AT THE BOUNDARY OF MINKOWSKI SPACE

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Abstract. The Cayley transform compactifies Minkowski space $\mathbb{M}$, realized as self-adjoint $2 \times 2$ complex matrices following Penrose, as the unitary group $\mathbb{U}(2)$. Its complement is a compactification of a copy of a light-cone as it is usually drawn, constructed by adjoining a bubble or $\mathbb{C}P_1$ of unitary matrices with eigenvalue $\pm 1$ at the ends of a lightcone at infinity.

The Brauer-Wall group of $\mathbb{U}(2)$ (i.e. of fields of certain kinds of graded $C^*$-algebras, up to projective equivalence) is $\mathbb{Z}_2 \times \mathbb{Z}$, defining an interesting class of nontrivial examples of Araki-Haag-Kastler backgrounds for quantum field theories on compactified Minkowski space. The second part of this paper extends such models to link presentations of more general spin four-manifolds.

PART I : THE WEYL/CAYLEY TRANSFORM

This work began as an exercise in linear algebra, i.e. to interpret stereographic projection

$$\mathbb{M} \ni \mathbf{X} \mapsto \mathcal{C} (\mathbf{X}) := \frac{\mathbf{X} - i \mathbf{1}}{\mathbf{X} + i \mathbf{1}} \in \mathbb{U}(2) \cong \mathbb{T} \times_{\pm 1} \mathbb{SU}(2) \cong \text{Spin}^c(3)$$

(regarded as defined on the Penrose-Minkowski space of self-adjoint $2 \times 2$ Hermitian matrices

$$\mathbf{X} := \begin{bmatrix} x_0 + x_1 & x_2 - i x_3 \\ x_2 + i x_3 & x_0 - x_1 \end{bmatrix}$$

with $x_s \in \mathbb{R}^{1,3}$) as a compactification. It was precipitated by David Mumford’s recent review of current cosmological literature, in particular by his beautiful image [24](Fig 1) of our past light-cone.

In §1 we show that this Cayley compactification has a stratification

$$\mathbb{U}(2) \cong \mathbb{M} \cup \mathbb{M}_\infty \cup \mathbb{B}$$

in which $\mathbb{M}_\infty$ is a ‘light-cone at infinity’, and $\mathbb{B} \cong \mathbb{C}P_1$ is a two-sphere of unitary matrices with eigenvalues $\pm 1$. The Cayley compactification of $\mathbb{M}$ maps to Penrose’s, with the point at infinity on the light-cone at infinity blown up as a two-sphere $S^2 = \mathbb{C}P^1$, providing a plausible keystone or
linchpin [26] for constructions involving the Bondi-Metzner-Sachs group [22] of classical general relativity.

Section 3 discusses fields of $C^*$-algebras over this stratification as a homotopy-theoretic setting for algebraic quantum field theory. The second part of this paper goes on to argue that both the geometric categories of three-manifolds and the algebraic categories of Hilbert space operators have homological dimensions roughly three, and pair in ways evoking a duality between differential topology and quantum physics.

§1 Recollections and calculations

1.1 Let $\text{SL}_2(\mathbb{C}) \subset M_2(\mathbb{C})^\times$ be the subgroup of $2 \times 2$ complex matrices $T$ with determinant one; note that the map $T \mapsto T^*$ which sends a matrix to its conjugate transpose or adjoint is an antihomomorphism, and that the determinant of the conjugate transpose of a matrix is the complex conjugate of the determinant of the original matrix. Then $\text{SU}(2) \subset \text{SL}_2(\mathbb{C})$ is the maximal compact subgroup, composed of matrices of the form

$$T = \begin{bmatrix} u & v \\ -\bar{v} & \bar{u} \end{bmatrix}$$

with $u = u_0 + iu_1, \ v = v_0 + iv_1 \in \mathbb{C}$ such that $\det T = |u|^2 + |v|^2 = 1$ (i.e. unit length elements of the quaternions $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$), and let $U(2)$ be the group of invertible $2 \times 2$ complex unitary matrices $U$ (such that $U^* = U^{-1}$); its Lie algebra $\mathfrak{u}$ consists of antiHermitian complex $2 \times 2$ matrices. The exponential map of a connected compact Lie group is surjective, and any element of $U \in U(2)$ can be expressed uniquely [3](Ch 9) as

$$U = \begin{bmatrix} u & v \\ -\lambda \bar{v} & \lambda \bar{u} \end{bmatrix}$$

with $|u|^2 + |v|^2 = 1, \ \det U = \lambda = e^{i\alpha} \in \mathbb{T}, \ \alpha \in [-\pi, +\pi],$ defining a homeomorphism of $U(2)$ with $S^1 \times S^3$. However, the group extension

$$1 \longrightarrow \text{SU}(2) \longrightarrow \text{SU}(2) \times \pm_1 \mathbb{T} \cong U(2) \cong \text{Spin}^c(3) \overset{\det}{\longrightarrow} \mathbb{T} \longrightarrow 1$$

is nontrivial.

1.2 Let

$$M := \{M \in M_2(\mathbb{C}) \mid M = M^*\}$$

denote the real vector space of $2 \times 2$ complex Hermitian (self-adjoint) matrices

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} \bar{M}_{11} & \bar{M}_{21} \\ \bar{M}_{12} & \bar{M}_{22} \end{bmatrix} = \begin{bmatrix} \rho_+ & w \\ \bar{w} & \rho_- \end{bmatrix} = \begin{bmatrix} x_0 + x_1 & x_2 - ix_3 \\ x_2 + ix_3 & x_0 - x_1 \end{bmatrix}$$
(with \( w \in \mathbb{C} \), and \( \rho_{\pm}, x_i \in \mathbb{R} \)). Penrose coordinates \( \mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^{1,3} \to \mathbb{M} \) identify

\[
\det M := q(M) = x_0^2 - (x_1^2 + x_2^2 + x_3^2) \in \mathbb{R}
\]

with the Lorentz-Einstein pseudometric of Minkowski space.

If \( M \in \mathbb{M} \) then its eigenvalues are real, so \( i1 \pm M \) is invertible. Let

\[
\sigma : \mathbb{M} \ni M \mapsto \frac{M - i1}{M + i1} = \sigma(M) \in \mathbb{U}(2)
\]

denote the Cayley transform: essentially, \(-i\) times Riemannian stereographic projection. This clearly satisfies \( \sigma(M)^* = \sigma(M)^{-1} \), and because

\[
1 - \sigma(M) = 1 - \frac{M - i1}{M + i1} = \frac{2i1}{M + i1},
\]

is invertible, a matrix in the image of \( \sigma \) cannot have 1 as an eigenvalue, so \( \sigma \) has a well-defined inverse

\[
\sigma^{-1}(U) := \frac{1 + U}{1 - U} \in \mathbb{M}
\]
on that image, guaranteeing that \( \sigma \) is an embedding.

The complement \( \mathbb{M}_\infty = \mathbb{U}(2) - \sigma(\mathbb{M}) \cong S^3/S^0 \) consists of unitary matrices which do have 1 as an eigenvalue; in particular, they can be written as \( \exp(iZ) \) with \( Z \) self-adjoint and zero as an eigenvalue.

1.3 If \( U \in \mathbb{M}_\infty \) then

\[
\det(U - 1) = \det \begin{bmatrix} u - 1 & v \\ -\lambda \bar{v} & \lambda \bar{u} - 1 \end{bmatrix} = 1 - (u + \lambda \bar{u}) + \lambda = 0,
\]

so Trace \( U = u + \lambda \bar{u} = 1 + \lambda = 1 + \det U \). For example, \( \lambda = 1 \) implies \( U = 1 \), but if \( \lambda = -1 \) then

\[
U = \begin{bmatrix} u & v \\ \bar{v} & -\bar{u} \end{bmatrix}
\]

has trace zero, so \( u = u_0 \) is real. There is thus a ‘bubble’, a two-sphere \( \mathbb{B} \subset \mathbb{U}(2) \)

\[
u_0^2 + v_0^2 + v_1^2 = 1,
\]
of such matrices.

1.4 The light-cone is the subset

\[
\mathbb{M}_0 := \{ M \in \mathbb{M} \mid \det M = 0 \} \cong \mathbb{R} \times \mathbb{C}_+ = (\mathbb{R} \times \mathbb{C}_+)/(0 \times \mathbb{C}_+)
\]
of Minkowski space. It can be parametrized by stereographic projection

\[
(x_0, z) \mapsto x_0(1, s(z)) = M_0(x_0, z)
\]
where

\[
\mathbb{C}_+ \ni z \mapsto s(z) := (1 + |z|^2)^{-1}(|z|^2 - 1, 2z) \in \mathbb{R}^3 \cong \mathbb{R} \times \mathbb{C},
\]

We regard \( \mathbb{R} \cong 0, \mathbb{C} \cong 0 \) as basepointed spaces, with one-point compactifications \( \mathbb{R}_+ = \mathbb{P}^1(\mathbb{R}) \cong S^1, \mathbb{C}_+ = \mathbb{P}^1(\mathbb{C}) \cong S^2 \)
\[ (x_0, z) \mapsto M_0(x_0, z) = k \begin{bmatrix} |z| & u \\ \bar{u} & |z|^{-1} \end{bmatrix} \in \mathcal{M}_0 \]

with \( u = |z|^{-1}z \) and \( k = 2(|z| + |z|^{-1})^{-1}x_0 \).

**Claim** The composition \( \sigma^\perp := -\sigma \circ M_0 \),

\[
\sigma^\perp : \mathbb{R} \times \mathbb{C} \ni (x_0, z) \mapsto \frac{1 + iM_0}{1 - iM_0} \in \mathcal{M}_\infty \subset U(2)
\]

is an embedding, with the light-cone \( \mathcal{M}_\infty \) at infinity as its image, disjoint from \( \sigma(M) \).

In particular, \( \sigma^\perp(0, z) = 1 \). The map is well-defined, for

\[
\det(1-iM_0(x_0, z)) = \det \begin{bmatrix} 1 - ik|z| & -iku \\ -iku & 1 - ik|z|^{-1} \end{bmatrix} = (1-ik|z|)(1-ik|z|^{-1})+k^2
\]

\[
= 1 - ik(|z| + |z|^{-1}) = 1 - 2ix_0 \neq 0.
\]

This implies that

\[
1 - \sigma M_0 = 2(1 - iM_0)^{-1}
\]

is invertible, and hence that \( \sigma^\perp \) is an embedding since

\[
M_0 = i\frac{1 + \sigma M_0}{1 - \sigma M_0}.
\]

The image of \( \sigma^\perp \) is disjoint from \( \sigma(M) \), because

\[
\det(1 + \sigma M_0) = \det \frac{2iM_0}{1 - iM_0} = 0
\]

implies \( -\sigma M_0 \) has 1 as an eigenvalue.

**1.5** Calculation now shows that

\[
(1-2ix_0)\sigma^\perp(x_0, z) = \left[ \begin{array}{cc} 1 + ik|z| & iku \\ ik\bar{u} & 1 + ik|z|^{-1} \end{array} \right] \left[ \begin{array}{cc} 1 - ik|z|^{-1} & iku \\ ik\bar{u} & 1 - ik|z| \end{array} \right] =
\]

\[
\left[ \begin{array}{cc} 1 + ik(|z| - |z|^{-1}) & 2iku \\ 2ik\bar{u} & 1 + ik(|z|^{-1} - |z|) \end{array} \right] = 1 + 2ix_0V(z),
\]

where

\[
V(z) = (|z| + |z|^{-1})^{-1} \begin{bmatrix} |z| - |z|^{-1} & 2u \\ 2\bar{u} & |z|^{-1} - |z| \end{bmatrix}
\]

is Hermitian, satisfying \( V^2 = 1 \) and \( \text{Trace } V = 0 \). If \( z = re^{i\theta} \), then

\[
V(re^{i\theta}) = (r^2 + 1)^{-1} \begin{bmatrix} r^2 - 1 & 2re^{i\theta} \\ 2re^{-i\theta} & 1 - r^2 \end{bmatrix}.
\]
Evidently $P = \frac{1}{2}(1 + V)$ is an element of the space $\mathbb{D}$ of projections with Trace $P = 1$ and $e = (z, 1) \in \mathbb{C}^2$ as eigenvector. We have

$$\sigma^\perp(x_0, z) = \frac{1 + 2ix_0V(z)}{1 - 2ix_0} = 1 + \frac{4ix_0}{1 - 2ix_0}P,$$

so Trace $\sigma^\perp = (1 - 2ix_0)^{-1} = 1 + \det \sigma^\perp$, i.e.

$$\det \sigma^\perp(x_0, z) = \frac{1 + 2ix_0}{1 - 2ix_0} = e^{i\alpha(x_0)} \in \mathbb{T}$$

with

$$x_0 = -\frac{1}{2}\tan \frac{1}{2}\alpha, \alpha(\pm \infty) = \pm \pi.$$

If we write $-\beta$ for $\frac{4ix_0}{1 - 2ix_0} = e^{-i\alpha} - 1$, then $\sigma^\perp(x_0, z) = 1 - \beta P$, so

$$\log(1 - \beta P) = -\sum_{n \geq 1} \frac{(\beta P)^n}{n} = \log(1 - \beta) \cdot P = -i\alpha P$$

and hence

$$\sigma^\perp(x_0, z) = \exp(-i\alpha P).$$

This identifies the space $\mathbb{D}$ of projections with the bubble of unitary matrices with eigenvalues $\pm 1$.

Let

$$\varepsilon := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

then $V(z) \to \mp \varepsilon$ as $z \to 0$ resp $\infty$. Similarly, as $x_0 \to 0$, $\sigma^\perp(x_0, z) \to 1$, while

$$\sigma^\perp(x_0, z) \to -V(z) \in \mathbb{B} = \overline{\mathbb{M}}_\infty - \mathbb{M}_\infty$$

as $x_0 \to \pm \infty$, so $\sigma^\perp(x_0, z) \to \varepsilon$ as $(x_0, z) \to (\infty, \infty)$.

**Remark** If $\mathbb{B} = \begin{bmatrix} b_+ & w \\ \overline{w} & b_- \end{bmatrix} \in M_2(\mathbb{C})$ is Hermitian, with determinant zero and trace one, then it is a projection. Setting $r = (1 - b_+)^{-1}|w|$ identifies it with $\mathbb{D}$.

1.6 It follows that $\sigma^\perp$ extends to a homeomorphism

$$\bar{\sigma}^\perp : \mathbb{R}_+ \times \mathbb{C}_+ \cong (\mathbb{R}_+ \times \mathbb{C}_+)/ (0 \times \mathbb{C}_+) \to \overline{\mathbb{M}}_\infty.$$

Note that the domain of this map can be expressed as

$$\mathbb{R}_+ \wedge (\ast \cup \mathbb{C}_+) \cong \Sigma(S^0 \vee S^2),$$

where $\Sigma$ denotes the reduced suspension used in homotopy theory.
Corollary 2 The obvious inclusion induces an isomorphism $H^*(U(2), \mathbb{Z}) \cong H^*(\overline{M}_\infty, \mathbb{Z})$ in degrees below four; moreover, $\overline{M}_\infty - M_\infty \cong S^2$, $U/M_\infty \cong S^4$, while $H^*(U(2)/\mathbb{B}, \mathbb{Z}) = \mathbb{Z}$ when $* = 3, 4$ and is zero otherwise.

An exercise, with most grateful thanks to David Mumford:

As $t \to \infty$, a light ray $x_*(t) = (0, x) + t(1, v) \in \mathbb{R} \times \mathbb{R}^3$ (with $|v| = 1$) approaches
$$\begin{bmatrix} u \\ -\lambda \bar{v} \\ \lambda \bar{u} \end{bmatrix} = \frac{1}{1 - i\omega} \begin{bmatrix} z \\ -\bar{v} \\ -\bar{z} \end{bmatrix} \in U(2)$$
as above, with $\omega := x \cdot v$, $\lambda = C(\omega)$, $z = v_1 + i\omega$, $\nu = v_2 + iv_3$, ending on the line
$$v = -\frac{v_2 + iv_3}{1 + iv_1} (1 + iu), \quad |u|^2 + |v|^2 = 1.$$

§2 Some group actions

Definition $\text{Sl}_2(\mathbb{C}) \times \mathbb{M} \ni T, M \mapsto T(M) := TM^* \in \mathbb{M}$ defines a group action: for
$$ (T(M))^* = (TM^*)^* = TM^*T^* = T(M), $$
while
$$ S(T(M)) = S(TM^*)S^* = (ST)M(ST)^* = (ST)(M). $$

Moreover,
$$ \det(T(M)) = \det(TM^*) = \det T \cdot \det M \cdot \det T^* = \det M. $$

Corollary $\text{Sl}_2(\mathbb{C})$ is the double cover of the identity component of the (Lorentz) group of isometries of $(\mathbb{M}, q)$.

The action of the subgroup SU(2) on $\mathbb{M}$ preserves the decomposition of $\mathbb{M}$ into $(\text{Time}) \times (\text{Space})$, factoring through the action of the rotation group $\text{SU}(2) \to \text{SO}(3)$ on the second term. Moreover, the conjugation action of $\text{SU}(2)$ on $M_2(\mathbb{C})$ defined by the composition
$$ \text{SU}(2) \to \text{Sl}_2(\mathbb{C}) \to \text{PGL}_2(\mathbb{C}) $$
preserves the matrix algebra structure.

By the remarks in the previous section, $\sigma$ is equivariant with respect to the action of $\text{SU}(2)$ on $U(2)$ by conjugation.

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2It is not clear to me how well this is understood in the physics community; cf. [12](§5.1). I learned of [10] only after posting an earlier version of this paper.
The action of SU(2) on $\mathbb{B} = \mathbb{M}_\infty - \mathbb{M}_\infty$, regarded as the space of projections in $M_2(\mathbb{C})$ with determinant zero and trace one, can be identified with its action via $\text{PGL}_2(\mathbb{C})$ on the space of projections with eigenvector $e = [z : 1] \in \mathbb{P}_1(\mathbb{C})$, defining a Hopf bundle at time-like infinity. This is reminiscent of (the other kind of Hopf) bifurcation.

§ 3 A sandbox for entanglement

3.1 The Brauer-Wall/Maycock group

$$0 \rightarrow H^3(Z, Z) \rightarrow (\text{BW} \cong \text{MC})(Z) \rightarrow H^1(Z, \mathbb{Z}_2) \rightarrow 0$$

(with composition $(b, s) + (b', s') := (b + \beta(s \cdot s') + b', s + s')$, [39] (Prop 2.5), $\beta$ being the mod two Bockstein; represented by a truncation of the loopspace $\Omega^\infty k\mathbb{O}$) classifies Morita equivalence classes of fields of graded continuous trace class $C^*$ algebras over a CW-space $Z$.

Contractibility of the group of invertible Hilbert space operators implies that bundles $H^1(Z, \text{PGL}_2(\mathbb{H}))$ of projective Hilbert spaces over $Z$ – equivalently, locally coherent fields of quantum mechanical state spaces – are classified by elements of

$$H^3(Z, Z) \cong H^2(Z, B\mathbb{Z} \simeq \mathbb{T}) \cong H^1(Z, B\mathbb{T} \simeq \text{Gl}_2(\mathbb{H})/\mathbb{C}^\times).$$

Small $H$-spaces $H(V, 1) \rtimes_q H(\mathbb{Z}_2, 3)$ generalizing MC can be associated naturally to symmetric bilinear forms $q : V \times V \rightarrow \mathbb{Z}_2$ in characteristic two; cf. § 6.

For the purposes of this note, a Haag-Kastler background $[\mathcal{A}]$ on a connected locally compact space $Z$ will be the projective equivalence class of a bundle of complex Hilbert spaces trivialized at infinity on its one-point compactification $Z_+$, as a toy model for quantum mechanics. Compactly supported cohomology groups $H^*_c(Z) := H^*(Z_+, +)$ (i.e. defined by the one-point compactifications of the components of $Z$) are useful in this context; the resulting functors are natural with respect to proper, but not general, homotopy equivalence.

**Proposition** A connected oriented three-manifold $Y$ has a canonical Haag-Kastler background $[\mathcal{A}_Y]$ of $C^*$ algebras defined by its orientation or volume form $[\omega_Y] \in H^3(Y, \mathbb{Z})$.

The light-cone $\mathbb{M}_0$, for example, is contractible, but its two ends imply a serious amount of compactly supported cohomology:

$$H^*_c(\mathbb{M}_0, Z) \cong \mathbb{Z} \text{ if } *=1, \cong \mathbb{Z}^2 \text{ if } *=3$$

and is otherwise zero; and, similarly, by §1.4, for $\mathbb{M}_\infty$. A chiral structure on the light-cone [28] is defined by a choice of the isomorphism in degree three;
it is not clear to me that the two ends need necessarily to be glued by the identity map. Collapsing $M^\infty = M^\infty \cup B \to M^\infty+$ sends $H^3_c(M^\infty) \cong \mathbb{Z}^2 \to \mathbb{Z} \cong H^3_c(M^\infty)$. The decomposition $M^\infty = M^\infty - B$, together with the long exact sequence

$$\cdots \to H^*_c(X - Z, Z) \to H^*_c(X, Z) \to H^*_c(Z, Z) \to \cdots$$

for a closed subspace $Z \subset X$ then implies an exact sequence

$$0 \to H^2_c(B, Z) \cong \mathbb{Z} \to BW(M^\infty) \cong \mathbb{Z}^2 \to BW(M^\infty) \cong \mathbb{Z}_2 \times \mathbb{Z} \to H^2_c(B, \mathbb{Z}_2) \cong \mathbb{Z}_2 \to 0 .$$

The restriction of $A$ to $M$ is trivial since $BW(M^+) = 0$, but an algebra bundle of class $[A]$ over $U(2)$ nevertheless defines at least a precursor for a Haag-Kastler structure: it provides a sheaf of $C^*$-algebras and quantum-mechanical state spaces, though without any concerns about local causality. This is an issue of possible interest in questions of entanglement.

**Corollary** There is a canonical nontrivial equivalence class $[A]$ of bundles of $\mathbb{Z}_2$-graded $C^*$-algebras over $U(2)$, classified by

$$(-1, +1) \in \mathbb{Z}_2 \times \mathbb{Z} \cong H^1(U(2), \mathbb{Z}_2) \times H^3(U(2), \mathbb{Z}) \cong BW(U(2)) .$$

This bundle is supported on $M^\infty$, in the sense that the restriction map $BW(U(2)) \to BW(M^\infty)$ is an isomorphism.

The final arrow in the exact sequence above similarly suggests that the spin part of the structure is supported on the bubble $B$. The Bockstein homomorphisms for both spaces are trivial.

### 3.2 Some questions: This document is a working draft; it is intended to provide a framework for questions like the following:

- Is there an analytic construction for (a bundle of class) $[A]$?
- Does the class $[A]$ contain a smooth representative?
- Can the action of SU(2) on $U(2)$ be extended to some algebra bundle representing $[A]$?

[More precisely: can $[A]$ be realized as the bundle of automorphisms of a field of (projective) Hilbert space representations of SU(2) over $U(2)$? If so, could these be related to (projective) representations of $Sl_2(\mathbb{C})$?]

The Bondi-Metzner-Sachs group $[22]$ is a semi-direct product

$$0 \to V \to BMS \to Sl_2(\mathbb{C}) \to 1 ,$$
where $V$ is a vector space of real-valued functions on $\mathbb{CP}^1$ with the induced $\text{PGL}_2(\mathbb{C})$ action; it is the symmetry group of a generic asymptotically-flat solution of the equations of general relativity. It is tempting to imagine $V$ as the group of smooth functions on $\mathbb{R}$, interpreted as conformal deformations of its metric.

• [25] How is a principal bundle $\text{PGL}^* (A) \to \mathbb{U}(2)$ related to $\mathbb{T} \times_{\pm 1} S^3(3)$?

PART II AN OCEAN OF THREE-MANIFOLDS

...Nehwon is a giant bubble rising through the waters of eternity with continents, islands, and the great jewels that at night are the stars all orderly afloat on the bubble’s inner surface . . .

F Leiber, Swords of Lankhmar

4.1 Following N Strickland [29] (§12-13) and GA Swarup [30, 31], the category $(\text{III})$, with compact connected closed base-pointed oriented three-manifolds $Y$ as objects, and with degree one maps as morphisms, maps fully faithfully by $Y \mapsto \pi_1 Y$ to the category of groups $\pi$ endowed with the three-dimensional level $\text{H}_3$ structure $\text{H}_3 Y \to \text{H}_3 B\pi_1(Y)$, and with homomorphisms of such oriented groups as morphisms. The three-sphere $S^3 = \text{SU}(2)$ is a distinguished point of this generalized stack, as is $S^1 \times S^2$, but the generic example of a prime object under connected sum is an acyclic three-manifold with fundamental group satisfying three-dimensional Poincaré duality. There is also an archipelago of manifolds such as Lens spaces, which have finite fundamental groups.

For example, the collapse map $S^1 \times S^2 \to S^1 \wedge S^2 \cong S^3$ has degree one. It changes the Kervaire semicharacteristic mod two [10].

4.2 On another hand, the Morita equivalence classes $\text{MC}(Y)$ define a sheaf of abelian groups on $(\text{III})$, and the Grothendieck category $$(\text{HK}) := \int_{Y \in (\text{III})} \text{MC}(Y)$$ of compact three-manifolds, together with the $C^*$ algebra indexed by their orientations $[Y] \in H^3(Y, \mathbb{Z})$, defines an interesting class of background geometries for Araki-Haag-Kastler models. Bundle gerbes [17] and Deligne cohomology provide smooth versions of these things, in terms of connections and curvature.

\[ \]
An element \((b, s) \in \text{MC}(Y)\) defines the class \(s \in H^1(Y, \mathbb{Z}_2)\) of a spin or fermionic structure, together with a class \(b \in H^3(Y, \mathbb{Z})\) which could perhaps be called a boson or baryon number, but that may be misleading. From here on we’ll restrict our attention to the cross-section \((\text{HK})_1\) of the category of Haag-Kastler models defined by normalizing at \(b = 1\).

5.1 More generally, let us consider the category \((\text{IV})\) with pairs \((Y \cong \partial X \subset X)\) as objects, with \(X\) a connected oriented smooth four-manifold bounded by \(Y \in (\text{III})\), and smooth maps of pairs with boundary restrictions of degree one, as morphisms. Forgetting the spanning manifold defines a fibration

\[
\partial : \int_{X \in (\text{IV})} \text{MC}(X) \to \int_{Y \in (\text{III})} \text{MC}(Y)
\]

of some kind of categories.

If \(X\) is simply-connected, the homology exact sequence of \((X, Y)\) reduces (using the universal coefficient theorem and Lefschetz duality as in Hatcher \([\S 3.3]\)) to a free three-term resolution

\[
0 \to H_2Y \to H_2X \to H_2X/Y \to H_1Y \to 0
\]

of \(H_1Y\) (coefficients are integral if unspecified), and thereby a contravariant class

\[
Q_{Y:X} : \in \text{Ext}^2_{\mathbb{Z}[\pi]}(\pi_{\text{ab}}, \pi_{\text{ab}})\]

\([11]\)(§5.3.13f). Here \(A^\dagger := \text{Hom}(A, \mathbb{Z})\) for finitely generated abelian groups, and \(\pi = \pi_1Y\). The diagram

\[
\text{Hom}(H_2X/Y, \mathbb{Z}) \to \text{Hom}(H_2X, \mathbb{Z}) \\
\text{Hom}(H_2X/Y, \mathbb{Z}) \to \text{Hom}(H_2X, \mathbb{Z}) \\
H_2X/Y \to H_2X \\
H_2X/Y \to H_2X \\
H_1Y \to 0
\]

identifies the unimodular intersection form \(Q := Q_{X/Y}\) on \(H_2X\) with that defined by the cup product on \(H_2X/Y\), yielding a presentation

\[
H_1Y = \pi_{\text{ab}} \cong \text{coker} Q, \quad H_2Y = \pi_{\text{ab}}/\text{tors} \cong \text{ker} Q
\]

of \(H_4Y\) in terms of a quadratic form.

5.2 Link calculus \([11, 18, 21, 27]\)(Ch 9 §I) presents any \(Y \in (\text{III})\) as the boundary \(Y \cong Y_L\) of a simply-connected four-dimensional handlebody \(X_L\) defined by a framed oriented link

\[
L = \bigcup_{\lambda \in \pi_0L} \lambda \subset \mathbb{R}^3_+,
\]
together with an identification of the intersection matrix of $X_L$ and the $\pi_0 L \times \pi_0 L$ linking matrix of $L$.

It is helpful to know that the Stiefel-Whitney map

$$\text{Pic}_\otimes \otimes \mathbb{R}(Z) \ni \xi \mapsto w_1(\xi) \in H^1(Z, \mathbb{Z}_2)$$

classifies real line bundles, while Chern’s map

$$\text{Pic}_\otimes \otimes \mathbb{C}(Z) \ni \lambda \mapsto c_1(\lambda) \in H^2(Z, \mathbb{Z})$$
classifies complex line bundles. In a link presentation, equivalence classes $\lambda \in H_2 X_L \cong H^2(X/Y)_L \cong \text{Pic}_\otimes \otimes \mathbb{C}(X/Y)_L \cong \mathbb{Z}[\pi_0 L] := \Lambda \cong \mathbb{Z}^l$ correspond to line bundles $\lambda$ over $X$ trivialized on $Y$, or to the surfaces $[\sigma^{-1}(0)] \sim \delta \lambda \in H_2(X_L)$ defined by the Euler class of a generic section $\sigma$.

With $\mathbb{Z}_2$ coefficients, and in cohomology $H$ for convenience, the exact sequence of §3 becomes a symmetric biextension

$$
\begin{array}{cccccc}
0 & \longrightarrow & H^1 X & \delta & \longrightarrow & [H^2 X/Y \sim H^2 X] & \longrightarrow & H^2 Y & \longrightarrow & 0 \\
& & \cong & & \cong & & \cong & & \cong & \\
0 & \longrightarrow & \text{Pic}_\otimes \otimes \mathbb{R}(Y) & \longrightarrow & [\text{Pic}_\otimes \otimes \mathbb{C}(X/Y) \otimes \mathbb{Z}_2, q_{X/Y}] & \longrightarrow & \text{Pic}_\otimes \otimes \mathbb{R}(Y) \to \longrightarrow & 0
\end{array}
$$

of $\mathbb{Z}_2$-vector spaces (with $\to$ denoting vector space duality and $q := Q \otimes \mathbb{Z}_2$). The left-hand monomorphism sends a real vector bundle $\xi$ on the boundary $Y$ to a complex line bundle $\delta \xi$ on $X$; we may perhaps interpret it as bosonic $\mathbb{C}^\times$-gauge field on the interior created or supported by a fermionic field on the boundary:

A generic section $\sigma$ of a real line bundle over $Y$ defines a codimension one submanifold $\sigma^{-1}(0)$, whereas such a section of a complex line bundle over $X$ defines (mod two) a submanifold of codimension two [13](lemma 5.49) but these submanifolds are both surfaces, making it geometrically natural to think of a class in $H^2(X, \mathbb{Z}) \otimes \mathbb{Z}_2$ as extending a class in $H^1(\partial X, \mathbb{Z}_2)$ when its associated complex field turns on.

6.1 This leads to matters of spin and statistics, which suggests a pullback

$$
\begin{array}{ccc}
(GR) & \longrightarrow & (HK) \\
\downarrow & & \downarrow \\
(IV)_{\text{Spin}^c} & \longrightarrow & (III)_{\text{Spin}}
\end{array}
$$

of our fibered category. The geometry of link calculus on Spin and Spin$^c$ manifolds is rich enough to support (renormalizable [1] quantum) variational problems of Seiberg-Witten, Higgs-Yamabe [42] and Salam-Weinberg.
type; the latter model involves mysterious $\mathbb{T}$-valued ‘mixing angles’ which parametrize interactions between fermions and their gauge bosons.

This may be commensurable with Penrose’s memorable fancy, that at future infinity fermions decay into bosons, powering a new big bang. A cobordism $Y = \partial X$ can be regarded as a creation operator $X : \emptyset \to Y$ which thinks of the four-manifold $X$ as a bubble blown by its boundary $Y$, a solution extremizing a functional on a moduli space of membranes spanning a given boundary.

If $Y$ is $S^3$ then $X$ is a 4-ball with $X/Y = S^4$, and when $Y = S^1 \times S^2$ we have $X = S^1 \times B^3$, $X/Y \cong U(2)$, which recovers Penrose’s model. [In that case $X$ is not simply connected, but can be made so by allowing a codimension two singularity in $Y \sim S^1 \times \ast S^2$, cf §1.6.]

6.2 This marks a place for a discussion of spin links which we defer to a later draft. The following needs expansion and details:

In a link presentation $X_L$, a generic section of complex line bundle $\lambda$ defines the homology class $\delta_L$ of a (for example ‘weak neutral’) de Rham current, normal to its vanishing locus $\sigma^{-1}(0)$: a Dirac delta-function supported by the link, a model for a thunderbolt or crack of doom in the big bang.

Kirby and Taylor use the bilinear form $x, y \mapsto \langle x, 2y \rangle$ on $\text{Pic}_{\mathbb{R}} Y$ to show that the $\xi$-twisted Rokhlin (Theorem VI) invariant

$$\nu(\xi^* Y) \equiv \nu(Y) + 2\beta(\ast \xi) \pmod{16}$$

of a spin three-manifold is translated by a multiple of the EH Brown invariant $\nu(Y)$ (§4.2, 5.4), (§3.2, 4.11) of the surface Poincaré dual to $\xi$. Hopkins and Singer (App. E) study such refinements of the intersection matrix in terms of integral Wu classes; we hope to understand this better, in time.

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