A BERNSTEIN TYPE THEOREM FOR MINIMAL HYPERSURFACES
VIA GAUSS MAPS

QI DING

Abstract. Let \( M \) be an \( n \)-dimensional smooth oriented complete embedded minimal hypersurface in \( \mathbb{R}^{n+1} \) with Euclidean volume growth. We show that if the image under the Gauss map of \( M \) avoids some neighborhood of a half-equator, then \( M \) must be an affine hyperplane.

1. Introduction

The original Bernstein theorem says that each entire minimal graph in \( \mathbb{R}^3 \) must be a plane. The Bernstein theorem can be generalized to high dimensions as follows: each entire minimal graph in \( \mathbb{R}^{n+1} \) must be a plane provided \( n \leq 7 \), which were achieved by successive efforts of W. Fleming [7], E. De Giorgi [2], F. J. Almgren [1], and finally completely settled by J. Simons [14]. For \( n \geq 8 \), Bombieri-De Giorgi-Giusti provided a counterexample by constructing a nontrivial entire minimal graph in \( \mathbb{R}^{n+1} \), whose tangent cone at infinity is a vertical stable minimal cone, a non-warped product of a Simons’ cone and line. Under some conditions on graphic functions, all entire minimal graphs could be affine (see [5, 9]). In particular, all minimal graphs are stable minimal hypersurfaces. In \( \mathbb{R}^3 \), all oriented complete stable minimal surfaces in \( \mathbb{R}^3 \) are affine plane shown by Fischer-Colbrie and Schoen [6], and do Carmo-Peng [4]. For \( n \leq 5 \), with integral curvature estimates Schoen-Simon-Yau proved that all oriented complete stable minimal hypersurfaces with Euclidean volume growth in \( \mathbb{R}^{n+1} \) must be affine [12]. With the embedded condition, Schoen-Simon can show it for the case \( n \leq 6 \) by their regularity theorem [11].

Let \( M \) be an \( n \)-dimensional smooth oriented complete minimal hypersurface in \( \mathbb{R}^{n+1} \). The Ruh-Vilms theorem tells us that the Gauss map \( \gamma : M \to \mathbb{S}^n \) is a harmonic map [10]. In [15], Solomon showed that if \( S \) is an area-minimizing hypersurface in \( \mathbb{R}^{n+1} \) with \( \partial S = 0 \), the first Betti number of \( \text{reg} S \) vanishes and the Gauss map of \( S \) omits some neighborhood of \( \mathbb{S}^{n-2} \) in \( \mathbb{S}^n \), then each component of \( \text{spt} S \) is a hyperplane. The condition on the first Betti number is necessary by the example of Simons’ cones (see also section 6 in [15] for instance). In [8], Jost-Xin-Yang found a maximal open convex supporting subset \( S^n \setminus \mathbb{S}^{n-1}_+ \) of \( S^n \), where \( \mathbb{S}^{n-1}_+ \) is the hemisphere of \( S^{n-1} \) in \( S^n \). They constructed a smooth bounded strictly convex function on any compact set \( K \) in \( S^n \setminus \mathbb{S}^{n-1}_+ \), and then studied the regularity of harmonic maps to \( K \). As an application, they got a Bernstein type theorem as follows (see Theorem 6.5 in [8]).

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Theorem 1.1. Let \( M^n \subset \mathbb{R}^{n+1} \) be a complete minimal embedded hypersurface with Euclidean volume growth. Assume that there is a positive constant \( C \), such that for arbitrary \( y \in M \) and \( R > 0 \), the Neumann-Poincaré inequality

\[
\int_{M \cap B_R(y)} |v - \bar{v}_{R,y}|^2 \leq CR^2 \int_{M \cap B_R(y)} |\nabla v|^2
\]

holds for each function \( v \in C^\infty(B_R(y)) \), where \( B_R(y) \) is the ball in \( \mathbb{R}^{n+1} \) with the radius \( R \) and centered at \( y \), \( \bar{v}_{R,y} \) is the average value of \( v \) on \( B_R(y) \). If the image under the Gauss map omits a neighborhood of \( \mathbb{S}_+^{n-1} \), then \( M \) has to be an affine linear space.

The necessity of the Neumann-Poincaré inequality in the above theorem is not clear as they said in [8]. Later, Yang further proved that the above conclusion holds provided \( 1 \) of the distance between the Gauss image of \( M \cap B_R(y) \) and \( \mathbb{S}_+^{n-1} \) is less than \( o(\log \log R) \) for the large \( R \) in [18].

In this paper, we remove the condition on the Neumann-Poincaré inequality in Theorem 6.5 of [8] instead by the oriented condition, and obtain the following Bernstein type theorem.

Theorem 1.2. Let \( M \) be an \( n \)-dimensional smooth oriented complete embedded minimal hypersurface in \( \mathbb{R}^{n+1} \) with Euclidean volume growth. If the image under the Gauss map omits a neighborhood of \( \mathbb{S}_+^{n-1} \), then \( M \) must be an affine hyperplane.

One of the important ingredients in the proof of our theorem is to show that the Gauss map of \( M \) takes actually values in a compact subset of an open hemisphere of \( \mathbb{S}^n \) provided the support of one of tangent cones of \( M \) at infinity is the Euclidean space.

2. NEW BOUNDED SUBHARMONIC FUNCTIONS ON MINIMAL HYPERSURFACES

Let \( P \) denote the projection from \( \mathbb{S}^n \) onto \( \overline{\mathbb{D}^2} \) (2-dimensional closed unit disk) by

\[
P : \mathbb{S}^n \rightarrow \overline{\mathbb{D}^2} \quad (x_1, \ldots, x_{n+1}) \mapsto (x_1, x_2).
\]

For any \( x = (x_1, \ldots, x_{n+1}) \in \mathbb{S}^n \setminus \{(x_1, \ldots, x_{n+1}) \in \mathbb{S}^n \mid x_1 = 0, \ x_2 = 0\} \), there is a polar coordinate system in \( \overline{\mathbb{D}^2} \). Namely,

\[
P(x) = (r(x) \sin \theta(x), r(x) \cos \theta(x)).
\]

with \( r(x) = \sqrt{x_1^2 + x_2^2} \in (0, 1] \) and the unique \( \theta(x) \in \left[ \frac{\pi}{2}, \frac{\pi}{2} \right) \). In other words, we have defined two functions \( r, \theta \) on \( \mathbb{S}^n \setminus \{(x_1, \ldots, x_{n+1}) \in \mathbb{S}^n \mid x_1 = 0, \ x_2 = 0\} \). Let \( \sigma_3 \) be the standard metric on \( \mathbb{S}^n \), and Hess be the Hessian matrix on \( \mathbb{S}^n \) with the respect to \( \sigma_3 \). From [8], we have

\[
Hess \ r = -r \sigma_3 + r d\theta \otimes d\theta,
\]

and

\[
Hess \ \theta = -r^{-1} (dr \otimes d\theta + d\theta \otimes dr).
\]

Let \( \mathbb{S}_+^{n-1} \) be a hemisphere of \( \mathbb{S}^{n-1} \) defined by

\[
\{(x_1, \ldots, x_{n+1}) \in \mathbb{S}^n \mid x_1 = 0, \ x_2 \geq 0\},
\]

and \( K \) be a compact set in \( \mathbb{S}^n \setminus \mathbb{S}_+^{n-1} \). Clearly, there is a constant \( \delta_K > 0 \) such that \( r(x) \geq \delta_K \) for all \( x \in K \).
Let $B_r(\mathbb{S}^{n-2})$ denote the $\tau$-neighborhood of \{(0,0,x_3,\cdots,x_{n+1}) | x_3^2 + \cdots + x_{n+1}^2 = 1\} in \mathbb{S}^n$, i.e.,\[ B_r(\mathbb{S}^{n-2}) = \{(x_1,\cdots,x_{n+1}) \in \mathbb{S}^n | x_1^2 + x_2^2 < \sin^2\tau\}.

We choose $0 < \tau < 1/2$ sufficiently small such that $B_r(\mathbb{S}^{n-2}) \cap K = \emptyset$. For each pair $x_s,-x_s \in \mathbb{S}^n \setminus B_r(\mathbb{S}^{n-2})$, let $\theta_* \in \left[\frac{3}{2}\pi, \frac{5}{2}\pi\right)$ be a constant such that $\theta(x_s) = \theta_*$ and $\theta(-x_s) = \theta_* + \pi$. Then we define a function\[ \Theta = \lambda \phi \Theta(x_s),\]

where $k$ is an arbitrary constant $\geq 1$. In particular, $\phi(x_s) = \phi(-x_s)$, and $\phi$ is smooth on $K$. Combining (2.1) and (2.2), we have\[ \Theta(x) = \Theta(-x) \]

Lemma 2.1. Let $K$ be a compact set in $\mathbb{S}^n \setminus \mathbb{S}^{n-1}_+$. For each pair $x_s,-x_s \in \mathbb{S}^n \setminus B_r(\mathbb{S}^{n-2})$ with $K \cap B_r(\mathbb{S}^{n-2}) = \emptyset$ for some $\tau > 0$, there is a bounded function $\Theta$ on $\mathbb{S}^n \setminus B_r(\mathbb{S}^{n-2})$ with $\Theta(x_s) = \Theta(-x_s)$ such that $\Theta$ is smooth strictly convex on $K$.

Proof. Let $\phi$ be the function defined on $K$ as above with $\phi(x_s) = \phi(-x_s)$. Enlightened by Lemma 2.1 in [8], we define $\Theta = \lambda^{-1}e^{\lambda\phi}$ on $K$ with the positive constant $\lambda$ to be defined later. Then $\phi(x_s) = \phi(-x_s)$ implies $\Theta(x_s) = \Theta(-x_s)$. By the definition of $\Theta$, at any considered point $p \in \mathbb{S}^n$ we have\[ \Theta(\xi,\xi) = e^{\lambda\phi} (\Theta(\xi,\xi) + \lambda|d\phi(\xi)|^2)\]

for any unit vector $\xi \in T_p(\mathbb{S}^n)$. We denote $\tilde{\theta} = \theta - \theta_* - \frac{\pi}{2}$ for convenience. Combining (2.5), we have\[ e^{-\lambda\phi} \Theta(\xi,\xi) = \frac{\delta K}{r} + \left(k - \frac{\delta K}{r}\right) (d\theta(\xi))^2 + \frac{2\delta K}{r} (dr(\xi))^2 - \frac{2k}{r} \tilde{\theta}d\theta(\xi)d\theta(\xi)
\]

\[ + \lambda \left| \frac{\delta K}{r^2} dr(\xi) - \tilde{\theta}d\theta(\xi) \right|^2
\]

\[ = \frac{\delta K}{r} + \left(k + \frac{\delta K}{r}\right) (d\theta(\xi))^2 + \left(\frac{2\delta K}{r^3} + \lambda \frac{\delta K}{r^4}\right) (dr(\xi))^2
\]

\[ - 2 \left(1 + \frac{\delta K}{r^2}\right) k\tilde{\theta}d\theta(\xi)d\theta(\xi).
\]

From Cauchy-Schwarz inequality, we have\[ 2 \left(1 + \frac{\delta K}{r^2}\right) k \left| \tilde{\theta}d\theta(\xi)d\theta(\xi) \right| \leq k^2 \tilde{\theta}^2 (d\theta(\xi))^2 + \left(\frac{1}{\sqrt{\lambda}r^2} + \sqrt{\lambda} \frac{\delta K}{r^2}\right)^2 (dr(\xi))^2
\]

\[ = k^2 \tilde{\theta}^2 (d\theta(\xi))^2 + \left(\frac{1}{\lambda r^2} + \frac{2\delta K}{r^3} + \lambda \frac{\delta K}{r^4}\right) (dr(\xi))^2.
\]

Substituting the above inequality into (2.7) infers\[ e^{-\lambda\phi} \Theta(\xi,\xi) \geq \frac{\delta K}{r} + \left(k - \frac{\delta K}{r}\right) (d\theta(\xi))^2 - \frac{1}{\lambda r^2} (dr(\xi))^2 \geq \frac{\delta K}{r} - \frac{1}{\lambda r^2} (dr(\xi))^2
\]

as $k \geq 1$. 
In the upper hemisphere
\begin{equation}
\{(x_1, \cdots, x_{n+1}) \in S^n | x_{n+1} > 0\},
\end{equation}
let $T$ denote a tensor defined by
\begin{equation}
T = \sum_{i,j=1}^{n} \frac{x_i x_j}{1 - \sum_{k=1}^{n} |x_k|^2} dx_i \otimes dx_j.
\end{equation}
Then the metric $\sigma$ can be written as
\begin{equation}
\sum_{i=1}^{n} dx_i \otimes dx_i + T = dr \otimes dr + r^2 d\theta \otimes d\theta + \sum_{i=3}^{n} dx_i \otimes dx_i + T.
\end{equation}
Since all the eigenvalues of $T$ are nonnegative, then $\sigma (\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) \geq 1$. Hence at any point $q$ in $\{x_1, \cdots, x_{n+1}, x_{n+1} > 0\}$, and $q \in K$, we have
\begin{equation}
\sup_{\eta \in T_q(S^n)} |dr(\eta)| \leq |\eta|.
\end{equation}
Similarly, (2.13) holds at any $q \in \{(x_1, \cdots, x_{n+1}) \in S^n | x_{n+1} < 0\} \cap K$. By the differentiability of $r$ on $S^n \setminus \{(x_1, \cdots, x_{n+1}) \in S^n | x_1 = 0, x_2 = 0\}$, (2.13) holds at any $q \in K$. Therefore, if we choose $\lambda \geq \frac{2}{\delta_k^2}$, then from (2.9) we get
\begin{equation}
\text{Hess } \Theta(\xi, \xi) \geq \kappa \frac{\delta_k}{2r} \quad \text{on } K.
\end{equation}
This completes the proof. 

Let $\Delta$ and $\nabla$ be the Laplacian and Levi-Civita connection on $S^n$ with the respect to the metric $\sigma$, respectively. From Lemma 2.1 there is a constant $\kappa > 0$ depending only on $n, K$ such that the Hessian of $\Theta$ on $K \subset S^n$ satisfies
\begin{equation}
\text{Hess } \Theta \geq \kappa.
\end{equation}
By the construction of $\Theta$, there is a constant $c_K$ depending only on $n, K$ such that
\begin{equation}
|\Theta(\nu) - \Theta(\nu')| \leq c_K|\nu - \nu'|
\end{equation}
for any $\nu, \nu' \in K$ from Newton-Leibnitz formula.

Let $M$ be an $n$-dimensional complete oriented smooth minimal hypersurface in $\mathbb{R}^{n+1}$. Namely, there are a smooth $n$-dimensional Riemannian manifold $M'$, and an isometric mapping $X : M' \to M$ with $X(M') = M \subset \mathbb{R}^{n+1}$. Let $\gamma : M' \to \mathbb{R}^n$ be the Gauss map defined by $\gamma(p) = X_*(T_pM') \in \mathbb{R}^n$ via the parallel translation in $\mathbb{R}^{n+1}$ for all $p \in M'$. For convenience, we identify $M$ and $M'$ by viewing $X(p)$ as $p$. Suppose that the Gauss image of $M$ satisfies $\gamma(M) \subset K$, then we define the function $v = \Theta \circ \gamma$ on $M$. Let $\nabla$ be the Levi-Civita connection of $M$ with the induced metric from $\mathbb{R}^{n+1}$. We choose a local orthonormal frame $\{e_i\}$ on $M$ such that $\nabla e_i = 0$ at the considered point. Then (see formula (2.9) in [3] for instance)
\begin{equation}
\sum_{i=1}^{n} \langle \gamma_* e_i, \gamma_* e_i \rangle = |A|^2.
\end{equation}
As $\gamma$ is a harmonic map from $M$ to $\mathbb{S}^n$, combining (2.15) we get
\begin{equation}
\Delta_M v = \sum_{i=1}^{n} \text{Hess } \Theta(\gamma_* e_i, \gamma_* e_i) \geq \kappa \sum_{i=1}^{n} \langle \gamma_* e_i, \gamma_* e_i \rangle = \kappa |A|^2,
\end{equation}
where $\Delta_M$ is the Laplacian of $M$ with the induced metric from $\mathbb{R}^{n+1}$, $A$ is the second fundamental form of $M$. In particular, for any $x_+, -x_+ \in \overline{\gamma(M)} \subset K$, we can assume $\Theta(x_+) = \Theta(-x_+)$.

We say that $M$ is $\delta$-stable if
\begin{equation}
(2.19) \quad \int_M (|\nabla f|^2 - \delta |A|^2 f^2) \geq 0
\end{equation}
for each smooth function $f : M \to \mathbb{R}$ with compact support.

**Lemma 2.2.** Let $M$ be an $n$-dimensional complete oriented minimal hypersurface in $\mathbb{R}^{n+1}$. If there is a bounded function $\nu$ on $M$ satisfying (2.18) for some positive number $\kappa > 0$, then $M$ is $\delta_\kappa$-stable for some $\delta_\kappa > 0$.

**Proof.** Let $\nu_{\text{sup}}$ and $\nu_{\text{inf}}$ be the supremum and infimum of $\nu$ on $M$, respectively. From (2.18), one has
\begin{equation}
(2.20) \quad \kappa |A|^2 \leq -\Delta_M (\nu_{\text{sup}} - \nu) \leq -\frac{\nu_{\text{sup}} - \nu_{\text{inf}}}{\nu_{\text{sup}} - \nu} \Delta_M (\nu_{\text{sup}} - \nu).
\end{equation}
Taking $\delta_\kappa = \frac{\kappa}{\nu_{\text{sup}} - \nu_{\text{inf}}}, \tilde{\nu} = \nu_{\text{sup}} - \nu$, then
\begin{equation}
(2.21) \quad \delta_\kappa |A|^2 \leq \frac{1}{\nu} \Delta_M \tilde{\nu}.
\end{equation}

For any smooth function $\phi$ on $M$ with compact support, we have (see also the proof of proposition 6.2.2 of [17])
\begin{equation}
(2.22) \quad \int_M (|\nabla \phi|^2 - \delta_\kappa |A|^2 \phi^2) \geq \int_M \left( |\nabla \phi|^2 + \frac{\phi^2}{\nu} \Delta_M \tilde{\nu} \right) = \int_M \left( |\nabla \phi|^2 - 2 \frac{\phi}{\nu} \nabla \tilde{\nu} \cdot \nabla \phi + \frac{\phi^2}{\nu^2} |\nabla \tilde{\nu}|^2 \right) \geq 0,
\end{equation}
where we have used Cauchy-Schwartz inequality in the last step. Namely, $M$ is a smooth $\delta_\kappa$-stable minimal hypersurface in $\mathbb{R}^{n+1}$. $\square$

As a corollary, an $n$-dimensional complete oriented minimal hypersurface $M$ in $\mathbb{R}^{n+1}$ is $\delta$-stable for some $\delta > 0$ provided the Gauss image of $M$ is contained in $K$, where $K$ is defined as above.

### 3. Rigidity of minimal hypersurfaces with constraints

Let $\mathcal{B}_r(X)$ denote the ball in $\mathbb{R}^{n+1}$ with radius $r > 0$ and centered at $X \in \mathbb{R}^{n+1}$. Denote $\mathcal{B}_r = \mathcal{B}_r(0)$ for simplicity. Let $M$ be a smooth oriented complete minimal hypersurface in $\mathbb{R}^{n+1}$ with Euclidean volume growth and $\gamma(M) \subset K$, where $K$ is a compact set in $\mathbb{S}^n \setminus \mathbb{S}^{n-1}_+$ and $\mathbb{S}^{n-1}_+$ is defined as (23). Now we assume that the support of one of tangent cones of $M$ at infinity is an $n$-Euclidean space. Namely, there is a sequence $R_i \to \infty$ such that $\frac{1}{R_i} M$ converges to an integer varifold $T$ with the support $\text{spt} T$ being the $n$-Euclidean space. Let $\nu_+, -\nu_+$ denote the unit normal vectors of $\text{spt} T$. From the definition of the function $\theta$ in the last chapter, without loss of generality, we assume that $\theta(\nu_+) \in \left[\frac{1}{2}\pi, \frac{3}{2}\pi\right)$. Let $\nu$ denote the unit normal vector field of $M$. From [11] (or 22.2 in [13]), the *unoriented excess* satisfies
\begin{equation}
(3.1) \quad \lim_{i \to \infty} R_i^{-n} \int_{\mathcal{B}_{R_i} \cap M} (1 - \langle \nu, \nu_+ \rangle)^2 = 0.
\end{equation}
It is easy to see
\[(3.2) \min \{1 - \langle \nu, \nu_* \rangle, 1 + \langle \nu, \nu_* \rangle \} \leq 1 - \langle \nu, \nu_* \rangle^2.\]
With Cauchy inequality, we have
\[(3.3) \lim_{i \to \infty} R_i^{-n} \int_{B_{R_i} \cap M} \min \{|\nu - \nu_*|, |\nu + \nu_*|\} = 0.\]
From Lemma 2.1, there is a bounded function \(\Theta\) on \(\mathbb{S}^n \setminus B_r(\mathbb{S}^{n-2})\) with \(\Theta(\nu_*) = \Theta(-\nu_*)\) and \(B_r(\mathbb{S}^{n-2}) \cap K = \emptyset\) such that \(\Theta\) is smooth strictly convex on \(K\). Put \(\gamma : M \to \mathbb{S}^n\) be the Gauss map and \(v = \Theta \circ \gamma\) as before. Denote \(v_* = \Theta(\nu_*)\).

**Lemma 3.1.** We have
\[(3.4) \lim_{i \to \infty} R_i^{-n} \int_{B_{R_i} \cap M} |v - v_*| = 0.\]

**Proof.** Let us prove it by dividing into 3 cases.

Case 1: \(\nu_* - \nu_* \in \overline{\gamma(M)} \subset K\). From (2.16), we have
\[(3.5) |\Theta(\nu) - \Theta(\nu_*)| = |\Theta(\nu) - \Theta(-\nu_*)| \leq c_K \min \{|\nu - \nu_*|, |\nu + \nu_*|\}.\]
Combining (3.3) and (3.5), we can get (3.4).

Case 2: \(-\nu_* \in K\) and \(\nu_* \in \mathbb{S}^n \setminus K\), then (3.3) infers
\[(3.6) \lim_{i \to \infty} R_i^{-n} \int_{B_{R_i} \cap M} |\nu + \nu_*| = 0.\]
From (2.16) and \(\Theta(\nu_*) = \Theta(-\nu_*)\), one has \(|\Theta(\nu) - \Theta(\nu_*)| \leq c_K |\nu + \nu_*|\) for any \(\nu \in K\). Combining (3.6), we can get (3.4).

Case 3: \(\nu_* \in K\) and \(-\nu_* \in \mathbb{S}^n \setminus K\), then (3.3) infers
\[(3.7) \lim_{i \to \infty} R_i^{-n} \int_{B_{R_i} \cap M} |\nu - \nu_*| = 0.\]
Combining (2.16) and (3.7), we can get (3.4) analog to the case 2, and complete the proof. \(\square\)

**Lemma 3.2.** The supermum of the function \(v\) on \(M\) is \(v_*\).

**Proof.** From Euclidean volume growth of \(M\) and monotonicity of \(\rho^{-n} \mathcal{H}^n (M \cap B_\rho(p))\), \(M \cap B_\rho(p)\) has volume doubling condition (independent of \(p, \rho\)). Combining Sobolev inequality on \(M\) (see [13]), there is a positive constant \(c_M\) depending only on \(n\) and the limit of \(\rho^{-n} \mathcal{H}^n (M \cap B_\rho(p))\) such that for each \(p \in M\), each smooth nonnegative subharmonic function \(f\) on \(M\), one has the mean value inequality
\[(3.8) f(p) \leq \frac{c_M}{\rho^n} \int_{M \cap B_\rho(p)} f.\]
Now let us prove the lemma by contradiction. Namely, we assume \(\sup_M v > v_*\). For each \(0 < \epsilon < \frac{1}{2} (\sup_M v - v_*)\), we define
\[v_\epsilon = \left(\sup_M v + \epsilon - v\right)^{-1}.\]
Then
\[(3.9) \Delta_M v_\epsilon \geq v_*^2 \Delta_M v \geq 0.\]
There is a point \( p \in M \) such that \( v(p) \geq \sup_M v - \epsilon \). From (3.8), we have
\[
\frac{1}{2\epsilon} \leq v_\epsilon(p) \leq \frac{cM}{\rho^n} \int_{M \cap B_\rho(p)} v_\epsilon
\]
(3.10)
\[
\leq \frac{cM}{\rho^n} \int_{M \cap B_\rho(p)} \left( \frac{1}{\sup_M v + \epsilon - v} - \frac{1}{\sup_M v + \epsilon - v_*} + \frac{1}{\sup_M v + \epsilon - v_*} \right)
\]
\[
\leq \frac{cM}{\rho^n} \int_{M \cap B_\rho(p)} \left| v - v_* \right| + \frac{cM}{\sup_M v - v_*} \rho^{-n} \mathcal{H}^n (M \cap B_\rho(p)).
\]
Hence combining (3.11), one has
\[
\frac{1}{2\epsilon} \leq \lim_{R_i \to \infty} \frac{cM}{\epsilon^2 R_i^n} \int_{M \cap B_{R_i}(p)} \left| v - v_* \right| + \frac{cM}{\sup_M v - v_*} \lim_{\rho \to \infty} \rho^{-n} \mathcal{H}^n (M \cap B_\rho(p))
\]
(3.12)
\[
= \frac{cM}{\sup_M v - v_*} \lim_{\rho \to \infty} \rho^{-n} \mathcal{H}^n (M \cap B_\rho(p)).
\]
The above inequality fails for the sufficiently small \( \epsilon > 0 \). Hence we complete the proof. \( \square \)

For each small \( t > 0 \), we define a closed set \( E_t \subseteq \mathbb{D}^2 \) by
\[
E_t = \{ (r \sin \theta, r \cos \theta) \mid r \in [\delta_K, 1], \; \theta_* - t \leq \theta \leq \theta_* + \pi + t \}.
\]
We define a closed set \( \widetilde{E}_t \subseteq S^n \) as the inverse image of \( E_t \) under the mapping \( P \), i.e.,
\[
\widetilde{E}_t = P^{-1}(E_t).
\]
From \( v_* = \sup_M v \), we get \( \Theta(v_*) = \Theta(-v_*) = \sup_{\gamma(M)} \Theta \). With the definition of \( \phi \) in (2.4), for each small \( t > 0 \) we have \( \gamma(M) \subseteq \widetilde{E}_t \) when \( k \geq 2 \) is sufficiently large, which implies
\[
\gamma(M) \subseteq \lim_{t \to 0} \widetilde{E}_t.
\]
In particular, \( \gamma(M) \) is contained in a closed hemi-sphere of \( S^n \), denoted by
\[
\{ \xi \in S^n \mid \langle \xi, \nu_0 \rangle \geq 0 \}
\]
for the unique unit vector \( \nu_0 \in S^n \) with \( P(\nu_0) = (\sin(\theta(\nu_*) + \pi/2), \cos(\theta(\nu_*) + \pi/2)) \). From the well-known formula (see formula (1.3.8) in [17] for instance)
\[
\Delta_M \langle \nu, \nu_0 \rangle = -|A|^2 \langle \nu, \nu_0 \rangle,
\]
(3.13)
and the strong maximum principle, we have \( \langle \nu, \nu_0 \rangle > 0 \) on \( M \), or \( \langle \nu, \nu_0 \rangle \equiv 0 \) on \( M \). However, the later case only occurs for \( M \) being a Euclidean space. Now we consider the case \( \langle \nu, \nu_0 \rangle > 0 \) on \( M \). Hence, it follows that
\[
\gamma(M) \subseteq K \cap \{ \xi \in S^n \mid \langle \xi, \nu_0 \rangle > 0 \}.
\]
From (3.13) and the argument of Lemma 2.2 \( M \) is a stable minimal hypersurface in \( \mathbb{R}^{n+1} \). With [12], we have gotten the flatness of \( M \) for \( n \leq 5 \). For general \( n \), let us prove the flatness by constructing new bounded subharmonic functions on \( M \).

Recall
\[
B_\tau(S^{n-2}) = \{ (x_1, \cdots, x_{n+1}) \in S^n \mid x_1^2 + x_2^2 < \sin^2 \tau \}.
\]
We fix the sufficiently small constant \( \tau \in (0, 1/2) \) such that \( B_\tau(S^{n-2}) \cap K = \emptyset \). Let \( \Omega \) be the closed set in \( S^n \) defined by
\[
\{ \xi \in S^n \mid \langle \xi, \nu_0 \rangle \geq 0 \} \setminus B_\tau(S^{n-2})
\]
(3.14)
Then $\gamma(M) \subset \Omega$.

**Lemma 3.3.** For any positive constant $\tau_n < 1$, there is a positive smooth function $\Lambda$ on $\Omega$ such that

$$
(3.15) \quad \text{Hess } \Lambda \leq -(1 - \tau_n)A\sigma_S \quad \text{on } \Omega.
$$

**Proof.** By the choice of coordinates of $\mathbb{S}^n$ and the definition of $\nu_0$, (in this proof) we can allow

$$
\Omega = \{(x_1, \ldots, x_{n+1}) \in \mathbb{S}^n \mid x_1 \geq 0\} \setminus B_{\epsilon}(\mathbb{S}^{n-2}).
$$

Now we define a smooth cut-off function $\eta$ on $(-\infty, 1]$ by $\eta \equiv 1$ on $(-\infty, \tau/4]$, $\eta \equiv 0$ on $[\tau/2, 1]$, and $0 \leq \eta \leq 1$, $|\eta'| \leq c/\tau$, $|\eta''| \leq c/\tau^2$. Here, $c$ is an absolute positive constant.

For each $\epsilon > 0$, we define a positive function $\varphi_\epsilon$ on $\Omega$ by

$$
\varphi_\epsilon = x_1 + \epsilon \eta(x_1)|x_2|.
$$

Since $\sin \tau \geq \frac{2}{3}\tau$ on $[0, \frac{2}{3}]$, then obviously $\varphi_\epsilon$ is well-defined and smooth on $\Omega$. Put $\Omega^+_t = \Omega \cap \{x_2 > 0\} \cap \{0 \leq x_1 < t\}$ and $\Omega^-_t = \Omega \cap \{x_2 < 0\} \cap \{0 \leq x_1 < t\}$ for each $0 < t < \sin \tau$. From [3], $\text{Hess } x_i = -x_i\sigma_S$ on $\mathbb{S}^n$ for each $1 \leq i \leq n + 1$. Hence there is an absolute positive constant $c' > 0$ such that

$$
(3.16) \quad |\text{Hess}(\eta(x_1)x_2)| \leq \frac{c'}{\tau^2}.
$$

On $\Omega^+_{\tau/4}$ one has

$$
(3.17) \quad \text{Hess } \varphi_\epsilon = \text{Hess } x_1 + \epsilon \text{Hess } x_2 = -x_1\sigma_S - \epsilon x_2\sigma_S = -\varphi_\epsilon\sigma_S,
$$

and similarly on $\Omega^-_{\tau/4}$ one has

$$
(3.18) \quad \text{Hess } \varphi_\epsilon = \text{Hess } x_1 - \epsilon \text{Hess } x_2 = -x_1\sigma_S + \epsilon x_2\sigma_S = -\varphi_\epsilon\sigma_S.
$$

Note that $0 < \tau < 1/2$ is fixed. For each $0 < \tau_n < 1$, we choose $\epsilon$ sufficiently small such that $\frac{c'}{\epsilon} \leq \frac{2}{3}\tau_n - \epsilon$. On $\Omega^+_{\tau/2} \setminus \Omega^+_{\tau/4}$, combining (3.16) we have

$$
(3.19) \quad \text{Hess } \varphi_\epsilon = \text{Hess } x_1 + \epsilon \text{Hess}(\eta x_2) \leq -x_1\sigma_S + \frac{c'}{\tau^2}\sigma_S
$$

$$
\leq \left(-x_1 + \frac{c'}{\tau} \right)\sigma_S \leq (1 - \tau_n)\varphi_\epsilon\sigma_S,
$$

and similarly on $\Omega^-_{\tau/2} \setminus \Omega^-_{\tau/4}$ one has

$$
(3.20) \quad \text{Hess } \varphi_\epsilon = \text{Hess } x_1 - \epsilon \text{Hess}(\eta x_2) \leq -x_1\sigma_S + \frac{c'}{\tau^2}\sigma_S \leq (1 - \tau_n)\varphi_\epsilon\sigma_S.
$$

This is sufficient to complete the proof. $\square$

Note that $\Lambda$ in Lemma 3.3 is strictly bounded away from zero as $\Omega$ is closed. Put $\Phi = \frac{\Lambda}{\Delta M}$, then it is a smooth positive bounded function on $M$. Let $\nabla$ be the Levi-Civita connection of $M$ as before. We choose a local orthonormal tangent frame $\{e_i\}$ on $M$ such that $\nabla e_i = 0$ at the considered point. As $\gamma$ is a harmonic map from $M$ to $\mathbb{S}^n$, combining (2.17) and Lemma 3.3 we conclude

$$
(3.21) \quad \Delta_M \phi^{-1} = \sum_{i=1}^{n} \text{Hess } \Lambda(\gamma_s e_i, \gamma_s e_i) \leq -(1 - \tau_n)\phi^{-1} \sum_{i=1}^{n} \langle \gamma_s e_i, \gamma_s e_i \rangle = -(1 - \tau_n)\phi^{-1}|A|^2.
$$
Theorem 3.4. Let $\Omega$ be the set defined in (3.14). Let $M$ be a smooth oriented complete minimal hypersurface in $\mathbb{R}^{n+1}$ with Euclidean volume growth and $\gamma(M) \subset \Omega$. Then $M$ is an affine hyperplane in $\mathbb{R}^n$.

Proof. From (3.21), there is a smooth positive bounded function $\Phi$ on $M$ satisfying

(3.22) \[ \Delta_M \Phi^{-1} \leq - \left(1 - \frac{1}{2n}\right) \Phi^{-1} |A|^2 \quad \text{on } M. \]

Namely,

(3.23) \[ \Delta_M \Phi \geq \left(1 - \frac{1}{2n}\right) \Phi |A|^2 + 2 \Phi^{-1} |\nabla \Phi|^2 \quad \text{on } M. \]

Recall Simons’ inequality [14] (see also the formula (4) in [5] for instance):

(3.24) \[ \Delta_M |A|^2 \geq -2 |A|^4 + 2 \left(1 + \frac{2}{n}\right) |\nabla |A||^2 \quad \text{on } M. \]

Now we can use the idea in [5] by Ecker-Huisken to show the flatness of $M$. For any positive constants $p, q > 0$, from (3.23)-(3.24) one has

(3.25) \[ \Delta_M (|A|^p \Phi^q) \geq \left( \left(1 - \frac{1}{2n}\right) q - p \right) |A|^{p+2} \Phi^q + p \left( p - 1 + \frac{2}{n}\right) |A|^{p-2} \Phi^q |\nabla |A||^2 
+ q (q + 1) |A|^p \Phi^{q-2} |\nabla \Phi|^2 + 2pq |A|^{p-1} \Phi^{q-1} \langle |\nabla |A|, \nabla \Phi \rangle. \]

With Young’s inequality we derive

(3.26) \[ \Delta_M (|A|^p \Phi^q) \geq \left( \left(1 - \frac{1}{2n}\right) q - p \right) |A|^{p+2} \Phi^q + p \left( \frac{p}{q + 1} - 1 + \frac{2}{n}\right) |A|^{p-2} \Phi^q |\nabla |A||^2. \]

For $p = n, q = n + 1$, we have

(3.27) \[ \Delta_M (|A|^n \Phi^{n+1}) \geq \left( \frac{1}{2} - \frac{1}{2n}\right) |A|^{n+2} \Phi^{n+1}. \]

For $p = 2n + 2, q = 2n + 4$, we have

(3.28) \[ \Delta_M (|A|^{2n+2} \Phi^{2n+4}) \geq 0. \]

Let $\Phi_*$ be a positive constant $> 1$ so that $\Phi_*^{-1} < \Phi < \Phi_*$ on $M$. From (3.27) and (3.28), for each $z \in M$ we have

(3.29) \[ |A|^{2n+2} \Phi^{2n+4}(z) \leq \frac{CM}{\rho^n} \int_{M \cap B_{\rho}(z)} |A|^{2n+2} \Phi^{2n+4} \leq \frac{CM \Phi^2_*}{\rho^n} \int_{M \cap B_{\rho}(z)} |A|^{2n+2} \Phi^{2n+2}. \]

Let $\zeta$ be a smooth positive function in $[0, \infty)$ by $\zeta(r) = 1$ for $0 \leq r \leq \rho$, $\zeta(r) = 0$ for $r \geq 2\rho$, and $|\zeta'| \leq C_n \rho^{-1}$ for $\rho \leq r \leq 2\rho$. Here, $C_n$ is a positive constant depending only on $n$. We multiply (3.27) on both sides by $|A|^n \Phi^{n+1} \zeta^{2n+2}(|X|)$, and integrate by parts in
conjunction with Young’s inequality, then
\[
\left(\frac{1}{2} - \frac{1}{2n}\right) \int_M |A|^{2n+2} \Phi^{2n+2} \zeta^{2n+2} \leq \int_M |A|^n \Phi^{n+1} \zeta^{2n+2} \Delta_M (|A|^n \Phi^{n+1})
\]
\[-\int_M |\nabla (|A|^n \Phi^{n+1})|^2 \zeta^{2n+2} - 2(n+1) \int_M |A|^n \Phi^{n+1} \zeta^{2n+1} \langle \nabla \zeta, \nabla (|A|^n \Phi^{n+1}) \rangle \]
\[
\leq (n+1)^2 \int_M |A|^{2n+2} \Phi^{2n+2} \zeta^{2n+2} |\nabla \zeta|^2
\]
\[
\leq (n+1)^2 \left( \int_M |A|^{2n+2} \Phi^{2n+2} \zeta^{2n+2} \right)^{\frac{1}{n+2}} \left( \int_M \Phi^{2n+2} |\nabla \zeta|^{2n+2} \right)^{\frac{1}{n+2}}.
\]
(Note \(n \geq 2\). The above inequality implies
\[
\text{(3.31)} \quad \int_M |A|^{2n+2} \Phi^{2n+2} \zeta^{2n+2} \leq 2^{n+2}(n+1)^{2n+2} \int_M \Phi^{2n+2} |\nabla \zeta|^{2n+2}.
\]
Combining (3.29) and (3.31) and \(\Phi^{-1} < \Phi < \Phi, |\zeta'| \leq c_n \rho^{-1}\), we have
\[
\Phi^{-2n-4} |A|^{2n+2}(z) \leq \frac{c_M \Phi^2}{\rho^n} \int_M |A|^{2n+2} \Phi^{2n+2} \zeta^{2n+2}
\]
\[
\text{(3.32)} \quad \leq 2^{n+2}(n+1)^{2n+2} c_M \Phi_n^{2n+4} \rho^{-n} \int_M |\nabla \zeta|^{2n+2}
\]
\[
\leq 2^{n+2}(n+1)^{2n+2} c_M \zeta_n^{2n+4} \rho^{-n\zeta_n-2} \mathcal{H}^n (M \cap B_{2\rho}(z) \setminus B_{\rho}(z)).
\]

Letting \(\rho \to \infty\), we get \(|A| = 0\) at \(z\), which completes the proof. \(\square\)

In all, we have proven the following rigidity result in this chapter.

**Corollary 3.5.** Let \(M\) be a smooth oriented complete minimal hypersurface in \(\mathbb{R}^{n+1}\) with Euclidean volume growth and the Gauss image \(\gamma(M) \subset K\), where \(K\) is a compact set in \(S^n \setminus \mathbb{S}^n_+\). If the support of one of tangent cones of \(M\) at infinity is the Euclidean space, then \(M\) is an affine linear space.

### 4. Regularity of Minimal Hypersurfaces

Let \(M\) be a smooth oriented minimal hypersurface in \(B_{2\rho}(0) \subset \mathbb{R}^{n+1}\) with \(\partial M \subset \partial B_{2\rho}(0)\) and the Gauss image \(\gamma(M) \subset K\), where \(K\) is a compact set in \(S^n \setminus \mathbb{S}^n_+\). Moreover, we assume
\[
\mathcal{H}^n (M \cap B_{2\rho}(0)) < \alpha \rho^n
\]
for some constant \(\alpha > 0\). For each \(X_0 \in M \cap B_\rho(0)\) and each \(0 < \rho_1 \leq \rho\), by the monotonicity formula, we have
\[
\rho_1^{-n} \mathcal{H}^n (M \cap B_{\rho_1}(X_0)) \leq \rho^{-n} \mathcal{H}^n (M \cap B_\rho(X_0)) < \rho^{-n} \mathcal{H}^n (M \cap B_{2\rho}(0)) \leq \alpha.
\]
For simplicity, we denote \(B_r = B_r(0)\) for all \(r > 0\). Now let us use the technique in the last chapter to show the following regularity theorem.

**Theorem 4.1.** Let \(K, \alpha\) be as above. There is a positive constant \(\delta_{K, \alpha}\) depending only on \(n, K, \alpha\) such that if
\[
\text{(4.3)} \quad \inf_{\xi \in S^n} \sup_{X \in M \cap B_{2\rho}} |\langle X, \xi \rangle| < \delta_{K, \alpha} \rho,
\]
then \(|A| \leq 1/\rho\) on \(M \cap B_{\rho/2}\). Here, \(A\) is the second fundamental form of \(M\).
Proof. Without loss of generality, we can assume that \( M \) is a connected smooth manifold. There is a small constant \( \tau_K > 0 \) depending only on \( K \) such that \( B_{\tau_K}(K) \), the \( \tau_K \)-neighborhood of \( K \) in \( S^n \), is still contained in \( S^n \setminus \overline{S}^{n-1}_+ \). Let \( \hat{K} \) denote \( B_{\tau_K/2}(K) \), the \( \tau_K/2 \)-neighborhood of \( K \) in \( S^n \). Let \( \Psi(t|E, \alpha) \) denote a general positive function on \((0, 1) \times \left( S^n \setminus \overline{S}^{n-1}_+ \right) \times (0, \infty) \) with \( \lim_{t \to 0} \Psi(t|E, \alpha) = 0 \) for each \( E \in S^n \setminus \overline{S}^{n-1}_+ \) and \( \alpha > 0 \). For simplicity, we fix \((\hat{K}, \alpha) \in \left( S^n \setminus \overline{S}^{n-1}_+ \right) \times (0, \infty) \), and assume \( \Psi(t) = \Psi(t|\hat{K}, \alpha) \). We allow that \( \Psi(t) \) changes from the line to the line.

We assume that there are a unit vector \( \xi \in S^n \) and a constant \( \delta > 0 \) such that \(|\langle X, \xi \rangle| \leq \delta \rho \) for each \( X \in M \cap B_{2\rho} \). From [11] (or 22.2 in [13]), the unoriented excess satisfies

\[
\int_{B_{\frac{3}{2}\rho} \cap M} (1 - \langle \nu, \xi \rangle^2) < \Psi(\delta)\rho^n, \tag{4.4}
\]

where \( \nu \) denotes the unit normal vector field of \( M \). Using Cauchy inequality, one has (compared with \( \Psi(\delta) \))

\[
\int_{B_{\frac{3}{2}\rho} \cap M} \min\{|\nu - \xi|, |\nu + \xi|\} < \Psi(\delta)\rho^n. \tag{4.5}
\]

There is a constant \( \tau > 0 \) depending only on \( K \) such that \( B_{\tau}(S^{n-2}) \cap \hat{K} = \emptyset \), where \( B_{\tau}(S^{n-2}) \) denotes the \( \tau \)-neighborhood of \( \{(0,0,x_3,\cdots,x_{n+1})| x_3^2 + \cdots + x_{n+1}^2 = 1 \} \) in \( S^n \), i.e., \( B_{\tau}(S^{n-2}) = \{(x_1,\cdots,x_{n+1}) \in S^n| x_1^2 + x_2^2 < \sin^2 \tau \} \). From (4.5), at least one of \( \xi \) and \( -\xi \) is in \( \hat{K} \subset S^n \setminus B_{\tau}(S^{n-2}) \) for the sufficiently small \( \delta > 0 \). Then both of \( \xi \) and \( -\xi \) are in \( S^n \setminus B_{\tau}(S^{n-2}) \). From Lemma 2.1 there is a bounded function \( \Theta \) on \( S^n \setminus B_{\tau}(S^{n-2}) \) with \( \Theta(\xi) = \Theta(-\xi) \) such that \( \Theta \) is smooth strictly convex on \( \hat{K} \subset M \). Put \( v = \Theta \circ \gamma \), then \( v \) is a smooth subharmonic function on \( M \) with \( 0 < v \leq v_{\sup} \) for a constant \( v_{\sup} \) depending only on \( n, K \). Denote \( v^* = \Theta(\xi) \). Combining (2.16) and (4.5), by following the argument of Lemma 3.1 for the sufficiently small \( \delta > 0 \) one has

\[
\int_{B_{\frac{3}{2}\rho} \cap M} |v - v^*| < \Psi(\delta)\rho^n. \tag{4.6}
\]

Now we claim

\[
\sup_{B_{\rho} \cap M} v \leq v^* + \Psi(\delta). \tag{4.7}
\]

Or else, there is a large positive integer \( m \) independent of \( \delta \) such that \( \sup_{B_{\rho} \cap M} v > v^* + m^{-\frac{1}{3}} \). Note that \( v_{\sup} \) does not depend on \( M \). By the monotonicity of \( \sup_{B_{\rho} \cap M} v \) on \( \rho \), there is a constant \( \rho \leq \rho_0 < \frac{\delta}{4}\rho \) such that

\[
\sup_{B_{\rho_0} \cap M} v \leq \sup_{B_{\rho_0} \cap M} v + \frac{v_{\sup}}{m}. \tag{4.8}
\]

Or else, if for all \( j \in \{0, \cdots, m - 1\} \)

\[
\sup_{B_{\rho + \frac{j\rho}{m}} \cap M} v > \sup_{B_{\rho + \frac{j\rho}{m}} \cap M} v + \frac{v_{\sup}}{m}, \tag{4.9}
\]

then

\[
\sup_{B_{\frac{3}{4}\rho} \cap M} v - \sup_{B_{\rho} \cap M} v = \sum_{j=0}^{m-1} \left( \sup_{B_{\rho + \frac{j\rho}{m}} \cap M} v - \sup_{B_{\rho + \frac{j\rho}{m}} \cap M} v \right) \geq \sum_{j=0}^{m-1} \frac{v_{\sup}}{m} = v_{\sup}. \tag{4.10}
\]
However, it’s a contradiction by the definition of \(v_{\text{sup}}\), and we obtain (4.8). Let \(q\) be a point in \(\overline{B}_{\rho_0} \cap M\) with \(v(q) = \sup_{\overline{B}_{\rho_0} \cap M} v\). Then \(v(q) + \frac{v_{\text{sup}}}{m} \geq v\) on \(\overline{B}_{\frac{\rho_0}{m}}(q) \cap M\) and \(v(q) = \sup_{\overline{B}_{\rho_0} \cap M} v > v^* + m^{-\frac{1}{2}}\). Hence, \(\left(\frac{2v_{\text{sup}}}{m} + v(q) - v^*\right)^{-1}\) is a bounded nonnegative subharmonic function on \(\overline{B}_{\frac{\rho_0}{m}}(q) \cap M\). From the mean value inequality (3.8), one has

\[
\frac{m}{2v_{\text{sup}}} \leq \frac{c_\alpha m^n}{\rho^n} \int_{M \cap \overline{B}_{\frac{\rho}{m}}(q)} \left(\frac{2v_{\text{sup}}}{m} + v(q) - v\right)^{-1},
\]

where \(c_\alpha\) is a constant depending only on \(n, \alpha\). Analog to (3.10), combining the choice of \(q\) and (4.6) one has

\[
\frac{m}{2v_{\text{sup}}} \leq \frac{c_\alpha m^n}{\rho^n} \int_{M \cap \overline{B}_{\frac{\rho}{m}}(q)} \left(\frac{m^2 v_{\text{sup}}^2}{v_{\text{sup}}^2} |v - v^*| + \left(\frac{2v_{\text{sup}}}{m} + v(q) - v^*\right)^{-1}\right)
\]

\[
\leq \frac{c_\alpha m^{n+2}}{v_{\text{sup}}^2} \Psi(\delta) + \frac{c_\alpha m^n}{\rho^n} \int_{M \cap \overline{B}_{\frac{\rho}{m}}(q)} \frac{m^2}{2}.
\]

However, (4.12) fails for the sufficiently large integer \(m\) and sufficiently small \(m^2 \Psi(\delta)\). Hence we complete the proof of (4.7).

By the definition of \(\Theta\) with the sufficiently large \(k > 0\), \(\gamma(B_{\rho} \cap M) \subset \overline{E}_{\Psi(\delta)}\), where one can find the definition of \(\overline{E}_t\) in (3.12). Namely, there is a unit constant vector \((\eta_1, \eta_2) \in \mathbb{R}^2\) such that \(\gamma(B_{\rho} \cap M)\) is contained in the closed set \(\Omega_{\Psi(\delta)}\) defined by

\[
\{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^n | x_1 \eta_1 + x_2 \eta_2 \geq -\Psi(\delta)\} \setminus B_r(\mathbb{S}^{n-2}).
\]

Let \(\Omega_0\) denote the closed set \(\lim_{t \to 0^+} \Omega_t\). From the proof of Lemma 3.3, there is a positive smooth function \(\Lambda\) on a neighborhood of \(\Omega_0\) such that \(\text{Hess} \Lambda \leq -\left(1 - \frac{1}{3n}\right) \Lambda \sigma_S\) on \(\Omega_0\). Hence, for the sufficiently small \(\delta > 0\), \(\Lambda\) is positive on \(\Omega_{\Psi(\delta)}\) satisfying

\[
\text{Hess} \Lambda \leq -\left(1 - \frac{1}{2n}\right) \Lambda \sigma_S \quad \text{on} \ \Omega_{\Psi(\delta)},
\]

and \(\Lambda, \frac{1}{\Lambda}\) are bounded on \(\Omega_{\Psi(\delta)}\) by a constant depending on \(n, K\), but independent of \(\delta\).

By following the proof of Theorem 3.4, there is a positive constant \(C_{K, \alpha}\) depending only on \(n, K, \alpha\) such that \(|\Lambda| \leq C_{K, \alpha}/\rho\) on \(M \cap \overline{B}_{3\rho/4}\). Combining (4.5), we get

\[
\sup_{M \cap \overline{B}_{2\rho/3}} \min\{|\nu - \xi|, |\nu + \xi|\} < \Psi(\delta).
\]

Note that \(M\) is a connected smooth manifold. Hence, there is a smooth function \(w\) on a subset of \(n\)-Euclidean ball \(B_{2\rho/3}(0)\) in the hyperplane perpendicular to \(\xi\) so that \(M \cap \overline{B}_{2\rho/3}(0)\) can be written as a graph of \(w\) with \(\sup_{B_{3\rho/5}(0)} |Dw| < \Psi(\delta)\). Then combining the Schauder estimates of elliptic equations, we complete the proof. \(\square\)

**Remark.** It is interesting to compare Theorem 4.1 with the regularity theorem for stable minimal hypersurfaces by Schoen-Simon (Theorem 1 in [11].)

5. **Benstein theorem for minimal hypersurfaces**

**Lemma 5.1.** Let \(M\) be a sequence of \(n\)-dimensional smooth complete oriented embedded minimal hypersurfaces in \(\mathbb{R}^{n+1}\) with uniform Euclidean volume growth. If the Gauss image
\( \gamma(M_i) \subset K \), \( K \) is a compact set in \( \mathbb{S}^n \setminus \mathbb{S}^{n-1}_+ \), and \( M_i \) converges to a nontrivial minimal variety \( T \times \mathbb{R}^{n-1} \) in the varifold sense, then \( \text{spt} T \) is a line in \( \mathbb{R}^2 \).

**Proof.** From the definition of \( M_i \), there is a constant \( \alpha > 0 \) such that
\[
(5.1) \quad \rho^{-n} \mathcal{H}^n (M_i \cap B_{\rho}(p)) < \alpha
\]
for each \( i, \rho > 0 \), and \( p \in \mathbb{R}^{n+1} \). From (2.18), integrating by parts infers that there is a constant \( c_{\alpha} \) depending only on \( n \) and \( \alpha \) such that
\[
(5.2) \quad \int_{M_i \cap B_{\rho}(p)} |A_i|^2 \leq c_{\alpha} \rho^{n-2}
\]
for all \( \rho > 0 \). Here, \( A_i \) is the second fundamental form of \( M_i \). Now we can complete the proof by following the steps in the proof of Theorem 2 of [13], in which Theorem 1 of [13] is replaced by Theorem 1.1.

Recall that \( \mathbb{S}^{n-1}_+ \) is the hemisphere of \( \mathbb{S}^{n-1} \) defined by
\[
\{(x_1, \cdots, x_{n+1}) \in \mathbb{S}^n | x_1 = 0, \ x_2 \geq 0 \}.
\]

**Theorem 5.2.** Let \( M \) be an \( n \)-dimensional smooth oriented complete embedded minimal hypersurface in \( \mathbb{R}^{n+1} \) with Euclidean volume growth. If the image under the Gauss map omits a neighborhood of \( \mathbb{S}^{n-1}_+ \), then \( M \) must be an affine hyperplane.

**Proof.** Let us prove it by contradiction. Assume that \( M \) is not affine. There is a sequence \( r_i \to \infty \) such that \( r_i^{-1} M \) converges to a minimal cone \( C \) with integer multiplicity in the varifold sense. From Corollary 3.3, the support of \( C \) is not a hyperplane. Namely, the singular set of \( C \) is not empty.

If there is a singular point \( x \in \text{spt} C \setminus \{0\} \), then we blow up the cone \( C \) at \( x \) and obtain \( C' \times \mathbb{R} \), where \( C' \) is a minimal cone with a singular point at the origin. By dimension reduction argument, there is a sequence of smooth minimal hypersurfaces \( M_i \), which is obtained from \( M \) by scalings and translations, such that \( M_i \) converges to a minimal cone \( C_s \times \mathbb{R}^{n-k} \) in the varifold sense with \( 1 \leq k \leq n \), such that \( C_s \) is a \( k \)-minimal cone in \( \mathbb{R}^{k+1} \) with the only one singular point at the origin. From Lemma 5.1, \( C_s \) has dimension \( k \geq 2 \). Since \( C_s \) is a \( k \)-minimal cone in \( \mathbb{R}^{k+1} \) with the only one singular point, then \( k \) can not be equal to 2. Namely, \( k \geq 3 \). Let \( B_1^{k+1} \) be the unit ball in \( \mathbb{R}^{k+1} \) centered at the origin \( 0^{k+1} \). Here \( 0^j \) denotes the origin of \( \mathbb{R}^j \) for each \( j \geq 1 \). Let \( \Sigma = \text{spt} C_s \cap \partial B_1^{k+1} \), then \( \Sigma \) is a smooth complete embedded hypersurface in \( \partial B_1^{k+1} \).

From the assumption of \( M \), there is a compact set \( K \) contained in the open set \( \mathbb{S}^n \setminus \mathbb{S}^{n-1}_+ \) such that the Gauss image \( \gamma(M) \subset K \). By the definition of \( M_i \), we have \( \gamma(M_i) \subset K \). So the Gauss image of the regular set of the limit \( C_s \times \mathbb{R}^{n-k} \) is contained in \( K \), namely, \( \gamma((C_s \setminus \{0^{k+1}\}) \times \mathbb{R}^{n-k}) \subset K \). Let \( \Theta \) be a smooth strictly convex function on \( K \) defined in Lemma 2.1 (see also [8]). Then from (2.18) the function \( v = \Theta \circ \gamma \) satisfies
\[
(5.3) \quad \Delta_{C_s \times \mathbb{R}^{n-k}} v \geq \kappa |A_{C_s \times \mathbb{R}^{n-k}}|^2
\]
on the regular set of \( C_s \times \mathbb{R}^{n-k} \) for some positive constant \( \kappa > 0 \), where \( \Delta_{C_s \times \mathbb{R}^{n-k}}, A_{C_s \times \mathbb{R}^{n-k}} \) are the Laplacian and the second fundamental form of \( C_s \times \mathbb{R}^{n-k} \) on the regular set of \( C_s \times \mathbb{R}^{n-k} \), respectively. In particular, \( v \) is uniformly bounded.
Let $\Delta_{C^*}, A_{C^*}$ be the Laplacian and the second fundamental form of $C^*$ on the regular set of $C^*$, respectively. Through restricting $v$ on $\text{spt} C^* \times \{0^{n-k}\}$, from (5.3) one has

(5.4) $\Delta_{C^*} v \geq \kappa |A_{C^*}|^2$

on $\text{spt} C^* \setminus \{0^{k+1}\}$. Let $\Delta_\Sigma, A_\Sigma$ be the Laplacian and the second fundamental form of $\Sigma$, respectively. Then (5.4) infers

(5.5) $\Delta_\Sigma v \geq \kappa |A_\Sigma|^2$ on $\Sigma$.

The above inequality contradicts to the maximum principle. This is sufficient to complete the proof. $\square$

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