CHARACTERIZING TOTAL NEGATIVITY AND TESTING THEIR INTERVAL HULLS

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Abstract. A matrix is called totally negative (totally non-positive) of order \( k \), if all its minors of size at most \( k \) are negative (non-positive). The objective of this article is to provide several novel characterizations of total negativity via the (a) sign non-reversal property, (b) variation diminishing property, and (c) Linear Complementarity Problem. More strongly, each of these three characterizations uses a single test vector. As an application of the sign non-reversal property, we study the interval hull of two rectangular matrices. In particular, we identify two matrices \( C^{\pm}(A, B) \) in the interval hull of matrices \( A \) and \( B \) that test total negativity of order \( k \), simultaneously for the entire interval hull. We also show analogous characterizations for totally non-positive matrices. These novel characterizations may be considered similar in spirit to fundamental results characterizing totally positive matrices by Brown–Johnstone–MacGibbon [J. Amer. Statist. Assoc. 1981] (see also Gantmacher–Krein, 1950), Choudhury–Kunnan–Khare [Bull. London Math. Soc., in press] and Choudhury [2021 preprint]. Finally using a 1950 result of Gantmacher–Krein, we show that totally negative/non-positive matrices can not be detected by (single) test vectors from orthants other than the open bi-orthant that have coordinates with alternating signs, via the sign non-reversal property or the variation diminishing property.

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1. Introduction and main results

Given integers \( m, n \geq k \geq 1 \), a real matrix \( A \in \mathbb{R}^{m \times n} \) is called totally negative of order \( k \) if all minors of \( A \) of order at most \( k \) are negative, and \( A \) is called totally negative if \( k = \min\{m, n\} \). Similarly, one defines totally non-positive matrices (including order \( k \)).

In prior discussions of totally negative (or totally non-positive) matrices, their analogy to totally positive/TP (or totally non-negative/TN) matrices, which are matrices with all positive (or non-negative) minors, perpetually comes up first (and which we refer to henceforth by the standard abbreviations TP and TN, and \( TP_k \) and \( TN_k \) to distinguish them from the matrices studied in the present paper.) Indeed, TP and TN are widely studied because of their numerous applications in a variety of topics in mathematics, including, analysis, combinatorics, cluster algebras, differential equations, Gabor analysis, matrix theory, probability, and representation theory [2, 3, 4, 15, 16].

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In 1937, Gantmacher–Krein [18] characterized TP matrices using positivity of the spectra of all square submatrices. A similar characterization for totally negative matrices was given by Fallat–van den Driessche in 2000 [14]. They proved that if $A$ is a totally negative matrix then all the eigenvalues of each square submatrix of $A$ are real, distinct, nonzero and exactly one eigenvalue is negative. (The converse is immediate.) Similar spectral characterizations hold for TN and totally non-positive matrices, using the density of TP and totally negative matrices in these classes and the continuity of roots. Totally negative/non-positive matrices are also studied in the context of matrix factorization and spectral perturbation, see [3][7][23].

Another interesting property exhibited by TP and TN matrices is the variation diminishing property. The variation diminishing property is also satisfied by other matrices, e.g. $-A$ for a TP/TN. This was explored by Motzkin [28] in his PhD thesis in 1936, and he showed that the class of matrices satisfying the variation diminishing property are precisely the sign-regular matrices – in other words, there exists a sequence of signs $\varepsilon_k \in \{\pm 1\}$ such that every $k \times k$ minor of $A$ has sign $\varepsilon_k$, for all $k \geq 1$. A systematic treatment of sign-regular matrices and kernels can be found in Karlin’s comprehensive monograph [26].

The goal of this note is to provide similar characterization results for the class of totally negative/totally non-positive matrices, i.e. ones for which all minors are negative/non-positive. More precisely, we provide novel characterizations of such matrices using (a) sign non-reversal and (b) Linear Complementarity, to go along with Motzkin’s classical characterization using (c) variation diminution. In addition, we strengthen all three tests (a), (b), and (c) to work with only a single test vector, in parallel to our recent works [8][9] for TP matrices.

To state these results, we need some preliminary definitions, which are used below without further reference.

**Definition 1.1.** Let $m, n \geq 1$ be integers.

1. A square matrix $A \in \mathbb{R}^{n \times n}$ has the *sign non-reversal property* on a set of vectors $T \subseteq \mathbb{R}^n$, if for all vectors $0 \neq x \in T$, there exists a coordinate $i \in [1,n]$ such that $x_i(Ax)_i > 0$.

2. A matrix $A \in \mathbb{R}^{n \times n}$ has the *non-strict sign non-reversal property* on a set of vectors $T \subseteq \mathbb{R}^n$ if for all nonzero vectors $x \in T$, there exists a coordinate $i \in [1,n]$ such that $x_i \neq 0$ and $x_i(Ax)_i \geq 0$.

3. A *contiguous submatrix* is a submatrix whose rows and columns are indexed by sets of consecutive integers.

4. Let $\mathbb{R}_{alt}^{n} \subseteq \mathbb{R}^n$ denote the set of real vectors whose coordinates are nonzero and have alternating signs. Also define $\mathbb{R}^{n}_{+,-}$ to comprise the vectors in $\mathbb{R}^n$ with at least one positive and one negative coordinate.

5. Given a matrix $A \in \mathbb{R}^{n \times n}$ and $i, j \in [1,n]$, let $A_{ij}$ denote the determinant of the submatrix of $A$ of size $n - 1$ formed by deleting the $i$th row and $j$th column of $A$.

6. If $n = 1$, then define $A^{11} := -1$ to be the determinant of the empty matrix. (This, convention will only be used in Theorem [E] Proposition [5,8] and their proofs.)

7. Let $\text{adj}(A)$ denote the adjugate matrix of $A$ and let $e^i \in \mathbb{R}^n$ denote the vector whose $i$-th component is 1, and other components are zero.

8. We say that an array $X$ is $\geq 0$ (respectively $X > 0$, $X \leq 0$, $X < 0$) if all coordinates of $X$ are $\geq 0$ (respectively $> 0$, $\leq 0$, $< 0$).

9. Define the vector $d^{[n]} := (1,-1,\ldots,(-1)^{n-1})^T \in \mathbb{R}^n$.

10. Given $z = (z_1,\ldots,z_n)^T \in \{\pm 1\}^n$, define the diagonal matrix $D_z$ whose $i$th diagonal entry is $z_i$.

11. Given two matrices $A, B \in \mathbb{R}^{m \times n}$, and tuples of signs $z \in \{\pm 1\}^m, \bar{z} \in \{\pm 1\}^n$, define the $m \times n$ matrices $|A|$, $I_{z,\bar{z}}(A,B)$, and $C^\pm(A,B)$ via:

$$|A|_{ij} := |a_{ij}|, \quad I_{z,\bar{z}}(A,B) := \frac{A + B}{2} - D_z \frac{|A - B|}{2} D_{\bar{z}}, \quad C^\pm(A,B) := I_{d^{[m]}, \pm d^{[n]}}(A,B).$$
(12) Given a vector $x \in \mathbb{R}^n$, let $S^-(x)$ denote the number of sign changes in $x$ after removing all zero entries. Next, assign a value of $\pm 1$ to each zero entry of $x$, and let $S^+(x)$ denote the maximum possible number of sign changes in the resulting sequence. For $0 \in \mathbb{R}^n$, we define $S^+(0) := n$ and $S^-(0) := 0$.

Now we state our main results only for totally negative matrices. The penultimate section contains the analogous theorems for totally non-positive matrices. Our first result provides a characterization of total negativity in terms of sign non-reversal phenomena.

**Theorem A.** Let $m, n \geq k \geq 2$ be integers, and $A \in \mathbb{R}^{m \times n}$ such that $A < 0$. The following statements are equivalent:

1. The matrix $A$ is totally negative of order $k$.
2. Every square submatrix of $A$ of size $r \in [2, k]$ has the sign non-reversal property on $\mathbb{R}^r_{+,-}$.
3. Every contiguous square submatrix of $A$ of size $r \in [2, k]$ has the sign non-reversal property on $\mathbb{R}^r_{\text{alt}}$.

In fact, these conditions are further equivalent to sign non-reversal at a single vector:

4. For every contiguous submatrix $A_r$ of $A$ of size $r \in [2, k]$ and for any fixed nonzero vector $\alpha := (\alpha_1, -\alpha_2, \ldots, (-1)^{r-1}\alpha_r)^T \in \mathbb{R}^r$ with all $\alpha_i \geq 0$ (or $\leq 0$), define the vector
   
   $$x^{A_r} := \text{adj}(A_r)\alpha.$$  

Then $A_r$ has the sign non-reversal property for $x^{A_r}$.

**Remark 1.2.** A careful look at the characterization of TP in [8, 9] reveals the importance of the signs of the entries of $A$. Indeed, if $A > 0$ in Theorem A, the assertions (3), (4) are instead equivalent to $TP_k$.

As an application of Theorem A, we simultaneously detect the total negativity of an entire interval hull of matrices by reducing it to a set of two test matrices. Recall that the interval hull of two matrices $A, B \in \mathbb{R}^{m \times n}$, denoted by $I(A, B)$, is defined as follows:

$$I(A, B) = \{ C \in \mathbb{R}^{m \times n} : c_{ij} = t_{ij}a_{ij} + (1 - t_{ij})b_{ij}, t_{ij} \in [0, 1] \}.$$  

If $A \neq B$, then $I(A, B)$ is an uncountable set. We say that an interval hull is totally negative of order $k$ if all the matrices in it are totally negative of order $k$. For more details about interval hulls of matrices, we refer to [21]. One of the interesting questions, related to interval hulls of matrices, is to find a minimal test set which would determine if an entire interval hull $I(A, B)$ is totally negative. In [20], Garloff answered this question when (i) the interval hull consists of square matrices ($m = n$), and (ii) the order of total negativity equals the size of the matrices ($k = n$). In the next result, we drop these two constraints and answer this question completely.

**Theorem B.** Let $m, n \geq k \geq 1$ be integers, and $A, B \in \mathbb{R}^{m \times n}$. Then $I(A, B)$ is totally negative of order $k$ if and only if the matrices $C^\pm(A, B)$ are totally negative of order $k$.

**Remark 1.3.** Note that $C^+(A, B)$ and $C^-(A, B)$ are independent of the choice of $k$.

We now turn our attention to one of the most well known and widely used properties of TP matrices: their variation diminution on the test set of all nonzero real vectors [5]. Recently in [8], we characterized total positivity via the variation diminishing property by providing a finite set of test vectors — in fact a single vector for each contiguous submatrix. Our next main result similarly characterizes total negativity using the variation diminishing property.

**Theorem C.** Let $m, n \geq 2$ be integers. Given a real $m \times n$ matrix $A$ with $A < 0$, the following statements are equivalent:

1. $A$ is totally negative.
Theorem E. Let \( A \in \mathbb{R}^{m \times n} \) and contiguous submatrix \( A_r \) of \( A \) define the vectors
\[
 x^{A_r} := (A_{11}^{r}, 0, A_{13}^{r}, 0, \ldots)^T, \quad q^{A_r} := A_r x^{A_r}.
\]
Then 0 and \(-x^{A_r}\) are the only solutions of LCP(\( A_r, q^{A_r} \)).
We end by explaining the organization of the paper. The next three sections contain the proofs of our main results above: the sign non-reversal and interval hull result; the variation diminishing property; and the results involving the LCP. Also note that it is natural to ask for ‘totally non-positive’ analogues of the above main results, in the spirit of TN analogues of results for TP matrices. In the penultimate section of the paper, we indeed provide these – see Theorems 5.2, 5.3, 5.5, and Propositions 5.7 and 5.8, respectively. In the final section, we show that test vectors from open orthants other than the alternating bi-orthant cannot be used in the above characterizations of total negativity/non-positivity.

2. Theorems A and B: Sign non-reversal property and interval hulls for totally negative matrices

In this section we prove Theorems A and B. To prove Theorem A, we begin with two basic results. The first is a 1968 result of Karlin for totally negative matrices (in fact strictly sign-regular matrices) which was first analogously proved by Fekete in 1912 for TP matrices, and subsequently extended by Schoenberg to TP$_k$ in 1955.

**Theorem 2.1** (Karlin [29]). Let $m, n \geq k \geq 1$ be integers. Then $A \in \mathbb{R}^{m \times n}$ is totally negative of order $k$ if and only if every $r \times r$ contiguous submatrix of $A$ has negative determinant, for $r \in [1, k]$.

The next preliminary result establishes the sign non-reversal property for matrices with negative principal minors (these are known as N-matrices):

**Theorem 2.2.** [29] Let $n \geq 2$ be an integer and $A \in \mathbb{R}^{n \times n}$ such that $A < 0$. Then $A$ has all principal minors negative if and only if for all $x \in \mathbb{R}^n$, there exists $i \in [1, n]$ such that $x_i(Ax)_i > 0$.

**Proof of Theorem A**. That (1) $\Rightarrow$ (2) follows from Theorem 2.2 and that (2) $\Rightarrow$ (3) $\Rightarrow$ (4) is immediate. Finally, suppose (4) holds. By Theorem 2.1, it suffices to show the determinants of all $r \times r$ contiguous submatrices of $A$ are negative, for $r \in [1, k]$. We prove this by induction on $r$, where the base case holds by the hypothesis. Fix $r \in [2, k]$ and assume that all $(r - 1) \times (r - 1)$ and smaller contiguous minors of $A$ are negative. Let $A_r$ be an $r \times r$ contiguous submatrix of $A$. By the induction hypothesis, all the proper contiguous minors of $A_r$ are negative. In fact, all the proper minors of $A_r$ are negative by Theorem 2.1.

For any choice of nonzero vector $\alpha := (\alpha_1, -\alpha_2, \ldots, (-1)^{r-1}\alpha_r)\trans$ with all $\alpha_i \geq 0$ (or $\leq 0$), define the vector $x^{A_r}$ as in (2.1).

\[
x_i^{A_r} = \sum_{j=1}^{r} (-1)^{j-1}\alpha_j A_{ij}^{\beta_j}
\]  

(2.1)

and if the $\alpha_i$s are non-negative (or non-positive), then the summand on the right is negative (or positive) for odd $i$ and positive (or negative) for even $i$. Thus $x^{A_r} \in \mathbb{R}_{alt}^r$, and

\[
A_r x^{A_r} = (\det A_r)\alpha.
\]  

(2.2)

By the hypothesis, (2.1), and (2.2), we have $i_0 \in [1, r]$ such that

\[
0 < x_{i_0}^{A_r}(A_r x^{A_r})_{i_0} = (-1)^{i_0-1}(\det A_r)\alpha_{i_0} x_{i_0}^{A_r}.
\]  

(2.3)

From this, it follows that $\det A_r < 0$, which completes the induction step. □

To prove Theorem B we need two preliminary lemmas; the first is straightforward.

**Lemma 2.3.** Let $m, n \geq 1$ and $A, B \in \mathbb{R}^{m \times n}$. Then $I_{\bar{z}, \bar{z}}(A, B) \in \mathbb{I}(A, B)$ for all $z \in \{\pm 1\}^m, \bar{z} \in \{\pm 1\}^n$.

**Lemma 2.4.** [32] Let $A, B \in \mathbb{R}^{n \times n}$ and $x \in \mathbb{R}^n$. Let the vector $z \in \{\pm 1\}^n$ be such that $z_i = 1$ if $x_i \geq 0$ and $z_i = -1$ if $x_i < 0$. If $C \in \mathbb{I}(A, B)$, then

\[
x_i(Cx)_i \geq x_i(I_{z, z}(A, B)x)_i, \quad \forall i \in [1, n].
\]
With these preliminaries at hand, we now prove Theorem $B$.

**Proof of Theorem $B$** The result is immediate for $k = 1$, so we henceforth suppose $m, n \geq k \geq 2$. Let $I(A, B)$ be totally negative of order $k$. By Lemma 2.3, $C^\pm(A, B)$ are totally negative of order $k$.

Conversely, suppose $C^\pm(A, B)$ are totally negative of order $k$, and let $M \in I(A, B)$. Then $M < 0$, since $C^+(A, B), C^-(A, B) < 0$. Fix $r \in [2, k]$ and a vector $x \in \mathbb{R}^r$. Let $M' = M_{J \times K}$ be an $r \times r$ contiguous submatrix of $M$, where $J \subseteq [1, m]$ and $K \subseteq [1, n]$ are contiguous sets of indices with $|J| = |K| = r$. By Theorem $A(3)$, it suffices to show that there exists a coordinate $i \in [1, r]$ such that $x_i(M'x)_i > 0$.

To proceed further, we need some notation. Let $A', B'$ be contiguous submatrices of $A, B$ respectively, with the rows and columns indexed by $J$ and $K$ respectively. Since $M' \in I(A', B')$, Lemma 2.3 implies there exists $z' \in \{\pm 1\}^r \cap \mathbb{R}^r$ such that

$$x_i(M'x)_i \geq x_i((I_{z', z'})'(A', B')x)_i, \quad \forall i \in [1, r].$$

Since $M' = M_{J \times K}$, we extend $z'$ to $z \in \mathbb{R}^n \cap \{\pm 1\}^m (\tilde{z} \in \mathbb{R}^n \cap \{\pm 1\}^n)$ by embedding in contiguous positions $J \subseteq [1, m]$ ($K \subseteq [1, n]$) and uniquely padding by $\pm 1$ elsewhere. Then $I_{z, z'}(A', B')$ is a $r \times r$ contiguous submatrix of

$$I_{z, z}(A, B) := \frac{A + B}{2} - D_z \left(\frac{|A - B|}{2}\right)D_z \in \{C^+(A, B), C^-(A, B)\}.$$  

By hypothesis, this matrix is totally negative of order $k$. Thus $I_{z', z'}(A', B')$ is totally negative. Using Theorem $A(3)$ and (2.4), for some $i \in [1, r]$, we have

$$x_i((I_{z', z'})'(A', B')x)_i > 0.$$  

Thus $M'$ has the sign non-reversal property on $\mathbb{R}^r$. Since $M < 0$, by Theorem $A$, $M$ is totally negative of order $k$. \hfill $\Box$

3. **Theorem $C$** Characterization of total negativity using variation diminution

We next prove Theorem $C$—i.e., to provide a characterization of totally negative matrices using variation diminution.

**Proof of Theorem $C$** We begin by showing (1) $\implies$ (2). Let $A \in \mathbb{R}^{m \times n}$ be a totally negative matrix and $x \in \mathbb{R}^n_+$ with $1 < S^-(x) = k \leq n - 1$. Then $x$ can be partitioned into $k + 1$ components of contiguous coordinates with like signs:

$$(x_1, \ldots, x_{s_1}), (x_{s_1+1}, \ldots, x_{s_2}), \ldots (x_{s_k+1}, \ldots, x_n),$$

with at least one coordinate in each component nonzero and all nonzero coordinates in the $i$th component having the same sign $(1)^i$, without loss of generality. Moreover, we set $s_0 = 0$ and $s_{k+1} = n$. Denote the columns of $A$ by $a^1, \ldots, a^n \in \mathbb{R}^n$, and define

$$b^i := \sum_{j=s_{i-1}+1}^{s_i} |x_j|a^i, \text{ for } i \in [1, k + 1].$$

One can verify that the matrix $B := [b^1, \ldots, b^{k+1}] \in \mathbb{R}^{m \times (k+1)}$ is totally negative and $Bd^{[k+1]} = Ax$.

With this information in hand, we now prove $S^+(Ax) \leq S^-(x) = k$. For ease of exposition, we split the remainder of the proof into two cases.

**Case 1.** $m \leq k + 1$.

If $Ax \neq 0$, then $S^+(Ax) \leq m - 1 \leq S^-(x)$. If $Ax = 0$, then $m \leq k$, since $Ax = Bd^{[k+1]}$ and $B$ is invertible if $m = k + 1$. Thus $S^+(Ax) = m \leq S^-(x)$.

**Case 2.** $m > k + 1$.
Set $y := Ax = Bd^{k+1}$. If $S^+(Ax) > k$, then there exist indices $i_1 < i_2 < \cdots < i_{k+2} \in [1, m]$ and a sign $\epsilon \in \{+, -\}$ such that $(-1)^{r-1}\epsilon y_{r_1} \geq 0$ for $r \in [1, k + 2]$. Since $B$ is totally negative, at least two of the $y_{r_1}$ are nonzero. Let $I = \{i_1, \ldots, i_{k+2}\}$ and define the $(k + 2) \times (k + 2)$ matrix

$$M := [y_I | B_{I \times [1, k+1]}]$$

Then $\det M = 0$, since the first column of $M$ is a linear combination of the rest. Moreover, expanding along the first column gives

$$0 = \sum_{r=1}^{k+2} (-1)^{r-1} y_{i_r} \det B_{I \setminus \{i_r\} \times [1, k+1]},$$

(3.2)

a contradiction, since $B$ is totally negative, all $(-1)^{r-1} y_{i_r}$ have the same sign, and at least two $y_{i_r}$ are nonzero. Thus $S^+(Ax) \leq k = S^-(x)$.

It remains to prove the second part of the assertion (2). We continue our discussion using the notation in the preceding analysis. We claim that, if $S^+(Ax) = S^-(x) = k$ with $Ax \neq 0$, and $(-1)^{r-1}\epsilon y_{r_1} \geq 0$ for $r \in [1, k + 2]$, then $\epsilon = 1$. Since $B$ is totally negative and $Bd^{k+1} = Ax$, the submatrix $B_{I \times [1, k+1]}$ is invertible and $B_{I \times [1, k+1]}d^{k+1} = y_I$. By Cramer’s rule, the first coordinate of $d^{k+1}$ is

$$1 = \frac{\det[y_I | B_{I \times [2, k+1]}]}{\det B_{I \times [1, k+1]}}.$$ 

Multiplying both sides by $\epsilon \det B_{I \times [1, k+1]}$ and expanding the numerator along the first column, we have

$$\epsilon \det B_{I \times [1, k+1]} = \sum_{r=1}^{k+1} (-1)^{r-1} \epsilon y_{i_r} \det B_{I \setminus \{i_r\} \times [2, k+1]}.$$ 

(3.3)

Since the summation on the right side is negative and $B$ is totally negative, we obtain $\epsilon = 1$.

We next show that (2) $\implies$ (3). Fix $r \in [2, k]$ and suppose $A_r = A_{I \times J}$ is an $r \times r$ contiguous submatrix of $A$, for contiguous sets of indices $I \subseteq [1, m]$ and $J \subseteq [1, n]$ with $|I| = |J| = r$. Define $x^{A_r}$ as in (1.3); note this lies in $\mathbb{R}^r_{+, \cdot}$ by (2). We define $x \in \mathbb{R}^n_{+, \cdot}$ to have $x^{A_r}$ in contiguous positions $J \subseteq [1, m]$ and 0 elsewhere. Then

$$S^+(A_r x^{A_r}) \leq S^+(Ax) \quad \text{and} \quad S^-(x) = S^-(x^{A_r}).$$ 

(3.4)

By (2), we have $S^+(A_r x^{A_r}) \leq S^-(x^{A_r})$.

Next, suppose that $S^+(A_r x^{A_r}) = S^-(x^{A_r})$. Then $S^+(Ax) = S^-(x)$. Let the first and last nonzero component of $x^{A_r}$ occur in positions $s, t \in [1, r]$, respectively. List the indices in $I$ by $i_1 < i_2 < \ldots < i_r$. By (2), it follows that all coordinates of $Ax$ in positions $1, 2, \ldots, i_r$ (respectively, in positions $i_r, \ldots, m$) are all nonzero, and with the same sign as that of $x^{A_r}_s$ (respectively $x^{A_r}_t$). This shows (2) $\implies$ (3).

Finally, we show that (3) $\implies$ (1). By Theorem 2.1 it suffices to show for all $r \times r$ $(r \in [2, k])$ contiguous submatrices $A_r$ of $A$ that $\det A_r < 0$. The proof is by induction on $r \in [1, k]$. The base case holds by the hypothesis.

Let $A_r$ be an $r \times r$ contiguous submatrix of $A$ with $r \in [2, k]$ and suppose that all contiguous minors of $A$ of size at most $(r - 1)$ are negative. The same holds for all proper minors of $A$ by Theorem 2.1. Define the vector $x^{A_r}$ as in (1.3). Then

$$x^{A_r}_i = \sum_{j=1}^{r} (-1)^{j-1} \alpha_j A_r^{ij}$$

(3.5)
and if the $\alpha$s are non-negative (or non-positive), then the summation on the right is negative (or positive) for odd $i$ and positive (or negative) for even $i$. It follows that

$$x^{A_r} \in \mathbb{R}^r_{\text{alt}}, \quad A_rx^{A_r} = (\det A_r)\alpha. \quad (3.6)$$

We first claim that $A_r$ is invertible. Suppose instead that $A_r$ is singular. Then $r = S^+(A_rx^{A_r}) > S^-(x^{A_r}) = r - 1$, a contradiction by (3), and hence $A_r$ is invertible as claimed.

Next we show that $\det A_r < 0$. By (3.6), we have

$$r - 1 = S^+(A_rx^{A_r}) = S^-(x^{A_r}),$$

since the condition on the $\alpha$s implies that $S^+(\alpha) = r - 1$. Also, the sign of the first (last) component of $A_rx^{A_r}$ (if zero, the sign that attains $S^+(A_rx^{A_r})$) is positive or negative if the $\alpha$s are non-negative or non-positive respectively. The hypothesis in (3) and (3.5) imply that $\det A_r < 0$. This completes the induction step and also the proof. $\square$

4. Theorems D and E: Characterization of total negativity via the LCP

In this section, we show Theorems D and E. To proceed, we recall a 1990 result which establishes a connection between the LCP and matrices with negative principal minors.

**Theorem 4.1.** [29] A matrix $A \in \mathbb{R}^{n \times n}$ with $A < 0$ has all principal minors negative if and only if LCP$(A,q)$ has exactly two solutions for all $q \in \mathbb{R}^n$ with $q > 0$.

**Proof of Theorem 4.1** That (1) $\implies$ (2) follows from Theorem 4.1 while (2) $\implies$ (3) $\implies$ (4) is immediate. We now show (4) $\implies$ (1). We first claim that $A < 0$. Indeed, if $a_{ij} \geq 0$ for some $i \in [1,m]$ and $j \in [1,n]$, then (0) is the only solution of LCP$((a_{ij})_{1 \times 1}, q)$ for any scalar $q > 0$.

Next, we show that the determinants of all $r \times r$ submatrices of $A$ are negative, for $r \in [2,k]$. By Theorem A it suffices to show that every contiguous square submatrix of $A$ of size $r \in [2,k]$ has the sign non-reversal property on $\mathbb{R}^r_{\text{alt}}$. Let $r \in [2,k]$ and $A_r$ be an $r \times r$ contiguous submatrix of $A$ which does not satisfy this property. Then there exists $x \in \mathbb{R}^r_{\text{alt}}$ such that $x_i(A_r)x_i \leq 0$ for all $i \in [1,r]$. Let $A_rx = v$. Define $x^\pm := \frac{1}{2}(|x| \pm x)$ and $v^\pm := \frac{1}{2}(|v| \pm v)$. Then $x^+, x^-$ have sign patterns

\[
\begin{pmatrix}
+ \\
0 \\
+ \\
0 \\
\vdots \\
\vdots
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
\vdots \\
\vdots
\end{pmatrix}
\]

respectively and $(x^+)^Tv^+ = (x^-)^Tv^- = 0$, since $x_iv_i \leq 0$ for all $i \in [1,r]$.

Define

$$q := v^+ - A_rx^+ = v^- - A_rx^- . \quad (4.1)$$

Then $q > 0$, since $A < 0$. Thus LCP$(A_r,q)$ has solutions with sign patterns

\[
\begin{pmatrix}
+ \\
0 \\
+ \\
0 \\
\vdots \\
\vdots
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
\vdots \\
\vdots
\end{pmatrix}
\]

a contradiction. Hence every contiguous square submatrix of $A$ of size $r \in [2,k]$ has the sign non-reversal property on $\mathbb{R}^r_{\text{alt}}$ and the result follows. $\square$

**Proof of Theorem 4.1** That (1) $\implies$ (2) follows from Theorem 4.1. To show (2) $\implies$ (1), by Theorem 2.1 it suffices to show, for $r \in [1,k]$, that all contiguous minors of $A$ of size $r$ are negative. The proof is by induction on $r$. Let $r = 1$ and $A_1 = (a_{ij})$ for some $i \in [1,m]$ and $j \in [1,n]$. Then $x^{A_1} = (-1)$ by convention, and $q^{A_1} = -a_{ij}$. If $a_{ij} = 0$, then LCP$(A_1,q^{A_1})$ has infinitely many solutions, while if $a_{ij} > 0$, then LCP$(A_1,q^{A_1})$ has only one solution, a contradiction. Thus all entries of $A$ are negative.
For the induction step, suppose \( A_r \) is an \( r \times r \) contiguous submatrix of \( A \), with \( r \in [2, k] \) and all contiguous minors of \( A \) of size at most \( r - 1 \) are negative. By Theorem 2.1, all proper minors of \( A \) are negative. Define

\[ x^{A_r} := (A_{r1}^{11}, 0, A_{r1}^{13}, 0, \ldots)^T, \quad z^{A_r} := (0, A_{r1}^{12}, 0, A_{r1}^{14}, \ldots)^T, \quad q^{A_r} := A_r x^{A_r}. \]  

Then \( x^{A_r}, z^{A_r} \leq 0 \) and \( q^{A_r} > 0 \). Thus 0 and \(-x^{A_r}\) are two solutions of LCP(\( A_r, q^{A_r} \)).

We next claim that \( A_r \) is invertible. If not, then \( A x^{A_r} = A z^{A_r} \). So \(-z^{A_r}\) is also a solution of LCP(\( A_r, q^{A_r} \)), a contradiction. This shows the claim.

Finally we show that \( \det A_r < 0 \). Indeed, suppose \( \det A_r > 0 \). Then

\[ y = -A_r z^{A_r} + q^{A_r} = (\det A_r) e^1 \geq 0 \]  

and \( y^T z^{A_r} = 0 \). Thus \(-z^{A_r}\) is also a solution of LCP(\( A_r, q^{A_r} \)), a contradiction by the hypothesis (2). Thus \( \det A_r < 0 \). This completes the induction step and hence the proof.

**Remark 4.2.** In Theorem 4.1, in place of the vector \( x^{A_r} = (A_{r1}^{11}, 0, A_{r1}^{13}, 0, \ldots)^T \), one can take \( x^{A_r} := (A_{r1}^{1i}, 0, A_{r1}^{13}, 0, \ldots)^T \) for odd \( i \in [1, r] \), or a positive linear combination of these vectors. The result still holds with a similar proof, where we define \( q^{A_r} \) similarly as in (1.3).

## 5. Characterizations for totally non-positive matrices

Our final section presents the counterparts of the above main results for totally non-positive matrices as promised in the introduction. We first derive the sign non-reversal property and interval hull tests for totally non-positive matrices. For this, we require a 1950 density result of Gantmacher–Krein for totally negative matrices (in fact strictly sign regular matrices).

**Theorem 5.1.** [19] Given integers \( m, n \geq k \geq 1 \), the set of \( m \times n \) totally negative matrices of order \( k \) is dense in the set of \( m \times n \) totally non-positive matrices of order \( k \).

**Theorem 5.2.** Let \( m, n \geq k \geq 2 \) be integers. Given \( A \in \mathbb{R}^{m \times n} \) with \( A \leq 0 \), the following statements are equivalent:

1. The matrix \( A \) is totally non-positive of order \( k \).
2. Every square submatrix of \( A \) of size \( r \in [2, k] \) has the non-strict sign non-reversal property on \( \mathbb{R}_{r+,-} \).
3. Every square submatrix of \( A \) of size \( r \in [2, k] \) has the non-strict sign non-reversal property on \( \mathbb{R}_{r}^{alt} \).
4. For every \( r \in [2, k] \) and \( r \times r \) submatrix \( A_r \) of \( A \), the matrix \( A_r \) has the non-strict sign non-reversal property for the vector \( x^{A_r} = \text{adj}(A_r) \alpha \), for any choice of \( \alpha \in \mathbb{R}_{alt}^{r} \).

**Proof.** First suppose \( A \in \mathbb{R}^{m \times n} \) is totally non-positive of order \( k \). By Theorem 5.1, there exists a sequence \( A^{(l)} \) of totally negative matrices of order \( k \) such that \( A^{(l)} \to A \) entrywise. Fix an \( r \times r \) submatrix \( A_r \) of \( A \) for \( r \in [2, k] \), and let \( A_r^{(l)} \) be the submatrix of \( A^{(l)} \) indexed by the same rows and columns of \( A_r \). Now fix a vector \( x \in \mathbb{R}_{r+,-}^r \), and let \( I \subseteq [1, r] \) index the nonzero coordinates of \( x \). Since \( A_r^{(l)} \) is totally negative, so by Theorem [A1] there exists \( i_l \in I \) such that \( x_{i_l} A_r^{(l)} x_{i_l} > 0 \). Since \( I \) is finite, there exists an increasing subsequence of positive integers \( l_s \) such that \( i_{l_s} = i_0 \) for some \( i_0 \in I \) and for all \( s \geq 1 \). Hence

\[ x_{i_0} A_r x_{i_0} = \lim_{s \to \infty} x_{i_{l_s}} (A_r^{(l_s)} x_{i_{l_s}}) \geq 0, \quad x_{i_0} \neq 0. \]

This completes the proof of (1) \( \implies \) (2).

Next, that (2) \( \implies \) (3) \( \implies \) (4) is immediate. Now assume (4) holds. We show by induction on \( r \in [1, k] \) that the determinant of every \( r \times r \) submatrix of \( A \) is non-positive. The base case \( r = 1 \) follows from the hypothesis. For the induction step, suppose \( A_r \) is a square submatrix of \( A \) of size \( r \in [2, k] \) and all the proper minors of \( A \) are non-positive. If \( \det A_r = 0 \) then we are done;
else $\det(A_r) \neq 0$. Let $\alpha = (\alpha_1, \ldots, \alpha_r)^T \in \mathbb{R}_{alt}^r$ and define the vector $x^{A_r} := \text{adj}(A_r)\alpha$. Since no row of $\text{adj}(A_r)$ is zero, by (2.1), $x^{A_r} \in \mathbb{R}_{alt}^r$, and $x^{A_r}_{i_0} > 0$ if $\alpha_i > 0$ for all $i \in [1, r]$. Now a similar calculation as in (2.3), implies that for some $i_0 \in [1, r]$, we have

$$0 \leq x^{A_r}_{i_0}(A_r x^{A_r})_{i_0} = (\det A_r)\alpha_{i_0} x^{A_r}_{i_0}.$$ 

Since $x^{A_r}_{i_0}$ and $\alpha_{i_0}$ have from opposite signs, it follows that $\det A_r < 0$, which completes the proof by induction.

Given Theorem 5.2 which is a totally non-positive of order $k$ analogue of Theorem A, a natural question is to seek a similar totally non-positive (of order $k$) analogue of Theorem B. In [1], the authors show that under certain technical constraints, a minimal test set of two matrices suffices to check the total non-negativity of the entire interval hull. Our next result removes these technical constraints from [1], and holds for totally non-positive (of order $k$) interval hulls for arbitrary $k \geq 1$ – at the cost of working with a larger (but nevertheless finite) test set:

**Theorem 5.3.** Let $m, n \geq k \geq 1$ be integers, and $A, B \in \mathbb{R}_{m \times n}^r$. Then $I(A, B)$ is totally non-positive of order $k$ if and only if the matrices $\{I_{z, \tilde{z}}(A, B) : z \in \{\pm 1\}^m, \tilde{z} \in \{\pm 1\}^n\}$ are all totally non-positive of order $k$.

**Remark 5.4.** As in Remark 1.3 note that this set of test vectors is independent of the choice $k$.

**Proof.** We assume $m, n \geq k \geq 2$, since the result is immediate for $k = 1$. If $I(A, B)$ is totally non-positive of order $k$, then so are $I_{z, \tilde{z}}(A, B)$ by Lemma 2.3.

To show the converse, we repeat the proof of Theorem B but we work with arbitrary submatrices $M'$ of $M \in I(A, B)$ of size $r \in [2, k]$ instead of contiguous submatrices. We once again have (2.4), but the matrix $I_{z', \tilde{z}'}(A', B')$ is a submatrix of $I_{z, \tilde{z}}(A, B)$ for some $z \in \{\pm 1\}^m$, $\tilde{z} \in \{\pm 1\}^n$, which is totally non-positive of order $k$ by the hypothesis. The remainder of the proof is unchanged, except for the fact that we use Theorem 5.2 in place of Theorem A and the matrix $M$ is non-positive (since $I_{z, \tilde{z}}(A, B)$ are non-positive).

Analogous to Theorem C we next provide a characterization in terms of variation diminution for totally non-positive matrices.

**Theorem 5.5.** Given a real $m \times n$ matrix $A \leq 0$ and $m, n \geq 2$, the following statements are equivalent:

1. $A$ is totally non-positive.
2. For all $x \in \mathbb{R}_{1, -}^n$, $S^-(Ax) \leq S^-(x)$. Moreover, if equality occurs and $Ax \neq 0$, then the first (last) nonzero component of $Ax$ has the same sign as the first (last) nonzero component of $x$.
3. Let $k = \min\{m, n\}$. For every square submatrix $A_r$ of $A$ of size $r \in [2, k]$ define the vector $y^{A_r} := \text{adj}(A_r)\alpha$, for some $\alpha \in \mathbb{R}_{alt}^r$. Then $S^-(A_r y^{A_r}) \leq S^-(y^{A_r})$. If equality occurs here and $A_r y^{A_r} \neq 0$, then the first (last) nonzero component of $A_r y^{A_r}$ has the same sign as the first (last) nonzero component of $y^{A_r}$.

Apart from Gantmacher–Krein’s density Theorem 5.1, the proofs require another preliminary lemma on sign changes of limits of vectors; see e.g. 30 for the proof.

**Lemma 5.6.** Given $x = (x_1, \ldots, x_n)^T \in \mathbb{R}_n \setminus \{0\}$, define the vector $\overline{x} := (x_1, -x_2, x_3, \ldots, (-1)^{n-1} x_n) \in \mathbb{R}_n$. Then

$$S^+(x) + S^-(\overline{x}) = n - 1.$$
Proposition 5.7. Let $m, n \geq k \geq 2$ be integers and let $A \in \mathbb{R}^{m \times n}$ with $A \leq 0$. Then $A$ is totally non-positive of order $k$ if for every $r \times r$ submatrix $A_r$ of $A$ and for all $q \in \mathbb{R}^n$ with $q \geq 0$, if $z^1 = (x_1, 0, x_3, \ldots)^T$ and $z^2 = (0, x_2, 0, x_4, \ldots)^T$ (where all $x_i > 0$) are two solutions of $\text{LCP}(A_r, q)$, then $A_r z^1 = A_r z^2$.

Proof. By Theorem 5.2 it is enough to show that every $r \times r$ submatrix $A_r$ of $A$ has the non-strict sign non-reversal property with respect to $\mathbb{R}^r_{\text{alt}}$, for $r \in [2, k]$. Fix $r \in [2, k]$. Suppose that there exist an $r \times r$ submatrix $A_r$ of $A$ and a vector $x \in \mathbb{R}^r_{\text{alt}}$ such that $x_i (A_r x)_i < 0$ for all $i \in [1, r]$. Defining $x^+, x^-, v^+, v^-$ and $q$ as in the proof of Theorem 1, we conclude that $q \geq 0$ and $x^+, x^-$ are two solutions of $\text{LCP}(A_r, q)$. Also, $x^+, x^-$ are with sign patterns

$$
\begin{pmatrix} + & + & 0 \\ 0 & + & + \\ \vdots & \vdots & \vdots 
\end{pmatrix}
$$

and

$$
\begin{pmatrix} + \\ - \\ \vdots 
\end{pmatrix}
$$

respectively.

By (4.1), we have
\[ A_r x^+ + q = v^+ \neq v^- = A_r x^- + q, \]
since \( Ax \in \mathbb{R}_{alt}^r \). Thus \( A_r x^+ \neq A_r x^- \), a contradiction. Hence \( A \) is totally non-positive of order \( k \).

The next result provides an improvement, via Linear Complementarity at a single vector \( q \) (for each square submatrix of \( A \)).

**Proposition 5.8.** Let \( m, n \geq k \geq 1 \) be integers and \( A \in \mathbb{R}^{m \times n} \). Then \( A \) is totally non-positive of order \( k \) if for every square submatrix \( A_r \) of \( A \) of size \( r \in [1, k] \) and \( q^{A_r} \) as defined in (1.5), the following holds:

(i) \( 0 \) is the solution of LCP\( (A_r, q^{A_r}) \).

(1) If \( z^1 \) and \( z^2 \) are two nonzero solutions of LCP\( (A_r, q^{A_r}) \) then \( A_r z^1 = A_r z^2 \).

**Proof.** We prove by induction on \( r \in [1, k] \) that \( A_r \leq 0 \) for all \( r \times r \) submatrices \( A_r \) of \( A \). Let \( r = 1 \) and \( A_1 = (a_{ij}) \) for some \( i \in [1, m] \), \( j \in [1, n] \). Then \( x^{A_1} = (-1) \) by convention, and \( q^{A_1} = -(a_{ij}) \). If \( a_{ij} > 0 \) then \( 0 \) is not a solution of LCP\( (A_1, q^{A_1}) \). Thus \( A \leq 0 \).

Let \( r \in [2, k] \) and suppose that all the \((r-1) \times (r-1)\) and smaller minors of \( A \) are non-positive, and let \( A_r \) be a submatrix of \( A \) of size \( r \). If \( A_r = 0 \), then we are done, else assume \( A_r \) is non-singular. If \( \det A_r > 0 \), repeating the proof of Theorem [2] once again we have \(-x^{A_r} \) and \(-z^{A_r}\) (as defined in (1.2)) are two distinct nonzero solutions of LCP\( (A_r, q^{A_r}) \), but \( Ax^{A_r} \neq Az^{A_r} \). Thus \( \det A_r < 0 \) and hence \( A \) is totally non-positive of order \( k \).

**Example 5.9.** The converse of the previous theorem does not hold. We explain this with an example. Consider the totally non-positive matrix \( A = \begin{pmatrix} 0 & 0 & 0 \\ -1 & -3 & -3 \\ -1 & -1 & -1 \end{pmatrix} \). Then \( q = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \) and \( z^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \), \( z^2 = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} \), where \( t > 1 \), are two solutions of LCP\( (A, q) \), but \( Az^1 \neq Az^2 \).

### 6. Other orthants cannot yield test vectors

In the preceding sections, we have characterized a negative/non-positive matrix to be totally negative/totally non-positive of order \( k \), by its square submatrices \( A_r \) with \( r \in [2, k] \) satisfying the sign non-reversal or variation diminishing property on all of the open bi-orthant \( \mathbb{R}_{alt}^r \) – or equivalently, on single test vectors which are drawn from this bi-orthant. Our next result explains that these results are ‘best possible’ in the following sense: if \( x \in \mathbb{R}_{+, -}^r \setminus \mathbb{R}_{alt}^r \) with all \( x_i \neq 0 \) (i.e., there are two successive \( x_i \) of the same sign), then every \( r \times r \) totally negative matrix of order \( r - 1 \) satisfies the sign non-reversal property and the variation diminishing property for \( x \). Thus the above characterizations cannot hold with test vectors in any open bi-orthant other than in \( \mathbb{R}_{alt}^r \), since one shows that there exist totally negative matrices with positive determinants (see e.g. [19]).

**Theorem 6.1.** Let \( A_r \in \mathbb{R}_{+, -}^r \) be totally negative matrix of order \( r - 1 \) and let \( x \in \mathbb{R}_{+, -}^r \setminus \mathbb{R}_{alt}^r \) with all \( x_i \) are nonzero. Then:

(1) There exists a coordinate \( i \in [1, r] \) such that \( x_i(A_r x)_i > 0 \). In other words, \( A \) satisfies the sign non-reversal property.

(2) \( S^+(A_r x) \leq S^-(x) \). Moreover, if equality occurs, then the first (last) nonzero component of \( Az \) (if zero, the unique sign required to determine \( S^+(Az) \)) has the same sign as the first (last) component of \( x \). In other words, \( A \) satisfies the variation diminishing property.

**Proof.** Suppose \( x \in \mathbb{R}_{+, -}^r \setminus \mathbb{R}_{alt}^r \) with all \( x_i \neq 0 \). One can partition \( x \) as in (3.1) and the subsequent lines, with \( k \) replaced by some \( p \leq r - 2 \). By the argument following (3.1), the matrix \( B \in \mathbb{R}^{r \times (p+1)} \) is totally negative and \( Ax = Bd^{p+1} \). We define \( y := Ax = Bd^{p+1} \).
To show (1), suppose for contradiction that $x_i y_i \leq 0$ for all $i \in [1, r]$. Now consider the index set
$$I = \{i_1, \ldots, i_{k+1}\},$$
where $i_t \in [s_{t-1} + 1, s_t]$. Then the square submatrix $B_{I \times [p+1]}$ is totally negative and it reverses the signs of $d^{[p+1]}$, since $B_{I \times [k+1]} d^{[k+1]} = y_I$. This gives the desired contradiction by Theorem A.

The proof of (2) is similar to that of case (2) of Theorem C.

An analogue of Theorem 6.1 also holds for the class of totally non-positive matrices:

**Proposition 6.2.** If $A \in \mathbb{R}^{r \times r}$ is a totally non-positive matrix of order $r - 1$ and the vector $x$ is defined as in Theorem 6.1, then:

1. There exists a coordinate $i \in [1, r]$ such that $x_i \neq 0$ and $x_i(A, x)_i \geq 0$.
2. $S^-(A, x) \leq S^-(x)$. Moreover, if equality occurs and $Ax \neq 0$, then the first (last) nonzero component of $Ax$ has the same sign as the first (last) nonzero component of $x$.

**Proof.** Repeat the proofs of Theorems 5.2 and 5.5 except for the use of Theorem 6.1 in place of Theorems A and C respectively.

We conclude with a similar observation about the LCP: Theorem D shows that for a totally negative matrix $A$ of order $k$, the solution sets $SOL(A, q)$ with $r \in [2, k]$ do not contain two vectors with alternately positive and zero coordinates (and disjoint supports) simultaneously, for all $q \in \mathbb{R}^r$ with $q > 0$. Our final result shows that if $A \in \mathbb{R}^{r \times r}$ is totally negative of order $k$, the same holds when ‘alternating’ is replaced by ‘not always alternating’. Hence by the existence result alluded to in the first paragraph of this section, totally negative matrices can not be detected by the solution sets $SOL(A, q)$ of LCP, which do not contain two vectors with disjoint supports, at least one of which has two consecutive positive coordinates. Thus, the LCP-characterization of total negativity also distinguishes the above ‘alternation’.

**Proposition 6.3.** If $A \in \mathbb{R}^{r \times r}$ with $r \geq 2$ is a totally negative matrix of order $r - 1$, then $SOL(A, r)$ with $q > 0$ does not simultaneously contain two vectors which have disjoint supports, and at least one of which has two consecutive positive coordinates.

The proof is analogous to Theorem D using Theorem 6.1 (1).

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