Stabilizing an Attractive Bose-Einstein Condensate by Driving A Surface Collective Mode

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Bose-Einstein condensates of \(^7\)Li have been limited in number due to attractive interatomic interactions. Beyond this number, the condensate undergoes collective collapse. We study theoretically the effect of driving low-lying collective modes of the condensate by a weak asymmetric sinusoidally time-dependent field. We find that driving the radial breathing mode further destabilizes the condensate, while excitation of the quadrupolar surface mode causes the condensate to become more stable by imparting quasi-angular momentum to it. We show that a significantly larger number of atoms may occupy the condensate, which can then be sustained almost indefinitely. All effects are predicted to be clearly visible in experiments and efforts are under way for their experimental realization.

Bose-Einstein condensation (BEC) in cold and dilute atomic gases [1], provide a new domain for studying nonlinear many-body quantum systems [2]. Consider the condensate wavefunction in the mean-field picture: First, the zero-point kinetic energy of the atoms tends to increase the size of the condensate wavefunction. Second, this outward pressure is balanced by the effect of the confining harmonic trap potential \(V(\vec{r})\). Finally, the nonlinearity enters through the interatomic interactions scaling as \(aN_0|\psi|^2\), where \(a\) is the s-wave scattering length for the gas and \(N_0\) is the number of atoms participating in the condensate. The condensate is then well-characterized by the product wavefunction \(\phi = \sqrt{N_0}\psi\) where the Gross-Pitaevskii (GP) equation \(\psi\) governs the single-particle wavefunction \(\psi\) of this weakly-interacting Bose condensate at zero temperature.

Among the systems in which BEC has been observed, \(^7\)Li is unique in that it has a negative value for \(a\); i.e., the effective interaction between atoms is attractive. Such attractive interactions were initially argued to prevent BEC [3], but it is now established that a static metastable condensate exists as long as the number of participating atoms \(N_0\) is less than a maximum \(N_s\) [4]. This is understood [5] to unfold as follows: As the \(^7\)Li Bose gas is cooled below the critical temperature for BEC, the number of atoms in the condensate increases steadily with a kinetically determined ‘fill rate’. As \(N_0\) increases the nonlinear attractive interactions grow and the condensate becomes more localized. At \(N_0 \approx N_s\), the condensate undergoes collective collapse and there is a rapid increase in condensate density. The rates for inelastic collision processes, including two-body dipolar collisions and three-body recombination, are density dependent and increase rapidly during collapse. The atoms participating in these processes acquire large kinetic energies and are ejected from the condensate. Although the quantitative details of this collapse are not yet fully understood, experimental evidence for this process has been obtained [3].

In this Letter, we analyze theoretically the effect of weakly perturbing the harmonic trapping potential for an attractive condensate. We have considered general potentials \(V(\vec{r}) = \frac{1}{2}m\sum_{i=1}^{3}\omega_i^2x_i^2\) where \(\vec{r} \equiv (x_1, x_2, x_3) \equiv (x, y, z)\) and the parameters \(\omega_i\) characterize the trapping potential along the three axes. For the case presented in detail below, \(\omega_i^2 = \omega_0^2[1 + \alpha_i \cos(\omega t)]\) where \(\omega\) is the frequency of the forcing and the \(\alpha_i\) are its amplitudes. This corresponds to a spherically symmetric trap driven asymmetrically by a weak, sinusoidally time-dependent field. We employ a Gaussian time-dependent variational principle approximation [6] (GVA) analysis of the GP equation to solve for the condensate dynamics. The Gaussian ansatz is motivated by the shape of the ground-state for non-interacting particles in a harmonic trap and constrains the condensate wavefunction to be of the form \(\psi(\vec{r}, t) = \psi_x\psi_y\psi_z\) where \(\psi_i = \left[2\pi a_i(t)\right]^{-\frac{1}{4}}\exp\left\{-ax_i^2(\frac{1}{4a_i(t)}) + ib_i(t)\right\}\). This ansatz when used in Eq. [1] yields that the condensate dynamics correspond [6] to a classical effective Hamiltonian. With the transformation \(a_i(t) = \rho_i^2(t)\) and \(b_i(t) = \frac{\Pi_i(t)}{2\rho_i(t)}\) this

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Hamiltonian has the transparent form $H_{\text{var}} = \sum_i \left[ \frac{\Pi_i^2}{2m} + \frac{\hbar^2}{8m\rho_i} + \frac{m \omega_i^2 \rho_i^2}{2} \right] + \frac{\hbar^2 a N_0}{4 \sqrt{\beta} \rho_i \rho_2 \rho_3}$. The parameters $\rho_i$ and $\Pi_i$ are given by $\langle \hat{x}_i^2 \rangle = \rho_i^2$ and $\langle \hat{x}_i \hat{p}_i + \hat{p}_i \hat{x}_i \rangle = 2 \rho_i \Pi_i$. Note that $\langle \hat{x}_i \rangle = 0 = \langle \hat{p}_i \rangle$ for this wavefunction. The centroid of the condensate thus sits at the trap minimum and all the dynamics is restricted to changes in the condensate width $\rho_i$ in the three directions. That is, the dynamics of a quasi-particle with canonically conjugate position and momentum variables $\rho_i$ and $\Pi_i$ characterizes the condensate. A further scaling by the length of the trap $t_0 = \sqrt{\hbar/(m \omega_0)}$, the time-scale $\omega_0^{-1}$, and energy $h \omega_0$ yields the dimensionless Hamiltonian

$$H_{\text{var}} = \frac{1}{4} \sum_i \left[ \frac{\Pi_i^2}{2} + \frac{1}{\rho_i^2} + (1 + \alpha_i \cos(\omega t)) \rho_i^2 \right] - \frac{3 \beta}{4 \rho_i \rho_2 \rho_3}$$

(2)

where $\beta = \frac{4}{3 \sqrt{2 \pi}} \frac{N_0 |a|}{\rho_i t_0}$. The kinetics of the condensate, including the fill rate and dissipative losses, affect the dynamics through the time-dependence of $N_0$. We set the fill rate $G_0$ to a constant $\frac{G_0}{\rho_i}$, which is a good approximation to the results from the Boltzmann equation $\frac{G_0}{\rho_i}$ over the time-scales considered. Further, dipolar relaxation scales as $\phi^2$, while molecular recombination is a three-body process scaling as $\phi^3$. In particular, with the Gaussian approximation for the wavefunction, we can write

$$N_0 = G_0 - \frac{N_0^2}{\pi^{3/2} \rho_i \rho_2 \rho_3} \left( \frac{G_2}{2 \sqrt{2}} + \frac{N_0 G_3}{6 \sqrt{\pi} \rho_i \rho_2 \rho_3} \right)$$

(3)

where $G_1$ and $G_2$ are the appropriate rate constants $\frac{G_0}{\rho_i}$. The Hamiltonian Eq. (3) and the kinetics Eq. (3) define the condensate dynamics completely.

To understand these dynamics consider the fixed points given by $\partial H_{\text{var}}/\partial \Pi_i = 0 = \partial H_{\text{var}}/\partial \rho_i$. This yields a solution with $\Pi_i = 0$ and non-zero $\rho_i$ corresponding to the width of a metastable condensate. This solution $\rho_\beta = |\bar{\rho}|$ is strongly affected by the attractive hole at the origin ($\bar{\rho} = 0$ or infinite condensate density) due to the term $\beta / (\rho_1 \rho_2 \rho_3)$ in the Hamiltonian. The condensate is only stable when the quasi-particle avoids this attractive hole. As $N_0$ and correspondingly $\beta$ increase, the metastable minimum moves closer to the origin, becomes shallower until it becomes an inflection point, and finally vanishes. Correspondingly, the width $\rho_\beta$ decreases abruptly when the minimum vanishes, leading to collapse via the kinetics. The critical values $\rho_m$ and $\beta_m$ for collapse are determined by further imposing the inflection condition $\frac{d^2 H_{\text{var}}}{dr^2} = 0$. This gives $\rho_m = 5^{-1/4}$ and $\beta_m = \frac{5}{8} 5^{-5/4}$, corresponding to $N_0 \approx 1400$ atoms for the conditions of the Rice experiment $\frac{G_0}{\rho_i}$.

The normal modes of the quasi-particle dynamics around this minimum (when it exists) are the lowest-lying even collective excitations of the condensate $\frac{G_0}{\rho_i}$. These normal mode frequencies are obtained by computing the matrices $\mathcal{V}, \mathcal{T}$ with elements $V_{ij} = \frac{\partial^2 H_{\text{var}}}{\partial \rho_i \partial \rho_j}$, $T_{ij} = \frac{\partial^2 H_{\text{var}}}{\partial \Pi_i \partial \Pi_j}$, and then solving for the roots $\Omega_i$ of the equation $\det(\mathcal{V} - \Omega^2 \mathcal{T}) = 0$.

One solution is $\Omega_1 = 1 = \sqrt{5 - \rho_\beta^4}$ which is the frequency at which the quasi-particle vibrates radially, corresponding to the condensate’s ‘breathing’ mode. The other two solutions are the degenerate solutions $\Omega_{2,3} = 1 = \sqrt{2 + 2 \rho_\beta^4}$, corresponding to vibrational motion perpendicular to the radial direction. This can be visualized as being along the surface of a sphere of fixed radius – these are then quadrupolar surface modes $\frac{G_0}{\rho_i}$ of the condensate. Since $\rho_\beta$ decreases as a function of $N_0$, $\Omega_1$ decreases and $\Omega_{2,3}$ increases with increasing $N_0$. Also note that $\Omega_1 = \Omega_2 = 2$ for $\rho_\beta = 1$, i.e. for $\beta = 0$ (non-interacting limit) in these natural units.

We now turn to the numerical solution of this system of equations. As expected, these lowest-lying even collective modes of the condensate are excited by the driving. Driving the lowest such mode, a radial breathing mode, further destabilizes the condensate and decreases the maximum number of atoms below the static maximum $N_s$. Remarkably, driving the next-highest excitations, which are quadrupolar surface modes, causes the condensate to become more stable. That is, for certain frequencies, atoms occupy the condensate in numbers significantly greater than $N_s$ and the condensate is sustained for correspondingly longer times. We have studied in particular the frequency-dependent value of the maximum number of atoms $N_\omega$ that can be sustained in the presence of the driving. We find that this frequency response is broad and qualitatively robust, with similar features for wide range of tested parameters and many alternative configurations for both the confining and perturbing fields. An example of these responses is shown in Fig. (1), along with a theoretical estimate $N_{\text{ang}}$ as explained below. In obtaining it, we use $\alpha_3 = 0.02 = -2 \alpha_3 = -2 \alpha_2$ which represents the effect of a Helmholtz field oriented along the axis of an Ioffe-Pritchard magnetic trap. The response is shown as the ratio $N_m = N_\omega / N_s$. The broad stabilization effect is clear in this figure; further, it can be seen that the curve has some interesting structure, including in particular a signature dip corresponding to a parametric resonance between the radial and quadrupolar modes as explained below. The stabilization effect is so strong that in some cases it is no longer obvious whether and when a collapse occurs. It is possible for $N_0$ to grow so
large that the loss rate equals the fill rate, even for relatively large $\rho$. The values in Fig. (1) are therefore calculated using a ‘worst case’ definition of the collapse, as the $N_0$ value at which the inelastic decay terms in Eq. (3) exceed the growth rate $G_0$. Although it is unclear whether the physics of the true dynamical situation in these cases is fully captured by the simple kinetics of Eq. (3), the stabilization is nonetheless remarkable.

To understand the detailed structure of Fig. (1), consider the case of driving the condensate with $\omega < \Omega_0$. Note that the dynamics of the quasi-particle are always restricted to the $\rho_1 = \rho_2$ plane by our choice of driving. At an arbitrary value of $\omega$, the driving is not initially resonant with either of the excitations and there is a negligible response from the condensate. As $N_0$ increases with time, however, $\Omega_1(\beta)$ will ultimately equal any $\omega < \Omega_0$. The breathing mode is excited as a result and the quasi-particle oscillates along the radial direction. As this oscillation increases in amplitude, the quasi-particle is driven into the attractive hole and the condensate collapses. The net effect is to decrease $N_m(\omega)$, with the impact being greatest for $\omega$ slightly less than $\Omega_0$. The more interesting case is for $\omega > \Omega_0$ where again, the driving is not initially resonant. As $N_0$ increases, however, the quadrupolar surface modes come into resonance. The quasi-particle then oscillates with increasingly greater amplitude in a direction perpendicular to the radial direction, still in the $\rho_1 = \rho_2$ plane. An angular momentum vector $\vec{l}_q(t) = \vec{\rho} \times \vec{\Pi}$ pointing in the $\hat{q}$ direction can be associated with this motion, where $\hat{q} \equiv \frac{1}{\sqrt{2}}(1,-1,0)$ is the normal to the $\rho_1 = \rho_2$ plane. This angular momentum vector reverses orientation during the oscillation, with its magnitude going to zero at the turning points of the oscillation. However, the mean-square value of this angular momentum increases as the surface mode acquires increasing energy. By virtue of the energy in this oscillation, the quasi-particle avoids the chasm of the attractive ‘hole’, and the condensate is stabilized, avoiding collapse even when the attractive interactions are significant. The above implies that the stabilizing effect may be quantitatively estimated by adding to the kinetic energy pressure terms an angular momentum term $\vec{l}^2$; this estimate is plotted in Fig. (1) with $\vec{l}^2$ computed directly from the dynamics. It is clear from comparing the curves that this simple angular momentum argument captures the essential features of the stabilization. In particular, the region of discrepancy is precisely where the naive kinetics and collapse criteria render suspect the value of $N_0$ shown, as argued above.

This resonant driving is easier to sustain for frequencies close to $\Omega_0$ since $\Omega_{2,3}$ has a weaker dependence on $N_0$ in that regime. As $N_0$ increases, the condensate ultimately falls out of resonance with the driving. Consider for example the condensate radius $\rho(t) = |\vec{\rho}(t)|$: It is initially unaffected by the driving and decreases slowly as in the static case; this is followed by a resonantly driven increase that ultimately saturates. An example of this mechanism is shown in Fig. (2a) for $\omega = 2.3$ (corresponding to a driving frequency of $\approx 308$ Hz for the Rice experiment). In Fig. (2b) we show a numerical solution for the GP equation corresponding to the situation in Fig. (2a); the two curves agree qualitatively. We note here that the numerical solution to the nonlinear Schrödinger equation in the GP form for cylindrical symmetry is extremely difficult and a full scan of frequencies is computationally prohibitive. However, simulations including filling and inelastic decay similar to those in the literature have been performed to qualitatively validate the GVA results.

As $\omega$ increases, the resonance occurs at increasingly later times and for decreasing windows of time, and hence the greatest stabilization happens for $\omega$ slightly greater than $\Omega_0$. There is no such stabilization for $\omega > \omega_2 = \sqrt{12} = 3.46$; this critical frequency is obtained by substituting the maximum value for the equilibrium $\rho_m$ into the expression for $\Omega_{2,3} - i.e., the driving cannot resonate with the condensate modes if the metastable minimum does not exist in the first place. This cut-off can be seen clearly in Fig. (1). Finally, we point out the significant dip in the curves at $\omega = \omega_1 = \sqrt{8}$. This frequency corresponds to the situation where $\Omega_{2,3}(\beta) = 2\Omega_1(\beta)$ i.e. for $\rho_\beta^4 = 3$. In this case, the driving and the radial mode parametrically excite the radial mode through the $1:2$ resonance between the frequencies. Thus, the condensate is destabilized even for a frequency greater than $\Omega_0$. Similar resonances account for the other detailed structure in the frequency response curves. All these features should be clearly visible experimentally, since the experimental resolution is of the order of 50 condensate atoms, and the predicted features exceed this resolution significantly. Moreover, the details of the curves are sensitive to the precise models and parameters used. Comparison of experimental results with these theoretical predictions will therefore help a) define the regime of validity of a given model and b) improve our understanding of the details of the rich interplay between kinetics and non-linear many-body quantum dynamics in the attractive condensate. Efforts are under way for experimental realization of these phenomena.

The instability and finite lifetimes of attractive condensates have been a constraint in the push to understand these systems. It is expected that the significantly increased stability through weak driving will open the way to further novel experimental phenomena and interesting theory and ultimately a deeper understanding of this nonlinear quantum regime.

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[1] M. H. Anderson et al., Science 269, 198 (1995); C. C. Bradley, C. A. Sackett, J. J. Tollet and R. G. Hulet, Phys. Rev. Lett. 75, 1687 (1995); K. B. Davis et al., ibid 75, 3969 (1995).
[2] F. Dalfovo, S. Giorgini, L. P. Pitaevskii and S. Stringari, Rev. Mod. Phys. 71, 463 (1999) and extensive references therein.
[3] See for example H. T. C. Stoof, Phys. Rev. A 49, 3824 (1994); N. Bogolubov, J. of Phys. XI, 23 (1947).
[4] P. A. Ruprecht, et al., Phys. Rev. A 51, 4704 (1995); C. C. Bradley, C. A. Sackett and R.G. Hulet, Phys. Rev. Lett. 78, 985 (1997).
[5] C. A. Sackett, H. T. C. Stoof and R. G. Hulet, Phys. Rev. Lett. 80, 2031 (1998).
[6] C. A. Sackett, J. M. Gerton, M. Welling and R. G. Hulet, Phys. Rev. Lett. 82, 876 (1999).
[7] The qualitative results have been verified to not depend on this assumption. The quantitative effects will be considered in detail elsewhere.
[8] J. M. Gerton, C. A. Sackett,B. J. Frew and R. G. Hulet, Phys. Rev. A 59, 1514 (1999).
[9] V. M. Perez-Garcia et al., Phys. Rev. Lett. 77, 5320 (1996).
[10] K. Singh and D. Rokhsar, Phys. Rev. Lett. 77, 1667 (1996).
[11] M. Edwards et al., Phys. Rev. Lett. 77, 1671 (1996).
[12] A. K. Pattanayak and W. C. Schieve, Phys. Rev. E 50, 3601 (1994).
[13] Yu. Kagan, A. E. Muryshev and G. V. Shlyapnikov, Phys. Rev. Lett. 81, 933 (1998).
[14] Note that this is not a central force problem and as such the angular momentum is not conserved.

FIG. 1. Stabilization of an attractive condensate by weak driving as measured by the change in the maximum number of atoms participating in the condensate as a function of the driving frequency. The ratio $N_m$ between the frequency-dependent maximum number of atoms $N_m$ and the maximum value $N_s$ for the static condensate is shown. The figure is truncated at $N_m \approx 4.8$ due to the excessive ($t > 100s$) amount of time taken to reach this value. An estimated value $N_{ang}$ for $N_m$ obtained from an angular momentum analysis is also shown. See text for details.

FIG. 2. (a) Time-dependence of the radially-averaged width of the condensate $\rho(t)$. There are rapid oscillations, as expected, on the time-scale of the driving (308 Hz) that cannot be easily resolved on the scale of this figure. (b) As in (a) except as computed with a numerical solution of the GP equation with added dissipation. See text for details.
Number of atoms (in units of $N_s$)

$\omega$ (in units of $\omega_0$)
(a) $\rho$ (units of $\rho_0$) vs. Time (s)

(b) $\rho$ (units of $\rho_0$) vs. Time (s)