INTERIOR AND BOUNDARY REGULARITY CRITERIA FOR THE
6D STEADY NAVIER-STOKES EQUATIONS

SHUAI LI, WENDONG WANG

Abstract. It is shown in this paper that suitable weak solutions to the 6D steady incompressible Navier-Stokes are Hölder continuous at 0 provided that
\[ \int_{B_1} |u(x)|^3 \, dx + \int_{B_1} |f(x)|^q \, dx \]
\[ \text{or} \]
\[ \int_{B_1} |\nabla u(x)|^2 \, dx + \int_{B_1} |\nabla u(x)|^2 \, dx \left( \int_{B_1} |u(x)| \, dx \right)^2 + \int_{B_1} |f(x)|^q \, dx \]
with \( q > 3 \) is sufficiently small, which implies that the 2D Hausdorff measure of the set of singular points is zero. For the boundary case, we obtain that 0 is regular provided that
\[ \int_{B_1^+} |u(x)|^3 \, dx + \int_{B_1^+} |f(x)|^q \, dx \]
\[ \text{or} \]
\[ \int_{B_1^+} |\nabla u(x)|^2 \, dx + \int_{B_1^+} |f(x)|^q \, dx \]
is sufficiently small. These results improve previous regularity theorems by Dong-Strain [8], Indiana Univ. Math. J., 2012, Dong-Gu [7], J. Funct. Anal., 2014, and Liu-Wang [29], J. Differential Equations, 2018, where either the smallness of the pressure or the smallness on all balls is necessary.

Keywords: steady Navier-Stokes equations, local suitable weak solutions, interior regularity criteria, boundary regularity criteria.

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1. Introduction

Consider the following 6D steady incompressible Navier-Stokes equations on \( \Omega \subset \mathbb{R}^6 \) as follows:
\[ \text{(SNS)} \left\{ \begin{array}{l}
- \Delta u + u \cdot \nabla u = -\nabla \pi + f, \\
\nabla \cdot u = 0,
\end{array} \right. \tag{1.1} \]
where \( u \) represents the fluid velocity field, \( \pi \) is a scalar pressure.

The \( \varepsilon \)--regularity analysis of the above equations is started by Struwe’s question in [36, 37], where he obtained partial regularity for \( N = 5 \) by regularity methods of elliptic systems (c.f. Morrey [30] and Giaqinta [18]) and asked if analogous partial regularity results hold in spacial dimension \( N > 5 \). Later, the result of Struwe was extended to the boundary case by Kang [23]. Recently interior regularity results in 6D are obtained by Dong-Strain [8], and they proved 0 is regular if
\[ \limsup_{r \to 0} r^{-2} \int_{B_r} |\nabla u|^2 \, dx \leq \varepsilon_0. \]

Moreover, similar boundary regularity results are obtained in Dong-Gu [7] and Liu-Wang [29] by different methods, respectively. For more developments, in a series of papers by Frehse and Ruzicka [10, 11, 12, 13], the existence on a class of special regular solutions of (1.1) was obtained for the five-dimensional and higher dimensional case. Gerhardt [17] obtained the regularity of weak solutions under the four-dimensional case. More references, we refer to Li-Yang [28] for the existence of regular solutions.
of high dimensional Navier-Stokes equations. At last, we refer to [14] by Farwig-Sohr for existence and regularity criteria for weak solutions to inhomogeneous Navier-Stokes equations.

Recall that these so-called ε-regularity criteria can be traced back to the well-known work by Caffarelli-Kohn-Nirenberg [1] for the analysis of suitable weak solutions of the three dimensional time-dependent Navier-Stokes equations, where they showed that the set $S$ of possible interior singular points of a suitable weak solution is one-dimensional parabolic Hausdorff measure zero by improving Scheffer’s results in [33, 34, 35]. More references on simplified proofs and improvements, we refer to Lin [26], Ladyzhenskaya-Seregin [27], Tian-Xin [39], Seregin [31], Gustafson-Kang-Tsai [21], Vasseur [40], Kukavica [25], Wang-Zhang [42] and the references therein. Motivated by the recent interior regularity by Wolf [43], where the author proved $\int_{Q_1} |u(x)|^3 dx \leq \varepsilon_0$ in one scale can imply the regularity via pressure decomposition of Stokes equation. Also, we refer to Chae-Wolf [2] and [22, 41] for some recent progress. One can ask naturally:

“Whether the smallness of the velocity in a ball can ensure the interior or boundary regularity of the 6D steady Navier-Stokes equations?”

In this note, we try to investigate this issue and answer these questions.

After finishing this paper, the authors have become to know that, very recently, Cui [4] showed that local interior regularity and boundary regularity in one scale for the 5D steady Navier-Stokes equations via Campanatos method as Dong-Wang [9]. However, we considered the 6D case, which is the largest dimension, and used the Wolf’s decomposition of the pressure for the interior estimate and Liu-Wang’s line for the boundary case.

At first, let us introduce the definition of suitable weak solutions in the interior domain.

**Definition 1.1.** Let $\Omega \subset \mathbb{R}^6$ be an open domain. $(u, \pi)$ is said to be a suitable weak solution to the steady Navier-Stoks equations (1.1) in $\Omega$, if the following conditions hold.

(i) $u \in H^1(\Omega)$, $\pi \in L^{\frac{6}{5}}(\Omega)$, $f \in L^q(\Omega)$, $q > 3$;

(ii) $(u, \pi)$ satisfies the equations (1.1) in the sense of distribution sense;

(iii) $u$ and $\pi$ satisfy the local energy inequality

$$2 \int_{\Omega} |\nabla u|^2 \phi dx \leq \int_{\Omega} \left[ |u|^2 \Delta \phi + u \cdot \nabla \phi(|u|^2 + 2\pi) \right] + 2fu \phi dx,$$

(1.2)

for any nonnegative $C^\infty$ test function $\phi$ vanishing at the boundary $\partial \Omega$.

The existence of such a suitable weak solution can be found in [12]. The major concern of this paper is the regularity and the main results can be stated as follows:

**Theorem 1.2.** Let $(u, \pi)$ be a suitable weak solution to (1.1) in $B_1$. Then $0$ is a regular point of $u$, if there exists a small positive constant $\varepsilon$ such that the following conditions hold,

$$r^{-3} \int_{B_r} |u(x)|^3 dx + r^{3q-6} \int_{B_r} |f(x)|^q dx < \varepsilon,$$
for some \( r \in (0, 1) \).

**Remark 1.3.** The regularity criteria above for the 6D steady Navier-Stokes equations generalize recent interior regularity results by Dong-Strain [8], where the pressure is small:

\[
\int_{B_1} |u(x)|^3 + |\pi(x)|^{\frac{3}{2}} dx + \int_{B_1} |f(x)|^2 dx \leq \varepsilon_0.
\]

Although the authors [43, 2, 22, 41] proved \( \int_{Q_1} |u(x)|^3 dx \leq \varepsilon_0 \) in one scale can imply the regularity for the time-dependent Navier-Stokes equations, however it seems to be difficult for the regularity by only assuming \( \int_{Q_1} |\nabla u(x)|^2 dx \leq \varepsilon_0 \) in one scale. Here for the steady equations, we have the following criterion:

**Theorem 1.4.** Let \((u, \pi)\) be a suitable weak solution to (1.1) in \( B_1 \). Then 0 is a regular point of \( u \), if there exists a small positive constant \( \varepsilon \) such that the following conditions hold,

\[
\left( r^{-5} \int_{B_r} |u(x)| dx \right)^2 \left( r^{-2} \int_{B_r} |\nabla u(x)|^2 dx \right) + r^{-2} \int_{B_r} |\nabla u(x)|^2 dx + r^{3q-6} \int_{B_r} |f(x)|^q dx < \varepsilon,
\]

for some \( r \in (0, 1) \).

The theorem immediately implies the 2D Hausdorff measure of the set of singular points of \((u, \pi)\) in \( B_1 \) is equal to zero, and we omitted the proof, since it’s standard as in [8].

Second, let us introduce the definition of suitable weak solutions near the boundary.

**Definition 1.5.** Let \( \Omega \subset \mathbb{R}^6 \) be an open domain, and \( \Gamma \subset \partial \Omega \) be an open set. \((u, \pi)\) is said to be a suitable weak solution to the steady Navier-Stokes equations (1.1) in \( \Omega \) near the boundary \( \Gamma \), if the following conditions hold.

(i) \( u \in H^1(\Omega) \), \( \pi \in L^{\frac{3}{2}}(\Omega) \), \( f \in L^6(\Omega) \);

(ii) \((u, \pi)\) satisfies the equations (1.1) in the sense of distribution sense and the boundary condition \( u|_{\Gamma} = 0 \) holds;

(iii) \( u \) and \( \pi \) satisfy the local energy inequality

\[
2 \int_{\Omega} |\nabla u|^2 \phi dx \leq \int_{\Omega} \left[ |u|^2 \Delta \phi + u \cdot \nabla \phi(|u|^2 + 2\pi) \right] + 2fu \phi dx
\]

for any nonnegative \( C^\infty \) test function \( \phi \) vanishing at the boundary \( \partial \Omega \setminus \Gamma \).

Recall boundary regularity criteria in [29] stated as follows:

**Proposition 1.6** (Theorem 1.2., Proposition 1.6., [29]). Let \((u, \pi)\) be a suitable weak solution to (1.1) in \( B_1^+ \) near the boundary \( \{x \in B_1, x_6 = 0\} \). Then 0 is a regular point of \( u \), if there exists a small positive constant \( \varepsilon_1 \) such that one of the following conditions holds

(i) There exists \( \rho_0 > 0 \) such that such that

\[
\rho_0^{-3} \|u\|_{L^3(B_{\rho_0}^+)}^3 + \rho_0^{-2} \|\nabla \pi\|_{L^{6/5}(B_{\rho_0}^+)} + \rho_0^3 \|f\|_{L^3(B_{\rho_0}^+)}^3 \leq \varepsilon_1,
\]
\[(ii)\]
\[
\limsup_{r \to 0} r^{-2} \int_{B_r^+} \|
abla u(x)\|^2 dx \leq \varepsilon_1,
\]

\[(iii)\]
\[
\limsup_{r \to 0} r^{-3} \int_{B_r^+} |u(x)|^3 dx \leq \varepsilon_1.
\]

The above result can be improved as follows:

**Theorem 1.7.** Let \((u, \pi)\) be a suitable weak solution to \((1.1)\) in \(B_1^+\) near the boundary \(\{x \in B_1, x_6 = 0\}\). Then 0 is a regular point of \(u\), if there exists a small positive constant \(\varepsilon\) such that one of the following conditions holds,

\[(i)\]
\[
r^{-3} \int_{B_r^+} |u(x)|^3 dx + r^3 \int_{B_r^+} |f(x)|^3 dx < \varepsilon,
\]
for some \(r \in (0, 1)\);

\[(ii)\]
\[
r^{-2} \int_{B_r^+} |\nabla u(x)|^2 dx + r^3 \int_{B_r^+} |f(x)|^3 dx < \varepsilon,
\]
for some \(r \in (0, 1)\).

**Remark 1.8.** The regularity criteria above for the 6D steady Navier-Stokes equations improve recent interior regularity results in [29] by removing the condition of the pressure, which also improve the result of [7].

The rest of the paper is organized as follows. In Section 2, we introduce some notations, some technical lemmas and local energy estimates. In Section 3 and 4, we prove Theorem 1.2 and Theorem 1.4, respectively. Section 5 is devoted to the proof of Theorem 1.7. In Section 6, we show that any suitable weak solution to the steady Navier-Stokes equations is a local suitable weak solution.

Throughout this article, \(C_0\) denotes an absolute constant independent of \(u, \rho, r\) and may be different from line to line.

## 2. Notations and some technical lemmas

Let \((u, \pi)\) be a solution to the steady Navier-Stokes equations \((1.1)\). Set the following scaling:

\[u^\lambda(x) = \lambda u(\lambda x), \quad \pi^\lambda(x) = \lambda^2 \pi(\lambda x), \quad f^\lambda(x) = \lambda^3 f(\lambda x),\]

for any \(\lambda > 0\), then the family \((u^\lambda, \pi^\lambda)\) is also a solution of \((1.1)\) with \(f\) replaced by \(f^\lambda\). Now define some quantities which are invariant under the scaling \((2.1)\):

\[A(r) = r^{-4} \int_{B_r} |u(x)|^2 dx, \quad C(r) = r^{-3} \int_{B_r} |u(x)|^3 dx;\]
\[ E(r) = r^{-2} \int_{B_r} |\nabla u(x)|^2 dx; \]
\[ D(r) = r^{-3} \int_{B_r} |\pi - (\pi)_{B_r}|^2 dx, \quad (\pi)_{B_r} = \frac{1}{|B_r|} \int_{B_r} \pi dx; \]
\[ F(r) = r^{3q-6} \int_{B_r} |f(x)|^q dx, \]
where \( B_r(x_0) \) is the ball of radius \( r \) centered at \( x_0 \), and we denote \( B_r(0) \) by \( B_r \). Moreover, a solution \( u \) is said to be regular at \( x_0 \) if \( u \in L^\infty(B_r(x_0)) \) for some \( r > 0 \).

Let us introduce Wolf’s pressure decomposition as in [43]. Given a bounded \( C^2 \)-domain \( G \subset \mathbb{R}^n \) and \( 1 < s < \infty \), define the operator \( E_G : W^{-1,s}(G) \to W^{-1,s}(G) \) as follows. By the \( L^p \)-theory of the steady Stokes system [16], for any \( F \in W^{-1,s}(G) \) there exists a unique pair \((v, \pi)\) \( \in W_0^{1,s} \times L_0^s(G) \) which solves the steady Navier-Stokes equations in the weak sense
\[
\begin{align*}
-\Delta v + \nabla \pi &= F, \quad \text{in } G \\
\text{div } v &= 0, \quad \text{in } G \\
v &= 0, \quad \text{on } \partial G,
\end{align*}
\]
where \( \pi \in L_s^s(G) \) denotes
\[ \int_G \pi dx = 0, \quad \pi \in L^s(\Omega). \]
Then let \( E_G(F) = \nabla \pi \), where \( \nabla \pi \) denotes the gradient functional in \( W^{-1,s}(G) \) defined by
\[ < \nabla p, \psi >= - \int_G p \nabla \cdot \psi dx, \quad \psi \in W_0^{1,s}(G). \]
The operator \( E_G \) is bounded from \( W^{-1,s}(G) \) into itself with \( E_G(\nabla \pi) = \nabla \pi \) for all \( \pi \in L_0^s(G) \), and
\[ \| \pi \|_{L^s(G)} \leq C \| F \|_{W^{-1,s}(G)}. \]
The norm of \( E_G \) depends only on \( s \) and the geometric properties of \( G \), and is independent of \( G \), if \( G \) is a ball or an annulus, which is due to the scaling properties of the Stokes equation.

Let us introduce the definition of local suitable weak solutions.

**Definition 2.1.** Let a bounded \( C^2 \)-domain \( \Omega \subset \mathbb{R}^6 \). \((u, \pi)\) is said to be a local suitable weak solution to the steady Navier-Stokes equations (1.1) in \( \Omega \), if the following conditions hold.

(i) \( u \in H^1(\Omega) \), \( \pi \in L^2(\Omega) \), \( f \in L^q(\Omega) \), \( q > 3 \);
(ii) \((u, \pi)\) satisfies the equations (1.1) in the sense of distribution sense;
(iii) for any ball \( B \subset \Omega \), let \( u \) and \( \pi \) satisfy the local energy inequality
\[ 2 \int_{\Omega} |\nabla u|^2 \phi dx \leq \int_{\Omega} \left[ |u|^2 \Delta \phi + u \cdot \nabla \phi(|u|^2 + 2\pi_1 + 2\pi_2) \right] + 2fu \phi dx \quad (2.4) \]
for any nonnegative $C^\infty$ test function $\phi$ vanishing at the boundary $\partial B$, where
\[ \nabla \pi_1 = -E_B(u \cdot \nabla u), \quad \nabla \pi_2 = E_B(\Delta u). \]

**Remark 2.2.** A suitable weak solution $(u, \pi)$ of (1.1) is a local suitable weak solution under the Definition 2.1. We prove this remark on Sec.6.

More precisely, we will prove the following proposition, which implies Theorem 1.2

**Proposition 2.3.** Let $(u, \pi)$ be a local suitable weak solution in $B_1$ to the Navier-Stokes equations (1.1). There exists absolute positive numbers $C_*$ and $\varepsilon$ such that if
\[ \int_{B_1} |u|^3 dx + \left( \int_{B_1} |f|^q dx \right)^{\frac{3}{q}} \leq \varepsilon^3, \]
then we have
\[ r_k^{-6} \int_{B_{r_k}} |u|^3 dz \leq C_* \varepsilon^3, \quad \forall \ k \in \mathbb{N}, \quad (2.5) \]
where $r^k = 2^{-k}$.

Under the scaling (2.1), we also can define some quantities as follow:
\[ A^+(r) = r^{-4} \int_{B^+_r} |u(x)|^2 dx, \quad C^+(r) = r^{-3} \int_{B^+_r} |u(x)|^3 dx; \]
\[ E^+(r) = r^{-2} \int_{B^+_r} |\nabla u(x)|^2 dx; \]
\[ D^+(r) = r^{-3} \int_{B^+_r} |\pi - \pi_{B^+_r}|^2 dx, \quad \pi_{B^+_r} = \frac{1}{|B^+_r|} \int_{B^+_r} \pi dx; \]
\[ F^+(r) = r^3 \int_{B^+_r} |f(x)|^3 dx, \]
We need the following revised local energy inequality stated in [29].

**Proposition 2.4.** Let $0 < 16r < \rho \leq r_0$. It holds
\[ k^{-2} A^+(r) + E^+(r) \]
\[ \leq C k^4 \left( \frac{r}{\rho} \right)^2 A^+(\rho) + C k^{-1} \left( \frac{\rho}{r} \right)^3 \left[ C^+(\rho) + (C^+(\rho))^{\frac{3}{2}} (D^+(\rho))^{\frac{2}{3}} \right] \]
\[ + C \left( \frac{\rho}{r} \right)^2 (C^+(\rho))^{\frac{2}{3}} (F^+(\rho))^{\frac{1}{3}}. \]
Here $1 \leq k \leq \frac{\rho}{r}$ and constant $C$ is independent on $k, r, \rho$. 

3. Interior regularity: proof of Theorem 1.2

In this section, we present the proof of Proposition 2.3, whose proof is divided into several steps, which implies Theorem 1.2. In details, we shall prove the key inequality (2.5) in Proposition 2.3 by using a strong induction argument on \( k \). Let \( C_* \) be a constant which will be specified at the final moment. From the definition of a local suitable weak solution the following local energy inequality holds true for every nonnegative \( \phi \in C^\infty_0(B_{\frac{3}{4}}) \),

\[
2 \int_{B_{\frac{3}{4}}} |\nabla u|^2 \phi \, dx \leq \int_{B_{\frac{3}{4}}} \left[ |u|^2 \Delta \phi + u \cdot \nabla \phi (|u|^2 + 2\pi_1 + 2\pi_2) \right] \, dx + 2fu \phi \, dx. \tag{3.6}
\]

First, we introduce the following lemmas.

**Lemma 3.1** (Caccioppoli type inequality). Let \((u, \pi)\) be a local suitable weak solution in \( B_1 \) to the Navier-Stokes equations (1.1). Then for any \( 0 < R \leq 1 \) there holds

\[
\| \nabla u \|_{L^2(B_{R/2})}^2 \leq CR^{-2} \| u \|_{L^2(B_R)}^2 + CR^{-1} \| u \|_{L^3(B_R)}^3 + CR^2 \| f \|_{L^2(B_R)}^2. \tag{3.7}
\]

**Proof of Lemma 3.1.** For any \( 0 < R \leq \frac{3}{4} \), choose \( \phi = 1 \) in \( B_\tau \) and \( \phi = 0 \) on \( B_\rho \) with \( \frac{R}{2} \leq \tau < \rho \leq R \) and

\[
\nabla \pi_1 = -E_{B_\rho}(u \cdot \nabla u), \quad \nabla \pi_2 = E_{B_\rho}(\triangle u).
\]

It follows from (3.6) and (2.3) that

\[
\int_{B_\tau} |\nabla u|^2 \, dx \leq C(\rho - \tau)^{-2} \int_{B_R} |u|^2 \, dx + C(\rho - \tau)^{-1} \int_{B_R} |u|^3 \, dx
\]

\[
+ C(\rho - \tau)^{-1} \left( \int_{B_R} |u|^2 \, dx \right)^{1/2} \left( \int_{B_\rho} |\pi_1|^2 \, dx \right)^{1/2}
\]

\[
+ C(\rho - \tau)^{-1} \left( \int_{B_R} |u|^2 \, dx \right)^{1/2} \left( \int_{B_\rho} |\pi_2|^2 \, dx \right)^{1/2} + C \int_{B_R} |u| \| f \| \, dx
\]

\[
\leq C(\rho - \tau)^{-2} \int_{B_R} |u|^2 \, dx + C(\rho - \tau)^{-1} \int_{B_R} |u|^3 \, dx
\]

\[
+ \frac{1}{2} \int_{B_\rho} |\nabla u|^2 \, dx + C \int_{B_R} |u| \| f \| \, dx.
\]

By a standard iteration argument, the proof is complete.

Similar as Lemma 2.9 in [2] or Lemma 2.3 in [22], we have the decay estimate of the pressure part \( \pi_1 \).

**Lemma 3.2** (The pressure estimate). Let \((u, \pi)\) be a local suitable weak solution in \( B_1 \) to the Navier-Stokes equations (1.1). Assume that for any \( x_0 \in B_{\frac{1}{2}} \) and \( 0 < r \leq \frac{1}{2} \) there holds

\[
\int_{B_r(x_0)} |u \otimes u - (u \otimes u)_{B_r(x_0)}|^\frac{3}{2} \, dx \leq CC_* r^6 \int_{B_1} |u|^3 \, dx
\]
then for all $x_0 \in B_{\frac{1}{2}}$ and $0 < r < \frac{1}{8}$,
\[
\int_{B_r(x_0)} |\pi_1 - (\pi_1)_{B_r(x_0)}|^\frac{2}{3} dx \leq C C^3 r^6 \int_{B_1} |u|^3 dx,
\]
where $\nabla \pi_1 = -E_{B_{\frac{3}{4}}}(u \cdot \nabla u)$.

**Proof of Lemma 3.2.** Without loss of generality, let $x_0 = 0$. Assume that $0 < \theta < \frac{1}{4}$ and $r \in (0, \frac{1}{8})$ fixed. Since $\nabla \pi_1 = -E_{B_{\frac{3}{4}}}(u \cdot \nabla u)$, we can write
\[
-\Delta v + \nabla (\pi_1 - (\pi_1)_{B_r}) = \nabla \cdot (u \otimes u) \quad \text{in} \quad B_{\frac{3}{4}}.
\]

Let
\[
p_{0,r} = \frac{\partial_i \partial_j}{\Delta} ((u_i u_j - (u \otimes u)_{B_r})\zeta),
\]
where $\zeta = 1$ in $B_{r/2}$, $\zeta = 0$ on $B_r^c$, and
\[
p_{h,r} = \pi_1 - (\pi_1)_{B_r} - p_{0,r}.
\]

Then $\pi_1 - (\pi_1)_{B_r} = p_{h,r} + p_{0,r}$ in $B_r$. Using triangle inequality, we have
\[
\int_{B_{\theta r}} |\pi_1 - (\pi_1)_{B_{\theta r}}|^\frac{2}{3} dx \leq C \int_{B_{\theta r}} |p_{h,r} - (p_{h,r})_{B_{\theta r}}|^\frac{2}{3} dx
\]
\[
+ C \int_{B_{\theta r}} |p_{0,r} - (p_{0,r})_{B_{\theta r}}|^\frac{2}{3} dx
\]
\[
:= J_1 + J_2.
\]

For $J_1$, it follows from the properties of harmonic functions that
\[
J_1 \leq C(\theta r)^{\frac{5}{2}} \int_{B_{\theta r}} |\nabla p_{h,r}|^\frac{2}{3} dx \leq C \theta^{\frac{15}{2}} \int_{B_r} |p_{h,r}|^\frac{2}{3} dx.
\]

For $J_2$, using Calderón-Zygmund estimate, we have
\[
J_2 \leq C \int_{B_{\theta r}} |p_{0,r}|^\frac{2}{3} dx \leq C \int_{\mathbb{R}^6} |p_{0,r}|^\frac{2}{3} dx
\]
\[
\leq C \int_{\mathbb{R}^6} |(u \otimes u - (u \otimes u)_{B_r})\zeta|^\frac{2}{3} dx
\]
\[
\leq C \int_{B_r} |u \otimes u - (u \otimes u)_{B_r}|^\frac{2}{3} dx.
\]

Combining these estimates, we get
\[
\int_{B_{\theta r}} |\pi_1 - (\pi_1)_{B_{\theta r}}|^\frac{2}{3} dx \leq C \theta^{\frac{15}{2}} \int_{B_r} |p_{h,r}|^\frac{2}{3} dx + C \int_{B_r} |u \otimes u - (u \otimes u)_{B_r}|^\frac{2}{3} dx
\]
\[
\leq C \theta^{\frac{15}{2}} \int_{B_r} |\pi_1 - (\pi_1)_{B_r}|^\frac{2}{3} dx + C \int_{B_r} |u \otimes u - (u \otimes u)_{B_r}|^\frac{2}{3} dx.
\]

Using standard iteration argument,
\[
\int_{B_r} |\pi_1 - (\pi_1)_{B_r}|^\frac{2}{3} dx \leq C r^6 \int_{B_1} |\pi_1|^\frac{2}{3} dx + C C^3 r^6 \int_{B_1} |u|^3 dx.
\]

(3.8)
Noting that $\nabla \pi_1 = -E_{B_+}(\nabla \cdot (u \otimes u))$, we have
\[
\int_{B_+^3} |\pi_1|^\frac{3}{2} dx \leq C \int_{B_+^3} |u \otimes u - (u \otimes u)_{B_r}|^\frac{3}{2} dx.
\tag{3.9}
\]
Combining (3.8) and (3.9), the proof is complete. \hfill \Box

**Proof of Proposition 2.3.** Let $r_n = 2^{-n}$ and we introduce a smooth function as
\[
\Gamma_{n+1}(x) = \frac{1}{(r_{n+1}^2 + |x-x_0|^2)^2},
\]
which clearly satisfies
\[
\triangle \Gamma_{n+1} = -\frac{24r_{n+1}^2}{(r_{n+1}^2 + |x-x_0|^2)^4} < 0.
\]
Moreover, let
\[
\chi(x) = 1, \quad \text{as} \quad x \in B_{r_4}(x_0),
\]
and
\[
\chi(x) = 0, \quad \text{as} \quad x \in B_{r_3}(x_0).
\]
Obviously, the estimate of (2.5) holds for $k = 1$. Next we assume that (2.5) holds for $k = 1, \cdots, n$. Taking the test function $\phi = \Gamma_{n+1} \chi$ in the local energy inequality (3.6), we obtain that
\[
\begin{align*}
- \int_{B_{r_3}(x_0)} |u|^2 \chi \triangle \Gamma_{n+1} dx + 2 \int_{B_{r_3}(x_0)} |\nabla u|^2 \chi \Gamma_{n+1} dx \\
\leq \int_{B_{r_3}(x_0)} |u|^2 (\Gamma_{n+1} \triangle \chi + 2 \nabla \Gamma_{n+1} \cdot \nabla \chi) dx \\
+ \int_{B_{r_3}(x_0)} u \cdot \nabla \phi |u|^2 dx + 2 \int_{B_{r_3}(x_0)} u \cdot \nabla \phi \pi_1 dx + 2 \int_{B_{r_3}(x_0)} u \cdot \nabla \phi \pi_2 dx \\
+ 2 \int_{B_{r_3}(x_0)} fu \chi \Gamma_{n+1} dx = I_1 + \cdots + I_5.
\end{align*}
\]
It follows from some straightforward computations that
\[
\begin{align*}
i) \quad \chi \Gamma_{n+1}(x,t) \geq C_0(r_{n+1})^{-4}, \quad -\chi \triangle \Gamma_{n+1}(x,t) \geq C_0(r_{n+1})^{-6} \quad \text{in} \ B_{r_{n+1}},
\end{align*}
\]
\[
\begin{align*}
ii) \quad |\nabla \phi| \leq |\nabla \Gamma_{n+1} |\chi + \Gamma_{n+1} |\nabla \chi| \leq C_0(r_{n+1})^{-5} \quad \text{in} \ B_{\rho},
\end{align*}
\]
\[
\begin{align*}
iii) \quad |\Gamma_{n+1} \triangle \chi| + 2 |\nabla \Gamma_{n+1} \cdot \nabla \chi| \leq C_0\rho^{-6} \quad \text{in} \ B_{\rho}.
\end{align*}
\tag{3.10}
\]

**Estimate of $I_1$.** It follows from iii) of (3.10) that
\[
I_1 \leq C_\ast \left( \int_{B_1} |u|^3 dx \right)^\frac{2}{3}.
\]
\textbf{Estimate of }I_2.\textbf{ Due to }|\nabla \phi| \leq Cr_k^{-5} \text{ in } B_{r_k}(x_0) \setminus B_{r_{k+1}}(x_0), \text{ we have}

\[ I_2 = \int_{B_{r_3}(x_0)} u \cdot \nabla \phi |u|^2 \leq \sum_{k=3}^{n} \int_{B_{r_k}(x_0) \setminus B_{r_{k+1}}(x_0)} |u|^3 |\nabla \phi| + \int_{B_{r_{n+1}}(x_0)} |u|^3 |\nabla \phi| \]

\[ \leq C \sum_{k=3}^{n+1} r_k^{-5} \int_{B_{r_k}(x_0)} |u|^3 dx \]

\[ \leq CC_*^3 \int_{B_1} |u|^3 dx. \]

\textbf{Estimate of }I_3.\textbf{ As in [1], we choose a series of cut-off functions }\chi_k \text{ satisfying}

\[ \chi_k(x) = \begin{cases} 1, & x \in B_{r_{k+1}}(x_0), \\ 0, & x \in B_r(x_0)^c, \end{cases} \]

for \( k = 3, \ldots, k + 1 \). Then

\[ \frac{1}{2}I_3 = \int_{B_{r_3}(x_0)} u \cdot \nabla \phi_1 dx \]

\[ \leq \sum_{k=3}^{n} \int_{B_{r_k}(x_0) \setminus B_{r_{k+2}}(x_0)} (\pi_1 - (\pi_1)_{B_{r_k}(x_0)}) u \cdot \nabla \phi(\chi_k - \chi_{k+1}) \]

\[ + \int_{B_{r_2}(x_0)} (\pi_1) u \cdot \nabla \phi(1 - \chi_3) \]

\[ + \int_{B_{r_{n+1}}(x_0)} (\pi_1 - (\pi_1)_{B_{r_{n+1}}(x_0)}) u \cdot \nabla \phi \chi_{n+1} = J_1 + J_2 + J_3. \]

Due to \( |\nabla (\phi(\chi_k - \chi_{k+1}))| \leq Cr_k^{-5} \text{ in } B_{r_k}(x_0) \setminus B_{r_{k+2}}(x_0), \) we have

\[ J_1 \leq C_* C \sum_{k=3}^{n} r_k^{-5} r_k^2 \left( \int_{B_1} |u|^3 dx \right)^{1/3} \|(\pi_1 - (\pi_1)_{B_{r_k}(x_0)})\|_{L^2(B_{r_k}(x_0))}. \]

Since \( \nabla \pi = E_{B_{r/2}} (-u \cdot \nabla u) \) and

\[ \int_{B_r(x_0)} |u \otimes u - (u \otimes u)_{B_r(x_0)}|^2 dx \leq CC_*^3 r^6 \int_{B_1} |u|^3 dx, \]

then Lemma 3.2 implies

\[ \int_{B_r(x_0)} |\pi_1 - (\pi_1)_{B_r(x_0)}|^2 dx \leq CC_*^3 r^6 \int_{B_1} |u|^3 dx, \]

and

\[ \|(\pi_1 - (\pi_1)_{B_{r_k}(x_0)})\|_{L^2(B_{r_k}(x_0))} \leq Cr_k^4 C_*^2 \|u\|_{L^6(B_1)}. \]

Hence we have

\[ J_1 \leq CC_*^3 \int_{B_1} |u|^3 dx, \]

and the other terms are similar.
Estimate of $I_4$. We still use the functions $\chi_k$.

\[
I_4 = \int_{B_2(x_0)} u \cdot \nabla \phi \pi_2 dx
\]

\[
\leq \sum_{k=3}^{n} \int_{B_{r_k}(x_0) \setminus B_{r_{k+2}}(x_0)} (\pi_2 - (\pi_2)_{B_{r_k}(x_0)}) u \cdot \nabla [\phi (\chi_k - \chi_{k+1})]
\]

\[
+ \int_{B_{r_2}(x_0)} \pi_2 u \cdot \nabla [\phi (1 - \chi_3)]
\]

\[
+ \int_{B_{r_{n+1}}(x_0)} (\pi_2 - (\pi_2)_{B_{r_{n+1}}(x_0)}) u \cdot \nabla [\phi \chi_{n+1}] = J'_1 + J'_2 + J'_3.
\]

and by the induction assumption we get

\[
J'_1 \leq C \sum_{k=3}^{n} r_k^{-5} r_k^2 \left( \int_{B_1} |u|^3 dx \right)^{1/3} r_k \pi_2 \|\pi_2 - (\pi_2)_{B_{r_k}(x_0)}\|_{L^2(B_{r_k}(x_0))}.
\]

Due to the harmonic property of $\pi_2$, we have

\[
\|\pi_2 - \pi_2_{B_{r_k}(x_0)}\|_{L^2(B_{r_k}(x_0))} \leq C r_k^{4} \|\pi_2\|_{L^2(B_{1/2})} \leq C r_k^{4} \|u\|_{L^3(B_{1/2})} + \|u\|_{L^3(B_{1/4})} + \|f\|_{L^q(B_{1/2})},
\]

where we used the local energy inequality. And the other terms are similar.

Hence, we have

\[
I_4 \leq C \sum_{k=3}^{n} r_k^{-5} r_k^2 \left( \int_{B_1} |u|^3 dx \right)^{1/3} r_k \pi_2 \|\pi_2 - (\pi_2)_{B_{r_k}(x_0)}\|_{L^2(B_{r_k}(x_0))}.
\]

Estimate of $I_5$. Since $|\chi_k - \chi_{k+1}| B_{r_k}(x_0) \setminus B_{r_{k+2}}(x_0)$, we have

\[
\frac{1}{2} I_5 = \int_{B_{r_3}(x_0)} f u \chi \Gamma_{n+1} dx
\]

\[
\leq \sum_{k=3}^{n} \int_{B_{r_k}(x_0) \setminus B_{r_{k+2}}(x_0)} f u \chi (\chi_k - \chi_{k+1}) \Gamma_{n+1} dx + \int_{B_{r_3}(x_0)} f u \chi (1 - \chi_3) \Gamma_{n+1}
\]

\[
+ \int_{B_{r_2}(x_0)} f u \chi (\chi_{n+1}) \Gamma_{n+1} = J''_1 + J''_2 + J''_3.
\]

Since $q > 3$, we have

\[
J''_1 \leq C \sum_{k=3}^{n} r_k^{-4} r_k^{6 - \frac{6}{q}} \|u\|_{L^3(B_1)} \|f\|_{L^q(B_1)} \leq C \|u\|_{L^3(B_1)} \|f\|_{L^q(B_1)}.
\]

The estimate of other term is similar as $J''_1$. Hence we have

\[
I_5 \leq C \|u\|_{L^3(B_1)} \|f\|_{L^q(B_1)}.
\]

Combining $I_1, \ldots, I_5$, we have

\[
r_{n+1}^{-6} \int_{B_{r_{n+1}}(x_0)} |u|^2 dx + r_{n+1}^{-4} \int_{B_{r_{n+1}}(x_0)} |\nabla u|^2 dx
\]

\[
\leq C \|u\|_{L^3(B_1)}^3 + \|u\|_{L^3(B_{1/2})}^2 + C \|u\|_{L^3(B_1)} \|f\|_{L^q(B_1)}.
\]
which implies that
\[ r_{n+1}^{-6} \int_{B_{r_{n+1}}(x_0)} |u|^3 dx \leq C C_*^3 \left[ C_*^3 \|u\|_{L^3(B_1)}^3 + \|u\|_{L^3(B_1)}^3 \right] + C C_*^3 \|u\|_{L^3(B_1)}^3 \|f\|_{L^q(B_1)}. \]

Then by choosing \( \frac{5}{2} C \leq C_*^3 \) and \( \varepsilon \) small such that \( C_*^3 \|u\|_{L^3(B_1)}^3 \leq \frac{1}{2} \), we have
\[ r_{n+1}^{-6} \int_{B_{r_{n+1}}(x_0)} |u|^3 dx \leq C C_*^3 \left( \int_{B_1} |u|^3 dx + \|f\|_{L^3(B_1)}^3 \right). \]

The proof is complete.

4. PROOF OF THEOREM 1.4

Proof of Theorem 1.4. By Sobolev’s embedding theorem, for \( 0 < r < \rho \) we have
\[ r^{-3} \int_{B_r} |u|^3 dx \leq C r^{-3} \int_{B_r} |u - (u)_{B_r}|^3 dx + C r^{-3} \int_{B_r} |(u)_{B_r}|^3 dx \leq C r^{-3} \left( \int_{B_{\rho r}} |\nabla u|^2 dx \right)^{\frac{3}{2}} + C r^3 \rho^{-18} \left( \int_{B_{\rho r}} |u| dx \right)^3 \leq C \left( \frac{\rho}{r} \right)^3 E(\rho)^{\frac{3}{2}} + C \left( \frac{r}{\rho} \right)^3 \left( \rho^{-5} \int_{B_{\rho r}} |u| dx \right)^3. \]

Case I: If \( \rho^{-5} \int_{B_{\rho r}} |u| dx \leq E^{\frac{3}{2}}(\rho) \), we have
\[ r^{-3} \int_{B_r} |u|^3 dx \leq C \left[ \left( \frac{\rho}{r} \right)^3 + \left( \frac{r}{\rho} \right)^3 \right] E^{\frac{3}{2}}(\rho). \]

Choosing \( r = \frac{1}{2} \rho \), noting that the assumption of Theorem 1.3, we have \( C(r) \leq \varepsilon \).

Case II: If \( \rho^{-5} \int_{B_{\rho r}} |u| dx > E^{\frac{3}{2}}(\rho) \), let \( r = \theta \rho \), we have
\[ C(\theta \rho) \leq C \theta^{-3} E(\rho)^{\frac{3}{2}} + C \theta^3 \left( \rho^{-5} \int_{B_{\rho r}} |u| dx \right)^3. \]

Choosing \( \theta^6 = \frac{\rho^{\frac{3}{2}} E(\rho)^{\frac{3}{2}}}{\left( \rho^{-5} \int_{B_{\rho r}} |u| dx \right)^{\frac{3}{2}}} \), we have \( \theta < 1 \) and
\[ E(\theta \rho) \leq C E(\rho)^{\frac{3}{2}} \left( \rho^{-5} \int_{B_{\rho r}} |u| dx \right)^{\frac{3}{2}}. \]

Applying Theorem 1.2, the proof of Theorem 1.4 is complete.
In this section, we follow the same line as in [29] to prove the boundary regularity, and one major difference is that we only need to assume that the smallness condition holds on a fixed ball without the pressure term. Our new observation is based on the suitable pressure decomposition and the new estimates of the Stokes system including global and interior estimates (for example, see [38], [23]).

First of all, let us recall global Stokes estimates with zero boundary condition.

**Lemma 5.1** (Theorem 2.13, [38]). Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$ and $q \in (1, \infty)$. For every $f = (f_{ij}) \in L^q(\Omega)$, there is a unique $q$–weak solution $v \in W^{1,q}(\Omega)$ of

$$-\Delta v_j + \nabla_j p = \partial_i (f_{ij}); \quad \text{div } v = 0,$$

satisfying

$$||v||_{W^{1,q}(\Omega)} + ||p-(p)\alpha||_{L^q(\Omega)} \leq C ||f||_{L^q(\Omega)},$$

where the constant $C$ only depends on $q$ and $\Omega$. For every $g \in L^q(\Omega)$, there is a unique $q$–weak solution $v \in W^{2,q} \cap W^{1,q}_0(\Omega)$ of

$$-\Delta v + \nabla p = g; \quad \text{div } v = 0,$$

satisfying

$$||v||_{W^{2,q}(\Omega)} + ||\nabla p||_{L^q(\Omega)} \leq C ||g||_{L^q(\Omega)},$$

where the constant $C$ only depends on $q$ and $\Omega$.

**Remark 5.2.** A vector field $v$ defined on $\Omega$ is a very weak solution of (5.11), if $v \in L^2_{\text{loc}}(\Omega)$ and $v$ satisfies

$$\int_{\Omega} \nabla v : \nabla \zeta = -\langle f, \nabla \zeta \rangle; \quad \int_{\Omega} v \cdot \nabla \phi = 0;$$

for all $\zeta \in C^\infty_0(\Omega)$ with $\nabla \cdot \zeta = 0$ and for all $\phi \in C^\infty_0(\Omega)$. A very weak solution $v$ is a $q$–weak solution of (5.11), if $v \in W^{1,q}(\Omega)$. The $q$–weak solution of (5.12) is similar.

The interior estimate for the pressure plays an important role in the following arguments. At this time, one main feature is that the velocity is zero only on part of the boundary, but the estimation of the higher derivative of pressure can not depend on the lower derivative of pressure. We recall a theorem by Kang in [23] as follows.

**Lemma 5.3** (Theorem 3.8, [23]). Let $\Omega \subset \mathbb{R}^n$ be a domain of class $C^{k+2}$ and $k$ be an integer with $-1 \leq k < \infty$ and $1 < q < \infty$. Suppose $g \in W^{k,q}(\Omega_{r_0})$ and $u \in W^{1,q}(\Omega_{r_0})$ with a unique pressure satisfying $\int_{\Omega_{r_0}} p = 0$ solve the following Stokes system:

$$\begin{cases}
-\Delta u + \nabla p = g, & \text{in } \Omega_{r_0} \\
\nabla \cdot u = 0, & \text{in } \Omega_{r_0} \\
u = 0, & \text{on } B_{r_0} \cap \partial \Omega,
\end{cases}$$
in a weak sense. Let \( r, s \) be positive numbers with \( 0 \leq r < s \leq r_0 \). Then the following estimate holds:

\[
\|u\|_{W^{k+2,q}(\Omega_r)} + \|p\|_{W^{k+1,q}(\Omega_r)} \leq C \left( \|g\|_{W^{k,q}(\Omega_0)} + \|u\|_{L^1(\Omega_s)} \right),
\]

where \( C = C(k, n, q, r, s, \Omega) \) and \( \Omega_r = \Omega \cap B_r \) with \( r \leq r_0 \). Here \( r_0 \) is comparable to the radius of the sphere contained within this domain \( \Omega \).

Next we prove Theorem 1.7.

**Proof of Theorem 1.7.** The proof of this theorem is divided into three parts.

**Step I: The pressure estimate.** First, we choose a domain \( \tilde{B}^+ \) with a smooth boundary such that \( \tilde{B}^+_\frac{3}{4} \subset \tilde{B}^+ \subset B^+_1 \). Let \( \tilde{B}^+_\rho = \{ \rho x : x \in \tilde{B}^+ \} \), which implies \( \tilde{B}^+_\rho \) is also smooth. For \( 0 < \rho < 1 \), let \( v \) and \( \pi_1 \) be the unique solution to the following boundary value problem of Stokes system

\[
\begin{align*}
-\Delta v + \nabla \pi_1 &= f - u \cdot \nabla u \quad \text{in} \quad \tilde{B}^+_\rho, \\
\text{div} \ v &= 0 \quad \text{in} \quad \tilde{B}^+_\rho, \\
v &= 0 \quad \text{on} \quad \partial \tilde{B}^+_\rho, \\
(\pi_1)_{\tilde{B}^+_\rho} &= \int_{\tilde{B}^+_\rho} \pi_1 dx = 0.
\end{align*}
\]

Due to the uniqueness of the linear Stokes system, we decompose \( v = v_1 + v_2 \) and \( \pi_1 = \pi_{11} + \pi_{12} \) in this way, which satisfy

\[
\begin{align*}
-\Delta v_1 + \nabla \pi_{11} &= f \quad \text{in} \quad \tilde{B}^+_\rho, \\
\text{div} \ v_1 &= 0 \quad \text{in} \quad \tilde{B}^+_\rho, \\
v_1 &= 0 \quad \text{on} \quad \partial \tilde{B}^+_\rho, \\
(\pi_{11})_{\tilde{B}^+_\rho} &= \int_{\tilde{B}^+_\rho} \pi_{11} dx = 0,
\end{align*}
\]

and

\[
\begin{align*}
-\Delta v_2 + \nabla \pi_{12} &= -u \cdot \nabla u \quad \text{in} \quad \tilde{B}^+_\rho, \\
\text{div} \ v_2 &= 0 \quad \text{in} \quad \tilde{B}^+_\rho, \\
v_2 &= 0 \quad \text{on} \quad \partial \tilde{B}^+_\rho, \\
(\pi_{12})_{\tilde{B}^+_\rho} &= \int_{\tilde{B}^+_\rho} \pi_{12} dx = 0.
\end{align*}
\]

With the help of Lemma 5.11 we get

\[
\begin{align*}
\rho^{-2}||v_1||_{L^2(\tilde{B}^+_\rho)} + \rho^{-1}||\pi_{11}||_{L^2(\tilde{B}^+_\rho)} &\leq C||f||_{L^2(\tilde{B}^+_\rho)}, \\
\rho^{-1}||v_2||_{L^2(\tilde{B}^+_\rho)} + ||\pi_{12}||_{L^2(\tilde{B}^+_\rho)} &\leq C||u||^2_{L^2(\tilde{B}^+_\rho)},
\end{align*}
\]

(5.13)

where the constant \( C \) is independent of \( \rho \) due to the scaling transform. Then it follows that

\[
\begin{align*}
||\pi_1||_{L^2(\tilde{B}^+_\rho)} &\leq ||\pi_{11}||_{L^2(\tilde{B}^+_\rho)} + ||\pi_{12}||_{L^2(\tilde{B}^+_\rho)} \\
&\leq C \rho^3 ||f||_{L^2(\tilde{B}^+_\rho)} + C ||u||^2_{L^2(\tilde{B}^+_\rho)}, \quad (5.14)
\end{align*}
\]

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and

\[ ||v||_{L^2(B_0^+)} \leq ||v_1||_{L^2(B_0^+)} + ||v_2||_{L^2(B_0^+)} \]
\[ \leq C \rho^4 ||f||_{L^4(B_0^+)} + C \rho ||u||_{L^4(B_0^+)}^2. \]

On the other hand, let \( w = u - v, \pi_2 = \pi - (\pi)_B^+ - \pi_1, \) then \( \int_B^+ \pi_2 = 0. \) Moreover, \((w, \pi_2)\) solves the following boundary value problem:

\[
\begin{cases}
-\Delta w + \nabla \pi_2 = 0 & \text{in } B_0^+,
\text{div } w = 0 & \text{in } B_0^+,
\end{cases}
\]

\[ w = 0 \text{ on } \partial B_0^+ \cap \{x_6 = 0\}. \]

Using Lemma 5.3 by choosing \( r_0 = \rho, r = \frac{1}{4}\rho \) and \( s = \frac{1}{2}\rho, \) we have

\[ \rho^{3-\frac{q}{2}} ||\nabla \pi_2||_{L^q(B_0^+)} \leq C \rho^{-5} ||w||_{L^q(B_0^+)} \leq C \rho^{-5} ||u||_{L^q(B_0^+)} + C \rho^{-5} ||v||_{L^q(B_0^+)} \]
\[ \leq C \rho^{-5} ||u||_{L^q(B_0^+)} + C \rho ||f||_{L^q(B_0^+)} + C \rho^{-2} ||u||_{L^q(B_0^+)}^2, \]

(5.15)

where the constant \( C \) is independent of the radius \( \rho. \)

Combining (5.14) and (5.15), for \( 0 < 4r < \rho \) we have

\[ ||\pi - (\pi)_B^+||_{L^2(B_0^+)} \leq ||\pi_1 - (\pi)_B^+||_{L^2(B_0^+)} + ||\pi_2 - (\pi)_B^+||_{L^2(B_0^+)} \]
\[ \leq C ||\pi||_{L^2(B_0^+)} + C^\frac{3}{2} ||\nabla \pi_2||_{L^q(B_0^+)} \]
\[ \leq C \rho^3 ||f||_{L^q(B_0^+)} + C ||u||_{L^q(B_0^+)} + C \left( \frac{r}{\rho} \right)^{\frac{3}{2}} \rho^3 ||f||_{L^q(B_0^+)} + C \left( \frac{r}{\rho} \right)^{\frac{3}{2}} ||u||_{L^q(B_0^+)}^2, \]

which yields that for \( 0 < 4r < \rho < 1 \) there holds

\[ D^+(r) \leq C \left( \frac{r}{\rho} \right)^{\frac{3}{2}} \left( C^+(\rho) \right)^{\frac{3}{2}} + C \left( \frac{r}{\rho} \right)^{\frac{3}{2}} (F^+(\rho))^{\frac{3}{2}} + C \left( \frac{r}{\rho} \right)^{\frac{3}{2}} C^+(\rho), \]

(5.16)

**Step II: Theorem 1.7 under the assumption (i).** Proposition 2.3 and (5.10) tell us that for all \( 0 < \theta < \frac{1}{4}, \) the following estimate holds:

\[ k^{-2} A^+(\theta \rho) + E^+(\theta \rho) \leq C_0 k^\theta \theta^2 A^+(\rho) + C_0 k^{-1} \theta^{-3} C^+(\rho) + C_0 k^{-1} \theta^{-3} (C^+(\rho))^{\frac{1}{2}} (D^+(\rho))^{\frac{3}{2}} + C_0 \theta^{-2} (C^+(\rho))^{\frac{1}{2}} (F^+(\rho))^{\frac{3}{2}}, \]

and

\[ D^+(\theta \rho) \leq C_0 \theta^2 \left( C^+(\rho) \right)^{\frac{3}{2}} + C_0 \theta^{-3} (F^+(\rho))^{\frac{3}{2}} + C_0 \theta^{-3} C^+(\rho) \]

where \( k \in [1, \theta^{-1}] \) and \( C_0 \) is a constant independent of \( \rho \) and \( \theta. \)

Let \( G(r) = k^{-2} A^+(r) + E^+(r) + \gamma^{-1} (D^+(r))^{\frac{3}{2}}, \) where \( \gamma > 0 \) to be decided. By the embedding inequality

\[ C^+(\rho) \leq C_0 \left( E^+(\rho) \right)^{\frac{3}{4}}, \]

(5.17)
since \( u = 0 \) on partial boundaries, we have
\[
G(\theta \rho) \leq C_0 k^4 \theta^2 A^+(\rho) + C_0 k^{-1} \theta^{-3} C^+(\rho) + C_0 k^{-1} \theta^{-3} (C^+(\rho))^\frac{1}{2} (D^+(\rho))^\frac{2}{3}
\]
\[
+ C_0 \theta^{-2} (C^+(\rho))^\frac{1}{2} (F^+(\rho))^\frac{1}{3} + C_0 \gamma^{-1} \theta^{6 - \frac{12q}{q}} E^+(\rho)
\]
\[
+ C_0 \gamma^{-1} \theta^{-4} (F^+(\rho))^\frac{2}{3} + C_0 \gamma^{-1} \theta^{-4} (C^+(\rho))^\frac{2}{3}
\]
\[
\leq C_0 k^4 \theta^2 A^+(\rho) + C_0 k^{-1} \theta^{-3} C^+(\rho) + \frac{1}{4} G(\rho) + C_0 \gamma k^{-2} \theta^{-6} (C^+(\rho))^\frac{2}{3}
\]
\[
+ C_0 \theta^{-4} (F^+(\rho))^\frac{2}{3} + C_0 \gamma^{-1} \theta^{6 - \frac{12q}{q}} E^+(\rho) + C_0 \gamma^{-1} \theta^{-4} (F^+(\rho))^\frac{2}{3}
\]
\[
+ C_0 \gamma^{-1} \theta^{-4} (C^+(\rho))^\frac{2}{3}
\]
Letting \( k = \theta^{-\frac{3}{4}} \) and \( \gamma = \theta^{\frac{1}{4}} \), we have
\[
k^6 \theta^2 + \gamma k^{-2} \theta^{-6} + \gamma^{-1} \theta^{6 - \frac{12q}{q}} \leq \theta^\frac{1}{2} + \theta^\frac{1}{4} + \theta^\frac{1}{4} - \frac{12q}{q}.
\]
Take \( q > 48 \) and \( \theta = \theta_0 \), which satisfy
\[
C_0 \left( \theta_0^\frac{1}{2} + \theta_0^\frac{1}{4} + \theta_0^\frac{1}{4} - \frac{12q}{q} \right) < \frac{1}{4}, \tag{5.18}
\]
which implies
\[
G(\theta_0 \rho) \leq \frac{1}{2} G(\rho) + C_1 \left( C^+(\rho) + (C^+(\rho))^\frac{1}{2} + (F^+(\rho))^\frac{2}{3} \right), \tag{5.19}
\]
where the constant \( C_1 \) is only dependent on \( C_0 \) and \( \theta_0 \).

Without loss of generality, under the condition of (i), assume that \( C^+(\rho_0) + F^+(\rho_0) < \varepsilon_0 \) with \( 0 < \rho_0 < 1 \). Then (5.16) implies that \( D^+(\rho_0) < C_1(\varepsilon_0^\frac{1}{2} + \varepsilon_0) \) with \( C_1 \) only depending on \( \theta_0 \). On the other hand, by Proposition 2.3
\[
A^+(r) + E^+(r) \leq C_0 \frac{r^2}{\rho^2} A^+(\rho)
\]
\[
+ C_0 \left( \frac{\rho}{r} \right)^3 \left[ C^+(\rho) + (C^+(\rho))^\frac{1}{2} (D^+(\rho))^\frac{2}{3} \right] + C_0 \left( \frac{\rho}{r} \right)^2 (C^+(\rho))^\frac{1}{2} (F^+(\rho))^\frac{2}{3},
\]
which, the Hölder inequality of \( A^+(\rho) \leq C_0 (C^+(\rho))^\frac{2}{3} \) and the estimate of \( D^+(\theta_0^\frac{1}{2} \rho_0) \) imply that
\[
G(\rho_1) = G(\theta_0 \rho) = \theta_0^\frac{1}{2} A^+(\theta_0 \rho) + E^+(\theta_0 \rho) + \theta_0^{-\frac{23}{4}} (D^+(\theta_0 \rho))^\frac{7}{4}
\]
\[
\leq C_1(\varepsilon_0^\frac{1}{2} + \varepsilon_0^\frac{3}{2} + \varepsilon_0) \leq \varepsilon_1 \tag{5.20}
\]
where \( \rho_1 = \theta_0 \rho_0 \), \( \theta_0 \) is decided by (5.18) and \( C_1 \) only depending on \( \theta_0 \).

It follows from (5.19) and (5.20) that
\[
G(\theta_0 \rho_1) \leq \frac{1}{2} G(\rho_1) + C_1(\varepsilon_1^\frac{1}{2} + \varepsilon_1^\frac{3}{2} + \varepsilon_0^\frac{1}{2}).
\]
where we used (5.17) and the monotonicity of $F^+(\rho)$ with respect to $\rho$. Here $C_1$ only depends on $\theta_0$. Choosing $\varepsilon_1 = \varepsilon_1(\varepsilon_0)$ small enough, we have
\[
G(\theta_0\rho_1) \leq \frac{1}{2}\varepsilon_1 + C_1(\varepsilon_1^2 + \varepsilon_1^\frac{3}{2} + \varepsilon_1) \leq \varepsilon_1^\frac{3}{2}.
\]
Assume that $G(\theta_0^j\rho_1) \leq \varepsilon_1^\frac{3}{2}$ holds for $1 \leq j \leq k$, next we verify the case of $k + 1$. Then there holds
\[
G(\theta_0^{k+1}\rho_1) \leq \frac{1}{2}G(\theta_0^k\rho_1) + C_1(\varepsilon_1 + \varepsilon_1^\frac{3}{2} + \varepsilon_0^2) \leq \varepsilon_1^\frac{3}{2},
\]
where we used (5.19), (5.17) and the monotonicity of $F^+(\rho)$ again. Hence by mathematical induction, for all $j \in \mathbb{N}$ there holds
\[
G(\theta_0^j\rho_1) \leq \varepsilon_1^\frac{3}{2}. \tag{5.21}
\]
Consequently, for any $r \in (0, \theta_0\rho_1)$, there exists a constant $j$ such that $\theta_0^j\rho_1 \leq r < \theta_0^{j-1}\rho_1$. Thus
\[
E^+(r) \leq \theta_0^{-2}E^+(\theta_0^{j-1}\rho_1) \leq \theta_0^{-2}G(\theta_0^{j-1}\rho_1)
\]
Noting that (5.21), we have
\[
E^+(r) \leq C_3\varepsilon_1^\frac{3}{2}, \quad \forall \ r \in (0, \theta_0\rho_1),
\]
where $C_3$ only dependent on $\theta_0$ is a absolute constant. Using Proposition 1.6, the proof of Theorem 1.7 is complete under the assumption (i).

**Step III: Theorem 1.7 under the assumption (ii).**

For the assumption (ii), it’s obvious from (5.17) that
\[
C^+(\rho) \leq C_0(E^+(\rho))^{\frac{3}{2}},
\]
which implies the assumption (i). \qedhere

6. **Appendix: the proof of Remark 2.2**

We clarify that any suitable weak solution to the steady Navier-Stokes equations is a local suitable weak solution.

**Lemma 6.1.** Let $(u, \pi)$ as in Definition 1.1 be a suitable weak solution to the Navier-Stokes equations (1.1). Then $u$ is a local suitable weak solution in the sense of Definition 2.1.

**Proof of Lemma 6.1.** Let $B \subset \mathbb{R}^6$ be a fixed ball. Without loss of generality, we assume that $f = 0$. Define
\[
\nabla\pi_{0,B} = E_B(-u \cdot \nabla u + \Delta u).
\]
Since $E_B$ is a bounded operator and $\nabla\pi \in W^{-1,q}(B)$, we have
\[
\nabla\pi = E_B(\nabla\pi) = E_B(-u \cdot \nabla u + \Delta u) = \nabla\pi_{0,B} = E_B(-u \cdot \nabla u) + E_B(\Delta u) = \nabla\pi_1 + \nabla\pi_2.
\]
Since \((u, \pi)\) is a suitable weak solution, we have
\[
2 \int_{\Omega} |\nabla u|^2 \phi dx \leq \int_{\Omega} \left[ |u|^2 \Delta \phi + u \cdot \nabla \phi(|u|^2 + 2\pi) \right] dx.
\]
Applying integration by parts, it follows that
\[
2 \int_{\Omega} |\nabla u|^2 \phi dx \leq \int_{\Omega} \left[ |u|^2 \Delta \phi + u \cdot \nabla \phi(|u|^2 + 2\pi_0, B) \right] dx
\leq \int_{\Omega} \left[ |u|^2 \Delta \phi + u \cdot \nabla \phi(|u|^2 + 2\pi_1 + 2\pi_2) \right] dx.
\]
Thus the proof is complete. \(\square\)

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