Abstract

In his 1930 paper \cite{7}, Kuratowski categorized planar graphs, proving that a finite graph $\Gamma$ is planar if and only if it does not contain a subgraph that is homeomorphic to $K_5$, the complete graph on 5 vertices, or $K_{3,3}$, the complete bipartite graph on six vertices. This result is also attributed to Pontryagin \cite{6}. In their 2001 paper \cite{4}, Davis and Okun point out that the $K_{3,3}$ graph can be understood as the nerve of a right-angled Coxeter system and prove that this graph is not planar using results from $\ell^2$-homology. In this paper, we employ a similar method using results from \cite{9} to prove $K_5$ is not planar.

1 Introduction

Let $S$ be a finite set of generators. A Coxeter matrix on $S$ is a symmetric $S \times S$ matrix $M = (m_{st})$ with entries in $\mathbb{N} \cup \{\infty\}$ such that each diagonal entry is 1 and each off diagonal entry is $\geq 2$. The matrix $M$ gives a presentation of an associated Coxeter group $W$:

$$W = \langle S \mid (st)^{m_{st}} = 1, \text{ for each pair } (s, t) \text{ with } m_{st} \neq \infty \rangle. \quad (1.1)$$

The pair $(W, S)$ is called a Coxeter system. Denote by $L$ the nerve of $(W, S)$, $L$ is a simplicial complex with vertex set $S$, the precise definition will be given in section 2. In several papers (e.g., \cite{1}, \cite{2}, and \cite{3}), M. Davis describes a construction which associates to any Coxeter system $(W, S)$, a simplicial complex $\Sigma(W, S)$, or simply $\Sigma$ when the Coxeter system is clear, on which $W$ acts properly and cocompactly. The two salient features of $\Sigma$ are that (1) it is contractible and (2) that it admits a cellulation under which the nerve of each vertex is $L$. It follows that if $L$ is a triangulation of $\mathbb{S}^{n-1}$, $\Sigma$ is an aspherical $n$-manifold. Hence, there is a variation of Singer’s Conjecture, originally regarding the (reduced) $\ell^2$-homology of aspherical manifolds, for such Coxeter groups.

Singer’s Conjecture for Coxeter groups 1.1. Let $(W, S)$ be a Coxeter group such that its nerve, $L$, is a triangulation of $\mathbb{S}^{n-1}$. Then $H_i(\Sigma_L) = 0$ for all $i \neq n/2$. 

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1
For details on $\ell^2$-homology theory, see [3], [4] and [5]. Conjecture 1.1 holds for elementary reasons in dimensions 1 and 2. In [4], Davis and Okun prove that if Conjecture 1.1 holds for right-angled Coxeter systems in dimension $n$, then it also holds in dimension $n + 1$. (Here, right-angled means generators either commute, or have no relation). They also prove directly that Conjecture 1.1 holds for right-angled systems in dimension 3, and thus also in dimension 4. This result also follows from work by Lott and Lück ([8]) and Thurston ([10]) regarding Haken Manifolds. In [9], the author proves that Conjecture 1.1 holds for arbitrary Coxeter systems with nerve $S^2$.

Also in [4], Davis and Okun use their low dimensional results to prove the following generalization of Conjecture 1.1.

**Lemma 1.2.** (Lemma 9.2.3, [4]) Suppose $(W, S)$ is a right-angled Coxeter system with nerve $L$, a flag triangulation of $S^2$. Let $A$ be a full subcomplex of $L$. Then

$$H_i(W \Sigma_A) = 0 \text{ for } i > 1.$$ 

Here, $\Sigma_A$ is the Davis complex associated with the Coxeter system $(W_A, A^0)$, where $W_A$ is the subgroup of $W$ generated by vertices in $A$, with nerve $A$. It is a subcomplex of $\Sigma$.

Lemma 1.2 is the key to what Davis and Okun call “a complicated proof of the classical fact that $K_{3,3}$ is not planar,” (See section 11.4.1, [4]). We outline that argument in Section 3. The purpose of this paper is to employ a similar argument to prove that $K_5$ is not planar.

The key step for us is proving a result analogous to Lemma 1.2 but for subcomplexes of arbitrary Coxeter systems.

**Main Theorem.** (See Theorem 4.5) Let $(W, S)$ be a Coxeter system with nerve $L$, a triangulation of $S^2$. Let $A$ be full subcomplex of $L$ with right-angled complement. Then

$$H_i(W \Sigma_A) = 0 \text{ for } i > 1.$$ 

Here $A$ having a “right-angled complement” means that for generators $s$ and $t$, the Coxeter relation $m_{st} \neq 2$ nor $\infty$ implies that the vertices corresponding to $s$ and $t$ are both in $A$. From the Main Theorem, it follows that $K_5$ is not planar.

## 2 The Davis complex

Let $(W, S)$ be a Coxeter system. Given a subset $U$ of $S$, define $W_U$ to be the subgroup of $W$ generated by the elements of $U$. A subset $T$ of $S$ is spherical if $W_T$ is a finite subgroup of $W$. In this case, we will also say that the subgroup $W_T$ is spherical. Denote by $S$ the poset of spherical subsets of $S$, partially ordered by inclusion. Given a subset $V$ of $S$, let $S_{\geq V} := \{ T \in S | V \subseteq T \}$. Similar definitions exist for $<, >, \leq$. For any $w \in W$ and $T \in S$, we call the coset $wW_T$ a spherical coset. The poset of all spherical cosets we will denote by $WS$. 

2
Let $K = \lvert S \rvert$, the geometric realization of the poset $S$. It is a finite simplicial complex. Denote by $\Sigma(W, S)$, or simply $\Sigma$ when the system is clear, the geometric realization of the poset $WS$. This is the Davis complex. The natural action of $W$ on $WS$ induces a simplicial action of $W$ on $\Sigma$ which is proper and cocompact. $K$ includes naturally into $\Sigma$ via the map induced by $T \to WT$. So we view $K$ as a subcomplex of $\Sigma$, and note that $K$ is a strict fundamental domain for the action of $W$ on $\Sigma$.

The poset $S_{>0}$ is an abstract simplicial complex. This simply means that if $T \in S_{>0}$ and $T'$ is a nonempty subset of $T$, then $T' \in S_{>0}$. Denote this simplicial complex by $L$ and call it the nerve of $(W, S)$. The vertex set of $L$ is $S$ and a non-empty subset of vertices $T$ spans a simplex of $L$ if and only if $T$ is spherical.

Define a labeling on the edges of $L$ by the map $m : \text{Edge}(L) \to \{2, 3, \ldots\}$, where $\{s, t\} \mapsto m_{st}$. This labeling accomplishes two things: (1) the Coxeter system $(W, S)$ can be recovered (up to isomorphism) from $L$ and (2) the 1-skeleton of $L$ inherits a natural piecewise spherical structure in which the edge $\{s, t\}$ has length $\pi - \pi/m_{st}$. $L$ is then a metric flag simplicial complex (see Definition [2, I.7.1]). This means that any finite set of vertices, which are pairwise connected by edges, spans a simplex of $L$ if and only if it is possible to find some spherical simplex with the given edge lengths. In other words, $L$ is “metrically determined by its 1-skeleton.”

Recall that a simplicial complex $L$ is flag if every nonempty, finite set of vertices that are pairwise connected by edges spans a simplex of $L$. Thus, it is clear that any flag simplicial complex can correspond to the nerve of a right-angled Coxeter system. For the purpose of this paper, we will say that labeled (with integers $\geq 2$) simplicial complexes are metric flag if they correspond to the labeled nerve of some Coxeter system. We then treat vertices of metric flag simplicial complexes as generators of a corresponding Coxeter system. Moreover, for a metric flag simplicial complex $L$, we write $\Sigma_L$ to denote the associated Davis complex.

**A cellulation of $\Sigma$ by Coxeter cells.** $\Sigma$ has a coarser cell structure: its cellulation by “Coxeter cells.” (References include [2] and [4] for the features of this. The cellulation is summarized by [2] Proposition 7.3.4. We point out that under this cellulation the link of each vertex is $L$. It follows that if $L$ is a triangulation of $S^{n-1}$, then $\Sigma$ is a topological $n$-manifold.

**Full subcomplexes.** Suppose $A$ is a full subcomplex of $L$. Then $A$ is the nerve for the subgroup generated by the vertex set of $A$. We will denote this subgroup by $W_A$. (This notation is natural since the vertex set of $A$ corresponds to a subset of the generating set $S$.) Let $S_A$ denote the poset of the spherical subsets of $W_A$ and let $\Sigma_A$ denote the Davis complex associated to $(W_A, A^0)$. The inclusion $W_A \hookrightarrow W_L$ induces an inclusion of posets $W_A S_A \hookrightarrow W_L S_L$ and thus an inclusion of $\Sigma_A$ as a subcomplex of $\Sigma_L$. Note that $W_A$ acts on $\Sigma_A$ and that if $w \in W_L \setminus W_A$, then $\Sigma_A$ and $w\Sigma_A$ are disjoint copies of $\Sigma_A$. Denote by
Let $L$ be a metric flag simplicial complex, and let $A$ be a full subcomplex of $L$. The following notation will be used throughout.

\begin{align*}
    h_i(L) &:= \mathcal{H}_i(\Sigma L) \quad (3.1) \\
    h_i(A) &:= \mathcal{H}_i(W_L \Sigma A) \quad (3.2) \\
    \beta_i(A) &:= \dim_{W_L}(h_i(A)). \quad (3.3)
\end{align*}

Here $\dim_{W_L}(h_i(A))$ is the von Neumann dimension of the Hilbert $W_L$-module $W_L \Sigma A$ and $\beta_i(A)$ is the $i$th $\ell^2$-Betti number of $W_L \Sigma A$. The notation in 3.2 and 3.3 will not lead to confusion since $\dim_{W_L}(W_L \Sigma A) = \dim_{W_A}(\Sigma A)$. (See [4] and [5]).

**0-dimensional homology.** Let $\Sigma A$ be the Davis complex constructed from a Coxeter system with nerve $A$, so $W_A$ acts geometrically on $\Sigma A$. The reduced $\ell^2$-homology groups of $\Sigma A$ can be identified with the subspace of harmonic $i$-cycles (see [3] or [4]). That is, $x \in h_i(A)$ is an $i$-cycle and $i$-cocycle. 0-dimensional cocycles of $\Sigma A$ must be constant on all vertices of $\Sigma A$. It follows that if $W_A$ is infinite, and therefore the 0-skeleton of $\Sigma A$ is infinite, $\beta_0(A) = 0$.

**Singer Conjecture in dimensions 1 and 2.** As mentioned in Section 1, Conjecture 1.1 is true in dimensions 1 and 2. Indeed, let $L$ be $S^0$ or $S^1$, the nerve of a Coxeter system $(W, S)$. Then $W$ is infinite and so, as stated above, $\beta_0(L) = 0$. Poincaré duality then implies that the top-dimensional $\ell^2$-Betti numbers are also 0.

**Orbihedral Euler Characteristic.** $\Sigma L$ is a geometric $W$-complex. So there are only finite number of $W$-orbits of cells in $\Sigma L$, and the order of each cell stabilizer is finite. The *orbihedral Euler characteristic* of $\Sigma L/W = K$, denoted $\chi^\text{orb}(\Sigma L/W)$, is the rational number defined by

\[ \chi^\text{orb}(\Sigma L/W) = \chi^\text{orb}(K) = \sum_{\sigma} (-1)^{\dim \sigma} \frac{1}{|W_\sigma|}, \quad (3.4) \]

where the summation is over the simplices of $K$ and $|W_\sigma|$ denotes the order of the stabilizer in $W$ of $\sigma$. Then, if the dimension of $L$ is $n-1$, a standard argument (see [5]) proves Atiyah’s formula:

\[ \chi^\text{orb}(K) = \sum_{i=0}^{n} (-1)^i \beta_i(L). \quad (3.5) \]
Joints. If \( L = L_1 \ast L_2 \), the join of \( L_1 \) and \( L_2 \), where each edge connecting a vertex of \( L_1 \) with a vertex of \( L_2 \) is labeled 2, we write \( L = L_1 \ast_2 L_2 \) and then \( W_L = W_{L_1} \times W_{L_2} \) and \( \Sigma_L = \Sigma_{L_1} \times \Sigma_{L_2} \). We may then use Künneth formula to calculate the (reduced) \( \ell^2 \)-homology of \( \Sigma_L \), and the following equation from [4, Lemma 7.2.4] extends to our situation:

\[
\beta_k(L_1 \ast L_2) = \sum_{i+j=k} \beta_i(L_1) \beta_j(L_2). \tag{3.6}
\]

If \( L = P \ast_2 L_2 \), where \( P \) is one point, then we call \( L \) a right-angled cone. \( \Sigma_P = [-1,1] \), so there are no 1-cycles, and so \( \beta_1 = (P) = 0 \). But, \( \chi^{\text{orb}}(\Sigma_P/W_P) = 1/2 \) so by equation 3.5, \( \beta_0(P) = 1/2 \). Thus, in reference to the right-angled cone \( L \), equation 3.6 implies that

\[
\beta_i(L) = \frac{1}{2} \beta_i(L_2) \tag{3.7}
\]

The \( K_{3,3} \) case. Along with Lemma 1.2, the above gives us enough to prove that \( K_{3,3} \) is not planar. Indeed, let \( P_3 \) denote 3 disjoint points. Then \( K_{3,3} = P_3 \ast_2 P_3 \) is the nerve of a right-angled Coxeter system. Since \( W_{K_{3,3}} \) is infinite, so \( \beta_0(K_{3,3}) = 0 \), and equations 3.4 and 3.5 give us that \( \beta_1(P_3) = 1/2 \). It then follows from equation 3.6 that \( \beta_2(K_{3,3}) = 1/4 \). Thus, if \( K_{3,3} \) were a planar graph, it could be embedded as a full-subcomplex of a flag triangulation of \( S^2 \), where each edge is labeled 2. This triangulation of \( S^2 \) corresponds to the nerve of a right-angled Coxeter system. But this contradicts Lemma 1.2. For details on this proof see [4, Sections 8, 9 and 11].

4 The \( K_5 \) Graph

Let \( K_5 \) denote the complete graph on 5 vertices. The right-angled methods above cannot be applied to \( K_5 \) because, if the edges are labeled with 2’s, then \( K_5 \) cannot be embedded as a full subcomplex of a metric flag triangulation of \( S^2 \). However, \( K_5 \) is metric flag if the edges are labeled with 3’s. For if \( r, s \) and \( t \) are generators of a Coxeter system such that \( m_{rs} = m_{st} = m_{rt} = 3 \), then \( \{r,s,t\} \) is not a spherical subset and this set does not span a 2-simplex in the nerve of the corresponding Coxeter system. This simple observation leads to the following definition.

Definition 4.1. We say a full subcomplex \( A \) of a metric flag simplicial complex \( L \) has a right-angled complement if the label on all edges not in \( A \) is 2.

The following two Lemmas will be used in the set-up and proof of our Main theorem.

Lemma 4.2. Let \( L \) be a metric flag simplicial complex, \( A \subseteq L \) a full subcomplex with a right-angled complement. Let \( B \) be a full subcomplex of \( L \) such that \( A \subseteq B \) and let \( v \in B - A \) be a vertex. Then \( B_v \), the link of \( v \) in \( B \), is a full subcomplex of \( L \).
Proof. Let $T$ be a subset of vertices contained in $B_v$ and the vertex set of a simplex $\sigma$ of $L$. Then $T$ defines a spherical subset of the corresponding Coxeter system. Since the of $T$ are in $B_v$, $v$ commutes with each vertex of $T$. Thus $T \cup \{v\}$ is a spherical subset and therefore $\sigma$ is in $B_v$. \hfill \Box

**Lemma 4.3.** Let $L$ be a metric flag triangulation of $S^1$, let $A$ be a full subcomplex of $L$. Then $\beta_i(A) = 0$ for $i > 1$.

**Proof.** Consider the long exact sequence of the pair $(\Sigma_L, W \Sigma_A)$:

$$0 \rightarrow h_2(A) \rightarrow h_2(L) \rightarrow h(L, A) \rightarrow ...$$

Since Conjecture 1.1 is true in dimension 2, $h_2(L) = 0$ and exactness implies the result. \hfill \Box

For convenience, we restate the relevant result from [9] needed to prove $K_5$ is non-planar.

**Theorem 4.4.** (See Corollary 4.4, [9]) Let $L$ be a metric flag triangulation of $S^2$. Then

$$h_i(L) = 0 \text{ for all } i$$

We are now ready to prove our main theorem, analogous to Lemma 4.2.

**Theorem 4.5.** Let $L$ be a metric flag triangulation of $S^2$, $A \subseteq L$ a full subcomplex with right-angled complement. Then

$$\beta_i(A) = 0 \text{ for } i > 1$$

**Proof.** Let $B$ be a full subcomplex of $L$ such that $A \subseteq B \subseteq L$. We induct on the number of vertices of $L - B$, the case $L = B$ given by Theorem 4.4. Assume $h_i(B) = 0$ for $i > 1$. Let $v$ be a vertex of $B - A$ and set $B' = B - v$. Then $B = B' \cup C_2 B_v$ where $B_v$ (by Lemma 4.3) and $B'$ are full subcomplexes. We have the following Mayer-Vietoris Sequence:

$$\ldots \rightarrow h_i(B_v) \rightarrow h_i(B') \oplus h_i(C_2 B_v) \rightarrow h_i(B) \rightarrow \ldots$$

$B_v$ is a full subcomplex of $L_v$, the link of $v$ in $L$, a metric flag triangulation of $S^1$. So Lemma 4.3 implies $h_i(B_v) = 0$, for $i > 1$. Thus, by equation 3.7 $h_i(C_2 B_v) = 0$ for $i > 1$. It follows from exactness that $h_i(B') = 0$. \hfill \Box

The above Theorem can be restated as follows, cf. [4] Theorem 11.4.1.

**Theorem 4.6.** Let $A$ be a metric flag complex of dimension $\leq 2$. Suppose $A$ is planar (that is, it can be embedded as a subcomplex of the 2-sphere). Then

$$\beta_2(A) = 0.$$
Proof. By Mayer-Vietoris, we may assume $A$ is connected. Suppose $A$ is piecewise linearly embedded in $S^2$. By introducing a new vertex in the interior of each complementary region, and coning off the boundary of each region labeling each new edge with 2, we obtain a metric flag triangulation of $S^2$ in which every edge not in $A$ is labeled 2, that is, $A$ has a right-angled complement. The result follows from the proof of Theorem 4.6.

We are now ready to prove $K_5$ is not planar.

Corollary 4.7. $K_5$ is not planar.

Proof. Label each edge of $K_5$ with 3, and thus $K_5$ is a metric flag complex. In this case, $\chi^{orb}(K_5) = \frac{1}{6}$. Then Atiyah’s formula, equation 3.5 and the fact that $\beta_0(K_5) = 0$ imply that $\beta_2(K_5) > 0$, contradicting Theorem 4.6.