A Polynomial-time Fragment of Epistemic Probabilistic Argumentation (Technical Report)

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ABSTRACT
Probabilistic argumentation allows reasoning about argumentation problems in a way that is well-founded by probability theory. However, in practice, this approach can be severely limited by the fact that probabilities are defined by adding an exponential number of terms. We show that this exponential blowup can be avoided in an interesting fragment of epistemic probabilistic argumentation and that some computational problems that have been considered intractable can be solved in polynomial time. We give efficient convex programming formulations for these problems and explore how far our fragment can be extended without losing tractability.

KEYWORDS
Probabilistic Argumentation, Algorithms for Probabilistic Argumentation, Complexity of Probabilistic Argumentation

1 INTRODUCTION
Abstract argumentation [18] deals with the question what arguments a rational agent can accept. This question is answered independent of the content of the arguments, just based on their relationships. To this end, abstract argumentation problems can be modeled as graphs, where nodes correspond to arguments and edges to special relations like attack or support. In the basic setting introduced in [18] only attack relations were considered. In bipolar argumentation, this framework is extended with support relations [3, 8, 13, 14]. Another useful extension is to go beyond the classical two-valued view that arguments can only be accepted or rejected. Examples include ranking frameworks that can be based on fixed point equations [5, 7, 15, 37] or the graph structure [1, 12] and weighted argumentation frameworks [2, 4, 40, 48, 52]. Probabilistic argumentation frameworks express uncertainty by building up on probability theory and probabilistic reasoning methods. Uncertainty can be introduced, for example, over possible worlds, over subgraphs of the argumentation graph or over classical extensions [17, 19, 27, 36, 38, 44, 53–55, 57, 58]. For the subgraph-based approach, the computational complexity has been studied extensively in [20, 21].

Our focus here is on the epistemic approach to probabilistic argumentation that has evolved from work in [26, 56]. The idea is to consider probability functions over possible worlds in order to assign degrees of beliefs to arguments. Based on the relationships between arguments, the possible degrees of beliefs are then restricted by semantical constraints. Two basic computational problems have been introduced in [31]. The satisfiability problem asks whether a given set of semantical constraints over an argumentation graph can be satisfied by a probability function. The entailment problem is to answer queries about the probability of arguments. To this end, probability bounds on the probability of the argument are computed based on the probability functions that satisfy the given semantical constraints. Based on their close relationship to problems considered in probabilistic reasoning, it has been conjectured that these problems are intractable. However, as we will explain, both problems can actually be solved in polynomial time. Intuitively, the reason is that the semantical constraints can only talk about atomic probability statements. For this reason, reasoning with probability functions over possible worlds turns out to be equivalent to reasoning with functions that assign probabilities to arguments directly. We call these functions probability labellings as they can be seen as generalizations of labellings in classical abstract argumentation [11] that, intuitively, label arguments as rejected (probability 0), accepted (probability 1) or undecided (probability 0.5).

We explain the epistemic probabilistic argumentation approach from [26, 31, 56] in more detail in Section 2 and introduce a slight generalization of the computational problems considered in [31]. Even more general variants of these problems have been considered in [29], but these variants are too general to obtain polynomial runtime guarantees as we will explain in Section 2. In Section 3, we show that reasoning with probability labellings is equivalent to reasoning with probability functions when only atomic probability statements are considered and use this observation to show that both the satisfiability and the entailment problem considered in [31] and their generalizations can be solved in polynomial time. We then look at how far we can extend our language towards the language considered in [29] by allowing connecting arguments or constraints with logical connectives. In Section 4, we look at more expressive queries. We cannot avoid an exponential blowup when considering arbitrary queries. However, we show that when applying the principle of maximum entropy, conjunctive queries can still be answered in polynomial time. In particular, we show that a compact representation of the maximum entropy probability function that satisfies the constraints can be computed in polynomial time. In Section 5, we look at more expressive constraints. We find that the constraint language cannot be extended much further. If we only allow connecting two arguments or their negation by only conjunction or disjunction in probability statements or if we allow connecting two constraints disjunctively, the satisfiability problem becomes intractable.

2 BACKGROUND
We consider bipolar argumentation frameworks (BAFs) \((\mathcal{A}, \mathcal{R}, \mathcal{S})\) consisting of a set of arguments \(\mathcal{A}\), an attack relation \(\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}\) and a support relation \(\mathcal{S} \subseteq \mathcal{A} \times \mathcal{A}\). \(\text{Att}(A) = \{B \in \mathcal{A} \mid (B, A) \in \mathcal{R}\}\) denotes the set of attackers of an argument \(A\) and \(\text{Sup}(A) = \{B \in \mathcal{A} \mid (A, B) \in \mathcal{S}\}\) denotes the set of supporters of an argument \(A\).
A | (B, A) ∈ S denotes its supporters. We visualize bipolar argumentation frameworks as graphs, where arguments are denoted as nodes, solid edges denote attack relations and dashed edges denote support relations. Figure 1 shows an example BAF with four arguments A, B, C, D.

We define a possible world as a subset of arguments w ⊆ A. Intuitively, w contains the arguments that are accepted. As usual, 2^A denotes the set of all subsets of A, that is, the set of all possible worlds. In order to talk about an agent’s beliefs in arguments acceptance state we can consider probability functions P : 2^A → [0, 1] such that \( \sum_{w \in 2^A} P(w) = 1 \). We denote the set of all probability functions over A by P_A. The probability of an argument A ∈ A under P is defined by adding the probabilities of all worlds in which A is accepted, that is, \( P(A) = \sum_{w \in 2^A, A \subseteq w} P(w) \). P(A) can be understood as a degree of belief of an agent, where P(A) = 1 means complete acceptance and P(A) = 0 means complete rejection.

Given an argumentation graph, a probability function should maintain reasonable relationships between the probabilities of arguments based on their relationships in the graph. For example, if an argument is accepted, its attackers should not be accepted. To capture this intuition, several constraints have been introduced in the literature that can be imposed on the probability functions. For the satisfaction and entailment problem in [31], the following constraints have been considered (for attack-only graphs).

**COH**: P is called coherent if for all A, B ∈ A with (A, B) ∈ R, we have P(B) ≤ 1 − P(A).

**SFou**: P is called semi-founded if P(A) ≥ 0.5 for all A ∈ A with Att(A) = ∅.

**FOU**: P is called founded if P(A) = 1 for all A ∈ A with Att(A) = ∅.

**SOPT**: P is called semi-optimistic if P(A) ≥ 1 − \( \sum_{B \in Att(A)} P(B) \) for all A ∈ A with Att(A) = ∅.

**OPT**: P is called optimistic if P(A) ≥ 1 − \( \sum_{B \in Att(A)} P(B) \).

**JUS**: P is called justifiable if P is coherent and optimistic.

The intuition for these constraints comes from the idea that probability 0.5 represents indifference, whereas probabilities smaller (larger) than 0.5 tend towards rejection (acceptance) of the argument. Coherence imposes an upper bound on the beliefs in arguments based on the beliefs in their attackers. Semi-Foundedness says that an agent should not tend to reject an argument if there is no reason for this. Foundedness even demands that the argument should be fully accepted in this case. Semi-optimistic and Optimistic give upper bounds on the beliefs in an argument based on its attackers and supporters. Usually, not all constraints are employed, but a subset is selected that seems reasonable for a particular application.

**Example 2.1.** If we demand COH and FOU for the BAF in Figure 1, we get P(C) = 1 and P(D) = 1 from FOU. From COH, we get P(A) ≤ 1 − P(B), P(B) ≤ 1 − P(A) and P(B) ≤ 1 − P(D). Since P(D) = 1, the last inequality implies P(B) = 0.

One could define natural dual constraints for support-only graphs, for example:

**S-COH**: P is called s-coherent if for all A, B ∈ A with (A, B) ∈ S, we have P(B) ≥ P(A).

**PES**: P is called pessimistic if P(A) ≤ \( \sum_{B \in \text{Sup}(A)} P(B) \).

**Example 2.2.** If we add S-COH to our previous example, we get P(C) ≥ P(D) and P(A) ≥ P(C). Since we already know that P(C) = 1, we can conclude P(A) = 1. Overall, the constraints imply P(A) = P(C) = P(D) = 1 and P(B) = 0.

If both attack and support relations are present, one may want to further refine constraints like Optimism and Pessimism to take account of both attackers and supporters simultaneously. In order to provide more flexibility, a general constraint language has been considered recently that captures all of the previous examples [29]. This language allows constraints over complex formulas of arguments and connecting constraints via logical connectives. However, for now, we consider only a simple fragment here for which we can obtain polynomial performance guarantees.

**Definition 2.3 (Linear Atomic Constraint, Satisfiability).** A linear atomic constraint is an expression of the form \( \sum_{i=1}^{n} c_i \cdot \pi(A_i) \leq c_0 \), where \( A_i \in A \) and \( c_i \in \mathbb{R} \). A probability function P satisfies a linear atomic constraint iff \( \sum_{i=1}^{n} c_i \cdot P(A_i) \leq c_0 \). P satisfies a set of linear atomic constraints C, denoted as P |= C, if it satisfies all \( l \in C \). In this case, C is called satisfiable.

Note that \( ≥, \leq \) and = can be expressed as well. For \( ≥, \) just note that \( \sum_{i=1}^{n} c_i \cdot P(A_i) \leq c_0 \) is equivalent to \( \sum_{i=1}^{n} c_i \cdot P(A_i) \geq c_0 \). For equality, note that \( \sum_{i=1}^{n} c_i \cdot P(A_i) \leq c_0 \) and \( \sum_{i=1}^{n} c_i \cdot P(A_i) \geq c_0 \) together are equivalent to \( \sum_{i=1}^{n} c_i \cdot P(A_i) = c_0 \). We merely restrict our language to constraints with ≤ in order to keep the notation simple. Notice that this restriction is also not important for complexity considerations because the number of constraints just changes by a constant factor. All semantical constraints that we mentioned before are indeed linear atomic constraints.

Inspired by the probabilistic entailment problem from probabilistic logic [24, 25, 41], the authors in [31] considered the following reasoning problems: Given a partial probability assignment (constraints of the form P(A) = x for some A ∈ A) and a subset of the semantical constraints (linear atomic constraints),

1. decide whether there is a probability function that satisfies the partial probability assignment and the semantical constraints,
2. compute lower and upper bounds on the probability of an argument among the probability functions that satisfy the partial probability assignment and the semantical constraints,
3. decide whether given lower and upper bounds on the probability of an argument are taken by probability functions that satisfy the partial probability assignment and the semantical constraints.

Because of their similarity to intractable probabilistic reasoning problems, it was conjectured that these problems are intractable as well. However, as we will explain in the next section, all three problems can be solved in polynomial time.
Before doing so, we make the computational problems more precise. We will only generalize the first (satisfiability) and second (entailment) problem from [31]. Since the second problem can be solved in polynomial time, there is no need to look at the third problem, which is just a decision variant of the second problem. Formally, we consider the following two problems:

**PArgSAT**: Given a finite set of linear atomic constraints \( C \), decide whether it is satisfiable.

**PArgENT**: Given a finite set of satisfiable linear atomic constraints \( C \) and an argument \( A \), compute lower and upper bounds on the probability of \( A \) among the probability functions that satisfy \( C \). More precisely, solve the two optimization problems

\[
\min_{P \in \mathcal{P}_A} \max_{P \in \mathcal{P}_A} P(A) \quad \text{such that} \quad P \models C.
\]

PArg stands for probabilistic argumentation. At for the restriction to linear atomic constraints and SAT and ENT stand for satisfiability and entailment, respectively. Notice that the computational problems from [29] are indeed a special case because the partial probability assignments can just be encoded as constraints in \( C \).

**Example 2.4.** Consider the BAF in Figure 1. Say our partial probability assignment assigns probability 1 to \( B \) and 0 to \( C \). These assignments correspond to the two linear constraints \( \pi(B) = 1 \) and \( \pi(C) = 0 \). Say we also impose COH. Then, we additionally have the constraints \( \pi(A) + \pi(B) \leq 1 \) and \( \pi(B) + \pi(D) \leq 1 \). Together, these constraints imply that every probability function \( P \) that satisfies all constraints, must satisfy \( P(B) = 1, P(C) = 0 \) (partial assignment constraints), \( P(A) = 0 \) and \( P(D) = 0 \) (follow with coherence constraints). Note that when also adding the foundedness constraints \( \pi(C) = 1 \) and \( \pi(D) = 1 \), the set of constraints becomes unsatisfiable.

### 3 PROBABILITY LABELLINGS AND TWO POLYNOMIAL-TIME ALGORITHMS

We define a probability labelling as a function \( L : \mathcal{A} \to [0,1] \). That is, a probability labelling assigns a degree of belief to arguments directly, rather than in an indirect way using possible worlds. \( L_{\mathcal{A}} \) denotes the set of all probability labellings over \( \mathcal{A} \). We will now show that probability labellings correspond to equivalence classes of probability functions and that by restricting to these equivalence classes (represented by probability labellings), we can solve PArgSAT and PArgENT in polynomial time.

We call two probability functions \( P_1, P_2 \) atomically equivalent, denoted as \( P_1 \equiv P_2 \), if \( P_1(A) = P_2(A) \) for all \( A \in \mathcal{A} \). Atomic equivalence is an equivalence relation. \( [P] = \{P' \in \mathcal{P}_{\mathcal{A}} | P' \equiv P\} \) denotes the equivalence class of \( P \) and \( \mathcal{P}_{\mathcal{A}}/\equiv = \{[P] | P \in \mathcal{P}_{\mathcal{A}}\} \) denotes the set of all equivalence classes. We first note that there is a one-to-one relationship between \( \mathcal{P}_{\mathcal{A}}/\equiv \) and \( L_{\mathcal{A}} \).

**Proposition 3.1.** The function \( r : \mathcal{P}_{\mathcal{A}}/\equiv \to L_{\mathcal{A}} \) defined by \( r([P]) = L_P \), where \( L_P(A) = P(A) \) for all \( A \in \mathcal{A} \) is a bijection.

In particular, for every labelling \( L \in L_{\mathcal{A}} \), there is a probability function \( P_L \) such that \( P_L(w) = \prod_{A \in L} L(A) \cdot \prod_{A \notin \mathcal{A} \backslash \{w\}} (1 - L(A)) \) and \( r([P_L]) = L \).

**Proof.** First note that \( r \) is well-defined: this is because, for all \( P' \in [P] \), we have \( L_P(A) = P'(A) = P(A) = L_P(A) \) for all \( A \in \mathcal{A} \) by definition of \( \equiv \).

\( r \) is injective for if \( r([P_1]) = r([P_2]) \), then \( P_1(L) = P_2(L) = L_1(A) = L_2(A) \) for all \( A \in \mathcal{A} \). That is, \( P_1 \equiv P_2 \) and \( [P_1] = [P_2] \).

\( r \) is also surjective. To see this, consider an arbitrary \( L \in L_{\mathcal{A}} \). Define \( \mathcal{P}_L : 2^{\mathcal{A}} \to [0,1] \) via \( \mathcal{P}_L(w) = \prod_{A \in L} L(A) \cdot \prod_{A \notin \mathcal{A} \backslash \{w\}} (1 - L(A)) \) for all \( w \in 2^{\mathcal{A}} \). We prove by induction over the number of arguments that \( \sum_{w \in 2^\mathcal{A}} \mathcal{P}_L(w) = 1 \). For the base case, consider \( \mathcal{A} = \{ A \} \). Then \( \mathcal{P}_L(\emptyset) + \mathcal{P}_L(\{A\}) = (1 - L(A)) + L(A) = 1 \). For the induction step, consider \( |\mathcal{A}| = n + 1 \) and let \( B \in \mathcal{A} \). Then

\[
\sum_{w \in 2^n} \mathcal{P}_L(w) = \sum_{w \in 2^n} \prod_{A \in w} L(A) \cdot \prod_{A \notin \mathcal{A} \backslash \{w\}} (1 - L(A)) = (1 - L(B)) \sum_{w \in 2^n \backslash \{B\}} \prod_{A \in w} L(A) \cdot \prod_{A \notin \mathcal{A} \backslash \{w\}} (1 - L(A)) + L(B) \sum_{w \in 2^n \backslash \{B\}} \prod_{A \in w} L(A) \cdot \prod_{A \notin \mathcal{A} \backslash \{w\}} (1 - L(A)) = (1 - L(B)) + L(B) = 1.
\]

In the second and third row, we partitioned the worlds in those that reject \( B \) (second row) and those that accept \( B \) (third row). Notice that the sums in the second and third row correspond to possible worlds over a set of arguments of length \( n \), so that our induction hypothesis implies that they sum up to 1. Hence, \( P_L \) is a probability function. Furthermore, for all \( B \in \mathcal{A} \), we have

\[
P_L(B) = \sum_{w \in 2^n, B \in w} P(w) = L(B) \sum_{w \in 2^n \backslash \{B\}} \prod_{A \in w} L(A) \cdot \prod_{A \notin \mathcal{A} \backslash \{w\}} (1 - L(A)) = L(B),
\]

where we used again the fact that the sum in the second row has to sum up to 1. Hence, \( r([P_L]) = L \) and \( r \) is also surjective and thus bijective.

Intuitively, \( r \) determines a compact representative for the equivalence class \([P]\), namely the probability labelling \( L_P = r([P]) \). We say that a probability labelling \( L \) satisfies a linear atomic constraint \( \sum_{i=1}^n c_i \cdot \pi(A_i) \leq c_0 \) if \( r(\sum_{i=1}^n c_i \cdot L(A_i)) \leq c_0 \). The following proposition explains that we can capture the set of all probability functions that satisfy a constraint by the set of labellings that satisfy the constraint.

**Proposition 3.2.** A linear atomic constraint \( \sum_{i=1}^n c_i \cdot \pi(A_i) \leq c_0 \) is satisfied by a probability function \( P \) if and only if the probability labelling \( L_P = r([P]) \) and all \( P' \in [P] \) satisfy the constraint.

**Proof.** This follows immediately from the satisfaction definition and the observation that \( \sum_{i=1}^n c_i \cdot L_P(A_i) = \sum_{i=1}^n c_i \cdot L_P(A_i) = \sum_{i=1}^n c_i \cdot L_P(A_i) \) for all \( P' \in [P] \).

To begin with, we show that PArgSAT can be decided in polynomial time.

**Proposition 3.3.** PArgSAT can be solved in polynomial time. In particular, when given \( n \) arguments \( \mathcal{A} = \{A_1, \ldots, A_n\} \) and \( m \) constraints \( C = \left\{ \sum_{i=1}^n c_{ij} \cdot \pi(A_i) \mid 1 \leq j \leq m \right\} \), then \( C \) is
satisfiable if and only if the linear optimization problem

\[
\min_{(x,s) \in \mathbb{R}^{n+m}} \sum_{i=1}^{m} s_i \\
\text{such that } \sum_{i=1}^{n} c_i^{(j)} \cdot x_i \leq c_0^{(j)} + s_j, \quad 1 \leq j \leq m, \\
0 \leq x \leq 1, \quad s \geq 0,
\]

has minimum 0.

PROOF. First notice that every probabilistic labelling \( L \) corresponds to a vector \( x \in [0,1]^n \) such that \( x_i = L(A_i) \). The points \((x,s) \in \mathbb{R}^{n+m}\) are intuitively composed of a labelling \( x \) and a vector of slack variables \( s \) that relax the constraints.

To begin with, we show that the optimization problem always has a minimum. Let \( s \in \mathbb{R}^m \) be defined by \( s_j = \max(0,-c_0^{(j)}) \). Then \((0,s) \in \mathbb{R}^{n+m}\) is a feasible solution because for all constraints, we get \( \sum_{i=1}^{n} c_i^{(j)} \cdot 0 = 0 \leq c_0^{(j)} + \max(0,-c_0^{(j)}) \). Hence, the feasible region is non-empty and the theory of linear programming implies that the minimum exists [6]. In particular, since \( s \) is non-negative, it is clear that the minimum can never be smaller than 0.

We show next that the minimum is 0 if and only if there is a labelling that satisfies \( C \). Assume first that the minimum is 0 and let \((x^*,0) \in \mathbb{R}^{n+m}\) be an optimal solution. Consider \( L^* \) defined by \( L^*(A_i) = x_i^* \). We have \( \sum_{i=1}^{n} c_i^{(j)} \cdot L^*(A_i) = \sum_{i=1}^{n} c_i^{(j)} \cdot x_i^* \leq c_0^{(j)} + 0 \) for all \( 1 \leq j \leq m \). Hence, \( L^* \) satisfies \( C \).

Conversely, assume that there is a probability labelling that satisfies \( C \). Let \( x^* \in [0,1]^n \) be defined by \( x_i^* = L(A_i) \) and consider the point \((x^*,0) \in \mathbb{R}^{n+m}\). We have \( \sum_{i=1}^{n} c_i^{(j)} \cdot x_i^* = \sum_{i=1}^{n} c_i^{(j)} \cdot L^*(A_i) \geq c_0^{(j)} \). Therefore, \((x^*,0)\) is a feasible solution. In particular, it yields 0 for the objective function and hence is minimal.

We know from the theory of linear programming that linear optimization problems can be solved in polynomial time with respect to the number of optimization variables and constraints [6]. We have \( n + m \) optimization variables and \( m \) constraints (non-negativity constraints are free). Hence, we can decide in polynomial time whether there exists a probability labelling that satisfies \( C \). If there is such a labelling \( L \), then the probability function \( P_L \) from Proposition 3.1 satisfies \( C \) according to Proposition 3.2. Conversely, if there is no probability function that satisfies \( C \), then there can be no labelling that satisfies it either. For if there was such a labelling \( L \), then \( P_L \) would satisfy \( C \) as well. Hence, \( C \) is satisfiable by a probability function \( P \) if and only if the minimum of our linear optimization problem is 0. Hence, \( \text{PAArgAtENT} \) can be solved in polynomial time by the given linear program.

We can apply similar ideas to show that \( \text{PAArgAtENT} \) can be solved in polynomial time.

PROPOSITION 3.4. \( \text{PAArgAtENT} \) can be solved in polynomial time. In particular, when given \( n \) arguments \( \mathcal{A} = \{A_1, \ldots, A_n\} \) and \( m \) constraints \( C = \{\sum_{i=1}^{n} c_i^{(j)} \cdot \pi(A_i) \leq c_0^{(j)} \mid 1 \leq j \leq m\} \) such that \( C \) is satisfiable, then the lower and upper bounds on the probability of \( A_k \) are the results of the following linear optimization problems:

\[
\min_{x \in \mathbb{R}^n} \max_{x \in \mathbb{R}^n} x_k \\
\text{such that } \sum_{i=1}^{n} c_i^{(j)} \cdot x_i \leq c_0^{(j)}, \quad 1 \leq j \leq m, \\
0 \leq x \leq 1.
\]

PROOF. For concreteness and w.l.o.g assume that we want to compute bounds on the probability of \( A_1 \). We look only at the minimization problem for computing the lower bound (for the maximization problem everything is completely analogous). That is, we consider the following linear optimization problem:

\[
\min_{x \in \mathbb{R}^n} x_1 \\
\text{such that } \sum_{i=1}^{n} c_i^{(j)} \cdot x_i \leq c_0^{(j)}, \quad 1 \leq j \leq m, \\
0 \leq x \leq 1.
\]

By assumption, \( C \) is satisfiable. Hence, the feasible region is non-empty and the theory of linear programming implies that the minimum exists and can be computed in polynomial time [6]. The minimum found corresponds exactly to the smallest probability that is assigned to \( A_1 \) by a probability labelling that satisfies the constraints. To see this, note that if we take a minimal solution \( x^* \), we can construct a labelling \( L \) that satisfies the constraints as in the previous proof. In particular, \( x_1^* = L(A_1) \). There can be no probability labelling that satisfies the constraints and assigns a smaller probability to \( A_1 \) because each such labelling yields a feasible vector \( x \in \mathbb{R}^n \) with \( x_1 = L(A_1) \).

Similar to before, it follows that the minimum also corresponds to the smallest probability that is assigned to \( A_1 \) by a probability function that satisfies \( C \). If the minimum is taken by a labelling \( L \), we know that the corresponding probability function \( P_L \) from Proposition 3.1 yields the same probability and satisfies \( C \) according to Proposition 3.2. Hence, the minimum cannot be smaller than the probability taken by probability functions that satisfy \( C \). Conversely, if there is a probability function that satisfies \( C \) and gives \( P(A_1) = p \), then the labelling \( L_P \) gives \( L_P(A_1) = p \) as well and satisfies \( C \) according to Proposition 3.2. Hence, the minimum cannot be larger than the probability taken by probability functions that satisfy \( C \) either, and so it must be indeed equal. Hence, \( \text{PAArgAtENT} \) can be solved in polynomial time by the given linear program. □

4 COMPLEX QUERIES

Until now, we only looked at probabilities of arguments. However, the real power of probability functions is that they allow computing probabilities for arbitrary formulas over arguments. By a formula over a set of arguments \( \mathcal{A} \), we mean an expression that is formed by connecting the arguments in \( \mathcal{A} \) via logical connectives. Satisfaction of formulas by possible worlds is explained in the usual recursive way. For example, \( w \models \neg \phi \) if \( w \) does not satisfy \( \phi \) and \( w \models \phi \& \psi \) if \( w \) satisfies both \( \phi \) and \( \psi \) The probability of a formula \( \phi \) under \( P \) is defined by adding the probabilities of all worlds that satisfy \( \phi \), that is, \( P(\phi) = \sum_{w \models \phi} P(w) \). Unfortunately, we now have to add an exponential number of terms. There is probably no general way to avoid this problem because the entailment problem can now be used to solve the propositional satisfiability problem. In order to make this precise, we define a 3CNF-Query as a formula \( Q = \)
$A_{i,j}^0 \Delta \left( \bigvee_{j=1}^m A_{i,j}^{h_{i,j}} \right)$ over arguments, where $h_{i,j} \in \{0,1\}$, $A_{i,j}^0 := \neg A_i$ and $A_{i,j}^1 = A_i$.

**Proposition 4.1.** Let $C$ be a satisfiable set of linear atomic constraints over $\mathcal{A} = \{A_1, \ldots, A_n\}$ and let $Q$ be a 3CNF-query. Then the following problem is NP-complete: decide whether the upper bound on the probability of $Q$ among the probability functions that satisfy $C$ is non-zero.

**Proof.** For membership, we need a result from Linear Programming theory. Among the optimal solutions of an $N$-dimensional linear program, there must be one that satisfies $N$ constraints with equality [6]. In our context, this means that $2^n - |C|$ non-negativity constraints $P(w) \geq 0$ must be satisfied with equality. That is, among the optimal probability functions, there must be one that has at most $|C|$ non-zero probabilities. Let $W = \{w \mid w \in 2^\mathcal{A}, P(w) > 0\}$. Then for arbitrary formulas $F$ over $\mathcal{A}$, $P(F) = \sum_{w \in \mathcal{A}, w \models F} P(w) = \sum_{w \in W, w \models F} P(w)$. Hence, the set of pairs $(w, P(w)) \mid w \in \mathcal{A}$ provides a certificate of polynomial size such that checking the constraints and $P(Q) > 1$ can be done in polynomial time.

For hardness, we give a polynomial-time reduction from 3SAT. Given a propositional 3CNF formula $F$ with $n$ atoms $\alpha_i$, we introduce corresponding arguments $A_i$. Let $Q$ be the query obtained from $F$ by replacing $\alpha_i$ with $A_i$ for $i = 1, \ldots, n$. We do not add any constraints, so that all $P \in \mathcal{P}_\mathcal{A}$ satisfy our constraints. Then the upper bound on the probability of $Q$ is non-zero if $F$ is satisfiable. To see this, note that if $F$ is satisfiable, there is an interpretation that satisfies $F$ and a corresponding possible world $w$ that satisfies $Q$. Then the probability function $P_w$ with $P_w(w) = 1$ and $P_w^i(w) = 0$ for all other possible worlds gives $P_w(Q) = P_w(w) = 1 > 0$. Conversely, if $F$ is not satisfiable, $Q$ is not satisfiable either and $P(Q) = \sum_{w \in \mathcal{A}, w \models Q} P(w) = 0$ because the sum does not contain any terms for any $P \in \mathcal{P}_\mathcal{A}$.

There are, however, some interesting special cases that can be solved efficiently. One case is answering conjunctive queries under the principle of maximum entropy.

The entropy of a probability function $P$ over $\mathcal{A}$ is defined as $H(P) = -\sum_{w \in \mathcal{A}} P(w) \cdot \log P(w)$. It can be seen as a measure of uncertainty. Indeed, the entropy is always non-negative and maximal if $P$ is the uniform distribution. Intuitively, by maximizing entropy among the probability functions that satisfy a set of constraints, we select the probability distribution that adds as little information as possible. The principle of optimum entropy has been justified by several characterizations with common-sense properties [32–34, 43].

For a probability labelling $L$ over $n$ arguments $\mathcal{A} = \{A_1, \ldots, A_n\}$, we define its entropy as $H(L) = \sum_{w \in \mathcal{A}} (1 - L(A)) \cdot \log L(A) + (1 - (1 - L(A))) \cdot \log (1 - L(A))$. As we show next, $H(L)$ corresponds to the maximum entropy taken in the equivalence class $[P_L]$ and the maximum is taken by the corresponding probability function $P_L$.

**Proposition 4.2.** For every labelling $L \in \mathcal{L}_\mathcal{A}$, the probability function $P_L$ defined by $P_L(w) = \prod_{A \in \mathcal{A}} L(A) \cdot \prod_{A \in \mathcal{A}, A \cap \{w\} \neq \emptyset} (1 - L(A))$, maximizes entropy among all $P \in [P_L]$. In particular, $H(P_L) = H(L)$.

**Proof.** Consider an arbitrary probability function $P \in [P_L]$. For all formulas $F$, we let $1_F : 2^\mathcal{A} \to \{0, 1\}$ denote the indicator function that yields 1 iff $w \models F$. Then we have

$$H(L) - H(P) = -\sum_{i=1}^n (L(A_i) \cdot \log L(A_i) + (1 - L(A_i)) \cdot \log (1 - L(A_i))) + \sum_{w \in \mathcal{A}} P(w) \cdot \log P(w)$$

$$= -\sum_{i=1}^n (P(A_i) \cdot \log L(A_i) + (1 - P(A_i)) \cdot \log (1 - L(A_i))) + \sum_{w \in \mathcal{A}} P(w) \cdot \log P(w)$$

$$= -\sum_{w \in \mathcal{A}} P(w) \sum_{i=1}^n \left( 1_{A_i}(w) \cdot \log L(A_i) + 1 - 1_{A_i}(w) \cdot \log (1 - L(A_i)) \right) + \sum_{w \in \mathcal{A}} P(w) \cdot \log P(w)$$

$$= -\sum_{w \in \mathcal{A}} P(w) \sum_{i=1}^n \left( L(A_i) \cdot \prod_{A \in \mathcal{A}, A \cap \{w\} \neq \emptyset} (1 - L(A)) + (1 - L(A)) \cdot \prod_{A \in \mathcal{A}, A \cap \{w\} \neq \emptyset} (1 - L(A)) \right) + \sum_{w \in \mathcal{A}} P(w) \cdot \log P(w)$$

$$= \sum_{w \in \mathcal{A}} P(w) \log \frac{P(w)}{\prod_{A \in \mathcal{A}, A \cap \{w\} \neq \emptyset} (1 - L(A))} = \sum_{w \in \mathcal{A}} P(w) \log \frac{P(w)}{P_L(w)} = KL(P, P_L) \geq 0.$$ 

where, for the second equality, we used the fact that $P(A) = L(A)$ for all $A \in \mathcal{A}$ and in the last row, we used the observation that the previous formula corresponds to the KL-divergence between two probability functions that is always non-negative. Furthermore, the KL-divergence is 0 if and only if both arguments are equal [59], that is, $KL(P, P_L) = 0$ if and only if $P = P_L$. Therefore, $H(L) = H(P_L)$ and $H(L) > H(P)$ whenever $P \neq P_L$. In particular, $H(P_L) > H(P)$ for all $P \in [P_L] \setminus \{P_L\}$.

Hence, in order to compute the probability function with maximum entropy, we can just compute the labelling $L^*$ with maximum entropy. The corresponding probability function $P_{L^*}$ then maximizes entropy. This is the basic idea of the following proposition.

**Proposition 4.3.** Given a satisfiable finite set of linear atomic constraints $C$, the optimization problem

$$\arg \max_{P \in \mathcal{P}_\mathcal{A}} H(P)$$

such that $P \models C$. 

has a unique solution $P^*$ and $L_{P^*}$ is the unique solution of the optimization problem
\[
\arg\max_{L \in L_A} H(L)
\quad\text{such that}\quad L \models C.
\]
In particular, $L_{P^*}$ can be computed in polynomial time.

**Proof.** Both optimization problems have a strictly concave and continuous objective function. Maximizing such a function subject to consistent linear constraints yields a unique solution [42]. In particular, these problems can be solved by interior-point methods in polynomial time in the number of optimization variables and constraints [9]. For the first problem, the number of optimization variables is exponential in the number of arguments, but for the second problem the number of optimization variables equals the number of arguments. Hence, the problem can be solved in polynomial time. Since the solution $L^*$ of the second problem maximizes entropy among all probability labelings, and the probability distributions corresponding to the labelings maximize entropy among their equivalence classes according to Proposition 4.2, $L^*$ must equal $L_{P^*}^*$.

Having computed $L_{P^*}$, we can compute a compact representation of $P^*$. Of course, constructing $P^*$ explicitly would take exponential time again. Fortunately, for some queries, we can just work with the compact representation directly. This includes, in particular, conjunctive queries, as we explain in the following proposition.

**Proposition 4.4.** Let $\mathcal{A} = \{A_1, \ldots, A_w\}$, let $C$ be a satisfiable set of linear atomic constraints and let $Q$ be a conjunction of literals, that is, $Q = \bigwedge_{i \in I} A_i^{b_i}$, where $I \subseteq \{1, \ldots, n\}$, $b_i \in \{0, 1\}$. Let $P^*$ be the probability function that maximizes entropy among all probability functions that satisfy $C$. Then $P^*(Q)$ can be computed in polynomial time even if $P^*$ is unknown. In particular,
\[
P^*(\bigwedge_{i \in I} A_i^{b_i}) = \prod_{i \in I} L(A_i)^{b_i} \cdot (1 - L(A_i))^{1-b_i},
\]
where $L^*$ is the probability labelling that maximizes entropy among all probability labelings that satisfy $C$.

**Proof.** First note that $L(A_i)^{b_i} = L(A_i)$ if $b_i = 1$ and $L(A_i)^{b_i}$ is 1 otherwise. Dually, $\prod_{i \in I} (1 - L(A_i))^{1-b_i} = 1 - L(A_i)$ if $b_i = 0$ and $(1 - L(A_i))^{1-b_i}$ is 1 otherwise. Let $S = \bigcup_{i \in I} \{A_i\}$ denote the atoms occurring in $Q$. We know from Proposition 4.2 that for all $w \in 2^\mathcal{A}$ with $w \models Q$, we have
\[
P^*(w) = \prod_{A \in w} L(A) \cdot \prod_{A \in \mathcal{A} \setminus \{w\}} (1 - L(A))
\]
\[
= \left( \prod_{i \in I} L(A_i)^{b_i}(1 - L(A_i))^{1-b_i} \right) \cdot \prod_{A \in w \setminus S} L(A) \cdot \prod_{A \in \mathcal{A} \setminus \{w\}} (1 - L(A)),
\]
where we split up the arguments in $S$ (indexed by $i$) since we know their interpretation (because $w \models Q$). Therefore,
\[
P^*(Q) = \sum_{w \in 2^\mathcal{A}, w \models Q} P(w)
\]
\[
= \left( \prod_{i \in I} L(A_i)^{b_i}(1 - L(A_i))^{1-b_i} \right) \sum_{w \in 2^\mathcal{A} \setminus S} \prod_{A \in w} L(A) \cdot \prod_{A \in \mathcal{A} \setminus \{S \cup \{w\}\}} (1 - L(A))
\]
where we used the fact that $\prod_{i \in I} L(A_i) \prod_{A \in \mathcal{A} \setminus \{S \cup \{w\}\}} (1 - L(A)) = 1$ as we explained in the proof of Proposition 3.1 (the products correspond to probabilities of a probability function over $\mathcal{A} \setminus S$).

$\prod_{i \in I} L(A_i)^{b_i} \cdot (1 - L(A_i))^{1-b_i}$ can be computed in linear time when we know $L^*$. We can compute $L^*$ in polynomial time as explained in Proposition 4.3. Hence, we can compute $P^*(Q)$ in polynomial time.

\(\square\)

However, even under the principle of maximum entropy, queries cannot become arbitrarily complex. In this case, 3CNF-queries are even sufficient to solve #3SAT.

**Proposition 4.5.** The following problem is #P-hard: Given a satisfiable set of linear atomic constraints $C$ over $\mathcal{A}$ and a 3CNF-query $Q$, compute $P^*(Q)$, where $P^*$ is the probability function that maximizes entropy among all probability functions that satisfy $C$.

**Proof.** We give a polynomial-time reduction from #3SAT. Given a propositional 3CNF formula $F$, we construct a corresponding argument query $Q$ as in the proof of Proposition 4.1. We let $C = \emptyset$ so that $P^*$ is just the uniform distribution with $P^*(w) = \frac{1}{2^n}$ for all $w \in 2^\mathcal{A}$. Then $P^*(Q) = \sum_{w \in 2^\mathcal{A}, w \models Q} P^*(w) = \frac{|\{w \in 2^\mathcal{A} | w \models Q\}|}{2^n}$ and $P^*(Q)$ is the number of possible worlds that satisfy $Q$, which equals the number of propositional interpretations that satisfy $F$. \(\square\)

Similar to the proof of Proposition 4.1, it can be seen that the corresponding decision problem that asks whether the query has a non-zero probability, is NP-complete. While queries can be difficult to compute in general, there are still some special cases that can be solved efficiently. For example, consider the query $A \lor B$ that asks for the probability that $A$ or $B$ (or both) are accepted. Then the query is equivalent to $(A \land B) \lor (A \land \lnot B) \lor (\lnot A \land B)$. Since the three conjunctions are exclusive (they cannot be satisfied by the same worlds), we have $P(A \lor B) = P(A \land B) + P(A \land \lnot B) + P(\lnot A \land B)$. Hence, we can answer the disjunctive query by three conjunctive queries that can be computed in polynomial time. More generally, if we can rewrite a query efficiently as a disjunction of $k$ exclusive conjunctions, the query can be answered by $k$ conjunctive queries. However, in general, $k$ can grow exponentially with the number of atoms in the query.

## 5 Complex Constraints

In this section, we look at how far we can extend the expressiveness of our constraint language. Unfortunately, there are strong limitations. As soon as we only allow constraining the probability of the disjunction of two literals, the satisfiability problem becomes intractable. We define a linear 2DN constraint as an expression of the form $\sum_{i=1}^n c_i \cdot \pi(A_{i,1}^{b_{i,1}} \lor A_{i,2}^{b_{i,2}}) \leq c_0$, where $A_{i,1}, A_{i,2} \in \mathcal{A}$, $b_{i,1}, b_{i,2} \in \{0, 1\}$ and $c_i \in \mathbb{R}$. We say that a probability function $P$ satisfies such a constraint iff $\sum_{i=1}^n c_i \cdot P(A_{i,1}^{b_{i,1}} \lor A_{i,2}^{b_{i,2}}) \leq c_0$.

**Proposition 5.1.** The satisfiability problem for Linear 2DN Constraints is NP-complete.

**Proof.** For membership, we can construct a polynomial certificate like the one in Proposition 4.1.
We say that a probability function $P$ is a $2PSAT$, the problem of deciding whether a set of propositional statements of the form $\pi_i = a_i \cdot \pi_i$ is satisfi-able. As shown in [24], $2PSAT$ is NP-complete. We can introduce an argument for every propositional atom as in the previous proofs and represent every statement $\pi_i$ as $P(a_i \cdot \pi_i) = p$ with a linear $2D$ constraint $P(A_{1,1} \cdot \pi_1 + A_{2,1} \cdot \pi_2) = p$. Then, clearly, the set of linear constraints can be satisfied if and only if the $2PSAT$ instance can be satisfied. □

The problem does not get significantly easier when considering conjunction instead of disjunction. We define a linear $2CN$ con-straint as an expression of the form $\sum_{i=1}^{n} c_i \cdot P(A_{1,1} \cdot \pi_1 + A_{2,1} \cdot \pi_2) \leq c_0$ and say that $P$ satisfies such a constraint iff $\sum_{i=1}^{n} c_i \cdot P(A_{1,1} \cdot \pi_1 + A_{2,1} \cdot \pi_2) \leq c_0$.

PROPOSITION 5.2. The satisfiability problem for Linear $2CN$ Constraints is NP-complete.

Proof. Membership follows as in the previous proposition. For hardness, we can give a polynomial-time reduction from satisfiability of linear $2DN$ constraints that we considered before. Consider an arbitrary linear $2DN$ constraint $\sum_{i=1}^{n} c_i \cdot P(A_{1,1} \cdot \pi_1 + A_{2,1} \cdot \pi_2) \leq c_0$. Notice that every formula $A_{1,1} \cdot \pi_1 + A_{2,1} \cdot \pi_2$ can be equivalently ex-pressed as a disjunction of three exclusive conjunctions of length 2. For example $A \land \neg B \equiv (A \land \neg B) \lor (\neg A \land B)$. Since the conjunctions cannot be satisfied simultaneously, we have $P(A \land \neg B) = P(A \land \neg B) + P(\neg A \land B) + P(A \land B)$. In general, every linear $2DN$ constraint $\sum_{i=1}^{n} c_i \cdot P(A_{1,1} \cdot \pi_1 + A_{2,1} \cdot \pi_2) \leq c_0$ can be equivalently represented by a linear $2CN$ constraint $\sum_{i=1}^{n} c_i \cdot (\sum_{i=1}^{3} \pi_i) \leq c_0$ where the $C_i$ are exclusive conjunctions of two literals chosen as to satisfy $(A_{1,1} \cdot \pi_1 + A_{2,1} \cdot \pi_2) \equiv \lor_i c_i$. The number of constraints remains unchanged and their size changes only by a constant factor. In particular, a set of linear $2DN$ con-straints is satisfiable if and only if the corresponding set of linear $2CN$ constraints is satisfiable. □

So talking about the probability of formulas in constraints is diffi-cult. However, instead of allowing logical connectives in probability statements, we could consider logical connections of constraints as considered in [29]. Note that connecting constraints conjunctively does not add anything semantically. This is because there is no difference between adding two constraints or their conjunction to a knowledge base when the usual interpretation of conjunction is used. Adding negation basically means allowing for strict inequalities. Negation alone does not any additional difficulties and the problem can be reduced to the case without negation with constant cost [28]. The most interesting case is allowing for connecting constraints disjunctively. We define a $2D linear atomic constraint$ as an expression of the form $l_1 \lor l_2$, where $l_1, l_2$ are linear atomic atoms. We say that a probability function $P$ satisfies such a constraint iff it satisfies $l_1$ or $l_2$. Unfortunately, the satisfiability problem for $2D$ linear atomic constraints is intractable.

PROPOSITION 5.3. The satisfiability problem for $2D$ Linear Atomic Constraints is NP-complete. In particular, the problem remains NP-complete even when the linear atomic constraints are restricted to
the form $\sum_{i=1}^{k} c_i \cdot P(A_j) \leq c_0$ with $k \leq 2$, that is, even when they can contain at most 2 probability terms.

Proof. Membership follows again from noticing that a labelling that satisfies the constraints is a certificate that can be checked in polynomial time.

For hardness, we give a polynomial-time reduction from $3SAT$ to satisfiability of $2D Linear Atomic Constraints$. As before, for every propositional atom, we introduce a corresponding argument. Consider a clause $(a_1 \lor a_2 \lor a_3)$. We introduce three additional auxiliary arguments $X_1, X_2, X_3$ and encode the clause by four $2D$ linear atomic constraints. We use the constraints $(\pi(A_1) = b_1) \lor (\pi(X_1) = 1)$ for $i = 1, \ldots, 3$ and $(\pi(X_1) + \pi(X_2) \leq 1) \lor (\pi(X_2) + \pi(X_3) \leq 1)$. Notice that $\pi(A) = 1$ is equivalent to $\neg \pi(A) \leq -1$ and $\pi(A) = 0$ is equivalent to $\pi(A) \leq 0$. The constraint $\pi(X_1) = 1$ must be satisfied if the i-th literal is not satisfied. The last constraint expresses that at most one literal is allowed not to be satisfied. If all three literals are falsified, the last constraint is not satisfied. If the first or second literal are satisfied, the first atom in the disjunction will be satisfied, if the third literal is satisfied, the second atom will be satisfied. Our reduction introduces $3m$ new arguments and $3m$ additional constraints, so the size is polynomial.

If $F$ is satisfiable, then there is a possible world $w$ (interpretation) that satisfies $F$. Consider the probability function $P_w$ that assigns probability 1 to $w$ and 0 to all other worlds. Let $L$ denote the probability labelling corresponding to $P_w$. We extend $L$ to a probability labelling $L'$ over the $n + 3m$ arguments. For every clause $(A_{1} \lor A_{2} \lor A_{3})$, $w$ satisfies one literal $A_{1}$ and we set the corre-sponding auxiliary argument $X_{1}$ to 0 and the other two to 1. Then the $2D$ linear atomic constraints are satisfied by $L'$ and the corre-sponding probability function $P_{L'}$ satisfies the constraint as well as shown before. Hence, the $2D$ linear atomic constraints are satisfiable.

Conversely, if all $2D$ linear atomic constraints are satisfied by a probability function $P$, then every world $w$ with $P(w) > 0$ must satisfy $F$ (strictly speaking, $w$ also interprets the auxiliary arguments, but those can be ignored). For the sake of contradiction, assume that this is not the case. That is, there is a world $w$ with $P(w) > 0$ that does not satisfy $F$. Then there is a clause $(A_{1} \lor A_{2} \lor A_{3})$ in $F$ such that $w$ satisfies neither $A_{1}$ nor $A_{2}$ nor $A_{3}$. If $b_{1} = 1$, then $P(A_1) = \sum_{v \in \mathcal{A}, A_{1} \in v} P(v) \leq \sum_{v \in \mathcal{A}, w \in \mathcal{V}} P(v) = 1 - P(w) < 1$ and if $b_{1} = 0$, then $P(A_1) > P(w) > 0$. Then the constraints $(\pi(A_1) = b_1) \lor (\pi(X_1) = 1)$ can only be satisfied if $P(X_1) = 1$ for $i = 1, \ldots, 3$. But then $P(X_1) + P(X_2) = P(X_2) + P(X_3) = 2$ and the constraint $(\pi(X_1) + \pi(X_2) \leq 1) \lor (\pi(X_2) + \pi(X_3) \leq 1)$ is violated, which contradicts our assumption that $P$ satisfies the constraints. Hence, indeed every world $w$ with $P(w) > 0$ must satisfy $F$ and since there must be at least one world with non-zero probability (otherwise, $P$ cannot be a probability function), $F$ is satisfiable. □

However, it may still be possible to extend our fragment to statements of the form $(\pi(A_1) \leq c_1) \lor (\pi(A_2) \leq c_2)$, where every linear atomic constraint can only contain a single probability term. This would allow making conditional statements like in the rationality property [30]:

RAT: $P$ is called rational if for all $A, B \in \mathcal{A}$ with $(A, B) \in \mathcal{R}$, we have $P(A) > 0.5$ implies $P(B) \leq 0.5$. 

We may reuse ideas for 2SAT in order to handle such constraints efficiently. However, we currently cannot say for certain if this is possible in polynomial time and leave this question for future work.

6 RELATED WORK

As mentioned in the introduction, there is a large variety of other probabilistic argumentation frameworks [17, 19, 27, 36, 38, 44, 53–55, 57, 58]. We sketch three early works here to give an impression of some ideas. [19] consider probability functions over possible worlds as well, but the mechanics are very different from what we saw here. Instead of considering all possible probability functions that satisfy particular constraints, a single probability function is derived from a set of probabilistic rules. Roughly speaking, these rules express the likelihood of assumptions under given preconditions. Multiple rules for one assumption are only allowed if they can be ordered by specificity. [38] consider functions that assign probabilities to arguments (like probability labellings) and attack relations. The functions are supposed to be given and allow assigning a probability to subgraphs of the given argumentation framework using common independency assumptions. Then the probability of an argument is defined by taking the probability of every subgraph and adding those probabilities for which the argument is accepted in the subgraph under a particular semantics. Since the number of subgraphs is exponential, the authors present a Monte-Carlo algorithm to approximate the probability of an argument. In [53], probabilities are again introduced over possible worlds. Again, a single probability distribution is derived from rules. However, in contrast to [19], these rules are probabilistic extensions of a light form of ASPIC rules [10, 51]. They are also more flexible in that they do not need to be ordered according to specificity.

[55] recently introduced a very general probabilistic argumentation framework that generalizes many ideas that have been considered before in the literature. The authors consider probability functions over subsets of defeasible theories or over subgraphs. The latter approach can then be seen as a generalization of the former, which abstracts from the structure of arguments. The authors discuss probabilistic labellings that should not be confused with probability labellings that we considered here. Roughly speaking, in [55], a probabilistic labelling frame corresponds to a probability function over subsets of possible classical labellings over an argumentation framework. These probabilistic labelling frames can then be used to assign probabilities to arguments. In this sense, a probabilistic labelling considered in [55] induces a probability labelling as considered here. However, the focus in [55] is on conceptual questions and computational problems are not discussed.

Our polynomial-time algorithms are based on a connection between probability functions and probability labellings. The relationship is established by considering an equivalence relation over probability functions. Conceptual similar ideas have been considered in probabilistic-logical reasoning. However, in this area, equivalence relations are introduced over possible worlds. Roughly speaking, the possible worlds are partitioned into equivalence classes that interpret the formulas that appear in the knowledge base in the same manner. Reasoning algorithms can then be modified to work on probability functions over equivalence classes [22, 23, 35, 47]. If the number of equivalence classes is small, a significant speedup can be obtained. However, identifying compact representatives for these equivalence classes is intractable in general [49]. In particular, in general, the number of equivalence classes over possible worlds can still be exponential. Indeed, many polynomial cases that we found here cannot be solved in polynomial time with this approach. For example, if one atomic constraint \( P(A) = P_B \) is given for every argument \( A \), every equivalence class of possible worlds will contain exactly one possible world, so that actually nothing is gained.

[31] also considered an inconsistency-tolerant generalization of the entailment problem that still works when there are conflicts between the partial probability assignment constraints and the semantic constraints. We can probably derive similar polynomial runtime guarantees for this problem. However, the approach in [31] is based on the assumption that the semantical constraints are consistent. This is no problem for the semantical constraints considered in [31] because the probability of attacked arguments is only bounded from above and the probability of non-attacked arguments is only bounded from below. However, in bipolar argumentation frameworks, we want to consider more complicated relationships and the constraints can easily become inconsistent. Therefore, it is interesting to also analyze other variants that use ideas for parallel consistent probabilistic reasoning [16, 50] or reasoning with priorities [46]. It is also interesting to note that our satisfiability test from Proposition 3.3 actually corresponds to an inconsistency measure. If the knowledge base is inconsistent, the returned value will be 0, otherwise it measures by how much probability functions must violate the constraints numerically [45].

7 DISCUSSION AND FUTURE WORK

We showed that the satisfiability and entailment problem for the epistemic probabilistic argumentation approach considered in [31] can be solved in polynomial time. In fact, arbitrary linear atomic constraints can be considered. For the query language, we found that conjunctive queries can still be answered in polynomial time under the principle of maximum entropy. General disjunctive constraints are intractable, but if they can be expressed as a compact disjunction of exclusive conjunctions, the query can be reduced to a sequence of conjunctive queries. We found that the constraint language cannot be extended significantly. However, it may still be possible to allow disjunctions of two probability statements, which would allow expressing conditional constraints like RAT. Another interesting question for future work is whether we can compute conjunctive queries for the entailment problem in polynomial time even without using the principle of maximum entropy.

We focussed mainly on complexity results and did not speak much about the runtime guarantees of our convex programming formulations. In general, interior-point methods can solve convex programs in cubic time in the number of optimization variables and optimization constraints [9]. This means that all convex programs that we introduced here can be solved in cubic time in the size of the argumentation problem in the worst-case. Our linear programs for satisfiability and entailment can often be solved faster by using the Simplex algorithm. Even though the Simplex algorithm has exponential worst-case runtime, in practice, the runtime usually depends only linearly on the number of optimization variables and quadratically on the number of constraints [39]. Implementations
for satisfiability and entailment can be found in the Java-library ProBabble. You have to install IBM CPLEX in order to use ProBabble, but IBM offers free licenses for academic purposes. Problems with thousands of arguments can usually be solved within a few hundred milliseconds. Without the labelling approach, the same amount of time would be needed for 10-15 arguments already because the number of possible worlds grows exponentially.

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