REPRESENTATIONS OF THE FINITE-DIMENSIONAL POINT DENSITIES IN ARRATIA FLOWS WITH DRIFT

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Abstract. We derive representations for the finite-dimensional densities of the point processes associated with an Arratia flow with drift in terms of conditional expectations of the stochastic exponentials appearing in the analog of the Girsanov theorem for the Arratia flow.

1. Introduction

The study of the point process associated with an Arratia flow \( \{X^a(u, t) | u \in [0; 1], t \in [0; T]\} \) with drift \( a \) [1, Section 7] is carried out in the paper by means of special \((n, k)\)-point densities \( p_{t, n, k}^a \), \( k \leq n \). Such densities constitute a generalization of those discussed in [8, 9, 10] and are informally defined via the formula

\[
P\left( \forall i = 1, n X^a(u_i, t) \in \bigcup_{j=1}^{k} [y_j; y_j + dy_j], \forall j = 1, k \right) = p_{t, n, k}^a(u; y)dy_1 \ldots dy_k,
\]

the strict definition to be provided later in the text.

We find the Radon–Nikodym representation for \( p_{t, n, k}^a \) in terms of \( p_{t, 0, n, k}^0 \). It is known [1, Theorem 7.3.1] that the distribution of an Arratia flow with bounded Lipschitz continuous drift \( a \) is absolutely continuous in \( D([0; 1], C([0; T])) \) w.r.t. the distribution of the Arratia flow with zero drift. As a consequence, the distribution of the random process \( (X^a(u_1, \cdot), \ldots, X^a(u_n, \cdot)) \) is absolutely continuous in \( (C([0; T]))^n \) w.r.t. the distribution of \( (X^0(u_1, \cdot), \ldots, X^0(u_n, \cdot)) \) with density

\[
\exp \left\{ \sum_{k=1}^{n} \int_0^{\tau_k} a(X^0(u_k, t))dX^0(u_k, t) - \frac{1}{2} \sum_{k=1}^{n} \int_0^{\tau_k} a^2(X^0(u_k, t))dt \right\},
\]

where \( \tau_1 = T \) and

\[
\tau_k = \inf \left\{ T; t \left| \prod_{j=1}^{k-1} (X^0(u_j, t) - X^0(u_k, t)) = 0 \right. \right\}, \quad k = 1, n.
\]

Since the definition of the densities \( p_{t, n, k}^a \) contains the condition for the flow to hit the neighborhoods of certain points at time \( t \), we firstly investigate the distribution of \( \{X^0(u_1, t), \ldots, X^0(u_n, t)\} \) conditional on \( (X^0(u_1, T), \ldots, X^0(u_n, T)) \).

Hereinafter the superscript \( a = 0 \) is dropped in the case of zero drift, and \( a \) is always assumed to be bounded and Lipschitz continuous. We write \( x = (x_1, \ldots, x_n) \) for points in \( \mathbb{R}^n, n \in \mathbb{N} \).

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2. ON BROWNIAN BRIDGES AND RELATED CONDITIONAL DISTRIBUTIONS

The following constructive scheme is used. Assume \( w = (w_1, \ldots, w_n) \) to be a standard Wiener process in \( \mathbb{R}^n \) started at 0. Put \( \tilde{w}_1 = w_1, \theta_1 = T \) and define

\[
\theta_k = \inf \{ T; t \mid \tilde{w}_{k-1}(t) + u_{k-1} = w_k(t) + u_k \}, \\
\tilde{w}_k(t) = -u_k + w_k(t)\mathbb{I}(t < \theta_k) + (u_{k-1} + \tilde{w}_{k-1}(t))\mathbb{I}(t \geq \theta_k), \quad k = 1, n,
\]

where \( u \in \mathbb{R}^n \). Then one can easily verify the following statement.

**Lemma 2.1.** In \( (C([0; T]))^n \)

\[
(X(u_1, \cdot), \ldots, X(u_n, \cdot)) \overset{d}{=} u + \tilde{w} = (u_1 + \tilde{w}_1, \ldots, u_n + \tilde{w}_n),
\]

and the expression in (1) has the same distribution as

\[
\mathcal{E}_{T,n}^a(w, u) = \exp \left\{ \sum_{k=1}^n \int_0^{\theta_k} a(u_k + w_k(t))dw_k(t) - \frac{1}{2} \sum_{k=1}^n \int_0^{\theta_k} a^2(u_k + w_k(t))dt \right\}.
\]

The Ito stochastic integrals that participate in the definition of \( \mathcal{E}_{T,n}^a \) can be expressed in terms of stochastic integrals w.r.t. the Brownian bridges \( \eta = (\eta_1, \ldots, \eta_n) \) associated with \( w \):

\[
w_k(t) = \frac{t}{T}w_k(T) + \eta_k(t), \quad t \in [0; T], k = 1, n.
\]

We refer to [5, §5.6.B] for the general exposition of the theory of Brownian bridges.

Define the filtration

\[ \mathcal{F}_t = \sigma (\eta_k(s), s \leq t, k = 1, n), \quad t \in [0; T], \]

which is further supposed to be augmented in a standard way. Each \( \eta_k \) is a solution to

\[
d\eta_k(t) = d\beta_k(t) + \frac{\eta_k(t)}{T-t}dt, \quad t \in [0; T],
\]

\[
\eta_k(0) = \eta_k(T) = 0,
\]

where each \( \beta_k \) is a \( (\mathcal{F}_t)_{t \in [0; T]} \)-Wiener process. At the same time, every \( \eta_k \) admits the representation

\[
\eta_k(t) = (T-t)b_k \left( \frac{t}{T(T-t)} \right), \quad t \in [0; T),
\]

with \( b_1, \ldots, b_n \) being independent standard Wiener processes.

We summarize the properties of integration w.r.t. the bridges \( \eta_1, \ldots, \eta_n \) that we need as follows (cf. [3]).

**Lemma 2.2.** Assume that \( f_n : [0; T] \times \Omega \rightarrow \mathbb{R}, n \in \mathbb{N} \), are random functions such that

1. \( \exists C \in \mathbb{R}_+ \), 
   \[ \sup_{n \in \mathbb{N}} |f_n(t, \omega)| \leq C \quad \text{Leb} \otimes P\text{-a.e.} ; \]
2. there exist partitions \( 0 = t_0^n < \ldots < t_n^n = T, n \in \mathbb{N} \), and random variables \( c_{nk}, k = 0, N_n, n \in \mathbb{N}, \) such that every \( c_{nk} \) is \( \mathcal{F}_{\theta_k}^a \)-measurable and
   \[ f_n(t) = c_{n0} \mathbb{I}(t = 0) + \sum_{j=0}^{N_n-1} c_{nj+1} \mathbb{I}(t \in [t_j^n, t_{j+1}^n])(t). \]
If there exists a progressively measurable w.r.t. \((\mathcal{F}_t)_{t \in [0;T]}\) random process \(f : [0;T] \times \Omega \to \mathbb{R}\) with a.s. càdlàg trajectories such that on a set of full probability \(\Omega'\)

\[ f_n(t, \omega) \to f(t, \omega) \text{ for almost all } t \in [0;T], \]

then for any \(k = \overline{1,n}\)

\[
\sum_{j=0}^{N_n-1} c_{nj+1} \left( \eta_k(t_{j+1}^n) - \eta_k(t_j^n) \right) \xrightarrow{P} \int_0^T f(t) d\beta_k(t) - \frac{1}{2} \int_0^T f(t) \frac{\eta_k(t)}{T-t} dt,
\]

where the first integral in the RHS is the Ito integral while the second one is the pathwise Lebesque integral on \(\Omega'\).

**Proof.** Due to the boundness of \(f_n, n \in \mathbb{N}\),

\[
E \int_0^T f_n^2(t) dt < +\infty,
\]

and

\[
E \int_0^T |f_n(t)| \frac{\eta_k(t)}{T-t} dt \leq \text{const} \int_0^T \left( \frac{E f_n^2(t)}{T-t} \right)^{1/2} dt \leq \text{const} \int_0^T \left( \frac{t}{(T-t)T} \right)^{1/2} dt < +\infty,
\]

so both integrals in the RHS of the statement of the lemma are well defined. By (3),

\[
\sum_{j=0}^{N_n-1} c_{nj} \left( \eta_k(t_{j+1}^n) - \eta_k(t_j^n) \right) = \int_0^T f_n(t) d\beta_k(t) - \int_0^T f_n(t) \frac{\eta_k(t)}{T-t} dt,
\]

where, by the dominated convergence theorem,

\[
E \int_0^T \left( (f_n(t) - f(t))^2 + |f_n(t) - f(t)| \frac{\eta_k(t)}{T-t} \right) dt \to 0, n \to \infty,
\]

since the functions \(t \mapsto E(f_n(t) - f(t))^2\) and \(t \mapsto E |f_n(t) - f(t)| \frac{\eta_k(t)}{T-t}\) belong to \(L_1([0;T])\) and converge to 0 a.e.. This finishes the proof. \(\square\)

**Define**

\[
\theta_{ij} = \inf \{ T; s \mid w_i(s) + u = w_j(s) + u_j \},
\]

\[
\theta_{ij}(y) = \inf \left\{ T; t \mid \eta_i(t) + u_i + \frac{t}{T}(y_i - u_i) = \eta_j(t) + u_j + \frac{t}{T}(y_j - u_j) \right\},
\]

\[
j = 1, i - 1, i = 1, 2, y \in \mathbb{R}^p.
\]

Additionally, put

\[
\theta_{kk} = \theta_{jj} = T, \quad k, j = \overline{1,n}.
\]

Then a.s.

\[
\theta_{ij} = \theta_{ij}(w(T) + u), \quad i, j = \overline{1,n}.
\]

**Lemma 2.3.** With probability 1 all \(\theta_{ij}\) are distinct. For any \(y \in \mathbb{R}^n\) with probability 1 all \(\theta_{ij}(y)\) that are less than \(T\) are distinct.

**Proof.** The first assertion is trivial. To prove the second one, one uses (1). \(\square\)
To describe sequences of collisions in finite-dimensional motions of the Arratia flow, the following additional notation is introduced.

Put

$$\mathcal{S}_n, k = \{(j_1, \ldots, j_k) \mid j_i \in \{1, \ldots, n - i\}, i = 1, k, \ k = 1, n, \$$

$$\mathcal{S}_n = \emptyset \vee \bigcup_{k = 1, n - 1} \mathcal{S}_{n, k}, \ n \in \mathbb{N}. \$$

Recall $\tilde{w} + u = (\tilde{w}_1 + u_1, \ldots, \tilde{w}_n + u_n)$ to be coalescing Wiener processes constructed from the process $w + u$. Let $n - \varepsilon$ be the number of distinct values in the sequence $\{\tilde{w}_i(T) + u_i \mid i = 1, n\}$, $\varepsilon$ ranging in $\{0, \ldots, n - 1\}$. Let $\tau_1 < \tau_2 < \ldots < \tau_{\varepsilon}$ be random moments such that

$$\{\tau_1, \ldots, \tau_{\varepsilon}\} = \{\theta_k \mid \theta_k < T, k = 1, n\}. \$$

By virtue of Lemma 2.3 such $\tau_1, \ldots, \tau_{\varepsilon}$ exist a.s.. Put $j_1 = \min\{i \mid \exists j \neq i \tilde{w}_j(\tau_1) + u_j = \tilde{w}_i(\tau_1) + u_i\}$ and define the process $\tilde{W}^{n-1}$ by excluding the $j_1$-th coordinate from the vector $\tilde{w} + u$. Then put $j_2 = \min\{i \mid \exists j \neq i \tilde{w}_j^{n-1}(\tau_2) = \tilde{w}_i^{n-1}(\tau_2)\}$, define $\tilde{W}^{n-2}$ by excluding the $j_2$-th coordinate from the process $\tilde{W}^{n-1}$ and repeat the procedure until a random collection $S(w) = (j_1, \ldots, j_{\varepsilon}) \in \mathcal{S}_{n, \varepsilon}$ appears. We will call $S(w)$ the coalescing scheme for the process $w$ given fixed $u$.

Using Lemma 2.3, one finds random numbers

$$\lambda_{ij}(s), i = 1, 2, j = 1, n, \lambda_{ip}(s) \in \{1, \ldots, p\}, \lambda_{2p}(s) \in \{1, \ldots, \lambda_{1p}(s)\}, p = 1, n, \$$

such that on the set $\{S(w) = s\}$

$$\theta_k = \theta_{\lambda_{ik}(s)\lambda_{2k}(s)}, k = 1, n, \$$

up to sets of zero probability.

For ease of the further presentation, put

$$a_k(t, y, s) = \mathbb{I}(t < \theta_{\lambda_{ik}(s)\lambda_{2k}(s)}(y)) \cdot a\left(\eta_k(t) + u_k + \frac{t}{T}(y_k - u_k)\right), \$$

$$t \in [0; T], \ k = 1, n, \ y \in \mathbb{R}^n, \ s \in \mathcal{S}_n; \$$

$$\mathcal{E}_{T, n}^a(y, s) = \exp\left\{\sum_{k = 1}^n \int_0^T a_k(t, y, s)d\beta_k(t) + \sum_{k = 1}^n \int_0^T a_k(t, y, s)\left(\frac{y_k - u_k}{T} - \frac{\eta_k(t)}{T - t} - \frac{1}{2}a_k(t, y, s)\right)ds\right\}, \ y \in \mathbb{R}^n, \ s \in \mathcal{S}_n. \$$

**Lemma 2.4.** $\forall C > 0 \ \forall k = 1, n$

$$\mathbb{E} e^{C \int_0^T \frac{\mid \eta_k(t) \mid}{T - t} dt} < +\infty. \$$

**Proof.** As shown in the proof of Lemma 2.2, the process $t \mapsto \frac{\eta_k(t)}{T - t}$ is a Gaussian random element in $L_1([0; T])$, therefore the claim follows from the Fernique theorem [7, Theorem 3.1]. \hfill \Box

**Lemma 2.5.** $\forall y \in \mathbb{R}^n \ \forall s \in \mathcal{S}_n \ \forall p \geq 0$

$$\mathbb{E}\left(\mathcal{E}_{T, n}^a(y, s)\right)^p \leq C_1 e^{C_2\|y\|}, \$$
where
\[ C_1 = e^{n^p(2p - 1/2) ||a||^2 L_{\infty}(\mathbb{R})} + \eta^n ||u|| L_{\infty}(\mathbb{R}) \left( E e^{2p ||a|| L_{\infty}(\mathbb{R}) \int_0^T |\eta_k(t)| dt} \right)^{n/2}, \]

\[ C_2 = \eta^n ||a|| L_{\infty}(\mathbb{R}). \]

Proof.

\[ (E (\mathcal{E}_{T,n}^a(y,s))^2) \leq E \exp \left\{ \sum_{k=1}^{n} \int_0^T 2p a_k(t, y, s) d\beta_k(t) - \frac{1}{2} \sum_{k=1}^{n} \int_0^T (2p a_k(t, y, s))^2 dt \right\} \times \]

\[ \times E \exp \left\{ \sum_{k=1}^{n} \int_0^T \left( \frac{p(2p - 1)a_k^2(t, y, s) - 2p a_k(t, y, s) \eta_k(t)}{T - t} + \left( \frac{2p(y_k - u_k)}{T} \right) a_k(t, y, s) \right) dt \right\} \leq \]

\[ \leq \exp \left\{ 2C_2 ||y|| + 2n^p Y^n(2p - 1) T \cdot ||a||^2 L_{\infty}(\mathbb{R}) + 2n^p Y^n ||a|| L_{\infty}(\mathbb{R}) \right\} \times \]

\[ \times E \exp \left\{ 2p ||a|| L_{\infty}(\mathbb{R}) \sum_{k=1}^{n} \int_0^T \frac{|\eta_k(t)|}{T - t} dt \right\}, \]

thus the application of Lemma 2.4 finishes the proof. \[\square\]

In parallel to \( S(w) \) one defines random elements \( S(\eta, y), y \in \mathbb{R}^n, \) in \( Sh_n \) by using the same recursive procedure applied to the process \( t \mapsto \eta + u + \frac{1}{T} (y - u) \) and the times \( \theta_{ij}(y - u), j, i = 1, n. \)

**Theorem 2.1.** \( \forall y \in \mathbb{R}^n \) \( \forall s \in Sh_n \) a.s.

\[ E \left( \mathbb{I}(S(w) = s) \mathcal{E}_{T,n}^a(w, u)/w(T) = y - u \right) = E \mathbb{I}(S(\eta, y) = s) \mathcal{E}_{T,n}^a(y, s). \]

Proof. Define

\[ a_{k,j}^m(y, s) = \int_{T^{-1}t}^T a_k(t, y, s) dt, \quad k = 1, n, j = 1, m - 1, \]

\[ a_{k}^m(t, y, s) = \sum_{j=1}^{m-1} \mathbb{I}(\frac{1}{m} t, \frac{j+1}{m} T) a_{k,j}^m(y, s), \quad t \in [0; T], m \in \mathbb{N}. \]

Note that on the set \( \{ S(w) = s \} \)

\[ a(w_k(t) + u_k)(t \leq \theta_k) = a_k(t, w(T) + u, s), \quad k = 1, n, t \in [0; T]. \]

Due to the representation (2)

\[ \int_0^T a_{k}^m(t, w(T) + u, s) dw_k(t) = \frac{w_k(T)}{T} \int_0^T a_{k}^m(t, w(T) + u, s) dt + \]

\[ + \sum_{j=1}^{m-1} a_{k,j}^m(w(T) + u, s) \left( \eta_k \left( \frac{j+1}{m} T \right) - \eta_k \left( \frac{j}{m} T \right) \right). \]

(5)
Since \( w(T) \) and \( \eta \) are independent, the application of the disintegration theorem \([4, \text{Theorem 6.4}]\) gives, due to \((5)\), that
\[
E \left( \mathbb{I}(S(w) = s) \exp \left\{ \sum_{k=1}^{n} \int_0^T a_k^m(t, w(T) + u, s)\, dw_k(t) - \frac{1}{2} \sum_{k=1}^{n} \int_0^T (a_k^m(t, w(T) + u, s))^2 \, dt \right\} / w(T) = y - u \right) = \\
E \left( \mathbb{I}(S(\eta, y) = s) e_m(y, s) \right),
\]
where
\[
e_m(y, s) = \exp \left\{ \sum_{k=1}^{n} \sum_{j=1}^{m-1} a_{k,j}^m(y, s) \left( \eta_k \left( \frac{j+1}{m} \right) - \eta_k \left( \frac{j}{m} \right) \right) + \sum_{k=1}^{n} \int_0^T a_k^m(t, y, s) \left( \eta_k(t) - \eta_j(t) \right) \, dt \right\}.
\]
Since the functions \( a_k, k = 1, \ldots, n \) are piecewise continuous a.s., there exists a set \( \Omega' \) of full probability such that \( \forall \omega \in \Omega' \) for almost all \( t \in [0; T] \)
\[
a_k^m(t, w(T) + u, s) \to a_k(t, w(T) + u, s) \text{ in } \mathbb{R}.
\]
Therefore one can use the dominated convergence theorem for conditional expectations to prove that, due to \((6)\),
\[
E \left( \mathbb{I}(S(\eta, y) = s) e_m(y, s) \right) \bigg|_{y = w(T) + u} \xrightarrow{P} \lim_{m \to \infty} \mathbb{E}_{T, n}^a (w, u) / w(T) = y - u.
\]
Lemmas \(2.2\) and \(2.5\) yield the existence of
\[
P \lim_{m \to \infty} \left( E(\mathbb{I}(S(\eta, y) = s) e_m(y, s)) \right) \bigg|_{y = w(T) + u}.
\]
Using \((5)\) one establishes an explicit expression for this limit, which, when combined with \((7)\), completes the proof.

**Lemma 2.6.** \( \forall \ y \in \mathbb{R}^n \ \forall s \in Sh_n \)
\[
\mathbb{I}(S(\eta, y_m) = s) \xrightarrow{a.s.} \mathbb{I}(S(\eta, y) = s),
\]
whenever \( y_m \to y, m \to \infty \).

**Proof.** Due to Lemma \(2.3\) it is sufficient to check that a particular ordering of the moments \( \theta_{ij}(y) \) is preserved in a sufficiently small random neighborhood of the point \( y \). Using \((4)\), one gets
\[
P \left( \exists k \neq j : \eta_k(t) - \eta_j(t) \geq \left( 1 - \frac{t}{T} \right) + \frac{t}{T} (y_k - y_j) \neq 0, t \in [0; T], \right)
\]
\[
\eta_k(T) - \eta_j(T) + y_k - y_j = 0 \right) \leq \right.
\]
\[
\leq P \left( \exists k \neq j : \lim_{s \to \infty} \left( b_k(s) - b_j(s) + \frac{T_s}{T_s + 1} (y_k - y_j) \right) = 0 \right) = 0,
\]
\[
P \left( \exists k \neq j : \eta_k(t) - \eta_j(t) + (u_k - u_j) \left( 1 - \frac{t}{T} \right) + \frac{t}{T} (y_k - y_j) \neq 0, t \in [0; T], \right)
\]
\[
\eta_k(T) - \eta_j(T) + y_k - y_j = 0 \right) \leq \right.
\]
\[
\leq P \left( \exists k \neq j : \lim_{s \to \infty} \left( b_k(s) - b_j(s) + \frac{T_s}{T_s + 1} (y_k - y_j) \right) = 0 \right) = 0,
\]
which implies that with probability 1 for each \((k, j)\) either \(\theta_{kj}(y) < T\) or
\[
\inf_{t \in [0; T]} \left| \eta_k(t) - \eta_j(t) + (u_k - u_j) \left(1 - \frac{t}{T}\right) + \frac{t}{T}(y_k - y_j) \right| > 0.
\]
From now on, only a set of full probability which the condition \(\Box\) or its counterpart holds for is considered.

Fix a pair \((k, j)\) and some positive \(\varepsilon < 1\). Suppose \(\theta_{kj} < T\). Obviously, there exists random \(r > 0\) such that the condition \(\|y - y'\| < r\) implies \(\theta_{kj}(y') \geq \theta_{kj}(y) - \varepsilon\). The moment \(\theta_{kj}(y)\) is a Markov time w.r.t. the filtration generated by the process \(\eta\), therefore it follows from the iterated logarithm law for the Wiener process that there exist random \(\varepsilon_1, \varepsilon_2 : 0 < \varepsilon_1, \varepsilon_2 < \varepsilon\) such that
\[
\eta_k(\theta_{kj}(y) + \varepsilon_1) - \eta_j(\theta_{kj}(y) + \varepsilon_1) + (u_k - u_j) \left(1 - \frac{\theta_{kj}(y)}{T}\right) + (y_k - y_j) \frac{\theta_{kj}(y)}{T} + \varepsilon_1 \left(\frac{y_k - y_j}{T} + \frac{u_j - u_k}{T}\right) > 0
\]
and
\[
\eta_k(\theta_{kj}(y) + \varepsilon_2) - \eta_j(\theta_{kj}(y) + \varepsilon_2) + (u_k - u_j) \left(1 - \frac{\theta_{kj}(y)}{T}\right) + (y_k - y_j) \frac{\theta_{kj}(y)}{T} + \varepsilon_2 \left(\frac{y_k - y_j}{T} + \frac{u_j - u_k}{T}\right) < 0.
\]
Thus in order to have \(\theta_{kj}(y') \leq \theta_{kj}(y) + \varepsilon\) when \(\|y - y'\| \leq \delta\) it is sufficient to choose \(\delta\) in such a way that the signs in \((9)\) and \((10)\) do not change when \(y_k\) and \(y_j\) are replaced with \(y'_k\) and \(y'_j\), respectively. \(\Box\)

By using Lemmas 2.5 and 2.6 one can establish the following result.

**Lemma 2.7.** \(\forall s \in Sh_n\) the function \(\mathbb{R}^n \ni y \mapsto \mathbb{E}\mathbb{I}(S(y, y) = s)\mathcal{E}^n_{T,n}(y, s)\) is continuous.

### 3. Finite-dimensional densities for the point process in the Arratia flow

The section is devoted to the study of the connection between the finite-dimensional densities for the Arratia flow and the stochastic exponentials that arise when addressing the Girsanov theorem for Arratia flows with drift [11 Section 7]. We start with the corresponding results from [11 §7.2-7.3]. Consider a dense subset of \([0; 1]\), \(U = \{u_k \mid k \in \mathbb{N}\}\). Put \(\Delta_n = \{u \in \mathbb{R}^n \mid u_1 < \ldots < u_n\}, \ n \in \mathbb{N}\). Define given an Arratia flow \(X\)
\[
\tau_1 = T,
\]
\[
\tau_k = \inf \left\{ T; s \mid \prod_{j=1}^{k-1} (X(u_k, s) - X(u_j, s)) = 0 \right\}, \ k \geq 2,
\]
and put, for \(u^{(N)} = (u_1, \ldots, u_N) \in \Delta_N\),
\[
I_N(u^{(N)}) = \sum_{k=1}^{N} \int_{0}^{\tau_k} a(X(u_k, t))dX(u_k, t),
\]
\[
J_N(u^{(N)}) = \sum_{k=1}^{N} \int_{0}^{\tau_k} a^2(X(u_k, t))dt, \ \ N \in \mathbb{N}.
\]
The integrals in the expression for $I_N$ are ordinary Ito integrals w.r.t. the Wiener processes $X(u_j, \cdot), j \in \mathbb{N}$. Note $I_N$ and $J_N$ are well defined as function on $\Delta_N$. There exist limits

$$I = L_2^* \lim_{n \to \infty} I_n(u^{(n)}),$$

$$J = L_2^* \lim_{n \to \infty} J_n(u^{(n)}),$$

which do not depend on the set $U$. The distribution of an Arratia flow with drift $a$ as a random element in the Skorokhod space $D([0; 1], C([0; T]))$ is absolutely continuous w.r.t. the distribution of $X$ with density

$$\hat{\mathcal{E}}_T^n = \exp \left\{ I - \frac{1}{2} J \right\}.$$ 

Define $\hat{\mathcal{E}}_{T,n}^a(u^{(n)}) = \exp\{I_n(u^{(n)}) - J_n(u^{(n)})\}, n \in \mathbb{N}.$

**Remark 3.1.** It is easy to see that $\hat{\mathcal{E}}_{T,n}^a(u)$ is well defined for any $u \in \Delta_n$.

**Lemma 3.1.** $\forall n \in \mathbb{N}$

$$E\left( \hat{\mathcal{E}}_{T,m}^a(u^{(m)}) / X(u_1, \cdot), \ldots, X(u_n, \cdot) \right) = \hat{\mathcal{E}}_{T,n}^a(u^{(n)}), \ m \geq n,$$

$$E\left( \hat{\mathcal{E}}_T^a / X(u_1, \cdot), \ldots, X(u_n, \cdot) \right) = \hat{\mathcal{E}}_{T,n}^a(u^{(n)}).$$

**Proof.** The random variables $\{\hat{\mathcal{E}}_{T,n}^a(u^{(n)})\}_{n \in \mathbb{N}}$ form a uniformly integrable sequence (see [1, the proof of Theorem 7.3.1, pp. 268-270]), therefore it is sufficient to prove

$$E\left( \hat{\mathcal{E}}_{T,m}^a(u^{(m)}) / \mathcal{G}_n \right) = \hat{\mathcal{E}}_{T,n}^a(u^{(n)}), m \geq n,$$

where $\mathcal{G}_n = \sigma(X(u_1, \cdot), \ldots, X(u_n, \cdot)).$ Suppose $n$ and $m > n$ are fixed. Put

$$e_k(t) = \exp \left\{ \int_0^t a(X(u_k, s))dX(u_k, s) - \frac{1}{2} \int_0^t a^2(X(u_k, s))ds \right\}, \quad t \in [0; T], k \in \mathbb{N},$$

then, using the Ito formula one can verify that, for $k \in \mathbb{N},$

$$e_k(\tau_k) = 1 + \int_0^{\tau_k} e_k(t)a(X(u_k, t))dt,$$

since every $\tau_k$ is a stopping time w.r.t. the filtration generated by the processes $X(u_j, \cdot), j \in \mathbb{N}.$ Therefore

$$E\left( \hat{\mathcal{E}}_{T,m}^a(u^{(m)}) / \mathcal{G}_n \right) = \hat{\mathcal{E}}_{T,n}^a(u^{(n)}) E\left( \prod_{j=n+1}^m e_j / \mathcal{G}_n \right) =$$

$$= \hat{\mathcal{E}}_{T,n}^a(u^{(n)}) \left( 1 + E\left( \sum_{k=1}^{m-n} \sum_{j_1 < \ldots < j_k} A_{j_1\ldots j_k} / \mathcal{G}_n \right) \right),$$

where

$$A_{j_1\ldots j_k} = \prod_{l=1}^k \int_0^{\tau_{j_l}} a(X(u_{j_l}, t))e_{j_l}(t)dt.$$ 

Rewriting the internal stochastic integrals in $A_{j_1\ldots j_k}$ as

$$\int_0^T a_{j_l}(t)dX(u_{j_l}, t), \quad l = 1, k,$$
for some progressively measurable w.r.t. the filtration generated by $X(u_j, \cdot), j \in \mathbb{N},$ processes
\[ a_{j_i}(t) = \mathbb{1}(t \leq \tau_{j_i}) a(X(u_{j_i}, t))e_{j_i}(t), \]
applying the same approximation scheme as the one used in the proof of Theorem 2.1 and utilizing the fact that the joint covariance of $X(u_i, \cdot)$ and $X(u_j, \cdot)$ equals
\[ \left( t - \inf \{ s \mid X(u_i, s) = X(u_j, s) \} \right)_+, \quad t \geq 0, i, j \in \mathbb{N}, \]
one proves via standard reasoning that
\[ \mathbb{E}(A_{j_1 \ldots j_n} / \mathcal{G}_n) = 0, \]
which concludes the proof.

The following notation will be needed further. Let $\{X^a(u, t) \mid u \in [0; 1], t \in [0; T]\}$ be an Arratia flow with drift $a.$ Define
\[ X^a(u, t) = (X^a(u_1, t), \ldots, X^a(u_n, t)), \]
\[ X^a_t(u) = \{X^a(u_1, t), \ldots, X^a(u_n, t)\}, \quad t \in [0; T], u \in \Delta_n, n \in \mathbb{N}. \]
Analogously to the case of coalescing Wiener processes in Section 2 one defines the coalescence scheme $S(X^a, u)$ for the family $(X^a(u_1, \cdot), \ldots, X^a(u_n, \cdot)).$

Given a set $K = \{k_1, \ldots, k_m\} \subset \{1, \ldots, n\}$ and a point $z \in \mathbb{R}^n$ we denote by $z^{-K}$ the vector obtained by removing in the vector $z$ all the coordinates whose numbers are in $K;$ by $z^K,$ the vector obtained by removing all coordinates except those in $K.$ We write $z^{K_1, \pm K_2}$ for $(z^{K_1})^{\pm K_2}.$

The following definitions of the finite-dimensional densities were introduced in [2] and represent a further development of the notions used in [8, 9].

Given an Arratia flow $X^a,$ a start point $u \in \Delta_n$ and a coalescence scheme $s \in Sh_{n,k}$ for some $k$ the corresponding $(n - j)$-point density $p_t^{a,s,n-j}(u; \cdot), j \geq k,$ is a measurable function on $\mathbb{R}^{n-j}$ such that for any bounded nonnegative measurable $f: \mathbb{R}^{n-j} \to \mathbb{R}$
\[ \mathbb{E} \sum_{v_1, \ldots, v_{n-j} \in X^a_t(u), v_1, \ldots, v_{n-j} \text{ are distinct}} f(v_1, \ldots, v_{n-j}) \mathbb{1}(S(X^a, u) = s) = \int_{\mathbb{R}^{n-j}} p_t^{a,s,n-j}(u; y)f(y)dy. \]
The existence of $p_t^{a,s,n-j}(u; \cdot)$ is shown in [2, Lemma 3.1] whereas an explicit expression in a particular case $k = j$ is obtained in [2, Theorem 3.1).

**Lemma 3.2.** For all $s \in Sh_{n,k}, u \in \Delta_n$ and $j \leq n - k$ the density $p_t^{a,s,j}(u; \cdot)$ exists.

**Proof.** Let $A$ be a Borel subset of $\Delta_j.$ By the Girsanov theorem for the Arratia flow
\[ \int_{\mathbb{R}^j} p_t^{a,s,j}(u; v) \mathbb{1}_A(v)dv = \]
\[ = \mathbb{E} \sum_{v_1, \ldots, v_j \in X^a_t(u), v_1, \ldots, v_j \text{ are distinct}} \mathbb{1}_A(v_1, \ldots, v_j) \mathbb{1}(S(X, u) = s) \hat{E}_{T,n}^a(u), \]
\[ \leq \sum_{L = \{i_1, \ldots, i_l\}, l \in \{1, \ldots, n-k\}, i = 1, j} \mathbb{E} \mathbb{1}_A \left( X(u, t)^L \right) \mathbb{E} \left( \hat{E}_{T,n}^a(u) / X(u, t)^L \right) \leq \]
\[ \leq \sum_{L = \{i_1, \ldots, i_l\}, l \in \{1, \ldots, n-k\}, i = 1, j} \int_{A} p_t^{0,n,j}(u^L; y) \mathbb{E} \left( \hat{E}_{T,n}^a(u) / X(u, t)^L = y \right)dy. \]
The case when $A$ is not a subset of $\Delta_j$ is treated similarly. □
Let \( g_T^n(u; \cdot) \) be the standard \( m \)-dimensional Gaussian density with mean \( u \) and variance \( T \operatorname{Id}_{m \times m} \), where \( \operatorname{Id}_{p \times p} \) is the unit square matrix of size \( p \).

For \( u \in \mathbb{R}^n \) and a coalescence scheme \( s \in Sh_{n,k} \) the set \( I(s) \) is defined as the set of all indexes such that
\[
|\{X^a(u_i, T) \mid i \in I(s)\}| = n - k
\]
on \( \{S(X^a, u) = s\} \). Obviously, the coalescence scheme deterministically and uniquely defines \( I(s) \), which does not depend on \( u \) and a specific realization of the flow \( X^a \).

**Theorem 3.1** (cf. [1]). Assume \( u \in \Delta_n \) and \( S \in Sh_{n,n-k} \) for some \( k \in \{0, \ldots, n-1\} \). Then for each \( j \in \{1, \ldots, k\} \) for all \( y \in \Delta_k \)
\[
p^{a,s,j}_T(u; y) = \sum_{L=\{1, \ldots, l\} \subset \{1, \ldots, k\}} \frac{g_T^l(u^{I(s)L}; z^{I(s)L})}{\int_{\mathbb{R}^{n-k}} dz^{I(s)L}} \int_{\mathbb{R}^{n-k}} dz^{I(s)L} g_T^{k-j}(u^{I(s)L}; z^{I(s)L})
\]

\[
\cdot (E(\|S(y, z) = s\|) E^a_{T,n}(z, s)) \bigg)|_{z \in \mathbb{R}^n, l = y}.
\]

**Proof.** Define
\[
B^+_\delta(v) = [v; v + \delta], \quad v \in \mathbb{R}, \delta > 0,
\]
\[
\xi = \tilde{w}(T) + u,
\]
\[
\mathcal{W} = \{\tilde{w}_1(T) + u_1, \ldots, \tilde{w}_n(T) + u_n\}
\]

We have, due to the Lebesgue differentiation theorem and [1, Theorem 7.3.1], for almost all \( y \)
\[
p^{a,s,j}_T(u; y) = \lim_{\delta \to 0^+} \delta^{-j} \operatorname{E} \prod_{i=1}^{j} |B^+_\delta(y_i) \cap X^a_T(u)| \cdot \|S(X^a, u) = s\|
\]
\[
= \lim_{\delta \to 0^+} \delta^{-j} \operatorname{E} \prod_{i=1}^{j} |B^+_\delta(y_i) \cap \mathcal{W}| \cdot \|E^a_{T,n}(w, u) \|\|S(w) = s\|
\]
since by Lemma [2,1] \( (X(u_1, \cdot), \ldots, X(u_n, \cdot)) \overset{d}{=} \tilde{w} + u \). The reasoning of [8, Appendix B], combined with Lemma [2,5] allows one to replace \( |B^+_\delta(y_i) \cap \mathcal{W}| \) with \( \|B^+_\delta(y_i) \cap \mathcal{W} \neq \emptyset \) for all \( i \) in the latter expression, so that
\[
p^{a,s,j}_T(u; y) = \lim_{\delta \to 0^+} \delta^{-j} \operatorname{E} \sum_{L=\{1, \ldots, l\} \subset \{1, \ldots, k\}} \prod_{i=1}^{j} \|B^+_\delta(y_i) \| \cdot E^a_{T,n}(w, u) \|S(w) = s\|
\]

Consider a separate term
\[
A_{\delta,L} = \operatorname{E} \prod_{i=1}^{j} \|B^+_\delta(y_i) \| \cdot E^a_{T,n}(w, u) \|S(w) = s\|
\]

Since on the set \( \{S(w) = s\} \) by the definition of \( I(s) \)
\[
\mathcal{W} = \{w_i(T) + u_i \mid i \in I(s)\}
\]
Theorem 3.2.\[ A_{δ,L} = E \left( \prod_{i=1}^{j} \mathbb{I}_{B^+_{δ} (y_i)} (ξ_i) \mathbb{I}(S(w) = s) \mathcal{E}_{T,n}^a (w, u)/w(T) \right) = \]

\[ = E \left( \prod_{i=1}^{j} \mathbb{I}_{B^+_{δ} (y_i)} \left( (w(T) + u)^{I(s,L)}_i \right) \right) \left( E \mathbb{I}(S(η, z) = s) \mathcal{E}_{T,n}^a (z, s) \right) \bigg|_{z=w(T)+u}. \]

The vectors \( w(T)^{I(s,L)}, w(T)^{I(s,-L)} \) and \( w(T)^{-I(s)} \) being independent given fixed \( s \), the local property of conditional expectation [4, Lemma 6.2] implies that the analog of (11) holds with Lemma 3.3. For any \( p \)

\[ (11) E \mathcal{E}_{T,n}^a (w, u)/w(T) \]

then for all \( w \)

\[ \text{The vectors } w(T)^{I(s,L)}, w(T)^{I(s,-L)} \text{ and } w(T)^{-I(s)} \text{ being independent given fixed } s, \text{ the local property of conditional expectation } [4, \text{Lemma 6.2}] \text{ implies that} \]

\[ p_{T}^{a,s,j}(u; y) = \lim_{δ \to 0+} \delta^{-j} E \left( \prod_{L=\{1,\ldots,l\} \subset \{1,\ldots,k\}} \prod_{i=1}^{j} \mathbb{I}_{B^+_{δ} (y_i)} \left( (w(T) + u)^{I(s,L)}_i \right) \times \left( E \mathbb{I}(S(η, z) = s) \mathcal{E}_{T,n}^a (z, s) \right) \bigg|_{z=w(T)+u}^{I(s,L)}, \]

where \( w(T)^{I(s,L)} \) and the Gaussian random variables \( α \sim \mathcal{N}(u^{I(s)}, T \cdot \text{Id}_{(k-j)\times(k-j)}), \beta \sim \mathcal{N}(u^{-I(s)}, T \cdot \text{Id}_{(n-k)\times(n-k)}) \) are jointly independent. With Lemmas 2.6 and 2.7, the rest of the proof follows by standard reasoning. We omit the details. \[ \square \]

**Remark 3.2.** The expression for the density \( p_{T}^{a,s,j}(u; \cdot) \) is given in Theorem 3.1 only for \( y \in Δ_j \), however it can be extended onto the whole \( \mathbb{R}^J \) by symmetry.

Given an Arratia flow \( X^a, a \) start point \( u \in Δ_n \) and \( k \in \{1, \ldots, n\} \) the corresponding \((n, k)\)-point density is a measurable function \( p_{T}^{a,k}(u; \cdot) \) on \( \mathbb{R}^k \) such that for any bounded nonnegative measurable \( f: \mathbb{R}^k \to \mathbb{R} \)

\[ (11) E \sum_{v_1, \ldots, v_k \in X^a_t (u), v_1, \ldots, v_k \text{ are distinct}} f(v_1, \ldots, v_k) \mathbb{I}(\{ |X^a_t (u)| \geq k \}) = \int_{\mathbb{R}^k} p_{T}^{a,k}(u; y) f(y)dy. \]

The next consequence of the formula of total probability gives a relation between \( p_{T}^{a,k} \) and \( p_{T}^{a,s,k} \).

**Lemma 3.3.** For any \( n \in \mathbb{N}, u \in Δ_n \) and \( k \in \{1, \ldots, n\} \) a.e.

\[ p_{T}^{a,k}(u, \cdot) = \sum_{l=0}^{n-k} \sum_{s \in S_h, n} p_{T}^{a,s,k} (u; \cdot). \]

The \( k \)-point density \( p_{T}^{a,k}(\cdot) \) (cf [8, 9]) is defined as a measurable function on \( \mathbb{R}^k \) such that the analog of (11) holds with \( X^a_t (u) \) replaced with \( \{ X^a_t (v, T) \mid v \in [0; 1] \} \) and the condition \( |X^a_t (u)| \geq k \) dropped. The result of [2, Theorem 3.2] admits the following extension, the proof being the same with some minor changes having been made.

**Theorem 3.2.** Let \( u^{(n)} = (u_1^{(n)}, \ldots, u_n^{(n)}) \in Δ_n, n \in \mathbb{N}, \) be such that \( u_1^{(n)} = 0, u_n^{(n)} = 1, n \in \mathbb{N}, \) and

\[ \max_{j=\max_{n-1}} (u_j^{(n)} - u_{j+1}^{(n)}) \to 0, n \to \infty. \]

Then for all \( k \in \mathbb{N} \) a.e.

\[ p_{T}^{a,k} (u^{(n)}; \cdot) \to p_{T}^{a,k}, n \to \infty. \]
Due to Lemma 3.3 Theorem 3.1 provides an explicit expression for the densities $p_T^{a,k}(u;\cdot)$ in terms of conditional expectations of certain stochastic exponentials discussed in Section 2.

Consider $u \in \Delta_n$. Suppose that elements of the set $\mathcal{X}_T(u)$ are listed in ascending order. Let $\mathcal{N}$ be the cemetery state. Given a set $L = \{l_1, \ldots, l_k\}, l_i \in \mathbb{N}, i = 1, k$, for some $k$, put the random vector $X_T(u)$ to be equal to

$$((\mathcal{X}_T(u))_{l_1}, \ldots, (\mathcal{X}_T(u))_{l_k}),$$

if $\max_{i, l_k} l_i \leq |\mathcal{X}_T(u)|$, and $\mathcal{N}$, otherwise. We denote the density of $X_T(u)$ in $\mathbb{R}^k$ by $q_T^{u}(u;\cdot)$. Note that always

$$\int_{\mathbb{R}^k} q_T^{u}(u;y)dy < 1.$$

**Theorem 3.3 (cf. [6]).** For any $u \in \Delta_n$ and any $k \in \{1, \ldots, n\}$ a.e.

$$p_T^{a,k}(u; y) = \sum_{L = \{l_1, \ldots, l_k\}, l_i \in \mathbb{N}, i = 1, k} q_T^{l}(u;y) \cdot \mathbb{E}\left(\hat{\mathcal{E}}^{a}_T / X_T(u) = y\right).$$

**Proof.** Proceeding similarly to the proof of Theorem 3.1 one obtains a.e.

$$p_T^{a,k}(u; y) = \lim_{\delta \to 0^+} \delta^{-k} \sum_{L = \{l_1, \ldots, l_k\}, l_i \in \mathbb{N}, i = 1, k} \mathbb{E}\left(\hat{\mathcal{E}}^{a}_T | X_T(u) \neq \mathcal{N}\right) \prod_{i=1}^{k} \mathbb{I}_B^T(y_i) ((X_T(u))_{i}) =$$

$$= \lim_{\delta \to 0^+} \delta^{-k} \sum_{L = \{l_1, \ldots, l_k\}, l_i \in \mathbb{N}, i = 1, k} \mathbb{E}\left(\mathbb{I}(X_T(u) \neq \mathcal{N}) \times \mathbb{I}(X_T(u)_{i} \in B^T_{\delta}(y_i)) \right) \times$$

$$\times \mathbb{E}\left(\hat{\mathcal{E}}^{a}_T / X_T(u)\right).$$

Using Lemma 3.1 we can check that

$$\mathbb{E}\left(\hat{\mathcal{E}}^{a}_T / X_T(u)\right) = \mathbb{E}\left(\hat{\mathcal{E}}^{a}_T / X_T(u)\right).$$

Finally, as in Theorem 3.1 and Lemma 2.7 one shows that, for fixed $L,$

$$\mathbb{E}\left(\hat{\mathcal{E}}^{a}_T(u)/X_T(u) = y_m\right) \to \mathbb{E}\left(\hat{\mathcal{E}}^{a}_T(u)/X_T(u) = y\right),$$

when $y_m \to y, m \to \infty.$ Thus the application of the Lebesgue differentiation theorem finishes the proof. We skip the details. \hfill \Box

Replacing in (12) the set $\mathcal{X}_T(u)$ with the set $\{X(v, T) | v \in [0, 1]\}$ one defines, analogously to $X_T(u)$, the random vector $X_T(u)$ with values in $\mathbb{R}^k \cup \{\mathcal{N}\}$. The corresponding density being denoted by $q_T^{l}(\cdot)$, the following result holds.

**Theorem 3.4.** For any $k \in \mathbb{N}$ a.e.

$$p_T^{a,k}(y) = \sum_{L = \{l_1, \ldots, l_k\}, l_i \in \mathbb{N}, i = 1, k} q_T^{l}(y) \cdot \mathbb{E}\left(\hat{\mathcal{E}}^{a}_T / X_T(u) = y\right).$$

The arguments, which are based on the locality of conditional expectation, are repetetive and thus omitted.
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