Photonic processes in Born-Infeld theory

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Abstract:

We study the processes of photon-photon scattering and photon splitting in a magnetic field in Born-Infeld theory. In both cases we combine the terms from the tree-level Born-Infeld Lagrangian with the usual one-loop QED contributions, where those are approximated by the Euler-Heisenberg Lagrangian, including also the interference terms. For photon-photon scattering we obtain the total cross section in the low-energy approximation. For photon splitting we compute the total absorption coefficient in the hexagon (weak field) approximation, and also show that, due to the non-birefringence property of Born-Infeld theory, the selection rules found by Adler for the QED case continue to hold in this more general setting. We discuss the bounds on the free parameter of Born-Infeld theory that may be obtained from this type of processes.
1 Introduction

In their famous 1934 paper [1], Born and Infeld proposed the following Lagrangian as a nonlinear generalization of electrodynamics:

\[ \mathcal{L}^{BI} = -b^2 \sqrt{1 - \frac{2s}{b^2} - \frac{p^2}{b^4} + b^2}. \] (1.1)

Here \( s \) and \( p \) are the two invariants of the Maxwell field,

\[ s \equiv -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2} (E^2 - B^2), \] (1.2)

\[ p \equiv -\frac{1}{4} \tilde{F}^{\mu\nu} F_{\mu\nu} = \vec{E} \cdot \vec{B}, \] (1.3)

(\( p \) is only a pseudo-invariant and thus must appear squared.)

Alternatively, the Born-Infeld Lagrangian (‘BIL’) can also be written in determinantal form,

\[ \mathcal{L}^{BI} = -b^2 \sqrt{-\det(\eta^{\mu\nu} + \frac{1}{b} F_{\mu\nu})} + b^2 \sqrt{-\det(\eta_{\mu\nu})}, \] (1.4)

where \( F \) is the field strength tensor.

In the limit of large \( b \) the Lagrangian (1.1) reduces to the Maxwell Lagrangian \( \mathcal{L}^M = s = \frac{1}{2} (E^2 - B^2) \), as can be seen by expanding \( \mathcal{L}^{BI} \) in powers of \( 1/b \):

\[ \mathcal{L}^{BI} = s + \frac{1}{2b^2} \left( s^2 + p^2 \right) + \frac{1}{2b^4} \left( s^3 + sp^2 \right) + \ldots. \] (1.5)

The higher order terms correspond to quartic, sextic, etc. photon vertices.

Originally Born and Infeld attempted to fix \( b \) by equating the electromagnetic self energy of the electron, which is finite in their theory, with its mass energy. This leads to the following numerical value of \( b \),

\[ b = 1.2 \times 10^{20} \frac{V}{m}. \] (1.6)

Nowadays such an interpretation is hardly viable, but Born–Infeld theory, with \( b \) as a free parameter, is still considered the prototypical example of a nonlinear generalization of electrodynamics. This is not only because of its concise determinant formulation (1.4), but also because it shares with Maxwell theory the important property of not leading to birefringence; that is, the velocity of photon propagation, although dependent on the frequency,
does not depend on the photon polarization. This property is not shared by more general nonlinear theories of electromagnetism.

Although Born and Infeld thought of their theory as a substitute for Maxwell theory at the fundamental level, as a quantum field theory it is not renormalizable, so that nowadays it seems more natural to think of the Born–Infeld Lagrangian (‘BIL’) as an effective one. And indeed, in 1985 the Born–Infeld theory acquired new relevance through the discovery by Fradkin and Tseytlin [2] that the same Lagrangian appears also as a low-energy effective Lagrangian in open string theory (on the other hand, it has been shown [3] that no combination of scalar, spinor, and vector particles alone can, assuming the standard couplings to an abelian gauge field, generate the BIL as a one-loop effective Lagrangian).

More recently, the BIL has become also an important ingredient for brane theories (see, e.g., [4], which contains also an excellent introduction to Born–Infeld theory and nonlinear electrodynamics). In this context the determinantal definition (1.4) is very convenient, since it admits an immediate generalization to other (even) dimensions.

Despite of the present ubiquitous appearance of the Born–Infeld Lagrangian in field theory, relatively little effort has been devoted to taking it seriously as an alternative to Maxwell electrodynamics. In the absence of experimental evidence for nonlinear electrodynamics, here the goal must be to establish successively stricter lower bounds on the parameter $b$. Since this parameter also represents the maximal possible field strength in Born–Infeld theory, it seems logical in this context to focus on strong-field atomic physics [5, 6, 7, 8, 9, 10]. And indeed, as far as is known to the authors the strongest bound on $b$ presently available is the one obtained by Soff, Rafelski and Greiner [7] from muonic transitions in lead,

$$b \geq 1.7 \times 10^{22} \frac{V}{m}.$$  \hspace{1cm} (1.7)

This is already considerably beyond the original Born-Infeld value (1.6). However, the mixture of nonlinear electrodynamics, strong-field physics and quantum mechanics involved in this type of estimate is a subtle one, and it is natural to ask what bounds on $b$ can be obtained by purely photonic processes not involving the electrostatic potential between point charges. In this paper we will study two such processes that are the most obvious ones for testing the four-point and six-point vertices contained in the weak-field expansion of the Born–Infeld Lagrangian (1.5), namely photon-photon scattering and photon splitting in a magnetic field.
In the context of such purely photonic processes the classical Born–Infeld Lagrangian enters naturally in competition with the quantum Euler-Heisenberg Lagrangian (‘EHL’), the one-loop low-energy effective Lagrangian of QED \[11\] in the constant field approximation. We recall the standard proper-time representation of this Lagrangian (see, e.g., \[12, 13\])

\[
\mathcal{L}_{EH} = -\frac{1}{8\pi^2} \int_0^\infty \frac{dT}{T^3} e^{-m^2T} \left[ \frac{(eaT)(ebT)}{\tanh(eaT) \tan(ebT)} - \frac{e^2}{3} (a^2 - b^2)T^2 - 1 \right].
\]

Here \(T\) is the proper-time of the loop fermion, \(m\) its mass, and \(a, b\) are related to the invariants \(s\) and \(p\) by \(b^2 - a^2 = 2s\) and \(ab = p\). The two subtraction terms implement the renormalization of charge and vacuum energy.

The EHL contains a wealth of information on photonic processes involving a single virtual electron-positron pair in the vacuum \[14, 15\]. However, it can be expected to be a good approximation only for photon energies \(\omega\) much smaller than the electron mass, since otherwise derivative corrections to the EHL come into play which carry factors of \(\omega/m\). If to this constant field (resp. low photon energy) approximation one adds the approximation of a weak field (resp. low photon beam intensity), then one can further simplify (1.8) by expanding out in powers of the field. The first two terms of this expansion are

\[
\mathcal{L}_{EH} = \frac{2\alpha^2}{45m^4} \left( 4s^2 + 7p^2 \right) + \frac{32\pi\alpha^3}{315m^8} \left( 8s^3 + 13sp^2 \right) + \ldots ,
\]

where \(\alpha = \frac{e^2}{4\pi}\) is the fine structure constant.

Here the first term holds the information on the one-loop photon-photon scattering amplitude in the low energy limit, see fig. 1 left.

Here and in the following a diagram is understood to include also all relevant permutations of the external legs. The second term in (1.9) corresponds to the low energy limit of various photonic processes, of which the most important one is the splitting of one photon into two in a magnetic field \[16, 17\]. In vacuum this process has, despite of its very special kinematics - energy-momentum conservation forces all three photons to be collinear - a non vanishing two-body phase space, but the matrix element vanishes on account of Furry’s theorem. In an external field it becomes possible, since the
odd number of photons can be balanced by an odd number of interactions with the field. For a generic magnetic field, the lowest order contribution to the matrix element would again be given by the photon-photon scattering diagram (fig. 1 left), now with one of the photons replaced by an interaction with the field. In the case of a constant magnetic field, the most important one for phenomenology, it turns out that this box diagram contribution to the matrix element vanishes due to the collinear kinematics; therefore the leading contribution is presented by the hexagon diagram, shown in fig. 2.

In this diagram the crosses denote the interactions with the magnetic field. This contribution is non-vanishing, but still has the very special property of being exactly given by its low-energy (small $\omega/m$) limit. It thus can be exactly computed from the Euler-Heisenberg Lagrangian, which was done by Z. and I. Bialynicka-Birula [16]. A full calculation of the photon splitting process, summing all diagrams with an arbitrary (odd) number of interactions with the field, requires more advanced methods, and was first achieved.
by Adler [17] (see also [18, 19, 14]).

A full treatment of the photon splitting process must, however, also take into account the fact that the presence of the magnetic field will modify the photon dispersion relations. The vacuum will acquire a nontrivial index of refraction, which moreover turns out to depend on the photon polarization, leading to birefringence [20, 21, 17]. The effect of this on the photon splitting process is that, depending on the chosen combination of photon polarizations, either the collinearity of the photons is slightly modified, or else the process becomes impossible altogether. As shown in [17], combining this dispersion-induced selection rule with CP invariance leaves, up to order α corrections, the splitting of a perpendicularly polarized photon into two parallely polarized ones as the only possible polarization choice. Photon splitting thus naturally has a polarizing effect.

Although physically the meanings of the BIL and the EHL are very different, as seen by a comparison of (1.5) and (1.9) their weak field expansions differ, apart from the replacement of the electron mass m by √b as the intrinsic mass scale, only by a change of the numerical coefficients (as a historical curiosity, let us mention that Infeld [22] attempted to determine the fine structure constant by equating $\mathcal{L}_{\text{EH}}^4$ and $\mathcal{L}_{\text{BI}}^4$, and with $b$ as given in (1.6) obtained its correct order of magnitude). Therefore any physical process encoded in the EHL is expected to have an analogous contribution induced by the BIL, and can potentially be used for obtaining a bound on $b$. Constraining the Born-Infeld parameter through low-energy photonic processes provides not only the most direct possible way of testing the presence of the nonlinear photon vertices, but has also the advantage that the obtained bounds will hold irrespectively of whether the BIL is considered as fundamental or effective; derivative corrections, which would constitute the difference between the both scenarios, would come with factors of $\omega^2/b$ that will be negligible for the values of $b$ that are presently still viable.

Photon amplitudes in Born-Infeld theory have already been discussed by many authors, both from a theoretical and a phenomenological perspective. On the theory side, there has been interest in the fact that the $N$-photon helicity amplitudes in Born-Infeld theory are, at least at the tree-level, helicity conserving. This was shown in [23] to be a consequence of the invariance under the $U(1)$ duality rotation of $F$ and $\tilde{F}$. When viewing the BIL as induced by open string theory this mechanism relates to S-duality [24].

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1Here by $\perp (\parallel)$ we denote a polarization vector which is perpendicular (parallel) to the plane spanned by the photon momentum and the direction of the magnetic field (this convention is opposite to the one used in [17]), but agrees with [14].
Phenomenologically, the recent interest in Born-Infeld theory, and more generally nonlinear electrodynamics, is due to the construction of high-power laser facilities such as POLARIS, HERCULES, VIRGO and ELI, which will allow one to test QED in hitherto completely unexplored sectors.

Despite of all this recent activity, it seems that the cross section for photon-photon scattering in BI theory is not available in the literature. In section 2 we will calculate this cross section, combining the standard QED contribution (Fig. 1, left) with the one of the four-photon vertex of Born-Infeld theory (Fig. 1, right), including also the interference term between them.

In section 3 we analogously study the photon splitting process in a constant magnetic field for Born-Infeld theory, combining the QED contribution with the one induced by the sextic Born-Infeld vertex, Fig. 3.

We also show that the six-point diagrams with two quartic vertices can be omitted, and that in BI theory the same selection rules hold for this process as in QED. In the photon splitting case, we are not aware of any previous discussions for Born-Infeld theory.

In section 4 we discuss the use of these results for obtaining lower bounds on $\delta$ from laser physics respectively neutron star physics. We summarize our findings in section 5.

### 2 Photon-photon scattering

In BI theory, photon-photon scattering occurs already at the tree-level, due to the quartic vertex implied by the first nontrivial term on the right hand
side of (1.5) and depicted on the right in Fig. 1. The calculation of the cross section from this vertex is analogous to the one of the photon-photon cross section in QED in the low-energy limit from the quartic term of the EHL. We will thus first retrace this textbook calculation, following [25].

2.1 Photon-photon scattering in QED

Let us denote the quartic term in (1.9) by $L_{EH}^4$, and rewrite it using the identity

$$ (F \cdot \tilde{F})^2 = 4 \text{tr} (F^4) - 2 (\text{tr} (F^2))^2. $$

It then reads

$$ L_{EH}^4 = \frac{2 \alpha^2}{45 m^4} \left[ \frac{7}{4} (\text{tr} (F^4)) - \frac{5}{8} (\text{tr} (F^2))^2 \right]. $$

We introduce for each photon leg its field strength tensor,

$$ F_{i\mu
uepsilon} \equiv (k_i \epsilon_{i\mu} - k_i \epsilon_{i\nu}). $$

We replace in $L_{EH}^4$ each $F$ by the sum $F_1 + F_2 + F_3 + F_4$, and keep only the terms containing each $F_1, \ldots, F_4$. This gives the photon scattering amplitude $\mathcal{M}$ as

$$ \mathcal{M} = -\frac{2 \alpha^2}{45 m^4} \left[ 5 (\text{tr} (F_1 F_2) \text{tr} (F_3 F_4)) + \text{tr} (F_1 F_3) \text{tr} (F_2 F_4) + \text{tr} (F_1 F_4) \text{tr} (F_2 F_3)) \\
-7 \text{tr} (F_1 F_2 F_3 F_4) + F_3 F_1 F_2 F_4 + F_2 F_3 F_1 F_4 + F_3 F_2 F_1 F_4 + F_1 F_3 F_2 F_4 + F_2 F_1 F_3 F_4 \right]. $$

The unpolarized cross section in the center-of-mass-frame becomes

$$ d\sigma = \frac{1}{4 (k_1 \cdot k_2)} \int \frac{d^3 k_3}{2 \omega_3 (2 \pi)^3} \frac{d^3 k_4}{2 \omega_4 (2 \pi)^3} (2 \pi)^4 \delta^4(k_3 + k_4 - k_1 - k_2) |\mathcal{M}|^2 \\
= \frac{1}{64 \omega^2 (2 \pi)^2} |\mathcal{M}|^2 d\Omega, $$

$^2$In [25] it is stated that one should also divide by a combinatorial factor of 4!, however this is erroneous, as was confirmed to us by J.B. Zuber.
\[
|\mathcal{M}|^2 \equiv \frac{1}{4} \sum_{\varepsilon_i} |\mathcal{M}|^2. \tag{2.6}
\]

The sums over polarization can be done using, for each photon leg, the identity
\[
\sum_{\varepsilon} F_{\alpha}^{\ast \beta} F_{\mu}^{\nu} = -k^{3} k^{\nu} g_{\alpha \mu} + k_{\mu} k^{3} g_{\alpha}^{\nu} + k_{\alpha} k^{\nu} g_{\mu}^{\beta} - k_{\alpha} k_{\mu} g^{\beta \nu}, \tag{2.7}
\]
with the result
\[
|\mathcal{M}|^2 = \frac{32 \cdot 139}{(90)^2} \alpha^4 \left[ (k_1 \cdot k_2)^2 (k_3 \cdot k_4)^2 + (k_1 \cdot k_3)^2 (k_2 \cdot k_4)^2 + (k_1 \cdot k_4)^2 (k_2 \cdot k_3)^2 \right]. \tag{2.8}
\]

Using the kinematic relations
\[
\begin{align*}
    k_1 \cdot k_2 &= k_3 \cdot k_4 = 2 \omega^2, \\
    k_1 \cdot k_3 &= k_2 \cdot k_4 = \omega^2 (1 - \cos \theta), \\
    k_1 \cdot k_4 &= k_2 \cdot k_3 = \omega^2 (1 + \cos \theta), \tag{2.9}
\end{align*}
\]
one finds
\[
\frac{d\sigma}{d\Omega} = \frac{1}{(2\pi)^2} \frac{139}{(90)^2} \alpha^4 \left( \frac{\omega}{m} \right)^6 \frac{1}{m^2} (3 + \cos^2 \theta)^2 \tag{2.10}
\]
for the differential cross section, and
\[
\sigma = \frac{1}{2 \pi} \frac{139}{(90)^2} \left( \frac{56}{5} \right) \alpha^4 \left( \frac{\omega}{m} \right)^6 \frac{1}{m^2} \tag{2.11}
\]
for the total cross section. See [26, 27, 28, 29] for the generalization to arbitrary energies, [30] for the generalization to the \( N \) - photon case.
2.2 Photon-photon scattering in Born-Infeld theory

The calculation of the cross section for the Born-Infeld case is completely analogous. Using the identity (2.1) to eliminate $\tilde{F}$ from the quartic term of the BIL (1.5) gives

$$L_{BI}^{4} = \frac{1}{32b^2} \left[ 4\text{tr}(F^4) - (\text{tr}(F^2))^2 \right].$$

(2.12)

Proceeding in the same way as before one obtains, after a lengthy calculation,

$$|M_{BI}|^2 = \frac{2}{b^4} \left[ (k_1 \cdot k_2)^2(k_3 \cdot k_4)^2 + (k_1 \cdot k_3)^2(k_2 \cdot k_4)^2 + (k_1 \cdot k_4)^2(k_2 \cdot k_3)^2 \right].$$

(2.13)

The differential cross section in the center-of-mass frame becomes

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\omega^2(2\pi)^2} |M_{BI}|^2 = \frac{1}{16(2\pi)^2} \left( \frac{\omega}{b} \right)^4 \frac{\omega^2}{\left( 3 + \cos^2 \theta \right)^2}. \quad (2.14)$$

Note that the angular dependence is the same as in the QED case. For the total cross section one obtains

$$\sigma_{BI} = \frac{1}{2\pi} \frac{7}{10} \frac{\omega^6}{b^4}. \quad (2.15)$$

When contemplating Born-Infeld theory as a phenomenologically viable extension of QED, we must include also the standard coupling to fermions. Thus both of the diagrams of Fig. 1 must be taken into account, including the interference terms. The total photon-photon scattering differential and total cross sections become

$$\frac{d\sigma}{d\Omega} = \left( \frac{1}{64} \frac{11 \alpha^2}{b^4} + \frac{139 \alpha^4}{32400} \frac{\omega^6}{m^8} \right) \frac{\omega^6}{\pi^2} \left( 3 + \cos^2 \theta \right)^2, \quad (2.16)$$

and

$$\sigma = \left( \frac{7}{20} \frac{77 \alpha^2}{b^4} + \frac{973 \alpha^4}{10125} \frac{\omega^6}{m^8} \right) \frac{\omega^6}{\pi}. \quad (2.17)$$
3 Photon splitting

We proceed to the calculation of the photon splitting amplitude in a constant magnetic field in Born-Infeld theory, in the leading order in the weak-field expansion. It is easily checked that, for the same kinematical reasons as in the QED case, also in Born-Infeld theory there is no contribution to the photon splitting amplitude from the quartic vertex in the BIL [1.5]. Thus, apart from the same diagram as in QED, Fig. 2, we have to consider the basic sextic vertex, Fig. 3. As in the photon-photon scattering case, we will first retrace the calculation of the standard QED contribution, and then indicate the necessary modifications to get the Born-Infeld contributions to the amplitude.

3.1 Photon splitting in QED

Since a constant magnetic field cannot absorb four-momentum, the four-vectors $k_1, k_2, k_3$ of the initial and final photons must obey energy-momentum conservation by themselves,

$$k_1 = \omega_1(1, \hat{k}_1) = k_2 + k_3 = \omega_2(1, \hat{k}_2) + \omega_3(1, \hat{k}_3). \quad (3.1)$$

Neglecting at first the modification of the photon dispersion relation due to the presence of the magnetic field, it is easy to see that, in vacuum, this implies collinearity of the three photons,

$$\hat{k}_1 = \hat{k}_2 = \hat{k}_3. \quad (3.2)$$

Thus the photon four-vectors are proportional, and one has the vanishing scalar products

$$k_1^2 = k_2^2 = k_3^2 = k_1 \cdot k_2 = k_1 \cdot k_3 = k_2 \cdot k_3 = 0. \quad (3.3)$$

It is this lack of nonzero Lorentz invariants that leads to the vanishing of the box diagram contribution to the photon splitting amplitude, and to the already mentioned property of the leading hexagon diagram contribution to be given by its low-energy limit. Thus we can compute this hexagon contribution exactly from the sextic term in the expansion of the EHL [1.9]. Using the identity (2.1) this term becomes
\[ L_{EH}^{6} = 8 \frac{\alpha^{3}\pi}{m^{8}} \left[ \frac{13}{1260} \text{tr}(F^{2})\text{tr}(F^{4}) - \frac{1}{280}(\text{tr}(F^{2}))^{3} \right]. \]  \hspace{1cm} (3.4)

Similarly to the photon-photon scattering case, we can get the matrix element from this by substituting \( F = F_{1} + F_{2} + F_{3} + F + F + F \), where the \( F_{i} \) are the photon field strength tensors and \( F \) is the one of the external field, and selecting the terms of the form \( F_{1}F_{2}F_{3}FFF \). But before starting on this, let us first establish the polarization selection rules. A priori, there are eight possible combinations of polarizations in the photon splitting process, whose matrix elements we will denote by

\[ \mathcal{M} \left[ \begin{pmatrix} \parallel \\ \parallel \end{pmatrix} \rightarrow \begin{pmatrix} \parallel \\ \perp \end{pmatrix} + \begin{pmatrix} \parallel \\ \perp \end{pmatrix} \right]. \]  \hspace{1cm} (3.5)

As shown in [17], four of them vanish on account of CP invariance; those are \( \mathcal{M}\left[ \begin{pmatrix} \parallel \\ \parallel \end{pmatrix} \rightarrow \begin{pmatrix} \parallel \\ \parallel \end{pmatrix} + \begin{pmatrix} \parallel \\ \parallel \end{pmatrix} \right] \), \( \mathcal{M}\left[ \begin{pmatrix} \parallel \\ \parallel \end{pmatrix} \rightarrow \begin{pmatrix} \parallel \\ \parallel \end{pmatrix} + \begin{pmatrix} \parallel \\ \parallel \end{pmatrix} \right] \), \( \mathcal{M}\left[ \begin{pmatrix} \parallel \\ \parallel \end{pmatrix} \rightarrow \begin{pmatrix} \parallel \\ \parallel \end{pmatrix} + \begin{pmatrix} \parallel \\ \parallel \end{pmatrix} \right] \), and \( \mathcal{M}\left[ \begin{pmatrix} \parallel \\ \parallel \end{pmatrix} \rightarrow \begin{pmatrix} \parallel \\ \parallel \end{pmatrix} + \begin{pmatrix} \parallel \\ \parallel \end{pmatrix} \right] \). But, as was discussed already in the introduction, here it is important to take also into account the change of the photon dispersion relation due to the magnetic field. This change can be described by polarization-dependent indices of refraction [20, 17, 14]

\[ n_{\parallel,\perp}(\omega) = \frac{k}{\omega}. \]  \hspace{1cm} (3.6)

To lowest order in the field, those are induced by the diagram of Fig. 4.

Figure 4: Diagram modifying the photon dispersion relation in QED.

In the low-frequency limit, they are given explicitly by (see, e.g., [14])

\[ n_{\parallel}^{QED} = 1 + \frac{14}{45} \frac{\alpha^2}{m^4} B^2 \sin^2 \theta, \]

\[ n_{\perp}^{QED} = 1 + \frac{8}{45} \frac{\alpha^2}{m^4} B^2 \sin^2 \theta. \]
Here $\theta$ is the angle between the photon propagation direction and the magnetic field. It is easy to see [17] that, with these nontrivial dispersion relations, photon splitting is compatible with energy-momentum conservation only if the following condition is fulfilled:

$$\Delta \equiv \omega_2n(\omega_2) + \omega_3n(\omega_3) - (\omega_2 + \omega_3)n(\omega_2 + \omega_3) \geq 0. \quad (3.8)$$

In [17] it is then shown that, for photon frequencies up to the pair creation threshold $\omega = 2m$ and arbitrary field strengths $B$, of the four polarization choices allowed by CP invariance only the case $\mathcal{M}[(\perp)_1 \rightarrow (\parallel)_2 + (\parallel)_3]$ fulfills the condition (3.8).

Returning to the hexagon approximation, choosing polarization vectors corresponding to this particular component one obtains from (3.4), after a simple calculation (which moreover needs to be done only for $\sin \theta = 1$, by Lorentz invariance) the matrix element

$$\mathcal{M}[(\perp)_1 \rightarrow (\parallel)_2 + (\parallel)_3] = \frac{13}{315} e^3 \frac{(eB \sin \theta)^3}{\pi^2 m^8} \omega_1 \omega_2 \omega_3. \quad (3.9)$$

From this the absorption coefficient $\kappa$ is obtained as

$$\kappa[(\perp)_1 \rightarrow (\parallel)_2 + (\parallel)_3] = \frac{1}{32 \pi \omega_1^2} \int_0^{\omega_1} d\omega_2 \int_0^{\omega_1} d\omega_3 \delta(\omega_1 - \omega_2 - \omega_3) \times |\mathcal{M}[(\perp)_1 \rightarrow (\parallel)_2 + (\parallel)_3]|^2 = \left( \frac{13}{315} \frac{e^3}{\pi^2 m^8} \right)^2 \frac{\omega_1^5 (eB \sin \theta)^6}{960 \pi}. \quad (3.10)$$

For applications it is useful to introduce the “critical” field strength $B_{cr} \equiv m^2/e$, and rewrite $\kappa$ in terms of dimensionless ratios as

$$\frac{\kappa}{m} = \frac{13^2}{35 \times 5^3 \times 7^2 \pi^2} \frac{\alpha^3}{(B_{cr} \sin \theta)^6} \left( \frac{\omega_1}{m} \right)^5. \quad (3.11)$$

Once the modified dispersion relations are taken into account one needs to also reanalyze the contribution of the box diagram, as well as the CP-forbidden polarization choices [17]. It turns out that now those are in general nonzero, but the matrix elements of the former are still down by a factor of order $\alpha$ with respect to the hexagon contribution, and the ones of the latter even by a factor of the order of $\alpha(B/B_{cr})^2$. 

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3.2 Photon splitting in Born-Infeld theory

Proceeding to the Born-Infeld case, we can see immediately that, at least in the weak field approximation, there are no essential differences to the QED case. The lowest order contribution to the photon splitting amplitude now comes from the quartic term in the BIL (2.12) with one photon leg substituted by the interaction with the magnetic field, and with the vacuum dispersion relation it vanishes for the same kinematic reasons as in QED. The leading contribution to the matrix element thus comes from the sextic vertex, shown in Fig. 3, and the CP-induced part of the selection rules holds for it as well. As to the selection rules coming from the modified dispersion relation, these would not hold in pure Born-Infeld theory due to the absence of birefringence; the photon propagation is modified, but the corresponding index of refraction \( n_{\text{BI}}(\omega) \) does not depend on polarization, i.e. \( n_{\parallel}^{\text{BI}} = n_{\perp}^{\text{BI}} = n^{\text{BI}} \). For example, to lowest order in the field in pure Born–Infeld theory the index of refraction would come from the first of the diagrams shown in Fig. 5 where the quartic vertex now appears with two photon and two magnetic field legs.

\[
\begin{align*}
\frac{kk}{B} + \frac{kk}{B}
\end{align*}
\]

Figure 5: Diagrams modifying the photon dispersion relation in BI theory.

It is easily calculated that this diagram alone would yield a refraction index

\[
n_{\parallel,\perp}^{\text{BI}} = 1 + \frac{1}{2} \left( \frac{B \sin \theta}{b^2} \right)^2,
\]

confirming the independence of polarization. However, if as usual we assume also the presence of the QED diagrams, we will get a total refraction index from the sum of both diagrams of Fig. 5 that combines (3.7) and (3.12)

\[
n_{\parallel,\perp}^{\text{total}} = n_{\parallel,\perp}^{\text{QED}} + \frac{1}{2} \left( \frac{B \sin \theta}{b^2} \right)^2
\]

(3.13)
Since a shift of both refraction indices by the same amount drops out of the selection condition (3.8) we then find the same selection rules as above. Thus as in the QED case photon splitting is, at leading order in $\alpha$ and in the weak field expansion, possible only for the combination $(\|)_1 \rightarrow (\perp)_2 + (\|)_3$. The calculation of the matrix element from the sextic term in the BIL (1.5) again parallels the QED calculation, and we find

$$\mathcal{M}^{\text{BI}}[(\perp)_1 \rightarrow (\|)_2 + (\|)_3] = \frac{(B \sin \theta)^3}{h^4} \omega_1 \omega_2 \omega_3. \quad (3.14)$$

Adding this to the QED matrix element (3.9) leads to the total absorption coefficient

$$\kappa^{\text{total}}[(\perp)_1 \rightarrow (\|)_2 + (\|)_3] = \left[ \frac{1}{h^4} + \frac{13 \times 64 \alpha^3 \pi}{315 m^8} \right]^2 (B \sin \theta)^6 \frac{\omega_1^5 \omega_2 \omega_3}{960 \pi}. \quad (3.15)$$

As yet another analogy to the QED case, this result for the leading contribution to the photon splitting decay rate would, for the same kinematic reasons that make the QED hexagon contribution exact in the weak-field limit, not be affected by the addition of derivative corrections to the sextic vertex in the BIL. Thus, unlike the case of photon-photon scattering, here even away from the low energy limit $\omega/m \ll 1$ it does not make a difference whether we consider the BIL as fundamental or effective.

It should be noted that, at the same order of the hexagon diagram, in Born-Infeld theory there are also diagrams with two quartic vertices connected by a virtual photon, see Fig. 6.

![Figure 6: Photon splitting in a magnetic field in BI theory.](image)
However, for a constant field in this type of graph the momentum of the internal photon is either zero or on-shell, so that these contributions will be removed by renormalization (although Born-Infeld theory is not renormalizable, the photon propagator must, of course, be renormalized as for the QED case in any halfway realistic setting).

4 Comparison with experiment

Let us now discuss what possible bounds on the Born-Infeld parameter $b$ might be obtained experimentally through the processes which we have discussed here. Since we always have a Born-Infeld as well as a QED contribution, let us first establish the precise numerical relations between both contributions for the various cases.

For the case of the photon-photon cross section, we find from (2.17) that equality of the pure Born-Infeld and the pure QED contributions would fix the value of $b$ as

$$b = 189.3 \, m^2 = 7.59 \times 10^{19} \frac{V}{m}. \quad (4.1)$$

For the photon splitting absorption coefficient, the same requirement gives, from (3.15),

$$b = 23.6 \, m^2 = 9.46 \times 10^{18} \frac{V}{m}. \quad (4.2)$$

Let us also consider the case of the refraction indices $n_{\perp, \parallel}$. Here (3.7) and (3.13) lead to

$$b = \frac{650.0}{\sqrt{a_{\perp, \parallel}}} \, m^2 = \frac{2.61}{\sqrt{a_{\perp, \parallel}}} \times 10^{20} \frac{V}{m}. \quad (4.3)$$

where $a_{\perp} = 8$ and $a_{\parallel} = 14$. Note that all these values for $b$ are numerically close to the one of the original Born-Infeld theory, (1.6). For (4.1) and (4.3) this fact is, of course, related to Infeld’s observation about the four-point vertices mentioned in the introduction.

The fact that these numbers are two or three orders of magnitudes below the bound achieved by Soff et al. [7] makes it already clear that improving on that bound by low-energy photon processes will be difficult. One would
need to find a case where the pure QED contribution is not only experimentally accessible, but can be measured with sufficient precision to exclude an additional Born-Infeld contribution at the sub-percent level. Let us thus focus on the question for which of the processes induced by the four- and six-point vertices the QED contribution is measurable at present, or may become so in the near future.

As is well-known, the QED four-photon amplitude is tested in Delbrück scattering, the scattering of a photon in the electric field of the atomic nucleus (given by the diagram of fig.1, left, with two photon legs and two legs representing the interaction with the field), which was discovered as early as 1933 and thus even predated the Euler-Heisenberg calculation. However, Delbrück scattering is of limited relevance for our present investigation, since it is known that for this process derivative corrections to the effective QED Lagrangian cannot be neglected. Thus also in the Born-Infeld case we would have to assume either the total absence of such corrections, or come up with a more specific model. For the same reason we will not consider here processes involving the scattering of high-energy photons (see the recent [31] for a discussion of the prospects of a direct verification of photon-photon scattering at the LHC).

In our context, the cleanest possible test of the four-vertex would be by laser scattering, where the low-energy approximation is generally justified. But for optical frequencies the QED cross section (2.11) is extremely small, due to the factor of \((\omega m)^6\). The experimental state-of-the-art, presently defined by an experiment performed in 2000 by Bernard et al. [32], still leaves a gap of 18 orders of magnitude between the QED cross section and its experimental verification. However, with present-day laser technology one could presumably come much closer to the cross section. See [33, 34, 35, 36] for various projections of what may be achievable in laser scattering in the near future.

Alternatively one may also try to test the four-vertex through the low-energy dispersion relations (3.7), (3.12). Here the polarization-independence of the Born-Infeld correction (3.12) constitutes a disadvantage, since a measurement of the absolute magnitudes of \(n_{\parallel,\perp}\), or of their average, must be done rather than of the difference \(n_{\parallel} - n_{\perp}\), so that birefringence experiments such as PVLAS cannot be used (although birefringence can and has been used to place bounds on more general nonlinear electromagnetic theories [37]). As discussed by various authors [38, 39, 40, 41], the measurement of the QED refraction indices may soon become viable using large scale laser interferometers such as LIGO, GEO or VIRGO, simply by inducing interference through the application of a strong magnetic field perpendicularly...
to one of the two legs of the interferometer. The confirmation of the QED refraction indices would lead to a lower bound on $b$ roughly equal to (4.3).

Coming to the case of photon splitting, to the best of our knowledge its only definite observation so far was achieved for the electric field case, and again using the Coulomb field of heavy atoms [42]. However, this now corresponds to a calculation entirely different from the one above since the box diagram does not drop out in the Coulomb field case, so that the inclusion of derivative corrections becomes even more imperative than for Delbrück scattering. Keeping thus to the (constant) magnetic field case, due to the factor of $(B/B_{cr})^6$ in (3.15) measurable effects can be expected only for magnetic fields close to $B_{cr}$; in laboratory experiments such as PVLAS magnetic photon splitting in principle contributes to dichroism, but the effect is tiny. Thus the natural application of the magnetic effect, which was in fact already the motivation for Adler’s original calculation in 1971 [17], is to the physics of neutron stars, which are known to have surface magnetic fields close to or even exceeding $B_{cr}$ (for a recent review see [43]). Photon splitting could be seen in the spectra of neutron stars both through its softening and its polarizing effect, and, although no definite detection seems to have been reported as of date, recent results from magnetars are suggestive of such a softening effect [44]. Although the study of [44] concerns hard x-rays, due to the absence of derivative corrections to the hexagon diagram here this does not necessarily lead us out of the range of the applicability of the low photon-energy approximation. For $B/B_{cr} \sim 1$ higher-point corrections can be sizable, but according to the numerical studies of [17, 18] at least in the QED case do not tend to reduce the hexagon contribution (this is in contrast to the Coulomb field photon splitting where it has been shown that the leading four-point contribution significantly overestimates the amplitude [45]). Thus a confirmation in this context of the magnetic photon splitting effect as predicted by QED would effectively place a lower bound on $b$ of the order of magnitude of (4.2).

5 Conclusions

To summarize, we have obtained here the following tree-level quantities in Born-Infeld theory: the total photon-photon scattering cross section, the photon splitting amplitude in a constant magnetic field in the leading (hexagon) approximation, and also the related refractive indices in a weak constant field. We have also shown that Adler’s selection rules for magnetic photon splitting in QED hold for the Born-Infeld case unchanged.
Each of these tree-level quantities has a one-loop analogue in QED, and can in principle be used for constraining the Born-Infeld parameter $b$. Discussing various experimental options we have come to the conclusion that what is achievable along these lines in the near future is a lower bound on $b$ of about the value of the original Born theory, eq. (1.6). Finally, we would like to point out that photon splitting may turn out more useful for testing more general non-linear electromagnetic theories, since for those contrary to the Born-Infeld case Adler’s selection rules may be violated.

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