Orlicz Mixed Affine Quermassintegrals

Nikos Dafnis

Abstract

Lutwak’s notion of affine quermassintegrals of a convex body quickly became of great importance in convex and affine geometry and more recently, also in asymptotic geometric analysis. In this note, following the ideas in [30], we introduce the notion of the Orlicz mixed affine quermassintegrals of a convex body in $\mathbb{R}^n$, as a generalization of the affine quermassintegrals in the framework of the Orlicz-Brunn-Minkowski theory. We prove a Minkowski inequality for the Orlicz mixed and affine quermassintegrals, and an Orlicz-Brunn-Minkowski inequality, which provides of a direct generalization of Lutwak’s Brunn-Minkowski inequality for affine quermassintegrals, in the Orlicz space.

1 Introduction.

Let $\mathcal{K}^n$ be the class of all non-empty compact convex subsets of $\mathbb{R}^n$. The support function of a $K \in \mathcal{K}^n$ is defined by

$$h_K(x) = \sup_{y \in K} \langle x, y \rangle, \quad x \in \mathbb{R}^n.$$  

Support functions are sublinear i.e., subadditive and homogeneous of degree 1, and therefore often regarded as functions on $S^{n-1}$. Conversely, any sublinear real-valued function on $\mathbb{R}^n$, is the support function of a unique compact convex set. Consequently, any $K \in \mathcal{K}^n$ is uniquely determined by its support function.

The classical Brunn-Minkowski theory combine two basic concepts in geometry, volume and Minkowski (vector) addition. More precisely, if $K, L \in \mathcal{K}^n$, $a, b > 0$, their Minkowski linear combination $aK + bL \in \mathcal{K}^n$ can be defined by

$$h_{aK + bL}(u) = ah_K(u) + bh_L(u), \quad u \in S^{n-1}.$$  

The Brunn-Minkowski theory played a crucial role in the development of convex geometry and asymptotic geometric analysis, among other areas of mathematics. Fundamental within the theory, is the Brunn-Minkowski inequality:

$$V(K + L)^{\frac{1}{n}} \geq V(K)^{\frac{1}{n}} + V(L)^{\frac{1}{n}},$$  

for any $K, L \in \mathcal{K}^n$, and if $K$ and $L$ are non-trivial, then equality holds if and only if $K$ and $L$ are homothetic, or they lie in parallel hyperplanes.

Minkowski’s fundamental notion of mixed volume $V_1(K, L)$, $K, L \in \mathcal{K}^n$, is defined to be proportional to the first variation of the volume with respect to Minkowski linear combination, by the formula

$$V_1(K, L) = \frac{1}{n} \lim_{\varepsilon \to 0^+} \frac{V(K + \varepsilon L) - V(K)}{\varepsilon}.$$  

Aleksandrov [1] and Fenchel and Jensen [3] proved that or any $K \in \mathcal{K}^n$ there exists a unique Borel measure $S(K, \cdot)$ on the unit sphere $S^{n-1}$, called the surface area
measure of $K$, such that for any $L \in K^n$ their mixed volume has the following integral representation:

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u) \, dS(K, u).$$

By its definition the mixed volume satisfy that $V_1(K, K) = V(K)$ for any $K \in K^n$. The Minkowski inequality settle in general the relation between the volume and the mixed volume:

$$V_1(K, L)^n \geq V(K)^{n-1} V(L),$$

$K, L \in K^n$. If moreover $K$ and $L$ are non-trivial, then equality holds if and only if $K$ and $L$ are homothetic, or they lie in parallel hyperplanes.

$L_p$ Mixed Volume.

Let $K_0^n$ be the class of all nonempty compact convex subsets of $\mathbb{R}^n$, that contain the origin in their interior. Minkowski [14] introduced a new notion of linear combination of convex bodies in $K_0^n$. For $p \geq 1$, $K, L \in K_0^n$ and $a, b > 0$, their $L_p$ linear combination is the set $a \cdot_p K + b \cdot_p L \in K_0^n$, with support function

$$h_{a \cdot_p K + b \cdot_p L}(u)^p = ah_K(u)^p + bh_L(u)^p, \quad u \in S^{n-1}.$$ 

See also [22] for an extension on non-convex sets. Using Minkowski’s linear combination, Lutwak [16], [17] gave rise to the $L_p$-Brunn-Minkowski theory as an extension of the Brunn-Minkowski theory, which strengthened many of the classical results. In his setting, the $L_p$ mixed volume of $K, L \in K_0^n$ can be defined by

$$V_p(K, L) = \frac{p}{n} \lim_{\varepsilon \to 0^+} \frac{V(K + \varepsilon \cdot_p L) - V(K)}{\varepsilon},$$

and has the following integral representation

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L(u)^p \, h_K(u)^{1-p} \, dS(K, u).$$

Note that $V_p(K, K) = V(K)$, for any $K \in K_0^n$, and in correspondence to the classical theory, the $L_p$-Minkowski inequality asserts that

$$V_p(K, L)^n \geq V(K)^{n-p} V(L)^p,$$

for any $K, L \in K_0^n$, with equality if and only if $K$ and $L$ are dilates of each other, and as a consequence we get the following $L_p$-Brunn-Minkowski inequality:

$$V(K + \varepsilon \cdot_p L)^{\frac{n}{p}} \geq V(K)^{\frac{n}{p}} + V(L)^{\frac{n}{p}},$$

for any $K, L \in K_0^n$, with equality if and only if $K$ and $L$ are homothetic.

For further details on Brunn-Minkowski and $L_p$-Brunn-Minkowski theories, we refer to the book of Schneider [26].

Orlicz Mixed Volume.

Lutwak, Yang and Zhang [21], [20] initiated a further extension of Brunn-Minkowski theory to an Orlicz setting. This involves the replacement of the function $t^p$, by an increasing convex function in $\varphi : [0, \infty) \to [0, \infty)$. The new Orlicz-Brunn-Minkowski theory studied systematically by Gardner, Hug & Weil in [7], where the authors constructed
a solid framework and indicate its relation to the Orlicz spaces. Following their point of view, we denote by $\mathcal{C}$, the class of all increasing convex functions $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and $\varphi(1) = 1$.

For every $K, L \in \mathcal{K}_n^0$, $a, b > 0$ and $\varphi \in \mathcal{C}$, the Orlicz linear combination $a \cdot \varphi K + b \cdot \varphi L \in \mathcal{K}_n^0$ or simply $a \cdot K + b \cdot L$ is defined by

$$h_{a \cdot K + b \cdot L}(x) = \inf \left\{ \lambda > 0 : a \varphi \left( \frac{h_K(x)}{\lambda} \right) + b \varphi \left( \frac{h_L(x)}{\lambda} \right) \leq 1 \right\},$$

or equivalently by the implicit equation

$$a \varphi \left( \frac{h_K(x)}{h_{a \cdot K + b \cdot L}(x)} \right) + b \varphi \left( \frac{h_L(x)}{h_{a \cdot K + b \cdot L}(x)} \right) = 1.$$

The Orlicz linear combination is continuous with respect to the Hausdorff metric. In particular, for any $K, L \in \mathcal{K}_n^0$, we have that

$$K + \varphi \varepsilon \cdot L \to K,$$

as $\varepsilon \to 0^+$, in the Hausdorff metric

$$\delta(K, L) = \sup_{u \in S^{n-1}} \left| h_K(u) - h_L(u) \right|.$$

The Orlicz mixed volume of $K, L \in \mathcal{K}_n^0, \varphi \in \mathcal{C}$, can be defined by

$$V_{\varphi}(K, L) = \frac{1}{n} \int_{S^{n-1}} \varphi \left( \frac{h_L(u)}{h_K(u)} \right) h_K(u) dS(K, u),$$

As in the $L_p$ case, the Orlicz mixed volume of $K, L \in \mathcal{K}_n^0$, is proportional to the first variation of volume with respect to their Orlicz linear combination:

$$V_{\varphi}(K, L) = \frac{\varphi'(1^-)}{n} \lim_{\varepsilon \to 0^+} \frac{V(K + \varphi \varepsilon \cdot L) - V(K)}{\varepsilon},$$

where $\varphi'(1^-)$ is the left derivative of $\varphi$ at 1. Similarly to the $L_p$ case, we have that $V_{\varphi}(K, K) = V(K)$ for any $K \in \mathcal{K}_n^0$, and more generally the Orlicz-Minkowski inequality asserts that

$$\frac{V_{\varphi}(K, L)}{V(K)} \geq \varphi \left( \left( \frac{V(L)}{V(K)} \right)^{1/n} \right),$$

for all $K, L \in \mathcal{K}_n^0$. If $\varphi$ is strictly convex then equality holds if and only if $K$ and $L$ are dilates of each other. We also have a Orlicz-Brunn-Minkowski inequality, which relates the volume with the Orlicz linear combination:

$$1 \geq \varphi \left( \left( \frac{V(K)}{V(K + \varphi L)} \right)^{1/n} \right) + \varphi \left( \left( \frac{V(L)}{V(K + \varphi L)} \right)^{1/n} \right),$$

for all $K, L \in \mathcal{K}_n^0$, and if $\varphi$ is strictly convex then equality holds if and only if $K$ and $L$ are dilates of each other. We refer to [7] for any further details on the Orlicz-Brunn-Minkowski theory.

Affine Quermassintegrals.
Lutwak [12] defined the affine quermassintegrals of a convex body \( K \) in \( \mathbb{R}^n \) by

\[
\Phi_{n-j}(K) = \omega_n \left( \int_{G_{n,j}} \text{Vol}_j(K|\xi)^{-n} d\nu_{n,j}(\xi) \right)^{-\frac{1}{n}},
\]

for \( 1 < j < n \), while \( \Phi_0(K) = V(K) \) and \( \Phi_n(K) = \omega_n \). Here and for the rest of this note, \( \nu_{n,j} \) is the Haar probability measure on the Grassmannian manifold \( G_{n,j} \) of the \( j \)-dimensional subspaces of \( \mathbb{R}^n \). The terminology “affine” was justified a few years later by Grinberg [9], who showed that these quantities are actually invariant under volume preserving linear transformations. Lutwak proved a Brunn-Minkowski inequality for the affine quermassintegrals:

\[
\Phi_{n-j}(K + L)^{1/j} \geq \Phi_{n-j}(K)^{1/j} + \Phi_{n-j}(L)^{1/j},
\]

and conjectured in [19] that they satisfy the inequalities

\[
\omega_n^{n-k} \Phi_{n-j}(K)^k \geq \omega_n^{n-j} \Phi_{n-k}(K)^j,
\]

for all \( 0 < j < k \leq n \). In particular, Lutwak asks if the following inequalities holds true:

\[
\Phi_{n-j}(K)^n \geq \omega_n^{n-j} V(K)^j,
\]

for every \( 0 \leq j < n \), with equality if and only if \( K \) is an ellipsoid. Most of these conjectures remain open. Note that two cases of (1.9) follow from classical results: when \( j = n - 1 \) this inequality is the Petty projection inequality and when \( j = 1 \) and \( K \) is origin symmetric then (1.9) is the Blaschke-Santaló inequality. For more details we refer to the book of Gardner [6] (see also [2] and [23], where an asymptotic version of (1.9) is proved).

In [11], an extension of affine quermassintegrals was considered, where the authors defined the Orlicz mixed affine quermassintegrals for \( K, L \in K_0^n \), \( \varphi \in \mathcal{C} \) and \( 0 < j \leq n \), by

\[
\Phi_{\varphi,n-j}(K, L) := \omega_n \left( \int_{G_{n,j}} V_{\varphi}^{(j)}(K|\xi, L|\xi)^{-n} d\nu_{n,j}(\xi) \right)^{-\frac{1}{n}}.
\]

These quantities are invariant under volume preserving linear transformations, and provide a generalization of affine quermassintegrals to the Orlicz spaces.

In this note, drawing our inspiration from [30], we introduce the following alternative definition. For any \( \varphi \in \mathcal{C} \) and \( 0 < j \leq n \), we define the Orlicz mixed affine quermassintegrals of \( K, L \in K_0^n \), by

\[
\Phi_{\varphi,n-j}(K, L) := \omega_n \left( \int_{G_{n,j}} V_{\varphi}^{(j)}(K|\xi, L|\xi) \text{Vol}_j(K|\xi)^{-n} d\nu_{n,j}(\xi) \right)^{-\frac{1}{n}}.
\]

Our definition of Orlicz mixed affine quermassintegrals combine and extend both Lutwak’s concepts of affine quermassintegrals and Orlicz mixed volume.

In section 2 we prove their invariance under volume preserving linear transformations and we show a first variation formula for them (see Proposition 2.4) with respect to the Orlicz linear combination:

\[
\Phi_{\varphi,n-j}(K, L)^{-n} = \frac{\varphi'(1^-)}{j \Phi_{n-j}(K)^{n+1}} \lim_{\varepsilon \to 0^+} \frac{\Phi_{n-j}(K + \varepsilon \varphi \varepsilon L) - \Phi_{n-j}(K)}{\varepsilon}.
\]
This can be seen as a generalization of the corresponding formula (1.4) for the Orlicz mixed volume.

In section 3 we use Hölder’s inequality to derive the following Orlicz-Minkowski inequality for the Orlicz mixed affine quermassintegrals
\[
\left( \frac{\Phi_{\varphi,n-j}(K,L)}{\Phi_{n-j}(K)} \right)^{-n} \geq \varphi \left( \left( \frac{\Phi_{n-j}(L)}{\Phi_{n-j}(K)} \right)^{1/j} \right),
\]
which generalize the Orlicz-Minkowski inequality (1.6) for the mixed volume.

In the last section 4 we prove an Orlicz-Brunn-Minkowski inequality for Lutwak’s affine quermassintegrals:
\[
1 \geq \varphi \left( \left( \frac{\Phi_{n-j}(K)}{\Phi_{n-j}(K+\varepsilon \cdot L)} \right)^{1/j} \right) + \varepsilon \varphi \left( \left( \frac{\Phi_{n-j}(L)}{\Phi_{n-j}(K+\varepsilon \cdot L)} \right)^{1/j} \right).
\]
Note that (1.12) generalize the corresponding Brunn-Minkowski inequalities for Orlicz mixed volumes (1.6), and for affine quermassintegrals (1.8).

**Acknowledgments.** The author is supported by the Austrian Science Fund (FWF): Lise Meitner [M 2338-N35].

## 2 Orlicz Mixed Affine Quermassintegrals

**Definition 2.1** (Orlicz mixed Affine Quermassintegrals). The Orlicz mixed affine quermassintegrals are defined for any \( K, L \in K_n^o \), \( \varphi \in \mathcal{E} \) and \( 1 \leq j \leq n \), by
\[
\Phi_{\varphi,n-j}(K,L) := \frac{\omega_n}{\omega_j} \left( \int_{G_{n,j}} V_{\varphi}^{(j)}(K|\xi,L|\xi) \operatorname{Vol}_j(K|\xi)^{-n-1} d\nu_{n,j}(\xi) \right)^{-\frac{1}{n}}.
\]

**Lemma 2.2.** For any \( K \in K_n^o \), \( \lambda > 0 \) and all \( 1 \leq j \leq n \), we have that
\[
\Phi_{\varphi,n-j}(K,\lambda K) = \varphi(\lambda)^{-1/n} \Phi_{n-j}(K).
\]

**Proof.** By the definition of Orlicz mixed volume (1.1), we have that
\[
V_{\varphi}(K,\lambda K) = \frac{1}{n} \int_{S_{n-1}} \varphi \left( \frac{h_{\lambda K}(u)}{h_K(u)} \right) h_K(u) dS(K,u)
\]
\[
= \varphi(\lambda) \frac{1}{n} \int_{S_{n-1}} h_K(u) dS(K,u)
\]
\[
= \varphi(\lambda) V_1(K, K) = \varphi(\lambda) V(K).
\]
Thus,
\[
\Phi_{\varphi,n-j}(K,\lambda K) = \frac{\omega_n}{\omega_j} \left( \int_{G_{n,j}} V_{\varphi}^{(j)}(K|\xi,\lambda K|\xi) \operatorname{Vol}_j(K|\xi)^{-n-1} d\nu_{n,j}(\xi) \right)^{-\frac{1}{n}}
\]
\[
= \varphi(\lambda)^{-1} \Phi_{n-j}(K).
\]

Note that \( \varphi(1) = 1 \), and so we have that \( \Phi_{\varphi,n-j}(K,K) = \Phi_{n-j}(K) \), for all \( K \in K_n^o \), and so in that sense, Orlicz mixed affine quermassintegrals provides a natural extension of Lutwak’s affine quermassintegrals in the Orlicz setting.
Lemma 2.3. Let \( \varphi \in \mathcal{C} \), \( K, L \in K_0^n \), \( \varepsilon > 0 \) and \( \xi \in G_{n,j} \). Then,

\[
(K + \varphi \varepsilon \cdot L)|\xi| = K|\xi + \varphi \varepsilon \cdot L|\xi.
\]

Proof. For every \( u \in S^{n-1} \cap \xi \), we have that

\[
h_Q(u) = h_Q|\xi|(u),
\]

(2.3)

for every \( Q \in K_0^n \). Thus, by Lemma 2.2 we have that for every \( u \in S \cap \xi \)

\[
\varphi \left( \frac{h_K(u)}{h(K + \varphi \varepsilon \cdot L)} \right) + \varepsilon \varphi \left( \frac{h_L(u)}{h(K + \varphi \varepsilon \cdot L)} \right) = \varphi \left( \frac{h_K(u)}{h(K + \varphi \varepsilon \cdot L)} \right) + \varepsilon \varphi \left( \frac{h_L(u)}{h(K + \varphi \varepsilon \cdot L)} \right) = 1,
\]

which means that \((K + \varphi \varepsilon \cdot L)|\xi| = K|\xi + \varphi \varepsilon \cdot L|\xi\).

Next we prove a first variation formula for the Orlicz mixed affine quermassintegrals with respect to the Orlicz addition.

Proposition 2.4. Let \( \varphi \in \mathcal{C} \), \( K, L \in K_0^n \) and \( 1 \leq j \leq n \). Then,

\[
\Phi_{n-j}^{(K)}(K)^{n-1} \Phi_{\varphi,n-j}(K, L)^{-n} = \frac{\varphi'(1^-)}{j} \lim_{\varepsilon \to 0^+} \frac{\Phi_{n-j}(K + \varphi \varepsilon \cdot L) - \Phi_{n-j}(K)}{\varepsilon}.
\]

(2.4)

Proof. By (1.5) we have

\[
\frac{d}{d\varepsilon} |_{\varepsilon=0^+} \int_{G_{n,j}} \text{Vol}_j ((K + \varphi \varepsilon \cdot L)|\xi|)^{-n} d\nu_{n,j}(|\xi|)
\]

\[
= -n \int_{G_{n,j}} \text{Vol}_j(K|\xi|)^{-n-1} \frac{d}{d\varepsilon} |_{\varepsilon=0^+} \text{Vol}_j((K + \varphi \varepsilon \cdot L)|\xi|) d\nu_{n,j}(\xi)
\]

\[
= -\frac{j n}{\varphi'(1^-)} \int_{G_{n,j}} V_{\varphi}^{(j)}(K|\xi|, L|\xi|) \text{Vol}_j(K|\xi|)^{-n-1} d\nu_{n,j}(\xi)
\]

\[
= -\frac{j n}{\varphi'(1^-)} \left( \frac{\omega_n}{\omega_j} \right)^n \Phi_{\varphi,n-j}(K, L)^{-n}.
\]

Thus,

\[
\lim_{\varepsilon \to 0^+} \frac{\Phi_{n-j}(K + \varphi \varepsilon \cdot L) - \Phi_{n-j}(K)}{\varepsilon}
\]

\[
= \frac{d}{d\varepsilon} |_{\varepsilon=0^+} \Phi_{n-j}(K + \varphi \varepsilon \cdot L)
\]

\[
= \frac{\omega_n}{\omega_j} \frac{d}{d\varepsilon} |_{\varepsilon=0^+} \left( \int_{G_{n,j}} \text{Vol}_j((K + \varphi \varepsilon \cdot L)|\xi|)^{-n} d\nu_{n,j}(\xi) \right)^{-\frac{1}{n}-1}
\]

\[
= -\frac{\omega_n}{\omega_j} \frac{1}{n} \left( \int_{G_{n,j}} \text{Vol}_j(K|\xi|)^{-n} d\nu_{n,j}(\xi) \right)^{-\frac{1}{n}-1}
\]

\[
= \frac{d}{d\varepsilon} |_{\varepsilon=0^+} \int_{G_{n,j}} \text{Vol}_j((K + \varphi \varepsilon \cdot L)|\xi|)^{-n} d\nu_{n,j}(\xi)
\]

\[
= \frac{j}{\varphi'(1^-)} \left[ \frac{\omega_n}{\omega_j} \left( \int_{G_{n,j}} \text{Vol}_j(K|\xi|)^{-n} d\nu_{n,j}(\xi) \right)^{-\frac{1}{n}} \right]^{n+1} \Phi_{\varphi,n-j}(K, L)^{-n}
\]

\[
= \frac{j}{\varphi'(1^-)} \Phi_{n-j}(K)^{n+1} \Phi_{\varphi,n-j}(K, L)^{-n}.
\]
Note. Definition 2.1 for \( j = n \) gives that
\[
\Phi_{\varphi,0}(K,L)^{-n} = V_{\varphi}(K,L) V(K)^{-n-1}. \tag{2.5}
\]

So in that case, formula \( \Phi_{\varphi,0}(K,L)^{-n} \) reads exactly as the corresponding first variation formula \( \Phi_{\varphi,0}(K,L)^{-n} \) for the Orlicz mixed volume.

The following lemma comes from [7, Theorem 5.2].

**Lemma 2.5.** Let \( \varphi \in \mathcal{C} \), \( K, L \in \mathcal{K}^n_0 \), \( \varepsilon > 0 \) and \( T \in GL(n) \). Then,
\[
T(K + \varepsilon \cdot L) = TK + \varepsilon \cdot TL.
\]

**Proof.** Note that for any \( u \in S^{n-1} \) and \( Q \in \mathcal{K}^n_0 \),
\[
h_{TQ}(u) = h_Q(T^*u).
\]

Thus, by definition (1.1) for the Orlicz linear combination, we have that for every \( u \in S^{n-1} \).
\[
\begin{align*}
\Phi_{\varphi,v}\in TL(u) &= \inf \left\{ \lambda > 0 : \varphi \left( \frac{h_{TK}(u)}{\lambda} \right) + \varepsilon \varphi \left( \frac{h_{TL}(u)}{\lambda} \right) \leq 1 \right\} \\
&= \inf \left\{ \lambda > 0 : \varphi \left( \frac{h_{K}(T^*u)}{\lambda} \right) + \varepsilon \varphi \left( \frac{h_{L}(T^*u)}{\lambda} \right) \leq 1 \right\} \\
&= \Phi_{\varphi,v\in L}(T^*u) = h_{T(K + \varepsilon \cdot L)}(u)
\end{align*}
\]

Using the first variation formula \( \Phi_{\varphi,v\in L}(T^*u) \), we can easily see that Orlicz mixed affine quermassintegrals are invariant under volume preserving linear transportations.

**Proposition 2.6.** Let \( \varphi \in \mathcal{C} \), \( K, L \in \mathcal{K}^n_0 \), \( 1 \leq j \leq n \), and \( T \in SL(n) \). Then,
\[
\Phi_{\varphi,v\in L}(TK,TL) = \Phi_{\varphi,v\in L}(K,L).
\]

**Proof.** By Proposition 2.4 and Lemma 2.5 and the \( SL(n) \)-invariant of Lutwak’s affine quermassintegrals, we get that
\[
\Phi_{\varphi,v\in L}(TK,TL)^{-n} = \Phi_{\varphi,v\in L}(TK)^{-n-1} \frac{\varphi'(1^-)}{j} \left. \frac{d}{dx} \right|_{x=0^+} \Phi_{\varphi,v\in L}(TK + \varepsilon \cdot TL) \]
\[
= \Phi_{\varphi,v\in L}(K)^{-n-1} \frac{\varphi'(1^-)}{j} \left. \frac{d}{dx} \right|_{x=0^+} \Phi_{\varphi,v\in L}(K + \varepsilon \cdot TL) \]
\[
= \Phi_{\varphi,v\in L}(K,L)^{-n}.
\]

We close the section with a definition, that comes of by choosing \( \varphi(t) = t^p \) in (2.1).

**Definition 2.7.** The \( L_p \) mixed affine quermassintegrals of \( K, L \in \mathcal{K}^n_0 \), \( p \geq 1 \), are defined by
\[
\Phi_{p,v\in L}(K,L) := \frac{\omega_n}{\omega_j} \left( \int_{G_{n,j}} V_p^{(j)}(K|L) \, d\nu_{n,j}(\xi) \right)^{-\frac{1}{p}}. \tag{2.6}
\]

In particular, the mixed affine quermassintegrals of \( K, L \in \mathcal{K}^n_0 \) are defined by
\[
\Phi_{1,v\in L}(K,L) := \frac{\omega_n}{\omega_j} \left( \int_{G_{n,j}} V_1^{(j)}(K|L) \, d\nu_{n,j}(\xi) \right)^{-\frac{1}{p}}. \tag{2.7}
\]
3 Orlicz-Minkowski Inequality for Orlicz Mixed Affine Quermassintegrals.

In this section we prove an Orlicz-Minkowski inequality for the Orlicz mixed affine quermassintegrals. For its proof we use the Orlicz-Minkowski inequality [10] and Hölder inequality, which we quote here for the reader’s convenience (see [10, Theorem 189]).

**Theorem** (Hölder’s inequality). Let \( f, g : X \rightarrow [0, \infty] \) be measurable functions on a measure space \((X, \mu)\). For every \( p \neq 0 \) we consider \( p' \neq 0 \) such that \( \frac{1}{p} + \frac{1}{p'} = 1 \).

(i) If \( p \geq 1 \), then
\[
\int fg \, d\mu \leq \left( \int f^p \, d\mu \right)^{1/p} \left( \int g^{p'} \, d\mu \right)^{1/p'},
\]
with equality if and only if \( f^p \) and \( g^{p'} \) are proportional.

(ii) If \( 0 < p < 1 \) or \( p < 0 \), then
\[
\int fg \, d\mu \geq \left( \int f^p \, d\mu \right)^{1/p} \left( \int g^{p'} \, d\mu \right)^{1/p'},
\]
with equality if and only if \( f^p \) and \( g^{p'} \) are proportional, or \( fg \equiv 0 \).

**Theorem 3.1** (Orlicz-Minkowski inequality for Orlicz mixed affine quermassintegrals). Let \( K, L \in \mathcal{K}_n^o \), \( \varphi \in \mathcal{C} \), and \( 1 \leq j \leq n \). Then,
\[
\left( \frac{\Phi_{\varphi,n-j}(K,L)}{\Phi_{n-j}(K)} \right)^{-n} \geq \varphi \left( \frac{\Phi_{n-j}(L)}{\Phi_{n-j}(K)} \right)^{1/j}.
\]

If \( \varphi \) is strictly convex, then equality holds if and only if \( K \) and \( L \) are dilates of each other.

**Proof.** For every \( K \in \mathcal{K}_n^o \), we define the Borer probability measure \( \mu_{K,n,j} \) on \( G_{n,j} \) by
\[
d\mu_{K,n,j}(\xi) = \frac{\text{Vol}_j(K|\xi)^{-n}}{\Phi_{n-j}(K)^{-n}} \, d\nu_{n,j}(\xi).
\]

Then, Orlicz-Minkowski inequality [10], and Jensen inequality for \( \mu_{n,j} \), imply
\[
\left( \frac{\Phi_{\varphi,n-j}(K,L)}{\Phi_{n-j}(K)} \right)^{-n} = \int_{G_{n,j}} \frac{V_{\varphi}^{(j)}(K|\xi,L|\xi)}{\text{Vol}_j(K|\xi)} \, d\mu_{K,n,j}(\xi)
\geq \int_{G_{n,j}} \varphi \left( \frac{\text{Vol}_j(L|\xi)}{\text{Vol}_j(K|\xi)} \right)^{1/j} \, d\mu_{K,n,j}(\xi)
\geq \varphi \int_{G_{n,j}} \left( \frac{\text{Vol}_j(L|\xi)}{\text{Vol}_j(K|\xi)} \right)^{1/j} \, d\mu_{K,n,j}(\xi)
= \varphi \left( \int_{G_{n,j}} \frac{\text{Vol}_j(K|\xi)^{-\frac{j(n+1)}{j}}}{{\Phi_{n-j}(K)}^{-n}} \, d\nu_{n,j} \right)^{1/j}.
\]
We use Hölder inequality (3.2) on \( G_{n,j} \), with exponents
\[
p = \frac{jn}{jn + 1} \quad \text{and} \quad p' = -jn < 0.
\]
Taking into account that \( \phi \) is increasing, we get that
\[
\int_{G_{n,j}} \Vol_j(K|\xi)^{-\frac{jn+1}{jn}} \Vol_j(L|\xi)^{\frac{1}{jn}} d\nu_{n,j}
\geq \left( \int_{G_{n,j}} \Vol_j(K|\xi)^{-n} d\nu_{n,j} \right)^{\frac{1}{jn+1}} \left( \int_{G_{n,j}} \Vol_j(L|\xi)^{-n} d\nu_{n,j} \right)^{-\frac{1}{jn}}
= \Phi_{n-j}(K)^{-n} \Phi_{n-j}(L)^{\frac{1}{jn}}.
\]
Thus, by (3.4) we have
\[
\left( \frac{\Phi_{\phi,n-j}(K,L)}{\Phi_{n-j}(K)} \right)^{-n} \geq \phi \left( \frac{\Phi_{n-j}(K)^{-n} \Phi_{n-j}(L)^{\frac{1}{jn}}}{\Phi_{n-j}(K)^{-n}} \right)^{-\frac{1}{jn}} = \phi \left( \frac{\Phi_{n-j}(L)}{\Phi_{n-j}(K)} \right)^{1/j}.
\]

For the equality condition, note that if \( K, L \) are dilates of each other, then it can be easily checked that equality holds in (3.3). Conversely, if we assume that \( \phi \) is strictly convex and that equality holds in (3.3), then all inequalities in the above proof should hold as equalities. Thus, equality must holds in the Orlicz-Minkowski inequality for \( K|\xi \) and \( L|\xi \), \( \xi \in G_{n,j} \), and so we must have that for every \( \xi \in G_{n,j} \) there exist \( \lambda(\xi) > 0 \) such that
\[
L|\xi = \lambda(\xi) L|\xi \quad \forall \xi \in G_{n,j}.
\]
Moreover we must have equality in Jensen’s and Hölder’s inequalities. This implies that the positive functions \( f(\xi) := \Vol_j(K|\xi) \) and \( g(\xi) := \Vol_j(L|\xi) \), \( \xi \in G_{n,j} \), must be proportional to each other, i.e., there exists \( \lambda > 0 \) such that
\[
\Vol_j(L|\xi) = \lambda \Vol_j(K|\xi) \quad \forall \xi \in G_{n,j}.
\]
By (3.5) and (3.6) we conclude that \( L = \lambda K \).

Next uniqueness criterion follows directly from Theorem 3.1.

**Proposition 3.2.** Let \( \phi \in \mathcal{C} \) be strictly convex, \( 1 \leq j \leq n \), and \( K, L \in \mathcal{M}_n \subseteq K^* \). If
\[
\Phi_{\phi,n-j}(M,K) = \Phi_{\phi,n-j}(M,L), \quad \forall M \in \mathcal{M}^n
\]
or
\[
\frac{\Phi_{\phi,n-j}(K,M)}{\Phi_{n-j}(K)} = \frac{\Phi_{\phi,n-j}(L,M)}{\Phi_{n-j}(L)}, \quad \forall M \in \mathcal{M}^n,
\]
then \( K = L \).

**Proof.** First we suppose that (3.7) holds, and we take \( M = K \). Then by (3.3) we have
\[
\Phi_{n-j}(K)^{-n} = \Phi_{\phi,n-j}(K,L)^{-n} \geq \Phi_{n-j}(K)^{-n} \varphi \left( \frac{\Phi_{n-j}(L)}{\Phi_{n-j}(K)} \right)^{1/j},
\]

By (3.5) and (3.6) we conclude that \( L = \lambda K \). \( \square \)
with equality if and only if \( K \) and \( L \) are dilates of each other. Thus we have,

\[
\varphi \left( \left( \frac{\Phi_{n-j}(L)}{\Phi_{n-j}(K)} \right)^{1/j} \right) \leq 1,
\]

with equality if and only if \( K \) and \( L \) are dilates of each other. Since \( \varphi \) is increasing and \( \varphi(1) = 1 \), we get that \( \Phi_{n-j}(L) \leq \Phi_{n-j}(K) \), with equality if and only if \( K \) and \( L \) are dilates of each other. Under the light of \( j \)-homogeneity of the affine quermassintegrals, this means that

\[
\Phi_{n-j}(L) \leq \Phi_{n-j}(K),
\]

with equality if and only if \( K = L \). Similarly, by taking \( M = L \) in (3.7), we get

\[
\Phi_{n-j}(K) \leq \Phi_{n-j}(L),
\]

with equality if and only if \( K = L \). Thus, we must have \( \Phi_{n-j}(K) = \Phi_{n-j}(L) \) and \( K = L \). The same arguments also show that if (3.8) holds, then \( K = L \).

\[\square\]

4 Olricz-Brunn-Minkowski Inequality for Affine Quermassintegrals.

**Lemma 4.1.** Let \( K, L \in \mathcal{K}_n^0 \), \( 1 \leq j \leq n \), and \( \varphi \in \mathcal{C} \). Then, for every \( \varepsilon > 0 \)

\[
1 = \left( \frac{\Phi_{n-j}(K + \varepsilon \cdot L, K)}{\Phi_{n-j}(K + \varepsilon \cdot L)} \right)^n + \varepsilon \left( \frac{\Phi_{n-j}(K + \varepsilon \cdot L, L)}{\Phi_{n-j}(K + \varepsilon \cdot L)} \right)^{-n}.
\]

(4.1)

**Proof.** We first prove the following fact: For every \( A, B \in \mathcal{K}_j \), \( 1 \leq j \leq n \) and \( \varepsilon > 0 \) one has that

\[
V_\varphi(A + \varepsilon \cdot B, A) + \varepsilon V_\varphi(A + \varepsilon \cdot B, B) = V(A + \varepsilon \cdot B).
\]

Indeed, if \( A_\varphi := A + \varepsilon \cdot B \), then (4.1) and (4.2) imply that

\[
V_\varphi(A_\varphi, A) + \varepsilon V_\varphi(A_\varphi, B)
= \frac{1}{n} \int_{\mathbb{S}^{n-1}} \varphi \left( \frac{h(A, u)}{h(A_\varphi, u)} \right) h(A_\varphi, u) dS(A_\varphi, u)
+ \varepsilon \frac{1}{n} \int_{\mathbb{S}^{n-1}} \varphi \left( \frac{h(B, u)}{h(A_\varphi, u)} \right) h(A_\varphi, u) dS(A_\varphi, u)
= \frac{1}{n} \int_{\mathbb{S}^{n-1}} \varphi \left( \frac{h(A, u)}{h(A_\varphi, u)} \right) + \varepsilon \varphi \left( \frac{h(B, u)}{h(A_\varphi, u)} \right) h(A_\varphi, u) dS(A_\varphi, u)
= \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(A_\varphi, u) dS(A_\varphi, u) = V(A_\varphi).
\]

Thus, by setting \( K_\varphi = K + \varepsilon \cdot L \), we get

\[
\Phi_{n-j}(K_\varphi, K)^{-n} + \varepsilon \Phi_{n-j}(K_\varphi, L)^{-n}
= \left( \frac{\omega_n}{\omega_j} \right)^{-n} \int_{G_{n,j}} V_\varphi^{(j)}(K_\varphi, K) - \varepsilon V_\varphi^{(j)}(K_\varphi, L) \cdot \operatorname{Vol}_j(K_\varphi, \xi)^{-n-1} d\nu_{n,j}(\xi)
= \Phi_{n-j}(K_\varphi)^{-n}.
\]

\[\square\]
Theorem 4.2 (Orlicz-Brunn-Minkowski inequality for affine quermassintegrals). Let $K, L \in \mathcal{K}_o^n$, $1 \leq j \leq n$, $\varphi \in \mathcal{C}$, and $\varepsilon > 0$. Then,

$$1 \geq \varphi \left( \frac{\Phi_{n-j}(K)}{\Phi_{n-j}(K + \varphi \varepsilon \cdot L)} \right)^{1/j} + \varepsilon \varphi \left( \frac{\Phi_{n-j}(L)}{\Phi_{n-j}(K + \varphi \varepsilon \cdot L)} \right)^{1/j}. \quad (4.2)$$

If in addition, $\varphi$ is strictly convex, then equality holds if and only if $K$ and $L$ are dilates of each other.

Proof. By (4.1) and the Orlicz-Minkowski inequality (3.3), we have that

$$1 = \left( \frac{\Phi_{\varphi,n-j}(K + \varphi \varepsilon \cdot L)}{\Phi_{n-j}(K + \varphi \varepsilon \cdot L)} \right)^n + \varepsilon \left( \frac{\Phi_{\varphi,n-j}(K + \varphi \varepsilon \cdot L)}{\Phi_{n-j}(K + \varphi \varepsilon \cdot L)} \right)^{-n} \geq \varphi \left( \frac{\Phi_{n-j}(K)}{\Phi_{n-j}(K + \varphi \varepsilon \cdot L)} \right)^{1/j} + \varepsilon \varphi \left( \frac{\Phi_{n-j}(L)}{\Phi_{n-j}(K + \varphi \varepsilon \cdot L)} \right)^{1/j}.$$

Let now suppose that $\varphi$ is strictly convex. Then, by the equality conditions in the Orlicz-Minkowski inequality (3.3), and since $K, L \in \mathcal{K}_o^n$, we get that equality holds in (4.2) if and only if $K$ and $K + \varphi \varepsilon \cdot L$ are dilates of each other and $L$ and $L$ and $K + \varphi \varepsilon \cdot L$ are dilates of each other. Thus, equality holds if and only if $K$ and $L$ are dilates of each other.

In the last proof, we saw that Theorem 4.2 is a consequence of Theorem 3.1. Actually, those two inequalities are equivalent.

Proposition 4.3. Inequalities (4.2) and (3.3) are equivalent.

Proof. We only have to show that (4.2) implies (3.3). Indeed, by Proposition 2.4, the Orlicz-Brunn-Minkowski inequality (4.2), and Lemma 1.3, we have

$$\frac{j}{\varphi'(1^-)} \Phi_{n-j}(K)^{n+1} \Phi_{\varphi,n-j}(K, L)^{-n} = \lim_{\varepsilon \to 0^+} \frac{\Phi_{n-j}(K + \varphi \varepsilon \cdot L) - \Phi_{n-j}(K)}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{1 - \varphi \left( \frac{\Phi_{n-j}(K)}{\Phi_{n-j}(K + \varphi \varepsilon \cdot L)} \right)^{1/j}}{\varepsilon} \cdot \lim_{\varepsilon \to 0^+} \frac{1 - \varphi \left( \frac{\Phi_{n-j}(K)}{\Phi_{n-j}(K + \varphi \varepsilon \cdot L)} \right)^{1/j}}{1 - \varphi \left( \frac{\Phi_{n-j}(K)}{\Phi_{n-j}(K + \varphi \varepsilon \cdot L)} \right)^{1/j}} \cdot \lim_{\varepsilon \to 0^+} \Phi_{n-j}(K + \varphi \varepsilon \cdot L) $$

$$\geq \lim_{\varepsilon \to 0^+} \varphi \left( \frac{\Phi_{n-j}(L)}{\Phi_{n-j}(K + \varphi \varepsilon \cdot L)} \right)^{1/j} \cdot \lim_{t \to 1^-} \frac{1 - t}{1 - \varphi(t^{1/j})} \cdot \lim_{\varepsilon \to 0^+} \Phi_{n-j}(K + \varphi \varepsilon \cdot L) = \varphi \left( \frac{\Phi_{n-j}(L)}{\Phi_{n-j}(K)} \right)^{1/j} \cdot \frac{j}{\varphi'(1^-)} \cdot \Phi_{n-j}(K).$$
[3] W. Fenchel and B. Jessen. *Mengenfunktionen und konvexe Körper*. Danske Vid. Selskab. Mat.-fys. Medd. 16 (1938), 1-31.

[4] W. J. Firey. *Polar means of convex bodies and a dual to the Brunn-Minkowski theorem*. Canadian Journal of Mathematics 13 (1961), 444-453.

[5] W. J. Firey. *p-means of convex bodies*. Math. Scand. 10 (1962), 17-24.

[6] R. J. Gardner. *Geometric Tomography*, Encyclopedia of Mathematics and its Applications 58, Cambridge University Press, Cambridge 2nd edition (2006).

[7] R. J. Gardner, D. Hug & W. Weil. *The Orlicz-Brunn-Minkowski theory: A general framework, additions, and inequalities*. Journal of Differential Geometry 97 (2014) 427-476.

[8] R. J. Gardner, D. Hug & W. Weil. *Dual Orlicz-Brunn-Minkowski theory*. Advances in Mathematics 264 (2014), 700-725.

[9] E. L. Grinberg. *Isoperimetric inequalities and identities for k-dimensional cross-sections of a convex bodies*. Mathematische Annalen 291 (1991) no. 1, 75-86.

[10] G. Hardy, J. Littlewood & G. Plya. *Inequalities*. Cambridge University Press, Cambridge, 1934.

[11] D.Y. Li, D. Zou & G.Xiong. *Orlicz mixed affine quermassintegrals*. Sci China Math, 58 (2015) 1715-1722, doi 10.1007/s11425-014-4965-1.

[12] E. Lutwak. *A general isepiphanic inequality*, Proceedings of the American Mathematical Society 90 (1984), 415-421.

[13] E. Lutwak. *Intersection bodies and dual mixed volumes*, Advances in Mathematics 71 (1988), 232-261.

[14] E. Lutwak. *Dual mixed volumes*. Pacific Journal of Mathematics 58 (1975) 531-538.

[15] E. Lutwak. *Intersection bodies and dual mixed volumes*. Advances in Mathematics 71 (1988) 232-261.

[16] E. Lutwak. *The Brunn-Minkowski-Firey Theory I: Mixed volumes and the Minkowski Problem*. J. Differential Geom. 38 (1993), 131-150.

[17] E. Lutwak. *The Brunn-Minkowski-Firey theory II: affine and geominimal surface areas*. Advances in Mathematics 118 (1996) 244-294.

[18] E. Lutwak. *Extended affine surface area*. Advances in Mathematics 85 (1991), 39-68.

[19] E. Lutwak. *Inequalities for Hadwiger’s harmonic Quermassintegrals*. Mathematische Annalen 280 (1988), 165-175.

[20] E. Lutwak, D. Yang & G. Zhang. *Orlicz projection bodies*. Advances in Mathematics 223 (2010), 220-242.

[21] E. Lutwak, D. Yang & G. Zhang. *Orlicz centroid bodies*. Journal of Differential Geometry 84 (2010), 365-387.

[22] E. Lutwak, D. Yang, G. Zhang. *The Brunn-Minkowski-Firey inequality for non-convex sets*. Advances in Applied Mathematics 48 (2012), 407-413.

[23] G. Paouris & P. Pivovarov. *Small ball probabilities for the volume of random convex sets*. Discrete and Comp. Geom. 49 no. 3 (2013), 601-646.

[24] C. M. Petty. *Affine isoperimetric problems*. Ann. N.Y. Acad. Sci. 440 (1985), 113-127.

[25] C. M. Petty. *Geominimal surface area*. Geom. Dedicata 3 (1974), 77-97.

[26] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, Second Expanded Edition Encyclopedia of Mathematics and its applications, Cambridge University Press (2014).

[27] C. Schütt. *On the affine surface area*. Proc. Amer. Math. Soc. 118 (1993), 1213-1218.

[28] C. Schütt and E. Werner *The convex floating body*. Math. Scand. 66 (1990), 275-290.

[29] E. Werner. *Illumination bodies and affine surface area*. Studia Math. 110 (1994), 257-269.

[30] Chang-Jian Zhao. *Orlicz dual affine quermassintegrals*. Forum Mathematicum 2017, DOI: https://doi.org/10.1515/forum-2017-0174.

[31] B. Zhu, J. Zhou, W. Xu. *Dual Orlicz-Brunn-Minkowski theory*. Advances in Mathematics 264, (2014) 700-725.

Nikos Dafnis
Vienna University of Technology,
Institute of Discrete Mathematics and Geometry.
Wiedner Hauptstrasse 8-10, 1040 Vienna, Austria.
Email address: nikdafnis@gmail.com