Gauge-Invariant Noncompact Lattice Simulations

Kevin Cahill

Department of Physics and Astronomy, University of New Mexico, Albuquerque, New Mexico 87131-1156, U.S.A.

Three techniques for performing gauge-invariant, noncompact lattice simulations of nonabelian gauge theories are discussed.

In the first method, the action is not itself gauge invariant, but a kind of lattice gauge invariance is restored by random compact gauge transformations during the successive sweeps of the simulation. This method has been applied to pure SU(2) gauge theory on a $12^4$ lattice, and Wilson loops have been measured at strong coupling, $\beta = 0.5$. These Wilson loops display a confinement signal not seen in simulations performed earlier with the same action but without the random gauge transformations.

In the second method, the action is gauge symmetrized by integrations over the group manifold.

The third method is based upon a new, noncompact form of the action that is exactly invariant under lattice gauge transformations. The action is a natural discretization of the classical Yang-Mills action.

1. INTRODUCTION

In Wilson’s formulation of lattice gauge theory, the basic variables are the elements of the gauge group, not the fields of the theory. Using this formulation Creutz in 1980 displayed evidence for confinement in both abelian and nonabelian gauge theories.

The Wilson action is exactly invariant under lattice gauge transformations and reduces to the classical action when the group elements are close to the identity. Wilson’s elegant formulation is ideal at weak coupling.

But at strong coupling, the group elements are often far from the identity, and the Wilson action is quite different from the classical Yang-Mills action. It is unknown whether Wilson’s formulation is accurate at the moderately strong couplings where it is really needed. Among the symptoms of trouble are the false vacua of the Wilson action which at strong coupling affect the string tension and the confinement signals exhibited by Creutz in his simulations of abelian gauge theories.

To examine these questions, some physicists have introduced lattice actions that are noncompact discretizations of the continuum action with the fields as the basic variables. For $U(1)$ these noncompact formulations are accurate for all coupling strengths; for $SU(2)$ they agree well with perturbation theory at very weak coupling.

Patrascioiu, Seiler, Stamatescu, Wolff, and Zwanziger performed the first noncompact simulations of $SU(2)$ by discretizing the classical action and fixing the gauge. They saw a Coulomb force.

Later simulations were carried out with an action free of spurious zero modes, for which it was not necessary to fix the gauge. The Wilson loops of these simulations showed no sign of quark confinement. A likely explanation of this negative result is that these noncompact actions lack an exact lattice gauge invariance.

The first gauge-invariant noncompact simulations were carried out by Palumbo, Polikarpov, and Veselov and were based on earlier work by Palumbo et al. They saw a confinement signal. Their action contains five terms, constructed from two invariants, and involves auxiliary fields and an adjustable parameter.

The present paper discusses three ways of performing gauge-invariant noncompact simulations with an action that is a discretization of the classical Yang-Mills action without extra terms or

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fields.

In the first technique, one subjects the fields to random compact gauge transformations during the Monte-Carlo updates \( \mathbb{E} \). The random gauge transformations restore a semblance of gauge invariance \( \mathbb{12} \). This method has been used to measure Wilson loops at strong coupling, \( \beta \equiv 4/g^2 = 0.5 \), on a 12\( ^4 \) lattice in a noncompact simulation of SU(2) gauge theory without gauge fixing or fermions. In this simulation the Wilson loops fall off exponentially with the area of the loop at a rate that is over two orders of magnitude greater than the negligible rate seen in an earlier simulation with the same action but without the random gauge transformations.

The second technique is to symmetrize the action itself by subjecting it to a full set of gauge transformations. Here one inserts a gauge transformation at each vertex and integrates over the transformations. Here one inserts a gauge transformation itself by subjecting it to a full set of gauge transformations. In the plaquette with vertices \( 0, a, n \), and \( n + e_\mu + e_\nu \), the field is

\[
A_\mu^a(x) = \left( \frac{x_\mu}{a} - n_\mu \right) A^a_\mu(n + e_\nu) + (n_\nu + 1 - \frac{x_\nu}{a}) A^a_\mu(n),
\]

and the field strength is

\[
F^a_{\mu\nu}(x) = \partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) + g f^{abc} A_\mu^b(x) A_\nu^c(x).
\]

The action \( S \) is the sum over all plaquettes of the integral over each plaquette of the squared field strength,

\[
S = \sum_{\mu\nu} \frac{a^2}{2} \int dx_\mu dx_\nu F^a_{\mu\nu}(x)^2.
\]

The mean value in the vacuum of a euclidean-time-ordered operator \( Q \) is approximated by a ratio of multiple integrals over the \( A_\mu^a(n) \)’s

\[
\langle T Q(A) \rangle \approx \frac{\int e^{-S(A)} Q(A) \prod_{\mu,a,n} dA_\mu^a(n)}{\int e^{-S(A)} \prod_{\mu,a,n} dA_\mu^a(n)}
\]

which one may compute numerically. Macsyma was used to write most of the Fortran code \( \mathbb{13} \) for the present simulation.

2.2. Gauge Invariance

To restore gauge invariance, the fields are subjected to random compact gauge transformations during every sweep, except those devoted exclusively to measurements. At each vertex \( n \) a random number \( r \) is generated uniformly on the interval \( (0,1) \); and if \( r \) is less than a fixed probability, set equal to 0.5 in this work, then a random group element \( U(n) \) is picked from the group \( SU(2) \). The fields on the four links coming out of the vertex \( n \) are then subjected to the compact gauge transformation

\[
e^{-igaA_\mu^a(n)T_\mu} = e^{-igaA_\mu^a(n)T_\mu} U(n)T_\mu
\]

and those on the links entering the vertex to the transformation

\[
e^{-igaA_\mu^a(n-e_\mu)T_\mu} = U(n)e^{-igaA_\mu^a(n-e_\mu)T_\mu}
\]

At weak coupling one should use random gauge transformations that are suitably close to the identity.

2.3. Wilson Loops

The quantity normally used to study confinement in quarkless gauge theories is the Wilson loop \( W(r,t) \) which is the mean value in the vacuum of the trace of a path-and-time-ordered exponential of a line integral of the connection around an \( r \times t \) rectangle

\[
W(r,t) = \langle \text{tr} \mathcal{P} e^{-ig \oint A_\mu^a(x) dx_\mu} \rangle
\]

where \( d \) is the dimension of the generators \( T_\mu \). Although Wilson loops vanish in the exact theory \( \mathbb{14} \), Creutz ratios \( y(r,t) \) of Wilson loops defined \( \mathbb{15} \) as double differences of logarithms of
Wilson loops are finite. For large $t$, $\chi(r, t)$ approximates $a^2$ times the force between a quark and an antiquark separated by the distance $r$.

In this simulation the data are not yet sufficient to allow one to determine the Creutz ratios beyond the $3 \times 4$ loop. The Wilson loops therefore have been fitted to an expression involving Coulomb, perimeter, scale, and area terms.

### 2.4. Measurements and Results

It will be useful to compare this simulation with an earlier one \[9\] in which the fields were not subjected to random gauge transformations. Both simulations were done on a $12^4$ periodic lattice with a heat bath. The earlier simulation consisted of 20 independent runs with cold starts. The first run had 25,000 thermalizing sweeps at inverse coupling $\beta = 2$ followed by 5000 at $\beta = 0.5$; the other nineteen runs began at $\beta = 0.5$ with 20,000 thermalizing sweeps. There were 59,640 Parisi-assisted \[16\] measurements, 20 sweeps apart.

The present simulation with random gauge transformations is very noisy. It consists of 21 runs of which 20 were from cold starts and one from a random start. Each began with 20,000 thermalizing sweeps. After the first few hundred thermalizing sweeps in each run, the average value of the action was stable.

Wilson loops have been measured every five sweeps for a total of 1,313,202 measurements. The values of the Wilson loops so obtained are listed in the table. The errors have been estimated by the jackknife method, with all measurements in bins of 4000 considered to be independent.

The Wilson loops of the gauge-invariant simulation fall off much faster with increasing loop size than do those of the earlier simulation. Because the data do not accurately determine all the Creutz ratios, I have fitted both sets of loops, including the non-diagonal loops, to the formula

$$W(r, t) \approx e^{a + b(r/t + t/r) - 2c(r + t) - d rt}$$

in which $a$ is a scale factor, $b$ a Coulomb term, $c$ a perimeter term, and $d$ an area term. For the simulation without random gauge transformations, I found $a \approx 0.26$, $b \approx 0.20$, $c \approx 0.39$, and $d \approx 0.00$. For the simulation with random gauge transformations, I found $a \approx 0.57$, $b \approx 0.15$, $c \approx 0.51$, and $d \approx 0.18$. In the gauge-invariant simulation, the coefficient of the area-law term is over two orders of magnitude larger than in the earlier simulation which lacked gauge invariance.

| $\frac{r}{n} \times \frac{t}{n}$ | Not invariant | Invariant |
|---|---|---|
| 1 × 1 | 0.402330(6) | 0.254558(5) |
| 1 × 2 | 0.208096(5) | 0.082528(6) |
| 2 × 2 | 0.085426(4) | 0.018709(4) |
| 1 × 3 | 0.111792(5) | 0.027715(5) |
| 2 × 3 | 0.040008(3) | 0.005224(3) |
| 3 × 3 | 0.018080(2) | 0.001431(5) |
| 1 × 4 | 0.060451(5) | 0.009352(3) |
| 2 × 4 | 0.019212(2) | 0.001502(5) |
| 3 × 4 | 0.008517(1) | 0.000407(3) |
| 4 × 4 | 0.003993(1) | 0.000121(5) |
| 1 × 5 | 0.032726(5) | 0.003152(3) |
| 2 × 5 | 0.009269(1) | 0.000434(5) |
| 3 × 5 | 0.004041(1) | 0.000118(3) |
| 4 × 5 | 0.001889(1) | 0.000039(5) |
| 5 × 5 | 0.000893(1) | 0.000013(3) |
| 1 × 6 | 0.017717(4) | 0.001064(5) |
| 2 × 6 | 0.004474(1) | 0.000125(1) |
| 3 × 6 | 0.001919(1) | 0.000033(1) |
| 4 × 6 | 0.000896(1) | 0.000009(1) |
| 5 × 6 | 0.000423(0) | 0.000003(1) |
| 6 × 6 | 0.000201(0) | 0.000003(1) |

To exhibit the renormalized quark-antiquark potential, I have plotted in the figure the negative logarithms $-\log_{10} [e^{-b(r/t + t/r) + 2c(r + t)} W(r, t)]$ of the Wilson loops with the Coulomb and perimeter terms removed. Apart from the uncertain value of $W(6, 6)$, the loops of the gauge-invariant simulation, represented by bullets, seem to display an area law; whereas the larger loops of the earlier simulation, represented by circles, show an essentially flat potential.
Figure 1. The negative logarithms of Wilson loops with the Coulomb and perimeter factors canceled are plotted against the area $rt$ of the loop. The loops of the gauge-invariant simulation are represented by bullets; those of the earlier simulation by circles.

3. THE SECOND METHOD

While the random gauge transformations of the first method can restore a measure of gauge invariance, it clearly would be better to use an action that is itself exactly gauge invariant or has been made so. In the second method of performing gauge-invariant noncompact simulations, one symmetrizes an action that is not itself gauge invariant. To make the action gauge invariant, one inserts a general gauge transformation at each vertex of the lattice and then independently integrates over the group manifold at each vertex using the Haar measure.

One may take any suitable noncompact action as a starting point, for instance the action used in the first method. Since the action is positive definite, the result of the integration over the parameters of the group will not be zero. In practice one need only integrate over the vertices of the plaquettes that contain the arbitrary link that is to be updated, fourteen in the case of the action (8). The relation between the gauge field before and after the gauge transformation is transcendental in the case of compact gauge transformations and affine in the case of noncompact gauge transformations. Obviously the latter kind would be easier to implement.

4. THE THIRD METHOD

The third method of performing gauge-invariant noncompact simulations is to start with a noncompact lattice action that is intrinsically gauge invariant.

4.1. Gauge Invariance

Let us first decide what constitutes a gauge transformation in this third method. A good way to do that is to start with the fermionic action density which in the continuum is

$$\bar{\psi}(i\gamma_{\mu}D_{\mu} - m)\psi.$$  \hspace{2cm} (9)

A suitable discretization of the free part of that action density is

$$\frac{i}{a}\bar{\psi}(n)\gamma_{\mu}\left[\psi(n + e_{\mu}) - \psi(n)\right]$$  \hspace{2cm} (10)

in which $n$ is a four-vector of integers representing an arbitrary vertex of the lattice, $e_{\mu}$ is a unit vector in the $\mu$th direction, and $a$ is the lattice spacing. The product of Fermi fields at the same point is gauge invariant as it stands. The other product of Fermi fields becomes gauge invariant if we insert a matrix $A_{\mu}(n)$ of gauge fields

$$\frac{i}{a}\bar{\psi}(n)\gamma_{\mu}\left[(1 + iagA_{\mu}(n))\psi(n + e_{\mu}) - \psi(n)\right]$$  \hspace{2cm} (11)

that transforms under a gauge transformation represented by the group elements $U(n)$ and $U(n + e_{\mu})$ in such a way that

$$1 + iagA'_{\mu}(n) = U(n)[1 + iagA_{\mu}(n)]U^{-1}(n + e_{\mu}).$$ \hspace{2cm} (12)

The required behavior is

$$A'_{\mu}(n) = U(n)A_{\mu}(n)U^{-1}(n + e_{\mu})$$

$$+ \frac{i}{ag}U(n)\left[U^{-1}(n) - U^{-1}(n + e_{\mu})\right].$$ \hspace{2cm} (13)

Under such a gauge transformation, the usual gauge-field matrix $A_{\mu}(n) = T_{a}A_{a}^{\mu}(n)$ remains in the Lie algebra only to first order in the lattice
spacing $a$. Three ways of coping with this problem are outlined in Sec. 4.2.

Let us define the lattice field strength $F_{\mu\nu}(n)$ as

$$F_{\mu\nu}(n) = \frac{1}{a} [A_\mu(n + e_\nu) - A_\mu(n)]$$

$$- \frac{1}{a} [A_\nu(n + e_\mu) - A_\nu(n)] + ig [A_\mu(n) A_\nu(n + e_\nu) - A_\mu(n) A_\nu(n + e_\mu)]$$

(14)

which reduces to the continuum Yang-Mills field strength in the limit $a \to 0$. Under the aforementioned gauge transformation (13), this field strength transforms as

$$F'_{\mu\nu}(n) = U(n) F_{\mu\nu}(n) U^{-1}(n + e_\mu + e_\nu).$$

(15)

The field strength $F_{\mu\nu}(n)$ is antisymmetric in the indices $\mu$ and $\nu$, but it is not hermitian. To make a positive plaquette action density, we write

$$\frac{1}{4k} \text{Tr} F_{\mu\nu}(n) F_{\mu\nu}(n),$$

(16)

in which it is assumed that the generators $T_a$ of the gauge group have been orthonormalized as $\text{Tr}(T_a T_b) = k \delta_{ab} i$. Because $F_{\mu\nu}(n)$ transforms covariantly (13), this action density is exactly invariant under the noncompact gauge transformation (13).

### 4.2. Three Interpretations

In general the gauge transformation (13) maps the usual matrix of gauge fields $A_\mu(n) = T_a A_{\mu a}(n)$ into a matrix that lies outside the Lie algebra of the gauge group. I shall now outline three responses to this problem.

The first response is to note that for group elements $U(n)$ of the form

$$U(n) = e^{-iga \omega a T_a},$$

(17)

the gauge transformation (13) to lowest (zeroth) order in the lattice spacing $a$, does keep the gauge-field matrix in the Lie algebra. Thus one may use the usual matrix $A_\mu(n) = T_a A_{\mu a}(n)$ of gauge fields and retain invariance under infinitesimal gauge transformations.

The second response is to accept the fact that the gauge-field matrix $A_\mu(n)$ will be mapped by the gauge transformation (13) into a matrix that lies outside the Lie algebra of the gauge group and to use this more-general matrix in the simulation. Thus one may use the action (14) in which the field strength (14) is defined in terms of gauge-field matrices that are of the more general form

$$A_\mu(n) = VA_\mu^0(n)W^{-1} + \frac{i}{ag} V \left( V^{-1} - W^{-1} \right)$$

(18)

where $A_\mu^0(n)$ is a matrix of gauge fields defined in the usual way, $A_\mu^0(n) \equiv T_a A_{\mu a}^0(n)$. Here the group elements $V$ and $W$ associated with the gauge field $A_\mu(n)$ are unrelated to those associated with the neighboring gauge fields $A_\mu(n + e_\nu)$, $A_\nu(n)$, and $A_\nu(n + e_\mu)$. I intend to test this method in the near future.

The third response is to take a cue from the transformation rule (13) and to represent the quantity $1 + i ga A_\mu(n)$ as a element $L_\mu(n)$ of the gauge group. In this case the matrix $A_\mu(n)$ of gauge fields is related to the link $L_\mu(n)$ by

$$A_\mu(n) = \frac{[L_\mu(n) - 1]}{iga}$$

(19)

and the action (16) defined in terms of the field strength (14) with gauge-field matrix (13) is, mirabile dictu, Wilson’s action

$$S = \frac{k - \Re(\text{Tr}[L_\mu(n) L_\nu(n + e_\mu)L_{\mu a}^+(n + e_\nu)L_{\mu b}^+(n)])}{2a^2 g^2 k}.$$  

(20)

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