Blurred maximal cyclically monotone sets and bipotentials

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Abstract
Let \( X \) be a reflexive Banach space and \( Y \) its dual. In this paper we find necessary and sufficient conditions for the existence of a bipotential for a blurred maximal cyclically monotone set. Equivalently, we find a necessary and sufficient condition on \( \phi \in \Gamma_0(X) \) for that the differential inclusion \( y \in B(\varepsilon) + \partial \phi(x) \) can be put in the form \( y \in \partial b(\cdot, y)(x) \), with \( b \) a bipotential.

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1 Introduction
Let \( \phi \in \Gamma_0(X) \), where \( X \) is a reflexive Banach space, with dual \( Y \), and \( \varepsilon > 0 \). We want to describe the set \( M(\phi, \varepsilon) \) of solutions \( (x, y) \in X \times Y \) of the following problem:

\[
\text{there is } a \in Y, \|a\| \leq \varepsilon \text{ such that } y + a \in \partial \phi(x) \quad (1.0.1)
\]

Our main result are proposition 6.1 and theorem 6.2 which imply the following (for the notion of bipotential see definition 2.1).

Corollary 1.1 If for any \( y \in Y \) the set \( \bigcup_{\|\bar{y} - y\| \leq \varepsilon} \partial \phi^* (\bar{y}) \) is convex then there exists a bipotential \( b : X \times Y \to [\bar{R}] \) such that \( M(\phi, \varepsilon) = M(b) \). Therefore, there is a lower semicontinuous and convex in each variable function \( b \), such that any of the following relations

(a) \( y \in \partial b(\cdot, y)(x) \),
(b) \( x \in \partial b(x, \cdot)(y) \),
(c) \( b(x, y) = \langle x, y \rangle \),

is equivalent with \( (x, y) \in M(\phi, \varepsilon) \).
Proof. Let \( M = \text{Graph}(\partial \phi) \) and \( A = \{0\} \times \overline{B}_Y(\varepsilon) \). Then \( M(\phi, \varepsilon) = M + A \). By proposition \([22]\) in the hypothesis of the corollary (which will appear further as relation \((1.0.1)\)) the set \( M + A \) is a BB-graph, therefore by theorem \( 3.1 \) there exists a bipotential \( b \) (definition \( 2.1 \)) such that

\[
M + A = M(b) = \{(x, y) \in X \times Y : b(x, y) = \langle x, y \rangle\}
\]

therefore \((1.0.1)\) is equivalent also with \( y \in \partial b(\cdot, y)(x) \), or with \( x \in \partial b(x, \cdot)(y) \). \( \Box \)

**Corollary 1.2** If the function \( b \) defined by

\[
b(x, y) = \phi(x) + \inf_{[a] \leq \varepsilon} \left[ \phi^*(y - a) + \langle x, a \rangle \right] \tag{1.0.2}
\]

is convex in the first variable, then \( b \) is a bipotential such that \( M(b) = M(\phi, \varepsilon) \).

**Proof.** If \( b \) defined by \((1.0.2)\) is convex then the function \( f : \overline{B}(\varepsilon) \times X \times Y \to \mathbb{R} \) defined by

\[
f(a, x, y) = \phi(x) + \phi^*(y - a) + \langle x, a \rangle
\]

is implicitly convex in the first two arguments, in the sense of definition \( 3.1 \). Indeed, suppose that for any \( y \in Y \), for any \( x_1, x_2 \in X \) and any \( \alpha, \beta \in [0, 1], \alpha + \beta = 1 \), such that \( b(x_i, y) < +\infty \), \( i = 1, 2 \), we have

\[
b(\alpha x_1 + \beta x_2, y) \leq \alpha b(x_1, y) + \beta b(x_2, y)
\]

The function \( f(\cdot, \alpha x_1 + \beta x_2, y) \) is weakly lower semicontinuous and \( \overline{B}(\varepsilon) \) is weakly compact, therefore there exists \( \bar{a} \in \overline{B}(\varepsilon) \) such that

\[
f(\bar{a}, \alpha x_1 + \beta x_2, y) = b(\alpha x_1 + \beta x_2, y)
\]

But then for any \( a_1, a_2 \in \overline{B}(\varepsilon) \) we have

\[
f(\bar{a}, \alpha x_1 + \beta x_2, y) \leq \alpha f(a_1, x_1, y) + \beta f(a_2, x_2, y)
\]

which proves the previous claim. We use one implication from theorem \( 6.2 \) in order to finish the proof. \( \Box \)

**Motivation of the problem \((1.0.1)\).** Recently \([7]\) we have found a new application of the bipotential method to blurred constitutive laws. This application led us to the mathematical problem \((1.0.1)\) of describing blurred maximal cyclically monotone sets by bipotentials.

The notion of bipotential (definition \( 2.1 \)) has been introduced in \([20]\), in order to formulate a large family of non associated constitutive laws in terms of convex analysis. The basic idea is explained further in few words. In Mechanics the associate constitutive laws are simply relations \( y \in \partial \phi(x) \), with \( \phi : X \to \mathbb{R} \cup \{+\infty\} \) a convex and lower semicontinuous function. By Fenchel inequality such a relation is equivalent with \( \phi(x) + \phi^*(y) = \langle x, y \rangle \), where \( \phi^* \) is the Fenchel conjugate of \( \phi \). It has been noticed that often in the mathematical study of problems related to associated constitutive laws enters not the function \( \phi \), but the expression

\[
b(x, y) = \phi(x) + \phi^*(y)
\]

which we call "separable bipotential". The idea is then to use as a basic notion the one of bipotential \( b : X \times Y \to \mathbb{R} \cup \{+\infty\} \), which is convex and lsc in each argument and satisfies a generalization of the Fenchel inequality. To non associated constitutive laws thus corresponds bipotentials which are not separable.

Examples of such laws which can be studied with the help of bipotentials are: non-associated Dr"ucker-Prager \([22]\) and Cam-Clay models \([23]\) in soil mechanics, cyclic Plasticity \((21, 2)\) and...
Viscoplasticity \cite{11} of metals with non-linear kinematical hardening rule, Lemaitre’s damage law \cite{1}, the coaxial laws (\cite{24}, \cite{26}), the Coulomb’s friction law \cite{20}, \cite{21}, \cite{3}, \cite{8}, \cite{9}, \cite{12}, \cite{22}, \cite{25}, \cite{14}. A complete survey can be found in \cite{24}.

Later we started in \cite{4}, \cite{5}, \cite{6} a mathematical study of bipotentials and their relation with convex analysis. This paper is another contribution along this subject.

Blurred constitutive laws (in mathematical terms blurred graphs of multivalued operators) appear in many practical situations, due either to experimental or numerical indeterminacies \cite{15}, \cite{16}, \cite{13}, \cite{10}, \cite{17}. It is then interesting to take the indeterminacy into account and to associate, for example, to a differential relation like \( y \in \partial \phi (x) \) another differential relation \( y \in \partial \Phi (\cdot , y)(x) \), where \( \Phi \) is a bipotential constructed from \( \phi \) and the indeterminacy.

We achieve this in the paper, by using lagrangian convex covers introduced in \cite{4}. For future study is left the more general problem of the existence and construction of a bipotential for a blurred graph of a multivalued operator which can be expressed by a bipotential. Such operators may be monotone, but not cyclically monotone, or even non monotone. For example in \cite{7} we considered the operator associated to Coulomb friction law, which is not even monotone, and we were able to construct a differential inclusion for the Coulomb friction law with indeterminacy.

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\section{Bipotentials and syncs}

\( X \) and \( Y \) are topological, locally convex, real vector spaces of dual variables \( x \in X \) and \( y \in Y \), with the duality product \( \langle , \cdot \rangle : X \times Y \to \mathbb{R} \). We shall suppose that \( X, Y \) have topologies compatible with the duality product, that is: any continuous linear functional on \( X \) (resp. \( Y \)) has the form \( x \mapsto \langle x, y \rangle \), for some \( y \in Y \) (resp. \( y \mapsto \langle x, y \rangle \), for some \( x \in X \)). We use the notations and conventions of Moreau \cite{18}:

- \( \mathbb{R} = \mathbb{R} \cup \{ +\infty \} \); by convention \( a + (+\infty) = +\infty \) for any \( a \in \mathbb{R} \) and \( a (+\infty) = +\infty \) for any \( a \geq 0 \);
- the domain of a function \( \phi : X \to \mathbb{R} \) is \( \operatorname{dom} \phi = \{ x \in X : \phi (x) \in \mathbb{R} \} \);
- \( \Gamma_0 (X) = \{ \phi : X \to \mathbb{R} : \phi \) is lsc and \( \operatorname{dom} \phi \neq \emptyset \} \);
- for any convex and closed set \( A \subset X \), its indicator function, \( \chi_A \), is defined by
  \[
  \chi_A (x) = \begin{cases} 
  0 & \text{if } x \in A \\
  +\infty & \text{otherwise}
  \end{cases}
  \]
- the subdifferential of a function \( \phi : X \to \mathbb{R} \) at a point \( x \in X \) is the (possibly empty) set:
  \[
  \partial \phi (x) = \{ u \in Y : \forall z \in X \ (z - x, u) \leq \phi (z) - \phi (x) \} .
  \]

A non empty set \( M \subset X \times Y \) is cyclically monotone if for any natural number \( n \geq 1 \) and for any collection \( \{ (x_k, y_k) \in M : k = 0, 1, \ldots, n \} \) we have

\[
\langle x_n - x_0, y_n \rangle + \sum_{k=1}^{n} \langle x_{k-1} - x_k, y_{k-1} \rangle \geq 0 .
\]
Definition 2.1 A **bipotential** is a function $b : X \times Y \to \bar{\mathbb{R}}$, with the properties:

(a) for any $x \in X$, if $\text{dom } b(x, \cdot) \neq \emptyset$ then $b(x, \cdot) \in \Gamma_0(X)$; for any $y \in Y$, if $\text{dom } b(\cdot, y) \neq \emptyset$ then $b(\cdot, y) \in \Gamma_0(Y)$;

(b) for any $x \in X, y \in Y$ we have $b(x, y) \geq \langle x, y \rangle$;

(c) for any $(x, y) \in X \times Y$ we have the equivalences:

$$y \in \partial b(\cdot, y)(x) \iff x \in \partial b(x, \cdot)(y) \iff b(x, y) = \langle x, y \rangle.$$  \hfill (2.0.1)

The **graph** of $b$ is

$$M(b) = \{ (x, y) \in X \times Y \mid b(x, y) = \langle x, y \rangle \}. \hfill (2.0.2)$$

If $X$ is a Banach space, $Y = X^*$ and $M \subset X \times X^*$ then, by an important result of Rockafellar Theorem B \[19\], it is cyclically maximal monotone if and only if

$$M = \text{Graph}(\partial \phi) = \{ (x, y) \in X \times X^* : y \in \partial \phi(x) \}$$

for some $\phi \in \Gamma_0(X)$. Remark that even in the general case of a pair of spaces $X, Y$ in duality, one implication from this result is still true, namely if $M$ is cyclically maximal monotone then there exists $\phi \in \Gamma_0(X)$ such that $M = \text{Graph}(\partial \phi)$. Moreover, by Fenchel inequality this means that

$$b(x, y) = \phi(x) + \phi^*(y) \hfill (2.0.3)$$

is a bipotential and $M(b) = \text{Graph}(\partial \phi)$. Such a bipotential is called separable.

**Remark 2.2** Although using the term "graph of $b$" for the set defined by (2.0.2) seems not adequate, this denomination is used repeatedly in previous articles concerning bipotentials, therefore we shall keep it for coherence reasons. A motivation for the use of word "graph" comes from the relation $M(b) = \text{Graph}(\partial \phi)$ in the case of a separable bipotential.

Bipotentials are related to synchronised convex functions, defined further.

Definition 2.3 A **sync** (synchronised convex function) is a function $c : X \times Y \to [0, +\infty]$ with the properties:

(a) for any $x \in X$, if $\text{dom } c(x, \cdot) \neq \emptyset$ then $c(x, \cdot) \in \Gamma_0(X)$; for any $y \in Y$, if $\text{dom } c(\cdot, y) \neq \emptyset$ then $c(\cdot, y) \in \Gamma_0(Y)$;

(b) for any $x \in X$, if $\text{dom } c(x, \cdot) \neq \emptyset$ and the minimum $\min \{ c(x, y) : y \in Y \}$ exists then this minimum equals 0; for any $y \in X$, if $\text{dom } c(\cdot, y) \neq \emptyset$ and the minimum $\min \{ c(x, y) : x \in X \}$ exists then this minimum equals 0.

Proposition 2.4 A function $b : X \times Y \to \bar{\mathbb{R}}$ is a bipotential if and only if the function $c : X \times Y \to \mathbb{R}$, $c(x, y) = b(x, y) - \langle x, y \rangle$ is a sync.
Proof. With the notations from the proposition, the property (b) definition 2.1 for the function $b$ is equivalent with $c : X \times Y \to [0, +\infty]$. The property (a) definition 2.1 for the function $b$ is equivalent with property (a) definition 2.3 for the function $c$. Finally, the string of equivalences (2.0.1) from property (c) definition 2.1 for the function $b$ is equivalent with the following property for the function $c$:

$$0 \in \partial c(\cdot, y)(x) \iff 0 \in \partial c(x, \cdot)(y) \iff c(x, y) = 0.$$  \hspace{1cm} (2.0.4)

But this is just a reformulation of the property (c) definition 2.3 for the function $b$. \hspace{1cm} \Box

Remark 2.5 The string of equivalences (2.0.4) justifies the name "synchronised convex function", as it expresses the fact that critical points of functions $c(x, \cdot)$ are related with critical points of functions $c(\cdot, y)$.

With the notations from proposition 2.4, we have $M(b) = c^{-1}(0)$. Also, for any $x \in X$ and $y \in Y$, property (a) definition 2.3 of syncs is equivalent with:

$$\text{epi}(c) \cap \{x\} \times Y \times \mathbb{R} \text{ and } \text{epi}(c) \cap X \times \{y\} \times \mathbb{R}$$

are closed convex sets, where $\text{epi}(c)$ is the epigraph of $c$:

$$\text{epi}(c) = \{(x, y, r) \in X \times Y \times \mathbb{R} : c(x, y) \leq r\}$$

For any graph $M \subset X \times Y$, we introduce the sections $M(x) = \{y \in Y \mid (x, y) \in M\}$ and $M^*(y) = \{x \in X \mid (x, y) \in M\}$. The following is definition 3.1 [4].

Definition 2.6 $M \subset X \times Y$ is a BB-graph (bi-convex and bi-closed) if for any $x \in X$ and $y \in Y$ the sections $M(x)$ and $M^*(y)$ are convex and closed.

For any non empty BB-graph $M$ the indicator function $\chi_M$ is obviously a sync.

3 Existence and non uniqueness of the bipotential

Let a constitutive law be given by a graph $M$. Does it admit a bipotential? The existence problem is easily settled by the following result (theorem 3.2 [4]).

Theorem 3.1 Given a non empty set $M \subset X \times Y$, there is a bipotential $b$ such that $M = M(b)$ if and only if $M$ is a BB-graph.

To any BB-graph $M$ is associated the sync $\chi_M$. To this sync corresponds the bipotential

$$b_\infty(x, y) = \langle x, y \rangle + \chi_M(x, y).$$  \hspace{1cm} (3.0.1)

In particular, this shows that to a BB-graph we may associate more than one bipotential. Indeed, if $M$ is maximal cyclically monotone then there are two different bipotentials associated to it. First there is the separable bipotential (2.0.3), thus $M = M(b) = \text{Graph}(\partial \phi)$. But if $M$ is a BB-graph then $\chi_M$ is a sync, which implies that $M = M(b_\infty)$ with $b_\infty$ the bipotential defined at (3.0.1). Therefore maximal cyclically monotone graphs admit at least two distinct bipotentials.

The graph alone is not sufficient to uniquely define the bipotential. However, remark that syncs expressed as indicator functions don’t seem useful as constitutive laws, as they are somehow trivial. Therefore we would like to be able to construct more interesting bipotentials, for example we want a method of construction of bipotentials which associates to a maximal cyclically monotone set a separable bipotential.

Nevertheless the trivial indicator functions will prove to be useful in connection to blurred constitutive laws.
4 Bipotential convex covers

Theorem 3.1 does not give a satisfying bipotential for a given multivalued constitutive law, because the bipotential \( b_\infty \) is somehow degenerate. We would like to be able to find a bipotential \( b \) which is not everywhere infinite outside the graph \( M \). We saw that the graph alone is not sufficient to construct interesting bipotentials. We need more information to start from. This is provided by the notion of bipotential convex cover.

Let \( Bp(X,Y) \) be the set of all bipotentials \( b : X \times Y \to \bar{\mathbb{R}} \). We shall need the following definitions.

**Definition 4.1** Let \( \Lambda \) be an arbitrary non-empty set and \( V \) a real vector space. The function \( f : \Lambda \times V \to \bar{\mathbb{R}} \) is **implicitly convex** if for any two elements \( (\lambda_1,z_1), (\lambda_2,z_2) \in \Lambda \times V \) and for any two numbers \( \alpha, \beta \in [0,1] \) with \( \alpha + \beta = 1 \) there exists \( \lambda \in \Lambda \) such that

\[
f(\lambda, \alpha z_1 + \beta z_2) \leq \alpha f(\lambda, z_1) + \beta f(\lambda, z_2).
\]

**Definition 4.2** A **bipotential convex cover** of the non-empty set \( M \) is a function \( \lambda \mapsto b_\lambda \) from \( \Lambda \) with values in the set \( Bp(X,Y) \), with the properties:

(a) The set \( \Lambda \) is a non-empty compact topological space,

(b) Let \( f : \Lambda \times X \times Y \to \mathbb{R} \cup \{+\infty\} \) be the function defined by

\[
f(\lambda, x, y) = b_\lambda(x,y).
\]

Then for any \( x \in X \) and for any \( y \in Y \) the functions \( f(\cdot, x, \cdot) : \Lambda \times Y \to \bar{\mathbb{R}} \) and \( f(\cdot, \cdot, y) : \Lambda \times X \to \bar{\mathbb{R}} \) are lower semi continuous on the product spaces \( \Lambda \times Y \) and respectively \( \Lambda \times X \) endowed with the standard topology,

(c) We have \( M = \bigcup_{\lambda \in \Lambda} M(b_\lambda) \).

(d) with the notations from point (b), the functions \( f(\cdot, x, \cdot) \) and \( f(\cdot, \cdot, y) \) are implicitly convex in the sense of Definition 4.1.

A bipotential convex cover is in some sense described by the collection \( \{b_\lambda : \lambda \in \Lambda\} \). This is this point of view that we will adopt in the sequel. The next result defines under which conditions the notion of bipotential convex cover is independent of the choice of the parameterization \([5]\).

**Proposition 4.3** Let \( \lambda \mapsto b_\lambda \) be a bipotential convex cover of the graph \( M \) and \( g : \Lambda \to \Lambda \) be a homeomorphism. Then \( \lambda \in \Lambda \mapsto b_{g(\lambda)} \in Bp(X,Y) \) is a bipotential convex cover.

The next theorem, (4.6 [6]), is the key result needed further.

**Theorem 4.4** Let \( \lambda \mapsto b_\lambda \) be a bipotential convex cover of the graph \( M \) and \( b : X \times Y \to \bar{\mathbb{R}} \) defined by

\[
b(x,y) = \inf \{b_\lambda(x,y) \mid \lambda \in \Lambda\}.
\]

Then \( b \) is a bipotential and \( M = M(b) \).

An inferior envelope of convex functions is not generally convex. The property (d) of the Definition 4.2 is essential to ensure the convexity properties of \( b \).
5 Blurred BB-graphs

In many practical situations, indeterminacies affect the mechanical behaviour. In other words, we tolerate indeterminacy of the constitutive law.

We shall represent the indeterminacy by a set $A \subset X \times Y, (0,0) \in A$ and we shall suppose that it is a BB-graph. This hypothesis is justified by the following examples.

Suppose $X$ is a reflexive Banach space and $(x,y) = y(x)$. We shall denote by $\| \cdot \|$ both norms, in $X$ and in $Y$. Let $A = \{0\} \times \bar{B}_Y(\varepsilon) = \{(0,y) : \|y\| \leq \varepsilon\}$. This set is a BB-graph and represents the indeterminacy $\varepsilon$ in the norm of $y$, for given $x$.

In the same setting we can see $X \times Y$ as a normed vector space with the norm

$$\| (x,y) \| = (\|x\|^p + \|y\|^p)^{1/p}$$

for a $p \geq 1$. Take then $A = \{(x,y) \in X \times Y : \|(x,y)\| \leq \varepsilon\}$. This set, which is a BB-graph, represents the indeterminacy $\varepsilon$ in the norm of the pair $(x,y)$.

Let $c : X \times Y \to [0, +\infty]$ be a sync and $M = c^{-1}(0)$ the graph of the associated bipoential $b$. Suppose that the graph $M$ represents a constitutive law. The constitutive law with indeterminacy represented by a BB-graph $A \subset X \times Y$ has then the graph

$$M + A = \{(x,y) \in X \times Y : x = x' + x'', y = y' + y'', (x', y') \in M, (x'', y'') \in A\}$$

For example, if $c(x,y) = \phi(x) + \phi^*(y) - \langle x, y \rangle$, with $\phi \in \Gamma_0(X)$ then $M$ is the graph of the subdifferential $\partial \phi$. If we take the indeterminacy $A = \{0\} \times \bar{B}_Y(\varepsilon)$ then

$$M + A = \{(x,y) \in X \times Y : \exists a \in Y, \|a\| \leq \varepsilon, y + a \in \partial \phi(x)\}$$

(5.0.1)

A natural function associated to $M + A$ is the inf-convolution

$$c_A(x,y) = (c\nabla \chi_A)(x,y) = \inf \{c(x', y') + \chi_A(x'', y'') : x' + x'' = x, y' + y'' = y\}$$

By the properties of the inf-convolution we have

$$\text{epi}(c_A) = \text{epi}(c) + \text{epi}(\chi_A) = \text{epi}(c) + (A \times (0, +\infty))$$

By definition of $c_A$ we have $c^{-1}(0) + A = M + A \subset c_A^{-1}(0)$. Under supplementary hypothesis on $A$ and $c$ we have the equality $M + A = c_A^{-1}(0)$. Let us present such hypothesis.

On the space $X \times Y$ we consider the convergence $(x_h, y_h) \to (x, y)$ (as $h \to \infty$) if $x_h$ converges weakly to $x$ and $y_h$ converges weakly* to $y$. If for any $(x,y) \in X \times Y$ the function $(x'', y'') \in A \mapsto c(x - x'', y - y'')$ is lsc with respect to this convergence and $A$ is compact with respect to this convergence then $M + A = c_A^{-1}(0)$. Indeed, for any $h \in \mathbb{N}^*$ there is $(x_h, y_h) \in A$ such that $0 \leq c(x - x_h, y - y_h) \leq 1/h$. Because $A$ is compact, we can extract a subsequence converging to a pair $(x', y') \in A$; by the lsc of $c$ we find that $0 \leq c(x - x', y - y') \leq \lim_{h \to \infty} c(x - x_h, y - y_h) = 0$, therefore $c(x - x', y - y') = 0$. This proves the inclusion $c_A^{-1}(0) \subset M + A$, therefore we get the desired equality of sets.

A particular case when this hypothesis is obviously true is the one mentioned in relation 6.0.1, presented further in section 5 that is $A = \{0\} \times \bar{B}_Y(\varepsilon)$ and $c(x,y) = \phi(x) + \phi^*(y) - \langle x, y \rangle$, with $\phi \in \Gamma_0(X)$. Another situation when the mentioned hypothesis is true is the one where $X$ and $Y$ are finite dimensional and $A$ is bounded.

Definition 5.1 Consider a BB-graph $M \subset X \times Y$ and another BB-graph $A \subset X \times Y$ such that $(0,0) \in A$. We say that $M$ admits the blurring $A$ if $M + A$ is a BB-graph.

Let $c : X \times Y \to [0, +\infty]$ be a sync and $A \subset X \times Y$ be a BB-graph such that $(0,0) \in A$. We say that $c$ admits the blurring $A$ if $c_A = c\nabla \chi_A$ is a sync and $c_A^{-1}(0) = c^{-1}(0) + A$. 

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Example 1. (Blurred elasticity.) We take $X = Y = \mathbb{R}^n$, the duality product is the usual scalar product in $\mathbb{R}^n$ and $\| \cdot \|$ is the usual norm. We consider the elastic linear law $y = Kx$ which is the most simple example of linear elastic law where the dual variables $x$ and $y$ are vectors, and the ”elastic modulus” $K > 0$. To this law is associated the graph:

$$M = \{(x, y) \in X \times Y : y = Kx\}$$

This graph is maximal cyclically monotone and it admits the sync:

$$c(x, y) = \frac{K}{2} \|x\|^2 + \frac{1}{2K} \|y\|^2 - \langle x, y \rangle$$

Let $\varepsilon > 0$ and $A = \{0\} \times \{y \in Y : \|y\| \leq \varepsilon\}$. Then we have:

$$M + A = \{(x, y) \in X \times Y : \|y - Kx\| \leq \varepsilon\}$$

which is a BB-graph, therefore $M$ admits the blurring $A$. Moreover, after some computations we get that

$$c_A(x, y) = (c \nabla \chi_A)(x, y) = \frac{1}{2K} \left(\|y - Kx\| - \varepsilon\right)_+^2$$

with the notation $z \in \mathbb{R} \mapsto (z)_+ = \max(z, 0)$. It is easy to check that $c_A$ is a sync. Therefore the graph $M$ and the sync $c$ admit the blurring $A$. A similar computation can be done in the case $K$ is a strictly positive definite matrix, only that the expression of $c_A$ is more complex.

Example 2. (A BB-graph made of two points.) In the setting of the previous example, consider this time $M = \{(x_1, y_1), (x_2, y_2)\}$, with $x_1 \neq x_2$ and $y_1 \neq y_2$. This is a BB-graph (although not a very interesting one). We take $A$ as previously. Remark that if $\|y_1 - y_2\| \leq 2\varepsilon$ then $M + A$ is not bi-convex. In this case $M$ does not admit the blurring $A$.

6 Bipotentials for blurred maximal cyclically monotone sets

In this section $X$ is a reflexive Banach space, $Y$ the dual space and the symbol $\| \cdot \|$ is used to denote the norm in $X$ or the dual norm in $Y$.

As the previous example looks somehow degenerate, we might hope that at least in the case $M$ is a maximal cyclically monotone set then $M$ admits the blurring $A$, where $A$ is defined as in the examples above.

The following proposition gives a necessary and sufficient condition for this.

Proposition 6.1 Let $\varepsilon > 0$, $M = \text{Graph}(\partial \phi)$ be a maximal cyclically monotone set with $\phi \in \Gamma_0(X)$, and $A \subset X \times Y$ be defined by $A = \{0\} \times \bar{B}_B(\varepsilon)$. The set $M$ admits the blurring $A$ if and only if the following condition is true:

for any $y \in Y$ the set $\bigcup_{\|\bar{y} - y\| \leq \varepsilon} \partial \phi^*(\bar{y})$ is convex \hspace{1cm} (6.0.1)

Proof. The expression of $M + A$ is given in (5.0.1). The set $M + A$ is a BB-graph if and only if for any $y \in Y$ the set $\{x \in X : (x, y) \in M + A\}$ is convex and closed. A simple computation shows that:

$$\{x \in X : (x, y) \in M + A\} = \bigcup_{\|\bar{y} - y\| \leq \varepsilon} \partial \phi^*(\bar{y}) \hspace{1cm} (6.0.2)$$
This set is closed, for any $\phi \in \Gamma_0(X)$ and $\varepsilon > 0$. Indeed, let $x_h$, $h \in \mathbb{N}$ be a sequence in this set, converging to $x \in X$ as $h$ goes to infinity. For any $h \in \mathbb{N}$ we have $y + a_h \in \partial \phi(x_h)$, with $\|a_h\| \leq \varepsilon$. Equivalently, for any $h \in \mathbb{N}$ there exists $a_h \in Y$, $\|a_h\| \leq \varepsilon$ such that $\phi(x_h) < +\infty$ and for any $x' \in X$ we have

$$\phi(x') - \langle x', y + a_h \rangle \geq \phi(x_h) - \langle x_h, y + a_h \rangle$$  \hspace{1cm} (6.0.3)

We may suppose without reduction of generality that $a_h$ weakly* converges to $a \in Y$, with $\|a\| \leq \varepsilon$. For a fixed, but arbitrary $x' \in X$ we pass to the limit in (6.0.3) and we get:

$$\phi(x') - \langle x', y + a \rangle = \lim_{h \to \infty} [\phi(x') - \langle x', y + a_h \rangle] \geq \liminf_{h \to \infty} [\phi(x_h) - \langle x_h, y + a_h \rangle] \geq \phi(x) - \langle x, y + a \rangle$$

We proved that $y + a \in \partial \phi(x)$, with $\|a\| \leq \varepsilon$.

The relation (6.0.2) lead us to the conclusion that condition (6.0.1) is necessary and sufficient for the set $M$ to be convex.

**Example 3.** The condition (6.0.1) is not true for any $\phi \in \Gamma_0(X)$, at least when $X = Y = \mathbb{R}^n$ and $n \geq 2$. Indeed, for $n = 2$ let us take $\phi^*(y) = \chi_F(y)$ where $F \subset Y$ is the closed convex cone

$$F = \{(y_1, y_2) \in \mathbb{R}^2 : \ |y_2| \leq \alpha y_1\}$$

with $\alpha \in (0, 1)$. Obviously then $2\alpha < 1 + \alpha^2$. Remark that the boundary of $F$ is made by two half lines: $h_1, h_2$ of equation $\pm y_2 = \alpha y_1$, $y_1 \geq 0$. These half lines have normals denoted by $n_1, n_2$, which are not proportional one with another.

Let $M = Graph(\partial \phi)$. We shall choose now $\varepsilon$ such that there exists $y_1 > 0$ with

$$\frac{2\alpha}{\sqrt{1 + \alpha^2}} y_1 < \varepsilon < y_1 \sqrt{1 + \alpha^2}$$

Let $y_2 = \alpha y_1$. We have then $(y_1, y_2) \in F$ and

$$(y_1, y_2) + \overline{B}_Y(\varepsilon) \cap h_1 \neq \emptyset$$ \hspace{0.5cm} , \hspace{0.5cm} $(y_1, y_2) + \overline{B}_Y(\varepsilon) \cap h_2 \neq \emptyset$$

and $(0, 0) \notin (y_1, y_2) + \overline{B}_Y(\varepsilon)$. Then

$$\bigcup_{\|\overline{y} - y\| \leq \varepsilon} \partial \phi^*(\overline{y}) = \{\lambda n_1, \lambda n_2 : \lambda \geq 0\}$$

which is not a convex set. Therefore $M + A$ is not a BB-graph.

In the following consider an arbitrary $\phi \in \Gamma_0(X)$ and $M = Graph(\partial \phi)$. To $\phi$ is associated the separable bipotential $b(x, y) = \phi(x) + \phi^*(y)$ and the sync $c(x, y) = b(x, y) - \langle x, y \rangle$. We have then $M = M(b) = c^{-1}(0)$.

Take $A \subset X \times Y$, $A = \{0\} \times \overline{B}_Y(\varepsilon)$ and let $c_A(x, y) = (c \nabla \chi_A)(x, y)$. We saw that $c_A^{-1}(0) = M + A$. We want to know if $c_A$ is a sync, equivalently if

$$b_A(x, y) = \langle x, y \rangle + \inf \{c(x', y') + \chi_A(x'', y'') : x' + x'' = x, y' + y'' = y\}$$

is a bipotential. Proposition 5.1 gives us the necessary condition (6.0.1) for this to be true, because if $M + A$ is not a BB-graph then $c_A$ cannot be a sync.

Further we shall give a necessary and sufficient condition for $c_A$ to be a sync. Before this, we want to comment on the form of the function $b_A$. We start from the remark that for any
\(a \in Y\) and any sync \(c : X \times Y \to [0, +\infty]\) the function defined by \(c_a(x, y) = c(x, y - a)\) is also a sync. In particular, for any \(\phi \in \Gamma_0(X)\) the function
\[
b_a(x, y) = \phi(x) + \phi^*(y - a) + \langle x, a \rangle
\]
is a (separable) bipotential.

We see that the function \(b_A\) has the expression
\[
b_A(x, y) = \inf \{\phi(x) + \phi^*(y - a) + \langle x, a \rangle : \|a\| \leq \varepsilon\}
\]therefore, \(b_A\) is an infimum of bipotentials:
\[
b_A(x, y) = \inf \{b_a(x, y) : \|a\| \leq \varepsilon\}
\]
If the function \(a \in \overline{B}_Y(\varepsilon) \mapsto b_a\) is a bipotential convex cover then \(b_A\) is a bipotential, by theorem 4.4.

**Theorem 6.2** With the previous notations \(b_A\) is a bipotential such that \(M(b_A) = M(\phi, \varepsilon)\) if and only if for any \(y \in Y\) the function \(f(\cdot, \cdot, y) : \overline{B}_Y(\varepsilon) \times X \to \mathbb{R}\) defined by
\[
f(a, x, y) = \phi(x) + \phi^*(y - a) + \langle x, a \rangle
\]
is implicitly convex.

**Proof.** If \(b_A\) is a bipotential then for any \(y \in Y\) the function \(b_A(\cdot, y)\) is convex. By reasoning as in the proof of corollary 1.2, it follows that \(f(\cdot, \cdot, y)\) is implicitly convex.

We have to prove now the inverse implication. We shall prove that the function \(a \in \overline{B}_Y(\varepsilon) \mapsto b_a\) is a bipotential convex cover of the set \(M(\phi, \varepsilon)\).

Let us denote \(\Lambda = \overline{B}_Y(\varepsilon)\). This set, endowed with the weak topology from \(Y\) is a compact topological space. Then the function \(f : \Lambda \times X \to \mathbb{R}\) defined by
\[
f(a, x, y) = \phi(x) + \phi^*(y - a) + \langle x, a \rangle
\]
has the following properties. For any \(x \in X\) the function \(f(\cdot, \cdot, \cdot)\) is convex and lower semicontinous, with respect to the product topology (weak topology on \(\Lambda\) times the strong topology on \(Y\)). Also, for any \(y \in Y\) the function \(f(\cdot, \cdot, y)\) is implicitly convex by hypothesis, and also lower semicontinous with respect to the product topology. Indeed, this is true because: \(\phi\) is strongly lower semicontinous, the function \(a \in \Lambda \mapsto \phi^*(y - a)\) is weakly lower semicontinous and for any sequence \((a_h, x_h) \in \Lambda \times X\) such that \(a_h\) weakly converges to \(a\) in \(\Lambda\) and \(x_h\) strongly converges to \(x\) in \(X\), then \(\langle x_h, a_h \rangle\) converges to \(\langle x, a \rangle\).

It follows that for any \((x, y) \in X \times Y\) there exists \(a \in \Lambda\) such that
\[
b_A(x, y) = \min_{a \in \Lambda} f(a, x, y)
\]
We deduce that \(M(\phi, \varepsilon) = \bigcup_{a \in \Lambda} M(b_a)\). We proved that \(a \in \overline{B}_Y(\varepsilon) \mapsto b_a\) is a bipotential convex cover of the set \(M(\phi, \varepsilon)\). By theorem 4.4 we obtain the conclusion. \(\Box\)

**Final remark.** Suppose \(X = \mathbb{R}\). Then condition 6.0.1 should take a simpler form, because in one dimension convex is the same as connected and \(6.0.1\) is just a kind of Darboux property for subgradients. Is it true for any \(\phi \in \Gamma_0(X)?\)
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