CUTTING ARCS FOR TORUS LINKS

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Abstract. Among all torus links, we characterize those arising as links of simple surface singularities, i.e. $A_n$, $D_4$, $E_6$ and $E_8$, by the property that their fibre surfaces admit only a finite number of cutting arcs that preserve fibredness.

1. Introduction

A fibred link is a link $L \subset S^3$ such that $S^3 \setminus L$ fibers over the circle, and where each fibre is the interior of a Seifert surface $S$ for $L$ in $S^3$. Cutting $S$ along a properly embedded interval $\alpha$ (an arc for short) results in another Seifert surface $S'$ for another link $\partial S' = L'$. If $L'$ is again a fibred link with fibre $S'$, we say that $\alpha$ preserves fibredness. For example, $\alpha$ could be the spanning arc of a plumbed Hopf band, and cutting along $\alpha$ amounts to deplumbing that Hopf band. In [BIRS], Buck et al. give a simple criterion for when an arc preserves fibredness in terms of the monodromy $\varphi: S \to S$. As a Corollary, they prove that each of the torus links of type $T(2, n)$ admits only a finite number of such arcs up to isotopy. It turns out that among torus links, this is an exception:

Theorem 1. Let $n, m \geq 4$ or $n = 3, m \geq 6$. Then the fibre surface $S$ of the torus link $T(n, m)$ contains infinitely many homologically distinct cutting arcs preserving fibredness.

In fact, for all torus links except $T(2, n)$, $T(3, 3)$, $T(3, 4)$ and $T(3, 5)$, an infinite family of homologically distinct cutting arcs preserving fibredness will be given in Section [3]. The mentioned exceptions happen to be exactly those torus links that are also simple surface singularities (namely, in the classification: $A_n$, $D_4$, $E_6$, $E_8$, see [Ar]). In addition, we show that every (prime) positive braid link with pseudo-Anosov monodromy admits infinitely many non-isotopic arcs preserving fibredness. This suggests the following question: Is it true that among all (non-split prime) positive braid links, the simple surface singularities are exactly those that admit just a finite number of fibredness preserving arcs up to isotopy?
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2. Right-veering surface diffeomorphisms and cutting arcs that preserve fibredness

In the sequel we would like to make statements on the relative position of two arcs $\alpha, \beta$ in a surface $S$ with boundary (that is, $\alpha, \beta$ are embedded intervals with endpoints on the boundary of $S$ that are nowhere tangent to $\partial S$). The following definition will simplify matters.

Definition 1. Let $S$ be an oriented surface with boundary and let $\alpha, \beta \subset S$ be two arcs. A property $P(\alpha, \beta)$ is said to hold after minimising isotopies on $\alpha$ and $\beta$, if $P(\tilde{\alpha}, \tilde{\beta})$ holds, where $\tilde{\alpha}$ and $\tilde{\beta}$ are obtained from $\alpha$, $\beta$ by two isotopies (fixed at the endpoints) that minimise the geometric number of intersections between the two arcs.

The remainder of this section will recall the fact that every positive braid link (that is, the closure of a braid word consisting only of the positive generators of the braid group, without their inverses) is fibred and has so-called right-veering monodromy (see below for a definition). The torus links $T(n, m)$ provide examples, since they can be viewed as the closures of the positive braids $(\sigma_1 \cdots \sigma_{n-1})^m$, where the $\sigma_i$ denote the (positive) standard generators of the braid group.

Definition 2 (see [HKM], Definition 2.1). Let $S$ be an oriented surface with boundary and $\phi : S \to S$ a self-diffeomorphism that restricts to the identity on $\partial S$. Then, $\phi$ is called right-veering if for every arc $\alpha : [0, 1] \to S$, the vectors $(\alpha'(0), (\phi \circ \alpha)'(0))$ form an oriented basis after minimising isotopies on $\alpha$ and $\phi \circ \alpha$. This means basically, that arcs starting at a boundary point of $S$ get mapped “to the right” by $\phi$.

It is known that every positive braid can be obtained as an iterated plumbing of positive Hopf bands (see [St]). Since a Hopf band is a fibre and plumbing preserves fibredness, every positive braid link is fibred. Moreover the monodromy is a product of positive Dehn twists, since the monodromy of a (positive) Hopf band is a (positive) Dehn twist and the monodromy of a plumbing is the composition of the monodromies of the plumbed surfaces (see [Ga]). A product of positive Dehn twists is right-veering [HKM]. So we conclude that every positive braid link is fibred with right-veering monodromy. Together with a theorem by Buck et al., this property implies the following simple geometric criterion for when an arc preserves fibredness.
**Theorem 2** (compare Theorem 1 in [BIRS]). Let $L$ be a fibred link with fibre surface $S$ and right-veering monodromy $\varphi : S \to S$. Then, a cutting arc $\alpha$ preserves fibredness if and only if $\alpha \cap \varphi(\alpha) = \partial \alpha$ after minimising isotopies on $\alpha$ and $\varphi(\alpha)$.

**Proof.** This is a special case of Theorem 1 in [BIRS], saying that the arc $\alpha$ preserves fibredness if and only if $\alpha$ is clean and alternating or once unclean and non-alternating (see Figure 1), without the assumption on $\varphi$ to be right-veering. But for a right-veering map, every arc is alternating, by definition. Finally, $\alpha$ is clean if and only if $\alpha \cap \varphi(\alpha) = \partial \alpha$ after minimising isotopies on $\alpha$ and $\varphi(\alpha)$.  

![Figure 1](Adapted from [BIRS])

3. **Monodromy of torus links and infinite families of cutting arcs**

The fibre surface $S$ of the torus link $T(n,m)$ can be constructed as thickening of a complete bipartite graph on $n$ and $m$ vertices in the following way, as in Figure 2:

![Figure 2](The complete bipartite graph on 2 and 3 vertices and blackboard framed thickening.)

Arrange $n$ collinear points $a_1, \ldots, a_n$ (in this order) in a plane and, similarly, another $m$ points $b_1, \ldots, b_m$ along a line parallel to the $a_i$. Connect $a_i$ and $b_j$ by a straight segment $k_{ij}$, for every $i \in \{1, \ldots, n\}$,
\( j \in \{1, \ldots, m\} \). Avoid intersections between the segments by letting \( k_{ij} \) pass slightly under \( k_{pq} \) if \( i > p \) and \( j < q \) (in a slight thickening of the plane containing the points \( a_i \) and \( b_j \)). Use the blackboard-framing to thicken \( a_i, b_j, k_{ij} \) to disks \( A_i, B_j \) and bands \( K_{ij} \). Choose the thickness of the bands \( K_{ij} \) so that they do not intersect outside the disks \( A_i \) and \( B_j \). It can be seen that \( S := \bigcup_i A_i \cup \bigcup_j B_j \cup \bigcup_{i,j} K_{ij} \subset \mathbb{R}^3 \subset S^3 \) is isotopic to the minimal Seifert surface of \( T(n, m) \) in \( S^3 \) (compare [Ba]). In addition, the monodromy \( \varphi : S \to S \) is given as a so-called \( \text{tête-à-tête twist} \) with twist length two along the above graph, a notion invented by A’Campo and further developed by Graf in his thesis [Gr]. This means that \( \varphi \) leaves the graph invariant and, when we cut \( S \) open along the graph, \( \varphi \) descends to certain twist-maps of the resulting annuli. More precisely, each of these annuli has one component of \( \partial S \) as one boundary circle and a cycle consisting of edges of the graph as the other. \( \varphi \) fixes \( \partial S \) pointwise and rotates the edge-cycles two edges to the right with respect to the orientation of \( S \). Using this description of \( \varphi \) as a \( \text{tête-à-tête twist} \), it is possible to see that on the graph, \( \varphi \) acts as follows: \( \varphi(a_i) = a_{i-1}, \varphi(b_j) = b_{j+1}, \varphi(k_{ij}) = k_{i-1,j+1} \), where the indices \( i, j \) are to be taken modulo \( n, m \) respectively. After an isotopy (fixing the boundary of \( S \)), we may assume that \( \varphi \) is periodic on a neighbourhood of the graph. It is thus easy to understand the effect of \( \varphi \) on an arc \( \alpha \), up to isotopy, because a subarc of \( \alpha \) that travels near \( k_{ij} \) will be mapped to a subarc of \( \varphi(\alpha) \) that travels near \( k_{i-1,j+1} \). The edges \( k_{ij} \) induce a decomposition of \( \partial A_i \) into circular arcs lying between points of the form \( k_{ij} \cap \partial A_i \) (and the same for \( B_j \)). If \( n, m \geq 3 \), it is hence meaningful to speak of points on \( \partial A_i \) between \( k_{ij} \) and \( k_{i,j+1} \).

**Theorem 1.** Let \( n, m \geq 4 \) or \( n = 3, m \geq 6 \). Then the fibre surface \( S \) of the torus link \( T(n, m) \) contains infinitely many homologically distinct cutting arcs preserving fibredness.

**Proof.** For \( n, m \geq 4 \) consider the following arcs in \( S \), using the notation from above (compare Figure 3):

- Let \( \gamma_1 \) be a straight segment starting at a point of \( \partial A_n \) between \( k_{n1} \) and \( k_{nm} \) and ending at the vertex \( a_n \).
- Let \( \gamma_2 \) start at \( a_n \), follow the edges \( k_{n,m-1} \) and \( k_{n-2,m-1} \), thus ending at \( a_{n-2} \).
- \( \gamma_3 \) starts at \( a_{n-2} \), runs along \( k_{n-2,1}, k_{n1}, k_{m-1}, k_{n-2,m-1} \) and ends again at \( a_{n-2} \).
- \( \gamma_4 \) is a straight segment from \( a_{n-2} \) to a point of \( \partial A_{n-2} \) between \( k_{n-2,1} \) and \( k_{n-2,m} \).
From $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ we can build an infinite family $(\alpha_r)_{r \in \mathbb{N}}$ of arcs in $S$, taking $\alpha_r = \gamma_1 * \gamma_2 * \gamma_3 * \ldots * \gamma_3 * \gamma_4$. Here, $*$ denotes concatenation of paths. Replacing the $r$ consecutive copies of $\gamma_3$ by $r$ parallel copies, the $\alpha_r$ can be thought of as embedded arcs. It is now easy to check that $\alpha_r$ and its image $\varphi \circ \alpha_r$ under the monodromy $\varphi$ have only their endpoints in common. Using Theorem 2 it follows that each $\alpha_r$ preserves fibredness. Finally, the $\alpha_r$ are homologically pairwise distinct. This can be seen in the following way: Let $[c] \in H_1(S, \mathbb{Z})$ be the cycle represented by a simple closed curve $c$ whose image is $k_{nm} \cup k_{n-1,m} \cup k_{n-1,m-1} \cup k_{n,m-1}$. After an isotopy, $c$ will intersect $\alpha_r$ transversely in $r+1$ points. Now, the linear form on $H_1(S, \partial S, \mathbb{Z})$ that sends $\alpha$ to $i(c, \alpha)$, the number of intersections with $c$ (counted with signs), defines an element $c^*$ of $H^1(S, \partial S, \mathbb{Z})$ such that $c^*(\alpha_r) = r + 1$, hence the claim.

If $n = 3, m \geq 6$, take the following arcs (compare Figure 4):

- $\gamma_1$ is a straight segment from a point of $\partial A_3$ between $k_{31}$ and $k_{3m}$ to $a_3$.
- $\gamma_2$ starts at $a_3$, follows the edges $k_{3,m-1}$ and $k_{2,m-1}$, thus ending at $a_2$.
- $\gamma_3$ starts at $a_2$, follows $k_{23}$, $k_{13}$, $k_{11}$, $k_{31}$, $k_{3,m-1}$ and $k_{2,m-1}$, ending at $a_2$.
- $\gamma_4$ is a straight segment from $a_2$ to a point of $\partial A_2$ between $k_{22}$ and $k_{23}$.
Figure 4. The arc $\alpha_1$ (solid line) and its image under the monodromy (dotted line) for a $T(3, m)$ torus link, $m \geq 6$. Again, the two arcs do not intersect.

As above, we get a family $(\alpha_r)_{r \in \mathbb{N}}$ of homologically distinct arcs preserving fibredness, where $\alpha_r = \gamma_1 * \gamma_2 * \gamma_3 * \ldots * \gamma_3 * \gamma_4$, using the curve with image $k_{3m} \cup k_{1m} \cup k_{1,m-2} \cup k_{3,m-2}$ to distinguish the $\alpha_r$.

4. The exceptional cases

In [BIRS, Corollary 2, p.19], it is shown that $T(2, n)$ admits only finitely many arcs preserving fibredness (up to isotopy). More precisely, they show that every clean arc is isotopic (free on the boundary) to an arc that is contained in one of the disks $A_1, A_2$. Apart from this infinite family of torus links, there are only three more torus links with just a finite number of arcs that preserve fibredness:

Proposition 1. The torus links $T(3, 3)$, $T(3, 4)$ and $T(3, 5)$ admit, up to isotopy (free on the boundary), only a finite number of cutting arcs that preserve fibredness.

The proof of Proposition 1 will be given in the appendix, for it is rather technical. Nevertheless, the idea is very simple: Let $S$ be the fibre surface of any torus link $T(n, m)$, given as thickening of a complete bipartite graph on $n + m$ vertices, as described in Section 3. An arc $\alpha \subset S$ is determined up to isotopy by its endpoints and by the sequence of bands $K_{ij}$ it passes through. Now start listing all possible such sequences that yield clean arcs, for increasing length of the sequence.
In order to prove finiteness of this list, we use three Lemmas, also given in the appendix. The intuitive meaning of Lemma 1 and Lemma 2 can be phrased as follows: If \( \alpha \) and \( \varphi(\alpha) \) intersect and this intersection seemingly cannot be removed by an isotopy, then \( \alpha \) is indeed unclean. Lemma 3 asserts that a clean arc cannot stay in the complement of the graph for a distance of more than two consecutive bands. This is made precise in the appendix, using a notion of arcs in normal position (cf. Definition 3). Along with this case-by-case analysis, one can find all possible fibred links obtained from \( T(3, 3) \), \( T(3, 4) \) and \( T(3, 5) \) by cutting along an arc:

| From       | one obtains by cutting along a clean arc |
|------------|----------------------------------------|
| \( T(2, n) \) | \( T(2, n - 1) \), \( T(2, m_1)\#T(2, m_2) \) for \( m_1 + m_2 = n \) |
| \( T(3, 3) \) | \( T(2, 4) \), \( (T(2, 2)\#T(2, 2)\#T(2, 2))^{*1} \) |
| \( T(3, 4) \) | \( D_5 \), \( T(2, 6) \), \( T(2, 5)\#T(2, 2) \), \( T(2, 3)\#T(2, 3)\#T(2, 2) \), \( (T(2, 3)\#T(2, 2)\#T(2, 2))^{*2} \) |
| \( T(3, 5) \) | \( D_7 \), \( E_7 \), \( T(2, 8) \), \( (D_5\#T(2, 3))^{*3} \), \( (T(2, 5)\#T(2, 2)\#T(2, 2))^{*4} \) |

\( K_1\#K_2 \) denotes the connected sum of \( K_1 \) and \( K_2 \).

\( D_n \) denotes the closure of the braid \( \sigma_1^{-2}\sigma_2\sigma_1^2\sigma_2 \), \( n \geq 3 \).
\( E_7 \) denotes the closure of the braid \( \sigma_1^4\sigma_2\sigma_1^3\sigma_2 \).

\(^{*1}\) chain of four successive unknots.
\(^{*2}\) both components of the Hopf link in the middle are summed to one trefoil knot each.
\(^{*3}\) both possible sums appear (trefoil summed with the unknot component of \( D_5 \) as well as trefoil summed with the trefoil component of \( D_5 \)).
\(^{*4}\) one component of the Hopf link in the middle is summed to \( T(2, 5) \) and the other is summed to the trefoil.

**Table 1.** Fibred links obtained from the exceptional torus links by cutting along an arc.

**Remark 1.** An arc \( \alpha \) is clean if and only if \( \varphi^k(\alpha) \) is clean, for all \( k \in \mathbb{Z} \). This is clear since \( \alpha \cap \varphi(\alpha) = \partial\alpha \) after minimising isotopies if and only if \( \varphi^k(\alpha) \cap \varphi^{k+1}(\alpha) = \partial\alpha \) after minimising isotopies. Since the monodromy \( \varphi \) permutes the vertices \( \{a_i\} \) cyclically as well as the vertices \( \{b_j\} \), it suffices to show that there are only finitely many clean arcs starting at a point of \( \partial A_1 \) or at a point of \( \partial B_1 \), up to isotopy. Note
that for torus links, $\varphi$ is periodic up to isotopy, but this isotopy cannot be taken to be fixed on $\partial S$.

5. ARCS FOR POSITIVE BRAID LINKS WITH PSEUDO-ANOsov MONODROMY

**Theorem 3.** Let $S$ be the fibre surface of a (prime) positive braid link and suppose the monodromy $\varphi : S \to S$ is pseudo-Anosov. Then, $S$ contains infinitely many non-isotopic cutting arcs preserving fibredness.

**Proof.** Think of $S$ as an iterated plumbing of positive Hopf bands (compare Section 2). The monodromy $\varphi$ is then a composition of right Dehn twists along the core curves of these Hopf bands. Let $\alpha$ be an arc dual to the core curve of the last plumbed Hopf band and such that $\alpha$ does not enter any of the previously plumbed Hopf bands. Then, in the product of Dehn twists representing $\varphi$, only the last factor affects $\alpha$. It follows that $\alpha$ is clean (and therefore $\varphi^n(\alpha)$ is also clean by Remark 1). Since $\varphi$ is pseudo-Anosov and $\alpha$ is essential, the length of $\varphi^n(\alpha)$ (with respect to an auxiliary Riemannian metric) grows exponentially as $n$ tends to infinity (compare [FM], Section 14.5). In particular, the arcs $\varphi^n(\alpha)$ are pairwise non-isotopic and clean. \qed

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Appendix. Proof of Proposition 1

The objective of this appendix is to give a proof of the subsequent proposition.

**Proposition 1.** The torus links $T(3,3)$, $T(3,4)$ and $T(3,5)$ admit, up to isotopy (free on the boundary), only a finite number of cutting arcs that preserve fibredness.

Before we begin with the proof (p. 14), some notation and remarks are necessary. Let $S$ be the fibre surface of any torus link $T(n,m)$, given as thickening of the complete bipartite graph on $n + m$ vertices (compare Section 3), and denote by $\varphi : S \to S$ the monodromy. Let $U$ be the union of all the disks $A_i$ and $B_j$, and let $N \subset S$ be the neighbourhood of the graph on which $\varphi$ is periodic.

**Definition 3.** An arc $\alpha \subset S$ is in normal position if the following conditions hold:

(a) The endpoints of $\alpha$ lie in $\partial U$.
(b) For every $(i,j)$, $\alpha \cap K_{ij} \setminus U$ consists of finitely many straight segments parallel to $k_{ij}$.
(c) The number of such segments in $K_{ij}$ is minimal among all arcs isotopic to $\alpha$.
(d) $\alpha$ intersects the graph transversely in finitely many points of $U$.
(e) $\alpha \setminus N \subset U$, that is, before $\alpha$ enters $N$ and after it leaves $N$, it stays in the disks that contain its endpoints.
(f) $\alpha \cap U$ consists of finitely many straight arcs.

**Remarks 2** (on normal position).

- Any arc can be brought into normal position by an isotopy (free on $\partial S$).
- If $\alpha$ is in normal position, then $\varphi(\alpha)$ can be brought into normal position keeping $N$ fixed. Indeed, it suffices to straighten the two subarcs $\varphi(\alpha) \setminus N$ (each of them traverses two bands), sliding the endpoints of $\varphi(\alpha)$ along $\partial S$, see Figure 5.

![Figure 5. How to bring $\varphi(\alpha)$ in normal position, keeping $N$ fixed.](slide)
• If \( \alpha \) and \( \varphi(\alpha) \) are in normal position as above, then we may isotope \( \varphi(\alpha) \) with endpoints fixed and keeping it in normal position, such that \( \alpha \) and \( \varphi(\alpha) \) intersect transversely in finitely many points of \( U \). In particular, the sets \( \alpha \setminus U \) and \( \varphi(\alpha) \setminus U \) are now disjoint (cf. Figure 6).

\[ \xrightarrow{\text{\textbullet}} \]

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\[ \xrightarrow{\text{\textbullet}} \]

**Figure 6.** How to make \( \alpha, \varphi(\alpha) \) intersect transversely, keeping normal position.

• Let \( \alpha \) be in normal position and suppose it passes through at least one band. Let \( K_{ij} \) be the first (last) band traversed by \( \alpha \) after (before) it starts (ends) at a point \( p \) of either \( \partial A_i \) or \( \partial B_j \). Then, \( p \) cannot lie between \( k_{ij} \) and \( k_{i,j+1} \) (if \( p \in A_i \)) nor between \( k_{ij} \) and \( k_{i+1,j} \) (if \( p \in B_j \)). Otherwise, an isotopy sliding the starting point (endpoint) of \( \alpha \) along \( \partial K_{ij} \) would decrease the number of segments in \( K_{ij} \), contradicting part (c) of Definition 3.

Remarks 3 (compare the bigon criterion, Prop. 1.7 in [FM]). Suppose \( \alpha \) intersects \( \varphi(\alpha) \). If \( \alpha \) is clean, there must be a bigon \( \Delta \subset S \) whose sides consist of a subarc of \( \alpha \) and a subarc of \( \varphi(\alpha) \). If \( \alpha, \varphi(\alpha) \) are in normal position, such \( \Delta \) takes a particularly simple form:

• \( \Delta \) cannot be contained in \( U \) (i.e., in one of the disks \( A_i \) or \( B_j \)). This would contradict part (i) of the above Definition 3.

• None of the two sides of \( \Delta \) is contained in \( U \), since the other side of \( \Delta \) would have to leave \( U \) through one of the bands \( K_{ij} \) and return through the same \( K_{ij} \). The disk \( \Delta \) would then yield an isotopy reducing the number of segments of \( \alpha \cap K_{ij} \) or \( \varphi(\alpha) \cap K_{ij} \), contradicting part (c) of Definition 3.

• For every \((i, j)\), \( \Delta \cap K_{ij} \setminus U \) consists of rectangles with two opposite sides parallel to \( k_{ij} \).

• \( \Delta \cap U \) consists of polygons, i.e. disks connected to at least one rectangle.

• Construct a spine \( \Gamma \) for \( \Delta \) as follows: put a vertex for each polygon and connect two vertices by an edge if the corresponding polygons connect to the same rectangle. \( \Gamma \) is a tree, for \( \Delta \) is contractible. Two of its vertices correspond to the vertices of the bigon \( \Delta \). Among the other vertices of \( \Gamma \), there is none of valence one because the adjacent edge would correspond to a rectangle in some \( K_{ij} \) whose sides parallel to \( k_{ij} \) both belong to the same arc \( (\alpha \text{ or } \varphi(\alpha)) \). In other words, either
\( \alpha \) or \( \varphi(\alpha) \) would pass through \( K_{ij} \) and immediately return through \( K_{ij} \) in the opposite direction. This would contradict part (c) of Definition 3. Therefore, \( \Gamma \) is a line consisting of some number of consecutive edges, and the two extremal vertices correspond to the vertices of \( \Delta \).

**Lemma 1.** Let \( \alpha, \varphi(\alpha) \) be in normal position and suppose they intersect in a point \( p \in D \), where \( D \) is one of the disks \( A_i, B_j \). Let \( \alpha', \alpha'' \) be the components of \( \alpha \cap D, \varphi(\alpha) \cap D \) containing \( p \). If no two of the four points \( \partial \alpha' \cup \partial \alpha'' \subset \partial D \) lie in the same band \( K_{ij} \), then \( \alpha \) cannot be clean.

![Figure 7. \( \alpha \) cannot be clean by Lemma 1](image)

**Remark 4.** Note that we did not exclude the possibility that one of the endpoints of \( \alpha \) or \( \varphi(\alpha) \) lie in \( \partial \alpha' \cup \partial \alpha'' \).

**Proof of Lemma 1.** If \( \alpha \) were clean, there would be a bigon. After possibly removing a certain number of such bigons, we are left with a bigon \( \Delta \) with vertex \( p \). By Remark 3, \( \Delta \) has to leave \( D \) through one of the adjacent bands. Since one of the sides of \( \Delta \) is a subarc of \( \alpha \) and the other side is a subarc of \( \varphi(\alpha) \), we find two points among \( \partial \alpha' \cup \partial \alpha'' \) that lie in this band, contradicting the assumption on \( \alpha', \alpha'' \).

**Lemma 2.** Let \( \alpha, \varphi(\alpha) \) be in normal position and let \( \alpha', \alpha'' \) be subarcs of \( \alpha, \varphi(\alpha) \) respectively (not necessarily contained in \( U \)). Suppose that the four endpoints of \( \alpha' \) and \( \alpha'' \) are contained in \( \partial U \) and that no two of them lie on the same band \( K_{ij} \). We assume further that \( \alpha' \) and \( \alpha'' \) intersect in exactly one point and that \( \alpha', \alpha'' \) run through the same bands (see Figure 8). Then \( \alpha \) cannot be clean.

**Proof.** Assume \( \alpha' \cap \alpha'' = \{p\} \), then \( p \in U \). As in the proof of Lemma 1, study a bigon \( \Delta \) that starts at \( p \). \( \Delta \) consists of a sequence of rectangles as described in Remarks 3. Starting at \( p \), it therefore has to pass through the same bands as \( \alpha' \) and \( \alpha'' \). Since \( p \) was the only intersection between \( \alpha' \) and \( \alpha'' \), \( \Delta \) has to pass through at least one more band. But this is impossible by the assumption on the endpoints of \( \alpha' \) and \( \alpha'' \).
Lemma 3. A clean arc in normal position cannot traverse more than two consecutive bands.

Here, a sequence of bands \( K^{(1)}, K^{(2)}, \ldots \in \{K_{ij}\} \) is consecutive, if the set \( (\bigcup_r K^{(r)} \cup \bigcup_i A_i \cup \bigcup_j B_j) \setminus (\bigcup_{i,j} k_{ij}) \) has a connected component that intersects all bands \( K^{(r)} \) of the sequence in this order, i.e. it is possible to stay on the same side of the graph when walking along the bands.

Proof. Suppose that \( \alpha \) is a clean arc in normal position that traverses \( n \) consecutive bands, \( n \geq 3 \). We may assume that \( n \) is the maximal number of consecutively traversed bands. In these bands as well as the adjacent disks, isotope \( \alpha \) such that it stays on one side of the graph, keeping it in normal position. Now bring \( \varphi(\alpha) \) into normal position transverse to \( \alpha \) as described in the Remarks 2. Recall the description of the monodromy \( \varphi \) as a têté-à-têté twist from Section 3.

Cutting the surface \( S \) open along the graph results in \( d \) annuli, where \( d \) is the number of components of \( \partial S = L \) and each annulus has a link component as one boundary and a cycle consisting of edges of the graph as the other boundary.

In one of these annuli we will see a subarc \( \alpha' \subset \alpha \) that has exactly its endpoints in common with the graph and that travels near the
edge boundary for a distance of \( n \) consecutive edges. (Note that \( \alpha' \) cannot have any endpoint on \( \partial S \). This would contradict part (c) of Definition 3). Let \( C \) be the disk bounded by \( \alpha' \) and the graph. The monodromy \( \varphi \) keeps the link-boundary of this annulus fixed and rotates the neighbourhood \( N \) of the graph boundary by two edges. Since \( n \geq 3 \), \( \varphi(\alpha') \) has one of its endpoints in \( C \) and the other outside of \( C \), so \( \alpha' \) has to intersect its image \( \varphi(\alpha') \) in a point \( p \in U \), and we may assume that \( p \) is the only intersection between \( \alpha' \) and \( \varphi(\alpha') \). Denote by \( q \) the endpoint of \( \varphi(\alpha') \) that lies in \( C \) and let \( D \) be the disk \( A_i \) or \( B_j \) containing \( q \). Then make sure that \( p \in D \) by an isotopy on \( \varphi(\alpha') \) preserving normal position if necessary (compare Figures 9 and 10). However \( \alpha \) is clean, so there must be a bigon in \( S \) whose sides consist of a subarc of \( \alpha \) and a subarc of \( \varphi(\alpha) \). After possibly removing a certain number of such bigons, we will be left with a bigon \( \Delta \) starting at \( p \). From the Remarks 3 we know that \( \Delta \) has to leave \( D \) and consists of a sequence of rectangles. Let \( R \) be the first rectangle in this sequence, i.e. \( R \) is contained in a band adjacent to \( D \). Let \( K^-, K^+ \in \{ K_{ij} \} \) be the two bands adjacent to \( D \) that contain segments of \( \alpha', \) \( K^+ \) being the one that also contains a segment of \( \varphi(\alpha') \) (see Figure 10). Let \( \beta \) be the component of \( \varphi(\alpha) \setminus \{ p \} \) that contains \( q \). We claim that \( \beta \) cannot leave \( D \) through \( K^- \) nor \( K^+ \). Indeed, if \( \beta \) would leave \( D \) through \( K^- \), \( \varphi(\alpha) \) would traverse \( n+1 \) consecutive bands, contradicting the assumption on \( n \) being maximal. On the other hand, if \( \beta \) would leave \( D \) through \( K^+ \), we could reduce the number of segments in \( \varphi(\alpha) \cap K^+ \), contradicting the normal position of \( \varphi(\alpha) \), i.e. part (c) of Definition 3. In contrast, \( \alpha \) leaves \( D \), starting from \( p \) in both directions, through \( K^- \) and \( K^+ \). Consider now the subarcs of \( \alpha \) and \( \varphi(\alpha) \) that constitute two opposite sides of the rectangle \( R \). Since \( R \) is contained in a band adjacent to \( D \), these two subarcs arrive at \( \partial D \) through the same band, and they connect directly to \( p \in D \). Therefore, we must have \( R \subset K^+ \), since \( K^+ \) is the only band containing two subarcs of \( \alpha \) and \( \varphi(\alpha) \) that directly connect to \( p \in D \). Furthermore, \( R \) has to be the region enclosed by

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure10.png}
\caption{Part of the mentioned annulus, where the arcs \( \alpha' \) and \( \varphi(\alpha') \) intersect in a point \( p \in D \).}
\end{figure}
\( \alpha' \cap K^+ \) and \( \varphi(\alpha') \cap K^+ \). Following \( \varphi(\alpha') \) in the direction from \( q \) to \( p \), we see that it leaves \( D \) through \( K^+ \) as one of the sides of \( R \) and continues staying on the same side of the graph for exactly \( n - 1 \) more edges. By assumption, \( p \) is the only intersection between \( \alpha' \) and \( \varphi(\alpha') \), so the bigon \( \Delta \) has to continue for at least \( n - 1 \) more rectangles through consecutive bands. Similarly, the sides of these rectangles that are subarcs of \( \alpha \) have to continue for at least \( n - 1 \) consecutive bands. We obtain a contradiction to the maximality of \( n \), because \( \alpha' \) ends after \( n - 2 \) bands starting from \( D \), since \( \varphi \) rotates the graph by two edges. This finishes the proof. \( \square \)

**Proof of Proposition 1.** We will concentrate on the most complicated case of the torus knot \( T(3, 5) \). It contains all difficulties appearing in the proofs for \( T(3, 3) \) and \( T(3, 4) \) which go along the same lines with fewer cases to consider. For each link appearing in Table 1 of Section 4, we will indicate one (but not every) possible choice of a cutting arc that yields the link in question. Let hence \( S \) be the fibre surface of \( T(3, 5) \) and let \( \alpha \subset S \) be any arc that preserves fibredness, i.e. a clean arc. Bring \( \alpha \) into normal position using an isotopy (not fixing the boundary), cf. Remarks 2. By Remark 1, it is enough to study the cases in which \( \alpha \) starts at a point of \( \partial A_1 \) or at a point of \( \partial B_1 \). We may further assume that \( \alpha \) starts either at a point of \( \partial A_1 \) between \( k_{11} \) and \( k_{15} \) or at a point of \( \partial B_1 \) between \( k_{21} \) and \( k_{31} \).

**Case A.** \( \alpha \) starts at \( \partial A_1 \), between \( k_{11} \) and \( k_{15} \). Then, \( \alpha \) cannot continue through either of the bands \( K_{11} \) nor \( K_{15} \) by the last item of Remarks 2. So, either \( \alpha \) stays in \( A_1 \) (and there are only four such arcs up to isotopy), or it continues through \( K_{12}, K_{13} \) or \( K_{14} \). If \( \alpha \) stays in \( A_1 \), the links obtained by cutting are \( E_7 \) (e.g. if \( \alpha \) ends between \( k_{24} \) and \( k_{25} \)), or it can continue through \( K_{23} \) or \( K_{24} \).

**Case A.1.** \( \alpha \) continues through \( K_{12} \). Arriving in \( B_2 \), there are three possibilities: Either \( \alpha \) ends at a point of \( \partial B_2 \) between \( k_{22} \) and \( k_{32} \) (and cutting along \( \alpha \) yields \( T(3, 4) \# T(2, 2) \)), or it continues through \( K_{22} \) or \( K_{32} \) (ending at other points of \( \partial B_2 \) is impossible by the last item of Remarks 2).

**Case A.1.1.** \( \alpha \) continues through \( K_{22} \). Arriving in \( A_2 \), \( \alpha \) can end at a point of \( \partial A_2 \) (cutting yields \( T(2, 7) \# T(2, 2) \)) if \( \alpha \) ends between \( k_{24} \) and \( k_{25} \), and \( T(2, 3) \) summed with the unknot component of \( D_5 \) if \( \alpha \) ends between \( k_{23} \) and \( k_{24} \), or it can continue through \( K_{23} \) or \( K_{24} \). \( \alpha \) cannot continue through \( K_{21} \), since \( K_{12}, K_{22}, K_{21} \) is a sequence of three consecutive bands, so \( \alpha \) would not be clean by Lemma 3. Finally, \( \alpha \) cannot continue through \( K_{25} \). If it did, \( \alpha \) and \( \varphi(\alpha) \) would intersect in a point of \( A_1 \), and Lemma 4 would imply that \( \alpha \) cannot be clean (see
Figure 11 (top left). Note that we do not know whether the mentioned intersection is the only one since we do not know how $\alpha$ ends.

Case A.1.1.1. $\alpha$ continues through $K_{23}$. From $B_3$, it cannot continue through $K_{13}$, for $K_{22}, K_{23}, K_{13}$ are consecutive (Lemma 3). If it continues through $K_{33}$ it cannot continue through any band adjacent to $A_3$. Indeed, $K_{23}, K_{33}, K_{32}$ are consecutive, so $\alpha$ cannot continue through $K_{32}$. If it would continue through $K_{34}$ or $K_{35}$ or $K_{31}$, we could apply Lemma 2 to the band $K_{33}$ to show that $\alpha$ is not clean (see Figure 11).

Case A.1.1.2. $\alpha$ continues through $K_{24}$. If it ends in $B_4$ between $k_{14}$ and $k_{34}$, we obtain $T(2, 5) \# T(2, 4)$ after cutting. Otherwise, it can continue from $B_4$ through $K_{14}$ or through $K_{34}$.

Case A.1.1.2.1. If it continues through $K_{14}$, it cannot go further. Firstly, $K_{24}, K_{14}, K_{15}$ are consecutive, so $K_{15}$ is no option (Lemma 3). Neither can it proceed through $K_{11}$ (this would produce a self-intersection of $\alpha$) nor $K_{12}$ (for otherwise we could apply Lemma 1 to an intersection between $\alpha$ and $\varphi(\alpha)$ in $A_1$). If it continues through $K_{13}$, it cannot go on through $K_{23}$ since $K_{14}, K_{13}, K_{23}$ are consecutive (Lemma 3). Suppose it continues through $K_{33}$. From $A_3$, it cannot proceed through any of $K_{31}, K_{35}, K_{34}$, for otherwise we could apply Lemma 2 to the bands $K_{13}$ and $K_{33}$, with an intersection between $\alpha$ and $\varphi(\alpha)$ occurring in $A_3$ (see Figure 11 left). However, $\alpha$ cannot continue through $K_{32}$ either, because we could again apply Lemma 2 this time for the band $K_{24}$ and an intersection in $A_2$ (see Figure 11 right).

Case A.1.1.2.2. $\alpha$ continues from $B_3$ through $K_{34}$. If it ends in $A_3$ between $k_{35}$ and $k_{31}$, cutting yields $T(2, 5) \# T(2, 2) \# T(2, 3)$. Otherwise, it cannot continue from $A_3$ through $K_{33}$ since $K_{24}, K_{34}, K_{33}$ are consecutive. Neither can it proceed through $K_{32}$ (apply Lemma 1 to $A_3$). So we have to study the cases where $\alpha$ continues through either of $K_{35}$ or $K_{31}$.

Case A.1.1.2.2.1. If it continues through $K_{35}$, the only option to go further is through $K_{15}$, since $K_{34}, K_{35}, K_{25}$ are consecutive. From $A_1$ (compare Figure 11), it cannot continue through $K_{11}$ nor $K_{12}$ (apply Lemma 2 to $K_{15}$ with an intersection occurring in $A_1$). Neither can it continue through $K_{14}$, since $K_{35}, K_{15}, K_{14}$ are consecutive. So it has to go through $K_{13}$. Arriving in $B_3$, it cannot continue through $K_{23}$ (apply Lemma 2 to $K_{34}$ with an intersection occurring in $B_3$). Therefore $\alpha$ has to continue through $K_{33}$. From $A_3$, it cannot proceed further. Firstly, $K_{32}$ is not an option (otherwise apply Lemma 2 to $K_{34}$ and $K_{24}$ with an intersection in $A_2$). Neither can it go through $K_{34}$ or $K_{35}$ (apply Lemma 2 to $K_{15}, K_{13}, K_{33}$ with an intersection occurring in $A_3$). Finally, it cannot pass through $K_{31}$ either (apply Lemma 2 to the bands $K_{15}, K_{13}, K_{33}$ with an intersection occurring in $A_3$).
Case A.1.1.2.2.2. If it continues through $K_{31}$ and arrives in $B_1$, it cannot proceed through $K_{11}$ (apply Lemma 2 to $K_{22}$ with an intersection occurring in $B_2$, see Figure 11 left). So it has to go through $K_{21}$.

**Figure 11.** Schematic illustration for a selection of the cases in the proof of Proposition 1. The arc $\alpha$ is drawn as a solid line, whereas $\varphi(\alpha)$ is shown as a dotted line.
From $A_2$, it cannot proceed through $K_{22}$, for $K_{31}, K_{21}, K_{22}$ are consecutive. Neither can it go through either of $K_{23}$ nor $K_{24}$ (apply Lemma 1 to an intersection occurring in $A_2$, see Figure 11 right). Finally, $K_{25}$ can be ruled out by Lemma 2 applied to the bands $K_{22}$ and $K_{12}$, with an intersection occurring in $A_1$.

**Case A.1.2.** $\alpha$ continues through $K_{32}$ (see Figure 11). Arriving in $A_3$, it cannot continue through any band. Firstly, $K_{12}, K_{32}, K_{33}$ are consecutive, so $\alpha$ cannot continue through $K_{33}$. If it would continue through any of the other bands adjacent to $A_3$, $\alpha$ would intersect $\varphi(\alpha)$ in a point of $A_3$ such that we could apply Lemma 1 to obtain a contradiction to $\alpha$ being clean.

**Case A.2.** $\alpha$ proceeds through $K_{13}$. If it ends in $B_3$ between $k_{23}$ and $k_{33}$, we obtain $T(2,8)$ after cutting. From $B_3$, it can continue through $K_{23}$ or through $K_{33}$.

**Case A.2.1.** $\alpha$ continues through $K_{23}$. It cannot go on via $K_{22}$, for $K_{13}, K_{23}, K_{22}$ are consecutive. Neither can it continue through $K_{21}$ or $K_{25}$ by Lemma 1 applied to an intersection in $A_1$. If it next passes through $K_{24}$, it cannot go on through $K_{14}$, because $K_{23}, K_{24}, K_{14}$ are consecutive. Proceeding through $K_{34}$, it can end in $A_3$ between $k_{31}$ and $k_{32}$ (this yields $T(2,5)\#T(2,3)\#T(2,2)$). However, the only possibility for $\alpha$ to go further is via $K_{32}$, for $K_{24}, K_{34}, K_{33}$ are consecutive (so $K_{33}$ is no option), and $\alpha$ cannot continue through $K_{35}$ nor $K_{31}$ by applying Lemma 2 to the band $K_{34}$ with an intersection of $\alpha, \varphi(\alpha)$ in $A_3$. So $\alpha$ continues through $K_{32}$ and arrives in $B_2$. From there, it cannot continue through $K_{32}$ and arrives in $B_2$. From there, it cannot continue through $K_{12}$ (apply Lemma 2 to $K_{22}$ and an intersection in $B_3$). If it continues through $K_{22}$, it cannot go further: $K_{23}$ is impossible because $K_{32}, K_{22}, K_{23}$ are consecutive, $K_{24}$ can be excluded by Lemma 1 applied to $A_3$, and $K_{31}$ as well as $K_{25}$ can be ruled out by Lemma 2 applied to $K_{23}$ and $K_{13}$ with an intersection occurring in $A_1$.

**Case A.2.2.** $\alpha$ continues through $K_{33}$. This is similar to Case A.2.1. Again there is always a single option to go on, until there is no possibility left after four more steps.

**Case A.3.** $\alpha$ continues through $K_{14}$. This is analogous to Case A.1.

**Case B.** $\alpha$ starts at $\partial B_1$ between $k_{21}$ and $k_{31}$. Then, it can only continue through $K_{11}$ by the last item of Remarks 2. From $A_1$, it can proceed through four distinct bands.

**Case B.1.** $\alpha$ continues through $K_{15}$. Since $K_{11}, K_{15}, K_{25}$ are consecutive, it can a priori only continue through $K_{35}$. But this is impossible as well by Lemma 1 applied to the intersection between $\alpha$ and $\varphi(\alpha)$ occurring in $B_1$.

**Case B.2.** $\alpha$ continues through $K_{12}$. This is analogous to Case B.1.
Case B.3. \( \alpha \) continues through \( K_{14} \). Arriving in \( B_4 \), it can end between \( k_{24} \) and \( k_{34} \) (this results in \( T(2,3) \) summed with the trefoil component of \( D_5 \)).

Case B.3.1. \( \alpha \) continues through \( K_{24} \). From \( A_2 \), it cannot continue through \( K_{23} \) because \( K_{14}, K_{24}, K_{23} \) are consecutive (Lemma 3). Neither can it go on through \( K_{22} \) nor \( K_{21} \) (apply Lemma 1 to \( A_1 \)). Suppose \( \alpha \) continues through \( K_{25} \). From \( B_5 \), it cannot go on via \( K_{35} \) since \( K_{24}, K_{25}, K_{15} \) are consecutive. If it proceeds via \( K_{35} \), we can apply Lemma 2 to the band \( K_{11} \) with an intersection in \( B_1 \) to obtain a contradiction.

Case B.3.2. \( \alpha \) continues through \( K_{34} \). From \( A_3 \), there are only two options for \( \alpha \) to proceed further. Indeed, \( K_{14}, K_{34}, K_{35} \) are consecutive, so \( K_{35} \) is out of the question. \( K_{31} \) can be ruled out by Lemma 1 for \( A_3 \). The remaining possibilities are \( K_{32} \) and \( K_{33} \).

Case B.3.2.1. \( \alpha \) continues through \( K_{32} \). From there, it cannot continue through \( K_{22} \) (apply Lemma 2 to \( K_{32} \)). So it has to branch off via \( K_{12} \) to \( A_1 \). From there, it cannot continue through \( K_{15} \) since otherwise \( \alpha \) would self intersect in \( A_1 \). \( K_{11} \) is impossible as well, for \( K_{32}, K_{12}, K_{11} \) are consecutive. \( K_{15} \) can be ruled out using Lemma 1 for \( A_3 \). So \( \alpha \) can only continue through \( K_{13} \), and from there only through \( K_{23} \) (\( K_{12}, K_{13}, K_{33} \) are consecutive). From \( A_2 \), it cannot go on through any band except \( K_{25} \). Indeed, \( K_{22} \) is impossible because \( K_{13}, K_{23}, K_{22} \) are consecutive. \( K_{21} \) and \( K_{24} \) can be ruled out by applying Lemma 2 to \( (K_{34}, K_{14}) \) and \( K_{23} \) respectively. After passing through \( K_{25} \), \( \alpha \) cannot go further: \( K_{15} \) is impossible by Lemma 2 (applied to \( K_{23}, K_{25} \)) and \( K_{35} \) can be ruled out by applying Lemma 2 to \( K_{34}, K_{14}, K_{11} \).

Case B.3.2.2. \( \alpha \) continues through \( K_{33} \). Then, \( K_{13} \) cannot be next since \( K_{34}, K_{32}, K_{13} \) are consecutive. Thus \( \alpha \) passes through \( K_{23} \). From \( A_2 \), it cannot go on via \( K_{24} \) for \( K_{33}, K_{23}, K_{24} \) are consecutive. \( K_{21} \) and \( K_{22} \) are impossible as well (apply Lemma 2 to \( K_{14} \)). So \( \alpha \) has to go through \( K_{25} \). Then, it cannot proceed through \( K_{15} \) (apply Lemma 2 to \( K_{25} \)). It cannot go via \( K_{35} \) either (apply Lemma 2 to \( K_{14}, K_{11} \)), so \( \alpha \) cannot continue at all.

Case B.4. \( \alpha \) continues through \( K_{13} \). This is analogous to Case B.3.

\( \square \)