The Classification of Spun Torus Knots

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ABSTRACT

S. Satoh has defined a construction to obtain a ribbon torus knot given a welded knot. This construction is known to be surjective. We show that it is not injective. Using the invariant of the peripheral structure, it is possible to provide a restriction on this failure of injectivity. In particular we also provide an algebraic classification of the construction when restricted to classical knots, where it is equivalent to the torus spinning construction.

Keywords: Surface knots, welded knots, ribbon knots, longitude group

1. Introduction

Throughout this paper we will work only with oriented, 1-component classical knots, oriented, 1-component welded knots, and oriented surface knots. Furthermore, all isotopies are assumed to be orientation-preserving.

S. Satoh has shown that there is an algorithm to produce an oriented ribbon torus knot from any oriented welded knot diagram. We follow his notation and designate this operation as Tube. Furthermore, this operation was shown to be independent of the particular representative of a welded-equivalence class of welded knots. This was proved by showing that any welded Reidemeister move induces an isotopy on the corresponding ribbon torus knots. Satoh also demonstrated that this operation is surjective, in the following sense: for any isotopy class of ribbon torus knots, there is some welded knot \( K \) such that Tube\((K)\) lies in that isotopy class.

It is natural to ask whether or not ribbon torus knots are classified by welded knots under this Tube operation; that is, if Tube\((K)\) and Tube\((L)\) are isotopic, must \( K \) and \( L \) be welded-equivalent? It will be shown that this is not the case, by exhibiting a specific example of inequivalent welded knots which are mapped to the same ribbon torus knot by Tube. We will consider knots which are not \((-\text{amphichiral})\) and show that for such knots, Tube fails to be injective.

We will then examine the peripheral structure for oriented, 1-component classical and welded knots, and extend this invariant to surface knots. Using this it is possible to determine that for classical knots, Tube\(^{-1}\)(Tube\((K)) = \{K, -K^*\}. This leads to an algebraic classification of oriented spun classical knots.
2. Welded Knots

Kauffman\textsuperscript{[2]} introduced virtual knots as a combinatoric generalization of classical knot diagrams, in which a third crossing type is allowed, a virtual crossing. Welded knots\textsuperscript{[3,1,4]} are presented via the same diagrams as virtual knots, but with one additional move, one of the 'forbidden' moves from virtual knot theory (see Fig. 1). Two welded knot diagrams which may be transformed into each other by a sequence of such moves are called \textit{welded equivalent} or \textit{w-equivalent}.

Any classical knot diagram may be interpreted as a welded knot diagram. Our first theorem shows that if two classical knot diagrams are welded equivalent, then they are equivalent as classical knots as well. In order prove this, we first recall the definition of the knot group and longitude for classical and welded knots. For both classical and welded knots, the knot group may be found in a Wirtinger presentation from the diagram. Each arc in the diagram is a generator, and each crossing yields a relation of the form $x = z^{-1}yz$, which may be abbreviated as $x = y^z$.

![Fig. 1. The welded Reidemeister moves. The move in the lower right is one of the 'forbidden' moves.](image)

![Fig. 2. Each crossing in a knot diagram yields a relation of the form $x = y^z$. Reversing the orientation of the overcrossing would change the relation to $y = x^z$.](image)
The longitude is defined as an element of the knot group corresponding to circling the knot exactly once in the direction of the orientation of the knot without algebraic linking; it is the canonical generator of the non-linking part of the peripheral group\[5,8\]. Thus, it is an ordered list of the arcs one crosses under, where the arc’s generator appears at right handed crossings and its inverse appears at left handed crossings, multiplied at the end by \(m^{-k}\) where \(k\) is the sum of the signs of the crossings, and \(m\) is the meridian, so that the linking number is 0.

This latter, combinatorial definition of the longitude extends naturally to welded knot diagrams. The longitude of a welded knot is defined to be the element of the knot group obtained by multiplying the generators of the arcs which one passes under in classical crossings, multiplied at the end by \(m^{-k}\) where \(k\) is the sum of the signs of the crossings, and \(m\) is the meridian. Welded crossings do not contribute to the longitude. We see that this definition is in fact invariant by considering their behavior under the welded Reidemeister moves. For example, we may check that under the ‘forbidden move,’ which moves an overcrossing over a welded crossing, the longitude does not change: the order in which the overcrossing will appear in the list remains unchanged.

Since the combinatorial definitions of the longitude are identical for both classical and welded knot diagrams, the longitude of a classical knot is the same whether computed using the welded definition or the classical definition. For the order in which we encounter overcrossings does not depend upon whether the knot diagram is being considered as a classical diagram or a welded diagram.

It is known\[14,2,5,6\] that the group system (the knot group, the meridians, and their corresponding longitudes) classifies (oriented, 1-component) classical knots.

**Theorem 2.1.** If \(K\) and \(L\) are classical (oriented, 1-component) knots whose diagrams are welded equivalent, then they are isotopic.

Proof (\[8,2\]): The group system is preserved under welded Reidemeister moves. Therefore \(K\) and \(L\) must have the same classifying invariant, and be classically equivalent. □

### 3. Satoh’s Tube Map

For full proofs and exposition of the following, we refer the reader to Satoh’s development\[1\]. We will review here only the essential points of the construction.

Satoh defined the operation \(Tube\) as follows. Given a welded knot diagram \(K\), we draw a broken surface diagram\[7\] by placing a thin tube wherever we see an edge in the welded knot diagram, orienting the surface as shown in Fig. 2. At welded crossings, the tubes pass over/under each other, and at classical crossings, they knot together as in Fig. 2. We orient the surface so that the normal vector in our broken surface diagram points outward.
The surface corresponding to the resulting broken surface diagram is defined to be \( \text{Tube}(K) \). We have from Satoh’s work the following important results.

**Theorem 3.1.** For any two welded knots \( K, K' \), if \( K \cong K' \), \( \text{Tube}(K) \cong \text{Tube}(K') \).

Satoh proves this by showing that any welded Reidemeister move can be mirrored on the broken surface diagram using Roseman moves.

**Theorem 3.2.** For any ribbon torus knot \( R \), there is a welded knot \( L \) such that \( \text{Tube}(L) \cong R \).

In the following the \(*\) operation refers to taking the mirror image of the surface knot.

**Theorem 3.3.** The following equivalences hold for any welded knot \( K \): \( \text{Tube}(K) \cong -\text{Tube}(K)^* \), \( -\text{Tube}(K) \cong \text{Tube}(-K) \).

From this it follows that \( -\text{Tube}(-K) \cong \text{Tube}(K) \). We recall here the operation on classical knots which Satoh denotes by \( \text{Spin}(K) \). Take a classical knot \( K \) and place it in a half-hyperplane copy of \( \mathbb{R}^3 \) within \( \mathbb{R}^4 \). Now rotate the half-hyperplane about its face. \( K \) will trace out a surface with the diffeomorphism class of a torus. We refer to this torus as \( \text{Spin}(K) \).

**Theorem 3.4.** For oriented classical knots \( K \), \( \text{Tube}(K) \cong \text{Spin}(K) \) for at least one orientation of \( \text{Spin}(K) \), and exactly one if \( \text{Spin}(K) \) is not reversible. We denote this orientation of \( \text{Spin}(K) \) by \( \text{OSpin}(K) \); that is, \( \text{OSpin}(K) \) is the orientation of \( \text{Spin}(K) \) which makes \( \text{Tube}(K) \cong \text{OSpin}(K) \) true.

Proof: We refer the reader to Satoh's construction of an explicit isotopy of the unoriented \( \text{Spin}(K) \) to the (unoriented) \( \text{Tube}(K) \). By requiring that \( \text{OSpin}(K) \cong \text{Tube}(K) \) as oriented surfaces, we induce an orientation on \( \text{OSpin}(K) \) via this
isotopy, which is clearly well defined, unless $Spun(K)$ is reversible, in which case $O\!Spun(K)$ is reversible as well, and hence it is well defined in this case as well. Note that if $Spun(K)$ is not reversible, then neither is $O\!Spun(K)$ nor $Tube(K)$, for if they were Satoh’s construction could be applied to reverse the orientation of $Spun(K)$. □

Theorem 3.5. Tube preserves the knot group and quandle.

This follows from a straightforward computation (see the diagram). We require an additional theorem, which slightly generalizes a theorem pointed out to the author by Dennis Roseman. Recall that vertical reflection of a welded knot diagram is performed by reflecting the planar graph across a vertical plane[11]. For classical knots this reduces to the usual reflection. We denote the vertical reflection of $K$ by $K^\uparrow$.

Theorem 3.6. For welded knots $K$, $Tube(K)^* \cong Tube(-K^\uparrow)$.

Proof: Draw the diagram of $Tube(K)$ and then draw the mirror, $Tube(K)^*$. Now look at the welded knot $K'$ which naturally yields $-Tube(K)^*$. This is precisely $-K^\uparrow$, so $-Tube(K)^* \cong Tube(-K^\uparrow)$. □

3.1. Tube is Not Injective

We are now ready to prove our main theorem about Satoh’s construction. In private correspondence Satoh has indicated that he became aware of this theorem, or the possibility of it, at some point after [11] was published.
Theorem 3.7. There is a ribbon torus knot $R$ with the property that $\text{Tube}^{-1}(R)$ contains inequivalent welded knots.

Proof: Let $K$ be a welded knot, with $K \not\cong -K^\uparrow$. Such knots exist; take for instance the right handed trefoil which is chiral as a welded knot, since classical knot theory embeds faithfully in welded knot theory. But we have shown that $\text{Tube}(K) \cong -\text{Tube}(K)^* \cong \text{Tube}(-K^\uparrow)$, so $\text{Tube}^{-1}(\text{Tube}(K))$ contains the inequivalent welded knots $K$ and $K^\uparrow$. □

Satoh has also defined arc diagrams [1], which are welded knots with endpoints, in order to describe ribbon 2-knots (which have the diffeomorphism type of a sphere). However, it is not clear whether this theorem extends to this case or not. In particular, the theorem must extend if there are arc diagrams which are not $(-)$ amphichiral.

It remains an open question to determine the extent to which $\text{Tube}$ fails to be injective. For example, is it possible to place an upper bound on the cardinality of $\text{Tube}^{-1}(R)$, or to describe this set precisely? By considering the peripheral structure we can place a partial bound on this set, and also show that the peripheral structure is a classifying invariant on the subset of ribbon torus knots consisting of oriented spun tori. To prove this we will need to carefully examine the peripheral group [2, 3, 4] classical knots.

4. The Longitude Group Invariant

Recall that a surface knot is an embedding of a surface, of arbitrary genus, into $S^4$. We define the longitude group invariant generally, for any surface knot regardless of genus, in a way which generalizes the longitude invariant of a classical knot.

Define the linking number of a loop with a surface knot as follows. Given a Wirtinger presentation of the knot group $G$ from the broken surface diagram of a surface knot there is a homomorphism $G \to \mathbb{Z}$ defined as the sum of the exponents in a word defining the element. This is well defined, as any changes to the word involve replacing a generator with a conjugate of a generator, or the reverse, and therefore do not change the sum of the exponents. Similarly, under Roseman moves on the diagram, the definition remains unchanged.
Let $R$ be a surface knot of genus $n$ embedded in a manifold $M$. Let $N(R)$ be a regular neighborhood of $R$, whose closure is contained in another regular neighborhood of $R$, and let $X = M - N(R)$. Let $B = \partial X$. Observe that $B = \partial N(X)$, as well. There is a natural embedding $i : B \hookrightarrow M$, which is the inclusion. The embedding induces a homomorphism $i_* : \pi_1(B) \to \pi_1(X)$. We refer to the image of this homomorphism as a peripheral group of $R$, $P(R)$. The longitude group is now defined to be those elements in the image which do not link with the knot. From the definition, the linking number is additive under composition, from which it follows that this set is in fact a group. We define this to be the longitude group, and denote it by $LG(R)$.

Now any element of the knot group for an oriented surface represents some homology class of curves. The first homology will be isomorphic to the integers, since the abelianization of any group with a Wirtinger presentation, such as the knot group, is isomorphic to the integers. The linking number of a curve representing an element in the knot group is defined to be the image in the integers of its homology class under the natural isomorphism.

We define an element of $P(R)$ with linking number 1 to be a meridian (of the knot group) if it bounds a hyperdisk inside $B$ about a point on the surface knot and has linking number 1. The meridian element of the peripheral group is denoted by $m(R)$. Uniqueness for the meridian follows from the fact that any regular neighborhood of an oriented knotted surface is a trivial disk bundle.

The triple $(P(R), LG(R), m(R))$ will be called the peripheral structure. Note also that this definition may be immediately generalized to other dimensions. However except in dimension 1 the longitude group may be acyclic, and even if it is cyclic there need not be a canonical generator.

Observe that one may choose a different meridian, which defines a different longitude group conjugate to the first one. We will use the term peripheral structure to refer to this collection and its conjugacy relations. Recall that the quandle of a surface knot may be defined in terms of nooses which link with the surface, as Joyce defined them for one dimensional knots[5]. By the topological definition of all these constructions,

**Theorem 4.1.** If $R, R'$ are surface knots which are ambiently isotopic, then there is an isomorphism of their quandles, which induces an isomorphism of their peripheral structure.

We can also extend a theorem of Joyce about quandles of classical knots and peripheral groups. Recall that the knot group has a natural right action on the quandle given by composing the quandle noose with the group loop.

**Theorem 4.2.** For a surface knot $R$ with group $G$ and quandle $Q$, let $q \in Q$, and let $G_q = \{g \in G | qg = q\}$. Then $G_q$ is a peripheral group of $R$. Conversely any peripheral group stabilizes some quandle element.
Proof: See [5], Thm. 16.1; the proof generalizes identically. We sketch the proof here. Given a peripheral group, choose a quandle element $q$ such that the meridian is representable by a path which follows the quandle noose and then comes back upon it. Then the meridian stabilizes $q$. Now given an element $g$ of the longitude group, we may perform a homotopy so that this element lies in a regular neighborhood of a surface. Acting on $q$ with this element, we may now pull the head of $q$ along a path on the surface parallel to $g$, which will end back at $q$ itself.

The other direction is similar. Given $q$ we choose the peripheral group whose meridian is parallel to $q$. Then that peripheral group stabilizes $q$. No other elements stabilize it, however. For if some element $g$ stabilizes it, then the homotopy from $qg$ to $q$ implies that we may push $g$ via a homotopy to make it into a path lying completely within some regular neighborhood of the surface conjugated by $q$. It must therefore be in the peripheral group with meridian $q$. See [5] for details. □

4.1. Computing Longitude Groups for Surface Knots

Consider an arbitrary surface knot $R$. Let us take a broken surface diagram (although the method will generalize to any desired presentation). Observe that the Wirtinger generators, those which loop through only one surface in the broken surface diagram, each have linking number 1.

Consider the boundary of a regular neighborhood, $B$. The regular neighborhood will be a trivial disk bundle $D^2 \times R$, as $R$ has codimension two and all orientable surfaces in $S^4$ have normal Euler number 0. Therefore the boundary of this disk bundle will be $B \cong R \times S^1$. Let us take a particular generating set for $\pi_1(B) \cong \pi_1(R) \times \mathbb{Z}$ consisting of $\{a_1, ..., a_n\}$ the generators of the $R$ component, and $m$ the generator of the $S^1$ component. We choose these such that $i_*(m)$ is the meridian, with linking number 1, and such that to find $i_*(a_j)$ we take a path on the surface of the broken surface diagram and perturb it off the surface. It follows then that to write down $i_*(a_j)$ we need only record the signed overcrossings which we encounter as we loop around the broken surface diagram.

Now we wish to change to a collection of generators which will make the longitude group and the meridian easily recognizable. To do so, let $k_j$ be the sum of the signs of the overcrossings encountered along $a_j$; that is, it is the linking number of $a_j$. Let us construct a new generating set for $\pi_1(B)$ consisting of $\{m, b_1, ..., b_n\}$, where $b_j = a_j m^{-k_j}$. It is straightforward to verify that this really is a generating set. Furthermore, all the images of the generators have linking number 0 except for $i_*(m)$ which has linking number 1. But this is a generating set of the peripheral group. Therefore, $LG(R)$ is generated by $\{i_*(b_1), ..., i_*(b_n)\}$, since these have linking number 0.

Now notice also that $P(R)$ is the image of a homomorphism from $\pi_1(R) \times \mathbb{Z}$, and that $m(R)$ is the image of the generator of $\mathbb{Z}$, which commutes with anything in $\pi_1(R)$ part of the product. Therefore,
Theorem 4.3. For any surface knot $R$ with the diffeomorphism type of $N$, $LG(R)$ is a quotient of $\pi_1(N)$, and $m(K)$ commutes with all elements of the peripheral group.

Note that this implies that the longitude group of any sphere-knot is trivial.

We now restrict our attention to the specific case of tori. For a torus, $B \cong S^1 \times S^1 \times S^1$. Following the above computation we see that the longitude group is isomorphic to a quotient of $\mathbb{Z} \oplus \mathbb{Z}$.

![Fig. 6. The three generators of the peripheral group for a torus.](image)

Theorem 4.4. If $R$ is a surface knot with the diffeomorphism class of a torus, then $LG(R)$ is a quotient of $\mathbb{Z} \oplus \mathbb{Z}$.

For welded knots, define the peripheral group to be the subgroup of the knot group generated by the longitude and meridian. Observe that this combinatorial definition agrees with the topological definition for classical knots. Now consider $R \cong Tube(K)$ where $K$ is a welded knot. Then we have the following important result about Satoh's construction.

Theorem 4.5. The peripheral group, meridian, and longitude group are preserved by $Tube$.

Proof: Let the generators of $\pi_1(B)$ be $\alpha, \beta, \gamma$. Map $\alpha$ to the meridian of the peripheral group. We may choose this to coincide with the meridian of the welded knot, which proves the second claim. We send $\beta$ to the path which loops around the meridian of the torus (not to be confused with the meridian of the peripheral group), which is a trivial loop in the knot group and therefore already has linking number 0. Finally we send $\gamma$ to the path traveling around the equator of the torus. $i_* (\gamma)$ is therefore given by recording the signed overcrossings it encounters. We take a new generating element $\delta = \gamma \alpha^{-k}$ where $k$ is the sum of these signs. As before, $\{\alpha, \beta, \delta\}$ form a generating set for $\pi_1(B)$, and furthermore $LG(R)$ is generated by $i_* (\delta)$, since any element in $LG(R)$ must have a reduced word consisting only of $i_* (\delta)$ and $i_* (\beta)$, the latter being trivial. But $i_* (\delta)$ is also the longitude of $K$, or possibly its inverse. The cyclic subgroup generated by $i_* (\delta)$ is the longitude group of $K$ in either case. Therefore the longitude group is preserved, and since the meridian is
also preserved, the peripheral group is as well. □

However, it is not possible to choose a preferred generator of the longitude group for ribbon tori, as one may for classical knots (for if it were possible, then Tube would be injective when restricted to classical knots). In particular, for classical knots the orientation induces a choice of canonical generator for the longitude group, whereas the orientation of a surface does not. However, there are only two choices for generators of the longitude group for the surface, \( i_\ast(\delta) \) and \( i_\ast(\delta)^{-1} \). Our computation shows that one of these is the longitude of the corresponding welded knot, while the other is its inverse. Note also that as a consequence of the above theorem and the fact that the peripheral group of any torus (surface) knot is abelian (being the image of a homomorphism of \( \pi_1(S^1 \times S^1 \times S^1) \)) that the peripheral group of a welded knot is abelian.

Define the peripheral structure of a welded knot to include all the peripheral groups and their conjugation relations; from these definitions it follows that

**Theorem 4.6.** Tube preserves the conjugacy relations among the peripheral groups, and hence the peripheral structure.

As an immediate consequence from our computation we have

**Theorem 4.7.** If a torus embedding \( R \) is ribbon, then \( LG(R) \) is cyclic.

However, this condition is not sufficient. Consider the so-called ”1-turned trefoil torus,” which is studied in [13]. This is constructed from a trefoil by rotating the classical knot around a \( S^2 \subset S^4 \) surface, while also turning it \( 2\pi \) around another \( S^2 \subset S^4 \). The resulting longitude group is isomorphic to that of the trefoil and hence cyclic. To compute this we choose one generator to be the one which is parallel to the trefoil itself, and the other to be one which spins and rotates with some point on the trefoil. The latter loop is contractible, whereas the former generates the longitude group of the 1-turned trefoil torus[12]. However this torus embedding fails to be ribbon [14]. Additionally not all nontrivial tori will have nontrivial longitude groups; indeed the connected sum of a trivial torus embedded in \( S^4 \) with any knotted sphere will have trivial longitude group. For the generators of the longitude group will be homotopic to paths parallel to the meridian and longitude of the trivial torus, both of which are contractible. To summarize our above results, then:

**Theorem 4.8.** Tube preserves the knot group, the meridian, and the longitude up to inverse, as well as the conjugacy relations between different peripheral groups.

### 5. A Classification of Oriented Spun Tori

Throughout this section we will work only with oriented 1-component classical knots. We will therefore denote the mirror image of a knot \( K \) by \( K^\ast \), since there is no ambiguity.
The triple \((\pi_1, P, m)\) (hereinafter referred to as the peripheral structure) classifies classical oriented knots up to mirror reverses\[14][9][15].

**Theorem 5.1.** If \(K\) and \(K'\) are oriented one-component classical knots with isomorphic peripheral structures, then either \(K \cong K'\) or \(K \cong -K'^*\).

From this theorem, together with Thm. 4.8, we have

**Theorem 5.2.** For classical oriented knots \(K, K'\), \(\text{Tube}(K) \cong \text{Tube}(K')\) (or equivalently \(\text{OSpun}(K) \cong \text{OSpun}(K')\)) iff \(K \cong K'\) or \(K \cong -K'^*\).

Proof: If \(\text{Tube}(K) \cong \text{Tube}(K')\) then \(K\) and \(K'\) have isomorphic peripheral structures, and therefore \(K \cong K'\) or \(K \cong -K'^*\). On the other hand if \(K \cong K'\) then \(\text{Tube}(K) \cong \text{Tube}(K')\) automatically. If \(K \cong -K'^*\), then \(\text{Tube}(K') \cong -\text{Tube}(K'^*).\)

As a consequence, we have an algebraic version of this theorem.

**Theorem 5.3.** Oriented spun tori are classified by their peripheral structures.

It is an open question to determine whether the peripheral structure suffices to classify ribbon torus knots generally. It is known that it does not classify torus embeddings up to isotopy generally\[12\]. A related problem is to determine whether the knot group, meridian, and longitude of an oriented welded knot suffice to determine the welded knot. A positive answer to the latter problem would imply a positive answer to the former question, using the same method as used above to classify the oriented spun tori. Conversely, if the peripheral structure classifies ribbon tori, this would imply that it classifies oriented welded knots up to reversed vertical reflection.

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