THE FINITE EMBEDDING PROPERTY FOR IP LOOPS
AND LOCAL EMBEDDABILITY OF GROUPS
INTO FINITE IP LOOPS

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Abstract. We prove that the class of all loops with the inverse property (IP loops) has the Finite Embedding Property (FEP). As a consequence, every group is locally embeddable into finite IP loops.

The Finite Embedding Property (briefly FEP), was introduced by Henkin [13] for general algebraic systems already in 1956. For groupoids (i.e., algebraic structures \((G, \cdot)\) with a single binary operation), which is sufficient for our purpose, it reads as follows: A class \(K\) of groupoids has the FEP if for every algebra \((G, \cdot) \in K\) and each nonempty subset \(X \subseteq G\) there is a finite algebra \((H, \ast) \in K\) extending \((X, \cdot)\), i.e., \(X \subseteq H\) and \(x \cdot y = x \ast y\) for all \(x, y \in X\), such that \(x \cdot y \in X\). Using this notion an earlier result of Henkin [12] can be stated as follows: The class of all abelian groups has the FEP (see also Grätzer [11]).

A more general notion of local embeddability can be traced back to even earlier papers by Mal’tsev [16], [17] (see also the posthumous monograph [18]). It was explicitly (re)introduced and studied in detail mainly for groups by Vershik and Gordon [20]: A groupoid \((G, \cdot)\) is locally embeddable into a class of groupoids \(M\) if for every \(X \subseteq G\) there is an \((H, \ast) \in M\) such that \(X \subseteq H\) and \(x \cdot y = x \ast y\) for all \(x, y \in X\) satisfying \(x \cdot y \in X\). Informally this means that every finite cut-out from the multiplication table of \((G, \cdot)\) can be embedded into an algebra from \(M\). A standard model-theoretic argument shows that this condition is equivalent to the embeddability of \((G, \cdot)\) into an ultraproduct of algebras from \(M\) (for the ultraproduct construction see, e.g., Chang, Keisler [3]).

Thus a class \(K\) has the FEP if and only if every \((G, \cdot) \in K\) is locally embeddable into the class \(K_{\text{fin}}\) of all finite members in \(K\). The groups locally embeddable into (the class of all) finite groups were called LEF groups in [20]. The authors also noticed that, unlike the abelian ones, not all groups are LEF, in other words, the class of all groups doesn’t have the FEP. This immediately raises the question of finding some classes of finite groupoids into which all the groups were locally embeddable and which, at the same time, would be “as close to groups as possible”. The question is of interest for various reasons: The class of all LEF groups properly extends the class of all locally residually finite groups and plays an important role, in dynamical systems, cellular automata, etc. (see, e.g., Ceccherini-Silberstein, Coornaert [2], Vershik, Gordon [20]).

Glebsky and Gordon [9] have shown that a group is locally embeddable into finite semigroups if and only if it is an LEF group. It follows that looking for a
class of finite groupoids into which one could locally embed all the groups one has
to sacrifice the associativity condition. They also noticed that the results about
extendability of partial latin squares to (complete) latin squares imply that every
group is locally embeddable into finite quasigroups. Refining slightly the original
argument they have shown that every group can even be locally embedded into
finite loops (see also their survey article [10]).

A further decisive step in this direction was done by Ziman [21]. Refining con-
siderably the methods of extension of partial latin squares and preserving some
symmetry conditions (see Lindner [15], Cruse [4]), he has shown that the class of
all loops with antiautomorphic inverses, i.e., loops with two-sided inverses satisfying
the identity

\[(xy)^{-1} = y^{-1}x^{-1}\]

(briefly AAI loops), has the FEP (though he didn’t use this notion explicitly). As
a consequence, every group is locally embeddable into finite AAI loops.

Quasigroups and loops experts consider the class of all AAI loops still as a
“rather far going extension” of the class of all groups. On the other hand, they find
the class of all loops with the inverse property, i.e., loops with two-sided inverses satisfying the identities

\[x^{-1}(xy) = y = (yx)x^{-1}\]

(briefly IP loops), which is a proper subclass of the class of all AAI loops, a much
more moderate extension of the class of all groups (Drápal [7]). In the present
paper we are going to show that Ziman’s result can indeed be strengthened in this
sense. Using mainly graph-theoretical methods and Steiner triple systems, we will
prove that the class of all IP loops still has the FEP. As a consequence, every group
is locally embeddable into finite IP loops.

For basic definitions and facts about quasigroups and loops the reader is referred
to the monographs Belousov and Belyavskaya [1] and Pflugfelder [19].

1. Formulation of the main results and plan of the proof

Our main results are the following two theorems, the second one of which is obvi-
ously an immediate consequence of the first one.

**Theorem 1.** The class of all IP loops has the Finite Embedding Property.

**Theorem 2.** Every group can be locally embedded into the class of all IP loops. Equivalently, every group can be embedded into some ultraprod-uct of finite IP loops.

We divide the proof of Theorem 1 into three steps consisting of the three propo-
sitions below. Their formulation requires some notions an notation.

A partial IP loop \((P, \cdot)\) is a set \(P\) endowed with a partial binary operation \(\cdot\)
defined on a subset \(D(P) \subseteq P \times P\), called the domain of the operation \(\cdot\), satisfying the following three conditions:

1. there is an element \(1 \in P\), called the unit of \(P\), such that \((1, x), (x, 1) \in D(P)\)
   and \(1x = x1 = x\) for all \(x \in P\);
2. for each \(x \in P\) there is a unique \(y \in P\), called the inverse of \(x\) and denoted by
   \(y = x^{-1}\), such that \((x, y), (y, x) \in D(P)\) and \(xy = yx = 1\);
3. for any \(x, y \in P\) such that \((x, y), (y, x) \in D(P)\) we have \((x^{-1}, xy), (xy, y^{-1}) \in D(P)\)
   and \(x^{-1}(xy) = y, (xy)y^{-1} = x\).
In most cases we will denote a partial IP loop \((P, \cdot)\) as \(P\), only; the more unambiguous notation \((P, \cdot)\) will be used just in case we need to distinguish the operations on two or more more (partial) IP loops.

Obviously, every subset \(P \subseteq L\) of an IP loop \(L\), such that \(1 \in P\) and \(x^{-1} \in P\) for each \(x \in P\), gives rise to a finite IP loop with domain
\[
D(P) = \{ (x, y) \in P \times P : xy \in P \}.
\]

A partial IP loop \((Q, *)\) is called an extension of a partial IP loop \((P, \cdot)\) if \(P \subseteq Q\), \(D(P) \subseteq D(Q)\) and \(x \cdot y = x * y\) for each pair \((x, y) \in D(P)\). Suppressing the signs of the operations, we write \(P \leq Q\) or \(Q \geq P\). Obviously, the relation \(\leq\) between partial IP loops is reflexive, antisymmetric and transitive.

In the absence of associativity there is no obvious way how to define the order of an element. Nonetheless, the sets of elements of order 2 and 3, respectively, can still be defined for any partial IP loop \(P\):
\[
O_2(P) = \{ x \in P : (x, x) \in D(P), x \neq 1 \text{ and } xx = 1 \},
\]
\[
O_3(P) = \{ x \in P : (x, xx) \in D(P), x \neq 1 \text{ and } x(xx) = 1 \}.
\]

In other words, for an \(x \neq 1\) in \(P\) we have \(x \in O_2(P)\) if and only if \(x^{-1} = x\), and \(x \in O_3(P)\) if and only if \(x^{-1} = xx\). The number of elements of the sets \(O_2(P)\), \(O_3(P)\) in a finite partial IP loop \(P\) will be denoted by \(o_2(P), o_3(P)\), respectively. In general, the number of elements of a finite set \(A\) is denoted by \(#A\).

**Proposition 1.** Let \((P, \cdot)\) be a finite partial IP loop. Then there exists a finite partial IP loop \((Q, *)\) such that \(P \leq Q\) and \(3 \mid o_3(Q)\).

A pair \((x, y)\) in a partial IP loop \(P\) will be called a gap if \((x, y) \notin D(P)\). The set of all gaps in \(P\) will be denoted by
\[
\Gamma(P) = (P \times P) \setminus D(P) = \{ (x, y) \in P \times P : (x, y) \notin D(P) \}
\]

Obviously, both \(D(P)\), \(\Gamma(P)\) are binary relation on the set \(P\), and a partial IP loop \(P\) is an IP loop if and only if it contains no gaps, i.e., \(\Gamma(P) = \emptyset\).

**Proposition 2.** Let \(P\) be a finite partial IP loop such that \(3 \mid o_3(P)\). Then there exists a finite partial IP loop \(Q\) satisfying the following four conditions:

\(4\) \quad \(3 \mid o_3(Q), \quad \#Q \geq 10, \quad \#Q \equiv 4 (\text{mod } 6)\) and \(\Gamma(Q) \subseteq O_2(Q) \times O_2(Q)\).

**Proposition 3.** Let \(P\) be a finite partial IP loop satisfying the above conditions \((4)\), such that \(\Gamma(P) \neq \emptyset\). Then there is a finite partial IP loop \(Q \geq P\) satisfying the conditions \((4)\), as well, such that \(\#\Gamma(Q) < \#\Gamma(P)\).

Theorem 1 follows from Propositions 1 \textsuperscript{2} 2 and 3 \textsuperscript{3}. Indeed, given an IP loop \(L\) and a finite set \(X \subseteq L\) (which is not a sub-IP loop of \(L\)) we can form the finite partial IP loop
\[
P = X \cup \{1\} \cup X^{-1},
\]
where \(X^{-1} = \{ x^{-1} : x \in X \}\), satisfying \(X \subseteq P \leq L\). Then, using Proposition 1 we can find a finite partial IP loop \(Q \geq P\) such that \(3 \mid o_3(Q)\). If \(\Gamma(Q) = \emptyset\) then \(Q\) is already a finite IP loop extending \(P\), and we are done. Otherwise, applying Proposition 2 we obtain a finite partial IP loop \(Q_1 \geq Q\) satisfying conditions \((4)\) from Proposition 2. If \(\Gamma(Q_1) = \emptyset\) then we are done, again. Otherwise, we can apply Proposition 3 and get a finite partial IP loop \(Q_2 \geq Q_1\) satisfying conditions \((4)\), as well, such that \(\#\Gamma(Q_2) < \#\Gamma(Q_1)\). Iterating this step finitely many times we
finally arrive at some finite partial IP loop \( Q_n \) extending \( P \) such that \( \Gamma(Q_n) = \emptyset \). Then \( Q_n \geq P \) is a finite IP loop we have been looking for.

Thus it is enough to prove the Propositions 1, 2 and 3. This will take place in the next four sections.

2. SOME PRELIMINARY RESULTS

In this section we list the auxiliary results we will use in the proofs of Propositions 1, 2 and 3.

Lemma 1. Let \( P \) be a partial IP loop and \( x, y, z \in P \). Then the following six conditions are equivalent:

(i) \((x, y) \in D(P) \) and \( xy = z \);
(ii) \((z, y^{-1}) \in D(P) \) and \( zy^{-1} = x \);
(iii) \((x^{-1}, z) \in D(P) \) and \( x^{-1}z = y \);
(iv) \((y, z^{-1}) \in D(P) \) and \( yz^{-1} = x^{-1} \);
(v) \((z^{-1}, x) \in D(P) \) and \( z^{-1}x = y^{-1} \);
(vi) \((y^{-1}, x^{-1}) \in D(P) \) and \( y^{-1}x^{-1} = z^{-1} \).

Proof. Applying the IP loop property (3) in a proper way and (if necessary) using the fact that \((a^{-1})^{-1} = a \) for any \( a \in P \), we can get the following cycle of implications:

\[(i) \Rightarrow (ii) \Rightarrow (v) \Rightarrow (iv) \Rightarrow (iii) \Rightarrow (i).\]

We show just the first implication, leaving the remaining ones to the reader. If \((x, y) \in D(P) \) and \( xy = z \) then, according to (3), \((z, y^{-1}) = (xy, y^{-1}) \in D(P) \) and \( zy^{-1} = x \).

The last lemma implies that if any three of the pairs above are gaps in \( P \) then so are the remaining three pairs, as well.

In the generic case all the six equivalent conditions above are different. There are just two kinds of exceptions: first the trivial ones, when at least one of the elements \( x, y, z \) equals the unit 1 (which never produce gaps), and second, if \( x = y \in O_3(P) \), when the six conditions reduce to just two:

- \((x, x) \in D(P) \) and \( xx = x^{-1} \),
- \((x^{-1}, x^{-1}) \in D(P) \) and \( x^{-1}x^{-1} = x \).

From now on we will preferably use a more relaxed language: when writing \( xy = z \) for elements \( x, y, z \) of some partial IP loop \( P \) we will automatically assume that \((x, y) \in D(P) \), without mentioning it explicitly.

The number of gaps in any finite IP loop \( P \) is related to the size of \( P \) and that of the set \( O_3(P) \) of order three elements through a congruence modulo 6.

Lemma 2. Let \( P \) be a finite partial IP loop. Then

\[# \Gamma(P) \equiv (#P - 1)(#P - 2) - o_3(P) \pmod{6} \]

Proof. We know that \((x, 1), (1, x), (x, x^{-1}) \in D(P) \) for any \( x \in P \). At the same time, \((a, a) \in D(P) \) for all \( a \in O_3(P) \). Except for these pairs, there are other \((#P - 1)(#P - 2) - o_3(P) \) pairs which can be either in \( D(P) \) or in \( \Gamma(P) \). Those which are in \( D(P) \) can be split into sixtuples according to Lemma 1 hence their number is divisible by 6, proving the above congruence. \( \square \)
We will also use one graph-theoretical result, namely the Dirac’s theorem\footnote{1} giving a sufficient condition for the existence of a Hamiltonian cycle in a graph. For our purpose, the term graph always means an undirected graph without loops and multiple edges. For the basic graph-theoretical concepts the reader is referred to Diestel\footnote{2}.

**Lemma 3.** Let \( G \) be a graph with \( n \geq 3 \) vertices in which every vertex has the degree at least \( n/2 \). Then \( G \) has a Hamiltonian cycle.

### 3. Extensions of partial IP loops and the proof of Proposition 1

All the three Propositions\footnote{1}\footnote{2}\footnote{3} deal with extensions of a partial IP loop \((P, \cdot)\), which can be combined using two more specific types of this construction: first, extensions preserving (the domain of) the binary operation \( \cdot \) on the original partial IP loop \( P \) and extending the base set of \( P \), and, second, extensions preserving the base set of \( P \) and extending (the domain of) the binary operation on \( P \). We start with the first type of extensions.

Let \( P, Q \) be two partial IP loops such that \( P \cap Q = \{1\} \), i.e., their base sets have just the unit element 1 in common. Then, obviously, the set \( P \cup Q \) can be turned into a partial IP loop, which we denote by \( P \sqcup Q \), extending both \( P \) and \( Q \), with domain

\[
D(P \sqcup Q) = D(P) \cup D(Q),
\]

i.e., preserving the original operations on both \( P \) and \( Q \), and leaving undefined all the products \( xy, yx \), for \( x \in P \setminus \{1\}, y \in Q \setminus \{1\} \). The partial IP loop \( P \sqcup Q \) is called the **direct sum** of the partial IP loops \( P \) and \( Q \).

Let us fix the notation for some particular cases of this construction, considered as extensions of the IP loop \( P \) fixed in advance. In all the particular cases below \( A \) denotes a nonempty set disjoint from \( P \) such that \((A \cup \{1\}, \cdot)\) is a partial IP loop.

Let \( \sigma: A \to A \) be an involution, i.e., \( \sigma(\sigma(a)) = a \) for any \( a \in A \). Then the **minimal partial IP loop** \([A, \sigma]\) has the base set \( A \cup \{1\} \) and the partial binary operation given by \( 1 \cdot 1 = 1 \), and

\[
1a = a1 = a, \quad a\sigma(a) = \sigma(a)a = 1,
\]

for any \( a \in A \), leaving the operation result \( ab \) undefined for any other pair of elements \( a, b \in A \). The reader is asked to realize that \([A, \sigma]\) is indeed a partial IP loop, and that it is minimal (concerning its domain) among all partial IP loops with the base set \( A \cup \{1\} \), which satisfy

\[
a^{-1} = \sigma(a)
\]

for any \( a \in A \). Then, obviously,

\[
O_2[A, \sigma] = \{a \in A: \sigma(a) = a\},
\]

i.e., the order two elements in \([A, \sigma]\) coincide with the fixpoints of the map \( \sigma \). The direct sum of the partial IP loops \( P \) and \([A, \sigma]\) is denoted by

\[
P[A, \sigma] = P \sqcup [A, \sigma].
\]

The order two elements in \( P[A, \sigma] \) split into two disjoint easily recognizable parts

\[
O_2(P[A, \sigma]) = O_2(P) \cup O_2[A, \sigma].
\]

If \( \sigma = \text{id}_A: A \to A \) is the identity on \( A \), we write

\[
P[A, \text{id}_A] = P[A],
\]
in which case

\[ O_2(\{\{\}) = O_2(\{\}) \cup A. \]

If \( A = \{a_1, \ldots, a_n\} \) is finite, we write

\[ P[A] = P[\{a_1, \ldots, a_n\}]. \]

In particular, if \( A = \{a\} \) is a singleton (and \( \sigma = \text{id}_A \) is the unique map \( A \to A \)), then

\[ P[\{a\}] = P[a]. \]

If \( A = \{a, a'\} \) where \( a \neq a' \), and \( \sigma \) is the transposition exchanging \( a \) and \( a' \), we denote

\[ P[A, \sigma] = P[a \leftrightarrow a']. \]

From among the second type of extensions of a partial IP loop \( P \), preserving its base set \( P \) and extending just (the domain of) its operation the simplest ones attempt at filling in just a single gap in \( P \). This type of extension will be called a simple extension through the relation \( xy = z \). More precisely, having \( x, y, z \in P \) such that \( (x, y) \in \Gamma(P) \), we want to put \( xy = z \). From Lemma 1 it follows that then we have to satisfy the remaining five relations, too. This is possible only if all the pairs \( (z, y^-1) \), \( (x^-1, z) \), \( (y^-1, x^-1) \) (or, equivalently, any other three pairs occurring there) are gaps in \( P \). This is a sufficient condition, as well, since in that case we can define all the products as required by Lemma 1. Thus filling in the gap \( (x, y) \) enforces to fill in some other related gaps, too. In that case we automatically assume that the remaining five relations are defined in accord with Lemma 4.

Iterating simple extensions through particular relations we have have to check in each step whether any new relation \( uv = w \) (and its equivalent forms) does not interfere not only with the pairs in \( D(P) \) but also with the gaps already filled in by previous simple extensions. In other words, we are interested in situations when we can fill in a whole set of gaps at once.

If \( (P, \cdot) \) is a partial IP loop and \( * \) is a partial operation on the set \( P \) with domain \( T \subseteq P \times P \), such that such that \( T \subseteq \Gamma(P) \) then, since \( D(P) \cap T = \emptyset \), we can extend the original operation \( \cdot \) to the set \( D(P) \cup T \) by putting \( xy = x \cdot y \) for \( (x, y) \in T \). The resulting structure will be called the extension of the IP loop \( P \) through the operation \( * \). The next lemma tells us when such an extension gives us an IP loop, again. In its formulation \( x^{-1} \) denotes the inverse of the element \( x \in P \) with respect to the original operation \( \cdot \) in \( P \).

**Lemma 4.** Let \( (P, \cdot) \) be a partial IP loop and \( * \) be a partial binary operation on the set \( P \) with domain \( T \subseteq \Gamma(P) \). Then the extension of the operation \( \cdot \) through the operation \( * \) to the set \( D(P) \cup T \) yields a partial IP loop extending \( P \) if and only if \( T \) and \( * \) satisfy the following condition:

\[(5) \text{ for any } x, y, z \in P, \text{ if } (x, y) \in T \text{ and } x \ast y = z \text{ then also all the pairs } (z, y^{-1}), (x^{-1}, z), (y, z^{-1}), (z^{-1}, x), (y^{-1}, x^{-1}) \text{ belong to } T \text{ and satisfy all the relations } z \ast y^{-1} = x, x^{-1} \ast z = y, y \ast z^{-1} = x^{-1}, z^{-1} \ast x = y^{-1}, y^{-1} \ast x^{-1} = z^{-1}.\]

**Proof.** In view of Lemma 1 condition (5) obviously is necessary. By the same reason, condition (5) implies that each of the particular relations \( xy = x \ast y \), for \( (x, y) \in T \), can be separately added to \( P \). Since \( T \subseteq \Gamma(P) \), no particular relation \( xy = x \ast y \) can interfere with the remaining added relations \( uv = u \ast v \). \( \square\)
The following simple combination of both the types of extensions will be used in the proof of Proposition \(1\).

Let \(A\) be a set (disjoint from \(P\)) and \(\sigma: A \rightarrow A\) be a fixpointfree involution (i.e., \(\sigma(a) \neq a\) for every \(a \in A\)). Then \([A, \sigma]_3\) denotes the extension of the minimal partial IP loop \([A, \sigma]\) through (just) the additional relations
\[
aa = \sigma(a)
\]
for any \(a \in A\). Formally, \([A, \sigma]_3\) is the extension of \([A, \sigma]\) through the operation \(\ast\) defined on the set \(T = \{(a, a) : a \in A\}\) by \(a \ast a = \sigma(a)\) for any \(a \in A\). It is clear that each pair \((a, a)\) is indeed a gap in \([A, \sigma]\) and that the condition (5) from Lemma \(3\) is satisfied. Hence \([A, \sigma]_3\) is a partial IP loop extending \([A, \sigma]\) in which
\[
a^{-1} = \sigma(a) = aa
\]
for each \(a \in A\), i.e., every element \(a \in A\) has the order three. For the direct sum
\[
P[A, \sigma]_3 = P \sqcup [A, \sigma]_3
\]
we have
\[
O_3(P[A, \sigma]_3) = O_3(P) \cup A.
\]
If \(A = \{a, a'\}\), where \(a \neq a'\), then the denotations \([A, a \leftrightarrow a']_3\) and
\[
P[a \leftrightarrow a']_3 = P[A, a \leftrightarrow a']_3
\]
are already self-explanatory, and similarly for \([A, a \leftrightarrow a', b \leftrightarrow b']_3\) and
\[
P[a \leftrightarrow a', b \leftrightarrow b']_3 = P[A, a \leftrightarrow a', b \leftrightarrow b']_3
\]
where the set \(A\) consists of four distinct elements \(a, a', b, b'\).

**Proof of Proposition 1.** Let \(a, a', b, b'\) be four distinct elements not belonging to \(P\). Let us form the extensions \(Q = P[a \leftrightarrow a']_3\) and \(R = P[a \leftrightarrow a', b \leftrightarrow b']_3\). Obviously,
\[
o_3(Q) = o_3(P) + 2 \quad \text{and} \quad o_3(R) = o_3(P) + 4.
\]
Since one of the numbers \(o_3(P)\), \(o_3(P) + 2\), \(o_3(P) + 4\) is divisible by 3, one of the partial IP loops \(P, Q, R\) has the desired property. \(\square\)

4. **The proof of Proposition 2**

A more subtle combination of the two types of extensions introduced in Section \(3\) will be required in the proof of Proposition \(2\).

**Proof of Proposition 2.** Let \(P\) be a finite partial IP loop such that \(3 | o_3(P)\), and \(A\) be a finite set disjoint from \(P\) with the number of its elements satisfying
\[
#A \geq \max\{5(\#P) - 1, \#(P)/2\} \quad \text{and} \quad 10 \leq \#P + \#A \equiv 4 \pmod{6}.
\]
First we construct the minimal extension \(P[A]\), in which every element \(a\) of \(A\) has the order two, while
\[
O_3(P[A]) = O_3(P).
\]
Hence the partial IP loop \(P[A]\) has the base set \(P \cup A\) with the required number of elements and the same number of elements of the order three as \(P\).

Next, we construct an extension of \(P[A]\) in which all the original gaps in \(\Gamma(P)\) will be filled. We take \(T = \Gamma(P) \subseteq \Gamma(P[A])\) and introduce a binary operation \(\ast\) on \(T\), assigning to each pair of gaps \((x, y), (x^{-1}, y^{-1}) \in T\) a (self-inverse) element
\[
x \ast y = y^{-1} \ast x^{-1} = (x \ast y)^{-1}
\]
As all the elements of $A$ suffice to manage that $(P \cup A)\setminus (P \times P) = \emptyset$.

At the same time, $ab \in P$ for any $(a, b) \in D(P[A]^*) \cap (A \times A)$.

Finally, we construct an extension $Q$ of $P[A]^*$ with the same base set $P \cup A$, such that

$$
\Gamma(Q) \subseteq O_2(Q) \times O_2(Q).
$$

As all the elements of $A$ are of order two, and $P[A]^*$ has no gap $(x, y) \in P \times P$, it suffices to manage that $(x, a), (a, x) \in D(Q)$ for all $a \in A, x \in P \setminus O_2(P), x \neq 1$.

We will proceed by an induction argument. To this end we represent the set

$$
P \setminus (O_2(P) \cup \{1\}) = \{x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}\},
$$

in such a way that each pair of mutually inverse elements $x, x^{-1} \in P \setminus (O_2(P) \cup \{1\})$ occurs in this list exactly once. To start with we put $Q_0 = P[A]^*$. Now we assume that, for some $0 \leq k < n$, we already have an IP loop $Q_k \geq P[A]^*$ with the same base set $P \cup A$, satisfying the following three conditions:

1. $au, va \in A$ for any $a \in A, u, v \in P \setminus \{1\}$ such that $(a, u), (v, a) \in D(Q_k)$
2. $ab \in P$ for any $(a, b) \in D(Q_k) \cap (A \times A)$, and
3. $(x_l, a), (a, x_l) \in D(Q_k)$ for all $0 \leq l \leq k, a \in A$.

Observe that $Q_0$ trivially satisfies all these conditions (with $k = 0$), and condition (8) jointly with Lemma 1 imply that $(x_l^{-1}, a), (a, x_l^{-1}) \in D(Q_k)$ for all $0 \leq l \leq k, a \in A$, too. For $x = x_{k+1}$, we have to fill in all the gaps in $Q_k$ in which $x$ occurs, preserving all the conditions (6), (7), (8) with $k$ replaced by $k + 1$. That way all the gaps in $Q_k$ containing $x^{-1}$ will be filled in, as well.

Let us introduce the sets

$$
L_x = \{a \in A: (a, x) \in \Gamma(Q_k)\} \quad \text{and} \quad R_x = \{a \in A: (x, a) \in \Gamma(Q_k)\}.
$$

Claim 1. We have $\#L_x = \#R_x$.

Proof. Since $xu = v$ implies $v^{-1}x = u^{-1}$ for any $u, v \in P \cup A$, we have a bijection between the sets

$$
(P \cup A) \setminus L_x = \{u \in P \cup A: (x, u) \in D(Q_k)\},
$$

$$
(P \cup A) \setminus R_x = \{v \in P \cup A: (v^{-1}, x) \in D(Q_k)\},
$$

which implies that the sets $\#L_x$ and $\#R_x$ have the same number of elements. □

Thus there exists a bijective map $\eta: L_x \rightarrow R_x$ (with inverse map $\eta^{-1}: R_x \rightarrow L_x$); latter on we will specify some additional requirements concerning it. We intend to use $\eta$ in defining the extending operation $*$ on the set

$$
T_x = (L_x \times \{x, x^{-1}\}) \cup \{(x, x^{-1}) \times R_x\} \cup \{(a, \eta(a)) : a \in L_x\} \cup \{(\eta(a), a) : a \in L_x\}
$$

by putting

$$
a * x = \eta(a)
$$
for any $a \in L_x$. Then we have to satisfy the remaining five conditions of Lemma 4, i.e. (remembering that the elements of $A$ are self-inverse),

$$a \ast \eta(a) = x, \quad x^{-1} \ast a = \eta(a), \quad \eta(a) \ast a = x^{-1}, \quad \eta(a) \ast x^{-1} = x \ast \eta(a) = a.$$

The substitution $b = \eta(a)$ into the last two relations yields

$$b \ast x^{-1} = x \ast b = \eta^{-1}(b)$$

for any $b \in R_x$. It follows that each pair $(a, \eta(a))$, where $a \in L_x$, must be a gap in $Q_k$. Since $(a, a) \in D(Q_k)$ for all $a \in A$, this implies that $\eta(a) \neq a$ for $a \in L_x \cap R_x$ (if any). Additionally, $\eta$ must avoid any “crossing”, i.e., the situation that

$$\eta(a) = b \quad \text{and} \quad \eta(b) = a$$

for some distinct $a, b \in L_x \cap R_x$. This namely, according to Lemma 4, would imply that $a \ast b = x = b \ast a$, and, since $(a \ast b)^{-1} = b \ast a$, produce a contradiction $x = x^{-1}$.

Now, it is clear, that the partial IP loop $Q_{k+1}$, to be obtained as the extension of $Q_k$ through the operation $\ast$ constructed from the bijection $\eta$ as described, will satisfy all the conditions (6), (7), (8) (with $k + 1$ in place of $k$). Thus it is enough to show that there is indeed a “crossing avoiding” bijection $\eta: L_x \to R_x$ such that

$$(a, \eta(a)) \in \Gamma(Q_k)$$

for each $a \in L_x$. To this end we denote the common value $\#L_x = \#R_x$ by $m$, enumerate the sets

$$L_x = \{a_1, \ldots, a_m\}, \quad R_x = \{b_1, \ldots, b_m\}$$

in such a way that $i = j$ whenever $a_i = b_j \in L_x \cap R_x$, and introduce the graph $G_x$ on the vertex set $V = \{1, \ldots, m\}$, joining two vertices $i, j$ by an edge if and only if $i \neq j$ and both $(a_i, b_j), (a_j, b_i) \in \Gamma(Q_k)$.

**Claim 2.** The graph $G_x$ has a Hamiltonian cycle.

**Proof.** According to Lemma 5, it suffices to show that $m \geq 3$ and that the minimal degree of vertices in $G_x$ is at least $m/2$. We keep in mind that both the right side and the left side multiplication in $Q_k$ by a fixed element are injective maps.

Since $ax \in P \setminus \{1\}$ for every $a \in A$ such that $(a, x) \in D(Q_k)$, there are at most $\#P - 1$ pairs $(a, x)$ in $D(Q_k)$. Hence

$$m = \#L_x \geq \#A - \#P + 1 \geq 4(\#P) > 3.$$

Let $i$ be any vertex in $G_x$. Then $i$ is not adjacent to a vertex $j$ if and only if at least one of the pairs $(a_i, b_j), (a_j, b_i)$ belongs to $D(Q_k)$. However, for fixed $a_i$ or $b_i$, all such products $a_ib$ or $ab_i$ belong to $P$ and, in both cases, every element of $P$ occurs as a result at most once. Thus there are at most $2(\#P)$ vertices in $G_x$ not adjacent to $i$. Therefore,

$$\deg(i) \geq m - 2(\#P) \geq m - \frac{m}{2} = \frac{m}{2}.$$

Let $\pi$ be a cyclic permutation of the set $V$ such that $(1, \pi(1), \ldots, \pi^{n-1}(1))$ is a Hamiltonian cycle in $G_x$. We define $\eta: L_x \to R_x$ by

$$\eta(a_i) = b_{\pi(i)}$$

for any $i \in V$. Obviously, $\eta$ is bijective, $(a_i, \eta(a_i)) \in \Gamma(Q_k)$ for each $i \in V$, and, since $m \geq 3$, it avoids any crossing.
It follows that in the extension $Q_{k+1}$ of the partial IP loop $Q_k$ through the operation $\ast$ all the gaps from the set $T_x$ are filled in, and the conditions (6), (7), (8) are preserved. The last partial IP loop $Q = Q_n$ satisfies already all the requirements of Proposition 2. \hfill $\square$

5. Steiner triples and the proof of Proposition 3

In the proof of Proposition 3 we will make use of Steiner loops and Steiner triple systems. A Steiner loop is an IP loop satisfying the identity $xx = 1$, i.e., an IP loop in which every element $x \neq 1$ has the order two. Steiner loops are closely related to Steiner triple systems, which are systems $S$ of three element subsets of a given base set $X$ such that each two element subset \{x, y\} of $X$ is contained in exactly one set \{x, y, z\} $\in S$. Namely, if $L$ is a Steiner loop then $X = L \setminus \{1\}$ becomes a base set of the Steiner triple system $S_L = \{\{x, y, xy\} : x, y \in X\}$.

Conversely, if $S$ is a Steiner triple system with the base set $X$ then, adjoining to $X$ a new element $1 \not\in X$, we obtain a Steiner loop with the base set $X^+ = X \cup \{1\}$, the unit 1 and the operation given by the casework

$$xy = \begin{cases} 1, & \text{if } x = y, \\ z, & \text{where } \{x, y, z\} \in S, \text{ if } x \neq y, \end{cases}$$

for $x, y \in X$. Based on this definition, we will call a Steiner triple any three-element set $\{x, y, z\} \subseteq O_2(P)$ in any partial IP loop $P$, such that the product of any two of its elements equals the third one.

It is well known that there exists a Steiner triple system $S$ on an $n$-element set $X$ if and only if $n \equiv 1$ or $n \equiv 3 \pmod{6}$ (see, e.g., Hwang [14]).

The construction reducing eventually the number of gaps in a given partial IP loop $P$, satisfying certain conditions which will be emerging gradually, is composed of several simpler steps, we are going to describe, now. At the same time, it depends on a six term progression $a = (a_0, a_1, a_2, a_3, a_4, a_5)$ of pairwise distinct order two elements of $P$ chosen in advance; the criteria for its choice will be clarified later on.

The first step is the triplication construction, which uses Steiner loops heavily. Given an arbitrary finite partial IP loop $P$ such that $\#P \equiv 2$ or $\#P \equiv 4 \pmod{6}$ we denote $n = \#P - 1$. Then Steiner triple systems on $n$-element sets, as well as Steiner loops on $n + 1$-element sets exist; assume that $Y, Z$ are two $n$-element sets, such $P, Y, Z$ are pairwise disjoint, and that both the sets $Y^+ = Y \cup \{1\}$, $Z^+ = Z \cup \{1\}$ are equipped with binary operations turning them into Steiner loops.

We denote by, in a fairly ambiguous way,

$$3P = P \sqcup Y^+ \sqcup Z^+$$

the direct sum of the partial IP loop $P$ with the Steiner loops $Y^+$ and $Z^+$ (see Section 4). It is a partial IP loop with the base set $P \cup Y \cup Z$, consisting of $3n + 1$ elements, and the domain

$$D(3P) = D(P) \cup (Y \times Y) \cup (Z \times Z) \cup \{(1) \times (Y \cup Z)\} \cup ((Y \cup Z) \times \{1\})$$

We will extend the partial operation on $3P$ by filling all the gaps consisting of pairs of elements of different sets $P$, $Y$, $Z$. That way we’ll obtain an extension $3P^*$ of
3\(P\) with the same base set \(P \cup Y \cup Z\), such that \(\Gamma(3P^*) = \Gamma(P)\). The extending operation \(*\) is defined on the set

\[ T = (P_0 \times (Y \cup Z)) \cup ((Y \cup Z) \times P_0) \cup (Y \times Z) \cup (Z \times Y) \subseteq \Gamma(P \cup Y^+ \cup Z^+), \]

where \(P_0 = P \setminus \{1\}\). It depends on some arbitrary fixed enumerations

\[ P_0 = \{x_0, \ldots, x_{n-1}\}, \quad Y = \{y_0, \ldots, y_{n-1}\}, \quad Z = \{z_0, \ldots, z_{n-1}\} \]

of the sets \(P_0, Y, Z\), respectively. Once having them we put

\[ y_i \ast z_{i+k} = x_{i+2k} \]

for \(0 \leq i, k < n\), with the addition of subscripts modulo \(n\). Then, in order to satisfy the conditions of Lemma 1, we define

\[ x_{i+2k} \ast z_{i+k} = y_i, \quad y_i \ast x_{i+2k} = z_{i+k}, \quad x_{i+2k}^{-1} \ast y_i = z_{i+k}, \]

\[ z_{i+k} \ast x_{i+2k}^{-1} = y_i, \quad z_{i+k} \ast y_i = x_{i+2k}^{-1}, \]

using the fact that all the elements of \(Y\) and \(Z\) are self-inverse. As all the pairs \((x, y), \ (y, x), \ (x, z), \ (z, x), \ (y, z), \ (z, y)\), where \(x \in P_0, \ y \in Y, \ z \in Z\), are gaps in \(3P\), Lemma 4 guarantees that the extension \(3P^*\) of the partial IP loop \(3P\) through the operation \(*\) is a partial IP loop, again. For lack of better terminology we will call it a Steiner triplication of the partial IP loop \(P\) and suppress the Steiner loops \(Y^+, Z^+\) and the particular enumerations in its notation.

The Steiner triplication \(3P^*\) of \(P\) satisfies \(\Gamma(3P^*) = \Gamma(P)\), hence it still has the same number of gaps as \(P\). However, Proposition 3 requires us to decrease this number. This will be achieved in a roundabout way. First we cancel some pairs in the domain \(D(3P^*)\), creating that way the potential to fill in more gaps than we have added. In order to allow for this next step, \(P\) has to satisfy some additional conditions, namely, \(#P \geq 10\) (i.e., \(n \geq 9\)) and \(a_2(P) \geq 6\). Though the enumerations of the sets \(P_0, Y, Z\), used in the definition of the extending operation \(*\), could have been arbitrary, we now assume that these sets were enumerated in such a way that the six term progression \(a = (a_0, a_1, a_2, a_3, a_4, a_5)\) chosen in advance coincides with the sixtuple \((x_0, x_2, x_1, x_3, x_{n-3})\) and that \(\{y_0, y_1, y_3\}\) is a Steiner triple in \(Y^+\). This artificial trick will facilitate us the description of the next step of our construction.

Now, necessarily, \(z_0 = y_0x_0 = y_3x_{n-3}\), \(z_1 = y_0x_2 = y_1x_1\) and \(z_2 = y_1x_3 = y_3x_3\), in other words, we have the following seven Steiner triples in \(3P^*\):

\[ \{x_0, y_0, z_0\}, \{x_2, y_0, z_1\}, \{x_1, y_1, z_1\}, \{x_5, y_1, z_3\}, \{x_3, y_3, z_3\}, \{x_{n-3}, y_3, z_0\}, \{y_0, y_1, y_3\}. \]

We delete these triples from the domain of \(3P^*\). More precisely, for any one of these three-element sets we delete from \(D(3P^*)\) all the six pairs consisting of its distinct elements. That way we obtain a partial IP loop \(3P^- \leq 3P^*\), called the reduction of \(3P^*\) which still is an extension of \(P\), however, it has 42 more gaps than \(P\) (6 for each Steiner triple).

Instead we introduce some new triples consisting of the same elements, namely

\[ \{x_0, x_2, y_0\}, \{x_2, x_1, z_1\}, \{x_1, x_3, y_1\}, \{x_5, x_3, z_3\}, \{x_3, x_{n-3}, y_3\}, \{x_{n-3}, x_0, z_0\}, \{y_0, y_1, z_1\}, \{y_1, y_3, z_3\}, \{y_3, y_0, z_0\}. \]
which are intended to become Steiner triples, after we define a partial operation
\(\circ\) on the set \(\{x_0, x_2, x_3, x_5, x_7, x_9, y_0, y_1, y_3, z_0, z_1, z_3\}\) by putting the product of any pair of distinct elements of a given three-element set from this list equal to the third one. That way we obtain an extending operation of the partial IP loop \(3P^-\) if and only if all the pairs entering this new operation are gaps in \(3P^-\). This is obviously true for the 18 pairs arising from the three triples in the last row above. However, this need not be the case for the pairs arising from the six triples in the first two rows. The problem can be reduced to the question which of the pairs
\((x_0, x_2), (x_2, x_1), (x_1, x_5), (x_5, x_3), (x_3, x_7), (x_7, x_9)\) belong to \(\Gamma(P)\). If, e.g.,
\((x_0, x_2) \notin \Gamma(P)\) then we cannot put \(x_0 \circ x_2 = y_0\), so that \((x_0, x_2, y_0)\) cannot become a Steiner triple.

Therefore, we include just those triples \(\{x_i, x_j, y_k\}\) or \(\{x_i, x_j, z_k\}\) for which the pair \((x_i, x_j)\) is a gap in \(P\). Every such a “good” triple results in filling in six gaps. We already have 18 gaps filled in thanks to the last row. Thus we need at least five “good” triples in the first two rows in order to fill in additional 30 gaps; this would give \(18 + 30 = 48\) gaps, while having just four “good” triples results in refilling back 42 gaps, only. In general, we can fill in \(6(3 + g)\) gaps, where \(0 \leq g \leq 6\) is the number of gaps \((x_i, x_j)\) in the list.

We refer to this last step of the construction as to “filling in the gaps along the path” \(a\) and denote the final resulting extension of the reduction \(3P^-\) by \(3P(a)\). Obviously, \(3P(a)\) is an extension of the original IP loop \(P\), as well, having by \(6(3 + g) - 42 = 6(g - 4)\) less gaps than \(P\). This number can be negative, 0 or positive, depending on whether \(g < 4\), \(g = 4\), or \(g > 4\). That’s why we are interested just in the case when \(g \geq 4\).

After all these preparatory accounts we can finally approach the proof of Proposition 3.

**Proof of Proposition 3.** Let \(P\) be a finite partial IP loop satisfying the conditions
\[3 \mid \#O_2(P), \#P \geq 10, \#P \equiv 4 \pmod{6}, \Gamma(P) \subseteq O_2(P) \times O_2(P),\]
such that \(\Gamma(P) \neq \emptyset\). We are to show that there is a finite partial IP loop \(Q \supseteq P\) satisfying these conditions, as well, with less gaps than \(P\).

Since \(\Gamma(P) \subseteq O_2(P) \times O_2(P)\), it is an antireflexive and symmetric relation on the set \(O_2(P)\). Thus we can form the gap graph \(G(P) = (V, E)\) with the set of vertices
\[V = \{x \in O_2(P) : (x, y) \in \Gamma(P) \text{ for some } y \in O_2(P)\}\]
and the set of edges
\[E = \{(x, y) : (x, y) \in \Gamma(P)\} .\]
From the definition of the set of vertices \(V\) it follows there are no isolated vertices in \(G(P)\). Let’s record some less obvious useful facts about this graph.

**Claim 3.** (a) The degree of each vertex in \(G(P)\) is even.
(b) The number of edges in \(G(P)\) is divisible by three.

**Proof.** (a) Let \(x \in O_2(P)\). Then the conditions \(xy = z\) and \(xz = y\) are equivalent for any \(y, z \in P\). Additionally, as \(x \neq 1\), from \(xy = z\) it follows that \(y \neq z\). Thus the elements \(y \in P\) such that \((x, y) \in D(P)\) can be grouped into pairs, hence their number is even. As \(#P\) is even, too, so is the degree
\[\deg(x) = \#\{y \in O_2(P) : (x, y) \in \Gamma(P)\} = #P - \#\{y \in P : (x, y) \in D(P)\} .\]
(b) By Lemma 2 we have
\[ \# \Gamma(P) \equiv (\#P - 1)(\#P - 2) - o_3(P) \equiv 0 \pmod{6}. \]
On the other hand, \( \#P \equiv 4 \pmod{6} \) and \( 3 \mid o_3(P) \), yielding \( 3 \mid \#\Gamma(P) \). Obviously, the number of edges in \( G(P) \) is half of the number of gaps \( \#\Gamma(P) \), hence the number of edges in \( G(P) \) must be divisible by three.

The structure of connected components in \( G(P) \) obeys the following alternative.

**Claim 4.** Let \( C \) be a connected component of the graph \( G(P) \). Then either \( C \) contains a triangle or a path of length five, or, otherwise, \( C \) is isomorphic to one of the following graphs: the cycle \( C_4 \) of length four, the cycle \( C_5 \) of length five or the complete bipartite graph \( K_{2,m} \) where \( m \geq 4 \) is even.

**Proof.** Let \( C \) be any connected component in \( G(P) \). As \( G(P) \) has no isolated vertices and the degree of every vertex in \( C \) is even (and therefore at least two), there is a cycle in \( C \). Assume that \( C \) contains no triangle and no path of length five. Then the length of this cycle must be bigger than three and less than six. Therefore, there are just the following two options:

(a) There is a cycle of length five in \( C \). Then there cannot be any other edge coming out from its vertices since then there would be a path of length five contained in \( C \). Thus \( C \) coincides with this cycle.

(b) There is a cycle of length four in \( C \); let us denote it by \( (v_0, v_1, v_2, v_3) \). Then, as \( G(P) \) contains no triangle, neither \( \{v_0, v_2\} \) nor \( \{v_1, v_3\} \) is an edge in \( G(P) \). If there are no more vertices in \( C \) then \( C \) is a cycle of length four.

Otherwise, we can assume, without loss of generality, that there is a fifth vertex \( u_0 \in C \) adjacent to \( v_0 \). As \( u_0 \) has an even degree, it must be adjacent to some other vertex, too. If it were adjacent to some vertex \( u_1 \), distinct from all the vertices \( v_0, v_1, v_2, v_3 \), there would be a path \( (u_1, u_0, v_0, v_1, v_2, v_3) \) of length five in \( C \). If \( u_0 \) were adjacent to \( v_1 \) or to \( v_3 \), there would be a triangle \( (u_0, v_0, v_1) \) or \( (u_0, v_0, v_3) \) in \( C \). That means that \( \{u_0, v_2\} \) must be an edge in \( G(P) \) and \( \deg(u_0) = 2 \).

It follows that every other vertex in \( C \) must have the degree two and it must be adjacent either to \( v_0 \) and \( v_2 \) or to \( v_1 \) and \( v_3 \). However, the second option is impossible, since in that case \((u_0, u_0, v_0, v_1, v_1, v_1, v_2, v_3)\) would be a path of length five. This means that \( C \) is isomorphic to the complete bipartite graph \( K_{2,m} \), where one term of this partition is formed by the set \( \{v_0, v_2\} \) and the second one by the rest of the vertices in \( C \). Since every vertex has an even degree, \( m \) must be even. At the same time, \( m \geq 4 \), as \( K_{2,2} \) has just four vertices (and it is isomorphic to the cycle \( C_4 \)).

Thus the proof of Proposition 3 will be complete once we show how to construct the extension \( Q \) in each of the cases listed in Claim 4.

(a) \( G(P) \) contains a triangle, i.e., a three-element set of vertices \( \{x, y, z\} \) such that all its two-element subsets are edges. Then we can extend \( P \) through the operation \( * \) turning \( \{x, y, z\} \) into a Steiner triple. The corresponding extension \( Q \) of \( P \) has all the properties required and by six less gaps than \( P \).

(b) \( G(P) \) contains a path \( a = (a_0, a_1, a_2, a_3, a_4, a_5) \) of length five. Then we can form the Steiner tripllication \( 3P^* \) of \( P \) and, filling in the gaps along the path \( a \) in its reduction \( 3P^- \), we obtain the final extension \( Q = 3P(a) \) satisfying the condition
(4), again. If \((a_5, a_0)\) is a gap in \(P\) (i.e., if \(a\) is a cycle of length five in \(G(P)\)) then \(Q\) has twelve gaps less than \(P\), otherwise it still has by six gaps less than \(P\).

We still have to prove Proposition 3 in the case there is neither any triangle nor any path of length five in \(G(P)\). To this end it is enough to construct, in each of the remaining cases listed in Claim 4 an extension \(Q\) of \(P\) such that the graph \(G(Q)\) has the same number of edges as \(G(P)\), however, there is a path \(b\) of length five in \(G(Q)\). From such a \(Q\) we can construct another extension \(3Q(b) \geq Q \geq P\) with a smaller number of gaps and still satisfying the condition (4), similarly as we did in the case (b). So let us have a closer look at the remaining cases.

(c) \(G(P)\) contains a connected component isomorphic to \(K_{2,m}\), where \(m \geq 4\). Let \(\{u_0, u_1\}\) be the two-element term of the partition and \(\{v_0, v_1, v_2, v_3\}\) be any four-element subset of the second partition term. We denote by \(a\) the six-term progression \((v_0, u_0, v_1, v_2, u_2, v_3)\) and construct the extension \(Q = 3P(a)\) of the partial IP loop \(P\) with the gap graph \(G(Q)\). Then \(\{v_0, u_0\}, \{u_0, v_1\}, \{v_2, u_2\}\), and \(\{v_2, v_3\}\) are edges in \(G(P)\), while \(\{v_1, v_0\}\) and \(\{v_3, v_0\}\) are not. Hence the new graph \(G(Q)\) has the same number of edges as \(G(P)\) and \(Q\) has the same number of gaps as \(P\). At the same time, there are two distinct new vertices \(y_1, z_3\) in \(G(Q)\), occurring in the enumerations of the sets \(Y, Z\), respectively. Now, one can easily verify that \(b = (v_0, u_1, v_1, v_0, v_2, u_0)\) is a path of length five in \(G(Q)\).

(d) There are two distinct connected components \(C\) and \(D\) in \(G(P)\), each of them isomorphic to the cycle \(C_4\) or \(C_5\). Let \(m\) and \(l\) denote any of the numbers 4 or 5. We assume that \((u_0, u_1, \ldots, u_{m-1})\) and \((v_0, v_1, \ldots, v_{l-1})\) are the cycles forming the components \(C \cong C_m\) and \(D \cong C_l\), respectively. Now we take the six term progression \(a = (u_0, u_1, u_2, u_0, v_1, u_2)\) and form the extension \(Q = 3P(a)\). Once again, \(\{u_0, u_1\}, \{u_1, u_2\}, \{v_0, v_1\}\) and \(\{v_1, v_2\}\) are edges in \(G(P)\), while \(\{u_2, v_0\}\) and \(\{v_2, u_0\}\) are not. Hence \(G(Q)\) has the same number of edges as \(G(P)\) and \(Q\) has the same number of gaps as \(P\). Now, picking the new distinct vertices \(y_1 \in Y\), \(z_3 \in Z\), we obtain the path \(b = (u_3, u_2, y_1, v_0, v_{l-1}, v_{l-2})\) of length five in \(G(Q)\).

(e) \(G(P)\) consists of a single connected component isomorphic either to \(C_4\) or to \(C_5\). However, this is impossible, since number of edges in \(G(P)\) is divisible by three.

This concludes the proof of Proposition 3 as well as of Theorems 1 and 2.

\[\square\]

6. Final remarks

The discussion from the introduction together with Theorem 2 naturally lead to the following question.

**Problem 1.** Is there some minimal (or even the least) axiomatic class \(K\) of IP loops such that every group is locally embeddable into \(K_{\text{fin}}\)? Does this class (if it exists) satisfy the Finite Embedding Property?

The first candidate which should be examined in this connection seems to be the class of all Moufang loops. One possible way how to define this concept reads as follows (see Pflugfelder [19], Gagola [8]): A Moufang loop is a loop satisfying the identity

\[x(y(xz)) = ((xy)x)z\]

It is well known that every Moufang loop is an IP loop.
The following is not the usual definition of the concept of a sofic group (see Ceccherini-Silberstein, Coornaert [2]), however, as proved by Gordon and Glebsky [10], it is equivalent to it. A group \((G, \cdot)\) is sofic if for every finite set \(X \subseteq G\) and every \(\varepsilon > 0\) there exists a finite quasigroup \((Q, *)\) such that \(X \subseteq Q\),

\[
\frac{|\{q \in Q : 1 \cdot q \neq q\}|}{|Q|} < \varepsilon,
\]

where 1 denotes the unit element in \(G\), and, for any \(x, y \in X\) such that \(x \cdot y \in X\), we have \(x \cdot y = x \ast y\), as well as

\[
\frac{|\{q \in Q : (x \ast y) \ast q \neq x \ast (y \ast q)\}|}{|Q|} < \varepsilon.
\]

Theorem 2 together with the above description of sofic groups indicate that the sofic groups could perhaps be characterized as groups locally embeddable into some “nice” subclass of the class of finite IP loops, fulfilling some “reasonable amount of associativity”. A natural candidate is the class of all finite Moufang loops, once again. For some additional reasons in favor of this choice see [8].

As already indicated, one should start with trying to clarify the following question.

**Problem 2.** Does the class of all Moufang loops have the FEP?

If the answer is negative then it would make sense to elaborate on the following problem.

**Problem 3.** Characterize those groups which are locally embeddable into finite Moufang loops.

Finally, let us formulate two possible responses to Problem 3 adding as a comment that we find the first of them (which would follow from the affirmative answer to Problem 2) more probable to be true than the second one.

**Conjecture 1.** Every group is locally embeddable into finite Moufang loops.

**Conjecture 2.** A group \(G\) is sofic if and only if it is locally embeddable into finite Moufang loops.

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