Remarks on regularity conditions of the Navier-Stokes equations

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Abstract

Let \( v \) and \( \omega \) be the velocity and the vorticity of the a suitable weak solution of the 3D Navier-Stokes equations in a space-time domain containing \( z_0 = (x_0, t_0) \), and \( Q_{z_0, r} = B_{x_0, r} \times (t_0 - r^2, t_0) \) be a parabolic cylinder in the domain. We show that if \( v \times \omega \| \omega \| \in L^{\gamma, \alpha}_{x,t}(Q_{z_0, r}) \) or \( \omega \times \frac{v}{|v|} \in L^{\gamma, \alpha}_{x,t}(Q_{z_0, r}) \), where \( L^{\gamma, \alpha}_{x,t} \) denotes the Serrin type of class, then \( z_0 \) is a regular point for \( v \). This refines previous local regularity criteria for the suitable weak solutions.

1 Introduction

The Navier-Stokes equations in a domain \( \Omega \in \mathbb{R}^3 \) are the following.

\[
\begin{cases}
\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p + \Delta v, & (x, t) \in \Omega \times (0, T) \\
\text{div} v = 0, & (x, t) \in \Omega \times (0, T) \\
v(x, 0) = v_0(x), & x \in \Omega
\end{cases}
\]

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where $v = (v_1, v_2, v_3)$, $v_j = v_j(x, t)$, $j = 1, 2, 3$, is the velocity of the flow, $p = p(x, t)$ is the scalar pressure, and $v_0$ is the given initial velocity satisfying $\text{div} \ v_0 = 0$. The global in time existence of a smooth solution to the system (NS) is an outstanding open problem in mathematics, and is chosen as one of the seven millennium problems by Clay Institute. One traditional approach to the problem is to prove global in time existence of weak solutions, and the prove their regularity. A notion of weak solution of (NS) was introduced, and its global in time existence in $\mathbb{R}^3$ was proved by Leray in [16]. Later, Hopf proved existence of weak solution in a bounded domain in [14]. After that there are numerous conditional regularity results on the weak solutions, imposing integrability conditions on the velocity or the vorticity, which guarantees regularity of the weak solutions (see e.g. [23, 19, 21, 15, 11, 24, 10, 18, 5, 6, 7]). For the local analysis of the regularity properties of weak solutions Caffarelli-Kohn-Nirenberg introduced the notion of suitable weak solutions and proved its partial regularity as well as global in time existence (14). A refined definition of suitable weak solutions, using a stronger condition for pressure, which we adopt here, was introduced by Lin in [17]. Let $Q_T = \Omega \times (0, T)$. For a point $z = (x, t) \in Q_T$, we denote below

$$B_{x,r} = \{ y \in \mathbb{R}^3 : |y - x| < r \}, \quad Q_{z,r} = B_{x,r} \times (t - r^2, t).$$

**Definition 1** A pair $(v, p)$ of measurable functions is a suitable weak solution of (NS) if the following conditions are satisfied:

(i) $v \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega))$, $\quad p \in L^\frac{3}{2}(Q_T)$.

(ii) The following integral identity holds

$$\int_{Q_T} [-v \cdot \partial_t \varphi + (v \cdot \nabla)v \cdot \varphi + \nabla v : \nabla \varphi] \, dx \, dt = \int_\Omega v_0 \cdot \varphi(x, 0) \, dx$$

for all vector test functions $\varphi \in [C_0^\infty(\Omega \times [0, T])]^3$.

(iii) The pair $(v, p)$ satisfies the local energy inequality,

$$\int_\Omega |v(x, t)|^2 \phi(x, t) \, dx + 2 \int_0^t \int_\Omega |\nabla v(x, \tau)|^2 \phi(x, \tau) \, dx \, d\tau$$

$$\int_\Omega \partial_t \phi(x, t) \, dx \leq 0$$

for all vector test functions $\phi \in [C_0^\infty(\Omega \times [0, T])]^3$. 

\[2\]
\[ \int_0^t \int_\Omega (|v|^2 (\partial_t \phi + \Delta \phi) + (|v|^2 + 2p) v \cdot \nabla \phi) \, dx \, dt \]

for almost all \( t \in (0, T) \) and for all nonnegative scalar test function \( \phi \in C^\infty_0(Q_T) \).

We say that a weak solution \( v \) is regular at \( z \), if \( v \) is bounded in \( Q_{z,r} \) for some \( r > 0 \). Such point \( z \) is called a regular point. A point in \( Q_T \), which is not regular, is called a singular point. Caffarelli-Kohn-Nirenberg showed that the one dimensional Hausdorff measure of the set \( S \) of possible interior singular points of suitable weak solutions is zero ([4]), which refines the previous results due to Scheffer ([22]).

In this paper our aim is to obtain refined versions of regularity conditions for velocity and vorticity for suitable weak solutions, incorporating the directions of each vector field as well as the magnitudes. Our conditions are not directly on the velocity or vorticity, but on the orthogonal component of velocity to vorticity direction, or on the orthogonal component of vorticity to velocity direction. The associated integral norms are scaling invariant. More precisely our main theorem is the following.

**Theorem 1.1** Let \( z_0 = (x_0, t_0) \in Q_T \) with \( Q_{z_0,r} \subset Q_T \), and \( (v, p) \) be a suitable weak solution of (NS) in \( Q_T \) with the vorticity \( \omega = \text{curl} \, v \), where the derivatives are in the sense of distribution. Suppose \( v \) and \( \omega \) satisfy one of the following conditions:

(i) The following inequality holds true,

\[
\left\| v \times \frac{\omega}{|\omega|} \right\|_{L_{x,t}^{3,\infty}(Q_{z_0,r})} \leq \varepsilon_0
\]

for an absolute constant \( \varepsilon_0 > 0 \) to be determined in the proof below.

(ii) There exist \( \gamma \in (3, \infty] \), \( \alpha \in [2, \infty) \) with \( 3/\gamma + 2/\alpha \leq 1 \) such that

\[
v \times \frac{\omega}{|\omega|} \in L_{x,t}^{\gamma,\alpha}(Q_{z_0,r}).
\]

(iii) There exist \( \gamma \in [2, 3] \), \( \alpha \in [2, 4] \) with \( 3/\gamma + 2/\alpha \leq 2 \) such that

\[
\omega \times \frac{v}{|v|} \in L_{x,t}^{\gamma,\alpha}(Q_{z_0,r}).
\]

Then, \( z_0 \) is a regular point.
Remark 1.1 In a recent preprint J. Wolf ([27]) proved an $\varepsilon$-regularity type of theorem related to (iii) of the above theorem as follows: There exists $\varepsilon_0^*$ such that if a suitable weak solution in $Q_{z_0,r}$ satisfies

$$\limsup_{\rho \to 0^+} \frac{1}{\rho} \int_{Q_{z_0,\rho}} \left| \frac{\omega \times v}{|v|} \right|^2 \, dx \, dt \leq \varepsilon_0^*,$$

then, $z_0$ is a regular point. If the condition (iii) holds true, then by Hölder’s inequality and the absolute continuity of the integral norms we have

$$\limsup_{\rho \to 0^+} \frac{1}{\rho} \int_{Q_{z_0,\rho}} \left| \frac{\omega \times v}{|v|} \right|^2 \, dx \, dt \leq \limsup_{\rho \to 0^+} \rho^{2(2-\frac{3}{4}-\frac{4}{\alpha})} \left\| \frac{\omega \times v}{|v|} \right\|_{L^{\gamma,\alpha}_{x,t}(Q_{z_0,\rho})} = 0.$$

Hence, the part (iii) of the above theorem is implied by the main theorem of [27]. The part (i), (ii) of the above theorem, however, have no direct implication relationships with that result. Moreover, the proof of part (iii) of the above theorem given below is much simpler than that of [27].

Remark 1.2 We say $v$ is a Beltrami flow in $Q_{z_0,r}$ if $v \times \omega = 0$ in $Q_{z_0,r}$. In the study of physics of turbulent flows the Beltrami structure has important roles (see e.g. [12, 20] and the references therein). The condition that $v \times \frac{\omega}{|\omega|}$ or $\omega \times \frac{v}{|v|}$ is controllable in a space-time region implies intuitively that the weak solutions are not far from the Beltrami flows in that region in an appropriate sense, and the above theorem says that this implies regularity of the flows in that region.

2 Proof of Theorem 1.1

Before starting our proof we recall previous results concerning the notion of an epoch of possible irregularity of the weak solution of the Navier-Stokes equations. It is known that for weak solutions there exists a set $E \subset I = [0,T]$ such that $E$ is closed, of 1/2-dimensional Hausdorff measure zero, and solutions are regular in $I \setminus E$ ([16, 13, 12]). Moreover, the set $E$ can be written as $I \setminus \bigcup_{i \in J} I_i$, where set $J$ is at most countable, and $I_i = (\alpha_i, \beta_i)$ are disjoint open intervals in $[0,T]$. Following [13], we call the instant time $\beta_i$ an epoch of possible irregularity. We recall a fact proved by Neustupa and Penel in [18] on the epoch of possible irregularity for suitable weak solutions.
Lemma 2.1 Let $z_0 = (x_0, t_0) \in Q_T$. Suppose $v$ is a suitable weak solution of (NS) in $Q_T$ and $t_0$ be an epoch of possible irregularity. Then there exist positive numbers $\tau$, $r_1$, and $r_2$ with $r_1 < r_2$ such that the followings are satisfied:

(a) $\tau$ is sufficiently small so that $t_0$ is only one epoch of possible irregularity in time interval $[t_0 - \tau, t_0]$.

(b) The closure of $B_{x_0,r_2} \times (t_0 - \tau, t_0)$ is contained in $Q_T$, i.e. $\overline{B_{x_0,r_2} \times [t_0 - \tau, t_0]} \subset Q_T$.

(c) $((B_{x_0,r_2} - B_{x_0,r_1}) \times [t_0 - \tau, t_0]) \cap S = \emptyset$, where $S$ is the set of possible singular points of $v$.

(d) $v$, $\nabla v$, and $p$ are, together with all their space derivatives, continuous on $(B_{x_0,r_2} - B_{x_0,r_1}) \times [t_0 - \tau, t_0]$.

Next we recall the following result proved in [4], a corollary of which will be used in the proof of our main theorem.

Proposition 2.1 There exists an absolute constant $\varepsilon_1 > 0$ with the following property. If $(v, p)$ is a suitable weak solution of (NS) near $z_0$ and if

\[
\limsup_{\rho \to 0+} \frac{1}{\rho} \int_{Q_{z_0,\rho}} |\nabla v|^2 dx dt \leq \varepsilon_1, \tag{2.1}
\]

then $z_0$ is a regular point.

As an immediate corollary we have the following local regularity criterion, which is a local version of the one obtained in [2].

Corollary 2.1 If $(v, p)$ is a suitable weak solution of (NS) near $z_0$, and if either

\[
\|\nabla v\|_{L_{x,t}^{\frac{3}{\gamma},\infty}(Q_{z_0,r})} \leq \varepsilon_1,
\]

where $\varepsilon_1$ is the constant in Proposition 2.1, or there exist $\gamma \in (3/2, \infty]$ and $\alpha \in [2, \infty)$ with $3/\gamma + 2/\alpha \leq 2$ such that

\[
\nabla v \in L_{x,t}^{\gamma,\alpha}(Q_{z_0,r}),
\]

then, $z_0$ is a regular point.
Proof. Similarly to Remark 1.1 we observe

\[
\limsup_{\rho \to 0^+} \int_{Q_{z_0, \rho}} |\nabla v|^2 \, dx \, dt \leq \limsup_{\rho \to 0^+} \rho^{3/2 - \frac{3}{\gamma} - \frac{2}{\alpha}} \|\nabla v\|_{L^\gamma_\alpha(Q_{z_0, \rho})}^2
\]

\[
\begin{cases}
0 & \text{if } \gamma > \frac{3}{2} \text{ and } 3/\gamma + 2/\alpha \leq 2 \\
\leq \varepsilon_1 & \text{if } (\gamma, \alpha) = (3/2, \infty)
\end{cases}
\]

by the Hölder inequality. Then the conclusion is immediate by Proposition 2.1. \(\square\)

Proof of Theorem 1.1 We first assume that \(t_0\) is an epoch of possible irregularity for \(v\) in \(Q_{z_0, r}\). Suppose that \(0 < r_1 < r_2 < r\), and \(r^2 < \tau\) are the positive numbers in Lemma 2.1. Below, we denote \(B_{r_1} = B_{x_0, r_1}\) and \(B_{r_2} = B_{x_0, r_2}\). Following [18], we choose a cut-off function \(\varphi \in C_0^\infty(B_2)\) such that \(\varphi = 1\) on \(B_1\), and set \(u = \varphi v - V\), where \(V \in C_0^\infty(B_2 \setminus B_1)\) satisfies \(\text{div } V = v \cdot \nabla \varphi\). In particular, all the spatial derivatives of \(V\) and \(\frac{\partial V}{\partial t}\) are smooth. Using the well-known form of the Navier-Stokes equations,

\[
\frac{\partial v}{\partial t} - v \times \omega = -\nabla (p + \frac{1}{2}|v|^2) + \Delta v,
\]

one can check easily that \(u\) satisfies the following equations:

\[
\frac{\partial u}{\partial t} - \varphi v \times \omega = h - \nabla \left( \varphi (p + \frac{1}{2}|v|^2) \right) + \Delta u, \quad \text{div } u = 0, \quad (2.2)
\]

where we set

\[
h = -\frac{\partial V}{\partial t} + (p + \frac{1}{2}|v|^2) \nabla \varphi - v \Delta \varphi - 2(\nabla \varphi \cdot \nabla)v + \Delta V.
\]

We observe that \(h(\cdot, t)\) is supported on \((B_2 \setminus B_1)\) for each \(t \in [t_0 - \tau, t_0]\), which is sufficiently smooth in the region. Operating \(D\) on (2.2), and taking \(L^2(B_{x_0, r_2})\) inner product it by \(Du\), we obtain, after integration by part

\[
\frac{1}{2} \frac{d}{dt} \|Du\|_{L^2}^2 + \|D^2 u\|_{L^2}^2 = -(\varphi v \times \omega, D^2 u)_{L^2} - (D^2 u, h)
\]

\[
\leq |(\varphi v \times \omega, D^2 u)_{L^2}| + \frac{1}{8} \|D^2 u\|_{L^2}^2 + C \|h\|_{L^2}^2, \quad (2.3)
\]

where (and below) we used simplified notation for the \(L^p\)–norm in \(B_2\),

\[
\|f\|_{L^p} = \|f\|_{L^p(B_2)}, \quad p \in [1, \infty],
\]

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unless other domain is specified. Let us set $\xi = \omega/|\omega|$. We estimate the nonlinear term as follows:

$$
|\langle \varphi v \times \omega, D^2 u \rangle_{L^2}| \leq \int_{B_2} |v \times \xi| |\varphi\omega||D^2 u|\,dx
$$

$$
\leq \int_{B_2} |v \times \xi| |\varphi Dv||D^2 u|\,dx
$$

$$
= \int_{B_2} |v \times \xi| |Du - v\nabla \varphi + DV||D^2 u|\,dx
$$

$$
\leq \int_{B_2} |v \times \xi| |Du||D^2 u|\,dx + \int_{B_2} |v \times \xi| |g||D^2 u|\,dx
$$

$$
= I_1 + I_2,
$$

where we set $g = v\nabla \varphi - DV$. Since $g$ is a smooth function supported on $(B_2 \setminus \bar{B}_1) \times (t_0 - \tau, t_0]$, we estimate $I_2$ simply as

$$
I_2 \leq \|g\|_{L^\infty} \|v\|_{L^2} \|D^2 u\|_{L^2} \leq C \|v\|_{L^2}^2 + \frac{1}{4} \|D^2 u\|_{L^2}^2.
$$

(2.4)

We first assume the condition (i) of Theorem 1.1 holds true. In this case we estimate

$$
I_1 \leq \|v \times \xi\|_{L^3} \|Du\|_{L^6} \|D^2 u\|_{L^2} \leq C \|v \times \xi\|_{L^3} \|D^2 u\|_{L^2}^2.
$$

(2.5)

Combining estimates (2.3)-(2.6) together, we have

$$
\frac{d}{dt} \|Du\|_{L^2}^2 + \|D^2 u\|_{L^2}^2 \leq C_1 \|v \times \xi\|_{L^3} \|D^2 u\|_{L^2}^2 + C \|h\|_{L^2}^2 + C \|v\|_{L^2}^2
$$

$$
\leq C_1 \varepsilon_0 \|D^2 u\|_{L^2}^2 + C \|h\|_{L^2}^2 + C \|v\|_{L^2}^2
$$

(2.7)

for $t \in (t_0 - r_2^2, t_0]$, and for an absolute constant $C_1$. If $C_1 \varepsilon_0 < 1$, then integrating (2.7) in time over $[t_0 - r_2^2, t_0]$, we can obtain

$$
\sup_{t_0 - r_2^2 < t < t_0} \|Du(\cdot, t)\|_{L^2}^2 \leq \|Du(\cdot, t_0 - r_2^2)\|_{L^2}^2 + C \int_{t_0 - r_2^2}^{t_0} \|v\|_{L^2}^2 \,dt
$$

$$
+ C \int_{t_0 - r_2^2}^{t_0} \|h\|_{L^2}^2 \,dt < \infty.
$$

Hence, $Du \in L_{x,t}^{2,\infty}(Q_{r_2})$, and therefore $Dv \in L_{x,t}^{2,\infty}(Q_{r_1})$. Applying Corollary 2.1, we conclude that $z_0$ is a regular point. Next, we assume that the
condition (ii) of Theorem 1.1 holds true, and estimate
\[ I_1 \leq \|v \times \xi\|_{L^\gamma} \|D u\|_{L^{\frac{2\gamma}{\gamma-2}}} \|D^2 u\|_{L^2} \]
\[ \leq C \|v \times \xi\|_{L^\gamma} \|D u\|_{L^2}^{1-\frac{3}{\gamma}} \|D^2 u\|_{L^2}^{1+\frac{3}{\gamma}} \]
\[ \leq C \|v \times \xi\|_{L^\gamma} \|D u\|_{L^2}^{1-\frac{2}{\gamma}} \|D^2 u\|_{L^2}^{1+\frac{2}{\gamma}} \]
\[ \leq C \|v \times \xi\|_{L^\gamma} \|D u\|_{L^2}^{1-\frac{3}{\gamma}} \|D^2 u\|_{L^2}^{1+\frac{3}{\gamma}} \]
where we used the interpolation inequality,
\[ \|D u\|_{L^{\frac{2\gamma}{\gamma-2}}} \leq C \|D u\|_{L^2}^{1-\frac{2}{\gamma}} \|D^2 u\|_{L^2}^{\frac{2}{\gamma}} \]
for 3 < γ ≤ ∞. Combining (2.8) and (2.5) with (2.3), we obtain
\[ \frac{d}{dt}\|D u\|_{L^2}^2 + \|D^2 u\|_{L^2}^2 \leq C \|v \times \xi\|_{L^\gamma} \|D u\|_{L^2}^2 + C \|v\|_{L^2}^2 + C \|h\|_{L^2}^2. \] (2.9)

By Gronwall’s lemma we have
\[ \|D u(\cdot, t)\|_{L^2}^2 + \|D^2 u(\cdot, t)\|_{L^2}^2 \leq \|D u(\cdot, t_0)\|_{L^2}^2 + C \int_{t_0}^{t_0} \|D^2 u(\cdot, t)\|_{L^2}^2 dt \]
\[ \leq \|D u(\cdot, t_0 - r_2^2)\|_{L^2}^2 \exp \left( C \int_{t_0}^{t_0} \|v \times \xi(\cdot, t)\|_{L^\gamma} \|D^2 u(\cdot, t)\|_{L^2}^2 dt \right) \]
\[ + C \int_{t_0}^{t_0} \|h(\cdot, t)\|_{L^2}^2 dt + C \int_{t_0}^{t_0} \|v(\cdot, t)\|_{L^2}^2 dt. \] (2.10)

Since v × ξ ∈ L_\gamma,\alpha(Q_{z_0,r_2}) with 3/γ + 2/α ≤ 1 and γ > 3, we estimate
\[ \int_{t_0}^{t_0} \|v \times \xi(\cdot, t)\|_{L^\gamma} \|D^2 u(\cdot, t)\|_{L^2}^2 dt \leq \|v \times \xi\|_{L^\gamma,\alpha(Q_{z_0,r_2})} \|D^2 u(\cdot, t)\|_{L^2}^{2\gamma/3 (1-3/\gamma - 2/\alpha)} < \infty. \] (2.11)

From (2.10) and (2.11) we find that Du ∈ L_\gamma,\infty(x_0,r_2), and hence Dv ∈ L_\gamma,\infty(x_0,r_1). Similarly to the previous case, we conclude that z_0 is a regular point for v.

Now, we assume (iii) of the theorem holds true, and set η = v/|v|. Then, we
estimate in the preliminary step

\[ |(\varphi v \times \omega, D^2 u)_{L^2}| \leq \int_{B_2} |\omega \times \eta||\varphi v||D^2 u| dx \]

\[ = \int_{B_2} |\omega \times \eta||u + V||D^2 u| dx \]

\[ \leq \int_{B_2} |\omega \times \eta||D^2 u| dx + \int_{B_2} |\omega \times \eta||V||D^2 u| dx \]

\[ \leq \|\omega \times \eta\|_{L^\gamma} \|u\|_{L^{2\gamma}} \|D^2 u\|_{L^2} + \|V\|_{L^\infty} \|\omega\|_{L^2} \|D^2 u\|_{L^2} \]

\[ = J_1 + J_2. \tag{2.12} \]

The estimate of \( J_2 \) is simple as follows.

\[ J_2 \leq C \|Dv\|_{L^2}^2 + \nu \|D^2 u\|_{L^2}^2. \tag{2.13} \]

For \( 2 < \gamma \leq 3 \), we have \( 6 \leq \frac{2\gamma}{\gamma - 2} < \infty \), and the following interpolation inequality is valid

\[ \|u\|_{L^{2\gamma}} \leq C \|u\|_{L^6}^{2 - \frac{2}{\gamma}} \|Du\|_{L^6}^{-1 + \frac{3}{\gamma}} \leq C \|Du\|_{L^2}^{2 - \frac{2}{\gamma}} \|D^2 u\|_{L^2}^{-1 + \frac{3}{\gamma}}. \tag{2.14} \]

If \( \gamma = 2 \), then, instead, we use the inequality

\[ \|u\|_{L^\infty} \leq C \|u\|_{L^6}^{\frac{3}{2}} \|Du\|_{L^6}^{\frac{3}{2}} \leq C \|Du\|_{L^2}^{\frac{3}{2}} \|D^2 u\|_{L^2}^{\frac{3}{2}}. \tag{2.15} \]

Substituting (2.14) or (2.15) into (2.12), we obtain

\[ J_1 \leq C \|\omega \times \eta\|_{L^\gamma} \|Du\|_{L^2}^{2 - \frac{2}{\gamma}} \|D^2 u\|_{L^2}^{\frac{3}{\gamma}} \]

\[ \leq C \|\omega \times \eta\|_{L^\gamma}^{\frac{2\gamma}{\gamma - 3}} \|Du\|_{L^2}^{\frac{2\gamma}{\gamma - 3}} \|D^2 u\|_{L^2}^2 + \frac{1}{8} \|D^2 u\|_{L^2}^2 \tag{2.16} \]

for \( 2 \leq \gamma \leq 3 \). Combining (2.16) and (2.13), we have

\[ |(\varphi v \times \omega, D^2 u)_{L^2}| \leq C \|\omega \times \eta\|_{L^\gamma}^{\frac{2\gamma}{\gamma - 3}} \|Du\|_{L^2}^{\frac{2\gamma}{\gamma - 3}} \|D^2 u\|_{L^2}^2 + \frac{1}{4} \|D^2 u\|_{L^2}^2 + C \|Dv\|_{L^2}^2. \]

Hence, from (2.12) we derive

\[ \frac{d}{dt} \|Du\|_{L^2}^2 + \|D^2 u\|_{L^2}^2 \leq C \|\omega \times \eta\|_{L^\gamma}^{\frac{2\gamma}{\gamma - 3}} \|Du\|_{L^2}^{\frac{2\gamma}{\gamma - 3}} \|D^2 u\|_{L^2}^2 + C \|h\|_{L^2}^2 + \|Dv\|_{L^2}^2. \]
By Gronwall’s lemma we have

$$
\|Du(\cdot, t)\|_{L^2}^2 + \int_{t_0 - r_2^2}^{t_0} \|D^2 u(\cdot, t)\|_{L^2}^2 dt
\leq \|Du(\cdot, t_0 - r_2^2)\|_{L^2}^2 \exp \left( C \int_{t_0 - r_2^2}^{t_0} \|\omega \times \eta\|_{L^2}^{2^{\gamma-2}} dt \right)
+ C \int_{t_0 - r_2^2}^{t_0} \|Dv(\cdot, t)\|_{L^2}^2 dt + C \int_{t_0 - r_2^2}^{t_0} \|h(\cdot, t)\|_{L^2}^2 dt. \tag{2.17}
$$

We observe that

$$
C \int_{t_0 - r_2^2}^{t_0} \|Dv(\cdot, t)\|_{L^2}^2 dt + C \int_{t_0 - r_2^2}^{t_0} \|h(\cdot, t)\|_{L^2}^2 dt < \infty. \tag{2.18}
$$

Since $\omega \times \eta \in L_{x,t}^{\gamma,\alpha}(Q_{z_0,r_2})$ with $3/\gamma + 2/\alpha \leq 2$ and $2 \leq \gamma \leq 3$ by hypothesis, we have

$$
\int_{t_0 - r_2^2}^{t_0} \|\omega \times \eta(\cdot, t)\|_{L^2}^{2^{\gamma-2}} dt \leq \|\omega \times \eta\|_{L_{x,t}^{\gamma,\alpha}(B_2 \times (t_0 - r_2^2, t_0))}^{2^{\gamma-2} \gamma - 2} \left(2 - \frac{4}{\gamma} \right) < \infty. \tag{2.19}
$$

From (2.17)-(2.19) we find that $Du \in L_{x,t}^{2,\infty}(Q_{z_0,r_2})$, and therefore $Dv \in L_{x,t}^{2,\infty}(Q_{z_0,r_1})$, and similar conclusion to the above case is obtained.

Next, we assume that $z_0$ is a singular point for which $t_0$ is not an epoch of possible irregularity. Then, there exists a time $t^* \in (t_0 - r^2, t_0)$ and $0 < \tilde{r}_1 < \tilde{r}_2 < r$ such that $v$ is regular on $(B_{x_0,\tilde{r}_2} \setminus B_{x_0,\tilde{r}_1}) \times [t^*, t_0]$. This is due to that fact that the one dimensional Hausdorff measure of the set of all possible singular space-time points is equal to zero. We claim $v$ is regular on $B_{x_0,\tilde{r}_1} \times [t^*, t_0]$. Suppose not, then there exists another time $s \in [t^*, t_0]$ such that the weak solution is regular on $B_{x_0,\tilde{r}_1} \times [t^*, s)$, and singularity occurs at $(y, s) \in B_{x_0,\tilde{r}_1} \times \{s\}$. We can repeat the above argument for the parabolic neighborhoods of $(y, s)$ to conclude that $(y, s)$ is actually a regular point.

Hence, there exists no space-time point of singularity in $B_{x_0,\tilde{r}_1} \times [t^*, t_0]$, and we are reduced to the already considered case that $t_0$ is an epoch of possible irregularity. This completes the proof. □

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