Semi-Robust Replication of Barrier-Style Claims on Price and Volatility

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ABSTRACT
We show how to price and replicate a variety of barrier-style claims written on the log price \(X\) and quadratic variation \(<X>\) of a risky asset. Our framework assumes no arbitrage, frictionless markets and zero interest rates. We model the risky asset as a strictly positive continuous semimartingale with an independent volatility process. The volatility process may exhibit jumps and may be non-Markovian. As hedging instruments, we use only the underlying risky asset, zero-coupon bonds, and European calls and puts with the same maturity as the barrier-style claim. We consider knock-in, knock-out and rebate claims in single and double barrier varieties.

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1. Introduction
Barrier options are the most liquid of the second generation options, (i.e., options whose payoffs are path-dependent). In his landmark work, Merton (1973) first valued a down-and-out call in closed form when the underlying stock follows geometric Brownian motion. So long as the instantaneous volatility is a known function of the stock price and time, one can replicate any barrier claim by dynamically trading the stock and a zero-coupon bond. If the volatility process is a continuous stochastic process driven by a second independent source of uncertainty, then one must also dynamically trade an option. As with any hedge, the hedging strategy is invariant to the expected rate of return of the underlying stock.

Bowie and Carr (1994) show how a static hedge in European options can be used to hedge down-and-out calls on futures in the Black model. Essentially, the payoff of a down-and-out call with barrier \(H\) can be replicated by buying a European call on the same underlying futures price with the same maturity \(T\) and strike \(K\) and also selling \(K/H\) puts with strike \(H^2/K\). Carr, Ellis, and Gupta (1998) make clear that this static hedge works in any model with deterministic local volatility, provided that the volatility function is symmetric in the log of the futures price relative to the barrier. We thus see a continuation of the pattern initiated by Merton in which hedging strategies are invariant to aspects of the statistical process.

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Roger Lee and Matthew Lorig dedicate this paper to the memory of their friend and colleague Peter Carr.
In Carr, Ellis, and Gupta (1998), increments in the instantaneous volatility process are conditionally perfectly correlated with increments in the underlying futures price. Andreasen (2001) points out that the above hedge also works when increments in the instantaneous volatility process are conditionally independent of increments in the forward price. Similarly, Bates (1997) observes that the above hedge works for ‘Hull and White-type stochastic volatility processes.’ While Bates does not define this terminology, it seems reasonable to assert that both authors are assuming that the instantaneous volatility process is a diffusion, i.e., that the volatility process is continuous over time and has the strong Markov property, as in Hull and White (1987). Furthermore, as in Hull and White (1987), the volatility process should be autonomous in that its evolution coefficients refer only to volatility and time, but not the price of the underlying asset.

Carr and Lee (2009) make clear that these conditions are merely sufficient, but not necessary. The hedge described above for a down-and-out call works perfectly provided that there are no jumps over the barrier and the call and put have the same implied volatility at the first passage time to the barrier, if any. We refer to the latter condition as Put Call Symmetry (PCS), which was introduced to finance by Bates (1988) as a way to measure skewness. Hence, the bivariate process for the futures price and its volatility need not be Markov in itself. Furthermore, jumps in price and volatility can occur and increments in volatility can be correlated with returns, although some restrictions are necessary. As a result, we refer to these hedge strategies as semi-robust.

Barrier options are not the only path-dependent claims for which semi-robust hedges exist. All of the barrier option hedges also extend to lookbacks. For lookbacks, the hedge is semi-static in that standard options are traded each time a new maximum is reached. Furthermore, assuming only no arbitrage, frictionless markets, zero interest rates, a positive continuous futures price process, and an independent volatility process, Carr, Lee and Lorig (2021) show how to replicate a variety of claims on the quadratic variation of returns experienced between initiation and a fixed maturity date. Their hedging instruments consist of the underlying futures and European options written on these futures at all strikes and with the same maturity as the claim. In contrast to the hedges of barrier and lookback claims, their trading strategy in options is fully dynamic. Examples of claims whose payoffs can be spanned include volatility swaps and options on realized variance.

The purpose of this manuscript is to synthesize the literature on semi-robust hedging of barrier claims and claims linked to quadratic variation. In particular, we show how to price and hedge claims on the log price $X$ and the quadratic variation $\langle X \rangle$ of a risky asset subject to certain barrier events either occurring or not occurring. Examples of such claims include (i) a barrier start variance or volatility swap for which the non-negative payoff is the variance or volatility of log price experienced between the first passage time and a fixed maturity date, (ii) a barrier start claim whose final payoff is the realized Sharpe ratio calculated between the first passage time and the fixed maturity date, (iii) single and double barrier knock-out claims that, in the event no knock-out occurs, pay the product of powers and exponentials of log price and quadratic variation, and (iv) a single barrier rebate claim that pays the product of powers and exponentials of quadratic variation if and when a barrier is reached prior to maturity.

Our analysis makes the same assumptions as Carr and Lee (2008). In particular, we consider a continuous time stochastic process for instantaneous volatility whose increments are uncorrelated with returns. Jumps in the volatility process are allowed and the evolution
coefficients of the volatility process can refer to past or present values of the instantaneous volatility, time, and other variables as well, provided that they are independent of the futures price (i.e., non-Markovian dynamics are allowed for the volatility process). Both foreign exchange and bond markets exhibit symmetric smiles, which, in a stochastic volatility setting, implies a volatility process that is uncorrelated with returns of the underlying (Carr and Lee 2009, Theorem 3.4). Thus, our results are particularly relevant for these markets.

We note that there is a vast and developing literature on robust model-free (super)-replication of path-dependent claims. For an overview of this field of research, we refer the reader to Henry-Labordère (2017) and references therein.

The rest of this paper proceeds as follows. In Section 2, we introduce a general market model for a single risky asset \( S_t = e^{X_t} \) and state our main assumptions. In Section 3 we review and extend the results from Carr and Lee (2008) for pricing and replicating claims on \((X_T, \langle X \rangle_T)\). These results will be needed for the barrier-style claims considered in Sections 4, 5 and 6. Section 4 focuses on knock-out claims, Section 5 examines knock-in claims and Section 6 studies rebate claims. Concluding remarks and directions for future research are offered in Section 7.

2. Model and Assumptions

We consider a frictionless market (i.e., no transaction costs) and fix an arbitrary but finite time horizon \( T < \infty \). We assume zero interest rates\(^1\), no arbitrage, and take as given an equivalent martingale measure (EMM) \( \mathbb{P} \) chosen by the market on a complete filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\). The filtration \( \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T} \) represents the history of the market. All stochastic processes defined below live on this probability space and all expectations are with respect to \( \mathbb{P} \) unless otherwise stated.

Let \( B = (B_t)_{0 \leq t \leq T} \) represent the value of a zero-coupon bond maturing at time \( T \). As the risk-free rate of interest is zero by assumption, we have \( B_t = 1 \) for all \( t \in [0, T] \). Let \( S = (S_t)_{0 \leq t \leq T} \) represent the value of a risky asset. We assume \( S \) is strictly positive and has continuous sample paths. To rule out arbitrage, it is well-known that the asset \( S \) must be a martingale under the pricing measure \( \mathbb{P} \). As such, there exists a non-negative, \( \mathbb{F} \)-adapted stochastic process \( \sigma = (\sigma_t)_{0 \leq t \leq T} \) such that

\[
dS_t = \sigma_t S_t dW_t, \quad S_0 > 0,
\]

where \( W \) is a Brownian motion with respect to the pricing measure \( \mathbb{P} \) and the filtration \( \mathbb{F} \). Henceforth, the process \( \sigma \) will be referred to as the volatility process. We assume that the volatility process \( \sigma \) is right-continuous and \( \mathbb{F} \)-adapted, that it evolves independently of \( W \) and that it satisfies

\[
\int_0^T \sigma_t^2 dt < c < \infty,
\]

for some arbitrarily large but finite constant \( c > 0 \). Note that \( \sigma \) may experience jumps and is not required to be Markovian. Define the log price process \( X = (X_t)_{0 \leq t \leq T} \) by

\[
X_t = \log S_t.
\]
As \( S > 0 \), the process \( X \) is well-defined and finite for all \( t \in [0, T] \). By Itô’s Lemma,
\[
dX_t = -\frac{1}{2} \sigma_t^2 dt + \sigma_t dW_t, \quad X_0 = \log S_0.
\]
Note that a claim on (the path of) \( S \) can always be expressed as a claim on (the path of) \( X = \log S \).

For any \( \mathbb{F} \)-stopping time \( \tau \), define its \( T \)-bounded counterpart, the stopping time
\[
\tau^* := \tau \wedge T.
\]
Let \( C_{\tau^*}(K) \) denote the time \( \tau^* \) price of a European call written on \( S \) with maturity date \( T \) and strike price \( K > 0 \), and let \( P_{\tau^*}(K) \) denote the price of a European put written on \( S \) with the same strike and maturity. By no-arbitrage arguments,
\[
C_{\tau^*}(K) = E_{\tau^*}(S_T - K)^+ = E_{\tau^*}(e^{X_T} - K)^+,
\]
\[
P_{\tau^*}(K) = E_{\tau^*}(K - S_T)^+ = E_{\tau^*}(K - e^{X_T})^+,
\]
where we have introduced the shorthand notation \( E_{\tau^*} \cdot := E[ \cdot | \mathcal{F}_{\tau^*} ] \). For convenience, we will sometimes refer to a European call or put written on \( X \) rather than \( S \) with the understanding that these are equivalent. We assume that a European call or put with maturity \( T \) trades at every strike \( K \in (0, \infty) \). As demonstrated by Breeden and Litzenberger (1978), this assumption is equivalent to knowing the distribution of \( X_T \) under \( P \). This assumption additionally guarantees, as Carr and Madan (1998) show, that any \( T \)-maturity European claim on \( X_T \) can perfectly hedged with a static portfolio of the bonds \( B \), shares of the underlying \( S \) and calls and puts. Although in reality, calls and puts trade at only finitely many strikes, our results retain relevance; Leung and Lorig (2016) show how to adjust static hedges optimally when calls and puts are traded at only discrete strikes in a finite interval.

3. European-style Claims

Under the assumptions of Section 2, Carr and Lee (2008) show how to price and replicate the real and imaginary parts of a claim with a payoff of the form \( e^{i \omega X_T + i s \langle X \rangle_T} \), where \( \omega, s \in \mathbb{C} \). They then use these exponential claims as building blocks to price and replicate more general claims with payoffs of the form \( \varphi(X_T, \langle X \rangle_T) \). In this section, we briefly review the main results from Carr and Lee (2008) and derive some extensions needed in subsequent sections.

Throughout this paper, we will distinguish between European claims, which have path-independent payoffs of the form \( \varphi(X_T) \), and European-style claims, which have path-dependent payoffs of the form
\[
\text{European — style : } \varphi(X_T, \langle X \rangle_T).
\]
We use the phrase ‘European-style’ to indicate that a claim payoff depends only on the terminal values \( X_T \) and \( \langle X \rangle_T \) and not on any barrier event (e.g., knock-in or knock-out).

3.1. Pricing and Replicating Power-exponential Payoffs

In what follows, we shall consider claims with \( \mathbb{C} \)-valued payoffs. The pricing and hedging results are understood to hold for the real and imaginary parts separately. We begin by relating the characteristic function of \( (X_T, \langle X \rangle_T) \) to the characteristic function of \( X_T \) only.
Theorem 3.1: Let $\omega, s \in \mathbb{C}$. Define $u : \mathbb{C}^2 \to \mathbb{C}$ as either of the following

$$u \equiv u_{\pm}(\omega, s) = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \omega^2 - i\omega + 2is}.$$  \hspace{1cm} (3)

Then, for any $\mathcal{F}$-stopping time $\tau$, we have

$$\mathbb{E}_{\tau^*}e^{i\omega X_T + is(X)_T} = e^{i(\omega - u)X_{\tau^*} + is(X)_{\tau^*}}\mathbb{E}_{\tau^*}e^{iuX_T}.$$  \hspace{1cm} (4)

Proof: A proof of Theorem 3.1 is given in Carr, Lee and Lorig (2021, Lemma 3.2). We repeat it here as the conditioning arguments below will be used in subsequent sections. Let $\mathcal{F}_T^\sigma$ denote the sigma-algebra generated by $(\sigma_t)_{0 \leq t \leq T}$. Then $(X)_T - (X)_{\tau^*}) \in \mathcal{F}_{\tau^*} \lor \mathcal{F}_T^\sigma$ and

$$X_T - X_{\tau^*} | \mathcal{F}_{\tau^*} \lor \mathcal{F}_T^\sigma \sim \mathcal{N}(m, \nu^2), \quad m = -\frac{1}{2}((X)_T - (X)_{\tau^*}), \quad \nu^2 = (X)_T - (X)_{\tau^*}. \hspace{1cm} (5)$$

Thus, by the characteristic function of a normal random variable

$$Z \sim \mathcal{N}(m, \nu^2), \quad \mathbb{E}e^{i\omega Z} = e^{im\omega - \frac{1}{2}\nu^2\omega^2}, \hspace{1cm} (6)$$

we have

$$\mathbb{E}_{\tau^*}e^{i\omega (X_T - X_{\tau^*}) + is((X)_T - (X)_{\tau^*})}$$

$$= \mathbb{E}_{\tau^*}e^{is((X)_T - (X)_{\tau^*})}\mathbb{E}[e^{i\omega (X_T - X_{\tau^*})} | \mathcal{F}_{\tau^*} \lor \mathcal{F}_T^\sigma]$$

$$= \mathbb{E}_{\tau^*}\mathbb{E}[e^{i\omega (X_T - X_{\tau^*})/(X)_T - (X)_{\tau^*})} | \mathcal{F}_{\tau^*} \lor \mathcal{F}_T^\sigma] \quad \text{(by } (5) \text{ and } (6))$$

$$= \mathbb{E}_{\tau^*}\mathbb{E}[e^{-i\omega (\omega + iu)/2}((X)_T - (X)_{\tau^*}) | \mathcal{F}_{\tau^*} \lor \mathcal{F}_T^\sigma] \quad \text{(by } (3))$$

$$= \mathbb{E}_{\tau^*}\mathbb{E}[e^{i\omega X_T - X_{\tau^*})} | \mathcal{F}_{\tau^*} \lor \mathcal{F}_T^\sigma] \quad \text{(by } (5) \text{ and } (6))$$

$$= \mathbb{E}_{\tau^*}e^{i\omega (X_T - X_{\tau^*})}. \hspace{1cm} (7)$$

Multiplying (7) by $e^{i\omega X_{\tau^*} + is(X)_{\tau^*}}$ yields (4). \hfill \blacksquare

Corollary 3.2: Fix $\omega, s \in \mathbb{C}$ and $n, m \in \{0\} \cup \mathbb{N}$. Assume $\frac{1}{4} - \frac{1}{2}\omega + 2is - \omega^2 \neq 0$. Let $u : \mathbb{C}^2 \to \mathbb{C}$ be as defined in (3). Then

$$\mathbb{E}_{\tau^*}X^n_T(X)^m_T e^{i\omega X_T + is(X)_T}$$

$$= \mathbb{E}_{\tau^*} \sum_{j=0}^{n} \sum_{k=0}^{m} \binom{n}{j} \binom{m}{k} (-\frac{1}{2}\partial_{\omega})^j (-\frac{1}{2}\partial_{s})^k e^{i(\omega - u(\omega, s))X_{\tau^*} + is(X)_{\tau^*}}$$

$$\times (-\frac{1}{2}\partial_{\omega})^{n-j} (-\frac{1}{2}\partial_{s})^{m-k} e^{i\omega X_T}. \hspace{1cm} (8)$$
Proof: We have
\[
\mathbb{E}_{\tau^*} X^n_T (X)_T e^{i \omega X_T + i s (X)_T} = (-i \partial_{\omega})^n (-i \partial_s)^m \mathbb{E}_{\tau^*} e^{i \omega X_T + i s (X)_T}
\]
\[
= (-i \partial_{\omega})^n (-i \partial_s)^m e^{i (\omega - u (\omega, s)) X_{\tau^*} + i s (X_{\tau^*})} \mathbb{E}_{\tau^*} e^{i u (\omega, s) X_T}
\]
\[
= \text{R.H.S. of (8)},
\]
where the first equality follows from the Leibniz integral rule, the second equality follows from Theorem 3.1, and the last equality follows from the Leibniz integral rule and algebra. The two applications of the Leibniz rule are justified as follows: for any \( n, m \in \{0\} \cup \mathbb{N} \) and \( \omega, s \in \mathbb{C} \) there exists a constant \( c_1 > 0 \) such that
\[
| \partial^n \omega \partial_s^m e^{i \omega x + i s y} | < c_1 e^{c_1 (|x| + |y|)}, \quad \mathbb{E}_0 c_1 e^{c_1 (|X_T| + |(X)_T|)} < \infty, \tag{9}
\]
where the finiteness of the expectation follows from (1).

Remark 3.3 (Notation): Throughout this manuscript, when it causes no confusion, we will omit the subscript \( \pm \) and the arguments \( (\omega, s) \) from \( u_{\pm} (\omega, s) \) (and other functions/processes) in order to ease notation.

We now recall a classical result from Carr and Madan (1998). Suppose a function \( f \) can be expressed as the difference of convex functions. Then \( f \) can be represented as a linear combination of call and put payoffs. Specifically, for any \( \kappa \in \mathbb{R}_+ \) we have
\[
f(s) = f(\kappa) + f'(\kappa) \left( (s - \kappa)^+ - (\kappa - s)^+ \right) + \int_0^\kappa f''(K)(K - s)^+ dK
\]
\[
+ \int_\kappa^\infty f''(K)(s - K)^+ dK. \tag{10}
\]
Here, \( f' \) is the left-derivative of \( f \) and \( f'' \) is the second derivative, which exists as a generalized function. Replacing \( s \) in (10) with the random variable \( S_T \), choosing \( \kappa = S_{\tau^*} \) and taking the \( \mathcal{F}_{\tau^*} \)-conditional expectation, one obtains
\[
\mathbb{E}_{\tau^*} f(S_T) = f(S_{\tau^*}) B_{\tau^*} + \int_0^{S_{\tau^*}} f''(K) P_{\tau^*}(K) dK + \int_{S_{\tau^*}}^\infty f''(K) C_{\tau^*}(K) dK,
\]
using \( B_{\tau^*} = 1 \) and (2). Choosing \( f \) so that \( f(e^{X_T}) \) is equal to the right-hand side of (8) one obtains the price of a European-style power-exponential claim \( \mathbb{E}_{\tau^*} X^n_T (X)_T e^{i \omega X_T + i s (X)_T} \) in terms of (observable) European call and put prices.

Having priced European-style power-exponential claims relative to calls and puts, we turn to replication. Given a \( \mathbb{C}^d \)-valued price process \( V = (V_t)_{0 \leq t \leq T} \), a \( \mathbb{C}^d \)-valued portfolio process \( \Delta = (\Delta_t)_{0 \leq t \leq T} \) is said to be self-financing if its value \( \Pi \) satisfies
\[
d\Pi_t = \sum_{i=1}^d \Delta^i_{t-} dV^i_t, \quad \text{where } \Pi_t := \sum_{i=1}^d \Delta^i_t V^i_t \tag{11}
\]
If both \( \Delta \) and \( V \) are \( \mathbb{R}^d \)-valued, this definition corresponds to the usual notion of a self-financing portfolio. The following theorem gives a self-financing replication strategy for European-style exponential claims.
Theorem 3.4 (Replication of European-style exponential claims): Fix \( \omega, s, \in \mathbb{C} \) and define processes \( N = (N_t)_{0 \leq t \leq T} \) and \( Q = (Q_t)_{0 \leq t \leq T} \) by
\[
N_t = N_t(\omega, s) := e^{i(\omega-u)X_t + is(X)_t}, \quad Q_t = Q_t(\omega, s) := \mathbb{E}_t e^{i u X_T},
\] (12)
where \( u = u(\omega, s) \) is as given in (3). Define \( \Pi = (\Pi_t)_{0 \leq t \leq T} \) by
\[
\Pi_t = \Pi_t(\omega, s) = N_t Q_t + \left( \frac{i(\omega-u)N_t Q_t -}{S_t} \right) S_t + (-i(\omega-u)N_t Q_t -) B_t.
\] (13)
Then the portfolio \( (N_t, i(\omega-u)N_t Q_t -/S_t, -i(\omega-u)N_t Q_t -) \) of assets \( (Q, S, B) \) is self-financing in the sense of (11), and it has terminal value
\[
\Pi_T = e^{i \omega X_T + is(X)_T}.
\] (14)

Proof: From (13), with \( B_t = 1 \), we have \( \Pi_t = N_t Q_t \) at any time \( t \in [0, T] \). In particular, at the maturity date \( T \), using (12), we have
\[
\Pi_T = N_T Q_T = e^{i (\omega-u)X_T + is(X)_T} \mathbb{E}_T e^{i u X_T} = e^{i \omega X_T + is(X)_T},
\]
which establishes (14). To prove that \( \Pi \) satisfies the self-financing condition (11), observe that
\[
\mathbb{E}_t e^{i \omega X_T + is(X)_T} = e^{i (\omega-u)X_T + is(X)_T} \mathbb{E}_t e^{i u X_T} = N_t Q_t = \Pi_t.
\] (15)
The left-hand side of (15) is a martingale by iterated conditioning. Thus, the process \( \Pi \) must also be a martingale. The process \( Q \) must be a martingale by the same reasoning. Next,
\[
d\Pi_t = d(N_t Q_t) = N_t dQ_t + Q_t - dN_t + d[N, Q]_t
= N_t dQ_t + \left( \frac{i(\omega-u)N_t Q_t -}{S_t} \right) dS_t + dA_t,
\]
where \( A = (A_t)_{0 \leq t \leq T} \) has finite variation. As \( \Pi, Q \) and \( S \) are martingales, it follows that \( A \) is a local martingale. Moreover, as sample paths of \( S \) are continuous, so too are the sample paths of \( N \) and, hence, the sample paths of \( A \). As a finite variation, continuous local martingale, \( A \) must be constant. Thus \( dA_t = 0 \) and
\[
d\Pi_t = N_t dQ_t + \left( \frac{i(\omega-u)N_t Q_t -}{S_t} \right) dS_t
= N_t dQ_t + \left( \frac{i(\omega-u)N_t Q_t -}{S_t} \right) dS_t + (-i(\omega-u)N_t Q_t -) dB_t,
\] (16)
using \( dB_t = 0 \). Comparing (13) with (16) establishes the self-financing condition.

Corollary 3.5 (Replication of European-style power-exponential claims): Fix \( \omega, s, \in \mathbb{C} \) such that \( 2i s - \omega^2 - i \omega + \frac{1}{4} \neq 0 \). For any \( n, m \in \{0\} \cup \mathbb{N} \) let the processes \( N^{(n, m)} = \)
\[(N_t^{(n,m)})_{0 \leq t \leq T} \text{ and } Q_t^{(n,m)} = (Q_t^{(n,m)})_{0 \leq t \leq T} \text{ be given by}\]

\[
N_t^{(n,m)}(\omega, s) := (-\hat{i} \partial_{\omega})^n (-\hat{i} \partial_s)^m e^{i(\omega - u)X_t + is(X)_t}, \tag{17}
\]

\[
Q_t^{(n,m)}(\omega, s) := E_t(-\hat{i} \partial_{\omega})^n (-\hat{i} \partial_s)^m e^{iuX_T}, \tag{18}
\]

where \( u \equiv u(\omega, s) \) is as given in (3). Define the process \( \Pi_t^{(n,m)} = (\Pi_t^{(n,m)})_{0 \leq t \leq T} \) by

\[
\Pi_t^{(n,m)}(\omega, s) = \sum_{j=0}^{n} \sum_{k=0}^{m} \binom{n}{j} \binom{m}{k} N_t^{(j,k)} Q_t^{(n-j,m-k)} + \left( (-\hat{i} \partial_{\omega})^n (-\hat{i} \partial_s)^m \frac{i(\omega - u)N_tQ_{t-}}{S_t} \right) S_t + \left( (-\hat{i} \partial_{\omega})^n (-\hat{i} \partial_s)^m i(\omega - u)N_tQ_{t-} \right) B_t, \tag{19}
\]

Then \( \Pi^{(n,m)} \) is the value of a self-financing portfolio in the sense of (11) and satisfies

\[
\Pi_T^{(n,m)} = X_T^n(X) T e^{i\omega X_T + is(X)_T}. \tag{20}
\]

**Proof:** Throughout this proof, all uses of the Leibniz rule are justified by (9). From (19), at any time \( t \in [0, T] \) we have

\[
\Pi_t^{(n,m)} = \sum_{j=0}^{n} \sum_{k=0}^{m} \binom{n}{j} \binom{m}{k} N_t^{(j,k)} Q_t^{(n-j,m-k)} = (-\hat{i} \partial_{\omega})^n (-\hat{i} \partial_s)^m N_tQ_t, \tag{21}
\]

using \( B_t = 1 \), Equations (17) and (18). In particular, at the maturity date \( T \), we have

\[
\Pi_T^{(n,m)} = (-\hat{i} \partial_{\omega})^n (-\hat{i} \partial_s)^m N_TQ_T = (-\hat{i} \partial_{\omega})^n (-\hat{i} \partial_s)^m e^{i(\omega - u)X_T + is(X)_T} E_T e^{iuX_T}
\]

\[
= (-\hat{i} \partial_{\omega})^n (-\hat{i} \partial_s)^m e^{i\omega X_T + is(X)_T} = X_T^n(X) T e^{i\omega X_T + is(X)_T},
\]

which establishes (20). To prove that \( \Pi^{(n,m)} \) satisfies the self-financing condition (11), observe that

\[
E_t X_T^n(X) T e^{i\omega X_T + is(X)_T} = (-\hat{i} \partial_{\omega})^n (-\hat{i} \partial_s)^m e^{i(\omega - u)X_T + is(X)_T} E_t e^{iuX_T}
\]

\[
= (-\hat{i} \partial_{\omega})^n (-\hat{i} \partial_s)^m N_tQ_t = \Pi_t^{(n,m)}. \tag{22}
\]

The left-hand side of (22) is a martingale by iterated conditioning, so \( \Pi^{(n,m)} \) must also be a martingale. For any \( j, k \in \{0\} \cup \mathbb{N} \) the process \( Q_{(i,k)}^{(j)} \) is a martingale by the same reasoning.
Next, by (21),

\[
\begin{align*}
\text{d} \Pi_t^{(n,m)} &= \sum_{j=0}^n \sum_{k=0}^m \binom{n}{j} \binom{m}{k} \\
&\quad \times \left( N_t^{(k,j)} \text{d} Q_t^{(n-j,m-k)} + Q_{t-}^{(n-j,m-k)} \text{d} N_t^{(j,k)} + \text{d}[N_t^{(j,k)}, Q_{t-}^{(n-j,m-k)}]_t \right) \\
&= \sum_{j=0}^n \sum_{k=0}^m \binom{n}{j} \binom{m}{k} A_t^{(j,k)} \text{d} Q_t^{(n-j,m-k)} \\
&\quad + \left( -\frac{\partial}{\partial \omega} \right)^n (-\frac{\partial}{\partial s})^m \frac{i(\omega - u)N_t Q_{t-}}{S_t} \text{d} S_t + \text{d} A_t^{(n,m)},
\end{align*}
\]

where \(A^{(n,m)} = (A_t^{(n,m)})_{0 \leq t \leq T}\) has finite variation. As \(\Pi_t^{(n,m)}, S,\) and \(Q^{(j,k)}\) for any \(j, k\) are martingales, it follows that \(A^{(n,m)}\) is a local martingale. Moreover, as sample paths of \(S\) are continuous, so too are the sample paths of \(N^{(j,k)}\) for any \(j, k\) and, hence, the sample paths of \(A^{(n,m)}\). As a finite variation, continuous local martingale, \(A^{(n,m)}\) must be constant. Thus \(\text{d} A_t^{(n,m)} = 0\) and

\[
\begin{align*}
\text{d} \Pi_t^{(n,m)} &= \sum_{j=0}^n \sum_{k=0}^m \binom{n}{j} \binom{m}{k} N_t^{(j,k)} \text{d} Q_t^{(n-j,m-k)} \\
&\quad + \left( -\frac{\partial}{\partial \omega} \right)^n (-\frac{\partial}{\partial s})^m \frac{i(\omega - u)N_t Q_{t-}}{S_t} \text{d} S_t \\
&= \sum_{j=0}^n \sum_{k=0}^m \binom{n}{j} \binom{m}{k} N_t^{(j,k)} \text{d} Q_t^{(n-j,m-k)} \\
&\quad + \left( -\frac{\partial}{\partial \omega} \right)^n (-\frac{\partial}{\partial s})^m \frac{i(\omega - u)N_t Q_{t-}}{S_t} \text{d} S_t \\
&\quad + \left( -\frac{\partial}{\partial \omega} \right)^n (-\frac{\partial}{\partial s})^m \frac{i(\omega - u)N_t Q_{t-}}{S_t} \text{d} B_t, \tag{23}
\end{align*}
\]

using \(\text{d} B_t = 0\). Comparing (19) and (23) establishes the self-financing condition. \(\blacksquare\)

**Example 3.6 (Sanity check: hedging a variance swap):** To replicate the floating leg of a variance swap which pays \(\langle X \rangle_T\), take \((n, m) = (0, 1)\) in (19), which yields

\[
\Pi_t^{(0,1)} = N_t Q_t + N_t Q_t^{(0,1)} \\
+ \frac{1}{S_t} \left( (\partial_s u) N_t Q_{t-} + (\omega - u)(\partial_s N_t) Q_{t-} + (\omega - u) N_t (\partial_s Q_{t-}) S_t \right) \\
- \left( (\partial_s u) N_t Q_{t-} + (\omega - u)(\partial_s N_t) Q_{t-} + (\omega - u) N_t (\partial_s Q_{t-}) S_t \right) B_t.
\]

In particular, for \(u = u_+\) and \((\omega, s) = (0, 0)\), we have

\[
u(0, 0) = 0, \quad \partial_s u(0, 0) = -2, \\
-\partial N_t(0, 0) = 2X_t + \langle X \rangle_T, \quad -\partial N_t(0, 0) = -2E_t X_T,
\]
and
\[ \Pi_t^{(0,1)}(0, 0) = (2X_t + \langle X \rangle_t) + \mathbb{E}_t(-2X_T) + \frac{2}{S_t}S_t - 2B_t \]
\[ = -2\mathbb{E}_t(X_T - X_0) + \frac{2}{S_t}S_t + (-2 + \langle X \rangle_t + 2X_t - 2X_0)B_t \]
\[ = -2\mathbb{E}_t \log \left( \frac{S_T}{S_0} \right) + \frac{2}{S_t}S_t + \left(-2 + \int_0^t \frac{2}{S_r}dS_r\right)B_t, \]

which recovers the classical hedging strategy for a variance swap: hold \(-2\) European log contracts, keep two units of currency in \(S\) at all times \(t \in [0, T]\) and finance the position with zero-coupon bonds.

### 3.2. Pricing and Replicating More General Payoffs

As previously mentioned, Carr and Lee (2008) use complex exponential claims as building blocks to construct prices and replication strategies for a variety of other more complicated claims, including claims that pay \(\langle X \rangle_T\) where \(-\infty < r < 1\) (see Carr, Lee and Lorig 2021, Propositions 5.1, and 5.5). For options on \(\langle X \rangle_T\) only, this is typically done via Laplace transforms. For options on \((X, \langle X \rangle_T)\) it will be helpful to introduce the generalized Fourier transform \(\mathcal{F}\) and inverse transform \(\mathcal{F}^{-1}\). For any functions \(f : \mathbb{R} \to \mathbb{C}\) and \(\tilde{f} : \mathbb{C} \to \mathbb{C}\) such that the following integrals exist, define

\[
\text{Fourier Transform : } \mathcal{F}[f](\omega) := \frac{1}{2\pi} \int_{\mathbb{R}} f(x)e^{-ix\omega} dx, \quad \omega \in \mathbb{C},
\]

\[
\text{Inverse Transform : } \mathcal{F}^{-1}[	ilde{f}](x) := \int_{\mathbb{R}} \tilde{f}(\omega)e^{ix\omega} d\omega, \quad \omega_r = \text{Re}(\omega),
\]

Consider now a European-style claim with a payoff of the form
\[ \phi(X_T, \langle X \rangle_T) = f(X_T)\langle X \rangle_T^m e^{i\alpha(X)_T}, \quad f : \mathbb{R} \to \mathbb{C}, \quad m \in \{0\} \cup \mathbb{N}, \quad s \in \mathbb{C}. \tag{24} \]

If \(f = \mathcal{F}^{-1}[	ilde{f}]\) where \(\tilde{f} = \mathcal{F}[f]\), then formally, we have
\[
\mathbb{E}_t \phi(X_T, \langle X \rangle_T) = \mathbb{E}_t \mathcal{F}^{-1}[	ilde{f}](X_T)\langle X \rangle_T^m e^{i\alpha(X)_T}
\]
\[ = \mathbb{E}_t \mathcal{F}^{-1}[	ilde{f}](X_T)(-i\partial_s)^m e^{i\alpha(X)_T}
\]
\[ = \int_{\mathbb{R}} \tilde{f}(\omega)(-i\partial_s)^m e^{i\alpha X_T + i\alpha(X)_T} d\omega_r \quad (\text{as } f = \mathcal{F}^{-1}[	ilde{f}])
\]
\[ = \int_{\mathbb{R}} \tilde{f}(\omega)(-i\partial_s)^m e^{i\omega(\alpha(\omega, s))X_T + i\alpha(X)_T} \mathbb{E}_t e^{i\alpha(\omega, s)X_T} d\omega_r,
\]
(by Theorem 3.1)
\[ = \mathbb{E}_t g(X_T, X_{\tau^*}, \langle X \rangle_{\tau^*}), \tag{25} \]

\[
g(X_T, X_{\tau^*}, \langle X \rangle_{\tau^*}) := \int_{\mathbb{R}} \tilde{f}(\omega)(-i\partial_s)^m e^{i(\omega - u(\omega, s))X_T + i\alpha(X)_T} e^{i\alpha(\omega, s)X_T} d\omega_r. \tag{26} \]

Assuming the various applications of Fubini and the Leibniz integral rule are justified, Equation (25) relates the value of a European-style claim with a payoff of the form (24)
to the value of a European claim with payoff (26). Moreover, as
\[ \varphi(X_T, \langle X \rangle_T) = \int_{\mathbb{R}} \hat{f}(\omega) \langle X \rangle_T^m e^{i\alpha X_T + i\omega \langle X \rangle_T} d\omega, \]
a replicating strategy for \( \varphi(X_T, \langle X \rangle_T) \) can be obtained by taking a (continuous) linear combination of replicating strategies for power-exponential claims with payoffs of the form \( \langle X \rangle_T^m e^{i\alpha X_T + i\omega \langle X \rangle_T} \).

4. Knock-out Claims

For any \( H \in \mathbb{R} \) define the first hitting time to level \( H \) as
\[ \tau_H := \inf\{ t \geq 0 : X_t = H \}, \quad H \in \mathbb{R}, \]
where \( \inf \emptyset := \infty \). Next, for any \( L, U \in \mathbb{R} \) with \( L < X_0 < U \), define the first hitting time to level \( L \) or \( U \) as
\[ \tau_{L,U} := \tau_L \land \tau_U, \quad L < X_0 < U. \]
Observe that \( \tau_H \) and \( \tau_{L,U} \) are \( \mathbb{F} \)-stopping times as are their \( T \)-bounded counterparts \( \tau_H^* \) and \( \tau_{L,U}^* \).

4.1. Single Barrier Knock-out Claims

This section considers single barrier knock-out claims with payoffs of the form

Single barrier knock-out : \( \mathbb{1}_{\{\tau_H > T\}} \varphi(X_T, \langle X \rangle_T) \).

The following strategy replicates a single barrier knock-out claim with a down-barrier \( L < X_0 \).

**Theorem 4.1 (Replication of single barrier knock-out claims):** Fix \( L < X_0 \). The following trading strategy replicates a single barrier knock-out claim with payoff

\[ \mathbb{1}_{\{\tau_L > T\}} \varphi(X_T, \langle X \rangle_T). \quad (27) \]

At time 0 hold a European-style claim with payoff
\[ \varphi^{ko}_L(X_T, \langle X \rangle_T) := \mathbb{1}_{\{X_T > L\}} \varphi(X_T, \langle X \rangle_T) - \mathbb{1}_{\{X_T < L\}} e^{X_T - L} \varphi(2L - X_T, \langle X \rangle_T). \quad (28) \]

If and when the claim knocks out, close the position in \( \varphi^{ko}_L(X_T, \langle X \rangle_T) \) at no cost.

**Proof:** If \( \tau_L > T \), then \( X_T > L \) and thus, both the knock-out claim (27) and the European-style claim (28) pay \( \varphi(X_T, \langle X \rangle_T) \). It remains to show that, when \( \tau_L \leq T \), the European-style claim (28) has zero value at time \( \tau_L \). First, we note from Carr and Lee (2009, Definition 2.6)
that $S = e^X$ satisfies geometric put-call symmetry. Thus, Carr and Lee (2009, Theorem 5.3) implies that

$$E_{\tau^*}G(X_T) = E_{\tau^*}e^{X_T - X_{\tau^*}}G(2X_{\tau^*} - X_T).$$

(29)

for any $\mathcal{F}$-stopping time $\tau$ and $G : \mathbb{R} \to \mathbb{C}$. Therefore

$$E_{\tau^*}G(X_T, (X)_T) = E_{\tau^*}E[G(X_T, (X)_T) \mid \mathcal{F}_{\tau^*} \vee \mathcal{F}_{\tau^*}']$$

$$= E_{\tau^*}E[e^{X_T - X_{\tau^*}}G(2X_{\tau^*} - X_T, (X)_T) \mid \mathcal{F}_{\tau^*} \vee \mathcal{F}_{\tau^*}']$$

$$= E_{\tau^*}e^{X_T - X_{\tau^*}}G(2X_{\tau^*} - X_T, (X)_T),$$

(30)

where the second equality follows from (29) and the fact that $S = e^X$, conditioned on the path of $\sigma$, satisfies geometric put-call symmetry. Using (30) with $G(x, v) = \mathbb{I}_{\{x > L\}}\varphi(x, v)$ and recalling that $\mathbb{I}_{\{\tau_L \leq T\}}(X_{\tau_L^*} - L) = 0$, we have $\mathbb{I}_{\{\tau_L \leq T\}}E_{\tau_L^*}\varphi_{\text{ko}}(X_T, (X)_T) = 0$. ■

**Remark 4.2:** For the single barrier knock-out claim $\mathbb{I}_{\{\tau > L\}}\varphi(X_T, (X)_T)$ with up-barrier $U > X_0$, the replication strategy is to hold at time 0 a European-style claim with payoff

$$\varphi_{\text{uo}}(X_T, (X)_T) := \mathbb{I}_{\{X < U\}}\varphi(X_T, (X)_T) - \mathbb{I}_{\{X > U\}}e^{X_T - U}\varphi(2U - X_T, (X)_T),$$

and clear the position at no cost if and when the barrier $U$ is hit.

**Proposition 4.3 (Price of a single barrier knock-out power-exponential claim):** Assume the distribution of $X_T$ has no point masses (a sufficient condition is that $\int_0^T \sigma_t^2 \, dt > \epsilon > 0$). Then for any $L < X_0$, $j, k \in \{0\} \cup \mathbb{N}$ and $p, s \in \mathbb{C}$ we have

$$E\mathbb{I}_{\{\tau_L > T\}}X_T^j(X)_T^k e^{i\rho X_T + is(X)_T} = \lim_{n \to \infty} E(g_n(X_T) - h_n(X_T)),$$

(31)

where the functions $g_n$ and $h_n$ are given by

$$g_n(X_T) = \int_{\mathbb{R}} (-i \partial_p)^j(-i \partial_s)^k\mathbb{H}_n(\omega - p)e^{-i(\omega - p)L + i(\omega - u(\omega, s))X_0}e^{i\rho X_T} \, d\omega,$$

$$h_n(X_T) = \int_{\mathbb{R}} (-i \partial_p)^j(-i \partial_s)^k\mathbb{H}_n(-i - \omega - p)e^{-i(\omega - p)L + i(\omega - u(\omega, s))X_0} \, d\omega,$$

$$\mathbb{H}_n(\omega) = \frac{-i}{4n} \text{csch} \left( \frac{\pi \omega}{2n} \right).$$

(32)

Here, the contour of integration in $g_n$ is chosen so that $-2n + p_i < \omega_i < p_i$ and $2i \omega - \omega^2 - i\omega + \frac{1}{4} \neq 0$, and the contour of integration in $h_n$ is chosen so that $-1 - p_i < \omega_i < 2n - 1 - p_i$ and $2i \omega - \omega^2 - i\omega + \frac{1}{4} \neq 0$. 
\textbf{Proof:} Let $H$ denote the Heaviside function and let $H_n$ ($n \in \mathbb{N}$) denote a smooth approximation of $H$. Specifically, let

$$
H(x) := \frac{1}{2}(1 + \text{sgn } x), \quad H_n(x) := \frac{1}{2}(1 + \tanh nx).
$$

Observe that $H_n \to H$ pointwise as $n \to \infty$. Now, as the price of any claim is equal to the price of its replicating portfolio, we have by Theorem 4.1 that

$$
E \mathbb{1}_{[\tau_L>T]} X_T^j \frac{\partial}{\partial p} e^{i p X_T + i s(X)_T}
$$

\[= E \left( \mathbb{1}_{[X_T>L]} X_T^j \frac{\partial}{\partial p} e^{i p X_T + i s(X)_T} - \mathbb{1}_{[X_T<L]} (2L - X_T)^j \frac{\partial}{\partial p} e^{X_T - L + i p (2L - X_T) + i s(X)_T} \right) \]

\[= \lim_{n \to \infty} E \left( H_n(X_T - L) X_T^j \frac{\partial}{\partial p} e^{i p X_T + i s(X)_T} - H_n(L - X_T) (2L - X_T)^j \frac{\partial}{\partial p} e^{X_T - L + i p (2L - X_T) + i s(X)_T} \right) \]

\[= \lim_{n \to \infty} \left( -i \partial p \right)^j (-i \partial s)^k E \left( H_n(X_T - L) e^{i p X_T + i s(X)_T} - H_n(L - X_T) e^{X_T - L + i p (2L - X_T) + i s(X)_T} \right), \quad (33)

where the second equality holds by absence of point masses, the third equality holds by Lebesgue’s dominated convergence theorem and the last equality follows from the Leibniz integral rule. Noting that $F[H_n] = \hat{H}_n$ where \( \hat{H}_n(\omega) \) is defined for $-2n < \omega_i < 0$, it follows that

\[-i \partial p \right)^j (-i \partial s)^k E \hat{H}_n(X_T - L) e^{i p X_T + i s(X)_T}

\[= (-i \partial p)^j (-i \partial s)^k \int_{\mathbb{R}} \hat{H}_n(\omega - p) e^{-i(\omega - p)L} e^{i \omega X_T + i s(X)_T} d\omega_r \]

\[\times (-2n + p_i < \omega_i < p_i) \]

\[= (-i \partial p)^j (-i \partial s)^k \int_{\mathbb{R}} \hat{H}_n(\omega - p) e^{-i(\omega - p)L} e^{i \omega X_T + i s(X)_T} d\omega_r \quad \text{(by Fubini)} \]

\[= (-i \partial p)^j (-i \partial s)^k \int_{\mathbb{R}} \hat{H}_n(\omega - p) e^{-i(\omega - p)L + i(\omega - u(\omega,s))X_0} e^{i u(\omega,s)X_T} d\omega_r \quad \text{(by (4))} \]

\[= (-i \partial p)^j (-i \partial s)^k \int_{\mathbb{R}} \hat{H}_n(\omega - p) e^{-i(\omega - p)L + i(\omega - u(\omega,s))X_0} e^{i u(\omega,s)X_T} d\omega_r \quad \text{(by Fubini)} \]

\[= E g_n(X_T), \quad \text{(by Leibniz)} \]

(34)
Figure 1. We plot the payoff $g_n(\log \cdot) - h_n(\log \cdot)$, whose expectation, in the limit as $n \to \infty$, prices a single barrier knock-out power-exponential claim; see Equation (31). In both plots, the following parameters are fixed: $L = \log 90$, $X_0 = \log 110$, $p = 0$, $s = 0$, $j = 0$, $k = 1$, $n = 25$. The vertical dashed lines are placed at $e^L = 90$ and $S_0 = e^{X_0} = 110$. Note that with $(p, s, j, k)$ as chosen, the European payoff function plotted above has expectation which, in the large $n$ limit, prices a single barrier knock-out variance swap, which pays $\mathbb{1}_{[\tau_L > T]}(X)_T$.

where the applications of Fubini twice and the Leibniz rule are justified as $|\partial_p^j \hat{H}_n(\omega - p)| = \mathcal{O}(e^{-|\omega_r|/n})$ and

\[
\mathbb{E}|\partial_p^j \partial_s^k e^{-i(\omega - p)L + \imath \alpha X_T + \imath s(X)_T}| = \mathcal{O}(1),
\]

\[
\mathbb{E}|\partial_p^j \partial_s^k e^{-i(\omega - p)L + \imath (\omega - u(\omega, s))X_0 + \imath u(\omega, s)X_T}| = \mathcal{O}(1),
\]

as $|\omega_r| \to \infty$ and as the contour of integration is chosen to avoid any singularities in the integrand. Similarly,

\[
(-\imath \partial_p^j (-\imath \partial_s)^k \mathbb{E}H_n(L - X_T)e^{X_T - L + i p(2L - X_T) + i s(X)_T} = \mathbb{E}h_n(X_T).
\]

Equation (31) follows from (33), (34) and (35).

Remark 4.4: The reason we must replace the Heaviside function $H$ by $\lim_{n \to \infty} H_n$ in Proposition 4.3 is that the Heaviside function’s Fourier transform $\hat{H}(\omega) = -\imath/(2\pi \omega)$ ($\omega_i > 0$) does not decay fast enough as $|\omega_r| \to \infty$ to justify the second use of Fubini in (34).

Remark 4.5: Equation (31) is an equation of the form

\[
\mathbb{E}F[X] = \lim_{n \to \infty} \mathbb{E}g_n(X_T),
\]

where $F$ is a functional of $X = (X_t)_{0 \leq t \leq T}$. Observe that $\mathbb{E}F[X]$ is the price of a path-dependent claim and $\mathbb{E}g_n(X_T)$ is the price of a European (i.e., path-independent) claim. Thus, we say that, in the limit as $n \to \infty$, the expected payoff $\mathbb{E}g_n$ prices the claim $F[X]$.

Figure 1 plots the function whose expectation, in the limit as $n \to \infty$, prices a single barrier knock-out variance swap, which pays $\mathbb{1}_{[\tau_L > T]}(X)_T$. 
4.2. Double Barrier Knock-out Claims

This section considers double barrier knock-out claims with payoffs of the form

\[ \text{Double barrier knock-out claim: } 1_{[\tau_L, \tau_U]}>T] \varphi(X_T, (X)_T). \] (36)

The following theorem gives a replication strategy for such claims.

**Theorem 4.6 (Replication of double barrier knock-out claims):** Suppose \( L < X_0 < U \). Let \( \varphi : (L, U) \times \mathbb{R}_+ \to \mathbb{C} \) be bounded. The following trading strategy replicates a double barrier knock-out claim with payoff (36). At time 0 hold a European-style claim with payoff

\[ \psi^{ko}_{L,U}(X_T, (X)_T) \]

\[ := \sum_{n=-\infty}^{\infty} e^{-n \Delta} \left( \varphi^*(2n \Delta + X_T, (X)_T) - e^{X_T-L} \varphi^*(2n \Delta + 2(L - X_T), (X)_T) \right), \] (37)

where \( \Delta := U - L \). If and when the claim knocks out, clear the position in \( \psi^{ko}_{L,U}(X_T, (X)_T) \) at no cost.

**Proof:** If \( \tau_L, \tau_U > T \), then \( L < X_T < U \) and thus, both the knock-out claim (36) and the European-style claim (37) pay \( \varphi(X_T, (X)_T) \). It remains to show that, if \( \tau_L, \tau_U \leq T \), the European-style claim (37) has zero value at time \( \tau_L, \tau_U \). Recalling once again that \( S = e^X \) satisfies geometric put-call symmetry, we have by Carr and Lee (2009, Theorem 5.18) that

\[ 1_{[\tau_L, \tau_U] > T]} \mathbb{E}^{\tau_L, \tau_U}_{L,U} \psi^{ko}_{L,U}(X_T, (X)_T) = 0, \]

which holds for any fixed \( \nu \in \mathbb{R}_+ \). Thus

\[ 1_{[\tau_L, \tau_U] \leq T]} \mathbb{E}

\[ \psi^{ko}_{L,U}(X_T, (X)_T) = 1_{[\tau_L, \tau_U] \leq T]} \mathbb{E}^{\tau_L, \tau_U}_{L,U} \mathbb{E}^{\tau_L, \tau_U}_{L,U} \mathbb{E} \left[ \varphi^{ko}_{L,U}(X_T, (X)_T) | \mathcal{F}_{\tau_L, \tau_U} \cup \mathcal{F}_{\tau_L, \tau_U} \right] = 0, \]

because \( (X)_T \in \mathcal{F}_{\tau_L, \tau_U} \) and the process \( S = e^X \), conditioned on the path of \( \sigma \), satisfies geometric put-call symmetry. ■

**Proposition 4.7 (Prices of double barrier knock-out power-exponential claims):** Assume the distribution of \( X_T \) has no point masses (a sufficient condition is that \( \int_0^T \sigma_t^2 dt \geq \varepsilon > 0 \)). Then for any \( L < X_0 < U, j, k \in \{0\} \cup \mathbb{N} \) and \( p, s \in \mathbb{C} \) we have

\[ \mathbb{E}^{1_{[\tau_L, \tau_U] > T]} X_T^j (X)_T^k e^{i p X_T + i s (X)_T} = \lim_{q \to \infty} \lim_{m \to \infty} \sum_{n=-q}^{q} e^{-n \Delta} (g_{n,m}(X_T) - h_{n,m}(X_T)), \] (38)
where the functions \(g_{n,m}\) and \(h_{n,m}\) are given by

\[
\begin{align*}
g_{n,m}(X_T) &= \int_{\mathbb{R}} e^{i\omega 2n\Delta} (-i\partial_p)^j (-i\partial_s)^k \left( e^{-i(\omega-p)L} - e^{-i(\omega-p)U} \right) \\
&\quad \times \hat{H}_m(\omega-p)e^{i(\omega-u(\omega,s))X_0+i u(\omega,s)X_T} \, d\omega, \\
h_{n,m}(X_T) &= \int_{\mathbb{R}} e^{i(1-i\omega)(2n\Delta-2L)+j-i\partial_p)^j (-i\partial_s)^k \left( e^{-i(-i-p-\omega)L} - e^{-i(-i-p-\omega)U} \right) \\
&\quad \times \hat{H}_m(-i-p-\omega)e^{i(\omega-u(\omega,s))X_0+i u(\omega,s)X_T} \, d\omega,
\end{align*}
\]

with \(\hat{H}_m\) as defined in (32). The contour of integration for \(g_{n,m}\) must be chosen so that \(-2m+p_i < \omega_i < p_i\) and \(2i\omega - \omega^2 - i\omega + \frac{1}{4} \neq 0\) and the contour of integration for \(h_{n,m}\) must be chosen so that \(-1 - p_i < \omega_i < 2m - 1 - p_i\) and \(2i\omega - \omega^2 - i\omega + \frac{1}{4} \neq 0\).

**Proof:** As many of the arguments for passing limits and derivatives through integrals and expectations are analogous to those given in the proof of Proposition 4.3, we shall not repeat them here. Noting that the value of any claim is equal to the value of its replication portfolio, we have from Theorem 4.6 that

\[
\mathbb{E}\mathbb{1}_{\{\tau_L > T\}} \varphi(X_T, \langle X \rangle_T) = \sum_{n=-\infty}^{\infty} e^{-n\Delta} e^{X_T-L} \varphi^*(2n\Delta + 2L - X_T, \langle X \rangle_T),
\]

where passing the expectation through the infinite sum is allowed by the arguments given in the proof of Carr and Lee (2009, Theorem 5.18).

Examining the first term in the expectation above, with \(\varphi(X_T, \langle X \rangle_T) = X_T^{i} (X)_T^k \ e^{i p X_T + i s (X)_T}\), we have

\[
\mathbb{E}\varphi^*(2n\Delta + X_T, \langle X \rangle_T)
\]

\[
= \mathbb{E}\mathbb{1}_{\{L < 2n\Delta + X_T < U\}} (2n\Delta + X_T)^j (X)_T^k \ e^{i p(2n\Delta + X_T) + i s (X)_T}
\]

\[
= \lim_{m \to \infty} \mathbb{E} (H_m(2n\Delta + X_T - L) - H_m(2n\Delta + X_T - U))
\]

\[
\times (2n\Delta + X_T)^j (X)_T^k \ e^{i p(2n\Delta + X_T) + i s (X)_T},
\]

\[
= \lim_{m \to \infty} (-i\partial_p)^j (-i\partial_s)^k \mathbb{E} (H_m(2n\Delta + X_T - L) - H_m(2n\Delta + X_T - U))
\]

\[
\times e^{i p(2n\Delta + X_T) + i s (X)_T},
\]

where \(H_m(x) := \frac{1}{2} (1 + \tanh mx)\). Next, using \(\mathbb{F}[H_m] = \hat{H}_m\), we compute

\[
(-i\partial_p)^j (-i\partial_s)^k \mathbb{E} (H_m(2n\Delta + X_T - L) - H_m(2n\Delta + X_T - U)) e^{i p(2n\Delta + X_T) + i s (X)_T}
\]

\[
= \int_{\mathbb{R}} e^{i\omega 2n\Delta} (-i\partial_p)^j (-i\partial_s)^k \left( e^{-i(\omega-p)L} - e^{-i(\omega-p)U} \right)
\]
Figure 2. We plot the payoff appearing on the right-hand side of (38), whose expectation, in the limit as $q, m \to \infty$, prices a double barrier knock-out power-exponential claim. In both plots, the following parameters are fixed: $L = \log 90$, $U = \log 110$, $X_0 = \log 100$, $p = 0$, $s = 0$, $j = 0$, $k = 1$, $m = 15$ and $q = 5$. The vertical dashed lines are placed at $L = 90$, $X_0 = 100$ and $U = \log 110$. Note that with $(p, s, j, k)$ as chosen, the European payoff function plotted above has expectation which, in the large $q$, $m$ limit, prices a double barrier knock-out variance swap, which pays $\mathbb{1}_{\{\tau_L, U > T\}} \langle X \rangle_T$.

\[
\frac{\hat{H}_m(\omega - p) e^{i\omega X_T + is(X) T}}{e^{i\omega L} - e^{i\omega U}}
\times \hat{H}_m(\omega - p) e^{i(\omega - u(\omega, s)) X_0} e^{i u(\omega, s) X_T} d\omega_r
\[
= e^{X_T - L} \frac{\varphi(2n\Delta + 2L - X_T, \langle X \rangle_T)}{\langle X \rangle_T - \langle X \rangle_T^*}
\]

Similarly, a straightforward computation shows

\[
E e^{X_T - L} \varphi^* (2n\Delta + 2L - X_T, \langle X \rangle_T) = \lim_{m \to \infty} E h_{n,m}(X_T).
\]

Thus

\[
E \mathbb{1}_{\{\tau_L, U > T\}} \varphi(X_T, \langle X \rangle_T) = \sum_{n = -\infty}^{\infty} \lim_{m \to \infty} e^{-n\Delta} E (g_{n,m}(X_T) - h_{n,m}(X_T))
\]

\[
= \lim_{q \to \infty} \lim_{m \to \infty} E \sum_{n = -q}^{q} e^{-n\Delta} (g_{n,m}(X_T) - h_{n,m}(X_T)),
\]

as claimed.

Figure 2 plots the function whose expectation, in the limit as $m, q \to \infty$, prices a double barrier knock-out variance swap, which pays $\mathbb{1}_{\{\tau_L, U > T\}} \langle X \rangle_T$.

5. Single Barrier Knock-in Claims

This section considers single barrier knock-in claims with payoffs of the form

\[
\text{Single barrier knock-in : } \mathbb{1}_{\{\tau_H \leq T\}} \varphi(X_T - X_{\tau_H^*}, \langle X \rangle_T - \langle X \rangle_{\tau_H^*}).
\]

The following trading strategy replicates a knock-in power-exponential claim with a single barrier $L < X_0$. 

Theorem 5.1 (Replication of single barrier knock-in power-exponential claims): Fix $L < X_0$, $n, m \in \{0\} \cup \mathbb{N}$ and $\omega, s \in \mathbb{C}$ and assume $2i\omega - \omega^2 - i\omega + \frac{1}{4} \neq 0$. The following trading strategy replicates the single barrier knock-in power-exponential payoff

$$I_{\{\tau_L \leq T\}}(X_T - X_{t_L^+})^n((X_T - (X_{t_L^+}^+))^m e^{i\omega(X_T - X_{t_L^+}) + is(X_T - (X_{t_L^+}^+)}. \tag{39}$$

At time 0 hold a European claim with payoff

$$(-i\partial_\omega)^n(-i\partial_s)^m\psi_{L^i}^k(X_T; \omega, s), \tag{40}$$

where we have defined

$$\psi_{L}^k(X_T) \equiv \psi_{L}^k(X_T; \omega, s) = I_{\{X_T < L\}}e^{(1-i\omega)(X_T-L)} + I_{\{X_T \geq L\}}e^{iu(X_T-L)}, \tag{41}$$

with $u \equiv u_\pm(\omega, s)$ as given in (3). If and when the claim knocks in, exchange the claim (40) for the knock-in claim (39) at no cost. After the exchange, the knock-in claim (39) can be replicated as a European-style power-exponential claim.

Proof: If $\tau_L > T$, then $X_T > L$ and thus both the knock-in claim (39) and the European claim (40) expire worthless. It remains to show that, if $\tau_L \leq T$, the claim (40) can be exchanged for the claim (39) at no cost. Recalling that $S = e^X$ satisfies geometric put-call symmetry, we have from Carr and Lee (2009, equation (5.7)) that

$$I_{\{\tau_L \leq T\}}E_{t_L^+}e^{iu(X_T-L)} = I_{\{\tau_L \leq T\}}E_{t_L^+}\psi_{L}^k(X_T). \tag{42}$$

Thus

$$I_{\{\tau_L \leq T\}}E_{t_L^+}(X_T - X_{t_L^+})^n((X_T - (X_{t_L^+}^+))^m e^{i\omega(X_T - X_{t_L^+}) + is(X_T - (X_{t_L^+}^+)})$$

$$= I_{\{\tau_L \leq T\}}(-i\partial_\omega)^n(-i\partial_s)^mE_{t_L^+}e^{i\omega(X_T - X_{t_L^+}) + is(X_T - (X_{t_L^+}^+)}) \quad \text{(by Leibniz)}$$

$$= I_{\{\tau_L \leq T\}}(-i\partial_\omega)^n(-i\partial_s)^mE_{t_L^+}e^{iu(X_T-L)} \quad \text{(by (4))}$$

$$= I_{\{\tau_L \leq T\}}(-i\partial_\omega)^n(-i\partial_s)^mE_{t_L^+}\psi_{L}^k(X_T) \quad \text{(by (42))}$$

$$= I_{\{\tau_L \leq T\}}E_{t_L^+}(-i\partial_\omega)^n(-i\partial_s)^m\psi_{L}^k(X_T), \quad \text{(by Leibniz)}$$

where the two uses of the Leibniz rule are justified by (9).

Remark 5.2: To replicate the single barrier knock-in power-exponential claim with payoff

$$I_{\{\tau_U \leq T\}}(X_T - X_{t_U^+})^n((X_T - (X_{t_U^+}^+))^m e^{i\omega(X_T - X_{t_U^+}) + is(X_T - (X_{t_U^+}^+)})$$

where $U > X_0$, one should hold at time 0 a the European claim with payoff $(-i\partial_\omega)^n(-i\partial_s)^m\psi_{U}^k(X_T; \omega, s)$ where

$$\psi_{U}^k(X_T) \equiv \psi_{U}^k(X_T; \omega, s) = I_{\{X_T > U\}}e^{(1-i\omega)(X_T-U)} + I_{\{X_T \geq U\}}e^{iu(X_T-U)},$$

and, if $\tau_U \leq T$, exchange the European claim at time $\tau_U$ for the knock-in claim at no cost.
Proposition 5.3 (Prices of single barrier knock-in claims on fractional powers of quadratic variation): For any $0 < r < 1$ and $L < X_0$ we have

$$
E1_{\{\tau_L \leq T\}}(\langle X \rangle_T - \langle X \rangle^*_L)^r = Eg(X_T),
$$

where

$$
g(x) := \frac{r}{\Gamma(1 - r)} \int_0^\infty \frac{1}{z^{r+1}} \left( \psi^k_L(x; 0, 0) - \psi^k_L(x; 0, iz) \right) \mathrm{d}z.
$$

(43)

Here, $\Gamma$ is the Euler Gamma function and $\psi^k_L(x; \omega, s)$ is defined in (41).

Proof: Following the proof of Carr, Lee and Lorig (2021, Proposition 5.1), we have from Schürger (2002, equation (1.2.3)) that

$$
v^r = \frac{r}{\Gamma(1 - r)} \int_0^\infty \frac{1 - e^{-zv}}{z^{r+1}} \mathrm{d}z, \quad v \geq 0, \quad 0 < r < 1.
$$

(44)

Therefore

$$
E1_{\{\tau_L \leq T\}}(\langle X \rangle_T - \langle X \rangle^*_L)^r
= \frac{r}{\Gamma(1 - r)} \int_0^\infty \frac{1}{z^{r+1}} E1_{\{\tau_L \leq T\}} \left( 1 - e^{-z(\langle X \rangle_T - \langle X \rangle^*_L)} \right) \mathrm{d}z \quad \text{(by (44) and Tonelli)}
= \frac{r}{\Gamma(1 - r)} \int_0^\infty \frac{1}{z^{r+1}} E \left( \psi^k_L(X_T; 0, 0) - \psi^k_L(X_T; 0, iz) \right) \mathrm{d}z \quad \text{(by Theorem 5.1)}
= Eg(X_T) \quad \text{(by (43) and Fubini)}
$$

where the use of Fubini is justified as follows. As $z \to \infty$ we have $\im u(\omega, iz) \to 1/2$ by (41), hence

$$
E|\psi^k_L(X_T; 0, 0) - \psi^k_L(X_T; 0, iz)| = O(1), \quad \text{as } z \to \infty.
$$

(45)

Turning to the $z \to 0$ behavior, consider the case $u = u_+$; the proof for $u = u_-$ is similar. We have

$$
\begin{align*}
|\psi^k_L(X_T; 0, 0) - \psi^k_L(X_T; 0, iz)| & \leq \mathbb{1}_{\{X_T < L\}} e^{X_T - L} |1 - e^{-iz(0,0)X_T(0,0)}| + \mathbb{1}_{\{X_T \leq L\}} |1 - e^{iz(0,0)X_T(0,0)}| \\
& = \mathbb{1}_{\{\tau_L \leq T\}} \left( \mathbb{1}_{\{X_T < L\}} e^{X_T - L} |1 - e^{-iz(0,0)X_T(0,0)}| + \mathbb{1}_{\{X_T \leq L\}} |1 - e^{iz(0,0)X_T(0,0)}| \right) \\
& \leq \mathbb{1}_{\{\tau_L \leq T\}} \left| 1 - e^{-iz(0,0)X_T(0,0)} \right| + \mathbb{1}_{\{\tau_L \leq T\}} \left| 1 - e^{iz(0,0)X_T(0,0)} \right|.
\end{align*}
$$

(46)
using \( u(0, 0) = 0 \) as well as \( \mathbb{I}_{\{\tau_L > T\}} \mathbb{I}_{\{X_T \leq L\}} = 0 \) and \( \mathbb{I}_{\{X_T < L\}} e^{X_T - L} \leq 1 \). For \( z \) small enough, we have \( iu(0, i z) = 1/2 - \sqrt{1/4 - 2z} \in \mathbb{R} \). Thus
\[
\left( \mathbb{E} \mathbb{I}_{\{\tau_L \leq T\}} \left| 1 - e^{-iu(0, i z)(X_T - X_{t_L^*})} \right|^2 \right) \\
\leq \mathbb{E} \mathbb{I}_{\{\tau_L \leq T\}} \left| 1 - e^{-iu(0, i z)(X_T - X_{t_L^*})} \right|^2 \\
= \mathbb{E} \mathbb{I}_{\{\tau_L \leq T\}} e_{x_L^*} \left( 1 - 2e^{-iu(0, i z)(X_T - X_{t_L^*})} + e^{-2iu(0, i z)(X_T - X_{t_L^*})} \right) \\
= \mathbb{E} \mathbb{I}_{\{\tau_L \leq T\}} e_{x_L^*} \left( 1 - 2e^{-a(z)(X_T - (X_{t_L^*})} + e^{-b(z)(X_T - (X_{t_L^*})} \right),
\]
where \( a(z) = -\frac{1}{2} + \frac{1}{2} \sqrt{1 - 8z} + 2z \)
and \( b(z) = -\frac{3}{2} + \frac{3}{2} \sqrt{1 - 8z} + 8z \),
using \( iu(0, i a(z)) = -iu(0, i z) \) and \( iu(0, i b(z)) = -2iu(0, i z) \). Now define \( f(d) := \mathbb{E} \mathbb{I}_{\{\tau_L \leq T\}} e^{-d(X_T - (X_{t_L^*})} \). The function \( f \) is analytic by (1). Hence
\[
\left( \mathbb{E} \mathbb{I}_{\{\tau_L \leq T\}} \left| 1 - e^{-iu(0, i z)(X_T - X_{t_L^*})} \right|^2 \right) \\
\leq f(0) - 2f(a(z)) + f(b(z)) \\
= (f(0) - 2f(a(0)) + f(b(0))) \\
+ (-2f'(a(0))a'(0) + f'(b(0))b'(0)) z + O(z^2) = O(z^2),
\]
(47)
because \( a(0) = b(0) = 0 \) and \(-2a'(0) + b'(0) = 0 \). A similar computation shows
\[
\left( \mathbb{E} \mathbb{I}_{\{\tau_L \leq T\}} \left| 1 - e^{-iu(0, i z)(X_T - X_{t_L^*})} \right|^2 \right) = O(z^2).
\]
(48)
Combining (46), (47) and (48), we have
\[
\mathbb{E} \left| \psi_L^{k_1}(X_T; 0, 0) - \psi_L^{k_1}(X_T; 0, i z) \right| = O(z) \quad \text{as} \quad z \to 0.
\]
(49)
From (45) and (49), we have
\[
\frac{r}{\Gamma(1 - r)} \int_0^\infty \frac{1}{z^{r+1}} \mathbb{E} \left| \psi_L^{k_1}(X_T; 0, 0) - \psi_L^{k_1}(X_T; 0, i z) \right| dz < \infty,
\]
which verifies the conditions of Fubini.

**Proposition 5.4 (Ratio claims):** For any \( r, \varepsilon > 0 \) and \( p \in \mathbb{C} \) we have
\[
\mathbb{E} \left( \mathbb{I}_{\{\tau_L \leq T\}} \frac{(X_T - X_{t_L^*} e^{ip(X_T - X_{t_L^*})}}{(X_T - (X_{t_L^*} + \varepsilon)^r}} \right) = \mathbb{E} g(X_T),
\]
where the function \( g \) is given by
\[
g(x) = \frac{1}{r \Gamma(r)} \int_0^\infty (-i \partial_p) \psi_L^{k_1}(x; p, i z^{1/r}) e^{-z^{1/r} \varepsilon} dz.
\]
(50)
Here, \( \Gamma \) is the Euler Gamma function and \( \psi_L^{k_1}(x; \omega, s) \) is as defined in (41).
**Proof:** Following the proof of Carr, Lee and Lorig (2021, Proposition 5.5), we have from Schürger (2002, equation (1.0.1)) that

\[ \frac{1}{\nu'} = \frac{1}{r\Gamma(r)} \int_0^\infty e^{-z^{1/r}} dz, \quad r > 0. \]  

(51)

from which

\[
\mathbb{E} \left( \mathbb{I}_{\{t_L \leq T\}} (X_T - X_{t_L^*}) e^{i p (X_T - X_{t_L^*})} / (\langle X_T \rangle - \langle X \rangle) \right) \\
= \frac{1}{r\Gamma(r)} \int_0^\infty \mathbb{E} \mathbb{I}_{\{t_L \leq T\}} (X_T - X_{t_L^*}) e^{i p (X_T - X_{t_L^*}) - z^{1/r} (\langle X_T \rangle - \langle X \rangle) + e} dz \\
= \frac{1}{r\Gamma(r)} \int_0^\infty (-i \partial_p) \mathbb{E} \mathbb{I}_{\{t_L \leq T\}} e^{i p (X_T - X_{t_L^*}) - z^{1/r} (\langle X_T \rangle - \langle X \rangle) + e} dz \\
= \frac{1}{r\Gamma(r)} \int_0^\infty (-i \partial_p) \mathbb{E} \psi^k_L(X_T; p, i z^{1/r}) e^{-z^{1/r} e} dz \\
= \mathbb{E} \mathbb{E} \log (X_T). \quad \text{(by Leibniz, (50) and Fubini)}
\]

The first use of Fubini is justified as, for all \( p \in \mathbb{C} \) and \( z \geq 0 \), we have

\[
\mathbb{E} \left| \mathbb{I}_{\{t_L \leq T\}} (X_T - X_{t_L^*}) e^{i p (X_T - X_{t_L^*})} \right| \\
\leq \mathbb{E} \left| \mathbb{I}_{\{t_L \leq T\}} (X_T - X_{t_L^*}) \right| < \infty,
\]

from which

\[
\int_0^\infty \mathbb{E} \left| \mathbb{I}_{\{t_L \leq T\}} (X_T - X_{t_L^*}) e^{i p (X_T - X_{t_L^*}) - z^{1/r} (\langle X_T \rangle - \langle X \rangle) + e} \right| dz < \infty.
\]

The two uses of the Leibniz rule are by (9). The second use of Fubini is justified as follows. By (41),

\[
- i \partial_p \psi^k_L(X_T; p, i z^{1/r}) \\
= \left( - \mathbb{I}_{\{X_T < L\}} e^{(X_T - L) - i u(p, i z^{1/r})(X_T - L)} + \mathbb{I}_{\{X_T \leq L\}} e^{i u(p, i z^{1/r})(X_T - L)} \right) \\
\times \partial_p u(p, i z^{1/r})(X_T - L).
\]

where, from (3),

\[
\partial_p u(\pm, i z^{1/r}) = \frac{\pm(1 - 2iz/p)}{\sqrt{1 - 4p(p + 1 - 8z^{1/r}}}.
\]  

(52)

As \( i u(p, i z^{1/r}) \to 1/2 \) and \( \partial_p u(p, i z^{1/r}) \to 0 \) as \( z \to \infty \), it follows that

\[
\mathbb{E} \left| - i \partial_p \psi^k_L(X_T; p, i z^{1/r}) \right| = O(1), \quad \text{as } z \to \infty.
\]
Figure 3. Left: we plot the payoff \( g(\log \cdot) \), given in (43), whose expectation prices a single-barrier knock-in claim on volatility with payoff (54). Right: we plot the payoff \( g(\log \cdot) \), given in (50), whose expectation prices a single-barrier knock-in claim on realized Sharpe ratio with payoff (55). In both plots, the following parameters are fixed: \( e^L = 90 \) and \( r = 1/2 \). For the ratio claim, we have additionally fixed \( \varepsilon = 0.001 \). The vertical line in both plots are placed at the knock-in barrier \( e^L = 90 \).

Therefore

\[
\int_0^\infty \mathbb{E}\left( -\hat{\varphi}_p \psi^k_i (X_T; p, \text{1}^{1/2}) \right) e^{-z^{1/2} \varepsilon} \, dz < \infty, \quad (53)
\]

where any possible singularity in the integrand of (53) due to the denominator in (52) will not cause the integral to explode as, for any \( a \in \mathbb{R}_+ \) we have

\[
\int_0^\infty \left| \frac{e^{-z^{1/2}}}{\sqrt{a - z^{1/2}}} \right| \, dz = \int_0^\infty r x^{r-1} e^{-\varepsilon x} \left| \frac{1}{\sqrt{a - x}} \right| \, dx < \infty,
\]

thus justifying the second use of Fubini.

Figure 3 plots the European claims (43) and (50) whose expectations price, respectively

single barrier knock-in volatility swap : \( \mathbb{I}_{\{\tau_L \leq T\}} \sqrt{\langle X \rangle_T - \langle X \rangle_{\tau^*_L}} \), \quad (54)

single barrier knock-in realized Sharpe ratio : \( \mathbb{I}_{\{\tau_L \leq T\}} \frac{X_T - X_{\tau^*_L}}{\sqrt{\langle X \rangle_T - \langle X \rangle_{\tau^*_L}} + \varepsilon} \). \quad (55)

6. Single Barrier Rebate Claims

This section considers single barrier rebate claims with payoffs of the form

Single barrier rebate : \( \mathbb{I}_{\{\tau_H \leq T\}} \psi(\langle X \rangle_{\tau^*_H}) \),

paid at time \( \tau^*_H \). Define \( v_\pm : \mathbb{C} \to \mathbb{C} \) by

\[
v_\pm(s) := \frac{1}{2} \left( -\frac{1}{2} \pm \sqrt{1 - 2i} \right).
\]

As with \( u_\pm(\omega, s) \), when it causes no confusion, we will omit the subscript \( \pm \) and the argument \( s \) from \( v_\pm(s) \). The following trading strategy replicates a single barrier rebate power-exponential claim.
Theorem 6.1 (Replication of single barrier rebate power-exponential claims): Fix \( s \in \mathbb{C} \setminus \{-i/8\} \), \( m \in \{0\} \cup \mathbb{N} \) and \( H \in \mathbb{R} \). Define
\[
\psi_{H}^{rb}(X_{T}, (X)_{T}) \equiv \psi_{H}^{rb}(X_{T}, (X)_{T}; s) = e^{i\nu(X_{T} - H) + i\hat{s}(X)_{T}},
\]
where \( \nu \equiv \nu(s) \) is defined in (56). Then the following trading strategy replicates a single barrier rebate power-exponential claim that pays
\[
\mathbb{I}_{\{\tau_{H} \leq T\}} \langle X \rangle_{\tau_{H}}^{m} e^{i\hat{s}(X)_{\tau_{H}}^{*}},
\]
at time \( \tau_{H}^{*} \). At time 0 hold one European-style claim with payoff \((-\hat{i}\partial_{s})^{m}\psi_{H}^{rb}(X_{T}, (X)_{T}; s)\) and sell one single barrier knock-out claim with payoff \(\mathbb{I}_{\{\tau_{H} > T\}}(-\hat{i}\partial_{s})^{m}\psi_{H}^{rb}(X_{T}, (X)_{T}; s)\). If and when \( X \) hits the level \( H \), the knock-out claim becomes worthless; sell the European-style claim for \(\langle X \rangle_{\tau_{H}^{*}}^{m} e^{i\hat{s}(X)_{\tau_{H}^{*}}}\).

**Proof:** If \( \tau_{H} > T \) the rebate claim expires worthless. Likewise, if \( \tau_{H} > T \), the long position in the European-style claim pays \((-\hat{i}\partial_{s})^{m}\psi_{H}^{rb}(X_{T}, (X)_{T}; s)\) while the short position in the single barrier knock-out claim pays \(-(-\hat{i}\partial_{s})^{m}\psi_{H}^{rb}(X_{T}, (X)_{T}; s)\), for a net payout of zero.

If \( \tau_{H} \leq T \), the knock-out claim becomes worthless at time \( \tau_{H}^{*} \). Thus it remains to show that the value of the European-style claim equals the payoff of the rebate claim at time \( \tau_{H}^{*} \). We have
\[
\mathbb{I}_{\{\tau_{H} \leq T\}} \mathbb{E}_{\tau_{H}^{*}}(-\hat{i}\partial_{s})^{m}\psi_{H}^{rb}(X_{T}, (X)_{T}; s) = \mathbb{I}_{\{\tau_{H} \leq T\}}(-\hat{i}\partial_{s})^{m} e^{-i\nu H} \mathbb{E}_{\tau_{H}^{*}} e^{i\nu X_{T} + i\hat{s}(X)_{T}} \quad \text{(by (57) and Leibniz)}
\]
\[
= \mathbb{I}_{\{\tau_{H} \leq T\}}(-\hat{i}\partial_{s})^{m} e^{-i\nu H} e^{i\nu X_{T} + i\hat{s}(X)_{T}^{*}}
\]
\[(M_{t} := e^{i\nu X_{t} + i\hat{s}(X)_{t}} = \mathbb{E}_{t} M_{T} \text{ is a martingale})
\]
\[
= \mathbb{I}_{\{\tau_{H} \leq T\}}(-\hat{i}\partial_{s})^{m} e^{i\hat{s}(X)_{\tau_{H}^{*}}} \quad \text{(\( \mathbb{I}_{\{\tau_{H} \leq T\}}(X_{\tau_{H}^{*}} - H) = 0 \))}
\]
\[
= \mathbb{I}_{\{\tau_{H} \leq T\}} \langle X \rangle_{\tau_{H}^{*}}^{m} e^{i\hat{s}(X)_{\tau_{H}^{*}}},
\]
where use of the Leibniz integral rule is justified by (9).

Proposition 6.2 (Prices of single barrier rebate power-exponential claims): Assume the distribution of \( X_{T} \) has no point masses (a sufficient condition is that \( \int_{0}^{T} \sigma_{t}^{2} \, dt > \varepsilon > 0 \)). Then for any \( L < X_{0} \), \( k \in \{0\} \cup \mathbb{N} \) and \( s \in \mathbb{C} \setminus \{-i/8\} \), we have
\[
\mathbb{E} \mathbb{I}_{\{t_{L} \leq T\}} \langle X \rangle_{t_{L}}^{k} e^{i\hat{s}(X)_{t_{L}}} = \lim_{n \to \infty} \mathbb{E} (g_{n}(X_{T}) + h_{n}(X_{T})),
\]
where the functions \( g_{n} \) and \( h_{n} \) are given by
\[
g_{n}(X_{T}) = \int_{\mathbb{R}} (-\hat{i}\partial_{s})^{k} \hat{H}_{n}(\nu(s) - \omega) e^{-i\omega L + i(\omega - u(\omega,s))X_{0} + i\mu(\omega,s)X_{T}} \, d\omega,
\]
\[
h_{n}(X_{T}) = \int_{\mathbb{R}} (-\hat{i}\partial_{s})^{k} \hat{H}_{n}(\nu(s) - \omega) e^{-i\omega L + i(\omega - u(\omega,s))X_{0} + i\mu(\omega,s)X_{T}} \, d\omega,
\]
with \( \hat{H}_{n} \) as defined in (32). Here, the contour of integration in \( g_{n} \) is chosen so that \( 2n + \text{Im} \, \nu(s) > \omega_{i} > \text{Im} \, \nu(s) \) and \( 2i\sigma - \omega^{2} - i\omega + \frac{1}{4} \neq 0 \) and the contour of integration in \( h_{n} \) is chosen so that \( 2n - 1 - \text{Im} \, \nu(s) > \omega_{i} > -1 - \text{Im} \, \nu(s) \) and \( 2i\sigma - \omega^{2} - i\omega + \frac{1}{4} \neq 0 \).
Proof: As in the proof of Proposition 4.3, let $H(x) := \frac{1}{2}(1 + \text{sgn} x)$ denote the Heaviside function and let $H_n(x) := \frac{1}{2}(1 + \tanh nx)$. Then

$$\mathbb{E}[\mathbb{1}_{\{T \leq T\}} (X)^k_L e^{is(X)}e^{\frac{i}{T}}]$$

$$= (-i\partial_s)^k \mathbb{E}[\mathbb{1}_{\{T \leq T\}} e^{is(X)}e^{\frac{i}{T}}]$$

$$= (-i\partial_s)^k \mathbb{E}\left( \psi_L^{rb}(X_T, (X)_T; s) - \mathbb{1}_{\{T > T\}} \psi_L^{rb}(X_T, (X)_T; s) \right)$$

$$= (-i\partial_s)^k \mathbb{E}\left( \mathbb{1}_{\{X_T \leq L\}} \psi_L^{rb}(X_T, (X)_T; s) + \mathbb{1}_{\{X_T < L\}} \psi_L^{rb}(2L - X_T, (X)_T; s) \right)$$

$$=\mathbb{E} \lim_{n \to \infty} H_n(L - X_T) \left( \psi_L^{rb}(X_T, (X)_T; s) + e^{X_T - L} \psi_L^{rb}(2L - X_T, (X)_T; s) \right)$$

$$=\mathbb{E} \lim_{n \to \infty} (-i\partial_s)^k \mathbb{E}[\psi_L^{rb}(X_T, (X)_T; s) + e^{X_T - L} \psi_L^{rb}(2L - X_T, (X)_T; s)] \tag{59}$$

where the second equality follows from Theorem 6.1, the third equality follows from Theorem 4.1, the fourth equality is algebra, the sixth equality follows from the fact that the distribution of $X_T$ has no point masses (by assumption) and the various exchanges of limits, derivatives and expectations are allowed by Lebesgue’s dominated convergence and the Leibniz integral rule. Using the expression (57) for $\psi_L^{rb}$ and the fact that $\mathbb{F}[H_n] = \hat{H}_n$ we have

$$(-i\partial_s)^k \mathbb{E} H_n(L - X_T) \psi_L^{rb}(X_T, (X)_T; s)$$

$$= (-i\partial_s)^k \mathbb{E} \int_{\mathbb{R}} \hat{H}_n(v(s) - \omega) e^{-i\omega L} e^{i\omega X_T + is(X)_T} d\omega$$

$$\times (2n + \text{Im} v(s) > \omega_i > \text{Im} v(s))$$

$$= (-i\partial_s)^k \mathbb{E} \int_{\mathbb{R}} \hat{H}_n(v(s) - \omega) e^{-i\omega L} E e^{i\omega X_T + is(X)_T} d\omega \quad \text{(Fubini)}$$

$$= (-i\partial_s)^k \mathbb{E} \int_{\mathbb{R}} \hat{H}_n(v(s) - \omega) e^{-i\omega L + i(\omega - u(\omega, s))X_0} e^{i\omega(\omega, s)X_T} d\omega \quad \text{(by (4))}$$

$$= (-i\partial_s)^k \mathbb{E} \int_{\mathbb{R}} \hat{H}_n(v(s) - \omega) e^{-i\omega L + i(\omega - u(\omega, s))X_0 + i\omega(\omega, s)X_T} d\omega \quad \text{(Fubini)}$$

$$= \mathbb{E} g_n(X_T), \tag{60}$$
Figure 4. We plot the payoff $g_n(\log \cdot) + h_n(\log \cdot)$, given by (58) (see also Remark 6.3) whose expectation, in the limit as $n \to \infty$, prices a single barrier rebate power-exponential claim. Left: the solid, dashed, and dotted lines correspond to $e^{X_0} = \{100, 100^{1.25}, 100^{1.50}\}$, respectively, and the other parameters are fixed: $e^L = 90, s = 0, k = 1, n = 25$. The vertical dashed line is placed at the barrier $e^L = 90$. Right: the solid, dashed, and dotted lines correspond to $e^{X_0} = \{80, 80^{2/3}, 80^{1/3}\}$, respectively, and the other parameters are fixed: $e^U = 90, s = 0, k = 1, n = 25$. The vertical dashed line is placed at the barrier $e^U = 90$. Note that with $(k, s)$ as chosen, the European payoffs plotted above have expectations which price, in the limit as $n \to \infty$, single barrier rebate variance swaps, all of which have a payoff $\mathbb{1}_{\{\tau_H \leq T\}} \langle X \rangle_{\tau_H^{*}}$.

where the two applications of Fubini’s theorem and the use of the Leibniz integral rule are justified by the $|\omega_r| \to \infty$ behaviors: $|\partial_s^k \hat{H}_n(v(s) - \omega)| = O(e^{-|\omega_r|/n})$ and

$$E|e^{-i\omega L} e^{(i\omega X_T + i\omega s) s}| = O(1), \quad E|\partial_s^k e^{-i\omega L + i\omega u(\omega, s) X_0 + i\omega u(\omega, s) X_T}| = O(1),$$

and by the choice of the integration contour to avoid any singularities in the integrand. Similarly,

$$(-i \partial_s^k) E H_n(X_T - L) e^{X_T - L} \psi_{L}^{rb}(2L - X_T, \langle X \rangle_{T}; s) = Eh_n(X_T). \quad (61)$$

The result (58) follows from (59), (60) and (61).

Remark 6.3: To price a single-barrier rebate power-exponential claim $\mathbb{1}_{\{\tau_U \leq T\}} \langle X \rangle_{\tau_U^{*}} e^{i\omega X_T + i\omega s} \psi_{L}^{rb}$ with up-barrier $U > X_0$, make the following changes to Proposition 6.2: replace

$L \to U, \quad \hat{H}(v(s) - \omega) \to \hat{H}(\omega - v(s)), \quad \hat{H}(-i\omega + v(s) - \omega) \to \hat{H}(\omega + i + v(s))$

and reflect the contours of integration over the real axis: $\omega_i \to -\omega_i$.

In Figure 4, for various values of $X_0$ and $H$, we plot the payoff (58) whose expectation, in the limit as $n \to \infty$, prices the single barrier rebate variance swap which pays $\mathbb{1}_{\{\tau_H \leq T\}} \langle X \rangle_{\tau_H^{*}}$.

7. Summary and Future Research

Assuming only that the price of a risky asset $S = e^X$ is strictly positive and continuous and driven by an independent volatility process $\sigma$, we have shown how to price and hedge a variety of barrier-style claims written on the log returns $X$ and the quadratic variation of...
log returns \( \langle X \rangle \). In particular, we have studied single and double barrier knock-in, knock-out, and rebate claims. The pricing formulas we obtain are semi-robust in that they do not specify the dynamics of \( \sigma \), which may be non-Markovian and may include jumps.

Future research will focus three areas (i) weakening the independence assumption on log returns and volatility, (ii) pricing and hedging when calls and puts are available only at discrete strikes or only within a finite interval, (iii) considering richer payoff structures, which may depend on the running maximum or minimum of the asset in addition to log returns and quadratic variation of log returns.

**Note**

1. In a non-zero interest rate environment, our results are valid for derivatives written on the the path of the \( T \)-forward price of a risky asset.

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