Relationships Between the Maximum Principle and Dynamic Programming for Infinite Dimensional Stochastic Control Systems

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Abstract

Pontryagin type maximum principle and Bellman’s dynamic programming principle serve as two of the most important tools in solving optimal control problems. There is a huge literature on the study of relationship between them. The main purpose of this paper is to investigate the relationships between Pontryagin type maximum principle and dynamic programming principle for control systems governed by stochastic evolution equations in infinite dimensional space, with the control variables appearing into both the drift and the diffusion terms. To do so, we first establish dynamic programming principle for those systems without employing the martingale solutions. Then we establish the desired relationships in both cases that value function associated is smooth and nonsmooth. For the nonsmooth case, in particular, by employing the relaxed transposition solution, we discover the connection between the superdifferentials and subdifferentials of value function and the first-order and second-order adjoint equations.

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1 Introduction

Pontryagin type maximum principle (PMP, for short) and Bellman’s dynamic programming (DPP, for short) serve as two of the most important tools in solving optimal control problems. Consequently, it is natural to ask the followin question:

Problem (R) Are there any relations between PMP and DPP?

Problem (R) is first studied for control systems governed by ordinary differential equations in the early 1960s with the assumption that the value function is continuous differentiable (e.g.,[23]). For a long time, the result was just formal (except for some special cases) due to the fact that the value function is not smooth enough. In the late 1980s, with the tools of viscosity solution and nonsmooth analysis, results for Problem (R) without assuming the smoothness of the value function are established in [11,17,26]. Soon after, these results are generalized to systems by partial differential equations and stochastic differential equations (e.g.,[3,4,27]). After that, Problem

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In this paper, \( C \) and the cost functional is \( \tilde{A} \). Let such systems.

establish relationships between PMP (in terms of relaxed transposition solutions) and DPP for stochastic control systems with deterministic coefficients under the strong formulation, and then partial differential equations, or more general, stochastic evolution equations (SEEs for short) in infinite dimensional space.

This paper is a first attempt to study Problem (R) for controlled SEEs in infinite dimensional space. More precisely, in this paper, for the first time, we obtain DPP for infinite dimensional stochastic control systems with deterministic coefficients under the strong formulation, and then establish relationships between PMP (in terms of relaxed transposition solutions) and DPP for such systems.

To be more specific, let us first introduce the control system studied in this paper. Let \( H \) and \( \tilde{H} \) be two separable Hilbert spaces, \( T > 0, (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) be a complete filtered probability space, on which an \( \tilde{H} \)-valued cylindrical Brownian motion \( W(\cdot) \) is defined and \( \mathbb{F} \) is the natural filtration generated by \( W(\cdot) \). Denote by \( \mathcal{F}_t \) the progressive \( \sigma \)-algebra w.r.t. \( \mathbb{F} \). Write \( \mathcal{L}^0_2 \) for the space of all Hilbert-Schmidt operators from \( \tilde{H} \) to \( H \), which is also a separable Hilbert space. Let \( A : D(A) \subset H \to H \) be a linear operator, which generates a \( C_0 \)-semigroup \( \{S(t)\}_{t \geq 0} \) on \( H \). Let \( U \) be a separable metric space with a metric \( d(\cdot, \cdot) \). For \( t \in [0, T) \), put

\[
U[t, T] = \{u : [t, T] \times \Omega \to U \mid u \text{ is } \mathbb{F}\text{-adapted}\}.
\]

In this paper, \( C \) is a generic constant which may vary from line to line.

The control system is

\[
\begin{align*}
  dX(t) &= (AX(t) + a(t, X(t), u(t)))dt + b(t, X(t), u(t))dW(t), \quad t \in (0, T], \\
  X(0) &= \eta \in H,
\end{align*}
\]

and the cost functional is

\[
\mathcal{J}(\eta; u(\cdot)) = \mathbb{E} \left( \int_0^T f(t, X(t), u(t))dt + h(X(t)) \right).
\]

We make the following assumptions for the control system (1.1) and cost functional (1.2).

**S1** Suppose that: i) \( a(\cdot, \cdot, \cdot) : [0, T] \times H \times U \to H \) is \( \mathcal{B}([0, T]) \times \mathcal{B}(H) \times \mathcal{B}(U) \times \mathcal{B}(H) \)-measurable and \( b(\cdot, \cdot, \cdot) : [0, T] \times H \times U \to \mathcal{L}^0_2 \) is \( \mathcal{B}([0, T]) \times \mathcal{B}(H) \times \mathcal{B}(U) \times \mathcal{B}(\mathcal{L}^0_2) \)-measurable; ii) for any \( (t, \eta) \in [0, T] \times H \), the maps \( a(t, \eta, \cdot) : U \to H \) and \( b(t, \eta, \cdot) : U \to \mathcal{L}^0_2 \) are continuous; and iii) for any \( (t, \eta_1, \eta_2, u) \in [0, T] \times H \times H \times U \),

\[
\begin{align*}
  |a(t, \eta_1, u) - a(t, \eta_2, u)|_H &\leq C|\eta_1 - \eta_2|_H, \\
  |b(t, \eta_1, u) - b(t, \eta_2, u)|_{\mathcal{L}^0_2} &\leq C|\eta_1 - \eta_2|_H, \\
  |a(t, 0, u)|_H &\leq C, \quad |b(t, 0, u)|_{\mathcal{L}^0_2} \leq C.
\end{align*}
\]

**S2** Suppose that: i) \( f(\cdot, \cdot, \cdot) : [0, T] \times H \times U \to \mathbb{R} \) is \( \mathcal{B}([0, T]) \times \mathcal{B}(H) \times \mathcal{B}(U) \times \mathcal{B}(\mathbb{R}) \)-measurable and \( h(\cdot) : H \to \mathbb{R} \) is \( \mathcal{B}(H) \times \mathcal{B}(\mathbb{R}) \)-measurable; ii) for any \( (t, \eta) \in [0, T] \times H \), the functional \( f(t, \eta, \cdot) : U \to \mathbb{R} \) is continuous; and iii) for any \( (t, \eta_1, \eta_2, u) \in [0, T] \times H \times H \times U \),

\[
\begin{align*}
  |f(t, \eta_1, u) - f(t, \eta_2, u)| &\leq C|\eta_1 - \eta_2|_H, \\
  |h(\eta_1) - h(\eta_2)| &\leq C|\eta_1 - \eta_2|_H \\
  |f(t, 0, u)| &\leq C, \quad |h(0)| \leq C.
\end{align*}
\]
(S3) The maps $a(t, \eta, u)$ and $b(t, \eta, u)$, and the functionals $f(t, \eta, u)$ and $h(x)$ are $C^2$ with respect to $x$, such that for $\phi(t, \eta, u) = a(t, \eta, u)$, $b(t, \eta, u)$ and $\Psi(t, \eta, u) = f(t, \eta, u)$, $h(x)$, it holds that $\phi_x(t, \eta, u)$, $\Psi_x(t, \eta, u)$, $\phi_{xx}(t, \eta, u)$, and $\Psi_{xx}(t, \eta, u)$ are continuous with respect to $u$. Moreover, there is a modulus of continuity $\bar{\omega} : [0, \infty) \to [0, \infty)$ such that for any $(t, \eta_1, \eta_2, u) \in [0, T] \times H \times H \times U$,

$$\begin{align*}
&\left|a_{xx}(t, \eta, u)\right|_{L(H,H,H;H)} + \left|b_{xx}(t, \eta, u)\right|_{L(H,H;L_2^0)} + \left|\Psi_{xx}(t, \eta, u)\right|_{L(H)} \leq C; \\
&\left|a_{xx}(t, \eta_1, u_1) - a_{xx}(t, \eta_2, u_2)\right|_{L(H,H,H;H)} + \left|b_{xx}(t, \eta_1, u_1) - b_{xx}(t, \eta_2, u_2)\right|_{L(H,H;L_2^0)} \\
&+ \left|\Psi_{xx}(t, \eta_1, u_1) - \Psi_{xx}(t, \eta_2, u_2)\right|_{L(H)} \leq \bar{\omega}(\|\eta_1 - \eta_2\|) + d(u_1, u_2).
\end{align*}$$

Under (S1), for any $u(\cdot) \in \mathcal{U}[0, T]$, the control system (1.1) has a unique mild solution $X(\cdot) \in C_{\mathbb{F}}([0, T]; L^2(\Omega; H))$ (see [13] Theorem 3.14 for example).

Consider the following optimal control problem:

**Problem (S_\eta).** For any given $\eta \in H$, find a $\bar{u}(\cdot) \in \mathcal{U}[0, T]$ such that

$$\mathcal{J}(\bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[0, T]} \mathcal{J}(u(\cdot)). \quad (1.3)$$

Any $\bar{u}(\cdot) \in \mathcal{U}[0, T]$ satisfying (1.3) is called an optimal control (of Problem (S_\eta)). The corresponding state $\bar{X}(\cdot)$ is called an optimal state, and $(\bar{X}(\cdot), \bar{u}(\cdot))$ is called an optimal pair.

To study Problem (S_\eta), people introduce two tools, that is, PMP (see Section 2 for details) and DPP (see Section 3 for details). In this paper, we will investigate the relationship between these two tools. The rest of this paper is as follows. In Section 2, we recall the PMP for Problem (S_\eta). In Section 3, unlike the formulation of stochastic dynamic programming principle in the literature, which employs martingale solutions to SEEs, we establish a stochastic dynamic programming for the classical mild solution solutions to SEEs. Section 4 is addressed to the relationship between PMP and DPP when the value function is smooth enough, while Section 5 is devoted to the same problem for general value functions.

## 2 Pontryagin type maximum principle for Problem (S_\eta)

For the convenience of readers, let us recall the PMP for Problem (S_\eta), which were established recently in several papers under different assumptions (e.g., [8, 14, 16, 17]).

To establish the PMP, we need to introduce the first-order adjoint equation:

$$\begin{align*}
dp(t) &= -A^*p(t)dt - (a_x(t, \bar{X}(t), \bar{u}(t))^* p(t) + b_x(t, \bar{X}(t), \bar{u}(t))^* q(t) \\
&- f_x(t, \bar{X}(t), \bar{u}(t)))dt + q(t)dW(t), \\
p(T) &= -h_x(\bar{X}(T)),
\end{align*} \quad (2.1)$$

and the second-order adjoint equation:

$$\begin{align*}
dP(t) &= -[(A^* + a_x(t, \bar{X}(t), \bar{u}(t))^*)P(t) + P(t)(A + a_x(t, \bar{X}(t), \bar{u}(t)) \\
&+ b_x(t, \bar{X}(t), \bar{u}(t))^* P(t))b_x(t, \bar{X}(t), \bar{u}(t)) + b_x(t, \bar{X}(t), \bar{u}(t))^* Q(t) \\
&+ Q(t)b_x(t, \bar{X}(t), \bar{u}(t)) + \mathbb{H}_{xx}(t, \bar{X}(t), \bar{u}(t), p(t), q(t))]dt + Q(t)dW(t), \\
P(T) &= -h_{xx}(\bar{X}(T)),
\end{align*} \quad (2.2)$$

with

$$\mathbb{H}(t, \eta, u, p, q) = \langle p, a(t, \eta, u) \rangle_H + \langle q, b(t, \eta, u) \rangle_{L_2^0} - f(t, \eta, u),$$

$$(t, \eta, u, p, q) \in [0, T] \times H \times U \times H \times L_2^0.$$
The equation (2.1) is an $H$-valued backward stochastic evolution equation (BSEE for short). It admits a unique mild solution $(p, q) \in L^2_p(\Omega; C[0, T]; H) \times L^2_p(0, T; H)$ (e.g., [3] Section 4.2).

When $H = \mathbb{R}^n$, (2.2) is an $\mathbb{R}^{n \times n}$-valued backward stochastic differential equation (which can be easily regarded as an $\mathbb{R}^{n^2}$ (vector)-valued backward stochastic differential equation), and therefore, its well-posedness follows from the one for backward stochastic evolution equations valued in Hilbert spaces (e.g., [3] Section 4.2). When $\dim H = \infty$, although $L(H)$ is still a Banach space, it is neither reflexive (needless to say to be a Hilbert space) nor separable. To the best of our knowledge, in the previous literatures there exists no such a stochastic integration/evolution equation theory in general Banach spaces that can be employed to treat the well-posedness of (2.2) in the usual sense. Then, a new notion of solution, i.e., the relaxed transposition solution, is introduced to (2.2). Let us briefly recall it.

For simplicity of notation, we put

$$\mathcal{L}_{pd}(L^2_p(0, T; L^4(\Omega, H)); L^2_p(0, T; L^4(\Omega, H)))$$

$$\Delta \{ L \in \mathcal{L}(L^2_p(0, T; L^4(\Omega, H)); L^2_p(0, T; L^4(\Omega, H))) \text{ for a.e. } (t, \omega) \in [0, T] \times \Omega, \text{ there is } \bar{L}(t, \omega) \in \mathcal{L}(H_1; H_2) \text{ such that } (Lu(\cdot))(t, \omega) = \bar{L}(t, \omega)v(t, \omega), \forall v(\cdot) \in L^2_p(0, T; L^4(\Omega, H)) \}.$$ 

In the sequel, if there is no confusion, we identify $L \in \mathcal{L}_{pd}(L^2_p(0, T; L^4(\Omega, H)); L^2_p(0, T; L^4(\Omega, H)))$ with $\bar{L}(\cdot, \cdot)$.

Let

$$\mathcal{P}[0, T] \Delta \{ P(\cdot, \cdot) \mid P(\cdot, \cdot) \in \mathcal{L}_{pd}(L^2_p(0, T; L^4(\Omega, H)); L^2_p(0, T; L^4(\Omega, H))),$$

$$P(\cdot, \cdot) \in \mathcal{D}_F([t, T]; L^4(\Omega, H))) \text{ and } |P(\cdot, \cdot)|_{\mathcal{D}_F([t, T]; L^4(\Omega, H))} \leq C|\xi|_{L^2_p(\Omega; H)} \text{ for every } t \in [0, T] \text{ and } \xi \in L^2_p(\Omega; H) \},$$

and

$$\mathcal{Q}[0, T] \Delta \{ (Q^{(i)}, \tilde{Q}^{(i)}) \mid Q^{(i)}, \tilde{Q}^{(i)} \in \mathcal{L}(\mathcal{H}_t; L^2_p(t, T; L^4(\Omega; L^2_p)))$$

$$\text{ and } Q^{(i)}(0, 0, \cdot)^* = \tilde{Q}^{(i)}(0, 0, \cdot) \text{ for any } t \in [0, T) \}$$

with

$$\mathcal{H}_t \Delta L^4_p(\Omega; H) \times L^2_p(\Omega; H) \times L^2_p(t, T; L^4(\Omega; L^2_p)), \forall t \in [0, T).$$

For $j = 1, 2$ and $t \in [0, T)$, consider the following equation:

$$\begin{aligned}
&d\varphi_j = (A + J)\varphi_j ds + u_j ds + Kx_j dW(s) + v_j dW(s) \quad \text{in } (t, T], \\
&\varphi_j(t) = \xi_j
\end{aligned}$$

(2.3)

where $\xi_j \in L^4_p(\Omega; H), u_j \in L^2_p(t, T; L^4(\Omega; H))$ and $v_j \in L^2_p(t, T; L^4(\Omega; L^2_p))$. By the classical well-posedness result for SEEs, we know that (2.3) has a unique mild solution $\varphi_j \in C_F([t, T]; L^4(\Omega; H))$ (e.g., [3] Section 3.2).
Definition 2.1 A 3-tuple \((P(\cdot), Q(\cdot), \hat{Q}(\cdot))\) \(\in \mathcal{P}[0,T] \times \mathcal{Q}[0,T]\) is called a relaxed transposition solution to the equation (2.2) if for any \(t \in [0,T]\), \(\xi_j \in L^4_{F^j}(\Omega;H)\), \(u_j(\cdot) \in L^2(t,T;L^4(\Omega;H))\) and \(v_j(\cdot) \in L^2(t,T;L^4(\Omega;L^2))\) \((j=1,2)\), it holds that

\[
\mathbb{E}(P_T \varphi_1(T), \varphi_2(T))_H - \mathbb{E} \int_t^T \langle F(s) \varphi_1(s), \varphi_2(s) \rangle_H ds = \mathbb{E}(P(t)\xi_1, \xi_2)_H + \mathbb{E} \int_t^T \langle P(s)u_1(s), \varphi_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle P(s)\varphi_1(s), u_2(s) \rangle_H ds
\]

\[
+ \mathbb{E} \int_t^T \langle P(s)K(s)\varphi_1(s), v_2(s) \rangle_{L^2} ds + \mathbb{E} \int_t^T \langle P(s)v_1(s), K(s)\varphi_2(s) + v_2(s) \rangle_{L^2} ds
\]

\[
+ \mathbb{E} \int_t^T \langle v_1(s), \hat{Q}^{(t)}(\xi_2, u_2, v_2) \rangle_{L^2} ds + \mathbb{E} \int_t^T \langle \hat{Q}^{(t)}(\xi_1, u_1, v_1), v_2(s) \rangle_{L^2} ds
\]

As an immediate corollary of [18, Theorem 12.9], we have the following well-posedness result for the equation (2.2).

Proposition 2.1 The equation (2.2) admits a unique relaxed transposition solution \((P(\cdot), Q(\cdot), \hat{Q}(\cdot))\). Furthermore,

\[
|P|_{L^2(\mathcal{L}(\bar{\mathcal{P}}[0,T];L^4(\Omega,H));L^2(\bar{\mathcal{P}}[0,T];L^4(\Omega,H)))} + \sup_{t \in [0,T]} |(Q^{(t)}, \hat{Q}^{(t)})|_{L(H;L^2(t,T;L^2(\Omega;L^2)))}^2
\]

\[
\leq C(|F|_{L^2(\mathcal{L}(\bar{\mathcal{P}}[0,T];L^2(\Omega;H)))} + |P_T|_{L^2_{\mathcal{F}_T}(\Omega;L^2(H)))})^2
\]

\((\bar{X}(\cdot), \bar{u}(\cdot), p(\cdot), q(\cdot), P(\cdot), Q(\cdot), \hat{Q}(\cdot))\) is called an optimal 7-tuple of Problem \((S_\eta)\). Now we can present the PMP for Problem \((S_\eta)\).

Theorem 2.1 Suppose that the assumptions \((S1)-(S3)\) hold. Let \((\bar{X}(\cdot), \bar{u}(\cdot), p(\cdot), q(\cdot), P(\cdot), Q(\cdot), \hat{Q}(\cdot))\) be an optimal 6-tuple of Problem \((S_\eta)\). Then, for a.e. \((t, \omega) \in [0,T] \times \Omega\) and for all \(p \in U\),

\[
\mathbb{H}(t, \bar{X}(t), \bar{u}(t), p(t), q(t)) - \mathbb{H}(t, \bar{y}(t), \rho, p(t), q(t)) - \frac{1}{2} \mathbb{E}[P(t)[b(t, \bar{X}(t), \bar{u}(t)) - b(t, \bar{X}(t), \rho)]^2, b(t, \bar{X}(t), \bar{u}(t)) - b(t, \bar{X}(t), \rho)]_H \geq 0.
\]

3 The Dynamic Programming Principle for Problem \((S_\eta)\)

In the literature, stochastic dynamic programming principle for Problem \((S_\eta)\) in weak formulation is already established. A nice treatise for that is [11]. In that formulation, probability spaces and Brownian motions vary with the controls. In other words, the probability space and Brownian motion are part of the control. Usually, optimal control problems for SEEs are formulated in strong formulation, i.e., the probability space and the Brownian motion are fixed. Hence, it is natural to ask whether the stochastic dynamic programming principle holds in strong formulation. This question is first answered in [22, Chapter 2, Section 5] when \(H = \mathbb{R}^n\). In this section, we generalize the results in [22] to Problem \((S_\eta)\).

First, we introduce a family of optimal control problems. For any \((t, \eta) \in [0,T] \times H\), the control system is

\[
\begin{aligned}
&dX(s) = (AX(s) + a(t, X(s), u(s)))dt + b(s, X(s), u(s))dW(s), \quad s \in (t, T], \\
&X(t) = \eta,
\end{aligned}
\]
Hence, the cost functional (3.2) is well-defined.

For any $u(\cdot) \in \mathcal{U}[t,T]$, it follow immediately from the classical well-posedness of SEEs (e.g., [18, Theorem 3.14]) that the control system (3.1) has a unique mild solution $X(\cdot) \in C_{\mathbb{F}}([t,T]; L^2(\Omega; H))$. Hence, the cost functional (3.2) is well-defined.

Consider the following optimal control problem:

**Problem $(S_{t\eta})$.** For any given $(t, \eta) \in [0,T] \times H$, find a $\bar{u}(\cdot) \in \mathcal{U}[t,T]$ such that

$$
J(t, \eta; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[t,T]} J(t, \eta; u(\cdot)).
$$

Any $\bar{u}(\cdot) \in \mathcal{U}[t,T]$ satisfying (3.3) is called an optimal control (of Problem $(S_{t\eta})$). The corresponding state $\overline{X}(\cdot)$ is called an optimal state, and $(\overline{X}(\cdot), \bar{u}(\cdot))$ is called an optimal pair.

We have the following Dynamic Programming Principle.

**Theorem 3.1** For any $(t, \eta) \in [0,T] \times H$,

$$
V(t, \eta) = \inf_{u(\cdot) \in \mathcal{U}[t,T]} \mathbb{E} \left( \int_t^T f(s, X(s; t, \eta, u), u(s)) ds + V(t, X(t; t, \eta, u)) \right), \quad \forall 0 \leq t \leq T. \tag{3.4}
$$

Further, it holds that

**Theorem 3.2** If $(\overline{X}(\cdot), \bar{u}(\cdot))$ is optimal for Problem $(S_{t\eta})$, then

$$
V(t, \eta) = \mathbb{E} \left\{ \int_t^T f(t, \overline{X}(\tau), \bar{u}(\tau)) d\tau + h(\overline{X}(T)) \bigg| \mathcal{F}_t \right\}, \quad \mathbb{P}\text{-a.s., } \forall r \in [t, T]. \tag{3.5}
$$

Further, we have the following regularity properties for $V$.

**Proposition 3.1** For each $t \in [0,T]$, $\eta$ and $\eta' \in H$, we have

$$
|V(t, \eta)| \leq C(1 + |\eta|_H) \tag{3.6}
$$

and

$$
|V(t, \eta) - V(t, \eta')| \leq C|\eta - \eta'|_H. \tag{3.7}
$$

**Proposition 3.2** The function $V(\cdot, \eta)$ is continuous.

The proof of Theorems 3.1, 3.2 and Propositions 3.1, 3.2 have their own interest. But they are lengthy and technical. We put them in the next section. The readers may skip them when they read this paper at the first time.

Next, we formally derive a partial differential equation satisfied by the value function $V(\cdot, \cdot)$.

**Proposition 3.3** Suppose that $V(\cdot, \cdot) \in C^{1,2}([0,T]; H)$, $V_x : [0,T] \times H \to D(A^*)$, $A^*V_x \in C([0,T] \times H; H)$. Then $V(\cdot, \cdot)$ is a solution to the following HJB equation:

$$
\begin{align*}
V_t + \langle A^*V_x, \eta \rangle_H + \inf_{\rho \in \mathcal{U}} G(t, \eta, \rho, V_x, V_{xx}) &= 0, \quad (t, \eta) \in [0,T] \times H, \\
V(t, \eta) &= h(\eta), \quad \eta \in H
\end{align*}
$$

where

$$
G(t, \eta, \rho, P) = \frac{1}{2} \langle Pb(t, \eta, \rho), b(t, \eta, \rho) \rangle_{\mathbb{R}^2} + \langle p, a(t, \eta, \rho) \rangle_H - f(t, \eta, \rho), \quad \forall (t, \eta, \rho, P) \in [0,T] \times H \times U \times S(H). \tag{3.9}
$$
Proof. Let \((t, \eta) \in [0, T] \times H\) be given and for any \(u \in U\), taking the constant control \(u(\cdot) = \rho\), we have
\[
0 \leq \mathbb{E}\left( \int_t^T f(r, X(r), \rho) dr + V(t + \varepsilon, X(t + \varepsilon)) - V(t, X(t)) \right) \\
= \mathbb{E}\left[ \int_t^T \left( f(r, X(r), \rho) + V_t(r, X(r)) + \langle A^* V_x(r, X(r)), X(r) \rangle_H + \langle V_x(r, X(r)), a(r, X(r), \rho) \rangle_H \right. \right. \\
\left. \left. + \frac{1}{2} (V_{xx}(r, X(r))b(r, X(r), \rho), b(r, X(r), \rho))_{L^2} \right) dr \right].
\]
Taking expectations, dividing both sides by \(\varepsilon\) and sending \(\varepsilon \to 0^+\), one can obtain that
\[
V_t + \langle A^* V_x, \eta \rangle_H + G(t, \eta, \rho, V_x, V_{xx}) \geq 0, \quad \forall \rho \in U
\]
Thus,
\[
V_t + \langle A^* V_x, \eta \rangle_H + \inf_{\rho \in U} G(t, \eta, \rho, V_x, V_{xx}) \geq 0.
\]
Next, for any \(\varepsilon > 0\) and \(s > 0\), there exists a \(u^{\varepsilon, s}(\cdot) \in U[t, T]\) with the corresponding state \(X^{\varepsilon, s}(\cdot)\) such that
\[
\varepsilon > \frac{1}{s} \mathbb{E}\left[ \int_t^{t+s} \left( f(r, X^{\varepsilon, s}(r), u^{\varepsilon, s}(r)) dr + V(t + s, X^{\varepsilon, s}(t + s)) - V(t, \eta) \right) \right] \\
= \frac{1}{s} \mathbb{E}\left[ \int_t^{t+s} \left( f(r, X^{\varepsilon, s}(r), u^{\varepsilon, s}(r)) + \langle A^* V_x(r, X^{\varepsilon, s}(r)), X^{\varepsilon, s}(r) \rangle_H \right. \right. \\
\left. \left. + \langle V_x(r, X^{\varepsilon, s}(r)), a(r, X^{\varepsilon, s}(r), u^{\varepsilon, s}(r)) \rangle_H \right. \right. \\
\left. \left. + \frac{1}{2} (V_{xx}(r, X^{\varepsilon, s}(r))b(r, X^{\varepsilon, s}(r), u^{\varepsilon, s}(r)), b(r, X^{\varepsilon, s}(r), u^{\varepsilon, s}(r)))_{L^2} \right) dr \right] \\
\geq \frac{1}{s} \mathbb{E}\left[ \int_t^{t+s} \left( V_t(r, X^{\varepsilon, s}(r)) + \langle A^* V_x(r, X^{\varepsilon, s}(r)), X^{\varepsilon, s}(r) \rangle_H \right. \right. \\
\left. \left. + \inf_{u \in U} G(r, X^{\varepsilon, s}(r), u, V_x(r, X^{\varepsilon, s}(r)), V_{xx}(r, X^{\varepsilon, s}(r))) \right) dr \right] \\
\to V_t(t, \eta) + \langle A^* V_x(t, \eta), \eta \rangle_H + \inf_{\rho \in U} G(t, \eta, \rho, V_x(t, \eta), V_{xx}(t, \eta)).
\]
Since \(\varepsilon > 0\) is arbitrary, we obtain (3.8). \(\square\)

4 A stochastic recursive optimal control problem

In this section, we consider a more general optimal control problem, i.e., a stochastic recursive optimal control problem. Compared with Problem (\(S_{t\eta}\)), it is more convenient to be handled since one can employ the theory of BSEEs to study it.

4.1 Formulation of the stochastic recursive optimal control problem

For any given \(u \in U[t, T]\), consider the following BSEE:
\[
\begin{align*}
\begin{cases}
\d sY(s; t, \zeta, u) = -g(s, X(s; t, \zeta, u), Y(s; t, \zeta, u), Z(s; t, \zeta, u), u(s)) ds + Z(s; t, \zeta, u) dW(s), \quad s \in [t, T), \\
Y(T; t, \zeta, u) = \Phi(X(T; t, \zeta, u)).
\end{cases}
\end{align*}
\]
Here, \(X(\cdot)\) is the mild solution to the equation (3.1) with \(X(t) = \zeta \in L^2_{\mathcal{F}_t}(\Omega; H)\), and \(\Phi : H \to \mathbb{R}\) and \(g : [0, T] \times H \times \mathbb{R} \times \tilde{H} \times U \to \mathbb{R}\) satisfy the following conditions:
(S4) \( g(\cdot, \eta, y, z, u) \) is Lebesgue measurable, \( g(t, \eta, y, z, \cdot) \) is continuous. For some \( L > 0, \) and all \( \eta, \eta' \in H, \) \( y, y' \in \mathbb{R}, \) \( z, z' \in \widetilde{H}, \) \( u, u' \in U \) and a.e. \( t \in [0, T], \)

\[
|g(t, \eta, y, z, u) - g(t, \eta', y', z', u')| + |\Phi(\eta) - \Phi(\eta')| \leq L(|\eta - \eta'|_H + |\eta - \eta'| + |z - z'|_{\widetilde{H}})
\]

and

\[
g(t, \eta, 0, 0, u) + \Phi(\eta) \leq L(1 + |\eta|_H).
\]

By the classical well-posedness result for BSDEs (e.g., [18 Theorem 4.10] ), the equation (3.11) admits a unique solution \( (Y, Z) \in L^2_p(\Omega; C([t, T])) \times L^2_p(t, T; \widetilde{H}). \)

We first give some estimates for solutions to the equations (3.11) and (4.1).

**Proposition 4.1** For all \( t \in [0, T], \) \( \zeta, \zeta' \in L^2_{\mathcal{F}_t}(\Omega; H), \) \( u(\cdot), u'(\cdot) \in U[t, T], \)

\[
\sup_{t \leq s \leq T} \mathbb{E}\left( |X(s)|^2_H \big| \mathcal{F}_t \right) \leq C(1 + |\zeta|^2_H) \tag{4.2}
\]

and

\[
\sup_{t \leq s \leq T} \mathbb{E}\left( |X(s; t, \zeta, u) - X(s; t, \zeta', u')|^2_H \big| \mathcal{F}_t \right) \leq C \left[ |\zeta - \zeta'|^2_H + \mathbb{E}\left( \int_t^T |u(s) - u'(s)|^2_U ds \big| \mathcal{F}_t \right) \right], \tag{4.3}
\]

where the constant \( C \) is independent of \( t \in [0, T]. \)

**Proof.** Since

\[
X(s) = S(s - t)\zeta + \int_t^s S(s - \tau)a(\tau, X(\tau), u(\tau))d\tau + \int_t^s S(s - \tau)b(\tau, X(\tau), u(\tau))dW(\tau),
\]

it follows from (S1) that

\[
\mathbb{E}\left( |X(s)|^2_H \big| \mathcal{F}_t \right) \leq C \left[ |\zeta|^2_H + \mathbb{E}\left( \left| \int_t^s S(s - \tau)a(\tau, X(\tau), u(\tau))d\tau \right|^2_H \big| \mathcal{F}_t \right) \right]
\]

\[
\quad + \mathbb{E}\left( \left| \int_t^s S(s - \tau)b(\tau, X(\tau), u(\tau))dW(\tau) \right|^2_H \big| \mathcal{F}_t \right)
\]

\[
\leq C \left[ |\zeta|^2_H + \int_t^s \left[ 1 + \mathbb{E}( |X(\tau)|^2_H \big| \mathcal{F}_t \right) \right].
\]

This, together with Gronwall’s inequality, implies that

\[
\mathbb{E}\left( |X(s)|^2_H \big| \mathcal{F}_t \right) \leq C(1 + |\zeta|^2_H), \quad \forall s \in [t, T],
\]

which deduces (4.2) immediately.

By a similar argument, we have

\[
\sup_{t \leq s \leq T} \mathbb{E}\left( |X(s; t, \zeta, u) - X(s; t, \zeta', u')|^2_H \big| \mathcal{F}_t \right)
\]

\[
\leq C \mathbb{E}\left( \left| \zeta - \zeta' \right|^2_H + \int_t^T \left| a(s, 0, u(s)) - a(s, 0, u'(s)) \right|^2_H ds + \int_t^T \left| b(s, 0, u(s)) - b(s, 0, u'(s)) \right|^2_U ds \big| \mathcal{F}_t \right)
\]

\[
\leq C \left[ |\zeta - \zeta'|^2_H + \mathbb{E}\left( \int_t^T |u(s) - u'(s)|^2_U ds \big| \mathcal{F}_t \right) \right],
\]

which concludes (4.3).
Proposition 4.2 For any $\zeta, \zeta' \in L^2_{\mathcal{F}_t}(\Omega; H)$, and $u, u' \in U[t, T]$,

$$\sup_{t \leq s \leq T} \mathbb{E}\left(|Y(s; t, \zeta, u)|^2 + \int_t^T |Z(s; t, \zeta, u)|^2_H ds \big| \mathcal{F}_t \right) \leq C(1 + |\zeta|_H^2), \quad (4.4)$$

and

$$\sup_{t \leq s \leq T} \mathbb{E}\left(|Y(s; t, \zeta, u) - Y(s; t, \zeta', u')|^2 + \int_t^T |Z(s; t, \zeta, u) - Z(s; t, \zeta', u')|^2_H ds \big| \mathcal{F}_t \right)$$

$$\leq C\left[|\zeta - \zeta'|_H^2 + \mathbb{E}\left(\int_t^T |u(s) - u'(s)|^2_H ds \big| \mathcal{F}_t \right)\right]. \quad (4.5)$$

Proof. By (S3), (4.2) and the classical well-posedness result for BSEEs (e.g., [18, Theorem 4.10]), we have

$$\sup_{t \leq s \leq T} \mathbb{E}\left(|Y(s; t, \zeta, u)|^2 + \int_t^T |Z(s; t, \zeta, u)|^2_H ds \big| \mathcal{F}_t \right)$$

$$\leq C\mathbb{E}\left(|\Phi(X(t))|^2 + \int_t^T |g(s, X(s), 0, 0, u(s))|^2 ds \big| \mathcal{F}_t \right)$$

$$\leq C\left[1 + \sup_{t \leq s \leq T} \mathbb{E}(|X(s)|_H^2 \big| \mathcal{F}_t)\right] \leq C(1 + |\zeta|_H).$$

This gives (4.4). Similarly,

$$\sup_{t \leq s \leq T} \mathbb{E}\left(|Y(s; t, \zeta, u) - Y(s; t, \zeta', u')|^2 + \int_t^T |Z(s; t, \zeta, u) - Z(s; t, \zeta', u')|^2_H ds \big| \mathcal{F}_t \right)$$

$$\leq C\mathbb{E}\left(|\Phi(X(T; t, \zeta, u)) - \Phi(X(T; t, \zeta', u'))|^2$$

$$+ \int_t^T |g(s, X(s; t, \zeta, u), 0, 0, u(s)) - g(s, X(s; t, \zeta', u'), 0, 0, u'(s))|^2 ds \big| \mathcal{F}_t \right)$$

$$\leq C\left[\sup_{t \leq s \leq T} \mathbb{E}(|X(s; t, \zeta, u) - X(s; t, \zeta', u')|^2_H \big| \mathcal{F}_t) + \mathbb{E}\left(\int_t^T |u(s) - u'(s)|^2_H ds \big| \mathcal{F}_t \right)\right]$$

$$\leq C\left[|\zeta - \zeta'|_H^2 + \mathbb{E}\left(\int_t^T |u(s) - u'(s)|^2_H ds \big| \mathcal{F}_t \right)\right].$$

This deduces (4.5). \qed

Given the control process $u(\cdot) \in U[t, T]$, we introduce the associated cost functional:

$$\tilde{J}(t, \eta; u(\cdot)) = Y(t; t, \eta, u), \quad (t, \eta) \in [0, T] \times H,$$  

(4.6)

and consider the following stochastic recursive optimal control problem:

**Problem (S_{t, \eta}).** For any given $(t, \eta) \in [0, T] \times H$, find a $\tilde{u}(\cdot) \in U[t, T]$ such that

$$\tilde{J}(t, \eta; \tilde{u}(\cdot)) = \text{ess inf}_{u(\cdot) \in U[t, T]} \tilde{J}(t, \eta; u(\cdot)).$$  (4.7)

Since $\tilde{J}(t, \eta; u(\cdot))$ may be a random variable, so we take $\text{ess inf}$ on the right hand side of (4.7). At a first glance, one may think that $\text{ess inf}_{u(\cdot) \in U[t, T]} \tilde{J}(t, \eta; u(\cdot))$ is a random variable. Fortunately, we have the following result.
Proposition 4.3 Under assumptions (S1) and (S4), \( \inf_{(t, \eta)} \overline{f}(t, \eta; u(\cdot)) \) is a deterministic function of \( (t, \eta) \in [0, T] \times H \).

Before proving Proposition 4.3, we give some preliminaries. For each \( t > 0 \), denote by \( \mathcal{F}_t^t \triangleq \{ \mathcal{F}_s^t \}_{t \leq s \leq T} \) the natural filtration of the Brownian motion \( \{ W(s) - W(t) \}_{t \leq s \leq T} \). Write \( \mathcal{F}^t \) for the progressive \( \sigma \)-algebra w.r.t. \( \mathcal{F}^t \). Let

\[
\mathcal{U}^t \triangleq \{ u(\cdot) \in \mathcal{U}(t, T) \mid u(s) \text{ is } \mathcal{F}^t\text{-adapted, } \forall t \leq s \leq T \},
\]

and

\[
\mathcal{U}_D^t = \left\{ u(s) = \sum_{j=1}^{N} u^j 1_{\Omega_j} \mid u^j(s) \in \mathcal{U}^t, \{ \Omega_j \}_{j=1}^{N} \subset \mathcal{F}_t \text{ is a partition of } \Omega \right\}.
\]

From [24] Lemma 4.12, we know that \( \mathcal{U}_D^t \) is dense in \( \mathcal{U}(t, T) \). Next, we give the following Lemma.

Lemma 4.1 For any \( \eta \in H \), and \( \{ u^j \}_{j=1}^{N} \subset \mathcal{U}_D^t \), the solutions to \( (3.1) \) and \( (4.1) \) satisfy

\[
\begin{align*}
X(\cdot, t, \eta, \sum_{j=1}^{N} \chi_{\Omega_j} u^j) & = \sum_{j=1}^{N} \chi_{\Omega_j} X(\cdot, t, \eta, u^j), \\
Y(\cdot, t, \eta, \sum_{j=1}^{N} \chi_{\Omega_j} u^j) & = \sum_{j=1}^{N} \chi_{\Omega_j} Y(\cdot, t, \eta, u^j), \\
Z(\cdot, t, \eta, \sum_{j=1}^{N} \chi_{\Omega_j} u^j) & = \sum_{j=1}^{N} \chi_{\Omega_j} Z(\cdot, t, \eta, u^j).
\end{align*}
\]

Proof. For every \( j = 1, 2, \cdots, N \), we denote

\[
(X^j(s), Y^j(s), Z^j(s)) \equiv (X(s; t, \eta, u^j), Y(s; t, \eta, u^j), Z(s; t, \eta, u^j)).
\]

Then

\[
X^j(s) = S(s - t)\eta + \int_t^s S(s - r)a(r, X^j(r), u^j(r))dr + \int_t^s S(s - r)b(r, X^j(r), u^j(r))dW(r), \quad s \in [t, T],
\]

and

\[
Y^j(s) = \Phi(X^j(T)) + \int_s^T g(r, X^j(r), Y^j(r), Z^j(r), u^j(r))dr - \int_s^T Z^j(r)dW(r), \quad s \in [t, T].
\]

Multiply \( \chi_{\Omega_j} \) on both sides of \( (4.9) \) and \( (4.10) \), and sum the equations. From the trivial fact that

\[
\sum_{j=1}^{N} \chi_{\Omega_j} \varphi(x_j) = \varphi\left( \sum_{j=1}^{N} \chi_{\Omega_j} x_j \right), \quad \forall \varphi \in C(H),
\]

we get

\[
\sum_{j=1}^{N} \chi_{\Omega_j} X^j(s) = S(s - t)\eta + \int_t^s S(s - r)a\left( r, \sum_{j=1}^{N} \chi_{\Omega_j} X^j(r), \sum_{j=1}^{N} \chi_{\Omega_j} u^j(r) \right)dr
\]

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\[ + \int_t^s S(s - r)b(r, \sum_{j=1}^N \chi_{\Omega_j}X^j(r), \sum_{j=1}^N \chi_{\Omega_j}u^j(r))dW(r), \]

and

\[ \sum_{j=1}^N \chi_{\Omega_j}Y^j(s) = \Phi\left(\sum_{j=1}^N \chi_{\Omega_j}X^j(T)\right) - \int_s^T \sum_{j=1}^N \chi_{\Omega_j}Z^j(r)dW(r) \]

\[ + \int_s^T g(r, \sum_{j=1}^N \chi_{\Omega_j}X^j(r), \sum_{j=1}^N \chi_{\Omega_j}Y^j(r), \sum_{j=1}^N \chi_{\Omega_j}u^j(r))dr. \]

Then from the uniqueness of the solution of \((3.11)\) and \((4.1)\), we get \((4.8)\).

**Proof of Proposition 4.3.** We divide the proof into three steps.

**Step 1.** In this step, we show that

\[ \text{ess inf}_{u(\cdot) \in \mathcal{U}[t,T]} \widetilde{J}(t, \eta; u(\cdot)) = \text{ess inf}_{u(\cdot) \in \mathcal{U}'_D} \widetilde{J}(t, \eta; u(\cdot)). \tag{4.11} \]

Since \(\mathcal{U}'_D \subset \mathcal{U}[t,T]\), we have

\[ \text{ess inf}_{u(\cdot) \in \mathcal{U}[t,T]} \widetilde{J}(t, \eta; u(\cdot)) \leq \text{ess inf}_{u(\cdot) \in \mathcal{U}'_D} \widetilde{J}(t, \eta; u(\cdot)). \]

Consequently, we only need to prove

\[ \text{ess inf}_{u(\cdot) \in \mathcal{U}'_D} \widetilde{J}(t, \eta; u(\cdot)) \leq \text{ess inf}_{u(\cdot) \in \mathcal{U}[t,T]} \widetilde{J}(t, \eta; u(\cdot)). \tag{4.12} \]

For any \(\varepsilon > 0\), there exists \(\tilde{u}(\cdot) \in \mathcal{U}[t,T]\) such that

\[ \mathbb{P}\left\{ \text{ess inf}_{u(\cdot) \in \mathcal{U}[t,T]} \widetilde{J}(t, \eta; u(\cdot)) < \text{ess inf}_{u(\cdot) \in \mathcal{U}[t,T]} \widetilde{J}(t, \eta; u(\cdot)) + \varepsilon \right\} = \delta > 0. \]

From Proposition 4.2, we know that for any \(\tilde{u}(\cdot) \in \mathcal{U}'_D\),

\[ \mathbb{E}|Y(t; t, \eta, \tilde{u}) - Y(t; t, \eta, \tilde{u})|^2 \leq C\mathbb{E}\int_t^T |\tilde{u}(s) - \tilde{u}(s)|^2 ds. \]

Since \(\mathcal{U}'_D\) is dense in \(\mathcal{U}[t,T]\), there exists a sequence \(\{u_n(\cdot)\}_{n=1}^{\infty} \in \mathcal{U}'_D\) such that

\[ \lim_{n \to \infty} \mathbb{E}|Y(t; t, \eta, u_n) - Y(t; t, \eta, \tilde{u})|^2 = 0. \]

Then, there exists a subsequence, without loss of generality, denoted by \(\{u_n(\cdot)\}_{n=1}^{\infty}\) also, such that

\[ \mathbb{P}\left( \bigcap_{m=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \left\{ |Y(t; t, \eta, u_n) - Y(t; t, \eta, \tilde{u})| < \frac{1}{m} \right\} \right) = 1, \]

which implies that

\[ \mathbb{P}\left( \bigcup_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \left\{ |Y(t; t, \eta, u_n) - Y(t; t, \eta, \tilde{u})| < \frac{1}{m} \right\} \right) = 1, \quad \forall m \in \mathbb{N}. \]
Therefore,
\[
\lim_{N \to \infty} \mathbb{P} \left( \bigcap_{n=1}^{\infty} \left\{ \left| Y(t; t, \eta, u_n) - Y(t; t, \eta, \tilde{u}) \right| < \frac{1}{m} \right\} \right) = 1, \quad \forall m \in \mathbb{N},
\]
which deduces that
\[
\lim_{n \to \infty} \mathbb{P} \left\{ \left| Y(t; t, \eta, u_n) - Y(t; t, \eta, \tilde{u}) \right| < \frac{1}{m} \right\} = 1, \quad \forall m \in \mathbb{N}.
\]
Let us choose \( m \in \mathbb{N} \) to be large enough such that \( 1/m < \epsilon \) and set
\[
\tilde{\Omega} = \{ \omega \in \Omega \mid Y(t; t, \eta, \tilde{u}) < \text{ess inf}_{u(\cdot) \in \mathcal{U}[t,T]} J(t, \eta; u(\cdot)) + \epsilon \};
\]
\[
\Omega_n = \{ \omega \in \Omega \mid \left| Y(t; t, \eta, u_n) - Y(t; t, \eta, \tilde{u}) \right| \leq \frac{1}{m} \}, \quad n = 1, 2, \cdots
\]
Then, \( \mathbb{P}(\tilde{\Omega}) = \delta > 0 \) and \( \lim_{n \to \infty} \mathbb{P}(\Omega_n) = 1 \). We select \( n \in \mathbb{N} \) large enough such that \( \mathbb{P}(\Omega_n) > 1 - \delta \), then
\[
\mathbb{P}(\tilde{\Omega} \cap \Omega_n) = \mathbb{P}(\tilde{\Omega}) + \mathbb{P}(\Omega_n) - \mathbb{P}(\tilde{\Omega} \cup \Omega_n) > \delta + (1 - \delta) - 1 = 0.
\]
It is easy to see that
\[
\mathbb{P} \left\{ Y(t; t, \eta, u) < \text{ess inf}_{u(\cdot) \in \mathcal{U}[t,T]} \tilde{J}(t, \eta; u(\cdot)) + 2\epsilon \right\} \geq \mathbb{P}(\tilde{\Omega} \cap \Omega_n) > 0.
\]
This implies
\[
\text{ess inf}_{u(\cdot) \in \mathcal{U}_D} \tilde{J}(t, \eta; u(\cdot)) \leq \text{ess inf}_{u(\cdot) \in \mathcal{U}[t,T]} \tilde{J}(t, \eta; u(\cdot)) + 2\epsilon.
\]
From the arbitrariness of \( \epsilon > 0 \), we get (4.12).

**Step 2.** In this step, we prove
\[
\text{ess inf}_{u(\cdot) \in \mathcal{U}_D} \tilde{J}(t, \eta; u(\cdot)) = \text{ess inf}_{u(\cdot) \in \mathcal{U}} \tilde{J}(t, \eta; u(\cdot)) \tag{4.13}
\]
Since \( \mathcal{U} \subset \mathcal{U}_D \), we have
\[
\text{ess inf}_{u(\cdot) \in \mathcal{U}_D} \tilde{J}(t, \eta; u(\cdot)) \leq \text{ess inf}_{u(\cdot) \in \mathcal{U}} \tilde{J}(t, \eta; u(\cdot)). \tag{4.14}
\]
Now we show the inverse inequality of (4.14). For all \( u(\cdot) \in \mathcal{U}_D \), we have
\[
\tilde{J}(t, \eta; u(\cdot)) = \tilde{J} \left( t, \eta, \sum_{j=1}^{N} \chi_{\Omega_j} u^j(\cdot) \right) = \sum_{j=1}^{N} \chi_{\Omega_j} \tilde{J}(t, \eta; u^j(\cdot)).
\]
For \( j = 1, 2, \cdots, N \), noting that \( u^j(\cdot) \) is \( \mathbb{F}^t \) measurable, we find that \( \tilde{J}(t, \eta; u^j(\cdot)) \) is deterministic. Without loss of generality, we assume that
\[
\tilde{J}(t, \eta; u^1(\cdot)) \leq \tilde{J}(t, \eta; u^j(\cdot)), \quad \forall j = 2, 3, \cdots, N.
\]
Thus, it holds that
\[
\tilde{J}(t, \eta; u(\cdot)) \geq \tilde{J}(t, \eta; u^1(\cdot)) \geq \text{ess inf}_{u(\cdot) \in \mathcal{U}} \tilde{J}(t, \eta; u(\cdot)).
\]
From the arbitrariness of \( u(\cdot) \), we get
\[
\text{ess inf}_{u(\cdot) \in \mathcal{U}_D} \tilde{J}(t, \eta; u(\cdot)) \geq \text{ess inf}_{u(\cdot) \in \mathcal{U}} \tilde{J}(t, \eta; u(\cdot)).
\]

Step 3. We finish the proof in this step. From (4.11) and (4.13), we see that
\[
es \
\inf_{u(\cdot) \in \mathcal{U}} \tilde{\mathcal{J}}(t, \eta; u(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}} \mathcal{J}(t, \eta; u(\cdot)). \tag{4.15}
\]
The right hand side of (4.15) is deterministic, so is the left hand side of (4.15) is deterministic. This completes the proof.

By Proposition 4.3, we know that \( \inf_{u(\cdot) \in \mathcal{U}[t,T]} \) on the right hand side of (4.17) can be replaced by \( \inf_{u(\cdot) \in \mathcal{U}[t,T]} \). Let us define the value function of the stochastic optimal control problem as follows:
\[
\tilde{V}(t, \eta) = \inf_{u(\cdot) \in \mathcal{U}[t,T]} \mathcal{J}(t, \eta; u(\cdot)), \quad (t, \eta) \in [0,T] \times H. \tag{4.16}
\]

### 4.2 Some properties of \( \tilde{V}(\cdot, \cdot) \)

In this subsection, we present some properties of \( \tilde{V}(\cdot, \cdot) \), which will be used to establish the DPP for the stochastic recursive optimal control problem. The first result reveals some reglarity of \( \tilde{V}(\cdot, \cdot) \) with respect to \( \eta \).

**Proposition 4.4** Suppose (S1) and (S4) hold. For each \( t \in [0,T] \), \( \eta \) and \( \eta' \in H \), we have
\[
|\tilde{V}(t, \eta)| \leq C(1 + |\eta|_H) \tag{4.17}
\]
and
\[
|\tilde{V}(t, \eta) - \tilde{V}(t, \eta')| \leq C|\eta - \eta'|_H. \tag{4.18}
\]

**Proof.** From Propositions 4.1 and 4.2 for \( u(\cdot) \in \mathcal{U}[t,T] \), we have
\[
|\tilde{\mathcal{J}}(t, \eta; u(\cdot))| \leq C(1 + |\eta|_H), \quad \forall \eta \in H \tag{4.19}
\]
and
\[
|\tilde{\mathcal{J}}(t, \eta; u(\cdot)) - \tilde{\mathcal{J}}(t, \eta'; u(\cdot))|^2 \leq C|\eta - \eta'|_H^2, \quad \forall \eta, \eta' \in H. \tag{4.20}
\]

On the other hand, for each \( \varepsilon > 0 \), there exist \( u(\cdot) \) and \( u'(\cdot) \in \mathcal{U}[t,T] \) such that
\[
\tilde{\mathcal{J}}(t, \eta; u(\cdot)) - \varepsilon \leq \tilde{V}(t, \eta) \leq \tilde{\mathcal{J}}(t, \eta; u(\cdot)),
\]
\[
\tilde{\mathcal{J}}(t, \eta'; u(\cdot)) - \varepsilon \leq \tilde{V}(t, \eta') \leq \tilde{\mathcal{J}}(t, \eta'; u(\cdot)).
\]
Then from (4.19), we get
\[
-C(1 + |\eta|_H) - \varepsilon \leq \tilde{\mathcal{J}}(t, \eta; u(\cdot)) - \varepsilon \leq \tilde{V}(t, \eta) \leq \tilde{\mathcal{J}}(t, \eta; u(\cdot)) \leq C(1 + |\eta|_H).
\]

From the arbitrariness of \( \varepsilon > 0 \), we obtain (4.17).

Similarly,
\[
\tilde{\mathcal{J}}(t, \eta; u'(\cdot)) - \tilde{\mathcal{J}}(t, \eta'; u'(\cdot)) - \varepsilon \leq \tilde{V}(t, \eta) - \tilde{V}(t, \eta') \leq \tilde{\mathcal{J}}(t, \eta; u(\cdot)) - \tilde{\mathcal{J}}(t, \eta'; u(\cdot)) + \varepsilon.
\]
Thus,
\[
|\tilde{V}(t, \eta) - \tilde{V}(t, \eta')| \leq \max \{|\tilde{\mathcal{J}}(t, \eta; u(\cdot)) - \tilde{\mathcal{J}}(t, \eta'; u(\cdot))|, |\tilde{\mathcal{J}}(t, \eta; u'(\cdot)) - \tilde{\mathcal{J}}(t, \eta'; u'(\cdot))|\} + \varepsilon,
\]

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which, together with (4.20), implies that
\[
|\tilde{V}(t, \eta) - \tilde{V}(t, \eta')|^2 \leq 2 \max \{ |\tilde{J}(t, \eta; u(\cdot)) - \tilde{J}(t, \eta'; u(\cdot))|^2, |\tilde{J}(t, \eta; u'(\cdot)) - \tilde{J}(t, \eta'; u'(\cdot))|^2 \} + 2\varepsilon^2
\leq 2C|\eta - \eta'|^2_H + 2\varepsilon^2.
\]
From the arbitrariness of \(\varepsilon > 0\), we obtain (4.18).

In the following, we will define \(\tilde{V}(t, \cdot)\) for any \(\zeta\) in \(L^2_{\mathcal{F}_t}(\Omega; H)\). Before this, we need to define \(\bar{\zeta}(t, \cdot; u(\cdot))\) for \(\zeta\) in \(L^2_{\mathcal{F}_t}(\Omega; H)\).

First, for each simple function \(\zeta = \sum_{i=1}^N 1_{\Omega_i}, \eta_i \in H\), the similar argument as Lemma 4.1 leads to
\[
\tilde{J}(t, \zeta; u(\cdot)) = Y(t; t, \zeta; u) = \sum_{i=1}^N 1_{\Omega_i} Y(t; t, \eta_i, u) = \sum_{i=1}^N 1_{\Omega_i} \bar{\zeta}(t, \eta_i; u(\cdot)).
\] (4.21)

Hence, we can put
\[
\bar{\zeta}(t, \zeta) \triangleq \sum_{i=1}^N 1_{\Omega_i} \bar{\zeta}(t, \eta_i) = \sum_{i=1}^N 1_{\Omega_i} \inf_{u(\cdot) \in U[t,T]} \tilde{J}(t, \eta_i; u(\cdot)).
\] (4.22)

Secondly, for any \(\zeta\) in \(L^2_{\mathcal{F}_t}(\Omega; H)\), there exists a sequence of simple functions \(\{\zeta_j\}_{j=1}^\infty\) such that 
\[\zeta = \lim_{j \to \infty} \zeta_j\] in \(L^2_{\mathcal{F}_t}(\Omega; H)\). From Proposition 4.4 we see that \(\{\tilde{V}(t, \zeta_j)\}_{j=1}^\infty\) is a Cauchy sequence in \(L^2_{\mathcal{F}_t}(\Omega)\). Hence, we can define
\[
\tilde{V}(t, \zeta) \triangleq \lim_{j \to \infty} \tilde{V}(t, \zeta_j) \quad \text{in} \quad L^2_{\mathcal{F}_t}(\Omega).
\] (4.23)

Assume \(\{\tilde{\zeta}_j\}_{j=1}^\infty\) is another sequence of simple functions such that \(\zeta = \lim_{j \to \infty} \tilde{\zeta}_j\) in \(L^2_{\mathcal{F}_t}(\Omega; H)\). By Proposition 4.4 we know that
\[
\lim_{j \to \infty} \tilde{V}(t, \zeta_j) = \lim_{j \to \infty} \tilde{V}(t, \tilde{\zeta}_j).
\]

Hence, \(\tilde{V}(t, \zeta)\) is independent of the choice of the sequence of simple functions.

**Proposition 4.5** Suppose (S1) and (S4) hold. Fix \(t \in [0, T)\) and \(\zeta \in L^2_{\mathcal{F}_t}(\Omega; H)\). For each \(u(\cdot) \in U[t, T]\), we have
\[
\tilde{V}(t, \zeta) \leq Y(t; t, \zeta, u).
\] (4.24)

On the other hand, for each \(\varepsilon > 0\), there exists an admissible control \(u_\varepsilon(\cdot) \in U[t, T]\) such that
\[
\tilde{V}(t, \zeta) \geq Y(t; t, \zeta, u_\varepsilon) - \varepsilon, \quad \mathbb{P}\text{-a.s.}
\] (4.25)

**Proof.** We first prove (4.24). When \(\zeta = \sum_{i=1}^N 1_{\Omega_i}, \eta_i \in H\), for all \(u(\cdot) \in U[t, T]\), we have
\[
Y(t; t, \zeta, u) = Y(t; t, \sum_{i=1}^N 1_{\Omega_i} \eta_i, u) = \sum_{i=1}^N 1_{\Omega_i} Y(t; t, \eta_i, u) \geq \sum_{i=1}^N 1_{\Omega_i} \tilde{V}(t, \eta_i) = \tilde{V}(t, \zeta).
\]
When $\zeta \in L^2_{\mathcal{F}_t}(\Omega;H)$, we can choose a sequence of simple functions $\{\zeta_j\}^\infty_{j=1}$ converging to $\zeta$ in $L^2_{\mathcal{F}_t}(\Omega;H)$. By Proposition 4.2, we have that

$$\lim_{j \to \infty} \mathbb{E}|Y(t; t, \zeta, u) - Y(t; t, \zeta_j, u)|^2 = 0.$$  

From the definition of $\tilde{V}(t, \zeta)$, we can find that

$$\lim_{j \to \infty} \mathbb{E}|\tilde{V}(t, \zeta) - \tilde{V}(t, \zeta_j)|^2 = 0.$$  

Then, there exists a subsequence $\{\zeta_{j_k}\}^\infty_{k=1}$ of $\{\zeta_j\}^\infty_{j=1}$, such that

$$\left\{ \begin{array}{l}
\lim_{k \to \infty} Y(t; t, \zeta_{j_k}, u) = Y(t; t, \zeta, u), \quad \mathbb{P}\text{-a.s.}, \\
\lim_{k \to \infty} \tilde{V}(t, \zeta_{j_k}) = \tilde{V}(t, \zeta), \quad \mathbb{P}\text{-a.s.}
\end{array} \right.$$  

This, together with $Y(t; t, \zeta_{j_k}, u) \geq \tilde{V}(t, \zeta_{j_k}), k = 1, 2, \ldots$, implies (4.24).

Now we turn to prove (4.25). Since $H$ is separable, there exists a dense subset $\{\xi_j\}^\infty_{j=1} \subset H$. Put

$$\bar{\Omega}_j = \left\{ \omega \in \Omega \mid |\zeta(\omega) - \xi_j|_H < \frac{\varepsilon}{3 \sqrt{C}} \right\}, \quad \forall j \in \mathbb{N},$$  

where $C$ is the larger one of the constants in (4.5) and (4.18). Let

$$\Omega_1 = \bar{\Omega}_1, \quad \Omega_k = \bar{\Omega}_k \setminus \bigcup_{j=1}^{k-1} \bar{\Omega}_j, \quad k \in \mathbb{N}.$$  

Then, $\{\Omega_j\}^\infty_{j=1} \subset \mathcal{F}_t$ is a partition of $\Omega$. For $\eta = \sum_{j=1}^\infty 1_{\Omega_j} \xi_j$, from (4.3) and (4.18), we have

$$|Y(t; t, \zeta, u) - Y(t; t, \eta, u)| \leq \frac{\varepsilon}{3}, \quad |\tilde{V}(t, \zeta) - \tilde{V}(t, \eta)| \leq \frac{\varepsilon}{3}, \quad \mathbb{P}\text{-a.s.}$$  

For each $\xi_j$, by Proposition 4.3 we can choose $u^j \in \mathcal{U}^t$ such that

$$\tilde{V}(t, \xi_j) \geq Y(t; t, \xi_j, u^j) - \frac{\varepsilon}{3}.$$  

Let $u(\cdot) = \sum_{j=1}^\infty 1_{\Omega_j} u^j(\cdot)$. Then

$$Y(t; t, \zeta, u) \leq |Y(t; t, \zeta, u) - Y(t; t, \eta, u)| + Y(t; t, \eta, u)$$  

$$\leq \frac{\varepsilon}{3} + \sum_{j=1}^\infty 1_{\Omega_j} Y(t; t, \xi_j, u^j)$$  

$$\leq \frac{\varepsilon}{3} + \sum_{j=1}^\infty 1_{\Omega_j} \left( \tilde{V}(t, \xi_j) + \frac{\varepsilon}{3} \right) = \frac{2}{3} \varepsilon + \tilde{V}(t, \eta)$$  

$$\leq \varepsilon + \tilde{V}(t, \zeta), \quad \mathbb{P}\text{-a.s.},$$

which illustrates (4.25). This completes the proof.  

Next, we devote ourselves to obtaining the continuity of $\tilde{V}(t, \eta)$ with respect to $t$.  

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Proposition 4.6 Suppose (S1) and (S4) hold. The function $\tilde{V}(\cdot, \eta)$ is continuous.

Proof. We define $Y(s; t, \eta, u)$ for all $s \in [0, T]$ by choosing $Y(s; t, \eta, u) \equiv Y(t; t, \eta, u)$ for $0 \leq s \leq t$. Fix $\eta \in H$, for $0 \leq t_1 \leq t_2 \leq T$ and every $\varepsilon > 0$, there exist $u_1(\cdot), u_2(\cdot) \in \mathcal{U}[t, T]$, such that

\[
Y(t_1; t_1, \eta; u_2) - \varepsilon \leq \tilde{V}(t_1, \eta) \leq Y(t_1; t_1, \eta, u_1),
\]
\[
Y(t_2; t_2, \eta, u_1) - \varepsilon \leq \tilde{V}(t_2, \eta) \leq Y(t_2; t_2, \eta, u_2).
\]

Then,

\[
Y(t_1; t_1, \eta, u_2) - Y(t_2; t_2, \eta, u_2) - \varepsilon \leq \tilde{V}(t_1, \eta) - \tilde{V}(t_2, \eta) \leq Y(t_1; t_1, \eta, u_1) - Y(t_2; t_2, \eta, u_1) + \varepsilon,
\]

which implies that

\[
|\tilde{V}(t_1, \eta) - \tilde{V}(t_2, \eta)| \leq \max\{|Y(t_1; t_1, \eta, u_1) - Y(t_2; t_2, \eta, u_1)|, |Y(t_1; t_1, \eta, u_2) - Y(t_2; t_2, \eta, u_2)|\} + \varepsilon.
\]

Here we only estimate $|Y(t_1; t_1, \eta, u_1) - Y(t_2; t_2, \eta, u_1)|$ and the estimate of $|Y(t_1; t_1, \eta, u_2) - Y(t_2; t_2, \eta, u_2)|$ is the same. From Proposition 4.2, we have

\[
|Y(t_1; t_1, \eta, u_1) - Y(t_2; t_2, \eta, u_1)|^2
\]
\[
= |Y(0; t_1, \eta, u_1) - Y(0; t_2, \eta, u_1)|^2
\]
\[
\leq CE_2 \Phi(X(T; t_1, \eta, u_1)) - \Phi(X(T; t_2, \eta, u_1))|^2
\]
\[
+ CE_2 \left( \int_0^T [1_{[t_1, T]}g(s, X(s; t_1, \eta, u_1), Y(s; t_1, \eta, u_1), Z(s; t_1, \eta, u_1), u_1(s)) - 1_{[t_2, T]}g(s, X(s; t_2, \eta, u_1), Y(s; t_1, \eta, u_1), Z(s; t_1, \eta, u_1), u_1(s))] ds \right)^2
\]
\[
= I_1 + I_2.
\]

From Propositions 4.1 and 4.2, we get

\[
I_1 \leq CE_2|X(T; t_1, \eta, u_1) - X(T; t_2, \eta, u_1)|_H^2 \leq C\|X(T; t_1, \eta, u_1) - \eta\|_H^2
\]

and

\[
I_2 \leq C(t_2 - t_1).
\]

From (4.26) - (4.28), we know that

\[
|Y(t_1; t_1, \eta, u_1) - Y(t_2; t_2, \eta, u_1)| \leq C(t_2 - t_1)^{1/2} + C(\|X(t_2; t_1, \eta, u_1) - \eta\|_H^2)^{1/2}.
\]

The same argument used to $|Y(t_1; t_1, \eta, u_2) - Y(t_2; t_2, \eta, u_2)|^2$ leads to

\[
|\tilde{V}(t_1, \eta) - \tilde{V}(t_2, \eta)| \leq C(t_2 - t_1)^{1/2} + C(\|X(t_2; t_1, \eta, u_1) - \eta\|_H^2)^{1/2} + \varepsilon.
\]

Due to the arbitrariness of $\varepsilon$, we get

\[
|\tilde{V}(t_1, \eta) - \tilde{V}(t_2, \eta)| \leq C(t_2 - t_1)^{1/2} + C(\|X(t_2; t_1, \eta, u_1) - x\|_H^2)^{1/2}.
\]

From the continuity of $x(\cdot)$ with respect to $t$, we get the continuity of $\tilde{V}(\cdot; \eta)$. The proof is completed.
4.3 DPP for the stochastic recursive optimal control problem

Now, borrowing some ideas in [22], we study the (generalized) dynamic programming principle for our recursive optimal control problem (4.16).

Given the initial condition \((t, \eta)\), an admissible control \(u(\cdot) \in \mathcal{U}[t, T]\), a positive number \(\delta \leq T - t\) and a real-valued random variable \(\zeta \in L^2_{\mathbb{F}^{t+\delta}}(\Omega)\), we denote

\[
G_{t,t+\delta}^{t,\eta,u}[\zeta] = Y(t),
\]

where \((Y(s), Z(s))_{t \leq s \leq t+\delta}\) is the solution of the following BSDE with time horizon \(t + \delta\)

\[
Y(s) = \zeta + \int_s^{t+\delta} g(r, X(r; t, \eta, u), Y(r), Z(r), u(r))dr - \int_s^{t+\delta} Z(r)dW(r), \quad t \leq s \leq t + \delta.
\]

**Theorem 4.1** Under the assumptions (S1) and (S4), the value function \(\bar{V}(\cdot, \cdot)\) obeys the following dynamic programming principle: For each \(0 < \delta \leq T - t\),

\[
\bar{V}(t, \eta) \leq \essinf_{u(\cdot) \in \mathcal{U}[t,T]} G_{t,t+\delta}^{t,\eta,u}(\bar{V}(t + \delta, X(t + \delta; t, \eta, u))).
\] (4.29)

**Proof.** By the uniqueness of the solution to (4.1), we have

\[
Y(s) = \Phi(X(T; t, \eta, u)) + \int_s^T g(r, X(r; t, \eta, u), Y(r), Z(r), u(r))dr - \int_s^T Z(r)dW(r)
= Y(t + \delta; t, \eta, u) + \int_s^{t+\delta} g(r, X(r; t, \eta, u), Y(r), Z(r), u(r))dr - \int_s^{t+\delta} Z(r)dW(r).
\]

Hence,

\[
G_{t,t+\delta}^{t,\eta,u}[\Phi(X(T; t, \eta, u))] = G_{t,t+\delta}^{t,\eta,u}[Y(t + \delta; t, \eta, u)].
\] (4.30)

By the uniqueness of the solution to (3.11) and (4.11), for \(s \geq t + \delta\), we have

\[
Y(s) = \Phi(X(T; t, \eta, u)) + \int_s^T g(r, X(r; t, \eta, u), Y(r), Z(r), u(r))dr - \int_s^T Z(r)dW(r)
= \Phi(X(T; t + \delta, X(t + \delta), u)) + \int_s^T g(r, X(r; t, X(t + \delta), u), Y(r), Z(r), u(r))dr - \int_s^T Z(r)dW(r).
\]

Consequently,

\[
G_{t,t+\delta}^{t,\eta,u}[Y(t + \delta; t, \eta, u)] = G_{t,t+\delta}^{t,\eta,u}[Y(t + \delta; t + \delta, X(t + \delta; t, \eta, u), u)].
\] (4.31)

From (4.30) and (4.31), we see that

\[
\bar{V}(t, \eta) = \essinf_{u(\cdot) \in \mathcal{U}[t,T]} G_{t,t}^{t,\eta,u}[\Phi(X(T; t, \eta, u))] = \essinf_{u(\cdot) \in \mathcal{U}[t,T]} G_{t,t+\delta}^{t,\eta,u}[Y(t + \delta; t, x; u)]
= \essinf_{u(\cdot) \in \mathcal{U}[t,T]} G_{t,t+\delta}^{t,\eta,u}[Y(t + \delta; t + \delta, X(t + \delta; t, x; u), u)].
\]

From the classical comparison theorem of BSDE (e.g., [10] Theorem 2.2), we have that

\[
\bar{V}(t, \eta) \geq \essinf_{u(\cdot) \in \mathcal{U}[t,T]} G_{t,t+\delta}^{t,\eta,u}[\bar{V}(t + \delta, X(t + \delta; t, \eta, u))].
\]

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On the other hand, for every $\varepsilon > 0$, we can find an admissible control $\tilde{u}(\cdot) \in \mathcal{U}[t, T]$ such that
\[
\tilde{V}(t + \delta, X(t + \delta; t, \eta; \tilde{u})) \geq Y(t + \delta; t + \delta, X(t + \delta; t, \eta, \tilde{u}), \tilde{u}) - \varepsilon.
\]

From this and the comparison theorem of BSDE, we get
\[
\tilde{V}(t, \eta) \leq \text{ess inf}_{u(\cdot) \in \mathcal{U}[t, T]} G_{t,t+\delta}^{t,\eta;u} [\tilde{V}(t + \delta, X(t + \delta; t, \eta, u)) + \varepsilon].
\]

From Proposition 4.2, there exists a positive constant $C_0$ such that
\[
\tilde{V}(t, \eta) \leq \text{ess inf}_{u(\cdot) \in \mathcal{U}[t, T]} G_{t,t+\delta}^{t,\eta;u} [\tilde{V}(t + \delta, X(t + \delta; t, \eta, u))] + C_0 \varepsilon.
\]

Therefore, letting $\varepsilon \to 0$, we obtain (1.29). Theorems 3.1 and 3.2 follows from Theorem 4.1 immediately.

5 Relationships between PMP and DPP: Smooth Case

In this section, we head to the relationship between PMP and DPP when the value function is smooth.

**Theorem 5.1** Let (S1)–(S3) hold and let $\eta \in H$ be fixed, $(\overline{X}(\cdot), \tilde{u}(\cdot), p(\cdot), q(\cdot))$ be an optimal 4-tuple of Problem (S$\eta$). Suppose that the value function $V \in C^{1,2}([0, T]; H)$ and $V_{xx}$ ranges in $D(A^*)$. Then
\[
G(t, \overline{X}(t), \tilde{u}(t), -V_x(t, \overline{X}(t)), -V_{xx}(t, \overline{X}(t))) = \max_{u \in \mathcal{U}} G(t, \overline{X}(t), -V_x(t, \overline{X}(t)), -V_{xx}(t, \overline{X}(t))), \quad \text{a.e.} \ (t, \omega) \in [0, T] \times \Omega,
\]

where $G$ is given in (3.3). Further, if $V \in C^{1,3}([0, T] \times H)$ with $V_{tx} \in H$ and $A^*V_{xx} \in L(H)$ being continuous on $(0, T) \times H$, then
\[
\begin{cases}
V_x(t, \overline{X}(t)) = -p(t), \\
V_{xx}(t, \overline{X}(t))b(t, \overline{X}(t), \tilde{u}(t)) = -q(t),
\end{cases} \quad \text{a.e.} \ (t, \omega) \in [0, T] \times \Omega.
\]

**Proof.** From Theorem 3.2 and the fact that $(\overline{X}(\cdot), \tilde{u}(\cdot))$ is an optimal pair, we have
\[
V(t, \overline{X}(t)) = \mathbb{E} \left( \int_t^T f(r, \overline{X}(r), \tilde{u}(r))dr + h(\overline{X}(T)) \bigg| \mathcal{F}_t \right), \quad \forall t \in [0, T], \ \mathbb{P}\text{-a.s.}
\]

Let
\[
m(t) \triangleq \mathbb{E} \left( \int_0^T f(r, \overline{X}(r), \tilde{u}(r))dr + h(\overline{X}(T)) \bigg| \mathcal{F}_t \right), \quad t \in [0, T].
\]

By (S4), we have
\[
\int_0^T \left[ \mathbb{E} \mathbb{E} \left( \int_0^T \left( f(r, \overline{X}(r), \tilde{u}(r))dr + h(\overline{X}(T)) \right) \bigg| \mathcal{F}_t \right) \right]^{1/2} dt \\
\leq T \left[ \mathbb{E} \left( \int_0^T |f(r, \overline{X}(r), \tilde{u}(r))|^2 dr + |h(\overline{X}(T))|^2 \right) \right]^{1/2} \\
\leq C \left\{ \mathbb{E} \left[ \int_0^T (1 + |\overline{X}(r)|_H^2) dr + 1 + |\overline{X}(T)|_H^2 \right] \right\}^{1/2} \\
\leq C|\overline{X}(\cdot)|_{C_p([0,T];L^2(\Omega;H))}^{1/2}.
\]
which implies that \( m(\cdot) \in L^2_x(0, T; L^2(\Omega, H)) \). Thus, by the martingale representation theorem (see [13] Corollary 2.145 for example), there is a unique \( K(\cdot, \cdot, \cdot) \in L^1(0, T; L^2_x(0, T; L^2_0)) \) such that
\[
m(t) = \mathbb{E}m(t) + \int_0^t K(t, r)dW(r), \quad t \in [0, T].
\]
Then we have
\[
V(t, \overline{X}(t)) = m(t) - \int_0^t f(r, \overline{X}(r), \bar{u}(r))dr
\]
\[
= V(0, \eta) - \int_0^t f(r, \overline{X}(r), \bar{u}(r))dr + \int_0^t K(t, r)dW(r). \tag{5.3}
\]
On the other hand, applying Itô’s formula to \( V(t, \overline{X}(t)) \), we obtain
\[
V(t, \overline{X}(t)) = V(0, \eta) + \int_0^t \left( V_r(r, \overline{X}(r)) + \langle A^*V_x(t, \overline{X}(t)), \overline{X}(t) \rangle_H + \langle V_x(r, \overline{X}(r)), a(r, \overline{X}(r), \bar{u}(r)) \rangle_H \right. \\
\left. + \frac{1}{2} \langle V_{xx}(r, \overline{X}(r)), b(r, \overline{X}(r), \bar{u}(r)), b(r, \overline{X}(r), \bar{u}(r)) \rangle_{L^2_0} \right)dr \\
+ \int_0^t \langle b(r, \overline{X}(r), \bar{u}(r)), dW(r), V_x(r, \overline{X}(r)) \rangle_H. \tag{5.4}
\]
From (5.3) and (5.4), we conclude that
\[
\begin{cases}
-f(t, \overline{X}(t), \bar{u}(t)) = V_t(t, \overline{X}(t)) + \langle A^*V_x(t, \overline{X}(t)), \overline{X}(t) + a(t, \overline{X}(t), \bar{u}(t)) \rangle_H \\
+ \frac{1}{2} \langle V_{xx}(t, \overline{X}(t)), b(t, \overline{X}(t), \bar{u}(t)), b(t, \overline{X}(t), \bar{u}(t)) \rangle_{L^2_0}, \\
b(r, \overline{X}(r), \bar{u}(r))V_x(r, \overline{X}(r)) = K(t, t).
\end{cases} \tag{5.5}
\]
This, together with the fact that \( V \) is a smooth solution of the HJB equation, implies (5.1). In addition, we have
\[
G(t, \overline{X}(t), \bar{u}(t), -V_x(t, \overline{X}(t)), -V_{xx}(t, \overline{X}(t))) - \langle A^*V_x(t, \overline{X}(t)), \overline{X}(t) \rangle_H - V_t(t, \overline{X}(t)) \\
\leq G(t, \eta, \bar{u}(t), -V_x(t, \eta), -V_{xx}(t, \eta)) - \langle A^*V_x(t, \eta), \eta \rangle_H - V_t(t, \eta), \quad \forall \eta \in H.
\]
Consequently, if \( V \in C^{1,3}(0, T; H) \) with \( V_{tx} \) and \( A^*V_{xx} \) being also continuous, then
\[
\frac{\partial}{\partial x}(G(t, \eta, \bar{u}(t), -V_x(t, \eta), -V_{xx}(t, \eta)) - \langle A^*V_x(t, \eta), \eta \rangle_H - V_t(t, \eta)) \big|_{x=\overline{X}(t)} = 0.
\]
This implies that
\[
0 = V_{tx}(t, \overline{X}(t)) + A^*V_{xx}(t, \overline{X}(t))\overline{X}(t) + A^*V_x(t, \overline{X}(t)) \\
+ a_x(t, \overline{X}(t), \bar{u}(t))V_x(t, \overline{X}(t)) + V_{xx}(t, \overline{X}(t))a(t, \overline{X}(t), \bar{u}(t)) \\
+ \frac{1}{2} \sum_{j=1}^{\infty} V_{xx}(t, \overline{X}(t))(b(t, \overline{X}(t), \bar{u}(t))e_j, b(t, \overline{X}(t), \bar{u}(t))e_j) \\
+ b(t, \overline{X}(t), \bar{u}(t))V_{xx}(t, \overline{X}(t))b_x(t, \overline{X}(t), \bar{u}(t))^* + f_x(t, \overline{X}(t), \bar{u}(t)), \tag{5.6}
\]
where
\[
\frac{\partial}{\partial x}(A^*V_x(t, \eta), \eta)_H = A^*V_{xx}(t, \overline{X}(t))\overline{X}(t) + A^*V_x(t, \overline{X}(t))
\]
is due to the fact that \( A^*V_x \) is continuous and \( A^* \) is closed, and \( \{e_j\}_{j=1}^{\infty} \) is an orthonormal basis of \( \overline{H} \). One the other hand, by (S4), we have that
\[
dV_x(t, \overline{X}(t)) = \left( V_{tx}(t, \overline{X}(t)) + A^*V_{xx}(t, \overline{X}(t))\overline{X}(t) + V_{xx}(t, \overline{X}(t))a(t, \overline{X}(t), \bar{u}(t)) \right)
\]
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+ \frac{1}{2} \sum_{j=1}^{\infty} V_{xxx}(t, \overline{X}(t))(b(t, \overline{X}(t), \bar{u}(t))e_j, b(t, \overline{X}(t), \bar{u}(t))e_j)dt
+ V_{xx}(t, \overline{X}(t))b(t, \overline{X}(t), \bar{u}(t))dW(t)
= (A^*V_x(t, \overline{X}(t)) + a_x(t, \overline{X}(t), \bar{u}(t))V_x(t, \overline{X}(t)) + f_x(t, \overline{X}(t), \bar{u}(t))
+ V_{xx}(t, \overline{X}(t))b_x(t, \overline{X}(t), \bar{u}(t))^*b(t, \overline{X}(t), \bar{u}(t)))dt
+ V_{xx}(t, \overline{X}(t))b(t, \overline{X}(t), \bar{u}(t))dW(t).
\tag{5.7}

Noting that $-V_x(T, \overline{X}(T)) = -h_x(\overline{X}(T))$, by the uniqueness of the solution to the first-order adjoint equation, we obtain (5.2).

\textbf{Remark 5.1} The second equality in (5.4) may be regarded as a “maximum principle” in terms of the derivatives of the value function. It is different from the PMP in Theorem 2.1, where no derivative of the value function is involved.

\section{Relationships between PMP and DPP: Nonsmooth Case}

In Section 5 we dealt with the case when the value function is smooth enough. However, in many cases, the value function associated to an optimal control problem is not smooth. Hence it is sensible that we drop the smoothness assumptions of value function in Theorem 5.1.

\subsection{Differential in Spatial Variable}

We shall use the notions of super and subdifferentials to approach to our desired result. The main difficulty in carrying out this construction is that in infinite dimensional setting, the second-order adjoint equation is an operator-valued BSEE, in this case, the method of duality relation by means of mild solution breaks down. Thanks to the introduction of relaxed transposition solution, we are now able to legitimate our discussion by substituting the duality relation by the expected relaxed transposition solution. Before stating the main theory of this section, let us recall the notion of first-order and second-order super and subdifferentials.

For $v \in C([0, T] \times H)$ and $(t, \eta) \in [0, T) \times H$, the second-order parabolic superdifferential of $v$ at $(t, \eta)$ is defined as follows:

$$D_{t,x}^{1,2,+} v(t, \eta) = \left\{ (r, p, P) \in \mathbb{R} \times H \times \mathcal{S}(H) \mid \lim_{\substack{s \downarrow t, s \in [0, T) \setminus A \\{ t \} \rightarrow \eta \rightarrow \eta}} \frac{1}{|s - t| + |\eta - y|^2_H} \left[ v(s, y) - v(t, \eta) - r(s - t) - \langle p, y - \eta \rangle_H - \frac{1}{2} \langle P(y, \eta), y - \eta \rangle_H \right] \leq 0 \right\}.$$  

Similarly, the second-order parabolic subdifferential of $v$ at $(t, \eta)$ is defined as follows:

$$D_{t,x}^{1,2,-} v(t, \eta) = \left\{ (r, p, P) \in \mathbb{R} \times H \times \mathcal{S}(H) \mid \lim_{\substack{s \downarrow t, s \in [0, T) \setminus A \\{ t \} \rightarrow \eta \rightarrow \eta}} \frac{1}{|s - t| + |\eta - y|^2_H} \left[ v(s, y) - v(t, \eta) - r(s - t) - \langle p, y - \eta \rangle_H - \frac{1}{2} \langle P(y, \eta), y - \eta \rangle_H \right] \geq 0 \right\}.$$  

It is important to point out that the limit in $t$ is from the right. This fits the general irreversibility of evolution equations.

Let us recall also the concept of regular conditional probability, which allows us to regard conditional expectations as merely expectations taken with respect to a conditional measure. More details can be found in [21], Chapter V, Section 8.
Lemma 6.1 Let $\mathcal{G}$ be a sub-$\sigma$-algebra of $\mathcal{F}$. Then there exists a map $p : \Omega \times \mathcal{F} \to [0, 1]$, called a regular conditional probability given $\mathcal{G}$, such that

(i) for each $\omega \in \Omega$, $p(\omega, \cdot)$ is a probability measure on $\mathcal{F}$;

(ii) for each $A \in \mathcal{F}$, the function $p(\cdot, A)$ is $\mathcal{G}$-measurable;

(iii) for each $B \in \mathcal{F}$, $p(\omega, B) = \mathbb{P}(B|\mathcal{G})(\omega) = \mathbb{E}(1_B|\mathcal{G})(\omega)$, $\mathbb{P}$-a.s.

We write $\mathbb{P}(.|\mathcal{G})(\omega)$ for $p(\omega, \cdot)$.

Theorem 6.1 Suppose (S1)–(S3) hold. Let $\eta \in H$ be fixed, and $(\overline{X}(), \overline{u}(), p(\cdot), q(\cdot), P(\cdot), Q(\cdot), \overline{Q}(\cdot))$ be an optimal 6-tuple of Problem (S$\eta$). Suppose that $V \in C([0, T] \times H)$ is the associated value function. Then

$$
\{ -p(t) \times [-P(t), \infty) \subset D^2_xV(t, \overline{\chi}(t)), \quad \forall t \in [0, T], \quad \mathbb{P}\text{-a.s.,}
$$

and

$$
D^2_xV(t, \overline{\chi}(t)) \subset \{ -p(t) \times (-\infty, -P(t)], \quad \forall t \in [0, T], \quad \mathbb{P}\text{-a.s.}
$$

Proof. We borrow some ideas in [3, 4, 5, 25]. The proof is split into six steps.

Step 1. Fix a $t \in [0, T]$. For any $z \in H$, consider the following SEE:

$$
\begin{aligned}
\begin{cases}
      dx^z(r) &= \left(Ax^z(r) + a(r, x^z(r), \overline{u}(r))\right)dr + b(r, x^z(r), \overline{u}(r))dW(r), & r \in (t, T], \\
      x^z(t) &= z.
\end{cases}
\end{aligned}
$$

Set $\xi^z(r) = x^z(r) - \overline{X}(r)$. In order to reach an estimate for the conditional expectation of $\xi^z(\cdot)$ w.r.t. $\mathcal{F}_t$, we regard equation (6.3) as an SEE on $(\Omega, \mathcal{F}, \mathbb{P}(\cdot|\mathcal{F}_t)(\omega))$, for $\mathbb{P}$-a.s. $\omega$. This can be done by virtue of Proposition 6.1. Similar to the proof of Proposition 6.1 for any $k \geq 1$, we can obtain the following estimate:

$$
\mathbb{E}(\sup_{t \leq r \leq T} |\xi^z(r)|^2_H|\mathcal{F}_t) \leq K|z - \overline{X}(t)|^2_H, \quad \mathbb{P}\text{-a.s.}
$$

(6.4)

Now we present the equation for $\xi^z(\cdot)$ in two different ways based on different orders of expansions:

$$
\begin{aligned}
\begin{cases}
      d\xi^z(r) &= \left(A\xi^z(r) + \overline{a}_x(r)\xi^z(r)\right)dr + \overline{b}_x(r)\xi^z(r)dW(r) + \epsilon_{x, a}(r)dr + \epsilon_{x, b}(r)dW(r), & r \in (t, T], \\
      \xi^z(t) &= z - \overline{X}(t),
\end{cases}
\end{aligned}
$$

(6.5)

and

$$
\begin{aligned}
\begin{cases}
      d\xi^z(r) &= \left(A\xi^z(r) + \overline{a}_x(r)\xi^z(r) + \frac{1}{2}\overline{a}_{xx}(r)(\xi^z(r), \xi^z(r))\right)dr \\
      &\quad + \left(\overline{b}_x(r)\xi^z(r) + \frac{1}{2}\overline{b}_{xx}(r)(\xi^z(r), \xi^z(r))\right)dW(r) + \hat{\epsilon}_{x, a}dr + \hat{\epsilon}_{x, b}dW(r), & r \in (t, T], \\
      \xi^z(t) &= z - \overline{X}(t),
\end{cases}
\end{aligned}
$$

(6.6)

where for $\varphi = a, b$,

$$
\begin{aligned}
\varphi_x(r) &= \varphi_x(r, \overline{X}(r), \overline{u}(r)), \quad \varphi_{xx}(r) = \varphi_{xx}(r, \overline{X}(r), \overline{u}(r)),
\end{aligned}
$$

(6.7)

and

$$
\begin{aligned}
\epsilon_{x, \varphi}(r) &= \int_0^1 (\varphi_x(r, \overline{X}(r) + \theta\xi^z(r), \overline{u}(r)) - \varphi_x(r))\xi^z(r)d\theta, \\
\hat{\epsilon}_{x, \varphi}(r) &= \int_0^1 (1 - \theta)(\varphi_{xx}(r, \overline{X}(r) + \theta\xi^z(r), \overline{u}(r)) - \varphi_{xx}(r))\xi^z(r)\xi^z(r)d\theta.
\end{aligned}
$$

(6.8)
Step 2. This step is devoted to showing that for any $k \geq 1$, there exists a deterministic continuous and increasing function $\delta : [0, \infty) \rightarrow [0, \infty)$, independent of $z \in H$, with $\delta(r) = o(r)$, such that

$$
\mathbb{E}\left( \int_0^T |\epsilon_{z,\varphi}(r)|_{\dot{H}^k}^2 dr \big| \mathcal{F}_t \right)(\omega) \leq \delta\left( |z - \overline{X}(t, \omega)|_{\dot{H}^k}^2 \right), \quad \mathbb{P}\text{-a.s.}, \quad (6.9)
$$

and

$$
\mathbb{E}\left( \int_0^T |\epsilon_{z,\varphi}(r)|_{\dot{H}^k}^4 dr \big| \mathcal{F}_t \right)(\omega) \leq \delta\left( |z - \overline{X}(t, \omega)|_{\dot{H}^k}^2 \right), \quad \mathbb{P}\text{-a.s.} \quad (6.10)
$$

By setting $\varphi_x(r, \theta) = \varphi_x(r, \overline{X}(r) + \theta \xi^z(r), \overline{u}(r))$, and using assumption (S3), we have

$$
\mathbb{E}\left( \int_0^T |\epsilon_{z,\varphi}(r)|_{\dot{H}^k}^4 dr \big| \mathcal{F}_t \right) \leq C \int_0^T \mathbb{E}\left( |\xi^z(r)|_{\dot{H}^k}^4 \big| \mathcal{F}_t \right) dr \leq C |z - \overline{X}(t, \omega)|_{\dot{H}^k}^4.
$$

Thus (6.9) holds for $\delta(x) = x^2$.

Let $\varphi_{xx}(r, \theta) = \varphi_{xx}(r, \overline{X}(r) + \theta \xi^z(r), \overline{u}(r))$. Similar to the proof of (6.9), we have

$$
\mathbb{E}\left( \int_0^T |\epsilon_{z,\varphi}(r)|_{\dot{H}^k}^4 dr \big| \mathcal{F}_t \right) \leq C \int_0^T \mathbb{E}\left( |\xi^z(r)|_{\dot{H}^k}^4 \big| \mathcal{F}_t \right) dr \leq C |z - \overline{X}(t, \omega)|_{\dot{H}^k}^4.
$$

This implies that (6.10) holds for some $\delta(\cdot)$. Thus, we can pick the largest $\delta(\cdot)$ in the above calculations, so that (6.9)–(6.10) follows for all $z \in H$ with such $\delta(\cdot)$.

Step 3. Let $f_x(r) = f_x(r, \overline{X}(r), \overline{u}(r))$. By Itô’s formula, we get

$$
\mathbb{E}\left( \int_0^T \langle f_x(r), \xi^z(r) \rangle_H dr + \langle h_x(\overline{X}(T)), \xi^z(T) \rangle_H \big| \mathcal{F}_t \right)
$$

$$
= \langle -p(t), \xi^z(t) \rangle_H - \frac{1}{2} \mathbb{E}\left[ \int_0^T \left( \langle p(r), \xi^z(r) \rangle \dot{a}_{xx}(r) \xi^z(r) \right) \big| \mathcal{F}_t \right] dr
$$

$$
- \int_0^T \left( \langle p(r), \epsilon_{z,a}(r) \rangle + \langle q(r), \epsilon_{z,b}(r) \rangle \right) \big| \mathcal{F}_t \right] dr, \quad \mathbb{P}\text{-a.s.} \quad (6.11)
$$

In Definition 2.1 relaxed transposition solution is defined by expectation instead of conditional expectation, hence, we use the notion of regular conditional probability and consider the relaxed transposition solution of (2.2) on probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{P}(\cdot|\mathcal{F}_t)(\omega))$, for a.e. $\omega \in \Omega$. For simplicity of notations, we set $\mathbb{E}^\omega_t = \mathbb{E}(\cdot|\mathcal{F}_t)(\omega)$, which is the probability expectation related to $\mathbb{P}(\cdot|\mathcal{F}_t)(\omega)$, for a.e. $\omega \in \Omega$. Then we obtain

$$
\mathbb{E}^\omega_t \left( \int_0^T \langle H_{xx}(r) \xi^z(r), \xi^z(r) \rangle_H dr - \langle h_{xx}(T) \xi^z(T), \xi^z(T) \rangle_H \right)
$$

$$
= \langle P(t) \xi^z(t), \xi^z(t) \rangle_H + \mathbb{E}^\omega_t \left( \int_0^T \langle P(r) \epsilon_{z,a}(r), \xi^z(r) \rangle_H dr \right.
$$

$$
+ \mathbb{E}^\omega_t \left( \int_0^T \langle P(r) \epsilon_{z,b}(r), \epsilon_{z,a}(r) \rangle_H + \langle P(r) \bar{b}_x(r) \xi^z(r), \epsilon_{z,b} \rangle_{\mathcal{L}_2^2} \right) dr
$$
\begin{align}
&+ \mathbb{E}_ω^t \int_t^T \langle P(r)ε_{z,b}(r), \tilde{b}_x(r)ξ^z(r) + ε_{z,b}(r) \rangle_{\mathcal{L}^2} dr \\
&+ \mathbb{E}_ω^t \int_t^T \langle ε_{z,b}(r), \tilde{Q}^{(t)}(r) \rangle_{c^2} + \langle Q^{(t)}(r), ε_{z,b}(r) \rangle_{c^2} \rangle dr, \quad \mathbb{P}\text{-a.s.}
\end{align}

**Step 4.** Compute \( V(t, z) - V(t, \overline{X}(t, ω)) \).

Let \( M \) be a countable dense subset of \( H \). One can find a subset \( Ω_0 \subset Ω \) with \( \mathbb{P}(Ω_0) = 1 \) such that for any \( ω_0 \in Ω_0 \),

\[
\begin{cases}
V(t, \overline{X}(t, ω_0)) = \mathbb{E} \left( \int_t^T f(r, \overline{X}(r), \bar{u}(r)) dr + h(\overline{X}(T)) \bigg| F_t \right)(ω_0), \\
\text{[6.3], [6.9] - [6.12]} \text{ hold for any } z \in M, \\
\sup_{s \leq r \leq T} |p(r, ω_0)| < +∞, \\
P(t, ω_0) \in \mathcal{L}(H), \ P(\cdot, ω_0)ξ \in L^2(r, T), \ \forall ξ \in L^2_r(Ω; H), \ \forall r \in [t, T].
\end{cases}
\]

The first equality holds because of the Bellman’s optimal principle, while the last two conditions are due to the facts that

\[
\mathbb{E} \sup_{0 \leq r \leq T} |p(r)|^2_H < +∞
\]

and that \( P(\cdot, \cdot) \in \mathcal{P}[0, T] \) respectively. Let \( ω_0 \in Ω_0 \) be fixed, and again set \( \mathbb{E}_{ω_0}^t = \mathbb{E}(\cdot| F_t)(ω_0) \).

Then for any \( z \in M \), by the definition of value function, we see that

\[
V(t, z) - V(t, \overline{X}(t, ω_0)) \\
\leq -\langle p(t, ω_0), ξ^z(t, ω_0) \rangle_H - \frac{1}{2} \mathbb{E}_{ω_0}^t \left[ \int_t^T \left( \langle p(r), \overline{a}_{xx}(r)(ξ^z(r), ξ^z(r)) \rangle_{H} + \langle q(r), \overline{b}_{xx}(r)(ξ^z(r), ξ^z(r)) \rangle_{c^2} \right) dr \\
- \int_t^T \langle p(r), \tilde{ε}_{z,a}(r) \rangle_H + \langle q(r), \tilde{ε}_{z,b}(r) \rangle_{c^2} \right) \langle ξ^z(r), ξ^z(r) \rangle_{c^2} dr \\
+ \frac{1}{2} \mathbb{E}_{ω_0}^t \left( \int_t^T \langle \tilde{f}_{xx}(r)ξ^z(r), ξ^z(r) \rangle_H dr + \langle h_{xx}(\overline{X}(T))(ξ^z(T), ξ^z(T))_H \rangle + o(|z - \overline{X}(t, ω_0)|^2_H) \\
= -\langle p(t, ω_0), ξ^z(t, ω_0) \rangle_H - \frac{1}{2} \mathbb{E}_{ω_0}^t \left( \int_t^T \langle Π_{xx}(r)ξ^z(r), ξ^z(r) \rangle_H dr - \langle h_{xx}(T)ξ^z(T), ξ^z(T) \rangle_H \right) \\
- \frac{1}{2} \mathbb{E}_{ω_0}^t \left( \langle p(r), \tilde{ε}_{z,a}(r) \rangle_H + \langle q(r), \tilde{ε}_{z,b}(r) \rangle_{c^2} \right) dr + o(|z - \overline{X}(t, ω_0)|^2_H) \\
= -\langle p(t, ω_0), ξ^z(t, ω_0) \rangle_H - \frac{1}{2} \langle P(t, ω_0)ξ^z(t, ω_0), ξ^z(t, ω_0) \rangle_H \right)
\]

From the above discussion, by the definition of the relaxed transposition solution to [2.2], and the fact that for all \( (t, η) \in [0, T] \times H \), \( V(t, η) \) is deterministic, we have

\[
\begin{align}
V(t, z) - V(t, \overline{X}(t, ω_0)) \\
\leq -\langle p(t, ω_0), ξ^z(t, ω_0) \rangle_H - \frac{1}{2} \mathbb{E}_{ω_0}^t \left[ \int_t^T \left( \langle p(r), \overline{a}_{xx}(r)(ξ^z(r), ξ^z(r)) \rangle_{H} + \langle q(r), \overline{b}_{xx}(r)(ξ^z(r), ξ^z(r)) \rangle_{c^2} \right) dr \\
- \int_t^T \langle p(r), \tilde{ε}_{z,a}(r) \rangle_H + \langle q(r), \tilde{ε}_{z,b}(r) \rangle_{c^2} \right) \langle ξ^z(r), ξ^z(r) \rangle_{c^2} dr \\
+ \frac{1}{2} \mathbb{E}_{ω_0}^t \left( \int_t^T \langle \tilde{f}_{xx}(r)ξ^z(r), ξ^z(r) \rangle_H dr + \langle h_{xx}(\overline{X}(T))(ξ^z(T), ξ^z(T))_H \rangle + o(|z - \overline{X}(t, ω_0)|^2_H) \\
= -\langle p(t, ω_0), ξ^z(t, ω_0) \rangle_H - \frac{1}{2} \mathbb{E}_{ω_0}^t \left( \int_t^T \langle Π_{xx}(r)ξ^z(r), ξ^z(r) \rangle_H dr - \langle h_{xx}(T)ξ^z(T), ξ^z(T) \rangle_H \right) \\
- \frac{1}{2} \mathbb{E}_{ω_0}^t \left( \langle p(r), \tilde{ε}_{z,a}(r) \rangle_H + \langle q(r), \tilde{ε}_{z,b}(r) \rangle_{c^2} \right) dr + o(|z - \overline{X}(t, ω_0)|^2_H) \\
= -\langle p(t, ω_0), ξ^z(t, ω_0) \rangle_H - \frac{1}{2} \langle P(t, ω_0)ξ^z(t, ω_0), ξ^z(t, ω_0) \rangle_H \right)
\end{align}
\]

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\[-\mathbb{E}_0^t \int_t^T \langle p(r), \tilde{\epsilon}_{z,a}(r) \rangle_H + \langle q(r), \tilde{\epsilon}_{z,b}(r) \rangle_{L_2^2} dr - \frac{1}{2} \mathbb{E}_0^t \int_t^T \langle P(r) \epsilon_{z,a}(r), \xi^z(r) \rangle_H dr \]
\[-\mathbb{E}_0^t \int_t^T \langle P(r) \xi^z(r), \epsilon_{z,a}(r) \rangle_H + \langle P(r) \tilde{b}_x(r) \xi^z(r), \epsilon_{z,b}(r) \rangle_{L_2^2} dr \]
\[-\mathbb{E}_0^t \int_t^T \langle P(r) \epsilon_{z,b}(r), \tilde{b}_x(r) \xi^z(r) + \epsilon_{z,b}(r) \rangle_{L_2^2} dr \]
\[-\mathbb{E}_0^t \int_t^T \langle \epsilon_{z,b}(r), \tilde{Q}(t) \rangle_{L_2^2} + \langle Q(t) \rho, \epsilon_{z,b}(r) \rangle_{L_2^2} dr \]
\[+ o(|z - \overline{X}(t, \omega_0)|_H^2). \]

**Step 5.** In this step, we get rid of terms containing $\epsilon_{z,}$ and $\tilde{\epsilon}_{z,}$ in (6.14). By (6.9) and (6.10), we have

\[
|\mathbb{E}_0^t \int_t^T \langle p(r), \tilde{\epsilon}_{z,a}(r) \rangle_H + \langle q(r), \tilde{\epsilon}_{z,b}(r) \rangle_{L_2^2} dr | \\
\leq \int_t^T \mathbb{E}_0^t |p(r)|_H |\tilde{\epsilon}_{z,a}(r)|_H + |q(r)|_{L_2^2} |\tilde{\epsilon}_{z,b}(r)|_{L_2^2} dr \\
\leq |P| L^2(\Omega, C([-t, T]; H)) \left( \mathbb{E}_0^t \int_t^T |\epsilon_{z,a}(r)|_{L_2}^2 dr \right)^{1/2} + |q|_{L^2(0,T;L_2^2)} \left( \mathbb{E}_0^t \int_t^T |\epsilon_{z,b}(r)|_{L_2^2}^2 dr \right)^{1/2} \\
\leq C \delta (|z - \overline{X}(t, \omega_0)|_H^2) = o(|z - \overline{X}(t, \omega_0)|_H^2).
\]

Noting that $P$ is a component of the relaxed transposition solution to the equation (2.2), we get that

\[
|\mathbb{E}_0^t \int_t^T \langle P(r) \epsilon_{z,a}(r), \xi^z(r) \rangle_H dr | \\
\leq \left[ \int_t^T \left( \mathbb{E}_0^t |P(r) \epsilon_{z,a}(r)|_{L_2^2}^4 \right)^{3/2} dr \right]^{1/2} \left( \int_t^T \mathbb{E}_0^t |\xi^z(r)|_H^4 dr \right)^{1/2} \\
\leq |P| L^2(\Omega, C([-t, T]; H)) \left( \mathbb{E}_0^t \int_t^T |\epsilon_{z,a}(r)|_{L_2^2}^4 dr \right)^{1/4} \left( \sup_{t \leq r \leq T} \mathbb{E}_0^t |\xi^z(r)|_H^4 \right) \\
= o(|z - \overline{X}(t, \omega_0)|_H) O(|z - \overline{X}(t, \omega_0)|_H) = o(|z - \overline{X}(t, \omega_0)|_H^2).
\]

Similarly, we can show that

\[
|\mathbb{E}_0^t \int_t^T \langle P(r) \xi^z(r), \epsilon_{z,a}(r) \rangle_H dr | = o(|z - \overline{X}(t, \omega_0)|_H^2).
\]

Since $\tilde{b}_x$ is bounded, it is also easy to see that

\[
|\mathbb{E}_0^t \int_t^T \langle P(r) \tilde{b}_x(r) \xi^z(r), \epsilon_{z,b}(r) \rangle_{L_2^2} dr | \\
\leq \left[ \int_t^T \left( \mathbb{E}_0^t |P(r) \tilde{b}_x(r) \xi^z(r)|_{L_2^2}^4 \right)^{3/2} dr \right]^{1/2} \left( \int_t^T \mathbb{E}_0^t |\epsilon_{z,b}(r)|_{L_2^2}^4 dr \right)^{1/2} \\
\leq |P| L^2(\Omega, C([-t, T]; H)) \left( \mathbb{E}_0^t \int_t^T |\tilde{b}_x(r)|_{L_2^2}^4 dr \right)^{1/4} \left( \sup_{t \leq r \leq T} \mathbb{E}_0^t |\xi^z(r)|_H^4 \right) \\
\leq C |\xi^z| L^2(\Omega, C([-t, T]; H)) \left( \int_t^T \mathbb{E}_0^t |\tilde{b}_x(r)|_{L_2^2}^4 dr \right)^{1/2} \\
\leq C (|z - \overline{X}(t, \omega_0)|_H^4)^{1/4} \delta (|z - \overline{X}(t, \omega_0)|_H^4) = o(|z - \overline{X}(t, \omega_0)|_H^2).
\]

Similarly,
\[
\left| \mathbb{E}_0^t \int_t^T (P(r)\epsilon_{x,b}(r), \bar{b}_x(r)\xi^z(r) + \epsilon_{x,b}(r))_{L^2_0} dr \right| = o(|z - \bar{X}(t, \omega_0)|^2_H). \tag{6.19}
\]

Finally, by (6.9) and noting that \(\hat{Q}(t)\) is part of the relaxed transposition solution of (2.2), we obtain
\[
\left| \mathbb{E}_0^t \int_t^T \langle \epsilon_{x,b}(r), \hat{Q}(t)(r) \rangle_{L^2_0} dr \right|
\leq |\hat{Q}(t)(0,0,\epsilon_{x,b}(\cdot))|_{L^2_2(0,T;L^4(\Omega;\mathcal{L}^0_2))} \left[ \mathbb{E}_0^t \left( \int_t^T |\epsilon_{x,b}(r)|_{L^2_2}^4 dr \right)^{1/2} \right]^{1/2}
\leq C|\epsilon_{x,b}|_{L^2_2(0,T;L^4(\Omega;\mathcal{L}^0_2))} \left[ \mathbb{E}_0^t \left( \int_t^T |\epsilon_{x,b}(r)|_{L^2_2}^4 dr \right)^{1/2} \right]^{1/2}
\leq C \left[ \mathbb{E}_0^t \left( \int_t^T |\epsilon_{x,b}(r)|_{L^2_2}^4 dr \right)^{1/2} \right] \leq (\delta(|z - \bar{X}(t, \omega_0)|^2_H))^{1/2}
= o(|z - \bar{X}(t, \omega_0)|^2_H). \tag{6.20}
\]

From a similar argument, we arrive at
\[
\left| \mathbb{E}_0^t \int_t^T \langle Q(t)(r), \epsilon_{x,b}(r) \rangle_{L^2_2} dr \right| = o(|z - \bar{X}(t, \omega_0)|^2_H). \tag{6.21}
\]

**Step 6.** In this step, we complete the proof.

From the above discussion, we obtain
\[
V(t, z) - V(t, \bar{X}(t, \omega_0))
= \langle V_x(t, \bar{X}(t, \omega_0)), z - \bar{X}(t, \omega_0) \rangle_H + \frac{1}{2} \langle V_{xx}(t, \bar{X}(t, \omega_0))(z - \bar{X}(t, \omega_0)), z - \bar{X}(t, \omega_0) \rangle_H \tag{6.22}
\leq -\langle p(t, \omega_0), \xi^z(t, \omega_0) \rangle_H - \frac{1}{2} \langle P(t, \omega_0)\xi^z(t, \omega_0), \xi^z(t, \omega_0) \rangle_H + o(|z - \bar{X}(t, \omega_0)|^2_H).
\]

Note that the term \(o(|z - \bar{X}(t, \omega_0)|^2_H)\) is independent of \(z\). Thus by the continuity of \(V(t, \cdot)\), we see that (6.22) holds for all \(x \in H\), which proves
\[(p(t), -P(t)) \in D_x^{2,2}V(t, \bar{X}(t)).\]

By the definition of \(D_x^{2,2}V(t, \bar{X}(t))\), we obtain (6.1).

Let us now show (6.2). Fix an \(\omega\) such that (6.22) holds for any \(z \in H\). For any \((p, P) \in D_x^{2,2}V(t, \bar{X}(t))\), by the definition we have
\[
0 \leq \lim_{z \to \bar{X}(t)} \frac{V(t, z) - V(t, \bar{X}(t)) - \langle p, z - \bar{X}(t) \rangle_H - \frac{1}{2} \langle P(z - \bar{X}(t)), z - \bar{X}(t) \rangle_H}{|z - \bar{X}(t)|^2}
\leq \lim_{z \to \bar{X}(t)} -\langle p + p(t), z - \bar{X}(t) \rangle_H - \frac{1}{2} \langle (P + P(t))(z - \bar{X}(t)), z - \bar{X}(t) \rangle_H, \tag{6.22}
\]
where the last inequality is due to (6.22). Then it is necessary that
\[p = -p(t), \quad P \leq P(t).\]

This completes the proof. \(\square\)

When \(V \in C^{1,2}([0, T] \times H)\), (6.1) + (6.2) is reduced to
\[
V_x(t, \bar{X}(t)) = -p(t), \quad V_{xx}(t, \bar{X}(t)) \leq -P(t).
\]
6.2 Differentials in the Time Variable

In this section, we proceed to studying the super- and subdifferential of the value function in the time variable $t$ along an optimal trajectory.

**Theorem 6.2** Under the assumption of Theorem 6.1, for any $t$ such that $\overline{X}(t) \in D(A)$ or $p(t) \in D(A^*)$, we have

$$\langle \langle A\overline{X}(t), p(t) \rangle \rangle + \mathcal{H}(t, \overline{X}(t), \overline{u}(t)) \in D_{\mathcal{Y}}(V'(t, \overline{X}(t)), \text{ a.e. } t \in [0, T], \mathbb{P}\text{-a.s.}$$

where

$$\langle \langle A\overline{X}(t), p(t) \rangle \rangle = \begin{cases} \langle A\overline{X}(t), p(t) \rangle_H, & \text{if } \overline{X}(t) \in D(A), \\ \langle \overline{X}(t), A^*p(t) \rangle_H, & \text{if } p(t) \in D(A^*), \end{cases}$$

and $\mathcal{H}(t, \eta, u) = G(t, \eta, u, p(t), P(t)) + \langle b(t, \eta, u), q(t) - P(t)b(t, \eta, \overline{u}(t)) \rangle_{\mathcal{L}^2}$.

**Proof.** For any $t \in (0, T)$, take $\tau \in (t, T]$. Denote by $x_{\tau}$ the solution to the following SEE:

$$\begin{cases} \text{d}x_{\tau}(r) = A x_{\tau}(r)\text{d}r + a(r, \xi_{\tau}(r), \bar{u}(r))\text{d}r + b(r, \xi_{\tau}(r), \bar{u}(r))\text{d}W(r), \quad r \in (\tau, T], \\ x_{\tau}(\tau) = \overline{X}(t). \end{cases}$$

(6.23)

Set $\xi_{\tau}(r) = x_{\tau}(r) - \overline{X}(r)$ for $r \in [\tau, T]$. Working under $\mathbb{P}(\cdot | \mathcal{F}_t)(\omega)$, $\mathbb{P}\text{-a.s.}$ $\omega$, we have

$$\mathbb{E}\left( \sup_{\tau \leq r \leq T} |\xi_{\tau}(r)|^2 \mathbb{I}_{\mathcal{F}_t} \right) \leq C|\overline{X}(\tau) - \overline{X}(t)|^2, \quad \mathbb{P}\text{-a.s.}$$

(6.24)

Taking $\mathbb{E}(\cdot | \mathcal{F}_t)$ on both sides and noting that $\mathcal{F}_t \subset \mathcal{F}_\tau$, we obtain

$$\mathbb{E}\left( \sup_{\tau \leq r \leq T} |\xi_{\tau}(r)|^2 \mathbb{I}_{\mathcal{F}_t} \right) \leq C|\tau - t|^k(|A\overline{X}(t)|_H + 1 + |\overline{X}(t)|_H)^k \leq C|\tau - t|^k, \quad \mathbb{P}\text{-a.s.}$$

(6.25)

From the definition of $\xi_{\tau}(\cdot)$, we know that it satisfies the following variational equations:

$$\begin{cases} \text{d}\xi_{\tau}(r) = \left( A \xi_{\tau}(r) + \overline{a}(r) \right)\text{d}r + \overline{b}(r)\xi_{\tau}(r)\text{d}W(r) + \epsilon_{\tau,a}(r)\text{d}r + \epsilon_{\tau,b}(r)\text{d}W(r), \\ \xi_{\tau}(\tau) = -[S(\tau - t) - 1]\overline{X}(t) - \int_t^\tau S(\tau - r)\overline{a}(r)\text{d}r - \int_t^\tau S(\tau - r)\overline{b}(r)\text{d}W(r), \end{cases}$$

(6.26)

and

$$\begin{cases} \text{d}\xi_{\tau}(r) = \left( A \xi_{\tau}(r) + \overline{a}(r) \right)\text{d}r \\ + \left( \overline{b}(r)\xi_{\tau}(r) + \frac{1}{2}\overline{b}(r)\right)\xi_{\tau}(r)\text{d}W(r) + \overline{\epsilon}_{\tau,a}(r)\text{d}r + \overline{\epsilon}_{\tau,b}(r)\text{d}W(r), \\ \xi_{\tau}(\tau) = -[S(\tau - t) - 1]\overline{X}(t) - \int_t^\tau S(\tau - r)\overline{a}(r)\text{d}r - \int_t^\tau S(\tau - r)\overline{b}(r)\text{d}W(r). \end{cases}$$

(6.27)

Here for $\varphi = a, b$,

$$\begin{cases} \epsilon_{\tau,\varphi}(r) = \int_0^1 \left( \varphi_{x}(r, X(r) + \theta \xi_{\tau}(r), \bar{u}(r)) - \varphi_{x}(r) \right)\xi_{\tau}(r)\text{d}\theta, \\ \overline{\epsilon}_{\tau,\varphi}(r) = \int_0^1 (1 - \theta)\overline{\xi}_{\tau}(r)^T \left( \varphi_{xx}(r, X(r) + \theta \xi_{\tau}(r), \bar{u}(r)) - \varphi_{xx}(r) \right)\xi_{\tau}(r)\text{d}\theta. \end{cases}$$

(6.28)
Similar to (6.9) and (6.10), one can prove that for any $k \geq 1$,
\[
\begin{align*}
\left\{ \begin{array}{l}
E \left( \int_{\tau}^{T} |\varepsilon_{\tau} \varphi(r)\|_{H}^{2k} dr \right| \mathcal{F}_{t} \right) \leq \delta(|\tau - t|^{k}), \\
E \left( \int_{\tau}^{T} |\tilde{\varepsilon}_{\tau} \varphi(r)\|_{H}^{k} dr \right| \mathcal{F}_{t} \right) \leq \delta(|\tau - t|^{k}),
\end{array} \right. \\
P\text{-a.s.,}
\end{align*}
\] (6.29)
for some deterministic continuous function $\delta : [0, \infty) \to [0, \infty)$ with $\frac{\delta(r)}{r} \to 0$ as $r \to 0$. By the definition of the value function $V$, we have
\[
V(\tau, \overline{X}(t)) \leq E \left( \int_{\tau}^{T} f(r, x_{\tau}(r), \bar{u}(r))dr + h(x_{\tau}(T)) \right| \mathcal{F}_{\tau} \right), \quad P\text{-a.s.} 
\] (6.30)
Taking $E(\cdot |\mathcal{F}_{t}^{\tau})$ on both sides of (6.30) and noting that $t \leq \tau$, we conclude that
\[
V(\tau, \overline{X}(t)) \leq E \left( \int_{\tau}^{T} f(r, x_{\tau}(r), \bar{u}(r))dr + h(x_{\tau}(T)) \right| \mathcal{F}_{t} \right), \quad P\text{-a.s.} 
\] (6.31)
Choose a subset $\Omega_{0} \in \Omega$, with $P(\Omega_{0}) = 1$ such that for any $\omega_{0} \in \Omega_{0}$,
\[
\left\{ \begin{array}{l}
V(t, \overline{X}(t, \omega_{0})) = E \left( \int_{\tau}^{T} f(r, \overline{X}(r), \bar{u}(r))dr + h(\overline{X}(T)) \right| \mathcal{F}_{t} \right)(\omega_{0}), \\
(6.25), (6.28) \text{ and } (6.31) \quad \text{are satisfied for any rational } \tau > t,
\end{array} \right. \\
\sup_{s \leq T} \left| P(t, \omega_{0}) \right| < +\infty, \\
P(t, \omega_{0}) \in L(H), \quad P(\cdot, \omega_{0}) \xi \in L^{2}(r; T; H), \quad \forall \xi \in L_{\mathcal{F}_{\tau}}^{2}(\Omega; H), \quad \forall r \in [0, T].
\]
Let $\omega_{0} \in \Omega_{0}$ be fixed, and set $\mathbb{E}_{\omega_{0}}^{t} = E(\cdot |\mathcal{F}_{t})(\omega_{0})$. Then for any rational $\tau > t$, we have
\[
V(\tau, \overline{X}(t, \omega_{0})) - V(\tau, \overline{X}(t, \omega_{0})) \\
\leq \mathbb{E}_{\omega_{0}}^{t} \left\{ - \int_{\tau}^{T} \tilde{f}(r)dr + \int_{\tau}^{T} \left[ f(r, x_{\tau}(r), \bar{u}(r)) - \tilde{f}(r) \right] dr + h(x_{\tau}(T)) - h(\overline{X}(T)) \right\} \\
= \mathbb{E}_{\omega_{0}}^{t} \left( - \int_{\tau}^{T} \tilde{f}(r)dr + \int_{\tau}^{T} \langle \tilde{f}_{x}(r), \xi_{\tau}(r) \rangle_{H} dr + \langle h_{x}(\overline{X}(T)), \xi_{\tau}(T) \rangle_{H} \\
+ \frac{1}{2} \int_{\tau}^{T} \left( \tilde{f}_{xx}(r)\xi_{\tau}(r), \xi_{\tau}(r) \right)_{H} dr + \frac{1}{2} \left( h_{xx}(\overline{X}(T))\xi_{\tau}(T), \xi_{\tau}(T) \right)_{H} \right) + o(|\tau - t|). 
\] (6.32)
Similar to the argument in the forth and fifth steps in the proof of Theorem 6.2, we can obtain
\[
V(\tau, \overline{X}(t, \omega_{0})) - V(t, \overline{X}(t, \omega_{0})) \\
\leq -\mathbb{E}_{\omega_{0}}^{t} \int_{\tau}^{T} \tilde{f}(r)dr - \mathbb{E}_{\omega_{0}}^{t} \left( \langle p(r), \xi_{\tau}(r) \rangle_{H} - \frac{1}{2} \langle P(r)\xi_{\tau}(r), \xi_{\tau}(r) \rangle_{H} \right) \\
- \mathbb{E}_{\omega_{0}}^{t} \int_{\tau}^{T} \left( \langle p(r), \tilde{e}_{\tau,a}(r) \rangle_{H} + \langle q(r), \tilde{e}_{\tau,b}(r) \rangle_{L^{2}} \right) dr - \frac{1}{2} \mathbb{E}_{\omega_{0}}^{t} \int_{\tau}^{T} \langle P(r)\tilde{e}_{\tau,a}(r), \xi_{\tau}(r) \rangle_{H} dr \\
- \mathbb{E}_{\omega_{0}}^{t} \int_{\tau}^{T} \left( \langle P(r)\xi_{\tau}(r), \epsilon_{\tau,a}(r) \rangle_{H} + \langle P(r)\tilde{b}_{x}(r)\xi_{\tau}(r), \epsilon_{\tau,b}(r) \rangle_{L^{2}} \right) dr \\
- \mathbb{E}_{\omega_{0}}^{t} \int_{\tau}^{T} \langle P(r)\epsilon_{\tau,a}(r), \tilde{b}_{x}(r)\xi_{\tau}(r) + \epsilon_{\tau,b}(r) \rangle_{L^{2}} dr 
\] (6.33)
By the definition of \( \xi \)
any \( \phi, \phi' \),
Thus, by (6.34) and (6.35),
\[
E^t_0 \int_I^T \phi(r)dr - E^t_0 \left( \langle p(t), \xi \rangle_H + \frac{1}{2} \langle P(\tau)\xi, \xi \rangle_H \right) + o(|\tau - t|).
\]

Now, let us estimate the terms on the right-hand side of (6.33).
To this end, we first note that for any \( \phi, \phi' \in L^2_{\xi}(0, T; H) \), and \( \psi \in L^2_{\xi}(0, T; \mathbb{L}^0_2) \), it holds that
\[
E^t_0 \left( \int_I^T \phi(r)dr, \int_I^T \psi(r)dW(r) \right)_H \leq \left( E^t_0 \left( \int_I^T |\phi(r)|^2_H \right) \right)^{\frac{1}{2}} \left( E^t_0 \left( \int_I^T |\psi(r)|^2_H \right) \right)^{\frac{1}{2}} \leq \tau - t \leq \| \phi \|_{L^2_{\xi}} \leq \| \phi' \|_{L^2_{\xi}} \leq o(|\tau - t|), \quad \text{as} \quad \tau \downarrow t, \quad \forall \ t \in [0, T), \ \mathbb{P}\text{-a.s.}(6.34)
\]
and due to the full Lebesgue measure for integrable functions and the fact that \( t \mapsto F_t \) is continuous w.r.t \( t \), we have
\[
E^t_0 \left( \int_I^T \phi(r)dr, \int_I^T \psi(r)dW(r) \right)_H \leq \left( E^t_0 \left( \int_I^T |\phi(r)|^2_H \right) \right)^{\frac{1}{2}} \left( E^t_0 \left( \int_I^T |\psi(r)|^2_H \right) \right)^{\frac{1}{2}} \leq \tau - t \leq \| \phi \|_{L^2_{\xi}} \leq \| \phi' \|_{L^2_{\xi}} \leq o(|\tau - t|), \quad \text{as} \quad \tau \downarrow t, \quad \text{a.e.} \ t \in [0, T), \ \mathbb{P}\text{-a.s.}(6.35)
\]

Thus, by (6.34) and (6.35),
\[
E^t_0 \langle p(t), \xi \rangle_H = E^t_0 \left( \langle p(t), \xi \rangle_H \right) + o(|\tau - t|)
\]
\[
= E^t_0 \left\{ \langle p(t), -[S(\tau-t) - I]X(t) - \int_I^T S(\tau-r)a(r)dr - \int_I^T S(\tau-r)b(r)dW(r) \rangle_H \
+ \langle [S(\tau-t) - I]S(T-t)h_x(X(T)) - \int_I^T S(r-t)[a_x(r)^*p(r) + b_x(r)^*q(r) - \bar{f}_x] \rangle dr \
- \int_I^T [S(\tau-t) - I]S(\tau-r)\bar{a}_x(r)^*p(r) + \bar{b}_x(r)^*q(r) - \bar{f}_x \rangle dr \
+ \int_I^T S(r-t)q(r)dW(r) - \int_I^T [S(\tau-t) - I]S(\tau-r)q(r)dW(r), 
- [S(\tau-t) - I]X(t) - \int_I^T S(\tau-r)a(r)dr - \int_I^T S(\tau-r)b(r)dW(r) \rangle_H \right\}
\]
\[
= E^t_0 \left[- \langle A^*p(t), (\tau-t)X(t) \rangle_H - \langle p(t), \int_I^T S(\tau-r)a(r)dr \rangle_H \
- \int_I^T \langle S(r-t)q(r), S(\tau-r)b(r) \rangle_{\mathbb{L}^2_2} + o(|\tau - t|). \right]
\]

By the definition of \( \xi \), we have
\[
E^t_0 \langle P(\tau)\xi, \xi \rangle_H 
= E^t_0 \left\langle P(\tau) \left\{ [S(\tau-r) - I]X(t) - \int_I^T S(\tau-r)a(r)dr - \int_I^T S(\tau-r)b(r)dW(r) \right\}, 
[S(\tau-r) - I]X(t) - \int_I^T S(\tau-r)a(r)dr - \int_I^T S(\tau-r)b(r)dW(r) \right\}_H \right.
\]
\[
= 28
\]
It follows from (6.33), (6.36) and (6.41) that for any rational \( \tau > t \) and \( \omega = \omega_0 \),
\[
\mathbb{E}_{\omega_0}^t \int_t^\tau \langle P(\tau) \bar{b}(r), \bar{b}(t) \rangle_{L_2^2} dr + o(\tau - t) \tag{6.37}
\]
\[
= \mathbb{E}_{\omega_0}^t \int_t^\tau \langle P(\tau) \bar{b}(r), \bar{b}(r) - \bar{b}(t) \rangle_{L_2^2} dr + \mathbb{E}_{\omega_0}^t \int_t^\tau \langle P(\tau)(\bar{b}(r) - \bar{b}(t)), \bar{b}(t) \rangle_{L_2^2} dr
\]
\[+ \mathbb{E}_{\omega_0}^t \int_t^\tau \langle (P(\tau) - P(t)) \bar{b}(t), \bar{b}(t) \rangle_{L_2^2} dr + \mathbb{E}_{\omega_0}^t \int_t^\tau \langle P(t) \bar{b}(t), \bar{b}(t) \rangle_{L_2^2} dr + o(\tau - t).\]

Let us estimate the first three terms on the right-hand side of (6.37). Since \( P(\cdot)\bar{b}(r) \in D_E([r, T]; L^{4/3}(\Omega; L_2^0)) \), we see that
\[
\mathbb{E}_{\omega_0}^t \int_t^\tau \langle P(\tau) \bar{b}(r), \bar{b}(r) - \bar{b}(t) \rangle_{L_2^2} dr
\]
\[\leq |P(\cdot)\bar{b}(r)|_{D_E([t, T]; L^{4/3}(\Omega; L_2^0))} \int_t^\tau \left( \mathbb{E}_{\omega_0}^t |\bar{b}(r) - \bar{b}(t)|_{L_2^0}^{4/3} \right)^2 dr \tag{6.38}
\]
\[= o(|\tau - t|), \quad \text{as } \tau \downarrow t, \quad \text{a.e. } t \in [0, T).\]

Similarly,
\[
\mathbb{E}_{\omega_0}^t \int_t^\tau \langle P(\tau)(\bar{b}(r) - \bar{b}(t)), \bar{b}(t) \rangle_{L_2^2} dr
\]
\[\leq \int_t^\tau \left[ \mathbb{E}_{\omega_0}^t |P(\tau)(\bar{b}(r) - \bar{b}(t))|_{L_2^0}^{4/3} \right]^2 dr |\bar{b}(t)|_{L_{F_1}^4(\Omega; L_2^0)} \tag{6.39}
\]
\[= o(|\tau - t|), \quad \text{as } \tau \downarrow t, \quad \text{a.e. } t \in [0, T).\]

and
\[
\mathbb{E}_{\omega_0}^t \int_t^\tau \langle (P(\tau) - P(t)) \bar{b}(t), \bar{b}(t) \rangle_{L_2^2} dr
\]
\[= (\tau - t) \mathbb{E}_{\omega_0}^t \langle (P(\tau) - P(t)) \bar{b}(t), \bar{b}(t) \rangle_{L_2^0}
\]
\[\leq (\tau - t) \left[ \mathbb{E}_{\omega_0}^t |(P(\tau) - P(t)) \bar{b}(t)|_{L_2^0}^{4/3} \right] |\bar{b}(t)|_{L_{F_1}^4(\Omega; L_2^0)} \tag{6.40}
\]
\[\leq (\tau - t) |(P(\tau) - P(t)) \bar{b}(t)|_{L_{F_1}^{4/3}(\Omega; L_2^0)} |\bar{b}(t)|_{L_{F_1}^4(\Omega; L_2^0)}
\]
\[= o(|\tau - t|), \quad \text{as } \tau \downarrow t,
\]

where the last equality if due to the right continuity of \( P(\cdot)\xi \) in \( L_{F_1}^{4/3}(\Omega; H) \), \( \forall \xi \in L_{F_1}^4(\Omega; H) \). Combining (6.37)–(6.40), we see that
\[
\mathbb{E}_{\omega_0}^t \langle P(t) \bar{b}(t), \bar{b}(t) \rangle_{L_2^0} dr + o(\tau - t) \tag{6.41}
\]
\[= (\tau - t) \langle P(t) \bar{b}(t), \bar{b}(t) \rangle_{L_2^0} + o(\tau - t), \quad \text{as } \tau \downarrow t, \quad \text{a.e. } t \in [0, T).\]

It follows from (6.33), (6.36) and (6.41) that for any rational \( \tau > t \) and at \( \omega = \omega_0 \),
\[
V(\tau, \underline{X}(t)) - V(t, \underline{X}(t))
\]
\[\leq \mathbb{E}_{\omega_0}^t \left\{ \langle p(t), [S(\tau - t) - I] \underline{X}(t) \rangle_H + \langle p(t), \int_t^\tau S(\tau - r) \bar{a}(r) dr \rangle_H \right. \]
\[+ \int_t^\tau \langle S(r - t) q(r), S(\tau - r) \bar{b}(r) \rangle_{L_2^0} dr - \frac{1}{2} (\tau - t) \langle P(t) \bar{b}(t), \bar{b}(t) \rangle_{L_2^0} - \int_t^\tau \bar{f}(r) dr \bigg\} + o(\tau - t)
\]
\[\leq (\tau - t) \left[ \langle \langle A \underline{X}(t), p(r) \rangle \rangle + \mathcal{H}(t, \underline{X}(t), \bar{u}(t)) \right] + o(\tau - t),
\]
which completes the proof.
6.3 Examples

In this section, we present two examples which fulfill the assumptions in Theorem 6.1 and/or 6.2.

Let \( G \subset \mathbb{R}^n \) be a bounded domain with the smooth boundary \( \partial G \). Let \( H = L^2(G) \) and \( U \) be a bounded close subset of \( L^2(G) \). Consider the following stochastic parabolic equation:

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\frac{dy}{dt} = (\Delta y + \tilde{a}(t,y,u))dt + \tilde{b}(t,y,u)dW(t) & \text{in } (0,T] \times G, \\
y = 0 & \text{on } (0,T] \times \partial G, \\
y(0) = \eta & \text{in } G.
\end{array} \right.
\end{aligned}
\] (6.42)

where \( \eta \in L^2(G) \), \( u(\cdot) \in U[0,T] \), and \( \tilde{a} \) and \( \tilde{b} \) satisfy the following condition:

\( (B1) \) For \( \varphi = \tilde{a}, \tilde{b} \), suppose that \( \varphi(\cdot,\cdot,\cdot) : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) satisfies: i) For any \( (r,u) \in \mathbb{R} \times \mathbb{R} \), the functions \( \varphi(\cdot,r,u) : [0,T] \to \mathbb{R} \) is Lebesgue measurable; ii) For any \( (t,r) \in [0,T] \times \mathbb{R} \), the functions \( \varphi(t,\cdot,u) : \mathbb{R} \to \mathbb{R} \) is continuous; and iii) For all \( (t_1,t_2,u) \in [0,T] \times \mathbb{R} \times \mathbb{R} \),

\[
|\varphi(t,r_1,u) - \varphi(t,r_2,u)| \leq C|r_1 - r_2|,
\]

\[
|\varphi(t,0,u)| \leq C; \tag{6.43}
\]

iv) For all \( (t,u) \in [0,T] \times \mathbb{R} \), \( \varphi(\cdot,t,u) \) are \( C^2 \), and for any \( (r,u) \in \mathbb{R} \times \mathbb{R} \) and a.e. \( t \in [0,T] \),

\[
|\varphi_r(t,r,u)| + |\varphi_{rr}(t,r,u)| \leq C.
\]

Consider the following cost functional:

\[
J(\eta;u(\cdot)) = \mathbb{E}\left[ \int_0^T \int_G \tilde{f}(t,y(t),u(t))dxdt + \int_G \tilde{h}(y(T))dx \right], \tag{6.44}
\]

where \( \tilde{f} \) and \( \tilde{h} \) satisfy the following condition:

\( (B2) \) \( \tilde{f}(t,\cdot,u) \) and \( \tilde{h}(\cdot) \) are \( C^2 \), such that \( \tilde{f}_r(t,r,\cdot) \) and \( \tilde{f}_{rr}(t,r,\cdot) \) are continuous, and for any \( (r,u) \in \mathbb{R} \times \mathbb{R} \) and a.e. \( t \in [0,T] \),

\[
|\tilde{f}_r(t,r,u)| + |\tilde{h}_r(r)| + |\tilde{f}_{rr}(t,r,u)| + |\tilde{h}_{rr}(r)| \leq C.
\]

Under \( (B1) \) and \( (B2) \), it is easy to see that \( (S1)-(S3) \) hold. Then we know that all assumptions in Theorem 6.1 are fulfilled. By the regularity of backward stochastic parabolic equations (e.g., [7]), we know that \( A^p(t) \in L^2(G) \) for a.e. \( (t,\omega) \in (0,T) \times \Omega \). Hence, all assumptions in Theorem 6.2 are fulfilled.

Next, let \( H = H^1_0(G) \times L^2(G) \) and \( U \) be a bounded close subset of \( L^2(G) \). Consider the following stochastic hyperbolic equation

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\frac{dy}{dt} = (\Delta y + \tilde{a}(t,y,u))dt + \tilde{b}(t,y,u)dW(t) & \text{in } (0,T] \times G, \\
y = 0 & \text{on } (0,T] \times \partial G, \\
y(0) = \eta_1, \ y_t(0) = \eta_2 & \text{in } G.
\end{array} \right.
\end{aligned}
\] (6.45)

where \( (\eta_1,\eta_2) \in H^1_0(G) \times L^2(G) \), \( u(\cdot) \in U[0,T] \), and \( \tilde{a} \) and \( \tilde{b} \) satisfy \( (B1) \).

Consider the following cost functional:

\[
J(\eta_1,\eta_2;u(\cdot)) = \mathbb{E}\left[ \int_0^T \int_G \tilde{f}(t,y(t),u(t))dxdt + \int_G \tilde{h}(y(T))dx \right], \tag{6.46}
\]

where \( \tilde{f} \) and \( \tilde{h} \) satisfy \( (B2) \). Under \( (B1) \) and \( (B2) \), it is easy to see that \( (S1)-(S3) \) hold. Then we know that all assumptions in Theorem 6.1 are fulfilled.
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