REGULAR COVERINGS IN FILTER AND IDEAL LATTICES

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Abstract. The Dedekind–Birkhoff theorem for finite-height modular lattices has previously been generalized to complete modular lattices, using the theory of regular coverings. In this paper, we investigate regular coverings in lattices of filters and lattices of ideals, and the regularization strategy—embedding the lattice into its lattice of filters or lattice of ideals, thereby possibly converting a covering which is not regular into a covering which is regular. One application of the theory is a generalization of the notion of chief factors, and of the Jordan-Holder Theorem, to cases where the modular lattice in question is of infinite height. Another application is a formalization of the notion of the steps in the proof of a theorem.

Introduction

The purpose of this paper is to further develop and apply the theory of regular coverings in a complete modular lattice, introduced in [3]. The point of that theory is to generalize, to complete modular lattices, some of the nice results available for finite-height modular lattices.

For example, given a finite-height module $M$ over a ring $R$ (i.e., a module having a finite-height lattice of submodules), a composition series of $M$ is a (necessarily finite) sequence of submodules

$$\{0\} = M_0 \subset M_1 \subset \ldots \subset M_n = M$$

such that each quotient $M_i/M_{i-1}$ is simple. The Jordan-Holder Theorem states that any two composition series $\{M_i\}^n_{i=1}, \{M'_i\}^{n'}_{i=1}$ are the same length $n = n'$ and the quotients $M_i/M_{i-1}$ can be paired with the quotients $M'_j/M'_{j-1}$ in such a way that corresponding quotients are isomorphic.

The Jordan-Holder Theorem is an algebraic version of the lattice-theoretic Dedekind-Birkhoff Theorem. The lattice-theoretic correlates of the composition series and the isomorphism of corresponding quotients are maximal chains (maximal linearly-ordered subsets) and projective equivalence of coverings. The Dedekind-Birkhoff Theorem states that in any two maximal chains in a finite-height modular lattice, the lengths of the chains are the same and coverings in the chains can be paired in such a way that corresponding coverings are projectively equivalent.

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Unfortunately, the Dedekind-Birkhoff Theorem can fail for infinite-height modular lattices. For example, consider the modular lattice

\[
\begin{array}{c}
y_1 \\
| \\
x_1 \\
| \\
x_2 \\
| \\
\vdots \\
| \\
\bot
\end{array}
\]

where $\bigwedge_i x_i = \bigwedge_i y_i = \bot$. There are two maximal chains

\[
C_1 : \quad \bot < \ldots < x_n < \ldots < x_2 < x_1 < y_1
\]

and

\[
C_2 : \quad \bot < \ldots < y_n < \ldots < y_2 < y_1
\]

such that the covering $x_1 < y_1$ appears in $C_1$, but there is no projectively equivalent covering in $C_2$.

The theory of regular coverings was created to try to remedy this situation. In the language of that theory, the covering $x_1 < y_1$ is not \textit{regular}, as defined in Section 2. If it were regular, such behavior would be impossible because of Theorem 4.
Now, note that if we embed the lattice $L$ into its lattice of filters $\text{Fil} L$, we obtain the lattice

\[
\begin{array}{c}
\text{Fg}\{y_1\} \\
\text{Fg}\{x_1\} \quad \text{Fg}\{y_2\} \\
\text{Fg}\{x_2\} \\
\vdots \\
\text{Fg}\{y_1, y_2, \ldots\} \\
\text{Fg}\{x_1, x_2, \ldots\} \\
\bot
\end{array}
\]

which contains new elements, $\text{Fg}\{x_1, x_2, \ldots\}$ and $\text{Fg}\{y_1, y_2, \ldots\}$, forming a covering equivalent to $\text{Fg}\{x\} \prec \text{Fg}\{y\}$. Now any two maximal chains in $\text{Fil} L$ have one covering equivalent to $\text{Fg}\{x\} \prec \text{Fg}\{y\}$. We have regularized the covering $x_1 \prec y_1$ by embedding $L$ into $\text{Fil} L$. In this paper, we explore this process and strategy of regularization further. Of course, this involves studying coverings in $\text{Fil} L$ (and in its dual, the lattice $\text{Idl} L$ of ideals of $L$) and trying to determine whether or not they are regular.

We also examine questions of multiplicity, since in a modular lattice that is not distributive, a maximal chain can contain more than one covering projectively equivalent to a given one.

We give two applications. One application is a generalization of the theory of chief factors of an algebra having a modular congruence lattice. The information supplied by these results is entirely lattice-theoretic; we leave for another time the algebraic correlates such as play roles in the Jordan-Holder Theorem. The other application is a way of defining the steps in the proof of a theorem. Any proof of the theorem from the same premises must cover, as we say, the same steps. Also, from any set of instances of rules of inference which covers the steps, a finite subset can be selected and used to construct a proof.

After this introduction and a section of preliminaries, this paper begins in Section 1 with some definitions relating to multiplicities. Given a maximal chain $C$ in the modular lattice $L$, and a covering $x \prec y$, there is a corresponding multiplicity of $x \prec y$ in $C$ which may vary with $C$, except in the important case when $x \prec y$ is weakly regular. We also define notations for upper and lower bounds on the multiplicity.

Section 2 discusses the theory of regular coverings, which, due to a generalization of the Dedekind-Birkhoff Theorem as given in [6], are also weakly regular.

Section 3 is a preliminary examination of coverings in filter and ideal lattices. As our strategy is to use regular coverings in such lattices for various purposes, we must understand their basic properties before attempting to determine whether or not they are regular. In
this section, among other things, we classify filter and ideal coverings into three categories: atomic, quasi-atomic, and anomalous coverings.

Section 3 gives proofs of the stability of regularity, multiplicity when regular, and in some cases the multiplicity upper bound, under the embedding from $L$ into $\text{Fil} L$ or $\text{Idl} L$.

Section 5 proves a relationship between the multiplicity upper bound of a covering $x \prec y$ in $L$ and the multiplicity lower bound of the corresponding filter or ideal covering. The important consequence of this is that under appropriate conditions, if the multiplicity bound is infinite, then any maximal chain in the filter or ideal lattice will have an infinite number of coverings equivalent to $\text{Fg}\{x\} \prec \text{Fg}\{y\}$ or $\text{Ig}\{x\} \prec \text{Ig}\{y\}$. This complements other theorems which describe the behavior when the multiplicity upper bound is finite.

Section 6 discusses upper regularity of filter coverings (and dually, lower regularity of ideal coverings). We show in this section that anomalous filter and ideal coverings cannot be regular. We also give an example of an atomic filter covering in an algebraic lattice that is not upper regular, and thus not regular.

Section 7 gives a proof that certain filter and ideal coverings are regular. In particular, we show that in a meet-continuous lattice, if the multiplicity upper bound of a covering $x \prec y$ is finite, then the corresponding filter covering is regular.

Section 8 discusses the application of these ideas to generalizing the Jordan-Holder Theorem.

Section 9 applies the theory to defining the steps in the proof of a proposition from given premises.

We will talk almost entirely about modular lattices, complete in most cases, except in Section 9, where we will talk about the distributive lattice underlying a boolean algebra $B$, and complete distributive lattices constructed from it.

0. Preliminaries

The reader should know about modular lattices and distributive lattices, and that distributive lattices are modular. The reader should also know about complete lattices.

We denote the least element of any lattice, if one exists, by $\bot$, and the greatest element by $\top$. If $x \leq y$ are elements of a lattice $L$, then we denote by $I_L[x, y]$, or simply $I[x, y]$, the interval sublattice of elements $z$ such that $x \leq z \leq y$.

A covering is a pair $\langle x, y \rangle$ of elements such that $x < y$ and $I[x, y]$ has only $x$ and $y$ as elements. We say that $x$ is covered by $y$, or $x \prec y$. We will often say $x \prec y$ not only to state that $x$ is covered by $y$, but also to denote a pair $\langle x, y \rangle$ satisfying the covering relation.

If $L$ is a lattice, we say that an element $m \in L$ is meet-irreducible if $x > m, y > m$ imply $x \wedge y > m$. If $L$ is complete, then we say that $m$ is strictly meet-irreducible if for all $S \subseteq L$ such that $s \in S$ implies $s > m$, $\bigwedge S > m$. Note that if $m$ is strictly meet-irreducible, then there is a unique element $m'$ such that $m \prec m'$.

If $x, y, z, and w \in L$ with $x \leq y$ and $z \leq w$, we write $\langle x, y \rangle \triangleright \langle z, w \rangle$ when $\langle x, y \rangle$ transposes up to $\langle z, w \rangle$, i.e., when $y \wedge z = x$ and $y \vee z = w$. When pairs $\langle x, y \rangle, \langle z, w \rangle$, such that $x \leq y$ and $z \leq w$, are related by the symmetric and transitive closure of $\triangleright$, we say that they are projectively equivalent, or $\langle x, y \rangle \sim \langle z, w \rangle$. 


Projective equivalence classes of coverings in modular lattices will be of fundamental importance to us. The projective equivalence class of a covering \( x \prec y \) will be denoted by \([x \prec y]\).

A lattice \( L \) is a \textit{chain} if the natural ordering in \( L \) is a total order. Also, if \( L \) is a lattice, and \( C \subseteq L \), then \( C \) is called a \textit{chain in} \( L \) if in the ordering inherited from \( L \), \( C \) is a chain. If \( C \) is a chain in \( L \), then we say \( C \) is \textit{maximal} if no larger subset of \( L \) is a chain in \( L \).

A complete lattice \( L \) is \textit{meet-continuous} if for all \( a \in L \) and \( D \subseteq L \) such that \( D \) is \textit{directed upward} (i.e., \( d, d' \in D \) imply there exists \( d'' \in D \) such that \( d \leq d'' \) and \( d' \leq d'' \)) we have

\[
a \wedge \bigvee_{d \in D} D = \bigvee_{d \in D} (a \wedge d).
\]

A lattice with the dual property is called \textit{join-continuous}.

For some other important concepts of lattice theory that we shall mention—in particular, lattices which are \textit{algebraic} or \textit{coalgebraic}—we refer to texts on lattice theory such as \cite{1} and \cite{2}. We will use the fact that algebraic lattices are meet-continuous, and coalgebraic lattices are join-continuous.

In section \cite{8}, we also assume an acquaintance with the basic concepts of Universal Algebra, as defined, for example, in \cite{3}. In particular, the concept of a \textit{congruence} will be used, and that of the \textit{congruence lattice} of an algebra. The reader should know that the congruence lattice of an algebra is always algebraic, and hence, meet-continuous.

The reader should know about cardinal and ordinal numbers, as used in transfinite induction. If \( \kappa \) is a cardinal number, then \( \text{Succ} \kappa \) will stand for the smallest cardinal number strictly greater than \( \kappa \).

1. Multiplicity and Multiplicity Bounds

\textbf{C-Multiplicity.} If \( L \) is a modular lattice, \( C \) is a chain in \( L \), and \( u, v \in C \) are such that \( u \prec v \), then we say that the covering \( u \prec v \) is \textit{in} \( C \). If \( x \prec y \) is a covering in \( L \), \( C \) is a chain in \( L \), and the set of coverings \( u \prec v \in C \) such that \( u \prec v \sim x \prec y \) has cardinality \( n \), then we say that the \textit{C-multiplicity} of \( x \prec y \) (in \( L \)), denoted by \( \mu_C[x \prec y] \), is \( n \).

\textbf{Weak regularity.} We say that a covering \( x \prec y \) is \textit{weakly regular} if \( \mu_C[x \prec y] \), for maximal chains \( C \), is a number \( \mu[x \prec y] \), the \textit{multiplicity} of \( x \prec y \), independent of \( C \).

If \( x \prec y \) is weakly regular, with finite multiplicity, then we can talk not only about the multiplicity of \( x \prec y \) in \( L \), but in any interval sublattice \( I[a, b] \) of \( L \) where \( a < b \):

\textbf{Theorem 1.} If \( L \) is a modular lattice and \( x \prec y \) is weakly regular in \( L \), with finite multiplicity, then given \( a, b \in L \) with \( a < b \), the number of coverings \( u \prec v \) equivalent to \( x \prec y \) in any maximal chain of elements in the interval sublattice \( I[a, b] \) is a number \( \mu^{a, b}[x \prec y] \) independent of the particular chain \( C \).

\textit{Proof.} Any two maximal chains \( C, C' \in I[a, b] \) can be completed to maximal chains in \( L \) by including the elements of the same maximal chains in \( I[\bot, a] \) and \( I[a, \top] \) (where we first adjoin a \( \bot \) and a \( \top \) to \( L \) if not already present). Then we use the fact that \( \mu_{C'}[x \prec y] = \mu_{C'}[x \prec y] \).
Multiplicity upper bounds and lower bounds. Let \( x \prec y \) be a covering in a modular lattice \( L \). We define \( \nu[x \prec y] \), the multiplicity upper bound of \( x \prec y \) in \( L \), to be the least cardinal number \( \nu \) such that for every chain \( C \) in \( L \), \( \mu_C[x \prec y] < \nu \). We define \( \lambda[x \prec y] \), the multiplicity lower bound of \( x \prec y \) in \( L \), to be the least cardinal \( \nu \) such that \( \mu_C[x \prec y] = \nu \) for some maximal chain \( C \).

**Proposition 2.** If \( x \prec y \) is weakly regular in \( L \), then
\[
\lambda[x \prec y] = \mu[x \prec y]
\]
and
\[
\nu[x \prec y] = \text{Succ} \, \mu[x \prec y].
\]

**Distributive lattices.** The \( C \)-multiplicity is severely constrained for distributive lattices:

**Theorem 3.** If \( L \) is a distributive lattice, \( C \) is a maximal chain in \( L \), and \( x \prec y \) is a covering in \( L \), then \( \mu_C[x \prec y] \) is 0 or 1.

**Proof.** Assume that \( u \prec v \prec z \prec w \) and \( u' \prec v' \prec z' \prec w \). Then we claim that \( u \land u' \prec v \land v' \) and \( u \land u' \prec v \land v' \prec u \prec v \). For,
\[
(v \land v') \land u = (v \land v') \land (v \land z) \\
= (v \land z) \land (v' \land z) \\
= u \land u',
\]
and
\[
(v \land v') \lor u = (v \lor u) \land (v' \lor u) \\
= v \land (v' \lor u) \\
= v \land (v' \lor (v \land z)) \\
= v \land (v' \lor v) \land (v' \lor z) \\
= v \land (v' \lor v) \land w \\
= v.
\]
Similarly, \( u \land u' \prec v \land v' \prec u' \prec v' \).

It follows that if \( u \prec v \sim u' \prec v' \), then we must have some covering \( z \prec w \) such that \( z \prec w \sim u \prec v \) and \( z \prec w \sim u' \prec v' \). Therefore, \( u, v, u', \) and \( v' \) cannot all be elements of the same chain \( C \).

**Remark.** As a result of his theorem, if \( L \) is a distributive lattice, and \( a \prec b \in L \), then we can talk about the set of weakly regular coverings \( x \prec y \) in \( I[a, b] \). We will do so in the last section of this paper.
2. The Theory of Regular Coverings

Upper regular and lower regular coverings. We say that a covering \( x \prec y \) in a complete modular lattice \( L \) is upper regular if, for every chain \( I \), and mapping taking elements \( i \in I \) to coverings \( x_i \prec y_i \) of \( L \), projectively equivalent to \( x \prec y \) and such that \( i < j \) implies \( x_i \prec y_i \not\succ x_j \prec y_j \), we have \( \bigvee_i x_i \prec \bigvee_i y_i \) (rather than the only other possibility, for a modular lattice, which would be \( \bigvee_i x_i = \bigvee_i y_i \)). The property dual to upper regularity, we call lower regularity. We say that a covering is regular if it is both upper regular and lower regular.

Clearly, whether or not a covering is upper regular, lower regular, or regular depends only on the projective equivalence class of the covering.

The importance of the concept of regularity comes from a generalization of the Dedekind-Birkhoff Theorem, proved in [3]:

Theorem 4. If \( L \) is a complete modular lattice, and \( C, C' \) are any two maximal chains in \( L \), then for every regular covering \( x \prec y \), \( \mu_C[x \prec y] = \mu_{C'}[x \prec y] \).

Thus, if \( x \prec y \) is regular, we can drop the \( C \) from \( C \)-multiplicity and speak of the multiplicity \( \mu[x \prec y] \) of \( x \prec y \) in \( L \). In other words, if \( x \prec y \) is regular, then \( x \prec y \) is weakly regular.

A partial converse to Theorem 4:

Theorem 5. Let \( L \) be a complete modular lattice. If \( x \prec y \) is weakly regular, and furthermore, \( \mu[x \prec y] \) is finite, then \( x \prec y \) is regular.

Proof. Let \( I \) be a chain, and let coverings \( x_i \prec y_i \sim x \prec y \) be indexed by \( I \), such that \( i < j \) implies \( x_i \prec y_i \not\succ x_j \prec y_j \). Then, for any arbitrary \( i \in I \), consider the chain \( x_i \leq \cdots \leq x_j \leq \cdots \leq \bigvee_i x_i \leq \bigvee_i y_i \) and the chain \( x_i \prec y_i \leq \cdots \leq y_j \leq \cdots \leq \bigvee_i y_i \). If we take any refinement of the first of these chains to a maximal chain \( C \), we can find a refinement of the second chain to a maximal chain \( C' \), by letting \( C' \) consist of the elements of \( C \) less than or equal to \( x_i \), the lattice elements \( c \lor y_i \) for \( c \in C \) such that \( x_i \leq c \leq \bigvee_i x_i \), and the elements of \( C \) greater than or equal to \( \bigvee_i y_i \). By the modular law, the coverings in \( C \) between \( x_i \) and \( \bigvee_i x_i \) correspond in a one-to-one fashion with the coverings in \( C' \) between \( y_i \) and \( \bigvee_i y_i \), and corresponding coverings are projectively equivalent. Since \( \mu_C[x \prec y] = \mu_{C'}[x \prec y] \), and that number is finite, we must have \( \bigvee_i x_i \prec \bigvee_i y_i \), proving that \( x \prec y \) is upper regular. Lower regularity is proved similarly.

In order to apply Theorem 4, it helps to know which coverings are regular. Some preliminary observations in this direction are as follows: If \( L \) is finite, or of finite height, then all coverings in \( L \) are regular. It is easy to see that in any complete, modular, meet-continuous lattice, every covering is upper regular. Dually, in any complete, modular, join-continuous lattice, every covering is lower regular.
3. Coverings in Filter and Ideal Lattices

In this section, we will explore coverings in filter and ideal lattices. If $L$ is a lattice, a filter in $L$ is a nonempty subset $F$ such that if $x \in F$, and $y \geq x$, then $y \in F$, and also, if $x, y \in F$, then $x \wedge y \in F$. If $F$ and $G$ are filters in $L$, we say that $F \leq G$ if $G \subseteq F$. With this partial ordering, the filters of a lattice $L$ with $\top$ form a lattice $\text{Fil}L$ which is complete and co-algebraic. We have $F \vee G = F \cap G$, while $F \wedge G = \{ z \in L : z \geq x \wedge y, \text{for some } x \in F \text{ and } y \in G \}$. If $S \subseteq L$ is a nonempty subset, we write $Fg(S)$ for the smallest (in the sense of set inclusion) filter containing $S$, called the filter generated by $S$. An important special case is $Fg\{ x \}$, the principal filter generated by $x \in L$, which is $\{ y \in L : y \geq x \}$. The mapping $x \mapsto Fg\{ x \}$ is a lattice homomorphism embedding $L$ into $\text{Fil}L$. As another important example of a filter, if $m$ is a meet-irreducible element, then we denote by $F_{>m}$ the set of elements of $L$ strictly greater than $m$. $F_{>m}$ is obviously a filter, and is principal iff $m$ is not just meet-irreducible, but strictly meet-irreducible.

The dual concept, that of an ideal, leads to the lattice of ideals $\text{Idl}L$, which is complete and algebraic. If $S \subseteq L$, we write $\text{Ig}S$ for the smallest ideal containing $S$, and call it the ideal generated by $S$. If $x \in L$, the principal ideal generated by $x$, $\text{Ig}\{ x \} = \{ y \in L : y \leq x \}$ is an important example. $\text{Idl}L$ is ordered by inclusion as opposed to $\text{Fil}L$, which is ordered by reverse inclusion. The mapping $x \mapsto \text{Ig}\{ x \}$ is a lattice homomorphism embedding $L$ into $\text{Idl}L$.

Both the lattices $\text{Fil}L$ and $\text{Idl}L$ satisfy every lattice-theoretic identity satisfied by $L$; in particular, they are modular if $L$ is modular.

For the most part, we will concentrate our attention on filters of a modular lattice $L$, leaving to the reader the dualization of the statements and proofs of the theorems to yield similar results about ideals of $L$.

Filter coverings $F \prec G$ and $\mathcal{M}(F - G)$. If $L$ is a lattice and $S \subseteq L$, then we denote by $\mathcal{M}(S)$ the set of maximal elements of $S$ (in the partial ordering of $L$). If $F \prec G$ is a covering in $\text{Fil}L$, or, as we say, a filter covering, we will be particularly interested in $\mathcal{M}(F - G)$. We have

Lemma 6. Let $L$ be a lattice, and $F, G, F', G' \in \text{Fil}L$ such that $F \prec G$, $F' \prec G'$, and $F \prec G \triangleright F' \prec G'$. Then

1. $F' - G' = F' \cap (F - G)$, and
2. $\mathcal{M}(F' - G') = F' \cap \mathcal{M}(F - G)$.

Proof. (1): $F' \cap (F - G) = (F' \cap F) - (F' \cap G) = F' - G'$.

(2): $x \in \mathcal{M}(F' - G') \implies x \in F'\cap(F-G)$ by (1). If, in addition, $y > x$, then $y \in G'$ which implies $y \in G$. Thus, $x \in F' \cap \mathcal{M}(F - G)$. On the other hand, if $x \in F' \cap \mathcal{M}(F - G)$, then $x \in F' - G'$ by (1), and $y > x \implies y \in G \implies y \in F' \cap G = G'$. Thus, $x \in \mathcal{M}(F' - G')$. 

Filter coverings and maximal based filters. If $F \prec G$ is a filter covering in $\text{Fil}L$, then any $x \in F - G$ determines a principal filter $Fg\{ x \}$. We have $F \prec G \triangleright Fg\{ x \} \prec (Fg\{ x \} \vee G)$. Let $H = Fg\{ x \} \vee G = Fg\{ x \} \cap G$. We say that a pair $(x, H)$, such that
Lemma 7. Let \( L \) be a lattice. Given two filter coverings \( F \prec G \) and \( H \prec K \) in \( \text{Fil} L \), and given \( x \in F - G \) and \( y \in H - K \), \( F \prec G \sim H - K \) iff \( \langle x, Fg\{ x \} \lor G \rangle \sim \langle y, Fg\{ y \} \lor K \rangle \).

Proof. It suffices to prove that if \( F \prec G \not\prec H \prec K \), then for any such \( x \in F - G \) and \( y \in H - K \), we have \( \langle x, Fg\{ x \} \lor G \rangle \sim \langle y, Fg\{ y \} \lor K \rangle \).

\( y \in H - K \) implies \( y \in F - G \), so we have \( F = G \lor Fg\{ y \} \). Thus, there exists \( g \in G \) such that \( g \lor y \leq x \). Then we have \( \langle g \lor y, Fg\{ g \lor y \} \lor G \rangle \not\prec \langle x, Fg\{ x \} \lor G \rangle \) and \( \langle g \lor y, Fg\{ g \lor y \} \lor G \rangle \not\prec \langle y, Fg\{ y \} \lor K \rangle \), whence \( \langle x, Fg\{ x \} \lor G \rangle \sim \langle y, Fg\{ y \} \lor K \rangle \).

Corollary 8. If \( x \prec y \) and \( z \prec w \), then \( x \prec y \sim z \prec w \) in \( L \) iff \( Fg\{ x \} \sim Fg\{ y \} \sim Fg\{ z \} \sim Fg\{ w \} \) in \( \text{Fil} L \).

Proof. This follows from Lemma 7 and from the fact that \( x \prec y \not\prec z \prec w \) iff \( \langle x, Fg\{ y \} \rangle \not\prec \langle z, Fg\{ w \} \rangle \).

Atomic filter coverings. Let \( \langle x, F \rangle \) be a maximal based filter. We say that \( \langle x, F \rangle \), or a filter covering \( F' \prec G' \) such that \( Fg\{ x \} \prec F \sim F' \prec G' \), is atomic if \( F \) is principal.

As an example, if \( m \in L \) is strictly meet-irreducible, then \( \langle m, F_{>m} \rangle \) is an atomic maximal based filter.

Theorem 9. Let \( L \) be a modular lattice. The set of atomic maximal based filters in \( L \), and the set of atomic filter coverings, are closed under projective equivalence. If \( F \prec G \) is a filter covering, and \( m \in F - G \) is strictly meet-irreducible, then \( F \prec G \) is atomic.

Proof. Let \( \langle x, F \rangle \not\prec \langle y, G \rangle \). If \( F \) is principal, say \( F = Fg\{ x' \} \), then \( G = Fg\{ y \} \cap Fg\{ x' \} = Fg\{ y \lor x' \} \) is also principal.

On the other hand, if \( G \) is principal, say \( G = Fg\{ y' \} \), then let \( \bar{x} \in F \) be such that \( x = \bar{x} \lor y \), and let \( x' = \bar{x} \lor y' \). We cannot have \( x' = x \) because both \( \bar{x} \) and \( y' \) belong to \( F \). Thus, by modularity, \( x' \not\sim x \). But, this implies that \( F = Fg\{ x' \} \). Thus, the set of atomic maximal based filters is closed under projective equivalence, and by Theorem 7, the same is true of the set of atomic filter coverings.

If \( F \prec G \) and \( m \in F - G \) is strictly meet-irreducible, then \( F \prec G \sim Fg\{ m \} \prec (G \cap Fg\{ m \}) \). However, \( F_{>m} \) is the unique cover of \( Fg\{ m \} \) and is principal. It follows that \( G \cap Fg\{ m \} = F_{>m} \), and \( F \prec G \) is atomic.
Theorem 10. Let $L$ be a complete, meet-continuous modular lattice, and let $F \prec G$ be an atomic filter covering. Then $\mathcal{M}(F - G)$ is nonempty and consists of strictly meet-irreducible elements.

Proof. Let $x \in F - G$. Then $F \prec G \supseteq F_g \{x\} \prec (F_g \{x\} \vee G) = F_g \{x'\}$ where $x' > x$, because $F \prec G$ is atomic. The set of elements $y$ such that $y \geq x$ and $y \wedge x' = x$ is closed under joins of chains, by meet-continuity. Then by Zorn’s Lemma, $\mathcal{M}(F_g \{x\} - F_g \{x'\}) = F_g \{x\} \cap \mathcal{M}(F - G)$ is nonempty. Thus, $\mathcal{M}(F - G)$ is nonempty. It is easy to see that because $F \prec G$ is atomic, $\mathcal{M}(F - G)$ consists of strictly meet-irreducible elements. \hfill \Box

Quasi-atomic filter coverings. We say that a maximal based filter $\langle x, F \rangle$, or a filter covering $F' \prec G'$ such that $F_g \{x\} \prec F \sim F' \prec G'$, is quasi-atomic if $F$ is not principal, but contains an element $y$ such that $x < z \leq y$ implies $z \in F$.

As an example, if $m \in L$ is meet-irreducible, but not strictly meet-irreducible, then $\langle m, F_{\geq m} \rangle$ is a quasi-atomic maximal based filter.

Theorem 11. Let $L$ be a modular lattice. The set of quasi-atomic maximal based filters, and the set of quasi-atomic filter coverings, are closed under projective equivalence. If $F \prec G$ is a filter covering, and $m \in F - G$ is meet-irreducible but not strictly meet-irreducible, then $F \prec G$ is quasi-atomic.

Proof. Let $\langle x, F \rangle \not\supseteq \langle y, G \rangle$. If $\langle x, F \rangle$ is quasi-atomic, then $F$ contains an element $x'$ such that $x < z \leq x' \implies z \in F$. Consider the element $y' = x' \vee y \in G = F_g \{y\} \cap F$. If $y < z \leq y'$, then $x' \wedge z > x$, because by modularity, $y \vee (x' \wedge z) = (y \vee x') \wedge z = y' \wedge z = z > y$. Thus, $y < z \leq y' \implies z \in G = F \cap F_g \{y\}$, and $\langle y, G \rangle$ is quasi-atomic because if it were atomic, then $\langle x, F \rangle$ would also be atomic by Theorem 10.

On the other hand, if $\langle y, G \rangle$ is quasi-atomic, then there is an element $y' \in G$ such that $y < z \leq y'$ implies $z \in G$. Let $\bar{x} \in F$ be such that $x = \bar{x} \wedge y$, and let $x' = \bar{x} \wedge y'$. We have $x' \in F$, so $y \vee x' \in G$. If $x < z \leq x'$, then by modularity, $z = z \vee (y \wedge x') = (z \vee y) \wedge x'$. But, $y < z \vee y \leq y'$ because if $y = z \wedge y$, then $z = (z \vee y) \wedge x'$. Thus, $z \vee y \in G$ and $z \in F$. It follows that $\langle x, F \rangle$ is atomic or quasi-atomic, but $\langle x, F \rangle$ cannot be atomic, because then $\langle y, G \rangle$ would also be atomic by Theorem 10.

Now, if $F \sim G$ and $m \in F - G$ is meet-irreducible, but not strictly meet-irreducible, we have $F \prec G \not\supseteq F_g \{m\} \sim (F_g \{m\} \vee G)$. However, $F_{\geq m}$ is the unique cover of $F_g \{m\}$ in $\text{Fil} \ L$. Thus, $F \prec G \sim F_g \{m\} \prec F_{\geq m}$, which is quasi-atomic. \hfill \Box

Theorem 12. Let $L$ be a complete, meet-continuous modular lattice. If $F \prec G$ is a filter covering in $L$ that is quasi-atomic, then $\mathcal{M}(F - G)$ is a nonempty set of elements of $L$ that are meet-irreducible, but not strictly meet-irreducible.

Proof. Similar to the proof of Theorem 10. Instead of $F_g \{x\} \vee G = F_g \{x'\}$, we have an $x' \in F_g \{x\} \vee G$ such that $F_g \{x\} \vee G = F_g \{z \mid x < z \leq x'\}$. The set of elements $y$ such that $y \geq x$ and $y \wedge x' = x$ is again closed under joins of chains by meet-continuity, and nonempty by Zorn’s Lemma. Thus $\mathcal{M}(F_g \{x\} - (F_g \{x\} \vee G))$ is nonempty, and so is $\mathcal{M}(F - G)$ by Lemma 11. \hfill \Box
Anomalous filter coverings. We say that a maximal based filter $\langle x, F \rangle$, or a filter covering $F' \prec G'$ such that $Fg\{x\} \prec F \sim F' \prec G'$, is anomalous if it is neither atomic nor quasi-atomic.

Recall that $x \in L$ is called finitely decomposable if $x$ is a finite meet of meet-irreducible elements. For an example of an anomalous filter covering, let $x \in L$ be an element which is not finitely decomposable. (This is possible only if $L$ does not satisfy the ascending chain condition.) Let $G$ be the filter generated by the set of finitely decomposable elements of $L$ that are greater than $x$. (This is the same as the filter generated by the set of meet-irreducible elements of $L$ that are greater than $x$.) We have $x \notin G$ because otherwise, $x$ would be finitely decomposable. By Zorn’s Lemma, there is a filter $F \leq G$ such that $Fg\{x\} \prec F \sim F' \prec G'$, and $F$ is minimal (in the ordering of $\text{Fil} L$) for that property. Then by Theorems [10] and [12] $\langle x, F \rangle$ is an anomalous maximal based filter, because it cannot be atomic or quasi-atomic. The following theorem shows, among other things, that this example is typical:

**Theorem 13.** The set of anomalous maximal based filters, and the set of anomalous filter coverings, are closed under projective equivalence. If $F \prec G$ is an anomalous filter covering, then $\mathcal{M}(F \prec G)$ is empty, and $F - G$ contains no elements that are finitely decomposable.

*Proof.* The sets of atomic and quasi-atomic filter coverings are closed under projective equivalence. Since the set of anomalous filter coverings comprises the rest of the filter coverings, it is also closed under projective equivalence. If $F \prec G$ and $x \in \mathcal{M}(F - G)$, then clearly $x$ ismeet-irreducible. Thus, if $F \prec G$ is anomalous, $\mathcal{M}(F - G)$ must be empty by Theorems [10] and [12]. Finally, if $x$ is finitely decomposable, then $x = \bigwedge_{i=1}^{n} m_i$ where the $m_i$ are meet-irreducible. If $x \in F - G$, then $m_i$ also belongs to $F - G$ for some $i$, because $G$ is closed under finite meets. This would imply that $F \prec G$ was atomic or quasi-atomic. □

A counterexample. In working with the $\triangleright$ relation and filters, we might make the following conjecture: Let $L$ be a modular lattice, and $F, G, H, K$ filters such that $F \prec G$, $H \prec K$, and $F \prec G \triangleright H \prec K$. If $x \in F - G$, then there exists $w \in H - K$ such that $x \leq w$. However, this is false:

**Example 14.** Consider the modular lattice known as $M_5$, with its elements labeled as follows:

```
   ┌─┐   ┌─┐   ┌─┐
  ⊤ / \ / \ / \ / \ a
  │   │   │   │
  b ┌─┐   ┌─┐   ┌─┐
  │   │   │   │
  c └─┘   └─┘   └─┘
  ⊥   ⊥   ⊥
```

Let $F = M_5$, $G = Fg\{b\} = \{b, \top\}$, $H = Fg\{c\} = \{c, \top\}$, and $K = Fg\{\top\} = \{\top\}$. Then $F \prec G \triangleright H \prec K$. Observe that we have $a \in F - G$, but no element $w \in H - K$ such that $a \leq w$. 
4. Stability Theorems

We will shortly begin to address the question of when a covering in $\Fil L$ is regular. First, however, we pose and answer some other important questions, such as, under what circumstances do regular coverings in $L$ remain regular after the embedding from $L$ into $\Fil L$ or $\Idl L$? We also examine stability of multiplicity, in case a covering is regular, and of the multiplicity upper bound. We continue to focus on $\Fil L$. A lemma:

**Lemma 15.** Let $L$ be a complete modular lattice, and $x < y$ a covering in $L$ which is lower regular. If $F \prec G$ is a covering in $\Fil L$ and $F \prec G \sim Fg\{x\} \prec Fg\{y\}$, then let $f = \bigwedge F$ and $g = \bigwedge G$; we have

1. $f < g$,
2. $f < g \sim x < y$, and
3. $\mathcal{M}(Fg\{f\} - Fg\{g\}) = \mathcal{M}(F - G)$.

**Proof.** For all $w \in F - G$, $Fg\{w\} \wedge G = F$. It follows that $w \wedge g = f$ and hence, $Fg\{w\} \wedge Fg\{g\} = Fg\{f\}$.

We must show that $f \neq g$, which we will show by showing that if $w \in F - G$, then we cannot have $g \leq w$, or in other words, we cannot have $\bigwedge \nu g \nu \leq w$ for any $\kappa$-tuple $\{g \nu\}_{\nu < \kappa}$ of elements of $G$, for any cardinal number $\kappa$, where $\nu$ runs through ordinals less than $\kappa$. Assume the contrary, where $\kappa$ is the least cardinal possible. By modularity, and the fact that $F \prec G$ is atomic, we can assume w.l.o.g. that $g \nu \wedge w < g \nu$ for each $\nu$. For each ordinal $\nu \leq \kappa$, let $h \nu = \bigwedge_{\nu' < \nu} g \nu'$. By the minimality of $\kappa$, we have $h \nu \not\leq w$ if $\nu < \kappa$, so we must have $h \nu \wedge w < h \nu$ for $\nu < \kappa$. Note $\kappa$ cannot be finite, because $G$ is a filter. By lower regularity, we have $h \kappa \wedge w < h \kappa$, contradicting the assumption that $h \kappa \leq w$.

We have $Fg\{g\} \not\leq F$, $Fg\{g\} \leq G$, and $Fg\{f\} = Fg\{g\} \wedge F$, whence $Fg\{f\} \prec Fg\{g\}$, proving (1). Also, $Fg\{f\} \prec Fg\{g\} \not\prec Fg\{w\} \prec Fg\{w\} \cap G$ for any $w \in F - G$. (2) follows by corollary 3.

We have $\mathcal{M}(F - G) \subseteq \mathcal{M}(Fg\{f\} - Fg\{g\})$. For, let $m \in \mathcal{M}(F - G)$. Then $m \in Fg\{f\}$, and we cannot have $m \in Fg\{g\}$, because then we would have $g \leq m$, contradicting the fact that $m \wedge g = f$. $m \in \mathcal{M}(Fg\{f\} - Fg\{g\})$ because $m$ is strictly meet-irreducible.

On the other hand, let $m \in \mathcal{M}(Fg\{f\} - Fg\{g\})$. Since $m \notin Fg\{g\}$, $m \notin G$. It suffices to show $m \in F$, because, $m$ being strictly meet-irreducible, $M \in F - G$ will imply $m \in \mathcal{M}(F - G)$. Let $\kappa$ be the least cardinal number such that some $\kappa$-tuple $\{f \nu\}_{\nu < \kappa}$ of elements of $F$, where $\nu$ runs through ordinals less than $\kappa$, satisfies $\bigwedge \nu f \nu \leq m$. $m$ is strictly meet-irreducible because $Fg\{f\} \prec Fg\{g\}$ is atomic. Let $m'$ be the unique cover of $m$, and for each $\nu \leq \kappa$, define $u \nu = \bigwedge_{\nu' < \nu} f \nu'$. We have $m \vee u \nu \geq m'$ if $\nu < \kappa$, by the minimality of $\kappa$. Thus, for each $\nu < \kappa$, we have by modularity

$$m \wedge u \nu < m' \wedge u \nu \not\prec m < m'.$$

If $\nu' < \nu < \kappa$, then it is easy to see that

$$m \wedge u \nu < m' \wedge u \nu \not\prec m \wedge u \nu' < m' \wedge u \nu'.$$
Proof. Let $\bar{x}$. It suffices to prove upper regularity, because Fil $C$.

Now, if $\kappa$ is infinite, then by the lower regularity of $x < y$, we must have $m \wedge u_\kappa < m' \wedge u_\kappa$. However, this is absurd because $u_\kappa \leq m$. Thus, $\kappa$ is finite. It follows that $m \in F$, proving (3).

**Theorem 16.** Let $L$ be a complete, meet-continuous modular lattice, and $x < y$ a covering in $L$ which is regular. Then $Fg\{x\} \prec Fg\{y\}$ is regular in Fill $L$.

Proof. It suffices to prove upper regularity, because Fill $L$ is coalgebraic, which implies that all coverings are automatically lower regular. Suppose given $F_i \prec G_i \rightsquigarrow F_j \prec G_j$ for $i < j$. We must show that $\bigvee_i F_i \prec \bigvee_i G_i$.

For each $i \in I$, let $M_i = \mathcal{M}(F_i - G_i)$. We have $M_j \subseteq M_i$ for $i < j$ by Lemma 6, and $\bigcap_i M_i \subseteq \bigvee_i F_i - \bigvee_i G_i$. For, if $x \in M_i$ for all $i$, then $x \in F_i$ for all $i$ so $x \in \bigvee_i F_i = \bigcap_i F_i$, and $x \notin G_i$ for all $i$, so $x \notin \bigvee_i G_i = \bigcap_i G_i$. Thus, it suffices to show that $\bigcap_i M_i$ is nonempty.

For each $i$, let $f_i = \bigwedge F_i$ and $g_i = \bigwedge G_i$. Since $x < y$ is lower regular, we have $f_i < g_i$ and $f_i < g_i \sim x < y$ by Lemma 15(1) and (2), and for $i < j$, we have $f_i < g_i \not\prec f_j < g_j$. For, $f_i \leq f_j$, $f_i < g_i$, and $g_i \not\leq f_j$, because for any $x \in F_j - G_i$, we have $F_i = G_i \wedge Fg\{x\}$ and consequently, $f_i = g_i \wedge x$. It follows that $f_i = g_i \wedge f_j$. We also have $g_i \leq g_j$ and $f_j \prec g_j$, whence $g_j = g_i \wedge f_j$.

Since $x < y$ is upper regular, we have $f \prec g$ where $f = \bigvee_i f_i$ and $g = \bigvee_i g_i$. This implies that $\mathcal{M}(Fg\{f\} - Fg\{g\})$ is nonempty, by Theorem 10. Let $x \in \mathcal{M}(Fg\{f\} - Fg\{g\})$. Then $x \geq f_i$ for all $i$ so $x \notin Fg\{f_i\}$ for all $i$. On the other hand, $x \not\geq g$ so there is an $i$ such that $x \notin Fg\{g_i\}$. If $j > i$ then $g_j \geq g_i$, so $x \notin Fg\{g_j\}$. If $y > x$, then $y \geq g$ and $y \in G_k$ for all $k$. Thus, $x \in \mathcal{M}(Fg\{f_j\} - Fg\{g_j\})$ for all $j \geq i$. On the other hand, if $j < i$, then by Lemma 8, $\mathcal{M}(Fg\{f_i\} - Fg\{g_i\}) = Fg\{f_i\} \cap \mathcal{M}(Fg\{f_j\} - Fg\{g_j\})$, implying that $x \in \mathcal{M}(Fg\{f_j\} - Fg\{g_j\})$ in this case as well. It follows that $\bigcap_i \mathcal{M}(Fg\{f_i\} - Fg\{g_i\})$ is nonempty. However, by Lemma 15(3), $\mathcal{M}(Fg\{f_i\} - Fg\{g_i\}) = M_i$ for each $i$. Thus, $\bigcap_i M_i$ is nonempty, and $F \prec G$ is regular.

Now, we consider the multiplicity:

**Lemma 17.** If $L$ is a complete modular lattice and $x < y$ is lower regular in $L$, then for any maximal chain $C \subseteq L$ and any maximal chain $C' \subseteq Fg\{x\}$ refining the image of $C$ in Fill $L$, $\mu_{\bar{C}}[Fg\{x\} \prec Fg\{y\}] = \mu_{\bar{C}}[x \prec y]$.

**Proof.** Let $C$ be a maximal chain in $L$, and $\bar{C}$ a maximal chain in Fill $L$ refining the image of $C$.

If we have a principal filter $Fg\{u\} \in \bar{C}$, then we must have $u \in C$. For, if $u \notin \bar{C}$, then there must exist $c \in C$ such that $u$ and $c$ are not comparable. But, $Fg\{c\} \in \bar{C}$, so either $Fg\{c\} < Fg\{u\}$, implying $c < u$, or $Fg\{u\} < Fg\{c\}$, implying $u < c$.

If $F, G \in \bar{C}$ are principal and such that $F \prec G$, say $F = Fg\{u\}$ and $G = Fg\{v\}$, then we must have $u, v \in C$ with $u \prec v$, and if $F \prec G \sim Fg\{x\} \prec Fg\{y\}$ then $u \prec v \sim x \prec y$.

Let $F, G \in \bar{C}$ be such that $F \prec G$ and it is not true that $F$ and $G$ are both principal. We will show that $F \prec G$ is not projectively equivalent to $Fg\{x\} \prec Fg\{y\}$. 

If $F$ is principal and $G$ is not, then $F \prec G$ is not of atomic type, is not projectively equivalent to $F_g \prec F_g$, and does not count in the multiplicity of $x \prec y$.

The case $G$ principal and $F$ non-principal cannot occur, because $F \prec G$.

The only remaining case is that both $F$ and $G$ are non-principal. We can assume that $F \prec G$ is atomic and lower regular, since otherwise $F \prec G \sim F_g \prec F_g$ is impossible. If we had $c \in C$ with $c \in F - G$, then we would have $F < F_g \{c\} < G$; thus, we must have $C \cap F = C \cap G$ in order to have $F \prec G$. Denote this set by $D$. Since $F \prec G$ is atomic, let $z \in F - G$ and $w \in G$ with $z \prec w$. For each $d \in D$, we have $d \land z \prec d \land w$, and for $d < d' \in D$, we have $d \land z \prec d \land w \wedge d' \land z \prec d' \land w$. Since $z \prec w$ is lower regular in $L$, we would have

$$(\bigwedge D) \land z = \bigwedge_{d \in D} d \land z < \bigwedge_{d \in D} d \land w = (\bigwedge D) \land w.$$  

Now, if we had $\bigwedge D \in C - D$, we would have $F_g \{\bigwedge D\} < F \prec G$, implying that $\bigwedge D \land z = \bigwedge D \land w$, which is impossible.

The only other possibility is $\bigwedge D \in D$. In this case, we claim that we must have $\bigwedge D \subseteq w$ and $\bigwedge D \land z \in C$. For, if $c \in C - D$, then $F_g \{c\} \leq F$, implying that $c \leq \bigwedge D \land z$. Then because $C$ is maximal, we must have $\bigwedge D \land z \in C - D$ and $\bigwedge D \land w = \bigwedge D \in D$. However, this contradicts the fact that $F \cap C = D$, because $\bigwedge D \land z \in F$. It follows that the case $F \prec G$ atomic, lower regular, and neither $F$ nor $G$ principal is impossible.

It follows from the Lemma that we have

**Theorem 18.** If $x \prec y$ is regular in $L$, then $\mu[F_g\{x\} \prec F_g\{y\}] = \mu[x \prec y]$.

Finally, we examine the stability properties of the multiplicity upper bound.

**Theorem 19.** If $x \prec y$ is a covering in a modular lattice $L$, then $v[x \prec y] \leq v[F_g\{x\} \prec F_g\{y\}]$, with equality if $v[x \prec y]$ is finite or countable. In any case, $v[x \prec y]$ infinite implies $v[F_g\{x\} \prec F_g\{y\}]$ infinite.

**Proof.** Let $C$ be a chain in $L$. Then the image of $C$ in $\text{Fil} L$ can be refined to a maximal chain $\bar{C}$ in $\text{Fil} L$, and it is clear that $\mu_C[x \prec y] = \mu_{\bar{C}}[F_g\{x\} \prec F_g\{y\}]$.

Now, let

$$F_1 \prec G_1 \leq F_2 \prec G_2 \leq \ldots \leq F_n \prec G_n$$

where $F_i \prec G_i \sim F_g\{x\} \prec F_g\{y\}$ for all $i$. Then $\exists u_i \in F_i - G_i$, $v_i \in G_i$ such that $u_i \prec v_i$ and $u_i \prec v_i \sim x \prec y$. For each $i$, define $c_i = \bigwedge_{j > i} u_i$, $d_i = v_i \land c_i$. Then $c_i = d_i \land u_i$, $d_i \in G_i$, $c_i \in F_i - G_i$, and $d_i \leq v_i$, implying that $c_i \prec d_i \sim x \prec y$ and

$$c_1 \prec d_1 \leq c_2 \prec d_2 \leq \ldots \leq c_n \prec d_n.$$  

It follows that $v[x \prec y] \geq n$, and combined with the fact that $v[x \prec y] \leq v[F_g\{x\} \prec F_g\{y\}]$, this implies that $v[x \prec y] = v[F_g\{x\} \prec F_g\{y\}]$ if $v[x \prec y]$ is finite or countable, and that $v[x \prec y]$ is infinite if $v[F_g\{x\} \prec F_g\{y\}]$ is.  


In this section, we consider the relationship between the multiplicity upper bound of a covering, and the multiplicity lower bound of the corresponding filter covering or ideal covering. As usual, we focus on filter coverings, leaving the dual result to be stated by the reader.

Consider the function \( \Lambda : \mathbb{N} \to \mathbb{N} \), where \( \mathbb{N} \) stands for the natural numbers, defined recursively as follows:

\[
\Lambda(n) = \begin{cases} 
0, & n = 0 \\
1 + \Lambda(\lfloor \sqrt{n} \rfloor - 1), & n > 0 
\end{cases}
\]

**Lemma 20.** We have

1. \( \Lambda(n) \geq 0 \) for all \( n \)
2. \( \Lambda \) is increasing; i.e., \( n < n' \implies \Lambda(n) \leq \Lambda(n') \)
3. \( \lim_{n \to \infty} \Lambda(n) = \infty \)
4. \( \Lambda(n) \leq \lfloor \sqrt{n} \rfloor \) for all \( n \).

**Proof.**

(1) is clear.

To prove (2), note that we have \( \Lambda(0) = 0 \) and \( \Lambda(1) = 1 \), so (2) is true for \( n < n' \leq 1 \). If (2) is true for \( n < n' \leq \bar{n} > 1 \) then for \( n \leq \bar{n} + 1, \lfloor \sqrt{n} \rfloor - 1 \leq \bar{n} \), and the square root function is also increasing, whence \( \Lambda(n') - \Lambda(n) = \Lambda(\lfloor \sqrt{n'} \rfloor - 1) - \Lambda(\lfloor \sqrt{n} \rfloor - 1) \geq 0 \) if \( n < n' \leq \bar{n} + 1 \). Thus, (2) follows by induction.

To prove (3), we use (2) and note that if \( \Lambda(n) = m \), then \( \Lambda((n + 1)^2) = m + 1 \).

A computation shows that the inequality (4) holds for all \( n \leq 10 \). Suppose (4) holds for \( n \leq \bar{n} > 10 \), and let us prove it is true for \( n = \bar{n} + 1 \). We have by the induction hypothesis

\[
\Lambda(\bar{n} + 1) = 1 + \Lambda(\lfloor \sqrt{\bar{n} + 1} \rfloor - 1) \leq 1 + \sqrt{\bar{n} + 1} - 1.
\]

Squaring, we have

\[
\Lambda(\bar{n} + 1)^2 = 1 + 2\sqrt{\bar{n} + 1} - 1 + \sqrt{\bar{n} + 1} - 1 \leq 3\sqrt{\bar{n} + 1} \leq \bar{n} + 1.
\]

Thus, (4) holds for \( n = \bar{n} + 1 \), and by induction, for all \( n \).

**Theorem 21.** Let \( L \) be a complete, meet-continuous modular lattice, and \( x \prec y \) a covering in \( L \). Let

\[
b_1 \triangleright a_1 \triangleright b_2 \triangleright a_2 \geq \ldots \geq b_n \triangleright a_n,
\]

where \( x \prec y \sim a_i \prec b_i \) for all \( i \). If \( \bar{C} \) is a maximal chain in \( \text{Fil} L \), then \( \mu_{\bar{C}}[\text{Fg}\{x\} \prec \text{Fg}\{y\}] \geq \Lambda(n) \).

**Proof.** Let \( m_1 \in L \) be maximal for the property that \( m_1 \geq a_1 \) but \( m_1 \not\geq b_1 \). (By meet-continuity, the set of such elements is closed under joins of chains, so a maximal such element exists by Zorn’s Lemma.) Then \( m_1 \) is strictly meet-irreducible, and has a unique cover, \( m'_1 = m_1 \vee b_1 \).
Now, let \( m_2 \in L \) be maximal for the property that \( m_2 \geq a_2, m_2 \not\geq b_2, \) and \( m_2 \leq m_1 \). We cannot have \( m_2 = m_1 \), because \( m_1 \geq a_1 \geq b_2 \). \( m_2 = m_2 \lor b_2 \) is the unique cover of \( m_2 \) in the interval \( I[\bot, m_1] \), because if \( x > m_2 \) and \( x \leq m_1 \), we must have \( x \geq b_2 \).

Similarly, we successively choose \( m_3, \ldots, m_n \) such that \( m_i \) is maximal among elements \( x \) such that \( x \geq a_i, x \not\geq b_i, \) and \( x \leq m_{i-1} \), and we obtain covers \( m'_i = m_i \lor b_i \). We have

\[
m'_1 \succ m_1 \geq m'_2 \succ m_2 \geq \ldots \geq m'_n \succ m_n.
\]

For each \( i \), let \( F_i \) be the join of all elements of \( \bar{C} \) containing \( m_i \), and \( G_i \) the meet of all elements of \( \bar{C} \) not containing \( m_i \). We have \( F_i, G_i \in \bar{C} \) because the maximal chain \( \bar{C} \) is closed under joins and meets. Clearly, \( F_i \prec G_i \) for all \( i \).

The mapping \( i \mapsto F_i \prec G_i \) sends each \( i \) to the unique covering in \( \bar{C} \) such that \( m_i \in F_i - G_i \), and thus partitions the ordered set \( \{1, \ldots, n\} \) into intervals. If \( \{i, i + 1, \ldots, j\} \) is one of these intervals, then we claim that \( m'_i \in G_i \). For, if \( i = 1 \) then \( m_i \) is strictly meet-irreducible, and so we must have \( m'_i \in G_1 \). If \( i > 1 \), then we have \( m_{i-1} \in F_{i-1} \), so \( m_{i-1} \in G_i \), because, \( \bar{C} \) being a chain, \( G_i \leq F_{i-1} \). If \( m'_i \) did not belong to \( G_i \), then there would be an element \( g \in G_i \) such that \( m'_i \land g = m_i \). Then we would have \( m'_i \land (g \land m_{i-1}) = m_i \), but \( m_i \neq g \land m_{i-1} \) because \( G_i \) is closed under meets and does not contain \( m_i \). This is impossible, because \( m_i \) is meet-irreducible in \( I[\bot, m_{i-1}] \). Thus, the claim that \( m'_i \in G_i \) is proved. It follows that \( F_i \prec G_i \sim Fg\{x\} \prec Fg\{y\} \).

Clearly, we have \( m'_{i+1}, m_{i+1}, \ldots, m'_j, m_j \in F_i - G_i \), as well as \( m_i \). Thus, since we have shown that \( F_i - G_i \) is atomic, \( m_{i+1}, \ldots, m_j \) have unique covers \( \bar{m}_{i+1}, \ldots, \bar{m}_j \in G_i \). Defining \( \bar{m}'_k = \bar{m}_k \lor m'_k \) for \( k = i + 1, \ldots, j \), we obtain

\[
\bar{m}'_{i+1} \succ \bar{m}_{i+1} \geq \ldots \geq \bar{m}'_j \succ \bar{m}_j
\]

and each \( \bar{m}_k \prec \bar{m}'_k \prec x \prec y \).

Now, either the number of intervals is \( \geq \lceil \sqrt{n} \rceil \), or the cardinality of the largest interval is \( \geq \lfloor \sqrt{n} \rfloor \). In the first case, we have \( \mu_{\bar{C}}[Fg\{x\} \prec Fg\{y\}] \geq \Lambda(n) \) by Lemma 20(4). In the second case, by induction on \( n \) (and noting that recursive application of the construction of this proof will find filter coverings above \( F_i \prec G_i \) and therefore distinct from it), we also have \( \mu_{\bar{C}}[Fg\{x\} \prec Fg\{y\}] \geq 1 + \Lambda(\sqrt{n} - 1) = \Lambda(n) \).

\[
\Box
\]

**Corollary 22.** Let \( L \) be a complete, meet-continuous modular lattice. If \( v[x \prec y] \) is infinite, then so is \( \lambda[Fg\{x\} \prec Fg\{y\}] \).

### 6. Upper Regularity of Filter Coverings and Joins of Chains

In this section, we consider the issue of upper regularity of filter coverings, and show anomalous filter coverings cannot be regular, because they cannot be upper regular. We also give an example of an atomic filter covering, in an algebraic lattice, which is not upper regular, showing that upper regularity alone of \( x \prec y \) does not imply upper regularity of \( Fg\{x\} \prec Fg\{y\} \), even if the lattice is meet-continuous.

Filter coverings are always lower regular, because \( \text{Fil} L \) is coalgebraic, thus join-continuous. Thus, if a filter covering is upper regular, it must be regular.
A necessary condition for a filter covering to be upper regular is easy to state:

**Theorem 23.** Let $L$ be a complete modular lattice, and $F, G \in \text{Fil} L$ such that $F \prec G$. If $F \prec G$ is upper regular, then $F \prec G$ is closed under joins of chains.

*Proof.* Suppose $C$ is a chain in $F - G$. For each $c \in C$, define $F_c = Fg\{c\}$ and $G_c = G \vee F_c = G \cap F_c$. Then for all $c \in C$, $F \prec G \Join F_c \prec G_c$, and if $c, c' \in C$ with $c \leq c'$, we have $F_c \prec G_c \Join F_{c'} \prec G_{c'}$. Since $F \prec G$ is upper regular, $\bigvee_c F_c \prec \bigvee_c G_c$. However, $\bigvee_c F_c = Fg\{\bigvee C\}$. If $\bigvee C \in G$, then we would have $\bigvee_c F_c = \bigvee_c G_c$. Thus, $\bigvee C \in F - G$. $\square$

**Corollary 24.** If $L$ is a complete modular lattice, $F, G \in \text{Fil} L$ with $F \prec G$, and $F \prec G$ is anomalous, then $F \prec G$ is not upper regular.

*Proof.* By the Theorem, if $F \prec G$ is upper regular, then $F - G$ is closed under joins of chains. Then, by Zorn’s Lemma, $F - G$ has maximal elements. However, this is impossible for anomalous $F \prec G$ by Theorem 13. $\square$

Some sufficient conditions for the preceding necessary condition to hold:

**Theorem 25.** Let $L$ be a complete, modular, meet-continuous lattice, and $F \prec G$ a covering in $\text{Fil} L$ which is atomic or quasi-atomic. Then $F - G$ is closed under joins of chains.

*Proof.* Use meet-continuity as in the proof of Theorem 11 or Theorem 12. $\square$

**Theorem 26.** Let $L$ be a complete modular lattice, and $x \prec y$ a covering in $L$ which is upper regular. If $H \prec K \sim Fg\{x\} \prec Fg\{y\}$, then $H - K$ is closed under joins of chains.

*Proof.* Let $C$ be a chain in $H - K$, and let $c \in C$. We have $Fg\{c\} \lor K = Fg\{q\}$ for some $q \in K$ such that $c \prec q$ and $c \prec q \sim x \prec y$. For each $c', c'' \in C$ such that $c \leq c' \leq c''$, we have $c' \prec q \lor c' \Join c'' \prec q \lor c''$. Then $\bigvee C = \bigvee_{c' \geq c} c' \prec \bigvee_{c' \geq c} (c' \lor q)$ by the upper regularity of $x \prec y$. If we had $\bigvee C \in K$, then we would have $c = q \land \bigvee C \in K$, which is absurd. It follows that $\bigvee C \in H - K$. $\square$

There follows an example of a meet-continuous lattice, having an atomic filter covering which is not regular:

**Example 27.** Let $V$ be the infinite-dimensional real vector space of sequences of real numbers, only a finite number of which are nonzero. Let $L$ be the lattice of subspaces of $V$. For each finite set $S \subseteq \mathbb{N}$ of cardinality $\geq 2$, consider the subspace

$$A_S = \{ \langle a_0, a_1, \ldots \rangle \in V : s \in S \implies a_s = 0 \},$$

and the subspace

$$B_S = \{ \langle b_0, b_1, \ldots \rangle \in V : s, s' \in S \implies b_s = b_{s'} \}.$$

Note that $A_S \prec B_S$ for each $S$, and if $S \subseteq S'$ then $A_{S'} \prec B_{S'} \Join A_S \prec B_S$.

For each $n$, let

$$U_n = \{ A_S : s \in S \implies s \geq n \},$$
and

\[ U'_n = \{ B_S : s \in S \implies s \geq n \}; \]

the sets \( U_n, U'_n \) are bases for filters \( F_n = F_g U_n \) and \( G_n = F_g U'_n \).

We have \( F_n \prec G_n \) for all \( n \). For, if \( H \in F_n - G_n \), then \( A_S \subseteq H \) for some \( S \) such that \( s \in S \implies s \geq n \), but there does not exist an \( S' \) such that \( s \in S' \implies s > n \) and \( B_{S'} \subseteq H \).

In particular, \( B_1 \) in positions \( 1 \) will show that \( H \) is trivial. If \( A \) is such that \( s \in F \), we have \( n \) and \( F \) is principal, and \( H \prec (H \lor B) \).

Thus, \( G \) is atomic or quasi-atomic. If every filter covering \( A \) is such that \( i < j \) implies \( A_i \prec A_j \), then there is an \( S \) such that \( s \in S \implies s > n + 1 \) and \( B_S \subseteq H \).

Thus, \( G \) is an atom of \( F \). We already proved that \( F \prec G_n \) is atomic.

Also, if \( n' > n \) then we claim that \( F_n \prec G_n \succ F_{n'} \prec G_{n'} \). To prove this, it suffices to prove \( F_n \prec G_n \succ F_{n+1} \prec G_{n+1} \). We have \( F_n \prec F_{n+1} \) and \( G_n \prec G_{n+1} \), and if \( S \) is such that \( s \in S \implies s > n \), then \( A_S \subseteq F_n \).

We may assume that card \( S > 2 \) since the \( A_S \) for such \( S \) and such that \( A_S \subseteq F_n \) form a base for \( F_n \). Then \( A_S = B_{\{i+j \}} \cap A_{\{j+1 \}} \), where \( j \neq n + 1 \) is any other element of \( S \).

Thus, \( F_n = G_n \lor F_{n+1} \). On the other hand, if \( H \in G_{n+1} \), then there is an \( S \) such that \( s \in S \implies s > n + 1 \) and \( B_S \subseteq H \).

Thus, \( G_n \lor F_{n+1} \subseteq G_{n+1} \). The claim follows.

We already proved that \( F_n \prec G_n \) is atomic for each \( n \). However, \( \sqcup F_n = \sqcup G_n = \{ V \} \).

Thus, the coverings \( F_n \prec G_n \) are not upper regular, and so are not regular.

7. Regularity of Filter Coverings

**Lemma 28.** Let \( L \) be a complete lattice and \( S \subseteq L \), where \( S \neq \emptyset \). Then the following are equivalent:

1. \( F_g S \) is principal, and
2. \( \sqcap S \in F_g S \).

Let \( L \) be a complete, meet-continuous modular lattice. If \( F \prec G \) is an atomic or quasi-atomic filter covering, such that the equivalent conditions of the Lemma are satisfied, then we say that \( F \prec G \) is \textit{principally bounded}.

**Theorem 29.** Let \( L \) be a complete, meet-continuous modular lattice, and \( F \prec G \) a covering in \( \text{Fil} L \) which is atomic or quasi-atomic. If every filter covering \( H \prec K \) such that \( H \prec K \sim F \prec G \) is principally bounded, then \( F \prec G \) is regular.

**Proof.** Let \( I \) be a chain, and \( F_i \prec G_i \) be filter coverings projectively equivalent to \( F \prec G \) and such that \( i < j \) implies \( F_i \prec G_i \succ F_j \prec G_j \). For each \( i \), let \( q_i = \bigwedge \mathcal{M}(F_i - G_i) \), and let \( q = \bigvee_i q_i \).

We have \( q_i \in F_i \) for each \( i \), so \( q \in \bigvee_i F_i \).

On the other hand, we have \( q \notin q \bigvee_i F_i \).

For, \( q \notin \bigvee_i G_i \). Also, \( G_j \geq G_i \) for \( j > i \), so \( q_i \notin G_j \) in that case.
If \( j < i \), then \( G_i = G_j \lor F_i = G_j \land F_i \). However, \( q_i \in F_i \). Thus, \( q_i \notin G_j \).

So, \( q_i \notin G_i \) for all \( i \) and \( j \). Now, \( F_j - G_j \) is closed under joins of chains, by Theorem 25. Thus, \( q = \bigvee_i q_i \notin G_j \) for all \( j \). It follows that \( q \notin \bigvee_i G_i \).

Thus, \( q \in \bigvee_i F_i - \bigvee_i G_i \), implying that \( \bigvee_i F_i < \bigvee_i G_i \), and that \( F < G \) is upper regular, hence regular. \( \square \)

**Corollary 30.** Let \( L \) be a complete, meet-continuous distributive lattice. If \( F < G \) is a covering in \( \text{Fil} L \) which is atomic or quasi-atomic, then \( F < G \) is regular.

**Proof.** If \( H < K \sim F < G \), then \( H < K \) is not anomalous, so \( \mathcal{M}(H - K) \) is nonempty by Theorem 10 and Theorem 12. Furthermore, by distributivity, it has cardinality one, proving that \( H < K \) is principally bounded. The Corollary then follows from Theorem 29. \( \square \)

**Theorem 31.** Let \( L \) be a complete, meet-continuous modular lattice. If \( x \prec y \) is a covering in \( L \) such that \( v[x \prec y] \) is finite, then \([F_g \{ x \}, F_g \{ y \}]\) is regular, with multiplicity equal to \( v[x \prec y] \).

**Proof.** If \( F_g \{ x \} < F_g \{ y \} \) were not principally bounded, \( \mathcal{M}(F_g \{ x \} - F_g \{ y \}) \) would contain a sequence of strictly meet-irreducible elements \( m_1, m_2, \ldots \) such that for all \( n \), \( \bigwedge_{i < n} m_i \neq \bigwedge_{i < n} m_i \). For each \( i \), let \( m'_i \) be the unique cover of \( m_i \). Then if \( y \in L \) is such that \( y \not\leq m_i \), we have \( y \land m_i < y \land m'_i \). For, we must have \( y \lor m_i \geq m'_i \), whence \( y \lor m'_i = y \lor m_i \). If we had \( y \land m_i = y \land m'_i \), then the elements \( y, m_i, m'_i, y \lor m_i \), and \( y \land m_i \) would form a sublattice isomorphic to the lattice \( N_5 \), which cannot happen in a modular lattice.

It follows that

\[
m_1 \land m'_2 \triangleright m_1 \land m_2 \geq m_1 \land m_2 \land m'_3 \triangleright m_1 \land m_2 \land m_3 \geq \ldots,
\]

with \( \bigwedge_{i \leq n} m_i \prec (\bigwedge_{i < n} m_i) \land m'_n \), the general covering in the chain, equivalent to \( x \prec y \) for all \( i \). This sequence of elements can be refined to a maximal chain \( C \) such that \( \mu_C[x \prec y] \) is infinite. However, this is contrary to the assumption that \( v[x \prec y] \) is finite.

Thus, \( F_g \{ x \} < F_g \{ y \} \) is principally bounded, and regular by Theorem 29. \( \square \)

8. Lattice-theoretic Chief Factors

Suppose we have an algebra \( A \) (in the sense of universal algebra) which has a modular congruence lattice. Then we can apply the preceding theory to the congruence lattice \( \text{Con} A \) and talk about the chief factors of \( A \), obtaining a generalization of the nice multiplicity result seen in the Jordan-Holder Theorem.

**Coverings of rank \( \mathcal{F} \) and lattice-theoretic chief factors of rank \( \mathcal{F} \).** If we have a regular covering \( x \prec y \) in the lattice \( \mathcal{F}(L) \), where the functor \( \mathcal{F} \) is some composite of the functors \( \text{Fil} \) and \( \text{Idl} \), then we say that \( x \prec y \) is a covering of \( L \) of rank \( \mathcal{F} \). Then, if \( L = \text{Con} A \), we say that a covering in \( L \) of rank \( \mathcal{F} \) is a **lattice-theoretic chief factor** of \( A \) of rank \( \mathcal{F} \).

In the theories of finite groups and finite-height modules, where the lattices involved have finite height, it is standard practice to assign a group or module to a covering, obtaining a **chief factor**, or, in case \( A \) is a module, a **composition factor**, of \( A \). In order to do something
similar for an arbitrary congruence-modular algebra $A$, it is necessary to assign some type of algebraic object to each lattice-theoretic chief factor. We leave to future investigations the question of the manner in which this may be done generally. (We have taken some small steps toward such a theory in [4], [5], and [7].) However, the lattice-theoretic chief factors themselves are of interest, because their multiplicities are invariants of the algebra.

Thus, in the remainder of this section, unless otherwise specified, $L$ will denote the lattice $\text{Con} A$, for some algebra $A$ such that $\text{Con} A$ is modular. Since $\text{Con} A$ is algebraic, we also are assuming that $L$ is meet-continuous.

The case when $L = \text{Con} A$ satisfies the descending chain condition. If $L$ satisfies the descending chain condition, then all coverings in $L$ are lower regular, all filters are principal, and all filter coverings are of atomic type. In fact, $\text{Fil} L \cong L$. Since $L$ is meet-continuous, coverings in $L$ (and $\text{Fil} L$) are also upper regular. Thus, in this situation, all coverings in $L \cong \text{Fil} L$ are regular. This result was stated but not proved in [6]; it must be admitted, however, that as an example for the application of the ideas in that paper, and of lattice-theoretic chief factors, it is vacuous.

The case when $L = \text{Con} A$ is distributive. A better example presents itself when $L$ is distributive. Then, Corollary 30 shows that every filter covering $F \prec G$ of atomic or strictly quasi-atomic type is regular. We do not know how many coverings there may be in $\text{Fil} L$ which are not of anomalous type. However, we note that $\text{Fil} L$ is provided with a profuse supply of coverings, by which we mean, somewhat informally, that if $\alpha, \beta \in L$ with $\alpha < \beta$, then there is at least one covering in $\text{Fil} L$ between $Fg\{\alpha\}$ and $Fg\{\beta\}$. (This is easy to prove using Zorn’s Lemma.) We can then apply the dual of Corollary 30 to $\text{Idl} \text{Fil} L$, because $\text{Fil} L$ is coalgebraic. The conclusion is that there is a profuse supply of regular coverings in $\text{Idl} \text{Fil} L$, i.e., a profuse supply of lattice-theoretic chief factors of $A$ of rank $\text{Idl} \circ \text{Fil}$. We have regularized coverings in $\text{Fil} L$ by the embedding into $\text{Idl} \text{Fil} L$.

Those coverings in $\text{Idl} \text{Fil} L$ which are not known to be regular can be regularized by considering them in $\text{Fil} \text{Idl} \text{Fil} L$, where they become regular by Corollary 30, as long as $L$ is distributive. And so on. By this method, any covering in $\mathcal{F}(L)$, for any functor $\mathcal{F}$ which is a nonempty composite of $\text{Idl}$ and $\text{Fil}$, can be regularized by applying either $\text{Fil}$, or $\text{Idl}$, and similarly, any covering in any distributive lattice whatever can be regularized by applying either $\text{Idl} \circ \text{Fil}$, or $\text{Fil} \circ \text{Idl}$.

Because of Theorem 3, all of the regular coverings that arise in this way have multiplicity one.

Modular but not distributive $L = \text{Con} A$. In this case, multiplicities higher than 1 are possible. If $x \prec y$ is a covering in $L$, such that $v[x \prec y]$ is finite, then $Fg\{x\} \prec Fg\{y\}$ is regular by Theorem 31. Thus, the embedding from $L$ into $\text{Fil} L$ regularizes such coverings, and the multiplicity of the corresponding filter covering is $v[x \prec y]$ by Theorem 14.

On the other hand, if $v[x \prec y]$ is infinite, then by Corollary 22, so is $\lambda[Fg\{x\} \prec Fg\{y\}]$. Thus, any maximal chain $\bar{C}$ in $\text{Fil} L$ has an infinite number of coverings in it that are equivalent to $Fg\{x\} \prec Fg\{y\}$, and we can say that the multiplicity of the filter covering
is infinite, even though we may not be able to say that that multiplicity is a well-defined cardinal number.

9. The Steps in the Proof of a Theorem

Another application of these ideas is a method of formalizing the steps necessary and sufficient to prove a given proposition from given premises. Our treatment of this will use a simple Logic framework.

Suppose we have a set $P$ of “propositions,” which can in principal be determined to either be true, or not. Then we have two truth values $T$ and $F$, and for any proposition $P$ we can say that $T(P)$ (the truth value of $P$) takes values $T$ and $F$. Given any $n$ propositions $P_1, \ldots, P_n$, and any $n$-ary function $f$ with arguments consisting of truth values, we can formulate a new, synthetic proposition $f(\vec{P})$ with truth value $f(T(P_1), \ldots, T(P_n))$. If we consider the truth values as elements of the two-element boolean algebra \{T, F\}, then the functions obtainable by compositions of the ordinary logical connectives give us this, because the two-element boolean algebra has the property of being primal – i.e., the property that every finitary function can be constructed from the basic operations. We will use the symbols $\land, \lor, \lnot, \rightarrow$ with their usual meanings, along with $T$ and $F$. In fact, it is convenient to replace our original set of propositions $P$ by a boolean algebra $B$ free on $P$ as set of generators. (Or, if $P$ already has some or all of the logical connectives, by a quotient of such a free boolean algebra.) The assignment $T$ of truth values can then be extended to a boolean algebra homomorphism from $B$ to the two-element boolean algebra. Henceforth, proposition shall mean an element of $B$.

We will write $\top$ and $\bot$ for the maximum and minimum elements of $B$. The underlying lattice of $B$ is just $B$, forgetting the unary operation $\lnot$. A filter of $B$ is the same as a filter of the underlying lattice.

Given some sort of calculus of proving propositions, consisting of finitary rules of inference, we assume that the rules of inference include a small set of trivial rules of inference, and otherwise we call them nontrivial rules of inference. The trivial rules of inference are the rule that we can infer $P \land Q$ from $P$ and $Q$, for any elements $P, Q \in B$, and the rule that for any $P, Q \in B$, if $P \leq Q$, then we can infer $Q$ from $P$. Note that modus ponens, the rule that we can infer $Q$ from $P$ and $P \rightarrow Q$ (or $\lnot P \lor Q$) will thus be considered a trivial rule of inference, because $P \land (\lnot P \lor Q) = (P \land \lnot P) \lor (P \land Q) = P \land Q \leq Q$.

Note that if the ordering of $B$ provides that $P \leq Q$ whenever $Q$ can be proved from $P$, then the second trivial rule of inference would actually subsume all the rules of inference, rendering our analysis of the situation vacuous. Thus, we want to consider a situation where $B$ does not have such an ordering.

We say $P \vdash Q$ ($S \vdash Q$, where $S \subseteq B$) if $Q$ can be proved from $P$ (from elements of $S$) using both trivial and nontrivial rules of inference.

**Theorem 32.** If $T \subseteq B$, then the following are equivalent:

1. $T$ is a filter of the underlying lattice of $B$, and
2. $T$ is closed under application of the trivial rules of inference, and contains $\top$. 

Theorem 35. If \( T \) is a pretheory and \( P \) is a proposition, then any proof of \( P \) from \( T \) covers the steps (of any order \( F \)) in the proof of \( P \) from \( T \).

Proof. Let \( N_i, i = 1, \ldots, n \) be the instances of rules of inference in a proof, in order. Let pretheories \( T_i, i = 0, \ldots, n \) be defined by \( T_0 = T \), \( T_i = T_{i-1} \land \text{Fg}\{Q_i\} \) for \( 0 < i \leq n \), where...
$Q_i$ is the conclusion of $N_i$. For each $i > 0$, let the set of steps of order $F$ in $I[\phi(T_i), \phi(T_{i-1})]$ be $E_i$, and the set of steps of order $F$ covered by $N_i$, by $E'_i$.

We have $E_i \subseteq E'_i$. For, if $N_i$ infers $Q_i$ from the finite set of propositions $S_i$, we have

$$\langle T_i, T_{i-1} \rangle \not\supseteq \langle \text{Fg}\{Q_i\}, \text{Fg}\{Q_i\} \cup T_{i-1} \rangle,$$

where $Q_i = Q_i \land \bigwedge S_i$. However, $\text{Fg}\{Q_i\} \cup T_{i-1} \leq \text{Fg}\{\bigwedge S_i\}$, because, the $n$-tuple $\langle N_1, \ldots, N_n \rangle$ being a proof, $S_i \subseteq T_{i-1}$.

Thus, $\bigcup_i E_i \subseteq \bigcup_i E'_i$, but the left side is the set of steps of order $F$ in the proof of $P$ from $T$, and the right side is the set of steps of order $F$ covered by the proof. □

Let $T$, $T'$ be pretheories such that $T' \leq T$, and let $N$ be a set of instances of rules of inference. For each $N \in \mathbf{N}$, let $S_N$ be the (finite) set of premises of $N$, and $Q_N$ the conclusion. If $T'$ is the join (intersection) of all pretheories $\bar{T} \leq T$ such that $N \in \mathbf{N}$ and $S_N \subseteq \bar{T}$ imply $Q_N \in \bar{T}$, then we say that $N$ generates $T'$ from $T$. In this case, $T'$ consists of all propositions provable from $T$ using the elements of $N$ as the only instances of nontrivial rules of inference.

**Theorem 36.** Let $T$, $T'$ be pretheories with $T' \leq T$, and let $N$ be a set of instances of rules of inference which generates $T'$ from $T$. Then $T'$ is the set of propositions $P$ such that there is a finite sequence of elements of $N$ that can be refined to a proof of $P$ from $T$ by adding instances of trivial rules of inference.

**Proof.** Let $\bar{T}$ be that set of propositions, and we will show that $T' = \bar{T}$. Since $T'$ is generated from $T$ by $N$, $T'$ is the intersection (join) of all pretheories $\bar{T} \leq T$ such that $N \in \mathbf{N}$ and $S_N \subseteq \bar{T}$ imply $Q_N \in \bar{T}$.

Clearly, $N \in \mathbf{N}$ and $S_N \subseteq \bar{T}$ imply $Q_N \in \bar{T}$, because we can construct a proof of $Q_N$ from proofs of the elements of $S_N$. Thus, $\bar{T} \leq T'$.

On the other hand, suppose that $\bar{T} \leq T$ is such that $N \in \mathbf{N}$ and $S_N \subseteq \bar{T}$ imply $Q_N \in \bar{T}$, and let $P \in \bar{T}$. The existence of a proof of $P$ from $T$ using instances from $N$ implies that $P \in \bar{T}$. Thus, $\bar{T} \leq T$, so $T' \leq \bar{T}$.

Thus, $T' = \bar{T}$. □

Finally, a theorem which shows that covering the steps in the proof of $P$ from $T$ is not only necessary, but sufficient:

**Theorem 37.** Given pretheories $T$, $T'$ such that $T' \leq T$, a set $N$ of instances of rules of inference that generates $T'$ from $T$, and a proposition $P$, then we have $P \in T'$ iff $N$ covers the steps in the proof of $P$ from $T$.

**Proof.** If $P \in T'$, then the conclusion follows from Theorems 35 and 36.

If $P \notin T'$, then we have

$$\langle \text{Fg}\{P\} \land T', T' \rangle \not\supseteq \langle \text{Fg}(P) \land T, (\text{Fg}\{P\} \land T) \lor T' \rangle$$

and we have $(\text{Fg}\{P\} \land T) \lor T' \leq T$. Thus, the steps in the interval $I[\text{Fg}\{P\} \land T', T']$ are a subset of the set of steps in the proof of $P$ from $T$. By Zorn’s Lemma, there is an ideal $J \in \text{IdlFilB}$ such that $\text{Ig}\{\text{Fg}\{P\} \land T'\} < J < \text{Ig}\{T'\}$. The covering $\text{Fg}\{J\} \prec \text{Fg}\{\text{Ig}\{T'\}\}$ is an step (of order $\text{Fil} \circ \text{Idl}$) in the proof of $P$ from $T$ that is not covered by $N$. □
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