GLOBAL WELL-POSEDNESS FOR KDV IN SOBOLEV SPACES OF NEGATIVE INDEX

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Abstract. The initial value problem for the Korteweg-deVries equation on the line is shown to be globally well-posed for rough data. In particular, we show global well-posedness for initial data in $H^s(\mathbb{R})$ for $-\frac{3}{10} < s$.

1. Introduction

Consider the initial value problem for the Korteweg-deVries (KdV) equation

$$
\begin{cases}
\partial_t u + \partial_x^3 u + \frac{1}{2} \partial_x (u^2) = 0, & x \in \mathbb{R}, \\
u(0) = \phi,
\end{cases}
$$

(1.1)

for rough initial data $\phi \in H^s(\mathbb{R})$, $s < 0$. This problem is known [3] to be locally well-posed provided $-\frac{1}{4} < s$. For $s \geq 0$, the local result and $L^2$ norm conservation imply (1.1) is globally well-posed [3]. Recently, a direct adaptation [2] of Bourgain’s high-low frequency technique [1] showed (1.1) is globally well-posed for $\phi \in H^s \cap H^a$ for certain $s, a < 0$. A modification of the high-low frequency technique, first used in [8], is presented in this paper which establishes global well-posedness of (1.1) in $H^s(\mathbb{R})$, $-\frac{3}{10} < s$.

A subsequent paper [6] will establish that (1.1) is globally well-posed in $H^s(\mathbb{R})$ for $-\frac{1}{4} < s$. The simplicity of the argument presented here may extend more easily to other situations, such as in our treatment [5] of cubic NLS on $\mathbb{R}^2$ and NLS with derivative in $\mathbb{R}$ [4].

The Multiplier operator $I$

Let $s < 0$ and $N \gg 1$ be fixed. Define the Fourier multiplier operator

$$
\hat{I} u(\xi) = m(\xi) \hat{u}(\xi), \quad m(\xi) = \begin{cases}
1, & |\xi| < N, \\
\frac{1}{N-|\xi|}, & |\xi| \geq 10N
\end{cases}
$$

(1.2)

with $m$ smooth and monotone. The operator $I$ (barely) maps $H^s(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$. Observe that on low frequencies $\{\xi : |\xi| < N\}$, $I$ is the identity operator. Note also that $I$ commutes with differential operators. The operator $I^{-1}$ is the Fourier multiplier operator with multiplier $\frac{1}{m(\xi)}$.

An almost $L^2$ conservation property of (1.1)

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Let $\phi \in H^s(\mathbb{R})$, $-\frac{3}{2} < s < 0$ in (1.1). There is a $\delta = \delta(||\phi||_{H^s}) > 0$ such that (1.1) is well-posed for $t \in [0, \delta]$. We observe using the Fundamental Theorem of Calculus, the equation, and integration by parts that

$$\|Iu(\delta)\|^2_{L^2} = \|Iu(0)\|^2_{L^2} + \int_0^\delta \frac{d}{d\tau}(Iu(\tau), Iu(\tau))d\tau,$$

$$= \|Iu(0)\|^2_{L^2} + 2 \int_0^\delta (Iu(\tau), Iu(\tau))d\tau,$$

$$= \|Iu(0)\|^2_{L^2} + 2 \int_0^\delta (I(-u_{xxx} - \frac{1}{2}\partial_x[u^2])Iu(\tau))d\tau,$$

$$= \|Iu(0)\|^2_{L^2} + \int_0^\delta (I(-\partial_x[u^2]), Iu)d\tau.$$

Finally, we add $0 = \int_0^\delta \partial_x(I(u)^2)I(u)d\tau$ to observe

$$(1.3) \quad \|Iu(\delta)\|^2_{L^2} = \|Iu(0)\|^2_{L^2} + \int_0^\delta \partial_x \{ (I(u))^2 - I(u^2) \} Iu dx d\tau.$$

This last step enables us to take advantage of some internal cancellation. We apply Cauchy-Schwarz as in [10] and bound the integral above by

$$(1.4) \quad \left\| \partial_x \{ (I(u))^2 - I(u^2) \} \right\|_{X_{\delta,0}^{-\frac{1}{2}+}} \||Iu||_{X_{\delta,0}^{\frac{1}{2}+}}.$$

Remark 1. An effort to find a term providing more cancellation than $\int_0^\delta \partial_x(I(u)^2)I(u)d\tau$ used above led to the general procedure described in [6].

**Proposition 1.** *(A variant of local well-posedness)* The initial value problem (1.1) is locally well-posed in the Banach space $I^{-1}L^2 = \{ \phi \in H^s \text{ with norm } \|I\phi\|_{L^2} \}$, with existence lifetime $\delta$ satisfying

$$(1.5) \quad \delta \gtrsim \|I\phi\|_{L^2}^{-\alpha}, \text{ for some } \alpha > 0,$$

and moreover

$$(1.6) \quad \|Iu\|_{X_0^{\delta,0}_{-\frac{1}{2}+}} \leq C\|I\phi\|_{L^2}.$$

This proposition is not difficult to prove using the argument in [6]. Using Duhamel’s formula and $X_{s,b}$ space properties reduces matters to proving the bilinear estimate

$$(1.7) \quad \|\partial_x I(uv)\|_{X_{0,-\frac{1}{2}+}} \leq C\|Iu\|_{X_{0,-\frac{1}{2}+}}\|Iv\|_{X_{0,-\frac{1}{2}+}}$$

to obtain the contraction. The space-time norm bound is then implied by the contraction estimate. The estimate (1.7) follows from the next proposition and the bilinear estimate of Kenig, Ponce and Vega [9].

**Proposition 2.** *(Extra smoothing)* The bilinear estimate

$$(1.8) \quad \|\partial_x \{ (I(u)I(v) - I(uv)) \}\|_{X_{0,-\frac{1}{2}-}} \leq CN^{-\frac{1}{2}+}\|Iu\|_{X_0^{\delta,0}_{\frac{1}{2}+}}\|Iv\|_{X_0^{\delta,0}_{\frac{1}{2}+}}.$$

holds.
Recall the bilinear estimate \( \|\partial_x (uv)\|_{X^{0,\frac{1}{2}+}} \leq C \|u\|_{X^{0,\frac{1}{2}+}} \|v\|_{X^{0,\frac{1}{2}+}} \) from (1.3). Proposition 2 reveals a smoothing beyond the recovery of the first derivative for the particular quadratic expression encountered above in (1.3). We prove Proposition 2 in the next section.

The required pieces are now in place for us to give the proof of global well-posedness of (1.1) in \( H^s(\mathbb{R}) \), \( \frac{3}{10} < s \). Global well-posedness of (1.1) will follow if we show well-posedness on \([0,T]\) for arbitrary \( T > 0 \). We renormalize things a bit via scaling. If \( u \) solves (1.1) then \( u_\lambda(x,t) = (\frac{1}{\lambda})^2 u(\frac{x}{\lambda}, \frac{t}{\lambda^2}) \) solves (1.1) with initial data \( \phi_\lambda(x,t) = (\frac{1}{\lambda})^2 \phi(\frac{x}{\lambda}) \). Note that \( u \) exists on \([0,T]\) if and only if \( u_\lambda \) exists on \([0,\lambda^2 T]\). A calculation shows that

\[
\|I \phi_\lambda\|_{L^2} \leq C \lambda^{-\frac{3}{2}-s} N^{-s}\|\phi\|_{H^s}.
\]

Here \( N = N(T) \) will be selected later but we choose \( \lambda = \lambda(N) \) right now by requiring

\[
C \lambda^{-\frac{3}{2}-s} N^{-s}\|\phi\|_{H^s} \sim 1 \implies \lambda \sim N^{-\frac{3}{4+2s}}.
\]

We now drop the \( \lambda \) subscript on \( \phi \) by assuming that

\[
\|I \phi\|_{L^2} = \epsilon_0 \ll 1
\]

and our goal is to construct the solution of (1.1) on the time interval \([0,\lambda^2 T]\).

The local well-posedness result of Proposition 1 shows we can construct the solution for \( t \in [0,1] \) if we choose \( \epsilon_0 \) small enough. The almost \( L^2 \) conservation property shows \( \|Iu(1)\|_2^2 \leq \|Iu(0)\|_2^2 + N^{-\frac{3}{2}+} \|Iu\|_{X^{0,\frac{1}{2}+}}^3 \). Using (1.6) and (1.11) gives

\[
\|Iu(1)\|_2^2 \leq \epsilon_0^2 + N^{-\frac{3}{2}+}.
\]

We can iterate this process \( N^{\frac{3}{2}+} \) times before doubling \( \|Iu(t)\|_{L^2} \). Therefore, we advance the solution by taking \( N^{\frac{3}{2}+} \) time steps of size \( O(1) \). We now restrict \( s \) by demanding that

\[
N^{\frac{3}{2}+} \gtrsim \lambda^3 T = N^{\frac{3}{4+2s}} T
\]

is ensured for large enough \( N \), so \( s > -\frac{3}{10} \).

2. Proof of the bilinear smoothing estimate

This section establishes Proposition 2. We distinguish the very low frequencies \( \{\xi : |\xi| \lesssim 1\} \), the low frequencies \( \{\xi : 1 \lesssim |\xi| \lesssim \frac{1}{2}N\} \) and the high frequencies \( \{\xi : \frac{1}{2}N \lesssim |\xi|\} \). Decompose the factor \( u \) in the bilinear estimate by writing \( u = u vl + u l + u h \) with \( \hat{u}_l \) supported on the low frequencies and similarly for the very low and high frequency pieces. We decompose \( v \) the same way. Since \( I \) is the identity operator on the low and very low frequencies, we can assume one of the factors \( u, v \) in the estimate to be shown has its Fourier transform supported in the high frequencies. Symmetry allows us to assume \( u = u_h \) and we need to consider the three possible interactions of \( u_h \) with \( v_{vl}, v_{l}, v_{h} \). Finally, since we are considering (weighted) \( L^2 \) norms, we can replace \( \hat{u} \) and \( \hat{v} \) by \( |\hat{u}| \) and \( |\hat{v}| \). Assume therefore that \( \hat{u}, \hat{v} \geq 0 \).

Very low/high interaction
An explicit calculation shows that
\[ (2.1) \quad \mathcal{F}(\partial_x\{I(u_h v_{cd}) - I(u_h v_{cd})\})(\xi) = \int \xi [m(\xi) - m(\xi_1)] \hat{u}_h(\xi_1) \hat{v}_{cd}(\xi_2), \]
where \( \mathcal{F} \) denotes the Fourier transform. The mean value theorem gives
\[ |m(\xi) - m(\xi_1)| \leq |m'(\xi_1)||\xi_1|, \]
which may be interpolated with the trivial estimate to give
\[ (2.2) \quad |m(\xi) - m(\xi_1)| \leq C N^{-s} |\xi_1|^{\alpha} |\xi_2|^\theta \]
for \( 0 \leq \theta \leq 1 \). Recall that \( m \) was defined to be smooth and monotone in \([1,2]\).

Therefore, upon defining \( \mathcal{F}(\nabla^\theta f)(\xi) = |\xi|^\theta \hat{f}(\xi) \), we can write
\[ |\mathcal{F}(\partial_x \{I(u_h v_{cd}) - I(u_h v_{cd})\})(\xi)| \leq |\mathcal{F}(\partial_x (\nabla^{-\theta} I(u_h)(\nabla^\theta v_{cd})))(\xi)|. \]

We now estimate the left side of the bilinear estimate in this interaction by
\[ (2.3) \quad \|\partial_x (\nabla^{-\theta} I(u_h))(\nabla^\theta v_{cd})\|_{X_{0,\frac{1}{4}+}} \]
and by the bilinear estimate of Kenig, Ponce and Vega
\[ (2.4) \quad \leq C \|\nabla^{-\theta} I(u_h)\|_{X_{0,\frac{1}{4}}} \|\nabla^\theta v_{cd}\|_{X_{0,\frac{1}{4}+}}. \]
The frequency support of \( v_{cd} \) shows that \( \|\nabla^\theta v_{cd}\|_{X_{0,\frac{1}{4}+}} \lesssim \|v_{cd}\|_{X_{0,\frac{1}{4}+}} \). A moments thought shows
\[ (2.5) \quad \|\nabla^{-\theta} I(u_h)\|_{X_{0,\frac{1}{4}+}} \leq N^{-\theta} \|I(u_h)\|_{X_{0,\frac{1}{4}+}} \]
and the claim of the Proposition follows for the (very low)(high) interaction by choosing \( \theta > \frac{3}{4} \).

**Low/high interaction**

The preceding calculations reduce matters to controlling
\[ (2.6) \quad \|\partial_x \nabla^{-\theta} I(u_h)(\nabla^\theta v_l)\|_{X_{0,\frac{1}{4}+}} \]
and we know that \( \hat{u}_h \) and \( \hat{v}_l \) are supported outside the very low frequencies.

**Lemma 1.** Assume \( \hat{u} \) and \( \hat{v} \) are supported outside \( \{|\xi| < 1\} \). Then
\[ (2.7) \quad \|\partial_x (uv)\|_{X_{-\gamma_1,\frac{1}{4}+}} \leq C \|u\|_{X_{-\gamma_1,\frac{1}{4}+}} \|v\|_{X_{-\gamma_2,\frac{1}{4}+}} \]
provided
\[ \alpha - (\gamma_1 + \gamma_2) < \frac{3}{4}, \]
\[ \alpha - \gamma_i < \frac{1}{2}, \quad i = 1, 2. \]

We will apply the lemma momentarily with \( \alpha = 0, \gamma_1 = \gamma_2 = -\frac{3}{4} \).

The proof of the lemma is contained in the proof of Theorem 2 in \([3]\). In particular, the support properties on \( \hat{u} \), \( \hat{v} \) reduce matters to considering Cases A.3, A.4, A.6, B.3, B.4, B.5 and B.6 in \([3]\). The restriction \( \alpha - (\gamma_1 + \gamma_2) < \frac{3}{4} \) arises in Case A.4.c.ii of \([3]\) which is the region containing the counterexample of \([3]\). Case B.4.b of \([3]\) requires the other condition \( \alpha - \gamma_i < \frac{1}{2} \).
The lemma applied to (2.6) gives
\[
\leq C \|\nabla^{-\theta} I(u_h)\|_{X_{-\frac{3}{8}+\frac{1}{4}}} \|\nabla^\theta v_l\|_{X_{-\frac{3}{8}+\frac{1}{4}}}.
\]
Setting \(\theta = \frac{3}{8}\) leaves
\[
C \|\nabla^{-\frac{3}{4}} I(u_h)\|_{X_0,1} \|v_l\|_{X_0,1} \leq CN^{-\frac{3}{4}} \|I(u_h)\|_{X_0,1} \|v_l\|_{X_0,1},
\]
which was to be shown.

**High/high interaction**

In this region of the interaction, we do not take advantage of any cancellation and estimate the difference with the triangle inequality
\[
\|\partial_x \{I(u_h)I(v_h)\}\|_{X_{0,-\frac{1}{4}}^1} + \|\partial_x \{I(u_hv_h)\}\|_{X_{0,-\frac{1}{4}}^1}.
\]
For the first contribution we use the lemma to get
\[
(I(u_h))\|_{-\frac{3}{8}+\frac{1}{4}} \|I(v_h)\|_{-\frac{3}{8}+\frac{1}{4}} \leq N^{-\frac{3}{4}} \|I(u_h)\|_{X_0,1} \|I(v_h)\|_{X_0,1},
\]
The second contribution is bounded by throwing away \(I\) and applying the lemma,
\[
\|\partial_x \{u_hv_h\}\|_{X_{0,-\frac{1}{4}}^1} \leq \|u_h\|_{X_{-\frac{3}{8}+\frac{1}{4}}} \|u_h\|_{X_{-\frac{3}{8}+\frac{1}{4}}},
\]
\[
\leq N^{-\frac{3}{4}+s} \|u_h\|_{X_{s,\frac{1}{4}}} N^{-\frac{3}{4}+s} \|v_h\|_{X_{s,\frac{1}{4}}},
\]
\[
\leq N^{-\frac{3}{4}} \|u_h\|_{X_{0,\frac{1}{4}}} \|v_h\|_{X_{0,\frac{1}{4}}},
\]

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