RELATIVE POSITION OF THREE SUBSPACES IN A HILBERT SPACE

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Abstract. We study the relative position of three subspaces in a separable infinite-dimensional Hilbert space. In the finite-dimensional case, Brenner described the general position of three subspaces completely. We extend it to a certain class of three subspaces in an infinite-dimensional Hilbert space. We also give a partial result which gives a condition on a system to have a (dense) decomposition containing a pentagon.

KEYWORDS: three subspaces, Hilbert space,

AMS SUBJECT CLASSIFICATION: 46C07, 47A15, 15A21, 16G20, 16G60.

1. Introduction

We study the relative position of three subspaces in a separable infinite-dimensional Hilbert space.

The relative position of one subspace of a Hilbert space is extremely simple and determined by the dimension and the co-dimension of the subspace. It is a well known fact that the relative position of two subspaces $E$ and $F$ in a Hilbert space $H$ can be described completely up to unitary equivalence as in Araki [1], Davis [4], Dixmier [5] and Halmos [13]. The Hilbert space is the direct sum of five subspaces:

$$H = (E \cap F) \oplus \text{(the rest)} \oplus (E \cap F^\perp) \oplus (E^\perp \cap F) \oplus (E^\perp \cap F^\perp).$$

In the rest part, $E$ and $F$ are in generic position and the relative position is described only by “the angles” between them.

We disregard “the angles” and study the still-remaining fundamental feature of the relative position of subspaces. This is the reason why we use bounded invertible operators instead of unitaries to define isomorphisms in our paper.

Let $H$ be a Hilbert space and $E_1, \ldots, E_n$ be $n$ subspaces in $H$. Then we say that $S = (H; E_1, \ldots, E_n)$ is a system of $n$ subspaces in $H$ or a $n$-subspace system in $H$. Let $T = (K; F_1, \ldots, F_n)$ be another system of $n$-subspaces in a Hilbert space $K$. We say that systems $S$ and $T$ are isomorphic if there is a bounded invertible operator $\varphi : H \rightarrow K$ satisfying that $\varphi(E_i) = F_i$ for $i = 1, \ldots, n$. See also Sunder [22] for other topics on $n$-subspaces.
In [3] S. Brenner gave a complete description of systems of three subspaces up to isomorphims when an ambient space $H$ is finite-dimensional. A system $S$ is called indecomposable if $S$ cannot be decomposed into a nontrivial direct sum. If the ambient Hilbert space $H$ is finite-dimensional, then any system of $n$ subspaces in $H$ is a finite direct sum of indecomposable systems.

Let $S = (H; E_1, E_2, E_3)$ be an indecomposable system of three subspaces in a finite-dimensional Hilbert space $H$. Then $S$ is isomorphic to one of the following eight trivial systems $S_1, \ldots, S_8$ and one non-trivial system $S_9$: 

$$S_1 = (\mathbb{C}; 0, 0, 0), \quad S_2 = (\mathbb{C}; 0, 0, 0), \quad S_3 = (\mathbb{C}; 0, 0, 0),$$ 

$$S_4 = (\mathbb{C}; 0, 0, 0), \quad S_5 = (\mathbb{C}; 0, 0, 0), \quad S_6 = (\mathbb{C}; 0, 0, 0),$$ 

$$S_7 = (\mathbb{C}; 0, 0, 0), \quad S_8 = (\mathbb{C}; 0, 0, 0), \quad S_9 = (\mathbb{C}; 0, 0, 0).$$

See, for example, [10], [12] or [6] on indecomposable systems of $n$ subspaces.

Therefore we have the following theorem of Brenner:

**Theorem 1.1** (Brenner [3]). Let $S = (H; E_1, E_2, E_3)$ be a system of three subspaces in a finite-dimensional Hilbert space $H$. Then $S$ is isomorphic to the following $T = (H; F_1, F_2, F_3)$ such that there exist subspaces $S, N_1, N_2, N_3, M_1, M_2, M_3, Q, L$ of $H$ satisfying that $Q$ has a form 

$$(Q; Q_1, Q_2, Q_3) := (K \oplus K; K \oplus 0, 0 \oplus K, \{(x, x) \mid x \in K\})$$

of double triangle and 

$$H = S \oplus N_1 \oplus N_2 \oplus N_3 \oplus M_1 \oplus M_2 \oplus M_3 \oplus Q \oplus L$$

$$F_1 = S \oplus O \oplus N_2 \oplus N_3 \oplus M_1 \oplus O \oplus O \oplus Q_1 \oplus O$$

$$F_2 = S \oplus N_1 \oplus O \oplus N_3 \oplus O \oplus M_2 \oplus O \oplus Q_2 \oplus O$$

$$F_3 = S \oplus N_1 \oplus N_2 \oplus O \oplus O \oplus O \oplus M_3 \oplus Q_3 \oplus O$$

**Remark.** In the above decomposition, we can choose $T$ such that $S = F_1 \cap F_2 \cap F_3, N_1 = F_1^1 \cap F_2 \cap F_3, N_2 = F_1 \cap F_2^1 \cap F_3, N_3 = F_1 \cap F_2 \cap F_3^1, M_1 = F_1 \cap F_2^1 \cap F_3^1, M_2 = F_1^1 \cap F_2 \cap F_3^1, M_3 = F_1 \cap F_2^1 \cap F_3$ and $L = F_1^1 \cap F_2^1 \cap F_3^1$. But we should be careful that the isomorphism by an invertible operator does not preserve the orthogonality.

The aim of our papape is to extend the Brenner’s theorem to a certain class of three subspaces in an infinite-dimensional Hilbert space.

The above Brenner’s theorem says that any system of three subspace of a finite-dimensional Hilbert space is decomposed as a direct sum of a distributive part (or Boolean part)

$$S \oplus N_1 \oplus N_2 \oplus N_3 \oplus M_1 \oplus M_2 \oplus M_3 \oplus L$$
and a non-distributive part $Q$. Furthermore the non-distributive part $Q = K \oplus K$ has a typical form

$$(K \oplus K; K \oplus 0, 0 \oplus K, \{(x, x) \mid x \in K\})$$

double triangle. The double triangle is the only obstruction of distributive law in finite-dimensional case. We study this type of decomposition for a certain class of systems of three subspaces for an infinite-dimensional Hilbert space. In order to proceed this type of decomposition, we should recall the following basic facts on the subspace lattice structure: In general, a lattice is distributive if and only if it has neither a double triangle nor a pentagon as a sublattice, see [11] for example. In the subspace lattices of an infinite-dimensional Hilbert space, there occur both double triangles and pentagons. A von Neumann algebra $M$ is commutative if and only if the lattice of the projections in $M$ is distributive. A von Neumann algebra $M$ is finite if and only if the lattice of the projections in $M$ has no pentagons if and only if the lattice of the projections in $M$ is modular. Therefore we understand that the general case is far beyond having a Brenner type decomposition.

For any bounded linear operator $A$ on a Hilbert space $K$, we can associate a system $S_A$ of four subspaces in $H = K \oplus K$ by

$$S_A = (H; K \oplus 0, 0 \oplus K, \text{graph } A, \{(x, x); x \in K\})$$

Two such systems $S_A$ and $S_B$ are isomorphic if and only if the two operators $A$ and $B$ are similar. The direct sum of such systems corresponds to the direct sum of the operators. In this sense the theory of operators is included into the theory of relative positions of four subspaces. In particular on a finite-dimensional space, Jordan blocks correspond to indecomposable systems. Moreover on an infinite-dimensional Hilbert space, the above system $S_A$ is indecomposable if and only if $A$ is strongly irreducible, which is an infinite-dimensional analog of a Jordan block, see, for example, a monograph by Jiang and Wang [17].

Halmos initiated the study of transitive lattices and gave an example of transitive lattice consisting of seven subspaces in [14]. Harrison-Radjavi-Rosenthal [15] constructed a transitive lattice consisting of six subspaces using the graph of an unbounded closed operator. Hadwin-Longstaff-Rosenthal found a transitive lattice of five non-closed linear subspaces in [12]. Any finite transitive lattice which consists of $n$ subspaces of a Hilbert space $H$ gives an indecomposable system of $n - 2$ subspaces by withdrawing 0 and $H$, but the converse is not true. It is still unknown whether or not there exists a transitive lattice consisting of five subspaces. Therefore it is also an interesting problem to know whether there exists an indecomposable system of three subspaces in an infinite-dimensional Hilbert space.

Throughout the paper a projection means an operator $e$ with $e^2 = e = e^*$ and an idempotent means an operator $p$ with $p^2 = p$. The direct sum $\oplus$ is the orthogonal direct sum and $\oplus_{\text{alg}}$ is the algebraic direct sum.
The subspace mostly means closed subspace except the algebraic direct sum.

There seems to be interesting relations with the study of representations of \(\ast\)-algebras generated by idempotents by S. Kruglyak and Y. Samoilenko \[20\] and the study on sums of projections by S. Kruglyak, V. Rabanovich and Y. Samoilenko \[19\]. But we do not know the exact implication, because their objects are different with ours.

In finite dimensional case, the classification of four subspaces is described as the classification of the representations of the extended Dynkin diagram \(D_{4}^{(1)}\). Recall that Gabriel \[9\] listed Dynkin diagrams \(A_n, D_n, E_6, E_7, E_8\) in his theory on finiteness of indecomposable representations of quivers. We discussed on indecomposable representations of quivers on infinite-dimensional Hilbert spaces \[7\]. We are also under the influence of subfactor theory by Jones \[18\].

Our study also has a relation with \(C^\ast\)-algebras generated by idempotents or projections. See Bottcher, Gohberg, Karlovich, Krupnik, Roch, Silbermann and Spittovsky \[2\], Hu and Xue \[16\] and references there.

The authors are supported by JSPS KAKENHI Grant number 25287019.

2. SYSTEMS OF \(n\) SUBSPACES

We introduce some basic definitions and facts on the relative position of \(n\) subspaces in a separable Hilbert space. Let \(H\) be a Hilbert space and \(E_1, \ldots, E_n\) be \(n\) subspaces in \(H\). Then we say that \(S = (H; E_1, \ldots, E_n)\) is a system of \(n\)-subspaces in \(H\) or an \(n\)-subspace system in \(H\). Let \(T = (K; F_1, \ldots, F_n)\) be another system of \(n\)-subspaces in a Hilbert space \(K\). Then \(\varphi : S \rightarrow T\) is called a homomorphism if \(\varphi : H \rightarrow K\) is a bounded linear operator satisfying that \(\varphi(E_i) \subset F_i\) for \(i = 1, \ldots, n\). And \(\varphi : S \rightarrow T\) is called an isomorphism if \(\varphi : H \rightarrow K\) is an invertible (i.e., bounded bijective) linear operator satisfying that \(\varphi(E_i) = F_i\) for \(i = 1, \ldots, n\). We say that systems \(S\) and \(T\) are isomorphic if there is an isomorphism \(\varphi : S \rightarrow T\). This means that the relative positions of \(n\) subspaces \((E_1, \ldots, E_n)\) in \(H\) and \((F_1, \ldots, F_n)\) in \(K\) are same under disregarding angles. We say that systems \(S\) and \(T\) are unitarily equivalent if the above isomorphism \(\varphi : H \rightarrow K\) can be chosen to be a unitary. This means that the relative positions of \(n\) subspaces \((E_1, \ldots, E_n)\) in \(H\) and \((F_1, \ldots, F_n)\) in \(K\) are same with preserving the angles between the subspaces. We are interested in the relative position of subspaces up to isomorphism to study the still-remaining fundamental feature of the relative position after disregarding “the angles”.

We denote by \(\text{Hom}(S, T)\) the set of homomorphisms of \(S\) to \(T\) and \(\text{End}(S) := \text{Hom}(S, S)\) the set of endomorphisms on \(S\).

Let \(G_2 = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \langle a_1, a_2 \rangle\) be the free product of the cyclic groups of order two with generators \(a_1\) and \(a_2\). For two subspaces \(E_1\) and \(E_2\) of a Hilbert space \(H\), let \(e_1\) and \(e_2\) be the projections onto \(E_1\) and \(E_2\). Then \(u_1 = 2e_1 - I\) and \(u_2 = 2e_2 - I\) are self-adjoint unitaries.
Thus there is a bijective correspondence between the set $\text{Sys}^2(H)$ of systems $\mathcal{S} = (H; E_1, E_2)$ of two subspaces in a Hilbert space $H$ and the set $\text{Rep}(G_2, H)$ of unitary representations $\pi$ of $G_2$ on $H$ such that $\pi(a_1) = u_1$ and $\pi(a_2) = u_2$. Similarly let $G_n = \mathbb{Z}/2\mathbb{Z} * \cdots * \mathbb{Z}/2\mathbb{Z}$ be the $n$-times free product of the cyclic groups of order two. Then there is a bijective correspondence between the set $\text{Sys}^n(H)$ of systems of $n$ subspaces in a Hilbert space $H$ and the set $\text{Rep}(G_n, H)$ of unitary representations of $G_n$ on $H$.

**Example 1.** Let $H = \mathbb{C}^2$. Fix an angle $\theta$ with $0 < \theta < \pi/2$. Put $E_1 = \{\lambda(1,0) \mid \lambda \in \mathbb{C}\}$ and $E_2 = \{\lambda(\cos\theta, \sin\theta) \mid \lambda \in \mathbb{C}\}$. Then $\mathcal{S}_1 = (H; E_1, E_2)$ is isomorphic to $\mathcal{S}_2 = (\mathbb{C}^2; \mathbb{C} \oplus 0, 0 \oplus \mathbb{C})$. But the corresponding two unitary representations $\pi_1$ and $\pi_2$ are not similar, because $\frac{1}{2}(\pi_1(a_1) + 1)\frac{1}{2}(\pi_1(a_2) + 1) \neq 0$ and $\frac{1}{2}(\pi_2(a_1) + 1)\frac{1}{2}(\pi_2(a_2) + 1) = 0$.

We start with known facts to recall some notation. See [6] for example.

Let $H$ be a Hilbert space and $H_1$ and $H_2$ be two subspaces of $H$. We write $H_1 \vee H_2 := H_1 + H_2$ and $H_1 \wedge H_2 := H_1 \cap H_2$. Then the set of (closed) subspaces of $H$ forms a lattice under these operations $\vee$ and $\wedge$. Two subspaces $H_1$ and $H_2$ is said to be topologically complementary if $H = H_1 \vee H_2$ and $H_1 \wedge H_2 = 0$. Two subspaces $H_1$ and $H_2$ is said to be algebraically complementary if $H = H_1 + H_2$ and $H_1 \cap H_2 = 0$.

**Lemma 2.1.** Let $H$ be a Hilbert space and $H_1$ and $H_2$ be two subspaces of $H$. Then the following are equivalent:

1. $H_1$ and $H_2$ are algebraically complementary, i.e., $H = H_1 + H_2$ and $H_1 \cap H_2 = 0$.
2. There exists a closed subspace $M \subset H$ such that $(H; H_1, H_2)$ is isomorphic to $(H; M, M^\perp)$
3. There exists an idempotent $P \in B(H)$ such that $H_1 = \text{Im} P$ and $H_2 = \text{Im}(1 - P)$.

**Lemma 2.2** ([6]). Let $H$ and $K$ be Hilbert spaces and $E \subset H$ and $F \subset K$ be closed subspaces of $H$ and $K$. Let $e \in B(H)$ and $f \in B(K)$ be the projections onto $E$ and $F$. Then the following are equivalent:

1. There exists an invertible operator $T : H \to K$ such that $T(E) = F$.
2. There exists an invertible operator $T : H \to K$ such that $e = (T^{-1}fT)e$ and $f = (TeT^{-1})f$.

Using the above lemma, we can describe an isomorphism between two systems of $n$ subspaces in terms of operators only as follows:

**Corollary 2.3.** Let $\mathcal{S} = (H; E_1, \cdots, E_n)$ and $\mathcal{S}' = (H'; E'_1, \cdots, E'_n)$ be two systems of $n$-subspaces. Let $e_i$ (resp. $e'_i$) be the projection onto $E_i$ (resp. $E'_i$). Then two systems $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic if and
only if there exists an invertible operator \( T : H \to H' \) such that \( e_i = (T^{-1}e'_i)T \) and \( e'_i = (Te_iT^{-1})' \) for \( i = 1, \ldots, n \).

**Remark.** If there exists an invertible operator \( T : H \to H' \) such that \( e'_i = Te_iT^{-1} \) for \( i = 1, \ldots, n \), then two systems \( S \) and \( S' \) are isomorphic. But the converse is not true as in Example 1.

3. INDECOMPOSABLE SYSTEMS

In this section we shall introduce a notion of indecomposable system, that is, a system which cannot be decomposed into a direct sum of smaller systems anymore.

**Definition** (direct sum). Let \( S = (H; E_1, \ldots, E_n) \) and \( S' = (H'; E'_1, \ldots, E'_n) \) be systems of \( n \) subspaces in Hilbert spaces \( H \) and \( H' \). Then their direct sum \( S \oplus S' \) is defined by

\[
S \oplus S' := (H \oplus H'; E_1 \oplus E'_1, \ldots, E_n \oplus E'_n).
\]

**Definition.** (indecomposable system). A system \( S = (H; E_1, \ldots, E_n) \) of \( n \) subspaces is called **indecomposable** if the system \( S \) is isomorphic to a direct sum of two non-zero systems. A system \( S = (H; E_1, \ldots, E_n) \) is said to be **indecomposable** if it is not decomposable.

**Example 2.** Let \( H = \mathbb{C}^2 \). Fix an angle \( \theta \) with \( 0 < \theta < \pi/2 \). Put \( E_1 = \{ \lambda(1, 0) \mid \lambda \in \mathbb{C} \} \) and \( E_2 = \{ \lambda(\cos \theta, \sin \theta) \mid \lambda \in \mathbb{C} \} \). Then \((H; E_1, E_2)\) is isomorphic to

\[
(\mathbb{C}^2; \mathbb{C} \oplus 0, 0 \oplus \mathbb{C}) \cong (\mathbb{C}; \mathbb{C}, 0) \oplus (\mathbb{C}; 0, \mathbb{C}).
\]

Hence \((H; E_1, E_2)\) is decomposable.

**Remark.** Let \( e_1 \) and \( e_2 \) be the projections onto \( E_1 \) and \( E_2 \) in the Example above. Then the \( C^* \)-algebra \( C^*\{e_1, e_2\} \) generated by \( e_1 \) and \( e_2 \) is exactly \( B(H) \cong M_2(\mathbb{C}) \). Therefore the irreducibility of \( C^*\{e_1, e_2\} \) does not imply the indecomposability of \((H; E_1, E_2)\). Thus seeking an indecomposable system of subspaces is much more difficult and fundamental task than showing irreducibility of the \( C^* \)-algebra generated by the corresponding projections for the subspaces.

We can characterize decomposability of systems inside the ambient Hilbert space as in [3]

Let \( H \) be a Hilbert space and \( S = (H; E_1, \ldots, E_n) \) a system of \( n \) subspaces. Then the following conditions are equivalent:

1. \( S \) is decomposable.
2. there exist non-zero closed subspaces \( H_1 \) and \( H_2 \) of \( H \) such that \( H_1 + H_2 = H \), \( H_1 \cap H_2 = 0 \) and \( E_i = E_i \cap H_1 + E_i \cap H_2 \) for any \( i = 1, \ldots, n \).
We give a condition of decomposability in terms of endomorphism algebras for the systems.

We put $\text{Idem}(S) := \{T \in \text{End}(S); T = T^2\}$.

Let $S = (H; E_1, \ldots, E_n)$ be a system of $n$ subspaces in a Hilbert space $H$. Then $S$ is indecomposable if and only if $\text{Idem}(S) = \{0, I\}$.

Let $S = (H; E_1, \ldots, E_n)$ be a system of $n$ subspaces in a Hilbert space $H$. Let $e_i$ be the projection of $H$ onto $E_i$ for $i = 1, \ldots, n$. If $S = (H; E_1, \ldots, E_n)$ is indecomposable, then the $C^*({\{e_1, \ldots, e_n\}})$ generated by $e_1, \ldots, e_n$ is irreducible. But the converse is not true.

**Definition.** Let $S = (H; E_1, \ldots, E_n)$ be a system of $n$ subspaces in a Hilbert space $H$. Let $e_i$ be the projection of $H$ onto $E_i$ for $i = 1, \ldots, n$. We say that $S$ is a commutative system if the $C^*({\{e_1, \ldots, e_n\}})$ generated by $e_1, \ldots, e_n$ is commutative. Be careful that commutativity is *not* an isomorphic invariant as shown in Example 1. But it makes sense that a system is isomorphic to a commutative system.

Let $S = (H; E_1, \ldots, E_n)$ be a system of $n$ subspaces in a Hilbert space $H$. Assume that $S$ is a commutative system. Then $S$ is indecomposable if and only if $\dim H = 1$. Moreover each subset $\Lambda \subset \{1, \ldots, n\}$ corresponds to a commutative system satisfying $\dim E_i = 1$ for $i \in \Lambda$ and $\dim E_i = 0$ for $i \notin \Lambda$.

**Example 3.** Let $H = \mathbb{C}^2$. Put $E_1 = \mathbb{C} \oplus 0$, $E_2 = 0 \oplus \mathbb{C}$ and $E_3 = \{(x, x) \mid x \in \mathbb{C}\}$. Then $S = (H; E_1, E_2, E_3)$ is indecomposable. The system $S$ is the lowest dimensional one among non-commutative indecomposable systems. In fact, the system the $S$ forms a double triangle in the sense below. We see that the distributive law fails:

$$(E_1 \vee E_2) \wedge E_3 \neq (E_1 \wedge E_2) \vee (E_1 \wedge E_3).$$

**Definition.** We say that a system $S = (H; E_1, E_2, E_3)$ of three subspaces in a Hilbert space $H$ forms a double triangle if the family $\{H, E_1, E_2, E_3, 0\}$ is a double triangle lattice, (which is also called a diamond), that is, $E_i \vee E_j = H$, and $E_i \wedge E_j = 0$, $(i \neq j, i, j = 1, 2, 3)$.

and each $E_i \neq H, E_i \neq 0$.

**Example 4.** Let $G = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \langle a_1, a_2, a_3 \rangle$ be the free product of the cyclic groups of order two with three generators $a_1, a_2$ and $a_3$. Let $\lambda$ be the left regular representation of $G$ on $H = \ell^2(G)$. Then the reduced group $C^*$-algebra $C^*_r(G)$ is generated by $\lambda_{a_1}$, $\lambda_{a_2}$ and $\lambda_{a_3}$. Since these three generators are self-adjoint unitaries, $e_i := (\lambda_{a_i} + I)/2$, $(i = 1, 2, 3)$ are projections. Let $E_i = \text{Im} e_i$. Then a system $S = (H; E_1, E_2, E_3)$ of three subspaces forms a double triangle. In fact, let $x = \sum_y x_y \delta_y \in E_1 \cap E_2$. Since $e_2 x = x$, we have $\lambda_{a_2} x = x$ for $i = 1, 2$. Therefore $x_{a_i g} = x_g$ for any $g \in G$. Since $\sum |x_h|^2 < \infty$, $x_g = 0$ for any $g$. Therefore $\sum x = 0$. Hence $E_1 \cap E_2 = 0$. The other conditions are similarly checked.
Definition. We say that a system $S = (H; E_1, E_2, E_3)$ of three subspaces in a Hilbert space $H$ forms a pentagon if the family $\{H, E_1, E_2, E_3, 0\}$ is a pentagon lattice with $E_3 \supset E_2$, that is,

$$E_1 \vee E_2 = H, \quad E_1 \wedge E_3 = 0, \text{ and } E_3 \supset E_2 \text{ with } E_1 \neq E_2,$$

and each $E_i \neq H, E_i \neq 0$. We also say that $S = (H; E_1, E_2, E_3)$ is a pentagon system.

Example 5. Let $K$ be a Hilbert space and $A : K \to K$ a bounded operator such that $\text{Im} A$ is dense in $K$ and not equal to $K$. Put $H = K \oplus K, E_1 = K \oplus 0$ and $E_2 = \{(x, Ax) | x \in K\}$. Let $M \neq 0$ be a finite-dimensional subspace of $K$ such that $M \cap \text{Im} A = 0$. Put $E_3 = E_2 + M$. Then $S = (H; E_1, E_2, E_3)$ is a pentagon system.

Recall that Halmos initiated the study of transitive lattices. A complete lattice of closed subspaces of a Hilbert space $H$ containing 0 and $H$ is called transitive if every bounded operator on $H$ leaving each subspace invariant is a scalar multiple of the identity. Halmos gave an example of transitive lattice consisting of seven subspaces in [14]. Harrison-Radjavi-Rosenthal [15] constructed a transitive lattice consisting of six subspaces using the graph of an unbounded operator. Any finite transitive lattice which consists of $n$ subspaces gives an indecomposable system of $n$-2 subspaces but the converse is not true. Following the study of transitive lattices, we shall introduce the notion of transitive system.

Definition. Let $S = (H; E_1, \ldots, E_n)$ be a system of $n$ subspaces in a Hilbert space $H$. Then we say that $S$ is transitive if $\text{End}(S) = \mathbb{C} I_H$. Recall that $S$ is indecomposable if and only if $\text{Idem}(S) = \{0, I\}$. Hence if $S$ is transitive, then $S$ is indecomposable. But the converse is not true. In fact, the system

$$S_S = (H; K \oplus 0, 0 \oplus K, \text{graph } A, \{(x, x) | x \in K\})$$

of four subspaces associated with a unilateral shift $S$ as above is indecomposable but is not transitive, because $\text{End}(S)$ contains $S \oplus S$.

Example 6. (Harrison-Radjavi-Rosenthal [15]) Let $K = \ell^2(\mathbb{Z})$ and $H = K \oplus K$. Consider a sequence $(\alpha_n)_n$ given by $\alpha_n = 1$ for $n \leq 0$ and $\alpha_n = \exp((-1)^n n!)$ for $n \geq 1$. Consider a bilateral weighted shift $S : D_T \to K$ such that $T(x_n)_n = (\alpha_{n-1} x_{n-1})_n$ with the domain $D_T = \{(x_n)_n \in \ell^2(\mathbb{Z}); \sum |\alpha_n x_n|^2 < \infty\}$. Let $E_1 = K \oplus 0, E_2 = 0 \oplus K, E_3 = \{(x, Tx) \in H; x \in D_T\}$ and $E_4 = \{(x, x) \in H; x \in K\}$. Harrison, Radjavi and Rosenthal showed that $\{0, H, E_1, E_2, E_3, E_4\}$ is a transitive lattice. Hence the system $S = (H; E_1, E_2, E_3, E_4)$ of four subspaces in $H$ is transitive and in particular indecomposable.

It is easy to see the case of indecomposable systems of one subspace even in an infinite-dimensional Hilbert space.
Let $H$ be a Hilbert space and $S = (H; E)$ a system of one subspace. Then $S = (H; E)$ is indecomposable if and only if $S \cong (\mathbb{C}; 0)$ or $S \cong (\mathbb{C}; \mathbb{C})$.

Let $S = (H; E)$ and $S' = (H'; E')$ be two systems of one subspace. Then $S$ and $S'$ are isomorphic if and only if $\dim E = \dim E'$ and $\text{codim } E = \text{codim } E'$.

It is a well known fact that the relative position of two subspaces $E_1$ and $E_2$ in a Hilbert space $H$ can be described completely up to unitary equivalence. The Hilbert space $H$ is the direct sum of five subspaces:

$$H = (E_1 \cap E_2) \oplus \text{(the rest)} \oplus (E_1 \cap E_2^\perp) \oplus (E_1^\perp \cap E_2) \oplus (E_1^\perp \cap E_2^\perp).$$

In the rest part, $E_1$ and $E_2$ are in generic position and the relative position is described only by “the angles” between them. In fact the rest part is written as $K \oplus K$ for some subspace $K$ and there exist two positive operators $c, s \in B(K)$ with null kernels with $c^2 + s^2 = 1$ such that

$$E_1 = (E_1 \cap E_2) \oplus \text{Im} \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \oplus (E_1 \cap E_2^\perp) \oplus 0 \oplus 0,$$

and

$$E_2 = (E_1 \cap E_2) \oplus \text{Im} \left( \begin{array}{cc} c^2 & cs \\ cs & s^2 \end{array} \right) \oplus 0 \oplus (E_1^\perp \cap E_2) \oplus 0.$$

By the functional calculus, there exists a unique positive operator $\theta$, called the angle operator, such that $c = \cos \theta$ and $s = \sin \theta$ with $0 \leq \theta \leq \frac{\pi}{2}$. We see that the algebraic sum $E_1 + E_2$ is closed if and only if $scK + s^2K = K$ if and only if $s$ is invertible. And $E_1 + E_2^\perp$ is closed if and only if $E_1^\perp + E_2$ is closed if and only if $c$ is invertible. We need the following fact:

**Lemma 3.1.** Let $E_1$ and $E_2$ be two subspaces in a Hilbert space $H$. Let $P_i$ be the projection of $H$ onto $E_i$. If $E_1 + E_2$ is closed, then $T := (P_1 + P_2)|_{(E_1+E_2)} : E_1 + E_2 \to E_1 + E_2$ is an onto invertible operator.

**Proof.** Since $E_1 + E_2$ is closed, $s$ is invertible. Then it is easy to see that

$$T = I \oplus \text{Im} \left( \begin{array}{cc} I + c^2 & cs \\ cs & s^2 \end{array} \right) \oplus I \oplus I$$

is invertible, because the non-trivial component has the operator determinant $(I + c^2)s^2 - cs cs = s^2$. \hfill \Box

Let $S = (H; E_1, E_2)$ be a system of two subspaces in a Hilbert space $H$. Then $S$ is indecomposable if and only if $S$ is isomorphic to one of the following four commutative systems:

$$S_1 = (\mathbb{C}; \mathbb{C}, 0), \quad S_2 = (\mathbb{C}; 0, \mathbb{C}), \quad S_3 = (\mathbb{C}; \mathbb{C}, \mathbb{C}), \quad S_4 = (\mathbb{C}; 0, 0).$$
4. Brenner type decomposition

We introduce a Brenner type decomposition which is a generalization of a Brenner decomposition of a system of three subspaces in a finite dimensional Hilbert space.

**Definition.** Let $\mathcal{S} = (H; E_1, E_2, E_3)$ be a system of three subspaces in a Hilbert space $H$. Then $\mathcal{S}$ is said to have a Brenner type decomposition if $\mathcal{S}$ is isomorphic to a system $\mathcal{T} = (H; F_1, F_2, F_3)$ satisfying that there exist subspaces $S, N_1, N_2, N_3, M_1, M_2, M_3, Q, L$ of $H$ such that $(Q; Q_1, Q_2, Q_3)$ forms a double triangle and

\[
\begin{align*}
H &= S \oplus N_1 \oplus N_2 \oplus N_3 \oplus M_1 \oplus M_2 \oplus M_3 \oplus Q \oplus L \\
F_1 &= S \oplus O \oplus N_2 \oplus N_3 \oplus M_1 \oplus O \oplus O \oplus Q_1 \oplus O \\
F_2 &= S \oplus N_1 \oplus O \oplus N_3 \oplus O \oplus M_2 \oplus O \oplus Q_2 \oplus O \\
F_3 &= S \oplus N_1 \oplus N_2 \oplus O \oplus O \oplus M_3 \oplus Q_3 \oplus O
\end{align*}
\]

We need the following Theorem after [8] Corollary 4.1 by Feshchenko who studies closedness of the sum of $n$ subspaces of a Hilbert space. Let $H_1, \ldots, H_n$ be subspaces of a Hilbert space. Then $H_1, \ldots, H_n$ are said to be linearly independent if for any $x_i \in H_i$ ($i = 1, \ldots, n$), if $x_1 + \ldots + x_n = 0$, then $x_1 = \cdots = x_n = 0$. They are linearly independent if and only if the representation $x = x_1 + \cdots + x_n$ for $x_i \in H_i$ ($i = 1, \ldots, n$) is unique if and only if

\[
H_i \cap \left( \sum_{\{j\mid j \neq i\}} H_j \right) = 0
\]

for any $i = 1, \ldots, n$.

**Theorem 4.1** ([8]). Let $H_1, \ldots, H_n$ be linear independent subspaces of a Hilbert space $H$. If $H = H_1 + \cdots + H_n$, then for any collection of subscripts $i(1), \ldots, i(k)$, the sum $H_{i(1)} + \cdots + H_{i(k)}$ is closed.

Using the Feshchenko’s Theorem above, we can extend Lemma [2.1] to $n$-subspaces.

**Theorem 4.2.** Let $H_1, \ldots, H_n$ be $n$-subspaces of a Hilbert space $H$. Then the following are equivalent:

1. $H = H_1 + \cdots + H_n$ and $H_1, \ldots, H_n$ are linear independent.
2. $H$ is isomorphic to an outer orthogonal sum $H_1 \oplus H_2 \ldots H_n$ by an invertible operator.

**Proof.** Assume (1). By the Feshchenko’s Theorem above, $H_1 + H_2$ is closed. Since $H_1$ and $H_2$ are linearly independent, $H_1 + H_2$ is isomorphic to an outer orthogonal sum $H_1 \oplus H_2$ by Lemma [2.1]. Since $(H_1 \oplus H_2) + H_3$ is closed by the Feshchenko’s theorem and $(H_1 \oplus H_2)$ and $H_3$ are linearly independent, $(H_1 \oplus H_2) + H_3$ is isomorphic to an outer orthogonal sum
It is trivial that (2) implies (1). Conversely, assume (1). Let

$$H_1 \oplus H_2 \oplus H_3$$

by Lemma 2.1. Inductively we can show (2). The converse
is clear.

The failure of the ditributive law is measured by the inclusions:

$$((E_i \land E_j) \lor (E_i \land E_k)) \subset (E_i \land (E_j \lor E_k))$$

Therefore the finite dimensionality of its quotient space is a slight
generalization of the finite dimensionality of the ambient space $H$.

**Theorem 4.3.** Let $S = (H; E_1, E_2, E_3)$ be a system of three subspaces
in a Hilbert space $H$. Then the followings are equivalent:

1. Linear sums $E_i + E_j$ and $(E_i \cap E_k) + (E_j \cap E_k)$ are closed for
   $i, j, k \in \{1, 2, 3\}$ with $i \neq j \neq k \neq i$ and the quotient space
   $(E_3 \cap (E_1 \lor E_2))/((E_3 \land E_1) \lor (E_3 \land E_2))$ is finite-dimensional.

2. $S$ has a Brenner type decomposition with a finite-dimensional
double triangle part $Q$.

**Proof.** It is trivial that (2) implies (1). Conversely, assume (1). Let

$$Q_3 = (E_3 \land (E_1 \lor E_2)) \cap ((E_3 \land E_1) \lor (E_3 \land E_2))^{\perp}$$

Then $Q_3$ is finite-dimensional by the assumption and

$$(E_3 \land (E_1 \lor E_2)) = ((E_3 \land E_1) \lor (E_3 \land E_2)) \oplus Q_3$$

Let $P_1$ be the projection of $H$ on $E_i$. Since $E_1 + E_2$ is closed, $T :=
(P_1 + P_2)|_{E_1 + E_2}$ : $E_1 + E_2 \to E_1 + E_2$ is an onto invertible operator
by Lemma 3.1. Put $A_1 = P_1T^{-1}$ and $A_2 = P_2T^{-1}$. Then $A_1 + A_2 =
\text{id}|_{E_1 + E_2}$. Put $Q_1 := A_1(Q_3) \subset E_1$ and $Q_2 := A_2(Q_3) \subset E_2$. Then $Q_1$
and $Q_2$ are finite-dimensional. For any $q_3 \in Q_3$, put $q_1 = A_1q_3 \in Q_1$
and $q_2 = A_2q_3 \in Q_2$. Then $q_1 + q_2 = q_3$. Let $Q := Q_1 + Q_2$. Then

$$Q = Q_1 + Q_2 = Q_2 + Q_3 = Q_3 + Q_1.$$

Moreover $Q_2 \cap Q_3 = 0$. In fact, $Q_2 \cap Q_3 \subset E_2 \cap E_3$ and $Q_2 \cap Q_3 \subset
Q_3 \subset (E_2 \cap E_3)^{\perp}$. Similarly we have $Q_1 \cap Q_3 = 0$. Let $q \in Q_1 \cap Q_2$.
Then there exists $q_3 \in Q_3$ such that $q = A_1q_3$ and

$$q_3 = A_1q_3 + A_2q_3 = q + A_2q_3 \in Q_2 + Q_2 = Q_2$$

Thus $q_3 \in Q_3 \cap Q_2 = 0$. Hence $q = A_1q_3 = 0$. This shows that
$Q_1 \cap Q_2 = 0$. Therefore $(Q; Q_1, Q_2, Q_3)$ forms a double triangle.

We shall show that

$$(E_1 \land (E_2 + E_3)) = ((E_1 \land E_2) + (E_1 \land E_3)) \oplus_{\text{alg}} Q_1$$

Since $Q_1 \subset E_1$ and $Q_1 \subset Q_2 + Q_3 \subset (E_2 + E_3)$,

$$(E_1 \land (E_2 + E_3)) \subset ((E_1 \land E_2) + (E_1 \land E_3)) + Q_1$$

Conversely let $x_1 \in (E_1 \land (E_2 + E_3))$. Then there exist $x_2 \in E_2$
and $x_3 \in E_3$ such that $x_1 = x_2 + x_3$. Since

$$x_3 = x_1 - x_2 \in E_3 \land (E_1 + E_2) = ((E_3 \land E_1) + (E_3 \land E_2)) + Q_3,$$
there exist \( y_1 \in E_3 \cap E_1 \), \( y_2 \in E_3 \cap E_2 \) and \( q_3 \in Q_3 \) such that \( x_3 = y_1 + y_2 + q_3 \). Since \( Q_3 \subset Q_1 + Q_2 \), there exist \( q_1 \in Q_1 \) and \( q_2 \in Q_2 \) such that \( q_3 = q_1 + q_2 \). Then we have that

\[
x_1 - x_2 = x_3 = y_1 + y_2 + q_3 = y_1 + y_2 + q_1 + q_2.
\]

Put

\[
z_{12} := x_1 - y_1 - q_1 = y_2 + x_2 + q_2 \in E_1 \cap E_2.
\]

Then

\[
x_1 = q_1 + y_1 + z_{12} \in Q_1 + E_3 \cap E_1 + E_1 \cap E_2
\]

This implies that

\[
(E_1 \cap (E_2 + E_3)) \subset ((E_1 \cap E_2) + (E_1 \cap E_3)) + Q_1.
\]

We shall show that \(((E_1 \cap E_2) + (E_1 \cap E_3)) \cap Q_1 = 0\). Let \( q_1 \in ((E_1 \cap E_2) + (E_1 \cap E_3)) \cap Q_1 \). Then there exist \( y \in E_1 \cap E_2 \) and \( z \in E_1 \cap E_3 \) such that \( q_1 = y + z \). Since \( q_1 \in Q_1 \), there exists \( q_3 \in Q_3 \) such that \( q_1 = A_1 q_3 \). Put \( q_2 = A_2 q_3 \). Then \( q_3 = q_1 + q_2 \). Hence \( y + z = q_1 = q_3 - q_2 \). Put

\[
s := z - q_3 = -y - q_2 \in E_3 \cap E_2.
\]

Then \( q_3 = z - s \in (E_3 \cap E_1) + (E_3 \cap E_2) \). Hence \( q_3 \in Q_3 \cap ((E_3 \cap E_1) + (E_3 \cap E_2)) = 0 \). Thus \( q_1 = A_1 q_3 = 0 \). Therefore we have that

\[
(E_1 \cap (E_2 + E_3)) = ((E_1 \cap E_2) + (E_1 \cap E_3)) \oplus_{alg} Q_1
\]

Similarly we have that

\[
(E_2 \cap (E_1 + E_3)) = ((E_2 \cap E_1) + (E_2 \cap E_3)) \oplus_{alg} Q_2
\]

Put

\[
M_1 := E_1 \cap (E_1 \cap (E_2 + E_3))^\perp
\]

\[
M_2 := E_2 \cap (E_2 \cap (E_3 + E_1))^\perp
\]

\[
M_3 := E_3 \cap (E_3 \cap (E_1 + E_2))^\perp
\]

Then we have that

\[
E_1 = M_1 \oplus (E_1 \cap (E_2 + E_3)), \quad E_2 = M_2 \oplus (E_2 \cap (E_3 + E_1))
\]

and

\[
E_3 = M_3 \oplus (E_3 \cap (E_1 + E_2))
\]

Put \( S := E_1 \cap E_2 \cap E_3 \) and

\[
N_1 := E_2 \cap E_3 \cap S^\perp, \quad N_2 := E_3 \cap E_1 \cap S^\perp \quad \text{and} \quad N_3 := E_1 \cap E_2 \cap S^\perp.
\]

Then we have that

\[
E_2 \cap E_3 = S \oplus N_1, \quad E_3 \cap E_1 = S \oplus N_2 \quad \text{and} \quad E_1 \cap E_2 = S \oplus N_3.
\]

Put \( L := (E_1 + E_2 + E_3)^\perp \cap H \). Moreover

\[
E_1 = M_1 + ((E_1 \cap E_2) + (E_1 \cap E_3)) + Q_1 = S + N_2 + N_3 + M_1 + Q_1
\]

Similarly we also have that

\[
E_2 = S + N_1 + N_3 + M_2 + Q_2, \quad \text{and} \quad E_3 = S + N_1 + N_2 + M_3 + Q_3.
\]
Therefore
\[ E_1 + E_2 + E_3 = S + N_1 + N_2 + N_3 + M_1 + M_2 + M_3 + Q \]
and
\[ H = (E_1 + E_2 + E_3) + L = S + N_1 + N_2 + N_3 + M_1 + M_2 + M_3 + Q + L. \]
Finally we shall show that the linear sum of the right-hand side is in fact an algebraic direct sum. We need to show that \( S, N_1, N_2, N_3, M_1, M_2, M_3, Q \) and \( L \) are linearly independent. Let
\[ s + n_1 + n_2 + n_3 + m_1 + m_2 + m_3 + q_1 + q_2 + \ell = 0 \]
for \( s \in S, n_1 \in N_1, n_2 \in N_2, n_3 \in N_3, m_1 \in M_1, m_2 \in M_2, m_3 \in M_3, q_1 \in Q_1, q_2 \in Q_2 \) and \( \ell \in L \). Then it is clear that \( \ell = 0 \). Therefore
\[ -m_3 = (n_2 + m_1 + q_1) + (n_1 + m_2 + q_2) + (n_3 + s) \in E_1 + E_2 + E_1 \cap E_2 \subseteq E_1 + E_2. \]
Therefore \( m_3 \in M_3 \cap (E_3 \cap (E_1 + E_2)) = 0 \). Thus \( m_3 = 0 \). Since
\[ q_1 + q_2 = q'_2 + q''_3 = q''_1 \]
for some \( q_2 \in Q_2, q'_3, q''_3 \in Q_3 \) and \( q''_1 \in Q_1 \), we similarly have that \( m_1 = m_2 = 0 \). Hence
\[ s + n_1 + n_2 + n_3 + q_1 + q_2 = 0. \]
Put \( w := n_1 + q_2 = -(n_2 + n_3 + q_1) \in E_2 \cap E_1 \). Then
\[ q_2 = w - n_1 \in (E_2 \cap E_1) + (E_2 \cap E_3) \]
Since \( q_2 \in Q_2 \cap ( (E_2 \cap E_1) + (E_2 \cap E_3) ) = 0 \), we have that \( q_2 = 0 \). Similarly we have that \( q_1 = 0 \). Therefore \( s + n_1 + n_2 + n_3 = 0 \). Since
\[ E_1 \cap E_2 \ni s + n_3 = -n_1 - n_2 \in E_3, \]
\( s + n_3 \in (E_1 \cap E_2) \cap E_3 = S \). Thus \( n_3 \in S \cap N_3 = 0 \). Therefore \( n_3 = 0 \). Similarly we have that \( n_1 = n_2 = 0 \). Hence \( s = 0 \). Finally Theorem 12 implies the conclusion.

As a Corollary, we get the original Brenner’s theorem.

**Corollary 4.4** ([3], [21]). Let \( S = (H; E_1, E_2, E_3) \) be a system of three subspaces in a finite dimensional Hilbert space \( H \). Then \( S \) has a Brenner type decomposition.

**Remark.** Even if an ambient space \( H \) is finite-dimensional, a double triangle part is not uniquely determined in a Brenner type decomposition. In fact, let \( H = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \), \( E_1 = \mathbb{C} \oplus O \oplus \mathbb{C} \), \( E_2 = O \oplus \mathbb{C} \oplus \mathbb{C} \) and \( E_3 = \{ x(1, 1, 1) \in H \mid x \in \mathbb{C} \} \). Put \( N_3 = O \oplus O \oplus \mathbb{C} \). Put \( S = N_1 = N_2 = M_1 = M_2 = M_3 = L = O \). Let \( Q_3 = E_3 \), \( Q_1 = \{ x(1, 0, 1/2) \in H \mid x \in \mathbb{C} \} \) and \( Q_2 = \{ x(0, 1, 1/2) \in H \mid x \in \mathbb{C} \} \). This gives a Brenner type decomposition. We have another Brenner type decomposition by \( Q'_3 = E_3 \), \( Q'_1 = \{ x(1, 0, 1/3) \in H \mid x \in \mathbb{C} \} \) and \( Q'_2 = \{ x(0, 1, 2/3) \in H \mid x \in \mathbb{C} \} \) and the others are the same as the first.
one. Since $Q := Q_1 + Q_2 \neq Q' := Q'_1 + Q'_2$, they provide two kinds of Brenner type decompositions.

Let $S = (H; E_1, E_2, E_3)$ be a system of three subspaces which has a Brenner type decomposition. Then it is clear that for any $i, j, k = 1, 2, 3$ with $i \neq j, j \neq k$ and $k \neq i$,

$$E_i + E_j + E_k \quad \text{and} \quad (E_i \cap E_j) + E_k$$

are closed in $H$. We shall show that this topological property characterize a system of three subspaces which have a Brenner type decomposition.

**Example 7.** Let $S = (H; E_1, E_2, E_3)$ be a system of three subspaces in a Hilbert space $H$. If $S = (H; E_1, E_2, E_3)$ forms a pentagon (with $E_1 \cup E_2$), then neither $E_1 + E_3$ nor $E_2 + E_3$ are closed. Therefore $E_1 + E_2 + E_3 = E_1 + E_2$ is not closed and $(E_1 \cap E_2) + E_3 = E_2 + E_3$ is not closed. Hence $S = (H; E_1, E_2, E_3)$ does not have a Brenner type decomposition. Thus this closedness property excludes pentagons to have a Brenner type decomposition.

We shall split out a distributive part and a double triangle part by step by step.

**Lemma 4.5.** Let $S = (H; E_1, E_2, E_3)$ be a system of three subspaces in a Hilbert space $H$. Suppose that $(E_1 \cap E_2) + E_3$ is closed. Then there exist systems $S' = (H'; E'_1, E'_2, E'_3)$ and $S'' = (H''; E''_1, E''_2, E''_3)$ of three subspaces such that

1. $S = (H; E_1, E_2, E_3) \cong (H'; E'_1, E'_2, E'_3) \oplus (H''; E''_1, E''_2, E''_3)$.
2. $E'_1 \cap E'_2 = 0$
3. $E'_3 \subseteq E'_1 = E'_2$ ("distributive component ")

**Proof.** Consider two subspace decomposition for $F := E_1 \cap E_2$ and $E_3$. Since $(E_1 \cap E_2) + E_3$ is closed, we may and do assume that there exits no angle part for $F := E_1 \cap E_2$ and $E_3$ up to isomorphism. Therefore we have the following decomposition:

$$H = (E_1 \cap E_2) \cap E_3 \oplus (E_1 \cap E_2) \cap E_3^\perp \oplus (E_1 \cap E_2) \cap E_3 \oplus (E_1 \cap E_2) \cap E_3^\perp \cap E_3 \cap E_3^\perp$$

$$E_1 \cap E_2 = (E_1 \cap E_2) \cap E_3 \oplus (E_1 \cap E_2) \cap E_3^\perp \oplus O \oplus O$$

$$E_3 = (E_1 \cap E_2) \cap E_3 \oplus O \oplus (E_1 \cap E_2) \cap E_3 \oplus O$$

Then put $H' := E_1 \cap E_2$ and $H'' := (E_1 \cap E_2)^\perp$. Consider corresponding decompositions for $H = H' \oplus H''$. By taking intersections with $H'$ and $H''$, let

$$E'_1 = E_1 \cap (E_1 \cap E_2) = E_1 \cap E_2, \quad E'_2 = E_2 \cap (E_1 \cap E_2) = E_1 \cap E_2,$$

and

$$E'_3 = E_3 \cap (E_1 \cap E_2) = E_1 \cap E_2 \cap E_3$$
Then clearly we have that
\[ E'_1 + E''_1 = E_1, \quad E'_2 + E''_2 = E_2 \quad \text{and} \quad E'_3 + E''_3 = E_3 \]
Moreover we have that \( E'_3 \subset E'_1 \cap E'_2 = E_1 \) and \( E''_1 \cap E''_2 = O \).

\[ \square \]

**Lemma 4.6.** Let \( S = (H; E_1, E_2, E_3) \) be a system of three subspaces in a Hilbert space \( H \). Suppose that \( (E_1 \vee E_2) + E_3 \) is closed. Then there exist systems \( S' = (H'; E'_1, E'_2, E'_3) \) and \( S'' = (H''; E''_1, E''_2, E''_3) \) of three subspaces such that
\begin{enumerate}
\item \( S = (H; E_1, E_2, E_3) \cong (H'; E'_1, E'_2, E'_3) \oplus (H''; E''_1, E''_2, E''_3) \).
\item \( E'_1 \vee E''_2 = H'' \).
\item \( E'_1 = E'_2 = O \subset E'_3 \quad \text{("distributive component ")} \)
\end{enumerate}

**Proof.** Consider two subspace decomposition for \( F := E_1 \vee E_2 \) and \( E_3 \).
Since \( (E_1 \vee E_2) + E_3 \) is closed, we may and do assume that there exits no angle part for \( F := E_1 \vee E_2 \) and \( E_3 \) up to isomorphism. Therefore we have the following decomposition:
\[ H = (E_1 \vee E_2) \cap E_3 \oplus (E_1 \vee E_2) \cap E_3^\perp \oplus (E_1 \vee E_2)^\perp \cap E_3 \oplus (E_1 \vee E_2)^\perp \cap E_3^\perp \]
\[ E_1 \vee E_2 = ((E_1 \vee E_2) \cap E_3) \oplus (E_1 \vee E_2) \cap E_3^\perp \oplus O \oplus O \]
\[ E_3 = ((E_1 \vee E_2) \cap E_3) \oplus O \oplus (E_1 \vee E_2)^\perp \cap E_3 \oplus O \]

Then put \( H' := (E_1 \vee E_2)^\perp \) and \( H'' := (E_1 \vee E_2) \cap E_3 \). Consider corresponding decompositions for \( H = H' \oplus H'' \). By taking intersections with \( H' \) and \( H'' \), let
\[ E'_1 = (E_1 \vee E_2)^\perp \cap E_1 = O, \quad E'_2 = (E_1 \vee E_2)^\perp \cap E_2 = O, \quad E'_3 = (E_1 \vee E_2)^\perp \cap E_3 \]
\[ E''_1 = (E_1 \vee E_2) \cap E_1 = E_1, \]
\[ E''_2 = (E_1 \vee E_2) \cap E_2 = E_2, \]
\[ E''_3 = (E_1 \vee E_2) \cap E_3, \]

Then clearly we have that
\[ E'_1 + E''_1 = E_1, \quad E'_2 + E''_2 = E_2 \quad \text{and} \quad E'_3 + E''_3 = E_3 \]
Moreover we have that \( E'_1 = E'_2 = O \subset E'_3 \) and \( E''_1 \vee E''_2 = H'' \).

\[ \square \]

**Lemma 4.7.** Let \( H \) be a Hilbert space and \( H_1 \) and \( H_2 \) be two closed subspaces of \( H \). Assume that \( H = H_1 + H_2 \) and \( H_1 \cap H_2 = 0 \). Let \( F_i \) be an algebraic linear subspace of \( H_i \) for \( i = 1, 2 \). Consider their algebraic linear sum \( F := F_1 + F_2 \) in \( H \). Then \( F \) is closed if and only if \( F_1 \) and \( F_2 \) are closed.
Proof. There exists a closed subspace $M \subset H$ such that $(H; H_1, H_2)$ is isomorphic to $(H; M, M^\perp)$. Then the statement is reduced to this case where the statement is clear. □

By the Lemma above, we have immediately have the followings:

**Lemma 4.8.** Let $S = (H; E_1, E_2, E_3)$ be a system of three subspaces in a Hilbert space $H$. Suppose that there exist systems $S' = (H'; E'_1, E'_2, E'_3)$ and $S'' = (H''; E''_1, E''_2, E''_3)$ of three subspaces such that

$$S = (H; E_1, E_2, E_3) \cong (H'; E'_1, E'_2, E'_3) \oplus (H''; E''_1, E''_2, E''_3).$$

Then the followings hold:

1. $(E_1 \cap E_2) + E_3$ is closed in $H$ if and only if $(E'_1 \cap E'_2) + E'_3$ is closed in $H'$ and $(E''_1 \cap E''_2) + E''_3$ is closed in $H''$.
2. $(E_1 \vee E_2) + E_3$ is closed in $H$ if and only if $(E'_1 \vee E'_2) + E'_3$ is closed in $H'$ and $(E''_1 \vee E''_2) + E''_3$ is closed in $H''$.
3. $E_1 \cap E_2 = 0$ if and only if $E'_1 \cap E'_2 = 0$ and $E''_1 \cap E''_2 = 0$.
4. $E_1 \vee E_2 = H$ if and only if $E'_1 \vee E'_2 = H'$ and $E''_1 \vee E''_2 = H''$.

**Theorem 4.9.** Let $S = (H; E_1, E_2, E_3)$ be a system of three subspaces in a Hilbert space $H$. Then the followings are equivalent:

1. Linear sums $(E_i \vee E_j) + E_k$ and $(E_i \cap E_j) + E_k$ are closed for $i, j, k \in \{1, 2, 3\}$ with $i \neq j \neq k \neq i$.
2. $S$ has a Brenner type decomposition.

Proof. It is trivial that (2) implies (1). Conversely, assume (1). Since $(E_1 \cap E_2) + E_3$ is closed, there exist systems $S' = (H'; E'_1, E'_2, E'_3)$ and $S'' = (H''; E''_1, E''_2, E''_3)$ of three subspaces such that

1. $S = (H; E_1, E_2, E_3) \cong (H'; E'_1, E'_2, E'_3) \oplus (H''; E''_1, E''_2, E''_3)$.
2. $E''_1 \cap E''_2 = 0$
3. $E'_3 \subset E'_1 = E'_2$ ($\text{"{distributive component"}$}$)

Since $(E'_1 \vee E'_2) + E'_3$ is closed. Then there exist systems $S''' = (H'''; E'''_1, E'''_2, E'''_3)$ and $S'''' = (H''''; E''''_1, E''''_2, E''''_3)$ of three subspaces such that

1. $S''' = (H'''; E'''_1, E'''_2, E'''_3) \cong (H'''; E'''_1, E'''_2, E'''_3) \oplus (H''''; E''''_1, E''''_2, E''''_3)$.
2. $E'''''_1 \vee E'''''_2 = H'''''$
3. $E'''''_1 = E'''''_2 = O \subset E''''$ (\text{"{distributive component"}$}$)

Therefore

$$S = (H; E_1, E_2, E_3) \cong (H'; E'_1, E'_2, E'_3) \oplus (H'''; E'''_1, E'''_2, E'''_3) \oplus (H''''; E''''_1, E''''_2, E''''_3).$$

Moreover $E'''''_1 \vee E'''''_2 = H'''''$ and $E'''''_1 \cap E'''''_2 = 0$. We shall split out a distributive part and a double triangle part step by step using Lemma 4.5, Lemma 4.6, Lemma 4.7 and Lemma 4.8. Since closedness property is preserved after we split out one step, we can proceed the next step.
Finally we can split out a double triangle part \((Q; Q_1, Q_2, Q_3)\), because in the final step we have that
\[ Q_i \lor Q_j = Q, \quad \text{and} \quad Q_i \land Q_j = 0, \quad (i \neq j, i, j = 1, 2, 3). \]
and the rest part consists of finite direct sum of distributive systems of three subspaces. Hence we have (2).

5. Dense Decomposition

In an infinite-dimensional Hilbert space \(H\), the algebraic linear sum \(H' + H''\) of closed subspaces \(H'\) and \(H''\) is not necessary closed. Therefore we cannot expect direct sum decomposition in general.

**Example 8.** Let \(S = (H; E_1, E_2, E_3)\) be a system of three subspaces in a Hilbert space \(H\). Suppose that \(E_1 \cap E_2 = O\) and \(E_1 + E_2\) is not closed and \(E_1 \lor E_2 = E_1 + E_2 = E_3 = H\). Put \(H' = E_1\) and \(H'' = E_2\). Then \(H\) has a "dense decomposition" \(H = H' + H''\) such that \(H' \cap H'' = O\).

For example, any system of two subspaces \(S = (H; E_1, E_2)\) has a dense decomposition satisfying distributive law. In fact,
\[
(E_1 \cap E_2) \oplus_{alg} (K \oplus O) \oplus_{alg} \text{Im} \left( \begin{pmatrix} c^2 & c s \\ c s & s^2 \end{pmatrix} \right) \oplus_{alg} (E_1 \cap E_2^2) \oplus_{alg} (E_1 \cap E_2) \oplus_{alg} (E_1 \cap E_2^2) \oplus_{alg} (E_1 \cap E_2) \oplus_{alg} (E_1 \cap E_2^2).
\]
is a dense decomposition of \(H\). We expect that a certain class of systems \(S\) of three subspaces has a dense decomposition with a distributive part \(H^{dis}\), a double triangle part \(Q\) and six kinds of pentagon parts

\[
H^\sigma = \sum_{\sigma \in S_3} (E^\sigma_{\sigma(1)} \oplus_{alg} E^\sigma_{\sigma(3)})
\]
(with \(E^\sigma_{\sigma(3)} \supset E^\sigma_{\sigma(2)}\)), for a permutation \(\sigma \in S_3\) on three letters \(\{1, 2, 3\}\).

A distributive part is an algebraic sum of \(2^3 = 8\) components
\[
H^{dis} = S \oplus_{alg} N_1 \oplus_{alg} N_2 \oplus_{alg} N_3 \oplus_{alg} M_1 \oplus_{alg} M_2 \oplus_{alg} M_3 \oplus_{alg} L
\]
and a double triangle part is a Hilbert space \(Q\) with
\[
Q = \frac{Q_1 \oplus_{alg} Q_2}{17} = \frac{Q_2 \oplus_{alg} Q_3}{Q_3 \oplus_{alg} Q_1}.
\]
Then \((H^\sigma; E_1^\sigma, E_2^\sigma, E_3^\sigma)\) form pentagons (with \(E_{\sigma(3)}^\sigma \supset E_{\sigma(2)}^\sigma\)), so that

\[ E_{\sigma(1)}^\sigma \lor E_{\sigma(2)}^\sigma = H^\sigma, \quad E_{\sigma(1)}^\sigma \land E_{\sigma(3)}^\sigma = 0, \quad \text{and} \quad E_{\sigma(3)}^\sigma \supset E_{\sigma(2)}^\sigma \quad \text{with} \quad E_{\sigma(1)}^\sigma \neq E_{\sigma(2)}^\sigma. \]

In this way

\[ S \oplus_{\text{alg}} N_1 \oplus_{\text{alg}} N_2 \oplus_{\text{alg}} N_3 \oplus_{\text{alg}} M_1 \oplus_{\text{alg}} M_2 \oplus_{\text{alg}} M_3 \oplus_{\text{alg}} L \oplus_{\text{alg}} Q \oplus_{\text{alg}} \sum_{\sigma \in S_3} (E_{\sigma(1)}^\sigma \oplus_{\text{alg}} E_{\sigma(3)}^\sigma) \]

is dense in \(H\).

But we do not know whether this kinds of decomposition hold or not in general.

Finally we give a partial result which gives a condition on a system to have a (dense) decomposition containing a pentagon.

**Lemma 5.1.** Let \(S = (H; E_1, E_2, E_3)\) be a system of three subspaces in a Hilbert space \(H\). Suppose that there exist systems \(S' = (H'; E'_1, E'_2, E'_3)\) and \(S'' = (H''; E''_1, E''_2, E''_3)\) of three subspaces such that

1. \(S = (H; E_1, E_2, E_3) \cong (H'; E'_1, E'_2, E'_3) \oplus (H''; E''_1, E''_2, E''_3)\).
2. \((H'; E'_1, E'_2, E'_3)\) forms a pentagon (with \(E'_3 \supset E'_2\)).
3. (distributive component) there exist subspaces \(N_1, N_2\) and \(M_1\) of \(H''\) such that

\[ H'' = N_1 \oplus N_2 \oplus M_1, \]

\[ E''_1 = O \oplus N_2 \oplus M_1, \quad E''_2 = N_1 \oplus O \oplus O \quad \text{and} \quad E''_3 = N_1 \oplus N_2 \oplus O. \]

Then \(E_1 \land E_2 = O, \quad E_1 \lor E_2 = H \quad \text{and} \quad E_2 \not\subseteq E_3\).

**Proof.** It is clear. \(\square\)

We can rearrange the above decomposition such that \((H'; E'_1 \oplus M_1, E'_2 \oplus N_1, E'_3 \oplus N_1)\) forms a pentagon (with \(E'_3 \supset E'_2\)) and \((N_2; N_2, O, N_2)\) is a distributive part.

We shall show that the converse of the Lemma above holds in the sense of dense decomposition.

**Proposition 5.2.** Let \(S = (H; E_1, E_2, E_3)\) be a system of three subspaces in a Hilbert space \(H\). Suppose that

\[ E_1 \land E_2 = O, \quad E_1 \lor E_2 = H \quad \text{and} \quad E_2 \not\subseteq E_3. \]

We assume that \(E_3/E_2\) is finite dimensional. Then we have the following:

(i) If \(E_3 \neq E_3 \cap (E_1 + E_2)\), then there exist subspaces \(E'_1, E'_2, E'_3\) and \(N_2\) of \(H\) such that

1. \(H \supset E'_1 \oplus_{\text{alg}} E'_3 \oplus_{\text{alg}} N_2\) (dense)
2. \((E'_1 \lor E'_3, E'_1, E'_2, E'_3)\) forms a pentagon (with \(E'_3 \supset E'_2\)).
3. \(E_1 = E'_1 \oplus_{\text{alg}} N_2, \quad E_2 = E'_2 \quad \text{and} \quad E_3 = E'_3 \oplus_{\text{alg}} N_2\).
Moreover \((N_2; N_2, O, N_2)\) is a distributive part.
(ii) If \(E_3 = E_3 \cap (E_1 + E_2)\), then there exist subspaces \(N_1, N_2\) and \(M_1\) of \(H\) such that
\[
H \supset N_1 \oplus_{\text{alg}} N_2 \oplus_{\text{alg}} M_1 \quad (\text{dense}),
\]
\(E_1 = O + N_2 + M_1, E_2 = N_1 + O + O\) and \(E_3 = N_1 + N_2 + O\).

Proof. Case (i): Assume that \(E_3 \neq E_3 \cap (E_1 + E_2)\).

Put \(F_3 = E_3 \cap (E_3 \cap (E_1 + E_2))^\perp \neq 0\). Then \(E_3 = (E_3 \cap (E_1 + E_2)) \oplus F_3\).

Since \(E_3 \supset E_2, F_3\) is orthogonal to \(E_2\). We shall show that
\[
E_1 \cap (E_2 + F_3) = 0
\]

In fact, let \(x_1 = x_2 + f_3 \in E_1 \cap (E_2 + F_3)\) for \(x_1 \in E_1, x_2 \in E_2\) and \(f_3 \in F_3\). Then \(f_3 = x_1 - x_2 \in E_3 \cap (E_1 + E_2)\). But \(f_3\) is also in \((E_3 \cap (E_1 + E_2))^\perp\). Hence \(f_3 = 0\). Then \(x_1 = x_2\) is in \(E_1 \cap E_2 = 0\).

Therefore \(x_1 = x_2 = 0\).

Since \(E_3/E_2\) is finite dimensional, we can find
\[
\nu_1, \ldots, \nu_n \in E_3 \cap (E_1 + E_2) \cap E_2
\]
such that the quotient image \(\overline{\nu_1}, \ldots, \overline{\nu_n}\) are linearly independent in the quotient \(E_3/E_2\) and
\[
E_3 \cap (E_1 + E_2) = E_2 + [\nu_1, \ldots, \nu_n],
\]
where \([\nu_1, \ldots, \nu_n]\) is a linear span of \(\nu_1, \ldots, \nu_n\). In particular, \(E_3 \cap (E_1 + E_2)\) is closed in \(H\). Choose \(v_1, \ldots, v_n \in E_1\) and \(w_1, \ldots, w_n \in E_2\) such that
\[
v_k = v_k + w_k \quad (k = 1, \ldots, n).
\]
Since the quotient image \(\overline{v_k} = \overline{v_k} \quad (k = 1, \ldots, n)\), we have that \(v_1, \ldots, v_n\) are linearly independent.

Put \(N_2 = [v_1, \ldots, v_n]\). Since \(E_2 \subset E_3, v_k = u_k - w_k\) is also in \(E_3\).

Hence \(N_2 \subset E_1 \cap E_3\). Since \(u_1, \ldots, u_n\) are in \(E_2\), we have that
\[
E_3 \cap (E_1 + E_2) = E_2 + [v_1, \ldots, v_n],
\]
Put \(E'_1 = E_1 \cap [v_1, \ldots, v_n]^\perp, E'_2 = E_2\) and \(E'_3 = E_2 \oplus F_3\).

Then
\[
E_3 = (E_3 \cap (E_1 + E_2)) \oplus F_3 = (E_2 + [v_1, \ldots, v_n]) \oplus F_3.
\]

We shall show that \(N_2, E'_1, E_2\) and \(F_3\) are linearly independent. In fact, let \(n_2 + x_1 + x_2 + f_3 = 0\) for \(n_2 \in N_2, x_1 \in E'_1, x_2 \in E_2\) and \(f_3 \in F_3\). Then
\[
n_2 + x_1 = -x_2 - f_3 \in E_1 \cap (E_2 + F_3) = 0.
\]
Therefore \(n_2 + x_1 = 0\) and \(x_2 + f_3 = 0\). Since \(N_2 \perp E'_1\) and \(F_3 \perp E_2\), we have that \(n_2 = x_1 = x_2 = f_3 = 0\). Since
\[
(E'_1 \cap E'_3) \subset E_1 \cap (E_2 + F_3) = 0,
\]
\((E'_1 \cap E'_3, E'_1, E'_2, E'_3)\) forms a pentagon (with \(E'_3 \supset E'_2\)). And \(E_1 = E'_1 \oplus_{\text{alg}} N_2, E_2 = E'_2\) and \(E_3 = E'_3 \oplus_{\text{alg}} N_2\).

Case (ii): Assume that \(E_3 = E_3 \cap (E_1 + E_2)\). Then \(F_3 = E_3 \cap (E_3 \cap (E_1 + E_2)) = 0\) and
\((E_1 + E_2)^\perp = 0\). We can similarly define \(u_k, v_k, w_k\) as above. Put \(N_2 = \langle v_1, \ldots, v_n \rangle\). Define \(M_1 = E_1 \cap \langle v_1, \ldots, v_n \rangle^\perp\), \(N_1 = E_2\). Then \(E_1 = N_2 \oplus M_1\) and \(E_3 = N_1 + N_2\). Moreover \(N_1 + N_2 + M_1 = E_2 + E_1\) is dense in \(H\). And \(N_1, N_2\) and \(M_1\) are linearly independent. In fact, let \(n_1 + n_2 + m_1 = 0\) for \(n_1 \in N_1, n_2 \in N_2\) and \(m_1 \in M_1\). Then \(n_2 + m_1 = -n_1 \in E_1 \cap E_2 = 0\). Hence \(n_2 + m_1 = n_1 = 0\). Since \(N_2\) and \(M_1\) are orthogonal, \(n_2 = m_1 = 0\). Hence
\[
H \supset N_1 \oplus_{\text{alg}} N_2 \oplus_{\text{alg}} M_1 \ (\text{dense}) ,
\]

\(\square\)

**Example 9.** Let \(K = l^2(\mathbb{N})\) be the Hilbert space of square summable sequences. Let \(A : K \to K\) be a diagonal operator such that \((Ax)_n = \frac{1}{n} x_n\) for \(x = (x_n)_n \in K\). Then \(\text{Im } A\) is dense in \(K\) and not equal to \(K\). Put \(f = (1, 1/2, 1/3, \ldots, 1/n, \ldots) \in K\) and \(v = (0, 1/2, 1/3, \ldots, 1/n, \ldots) \in K\). Then \(f\) and \(v\) are not in \(\text{Im } A\). Define \(N_2 = \mathbb{C}\{0, v\}\), \(H = K \oplus K\), \(E_1 = (K \oplus O) + N_2\), \(E_2 = \{(x, Ax) \mid x \in K\}\) and \(E_3 = E_2 + \mathbb{C}\{0, f\} + N_2\). Put \(E'_1 = K \oplus O\), \(E'_2 = \{(x, Ax) \mid x \in K\}\) and \(E'_3 = E_2 + \mathbb{C}\{0, f\}\). Then \((E'_1 \vee E'_3; E'_1, E'_2, E'_3)\) forms a pentagon (with \(E'_3 \supset E'_2\)).

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