One helpful property of functions generating Pólya frequency sequences*

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In this work we study the solutions of the equation $z^p R(z^k) = \alpha$ with nonzero complex $\alpha$, integer $p$, $k$ and $R(z)$ generating a (possibly doubly infinite) totally positive sequence. It is shown that the zeros of $z^p R(z^k) - \alpha$ are simple (or at most double in the case $\text{Im}\, \alpha^k \neq 0$) and split evenly among the sectors $\left\{ \frac{j}{k} \pi \leq \text{Arg} \, z \leq \frac{j+1}{k} \pi \right\}$, $j = 0, \ldots, 2k - 1$. Our approach rests on the fact that $z(\ln z^p R(z))'$ is an $\mathcal{R}$-function (i.e. maps the upper half of the complex plane into itself).

This result guarantees the same localization to zeros of entire functions $f(z^k) + z^p g(z^k)$ and $g(z^k) + z^p f(z^k)$ provided that $f(z)$ and $g(-z)$ have genus 0 and only negative zeros. As an application, we deduce that functions of the form $\sum_{n=0}^{\infty} (\pm i)^n(n-1)/2 a_n z^n$ have simple zeros distinct in absolute value under a certain condition on the coefficients $a_n \geq 0$. This includes the "disturbed exponential" function corresponding to $a_n = q^{n(n-1)/2}/n!$ when $0 < q < 1$, as well as the partial theta function corresponding to $a_n = q^{n(n-1)/2}$ when $0 < q < q_\ast \approx 0.7457224107$.

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1 Introduction

The present paper studies quite a general equation of the form $z^p R(z^k)$, however the simple case $k = 2$ considered in Sections 8 and 9 has the most interesting applications. Corollary 32 introduces sufficient conditions on a function of the form $\sum_{n=0}^{\infty} i^\pm(n-1)/2 f_n z^n$, where $f_0 \neq 0$ and $f_n > 0$ for all $n$, which assure the simplicity of its zeros. It turns out that such functions as

$$F(z; \pm iq) = \sum_{n=0}^{\infty} \frac{1}{n!} (\pm iq)^{(n-1)/2} z^n, \quad \text{where} \quad 0 < q \leq 1,$$

and

$$\Theta_0(z; \pm iq) = \sum_{n=0}^{\infty} (\pm iq)^{(n-1)/2} z^n, \quad \text{where} \quad 0 < q \leq q_\ast \approx 0.7457224107,$$
have zeros which are simple and distinct in absolute value. The former function \( F(z; q) \) gives a solution to the functional-differential problem

\[
F'(z) = F(qz), \quad F(0) = 1,
\]

while the latter is the partial theta function satisfying

\[
\Theta_0(z; q) = 1 + z\Theta_0(zq; q).
\]

The partial theta function participates in a number of beautiful Ramanujan-type relations ([3, Chapter 6], [24]), it is related to \( q \)-series and some types of modular forms. Both \( F \) and \( \Theta_0 \) appear in problems of statistics and combinatorics (see e.g. [20, 21]) and their zeros are the subjects of conjectures by Alan Sokal. The details can be found in Section 9.

Nevertheless, general statements offer a better insight into the problem, give an opportunity to determine factors on which the result depends and to find possible generalizations. Their main drawback is an excessive amount of specific cases in Sections 4, 5 and 6. To give a survey of our results, we briefly introduce two special classes of functions and definitions of \( \alpha \)-sets and \( \alpha \)-points.

Definitions. A doubly infinite sequence \( (\rho_n)_{n=-\infty}^{\infty} \) is called totally positive if all of the minors of the (four-way infinite) Toeplitz matrix \( (\rho_{n-j})_{n,j=-\infty}^{\infty} \) are nonnegative (i.e. the matrix is totally nonnegative). The paper [9] answers the question of convergence of the correspondent power series \( \sum_{n=-\infty}^{\infty} \rho_n z^n \). Unless we have \( \rho_n = \rho_0^{-n} \rho_1^n \) for every \( n \), this series converges in some annulus to a function of the following form

\[
C z^p e^{Az} + \frac{\rho_0}{\rho_1} \prod_{\nu>0} \left( 1 + \frac{z}{\rho_\nu} \right) \prod_{\mu>0} \left( 1 + \frac{-z}{\rho_\mu} \right) \prod_{\nu>0} \left( 1 + \frac{z+1}{\nu} \right) \prod_{\mu>0} \left( 1 - \frac{-z}{\mu} \right)
\]

with absolutely convergent products, integer \( p \) and coefficients satisfying \( A, A_0, C \geq 0, \rho_\nu, c_\nu, d_\mu > 0 \) for all \( \nu, \mu \). The converse is also true: every function with the representation (1) generates (i.e. its Laurent coefficients form) a doubly infinite totally positive sequence.

In the case of \( \cdots = \rho_{j-2} = \rho_{j-1} = 0 \neq \rho_j \), we assume the sequence to be terminating on the left of \( \rho_j \) and call it totally positive. A totally positive sequence can be infinite when it contains no zeros to the right of \( \rho_j \) or finite otherwise. These sequences were studied earlier than doubly infinite ones in [1]. They are generated by functions of the form (1), where the products in the last fraction are empty and \( A_0 = 0 \). Note that the term Pólya frequency sequence is often used as a synonym for totally positive sequence.

Herein, it is convenient to use the notion of \( \alpha \)-point. Given a complex number \( \alpha \), the \( \alpha \)-set of a function \( f(z) \) is the set \( \{ z \in \mathbb{C} : f(z) = \alpha \} \) and points of this set are called \( \alpha \)-points. A non-constant meromorphic function can clearly only have isolated \( \alpha \)-points. We say that an \( \alpha \)-point \( z_\alpha \) of a function \( f \) has multiplicity \( n \in \mathbb{N} \) whenever \( f'(z_\alpha) = \cdots = f^{(n-1)}(z_\alpha) = 0 \neq f^{(n)}(z_\alpha) \). The \( \alpha \)-point is simple if its multiplicity equals one.

The present work aims at describing the behaviour of \( \alpha \)-points of functions which can be represented as \( z^p R(z^k) \), where \( p \) is an integer, \( k \) is a positive integer and \( R(z) \) is not constant and generates a (possibly doubly infinite) totally positive sequence.\(^1\) We confine ourselves to the case when \( \gcd(|p|, k) = 1 \): other cases can be treated by introducing the variable \( \eta := z^{\gcd(|p|, k)} \). As a main tool, we use a relation of such functions to the so-called \( R \)-functions (also known as the Pick or Nevanlinna functions). By definition, \( R \)-functions are the real (i.e. real at every real point of continuity) functions mapping the upper half of the complex plane into itself.

Results. Our first goal is to describe the \( \alpha \)-set of the expression \( z^p R(z) \) in the upper half of the complex plane \( \mathbb{C}_+ \), where \( R(z) \) is as above, \( B \) is real and \( \alpha \in \mathbb{C} \setminus \{0\} \). Theorem 11 states that if the equa-

\(^1\)Functions of this form are the \( k \)th root transforms of \( z^p R(z^k) \). In the particular case when \( R(z) \) and \( R'(z) \) are holomorphic and nonzero at \( z = 0 \), the function \( z^p R(z^k) \) is univalent in some disk centred at the origin. Then, \( z^p R(z^k) \) will be a univalent function with \( k \)-fold symmetry in this disk in the sense that the image of \( z^p R(z^k) \) will be \( k \)-fold rotationally symmetric (see e.g. [7, § 2.1] for the details). The term "functions with \( k \)-fold symmetry" could be good under the posed narrower conditions, however we study a more general case assuming no such regularity at the origin and allowing any integer \( p \) satisfying \( \gcd(|p|, k) = 1 \).
tion $z^B R(z) = \alpha$ has solutions in $C_+$, then the $\alpha$-points are simple and distinct in absolute value. The $\alpha$-points on the real line (excepting the origin) may be either simple or double. Furthermore, for real constants $a$ and $b_1 \neq b_2$ the machinery developed in Lemma 1 implies that solutions to $z^B R(z) = ae^{ib_1}$ and to $z^B R(z) = ae^{ib_2}$ alternate when ordered in absolute value (under the additional condition that none of them fall onto the real line — see Theorem 15 for the details). The corresponding properties of $\alpha$-points in the whole complex plane are described in Theorem 14 and Remark 15.

Our approach is based on Lemma 1: a function $\psi(z)$ is univalent in the upper half of the complex plane provided that $z \psi'(z)$ is an $R$-function. In fact, this Lemma is an “appropriate” reformulation of classical results, however we need a construction from its proof. Then we study the properties of $\psi(z)$ on the real line under the additional assumption that $\psi(z)$ is meromorphic in $\overline{C_+ \setminus \{0\}}$ (Section 3). This assumption can be relaxed: poles can have condensation points on the real line. However, we are interested in the narrower case $\psi(z) := \ln z^B R(z)$ with $R(z)$ of the form (1), which appears in Theorem 11 and Theorem 13. The reason is that the function $R(z)$ can be represented as in (1) if and only if the sequence of its Laurent coefficients has the above property of total positivity.

The second goal of the present work is to study $\alpha$-points of $z^p R(z^k)$, which is done by tracking the solutions to $z^{p/k} R(z) = \alpha \cdot \exp \left( i \frac{2\pi n}{k} \right)$ and $z^{p/k} R(z) = \pi \cdot \exp \left( i \frac{2\pi n}{k} \right)$, $n \in \mathbb{Z}$, under the change of variable $z \mapsto z^k$. This result is presented by Theorems 19, 21, 22 and 23. If we split the complex plane into $2k$ sectors $Q_j = \left\{ \frac{j}{k} \pi < \text{Arg } z < \frac{j+1}{k} \pi \right\}$, $j = 0$, \ldots, $2k-1$, then Theorem 19 states that for $\text{Im } \alpha^k \neq 0$ all $\alpha$-points are inner points of the sectors, simple, and those in distinct sectors strictly interlace with respect to their absolute value. Put in other words, if $\alpha$-points of $z^p R(z^k)$ are denoted by $z_i$, then $\cdots \leq |z_{-1}| \leq |z_0| \leq |z_1| \leq \cdots$, then $\cdots < |z_{-1}| < |z_0| < |z_1| < \cdots$ and $z_1 \in Q_n$ implies that $z_{i+1}, \ldots, z_{i+k-1} \notin Q_n$ and (as soon as $R(z_{i+k}) = \alpha$) that $z_{i+k} \in Q_n$. In fact, there is a formula for $m$ such that $z_{i+1} \in Q_m$, which is trivial for $p = \pm 1$ or $k = 2$. Theorem 21 provides analogous properties in the case $\text{Im } \alpha^k = 0$. In particular, it asserts that there are at most two $\alpha$-points sharing the same absolute value, which are simple unless they occur at a sector boundary where they may collapse into a double $\alpha$-point.

In turn, Theorem 22 and Theorem 23 answer the question which sector contains the $\alpha$-point that is minimal in absolute value for a meromorphic function $R(z)$. This automatically extends to the $\alpha$-point that is the maximal in absolute value when $R \left( \frac{1}{z} \right)$ is meromorphic.

Theorems 19 and 21–23 describe zeros of entire functions of the form

$$f(z^k) + z^j g(z^k) \quad \text{or} \quad g(z^k) + z^j f(z^k), \quad j, k \in \mathbb{N},$$

(2)

where (complex) entire functions $f(z)$ and $g(-z)$ are of genus 0 and have only negative zeros. Since $f(z^k)/f(0)$ and $g(z^k)/g(0)$ become real functions, the correspondence is provided by

$$f(z^k) + z^p g(z^k) = 0 \iff z^{-p} \left( g(z^k) + z^p f(z^k) \right) = 0 \iff z^p \frac{f(z^k)}{g(z^k)} \frac{f(0)}{g(0)} = -\frac{g(0)}{f(0)},$$

(3)

on setting $p := \pm j$. We can allow $f(z)$ and $g(-z)$ to be any functions generating totally positive sequences up to constant complex factors. Then the functions of the form (2) can be identified by the condition on their Maclaurin or Laurent coefficients. See Section 7 for further details.

Our third goal is attained in the two last sections. It consists in applying the above results in the setting $k = 2$, which is summarized in Theorems 27–28. For a (complex) entire function $H$ of the complex variable $z$ consider its decomposition into odd and even parts such that $H(z) = f(z^2) + zg(z^2)$. In other words, Theorem 27 from Section 8 answers the following question: how are the zeros of the function $H(z)$ distributed if the ratio $\frac{f(z)}{g(z)}$ has only negative zeros and positive poles? The case when the ratio $\frac{f(z)}{g(z)}$ has only negative poles and positive zeros is treated by Theorem 28. The question appears to be connected to the Hermite-Biehler theorem. This is a well-known fact asserting that if the function $H(z)$ is a real
polynomial, then its stability \(^2\) is equivalent to that \(f(z)\) and \(g(z)\) only have simple negative interlacing\(^3\) zeros and \(f(0) \cdot g(0) > 0\). This correspondence (expressed as conditions on the Hurwitz matrix) is at the heart of the Routh–Hurwitz theory (see, e.g., [13, Ch. XV], [22, 4, 6]). With a proper extension of the notion of stability, this criterion extends to entire (see [6]), rational (see [4]) and further towards meromorphic functions. Furthermore, if \(H(z)\) is a polynomial and we additionally allow the ratio \(\frac{f(z)}{g(z)}\) to have positive zeros and poles, then we will obtain the "generalized Hurwitz" polynomials as introduced in [22]. In the same paper [22, Subsection 4.6], its author describes "strange" polynomials (related to stable polynomials) with interesting behaviour. Item (ii) of our Theorem 27 explains the nature of their "strangeness".

2 Connection between \(\mathcal{R}\)-functions and univalent functions

Let us use the notation "\(\arg\)" for the multivalued argument function and "\(\text{Arg}\)" for the principal branch of argument, \(-\pi < \text{Arg} z \leq \pi\) for any \(z\). We are starting from the following short but useful observation.\(^4\)

**Lemma 1.** Let \(\psi\) be a function holomorphic in \(\mathbb{C}_+ := \{z \in \mathbb{C} : \text{Im} z > 0\}\) with values in \(\mathbb{C}_+\) and let \(\psi\) be a fixed holomorphic branch of \(\int \frac{\phi(z)}{z} \, dz\). Then the function \(\psi\) is univalent in \(\mathbb{C}_+\). Moreover, if for some \(z_1, z_2 \in \mathbb{C}_+\) we have

\[
\text{Re} \, \psi(z_1) = \text{Re} \, \psi(z_2) =: a \quad \text{and} \quad \text{Im} \, \psi(z_1) =: b < \tilde{b} := \text{Im} \, \psi(z_2),
\]

then \(|z_1| < |z_2|\).

**Proof.** First let us approximate the upper half-plane \(\mathbb{C}_+\) by the set

\[
\mathbb{C}_\delta := \{z \in \mathbb{C} : \delta < \text{Arg} z < \pi - \delta, |z| > \delta\}, \quad \delta > 0.
\]

For \(z = re^{i\theta}\) we have \(\frac{\partial z}{\partial r} = \frac{z}{|z|}\) and \(\frac{\partial z}{\partial \theta} = iz\), so

\[
\frac{r}{\partial r} \text{Im} \, \psi(z) = \text{Im} \left( \frac{zr}{r} \psi'(z) \right) = \text{Im} \, \psi(z) = -\frac{\partial}{\partial \theta} \text{Re} \, \psi(z),
\]

which is the matter of the Cauchy-Riemann equation. The lemma’s hypothesis \(\text{Im} \, \phi(z) > 0\) for \(z \in \mathbb{C}_\delta\) yields that

\[
\frac{\partial}{\partial \theta} \text{Im} \, \psi(z) > 0 \quad \text{and} \quad \frac{\partial}{\partial \theta} \text{Re} \, \psi(z) < 0.
\]

The latter inequality implies that for each \(r > 0\) there can be at most one value of \(\theta \in [\delta, \pi - \delta]\) such that \(\text{Re} \, \psi(re^{i\theta}) = a\). Moreover, the set \(\Gamma_\delta := \{re^{i\theta} \in \mathbb{C}_\delta : \text{Re} \, \psi(re^{i\theta}) = a\}\) only consists of analytic arcs because \(\text{Re} \, \psi\) is a function harmonic in \(\overline{\mathbb{C}_\delta}\). In other words, we obtained the following.

(a) For every \(r > 0\) there is at most one point \(z \in \Gamma_\delta\) satisfying \(|z| = r\). That is, the arc \(\gamma_i\) in polar coordinates \((r, \theta)\) can be set by a function \(\theta(r)\).

Denote by \(\gamma_1, \gamma_2, \ldots\) the connected components of \(\Gamma_\delta\) according to their distance to the origin, so that \(\text{dist}(0, \gamma_1) \leq \text{dist}(0, \gamma_2) \leq \cdots\). To count the arcs in this manner is possible due to the regularity of the

\(^2\)The polynomial is called (Hurwitz) **stable** if all of its roots have negative real parts.

\(^3\)The zeros of two functions are called **interlacing** if in between two consecutive zeros of the first function there is a zero of the second function and vice versa.

\(^4\)There are many akin facts well known. For example, considering functions \(\Phi(\zeta) := \phi(e^{-\zeta})\) gives the problem from [25] but in a strip. However, we place it here since we need the relation between \(|z_1|\) and \(|z_2|\) satisfying (4) rather than the univalence itself.
function $\Re \psi$ in a neighbourhood of $\overline{C_\delta}$ (so every bounded subdomain of $C_\delta$ contains only a finite number of the arcs\(^5\)). It is enough to justify two additional statements, which together with (a) imply the lemma. 

(b) On each arc $\gamma_i$, $i = 1, 2, \ldots$, the value of $\Im \psi$ increases (strictly) for increasing $|z|$. 

(c) If we pass from $\gamma_i$ to $\gamma_{i+1}$ (due to (a) it corresponds to the grow of $|z|$), then $\Im \psi$ cannot decrease. (In fact, we will show that these arcs can be connected by a line segment of $\partial C_\delta$ where $\Im \psi$ increases.)

To wit, the assertions (a)–(c) provide that any distinct points of $C_\delta$ giving the same $\Re \psi$ give distinct $\Im \psi$ such that the conditions (4) imply $|z_1| < |z_2|$. In particular, this yields the univalence of $\psi$ in $C_\delta$. Furthermore, since $\delta$ is an arbitrary positive number, the lemma will hold in the whole open half-plane $C_+$. 

For the arc $\gamma_i$, $i = 1, 2, \ldots$, consider its natural parameter $\tau$. Orienting the arc according to the growth of $r$, we obtain $\frac{\partial \tau}{\partial r} > 0$. In addition, let us consider a coordinate $\nu$ changing in a direction orthogonal to $\tau$, i.e. such that $(\tau, \nu)$ form an orthogonal coordinate system. Then, with the help of the inequality (5) and one of the Cauchy-Riemann equations, we deduce that\(^6\) 

$$0 < \frac{\partial \Im \psi(z)}{\partial r} = \frac{\partial \Im \psi(z)}{\partial \tau} \frac{\partial \tau}{\partial r} + \frac{\partial \Im \psi(z)}{\partial \nu} \frac{\partial \nu}{\partial r} = \frac{\partial \Re \psi(z)}{\partial \tau} \frac{\partial \tau}{\partial r} + \frac{\partial \Re \psi(z)}{\partial \nu} \frac{\partial \nu}{\partial r} = \frac{\partial \Im \psi(z)}{\partial \tau} \frac{\partial \tau}{\partial r}.$$ 

Therefore, it is true that $z_1, z_2 \in \gamma_i$ and $|z_1| < |z_2|$ imply $\Im \psi(z_1) < \Im \psi(z_2)$, which is equivalent to (b).

Now, given two consecutive arcs $\gamma_i$ and $\gamma_{i+1}$ consider the arguments $\theta_1$ and $\theta_2$ of their adjacent points, i.e. 

$$\theta_1 := \lim_{|z| \to r_1; z \in \gamma_i} \arg z \quad \text{and} \quad \theta_2 := \lim_{|z| \to r_2; z \in \gamma_{i+1}} \arg z, \quad \text{where} \quad r_1 = \sup_{z \in \gamma_i} |z|, \quad r_2 = \inf_{z \in \gamma_{i+1}} |z|.$$ 

The arguments can be either $\pi - \delta$ or $\delta$ since the arcs are regular, and hence can only end at the boundary of $C_\delta$. Observe that $\theta_1 = \theta_2$. Indeed, let for example $\theta_1 = \pi - \delta$. Then $\Re \psi(z) < a$ as $|z| = r_1$. However, $\theta_2 = \delta$ in its turn would imply $\Re \psi(z) > a$ when $|z| = r_2$. So, in the "semi-annulus" $\{z \in C_\delta : r_1 < |z| < r_2\}$ there would be such $z$ that $\Re \psi(z) = a$, i.e. $z \in \Gamma_\delta$ which contradicts the fact that $\gamma_i$ and $\gamma_{i+1}$ are consecutive arcs of $\Gamma_\delta$.

Since $\theta_1 = \theta_2$, the ray $\Theta := \{re^{i\theta_1}, r > \delta\}$ meets both arcs $\gamma_i$ and $\gamma_{i+1}$ in the limiting points $r_1e^{i\theta_1}$ and $r_2e^{i\theta_2}$, respectively. As a consequence, we obtain that $\Im \psi(r_1e^{i\theta_1}) < \Im \psi(r_2e^{i\theta_2})$ since $\Im \psi$ grows everywhere on $\Theta$ by the condition (5). Then (b) implies that $\sup_{z \in \gamma_i} \Im \psi(z) = \inf_{z \in \gamma_{i+1}} \Im \psi(z)$. Thus, the condition (c) is satisfied as well.

3 Properties of $\alpha$-points on the real line

**Lemma 2.** Under the conditions of Lemma 1, let the function $\phi$ admit an analytic continuation through the interval $(x_1, x_2) \subset \mathbb{R} \setminus \{0\}$. Then the function $\psi$ defined as in Lemma 1 has no triple $\alpha$-points in $(x_1, x_2)$.

**Proof.** The assertion of this lemma is exactly that $\phi(z) = z\psi(z)$ has no double zeros on $(x_1, x_2)$. However, if $\phi$ could have a double zero $x_0$, then $\Im \phi(z)$ in the semi-disk $\{z \in C_+ : |z - x_0| < \varepsilon \ll 1\}$ must have values of both signs (since $\phi(z)$ is close to $(z - x_0)^2$ for such $z$). In its turn, this contradicts $\phi(C_+) \subset C_+$. \(\square\)

Further in this section, we restrict the $\mathcal{R}$-functions $\phi_1, \phi_2$ to be meromorphic in $C$ and real on the real line (where finite), i.e. to have the (absolutely convergent) Mittag-Leffler representation\(^7\)

$$B + Az - \frac{A_0}{z} - \sum_{v \neq 0} \frac{zA_v/a_v}{z - a_v}, \quad \text{where} \quad B, a_v \in \mathbb{R}, \quad a_v \neq 0, \quad A, A_0 \geq 0 \quad \text{and} \quad A_v > 0 \quad \text{for all} \quad v \neq 0.$$ 

\(^5\) If it is not, then the ray $(re^{i\theta} : r > 0)$ for appropriate fixed $\theta \in [\delta, \pi - \delta]$ meets $\Gamma_\delta$ in an infinite number of points of the subdomain (it follows from (a)). The function $\Re \psi(re^{i\theta})$ is analytic in $r > 0$ (as a function of two variables $\theta$ and $r$ with $\theta$ fixed).

\(^6\) Consequently, $\Re \psi(re^{i\theta})$ must be constant on that ray, because it attains the same value on a point sequence converging to an internal point of its domain of analyticity. So, we have a contradiction unless $\Gamma_\delta = \{re^{i\theta} : r > 0\}$.

\(^7\) In fact we have more: $\Im \phi(z)/\partial \nu = 0$ implies that the gradient of $\Im \psi$ on $\gamma$, is tangential to $\gamma_1$.

\(^7\) Non-constant meromorphic functions of this form (and only of this form) map $C_+$ into $C_+$, see [6, § V Thm. 1].
such that \( \phi_1, \phi_2 \neq B \). For our purposes, we need functions of more general form. Let us take a non-constant function \( \phi(z) \) with the representation \( \phi_1(z) - \phi_2(1/z) \), where \( \phi_1(z) \) and \( \phi_2(z) \) are as given by (7). Note that both mappings \( z \mapsto \frac{1}{2} \) and \( z \mapsto -z \) map the upper half of the complex plane \( \mathbb{C}_+ \) into the lower half-plane. Therefore, \( \phi \) is necessarily an \( \mathcal{R} \)-function.

**Remark 3.** If \( \psi'(z) \) has the form (7), then \( \psi(z) \) can be represented as

\[
\psi(z) = \int \frac{\psi'(z)}{z} dz = C + B \ln z + A_0 \frac{1}{z} - \sum \frac{A_v}{a_v} \ln \left( 1 - \frac{z}{a_v} \right)
\]

for some complex constant \( C \). This implies the equality

\[
\text{Re } \psi(z) = \text{Re } C + B \ln |z| + \left( A + \frac{A_0}{|z|^2} \right) \text{Re } z - \sum \frac{A_v}{a_v} \ln \left| 1 - \frac{z}{a_v} \right|.
\]

**Remark 4.** If \( \psi'(z) = \phi(z) = \phi_1(z) - \phi_2(1/z) \), then we introduce two auxiliary functions \( \psi_1 \) and \( \psi_2 \) (single-valued in \( \mathbb{C}_+ \) where regular) so that \( \psi_1'(z) = \phi_1(z) \) and \( \psi_2'(z) = \phi_2(z) \). These settings then imply \( \psi_2(z) = -\phi_2(\frac{1}{z}) = z^2 \left( \frac{1}{z} \right)' \cdot \phi_2 \left( \frac{1}{z} \right) \), that is \( \psi_2 \left( \frac{1}{z} \right) = \int \frac{\phi(z)}{z} dz \). Both \( \psi_1 \) and \( \psi_2 \) satisfy (7), therefore

\[
\text{Re } \psi(z) = \text{Re } \psi_1(z) + \text{Re } \psi_2(1/z), \quad \text{where both } \text{Re } \psi_1 \text{ and } \text{Re } \psi_2(1/z) \text{ have the form (8)}.
\]

In particular, in each pole \( x_+ \neq x_0 \) of \( \phi \) the function \( \psi \) has a logarithmic singularity and \( \text{Re } \psi(z) \to +\infty \cdot x_+ \) when \( z \to x_+ \).

**Lemma 5.** If \( x \psi'(x) = \phi(x) = \phi_1(x) - \phi_2(1/x) \), where \( x \in \mathbb{R} \) and \( \phi_1(x), \phi_2(x) \) have the form (7), then the following assertions are true.

(a) The function \( \text{Im } \psi(x) \) does not change its value possibly except at the origin and poles of \( \phi \).

(b) Between every two consecutive negative poles \( x_2 < x_1 \) of \( \phi \), there is exactly one local minimum of \( \text{Re } \psi \).

(c) Between every two consecutive positive poles \( x_1 < x_2 \) of \( \phi \), there is exactly one local maximum of \( \text{Re } \psi \).

(d) In (b) and (c), \( x_1 \) can be set to zero provided that \( \phi \) is regular between 0 and \( x_2 \), and \( \lim_{t \to +0} |\phi(tx_2)| = \infty \).

In this case we have \( \text{Re } \psi(tx_2) \to +\infty \cdot x_2 \) as \( t \to +0 \).

**Proof.** Take a real \( x \neq 0 \) such that both functions \( \phi_1(x) \) and \( -\phi_2(1/x) \) are regular. Since their values are real on the real line, the condition

\[
x \frac{\partial \text{Im } \psi(x)}{\partial x} = r \frac{\partial \text{Im } \psi(x)}{\partial r} = \text{Im } \phi(x) = \text{Im } \phi_1(x) - \text{Im } \phi_2(1/x) = 0
\]

is satisfied. So the assertion (a) is true.

The function \( x \frac{\partial \text{Re } \psi(x)}{\partial x} = \text{Re } \phi(x) = \phi_1(x) - \phi_2(1/x) \) strictly increases form \( -\infty \) to \( +\infty \) between the points \( x_1 \) and \( x_2 \) and hence it changes its sign exactly once in the interval \( (\min(x_1, x_2), \max(x_1, x_2)) \). That is, sign \( x \cdot \text{Re } \psi(x) \) changes from decreasing to increasing on this interval, which is giving us the assertions (b) and (c) for both zero and nonzero \( x_1 \).

Suppose that the function \( \phi \) is regular between 0 and \( x_2 \) and \( \lim_{t \to +0} |\phi(tx_2)| = \infty \) is infinite. Then \( \phi \) increases in this interval, so \( \lim_{t \to +0} \phi(tx_2) = -\infty \cdot x_2 \). Therefore, \( -\psi'(tx_2) = -\frac{\phi(tx_2)}{tx_2} > \frac{1}{t} \) for small enough \( t > 0 \) and

\[
\text{Re } \psi(tx_2) = \text{Re } \psi \left( \frac{1}{2} x_2 \right) + \int_{\frac{1}{2} x_2}^{tx_2} \frac{\phi(x)}{x} dx = \text{Re } \psi \left( \frac{1}{2} x_2 \right) + x_2 \int_{t}^{\frac{1}{t} x_2} \left( -\frac{\phi(sx_2)}{sx_2} \right) ds \to +\infty \cdot x_2 \quad \text{as } t \to +0,
\]

which is (d). \( \square \)
**Lemma 6.** In addition to the conditions of Lemma 5, suppose that \( \phi \) is a regular function in the interval \( \mathcal{I} = (\min\{0,x_2\}, \max\{0,x_2\}) \subset \mathbb{R} \), \( x_2 \) is a pole of \( \phi \) and the limit \( \mathcal{B} := \lim_{t \to +0} \phi(tx_2) \) is finite.\(^8\)

(a) If \( \mathcal{B}x_2 > 0 \), then \( \Re \psi(x) \) is an increasing function in \( \mathcal{I} \) such that \( \Re \psi(\mathcal{I}) = \mathbb{R} \), and furthermore, \( \Re \psi(z) \neq \Re \psi(x) \) on condition that \( |z| \leq |x| \) with \( x \in \mathcal{I} \) and \( z \in \mathbb{C}_+ \setminus \{x\} \).

(b) If \( \mathcal{B}x_2 < 0 \), then \( \Re \psi(x) \) has exactly one local extremum in \( \mathcal{I} \) and tends to \(+\infty \cdot x_2\) as \( x \) approaches \( 0 \) or \( x_2 \).

(c) If \( \mathcal{B} = 0 \) then \( \Re \psi(x) \) is an increasing function in \( \mathcal{I} \) and the inequality \( \Re \psi(z) \neq \Re \psi(x) \) holds provided that \( |z| \leq |x| \) with \( z \in \mathbb{C}_+ \), \( x \in \mathcal{I} \). Moreover, \( \lim_{t \to +0} \frac{\phi(tx_2)}{tx_2} \) is positive or \(+\infty\). If additionally \( \Re \psi(tx_2) \) is unbounded as \( t \to +0 \), then \( \Re \psi(\mathcal{I}) = \mathbb{R} \).

**Proof.** In the interval \( \mathcal{I} \), the function \( x^2 \frac{\partial \Re \psi(x)}{\partial x} = \phi(x) \) strictly increases, and hence changes its sign at most once. Therefore, \( \Re \psi(x) \) has at most one local extremum: maximum for \( x_2 < 0 \) and minimum for \( x_2 > 0 \). Suppose that \( 0 < |\mathcal{B}| < \infty \). Then the equality \( x^2 \frac{\partial \Re \psi(x)}{\partial x} = \phi(x) \) yields the following relation

\[
\Re \psi(tx_2) = \Re \psi\left(\frac{1}{2}tx_2\right) + \int_{\frac{1}{2}tx_2}^{tx_2} \frac{\phi(x) - \mathcal{B}}{x} dx + \mathcal{B} \ln \frac{tx_2}{\frac{1}{2}tx_2} \sim \mathcal{B} \ln t \to -\infty \cdot \mathcal{B} \quad \text{when} \quad t \to +0.
\]

On account of \( \Re \psi(x) \to +\infty \cdot x_2 \) when \( x \to x_2 \) (see Remark 4) this relation implies the assertion (b) and that \( \Re \psi \) increases in \( \mathcal{I} \) from \(-\infty\) to \(+\infty\) if \( \mathcal{B}x_2 > 0 \). Therefore, to obtain (a) it is enough to use the inequality

\[
\Re \psi(-|z|) < \Re \psi(z) < \Re \psi(|z|), \quad \text{where} \quad \Im z > 0,
\]

which is a consequence of (6). Indeed, if for example \( x_2 < 0 \), then we have \( \Re \psi(x) \leq \Re \psi(-|z|) < \Re \psi(z) \) for each \( x \in \mathcal{I} \) satisfying \( |x| \geq |z| \).

The condition \( \mathcal{B} = 0 \) implies \( \frac{\phi(x)}{x} > 0 \) in the interval \( \mathcal{I} \), i.e. that \( \Re \psi \) is growing independently of the sign of \( x_2 \). The inequality \( \lim_{t \to +0} \frac{\phi(tx_2)}{tx_2} \neq 0 \) is provided by the fact that \( \mathcal{R} \)-functions cannot vanish faster then linearly. Furthermore, \( \Re \psi \) runs through the whole \( \mathbb{R} \) on condition that it is unbounded near the origin, as asserted in (c). If \( |z| \leq |x| \) with \( z \in \mathbb{C}_+ \) and \( x \in \mathcal{I} \), then the inequality (9) provides \( \Re \psi(z) \neq \Re \psi(x) \).

**Remark 7.** In Lemma 5 and Lemma 6, the value of \( x_2 \) can be taken equal to \(+\infty\) or \(-\infty\) at the cost of some of the conclusions. With such a choice, the condition \( \Re \psi(x) \to +\infty \cdot x_2 \) as \( x \to x_2 \) may be violated. This, in turn, implies that the function \( \psi(x) \) in (b), (c) and (d) of Lemma 5 and (b) of Lemma 6 may lose the extremum and become monotonic. In cases (a) and (c) of Lemma 6, \( \Re \psi(\mathcal{I}) \) is only a semi-infinite interval of the real line, instead of the equality \( \Im \psi(\mathcal{I}) = \mathbb{R} \).

4 Location of \( \alpha \)-points in the closed upper half-plane

**Lemma 8.** Let functions \( \phi_1(z), \phi_2(z) \) be of the form (7) and let \( \psi(z) \) be a smooth branch of \( \int \frac{\phi_1(z) - \phi_2(1/z)}{z} \, dz \). If two points \( z_1, z_2 \in \mathbb{C}_+ \) that are regular for \( \psi \) satisfy \( |z_1| < |z_2| \) and \( \Re \psi(z_1) = \Re \psi(z_2) = : a \), then

(a) \( \Im \psi(z_1) \leq \Im \psi(z_2) \);

(b) for each \( \epsilon \in (\Im \psi(z_1), \Im \psi(z_2)) \) there exists \( z \in \mathbb{C}_+ \) such that \( |z_1| < |z| < |z_2| \) and \( \psi(z) = a + i\epsilon \);

(c) \( z_1 \) and \( z_2 \) can be connected by a piecewise analytic curve of a finite length, on which \( \psi \) is smooth and \( \Im \psi(z) \) is a non-decreasing function of \( |z| \); the curve is a subinterval of \( \mathbb{R} \) if and only if equality holds in (a);

(d) furthermore, equality holds in (a) if and only if \( z_1, z_2 \in \mathbb{R} \), \( z_1 \cdot z_2 \geq 0 \) and \( \psi(z) \neq a \) for all \( z \in \mathbb{C}_+ \) such that \( |z_1| < |z| < |z_2| \).

**Proof.** Recall that the function \( \phi(z) = \phi_1(z) - \phi_2(1/z) \) maps \( \mathbb{C}_+ \to \mathbb{C}_+ \), i.e. satisfies Lemma 1. So if \( z_1, z_2 \in \mathbb{C}_+ \), then by Lemma 1 the condition \( \Im \psi(z_1) > \Im \psi(z_2) \) induces \( |z_1| > |z_2| \), and \( \Im \psi(z_1) < \Im \psi(z_2) \) induces \( |z_1| < |z_2| \). As a consequence, the assertions (a) and (d) holds for non-real \( z_1, z_2 \).

---

\(^8\)This limit exists since the function \( \phi \) increases in \( \mathcal{I} \).
The real part of \( \psi \) goes to \( \pm \infty \) on approaching a (nonzero) pole of \( \phi \), as stated in Remark 4. Consequently, it is impossible for a pole of \( \phi \) to be a limiting point of the set

\[
\Gamma := \{ re^{i\theta} \in \mathbb{C}_+ : \text{Re } \psi(re^{i\theta}) = a, \ |z_1| < |z| < |z_2| \},
\]

so the function \( \psi \) is regular in a neighbourhood of \( \Gamma \). (Recall that \( z_1 = 0 \) is allowed by the lemma’s condition only if \( \psi \) is regular at the origin.) Analogously to the proof of Lemma 1, consider arbitrary consecutive (i.e. there are no points of \( \Gamma \) in between) arcs \( \gamma_1 \) and \( \gamma_2 \) such that \( \sup_{z \in \gamma_1} |z| =: r_1 \leq r_2 := \sup_{z \in \gamma_2} |z| \). Within these settings, the limits

\[
\zeta_1 := \lim_{|z| \to r_1 : z \in \gamma_1} z \quad \text{and} \quad \zeta_2 := \lim_{|z| \to r_1 : z \in \gamma_2} z
\]

exist, are real and of the same sign. Let us show now that the function \( \phi \) is regular on the line segment \( \mathcal{J} := [\min\{\zeta_1, \zeta_2\}, \max\{\zeta_1, \zeta_2\}] \). Indeed, in the case \( \zeta_1 > 0 \) we have \( \text{Re } \psi(z) \to +\infty \) as \( z \) tends to a positive pole in contrast to \( \text{Re } \psi(z) < a \) on the whole \( \mathcal{J} \). Similarly, \( \text{Re } \psi(z) \to -\infty \) for \( z \) tending to a negative pole, although \( \text{Re } \psi(z) > a \) when \( z \in \mathcal{J} \) and \( \zeta_1 < 0 \). Hence, the function \( \phi \) is regular on \( \mathcal{J} \). By Lemma 5, \( \text{Im } \psi(z) \) is constant on the segment \( \mathcal{J} \) (this implies the equality in (d) for \( z_1 = \zeta_1 \neq 0 \) and \( z_2 = \zeta_2 \)). Summarizing, we obtain that the function \( \psi \) is regular on \( \gamma_1 \cup \mathcal{J} \cup \gamma_2 \), and \( \text{Im } \psi(z) \) is continuous and non-decreasing as \( |z| \) grows and thus attains all intermediate values. This reasoning is applicable for each pair of consecutive arcs constituting the set \( \Gamma \). That is, any two points \( z_1, z_2 \in \Gamma \) can be connected by a piecewise analytic curve of a finite length, so that \( \text{Im } \psi \) is continuous and has a uniformly bounded growth on it. This implies the first two assertions of the lemma. Furthermore, we necessarily have \( \text{Im } \psi(z_1) < \text{Im } \psi(z_2) \) unless this piecewise analytic curve is a segment of the real line.

It remains to check the case when \( z_1 = 0 \). In this case, \( \psi(z) \) is regular at the origin, and thus (see the assertion \( c \) of Lemma 6) it is strictly increasing in some real interval enclosing \( z_1 \). Then (9) shows that \( z_1 \) is the end of some arc from \( \Gamma \). Choosing this arc as \( \gamma_1 \) allows us to apply the previous part of the proof, and therefore yields \( \text{Im } \psi(z_1) < \text{Im } \psi(z_2) \).

\[\square\]

Lemma 9. If \( f(z) \) is holomorphic at \( z_0 \), \( g(z) \) is holomorphic at \( f_0 = f(z_0) \) such that \( g'(f_0) \neq 0 \) and \( n \) is a natural number, then \( f^n(z_0) = f''(z_0) = \cdots = f^{(n)}(z_0) = 0 \) if and only if

\[
\frac{d^k g(f(z))}{dz^k} \bigg|_{z=z_0} = \frac{d^{k+1} g(f(z))}{dz^{k+1}} \bigg|_{z=z_0} = \cdots = \frac{d^n g(f(z))}{dz^n} \bigg|_{z=z_0} = 0. \tag{10}
\]

Analogously, if \( f(z) \) is holomorphic at \( z_0 \) such that \( f'(z_0) \neq 0 \) and \( g(z) \) is holomorphic at \( f_0 = f(z_0) \), then the condition (10) is equivalent to \( g'(f_0) = g''(f_0) = \cdots = g^{(n)}(f_0) = 0 \).

**Proof.** Both facts follow from solving equations provided by the chain rule sequentially:

\[
\frac{d^k g(f(z))}{dz^k} \bigg|_{z=z_0} = g'(f_0) f'(z_0),
\]

\[
\frac{d^{k+1} g(f(z))}{dz^{k+1}} \bigg|_{z=z_0} = g''(f_0)(f'(z_0))^2 + g'(f_0)f''(z_0),
\]

\[
\frac{d^n g(f(z))}{dz^n} \bigg|_{z=z_0} = g^{(n)}(f_0)(f'(z_0))^n + \cdots + g'(f_0)f^{(n)}(z_0).
\]

---

9The arcs \( \gamma_1 \) and \( \gamma_2 \) are consecutive, thus one of the inequalities \( \text{Re } \psi(z) > a \) and \( \text{Re } \psi(z) < a \) holds for all \( z \in \mathbb{C}_+ \) satisfying \( r_1 < |z| < r_2 \). The former inequality corresponds to the positive sign of \( \zeta_1, \zeta_2 \), while the latter corresponds to the negative sign. See the proof of Lemma 1 for the details.

10See footnote 9.

11Poles of \( \phi \) can concentrate only at the origin and each interval between poles contains at most two ends of arcs from \( \Gamma \). That is, \( \Gamma \) contains a finite number of arcs. Each of the arcs has a finite length since \( \psi \) is smooth in a neighbourhood of \( \Gamma \). Therefore, the length of the curve is finite.
which is in fact (9). Put in other words, solutions to

$$W(z) = e^{Az + C + \frac{A_0}{2} z^2} \prod_{\nu > 0} \left(1 + \frac{z}{\nu} \right)^{\xi_\nu} \prod_{\mu > 0} \left(1 - \frac{z}{\mu} \right)^{\lambda_\mu}$$

considered as a single-valued function in $\mathbb{C}_+$, (11)

where $C \in \mathbb{C}$, $B \in \mathbb{R}$, $A, A_0 \geq 0$, and $a_\nu, \kappa_\nu, b_\mu, \lambda_\mu$ are positive reals for all $\nu, \mu$. The function $V$, as well as the more general

$$W(z) = e^{Az + C + \frac{A_0}{2} z^2} \prod_{\nu > 0} \left(1 + \frac{z}{\nu} \right)^{\xi_\nu} \prod_{\mu > 0} \left(1 - \frac{z}{\mu} \right)^{\lambda_\mu} = V_1(z) \cdot V_2 \left( \frac{1}{z} \right),$$

where both functions $V_1$ and $V_2$ admit the representation (11), then has an analytic continuation in a neighbourhood of each its real $\alpha$-point (excluding the origin). This allows us to determine the multiplicity of such $\alpha$-points. Another straightforward fact is that both $V$ and $W$ never vanish outside the real line.

**Theorem 11.** If a function $W$ defined in $\overline{\mathbb{C}}_+$ has the form (12) such that $W(z) \neq e^C z^B$, then for any $\alpha \in \mathbb{C} \setminus \{0\}$ the $\alpha$-points of $W(z)$ inside $\mathbb{C}_+$ (if they exist) are simple and distinct in absolute value from all other solutions to $|W(z)| = |\alpha|$ lying in $\overline{\mathbb{C}}_+$. The $\alpha$-points on the real line (except at 0) may be either simple or double.

**Remark 12.** If $W(z)$ is regular and nonzero for $z = 0$, then it has the form (11) with $A_0 = B = 0$. Therefore, the equality $W(0) = \alpha \neq 0$ necessarily gives $(\ln W)'(0) \neq 0$ by the case (c) of Lemma 6, that is the $\alpha$-point $z = 0$ can be only simple.

**Proof of Theorem 11.** For $z \psi'(z) = \phi(z) = z(\ln W(z))'$ and $z \in \mathbb{C}_+$

$$\phi(z) \neq 0 \implies \psi(z) \neq 0 \implies W'(z) = \frac{\psi'(z)}{\psi(z)} \neq 0,$$

which gives us that the non-real $\alpha$-points are simple. If $W(z) = \alpha$ then

$$|W(-|z|)| < |W(z)| = |\alpha| < |W(|z|)| \quad \text{unless} \quad z = \pm |z|,$$

which is in fact (9). Put in other words, solutions to $|W(z)| = |\alpha|$ (which include all $\alpha$-points) in $\overline{\mathbb{C}}_+$ have distinct absolute values. By Lemma 9, $\alpha$-points of $W$ and (in $\ln \alpha$-)points of $\ln W$ (with the same branch of logarithm) have equal multiplicities. Lemma 2 hence justify that the multiplicity of real $\ln \alpha$-points of $\psi$ is at most 2.

**Theorem 13.** Under the assumptions of Theorem 11, if $|z_1| < |z_2|$, $W(z_1) = \alpha$ and $W(z_2) = a e^{i \theta}$ with a real $\theta > 0$, then for every $q \in (0, \theta)$ there exists $z_\ast \in C_{12} := \{z \in \mathbb{C}_+: |z_1| < |z| < |z_2|\}$ such that $W(z_\ast) = a e^{i \theta}$, unless simultaneously $\theta = 0$ (mod $2\pi k$), both $z_1$ and $z_2$ are real of the same sign and $|W(z)| \neq |\alpha|$ in the semi-annulus $C_{12}$.

**Proof.** This is a straightforward corollary of Lemma 8 for $\psi(z)$ being a branch of $\ln W(z)$. We only need to account that the exponential function maps $\alpha + 2\pi n$ for all integer $n$ to the same point $e^\alpha$.

If the $\alpha$-set of $W$ is not empty, then $\alpha$-points of $W$ are assumed to be numbered according to the growth of their absolute values, i.e. $\cdots < |z_0| \leq |z_1| \leq |z_2| \leq \cdots$ and $W(z) = \alpha \iff z \in \bigcup_k z_k$. At that, each multiple $\alpha$-point we count only once. In the sequel we consider only the case of $C = 0$: otherwise, the equality $W(z) = \alpha$ can be replaced with $W(z) e^{-C} = a e^{-C}$. □
Theorem 14. Let $W(z)$ be of the form (12) with natural $\kappa, \tilde{\kappa}, \lambda, \tilde{\lambda}$, and $C = 0$. Choose the branch of $z^B$ which is smooth in $\mathbb{C}_+ \setminus \{0\}$ and positive for $z > 0$. Given a complex number $\alpha \notin \mathbb{R}$ such that $\alpha e^{\pm iB\pi} \notin \mathbb{R}$, each $\alpha$-point of $W(z)$ in $\mathbb{C} \setminus \mathbb{R}$ is simple and distinct in absolute value from other $\alpha$-points. If $z_i, z_{i+1}$ are two consecutive points of the $\alpha$-set, then $\text{Im } z_i \cdot \text{Im } z_{i+1} < 0$.

Moreover, the equations $W(x) = \alpha$ and $W_\pm(-x) := \lim_{y \to \pm 0} W(-x + iy) = \alpha$ have no solution for $x > 0$.

Note that in the case of integer $B$, the conditions $\alpha e^{\pm iB\pi} \notin \mathbb{R}$ and $\alpha \notin \mathbb{R}$ of this theorem are equivalent; furthermore, the function $W(x)$ is defined for $x < 0$ and equal to $W_-(x) = W_+(x)$.

Proof. On the one hand, for $x > 0$ the functions $W(x), e^{-iB\pi}W_+(x)$ and $e^{iB\pi}W_-(x)$ are real. On the other hand, both $\alpha$ and $\alpha e^{\pm iB\pi}$ are non-real. Therefore, there is no solution to $W(x) = \alpha$ and to $W_\pm(-x) = \alpha$ when $x > 0$. Since $W(\mathbb{C}) = \mathbb{W}(\mathbb{Z})$, we can find the solutions to $W(z) = \alpha$ in the rest part of the complex plane $\mathbb{C} \setminus \mathbb{R}$ from the equations $W(z) = \alpha$ and $W(z) = \overline{\alpha}$ in the upper half-plane.

Now assume that the argument of $W$ varies in $\mathbb{C}_+$. Theorem 11 implies that all $\alpha$-points (as well as all $\tilde{\alpha}$-points) of the function $W$ are simple and distinct in absolute value. Furthermore, according to (13) absolute values of $\alpha$-points and of $\tilde{\alpha}$-points cannot coincide (due to $\alpha \neq \overline{\alpha}$). On account of $\overline{\alpha} = \alpha e^{-2i\arg \alpha}$, Theorem 15 (with the setting $\theta = 2\pi$) induces that if we have two solutions $z_i, z_{i+k}$ to $W(z) = \alpha$, then there is a solution $z_*$ to $W(\mathbb{R}) = \alpha$ such that $|z_i| < |z_*| < |z_{i+k}|$. Conversely, between each pair of $\alpha$-points there is an $\alpha$-point by the same theorem. That is, the absolute values of $\alpha$- and $\tilde{\alpha}$-points interlace each other. This fact provides the theorem.

Remark 15. If in Theorem 14 we take the number $\alpha \neq 0$ real, then the equations $W(z) = \alpha$ and $W(\mathbb{C}) = \alpha$ are satisfied simultaneously. As a result, each $\alpha$-point of $W(z)$ in $\mathbb{C} \setminus \mathbb{R}$ is simple and there is a unique $\alpha$-point with the matching absolute value (which is the complex conjugate). For an $\alpha$-point $z_i$ on the real line (such points are positive unless $B$ is integer) there are only the following possibilities.

(a) The point $z_i$ belongs to an interval between two consecutive positive poles or negative zeros of $W$. If $z_i$ is double, then the interval contains no other $\alpha$-points of $W$. If $z_i$ is simple, then the interval contains exactly one another $\alpha$-point: either $z_{i-1}$ or $z_{i+1}$.

(b) The point $z_i$ belongs to an interval between the origin and the maximal negative zero or the minimal positive pole. Then exactly one another $\alpha$-point (if $z_i$ is simple) or no other $\alpha$-points (if $z_i$ is double) lies on the same interval provided that $A_0 > 0$ or $Bz_i < 0$ in (12). If $A_0 = 0$ and $Bz_i \geq 0$, then $z_i$ is the $\alpha$-point minimal in absolute value and the same interval contains no other $\alpha$-points.

(c) The point $z_i$ lies on a ray of the real line, in which $W$ has no poles or zeros. Then this ray contains at most one another $\alpha$-point of $W$. If $A_0 = 0$, $Bz_i \geq 0$ and one end of this ray is the origin, then $z_i$ is the only $\alpha$-point on the ray and its absolute value is minimal among all solutions to $W(z) = \alpha$.

All assertions stated here are straightforward corollaries to Lemma 5, Lemma 6 and Remark 7. The $\alpha$-set of $W$ for $\alpha e^{\pm iB\pi} \in \mathbb{R}$ and $B \notin \mathbb{Z}$ can be studied similarly; the main distinction is that $W$ is not continuous on the negative semi-axis, so the corresponding result will be concerned with the limiting values $W_+$ or $W_-.$

Remark 16. Functions of the form (11) are meromorphic exactly when the exponents $\kappa, \lambda$ are positive integers, $A_0 = 0$ and $B \in \mathbb{Z}$. These functions generate infinite totally positive sequences. Functions of the form (12) generate doubly infinite totally positive sequences if $\kappa, \lambda, \tilde{\kappa}, \tilde{\lambda} \in \mathbb{N}$ and $B \in \mathbb{Z}$. See the subsection “Definitions” of Section 1 for the further details.

Hereinafter we concentrate on the case $B = \frac{p}{k}$ of (12) with positive integers $\kappa, \tilde{\kappa}, \lambda, \tilde{\lambda}, \mu, \tilde{\mu}$, integer $k \geq 2$ and $p \neq 0$. We assume that $\gcd(|p|, k) = 1$, i.e. the fraction $\frac{p}{k}$ is irreducible. The $k$-th root is a $k$-valued holomorphic function on the punctured plane $\mathbb{C} \setminus \{0\}$. So, let $\sqrt[k]{\cdot}$ denote its branch that is holomorphic in $\mathbb{C}_+ \setminus \{0\}$ and maps positive semi-axis into itself. Then

\[
R(w) = (\sqrt[k]{w})^p e^{\lambda w + A_0 w^{-1}} \prod_{\nu > 0} \left(1 + \frac{w}{\alpha \nu}ight) \prod_{\nu > 0} \left(1 + \frac{1}{\alpha \nu}ight) \prod_{\mu > 0} \left(1 - \frac{w}{\beta \mu}ight) \prod_{\mu > 0} \left(1 - \frac{1}{\beta \mu}ight),
\] (14)
in which the coefficients satisfy $A, A_0 \geq 0$ and $a_v, b_v, c_v, d_v > 0$ for all $v, \mu$, is a single-valued meromorphic function in $\mathbb{C}_+ \setminus \{0\}$ regular for $\text{Im } w \neq 0$.

### 5 Composition with $k$th power function

In the current section we assume that a function $G \neq z^p$ has the representation

$$G(z) := e^{A z^k + A_0 z^{-k}} z^p \prod_{\nu > 0} \left(1 + \frac{z^k}{a_{\nu}}\right) \prod_{\mu > 0} \left(1 + \frac{z^{-k}}{c_{\mu}}\right) \prod_{\nu > 0} \left(1 - \frac{z^k}{b_{\nu}}\right) \prod_{\mu > 0} \left(1 - \frac{z^{-k}}{d_{\mu}}\right)$$

for some natural $k \geq 2$ and integer $p$, $\gcd(|p|, k) = 1$, in which the coefficients satisfy $A, A_0 \geq 0$ and $a_v, b_v, c_v, d_v > 0$ for all $v, \mu$. As we noted above, the case when $|p|$ and $k$ are not coprime does not need any additional study: it can be treated by introducing the new variable $\eta := z^{1/\gcd(|p|, k)}$. Furthermore, the location of zeros and poles of $G(z)$ is clear from the expression (15), so we concentrate on the equation $G(z) = \alpha$ where $\alpha \in \mathbb{C} \setminus \{0\}$.

For the sake of brevity denote $e_m := \exp \left(\frac{i m \pi}{k}\right)$. The condition $\gcd(|p|, k) = 1$ implies that

- $(e_{mp})^{k-1}_{m=-\infty} \subset \mathbb{C}$ is a cyclic group of order $2k$ generated by $e_p$ when $p$ is odd (thus $e_{mp} = e_n$ for $n \in \mathbb{Z}$ if and only if $mp \equiv n \pmod{2k}$);
- $(e_{mp})^{k-1}_{m=0}$ and $(e_{mp+1})^{k-1}_{m=0}$ are two disjoint cyclic groups of order $k$ generated by $e_p$ when $p$ is even (the former group contains $e_0 = 1$ and the latter one contains $e_k = -1$).

Denote the sectors of the complex plane with the central angle $\frac{\pi}{k}$ by

$$Q_s := \left\{ z \in \mathbb{C} \setminus \{0\} : 0 < \text{Arg } z e^{-s} < \frac{\pi}{k} \right\}, \quad \bar{Q}_s := \left\{ z \in \mathbb{C} \setminus \{0\} : 0 \leq \text{Arg } z e^{-s} < \frac{\pi}{k} \right\}, \quad s \in \mathbb{Z},$$

so that they are numbered in an anticlockwise direction and $Q_s = Q_{2k+s}$, $\bar{Q}_s = \bar{Q}_{2k+s}$.

The equation $G(z) = \alpha$ with a fixed $\alpha \neq 0$ and $w \in \mathbb{C}_+ \cup \{z > 0\}$. Substituting $z = \sqrt[k]{\overline{w} e_{2m}}$ into (15) shows that if

$$R(w) = a e_{-2pm}, \quad \text{where } m = 0, \ldots, k - 1,$$

then $z = \sqrt[k]{\overline{w} e_{2m}} \in \bar{Q}_{2m}$ solves the equation $G(z) = \alpha$. Analogously, if the equality

$$R(w) = \overline{a} e_{2pm}, \quad \text{where } m = 0, \ldots, k - 1,$$

holds for some $m$, then $z = \sqrt[k]{\overline{w} e_{2m}} \in \bar{Q}_{2m-1}$ solves $G(z) = \alpha$. Conversely, for each solution of $G(z) = \alpha$ there exists an integer $m$ (unique under the condition $0 \leq m < k$) such that $R(z^k) = a e_{-2pm}$ on condition that $z^k \in \mathbb{C}_+ \cup \{z > 0\}$, or $R(z^k) = \overline{a} e_{2pm}$ on condition that $z^k \notin \mathbb{C}_+ \cup \{z > 0\}$. In this sense, the equation $G(z) = \alpha$ can be replaced with the relation

$$R(w) \in \Omega, \quad \text{where } \Omega := \{a e_{-2pm}\}^{k-1}_{m=0} \cup \{\overline{a} e_{2pm}\}^{k-1}_{m=0}$$

for $w \in \overline{\mathbb{C}_+}$, and then all $\alpha$-points of $G(z)$ can be determined from the solutions to (18).

**Remark 17.** Let $G(z)$ and $R(\omega)$ be as in (15) and (14), respectively, $\alpha \neq 0$ and $w \in \mathbb{C}_+ \cup \{z > 0\}$. Substituting $z = \sqrt[k]{\overline{w} e_{2m}}$ into (15) shows that if

$$R(w) = a e_{-2pm}, \quad \text{where } m = 0, \ldots, k - 1,$$

then $z = \sqrt[k]{\overline{w} e_{2m}} \in \bar{Q}_{2m}$ solves the equation $G(z) = \alpha$. Analogously, if the equality

$$R(w) = \overline{a} e_{2pm}, \quad \text{where } m = 0, \ldots, k - 1,$$

holds for some $m$, then $z = \sqrt[k]{\overline{w} e_{2m}} \in \bar{Q}_{2m-1}$ solves $G(z) = \alpha$. Conversely, for each solution of $G(z) = \alpha$ there exists an integer $m$ (unique under the condition $0 \leq m < k$) such that $R(z^k) = a e_{-2pm}$ on condition that $z^k \in \mathbb{C}_+ \cup \{z > 0\}$, or $R(z^k) = \overline{a} e_{2pm}$ on condition that $z^k \notin \mathbb{C}_+ \cup \{z > 0\}$. In this sense, the equation $G(z) = \alpha$ can be replaced with the relation

$$R(w) \in \Omega, \quad \text{where } \Omega := \{a e_{-2pm}\}^{k-1}_{m=0} \cup \{\overline{a} e_{2pm}\}^{k-1}_{m=0}$$

for $w \in \overline{\mathbb{C}_+}$, and then all $\alpha$-points of $G(z)$ can be determined from the solutions to (18).

**Remark 18.** The relation (18) shows that the equation $G(z) = \alpha$ has different properties depending whether $\text{Im } \alpha^k$ is zero or not. The case of $\overline{a} \in \{a e_{-2pm}\}^{k-1}_{m=0}$ is equivalent to $\text{Im } a e_{ps} = 0$ for some $s = 0, \ldots, k - 1$, and hence to $\text{Im } \alpha^k = 0$. If it occurs, then the change of variable $z \mapsto \zeta e_s$ in the equation $G(z) = \alpha$ produces the real equation $G(\zeta) = a e_{ps}$, which has solutions symmetric with respect to the
real line. Consequently, each solution to \( G(z) = \alpha \) has a pair: the reflected solution \( \bar{z}e^{-2\pi} \) with the same absolute value (unless they coincide). In the case of \( \bar{z} \notin \{\alpha e^{-2\pi m}\}_{m=0}^{k-1} \), which is equivalent to \( \text{Im} \alpha^k \neq 0 \), the relation (18) has no real solutions, and solutions to (16) and (17) have distinct absolute values, as is shown in Theorem 14. Accordingly, all solutions of \( G(z) = \alpha \) are distinct in absolute value.

We examine these cases in detail in Theorem 21 and Theorem 19, respectively.

**Definition.** Denote by \( \Xi \) the set of absolute values of all solutions to \( G(z) = \alpha \) with \( G \) of the form (15), that is

\[
\Xi := \{ \xi > 0 : \exists \theta \in (-\pi, \pi] \text{ such that } G(\xi e^{i\theta}) = \alpha \}.
\]

Let \( \cdots < \tilde{z}_i < \tilde{z}_{i+1} < \cdots \) be the entries of \( \Xi \), such that \( \Xi = \{\tilde{z}_i\}_i \), and let \( \ldots, z_i, z_{i+1}, \ldots \) be the corresponding \( \alpha \)-points or, more precisely, \( |z_i| = \tilde{z}_i \) and \( G(z_i) = \alpha \) for all \( i \) (that is, \( z_i \) stands for any of the \( \alpha \)-points which correspond to the value of \( \tilde{z}_i \)).

**Theorem 19.** If \( \text{Im} \alpha^k \neq 0 \), then for each \( i \) the \( \alpha \)-point \( z_i \) is simple, satisfies \( \text{Im} z_i^k \neq 0 \) and distinct in absolute value from other \( \alpha \)-points of \( G \) (i.e. \( G(z) = \alpha \) and \( |z| = |z_i| \implies z = z_i \)).

If \( z_i, z_{i+1} \) are two consecutive \( \alpha \)-points, then the inclusions \( \alpha \in Q_{2q-\kappa} \) and \( z_i \in Q_{2m-\sigma} \) with \( q, m \in \mathbb{Z} \) and \( \kappa, \sigma \in \{0, 1\} \) imply that \( z_{i+1} \in Q_{2\xi-1,\sigma} \), where \( l \) is an integer solution of \( p(l + m) \equiv 2q + 1 - \kappa - \sigma \pmod{k} \).

**Proof.** The expression (14) for real \( w \) yields that \( \text{Arg} R(w) \in \{0, \pi, \pm \frac{\pi}{2}, \pi\} \), and hence \( R(w) \notin \Omega \). Consequently, all solutions to (18) lie in the open upper half-plane \( \mathbb{C}_+ \). That is \( G(z) = \alpha \) and \( |z| = |z_i| \) for some \( i \), then \( z = z_i \).

Now let \( |z_i|, |z_{i+1}| \in \Xi \). Take integer numbers \( q, m, l = 0, \ldots, k-1 \) and \( \kappa, \sigma, \tau = 0, 1 \) so that \( \alpha \in Q_{2q-\kappa} \), \( z_i \in Q_{2m-\sigma} \), and \( z_{i+1} \in Q_{2\xi-\tau} \). Then the points \( z_i \in Q_{2m} \) and \( z_{i+1} \in Q_{2\xi} \) correspond to solutions of (16), while \( z_i \in Q_{2m-1} \) and \( z_{i+1} \in Q_{2\xi-1} \) correspond to solutions of (17).

First, assume that \( \text{Im} \alpha^k > 0 \), i.e. \( \kappa = 0 \) and \( \alpha \in Q_{2q} \). Then the points \( \alpha e^{-2\pi m} \in Q_{2q-2\pi m} \) of the set \( \Omega \) occur exactly once in each sector \( Q_j \) with the even indices \( j = 0, 2, \ldots, 2k-2 \) when \( m \) runs over the integers \( 0, \ldots, k-1 \). Analogously, the points \( \bar{\alpha}e^{-2\pi m} \in Q_{-2q-1+2\pi m} \) of the set \( \Omega \) occur exactly once in each sector \( Q_j \) with the odd indices \( j = 1, 3, \ldots, 2k-1 \) when \( m = 0, \ldots, k-1 \). Consequently, \( \sigma = 0 \) induces the equation

\[
R(w_i) = \alpha e^{-2\pi m} \in Q_{2q-2\pi m}, \quad \text{while } \sigma = 1 \text{ induces } R(w_i) = \bar{\alpha}e^{-2\pi m} \in Q_{-2q-1+2\pi m}.
\]

Combining these equalities together gives

\[
R(w_i) \in \Omega \cap Q_{(-1)^{\sigma}(2q+\sigma)-2m},
\]

the same reasoning for \( w_{i+1} \) provides us with the condition

\[
R(w_{i+1}) \in \Omega \cap Q_{(-1)^{\sigma}(2q+\tau)-2m}.
\]

Since \( R(w_{i+1}) = R(w_i)e^{i\theta} \) with an appropriate real \( \theta \), for all \( \theta \in (0, \pi) \) the quantity \( R(w_i)e^{i\theta} \) cannot belong to \( \Omega \) by Theorem 13. However, \( \Omega \) has exactly one point in each sector of the complex plane, so we necessarily have \( R(w_{i+1}) \in \Omega \cap Q_{(-1)^{\sigma}(2q+\sigma)-2m}+1 \) from (19). Thus, on account of the relation (20),

\[
(-1)^{\tau}(2q + \tau) - 2pl \equiv (-1)^{\sigma}(2q + \sigma) - 2p + 1 \pmod{2k}.
\]

Checking the parity immediately gives \( \tau = 1 - \sigma \). As a result,

\[
\sigma = 0 \quad \Rightarrow \quad (2q + 1) - 2pl \equiv -(2q - 2pm + 1) = 2q + 1 - 2(1 + 2q - pm) \pmod{2k}
\]

and

\[
\sigma = 1 \quad \Rightarrow \quad 2q - 2pl \equiv -(2q + 1 - 2pm) + 1 = 2q - 2(2q - pm) \pmod{2k}.
\]

These relations imply that \( 2pl \equiv 2((1 - \sigma) + 2q - pm) \pmod{2k} \) or equivalently \( p(l + m) \equiv 2q + 1 - \sigma \pmod{k} \).
Composition with $k$th power function

$$\frac{f(w)}{\sqrt[k]{w}g(w)} = \alpha + e_m$$

Legend:
- $\frac{f(w)}{\sqrt[k]{w}g(w)} = \alpha + e_m$
- $\frac{|f(w)|}{\sqrt[k]{|w|}g(w)} = |\alpha|$
- $\frac{|f(w)|}{\sqrt[k]{|w|}g(w)} = |\alpha|$
- $e_m := \exp \frac{2\pi mi}{3}$ and $m = 0, 1, 2$

Enlargement of the neighbourhood of the origin

Figure 1: Illustration to Theorems 19, 21, 22 and 23.

The $\alpha$-points of the function $F(z) = \frac{f(z^3)}{zg(z^3)} = \frac{(z^3 + 0.1)(z^3 + 1)(z^3 + 4)}{z(z^3 - 1)(z^3 - 5)}$, $\alpha = -1 - i$, are presented in the bottom graph. The plot of the corresponding intermediary function $R(w)$ is in the top graph. The $\alpha$-points of $F(z)$ coincide with zeros of the polynomial $z^9 + (1 + i)z^7 + 5.1z^6 - 6(1 + i)z^4 + 4.5z^3 + 5(1 + i)z + 0.4$. 

Legend:
- $f(z^3) = 0$
- $zg(z^3) = 0$
- $|f(z^3)| = |\alpha|$
- $\{z : ze_m \in \mathbb{R}\}$
- $\frac{f(w)}{\sqrt[k]{w}g(w)} = \alpha$
- $\{z : \exists \theta \in [0, 2\pi) F(z \exp i\theta) = \alpha\}$
Now let $\text{Im } \alpha^k < 0$ — that is to say, $\kappa = 1$ and $\alpha \in Q_{2q-1}$, so consequently $\alpha e^{-2pm} \in Q_{2q-1-2pm}$ and $\text{Re}e^{2pm} \in Q_{-2q+2pm}$. It implies that
\[
R(w_i) \in \Omega \cap Q(-1)^r(2q-(1-r)-2pm) \quad \text{and} \quad R(w_{i+1}) \in \Omega \cap Q(-1)^r(2q-(1-r)-2pl) = \Omega \cap Q(-1)^r(2q-(1-r)-2pm)+1
\]
analogously to the case of positive $\text{Im } \alpha^k$. Due to the parity, we have $\tau = 1 - \sigma$, and thus
\[
\sigma = 0 \implies -(2q-2pl) \equiv 2q - 1 - 2pm + 1 = 2q - 2pm \pmod{2k}
\]
and
\[
\sigma = 1 \implies 2q - 1 - 2pl \equiv -(2q - 2pm) + 1 = -2q + 2pm + 1 \pmod{2k}.
\]
The last two equations are equivalent to $2pl \equiv 4q - 2\sigma - 2pm \pmod{2k}$, which coincides with $p(l + m) \equiv 2q - \sigma \pmod{k}$.

Remark 20. The rays of the line $\{z \in \mathbb{C} : \text{Im } ze_s = 0\}$, which is given by $z = ze_{-2s}$, can be expressed via the sectors $Q_i$ of the complex plane by the formula
\[
\mathbb{Q}_{2m} \cap \mathbb{Q}_{-2s-2m-1} \setminus \{0\} = \begin{cases} \{z \in \mathbb{C} : ze_s > 0\}, \quad \text{if } m \equiv -\left\lceil \frac{s}{2} \right\rceil \pmod{k}, \\ \{z \in \mathbb{C} : ze_s < 0\}, \quad \text{if } m \equiv -\left\lfloor \frac{s-k}{2} \right\rfloor \pmod{k}, \\ \emptyset \quad \text{otherwise}; \end{cases}
\]
the notation $\lceil a \rceil$ stands for the minimal integer which is greater or equal to a real number $a$.

Theorem 21. Let $\text{Im } \alpha^k = 0$, $\alpha \neq 0$ and the integers $s,l$ be such that $\text{Im } \alpha e_{ps} = 0$ and $p(m - l) \equiv 1 \pmod{k}$.

(a) A point $z$ satisfies the conditions $G(z) = \alpha$ and $|z| = |z_i|$ if and only if $z \in \{z_i, z_i^*\}$, where $z_i^* := z_ie_{2s}$.

(b) The inclusion $z_i \in Q_{2m} \cup Q_{-2s-2m-1}$ for some integer $m$ implies that both $z_i^* \neq z_i$ are simple $\alpha$-points and $z_{i+1} \in \mathbb{Q}_{2l} \cup \mathbb{Q}_{-2s-2l-1}$ (when $|z_{i+1}| \in \mathbb{Z}$).

(c) The conditions $z_i^* = z_i$ and $\arg z_i = \arg z_{i+1}$ imply that both $z_i$, $z_{i+1}$ are simple, $\arg z_i \neq \arg z_{i-1}$ provided that $|z_{i-1}| \in \mathbb{Z}$ and $\arg z_{i+1} \neq \arg z_{i+2}$ provided that $|z_{i+2}| \in \mathbb{Z}$.

(d) If $z_i^* = z_i$ and $\arg z_i = \arg z_{i+1}$, then $z_i$ is simple or double (which corresponds to, respectively, $\arg z_i = \arg z_{i-1}$ or $\arg z_i \neq \arg z_{i-1}$ on condition that $|z_{i-1}| \in \mathbb{Z}$). Furthermore, $z_i \in \mathbb{Q}_{2m} \cap \mathbb{Q}_{-2s-2m-1}$ with $m$ given by (21) implies $z_{i+1} \in \mathbb{Q}_{2l} \cup \mathbb{Q}_{-2s-2l-1}$.

(e) If $z_i^* = z_i$ and $|z_{i+1}| \notin \mathbb{Z}$, then the multiplicity of $z_i$ is at most 2.

In other words, if $\text{Im } z_i e_s \neq 0$, then $z_i$ is simple, $\text{Im } z_i^* \neq 0$ and the reflected point $z_i^* = z_ie_{-2s}$ also solves $G(z) = \alpha$; no other $\alpha$-points share the same absolute value. Furthermore, $z_i \in Q_{2m}$ and $z_i^* \in Q_{-2s-2m-1}$ for some integer $m$ (probably after exchanging $z_i \leftrightarrow z_i^*$).

If $\text{Im } z_i e_s = 0$ i.e. $z_i \in \mathbb{Q}_{2m} \cap \mathbb{Q}_{-2s-2m-1}$ for some $m$ satisfying (21), then Theorem 21 asserts that $z_i$ is simple or double, and there are no other solutions of $G(z) = \alpha$ sharing the same absolute value. If $z_i$ is not the first or the last $\alpha$-point (with respect to the absolute value), then either $z_i$ is double or exactly one another $\alpha$-point adjacent to $z_i$ has the same argument (in fact, it belongs to the same interval between two consequent singularities of $\ln G$).

Proof. The equality $G(z_i) = \alpha$ is equivalent to $G(z_i e_{-2s}) = \alpha$ since
\[
G(z_i e_{-2s}) = \overline{G(z_i e_{2s})} = \overline{\alpha e_{2ps}} = \overline{\alpha e_{ps} e_{-ps}} = \alpha e_{ps} e_{-ps} = \alpha.
\]
Consequently, \( G(z_i) = \alpha \) if and only if \( G(z_i^*) = \alpha \), where \( z_i^* = z_i e^{-2\pi \alpha} \). The points \( z_i \) and \( z_i^* \) coincide exactly when \( z_i e^{\alpha} \) is a real number (cf. Remark 18).

Choose the integer \( m \) satisfying \( z_i \in \tilde{Q}_{2m} \cup \tilde{Q}_{-2m-2} \), which implies the same inclusion for \( z_i^* \). We constrain ourselves to the case \( z_i \in \tilde{Q}_{2m} \) and thus \( z_i^* \in \tilde{Q}_{-2m-2} \): this causes no loss of generality since \( z_i \) and \( z_i^* \) are interchangeable with each other. The closed sector \( \tilde{Q}_{2m} \) replaces \( \tilde{Q}_{2m} \) since it is possible that \( z_i = z_i^* \in \tilde{Q}_{2m} \cap \tilde{Q}_{-2m-2} \), satisfying the equality \( R(w_i) = a e^{-2\pi \alpha} \). Conversely, if \( R(w_i) = a e^{-2\pi \alpha} \) then \( z_i = \sqrt[2]{\alpha} e^{2\pi \alpha} \) and \( z_i^* = \sqrt[2]{\alpha} e^{-2\pi \alpha} \) are \( \alpha \)-points of \( G \).

Now, the function \( R(w) \) satisfies the conditions of Theorem 11. Therefore, solutions of \( R(w) \in \Omega \) in the closed upper half-plane \( \mathbb{C}_+ \) are distinct in absolute value; those in \( \mathbb{C}_+ \) are additionally simple, and those on the real line are simple or double. In particular, if \( R(w) \in \Omega \) and \( |w| = |w_i| \) then \( w = w_i \), which implies the assertion (a). Moreover, by Lemma \ref{lem:zeta} the multiplicity of \( z \) equals one in the assertion (b) and is at most two in the assertions (c)-(e) (and therefore (e) is proved).

Now let \( |z_{i+1}| \in \mathbb{E} \). Then, analogously to \( z_i \), the points \( z_{i+1} \) and \( z_{i+1} e^{-2\pi \alpha} \) are the only two solutions of the equation \( G(z) = \alpha \) which satisfy \( |z| = |z_{i+1}| \). Furthermore, we can assume that \( z_{i+1} \in \tilde{Q}_{2l} \) for some integer \( l \) without loss of generality. Then \( w_{i+1} := z_{i+1}^k \in \mathbb{C}_+ \) implies \( z_{i+1} = \sqrt[2]{\alpha} e^{2\pi \alpha} \), and consequently \( R(w_{i+1}) = a e^{-2\pi \alpha} \).

Observe that the points \( w_i, w_{i+1} \in \mathbb{C}_+ \) satisfy the conditions \( |w_i| < |w_{i+1}| \), \( R(w_i) = a e^{-2\pi \alpha} \) and \( R(w_{i+1}) = a e^{-2\pi \alpha} = a e^{-2\pi \alpha} + 2\pi \alpha \) for appropriate integers \( m, \delta \) and the quantity \( a e^{2\pi \alpha} \) cannot belong to \( \Omega \) for all \( \rho \in (0, \frac{2\pi}{\delta}) \). By Theorem \ref{thm:main}, this is possible only in two cases: if \( \delta = 1 \) or if simultaneously: \( \delta = 0 \), \( \text{Arg} w_i = \text{Arg} w_{i+1} \in (0, \pi) \) and \( |R(w_i)| \neq |z| \) provided that \( |w_i| < |w| < |w_{i+1}| \). In the former case, we necessarily obtain the equation \( -2\pi \delta = -2\pi - 2\delta \mod 2k \) with respect to the unknown \( l \), that is \( p(m-l) \equiv \delta = 1 \mod k \). This proves the assertion (b).

Lemma \ref{lem:zeta}, Lemma \ref{lem:increment} and Remark \ref{rem:increment} together yield that, on each subinterval of the real line containing no singularities of \( \ln R \), \( \text{arg} R(w) \) is constant and there are at most two solutions (counting with multiplicities) to \( |R(w)| = |\alpha \rangle \). Moreover, there are exactly one double or two simple solutions provided that both ends of the subinterval are positive poles or negative zeros of \( R \). Therefore, \( \text{arg} z_i = \text{arg} z_{i+1} \) (this condition corresponds to the case \( \delta = 0 \) above) implies that both \( z_i \) and \( z_{i+1} \) are simple.

Let \( \text{arg} z_i = \text{arg} z_{i+1} \). The assertion (c) will be proved if we show that \( \text{arg} z_i \neq \text{arg} z_{i-1} \) when \( |z_{i-1}| \in \mathbb{E} \) and that \( \text{arg} z_{i+1} \neq \text{arg} z_{i+2} \) when \( |z_{i+2}| \in \mathbb{E} \). We do it by contradiction: suppose that, for example, \( \text{arg} z_i = \text{arg} z_{i-1} \). Then \( R(w_{i-1}) = R(w_i) = a e^{-2\pi \alpha} \) with \( w_{i-1} := z_{i-1}^k \). On the one hand, the interval between \( w_{i-1} \) and \( w_i \) contains at least one singularity of \( \ln R \), and thus it cannot be a curve provided by (c) of Lemma \ref{lem:zeta}. Along that curve, \( \arg R(w) \) is continuous and must have a nonzero increment. Then the increment is at least \( 2\pi \) due to the condition \( R(w_{i+1}) = R(w_i) \). On the other hand, Theorem \ref{thm:main} then implies that there exists a point \( w_s \in \mathbb{C}_+ \) such that \( R(w_s) \in \Omega \) and \( |w_{i-1}| < |w_s| < |w_i| \) since \( \Omega \) contains at least two points. Therefore, \( z_s = \sqrt[2]{\alpha} e^{2\pi \alpha} \) with a proper choice of \( m \) satisfies the relations \( G(z_s) = \alpha \) and \( |z_{i-1}| < |z_s| < |z_i| \), which contradicts to the definitions of the points \( z_{i-1} \) and \( z_i \). As a consequence, \( \text{arg} z_i \neq \text{arg} z_{i+1} \). On condition that \( |z_{i+2}| \in \mathbb{E} \), the inequality \( \text{arg} z_{i+1} \neq \text{arg} z_{i+2} \) can be obtained analogously. Therefore, (c) is true. Furthermore, the similar proof gives us that if \( \text{arg} z_i \neq \text{arg} z_{i+1} \) (which corresponds to the case \( \delta = 1 \) above), \( |z_{i+1}| \in \mathbb{E} \) and \( z_i \) is a double \( \alpha \)-point, then \( \text{arg} z_i \neq \text{arg} z_{i+1} \).

The case when the \( \alpha \)-point \( z_i \) is real and simple, \( \text{arg} z_i \neq \text{arg} z_i \) and \( \text{arg} z_i \neq \text{arg} z_{i+1} \) is impossible. Indeed, let \( w_{i+1} \) be introduced in a way similar to \( w_i \) and \( w_{i+1} \). Suppose firstly that \( w_i > 0 \). Then we have \( |R(w_{i+1})| = |R(w_i)| \) according to the inequality (13). If \( \ln R \) is regular in the interval \( \mathcal{J} := ([w_i], |w_i|) \), then there exists a point \( w_s \) of this interval such that \( |R(w_s)| = |\alpha \rangle \) and, since \( w_s \) is simple, \( w_s \neq w_i \). At the same time, \( \text{arg} R \) does not change in \( \mathcal{J} \), which gives us the contradiction \( R(w_s) = R(w_i) = \alpha \). If \( \ln R \) has singularities in the interval \( \mathcal{J} \), then, instead of \( \mathcal{J} \), it is enough to

\footnote{More specifically, the difference \( \text{arg} R(w_i) - \text{arg} R(w_{i-1}) \) is positive for any fixed branch of \( \text{arg} R \), which is continuous on the curve.}
consider the maximal subinterval of $\Omega$ containing $w_k$, in which $R$ is regular. The case $w_i < 0$ is proved analogously with the help of the inequality $|R(-|w_i|)| > |R(w_i)|$, which is provided by (13). The assertion (d) holds, so the theorem is completely proved.

6 Location of the $\alpha$-point that is minimal or maximal in absolute value

Let a function $F$ have the form

$$F(z) := z^p e^{\alpha z},$$  \hspace{1cm} (22)

where $k$ and $p$ are integer such that $k \geq 2$ and $\gcd(|p|, k) = 1$, $0 < \omega_1, \omega_2 \leq +\infty$, $A \geq 0$ and $a_\nu, b_\mu > 0$ for all $\nu, \mu$. Such functions are the particular case of (15) and, therefore, satisfy conditions of Theorem 19 and Theorem 21. The next two theorems reveal another property of the $\alpha$-set of $F$. Assuming that the $\alpha$-set is nonempty, they answer which of the sectors contains the $\alpha$-point (or $\alpha$-points) of the function $F$ that is minimal in absolute value.

**Theorem 22.** Consider a complex number $\alpha \neq 0$ and a function $F$ of the form (22) with $p > 0$. Let $q = 0, \ldots, k - 1$ and $\kappa = 0, 1$ be chosen so that $\alpha \in \mathbb{Q}_{2q - \kappa}$, and the integer $m$ be such that $pm \equiv q \pmod{k}$.

If $\alpha^k \neq 0$, then the $\alpha$-point $z_*$ of $F(z)$ closest to the origin is simple and distinct in argument and absolute value from the succeeding $\alpha$-points. Moreover, $\alpha \in \mathbb{Q}_{2q - \kappa}$ implies $z_* \in \mathbb{Q}_{2m - \kappa}$. If $\alpha e^{-2q} > 0$, then $z_* e^{-2m} > 0$.

If $\alpha^k < 0$, that is $\alpha e^{-2q} > 0$, then the two zeros of $F(z) - \alpha$ closest to the origin (counting double zeros as two) are equal in absolute value or in argument. The latter case is possible only when $p = 1$; if it occurs, both zeros belong to the ray $\{ze^{-2m+1} > 0\}$. In the former case, one of them belongs to $\mathbb{Q}_{2m-1}$ and another belongs to $\mathbb{Q}_{2m} = \mathbb{Q}_{-2m-2s}$ where $m$ satisfies $pm \equiv q - 1 \pmod{k}$ and $s$ is introduced in Remark 18.

**Proof.** Let $z_0$ denote the solution of the equation $F(z) = \alpha$ that is minimal in absolute value. Consider the corresponding point $w_0 \in \mathbb{C}_+$ determined by $w_0 = z_0^k$ if $\text{Im} z_0^k \geq 0$ and by $w_0 = z_0^k$ if $\text{Im} z_0^k \leq 0$. Recall that (see Remark 17) the equality $F(z_0) = \alpha$ is equivalent to $R(w_0) \in \Omega$, where the function $R(w) = F(\sqrt[k]{w})$ is defined in $\mathbb{C}_+ \setminus \{0\}$ by the equality (14) and $\Omega = \{\alpha e^{2\pi i m/k} \setminus \{1\} \forall m = 0\}$.

The key moment here is to implement the assertion (a) of Lemma 6 with the setting $\psi = \ln R$. It implies that the point $w_*$ of set $\Gamma := \{w \in \mathbb{C}_+ : |R(w)| = |\alpha|\}$ which is closest to the origin satisfies $0 < w_* < b_1$, because

$$\mathbb{B} = \lim_{t \to +0} \phi(t b_1) = \lim_{t \to +0} t \psi'(t) = \frac{p}{k} > 0.$$  \hspace{1cm} (Moreover, $R(w)$ cannot be uniformly bounded in $\{w : w > 0\}$, so $w_*$ necessarily exists.) Note that $R(w)$ has the form (14), so $R(w_*) > 0$, that is $R(w_*) = |\alpha|$. Putting $z_* := \sqrt[k]{w_*} e^{-2m}$ we obtain $F(z_*) = |\alpha| e^{2pm}$. As suggested by the theorem’s statement, the integer $m$ satisfies $pm \equiv q \pmod{k}$. Consequently, if $\alpha e^{-2q} > |\alpha|$ then the point $z_0 := z_*$ satisfying the inequality $z_0 e^{-2m} > 0$ is the zero of $F(z) - \alpha$ we are looking for; it is simple by Lemma 9. \hspace{1cm} (The example is given in Figure 2a, $\alpha = e^{2\pi i/3}$.)

Suppose now that $\alpha \in \mathbb{Q}_{2q}$. Then the increment $\arg R(w_0) - \arg R(w_*) = \arg(\alpha e^{-2q})$ is positive and less than $\frac{\pi}{k}$. Theorem 13 implies that $R(w) \notin \Omega$ on condition that $|w_*| < |w| < |w_0|$, and moreover, $R(w_0) = \alpha e^{-2q} = \alpha e^{-2pm}$. Therefore, $z_0 := \sqrt[k]{w_0} e^{2m} \in \mathbb{Q}_{2m}$ is the required $\alpha$-point.
Analogously to the previous case,\( z_0 = \frac{\sqrt{|w_0|}e^{2\arg w_0}}{|w_0|} \) is the point of \( \Omega \) inducing the positive increment less than \( \frac{\pi}{6} \). By Theorem 13, \( R(w) \notin \Omega \) on condition that \( |w_0| < |w| < |w_0| \), and moreover, \( R(w_0) = \frac{\bar{\alpha}e_{\frac{2\pi}{3}}}{\bar{\alpha}e_{-\frac{2\pi}{3}}} = \bar{\alpha}e_{2\pi n} \). Consequently, \( z_0 = \frac{\sqrt{|w_0|}e^{2\arg w_0}}{|w_0|} \in Q_{2m-1} \) as stated in Remark 17 (for the illustration see Figure 2a with \( \alpha = e^{i\pi/2} \)). Combining two last cases gives the implication \( \alpha = Q_{2l-\pi} \Rightarrow z_0 = Q_{2m-\pi} \) and the simplicity of \( z_0 \) by Theorem 19.

If \( \alpha = 2q+1 > 0 \), then \( \bar{\alpha}e_{\frac{2\pi}{3}} = \bar{\alpha}e_{-\frac{2\pi}{3}} \), and therefore the equality \( z_0 = \frac{\sqrt{|w_0|}e^{2\arg w_0}}{|w_0|} \in Q_{2m-1} \) determines the \( \alpha \)-point with the smallest absolute value. The situation when \( z_0 = Q_{2m-1} \) appears in Figure 3, \( \alpha = i \), and Figure 2a, \( \alpha = e^{i\pi/3} \). Theorem 21 yields that there exists one another \( \alpha \)-point of \( F \) with the same absolute value, namely \( z_0^* := z_{0e^{2\pi}} \in Q_{-2m-2} \).

The case of \( z_0e^{2\pi} > 0 \), that is \( w_0 < 0 \), needs additional attention. The interval \((-a_1,0)\) contains one double (namely \( w_0 \)) or two simple (\( w_0 \) and \( w_1 \)) solutions to \( R(w_0) \in \Omega \) as provided by (b) of Lemma 6 with \( \psi = \ln R \). In addition, \( R(w) \notin \Omega \) on condition that \( |w_0| < |w| < |w_1| \), which is given by the inequality (9). These solutions determine the corresponding properties of the double \( \alpha \)-point \( z_0 \) or, respectively, of the simple pair \( z_0, z_1 \) with \( z_1e^{2\pi} > 0 \) (as it is shown in Figure 3 for \( \alpha = i \)).

In addition, the case when \( z_0e^{2\pi} > 0 \) is possible only if \( p = 1 \). Indeed, let \( w_0 < 0 \) and \( p \geq 2 \).
There exist a piecewise-smooth curve \( \gamma_1 \subset \mathbb{C}_+ \) connecting \( w \), and \( w_0 \), such that \( \psi \) is holomorphic in its neighbourhood (see Lemma 8). Consider the closed contour \( \gamma_2 = \gamma_1 \cup \{ w \in \mathbb{C} : \overline{w} \in \gamma_1 \} \) and the enclosed domain \( D \) with the boundary \( \gamma_2 \). The point \( w_0 \), lies between the origin and the minimal pole, therefore the domain \( D \) contains no poles of \( R \). Consequently, there are no poles of the function \( S(w) := \sqrt{w^p R(w)} \) in \( D \) as well. Since \( S(w) \) has the form (11) with \( B = A_0 = 0 \), it is holomorphic in \( D \) and its argument on \( \gamma_2 \) has the increment \( 2 \pi \) with some nonnegative integer \( l \) (we assume that \( \gamma_1 \) and \( \gamma_2 \) are directed anticlockwise).

The increment on \( \gamma_1 \) then must be \( l \pi \geq 0 \), because \( S(\overline{w}) = \overline{S(w)} \). At the same time, \( \arg \sqrt{w^p} = \arg \frac{R(w)}{S(w)} \) has the increment \( \frac{p}{k} \pi \) on \( \gamma_1 \), which implies \( \arg R(w_0) - \arg R(w_s) = \left( \frac{p}{k} + l \right) \pi \geq \frac{p}{k} \pi \). Therefore, the condition \( \arg R(w_0) - \arg R(w_s) = \frac{p}{k} \), which is required for \( w_0 < 0 \), fails to hold unless \( p = 1 \).

Observe that the change of variable \( z \mapsto \zeta e^{-1} \) implies \( z^k \mapsto -\zeta^k \). Hence, the function

\[
\tilde{F}(\zeta) := \frac{e^{-p}}{F(\zeta e^{-1})} = \zeta^{-p} e^{A_1 \zeta} \prod_{\mu=1}^{\infty} \left( 1 + \frac{z^k}{b_{\mu}} \right) / \left( 1 - \frac{\zeta^k}{a_{\nu}} \right)
\]

has the form (22) with a positive power of \( \zeta \) as the first factor provided that \( p < 0 \). Moreover,

\[
F(z) = \alpha \iff \tilde{F}(\tilde{\zeta}) = \frac{e^{-p}}{\alpha} =: \tilde{\alpha},
\]

\[
\alpha \in Q_{2q} \iff \tilde{\alpha} \in Q_{-2q+p-1} \quad \text{and} \quad \alpha e^{-2q+p} > 0 \iff \tilde{\alpha} e^{-2q+p} > 0.
\]

(23)

This way the case of \( p < 0 \) can be reduced to the situation studied in the previous theorem. Unfortunately, the notation convenient in Theorem 22 suits this case worse inducing more complicated relations.

**Theorem 23.** Suppose that all conditions of Theorem 22 hold excepting that \( p < 0 \).

If \( \alpha \in Q_{2q} \), then the \( \alpha \)-point \( z_0 \) of \( F(z) \) closest to the origin is simple and distinct in argument and absolute value from the succeeding \( \alpha \)-point. Furthermore, \( z_0 \in Q_{2m-\sigma} \) where \( \sigma := \kappa \) for even \( p \), \( \sigma := 1 - \kappa \) for odd \( p \), and the integer \( m \) satisfies\(^{13} \) \( pm \equiv q - (-1)^{\kappa} \left( \frac{p}{2} \right) \) (mod \( k \)).

If \( \alpha e^{-2q+p} > 0 \), where \( p \) and \( \kappa \) have the same parity, then the \( \alpha \)-point \( z_0 \) of \( F(z) \) closest to the origin is simple and distinct in argument and absolute value from the succeeding \( \alpha \)-point. Moreover, \( z_0 e^{-2m+1} > 0 \) for \( pm \equiv q + \left( \frac{p+1}{2} \right) \) (mod \( k \)).

If \( \alpha e^{-2q+p} > 0 \), where \( p \) and \( \kappa \) have distinct parity, then the two zeros of \( F(z) - \alpha \) closest to the origin (counting double zeros as two) are equal in absolute value or in argument. In the latter case, which it possible only when \( p = -1 \), both the zeros belong to the ray \( \{ ze^{-2m} > 0 \} \). In the former case, one of them belongs to \( Q_{2m} \) and another belongs to \( Q_{-2m-1} \). Here \( m \) solves \( pm \equiv q - \left( \frac{p+1}{2} \right) \) (mod \( k \)) and \( s \) is as in Remark 18.

**Proof.** With the notation

\[
\tilde{\alpha} := -q + \left[ \frac{\kappa - p - 1}{2} \right] \quad \text{and} \quad \tilde{\kappa} := 2\tilde{\alpha} + 2\kappa - \kappa + p + 1,
\]

the relations (23) immediately yield

\[
\alpha \in Q_{2q} \iff \tilde{\alpha} \in Q_{2\tilde{\alpha} - \tilde{\kappa}} \iff \tilde{\kappa} \in Q_{2\tilde{\kappa} - \tilde{\kappa}} \iff z_0 \in Q_{2\tilde{m} - \tilde{\kappa} - 1},
\]

where \( \tilde{m} \) satisfies \( \tilde{m} \equiv \tilde{q} \) (mod \( k \)) and \( \tilde{z}_0 \) is the solution to \( \tilde{F}(\tilde{\zeta}) = \tilde{\alpha} \) minimal in absolute value. That is, modulo \( k \) we have

\[
p\tilde{m} \equiv q - \left[ \frac{\kappa - p - 1}{2} \right] = \begin{cases} q + \frac{p}{2}, & \text{if } p \text{ is even} \\ q + \frac{p+1}{2} - \kappa, & \text{if } p \text{ is odd} \end{cases}
\]

\( ^{13} \)Recall that \( \left( \frac{p}{2} \right) \) stands for the minimal integer greater than or equal to \( \frac{p}{2} \). Here \( \left\lfloor \frac{p}{2} \right\rfloor \leq \frac{p}{2} \) since \( p < 0 \).
Let \( m \) denote such an integer that \( z_0 \in Q_{2m - \sigma} \) for some \( \sigma \in \{0, 1\} \). Then necessarily \( 2m - \sigma \equiv 2\bar{m} - \bar{z} - 1 \mod{2k} \), which is satisfied by \( m = \bar{m} - \bar{z} \) and \( \sigma = 1 - \bar{z} \). At that, the second of the expressions (24) yields \( \bar{z} = 1 - \pi \) if \( p \) is even and \( \bar{z} = \pi \) if \( p \) is odd. The relation (25) within these settings becomes
\[
\begin{align*}
\pm & \equiv \begin{cases} 
q + \frac{p}{2} - \bar{z}, & \text{if } \pi \text{ is even;} \\
q + \frac{p+1}{2} - \bar{z}(p + 1), & \text{if } \pi \text{ is odd}
\end{cases} \\
& \begin{cases} 
q + \left(-1\right)^{p} \frac{p}{2}, & \text{if } \pi \text{ is even;} \\
q + \left(-1\right)^{p} \frac{p+1}{2}, & \text{if } \pi \text{ is odd}
\end{cases}
\end{align*}
\]
modulo \( \bar{k} \). Since \( p < 0 \), the last equality implies
\[
\pm \equiv q + \left(-1\right)^{\pi} \left\lfloor \frac{p}{2} \right\rfloor = q - \left(-1\right)^{\pi} \left\lfloor \frac{p}{2} \right\rfloor \quad \text{(mod } \bar{k}).
\]
However, this relation coincides with the relation for \( \pi \) suggested by the theorem’s statement. For the corresponding illustration see Figure 2b, \( \alpha = e^{i\pi/2} \).

When \( \tilde{\alpha} \) satisfies \( \tilde{\alpha} e^{-2\pi} \equiv 0 \), from (25) we have the relation \( -2\bar{q} \equiv 2q - \pi + \pi \mod{2k} \) instead of (24), which determines the pair \( q, \pi \) satisfying the inequality \( \alpha e^{-2q + \pi} > 0 \). So, \( z_0 e^{-2\bar{m} - 1} > 0 \) for
\[
\pm = -\left(-p\right) \cdot m \equiv -\bar{\pi} \equiv q + \frac{p - \pi}{2} = q + \left\lfloor \frac{p - 1}{2} \right\rfloor \quad \text{(mod } \bar{k}).
\]
on account of the equal parity of \( p \) and \( \pi \). The corresponding plot can be found in Figure 2b, \( \alpha = e^{i\pi/3} \).

When \( \tilde{\alpha} e^{-2\pi+1} > 0 \), the relation \( -2\bar{q} + 1 \equiv 2q - \pi + \pi \mod{2k} \) provides another pair \( q, \pi \) making the inequality \( \alpha e^{-2q + \pi} > 0 \) true. This gives us that \( z_0 \in Q_{2\bar{m} - 2} \) or \( z_0 e^{-2\bar{m} + 1} > 0 \) (the latter is possible only for \( p = -1 \)) whenever
\[
\pm \equiv q + \frac{p - 1 - \pi}{2} \quad \text{(mod } \bar{k}).
\]
The change \( m := \bar{m} - 1 \) gives \( z_0 \in Q_{2m} \) or \( z_0 e^{-2m} > 0 \) whenever
\[
\pm \equiv q - p + \frac{p - 1 - \pi}{2} = q - p + 1 + \pi \quad \text{(mod } \bar{k}).
\]
For \( z_0 \in Q_{2m} \), the integer \( s \) defined as in Remark 18 provides the expression \( z_0 e^{-2s} \) for the \( \alpha \)-point of \( F(z) \) which is equidistant with \( z_0 \) from the origin. See the relevant example in Figure 2b, \( \alpha = e^{2i\pi/3} \).

**Remark 24.** Note that the last two theorems can be applied to the function \( H(1/z) \) when it has the form (22). This way one obtains a straightforward conclusion concerning the most distant from 0 solution of the equation \( H(z) = \alpha \). It is of special interest for rational \( H(z) \): then both \( H(z) \) and \( H(1/z) \) can be represented as in (22).

### 7 Zeros of entire functions

Let the natural numbers \( j \) and \( k \) be coprime and \( k \geq 2 \). Theorems 19, 21–23 admit a transition to describing the zeros of functions of the forms
\[
H_1(z) := f(z^k) + z^j g(z^k) \quad \text{and} \quad H_2(z) := g(z^k) + z^j f(z^k),
\]
where the functions \( f(z) \) and \( g(-z) \) are entire, have genus 0 and only negative zeros. At least one of the functions \( f \) and \( g \) needs to be not a constant to exclude the trivial case. Furthermore, both functions \( f(z^k)/f(0) \) and \( g(z^k)/g(0) \) must be real. They have no common zeros, therefore \( f(z^k) \neq 0 \), \( g(z^k) \neq 0 \) and \( z \neq 0 \) when \( H_1(z) = 0 \) or \( H_2(z) = 0 \).

To adapt the facts stated in Sections 5 and 6 for studying zeros of the functions \( H_1 \) and \( H_2 \), put
\[
F_1(z) := z^{-j} \frac{f(z^k)/f(0)}{g(z^k)/g(0)}, \quad F_2(z) := z^j \frac{f(z^k)/f(0)}{g(z^k)/g(0)} \quad \text{and} \quad \alpha := -\frac{g(0)}{f(0)}.
\]
Then the following identities hold:

\[ H_1(z) = \left(1 - \frac{F_1(z)}{\alpha}\right) z^j g(z^k) \quad \text{and} \quad H_2(z) = \left(1 - \frac{F_2(z)}{\alpha}\right) g(z^k). \]  

Recall that \( z^j g(z^k) \) and \( H_1(z) \) have no common zeros for \( i = 1, 2 \). Therefore, the equalities (27) imply that \( F_1(z) = \alpha \iff H_1(z) = 0 \) and \( F_2(z) = \alpha \iff H_2(z) = 0 \). That is, the zero set of \( H_i(z) \) coincides with the \( \alpha \)-set of \( F_i(z) \) for \( i = 1, 2 \). Moreover, (27) give that each \( \alpha \)-point \( z_\ast \) of the function \( F_i(z) \) is the zero of \( H_i(z) \) with the same multiplicity.

Since the functions \( F_i(z) \) have the form (22), the zeros of \( H_i(z) \) for \( i = 1, 2 \) are located as it is asserted about \( \alpha \)-points of \( F_i(z) \) by Theorems 19, 21–23 with \( \alpha = -\frac{g(0)}{f(0)} \) and \( p = (−1)^j \cdot j \).

**Remark 25.** Some extensions of the fact proposed in the current section are possible. Here we give two examples. However, it is unclear whether studying such functions is well-motivated.

1. Assume that \( f(z) \) and \( g(z) \) are functions regular and nonzero at the origin, and that \( \frac{f(z)}{g(z)} \) does not coincide with \( z^n \) up to a constant (to suppress the trivial case). From the comparison of the formulae (26) with (15) and (22) it is seen that \( f(z) / f(0) \) and \( g(−z) / g(0) \) can be allowed to have the form

\[ e^{Az} \cdot \prod_{\nu > 0} \left(1 + \frac{z}{a_\nu}\right) / \prod_{\mu > 0} \left(1 - \frac{z}{b_\mu}\right), \quad \text{where} \quad A \geq 0 \quad \text{and} \quad a_\nu, b_\mu > 0 \quad \text{for all} \quad \mu, \nu. \]  

Put in other words, if \( f(0), g(0) \in C \setminus \{0\} \) then \( f(z) / f(0) \) and \( g(−z) / g(0) \) can generate any **totally positive sequences** which start with 1. Indeed, after multiplying \( H_i \) by the denominators of \( f(z^k) \) and \( g(z^k) \) we obtain the entire function \( \overline{H}_i \) with the same zeros as \( H_i \). Then it is enough to note that the exponential factor originating from those of \( f(z^k) \) and \( g(z^k) \) is allowed in the representation (22). So, the result of the current section extends to such functions without any changes.

2. Let \( f(z) \) and \( g(−z) \) be nontrivial functions generating **doubly infinite totally positive sequences** up to a complex constant factor, i.e. of the form (1) with a complex \( C \neq 0 \). In addition let \( f(z) \neq \text{const} \cdot z^n g(−z) \).

With the help of analogous manipulations we still can obtain a transition of Theorems 19 and 21. After taking out the factors \( z^n \) of (1), it is enough additionally to factor some power of \( z \) out of \( H_i \) (this cannot change the zero set excepting the origin).

**Remark 26.** Allowing \( f(z) \) and \( g(−z) \) to be arbitrary functions of the forms (28) or (1) with \( C \in C \setminus \{0\} \) can be useful in the following sense. Consider the power series

\[ f(z) = \sum_{n=-\infty}^{\infty} f_n z^n \quad \text{and} \quad g(z) = \sum_{n=-\infty}^{\infty} g_n z^n \]

such that \( f_n \neq f_0^{1−n} f_1^n \) and \( g_m \neq g_0^{1−m} g_1^m \) for some \( n, m \in \mathbb{Z} \). Then (see the discussion on Page 2) the series converge and the functions \( f(z) / f_0 \) and \( g(−z) / g_0 \) generate **totally positive sequences** (possibly doubly infinite) if and only if the Toeplitz matrices \( (f_{n−i} / f_0)_{i,n=-\infty}^{\infty} \) and \( ((−1)^{n−i} g_{n−i} / g_0)_{i,n=-\infty}^{\infty} \) have all their minors nonnegative [1, 9]. However, then

\[ H_1(z) = \sum_{n=-\infty}^{\infty} \left(f_n + z^j g_n\right) z^{kn} \quad \text{and} \quad H_2(z) = \sum_{n=-\infty}^{\infty} \left(g_n + z^j f_n\right) z^{kn} \]

are the Laurent series. This gives us the conditions in terms of the Laurent coefficients of \( H_1(z) \) and \( H_2(z) \) which provide that the zeros of \( H_1(z) \) and \( H_2(z) \) are localized according to Theorems 19 and 21 (and to Theorems 22–23 when the series \( f(z), g(z) \) are not doubly infinite, that the limiting functions are are meromorphic).
8 Conclusions for the case $k = 2$

Note that in the particular case of $p = \pm 1$ the relations modulo $k$ from Theorems 19, 21–23 have obvious solutions. The setting $k = 2$ (implying that $p$ is odd) also provides us with simple (and very useful) solutions. Let us restate the facts of the previous section for this particular situation.

Denote $p = 2j + 1$. The congruence modulo $k$ (a linear Diophantine equation) from Theorem 19 becomes $l \equiv 1 + x + \sigma + m \pmod{2}$. If $\alpha \in \mathbb{R}$ (or $i \alpha \in \mathbb{R}$), then the constant $s$ in Theorems 21–23 equals 0 (or 1, respectively). The congruence from Theorem 21 turns into $l \equiv 1 + m \pmod{2}$. The equation from Theorem 22 becomes $m = q \pmod{2}$, and those from Theorem 23 become

$$m \equiv \begin{cases} q + j + 1 \pmod{2}, & \text{if } \alpha \in Q_{2q-x} \text{ or } \alpha e^{-2q} > 0; \\ q + j \pmod{2}, & \text{if } \alpha e^{-2q} > 0. \end{cases}$$

Let $N = (z_k)_{k=1}^\omega$ be the set of all $\alpha$-points of $F(z)$, where $|z_k-1| \leq |z_k|$ for all $k$ and each $\alpha$-point counts so many times which multiplicity it has. Then we have the following two theorems as a summary.

**Theorem 27 (cf. [8]).** Let a function $F(z)$ have the form (22), $p = 2j + 1$, $j < 0$ and $k = 2$. then the $\alpha$-points $N = (z_k)_{k=1}^\omega$ of $F(z)$ are distributed as follows:

(i) If $\text{Im} \alpha^2 \neq 0$, then all zeros are simple and satisfying $0 < |z_1| < |z_2| < \ldots$; $\text{Im} z_n^2 \neq 0$ for every natural $n$ and $z_n \in Q_1$ implies $z_{n+1} \in Q_{\text{sign}(|\text{Im} \alpha^2|)}$. Moreover, $(-1)^j \text{Im} \alpha \text{Im} z_1 > 0$ and $\text{Im} \alpha^2 \text{Re} z_1 \text{Im} z_1 < 0$.

(ii) If $\text{Im} \alpha = 0$, then $0 < |z_1| \leq |z_2| \leq |z_3| \leq |z_4| \leq |z_5| \ldots$, there are no purely imaginary zeros, real zeros can be simple or double, other zeros are simple. Moreover, for each natural $n$ the following five conditions hold

$$|z_{2n-1}| = |z_{2n}| \implies z_{2n-1} = \bar{z}_{2n}, \quad |z_{2n-1}| < |z_{2n}| \implies \text{Arg} z_{2n-1} = \text{Arg} z_{2n} \in \{0, \pi\}, \quad \text{Re} z_{2n} \text{Re} z_{2n+1} < 0, \quad (-1)^j \text{Re} z_1 < 0 \quad \text{and} \quad |z_1| < |z_2| \implies j = -1.$$

(iii) If $\text{Re} \alpha = 0$, then $0 < |z_1| < |z_2| < |z_3| < |z_4| < |z_5| \ldots$, there are no real zeros, purely imaginary zeros can be simple or double, other zeros are simple. Moreover, for each natural $n$ the following five conditions hold

$$|z_{2n}| = |z_{2n+1}| \implies z_{2n} = \bar{z}_{2n+1}, \quad |z_{2n}| < |z_{2n+1}| \implies \text{Arg} z_{2n} = \text{Arg} z_{2n+1} \in \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}, \quad \text{Im} z_{2n-1} \text{Im} z_{2n} < 0, \quad (-1)^j \text{Im} \alpha \text{Im} z_1 > 0 \quad \text{and} \quad \text{Re} z_1 = 0.$$

**Theorem 28.** Let a function $F(z)$ have the form (22), $p = 2j + 1$, $j \geq 0$ and $k = 2$. Then the $\alpha$-points $N = (z_k)_{k=1}^\omega$ of $F(z)$ are distributed as follows:

(iv) If $\text{Im} \alpha^2 \neq 0$, then all zeros are simple and satisfying $0 < |z_1| < |z_2| < \ldots$; $\text{Im} z_n^2 \neq 0$ for every natural $n$, $z_n \in Q_1$ implies $z_{n+1} \in Q_{\text{sign}(|\text{Im} \alpha^2|)}$. Moreover, $\text{Im} \alpha \text{Im} z_1 > 0$ and $\text{Re} \alpha \text{Re} z_1 > 0$.

(v) If $\text{Im} \alpha = 0$, then $0 < |z_1| < |z_2| < |z_3| < |z_4| < |z_5| \ldots$, there are no purely imaginary zeros, real zeros can be simple or double, other zeros are simple. Moreover, for each natural $n$ the following five conditions hold

$$|z_{2n}| = |z_{2n+1}| \implies z_{2n} = \bar{z}_{2n+1}, \quad |z_{2n}| < |z_{2n+1}| \implies \text{Arg} z_{2n} = \text{Arg} z_{2n+1} \in \{0, \pi\}, \quad \text{Re} z_{2n-1} \text{Re} z_{2n} < 0, \quad \text{Re} z_1 = 0 \quad \text{and} \quad \alpha z_1 > 0.$$
Consequently, the following fact is true. At that, the interlacing property of distributed as stated in Theorem 27 has only positive zeros and the genus equal to 

Denote a strongly stable functions like polynomials, which are close to them in some sense (for example, the closed right half of the complex plane for all complex numbers). Similarly, zeros of 

Since an analytic function 

Let us turn to zeros of entire functions by applying the idea of Section 7. An entire function 

Remark 29. Two last theorems have analogous statements if, instead of 

Remark 30. Note that in all cases (i)–(vi) the 

Let us turn to zeros of entire functions by applying the idea of Section 7. An entire function 

Since 

A real entire function 

A real entire function 

Remark 31. Chebotarev’s Theorem (see [5] or e.g. [6, 19]) implies that if 

Observe that if a complex number 

Consequently, the following fact is true.
Corollary 32. Consider the functions
\[ h(z) = \sum_{k=0}^{\infty} i^{\frac{k-1}{2}} f_k z^k \quad \text{and} \quad \bar{h}(z) = \sum_{k=0}^{\infty} i^{-\frac{k-1}{2}} f_k z^k, \]
where \( f(z) = \sum_{n=0}^{\infty} f_{2n} z^n \) and \( g(z) = \sum_{n=0}^{\infty} f_{2n+1} z^n \) are entire functions of genus 0 and have only negative zeros. For \( \mu = \sqrt{i} \), zeros of the function \( h(\bar{\mu} z) \) distributed as claimed in Theorem 27 for \( \alpha = \bar{\mu} f_1 / f_0 \) and the entire function \( \bar{h}(\mu z) \) distributed as claimed in Theorem 27 for \( \alpha = \mu f_1 / f_0 \).

9 Two problems by A. Sokal

"Disturbed exponential" function. Alan Sokal in his talk at Institut Henri Poincaré (see [20]) put forward the hypothesis that

**Conjecture A.** The entire function
\[ F(z; q) = \sum_{n=0}^{\infty} q^\frac{n(n-1)}{2} z^n, \]  
where \( q \) is a complex number, \( 0 < |q| < 1 \), can have only simple zeros.

The stronger version of the conjecture claims that

**Conjecture B.** The function \( F(z; q) \) for \( q \in \mathbb{C}, \ 0 < |q| < 1 \), can have only simple zeros with distinct absolute values.

The following facts on \( F(z; q) \) are known. The function \( F \) is the unique solution to the following Cauchy problem
\[ F'(z) = F(qz), \quad F(0) = 1, \]  
which can be checked by substitution. Moreover, when \( |q| = 1 \) this function has the exponential type 1, for \( q \) lower in absolute value the function \( F \) is of zero genus.

The case of positive \( q \) was studied extensively. It is known that the zeros of \( F \) are negative (see [17]), simple and satisfy Conjecture B as well as certain further conditions [16, 15]. Conjecture B holds true for negative \( q \) as well, see e.g. [8]. The properties of \( F(z; q) \) for complex \( q \) were studied in [2, 23, 10]. According to [20], Conjecture B is true if \( |q| < 1 \) and the zeros of \( F(z; q) \) are big enough in absolute value (A. Eremenko) as well as for small \( |q| \).

Let us prove that Conjecture B also holds true for purely imaginary values of the parameter. As we pointed out, for positive \( q \leq 1 \) the function \( F(z; q) = f(z^2) + z^2 g(z^2) \) has only negative zeros. In particular, it is stable. The Hermite-Biehler theory (e.g. [6, 19]) implies that the zeros of \( f(z) \) and \( g(z) \) are negative and interlacing. Therefore, by Corollary 32 the zeros of \( F(z; \pm i q) \) with \( 0 < q \leq 1 \) are simple and their absolute values are distinct.

The family of polynomials
\[ P_N(z; q) = \sum_{n=0}^{N} \binom{N}{n} z^n q^{\frac{n(n-1)}{2}} \]  
is connected tightly to the function \( F(z; q) \); it approximates this function in the sense that
\[ P_N(\pm i q N^{-1}; q) \xrightarrow{N\to\infty} F(z; q). \]  
The polynomial version of this conjecture has the following form.
Conjecture C. For all $N > 0$ the polynomial $P_N(z; q)$ where $|q| < 1$, can have only simple roots, separated in absolute value by at least the factor $|q|$.

The original statement (which is equivalent to the given here) is concerned with the family of polynomials

$$\{ P_N(zw^{N-1}; w^{-2}) \}_{N \in \mathbb{N}} ,$$

where $w^{-2} = q$. Observe that $C \implies B \implies A$.

The approach for $F(z; q)$ extends to the polynomials $P_N(z; q)$ without changes. Their zeros are negative for positive $q$ provided that the polynomials coincide with the action of the multiplier sequence\textsuperscript{14} ($q^{n(n-1)/2})_{n=0}^\infty$ on the polynomial $(z + 1)^N$. This justifies the assertion of Conjecture C for purely imaginary $q$ without bounds on the ratio of subsequent (by the absolute value) roots.

Partial theta function. An analogous problem by A. Sokal appears in [12]. The partial theta function

$$\Theta_0(z; q) = \sum_{n=0}^{\infty} q^{n(n-1)/2} z^n$$

has only negative zeros whenever $0 < q < \tilde{q} \approx 0.3092493386$, which is shown in [12] (see also [11]). Splitting it into the even part $f(z^2)$ and the odd part $zg(z^2)$ gives

$$\Theta_0(z; q) = f(z^2) + zg(z^2),$$

$$f(z) = \sum_{n=0}^{\infty} q^{n(2n-1)} z^n = \sum_{n=0}^{\infty} q^{n(2n-2)} (qz)^n = \sum_{n=0}^{\infty} (q^4)^{n(n-1)/2} (qz)^n = \Theta_0(qz; q^4) \text{ and}$$

$$g(z) = \sum_{n=0}^{\infty} q^{n(2n+1)} z^n = \sum_{n=0}^{\infty} q^{n(2n-2)} (q^3z)^n = \Theta_0(q^3z; q^4).$$

Thus, both $f(z)$ and $g(z)$ have only negative zeros whenever $0 < q^4 \leq \tilde{q}$. Therefore, by Corollary 32 all zeros of $\Theta_0(z; iq)$ are simple and distinct in absolute value if $0 < q^4 \leq \tilde{q}$, that is if $0 < q \leq q_* \approx 0.7457224107$. This is a partial confirmation of the following assertion.

Conjecture D (see [12, p. 832]). *Is it true that all zeros of $\Theta_0(z; q)$ remain simple within the open disk $|q| < \tilde{q}$?*

With the help of exactly the same manipulations we could deduce that, for example, the (Jacobi) theta function

$$\Theta(z; iq) = \sum_{n=-\infty}^{\infty} (iq)^{n(n-1)} z^n, \quad 0 < q < 1,$$

also has its zeros simple, distinct in absolute value and residing in the quadrants of the complex plane rotated by $\pi/4$ (according to the Remark 29). However, this is redundant (although yet instructive) because the exact information is provided by the Jacobi triple product formula (see e.g. [14, Theorem 352]) which is valid for any complex $z \neq 0$ and $|q| < 1$:

$$\sum_{n=-\infty}^{\infty} q^{n(n-1)} z^n = \prod_{j=1}^{\infty} (1 - q^j)(1 + zq^{-j}) \left(1 + \frac{q^j}{z} \right).$$

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\textsuperscript{14}See the definition and properties e.g. in the book [18]. The sequence $(q^{n(n-1)/2})_{n=0}^\infty$ is formed by values of the function $e^{\frac{1}{2} \ln |q| z(z-1)}$ at integer points; since $\ln |q| < 0$, it is a multiplier sequence according to Satz 10.1 from [18, p. 42].
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