General \((\alpha, \beta)\) metrics with relatively isotropic mean Landsberg curvature

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Abstract

In this paper, we study a new class of Finsler metrics, \(F = \alpha \phi(b^2, s), \quad s := \frac{\beta}{\alpha}\), defined by a Riemannian metric \(\alpha\) and 1-form \(\beta\). It is called general \((\alpha, \beta)\) metric. In this paper, we assume \(\phi\) be coefficient by \(s\) and \(\beta\) be closed and conformal. We find a necessary and sufficient condition for the metric of relatively isotropic mean Landsberg curvature to be Berwald.

Keywords: Finsler geometry, Relatively isotropic mean Landsberg curvature , General \((\alpha, \beta)\)-metrics.

1 Introduction

The \((\alpha, \beta)\) metrics were first introduced by Matsumoto [2]. They are Finsler metrics built from a Riemannian metric \(\alpha = \sqrt{a_{ij}y^iy^j}\) and 1-form \(\beta = b_i(x)y^i\) and a \(C^\infty\) function \(\phi(s)\) on a manifold \(M\). A Finsler metric of \((\alpha, \beta)\) metrics is given by the form

\[ F := \alpha \phi(s), \quad s := \frac{\beta}{\alpha} \]

It is known that \(F\) is positive and strongly convex on \(TM \setminus \{0\}\) if and only if

\[ \phi(s) > 0, \quad \phi(s) - s \phi'(s) + (b^2 - s^2) \phi''(s) > 0, \quad |s| \leq b < b_0, \]

where \(b = \|\beta\|_\alpha\).

The aim of this paper is to study a new class of Finsler metrics given by

\[ F := \alpha \phi(b^2, s), \quad s := \frac{\beta}{\alpha} \quad (1) \]

where \(\phi = \phi(b^2, s)\) is a \(C^\infty\) positive function and \(b = \|\beta\|_\alpha\) is its norm[8]. It is called general \((\alpha, \beta)\) metrics. This kind of metrics is first discussed by Yu and Zhu [3]. Many well-known Finsler metrics are general \((\alpha, \beta)\) metrics.

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Example 1. \[8\] The Randers metrics and the square metrics are defined by functions \( \phi = \phi(b^2, s) \) in the following form:

\[
\phi = \frac{\sqrt{1 - b^2 + s^2} + s}{1 - b^2}
\]

(2)

\[
\phi = \frac{\left(\sqrt{1 - b^2 + s^2} + s\right)^2}{(1 - b^2)^2 \sqrt{1 - b^2 + s^2}}
\]

(3)

Example 2. \[9\] One Important example of \((\alpha, \beta)\) metric was given by L. Berwald:

\[
F = \frac{\left(\sqrt{(1 - |x|^2)|y|^2 + \langle x, y \rangle}^2\right)^2}{(1 - |x|^2)^2 \sqrt{(1 - |x|^2)|y|^2 + \langle x, y \rangle}^2}.
\]

(4)

It is a projectively flat Finsler metrics on \( B \subset \mathbb{R}^n \) with flag curvature \( K = 0 \). Berwald’s metric can be expressed in form

\[
F = \alpha \phi(b^2, s) = \alpha \left(\sqrt{1 + b^2} + s\right)^2
\]

(5)

where

\[
\alpha = \frac{\sqrt{(1 - |x|^2)|y|^2 + \langle x, y \rangle}^2}{1 - |x|^2}, \quad \beta = \frac{\langle x, y \rangle}{(1 - |x|^2)^{3/2}}
\]

(6)

\[
s = \frac{\beta}{\alpha}, \quad b^2 = \frac{|x|^2}{1 - |x|^2}
\]

(7)

Example 3. \[8\] There is a special class of general \((\alpha, \beta)\) metrics called spherically symmetric metrics, which are defined on an open subset of \( \mathbb{R}^n \) with \( \alpha = |y| \) and \( \beta = \langle x, y \rangle \),

\[
F = |y| \phi \left( |x|^2, \frac{\langle x, y \rangle}{|y|} \right).
\]

In Finsler geometry, there are several very important non-Riemannian quantities. The Cartan torsion \( C \) is a primary quantity. There is another quantity which is determined by the Busemann-Hausdorff volume form, that is the so-called distortion \( \tau \). The vertical differential of \( \tau \) on each tangent space gives rise to mean Cartan torsion \( I := \tau_{y_k} dx^k \). \( C \), \( \tau \) and \( I \) are the basic geometric quantities which characterize Riemannian metrics among Finsler metrics. Differentiating \( C \) along geodesics gives rise to the Landsberg curvature \( L \). The horizontal derivative of \( \tau \) along geodesics is the so-called \( S \)-curvature \( S := \tau_{y_k} y^k \). The horizontal derivative of \( I \) along geodesics is called the mean Landsberg curvature \( J := I_{y_k} y^k \).

By the definition, \( J/I \) can be regarded as the relative growth rate of the mean Cartan torsion along geodesics. We call a Finsler metric \( F \) is of relatively isotropic mean Landsberg curvature if \( F \) satisfies \( J + \tilde{c} F I = 0 \), where \( \tilde{c} = \tilde{c}(x) \) is a scalar function on the Finsler manifold. In particular, when \( \tilde{c} = 0 \), Finsler metrics with \( J = 0 \) are called weak Landsberg metrics \[4\].
We study general \((\alpha, \beta)\) metrics with relatively isotropic mean Landsberg curvature, where \(\beta\) is a closed and conformal 1-form, i.e.

\[ b_{ij} = c(x)a_{ij}, \quad (8) \]

where \(b_{ij}\) is the covariant derivation of \(\beta\) with respect to \(\alpha\) and \(c = c(x) \neq 0\) is a scalar function on \(M\). In [7], Zohrehvand and Maleki proved that, every Landsberg general \((\alpha, \beta)\) metric is a Berwald metric with condition (8). In [1], the authors showed that this result for the metric of mean Landsberg curvature.

In this paper, we prove the following

**Theorem 1.** Let \(F = \alpha\phi(b^2, s), s := \beta/\alpha\), be a non-Riemannian general \((\alpha, \beta)\) metric on an \(n\)-dimensional manifold \(M\). Suppose that \(\beta\) satisfies (8). If \(\phi = \phi(b^2, s)\) is a polynomial in \(b^2\) and \(s\), then \(F\) is of relatively isotropic mean Landsberg curvature, \(\mathbf{J} + \tilde{c}(x)F\mathbf{I} = 0\), if and only if it is a Berwald metric. In this case,

\[ \phi(b^2, s) = c_0(b^2) + c_1(b^2)s + c_2(b^2)s^2 + \cdots + c_m(b^2)s^m, \quad (9) \]

where

\[ c_0(b^2) = \frac{a_0}{\sqrt{b^2}}, \quad c_1(b^2) = \frac{a_1}{b^2}, \quad c_2(b^2) = \frac{a_2}{(b^2)^{\frac{3}{2}}}, \quad \ldots, \quad c_m(b^2) = \frac{a_m}{(b^2)^{\frac{m+1}{2}}} \]

and \(a_i, 1 \leq i \leq n\) are constants.

Because every analytic function can be approximated by a series polynomials, we can assume that \(\phi\) is a polynomial in \(b^2\) and \(s\).

## 2 Preliminary

Let \(F = F(x, y)\) be a Finsler metric on an \(n\)-dimensional manifold \(M\). Let

\[ g_{ij} := \frac{1}{2}[\mathbf{F}^2]_{y^iy^j}(x, y) \]

and \((g^{ij}) := (g_{ij})^{-1}\). For a non-zero vector \(y = y^i \frac{\partial}{\partial x^i} |_{x \in T_xM}\), \(F\) induces an inner product on \(T_xM\)

\[ g_y(u, v) = g_{ij}u^iv^j, \]

where \(u = u^i \frac{\partial}{\partial x^i}, v = v^j \frac{\partial}{\partial x^j} \in T_xM\). \(g = \{g_y\}\) is called the fundamental tensor of \(F\).

Let

\[ C_{ijk} := \frac{1}{4} [\mathbf{F}^2]_{y^iy^jy^k} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}. \]

Define symmetric trilinear form \(\mathbf{C} := C_{ijk}(x, y)dx^i \otimes dx^j \otimes dx^k\) on \(TM \setminus \{0\}\). We call \(\mathbf{C}\) the **Cartan torsion**. The **mean Cartan torsion** \(\mathbf{I} = I_idx^i\) is defined by

\[ I_i := g^{jk}C_{ijk}. \]
Further, we have \((4), (5)\)

\[ I_i = g^{jk} C_{ijk} = \frac{\partial}{\partial y^j} \left[ \ln \sqrt{\det(g_{jk})} \right] \]  

(10)

For a Finsler metric \(F\), the geodesics are characterized locally by a system of 2nd ODEs:

\[ \frac{d^2 x^i}{dt^2} + 2G^i \left( x, \frac{dx}{dt} \right) = 0, \]  

where

\[ G^i = \frac{1}{4} g^{il} \left\{ [F^2]_{x^m y^i y^m} - [F^2]_{x^i} \right\}. \]  

(11)

\(G^i\) are called the geometric coefficients of \(F\).

For a tangent vector \(y = y^i \partial/\partial x^i \in T_x M\), the Berwald curvature \(B := B^i_{jkl} dx^j \otimes \partial/\partial x^i \otimes dx^k \otimes dx^l\) can be expressed by

\[ B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}. \]  

(12)

\(F\) is a Berwald metric if \(B = 0\). The Landsberg curvature \(L = L_{ijk}(x, y) dx^i \otimes dx^j \otimes dx^k\) is a horizontal tensor on \(TM \setminus \{0\}\) defined by \([6]\)

\[ L_{ijk} := -\frac{1}{2} FF_{y^m} [G^m]_{y^i y^j y^k}. \]  

(13)

\(F\) is called a Landsberg metric if \(L = 0\). The mean Landsberg curvature \(J = J_i dx^i\) is defined by

\[ J_i := g^{jk} L_{ijk}. \]  

(14)

We call \(F\) a weak Landsberg metric if \(J = 0\). We say that \(F\) is of relatively isotropic mean Landsberg curvature if \(J_i + \tilde{c}(x) FI_i = 0\) for a scalar function \(\tilde{c} = \tilde{c}(x)\) on \(M\).

Now we consider a general \((\alpha, \beta)\) metric:

**Definition 1.** Let \(F\) be a Finsler metric on an \(n\)-dimensional manifold \(M\). \(F\) is called a general \((\alpha, \beta)\) metric if it can be expressed as the form \([1]\) where \(||\beta||_\alpha \leq b_0\) and \(\phi = \phi(b^2, s)\) is a positive \(C^\infty\) function.

**Proposition 1.** Let \(M\) be an \(n\)-dimensional manifold. A function \(F = \alpha \phi(b^2, s)\) on \(TM\) is a Finsler metric on \(M\) for any Riemannian metric \(\alpha\) and 1-form \(\beta\) with \(||\beta||_\alpha < b_0\) if and only if \(\phi = \phi(b^2, s)\) is a positive \(C^\infty\) function satisfying

\[ \phi > 0, \quad \phi - s \phi_2 + (b^2 - s^2) \phi_{22} > 0, \]  

(15)

where \(s\) and \(b\) are arbitrary numbers with \(||s||_s \leq b < b_0||\).
Proof. It is easy to verify $F$ is a function with regularity and positive homogeneity. In the following we will verify strong convexity: The $n \times n$ Hessian matrix
\[
(g_{ij}) := \left( \frac{1}{2} [F^2]_{y'_{y'}} \right).
\]
For the general $(\alpha, \beta)$ metric $F = \alpha \phi(b^2, \frac{b^2}{\beta})$, direct computations yield
\[
[F^2]_{y'_{y'}} = [\alpha^2]_{y'} \phi^2 + 2 \alpha^2 \phi \phi_2 s_i^i
\]
(16)
\[
[F^2]_{y'_{y'}} = [\alpha^2]_{y'} \phi^2 + 2 [\alpha^2]_{y'} \phi \phi_2 s_i^i + 2 [\alpha^2]_{y'} \phi \phi_2 s_i^j s_j^i + 2 \alpha^2 [\phi^2]_{y'_{y'}}
\]
(17)
Direct computations yield
\[
g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_1 (b_i \alpha_{y^i} + b_j \alpha_{y^j}) - s \rho_1 \alpha_{y^i} \alpha_{y^j},
\]
(18)
where
\[
\rho = \phi(\phi - s \phi_2), \quad \rho_0 = \phi \phi_2 + (\phi_2)^2,
\]
(19)
By Lemma 1.1.1 in [4], we find a formula for $\det(g_{ij})$
\[
\det(g_{ij}) = \phi^{n+1}(\phi - s \phi_2)^{n-2}(\phi - s \phi_2 + (b^2 - s^2) \phi_2) \det(a_{ij}).
\]
(20)
Assume that (15) is satisfied. Then by taking $b = s$ in (15), we see that the following inequality holds for any $s$
\[
\phi - s \phi_2 > 0, \quad |s| < b_0.
\]
(21)
Using (16), (20) and (21), we get $\det(g_{ij}) > 0$, namely $(g_{ij})$ is positive-definite. The converse is obvious, so the proof is omitted here.

Remark. Note that $\phi_1$ means the derivation of $\phi$ with respect to the first variable $b^2$. In this paper, $\beta$ is closed and conformal 1-form, i.e. $b_{ij} = c(x)a_{ij}$. Let [3]
\[
r_{ij} := \frac{1}{2}(b_{ij} + b_{ji}), \quad r_{00} := r_{ij} y^j y^j, \quad r_i := b^i r_{ji}, \quad r_0 := r_{ij} y^i, \quad r^i := a_{ij} r_j, \quad r := b^i r_i,
\]
\[
s_{ij} := \frac{1}{2}(b_{ij} - b_{ji}), \quad s_0 := a^{ij} s_{kj} y^j, \quad s_i := b^i s_{ji}, \quad s_0 := s_i y^i, \quad s^i := a_{ij} s_j
\]
For a general \((\alpha, \beta)\) metric, its spray coefficients \(G^i\) are related to the spray coefficients \(G^i_\alpha\) of \(\alpha\) by [13]

\[
G^i = G^i_\alpha + \alpha Q s^i_0 + \{\Theta(-2\alpha Q s_0 + r_0 + 2\alpha^2 R r) + \alpha \Omega(r_0 + s_0)\} l^i
+ \{\Psi(-2\alpha Q s_0 + r_0 + 2\alpha^2 R r) + \alpha \Pi(r_0 + s_0)\} b^i - \alpha^2 R (r^i + s^i),
\]

where \(l^i := \frac{y^i}{\alpha}\) and

\[
\begin{align*}
Q &:= \frac{\phi_2}{\phi - s \phi_2}, & R &:= \frac{\phi_1}{\phi - s \phi_2}, \\
\Theta &:= \frac{(\phi - s \phi_2) \phi_2 - s \phi \phi_{22}}{2(\phi - s \phi_2 + (b^2 - s^2) \phi_{22})}, & \Psi &:= \frac{\phi_{22}}{2(\phi - s \phi_2 + (b^2 - s^2) \phi_{22})}, \\
\Pi &:= \frac{(\phi - s \phi_2) \phi_{12} - s \phi_{1} \phi_{22}}{(\phi - s \phi_2)(\phi - s \phi_2 + (b^2 - s^2) \phi_{22})}, & \Omega &:= \frac{2 \phi_1}{\phi} - \frac{s \phi + (b^2 - s^2) \phi_2}{\phi} \Pi
\end{align*}
\]

When \(\beta\) is closed and conformal one-form, i.e. satisfies [8], then

\[
r_0 = \alpha^2, \quad r_0 = c_\beta, \quad r = cb^2, \quad r^i = cb^i, \quad s_0 = s = i = 0.
\]

Substituting this into [24] yields [9]

\[
G^i := G^i_\alpha + \alpha^2 \{\Theta(1 + 2RB^2) + s \Omega\} l^i + \alpha^2 \{\Psi(1 + 2RB^2) + s \Pi - R\} b^i
\]

If we have

\[
\begin{align*}
E &:= \frac{\phi_2 + 2s \phi_1}{2 \phi} - H \frac{s \phi + (b^2 - s^2) \phi_2}{\phi}, \\
H &:= \frac{\phi_{22} - 2(\phi_1 - s \phi_{12})}{2(\phi - s \phi_2 + (b^2 - s^2) \phi_{22})},
\end{align*}
\]

then from [25]

\[
G^i := G^i_\alpha + \alpha^2 E l^i + \alpha^2 H b^i.
\]

**Proposition 2.** [11] Let \(F = \alpha \phi(b^2, s), \quad s = \beta/\alpha, \) be a general \((\alpha, \beta)\)-metric on an \(n\)-dimensional manifold \(M\). Suppose that \(\beta\) satisfies [8], then the weak Landsberg curvature of \(F\) is given by

\[
J_j = -\frac{c \phi}{2 \rho} W_j,
\]

where

\[
W_j := \{(E - sE_2)(n + 1) \phi_2 + 3E_{22} \phi_2 (b^2 - s^2) - sE_{22}(n + 1) \phi + E_{222} \phi (b^2 - s^2)
+ \{(H_2 - sH_{22})(n + 1) + H_{222}(b^2 - s^2)\} (s \phi + (b^2 - s^2) \phi_2)
+ 3s(E - sE_2) \phi_2 (b^2 - s^2) + 3E_{222} \phi_2 (b^2 - s^2)^2 - 3sE_{22} \phi (b^2 - s^2)
+ \eta E_{222} (b^2 - s^2)^2 \phi + \eta [3(H_2 - sH_{22})(b^2 - s^2) + H_{222}(b^2 - s^2)^2]
\times (s \phi + (b^2 - s^2) \phi_2)\} (b_j - s l_j)
\]

where \(\rho\) and \(\eta\) is defined in [14] and [23] and \(l_j := a_{ij} l^i\).
Proposition 3. Let $F = \alpha\phi(b^2, s)$, $s = \beta/\alpha$, be a general $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M$. Suppose that $\beta$ satisfies (8), then $F$ is weak Landsberg metric if and only if the following equations hold:

$$E_{22} = 0, \quad H_{222} = 0,$$

(31)

$$(E - sE_2)\phi_2 + (H_2 - sH_{22})(s\phi + (b^2 - s^2)\phi_2 = 0$$

(32)

Theorem 2. Let $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ be a non-Riemannian general $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M$ and $\beta$ satisfies (8). Then $F$ is a weak Landsberg metric if and only if it is Landsberg metric.

3 Proof of Theorem

In this section, we prove Theorem. From (10) and (18), we have

$$I_j = \frac{\partial}{\partial y^j} \left[ \ln \sqrt{\det (g_{kl})} \right]$$

$$= \frac{1}{2\alpha} \left\{ (n + 1) \phi_2 - (n - 2) \frac{s\phi_{22}}{\phi - s\phi_2} + \frac{(b^2 - s^2)\phi_{222} - 3s\phi_{22}}{\phi - s\phi_2 + (b^2 - s^2)\phi_{22}} \right\} (b_j - s l_j)$$

$$= \frac{1}{2\alpha} \left\{ \frac{(b^2 - s^2)(\phi - s\phi_2)\phi_{222} + (n + 1)(\phi - s\phi_2)^2\phi}{\phi - s\phi_2 + (b^2 - s^2)\phi_{22}} ight.$$

$$- (n - 2)(b^2 - s^2)s\phi_{222}\eta + (n + 1)(\phi - s\phi_2)[(b^2 - s^2)\phi_2 - s\phi]\eta \right\} (b_j - s l_j).$$

(33)

We must mention the following lemmas firstly.

Lemma 1. Let $F = \alpha\phi(b^2, s)$, $s = \beta/\alpha$, be a general $(\alpha, \beta)$-metric on an $n$-dimensional manifold $M$ and $\beta$ satisfies (8). Then $F$ is of relatively isotropic mean Landsberg curvature if and only if $\phi$ satisfies the following ODE:

$$\phi_{2\rho} (cW_j + \tilde{c}V_j) = 0,$$

(34)

where $W_j$ is defined in (30), and

$$V_j := \frac{(b^2 - s^2)(\phi - s\phi_2)\phi_{222} + (n + 1)(\phi - s\phi_2)^2\phi}{\phi - s\phi_2 + (b^2 - s^2)\phi_{22}}$$

$$- (n - 2)(b^2 - s^2)s\phi_{222}\eta + (n + 1)(\phi - s\phi_2)[(b^2 - s^2)\phi_2 - s\phi]\eta$$
**Proof.** By Proposition 2,

\[ J_j + \tilde{c} F I_j = \frac{c_0}{2 \rho} \left\{ [1 + n + 3(b^2 - s^2)\eta][(E - sE_2)\phi_2 + (H_2 - sH_{22})(s\phi + (b^2 - s^2)\phi_2)] \right. \\
+ (b^2 - s^2)[1 + (b^2 - s^2)\eta][s\phi + (b^2 - s^2)\phi_2]H_{222} \\
+ \{3(b^2 - s^2)[1 + (b^2 - s^2)\eta]\phi_2 - [1 + n + 3(b^2 - s^2)\eta]s\phi\}E_{22} \\
+ (b^2 - s^2)[1 + (b^2 - s^2)\eta]\phi E_{222}(b_j - s l_j) \\
+ \frac{\tilde{c}}{2 \rho} \left\{ (b^2 - s^2)(\phi - s\phi_2)\phi \phi_{222} + (n + 1)(\phi - s\phi_2)^2\phi \right. \\
\left. \times s\phi\phi_{222}\eta - (n + 1)(\phi - s\phi_2) [(b^2 - s^2)\phi_2 - s\phi] \eta \right\}(b_j - s l_j). \quad (35) \]

By use of Maple program, we can immediately get the following lemma.

**Lemma 2.** Let \( NJFI \) denote the numerator of the left of (34), then (34) holds if and only if

\[ NJFI = 0. \quad (36) \]

Also, Let \( NE_{22} \), \( NH_{222} \) and \( NP \) denote the numerators of \( E_{22} \), \( H_{222} \) and Eq. (32), respectively. Then from (31) and (32), we have

\[ NE_{22} = 0, \quad NH_{222} = 0 \]
\[ NP = 0. \quad (37) \]

By assumption, \( F \) is of relatively isotropic mean Landsberg curvature. Express \( \phi(b^2, s) \) as below.

\[ \phi(b^2, s) = c_0(b^2) + c_1(b^2)s + c_2(b^2)s^2 + \cdots + c_m(b^2)s^m, \quad m \geq 1. \quad (39) \]

Plugging (39) to \( NJFI \) yields a polynomial in \( s \). Denote the order of \( NJFI \) by \( r \). Then (36) can be rewritten as follows.

\[ v_i s^i h_j = 0, \quad 0 \leq i \leq r, \quad (40) \]

where \( v_i \) (\( 0 \leq i \leq r \)) are independent of \( s \).

By using Maple program, we can get following results:

**Case 1.** \( m = 1 \): \( \phi(b^2, s) = c_0(b^2) + c_1(b^2)s \), where \( c_1(b^2) \neq 0 \). We can get \( r = 2 \) and

\[ v_2 := (n + 1) \left\{ 2 c c_1(b^2) \left[ 2 c_1(b^2)c_0'(b^2) - c_0(b^2)c_1'(b^2) \right] + \tilde{c} c_1^3(b^2) \right\}, \quad (41) \]

In this case, because \( v_2 = 0 \), so \( \tilde{c} \) must be zero.
By solving the above ODE, we have

From (42) and (43), we obtain the following ODE:

\[ 2c_0^2(b^2) \left\{ -c_0(b^2)c'_1(b^2) + 2c_1(b^2)c'_0(b^2) \right\} c_1(b^2)s^2 \\
+ 2c_1^2(b^2) \left\{ 2b^2c'_0(b^2) + c_0(b^2) \right\} s + c_0(b^2)c'_0(b^2) \left\{ 2b^2c'_0(b^2) + c_0(b^2) \right\} = 0, \tag{43} \]

By solving the above ODE, we have

\[ c_0(b^2) = \frac{a_0}{\sqrt{b^2}}, \quad c_1(b^2) = \frac{a_1}{b^2}, \tag{46} \]

**Case 2.** \( m = 2 \). \( \phi(b^2, s) = c_0(b^2) + c_1(b^2)s + c_2(b^2)s^2 \), where \( c_2(b^2) \neq 0 \). By using Maple, We can get \( r = 17 \) and

\[ v_{17} = cf_{17c} + \hat{c}f_{17c}, \tag{47} \]

where \( f_{17c} \) is independent of \( s \) and

\[ f_{17c} := 927n\hat{c}c_0^2(b^2). \tag{48} \]

In this case, because \( v_{17} = 0 \), so \( \hat{c} \) must be zero.

We can obtain the form of \( c_0(b^2), c_1(b^2) \) and \( c_2(b^2) \) too. Plugging the \( \phi \) into (37) and (38) and similar argument yields the following ODE:

\[ 2b^2c_2(b^2) + c_0(b^2) = 0, \tag{49} \]

\[ 2c_1(b^2)c'_0(b^2) - 3c_0(b^2)c'_1(b^2) = 0, \tag{50} \]

\[ c_2(b^2) \left\{ 2b^2c'_2(b^2) + 3c_2(b^2) - 3c'_0(b^2) \right\} + c_0(b^2)c'_2(b^2) = 0, \tag{51} \]

Then

\[ c_0(b^2) = \frac{a_0}{\sqrt{b^2}}, \quad c_1(b^2) = \frac{a_1}{b^2}, \quad c_2(b^2) = \frac{a_2}{(b^2)^{\frac{3}{2}}}. \tag{52} \]

It is not hard to prove by induction that given any \( m \geq 3 \) in (39), the function \( \hat{c} \) must vanish.

In sum, we have proved that, if \( F = \alpha \phi(b^2, s) \) is of relatively isotropic mean Landsberg curvature and \( \phi \) be polynomial in \( b^2 \) and \( s \), then \( F \) must be a weak Landsberg metric. Then \( F \) is a Berwald metric by Theorem 2.

In this case, if

\[ \phi(b^2, s) = c_0(b^2) + c_1(b^2)s + c_2(b^2)s^2 + \cdots + c_m(b^2)s^m, \]

then

\[ c_0(b^2) = \frac{a_0}{\sqrt{b^2}}, \quad c_1(b^2) = \frac{a_1}{b^2}, \quad c_2(b^2) = \frac{a_2}{(b^2)^{\frac{3}{2}}}, \quad \ldots, \quad c_m(b^2) = \frac{a_m}{(b^2)^{\frac{m}{2}}}, \]

and \( a_i, 1 \leq i \leq n \) are constants.
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