Large-scale Ising spin network based on degenerate optical parametric oscillators

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Solving combinatorial optimization problems is becoming increasingly important in modern society, where the analysis and optimization of unprecedentedly complex systems are required. Many such problems can be mapped onto the ground-state-search problem of the Ising Hamiltonian, and simulating the Ising spins with physical systems is now emerging as a promising approach for tackling such problems. Here, we report a large-scale network of artificial spins based on degenerate optical parametric oscillators (DOPOs), paving the way towards a photonic Ising machine capable of solving difficult combinatorial optimization problems. We generate >10,000 time-division-multiplexed DOPOs using dual-pump four-wave mixing in a highly nonlinear fibre placed in a cavity. Using those DOPOs, a one-dimensional Ising model is simulated by introducing nearest-neighbour optical coupling. We observe the formation of spin domains and find that the domain size diverges near the DOPO threshold, which suggests that the DOPO network can simulate the behaviour of low-temperature Ising spins.

Combinatorial optimization problems are becoming ever more important in our society, for example in applications such as artificial intelligence, drug discovery, optimization of cognitive wireless networks, and analysis of social networks. Many such problems are classified as non-deterministic polynomial time (NP)-hard or NP-complete problems, which are considered to be hard to solve efficiently with modern computers. It is well known that many combinatorial optimization problems can be mapped onto the ground-state-search problems of the Ising Hamiltonian. Various schemes have been proposed and demonstrated that simulate the Ising Hamiltonian with physical systems, such as superconducting circuits, trapped ions, CMOS devices and electromechanical oscillators. Among these schemes, the coherent Ising machine (CIM) is increasingly becoming the focus of research. A CIM simulates the Ising model using a network of lasers with binary oscillation conditions as artificial spins, and is expected to have a significant advantage in terms of computation time over conventional schemes such as simulated annealing and semi-definite programming. Recently, Marandi et al. demonstrated a CIM using degenerate optical parametric oscillators (DOPOs). A DOPO can be utilized as a stable artificial spin because it takes only the 0 or π phase relative to the pump phase, and is realized using second- and third-order optical nonlinear processes. The spin–spin interaction can be implemented simply with mutual injections of DOPO lights using delay interferometers. In ref. 11, a spin system composed of four DOPOs was used for a proof-of-principle CIM experiment. However, to simulate a more complex Ising Hamiltonian to verify the advantages of the CIM over existing methods, we need to implement a CIM with a much larger number of spins. Here, we report a large-scale network of artificial spins realized with as many as 10,000 time-division-multiplexed DOPOs generated via dual-pump four-wave mixing (FWM) in a highly nonlinear fibre (HNLF) placed in a fibre cavity. We successfully simulate the ferro- and anti-ferromagnetic-like behaviour of a one-dimensional (1D) Ising spin chain by introducing unidirectional nearest-neighbour coupling between DOPOs. In addition, we observe the formation of domain walls, with which we can obtain information on how much the state of the spin network is excited from the ground state. We believe the present result will provide a promising platform on which to realize an efficient machine for solving the Ising model based on the CIM concept.

Coherent Ising machine

A dimensionless Hamiltonian of an N-spin Ising model without an external magnetic field (Fig. 1a) is given by

$$H = - \sum_{1 \leq i < j \leq N} J_{ij} \sigma_i \sigma_j$$  \hspace{1cm} (1)

where $J_{ij}$ is the coupling coefficient between the $i$th and $j$th spins, and $\sigma_i$ ($i \in \{1, 2, \ldots, N\}$) denotes the $z$ projection of the $i$th spin, which can take $\pm 1$ values. The purpose of an Ising machine is to find the ground state of the above Hamiltonian with a given set of $J_{ij}$ using a physical system. To realize an Ising machine, we need elements with a binary degree of freedom to represent the spins and a method for realizing programmable coupling between spins. In a CIM, we use two-mode laser oscillators or DOPOs as artificial spins. The spin coupling can be implemented by injecting a portion of light from the $i$th spin into the $j$th spin, and vice versa. We can therefore set $J_{ij}$ by changing the phase and transmittance of the optical paths that connect the $i$th and $j$th spins. For CIM operation, we start with zero pump power for all the oscillators, and set the $J_{ij}$ values by establishing optical paths between the spins. We then gradually increase the pump. With $N$ spins, there are $2^N$ combinations of spin configurations. In other words, we are operating a multimode oscillator with $2^N$ modes. As we increase the pump, the network reaches the threshold, and an oscillation starts, most probably at the mode (or spin configuration) with the lowest loss, which will give the ground state of the Ising Hamiltonian.
A DOPO can be realized by placing a phase-sensitive amplifier (PSA) in a cavity where only a signal with phase 0 or π relative to the phase of the pump for the parametric amplification process is amplified. Details about the principle of PSA with dual-pump FWM are provided in the Methods. When a PSA is installed in a cavity and driven with a below-threshold pump, we observe a quadrature-squeezed noise generated by spontaneous parametric downconversion or spontaneous FWM. As the pump power is increased, the noise light undergoes phase-sensitive amplification, which leads to phase bifurcation as a result of spontaneous symmetry breaking. When the pump power reaches threshold, DOPOs are obtained with phases that can take only 0 or π. Because oscillation is initiated with noise photons generated by spontaneous parametric processes, the emergence probabilities of 0 and π phases are inherently equal. If we use a pulsed pump with temporal separation Δτ, we can generate N independent DOPOs with a single cavity by satisfying the condition Tc = NΔτ, where Tc denotes the cavity roundtrip time. The characteristics of these DOPOs are essentially identical except for their phases, because they share the same cavity. Thus, we can increase the number of spins simply by increasing the pump repetition frequency or by increasing Tc.

Figure 1b shows the experimental set-up. Continuous waves from two lasers with wavelengths of 1,531 and 1,531 nm were modulated by EDFAs and passed through optical bandpass filters to suppress noise from the EDFAs, respectively. Inset: Wavelength allocation of pumps 1 and 2 and the signal/idler wave. MZI2 is inserted when simulating the 1D Ising model.

**Simulation of 1D Ising model with DOPOs**

We realized a simulator of a 1D Ising model by inserting a 1 bit Mach–Zehnder interferometer (MZI1) whose two outputs were each connected to a photodetector. The phase difference between the two arms of the interferometer was adjusted so that the light was detected by detector 1 (detector 2) if the phase difference between adjacent DOPOs was 0 (π). Figure 2a illustrates the phase difference measurement I(t) waveform shifted for every 10,320 pulses, which means that each DOPO is almost the same. The measured pulse width was ~30 ps. Figure 2c presents a histogram of the peak values of I(t), which clearly indicates the discretization of the DOPO phase into 0 and π. The ratio between the positive and negative pulses was 1:0.996, indicating that the probabilities of the emergence of 0 and π were the same.

Although the phases of the N DOPOs generated in our set-up are completely random, each OPO should preserve the phase once the pump power exceeds the threshold level. This means that the same random pattern should be observed in the phase difference measurement for every NΔτ. To confirm this, we took the phase difference measurement result for 2,000 pulses and calculated the autocorrelation (Supplementary Section 1). The result is shown in Fig. 2d, and the region around 0 pulse delay is enlarged in the inset. As seen, an identical phase pattern was obtained 93 times for every 10,320 pulses, which means that each DOPO was oscillating with the same phase for at least 93 circulations in the cavity. The red curve in Fig. 2b shows the I(t) waveform shifted by 10,320 pulses (with a 100 ps offset for clarity). Thus, we could confirm that an identical phase pattern was repeated after a circulation of the pulse train. These results indicate that our DOPOs could be operated stably at well above the threshold.
the power of the ith pulse and inject it into the (i + 1)th pulse for i < N, and a portion of the Nth pulse was launched into the first pulse. Accordingly, with this set-up, we simulated the Hamiltonian given by $H = -\sum_{i=1}^{N} J \sigma_{i} \sigma_{i+1}$, with a periodic boundary condition $\sigma_{N+1} = \sigma_{1}$, which corresponds to the 1D Ising Hamiltonian analysed in Ising’s original paper. The sign of $J$ can be changed by tuning the phase of the delayed interferometer: $\text{sgn}(J) = 1$ and $-1$ can be realized by setting the phase difference of the interferometer at 0 and $\pi$, respectively. We measured the phase difference of the DOPOs from the cavity with the set-up described in the previous section. $N$ was increased to 10,337 in this experiment, due to the increase in the fibre cavity length caused by the insertion of MZI2.

The phase differences between adjacent DOPOs, $\cos \theta_{i}$, for coupling phases 0 and $\pi$, are shown in Fig. 3a,b, respectively, at a normalized 1,551 nm pump amplitude of 1.40. When the coupling phase was set at 0, $\cos \Delta \theta_{i}$ was $\approx 1$ for the majority of pulses, implying that the phases of the DOPOs were now aligned so that they were in phase. With the $\pi$ coupling phase, $\cos \Delta \theta_{i}$ was mostly negative, which means that the adjacent pulses now had alternating phases. These phase configurations are analogous to ferromagnetic and anti-ferromagnetic spin configurations, respectively. We also measured the phase difference as we changed the coupling phase. We observed a sharp transition from the ferromagnetic to the anti-ferromagnetic spin configuration at a coupling phase of $(2k + 1)/2\pi$ (where $k$ is an integer), which confirmed that the DOPO phase was discretized even at the boundary between 0 and $\pi$ phase coupling (see Supplementary Section 2 for details).

It is well known that no phase transition occurs in a 1D Ising model at finite temperatures. Accordingly, when $N$ is large, all the spins are not aligned in the same value in a 1D Ising model, and instead we observe the formation of stable domains at a temperature greater than absolute zero. Interestingly, we observe several inverted peaks in the $\cos \theta_{i}$ measurement results in both Fig. 3a and b, suggesting that we are observing in- and anti-phase ‘domains’, and the inverted peaks correspond to the boundaries (domain walls). The domain length distribution histograms for various pump amplitudes are shown in Fig. 3c, which clearly suggests that the interaction length between spins became longer as the pump amplitude was set closer to threshold.

We can estimate the energy increase of the spins from the ground state by counting the number of domain walls, which we refer to as $N_{d}$ hereafter. In our 1D Ising system, one spin flip from the ground state increases the total energy by $2J$. The energy increase per spin from the ground state is therefore given by $2Jn_{d}$, where $n_{d}$ is the defect density $N_{d}/N$. The experimentally obtained defect densities $n_{d}$ are plotted as a function of pump amplitude in Fig. 3d. The result agrees well with a numerical simulation based on the discrete-time model described in the Methods (shown by the solid line).

We also took the autocorrelation of the phase difference measurement data for various $p$ values, and fitted it with the function $\exp(-\lambda x_{0})$, where $x_{0}$ is the correlation length. The $x_{0}$ values obtained as a function of $p$ are shown as circles in Fig. 3d. The correlation length diverges as $x_{0} \propto 1/(p - 1)$, implying that a longer-range order can be formed when $p$ approaches 1. On the other hand, it takes a longer time to reach the DOPO transitions when $p$ approaches 1. The time to reach the DOPO transitions, which we call the saturation time $T_{s}$, is analytically related to $x_{0}$ as $T_{s} \propto x_{0}^{2}$ (see Methods). Our results show that the ferro- or antiferromagnetic ground states of the 1D Ising model realized with the DOPO can in principle find the ground state of the $N$-spin systems ($N < x_{0}$) within a power-law time-scaling of $N^{2}$. Note that the 1D Ising model is in fact a hard problem to compute with a physical system because a 1D spin system suffers from larger spin fluctuations than those in a higher-dimensional spin system.

It is informative to estimate the normalized temperature $T_{n} := k_{B}T/J$ of the spin system, where $T$ and $k_{B}$ denote absolute temperature and the Boltzmann constant, respectively. The defect density $n_{d}$ and the correlation length $x_{0}$ can be related to $T_{n}$ with the following equations: $2n_{d} = 1 - \tanh(1/T_{n})$, $x_{0} = -1/\ln(\tanh(1/T_{n}))$. For example, at a normalized pump amplitude of 1.01 and with in-phase coupling, both of the above equations consistently gave $T_{n} \approx 0.5$. 

Figure 2 | DOPO measurement results (without optical coupling). a, DOPO output power as a function of normalized 1,551 nm pump amplitude. b, Temporal waveforms of phase measurement signal $\theta(t)$. The red curve shows a waveform with its temporal position shifted by 10,320Â. To distinguish the no-shift waveform shown by the black curve, a 100 ps offset has been inserted between the black and red curves. c, Histogram of pulse peak values for 10,320 DOPOs. Clear phase discretization is observed. d, Autocorrelation measurement result. Inset: Magnification of the area indicated by the red dashed rectangle.
state because the final state is projected to the $0$ or $\pi$ phase as a result of phase bifurcation, and thus the purpose of the CIM is concentrated on simulating the Ising Hamiltonian.

To solve more complex problems with the CIM, we need to implement a much larger number of spin couplings, and the present approach based on optical coupling with MZIs will not scale as the number of spin coupling increases. To realize flexible spin couplings, a ‘measurement-and-feedback’ scheme was proposed by Haribara and colleagues10. In that scheme, the phases of all DOPOs are measured with a coherent measurement for every circulation in the cavity. The measured phases are supplied to a field-programmable gate array (FPGA), where the coupling coefficients $J_{ij}$ are preprogrammed. Using the phase measurement results and $J_{pi}$, the FPGA calculates the feedback signal for the $i$th (where $i$ is an integer) DOPO in the next circulation. Optical pulses with wavelength exactly the same as that of the DOPOs are modulated using the feedback signal and injected into each DOPO. This procedure is repeated for each roundtrip of the DOPO pulses. With this scheme, we can achieve all possible combinations of DOPO couplings, and thus realize a CIM to solve real-world problems.

Methods

Methods and any associated references are available in the online version of the paper.

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Discussion

An interesting question is the difference between the CIM and other ‘quantum machines’ such as those based on superconducting devices3 and trapped ions5. These systems are based on quantum bits (qubits), which means that each spin element can represent a superposed state. Therefore, such quantum machines can in principle simulate a quantum Hamiltonian based on the Heisenberg model. On the other hand, a DOPO cannot represent a superposed state because the final state is projected to the $0$ or $\pi$ phase as a result of phase bifurcation, and thus the purpose of the CIM is concentrated on simulating the Ising Hamiltonian.

These results obtained from observation of the 1D Ising model show that our DOPOs have simulated the behaviour of a low-temperature spin system well. We expect that the normalized temperature of a 1D Ising system can be a useful index with which to evaluate the quality of both DOPOs and other physical systems that constitute Ising machines. How much further we can ‘cool’ the spins may be an important consideration in developing Ising systems in the future.
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**Author contributions**
T.I. and H.T. constructed the DOPO set-up and performed the experiments. R.H. and K.Inaba developed the theoretical model. T.I., K.Inaba, R.H. and H.T. analysed the data. H.T., K.Inoue and Y.Y. conceived the concept for the experiment. All authors discussed the results and wrote the paper.

**Additional information**
Supplementary information is available in the online version of the paper. Reprints and permissions information is available online at www.nature.com/reprints. Correspondence and requests for materials should be addressed to H.T.

**Competing financial interests**
The authors declare no competing financial interests.
Methods

Discrete-time model for simulating a DOPO based on dual-pump FWM. The DOPOs in this experiment were operated with high gain because of the relatively large cavity loss, so the continuous-time model for simulating DOPOs reported in refs 9 and 10 does not accurately simulate the present experiment. We therefore needed to use a discrete-time model. Here, we briefly describe the discrete-time simulation of the DOPOs. This model will be reported in detail elsewhere.

We assume that the complex amplitudes of a degenerate signal and two pumps are represented by $a$, $b$ and $c$. When the phase-matching condition is satisfied, the mode coupling equations for these amplitudes are expressed as:

$$\frac{da}{dz} = 2i \alpha a + \frac{1}{2} ab$$

$$\frac{db}{dz} = -i \alpha b - \frac{1}{2} ab$$

$$\frac{dc}{dz} = -ic$$

where $z$ is the position along the HNLF and $\alpha$ is the nonlinear coefficient of the HNLF. We recast the field amplitudes as $x = (2i \alpha \omega_0)^{-1/2} e^{i \omega_0 z} (x \in a, b, c)$, where $(L_{\text{eff}} = (1 - e^{-\alpha L})) \alpha$, and with the normalized distance $z = (1 - e^{-\alpha L})/(1 - e^{-\alpha L})$. We then obtain the following normalized equations:

$$\frac{d\bar{a}}{dz} = 2i \bar{a} b, \quad \frac{d\bar{b}}{dz} = -i \bar{a} c - \frac{1}{2} \bar{a} b$$

$$\frac{d\bar{c}}{dz} = -i \bar{a} c$$

The bounds are $0 \leq z \leq 1$. If we express the phase terms of $\bar{x}$ with $\phi_x$, we obtain the following equations for the signal amplitude and phase:

$$\frac{d|\bar{a}|}{dz} = |\bar{a}| |\bar{b}| |\bar{c}| \cos \theta$$

$$\frac{d\phi_x}{dz} = |\bar{b}| |\bar{c}| \sin \theta$$

$$\theta = \phi_x + \phi_a - 2 \phi_a + \pi / 2$$

Thus, we can realize a PSA where only the signal with phase $0$ or $\pi$ relative to the sum of the pump phases is amplified. We may assume $\phi_x = 0$ without loss of generality. Because we are interested in above-threshold behaviour, $\bar{a}$ may be presumed real because only the real quadrature of $\bar{x}$ experiences gain. So, in the following we treat $\bar{x}$ as real in equation (5).

Equation (5) has two constants of motion, $A_x^2 = \bar{a}^2 + 2\bar{b}^2$ and $A_x^2 = \bar{a}^2 + 2\bar{c}^2$, which arise from the detailed balance in the $a + a \rightarrow b + c$ process. On integrating, we find

$$\bar{a} = \frac{1 - \tanh^2 (A_x A_y (s + s_0)/2)}{A_x^2 - A_y^2 \tanh^2 (A_x A_y (s + s_0)/2)}$$

Given the initial conditions $\alpha_{na}, \beta_{na}, \gamma_{na}$, we may invert equation (9) to find the constant of integration $s_0$, and then directly compute $\alpha_{na}$. This gives the input–output relation. Rescaling to physical units, we define the input–output map $F[s]$ so that $\alpha_{na} = F[\alpha_{na}]e^{-\alpha_{na} L}$; that is, $F[s]$ accounts for the PSA gain but not for its loss.

Let $a_i(m)$ represent the $i$th pulse at roundtrip $m$. To calculate the field at $m + 1$, the pulse passes through the nonlinear fibre and is then split in the delay line. Defining $G_0$ as the total roundtrip (power) loss, we find

$$a_i(m + 1) = \frac{F[a_i(m)] + F[a_i(m + 1)]}{2 \sqrt{G_0}}$$

(Ferromagnetic interactions use a ‘+’ sign and antiferromagnetic interactions use a ‘−’ sign. To account for quantum noise, we work in a truncated Wigner picture, which is convenient and accurate when the threshold photon number is $\gg 1$. This procedure adds Gaussian noise terms to (10) to maintain the commutation relations $[a_i, a_j^*] = \delta_{ij}$ in the presence of cavity and coupling losses. Equation (10) was simulated numerically to obtain the curve shown in Fig. 3d. In the simulation, the pump amplitudes were turned on at $m = 0$ and kept at the same values until the end of the simulation at $m = 1000$. We performed the simulation for various pump amplitudes, and the defect density $n_d$ was derived from the final spin configuration for each pump amplitude.

The simulations show that the evolution is a two-stage process: in the ‘growth’ stage, the field is weak and pump depletion may be ignored, giving rise to exponential growth in the fields, $a_i$. This lasts for time $T_i$ and is followed by a ‘crystallization stage’, where the field saturates to one of two values, $a \rightarrow \pm a_{sat}$, and domain walls form. Over time, the domain walls attract each other and this causes smaller domains to evaporate.

With the reasonable approximation stated above, the dynamics in the growth state can be analytically solved as follows. At threshold, the fibre gain must be $G_0$ to compensate for loss. Above the threshold for $a = b = c$, the input–output relation can be linearized to give $F[a] = G_0 (a + a_{sat})$. Applying equation (10), the net effect of a single roundtrip is

$$a_i(t + 1) = G_0 (a_{sat} - 1) a_i(t) + a_{sat}(t) / 2$$

Linear map (11) is diagonalized by going to the Fourier domain. We then integrate the equation up to time $T_i = (p - 1) (\log (N_{sat}) / \log (G_0))$, namely the time it takes to reach saturation. $T_i$ depends only logarithmically on the saturation photon number $N_{sat}$, which is $O(10^2 - 10^3)$. The field amplitude at saturation is given by

$$\tilde{a}_i(T_i) \sim \sqrt{N_{sat}} e^{-(T_i / 10)(\ln N_{sat})}$$

This has a Gaussian power spectral density, which in turn gives a Gaussian autocorrelation function, $R(s) = \langle a_i a_{i+s} \rangle / \langle a_i \rangle = e^{-s^2 / 2}$, with the autocorrelation length given by $s_0 = \sqrt{10T_i} / 2$.

We analysed the crystallization stage with numerical simulations, and found that $a_i$ saturates to $\pm a_{sat}$ and, as a result, $a_i$ changes its form. We also found that the autocorrelation curves obtained in the simulations changed from the Gaussian to a function that was approximated by $\exp (-x^2 / 2)$ with $x_0 \approx x_0$, and thus the auto-correlation of the DOPOs may be reasonably approximated as the spin correlation of the 1D Ising spins.

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