Research Article

A Note on Reverse Minkowski Inequality via Generalized Proportional Fractional Integral Operator with respect to Another Function

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1. Introduction

Recently, the idea of nonlocal operators of differentiation has boarded out numerous analysts from practically all parts of sciences and engineering due to their abilities to include progressively complex characteristics into numerical conditions. Fractional calculus has also been comprehensively utilized in several instances, but the concept has been popularized and implemented in numerous disciplines of science, technology, and engineering as a mathematical model [1, 2]. Numerous distinguished generalized fractional integral operators consist of the Hadamard operator, Erdélyi–Kober operators, the Saigo operator, the Gaussian hypergeometric operator, the Marichev–Saigo–Maeda fractional integral operators, and so on, out of the which, the Riemann–Liouville fractional integral operator has been extensively utilized by researchers in theory as well as applications. For added information related to fractional calculus operators and their usefulness, one may also communicate to the expositions by Miller and Ross [3], Samko et al. [4], Kiryakova [5], and Baleanu et al. [6]. Almeida [7] proposed a new fractional derivative called Caputo derivative with respect to another function Φ, and Kilbas et al. [8] explored the concept of Riemann–Liouville fractional integrals with respect to another function Φ.

Within the structure of applied science and mathematical modeling, there exists an outstanding kind of operator known as generalized proportional fractional integral operator with respect to another function Φ in which the variable is a scaled according to proportionality index σ. This diversified operator was introduced by Rashid et al. [9], to conceivably role those physical problems for which classical physical law, for example, the well-known Mellin transform, Fourier transform, and probability theory, is suitable; such
physical issue is accepted to be founded on the fractional calculus and pertinent to the media of nonintegral fractional operators. Amongst others, we estimate real-world issues such as Porous media, aquifer, and turbulence; furthermore, progressively, other media regularly show fractional properties [10–22].

During the most recent decade, integral inequalities have been expanding enthusiasm to employ fractional techniques that have capacious significance to many fields, including neural networks, remote sensing, optimization of structures, optimization of electromagnetic systems, and many other applied sciences [23–33]. Lately, much consideration has been given to the fractional calculus of integral inequalities. We comment that fractional calculus is imperative for a few applications have been derived via the classical fractional operators [23–32].

Recently, fractional calculus of integral inequalities has been expanding enthusiasm to employ fractional techniques on reverse Minkowski inequality for generalized conformable fractional integral operators. In [46–48], many researchers have been focused on variants by contemplating Katugampola’s fractional techniques. In [45], the first fractional technique was employed on reverse Minkowski inequality in [43]. Lately, Anber et al. [44] have been introducing in this paper comprising exponential integral operators.

Additionally, Hardy’s type and reverse Minkowski inequalities were supplied by Bougoffa in [36]. The subsequent consequences concerning the reverse Minkowski inequalities are the inducement of labor finished to date, concerning the classical integrals.

**Theorem 1** (see [49]). Let $v \geq 1$, $Y \geq y > 0$, $y > x \geq 0$, and $G$ and $H$ be two positive functions defined on $[0, \infty)$ such that $y \leq (G(z)/H(z)) \leq Y$, for all $z \in [x, y]$. Then, one has

$$
\left( \int_{x}^{y} G^\sigma(\varphi)d\varphi \right)^{(1/\sigma)} + \left( \int_{x}^{y} H^\sigma(\varphi)d\varphi \right)^{(1/\sigma)} \leq \frac{1 + y (y + 2)}{(Y + 1) (y + 1)} \left( \int_{x}^{y} (G + H)^\sigma(\varphi)d\varphi \right)^{(1/\sigma)}.
$$

**Theorem 2** (see [49]). Let $v \geq 1$, $Y \geq y > 0$, $y > x \geq 0$, and $G$ and $H$ be two positive functions defined on $[0, \infty)$ such that $y \leq (G(z)/H(z)) \leq Y$, for all $z \in [x, y]$. Then, the inequality

$$
\left( \int_{x}^{y} G^\sigma(\varphi)d\varphi \right)^{(2/\sigma)} + \left( \int_{x}^{y} H^\sigma(\varphi)d\varphi \right)^{(2/\sigma)} \geq \left( \frac{(1 + y)(Y + 1)}{Y} - 2 \right) \left( \int_{x}^{y} G^\sigma(\varphi)d\varphi \right)^{(1/\sigma)} \left( \int_{x}^{y} H^\sigma(\varphi)d\varphi \right)^{(1/\sigma)},
$$

holds.

In [43], Dahmani used the Riemann–Liouville fractional integral operators to prove the subsequent reverse Minkowski inequalities.

**Theorem 3** (see [43]). Let $\delta > 0$, $v \geq 1$, $Y \geq y > 0$, $y > x \geq 0$, and $G$ and $H$ be two positive functions defined on $[0, \infty)$ such that $\mathcal{F}_{x}^{\delta} G^\sigma(\varphi) < \infty$ and $\mathcal{F}_{x}^{\delta} H^\sigma(\varphi) < \infty$, for all $\varphi > 0$. Then, the inequality

$$
\left( \mathcal{F}_{x}^{\delta} G^\sigma(\varphi) \right)^{(1/\sigma)} + \left( \mathcal{F}_{x}^{\delta} H^\sigma(\varphi) \right)^{(1/\sigma)} \leq \frac{1 + y (y + 2)}{(Y + 1) (y + 1)} \left( \mathcal{F}_{x}^{\delta} (G + H)^\sigma(\varphi) \right)^{(1/\sigma)},
$$

holds if $0 < y \leq (G(z)/H(z)) \leq Y$, for all $z \in [x, y]$.

**Theorem 4** (see [43]). Let $\delta > 0$, $v \geq 1$, $Y \geq y > 0$, $y > x \geq 0$, and $G$ and $H$ be two positive functions defined on $[0, \infty)$ such that $\mathcal{F}_{x}^{\delta} G^\sigma(\varphi) < \infty$ and $\mathcal{F}_{x}^{\delta} H^\sigma(\varphi) < \infty$, for all $\varphi > 0$. Then, the inequality

$$
\left( \mathcal{F}_{x}^{\delta} G^\sigma(\varphi) \right)^{(2/\sigma)} + \left( \mathcal{F}_{x}^{\delta} H^\sigma(\varphi) \right)^{(2/\sigma)} \geq \left( \frac{(1 + y)(Y + 1)}{Y} - 2 \right) \left( \mathcal{F}_{x}^{\delta} G^\sigma(\varphi) \right)^{(1/\sigma)} \left( \mathcal{F}_{x}^{\delta} H^\sigma(\varphi) \right)^{(1/\sigma)},
$$

takes place if $0 < y \leq (G(z)/H(z)) \leq Y$, for all $z \in [x, y]$.

Now, we present a new nonlocal fractional operator which is known as the generalized proportional fractional integral operator of a function with respect to another function $\Phi$ introduced by Rashid et al. [9].

**Definition 1** (see [9]). Let $\delta > 0$, $\sigma \in (0, 1]$, $x, y \in \mathbb{R}$ with $x < y$, and $\Phi$ be an increasing and positive monotone

2 Preliminaries

This segment is dedicated to some recognized definitions and outcomes associated with the generalized conformable fractional integral operators and their generalization related to the generalized conformable fractional integral operators. Set et al., in [49], launched the fractional version of the Hermite–Hadamard and reverse Minkowski inequality. Additionally, Hardy’s type and reverse Minkowski inequalities were supplied by Bougoffa in [36]. The subsequent consequences concerning the reverse Minkowski
function on \((x, y]\) such that \(\Phi'\) is continuous on \((x, y]\) and \(\Phi(0) = 0\). Then, the left and right generalized proportional fractional integral operators \((\mathcal{T}_{x,\Phi}^\delta,\mathcal{F})(\varphi)\) and \((\mathcal{T}_{y,\Phi}^\delta,\mathcal{F})(\varphi)\) of the function \(\mathcal{F}\) with respect to the function \(\Phi\) of order \(\delta > 0\) are defined by

\[
(\mathcal{T}_{x,\Phi}^\delta,\mathcal{F})(\varphi) = \frac{1}{\sigma^\delta \Gamma(\delta)} \int_x^\varphi \frac{\exp\left[\left(\frac{1}{\sigma} - 1\right)(\Phi(\varphi) - \Phi(z))\right]}{(\Phi(\varphi) - \Phi(z))^{1-\delta}} \mathcal{F}(z) \, dz, \quad x < \varphi, \tag{5}
\]

\[
(\mathcal{T}_{y,\Phi}^\delta,\mathcal{F})(\varphi) = \frac{1}{\sigma^\delta \Gamma(\delta)} \int_\varphi^y \frac{\exp\left[\left(\frac{1}{\sigma} - 1\right)(\Phi(z) - \Phi(\varphi))\right]}{(\Phi(z) - \Phi(\varphi))^{1-\delta}} \mathcal{F}(z) \, dz, \quad \varphi < y, \tag{6}
\]

respectively, where \(\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt\) is the Gamma function \([50–52]\).

Remark 1. Many fractional integral operators are the special cases of (5) and (6). For example,

\[
(\mathcal{T}_{x,\Phi}^\delta,\mathcal{F})(\varphi) = \frac{1}{\sigma^\delta \Gamma(\delta)} \int_x^\varphi \frac{\exp\left[\left(\frac{1}{\sigma} - 1\right)(\varphi - z)\right]}{(\varphi - z)^{1-\delta}} \mathcal{F}(z) \, dz, \quad x < \varphi, \tag{7}
\]

\[
(\mathcal{T}_{y,\Phi}^\delta,\mathcal{F})(\varphi) = \frac{1}{\sigma^\delta \Gamma(\delta)} \int_\varphi^y \frac{\exp\left[\left(\frac{1}{\sigma} - 1\right)(z - \varphi)\right]}{(z - \varphi)^{1-\delta}} \mathcal{F}(z) \, dz, \quad \varphi < y. \tag{8}
\]

(2) If \(\sigma = 1\), then (5) and (6) reduce to the left and right generalized Riemann–Liouville fractional integral operators introduced by Kilbas et al. \([8]\) as follows:

\[
(\mathcal{T}_{x,\Phi}^\delta,\mathcal{F})(\varphi) = \frac{1}{\Gamma(\delta)} \int_x^\varphi \frac{\Phi'(z)\mathcal{F}(z)}{(\Phi(\varphi) - \Phi(z))^{1-\delta}} \, dz, \quad x < \varphi, \tag{8}
\]

\[
(\mathcal{T}_{y,\Phi}^\delta,\mathcal{F})(\varphi) = \frac{1}{\Gamma(\delta)} \int_\varphi^y \frac{\mathcal{F}(z)\Phi'(z)}{(\Phi(z) - \Phi(\varphi))^{1-\delta}} \, dz, \quad \varphi < y. \tag{9}
\]

(3) Let \(\Phi(\varphi) = \ln \varphi\). Then, (5) and (6) become the left and right generalized proportional Hadamard fractional integral operators \([54]\):

\[
(\mathcal{T}_{x,\Phi}^\delta,\mathcal{F})(\varphi) = \frac{1}{\sigma^\delta \Gamma(\delta)} \int_x^\varphi \frac{\exp\left[\left(\frac{1}{\sigma} - 1\right)(\ln(\varphi/z))\right]}{(\ln(\varphi/z))^{1-\delta}} \mathcal{F}(z) \, dz, \quad x < \varphi, \tag{9}
\]

\[
(\mathcal{T}_{y,\Phi}^\delta,\mathcal{F})(\varphi) = \frac{1}{\sigma^\delta \Gamma(\delta)} \int_\varphi^y \frac{\exp\left[\left(\frac{1}{\sigma} - 1\right)(\ln(z/\varphi))\right]}{(\ln(z/\varphi))^{1-\delta}} \mathcal{F}(z) \, dz, \quad \varphi < y. \tag{9}
\]
(4) If \( \Phi(\phi) = \ln \phi \) and \( \sigma = 1 \), then, (5) and (6) lead to the left and right Hadamard fractional integral operators [8]:

\[
\mathcal{T}_x^\delta \mathcal{F}(\phi) = \frac{1}{\Gamma(\delta)} \int_x^\infty \frac{F(z_1)}{(z_1 - x)^{1-\delta}} \, dz_1, \quad x < \phi,
\]

\[
\mathcal{F}_y^\delta \mathcal{F}(\phi) = \frac{1}{\Gamma(\delta)} \int_y^\infty \frac{\mathcal{F}(z_1)}{(z_1 - y)^{1-\delta}} \, dz_1, \quad \phi < y.
\]  

(10)

(5) Let \( \Phi(\phi) = \phi \) and \( \sigma = 1 \). Then, (5) and (6) become the left and right Riemann–Liouville fractional integral operators:

\[
\mathcal{T}_x^\delta \mathcal{F}(\phi) = \frac{1}{\Gamma(\delta)} \int_x^\phi \frac{\mathcal{F}(z_1)}{(z_1 - x)^{1-\delta}} \, dz_1, \quad x < \phi,
\]

\[
\mathcal{F}_y^\delta \mathcal{F}(\phi) = \frac{1}{\Gamma(\delta)} \int_y^\phi \frac{\mathcal{F}(z_1)}{(z_1 - y)^{1-\delta}} \, dz_1, \quad \phi < y.
\]

(11)

3. Reverse Minkowski Inequalities via Generalized Proportional Fractional Integral Operator with respect to Another Function

This segment will consist of several generalizations by using generalized nonlocal proportional fractional integral operator with respect to another function \( \Phi \) to derive reverse Minkowski integral inequalities.

**Theorem 5.** Let \( \sigma \in (0, 1], \delta > 0, v \geq 1, Y \geq y > 0 \), and \( G \) and \( H \) be two positive functions defined on \([0, \infty)\) such that \( \mathcal{T}_{x, \Phi}^{\delta, \sigma} G^v(\phi) < \infty \) and \( \mathcal{F}_{x, \Phi}^{\delta, \sigma} H^v(\phi) < \infty \), for all \( \phi > 0 \), and \( \Phi \) be an increasing and positive function defined on \([0, \infty)\) such that \( \Phi' \) is continuous on \([0, \infty)\) and \( \Phi(0) = 0 \). Then,

\[
\left( \mathcal{T}_{x, \Phi}^{\delta, \sigma} G^v(\phi) \right)^{(1/v)} + \left( \mathcal{F}_{x, \Phi}^{\delta, \sigma} H^v(\phi) \right)^{(1/v)} \leq \left( \frac{1 + Y}{Y + 1} \right) \left( \mathcal{T}_{x, \Phi}^{\delta, \sigma} G^v(\phi) \right)^{(1/v)}.
\]  

(12)

if \( 0 < y \leq (G(z)/H(z)) \leq Y \), for all \( z \in [x, \phi] \subseteq [0, \infty) \).

**Proof.** It follows from \( (G(z)/H(z)) \) is bounded by \( y \) and \( v \) for \( z_1 \in [x, \phi] \) that

\[
(Y + 1)^v G^v(z_1) \leq Y^v (G + H)^v(z_1).
\]  

(13)

Multiplying both sides of (13) by

\[
\frac{1}{\sigma^\Gamma(\delta)} \exp\left([((\sigma - 1)/\sigma) (\Phi(\phi) - \Phi(z_1))]\Phi'(z_1) \right)
\]

and integrating with respect to \( z_1 \) on \((x, \phi)\), we obtain

\[
\frac{(Y + 1)^v}{\sigma^\Gamma(\delta)} \int_x^\phi \exp\left([((\sigma - 1)/\sigma) (\Phi(\phi) - \Phi(z_1))]\Phi'(z_1) \right) G^v(z_1) \, dz_1
\]

\[
\leq \frac{Y^v}{\sigma^\Gamma(\delta)} \int_x^\phi \exp\left([((\sigma - 1)/\sigma) (\Phi(\phi) - \Phi(z_1))]\Phi'(z_1) \right) (G + H)^v(z_1) \, dz_1.
\]  

(15)

which can be written as

\[
\mathcal{T}_{x, \Phi}^{\delta, \sigma} G^v(\phi) \leq \frac{Y^v}{(Y + 1)^v} \mathcal{T}_{x, \Phi}^{\delta, \sigma} (G + H)^v(\phi),
\]  

(16)

that is,

\[
\left( \mathcal{T}_{x, \Phi}^{\delta, \sigma} G^v(\phi) \right)^{(1/v)} \leq \frac{Y^v}{(Y + 1)^v} \left( \mathcal{T}_{x, \Phi}^{\delta, \sigma} (G + H)^v(\phi) \right)^{(1/v)}.
\]  

(17)

On the contrary, from \( Y H(z_1) \leq G(z_1) \), one has

\[
\left( 1 + \frac{1}{y} \right) H(z_1) \leq \left( 1 + \frac{1}{y} \right) (G(z_1) + H(z_1)),
\]  

(18)

which leads to

\[
\left( 1 + \frac{1}{y} \right)^v H^v(z_1) \leq \left( 1 + \frac{1}{y} \right)^v (G(z_1) + H(z_1))^v.
\]  

(19)

Multiplying both sides of (19) by

\[
\frac{1}{\sigma^\Gamma(\delta)} \exp\left([((\sigma - 1)/\sigma) (\Phi(\phi) - \Phi(z_1))]\Phi'(z_1) \right)
\]

and integrating with respect to \( z_1 \) on \((x, \phi)\), we obtain

\[
\left( \mathcal{T}_{x, \Phi}^{\delta, \sigma} G^v(\phi) \right)^{(1/v)} \leq \frac{1}{(Y + 1)^v} \left( \mathcal{T}_{x, \Phi}^{\delta, \sigma} (G + H)^v(\phi) \right)^{(1/v)}.
\]  

(21)

Adding inequalities (17) and (21) yields the desired inequality (12).

**Remark 2.** If \( \sigma = 1 \), then Theorem 5 leads to Theorem 3.1 in [47]. If \( \Phi(z_1) = z_1 \) and \( \sigma = 1 \), then Theorem 5 reduces to inequality (3). If \( \Phi(z_1) = z_1 \) and \( \delta = \sigma = 1 \), then Theorem 5 becomes inequality (1).
Theorem 6. Let $\sigma \in (0, 1]$, $\delta > 0$, $v \geq 1$, $Y \geq \gamma > 0$, $G$ and $H$ be two positive functions defined on $[0, \infty)$ such that $\mathcal{F}_{x, \eta}(G^\eta) < \infty$ and $\mathcal{F}_{x, \eta}(H^\eta) < \infty$, for all $\eta > 0$, and let $\Phi$ be an increasing and positive function defined on $[0, \infty)$ such that $\Phi'$ is continuous on $[0, \infty)$ and $\Phi(0) = 0$. Then,

$$\left( \mathcal{F}_{x, \sigma} G^\sigma(x) \right)^{2/\nu} + \left( \mathcal{F}_{x, \sigma} H^\sigma(x) \right)^{2/\nu} \geq \left( \frac{(Y + 1)(y + 1)}{Y} - 2 \right) \left( \mathcal{F}_{x, \sigma} G^\sigma(x) \right)^{1/\nu} \left( \mathcal{F}_{x, \sigma} H^\sigma(x) \right)^{1/\nu},$$

which complete the proof of Theorem 6.

Remark 3. If $\sigma = 1$, then Theorem 6 leads to Theorem 3.2 of [47]; if $\Phi(z_1) = z_1$ and $\sigma = 1$, then Theorem 6 reduces to inequality (4); if $\Phi(z_1) = z_1$ and $\delta = \sigma = 1$, then Theorem 6 becomes inequality (2).

4. Some Estimates for the Generalized Proportional Fractional Integral Operator with Respect to Another Function

This section is consisted to establishing several associated variants concerning to the generalized proportional fractional integral operator with respect to another function $\Phi$.

Theorem 7. Let $\sigma \in (0, 1]$, $\delta > 0$, $Y \geq \gamma > 0$, $v_1$, $v_2 > 1$ with $(1/v_1) + (1/v_2) = 1$, $G$ and $H$ be two positive functions defined on $[0, \infty)$ such that $\mathcal{F}_{x, \sigma} G^\eta < \infty$ and $\mathcal{F}_{x, \eta}(H^\eta) < \infty$ for $\eta > 0$, and $\Phi$ be an increasing and positive function defined on $[0, \infty)$ such that $\Phi'$ is continuous on $[0, \infty)$ and $\Phi(0) = 0$. Then, one has

$$\left( \mathcal{F}_{x, \sigma} G^\sigma(x) \right)^{1/v_1} \left( \mathcal{F}_{x, \sigma} H^\sigma(x) \right)^{1/v_2} \leq \left( \frac{Y}{\gamma} \right)^{1/v_1 v_2} \left( \mathcal{F}_{x, \sigma} G^\sigma(x) \right)^{1/v_1} \left( \mathcal{F}_{x, \sigma} H^\sigma(x) \right)^{1/v_2},$$

if $0 < y \leq (G(z)/H(z)) \leq Y$, for all $z \in [x, \varphi] \subseteq [0, \infty)$.

Proof. It follows from $(G(z_1)/H(z_1)) \leq Y$ for $z_1 \in [x, \varphi]$ that

$$H(z_1)^{1/v_2} \geq Y^{-(1/v_2)} (G(z_1))^{1/v_2}.$$ (28)

Multiplying both sides of (28) by $G^{1/v_2}(z_1)$ leads to

$$\left( G^{1/v_2}(z_1) \right) (H^{1/v_2}(z_1)) \geq Y^{-(1/v_2)} (G(z_1)).$$ (29)

Multiplying on both sides of (28) by

$$\frac{1}{\sigma^\Omega(\delta)} \exp\left( \frac{1}{\sigma^\Omega(\delta)} (\Phi(\Phi) - \Phi(z_1)) \right) \Phi'(z_1)^{1-\delta}$$ (30)

and integrating with respect to $z_1$ on $(x, \varphi)$, we obtain
\[ \frac{\gamma^{-1(1/\nu_1)}}{\sigma^d \Gamma(\delta)} \int_x^y \exp\left[\frac{(\sigma - 1)/\sigma (\Phi(\varphi) - \Phi(\varphi_1))}{\Phi'(\varphi_1)} (G(z_1)) dz_1 \right] \]

\[ \leq \frac{1}{\sigma^d \Gamma(\delta)} \int_x^y \exp\left[\frac{(\sigma - 1)/\sigma (\Phi(\varphi) - \Phi(\varphi_1))}{\Phi'(\varphi_1)} (G^{1/\nu_1}(z_1)) (H^{1/\nu_2}(z_1)) dz_1. \right] \]

Inequality (31) can be written as

\[ \gamma^{-1(1/\nu_1)} [\mathcal{G}_{\delta, \sigma}^d G(\varphi)]^{1/\nu_1} \leq \left[ \mathcal{G}_{\delta, \sigma}^d [G(\varphi)]^{1/\nu_1} [H(\varphi)]^{1/\nu_2} \right]^{1/\nu_1}. \]

(32)

On the contrary, \( \gamma H(\varphi_1) \leq G(\varphi_1) \) leads to

\[ \gamma^{1/\nu_1} H^{1/\nu_1}(z_1) \leq G^{1/\nu_1}(z_1). \]

(33)

Multiplying on both sides of (33) by \( H^{1/\nu_2}(z_1) \) and using the identity \( v_1^{-1} + v_2^{-1} = 1 \), we have

\[ \gamma^{1/\nu_1} \left( \mathcal{G}_{x, \delta}^d (\Phi(\varphi_1)) G^{1/\nu_2}(\varphi) \right)^{1/\nu_1}. \]

(36)

Inequality (36) leads to

\[ \gamma^{1/\nu_1}(\mathcal{G}_{x, \delta}^d H(\varphi))^{1/\nu_1} \leq \left( \mathcal{G}_{x, \delta}^d H^{1/\nu_1}(\varphi) G^{1/\nu_2}(\varphi) \right)^{1/\nu_1}. \]

(37)

which completes the proof of inequality (27).

\[ \square \]

**Theorem 8.** Let \( \sigma \in (0, 1], \delta > 0, \gamma \geq 0, \nu_1, \nu_2 > 1 \) with \( (1/\nu_1) + (1/\nu_2) = 1, G \) and \( H \) be two positive functions defined on \( [0, \infty) \) such that \( \mathcal{G}_{x, \delta}^d G(\varphi) < \infty \) and \( \mathcal{G}_{x, \delta}^d H(\varphi) < \infty \) for \( \varphi > 0 \), and \( \Phi \) be an increasing and positive function defined on \([0, \infty)\) such that \( \Phi' \) is continuous on \([0, \infty)\) and \( \Phi(0) = 0 \). Then, one has

\[ \mathcal{G}_{x, \delta}^d (G(\varphi)H(\varphi)) \leq \frac{2^{\nu_1 - 1} \gamma^{\nu_1}}{v_1 (Y + 1)^{\nu_1}} \mathcal{G}_{x, \delta}^d (G^{\nu_1} + H^{\nu_1})(\varphi) + \frac{2^{\nu_2 - 1}}{v_2 (Y + 1)^{\nu_2}} \mathcal{G}_{x, \delta}^d (G^{\nu_2} + H^{\nu_2})(\varphi), \]

(39)

if \( 0 < \gamma \leq (G(z)/H(z)) \leq Y, \) for all \( z \in [x, \varphi] \subseteq [0, \infty) \).

**Proof.** By the given conditions, we have the following inequality:

\[ (Y + 1)^{\nu_1} G^{\nu_1}(z_1) \leq Y^{\nu_1} (G + H)^{\nu_1}(z_1). \]

(40)

Multiplying both sides of (40) by

\[ \frac{1}{\sigma^d \Gamma(\delta)} \exp\left[\frac{(\sigma - 1)/\sigma (\Phi(\varphi) - \Phi(\varphi_1))}{\Phi'(\varphi_1)} (G^{1/\nu_1}(z_1)) (H^{1/\nu_2}(z_1)) dz_1. \right] \]

and integrating with respect to \( z_1 \) on \( (x, \varphi) \) lead to
\[
\frac{(Y + 1)^{\nu_1}}{\sigma^F(\delta)} \int_x^y \exp \left[ \frac{((\sigma - 1)/\sigma)(\Phi(\varphi) - \Phi(z_1))}{(\Phi(\varphi) - \Phi(z_1))^{1-\delta}} \right] \Phi'(z_1) G^{\nu_1}(z_1) dz_1 \\
\leq \frac{Y^{\nu_1}}{\sigma^F(\delta)} \int_x^y \exp \left[ \frac{((\sigma - 1)/\sigma)(\Phi(\varphi) - \Phi(z_1))}{(\Phi(\varphi) - \Phi(z_1))^{1-\delta}} \right] (G + H)^{\nu_1}(z_1) dz_1.
\]
(42)

Inequality (42) can be written as
\[
\mathcal{T}_{x,\Phi} \varphi^{\nu_1}(\varphi) \leq \frac{Y^{\nu_1}}{(Y + 1)^{\nu_1}} \mathcal{T}_{x,\Phi} (G + H)^{\nu_1}(\varphi). \tag{43}
\]

On the contrary, it follows from \((G(z_1)/H(z_1)) > Y\) that
\[
(Y + 1)^{\nu_1}H^{\nu_1}(z_1) \leq (G + H)^{\nu_1}(z_1). \tag{44}
\]

Multiplying on both sides of (44) by
\[
\frac{1}{\sigma^F(\delta)} \exp \left[ \frac{((\sigma - 1)/\sigma)(\Phi(\varphi) - \Phi(z_1))}{(\Phi(\varphi) - \Phi(z_1))^{1-\delta}} \right] \Phi'(z_1) \tag{45}
\]
and integrating with respect to \(z_1\) on \((x, \varphi)\), we obtain
\[
\mathcal{T}_{x,\Phi} \varphi^{\nu_1}(\varphi) \leq \frac{1}{(Y + 1)^{\nu_1}} \mathcal{T}_{x,\Phi} (G + H)^{\nu_1}(\varphi). \tag{46}
\]

The well-known Young's inequality states that
\[
\frac{G^{\nu_1}(z_1)}{v_1} + \frac{H^{\nu_1}(z_1)}{v_2} \geq G(z_1)H(z_1). \tag{47}
\]

Multiplying both sides of (47) with
\[
\frac{1}{v_1} \exp \left[ \frac{((\sigma - 1)/\sigma)(\Phi(\varphi) - \Phi(z_1))}{(\Phi(\varphi) - \Phi(z_1))^{1-\delta}} \right] \Phi'(z_1) \tag{48}
\]
and integrating with respect to \(z_1\) on \((x, \varphi)\) give
\[
\frac{1}{v_1} \left( \mathcal{T}_{x,\Phi} \varphi^{\nu_1}(\varphi) \right) + \frac{1}{v_2} \left( \mathcal{T}_{x,\Phi} \varphi^{\nu_1}(\varphi) \right) \geq \mathcal{T}_{x,\Phi} (G(\varphi)H(\varphi)). \tag{49}
\]

From (43), (46), and (49), we clearly see that
\[
\mathcal{T}_{x,\Phi} \varphi^{\nu_1}(\varphi) \leq \frac{1}{v_1} \left( \mathcal{T}_{x,\Phi} \varphi^{\nu_1}(\varphi) \right) + \frac{1}{v_2} \left( \mathcal{T}_{x,\Phi} \varphi^{\nu_1}(\varphi) \right). \tag{50}
\]

Making use of the inequality \((a_1 + a_2)^n \leq 2^{n-1}(a_1^n + a_2^n)\), for \(a_1, a_2 > 0\) and \(q > 1\), we can obtain
\[
\mathcal{T}_{x,\Phi} (G + H)^{\nu_1}(\varphi) \leq 2^{\nu_1 - 1} \mathcal{T}_{x,\Phi} (G^{\nu_1} + H^{\nu_1})(\varphi). \tag{51}
\]
\[
\mathcal{T}_{x,\Phi} (G + H)^{\nu_1}(\varphi) \leq 2^{\nu_1 - 1} \mathcal{T}_{x,\Phi} (G^{\nu_1} + H^{\nu_1})(\varphi). \tag{52}
\]

Therefore, inequality (39) follows easily from inequalities (50)–(52).

**Theorem 9.** Let \(\sigma \in (0, 1], \delta > 0, Y \geq y > 0, \nu \geq 1, G \) and \(H\) be two positive functions defined on \([0, \infty)\) such that \(\mathcal{T}_{x,\Phi} G(\varphi) < \infty\) and \(\mathcal{T}_{x,\Phi} H(\varphi) < \infty\) for \(\varphi > 0\), and \(\Phi\) be an increasing and positive function defined on \([0, \infty)\) such that \(\Phi'\) is continuous on \([0, \infty)\) and \(\Phi(0) = 0\). Then, one has
\[
Y + \left( \frac{(\mathcal{T}_{x,\Phi} \varphi^{\nu_1}(\varphi))^{1/\nu}}{1 - \omega} \right)^{1/\nu} \leq \left( \frac{\mathcal{T}_{x,\Phi} G(\varphi - \omega H(\varphi))^{1/\nu}}{1 - \omega} \right)^{1/\nu} \tag{53}
\]
\[
\leq \frac{Y + 1}{Y - \omega} \left( \frac{\mathcal{T}_{x,\Phi} \varphi^{\nu_1}(\varphi)}{1 - \omega} \right)^{1/\nu}, \tag{54}
\]
if \(0 < \omega < \gamma \leq (G(z_1)/H(z_1)) \leq Y\), for all \(z_1 \in [x, \varphi] \leq [0, \infty).

**Proof.** It follows from \(0 < \omega < \gamma \leq (G(z_1)/H(z_1)) \leq Y\) that
\[
y \omega \leq Y \omega, \tag{55}
\]
\[
y \omega + \gamma \leq y \omega + Y \omega \leq Y \omega + Y, \tag{56}
\]
\[
(Y + 1)(\gamma - \omega) \leq (Y + 1)(Y - \omega), \tag{57}
\]
\[
\frac{Y + 1}{Y - \omega} \leq \frac{y + 1}{y - \omega}, \tag{58}
\]
\[
\gamma - \omega \leq \frac{G(z_1) - \omega H(z_1)}{H(z_1)} \leq Y - \omega, \tag{59}
\]
\[
\frac{(G(z_1) - \omega H(z_1))^\nu}{(Y - \omega)^\nu} \leq H(z_1) \leq \frac{(G(z_1) - \omega H(z_1))^\nu}{(\gamma - \omega)^\nu}, \tag{60}
\]
\[
\frac{1}{Y} \leq \frac{H(z_1)}{G(z_1)} \leq \frac{1}{\gamma}, \tag{61}
\]
\[
\frac{\gamma - \omega}{\omega} \leq \frac{G(z_1) - \omega H(z_1)}{\omega G(z_1)} \leq \frac{Y - \omega}{\omega Y}, \tag{62}
\]
\[
\left( \frac{Y}{Y - \omega} \right)^\nu \frac{G(z_1) - \omega H(z_1)}{\omega G(z_1)} \leq \left( \frac{\gamma}{\gamma - \omega} \right)^\nu (G(z_1) - \omega H(z_1))^\nu, \tag{63}
\]
(55)
Proof. It follows from the conditions given in Theorem 10 that
\[
1 \leq \frac{1}{H(z_1)} \leq \frac{1}{\mathcal{M}}.
\]
(62)

Inequality (62) and 0 ≤ κ ≤ G(z_1) ≤ \mathcal{K} lead to the conclusion that
\[
0 < \kappa \leq \frac{G(z_1)}{\mathcal{K}} \leq \frac{G(z_1)}{\mathcal{M}}.
\]
(63)
From (63), we clearly see that
\[
H^\nu(z_1) \leq \left(\frac{\mathcal{M}}{\mathcal{K} + \mathcal{M}}\right)^\nu (G(z_1) + H(z_1))^\nu,
\]
(64)
and
\[
G^\nu(z_1) \leq \left(\frac{\mathcal{K}}{\mathcal{K} + \mathcal{M}}\right)^\nu (G(z_1) + H(z_1))^\nu.
\]
(65)

Multiplying both sides of (64) with
\[
\frac{1}{\sigma^\delta(\delta)} \exp\left(\frac{(\sigma - 1)/\sigma)\left(\Phi(\varphi) - \Phi(z_1)\right)\Phi'(z_1)}{\Phi(\varphi) - \Phi(z_1)}\right)^{1-\delta}
\]
by
\[
\frac{1}{Y - \omega} \sigma^\delta(\delta) \int_x^y \exp\left(\frac{(\sigma - 1)/\sigma)\left(\Phi(\varphi) - \Phi(z_1)\right)\Phi'(z_1)}{\Phi(\varphi) - \Phi(z_1)}\right)^{1-\delta}
\]
and integrating with respect to z_1 on (x, \varphi) lead to
\[
1 \leq \frac{1}{\sigma^\delta(\delta)} \int_x^y \exp\left(\frac{(\sigma - 1)/\sigma)\left(\Phi(\varphi) - \Phi(z_1)\right)\Phi'(z_1)}{\Phi(\varphi) - \Phi(z_1)}\right)^{1-\delta}
\]
(56)
and integrating with respect to \( z_1 \) on \((x, \varphi)\), we obtain

\[
\left( \frac{\mathcal{T}_{x, \varphi} G^\nu (\varphi) \left( \mathcal{T}_{x, \varphi} (G(z_1) + H(z_1)) \right)^{1/\nu}}{\mathcal{M}} \right)^{1/\nu} \leq \left( \frac{\mathcal{T}_{x, \varphi} G^\nu (\varphi) \left( \mathcal{T}_{x, \varphi} (G(z_1) + H(z_1)) \right)^{1/\nu}}{\mathcal{M} + \mathcal{M}} \right)^{1/\nu}.
\]

Therefore, inequality (61) follows from (67) and (68).

\[\square\]

**Theorem 11.** Let \( \sigma \in (0, 1], \delta > 0, \nu \geq 1, 0 < \gamma \leq Y, G \) and \( H \) be two positive functions defined on \([0, \infty)\) such that \( \mathcal{T}_{x, \varphi} G^\nu (\varphi) < \infty \) and \( \mathcal{T}_{x, \varphi} H^\nu (\varphi) < \infty \) for all \( \varphi > 0 \), and \( \Phi \) be an increasing and positive function defined on \([0, \infty)\) such that \( \Phi' \) is continuous on \([0, \infty)\) and \( \Phi(0) = 0 \). Then, the double inequality

\[
\frac{1}{Y \sigma^\Omega (\delta)} \int_x^Y \exp\left[\left(\frac{(z - 1)}{\nu} (\Phi(\varphi) - \Phi(z_1)) \right] \Phi'(z_1) G(z_1) H(z_1) dz_1
\]

\[
\leq \frac{1}{(y + 1)(Y + 1) \sigma^\Omega (\delta)} \int_x^Y \exp\left[\left(\frac{(z - 1)}{\nu} (\Phi(\varphi) - \Phi(z_1)) \right] \Phi'(z_1) (G(z_1) + H(z_1))^{1/\nu} dz_1
\]

\[
\leq \frac{1}{\gamma \sigma^\Omega (\delta)} \int_x^Y \exp\left[\left(\frac{(z - 1)}{\nu} (\Phi(\varphi) - \Phi(z_1)) \right] \Phi'(z_1) G(z_1) H(z_1) dz_1.
\]

Inequality (74) can be rewritten as

\[
\frac{\mathcal{T}_{x, \varphi} G^\nu (\varphi) H(\varphi)}{Y} \leq \frac{\mathcal{T}_{x, \varphi} (G + H)^\nu (\varphi)}{(y + 1)(Y + 1)} \leq \frac{\mathcal{T}_{x, \varphi} G^\nu (\varphi) H(\varphi)}{\gamma}.
\]

\[\square\]

**Theorem 12.** Let \( \sigma \in (0, 1], \delta > 0, \nu \geq 1, 0 < \gamma \leq Y, G \) and \( H \) be two positive functions defined on \([0, \infty)\) such that \( \mathcal{T}_{x, \varphi} G^\nu (\varphi) < \infty \) and \( \mathcal{T}_{x, \varphi} H^\nu (\varphi) < \infty \) for all \( \varphi > 0 \), and \( \Phi \) be an increasing and positive function defined on \([0, \infty)\) such that \( \Phi' \) is continuous on \([0, \infty)\) and \( \Phi(0) = 0 \). Then, the inequality

\[
\left( \frac{\mathcal{T}_{x, \varphi} G^\nu (\varphi)}{\mathcal{M} + \mathcal{M}} \right)^{1/\nu} \leq \left( \frac{\mathcal{T}_{x, \varphi} G^\nu (\varphi) H(\varphi)}{\gamma} \right)^{1/\nu} \leq \left( \frac{\mathcal{T}_{x, \varphi} G^\nu (\varphi) H(\varphi)}{\gamma} \right)^{1/\nu}
\]

holds if \( \gamma \leq (G(z_1)/H(z_1)) \leq Y \) for all \( z_1 \in [x, \varphi] \), where

\[
\Delta(G(\varphi), H(\varphi)) = \max\left\{ Y \left[ \frac{Y}{Y + 1}G(z_1) - \frac{Y}{Y + 1}H(z_1) \right], \frac{(y + 1)H(z_1) - G(z_1)}{y} \right\}.
\]

\[\square\]
From (80) and (85), we clearly see that
\[
H(z_1) < \frac{(Y + \gamma)H(z_1) - G(z_1)}{\gamma} \leq \Delta(G(\varphi), H(\varphi)),
\]  
where
\[
\Delta(G(\varphi), H(\varphi)) = \max \left\{ \frac{Y}{2} \left[ \frac{Y}{Y + 1} G(z_1) - \frac{1}{Y} Y H(z_1) \right], \frac{(Y + \gamma)H(z_1) - G(z_1)}{\gamma} \right\}. \tag{81}
\]

Similarly, from \(0 < (1/Y) \leq (H(z_1)/G(z_1)) \leq (1/\gamma),\) we have
\[
\frac{1}{Y} \leq \frac{1}{Y + 1} G(z_1) - \frac{1}{Y} Y H(z_1) \leq \frac{1}{Y} \leq \frac{1}{Y + 1} G(z_1) - \frac{1}{Y} Y H(z_1) \tag{82}
\]
and
\[
G(z_1) = \frac{Y}{Y + 1} G(z_1) - Y H(z_1) = \frac{Y(Y + \gamma)G(z_1) - Y^2 \gamma H(z_1)}{\gamma Y} = \frac{Y}{Y + 1} G(z_1) - Y H(z_1) \tag{85}
\]
\[
\leq \frac{Y}{Y + 1} G(z_1) - Y H(z_1) \leq \Delta(G(\varphi), H(\varphi)).
\]

From (80) and (85), we clearly see that
\[
\Gamma^\alpha(z_1) \leq \Delta^\alpha(G(\varphi), H(\varphi)), \tag{86}
\]
\[
\Gamma^\sigma(z_1) \leq \Delta^\sigma(G(\varphi), H(\varphi)). \tag{87}
\]

Multiplying both sides of (86) with
\[
\int_x^\varphi \frac{1}{\sigma^\gamma(\delta)} \frac{\exp\left\{ ((\sigma - 1)/\sigma)(\Phi(\varphi) - \Phi(z_1)) \right\} \Phi'(z_1)}{(\Phi(\varphi) - \Phi(z_1))^{1-\delta}} G^\alpha(z_1)dz_1 \leq \frac{1}{\sigma^\gamma(\delta)} \tag{88}
\]
and integrating with respect to \(z_1\) on \(x, \varphi\), we obtain
\[
\int_x^\varphi \frac{1}{\sigma^\gamma(\delta)} \frac{\exp\left\{ ((\sigma - 1)/\sigma)(\Phi(\varphi) - \Phi(z_1)) \right\} \Phi'(z_1)}{(\Phi(\varphi) - \Phi(z_1))^{1-\delta}} \Delta^\alpha(G(\varphi), H(\varphi))dz_1.
\]

Inequality (89) can be written as
\[
(\mathcal{S}_{x,\Phi}^{\delta,\sigma} G^\alpha(\varphi))^{1/\sigma} \leq (\mathcal{S}_{x,\Phi}^{\delta,\sigma} \Delta^\sigma G(\varphi), H(\varphi))^{1/\sigma}. \tag{90}
\]

Similarly, from (87), we obtain
\[
(\mathcal{S}_{x,\Phi}^{\delta,\sigma} H^\sigma(\varphi))^{1/\sigma} \leq (\mathcal{S}_{x,\Phi}^{\delta,\sigma} \Delta^\sigma G(\varphi), H(\varphi))^{1/\sigma}. \tag{91}
\]

Therefore, inequality (76) follows easily from (90) and (91).

\section{Conclusion}

In this paper, we introduce a nonlocal generalized proportional fractional integral operator with respect to another function \(\Phi\), and then we derived several variants concerning to the reverse Minkowski inequality by involving the generalized proportional fractional integral operator with respect to another function \(\Phi\); as a particular case, the inequality involving fractional integrals in the
Riemann–Liouville, Hadamard, and Katugampola sense can be found by choosing appropriate and suitable substitutions in the proportionality index $\sigma$ and $\Phi$. The variants obtained in this research will lead to the inequalities which are established earlier by Rahman et al. [47] and numerous outcomes can be generalized for the application of these newly introduced fractional integral operators by utilizing Remark 1. Note that the outcomes in this paper are like hypothetically surely understood proliferation properties of fractional Schrödinger equation [55, 56]. Besides, our outcomes are practically identical to equality-time evenness in a fractional Schrödinger equation [57]. Indeed, the work established in the given arrangement is new and contributes suggestively to the study of integrodifferential and difference equations.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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