Fully-Quantum-Theoretic Numerical Study on Quantum Phase Sensing and Ghost Imaging Systems Operating with Multimode N00N States

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1. INTRODUCTION

The resolution of classical sensing and imaging systems is restricted by the Rayleigh diffraction limit that, roughly speaking, a subwavelength object less than half wavelength of light cannot be identified. Quantum sensing and imaging technologies are of great interest since the use of entangled photons overcomes such fundamental resolution limit, viz., super-resolution. Among various types of entanglement encoded in lights, the photon-number entanglement along different paths, called N00N states, is a promising candidate for quantum metrology. Experimental works have verified the super-resolution in phase measurements using N00N states [3] and shown the enhanced performance in quantum imaging systems operating with multimode N00N states beyond the Rayleigh diffraction limit. Our computational simulations are based on the canonical quantization via numerical mode-decomposition (CQ-NMD) [1, 2], in which normal (eigen) modes of electromagnetic fields in inhomogeneous dielectric media are numerically found using computational electromagnetics methods. In the CQ-NMD framework and the Heisenberg picture, the expectation value of arbitrary observables with respect to initial quantum states of various non-classical lights can be evaluated with the use of Wick’s theorem. The present numerical framework has a great potential to deal with scattering problems of entangled photons due to arbitrary dielectric objects. © 2022 Optical Society of America

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2. FUNDAMENTAL MATH/PHYSICS MODEL

A. Quantum Maxwellian operator and quantum state

Quantum optics describes the random behaviors of EM fields in the quantum regime. To do this, one needs to perform the canonical quantization where classical Maxwellian field and source variables can be elevated into (infinite-dimensional) operators while quantum states are introduced spanning (infinite-dimensional) Hilbert spaces. Quantum Maxwellian operators and quantum states are solutions to (1) quantum Maxwell’s equations (QME) and (2) quantum state equation (QSE), respectively [9, 10]. This study adopts the Heisenberg picture where observable operators are time-dependent, whereas quantum states are not. After performing the canonical quantization, one can express the positive frequency part of the quantized vector potential by the normal mode expansion, such as,

\[ \mathbf{A}^{(+)}(r,t) = \sum_{\lambda} \int_{\Omega} d\omega \sqrt{\frac{\hbar}{2\omega}} \mathbf{\Phi}_{\omega,\lambda}(r) \hat{a}_{\omega,\lambda} e^{-i\omega t}. \]  

(1)

where the hat symbol ‘\( \hat{\} \)’ denotes an operator, \( \hbar \) is the reduced Planck constant, \( \omega \) is eigenfrequency, \( \Omega^+ \) is a set of positive eigenfrequencies, \( \lambda \) denotes a generic degeneracy index, \( \Phi_{\omega,\lambda}(r) \) is time-harmonic vectorial normal mode for \( (\omega, \lambda) \), and \( \hat{a}_{\omega,\lambda} \) (\( \hat{a}^\dagger_{\omega,\lambda} \)) is an annihilation (creation) operator. The annihilation and creation operator obey the standard bosonic commutator relation. Using the orthonormality of the normal modes, one can diagonalize the Hamiltonian operator written
by \( \hat{H} = \sum_{\omega, \lambda} \omega_a \left( \hat{a}_{\omega, \lambda} + \frac{1}{2} \right) \) where \( \hat{a}_{\omega, \lambda} = a^\dagger_{\omega, \lambda} a_{\omega, \lambda} \) is called number operator. The eigenstates of number operators are known as Fock states, i.e., \( \hat{a}_{\omega, \lambda} n_{\omega, \lambda} = n n_{\omega, \lambda} \) where \( n_{\omega, \lambda} \) is the Fock state representing that \( n \) number of photons are occupied in \((\omega, \lambda)\)-th normal mode. Thus, the mode-decomposition enables one to easily model arbitrary quantum states by the linear superposition of multimode Fock states. One can refer to [11] for more details.

B. Numerical mode-decomposition [1, 2]

To extract normal modes for EM fields in inhomogeneous dielectric media, one should solve the following vector wave equation

\[
\nabla \times \mu_0^{-1} \nabla \times \Phi_{\omega, \lambda}(\mathbf{r}) - \omega^2 \varepsilon(\mathbf{r}) \Phi_{\omega, \lambda}(\mathbf{r}) = 0.
\]

Note that this study considers lossless and dispersionless dielectric media for simplicity. To model arbitrary geometric and medium complexity in dielectric scatterers, we can utilize numerical methods in CEM such as finite-difference or finite-element methods. The resulting discrete counterpart of Eq. (2) with Bloch periodic boundary conditions becomes a finite-dimensional generalized Hermitian eigenvalue problem written by \( \mathbf{S} \cdot \Phi = \mathbf{M} \cdot \Phi - \mathbf{\omega}^2 \) where \( \mathbf{S} \) and \( \mathbf{M} \) are (sparse) stiffness and mass matrices, which encodes double-curl operator and medium and metric information, \( \mathbf{\Phi} \) is a (full) matrix including numerical normal modes, and \( \mathbf{\omega} \) is a diagonal matrix including relevant eigenfrequencies.

C. Representation based on CQ-NMD framework

With the use of numerical normal modes, the continuum modal index \( (\omega, \lambda) \) is replaced by a single discrete modal index \( i \). Thus, the resulting vector potential operator at \( i \)-grid point, denoted by \( \mathbf{r}_i \), can be rewritten by

\[
\hat{A}^{(+)}(\mathbf{r}_i, t) \approx \sum_{m=1}^{N_{\text{det}}} \sqrt{\frac{\hbar}{2 \omega_i}} \Phi_i(\mathbf{r}_i) a_m e^{-i \omega_i t}.
\]

where \( N_{\text{det}} \) denotes the total number of normal modes. The resulting Hamiltonian can be also written by \( \hat{H} \approx \sum_{i=1}^{N_{\text{det}}} \hbar \omega_i \left( a_m^\dagger a_m + \frac{1}{2} \right) \).

D. Modeling multimode N00N states

A typical N00N state takes the form of

\[
|\psi\rangle_{\text{N00N}} = \frac{1}{\sqrt{2}} \left( |\phi_1^{(N)}\rangle |\phi_2\rangle + |\phi_2^{(N)}\rangle |\phi_1\rangle \right)
\]

where \( |\phi_m\rangle \) and \( |\phi_1^{(N)}\rangle \) represent quantum states of no-photon and \( N \) number of (monochromatic) photons along \( m \)-th path for \( m = 1, 2 \), respectively. Furthermore, we assume that each photon is riding on a wavepacket (i.e., quasi-monochromatic) such that the resulting quantum state should be expanded by multimode Fock states [12] \( |\psi_i^{(N)}\rangle \approx \sum_{m=1}^{N_{\text{det}}} g_i^m |\phi_m\rangle |\phi_1^{(N)}\rangle \) where \( g_i^m \) represents a probability amplitude of \( i \)-th single-photon Fock state that encodes the spectrum of a wavepacket along \( m \)-th path. Similarly, a quantum state of \( N \)-quasi-monochromatic photons occupied in \( m \)-th path can be explicitly expressed by

\[
|\psi_i^{(N)}\rangle \approx \frac{1}{\sqrt{N!}} \left( \sum_{m=1}^{N_{\text{det}}} g_i^m |\phi_m\rangle |\phi_1^{(N)}\rangle \right)^N |0\rangle.
\]

Note that the multimode N00N state fulfills the normalization condition rigorously, proven by using Wick’s theorem (See the supplementary material).

E. Modeling coincidence counting

Photon statistics is the theoretical and experimental study to identify the statistical distributions of photons produced in a light source via photon counting experiments. Particularly, coincidence counting, referring to the simultaneous detection of two or more photons at photodetectors, is of cardinal importance in quantum optics widely used to study the quantum state of non-classical lights. Here, we define N-th order correlation function (CF) for a pair of photodetectors (indexed by \( \alpha \) and \( \beta \) [12] as

\[
\text{N-th order CF} = \frac{\langle \psi \hat{A}^{(+)\dagger}(\mathbf{r}_1, t_1) \hat{A}^{(+)\dagger}(\mathbf{r}_2, t_2) |\psi\rangle \langle \psi | \hat{A}^{(+)\dagger}(\mathbf{r}_3, t_3) \hat{A}^{(+)}(\mathbf{r}_4, t_4) \rangle}{\langle \psi \hat{A}^{(+)\dagger}(\mathbf{r}_1, t_1) \hat{A}^{(+)\dagger}(\mathbf{r}_2, t_2) |\psi\rangle \langle \psi | \hat{A}^{(+)\dagger}(\mathbf{r}_3, t_3) \hat{A}^{(+)\dagger}(\mathbf{r}_4, t_4) \rangle}
\]

where \( |\psi\rangle \) is an initial quantum state, and, for \( \zeta = \alpha \) or \( \beta \), \( \xi_{\zeta}^{(+)\dagger} = (\xi_{\zeta}^{(+)} \xi_{\zeta}^{(+)})^\dagger \) and \( \xi_{\zeta}^{(+)} = \prod_{i=1}^{N/2} \hat{A}^{(+)}(\mathbf{r}_i, t_i) \) where \( r_i \) and \( t_i \) denote \( \zeta \)-th photodetector’s location and time, respectively, and subscript \( l = x, y, z \) denotes a component of a vectorial field operator \( \hat{A}^{(+)} \). Physically speaking, the numerator represents \( N\)-fold coincidence count and two terms in the denominator are normalization factors, which are associated with the photodetection probability at each photodetector independently. Note that for simplicity we assume the photodetection number at each photodetector to be equal, i.e., \( N/2 \)-photodetection per photodetector. \(^3\) To calculate Eq. (6), we translate it to products of ladder

\(^3\)This is one of possible configurations to calculate the N-order CF. Unlike the case \( N = 2 \), if \( N \) is large, there are many possible photodetection configurations for the N-order CF when using a pair of photodetectors. For example, when \( N = 4 \) for binary paths (say \( a \) and \( \beta \)), possible non-entangled quantum states are \( |1\rangle, |3\rangle, |2\rangle, |2\rangle, |3\rangle, |1\rangle \); therefore, one needs to perform \( (1,3), (2), (3,1) \)-times photodetections at photodetectors \( a \) and \( \beta \) to identify above non-entangled quantum states, respectively.
operators by substituting $|\psi\rangle_{\text{N00N}}$ and Eq. (3) into Eq. (6). Then, one can apply Wick’s theorem [8] to have the normal order of the products of ladder operators and then sum up full-contraction terms (See the supplementary material for the details).

3. SIMULATION RESULTS

A. Quantum phase sensing system

Consider a quantum phase sensing system, consisting of a N00N state generator, phase shifter, beam splitter, and coincidence counting measurement circuit, as illustrated in Fig. 1. The phase shifter ($\theta$), inserted on the upper path, adds the phase $\theta$ into the probability amplitude of a quantum state on that arm, and the divided N00N state is self-interfered through a 50:50 beam splitter. We can observe a correlation pattern with respect to $\theta$ from calculating Eq. (6). Here, instead of modeling the entire quantum phase sensing system, we consider the beam splitter part only with a proper initial N00N state that incorporates the phase shifter effect. Note that we assume that the overall interaction potential operators are polarized along $z$ direction. An object to be imaged is a dielectric slab (the relative dielectric constant is $\varepsilon_r = 4$) including a subwavelength slit. The slit width $L_B = 0.34\lambda_0 = 4.33 \times 10^{-2}$ [m], the length of each side of the dielectric slab $L_I = 1.70 \times 10^{-1}$ [m], and the slab thickness $L_I = 9.67 \times 10^{-2}$ [m]. We place a single-pixel (or bucket) photodetector on the left behind the object while locating a multi-pixel photodetector on the right. The entangled photons propagating toward the right hit the target object and are collected at the single-pixel detector. In contrast, the rest entangled-photon propagating toward the right is measured at the multi-pixel detector without having any interference with the target object. It is assumed that each photodetector has the photon number resolving capability. Repeating the above tasks at each scanning position parameter $s$ (see Fig. 5), we can calculate the N-th order CF in terms of $s$ for various $N$. Fig. 5 shows normalized N-th order CF versus scanning position parameter $s$ for $N = 2, 4, 8$. In an ideal case, as illustrated by a green-solid-line in Fig. 5, N-th order CF should be zero in the absence of the object; otherwise, it is to be unity. This behavior can be deduced from the definition of N-th order CF in Eq. (6). The numerator measures the degree of coincidence, whereas terms in the denominator measure the photodetection probability at each photodetector independently. In the absence of the dielectric object, original entanglement is preserved, implying that the numerator goes agree with the theoretical prediction [3]. This super-resolution comes from the fact that when a pure photon number state $|N\rangle$ passes through a phase shifter $\theta$, the coherent accumulation of the phase shift $\theta$ experienced by each single photon is possible. The net phase delay, i.e., $e^{iN\theta}$, is then transformed into the probability amplitude of a N00N state. Thus, correlation patterns in the Mach-Zehnder interferometer can oscillate depending the net phase delay. On the other hand, the action of the phase shifter on a coherent (classical) state with the average $N$ photons averages out all phase shifts experienced by different photon number states (due to the linear superposition); consequently, the probability amplitude can only gain the phase delay $e^{i\theta}$. One can find more details in supplementary material.

B. Quantum Ghost-Imaging

Next, consider a quantum ghost imaging system whose two-dimensional simulation scenario is illustrated in Fig. 4. A multimode N00N state including $N$ quasi-monochromatic photons is initialized in the middle. The center frequency and full width at half maximum (FWHM) of each photon are assumed to be $\omega_0 = 50 c$ [rad/s] ($\lambda_0 \approx 1.26 \times 10^{-1}$ [m]) and $\Delta \omega_{\text{FWHM}} = 1.56 c$ [rad/s]. We assume that vector potential operators are polarized along $z$ direction. An object to be imaged is a dielectric slab (the relative dielectric constant is $\varepsilon_r = 4$) including a subwavelength slit. The slit width $L_B = 0.34\lambda_0 = 4.33 \times 10^{-2}$ [m], the length of each side of the dielectric slab $L_I = 1.70 \times 10^{-1}$ [m], and the slab thickness $L_I = 9.67 \times 10^{-2}$ [m]. We place a single-pixel (or bucket) photodetector on the left behind the object while locating a multi-pixel photodetector on the right. The entangled photons propagating toward the left hit the target object and are collected at the single-pixel detector. In contrast, the rest entangled-photon propagating toward the right is measured at the multi-pixel detector without having any interference with the target object. It is assumed that each photodetector has the photon number resolving capability. Repeating the above tasks at each scanning position parameter $s$ (see Fig. 5), we can calculate the N-th order CF in terms of $s$ for various $N$. Fig. 5 shows normalized N-th order CF versus scanning position parameter $s$ for $N = 2, 4, 8$. In an ideal case, as illustrated by a green-solid-line in Fig. 5, N-th order CF should be zero in the absence of the object; otherwise, it is to be unity. This behavior can be deduced from the definition of N-th order CF in Eq. (6). The numerator measures the degree of coincidence, whereas terms in the denominator measure the photodetection probability at each photodetector independently. In the absence of the dielectric object, original entanglement is preserved, implying that the numerator goes

Fig. 3. N-th order correlation function (CF) versus $\theta$ (phase shifter) for $N = 2, 4, 6$, compared with the classical case.

Fig. 4. Two-dimensional simulation scenario of a quantum ghost imaging system using a multimode N00N state.
We have performed fully-quantum-theoretic computational simulations of quantum phase sensing and ghost imaging systems operating with multimode N00N states to observe the super-resolution based on the canonical quantization via numerical mode-decomposition (CQ-NMD) approach [1, 2]. The simulation results agreed well with both theoretical estimates and experimental observations that the increase of the photon number $N$ achieves $N$-times higher sensitivity and resolution. The present study has shown great promise of utilizing the conventional computational electromagnetic methods together with quantum Maxwell’s equations and quantum state equation for quantum metrology applications. Although our simulations assumed an ideal condition, three practical issues need to be resolved for fully taking the quantum advantages: (1) generating arbitrary high N00N states, (2) having the photon-number-resolving photodetection capability and (2) improving the extreme fragility to the interaction with environment [14]. For our future studies, we plan to account for dissipation and dispersion effects of media on the performance of quantum sensing and imaging systems. Moreover, we plan to consider various kinds of non-classical states of light for quantum metrology applications, such as, squeezed states, entangled coherent states [15], and N00N-like states [16], which are relatively easier to create as well as detect in practice while still exhibiting the quantum advantages from the metrology aspect.

4. CONCLUSION

We have performed fully-quantum-theoretic computational simulations of quantum phase sensing and ghost imaging systems operating with multimode N00N states to observe the super-resolution based on the canonical quantization via numerical mode-decomposition (CQ-NMD) approach [1, 2]. The simulation results agreed well with both theoretical estimates and experimental observations that the increase of the photon number $N$ achieves $N$-times higher sensitivity and resolution. The present study has shown great promise of utilizing the conventional computational electromagnetic methods together with quantum Maxwell’s equations and quantum state equation for quantum metrology applications. Although our simulations assumed an ideal condition, three practical issues need to be resolved for fully taking the quantum advantages: (1) generating arbitrary high N00N states, (2) having the photon-number-resolving photodetection capability and (2) improving the extreme fragility to the interaction with environment [14]. For our future studies, we plan to account for dissipation and dispersion effects of media on the performance of quantum sensing and imaging systems. Moreover, we plan to consider various kinds of non-classical states of light for quantum metrology applications, such as, squeezed states, entangled coherent states [15], and N00N-like states [16], which are relatively easier to create as well as detect in practice while still exhibiting the quantum advantages from the metrology aspect.

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