamma_p(t/T_p, t/\tau_{\text{dis}}, Q) \), \( \phi_p = \phi_p(t/T_p, t/\tau_{\text{dis}}, Q) \), with \( T_p = T_p(t/\tau_{\text{dis}}) \).

To separate the two motions completely and to avoid errors related to inaccuracy in \( Q \) we use, instead of \( \phi_p \) and \( \gamma_p \), the polhode parameters of a symmetric \((Q = 0)\) gyroscope with the same time history of the polhode period, \( \gamma_0 = \gamma_0(q=0) \), \( \phi_0 = \phi_0(q=0) \). They are given by simple formulas:

\[
\cos \gamma_0 = \frac{T_p}{T_p(t/\tau_{\text{dis}})}, \quad \phi_0 = 2\pi \int_0^t \frac{\text{d}t'}{T_p(t'/\tau_{\text{dis}})},
\]

(A.26)

where \( T_p \) is the asymptotic value of the polhode period. Note that \( \gamma_0 \) is constant, and \( \phi_0 \) is linear in time up to a slow dissipative change. We now transform representation (A.24), (A.25) to these variables.

**A.5. Final form of \( \mathcal{C}_T(t) \) model using polhode phase and angle of a symmetric rotor**

We start by substituting expressions (A.25) in the basic formula (A.24):

\[
\mathcal{C}_T(\gamma_p, \phi_p) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left[ c_{nk}^p \epsilon_p^{n+2k} \cos n\phi_p + c_{nk}^p \epsilon_p^{n+2k} \sin n\phi_p \right].
\]

(A.27)

Locally, that is, over times much shorter than \( \tau_{\text{dis}} \) (see few polhode periods), harmonics of \( \phi_p \) are periodic with period \( T_p(t/\tau_{\text{dis}}) \), and \( \gamma_p \) and \( \epsilon_p \) are periodic with the period \( 0.5T_p(t/\tau_{\text{dis}}) \).

The functions \( \epsilon_p^{n+2k} \cos n\phi_p \), \( \epsilon_p^{n+2k} \sin n\phi_p \) are also periodic with the period \( T_p(t/\tau_{\text{dis}}) \), so they can be represented by Fourier series in \( \phi_0 \):

\[
\epsilon_p^{n+2k} \cos n\phi_p = \sum_{m=0}^{\infty} A_{nk} \cos m\phi_0, \quad \epsilon_p^{n+2k} \sin n\phi_p = \sum_{m=0}^{\infty} B_{nk} \sin m\phi_0.
\]

(A.28)

The Fourier coefficients

\[
A_{nk} = \frac{1}{\pi(1 + \delta_{n0})} \int_0^{2\pi} \epsilon_p^{n+2k} \cos n\phi_p \cos m\phi_0 \, \text{d}\phi_0,
\]

\[
B_{nk} = \frac{1}{\pi} \int_0^{2\pi} \epsilon_p^{n+2k} \sin n\phi_p \sin m\phi_0 \, \text{d}\phi_0
\]

change slowly with time, \( A_{nk}(t/\tau_{\text{dis}}) \), \( B_{nk}(t/\tau_{\text{dis}}) \). They can be equivalently written as \( A_{nk}(\epsilon_0) \), \( B_{nk}(\epsilon_0) \), where \( \epsilon_0 = \tan(\gamma_0/2) \) goes to zero monotonically. Combining expansions (A.27) and (A.28) gives

\[
\mathcal{C}_T = \mathcal{C}_T(\gamma_0, \phi_0) = \sum_{m=0}^{\infty} \left[ C_m^c(\epsilon_0) \cos m\phi_0 + C_m^s(\epsilon_0) \sin m\phi_0 \right].
\]

(A.29)

\[
C_m^c(\epsilon_0) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_{nk}^p A_{nk}(\epsilon_0), \quad C_m^s(\epsilon_0) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_{nk}^p B_{nk}(\epsilon_0).
\]

(A.30)
The functions \( C_m^e(\epsilon_0) \) and \( C_m^s(\epsilon_0) \) are regular in \( \epsilon_0 \) at least near \( \epsilon_0 = 0 \), and, for \( m \neq 0 \), tend to zero in the asymptotic limit,

\[
C_m^e(\epsilon_0) \to 0, \quad C_m^s(\epsilon_0) \to 0 \quad \text{for} \quad \epsilon_0 \to +0, \quad m = 1, 2, \ldots.
\]

Therefore they are represented by the series

\[
C_m^e(\epsilon_0) = \sum_{j=0}^{\infty} C_{mj}^e \epsilon_0^j; \quad C_m^s(\epsilon_0) = \epsilon_0 \sum_{j=0}^{\infty} C_{mj}^s \epsilon_0^j, \quad m = 1, 2, \ldots.
\]

(A.31)

Coefficients \( C_{mj}^e \) are completely determined by the fluxon distribution and the asymmetry parameter \( Q \) (through \( A_{nkm}, B_{nkm} \) in (A.30)). Equalities (A.29) and (A.31) constitute the final model for \( C_{Tf} \) used in section 4.2.

Appendix B. Angle \( \beta \) between the spin axis and pick-up loop plane during guide star invalid

By formula (39) of section 4.4, this angle is determined by the scalar product of the spin unit vector \( \hat{s} \) and the unit normal to the pick-up loop plane \( \hat{n} \):

\[ \beta \approx \sin \beta = \hat{s} \cdot \hat{n}. \]

The dot product is already calculated for GSV periods using the inertial expression (33) for the normal to the pick-up loop. The derivation and result is slightly different for the GSI part of an orbit. As compared to GSV, the main complications for GSI are: (1) the roll axis generally does not coincide with the telescope axis, \( \hat{\tau} \neq \hat{z}_b \), forming a non-zero angle, \( \nu \), with it; and (2) the roll axis generally moves relative to the spacecraft, so that \( \nu = \nu(t) \).

On the other hand, the situation simplifies because \( \nu \) was small during the flight, \( \nu \lesssim 2 \times 10^{-5} \), so we need only linear terms in \( \nu \). Still, the conversion formulas (30)–(32) relating the body-fixed and the roll axis frames should be corrected. The corrections change the inertial representation (33) of the pick-up loop normal, thus of the angle \( \beta \).

An arbitrary vector radius, \( \vec{r} \), in the roll axis frame is related to \( \vec{r}_b \) in the body–fixed frame by the equalities:

\[
\vec{r} = R_3(\phi_f)R_2(\nu)R_3(\delta_b)\vec{r}_b, \quad \vec{r}_b = R_3(-\delta_b)R_2(-\nu)R_3(-\phi_f)\vec{r}, \quad (B.1)
\]

where \( R_i \) is the rotation matrix introduced (see section 2), and \( \phi_f \) is the roll phase. The notation \( \delta_b \) here has a slightly different meaning than in section 2.5: it is the angle between the \( x_b \)-axis and the projection of the instant position of the roll axis on the body–fixed plane \{\( \hat{x}_b, \hat{y}_b \)\}, so that generally \( \delta_b = \delta_b(t) \). For \( \nu = 0 \) angle \( \delta_b \) reacquires its meaning of the roll phase offset.

Therefore by setting \( \vec{r} = \hat{x}_b, \hat{y}_b, \hat{z}_b \) in the transformation (B.1) one finds these vectors through \( \vec{x}^t, \vec{y}^t, \vec{z}^t \) in a straightforward, although cumbersome, way. Of course, for \( \nu = 0 \) the result exactly coincides with expressions (30)–(32), so we give only the corrections, namely,

\[
\Delta \hat{x}_b \equiv \hat{x}_b \big|_{\nu=0}, \quad \Delta \hat{y}_b \equiv \hat{y}_b \big|_{\nu=0}, \quad \Delta \hat{z}_b \equiv \hat{z}_b \big|_{\nu=0}.
\]
Keeping only linear and quadratic terms in all the small angles involved, we obtain:
\[
\Delta \hat{x}_b = -\nu \cos \delta_r \left[ 0.5\nu \left( \cos \phi \ \hat{x}^r + \sin \phi \ \hat{y}^r \right) + \hat{z} \right] + \text{cubic terms in } \tau \text{ and } \nu;
\]
\[
\Delta \hat{z}_b = \nu \left[ \left( \cos \phi \ \hat{x}^r - \sin \phi \ \hat{y}^r \right) - 0.5\nu \ \hat{z} \right] + \text{cubic terms in } \tau \text{ and } \nu.
\]
(B.2)

We do not give \( \Delta \hat{y}_b \) because it is not involved in \( \Delta \hat{n} \), which we only need for the answer. Using formula (29) for normal in the body-fixed frame, we obtain
\[
\Delta \hat{n} \equiv \hat{n} - \hat{n}_{|_{\omega = 0}} = (1 - 0.5\alpha^2)\Delta \hat{x}_b + \alpha \Delta \hat{z}_b + \text{cubic terms in } \alpha, \ \tau \text{ and } \nu
\]
\[
= \Delta \hat{x}_b + \alpha \Delta \hat{z}_b + \text{cubic terms in } \alpha, \ \tau \text{ and } \nu,
\]
(B.3)

where the last representation is true because, by (B.2), \( \Delta \hat{x}_b \) itself is proportional to \( \nu \). In addition, \( \Delta \hat{z}_b \) is needed only to linear order, due to the small factor \( \alpha \) in front of it in the utmost right of the previous formula.

With that in mind, we use formulas (18)–(19) (roll frame unit vectors in the GS inertial frame) in the formulas (B.2) to find the inertial corrections:
\[
\Delta \hat{x}_b = -\nu \cos \delta_r \left\{ \hat{e}_{ps} + 0.5\nu \left[ (\tau_{ms} - \cos \phi \hat{f}) \hat{e}_{ms} + (\tau_{we} - \sin \phi \hat{f}) \hat{e}_{we} \right] \right\}
\]
+ cubic terms in \( \alpha, \ \tau \text{ and } \nu; \)
\[
\Delta \hat{z}_b = \nu \left( \cos \phi \hat{f} \hat{e}_{ms} + \sin \phi \hat{f} \hat{e}_{we} \right) + \text{quadratic terms in } \alpha, \ \tau \text{ and } \nu.
\]
(B.4)

The inertial expression for \( \Delta \hat{n} \) follows from introducing these results to the equation (B.3). In the process we drop a lot of terms because in the correction \( \Delta \beta = \hat{s} \cdot \Delta \hat{n} \) we multiply \( \Delta \hat{n} \) by the spin whose GS component is of order unity, and the other two are small. Therefore the combination of (B.3) and (B.4) is remarkably simple:
\[
\Delta \hat{n} = -\nu \cos \delta_r \hat{e}_{ps} + \text{cubic terms in } \alpha, \ \tau, \ \nu \text{ orthogonal to } \hat{e}_{ps},
\]
(B.5)

the dropped terms make at most a cubic contribution to the angle \( \beta \). This is the needed GSI correction to the inertial expression (33) for the pick-up loop normal. To avoid any misunderstanding, let us emphasize that the vectors \( \hat{e}_{ps}, \ \hat{z}_b, \ \text{and} \ \hat{n} \) are fixed in the spacecraft body, so they are, of course, the same for both GSV and GSI. However, they are different for GSV and GSI in the inertial space due to the different ATC performance in keeping the S/C attitude. Exactly this difference is seen in the formulas (B.4), (B.5) (as usual, ATC stands for the attitude and translation control system).

By (B.5) the GSI correction to the GSV value of angle \( \beta \) is:
\[
\beta - \beta_{|_{\omega = 0}} = \hat{s} \cdot \Delta \hat{n} = -\nu \cos \delta_r + \text{cubic terms in } \alpha, \ \tau, \ \nu \text{ and } s.
\]
(B.6)

Introducing the expression (49) for \( \beta_{|_{\omega = 0}} \), we arrive at the answer:
\[
\beta = (\tau_{ms} - s_{ms}) \cos \left( \phi \hat{f} + \delta_r \right) + (\tau_{we} - s_{we}) \sin \left( \phi \hat{f} + \delta_r \right) + \alpha - \nu \cos \delta_r
\]
+ cubic terms in \( \alpha, \ \tau, \ \nu \text{ and } s
\]
\[
= \mu_{ms} \cos \left( \phi \hat{f} + \delta_r \right) + \mu_{we} \sin \left( \phi \hat{f} + \delta_r \right) + \alpha - \nu \cos \delta_r + \text{cubic terms}.
\]
(B.7)

All quadratic terms have nicely disappeared once again. The on-orbit magnitude of the corrections is almost the same as for the GSV: \( \sim 10^{-11} \) for any GSI outside the anomalies, and \( \lesssim 10^{-9} \) in the majority of anomalies with the pointing \( \lesssim 500 \) as. Expression (B.7) turns into the GSV formula (49) when \( \nu = 0 \). The model (B.7) was further developed and used to determine
Appendix C. PE torques

C.1. Electrostatic potential and energy in the small gap between concentric spherical rotor and housing with arbitrary potential distributions on them

The electrostatic potential, $\Psi_{el}$, related to the field in the usual way, $\vec{E} = -\nabla \Psi_{el}$, satisfies the following boundary value problem in the gap between the rotor and its housing:

$$\Delta \Psi_{el} = 0, \quad a < r < b;$$
$$\Psi_{el}|_{r=a} = V_a(\theta, \varphi) = \sum_{l,m} R_{lm} Y_{lm}(\theta, \varphi);$$
$$\Psi_{el}|_{r=b} = V_b(\theta, \varphi) = \sum_{l,m} H_{lm} Y_{lm}(\theta, \varphi). \quad (C.1)$$

Here $a$, $b$ are the rotor and housing radii, $Y_{lm}(\theta, \varphi)$ are the orthonormal spherical harmonics (A.4), and we are using the notation (A.2) for the proper sums in $l$ and $m$. The coefficients $R_{lm}$ and $H_{lm}$ specify the electrical patch patterns on the rotor and housing, respectively. It is important that the voltage distributions are fixed here in an arbitrary centered frame, but the same one for the rotor and housing.

The standard separation of variables in the Laplace equation (C.1) in the gap provides the solution representation

$$\Psi_{el} = \sum_{l,m} \left[ C_{lm} \left( \frac{r}{a} \right)^l + D_{lm} \left( \frac{b}{r} \right)^{l+1} \right] Y_{lm}(\theta, \varphi), \quad (C.2)$$

whose coefficients are readily found from the boundary conditions:

$$C_{lm} = \frac{(b/a)^{l+1}H_{lm} - R_{lm}}{(b/a)^{2l+1} - 1}, \quad D_{lm} = \frac{(b/a)^lR_{lm} - H_{lm}}{(b/a)^{2l+1} - 1}.$$  

The gap, $d = b - a$, is very small as compared to the radii, $d/a \sim 10^{-4}$, so we need the solution only to l.o. in this small parameter. Replacing the ratio $b/a$ with unity in the numerators and transforming the denominators as

$$(b/a)^{2l+1} - 1 = (d/a)\left( (b/a)^2 + (b/a)^{2l-1} + \ldots + (b/a) + 1 \right),$$

we arrive at the following l.o. expressions for the coefficients:

$$C_{lm} = -D_{lm} = \frac{a}{d} \frac{H_{lm} - R_{lm}}{2l + 1}. \quad (C.3)$$

Thus by the equations (C.2) and (C.3) the potential to l.o. in $d/a$ is:

$$\Psi_{el} = \frac{a}{d} \sum_{l,m} \left( H_{lm} - R_{lm} \right) \left( \frac{r}{a} \right)^l - \left( \frac{b}{r} \right)^{l+1} \frac{Y_{lm}(\theta, \varphi)}{2l + 1}. \quad (C.4)$$

Moreover, using once again $a \approx b$ after differentiation, we obtain, in the same approximation, the normal field at the boundaries:
\[
\frac{\partial \psi_{cl}}{\partial r} \bigg|_{r=a} = \frac{\partial \psi_{cl}}{\partial r} \bigg|_{r=b} = \frac{1}{d} \sum_{l,m} (H_{lm} - R_{lm}) Y_{lm}(\theta, \varphi) = \frac{1}{d} \left[ V_b(\theta, \varphi) - V_a(\theta, \varphi) \right];
\] (C.5)

This result is anticipated by the usual analogy with a capacitor.

To calculate the electrostatic energy in the gap, we multiply the Laplace equation (C.1) by \( \psi_{cl} \) and integrate by parts:

\[
0 = \int_{\text{gap}} \psi_{cl} \Delta \psi_{cl} \, dV = \int_{\text{gap}} (\nabla \psi_{cl})^2 \, dV - \left[ \int_{r=b} \psi_{cl} \frac{\partial \psi_{cl}}{\partial r} \, dA - \int_{r=a} \psi_{cl} \frac{\partial \psi_{cl}}{\partial r} \, dA \right].
\]

Using this and the expression (C.5), the energy to l.o. is found to be

\[
W_{\text{tot}} = \frac{\varepsilon_0}{2} \int_{\text{gap}} (\nabla \psi_{cl})^2 \, dV = \frac{\varepsilon_0}{2} \left[ \int_{r=b} \psi_{cl} \frac{\partial \psi_{cl}}{\partial r} \, dA - \int_{r=a} \psi_{cl} \frac{\partial \psi_{cl}}{\partial r} \, dA \right]
= \frac{\varepsilon_0 a^2}{2d} \sum_{l,m} (H_{lm} - R_{lm}) \left( H_{lm}^* - R_{lm}^* \right).
\] (C.6)

Here \( d\Omega \) is the element of the solid angle, and the last equality holds by the Parceval theorem. The part of the total energy \( W_{\text{tot}} \) proportional to \( ||V_a||^2 + ||V_b||^2 = \sum_{l,m} (|R_{lm}|^2 + |H_{lm}|^2) \) does not change when the rotor is rotated relative to the housing. So only

\[
W = -\frac{\varepsilon_0 a^2}{2d} \sum_{l,m} (H_{lm} R_{lm}^* + H_{lm}^* R_{lm}),
\]

is relevant to the electrostatic torque calculation; it is called just ‘energy’ below. Note that the property (A.4) of spherical harmonics and the fact that the potential distributions \( V_a \) and \( V_b \) in (C.1) are real imply the same relations for \( R_{lm} \) and \( H_{lm} \) as in equation (A.5) for \( A_{lm} \). That is why the energy can be also written as

\[
W = -\frac{\varepsilon_0 a^2}{d} \sum_{l,m} H_{lm} R_{lm},
\] (C.7)
a slightly more convenient form.

C.2. PE torques on the spinning rotor inside rolling housing: general expressions

By energy conservation, the torque \( \vec{T}_\eta \) on the rotor rotated by the angle \( \eta \) about the axis \( \hat{\eta} \) is (see [47], p 59):

\[
\vec{T}_\eta = -\frac{\partial W}{\partial \eta} \hat{\eta},
\] (C.8)

where \( W \) is the electrostatic energy that depends on the mutual position of the rotor and housing. To find the torque effectively, we cannot immediately use the energy value (C.7), because it lacks the explicit configuration dependence hidden in the coefficients \( R_{lm} \) and \( H_{lm} \).

Indeed, the real patch distribution on the rotor and housing is specified in the rotor-fixed (primed) and S/C-fixed (double primed) frames, respectively:
\[ V_a(\theta', \varphi') = \sum_{l,m} R_{lm}^a Y_{lm}(\theta', \varphi'); \quad V_b(\theta'', \varphi'') = \sum_{l,m} H_{lm}^b Y_{lm}(\theta'', \varphi''); \]  

(C.9)

in these frames the coefficients \( R_{lm}^a, H_{lm}^b \) are constant, and they also satisfy relations (A.5). We need to convert this into representations (C.1) in the same frame, i.e., to express the unprimed coefficients \( R_{lm}^a, H_{lm}^b \) in terms of primed ones.

For the housing, let us fix the original distribution (C.9) in the frame where the \( z \)-axis coincides with the S/C axis, \( \hat{z}_b \), from section 2.5. Neglecting any small difference between \( z_b \) and the roll axis, we see that transforming to the (quasi-) inertial frame with \( \hat{z} = \hat{z}_b = \hat{z} \) involves only a rotation by the satellite roll phase,

\[ H_{lm} = e^{im\theta} H_{lm}^b. \]  

(C.10)

The potential on the rotor is naturally specified in the frame of its principal axes of inertia from section 2.2.3, whose \( 3 \rightarrow 2 \rightarrow 3 \) Euler rotation to the spin-frame, \( \hat{z} = \hat{z}_s \), was described in section A.1 of appendix A (see formulas (A.9) and (A.10)). To go from this frame to the roll axis frame, another \( 2 \rightarrow 3 \) rotation is needed: first by the spin-to-roll misalignment angle (\(-\psi\)) (section 2.4), to align with the roll axis, and then by an angle (\(-\xi\)); these two angles specify the position of the spin axis in the roll axis frame. Applying the spherical harmonics transformation (A.7) corresponding to these rotations twice, we obtain

\[ R_{lm} = e^{-im\psi} \sum_{n=-l}^{l} \sum_{q=-l}^{l} R_{nq}^s d_{nm}^l(\psi) d_{mq}^l(-\psi) e^{i(n\theta+q\phi)}. \]  

(C.11)

It is now straightforward to get the needed representation of energy by introducing expressions (C.10) and (C.11) to the formula (C.7):

\[ W = -e_0a^2 \frac{d}{d} \sum_{l,m,n=-l}^{l} \sum_{q=-l}^{l} H_{nm}^s R_{nq}^s d_{nm}^l(\psi) d_{mq}^l(-\psi) e^{i(n\theta+q\phi)} e^{-im(\xi+\phi)}, \]

(C.12)

Here the mutual orientation of the rotor and housing is given explicitly by the angles \( \xi + \phi_\xi \), \( \psi \) and \( \phi_\psi \), and the three torque components \( T_x, T_y, T_z \) are obtained from expression (C.12) by differentiating it by the proper angles according to the formula (C.8) (note that the same result for \( T_z \) follows if one differentiates in \( \xi \), or \( \phi_\xi \), or \( \xi + \phi_\xi \)). These torques are fixed in the frame with \( \hat{z} = \hat{z} \) along the apparent direction to the GS, and their cartesian components are given by the formulas:

\[ T_x = -T_z \cos \xi \cot \psi - T_y \sin \xi, \quad T_x = -T_z \sin \xi \cot \psi + T_y \cos \xi, \]
\[ T_z = T_x = T_y, \]

(C.13)

Neglecting the small difference between the apparent and the true direction to the GS (see section 2.4), we can identify \( \{ \hat{z} = \hat{z}, \hat{z}_s, \hat{z}_b \} \) with \( \{ \hat{e}_{gs}, \hat{e}_{ns}, \hat{e}_{ws} \} \), so that in particular \( T_i = T_{ns}, \ T_i = T_{ws}, \) with sufficient accuracy.

The energy (C.12) and the three torques generated by it oscillate with the spin frequency which is rather high relative to all other frequencies involved, including the data sampling rates. For this reason, only the spin-averaged values of the torques are needed for the analysis, and, since they are obtained by the linear operation of differentiation, the energy itself can be averaged over \( \phi_i \), resulting in
Allowing for some notation, this exactly coincides with formula (7) of our paper [41]; the spin-averaged torques, for which we use the same notation as above, are:

\[ T_{\psi} = -\frac{\epsilon_0 a^2}{d} \sum_{l, m, q=-l}^{l} H_{lm}^* R_{i\theta} d_{m0}^{l}(\psi) d_{i\theta}^{l}(-\gamma_p)e^{i\phi_p}e^{-im(\zeta+\phi)}. \]  

(C.14)

Here prime denotes the derivative in \( \psi \); note that in \( T_{\xi} \) there is no \( m = 0 \) contribution independent of roll. The corresponding cartesian torque components are derived from this by the two first equations (C.13).

**C.3. PE torques to linear order in small spin-to-roll misalignment**

The above results are dramatically simplified by the small value of the spin-to-roll misalignment angle, \( \psi \lesssim 10^{-4} \); PE torques on GP-B rotors are calculated just to linear order in \( \psi \). The dependence (C.14) on \( \psi \) is given by

\[ d_{m0}^{l}(\psi) = O(\psi^{|m|}), \quad \psi \to 0; \]

these functions are regular around zero, as seen c.f. from formulas (4.14), (4.15) in [46]. Thus only the terms \( |m| = 0, 1, 2 \) are considered; the \( |m| = 2 \) term is needed: \( T_{\psi} \) contains the derivative in \( \psi \), and the torque components (C.16) to l.o.,

\[ T_{\xi} = -\frac{1}{\psi} T_{\xi} \cos \xi - T_{\psi} \sin \xi, \quad T_{\psi} = -\frac{1}{\psi} T_{\xi} \sin \xi + T_{\psi} \cos \xi, \]  

(C.16)

contain only the combination \( T_{\xi}/\psi \). From the general formulas cited above the \( \psi \to 0 \) asymptotics follows: for \( T_{\xi} \),

\[ d_{\pm 10}^{l}(\psi) = \mp \frac{l(l+1)}{2} \psi + O(\psi^3); \]

\[ d_{\pm 20}^{l}(\psi) = \frac{1}{8} (l-1)(l+1)(l+2)\psi^2 + O(\psi^4). \]  

(C.17)

(recall that there is no \( m = 0 \) term in \( T_{\xi}^{+} \)), and for \( T_{\psi} \),

\[ \left[ d_{00}^{l}(\psi) \right] = -\frac{l(l+1)}{2} \psi + O(\psi^3); \]

\[ \left[ d_{\pm 10}^{l}(\psi) \right] = \frac{1}{2} \frac{l(l+1)}{2} + O(\psi^2); \]

\[ \left[ d_{\pm 20}^{l}(\psi) \right] = \frac{1}{4} (l-1)(l+1)(l+2)\psi + O(\psi^3). \]  

(C.18)

These expressions allow us to get the needed result.

**C.3.1. Misalignment torque.** Let us start with the first order in \( \psi \). One linear term in \( T_{\psi} \) comes from the first line of equations (C.18) corresponding to \( m = 0 \). In this case one does not even
need formulas (C.16) to get the inertial torque components, applying instead the general torque definition (C.8) and equations (C.12) and (C.18) as follows:

\[
\tilde{T}_\psi = -\frac{\partial \tilde{W}_\psi}{\partial \tilde{\psi}} |_{m=0} \tilde{\psi} = \psi \kappa \tilde{\psi} = (\psi/\mu) \kappa \left( \mu_w \tilde{e}_{ns} - \mu_e \tilde{e}_{we} \right) = \kappa \tilde{\mu} \times \tilde{e}_p, \tag{C.19}
\]

\(\tilde{\mu}\) being the misalignment vector (21). Here we used expression (27) for \(\tilde{\psi}\), took \(\psi\) equal to \(\mu\) according to equation (26), and introduced the torque coefficient

\[
\kappa = \kappa(\gamma_p, \phi_p) = -\frac{e_0 a^2}{2d} \sum_{l=1}^{\infty} \sum_{q=-l}^{l} R_{pq} \tilde{H}_{[0]} P^q(\cos \gamma_p) e^{i q \phi_p}, \tag{C.20}
\]

taking into account that \(H_{[0]}\) is real.

Apparently, any part of \(\tilde{T}_\psi\) is perpendicular to the misalignment vector \(\tilde{\mu}\), but the torque (C.19) is also proportional to \(\mu = |\tilde{\mu}|\), which is why it was named the misalignment torque, \(\tilde{T}_{\text{mis}}\). The corresponding drift rate is given by

\[
r_{\text{mis}}^{\psi} = T_{\text{mis}}^{\psi} / (I_{\Omega_0}) = k \mu_{we}, \quad r_{\text{mis}}^{\psi} = T_{\text{mis}}^{\psi} / (I_{\Omega_0}) = -k \mu_{we};
\]

\[
k(\gamma_p, \phi_p) = \kappa(\gamma_p, \phi_p) / (I_{\Omega_0}), \tag{C.21}
\]

exactly as used in the gyro motion equations (58).

Note that the same result (C.19) and (C.20) can be also obtained by formally averaging of the energy (C.14) over the roll phase, \(\phi_p\). Moreover, averaging the torque coefficient (C.20) over the polhode phase, \(\phi_p\), after replacing \(d_{[0]} \gamma_p(\gamma_p)\) with \(P(\cos \gamma_p)\), leads to the same expression as in formula (9) of paper [41] to l. o. in \(\psi\) (denoted \(\beta\) there). The replacement is based on the formula (4.30) of the book [46], which has already helped with getting the relation (A.15).

Note also that another torque of order \(\psi\) is produced by the \(|m| = 2\) terms from the last lines of equations (C.17) and (C.18). However, it is proportional to the harmonics of twice roll, so its contribution normally averages out over each half-roll period and can thus be neglected. Only very high order harmonics of polhode frequency can resonate with twice roll frequency, and these are of too small amplitude to make the resonance observable. The additional factor \(\psi \approx \mu\) reduces the contribution by at least another four orders of magnitude as compared to the roll–polhode resonances treated below, to entirely negligible values \(\lesssim 10 \mu\) as; in any case, no trace of such resonances was found in the GP-B data.

### C.3.2. Roll-polhode resonance torque

The remaining two terms, from the first line of equations (C.17) and the second line of equations (C.18), give torques of the order zero in \(\psi\): their substitution in formulas (C.15) yields
Putting them in the inertial expressions (C.16) and switching from torques to the corresponding drift rates, we obtain (see notations in section 4.6, equation (64)):

\[
D_{\text{rot}}^{\text{cos}} = T_{\text{rot}}^{\text{cos}} / (I_0 \omega) = A^+ \cos \phi_p + A^- \sin \phi_p;
\]

\[
D_{\text{rot}}^{\text{sin}} = T_{\text{rot}}^{\text{sin}} / (I_0 \omega) = -A^+ \sin \phi_p + A^- \cos \phi_p,
\]

where the polhode-dependent torque coefficients are:

\[
A^+(\gamma_p, \phi_p) = \frac{\epsilon_0 \alpha^2}{d} \sum_{l,q} l (l + 1) \text{Re} \left( H_1 e^{i(\gamma_p + \phi)} \right) \sum_{q=-l}^{l} R_{lq} d_{lq} \left( -\gamma_p \right) e^{i\phi};
\]

\[
A^- (\gamma_p, \phi_p) = \frac{\epsilon_0 \alpha^2}{d} \sum_{l,q} l (l + 1) \text{Im} \left( H_1 e^{i(\gamma_p + \phi)} \right) \sum_{q=-l}^{l} R_{lq} (\omega) d_{lq} \left( -\gamma_p \right) e^{i\phi}.
\]

Remarkably, the angle \( \xi \) drops out of expressions (C.22) and (C.23) completely.

Without a small factor \( \mu \) in them, the classical drift rates (C.22) could be huge but for the sines and cosines of roll in their expressions. Since both the polhode and dissipative variations are negligible over a roll period of 77.5 s, the coefficients \( A^\pm \) are practically constant, so the whole contribution averages to zero. This picture becomes invalid at times because of the changing polhode period and the noticeable presence of rather high polhode harmonics (depending on the gyroscope, \( q = 40 \) and larger). Around times \( t_s \) when one of the multiple polhode frequencies in the coefficients \( A^\pm \) coincides with the roll frequency (resonance condition), the oscillating signal (C.22) rectifies and its contribution becomes non-negligible, as explained in section 4, and, in more detail, in section C.5 below. This feature prompted the torque’s name.

C.4. Models of torque coefficients

All the torque coefficients \( k_A \), \( A^\pm \) in equations (C.20), (C.19), and (C.22) have practically the same structure as the TF scale factor (A.22), which is not convenient for the LF SQUID signal analysis (see section A.4). Barring the ‘mute’ summation index (\( q \) here, \( m \) there), the only differences are weight factors depending on \( l \), and products of housing and rotor patch harmonics coefficients \( H_{l0} R_{l0} \), etc, in place of the fluxon harmonics coefficients \( A_{lm} \). None of these prevent the transformations carried out in sections A.4, A.5. Accordingly, the torque coefficients are first represented by a series of harmonics of the polhode phase \( \phi_p(t) \) with the amplitudes depending on the polhode angle through \( \epsilon_p(t) = \tan(\gamma_p(t)/2) \), as in formulas (A.24), (A.25). They are still in a form not useful for estimation purposes, so we transform them further as in section A.5. This results in Fourier series with the harmonics of the symmetric rotor polhode phase \( \phi_p^s \); coefficients at the harmonics depend on \( \epsilon_0(t) = \tan(\gamma_p(t)/2) \), i.e., they change only slowly due to energy dissipation. This representation for the misalignment torque is (recall
\( \epsilon_0(t) = \tan \left[ \frac{\gamma_0(t)/2}{2} \right] \to 0, \quad t \to \infty): \)

\[
k(t) = \sum_{n=0}^{\infty} \left[ k_n^c(\epsilon_0) \cos n \phi_0 + k_n^s(\epsilon_0) \sin n \phi_0 \right];
\]

\[
k_0(\epsilon_0) = \sum_{j=0}^{\infty} k_{0,j} \epsilon_0^j; \quad k_n(\epsilon_0) = \epsilon_0 \sum_{j=0}^{\infty} k_{n,j} \epsilon_0^j \quad n = 1, 2, \ldots; \quad (C.24)
\]

for the roll-polhode resonance torque it is

\[
A^2 = \sum_{n=0}^{\infty} \left[ a_n^\pm(\epsilon_0) \cos n \phi_0 + b_n^\pm(\epsilon_0) \sin n \phi_0 \right];
\]

\[
a_n^\pm(\epsilon_0) = \sum_{j=0}^{\infty} a_{n,j}^\pm \epsilon_0^j; \quad a_n^\pm(\epsilon_0) = \epsilon_0 \sum_{j=0}^{\infty} a_{n,j}^\pm \epsilon_0^j \quad n = 1, 2, \ldots,
\]

\[
b_n^\pm(\epsilon_0) = \sum_{j=0}^{\infty} b_{n,j}^\pm \epsilon_0^j; \quad b_n^\pm(\epsilon_0) = \epsilon_0 \sum_{j=0}^{\infty} b_{n,j}^\pm \epsilon_0^j \quad n = 1, 2, \ldots; \quad (C.25)
\]

The coefficients \( a_n^\pm, b_n^\pm \) are completely determined by the patch patterns on the rotor and its housing and the rotor asymmetry parameter \( Q \).

We now introduce expressions (C.25) to the drift rate formulas (C.22), move the cosine and sine of roll inside the sums over \( n \), and convert the products of trigonometric functions into sums. This leads to the following expressions:

\[
D_{ns}^{\text{res}} = \sum_{n=0}^{\infty} \left( a_n \cos \Delta \phi_0 - b_n \sin \Delta \phi_0 + \alpha_n \cos \Delta \phi_0^+ + \beta_n \sin \Delta \phi_0^+ \right);
\]

\[
D_{we}^{\text{res}} = \sum_{n=0}^{\infty} \left( b_n \cos \Delta \phi_0 + a_n \sin \Delta \phi_0 + \beta_n \cos \Delta \phi_0^+ - \alpha_n \sin \Delta \phi_0^+ \right), \quad (C.26)
\]

where

\[
\Delta \phi_0(t) = n \phi_0(t) - \dot{\phi}(t), \quad \Delta \phi_0^+(t) = n \phi_0(t) + \dot{\phi}(t), \quad (C.27)
\]

and

\[
a_n = a_n^+ + b_n^-; \quad b_n = a_n^- - b_n^+; \quad \alpha_n = a_n^+ - b_n^-, \quad \beta_n = a_n^- + b_n^+; \quad (C.28)
\]

As with \( a_n^\pm, b_n^\pm \), these are functions of \( \epsilon_0(t) \), they change only slowly due to dissipation, and are represented by the same models as (C.25). Apparently terms with \( \Delta \phi_0^+(t) \) in the drift rates (C.26) do not undergo any resonances. They are negligible and thus not included in the gyro motion equations (58).

**C.5. Gyro motion through a roll-polhode resonance**

Let us finally clarify the picture of gyro motion through a roll-polhode resonance assuming, first of all, that other drifts are negligible as compared to the one generated by the resonance. Then all terms but the resonance term (number \( m \), say,) can be dropped from the gyro motion equations (58), which become:

\[
\dot{s}_m = a_m \cos \Delta \phi_m - b_m \sin \Delta \phi_m, \quad \dot{s}_m = a_m \sin \Delta \phi_m + b_m \cos \Delta \phi_m, \quad (C.29)
\]
with $\Delta \phi_p$ defined by the formula (C.27). Expanding the polhode angular velocity $\omega_p(t)$ in Taylor series around the resonance time $t_m$ and using the resonance condition $m a(t_m) - \omega_r = 0$ we obtain:

$$m \omega_p(t) - \omega_r = m \omega_p(t_m) - \omega_r + m \frac{d\omega_p}{dt} \bigg|_{t=t_m} (t - t_m) + \ldots$$

integrating this provides the phase:

$$\Delta \phi_p(t) = \Delta \phi_p(t_m) = m \frac{d\omega_p}{dt} \bigg|_{t=t_m} (t - t_m)^2 + \ldots$$

So our second approximation is that we drop all the higher order corrections here, taking the phase to be a quadratic function of time:

$$\Delta \phi_p(t) = \theta_m + r_m (t - t_m)^2; \quad \theta_m \equiv \Delta \phi_p(t_m), \quad r_m \equiv m \frac{d\omega_p}{dt} \bigg|_{t=t_m}. \quad (C.30)$$

Using this we rewrite the equations of motion (C.29) as:

$$s_{ns} = a_m \cos r_m (t - t_m)^2 - \tilde{b}_m \sin r_m (t - t_m)^2,$$

$$s_{we} = a_m \sin r_m (t - t_m)^2 + \tilde{b}_m \cos r_m (t - t_m)^2, \quad (C.31)$$

where the new coefficients are

$$a_m(\theta_m) \equiv a_m \cos \theta_m - b_m \sin \theta_m, \quad \tilde{b}_m(\theta_m) \equiv a_m \sin \theta_m + b_m \cos \theta_m. \quad (C.32)$$

Our third and final assumption is that the coefficients $a_m$, $b_m$ (and hence $\tilde{a}_m$, $\tilde{b}_m$) remain constant during the resonance. This is rather accurate, since the characteristic time of their variation is the dissipative time close to either 1 or 2 months, and the resonance duration is from few hours to a day. Under these three assumptions, equations (C.31) can be immediately integrated providing $s_{ns}(t)$, $s_{we}(t)$ as a linear combination of Fresnel integrals [49]. In the NS–WE plane such motion is described by a Cornu spiral plotted in figure 4 of our paper [41]; figure 3 there shows a similar pattern obtained from the GP-B data. So, the gyro spin axis during a resonance makes a ‘step-over’ in the NS–WE plane following, to l.o., a Cornu spiral winding out from its initial direction before the resonance, moving across, then winding back in to the new direction. We now evaluate the magnitude of this resonance ‘step’.

Let us integrate equations (C.31) from $-\infty$ to $\infty$, obtaining on the left the ‘jumps’ $(\Delta s_{ns})_m$, $(\Delta s_{we})_m$ in the gyro orientation projections due to the $m$th resonance. On the right we use the well-known value of the Fresnel integrals over the whole axis, $\sqrt{\pi/2r_m}$, to obtain:

$$(\Delta s_{ns})_m = (a_m - \tilde{b}_m) \sqrt{\pi/2r_m}, \quad (\Delta s_{we})_m = (a_m + \tilde{b}_m) \sqrt{\pi/2r_m}. \quad (C.33)$$

Thus the total change in the gyro orientation, $\Delta s_m$, due to the resonance is

$$\Delta s_m \equiv \sqrt{(\Delta s_{ns})_m^2 + (\Delta s_{we})_m^2}$$

$$= \sqrt{(a_m - \tilde{b}_m)^2 + (a_m + \tilde{b}_m)^2} \sqrt{\pi/r_m} = \sqrt{a_m^2 + b_m^2} \sqrt{\pi/r_m}, \quad (C.34)$$

since, by the definitions (C.32), $\tilde{a}_m^2 + \tilde{b}_m^2 = a_m^2 + b_m^2$. 
So the resonance ‘jump’ $\Delta s_m$ is proportional to $\sqrt{a_m^2 + b_m^2}$, which is naturally called the height, or strength, of the resonance; it has the drift rate dimension of inverse time. The second factor, $\sqrt{\pi/r_m}$, is a time characterizing the resonance duration; we call it the resonance width. Its value for each resonance can be determined from the polhode period time history, however, some universal expression for it would be valuable.

To get it, we first note that the resonance condition implies $m = \omega_r / \omega_p(t_m) = T_p(t_m) / T_r$. Then from the definition (C.30) of the quantity $r_m$ we find:

$$r_m = \frac{m}{2} \frac{d\omega_p}{dt} \bigg|_{t=t_m} = \frac{T_p}{2T_r} \frac{d\omega_p}{dt} \bigg|_{t=t_m} = \frac{T_p}{2T_r} \left( -2\pi \frac{dT_p}{T^2} \frac{dt}{dt} \bigg|_{t=t_m} \right) = -\pi \frac{1}{T_r} \frac{dT_p}{T_p} \frac{dt}{dt} \bigg|_{t=t_m}. $$

However, for all the GP-B gyroscopes,

$$\frac{dT_p}{dr} = -\frac{T_p - T_{pa}}{\tau_{dis}}$$

with at worst a few percent accuracy for the whole science mission. Therefore

$$r_m = \frac{\pi}{T_r \tau_{dis}} \frac{T_p(t_m) - T_{pa}}{T_p(t_m)};$$

using this fourth approximation, the total resonance jump (C.34) becomes

$$\Delta s_m = \sqrt{a_m^2 + b_m^2} \frac{T_r \tau_{dis}}{1 - T_{pa}/T_p(t_m)}. $$

The resonance width, i.e., the second factor here, is larger, the closer the resonance polhode period is to its asymptotic value. This is natural, because a resonance with a constant polhode period ($T_{pa}$) would last forever. However, in the asymptotic limit $T_p(t) \to T_{pa}$ the strength of a resonance goes to zero so that the total jump $\Delta s_m$, which is the product of the strength and width, remains finite. This is seen from the formulas (C.25) and (A.26) that allow for the following chain:

$$\sqrt{a_m^2 + b_m^2} \propto \epsilon_0 = \tan(\gamma_0/2) = \sqrt{1 - \cos \gamma_0} \propto \sqrt{1 - T_{pa}/T_p(t)} \propto \sqrt{1 - T_{pa}/T_p(t)}.$$  

$$T_p(t) \to T_{pa}, \quad m \geq 1.$$  

If some polhode harmonics resonates not with the roll frequency but with another (e.g., the orbital frequency), then only the roll period $T_r$ should be replaced with the proper period $T$ in the formulas (C.34) and (C.35).

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