A NOTE ON THE CONTINUITY OF MINORS IN GRAND LEBESGUE SPACES

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Abstract. We present a simple proof of the continuity, in the sense distributions, of the minors of the differential matrices of mappings belonging to grand Sobolev spaces. Such function spaces were introduced in connection with a problem on minimal integrability of the Jacobian and are useful in certain aspects of geometric function theory and partial differential equations.

1. Introduction

The academic literature on enlarged function spaces has grown considerably in recent times. Authors are typically concerned with the general theory of function spaces and applications in PDEs. An attractive feature of such function spaces is that they require a minimum of a priori assumptions, while member functions retain specific attractive properties such as continuity or regularity. Particular examples of these spaces are the so-called grand Lebesgue and grand Sobolev spaces. These spaces first appear in a paper by T. Iwaniec and C. Sbordone [24] in which they investigate minimal conditions for the integrability of the Jacobian of an orientation-preserving Sobolev mapping. Fundamental properties of these spaces have since been established such as duality and reflexivity [11, 14], as well as the boundedness of various integral operators [26, 29, 30]. For further discussion of grand spaces, the interested reader is referred to [2, 6, 9, 13, 15, 16, 17, 21, 27, 35, 36].

It is well known that if a sequence of mappings $f_m$ converges weakly in the Sobolev space $W^{1,n}_\text{loc}$ to a mapping $f_0$, then all $k \times k$-minors, $k = 1, \ldots, n$, of matrices $Df_m$ tend to the corresponding $k \times k$-minors of the matrix $Df_0$, in the sense of distributions (in $D'$), see [33, Ch. 9] for the particular case $n = 2$, [34, §4.5] and [8, Theorem 8.20] for $n \geq 2$. The weak continuity of such minors plays a key role in the calculus of variations respecting the lower semicontinuity problem, see [1, 8] and references therein for more information. The related question of the integrability of the Jacobian (which is a particular case of a minor) under minimal assumptions, is partially motivated by applications such as nonlinear elasticity theory [7]. Significant results were obtained for mappings with nonnegative Jacobians, which are sometimes called ‘orientation-preserving’ mappings. Specifically, S. Müller proved that if $|Df| \in L^n$ and $J_f(x) \geq 0$, then the Jacobian possesses the higher integrability $J_f(x) \in L \log L$ [32]. Further generalizations can be found in [20, 31] and associated

1991 Mathematics Subject Classification. Primary 46E30; Secondary 46E35.

Key words and phrases. Grand Sobolev space, weak continuity.

This work was supported by a Grant of the Russian Foundation of the Russian Science Foundation (Agreement No. 16-41-02004).

1
references. Following integrability, continuity theorems for Jacobians in corresponding spaces are the next natural step towards more general approximation results. In this way T. Iwaniec and A. Verde obtained, in [25], the strong continuity of Jacobians in $L_{\log L}$, while L. D’Onofrio and R. Schiattarella in [10] proved a continuity theorem for orientation preserving mappings $f_k$ belonging to the grand Sobolev space $W^{1,n})$. Provided that we have the additional requirement of uniformly vanishing $n$-modulus, i.e.

$$\lim_{\varepsilon \to 0^+} \varepsilon \sup_{k \geq 1} \int_{\Omega} |Df_k(x)|^{n-n\varepsilon} \, dx = 0,$$

the weak continuity of Jacobians is obtained by L. Greco, T. Iwaniec, and U. Subramanian [22].

This paper proves continuity theorems for the minors of the differential matrix of mappings belonging to grand Sobolev spaces (see Section 2 for the definitions). More precisely,

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^n$ and $f_m = (f_1^m, \ldots, f_k^m), 1 \leq k \leq n, m \in \mathbb{N}$, be a sequence of mappings locally bounded in $W^{1,p)}(\Omega)$ with $p > k$. Assume that $f_m$ converges in $L^1_{\text{loc}}$ to $f_0 = (f_1^0, \ldots, f_k^0)$ as $m \to \infty$, then the sequence of forms $\omega_m = df_1^m \wedge \cdots \wedge df_k^m$ converges to $\omega_0 = df_1^0 \wedge \cdots \wedge df_k^0$ in $D'$ and is locally bounded in $L^1_{\text{loc}}(\Omega)$.

The case $p = k$ requires some additional conditions, since it makes use of the property of the coincidence between the distributional Jacobian and the point-wise Jacobian (Theorem 2.4 below). The same technique used in obtaining proof of the main result, with minor changes, allows us to prove the following results.

**Theorem 1.2.** Let $f_m = (f_1^m, \ldots, f_k^m), 1 \leq k \leq n, m \in \mathbb{N}$, be a sequence of mappings locally bounded in $W^{1,k)}$ and with $Df_m \in L^k_{\text{loc}}$. Assume that $f_m$ converges in $L^1_{\text{loc}}$ to $f_0 = (f_1^0, \ldots, f_k^0)$ as $m \to \infty$ and forms $\omega_m = df_1^m \wedge \cdots \wedge df_k^m$ and $\omega_0 = df_1^0 \wedge \cdots \wedge df_k^0$ are locally integrable. It follows that $\omega_m$ converges to $\omega_0$ in $D'$ and is locally bounded in $L^1$.

**Theorem 1.3.** Let $f_m = (f_1^m, \ldots, f_k^m), 1 \leq k \leq n, m \in \mathbb{N}$, be a sequence of mappings locally bounded in $W^{1,k})$ and with $Df_m \in L^k_{\text{loc}}$. Assume that $f_m$ converges in $L^1_{\text{loc}}$ to $f_0 = (f_1^0, \ldots, f_k^0)$ as $m \to \infty$ and all $k$-minors of matrix $Df_m$ are nonnegative. It follows that $\omega_m = df_1^m \wedge \cdots \wedge df_k^m$ converges to $\omega_0 = df_1^0 \wedge \cdots \wedge df_k^0$ in $D'$ and is locally bounded in $L^1$.

The stated results are similar to those of [22] but the proof, based on a technique used by Yu. Reshetnyak [34], is comparatively simple and requires us to know only basic properties of the theory of differential forms and Sobolev spaces. Moreover, this method allows us to easily extend the results for grand Sobolev spaces $W^{1,p})$ to grand Sobolev spaces with respect to measurable functions $W^{1,p),\delta}$, as stated in Theorem 1.1.
2. Preliminaries

For a bounded open subset $\Omega$ in $\mathbb{R}^n$, $n \geq 1$, vector functions $f = (f^1, \ldots, f^n) : \Omega \to \mathbb{R}^n$ are called mappings of the Sobolev class $W^{1,p}(\Omega, \mathbb{R}^n)$, $1 \leq p \leq \infty$, if all coordinate functions $f^i$, $i = 1, 2, \ldots, n$, belong to $W^{1,p}(\Omega, \mathbb{R})$. Throughout this paper the symbol $Df$ stands for the differential matrix and $J_f$ denotes its determinant, the Jacobian.

**Definition 2.1.** For $0 < q < \infty$ the grand Lebesgue space $L^q(\Omega)$ consists of all measurable functions $f : \Omega \to \mathbb{R}$ such that

$$
\|f\|_{L^q} = \sup_{0 < \varepsilon < \varepsilon_0} \left( \frac{\varepsilon}{\Omega} \int_{\Omega} |f(x)|^{q-\varepsilon} \, dx \right)^{\frac{1}{q-\varepsilon}} < \infty,
$$

where $\varepsilon_0 = q - 1$ if $q > 1$ and $\varepsilon_0 \in (0, q)$ if $0 < q < 1$.

Grand Lebesgue spaces have been thoroughly studied by many different authors. We refer the interested reader to the reviews given in articles [9, 12, 27] and [6, §7.2]. However, we now state some basic properties of these spaces which will be useful for the results that follow.

For the case $q > 1$, the continuous embeddings

$$
L^q \subset L^q(\Omega) \subset L^{q-\varepsilon}, \quad \text{for } 0 < \varepsilon < q - 1,
$$

hold, and are strict. This can be easily seen by considering a unit ball $B(0,1)$ and the function $f(x) = |x|^{-\frac{n}{q}}$. In this case $f$ belongs to $L^q(B(0,1))$ but not $L^q(B(0,1))$.

Spaces $L^q$ for $q > 1$ are known to be non-reflexive Banach spaces [11].

**Definition 2.2.** The space $L^q_0$ consists of all functions $f \in L^q$ such that

$$
\lim_{\varepsilon \to 0^+} \varepsilon \int_{\Omega} |f(x)|^{q-\varepsilon} \, dx = 0.
$$

The space $L^q_0$ is the closure of $L^q$ in the norm $\| \cdot \|_{L^q}$ and $L^q_0 \neq L^q$ see [5, 19]. The validity of this latter claim is easy to see by considering once again the function $f(x) = |x|^{-\frac{n}{q}}$ on the unit ball $B(0,1)$, for which $f \notin L^q_0(B(0,1))$, since

$$
\varepsilon \int_{\Omega} |f(x)|^{q-\varepsilon} \, dx = \frac{q}{n} |B(0,1)| \not\to 0 \text{ as } \varepsilon \to 0^+.
$$

The embeddings $L^{q,p} \subset L^{q,\infty} \subset L^q$ and $L^q(\log L)^{-1} \subset L^q_0 \subset L^q$ also hold, where $L^{q,p}$ are Lorenz spaces, and $L^q(\log L)^{-1}$ are Orlicz spaces. For further discussions of embeddings of these spaces, we refer the reader to [9, 18, 19, 24].

As is seen from [11], grand Lebesgue spaces can be characterized as controlling the blow-up of the Lebesgue norm by the parameter $\varepsilon$. Indeed, the norm of the function $f$, belonging to $\bigcap_{0<\varepsilon<q-1} L^{q-\varepsilon}$ but not $L^q$, must blow up, i.e., $\|f\|_{L^{q-\varepsilon}} \to \infty$, when $\varepsilon \to 0$. Thus, a natural generalization is to substitute for $\varepsilon$ a measurable function $\delta(\varepsilon)$, which is positive a.e. [3].
Definition 2.3. For $0 < q < \infty$ the grand Lebesgue space $L^{q,\delta}(\Omega)$ with respect to $\delta$ consists of all measurable functions $f : \Omega \to \mathbb{R}$ such that

$$
\|f\|_{L^{q,\delta}} = \sup_{0 < \varepsilon < \varepsilon_0} \left( \frac{\delta(\varepsilon)}{|\Omega|} \int_{\Omega} |f(x)|^{q-\varepsilon} \, dx \right)^{\frac{1}{q-\varepsilon}} < \infty,
$$

where $\delta \in L^\infty((0,\varepsilon_0),(0,1))$ is a left continuous function such that $\lim_{\varepsilon \to 0^+} \delta(\varepsilon) = 0$ and $\delta^{\frac{1}{q-\varepsilon}}(\varepsilon)$ is nondecreasing, $\varepsilon_0 = q - 1$ if $q > 1$ and $\varepsilon_0 \in (0,q)$ if $0 < q \leq 1$.

If $\delta(\varepsilon) = \varepsilon$, the space $L^{q,\delta}$ is equivalent to $L^q$. If $\delta(\varepsilon) = \varepsilon^\theta$ with $\theta > 0$, we denote the resulting space by $L^{q,\theta}$. It was first introduced and studied in [21]. In [3] it was also shown that for $q > 1$

$$
L^q \subset L^{q,\delta} \subset L^{q-\varepsilon} \quad \text{for} \ 0 < \varepsilon \leq q - 1.
$$

The definition of convergence in the sense of distributions is standard. We say that the sequence $f_m \in X(\Omega)$ converges in the sense of distributions (in $D'$) to $f_0$ if, for every function $\varphi \in C^\infty_0(\Omega)$,

$$
\int_{\Omega} f_m(x) \varphi(x) \, dx \to \int_{\Omega} f_0(x) \varphi(x) \, dx \quad \text{as} \ m \to \infty.
$$

It is well-known that $f_m$ converges to $f_0$ weakly in $L^p$ if and only if the sequence $\{f_m\}_{m \in \mathbb{N}}$ is bounded in $L^p$ and $f_m$ converges in the sense of distributions to $f_0$.

We now make some brief comments on exterior algebra that will be useful for the results that follow. Let $\omega$ be differential $k$-forms, where $1 \leq k \leq n$. If $I = (i_1,i_2,\ldots,i_k)$ is a $k$-tuple with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, a differential form $\omega$ can be represented as

$$
\omega = \sum_I \omega_I(x) \, dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_I \omega_I(x) \, dx^I.
$$

Note that the sequence of $k$-forms $\omega_m$ converges to $\omega_0$ in $D'$ as $m \to \infty$ if the coefficients of the forms $\omega_m$ converge in $D'$ to the corresponding coefficients of $\omega_0$.

The calculus of differential forms is a powerful tool in the study of the analytical and geometrical properties of mappings. Thus, for mappings $f$ in Sobolev class $W^{1,p}$, with $p \geq n$, the Jacobian can be represented by the $n$-form

$$
J_f = df_1 \wedge \cdots \wedge df_n.
$$

To deal with the borderline case $p = k$ we need the integration-by-parts formula,

$$
(2) \quad \int_{\Omega} \varphi(x) J_f(x) \, dx = - \int_{\Omega} f^n \, df^1 \wedge df^2 \cdots \wedge df^{n-1} \wedge d\varphi.
$$

It is easy to see that (2) holds for $f \in W^{1,n}(\Omega)$. In general, Sobolev embeddings and the Hölder inequality ensure that for $f \in W^{1,\frac{n}{n+1}}_{loc}(\Omega)$, the right-hand-side of (2) can be
considered as a distribution, called the \textit{distributional Jacobian} $J_f$, and defined by the rule

$$J_f[\varphi] = - \int_{\Omega} f^n \, df^1 \wedge df^2 \ldots \wedge df^{n-1} \wedge d\varphi$$

for every test function $\varphi \in C_0^\infty(\Omega)$. A function $f = x + x |x|$, with $\Omega$ being a unit ball, shows that (2) fails as soon as $f \in W^{1,p}(\Omega)$, $p < n$. The natural question of the coincidence of the distributional and the point-wise Jacobians is thoroughly studied in [19, 24, 32], as well as in [23, §7.2] and [28, §6.2]. We need the following results for grand Lebesgue spaces.

\textbf{Lemma 2.4} ([19, Theorem 4.1]). Let $f = (f^1, \ldots, f^n) \in W^{1,1}_{\text{loc}}(\Omega)$ be a function such that $J_f \in L^1_{\text{loc}}(\Omega)$ and $|Df| \in L^n_0(\Omega)$. Then (2) holds for all compactly supported test functions $\varphi \in C_0^\infty(\Omega)$.

\textbf{Lemma 2.5} ([19, Corollary 4.1]). Let $f = (f^1, \ldots, f^n) \in W^{1,1}_{\text{loc}}(\Omega)$ be a function such that $J_f(x) \geq 0$ a.e. in $\Omega$ and $|Df| \in L^n_0(\Omega)$. Then (2) holds for all compactly supported test functions $\varphi \in C_0^\infty(\Omega)$.

Before we proceed to the proof of the main results, we need the following auxiliary lemma, which can be found in [34, §4.5], and for which we now provide a proof for the convenience of the reader.

\textbf{Lemma 2.6}. Let $\omega_m$ be a sequence of differential $k$-forms, bounded in $L^1_{\text{loc}}(\Omega)$, that converges in $D'$ to a form $\omega_0$ as $m \to \infty$. Assume that each of the forms $\omega_m$, $m \in \mathbb{N}$, has in $\Omega$ a generalized differential, and that the sequence $d\omega_m$ is bounded in $L^1_{\text{loc}}(\Omega)$. It follows that the forms $d\omega_m$ converge to $d\omega_0$ in $D'$ as $m \to \infty$.

\textit{Proof.} Consider an arbitrary $C^\infty$-smooth, compactly supported $(n-k-1)$-form $\alpha$. From the definition of a generalized differential we have

$$\int_{\Omega} \omega_m \wedge d\alpha = (-1)^{k-1} \int_{\Omega} d\omega_m \wedge \alpha.$$ 

Since $\omega_m \to \omega_0$ in $D'$ and $d\alpha$ is a $(n-k)$-form of the class $C_0^\infty(\Omega)$, we obtain

$$\int_{\Omega} \omega_m \wedge d\alpha \underset{m \to \infty}{\longrightarrow} \int_{\Omega} \omega_0 \wedge d\alpha = (-1)^{k-1} \int_{\Omega} d\omega_0 \wedge \alpha.$$ 

And finally

$$\int_{\Omega} d\omega_m \wedge \alpha \underset{m \to \infty}{\longrightarrow} \int_{\Omega} d\omega_0 \wedge \alpha$$

for all test $(n-k-1)$-forms $\alpha \in C_0^\infty(\Omega)$. \hfill \Box

We now make use of Lemma [2.6] for grand Lebesgue spaces.
Lemma 2.7. Let $\omega_m$ be a sequence of differential $k$-forms, locally bounded in $L^p,\delta(\Omega)$, that converges in $D'$ to a form $\omega_0$ as $m \to \infty$. Assume that each of the forms $\omega_m$, $m \in \mathbb{N}$, has a generalized differential in $\Omega$, and that the sequence $d\omega_m$ is locally bounded in $L^q,\delta(\Omega)$. It follows that the forms $d\omega_m$ converge to $d\omega_0$ in $D'$ as $m \to \infty$.

For a mapping $f = (f^1, \ldots, f^n): \Omega \to \mathbb{R}^n$, we define the $k \times k$-minors of the differential matrix as
\[
\frac{\partial f^I}{\partial x^J} = \frac{\partial (f^{i_1}, \ldots, f^{i_k})}{\partial (x^{j_1}, \ldots, x^{j_k})}
\]
for ordered $k$-tuples $I = (i_1, i_2, \ldots, i_k)$ and $J = (j_1, j_2, \ldots, j_k)$. The representation
\[
df^{i_1} \wedge \cdots \wedge df^{i_k} = \sum J \frac{\partial f^I}{\partial x^J} dx^{j_1} \wedge \cdots \wedge dx^{j_k}
\]
is valid.

Since in the proofs we investigate the properties of a particular $k \times k$ minor, it suffices to consider mappings $f: \Omega \to \mathbb{R}^k$ instead of maps into $\mathbb{R}^n$; also, this makes the notation simpler. Moreover, the condition “$f_m$ converges in $L^1_{loc}$ to $f_0$ as $m \to \infty$” results from the statement “there exists a subsequence converging weakly in $W^{1,q}_{loc}$ to $f_0$ for all $1 \leq q < p$”. Indeed, by the Sobolev embeddings we can find a subsequence $f_{m_l}$, which converges to $f_0$ in $L^s_{loc}$, for some $1 \leq s < \frac{np}{n^p}$. The Hölder inequality and boundedness of $\Omega$ then guarantee that $f_0$ is also an $L^1_{loc}$-limit of $f_{m_l}$.

3. Proof of the main results

We will prove Theorem 1.1 by induction on $k$. The case of $k = 1$ follows directly from Lemma 2.7. Assume that the lemma has been proven for some general $k$, and let $f_m: \Omega \to \mathbb{R}^{k+1}$ be a sequence of mappings of class $W^{1,p},\delta(\Omega)$, $p > k+1$. The sequence $f_m$ is locally bounded in $W^{1,p},\delta(\Omega)$, consequently, also bounded in $W^{1,p-\varepsilon}(\Omega)$ for $0 < \varepsilon < p - 1$, and is locally convergent in $L^1$ to $f_0$. From the Sobolev embedding theorem we obtain that $f_m \to f_0$ in $L^s$ for $s < \frac{n(p-\varepsilon)}{n^p}$.

Step I. Let us consider the forms
\[
u = (-1)^k y^{k+1} u = (-1)^k y^{k+1} dy^1 \wedge \cdots \wedge dy^k,
\]
\[
u = u \wedge dy^{k+1} = dy^1 \wedge dy^2 \wedge \cdots \wedge dy^k \wedge dy^{k+1}
\]
in $\mathbb{R}^{k+1}$.

It is easy to see that $w = dv$.

Consider also the pull-backed forms
\[
\tilde{\omega}_m = f_m^* u = df^1_m \wedge df^2_m \wedge \cdots \wedge df^k_m,
\]
\[
\psi_m = f_m^* \nu = (-1)^k f^k_m \tilde{\omega}_m,
\]
\[
\omega_m = f_m^* w = df^1_m \wedge df^2_m \wedge \cdots \wedge df^{k+1}_m.
\]
Then $\omega_m = d\psi_m$ for each $m$. In fact $\omega_m, \psi_m \in L^1$, since for each of $j$, the functions $f^j_m, df^j_m$ lie in $L^{\tilde{p}}$, where $p > \tilde{p} \geq k + 1$. Thus, for any $(n - k - 1)$-form $\eta \in C_0^\infty(\Omega)$,

\[(5) \quad \int_{\Omega} \omega_m \wedge \eta = (-1)^{k-1} \int_{\Omega} \psi_m \wedge d\eta.\]

By the induction hypothesis $\tilde{\omega}_m \to \tilde{\omega}_0$ in $D'$ and $\tilde{\omega}_m$ is locally bounded in $L^{p/k}$.

**STEP II.** Let $\xi$ be an arbitrary $C^\infty$-smooth, compactly supported $(n - k)$-form. Let us show that

\[(6) \quad \int_{\Omega} f^{k+1}_m \tilde{\omega}_m \wedge \xi \to \int_{\Omega} f^{k+1}_0 \tilde{\omega}_0 \wedge \xi.\]

Indeed, fix $0 < \varepsilon = \frac{k}{n+1} < p - 1$. Then, by the Sobolev embedding theorem, $f^{k+1}_m \to f^{k+1}_0$ in $L^s$, $s < \frac{n(p-\varepsilon)}{n-p+\varepsilon}$. Put $s' = \frac{p-\varepsilon}{k}$ and $s = \frac{p-\varepsilon}{p-k-\varepsilon}$, then $\frac{1}{s'} + \frac{1}{s} = 1$. Hence

\[(7) \quad \left| \int_{\Omega} f^{k+1}_m \tilde{\omega}_m \wedge \xi - \int_{\Omega} f^{k+1}_0 \tilde{\omega}_0 \wedge \xi \right| \leq C \|\tilde{\omega}_m\|_{L^{s'}(A)} \|f^{k+1}_m - f^{k+1}_0\|_{L^s(A)} \to 0,
\]

where $A = \text{supp} \xi$. Further, for any $\gamma > 0$ let $f \in C_0^\infty(\Omega)$ be such that $\|f - f^{k+1}_0\|_{L^s(\Omega)} < \gamma$. Then

\[
\left| \int_{\Omega} f^{k+1}_0 \tilde{\omega}_m \wedge \xi - \int_{\Omega} f^{k+1}_0 \tilde{\omega}_0 \wedge \xi \right| \leq \left| \int_{\Omega} (f^{k+1}_0 - f) \tilde{\omega}_m \wedge \xi \right| \\
+ \left| \int_{\Omega} f (\tilde{\omega}_m \wedge \xi - \tilde{\omega}_0 \wedge \xi) \right| + \left| \int_{\Omega} (f - f^{k+1}_0) \tilde{\omega}_0 \wedge \xi \right| \to 0
\]

as $m \to \infty$. The first and the third terms are less than $C\gamma$ due to the choice of $f$, the second one tends to zero by the induction hypothesis. Since $\gamma$ is arbitrary, this implies that

\[(8) \quad \int_{\Omega} f^{k+1}_0 \tilde{\omega}_m \wedge \xi \to \int_{\Omega} f^{k+1}_0 \tilde{\omega}_0 \wedge \xi.
\]

The convergence [8] follows from [7] and [8]. This means that the sequence of forms $\psi_m = f^{k+1}_m \tilde{\omega}_m$ converges to the form $\psi_0 = f^{k+1}_0 \tilde{\omega}_0$ in $D'$.

It remains to show that the sequences of forms $\psi_m$ and $d\psi_m$ are bounded in $L^{q,\delta}$ for $q = \frac{p}{k+1}$. Indeed, the Hölder inequality provides

\[
\left( \int_{\Omega} |\psi_m|^{q-\varepsilon} \, dx \right)^{\frac{1}{q-\varepsilon}} = \left( \int_{\Omega} |f^{k+1}_m \tilde{\omega}_m|^{q-\varepsilon} \, dx \right)^{\frac{1}{q-\varepsilon}} \\
\leq \left( \int_{\Omega} |f^{k+1}_m|^{(q-\varepsilon)(p-\varepsilon)} \, dx \right)^{\frac{1}{p-\varepsilon}} \left( \int_{\Omega} |\tilde{\omega}_m|^{(q-\varepsilon)(p-\varepsilon)} \, dx \right)^{\frac{p-\varepsilon}{(p-\varepsilon)(q-\varepsilon)}}
\]

Here $\frac{p-\varepsilon}{q-\varepsilon} > 1$ as $p - \varepsilon > q - \varepsilon$. 


Multiplying by $\delta(\varepsilon)$ and taking the supremum, we obtain

\begin{equation}
\|\psi_m\|_{L^q(\Omega)}
\leq \sup_{0<\varepsilon<q-1} \left( \delta(\varepsilon) \int_\Omega |f_m^{k+1}(q-\varepsilon)_{\frac{p-\varepsilon}{q-\varepsilon}} dx \right)^{\frac{1}{p-\varepsilon}} \left( \delta(\varepsilon) \int_\Omega |\tilde{\omega}_m|^{(q-\varepsilon)\frac{p-\varepsilon}{p-q}} dx \right)^{\frac{p-q}{(p-\varepsilon)(q-\varepsilon)}}
\leq \sup_{0<\varepsilon<p-1} \left( \delta(\varepsilon) \int_\Omega |f_m^{k+1}|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}} \sup_{0<\varepsilon<p-1} \left( \delta(\varepsilon') \int_\Omega |\tilde{\omega}_m|^{\frac{p}{k-\varepsilon'}} dx \right)^{\frac{1}{p/k-\varepsilon'}}
\leq \|f_m^{k+1}\|_{L^p(\Omega)} \|\tilde{\omega}_m\|_{L^{p/k}(\Omega)}.
\end{equation}

The last inequality is valid for $\varepsilon' = \frac{\varepsilon(2p+pk-\varepsilon)}{pk}$, which satisfies $\frac{(q-\varepsilon)(p-\varepsilon)}{p-q} = \frac{p}{k} - \varepsilon'$. It is easy to check that $\varepsilon < \varepsilon'$, and from Definition 2.3 we can deduce that $\delta$ is a nondecreasing function, and thus $\delta(\varepsilon')^{\frac{1}{p/k}} \leq \delta(\varepsilon)^{\frac{1}{p/k}}$.

In order to make sure that $0 < \varepsilon' < \frac{p}{k} - 1$, we show that

$$h(\varepsilon) = pk \left( \frac{p}{k} - 1 - \varepsilon' \right) = (k+1)\varepsilon^2 - (2p+pk)\varepsilon + p^2 - pk > 0.$$ 

First, note that $h(0) > 0$ and $h \left( \frac{p}{k+1} - 1 \right) > 0$. Moreover, $h'(\varepsilon) = 2(k+1)\varepsilon - (2p+pk) < 0$ if $\varepsilon < \frac{2p+pk}{2(k+1)}$ with $\frac{p}{k+1} - 1 < \frac{2p+pk}{2(k+1)}$, i.e., $h(\varepsilon)$ decreases for $0 < \varepsilon < \frac{p}{k+1} - 1$ and takes positive values at the boundary points. Thus, $h(\varepsilon) > 0$ for all $\varepsilon \in (0, \frac{p}{k+1} - 1)$, and so it follows that $0 < \varepsilon' < \frac{p}{k} - 1$.

In view of this, we can consider the supremum over all $0 < \varepsilon' < \frac{p}{k} - 1$, and its value is not less than the supremum over all $0 < \varepsilon < q - 1 = \frac{p}{k+1} - 1$. This completes the proof of (9).

The same arguments show that $d\psi_m = \omega_m = \tilde{\omega}_m \wedge df_m^{k+1}$ is bounded in $L^{p/k+1}(\Omega)$. By Lemma 2.4 this implies that $\omega_m \to \omega_0$ in $D'$.

**Proof of Theorem 1.2 and Theorem 1.3.** Here, we need some modifications of the proof of Theorem 1.1. At Step I we use Lemma 2.4 to obtain the relation (5). Note that Lemma 2.4 can be modified for k-forms by considering $f^I = (f^1, f^2, \ldots, f^k, x^{k+1}, \ldots, x^n)$, where $x^i$ is a corresponding coordinate function.

**Step I.** Recall that $p = k + 1$. Let us consider the forms $u$, $v$, $w$ and their pullbacks $\tilde{\omega}_m$, $\psi_m$, and $\omega_m$ defined by (3) and (4), correspondingly. Now we use Lemma 2.4 to obtain $\omega_m = d\psi_m$ for each $m$.

Indeed, $\omega_m = df_m^1 \wedge df_m^2 \wedge \cdots \wedge df_m^{k+1} \in L^1_{\text{loc}}$ by the hypothesis of Theorem 1.2, the local integrability of $\psi_m = (-1)^k f_m^{k+1} df_m^1 \wedge df_m^2 \wedge \cdots \wedge df_m^k$ follows from $f \in W^{1,\frac{n(k+1)}{n+1}}_{\text{loc}}$, as $\frac{n(k+1)}{n+1} < k+1$. Then we have $df_m^1 \wedge df_m^2 \wedge \cdots \wedge df_m^k \in L^n_{\text{loc}}$ and, from the Sobolev embedding theorem $f_m^{k+1} \in L^{n(k+1)}_{\text{loc}}$. The Hölder inequality provides the required integrability, as $\frac{n(k+1)}{n(k+1)} + \frac{n-k}{n(k+1)} = 1$. 

Hence, for any \((n - k - 1)\)-form \(\eta \in C^\infty_0(\Omega)\),
\[
\int_\Omega \omega_m \wedge \eta = (-1)^{k-1} \int_\Omega \psi_m \wedge d\eta.
\]
By the induction hypothesis \(\tilde{\omega}_m \to \tilde{\omega}_0\) in \(D'\) and the sequence \(\tilde{\omega}_m\) is locally bounded in \(L^{p/k}\).

**Step II.** All the estimates of Step II in the proof of Theorem 1.1 are satisfied if we consider in the definition of the grand Lebesgue norm \(\varepsilon_0 = \frac{k+2-\sqrt{k^2+4k}}{2} < 1\). According to Lemmas 2.4 and 2.5, we can replace the local integrability condition of \(\omega_m\) by non-negativity of all \(k\)-minors of the matrix \(Df_m\).

Let \(\xi\) be an arbitrary \(C^\infty\)-smooth, compactly supported \((n - k)\)-form. Let us show that
\[
(11) \quad \int_\Omega f_m^{k+1} \tilde{\omega}_m \wedge \xi \to \int_\Omega f_0^{k+1} \tilde{\omega}_0 \wedge \xi.
\]
To this end, fix \(0 < \varepsilon = \frac{k}{n+1} < k = p - 1\). From the Sobolev embedding theorem \(f_m^{k+1} \to f_0^{k+1}\) in \(L^s\), \(s < \frac{n(k+1-\varepsilon)}{n-k+1-\varepsilon}\). Put \(s' = \frac{k+1-\varepsilon}{k}\) and \(s = \frac{k+1-\varepsilon}{1-\varepsilon}\), then \(\frac{1}{s'} + \frac{1}{s} = 1\). Hence
\[
(11) \quad \left| \int_\Omega f_m^{k+1} \tilde{\omega}_m \wedge \xi - \int_\Omega f_0^{k+1} \tilde{\omega}_0 \wedge \xi \right| \leq C \|\tilde{\omega}_m\|_{L^{s'}(A)} \|f_m^{k+1} - f_0^{k+1}\|_{L^s(A)} \to 0,
\]
where \(A = \text{supp} \xi\). Furthermore, for any \(\gamma > 0\) let \(f \in C^\infty_0(\Omega)\) be such that \(\|f - f_0^{k+1}\|_{L^s(\Omega)} < \gamma\), then
\[
\left| \int_\Omega f_0^{k+1} \tilde{\omega}_m \wedge \xi \right| \leq \left| \int_\Omega (f - f_0^{k+1}) \tilde{\omega}_m \wedge \xi \right| \to 0
\]
as \(m \to \infty\). The first and the third terms are less than \(C\gamma\) due to the choice of \(f\), and the second one tends to zero by the induction hypothesis. Since \(\gamma\) is arbitrary, this implies that
\[
(12) \quad \int_\Omega f_0^{k+1} \tilde{\omega}_m \wedge \xi \to \int_\Omega f_0^{k+1} \tilde{\omega}_0 \wedge \xi.
\]
The relation indicated in (11) follows from (11) and (12). This means that the sequence of forms \(\psi_m = f_m^{k+1} \tilde{\omega}_m\) converges to the form \(\psi_0 = f_0^{k+1} \tilde{\omega}_0\) in \(D'\).

It remains to check that the sequences of forms \(\psi_m\) and \(d\psi_m\) are bounded in \(L^1\). The Hölder inequality provides
\[
\left( \int_\Omega |\psi_m|^{1-\varepsilon} \, dx \right)^{\frac{1}{1-\varepsilon}} \leq \left( \int_\Omega |f_m^{k+1} \tilde{\omega}_m|^{1-\varepsilon} \, dx \right)^{\frac{1}{1-\varepsilon}} \leq \left( \int_\Omega |f_m^{k+1}|(1-\varepsilon)^{k+1-\varepsilon} \, dx \right)^{\frac{k+1-\varepsilon}{k}} \left( \int_\Omega |\tilde{\omega}_m|^{(1-\varepsilon)^{k+1-\varepsilon}} \, dx \right)^{\frac{k}{(k+1-\varepsilon)(1-\varepsilon)}},
\]
here \(\frac{k+1-\varepsilon}{1-\varepsilon} > 1\).
Multiplying by $\varepsilon$ and taking the supremum, we obtain

$$
\|\psi_m\|_{L^1} \leq \sup_{0<\varepsilon<\varepsilon_0} \left( \varepsilon \int_{\Omega} \left| f_{m}^{k+1} \right|^{k+1-\varepsilon} |\omega|^{\frac{1}{k+1-\varepsilon}} \, dx \right)^{\frac{1}{k+1-\varepsilon}} \left( \varepsilon \int_{\Omega} \left| \bar{\omega}_m \right|^{(1-\varepsilon)\frac{1}{k+1-\varepsilon}} \, dx \right)^{\frac{k}{k-1}} \leq \sup_{0<\varepsilon<\varepsilon_0} \left( \varepsilon \int_{\Omega} \left| f_{m}^{k+1} \right|^{k+1-\varepsilon} |\omega|^{\frac{1}{k+1-\varepsilon}} \, dx \right)^{\frac{1}{k+1-\varepsilon}} \sup_{0<\varepsilon<\varepsilon_0} \left( \varepsilon' \int_{\Omega} \left| \bar{\omega}_m \right|^{k+1-\varepsilon'} \, dx \right)^{\frac{k+1}{k+1-\varepsilon'}} \leq \|f_{m}^{k+1}\|_{L^{k+1}} \left\| \bar{\omega}_m \right\|_{L^{\frac{k+1}{k+1-\varepsilon}}}.
$$

The last inequality is valid for $\varepsilon' = \frac{\varepsilon(2+k-\varepsilon)}{k}$, which satisfies $\frac{(1-\varepsilon)(k+1-\varepsilon)}{k} = \frac{k+1}{k} - \varepsilon'$. It is easy to check that $\varepsilon < \varepsilon'$. In order to make sure that $0 < \varepsilon' < \frac{k+1}{k} - 1 = \frac{1}{k}$, note that the roots of $h(\varepsilon) = \varepsilon^2 - (2+k)\varepsilon + 1$, $\varepsilon_{1,2} = \frac{k+2 \pm \sqrt{k^2-4k}}{2}$ are not less than $\varepsilon_0 = \frac{k+2 - \sqrt{k^2-4k}}{2}$, and $h(0) = 1 > 0$.

In view of this, we can consider the supremum over all $0 < \varepsilon' < \frac{1}{k}$, and, by doing so, its value is seen to increase. This completes the proof of the estimate (13).

The same arguments show that $d\psi_m = \omega_m = \bar{\omega}_m \wedge df_{m}^{k+1}$ is bounded in $L^{\frac{k}{k+1}}$. By Lemma 2.7 this implies that $\omega_m \to \omega_0$ in $D'$. □

Acknowledgment. The author warmly thanks professor Sergey Vodopyanov and my great friend Dr. Ian McGregor for the numerous discussions on, and useful comments about this paper.

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