COMBINATORIAL IDENTITIES VIA PHI FUNCTIONS AND RELATIVELY PRIME SUBSETS

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ABSTRACT. Let $n$ be a positive integer and let $A$ be nonempty finite set of positive integers. We say that $A$ is relatively prime if $\gcd(A) = 1$ and that $A$ is relatively prime to $n$ if $\gcd(A, n) = 1$. In this work we count the number of nonempty subsets of $A$ which are relatively prime and the number of nonempty subsets of $A$ which are relatively prime to $n$. Related formulas are also obtained for the number of such subsets having some fixed cardinality. This extends previous work for the cases where $A$ is an interval or a set in arithmetic progression. Applications include:

a) An exact formula is obtained for the number of elements of $A$ which are co-prime to $n$; note that this number is $\phi(n)$ if $A = [1, n]$.

b) Algebraic characterizations are found for a nonempty finite set of positive integers to have elements which are all pairwise co-prime and consequently a formula is given for the number of nonempty subsets of $A$ whose elements are pairwise co-prime.

c) We provide combinatorial formulas involving Mertens function.

1. Introduction

Throughout let $n$ and $\alpha$ be positive integers and let $A$ be a nonempty finite set of positive integers. Let $\mu$ be the Möbius function and let $\lfloor x \rfloor$ be the floor of $x$. If $a$ and $b$ are positive integers such that $a \leq b$, then we let $[a, b] = \{a, a+1, \ldots, b\}$. Further we need the following set theoretical notation. Let $\#X = |X|$ denote the cardinality of a set $X$, let $\mathcal{P}(X)$ denote the power set of $X$, let $\mathcal{P}^*(X)$ denote the set of nonempty subsets of $X$, and let $\mathcal{P}_\alpha(X)$ denote the set of subsets of $X$ whose cardinality is $\alpha$. Multiples of integers and their cardinality are crucial in this paper.

Definition 1. For any positive integer $d$ let $V(A, d)$ be the set of multiples of $d$ in $A$ and let $v(A, d) = |V(A, d)|$.

Theorem 1. We have

$$v(A, d) = \sum_{a \in A} (\lfloor a/d \rfloor - \lfloor (a-1)/d \rfloor).$$

Proof. The result follows since

$$\lfloor a/d \rfloor - \lfloor (a-1)/d \rfloor = \begin{cases} 1, & \text{if } a \in V(A, d) \\ 0, & \text{if } a \notin V(A, d). \end{cases}$$

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We now state some of the results which we shall prove in this work.

1) While the number of elements in the set \([1, n]\) which are co-prime to \(n\) is \(\phi(n)\), to the author’s knowledge no such formula exists for the number of elements in an arbitrary nonempty finite set \(A\) of positive integers. In this paper we will show in Corollary 1 that such a number is

\[
\sum_{d|n} \mu(d) v(A, d).
\]

2) Cameron and Erdős in [4] considered for any positive real number \(x\) the sets of positive integers whose elements are \(\leq x\) and pairwise co-prime and considered the related sets of positive integers with elements \(\leq x\) and which are free of co-prime pairs. The authors gave asymptotic lower and upper bounds for the numbers of these sets. Calkin and Granville in [3] gave asymptotic formulas for such numbers and improved the result of Cameron and Erdős. In this paper we shall prove in Corollary 5 that the number of subsets of \(A\) whose elements are pairwise co-prime is:

\[
\# \left\{ B \subseteq A : B \neq \emptyset \text{ and } 2|B| - 1 = \left( 1 + 4 \sum_{d=1}^{\sup B} \mu(d) v(B, d) (v(B, d) - 1) \right)^{1/2} \right\}.
\]

3) From Corollary 4, for any nonempty subset \(B\) of \(A\) satisfying \(\alpha \leq |B|\) the identity

\[
\binom{|A|}{\alpha} = \sup_{d=1}^{A} \mu(d) \binom{v(A, d)}{\alpha},
\]

yields

\[
\binom{|B|}{\alpha} = \sup_{d=1}^{B} \mu(d) \binom{v(B, d)}{\alpha}.
\]

As a consequence, for any nonempty subset \(B\) of \(A\), if

\[
1 + 4 \sum_{d=1}^{\sup A} \mu(d) v(A, d) (v(A, d) - 1)
\]

is a square, then so is

\[
1 + 4 \sum_{d=1}^{\sup B} \mu(d) v(B, d) (v(B, d) - 1).
\]

4) In Theorem 6 we will show that for any positive integers \(1 < m \leq n\)

\[
\sum_{d=1}^{n} \mu(d) (2^{\lfloor \frac{m}{d} \rfloor} - \lfloor \frac{n-1}{d} \rfloor + \lfloor \frac{n}{d} \rfloor - \lfloor \frac{m-1}{d} \rfloor) = \begin{cases} M(n) & \text{if } (m, n) > 1, \\ 1 + M(n) & \text{if } (m, n) = 1, \end{cases}
\]

where \(M\) is Mertens function given by \(M(n) = \sum_{d=1}^{n} \mu(d)\).

5) A direct consequence of Theorem 8 is the following combinatorial identity. For simplicity of notation if \(b\) is a positive integer, then

\[bA = \{ ba : a \in A \}.
\]
Let $c$ be a composite positive integer, say $c = ab$ with $a > 1$ and $b > 1$. Then for any finite sets $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_l\}$ of positive integers with $\text{sup} A = a$ and $\text{sup} B = b$ we have

$$M(c) = \sum_{d=1}^{c} \mu(d)2^{v(bA,d)} = \sum_{d=1}^{c} \mu(d)2^{v(aB,d)}.$$ 

Our main tools are relatively prime sets and phi functions for sets of positive integers. The set $A$ is called relatively prime if $\text{gcd}(A) = 1$ and it is called relatively prime to $n$ if $\text{gcd}(A \cup \{n\}) = \text{gcd}(A,n) = 1$. From now on we assume that $\alpha \leq |A|$.

**Definition 2.**

Let $f(A) = \#\{X \subseteq A : X \neq \emptyset \text{ and } \text{gcd}(X) = 1\}$,

$f_{\alpha}(A) = \#\{X \subseteq A : \#X = \alpha \text{ and } \text{gcd}(X) = 1\}$,

$\Phi(A, n) = \#\{X \subseteq A : X \neq \emptyset \text{ and } \text{gcd}(X, n) = 1\}$,

$\Phi_{\alpha}(A, n) = \#\{X \subseteq A : \#X = \alpha \text{ and } \text{gcd}(X, n) = 1\}$.

Nathanson in [9] introduced among others the functions $f(n)$, $f_{\alpha}(n)$, $\Phi(n)$, and $\Phi_{\alpha}(n)$ (in our terminology $f([1,n])$, $f_{\alpha}([1,n])$, $\Phi([1,n], n)$, and $\Phi_{\alpha}([1,n], n)$ respectively) and found exact formulas along with asymptotic estimates for each of these functions. Formulas for these functions are found in [5, 10] for $A = [m,n]$ and in [6] for $A = [1,m]$. Ayad and Kihel in [1, 2] considered extensions to sets in arithmetic progression and obtained identities for these functions for $A = [l,m]$ as consequences. Recently in [7, 8] these functions have been studied for the union of two intervals, the special case of $A = [l,m]$ has been obtained, and various combinatorial identities involving Möbius and Mertens functions have been found as applications. For the purpose of this work we give these functions for $A = [l,m]$.

**Theorem 2.** We have

(a) $f([l,m]) = \sum_{d=1}^{m} (2^\left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{l-1}{d} \right\rfloor) - 1$,

(b) $f_{\alpha}([l,m]) = \sum_{d=1}^{m} \left( \left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{l-1}{d} \right\rfloor \right) \alpha$,

(c) $\Phi([l,m], n) = \sum_{d|n} 2^\left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{l-1}{d} \right\rfloor$,

(d) $\phi_{\alpha}([l,m], n) = \sum_{d|n} \left( \left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{l-1}{d} \right\rfloor \right) \alpha$.

An analysis of the functions $f$, $f_{\alpha}$, $\Phi$, and $\Phi_{\alpha}$ obtained for different cases of the set $A$ lead us to more general formulas for any nonempty finite set of positive integers. See Sections Section 2 and 3 below.

2. Phi functions for integer sets

**Theorem 3.** We have

(a) $\Phi(A, n) = \sum_{d|n} \mu(d)2^{v(A,d)} = \sum_{d|n} \mu(d)|P(V(A,d))|$, 

(b) $\Phi_{\alpha}(A, n) = \sum_{d|n} \mu(d)2^{v(A,d)}$.
(b) \( \Phi_\alpha(A, n) = \sum_{d|n} \mu(d) \left( \frac{v(A, d)}{\alpha} \right) = \sum_{d|n} \mu(d) |P_\alpha(\mathcal{V}(A, d))| \).

**Proof.** The second identities in (a) and (b) are trivial. As to the first identities we use induction on \(|A|\). If \( A = \{a\} = [a, a] \), then by Theorem 2 (c, d)

\[
\Phi(A, n) = \sum_{d|n} \mu(d) 2^{\lceil \frac{n}{d} \rceil - \lfloor \frac{n}{d} \rfloor} \quad \text{and} \quad \Phi_\alpha(A, n) = \sum_{d|n} \mu(d) \left( \left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{n-1}{d} \right\rfloor \right) \).
\]

Assume that \( A = \{a_1, a_2, \ldots, a_k\} \) and that the identities hold for \( \{a_2, \ldots, a_k\} \). Then as to (a) we have

\[
\Phi(\{a_1, \ldots, a_k\}, n) = \Phi(\{a_2, \ldots, a_k\}, n) + \Phi(\{a_2, \ldots, a_k\}, \gcd(a_1, n))
\]

\[
= \sum_{d|n} \mu(d) 2^{\sum_{i=1}^{k} \left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{n-a_i}{d} \right\rfloor} + \sum_{d|(a_1, n)} \mu(d) 2^{\sum_{i=2}^{k} \left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{n-a_i}{d} \right\rfloor} \]

\[
= \sum_{d|(a_1, n)} \mu(d) 2^{\sum_{i=1}^{k} \left\lfloor \frac{n-a_i}{d} \right\rfloor} + \sum_{d|n} \mu(d) 2^{\sum_{i=2}^{k} \left\lfloor \frac{n-1}{d} \right\rfloor - \left\lfloor \frac{n-a_i}{d} \right\rfloor} \]

As to (b) we have

\[
\Phi_\alpha(\{a_1, \ldots, a_k\}, n) = \Phi_\alpha(\{a_2, \ldots, a_k\}, n) + \Phi_{\alpha-1}(\{a_2, \ldots, a_k\}, \gcd(a_1, n))
\]

\[
= \sum_{d|n} \mu(d) \left( \sum_{i=2}^{k} \left\lfloor \frac{n-a_i}{d} \right\rfloor \right) + \sum_{d|(a_1, n)} \mu(d) \left( \sum_{i=2}^{k} \left\lfloor \frac{n-a_i}{d} \right\rfloor \right) \]

\[
= \sum_{d|(a_1, n)} \mu(d) \left( \sum_{i=1}^{k} \left\lfloor \frac{n-a_i}{d} \right\rfloor \right) + \sum_{d|(a_1, n)} \mu(d) \left( \sum_{i=2}^{k} \left\lfloor \frac{n-a_i}{d} \right\rfloor \right) \]

\[
= \sum_{d|(a_1, n)} \mu(d) \left( \sum_{i=1}^{k} \left\lfloor \frac{n-a_i}{d} \right\rfloor \right) + \sum_{d|(a_1, n)} \mu(d) \left( \sum_{i=1}^{k} \left\lfloor \frac{n-a_i}{d} \right\rfloor \right) \]

\[
= \sum_{d|(a_1, n)} \mu(d) \left( \sum_{i=1}^{k} \left\lfloor \frac{n-a_i}{d} \right\rfloor \right) + \sum_{d|(a_1, n)} \mu(d) \left( \sum_{i=1}^{k} \left\lfloor \frac{n-a_i}{d} \right\rfloor \right) \]

\[
= \sum_{d|n} \mu(d) \left( \sum_{i=1}^{k} \left\lfloor \frac{n-a_i}{d} \right\rfloor \right) \].
This completes the proof. □

**Corollary 1.** The number of positive integers in the set \( A \) which are co-prime to \( n \) is

\[
\sum_{d|n} \mu(d)v(A, d).
\]

**Proof.** Apply Theorem 3(b) with \( \alpha = 1 \). □

**Corollary 2.** If \( n \in A \), then \( \Phi(A, n) \equiv 0 \mod 2 \).

**Proof.** If \( n \in A \), then evidently \( v(A, d) > 0 \) for all divisor \( d \) of \( n \) and thus the required congruence follows by Theorem 3(a). □

### 3. Relatively prime subsets of integer sets

**Theorem 4.** We have

\[
(a) \quad f(A) = \sum_{d=1}^{\sup A} \mu(d) \left( 2^{v(A, d)} - 1 \right) = \sum_{d=1}^{\sup A} \mu(d)|P^*(V(A, d))|,
\]

\[
(b) \quad f_{\alpha}(A) = \sum_{d=1}^{\sup A} \mu(d)\left( \frac{v(A, d)}{\alpha} \right) = \sum_{d=1}^{\sup A} \mu(d)|P_{\alpha}(V(A, d))|.
\]

**Proof.** The second equalities in (a) and (b) are evident. As to the first identities, we use induction on \( |A| \). If \( A = \{a\} = [a, a] \), then by Theorem 3(a, b)

\[
f(A) = \sum_{d=1}^{a} \mu(d) \left( 2^{\left\lfloor \frac{a}{d} \right\rfloor - \left\lfloor \frac{a-1}{d} \right\rfloor} - 1 \right) \quad \text{and} \quad f_{\alpha}(A) = \sum_{d=1}^{a} \mu(d)\left( \frac{\left\lfloor \frac{a}{d} \right\rfloor - \left\lfloor \frac{a-1}{d} \right\rfloor}{\alpha} \right).
\]

Assume now that \( A = \{a_1, a_2, \ldots, a_k\} \) and that the identities are true for \( \{a_2, \ldots, a_k\} \). Without loss of generality we may assume that \( a_1 < \sup A \). Then as to (a) we have

\[
f(\{a_1, \ldots, a_k\}) = f(\{a_2, \ldots, a_k\}) + \Phi(\{a_2, \ldots, a_k\}, a_1)
\]

\[
= \sum_{d=1}^{\sup A} \mu(d) \left( 2^{\sum_{i=2}^{k} \left\lfloor \frac{a_i}{d} \right\rfloor - \left\lfloor \frac{a_i-1}{d} \right\rfloor} - 1 \right) + \sum_{d|a_1} \mu(d)2^{\sum_{i=2}^{k} \left\lfloor \frac{a_i}{d} \right\rfloor - \left\lfloor \frac{a_i-1}{d} \right\rfloor}
\]

\[
\qquad = \sum_{d|a_1} \mu(d) \left( 2^{\sum_{i=2}^{k} \left\lfloor \frac{a_i}{d} \right\rfloor - \left\lfloor \frac{a_i-1}{d} \right\rfloor} - 1 \right) + \sum_{d|a_1} \mu(d) \left( \sum_{d=1}^{\sup A} \mu(d) \left( 2^{\sum_{i=2}^{k} \left\lfloor \frac{a_i}{d} \right\rfloor - \left\lfloor \frac{a_i-1}{d} \right\rfloor} - 1 \right) \right)
\]

\[
= \sum_{d|a_1} \mu(d)2^{\sum_{i=2}^{k} \left\lfloor \frac{a_i}{d} \right\rfloor - \left\lfloor \frac{a_i-1}{d} \right\rfloor} - \sum_{d|a_1} \mu(d) + \sum_{d=1}^{\sup A} \mu(d) \left( 2^{\sum_{i=2}^{k} \left\lfloor \frac{a_i}{d} \right\rfloor - \left\lfloor \frac{a_i-1}{d} \right\rfloor} - 1 \right)
\]

\[
= \sum_{d|a_1} \mu(d) \left( 2^{\sum_{i=1}^{k} \left\lfloor \frac{a_i}{d} \right\rfloor - \left\lfloor \frac{a_i-1}{d} \right\rfloor} - 1 \right) + \sum_{d=1}^{\sup A} \mu(d) \left( 2^{\sum_{i=1}^{k} \left\lfloor \frac{a_i}{d} \right\rfloor - \left\lfloor \frac{a_i-1}{d} \right\rfloor} - 1 \right)
\]

\[
= \sum_{d=1}^{\sup A} \mu(d) \left( 2^{\sum_{i=1}^{k} \left\lfloor \frac{a_i}{d} \right\rfloor - \left\lfloor \frac{a_i-1}{d} \right\rfloor} - 1 \right).
\]
As to (b) we find
\[ f_\alpha(a_1, \ldots, a_k) = f_\alpha(a_2, \ldots, a_k) + \Phi(\alpha_{a_1}(a_2, \ldots, a_k), a_1) \]
\[ = \sup_{d \mid a_1} \mu(d) \left( \sum_{i=2}^k \left( \frac{a_i}{d} - \left\lfloor \frac{a_i - 1}{d} \right\rfloor \right) \right) + \sum_{d \mid a_2} \mu(d) \left( \sum_{i=2}^k \left( \frac{a_i}{d} - \left\lfloor \frac{a_i - 1}{d} \right\rfloor \right) \right) \]
\[ = \sum_{d \mid a_1} \mu(d) \left( \sum_{i=2}^k \left( \frac{a_i}{d} - \left\lfloor \frac{a_i - 1}{d} \right\rfloor \right) \right) + \sup_{d \mid a_2} \mu(d) \left( \sum_{i=2}^k \left( \frac{a_i}{d} - \left\lfloor \frac{a_i - 1}{d} \right\rfloor \right) \right) \]
\[ = \sum_{d \mid a_1} \mu(d) \left( \sum_{i=1}^k \left( \frac{a_i}{d} - \left\lfloor \frac{a_i - 1}{d} \right\rfloor \right) \right) + \sum_{d \mid a_2} \mu(d) \left( \sum_{i=1}^k \left( \frac{a_i}{d} - \left\lfloor \frac{a_i - 1}{d} \right\rfloor \right) \right) \]
\[ = \sup_{d \mid a} \mu(d) \left( \sum_{i=1}^k \left( \frac{a_i}{d} - \left\lfloor \frac{a_i - 1}{d} \right\rfloor \right) \right) \]
This completes the proof. \[\square\]

Alternatively, we have the following formulas for \( f(A) \) and \( f_\alpha(A) \).

**Theorem 5.** Let \( A = \{a_1, a_2, \ldots, a_k\} \), let \( \tau \) be a permutation of \( \{1, 2, \ldots, k\} \), and let \( A_{\tau(j)} = \{a_{\tau(1)}, a_{\tau(2)}, \ldots, a_{\tau(j)}\} \) for \( j = 1, 2, \ldots, k \). Then

\[
(a) \quad f(A) = \sum_{j=1}^k \sum_{d \mid \tau(j)} \mu(d) v(A_{\tau(j)-1}, d),
\]

\[
(b) \quad f_\alpha(A) = \sum_{j=1}^k \sum_{d \mid \tau(j)} \mu(d) \left( \frac{v(A_{\tau(j)-1}, d)}{\alpha - 1} \right).
\]

**Proof.** For simplicity we assume that \( \tau \) is the identity permutation. As to part (a) we have
\[
f(a_1, \ldots, a_k) = f(a_1, \ldots, a_{k-1}) + \Phi(a_1, \ldots, a_{k-1}, a_k)
\]
\[
= f(a_1) + \Phi(a_1, a_2, \ldots, \Phi(a_1, \ldots, a_{k-1}, a_k)
\]
\[
= \sum_{d \mid a_1} \mu(d) + \sum_{d \mid a_2} \mu(d) 2^\left\lfloor \frac{a_2}{d} - \left\lfloor \frac{a_2 - 1}{d} \right\rfloor \right\rfloor + \ldots + \sum_{d \mid a_k} \mu(d) 2^\left\lfloor \frac{a_k}{d} - \left\lfloor \frac{a_k - 1}{d} \right\rfloor \right\rfloor
\]
\[
= \sum_{d \mid a_1} \mu(d) 2^\left\lfloor \frac{a_1}{d} - \left\lfloor \frac{a_1 - 1}{d} \right\rfloor \right\rfloor,
\]
where the third formula follows from Theorem 3. Part (b) follows similarly. This completes the proof. \[\square\]
4. Combinatorial identities involving Mertens function

We now give some identities which involves Mertens function.

**Theorem 6.** Let \( m \) and \( n \) be positive integers such that \( 1 < m \leq n \). Then

\[
\sum_{d=1}^{n} \mu(d)(2^{\left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{n-1}{d} \right\rfloor} + \left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{m-1}{d} \right\rfloor) = \begin{cases} 
M(n) & \text{if } (m, n) > 1, \\
1 + M(n) & \text{if } (m, n) = 1.
\end{cases}
\]

**Proof.** Apply Theorem 4 (a) to the set \( \{m, n\} \). \( \square \)

**Theorem 7.** Let \( l, m, \) and \( n \) be positive integers such that \( 1 < l < m \leq n \). Then

\[
\sum_{d=1}^{n} \mu(d)(2^{\left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{n-1}{d} \right\rfloor} + \left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{m-1}{d} \right\rfloor + \left\lfloor \frac{l}{d} \right\rfloor - \left\lfloor \frac{l-1}{d} \right\rfloor) = \begin{cases} 
4 + M(n) & \text{if } (l, m) = (l, n) = (m, n) = 1, \\
3 + M(n) & \text{if exactly two pairs from } \{l, m, n\} \text{ are co-prime}, \\
2 + M(n) & \text{if exactly one pair from } \{l, m, n\} \text{ is co-prime}, \\
1 + M(n) & \text{if no pair from } \{l, m, n\} \text{ is co-prime and } (l, m, n) = 1, \\
M(n) & \text{Otherwise}.
\end{cases}
\]

**Proof.** Apply Theorem 4 (a) to the set \( \{l, m, n\} \). \( \square \)

**Theorem 8.** We have

\[
M(\sup A) \leq \sum_{d=1}^{\sup A} \mu(d)2^{v(A,d)},
\]

and equality occurs if and only if \( \gcd(A) > 1 \).

**Proof.** The inequality follows Since by Theorem 4 (a) the identity

\[
\sum_{d=1}^{\sup A} \mu(d)2^{v(bA,d)} - M(\sup A)
\]

counts the number of relatively prime subsets of \( A \). Moreover, we clearly have that \( \gcd(A) > 1 \) if and only if \( f(A) = 0 \), which by Theorem 4 (a) means that

\[
\sum_{d=1}^{\sup A} \mu(d)2^{v(bA,d)} = M(\sup A),
\]

as desired. \( \square \)

5. \( \alpha \)-relatively prime and \( \alpha \)-free relatively prime sets

Motivated by the work of Cameron and Erdős in 4 and the work of Calkin and Granville in 3 we introduce the following notions.

**Definition 3.** We say that the set \( A \) is:

- **\( \alpha \)-relatively prime** if every subset of \( A \) of cardinality \( \alpha \) is relatively prime,
- **\( \alpha \)-relatively prime to \( n \)** if every subset of \( A \) of cardinality \( \alpha \) is relatively prime to \( n \),
- **\( \alpha \)-free relatively prime** if no subset of \( A \) of cardinality \( \alpha \) is relatively prime,
• \( \alpha \)-free relatively prime to \( n \) if no subset of \( A \) of cardinality \( \alpha \) is relatively prime to \( n \).

We have the following algebraic characterizations.

**Theorem 9.** (a) The set \( A \) is \( \alpha \)-relatively prime if and only if \( \left( \frac{|A|}{\alpha} \right) = f_{\alpha}(A) = 0 \).

(b) The set \( A \) is \( \alpha \)-relatively prime to \( n \) if and only if \( \left( \frac{|A|}{\alpha} \right) = \Phi_{\alpha}(A, n) \).

(c) The set \( A \) is \( \alpha \)-free relatively prime if and only if \( f_{\alpha}(A) = 0 \).

(d) The set \( A \) is \( \alpha \)-free relatively prime to \( n \) if and only if \( \Phi_{\alpha}(A, n) = 0 \).

**Proof.** Immediate from the definitions. \( \square \)

**Corollary 3.** Let \( \beta \) be a positive integer such that \( \alpha \leq \beta \leq |A| \). Then we have the following implications.

(a) If \( \left( \frac{|A|}{\alpha} \right) = \sum_{d=1}^{\sup A} \mu(d) \left( \frac{v(A, d)}{\alpha} \right) \), then \( \left( \frac{|A|}{\beta} \right) = \sum_{d=1}^{\sup A} \mu(d) \left( \frac{v(A, d)}{\beta} \right) \).

(b) If \( \left( \frac{|A|}{\alpha} \right) = \sum_{d|n} \mu(d) \left( \frac{v(A, d)}{\alpha} \right) \), then \( \left( \frac{|A|}{\beta} \right) = \sum_{d|n} \mu(d) \left( \frac{v(A, d)}{\beta} \right) \).

(c) If \( \sum_{d=1}^{\sup A} \mu(d) \left( \frac{v(A, d)}{\beta} \right) = 0 \), then \( \sum_{d|n} \mu(d) \left( \frac{v(A, d)}{\alpha} \right) = 0 \).

(d) If \( \sum_{d|n} \mu(d) \left( \frac{v(A, d)}{\beta} \right) = 0 \), then \( \sum_{d|n} \mu(d) \left( \frac{v(A, d)}{\alpha} \right) = 0 \).

**Proof.** (a) Clearly if \( A \) is \( \alpha \)-relatively prime and \( \alpha \leq \beta \leq |A| \), then \( A \) is \( \beta \)-relatively prime. Combining this fact with Theorem 9(a) and Theorem 3(b) gives the desired implication.

(b) Similarly, if \( A \) is \( \alpha \)-relatively prime to \( n \) and \( \alpha \leq \beta \leq |A| \), then \( A \) is \( \beta \)-relatively prime to \( n \). Now combine this fact with Theorem 9(b) and Theorem 3(b) to get the desired implication.

(c) As to part (c), we use the fact that \( A \) is \( \alpha \)-free relatively prime whenever \( A \) is \( \beta \)-free relatively prime and \( \alpha \leq \beta \).

(d) As to part (d), we use the fact that \( A \) is \( \alpha \)-free relatively prime to \( n \) whenever \( A \) is \( \beta \)-free relatively prime to \( n \) and \( \alpha \leq \beta \). \( \square \)

Using similar ideas we have:

**Corollary 4.** Let \( B \) be a nonempty subset of \( A \) such that \( \alpha \leq |B| \). Then we have the following implications.
(a) If \( |A| = \frac{\sum_{d = 1}^{\alpha} \mu(d) \left( v_A(d) \right)}{\alpha} \), then \( |B| = \frac{\sum_{d = 1}^{\alpha} \mu(d) \left( v_B(d) \right)}{\alpha} \).

(b) If \( |A| = \sum_{d \mid n} \mu(d) \left( v_A(d) \right) \), then \( |B| = \sum_{d \mid n} \mu(d) \left( v_B(d) \right) \).

(c) If \( \sum_{d = 1}^{\alpha} \mu(d) \left( v_A(d) \right) = 0 \), then \( \sum_{d \mid n} \mu(d) \left( v_B(d) \right) = 0 \).

(d) If \( \sum_{d \mid n} \mu(d) \left( v_A(d) \right) = 0 \), then \( \sum_{d \mid n} \mu(d) \left( v_B(d) \right) = 0 \).

Note that \( A \) is 2-relatively prime if and only if the elements of \( A \) are pairwise co-prime. We have the following two characterizations for 2-relatively prime sets.

**Theorem 10.** Let \( A = \{a_1, a_2, \ldots, a_k\} \), let \( \tau \) be a permutation of \( \{1, 2, \ldots, k\} \), and let \( A_{\tau(j)} = \{a_{\tau(1)}, a_{\tau(2)}, \ldots, a_{\tau(j)}\} \) for \( j = 1, 2, \ldots, k \). Then the following are equivalent:

(a) The set \( A \) is 2-relatively prime.

(b) \( 2|A| - 1 = \left( 1 + 4 \sum_{d = 1}^{\alpha} \mu(d) v_A(d) (v_A(d) - 1) \right)^{1/2} \).

(c) \( 2|A| - 1 = \left( 1 + 8 \sum_{j = 1}^{k} \sum_{d \mid a_{\tau(j)}} \mu(d) v_A(a_{\tau(j)-1}, d) \right)^{1/2} \).

**Proof.** As before we assume that \( \tau \) is the identity permutation. By Theorem 9, Theorem 4, and Theorem 5, the set \( A \) is 2-relatively prime means that

\[ \left( \frac{|A|}{2} \right) = \sum_{d = 1}^{\alpha} \mu(d) \left( v_A(d) \right) \]

or equivalently

\[ \left( \frac{|A|}{2} \right) = \sum_{j = 1}^{k} \sum_{d \mid a_j} \mu(d) \left( v_A(a_j-1), d \right) . \]

The former identity is equivalent to the quadratic equation

\[ |A|^2 - |A| - \sum_{d = 1}^{\alpha} \mu(d) v_A(d) (v_A(d) - 1) = 0 \]

which means that

\[ |A| = \frac{1}{2} \left( 1 + \left( 1 + 4 \sum_{d = 1}^{\alpha} \mu(d) v_A(d) (v_A(d) - 1) \right)^{1/2} \right) , \]
showing the equivalence of (a) and (b). The latter identity is equivalent to the quadratic equation

\[ |A|^2 - |A| - 2 \sum_{j=1}^{k} \sum_{d|a_j} \mu(d)v(A_{j-1}, d) = 0 \]

which is equivalent to

\[ |A| = \frac{1}{2} \left( 1 + \left( 1 + \frac{1}{2} \sum_{d=1}^{\sup B} \mu(d)v(B, d)(v(B, d)-1) \right)^{1/2} \right), \]

showing the equivalence of (a) and (c). This completes the proof. \qed

**Corollary 5.** The number of nonempty subsets of \( A \) whose elements are pairwise co-prime is

\[ \# \left\{ B \subseteq A : B \neq \emptyset \text{ and } 2|B| - 1 = \left( 1 + \sum_{d=1}^{\sup B} \mu(d)v(B, d)(v(B, d)-1) \right)^{1/2} \right\}. \]

Similar to Theorem 10 we have:

**Theorem 11.** Let \( A = \{a_1, a_2, \ldots, a_k\} \), let \( \tau \) be a permutation of \( \{1, 2, \ldots, k\} \), and let \( A_{\tau(j)} = \{a_{\tau(1)}, a_{\tau(2)}, \ldots, a_{\tau(j)}\} \) for \( j = 1, 2, \ldots, k \). Then the following are equivalent:

(a) The set \( A \) is 2-free relatively prime.

(b) \[ \sup_{d=1}^{\sup A} \mu(d)v(A, d)(v(A, d)-1) = 0. \]

(c) \[ \sum_{j=1}^{k} \sum_{d|a_{\tau(j)}} \mu(d)v(A_{\tau(j-1)}, d) = 0. \]

**Corollary 6.** The number of nonempty subsets of \( A \) which are co-prime free is

\[ \# \{ B \subseteq A : B \neq \emptyset \text{ and } \sup_{d=1}^{\sup B} \mu(d)v(B, d)(v(B, d)-1) = 0 \}. \]

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