Equation of State and Viscosities from a Gravity Dual of the Gluon Plasma

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Abstract

Employing new precision data of the equation of state of the SU(3) Yang-Mills theory (gluon plasma) the dilaton potential of the gravity dual is adjusted in the temperature range $(1-10) T_c$ in a bottom-up approach. The ratio of bulk viscosity to shear viscosity follows then as $\zeta/\eta \approx \pi \Delta v_s^2$ for $\Delta v_s^2 < 0.2$ and achieves a maximum value of 0.95 at $\Delta v_s^2 \approx 0.32$, where $\Delta v_s^2$ is the non-conformality measure, while the ratio of shear viscosity to entropy density is known as $(4\pi)^{-1}$ for the considered special set-up with Hilbert action on the gravity side.

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I. INTRODUCTION

With the advent of new precision data [1], which extend previous lattice QCD gauge theory evaluations [2, 3] for the pure gluon plasma to a larger temperature range, a tempting task is to seek for an appropriate gravity dual. While such an approach does not provide new insights in the pure SU(3) Yang-Mills equation of state above the deconfinement temperature $T_c$, it however allows to calculate, without additional ingredients, further observables, e.g., transport coefficients. (This is in contrast to quasiparticle approaches which require additional input to access transport coefficients [4].) In considering gravity duals to pure non-abelian gauge thermo-field theories within a bottom-up approach one has to adjust either the potential of the dilaton field (which is the dual to $\text{Tr} F^2$ of the gauge field $F$ according to the AdS/CFT dictionary) [5, 6] or a metric function [7] or the dilaton profile; one may also take the $\beta$ function as input [8, 9]. The previous benchmark lattice data [2] and further SU($N_c$) data for $N_c \leq 8$ [10] in a narrow temperature range above $T_c$ allowed in fact to constrain the dilaton field and to study implications [5–8]. Here we are going to adjust precisely the dilaton potential to the new lattice data [1] in a somewhat larger temperature range uncovering the strong-coupling regime.

While for $N_c = 3$ one leaves already the grounds of the original AdS/CFT correspondence, the extension towards a larger temperature interval stretches further the range of employing an AdS/QCD duality for application purposes. Having in mind a simple shape function of the dilaton potential, which reproduces the lattice data in a chosen temperature range above $T_c$, one could consider such an approach as a convenient parameterization of the equation of state. Once such a potential is adjusted, it qualifies for further studies, e.g. of transport coefficients, as mentioned above. Our goal is accordingly the quantification of the bulk viscosity in the LHC relevant region, in particular near to $T_c^\pm$.

Transport properties of the matter produced in relativistic heavy-ion collisions at RHIC and LHC are important to characterize precisely such novel states of a strongly interacting medium besides the equation of state. The impact of the bulk viscosity on the particle spectra and differential elliptic flows has been recently discussed in [11] and found to be sizeable in [12], in particular for higher-order collective flow harmonics. The bulk viscosity enters also a new soft-photon emission mechanism [13] via the conformal anomaly, thus offering a solution to the photon-$v_2$ puzzle (cf. [13] for details and references). Compilations
of presently available lattice QCD results of viscosities can be found in [4].

II. THE SET-UP

The action

$$S = \frac{1}{16\pi G_5} \int d^5 x \sqrt{-g} \left\{ R - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right\}$$

(the Hawking-Gibbons term is omitted) leads, with the ansatz for the infinitesimal line element squared in Riemann space

$$ds^2 = \exp\{2A\}(d\vec{x}^2 - hdt^2) + \exp\{2B\}h^{-1}L^2d\phi^2,$$

to the field equations quoted in [5] under (25a - 25c); the equation of motion (25d) follows from the derivative of (25c) with insertion of (25a - 25c). Here, the coordinate transformation

$$dz = L(e^{2B} - e^{2A})d\phi$$

has been employed to go from the Fefferman-Graham (holographic) coordinate $z$ in the infinitesimal line element squared

$$ds^2 = \exp\{A\}(-hd\tau^2 + d\vec{x}^2 + h^{-1}dz^2)$$

to a gauged radial coordinate expressed by the dilaton field $\phi$ which requires the introduction of a length scale $L$. The metric functions are thus to be understood as $A(\phi)$, $B(\phi)$ and $h(\phi)$, and a prime means in the following the derivative with respect to $\phi$. These equations can be rearranged to change the mixed boundary value problem into an initial value problem. With $U \equiv V/(3V')$ we get

$$Y_1' = Y_2 - U,$$

$$Y_2' = Y_3 + \frac{1}{2}U',$$

$$Y_3' = \frac{1}{2}U'' + \frac{Y_3 - \frac{1}{2}U'}{(Y_2 - U)Y_2} \left( (Y_3 - \frac{1}{2}U')(3Y_2 - 2U) + (4Y_2 - U')U)(Y_2 - U)^2 + \frac{Y_2}{6U}(2Y_2 - U) \right),$$

$$Y_4' = \frac{6(Y_3 - \frac{1}{2}U') + 1}{6(Y_2 - U)},$$

$$Y_5' = \exp\{-4Y_1 + Y_4\}$$

which is integrated from $\phi_H - \epsilon$ to $\epsilon$ with the initial values $Y_i = 0$ at $\phi_H - \epsilon$. The limit $\epsilon \to 0^+$ has to be taken to obtain the entropy density $s$ and the temperature $T$

$$sG_5 = \frac{1}{4} \exp(3A_H),$$

$$TL = -\frac{1}{4\pi} \frac{\exp(A_H - B_H)}{Y_5(\epsilon)},$$

where $A_H = \log_{\Delta - 4} - Y_1(\epsilon)$ and $B_H = -\log(-\epsilon[\Delta - 4]) - Y_4(\epsilon)$. This set ensures the boundary conditions $h(\phi = 0) = 1$ and $h(\phi_H) = 0$ as well as the AdS asymptotic limits $A(\phi) = \frac{\log\phi}{\Delta - 4}$ and $B(\phi) = -\log(-\phi[\Delta - 4])$ at $\phi \to 0^+$. The quantities $Y_i(\epsilon)$ depend on the horizon
position \( \phi_H \), implying in particular \( s(\phi_H) \) and \( T(\phi_H) \), thus providing the equation of state \( s(T) \) in parametric form. The quantity \( \Delta \) is the dimension of the gauge theory operator being dual to the dilaton \( \phi \).

III. EQUATION OF STATE

To compare with the lattice results \([1]\) of the relevant thermodynamical quantities (i) sound velocity squared \( v_s^2 = \frac{d \log T}{d \log s} \), (ii) scaled entropy density \( s/T^3 \), (iii) scaled pressure \( p/T^4 \), and (iv) scaled interaction measure \( I/T^4 = s/T^3 - 4p/T^4 \) (all as a function of \( T/T_c \)) one must adjust the scale \( T_c \) and the scale parameters \( L \) and \( G_5 \) (actually, the dimensionless combinations \( LT_c \) and \( G_5/L^3 \) are needed). In the present bottom-up approach, we employ a distorted Gubser potential \([6]\)

\[
V(\phi)L^2 = -12 \cosh(\gamma \phi) + (6\gamma^2 + \frac{1}{2} \Delta[\Delta - 4])\phi^2 + \sum_{i=2}^{5} c_{2i}\phi^{2i} \tag{8}
\]

as trial ansatz and optimize the parameters \( G_5, \gamma, \Delta \) within the interval \( 2^+ \cdots 4^- \) (to ensure the renormalizability of the gauge field operator, and selecting the upper branch of the mass dimension relation \( M^2L^2 \equiv \Delta(\Delta - 4) \) to respect the Breitenlohner-Freedman bound), and \( c_{2i} \). The critical temperature \( T_c \), and thus the scale \( L \), are determined by either the minimum or the the inflection/turning point of \( T(s/T^3) \). In case of a multitude of such points the one with the smallest value of \( \phi_H \) is to be selected to ensure a smooth equation of state towards the asymptotic region, i.e. \( \phi_H \to 0 \).

The Gubser potential \([5]\) follows from (8) when omitting the polynomial distortions, i.e. for \( c_{2i} = 0 \). Within the range \( 3.5 \leq \Delta \leq 3.9 \) the turning points of \( T(s/T^3) \) are then on the curve \( \gamma \approx -0.6214 + 0.8296\Delta - 0.12947\Delta^2 \); for larger (smaller) values of \( \gamma \), at given \( \Delta \), \( T(s/T^3) \) displays a minimum (inflection point). Non-zero values of \( c_{2i} \) improve \( \chi^2_{d.o.f.} \) (e.g., for the entropy density) by a factor of two, however, with only marginal deviations of the other equation-of-state variables compared to optimized sets with \( c_{2i} = 0 \). In the latter case, \( \chi^2_{d.o.f.} \) is minimized on the curve \( \gamma \approx -1.4486 + 1.3174\Delta - 0.2006\Delta^2 \). On this curve, typical values are \( \phi_H \sim 4.2 \) and \( G_5/L^3 \sim 1.15 \).

Examples of the equation of state are exhibited in Fig. 1. The velocity of sound is independent of \( G_5 \) which steers the number of degrees of freedom, thus being important for entropy density, energy density \( e \), pressure and interaction measure. In asymptotically free
FIG. 1: The sound velocity squared $v_s^2$ (left top panel), scaled entropy density $s/T^3$ (right top panel), scaled pressure $p/T^4$ (left bottom panel), and scaled interaction measure $I/T^4$ (right bottom panel) as a function of $T/T_c$. $V_I$ (solid curves): $\gamma = 0.6938$, $\Delta = 3.5976$, $G_5/L^3 = 1.1753$, $LT_c = 0.148$, $V_{IV}$ (dashed curves): $\gamma = 0.6580$, $\Delta = 3.7560$, $c_4 = 0.2514$, $c_6 = -0.3760$, $c_8 = -0.0076$, $c_{10} = -0.0188$, $G_5/L^3 = 1.1443$, $LT_c = 0.099$. For both parameter sets, $LT_c$ is determined by the minimum of $T(s/T^3)$. The lattice data (symbols) are from [1]. The horizontal lines in the upper right corners depict the respective Stefan-Boltzmann limits.

Theories, the $T^4$ term dominates $s$, $e$ and $p$ at large temperatures; it is subtracted in the interaction measure making it a sensible quantity. The appearance of a maximum of $I/T^4$ at $T/T_c \approx 1.1$ is related to a turning point of $p/T^4$ as a function of $\log T$. Position and height of $I/T^4$ – the primary quantity in lattice calculations – are sensible characteristics of the equation of state. The dropping of $I/T^4$ at larger temperatures signals the approach towards conformality. (Since in conformal theories $v_s^2 = 1/3$, the quantity $\Delta v_s^2 = 1/3 - v_s^2$ is termed non-conformality measure; also here, the dominating $T^4$ terms at large temperatures
drop out.) Inspection of Fig. 1 unravels the nearly perfect description of the lattice data\[1\], where the polynomial distortions (included in \(V_{IV}\) corresponding to (8)) of the plain Gubser potential (\(V_I\) where \(c_2i = 0\)) allow for a better fit. Note that our determination of the pressure from \(p = p(T_0) + \int_{T_0}^{T} dT' s(T')\) uses \(p(T_c)/T_c^4\) from [1].

### IV. VISCOSITIES

Irrespectively of the dilaton potential \(V(\phi)\), the present set-up with Hilbert action \(R\) for the gravity part delivers \(\eta/s = (4\pi)^{-1}\) \[13\] for the shear viscosity \(\eta\), often denoted as KSS (bound) value \[15\]. In contrast, the scaled bulk viscosity \(\zeta/T^3\) has a pronounced temperature dependence. Following \[16\] we calculate \(\zeta\) from the relation

\[
\frac{\zeta}{\eta} = \frac{1}{9U(\phi_H)^2} \frac{1}{|p_{11}(\epsilon)|^2},
\]

where the asymptotic value \(p_{11}(\epsilon)\) of the perturbation \(p_{11}\) of the 11-metric coefficient is obtained by integrating

\[
p_{11}'' + \left( \frac{1}{3(Y_2 - U)} + 4(Y_2 - U) - 3Y_4' + \frac{Y_5'}{Y_5} \right) p_{11}' + \frac{Y_5' Y_3 - \frac{1}{2}U'}{Y_5 Y_2 - U} p_{11} = 0
\]

from \(\phi_H - \epsilon\) to \(\epsilon\) with initial conditions \(p_{11}(\phi_H - \epsilon) = 1\) and \(p_{11}'(\phi_H - \epsilon) = 0\) and \(\epsilon \to 0^+\) (cf. \[17\] for a discussion of this calculation scheme vs. the null horizon focusing equation \[18\]).

Our results are exhibited in Fig. 2. The scaled bulk viscosity \(\zeta/T^3\) has a maximum slightly above \(T_c\) at \(1.05T_c\) (which is slightly below the maximum of \(I/T^4\)) and drops rapidly for increasing temperatures, see left panel of Fig. 2. Remarkable is the almost linear section of \(\zeta/\eta\) as a function of the non-conformality measure \(\Delta v_s^2\) (see right panel), as already pointed out in \[19\] and further argued in \[16\]: for further reasoning on such a linear behavior within holography approaches cf. \[20\]. A non-linear behavior occurs in a small temperature interval \(1 \leq T/T_c < 1.05\), i.e. for \(\Delta v_s^2 > 0.22\), see right panel of Fig. 2. The maximum value of \(\zeta/\eta \approx 0.95\) depends on the actual values of \(\gamma\) and \(\Delta\). The shape of \(\zeta/\eta\) vs. \(\Delta v_s^2\) is also not universal as it depends fairly sensitively on \(\gamma\) and \(\Delta\). (That is why a precise adjustment of the equation of state is required to address properly the quark-gluon plasma.)

Interesting is the relation \(\zeta/\eta \propto 1.2\pi \Delta v_s^2\) for \(0.025 < \Delta v_s^2 < 0.2\) (corresponding to the temperature interval \(1.07 < T/T_c < 2\); extending the fit to \(1.07 < T/T_c < 10\) we find \(\zeta/\eta \approx \pi \Delta v_s^2\) which follows numerically and is specific for the selected potential parameters.
FIG. 2: The scaled bulk viscosity $\zeta/T^3$ as a function of the temperature (left panel) and the ratio $\zeta/\eta$ as a function of the non-conformality measure (right panel; in addition, the temperature scale for "$V_I"$ is provided at the top axis, and the Buchel bound $\zeta/\eta = 2\Delta v_s^2$ as well as a linear fit are depicted). Line codes and parameters as in Fig. 1.

It accommodates the Buchel bound [19] and agrees quantitatively with the result of [4]. There, a quasi-particle approach has been employed which needs, beyond the equation-of-state adjustment, further input: In [4] it is the dependence of the relaxation time on the temperature which causes a deviation from the linear relation $\zeta/\eta \propto \Delta v_s^2$ at large temperatures corresponding to small values of $\Delta v_s^2$. Note also the shift of the linear section of $\zeta/\eta$ in [4] by a somewhat larger off-set which can cause a descent violation of the Buchel bound, which is not unexpected with respect to [21].

V. DISCUSSION AND SUMMARY

Inspired by the AdS/CFT correspondence we employ an AdS/QCD hypothesis and adjust, in a bottom-up approach, the dilaton potential at lattice QCD gauge theory thermodynamics data for the pure $SU(3)$ gauge field sector. With appropriate scale settings one can fairly accurately reproduce the data of [11] in the LHC relevant temperature region from $T_c$ up to $10T_c$ (having in mind that with quarks included the value of $T_c$ significantly drops). While further fine tuning is possible, we find that the two-parameter ($\gamma$, $\Delta$) Gubser potential [5] is sufficient; least-square minimization is accomplished in a narrow banana type region in the
\(\gamma - \Delta\) plane. Without additional input, one can calculate directly further observables. We focus on the bulk viscosity \((\zeta/T^3)\) which displays a strong increase when approaching \(T_c^+\) from above and acquires a maximum slightly before that. Within the non-conformal region \(1 \leq T/T_c \leq 10\), where the non-conformality measure \(0.2 > \Delta v_s^2 > 0.004\) and the interaction measure \(2.48 > I/T^4 > 0.07\), an almost linear dependence \(\zeta/\eta \approx \pi \Delta v_s^2\) on the non-conformality measure \(\Delta v_s^2\) is observed, already argued qualitatively in [16] within a holographic approach and in [4] quantitatively within a quasi-particle approach to the pure gauge sector of QCD.

Extensions towards including quark degrees of freedom and subsequently non-zero baryon density, i.e. to address full QCD, have been outlined and explored in [22]. Incorporating additional degrees of freedom (which are aimed at mimicking an equal number of quarks and anti-quarks) within the present set-up, one essentially has to lower \(G_5/L^3\) in adjusting the extensive and intensive densities. Since the viscosities scale with \(L^3/G_5\) [16] (as the entropy density does, too) the corresponding ratios \(\zeta/s\) and \(\zeta/\eta\) would stay unchanged, if the same potential would apply and the same behavior of the sound velocity would be used as input. However, as stressed above, \(\zeta/\eta\) depends rather sensitively on the actual potential \(V(\phi)\) and its parameters. Therefore, in contrast to the consistent results of [13] for the \(SU(3)\) gluon plasma, the lattice QCD results for the quark-gluon plasma need consolidation before similar adjustments of \(V(\phi)\) can be attempted on firm grounds.

On the gravity side, inclusion of terms beyond the Hilbert action cause a temperature dependence of the ratio \(\eta/s\) [23] which is needed to furnish the transition into the weak-coupling regime at large temperatures.

In summary we adjust the dilaton potential exclusively at new lattice data for \(SU(3)\) gauge theory thermodynamics and calculate the bulk viscosity. The ratio of the bulk to shear viscosity obeys, in the strong-coupling regime, a linear dependence on the non-conformality measure for temperatures above \(1.05 T_c\), while at \(1.004 T_c\) it has a maximum of 0.95.

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