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Model reduction analysis using Hankel Norm Approximation on discrete-time linear system with the special shape of matrix $A$

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Abstract. A system can describe real problems that occur in the environment. The system is the result of modeling of real problems. Based on the results of real problem modeling, it often produces a linear system with a special shape of matrix $A$. In general, to represent the actual conditions of the real problem accurately, a system with a large order is required. While on the other hand, in terms of analysis and computation of the system, it is not desired to have a system with a large order. Therefore, we propose to use a model reduction that produces a system with small orders but without significant errors. One of the commonly used model reduction methods is Hankel Norm Approximation (HNA). In this research, we analyze the process of model reduction with the HNA method in the discrete-time linear system which has special forms of the matrix $A$. The process of reduction of this model begins with the formulation of the initial system with a special shaped matrix $A$. Next, we analyze the nature of stability, controllability and observability of the initial system. Then a balanced system is formed and followed by model reduction using HNA. Based on the results of the model reduction simulation using HNA, it was found that the reduced system with HNA has the same performance and properties as the initial system, which is stable, controllable, and observable. Besides that, HNA model reduction is suitable for use at high frequencies and has a fast computation time.

1. Introduction
Mathematical science has a very important role in solving real-life problems. One of the problems is using system modeling. System modeling from real problems often produces discrete-time linear system with some special shape of matrix $A$ such as tridiagonal [1], modelling system of high river level will produce a linear system with matrix $A$ in the form of hexa-diagonal [2] etc. In the establishment of system model expected resembles phenomenon in nature, we often get system model with many state variables. System model with many state variables usually called system with a large order. A system with a large order will influence the complexity in analyzing dan solving system properties, and also takes a long computation time. Therefore, we need to simplify the system with a large order so that system has smaller order without significant errors. That simplification of a system is called model reduction [1]. Now, many methods has been widely developed for model reduction, such as Balanced Truncation (BT) method [3, 4, 5], Singular Perturbation Approximation (SPA) method [6, 11]. In model reduction using SPA method, all state variables from balanced system are partitioned into fast mode and slow mode.
State variables corresponding to small Hankel singular value are defined as fast mode. State variables corresponding to large Hankel singular value are defined as slow mode. Hankel singular value is a value which represents the influence of every state variable to a system. Next, the reduced model is obtained by taking a rate from fast mode equal to zero [6]. While on a reduced model with Balanced Truncation (BT), all state variables from balanced system are partitioned into fast mode and slow mode. State variables corresponding to small Hankel singular value are defined as fast mode. State variables corresponding to large Hankel singular value are defined as slow mode. Then, the reduced model is obtained by eliminating rate from fast mode [5]. Based on this background, on this research we study the model reduction using Hankel Norm Approximation on a linear system where the matrix A has special shape. Next, we will analyze properties of reduced system using HNA and the initial system. To support analysis result, in this paper, we will use heat conduction system as the case study.

2. Discrete-Time Linear Systems

2.1. Initial Systems

Given a discrete-time linear system as follows [7]:

\[
\begin{align*}
    x_{k+1} &= Ax_k + Bu_k \\
    y_k &= Cx_k + Du_k
\end{align*}
\]  

(1)

where \( x_k \in \mathbb{R}^n \) is state variable at time k, \( u_k \in \mathbb{R}^m \) is deterministic input vector at time k, and \( y_k \in \mathbb{R}^p \) is output vector at time k. Next, (1) can be expressed as system \((A, B, C, D)\) where \( A, B, C, D \) is a constant matrix of the corresponding size and we assume \( A \) is a non-singular matrix.

Behavior of a system (1) hence can be seen from its transfer function. Transfer function of the system \((A, B, C, D)\) is defined as follows [3]:

\[
G(z) = C(zI - A)^{-1}B + D
\]  

(2)

The following theorems are used to check properties of system (1).

**Theorem 1.** [8] Discrete linear system (1) is asymptotically stable if and only if \( |\lambda_i(A)| < 1 \) for \( i = 1, \ldots, n \) with \( \lambda_i(A) \) is an eigenvalue of matrix A. If \( |\lambda_i(A)| < 1 \), discrete system is stable.

**Theorem 2.** [7] Given a controllability matrix \( M_c \) as:

\[
M_c = (B \ AB ... A^{n-1}B)
\]

Discrete system on (1) is controllable if and only if the rank of controllability matrix \( M_c \) is equal to \( n \).

**Theorem 3.** [7] Given observability matrix \( M_o \) as:

\[
M_o = (C^T \ A^T C^T ... (A^T)^{n-1}C^T)
\]

Discrete system on (1) is observable if and only if the rank of observability matrix \( M_o \) is equal to \( n \).

Relationship between stability, controllability, and observability of system with controllability grammian \( W \), and observability grammian \( M \) is discussed in next theorem.
Theorem 4. [1] Given system $(A, B, C, D)$ that is stable, controllable, and observable. Controllability grammian $W$, and observability grammian $M$, respectively is a positive definite matrix that is the unique solution of the following Lyapunov equation

$$ AWAT + BB^T - W = 0 $$  \hspace{1cm} (3)

$$ A^T MA + CTC - M = 0 $$  \hspace{1cm} (4)

2.2. Model Reduction using HNA

The steps in model reduction using HNA are almost same with BT or SPA, i.e the first step is constructing balanced system. The balanced system is a new system which is obtained from the initial system through a transformation matrix $T$ such that the new system has the following property: the controllability grammian is the same with the observability grammian. The new system is called as a balanced system and denoted by $(\tilde{A}_s, \tilde{B}_s, \tilde{C}_s, \tilde{D}_s)$. While controllability grammian and observability grammian on balanced system are denoted by $W$ and $M$.

Definition 5. [10] System $(\tilde{A}_s, \tilde{B}_s, \tilde{C}_s, \tilde{D}_s)$ is called balanced system from system $(A, B, C, D)$, if system $(\tilde{A}_s, \tilde{B}_s, \tilde{C}_s, \tilde{D}_s)$ has controllability grammian $\tilde{W}$, and observability grammian $\tilde{M}$ which is the unique solution of the following Lyapunov equation

$$ \tilde{A}_s\tilde{W}\tilde{A}_s^T + \tilde{B}_s\tilde{B}_s^T - \tilde{W} = 0 $$  \hspace{1cm} (5)

$$ \tilde{A}_s^T MA_s + \tilde{C}_s^T \tilde{C}_s - \tilde{M} = 0 $$  \hspace{1cm} (6)

such that it satisfies

$$ \tilde{W} = \tilde{M} = \text{diag}(\sigma_1, \sigma_2, ..., \sigma_n), $$

$$ \sigma_1 \geq ... \geq \sigma_r \geq ... \geq \sigma_n > 0 $$

where $\sigma_i$ is the Hankel singular value from system $(A, B, C, D)$ which can be defined as

$$ \sigma_i = \sqrt{\lambda_i(WM)} , i = 1, ..., n $$

where $\lambda_i$ is eigenvalues of $WM$.

Next, $\Sigma$ is called as an equilibrium grammian and it can be partitioned into $\Sigma = \text{diag}(\Sigma_1, \Sigma_2)$, where $\Sigma_1 = \text{diag}(\sigma_1, \sigma_2, ..., \sigma_r)$ and $\Sigma_2 = \text{diag}(\sigma_{r+1}, \sigma_{r+2}, ..., \sigma_n)$. Therefore, balanced realizations systems $(\tilde{A}_s, \tilde{B}_s, \tilde{C}_s, \tilde{D}_s)$ can be written as,

$$ \begin{bmatrix} \tilde{x}_1(k+1) \\ \tilde{x}_2(k+1) \end{bmatrix} = \begin{bmatrix} \tilde{A}_s11 & \tilde{A}_s12 \\ \tilde{A}_s21 & \tilde{A}_s22 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{bmatrix} + \begin{bmatrix} \tilde{B}_s1 \\ \tilde{B}_s2 \end{bmatrix} u(k) $$  \hspace{1cm} (7)

$$ \tilde{y}(k) = \begin{bmatrix} \tilde{C}_s1 \\ \tilde{C}_s2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{bmatrix} + \tilde{D}_s u(k) $$  \hspace{1cm} (8)

where, $\tilde{x}_1(k) \in \mathbb{R}^r$ and $\tilde{A}_{s11} \in \mathbb{R}^{r \times r}$ corresponds to $\Sigma_1$ and $\tilde{x}_2(k) \in \mathbb{R}^{n-r}$ corresponds to $\Sigma_2$. And then we obtain a model reduction using HNA as follows [6]:

$$ \begin{align*}
\tilde{A}_{sr} & \triangleq \Gamma^{-1}(\sigma_{k+1}^2 A_{s11}^T + \Sigma_1 A_{s11} \Sigma_1 - \sigma_{k+1} C_{s1}^T U B_{s1}^T) \\
\tilde{B}_{sr} & \triangleq \Gamma^{-1}(\Sigma_1 B_{s1} + \sigma_{k+1} C_{s1}^T U) \\
\tilde{C}_{sr} & \triangleq C_{s1} \Sigma_1 + \sigma_{k+1} U B_{s1}^T \\
\tilde{D}_{sr} & \triangleq D_s - \sigma_{k+1} U
\end{align*} $$
where,
\[ \Gamma = \Sigma_k^2 - \sigma_{k+1}^2 I \]
\[ U = - (\tilde{C}_r \tilde{B}^T_{12})^\perp \]

where \( \perp \) is a pseudo-inverse operator.

Thus, the reduced system can be written as follows:
\[
\begin{align*}
\tilde{x}_{r+1} &= \tilde{A}_r \tilde{x}_r + \tilde{B}_r \tilde{u}_k, \\
\tilde{y}_r &= \tilde{C}_r \tilde{x}_r + \tilde{D}_r \tilde{u}_k
\end{align*}
\]

(9)

The system reduced using HNA has order \( r \) and it has properties which are stable, controllable, observable and complementary \( \| G_s - G_{sr} \|_{\infty} \geq \sigma_{r+1} \) with \( G_s \) and \( G_{sr} \) is transfer function of initial system \( (A_s, B_s, C_s, D_s) \) and transfer function of reduced system [9].

The construction of reduced model using HNA is very dependent on the form of initial system. While the initial system is obtained by the result of modelling of real problems which usually declared in differential equation and then we discretized it such that we finally obtained linear system. Based on a result of constructing initial system model which is originated from modeling the real problems, we often meets that initial system indicates a tendency that matrix \( A \) has special form, such as diagonal, tridiagonal, hexa-diagonal, and etc. Therefore, we need to simulate in order to measure the performance of HNA if matrix \( A \) has special form.

3. Simulation

3.1. Simulation 1: Diagonal Matrix \( A \)

In this simulation, we use a discrete-time linear system with \( n = 10 \) and matrix \( A \) is diagonal, \( B, C, \) and \( D \) in Equation (10)

\[
A = \begin{pmatrix}
0.9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05
\end{pmatrix},
\]

(10)

\[
B = \begin{pmatrix}
0.9 \\
0.8 \\
0.7 \\
0.6 \\
0.5 \\
0.4 \\
0.3 \\
0.2 \\
0.1 \\
0.05
\end{pmatrix},
\]

\[
C = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix},
\]

\[
D = \begin{pmatrix}
1
\end{pmatrix}
\]

Based on initial system \( (A, B, C, D) \) in Equation (10), we can analyze properties from an initial system such as stability, controllability, observability. Based on eigenvalues of the initial system, we know that the system is controllable and observable. And then we construct balanced system from stable system, we obtain a balanced system \( (\tilde{A}_s, \tilde{B}_s, \tilde{C}_s, \tilde{D}_s) \). Then we obtain the Hankel singular value. After that the balanced system \( (A_s, B_s, C_s, D_s) \) will be reduced. But there are some requirements to do the reduction. The requirements to be able to do the reduction using HNA are as mentioned in Table 1. We can do reduction starting from order 1 until order 9.

The reduced model of order 5\(^{th}\) obtained using HNA can be expressed by matrices:

\[
\tilde{A}_{sr} = \begin{pmatrix}
0.762139 & -0.02094 & -0.000541 & -7.10^{-6} & -5.10^{-8} \\
-1.378084 & 0.605920 & -0.021478 & -3.65.10^{-4} & -3.10^{-6} \\
-3.329490 & -2.091121 & 0.536021 & -0.018715 & -2.07.10^{-4} \\
-5.713535 & -4.380093 & -2.405394 & 0.490214 & -0.014683 \\
-8.163831 & -7.064122 & -5.140390 & -2.832811 & 0.461508
\end{pmatrix},
\]

\[
\tilde{B}_{sr} = \begin{pmatrix}
0.206326 \\
-0.572352 \\
-1.019334 \\
1.741187 \\
-2.336720
\end{pmatrix}
\]

\[
\tilde{C}_{sr} = \begin{pmatrix}
-19.660945 & -0.828392 & -0.017092 & -2.1.10^{-4} & -10^{-6}
\end{pmatrix},
\]

\[
\tilde{D}_{sr} = (1.000044)
\]
Table 1. The requirements for model reduction using HNA for Diagonal Matrix

| Orde | \( \| G_s - G_{sr} \|_{\infty} \) | \( \sigma_{r+1} \) | \( \| G_s - G_{sr} \|_{\infty} \geq \sigma_{r+1} \) | time |
|------|-----------------|-------------|----------------|-----|
| 1    | 2.33988         | 1.20305     | Fulfill        | 0.067727 |
| 2    | 0.36212         | 0.12454     | Fulfill        | 0.041889 |
| 3    | 0.04151         | 0.01098     | Fulfill        | 0.044081 |
| 4    | 0.00339         | 0.00079     | Fulfill        | 0.042605 |
| 5    | 0.00020         | 0.00004     | Fulfill        | 0.040412 |
| 6    | 8 \times 10^{-6}| 0.000001    | Fulfill        | 0.044944 |
| 7    | 2 \times 10^{-7}| 5 \times 10^{-8} | Fulfill        | 0.044851 |
| 8    | 5 \times 10^{-8}| 1.10^{-9}  | Fulfill        | 0.049357 |
| 9    | 4 \times 10^{-7}| 2.10^{-11} | Fulfill        | 0.048357 |

The stability of reduced system of order 5 \((\tilde{A}_s, \tilde{B}_s, \tilde{C}_s, \tilde{D}_s)\) can be found from the eigenvalues that we obtained. Based on eigenvalues of the initial system, we can infer that the system is asymptotically stable and based on rank of controllability and observability matrix, we obtain that the system is controllable and observable. And then the frequency response of the initial system and reduced system using HNA are shown in Figure 1. Based on the simulation in Figure 1, we can see that the frequency response of reduced model in low frequency to high frequency is approximately the same with the frequency response of the initial system.

3.2. Simulation 2 (Upper Triangular Matrix A)

In this simulation, we use a discrete-time linear with \( n = 10 \) and an upper triangular matrix A. The initial system \((A, B, C, D)\) is given in (11).

\[
A = \begin{bmatrix} 0.2 & 0.4 & 0.6 & 0.2 & 0.3 & 0.8 & 0.9 & 0.6 & 0.1 & 0.3 \\ 0 & 0.4 & 1 & 0.5 & 3 & 0.8 & 3 & 8 & 0.9 & 5 \\ 0 & 0 & 0.6 & 0.4 & 0.7 & 7 & 0.3 & 7 & 0.5 & 8 \\ 0 & 0 & 0 & 0.9 & 0.1 & 4 & 0.6 & 4 & 0.7 & 0.1 \\ 0 & 0 & 0 & 0 & 0 & 0.7 & 2 & 7 & 9 & 0.3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0.8 & 0.1 & 0.3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0.5 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 \\
\end{bmatrix}, \quad B = \begin{bmatrix} 0.3 \\ 0.7 \\ 0.6 \\ 0.9 \\ 0.3 \\ 0.2 \\ 0.1 \\ 0.5 \\ 0.3 \\
\end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}, \quad D = \begin{bmatrix} 1 \\
\end{bmatrix}
\] (11)

Based on initial system \((A, B, C, D)\) in Equation (11), we can analyze properties from the initial system such as stability, controllability, observability. Based on eigenvalues of the initial system,
we can infer that the system is asymptotically stable, and based on rank of controllability and observability matrix, we know that the system is controllable and observable. Then we construct balanced system from stable system, we obtain a balanced system \((\tilde{A}_s, \tilde{B}_s, \tilde{C}_s, \tilde{D}_s)\). After that the balanced system \((\tilde{A}_s, \tilde{B}_s, \tilde{C}_s, \tilde{D}_s)\) will be reduced using HNA as shown in Table 2.

| Orde | \(\|G_s - G_{sr}\|_\infty\) | \(\sigma_{r+1}\) | \(\|G_s - G_{sr}\|_\infty \geq \sigma_{r+1}\) | time |
|------|-----------------------|-------------|---------------------------------|------|
| 1    | 1131124.783           | 221575.0469 | Fulfill                          | 0.072889 |
| 2    | 1383272.454           | 47905.32144 | Fulfill                          | 0.050035 |
| 3    | 24429.8369            | 7194.871223 | Fulfill                          | 0.054208 |
| 4    | 2321.909049           | 731.007697 | Fulfill                          | 0.050951 |
| 5    | 141.943731            | 32.430340  | Fulfill                          | 0.050897 |
| 6    | 5.33563795            | 1.19130464 | Fulfill                          | 0.051191 |
| 7    | 0.000680661           | 0.00053833 | Fulfill                          | 0.057349 |
| 8    | 0.000680661           | 0.00053833 | Fulfill                          | 0.057349 |
| 9    | 0.000680661           | 0.00053833 | Fulfill                          | 0.088375 |

The stability of reduced system order 6 \((\tilde{A}_s, \tilde{B}_s, \tilde{C}_s, \tilde{D}_s)\) can be found from the eigenvalues. Based on eigenvalues of the initial system, we can infer that the system is asymptotically stable and based on rank of controllability and observability matrix, we obtain that the system is controllable and observable. Then, the frequency response between the initial system and reduced system using HNA can be seen in Figure 2. From the second simulation, we can see that the frequency response of reduced model in low frequency is quite close to the frequency response of the initial system. But in high frequency, it gets slightly away from frequency response of the initial system.

3.3. Simulation 3 (Lower Triangular Matrix \(A\))

In the next simulation, we use a discrete-time linear system with \(n = 10\) and a lower triangular matrix \(A\). More specifically, we use the initial system \((A, B, C, D)\) in Equation (12). From the equation, we can analyze properties from an initial system for example stability, controllability,
or observability.

\[
A = \begin{bmatrix}
0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.4 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.8 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.3 & 0.8 & 0.9 & 0.9 & 0 & 0 & 0 & 0 & 0 \\
0.6 & 0.1 & 0.3 & 1 & 0.3 & 0 & 0 & 0 & 0 \\
0.5 & 3 & 0.8 & 3 & 8 & 0.7 & 0 & 0 & 0 \\
0.9 & 5 & 0.4 & 0.7 & 7 & 0.3 & 0.1 & 0 & 0 \\
7 & 0.5 & 8 & 0.1 & 4 & 0.6 & 4 & 0.5 & 0 \\
7 & 9 & 0.3 & 0.8 & 0.1 & 0.3 & 5 & 9 & 2 \end{bmatrix}, \\
B = \begin{bmatrix}
0.3 \\
0.7 \\
0.6 \\
0.9 \\
0.4 \\
0.8 \\
0.2 \\
0.1 \\
0.5 \\
0.3 \end{bmatrix}, \\
C = (1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1), \ D = (1) \tag{12}
\]

Based on eigenvalues of the initial system, we can infer that the system is asymptotically stable and based on rank of controllability and observability matrix, we know that the system is controllable and observable. Then we construct balanced system from stable system, we obtain a balanced system \((\tilde{A}_s, \tilde{B}_s, \tilde{C}_s, \tilde{D}_s)\). After that the balanced system \((\tilde{A}_s, \tilde{B}_s, \tilde{C}_s, \tilde{D}_s)\) will be reduced. But there are some requirements for the model reduction. The requirements to be able to apply the model reduction using HNA for lower triangular matrix are listed in Table 3.

| Orde | \(\| G_s - G_{sr} \|_{\infty} \) | \(\sigma_{r+1} \) | \(\| G_s - G_{sr} \|_{\infty} \geq \sigma_{r+1} \) | time |
|------|----------------|--------|----------------|------|
| 1    | 67674.64890   | 12880.3219 | Fulfill        | 0.058161 |
| 2    | 81675.73666   | 3793.146723 | Fulfill        | 0.039448 |
| 3    | 1900.258641   | 653.1484283 | Fulfill        | 0.047457 |
| 4    | 995.0434268   | 250.0838604 | Fulfill        | 0.045196 |
| 5    | 294.818304    | 62.09376204 | Fulfill        | 0.044226 |
| 6    | 0.99966401    | 0.242579326 | Fulfill        | 0.043891 |
| 7    | 0.014208426   | 0.002859514 | Fulfill        | 0.043803 |
| 8    | 0.001562552   | 0.000337961 | Fulfill        | 0.051463 |
| 9    | 0.000012108   | 9.043.10^{-8} | Fulfill        | 0.043964 |

We can do reduction starting from first order to ninth order. The 7th order model reduced using HNA can be expressed by following matrices:

**Figure 2.** Frequency response of reduced system using 6th order HNA
The stability of reduced system of order 7 \((A_s, B_s, C_s, D_s)\) can be found from eigenvalues. From the eigenvalues, we can infer that the system is asymptotically stable. Based on rank of controllability and observability matrix, we obtain that the system is controllable and observable. Figure 3 shows frequency response of the initial system and reduced system using HNA. Based on the simulation, we can see that the frequency response of reduced model in low frequency until high-frequency is quite close to the frequency response of the initial system.

![Figure 3. Frequency response of reduced system using 7th order HNA](image)

3.4. Simulation 4 (Tridiagonal Matrix A)
In this simulation, we use a discrete-time linear with \(n = 10\) and tridiagonal matrix A. Then we obtain initial system \((A, B, C, D)\) as shown in (13)

\[
A = \begin{pmatrix}
0.3 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.3 & 0.4 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.3 & 0.1 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.3 & 0.1 & 0.3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.3 & 0.3 & 0.4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.3 & 0.4 & 0.4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.1 & 0.4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0.1 & 0.4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0.1 & 0.3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
0.8 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix},
\]

\[
C = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
D = \begin{pmatrix}
0
\end{pmatrix}
\]

Based on initial system \((A, B, C, D)\) in Equation (13), we can analyze the properties of the initial system such as stability, controllability, observability. All eigenvalues that we obtained hinted that the system is asymptotically stable. Based on rank of controllability and observability matrix, we obtain that the system is controllable and observable. Then we construct balanced system from stable system, we obtain a balanced system \((A_s, B_s, C_s, D_s)\). After that the balanced system \((A_s, B_s, C_s, D_s)\) will be reduced. The requirements to be able to do the model reduction using HNA are shown in Table 4.
Table 4. The requirements for model reduction using HNA on Tridiagonal Matrix

| Orde | $\| G_s - G_{sr} \|_\infty$ | $\sigma_{r+1}$ | $\| G_s - G_{sr} \|_\infty \geq \sigma_{r+1}$ | time  |
|------|-----------------|--------------|---------------------------------|-------|
| 1    | 0.6233466602   | 0.230476127540 | Fulfill                          | 0.071705 |
| 2    | 0.0861930667   | 0.019234063974 | Fulfill                          | 0.048399 |
| 3    | 0.0128603787   | 0.002535615580 | Fulfill                          | 0.050056 |
| 4    | 0.0017036974   | 0.000339266678 | Fulfill                          | 0.033953 |
| 5    | 0.0002235332   | 0.000045229243 | Fulfill                          | 0.042797 |
| 6    | 0.0000291591   | 0.00005898205  | Fulfill                          | 0.036280 |
| 7    | 0.0000035212   | 0.00000705283  | Fulfill                          | 0.035510 |
| 8    | 0.0000003433   | 0.0000067655   | Fulfill                          | 0.039167 |
| 9    | 0.0000000215   | 0.00000004166  | Fulfill                          | 0.075809 |

We can do reduction starting from first order to ninth order. The $4^{th}$ order model reduced using HNA can be expressed by following matrices:

$$\tilde{A}_{sr} = \begin{pmatrix}
0.497229 & 0.079723 & -0.001147 & 0.000007 \\
1.558421 & 0.423133 & 0.020646 & -0.000264 \\
-3.223141 & 2.965476 & 0.209255 & 0.045713 \\
1.156667 & -2.224644 & 2.677226 & 0.152185
\end{pmatrix}$$

$$\tilde{B}_{sr} = \begin{pmatrix}
-0.830866 \\
1.072515 \\
-0.674209 \\
-0.396237
\end{pmatrix}, \tilde{C}_{sr} = \begin{pmatrix}
1.000339
\end{pmatrix}$$

$$\tilde{D}_{sr} = \begin{pmatrix}
1.000339
\end{pmatrix}$$

The stability of reduced system of order 7 ($\tilde{A}_{sr}, \tilde{B}_{sr}, \tilde{C}_{sr}, \tilde{D}_{sr}$) can be found from eigenvalues. From the eigenvalues, we can infer that the system is asymptotically stable. Based on rank of controllability and observability matrix, we obtain the system is controllable and observable. Figure 4 shows frequency response between the initial system and reduced system using HNA. Based on the simulation, we can see that the frequency response of reduced model in low frequency until high-frequency is quite close to the frequency response of the initial system.

![Frequency response](image.png)

Figure 4. Frequency response of reduced system using $4^{th}$ order HNA

4. Case Study: Heat Conduction Systems

In this research, we will discuss a case study about heat conduction system. Heat exchanger system is one example of a system where the matrix is tridiagonal. The following equation is a
Figure 5. Frequency response of reduced system using $r^{th}$ order HNA

common form of heat exchanger system after discretization.

$$x_{k+1} = Ax_k + Bu_k$$

Then we set the input parameter $\gamma = 0.4$ and the initial system has $10^{th}$ order so that we obtain initial system $(A, B, C, D)$ as follows:

$$A = \begin{pmatrix} 0.2 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0.4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Based on initial system $(A, B, C, D)$, we can analyze properties from initial system for example stability, controllability, observability. All eigenvalues that we have obtained hinted that the system is asymptotically stable. Based on rank of controllability and observability matrix, we obtain the system is controllable and observable. Then we construct balanced system from stable system, we obtain a balanced system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$. After that the balanced system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ will be reduced. The requirements to be able to do the reduction using HNA are listed in Table 5.

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We can do reduction starting from first order to ninth order. The $7^{th}$ order model reduced using HNA is as follows:
Table 5. The requirements for model reduction using HNA

| Order | $\|G_s - G_{sr}\|_\infty$ | $\sigma_{r+1}$ | $\|G_s - G_{sr}\|_\infty \geq \sigma_{r+1}$ | time |
|-------|-----------------|--------------|-----------------------------------|------|
| 1     | 0.382443        | 0.128039     | Fulfill                           | 0.9694 |
| 2     | 0.350568        | 0.039566     | Fulfill                           | 0.01492 |
| 3     | 0.200753        | 0.010430     | Fulfill                           | 0.04226 |
| 4     | 0.013513        | 0.002241     | Fulfill                           | 0.04414 |
| 5     | 0.001943        | 0.0003942    | Fulfill                           | 0.04443 |
| 6     | 0.000276        | 0.000056     | Fulfill                           | 0.04488 |
| 7     | 0.000031        | 0.000006     | Fulfill                           | 0.03399 |
| 8     | 0.000002        | 0.00000052   | Fulfill                           | 0.04378 |
| 9     | $14.10^{-8}$    | 0.00000002   | Fulfill                           | 0.10072 |

\[
\tilde{A}_{sr} = \begin{pmatrix}
0.379 & 0.1058 & -0.0131 & 0.0006 & 0.00001 & -0.000001 & -0.0000001 \\
1.3053 & 0.5012 & 0.1156 & -0.0113 & 0.00026 & 0.000031 & -0.00000037 \\
1.2108 & 0.4674 & 0.1083 & -0.00572 & 0.000022 & 0.0000039 \\
1.2051 & -1.7042 & 1.4442 & 0.3798 & 0.0026 & -0.002559 & -0.0000187 \\
0.695 & 0.8487 & -1.7836 & 1.7992 & 0.293292 & 0.066306 & -0.00101096 \\
-2.1809 & 1.1693 & 0.2284 & -1.7921 & -2.14697 & 0.208082 & 0.05120747 \\
0.9603 & -1.9508 & 1.9572 & -0.6509 & -1.62329 & 2.548140 & 0.12133406
\end{pmatrix}
\]

\[
\tilde{B}_{sr} = \begin{pmatrix}
-1.3229 \\
1.1438 \\
-0.4755 \\
-0.0681 \\
1.4845 \\
-0.7438 \\
-1.0675
\end{pmatrix}, \quad \tilde{D}_{sr} = (1)
\]

\[
\tilde{C}_{sr} = \begin{pmatrix}
-0.20858 & 0.0197 & -0.0007 & -0.00072 & 0.00007 & -10^{-7} & -3.10^{-9}
\end{pmatrix}
\]

The stability of reduced system order 7 ($\tilde{A}_s, \tilde{B}_s, \tilde{C}_s, \tilde{D}_s$) can be found from the eigenvalues. We can infer that the system is asymptotically stable. Based on rank of controllability and observability matrix, we obtain that the system is controllable and observable. Figure 5 shows frequency response between the initial system and reduced system using HNA. Based on the simulation, we can see the frequency response of reduced model in low frequency until high-frequency is quite close to the frequency response of the initial system.

5. Conclusions
The process of model reduction with HNA begins with an initial system which is stable, controllable, and observable. Then we construct the balanced system. And then we decompose the system by using Hankel singular value. And finally we construct the reduced system using HNA. Based on analysis of the system, the reduced systems using HNA have the same properties with the initial system (stable, controllable, and observable). Based on analysis of the simulation with several matrices A, the model reduced using HNA is suitable for use on systems with high frequency because it is accurate and the computation time is fast. But upper triangular matrix A is suitable for use on systems with low frequency.

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