Intrinsic and extrinsic geometries of a tidally deformed black hole

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Received 9 June 2011
Published 26 July 2011
Online at stacks.iop.org/CQG/28/175006

Abstract
A description of the event horizon of a perturbed Schwarzschild black hole is provided in terms of the intrinsic and extrinsic geometries of the null hypersurface. This description relies on a Gauss–Codazzi theory of null hypersurfaces embedded in spacetime, which extends the standard theory of spacelike and timelike hypersurfaces involving the first and second fundamental forms. We show that the intrinsic geometry of the event horizon is invariant under a reparameterization of the null generators, and that the extrinsic geometry depends on the parameterization. Stated differently, we show that while the extrinsic geometry depends on the choice of gauge, the intrinsic geometry is gauge invariant. We apply the formalism to solutions to the vacuum field equations that describe a tidally deformed black hole. In a first instance, we consider a slowly varying, quadrupolar tidal field imposed on the black hole, and in a second instance, we examine the tide raised during a close parabolic encounter between the black hole and a small orbiting body.

PACS numbers: 04.20.−q, 04.25.−g, 04.25.Nx, 04.70.−s, 04.70.Bw, 97.60.Lf

(Some figures in this article are in colour only in the electronic version)

1. Introduction
The tidal dynamics of inspiralling compact binaries (involving neutron stars and/or black holes) has been the subject of vigorous investigation in recent years, motivated by the exciting prospect of measuring tidal signatures in the gravitational waves emitted by such systems. Some of this work has focused on calculating the influence of the tidal coupling on the gravitational waves, and estimating the accuracy with which the tidal deformation of each body can be measured [1–5]. Some has focused on calculating the tidal deformation of neutron stars in the post-Newtonian approximation to general relativity [6–8] and in the full theory [4, 9, 10]. And some has focused on the tidal deformation of nonrotating black holes [11–15].
An issue that is central to all these investigations is the dependence of adopted measures of tidal deformation on the coordinates employed to describe the spacetime geometry. In the case of neutron stars, the coordinate independence of the relativistic Love numbers which measure the tidal deformation of the body’s external gravitational field was firmly established by Damour and Nagar [9] and Binnington and Poisson [10]. In the case of nonrotating black holes, however, these gauge-invariant Love numbers were shown to vanish [10], and the identification of nonvanishing, coordinate-independent measures of tidal deformation has remained an open problem. For example, Poisson and Vlasov [15] rely on light-cone coordinates to describe the geometry of a deformed black hole, while Damour and Lecian [14] rely on Weyl coordinates in a context of stationary and axisymmetric tides. Our main objective with this paper is to remedy this situation by providing a complete description of the intrinsic and extrinsic geometries of a tidally deformed event horizon, and fully clarifying the coordinate dependence of all horizon quantities. In particular, we introduce meaningful and practical measures of the tidal deformation of an event horizon.

The central assumptions in our work are that the unperturbed black hole is nonrotating and described by the Schwarzschild solution to the Einstein field equations, and that the tidal deformation is sufficiently small that it can be described accurately to first order in a perturbative treatment. Otherwise our formulation is completely general: the tide can be either static, slowly varying or fully dynamical, and there is no requirement that it be axisymmetric. Our description of a tidally deformed event horizon relies on two major theoretical foundations. The first is a Gauss–Codazzi theory of null hypersurfaces embedded in spacetime, an extension of the standard theory of (spacelike and timelike) hypersurfaces formulated in terms of first and second fundamental forms. This material is developed here ab initio, in spite of the fact that similar formalisms are extant in the literature (for example, in [16–18]); our version is presented in a form directly suited to our application to perturbed event horizons. The second foundation is a covariant and gauge-invariant formulation of black-hole perturbation theory, as summarized in the work of Martel and Poisson [19].

Our description of a null hypersurface embedded in spacetime is tied to its generators, the congruence of null geodesics that trace the hypersurface. We label each generator with two comoving coordinates $a^A = (\alpha, \beta)$ (with the index $A$ running over the values 2 and 3), and we let $\lambda$ be a running parameter on each generator. The hypersurface is charted with the intrinsic coordinates $(\lambda, \alpha^A)$, and its (degenerate) intrinsic geometry is fully characterized by the explicitly two-dimensional metric $\gamma_{AB}$, the analogue of the first fundamental form of a (spacelike or timelike) hypersurface. The extrinsic geometry, on the other hand, is characterized by a scalar $\kappa$ (a generalization of the black hole’s surface gravity), a vector $\omega_A$ and a tensor $K_{AB}$; these are analogous to the second fundamental form of a (spacelike or timelike) hypersurface. We examine how these quantities transform under reparameterizations $\lambda \rightarrow \bar{\lambda}(\lambda, \alpha^A)$ of the generators, and show that while the extrinsic geometry of the null hypersurface depends on the parameterization, the intrinsic geometry is independent of the parameterization. When applied to an event horizon, this observation becomes one of the central results of this paper: the intrinsic geometry of a black-hole horizon is invariant under a reparameterization of the horizon’s null generators. This statement implies that any measure of tidal deformation that derives from the induced metric $\gamma_{AB}$ is necessarily invariant under reparameterizations.

This result can be restated in terms of gauge transformations, small deformations $x^\alpha \rightarrow x^\alpha + f^\alpha$ of the coordinates employed in the unperturbed spacetime. With regards to transformations of the spacetime coordinates $x^\alpha$, the horizon quantities $\gamma_{AB}$, $\kappa$, $\omega_A$ and $K_{AB}$ are a collection of scalar fields expressed entirely in terms of the hypersurface coordinates $(\lambda, \alpha^A)$. As such they are independent of the spacetime coordinates and therefore immune
to gauge transformations. As a matter of principle, therefore, all horizon quantities are
gauge-invariant quantities. The situation, however, is made more subtle by a matter of
practice, our identification of the generator parameter $\lambda$ with the advanced-time coordinate
$v$ of the underlying spacetime. With this identification, a transformation of the spacetime
coordinates is necessarily associated with a reparameterization of the null generators, and
the horizon quantities acquire a gauge dependence that is inherited from their dependence on
reparameterizations. In this context, the results summarized in the preceding paragraph can
be stated as follows: while the extrinsic geometry of a perturbed event horizon depends on the
choice of gauge, the intrinsic geometry is gauge invariant.

Our Gauss–Codazzi theory of null hypersurfaces is developed in section 2. In
section 3, we examine a nonrotating black hole deformed by an arbitrary distribution of
matter, describe its geometry in terms of a perturbed Schwarzschild metric and compute the
horizon quantities $\gamma_{AB}$, $\kappa$, $\omega_A$ and $K_{AB}$ to first order in perturbation theory. In section 4,
we specialize our results to tidal deformations produced by a remote distribution of matter.
Adopting a specific choice of gauge (the ‘Killing gauge’), we integrate the vacuum field
equations near the horizon to express the horizon quantities in terms of the well-known master
functions $\Psi_{\text{even}}$ (the Zerilli–Moncrief function) and $\Psi_{\text{odd}}$ (the Cunningham–Price–Moncrief
function) of black-hole perturbation theory. In section 5, we consider two applications of our
formalism, the first involving a slowly varying, quadrupolar tidal field imposed on the black
hole and the second involving a close parabolic encounter between the black hole and a small
orbiting body. The appendix contains mathematical developments regarding the late-time
behaviour of the horizon quantities.

2. Differential geometry of null hypersurfaces

To guide the development of a theory of perturbed event horizons, it is helpful to formulate a
differential geometry of embedded null hypersurfaces. The main goal is to arrive at a set of
Gauss–Codazzi equations that apply to the null case instead of being restricted to usual cases
of timelike or spacelike hypersurfaces. The developments of this section rely on material
presented in sections 3.1 and 3.11 of [20].

2.1. Generators, vector basis and intrinsic coordinates

A null hypersurface is generated by a congruence of null geodesics that are described by the
parametric equations $x^\alpha = x^\alpha(\lambda, \alpha^A)$, in which $\lambda$ is a running parameter on each generator,
and $\alpha^A = (\alpha, \beta)$ are generator labels that stay constant on each generator; uppercase latin
indices such as $A$ run from 2 to 3. The null vector field

$$k^\alpha = \left( \frac{\partial x^\alpha}{\partial \lambda} \right)_A$$  (2.1)

is tangent to the congruence of null generators, and

$$e^\alpha_A = \left( \frac{\partial x^\alpha}{\partial \alpha^A} \right)_\lambda$$  (2.2)

are spacelike displacements vectors that point from one generator to another. These are
orthogonal to $k^\alpha$, $k^\alpha e^\alpha_A = 0$, and their mutual inner products are

$$\gamma_{AB} := g_{\alpha\beta} e^\alpha_A e^\beta_B.$$ (2.3)

The definitions imply that the vectors satisfy the Lie-transport equations

$$k^\alpha_{,\beta} e^\beta_A = e^\alpha_{A,\beta} k^\beta_A, \quad e^\alpha_{A,\beta} e^\beta_B = e^\alpha_{B,\beta} e^\beta_A.$$ (2.4)
in which a semicolon indicates covariant differentiation in spacetime, with a connection compatible with $g_{\alpha\beta}$. The basis is completed with a second null vector $N^\alpha$ that cuts across the hypersurface; its normalization is chosen so that $N_\alpha k^\alpha = -1$, and the vector is also required to satisfy $N_\alpha e^\alpha_A = 0$.

We select $(\lambda, \alpha_A)$ as intrinsic coordinates on the hypersurface. In the spacetime coordinates, a displacement within the hypersurface is described by $dx^\alpha = k^\alpha d\lambda + e^\alpha_A d\alpha_A$, and the intrinsic line element is

$$ds^2 = \gamma_{\alpha\beta} d\alpha^\alpha d\alpha^\beta.$$  \hfill (2.5)

This shows that $\gamma_{\alpha\beta}(\lambda, \alpha_A)$ acts as a metric on the hypersurface. In this generator-adapted coordinate system, the degenerate metric is explicitly two dimensional. We let $\gamma^{\alpha\beta}$ denote the matrix inverse to $\gamma_{\alpha\beta}$ and $\Gamma^\alpha_{BC}$ be the connection compatible with the two-dimensional metric; the associated covariant-derivative operator is denoted $\nabla_A$. We use $\gamma_{\alpha\beta}$ and its inverse to lower and raise uppercase latin indices.

2.2. Gauss–Weingarten equations

The tangent vector fields admit the following set of Gauss–Weingarten equations:

$$k^\alpha;\beta k^\beta = \kappa k^\alpha,$$  \hfill (2.6)

$$k^\alpha;\beta e^\beta_A = \omega^A k^\alpha + B^A_{\beta} e^\alpha_B = e^\alpha_A k^\beta,$$  \hfill (2.7)

$$e^\alpha_A;\beta e^\beta_B = B_{AB} N^\alpha + K_{AB} k^\alpha + \Gamma^\alpha_{BC} e^\alpha_C = e^\alpha_B k^\beta.$$  \hfill (2.8)

These equations define $\kappa$, $\omega_A$, $B_{AB}$, $K_{AB}$ and $\Gamma^\alpha_{AB}$. Explicitly,

$$\kappa = -N_\alpha k^\alpha;\beta k^\beta,$$  \hfill (2.9)

$$\omega_A = -N_\alpha k^\alpha;\beta e^\beta_A,$$  \hfill (2.10)

$$B_{AB} = k^\alpha;\beta e^\alpha_A e^\beta_B = B_{BA},$$  \hfill (2.11)

$$K_{AB} = -N_\alpha e^\alpha_A;\beta e^\beta_B = K_{BA},$$  \hfill (2.12)

$$\Gamma^\alpha_{CAB} = e^\alpha_A e^\beta_C e^\alpha_B = \Gamma^\alpha_{CBA},$$  \hfill (2.13)

where $\Gamma^\alpha_{CAB} := \gamma_{CD} \Gamma^D_{AB}$. These equations reveal that while $\gamma_{AB}$, $B_{AB}$ and $\Gamma^\alpha_{AB}$ characterize the intrinsic geometry of the hypersurface, $\kappa$, $\omega_A$ and $K_{AB}$ characterize its extrinsic geometry.

Each equation in the set (2.6)–(2.8) can be viewed as an expansion of a vector field (defined on the left-hand side) in terms of the hypersurface basis $N^\alpha$, $k^\alpha$ and $e^\alpha_A$. The absence of terms proportional to $N^\alpha$ in equations (2.6) and (2.7) is a consequence of the fact that $k^\alpha$ is null everywhere on the hypersurface. The absence of a term proportional to $e^\alpha_A$ in equation (2.6) follows from the identity $e^\alpha_A k^\alpha;\beta k^\beta = -k_\alpha e^\alpha_A k^\beta = 0$, which itself follows from the orthogonality of $k^\alpha$ and $e^\alpha_A$. Equality of $B_{AB}$ as defined by equation (2.7) and $B_{AB}$ as defined by equation (2.8) is confirmed by a similar calculation. Equation (2.6) states that $k^\alpha$ is a geodetic vector field, and $\kappa$ measures the failure of $\lambda$ to be an affine parameter.

The definition of equation (2.3) and the Gauss–Weingarten equations imply that

$$\partial_\lambda \gamma_{AB} = 2 B_{AB}.$$  \hfill (2.14)

It is customary to decompose $B_{AB}$ into irreducible components,

$$B_{AB} = \frac{1}{2} \Theta \gamma_{AB} + \sigma_{AB}.$$  \hfill (2.15)
with the trace term \( \Theta := \gamma^{AB} B_{AB} \) representing the rate of expansion of the congruence of null generators, and the tracefree term \( \sigma_{AB} := B_{AB} - \frac{1}{2} \Theta \gamma_{AB} \) representing the rate of shear. A similar decomposition could also be introduced for \( \mathcal{K}_{AB} \).

The Gauss–Weingarten equations also imply that

\[
N^\alpha : \beta k^\beta = -\kappa N^\alpha + \omega^A e_A^\alpha,
\]

(2.16)

\[
N^\alpha : \beta e_A^\alpha = -\omega_A N^\alpha + \mathcal{K}_A^B e_B^\alpha.
\]

(2.17)

These equations govern the behaviour of the transverse vector on the hypersurface.

### 2.3. Gauss–Codazzi equations

It is straightforward, following the methods described in section 3.5 of [21], to derive from equations (2.6)–(2.8) a set of Gauss–Codazzi equations which express projections of the spacetime Riemann tensor in terms of geometric quantities defined on the null hypersurface. We have

\[
R_{\mu\nu\alpha\beta} k^\mu k^\nu e_A^\alpha e_B^\beta = \partial_\lambda \omega_A - \partial_A \kappa + B_A^B \omega_B,
\]

(2.18)

\[
R_{\mu\nu\alpha\beta} N^\nu e^\alpha A e^\beta B = \nabla_A B_{AB} - \nabla_B \omega_A - B_A^C K_{CB} + B_C^B K_{CA},
\]

(2.19)

\[
R_{\mu\nu\alpha\beta} e_A^\alpha N^\nu e^\beta B = -\partial_\lambda K_{AB} - \kappa K_{AB} + \nabla_A \omega_B + \omega_A \omega_B + K_A^C B_{CB},
\]

(2.20)

\[
R_{\mu\nu\alpha\beta} e_A^\alpha e^\beta B = -\partial_\lambda B_{AB} + \kappa B_{AB} + B_A^C B_{CB},
\]

(2.21)

\[
R_{\mu\nu\alpha\beta} N^\nu e^\alpha A e^\beta C = \nabla_C B_{AB} - \nabla_B \omega_C - \omega_C B_{AB} + \omega_B B_{AC},
\]

(2.22)

\[
R_{\mu\nu\alpha\beta} e_A^\alpha e^\beta C = \nabla_C K_{AB} - \nabla_B K_{AC} + \omega_C K_{AB} - \omega_B K_{AC},
\]

(2.23)

\[
R_{\mu\nu\alpha\beta} e_A^\alpha e^\beta B e^\gamma C = \frac{1}{2} \mathcal{R} (\gamma_{AC} \gamma_{BD} - \gamma_{AD} \gamma_{BC}) + B_{AC} K_{BD} - B_{AD} K_{BC} + K_{AC} B_{BD} - K_{AB} B_{BC},
\]

(2.24)

where \( \mathcal{R} \) is the Ricci scalar associated with the two-dimensional metric \( \gamma_{AB} \). To arrive at these equations, we used the fact that the Riemann tensor on a two-dimensional metric space can always be expressed as \( R_{ABCD} = \frac{1}{2} \mathcal{R} (\gamma_{AC} \gamma_{BD} - \gamma_{AD} \gamma_{BC}) \). We also relied on the identity \( \gamma_{CD} \partial_\lambda \gamma_{AB} = \nabla_A B_{BC} + \nabla_B \omega_{AC} - \nabla_C B_{AB} \), which can be derived on the basis of equation (2.14).

Insertion of the Gauss–Codazzi equations within the identity

\[
R_{\mu\nu} = g^{\alpha\beta} R_{\mu\alpha\beta\nu} = (-k^\alpha N^\beta - N^\alpha k^\beta + \gamma^{AB} e_A^\alpha e_B^\beta) R_{\mu\alpha\beta\nu}
\]

(2.25)

produces the following components of the Ricci tensor:

\[
R_{\mu\nu} = g^{\alpha\beta} R_{\mu\alpha\beta\nu} = (-k^\alpha N^\beta - N^\alpha k^\beta + \gamma^{AB} e_A^\alpha e_B^\beta) R_{\mu\alpha\beta\nu}
\]

(2.26)

\[
R_{\mu\alpha} k^\nu = -\partial_\gamma \omega_A - \partial_A \kappa - B_{AB} B^\alpha B^\beta + \Theta \omega_A - \omega_A \omega_B + \kappa \kappa_{AB} - \gamma_{AC} \gamma_{BD} - \gamma_{AD} \gamma_{BC}
\]

(2.27)

\[
R_{\mu\nu} e_A^\alpha = \partial_\gamma \omega_A - \partial_A \kappa + \nabla_A \omega_B + \Theta \omega_A + \kappa \kappa_{AB} - \gamma_{AC} \gamma_{BD} + \gamma_{BD} \gamma_{AC} - \gamma_{AD} \gamma_{BC} + \gamma_{AB} \gamma_{CD}
\]

(2.28)

where \( \kappa := \gamma^{AB} K_{AB} \).
By involving the Einstein field equations, equation (2.26) can be turned into Raychaudhuri’s equation,
\[ \partial_\lambda \Theta = \kappa \Theta - \frac{1}{2} \Theta^2 - \sigma_{A\beta} \sigma^{A\beta} - 8\pi T_{\alpha\beta} \kappa^\alpha k^\beta, \]
(2.29)
with \( T_{\alpha\beta} \kappa^\alpha k^\beta \) representing the flux of matter across the null hypersurface. By extracting the tracefree piece of equation (2.21), we obtain an analogous equation for the shear tensor,
\[ \partial_\lambda \sigma_{A\beta} = (\kappa - \Theta) \sigma^{A\beta} - C^A_{B\lambda}, \]
(2.30)
where \( C^A_{B\lambda} \) are the components of the Weyl tensor.

2.4. Reparameterizations

The geometric quantities \( \gamma_{AB}, B_{AB}, \kappa, \omega_A \) and \( K_{AB} \) all refer to a selected parameterization \((\lambda, \alpha^A)\) of the null generators. We first examine how these quantities change under a reparameterization of the form
\[ \lambda \rightarrow \tilde{\lambda}(\lambda, \alpha^A), \]
(2.31)
which represents an independent change of parameter on each generator. The differential form of the transformation is expressed as
\[ d\tilde{\lambda} = e^{-\beta}(d\lambda - c_A d\alpha^A), \]
(2.32)
with
\[ e^{-\beta} := \left( \frac{\partial \tilde{\lambda}}{\partial \lambda} \right)_{\alpha^A}, \quad -e^{-\beta}c_A := \left( \frac{\partial \tilde{\lambda}}{\partial \alpha^A} \right)_{\lambda}. \]
(2.33)
These are the functions of \((\lambda, \alpha^A)\) on the hypersurface, and the notation was chosen so as to simplify our expressions below. The inverse transformation is \( d\lambda = e^\beta d\tilde{\lambda} + c_A d\alpha^A \).

As we saw previously, a displacement on the hypersurface is described by \( dx^A = k^A d\lambda + e^A_A d\alpha^A \), but the reparameterization brings this to the new form \( d\tilde{x}^A = \tilde{k}^A d\tilde{\lambda} + \tilde{e}^A_A d\alpha^A \), with
\[ \tilde{k}^A = e^\beta k^A, \quad \tilde{e}^A_A = e^A_A + c_A k^A. \]
(2.34)
These vectors have the same interpretation as the old vectors: \( \tilde{k}^A \) is still tangent to the congruence of null generators, but is renormalized so as to reflect the new parameterization, and \( \tilde{e}^A_A \) still points from generator to generator. It is easy to show that the new transverse vector must be given by
\[ \tilde{N}^A = e^{-\beta} \left( N^A + \frac{1}{2} c^A_A c^B k^B + c^A \tilde{e}^A_A \right), \]
(2.35)
to satisfy its defining relations. The inverse transformations are \( k^A = e^{-\beta} \tilde{k}^A, \quad e^A_A = \tilde{e}^A_A - e^{-\beta} c^A_A \tilde{k}^A \) and \( N^A = e^\beta \tilde{N}^A + \frac{1}{2} e^{-\beta} c^A_A \tilde{k}^A - c^A \tilde{e}^A_A \).

The reparameterization produces the following changes in the geometric quantities:
\[ \tilde{\gamma}_{AB} = \gamma_{AB}, \]
(2.36)
\[ \tilde{B}_{AB} = e^\beta B_{AB}, \]
(2.37)
\[ \tilde{\kappa} = e^{\beta}(\kappa + \partial_\lambda \beta), \]
(2.38)
\[ \tilde{\omega}_A = \omega_A - B^B_B c_B + \kappa c_A + c_A \partial_\lambda \beta + \partial_\lambda \beta, \]
(2.39)
\[ \tilde{K}_{AB} = e^{-\beta} \left( K_{AB} + \omega_A c_B + \omega_B c_A + \kappa A c_B + c_B \partial_\lambda c_A + \nabla_B c_A \right. \]
\[ \left. + \frac{1}{2} c^C c C B_{AB} - B^C_A c C B_{AB} - B^C_B c C B_{AB} \right). \]
(2.40)
The term \( c_B \partial_\lambda c_A + \nabla_B c_A \) in the last equation is not manifestly symmetric in the pair of indices \( AB \). With the definitions of equations (2.33), however, we find that this can be expressed in the form

\[
c_B \partial_\lambda c_A + \nabla_B c_A = -e^\beta \frac{\partial^2 \lambda}{\partial \alpha^A \partial \alpha^B} + e^{2\beta} \frac{\partial^2 \lambda}{\partial \lambda \partial \alpha^A \partial \alpha^B} \frac{\partial \lambda}{\partial \alpha^A} \frac{\partial \lambda}{\partial \alpha^B} - e^{2\beta} \frac{\partial^2 \lambda}{\partial \lambda \partial \alpha^A \partial \alpha^B} \frac{\partial \lambda}{\partial \alpha^B} - e^{3\beta} \frac{\partial^2 \lambda}{\partial \lambda^2 \partial \alpha^A \partial \alpha^B} - \Gamma^{c}_{AB} c^C, \tag{2.41}
\]

which reveals the required symmetry. An additional change produced by the reparameterization is \( \Gamma^{c}_{AB} c^C = \Gamma^{c}_{AB} c^C + B^C_{AB} c^B + B^C_{BA} c^A - K^C_{AB} c^C \).

In the case of infinitesimal transformations described by \( \bar{\lambda} = \lambda + \delta \lambda(\lambda, \alpha^A) \), the partial derivatives are captured by \( \delta \beta := -\partial_\lambda \delta \lambda \) and \( \delta c_A := -\partial_\lambda \delta \lambda \), and the transformations of equations (2.36)–(2.40) simplify. For the purposes of an application of the formalism presented below, we assume that the geometric quantities can be expressed as

\[
\gamma_{AB} = \gamma^0_{AB} + \delta \gamma_{AB}, \tag{2.42}
\]

\[
B_{AB} = \delta B_{AB}, \tag{2.43}
\]

\[
\kappa = \kappa^0 + \delta \kappa, \tag{2.44}
\]

\[
\omega_A = \delta \omega_A, \tag{2.45}
\]

\[
K_{AB} = K^0_{AB} + \delta K_{AB}, \tag{2.46}
\]

where the ‘background quantities’ \( \gamma^0_{AB}, \kappa^0 \) and \( K^0_{AB} \) are assumed to be \( \lambda \)-independent, and where \( \delta \gamma_{AB}, \delta B_{AB}, \delta \kappa, \delta \omega_A \) and \( \delta K_{AB} \) are \( \lambda \)-dependent ‘perturbations’.

In the restricted context, the transformations reduce to

\[
\Delta \gamma_{AB} = \delta \gamma_{AB}, \tag{2.47}
\]

\[
\Delta B_{AB} = \delta B_{AB}, \tag{2.48}
\]

\[
\Delta \kappa = \delta \kappa + \partial_\lambda \delta \beta + \kappa_0 \delta \beta, \tag{2.49}
\]

\[
\Delta \omega_A = \delta \omega_A + \partial_\lambda \delta \beta + \kappa_0 \delta c_A, \tag{2.50}
\]

\[
\Delta K_{AB} = \delta K_{AB} - k^0_{AB} \delta \beta + \nabla_B \delta c_A. \tag{2.51}
\]

In the last equation, the covariant derivative \( \nabla_B \) is evaluated with a connection compatible with the background metric \( \gamma^0_{AB} \).

We next examine the possibility of transforming the generator labels. A general transformation of the form \( \alpha^A \to \tilde{\alpha}^A(\lambda, \alpha^B) \) is excluded because the dependence upon \( \lambda \) would imply that \( \tilde{\alpha}^A \) is not constant on each generator, in violation of its defining property. The remaining freedom is a rigid transformation of the form \( \alpha^A \to \tilde{\alpha}^A(\alpha^B) \), upon which scalars such as \( \kappa \) remain invariant, while tensors such as \( \omega_A \) and \( \gamma_{AB} \) transform in the usual way. In particular, for infinitesimal transformations of the form \( \tilde{\alpha}^A = \alpha^A + \delta \alpha^A \), the metric tensor transforms as

\[
\gamma_{AB}(\tilde{\alpha}^C) = \gamma_{AB}(\alpha^C) - \nabla_A \delta \alpha_B - \nabla_B \delta \alpha_A, \tag{2.52}
\]

where \( \nabla_A \) refers to \( \gamma_{AB} \), and \( \delta \omega_A = \gamma_{AB} \delta \alpha^B \). In this formulation, the original metric is expressed as a function of the new coordinates (instead of the original coordinates), and the transformation takes the standard appearance of a gauge transformation.
3. Deformed black hole

We consider a nonrotating black hole perturbed by a distribution of matter. The perturbation is sufficiently small that we can describe it within linearized perturbation theory, and to achieve this, we rely on the formulation of the theory provided in [19]. The matter is either either flowing across the event horizon, in which case the perturbation is sourced by matter, or it is situated outside the black hole’s immediate neighbourhood, in which case the perturbation is in vacuum and describes a tidal deformation of the black hole.

3.1. Spacetime metric

The metric of the unperturbed spacetime is Schwarzschild’s solution expressed in Eddington–Finkelstein coordinates,

\[ g^0_{\alpha\beta} \, dx^\alpha \, dx^\beta = - f \, dv^2 + 2 \, dv \, dr + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2), \]

with \( f := 1 - 2M/r \). We let \( x^a = (v, r) \) and \( \theta^A = (\theta, \phi) \). The metric on the unit two-sphere is \( \Omega_{AB} \, d\theta^A \, d\theta^B = d\theta^2 + \sin^2 \theta \, d\phi^2 \), and its inverse is denoted \( \Omega^{AB} \); covariant differentiation compatible with \( \Omega_{AB} \) is denoted \( D_A \).

The metric perturbation is denoted \( p_{a\beta} \), and it is decomposed in tensorial spherical harmonics (as defined in [19]). In the even-parity, sector we have

\[ p_{a\beta} = h_{ab}(v, r)Y(\theta^A), \]

\[ p_{a\beta} = j_a(v, r)Y_B(\theta^A), \]

\[ p_{AB} = r^2 K(v, r) \Omega_{AB} Y(\theta^A) + r^2 G(v, r) Y_{AB}(\theta^A), \]

with \( Y(\theta^A) \) denoting standard spherical-harmonic functions, \( Y_A := D_A Y \) and \( Y_{AB} := [D_A D_B + \frac{1}{2}\ell(\ell + 1)\Omega_{AB}]Y \). In the odd-parity sector, we have

\[ p_{ab} = 0, \]

\[ p_{aB} = h_a(v, r) X_B(\theta^A), \]

\[ p_{AB} = h_2(v, r) X_{AB}(\theta^A), \]

where \( X_A := -\epsilon^B_A \Omega_B Y \) and \( X_{AB} := \frac{1}{2}(D_A X_B + D_B X_A) \), with \( \epsilon_{AB} \) denoting the Lévi-Civita tensor (with component \( \epsilon_{\theta\phi} = \sin \theta \)) on the unit two-sphere. The tensorial harmonics \( Y_{AB} \) and \( X_{AB} \) are both symmetric and tracefree. The spherical-harmonic labels \( \ell m \) are suppressed, and so is summation over these labels. The complete metric of the perturbed spacetime is \( g^0_{\alpha\beta} = g^0_{\alpha\beta} + p_{a\beta} \).

Under an even-parity gauge transformation generated by the vector fields \( f_v = \eta_v(v, r)Y \) and \( f_A = r^2 \eta_{\text{even}}^v(v, r) Y_A \), the perturbation fields change according to

\[ \Delta h_{vv} = -2\partial_v \eta_v + \frac{2M}{r^2} \eta_v + \frac{2Mf}{r^2} \eta_r, \]

\[ \Delta h_{vr} = -\partial_r \eta_v - \partial_v \eta_r - \frac{2M}{r^2} \eta_r, \]

\[ \Delta h_{rr} = -2\partial_r \eta_r, \]

\[ \Delta j_v = -r^2 \partial_v \eta_{\text{even}}^v - \eta_v, \]

\[ \Delta j_r = -r^2 \partial_r \eta_{\text{even}}^v - \eta_r. \]
\[
\Delta K = -\frac{2f}{r} \eta_r - \frac{2}{r} \eta_v + \ell(\ell + 1)\eta_{\text{even}},
\]
\[
\Delta G = -2\eta_{\text{even}}.
\]

Under an odd-parity gauge transformation generated by the vector fields \( f_a = 0 \) and \( f_A = r^2 \eta_{\text{odd}}(v, r) X_A \), the perturbation fields change according to

\[
\Delta h_v = -r^2 \partial_v \eta_{\text{odd}},
\]
\[
\Delta h_r = -r^2 \partial_r \eta_{\text{odd}},
\]
\[
\Delta h_2 = -2r^2 \eta_{\text{odd}}.
\]

These transformations will play a role in the forthcoming developments.

3.2. Deformed horizon

The description of the deformed horizon relies on the geometrical methods reviewed in section 2. The event horizon is traced by its null generators, which are identified by constant labels \( \alpha^A = (\alpha, \beta) \); we use \( \lambda = v \) as a running parameter on each generator, and \((v, \alpha^A)\) forms a system of intrinsic coordinates on the horizon. The parametric equations that describe the horizon’s position in the unperturbed spacetime are \( v = v, r = 2M \) and \( \theta^A = \alpha^A \). In the perturbed spacetime, we have instead

\[
v = v, \quad r = 2M[1 + B(v, \alpha^A)], \quad \theta^A = \alpha^A + \Xi^A(v, \alpha^A),
\]

where \( 2MB \) and \( \Xi^A \) are the components of a Lagrangian displacement vector. This vector takes the horizon point identified by \((v, \alpha^A)\) in the original spacetime to a point also identified by \((v, \alpha^A)\) in the perturbed spacetime. We express the displacement fields as

\[
B = b(v) Y(\alpha^A), \quad \Xi^A = \Omega^{AB}[\ddot{\xi}_{\text{even}}(v) Y_B(\alpha^A) + \ddot{\xi}_{\text{odd}}(v) X_B(\alpha^A)],
\]

in which \( \Omega^{AB} \) is expressed in terms of the intrinsic coordinates \( \alpha^A \). As previously, we suppress the \( \ell m \) labels, as well as summation over these labels.

The parametric equations (3.18) allow us to calculate the basis vectors

\[
k^\alpha := \left( \frac{\partial x^\alpha}{\partial v} \right)_\alpha, \quad e_1^A := \left( \frac{\partial x^\alpha}{\partial \alpha^A} \right)_v.
\]

In this section, we place brackets around a basis index (which refers to the intrinsic coordinates \( \alpha^A \)) to distinguish it from a coordinate index (which refers to the spacetime coordinates \( \theta^A \)). Explicitly,

\[
k^v = 1,
\]
\[
k^r = 2M \partial_v B = 2Mb Y,
\]
\[
k^A = \partial_v \Xi^A = \Omega^{AB} \ddot{\xi}_{\text{even}} Y_B + \ddot{\xi}_{\text{odd}} X_B,
\]

in which an overdot indicates differentiation with respect to \( v \), and

\[
e_1^v = 0, \quad e_1^r = 2M \partial_v B, \quad e_1^A = \delta^A_B + \partial_v \Xi^A.
\]

The null condition \( k^\alpha k^\alpha = 0 \) gives rise to the first horizon equation,

\[
b - 4Mb = h_{vv}(v, 2M),
\]

and the conditions \( k^\alpha e_1^\alpha = 0 \) give rise to a second set of horizon equations,

\[
\ddot{\xi}_{\text{even}} = -(2M)^{-2}[j_{\text{ev}}(v, 2M) + 2Mb(v)].
\]
\[ \dot{\xi}^{\text{odd}} = -(2M)^{-2} h_v(v, 2M). \] (3.27)

These equations, along with appropriate choices of boundary conditions, fully determine the description of the deformed horizon.

The basis can be completed with a transverse vector \( N^a \) that satisfies the relations \( N_v N^a = 0, N_v k^a = -1 \) and \( N_a e^a_{(A)} = 0 \). A simple computation reveals that the components of this vector are given by

\[ N^v = \frac{1}{4} h_{rr} Y, \] (3.28)
\[ N^r = -1 + h_{rr} Y, \] (3.29)
\[ N^A = (2M)^{-2} \Omega^{AB} (j^r Y_B + h_r X_B), \] (3.30)

where all perturbation fields are evaluated at \( r = 2M \). The covariant components of the vector are \( N_v = -1, N_r = -\frac{1}{2} h_{rr} Y \) and \( N_A = 0 \).

To identify the correct solutions to the horizon equations, we imagine first an artificial situation in which the perturbation is switched off at times larger than \( v_1 \). The spacetime for \( v > v_1 \) is described by the Schwarzschild metric, and for these times the event horizon is correctly identified with the hypersurface \( r = 2M \). To locate the event horizon at times \( v < v_1 \), we must smoothly extend \( r = 2M \) backwards in time, to a null hypersurface in the perturbed spacetime. This surface is described by equation (3.18), with \( b(v) \) restricted to vanish for \( v > v_1 \). The appropriate solution to equation (3.25) is therefore

\[ b(v) = \kappa_0 \int_{v_1}^{v} e^{\kappa_0 (v-v')} h_{vv}(v', 2M) \, dv', \] (3.31)

where \( \kappa_0 \) is the surface gravity of the unperturbed black hole. The upper limit of integration was extended to \( v = \infty \) because, by the stated assumptions on the perturbation, \( h_{vv} \) is zero in the interval \( v_1 < v' < \infty \). At this stage, however, the artifice can be removed and equation (3.31) be adopted as the appropriate solution to equation (3.25) even when the perturbation does not switch off at \( v = v_1 \). The perturbation must still fall off sufficiently fast that the integral converges, and under these conditions \( b(v) \) will approach zero as \( v \to \infty \).

Because equation (3.31) reflects a choice of final condition, it is known as a teleological solution to the horizon equation.

The teleological solutions to equations (3.26) and (3.27) are

\[ \xi^{\text{even}}(v) = (2M)^{-2} \int_v^{v_1} [j_v(v', 2M) + 2M b(v')] \, dv', \] (3.32)
\[ \xi^{\text{odd}}(v) = (2M)^{-2} \int_v^{\infty} h_v(v', 2M) \, dv'. \] (3.33)

The behaviour of the horizon generators in the perturbed spacetime is now completely determined. The solutions to the horizon equations imply that, in general, the event horizon leads the perturbation by a time interval of order \( \kappa_0^{-1} = 4M \).

3.3. Horizon’s intrinsic geometry

As described in section 2, the intrinsic geometry of the event horizon is characterized by the induced metric \( \gamma_{AB} \), which is expressed in the intrinsic coordinates \((v, \alpha^A)\) attached to the null generators. According to equation (2.14), the \( v \)-derivative of the induced metric satisfies

\[ \partial_v \gamma_{AB} = 2B_{AB} = \Theta_{AB} + 2\sigma_{AB}, \] (3.44)
and this equation defines the expansion scalar $\Theta$ and shear tensor $\sigma_{AB}$ associated with the congruence of null generators. The expansion, in particular, can be computed as $\Theta = \frac{1}{2} \dot{\gamma}^{-1} \delta_{v} \gamma$, where $\gamma := \text{det}[\gamma_{AB}]$.

A computation of the horizon metric involves the substitution of equation (3.24) into equation (2.3). The computation must account for the fact that while the spacetime metric is expressed in terms of the coordinates $(v, r, \theta^A)$, the horizon metric will be expressed in terms of the intrinsic coordinates $(v, \alpha^A)$. A piece of the computation that requires some care involves $\Omega_{AB}(\theta^A)$, which must be written as $\Omega_{AB}(\alpha^A + \Xi^A) = \Omega_{AB} + \Xi^C \partial_C \Omega_{AB}$, with the right-hand side expressed in terms of $\alpha^A$. With this accounted for, we find that the horizon metric is

$$\gamma_{AB} = (2M)^2 \Omega_{AB} + (2M)^2 (2B \Omega_{AB} + \Omega_{BC} D_A \Xi^C + \Omega_{AC} D_B \Xi^C) + p_{AB}. \quad (3.35)$$

With equations (3.2)–(3.7) and (3.19), this becomes

$$\gamma_{AB} = (2M)^2 (\Omega_{AB} + \gamma^{\text{trace}} \Omega_{AB} Y + \gamma^{\text{even}} Y_{AB} + \gamma^{\text{odd}} X_{AB}), \quad (3.36)$$

where

$$\gamma^{\text{trace}} := 2b(v) - \ell(\ell + 1) \xi^{\text{even}}(v) + K(v, 2M), \quad (3.37)$$

$$\gamma^{\text{even}} := 2\xi^{\text{even}}(v) + G(v, 2M), \quad (3.38)$$

$$\gamma^{\text{odd}} := 2\xi^{\text{odd}}(v) + (2M)^{-\ell} h^2(v, 2M). \quad (3.39)$$

The square root of the metric determinant is given by $\sqrt{\gamma} = (2M)^2 \sin \alpha (1 + \gamma^{\text{trace}} Y)$, with $\alpha$ denoting the intrinsic polar angle on the horizon.

It follows from these equations that the expansion scalar is

$$\Theta = \gamma^{\text{trace}} Y, \quad (3.40)$$

while the shear tensor is

$$\sigma_{AB} = \frac{1}{2} (2M)^2 (\gamma^{\text{even}} Y_{AB} + \gamma^{\text{odd}} X_{AB}). \quad (3.41)$$

The expressions for $\gamma^{\text{trace}}$, $\gamma^{\text{even}}$ and $\gamma^{\text{odd}}$ can be simplified with the help of equations (3.25)–(3.27). We obtain

$$\gamma^{\text{trace}} = (2M)^{-1} \left\{ \ell(\ell + 1) b(v) - h_{v^2}(v, 2M) + \ell(\ell + 1)(2M)^{-1} j_v(v, 2M) + (2M) \partial_v K(v, 2M) \right\}, \quad (3.42)$$

$$\gamma^{\text{even}} = (2M)^{-1} \left\{ -2b(v) - 2(2M)^{-1} j_v(v, 2M) + 2M \partial_v G(v, 2M) \right\}, \quad (3.43)$$

$$\gamma^{\text{odd}} = (2M)^{-1} \left\{ -2(2M)^{-1} h_{v^2}(v, 2M) + (2M)^{-1} \partial_v h^2(v, 2M) \right\}. \quad (3.44)$$

The Ricci curvature associated with the metric of equation (3.36) is given by

$$R = \frac{1}{2M^2} \left\{ 1 + \frac{1}{2}(\ell - 1)(\ell + 2) \left[ \gamma^{\text{trace}} + \frac{1}{2} \ell(\ell + 1) \gamma^{\text{even}} \right] Y(\alpha^A) \right\}. \quad (3.45)$$

This indicates that the metric’s geometrical information is contained within $\gamma^{\text{trace}} + \frac{1}{2} \ell(\ell + 1) \gamma^{\text{even}} = 2b + K + \frac{1}{2} \ell(\ell + 1) G$; the remaining information is entirely about the choice of intrinsic coordinates, in particular, the fact that they are attached to the horizon’s null generators. To flesh out this last point, we recall that, according to the discussion near the end of section 2.4, the freedom to change the generator labels $\alpha^A$ is limited to a $v$-independent rotation of the form $\alpha^A \rightarrow \tilde{\alpha}^A(\alpha^B)$. For infinitesimal changes $\delta \alpha^A = \alpha^A + \delta \alpha^A$, the transformation is described by equation (2.52). If we choose

$$\delta \alpha^A = \Omega^{AB}(\xi^{\text{even}} Y_B + \xi^{\text{odd}} X_B), \quad (3.46)$$
where \( \xi^{\text{even}} \) and \( \xi^{\text{odd}} \) are constants, we find that \( \gamma^{\text{trace}} \), \( \gamma^{\text{even}} \) and \( \gamma^{\text{odd}} \) change according to

\[
\begin{align*}
\gamma^{\text{trace}} &\rightarrow \gamma^{\text{trace}} + \ell(\ell + 1)\xi^{\text{even}}, \\
\gamma^{\text{even}} &\rightarrow \gamma^{\text{even}} - 2\xi^{\text{even}}, \\
\gamma^{\text{odd}} &\rightarrow \gamma^{\text{odd}} - 2\xi^{\text{odd}}.
\end{align*}
\] (3.47)

We observe that the combination \( \gamma^{\text{trace}} + \frac{1}{2} \ell(\ell + 1)\gamma^{\text{even}} \) is unaffected by the transformation, which confirms its role as carrier of geometric information. At any given time (but at only one such time), \( \xi^{\text{even}} \) and \( \xi^{\text{odd}} \) can be chosen so as to make \( \gamma^{\text{even}} \) and \( \gamma^{\text{odd}} \) vanish. At this time, say \( v = v_0 \), we have that the horizon metric is given by

\[
\gamma_{AB}(v_0, \alpha A) = \left(\frac{2}{M}\right)^2 \Omega_{AB}^1 \left[1 + \gamma^{\text{trace}} Y(\alpha A)\right],
\] (3.50)

with \( \gamma^{\text{trace}} := \gamma^{\text{new}} = \gamma^{\text{old}} + \frac{1}{2} \ell(\ell + 1)\gamma^{\text{old}} \), or

\[
\gamma^{\text{trace}}(v_0) = 2b(v_0) + K(v_0, 2M) + \frac{1}{2} \ell(\ell + 1)G(v_0, 2M).
\] (3.51)

At other times \( v \neq v_0 \), the tracefree terms proportional to \( Y_{AB} \) and \( X_{AB} \) will no longer vanish, and the metric will return to its general form of equation (3.36).

### 3.4. Horizon’s extrinsic geometry

The horizon’s extrinsic geometry is characterized by \( \kappa, \omega_A \) and \( K_{AB} \), as defined by equations (2.6)–(2.8). Computation reveals that

\[
\kappa = \kappa_0(1 + kY),
\] (3.52)

where \( \kappa_0 = (4M)^{-1} \) is the unperturbed surface gravity, and

\[
k = -(2M \partial_r h_{vv} - 4M \partial_j h_{vr} + h_{vr} + 2b).
\] (3.53)

We also obtain

\[
\omega_A = \omega^{\text{even}} Y_A + \omega^{\text{odd}} X_A,
\] (3.54)

with

\[
\omega^{\text{even}} = \frac{1}{2} h_{vr} - \frac{1}{2} \partial_r j_v + (2M)^{-1} j_v + \frac{1}{2} \partial_j j_r + b,
\] (3.55)

\[
\omega^{\text{odd}} = -\frac{1}{2} \partial_r h_v + (2M)^{-1} h_r + \frac{1}{2} \partial_j h_r.
\] (3.56)

And finally, we obtain

\[
K_{AB} = -2M \Omega_{AB} + k^{\text{trace}} \Omega_{AB} Y + k^{\text{even}} Y_{AB} + k^{\text{odd}} X_{AB},
\] (3.57)

with

\[
k^{\text{trace}} = \ell(\ell + 1)2M^2 \xi^{\text{even}} + 2M h_{vr} - \frac{1}{2} \ell(\ell + 1)j_v - \frac{1}{2} (2M)^2 \partial_r K - 2MK - 2Mb,
\] (3.58)

\[
k^{\text{even}} = -4M \xi^{\text{even}} + j_v - \frac{1}{2} (2M)^2 \partial_j G - 2MG,
\] (3.59)

\[
k^{\text{odd}} = -4M^2 \xi^{\text{odd}} + h_r - \frac{1}{2} \partial_r h_2.
\] (3.60)

In these expressions, all perturbation fields and their derivatives are evaluated at \( r = 2M \), and all spherical harmonics are expressed as functions of \( \alpha A \). We note that the computation of \( K_{AB} \) requires the same level of care as the previous computation of \( \gamma_{AB} \): the unperturbed expression \( -r \Omega_{AB} \) must be evaluated at \( r = 2M(1 + B) \) and \( \theta^A = \alpha^A + \Xi^A \) and combined with the terms that arise from the metric perturbation.
3.5. Gauge transformations

We next work out how the various horizon quantities introduced previously are affected by a gauge transformation of the form

\[ x^a \rightarrow x^a + f^a, \quad f^a = \eta^a(v, r)Y(\theta^A) \]  

(3.61)

and

\[ \theta^A \rightarrow \theta^A + f^A, \quad f^A = \Omega^{AB}[\eta^{\text{even}}(v, r)Y_B(\theta^A) + \eta^{\text{odd}}(v, r)X_B(\theta^A)]. \]  

(3.62)

We also have that

\[ f^A = r^2 \Omega^{AB}f^B = (r^2 \eta^{\text{even}})Y_A + (r^2 \eta^{\text{odd}})X_A. \]  

The gauge transformation affects the coordinate description of the horizon. Recalling equation (3.18), we find that \( b, \xi^{\text{even}} \) and \( \xi^{\text{odd}} \) change according to

\[ \Delta b = (2M)^{-1} \eta_r(v, 2M), \]  

(3.63)

\[ \Delta \xi^{\text{even}} = \eta^{\text{even}}(v, 2M), \]  

(3.64)

\[ \Delta \xi^{\text{odd}} = \eta^{\text{odd}}(v, 2M). \]  

(3.65)

A complete listing of the corresponding changes in the metric perturbation can be found in section 3.1.

With these rules, it is easy to show that the quantities associated with the horizon’s intrinsic geometry change according to

\[ \Delta \gamma^{\text{trace}} = \Delta \gamma^{\text{even}} = \Delta \gamma^{\text{odd}} = 0. \]  

(3.66)

These results imply that \( \gamma_{AB}, \Theta \) and \( \sigma_{AB} \) are all gauge invariant, and we conclude that the horizon’s intrinsic geometry is gauge invariant. This is not a surprising conclusion. The intrinsic metric is a collection of scalar fields with regards to transformations of the spacetime coordinates \( x^a \), and it is expressed entirely in terms of the intrinsic coordinates \( (\lambda, \alpha^A) \). As such it is as a matter of principle immune to a gauge transformation. The fact that \( \lambda \) is identified with the spacetime coordinate \( v \) adds a small complication to this argument, because \( \gamma_{AB} \) could in principle be sensitive to a change in \( v \). The identification associates a gauge transformation on \( v \) with a reparameterization of the generators, as described in section 2.4. But \( \gamma_{AB}^0 = (2M)^2 \Omega_{AB} \), the induced metric on the unperturbed horizon, is independent of \( v \), and the results displayed in equations (2.47)–(2.51) reveal that an infinitesimal reparameterization has no effect on \( \delta \gamma_{AB} \), the metric perturbation. The conclusion, therefore, remains valid regardless of the identification \( \lambda \equiv v \). As an additional remark, we recall that the invariance of \( \gamma_{AB} \) under general (large) reparameterizations was established in equations (2.36)–(2.40).

On the other hand, the quantities associated with the horizon’s extrinsic geometry change according to

\[ \Delta k = -4M \partial_\nu(\partial_\nu \eta_r + \kappa_0 \eta_r), \]  

(3.67)

\[ \Delta \omega^{\text{even}} = -\partial_\nu \eta_r + \kappa_0 \eta_r, \]  

(3.68)

\[ \Delta \kappa^{\text{trace}} = -2M \partial_\nu \eta_r + \frac{1}{2} \ell (\ell + 1) \eta_r, \]  

(3.69)

\[ \Delta \kappa^{\text{even}} = -\eta_r, \]  

(3.70)

and

\[ \Delta \omega^{\text{odd}} = 0 = \Delta \kappa^{\text{odd}}. \]  

(3.71)

In the even-parity sector, the changes in the extrinsic geometry are all associated with \( \eta_r(v, 2M) = \eta^r(v, 2M) \), which describes a change in \( v \); there are no changes in the odd-parity sector. As before, we can observe that since \( \kappa, \omega_A \) and \( \kappa_{AB} \) are all spacetime scalars...
expressed entirely in terms of the intrinsic coordinates, they should all be immune to a gauge transformation. But as before we can identify a change in $v$ with a reparameterization of the generators, and infer from equations (2.47)–(2.51) the effect of the reparameterization on the extrinsic geometry. With $\delta \lambda$ identified with $\eta_{\lambda}(v, 2M)Y(\alpha^{-1})$, we quickly find that equations (2.47)–(2.51) reproduce the statements of equations (3.67)–(3.71).

It is easy to identify four linearly independent quantities, formed from $k$, $\omega^{\text{even}}$, $K^{\text{trace}}$ and $K^{\text{even}}$, that are invariant under infinitesimal reparameterizations. We choose

$$\psi_1 := \dot{\omega}^{\text{even}} - \kappa_0 k,$$

$$\psi_2 := \dot{K}^{\text{even}} + \kappa_0 K^{\text{even}} - \omega^{\text{even}},$$

$$\psi_3 := \dot{\gamma}^{\text{trace}} + \kappa_0 \gamma^{\text{trace}} + \frac{1}{2}(\ell + 1)\omega^{\text{even}} - \frac{1}{2}k,$$

$$\psi_4 := \dot{\gamma}^{\text{even}} + \frac{1}{2}[\ell(\ell + 1) + 1]K^{\text{even}} - 2M\omega^{\text{even}}.$$  

The first three combinations can be shown to be pieces of the spacetime Riemann tensor evaluated on the deformed horizon. Indeed, inserting the results obtained in section 3.4 within equation (2.18) yields

$$R_{\mu \nu \lambda \rho} k^\mu N^\nu k^\lambda e_A^\rho = (\dot{\omega}^{\text{even}} - \kappa_0 k)Y_A + \dot{\omega}^{\text{odd}} X_A.$$  

It is easy to show that the left-hand side is invariant under infinitesimal reparameterizations, and this guarantees that $\psi_1$ and $\omega^{\text{odd}}$ also must be invariant. Similarly, we find from equation (2.28) that

$$\frac{1}{2} R_{\mu \rho \lambda} e_A^\mu e_A^\rho = \left[\dot{K}^{\text{trace}} + \kappa_0 K^{\text{trace}} + \frac{1}{2}(\ell + 1)\omega^{\text{even}} - \frac{1}{2}k \right.$$

$$\left. + \frac{1}{2}\ell(\ell + 1)\gamma^{\text{trace}} + \frac{1}{8}(\ell - 1)(\ell + 1)(\ell + 2)\gamma^{\text{even}} \right] \Omega_{A B} \gamma X_A +$$

$$\left(\dot{K}^{\text{even}} + \kappa_0 K^{\text{even}} - \omega^{\text{even}} + M\gamma^{\text{even}} + \frac{1}{2}\gamma^{\text{even}} \right) Y_{A B} +$$

$$\left(\dot{\gamma}^{\text{odd}} + \kappa_0 \gamma^{\text{odd}} - \omega^{\text{odd}} + M\gamma^{\text{odd}} + \frac{1}{2}\gamma^{\text{odd}} \right) X_{A B};$$  

invariance of $R_{\mu \rho \lambda} e_A^\mu e_A^\rho$ and $\gamma_{A B}$ under infinitesimal parameterizations guarantees that $\psi_2$ and $\psi_3$ also must be invariant. The fourth quantity, $\psi_4$, does not appear to be related in a similar way to a piece of the spacetime Riemann tensor.

4. Tidal deformations

The formalism developed in the preceding section is very general, and it can accommodate black-hole deformations created by matter flowing across the event horizon, or by matter situated outside the black hole’s immediate neighbourhood. The formalism is also general relative to the choice of gauge because the relations between the horizon quantities (such as $\gamma^{\text{trace}}$, $\gamma^{\text{even}}$, $\gamma^{\text{odd}}$, $k$, $\omega^{\text{even}}$, $\omega^{\text{odd}}$, $K^{\text{trace}}$, $K^{\text{even}}$ and $K^{\text{odd}}$) and the metric perturbation are valid in any gauge. How the horizon quantities change under gauge transformations (or better stated, reparameterizations of the horizon’s null generators) was described in section 3.5.

In this section, we specialize the situation to a tidal deformation of a black hole created by a remote distribution of matter. We incorporate the vacuum field equations into our analysis to relate the horizon quantities to the well-known master functions $\Psi^{\text{even}}$ and $\Psi^{\text{odd}}$ of black-hole perturbation theory (defined below). We next introduce a geometric notion of tidal displacement on the event horizon and describe how the tidal bulge is related to the applied tidal field.
4.1. Master functions

Gauge-invariant definitions of the master functions were provided in [19]. In the even-parity sector, $\Psi_{\text{even}}$ is the Zerilli–Moncrief function [22, 23] defined by

$$
\Psi_{\text{even}} := \frac{2r}{\lambda} \left( \tilde{K} + \frac{2}{\mu + 6M/r} \left( r^a r^b \tilde{h}_{ab} - rr^a \nabla_a \tilde{K} \right) \right),
$$

(4.1)

where $\lambda := \ell(\ell+1) = \mu + 2, \mu := (\ell - 1)(\ell + 2) = \lambda - 2$, and where $\tilde{h}_{ab} := h_{ab} - \nabla_a \eta_b - \nabla_b \eta_a, \tilde{K} := K + \frac{1}{2} \lambda G - 2r^a \epsilon_{ab}/r$, with $\epsilon_a := j_a - \frac{1}{2} r^2 \nabla_a G$, are gauge-invariant combinations of metric perturbations. We use the notation $r_a := \partial_r/\partial x^a, \nabla_a$ is the covariant-derivative operator compatible with the two-dimensional metric $g_{0}^{ab} d x^a d x^b = -f d \nu^2 + 2 d \nu d r$ and as usual the spherical-harmonic labels $\ell m$ are omitted. The Zerilli–Moncrief function is known to satisfy the Zerilli equation [22], which is a two-dimensional wave equation with an effective potential and a source term constructed from the energy-momentum tensor of the matter distribution.

In the odd-parity sector, $\Psi_{\text{odd}}$ is the Cunningham–Price–Moncrief function [24, 25] defined by

$$
\Psi_{\text{odd}} := \frac{2r}{\mu} \epsilon^{ab} \left( \nabla_a \tilde{h}_{lm}^b - \frac{2}{r} r_a \tilde{h}_{lm}^b \right),
$$

(4.2)

where $\epsilon_{ab}$ is the Lévi-Civita tensor on the two-dimensional manifold with metric $g_{0}^{ab}$, and $\tilde{h}_a := h_a - \frac{1}{4} \nabla_a h_2 + r_a h_2/r$ is a gauge-invariant combination of metric perturbations. The master function is known to satisfy the Regge–Wheeler equation [26], another two-dimensional wave equation with an effective potential and a source term. The Regge–Wheeler equation is also satisfied by another choice of master function, the original Regge–Wheeler function [26]; in vacuum this is equal to half the time derivative of the Cunningham–Price–Moncrief function.

4.2. Killing gauge

To relate the horizon quantities to $\Psi_{\text{even}}$ and $\Psi_{\text{odd}}$, it is convenient to adopt a ‘Killing gauge’ defined by

$$
p_{\mu \beta} t^\beta = 0,
$$

(4.3)

where $t^a$ is the timelike Killing vector of the Schwarzschild spacetime. In the coordinates $(\nu, r, \theta, \phi)$, we have that $t^a = (1, 0, 0, 0)$, and the gauge conditions translate to

$$
h_{\nu \nu} = h_{\nu r} = j_\nu = 0
$$

(4.4)

in the even-parity sector, and

$$
h_{\nu} = 0
$$

(4.5)

in the odd-parity sector. These conditions apply in a neighbourhood of the event horizon.

An immediate virtue of the Killing gauge is that it preserves the coordinate description of the event horizon, which continues, even in the perturbed spacetime, to be described by $r = 2M$ and $\theta^A = \alpha^A$. In the terminology of Poisson and Vlasov [15], the Killing gauge is a horizon-locking gauge. This can be seen at once from equations (3.31)–(3.33), which imply that

$$
b(\nu) = \xi_{\text{even}} = \xi_{\text{odd}} = 0
$$

(4.6)

whenever $h_{\nu \nu} = j_\nu = h_\nu = 0$ at $r = 2M$. We remark that while the light-cone gauge adopted by Poisson and Vlasov also has the property of being a horizon-locking gauge, the Killing gauge adopted here is quite distinct from the light-cone gauge.
4.3. Near-horizon analysis

To calculate the horizon quantities, we must integrate the perturbation equations in a neighbourhood of the event horizon; these are listed in sections IV B and V B of [19].

In the even-parity sector, this can be accomplished by inserting the expansions:

\[ h_r = h_0(v) + h_1(v)(r-2M) + h_2(v)(r-2M)^2 + \cdots, \]
\[ j_r = j_0(v) + j_1(v)(r-2M) + j_2(v)(r-2M)^2 + \cdots, \]
\[ K = K_0(v) + K_1(v)(r-2M) + K_2(v)(r-2M)^2 + \cdots, \]
\[ G = G_0(v) + G_1(v)(r-2M) + G_2(v)(r-2M)^2 + \cdots. \]

Such an analysis reveals that \( j_0(v) \) is unconstrained by the field equations, that \( K_0(v) = 0 \) and \( G_0(v) = 2\mu^{-1} d j_0 / dv \), and that \( K_1(v) \) and \( G_1(v) \) must satisfy the differential equations

\[
\frac{dK_1}{dv} + \kappa_0 K_1 = -4\lambda \kappa_0^3 j_0
\]

and

\[
\frac{dG_1}{dv} + \kappa_0 G_1 = -\frac{4\kappa_0}{\mu} \left( \frac{d^2 j_0}{dv^2} - \mu \kappa_0 \frac{d j_0}{dv} - 2\mu \kappa_0^2 j_0 \right).
\]

where \( \kappa_0 := (4M)^{-1} \) is the surface gravity of the unperturbed horizon.

The fact that the differential operator acting on \( K_1 \) and \( G_1 \) is \( d/dv + \kappa_0 \), instead of \( d/dv - \kappa_0 \) as in equation (3.25), implies that one should not look for teleological solutions to these equations: the presence of \( e^{\kappa_0 v} \) instead of \( e^{-\kappa_0 v} \) within the integrals would prevent them from converging if the integrations were unbounded. We work instead with the most general solutions

\[
K_1(v) = K_1(v_0) e^{-\kappa_0(v-v_0)} - 4\lambda \kappa_0^3 \int_{v_0}^{v} e^{-\kappa_0(v'-v_0)} j_0(v') \, dv'
\]

and

\[
G_1(v) = \left\{ G_1(v_0) + \frac{4\kappa_0}{\mu} \left[ \frac{d j_0}{dv}(v_0) - (\lambda - 1) \kappa_0 j_0(v_0) \right] \right\} e^{-\kappa_0(v-v_0)}
\]

\[ - \frac{4\kappa_0}{\mu} \left[ \frac{d j_0}{dv}(v) - (\lambda - 1) \kappa_0 j_0(v) - (\lambda - 3) \kappa_0^2 \int_{v_0}^{v} e^{-\kappa_0(v'-v_0)} j_0(v') \, dv' \right].
\]

in which the initial values \( K_1(v_0) \) and \( G_1(v_0) \) are not determined by the requirements that \( K_1(v \to \infty) \to 0 \) and \( G_1(v \to \infty) \to 0 \). (We explore these issues further in the appendix below.) We do not need expressions for \( h_0(v) \), \( h_1(v) \), \( j_1(v) \) (nor other coefficients in the expansions), and the master function can be shown to be given by \( \Psi_{even} = (\mu \kappa_0)^{-1} d h_0 / dv \) at \( r = 2M \).

In the odd-parity sector, we substitute the expansions:

\[
h_0 = h_{r0}(v) + h_{s1}(v)(r-2M) + h_{s2}(v)(r-2M)^2 + \cdots
\]
\[ h_2 = h_{r0}(v) + h_{s1}(v)(r-2M) + h_{s2}(v)(r-2M)^2 + \cdots \]

into the perturbation equations and solve order by order in \( r-2M \). Such an analysis reveals that \( h_{r0}(v) \) is unconstrained by the field equations, that \( h_{r0}(v) = 2\mu \kappa_0^{-1} d h_{r0} / dv \) and that \( h_{s1}(v) \) satisfies the differential equation

\[
\frac{dh_{s1}}{dv} + \kappa_0 h_{s1} = \frac{1}{\mu \kappa_0} \left[ \frac{d^2 h_{r0}}{dv^2} + \lambda \kappa_0 \frac{dh_{r0}}{dv} + 2\mu \kappa_0^2 h_{r0} \right].
\]

The general solution is

\[
h_{s1}(v) = \left\{ h_{s1}(v_0) - \frac{1}{\mu \kappa_0} \left[ \frac{dh_{r0}}{dv}(v_0) + (\lambda - 1) \kappa_0 h_{r0}(v_0) \right] \right\} e^{-\kappa_0(v-v_0)}
\]

\[ + \frac{1}{\mu \kappa_0} \left[ \frac{dh_{r0}}{dv}(v) + (\lambda - 1) \kappa_0 h_{r0}(v) + (\lambda - 3) \kappa_0^2 \int_{v_0}^{v} e^{-\kappa_0(v'-v_0)} h_{r0}(v') \, dv' \right],
\]

and the master function can be shown to be given by \( \Psi_{odd} = -(\mu \kappa_0)^{-1} d h_{r0} / dv \) at \( r = 2M \).
4.4. Horizon quantities

With the results obtained in the preceding subsection, we find that the horizon quantities defined in sections 3.3 and 3.4 are given by

\[
\gamma^{\text{trace}} = 0, \tag{4.13}
\]

\[
\gamma^{\text{even}} = 2\kappa_0 \Psi_{\text{even}}(v, 2M), \tag{4.14}
\]

\[
\gamma^{\text{odd}} = -2\kappa_0 \Psi_{\text{odd}}(v, 2M), \tag{4.15}
\]

\[
k = 0, \tag{4.16}
\]

\[
\omega^{\text{even}} = \frac{1}{2} \mu \kappa_0 \Psi_{\text{even}}(v, 2M), \tag{4.17}
\]

\[
\omega^{\text{odd}} = -\frac{1}{2} \mu \kappa_0 \Psi_{\text{odd}}(v, 2M), \tag{4.18}
\]

\[
K^{\text{trace}} = K^{\text{trace}}(v_0)e^{-\kappa_0(v-v_0)} - \frac{1}{2} \lambda \kappa_0 \int_{v_0}^{v} e^{-\kappa_0(v-v')} \Psi_{\text{even}}(v', 2M) \, dv', \tag{4.19}
\]

\[
K^{\text{even}} = \left[ K^{\text{even}}(v_0) + \frac{1}{2} \Psi_{\text{even}}(v_0, 2M) \right] e^{-\kappa_0(v-v_0)} - \frac{1}{2} \Psi_{\text{even}}(v, 2M)
+ \frac{1}{2} (\lambda - 3) \kappa_0 \int_{v_0}^{v} e^{-\kappa_0(v-v')} \Psi_{\text{even}}(v', 2M) \, dv', \tag{4.20}
\]

\[
K^{\text{odd}} = \left[ K^{\text{odd}}(v_0) - \frac{1}{2} \Psi_{\text{odd}}(v_0, 2M) \right] e^{-\kappa_0(v-v_0)} + \frac{1}{2} \Psi_{\text{odd}}(v, 2M)
- \frac{1}{2} (\lambda - 3) \kappa_0 \int_{v_0}^{v} e^{-\kappa_0(v-v')} \Psi_{\text{odd}}(v', 2M) \, dv', \tag{4.21}
\]

where \(K^{\text{trace}}(v_0), K^{\text{even}}(v_0),\) and \(K^{\text{odd}}(v_0)\) can be expressed in terms of \(K_1(v_0), G_1(v_0),\) and \(j_0(v_0).\) We recall that \(\lambda := \ell (\ell + 1) = \mu + 2\) and \(\mu := (\ell - 1)(\ell + 2) = \lambda - 2.\)

The results display a pleasing symmetry (up to signs, which are inherited from the definitions of the master functions) between the even-parity and odd-parity sectors. In the case of the intrinsic-geometry quantities \(\gamma^{\text{even}}\) and \(\gamma^{\text{odd}},\) the symmetry is gauge invariant; in the case of the extrinsic-geometry quantities \(\omega^{\text{even}}\) and \(\omega^{\text{odd}},\) \(K^{\text{even}}\) and \(K^{\text{odd}},\) the symmetry is a property of the Killing gauge adopted here (it is not, in particular, a property of the light-cone gauge [15]). Another remarkable property of the Killing gauge is the fact that \(k = 0,\) so that the surface gravity of the perturbed black hole is \(\kappa = \kappa_0 = (4M)^{-1}.\)

The expressions for \(K^{\text{trace}}\) and \(K^{\text{even}}\) given previously were simplified relative to the more primitive expressions obtained in terms of \(j_0.\) These, however, involved the combination

\[
j_0 - \kappa_0 \int_{v_0}^{v} e^{-\kappa_0(v-v')} j_0(v') \, dv' = j_0(v_0) e^{-\kappa_0(v-v_0)} + \int_{v_0}^{v} e^{-\kappa_0(v-v')} d_j j_0 \, dv', \tag{4.22}
\]

which could readily be expressed in terms of \(\Psi_{\text{even}} = (\mu \kappa_0)^{-1} d_j j_0 / dv.\) A very similar simplification was achieved in the case of \(K^{\text{odd}}.\)

The gauge-invariant quantities defined by equations (3.72)–(3.75) are easily shown to be given by

\[
\psi_1 = \frac{1}{2} \mu \kappa_0 \partial_v \Psi_{\text{even}}(v, 2M), \tag{4.23}
\]
\(\psi_2 = -\frac{1}{2} \partial_v \Psi_{\text{even}}(v, 2M) - \kappa_0 \Psi_{\text{even}}(v, 2M),\) \hspace{1cm} (4.24)

\(\psi_3 = -\frac{1}{4} \lambda \mu \kappa_0 \Psi_{\text{even}}(v, 2M),\) \hspace{1cm} (4.25)

\[\psi_4 = \left[ \kappa^{\text{trace}}(v_0) + \frac{1}{2} (\lambda + 1) \kappa_{\text{even}}(v_0) + \frac{1}{4} (\lambda + 1) \Psi_{\text{even}}(v_0, 2M) \right] e^{-\kappa_0 (v-v_0)} - \frac{1}{4} (2\lambda - 1) \Psi_{\text{even}}(v, 2M) - \frac{1}{4} (\lambda + 3) \kappa_0 \int_{v_0}^{v} e^{-\kappa_0 (v - v')} \Psi_{\text{even}}(v', 2M) \, dv'.\] \hspace{1cm} (4.26)

4.5. Intrinsic geometry and tidal displacement

The results obtained in the preceding subsection imply that the induced metric on the event horizon simplifies to

\[\gamma_{AB} = (2M)^2 (\Omega_{AB} + \delta_{AB}),\] \hspace{1cm} (4.27)

\[\delta_{Y_{AB}} = 2\kappa_0 [\Psi_{\text{even}}(v, 2M) Y_{AB} - \Psi_{\text{odd}}(v, 2M) X_{AB}],\] \hspace{1cm} (4.28)

in the case of a tidally deformed black hole. From equations (3.40) and (3.41) we also obtain

\[\Theta = 0\] \hspace{1cm} (4.29)

and

\[\sigma_{AB} = M [\partial_v \Psi_{\text{even}}(v, 2M) Y_{AB} - \partial_v \Psi_{\text{odd}}(v, 2M) X_{AB}].\] \hspace{1cm} (4.30)

The fact that the expansion vanishes to leading order in perturbation theory can be derived directly from Raychaudhuri’s equation: the reduction of equation (2.29) to vacuum and to first-order perturbation theory is

\[\partial_v \Theta = \frac{\kappa_0}{\Theta},\] and this implies (with an appropriate choice of final condition) that \(\Theta\) must vanish.

The reduction of equation (2.30) gives an expression for the Weyl tensor evaluated on the event horizon:

\[C_{AB} = (\kappa_0 - \partial_v) \sigma_{AB},\] \hspace{1cm} (4.31)

where \(C_{AB} := C_{\mu \alpha \nu \beta} k^\alpha e^\mu (A) k^\nu e^\beta (B).\) This equation can be integrated to relate the shear tensor to the Weyl tensor; the appropriate teleological solution is

\[\sigma_{AB}(v, \alpha^A) = \int_{v_0}^{v} e^{-\kappa_0 (v - v')} C_{AB}(v', \alpha^A) \, dv'.\] \hspace{1cm} (4.32)

This equation implies that the shear tensor anticipates the behaviour of the Weyl tensor by a time interval of order \(\kappa_0^{-1} = 4M.\) If the Weyl tensor is identified with the tidal field acting on the black hole, and if the shear tensor is adopted as a measure of tidal deformation, then we have the statement that the tide leads the applied field by a time interval of order \(4M.\) This observation was already made by Fang and Lovelace [13] in a more restricted context (and by Hartle [27] in the case of a rotating black hole), and we find here that it holds in all generality as a consequence of the teleological nature of the event horizon. We remark that in the case of a Newtonian body made up of a viscous fluid, the tide would be lagging instead of leading (when the body is nonrotating), and that the time interval would be proportional to \(R\nu / M,\) with \(R\) denoting the body’s averaged radius, \(\nu\) its kinematic viscosity and \(M\) its mass.

Another meaningful measure of tidal deformation comes from the Ricci curvature scalar associated with the metric of equations (4.27) and (4.28). This is given by equation (3.45) with \(\gamma^{\text{trace}} = 0\) and \(\gamma^{\text{even}} = 2\kappa_0 \Psi_{\text{even}}(v, 2M):\)

\[\mathcal{R} = \frac{1}{2M^2} \left[ 1 + \frac{1}{2} (\ell - 1) (\ell + 1) (\ell + 2) \kappa_0 \Psi_{\text{even}}(v, 2M) Y(\alpha^A) \right].\] \hspace{1cm} (4.33)
It is helpful to convert this into a dimensionless tidal displacement field \( \rho(v, \alpha^A) \) by identifying \( R \) with the curvature of a two-dimensional surface embedded in a flat, three-dimensional space. We describe this surface in spherical coordinates \((r, \alpha^A)\) by the parametric equation
\[
r = \frac{2}{M} \left[ 1 + \rho(v, \alpha^A) \right]
\]
and demand that its curvature be equal to \( R \). We thus obtain
\[
2M^2R = \left[ 1 + (\ell - 1)(\ell + 2)\epsilon Y(\alpha^A) \right],
\]
and the identification
\[
\rho(v, \alpha^A) = \frac{1}{2} \ell(\ell + 1)\kappa_0 \Psi_{even}(v, 2M) Y(\alpha^A)
\]
follows immediately. Once more summation over the omitted spherical-harmonic labels \( \ell m \) is understood.

The absence of trace terms in equations (4.28), (4.30) and (4.31) implies that each tensor \( \delta_{\gamma AB}, \sigma_{AB} \) and \( C_{AB} \) possesses only two independent components. Introducing the basis vectors
\[
\alpha^A = [1, 0], \quad \beta^A = [1, 1/\sin \alpha]
\]
in the horizon coordinates \( \alpha^A = (\alpha, \beta) \), we take the independent components of \( \delta_{\gamma AB} \) to be
\[
\gamma^+ := \frac{1}{2} (\alpha^A \alpha^B - \beta^A \beta^B) \delta_{\gamma AB},
\]
\[
\gamma^\times := \frac{1}{2} (\alpha^A \beta^B + \beta^A \alpha^B) \delta_{\gamma AB}.
\]
The independent components \( \sigma^{+, \times} \) and \( C^{+, \times} \) of the shear and Weyl tensors are defined in a similar manner. These quantities are closely analogous to the gravitational-wave polarizations \( h^{+, \times} \) that can be defined in the wave zone of an asymptotically flat spacetime.

5. Applications

5.1. Slowly varying quadrupolar tidal field

As an application of the general formalism developed here, we revisit the situation examined by Poisson and Vlasov [15], that of a black hole deformed by a slowly varying tidal field. To simplify the discussion, we neglect the nonlinear terms included in [15] and specialize the tidal field to a pure quadrupolar form.

As described in section II of [15], the black hole’s tidal environment is described by the tidal moments \( E_{jk}(v) \) and \( B_{jk}(v) \). These quantities are symmetric-tracefree Cartesian tensors that represent the components of the spacetime Weyl tensor evaluated far away from the black hole; latin indices \( j \) and \( k \) (and so on) run over the values 1, 2 and 3. The tidal moments give rise to the tidal potentials
\[
E^+ := E_p^q \Omega^p \Omega^q, \quad (5.1)
\]
\[
E^\times := P^p_j E_p^q \Omega^k, \quad (5.2)
\]
\[
E^{+}_{jk} := 2P^p_j P^q_k E_p^q + P_{jk} E^+, \quad (5.3)
\]
and
\[
B^+ := \epsilon_{jqp} \Omega^p B^q_n \Omega^n, \quad (5.4)
\]
\[
B^{+}_{jk} := \epsilon_{jqp} \Omega^p B^q_n P^n + \epsilon_{knq} \Omega^q B^p_n P^q_j, \quad (5.5)
\]
where the label ‘q’ stands for ‘quadrupolar’, \( \Omega^j := [\sin \alpha \cos \beta, \sin \alpha \sin \beta, \cos \alpha] \) is a Cartesian unit vector constructed from the generator labels \( \alpha^A = (\alpha, \beta) \), \( P_{jk} := \delta_{jk} - \Omega^k \Omega_k \) is a projection operator to the subspace transverse to \( \Omega^j \) and \( \epsilon_{jkn} \) is the Cartesian permutation symbol. The vector potentials \( E^+ \) and \( B^+ \) are transverse, in the sense that \( E^+_j \Omega^j = 0 = B^+_j \Omega^j \). In addition to being transverse, the tensor potentials \( E^{+}_{jk} \) and \( B^{+}_{jk} \) are also tracefree, in the sense
that \( \delta^{jk} \epsilon^l_{jk} = 0 = \delta^{jk} B^l_{jk} \). In all manipulations involving Cartesian tensors, latin indices are lowered and raised with the Euclidean metric \( \delta_{jk} \).

The vectorial and tensorial potentials can be converted to angular components by means of the transformation matrix \( \Omega_{AB}^l := \partial \Omega^l / \partial \alpha^A \). We thus introduce
\[
\epsilon^l_A := \epsilon^l_j \Omega_j^A, \quad \epsilon_{AB}^l := \epsilon^l_{jk} \Omega^j_A \Omega^k_B.
\]

and
\[
B^l_A := B^l_j \Omega^j_A, \quad B_{AB}^l := B^l_{jk} \Omega^j_A \Omega^k_B.
\]

As shown in section II of [15], these angular potentials can be expressed as expansions in spherical harmonics of degree \( \ell = 2 \). We have
\[
\epsilon^l = \sum_m \epsilon_m Y^{2,m}_l, \quad \epsilon^l_A = \frac{1}{2} \sum_m \epsilon_m Y^{2,m}_A, \quad \epsilon_{AB}^l = \sum_m \epsilon_m Y^{2,m}_{AB}
\]

and
\[
B^l_A = \frac{1}{2} \sum_m B_m X^{2,m}_A, \quad B_{AB}^l = \sum_m B_m X^{2,m}_{AB}.
\]

The sums are carried out from \( m = -2 \) to \( m = 2 \), the coefficients \( \epsilon_m \) and \( B_m \) are related to \( \epsilon_{jk} \) and \( B_{jk} \) and depend on \( v \) only; the spherical harmonics are functions of \( \alpha^A \). These expansions reveal that \( \epsilon_{jk}(v) \) gives rise to a perturbation of even parity, while \( B_{jk}(v) \) gives rise to a perturbation of odd parity.

Solutions to the perturbation equations corresponding to a black hole deformed by a quadrupolar tidal field were constructed by Poisson and Vlasov [15]. The construction assumes that the tidal moments vary slowly, in the sense that the timescale \( \tau \) associated with these variations (denoted \( R \) in [15]) is very long compared with the black-hole mass. The solutions were provided in the light-cone gauge, but it is easy from these results to obtain the gauge-invariant master functions. The relations, in fact, are the same as in the Killing gauge adopted in section 4.2: we have that

\[
\delta_{jk} \epsilon^l_{jk} = 0 = \delta_{jk} B^l_{jk}.
\]

where \( p_1, p_2, q_1 \) and \( q_2 \) are arbitrary numbers.

It is a simple matter to insert equations (5.10) and (5.11) within equations (4.13)–(4.21) and to calculate the horizon quantities. Because the tidal moments \( \epsilon_{jk} \) and \( B_{jk} \) vary slowly, the integrations can be carried out as in the appendix, by repeated integration by parts. After discarding the transient terms that decay exponentially, we arrive at

\[
\begin{align*}
\gamma^{\text{trace}} &= 0, \quad (5.12) \\
\gamma^{\text{even}} &= -\frac{i}{2} M^2 \epsilon_m, \quad (5.13) \\
\gamma^{\text{odd}} &= -\frac{i}{2} M^2 B_m. \quad (5.14)
\end{align*}
\]
\( k = 0, \)  
\( \omega_{\text{even}} = -\frac{2}{3} M^2 \epsilon_m, \)  
\( \omega_{\text{odd}} = -\frac{2}{3} M^2 \beta_m, \)  
\( \kappa_{\text{trace}} = 16 M^3 (\epsilon_m - 4 M \dot{\epsilon}_m + 16 M^2 \ddot{\epsilon}_m), \)  
\( \kappa_{\text{even}} = -\frac{4}{3} M^3 (\epsilon_m - 6 M \dot{\epsilon}_m + 24 M^2 \ddot{\epsilon}_m), \)  
\( \kappa_{\text{odd}} = -\frac{4}{3} M^3 (\beta_m - 6 M \dot{\beta}_m + 24 M^2 \ddot{\beta}_m). \)  
(5.15)  
(5.16)  
(5.17)  
(5.18)  
(5.19)  
(5.20)

These expressions are valid up to correction terms of fractional order \( (M/\tau)^{3}; \) they are given in the Killing gauge introduced in equation (4.3).

These results, together with equations (5.8) and (5.9), imply that in the Killing gauge,
\[
\gamma_{AB} = (2M)^2 \left[ \Omega_{AB} - \frac{2}{3} M^2 (\epsilon^s_{AB} + \beta^s_{AB}) \right].
\]
(5.21)
\[
\kappa = \frac{1}{4M},
\]
(5.22)
\[
\omega_A = -\frac{4}{3} M^2 (\epsilon^s_{AB} + \beta^s_{AB}).
\]
(5.23)
\[
\kappa_{AB} = -2M \Omega_{AB} + 16 M^3 (\epsilon^s - 4 M \dot{\epsilon}^s + 16 M^2 \ddot{\epsilon}^s) \Omega_{AB}
= -\frac{4}{3} M^3 \left[ (\epsilon^s_{AB} + \beta^s_{AB}) - 6 M (\dot{\epsilon}^s_{AB} + \dot{\beta}^s_{AB}) + 24 M^2 (\ddot{\epsilon}^s_{AB} + \ddot{\beta}^s_{AB}) \right];
\]
(5.24)
as before, these expressions are accurate up to terms involving the third derivative of the tidal moments. We showed in section 3.5 that \( \gamma_{AB} \) is gauge invariant, while \( \kappa, \omega_A \) and \( \kappa_{AB} \) are affected by a reparameterization of the horizon’s null generators. Gauge-invariant combinations of these quantities were identified, and in particular we have that
\[
R_{\mu\nu\lambda\kappa} k^\mu N^\nu k^\lambda e^\alpha_A = -\frac{4}{3} M^2 (\dot{\epsilon}^s_A + \dot{\beta}^s_A)
\]
(5.25)
is invariant under infinitesimal reparameterizations; because it originates from \( \dot{\omega}_{\text{even}} \) and \( \dot{\omega}_{\text{even}}, \) this expression is accurate up to the fourth derivative of the tidal moments.

5.2. Parabolic encounter

As a second application of the formalism, we consider a parabolic encounter between a particle of mass \( m \) and a black hole of mass \( M. \) We take \( m \) to be much smaller than \( M \) and the motion of the particle to be a geodesic in the Schwarzschild spacetime. We give the orbit a semi-latus rectum \( p = 8.1M \) and an eccentricity \( e = 1; \) the parameterization is such that the radial turning points are situated at \( r_{\min} = p/(1 + e) = 4.05M \) and \( r_{\max} = p/(1 - e) = \infty. \) The orbit has a Killing energy \( E = m \) and a Killing angular momentum \( L \approx 4.0003 mM. \) The particle begins from rest at infinity, moves inward, circles approximately twice around the black hole, moves outward and returns to rest at infinity; the shape of the orbit is displayed in figure 1. Because the turning point is so close to the black hole, the motion is highly relativistic when the particle revolves around the black hole, and the tidal interaction is highly dynamical.

We calculate the gravitational perturbations created by the orbiting particle by integrating the Zerilli and Regge–Wheeler equations for the master functions \( \Psi_{\text{even}} \) and \( \Psi_{\text{odd}}. \) This must
Figure 1. Orbit of a parabolic encounter between a small body of mass $m$ and a black hole of mass $M$. The orbit’s semi-latus rectum is $p = 8.1M$ and its eccentricity is $e = 1$. The particle begins from rest at infinity, reaches a radial turning point at $r = 4.05M$ and returns to rest at infinity. The orbit is displayed in a $x$–$y$ plane constructed in the usual way from the Schwarzschild coordinates $r$ and $\phi$, so that $x = r \cos \phi$ and $y = r \sin \phi$. The coordinates are rescaled by a factor of $2M$ to make them dimensionless; in these units the unperturbed horizon (shown in black) is described by a circle of unit radius. The orbital motion is calibrated so that $\phi = 0$ when $r = 4.05M$.

be accomplished numerically, and we rely on the time-domain, finite-difference code written by Karl Martel; the details of the code are described in [28, 29]. Martel’s original code had to be modified to account for a different choice of odd-parity master function: while Martel’s code integrates the Regge–Wheeler equation for the original Regge–Wheeler function (which is equal to $\frac{1}{2} \partial_t \Psi_{\text{odd}}$), our modified version of the code calculates instead the Cunningham–Price–Moncrief function $\Psi_{\text{odd}}$. The code returns the master functions evaluated as functions of $v$ at a fixed radial position $r = 2M(1 + \epsilon)$ close to the event horizon; in our runs, we chose $\epsilon \simeq 10^{-5}$.

In figure 2, we plot the polarization $\gamma_+$ associated with the horizon’s intrinsic geometry, as defined by equation (4.36); this is shown as a function of advanced-time $v$ at azimuthal position $\beta = 0$ on the orbital plane $\alpha = \frac{\pi}{2}$; for this orientation, we have that $\gamma_+ = 0$. The calculation involves a summation over all multipoles up to (and including) $\ell = 4$; multipoles with $\ell \geq 5$ give contributions that are too small to be visible in the plot. Most of the signal is produced when the particle revolves around the black hole, and the plot reveals the rich harmonic structure that a parabolic encounter imprints on the tidal deformation of an event horizon.

In figure 3, we plot the polarizations $\sigma_+ = \frac{1}{2} \partial_v \gamma_+$ and $C_+ = (\kappa_0 - \partial_v)\sigma_+$ of the shear and Weyl tensors, respectively; these also are displayed as functions of $v$ at position $\alpha = \frac{\pi}{2}$ and $\beta = 0$ on the event horizon. The figure reveals very clearly that the horizon tide (as measured by the shear tensor) leads the tidal field (as measured by the Weyl tensor) by a time interval of order $\kappa_0^{-1} \approx 4M$; this feature of the tidal dynamics of a nonrotating black hole was discussed in section 4.5.
Figure 2. Polarization $\gamma_+$ associated with the intrinsic geometry of a black-hole horizon perturbed by a parabolic encounter, calculated at azimuthal position $\beta = 0$ on the horizon’s equatorial plane $\alpha = \frac{\pi}{2}$, which coincides with the orbital plane. The polarization is displayed as a function of $v/(2M)$ and is rescaled by a factor of $m/(2M)$. All relevant multipoles up to $\ell = 4$ are included in the computation.

Figure 3. Shown in solid red is the polarization $\sigma_+$ associated with the shear tensor of the horizon’s intrinsic geometry, rescaled by a factor of $m$, as a function of $v/(2M)$. Shown in dashed blue is the polarization $C_+$ associated with the Weyl tensor, rescaled by a factor of $m/(2M)$, as a function of $v/(2M)$. Both quantities are calculated at position $\alpha = \frac{\pi}{2}$ and $\beta = 0$ on the horizon. The plots show clearly that the horizon tide (as measured by the shear tensor) leads the tidal field (as measured by the Weyl tensor) by a time interval of order $\kappa^{-1} = 4M$. The Weyl tensor is noisy for early and late times because it is inaccurately computed by estimating the second derivative of $\gamma_+$ with respect to $v$ with finite-difference techniques.
Figure 4. Snapshot of the tidal bulge at $v/(2M) = 4.4874$ as described by the dimensional tidal displacement $\rho$ evaluated as a function of $\beta$ on the black hole’s equatorial plane $\alpha = \pi/2$. The figure shows, in the same $x$-$y$ plane as in figure 1, the surface $r = 2M$ of the unperturbed horizon (in thin black) as well as the surface $r = 2M[1 + (M/m)\rho]$ (in thick red), which grossly exaggerates the horizon deformation by a factor of $M/m$ to make it visible. The figure also shows (red disk) the position of the orbiting body at this value of advanced time $v$; we have $r/(2M) \simeq 2.0273$ and $\phi \simeq 0.6005$, leading to the Cartesian positions $x/(2M) \simeq 1.6726$ and $y/(2M) \simeq 1.1455$. The tidal bulge and orbiting body are intersected by the same light cone $v = constant$, and here also we see the tidal bulge leading the source of the tidal field.

Finally, in figure 4 we display the shape of the tidal bulge at a selected value of $v$ in relation to the position of the orbiting body. The tidal bulge is described geometrically in terms of the tidal displacement field $\rho(v, \alpha A)$ defined by equation (4.34), and the body’s position is evaluated on the past light cone $v = constant$ so as to yield a meaningful comparison. Here also we find that the horizon tide (as measured by the displacement field) leads the source of the tide (as measured by the orbital position on the light cone).

Acknowledgments

This work was supported by the Natural Sciences and Engineering Research Council of Canada.

Appendix. Late-time behaviour of horizon quantities

Some of the horizon quantities (such as $\gamma^{even}$, $\gamma^{odd}$, $\omega^{even}$ and $\omega^{odd}$) can be expressed purely in terms of the current value of the master functions, while others (such as $K^{trace}$, $K^{even}$ and $K^{odd}$) involve integrals of the master functions. We wish to verify that all horizon quantities properly vanish at $v = \infty$, assuming that $\Psi_{even}(v, 2M)$ and $\Psi_{odd}(v, 2M)$ decay at least as fast as an inverse power law in $v$; this is the late-time behaviour expected of radiative tails that linger on after the external processes that produce the perturbation have shut down.
The general structure of the integrals is
\[ x(v) = x(v_0) e^{-\kappa_0 (v-v_0)} - \int_{v_0}^{v} e^{-\kappa_0 (v-v')} F(v') \, dv', \]  
(A.1)
and for our purposes, here we assume that the source function \( F(v) \) varies over a timescale \( \tau \) that is very long compared with \( \kappa_0^{-1} = 4M \). In the case of an inverse-power falloff, for example, we assume that \( v_0 \) is sufficiently large that \( F(v') \propto (v')^{-p} \) within the integral, with \( p > 0 \). Then \( F \propto (v')^{-p-1} \) and the timescale \( \tau \) can be identified with \( F/F_0 \propto v' \); this is indeed much larger than \( 4M \) for the specified domain of integration. In these circumstances, we can evaluate the integral and express it as an asymptotic series in powers of \( (\kappa_0 \tau)^{-1} \ll 1 \).

If we let
\[ I[F] := \int_{v_0}^{v} e^{-\kappa_0 (v-v')} F(v') \, dv', \]  
(A.2)
then the identity
\[ I[F] = \frac{1}{\kappa_0} \left[ F(v_0) e^{-\kappa_0 (v-v_0)} - I[F] \right] \]  
(A.3)
follows immediately by integration by parts. Repeated applications yield
\[ I[F] = -\frac{1}{\kappa_0^2} \left[ F(v_0) - \kappa_0^{-1} F(v_0) + \kappa_0^{-2} F(v_0) + \cdots \right] e^{-\kappa_0 (v-v_0)} \]
\[ + \frac{1}{\kappa_0^2} \left[ F(v) - \kappa_0^{-1} F(v) + \kappa_0^{-2} F(v) + \cdots \right], \]  
(A.4)
in which each term within the square brackets is smaller than the preceding one by a factor of order \( (\kappa_0 \tau)^{-1} \). With this we arrive at
\[ x(v) = \frac{1}{\kappa_0} \left[ \kappa_0 x(v_0) + F(v_0) - \kappa_0^{-1} F(v_0) + \kappa_0^{-2} F(v_0) + \cdots \right] e^{-\kappa_0 (v-v_0)} \]
\[ - \frac{1}{\kappa_0^2} \left[ F(v) - \kappa_0^{-1} F(v) + \kappa_0^{-2} F(v) + \cdots \right]. \]  
(A.5)
At large \( v \), the first set of terms decay exponentially, and \( x(v) \) is dominated by the second set of terms. A good approximation is then \( x(v) \approx -\kappa_0^{-1} F(v) \), and \( x \) is seen to decay at the same rate as \( F(v) \). This shows that our integrals are indeed well behaved in the limit \( v \to \infty \), and that the horizon quantities decay at the same rate as the master functions.

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