Expansivity and Shadowing in Linear Dynamics

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Abstract

In the early 1970’s Eisenberg and Hedlund investigated relationships between expansivity and spectrum of operators on Banach spaces. In this paper we establish relationships between notions of expansivity and hypercyclicity, supercyclicity, Li-Yorke chaos and shadowing. In the case that the Banach space is $c_0$ or $\ell_p$ ($1 \leq p < \infty$), we give complete characterizations of weighted shifts which satisfy various notions of expansivity. We also establish new relationships between notions of expansivity and spectrum. Moreover, we study various notions of shadowing for operators on Banach spaces. In particular, we solve a basic problem in linear dynamics by proving the existence of nonhyperbolic invertible operators with the shadowing property. This also contrasts with the expected results for nonlinear dynamics on compact manifolds, illuminating the richness of dynamics of infinite dimensional linear operators.

1 Introduction

The study of the dynamics of continuous linear operators on infinite dimensional Banach (or Fréchet) spaces has witnessed a great development during the last three decades and many links between this area and other areas of mathematics, such as ergodic theory, number theory and geometry of Banach spaces, have been established. We refer the reader to the books [2, 19] and to the more recent papers [3, 6, 7, 8, 17, 18], where many additional references can be found.

On the other hand, the notions of expansivity and shadowing play important roles in many branches of the area of dynamical systems, including topological dynamics, differentiable dynamics and ergodic theory; see [1, 24, 25, 31], for instance.

Our goal in this paper is to investigate the notions of expansivity and shadowing in the context of linear dynamics, thereby complementing previous works by various authors. In particular, we give a class of examples of linear operators exhibiting a shadowing property which are neither hyperbolic nor expansive, however they are chaotic. These types of examples show the richness of linear dynamics and its difference from finite dimensional nonlinear dynamics, yielding counterintuitive results to the corresponding ones from finite dimensional smooth dynamics.

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Let us now discuss our results in detail and describe the organization of the article.

In Section 2, we investigate relationships between various notions of expansivity and some popular notions in linear dynamics, namely: hypercyclicity, supercyclicity and Li-Yorke chaos (Definition 7). In particular, we prove that a uniformly expansive operator cannot be Li-Yorke chaotic and hence it cannot be hypercyclic (Theorem A), but we observe that every infinite-dimensional separable Banach space supports a supercyclic uniformly expansive operator (Remark 9). On the other hand, we prove that a hyperbolic operator with nontrivial hyperbolic splitting cannot be supercyclic (Proposition 11).

In Section 3, we consider weighted shifts. Due to their importance in operator theory and its applications, the study of the dynamics of weighted shifts has received special attention from the specialists in linear dynamics. Many dynamical properties have been extensively studied and, in some cases, complete characterizations have been obtained. For instance, Salas [30] characterized hypercyclicity and weak mixing whereas Costakis and Sambarino [12] characterized mixing for unilateral and bilateral weighted shifts on \( \ell_2(\mathbb{N}) \) and \( \ell_2(\mathbb{Z}) \), respectively. We obtain here complete characterizations of various notions of expansivity for unilateral and bilateral weighted shifts on the Banach spaces \( c_0(A) \) and \( \ell_p(A) \) \( (1 \leq p < \infty) \), where \( A = \mathbb{N} \) or \( \mathbb{Z} \) (Theorem B and Propositions 14 and 15). As applications we obtain examples of hypercyclic positively expansive operators and of supercyclic uniformly positively expansive operators (Examples 18 and 22).

In Section 4, we investigate the relationship between expansivity of an operator and its spectrum. In particular, we expand earlier results of Eisenberg and Hedlund [15, 16] and Mazur [22]. In 1966 Eisenberg [15] proved that if \( T \) is an invertible operator on \( \mathbb{C}^n \), then \( T \) is expansive if and only if \( T \) has no eigenvalue on the unit circle. Subsequently, Eisenberg and Hedlund [16] studied expansive and uniformly expansive operators on Banach spaces. They showed that if \( T \) is uniformly expansive, then \( \sigma_a(T) \), the approximate point spectrum of \( T \), does not intersect \( T \). The converse was shown for invertible operators by Hedlund [20]. As a corollary, they obtained that invertible hyperbolic operators are uniformly expansive. Relations between hyperbolicity and the shadowing property for operators were studied by Ombach [23] and Mazur [22]. In [22] it was also shown that an invertible normal operator \( T \) on a Hilbert space \( H \) is expansive if and only if \( \sigma_p(T\ast T) \), the point spectrum of \( T\ast T \), does not intersect \( T \). We show that for a uniformly positively expansive operator \( T \) on a Banach space \( X \), \( \sigma_a(T) \) does not intersect the closed unit disc \( \mathbb{D} \), and the converse holds if \( T \) is invertible (Theorem C). Moreover, we expand Mazur’s result by giving a necessary and sufficient condition for a normal operator to be positively expansive (Proposition 27). Our techniques also yield a simpler proof of his result (Theorem 23).

In Section 5, we investigate the notions of shadowing, limit shadowing and \( \ell_p \) shadowing for invertible operators on Banach spaces. It is well-known that invertible hyperbolic operators have the shadowing property and that the converse holds for invertible operators on finite dimensional euclidean spaces [24] and for invertible normal operators on Hilbert spaces [22]. Moreover, the converse also holds for certain sequences of finite dimensional operators considered in [20]. This implies that for \( C^1 \) diffeomorphisms of \( m \)-dimensional closed smooth manifolds, hyperbolicity is equivalent to expansivity plus Lipschitz shadowing [26]. A basic question in linear dynamics is whether the shadowing property implies hyperbolicity for invertible operators on Banach (or Hilbert) spaces. This question appeared explicitly in [22, Page 148], for instance. In Theorem D, we answer this question in the negative by proving the existence of operators with the shadowing property that exhibit several types of chaotic behaviors (they are simultaneously frequently hypercyclic, Devaney chaotic, mixing and densely distributionally chaotic) and, in particular, are not
even expansive. Moreover, such type of examples are robust inside the weighted shifted class, meaning that the same properties are share by the perturbed ones.

We also establish a generalization of the aforementioned result from \[22\] (Theorem 41) and prove that every expansive operator with the shadowing property is uniformly expansive (Proposition 43).

In the final Section 6, we investigate analogous results for operators which are not necessarily invertible. In Theorem E, we show that hyperbolic operators always have the positive shadowing property for all \(1 \leq p < \infty\). The converse is not true in general (Remark 44). We also show that all these notions are equivalent for compact operators (Theorem 46) and that positive shadowing and hyperbolicity coincide for normal operators (Theorem 47).

## 2 Expansive behavior of operators

As usual, \(\mathbb{N}\) denotes the set of all positive integers and \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\). Given a (real or complex) Banach space \(X\), \(S_X\) denotes the unit sphere of \(X\), i.e., \(S_X = \{x \in X : \|x\| = 1\}\). Moreover, by an operator on \(X\) we mean a bounded linear map \(T\) from \(X\) into \(X\). The spectrum of an operator \(T\) on a complex Banach space \(X\) is the set

\[
\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}.
\]

It is well-known that \(\sigma(T)\) is a nonempty compact subset of \(\mathbb{C}\). In the case \(T\) is an operator on a real Banach space \(X\), we define the spectrum of \(T\) as the spectrum of its complexification \(T_\mathbb{C}\), that is,

\[
\sigma(T) = \sigma(T_\mathbb{C}).
\]

It is well-known that the spectrum can be divided into three disjoint sets: the point spectrum \(\sigma_p(T)\), the continuous spectrum \(\sigma_c(T)\) and the residual spectrum \(\sigma_r(T)\). Recall that \(\lambda\) belongs to \(\sigma_p(T)\) if \(T - \lambda I\) is not one-to-one. If \(T - \lambda I\) is one-to-one but not onto, then \(\lambda \in \sigma_c(T)\) if \((T - \lambda I)(X)\) is dense in \(X\) and \(\lambda \in \sigma_r(T)\) otherwise. We also recall that the approximate point spectrum of \(T\), denoted by \(\sigma_a(T)\), is the set of all \(\lambda \in \mathbb{C}\) for which there is a sequence \((x_n)\) in \(S_X\) with \(\|\lambda x_n - Tx_n\| \to 0\) as \(n \to \infty\). It is classical (see [33]) that

\[
\partial \sigma(T) \subset \sigma_a(T) \subset \sigma(T), \quad \text{\(\sigma(T) = \sigma_c(T) \cup \sigma_a(T)\)} \quad \text{and} \quad \text{\(\sigma_r(T) \subset \sigma_p(T^*)\)},
\]

(1)

where \(\partial \sigma(T)\) is the boundary of \(\sigma(T)\) and \(T^*\) is the adjoint of \(T\) acting on the dual space \(X^*\) of \(X\).

**Definition 1.** An invertible operator \(T\) on a Banach space \(X\) is said to be expansive (positively expansive) if for every \(z \in S_X\), there exists \(n \in \mathbb{Z}\) \((n \in \mathbb{N})\) such that \(\|T^n z\| \geq 2\).

**Definition 2.** An invertible operator \(T\) on a Banach space \(X\) is said to be uniformly expansive (uniformly positively expansive) if there exists \(n \in \mathbb{N}\) such that

\[
z \in S_X \implies \|T^n z\| \geq 2 \text{ or } \|T^{-n} z\| \geq 2 \quad (z \in S_X \implies \|T^n z\| \geq 2).
\]

We remark that for the definitions of positive expansivity and uniform positive expansivity, \(T\) need not be invertible. Also, there is nothing special about the number 2 in the above definitions. One can replace 2 by any number \(e > 1\). Moreover, the above definition of expansivity agrees with the usual definition of expansivity in metric spaces, since it is equivalent to the existence of a constant \(e > 0\) such that, for any pair \(x, y\) of distinct points in \(X\), there exists \(n \in \mathbb{Z}\) with \(\|T^n x - T^n y\| \geq e\).
Remark 3. In the case $T$ is an operator on a real Banach space $X$, the (uniform) (positive) expansivity of $T$ is equivalent to the corresponding property for its complexification $T_{\mathbb{C}}$.

**Definition 4.** An operator $T$ on a Banach space $X$ is said to be hyperbolic if

$$\sigma(T) \cap \mathbb{T} = \emptyset,$$

where $\mathbb{T}$ denotes the unit circle in the complex plane $\mathbb{C}$.

It is classical that $T$ is hyperbolic if and only if there are an equivalent norm $\| \cdot \|$ on $X$ and a splitting

$$X = X_s \oplus X_u, \quad T = T_s \oplus T_u$$

(the hyperbolic splitting of $T$), where $X_s$ and $X_u$ are closed $T$-invariant subspaces of $X$ (the stable and the unstable subspaces for $T$, respectively), $T_s = T|_{X_s}$ is a proper contraction (i.e., $\|T_s\| < 1$), $T_u = T|_{X_u}$ is invertible and is a proper dilation (i.e., $\|T_u^{-1}\| < 1$), and the identification of $X$ with the product $X_s \times X_u$ identifies $\| \cdot \|$ with the max norm on the product.

It is also known [20] that $T$ is uniformly expansive if and only if $\sigma_a(T) \cap \mathbb{T} = \emptyset$.

Hence, every invertible hyperbolic operator is uniformly expansive.

The next result gives simple characterizations of the notions of expansivity by means of the behaviors of orbits.

**Proposition 5.** Let $T$ be an operator on a Banach space $X$. Then:

(a) $T$ is positively expansive $\iff \sup_{n \in \mathbb{N}} \|T^n x\| = \infty$ for every nonzero $x \in X$.

(b) $T$ is uniformly positively expansive $\iff \lim_{n \to \infty} \|T^n x\| = \infty$ uniformly on $S_X$.

If, in addition, $T$ is invertible, then:

(c) $T$ is expansive $\iff \sup_{n \in \mathbb{Z}} \|T^n x\| = \infty$ for every nonzero $x \in X$.

(d) $T$ is uniformly expansive $\iff S_X = A \cup B$ where $\lim_{n \to \infty} \|T^n x\| = \infty$ uniformly on $A$ and $\lim_{n \to \infty} \|T^{-n} x\| = \infty$ uniformly on $B$.

**Proof.** (a) and (c) follow easily from the fact, already mentioned, that the constant 2 that appears in Definitions 1 and 2 can be replaced by any constant $c > 1$.

Let us prove (d). Since the sufficiency of the condition is clear, we have only to prove its necessity. Suppose that $T$ is uniformly expansive and let $n \in \mathbb{N}$ be as in Definition 2. Let

$$A = \{x \in S_X : \|T^n x\| \geq 2\} \quad \text{and} \quad B = \{x \in S_X : \|T^{-n} x\| \geq 2\}.$$

Then, $S_X = A \cup B$. We claim that

$$\frac{T^n x}{\|T^n x\|} \in A \quad \text{whenever} \quad x \in A.$$

Indeed, if $x \in A$ and $y = \frac{T^n x}{\|T^n x\|} \notin A$, then $y \in B$ and so $\|T^{-n} y\| \geq 2$, implying that $\|x\| \geq 2\|T^n x\| \geq 4$, a contradiction. Hence, given $x \in A$, we can define inductively a
sequence \( (x_k)_{k \in \mathbb{N}} \) in \( A \) by putting \( x_1 = x \) and \( x_k = \frac{T^{n}x_{k-1}}{\|T^{n}x_{k-1}\|} \) for \( k \geq 2 \). It follows from the definition that
\[
x_k = \frac{T^{(k-1)n}x}{\|T^n x_1\| \cdots \|T^n x_{k-1}\|} \quad \text{for all } k \in \mathbb{N}.
\]
Since \( \|T^n x_k\| \geq 2 \) for all \( k \in \mathbb{N} \), we obtain
\[
\|T^{kn}x\| \geq \frac{2^{k_m}}{C} \quad \text{for all } k \in \mathbb{N}.
\]

**Remark 6.** The sets \( A \) and \( B \) in Proposition 5(d) can be chosen to be disjoint or to be both closed in \( S_X \) or to be both open in \( S_X \).

We shall now show that uniformly (positively) expansive operators do not exhibit chaotic behavior. First, let us recall a few definitions.

**Definition 7.** An operator \( T \) on a Banach space \( X \) is said to be Li-Yorke chaotic if it has an uncountable scrambled set \( U \), i.e., for all \( x, y \in U \) with \( x \neq y \), we have that
\[
\liminf_{n \to \infty} \|T^n x - T^n y\| = 0 \quad \text{and} \quad \limsup_{n \to \infty} \|T^n x - T^n y\| > 0.
\]
We say that \( T \) is hypercyclic if it has a dense orbit, i.e.,
\[
\{T^n x : n \geq 0\}
\]
is dense in \( X \) for some \( x \in X \). Finally, \( T \) is supercyclic if there exists \( x \in X \) whose projective orbit
\[
\{\lambda T^n x : n \geq 0, \lambda \text{ scalar}\}
\]
is dense in \( X \).

**Theorem A.** A uniformly (positively) expansive operator on a Banach space cannot be Li-Yorke chaotic. In particular, it cannot be hypercyclic.

**Proof.** Let us consider the case of a uniformly expansive (necessarily invertible) operator \( T \) on a Banach space \( X \) (the case of a uniformly positively expansive (not necessarily invertible) operator is simpler). Write \( S_X = A \cup B \) as in Proposition 5(d). It was proved in [5, Theorem 5] that \( T \) is Li-Yorke chaotic if and only if \( T \) admits an irregular vector, that is, a vector \( x \in X \) such that
\[
\inf_{n \in \mathbb{N}} \|T^n x\| = 0 \quad \text{and} \quad \sup_{n \in \mathbb{N}} \|T^n x\| = \infty.
\]
Suppose that $T$ is Li-Yorke chaotic and let $y \in S_X$ be an irregular vector for $T$. We must have
\[ \frac{T^k y}{\|T^k y\|} \in B \quad \text{for all } k \in \mathbb{N}. \]
Indeed, if $\frac{T^k y}{\|T^k y\|} \in A$ for some $k \in \mathbb{N}$, then $\lim_{n \to \infty} \|T^n \left( \frac{T^k y}{\|T^k y\|} \right)\| = \infty$, which implies that $\lim_{n \to \infty} \|T^n y\| = \infty$ and contradicts the fact that $y$ is an irregular vector for $T$. Since $\lim_{n \to \infty} \|T^{-n} x\| = \infty$ uniformly on $B$, there exists $n_0 \in \mathbb{N}$ such that
\[ \|T^{-n} x\| \geq 2 \quad \text{whenever } x \in B \text{ and } n \geq n_0. \] (2)

Since $y$ is an irregular vector for $T$, we can choose $k_0 \geq n_0$ such that $\|T^{k_0} y\| \geq 1$. Now, by choosing $n = k_0 \geq n_0$ and $x = \frac{T^{k_0} y}{\|T^{k_0} y\|} \in B$ in (2), we obtain $\|y\| \geq 2\|T^{k_0} y\| \geq 2$. This contradiction proves the theorem.

Remark 8. The fact that a uniformly expansive operator $T$ on a Banach space $X$ cannot be hypercyclic can be seen by means of a spectral argument. Indeed, if $T$ is hypercyclic, then its spectrum $\sigma(T)$ intersects the unit circle and the point spectrum $\sigma_p(T^*)$ of the adjoint operator $T^*$ is empty (see [2]). Since $\sigma_r(T) \subset \sigma_p(T^*)$, we deduce that $\sigma_r(T)$ is empty. Since $\sigma(T) = \sigma_r(T) \cup \sigma_a(T)$, we have that $\sigma_a(T)$ intersects the unit circle and thus $T$ is not uniformly expansive.

On the other hand, there exist supercyclic uniformly (positively) expansive operators, as we shall see in the remark below. A simple concrete example of such an operator on the Hilbert space $\ell_2$ will be given in Example 22.

Remark 9. Every infinite-dimensional separable Banach space admits an invertible operator which is uniformly positively expansive and supercyclic.

Indeed, it is well-known that every infinite-dimensional separable Banach space $X$ supports a hypercyclic invertible operator $S$ (see [19, Section 8.2]). Since any nonzero scalar multiple of a supercyclic operator is a supercyclic operator,
\[ T = 2\|S^{-1}\|S \]
is a supercyclic operator on $X$. Moreover, since $\|T^{-1}\| = \frac{1}{2} < 1$, $T$ is a proper dilation. In particular, $T$ is uniformly positively expansive.

Remark 10. A positively expansive operator can be Li-Yorke chaotic. For example, Beausamy [1] and Prâjiurâ [28] constructed examples of completely irregular operators on the Hilbert space $\ell_2$. These are operators with the property that every nonzero vector is irregular. It follows from Proposition 5(a) and [7, Theorem 34] that every completely irregular operator on a Banach space is simultaneously positively expansive and generically Li-Yorke chaotic. Also, Read [22] constructed an operator $T$ on $\ell_1$ with all nonzero vectors hypercyclic. This operator is simultaneously positively expansive, generically Li-Yorke chaotic and hypercyclic. Moreover, we shall see later an example of an invertible operator on the Hilbert space $\ell_2$ which is positively expansive (hence expansive) and hypercyclic (Example 18).

As we saw in Remark 9, a uniformly expansive operator can be supercyclic. However, this is not the case for hyperbolic operators with nontrivial hyperbolic splittings.
Proposition 11. If $T$ is a hyperbolic operator with nontrivial hyperbolic splitting, then $T$ is not supercyclic.

Proof. By hypothesis, there is a splitting

$$X = X_s \oplus X_u, \quad T = T_s \oplus T_u,$$

as above, with $X_s \neq \{0\}$ and $X_u \neq \{0\}$. By renorming $X$ we may assume that $\|T_s\| < 1$ and $\|T_u^{-1}\| < 1$. Each vector $x \in X$ has a unique decomposition $x = x_s + x_u$ with $x_s \in X_s$ and $x_u \in X_u$. Moreover, by the open mapping theorem, the mapping $x \in X \mapsto (x_s, x_u) \in X_s \times X_u$ is an isomorphism. Suppose that $T$ admits a supercyclic vector $y \in X$. It must be true that $y_s \neq 0$ and $y_u \neq 0$. Since $\|T^n y_s\| \to 0$ and $\|T^n y_u\| \to \infty$, there exists $n_0 \in \mathbb{N}$ such that

$$\|T^n y_u\| \geq 2 \|T^n y_s\| \quad \text{whenever } n \geq n_0.$$

On one hand, the set

$$D = \{ \lambda T^n y : \lambda \text{ is a scalar and } n \geq n_0 \}$$

is dense in $X$. But on the other hand, each element $z = \lambda T^n y \in D$ has decomposition $z = z_s + z_u = \lambda T^n y_s + \lambda T^n y_u$ satisfying $\|z_u\| \geq 2 \|z_s\|$, and so $D$ cannot be dense in $X$. This contradiction proves the proposition.

Remark 12. Another way to see the last result in the case of complex scalars comes from the fact that if $T$ is supercyclic, then there exists $R \geq 0$ such that each connected component of the spectrum of $T$ intersects the (possibly degenerate) circle $\{ z \in \mathbb{C} : |z| = R \}$ (see [2]). This is impossible if $T$ is a hyperbolic operator with nontrivial hyperbolic splitting, since the unit circle separates at least two connected components of $\sigma(T)$.

3 Expansive weighted shifts

In this section we characterize the various notions of expansivity for weighted shifts by looking at their weights.

For each real number $p \in [1, \infty)$, we denote by $\ell_p(\mathbb{Z})$ the Banach space of all sequences $x = (x_n)_{n \in \mathbb{Z}}$ of scalars such that $\sum_{n \in \mathbb{Z}} |x_n|^p < \infty$, endowed with the norm

$$\|x\| = \left( \sum_{n \in \mathbb{Z}} |x_n|^p \right)^{\frac{1}{p}}.$$

In particular, $\ell_2(\mathbb{Z})$ is a Hilbert space with respect to the inner product

$$\langle x, y \rangle = \sum_{n \in \mathbb{Z}} x_n \overline{y_n}.$$

Moreover, $c_0(\mathbb{Z})$ denotes the Banach space of all sequences $x = (x_n)_{n \in \mathbb{Z}}$ of scalars such that $\lim_{n \to \pm \infty} x_n = 0$, endowed with the norm

$$\|x\| = \sup_{n \in \mathbb{Z}} |x_n|.$$

The Banach spaces $\ell_p(\mathbb{N})$ ($1 \leq p < \infty$) and $c_0(\mathbb{N})$ are defined analogously.
If \( X = \ell_p(\mathbb{Z}) \) \((1 \leq p < \infty)\) or \( X = c_0(\mathbb{Z}) \), then \( F_w : X \to X \) \((B_w : X \to X)\) denotes the bilateral weighted forward (backward) shift on \( X \) given by

\[
F_w((x_n)_{n \in \mathbb{Z}}) = (w_{n-1}x_{n-1})_{n \in \mathbb{Z}} \quad (B_w((x_n)_{n \in \mathbb{Z}}) = (w_{n+1}x_{n+1})_{n \in \mathbb{Z}}),
\]

where \( w = (w_n)_{n \in \mathbb{Z}} \) is a bounded sequence of scalars, called a weight sequence. Recall that

\[
F_w(B_w) \text{ is invertible } \iff \inf_{n \in \mathbb{Z}} |w_n| > 0.
\]

In the case \( X = \ell_p(\mathbb{N}) \) \((1 \leq p < \infty)\) or \( X = c_0(\mathbb{N}) \), we also denote by \( F_w : X \to X \) \((B_w : X \to X)\) the unilateral weighted forward (backward) shift on \( X \) with weight sequence \( w = (w_n)_{n \in \mathbb{N}} \), which is defined by

\[
F_w((x_1, x_2, \ldots)) = (0, w_1x_1, w_2x_2, \ldots) \quad (B_w((x_1, x_2, \ldots)) = (w_2x_2, w_3x_3, \ldots)).
\]

We remark that in this case the weight sequence \( w \) is also assumed to be bounded.

We begin by characterizing (uniform) expansivity for invertible bilateral weighted forward shifts. For this purpose, we will need the following fact.

**Lemma 13.** If \( \{I, J\} \) is a nontrivial partition of \( \mathbb{Z} \) (that is, \( I \cup J = \mathbb{Z}, I \cap J = \emptyset, I \neq \emptyset \) and \( J \neq \emptyset \)), \( \varphi : \mathbb{Z} \to [0, \infty) \) is a map,

\[
\lim_{n \to \infty} \left[ \inf_{k \in I} \left( \varphi(k) \ldots \varphi(k + n - 1) \right) \right] > 1
\]

and

\[
\lim_{n \to \infty} \left[ \sup_{k \in J} \left( \varphi(k - n) \ldots \varphi(k - 1) \right) \right] < 1,
\]

then there exist \( i, j \in \mathbb{Z} \) such that

\[
(-\infty, j] \cap \mathbb{Z} \subset J \quad \text{and} \quad [i, \infty) \cap \mathbb{Z} \subset I.
\]

**Proof.** By hypothesis, there exists \( n_0 \in \mathbb{N} \) such that

\[
\varphi(k) \ldots \varphi(k + n - 1) > 1 \quad \text{for all } k \in I
\]

and

\[
\varphi(k - n) \ldots \varphi(k - 1) < 1 \quad \text{for all } k \in J,
\]

whenever \( n \geq n_0 \). We claim that

\[
k \in I \Rightarrow k + n \in I \quad \text{for all } n \geq n_0. \tag{3}
\]

Indeed, suppose that \( k \in I \) but \( k + n \in J \) for a certain \( n \geq n_0 \). Then,

\[
\varphi(k) \ldots \varphi(k + n - 1) = \varphi((k + n) - n) \ldots \varphi((k + n) - 1)
\]

is simultaneously > 1 and < 1, because \( k \in I, k + n \in J \) and \( n \geq n_0 \). This contradiction proves \( \text{[3]} \). Analogously, we have that

\[
k \in J \Rightarrow k - n \in J \quad \text{for all } n \geq n_0. \tag{4}
\]

Since \( I \neq \emptyset \) and \( J \neq \emptyset \), it is clear that \( \text{[3]} \) and \( \text{[4]} \) imply the existence of \( i, j \in \mathbb{Z} \) with the desired properties. \( \square \)
Theorem B. Let $X = \ell_p(\mathbb{Z})$ $(1 \leq p < \infty)$ or $X = c_0(\mathbb{Z})$, and consider a weight sequence $w = (w_n)_{n \in \mathbb{Z}}$ with $\inf_{n \in \mathbb{Z}} |w_n| > 0$.

(a) The following assertions are equivalent:

(i) $F_w : X \to X$ is expansive;

(ii) $F_w : X \to X$ or $F_w^{-1} : X \to X$ is positively expansive;

(iii) $\sup_{n \in \mathbb{N}} |w_1 \cdot \ldots \cdot w_n| = \infty$ or $\sup_{n \in \mathbb{N}} |w_{-n} \cdot \ldots \cdot w_{-1}|^{-1} = \infty$.

(b) The following assertions are equivalent:

(i) $F_w : X \to X$ is uniformly expansive;

(ii) One of the following conditions holds:

- $\lim_{n \to \infty} \left( \inf_{k \in \mathbb{Z}} |w_k \cdot \ldots \cdot w_{k+n-1}| \right) = \infty$, or
- $\lim_{n \to \infty} \left( \inf_{k \in \mathbb{Z}} |w_{k-n} \cdot \ldots \cdot w_{k-1}|^{-1} \right) = \infty$, or
- $\lim_{n \to \infty} \left( \inf_{k \in \mathbb{Z}} |w_k \cdot \ldots \cdot w_{k+n+1}| \right) = \infty$ and $\lim_{n \to \infty} \left( \inf_{k \in \mathbb{Z}} |w_{k-n} \cdot \ldots \cdot w_{k-1}|^{-1} \right) = \infty$.

Proof. Let $e_j$, $j \in \mathbb{Z}$, denote the canonical unit vectors in $X$.

(a): If $F_w$ is expansive, then Proposition 5(c) implies that

$$ \sup_{n \in \mathbb{N}} \|F_w^n(e_1)\| = \infty \quad \text{or} \quad \sup_{n \in \mathbb{N}} \|F_w^{-n}(e_1)\| = \infty. $$

The first equality means that $\sup_{n \in \mathbb{N}} |w_1 \cdot \ldots \cdot w_n| = \infty$, whereas the second one means that $\sup_{n \in \mathbb{N}} |w_{-n} \cdot \ldots \cdot w_0|^{-1} = \infty$, which is clearly equivalent to $\sup_{n \in \mathbb{N}} |w_{-n} \cdot \ldots \cdot w_{-1}|^{-1} = \infty$. This shows that (i) implies (iii). Now, assume that $\sup_{n \in \mathbb{N}} |w_1 \cdot \ldots \cdot w_n| = \infty$. Let $x = (x_j)_{j \in \mathbb{Z}}$ be any nonzero vector in $X$ and choose $k \in \mathbb{Z}$ such that $x_k \neq 0$. Then,

$$ \sup_{n \in \mathbb{N}} \|F_w^n(x)\| \geq \sup_{n \in \mathbb{N}} \left( |w_k \cdot \ldots \cdot w_{k+n-1}| x_k \right) = \frac{|x_k| \prod_{j=k}^{0} |w_j|}{\prod_{j=1}^{k-1} |w_j|} \sup_{n \in \mathbb{N}} |w_{1} \cdot \ldots \cdot w_{k+n-1}| = \infty, $$

where a product over an empty set of indices has value 1, by definition. Hence, by Proposition 5(a), $F_w$ is positively expansive. Analogously, the relation $\sup_{n \in \mathbb{N}} |w_{-n} \cdot \ldots \cdot w_{-1}|^{-1} = \infty$ implies that $F_w^{-1}$ is positively expansive. Thus, (iii) implies (ii). Finally, it is trivial that (ii) implies (i).

(b): Suppose that $F_w$ is uniformly expansive. By Proposition 5(d), there is a partition $\{A, B\}$ of $S_X$ such that

$$ \lim_{n \to \infty} c_n = \infty \quad \text{and} \quad \lim_{n \to \infty} d_n = \infty, $$

where

$$ c_n = \inf_{x \in A} \|F_w^n(x)\| \quad \text{and} \quad d_n = \inf_{x \in B} \|F_w^{-n}(x)\| \quad (n \in \mathbb{N}). $$

We remark that an infimum over an empty set of indices has value $\infty$, by definition. Let

$$ I = \{k \in \mathbb{Z} : e_k \in A\} \quad \text{and} \quad J = \{k \in \mathbb{Z} : e_k \in B\}. $$
Then \( \{I, J\} \) is a partition of \( \mathbb{Z} \). Since, for all \( n \in N \),
\[
\inf_{k \in I} |w_k \cdots w_{k+n-1}| = \inf_{k \in I} \|F_w^n(e_k)\| \geq c_n
\]
and
\[
\inf_{k \in J} |w_{k-n} \cdots w_{k-1}|^{-1} = \inf_{k \in J} \|F_w^{-n}(e_k)\| \geq d_n,
\]
we conclude that
\[
\lim_{n \to \infty} \left( \inf_{k \in I} |w_k \cdots w_{k+n-1}| \right) = \infty \quad \text{and} \quad \lim_{n \to \infty} \left( \inf_{k \in J} |w_{k-n} \cdots w_{k-1}|^{-1} \right) = \infty. \tag{5}
\]
Thus, \( J = \emptyset \) gives the first possibility in (ii) while \( I = \emptyset \) gives the second one. Assume that \( I \neq \emptyset \) and \( J \neq \emptyset \). By Lemma 13 there exist \( i, j \in \mathbb{Z} \) such that
\[
(-\infty, j] \cap \mathbb{Z} \subset J \quad \text{and} \quad [i, \infty) \cap \mathbb{Z} \subset I. \tag{6}
\]
Since \( w_k \neq 0 \) for all \( k \in \mathbb{Z} \), it is easy to see that (5) and (6) imply the third possibility in (ii).

Conversely, assume that (ii) holds. Let \( I = \mathbb{Z} \) and \( J = \emptyset \), or \( I = \emptyset \) and \( J = \mathbb{Z} \), or \( I = \mathbb{N} \) and \( J = -\mathbb{N}_0 \), depending on whether the first, the second, or the third possibility in (ii) holds, respectively. Then, in any case, (5) holds. Let \( n \in N \) be such that
\[
\inf_{k \in I} |w_k \cdots w_{k+n-1}| \geq 4 \quad \text{and} \quad \inf_{k \in J} |w_{k-n} \cdots w_{k-1}|^{-1} \geq 4.
\]
Given \( x = (x_k)_{k \in \mathbb{Z}} \in S_X \), we can write \( x = a + b \) where \( a = (a_k)_{k \in \mathbb{Z}} \) and \( b = (b_k)_{k \in \mathbb{Z}} \) satisfy \( a_k = 0 \) whenever \( k \in J \) and \( b_k = 0 \) whenever \( k \in I \). Since \( 1 = \|x\| \leq \|a\| + \|b\| \), we have that \( \|a\| \geq \frac{1}{2} \) or \( \|b\| \geq \frac{1}{2} \). If \( \|a\| \geq \frac{1}{2} \) then
\[
\|F_w^n(x)\| \geq \|F_w^n(a)\| = \|(w_k \cdots w_{k+n-1})a_k\|_{k \in \mathbb{Z}} \geq 4\|a\| \geq 2,
\]
and if \( \|b\| \geq \frac{1}{2} \) then
\[
\|F_w^{-n}(x)\| \geq \|F_w^{-n}(b)\| = \|(w_{k-n} \cdots w_{k-1})^{-1}b_k\|_{k \in \mathbb{Z}} \geq 4\|b\| \geq 2.
\]
Hence, by definition, \( F_w \) is uniformly expansive. \( \square \)

By using analogous (but simpler) arguments, we can establish the following characterizations of (uniform) positive expansivity for weighted forward shifts.

**Proposition 14.** Let \( A = N \) or \( A = \mathbb{Z} \), let \( X = \ell_p(A) \) (1 \( \leq p < \infty \)) or \( X = c_0(A) \), and consider a weight sequence \( w = (w_n)_{n \in A} \).

(a) The following assertions are equivalent:

(i) \( F_w : X \to X \) is positively expansive;

(ii) \( \sup_{n \in N} |w_1 \cdots w_n| = \infty \) and \( w_j \neq 0 \) for all \( j \in A \).

(b) The following assertions are equivalent:

(i) \( F_w : X \to X \) is uniformly positively expansive;

(ii) \( \sup_{n \in N} \left( \inf_{k \in A} |w_k \cdots w_{k+n-1}| \right) = \infty \);
(iii) \( \lim_{n \to \infty} \left( \inf_{k \in A} |w_k \cdot \ldots \cdot w_{k+n-1}| \right) = \infty. \)

It is clear that a unilateral weighted backward shift cannot be positively expansive, but for bilateral weighted backward shifts we have the following characterizations.

**Proposition 15.** Let \( X = \ell_p(\mathbb{Z}) \) (\( 1 \leq p < \infty \)) or \( X = c_0(\mathbb{Z}) \), and consider a weight sequence \( w = (w_n)_{n \in \mathbb{Z}} \).

(a) The following assertions are equivalent:

(i) \( B_w : X \to X \) is positively expansive;

(ii) \( \sup_{n \in \mathbb{N}} |w_{-n} \cdot \ldots \cdot w_{-1}| = \infty \) and \( w_j \neq 0 \) for all \( j \geq 0 \).

(b) The following assertions are equivalent:

(i) \( B_w : X \to X \) is uniformly positively expansive;

(ii) \( \sup_{n \in \mathbb{N}} \left( \inf_{k \in \mathbb{Z}} |w_{k-n+1} \cdot \ldots \cdot w_k| \right) = \infty; \)

(iii) \( \lim_{n \to \infty} \left( \inf_{k \in \mathbb{Z}} |w_{k-n+1} \cdot \ldots \cdot w_k| \right) = \infty. \)

**Remark 16.** If \( T \) is an invertible operator on a Banach space \( X \), it is clear that

\[ T \text{ or } T^{-1} \text{ positively expansive } \Rightarrow T \text{ expansive.} \]

We saw in Theorem B(a) that the converse holds for the operators \( F_w \) on the spaces \( \ell_p(\mathbb{Z}) \) (\( 1 \leq p < \infty \)) or \( c_0(\mathbb{Z}) \). Of course, the converse is not true in general. For instance, if \( T \) is any invertible hyperbolic operator with nontrivial hyperbolic splitting, then \( T \) is uniformly expansive, but neither \( T \) nor \( T^{-1} \) is positively expansive.

**Remark 17.** (a) It follows from the equivalence (ii) \( \iff \) (iii) in Proposition 14(b) that all limits in Theorem B(b)(ii) can be replaced by the supremum over \( n \in \mathbb{N} \).

(b) The first possibility in Theorem B(b)(ii) means that \( F_w \) is uniformly positively expansive (by Proposition 14(b)), whereas the second one means that \( F_w^{-1} \) is uniformly positively expansive (by Proposition 15(b)). However, the third possibility can indeed happen, as can be seen by choosing \( w = (\ldots, \frac{1}{2}, \frac{1}{2}, 2, 2, 2, \ldots) \). This shows that \( F_w \) can be uniformly expansive without \( F_w \) or \( F_w^{-1} \) being uniformly positively expansive, in contrast to what happens in the case of expansivity (see Theorem B(a)).

Let us now see an example of an invertible operator on the Hilbert space \( \ell_2(\mathbb{Z}) \) which is positively expansive and hypercyclic.

**Example 18.** Fix a real number \( t > 1 \) and consider the weight sequence \( w = (w_n)_{n \in \mathbb{Z}} \) given by

\[ w_n = t \quad \text{for all } n \geq 0 \]

and

\[ (w_{-1}, w_{-2}, w_{-3}, \ldots) = (t, t, t, t, t, t, t, \ldots), \]

where the successive blocks of \( t \)'s and \( \frac{1}{t} \)'s have lengths \( 2^0, 2^1, 2^2, \ldots \). Let

\[ m_k = 2^0 + 2^1 + \cdots + 2^{2k-1} \quad \text{and} \quad n_k = 2^0 + 2^1 + \cdots + 2^{2k} \quad (k \in \mathbb{N}). \]
A simple induction argument shows that
\[ w_{-m_k} \cdot \ldots \cdot w_1 \leq \frac{1}{t_k} \quad \text{and} \quad w_{-n_k} \cdot \ldots \cdot w_{-1} \geq t_k \quad \text{for all } k \in \mathbb{N}. \]
In particular, \( \sup_{n \in \mathbb{N}}(w_{-n} \cdot \ldots \cdot w_1) = \infty. \) Hence, by Proposition 15(a), the bilateral weighted backward shift
\[ B_w : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z}) \]
is positively expansive. Since \( \inf_{n \in \mathbb{Z}} w_n > 0, B_w \) is invertible. Hence, \( B_w \) is also expansive. By [2, Corollary 1.39], \( B_w \) is hypercyclic if and only if, for any \( q \in \mathbb{N}, \)
\[ \lim \inf \max \left\{ (w_1 \cdot \ldots \cdot w_{n+q})^{-1}, (w_0 \cdot \ldots \cdot w_{-n+q+1}) \right\} = 0. \]
But this condition follows from the fact that
\[ \max \left\{ (w_1 \cdot \ldots \cdot w_{(m+q+1)+q}^{-1}, (w_0 \cdot \ldots \cdot w_{-(m+q+1)+q+1}) \right\} \leq \frac{1}{t_{k-1}} \quad \text{for all } k \in \mathbb{N}. \]
Thus, the operator \( B_w \) is also hypercyclic.

**Remark 19.** (a) Let \( X = \ell_p(\mathbb{Z}) (1 \leq p < \infty) \) or \( X = c_0(\mathbb{Z}), \) and consider a weight sequence \( w = (w_n)_{n \in \mathbb{Z}} \) with \( \inf_{n \in \mathbb{Z}} |w_n| > 0. \) It is known (see [10]) that the spectrum of the invertible bilateral weighted forward shift \( F_w : X \to X \) is the annulus \( \{ \lambda \in \mathbb{C} : \frac{1}{r(F_w^{-1})} \leq |\lambda| \leq r(F_w) \}, \) where \( r(S) = \lim_{n \to \infty} \|S^n\|^\frac{1}{n} \) denotes the spectral radius of the operator \( S : X \to X. \) Since
\[ \|F_w^n\| = \sup_{k \in \mathbb{Z}} |w_k \cdot \ldots \cdot w_{k+n-1}| \quad \text{and} \quad \|F_w^{-n}\| = \sup_{k \in \mathbb{Z}} |w_k \cdot \ldots \cdot w_{k+n-1}|^{-1}, \]
we deduce that the following assertions are equivalent:

(i) \( F_w \) is hyperbolic;
(ii) \( \sigma(F_w) \subset \mathbb{D} \) or \( \sigma(F_w^{-1}) \subset \mathbb{D}; \)
(iii) \( \lim \sup_{n \to \infty} |w_k \cdot \ldots \cdot w_{k+n-1}|^{\frac{1}{n}} < 1 \) or \( \lim \sup_{n \to \infty} |w_k \cdot \ldots \cdot w_{k+n-1}|^{-\frac{1}{n}} < 1. \)

(b) Let \( A = \mathbb{N} \) or \( A = \mathbb{Z}, \) let \( X = \ell_p(A) (1 \leq p < \infty) \) or \( X = c_0(A), \) and consider a weight sequence \( w = (w_n)_{n \in A}. \) Let \( T \) be either the weighted forward shift \( F_w : X \to X \) or the weighted backward shift \( B_w : X \to X. \) Assume that \( T \) is not invertible (this is automatically the case if \( A = \mathbb{N} \)). Since \( \sigma(T) \) is equal to the disc \( \{ \lambda \in \mathbb{C} : |\lambda| \leq r(T) \} \) (see [10]), we deduce that the following assertions are equivalent:

(i) \( T \) is hyperbolic;
(ii) \( \sigma(T) \subset \mathbb{D}; \)
(iii) \( \lim \sup_{n \to \infty} |w_k \cdot \ldots \cdot w_{k+n-1}|^{\frac{1}{n}} < 1. \)

**Remark 20.** It follows immediately from the definitions that an invertible operator \( T \) is expansive (uniformly expansive, hyperbolic) if and only if so is its inverse operator \( T^{-1}. \) Hence, the study of these notions for invertible bilateral weighted backward shifts can be reduced to the corresponding case of forward shifts (see Theorem B and Remark 15(a)).
Remark 21. (a) As mentioned before, it was proved in [16] that every invertible hyperbolic operator is uniformly expansive. Examples of uniformly expansive nonhyperbolic operators were also obtained in [16]. We observe that such examples can be easily obtained by using the characterizations given in Theorem B(b) and Remark 19(a).

(b) In the case of noninvertible operators, we observe that there is no relation between hyperbolicity and uniform positive expansivity in general. For instance, it follows from Proposition 14(a) and Remark 19(b) that in the class of unilateral weighted forward shifts on \( \ell_p(\mathbb{N}) \) \((1 \leq p < \infty)\) or on \(c_0(\mathbb{N})\), the set of hyperbolic shifts is disjoint from the set of positively expansive shifts.

Let us now see a concrete example of an invertible operator on the Hilbert space \( \ell_2(\mathbb{Z}) \) which is uniformly positively expansive and supercyclic.

Example 22. Fix real numbers \( \alpha > \beta > 1 \) and consider the weight sequence

\[
w = (w_n)_{n \in \mathbb{Z}} = (\ldots, \beta, \beta, \beta, \alpha, \alpha, \alpha, \ldots),
\]

where the first \( \alpha \) appears at position 1. Consider the bilateral weighted backward shift

\[B_w : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z}).\]

Since \( \|B_w(x)\| \geq \beta \|x\| \) for all \( x \in \ell_2(\mathbb{Z}) \), we have that \( B_w \) is uniformly positively expansive. Since \( B_w \) is invertible, \( B_w \) is also uniformly expansive. By [2, Corollary 1], \( B_w \) is supercyclic if and only if, for any \( q \in \mathbb{N} \),

\[
\liminf_{n \to \infty} \frac{w_0 \cdot \ldots \cdot w_{-n+q+1}}{w_1 \cdot \ldots \cdot w_{n+q}} = 0.
\]

But, by our choice of \( w \),

\[
\liminf_{n \to \infty} \frac{w_0 \cdot \ldots \cdot w_{-n+q+1}}{w_1 \cdot \ldots \cdot w_{n+q}} = \lim_{n \to \infty} \frac{\beta^{n-q}}{\alpha^{n+q}} = \lim_{n \to \infty} \frac{\beta^n}{\alpha^n} = 0.
\]

Thus, the operator \( B_w \) is also supercyclic.

4 Expansivity and spectrum

We denote by \( \mathbb{D} \) the open unit disc in the complex plane \( \mathbb{C} \). Moreover, \( \rho(T) \) denotes the resolvent set of the operator \( T \). In the case \( T \) is an operator on a real Banach space, we define

\[
\rho(T) = \rho(T_C), \quad \sigma_p(T) = \sigma_p(T_C) \quad \text{and} \quad \sigma_a(T) = \sigma_a(T_C).
\]

It is known that if \( T \) is a self-adjoint operator on a complex Hilbert space \( H \) and \( x \in H \), then there exists a unique positive Radon measure \( \mu \) on \( \sigma(T) \) such that

\[
\langle f(T)x, x \rangle = \int_{\sigma(T)} f(t)d\mu(t) \quad \text{for all } f \in C(\sigma(T)).
\]

In particular, \( \mu(\sigma(T)) = \|x\|^2 \). The measure \( \mu \) is called the spectral measure associated to \( T \) and \( x \).

We refer the reader to the books [14] and [33] for more informations concerning spectrum.
Theorem C. If $T$ is an operator on a Banach space $X$, then

$T$ uniformly positively expansive $\Rightarrow \sigma_a(T) \cap \overline{D} = \emptyset$. Moreover, the converse holds if $\rho(T) \cap \mathbb{D} \neq \emptyset$. In particular, the converse holds if $T$ is invertible.

Proof. It is enough to consider the case of complex scalars. Suppose that there is a point $\lambda \in \sigma_a(T) \cap \overline{D}$. Let $(x_k)_{k \in \mathbb{N}}$ be a sequence in $S_X$ such that $\lim_{k \to \infty} \|\lambda x_k - T x_k\| = 0$. Since $\| \lambda^n x_k - T^n x_k \| = |\lambda| \| \lambda^{n-1} x_k - T^{n-1} x_k \| + \| T^{n-1} \| \| \lambda x_k - T x_k \|$, it follows by induction that

$$\lim_{k \to \infty} \| \lambda^n x_k - T^n x_k \| = 0 \quad \text{for all } n \in \mathbb{N}.$$ 

Since $\| \lambda^n x_k \| \leq 1$ for all $k, n \in \mathbb{N}$, we conclude from Proposition 5(b) that $T$ is not uniformly positively expansive.

Now, assume that $\rho(T) \cap \mathbb{D} \neq \emptyset$ and $\sigma_a(T) \cap \overline{D} = \emptyset$. Since $\sigma_a(T)$ contains the boundary of $\sigma(T)$, we must have $\sigma(T) \cap \overline{D} = \emptyset$. Hence,

$$\sigma(T^{-1}) = \{ \lambda^{-1} : \lambda \in \sigma(T) \} \subset \mathbb{D},$$

that is, $r(T^{-1}) < 1$. Choose $R \in \mathbb{R}$ such that $r(T^{-1}) < R < 1$. It follows from the spectral radius formula that there exists $n_0 \in \mathbb{N}$ such that

$$\| T^n x \| \geq R^{-n} \| x \| \quad \text{for all } x \in X \text{ and } n \geq n_0,$$

which implies that $T$ is uniformly positively expansive.

Let us now give a short direct proof of the following result from [22].

Theorem 23. If $T$ is an invertible normal operator on a Hilbert space $H$, then $T$ is expansive if and only if $\sigma_p(T^*T) \cap \mathbb{T} = \emptyset$.

Proof. We may assume complex scalars. Suppose that $\sigma_p(T^*T) \cap \mathbb{T} \neq \emptyset$ and let $\lambda$ be a point in this intersection. There exists $x \in H \setminus \{0\}$ such that $T^*T x = \lambda x$. Hence, for every $n \in \mathbb{Z}$, $\| T^n x \|^2 = \langle (T^*T)^n x, x \rangle = \lambda^n \| x \|^2$, implying that $\| T^n x \| = \| x \|$. Thus, $T$ is not expansive.

Conversely, assume that $T$ is not expansive and consider the positive operator $S = T^*T$. There exists $x \in S_H$ with $\| T^n x \| < 2$ for all $n \in \mathbb{Z}$. Since $T$ is normal,

$$\| S^n x \| = \| (T^*)^n T^n x \| = \| T^{2n} x \| < 2 \quad \text{for all } n \in \mathbb{Z}.$$ 

Let $\mu$ be the spectral measure associated to $S$ and $x$. Since $S$ is an invertible positive operator, $\sigma(S) \subset (0, \infty)$. Thus, by the Cauchy-Schwartz inequality,

$$0 \leq \int_{\sigma(S)} t^n d\mu(t) = \langle S^n x, x \rangle \leq \| S^n x \| \| x \| < 2 \quad \text{for all } n \in \mathbb{Z}.$$ 

For each $\alpha < 1$ and each $\beta > 1$, let $A_\alpha = \sigma(S) \cap (0, \alpha]$ and $B_\beta = \sigma(S) \cap [\beta, \infty)$. Since

$$\alpha^{-n} \mu(A_\alpha) \leq \int_{\sigma(S)} t^{-n} d\mu(t) < 2 \quad \text{and} \quad \beta^n \mu(B_\beta) \leq \int_{\sigma(S)} t^n d\mu(t) < 2,$$
for all \( n \in \mathbb{N} \), we conclude that \( \mu(A_n) = \mu(B_{\beta}) = 0 \). This implies that \( \sigma(S) \setminus \{1\} \) has \( \mu \)-measure zero. Therefore,
\[
\|Sx - x\|^2 = \langle (S - I)^2 x, x \rangle = \int_{\sigma(S)} (t - 1)^2 d\mu(t) = 0,
\]
and so \( 1 \in \sigma_p(S) \). \( \square \)

Recall that a Hilbert space operator \( T \) is said to be hyponormal if
\[
\|T^*x\| \leq \|Tx\| \quad \text{for all } x.
\]
It is natural to ask if the previous theorem can be generalized to hyponormal operators. Let us now show that the implication
\[
\sigma_p(T^*T) \cap \mathbb{T} = \emptyset \implies T \text{ expansive}
\]
holds for hyponormal weighted shifts, but it is not true in general, and that the converse implication may fail even for hyponormal weighted shifts.

**Proposition 24.** Let \( w = (w_n)_{n \in \mathbb{Z}} \) be a weight sequence with \( \inf_{n \in \mathbb{Z}} |w_n| > 0 \) and consider the bilateral weighted forward shift
\[
F_w : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z}).
\]
Assume that \( F_w \) is hyponormal. If \( \sigma_p(F_w^*F_w) \cap \mathbb{T} = \emptyset \), then \( F_w \) is expansive.

**Proof.** Since \( F_w(e_n) = w_ne_{n+1} \) and \( F_w^*(e_n) = \overline{w_{n-1}}e_n \),
\[
F_w^*F_w(e_n) = |w_n|^2e_n \quad (n \in \mathbb{Z}),
\]
which implies that
\[
\sigma_p(F_w^*F_w) \cap \mathbb{T} = \emptyset \iff |w_n| \neq 1 \text{ for all } n \in \mathbb{Z}. \tag{7}
\]
Since \( F_w \) is hyponormal, it is well-known that the sequence \( (|w_n|)_{n \in \mathbb{Z}} \) is increasing. Therefore, \( \sup_{n \in \mathbb{N}} |w_1 \ldots w_n| = \infty \) if \( |w_0| > 1 \), while \( \sup_{n \in \mathbb{N}} |w_{-n} \ldots w_{-1}|^{-1} = \infty \) if \( |w_0| < 1 \). Anyway, it follows from Theorem B(a) that \( F_w \) is expansive. \( \square \)

**Remark 25.** (a) We cannot remove the hyponormality hypothesis in Proposition 24. To see this, it is enough to choose \( w \) so that \( |w_n| \neq 1 \) for all \( n \in \mathbb{Z} \), \( \sup_{n \in \mathbb{N}} |w_1 \ldots w_n| < \infty \) and \( \sup_{n \in \mathbb{N}} |w_{-n} \ldots w_{-1}|^{-1} < \infty \). Then, \( \sigma_p(F_w^*F_w) \cap \mathbb{T} = \emptyset \) (by (7)), but \( F_w \) is not expansive (by Theorem B(a)).

(b) The converse of the conclusion of Proposition 24 is not true in general. For instance, if \( w = (\ldots, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, 1, 2, 2, 2, \ldots) \) then \( F_w \) is hyponormal and uniformly expansive, but \( \sigma_p(F_w^*F_w) \cap \mathbb{T} \neq \emptyset \).

**Remark 26.** Let \( T \) be a normal operator on a complex Hilbert space \( H \). In view of the previous theorem, it is natural to make the following question: *Is it true that \( T \) is positively expansive if and only if \( \sigma_p(T^*T) \cap \mathbb{T} = \emptyset ? \)*

The direct implication is true, since the relation \( T^*Tx = \lambda x \), with \( \lambda \in \mathbb{T} \) and \( x \neq 0 \), implies that \( \|T^*x\|^2 = \langle (T^*)^nx, x \rangle = \lambda^n\|x\|^2 \leq \|x\|^2 \) for all \( n \in \mathbb{N} \), and so \( T \) is not positively expansive.
However, the converse is not true in general, even if $T$ is invertible. Indeed, let $T : L^2[0,1] \to L^2[0,1]$ be defined by 
\[(T f)(t) = \frac{t + 1}{2} f(t) \text{ for all } f \in L^2[0,1].\]
It is not difficult to see that $T$ is invertible, self-adjoint, not positively expansive, and $\sigma_p(T^*T) = \emptyset$.

Nevertheless, we have the following characterization.

**Proposition 27.** Let $T$ be a normal operator on a complex Hilbert space $H$. Then, $T$ is positively expansive if and only if $\mu(\sigma(T^*T) \cap (1, \infty)) > 0$ for every spectral measure $\mu$ associated to $T^*T$.

**Proof.** Let $S = T^*T$. If $T$ is not positively expansive, then there exists $x \in S_H$ such that $\|T^n x\| < 2$ for all $n \in \mathbb{N}$. By letting $\mu$ be the spectral measure associated to $S$ and $x$, we obtain
\[0 \leq \int_{\sigma(S)} t^n d\mu(t) = \langle S^n x, x \rangle < 2 \text{ for all } n \in \mathbb{N},\]
which implies that $\mu(\sigma(S) \cap (1, \infty)) = 0$.

Conversely, suppose that for some $x \neq 0$, the spectral measure $\mu$ associated to $S$ and $x$ satisfies $\mu(\sigma(S) \cap (1, \infty)) = 0$. Then,
\[\|T^{2n} x\|^2 = \|S^n x\|^2 = \langle S^{2n} x, x \rangle = \int_{\sigma(S)} t^{2n} d\mu(t) \leq \|x\|^2 \text{ for all } n \in \mathbb{N},\]
implying that $T$ is not positively expansive. \[\square\]

## 5 Shadowing

Given a metric space $M$ and a homeomorphism $h : M \to M$, recall that a sequence $(x_n)_{n \in \mathbb{Z}}$ is a $\delta$-pseudotrajectory of $h$ ($\delta > 0$) if
\[d(h(x_n), x_{n+1}) \leq \delta \text{ for all } n \in \mathbb{Z}.\]
The homeomorphism $h$ has the **shadowing property** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every $\delta$-pseudotrajectory $(x_n)_{n \in \mathbb{Z}}$ of $h$ is $\varepsilon$-shadowed by a real trajectory of $h$, that is, there exists $x \in M$ such that
\[d(x_n, h^n(x)) < \varepsilon \text{ for all } n \in \mathbb{Z}.\]
Moreover, the homeomorphism $h$ has the **Lipschitz shadowing property** if there exists $K > 0$ such that $\delta$ can be choosen satisfying that $\varepsilon < K\delta$. More generally, we call it $\alpha$-Hölder shadowing property, $0 < \alpha \leq 1$, if $\delta$ can be choosen so that $\varepsilon < K\delta^\alpha$.

**Remark 28.** In the case of operators, it is enough to check the above condition for a single $\varepsilon > 0$. More precisely, if $T$ is an invertible operator on a Banach space $X$ and if for some $\varepsilon > 0$ there exists $\delta > 0$ such that every $\delta$-pseudotrajectory of $T$ is $\varepsilon$-shadowed by a real trajectory of $T$, then $T$ has the shadowing property. It is also true that any linear operator satisfying the shadowing property trivially satisfies the Lipschitz shadowing property. These facts follows easily from the linearity of $T$. 

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Many variations of the notion of shadowing have been introduced and studied by several authors, e.g., [24, 25, 32]. We shall consider here the notions of limit shadowing and \(\ell_p\) shadowing. Let \(M\) and \(h\) be as above. The homeomorphism \(h\) is said to have the limit shadowing property if for every sequence \((x_n)_{n \in \mathbb{Z}}\) in \(M\) with
\[
\lim_{|n| \to \infty} d(h(x_n), x_{n+1}) = 0,
\]
there exists \(x \in M\) such that
\[
\lim_{|n| \to \infty} d(x_n, h^n(x)) = 0.
\]
Moreover, \(h\) is said to have the \(\ell_p\) shadowing property \((1 \leq p < \infty)\) if for every sequence \((x_n)_{n \in \mathbb{Z}}\) in \(M\) with
\[
\sum_{n \in \mathbb{Z}} d(h(x_n), x_{n+1})^p < \infty,
\]
there exists \(x \in M\) such that
\[
\sum_{n \in \mathbb{Z}} d(x_n, h^n(x))^p < \infty.
\]

**Proposition 29.** Let \(T\) be an invertible operator on a Banach space \(X\). Suppose that \(X = M \oplus N\), where \(M\) and \(N\) are closed \(T\)-invariant subspaces of \(X\). Then \(T\) has the shadowing property (the limit shadowing property, the \(\ell_p\) shadowing property) if and only if so do \(T|M\) and \(T|N\).

**Proof.** By the open mapping theorem, there is a constant \(\beta > 0\) such that
\[
\|a\| \leq \beta\|x\| \quad \text{and} \quad \|b\| \leq \beta\|x\|,
\]
whenever \(x = a + b\) with \(a \in M\) and \(b \in N\). Let \(x = a + b\) and \(x_n = a_n + b_n\) \((n \in \mathbb{Z})\), where \(a, a_n \in M\) and \(b, b_n \in N\). The direct implication follows easily from the inequalities
\[
\|a_n - (T|M)^n(a)\| \leq \beta\|a_n - T^n x\| \quad \text{and} \quad \|b_n - (T|N)^n(b)\| \leq \beta\|b_n - T^n x\|,
\]
wheras the inverse implication follows easily from the inequalities
\[
\|(T|M)(a_n) - a_{n+1}\| \leq \beta\|T x_n - x_{n+1}\|, \quad \|(T|N)(b_n) - b_{n+1}\| \leq \beta\|T x_n - x_{n+1}\|
\]
and
\[
\|x_n - T^n x\| \leq \|a_n - (T|M)(a)\| + \|b_n - (T|N)(b)\|.
\]

**Corollary 30.** If \(T\) is an invertible operator on a real Banach space \(X\), then \(T\) has the shadowing property (the limit shadowing property, the \(\ell_p\) shadowing property) if and only if so does its complexification \(T_C\).

In view of the above corollary, we will tacitly assume complex scalars in all the proofs that follow.

It is well-known that any invertible hyperbolic operator on any Banach space has the shadowing property (see [23], for instance). We shall prove that it also has the limit shadowing property and the \(\ell_p\) shadowing property for every \(1 \leq p < \infty\). In fact, we will derive this from a more general theorem which will also imply the existence of nonhyperbolic operators with these shadowing properties. For this purpose, we shall need two lemmas.
Lemma 31. (see [27]) An invertible operator $T$ on a Banach space $X$ has the shadowing property if and only if there is a constant $K > 0$ such that for every bounded sequence $(z_n)_{n \in \mathbb{Z}}$ in $X$, there is a sequence $(y_n)_{n \in \mathbb{Z}}$ in $X$ such that

$$\sup_{n \in \mathbb{Z}} \|y_n\| \leq K \sup_{n \in \mathbb{Z}} \|z_n\|$$

(8)

and

$$y_{n+1} = Ty_n + z_n \quad \text{for all } n \in \mathbb{Z}.$$ 

(9)

Proof. Assume that $T$ has the shadowing property and let $\delta > 0$ be the constant that appears in the definition of shadowing associated to $\varepsilon = 1$. Consider a bounded sequence $(z_n)_{n \in \mathbb{Z}}$ and put $L = \sup_{n \in \mathbb{Z}} \|z_n\|$. Let $(x_n)_{n \in \mathbb{Z}}$ be such that $x_{n+1} = Tx_n + \frac{1}{L} z_n$ for all $n \in \mathbb{Z}$. Observe that $(x_n)_{n \in \mathbb{Z}}$ is completely determined by $x_0$. Then $(x_n)_{n \in \mathbb{Z}}$ is a $\delta$-pseudotrajectory of $T$. By hypothesis, there exists $x \in X$ such that

$$\|x - T^n x\| < 1 \quad \text{for all } n \in \mathbb{Z}.$$ 

By putting $y_n = \frac{L}{\delta}(x_n - T^n x)$, we have that (8) holds with $K = 1/\delta$ and (9) also holds.

For the converse, it is enough to consider $\varepsilon = 1$ (Remark 28). Put $\delta = \frac{1}{2K}$ and let $(x_n)_{n \in \mathbb{Z}}$ be a $\delta$-pseudotrajectory of $T$. Put $z_n = x_{n+1} - Tx_n$ for all $n \in \mathbb{Z}$. By hypothesis, there exists $(y_n)_{n \in \mathbb{Z}}$ such that (8) and (9) hold. Since $x_{n+1} - y_{n+1} = T(x_n - y_n)$ for all $n \in \mathbb{Z}$, it follows that $x_n - y_n = T^n(x_0 - y_0)$ for all $n \in \mathbb{Z}$. Thus,

$$\|x_n - T^n(x_0 - y_0)\| = \|y_n\| \leq K \sup_{j \in \mathbb{Z}} \|z_j\| \leq K\delta = 1,$$

for all $n \in \mathbb{Z}.$

Lemma 32. An invertible operator $T$ on a Banach space $X$ has the limit shadowing property (the $\ell_p$ shadowing property) if and only if for every sequence $(z_n)_{n \in \mathbb{Z}}$ in $X$ with

$$\lim_{|n| \to \infty} \|z_n\| = 0 \quad \left( \sum_{n \in \mathbb{Z}} \|z_n\|^p < \infty \right),$$

(10)

there exists a sequence $(y_n)_{n \in \mathbb{Z}}$ in $X$ such that

$$\lim_{|n| \to \infty} \|y_n\| = 0 \quad \left( \sum_{n \in \mathbb{Z}} \|y_n\|^p < \infty \right)$$

(11)

and

$$y_{n+1} = Ty_n + z_n \quad \text{for all } n \in \mathbb{Z}.$$ 

(12)

Proof. Assume that $T$ has the limit shadowing property (the $\ell_p$ shadowing property) and consider a sequence $(z_n)_{n \in \mathbb{Z}}$ satisfying (10). Let $(x_n)_{n \in \mathbb{Z}}$ be such that $x_{n+1} = Tx_n + z_n$ for all $n \in \mathbb{Z}$. Then, by hypothesis, there exists $x \in X$ such that $\lim_{|n| \to \infty} \|x_n - T^n x\| = 0$ (since $\|x_n - T^n x\|^p < \infty$). Hence, by putting $y_n = x_n - T^n x$ for all $n \in \mathbb{Z}$, we have that (11) and (12) hold.

For the converse, consider $(x_n)_{n \in \mathbb{Z}}$ such that

$$\lim_{|n| \to \infty} \|x_{n+1} - Tx_n\| = 0 \quad \left( \sum_{n \in \mathbb{Z}} \|x_{n+1} - Tx_n\|^p < \infty \right).$$

Put $z_n = x_{n+1} - Tx_n$ for all $n \in \mathbb{Z}$. Then $(z_n)_{n \in \mathbb{Z}}$ satisfies (10). Hence, by hypothesis, there exists $(y_n)_{n \in \mathbb{Z}}$ such that (11) and (12) hold. Since $x_n - T^n(x_0 - y_0) = y_n$ for all $n \in \mathbb{Z}$, we are done. \qed
Theorem 33. Let $T$ be an invertible operator on a Banach space $X$. Suppose that
\[ X = M \oplus N, \tag{13} \]
where $M$ and $N$ are closed subspaces of $X$ with $T(M) \subset M$ and $T^{-1}(N) \subset N$. If
\[ \sigma(T|M) \subset \mathbb{D} \quad \text{and} \quad \sigma(T^{-1}|N) \subset \mathbb{D}, \tag{14} \]
then $T$ has the shadowing property, the limit shadowing property and the $\ell_p$ shadowing property for all $1 \leq p < \infty$.

Proof. By (13), for each $x \in X$, there are unique $x^{(1)} \in M$ and $x^{(2)} \in N$ satisfying $x = x^{(1)} + x^{(2)}$. Moreover, there is a constant $\beta > 0$ such that
\[ \|x^{(1)}\| \leq \beta\|x\| \quad \text{and} \quad \|x^{(2)}\| \leq \beta\|x\| \quad \text{for all} \ x \in X. \]

By (14), $r(T|M) < 1$ and $r(T^{-1}|N) < 1$. Choose $t \in \mathbb{R}$ such that
\[ \max\{r(T|M), r(T^{-1}|N)\} < t < 1. \]
It follows from the spectral radius formula that there exists a constant $C \geq 1$ such that
\[ \|(T|M)^n\| \leq Ct^n \quad \text{and} \quad \|(T^{-1}|N)^n\| \leq Ct^n \quad \text{for all} \ n \in \mathbb{N}_0. \]

Consider a bounded sequence $(z_n)_{n \in \mathbb{Z}}$ in $X$. For each $n \in \mathbb{Z}$, we define
\[ y_n^{(1)} = \sum_{k=0}^{\infty} T^k z_{n-k}^{(1)} \in M, \quad y_n^{(2)} = -\sum_{k=1}^{\infty} T^{-k} z_{n+k-1}^{(2)} \in N \]
\[ y_n = y_n^{(1)} + y_n^{(2)} \in X. \]

An easy computation shows that (9) (which is the same as (12)) holds. Moreover,
\[ \|y_n^{(1)}\| \leq C \sum_{k=0}^{\infty} t^k \|z_{n-k}^{(1)}\| \quad \text{and} \quad \|y_n^{(2)}\| \leq C \sum_{k=1}^{\infty} t^k \|z_{n+k-1}^{(2)}\|. \tag{15} \]

Hence,
\[ \sup_{n \in \mathbb{Z}} \|y_n\| \leq \left( \frac{2\beta C}{1-t} \right) \sup_{n \in \mathbb{Z}} \|z_n\|, \]
which proves that $T$ has the shadowing property by Lemma 31.

Now, assume that $(z_n)_{n \in \mathbb{Z}}$ satisfies (10). By Lemma 32, it remains to show that (11) holds. By (15), for each $j \in \mathbb{N}$ and each $i \in \{1, 2\},$
\[ \|y_n^{(i)}\| \leq C \left( \sum_{k=0}^{j} t^k \right) \left( \sup_{0 \leq k \leq j} \|z_{n+k-1}^{(i)}\| \right) \quad \text{and} \quad C \left( \sum_{k=j}^{\infty} t^k \right) \left( \sup_{k \in \mathbb{Z}} \|z_k^{(i)}\| \right), \]
which shows that $\lim_{|n| \to \infty} \|y_n\| = 0$ whenever $\lim_{|n| \to \infty} \|z_n\| = 0$. In the case $p = 1$, it is enough to use the estimates
\[ \sum_{n \in \mathbb{Z}} \|y_n^{(i)}\| \leq C \sum_{n \in \mathbb{Z}} \sum_{k=0}^{\infty} t^k \|z_{n+(i-1)k-1}^{(i)}\| = C' \left( \sum_{k=0}^{\infty} t^k \right) \left( \sum_{n \in \mathbb{Z}} \|z_n\| \right) \quad (i \in \{1, 2\}). \]
Finally, in the case $1 < p < \infty$, we consider its conjugate exponent $q$ (i.e., $1/p + 1/q = 1$) and apply Hölder’s inequality to obtain

$$\|y^{(i)}_n\| \leq C \left( \sum_{k=0}^{\infty} t^{\frac{p}{q} k} \right)^{\frac{1}{p}} \left( \sum_{k=0}^{\infty} t^{\frac{q}{q} k} \|z^{(i)}_{n+(i-1)k-1}\|^p \right)^{\frac{1}{p}} \quad (i \in \{1, 2\}).$$

As a consequence,

$$\sum_{n \in \mathbb{Z}} \|y^{(i)}_n\|^p \leq C^p \left( \sum_{k=0}^{\infty} t^{\frac{p}{q} k} \right)^{\frac{q}{p}} \left( \sum_{k=0}^{\infty} t^{\frac{q}{q} k} \right) \left( \sum_{n \in \mathbb{Z}} \|z^{(i)}_n\|^p \right) \quad (i \in \{1, 2\}).$$

Thus, in all cases, (11) holds.

As an immediate consequence, we have the following:

**Corollary 34.** Every invertible hyperbolic operator $T$ on a Banach space $X$ has the shadowing property, the limit shadowing property and the $\ell_p$ shadowing property for all $1 \leq p < \infty$.

It is well-known that the shadowing property implies hyperbolicity in the cases of invertible operators on finite-dimensional euclidean spaces [23] and invertible normal operators on Hilbert spaces [22]. It is a basic question in linear dynamics whether or not this implication is always true, that is, whether or not shadowing and hyperbolicity coincide for invertible operators on Banach (or Hilbert) spaces. This question appeared explicitly in [22, Page 148], for instance. Let us now answer this question in the negative as an application of Theorem 33. The following is much stronger.

**Theorem D.** Let $X = \ell_q(\mathbb{Z})$ ($1 \leq q < \infty$) or $X = c_0(\mathbb{Z})$. There exists an invertible weighted shift $T$ on $X$ which satisfies the frequent hypercyclicity criterion (hence it is not hyperbolic) and has the shadowing property, the limit shadowing property and the $\ell_p$ shadowing property for all $1 \leq p < \infty$. Moreover any weighted shift operator sufficiently close to $T$ also satisfies the thesis of previous statement.

Recall that an operator $T$ on a separable Banach space $X$ is said to satisfy the frequent hypercyclicity criterion if there exist a dense subset $X_0$ of $X$ and a map $S : X_0 \to X_0$ such that the following properties hold for every $x \in X_0$:

- $\sum_{n=0}^{\infty} T^n x$ converges unconditionally;
- $\sum_{n=0}^{\infty} S^n x$ converges unconditionally;
- $T S x = x$.

If $T$ satisfies this criterion, then $T$ is frequently hypercyclic, Devaney chaotic, mixing and densely distributionally chaotic; see [19, Section 9.2] and [6, Corollary 20]. Let us also recall that $T$ is said to be Devaney chaotic if it is hypercyclic and has a dense set of periodic points. Of course, an invertible operator which has a nontrivial periodic point is not expansive and, in particular, it is not hyperbolic. Thus, the above theorem implies the existence of operators with the shadowing property that are not hyperbolic and exhibit several types of chaotic behaviors.
Proof of theorem D Fix a real number $\alpha > 1$ and let $T$ be the bilateral weighted forward shift on $X$ whose weight sequence $(w_n)_{n \in \mathbb{Z}}$ is given by $w_n = \alpha$ if $n < 0$ and $w_n = 1/\alpha$ if $n \geq 0$. By applying Theorem 33 with

$$M = \{(x_n)_{n \in \mathbb{Z}} : x_n = 0 \text{ for all } n < 0\}$$

and

$$N = \{(x_n)_{n \in \mathbb{Z}} : x_n = 0 \text{ for all } n \geq 0\},$$

we see that $T$ has all the above-mentioned shadowing properties. Moreover, in order to see that $T$ satisfies the frequent hypercyclicity criterion, it is enough to consider $X_0$ as the set of all sequences $(x_n)_{n \in \mathbb{Z}}$ with finite support and $S = T^{-1}$. To conclude the second part of the thesis, observe that any weighted shift closed to $T$ also fits in Theorem 33 (with the same $M$ and $N$) and satisfies the frequent hypercyclicity criterion.

The next remarks highlight the differences between nonlinear finite dimensional dynamics and infinite dimensional linear dynamics, explaining the status of the shadowing property for finite dimensional diffeomorphisms and raising a series of questions.

Remark 35. As commented in the introduction, for $C^1$ diffeomorphisms on a finite dimensional manifold (see [26]), Lipschitz shadowing is equivalent to hyperbolicity. Our example proves this is not the case for infinite dimensional linear dynamics. In some sense, this shows that when one considers infinite dimensional spaces, even linear dynamics is richer than nonlinear finite dimensional dynamics.

Remark 36. The previous theorem resembles a result in [21] where the existence of a non-hyperbolic yet having the shadowing property $C^\infty$ diffeomorphism on a surface is exhibited. In view of the results in [26], the shadowing property can not be Lipschitz shadowing. Indeed, in examples provided in [21] the shadowing property is only $\alpha$- Hölder for some $\alpha < 1$.

Remark 37. It is worth pointing out that finite dimensional diffeomorphisms induce infinite dimensional operators: for any diffeomorphism one obtains finite dimensional linear cocycles provided by the derivative of that diffeomorphism and those linear cocycles can be recast as an infinite dimensional linear map (see for instances [9] for discussions of linear cocycles). In particular, the proof in [26] is based on analyzing the dynamic of a diffeomorphisms as a linear cocycle, showing that the Lipschitz shadowing implies shadowing for the cocycle of linear maps and from there concluding hyperbolicity using the results in [27]. On the other hand, the spectrum problem related to certain nonlinear infinite dimensional operators, as the discrete Schrödinger operator can be reduced to a linear cocycle (see [13] for instance).

Remark 38. It is shown in theorem D that the shadowing property is satisfied for an open set of weighted shifts. It is natural to wonder if the same holds when one considers open sets of linear operators; in particular, is the shadowing property satisfied for any linear operator close to the ones that satisfies theorem D?

As we have just seen, an invertible bilateral weighted shift can have the shadowing property without being expansive. The next result presents an additional condition which guarantees expansivity.

Proposition 39. Let $X = \ell^p_p(Z) \ (1 \leq p < \infty)$ or $X = c_0(Z)$, and consider a weight sequence $w = (w_n)_{n \in \mathbb{Z}}$ with $\inf_{n \in \mathbb{Z}} |w_n| > 0$. If $F_w : X \to X$ has the shadowing property and the sequence $(nF_w^n(e_0))_{n \in \mathbb{Z}}$ is not bounded, then $F_w$ is expansive.
We will prove at the end of this section that every expansive operator with the shadowing property is uniformly expansive. So, in the above proposition we can actually conclude that $F_w$ is uniformly expansive.

**Proof.** Assume that $F_w$ is not expansive. By Theorem B(a),

$$\sup_{n \in \mathbb{N}} |w_1 \cdot \ldots \cdot w_n| < \infty \quad \text{and} \quad \sup_{n \in \mathbb{N}} |w_{-n} \cdot \ldots \cdot w_{-1}|^{-1} < \infty.$$  

Thus, the sequence $(z_n)_{n \in \mathbb{Z}}$ given by $z_n = F_w^{n+1}(e_0)$ $(n \in \mathbb{Z})$ is bounded. Since $F_w$ has the shadowing property, Lemma 31 guarantees the existence of a bounded sequence $(y_n)_{n \in \mathbb{Z}}$ in $X$ such that

$$y_{n+1} = F_w(y_n) + z_n \quad \text{for all} \quad n \in \mathbb{Z}.$$  

For each $n \in \mathbb{N}$, note that

$$y_n = F_w^n(y_0) + F_w^{n-1}(z_0) + \cdots + F_w(z_{n-1}) + z_{n-1} = F_w^n(y_0) + n F_w^n(e_0)$$

and

$$y_{-n} = F_w^{-n}(y_0) - F_w^{-n}(z_{-1}) - F_w^{-n+1}(z_{-2}) - \ldots - F_w^{-1}(z_{-n}) = F_w^{-n}(y_0) - n F_w^{-n}(e_0).$$

Write $y_0 = (a_n)_{n \in \mathbb{Z}}$. Then

$$\|y_n\| \geq |a_0 + n||w_0 \cdot \ldots \cdot w_{n-1}| \quad \text{and} \quad \|y_{-n}\| \geq |a_0 - n||w_{-n} \cdot \ldots \cdot w_{-1}|^{-1},$$

for every $n \in \mathbb{N}$. Since we are assuming that the sequence $(n F_w^n(e_0))_{n \in \mathbb{Z}}$ is not bounded, these estimates imply that the sequence $(y_n)_{n \in \mathbb{Z}}$ is not bounded, a contradiction. 

Let us now see that all notions of shadowing considered here coincide in the finite dimensional setting. As mentioned before the equivalence (i) $\iff$ (ii) below is already known (see [23], where further references can be found).

**Proposition 40.** Fix $p \in [1, \infty]$. If $T$ is an invertible operator on a finite-dimensional Banach space $X$, then the following assertions are equivalent:

(i) $T$ is hyperbolic;

(ii) $T$ has the shadowing property;

(iii) $T$ has the limit shadowing property;

(iv) $T$ has the $\ell_p$ shadowing property.

**Proof.** Suppose that $T$ is not hyperbolic. Since it is easy to see that if $T$ has the shadowing property (the limit shadowing property, the $\ell_p$ shadowing property), then so does $\lambda T$ whenever $|\lambda| = 1$, we may assume that $1 \in \sigma(T)$. Hence, by Proposition [22] and the Jordan canonical form, it is enough to consider the cases in which $T$ is an operator on $\mathbb{C}^k$ (with the euclidean norm) whose canonical matrix has the form

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$  

(16)
We shall handle both cases simultaneously, but we remark that in the first case the vectors we are going to define have only the first coordinate.

Assume $p > 1$ and consider the sequence $(x_n)_{n \in \mathbb{Z}}$ in $\mathbb{C}^k$ given by $x_n = (0, \ldots, 0)$ for $n \leq 0$ and

$$x_{n+1} = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n+1}, (Tx_n)_2, \ldots, (Tx_n)_k\right) \quad \text{for } n \geq 0,$$

where $(x)_j$ denotes the $j$th coordinate of the vector $x \in \mathbb{C}^k$. Then

$$\sum_{n \in \mathbb{Z}} \|Tx_n - x_{n+1}\|^p = \sum_{n=0}^{\infty} \left(\frac{1}{n+1}\right)^p < \infty.$$

However, it is not possible to find an $x \in X$ such that

$$\lim_{|n| \to \infty} \|x_n - T^n x\| = 0,$$

since

$$\|x_n - T^n x\| \geq |(x_n - T^n x)_1| = \left|\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) - (x)_1\right| \to \infty \quad \text{as } n \to \infty,$$

for every $x \in \mathbb{C}^k$. This shows that $T$ does not have the limit shadowing property nor the $\ell_p$ shadowing property for any $p > 1$.

Now, assume $p = 1$. Consider the sequence $(x_n)_{n \in \mathbb{Z}}$ in $\mathbb{C}^k$ given by $x_n = (0, \ldots, 0)$ for $n \leq 0$ and

$$x_{n+1} = \left(1 + \frac{1}{n+1}, (Tx_n)_2, \ldots, (Tx_n)_k\right) \quad \text{for } n \geq 0.$$

Then

$$\sum_{n \in \mathbb{Z}} \|Tx_n - x_{n+1}\| = 1 + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} < \infty.$$

Nevertheless,

$$\sum_{n \in \mathbb{Z}} \|x_n - T^n x\| \geq \sum_{n \in \mathbb{Z}} |(x_n - T^n x)_1| \geq \sum_{n=1}^{\infty} \frac{1}{n} - (x)_1 = \infty,$$

for every $x \in \mathbb{C}^k$. Thus, $T$ does not have the $\ell_1$ shadowing property.

We say that a sequence $(t_n)_{n \in \mathbb{Z}}$ of scalars is $O(|n|)$ if there exist $\alpha > 0$, $\beta > 0$ and $n_0 \in \mathbb{N}$ such that

$$\alpha |n| \leq |t_n| \leq \beta |n| \quad \text{whenever } |n| \geq n_0.$$

Let us now establish a result which will imply a much simpler and shorter proof of the main result in [22] (see Corollary [42]).

**Theorem 41.** Let $T$ be an invertible operator on a Banach space $X$ such that for all $z \in X$, the sequence $(\|T^n z\|)_{n \in \mathbb{Z}}$ is not $O(|n|)$. If $T$ has the shadowing property, then $T$ is uniformly expansive.
Proof. Suppose that \(T\) has the shadowing property and let \(\delta > 0\) be the constant that appears in the definition of shadowing associated to \(\varepsilon = 1\). Assume that \(T\) is not uniformly expansive. Then, by [20, Theorem 1], the intersection \(\sigma_\alpha(T) \cap T\) is nonempty. Take a scalar \(\lambda\) in this intersection. Hence, there exists \(x_0 \in X\) such that
\[
\|x_0\| = 1 \quad \text{and} \quad \|\lambda x_0 - Tx_0\| < \frac{\delta}{2}.
\]
For each \(n \in \mathbb{Z}\), let \(y_n = 2\lambda^n x_0\). Then \((y_n)_{n \in \mathbb{Z}}\) is a \(\delta\)-pseudotrajectory of \(T\), and so there exists \(y \in X\) such that \(\|y_n - T^n y\| < 1\) for all \(n \in \mathbb{Z}\). Therefore,
\[
1 < \|T^n y\| < 3 \quad \text{for all} \quad n \in \mathbb{Z}.
\]
Now, consider the sequence \((z_n)_{n \in \mathbb{Z}}\) defined by \(z_n = \frac{\lambda^n y_0}{3}\) for all \(n \in \mathbb{Z}\). Since \((z_n)_{n \in \mathbb{Z}}\) is a \(\delta\)-pseudotrajectory of \(T\), there exists \(z \in X\) such that \(\|z_n - T^n z\| < 1\) for all \(n \in \mathbb{Z}\). Thus,
\[
\frac{|n|\delta}{3} - 1 < \|T^n z\| < |n|\delta + 1 \quad \text{for all} \quad n \in \mathbb{Z}.
\]
This contradicts the fact that \((\|T^n z\|)_{n \in \mathbb{Z}}\) is not \(O(|n|)\).

Corollary 42. If \(T\) is an invertible normal operator on a Hilbert space \(H\), then \(T\) has the shadowing property if and only if \(T\) is hyperbolic.

Proof. Suppose that \(T\) has the shadowing property but is not hyperbolic. Since \(T\) is normal, \(\sigma(T) = \sigma_\alpha(T)\), and so \(T\) is not uniformly expansive. Hence, by Theorem 41 there exists \(z \in H\) such that \((\|T^n z\|)_{n \in \mathbb{Z}}\) is \(O(|n|)\). Let \(\alpha > 0\), \(\beta > 0\) and \(n_0 \in \mathbb{N}\) be such that
\[
\alpha |n| \leq \|T^n z\| \leq \beta |n| \quad \text{whenever} \quad |n| \geq n_0.
\]
(17)
Consider the invertible positive operator \(S = T^* T\) and let \(\mu\) be the spectral measure associated to \(S\) and \(z\). Then,
\[
0 \leq \int_{\sigma(S)} t^n d\mu(t) = \langle S^n z, z \rangle = \|T^n z\|^2 \leq \beta^2 |n|^2 \quad \text{whenever} \quad |n| \geq n_0.
\]
By arguing as in the proof of Theorem 20 (with the sets \(A_\alpha\) and \(B_\beta\)), we see that \(\sigma(S) \setminus \{1\}\) has \(\mu\)-measure zero and so \(Sz = z\). This implies that \(\|T^n z\| = \|z\|\) for all \(n \in \mathbb{Z}\), which contradicts the first inequality in (17).

The next proposition gives another additional condition under which shadowing implies uniform expansivity (compare with Theorem 11).

Proposition 43. Let \(T\) be an invertible operator on a Banach space \(X\). If \(T\) is expansive and has the shadowing property, then \(T\) is uniformly expansive.

Proof. Suppose that \(T\) is not uniformly expansive and fix a scalar \(\lambda \in \sigma_\alpha(T) \cap T\). By arguing as in the proof of Theorem 11 we obtain a vector \(y \in X\) such that
\[
1 < \|T^n y\| < 3 \quad \text{for all} \quad n \in \mathbb{Z}.
\]
This contradicts the hypothesis that \(T\) is expansive.

\(\square\)
6 Positive shadowing

In the case \( h \) is a continuous self-map of a metric space \( M \), we can define the notion of positive shadowing simply by replacing the set \( \mathbb{Z} \) by the set \( \mathbb{N}_0 \) in the definition of shadowing. Similarly, we can define the notions of positive limit shadowing and positive \( \ell_p \) shadowing for such a map \( h \).

Remark 28, Proposition 29 and Corollary 30 have analogous statements for not necessarily invertible operators if we add the word “positive” to the corresponding notions of shadowing.

Theorem E. Let \( T \) be a (not necessarily invertible) operator on a Banach space \( X \). If \( T \) is hyperbolic, then \( T \) has the positive shadowing property, the positive limit shadowing property and the positive \( \ell_p \) shadowing property for all \( 1 \leq p < \infty \).

Proof. We divide the proof in three cases.

Case 1. \( \sigma(T) \subset \mathbb{D} \).

Then there exist \( t \in (0, 1) \) and \( C \geq 1 \) such that
\[
\|T^n\| \leq Ct^n \quad \text{for all } n \in \mathbb{N}_0.
\]

Given \( \varepsilon > 0 \), put \( \delta = \frac{(1-t)\varepsilon}{C} \). Let \( (x_n)_{n \in \mathbb{N}_0} \) be a \( \delta \)-pseudotrajectory of \( T \) and define \( y_n = x_n - Tx_{n-1} \) for \( n \in \mathbb{N} \). Then
\[
x_n = T^n x_0 + T^{n-1} y_1 + T^{n-2} y_2 + \cdots + Ty_{n-1} + y_n \quad \text{for all } n \in \mathbb{N}.
\] (18)

Since \( \|y_n\| \leq \delta \) for all \( n \in \mathbb{N} \), we conclude that
\[
\|x_n - T^n x_0\| \leq Ct^{n-1} \delta + Ct^{n-2} \delta + \cdots + Ct \delta + \delta \leq \frac{C\delta}{1-t} = \varepsilon \quad (n \in \mathbb{N}).
\]
Hence, \( (x_n)_{n \in \mathbb{N}_0} \) is \( \varepsilon \)-shadowed by \( (T^n x_0)_{n \in \mathbb{N}_0} \). This proves that \( T \) has the positive shadowing property.

Let \( (x_n)_{n \in \mathbb{N}_0} \) be a sequence in \( X \) with
\[
\lim_{n \to \infty} \|Tx_n - x_{n+1}\| = 0.
\]

Let \( y_n \) be defined as above. By (18),
\[
\|x_n - T^n x_0\| \leq Ct^{n-1} \|y_1\| + Ct^{n-2} \|y_2\| + \cdots + Ct \|y_{n-1}\| + \|y_n\|
\]
\[
\leq C \left( \sum_{k=0}^{j} t^k \right) \left( \sup_{0 \leq k \leq j} \|y_{n-k}\| \right) + C \left( \sum_{k=j+1}^{n-1} t^k \right) \left( \sup_{k \in \mathbb{N}} \|y_k\| \right),
\]
whenever \( 0 < j < n \). Since \( \|y_n\| \to 0 \), the above estimate implies that \( \|x_n - T^n x_0\| \to 0 \) as well. Thus, \( T \) has the positive limit shadowing property.

Now, suppose that
\[
\sum_{n=0}^{\infty} \|Tx_n - x_{n+1}\| < \infty.
\]
Then, by \( (18) \),
\[
\sum_{n=0}^\infty \| x_n - T^n x_0 \| \leq C \sum_{n=1}^\infty \sum_{k=0}^{n-1} t^k \| y_{n-k} \| = C \sum_{k=0}^\infty \sum_{n=k+1}^\infty t^k \| y_{n-k} \|
\]
\[
= C \left( \sum_{k=0}^\infty t^k \right) \left( \sum_{n=1}^\infty \| y_n \| \right) < \infty. 
\]

Finally, suppose that \( 1 < p < \infty \) and that
\[
\sum_{n=0}^\infty \| x_n - T^n x_0 \| \leq C \sum_{k=0}^\infty \sum_{n=0}^{n-1} t^k \| y_{n-k} \| < \infty. 
\]

By \( (18) \) and Hölder’s inequality,
\[
\| x_n - T^n x_0 \| \leq C \left( \sum_{k=0}^\infty \| y_{n-k} \| \right) \leq C \left( \sum_{k=0}^{n-1} t^k \| y_{n-k} \| \right)^{\frac{1}{q}} \left( \sum_{k=0}^\infty \| y_n \| \right)^{\frac{1}{p}},
\]
where \( q \) is the conjugate exponent to \( p \). Thus,
\[
\sum_{n=0}^\infty \| x_n - T^n x_0 \|^p \leq C^p \left( \sum_{k=0}^\infty \frac{t^k}{t^p} \right)^{\frac{p}{q}} \left( \sum_{k=0}^\infty \frac{t^k}{t^p} \| y_{n-k} \|^p \right)^{\frac{1}{p}} \left( \sum_{n=1}^\infty \| y_n \|^p \right)^{\frac{1}{p}} < \infty.
\]

Therefore, \( T \) also has the positive \( \ell_p \) shadowing property.

**Case 2.** \( \sigma(T) \subset \mathbb{C} \setminus \mathbb{D} \).

Then \( T \) is invertible and we can apply Corollary 34.

**Case 3.** \( \sigma(T) \cap \mathbb{D} \neq \emptyset \) and \( \sigma(T) \cap (\mathbb{C} \setminus \mathbb{D}) \neq \emptyset \).

In this case, the sets \( \sigma_1 = \sigma(T) \cap \mathbb{D} \) and \( \sigma_2 = \sigma(T) \cap (\mathbb{C} \setminus \mathbb{D}) \) form a partition of \( \sigma(T) \) into two nonempty closed sets. By the Riesz decomposition theorem \([19, \text{Theorem B.9}]\), there are nontrivial \( T \)-invariant closed subspaces \( M_1 \) and \( M_2 \) of \( X \) such that
\[
X = M_1 \oplus M_2, \quad \sigma(T|_{M_1}) = \sigma_1 \quad \text{and} \quad \sigma(T|_{M_2}) = \sigma_2.
\]

By Cases 1 and 2, both \( T|_{M_1} \) and \( T|_{M_2} \) have the positive shadowing property, the positive limit shadowing property and the positive \( \ell_p \) shadowing property for all \( 1 \leq p < \infty \), from which it follows easily that \( T \) also has these properties.

**Remark 44.** The converse to Theorem E is not true in general. Indeed, the operator constructed in the proof of Theorem D has the positive shadowing property, the positive limit shadowing property and the positive \( \ell_p \) shadowing property for all \( 1 \leq p < \infty \), but it is not hyperbolic.

Let us now prove that all notions of positive shadowing considered here coincide with hyperbolicity in the case of compact operators. First, let us consider the case of finite-dimensional spaces.

**Lemma 45.** Fix \( p \in [1, \infty) \). If \( T \) is an operator on a finite-dimensional Banach space \( X \), then the following assertions are equivalent:

\( \square \)
(i) $T$ is hyperbolic;
(ii) $T$ has the positive shadowing property;
(iii) $T$ has the positive limit shadowing property;
(iv) $T$ has the positive $\ell_p$ shadowing property.

Proof. Suppose that $T$ is not hyperbolic. We have to prove that (ii), (iii) and (iv) are all false. By arguing as in the proof of Theorem 40, we see that it is enough to consider the cases in which $T$ is an operator on $C^k$ (with the euclidean norm) whose canonical matrix has one of the forms in (16).

Fix $\delta > 0$ and let $(x_n)_{n \in \mathbb{N}_0}$ in $C^k$ be given by $x_0 = (0, \ldots, 0)$ and
\[
x_{n+1} = ((n+1)\delta, (Tx_n)_2, \ldots, (Tx_n)_k)
\]
for $n \in \mathbb{N}_0$.

Then $(x_n)_{n \in \mathbb{N}_0}$ is a $\delta$-pseudotrajectory of $T$ that cannot be shadowed by the real trajectory of any point $x \in C^k$, because
\[
\|x_n - T^n x\| \geq |(x_n - T^n x)_1| = |n\delta - (x)_1| \to \infty \quad \text{as } n \to \infty.
\]

So, $T$ does not have the positive shadowing property.

The proofs that $T$ does not have the positive limit shadowing property and does not have the positive $\ell_p$ shadowing property are similar to the corresponding proofs in Theorem 40 and so we omit them.

**Theorem 46.** Fix $p \in [1, \infty)$. If $T$ is a compact operator on a Banach space $X$, then the following assertions are equivalent:

(i) $T$ is hyperbolic;
(ii) $T$ has the positive shadowing property;
(iii) $T$ has the positive limit shadowing property;
(iv) $T$ has the positive $\ell_p$ shadowing property.

Proof. Suppose that $T$ has the positive shadowing property (the positive limit shadowing property, the positive $\ell_p$ shadowing property). We have to prove that $T$ is hyperbolic. We may assume that $X$ is infinite-dimensional (because of Lemma 45) and that $\sigma(T)$ is not contained in $\mathbb{D}$. Since $T$ is a compact operator, it follows that the sets $\sigma_1 = \sigma(T) \cap \mathbb{D}$ and $\sigma_2 = \sigma(T) \setminus \mathbb{D}$ form a partition of $\sigma(T)$ into two nonempty closed sets. By the Riesz decomposition theorem, there are nontrivial $T$-invariant closed subspaces $M_1$ and $M_2$ of $X$ such that
\[
X = M_1 \oplus M_2, \quad \sigma(T|_{M_1}) = \sigma_1 \quad \text{and} \quad \sigma(T|_{M_2}) = \sigma_2.
\]

The compactness of $T$ also implies that $M_2$ is finite-dimensional. Hence, since $T|_{M_2}$ has the positive shadowing property (the positive limit shadowing property, the positive $\ell_p$ shadowing property), Lemma 45 tell us that $\sigma_2 \cap \mathbb{T} = \emptyset$. Thus, $\sigma(T) \cap \mathbb{T} = \emptyset$. 

Let us now show that the notions of hyperbolicity and positive shadowing coincide for normal operators.
Theorem 47. If $T$ is a normal operator on a Hilbert space $H$, then $T$ has the positive shadowing property if and only if $T$ is hyperbolic.

Proof. Suppose that $T$ has the positive shadowing property. Assume that $T$ is not hyperbolic and argue as in the proof of Theorem [11] to obtain a vector $z \in H$ such that

$$\frac{n\delta}{3} - 1 < \|T^n z\| < n\delta + 1 \quad \text{for all } n \in \mathbb{N}_0. \quad (19)$$

Consider the positive operator $S = T^* T$ and let $\mu$ be the spectral measure associated to $S$ and $z$. Since

$$0 \leq \int_{\sigma(S)} t^n d\mu(t) = \langle S^n z, z \rangle = \|T^n z\|^2 \leq (n\delta + 1)^2 \quad \text{for all } n \in \mathbb{N}_0,$$

it follows that $\sigma(S) \cap (1, \infty)$ has $\mu$-measure zero. Hence, $\|T^n z\| \leq (\mu(\sigma(S)))^{1/2}$ for all $n \in \mathbb{N}_0$, which contradicts (19). \qed

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