ON THE AMPLENESS
OF THE COTANGENT BUNDLES
OF COMPLETE INTERSECTIONS

SONG-YAN XIE

ABSTRACT. Based on a geometric interpretation of Brotbek’s symmetric differential forms, for the intersection family \( X \) of generalized Fermat-type hypersurfaces in \( \mathbb{P}_K^N \) defined over any field \( K \), we construct/reconstruct explicit symmetric differential forms by applying Cramer’s rule, skipping cohomology arguments, and we further exhibit unveiled families of lower degree symmetric differential forms on all possible intersections of \( X \) with coordinate hyperplanes.

Thereafter, we develop what we call the ‘moving coefficients method’ to prove a conjecture made by Olivier Debarre: for generic \( c \geq N/2 \) hypersurfaces \( H_1, \ldots, H_c \subset \mathbb{P}_C^N \) of degrees \( d_1, \ldots, d_c \) sufficiently large, the intersection \( X := H_1 \cap \cdots \cap H_c \) has ample cotangent bundle \( \Omega_X \), and concerning effectiveness, the lower bound \( d_1, \ldots, d_c \geq N^{N'} \) works.

Lastly, thanks to known results about the Fujita Conjecture, we establish the very-amenality of 
\( \text{Sym}^\kappa \Omega_X \) for all \( \kappa \geq 64 \left( \sum_{i=1}^c d_i \right)^2 \).

1. Introduction

In 2005, Debarre established that, in a complex abelian variety of dimension \( N \), for \( c \geq N/2 \) sufficiently ample generic hypersurfaces \( H_1, \ldots, H_c \), their intersection \( X := H_1 \cap \cdots \cap H_c \) has ample cotangent bundle \( \Omega_X \), thereby answering a question of Lazarsfeld (cf. [18]). Then naturally, by thoughtful analogies between geometry of Abelian varieties and geometry of projective spaces, Debarre proposed the following conjecture in Section 3 of [18], extending in fact an older question raised by Schneider [58] in the surface case:

**Conjecture 1.1. [Debarre Ampleness Conjecture]** For all integers \( N \geq 2 \), for every integer \( N/2 \leq c < N \), there exists a positive lower bound:

\[ d \gg 1 \]

such that, for all positive integers:

\[ d_1, \ldots, d_c \geq d, \]

for generic choices of \( c \) hypersurfaces:

\[ H_i \subset \mathbb{P}_C^N \quad (i = 1 \cdots c) \]

with degrees:

\[ \deg H_i = d_i, \]

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the intersection:
\[ X := H_1 \cap \cdots \cap H_c \]
has ample cotangent bundle \( \Omega_X \).

Precisely, according to a ground conceptualization due to Hartshorne [36], the expected ampleness is that, for all large degrees \( k \geq k_0 \gg 1 \), the global symmetric \( k \)-differentials on \( X \):
\[
\Gamma(X, \text{Sym}^k \Omega_X)
\]
are so abundant and diverse, that firstly, at every point \( x \in X \), the first-order jet evaluation map:
\[
\Gamma(X, \text{Sym}^k \Omega_X) \twoheadrightarrow \text{Jet}_1 \text{Sym}^k \Omega_X |_x
\]
is surjective, where for every vector bundle \( E \to X \) the first-order jet of \( E \) at \( x \) is defined by:
\[
\text{Jet}_1 E |_x := \mathcal{O}_x (E) / (m_x)^2 \mathcal{O}_x (E),
\]
and that secondly, at every pair of distinct points \( x_1 \neq x_2 \) in \( X \), the simultaneous evaluation map:
\[
\Gamma(X, \text{Sym}^k \Omega_X) \twoheadrightarrow \text{Sym}^k \Omega_X |_{x_1} \oplus \text{Sym}^k \Omega_X |_{x_2}
\]
is also surjective.

The hypothesis:
\[
c \geq n
\]
appears optimal, for otherwise when \( c < n \), there are no nonzero global sections for all degrees \( k \geq 1 \):
\[
\Gamma(X, \text{Sym}^k \Omega_X) = 0,
\]
according to Brückmann-Rackwitz [8] and Schneider [58], whereas, in the threshold case \( c = n \), nonzero global sections are known to exist.

As highlighted in [18], projective varieties \( X \) having ample cotangent bundles enjoy several fascinating properties, for instance the following ones.

- All subvarieties of \( X \) are all of general type.
- There are finitely many nonconstant rational maps from any fixed projective variety to \( X \) ([51]).
- If \( X \) is defined over \( \mathbb{C} \), then \( X \) is Kobayashi-hyperbolic, i.e. every holomorphic map \( \mathbb{C} \to X \) must be constant ([20, p. 16, Proposition 3.1], [27, p. 52, Proposition 4.2.1]).
- If \( X \) is defined over a number field \( K \), the set of \( K \)-rational points of \( X \) is expected to be finite (Lang’s conjecture, cf. [38], [48]).

Since ampleness of cotangent bundles potentially bridges Analytic Geometry and Arithmetic Geometry in a deep way, it is interesting to ask examples of such projective varieties. In one-dimensional case, they are in fact our familiar Riemann surfaces/algebraic curves with genus \( \geq 2 \). However, in higher dimensional case, not many examples were known, even though they were expected to be reasonable abundant.

In this aspect, we would like to mention the following nice construction of Bogomolov, which is written down in the last section of [18]. If \( X_1, \ldots, X_\ell \) are smooth complex projective varieties having positive dimensions:
\[
\dim X_i \geq d \geq 1 \quad (i = 1 \cdots \ell),
\]
all of whose Serre line bundles \( \mathcal{O}_{\mathbb{P}(T_X)}(1) \to \mathbb{P}(T_X) \) enjoy bigness:

\[
\dim \Gamma(\mathbb{P}(T_X), \mathcal{O}_{\mathbb{P}(T_X)}(k)) = \dim \Gamma(X, \text{Sym}^k\Omega_X) \geq \frac{\text{constant} \cdot k^{2\dim X - 1}}{k \to \infty}.
\]

then a generic complete intersection:

\[
Y \subset X_1 \times \cdots \times X_\ell
\]

having dimension:

\[
\dim Y \leq \frac{d(\ell + 1) + 1}{2(d + 1)}
\]

has ample cotangent bundle \( \Omega_Y \).

In his Ph.D. thesis under the direction of Mourougane, Brotbek [5] reached an elegant proof of the Debarre Ampleness Conjecture in dimension \( n = 2 \), in all codimensions \( c \geq 2 \), for generic complete intersections \( X^2 \subset \mathbb{P}^{2+c}(\mathbb{C}) \) having degrees:

\[
d_1, \ldots, d_c \geq \frac{8(n + c) + 2}{n + c - 1},
\]

by extending the techniques of Siu [62, 63, 64], Demailly [20, 23, 22], Rousseau [56], Păun [52, 53], Merker [43], Diverio-Merker-Rousseau [26], Mourougane [50], and by employing the concept of \emph{ampleness modulo a subvariety} introduced by Miyaoka in [47]. Also, for smooth complete intersections \( X^n \subset \mathbb{P}^{n+c}(\mathbb{C}) \) with \( c \geq n \geq 2 \), Brotbek showed using holomorphic Morse inequalities that when:

\[
d_1, \ldots, d_c \geq \left[ \frac{2^{n-1}(2n-2) \frac{n^2}{n + c + 1} \left( \frac{2n-1}{n} \right) + 1}{\left( \frac{2n-1}{n} \right)^{n+c+1} (2n+c)! (c-n)!} \right],
\]

bigness of the Serre line bundle \( \mathcal{O}_{\mathbb{P}(T_X)}(1) \to \mathbb{P}(T_X) \) holds:

\[
\dim \Gamma(\mathbb{P}(T_X), \mathcal{O}_{\mathbb{P}(T_X)}(k)) = \dim \Gamma(X, \text{Sym}^k\Omega_X) \geq \frac{1}{2\chi_{\text{Euler}}(X, \text{Sym}^k\Omega_X)} \geq \frac{\text{constant} \cdot k^{2n-1}}{k \to \infty}
\]

whereas a desirable control of the base locus of the inexplicitly given nonzero holomorphic sections seems impossible by means of currently available techniques.

To find an alternative approach, a key breakthrough happened in 2014, when Brotbek [7] obtained explicit global symmetric differential forms in coordinates by an intensive cohomological approach. More specifically, under the assumption that the ambient field \( \mathbb{K} \) has characteristic zero, using exact sequences and the snake lemma, Brotbek firstly provided a key series of long injective cohomology sequences, whose left initial ends consist of the most general global twisted symmetric differential forms, and whose right target ends consist of huge dimensional linear spaces well understood. Secondly, Brotbek proved that the image of each left end, going through the full injections sequence, is exactly the kernel of a certain linear system at the right end. Thirdly, by focusing on pure Fermat-type hypersurface equations ([7, p. 26]):

\[
F_j = \sum_{i=0}^N s_i^j Z_i^e \quad (j = 1 \cdots c),
\]

with integers \( c \geq N/2, e \geq 1 \), where \( s_i^j \) are some homogeneous polynomials of the same degree \( e \geq 0 \), Brotbek step-by-step traced back some kernel elements from each right end all the way to
the left end, every middle step being an application of Cramer’s rule, and hence he constructed global twisted symmetric differential forms with neat determinantal structures ([7, p. 27–31]).

Thereafter, by employing the standard method of counting base-locus-dimension in two ways in algebraic geometry (see e.g. Lemma 8.15 below), Brotbek established that the Debarre Ampleness Conjecture holds when:

\[ 4c \geq 3N - 2, \]

for equal degrees:

\[ d_1 = \cdots = d_c \geq 2N + 3, \quad (2) \]

the constructions being flexible enough to embrace ‘approximately equal degrees’, in the same sense as Theorem 5.2 below.

Inspired much by Brotbek’s works, we propose the following answer to the Debarre Ampleness Conjecture.

**Theorem 1.2.** The cotangent bundle of the intersection in \( \mathbb{P}^N_C \) of at least \( N/2 \) generic hypersurfaces with degrees \( \geq N^{N^2} \) is ample.

In fact, we will prove the following main theorem, which coincides with the above theorem for \( r = 0 \) and \( K = \mathbb{C} \), and whose effective bound \( d_0 = N^{N^2} \) will be obtained in Theorem 11.2.

**Theorem 1.3 (Ampleness).** Over any field \( K \) which is not finite, for all positive integers \( N \geq 1 \), for any nonnegative integers \( c, r \geq 0 \) with:

\[ 2c + r \geq N, \]

there exists a lower bound \( d_0 \gg 1 \) such that, for all positive integers:

\[ d_1, \ldots, d_c, d_{c+1}, \ldots, d_{c+r} \geq d_0, \]

for generic \( c + r \) hypersurfaces:

\[ H_i \subset \mathbb{P}^N_K \quad (i = 1, \ldots, c + r) \]

with degrees:

\[ \deg H_i = d_i, \]

the cotangent bundle \( \Omega_V \) of the intersection of the first \( c \) hypersurfaces:

\[ V := H_1 \cap \cdots \cap H_c \]

restricted to the intersection of all the \( c + r \) hypersurfaces:

\[ X := H_1 \cap \cdots \cap H_c \cap H_{c+1} \cap \cdots \cap H_{c+r} \]

is ample.

First of all, remembering that ampleness (or not) is preserved under any base change obtained by ambient field extension, one only needs to prove the Ampleness Theorem 1.3 for algebraically closed fields \( K \).

Of course, we would like to have \( d_0 = d_0(N, c, r) \) as small as possible, yet the optimal one is at present far beyond our reach, and we can only get exponential ones like:

\[ d_0 = N^{N^2} \quad (\text{Theorem 11.2}), \]

which confirms the large degree phenomena in Kobayashi hyperbolicity related problems ([26, 2, 22, 16, 64]). When \( 2(2c + r) \geq 3N - 2 \), we obtain linear bounds for equal degrees:

\[ d_1 = \cdots = d_{c+r} \geq 2N + 3, \]
hence we recover the lower bounds (2) in the case \( r = 0 \), and we also obtain quadratic bounds for all large degrees:

\[
d_1, \ldots, d_{c+r} \geq (3N + 2)(3N + 3).
\]

Better estimates of the lower bound \( d_0 \) will be explained in Section 12.

Concerning the proof, primarily, as anticipated/emphasized by Brotbek and Merker ([7, 44]), it is essentially based on constructing sufficiently many global negatively twisted symmetric differential forms, and then inevitably, one has to struggle with the overwhelming difficulty of clearing out their base loci, which seems, at the best of our knowledge, to be an incredible mission.

In order to bypass the complexity in these two aspects, the following seven ingredients are indispensable in our approach:

1. generalized Brotbek’s symmetric differential forms (Subsection 6.10);
2. global moving coefficients method (MCM) (Subsection 7.2);
3. ‘hidden’ symmetric forms on intersections with coordinate hyperplanes (Subsection 6.4);
4. MCM on intersections with coordinate hyperplanes (Subsection 7.3);
5. Algorithm of MCM (Subsection 7.1);
6. Core Lemma of MCM (Section 10);
7. product coup (Subsection 5.3).

In fact, 1 is based on a geometric interpretation of Brotbek’s symmetric differential forms ([7, Lemma 4.5]), and has the advantage of producing symmetric differential forms by directly copying hypersurface equations and their differentials. Facilitated by 2, which is of certain combinatorial interest, 1 amazingly cooks a series of global negatively twisted symmetric differential forms, which are of nice uniform structures. However, unfortunately, one still has the difficulty that all these obtained global symmetric forms happen to coincide with each other on the intersections with any two coordinate hyperplanes, so that their base locus stably keeps positive (large) dimension, which is an annoying obstacle to ampleness.

Then, to overcome this difficulty enters 3, which is arguably the most critical ingredient in harmony with MCM, and whose importance is much greater than its appearance as somehow a corollary of 1. Thus, to compensate the defect of 1-2, it is natural to design 4 which completes the framework of MCM. And then, 5 is smooth to be devised, and it provides suitable hypersurface equations for MCM. Now, the last obstacle to ampleness is about narrowing the base loci, an ultimate difficulty solved by 6. Thus, the Debarre Conjecture is settled in the central cases of almost equal degrees. Finally, the magical coup 7 thereby embraces all large degrees for the Debarre Conjecture, and naturally shapes the formulation of the Ampleness Theorem.

Lastly, taking account of known results about the Fujita Conjecture in Complex Geometry (cf. survey [21]), we will prove in Section 13 the following

**Theorem 1.4 (Effective Very Ampleness).** Under the same assumption and notation as in the Ampleness Theorem 1.3, if in addition the ambient field \( K \) has characteristic zero, then for generic choices of \( H_1, \ldots, H_{c+r} \), the restricted cotangent bundle \( \text{Sym}^\kappa \Omega^1_V|_X \) is very ample on \( X \), for every \( \kappa \geq \kappa_0 \), with the uniform lower bound:

\[
\kappa_0 = 16 \left( \sum_{i=1}^c d_i + \sum_{i=1}^{c+r} d_i \right)^2.
\]
In the end, we would like to propose the following

**Conjecture 1.5.** (i) Over an algebraically closed field \( \mathbb{K} \), for any smooth projective \( \mathbb{K} \)-variety \( P \) with dimension \( N \), for any integers \( c, r \geq 0 \) with \( 2c + r \geq N \), for any very ample line bundles \( L_1, \ldots, L_{c+r} \) on \( P \), there exists a lower bound:

\[
d_0 = d_0(P, \mathcal{L}) \gg 1
\]

such that, for all positive integers:

\[
d_1, \ldots, d_c, d_{c+1}, \ldots, d_{c+r} \geq d_0,
\]

for generic choices of \( c + r \) hypersurfaces:

\[
H_i \subset P \quad (i = 1 \cdots c+r)
\]

defined by global sections:

\[
F_i \in H^0(P, \mathcal{L}_i^{\otimes d_i})
\]

the cotangent bundle \( \Omega_V \) of the intersection of the first \( c \) hypersurfaces:

\[
V := H_1 \cap \cdots \cap H_c
\]

restricted to the intersection of all the \( c + r \) hypersurfaces:

\[
X := H_1 \cap \cdots \cap H_c \cap H_{c+1} \cap \cdots \cap H_{c+r}
\]

is ample.

(ii) There exists a uniform lower bound:

\[
d_0 = d_0(P) \gg 1
\]

independent of the chosen very ample line bundles \( \mathcal{L} \).

(iii) There exists a uniform lower bound:

\[
kappa_0 = \kappa_0(P) \gg 1
\]

independent of \( d_1, \ldots, d_{c+r} \), such that for generic choices of \( H_1, \ldots, H_{c+r} \), the restricted cotangent bundle \( \text{Sym}^k \Omega_V|_X \) is very ample on \( X \), for every \( k \geq \kappa_0 \).

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2. Preliminaries and Restatements of the Ampleness Theorem 1.3

2.1. Two families of hypersurface intersections in \( \mathbb{P}^N_\mathbb{K} \). Fix an arbitrary algebraically closed field \( \mathbb{K} \). Now, we introduce the fundamental object of this paper: the intersection family of \( c + r \) hypersurfaces with degrees \( d_1, \ldots, d_{c+r} \geq 1 \) in the \( \mathbb{K} \)-projective space \( \mathbb{P}^N_\mathbb{K} \) of dimension \( N \), equipped with homogeneous coordinates \( [z_0 : z_1 : \cdots : z_N] \).

Recalling that the projective parameter space of degree \( d \geq 1 \) hypersurfaces in \( \mathbb{P}^N_\mathbb{K} \) is:

\[
\mathbb{P} \left( H^0(\mathbb{P}^N_\mathbb{K}, \mathcal{O}_{\mathbb{P}^N_\mathbb{K}}(d)) \right) = \mathbb{P} \left\{ \sum_{|\alpha| = d} A_\alpha z^\alpha : A_\alpha \in \mathbb{K} \right\},
\]

we may denote by:

\[
\mathbb{P} \left( \bigoplus_{i=1}^{c+r} H^0(\mathbb{P}^N_\mathbb{K}, \mathcal{O}_{\mathbb{P}^N_\mathbb{K}}(d_i)) \right) = \mathbb{P} \left\{ \bigoplus_{i=1}^{c+r} \sum_{|\alpha| = d_i} A_\alpha^i z^\alpha : A_\alpha^i \in \mathbb{K} \right\}
\]

the projective parameter space of \( c + r \) hypersurfaces with degrees \( d_1, \ldots, d_{c+r} \). This \( \mathbb{K} \)-projective space has dimension:

\[
\phi := \sum_{i=1}^{c+r} \left( N + d_i \right) - 1,
\]

hence we write it as:

\[
\mathbb{P}^\phi_\mathbb{K} = \text{Proj} \, \mathbb{K} \left\[ \left\{ A_\alpha^i \right\}_{|\alpha| = d_i} \right\],
\]

where, as shown above, \( A_\alpha^i \) are the homogeneous coordinates indexed by the serial number \( i \) of each hypersurface and by all multi-indices \( \alpha \) with the weight \( |\alpha| = d_i \) associated to the degree \( d_i \) monomials \( z^\alpha \in \mathbb{K}[z_0, \ldots, z_N] \).

Now, we introduce the two subschemes:

\[ \mathcal{X} \subset \mathcal{Y} \subset \mathbb{P}^\phi_\mathbb{K} \times_\mathbb{K} \mathbb{P}^N_\mathbb{K}, \]

where \( \mathcal{X} \) is defined by ‘all’ the \( c + r \) bihomogeneous polynomials:

\[
\mathcal{X} := \mathbb{V} \left( \sum_{|\alpha| = d_1} A_\alpha^1 z^\alpha, \ldots, \sum_{|\alpha| = d_c} A_\alpha^c z^\alpha, \sum_{|\alpha| = d_{c+1}} A_\alpha^{c+1} z^\alpha, \ldots, \sum_{|\alpha| = d_{c+r}} A_\alpha^{c+r} z^\alpha \right),
\]

and where \( \mathcal{Y} \) is defined by the ‘first’ \( c \) bihomogeneous polynomials:

\[
\mathcal{Y} := \mathbb{V} \left( \sum_{|\alpha| = d_1} A_\alpha^1 z^\alpha, \ldots, \sum_{|\alpha| = d_c} A_\alpha^c z^\alpha \right).
\]

Then we view \( \mathcal{X}, \mathcal{Y} \subset \mathbb{P}^\phi_\mathbb{K} \times_\mathbb{K} \mathbb{P}^N_\mathbb{K} \) as two families of closed subschemes of \( \mathbb{P}^N_\mathbb{K} \) parametrized by the projective parameter space \( \mathbb{P}^\phi_\mathbb{K} \).

2.2. The relative cotangent sheaves family of \( \mathcal{Y} \). A comprehensive reference on sheaves of relative differentials is [42, Section 6.1.2].

Let \( \text{pr}_1, \text{pr}_2 \) be the two canonical projections:

\[
\begin{array}{c}
\mathbb{P}^\phi_\mathbb{K} \\
\downarrow \text{pr}_1 \\
\mathbb{P}^\phi_\mathbb{K} \times_\mathbb{K} \mathbb{P}^N_\mathbb{K} \\
\downarrow \text{pr}_2 \\
\mathbb{P}^N_\mathbb{K}
\end{array}
\]
Then, by composing with the subscheme inclusion:

\[ i : \mathcal{V} \hookrightarrow \mathbb{P}^0_\mathbb{K} \times_{\mathbb{K}} \mathbb{P}^N_\mathbb{K}, \]

we receive a morphism:

\[ \text{pr}_1 \circ i : \mathcal{V} \longrightarrow \mathbb{P}^0_\mathbb{K}, \]

together with a sheaf \( \Omega^1_{\mathcal{V}/\mathbb{P}^0_\mathbb{K}} \) of relative differentials of degree 1 of \( \mathcal{V} \) over \( \mathbb{P}^0_\mathbb{K} \).

Since \( \text{pr}_1 \) is of finite type and \( \mathbb{P}^0_\mathbb{K} \) is noetherian, a standard theorem ([42, p. 216, Proposition 1.20]) shows that the sheaf \( \Omega^1_{\mathcal{V}/\mathbb{P}^0_\mathbb{K}} \) is coherent.

We may view \( \Omega^1_{\mathcal{V}/\mathbb{P}^0_\mathbb{K}} \) as the family of the cotangent bundles for the intersection family \( \mathcal{V} \), since the coherent sheaf \( \Omega^1_{\mathcal{V}/\mathbb{P}^0_\mathbb{K}} \) is indeed locally free on the Zariski open set that consists of smooth complete intersections.

2.3. The projectivizations and the Serre line bundles. We refer the reader to [37, pp. 160-162] for the considerations in this subsection.

Starting with the noetherian scheme \( \mathcal{V} \) and the coherent degree 1 relative differential sheaf \( \Omega^1_{\mathcal{V}/\mathbb{P}^0_\mathbb{K}} \) on it, we consider the sheaf of relative \( \partial_{\mathcal{V}} \)-symmetric differential algebras:

\[ \text{Sym}^* \Omega^1_{\mathcal{V}/\mathbb{P}^0_\mathbb{K}} := \bigoplus_{i \geq 0} \text{Sym}^i \Omega^1_{\mathcal{V}/\mathbb{P}^0_\mathbb{K}}. \]

According to the construction of [37, p. 160], noting that this sheaf has a natural structure of graded \( \partial_{\mathcal{V}} \)-algebras, and moreover that it satisfies the condition (\dagger) there, we receive the projectivization of \( \Omega^1_{\mathcal{V}/\mathbb{P}^0_\mathbb{K}} \):

\[ \mathbb{P}(\Omega^1_{\mathcal{V}/\mathbb{P}^0_\mathbb{K}}) := \text{Proj} \left( \text{Sym}^* \Omega^1_{\mathcal{V}/\mathbb{P}^0_\mathbb{K}} \right). \]

As described in [37, p. 160], \( \mathbb{P}(\Omega^1_{\mathcal{V}/\mathbb{P}^0_\mathbb{K}}) \) is naturally equipped with the so-called Serre line bundle \( \mathcal{O}_{\mathbb{P}(\Omega^1_{\mathcal{V}/\mathbb{P}^0_\mathbb{K}})}(1) \) on it.

Similarly, replacing \( \mathcal{V} \) by \( \mathbb{P}^0_\mathbb{K} \times_{\mathbb{K}} \mathbb{P}^N_\mathbb{K} \), we obtain the relative differentials sheaf of \( \mathbb{P}^0_\mathbb{K} \times_{\mathbb{K}} \mathbb{P}^N_\mathbb{K} \) with respect to \( \text{pr}_1 \) in (7):

\[ \Omega^1_{\mathbb{P}^0_\mathbb{K} \times_{\mathbb{K}} \mathbb{P}^N_\mathbb{K}} \cong \text{pr}_2^* \Omega^1_{\mathbb{P}^N_\mathbb{K}}, \]

and we thus obtain its projectivization:

\[ \mathbb{P}(\Omega^1_{\mathbb{P}^0_\mathbb{K} \times_{\mathbb{K}} \mathbb{P}^N_\mathbb{K}}) := \text{Proj}(\text{Sym}^* \Omega^1_{\mathbb{P}^0_\mathbb{K} \times_{\mathbb{K}} \mathbb{P}^N_\mathbb{K}}) \cong \mathbb{P}^0_\mathbb{K} \times_{\mathbb{K}} \text{Proj}(\text{Sym}^* \Omega^1_{\mathbb{P}^N_\mathbb{K}}). \]

We will abbreviate \( \text{Proj}(\text{Sym}^* \Omega^1_{\mathbb{P}^N_\mathbb{K}}) \) as \( \mathbb{P}(\Omega^1_{\mathbb{P}^N_\mathbb{K}}) \), and denote its Serre line bundle by \( \mathcal{O}_{\mathbb{P}(\Omega^1_{\mathbb{P}^N_\mathbb{K}})}(1) \).

Then, the Serre line bundle \( \mathcal{O}_{\mathbb{P}(\Omega^1_{\mathbb{P}^N_\mathbb{K}})}(1) \) on the left hand side of (9) is nothing but the line bundle \( \mathcal{P}_2^* \mathcal{O}_{\mathbb{P}(\Omega^1_{\mathbb{P}^N_\mathbb{K}})}(1) \) on the right hand side, where \( \mathcal{P}_2 \) is the canonical projection:

\[ \mathcal{P}_2 : \mathbb{P}^0_\mathbb{K} \times_{\mathbb{K}} \mathbb{P}(\Omega^1_{\mathbb{P}^N_\mathbb{K}}) \rightarrow \mathbb{P}(\Omega^1_{\mathbb{P}^N_\mathbb{K}}). \]

Now, note that the commutative diagram:

\[ \begin{array}{ccc}
\mathcal{V} & \xrightarrow{i} & \mathbb{P}^0_\mathbb{K} \times_{\mathbb{K}} \mathbb{P}^N_\mathbb{K} \\
\downarrow{\text{pr}_1 \circ i} & & \downarrow{\text{pr}_1} \\
\mathbb{P}^0_\mathbb{K} & \end{array} \]
induces the surjection (cf. [37, p. 176, Proposition 8.12]):

\[ i^* \Omega^1_{P_K \times K} \rightarrow \Omega^1_{\mathcal{V}/P_K} \]

and hence yields the surjection:

\[ i^* \text{Sym}^* \Omega^1_{P_K \times K} \rightarrow \text{Sym}^* \Omega^1_{\mathcal{V}/P_K} \]

Taking ‘Proj’, thanks to (9), we obtain the commutative diagram:

\[ \begin{array}{ccc}
P(\Omega^1_{\mathcal{V}/P_K}) & \xrightarrow{i} & P_P^\phi \times_k P(\Omega^1_{P_K}) \\
\downarrow & & \downarrow \\
\mathcal{V} & \xrightarrow{i} & P_P^\phi \times_k P^N \\
\end{array} \]  \tag{11}

Thus, the Serre line bundle \( \mathcal{O}_{P(\Omega^1_{\mathcal{V}/P_K})} \) becomes exactly the pull back of ‘the Serre line bundle’ \( \mathcal{O}_{P(\Omega^1_{P_K})} \) under the inclusion \( \iota \):

\[ \mathcal{O}_{P(\Omega^1_{\mathcal{V}/P_K})} = \iota^*(\mathcal{O}_{P(\Omega^1_{P_K})}) = (\iota_2 \circ \pmb{\iota})^* \mathcal{O}_{P(\Omega^1_{P_K})} \]  \tag{12}

2.4. Restatement of Theorem 1.3. Let \( \pi \) be the canonical projection:

\[ \pi: P_P^\phi \times_k P(\Omega^1_{P_K}) \rightarrow P_P^\phi \times_k P^N, \]

and let \( \pi_1, \pi_2 \) be the compositions of \( \pi \) with \( \text{pr}_1, \text{pr}_2 \):

\[ \begin{array}{ccc}
P_P^\phi \times_k P(\Omega^1_{P_K}) & \xrightarrow{\pi_1:=\pi_1 \circ \pi} & P_P^\phi \times_k P^N \\
& & \downarrow \text{pr}_1 \\
P_P^\phi \times_k P^N & \xrightarrow{\pi_2:=\pi_2 \circ \pi} & P^N \\
\end{array} \]  \tag{13}

Let:

\[ \mathbf{P} := \pi^{-1}(\mathbf{\mathcal{V}}) \cap P(\Omega^1_{\mathcal{V}/P_K}) \subset P(\Omega^1_{\mathcal{V}/P_K}) \subset P_P^\phi \times_k P(\Omega^1_{P_K}) \]  \tag{14}

be ‘the pullback’ of:

\[ \mathbf{\mathcal{V}} \subset \mathcal{V} \subset P_P^\phi \times_k P^N \]

under the map \( \pi \), and let:

\[ \mathcal{O}_{\mathbf{P}}(1) := \mathcal{O}_{P(\Omega^1_{\mathcal{V}/P_K})}(1)|_{\mathbf{P}} = \pi_2^* \mathcal{O}_{P(\Omega^1_{P_K})}(1)|_{\mathbf{P}} \quad \text{[see (12)]} \]  \tag{15}

be the restricted Serre line bundle.

Now, we may view \( \mathbf{P} \) as a family of subschemes of \( P(\Omega^1_{P_K}) \) parametrized by the projective parameter space \( P_P^\phi \) under the restricted map:

\[ \pi_1: \mathbf{P} \rightarrow P_P^\phi. \]  \tag{16}
Thus Theorem 1.3 can be reformulated as below, with the assumption that the hypersurface degrees \(d_1, \ldots, d_{c+r}\) are sufficiently large:

\[d_1, \ldots, d_{c+r} \gg 1.\]

**Theorem 1.3 (Version A).** For a generic point \(t \in \mathbb{P}_{\mathbb{K}}^\phi\), over the fibre:

\[\mathbb{P}_t := \pi_1^{-1}(t) \cap \mathbb{P},\]

the restricted Serre line bundle:

\[\mathcal{O}_{\mathbb{P}_t}(1) := \mathcal{O}_{\mathbb{P}}(1)|_{\mathbb{P}_t}\]  \(\text{(17)}\)

is ample.

From now on, every closed point:

\[t = \left[[A_i^j]_{1 \leq j \leq c+r}\right] \in \mathbb{P}_{\mathbb{K}}^\phi\]

will be abbreviated as:

\[t = [F_1 : \cdots : F_{c+r}],\]

where:

\[F_i := \sum_{|\alpha| = d_i} A_i^j z^\alpha \quad (i = 1 \cdots c+r).\]

Then we have:

\[\mathbb{P}_t = \{t\} \times_{\mathbb{K}} F_{c+1}, \ldots, F_{c+r} \mathbb{P}_{F_1, \ldots, F_c},\]

for a uniquely defined subscheme:

\[F_{c+1}, \ldots, F_{c+r} \mathbb{P}_{F_1, \ldots, F_c} \subset \mathbb{P}(\mathcal{O}_{\mathbb{P}_{\mathbb{K}}}^{1}).\]  \(\text{(18)}\)

**Theorem 1.3 (Version B).** For a generic closed point:

\[[F_1 : \cdots : F_{c+r}] \in \mathbb{P}_{\mathbb{K}}^\phi,\]

the Serre line bundle \(\mathcal{O}_{\mathbb{P}\mathcal{O}_{\mathbb{P}_{\mathbb{K}}}^{1}}(1)\) is ample on \(F_{c+1}, \ldots, F_{c+r} \mathbb{P}_{F_1, \ldots, F_c}.\)

To have a better understanding of the above statements, we now investigate the geometry behind.

3. **The background geometry**

Since \(\mathbb{K}\) is an algebraically closed field, throughout this section, we view each scheme in the classical sense (cf. [37, Chapter 1]), i.e. its underlying topological space (\(\mathbb{K}\)-variety) consists of all the closed points.

3.1. **The geometry of \(\mathbb{P}_{\mathbb{K}}^N\) and \(\mathcal{O}_{\mathbb{P}_{\mathbb{K}}}^{N}(1)\).** Recall that, the projective \(N\)-space \(\mathbb{P}_{\mathbb{K}}^N\) is obtained by projectivizing the Euclidian \((N+1)\)-space \(\mathbb{K}^{N+1}\), i.e. is defined as the set of lines passing through the origin:

\[\mathbb{P}_{\mathbb{K}}^N := \mathbb{P}(\mathbb{K}^{N+1}) := \mathbb{K}^{N+1}\{0\} / \sim,\]  \(\text{(19)}\)

where the quotient relation \(\sim\) for \(z \in \mathbb{K}^{N+1}\{0\}\) is:

\[z \sim \lambda z \quad (\forall \lambda \in \mathbb{K}^\times).\]

On \(\mathbb{P}_{\mathbb{K}}^N\), there is the so-called tautological line bundle \(\mathcal{O}_{\mathbb{P}_{\mathbb{K}}}^{\mathcal{O}_{\mathbb{P}_{\mathbb{K}}}^{N}(1)}(-1)\), which at every point \([z] \in \mathbb{P}_{\mathbb{K}}^N\) has fibre:

\[\mathcal{O}_{\mathbb{P}_{\mathbb{K}}}^{\mathcal{O}_{\mathbb{P}_{\mathbb{K}}}^{N}(1)}|_z \ := \mathbb{K} \cdot z \subset \mathbb{K}^{N+1}.\]
Its dual line bundle is the well known:

\[ \mathcal{O}_{\mathbb{P}^N}(-1)^{\vee} = \mathcal{O}_{\mathbb{P}^N}(1). \]

3.2. The geometry of \( \mathbb{P}(\Omega^1_{\mathbb{P}^N}) \) and \( \mathcal{O}_{\mathbb{P}(\Omega^1_{\mathbb{P}^N})}(1) \). For every point \([z] \in \mathbb{P}^N\), the tangent space of \( \mathbb{P}^N \) at \([z]\) is:

\[ T_{\mathbb{P}^N}|_{[z]} = \mathbb{K}^{N+1}/\mathbb{K} \cdot z, \]

and the total tangent space of \( \mathbb{P}^N \):

\[ T_{\mathbb{P}^N} := T_{\text{hor}}\mathbb{K}^{N+1}/\sim, \]

is the quotient space of the horizontal tangent space of \( \mathbb{K}^{N+1} \setminus \{0\} \):

\[ T_{\text{hor}}\mathbb{K}^{N+1} := \{(z, [\xi]) : \, z \in \mathbb{K}^{N+1} \setminus \{0\} \text{ and } [\xi] \in \mathbb{K}^{N+1}/\mathbb{K} \cdot z\}, \]

by the quotient relation \( \sim \):

\[(z, [\xi]) \sim (\lambda z, [\lambda \xi]) \quad (\forall \lambda \in \mathbb{K}^\times).\]

Now, the \( \mathbb{K} \)-variety associated to \( \mathbb{P}(\Omega^1_{\mathbb{P}^N}) \) is just the projectivized tangent space \( \mathbb{P}(T_{\mathbb{P}^N}) \), which is obtained by projectivizing each tangent space \( T_{\mathbb{P}^N}|_{[z]} \) at every point \([z] \in \mathbb{P}^N\):

\[ \mathbb{P}(T_{\mathbb{P}^N})|_{[z]} := \mathbb{P}(T_{\mathbb{P}^N}|_{[z]}). \]

And the Serre line bundle \( \mathcal{O}_{\mathbb{P}(\Omega^1_{\mathbb{P}^N})}(1) \) on \( \mathbb{P}(\Omega^1_{\mathbb{P}^N}) \) corresponds to the ‘Serre line bundle’ \( \mathcal{O}_{\mathbb{P}(T_{\mathbb{P}^N})}(1) \) on \( \mathbb{P}(T_{\mathbb{P}^N}) \), which after restricting on \( \mathbb{P}(T_{\mathbb{P}^N})|_{[z]} \) becomes \( \mathcal{O}_{\mathbb{P}(T_{\mathbb{P}^N})|_{[z]}}(1) \). In other words, the Serre line bundle \( \mathcal{O}_{\mathbb{P}(T_{\mathbb{P}^N})}(1) \) is the dual of the tautological line bundle \( \mathcal{O}_{\mathbb{P}(T_{\mathbb{P}^N})}(-1) \), where the latter one, at every point \(([z], [\xi]) \in \mathbb{P}(T_{\mathbb{P}^N})\), has fibre:

\[ \mathcal{O}_{\mathbb{P}(T_{\mathbb{P}^N})}(-1)|_{([z], [\xi])} := \mathbb{K} \cdot [\xi] \subset T_{\mathbb{P}^N}|_{[z]} = \mathbb{K}^{N+1}/\mathbb{K} \cdot z. \]
3.3. The geometry of $\mathbf{P}(\Omega^1_{V/P_{K}^0})$, $\mathbf{P}$ and $\mathbf{P}_t$. Recalling (6), the $\mathbb{K}$-variety $V$ associated to $V \subset P^0_{K} \times_{\mathbb{K}} P^N_{K}$ is:

$$V := \{([F_1, \ldots, F_{c+r}], [z]) \in P^0_{K} \times P^N_{K} : F_i(z) = 0, \forall i = 1 \cdots c\}.$$  

Moreover, recalling (11), the $\mathbb{K}$-variety:

$$P(T_{V/P_{K}^0}) \subset P^0_{K} \times P(T_{P_{K}^N})$$

associated to $\mathbf{P}(\Omega^1_{V/P_{K}^0}) \subset P^0_{K} \times \mathbf{P}(\Omega^1_{P_{K}^N})$ is:

$$P(T_{V/P_{K}^0}) := \{([F_1, \ldots, F_{c+r}], ([z], [\xi])) : F_i(z) = 0, dF_j|_{\xi}(\xi) = 0, \forall i = 1 \cdots c \}.$$  

Similarly, the $\mathbb{K}$-variety:

$$P := P(T_{V/P_{K}^0})$$

associated to $P \subset \mathbf{P}(\Omega^1_{V/P_{K}^0})$ is:

$$P := \{([F_1, \ldots, F_{c+r}], ([z], [\xi])) : F_i(z) = 0, dF_j|_{\xi}(\xi) = 0, \forall i = 1 \cdots c + r, \forall j = 1 \cdots c \},$$  

and the $\mathbb{K}$-variety:

$$F_{c+1, \ldots, c+r} P_{F_1, \ldots, F_c} \subset P(T_{P_{K}^N})$$

associated to (18) is:

$$F_{c+1, \ldots, c+r} P_{F_1, \ldots, F_c} := \{([z], [\xi]) : F_i(z) = 0, dF_j|_{\xi}(\xi) = 0, \forall i = 1 \cdots c + r, \forall j = 1 \cdots c \}. \quad (22)$$

Now, the $\mathbb{K}$-variety $P_t$ of $\mathbf{P}_t$ is:

$$P_t := \{t\} \times_{F_{c+1, \ldots, c+r}} P_{F_1, \ldots, F_c}.$$

4. Some hints on the Ampleness Theorem 1.3

The first three Subsections 4.1–4.3 consist of some standard knowledge in algebraic geometry, and the last Subsection 4.4 presents a helpful nefness criterion which suits our moving coefficients method.

4.1. Ampleness is Zariski open. The foundation of our approach is the following classical theorem of Grothendieck (cf. [33, III.4.7.1] or [40, p. 29, Theorem 1.2.17]).

**Theorem 4.1. [Amplitude in families]** Let $f : X \to T$ be a proper morphism of schemes, and let $\mathcal{L}$ be a line bundle on $X$. For every point $t \in T$, denote by:

$$X_t := f^{-1}(t), \quad \mathcal{L}_t := \mathcal{L}|_{X_t}.$$  

Assume that, for some point $0 \in T$, $\mathcal{L}_0$ is ample on $X_0$. Then in $T$, there is a Zariski open set $U$ containing 0 such that $\mathcal{L}_t$ is ample on $X_t$, for all $t \in U$.

Note that in (13), $\pi_1 = \text{pr}_1 \circ \pi$ is a composition of two proper morphisms, hence is proper, and so is (16). Therefore, by virtue of the above theorem, we only need to find one (closed) point $t \in P^0_{K}$ such that:

$$\mathcal{O}_{P_t}(1) \text{ is ample on } P_t. \quad (23)$$
4.2. Largely twisted Serre line bundle is ample. Let:

$$\pi_0 : \mathbb{P}(\Omega^{1}_{\mathbb{P}^{N}}) \rightarrow \mathbb{P}^{N}$$

be the canonical projection. [37, p. 161, Proposition 7.10] yields that, for all sufficiently large integer $\ell$ \(^1\), the twisted line bundle below is ample on $\mathbb{P}(\Omega^{1}_{\mathbb{P}^{N}})$:

$$\mathcal{O}_{\mathbb{P}(\Omega^{1}_{\mathbb{P}^{N}})}(1) \otimes \pi_0^* \mathcal{O}_{\mathbb{P}^{N}}(\ell).$$

(24)

Recalling (10) and (13), and noting that:

$$\pi_2 = \pi_0 \circ \pi_2,$$

(25)

for the following ample line bundle $\mathcal{H}$ on the scheme $\mathbb{P}^{N} \times_{\mathbb{K}} \mathbb{P}^{N}$:

$$\mathcal{H} := \text{pr}_1^* \mathcal{O}_{\mathbb{P}^{N}}(1) \otimes \text{pr}_2^* \mathcal{O}_{\mathbb{P}^{N}}(1),$$

the twisted line bundle below is ample on $\mathbb{P}^{N} \times_{\mathbb{K}} \mathbb{P}(\Omega^{1}_{\mathbb{P}^{N}})$:

\[
\begin{align*}
\overline{\pi}_2 \mathcal{O}_{\mathbb{P}(\Omega^{1}_{\mathbb{P}^{N}})}(1) \otimes \overline{\pi}_2^* \mathcal{H}^{\ell} &= \overline{\pi}_2 \mathcal{O}_{\mathbb{P}(\Omega^{1}_{\mathbb{P}^{N}})}(1) \otimes \overline{\pi}_2 \left(\text{pr}_1^* \mathcal{O}_{\mathbb{P}^{N}}(\ell) \otimes \text{pr}_2^* \mathcal{O}_{\mathbb{P}^{N}}(\ell)\right) \\
&= \overline{\pi}_2 \mathcal{O}_{\mathbb{P}(\Omega^{1}_{\mathbb{P}^{N}})}(1) \otimes (\text{pr}_1 \circ \overline{\pi})^* \mathcal{O}_{\mathbb{P}^{N}}(\ell) \otimes (\text{pr}_2 \circ \overline{\pi})^* \mathcal{O}_{\mathbb{P}^{N}}(\ell) \\
\text{[use (13)]} \quad &\quad = \overline{\pi}_2 \mathcal{O}_{\mathbb{P}(\Omega^{1}_{\mathbb{P}^{N}})}(1) \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^{N}}(\ell) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^{N}}(\ell) \\
\text{[use (25)]} \quad &\quad = \overline{\pi}_2 \mathcal{O}_{\mathbb{P}(\Omega^{1}_{\mathbb{P}^{N}})}(1) \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^{N}}(\ell) \otimes \overline{\pi}_2 \left(\pi_0^* \mathcal{O}_{\mathbb{P}^{N}}(\ell)\right) \\
&= \pi_1^* \mathcal{O}_{\mathbb{P}^{N}}(\ell) \otimes \overline{\pi}_2 \left(\mathcal{O}_{\mathbb{P}(\Omega^{1}_{\mathbb{P}^{N}})}(1) \otimes \pi_0^* \mathcal{O}_{\mathbb{P}^{N}}(\ell)\right). \\
\end{align*}
\]

(26)

In particular, for every point $t \in \mathbb{P}^{N}$, recalling (15), (17), restricting (26) to the subscheme:

$$\mathbb{P}_t = \pi_1^{-1}(t) \cap \mathbb{P},$$

we receive an ample line bundle:

$$\mathcal{O}_{\mathbb{P}_t}(1) \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^{N}}(\ell) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^{N}}(\ell) = \mathcal{O}_{\mathbb{P}_t}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^{N}}(\ell).$$

(27)

4.3. Nefness of negatively twisted cotangent sheaf suffices. As we mentioned at the end of Subsection 4.1, our goal is to show the existence of one such (closed) point $t \in \mathbb{P}^{N}$ satisfying (23). In fact, we can relax this requirement thanks to the following theorem.

**Theorem 4.2.** For every point $t \in \mathbb{P}^{N}$, the following properties are equivalent.

(i) $\mathcal{O}_{\mathbb{P}_t}(1)$ is ample on $\mathbb{P}_t$.

(ii) There exist two positive integers $a, b \geq 1$ such that $\mathcal{O}_{\mathbb{P}_t}(a) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^{N}}(-b)$ is ample on $\mathbb{P}_t$.

(iii) There exist two positive integers $a, b \geq 1$ such that $\mathcal{O}_{\mathbb{P}_t}(a) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^{N}}(-b)$ is nef on $\mathbb{P}_t$.

\(^1\) In fact, $\ell \geq 3$ is enough, see (222) below.
Proof. It is clear that (i) \(\implies\) (ii) \(\implies\) (iii), and we now show that (iii) \(\implies\) (i).

In fact, the nefness of the negatively twisted Serre line bundle:

\[
\mathcal{S}^a_t(-b) := \mathcal{O}_{\mathbb{P}_t}(a) \otimes \pi_* \mathcal{O}_{\mathbb{P}_{\mathbb{K}}}(b)
\]  

implies that:

\[
(27)^{\otimes b} \otimes (28)^{\otimes \ell} = \mathcal{O}_{\mathbb{P}_t}(b + a \ell) = \mathcal{O}_{\mathbb{P}_t}(1)^{\otimes (b + a \ell)}
\]

is also ample, because of the well known fact that “ample \(\otimes\) nef = ample” (cf. [40, p. 53, Corollary 1.4.10]).

By definition, the nefness of (28) means that for every irreducible curve \(C \subset \mathbb{P}_t\), the intersection number \(C \cdot \mathcal{S}^a_t(-b)\) is \(\geq 0\). Recalling now the classical result [37, p. 295, Lemma 1.2], we only need to show that the line bundle \(\mathcal{S}^a_t(-b)\) has a nonzero section on the curve \(C\):

\[
H^0(C, \mathcal{S}^a_t(-b)) \neq \{0\}.  
\]  

To this end, of course we like to construct sufficiently many global sections:

\[
s_1, \ldots, s_m \in H^0(\mathbb{P}_t, \mathcal{S}^a_t(-b))
\]

such that their base locus is empty or discrete, whence one of \(s_1|_C, \ldots, s_m|_C\) suffices to conclude (29).

More flexibly, we have:

**Theorem 4.3.** Suppose that there exist \(m \geq 1\) nonzero sections of certain negatively twisted Serre line bundles:

\[
s_i \in H^0(\mathbb{P}_t, \mathcal{S}_t^a(-b_i)) \quad (i = 1, \ldots, m, a_i, b_i \geq 1)
\]

such that their base locus is discrete or empty:

\[
\dim \bigcap_{i=1}^m \text{BS}(s_i) \leq 0,
\]

then for all positive integers \(a, b\) with:

\[
\frac{a}{b} \geq \max\{\frac{a_1}{b_1}, \ldots, \frac{a_m}{b_m}\},
\]

the twisted Serre line bundle \(\mathcal{S}_t^a(-b)\) is nef.

**Proof.** For every irreducible curve \(C \subset \mathbb{P}_t\), noting that:

\[
\bigcap_{\dim = 1}^\infty \bigcap_{i=1}^m \text{BS}(s_i),
\]

there exists some integer \(1 \leq i \leq m\) such that:

\[
C \not\subset \text{BS}(s_i).
\]

Therefore \(s_i|_C\) is a nonzero section of \(\mathcal{S}_t^a(-b_i)\) on the curve \(C\):

\[
s_i \in H^0(C, \mathcal{S}_t^a(-b_i)) \setminus \{0\},
\]

and hence:

\[
C \cdot \mathcal{S}_t^a(-b_i) \geq 0.
\]
Thus we have the estimate:

\[ 0 \leq C \cdot (\mathcal{I}_t^a (-b))^{\otimes a} \quad [= a C \cdot \mathcal{I}_t^a (-b)] \]

\[ = C \cdot \mathcal{O}_{\mathbb{P}_t} (a_i a) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}_{X_2}^n} (-b_1 a) \quad \text{[see (28)]} \]

\[ = a_i C \cdot (\mathcal{O}_{\mathbb{P}_t} (a) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}_{X_2}^n} (-b)) - (b_1 a - a_i b) C \cdot \pi_2^* \mathcal{O}_{\mathbb{P}_{X_2}^n} (1) \]

\[ = a_i C \cdot (\mathcal{I}_t^a (-b)) - b b_1 (a/b - a_i/b_i) C \cdot \pi_2^* \mathcal{O}_{\mathbb{P}_{X_2}^n} (1). \]

Noting that $\mathcal{O}_{\mathbb{P}_{X_2}^n} (1)$ is nef and hence is $\pi_2^* \mathcal{O}_{\mathbb{P}_{X_2}^n} (1)$ (cf. [40, p. 51, Example 1.4.4]), the above estimate immediately yields:

\[ C \cdot (\mathcal{I}_t^a (-b)) \geq \frac{b b_1}{a_i} (a/b - a_i/b_i) C \cdot \pi_2^* \mathcal{O}_{\mathbb{P}_{X_2}^n} (1) \geq 0. \]

Repeating the same reasoning as in the above two theorems, we obtain:

**Proposition 4.4.** For every point $t \in \mathbb{P}_{X_2}^n$, if $\mathcal{O}_{\mathbb{P}_t} (\ell_1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}_{X_2}^n} (-\ell_2)$ is nef on $\mathbb{P}_t$ for some positive integers $\ell_1, \ell_2 \geq 1$, then for any positive integers $\ell_1', \ell_2' \geq 1$ with $\ell_2'/\ell_1' < \ell_2/\ell_1$, the twisted line bundle $\mathcal{O}_{\mathbb{P}_t} (\ell_1') \otimes \pi_2^* \mathcal{O}_{\mathbb{P}_{X_2}^n} (-\ell_2')$ is ample on $\mathbb{P}_t$.

4.4. **A practical nefness criterion.** However, in practice, it is often difficult to gather enough global sections (with discrete base locus) to guarantee nefness of a line bundle. We need to be more clever to improve such a coarse nefness criterion with the help of nonzero sections of the same bundle restricted to proper subvarieties. First, let us introduce the theoretical reason behind.

**Definition 4.5.** Let $X$ be a variety, and let $Y \subset X$ be a subvariety. A line bundle $\mathcal{L}$ on $X$ is said to be nef outside $Y$ if, for every irreducible curve $C \subset X$ with $C \not\subset Y$, the intersection number $C \cdot \mathcal{L} \geq 0$.

Of course, $\mathcal{L}$ is nef on $X$ if and only if $\mathcal{L}$ is nef outside the empty set $\emptyset \subset X$.

**Theorem 4.6 (Nefness Criterion).** Let $X$ be a noetherian variety, and let $\mathcal{L}$ be a line bundle on $X$. Assume that there exists a set $\mathcal{V}$ of closed subvarieties of $X$ satisfying:

(i) $\emptyset \in \mathcal{V}$ and $X \in \mathcal{V}$;

(ii) for every element $Y \in \mathcal{V}$ with $Y \neq \emptyset$, there exist finitely many elements $Z_1, \ldots, Z_n \in \mathcal{V}$ with $Z_1, \ldots, Z_n \subsetneq Y$ such that the restricted line bundle $\mathcal{L}\mid_Y$ is nef outside the union $Z_1 \cup \cdots \cup Z_n$.

Then $\mathcal{L}$ is nef on $X$.

**Proof.** For every irreducible curve $C \subset X$, we have to show that $C \cdot \mathcal{L} \geq 0$.

Assume on the contrary that $C \cdot \mathcal{L} < 0$. Then introduce the subset $\mathcal{N} \subset \mathcal{V}$ consisting of all subvarieties $Y \in \mathcal{V}$ which contain the curve $C$. Clearly, $\mathcal{N} \ni X$, so $\mathcal{N}$ is nonempty. Note that there is a natural partial order `$\subset$' on $\mathcal{N}$ given by the strict inclusion relation `$\subsetneq$'. Since $X$ is noetherian, $\mathcal{N}$ has a minimum element $M \ni C$. We now show a contradiction.

In fact, according to (ii), there exist some elements $\mathcal{V} \ni Z_1, \ldots, Z_n \subsetneq M$ such that $\mathcal{L}\mid_M$ is nef outside $Z_1 \cup \cdots \cup Z_n$. Rembering that:

\[ 0 > C \cdot \mathcal{L} = C \cdot \mathcal{L}\mid_M. \]
the curve \( C \) is forced to lie in the union \( Z_1 \cup \cdots \cup Z_p \), and thanks to irreducibility, it is furthermore contained in one certain:

\[
\exists M \ni \frac{Z_i}{M} \in \mathcal{N},
\]

which contradicts the minimality of \( M \)!

Now, using the same idea as around (29), we may realize \( \text{(ii)} \) above with the help of sections over proper subvarieties.

**Corollary 4.7.** Let \( X \) be a hnoetherian variety, and let \( \mathcal{L} \) be a line bundle on \( X \). Assume that there exists a set \( \mathcal{V} \) of closed subvarieties of \( X \) satisfying:

\( \text{(i)} \) \( \emptyset \in \mathcal{V} \) and \( X \in \mathcal{V} \);

\( \text{(ii')} \) every element \( \emptyset \neq Y \in \mathcal{V} \) is a union of some elements \( Y_1, \ldots, Y_c \in \mathcal{V} \) such that the union of base loci:

\[
\bigcup_{s=1}^n \left( \bigcap_{s \in \mathcal{H}(Y, \mathcal{L}, \mathcal{V})} \{s = 0\} \right)
\]

is contained in a union of some elements \( \mathcal{V} \ni Z_1, \ldots, Z_{\flat} \subseteq Y \), except discrete points.

Then \( \mathcal{L} \) is nef on \( X \). \( \square \)

5. **A proof blueprint of the Ampleness Theorem 1.3**

5.1. **Main Nefness Theorem.** Recalling Theorem 4.2, the Ampleness Theorem 1.3 is a consequence of the theorem below, whose effective bound \( d_0(\mathfrak{v}) \) for \( \mathfrak{v} = 1 \) will be given in Theorem 11.2.

**Theorem 5.1.** Given any positive integer \( \mathfrak{v} \geq 1 \), there exists a lower degree bound \( d_0(\mathfrak{v}) \gg 1 \) such that, for all degrees \( d_1, \ldots, d_c, d_{\mathfrak{v}} \geq d_0(\mathfrak{v}) \), for a very generic \( t \in \mathbb{P}^{\mathfrak{v}} \), the negatively twisted Serre line bundle \( \mathcal{O}_{\mathbb{P}}(1) \otimes \pi^* \mathcal{O}_{\mathbb{K}}(-\mathfrak{v}) \) is nef on \( \mathbb{P} \).

It suffices to find one such \( t \in \mathbb{P}^{\mathfrak{v}} \) to guarantee ‘very generic’ (cf. [40, p. 56, Proposition 1.4.14]).

We will prove Theorem 5.1 in two steps. At first, in Subsection 5.2, we sketch the proof in the central cases when all \( c + r \) hypersurfaces are approximately of the same large degrees. Then, in Subsection 5.3, we play a product coup to embrace all large degrees.

5.2. **The central cases of relatively the same large degrees.**

**Theorem 5.2.** For any fixed \( c + r \) positive integers \( \epsilon_1, \ldots, \epsilon_{c+r} \geq 1 \), for every sufficiently large integer \( d \gg 1 \), Theorem 5.1 holds with \( d_i = d + \epsilon_i, i = 1 \cdots c + r \).

When \( c + r \geq N \), generically \( X \) is discrete or empty, so there is nothing to prove. Assuming \( c + r \leq N - 1 \), we now outline the proof.

**Step 1.** In the entire family of \( c + r \) hypersurfaces with degrees \( d + \lambda_1, \ldots, d + \lambda_{c+r} \), whose projective parameter space is \( \mathbb{P}^{\mathfrak{v}} \) (see (3)), we select a specific subfamily which best suits our moving coefficients method, whose projective parameter space is a subvariety:

\[
\mathbb{P}^\mathfrak{v} \subset \mathbb{P}^{\mathfrak{v}} \quad \text{[see (140)]}.
\]

For the details of this subfamily, see Subsection 7.1.

---

\(^2 t \in \mathbb{P}^{\mathfrak{v}} \setminus \bigcup_{i=1}^{\infty} Z_i \) for some countable proper subvarieties \( Z_i \not\subseteq \mathbb{P}^{\mathfrak{v}} \).
Recalling (5) and (7), we then consider the subfamily of intersections \( \mathcal{Y} \subset \mathcal{X} \):

\[
\text{pr}_1^{-1}(\mathcal{F}_\mathcal{X}^\bullet) \cap \mathcal{X} =: \mathcal{Y} \subset \mathcal{F}_\mathcal{X}^\bullet \times_{\mathcal{X}} \mathcal{F}_\mathcal{X}^N = \text{pr}_1^{-1}(\mathcal{F}_\mathcal{X}^\bullet).
\]

Recalling (13), (14), we introduce the subscheme of \( \mathcal{P} \):

\[
\mathcal{P}'} := \tilde{\pi}^{-1}(\mathcal{Y}) \cap \mathcal{P} \subset \mathcal{F}_\mathcal{X}^\bullet \times_{\mathcal{X}} \mathcal{P}(\Omega^1_{\mathcal{X}^\bullet}),
\]

which is parametrized by \( \mathcal{Y} \). By restriction, (13) yields the commutative diagram:

\[
\begin{array}{ccc}
\mathcal{P}'} & \xrightarrow{\tilde{\pi}} & \mathcal{Y} \\
\text{pr}_1 & & \text{pr}_2 \\
\mathcal{F}_\mathcal{X}^\bullet & & \mathcal{F}_\mathcal{X}^N \\
\end{array}
\]

Introducing the restricted Serre line bundle \( \mathcal{O}_{\mathcal{P}}(1) := \mathcal{O}_{\mathcal{P}}(1)|_{\mathcal{P}'} \) over \( \mathcal{P}' \), in order to establish Theorem 5.2, it suffices to provide one such example. In fact, we will prove

**Theorem 5.3.** For a generic closed point \( t \in \mathcal{F}_\mathcal{X}^\bullet \), the bundle \( \mathcal{O}_{\mathcal{P}}(1) \otimes_{\mathcal{P}'} \mathcal{O}_{\mathcal{P}}(\mathcal{Y}) \) is nef on \( \mathcal{P}'} := \mathcal{P} \).

**Step 2.** The central objects now are the universal negatively twisted Serre line bundles:

\[
\mathcal{O}_{\mathcal{P}'}(a, b, -c) := \mathcal{O}_{\mathcal{P}'}(a) \otimes_{\mathcal{P}'} \mathcal{O}_{\mathcal{P}'}(b) \otimes_{\mathcal{P}'} \mathcal{O}_{\mathcal{P}'}(\mathcal{Y}),
\]

where \( a, c \) are positive integers such that \( c/a \geq \heartsuit \), and where \( b \) are any integers.

Taking advantage of the moving coefficients method, firstly, we construct a series of global universal negatively twisted symmetric differential \( n \)-forms:

\[
S_\ell \in \Gamma(\mathcal{P}', \mathcal{O}_{\mathcal{P}'}(n, -\varpi_\ell)) \quad (\ell = 1 \cdots \#),
\]

where \( n := N - (\ell + r) \geq 1 \) and all \( \varpi_\ell/n \geq \heartsuit \), and where we always use the symbol “\( \# \)” to denote auxiliary positive integers, which vary according to the context.

Secondly, for every integer \( 1 \leq \eta \leq n - 1 \), for every sequence of ascending indices:

\[
0 \leq \nu_1 < \cdots < \nu_\eta \leq N,
\]

considering the vanishing part of the corresponding \( \eta \) coordinates:

\[
\nu_1, \ldots, \nu_\eta \mathcal{P}' := \mathcal{P}' \cap \pi_2^{-1}\{z_{\nu_1} = \cdots = z_{\nu_\eta} = 0\},
\]

we construct a series of universal negatively twisted symmetric differential \( (n - \eta) \)-forms on it:

\[
\nu_1, \ldots, \nu_\eta S_\ell \in \Gamma(\nu_1, \ldots, \nu_\eta \mathcal{P}', \mathcal{O}_{\mathcal{P}'}(n - \eta, N - \eta, -\varpi_{\nu_1, \ldots, \nu_\eta})) \quad (\ell = 1 \cdots \#),
\]

where all \( \nu_1, \ldots, \nu_\eta \varpi_\ell/(n - \eta) \geq \heartsuit \).

This step will be accomplished in Sections 6 and 7.

**Step 3.** From now on, we view every scheme as its \( \mathbb{K} \)-variety.

Firstly, we control the base locus of all the global sections obtained in (31):

\[
\text{BS} := \text{Base Locus of } \{S_\ell\}_{1 \leq \ell \leq \#} \subset \mathcal{P}'.
\]
In fact, on the coordinates nonvanishing part of \( P' \):
\[
\hat{P}' := P' \cap \pi_2^{-1}\{z_0 \cdots z_N \neq 0\}
\]
we prove that:
\[
\dim \text{BS} \cap \hat{P}' \leq \dim \mathbb{P}_K^*.
\] (33)

Secondly, we control the base locus of all the sections obtained in (32):
\[
{v_1, \ldots, v_\eta}_{\text{BS}} := \text{Base Locus of } \{v_1, \ldots, v_\eta \}\{S_\ell\}_{1 \leq \ell \leq \eta} \subset v_1, \ldots, v_\eta P'.
\]
In fact, on the corresponding ‘coordinates nonvanishing part’ of \( v_1, \ldots, v_\eta P' \):
\[
{v_1, \ldots, v_\eta}_{\hat{P}'} := v_1, \ldots, v_\eta P' \cap \pi_2^{-1}\{z_{r_0} \cdots z_{r_{N-\eta}} \neq 0\},
\]
where:
\[
\{r_0, \ldots, r_{N-\eta}\} := \{0, \ldots, N\} \setminus \{v_1, \ldots, v_\eta\},
\] (34)
we prove that:
\[
\dim {v_1, \ldots, v_\eta}_{\text{BS}} \cap {v_1, \ldots, v_\eta}_{\hat{P}'} \leq \dim \mathbb{P}_K^*.
\] (35)

This crucial step will be accomplished in Sections 9 and 10. Anticipating, we would like to emphasize that, in order to lower down dimensions of base loci for global symmetric differential forms (or for higher order jet differential forms in Kobayashi hyperbolicity conjecture), a substantial amount of algebraic geometry work is required, mainly because some already known/constructed sections have the annoying tendency to proliferate by multiplying each other without shrinking their base loci (\( 0 \times \text{anything} = 0 \)). Hence the first main difficulty is to devise a wealth of independent symmetric differential forms, which the Moving Coefficients Method is designed for, and the second main difficulty is to establish the emptiness/discreteness of their base loci, an ultimate difficulty that will be settled in the Core Lemma 9.5.

Step 4. Firstly, for the regular map:
\[
\pi_1: P' \rightarrow \mathbb{P}_K^*,
\]
noting the dimension estimates (33), (35) of the base loci, applying now a classical theorem [35, p. 132, Theorem 11.12], we know that there exists a proper closed algebraic subvariety:
\[
\Sigma \subset \mathbb{P}_K^*
\]
such that, for every closed point \( t \) outside \( \Sigma \):
\[
t \in \mathbb{P}_K^* \setminus \Sigma,
\]

(i) the base locus of the restricted symmetric differential \( n \)-forms:
\[
\text{BS}_t := \text{Base Locus of } \{S_\ell(t) := S_\ell|_{P_t}\}_{1 \leq \ell \leq \eta} \subset P'_t
\]
is discrete or empty over the coordinates nonvanishing part:
\[
\dim \text{BS}_t \cap \hat{P}'_t \leq 0,
\] (36)
where:
\[
\hat{P}'_t := \hat{P}' \cap \pi_1^{-1}(t);
\]
(ii) the base locus of the restricted symmetric differential \((n - \eta)\)-forms:

\[
\varphi_{v_1, \ldots, v_\eta} := \text{Base Locus of } \{v_1, \ldots, v_\eta S_{\ell}(t) := \left. v_1, \ldots, v_\eta S_{\ell} \right|_{v_1, \ldots, v_\eta P'_{\ell}} \}_{1 \leq \ell \leq r} \subset v_1, \ldots, v_\eta P'_{v}
\]

is discrete or empty over the corresponding ‘coordinates nonvanishing part’:

\[
\dim_{v_1, \ldots, v_\eta} \varphi_{v_1, \ldots, v_\eta} P'_{v} \leq 0 ,
\]

where:

\[
v_1, \ldots, v_\eta P'_{v} := v_1, \ldots, v_\eta P' \cap \pi^{-1}(t).
\]

Secondly, there exists a proper closed algebraic subvariety:

\[
\Sigma' \subseteq \mathbb{P}_K^r
\]

such that, for every closed point \(t\) outside \(\Sigma'\):

\[
t \in \mathbb{P}_K^r \setminus \Sigma',
\]

the fibre:

\[
\mathcal{B}_t := \mathcal{Y} \cap \text{pr}^{-1}_1(t)
\]

is smooth and of dimension \(n = N - (c + r)\), and it satisfies:

\[
\dim \mathcal{B}_t \cap \text{pr}^{-1}_2(v_1, \ldots, v_\eta \mathbb{P}^N) = 0 \quad (0 \leq v_1 < \cdots < v_\eta \leq N),
\]

i.e. the intersection of \(\mathcal{B}_t\) — (under the regular map \(\text{pr}_2\)) viewed as a dimension \(n\) subvariety in \(\mathbb{P}^N\) — with every \(n\) coordinate hyperplanes:

\[
\varphi_{v_1, \ldots, v_\eta} \mathbb{P}^N := \{z_{v_1} = \cdots = z_{v_\eta} = 0\}
\]

is just finitely many points, which we denote by:

\[
\varphi_{v_1, \ldots, v_\eta} \mathcal{B}_t \subset v_1, \ldots, v_\eta \mathbb{P}^N.
\]

Now, we shall conclude Theorem 5.3 for every closed point \(t \in \mathbb{P}_K^r \setminus (\Sigma \cup \Sigma')\).

**Proof of Theorem 5.3.** For the line bundle \(\mathcal{L} = \mathcal{O}_{P'}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^n}(\cdot \neg \bigotimes)\) over the variety \(P'\), we claim that the set of subvarieties:

\[
\mathcal{Y} := \left\{0, P', v_1, \ldots, v_\eta P' \right\}_{1 \leq \eta \leq n} \text{ s.t. } 0 \leq v_1 < \cdots < v_\eta \leq N
\]

satisfies the conditions of Theorem 4.6.

Indeed, firstly, recalling (36), the sections \(\{S_\ell(t)\}_{\ell=1, \ldots, r}\) have empty/discrete base locus over the coordinates nonvanishing part, i.e. outside \(\cup_{j=0}^N P'_{j}\). Hence, using an adaptation of Theorem 4.3, remembering \(\neg \bigotimes/1 \leq \min\left\{\neg \bigotimes/n\right\}_{1 \leq \ell \leq r}\), the line bundle \(\mathcal{O}_{P'}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^n}(\neg \bigotimes)\) is nef outside \(\cup_{j=0}^N P'_{j}\).

Secondly, for every integer \(\eta = 1 \cdots n - 1\), recalling the dimension estimate (37), again by Theorem 4.3, remembering \(\neg \bigotimes/1 \leq \min\left\{v_1, \ldots, v_\eta \neg \bigotimes/(n - \eta)\right\}_{1 \leq \ell \leq r}\), the line bundle \(\mathcal{O}_{P'}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^n}(\neg \bigotimes)\) is nef on \(v_1, \ldots, v_\eta P'\) outside \(\cup_{j=0}^{N-\eta} P'_{v_j}\) (see (34)).

Lastly, for \(\eta = n\), noting that under the projection \(\pi: P' \rightarrow \mathcal{B}\), thanks to (38), every subvariety \(v_1, \ldots, v_\eta P'\) contracts to discrete points \(v_1, \ldots, v_\eta \mathcal{B}\), we see that on \(v_1, \ldots, v_\eta P'\), the line bundle \(\mathcal{O}_{P'}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^n}(\neg \bigotimes)\) is not only nef, but also ample!

Summarizing the above three parts, by Theorem 4.6, we conclude the proof. \(\square\)
5.3. **Product Coup.** We will use in an essential way Theorem 5.2 with all $\epsilon_i$ equal to either 1 or 2. To begin with, we need an elementary

**Observation 5.4.** For all positive integers $d \geq 1$, every integer $d_0 \geq d^2 + d$ is a sum of nonnegative multiples of $d + 1$ and $d + 2$.

**Proof.** According to the Euclidian division, we can write $d_0$ as:

$$d_0 = p(d + 1) + q$$

for some positive integer $p \geq 1$ and residue number $0 \leq q \leq d$. We claim that $p \geq q$.

Otherwise, we would have:

$$p \leq q - 1 \leq d - 1,$$

which would imply the estimate:

$$d = p(d + 1) + q \leq (d - 1)(d + 1) + d = d^2 + d - 1,$$

contradicting our assumption.

Therefore, we can write $d_0$ as:

$$d_0 = (p - q)(d + 1) + q(d + 2),$$

which concludes the proof. \qed

**Proof of Theorem 5.1.** Take one sufficiently large integer $d$ such that Theorem 5.2 holds for any integers $\epsilon_i \in \{1, 2\}$, $i = 1 \cdots c + r$. Now, the above observation says that all large degrees $d_1, \ldots, d_{c+r} \geq d^2 + d$ can be written as $d_i = p_i(d + 1) + q_i(d + 2)$, with some nonnegative integers $p_i, q_i \geq 0$, $i = 1 \cdots c + r$. Let $F_i := f_1^{i} \cdots f_p^{i} p_{i+1} \cdots f_{i+q_i}$ be a product of some $p_i$ homogeneous polynomials $f_i^{j}, \ldots, f_{p_i}$ each of degree $d + 1$ and of some $q_i$ homogeneous polynomials $f_{p_i+1}^{j}, \ldots, f_{p_i+q_i}$ each of degree $d + 2$, so that $F_i$ has degree $d_i$.

Recalling (22), a point $([z], [\xi]) \in \mathbb{F}(T_{\mathbb{F}_p}[z])$ lies in $F_{c+1, \ldots, c+r} \mathbb{F}_{1, \ldots, c}$ if and only if:

$$F_i(z) = 0, dF_j|_{z}(\xi) = 0 \quad (\forall i = 1 \cdots c+r, \forall j = 1 \cdots c).$$

Note that, for every $j = 1 \cdots c$, the pair of equations:

$$F_j(z) = 0, dF_j|_{z}(\xi) = 0$$

is equivalent to either:

$$\exists 1 \leq v_j \leq p_j + q_j \quad \mathrm{s.t.} \quad f_{v_j}^{j}(z) = 0, dF_j|_{z}(\xi) = 0,$$

or to:

$$\exists 1 \leq w_j \leq w_j^2 \leq p_j + q_j \quad \mathrm{s.t.} \quad f_{w_j}^{j}(z) = 0, dF_j|_{z}(\xi) = 0.$$
Similarly, we can show that the scheme \( F_{c+1,\ldots,F_{c+r}} P_{F_{1},\ldots,F_{c}} \) also decomposes into a union of subschemes:

\[
F_{c+1,\ldots,F_{c+r}} P_{F_{1},\ldots,F_{c}} = \bigcup_{k=0}^{c} \bigcup_{1 \leq i_{1} < \cdots < i_{k} \leq c} \bigcup_{1 \leq \nu_{i_{j}} \leq p_{j} + q_{j}} \bigcup_{1 \leq w_{i_{j}} \leq p_{j} + q_{j}} \bigcup_{1 \leq l \leq c-k} f_{w_{i_{1}}^{-1}}^{1} f_{w_{i_{2}}^{-1}}^{2} \cdots f_{w_{i_{k-1}}^{-1}}^{k-1} f_{w_{i_{k}}^{-1}}^{k} P_{f_{v_{1}}^{1},\ldots,f_{v_{r}}^{k}} P_{f_{v_{1}}^{1},\ldots,f_{v_{r}}^{k}},
\]

(43)

Note that, for each subscheme on the right hand side, the number of polynomials on the lower-left of ‘\( P \)’ is \#_L = 2(c-k)+r, and the number of polynomials on the lower-right is \#_R = k, whence \#_L + \#_R = 2c + r \geq N. Now, applying Theorem 5.2, we can choose one \( \{ f_{v}^{*} \}_{\bullet,\bullet} \) such that the twisted Serre line bundle \( \mathcal{O}_{\mathbb{P}^{N}_{K}}(1) \otimes \pi_{0}^{\ast} \mathcal{O}_{\mathbb{P}^{N}_{K}}(-\bullet) \) is nef on each subscheme \( f_{w_{i_{1}}^{-1}}^{1} f_{w_{i_{2}}^{-1}}^{2} \cdots f_{w_{i_{k-1}}^{-1}}^{k-1} f_{w_{i_{k}}^{-1}}^{k} P_{f_{v_{1}}^{1},\ldots,f_{v_{r}}^{k}} P_{f_{v_{1}}^{1},\ldots,f_{v_{r}}^{k}}, \)
and therefore is also nef on their union \( F_{c+1,\ldots,F_{c+r}} P_{F_{1},\ldots,F_{c}} \). Since nefness is a very generic property in family, we conclude the proof.

\[
\Box
\]

6. Generalization of Brotké’s symmetric differentials forms

6.1. Preliminaries on symmetric differential forms in projective space. For a fixed algebraically closed field \( K \), for three fixed integers \( N, c, r > 0 \) such that \( N \geq 2, 2c + r \geq N \) and \( c + r \leq N - 1 \), for \( c + r \) positive integers \( d_{1}, \ldots, d_{c+r} \), let:

\[
H_{i} \subset \mathbb{P}^{N}_{K} \quad (j = 1 \cdots c+r)
\]

be \( c + r \) hypersurfaces defined by some degree \( d_{i} \) homogeneous polynomials:

\[
F_{i} \in \mathbb{K}[z_{0}, \ldots, z_{N}],
\]

let \( V \) be the intersection of the first \( c \) hypersurfaces:

\[
V := H_{1} \cap \cdots \cap H_{c} = \{ [z] \in \mathbb{P}^{N}_{K} : F_{i}(z) = 0, \forall i = 1 \cdots c \},
\]

(44)

and let \( X \) be the intersection of all the \( c + r \) hypersurfaces:

\[
X := V \cap H_{c+1} \cap \cdots \cap H_{c+r} = \{ [z] \in \mathbb{P}^{N}_{K} : F_{i}(z) = 0, \forall i = 1 \cdots c + r \}.
\]

(45)

It is well known that, for generic choices of \( \{ F_{i} \}_{i=1}^{c+r} \), the intersection \( V = \cap_{i=1}^{c} H_{i} \) and \( X = \cap_{i=1}^{c+r} H_{i} \) are both smooth complete, and we shall assume this henceforth. In Subsections 6.1–6.4, we focus on smooth \( \mathbb{K} \)-varieties to provide a geometric approach to generalize Brotké’s symmetric differential forms, where the ambient field \( \mathbb{K} \) is assumed to be algebraically closed. In addition, in Subsection 6.5, we will give another quick algebraic approach, without any assumption on the ambient field \( \mathbb{K} \).

Recalling (19), let us denote by:

\[
\pi : \mathbb{K}^{N+1} \setminus \{ 0 \} \longrightarrow \mathbb{P}^{N}_{K}
\]

the canonical projection.

For every integer \( k \), the standard twisted regular function sheaf \( \mathcal{O}_{\mathbb{P}^{N}_{K}}(k) \), geometrically, can be defined as, for all Zariski open subset \( U \) in \( \mathbb{P}^{N}_{K} \), the corresponding section set \( \Gamma(U, \mathcal{O}_{\mathbb{P}^{N}_{K}}(k)) \) consists

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of all the regular functions \( \hat{f} \) on \( \pi^{-1}(U) \) satisfying:

\[
\hat{f}(\lambda z) = \lambda^k \hat{f}(z) \quad (\forall z \in \pi^{-1}(U), \lambda \in \mathbb{K}^\times).
\]

For the cone \( \hat{V} := \pi^{-1}(V) \) of \( V \):

\[
\hat{V} = \{ z \in \mathbb{K}^{N+1} \setminus \{0\} : F_i(z) = 0, \forall i = 1 \cdots c \},
\]

recalling (20), (21), we can similarly define its horizontal tangent bundle \( T_{\text{hor}} \hat{V} \) which has fibre at any point \( z \in \hat{V} \):

\[
T_{\text{hor}} \hat{V}|_z = \{ [\xi] \in \mathbb{K}^{N+1}/\mathbb{K} \cdot z : dF_{\lambda z}(\xi) = 0, \forall i = 1 \cdots c \}.
\]

Its total space is:

\[
T_{\text{hor}} \hat{V} := (z, [\xi]) : z \in \hat{V}, [\xi] \in \mathbb{K}^{N+1}/\mathbb{K} \cdot z, dF_{\lambda z}(\xi) = 0, \forall i = 1 \cdots c \}.
\]

Then similarly we receive the total tangent bundle \( T_V \) of \( V \) as:

\[
T_V = T_{\text{hor}} \hat{V}/ \sim, \quad \text{where } (z, [\xi]) \sim (\lambda z, [\lambda \xi]), \forall \lambda \in \mathbb{K}^\times.
\]

Let \( \Omega_V \) be the dual bundle of \( T_V \), i.e. the cotangent bundle of \( V \), and let \( \Omega_{\text{hor}} \hat{V} \) be the dual bundle of \( T_{\text{hor}} \hat{V} \). For all positive integers \( l \geq 1 \) and all integers \( \varpi \in \mathbb{Z} \), we use the standard notation \( \text{Sym}^l \Omega_V \) to denote the symmetric \( l \)-tensor-power of the vector bundle \( \Omega_V \), and we use \( \text{Sym}^l \Omega_V(\varpi) \) to denote the twisted vector bundle \( \text{Sym}^l \Omega_V \otimes \mathcal{O}_V(\varpi) \).

**Proposition 6.1.** For two fixed integers \( l \geq 1, \varpi \in \mathbb{Z} \), and for every Zariski open set \( U \subset V \) together with its cone \( \hat{U} := \pi^{-1}(U) \), there is a canonical injection:

\[
\Gamma(U, \text{Sym}^l \Omega_V(\varpi)) \hookrightarrow \Gamma(\hat{U}, \text{Sym}^l \Omega_{\text{hor}} \hat{V}),
\]

whose image is the set of sections \( \Phi \) enjoying:

\[
\Phi(\lambda z, [\lambda \xi]) = \lambda^\varpi \Phi(z, [\xi]),
\]

for all \( z \in \hat{U} \), for all \( [\xi] \in T_{\text{hor}} \hat{V}|_z \) and for all \( \lambda \in \mathbb{K}^\times \).

**Proof.** Note that we have two canonical injections of vector bundles:

\[
\pi^* \text{Sym}^l \Omega_V \hookrightarrow \text{Sym}^l \Omega_{\text{hor}} \hat{V},
\]

\[
\pi^* \mathcal{O}_V(\varpi) \hookrightarrow \mathcal{O}_\hat{V},
\]

since the tensor functor is left exact (torsion free) in the category of \( \mathbb{K} \)-vector bundles, the tensoring of the above two injections remains an injection:

\[
\pi^* \text{Sym}^l \Omega_V \otimes \pi^* \mathcal{O}_V(\varpi) \hookrightarrow \text{Sym}^l \Omega_{\text{hor}} \hat{V} \otimes \mathcal{O}_\hat{V}.
\]

Recalling that:

\[
\text{Sym}^l \Omega_V(\varpi) = \text{Sym}^l \Omega_V \otimes \mathcal{O}_V(\varpi),
\]

we can rewrite the above injection as:

\[
\pi^* \text{Sym}^l \Omega_V(\varpi) \hookrightarrow \text{Sym}^l \Omega_{\text{hor}} \hat{V}.
\]

With \( U \subset X \) Zariski open, applying the global section functor \( \Gamma(\hat{U}, \cdot) \), which is left exact, we receive:

\[
\Gamma(\hat{U}, \pi^* \text{Sym}^l \Omega_V(\varpi)) \hookrightarrow \Gamma(\hat{U}, \text{Sym}^l \Omega_{\text{hor}} \hat{V}).
\]
Lastly, we have an injection:
\[ \Gamma(U, \text{Sym}^l \Omega_V(\heartsuit)) \hookrightarrow \Gamma(\hat{U}, \pi^* \text{Sym}^l \Omega_V(\heartsuit)), \]
whence, by composing the previous two injections, we conclude:
\[ \Gamma(U, \text{Sym}^l \Omega_V(\heartsuit)) \hookrightarrow \Gamma(\hat{U}, \text{Sym}^l \Omega_{\text{hor} \hat{V}}). \]

To view explicitly the image of this injection, notice that in the case \( l = 0 \), it is the standard injection:
\[ \Gamma(U, \text{Sym}^l \Omega_V(\heartsuit)) \hookrightarrow \Gamma(\hat{U}, \text{Sym}^l \Omega_{\text{hor} \hat{V}}) \]
\[ f \mapsto \pi^* f, \]
whose image consists of, as a consequence of the definition (46) above, all functions \( \tilde{f} \) on \( \hat{U} \) satisfying \( \tilde{f}(\lambda z) = \lambda \^ \heartsuit \tilde{f}(z) \), for all \( z \in \hat{U} \) and for all \( \lambda \in \mathbb{K}^\times \).

Furthermore, in the case \( \heartsuit = 0 \), the image of the injection:
\[ \Gamma(U, \text{Sym}^l \Omega_V) \hookrightarrow \Gamma(\hat{U}, \text{Sym}^l \Omega_{\text{hor} \hat{V}}) \]
\[ \omega \mapsto \pi^* \omega, \]
consists of sections \( \tilde{\omega} \) on \( \hat{U} \) satisfying:
\[ \tilde{\omega}(z, [\xi]) = \tilde{\omega}(\lambda z, [\lambda \xi]), \]
for all \( z \in \hat{U} \), all \( [\xi] \in T_{\text{hor} \hat{V}} \big|_z \) and all \( \lambda \in \mathbb{K}^\times \).

As \( \text{Sym}^l \Omega_V(\heartsuit) = \text{Sym}^l \Omega_V \otimes \mathcal{O}_V(\heartsuit) \), composing the above two observations by tensoring the corresponding two injections, we see that any element \( \Phi \) in the image of the injection:
\[ \Gamma(U, \text{Sym}^l \Omega_V(\heartsuit)) \hookrightarrow \Gamma(\hat{U}, \text{Sym}^l \Omega_{\text{hor} \hat{V}}), \]
automatically satisfies (48). On the other hand, for every element \( \Phi \) in \( \Gamma(\hat{U}, \text{Sym}^l \Omega_{\text{hor} \hat{V}}) \) satisfying (48), we can construct the corresponding element \( \phi \) in \( \Gamma(U, \text{Sym}^l \Omega_V(\heartsuit)) \), which maps to \( \Phi \) under the injection (49).

Let \( Y \subset V \) be a regular subvariety. Replacing the underground variety \( V \) by \( Y \), in much the same way we can show:

**Proposition 6.2.** For two fixed integers \( l \geq 1, \heartsuit \in \mathbb{Z} \), and for every Zariski open set \( U \subset Y \) together with its cone \( \hat{U} := \pi^{-1}(U) \), there is a canonical injection:
\[ \Gamma(U, \text{Sym}^l \Omega_V(\heartsuit)) \hookrightarrow \Gamma(\hat{U}, \text{Sym}^l \Omega_{\text{hor} \hat{V}}), \]
whose image is the set of sections \( \Phi \) enjoying:
\[ \Phi(\lambda z, [\lambda \xi]) = \lambda \heartsuit \Phi(z, [\xi]), \]
for all \( z \in \hat{U} \), for all \( [\xi] \in T_{\text{hor} \hat{V}} \big|_z \) and for all \( \lambda \in \mathbb{K}^\times \).

In future applications, we will mainly interest in the sections:
\[ \Gamma(Y, \text{Sym}^l \Omega_V(\heartsuit)), \]
where \( Y = X \) or \( Y = X \cap \{ z_{v_1} = 0 \} \cap \cdots \cap \{ z_{v_\eta} = 0 \} \) for some vanishing coordinate indices \( 0 \leq v_1 < \cdots < v_\eta \leq N \).
6.2. **Global regular symmetric horizontal differential forms.** In our coming applications, we will be mainly concerned with *Fermat-type hypersurfaces* $H_i$ defined by some homogeneous polynomials $F_i$ of the form:

$$ F_i = \sum_{j=0}^{N} A_{i}^{j} z_{j}^{\lambda_{j}} \quad (i = 1 \cdots c+r), $$

where $\lambda_{0}, \ldots, \lambda_{N}$ are some positive integers and where $A_{i}^{j} \in \mathbb{K}[z_{0}, z_{1}, \ldots, z_{N}]$ are some homogeneous polynomials, with all terms of $F_i$ having the same degree:

$$ \text{deg} A_{i}^{j} + \lambda_{j} = \text{deg} F_i =: d_i \quad (i = 1 \cdots c+r; j = 0 \cdots N). $$

Differentiating $F_i$, we receive:

$$ dF_i = \sum_{j=0}^{N} B_{i}^{j} z_{j}^{\lambda_{j}-1}, $$

where:

$$ B_{i}^{j} := z_{j} dA_{i}^{j} + \lambda_{j} A_{i}^{j} dz_{j} \quad (i = 1 \cdots c+r; j = 0 \cdots N). $$

To make the terms of $F_i$ have the same structure as that of $dF_i$, let us denote:

$$ \tilde{A}_{i}^{j} := A_{i}^{j} z_{j}, $$

so that:

$$ F_i = \sum_{j=0}^{N} \tilde{A}_{i}^{j} z_{j}^{\lambda_{j}-1}. $$

Recalling (45), we denote the cone of $X$ by:

$$ \hat{X} = \{ z \in \mathbb{K}^{N+1} \setminus \{ 0 \} : F_i(z) = 0, \forall i = 1 \cdots c+r \}. $$

For all $z \in \hat{X}$ and $[\xi] \in T_{\text{hor}} \hat{V},$ by the very definition (47) of $T_{\text{hor}} \hat{V},$ we have:

$$ \begin{cases} 
\sum_{j=0}^{N} \tilde{A}_{i}^{j} z_{j}^{\lambda_{j}-1}(z) = 0 \quad (i = 1 \cdots c+r), \\
\sum_{j=0}^{N} B_{i}^{j}(z, \xi) z_{j}^{\lambda_{j}-1}(z) = 0 \quad (i = 1 \cdots c).
\end{cases} $$

For convenience, dropping $z, \xi$, we rewrite the above equations as:

$$ \begin{cases} 
\sum_{j=0}^{N} \tilde{A}_{i}^{j} z_{j}^{\lambda_{j}-1} = 0 \quad (i = 1 \cdots c+r), \\
\sum_{j=0}^{N} B_{i}^{j} z_{j}^{\lambda_{j}-1} = 0 \quad (i = 1 \cdots c),
\end{cases} $$

and formally, we view them as a system of linear equations with respect to the unknown variables $z_{0}^{\lambda_{0}-1}, \ldots, z_{N}^{\lambda_{N}-1}$, of which the associated coefficient matrix, of size $(c + r + c) \times (N + 1)$, is:

$$ C := \begin{pmatrix}
\tilde{A}_{0}^{0} & \cdots & \tilde{A}_{0}^{N} \\
\vdots & \ddots & \vdots \\
\tilde{A}_{c+r}^{0} & \cdots & \tilde{A}_{c+r}^{N} \\
B_{0}^{0} & \cdots & B_{0}^{N} \\
\vdots & \ddots & \vdots \\
B_{c}^{0} & \cdots & B_{c}^{N}
\end{pmatrix}. $$
so that the system reads as:

\[
C \begin{pmatrix}
    z_0^{c_0-1} \\
    \vdots \\
    z_N^{c_N-1}
\end{pmatrix} = 0.
\]

(58)

Recalling our assumption:

\[
n = N - (c + r) \geq 1, \quad \text{since } 2c + r \geq N,
\]

we have \(1 \leq n \leq c\).

Let now \(D\) be the upper \((c + r + n) \times (N + 1)\) submatrix of \(C\), i.e. consisting of the first \((c + r + n)\) rows of \(C\):

\[
D := \begin{pmatrix}
    \tilde{\mathbf{A}}^0_1 & \cdots & \tilde{\mathbf{A}}^N_1 \\
    \vdots & \vdots & \vdots \\
    \tilde{\mathbf{A}}^{c+r}_1 & \cdots & \tilde{\mathbf{A}}^{c+r}_N \\
    \tilde{\mathbf{B}}^0_1 & \cdots & \tilde{\mathbf{B}}^N_1 \\
    \vdots & \vdots & \vdots \\
    \tilde{\mathbf{B}}^0_n & \cdots & \tilde{\mathbf{B}}^N_n
\end{pmatrix}.
\]

(59)

For \(j = 0 \cdots N\), let \(\tilde{D}_j\) denote the submatrix of \(D\) obtained by omitting the \((j + 1)\)-th column:

\[
\tilde{D}_j := \begin{pmatrix}
    \tilde{\mathbf{A}}^0_1 & \cdots & \tilde{\mathbf{A}}^j_1 & \cdots & \tilde{\mathbf{A}}^N_1 \\
    \vdots & \vdots & \vdots & \vdots & \vdots \\
    \tilde{\mathbf{A}}^{c+r}_1 & \cdots & \tilde{\mathbf{A}}^{c+r}_j & \cdots & \tilde{\mathbf{A}}^{c+r}_N \\
    \tilde{\mathbf{B}}^0_1 & \cdots & \tilde{\mathbf{B}}^j_1 & \cdots & \tilde{\mathbf{B}}^N_1 \\
    \vdots & \vdots & \vdots & \vdots & \vdots \\
    \tilde{\mathbf{B}}^0_n & \cdots & \tilde{\mathbf{B}}^j_n & \cdots & \tilde{\mathbf{B}}^N_n
\end{pmatrix},
\]

(60)

and let \(D_j\) denote the \((j + 1)\)-th column of \(D\).

Denote:

\[
W_j := \{z_j \neq 0\} \subset \mathbb{R}^N \quad (j = 0 \cdots N)
\]

(61)

the canonical affine open subsets, whose cones are:

\[
\tilde{W}_j := \pi^{-1}(W_j) \subset \mathbb{R}^{N+1} \setminus \{0\}.
\]

(62)

Denote also:

\[
U_j := W_j \cap X
\]

(63)

the open subsets of \(X\), whose cones are:

\[
\tilde{U}_j := \pi^{-1}(U_j) \subset \tilde{X}.
\]

(64)

Recalling the horizontal tangent bundle of \(\mathbb{R}^{N+1}\):

\[
T_{\text{hor}} \mathbb{R}^{N+1} = \{(z, [\xi]) : z \in \mathbb{R}^{N+1} \setminus \{0\} \text{ and } [\xi] \in \mathbb{R}^{N+1}/\mathbb{R} \cdot z\},
\]

now let \(\Omega_{\text{hor}} \mathbb{R}^{N+1}\) be its dual bundle.
Proposition 6.3. For every $j = 0 \cdots N$, on the affine set:

$$\tilde{W}_j = \{z_j \neq 0\} \subset \mathbb{K}^{N+1} \setminus \{0\},$$

the following affine symmetric horizontal differential n-form is well defined:

$$\tilde{\omega}_j := \frac{(-1)^j}{z_j^{N-j+1}} \det(\tilde{D}_j) \in \Gamma(\tilde{W}_j, \text{Sym}^n \Omega^\kappa_{\text{hor}} \mathbb{K}^{N+1}). \quad (65)$$

The essence of this proposition lies in the famous Euler’s Identity.

Lemma 6.4. [Euler’s Identity] For every homogeneous polynomial $P \in \mathbb{K}[z_0, \ldots, z_N]$, one has:

$$\sum_{j=0}^{N} \frac{\partial F}{\partial z_j} \cdot z_j = \deg F \cdot F,$$

where using differential writes as:

$$dF \bigg|_z = dF(z, z) = \deg F \cdot F(z), \quad (66)$$

at all points $z = (z_0, \ldots, z_N) \in \mathbb{K}^{N+1}$. □

Proof of Proposition 6.3. Without loss of generality, we only prove the case $j = 0$.

Recalling the notation (55) and (54) where all $\tilde{A}_j^i$ are regular functions and all $B_j^i$ are regular 1-forms on $\mathbb{K}^{N+1}$, we can see without difficulty that:

$$\tilde{\omega}_0 = \frac{1}{z_0^{N-1}} \det(\tilde{D}_0)$$

$$= \frac{1}{z_0^{N-1}} \det \begin{pmatrix} \tilde{A}_1^1 & \cdots & \tilde{A}_1^N \\ \vdots & \ddots & \vdots \\ B_1^1 & \cdots & B_1^N \end{pmatrix} \in \Gamma(\tilde{W}_0, \text{Sym}^n \Omega^\kappa \mathbb{K}^{N+1})$$

is a well defined regular symmetric differential n-form. Now we need an:

Observation 6.5. Let $N \geq 1$ be a positive integer, let $L$ be a field with $\text{Card} L = \infty$, and let $F$ be a polynomial:

$$F \in L[z_0, \ldots, z_N].$$

Then $F$ is a polynomial without the variable $z_0$:

$$F \in L[z_1, \ldots, z_N] \subset L[z_0, \ldots, z_N]$$

if and only if the evaluation map:

$$\text{ev}_F: \quad L^{N+1} \longrightarrow L$$

$$(x_0, \ldots, x_N) \longmapsto F(x_0, \ldots, x_N)$$

is independent of the first variable $x_0 \in L$. □
For the same reason as the above Observation, in order to show that $\hat{\omega}_0$ descends to a regular symmetric horizontal differential form in $\Gamma(\hat{V}_0, \text{Sym}^n \Omega_{\text{hor}}^{K^{N+1}})$, we only have to show, at every point $z \in \hat{V}_0$, for all $\xi \in T_z K^{N+1} \cong K^{N+1}$, $\lambda \in K^+$, that:

$$\hat{\omega}_0(z, \xi + \lambda z) = \hat{\omega}_0(z, \xi). \quad (67)$$

In fact, applying Euler’s Identity (66) to the above formula (54), we receive:

$$B^i_j(z, z) = \lambda_j A^i_j(z) dz_j(z, z) + dA^i_j(z, z) z_j(z)$$

$$= \lambda_j A^i_j(z) z_j(z) + \deg A^i_j(z) z_j(z)$$

$$= (\lambda_j + \deg A^i_j) \tilde{A}^i_j(z).$$

Since $B^i_j$ are 1-forms, we obtain:

$$B^i_j(z, \xi + \lambda z) = B^i_j(z, \xi) + \lambda B^i_j(z, z)$$

$$= B^i_j(z, \xi) + \lambda (\lambda_j + \deg A^i_j) \tilde{A}^i_j(z).$$

Therefore, the matrix:

$$\begin{pmatrix}
\tilde{A}^1_i & \cdots & \tilde{A}^N_i \\
\vdots & \ddots & \vdots \\
\tilde{A}^1_{c+r} & \cdots & \tilde{A}^N_{c+r} \\
B^1_i & \cdots & B^N_i \\
\vdots & \ddots & \vdots \\
B^1_n & \cdots & B^N_n
\end{pmatrix}(z, \xi + \lambda z)$$

not only has the same first $c + r$ rows as the matrix:

$$\begin{pmatrix}
\tilde{A}^1_i & \cdots & \tilde{A}^N_i \\
\vdots & \ddots & \vdots \\
\tilde{A}^1_{c+r} & \cdots & \tilde{A}^N_{c+r} \\
B^1_i & \cdots & B^N_i \\
\vdots & \ddots & \vdots \\
B^1_n & \cdots & B^N_n
\end{pmatrix}(z, \xi),$$

but also for $\ell = 1 \cdots n$, the $(c + r + \ell)$-th row of the former one equals to the $(c + r + \ell)$-th row of the latter one plus a multiple of the $\ell$-th row. Therefore both matrices have the same determinant, which verifies (67). □

Inspired by the explicit global symmetric differential forms in Lemma 4.5 of Brothbek’s paper [7], we carry out a simple proposition employing the above notation. First, let us recall the well known Cramer’s rule in a less familiar formulation (cf. [39, p. 513, Theorem 4.4]).

**Theorem 6.6.** [Cramer’s rule] *In a commutative ring $R$, for all positive integers $N \geq 1$, let:

$$A^0, A^1, \ldots, A^N \in R^N$$

be $(N + 1)$ column vectors, and suppose that $z_0, z_1, \ldots, z_N \in R$ satisfy:

$$A^0 z_0 + A^1 z_1 + \cdots + A^N z_N = 0. \quad (68)$$*
Then for all index pairs \(0 \leq i < j \leq N\), there holds the identity:
\[
(-1)^i \det (A^0, \ldots, \widehat{A^i}, \ldots, A^N) z_i = (-1)^j \det (A^0, \ldots, \widehat{A^j}, \ldots, A^N) z_j.
\]  

(69)

Proof. By permuting the indices, without loss of generality, we may assume \(i = 0\).

First, note that (68) yields:
\[
A^0 z_0 = - \sum_{\ell=1}^{N} A^\ell z_\ell.
\]  

(70)

Hence we may compute the left hand side of (69) as:
\[
(-1)^j \det (A^0, A^1, \ldots, \widehat{A^j}, \ldots, A^N) z_0 = (-1)^j \det \left( - \sum_{\ell=1}^{N} A^\ell z_\ell, A^1, \ldots, \widehat{A^j}, \ldots, A^N \right)
\]

[Substitute (70)]
\[
= (-1)^j+1 \sum_{\ell=1}^{N} \det (A^\ell, A^1, \ldots, \widehat{A^j}, \ldots, A^N) z_\ell
\]

[Only \(\ell = j\) is nonzero]
\[
= (-1)^{j+1} \det (A^1, A^1, \ldots, \widehat{A^j}, \ldots, A^N) z^j
\]

\[
= (-1)^0 \det (\widehat{A^0}, A^1, \ldots, A^N) z^j,
\]

which is exactly the right hand side. \(\square\)

**Proposition 6.7.** The following \((N + 1)\) affine regular symmetric horizontal differential \(n\)-forms:
\[
\hat{\omega}_j := \frac{(-1)^j}{z_j^{N+1}} \det (\overline{D}_j) \in \Gamma\left( \overline{U}_j, \Sym^n \Omega^\text{hor} \hat{V} \right) \quad (j = 0 \ldots N)
\]

glue together to make a regular symmetric horizontal differential \(n\)-form on \(\hat{X}\):
\[
\omega \in \Gamma\left( \hat{X}, \Sym^n \Omega^\text{hor} \hat{V} \right).
\]

**Proof.** Our proof divides into two parts.

**Part 1:** To show that these affine regular symmetric horizontal differential \(n\)-forms \(\hat{\omega}_0, \ldots, \hat{\omega}_N\) are well defined.

**Part 2:** To show that any two different affine regular symmetric horizontal differential \(n\)-forms \(\hat{\omega}_{j_1}\) and \(\hat{\omega}_{j_2}\) glue together along the intersection set \(\overline{U}_{j_1} \cap \overline{U}_{j_2}\).

**Proof of Part 1.** The Proposition 6.3 above shows that the:
\[
\hat{\omega}_j := \frac{(-1)^j}{z_j^{N+1}} \det (\overline{D}_j) \in \Gamma\left( \overline{W}_j, \Sym^n \Omega^\text{hor} \hat{V} \right) \quad (j = 0 \ldots N),
\]

are well defined, where:
\[
\overline{W}_j = \{z_j \neq 0\} \subset \mathbb{K}^{N+1} \setminus \{0\}.
\]

Thanks to the canonical inclusion embedding of vector bundles:
\[
\left( \overline{U}_j, T^\text{hor} \hat{V} \right) \hookrightarrow \left( \overline{W}_j, T^\text{hor} \hat{V} \right)
\]
a pullback of \(\hat{\omega}_j\) concludes the first part.
Proof of Part 2. Recalling the equations (58), in particular, granted that \(D\) consists of the first \((c + r + n)\) rows of \(C\), we have:

\[
D \begin{pmatrix}
\zeta_0^{l_0-1} \\
\vdots \\
\zeta_N^{l_N-1}
\end{pmatrix} = 0.
\]

Now applying the above Cramer’s rule to all the \((N + 1)\) columns of \(D\) and the \((N + 1)\) values \(\zeta_0^{l_0-1}, \ldots, \zeta_N^{l_N-1}\), for every index pair \(0 \leq j_1 < j_2 \leq N\), we receive:

\[
(-1)^{j_2} \det(\widehat{D}_{j_2}) \zeta_{j_1}^{l_{j_1}-1} = (-1)^{j_1} \det(\widehat{D}_{j_1}) \zeta_{j_2}^{l_{j_2}-1}.
\]

When \(z_{j_1} \neq 0, z_{j_2} \neq 0\), this immediately yields:

\[
\frac{(-1)^{j_1}}{\zeta_{j_1}^{l_{j_1}-1}} \det(\widehat{D}_{j_1}) = \frac{(-1)^{j_2}}{\zeta_{j_2}^{l_{j_2}-1}} \det(\widehat{D}_{j_2}),
\]

thus the two affine symmetric horizontal differential \(n\)-forms \(\widehat{\omega}_{j_1}\) and \(\widehat{\omega}_{j_2}\) glue together along their overlap set \(\widehat{U}_{j_1} \cap \widehat{U}_{j_2}\). \(\square\)

By permuting the indices, the above Proposition 6.7 can be trivially generalized to, instead of the particular upper \((c + r + n) \times (N + 1)\) submatrix \(D\), all \((c + r + n) \times (N + 1)\) submatrices of \(C\) containing the upper \(c + r\) rows, as follows.

For all \(n\) ascending positive integers \(1 \leq j_1 < \cdots < j_n \leq c\), denote \(C_{j_1, \ldots, j_n}\) the \((c + r + n) \times (N + 1)\) submatrix of \(C\) consisting of the first upper \(c + r\) rows and the rows \(c + r + j_1, \ldots, c + r + j_n\). Also, for \(j = 0 \cdots N\), let \(\widehat{C}_{j_1, \ldots, j_n; j}\) denote the submatrix of \(C_{j_1, \ldots, j_n}\) obtained by omitting the \((j + 1)\)-th column.

**Proposition 6.8.** The following \((N + 1)\) affine regular symmetric horizontal differential \(n\)-forms:

\[
\widehat{\omega}_{j_1, \ldots, j_n; j} := \frac{(-1)^j}{\zeta_{j}^{l_{j}-1}} \det(\widehat{C}_{j_1, \ldots, j_n; j}) \in \Gamma(\widehat{U}_j, \text{Sym}^n \Omega_{\text{hor}} \mathcal{V}) \quad (j = 0 \cdots N)
\]

glue together to make a global regular symmetric horizontal differential \(n\)-form on \(\widehat{X}\). \(\square\)

One step further, the above Proposition 6.8 can be generalized to a larger class of \((c + r + n) \times (N + 1)\) submatrices of \(C\), as follows.

For any two positive integers \(l_1 \geq l_2\) with:

\[
l_1 + l_2 = c + r + n,
\]

for any two ascending sequences of positive indices:

\[
1 \leq i_1 < \cdots < i_{l_1} \leq c + r,
\]

\[
1 \leq j_1 < \cdots < j_{l_2} \leq c
\]

satisfying:

\[
\{j_1, \ldots, j_{l_2}\} \subset \{i_1, \ldots, i_{l_1}\},
\]

denote \(C_{j_1, \ldots, j_{l_2}}^{l_1, \ldots, l_{l_2}}\) the \((c + r + n) \times (N + 1)\) submatrix of \(C\) consisting of the rows \(i_1, \ldots, i_{l_1}\) and the rows \(c + r + j_1, \ldots, c + r + j_{l_2}\). Also, for \(j = 0, \ldots, N\), let \(\widehat{C}_{j_1, \ldots, j_{l_2}; j}^{l_1, \ldots, l_{l_2}}\) denote the submatrix of \(C_{j_1, \ldots, j_{l_2}}^{l_1, \ldots, l_{l_2}}\) obtained by omitting the \((j + 1)\)-th column.
By much the same proof of Proposition 6.7, we obtain:

**Proposition 6.9.** The following $N + 1$ affine regular symmetric horizontal differential $l_2$-forms:

$$
\bar{\omega}_{j_1, \ldots, j_l_j} := \frac{(-1)^j}{\zeta_j^{l_j-1}} \det(\tilde{\omega}_{j_1, \ldots, j_l_j}) \in \Gamma(\tilde{U}_j, \text{Sym}^{l_2} \Omega_{\text{hor}} \tilde{V})
$$

(glue together to make a global regular symmetric horizontal differential $l_2$-form:

$$
\bar{\omega}_{j_1, \ldots, j_l_j} \in \Gamma(\tilde{X}, \text{Sym}^{l_2} \Omega_{\text{hor}} \tilde{V}).
$$

6.3. **Global twisted regular symmetric differential forms.** Now, using the structure of the above explicit global forms, and applying the previous Proposition 6.2, we receive a result which, in the case of pure Fermat-type hypersurfaces (1) where all $\lambda_0 = \cdots = \lambda_N = \epsilon$ are equal, with also equal $\deg F_1 = \cdots = \deg F_\epsilon = e + \epsilon$, coincides with Brotbek’s Lemma 4.5 in [7]; Brotbek also implicitly obtained such symmetric differential forms by his cohomological approach.

**Proposition 6.10.** Under the assumptions and notation of Proposition 6.9, the global regular symmetric horizontal differential $l_2$-form $\bar{\omega}_{j_1, \ldots, j_l_j}$ is the image of a global twisted regular symmetric differential $l_2$-form:

$$
\omega_{j_1, \ldots, j_l_j} \in \Gamma(X, \text{Sym}^{l_2} \Omega_V(\varnothing))
$$

under the canonical injection as a particular case of Proposition 6.2:

$$
\Gamma(X, \text{Sym}^{l_2} \Omega_V(\varnothing)) \hookrightarrow \Gamma(\tilde{X}, \text{Sym}^{l_2} \Omega_{\text{hor}} \tilde{V}),
$$

where the degree:

$$
\varnothing := \sum_{p=1}^{l_1} \deg F_{i_p} + \sum_{q=1}^{l_2} \deg F_{j_q} - \sum_{j=0}^{N} \lambda_j + N + 1.
$$

(71)

For all homogeneous polynomials $P \in \Gamma(P^N, \mathcal{O}_P(\deg P))$, by multiplication, one receives more global twisted regular symmetric differential $l_2$-forms:

$$
P \omega_{j_1, \ldots, j_l_j} \in \Gamma(X, \text{Sym}^{l_2} \Omega_V(\deg P + \varnothing)).
$$

(72)

It is worth to mention that, again by applying Cramer’s rule in linear algebra, one can construct determinantal shape sections concerning higher-order jet bundles on Fermat type hypersurfaces, as well as on their intersections.

**Proof.** According to the criterion (50) of Proposition 6.2, it is necessary and sufficient to show, for all $z \in \tilde{X}$, for all $[\xi] \in T_{\text{hor}} \tilde{V}|_z$, and for all $\lambda \in \mathbb{K}^\times$, that:

$$
\bar{\omega}_{j_1, \ldots, j_l_j}(\lambda z, [\lambda \xi]) = \lambda^\varnothing \bar{\omega}_{j_1, \ldots, j_l_j}(z, [\xi]).
$$

(72)
We may assume $z \in \tilde{U}_0 = \{z_0 \neq 0\}$ for instance. Now, applying Proposition 6.9, we receive:

$$\hat{\omega}_{i_1, \ldots, i_l 1, \ldots, j_l 2}^{(1)}(\lambda z, [\lambda \xi]) = \hat{\omega}_{i_1, \ldots, i_l 2}^{(1)}(\lambda z, [\lambda \xi]),$$

and therefore satisfies:

$$\tilde{A}_{i_p}^{(1)}(\lambda z) = \lambda^{\deg F_{i_p} - \lambda_j + 1} \tilde{A}_{i_p}^{(1)}(z). \quad (74)$$

Recalling also the notation (54), the entry $B_{j_q}^{(1)}$ is a 1-form satisfying:

$$B_{j_q}^{(1)}(\lambda z, [\lambda \xi]) = \lambda^{\deg F_{j_q} - \lambda_j + 1} B_{j_q}^{(1)}(z, [\xi]).$$

With the help of the above two entry identities (74) and (75), each term in the above sum equals to:

$$\text{sign}(\sigma) \tilde{A}_{i_1}^{(1)} \cdots \tilde{A}_{i_l}^{(1)} B_{j_1}^{(1)} \cdots B_{j_l}^{(1)} (\lambda z, \lambda \xi).$$
multiplied by $\lambda^j$, where:

$$
? = \sum_{p=1}^{l_1} (\deg F_{ip} - \lambda_{\sigma(p)} + 1) + \sum_{q=1}^{l_2} (\deg F_{jq} - \lambda_{\sigma(1+q)} + 1)
$$

$$
= \sum_{p=1}^{l_1} \deg F_{ip} + \sum_{q=1}^{l_2} \deg F_{jq} - (\sum_{p=1}^{l_1} \lambda_{\sigma(p)} + \sum_{q=1}^{l_2} \lambda_{\sigma(1+q)}) + l_1 + l_2
= \sum_{j=1}^{l+q} \lambda_j N
$$

[Use (71)]

$$
= \varphi + \lambda_0 - 1,
$$

therefore (76) factors as:

$$
\lambda^{\sigma, h_0 - 1} \sum_{\sigma \in S_N} \text{sign}(\sigma) \tilde{A}_{n_1}^{\sigma(1)} \cdots \tilde{A}_{n_1}^{\sigma(l)} \cdot \tilde{B}_{j_1}^{\sigma(l+1)} \cdots \tilde{B}_{j_2}^{\sigma(l+q)} (z, \xi) = \lambda^{\sigma, h_0 - 1} (z, \xi),
$$

and thus (73) becomes:

$$
\frac{1}{(\lambda \xi_0)^{h_0-1}} \lambda^{\sigma, h_0 - 1} \begin{pmatrix}
\tilde{A}_{n_1}^{i} & \cdots & \tilde{A}_{n_1}^{N} \\
\tilde{B}_{j_1}^{i} & \cdots & \tilde{B}_{j_1}^{N} \\
\vdots & \cdots & \vdots \\
\tilde{B}_{j_2}^{i} & \cdots & \tilde{B}_{j_2}^{N}
\end{pmatrix} (z, \xi) = \lambda^{\sigma} \frac{1}{\xi_0^{h_0-1}} \begin{pmatrix}
\tilde{A}_{n_1}^{i} & \cdots & \tilde{A}_{n_1}^{N} \\
\tilde{B}_{j_1}^{i} & \cdots & \tilde{B}_{j_1}^{N} \\
\vdots & \cdots & \vdots \\
\tilde{B}_{j_2}^{i} & \cdots & \tilde{B}_{j_2}^{N}
\end{pmatrix} (z, \xi)
$$

which is exactly our desired equality (72). □

Now, let $K$ be the $(c + r + c) \times (N + 1)$ matrix whose first $c + r$ rows consist of all $(N + 1)$ terms in the expressions of $F_1, \ldots, F_{c+r}$ in the exact order, i.e. the $(i, j)$-th entry of $K$ is:

$$K_{i,j} := A_i^{j-1} \tilde{z}_j^{l-1} \quad (i=1 \cdots c+r, j=1 \cdots N+1),
$$

and whose last $c$ rows consist of all $(N + 1)$ terms in the expressions of $dF_1, \ldots, dF_c$ in the exact order, i.e. the $(c + r + i, j)$-th entry of $K$ is:

$$K_{c+r+i,j} := d(A_i^{j-1} \tilde{z}_j^{l-1}) \quad (i=1 \cdots c, j=1 \cdots N+1).
$$

The $j$-th column $K_j$ of $K$ and the $j$-th column $C_j$ of $C$ are proportional:

$$K_j = C_j \tilde{z}_j^{l-1} \quad (j=1 \cdots N+1).
$$

(78)
In later applications, we will use Proposition 6.10 in the case:

\[ l_1 = c + r, \quad l_2 = n, \]

and in abbreviation, dropping the upper indices, we will write these global symmetric differential forms \( \omega^{1, \ldots, c+r}_{j_1, \ldots, j_n} \) as \( \omega_{j_1, \ldots, j_n} \). Since we will mainly consider the case where all coordinates are nonvanishing:

\[ z_0 \neq 0, \ldots, z_N \neq 0, \]

the corresponding symmetric horizontal differential \( n \)-forms of Proposition 6.8 read, in the set \( \{ z_0 \cdots z_N \neq 0 \} \), as:

\[
\tilde{\omega}_{j_1, \ldots, j_n} \cdot \omega = \frac{(-1)^j}{z_j^{l_j-1}} \det (\hat{C}_{j_1, \ldots, j_n})
\]

[Use (78)]

\[
= \frac{(-1)^j}{z_j^{l_j-1}} \left( \prod_{0 \leq l \leq n, l \neq j} \frac{1}{z_l^{l_l-1}} \right) \det (\hat{K}_{j_1, \ldots, j_n})
\]

(79)

where \( \hat{K}_{j_1, \ldots, j_n} \) is defined as an analog of \( \hat{C}_{j_1, \ldots, j_n} \) in the obvious way.

6.4. \textbf{Regular twisted symmetric differential forms with some vanishing coordinates.} Investigating further the construction of symmetric differential forms via Cramer’s rule, for every integer \( 1 \leq \eta \leq n - 1 \), for every sequence of ascending indices:

\[ 0 \leq v_1 < \cdots < v_\eta \leq N, \]

by focusing on the intersection of \( X \) with the \( \eta \) coordinate hyperplanes:

\[ v_1, \ldots, v_\eta X := X \cap \{ z_{v_1} = 0 \} \cap \cdots \cap \{ z_{v_\eta} = 0 \}, \]

we can also construct several twisted symmetric differential \((n - \eta)\)-forms:

\[ \Gamma(v_1, \ldots, v_\eta X, \text{Sym}^{n-\eta} \Omega_v(\omega)) \quad (? \text{ are twisted degrees}) \]

as follows, which will be essential ingredients towards the solution of the Debarre Ampleness Conjecture.

For every two positive integers \( l_1 \geq l_2 \) with:

\[ l_1 + l_2 = c + r + n - \eta = N - \eta, \]

for any two sequences of ascending positive integers:

\[ 1 \leq i_1 < \cdots < i_{l_1} \leq c + r \]

\[ 1 \leq j_1 < \cdots < j_{l_2} \leq c \]

such that the second one is a subsequence of the first one:

\[ \{ j_1, \ldots, j_{l_2} \} \subset \{ i_1, \ldots, i_{l_1} \}, \]

let us denote by \( v_1, \ldots, v_\eta \mathcal{C}_{j_1, \ldots, j_{l_2}}^{i_1, \ldots, i_{l_1}} \) the \((N - \eta) \times (N - \eta + 1)\) submatrix of \( \mathcal{C} \) determined by the \((N - \eta)\) rows \( i_1, \ldots, i_{l_1}, c + r + j_1, \ldots, c + r + j_{l_2} \) and the \((N - \eta + 1)\) columns which are complement to the columns \( v_1 + 1, \ldots, v_\eta + 1 \). Also, for every index \( j \in \{0, \ldots, N\} \setminus \{v_1, \ldots, v_\eta\} \), let \( v_1, \ldots, v_\eta \mathcal{C}_{j_1, \ldots, j_{l_2}}^{i_1, \ldots, i_{l_1}} \).
denote the submatrix of \( v_{1,\ldots, \eta} C_{j_1,\ldots, j_2}^{i_1,\ldots, i_l} \) obtained by deleting the column which is originally contained in the \((j+1)\)-th column of \( C \). Analogously to (61)-(64), we denote:

\[
v_{1,\ldots, \eta} W_j := \{z_{v_1} = 0\} \cap \cdots \cap \{z_{v_\eta} = 0\} \cap \{z_j \neq 0\} \subset \mathbb{P}^N,
\]

whose cone is:

\[
v_{1,\ldots, \eta} \mathcal{W}_j := \pi^{-1}(v_{1,\ldots, \eta} W_j) \subset \mathbb{R}^{N+1} \setminus \{0\},
\]

and we denote also:

\[
v_{1,\ldots, \eta} U_j := v_{1,\ldots, \eta} W_j \cap X \subset v_{1,\ldots, \eta} X,
\]

whose cone is:

\[
v_{1,\ldots, \eta} \mathcal{U}_j := \pi^{-1}(v_{1,\ldots, \eta} U_j) \subset v_{1,\ldots, \eta} \mathcal{X} := \pi^{-1}(v_{1,\ldots, \eta} X).
\]

Now we have two very analogs of Propositions 6.9 and 6.10.

First, write the \((N-\eta+1)\) remaining numbers of the set-minus:

\[
\{0, \ldots, N\} \setminus \{v_1, \ldots, \eta\}
\]

in the ascending order:

\[
r_0 < r_1 < \cdots < r_{N-\eta}.
\]

It is necessary to assume that \( \lambda_0, \ldots, \lambda_N \geq 2 \).

**Proposition 6.11.** For all \( j = 0 \cdot \cdots \cdot N-\eta \), the following \((N+1-\eta)\) affine regular symmetric horizontal differential 2-forms:

\[
v_{1,\ldots, \eta} \omega_{j_1,\ldots, j_2}^{i_1,\ldots, i_l} := \frac{(-1)^j}{z_{r_j}} \det(\mathcal{C}_{j_1,\ldots, j_2, r_j}^{i_1,\ldots, i_l}) \in \Gamma(v_{1,\ldots, \eta} \mathcal{U}_j, \text{Sym}^2 \Omega_{\text{hor} V})
\]

glue together to make a regular symmetric horizontal differential 2-form on \( v_{1,\ldots, \eta} \mathcal{X} \):

\[
v_{1,\ldots, \eta} \omega_{j_1,\ldots, j_2}^{i_1,\ldots, i_l} \in \Gamma(v_{1,\ldots, \eta} \mathcal{X}, \text{Sym}^2 \Omega_{\text{hor} V}).
\]

**Proposition 6.12.** Under the assumptions and notation of the above proposition, the regular symmetric horizontal differential 2-form \( v_{1,\ldots, \eta} \omega_{j_1,\ldots, j_2}^{i_1,\ldots, i_l} \) on \( v_{1,\ldots, \eta} \mathcal{X} \) is the image of a twisted regular symmetric differential 2-form on \( v_{1,\ldots, \eta} X \):

\[
v_{1,\ldots, \eta} \omega_{j_1,\ldots, j_2}^{i_1,\ldots, i_l} \in \Gamma(v_{1,\ldots, \eta} X, \text{Sym}^2 \Omega_{v}(v_{1,\ldots, \eta} \omega_{j_1,\ldots, j_2}^{i_1,\ldots, i_l}))
\]

under the canonical injection:

\[
\Gamma(v_{1,\ldots, \eta} X, \text{Sym}^2 \Omega_{v}(v_{1,\ldots, \eta} \omega_{j_1,\ldots, j_2}^{i_1,\ldots, i_l})) \hookrightarrow \Gamma(v_{1,\ldots, \eta} \mathcal{X}, \text{Sym}^2 \Omega_{\text{hor} V}),
\]

where the twisted degree is:

\[
\deg \omega_{j_1,\ldots, j_2}^{i_1,\ldots, i_l} := \sum_{p=1}^{l_1} \deg F_{p_1} + \sum_{q=1}^{l_2} \deg F_{q_1} - \left( \sum_{j=0}^{N} \lambda_j - \sum_{\mu=1}^{\eta} \lambda_{\mu} \right) + (N-\eta) + 1.
\]

Furthermore, for all homogeneous polynomials \( P \in \Gamma(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(\deg P)) \), by multiplication, one receives more twisted regular symmetric differential 2-forms:

\[
P \cdot v_{1,\ldots, \eta} \omega_{j_1,\ldots, j_2}^{i_1,\ldots, i_l} \in \Gamma(v_{1,\ldots, \eta} X, \text{Sym}^2 \Omega_{v}(\deg P + v_{1,\ldots, \eta} \omega_{j_1,\ldots, j_2}^{i_1,\ldots, i_l})).
\]
In our coming applications, we will use Proposition 6.12 in the case:

\[ l_1 = c + r, \quad l_2 = n - \eta, \]

and in abbreviation we write these symmetric differential forms \( v_1, \ldots, v_q \omega^1_{j_1, \ldots, j_{n-\eta}} \) as \( v_1, \ldots, v_q \omega_{j_1, \ldots, j_{n-\eta}} \). Since we will mainly consider the case when all coordinates but \( z_{v_1}, \ldots, z_{v_q} \) are nonvanishing:

\[ z_{r_0} \neq 0, \ldots, z_{r_{n-\eta}} \neq 0, \]

the corresponding symmetric horizontal differential \((n-\eta)\)-forms \( v_1, \ldots, v_q \omega_{j_1, \ldots, j_{n-\eta}} \) of Proposition 6.11 read, in the set \( \{ z_{r_0} \cdots z_{r_{n-\eta}} \neq 0 \} \), as:

\[
\begin{align*}
\det (v_1, \ldots, v_q \widehat{C}^{i_1, \ldots, i_k}_{j_1, \ldots, j_{n-\eta}, r_j}) & = \frac{(-1)^j}{z_{r_j}^{a_j - 1}} \det (v_1, \ldots, v_q \widehat{C}^{i_1, \ldots, i_k}_{j_1, \ldots, j_{n-\eta}, r_j}) \\
& = \frac{(-1)^j}{z_{r_0}^{a_0 - 1} \cdots z_{r_{n-\eta}}^{a_{n-\eta} - 1}} \det (v_1, \ldots, v_q \widehat{K}^{i_1, \ldots, i_k}_{j_1, \ldots, j_{n-\eta}, r_j}) \quad (j = 0 \cdots n-\eta),
\end{align*}
\]

where \( \det (v_1, \ldots, v_q \widehat{K}^{i_1, \ldots, i_k}_{j_1, \ldots, j_{n-\eta}, r_j}) \) is defined as an analog of \( \det (v_1, \ldots, v_q \widehat{C}^{i_1, \ldots, i_k}_{j_1, \ldots, j_{n-\eta}, r_j}) \) in the obvious way.

The two formulas (79), (82) will enable us to efficiently narrow the base loci of the obtained symmetric differential forms, as the matrix \( \widehat{K} \) directly copies the original equations/differentials of the hypersurface polynomials \( F_1, \ldots, F_{c+r} \). We will heartily appreciate such a formalism when a wealth of moving coefficient terms happen to tangle together.

6.5. A scheme-theoretic point of view. In future applications, we will only consider symmetric forms in coordinates. Nevertheless, in this subsection, let us reconsider the obtained symmetric forms in an algebraic way, dropping the assumption ‘algebraically-closed’ on the ambient field \( \mathbb{K} \).

Recalling (3), (4), we may denote the projective parameter space of the \( c + r \) hypersurfaces in (51) by:

\[
\mathbb{P}^* = \text{Proj} \mathbb{K} \left[ \left\{ A_{i, \alpha}^j \right\}_{i=1 \cdots c+r, \quad j=0 \cdots N, \quad |\alpha| = d_i - a_j} \right],
\]

so that the hypersurface coefficient polynomials \( A^j_i \) are written as:

\[
A^j_i := \sum_{|\alpha| = d_i - a_j} A_{i, \alpha}^j z^\alpha \quad (i = 1 \cdots c+r, j = 0 \cdots N).
\]

Now, we give a scheme-theoretic explanation of Proposition 6.3, firstly by expressing \( \widehat{\omega}_j \) in terms of affine coordinates.

For every index \( j = 0 \cdots N \), in each affine set:

\[
\widehat{W}_j = \{ z_j \neq 0 \} \subset \mathbb{K}^{N+1} \setminus \{0\},
\]

the \( c + r \) homogeneous hypersurface equations (51) in affine coordinates:

\[
\left( \frac{z_0}{z_j}, \ldots, \frac{\widehat{z}_j}{z_j}, \ldots, \frac{z_N}{z_j} \right)
\]
become:

\[
(F_i)_j = \sum_{k=0}^{N} (A_i^k)_j \left( \frac{z_k}{z_j} \right)^{l_k}
\]

[see (55)]

\[
= \sum_{k=0}^{N} (\tilde{A}_i^k)_j \left( \frac{z_k}{z_j} \right)^{l_k-1},
\]

where for any homogeneous polynomial \( P \), we dehomogenize:

\[
(P)_j := \frac{P}{z_j^{\deg P}}.
\]

Differentiating (84) for \( i = 1 \cdots c \), we receive:

\[
d(F_i)_j = \sum_{k=0}^{N} B_{i,j}^k \left( \frac{z_k}{z_j} \right)^{l_k-1},
\]

where:

\[
B_{i,j}^k := \frac{z_k}{z_j} d (A_i^k)_j + \lambda_k (A_i^k)_j \left( \frac{z_k}{z_j} \right) \quad (j, k = 0 \cdots N).
\]

Computing \( z_k d (A_i^k)_j \), we receive:

\[
z_k d (A_i^k)_j = z_k \left( \frac{A_i^k}{z_j^{d_i-A_k}} \right) \quad \text{[use (52)]}
\]

[Leibniz’s rule]

\[
= \frac{z_k d A_i^k}{z_j^{d_i-A_k}} - (d_i - \lambda_k) \frac{z_k A_i^k}{z_j^{d_i-A_k+1}} dz_j
\]

[use (53)]

\[
= (B_{i,j}^k - \lambda_k A_i^k dz_k) \frac{1}{z_j^{d_i-A_k}} - (d_i - \lambda_k) \frac{z_k A_i^k}{z_j^{d_i-A_k+1}} dz_j,
\]

therefore (85) become:

\[
B_{i,j}^k = (B_{i,j}^k - \lambda_k A_i^k dz_k) \frac{1}{z_j^{d_i-A_k+1}} - (d_i - \lambda_k) \frac{z_k A_i^k}{z_j^{d_i-A_k+2}} dz_j + \lambda_k (A_i^k)_j \left( \frac{z_k}{z_j} \right)
\]

\[
= \frac{B_{i,j}^k}{z_j^{d_i-A_k+1}} - \lambda_k (A_i^k)_j \frac{dz_k}{z_j} - (d_i - \lambda_k) (A_i^k)_j \frac{z_k}{z_j^2} dz_j + \lambda_k (A_i^k)_j \left( \frac{dz_k}{z_j} - \frac{z_k}{z_j^2} dz_j \right)
\]

\[
= \frac{1}{z_j^{d_i-A_k+1}} B_{i,j}^k - d_i \frac{z_k}{z_j^2} dz_j (A_i^k)_j
\]

\[
= \frac{1}{z_j^{d_i-A_k+1}} B_{i,j}^k - d_i \frac{z_k}{z_j^2} dz_j (\tilde{A}_i^k)_j.
\]

(86)
Recalling the matrix \( C \) in (57), which is obtained by copying the homogeneous hypersurface equations \( F_1, \ldots, F_{c+r} \) and the differentials \( dF_1, \ldots, dF_c \), we define the matrix:

\[
(C)_j := \begin{pmatrix}
(A_0^0)_j & \cdots & (A_1^N)_j \\
\vdots & & \vdots \\
(A_{c+r}^0)_j & \cdots & (A_{c+r}^N)_j \\
B_1^0 & \cdots & B_1^N \\
\vdots & & \vdots \\
B_{c,j}^0 & \cdots & B_{c,j}^N \\
\end{pmatrix},
\]

which is obtained by copying the dehomogenized hypersurface equations \( (F_1)_j, \ldots, (F_{c+r})_j \) and the differentials \( d(F_1)_j, \ldots, d(F_c)_j \). Recalling the matrices (59), (60), in the obvious way we also define \( (D)_j \), \( (\hat{D}_k)_j \) as:

\[
(D)_j := \begin{pmatrix}
(A_0^0)_j & \cdots & (A_1^N)_j \\
\vdots & & \vdots \\
(A_{c+r}^0)_j & \cdots & (A_{c+r}^N)_j \\
B_1^0 & \cdots & B_1^N \\
\vdots & & \vdots \\
B_{n,j}^0 & \cdots & B_{n,j}^N \\
\end{pmatrix},
\]

and:

\[
(\hat{D}_k)_j := \begin{pmatrix}
(A_0^0)_j & \cdots & (A_1^N)_j & \cdots & (A_1^N)_j \\
\vdots & & \vdots & & \vdots \\
(A_{c+r}^0)_j & \cdots & (A_{c+r}^N)_j & \cdots & (A_{c+r}^N)_j \\
B_1^0 & \cdots & B_1^N & & B_1^N \\
\vdots & & \vdots & & \vdots \\
B_{n,j}^0 & \cdots & B_{n,j}^N & & B_{n,j}^N \\
\end{pmatrix} \quad (k = 0 \cdots N).
\]

Recalling \( \hat{\omega}_j \) of Proposition 6.3, now thanks to (86), we have the following nice:

**Observation 6.13.** For every \( j = 0 \cdots N \), one has the identity:

\[
\hat{\omega}_j = \frac{(-1)^j}{z_j^{\lambda_j - 1}} \det (\hat{D}_j) = \frac{(-1)^j}{z_j^{\lambda_j}} \det ((\hat{D}_j)_j),
\]

where for the moment \( \heartsuit \) is defined in (71) for \( \omega_1^{1, \ldots, c+r} \):

\[
\heartsuit := \sum_{p=1}^{c+r} d_p + \sum_{q=1}^{N} d_q - \sum_{j=0}^{N} \lambda_j + N + 1.
\]

The proof is much the same as that of Proposition 6.10, hence we omit it here. \( \square \)
Now, let $p_1, p_2$ be the two canonical projections:

\[
\begin{array}{ccc}
\mathbb{P}_K^* \times \mathbb{P}_K^N & \xrightarrow{p_1} & \mathbb{P}_K^N \\
& & \\
\downarrow & & \\
& & \mathbb{P}_K^* \\
\end{array}
\]

Then thanks to the formula (90), we may view $\hat{\omega}_j$ as a section of the twisted sheaf:

\[
\text{Sym}^n \Omega^1_{\mathbb{P}_K^* \times \mathbb{P}_K^N} \otimes p_1^* \mathcal{O}_{\mathbb{P}_K^N}(N) \otimes p_2^* \mathcal{O}_{\mathbb{P}_K^N}(\nu)
\]

over the pullback:

\[
p_2^{-1}(W_j) \subset \mathbb{P}_K^* \times \mathbb{P}_K^N
\]

of the canonical affine scheme

\[
W_j := D(z_j) \subset \mathbb{P}_K^N.
\]

Using the same notation as (5), (6), recalling (51), (83), we now introduce the two subschemes:

\[
\mathcal{X} \subset \mathcal{V} \subset \mathbb{P}_K^* \times \mathbb{P}_K^N,
\]

where $\mathcal{X}$ is defined by ‘all’ the $c + r$ bihomogeneous polynomials:

\[
\mathcal{X} := V(\sum_{j=0}^{N} A_1^j z_j^{A_j}, \ldots, \sum_{j=0}^{N} A^j z_j^{A_j}, \ldots, \sum_{j=0}^{N} A_{c+r}^j z_j^{A_j}),
\]

and where $\mathcal{V}$ is defined by the ‘first’ $c$ bihomogeneous polynomials:

\[
\mathcal{V} := V(\sum_{j=0}^{N} A_1^j z_j^{A_j}, \ldots, \sum_{j=0}^{N} A^j z_j^{A_j}).
\]

Now, we may view each entry of the matrix (87) as a section in:

\[
\Gamma \left( \mathcal{X} \cap p_2^{-1}(W_j), \text{Sym}^n \Omega^1_{\mathcal{V}/\mathbb{P}_K^N} \otimes p_1^* \mathcal{O}_{\mathbb{P}_K^N}(1) \right),
\]

where the symmetric degrees are 0 for the first $c + r$ rows and 1 for the last $n$ rows. Noting that the $N + 1$ columns $C^0, \ldots, C^N$ of this matrix satisfy the relation:

\[
\sum_{k=0}^{N} C^k \frac{z_{k+1}^{A_k}}{z_{j+1}^{A_j}} = 0,
\]

in particular, so do the columns of the submatrix (88). Now, recalling the submatrices (89) of (88), an application of Cramer’s rule (Theorem 6.6) yields:

\[
(-1)^{k_1} \det \left( \left( D_{k_1} \right) \right) \frac{z_{k_2}^{A_{k+1}}}{z_j^{A_j}} = (-1)^{k_2} \det \left( \left( D_{k_2} \right) \right) \frac{z_{k_1}^{A_{k+1}}}{z_j^{A_j}}
\]

\[
\in \Gamma \left( \mathcal{X} \cap p_2^{-1}(W_j), \text{Sym}^n \Omega^1_{\mathcal{V}/\mathbb{P}_K^N} \otimes p_1^* \mathcal{O}_{\mathbb{P}_K^N}(N) \right) \quad (k = 0 \cdots N).
\]

Now, recalling (90), we may interpret Proposition 6.7 as follows. First, for $j = 0 \cdots N$, we view each:

\[
\hat{\omega}_j = \frac{(-1)^{j}}{z_j^{\nu}} \det \left( \left( \hat{D}_j \right) \right)
\]
as a section in:
\[ \Gamma \left( \mathcal{X} \cap \text{pr}_2^{-1}(W_j), \text{Sym}^n \Omega^1_{\mathcal{X}/\mathbb{P}^n_k} \otimes \text{pr}_1^* \mathcal{O}_{\mathbb{P}^n_k}(N) \otimes \text{pr}_2^* \mathcal{O}_{\mathbb{P}^2_k}(\nabla) \right). \]

Then, thanks to an observation below, for every different indices \( j_1 < j_2 \), over the open set:
\[ \mathcal{X} \cap \text{pr}_2^{-1}(W_{j_1} \cap W_{j_2}) \subset \mathcal{X}, \]
the twisted sheaf:
\[ \text{Sym}^n \Omega^1_{\mathcal{X}/\mathbb{P}^n_k} \otimes \text{pr}_1^* \mathcal{O}_{\mathbb{P}^n_k}(N) \otimes \text{pr}_2^* \mathcal{O}_{\mathbb{P}^2_k}(\nabla) \]
has the two coinciding sections:
\[
\tilde{\omega}_{j_1} = \frac{(-1)^j}{z_j^{d_j-1}} \det \left( (\tilde{D}_{j_1})_{j_1} \right) \quad \text{[Observation 6.13]}
\]
\[
\text{[ use (91) ]} = \frac{(-1)^j}{z_j^{d_j-1}} z_j^{d_j-1} \det \left( (\tilde{D}_{j_2})_{j_1} \right)
\]
\[
\text{[ Observation 6.14 below ]} = \frac{(-1)^j}{z_j^{d_j-1}} z_j^{d_j-1} \tilde{\omega}_{j_2} = \frac{(-1)^j}{z_j^{d_j-1}} \det \left( (\tilde{D}_{j_2})_{j_2} \right)
\]

Thus, the \( N + 1 \) sections \( \tilde{\omega}_0, \ldots, \tilde{\omega}_N \) glue together to make a global section:
\[ \tilde{\omega} \in \Gamma \left( \mathcal{X}, \text{Sym}^n \Omega^1_{\mathcal{X}/\mathbb{P}^n_k} \otimes \text{pr}_1^* \mathcal{O}_{\mathbb{P}^n_k}(N) \otimes \text{pr}_2^* \mathcal{O}_{\mathbb{P}^2_k}(\nabla) \right). \]

**Observation 6.14.** For all distinct indices \( 0 \leq j_1, j_2 \leq N \), one has the transition identities:
\[
\det \left( (\tilde{D}_{j_2})_{j_1} \right) = \frac{z_j^{d_j-1}}{z_j^{d_j-1}} \det \left( (\tilde{D}_{j_2})_{j_2} \right).
\]

The proof is but elementary computations, so we omit it here. \[\square\]

Next, repeating the same reasoning, using the obvious notation, we interpret Propositions 6.9 and 6.10 as:

**Proposition 6.15.** Each of the following \( N + 1 \) symmetric forms:
\[
\tilde{\omega}_{j_1, \ldots, j_l; j} = \frac{(-1)^j}{z_j^{d_j-1}} \det \left( \tilde{G}_{j_1, \ldots, j_l; j} \right)
\]
\[
\text{can be viewed as a section of:}
\Gamma \left( \text{pr}_2^{-1}(W_j), \text{Sym}^n \Omega^1_{\mathcal{X}/\mathbb{P}^n_k} \otimes \text{pr}_1^* \mathcal{O}_{\mathbb{P}^n_k}(N) \otimes \text{pr}_2^* \mathcal{O}_{\mathbb{P}^2_k}(\nabla) \right),
\]
with the twisted degree:
\[\nabla := \sum_{p=1}^{l_1} \deg F_{i_p} + \sum_{q=1}^{l_2} \deg F_{j_q} - \sum_{j=0}^{N} \lambda_j + N + 1.\]
Moreover, restricting on \( \mathcal{X} \), they glue together to make a global section:

\[
\hat{\omega}_{j_1, \ldots, j_l} \in \Gamma \left( \mathcal{X}, \text{Sym}^{|t/| \mathcal{E}} \otimes \text{pr}_1^* \mathcal{O}_{\mathcal{X}}(N) \otimes \text{pr}_2^* \mathcal{O}_{\mathcal{X}}(\varnothing) \right).
\]

We may view Propositions 6.11 and 6.12 in a similar way.

7. Moving Coefficients Method

7.1. Algorithm. As explained in Subsection 4.3, we wish to construct sufficiently many negatively twisted symmetric differential forms, and for this purpose we investigate the moving coefficients method as follows. We will be concerned only with the central cases that all positive integers negligible compared with the large integer \( F \).

Recall the integer \( \delta \geq 1 \) in Theorem 5.1. We start by setting:

\[
\delta_{c+r+1} \geq \max \{ \epsilon_1, \ldots, \epsilon_{c+r} \}. 
\]

For every integer \( l = c + r + 1 \cdots N \), in this step, we begin with choosing \( \mu_{l,0} \) that satisfies:

\[
\mu_{l,0} \geq l(\delta - 1) + 1 + (l - c - r) \delta, 
\]

then inductively we choose \( \mu_{l,k} \) with:

\[
\mu_{l,k} \geq \sum_{j=0}^{k-1} l \mu_{l,j} + (l - k) \delta l + 1 + (l - c - r) \delta \quad (k = 1 \cdots l). 
\]

If \( l < N \), we end this step by setting:

\[
\delta_{l+1} := l \mu_{l,l} 
\]

as the starting point for the next step \( l + 1 \). At the end \( l = N \), we demand the integer \( d \gg 1 \) to be big enough:

\[
d \geq (N + 1) \mu_{N,N}. 
\]

Roughly speaking, the Algorithm above is designed for the following three properties.

1. For every integer \( l = c + r + 1 \cdots N \), in this step, \( \mu_{l,\bullet} (\bullet = 0 \cdots l) \) grows so drastically that the former ones are negligible compared with the later ones:

\[
\mu_{l,0} \ll \mu_{l,1} \ll \cdots \ll \mu_{l,l}. 
\]

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For all integer pairs \((l_1, l_2)\) with \(c + r + 1 \leq l_1 < l_2 \leq N\), all the integers \(\mu_{l_1, *}\) chosen in the former step \(l_1\) are negligible compared with the integers \(\mu_{l_2, *}\) chosen in the later step \(l_2\):

\[
\mu_{l_1, *} \ll \mu_{l_2, *} \quad (\forall 0 < * \leq l_1; 0 < *_2 \leq l_2).
\]  

(iii) All integers \(\mu_{l, k}\) are negligible compared with the integer \(d\):

\[
\mu_{l, k} \ll d \quad (\forall c + r + 1 \leq l \leq N; 0 \leq k \leq l�).
\]

7.2. **Global moving coefficients method.** First, for all \(i = 1 \cdots c + r\), we write the polynomial \(F_i\) by extracting the terms for which \(l = N\):

\[
F_i = \sum_{j=0}^{N} A_i^j \cdot z_j^d + \sum_{l=c+r+1}^{N-1} \sum_{0 \leq j_0 < \cdots < j_l \leq N} \sum_{k=0}^{l} M_i^{j_0, \cdots, j_l; k} \cdot \mu_{j_0} \cdots \mu_{j_l} \cdot z_{j_0} \cdots z_{j_l} \cdot z_k^d - b_{l, k} +
\]

\[
+ \sum_{k=0}^{N} M_i^{0, \cdots, N; k} \cdot \mu_{N_k} \cdots \mu_{N_k} \cdot z_0 \cdots z_k \cdot z_k^d - N \cdot \mu_{N_k},
\]

and now this second line consists of exactly all the moving coefficient terms which associate to all variables \(z_0, \ldots, z_N\), namely of the form \(M_i^{*} \cdot z_0 \cdots z_N^*\).

To simplify the structure of the first line, associating each term in the second sums:

\[
M_i^{j_0, \cdots, j_l; k} \cdot \mu_{j_0} \cdots \mu_{j_l} \cdot z_{j_0} \cdots z_{j_l} \cdot z_k^d - b_{l, k}
\]

with the ‘corresponding’ term in the first sum:

\[
A_i^j \cdot z_j^d,
\]

and noting a priori the inequalities guaranteed by the Algorithm:

\[
d - l \cdot \mu_{l, k} \geq d - (N - 1) \cdot \mu_{N-1, N-1} = \delta_N \quad \text{[By (96)]}
\]

we rewrite the \(F_i\) as:

\[
F_i = \sum_{j=0}^{N} C_i^j \cdot z_j^{d - \delta_N} + \sum_{k=0}^{N} M_i^{0, \cdots, N; k} \cdot \mu_{N_k} \cdots \mu_{N_k} \cdot z_0 \cdots z_k \cdot z_k^{d - N \cdot \mu_{N_k}},
\]

where the homogeneous polynomials \(C_i^j\) are uniquely determined by gathering:

\[
C_i^j \cdot z_j^{d - \delta_N} = A_i^j \cdot z_j^d + \sum_{l=c+r+1}^{N-1} \sum_{0 \leq j_0 < \cdots < j_l \leq N} \sum_{k=0}^{l} M_i^{j_0, \cdots, j_l; k} \cdot \mu_{j_0} \cdots \mu_{j_l} \cdot z_{j_0} \cdots z_{j_l} \cdot z_k^{d - l \cdot \mu_{l, k}},
\]

namely, after dividing out the common factor \(z_j^{d - \delta_N}\) of both sides above:

\[
C_i^j := A_i^j \cdot z_j^d + \sum_{l=c+r+1}^{N-1} \sum_{0 \leq j_0 < \cdots < j_l \leq N} \sum_{k=0}^{l} M_i^{j_0, \cdots, j_l; k} \cdot \mu_{j_0} \cdots \mu_{j_l} \cdot z_{j_0} \cdots z_{j_l} \cdot z_k^{d - l \cdot \mu_{l, k}}.
\]

Next, we have two ways to manipulate the \((N + 1)\) remaining moving coefficient terms in (102):

\[
M_i^{0, \cdots, N; k} \cdot \mu_{N_k} \cdots \mu_{N_k} \cdot z_0 \cdots z_k \cdot z_k^{d - N \cdot \mu_{N_k}},
\]

in order to ensure the negativity of the symmetric differential forms to be obtained later.
The first kind of manipulations are, for every chosen index \( \nu = 0 \cdots N \), to associate all these \((N + 1)\) moving coefficient terms:

\[
\sum_{k=0}^{N} M_{i}^{0 \cdots N \mu_{nk}} z_{k}^{0} \cdots z_{k}^{N} \zeta_{k}^{N - N \mu_{nk}}
\]

with the term \( C_{i}^{\nu} z_{\nu}^{d - \delta_{N}} \) by rewriting \( F_{i} \) in (102) as:

\[
F_{i} = \sum_{j=0}^{N} C_{i}^{j} z_{j}^{d - \delta_{N}} + T_{i}^{\nu} z_{\nu}^{N}, \quad (105)
\]

where \( T_{i}^{\nu} \) is the homogeneous polynomial uniquely determined by solving:

\[
T_{i}^{\nu} z_{\nu}^{N} = C_{i}^{\nu} z_{\nu}^{d - \delta_{N}} + \sum_{k=0}^{N} M_{i}^{0 \cdots N \mu_{nk}} z_{k}^{0} \cdots z_{k}^{N} \zeta_{k}^{N - N \mu_{nk}}, \quad (106)
\]

in fact, guided by properties (98), (99), (100), our algorithm a priori implies:

\[
\mu_{N,0} \leq d - \delta_{N}, \quad \mu_{N,k}, \quad d - N \mu_{N,k} \quad (k = 0 \cdots N),
\]

thus the right hand side of (106) has a common factor \( z_{\nu}^{N} \).

The second kind of manipulations are, for every chosen integer \( \tau = 0 \cdots N - 1 \), for every chosen index \( \rho = \tau + 1 \cdots N \), to associate each of the first \((\tau + 1)\) moving coefficient terms:

\[
M_{i}^{0 \cdots N \mu_{nk}} z_{k}^{0} \cdots z_{k}^{N} \zeta_{k}^{N - N \mu_{nk}} \quad (k = 0 \cdots \tau)
\]

with the corresponding terms \( C_{i}^{k} z_{k}^{d - \delta_{N}} \) and to associate the remaining \((N - \tau)\) moving coefficient terms:

\[
\sum_{j=\tau+1}^{N} M_{i}^{0 \cdots N \mu_{nj}} z_{j}^{0} \cdots z_{j}^{N} \zeta_{j}^{N - N \mu_{nj}}
\]

with the term \( C_{i}^{\rho} z_{\rho}^{d - \delta_{N}} \) by rewriting \( F_{i} \) as:

\[
F_{i} = \sum_{k=0}^{\tau} E_{i}^{k} z_{k}^{d - N \mu_{nk}} + \sum_{j=\tau+1}^{N} C_{i}^{j} z_{j}^{d - \delta_{N}} + P_{i}^{\rho} z_{\rho}^{N}, \quad (107)
\]

where \( E_{i}^{k} \) and \( P_{i}^{\rho} \) are the homogeneous polynomials uniquely determined by solving:

\[
E_{i}^{k} z_{k}^{d - N \mu_{nk}} = C_{i}^{k} z_{k}^{d - \delta_{N}} + M_{i}^{0 \cdots N \mu_{nk}} z_{0}^{0} \cdots z_{k}^{N} \zeta_{k}^{N - N \mu_{nk}} \quad (k = 0 \cdots \tau),
\]

\[
P_{i}^{\rho} z_{\rho}^{N} = C_{i}^{\rho} z_{\rho}^{d - \delta_{N}} + \sum_{j=\tau+1}^{N} M_{i}^{0 \cdots N \mu_{nj}} z_{j}^{0} \cdots z_{j}^{N} \zeta_{j}^{N - N \mu_{nj}}, \quad (108)
\]

which is direct by the inequalities listed below granted by the Algorithm:

\[
d - N \mu_{N,k} \leq d - \delta_{N} \quad (k = 0 \cdots \tau),
\]

\[
\mu_{N,\tau+1} \leq \mu_{N,j}, \quad (j = \tau+1 \cdots N),
\]

\[
\mu_{N,\tau+1} \leq d - N \mu_{N,j} \quad (j = \tau+1 \cdots N),
\]

\[42\]
Now thanks to the above two kinds of manipulations (105), (107), applying Proposition 6.8, 6.10, we receive the corresponding twisted symmetric differential forms with negative degrees as follows.

Firstly, for every index \( \nu = 0 \cdots N \), applying Proposition 6.8, 6.10 with respect to the first kind of manipulation (105) on the hypersurface polynomial equations \( F_1, \ldots, F_{c+r} \), for every \( n \)-tuple \( 1 \leq j_1 < \cdots < j_n \leq c \), we receive a twisted symmetric differential \( n \)-form:

\[
\phi_{j_1, \ldots, j_n}^\nu \in \Gamma(X, \text{Sym}^n \Omega_X(\nabla_{j_1, \ldots, j_n}^\nu)),
\]

whose twisted degree \( \nabla_{j_1, \ldots, j_n}^\nu \), according to the formula (71), is negative:

\[
\begin{align*}
\text{[Use deg } F_i = d + \delta_i + \delta_{c+r+1}] & \quad \nabla_{j_1, \ldots, j_n}^\nu \leq N (d + \delta_{c+r+1}) - [N (d - \delta_N) + \mu_{N,0}] + N + 1 \\
\text{[Use (94) for } \ell = N] & \quad \leq -n. 
\end{align*}
\]

Secondly, for every integer \( \tau = 0 \cdots N - 1 \), for every index \( \rho = \tau + 1 \cdots N \), applying Proposition 6.8, 6.10 with respect to the second kind of manipulation (107) on the hypersurface polynomial equations \( F_1, \ldots, F_{c+r} \), for every \( n \)-tuple \( 1 \leq j_1 < \cdots < j_n \leq c \), we receive a twisted symmetric differential \( n \)-form:

\[
\psi_{j_1, \ldots, j_n}^{\tau \rho} \in \Gamma(X, \text{Sym}^n \Omega_X(\nabla_{j_1, \ldots, j_n}^{\tau \rho})),
\]

whose twisted degree \( \nabla_{j_1, \ldots, j_n}^{\tau \rho} \), according to the formula (71), is negative too:

\[
\begin{align*}
\nabla_{j_1, \ldots, j_n}^{\tau \rho} & \leq N (d + \delta_{c+r+1}) - \sum_{k=0}^{\tau} (d - N \mu_{N,k}) - (N - \tau - 1) (d - \delta_N) - \mu_{N,\tau+1} + N + 1 \\
& = \sum_{k=0}^{\tau} N \mu_{N,k} + (N - \tau - 1) \delta_N + N (\delta_{c+r+1} + 1) + 1 - \mu_{N,\tau+1} \\
& \leq -n \quad \text{[use (95) for } \ell = N, k = \tau + 1].
\end{align*}
\]

7.3. Moving coefficients method with some vanishing coordinates. To investigate further the moving coefficients method, for all integers \( 1 \leq \eta \leq n - 1 \), for every sequence of ascending indices:

\[
0 \leq v_1 < \cdots < v_\eta \leq N,
\]

take the intersection of \( X \) with the \( \eta \) coordinate hyperplanes:

\[
v_{1, \ldots, \eta} X := X \cap \{ z_{v_1} = 0 \} \cap \cdots \cap \{ z_{v_\eta} = 0 \}.
\]

 Applying Proposition 6.12, in order to obtain more symmetric differential \( (n - \eta) \)-forms having negative twisted degree, we carry on manipulations as follows, which are much the same as before.

First, write the \( (N - \eta + 1) \) remaining numbers of the set-minus:

\[
\{0, \ldots, N\} \setminus \{ v_1, \ldots, v_\eta \}
\]

in the ascending order:

\[
r_0 < \cdots < r_{N-\eta}.
\]

Note that in Proposition 6.12, the coefficient terms associated with the vanishing variables \( z_{v_1}, \ldots, z_{v_\eta} \) play no role, therefore we decompose \( F_i \) into two parts. The first part (the first two lines below)
is a very analog of (101) involving only the variables \(z_{v_0}, \ldots, z_{v_N} \), while the second part (the third line) collects all the residue terms involving at least one of the vanishing coordinates \(z_{v_1}, \ldots, z_{v_N} \):

\[
F_i = \sum_{j=0}^{N-\eta} A_i^{(j)} z_{r_j}^d + \sum_{l=c+r+1}^{N-\eta-1} \sum_{0<j_0<\cdots<j_l<N-\eta} \sum_{k=0}^{l} M_i^{r_{j_0},\ldots,r_{j_l}} \zeta_{r_0}^{\mu_{j_0}} \cdots \zeta_{r_k}^{\mu_{j_k}} \zeta_{r_{j_l}}^{d-\eta \mu_{j_k}} +
\sum_{k=0}^{N-\eta} M_i^{r_0,\ldots,r_{N-\eta}} \zeta_{r_0}^{c_{N-\eta}} \cdots \zeta_{r_k}^{c_{N-\eta}} \zeta_{r_{N-\eta}}^{d-(N-\eta) \mu_{N-\eta}} +
(\text{Residue Terms})_{v_1,\ldots,v_{\eta}}
\]

Moreover, using for instance the lexicographic order, we can write them as:

\[
(\text{Residue Terms})_{v_1,\ldots,v_{\eta}} = \sum_{j=1}^{\eta} R_i^{v_1,\ldots,v_{\eta}} z_{v_j}^2
\]

where \( R_i^{v_1,\ldots,v_{\eta}} \) are the homogeneous polynomials uniquely determined by solving:

\[
R_i^{v_1,\ldots,v_{\eta}} z_{v_j}^2 = A_i^{v_1,\ldots,v_{\eta}} + \sum_{l=c+r+1}^{N-\eta} \sum_{0<j_0<\cdots<j_l<N-\eta} \sum_{k=0}^{l} M_i^{r_{j_0},\ldots,r_{j_l}} \zeta_{r_0}^{\mu_{j_0}} \cdots \zeta_{r_k}^{\mu_{j_k}} \zeta_{r_{j_l}}^{d-\eta \mu_{j_k}}.
\]

Observing that the first two lines of (112) have exactly the same structure as (101), by mimicking the manipulation of rewriting (101) as (102), we can rewrite the first two lines of (112) as:

\[
\sum_{j=0}^{N-\eta} v_1,\ldots,v_{\eta} C_i^{(j)} z_{r_j}^d - \delta_{N-\eta} + \sum_{k=0}^{N-\eta} M_i^{r_0,\ldots,r_{N-\eta}} \zeta_{r_0}^{c_{N-\eta}} \cdots \zeta_{r_k}^{c_{N-\eta}} \zeta_{r_{N-\eta}}^{d-(N-\eta) \mu_{N-\eta}} +
\]

where the integer \( \delta_{N-\eta} \) was defined in (96) for \( l = N - \eta - 1 \):

\[
\delta_{N-\eta} = (N - \eta - 1) \mu_{N-\eta-1,N-\eta-1},
\]

and where the homogeneous polynomials \( v_1,\ldots,v_{\eta} C_i^{(j)} \) are obtained in the same way as \( C_i^{(j)} \) in (104):

\[
v_1,\ldots,v_{\eta} C_i^{(j)} := A_i^{v_1,\ldots,v_{\eta}} + \sum_{l=c+r+1}^{N-\eta} \sum_{0<j_0<\cdots<j_l<N-\eta} \sum_{j_c=j_l \text{ for some } 0 < c < l} M_i^{r_{j_0},\ldots,r_{j_l}} \zeta_{r_0}^{\mu_{j_0}} \cdots \zeta_{r_k}^{\mu_{j_k}} \zeta_{r_{j_l}}^{d-(N-\eta) \mu_{j_k}}.
\]

Now substituting (114), (113) into the equation (112), we rewrite \( F_i \) as:

\[
F_i = \sum_{j=0}^{N-\eta} v_1,\ldots,v_{\eta} C_i^{(j)} z_{r_j}^d + \sum_{k=0}^{N-\eta} M_i^{r_0,\ldots,r_{N-\eta}} \zeta_{r_0}^{c_{N-\eta}} \cdots \zeta_{r_k}^{c_{N-\eta}} \zeta_{r_{N-\eta}}^{d-(N-\eta) \mu_{N-\eta}} +
\sum_{j=1}^{\eta} R_i^{v_1,\ldots,v_{\eta}} z_{v_j}^2
\]

negligible in the coming applications.
Now, noting that the first line of \( F_i \) in (115) has exactly the same structure as (102), we repeat the two kinds of manipulations, as briefly summarized below.

The first kind of manipulations are, for every chosen index \( \nu = 0 \cdots N - \eta \), to associate all these \((N + 1 - \eta)\) moving coefficient terms:

\[
\sum_{k=0}^{N-\eta} M_i^{0,\cdots,N-\eta \cdot \tau_k} \mu_{\nu} \cdot \frac{d-\nu_{\eta}}{z_{\tau_k}} \cdots \frac{d-(N-\eta) \mu_{N-\eta \cdot k}}{z_{\tau_k}}
\]

with the term \( v_{1,\ldots,\nu} C_i^{\nu} \frac{d-\nu_{\eta}}{z_{r_1}} \) by rewriting (114) as:

\[
\sum_{j=0}^{\nu-\eta} v_{1,\ldots,\nu} \hspace{-0.5cm} \sum_{\tau=0}^{N-\eta} M_i^{0,\cdots,N-\eta \cdot \tau_j} \mu_{\nu} \cdot \frac{d-\nu_{\eta}}{z_{\tau_j}} + v_{1,\ldots,\nu} T_i^{\nu} \mu_{N-\eta \cdot 0}, \tag{116}
\]

where \( v_{1,\ldots,\nu} T_i^{\nu} \) is the homogeneous polynomial uniquely determined by solving:

\[
v_{1,\ldots,\nu}^{r_1} \frac{d^{\nu_{\eta}}}{z_{r_1}} = v_{1,\ldots,\nu} C_i^{r_1} \frac{d-\nu_{\eta}}{z_{r_1}} + \sum_{k=0}^{\nu-\eta} M_i^{0,\cdots,N-\eta \cdot \tau_k} \mu_{\nu} \cdot \frac{d-\nu_{\eta}}{z_{\tau_k}} \frac{d-(N-\eta) \mu_{N-\eta \cdot k}}{z_{\tau_k}} \cdot \tag{117}
\]

The second kind of manipulations are, for every integer \( \rho = 0 \cdots N - \eta - 1 \), for every index \( \rho = \tau + 1 \cdots N - \eta \), to associate each of the first \((\tau + 1)\) moving coefficient terms:

\[
M_i^{0,\cdots,N-\eta \cdot \tau_j} \mu_{\nu} \cdot \frac{d-\nu_{\eta}}{z_{\tau_j}} \frac{d-(N-\eta) \mu_{N-\eta \cdot j}}{z_{\tau_j}} (k = 0 \cdots \tau)
\]

with the corresponding term \( v_{1,\ldots,\nu} C_i^{r_1} \frac{d-\nu_{\eta}}{z_{r_1}} \) and to associate the remaining \((N - \eta - \tau)\) moving coefficient terms:

\[
\sum_{j=\tau+1}^{N-\eta} M_i^{0,\cdots,N-\eta \cdot \tau_j} \mu_{\nu} \cdot \frac{d-\nu_{\eta}}{z_{\tau_j}} \frac{d-(N-\eta) \mu_{N-\eta \cdot j}}{z_{\tau_j}} \cdot \tag{119}
\]

with the term \( v_{1,\ldots,\nu} T_i^{\nu} \frac{d-\nu_{\eta}}{z_{r_1}} \) by rewriting (114) as:

\[
\sum_{k=0}^{\tau} v_{1,\ldots,\nu} E_i^{r_k} \frac{d-(N-\eta) \mu_{N-\eta \cdot k}}{z_{r_k}} + \sum_{j=\tau+1}^{N-\eta} v_{1,\ldots,\nu} C_i^{r_j} \frac{d-\nu_{\eta}}{z_{r_j}} + v_{1,\ldots,\nu} P_i^{r_{\rho+1}} \frac{d-(N-\eta) \mu_{N-\eta \cdot \rho+1}}{z_{r_{\rho+1}}} \cdot \tag{118}
\]

where \( v_{1,\ldots,\nu} E_i^{r_k} \) and \( v_{1,\ldots,\nu} P_i^{r_{\rho+1}} \) are the homogeneous polynomials uniquely determined by solving:

\[
v_{1,\ldots,\nu}^{r_k} \frac{d-(N-\eta) \mu_{N-\eta \cdot k}}{z_{r_k}} = v_{1,\ldots,\nu} C_i^{r_k} \frac{d-\nu_{\eta}}{z_{r_k}} + M_i^{0,\cdots,N-\eta \cdot \tau_k} \mu_{\nu} \cdot \frac{d-\nu_{\eta}}{z_{\tau_k}} \frac{d-(N-\eta) \mu_{N-\eta \cdot k}}{z_{\tau_k}} \cdot \tag{119}
\]

which is possible by the Algorithm in subsection 7.1.

To summarize, taking the two forms (116), (118) of the first line of (115) into account, we can rewrite \( F_i \) in the following two ways. The first one is:

\[
F_i = \sum_{j=0}^{\eta} \sum_{j \neq \nu} v_{1,\ldots,\nu} C_i^{r_j} \frac{d-\nu_{\eta}}{z_{r_j}} + v_{1,\ldots,\nu} T_i^{\nu} \frac{d-\nu_{\eta}}{z_{r_1}} + \sum_{j=1}^{\eta} R_i^{v_{1,\ldots,\nu} v_j} \frac{d^2}{z_{v_j}^2} \cdot \tag{120}
\]

negligible in our coming applications.
and the second one is:

\[
F_i = \sum_{k=0}^{\tau} v_{1\ldots v_{\eta}} E_{i}^{\eta} \mu_{\eta}^{-1} \delta_{\eta}^{-(N-\eta)\mu_{\eta}-1} + \sum_{j=1}^{\tau} v_{1\ldots v_{\eta}} C_{j}^{i} \mu_{\eta}^{-\delta_{\eta}-(N-\eta)} + \sum_{j=1}^{\tau} P_{i}^{j} \mu_{\eta}^{-\delta_{\eta}-(N-\eta)} + \sum_{j=1}^{\tau} R_{i}^{v_{1\ldots v_{\eta}}v_{1\ldots v_{\eta}}} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot 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Proof. For every point $x \in X$, let $f(x) = z \in Y$, and denote the germ dimension of $Y$ at this point by:

$$d_z := \dim_C(Y, z).$$

Then we can find holomorphic function germs $g_1, \ldots, g_{d_z} \in \mathcal{O}_{Y, z}$ vanishing at $z$ such that:

$$(Y, z) \cap \{g_1 = \cdots = g_{d_z} = 0\} = \{z\}.$$

Pulling back by the holomorphic map $f$, we therefore realize:

$$(X, x) \cap \{g_1 \circ f = \cdots = g_{d_z} \circ f = 0\} = (f^{-1}(z), x).$$

Now, counting the germ dimension, we receive the estimate:

$$\dim_C(f^{-1}(z), x) \geq \dim_C(X, x) - d_z = \dim_C(X, x) - \dim_C(Y, z),$$

hence:

$$\dim_C(X, x) \leq \dim_C(Y, z) + \dim_C(f^{-1}(z), x) \leq \dim_C Y + \max_{y \in Y} \dim_C f^{-1}(y).$$

Finally, let $x \in X$ vary in the above estimate, thanks to:

$$\dim_C X = \max_{x \in X} \dim_C (X, x),$$

we receive the desired estimate (124). $$\qed$$

With the same proof (cf. [54, p. 169, Proposition 12.30; p. 140, Corollary 10.27]), here is an algebraic version of the analytic fibre dimension estimate above, for every algebraically closed field $\mathbb{K}$ and for the category of $\mathbb{K}$-varieties in the classical sense ([37, §1.3, p. 15]), where dimension is defined to be the Krull dimension ([37, §1.1, p. 6]).

**Theorem 8.2 (Algebraic Fibre Dimension Estimate).** Let $X, Y$ be two $\mathbb{K}$-varieties, and let $f : X \to Y$ be a morphism. Then the dimension of the source variety $X$ is bounded from above by the sum of the dimension of the target variety $Y$ plus the maximum fibre dimension:

$$\dim X \leq \dim Y + \max_{y \in Y} \dim f^{-1}(y). \tag{125}$$

In our future applications, $f$ will always be surjective, so one may also refer to [59, p. 76, Theorem 7]. The above theorem will prove fundamental in estimating every base locus involved in this paper.

**Corollary 8.3.** Let $X, Y$ be two $\mathbb{K}$-varieties, and let $f : X \to Y$ be a morphism such that every fibre satisfies the dimension estimate:

$$\dim f^{-1}(y) \leq \dim X - \dim Y \quad (\forall y \in Y).$$

Then for every subvariety $Z \subset Y$, its inverse image:

$$f^{-1}(Z) \subset X$$

satisfies the transferred codimension estimate:

$$\text{codim } f^{-1}(Z) \geq \text{codim } Z. \quad \Box$$
8.2. **Matrix-rank estimates.** This subsection recalls some elementary rank estimates in linear algebra.

**Lemma 8.4.** Let $\mathbb{K}$ be a field and let $W$ be a finite-dimensional $\mathbb{K}$-vector space generated by a set of vectors $\mathcal{B}$. Then every subset $\mathcal{B}_1 \subset \mathcal{B}$ that consists of $\mathbb{K}$-linearly independent vectors can be extended to a bigger subset $\mathcal{B}_2 \subset \mathcal{B}$ which forms a basis of $W$. □

**Lemma 8.5.** Let $\mathbb{K}$ be a field, and let $V$ be a $\mathbb{K}$-vector space. For all positive integers $e, k, l \geq 1$ with $k \geq l$, let $v_1, \ldots, v_e, v_{e+1}, \ldots, v_{e+k}$ be $(e + k)$ vectors such that:

(i) $v_1, \ldots, v_e$ are $\mathbb{K}$-linearly independent;

(ii) for every sequence of $l$ ascending indices between $e + 1$ and $e + k$:

$$e + 1 \leq i_1 < \cdots < i_l \leq e + k,$$

there holds the rank inequality:

$$\text{rank}_{\mathbb{K}} \{v_1, \ldots, v_e, v_{i_1}, \ldots, v_{i_l}\} \leq e + l - 1.$$

Then there holds the rank estimate:

$$\text{rank}_{\mathbb{K}} \{v_1, \ldots, v_e, v_{e+1}, \ldots, v_{e+k}\} \leq e + l - 1.$$

**Proof.** Assume on the contrary that:

$$\text{rank}_{\mathbb{K}} \{v_1, \ldots, v_e, v_{e+1}, \ldots, v_{e+k}\} = e + l_0 \geq e + l,$$

that is, $l_0 \geq l$.

Now applying the above lemma to:

$$W = \text{Span}_{\mathbb{K}} \{v_1, \ldots, v_e, v_{e+1}, \ldots, v_{e+k}\},$$

$$\mathcal{B}_1 = \{v_1, \ldots, v_e\},$$

we receive a certain basis of $V$:

$$\mathcal{B}_2 = \{v_1, \ldots, v_e, v_{i_1}, \ldots, v_{i_{l_0}}\}.$$

In particular, as $l_0 \geq l$, the first $(e + l)$ vectors in $\mathcal{B}_2$ are $\mathbb{K}$-linearly independent:

$$\text{rank}_{\mathbb{K}} \{v_1, \ldots, v_e, v_{i_1}, \ldots, v_{i_l}\} = e + l,$$

which contradicts condition (ii). □

Let $\mathbb{K}$ be a field, and let $p, q, e, l$ be positive integers with:

$$\min\{p, q\} \geq e + l.$$

Let $M \in \text{Mat}_{p \times q}(\mathbb{K})$ be a $p \times q$ matrix. For all sequences of ascending indices:

$$1 \leq i_1 < \cdots < i_k \leq p,$$

let us denote by $M_{i_1, \ldots, i_k}$ the $k \times q$ submatrix of $M$ that consists of the rows $i_1, \ldots, i_k$, and for all sequences of ascending indices:

$$1 \leq j_1 < \cdots < j_l \leq q,$$

let us denote by $M^{j_1, \ldots, j_l}_{i_1, \ldots, i_k}$ the $k \times l$ submatrix of $M_{i_1, \ldots, i_k}$ that consists of the columns $j_1, \ldots, j_l.$
Lemma 8.6. If the first \( e \) rows of the matrix \( M \) are of full rank:

\[ \text{rank}_K M_{1, \ldots, e} = e, \]

and if all the \((e + l) \times (e + l)\) submatrices always selecting the first \( e \) rows of \( M \) are degenerate:

\[ \text{rank}_K M_{j_1, \ldots, j_{e+l}} \leq e + l - 1 \quad (\forall e+1 \leq i_1 < \cdots < i_l \leq p; 1 \leq j_1 < \cdots < j_{e+l} \leq q), \]

then there holds the rank estimate:

\[ \text{rank}_K M \leq e + l - 1 \]

Proof. For every fixed sequence of ascending indices:

\[ e + 1 \leq i_1 < \cdots < i_l \leq p, \]

the rank inequalities (126) yields:

\[ \text{rank}_K M_{1, \ldots, e, i_1, \ldots, i_l} \leq e + l - 1 \]

Now applying the previous lemma to the rows of the matrix \( M \), we conclude the desired rank estimate. \( \square \)

Lemma 8.7. Let \( \mathbb{K} \) be a field and let \( e, m \) be positive integers. Let \( H \in \text{Mat}_{e \times m}(\mathbb{K}) \) be an \( e \times m \) matrix with entries in \( \mathbb{K} \) such that the sum of all \( m \) columns of \( H \) vanishes:

\[ H_1 + \cdots + H_m = 0, \tag{127} \]

where we denote by \( H_i \) the \( i \)-th column of \( H \). Then for every integer \( j = 1 \cdots m \), the \( e \times m \) submatrix \( \widehat{H}_j \) of \( H \) obtained by omitting the \( j \)-th column still has the same rank:

\[ \text{rank}_K \widehat{H}_j = \text{rank}_K H. \]

Proof. Note that (127) yields:

\[ H_j = -(H_1 + \cdots + H_{j-1} + H_{j+1} + \cdots + H_m), \]

therefore \( H_j \) lies in the \( \mathbb{K} \)-linear space generated by the columns of the matrix \( \widehat{H}_j \), thus we receive:

\[ \text{Span}_K \{ H_1, \ldots, \widehat{H}_j, \ldots, H_m \} = \text{Span}_K \{ H_1, \ldots, H_m \}. \]

Taking the dimension of both sides, we receive the desired rank equality. \( \square \)

8.3. Classical codimension formulas. In an algebraically closed field \( \mathbb{K} \), for all positive integers \( p, q \geq 1 \), denote by:

\[ \text{Mat}_{p \times q}(\mathbb{K}) = \mathbb{K}^{p \times q} \]

the space of all \( p \times q \) matrices with entries in \( \mathbb{K} \). For every integer \( 0 \leq \ell \leq \max(p, q) \), we have a classical formula (cf. [31, p. 247 exercise 10.10, and the proof in p. 733]) for the codimension of the subvariety:

\[ \Sigma^{p,q}_\ell \subset \text{Mat}_{p \times q}(\mathbb{K}) \]

which consists of all matrices with rank \( \leq \ell \).

Lemma 8.8. There holds the codimension formula:

\[ \text{codim} \Sigma^{p,q}_\ell = \max \{ (p - \ell) (q - \ell), 0 \}. \]

In applications, we will use the following two direct consequences.
Corollary 8.9. For every integer $0 \leq \ell \leq \max\{p, q-1\}$, the codimension of the subvariety:

$$0^\ell \Sigma^p,q \subset \Sigma^p,q,$$

which consists of matrices whose sum of all the columns vanish, is:

$$\text{codim } 0^\ell \Sigma^p,q = \max\{(p - \ell)(q - 1 - \ell), 0\} + p. \quad \Box$$

Proof. Since every matrix in $0^\ell \Sigma^p,q$ is uniquely determined by the first $(q - 1)$ columns, thanks to Lemma 8.7, the projection morphism into the first $(q - 1)$ columns:

$$\pi : 0^\ell \Sigma^p,q \rightarrow \Sigma^p,q-1$$

is an isomorphism. Remembering that:

$$\dim \Sigma^p,q = \dim \Sigma^p,q-1 + p,$$

now a direct application of the preceding lemma finishes the proof. \quad \Box

8.4. **Surjectivity of evaluation maps.** Given a field $\mathbb{K}$, for all positive integers $N \geq 1$, denote the affine coordinate ring of $\mathbb{K}^{N+1}$ by:

$$\mathcal{A}(\mathbb{K}^{N+1}) := \mathbb{K}[z_0, \ldots, z_N].$$

For all positive integers $\lambda \geq 1$, also denote by:

$$\mathcal{A}_\lambda(\mathbb{K}^{N+1}) \subset \mathcal{A}(\mathbb{K}^{N+1})$$

the $\mathbb{K}$-linear space spanned by all the degree $\lambda$ homogeneous polynomials:

$$\mathcal{A}_\lambda(\mathbb{K}^{N+1}) := \bigoplus_{\alpha_0 + \cdots + \alpha_N = \lambda} \mathbb{K} \cdot z_0^{\alpha_0} \cdots z_N^{\alpha_N} \cong \mathbb{K}(\mathbb{K}_N^{(N+1)}).$$

For every point $z \in \mathbb{K}^{N+1}$, denote by $v_z$ the $\mathbb{K}$-linear evaluation map:

$$v_z : \mathcal{A}(\mathbb{K}^{N+1}) \rightarrow \mathbb{K}$$

$$f \mapsto f(z),$$

and for every tangent vector $\xi \in T_z \mathbb{K}^{N+1} \cong \mathbb{K}^{N+1}$, denote by $d_z(\xi)$ the $\mathbb{K}$-linear differential evaluation map:

$$d_z(\xi) : \mathcal{A}(\mathbb{K}^{N+1}) \rightarrow \mathbb{K}$$

$$f \mapsto d f\big|_z (\xi).$$

For every polynomial $g \in \mathcal{A}(\mathbb{K}^{N+1})$, for every point $z \in \mathbb{K}^{N+1}$, denote by $(g \cdot v)_z$ the $\mathbb{K}$-linear evaluation map:

$$(g \cdot v)_z : \mathcal{A}(\mathbb{K}^{N+1}) \rightarrow \mathbb{K}$$

$$f \mapsto (gf)(z),$$

and for every tangent vector $\xi \in T_z \mathbb{K}^{N+1} \cong \mathbb{K}^{N+1}$, denote by $d_z(g \cdot)(\xi)$ the $\mathbb{K}$-linear differential evaluation map:

$$d_z(g \cdot)(\xi) : \mathcal{A}(\mathbb{K}^{N+1}) \rightarrow \mathbb{K}$$

$$f \mapsto d (gf\big|_z)(\xi).$$

The following Lemma was obtained by Brotbek in another affine coordinates version [7, p. 36, Proof of Claim 3].
Lemma 8.10. For all positive integers \( \lambda \geq 1 \), at every nonzero point \( z \in \mathbb{K}^{N+1} \setminus \{0\} \), for every tangent vector \( \xi \in T_z \mathbb{K}^{N+1} \cong \mathbb{K}^{N+1} \) which does not lie in the line of \( z \):

\[ \xi \notin \mathbb{K} \cdot z, \]

restricting on the subspace:

\[ \mathcal{A}_1(\mathbb{K}^{N+1}) \subset \mathcal{A}(\mathbb{K}^{N+1}), \]

the evaluation maps \( v_z \) and \( d_z(\xi) \) are \( \mathbb{K} \)-linearly independent. In other words, the map:

\[
\begin{pmatrix}
 v_z \\
 d_z(\xi)
\end{pmatrix} : \mathcal{A}_1(\mathbb{K}^{N+1}) \longrightarrow \mathbb{K}^2
\]

is surjective.

Proof. Step 1. For the case \( \lambda = 1 \), this lemma is evident. In fact, now every polynomials \( \ell \in \mathcal{A}_1(\mathbb{K}^{N+1}) \) can be viewed as, by evaluating \( \ell(z) \) at every point \( z \in \mathbb{K}^{N+1} \), a \( \mathbb{K} \)-linear form:

\[ \ell \in (\mathbb{K}^{N+1})^\vee, \]

thus there is a canonical \( \mathbb{K} \)-linear isomorphism:

\[ \mathcal{A}_1(\mathbb{K}^{N+1}) \cong (\mathbb{K}^{N+1})^\vee. \]

Moreover, it is easy to see:

\[ d\ell |_{z}(\xi) = \ell(\xi). \]

Since \( z, \xi \in \mathbb{K}^{N+1} \) are \( \mathbb{K} \)-linearly independent, now recalling the Riesz Representation Theorem in linear algebra:

\[ \mathbb{K}^{N+1} \cong ((\mathbb{K}^{N+1})^\vee)^\vee, \]

we conclude the claim.

Step 2. For the general case \( \lambda \geq 2 \), first, we choose a degree \((\lambda - 1)\) homogeneous polynomial

\[ g \in \mathcal{A}_{\lambda-1}(\mathbb{K}^{N+1}) \]

with \( g(z) \neq 0 \) (for instance, one of \( z_0^{l-1}, \ldots, z_N^{l-1} \) succeeds), and then we claim, restricting on the \( \mathbb{K} \)-linear subspace obtained by multiplying \( \mathcal{A}_1(\mathbb{K}^{N+1}) \) with \( g \):

\[ g \cdot \mathcal{A}_1(\mathbb{K}^{N+1}) \subset \mathcal{A}_1(\mathbb{K}^{N+1}), \]

that the evaluation maps \( v_z \) and \( d_z(\xi) \) are \( \mathbb{K} \)-linearly independent.

In fact, for all \( f \in \mathcal{A}(\mathbb{K}^{N+1}) \), we have:

\[
(g \cdot v_z)(f) = (gf)(z) \\
= g(z) f(z) \\
= g(z) v_z(f),
\]

and by Leibniz’s rule:

\[
d_z(g \cdot)(\xi)(f) = d (gf)|_{z}(\xi) \\
= g(z) d f|_{z}(\xi) + f(z) d g|_{z}(\xi) \\
= g(z) d_z(\xi)(f) + d g|_{z}(\xi) v_z(f),
\]

in other words:

\[
\begin{pmatrix}
 (g \cdot v_z) \\
 d_z(g \cdot)(\xi)
\end{pmatrix} = \begin{pmatrix}
 g(z) & 0 \\
 d g|_{z}(\xi) & g(z)
\end{pmatrix} \begin{pmatrix}
 v_z \\
 d_z(\xi)
\end{pmatrix}
\]

invertible, since \( g(z) \neq 0 \).
Now, restricting (129) on the $K$-linear subspace:

$$\mathcal{A}_1(K^{N+1}) \subset \mathcal{A}(K^{N+1}),$$

and recalling the result of Step 1 that the evaluation maps $v_z, d_z(\xi)$ are $K$-linearly independent, we immediately see that the evaluation maps $(g \cdot v)_z, d_z(g \cdot \xi)$ are $K$-linearly independent too. In other words, restricting on the $K$-linear subspace:

$$g \cdot \mathcal{A}_1(K^{N+1}) \subset \mathcal{A}(K^{N+1}),$$

the evaluation maps $v_z, d_z(\xi)$ are $K$-linearly independent.

**Lemma 8.11.** For all positive integers $\lambda \geq 1$, for all polynomials $g \in \mathcal{A}(K^{N+1})$, at every nonzero point $z \in K^{N+1} \setminus \{0\}$ where $g$ does not vanish:

$$g(z) \neq 0,$$

and for every tangent vector $\xi \in T_z K^{N+1} \cong K^{N+1}$ which does not lie in the line of $z$:

$$\xi \notin K \cdot z,$$

restricting on the subspace:

$$\mathcal{A}_\lambda(K^{N+1}) \subset \mathcal{A}(K^{N+1}),$$

the evaluation maps $(g \cdot v)_z$ and $d_z(g \cdot \xi)$ are $K$-linearly independent. In other words, the map:

$$\begin{pmatrix} (g \cdot v)_z \\ d_z(g \cdot \xi) \end{pmatrix} : \mathcal{A}_\lambda(K^{N+1}) \longrightarrow K^2$$

is surjective.

**Proof.** This is a direct consequence of formula (129) and of the preceding lemma. \qed

**8.5. Codimensions of affine cones.** Usually, it is more convenient to count dimension in an Euclidean space rather than in a projective space. Therefore we carry out the following lemma (cf. [37, p. 12, exercise 2.10]), which is geometrically obvious, as one point ($\dim K = 0$) in the projective space $\mathbb{P}_K^N$ corresponds to one $K$-line ($\dim K = 1$) in $K^{N+1}$.

**Lemma 8.12.** In an algebraically closed field $K$, let $\pi : K^{N+1} \rightarrow \mathbb{P}_K^N$ be the canonical projection, and let:

$$Y \subset \mathbb{P}_K^N$$

be a nonempty algebraic set defined by a homogeneous ideal:

$$I \subset K[z_0, \ldots, z_N].$$

Denote by $C(Y)$ the affine cone over $Y$:

$$C(Y) := \pi^{-1}(Y) \cup \{0\} \subset K^{N+1}.$$

Then $C(Y)$ is an algebraic set in $K^{N+1}$ which is also defined by the ideal $I$ (considered as an ordinary ideal in $K[z_0, \ldots, z_N]$), and it has dimension one more than $Y$:

$$\dim C(Y) = \dim Y + 1.$$

In other words, they have the same codimension:

$$\text{codim } C(Y) = \text{codim } Y.$$

\qed

The essence of the above geometric lemma is the following theorem in commutative algebra (cf. [42, p. 73, Cor. 5.21]):
Theorem 8.13. Let $B$ be a homogeneous algebra over a field $\mathbb{K}$, then:

$$\dim \text{Spec } B = \dim \text{Proj } B + 1.$$ 

8.6. Full rank of hypersurface equation matrices. In an algebraically closed field $\mathbb{K}$, for all positive integers $N \geq 2$, for all integers $e = 1 \cdots N$, for all positive integers $\epsilon_1, \ldots, \epsilon_e \geq 1$ and $d \geq 1$, consider the following $e$ hypersurfaces:

$$H_1, \ldots, H_e \subset \mathbb{P}^N,$$

each being defined as the zero set of a degree $(d + \epsilon_i)$ Fermat-type homogeneous polynomial:

$$F_i = \sum_{j=0}^{N} A_{i,j}^j z_j^{d_j} \quad (i = 1 \cdots e), \quad (130)$$

where all $A_{i,j}^j \in \mathcal{A}_e(\mathbb{K}^{N+1})$ are some degree $\epsilon_i$ homogeneous polynomials.

Now, denote by $H$ the $e \times (N + 1)$ matrix whose $i$-th row copies the $(N + 1)$ terms of $F_i$ in the exact order, i.e. the $(i, j)$-th entries of $H$ are:

$$H_{i,j} = A_{i,j}^{j-1} z_j^d \quad (i = 1 \cdots e; j = 1 \cdots N+1),$$

so $H$ writes as:

$$H := \begin{pmatrix} A_0^0 & \cdots & A_N^0 \\ \vdots & \ddots & \vdots \\ A_0^{N} & \cdots & A_N^{N} \end{pmatrix} \quad (131)$$

which we call the hypersurface equation matrix of $F_1, \ldots, F_e$. Passim, remark that by (130), the sum of all columns of $H$ vanishes at every point $[z] \in X := H_1 \cap \cdots \cap H_e$.

Also introduce:

$$\mathbb{P}(\mathcal{M}) := \mathbb{P}\left( \bigoplus_{0 \leq j \leq N} \mathcal{A}_e(\mathbb{K}^{N+1}) \bigg|_{\mathcal{M}} \right)$$

the projectivized parameter space of $(A_{i,j}^j)_{0 \leq i \leq e, 0 \leq j \leq N} \in \mathcal{M}$.

First, let us recall a classical theorem (cf. [59, p. 57, Theorem 2]) that somehow foreshadows Remmert’s proper mapping theorem.

**Theorem 8.14.** The image of a projective variety under a regular map is closed. 

The following lemma was proved by Brotbek in another version [7, p. 36, Proof of Claim 1], and the proof there is in affine coordinates $(\frac{z_0}{z_j}, \frac{z_1}{z_j}, \ldots, \frac{z_N}{z_j})$:

$$\mathbb{K}^N \cong \{z_j \neq 0\} \subset \mathbb{P}_\mathbb{K}^N \quad (j = 0 \cdots N).$$

Here, we may present a proof by much the same arguments in ambient coordinates $(z_0, \ldots, z_N)$:

$$\mathbb{K}^{N+1} \setminus \{0\} \longrightarrow \mathbb{P}_\mathbb{K}^N$$

$$(z_0, \ldots, z_N) \longmapsto \left[ z_0 : \cdots : z_N \right].$$

**Lemma 8.15.** In $\mathbb{P}(\mathcal{M})$, there exists a proper algebraic subset:

$$\Sigma \subset \mathbb{P}(\mathcal{M})$$
such that, for every choice of parameter outside $\Sigma$:
\[
\left\{ (A^i_j)_{0<j \leq N} \right\} \in \mathbb{P}(\mathcal{M}) \setminus \Sigma,
\]
on the corresponding intersection:
\[
X = H_1 \cap \cdots \cap H_e \subset \mathbb{P}^N_k,
\]
the matrix $H$ has full rank $e$ everywhere:
\[
\text{rank}_k H(z) = e \quad (\forall [z] \in X).
\]
Sharing the same spirit as the famous \textit{Fubini principle} in combinatorics, the essence of the proof below is to count dimension in two ways, which is a standard method in algebraic geometry having various forms (e.g. the proof of Bertini’s Theorem in [37, p. 179], main arguments in [18, 7], etc).

\textbf{Proof}. Now, introduce the universal family $\mathcal{X} \hookrightarrow \mathbb{P}(\mathcal{M}) \times \mathbb{P}^N_k$ of the intersections of such $e$ Fermat-type hypersurfaces:
\[
\mathcal{X} := \left\{ ([A^i_j], [z]) \in \mathbb{P}(\mathcal{M}) \times \mathbb{P}^N_k : \sum_{j=0}^N A^i_j z^d_j = 0, \text{ for } i = 1 \cdots e \right\},
\]
and then consider the subvariety $\mathcal{B} \subset \mathcal{X}$ that consists of all ‘bad points’ defined by:
\[
\text{rank}_k H \leq e - 1. \tag{132}
\]
Let $\pi_1, \pi_2$ below be the two canonical projections:
\[
\begin{array}{ccc}
\mathbb{P}(\mathcal{M}) \times \mathbb{P}^N_k & \xrightarrow{\pi_1} & \mathbb{P}(\mathcal{M}) \\
& \mathbb{P}^N_k \xrightarrow{\pi_2} & \\
\end{array}
\]
Since $\mathbb{P}(\mathcal{M}) \times \mathbb{P}^N_k \supset \mathcal{B}$ is a projective variety and $\pi_1$ is a regular map, now applying Theorem 8.14, we see that:
\[
\pi_1(\mathcal{B}) \subset \mathbb{P}(\mathcal{M})
\]
is an algebraic subvariety. Hence it is necessary and sufficient to show that:
\[
\pi_1(\mathcal{B}) \neq \mathbb{P}(\mathcal{M}). \tag{133}
\]
Our strategy is as follows.

\textit{Step 1}. To decompose $\mathbb{P}^N_k$ into a union of quasi-subvarieties:
\[
\mathbb{P}^N_k = \bigcup_{k=0}^N \mathbb{P}^N_k^k,
\]
where $\mathbb{P}^N_k^k$ consists of points $[z] = [z_0 : z_1 : \cdots : z_N] \in \mathbb{P}^N_k$ with exactly $k$ vanishing homogeneous coordinates, the other ones being nonzero.

\textit{Step 2}. For every integer $k = 0 \cdots N$, for every point $[z] \in \mathbb{P}^N_k^k$, to establish the fibre dimension identity:
\[
\dim \pi_2^{-1}([z]) \cap \mathcal{B} = \dim \mathbb{P}(\mathcal{M}) - (\max \{ N - k - e + 1, 0 \} + e). \tag{135}
\]
Proof of Step 2. Without loss of generality, we may assume that the last \( k \) homogeneous coordinates of \([z]\) vanish:
\[
\begin{align*}
  z_{N-k+1} = \cdots = z_N = 0,
\end{align*}
\]
and then by the definition of \( \mathcal{F}^N_{\mathbb{K}^e} \), none of the first \((N-k+1)\) coordinates \( z_0, \ldots, z_{N-k} \) vanish.

Noting that:
\[
\pi_2^{-1}([z]) \cap \mathcal{B} = \pi_1\left(\pi_2^{-1}([z]) \cap \mathcal{B}\right) \times \{[z]\},
\]
by Theorem 8.14 is an algebraic set one point set and considering the canonical projection:
\[
\bar{\pi}: \mathcal{M} \setminus \{0\} \longrightarrow \mathcal{P}(\mathcal{M}),
\]
we receive:
\[
\dim \pi_2^{-1}([z]) \cap \mathcal{B} = \dim \pi_1\left(\pi_2^{-1}([z]) \cap \mathcal{B}\right) = \dim \bar{\pi}^{-1}\left(\pi_1\left(\pi_2^{-1}([z]) \cap \mathcal{B}\right)\right) \cup \{0\} - 1. \tag{137}
\]

Now, observe that whatever choice of parameters:
\[
(A_{ij})_{1 \leq i \leq e, 0 \leq j \leq N} \in \mathcal{M},
\]
the vanishing of the last \( k \) coordinates of \([z]\) in \((136)\) makes the last \( k \) columns of \( H(z) \) in \((131)\) vanish. It is therefore natural to introduce the submatrix \( N-k+1 \) of \( H \) that consists of the remaining nonvanishing columns, i.e. the first \((N+1-k)\) ones. Since the sum of all columns of \( H(z) \) vanishes by \((130)\), the sum of all columns of \( N-k+1 H(z) \) also vanishes.

Observe that the set:
\[
\mathcal{M} \supset \bar{\pi}^{-1}\left(\pi_1\left(\pi_2^{-1}([z]) \cap \mathcal{B}\right)\right) \cup \{0\} = \left\{(A_{ij})_{1 \leq i \leq e, 0 \leq j \leq N} \in \mathcal{M} : \text{sum of all the columns of } N-k+1 H(z) \text{ vanishes, and } \right. \\
\left. \text{rank}_{\mathbb{K}} N-k+1 H(z) \leq e - 1 \right\}
\]
is nothing but the inverse image of:
\[
0_{\sum_{e-1}^{e,N+1-k}} \subset \text{Mat}_{e \times (N+1-k)}(\mathbb{K}) \tag{use notation of Lemma 8.9}
\]
under the \( \mathbb{K} \)-linear map:
\[
N-k+1 H_{z}: \mathcal{M} \longrightarrow \text{Mat}_{e \times (N+1-k)}(\mathbb{K}),
\]
\[
(A_{ij})_{i,j} \longmapsto N-k+1 H(z),
\]
which is surjective by Lemma 8.11.

Therefore we have the codimension identity:
\[
\text{codim } \bar{\pi}^{-1}\left(\pi_1\left(\pi_2^{-1}([z]) \cap \mathcal{B}\right)\right) \cup \{0\} = \text{codim } 0_{\sum_{e-1}^{e,N+1-k}} \left[N-k+1 H_{z} \text{ is linear and surjective}\right] \tag{138}
\]
\[
= \max \{N-k-e+1, 0\} + e,
\]
[use Lemma (8.9)]
and thereby we receive:

\[
\dim \pi_2^{-1}(\{z\}) \cap B = \dim \pi_2^{-1}\left(\pi_1(\pi_2^{-1}(\{z\}) \cap \mathcal{B})\right) \cup \{0\} - 1 \quad \text{[use (137)]}
\]

[by definition of codimension]

\[
= \dim \mathcal{M} - \operatorname{codim} \pi_2^{-1}\left(\pi_1(\pi_2^{-1}(\{z\}) \cap \mathcal{B})\right) \cup \{0\} - 1
\]

[exercise]

\[
= \dim \mathcal{P}(\mathcal{M}) - \operatorname{codim} \pi_2^{-1}\left(\pi_1(\pi_2^{-1}(\{z\}) \cap \mathcal{B})\right) \cup \{0\}
\]

[use (138)]

\[
= \dim \mathcal{P}(\mathcal{M}) - \max\{N - k - e + 1, 0\} + e,
\]

which is exactly our claimed fibre dimension identity (135). \(\square\)

**Step 3.** Applying Lemma 8.2 to the regular map:

\[
\pi_2: \pi_2^{-1}\left(k\mathbb{P}^N_k\right) \cap \mathcal{B} \longrightarrow k\mathbb{P}^N_k,
\]

remembering:

\[
\dim k\mathbb{P}^N_k = N - k \quad (k = 0 \ldots N),
\]

together with the identity (135), we receive the dimension estimate:

\[
\dim \pi_2^{-1}\left(k\mathbb{P}^N_k\right) \cap \mathcal{B} \leq \dim k\mathbb{P}^N_k + \dim \mathcal{P}(\mathcal{M}) - \max\{N - k - e + 1, 0\} - e
\]

\[
\leq (N - k) + \dim \mathcal{P}(\mathcal{M}) - (N - k - e + 1) - e \quad \text{(139)}
\]

\[
= \dim \mathcal{P}(\mathcal{M}) - 1.
\]

Note that \(\mathcal{B}\) can be written as the union of \((N + 1)\) quasi-subvarieties:

\[
\mathcal{B} = \pi_2^{-1}\left(\mathbb{P}^N_k\right) \cap \mathcal{B}
\]

\[
= \pi_2^{-1}\left( \bigcup_{k=0}^N k\mathbb{P}^N_k \right) \cap \mathcal{B}
\]

\[
= \left( \bigcup_{k=0}^N \pi_2^{-1}\left(k\mathbb{P}^N_k\right) \right) \cap \mathcal{B}
\]

\[
= \bigcup_{k=0}^N \pi_2^{-1}\left(k\mathbb{P}^N_k\right) \cap \mathcal{B},
\]

each one being, thanks to (139), of dimension less than or equal to:

\[
\dim \mathcal{P}(\mathcal{M}) - 1,
\]

and therefore we have the dimension estimate:

\[
\dim \mathcal{B} \leq \dim \mathcal{P}(\mathcal{M}) - 1.
\]

Finally, (133) follows from the dimensional comparison:

\[
\dim \pi_1(\mathcal{B}) \leq \dim \mathcal{B} \leq \dim \mathcal{P}(\mathcal{M}) - 1.
\]

\(\square\)

In the more general context of our moving coefficients method, we now want to have an everywhere full-rank property analogous to Lemma 8.15 just obtained.

Observing that in (92), the number of terms in each polynomial \(F_i\) is:

\[
(N + 1) + \sum_{\ell = e + r + 1}^N \binom{N + 1}{\ell + 1} (\ell + 1),
\]
and recalling that the $\mathbb{K}$-linear subspace $\mathcal{A}_\epsilon(\mathbb{K}^{N+1}) \subset \mathbb{K}[z_0, \ldots, z_N]$ spanned by all degree $\epsilon$ homogeneous polynomials is of dimension:

$$\dim_{\mathbb{K}} \mathcal{A}_\epsilon(\mathbb{K}^{N+1}) = \binom{N + \epsilon}{N},$$

we may denote by $\mathbb{P}^\bullet_{\mathbb{K}}$ the projectivized parameter space of such $c + r$ hypersurfaces, with the integer:

$$\bullet := \left( (N + 1) + \sum_{\ell = c + r + 1}^{N} \binom{N + 1}{\ell + 1} (\ell + 1) \right) \sum_{i=1}^{c+r} \binom{N + \epsilon}{N}. \quad (140)$$

Now, by mimicking the construction of the matrix $H$ in (131), employing the notation in Subsection 7.2, for every integer $\nu = 0 \cdots N$, let us denote by $H^\nu$ the $(c + r) \times (N + 1)$ matrix whose $i$-th row copies the $(N + 1)$ terms of $F_i$ in (105). Also, for every integer $\tau = 0 \cdots N - 1$, for every index $\rho = \tau + 1 \cdots N$, let us denote by $H^{\tau,\rho}$ the $(c + r) \times (N + 1)$ matrix whose $i$-th row copies the $(N + 1)$ terms of $F_i$ in (107).

**Lemma 8.16.** In $\mathbb{P}^\bullet_{\mathbb{K}}$, there exists a proper algebraic subset:

$$\Sigma \subsetneq \mathbb{P}^\bullet_{\mathbb{K}}$$

such that, for every choice of parameter outside $\Sigma$:

$$[A^*; M^*] \in \mathbb{P}^\bullet_{\mathbb{K}} \setminus \Sigma,$$

on the corresponding intersection:

$$X = H_1 \cap \cdots \cap H_{c+r} \subset \mathbb{P}^N_{\mathbb{K}},$$

all the matrices $H^\nu$, $H^{\tau,\rho}$ have full rank $c$:

$$\text{rank}_{\mathbb{K}} H^\nu(z) = c + r, \quad \text{rank}_{\mathbb{K}} H^{\tau,\rho}(z) = c + r \quad (\forall [z] \in X).$$

We can copy the proof of Lemma 8.15 without much modification and thus everything works smoothly. Alternatively, we may present a short proof by applying Lemma 8.15.

**Proof. Observation 1.** We need only prove this lemma separately for each matrix $H^\nu$ (resp. $H^{\tau,\rho}$), i.e. to show that there exists a proper algebraic subset:

$$\Sigma^\nu \subsetneq \mathbb{P}^\bullet_{\mathbb{K}}$$

outside of which every choice of parameter succeeds. Then the union of all these proper algebraic subsets works:

$$\Sigma := \bigcup_{\nu=0}^{N} \Sigma^\nu \cup \bigcup_{\rho=\tau+1}^{N-1} \Sigma^{\tau,\rho} \subsetneq \mathbb{P}^\bullet_{\mathbb{K}}.$$ 

**Observation 2.** For each matrix $H^\nu$ (resp. $H^{\tau,\rho}$), inspired by the beginning arguments in the proof of Lemma 8.15, especially (133), we only need to find one parameter:

$$[A^*; M^*] \in \mathbb{P}^\bullet_{\mathbb{K}} \setminus \Sigma$$

with the desired property.

**Observation 3.** Now, setting all the moving coefficients zero:

$$M^*_i := 0,$$
thanks to (103), (106), the equations (105) become exactly the equations (130), and therefore all the matrices $H^\nu$ become the same matrix $H$ of Lemma 8.15 (with $e = c + r$). Similarly, so do all the matrices $H^{\tau,\rho}$.

Observation 4. Now, a direct application of Lemma 8.15 clearly yields more than one parameter, an infinity!

Once again, by mimicking the construction of the matrix $H$ in Lemma 8.15, employing the notation in subsection 7.3, let us denote by $v_1,...,v_q$ (resp. $v_1,...,v_q H^{\tau,\rho}$) the $c \times (N + 1)$ matrix whose $i$-th row copies the $(N + 1)$ terms of $F_i$ in (120) (resp. (121)).

Lemma 8.17. In $\mathbb{P}^p_{\mathbb{K}}$, there exists a proper algebraic subset:

$$v_1,...,v_q \subseteq \mathbb{P}^p_{\mathbb{K}}$$

such that, for every choice of parameter outside $v_1,...,v_q$:

$$[A^\nu, M^\nu] \in \mathbb{P}^p_{\mathbb{K}} \setminus v_1,...,v_q \mathbb{P}^p_{\mathbb{K}}$$

on the corresponding intersection:

$$X = H_1 \cap \cdots \cap H_{e+r} \subset \mathbb{P}^N_{\mathbb{K}},$$

all the matrices $v_1,...,v_q H^\nu$ and $v_1,...,v_q H^{\tau,\rho}$ have full rank $c + r$:

$$\text{rank}_{\mathbb{K}} v_1,...,v_q H^\nu(z) = c + r, \quad \text{rank}_{\mathbb{K}} v_1,...,v_q H^{\tau,\rho}(z) = c + r \quad (\forall z \in X).$$

The proof goes exactly the same way as in the preceding lemma.

9. Controlling the base locus

9.1. Characterization of the base locus. Now, we are in a position to characterize the base locus of all the obtained global twisted symmetric differential $n$-forms in (110), (111):

$$\text{BS} := \text{Base Locus of } \{ \phi^\nu_{j_1,...,j_n}, \psi^{\tau,\rho}_{j_1,...,j_n} \}_{1 \leq j_1 < \cdots < j_n \leq c} \subseteq \mathbb{P}(T_V|_X),$$

(141)

where $\mathbb{P}(T_V|_X) \subset \mathbb{P}(T_{\mathbb{P}^p_{\mathbb{K}}})$ is given by:

$$\mathbb{P}(T_V|_X) := \{(\tau, \xi) : F_i(z) = 0, dF_j|_\xi(z) = 0, \forall i = 1 \cdots c + r, \forall j = 1 \cdots c \}$$

To begin with, for every $\nu = 0 \cdots N$, let us study the specific base locus:

$$\text{BS}^\nu := \text{Base Locus of } \{ \phi^\nu_{j_1,...,j_n} \}_{1 \leq j_1 < \cdots < j_n \leq c} \subseteq \mathbb{P}(T_V|_X)$$

associated with only the twisted symmetric differential forms obtained in (110).

For each sequence of ascending indices:

$$1 \leq j_1 < \cdots < j_n \leq c,$$

by mimicking the construction of the matrices $K, \widehat{K}_{\nu,\tau,\rho}$ at the end of Subsection 6.3, in accordance with the first kind of manipulation (105), we construct the $(c + r + c) \times (N + 1)$ matrix $K'$ in the obvious way, i.e. by copying terms, differentials, and then we define the analogous $\widehat{K}'_{\nu,\tau,\rho}$.

First, let us look at points $([z], [\xi]) \in \text{BS}^\nu$ having all coordinates nonvanishing:

$$z_0 \cdots z_N \neq 0.$$  

(142)
For each symmetric horizontal differential $n$-form $\phi^\nu_{j_1,\ldots,j_n}$ which corresponds to $\phi^\nu_{j_1,\ldots,j_n}$ in the sense of Propositions 6.10, 6.9, for every $j = 0 \cdots N$, we receive:

$$0 = \phi^\nu_{j_1,\ldots,j_n}(z, \xi) \quad [\text{since } ([z], [\xi]) \in \text{BS}^\nu]$$

$$= (-1)^j \frac{z_0^{\star} \cdots z_N^{\star}}{z_0 \cdots z_N} \det \left( K^\nu_{j_1,\ldots,j_n}(z, \xi) \right),$$

where all integers $\star$ are of no importance here. Indeed, we can drop the nonzero factor $\frac{(-1)^j}{z_0 \cdots z_N}$ and obtain:

$$\det \left( K^\nu_{j_1,\ldots,j_n}(z, \xi) \right) = 0.$$

In other words:

$$\text{rank}_{k=0} K^\nu_{j_1,\ldots,j_n}(z, \xi) \leq N - 1.$$ 

Now, letting the index $j$ run from 0 to $N$, we receive:

$$\text{rank}_{k=0} K^\nu_{j_1,\ldots,j_n}(z, \xi) \leq N - 1, \quad (143)$$

where $K^\nu_{j_1,\ldots,j_n}$ is defined analogously to the matrix $C^\nu_{j_1,\ldots,j_n}$ before Proposition 6.8 in the obvious way.

Note that the first $c + r$ rows of $K^\nu_{j_1,\ldots,j_n}$ constitute the matrix $H^\nu$ in Lemma 8.16, which asserts that for a generic choice of parameter:

$$\text{rank}_{k=0} H^\nu(z) = c + r.$$ 

Now, in (143), letting $1 \leq j_1 < \cdots < j_n \leq c$ vary, and applying Lemma 8.5, we immediately receive:

$$\text{rank}_{k=0} K^\nu(z, \xi) \leq N - 1.$$ 

Conversely, it is direct to see that any point $([z], [\xi]) \in \mathbb{P}(T_V)$ satisfying this rank inequality lies in the base locus $\text{BS}^\nu$.

Note that a point $([z], [\xi]) \in \mathbb{P}(T_P N)$ lies in $\mathbb{P}(T_V|_X)$ if and only if the sum of all columns of $K^\nu(z, \xi)$ vanishes. Summarizing the above analysis, restricting to the coordinates nonvanishing part of $\mathbb{P}(T_P N)$:

$$\widehat{\mathbb{P}}(T_P N) := \mathbb{P}(T_P N) \cap \{z_0 \cdots z_N \neq 0\},$$

we conclude the following generic characterization of:

$$\text{BS}^\nu \cap \widehat{\mathbb{P}}(T_P N),$$

where the exceptional locus $\Sigma$ just below is defined in Lemma 8.16.

**Proposition 9.1.** For every choice of parameter outside $\Sigma$:

$$[A^\nu, M^\nu] \in \mathbb{P}^n_d \setminus \Sigma$$

a point:

$$([z], [\xi]) \in \widehat{\mathbb{P}}(T_P N)$$

lies in the base locus:

$$([z], [\xi]) \in \text{BS}^\nu.$$
if and only if:
\[ \text{rank}_{\mathbb{K}} K^\nu (z, \xi) \leq N - 1, \text{ and the sum of all columns vanishes.} \]

Now, for every integer \( \tau = 0 \cdots N - 1 \) and for every index \( \rho = \tau + 1 \cdots N \), the base locus:
\[ BS^{\tau, \rho} := \text{Base Locus of } \{ \psi_{j_1, \ldots, j_n}^{\tau, \rho} \}_{j_1 < \cdots < j_n < c} \subseteq \mathbb{P}(T_V|_X) \]
associated with the twisted symmetric differential forms obtained in (111) enjoys the following generic characterization on the coordinates nonvanishing set \{\(z_0 \cdots z_N \neq 0\)\}. Of course, the matrix \( K^{\tau, \rho} \) is defined analogously to the matrix \( K^\nu \) in the obvious way. A repetition of the preceding arguments yields:

**Proposition 9.2.** For every choice of parameter outside \( \Sigma \):
\[ [A^*_0, M^*_1] \in \mathbb{P}_{X} \setminus \Sigma \]
a point:
\[ ([z], [\xi]) \in \mathbb{P}^\circ (T_P|_X) \]
lies in the base locus:
\[ ([z], [\xi]) \in BS^{\tau, \rho} \]
if and only if:
\[ \text{rank}_{\mathbb{K}} K^{\tau, \rho} (z, \xi) \leq N - 1, \text{ and the sum of all columns vanishes.} \]

It is now time to clarify the (uniform) structures of the matrices \( K^\nu, K^{\tau, \rho} \).

Thanks to the above two Propositions 9.1, 9.2, we may now receive a generic characterization of:
\[ BS \cap \mathbb{P}^\circ (T_P|_X). \]

Firstly, we construct the \((c + r + c) \times (2N + 2)\) matrix \( M \) such that, for \( i = 1 \cdots c + r, j = 1 \cdots c \), its \( i \)-row copies the \((2N + 2)\) terms of \( F_i \) in (102) in the exact order, and its \((c + r + j)\)-th row is the differential of the \( j \)-th row. In order to distinguish the first \((N + 1)\) ‘dominant’ columns from the last \((N + 1)\) columns of moving coefficient terms, we write \( M \) as:
\[ M = \left( A_0 | \cdots | A_N | B_0 | \cdots | B_N \right). \]

For every index \( \nu = 0 \cdots N \), comparing (105), (106) with (102), the matrix \( K^\nu \) is nothing but:
\[ K^\nu = \left( A_0 | \cdots | \widehat{A}_r | \cdots | A_N | A_0 + \sum_{j=0}^N B_j \right). \]  

Similarly, for every integer \( \tau = 0 \cdots N - 1 \) and for every index \( \rho = \tau + 1 \cdots N \), comparing (107), (108) with (102), the matrix \( K^{\tau, \rho} \) is nothing but:
\[ K^{\tau, \rho} = \left( A_0 + B_0 | \cdots | A_r + B_r | A_{r+1} | \cdots | \widehat{A}_\rho | \cdots | A_N | A_0 + \sum_{j=r+1}^N B_j \right). \]

Secondly, we introduce the algebraic subvariety:
\[ \mathcal{M}^{N,c+r}_{2c+r} \subset \text{Mat}_{(2c+r) \times 2(N+1)}(\mathbb{K}) \]
consisting of all \((c + r + c) \times 2(N + 1)\) matrices \((a_0 | a_1 | \cdots | a_N | b_0 | b_1 | \cdots | b_N)\) such that:
we also have to analyze the base locus of the twisted symmetric di
order, and whose (and more specifically, we focus on the ‘interior part’:

\[ \alpha_0 + \alpha_1 + \cdots + \alpha_N + \beta_0 + \beta_1 + \cdots + \beta_N = 0; \]  
\[(147)\]

**ii** for every index \( v = 0 \cdots N \), replacing \( \alpha_v \) with \( \alpha_v + (\beta_0 + \beta_1 + \cdots + \beta_N) \) in the collection of column vectors \( \{ \alpha_0, \alpha_1, \ldots, \alpha_N \} \), there holds the rank inequality:

\[ \text{rank} \subseteq \{ \alpha_0, \ldots, \alpha_N, \alpha_v + (\beta_0 + \beta_1 + \cdots + \beta_N) \} \leq N - 1; \]  
\[(148)\]

**iii** for every integer \( \tau = 0 \cdots N-1 \), for every index \( \rho = \tau + 1 \cdots N \), replacing \( \alpha_\rho \) with \( \alpha_\rho + (\beta_{\tau+1} + \cdots + \beta_N) \) in the collection of column vectors \( \{ \alpha_0 + \beta_0, \ldots, \alpha_\tau + \beta_\tau, \alpha_{\tau+1}, \ldots, \alpha_\rho, \ldots, \alpha_N \} \), there holds the rank inequality:

\[ \text{rank} \subseteq \{ \alpha_0 + \beta_0, \alpha_1 + \beta_1, \ldots, \alpha_\tau + \beta_\tau, \alpha_{\tau+1}, \ldots, \alpha_\rho, \ldots, \alpha_N, \alpha_\rho + (\beta_{\tau+1} + \cdots + \beta_N) \} \leq N - 1. \]  
\[(149)\]

**Proposition 9.3.** For every choice of parameter outside \( \Sigma \):

\[ [A^*, M^*] \in \mathbb{P}_K^* \setminus \Sigma \]

a point:

\[ ([z], [\xi]) \in \mathbb{P}_K^* (T_{\mathbb{P}^N}) \]

lies in the base locus:

\[ ([z], [\xi]) \in \text{BS} \]

if and only if:

\[ M(z, \xi) \in \mathcal{M}_{2c+r}. \]

Furthermore, for all integers \( 1 \leq \eta \leq n - 1 \), for every sequence of ascending indices:

\[ 0 \leq v_1 < \cdots < v_\eta \leq N, \]

we also have to analyze the base locus of the twisted symmetric differential forms (122), (123):

\[ v_{1, \ldots, v_\eta} \text{BS} := \text{Base Locus of } \{ v_{1, \ldots, v_\eta} \phi_{j_1, \ldots, j_{n-\eta}}^y, v_{1, \ldots, v_\eta} \psi_{j_1, \ldots, j_{n-\eta}}^{a^*} \}_{1 \leq j_1 < \cdots < j_{n-\eta} \leq c} \]
\[(150)\]

in the intersection of the \( \eta \) hyperplanes:

\[ v_{1, \ldots, v_\eta} \mathbb{P}(T_{\mathbb{P}^N}) := \mathbb{P}(T_{\mathbb{P}^N}) \cap \{ z_{v_1} = \cdots = z_{v_\eta} = 0 \}, \]

and more specifically, we focus on the ‘interior part’:

\[ v_{1, \ldots, v_\eta} \mathbb{P}(T_{\mathbb{P}^N}) := v_{1, \ldots, v_\eta} \mathbb{P}(T_{\mathbb{P}^N}) \cap \{ z_0 \cdots z_{r_{n-\eta}} \neq 0 \}; \]

[see (80) for the indices \( r_0, \ldots, r_{n-\eta} \)].

Firstly, we construct the \( (c + r + c) \times (2N + 2 - 2\eta) \) matrix \( v_{1, \ldots, v_\eta} M \), which will play the same role as the matrix \( M \), whose \( i \)-row \( (i = 1 \cdots c + r) \) copies the \( (2N + 2 - 2\eta) \) terms of (114) in the exact order, and whose \( (c + r + j) \)-th row \( (j = 1 \cdots c) \) is the differential of the \( j \)-th row.

Secondly, in correspondence with \( \mathcal{M}_{2c+r} \), by replacing plainly \( N \) with \( N - \eta \), we introduce the algebraic variety:

\[ \mathcal{M}_{2c+r}^{N-\eta} \subset \text{Mat}_{(2c+r) \times (2N-\eta+1)}(K). \]
\[(151)\]

Thirdly, let us recall the exceptional subvariety:

\[ v_{1, \ldots, v_\eta} \Sigma \subset \mathbb{P}_K^* \]

defined in Proposition 8.17.

By performing the same reasoning as in the preceding proposition, we get:
Proposition 9.4. For every choice of parameter outside \( v_1, \ldots, v_\eta \Sigma \):

\[
[A^*_1, M^*_1] \in \mathbb{P}^*_K \setminus v_1, \ldots, v_\eta \Sigma
\]

a point:

\[
([z], [\xi]) \in v_1, \ldots, v_\eta \mathbb{P}(T_{\mathbb{P}^N})
\]

lies in the base locus (150):

\[
([z], [\xi]) \in v_1, \ldots, v_\eta \mathbb{B}S
\]

if and only if:

\[
v_1, \ldots, v_\eta M(z, \xi) \in M^{N-\eta}_{2c+r}.
\]

\[\square\]

9.2. **Emptiness of the base loci.** First, for the algebraic varieties (146), (151), we claim the following codimension estimates, which serve as the engine of the moving coefficients method. However, we will not present it here but in the next section.

**Lemma 9.5 (Core Lemma of MCM).** (i) For every positive integers \( N \geq 1 \), for every integers \( c, r \geq 0 \) with \( 2c + r \geq N \), there holds the codimension estimate:

\[
codim M^{N}_{2c+r} \geq \dim \mathbb{P}(T_{\mathbb{P}^N}) = 2N - 1.
\]

(ii) For every positive integer \( \eta = 1 \cdots N - (c + r) - 1 \), for every sequence of ascending indices:

\[
0 \leq v_1 < \cdots < v_\eta \leq N,
\]

there holds the codimension estimate:

\[
codim M^{N-\eta}_{2c+r} \geq \dim v_1, \ldots, v_\eta \mathbb{P}(T_{\mathbb{P}^N}) = 2N - \eta - 1.
\]

\[\square\]

Now, let us show the power of this Core Lemma.

Bearing Proposition 9.3 in mind, by mimicking the proof of Proposition 8.15, it is natural to introduce the subvariety:

\[
M^{N}_{2c+r} \hookrightarrow \mathbb{P}^*_K \times \mathbb{P}(T_{\mathbb{P}^N}),
\]

which is defined ‘in family’ by:

\[
M^{N}_{2c+r} := \{([A^*_1, M^*_1]; [z], [\xi]) \in \mathbb{P}^*_K \times \mathbb{P}(T_{\mathbb{P}^N}) : M(z, \xi) \in M^{N}_{2c+r} \}.
\]

**Proposition 9.6.** There holds the dimension estimate:

\[
\dim M^{N}_{2c+r} \leq \dim \mathbb{P}^*_K.
\]

**Proof.** Let \( \pi_1, \pi_2 \) be the two canonical projections:

\[
\begin{array}{ccc}
\mathbb{P}^*_K \times \mathbb{P}(T_{\mathbb{P}^N}) & \xrightarrow{\pi_1} & \mathbb{P}^*_K \\
& \pi_2 \downarrow & \downarrow \\
& \mathbb{P}(T_{\mathbb{P}^N}) & \xrightarrow{\bar{\mathbb{P}}} & \mathbb{P}(T_{\mathbb{P}^N}).
\end{array}
\]

By mimicking Step 2 in Lemma 8.15, for every point \(([z], [\xi]) \in \mathbb{P}(T_{\mathbb{P}^N})\), we claim the fibre dimension estimate:

\[
\dim \pi^*_2([z], [\xi]) \cap M^{N}_{2c+r} = \dim \mathbb{P}^*_K - \text{codim } M^{N}_{2c+r}
\]

(152)
Proof. Noting that:
\[
\pi_2^{-1}([z], [\xi]) \cap M_{2c+r}^N = \pi_1\left(\pi_2^{-1}([z], [\xi]) \cap M_{2c+r}^N\right) \times \{[z], [\xi]\},
\]
by Theorem 8.14 is an algebraic set
one point set
and considering the canonical projection:
\[
\widehat{\pi} : \mathbb{K}^\bullet \setminus \{0\} \longrightarrow \mathbb{P}_{\mathbb{K}}^\bullet,
\]
we receive:
\[
\dim \pi_2^{-1}([z], [\xi]) \cap M_{2c+r}^N = \dim \pi_1\left(\pi_2^{-1}([z], [\xi]) \cap M_{2c+r}^N\right)
= \dim \widehat{\pi}^{-1}\left(\pi_1\left(\pi_2^{-1}([z], [\xi]) \cap M_{2c+r}^N\right) \cup \{0\}\right) - 1.
\]
(153)
Now, notice that the set:
\[
\mathbb{K}^\bullet \supset \pi_2^{-1}\left(\pi_1\left(\pi_2^{-1}([z], [\xi]) \cap M_{2c+r}^N\right) \cup \{0\}\right)
= \left\{(A^*_\bullet, M^*_\bullet) \in \mathbb{K}^\bullet : M(z, \xi) \in \mathcal{M}_{2c+r}^N\right\}
\]
is nothing but the inverse image of:
\[
\mathcal{M}_{2c+r}^N \subset \text{Mat}_{2c+r} \times 2(N+1)(\mathbb{K})
\]
under the \(\mathbb{K}\)-linear map:
\[
M_{c, \xi} : \mathbb{K}^\bullet \longrightarrow \text{Mat}_{2c+r} \times 2(N+1)(\mathbb{K})
(A^*_\bullet, M^*_\bullet) \longmapsto M(z, \xi),
\]
which is surjective by the construction of \(M\) — see (102), (103), and by applying Lemma 8.11 —
since \(z_0 \neq 0, \ldots, z_N \neq 0\) and \(\xi \notin \mathbb{K} \cdot z\).

Therefore, we have the codimension identity:
\[
\text{codim} \pi_2^{-1}\left(\pi_1\left(\pi_2^{-1}([z], [\xi]) \cap M_{2c+r}^N\right) \cup \{0\}\right) = \text{codim} \mathcal{M}_{2c+r}^N
\]
\[\text{[\(M_{c, \xi}\) is linear and surjective]},\]
(154)
and thereby we receive:
\[
\dim \pi_2^{-1}([z], [\xi]) \cap M_{2c+r}^N = \dim \pi_2^{-1}\left(\pi_1\left(\pi_2^{-1}([z], [\xi]) \cap M_{2c+r}^N\right) \cup \{0\}\right) - 1
= \dim \mathbb{K}^\bullet - \text{codim} \pi_2^{-1}\left(\pi_1\left(\pi_2^{-1}([z], [\xi]) \cap M_{2c+r}^N\right) \cup \{0\}\right) - 1
\]
\[\text{[by definition of codimension]},\]
\[
= \dim \mathbb{P}_{\mathbb{K}}^\bullet - \text{codim} \pi_2^{-1}\left(\pi_1\left(\pi_2^{-1}([z], [\xi]) \cap M_{2c+r}^N\right) \cup \{0\}\right)
\]
\[\text{[why?]},\]
\[
= \dim \mathbb{P}_{\mathbb{K}}^\bullet - \text{codim} \mathcal{M}_{2c+r}^N
\]
\[\text{[use (154)]},\]
which is exactly our claimed fibre dimension identity.

Lastly, by applying the Fibre Dimension Estimate 8.2, we receive:
\[
\dim M_{2c+r}^N \leq \dim \mathbb{P}_{\mathbb{T}_{\mathbb{P}^N}} + \dim \mathbb{P}_{\mathbb{K}}^\bullet - \text{codim} \mathcal{M}_{2c+r}^N
\]
\[\text{[use Core Lemma 9.5]},\]
\[
\leq \dim \mathbb{P}_{\mathbb{T}_{\mathbb{P}^N}} + \dim \mathbb{P}_{\mathbb{K}}^\bullet - \dim \mathbb{P}_{\mathbb{T}_{\mathbb{P}^N}}
= \dim \mathbb{P}_{\mathbb{K}}^\bullet,
\]
which is our claimed dimension estimate. \(\square\)
Now, restricting the canonical projection $\pi_1$ to $M^{N}_{2c+r}$:

$$\pi_1 : M^{N}_{2c+r} \rightarrow \mathbb{P}_k^c,$$

according to the dimension inequality just obtained, we gain:

**Proposition 9.7.** There exists a proper algebraic subset $\Sigma' \subsetneq \mathbb{P}_k^c$ such that, for every choice of parameter outside $\Sigma'$:

$$P = [A^*_i, M^*_i] \in \mathbb{P}_k^c \setminus \Sigma',$$

the intersection of the fibre $\pi_1^{-1}(P)$ with $M^{N}_{2c+r}$ is discrete or empty:

$$\dim \pi_1^{-1}(P) \cap M^{N}_{2c+r} \leq 0.$$ 

\[ \square \]

Combining Propositions 9.3 and 9.7, we receive:

**Proposition 9.8.** Outside the proper algebraic subset:

$$\Sigma \cup \Sigma' \subsetneq \mathbb{P}_k^c,$$

for every choice of parameter:

$$[A^*_i, M^*_i] \in \mathbb{P}_k^c \setminus (\Sigma \cup \Sigma'),$$

the base locus in the coordinates nonvanishing set:

$$\text{BS} \cap \{z_0 \cdots z_N \neq 0\}$$

is discrete or empty.  

\[ \square \]

Moreover, bearing in mind Proposition 9.4, by repeating the same reasoning as in the preceding proposition, consider the subvariety:

$$v_1, \ldots, v_{\eta} M^{N-\eta}_{2c+r} \hookrightarrow \mathbb{P}_k^c \times \mathbb{P}(T_{\mathbb{P}^N})$$

which is defined ‘in family’ by:

$$v_1, \ldots, v_{\eta} M^{N-\eta}_{2c+r} := \{(A^*_i, M^*_i); [z], [\xi] \in \mathbb{P}_k^c \times \mathbb{P}(T_{\mathbb{P}^N}); v_1, \ldots, v_{\eta} M(z, \xi) \in \mathcal{M}^{N-\eta}_{2c+r}\},$$

and hence receive a very analog of Proposition 9.6.

**Proposition 9.9.** There holds the dimension estimate:

$$\dim v_1, \ldots, v_{\eta} M^{N-\eta}_{2c+r} \leq \dim \mathbb{P}_k^c.$$ 

\[ \square \]

Again, restricting the canonical projection $\pi_1$ to $v_1, \ldots, v_{\eta} M^{N-\eta}_{2c+r}$:

$$\pi_1 : v_1, \ldots, v_{\eta} M^{N-\eta}_{2c+r} \rightarrow \mathbb{P}_k^c,$$

according to the dimension inequality above, we receive:

**Proposition 9.10.** There exists a proper algebraic subset $v_1, \ldots, v_{\eta} \Sigma' \subsetneq \mathbb{P}_k^c$ such that, for every choice of parameter outside $v_1, \ldots, v_{\eta} \Sigma'$:

$$P = [A^*_i, M^*_i] \in \mathbb{P}_k^c \setminus v_1, \ldots, v_{\eta} \Sigma',$$

the intersection of the fibre $\pi_1^{-1}(P)$ with $M^{N-\eta}_{2c+r}$ is discrete or empty:

$$\dim \pi_1^{-1}(P) \cap M^{N-\eta}_{2c+r} \leq 0.$$ 

\[ \square \]

Combining now Propositions 9.4 and 9.10, we receive:
Proposition 9.11. Outside the proper algebraic subset:

\[ v_1, \ldots, v_\eta \Sigma \cup v_1, \ldots, v_\eta \Sigma' \subset P^k_0 \]

for every choice of parameter:

\[ [A^*_1, M^*_1] \in P^k_0 \setminus (v_1, \ldots, v_\eta \Sigma \cup v_1, \ldots, v_\eta \Sigma'), \]

the base locus in the corresponding ‘coordinates nonvanishing’ set:

\[ v_1, \ldots, v_\eta BS \cap \{ z_{r_0} \cdots z_{r_\eta \eta} \neq 0 \} \]

is discrete or empty. \[\square\]

10. The Engine of MCM

10.1. Core Codimension Formulas. Our motivation of this section is to prove the Core Lemma 9.5, which will succeed in Subsection 10.6.

As an essential step, by induction on positive integers \( p \geq 2 \) and \( 0 \leq \ell \leq p \), we first estimate the codimension \( \ell C_p \) of the algebraic variety:

\[ \ell X_p \subset \text{Mat}_{p \times 2p}(K) \] \hspace{1cm} (155)

which consists of \( p \times 2p \) matrices \( X_p = (\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p) \) such that:

(i) the first \( p \) column vectors have rank:

\[ \text{rank} K \{ \alpha_1, \ldots, \alpha_p \} \leq \ell; \] \hspace{1cm} (156)

(ii) for every index \( v = 1 \cdots p \), replacing \( \alpha_v \) with \( \alpha_v + (\beta_1 + \cdots + \beta_p) \) in the collection of column vectors \( \{ \alpha_1, \ldots, \alpha_p \} \), there holds the rank inequality:

\[ \text{rank} K \{ \alpha_1, \ldots, \hat{\alpha}_v, \ldots, \alpha_p, \alpha_v + (\beta_1 + \cdots + \beta_p) \} \leq p - 1; \] \hspace{1cm} (157)

(iii) for every integer \( \tau = 1 \cdots p - 1 \), for every index \( \rho = \tau + 1 \cdots p \), replacing \( \alpha_\rho \) with \( \alpha_\rho + (\beta_{\tau + 1} + \cdots + \beta_p) \) in the collection of column vectors \( \{ \alpha_1 + \beta_1, \ldots, \alpha_\tau + \beta_\tau, \alpha_{\tau + 1}, \ldots, \alpha_\rho, \ldots, \alpha_p \} \), there holds the rank inequality:

\[ \text{rank} K \{ \alpha_1 + \beta_1, \ldots, \alpha_\tau + \beta_\tau, \alpha_{\tau + 1}, \ldots, \hat{\alpha}_\rho, \ldots, \alpha_p, \alpha_\rho + (\beta_{\tau + 1} + \cdots + \beta_p) \} \leq p - 1. \] \hspace{1cm} (158)

Let us start with the easy case \( \ell = 0 \).

Proposition 10.1. For every integer \( p \geq 2 \), the codimension value \( \ell C_p \) for \( \ell = 0 \) is:

\[ 0 C_p = p^2 + 1. \] \hspace{1cm} (159)

Proof. Now, (i) is equivalent to:

\[ \underbrace{\alpha_1 = \cdots = \alpha_p = 0}_{\text{codim} = p^2}. \]

Thus (ii) holds trivially, and the only nontrivial inequality in (iii) is:

\[ \underbrace{\text{rank} K \{ 0 + \beta_1, \ldots, 0 + \beta_p \}}_{\text{codim} = 1 \text{ by Lemma 8.8}} \leq p - 1, \]

which contributes one more codimension. \[\square\]

For the general case \( \ell = 1 \cdots p \), we will use Gaussian eliminations and do inductions on \( p, \ell \).

First, let us count the codimension of the exceptional locus of Gaussian eliminations.
**Proposition 10.2.** For every integer \( p \geq 2 \), the codimensions \( C^0_p \) of the algebraic varieties:

\[ \{ \alpha_1 + \beta_1 = 0 \} \cap \tau X_p \subset \text{Mat}_{p \times 2p}(K) \]

read according to the values of \( \ell \) as:

\[ C^0_p = \begin{cases} p + 2 & (\ell = p-1, p), \\ p + (p - \ell)^2 & (\ell = 0 \ldots p-2). \end{cases} \]

The following lemma is the key ingredient for the proof.

**Lemma 10.3.** In a field \( K \), let \( W \) be a \( K \)-vector space. Let \( p \geq 1 \) be a positive integer. For any \((p + 1)\) vectors:

\[ \alpha_1, \ldots, \alpha_p, \beta \in W, \]

the rank restriction:

\[ \text{rank}_K \{ \alpha_1, \ldots, \hat{\alpha}_\nu, \ldots, \alpha_p, \alpha_\nu + \beta \} \leq p - 1 \quad (\nu = 1 \ldots p), \tag{160} \]

is equivalent to either:

\[ \text{rank}_K \{ \alpha_1, \ldots, \alpha_p, \beta \} \leq p - 1, \]

or:

\[ \text{rank}_K \{ \alpha_1, \ldots, \alpha_p \} = p, \quad (\alpha_1 + \cdots + \alpha_p) + \beta = 0. \tag{161} \]

**Proof.** Since ‘\( \implies \)’ is clear, we only prove the direction ‘\( \impliedby \)’.

We divide the proof according to the rank of \( \{ \alpha_1, \ldots, \alpha_p \} \) into two cases.

**Case 1:** \( \text{rank}_K \{ \alpha_1, \ldots, \alpha_p \} \leq p - 1 \). Assume on the contrary that:

\[ \text{rank}_K \{ \alpha_1, \ldots, \alpha_p, \beta \} \geq p. \tag{162} \]

Since we have the elementary estimate:

\[ \text{rank}_K \{ \alpha_1, \ldots, \alpha_p, \beta \} \leq \text{rank}_K \{ \alpha_1, \ldots, \alpha_p \} + \text{rank}_K \{ \beta \} \leq (p - 1) + 1 = p, \tag{163} \]

the inequalities ‘\( \geq \)’ or ‘\( \leq \)’ in (161) and (162) are exactly equalities ‘\( = \)’, and thus we have:

\[ \beta \notin \text{Span}_K \{ \alpha_1, \ldots, \alpha_p \}, \tag{164} \]

Consequently, it is clear that we can find a certain index \( \nu \in \{1, \ldots, p\} \) such that:

\[ \text{rank}_K \{ \alpha_1, \ldots, \hat{\alpha}_\nu, \ldots, \alpha_p \} = p - 1, \]

whence the above rank inequality (160) implies:

\[ \alpha_\nu + \beta \in \text{Span}_K \{ \alpha_1, \ldots, \hat{\alpha}_\nu, \ldots, \alpha_p \}, \tag{165} \]

which contradicts the formula (163).

**Case 2:** \( \text{rank}_K \{ \alpha_1, \ldots, \alpha_p \} = p \). Here, inequalities (160) also yield (164) for every \( \nu \), whence:

\[ \beta + (\alpha_1 + \cdots + \alpha_p) \in \text{Span}_K \{ \alpha_1, \ldots, \hat{\alpha}_\nu, \ldots, \alpha_p \}. \tag{166} \]

Now, letting \( \nu \) run from 1 to \( p \), and noting that:

\[ \bigcap_{\nu=1}^{p} \text{Span}_K \{ \alpha_1, \ldots, \hat{\alpha}_\nu, \ldots, \alpha_p \} = \{0\}, \]

we immediately conclude the proof. \( \square \)
Proof of Proposition 10.2. For every matrix $X_p = (\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p)$ such that:

\[
\alpha_1 + \beta_1 = 0, \tag{165}
\]

the conditions (158) in (iii) is trivial, and the restriction (157), thanks to the lemma just obtained, is equivalent either to:

\[
\text{rank}_\mathbb{K} \{\alpha_1, \ldots, \alpha_p, \beta_1 + \cdots + \beta_p\} \leq p - 1, \tag{166}
\]

or to:

\[
\text{rank}_\mathbb{K} \{\alpha_1, \ldots, \alpha_p\} = p, \quad \beta_1 + \cdots + \beta_p = - (\alpha_1 + \cdots + \alpha_p). \tag{167}
\]

Now, since $\alpha_1 + \beta_1 = 0$, adding the first column vector of (166) to the last one, we get:

\[
\text{rank}_\mathbb{K} \{\alpha_1, \ldots, \alpha_p, \beta_2 + \cdots + \beta_p\} \leq p - 1, \quad \text{codim} = 2 \text{ by Lemma 8.8}
\]

and similarly, (167) is equivalent to:

\[
\text{rank}_\mathbb{K} \{\alpha_1, \ldots, \alpha_p\} = p, \quad (\alpha_2 + \cdots + \alpha_p) + (\beta_2 + \cdots + \beta_p) = 0. \quad \text{codim} = p
\]

Therefore, when $\ell = p - 1$ or $\ell = p$, we obtain the codimension formulas:

\[
p_{-1}C^0_p = p + 2, \quad pC^0_p = \min\{p + 2, p + p\} = p + 2.
\]

When $\ell = 0 \cdots p - 2$, the restriction (ii) is a consequence of (i):

\[
\text{rank}_\mathbb{K} \{\alpha_1, \ldots, \alpha_p, \alpha, \beta_1 + \cdots + \beta_p\} \leq \text{rank}_\mathbb{K} \{\alpha_1, \ldots, \alpha_p\} + \text{rank}_\mathbb{K} \{\alpha + (\beta_1 + \cdots + \beta_p)\} \leq \ell + 1 \leq p - 1.
\]

Lastly, applying Lemma 8.8, restriction (i) contributes codimension $(p - \ell)^2$. Together with (165), this finishes the proof. \qed

Now, we claim the following Codimension Induction Formulas, the proof of which will appear in Subsection 10.5. In order to make sense of $\ell - 2C_{p-1}$ in (170) when $\ell = 1$, we henceforth make a convention:

\[-1C_{p-1} := \infty.\]

Proposition 10.4 (Codimension Induction Formulas). (i) For every positive integer $p \geq 2$, for $\ell = p$, the codimension value $pC_p$ satisfies:

\[
pC_p = \min\{p, \ p_{-1}C_p\}. \tag{168}
\]

(ii) For every positive integer $p \geq 3$, for $\ell = p - 1$, the codimension value $\ell C_p$ satisfies:

\[
p_{-1}C_p \geq \min\{p_{-1}C^0_p, \ p_{-1}C_{p-1} + 2, \ p_{-2}C_{p-1} + 1, \ p_{-3}C_{p-1}\}. \tag{169}
\]

(iii) For all integers $\ell = 1 \cdots p - 2$, the codimension values $\ell C_p$ satisfy:

\[
\ell C_p \geq \min\{\ell C^0_p, \ \ell C_{p-1} + 2(p - \ell) - 1, \ \ell_{-1}C_{p-1} + (p - \ell), \ \ell_{-2}C_{p-1}\}. \tag{170}
\]
In fact, all the above inequalities \( \geq \) should be exactly equalities \( = \). Nevertheless, \( \geq \) are already adequate for our purpose.

Now, let us establish the initial data for the induction process.

**Proposition 10.5.** For the initial case \( p = 2 \), there hold the codimension values:
\[
0C_2 = 5, \quad 1C_2 = 3, \quad 2C_2 = 2.
\]

**Proof.** Recalling formulas (159) and (168), we only need to prove \( 1C_2 = 3 \).

For every matrix:
\[
(\alpha_1, \alpha_2, \beta_1, \beta_2) \in (X_2 \setminus 0X_2),
\]
we have:
\[
\text{rank}_\mathbb{K} \{\alpha_1, \alpha_2\} = 1, \quad (171)
\]
\[
\text{rank}_\mathbb{K} \{\alpha_1 + (\beta_1 + \beta_2), \alpha_2\} \leq 1, \quad (172)
\]
\[
\text{rank}_\mathbb{K} \{\alpha_1, \alpha_2 + (\beta_1 + \beta_2)\} \leq 1, \quad (173)
\]
\[
\text{rank}_\mathbb{K} \{\alpha_1 + \beta_1, \alpha_2 + \beta_2\} \leq 1. \quad (174)
\]

Either \( \alpha_1 \) or \( \alpha_2 \) is nonzero. Firstly, assume \( \alpha_1 \neq 0 \). Then (171) yields:
\[
\alpha_2 \in \mathbb{K} \cdot \alpha_1, \quad (175)
\]
and (173) yields:
\[
\alpha_2 + (\beta_1 + \beta_2) \in \mathbb{K} \cdot \alpha_1,
\]
whence by subtracting we receive:
\[
\beta_1 + \beta_2 \in \mathbb{K} \cdot \alpha_1. \quad (176)
\]

Next, adding the second column vector of (174) to the first one, we see:
\[
\text{rank}_\mathbb{K} \{\alpha_1 + \alpha_2 + (\beta_1 + \beta_2), \alpha_2 + \beta_2\} \leq 1. \quad (177)
\]
By (175) and (176):
\[
\alpha_1 + \alpha_2 + (\beta_1 + \beta_2) \in \mathbb{K} \cdot \alpha_1,
\]
therefore (177) yields two possible situations, the first one is:
\[
\alpha_1 + \alpha_2 + \beta_1 + \beta_2 = 0, \quad (178)
\]
and the second one is \( \alpha_1 + \alpha_2 + \beta_1 + \beta_2 \neq 0 \) plus:
\[
\alpha_2 + \beta_2 \in \mathbb{K} \cdot \alpha_1.
\]

Recalling (175), the latter case immediately yields:
\[
\beta_2 \in \mathbb{K} \cdot \alpha_1,
\]
and then (176) implies:
\[
\beta_1 \in \mathbb{K} \cdot \alpha_1,
\]
thus:
\[
\text{rank}_\mathbb{K} \{\alpha_1, \alpha_2, \beta_1, \beta_2\} = 1. \quad (179)
\]

Summarizing, the set:
\[
(X_2 \setminus 0X_2) \cap \{\alpha_1 \neq 0\}
\]
is contained in the union of two algebraic varieties, the first one is defined by (175), (176), (178), and the second one is defined by (179). Since both of the two varieties are of codimension 3, we get:

$$\text{codim}(1X_2 \setminus 0X_2) \cap \{\alpha_1 \neq 0\} \geq 3.$$  

Secondly, by symmetry, we also have:

$$\text{codim}(1X_2 \setminus 0X_2) \cap \{\alpha_2 \neq 0\} \geq 3.$$  

Hence the union of the above two sets satisfies:

$$\text{codim} 1X_2 \setminus 0X_2 \geq 3.$$  

Now, recalling (159):

$$\text{codim} 0X_2 = 5 > 3,$$

we immediately receive:

$$\text{codim} 1X_2 \geq 3.$$  

Finally, noting that \(1X_2\) contains the subvariety:

$$\{\text{rank} \{\alpha_1, \alpha_2, \beta_1, \beta_2\} \leq 1\} \subset \text{Mat}_{2 \times 4}(K),$$

it follows:

$$\text{codim} 1X_2 \leq 3.$$  

In conclusion, the above two estimates squeeze out the desired codimension identity. □

Admitting temporarily Proposition 10.4, it is now time to deduce the crucial

**Proposition 10.6 (Core Codimension Formulas).** For all integers \(p \geq 2\), there hold the codimension estimates:

$$\ell C_p \geq \ell + (p - \ell)^2 + 1 \quad (\ell = 0 \cdots p - 1),$$

and the codimension identity:

$$pC_p = p.$$  

**Proof.** The case \(p = 2\) is already done by the previous proposition.  

Reasoning by induction, assume the formulas (180) and (180′) hold for some integer \(p - 1 \geq 2\), and prove them for the integer \(p\).  

Firstly, formula (159) yields the case \(\ell = 0\).  

Secondly, for the case \(\ell = p - 1\), thanks to Proposition 10.2 and to the induction hypothesis, formula (169) immediately yields:

$$p_{-1}C_p \geq \min\{p_{-1}^0C_p, \ p_{-1}C_{p-1} + 2, \ p_{-2}C_{p-1} + 1, \ p_{-3}C_{p-1}\}$$

$$\geq \min\{p + 2, \ (p - 1) + 2, \ (p - 2) + 1 + 1, \ (p - 3) + 2^2 + 1\}$$

$$= p + 1$$

$$= (p - 1) + 1^2 + 1.$$  

Similarly, for \(\ell = 1 \cdots p - 2\), recalling formula (170):

$$\ell C_p \geq \min\{\ell^0C_p, \ \ell C_{p-1} + 2(p - \ell) - 1, \ \ell_{-1}C_{p-1} + (p - \ell), \ \ell_{-2}C_{p-1}\},$$
and computing:

\[ \ell C_p^0 = p + (p - \ell)^2 \]
\[ = \ell + (p - \ell)^2 + (p - \ell), \]
\[ \ell C_{p-1} + 2(p - \ell) - 1 \geq [\ell + (p - 1 - \ell)^2 + 1] + 2(p - \ell) - 1 \]
\[ = \ell + (p - \ell)^2 - 2(p - \ell) + 1 + 2(p - \ell) - 1 \]
\[ = \ell + (p - \ell)^2 - 1, \]

the desired lower bound!

\[ \ell C_{p-1} + (p - \ell) \geq [(\ell - 1) + (p - \ell)^2 + 1] + (p - \ell) \]
\[ = \ell + (p - \ell)^2 + (p - \ell) \]
\[ \geq 2, \]

\[ \ell C_{p-1} \geq (\ell - 2) + (p - \ell + 1)^2 + 1 \]
\[ = (\ell - 2) + [(p - \ell)^2 + 2(p - \ell) + 1] + 1 \]
\[ = \ell + (p - \ell)^2 + 2(p - \ell), \]

we distinguish the desired lower bound without difficulty.

Lastly, the formula (168) and (181) immediately yield (180′):

\[ pC_p = p, \]

which concludes the proof. □

**Remark 10.7.** In fact, the above estimates “\( \geq \)” in (180) are exactly identities “\( = \)”. By the same reasoning, in Section 12, we will generalize the Core Codimension Formulas to cases of less number of moving coefficients terms, and thus receive better lower bounds on the hypersurfaces degrees.

10.2. **Gaussian eliminations.** Following the notation in (155), we denote by:

\[ X_p = (\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p) \]

the coordinate columns of \( \text{Mat}_{p \times 2p}(\mathbb{K}) \), where each of the first \( p \) columns explicitly writes as:

\[ \alpha_i = (\alpha_{1,i}, \ldots, \alpha_{p,i})^T, \]

and where each of the last \( p \) columns explicitly writes as:

\[ \beta_i = (\beta_{1,i}, \ldots, \beta_{p,i})^T. \]

First, observing the structures of the matrices in (157), (158):

\[ X_{0,v}^0 := (\alpha_1 \mid \cdots \mid \alpha_p \mid \alpha_v + (\beta_1 + \cdots + \beta_p)), \]
\[ X_{r}^{0,v} := (\alpha_1 + \beta_1 \mid \cdots \mid \alpha_v + \beta_v \mid \alpha_{r+1} \mid \cdots \mid \alpha_p \mid \alpha_v + (\beta_{r+1} + \cdots + \beta_p)), \]

where, slightly differently, the second underlined columns are understood to appear in the first underlined removed places, we realize that they have the uniform shapes:

\[ X_{0,v}^0 = X_p^0, \]
\[ X_{r}^{0,v} = X_p^r, \]

(182)
where the $2p \times p$ matrices $I_p^{0,v}$ explicitly read as:

\[
\begin{bmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1
\end{bmatrix}_p
\]

the upper $p \times p$ submatrix being the identity, the lower $p \times p$ submatrix being zero except its $v$-th column being a column of 1, and where lastly, the $2p \times p$ matrices $I_p^\tau,\rho$ explicitly read as:

\[
\begin{bmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1
\end{bmatrix}_p
\]

the upper $p \times p$ submatrix being the identity, the lower $p \times p$ submatrix being zero except $\tau$ copies of 1 in the beginning diagonal and $p - \tau$ copies of 1 at the end of the $p$-th column.

**Observation 10.8.** For all $p \geq 3$, $\tau = 1 \cdots p - 1$, $\rho = \tau + 1 \cdots p$, the matrices $I_p^\tau,\rho$ transform to $I_{p-1}^{\tau-1,\rho-1}$ after deleting the first column and the rows 1, $p + 1$.

Next, observe that all matrices $X^{\tau,\rho}$ have the same first column:

\[
\alpha_1 + \beta_1 = (\alpha_{1,1} + \beta_{1,1}, \cdots, \alpha_{p,1} + \beta_{p,1})^T.
\]
Therefore, when $\alpha_{1,1} + \beta_{1,1} \neq 0$, operating Gaussian eliminations by means of the matrix:

$$G := \begin{pmatrix} 1 & \frac{-\alpha_{2,1} + \beta_{2,1}}{\alpha_{1,1} + \beta_{1,1}} & \cdots & \frac{-\alpha_{p,1} + \beta_{p,1}}{\alpha_{1,1} + \beta_{1,1}} \\ \frac{\alpha_{1,1}}{\alpha_{1,1} + \beta_{1,1}} & 1 & \cdots & \frac{-\alpha_{p,1} + \beta_{p,1}}{\alpha_{1,1} + \beta_{1,1}} \\ \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-\alpha_{p,1} + \beta_{p,1}}{\alpha_{1,1} + \beta_{1,1}} & \frac{-\alpha_{p,1} + \beta_{p,1}}{\alpha_{1,1} + \beta_{1,1}} & \cdots & 1 \end{pmatrix},$$

(183)

these matrices $X^{r,p}$ become simpler:

$$GX^{r,p} = \begin{pmatrix} \alpha_{1,1} + \beta_{1,1} & \bullet & \cdots & \bullet \\ 0 & \star & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \star & \cdots & \star \\ \end{pmatrix},$$

(184)

where the lower-right $(p - 1) \times (p - 1)$ star submatrices enjoy amazing structural properties. At first, we need an:

**Observation 10.9.** Let $p \geq 1$ be a positive integer, let $A$ be a $p \times 2p$ matrix, let $B$ be a $2p \times p$ matrix such that both its 1-st, $(p + 1)$-th rows are $(1, 0, \ldots, 0)$. Then there holds:

$$(AB)' = A'' B'' ,$$

where $(AB)'$ means the $(p - 1) \times (p - 1)$ matrix obtained by deleting the first row and column of $A B$, and where $A''$ means the $(p - 1) \times 2(p - 1)$ matrix obtained by deleting the first row and the columns $1, p + 1$ of $A$, and where $B''$ means the $2(p - 1) \times (p - 1)$ matrix obtained by deleting the first column and the rows $1, p + 1$ of $B$.  

Now, noting that:

$$GX^{r,p} = G (X^{r,p}_p) = (GX_p)^{r,p},$$

thanks to the above two observations, the $(p - 1) \times (p - 1)$ star submatrices enjoy the forms:

$$\begin{pmatrix} \star & \cdots & \star \\ \vdots & \ddots & \vdots \\ \star & \cdots & \star \end{pmatrix} = X^{r-1,p-1}_p,$$

(185)

where $X^{G}_p$ is the $(p - 1) \times 2(p - 1)$ matrix obtained by deleting the first row and the columns $1, p + 1$ of $GX_p$.

Comparing (185) and (182), we immediately see that the star submatrices have the same structures as $X^{0,v}_p, X^{r,p}_p$, which is the cornerstone of our induction approach.

### 10.3. Study the morphism of left-multiplying by $G$.

Let us denote by:

$$D(\alpha_{1,1} + \beta_{1,1}) \subset \text{Mat}_{p \times 2p}(\mathbb{K})$$

the Zariski open set where $\alpha_{1,1} + \beta_{1,1} \neq 0$. Now, consider the regular map of left-multiplying by the function matrix $G$:

$$L_G : D(\alpha_{1,1} + \beta_{1,1}) \longrightarrow D(\alpha_{1,1} + \beta_{1,1})$$

$$X_p \mapsto GX_p.$$  

Of course, it is not surjective, as (184) shows that its image lies in the variety:

$${\cap}_{i=2}^p {\{\alpha_{i,1} + \beta_{i,1} = 0\}}.$$
In order to compensate this loss of surjectivity, combing with the regular map:

$$
\mathcal{E}: \quad D(\alpha_{1,1} + \beta_{1,1}) \rightarrow \text{Mat}_{(p-1)\times 1}(\mathbb{K})
$$

$$
X_p \mapsto (\alpha_{2,1} + \beta_{2,1} | \cdots | \alpha_{p,1} + \beta_{p,1})^T,
$$

we construct a regular map:

$$
L \mathcal{G} \oplus \mathcal{E}: \quad D(\alpha_{1,1} + \beta_{1,1}) \rightarrow \left( \cap_{\ell=2}^p \{ \alpha_{i,1} + \beta_{i,1} = 0 \} \cap D(\alpha_{1,1} + \beta_{1,1}) \right) \oplus \text{Mat}_{(p-1)\times 1}(\mathbb{K}),
$$

which turns out to be an isomorphism. In fact, it has the inverse morphism:

$$
Y \oplus (s_2, \ldots, s_p)^T \mapsto -^1G \cdot Y,
$$

where the matrix $-^1G$ is the “inverse” of the regular function matrix $G$ in (183):

$$
\begin{pmatrix}
1 \\
\frac{s_2}{\alpha_{1,1} + \beta_{1,1}} \\
\vdots \\
\frac{s_p}{\alpha_{1,1} + \beta_{1,1}} \\
1
\end{pmatrix}.
$$

(186)

Now, let us denote by:

$$
\pi_p: \quad \text{Mat}_{p\times 2p}(\mathbb{K}) \rightarrow \text{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K})
$$

the projection map obtained by deleting the first row and the columns 1, $p + 1$. Let us denote also:

$$
\mathcal{L}_{\mathcal{G}} := \pi_p \circ L_{\mathcal{G}}.
$$

It is worth to mention that there is a natural isomorphism:

$$
\mathcal{R}: \quad \mathcal{G} \sim \rightarrow D(\alpha_{1,1} + \beta_{1,1}),
$$

$$
Y \oplus (s_2, \ldots, s_p)^T \mapsto ?
$$

where ? is $Y$ but replacing $(b_2, \ldots, b_{p,1})^T$ by $(s_2, \ldots, s_p)^T$, and thus we obtain a commutative diagram:

$$
\begin{array}{ccc}
D(\alpha_{1,1} + \beta_{1,1}) & \xrightarrow{L_{\mathcal{G}} \oplus \mathcal{E}} & D(\alpha_{1,1} + \beta_{1,1}) \\
\mathcal{L}_{\mathcal{G}} \downarrow & & \mathcal{L}_{\mathcal{G}} \\
\text{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K}), & \xrightarrow{\pi_{p} \oplus \mathcal{R}} & \text{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K}),
\end{array}
$$

(187)

where the horizontal maps are isomorphisms, and where the right vertical map is surjective with fibre:

$$
\ker \pi_p \cap D(\alpha_{1,1} + \beta_{1,1}).
$$

Recalling the end of Subsection 10.2, we in fact received the following key observation.
Observation 10.10. For every positive integer \( p \geq 3 \), for every integer \( \ell = 0 \cdots p - 1 \), the image of the variety:

\[ \ell X_p \cap D(\alpha_{1,1} + \beta_{1,1}) \subseteq D(\alpha_{1,1} + \beta_{1,1}) \]

under the map:

\[ \mathcal{L}_G : D(\alpha_{1,1} + \beta_{1,1}) \to \text{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K}) \]

is contained in the variety:

\[ \ell X_p - 1 \subseteq \text{Mat}_{(p-1)\times 2(p-1)}(\mathbb{K}). \]

\[ \square \]

10.4. A technical lemma. Now, we carry out one preliminary lemma for the final proof of Proposition 10.4.

For all positive integers \( p \geq 3 \), for every integer \( \ell = 0 \cdots p - 1 \), for every fixed \((p-1)\times(p-1)\) matrix \( J \) of rank \( \ell \), denote the space which consists of all the \( p \times p \) matrices of the form:

\[
\begin{pmatrix}
\begin{array}{cccc}
z_{1,1} & z_{1,2} & \cdots & z_{1,p} \\
\vdots & \ddots & \ddots & \vdots \\
z_{2,1} & & \ddots & \vdots \\
z_{p,1} & & & 1
\end{array}
\end{pmatrix}
\]

by \( jS_p,\ell \). For every integer \( j = \ell, \ell + 1 \), denote by \( jS_p,\ell \subseteq jS_p,\ell \) the subvariety that consists of all the matrices having rank \( \leq j \).

Lemma 10.11. The codimensions of \( jS_p,\ell \) are:

\[
\text{codim } jS_p,\ell = \begin{cases} 
2(p-1-\ell) + 1 & (j=\ell), \\
p - 1 - \ell & (j=\ell+1).
\end{cases}
\]

Proof. Step 1. We claim that the codimensions of \( jS_p,\ell \) are independent of the matrix \( J \).

Indeed, choose two invertible \((p-1)\times(p-1)\) matrices \( L \) and \( R \), which normalize the matrix \( J \) by multiplications on both sides:

\[
LJ = \begin{pmatrix}
0 & & & \\
\vdots & \ddots & \ddots & \\
0 & & 1 & \\
\vdots & & \ddots & \ddots \\
0 & & & 1
\end{pmatrix} =: J_0,
\]

where all the entries of \( J_0 \) are zeros except the last \( \ell \) copies of 1 in the diagonal. Therefore, we obtain an isomorphism:

\[ LR : jS_p,\ell \sim \sim J_0jS_p,\ell \]

\[ S \mapsto \left( \begin{array}{c} \begin{pmatrix} 1 \\
L \end{pmatrix} \\
S \end{array} \right) \]

whose inverse is:

\[ L^{-1}R^{-1} : J_0jS_p,\ell \sim \sim jS_p,\ell \]

\[ S \mapsto \left( \begin{array}{c} \begin{pmatrix} 1 \\
L^{-1} \end{pmatrix} \\
S \end{array} \right). \]
Since the map $LR$ preserves the rank of matrices, it induces an isomorphism between $J^j_{p,\ell}$ and $J^j_{0,p,\ell}$, which concludes the claim.

**Step 2.** For $J^j_{0}$, doing elementary row and column operations, we get:

\[
\begin{align*}
\text{rank}_K & \begin{pmatrix}
z_{1,1} & z_{1,2} & \cdots & z_{1,p} \\
z_{2,1} & \vdots \ & \ddots & \\
z_{p,1} & \end{pmatrix}
= J^j_{0} \\
& = \begin{pmatrix}
z_{1,1} & \cdots & z_{1,p-\ell} & z_{1,p-\ell+1} & \cdots & z_{1,p} \\
z_{2,1} & \vdots & \ddots & \vdots & \ddots & \\
z_{p-\ell,1} & \vdots & 0 & 1 & \ddots & \\
z_{p,1} & \end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
& = \begin{pmatrix}
z_{1,1} - \sum_{k=p-\ell+1}^{p} z_{k,1} z_{1,k} & z_{1,2} & \cdots & z_{1,p-\ell} & 0 & \cdots & 0 \\
z_{2,1} & 0 & \ddots & \ddots & \ddots & \cdots & \\
z_{p-\ell,1} & \vdots & 0 & 1 & \ddots & \cdots & 1 \\
z_{p,1} & \end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
& = \begin{pmatrix}
z_{1,1} - \sum_{k=p-\ell+1}^{p} z_{k,1} z_{1,k} & z_{1,2} & \cdots & z_{1,p-\ell} & 0 \\
z_{2,1} & 0 & \vdots & \vdots & \cdots & \cdots & 1 \\
z_{p-\ell,1} & \vdots & 0 & 1 & \ddots & \cdots & \cdots & 1 \\
z_{p,1} & \end{pmatrix}
\end{align*}
\]

\[+ \ell.
\]

**Step 3.** In the $\mathbb{K}$-Euclidian space $\mathbb{K}^{2N-1}$ with coordinates $(z_{1,1}, z_{1,2}, \ldots, z_{1,N}, z_{2,1}, \ldots, z_{N,1})$, the algebraic subvariety defined by the rank inequality:

\[
\begin{align*}
\text{rank}_K & \begin{pmatrix}
z_{1,1} - \sum_{k=p-\ell+1}^{p} z_{k,1} z_{1,k} & z_{1,2} & \cdots & z_{1,p-\ell} & 0 \\
z_{2,1} & 0 & \vdots & \vdots & \cdots & \cdots & 1 \\
z_{p-\ell,1} & \vdots & 0 & 1 & \ddots & \cdots & \cdots & 1 \\
z_{p,1} & \end{pmatrix}
\end{align*}
\]

\[\leq 0 \quad (\text{resp. } \leq 1)
\]

has codimension $2(p - 1 - \ell) + 1$ (resp. $p - 1 - \ell$).

\[\square
\]

10.5. **Proof of Proposition 10.4.** Recalling the definition (155), and applying Lemma 10.3, we receive:

**Corollary 10.12.** For every integers $p \geq 1$, the difference of the varieties:

\[pX_p \setminus p-1X_p \subset \text{Mat}_{px2p}(\mathbb{K})
\]
is exactly the quasi-variety:
\[
\{\alpha_1 + \cdots + \alpha_p + \beta_1 + \cdots + \beta_p = 0\}_{\text{codim} = p} \cap \{\text{rank}_\mathbb{K} \{\alpha_1, \ldots, \alpha_p\} = p\},
\]
whose codimension is \(p\).

\textbf{Proof.} For every \(p \times 2p\) matrix:
\[
p X_p \setminus p-1 X_p \ni X_p = (\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p),
\]
applying now Lemma 10.3 to condition (157):
\[
\text{rank}_\mathbb{K} \{\alpha_1, \ldots, \alpha_v, \ldots, \alpha_p, \alpha_v + (\beta_1 + \cdots + \beta_p)\}_{=\beta} \leq p - 1 \quad (v = 1 \ldots p),
\]
since:
\[
\text{rank}_\mathbb{K} \{\alpha_1, \ldots, \alpha_p\} = p,
\]
we immediately receive:
\[
\alpha_1 + \cdots + \alpha_p + \beta_1 + \cdots + \beta_p = 0.
\]
On the other hand, for all matrices:
\[
X_p = (\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p)
\]
satisfying the above identity, (ii) holds immediately. Noting that the \(p \times p\) matrix in (158) has a vanishing sum of all its columns, it has rank \(\leq p - 1\), i.e. (iii) holds too. \(\square\)

Now, we give a complete proof of the Codimension Induction Formulas.

\textbf{Proof of (168).} This is a direct consequence of the above corollary. \(\square\)

\textbf{Proof of (169).} By Observation 10.10, under the map:
\[
\mathcal{L}_G : D(\alpha_{1,1} + \beta_{1,1}) \to \text{Mat}_{(p-1)\times(2p-1)}(\mathbb{K}),
\]
the image of the variety:
\[
p-1 X_p \cap D(\alpha_{1,1} + \beta_{1,1})
\]
is contained in the variety:
\[
p-1 X_{p-1} \subset \text{Mat}_{(p-1)\times(2p-1)}(\mathbb{K}).
\]
Now, let us decompose the variety \(p-1 X_{p-1}\) into three pieces:
\[
p-1 X_{p-1} = p-3 X_{p-1} \cup (p-2 X_{p-1} \setminus p-3 X_{p-1}) \cup (p-1 X_{p-1} \setminus p-2 X_{p-1}), \tag{188}
\]
where each matrix \((\alpha_1, \ldots, \alpha_{p-1}, \beta_1, \ldots, \beta_{p-1})\) in the first (resp. second, third) piece satisfies:
\[
\text{rank}_\mathbb{K} (\alpha_1, \ldots, \alpha_{p-1}) \leq p-3 \quad \text{(resp.} = p-2 \text{, } = p-1 \text{).} \tag{189}
\]
Pulling back (188) by the map \(\mathcal{L}_G\), we see that:
\[
p-1 X_p \cap D(\alpha_{1,1} + \beta_{1,1})
\]
is contained in:
\[
\mathcal{L}_G^{-1} (p-3 X_{p-1}) \cup \mathcal{L}_G^{-1} (p-2 X_{p-1} \setminus p-3 X_{p-1}) \cup \mathcal{L}_G^{-1} (p-1 X_{p-1} \setminus p-2 X_{p-1}). \tag{190}
\]
Firstly, for every point in the first piece:
\[
Y \in p-3 X_{p-1},
\]
thanks to the commutative diagram (187), we receive the fibre dimension:
\[
\dim \mathcal{L}^{-1}_G(Y) = \dim \ker \pi_p \cap D(\alpha_{1,1} + \beta_{1,1})
= \dim \text{Mat}_{p \times 2p}(\mathbb{K}) - \dim \text{Mat}_{(p-1) \times 2(p-1)}(\mathbb{K}).
\]
Now, applying Corollary 8.3 to the regular map \( \mathcal{L} \) restricted on:
\[
\mathcal{L}^{-1}_G(p-3X_{p-1}) \subset \text{Mat}_{p \times 2p}(\mathbb{K})
\]
we receive the codimension estimate:
\[
\text{codim } \mathcal{L}^{-1}_G(p-3X_{p-1}) \geq \text{codim } p-3X_{p-1}.
\]
(191)

Secondly, for every point in the second piece:
\[
Y \in p-2X_{p-1} \setminus p-3X_{p-1},
\]
to look at the fibre of \( \mathcal{L}^{-1}_G(Y) \), thanks to the commutative diagram (187), we can use:
\[
\mathcal{L}^{-1}_G = (R \circ (L_G + \varepsilon))^{-1} \circ \pi_p^{-1},
\]
(192)
and obtain:
\[
\mathcal{L}^{-1}_G(Y) \cap (p-1X_p \cap D(\alpha_{1,1} + \beta_{1,1}))
\equiv R \circ (L_G + \varepsilon) \mathcal{L}^{-1}_G(Y) \cap R \circ (L_G + \varepsilon)(p-1X_p \cap D(\alpha_{1,1} + \beta_{1,1}))
\equiv \pi_p^{-1}(Y) \cap R \circ (L_G + \varepsilon)(p-1X_p \cap D(\alpha_{1,1} + \beta_{1,1})) \quad \text{[use (192)].}
\]

Observe now that every matrix:
\[
(\alpha_1 \mid \cdots \mid \alpha_p \mid \beta_1 \mid \cdots \mid \beta_p) \in \clubsuit
\]
satisfies the rank estimate:
\[
\text{rank}_\mathbb{K} (\alpha_1 \mid \cdots \mid \alpha_p) \leq p - 1.
\]
Moreover, noting that the lower-right \((p-1) \times (p-1)\) submatrix \(J\) of \((\alpha_1 \mid \cdots \mid \alpha_p)\) — which is the left \((p-1) \times (p-1)\) submatrix of \(Y\) — has rank:
\[
\text{rank}_\mathbb{K} J = p - 2 \quad \text{[see (189)]},
\]
by applying Lemma 10.11, we get that:
\[
\clubsuit \subset \pi_p^{-1}(Y)
\]
has codimension greater or equal to:
\[
\text{codim } J S^{p-1}_{p,p-2} = p - 1 - (p - 2) = 1.
\]
In other words:
\[
\mathcal{L}^{-1}_G(Y) \cap (p-1X_p \cap D(\alpha_{1,1} + \beta_{1,1})) \subset \mathcal{L}^{-1}_G(Y)
\]
has codimension \(\geq 1\). Thus, applying Corollary 8.3 to the map \( \mathcal{L}_G \) restricted on:
\[
\mathcal{L}^{-1}_G(p-2X_{p-1} \setminus p-3X_{p-1}) \cap (p-1X_p \cap D(\alpha_{1,1} + \beta_{1,1})) \subset \text{Mat}_{p \times 2p}(\mathbb{K}),
\]
\[
= \mathbb{II}
\]
we receive the codimension estimate:
\[
\text{codim } \Pi \geq \text{codim } \left( \frac{p-2}{p-3} X_{p-1} \right) + 1 \\
\geq \frac{\text{codim } p-2 X_{p-1} + 1}{p-2 C_{p-1} + 1}.
\] (193)

Thirdly, for every point in the third piece:
\[
Y \in \frac{p-1}{p-2} X_{p-1} \cap \frac{p-2}{p-2} X_{p-1},
\]
thanks to the diagram (187):
\[
\mathcal{L}_G^{-1} = (L_G \oplus \mathcal{E})^{-1} \circ (\pi_\rho \oplus 0)^{-1},
\] (194)
we receive:
\[
\mathcal{L}_G^{-1}(Y) \cap (p-1) X_p \cap D(\alpha_{1,1} + \beta_{1,1})
\approx (L_G \oplus \mathcal{E}) \mathcal{L}_G^{-1}(Y) \cap (L_G \oplus \mathcal{E}) (p-1) X_p \cap D(\alpha_{1,1} + \beta_{1,1})
\approx (\pi_\rho \oplus 0)^{-1}(Y) \cap (L_G \oplus \mathcal{E}) (p-1) X_p \cap D(\alpha_{1,1} + \beta_{1,1}) \tag{use (194)}.
\]
Recalling Corollary 10.12, the sum of all columns of $Y$ — the bottom $(p-1)$ rows of $(\alpha_2 | \cdots | \alpha_p | \beta_2 | \cdots | \beta_p)$ — is zero. Thus, every element:
\[
(\alpha_1 | \cdots | \alpha_p | \beta_1 | \cdots | \beta_p) \oplus (s_2, \ldots, s_p)^T \in \mathbf{1}
\]
not only satisfies:
\[
\text{rank}_\Sigma (\alpha_1 | \cdots | \alpha_p) \leq p - 1,
\] (195)
but also satisfies:
\[
\alpha_2 + \cdots + \alpha_p + \beta_2 + \cdots + \beta_p = (\alpha_{1,2} + \cdots + \alpha_{1,p} + \beta_{1,2} + \cdots + \beta_{1,p}, 0, \ldots, 0)^T.
\]
only this first entry could be nonzero
\[(p-1) \text{ copies}\]
Remembering that:
\[
\alpha_1 + \beta_1 = (\alpha_{1,1} + \beta_{1,1}, 0, \ldots, 0)^T, \tag{p-1 copies}
\]
summing the above two identities immediately yields:
\[
\alpha_1 + \cdots + \alpha_p + \beta_1 + \cdots + \beta_p = (\alpha_{1,1} + \cdots + \alpha_{1,p} + \beta_{1,1} + \cdots + \beta_{1,p}, 0, \ldots, 0)^T. \tag{p-1 copies}
\] (196)
Now, note that (157) (‘matrices ranks’) in condition (ii) are preserved under the map $L_G$ (‘Gaussian eliminations’), in particular, for $\nu = 1$, the image satisfies:
\[
\text{rank}_\Sigma \{\alpha_1 + (\beta_1 + \cdots + \beta_p), \alpha_2, \ldots, \alpha_p\} \leq p - 1,
\]
which, by adding the column vectors $2 \cdots p$ to the first one, is equivalent to:
\[
\text{rank}_\Sigma \{\alpha_1 + \cdots + \alpha_p + \beta_1 + \cdots + \beta_p, \alpha_2, \ldots, \alpha_p\} \leq p - 1.
\]
Remember (196), and recalling Corollary 10.12:
\[
\text{the bottom } (p-1) \times (p-1) \text{ submatrix of } (\alpha_2 | \cdots | \alpha_p) \text{ is of full rank } (p-1),
\] (197)
we immediately receive:

\[
\alpha_{1,1} + \cdots + \alpha_{1,p} + \beta_{1,1} + \cdots + \beta_{1,p} = 0. \tag*{\text{codim = 1}}
\]

Therefore, by applying Lemma 10.11, the restrictions \((195)\) and \((197)\) contribute one extra codimension:

\[
\text{codim } S_{p,p-1}^{p-1} = 1.
\]

Thus, we see that ‘the fibre in fibre’:

\[
(\pi_p \oplus \emptyset)^{-1}(\mathcal{Y}) \cap \bullet \subset (\pi_p \oplus \emptyset)^{-1}(\mathcal{Y})
\]

has codimension greater or equal to:

\[
1 + 1 = 2.
\]

Now, applying once again Corollary 8.3 to the map \(\mathcal{L}_G\) restricted on:

\[
\mathcal{L}_G^{-1}(p-1 \mathbf{X}_{p-1} \setminus p-2 \mathbf{X}_{p-1}) \cap (p-1 \mathbf{X}_p \cap D(\alpha_{1,1} + \beta_{1,1})) \subseteq \text{Mat}_{p \times 2p}(\mathbb{K}),
\]

we receive the codimension estimate:

\[
\text{codim III} \geq \text{codim } (p-1 \mathbf{X}_{p-1} \setminus p-2 \mathbf{X}_{p-1}) + 2 \geq \text{codim } p-1 \mathbf{X}_{p-1} + 2. \tag{198}
\]

Summarizing \((190), (191), (193), (198)\), we receive the codimension estimate:

\[
\text{codim } p-1 \mathbf{X}_p \cap D(\alpha_{i,1} + \beta_{1,i}) \geq \min \{\text{codim } p-3 \mathbf{X}_{p-1}, \text{codim } p-2 \mathbf{X}_{p-1} + 1, \text{codim } p-1 \mathbf{X}_{p-1} + 2\}.
\]

By permuting the indices, we know that all:

\[
p-1 \mathbf{X}_p \cap D(\alpha_{i,1} + \beta_{i,1}) \subseteq \text{Mat}_{p \times 2p}(\mathbb{K}) \quad (i = 1 \ldots p)
\]

have the same codimension, and so does their union:

\[
p-1 \mathbf{X}_p \cap D(\alpha_1 + \beta_1) = \bigcup_{i=1}^{p} (p-1 \mathbf{X}_p \cap D(\alpha_{i,1} + \beta_{i,1})) \subseteq \text{Mat}_{p \times 2p}(\mathbb{K}).
\]

Finally, taking codimension on both sides of:

\[
p-1 \mathbf{X}_p = (p-1 \mathbf{X}_p \cap V(\alpha_1 + \beta_1)) \cup (p-1 \mathbf{X}_p \cap D(\alpha_1 + \beta_1)),
\]

Proposition 10.2 and the preceding estimate conclude the proof.

\[\square\]

**Proof of (170).** If \(\ell \geq 2\), decompose the variety \(\ell \mathbf{X}_{p-1}\) into three pieces:

\[
\ell \mathbf{X}_{p-1} = \ell-2 \mathbf{X}_{p-1} \cup (\ell-1 \mathbf{X}_{p-1} \setminus \ell-2 \mathbf{X}_{p-1}) \cup (\ell \mathbf{X}_{p-1} \setminus \ell-1 \mathbf{X}_{p-1});
\]

and if \(\ell = 1\), decompose the variety \(\ell \mathbf{X}_{p-1}\) into two pieces:

\[
\ell \mathbf{X}_{p-1} = \ell-1 \mathbf{X}_{p-1} \cup (\ell \mathbf{X}_{p-1} \setminus \ell-1 \mathbf{X}_{p-1}).
\]

Now, by mimicking the preceding proof, namely by applying Lemma 10.11 and Corollary 8.3, everything goes on smoothly with much less effort, because there is no need to perform delicate codimension estimates such as \((198)\).
10.6. **Proof of Core Lemma 9.5.** If $N = 1$, there is nothing to prove. Assume now $N \geq 2$. Comparing (146) and (155), it is natural to introduce the projection:

$$
\pi_{2c+r,N} : \text{Mat}_{(2c+r) \times 2(N+1)}(\mathbb{K}) \to \text{Mat}_{N \times 2N}(\mathbb{K})
$$

$$(\alpha_0, \ldots, \alpha_p, \beta_0, \ldots, \beta_p) \mapsto (\hat{\alpha}_1, \ldots, \hat{\alpha}_p, \hat{\beta}_1, \ldots, \hat{\beta}_p),$$

where each widehat vector is obtained by extracting the first $N$ rows (entries) out of the original $2c + r$ rows (entries).

Now, for every point:

$$(\alpha_0, \ldots, \alpha_p, \beta_0, \ldots, \beta_p) \in \mathcal{M}^N_{2c+r} \subset \text{Mat}_{(2c+r) \times 2(N+1)}(\mathbb{K}),$$

in restriction (148), by setting $\nu = 0$, we receive:

$$\text{rank}_{\mathbb{K}} \{\alpha_1, \ldots, \alpha_N, \alpha_0 + (\beta_0 + \beta_1 + \cdots + \beta_N)\} \leq N - 1.$$

Dropping the last column and extracting the first $N$ rows, we get:

$$\text{rank}_{\mathbb{K}} \{\hat{\alpha}_1, \ldots, \hat{\alpha}_N\} \leq N - 1.$$

Similarly, in restriction (149), by dropping the first column and extracting the first $N$ rows, for all $\tau = 0 \cdots N - 1$ and $\rho = \tau + 1 \cdots N$, we obtain:

$$\text{rank}_{\mathbb{K}} \{\hat{\alpha}_1 + \hat{\beta}_1, \ldots, \hat{\alpha}_\tau + \hat{\beta}_\tau, \hat{\alpha}_{\tau+1}, \ldots, \hat{\alpha}_N, \hat{\alpha}_\rho + (\hat{\beta}_\tau + \cdots + \hat{\beta}_N)\} \leq N - 1,$$

where we omit the column vector $\hat{\alpha}_p$ in the box. Summarizing the above two inequalities, $(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p)$ satisfies the restriction (156) – (158):

$$\pi_{2c+r,N} (\alpha_0, \ldots, \alpha_p, \beta_0, \ldots, \beta_p) \in \mathcal{N}_{N-1} X_N \subset \text{Mat}_{N \times 2N}(\mathbb{K}).$$

Therefore:

$$\pi_{2c+r,N} (\mathcal{M}^N_{2c+r}) \subset \mathcal{N}_{N-1} X_N.$$

Moreover, for every point $Y \in \mathcal{N}_{N-1} X_N$, the ‘fibre in fibre’:

$$\pi_{2c+r,N}^{-1}(Y) \cap \mathcal{M}^N_{2c+r} \subset \pi_{2c+r,N}^{-1}(Y),$$

thanks to (147), has codimension $\geq 2c + r$. Thus a direct application of Corollary 8.3 yields:

$$\text{codim} \mathcal{M}^N_{2c+r} \geq \text{codim} \mathcal{N}_{N-1} X_N + 2c + r$$

[use (180)]

$$\geq N + 1 + 2c + r.$$

Repeating the same reasoning, we obtain:

$$\text{codim} \mathcal{M}^{N-\eta}_{2c+r} \geq \text{codim} \mathcal{N}_{N-\eta-1} X_{N-\eta} + 2c + r$$

[use (180)]

$$\geq N - \eta + 1 + 2c + r.$$

Remembering $2c + r \geq N$, we conclude the proof. □
10.7. ‘Macaulay2’, ‘Maple’ et al. vs. the Core Lemma. Believe it or not, concerning the Core Lemma or the Core Codimension Formulas, ‘Macaulay2’ – a professional software system devoted to supporting research in algebraic geometry and commutative algebra – is not strong enough to compute the precise codimensions of the involved determinantal ideals, even in small dimensions $p \geq 4$. And unfortunately, so do other mathematical softwares, like ‘Maple’...

This might indicate some weaknesses of current computers. Since the Core Lemma or a variation of it should be a crucial step in the constructions of ample examples, the dream of finding explicit examples with rational coefficients, firstly in small dimensional cases, could be kind of a challenge for a moment.

11. A rough estimate of lower degree bound

11.1. Effective results. Recalling Subsection 5.3, we first provide an effective

**Theorem 11.1.** For all $N \geq 3$, for any $\epsilon_1, \ldots, \epsilon_{c+r} \in \{1, 2\}$, Theorem 5.2 holds for $\Box = 1$. and for all $d \geq N^{N^2/2} - 1$.

*Proof.* Setting $\delta_{c+r+1} = 2$ in (93), and demanding all (94) – (96) to be equalities, we thus receive the desired estimate:

$$[\text{see (97)}] \quad (N + 1) \mu_{N,N} \leq N^{N^2/2} - 1.$$  

For the sake of completeness, we present all computational details in Subsection 11.2 below. □

Hence, the product coup in Subsection 5.3 yields

**Theorem 11.2.** In Theorem 5.1, for $\Box = 1$, the lower bound $d_0(-1) = N^{N^2}$ works. □

11.2. Computational details. We specify (94) – (97) as follows. Recalling that $\delta_{c+r+1} = 2$ and $\Box = 1$, for every integer $l = c + r + 1 \cdots N$, we choose:

$$\mu_{l,0} = l \delta_l + 4l + 1,$$  

(199)

and inductively we choose:

$$\mu_{l,k} = \sum_{j=0}^{k-1} l \mu_{l,j} + (l-k) \delta_l + 4l + 1 \quad (k = 1 \cdots l).$$  

(200)

Actually, we take the above values in purpose, because they also work in the degree estimates in our coming paper.

For every integer $l = c + r + 1 \cdots N$, for every integer $k = 0 \cdots l$, let:

$$S_{l,k} := \sum_{j=0}^{k} \mu_{l,j}.$$  

(201)

For $k = 1 \cdots l$, we have:

$$[\text{see (200)}] \quad S_{l,k} - S_{l,k-1} = (l + 1) S_{l,k-1} + (l-k) \delta_l + 4l + 1.$$  

Moving the term ‘$-S_{l,k-1}$’ to the right hand side, we receive:

$$S_{l,k} = (l + 1) S_{l,k-1} + (l-k) \delta_l + 4l + 1.$$
Dividing by \((l + 1)^k\) on both sides, we receive:

\[
\frac{S_{l,k}}{(l + 1)^k} = \frac{S_{l,k-1}}{(l + 1)^{k-1}} + \left(l \delta_l + 4l + 1\right) \frac{1}{(l + 1)^k} - \delta_l \frac{k}{(l + 1)^k}.
\]

Noting that the two underlined terms have the same structure, doing induction backwards \(k \cdots 1\), we receive:

\[
\frac{S_{l,k}}{(l + 1)^k} = \frac{S_{l,0}}{(l + 1)^0} + \left(l \delta_l + 4l + 1\right) \sum_{j=1}^{k} \frac{1}{(l + 1)^j} - \delta_l \sum_{j=1}^{k} \frac{j}{(l + 1)^j}.
\]

Now, applying the following two elementary identities:

\[
\sum_{j=1}^{k} \frac{1}{(l + 1)^j} = \frac{1}{l} \left(1 - \frac{1}{(l + 1)^k}\right),
\]

\[
\sum_{j=1}^{k} \frac{j}{(l + 1)^j} = \frac{l + 1}{l^2} \left(1 + \frac{k}{(l + 1)^{k+1}} - \frac{1 + k}{(l + 1)^k}\right),
\]

and recalling (199):

\[
S_{l,0} = \mu_{l,0} = l \delta_l + 4l + 1,
\]

we obtain:

\[
\frac{S_{l,k}}{(l + 1)^k} = l \delta_l + 4l + 1 + \left(l \delta_l + 4l + 1\right) \frac{1}{l} \left(1 - \frac{1}{(l + 1)^k}\right) - \delta_l \frac{l + 1}{l^2} \left(1 + \frac{k}{(l + 1)^{k+1}} - \frac{1 + k}{(l + 1)^k}\right).
\]

Next, multiplying by \((l + 1)^k\) on both sides, we get:

\[
S_{l,k} = \left(l \delta_l + 4l + 1\right) \left((l + 1)^k + \frac{(l + 1)^k}{l} - \frac{1}{l}\right) - \delta_l \frac{l + 1}{l^2} \left((l + 1)^{k+1} + k - (1 + k)(l + 1)\right)
\]

\[
= \left(l \delta_l + 4l + 1\right) \left(\frac{(l + 1)^{k+1}}{l} - \frac{1}{l}\right) - \delta_l \frac{l + 1}{l^2} \left((l + 1)^{k+1} + k - (1 + k)(l + 1)\right).
\]

Recalling (96), we have:

\[
\delta_{l+1} = l \mu_{l,0}
\]

[use (200) for \(k = l\)]

\[
= l \left(\sum_{j=0}^{l-1} l \mu_{l,j} + 4l + 1\right)
\]

[use (201) for \(k = l - 1\)]

\[
= \frac{l^2}{l} S_{l,l-1} + l(4l + 1)
\]

[use (202) for \(k = l - 1\)]

\[
= \left(l \delta_l + 4l + 1\right) \left(l \delta_l + 4l + 1\right) - \delta_l \left(l \delta_l + 4l + 1\right) + l(4l + 1)
\]

\[
= \delta_l \left(l^2(l + 1)^l - l(l + 1)^l + 1\right) + (4l + 1)l(l + 1)^l.
\]

Throwing away the first positive part, we receive the estimate:

\[
\delta_{l+1} > (4l + 1)l(l + 1)^l.
\]
Therefore, for all $l \geq c + r + 2$, we have the estimate of (203):

$$
\delta_{l+1} = \hat{H} (l + 1)^j \delta_l - \left( (l + 1)^j - 1 \right) \delta_l + (4l + 1) l (l + 1)^j
$$

[use (204)]

$$
< \hat{H} (l + 1)^j \delta_l - \left( (l + 1)^j - 1 \right) \left( 4l - 1 + 1 \right) (l - 1) \hat{H}^{l-1} + (4l + 1) l (l + 1)^j
$$

$$
= \hat{H} (l + 1)^j \delta_l - 4 \left[ ((l + 1)^j - 1) (l - 1) \hat{H}^{l-1} - \hat{H} (l + 1)^j \right]
$$

$$
- \left[ ((l + 1)^j - 1) (l - 1) \hat{H}^{l-1} - l (l + 1)^j \right].
$$

Since $2c + r \geq N \geq 1$, $c, r$ cannot be both zero, hence $l \geq c + r + 2 \geq 3$ above, thus we may realize that the first underlined bracket is positive:

$$
\left[ ((l + 1)^j - 1) (l - 1) \hat{H}^{l-1} - \hat{H} (l + 1)^j \right] = \hat{H} (l + 1)^j \left[ (1 - \frac{1}{l + 1}) \hat{H}^{l-3} - 1 \right]
$$

$$
\geq \hat{H} (l + 1)^j \left[ (1 - \frac{1}{3 + 1}) (3 - 1)^3 - 1 \right]
$$

$$
> 0,
$$

and that the second underlined bracket is also positive:

$$
\left[ ((l + 1)^j - 1) (l - 1) \hat{H}^{l-1} - l (l + 1)^j \right] = l (l + 1)^j \left[ (1 - \frac{1}{l + 1}) \hat{H}^{l-2} - 1 \right]
$$

$$
\geq l (l + 1)^j \left[ (1 - \frac{1}{3 + 1}) (3 - 1)^3 - 1 \right]
$$

$$
> 0.
$$

Consequently, we have the neat estimate suitable for the induction:

$$
\delta_{l+1} \leq \hat{H} (l + 1)^j \delta_l \quad (l = c + r + 2 \ldots N - 1),
$$

(205)

which for convenience, we may assume to be satisfied for $l = N$ by just defining $\delta_{N+1} := N\mu_{N,N}$.

In fact, using these estimates iteratively, we may proceed as follows:

$$
(N + 1) \mu_{N,N} = \frac{N + 1}{N} \frac{N \mu_{N,N}}{\delta_{N+1}}
$$

$$
= \frac{N + 1}{N} \delta_{N+1}
$$

(206)

[use (205)]

$$
< \frac{N + 1}{N} \delta_{c + r + 2} \prod_{l = c + r + 2}^{N} \hat{H} (l + 1)^j.
$$

Noting that (203) yields:

$$
\delta_{c + r + 2} = \left[ \delta_1 \left( \hat{H} (l + 1)^j - (l + 1)^j + 1 \right) + (4l + 1) l (l + 1)^j \right]_{l = c + r + 1}
$$

[recall $\delta_{c + r + 1} = 2$]

$$
< 6 \hat{H} (l + 1)^j \bigg|_{l = c + r + 1},
$$

$$
\frac{\delta_{c + r + 2}}{\delta_{c + r + 1}} < \frac{6 \hat{H} (l + 1)^j}{\delta_{c + r + 1}},
$$

$$
\frac{N \mu_{N,N}}{\delta_{N+1}} < \frac{6 \hat{H} (l + 1)^j}{\delta_{c + r + 1}}.
$$
thus the above two estimates yield:

\[(N + 1) \mu_{N,N} < \frac{N + 1}{N} 6 \prod_{l=c+r+1}^{N} l^2 (l + 1)^l. \]  

(207)

For the convenience of later integration, we prefer the term \((l + 1)^l\) to \((l + 1)^{l+1}\), therefore we firstly transform:

\[
\prod_{l=c+r+1}^{N} l^2 (l + 1)^l = \prod_{l=c+r+1}^{N} \frac{l}{l+1} l(l + 1)^{l+1}
\]

\[
= \prod_{l=c+r+1}^{N} \frac{l}{l+1} \prod_{l=c+r+1}^{N} l \prod_{l=c+r+1}^{N} (l + 1)^{l+1}
\]

\[
= \frac{c + r + 1}{N + 1} \prod_{l=c+r+1}^{N} l \prod_{l=c+r+1}^{N} (l + 1)^{l+1},
\]

whence (207) becomes:

\[(N + 1) \mu_{N,N} < 6 \frac{c + r + 1}{N} \prod_{l=c+r+1}^{N} l \prod_{l=c+r+1}^{N} (l + 1)^{l+1},\]

(208)

[recall \(c + r \leq N - 1\)]

Now, we estimate the dominant term:

\[
\prod_{l=c+r+1}^{N} l \prod_{l=c+r+1}^{N} (l + 1)^{l+1}.
\]

as follows.

Remembering \(2c + r \geq N\), we receive:

\[c + r \geq (2c + r)/2 \geq N/2,\]

and hence for \(N \geq 2\) we obtain:

\[
\ln \prod_{l=c+r+1}^{N} l = \ln N + \sum_{l=c+r+1}^{N-1} \ln l
\]

\[
< \ln N + \int_{c+r+1}^{N} \ln x \, dx
\]

\[
\leq \ln N + \int_{N/2+1}^{N} \ln x \, dx
\]

\[
= \ln N + (x \ln x - x) \bigg|_{N/2+1}^{N}.
\]
Similarly, when \( N \geq 4 \), noting that \( N \geq N/2 + 2 \), we get the estimate:

\[
\ln \prod_{l=c+r+1}^{N} (l+1)^{l+1} = (N+1) \ln (N+1) + N \ln N + \sum_{l=c+r+2}^{N-1} l \ln l \\
\leq (N+1) \ln (N+1) + N \ln N + \int_{N/2+2}^{N} x \ln x \, dx \\
= (N+1) \ln (N+1) + N \ln N + \left( \frac{1}{2} x^2 \ln x - \frac{x^2}{4} \right) \bigg|_{N/2+2}^{N}.
\]

Summing the above two estimates, for \( N \geq 4 \) we receive:

\[
\ln \prod_{l=c+r+1}^{N} l + \ln \prod_{l=c+r+1}^{N} (l+1)^{l+1} \\
\leq \ln N + \left( x \ln x - x \right) \bigg|_{N/2+1}^{N} + (N+1) \ln (N+1) + N \ln N + \left( \frac{1}{2} x^2 \ln x - \frac{x^2}{4} \right) \bigg|_{N/2+2}^{N} \\
= \frac{1}{2} N^2 \ln N - \frac{1}{2} (N/2 + 2)^2 \ln (N/2 + 2) - \left( \frac{3}{16} N^2 - 2 \right) - (N/2 + 1) \ln (N/2 + 1) + (N+1) \ln (N+1) + (2N + 1) \ln N \\
= \frac{3}{8} N^2 \ln N - O(N^2), \quad \text{as } N \to \infty.
\]

In order to have a neat lower bound, we would like to have:

\[
(N+1) \mu_{N,N} \leq N^{N^2/2} - 1.
\]

In fact, using the estimates (208), (209), when \( N \geq 48 \), we can show by hand that (211) holds true. For \( N = 14 \cdots 47 \), we can use a mathematical software ‘Maple’ to check the above estimate. Finally, for \( N = 3 \cdots 13 \), we ask ‘Maple’ to compute \( \delta_{N+1} \) explicitly, and thereby, thanks to (206), we again prove the estimate (211).

12. Some Improvements of MCM

12.1. General core codimension formulas. In order to lower the degree bound \( d_0 \) of MCM, we will modify the hypersurface constructions. Of course, we would like to reduce the number of moving coefficient terms, and this will be based on the General Core Lemma 12.6 below.

For every integers \( p \geq q \geq 2 \), for every integer \( 0 \leq \ell \leq q \), we first estimate the codimension \( t_{C_{p,q}} \) of the algebraic variety:

\[
\{X_{p,q} \subset \mathbf{Mat}_{p \times 2q}(\mathbb{K})
\]

which consists of \( p \times 2q \) matrices \( X_{p,q} = (\alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_q) \) such that:

(i) the first \( q \) column vectors have rank:

\[
\text{rank}_{\mathbb{K}} \{\alpha_1, \ldots, \alpha_q\} \leq \ell;
\]

(ii) for every index \( v = 1 \cdots q \), replacing \( \alpha_v \) with \( \alpha_v + (\beta_1 + \cdots + \beta_q) \) in the collection of column vectors \( \{\alpha_1, \ldots, \alpha_q\} \), there holds the rank inequality:

\[
\text{rank}_{\mathbb{K}} \{\alpha_1, \ldots, \alpha_q, \alpha_v + (\beta_1 + \cdots + \beta_q)\} \leq p - 1;
\]

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(iii) for every integer $\tau = 1 \cdots q - 1$, for every index $\rho = \tau + 1 \cdots q$, replacing $\alpha_\rho$ with $\alpha_\rho + (\beta_{\tau + 1} + \cdots + \beta_q)$ in the collection of column vectors $\{\alpha_1 + \beta_1, \ldots, \alpha_\tau + \beta_\tau, \alpha_{\tau + 1}, \ldots, \alpha_q, \alpha_\rho, \ldots, \alpha_q, \alpha_\rho + (\beta_{\tau + 1} + \cdots + \beta_q)\}$, there holds the rank inequality:

$$\text{rank}_{\Sigma} \{\alpha_1 + \beta_1, \ldots, \alpha_\tau + \beta_\tau, \alpha_{\tau + 1}, \ldots, \alpha_q, \alpha_\rho, \ldots, \alpha_q, \alpha_\rho + (\beta_{\tau + 1} + \cdots + \beta_q)\} \leq q - 1.$$ 

Repeating the same reasoning as in Section 10, we may proceed as follows. Firstly, here is a very analogue of Proposition 10.1:

**Proposition 12.1.** For every integers $p \geq q \geq 2$, the codimension value $qC_{p,q}$ for $\ell = 0$ is:

$$qC_{p,q} = pq + p - q + 1.$$ 

Next, we obtain an analogue of Proposition 10.2:

**Proposition 12.2.** For every integers $p \geq q \geq 2$, the codimensions $iC_{p,q}^0$ of the algebraic varieties:

$$\{\alpha_1 + \beta_1 = 0\} \cap iX_{p,q} \subset \text{Mat}_{p\times2q}(K)$$

read according to the values of $\ell$ as:

$$iC_{p,q}^0 = \begin{cases} p + \min\{2(p - q + 1), p\} & (\ell = q), \\ p + 2(p - q + 1) & (\ell = q - 1), \\ p + (p - \ell)(q - \ell) & (\ell = 0 \cdots q - 2). \end{cases}$$

The last two lines are easy to obtain, while the first line is a consequence of Lemma 10.3.

Now, we deduce the analogue of Proposition 10.4:

**Proposition 12.3 (General Codimension Induction Formulas).** (i) For every positive integers $p \geq q \geq 2$, for $\ell = q$, the codimension value $qC_{p,q}$ satisfies:

$$qC_{p,q} = \min\{p, \ q_{-1}C_{p,q}\}.$$ 

(ii) For every positive integers $p \geq q \geq 3$, for $\ell = q - 1$, the codimension value $q_{-1}C_{p,q}$ satisfies:

$$q_{-1}C_{p,q} \geq \min\{q_{-1}C_{p,q}^0, \ q_{-1}C_{p-1,q-1} + p - q + 2, \ q_{-2}C_{p-1,q-1} + 1, \ q_{-3}C_{p-1,q-1}\}.$$ 

(iii) For all positive integers $p \geq q \geq 3$, for all integers $\ell = 1 \cdots q - 2$, the codimension values $iC_{p,q}$ satisfy:

$$iC_{p,q} \geq \min\{iC_{p,q}^0, \ iC_{p-1,q-1} + (p - \ell) + (q - \ell) - 1, \ i_{-1}C_{p-1,q-1} + (q - \ell), \ i_{-2}C_{p-1,q-1}\}.$$ 

Similar to Proposition 10.5, we have:

**Proposition 12.4.** For the initial cases $p \geq q = 2$, there hold the codimension values:

$$0C_{p,2} = 3p - 1, \quad 1C_{p,2} = 2p - 1, \quad 2C_{p,2} = p.$$ 

Finally, by the same induction proof as in Proposition 10.6, we get:

**Proposition 12.5 (General Core Codimension Formulas).** For all integers $p \geq q \geq 2$, there hold the codimension estimates:

$$iC_{p,q} \geq (p - \ell)(q - \ell) + p - q + l + 1 \quad (\ell = 0 \cdots q - 1),$$

and the core codimension identity:

$$qC_{p,q} = p.$$ 

Actually, we could prove that the above estimates are identities, yet it is not really necessary.
12.2. **General Core Lemma.** Similar to (146), for every integer \( k = 1 \cdots N - 1 \), we introduce the algebraic subvariety:

\[
\mathcal{M}_{2c+r}^{N,k} \subset \text{Mat}_{(2c+r) \times (N+1+k+1)}(\mathbb{K})
\]

consisting of all \((c + r + c) \times (N + 1 + k + 1)\) matrices \((\alpha_0 | \alpha_1 | \cdots | \alpha_N | \beta_0 | \beta_1 | \cdots | \beta_k)\) such that:

(i) the sum of these \((N + 1 + k + 1)\) columns is zero:

\[
\alpha_0 + \alpha_1 + \cdots + \alpha_N + \beta_0 + \beta_1 + \cdots + \beta_k = 0;
\]

(212)

(ii) for every index \( v = 0 \cdots k \), replacing \( \alpha_v \) with \( \alpha_v + (\beta_0 + \beta_1 + \cdots + \beta_k) \) in the collection of column vectors \( \{\alpha_0, \alpha_1, \ldots, \alpha_N\} \), there holds the rank inequality:

\[
\text{rank}_\mathbb{K} \{\alpha_0, \ldots, \alpha_v, \ldots, \alpha_N, \alpha_v + (\beta_0 + \beta_1 + \cdots + \beta_k)\} \leq N - 1;
\]

(iii) for every integer \( \tau = 0 \cdots k - 1 \), for every index \( \rho = \tau + 1 \cdots k \), replacing \( \alpha_\rho \) with \( \alpha_\rho + (\beta_{\tau+1} + \cdots + \beta_k) \) in the collection of column vectors \( \{\alpha_0 + \beta_0, \alpha_1 + \beta_1, \ldots, \alpha_\tau + \beta_\tau, 0 \} \), there holds the rank inequality:

\[
\text{rank}_\mathbb{K} \{\alpha_0 + \beta_0, \alpha_1 + \beta_1, \ldots, \alpha_\tau + \beta_\tau, \alpha_{\tau+1}, \ldots, \hat{\alpha}_\rho, \ldots, \alpha_N, \alpha_\rho + (\beta_{\tau+1} + \cdots + \beta_k)\} \leq N - 1.
\]

**Lemma 12.6 (Sharp Core Lemma of MCM).** For every positive integers \( N \geq 3 \), for every integers \( c, r \geq 0 \) with \( 2c + r \geq N \), for every integer \( k = 1 \cdots N - 1 \), there holds the codimension estimate:

\[
\text{codim} \mathcal{M}_{2c+r}^{N,k} \geq k+1 \text{C}_{c+r-N+k+1} + (2c + r) \geq 2 (2c + r) - N + k + 1.
\]

The term \((2c + r)\) comes from (212). When \( k = N - 1 \), there is nothing to prove. When \( k < N - 1 \), noting that all matrices in (ii) and (iii) have the same last column \( \alpha_N \), we may do Gaussian eliminations with respect to this column, and then by much the same argument as before, we receive the estimate. □

Actually, these two estimates are identities.

12.3. **Minimum necessary number of moving coefficient terms.** Firstly, letting:

\[
\text{codim} \mathcal{M}_{2c+r}^{N,k} \geq \text{dim}^0 \mathbb{P}(\mathcal{T}_{\mathbb{P}^N}) = 2N - 1,
\]

we receive the lower bound:

\[
k \geq 3N - 2 (2c + r) - 2,
\]

which indicates that at the step \( N \), the least number of moving coefficient terms, if necessary, should be:

\[
3N - 2 (2c + r) - 2 + 1.
\]

When \( 3N - 2 (2c + r) - 2 \leq 0 \), no moving coefficient terms are needed, thanks to the following:

**Lemma 12.7. (Elementary Core Lemma) Let:**

\[
\mathcal{M}_{2c+r}^{N,-1} \subset \text{Mat}_{(2c+r) \times (N+1)}(\mathbb{K})
\]

consist of all \((c + r + c) \times (N + 1)\) matrices \((\alpha_0 | \alpha_1 | \cdots | \alpha_N)\) such that:

(i) the sum of these \((N + 1)\) columns is zero:

\[
\alpha_0 + \alpha_1 + \cdots + \alpha_N = 0;
\]

(213)
(ii) there holds the rank inequality:

$$\text{rank}_{\mathbb{C}} \{ \alpha_0, \ldots, \alpha_r, \ldots, \alpha_N \} \leq N - 1 \quad (v = 0 \cdots N).$$

Then one has the codimension identity:

$$\text{codim} \mathcal{M}^{N, -1}_{2c+r} = 2(2c + r) - N + 1.$$

Next, for $\eta = 1 \cdots n - 1$, in the step $N - \eta$, letting:

$$\text{codim} \mathcal{M}^{N-\eta, k}_{2c+r} \geq \dim_{v_1, \ldots, v_p} \left( T_{\mathbb{P}^n} \right) = 2N - \eta - 1,$$

we receive:

$$2(2c + r) - (N - \eta) + k + 1 \geq 2N - \eta - 1,$$

that is:

$$k \geq 3N - 2(2c + r) - 2 - 2\eta,$$

which indicates that, at the step $N - \eta$, the least number of moving coefficient terms, if necessary, should be:

$$3N - 2(2c + r) - 2 - 2\eta + 1.$$ When $3N - 2(2c + r) - 2 - 2\eta \leq 0$, no moving coefficient terms are needed, thanks to the Elementary Core Lemma.

12.4. **Improved Algorithm of MCM.** When $3N - 2(2c + r) - 2 > 0$, in order to lower the degrees, we improve the hypersurface equations (92) as follows.

Firstly, when $3N - 2(2c + r) - 2 = 2p$ is even, the following hypersurface equations are suitable for MCM:

$$F_i = \sum_{j=0}^{N} A_i^j z_j^d + \sum_{\eta=0}^{p-1} \sum_{0 \leq j_0 \leq \cdots \leq j_{\eta} < N} \sum_{k=0}^{2p-2\eta} M_{j_0, \ldots, j_{\eta}; j_k}^{\mu_{N-q} z_{j_0} \cdots z_{j_{\eta}} \cdots z_{j_k}^{d-(2p-2\eta)\mu_{N-q}-2(N+\eta-2p)}, \cdots z_{j_{2p-2q+1}}^{2}, \cdots z_{j_{N-q}}^{2}}. \quad (214)$$

Secondly, when $3N - 2(2c + r) - 2 = 2p + 1$ is odd, the following hypersurface equations are suitable for MCM:

$$F_i = \sum_{j=0}^{N} A_i^j z_j^d + \sum_{\eta=0}^{p} \sum_{0 \leq j_0 \leq \cdots \leq j_{\eta} < N} \sum_{k=0}^{2p+1-2\eta} M_{j_0, \ldots, j_{\eta}; j_k}^{\mu_{N-q} z_{j_0} \cdots z_{j_{\eta}} \cdots z_{j_k}^{d-(2p+1-2\eta)\mu_{N-q}-2(N+\eta-2p-1)}, \cdots z_{j_{2p-2q+2}}^{2}, \cdots z_{j_{N-q}}^{2}}. \quad (215)$$

Of course, all integers $\mu_{\bullet, \bullet}$ and the degree $d$ are to be determined by some improved Algorithm, so that all the obtained symmetric forms are negatively twisted. And then we may estimate the lower bound $d_0$ accordingly. We leave this standard process to the interested reader.
12.5. **Why is the lower degree bound $d_0$ so large in MCM.** Because we could not enter the intrinsic difficulties, firstly of solving some huge linear systems to obtain sufficiently many (negatively twisted, large degree) symmetric differential forms (see [7, Theorem 2.7]), and secondly of proving that the obtained symmetric forms have discrete base locus. What we have done is only focusing on the extrinsic negatively twisted symmetric forms with degrees $\leq n$, obtained by some minors of the hypersurface equations/differentials matrix.

Our tool is coarse, based on some robust extrinsic geometric/algebraic structures, yet our goal is delicate, to certify the conjectured intrinsic ampleness. So a large lower degree bound $d_0 \gg 1$ is a price we need to pay.

13. **Uniform Very-Ampleness of $\text{Sym}^r \Omega_X$**

13.1. **A reminder.** In [34], Fujita proposed the famous:

**Conjecture 13.1. (Fujita)** Let $M$ be an $n$-dimensional complex manifold with canonical line bundle $\mathcal{K}$. If $\mathcal{L}$ is any positive holomorphic line bundle on $M$, then:

(i) for every integer $m \geq n + 1$, the line bundle $\mathcal{L}^{\otimes m} \otimes \mathcal{K}$ should be globally generated;

(ii) for every integer $m \geq n + 2$, the line bundle $\mathcal{L}^{\otimes m} \otimes \mathcal{K}$ should be very ample.

Recall that, given a complex manifold $X$ having ample cotangent bundle $\Omega_X$, the projectivized tangent bundle $\mathbb{P}(T_X)$ is equipped with the ample Serre line bundle $\mathcal{O}_{\mathbb{P}(T_X)}(1)$. Denoting $n := \dim X$, one has:

$$\dim \mathbb{P}(T_X) = 2n - 1.$$ 

Anticipating, we will show in Corollary 13.3 below that the canonical bundle of $\mathbb{P}(T_X)$ is:

$$\mathcal{K}_{\mathbb{P}(T_X)} \cong \mathcal{O}_{\mathbb{P}(T_X)}(-n) \otimes \pi^* \mathcal{K}_X \otimes \mathcal{H}_X^{\otimes 2},$$

where $\pi : \mathbb{P}(T_X) \to X$ is the canonical projection. Thus, for the complex manifold $\mathbb{P}(T_X)$ and the ample Serre line bundle $\mathcal{O}_{\mathbb{P}(T_X)}(1)$, the Fujita Conjecture implies:

(i) for every integer $m \geq 2n$, the line bundle $\mathcal{O}_{\mathbb{P}(T_X)}(m - n) \otimes \pi^* \mathcal{H}_X \otimes \mathcal{H}_X^{\otimes 2}$ is globally generated;

(ii) for every integer $m \geq 2n + 1$, the line bundle $\mathcal{O}_{\mathbb{P}(T_X)}(m - n) \otimes \pi^* \mathcal{H}_X \otimes \mathcal{H}_X^{\otimes 2}$ is very ample.

In other words, we receive the following by-products of the Fujita Conjecture.

**A Consequence of the Fujita Conjecture.** For any $n$-dimensional complex manifold $X$ having ample cotangent bundle $\Omega_X$, there holds:

(i) for every integer $m \geq n$, the twisted $m$-symmetric cotangent bundle $\text{Sym}^m \Omega_X \otimes \mathcal{H}_X^{\otimes 2}$ is globally generated;

(ii) for every integer $m \geq n + 1$, the twisted $m$-symmetric cotangent bundle $\text{Sym}^m \Omega_X \otimes \mathcal{H}_X^{\otimes 2}$ is very ample.

13.2. **The canonical bundle of a projectivized vector bundle.** In this subsection, we recall some classical results in algebraic geometry.

Let $X$ be an $n$-dimensional complex manifold, and let $E$ be a holomorphic vector bundle on $X$ having rank $e$. Let $\mathbb{P}(E)$ be the projectivization of $E$. Now, we compute its canonical bundle $\mathcal{K}_{\mathbb{P}(E)}$ as follows.

Let $\pi$ be the canonical projection:

$$\pi : \mathbb{P}(E) \to X.$$
First, recall the exact sequence which defines the relative tangent bundle $T_\pi$:

$$0 \rightarrow T_\pi \rightarrow T_{\mathbb{P}(E)} \rightarrow \pi^* T_X \rightarrow 0,$$

(216)

and recall also the well known Euler exact sequence:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}(E)} \rightarrow \mathcal{O}_{\mathbb{P}(E)(1)} \otimes \pi^* E \rightarrow T_\pi \rightarrow 0.$$  

(217)

Next, taking wedge products, the exact sequence (216) yields:

$$\wedge^{n+e-1} T_{\mathbb{P}(E)} \cong \wedge^{e-1} T_\pi \otimes \pi^* \wedge^n T_X,$$

(218)

and the Euler exact sequence (217) yields:

$$\mathcal{O}_{\mathbb{P}(E)(e)} \otimes \pi^* \wedge^e E \cong \mathcal{O}_{\mathbb{P}(E)} \otimes \wedge^{e-1} T_\pi \cong \wedge^{e-1} T_\pi.$$

(219)

Thus, we may compute the canonical line bundle as:

$$K_{\mathbb{P}(E)} = \mathcal{H}_{\mathbb{P}(E)} = \wedge^{n+e-1} \mathcal{O}_{\mathbb{P}(E)} \cong (\wedge^{e-1} T_\pi)^\vee \otimes (\pi^* \wedge^n T_X)^\vee$$

[use the dual of (218)]

$$\cong (\mathcal{O}_{\mathbb{P}(E)(e)})^\vee \otimes (\pi^* \wedge^e E)^\vee \otimes (\pi^* \wedge^n T_X)^\vee$$

[use the dual of (219)]

$$\cong \mathcal{O}_{\mathbb{P}(E)}(-e) \otimes \pi^* \wedge^e E^\vee \otimes \pi^* \wedge^n \Omega_X^1$$

$$\cong \mathcal{O}_{\mathbb{P}(E)}(-e) \otimes \pi^* \wedge^e E^\vee \otimes \pi^* \mathcal{K}_X,$$

where $\mathcal{K}_X$ is the canonical line bundle of $X$.

**Proposition 13.2.** The canonical line bundle $K_{\mathbb{P}(E)}$ of $\mathbb{P}(E)$ satisfies the formula:

$$K_{\mathbb{P}(E)} \cong \mathcal{O}_{\mathbb{P}(E)}(-e) \otimes \pi^* \wedge^e E^\vee \otimes \pi^* \mathcal{K}_X.$$

□

In applications, first, we are interested in the case where $E$ is the tangent bundle $T_X$ of $X$.

**Corollary 13.3.** One has the formula:

$$\mathcal{K}_{\mathbb{P}(T_X)} \equiv \mathcal{O}_{\mathbb{P}(T_X)}(-n) \otimes \pi^* \mathcal{K}_X \otimes \mathcal{K}_X.$$

□

More generally, we are interested in the case where $X \subset V$ for some complex manifold $V$ of dimension $n + r$, and $E = T_V|_X$.

**Corollary 13.4.** One has:

$$\mathcal{K}_{\mathbb{P}(T_V|_X)} \equiv \mathcal{O}_{\mathbb{P}(T_V|_X)}(-n-r) \otimes \pi^* \mathcal{K}_V|_X \otimes \pi^* \mathcal{K}_X.$$  

□

In our applications, $X, V$ are some smooth complete intersections in $\mathbb{P}^N_C$, so their canonical line bundles $\mathcal{K}_X, \mathcal{K}_V$ have neat expressions by the following classical theorem, whose proof is based on the Adjunction Formula.

**Theorem 13.5.** For a smooth complete intersection:

$$Y := D_1 \cap \cdots \cap D_k \subset \mathbb{P}^N_C$$

with divisor degrees:

$$\deg D_i = d_i \quad (i = 1 \cdots k),$$

the canonical line bundle $\mathcal{K}_X$ of $X$ is:

$$K_X \cong \mathcal{O}_X \left( -N - 1 + \sum_{i=1}^k d_i \right).$$

□
13.3. **Proof of the Very-Ampleness Theorem 1.4.** Assume for the moment that the ambient field \( \mathbb{K} = \mathbb{C} \). Recall that in our Ampleness Theorem 1.3, \( V = H_1 \cap \cdots \cap H_c \) and \( X = H_1 \cap \cdots \cap H_{c+r} \) with \( \dim_{\mathbb{C}} X = n = N - (c + r) \). Then the above Corollary 13.4 and Theorem 13.5 imply:

\[
\mathcal{K}_{\mathbb{P}(T\mathbb{V}|X)} \cong \mathcal{O}_{\mathbb{P}(T\mathbb{V}|X)}(-n - r) \otimes \pi_{\mathbb{P}(T\mathbb{V}|X)}^* \mathcal{O}_{\mathbb{P}(T\mathbb{V}|X)}(-2(N + 1) + \sum_{i=1}^c d_i + \sum_{i=1}^{c+r} d_i).
\]

Also, recalling Theorem 5.1 and Proposition 4.4, for generic choices of \( H_1, \ldots, H_{c+r} \), for any positive integers \( a > b \geq 1 \), the negatively twisted line bundle below is ample:

\[
\mathcal{O}_{\mathbb{P}(T\mathbb{V}|X)}(a) \otimes \pi_{\mathbb{P}(T\mathbb{V}|X)}^* \mathcal{O}_{\mathbb{P}(T\mathbb{V}|X)}(-b).
\]

Recall the Fujita Conjecture that, by subsequent works of Demailly, Siu et al. (cf. the survey [21]), it is known that \( \mathcal{L}^{\otimes m} \otimes \mathcal{K}^{\otimes 2} \) is very ample for all large \( m \geq 2 + \binom{3n+1}{n} \). Consequently, the line bundle:

\[
\mathcal{O}_{\mathbb{P}(T\mathbb{V}|X)}(ma - 2n - 2r) \otimes \pi_{\mathbb{P}(T\mathbb{V}|X)}^* \mathcal{O}_{\mathbb{P}(T\mathbb{V}|X)}(-mb - 4(N + 1) + 2 \sum_{i=1}^c d_i + 2 \sum_{i=1}^{c+r} d_i)
\] (220)

is very ample.

Also note that, for similar reason as the ampleness of (24), for all large integers \( \ell \geq \ell_0(N) \):

\[
\mathcal{O}_{\mathbb{P}(T\mathbb{V}|X)}(1) \otimes \pi_0^* \mathcal{O}_{\mathbb{P}(T\mathbb{V}|X)}(\ell)
\]

is very ample. Consequently, so is:

\[
\mathcal{O}_{\mathbb{P}(T\mathbb{V}|X)}(1) \otimes \pi_{\mathbb{P}(T\mathbb{V}|X)}^* \mathcal{O}_{\mathbb{P}(T\mathbb{V}|X)}(\ell).
\] (221)

Now, recall the following two facts:

(A) if \( \mathcal{O}_{\mathbb{P}(T\mathbb{V}|X)}(\kappa) \otimes \pi_{\mathbb{P}(T\mathbb{V}|X)}^* (\star) \) is very ample, then for every \( \star' \geq \star \), \( \mathcal{O}_{\mathbb{P}(T\mathbb{V}|X)}(\kappa) \otimes \pi_{\mathbb{P}(T\mathbb{V}|X)}^* (\star') \) is also very ample;

(B) the tensor product of any two very ample line bundles remains very ample.

Therefore, thanks to the very-ampleness of (220), (221), we can already obtain the very-ampleness of \( \mathcal{O}_{\mathbb{P}(T\mathbb{V}|X)}(\kappa) \) for all large integers \( \kappa \geq \kappa_0 \), for some non-effective \( \kappa_0 \). In other words, the restricted cotangent bundle \( \text{Sym}^r \Omega^1_{\mathbb{V}|X} \) is very ample on \( X \) for every \( \kappa \geq \kappa_0 \). But to reach an explicit \( \kappa_0 \), one may ask the

**Questions.** (i) Find one explicit \( \ell_0(N) \).

(ii) Find one explicit \( \kappa_0 \).

**Answer of (i).** The value \( \ell_0(N) = 3 \) works. One can check by hand that the following global sections:

\[
z_k \frac{z_j^\ell - 1}{z_j} \frac{d\left(\frac{z_i}{z_j}\right)}{z_j} (i, j, k = 0 \cdots N, i \neq j)
\] (222)

guarantee the very-ampleness of \( \mathcal{O}_{\mathbb{P}(T\mathbb{V}|X)}(1) \otimes \pi_0^* \mathcal{O}_{\mathbb{P}(T\mathbb{V}|X)}(\ell) \).

**Answer of (ii).** The second fact (B) above leads us to consider the semigroup \( \mathcal{G} \) of the usual Abelian group \( \mathbb{Z} \oplus \mathbb{Z} \) generated by elements \( (\ell_1, \ell_2) \) such that \( \mathcal{O}_{\mathbb{P}(T\mathbb{V}|X)}(\ell_1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}(T\mathbb{V}|X)}(\ell_2) \) is very
ample. Then, the following elements are contained in $\mathcal{G}$, for all $m \geq 2 + \binom{3n+1}{n}$:

\[
\text{[see (220), } \forall b \geq 1, a \geq b + 1] \quad (m a - 2n - 2r, -m b - 4 (N + 1) + 2 \sum_{i=1}^{c} d_i + 2 \sum_{i=1}^{c+r} d_i),
\]

\[
\text{[see (221), } \ell \geq \ell_0(N) = 3] \quad (1, \ell).
\]

Also, the first fact (A) above says that if $(\ell_1, \ell_2) \in \mathcal{G}$, then $(\ell_1, \ell_3) \in \mathcal{G}$ for all $\ell_3 \geq \ell_2$. Thus, Question (ii) becomes to find one explicit $\kappa_0$ such that $(\kappa, 0) \in \mathcal{G}$ for all $\kappa \geq \kappa_0$.

Paying no attention to optimality, taking:

\[
b = 1, \quad a = 2, \quad m = -4 (N + 1) + 2 \sum_{i=1}^{c} d_i + 2 \sum_{i=1}^{c+r} d_i + 3, \]

we receive that $(m a - 2n - 2r, -3) \in \mathcal{G}$. Adding $(1, 3) \in \mathcal{G}$, we receive $(m a - 2n - 2r + 1, 0) \in \mathcal{G}$. Now, also using $(m a - 2n - 2r, 0) \in \mathcal{G}$, recalling Observation 5.4, we may take:

\[
\kappa_0 = (m a - 2n - 2r - 1) (m a - 2n - 2r) \leq a^2 m^2,
\]

or the larger neater lower bound:

\[
\kappa_0 = 16 \left( \sum_{i=1}^{c} d_i + \sum_{i=1}^{c+r} d_i \right)^2.
\]

Thus, we have proved the Very-Ampleness Theorem 1.4 for $\mathbb{K} = \mathbb{C}$. Remembering that very-ampleness (or not) is preserved under any base change obtained by ambient field extension, and noting the field extensions $\mathbb{Q} \hookrightarrow \mathbb{C}$ and $\mathbb{Q} \hookrightarrow \mathbb{K}$ for any field $\mathbb{K}$ with characteristic zero, by some standard arguments in algebraic geometry, we conclude the proof of the Very-Ampleness Theorem 1.4.

When $\mathbb{K}$ has positive characteristic, by the same arguments, we could also receive the same very-ampleness theorem provided the similar results about the Fujita Conjecture hold over the field $\mathbb{K}$.

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Laboratoire de Mathématiques d’Orsay, Université Paris-Sud (France)
E-mail address: songyan.xie@math.u-psud.fr