Analytically derived limits on short-range fifth forces from quantum states of neutrons in the Earth’s gravitational field

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Recently, quantum states of ultra-cold neutrons in the Earth’s gravitational field have been observed for the first time. From the fact that they are consistent with Newtonian gravity on the 10%-level, analytical limits on \( \alpha \) and \( \lambda \) of short-range Yukawa-like additional interactions are derived between \( \lambda = 1 \) \( \mu \)m and 1 \( \text{mm} \). We arrive for \( \lambda \geq 10 \mu \text{m} \) at \( \alpha < 2 \cdot 10^{11} \) at 90\% confidence level. This translates into a limit \( g_s g_p / \hbar c < 2 \cdot 10^{-15} \) on the pseudo-scalar coupling of axions in the previously experimentally unaccessible astrophysical axion window.

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I. INTRODUCTION

A gravitational bound quantum system has been realized experimentally. In this experiment, ultra-cold neutrons (UCN) are confined in between a bottom mirror and the gravitational potential of the Earth\(^\[1,2,3\].\) The neutrons are slow enough that they are reflected from the mirror at all angles of incidence. Therefore, the mirror can be modeled by an infinite high potential step. The neutrons are found in discrete quantum states of the gravity potential. Between the UCN source and a UCN detector one places a quantum state absorber at a certain height above the mirror. No neutrons except those in sufficiently low quantum states as given by the absorber height can pass through the slit between the mirror and absorber, and higher, unwanted states are removed and scattered out of the experiment (see Fig. 1).

A side-effect of this experiment is its sensitivity to additional short ranged forces at length scales below \( 10 \mu \text{m} \)\(^\[4,5\]\), while all electromagnetic effects are extremely suppressed compared to gravity\(^\[4\]\). The quantum states probe Newtonian gravity between \( 10^{-9} \) and \( 10^{-3} \) m and the experiment places limits for gravity-like forces there. So far, significant limits on hypothetical forces from this experiment are mainly obtained from measurements of the ground state or from a fit to the data\(^\[4,5\]\). Other experimental limits on extra forces are derived from mechanical experiments and can be found, e.g., in\(^\[6,7,8,9,10\]\). In the light of recent theoretical developments in higher dimensional field theory\(^\[11,12,13,14,15\]\), gauge fields could mediate forces that are \( 10^6 \) to \( 10^{12} \) times stronger than gravity at sub-millimeter distances (depending on the size of the extra dimensions), exactly in the interesting range of this experiment and might give a signal in an improved setup. Recent theoretical developments support this original idea of strong forces with bulk gauge fields. Burgess et al.\(^\[16\]\) predict deviations from Newton’s law on the micron scale on the basis of supersymmetric large extra dimensions (SLED). The basic idea behind this proposal is to modify gravity at small distances in such a way as to explain the smallness of the observed cosmological constant. A radius \( R \) of 10 microns as well as the necessary interaction strength may turn out to be well-motivated. We presents limits for additional interactions using an elementary particle, the neutron. The limits are derived in regions of the parameter space, where these interactions can be treated perturbatively.

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II. ADDITIONAL SHORT RANGE FORCES

We begin with recalling the standard parametrization of a fifth force by means of a Yukawa potential describing the low energy limit of the exchange of massive particles, proceeding then to the action of a perturbatively weak Yukawa-like fifth force on the gravitational bound states in different regimes of its range $\lambda$. On the assumption that the form of the non-Newtonian potential is given by the Yukawa expression, for a source mass $m$ and distance $r$ the modified Newtonian potential $\phi(r)$ has the form

$$\phi(r) = -G_4 \frac{m}{r} (1 + \alpha \cdot e^{-r/\lambda}), \tag{1}$$

where $\lambda$ is the range over which the corresponding force acts and $\alpha$ is the strength normalized relative to Newtonian gravity. $G_4$ is the gravitational constant. The mass of this extended source will modify the Newtonian potential if strong non-Newtonian forces are present, and this can be seen if neutrons are present. For small distances $z$ from the mirror, say several micrometers, we consider the mirror as an infinite half-space with mass density $\rho$. By replacing the source mass $m$ by $\int \rho dV$, the Yukawa-modification of the potential $\phi(r)$ has the form

$$\Delta \phi(z, \lambda) = -2\pi \cdot \rho \alpha^2 G_4 \cdot e^{-z/\lambda}. \tag{2}$$

Thus, the effective gravitational potential close above the mirror (close means: heights $z \ll D$, $D$: diameter of the mirror) is given as:

$$\phi(z) = g \cdot z - 2\pi \cdot \frac{\alpha^2 \cdot G_4 \cdot \rho \cdot e^{-z/\lambda}}{\Delta \phi}. \tag{3}$$

where $\rho$ denotes the mass density of the mirror material (glass in our experiment).

The absorber has (besides its Fermi pseudopotential) in presence of a fifth force a Yukawa-like additional potential attached to it just like the bottom mirror. Then $\Delta \phi$ becomes

$$\Delta \phi(z) = -2\pi \cdot \alpha \cdot e^{-z/\lambda} \cdot G_4 \cdot \left(\rho_1 \cdot e^{-z/\lambda} + \rho_2 \cdot e^{-(h-z)/\lambda}\right). \tag{4}$$

where $h$ denotes as before the absorber height and $\rho_1, \rho_2$ the mass densities of the bottom mirror and the absorber, respectively. In some parts of this paper, we can ignore the absorber.

Within an unchanged Newtonian gravitational potential

$$\phi_0(z) = g \cdot z \tag{5}$$

the bound eigenstates of the UCN are given by Airy functions $Ai(z)$. This function behaves similar to a sine wave below $h_0^{(0)}$ and approaches zero exponentially above this classical turning point height. With

$$h_0^{(0)} = R \cdot \left(\frac{3\pi}{2} \cdot \left(n + \frac{3}{4}\right)^{2/3}\right) \tag{6}$$

one finds

$$\psi_{n,g}^{(0)}(z) = C_n \cdot Ai\left((z - h_0^{(0)})/R\right), \quad C_n = \frac{\tilde{C}_n}{\sqrt{R}} \tag{7}$$

$$\simeq C_n \cdot \frac{1}{2} \left(\frac{h_0^{(0)}}{R}\right)^{1/4} \cdot \frac{z}{R} + \mathcal{O}\left((z/R)^2\right) \tag{8}$$

Here, $h_0^{(0)}$ denotes the position of the last turning point of the bound state $\psi_{n,g}^{(0)}(z)$, which coincides with the turning point height of the classical motion of a particle with energy $E_n^{(0)} = mgh_0^{(0)}$. $h_0^{(0)}$ has been determined for the first two states to be

$$h_0^{(0)} \text{exp} = 12.2 \pm 0.7 \text{stat} \pm 1.8 \text{syst} \tag{9}$$

$$h_0^{(0)} \text{exp} = 21.6 \pm 0.7 \text{stat} \pm 2.2 \text{syst}. \tag{10}$$

With this analogy in mind, $h_0^{(0)}$ is considered as the height of the wave function. The ground state is opaque for neutrons as long as it sufficiently overlaps with the absorber above the mirror. This is not the case when the absorber position exceeds the height $h_0^{(0)}$ following the exponentially decaying tail of the bound state’s Airy function, and neutrons in that state are transmitted to the detector, see Fig. 1. In the WKB approximation the $\psi_{n,g}^{(0)}$ are:

$$\psi_{n,g}^{(0)}(z) \simeq C_n \cdot \frac{1}{2} \cdot \left(\frac{h_0^{(0)} - z}{R}\right)^{-1/4} \cdot \sin \left\{\frac{1}{R} \cdot \int_0^z du \sqrt{\frac{1}{R} \cdot h_0^{(0)} - u}\right\}. \tag{11}$$

III. OUR METHOD

We start from the observation that at macroscopic absorber heights of several microns, where light is easily transmitted, no neutrons pass through the gap between the mirror and the absorber. A neutron in state $n$ is only transmitted as long the absorber height is higher than $h_0^{(0)}$.

Taking additional gravity-like forces into account, first order perturbation theory predicts a shift of the energy eigenvalue of the ground state. The effect of the Yukawa term in eq. [3] in first order is

$$\Delta E_n^{(1)} = m \cdot \left|\psi_{n,g}^{(0)} \cdot \Delta \phi(z) \cdot n_{g}^{(0)}\right|. \tag{12}$$

This energy shift is accompanied by a shift of the turning point $h_0^{(0)}$ by some additional height $\delta h$, in this way changing the onset of the transmission in the experiment. Our measurement follows the Newtonian expectation.
The absorber consists of glass material, where the surface has an approximate gaussian roughness of 0.75 \( \mu m \). The height of the absorber has been calibrated with wire-spacing of known thickness, a mechanical comparator, and a long focus microscope. An absolute height calibration of better than 0.5 \( \mu m \) has been achieved. Next, we recall that the absorber forms a hard wall for the very slow neutrons. This results in a squeezing of the bound states (17) and a small energy shift, which is considered in the following way: The squeezing increases the bound state energy eigenvalue by an amount \( \Delta E_{\text{squeez}} = m \cdot g \cdot \Delta Sq \), where \( \Delta Sq \approx 0.8 \mu m \) denotes the corresponding shift of the turning point of the squeezed wave function. As a result, the turning point of the squeezed bound states in presence of a real absorber is given by \( h_n = h_n^{(0)} + \Delta Sq \). We now calculate the influence of a hypothetical fifth force on the turning points of the first two states (Eq[17]) and compare them with the experimental results given in Eq[20]. The actual neutron transmission of the \( n^{th} \) bound state is observed at a mean absorber height \( h_n + \Delta T \), \( \Delta T \) has been chosen to be 3\( \mu m \), since 3\( \mu m \) above the turning point, the slope of the transmission curve shows that state \( n \) is transmitted. Summarizing both effects, a bound state \( n \) transmits neutrons, which are clearly visible in the detector if the absorber is at a height \( h_n + \Delta T = h_n^{(0)} + \Delta T + \Delta Sq \).

In this paper, we derive limits on 5th forces, which are largely independent from a precise knowledge of the absorber height and unaffected by the features of the absorber. The limits are obtained from the fact that the ground state and the first excited state wave functions are differently affected by non-Newtonian forces. The fifth force would introduce the height shift \( \delta h \), so that we measure \( h = h_n^{(0)} + \Delta T + \Delta Sq + \delta h \). In particular, the difference of the turning point heights of the ground state and the first excited state

\[
\Delta h_n^{(0)} := h_0^{(0)} - h_1^{(0)} \approx 10.3 \mu m
\]

is consistent with the measurement \( \Delta h_{\text{exp}} = 9.4 \pm 1.2 \mu m \). This error includes in addition to the statistical uncertainty a 0.5\% calibration uncertainty \( \% \) and a 30\% uncertainty due to model dependence. (N.B.: Only the relative error of a height measurement enters.) Since this relative quantity is mostly insensitive to absolute offsets of the measurement process of \( h \), we will use it to derive limits on the presence of additional short-range fifth forces, i.e., we can exclude a fifth force induced shift

\[
\delta \Delta h > 1.64 \cdot \sigma_{\Delta h} = 2.0 \mu m
\]

at 90\% confidence level. In the case of Eq. (25), the choice of \( \Delta T \) and \( \Delta Sq \) enters, but our limits are largely independent from these quantities.

To apply this method it is essential to derive the connection between the perturbative correction to the energy eigenvalue of a given bound state, which is induced by the additional force, and a possible accompanying shift \( \delta h_n \) of the turning point of the \( n^{th} \) state’s wave function. The stationary Schrödinger equation for the bound states

\[
\frac{\partial^2 \psi_{n,g}(z)}{\partial z^2} = -\frac{2m}{\hbar^2} \cdot [E_n - m \cdot \phi(z)] \psi_{n,g}(z),
\]

implies a turning point condition given by

\[
\frac{\partial^2 \psi_{n,g}(z)}{\partial z^2} |_{z=h_n} = 0
\]

\[
= -\frac{2m}{\hbar^2} \cdot [E_n - m \cdot \phi(h_n)] \psi_{n,g}(h_n)
\]

\[
\Rightarrow E_n - m \cdot \phi(z) = 0 \text{ for } z = h_n. \quad (16)
\]

Write now for the height of the wave function under the presence of the additional force \( h_n = h_n^{(0)} + \delta h_n + \Delta Sq \), \( E_n = E_n^{(0)} + \Delta E_n^{(1)} + \Delta E_{\text{squeez}} \) and use the fact that \( E_n^{(0)} = mg \cdot h_n^{(0)} \). Then one arrives at

\[
\Delta E_n^{(1)} \bigg|_{h=h_n^{(0)}+\Delta T} -m \cdot g \cdot \delta h_n + \Delta \phi(h_n^{(0)} + \delta h + \Delta Sq) = 0 \quad (17)
\]

with \( \Delta \phi(z) \) given by eq. (3). Note, that the piece linear in \( \Delta Sq \) from the squeezing of the bound states between absorber and mirror cancels against the corresponding energy shift \( \Delta E_{\text{squeez}} \). It is immediately clear from here that concerning the situation with just a bottom mirror for \( \lambda < h_n^{(0)} \) one has \( \Delta \phi(h_n^{(0)}) \approx 0 \) and eq. (17) simplifies in this regime to

\[
\delta h_n = \frac{g}{\langle n_g(0) | \Delta \phi(z) | n_g(0) \rangle}. \quad (18)
\]

In the opposite case \( \lambda \gtrsim h_n^{(0)} \) one may linearize \( \Delta \phi(z) \) in \( z/\lambda \), which approximation then has to be used simultaneously in exploiting eq. (17) and eq. (12) to calculate \( \Delta E_n^{(1)} \).

For later use, let us note one further property of the above turning point condition. \( \Delta \phi(z) \) may contain a constant, position-independent part (for instance, the above linear approximation in the case \( \lambda \gtrsim h_n^{(0)} \) generically produces such a constant piece). Now, from considering the general behaviour of the exact solution it is clear, that changing the potential by an arbitrary constant must leave the whole bound state as well as its turning point unchanged though it does change the energy eigenvalue of the state. This fact is clearly contained in eq. (17): Imagine adding a constant \( \Delta \phi_{\text{const}} \) to \( \Delta \phi(z) \). Then its contribution to \( \Delta E_n^{(1)} \) is given by \( \Delta \phi_{\text{const}} \) and thus cancels out against the same term in \( \Delta \phi(z) \). Thus one may expand \( \Delta \phi(z) \) around any given convenient point \( z \) and drop the constant piece.

In either case we proceed then by extracting the turning point shifts \( \delta h_{0,1} \) from eq. (17) for the ground state and the first excited state, respectively. Forming the difference

\[
\delta \Delta h = \delta h_0 - \delta h_1 \quad (19)
\]

allows us then to extract limits on the strength \( \alpha \) of the additional fifth force as a function of its range \( \lambda \) by demanding eq. (14), the experimental constraint at 90\% C.L.
IV. BOTTOM MIRROR AND NO ABSORBER - CASE I: SMALL $\lambda \ll h_0^{(0)}$

Consider now the first case $\lambda \ll h_0^{(0)}$. Here, for a perfect absorber as said above, eq. (18) provides a good description of the shift of the ground state turning point (see the general discussion of Sect. III). Then the linear approximation of the bound states given in Sect. II suffices to calculate eq. (18) since $\Delta \phi$ then is confined to a region $\simeq \lambda \ll h_0^{(0)}$. Since it is the ground state which defines the non-penetration region and thus that feature of the measured neutron transmission function which responds most sensitively to energy or equivalently $h_n$-shifts, we evaluate eq. (18) for $n = 0$:

$$\delta h_n = \frac{1}{g} \left\langle n_g^{(0)} \right| \Delta \phi \left| n_g^{(0)} \right\rangle \approx -\tilde{C}_n^2 \cdot \frac{\pi \cdot \alpha \cdot \lambda^2 \cdot G_4 \cdot \rho}{2 \cdot g \cdot R^3} \cdot \sqrt{\frac{h_0^{(0)}}{R}} \cdot \int_0^\infty dz \cdot z^2 e^{-z/\lambda} \cdot \frac{\pi \cdot \alpha \cdot \lambda^5 \cdot G_4 \cdot \rho}{g \cdot R^3}$$

Forming $\delta \Delta h$ according to eq. (19) we demand the experimental constraint eq. (14). Therefore we arrive at an exclusion limit in $\alpha$-$\lambda$-space given by

$$|\alpha| \leq \frac{1}{\tilde{C}_n^2 \sqrt{\frac{h_0^{(0)}}{R} - \tilde{C}_n^2 \sqrt{\frac{h_0^{(0)}}{R}}}} \cdot \frac{g \cdot R^3}{\pi \cdot \lambda^5 \cdot G_4 \cdot \rho} \cdot \delta \Delta h \sim \lambda^{-5}.$$  \hspace{1cm} (20)

which behaves symmetrical for attractive and repulsive forces.

V. BOTTOM MIRROR AND NO ABSORBER - CASE II: LARGE $\lambda >> h_0^{(0)}$

In the second case $\lambda >> h_0^{(0)}$. Then $\Delta \phi \simeq \text{const.}$ over the whole range where $\psi_{n_g}^{(0)}$ is sizable. However, since constant pieces of the potential drop out from the turning point condition one has to apply it now in its precise form carefully expanding the fifth force potential to linear order in $z$ and $h_0$, respectively. This yields

$$\Delta \phi(z) = 2\pi \cdot \alpha \cdot \lambda^2 \cdot G_4 \cdot \rho \cdot \left(1 - \frac{z}{\lambda}\right) + O(z^2/\lambda^2).$$

Instead of $\Delta \phi$ we plug its contribution linear in $z$ into eq. (12) (recall that constant pieces of $\Delta \phi$ later will drop out of the turning point condition anyway)

$$\Delta E^{(1)}_n = m \left\langle n_g^{(0)} \right| \Delta \phi \left| n_g^{(0)} \right\rangle \approx -C_n^2 \cdot m \cdot 2\pi \cdot \alpha \cdot \lambda^2 \cdot G_4 \cdot \rho \cdot \int_0^\infty dz \cdot \left|\phi_{n_g}^{(0)}(z)\right|^2 \cdot \left(-z/\lambda\right) \approx m \cdot 2\pi \cdot \alpha \cdot \lambda \cdot G_4 \cdot \rho \cdot \langle z \rangle_n$$

where eq. (17) allows to compute the turning point shift $\delta h$ of the ground state given by

$$2\pi \cdot \alpha \cdot \lambda \cdot G_4 \cdot \rho \cdot \left(\langle z \rangle_n - \left<h_0^{(0)} + \Delta h_0\right>\right) - (g + 2\pi \cdot \alpha \cdot \lambda \cdot G_4 \cdot \rho) \cdot \delta h_n = 0.$$  \hspace{1cm} (21)

From this follows - via forming the quantity $\delta \Delta h$ again - a limit in the case of a repulsive interaction ($\alpha < 0$) given by

$$\alpha \geq -\frac{g}{2\pi \cdot G_4 \cdot \rho} \cdot \frac{1}{1 + \frac{\Delta h^{(0)} - \langle\langle z \rangle_0 - \langle z \rangle_1\rangle}{\lambda}} \sim \lambda^{-1}.$$ \hspace{1cm} (22)

For the attractive case ($\alpha > 0$) a smooth solution of eq. (17) exists for all $0 \geq \delta \Delta h > -(\langle\langle z \rangle_0 - \langle z \rangle_1\rangle)$ which yields a limit

$$\alpha \leq \frac{g}{2\pi \cdot G_4 \cdot \rho} \cdot \frac{1}{1 + \frac{\Delta h^{(0)} - \langle\langle z \rangle_0 - \langle z \rangle_1\rangle}{\lambda \Delta h}} - 1 \sim \lambda^{-1}.$$ \hspace{1cm} (23)

Here it is $\langle z \rangle_0 \approx 1.56 \cdot R \approx 9.15 \mu m$ and $\langle z \rangle_1 \approx 2.73 \cdot R \approx 16.0 \mu m$.

The transition between the two regimes of small and large $\lambda$ takes place just around $\lambda \approx 5 \ldots 7 \mu m$ as it can be seen by comparing eq.s (21) and (23).

VI. BOTTOM MIRROR AND REAL ABSORBER - LARGE $\lambda$

A real absorber now has (besides its Fermi pseudopotential) in presence of a fifth force a Yukawa-like additional potential attached to it just like the bottom mirror. Then $\Delta \phi$ becomes

$$\Delta \phi(z) = -2\pi \cdot \alpha \cdot \lambda^2 \cdot G_4 \cdot \left(\rho_1 e^{-z/\lambda} + \rho_2 e^{-(h-z)/\lambda}\right)$$

where $h$ denotes as before the absorber height and $\rho_1, \rho_2$ the mass densities of the bottom mirror and the absorber, respectively. For large $\lambda$ it makes sense to expand eq. (25) in $(z - h/2)/\lambda$ around $z = h/2$:

$$\Delta \phi(z) \approx -2\pi \cdot \alpha \cdot \lambda^2 \cdot G_4 \cdot e^{-h/2\lambda} \cdot \left(\rho_1 + \rho_2\right) - (\rho_1 - \rho_2) \cdot \frac{z - h/2}{\lambda} + \frac{(\rho_1 + \rho_2) \cdot (z - h/2)^2}{2 \cdot \lambda^2} + O\left(\frac{z^3}{\lambda^3}\right).$$

A. Special case $\rho_1 = \rho_2$

Now in our case both mirror and absorber are made from glass, so $\rho_1 = \rho_2 = \rho$. Then the direct linear term in
the former expansion vanishes - and thus all terms containing positive powers of \( \lambda \)! For large \( \lambda \) thus just the quadratic term remains since it is independent of \( \lambda \), all higher terms are suppressed by negative powers of \( \lambda \) and vanish in this limit. As an immediate consequence this means that for large \( \lambda \) the presence of an absorber with a density equal to that of the mirror leads to a \( \lambda \)-independent limit on the strength \( \alpha \) of an additional Yukawa-like interaction.

To determine this limit one computes the energy correction eq. (12) induced by the potential eq. (26) now to first order in perturbation theory

\[
\Delta E_n^{(1)} = -mn \cdot 2\pi \cdot G_4 \cdot \rho \cdot e^{-h/2\lambda} \cdot (z^2 - z \cdot h)_n ,
\]

where \( (z^2 - zh)_n = (z^2 - z^{(0)}h)_n - (z)_n (\Delta_T + \Delta_{\text{Sq}} + \delta h_n) \) for \( h = h^{(0)} + \delta h_n + \Delta_T + \Delta_{\text{Sq}} \) denotes the expectation value of the \( z \)-dependent part of the correction (for the \( z \)-independent part dropping out of the turning point condition eq. (17) see the general discussion above) to the potential with respect to the ground state \( \psi_{0;g}^{(0)} \). Then one inserts eq. (28) into the turning point condition eq. (17) carefully evaluating to linear order in \( \delta h_n \), uses the value of the turning point \( h_n = h_n^{(0)} + \delta h_n + \Delta_{\text{Sq}} \) in the experiment, and obtains

\[
\Delta E_n^{(1)} \bigg|_{h=h_n+\Delta_T} = \frac{-m \cdot g \cdot \delta h_n}{m \cdot 2\pi \cdot G_4 \cdot \rho \cdot e^{-h/2\lambda} \cdot \Delta_T (h_n^{(0)} + \delta h_n + \Delta_{\text{Sq}})} = 0 .
\]

Determining \( \delta h_n \) from these equations for \( n = 0, 1 \) allows us to form subsequently \( \delta \Delta h = \delta h_0 - \delta h_1 \). Demanding then \(-1.64\sigma_{\Delta h} < \delta \Delta h < 1.64\sigma_{\Delta h} \) yields then a limit on the interaction strength \( \alpha \) given by

\[
|\alpha| \leq g |\delta \Delta h| \cdot \left\{ \pi \left( e^{-\frac{h_0}{2\lambda}} \left( (z^2)_0 + \Delta_T (h_0^{(0)} + \Delta_{\text{Sq}}) - (z)_0 (h_0^{(0)} + \Delta_T + \Delta_{\text{Sq}}) \right) - e^{-\frac{h_1}{2\lambda}} \left( (z^2)_1 + \Delta_T (h_1^{(0)} + \Delta_{\text{Sq}}) - (z)_1 (h_1^{(0)} + \Delta_T + \Delta_{\text{Sq}}) \right) \right) + \delta \Delta h \left( -\frac{\rho G_4}{2} \right) \right\}^{-1} \approx 2.2 \mu m .
\]

This limit becomes essentially independent of \( \lambda \) for \( \lambda > h \approx h_1^{(0)} \). The result for the repulsive case contains a pole, which in the worst case, e.g. for \( \lambda = 500 \mu m \) is at

\[
\delta \Delta h \approx 2.2 \mu m ,
\]

and signals a breakdown of the perturbative approach. The pole in the attractive case above corresponds to the fact that for an attractive Yukawa potential at the mirror there is a \( |\delta h| \), above which the wave function begins to get sucked into the Yukawa potential. However, once the attractive potential gets strong enough for this to happen, the perturbative expansion around the original states ceases to be a good approximation, which explains such a pole: If one added the higher orders of perturbation theory to the above constraint, the above pole would occur at just that \( \delta h \), where the corresponding wave function becomes non-perturbatively deformed, which would then coincide with validity boundary of perturbation theory derived below in eq. (15).

Finally, for really large \( \lambda \) the constant term in the potential eq. (26) eventually becomes energetically dominant. Since this term in any case has to stay significantly smaller than the total kinetic energy \( E_{\text{UCN}} \) of the ultra-cold neutrons (velocity of \(~5 m/s\)) entering the waveguide (otherwise they cease to be transmitted entirely, which is not observed), the condition

\[
\Delta \phi(z) \approx -2\pi \cdot \alpha \cdot \lambda^2 \cdot G_4 \cdot \rho \lesssim 0.1 \cdot E_{\text{UCN}}
\]

limits in the case of an attractive interaction the validity of eq. (29) to \( \lambda \lesssim 10 mm \). For the case of a repulsive interaction this leads to a further decrease of the bound of
the strength $\alpha$ above $\lambda \sim 10 \, \text{mm}$ of the 5th force like
\[ |\alpha| \leq 3.8 \cdot 10^{12} \cdot \left(\frac{10 \, \text{mm}}{\lambda}\right)^2 . \tag{31} \]

**B. General case $\rho_1 \neq \rho_2$**

Let us now discuss what happens if $\rho_1 \neq \rho_2$. In this case the linear term in eq. (20) reappears. Since this term is $\sim \lambda$ it will dominate the quadratic term considered so far for large $\lambda$. Thus, if $\rho_1 - \rho_2$ is not tuned to be very small, the limit on the Yukawa interaction is again given through calculating $\delta \Delta h$ by the eq.s (23) and (21), however, with $\rho$ replaced by $\rho_1 - \rho_2$ and $\Delta \phi$ given by the piece linear in $z$ of eq. (20).

\[ |\alpha| \lesssim \frac{g}{2\pi \cdot G_4 \cdot (\rho_1 - \rho_2)} \cdot \frac{1}{\Delta \phi_{(0)} - ((z)_{0} - (z)_{1})} + \frac{h^{2/2\lambda}}{\lambda} \tag{32} \]

\[ \sim \lambda^{-1}, \lambda > \bar{h} . \]

Here we have defined $\bar{h} = (h_1 + h_0)/2 \approx 18.9 \, \mu m$ and neglected factors exp($\pm \Delta h / 2\lambda$) which are suppressed for $\lambda > \Delta h = - \Delta h_0/2 \approx 5.2 \, \mu m$. This treatment is perturbative.

We can cross-check this result by looking again at the potential eq. (29), and add its linear piece to the earth’s gravitational potential $\phi_0 = g \cdot z$

\[ \phi(z) = g \cdot z + 2\pi \cdot \alpha \cdot \lambda \cdot G_4 \cdot (\rho_1 - \rho_2) \cdot z . \]

Clearly, the case $\rho_1 - \rho_2$ amounts to a renormalization of $g$ given by

\[ g \rightarrow g' = g + \delta g \tag{33} \]

\[ \delta g = 2\pi \cdot \alpha \cdot \lambda \cdot G_4 \cdot (\rho_1 - \rho_2) . \]

Since the turning points of the bound state Airy functions are given by eq. (7), the turning point shifts in this situation, where $g$ renormalizes, using eq. (35) write as

\[ \delta h_n = \left[ \frac{3\pi}{2} \left( n + \frac{3}{4} \right) \right]^{2/3} \cdot \frac{\partial R}{\partial g} \cdot \delta g \\
= - \left[ \frac{3\pi}{2} \left( n + \frac{3}{4} \right) \right]^{2/3} \cdot R \cdot \frac{\delta g}{3g} \\
= -h_n^{(0)} \cdot \frac{2\pi \cdot \alpha \cdot \lambda \cdot G_4 \cdot (\rho_1 - \rho_2)}{3g} . \tag{34} \]

This relation then implies after forming $\delta \Delta h = \delta h_0 - \delta h_1$ through the experimental constraint eq. (11) $|\delta \Delta h| < 1.6 \mu m$ a symmetrical limit on the strength of the Yukawa interaction

\[ |\alpha| \leq \frac{3 \cdot g}{2\pi \cdot G_4 \cdot (\rho_1 - \rho_2)} \cdot \frac{\delta \Delta h}{\Delta h_{(0)}} \cdot \frac{1}{\lambda} \sim \lambda^{-1} \tag{35} \]

which is the same functional dependence as given in eq. (32) ($\Delta h_{(0)} - ((z)_{0} - (z)_{1}) \approx \Delta h_{(0)}/3$) for large $\lambda > \bar{h}$.

Let us shortly note now, that this limit for large $\lambda > 20 \, \mu m$ can be relatively easily converted into bounds of the strength of the matter couplings of axions. Axion interactions with a range within $20 \, \mu m < \lambda < 200 \, mm$ (corresponding to axion masses $10^{-6} \, eV < m_a < 10^{-2} \, eV$), the ‘axion window’, are still allowed by otherwise stringent constraints posed by cosmological data (see e.g. [21, 22]). They lead to a potential which is proportional to the 5th force potential eq. (2), however, the axion-induced potential changes sign with the direction of the neutron spin polarization relative to the mirror. Thus, the relevant limit is again given by eq. (35) with, however, one small but significant change induced by this pseudoscalar nature of the axion interaction, namely, that $\rho_2 \rightarrow -\rho_2$ (see [23]). This, in turn, implies that the bound on the scalar-pseudoscalar axion interaction could be derived using just eq. (34), however, with $\rho_2$ replaced as $\rho_2 \rightarrow -\rho_2$ (or the perturbative limit eq. (32), with again the replacement $\rho_2 \rightarrow -\rho_2$).

**C. $\rho_2 \rightarrow -\rho_2$ - axion limits in the astrophysical axion window**

To make the last statement more precise, note that an axion would feel a CP-violating spin-dependent interaction in presence of matter given by [23]

\[ V(r) = h g_s g_p \frac{\mathbf{\sigma} \cdot \mathbf{n}}{8\pi mc} \left( \frac{1}{\lambda r} + \frac{1}{r^2} \right) e^{-r/\lambda} . \tag{36} \]

Here, $\mathbf{\sigma}$ denotes the neutron spin and $\mathbf{n}$ is a unit vector presumably related to the geometry of the macroscopic matter configuration. Integrating this potential over the geometry of our mirror-absorber-system gives a Yukawa-potential contribution of the form [24]

\[ \Delta \phi(z) = -\alpha_a \cdot \frac{h_2 \rho_1 \lambda}{8m^3} e^{-z/\lambda} + \alpha_a \cdot \frac{h_2 \rho_2 \lambda}{8m^3} e^{-(h-z)/\lambda} \]

\[ = -2\pi \alpha_{\text{eff}} \cdot \lambda^2 \cdot G_4 \cdot (\rho_1 e^{-z/\lambda} - \rho_2 e^{-(h-z)/\lambda}) , \tag{37} \]

where we have used that

\[ \alpha_{\text{eff}} := \alpha_a \cdot \frac{h_2}{16\pi G_4 \cdot m^3} \cdot \lambda^{-1} , \alpha_a := \frac{g \cdot g_p}{h c} . \tag{38} \]

Note, how the switch in the sign of the two terms in eq. (38) arises from the fact that $\mathbf{\sigma} \cdot \mathbf{n}/|\mathbf{\sigma} \cdot \mathbf{n}| = +1$ for $\mathbf{n}$ the unit normal on the mirror but $\mathbf{\sigma} \cdot \mathbf{n}/|\mathbf{\sigma} \cdot \mathbf{n}| = -1$ for $\mathbf{n}$ the unit normal on the absorber. Plugging now eq. (38) into eq. (32) we arrive at a limit for the dimensionless axion coupling strength $\alpha_a$ given by

\[ |\alpha_a| \lesssim \frac{4 \cdot 3^2 \cdot g}{h^2 \rho} \cdot \frac{1}{\Delta h_{(0)} - ((z)_{0} - (z)_{1})} + 1 \cdot e^{h/2\lambda} \tag{39} \]

where we used that $\rho_1 = \rho_2 = \rho$ in our case. For $\lambda \geq \bar{h}$ this leads at 90% confidence level to a $\lambda$-independent upper limit on the axion interaction strength

\[ \frac{g \cdot g_p}{h c} \lesssim 2 \cdot 10^{-15} . \tag{40} \]
VII. BOTTOM MIRROR AND REAL ABSORBER - SMALL $\lambda$

This situation is similar to that of Sect. VI, except for two crucial differences: Firstly, due to $\lambda \ll \hbar$ we have to use full potential of eq. (25) instead of its expansion in eq. (26). Secondly, the presence of the absorber at $h = h_n + \Delta h$, the turning point, yields a relatively large contribution to $\Delta \phi(h_n)$ since there the exponential factor in the Yukawa potential is $e^{(h_n-h)/\lambda} = e^{-\Delta h/\lambda}$. This inserted into eq. (17) and using eq. (21) to compute $\Delta E_n^{(1)}$ yields an equation for $\Delta h$ given by

$$\Delta E_n^{(1)} - m \cdot g \cdot \delta h_n - m \cdot \Delta \phi_{Abs.}(h_n) = 0$$

with $\Delta E_n^{(1)} = -\tilde{C}^2 \sqrt{h_n} / R \cdot \pi \cdot m \cdot G_4 \cdot \lambda^5 / R^2$.

Here it has been made use of the exponentially decaying tail of a bound state Airy function for $z > h_n$ that suppresses the contribution of the absorber attached Yukawa potential to $\Delta E_n^{(1)}$. Therefore we neglect the absorber for the calculation of the energy shift. Calculating for the first two states $\delta \Delta h = \delta h_0 - \delta h_1$ and reshuffling of this equation to extract $\alpha$ then writes as a limit symmetrical in its sign which reads exactly the same as the simple one of eq. (21) (1):

$$|\alpha| \leq \frac{1}{\tilde{C}^2 \sqrt{h_n} / R - \tilde{C}^2 \sqrt{h_{00} / R}} \cdot \frac{g \cdot R^3}{\pi \cdot \lambda^5 \cdot G_4 \cdot \rho} \Delta T$$

$$\sim \lambda^{-5} \text{ for small } \lambda, \lambda \lesssim 5 \mu m.$$  

The fact that the limit is symmetrical rests on neglecting the absorber in the calculation of the energy shift. Were this taken into account it would render the limit asymmetrical. Thus, this limit is shown for the attractive case in Fig. 2 (thick red long-dashed line) for $\lambda \geq 1.2 \mu m$. For $\lambda < 1.2 \mu m$ this limit violates the general perturbativity bound discussed in the next Section. As we expect the limit - as before - for the repulsive case to be weaker than for the attractive case, it would be completely outside the general perturbativity bound for $\lambda < h$, which is why we do not display it.

\[
\Delta h_{0}^{(2)} = \frac{1}{m_n \cdot g} \sum_{m > 0} \left| \langle m_n | m_n \cdot \Delta \phi | (0,0) \rangle \right|^2 \]

\[
\simeq - \sum_{m > 0} C_m^2 C_m \cdot \frac{\pi^2 \cdot \alpha^2 \cdot \lambda^3 \cdot G_4^2 \cdot \rho^2}{2 \cdot g^2 \cdot R^3 \cdot \sqrt{h_0 / h_0} \cdot \lambda^5} \cdot \left[ \int_0^\infty dz \cdot \sin(\phi_m(z)) \sin(\phi_m(z)) \cdot e^{-z/\lambda} \right]^2
\]

with: $C_m \approx e^{\kappa_0 / \sqrt{R}} \cdot m^{-1/3}$, $\kappa_0 \approx 0.71$ and $\phi_m(z) = \frac{1}{R} \int_0^z du \sqrt{1 / R \left[ h_{m}^{00} - u \right]}$ (use eq. (11) around $z = 0$).

\[
\simeq 48^{1/3} \frac{2 \cdot \pi^2 \cdot \alpha^2 \cdot \lambda^{10} \cdot G_4^2 \cdot \rho^2}{g^2 \cdot R^7} \sum_{m > 0} \frac{1}{\left[ (m + 3/4)^2/3 \cdot \lambda^2 + 1 \right]^4} \cdot \frac{(1 + 3/4m)^{1/3}}{(m + 3/4)^{2/3} - (3/4)^{2/3}}
\]

\[
\equiv: \sigma(\lambda) \approx \zeta(3) : \text{convergent}
\]

VIII. VALIDITY OF PERTURBATION THEORY

There remains now the question of the range of validity of perturbation theory. The limit of eq. (21) for small $\lambda$ in principle might lead to values of $\alpha$ and $\lambda$ where $\Delta \phi < E_0 / m$ does not hold any more at least for $z = 0$. To estimate the validity of perturbation theory we will resort to comparing the 2nd order perturbation theory to the 1st order (see e.g. [22]). Then $\Delta E_0^{(2)} < \Delta E_0^{(1)}$ will provide us with a validity limit of perturbation theory. The 2nd order perturbation theory is given as seen in eq. (12). The sum $\sigma(\lambda)$ is convergent and can be evaluated numerically via integral approximations. This, in turn, then yields the validity range of perturbation theory for small and large $\lambda$, respectively, as:

$$\alpha \leq \begin{cases} \frac{3^{2/3} \cdot \pi^{4/3} \cdot g \cdot R^4}{4 \cdot \pi^2 \cdot 48^{1/3} \cdot G_4 \cdot \rho} \cdot \frac{1}{\sigma(\lambda)} & \approx 1 / \lambda^4, \lambda \text{ small} \\ \frac{g \cdot R^4}{\pi \cdot 48^{1/3} \cdot G_4 \cdot \rho} \cdot \frac{1}{\sigma(\lambda)} & \approx 5 \cdot 10^{13}, \lambda \gtrsim 10 \mu m \end{cases}\]

which is plotted in Fig. 2 (green dash-dotted line).

The matrix elements above have been calculated using the WKB approximation for the states. It can be shown, however, that using the exact Airy functions for the bound
states in numerically calculating the matrix elements produces results, that agree with the ones derived from the above approximate matrix elements to within 10% on the whole range of $\lambda$. One should note that we have been using here the Yukawa potential $\Delta \phi$ including its constant pieces to yield the full contributions to $\Delta E_0^{(2)}$ and $\Delta E_0^{(1)}$ (which for $\Delta E_0^{(1)}$ at large $\lambda$ leads to a behaviour of $\Delta E_0^{(1)} \sim \lambda^{-2}$ instead of $\lambda^{-1}$, see eq. (22) and the one before).

IX. CONCLUSION

The recent observation of quantum states of ultra-cold neutrons in the Earth's gravitational field allows one to derive bounds on the strength and range of an additional force from the experimental fact, that the crucial measured parameters of the ground state and the first excited state, their vertical extensions (which in turn are related to their energy eigenvalues), have been determined to be consistent with Newtonian gravity to within a differential positioning uncertainty of $\delta \Delta h = 2.0 \, \text{um}$ at 90% confidence level. Such an analysis is interesting from the point of view of systematic errors. Since the absence of electric charge and the weakness of its magnetic moment extremely suppresses the electromagnetic false effects a neutron is exposed to, bounds on additional force derived from neutron experiments can be given in a systematically quite clean way.

The analytical bounds derived here and shown in Fig. 2 form a consistency check for numerical analyses like the one of [1]. Following from the turning point shift criterion developed here, our limits are valid for $\lambda \gtrsim 1 \, \text{um}$ because of breakdown of the perturbation expansion for Yukawa-like forces of smaller range. Within a range of $\lambda = 3 \ldots 10 \, \text{um}$ we observe a change in the negative power of the $\lambda$-dependency of the limit from -2 towards -1. This invokes the interesting fact that the expectation of a turning point shift $\delta h \sim \Delta E$ holding for small $\lambda$ fails for longer ranges of the interaction. For even larger $\lambda$ the presence of the absorber leads to a $\lambda$-independent upper limit on $\alpha$ of about $2 \cdot 10^{11}$ in the attractive case for $\lambda \gtrsim 10 \, \text{um}$, which for this case above $\lambda \sim 10 \, \text{mm}$ further decreases like $\lambda^{-2}$.

Furthermore, note that the bound at large $\lambda > 5 \, \text{um}$ translates into a bound on the strength of CP-violating pseudoscalar couplings of the axion within the (previously experimentally unaccessible) astrophysical axion window which is $g_{p \phi} / \hbar c < 2 \cdot 10^{-15}$ for $5 \, \text{um} < \lambda < 500 \, \text{um}$.

The analytical limit on $\alpha$ and $\lambda$ for very small $\lambda < 1 \, \text{um}$ given by the analysis of [5] is beyond the scope of our analysis due to its non-perturbative nature. Other experimental limits on extra forces are derived from mechanical experiments and can be found, e.g., in [4, 6, 7, 8, 9, 10]. Those limits are derived from Casimir-force measurements or mechanical pendulum experiments. They are significantly better in numbers than the one derived here, however, one should stress the completely different nature of possible systematical effects present in these micro-mechanical experiments compared to the systematics at work in our

![Image](image_url)

FIG. 2: Red thick solid: Limit for large $\lambda$ with real absorber, attractive case eqs. (29) and (30). Red thick long-dash: Limit for small $\lambda$ in presence of a real absorber, attractive case eq. (26). Red thin wide-dash: Limit for large $\lambda$ with real absorber, repulsive case (eq. 29). Black solid: The 'no-new-state' limit of ref. [5]. Green dash-dot: Validity boundary of perturbation theory, eq. (33). Blue vertical dash: PVLAS signal [26]. We display this, as an axion interaction would generate a Yukawa-like interaction with $\alpha \sim g_p g_\phi$ [23], and the photon-axion coupling (if PVLAS sees an axion and not a scalar) is - up to a loop suppression - roughly of the same order as the direct axion-nucleon coupling we measure.

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