Cyclic structures for simplicial objects from comonads

preliminary notes

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The simplicial endofunctor induced by a comonad in some category may underly a cyclic object in its category of endofunctors. The cyclic symmetry is then given by a sequence of natural transformations. We write down the commutation relations the first cyclic operator has to satisfy with the data of the comonad. If we add a version of quantum Yang Baxter relation and another relation we actually get a sufficient condition for constructing a sequence of higher cyclic operators in a canonical fashion. A degenerate case of this construction comes from so-called trivial symmetry of an additive comonad.

We also consider weaker versions for paracyclic objects as well as some connections to the subject of distributive laws.

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Comonadic homology and monadic ('triple') cohomology are among the standard unifying frameworks in homological algebra. Despite great effort, I could not find any references giving instances of cyclic (co)homology derived completely in that same framework (that is, including the cyclic operator as well). There is maybe one exception: operads are often treated in monadic language, and one enriched version, called cyclic operads, is designed in part as a gadget to do the cyclic homology ([10, 16]).

We perform here a pretty straightforward general nonsense exercise to provide a (nontautological) additional structure \( t \) on an arbitrary comonad \( G \) in \( A \) generating the cyclic operators \( t_n \) in the sense of Connes ([8, 13, 22]) on the simplicial objects associated to \( G \). We also relate our data to some other functorial data, e.g. a class of distributive laws. Our procedure is just one of at least several conceivable ways one could utilize (co)monads to obtain cyclic-type constructions. This kind of production of cyclic objects is functorial. Although this was intended (I won’t dwell on motivating picture as it is largely still conjectural), at the present moment this is a drawback: most known kinds of cyclic cohomology do not have coefficients, hence should not fit into our framework. I hope that the proposed framework is transparent enough to enable us to spot new classes of examples, remedying the problem.

Prerequisites. Given categories \( A, B, C \), functors \( f_1, f_2, g_1, g_2 \) and natu-
ral transformations $F, G$ as in the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{f_1} & B \\
\uparrow F & & \uparrow G \\
\downarrow f_2 & & \downarrow g_2 \\
C,
\end{array}
$$

define the natural transformation $G \star F : g_2 \circ f_2 \Rightarrow g_1 \circ f_1$ by

$$(G \star F)_A := G_{f_1(A)} \circ g_2(F_A) = g_1(F_A) \circ G_{f_2(A)} : g_2(f_2(A)) \rightarrow g_1(f_1(A)).$$

$(F, G) \mapsto F \star G$ is called the Godement product (‘horizontal composition’).

It is associative for triples for which $F \star (G \star H)$ is defined.

A (co)monad in category $A$ is a (co)monoid in the monoidal category $\text{End} A$ of endofunctors in $A$ (II 15 22). The monoidal product of endofunctors is their composition and of natural transformations the Godement product. Equivalently, a comonad is a triple $G$ (co)monad consisting of an endofunctor $G : A \rightarrow A$ and the ‘counit’ $\epsilon : G \Rightarrow 1_A$ are natural transformations of functors, such that for every object $M$ in $A$ the coassociativity axiom $\delta_G \circ \delta_M = G(\delta_M) \circ \delta_M$ and the counit axiom $\epsilon_G \circ \delta_M = G(\epsilon_M) \circ \delta_M = 1_{GM}$ hold.

Let $\Delta$ be the ‘cosimplicial’ category: its objects are nonnegative integers viewed as finite ordered sets $n := \{0 < 1 \ldots < n\}$ and its morphisms are nondecreasing monotone functions. Given a category $A$, denote by $\text{Sim} A$ the category of simplicial objects in $A$, i.e. functors $F : \Delta^{\text{op}} \rightarrow A$. Represent $F$ in $\text{Sim} A$ as a sequence $F_n := F(n)$ of objects, together with the face maps $\partial^n_i : A_n \rightarrow A_{n-1}$ and degeneracy maps $\sigma^n_i : F_n \rightarrow F_{n+1}$ for $i \in n$ satisfying the familiar simplicial identities (15 22). Notation $F_\bullet$ for these data is standard.

To any comonad $G$ in $A$ one associates the sequence $G_\bullet$ of endofunctors $n \mapsto G_n := G^{n+1} := G \circ G \circ \ldots \circ G$, together with natural transformations $\partial^n_i : G^i \circ G^{n-i} : G^{n+1} \rightarrow G^n$ and $\sigma^n_i : G^i \circ \delta G^{n-i} : G^{n+1} \rightarrow G^{n+2}$, satisfying the simplicial identities. Hence any comonad $G$ canonically induces a simplicial endofunctor, i.e. a functor $G_\bullet : \Delta^{\text{op}} \rightarrow \text{End} A$, or equivalently, a functor $G_\bullet : A \rightarrow \text{Sim} A$. The counit $\epsilon$ of the comonad $G$ satisfies $\epsilon \circ \partial^0_1 = \epsilon \circ \partial^1_1$, what means that $\epsilon : G_\bullet \rightarrow \text{Id}_A$ is in fact an augmented simplicial endofunctor.

A $\mathbb{Z}$-cyclic (synonym: paracyclic) object in $A$ is a simplicial object $F_\bullet$ together with a sequence of isomorphisms $t_n : F_n \rightarrow F_n$, $n \geq 1$, such that

$$\begin{align*}
\partial_i t_n &= t_{n-1} \partial_{i-1}, & i > 0, \\
\sigma_i t_n &= t_{n+1} \sigma_{i-1}, & i > 0, \\
\partial_0 t_n &= \partial_n, \\
\sigma_0 t_n &= t_{n+1} \sigma_n.
\end{align*}
$$

(1)
A $\mathbb{Z}$-cocyclic (paracocyclic) object in $\mathcal{A}$ is a $\mathbb{Z}$-cyclic object in $\mathcal{A}^{\text{op}}$. A (co)cyclic object is (co)cyclic if, in addition, $t_n^{n+1} = 1$ ([8, 13, 22]). Equivalently, the category $\Delta^{\text{op}}$ may be upgraded to the cyclic category $\mathcal{C}$ of Connes ([8, 13]). It is the universal category containing $\Delta^{\text{op}}$ as a nonfull subcategory, identical on objects, and having minimal set of additional morphisms containing a sequence of “cyclic” morphisms $\tau_n : n \to n$ such that any simplicial object $F_\bullet$ in any category $\mathcal{A}$ is a cyclic if the operators $t_n := F(\tau_n)$ are declared cyclic.

**Bottom relations.** Let now $\mathbf{G} = (G, \delta, \epsilon)$ be a comonad on a category $\mathcal{A}$ and $t : GG \Rightarrow GG$ a natural transformation.

For every object $M$ in $\mathcal{A}$ we require

\[
\begin{align*}
G(\epsilon_M)t_M &= \epsilon_{GM} : G^2M \to GM, \\
\epsilon_{GM}t_M &= G(\epsilon_M) : G^2M \to GM, \\
G(\delta_M)t_M &= t_{GMG}(t_M)\delta_{GM} : G^2M \to G^3M, \\
\delta_{GM}t_M &= t_{GMG}(t_M)t_{GM}(t_M)G(\delta_M) : G^2M \to G^3M,
\end{align*}
\]

or, in a more schematic form of natural transformations,

\[
\begin{align*}
G(\epsilon)t &= \epsilon_G \\
\epsilon_Gt &= G(\epsilon) \\
G(\delta)t &= t_{GMG}(t_M)\delta_G \\
\delta_Gt &= [t_{GMG}(t_M)]^2G(\delta)
\end{align*}
\]

Notice that if $t_M \circ t_M = \text{Id}_{GGM}$ then the first two identities in (2) are equivalent (composing one by $t_M$ from the right). These relations are the bottom-part of the relations required for the cyclic symmetry. Now we show that with few additional properties they are sufficient, as well.

**Theorem 1.** Let $\mathbf{G} = (G, \delta, \epsilon)$ be a comonad on a category $\mathcal{A}$, where $\epsilon$ is the counit and $\delta$ the coproduct. Let $\mathbf{G}_\bullet$ be the associated simplicial endofunctor. Suppose an invertible natural transformation $t : GG \Rightarrow GG$ satisfies (3), and the quantum Yang Baxter equation (QYBE)

\[
G(t_M) \circ t_{GM} \circ G(t_M) = t_{GM} \circ G(t_M) \circ t_{GM}, \quad \forall M \in \text{Ob} \mathcal{A}.
\]

If we also assume the relation

\[
t_M \circ t_M \circ t_M = t_M \circ \delta_M, \quad \forall M \in \text{Ob} \mathcal{A},
\]

then setting

\[
t_nM := t_{G^{n-1}M} \circ G(t_{G^{n-2}M}) \circ \ldots \circ G^{n-1}(t_M),
\]

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defines paracyclic operators $t_n$ on the augmented simplicial endofunctor $G_\bullet \Rightarrow Id_A$ making it into an augmented paracyclic object in $\text{End} \ A$.

**Remark.** 1. We do not claim that every paracyclic or even every cyclic operator on $G_\bullet$ is of that kind.

2. Practical (weaker) form of the conclusion: for any fixed object $M$ in $A$, the pair $(G_\bullet M \xrightarrow{\epsilon^M} M, t_M)$ is an augmented paracyclic object in $A$.

**Observation.**

$$t_{n+1}M = t_nGM \circ G^n(t_M). \quad (7)$$

**Proof of the theorem.** Substituting $\partial_i = G^i(\epsilon_{G^{n-i}})$, $\sigma_i = G^i(\delta_{G^{n-i}})$ and $[\square]$ in $(\square)$ we obtain

- $(A_{n,i})$ for $i = 1, \ldots, n$,
  $$G^i(\epsilon_{G^{n-i}M})t_{G^{n-1}M} \cdots G^{n-1}(t_M) = t_{G^{n-2}M} \cdots G^{n-2}(t_M)G^{i-1}(\epsilon_{G^{n-i+1}M}),$$

- $(B_{n,i})$ for $i = 1, \ldots, n$,
  $$G^i(\delta_{G^{n-i}M})t_{G^{n-1}M} \cdots G^{n-1}(t_M) = t_{G^nM} \cdots G^n(t_M)G^{i-1}(\delta_{G^{n-i+1}M}),$$

- $(C_n)$ $\epsilon_{G^n M} t_{G^nM} \cdots G^{n-1}(t_M) = G^n(\epsilon_M)$,

- $(D_n)$ $\delta_{G^n M} t_{G^nM} \cdots G^{n-1}(t_M) = [t_{G^nM} \cdots G^n(t_M)]^2 G^n(\delta_M)$.

Of course, the factors involving $G^{n-2}$ in our notation appear only if $n > 1$. Basis of induction: $(A_{1,1})$ is the first and $(C_1)$ the second formula in $(\square)$ while $(B_1)$ is the third and $(D_1)$ the bottom formula there.

The rest will follow by inductive calculations (as usual RHS = right-hand side etc.). Cases A, B, C are very simple:

a) $(A_{n,i}) \Rightarrow (A_{n+1,i+1})$ Act by $G$ on both sides of equation $(A_{n,i})$, and compose both sides from the left by $t_{G^{n-1}M}$. Then use the naturality formula $t_{G^{n-1}M}G^{i+1}(\epsilon_{G^{n-i}M}) = G^{i+1}(\epsilon_{G^{n-i}M})t_{G^nM}$ on the left-hand side.

b) $(A_{n,i}) \Rightarrow (A_{n+1,i})$ Write down $(A_{n,i})$ on $GM$ instead of $M$:

$$G^i(\epsilon_{G^{n+1-i}M})t_{G^nM} \cdots G^{n-1}(t_{GM}) = t_{G^{n-1}M} \cdots G^{n-2}(t_{GM})G^{n-1}(\epsilon_{G^{n-i+2}M}),$$

and compose from right both sides by $G^n(t_M)$. At RHS use the naturality formula $G^{i-1}(\epsilon_{G^{n-i+2}M})G^n(t_M) = G^{n-1}(t_M)G^{i-1}(\epsilon_{G^{n-i+2}M})$, which holds for $n - i > 0$.

c) $(B_{n,i}) \Rightarrow (B_{n+1,i+1})$ – analogously to a).
d) \((B_{n,i}) \Rightarrow (B_{n+1,i})\) – Write down \((B_{n,i})\) on \(GM\) instead of \(M\) and compose from right both sides by \(G^n(t_M)\). At RHS use the naturality formula \(G^{n-1}(\delta_{G^{n-i+2}M})G^n(t_M) = G^{n+1}(t_M)G^{n-1}(\delta_{G^{n-i+2}M})\) for \(n - i > 0\).

e) Assume \((C_n)\). Then

\[
\begin{array}{c}
G^nG \xrightarrow{t_{n+1}} G^nG \xrightarrow{G^n(\epsilon)} G^nG \xrightarrow{G^n(\epsilon_G)} G^{n+1}G
\end{array}
\]

is commutative: the left triangle by \(G(\epsilon) = \epsilon_G(t)\) and the functoriality of \(G^n\), and the right triangle is commutative by the inductive hypothesis (composed by \(G\)). The external triangle is then \((C_{n+1})\).

f) Finally, the case \((D_n) \Rightarrow (D_{n+1})\) is somewhat more elaborate, and this is the point where we also need QYBE. For this, we first notice that \((D_1)\), together with QYBE, implies the following calculation:

\[
G^n(\delta_{GM})G^n(t_M) = G^n(\delta_{GM}t_M)
\]

\[
= G^n(t_{GM})G^n(t_{GM}G(t_M)G(\delta_M))
\]

\[
= G^{n+1}(t_M)G^{n+1}(t_{GM})G^{n+1}(t_{GM}G^{n+1}(t_M)G^{n+1}(t_M)G^{n+1}(\delta_M))
\]

\[
= G^{n+1}(t_M)G^n(t_{GM})G^{n+1}(t_{GM}G^{n+1}(t_M)G^{n+1}(\delta_M)).
\]

Thus,

\[
G^n(t_{GM})G^n(\delta_{GM})G^n(t_M) = G^n(t_{GM})G^{n+1}(t_{GM})G^{n+1}(t_{GM}G^{n+1}(t_{GM}G^{n+1}(\delta_M))
\]

\[
= (\text{use QYBE}) = G^{n+1}(t_M)G^n(t_{GM})G^{n+1}(t_{GM}G^{n+1}(\delta_M))
\]

\[
= (\text{use (8)}) = G^{n+1}(t_M)G^n(t_{GM})G^{n+1}(t_{GM}G^{n+1}(\delta_M))
\]

We write down the \((D_n)\) for \(GM\) instead of \(M\), and then compose both sides by \(G^n(t_M)\) from the right:

\[
\delta_{G^{n+1}M}t_{nGM}G^n(t_M) = [t_{n+1GM}]^2G^n(\delta_{GM})G^n(t_M),
\]

\[
\delta_{G^{n+1}M}t_{n+1M} = t_{n+1GM}G^n(t_{GM})G^n(\delta_{GM})G^n(t_M).
\]

Now we substitute the identity from (8) to get

\[
\delta_{G^{n+1}M}t_{n+1M} = t_{n+1GM}G^n(t_{GM})G^{n+1}(t_M)G^n(t_{GM})G^{n+1}(t_M)G^{n+1}(\delta_M).
\]
and then we notice that naturality and the definition of \( t_n \) imply that \( t_nG_M \) commutes with \( G^{n+1}(t_M) \). Hence

\[
\delta_{G^{n+1}M}t_nM = t_{n+1}GMG^{n+1}(t_M)t_nG_MG^n(t_{GM})G^{n+1}(t_M)G^{n+1}(\delta_M) \\
= [t_{n+2}M]^2G^{n+1}(\delta_M).
\]

**Theorem 2.** Assume in addition that \( t^2 = 1 \). Then \( t_n^{n+1} = 1 \) as well, hence \( G_\bullet \xrightarrow{\delta} \text{Id}_A \) is an augmented cyclic object in \( \text{End} \, A \).

**Remark.** More generally, under the conditions of Theorem 1, given a fixed object \( M \) in \( A \), for the augmented paracyclic object \((G_\bullet M \xrightarrow{\epsilon_M} M, t_\star M)\) to be a cyclic object in \( A \) it is sufficient that \( t^2_{G^kM} = \text{Id}_M \) for all \( k \geq 0 \).

**Proof of Theorem 2.** Theorem 1 being proved it remains to verify \( t_n^{n+1} = 1 \). However this is standard. Namely, QYBE and the naturality of \( t_n \) imply that transformations \( \alpha^n_i = G^n(t_{G^{n-i-1}}) : G^n \to G^n \) for \( 0 \leq i \leq n \) are the standard generators (braids) of the braid group \( B_{n+1} \) on \( n+1 \) letters in a representation into by natural autoequivalences of functor \( G^n \)

\[
B_n \to \text{Aut}(G^n).
\]

If \( t^2 = 1 \) then \( (\alpha^n_i)^2 = [G^n(t_{G^{n-i-1}})]^2 = 1 \) as well. It is then a standard result, that this representation factors through the symmetric group on \( n+1 \) letters. In this representation, \( t_n \) by its definition equals to \( \alpha^n_0\alpha^n_1 \ldots \alpha^n_n \), what is easily recognized as the image of a standard cycle in the symmetric group of order \( n+1 \). Alternatively, one could use inductive calculations to show \( (\alpha^n_k\alpha^n_1 \ldots \alpha^n_n)^{n+1} = 1 \) whenever \( k \geq n \). A convenient intermediate step is to show \( (\alpha^n_k\alpha^n_1 \ldots \alpha^n_{n-1}\alpha^n_k)^{n-1} = (\alpha^n_k\alpha^n_1 \ldots \alpha^n_{n-1})^{n-1}\alpha^n_k \ldots \alpha^n_1 \).

**Definition.** Given a comonad \( G \) in \( A \), a natural transformation \( t : GG \to GG \) is

- (\(1\)) a symmetry of \( G \) if it satisfies the QYBE (4) and also

\[
t^2 = 1_{GG}, \quad t\delta = \delta, \quad \epsilon_Gt = G(\epsilon), \quad \delta_Gt = G(t)\epsilon_GG(\delta) \quad (9)
\]

In that case, the pair \((G, t)\) is called a symmetric comonad.

- a strong braiding of \( G \) if it satisfies the QYBE (4) and also

\[
G(\epsilon)t = \epsilon_G, \quad \epsilon_Gt = G(\epsilon), \quad tGG(t)\delta_G = G(\delta)t, \quad \delta_Gt = G(t)tGG(\delta) \quad (10)
\]
Lemma. Every symmetry of a comonad is a strong braiding.

Proof. It is immediate that (9) imply \( t_G G(t) \delta_G = G(\delta) t \) and \( G(\epsilon) t = \epsilon_G \). Relation \( t^2 = 1_G G \) and QYBE imply together \( G(t) t_G = [t_G G(t)]^2 \).

Observation. Every symmetric comonad \((G, t)\) satisfies the conditions of Theorem 2, and hence it gives a rule for producing certain cyclic objects in \( \mathcal{A} \). Conversely, every \( t \) satisfying the conditions of the Theorem 2, and satisfying \( t^2 = 1 \) is a symmetry.

Remark. A strong braiding on a comonad does not imply the conditions of Theorem 1. Namely, \([t_G M G(M)]^2 G(\delta_M) \neq G(\delta_M) t_G M G(M)\) in general.

Remark. Symmetric comonads are related to symmetric simplicial sets (\[11\]).

Proposition 1. (cf. \[17\]) If \( \mathcal{A} \) is an additive category, and comonad \( G \) respects its additive structure, then \( \tau = \tau^G : G G \to G G \) defined by \( \tau_M := \delta_M \epsilon_G M + \delta_M G(\epsilon_M) - 1_G G M \) is a symmetry of \( G \).

\( \tau^G \) is called the trivial symmetry or simply the symmetry of \( G \).

Proof. \( t \delta = \delta \) and \( \epsilon_G t = G(\epsilon) \) are immediate by the counit axioms. Other relations are left to the reader – they follow by calculations involving many summands (particularly the QYBE). Use the naturality of \( \delta \) and \( \epsilon \) and axioms for comonad when collecting and comparing the summands. Q.E.D.

Now we compute \( t_n = \tau_{G^n-1} \circ G(t_{n-1}) \) where \( \tau \) is the trivial symmetry. As this is a combinatorial problem, we will introduce some helpful notation.

Define a small strict monoidal category \( \mathcal{P} \) as follows. Objects of \( \mathcal{P} \) are nonnegative integers (written with square brackets \([n]\)). Morphisms \( \mathcal{P}([m],[n]) \) are \( m \)-tuples \((k_1, \ldots, k_m)\) such that \( \sum_{i=1}^m k_i = n \). The tensor product on objects is the addition \(([m],[n]) \mapsto [m+n] \). The tensor product of morphisms is the concatenation. We denote it either by \( \ast \) or simply concatenate. If \( r = (r_1, \ldots, r_m), k = (k_1, \ldots, k_n) \) are tuples which are composable (i.e. \( \sum_{i=1}^n k_i = m \)) then we define \( r \circ k \) as follows. Represent all components \( r_i (k_j) \) by trees of height one with \( r_i (k_j) \) respectively leaves. Attach \( r_1 \) to the left most leaf in \( k \) (of course we jump over trees corresponding to zeros as they are leafless). Then attach \( r_2 \) to the next leaf to the right from that leaf (on the same or on the next tree) and so on. In other words, attach \( r_i \) to the \( i \)-th leaf from the left in \( k \) (this leaf is at the \((k_1 + \ldots + k_{i-1} + 1)\)-th tree from the left in the graphical presentation of \( k \)). Then in the resulting depth \( \leq 2 \) trees remove the internal nodes (in particular at the places where the zero is attached the whole leg disappears after that proces). The pictures with variants of such short trees are useful for extension of similar calculations to more complicated setups.
Equivalently, in non-graphical presentation, the composition can be described as follows. $r \circ k$ has as many nodes as $k$. The number at $i$-th place is the sum of $k_i$ components taken successively from $r$ starting at position $1 + \sum_{p=0}^{i-1} k_p$ and proceeding taking components to the right from there.

**Example.** $(2031041) \circ (021202) = (0, 2 + 0, 3, 1 + 0, 4 + 0) = (023105)$.

**Proposition 2.** $\mathcal{P}$ is a small strict monoidal category.

In particular, the composition and the tensor product are strictly associative, and we have the interchange law $(r \circ r') \star (s \circ s') = (r \star s) \circ (r' \star s')$ holds for morphisms. The proof is an exercise.

For any $i \geq 0$ introduce the natural transformation $\Delta_i : G \to G^i$ as follows: $\Delta_0 := \epsilon$, $\Delta_1 := \text{id}_G$, $\Delta_2 := \delta$, and inductively $\Delta_{k+1} := \delta_{G^{k-1}} \circ \Delta_k$ for $k > 2$. We define a strict monoidal functor $\Delta : \mathcal{P} \to \text{End} \mathcal{A}$ as follows. On objects $\Delta([n]) := G^n$. On morphisms, $\Delta(k) = \Delta_k$ for $k \in \mathcal{P}([1],[k])$ (in fact $(k)$). As any morphism $s$ in $\mathcal{P}([m],[n])$ is an $m$-tuple, hence a tensor product (concatenation) of $m$ $1$-tuples in $\mathcal{P}([1],[s_i])$, there is at most one extension of these formulas respecting monoidal structures, namely $\Delta(k) = \Delta_{k} := \Delta_{k_1} \times \cdots \times \Delta_{k_n}$ where $k = (k_1, \ldots, k_n)$. The reader may verify that $\Delta$ is in fact a functor i.e. $\Delta(s \circ s') = \Delta(s) \circ \Delta(s')$ where $\circ$ on the left-hand side is the composition of endofunctors. A crucial part of that statement is to use the coassociativity of $\delta$ and that the property that $\epsilon$ is a counit translates to the fact $01 \circ 2 = 10 \circ 2 = 1$. One also has $k' \circ k = k + k' - 1$ for $k > 0$, and in particular $2 \circ k = k + 1$. We conclude:

**Proposition 3.** $\Delta : \mathcal{P} \to \text{End} \mathcal{A}$ is a monoidal functor.

Notice that set $\coprod \mathcal{P}([m],[n]) = \coprod_{n \geq 0} \mathbb{Z}_{\geq 0}$ is an ordered monoid with respect to the concatenation and lexicographic ordering. $\Delta$ also encodes the map, also denoted by $\Delta : \coprod_{n \geq 0} \mathbb{Z}_{\geq 0} \to \text{End} \mathcal{A}$ which evaluate on $n$-tuples the same as the functor $\Delta$ evaluates on the corresponding morphisms in $\mathcal{P}$.

Most important part in the following are the endomorphism sets $S_n := \mathcal{P}([n],[n])$ which are monoids with respect to the composition. This are sets of $n$-tuples of nonnegative integers which moreover add up to $n$. The restriction $\Delta = \Delta| : \coprod_n S_n \to \cup_n \text{Nat}(G^n, G^n)$ is a map of monoids.

The composition in $\mathcal{P}$ restricted to $S_n \times S_n$ takes values in $S_n$. Hence it is an associative binary operation and it extends to a bilinear operation on $\mathbb{Z} S_n$.

Since $\mathcal{A}$ is an abelian, hence additive category, $\text{Nat}(G^n, G^n)$ is an abelian group in particular. Therefore $\Delta|_{S_n}$ extends to a unique homomorphism from the free abelian group $\mathbb{Z} S_n$ with basis $S_n$ to $\text{Nat}(G^n, G^n)$, also denoted by $\Delta$. 8
Let $\text{NA}l_{n}$ be the subset of $\mathbb{Z}S_n$ consisting of all $x$ of the alternating sign form $x = s_1 - s_2 + s_3 - \ldots \pm s_k$ where $k \geq 1$, all $s_i \in S_n$ and $s_1 < s_2 < \ldots < s_k$ with respect to the lexicographic ordering. For example, writing 0004 for $(0, 0, 0, 4)$ etc. the element 004004040 is in $\text{NA}l_{n}$.

We also form $S^*_n \subset S_n$ inductively, along with a linear map $\alpha : \mathbb{Z}S^*_n \to \mathbb{Z}\tilde{S}_n$ where $\tilde{S}$ is the set of all sequences of $n-1$ characters from the alphabet $\{a, b, c\}$. We set $S^*_2 := \{02, 11, 20\}$, $\alpha(02) := a$, $\alpha(11) := b$, $\alpha(20) := c$. If $S^*_{n-1}$ is defined, then

1. $(0, s_2, \ldots, s_n) \in S^*_n$ iff $(s_2 > 1$ and $(s_2 - 1, \ldots, s_n) \in S^*_n-1$).

In that case, $\alpha(0, s_2, \ldots, s_n) = a\alpha(s_2, \ldots, s_n)$.

2. $(1, s_2, \ldots, s_n) \in S^*_n$ iff $(s_2, \ldots, s_n) \in S^*_n-1$.

In that case, $\alpha(1, s_2, \ldots, s_n) = b\alpha(s_2, \ldots, s_n)$.

3. $(2, s_2, \ldots, s_n) \in S^*_n$ iff $(s_2 = s_3 = \ldots = s_{n-1} = 1$ and $s_n = 0$).

In that case, $\alpha(2, s_2, \ldots, s_n) = cc \ldots c$ $(n-1$ times).

Finally, $\text{NA}l^*_{n} := \text{NA}l_{n} \cap \mathbb{Z}S^*_n$.

A $k$-tuple of sequences $s_1, \ldots, s_k \in S_n$ form a twin $k$-tuple if they become identical after subtracting 1 from the first nonzero member of each sequence. For example, 0030 and 1020 form a twin pair, because they both become 0020 after performing this operation.

**Definition.** $x \in \mathbb{Z}S^*_n$ represents $u \in \text{Nat}(G^n, G^n)$ if $\Delta(x) = u$. Then $x$ is called below the representation and $\alpha(x)$ the abc-representation of $u$.

**Theorem 3.** 1. $S^*_n$ has exactly $2^{n-1} + 1$ elements.

2. A sequence in $\tilde{S}$ of $n-1$ characters belongs to the image of $\alpha$ iff characters $a$ and $b$ never appear after $c$.

3. Let $m = m_1 - m_2 + \ldots - m_{2^{n-1} - 1} \in \text{NA}l^*_{n}$, $m_1 < \ldots < m_{2^{n-1} - 1}$ be the unique element of $\text{NA}l^*_{n}$ of maximal length $2^{n-1} + 1$. Then $\Delta(m) = t_{n-1}$.

4. If $m_k$ is the $k$-th summand in $m$, beginning in 0 then $m_{k+2^{n-2}}$ is its twin which comes with an opposite sign in $m$.

5. The sign of $m_k$ is positive iff character $b$ appears even number of times in $\alpha(m_k)$.

**Example.** $t_4 = \Delta(00005 - 00041 + 00050 - 00302 + 00311 - 00320 + 00410 - 02003 + 02021 - 02030 + 02102 - 02111 + 02120 - 02210 + 03110 - 10004 + 10031 - 10040 + 10202 - 10310 + 11003 - 11021 + 11030 - 11102 + 11111 - 11120 + 11210 - 12110 + 21110)$.

**Proof of the theorem.** (sketch) We prove all assertions simultaneously by induction. The case $n = 2$: $t_1 = \Delta(02 - 11 + 20) = \epsilon \delta - \text{id} + \delta \epsilon$. Together with the definition of $\Delta$ this also implies that $t_{G^{n-1}} = \Delta(02(1^{n-1}) -$
11(1^{n-1}) + 20(1^{n-1}), where 1^{n-1} = 11\ldots1 \ (n - 1 \text{ times}).

Suppose all 5 assertions hold for \( n - 1 \). Then \( G(t_{n-1}) = \sum_{i=1}^{2^n-2+1} \Delta(1 \star m_i) \), i.e. at the level of \( \mathbb{Z}S_n^\tau \) we attach 1 from the left to each summand in \( m \) (the element representing \( t_{n-1} \)). For example, \( G(t_1) = \Delta(102 - 111 + 120) \). The reader should play enough to get comfortable calculating with the composition in \( \mathcal{P} \). In any case, composing 02(1^{n-1}) from the left, decrements the leftmost nonzero number in a sequence and increments the first next nonzero number to the right in it; and 20(1^{n-1}) does other way around.

As the leftmost number in the representation of \( G(t_{n-1}) \) is 1, it becomes 0 after that composition. Hence composing by 02(1^{n-1}) results in twins of the original summands representing \( G(t_{n-1}) \), with opposite sign. For 20(1^{n-1}), the increments and decrements are switched and we end with a sequence starting with 2 and the rest is truncated in a way to erase the difference between the twins; so all such terms, pairwise cancel except for the twinless 211\ldots10. As a result we end without any other sequences headed by 2; and all new sequences not headed by 2 are paired in twins. Verifying the rules for signs and remaining requirements is straightforward.

Let \( k \) be a commutative ring, \( p, r \in k \), \( \mathcal{A} \) a \( k \)-linear category, and \( G \) a \( k \)-linear comonad in \( \mathcal{A} \). Following \[5\] we define \( \theta := p\epsilon \delta - pid + r\delta \epsilon \). Then \( \theta \) generalizes the trivial symmetry (just set \( p = r = 1 \)), it still satisfies the QYBE, but it is not a symmetry of \( G \), and it is not even a strong braiding (as the reader should check all 4 equations in \[10\] fail in general). As it satisfies the QYBE, the braid cycle \( t_n \) may still be of interest to compute. The basic combinatorics from the \( \tau \)-case actually passes through!

If \( t_1 = \theta \) then \( t_\theta = t_n \) is described as follows. Suppose \( s_i \in S_n^\tau \) corresponds to the sequence \( a_1a_2\ldots a_kc^{n-k} \) where \( a_i \in \{a, b\} \) and \( k = k(i) \) are determined by \( s_i \) as explained above. Define a map \( \text{norm}_\theta : \mathbb{Z}S_n^\tau \rightarrow kS_n^\theta \) by setting \( s_i \mapsto p^{k(i)}r^{n-k(i)} \) and extending additively. Let \( S_n^\theta := \text{norm}_\theta(S_n^\tau) \), \( \text{NAlt}_n^\theta := \text{norm}_\theta(\text{NAlt}_n^\tau) \) and \( m^\theta = \theta(m) \) where \( m \in \text{NAlt}_n^\tau \) is the longest element as above. \( \Delta \) extends \( k \)-linearly to a map \( \Delta : kS_n \rightarrow \text{Nat}(G^n, G^n) \) and this map restricts down to \( kS_n^\theta \) and \( \text{NAlt}_n^\theta \).

Proposition 4. \( t_\theta^n = \Delta(m^\theta) \).

Proof. The proof of the Theorem 1, passes through, but one has to keep track of coefficients. It is easy to see that the coefficients of each summand are easily tracked in terms of the \( abc \)-representation. The key observation is that those twin pairs which cancel in the inductive step of Theorem 1 actually do come with exactly the same coefficients (but different signs, as
before) so they again cancel after applying $20(1)^{n-1}$ in the next step.

**Definition.** ([2,3]) A **distributive law** from a comonad $G = (G, \delta^G, \epsilon^G)$ to a comonad $F = (F, \delta^F, \epsilon^F)$ is a natural transformation $l : F \circ G \to G \circ F$ such that

\[
G(\epsilon^F) \circ l = (\epsilon^F)_G, \quad G(\delta^F) \circ l = l_F \circ F(l) \circ (\delta^F)_G, \\
(\epsilon^G)_F \circ l = F(\epsilon^G), \quad (\delta^G)_F \circ l = G(l) \circ l_G \circ F(\delta^G).
\]

Any distributive law $l$ induces a **composite comonad**

\[
F \circ_l G = (F \circ G, \delta^{l,F}, \epsilon^{l,F}),
\]

where the coproduct $\delta^{l,F}$ equals the composition

\[
FG \xrightarrow{\delta^G} FGG \xrightarrow{(\delta^F)_G} FFGG \xrightarrow{F(l)} FGFG,
\]

and the counit $\epsilon^{l,F}$ equals the composition $FG \xrightarrow{\epsilon^G} F \xrightarrow{F} 1$.

**Theorem 4.** Suppose $t_1 = t$ is a strong braiding on $G$. Define inductively $t_{n+1} = t_{n}G \circ t_{G}^{-1}(t_{M})$. Then $t_n : G^n \circ G \to G \circ G^n$ form a sequence of distributive laws from $G$ to $G^n = (G^n, \delta^{(n-1)}, \epsilon^{(n-1)}), \delta^{(n)}$ is inductively defined by

\[
\delta^{(n)} = G^{n-1}(t_{n-1,G})\delta^{(n-1)}_G G^{n-1} \delta = G^{n-1}(t_{n-1,G})G^{2n-2} \delta \delta^{(n-1)}, \quad (11)
\]

and $\delta^{(1)} = \delta$. The counit is $\epsilon^{(n)} = G^{(n-1)} \circ G^{n-1} \epsilon = \epsilon \circ G(\epsilon) \circ \cdots \circ G^{n-1}(\epsilon)$.

In formulas, the following holds for $n \geq 1$:

\[
G(\delta^{(n)})t_n = t_{n,Gn}G^n(t_n)\delta^{(n)}_G \quad (12) \\
\delta_{G^n}t_n = G(t_{n})t_{n,G}G^n(\delta) \quad (13) \\
(\epsilon^{(n)})_G = G(\epsilon^{(n)})t_n \quad (14) \\
G^n(\epsilon) = \epsilon_{G^n}t_n \quad (15)
\]

Actually, ([4,5]) is a case $l = n$ of a more general identity:

\[
G^{n-l+1}(\delta^{(l)})t_n = t_{n+l,G^n-l}(\delta^{(l)}_G), \quad 1 \leq l \leq n \quad (16)
\]

The converse does not hold: a family of distributive laws as above does not need to be coming from the first plus QYBE.
Lemma. (a) Let $t(1) = t : GG \Rightarrow GG$ be any natural transformation and let $t(n)$ be inductively defined by $t(n+1) := (t(n))G^n(t)$, $n \geq 1$, then

$$ t^{(n+i)} = t^{(n)}G^n(t^{(i)}) \text{ for all } 1 \leq i, 1 \leq n. \quad (17) $$

(b) Under the assumptions from (a) and the QYBE \cite{4},

$$ G(t_G^{(n-1)})t^{(n+1)} = t^{(n+1)}t_G^{(n-1)} \text{ holds for } n \geq 2. \quad (18) $$

More generally,

$$ G^p(t_G^{(l-1)})t^{(p+l)} = t^{(p+l)}G^{p-1}(t_G^{(l-1)}) \text{ holds for } l \geq 2, p \geq 1. \quad (19) $$

(c) If in addition $[t^{(1)}]^2 = \text{id}$, then $[t^{(n+1)}]^2 = G(t^{(n)})t^{(n)}_G$ holds for $n \geq 1$.

Proof. (a) For $i = 1$ this is the definition of $t_{n+1}$. Suppose the lemma holds for some $i$. Then $t_{n+i+1} = t_{n+i}G^{n+i}(t) = t_{n+i}G^n(t)G^{n+i}(t) = t_{n,G^{n+1}}G^n(t,G^{i}(t)) = t_{n,G^{n+1}}G^n(t_{i+1})$.

(b) These are (equivalent to) simple identities in the braid group. For completeness, we give a direct proof.

Base of induction: for $n = 2$ we have the identity

$$ G(t_G)t^{(3)} = t^{(3)}t_G^2, \quad (20) $$

which follows by applying the QYBE: $G(t_G)t^{(3)} = G(t_G)t_G^2G(t_G)G^2(t) = t_G^2G(t_G)t_G^2G^2(t) = t_G^2G(t_G)G^2(t)t_G^2 = t^{(3)}t_G^2$.

$\begin{align*}
G(t_G^{(n-1)})t^{(n+1)} & = G(t_G^{(n-2)})G^{n-1}(t_G^2)t_G^{(n-2)}G^{n-2}(t_G^2)G^{n-1}(t_G^2)G^n(t) \\
& = G(t_G^{(n-2)})G^{n-1}(t_G^2)G^n(t) \quad \text{QYBE} \\
& = G(t_G^{(n-1)})G^{n-2}(t_G^2)G^{n-1}(t_G^2)G^n(t) \\
& = G(t_G^{(n-1)})G^{n-2}(t_G^2)G^{n-1}(t_G^2)G^n(t) \\
& = G(t_G^{(n-2)})G^n(t)G^{n-2}(t_G^2) \\
& = \left[ G(t_G^{(n-2)})t^{(n)} \right]_G^n(t)G^{n-2}(t_G^2) \\
& = G(t_G^{(n-l+1)})t^{(n-l+1)}G^n(t)G^{n-2}(t_G^2), \ 1 \leq l \leq n-2, \\
& \text{(induction on $l$)} \\
& \text{(l = n-2)} \\
& = t^{(3)}_{G^{n-2}}G^3(t_G^{n-3}) \cdots G^n(t)G(t_G^{n-1}) \cdots G^{n-2}(t_G^2) \\
& = t^{(3)}_{G^{n-2}}G^3(t_G^{n-3}) \cdots G^n(t)G(t_G^{n-1}) \cdots G^{n-2}(t_G^2) \\
& = t^{(n+1)}t^{(n-1)}_G. \\
\end{align*}$
In calculations, we will write \( G^p(t^{(l-1)})t^{(p+l)} \).

\[ G^p(t^{(l-1)})t^{(p+l)} = G^p(t^{(l-1)})t^{(p-1)}G^{p-1}(t^{(l+1)}) \]
\[ = t_{G^{l+1}} G^p(t_G^{l-1})G^{p-1}(t^{(l+1)}) \]
\[ = t^{(p-1)} G^{p-1}(G(t^{(l-1)})t^{(l+1)}) \]
\[ = t^{(p-1)} G^{p-1}(t^{(l+1)})t^{(p-1)}(t^{(l+1)}) \]
\[ = t^{(p-1)} G^{p-1}(t^{(l-1)}) \]

(c) is a simple identity in the symmetric group \( \Sigma(n+2) \).

**Proof of Theorem 4.** To warm up, we start with \( n = 2 \) case of eq. (12). In calculations, we will write \( t^{(n)} \) for \( t_n \).

\[ \delta^{(2)} = G(t_G)\delta_{G^2}G(\delta) = G(t_G)G^2(\delta)\delta_G \]  
(21)

\[ G(\delta^{(2)})t^{(2)} \]
\[ = G^2(t_G)G^3(\delta)G(\delta_G)t_GG(t) \]
\[ = G^2(t_G)G^3(\delta)G(\delta_G)t_GG(t) \]
\[ = G^2(t_G)t_G^2G^3(\delta)G(t_G)G^2(t)\delta_{G^2} \]
\[ = t_G^3G^2(t_G)G(t_G)G^3(\delta)G^2(t)\delta_{G^2} \]
\[ = t_G^3G^2(t_G)G(t_G)G^2(t_G)G^2(t)\delta_{G_G} \]
\[ = t_G^3G^2(t_G)G(t_G)G^2(t_G)G^2(t)\delta_{G_G} \]
\[ = t_G^3G^2(t_G)G(t_G)G^2(t_G)G^2(t)\delta_{G_G} \]
\[ = t_G^3G^2(t_G)G(t_G)G^2(t_G)G^2(t)\delta_{G_G} \]
\[ = t_G^3G^2(t_G)G(t_G)G^2(t_G)G^2(t)\delta_{G_G} \]
\[ = t^{(4)}G(t_G)G(\delta_G)\delta_G \]
\[ = t^{(4)}\delta^{(2)}_G. \]

The equation (12) is just the special case of (16) when \( l = n \). Now we prove (16).
We write down the (13) pushed by the proof of Theorem 1, as well as the assumptions from Theorem 1 are also fulfilled. If \( t^2 = \text{id} \) then the lemma, part (c) holds and (13) is identical to \((D_n)\) in the proof of Theorem 1, as well as the assumptions from Theorem 1 are also fulfilled. In general, eq. (13) is different from \((D_n)\), but we will prove it by induction, following very similar path as for \((D_n)\).

\[
G^n(t_G)G^n(\delta_G)G^n(t) = G^n(t_G)G^n(\delta_G)G^n(t) = G^n(t_G)G^n[G(t)t_GG(\delta)] = G^n(t_G)G^{n+1}(t)G^n(t_G)G^{n+1}(\delta) \tag{22}
\]

We write down the (13) pushed by \(G\) from the right, and then compose both sides by \(G^n(t)\) also from the right:

\[
\delta_{G^n+1}t_G^{(n+1)}G^n(t) = [t_G^{(n+1)}]^2G^n(\delta_G)G^n(t),
\]

\[
\delta_{G^n+1}t_G^{(n+1)} = t_G^{(n+1)}t_G^{(n)}G^n(t_G)G^n(\delta_G)G^n(t).
\]

We substitute the identity proved in (22) to get

\[
\delta_{G^n+1}t_G^{(n+1)} = t_G^{(n+1)}t_G^{(n)}G^n(t_G)G^{n+1}(t)G^{n+1}(\delta).
\]

and then we notice that naturality and the definition of \(t^{(n)}\) imply that \(t_G^{(n)}\) commutes with \(G^{n+1}(t)\). Hence

\[
\delta_{G^n+1}t_G^{(n)} = t_G^{(n+1)}G^{n+1}(t)G^{n+1}(t)G^{n+1}(\delta) = [t_G^{(n+2)}]^2G^{n+1}(\delta).
\]
Eq. (14) follows by induction, and naturality. Assume (14) holds for \( n \). Then

\[
\begin{array}{c}
G^nG \\
G^n(t)
\end{array}
\rightarrow
\begin{array}{c}
G^nG \\
G^{n+1}(e)
\end{array}
\rightarrow
\begin{array}{c}
t_n^G \\
G^{n+1}(e)
\end{array}
\rightarrow
\begin{array}{c}
GG^n \\
G^nG
\end{array}
\] is commutative (upper right triangle by \( G(\epsilon)t = \epsilon_G \) and functoriality of \( G^n \), the parallelogram by naturality of \( t_n \), and the lower triangle by the induction hypothesis). The external triangle may easily be recognized as the identity \( G^n = G^n(\epsilon) \rightarrow t_n \rightarrow G^n+1(\epsilon) \) once \( \epsilon^{(n+1)} \) is expanded as \( \epsilon^{(n)} \circ G^n(\epsilon) \) and \( t_{n+1} \) as \( t_{n,G} \circ G^n(t) \).

Eq. (15) is identical to \( (C_{n+1}) \) from the proof of Theorem 1. Although the assumptions on \( t \) differ slightly, the proof stays unchanged. Q.E.D.

Notice that in the special case \( t^2 = \text{id} \), 2 out of 4 requirements for the distributive laws are the identities for the cyclic operator on the nose, but the other two differ.

Let \( f : R \rightarrow S \) be an extension of noncommutative unital rings. Consider the \( S \)-bimodule \( S^\otimes R^n = S \otimes_R S \otimes_R \cdots \otimes_R S \) and let \( \tau \) be the trivial symmetry on \( S \). Given an \( S \)-bimodule map \( r : S \otimes_R S \rightarrow SM_S \) let \( r_{ii+1} : S^\otimes_R S^\otimes {R^{i-1}} \otimes_R M \otimes S^\otimes_R (k-i)^{-1} \) be the \( S \)-bimodule map obtained by evaluating \( r \) on \( i \)-th and \( i+1 \)-st places and the identity on the rest of tensor factors.

**Theorem 5.** The formula (written in Nuss [18])

\[
\mu^{(n)} = (\mu_{12} \circ \mu_{34} \circ \cdots \circ \mu_{2n-1,2n}) \circ \tau_{2n-2,2n-1} \circ (\tau_{2n-4,2n-3} \circ \tau_{2n-3,2n-2}) \circ \cdots \circ (\tau_{2n-4,2n-3} \circ \tau_{2n-3,2n-2} \circ \tau_{2n-2,2n-1}) \circ \cdots \circ (\tau_{45} \circ \cdots \circ \tau_{n+1,n+2}) \circ (\tau_{23} \circ \cdots \circ \tau_{nn+1}),
\]

(23)

for the induced associative multiplication on \( S^\otimes R^n \), is (a special case of) the dualization of our formula for coproduct (13) expanded by induction.
Proof. We iterate formula (11) descending $n$:

\[
\delta^{(n)} = G^{n-1}(t_{G}^{(n-1)})G^{n-2}(t_{G^3}^{(n-2)})\delta_{G^4}^{(n-2)}G^{n-2}(\delta_{G^2})G^{n-1}(\delta)
\]

\[
= G^{n-1}(t_{G}^{(n-1)})G^{n-2}(t_{G^3}^{(n-2)}) \circ \ldots \circ G^{n-p}(t_{G^{2p-1}}^{(n-p)})G^{n-p}(\delta_{G^{2p}})G^{n-p+1}(\delta_{G^{2p-2}}) \circ \ldots \circ G^{n-1}(\delta)
\]

\[
= G^{n-1}(t_{G}^{(n-1)})G^{n-2}(t_{G^3}^{(n-2)}) \circ \ldots \circ G(\tau_{2n-3})\delta_{2n-2}G(\delta_{G^{2n-4}})G^{2}(\delta_{G^{2n-6}}) \circ \ldots \circ G^{n-1}(\delta)
\]

\[
= \prod_{j=1}^{n-1} G^{n-j}(t_{G^{2j-1}}^{(n-j)}) \prod_{p=0}^{n-1} G^{p}(\delta_{G^{2n-2p-2}})
\]

Now we need to dualize: the arrows and the composition will be hence backwards. Dualizing $G^{n-j}(t_{G^{2j-1}}^{(n-j)}) = G^{n-j}(t_{G^{2j-1}}^{(n-j)})G(\tau_{2j+1}) \circ \ldots \circ \tau_{n+j-1,n+j}$, will hence be $(\tau_{2j+1} \circ \ldots \circ \tau_{n+j-1,n+j})$. Similarly, the dualization of $G^{p}(\delta_{G^{2n-2p-2}})$ is $\mu_{2p-1,2n-2p}$. Putting these together, in proper order, we get (23).

As distributive laws were designed in 1960s for exactly that kind of reason, we believe that the formula and our explanation for it, must have been known to experts before.

Lemma. Let $\mathcal{C}, \mathcal{A}$ be any two categories, and $\mathcal{G} : \mathcal{C} \to \text{End} \mathcal{A}$, $\mathcal{X} : \mathcal{C} \to \mathcal{A}$ functors, either both covariant or both contravariant. Then the rule

\[
\forall_{C} := \mathcal{Y}(C) := \mathcal{G}(C)(\mathcal{X}(C)), \quad C \in \text{Ob} \mathcal{C},
\]

\[
\forall_{f} := \mathcal{Y}(f) := (\mathcal{G}(\mathcal{C}')(\mathcal{X}(f))(f \circ \mathcal{G}(f))_{\mathcal{X}(C)} = (\mathcal{G}(f))_{\mathcal{X}(C')} \circ \mathcal{G}(C)(\mathcal{X}(f)),
\]

for $f \in \mathcal{C}(C,C')$, defines a functor $\mathcal{Y} : \mathcal{C} \to \mathcal{A}$ of the same covariance.

Proof. This is basically the same trick which is involved in the definition of Godement’s product. Given a chain $C \xrightarrow{f} C' \xrightarrow{g} C''$ in $\mathcal{C}$, the diagram

\[
\begin{array}{ccc}
\mathcal{G}(C)(\mathcal{X}(C)) & \xrightarrow{\mathcal{G}(f)_{\mathcal{X}(C')}} & \mathcal{G}(C')(\mathcal{X}(C')) \\
\mathcal{G}(C)(\mathcal{X}(C')) & \xrightarrow{\mathcal{G}(g)_{\mathcal{X}(C')}} & \mathcal{G}(C')(\mathcal{X}(C'')) \\
\mathcal{G}(C)(\mathcal{X}(C'')) & \xrightarrow{\mathcal{G}(g)_{\mathcal{X}(C'')}} & \mathcal{G}(C'')(\mathcal{X}(C''))
\end{array}
\]

is commutative because $\mathcal{G}(f)$ and $\mathcal{G}(g)$ are natural transformation. The legs of the big triangle compose to $\mathcal{Y}(g \circ f)$ and the hypothesis is $\mathcal{Y}(g) \circ \mathcal{Y}(f)$. The equality $\mathcal{Y}(\text{id}) = \text{id}$ is easy. The contravariant case is analogous. Q.E.D.
Specialize the lemma to the case where $C$ is the (para)cyclic category. Using the contravariant case of Theorem 2 we get immediately

**Theorem 6.** (assumptions from Theorem 2) Let $X_\bullet$ be a cyclic object in $A$ with boundaries $\partial X_{n}$, degeneracies $\sigma^i_{X,n}$ and (para)cyclic operators $t_{X,n}$. Let natural transformations $t_n$ from (5) be denoted by $t^G_n$, with components $(t^G_n)_M := t^G_{n,M}$ at $M$ in $A$. Then the formulas

$$Y_n := G^{n+1}X_n,$$
$$\partial^i_{n,Y} := G^i(\varepsilon_{G^{n-1}X_n}) \circ G^{n+1}(\partial^i_{X,n}),$$
$$\sigma^i_{n,Y} := G^i(\delta_{G^{n-1}X_n}) \circ G^{n+1}(\sigma^i_{X,n}),$$
$$(t^Y)_n := G^{n+1}(t_{X,n}) \circ t^G_{n,X_n},$$

(24)

define a (para)cyclic object $Y_\bullet$ in $A$.

The simplicial part of the theorem is previously known (cf. e.g. [2]).

**Remark.** 1. Every object $M \in A$, gives rise to a constant simplicial object $X_\bullet$, where $X_n = M$ for all $n$ and all faces and degeneracies are identities. In that case, the assertion of Theorem 2 for $X_\bullet$ is simply the main part of Theorem 1. The other parts of the Theorem 1 generalize as well: functoriality, augmented version.

2. Monadic version of our results (giving cosimplicial objects) is obvious by dualization. We expect that dihedral etc. analogues of our analysis are possible.

3. For $A$ abelian, all flavors of the cyclic homology associated to the trivial symmetry $\tau^G$ on $G$ should not carry "really cyclic information". E.g. how do they compare to the 'underlying' cobar homology?

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**Versions.** We'll address nontrivial examples and extensions in a later version or a sequel to this preprint. This preprint is posted at an unusually early stage to facilitate the communication with a number of colleagues who expressed their interest in the very main construction of this article.

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