Phaseless compressive sensing using partial support information

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Abstract

We study the recovery conditions of weighted $\ell_1$ minimization for real signal reconstruction from phaseless compressed sensing measurements when partial support information is available. A Strong Restricted Isometry Property (SRIP) condition is provided to ensure the stable recovery. Moreover, we present the weighted null space property as the sufficient and necessary condition for the success of $k$-sparse phase retrieval via weighted $\ell_1$ minimization.

1 Introduction

Compressive Sensing (CS) aims to recover an unknown signal from the underdetermined linear measurements (see [4, 5] for a comprehensive view). And it is known as phase retrieval or phaseless compressive sensing when there is no phase information. Specifically, the goal of phaseless compressive sensing is to recover $x \in \mathbb{C}^N$ up to a unimodular scaling constant from noisy magnitude measurements $y = |Ax| + e \in \mathbb{C}^m$ with the measurement matrix $A = (a_1, \cdots, a_m)^T \in \mathbb{C}^{m \times N}$, $|Ax| = (|\langle a_1, x \rangle|, \cdots, |\langle a_m, x \rangle|)^T$ and the noise term $e$. In the real case, i.e., $x \in \mathbb{R}^N$ and when $x$ is sparse or compressible, the stable recovery can be guaranteed by using the $\ell_1$ minimization problem

$$\min \|z\|_1 \text{ subject to } \|Az - y\|_2 \leq \varepsilon,$$

provided that the measurement matrix $A$ satisfies the SRIP [7,9]. In the noiseless case, the first sufficient and necessary condition was presented in [10] by proposing a new version of null space property for the phase retrieval problem.

In this paper, we generalize the existing theoretical framework for phaseless compressive sensing to incorporate partial support information, that is we consider the case that an estimate of the support of the signal is available. We follow the similar notations and arguments in [6]. For an arbitrary signal $x \in \mathbb{R}^N$, let $x^k$ be its best $k$-term approximation, so that $x^k$ minimizes $\|x - f\|_1$ over all $k$-sparse vectors
Let $T_0$ be the support of $x^k$, where $T_0 \subset \{1, \cdots, N\}$ and $|T_0| \leq k$. Let $\hat{T}$, the support estimate, be a subset of $\{1,2,\cdots,N\}$ with cardinality $|\hat{T}| = \rho k$, where $\rho \geq 0$ and $|\hat{T} \cap T_0| = \alpha \rho k$ with $0 \leq \alpha \leq 1$. To incorporate prior support information $\hat{T}$, we adopt the weighted $\ell_1$ minimization

$$
\min_z \sum_{i=1}^{N} w_i |z_i|, \text{ subject to } ||As||_2 - \|y\|_2 \leq \varepsilon, \text{ where } w_i = \begin{cases} \omega & i \in \hat{T}, \\
1 & i \in T^c. \end{cases}
$$

(2)

We present the SRIP condition and weighted null space property condition to guarantee the success of the recovery via the weighted $\ell_1$ minimization problem above.

The paper is organized as follows. In Section 2, we introduce the definition of SRIP and present the stable recovery condition with this tool. In Section 3, the sufficient and necessary weighted null space property condition for the real sparse phase retrieval is given. Finally, Section 4 is devoted to the conclusion. Throughout the paper, for any vector $x \in \mathbb{R}^N$, we denote the $\ell_p$ norm by $||x||_p = (\sum_{i=1}^{p} |x_i|^p)^{1/p}$ for $p > 0$ and the weighted $\ell_1$ norm as $||x||_{1,w} = \sum_{i=1}^{N} w_i |x_i|$. For any set $T$, we denote its cardinality as $|T|$. The vector $x \in \mathbb{R}^N$ is called $k$-sparse if at most $k$ of its entries are nonzero, i.e., if $||x||_0 = |\text{supp}(x)| \leq k$, where $\text{supp}(x)$ denotes the index set of the nonzero entries. We denote the index set $[N] := \{1,2,\cdots,N\}$. For a matrix $A = (a_1,\cdots,a_m)^T \in \mathbb{R}^{m \times N}$ and an index set $I \subset [m]$, we denote $A_I$ the sub-matrix of $A$ where only rows with indices in $I$ are kept, i.e., $A_I = (a_j, j \in I)^T$.

2 SRIP

To recover sparse signals via $\ell_1$ minimization in the classical compressive sensing setting, [2] introduced the notion of Restricted Isometry Property (RIP) and established a sufficient condition. We say a matrix $A$ satisfies the RIP of order $k$ if there exists a constant $\delta_k \in [0,1)$ such that for all $k$-sparse vectors $x$ we have

$$(1-\delta_k)||x||_2^2 \leq ||Ax||_2^2 \leq (1+\delta_k)||x||_2^2.$$

(3)

[1] proved that the RIP condition with $\delta_k < \sqrt{\frac{t-1}{t}}$ where $t > 1$ can guarantee the exact recovery in the noiseless case and stable recovery in the noisy case via $\ell_1$ minimization. And this condition is sharp when $t \geq \frac{4}{3}$. Very recently, [3] generalized this sharp RIP condition to the weighted $\ell_1$ minimization problem when partial support information was incorporated. We first present the following useful lemma, which is a simple extension of the result in [3].

Lemma 1 Let $x \in \mathbb{R}^N$, $y = Ax + e \in \mathbb{R}^m$ with $\|e\|_2 \leq \zeta$, and $\eta \geq 0$. Suppose that $A$ satisfies RIP with $\delta_k < \sqrt{\frac{t-d}{t}}$ for some $t > d$, where $\gamma = \omega + (1 - \omega)\sqrt{1 + \rho - 2\alpha \rho}$ and

$$
d = \begin{cases} 1, & \omega = 1 \\
1 - \alpha \rho + \alpha, & 0 \leq \omega < 1 \end{cases}
$$

(4)
with \( a = \max\{\alpha, 1 - \alpha\} \rho \). Then for any
\[
\hat{x} \in \{z \in \mathbb{R}^N : \|z\|_{1,w} \leq \|x\|_{1,w} + \eta, \|Az - y\|_2 \leq \varepsilon\},
\]
we have
\[
\|\hat{x} - x\|_2 \leq C_1(\zeta + \varepsilon) + C_2 \frac{2(\omega\|x_{T_0^c}\|_1 + (1 - \omega)\|x_{T_0^{c_1}T_0^c}\|_1)}{\sqrt{k}} + C_2 \frac{\eta}{\sqrt{k}},
\]
where
\[
C_1 = \frac{\sqrt{2(t - d)(t - d + \gamma^2)(1 + \delta_{tk})}}{(t - d + \gamma^2)(\sqrt{\frac{t - d}{t - d + \gamma^2}} - \delta_{tk})},
\]
\[
C_2 = \frac{\sqrt{2} \delta_{tk} \gamma + \sqrt{(t - d + \gamma^2)(\sqrt{\frac{t - d}{t - d + \gamma^2}} - \delta_{tk}) \delta_{tk}}}{(t - d + \gamma^2)(\sqrt{\frac{t - d}{t - d + \gamma^2}} - \delta_{tk})} + \frac{1}{\sqrt{d}}.
\]

**Remark 1** Note that if \( x_{\hat{\ell}^2} \) is the solution of the weighted \( \ell_1 \) minimization problem:
\[
\min_{\hat{z}} \|z\|_{1,w}, \text{ subject to } \|Az - y\|_2 \leq \varepsilon,
\]
then \( x_{\hat{\ell}^2} \in \{z \in \mathbb{R}^N : \|z\|_{1,w} \leq \|x\|_{1,w} + \eta, \|Az - y\|_2 \leq \varepsilon\} \) with \( \eta = 0 \). Therefore, this lemma is an extension of Theorem 3.1 in [3] by letting \( \zeta = \varepsilon \) and \( \eta = 0 \). The proof follows from almost the same procedure for the proof of Theorem 3.1 in Section 4 of [3] via replacing the \( P = \frac{2(\omega\|x_{T_0^c}\|_1 + (1 - \omega)\|x_{T_0^{c_1}T_0^c}\|_1)}{\sqrt{k}\gamma} \) by \( P' = \frac{2(\omega\|x_{T_0^c}\|_1 + (1 - \omega)\|x_{T_0^{c_1}T_0^c}\|_1)}{\sqrt{k}\gamma} + \frac{\eta}{\sqrt{k}\gamma} \), and letting \( \zeta = \varepsilon \). In order not to repeat, we leave out all the details. In addition, this result also generalizes the Lemma 2.1 in [7], which is the special case with the noise term \( e = 0, \zeta = 0 \) and \( \omega = 1 \).

To address the phaseless compressive sensing problem [2], a stronger version of RIP is needed. Its definition is provided as follows.

**Definition 1** (SRIP [7, 9]) We say a matrix \( A = (a_1, \cdots, a_m)^T \in \mathbb{R}^{m \times N} \) has the Strong Restricted Isometry Property (SRIP) of order \( k \) with bounds \( \theta_-, \theta_+ \in (0, 2) \) if
\[
\theta_-\|x\|_2^2 \leq \min_{I \subseteq [m], |I| \geq m/2} \|A_I x\|_2^2 \leq \max_{I \subseteq [m], |I| \leq m/2} \|A_I x\|_2^2 \leq \theta_+\|x\|_2^2
\]
holds for all \( k \)-sparse vectors \( x \in \mathbb{R}^N \), where \([m] = \{1, \cdots, m\}\). We say \( A \) has the Strong Lower Restricted Isometry Property of order \( k \) with bound \( \theta_- \) if the lower bound in (6) holds. Similarly, we say \( A \) has the Strong Upper Restricted Isometry Property of order \( k \) with bound \( \theta_+ \) if the upper bound in (6) holds.
Next, we present the conditions for the stable recovery via weighted $\ell_1$ minimization by using SRIP.

**Theorem 1** Let $x \in \mathbb{R}^N, y = |Ax| + e \in \mathbb{R}^m$ with $\|e\|_2 \leq \zeta$. Adopt the notations in Lemma 1 and assume that $A \in \mathbb{R}^{m \times N}$ satisfies the SRIP of order $tk$ with bounds $\theta_-, \theta_+ \in (0, 2)$ such that

$$t \geq \max \left\{ d + \frac{\gamma^2(1 - \theta_-)^2}{2\theta_- - \theta_+^2}, d + \frac{\gamma^2(1 - \theta_+)^2}{2\theta_+ - \theta_+^2} \right\}. \quad (7)$$

Then any solution $x^\theta$ of (3) satisfies

$$\min\{\|x^\theta - x\|_2, \|x^\theta + x\|_2\} \leq C_1(\zeta + \varepsilon) + C_2 \frac{2(\omega\|x_\mathbb{R}^N\|_1 + (1 - \omega)\|x_{F \cap \mathbb{R}^N}\|_1)}{\sqrt{k}}. \quad (8)$$

where $C_1$ and $C_2$ are constants defined in Lemma 1.

**Remark 2** As it has been proved in [9] that Gaussian matrices with $m = O(tk \log(N/k))$ satisfy SRIP of order $tk$ with high probability, thus the stable recovery result (8) can be achieved by using Gaussian measurement matrix with appropriate number of measurements $m$.

**Remark 3** Note that when the weight $\omega = 1$, we have $\gamma = d = 1$. Then, by assuming $\zeta = \varepsilon = 0$ and $x$ is exactly $k$-sparse, our theorem reduces to Theorem 2.2 in [9]. That is, if $A$ satisfies the SRIP of order $tk$ with bounds $\theta_-, \theta_+$ and $t \geq \max\left\{ \frac{1}{2\theta_- - \theta_+^2}, \frac{1}{2\theta_+ - \theta_+^2} \right\}$, then for any $k$-sparse signal $x \in \mathbb{R}^N$ we have $\arg\min_{x \in \mathbb{R}^N} \{\|x\|_1 : |Ax| = |Ax|\} = \{\pm x\}$. Similarly, if we let the noise term $e = 0$, $\zeta = 0$ and $\omega = 1$, this theorem goes to Theorem 3.1 in [7].

**Remark 4** If $\alpha = \frac{1}{2}$, we have $\gamma = d = 1$. The sufficient condition (7) of Theorem 1 is identical to that of Theorem 2.2 in [9], and that of Theorem 3.1 in [7]. And the constants $C_1 = c_1 = \frac{\sqrt{2(t-1) - \delta_{tk}}}{\sqrt{t(t-1)-\delta_{tk}}}$, $C_2 = \frac{\sqrt{2(t-1) + \delta_{tk}}}{\sqrt{t(t-1)-\delta_{tk}}}$ (see Theorem 3.1 in [7]). In addition, if $0 \leq \omega < 1$ and $\alpha > \frac{1}{2}$, then $d = 1$ and $\gamma < 1$. The sufficient condition (7) in Theorem 1 is weaker than that of Theorem 2.2 in [9] and that of Theorem 3.1 in [7]. And in this case, the constants $C_1 < c_1, C_2 < c_2$.

Set $t^\omega = \max\left\{ d + \frac{\gamma^2(1 - \theta_-)^2}{2\theta_- - \theta_+^2}, d + \frac{\gamma^2(1 - \theta_+)^2}{2\theta_+ - \theta_+^2} \right\}$. We illustrate how the constants $t^\omega, C_1$ and $C_2$ change with $\omega$ for different values of $\alpha$ in Figure 1. In all the plots, we set $\rho = 1$. In the plot of $t^\omega$, we set $\theta_- = \frac{1}{2}$ and $\theta_+ = \frac{3}{2}$, then $t^\omega = d + \frac{\gamma^2}{3}$. In the plots of $C_1$ and $C_2$, we fix $t = 4$ and $\delta_{tk} = 0.3$. Note that if $\omega = 1$ or $\alpha = 0.5$, then $t^\omega \equiv 1 + \frac{1}{2} = \frac{3}{2}$, $C_1 \equiv c_1$ and $C_1 \equiv c_2$. And it shows that $t^\omega$ decreases as $\alpha$ increases, which means that the sufficient condition (7) becomes weaker as $\alpha$ increases. For instance, if 90% of the support estimate is accurate ($\alpha = 0.9$) and $\omega = 0.6$, we have $t^\omega = 1.2022$, while $t^\omega = 1.3333$ for standard $\ell_1$ minimization ($\omega = 1$). As $\alpha$ increases, the constant $C_1$ decreases with $t = 4$ and $\delta_{tk} = 0.3$. Meanwhile, the constant $C_2$ with $\alpha \neq 0.5$ is smaller than that with $\alpha = 0.5$. 

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Figure 1: Comparison of the constants $t^\omega$, $C_1$ and $C_2$ for various of $\alpha$. In all the plots, we set $\rho = 1$. In the plot of $t^\omega$, we set $\theta_- = \frac{1}{2}$ and $\theta_+ = \frac{3}{2}$. In the plots of $C_1$ and $C_2$, we fix $t = 4$ and $\delta_{tk} = 0.3$.

**Proof of Theorem 1.** For any solution $x^\sharp$ of (2), we have

$$\|x^\sharp\|_{1,w} \leq \|x\|_{1,w}$$

and

$$\|Ax^\sharp\| - |Ax| - e_2 \leq \varepsilon.$$

If we divide the index set $\{1, 2, \cdots, m\}$ into two subsets

$$T = \{j : \text{sign}(\langle a_j, x^\sharp \rangle) = \text{sign}(\langle a_j, x \rangle)\} \quad \text{and} \quad T^c = \{j : \text{sign}(\langle a_j, x^\sharp \rangle) = -\text{sign}(\langle a_j, x \rangle)\},$$

then it implies that

$$\|A_T x^\sharp - A_T x - e\|_2 + \|A_{T^c} x^\sharp + A_{T^c} x - e\|_2 \leq \varepsilon. \quad (9)$$

Here either $|T| \geq m/2$ or $|T^c| \geq m/2$. If $|T| \geq m/2$, we use the fact that

$$\|A_T x^\sharp - A_T x - e\|_2 \leq \varepsilon. \quad (10)$$

Then, we obtain

$$x^\sharp \in \{z \in \mathbb{R}^N : \|z\|_{1,w} \leq \|x\|_{1,w}, \|A_T z - A_T x - e\|_2 \leq \varepsilon\}.$$
therefore, the definition of SRIP implies that $A_T$ satisfies the RIP of order $tk$ with
\[ \delta_{tk} \leq \max\{1 - \theta_-, \theta_+ - 1\} \leq \sqrt{\frac{t - d}{t - d + \gamma^2}}. \] (11)

Thus, by using Lemma 1 with $\eta = 0$, we have
\[ \|x^\dagger - x\|_2 \leq C_1(\zeta + \varepsilon) + C_2 \frac{2(\omega\|x_{T_0^c}\|_1 + (1 - \omega)\|x_{T_0^c \cap T_0^c}\|_1)}{\sqrt{k}}. \]

Similarly, if $|T^c| \geq m/2$, we obtain the other corresponding result
\[ \|x^\dagger + x\|_2 \leq C_1(\zeta + \varepsilon) + C_2 \frac{2(\omega\|x_{T_0^c}\|_1 + (1 - \omega)\|x_{T_0^c \cap T_0^c}\|_1)}{\sqrt{k}}. \]

The proof of Theorem 1 is now completed.

3 Weighted Null Space Property

In this section, we consider the noiseless weighted $\ell_1$ minimization problem, i.e.,
\[ \min_z \|z\|_{1,w}, \text{ subject to } |Az| = |Ax|, \text{ where } w_i = \begin{cases} \omega & i \in \hat{T} \\ 1 & i \in T^c \end{cases}. \] (12)

We denote the kernel space of $A$ by $\mathcal{N}(A) := \{h \in \mathbb{R}^N : Ah = 0\}$ and denote the $k$-sparse vector space $\Sigma_k^N := \{x \in \mathbb{R}^N : \|x\|_0 \leq k\}$.

**Definition 2** The matrix $A$ satisfies the $w$-weighted null space property of order $k$ if for any nonzero $h \in \mathcal{N}(A)$ and any $T \subset [N]$ with $|T| \leq k$ it holds that
\[ \|h_T\|_{1,w} < \|h_{T^c}\|_{1,w}, \] (13)
where $T^c$ is the complementary index set of $T$ and $h_T$ is the restriction of $h$ to $T$.

**Remark 5** Obviously, when the weight $\omega = 1$, the weighted null space property reduces to the classical null space property. And according to the specific setting of $w_i$, the expression [13] is equivalent to
\[ \omega\|h_{T \cap \hat{T}}\|_1 + \|h_{T \cap T^c}\|_1 < \omega\|h_{T_0^c \cap \hat{T}}\|_1 + \|h_{T_0^c \cap T^c}\|_1 \Leftrightarrow \omega\|h_T\|_1 + (1 - \omega)\|h_G\|_1 < \|h_{T^c}\|_1, \]
where $G = (T \cap \hat{T}^c) \cup (T^c \cap \hat{T})$ (see [8] for more arguments).

It is known that a signal $x \in \Sigma_k^N$ can be recovered via the weighted $\ell_1$ minimization problem if and only if the measurement matrix $A$ has the weighted null space property of order $k$. We state it as follows (see [11]):
Lemma 2  Given $A \in \mathbb{R}^{m \times N}$, for every $k$-sparse vector $x \in \mathbb{R}^{N}$ it holds that

$$\arg\min_{z \in \mathbb{R}^{N}} \{ \|z\|_{1,w} : Az = Ax \} = x$$

if and only if $A$ satisfies the $w$-weighted null space property of order $k$.

Next, we extend Lemma 2 to the following theorem on phaseless compressive sensing for the real signal reconstruction.

Theorem 2  The following statements are equivalent:

(a) For any $k$-sparse $x \in \mathbb{R}^{N}$, we have

$$\arg\min_{z \in \mathbb{R}^{N}} \{ \|z\|_{1,w} : |Az| = |Ax| \} = \{ \pm x \}. \quad (14)$$

(b) For every $S \subseteq [m]$, it holds

$$\|u + v\|_{1,w} < \|u - v\|_{1,w} \quad (15)$$

for all nonzero $u \in \mathcal{N}(A_{S})$ and $v \in \mathcal{N}(A_{S^c})$ satisfying $\|u + v\|_{0} \leq k$.

Remark 6  If $\omega = 1$, then Theorem 2 reduces to Theorem 3.2 in [10]. And since $w_{i} = \omega$ when $i \in \tilde{T}$, and $w_{i} = 1$ otherwise, the expression (15) is equivalent to

$$\omega \|u + v\|_{1} + (1 - \omega)\|(u + v)_{\tilde{T}^c}\|_{1} < \omega \|u - v\|_{1} + (1 - \omega)\|(u - v)_{\tilde{T}^c}\|_{1}.$$ 

Proof of Theorem 2.  The proof follows from the proof of Theorem 3.2 in [10] with minor modifications. First we show $(a) \Rightarrow (b)$. Assume $(b)$ is false, that is, there exist nonzero $u \in \mathcal{N}(A_{S})$ and $v \in \mathcal{N}(A_{S^c})$ such that

$$\|u + v\|_{1,w} \geq \|u - v\|_{1,w}$$

and $u + v \in \Sigma^{N}_{k}$. Now set $x = u + v \in \Sigma^{N}_{k}$, obviously for $i = 1, \ldots, m$, we have

$$|\langle a_{i}, x \rangle| = |\langle a_{i}, u + v \rangle| = |\langle a_{i}, u - v \rangle|,$$

since either $\langle a_{i}, u \rangle = 0$ or $\langle a_{i}, v \rangle = 0$. In other words $|Ax| = |A(u - v)|$. Note that $u - v \neq -x$, for otherwise we would have $u = 0$, which is a contradiction. Then, it follows from (a) that we obtain

$$\|x\|_{1,w} = \|u + v\|_{1,w} < \|u - v\|_{1,w},$$

This is a contradiction. Thus, $(b)$ holds.
Next we prove \((b) \Rightarrow (a)\). Let \(b = (b_1, \ldots, b_m)^T = |Ax|\) where \(x \in \Sigma^N_k\). For a fixed \(\sigma = (\sigma_1, \cdots, \sigma_m)^T \in \{-1, 1\}^m\), we set \(b^\sigma = (\sigma_1 b_1, \cdots, \sigma_m b_m)^T\). We now consider the following weighted \(\ell_1\) minimization problem:

\[
\min_z \|z\|_{1,w} \text{ subject to } Az = b^\sigma.
\]

Its solution is denoted as \(x^\sigma\). Then, we claim that for any \(\sigma \in \{1, -1\}^m\), if \(x^\sigma\) exists (it may not exist), we have

\[
\|x^\sigma\|_{1,w} \geq \|x\|_{1,w}
\]

and the equality holds if and only if \(x^\sigma = \pm x\).

To prove the claim, we assume \(\sigma^* \in \{1, -1\}^m\) such that \(b^{\sigma^*} = Ax\). First note that the statement (b) implies the classical weighted null space property of order \(k\). To see this, for any nonzero \(h \in \mathcal{N}(A)\) and \(T \subseteq [N]\) with \(|T| \leq k\), we set \(u = h, v = h_T - h_{T^c}\) and \(S = [m]\). Then, we have \(u \in \mathcal{N}(AS)\) and \(v \in \mathcal{N}(AS^c)\). Therefore, the statement (b) now implies

\[
2\|h_T\|_{1,w} = \|u + v\|_{1,w} < \|u - v\|_{1,w} = 2\|h_{T^c}\|_{1,w}.
\]

As a consequence, we have \(x^{\sigma^*} = x\) by Lemma 2. And, similarly we have \(x^{-\sigma^*} = -x\). Next, for any \(\sigma \in \{-1, 1\}^m \neq \pm \sigma^*\), if \(x^\sigma\) doesn’t exist then we have nothing to prove. Assume it does exist, set \(S_* = \{i : \sigma_i = \sigma_i^*\}\). Then

\[
\langle a_i, x^\sigma \rangle = \begin{cases}
    \langle a_i, x \rangle & i \in S_*,
    
    -\langle a_i, x \rangle & i \in S^c_*.
\end{cases}
\]

Set \(u = x - x^\sigma\) and \(v = x + x^\sigma\). Obviously, \(u \in \mathcal{N}(A_{S_*})\) and \(v \in \mathcal{N}(A_{S^c_*})\). Furthermore, \(u + v = 2x \in \Sigma^N_k\). Then, by the statement (b), we have

\[
2\|x\|_{1,w} = \|u + v\|_{1,w} < \|u - v\|_{1,w} = 2\|x^\sigma\|_{1,w}.
\]

This proves (a) and the proof is completed.

4 Conclusion

In this paper, we establish the sufficient SRIP condition and the sufficient and necessary weighted null property condition for phaseless compressive sensing using partial support information via weighted \(\ell_1\) minimization. We only consider the real-valued signal reconstruction case. It is challenging to generalize the present results to the complex case. And it will be very interesting to construct the measurement matrix \(A \in \mathbb{R}^{m \times N}\) satisfying the weighted null space property given in [15].

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