Stabilization and limit theorems for geometric functionals of Gibbs point processes

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Abstract

Given a Gibbs point process $\mathcal{P}_\Psi$ on $\mathbb{R}^d$ having a weak enough potential $\Psi$, we consider the random measures

$$\mu_\lambda := \sum_{x \in \mathcal{P}_\Psi \cap Q_\lambda} \xi(x, \mathcal{P}_\Psi \cap Q_\lambda) \delta_{x/\lambda^{1/d}}$$

where $Q_\lambda := [-\lambda^{1/d}/2, \lambda^{1/d}/2]^d$ is the volume $\lambda$ cube and where $\xi(\cdot, \cdot)$ is a translation invariant stabilizing functional. Subject to $\Psi$ satisfying a localization property and translation invariance, we establish weak laws of large numbers for $\lambda^{-1} \mu_\lambda(f)$, $f$ a bounded test function on $\mathbb{R}^d$, and weak convergence of $\lambda^{-1/2} \mu_\lambda(f)$, suitably centered, to a Gaussian field acting on bounded test functions. The result yields limit laws for geometric functionals on Gibbs point processes including the Strauss and area interaction point processes as well as more general point processes defined by the Widom-Rowlinson and hard-core model. We provide applications to random sequential packing on Gibbsian input, to functionals of Euclidean graphs, networks, and percolation models on Gibbsian input, and to quantization via Gibbsian input.

1 Introduction

Functionals of large complex geometric structures often consist of sums of spatially dependent terms admitting the representation

$$\sum_{x \in \mathcal{X}} \xi(x, \mathcal{X}),$$

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where $\mathcal{X} \subset \mathbb{R}^d$ is locally finite and where the function $\xi$, defined on all pairs $(x, \mathcal{X})$, with $x \in \mathcal{X}$, represents the interaction of $x$ with respect to $\mathcal{X}$. When $\mathcal{X}$ is a random $n$ point set in $\mathbb{R}^d$ (i.e. a finite spatial point process), the asymptotic analysis of the suitably scaled sums (1.1) as $n \to \infty$ can often be handled by $M$-dependent methods, ergodic theory, or mixing methods. However there are situations where these classical methods are either not directly applicable, do not give explicit asymptotics in terms of underlying geometry and point densities, or do not easily yield explicit rates of convergence. Stabilization methods originating in [23] and further developed in [3, 24, 26], provide another approach for handling sums of spatially dependent terms.

There are several similar definitions of stabilization, but the essence is captured by the notion of stabilization of the functional $\xi$ with respect to a rate $\tau > 0$ homogeneous Poisson point process $\mathcal{P} := \mathcal{P}_\tau$ on $\mathbb{R}^d$, defined as follows. Say that $\xi$ is translation invariant if $\xi(x, \mathcal{X}) = \xi(x + z, \mathcal{X} + z)$ for all $z \in \mathbb{R}^d$. Let $B_r(x)$ denote the Euclidean ball centered at $x$ with radius $r \in \mathbb{R}^+_0 := [0, \infty)$. Letting $0$ denote the origin of $\mathbb{R}^d$, we say that a translation invariant $\xi$ is stabilizing on $\mathcal{P} = \mathcal{P}_\tau$ if there exists an a.s. finite random variable $R := R^{\xi}(\tau)$ (a ‘radius of stabilization’) such that

$$\xi(0, \mathcal{P} \cap B_R(0)) = \xi(0, \mathcal{P} \cap B_R(x) \cup A)$$

(1.2)

for all locally finite $A \subset \mathbb{R}^d \setminus B_R(0)$.

Consider the point measures

$$\mu_\lambda := \sum_{x \in \mathcal{P} \cap Q_\lambda} \xi(x, \mathcal{P} \cap Q_\lambda) \delta_{x/\lambda^{1/d}},$$

(1.3)

where $\delta_x$ denotes the unit Dirac point mass at $x$ whereas $Q_\lambda := [-\lambda^{1/d}/2, \lambda^{1/d}/2]^d$ is the $\lambda$-volume cube. Let $\mathcal{B}(Q_1)$ denote the class of all bounded $f : Q_1 \to \mathbb{R}$ and for all random point measures $\mu$ on $\mathbb{R}^d$ let $(f, \mu) := \int f d\mu$ and let $\bar{\mu} := \mu - \mathbb{E}[\mu]$.

Stabilization of translation invariant $\xi$ on $\mathcal{P}$, as defined in (1.2), together with stabilization of $\xi$ on $\mathcal{P} \cap Q_\lambda, \lambda \geq 1$, when combined with appropriate moment conditions on $\xi$, yields for all $f \in \mathcal{B}(Q_1)$ the law of large numbers [22, 25]

$$\lim_{\lambda \to \infty} \lambda^{-1} \langle f, \mu_\lambda \rangle = \tau \mathbb{E}[\xi(0, \mathcal{P})] \int_{Q_1} f(x) dx \quad \text{in } L^1 \text{ and in } L^2,$$

(1.4)

and, if the stabilization radii on $\mathcal{P}$ and $\mathcal{P} \cap Q_\lambda, \lambda \geq 1$, decay exponentially fast, then [3, 21]

$$\lim_{\lambda \to \infty} \lambda^{-1} \text{Var}[(f, \mu_\lambda)] = \tau V^{\xi}(\tau) \int_{Q_1} f(x) dx,$$

(1.5)

where for all $\tau > 0$

$$V^{\xi}(\tau) := \mathbb{E}[\xi(0, \mathcal{P})^2] + \tau \int_{\mathbb{R}^d} [\mathbb{E} \xi(0, \mathcal{P} \cup \{z\}) \xi(z, \mathcal{P} \cup \{0\}) - \mathbb{E}[\xi(0, \mathcal{P})^2]] dz.$$
Additionally, the finite-dimensional distributions \(\langle f_1, \lambda^{-1/2} \mu_\lambda^\xi \rangle, \ldots, \langle f_k, \lambda^{-1/2} \mu_\lambda^\xi \rangle\), \(f_1, \ldots, f_k \in B(Q_1)\), converge to a Gaussian field with covariance kernel

\[
(f, g) \mapsto \tau V^\xi(\tau) \int_{Q_1} f(x)g(x)dx.
\]  

The limits (1.4)-(1.6) establish asymptotics for functionals and measures defined in terms of independent input and one might expect analogous asymptotics for functionals of dependent input subject to weak long range dependence conditions. The main purpose of this paper is to show that this is indeed the case. We establish the analogs of (1.4)-(1.6) when \(\mathcal{P} = \mathcal{P}_\tau\) is replaced by a weak Gibbsian modification having an exponentially localized potential; see Theorems 3.1-3.3 for a precise statement of the limit theory for functionals of Gibbsian input. Gibbsian point processes covered by this generalization include, for low enough reference intensity \(\tau\), the Strauss process, the area interaction process, as well as point processes defined by the continuum Widom Rowlinson and hard-core models. Gibbsian point processes considered here are intrinsically algorithmic. Their computational efficiency yields numerical estimates for asymptotic limits appearing in our main results.

Functionals of geometric graphs over Gibbsian input on large cubes, as well as functionals of random sequential packing models defined by Gibbsian input on large cubes, consequently satisfy weak laws of large numbers and central limit theorems as the cube size tends to infinity. The precise limit theorems are provided in sections 6 and 7, which also includes asymptotics for functionals of communication networks and continuum percolation models over Gibbsian point sets, as well as asymptotics for the distortion error arising in Gibbsian quantization of probability measures.

2 Gibbs point processes and their stabilizing functionals

2.1 Gibbs point processes with localized potentials

Throughout \(\Psi\) denotes a translation invariant functional defined on locally finite collections of points \(\mathcal{X}\) in \(\mathbb{R}^d\) and admitting values in \(\mathbb{R}_+ \cup \{+\infty\}\). By translation invariant we mean \(\Psi(\mathcal{X}) = \Psi(y + \mathcal{X})\) for all \(y \in \mathbb{R}^d\). In the sequel we refer to \(\Psi\) as the potential, Hamiltonian or energy functional, with all three terms used interchangeably. For a locally finite point set \(\mathcal{X}\) in \(\mathbb{R}^d\) and an open bounded set \(D \subseteq \mathbb{R}^d\) we define \(\Psi_D(\mathcal{X}) := \Psi(\mathcal{X} \cap D)\). We shall always assume that for all open, bounded \(D\) the potential \(\Psi_D(\mathcal{P})\) admits finite values with non-zero probability, where we recall that \(\mathcal{P} := \mathcal{P}_\tau\) is the Poisson point process of some arbitrary but fixed intensity \(\tau > 0\) in \(\mathbb{R}^d\).
Moreover, we assume the Hamiltonian is hereditary, that is to say if \( \Psi(X) = +\infty \) for some \( X \) then \( \Psi(Y) = +\infty \) for all \( Y \supseteq X \). This puts us in a position to define the Gibbs point process \( \mathcal{P}^\Psi_D \) given in law by

\[
\frac{d\mathcal{L}(\mathcal{P}^\Psi_D)}{d\mathcal{L}(\mathcal{P})}(X) := \frac{\exp(-\Psi_D(X))}{Z[\Psi_D]},
\]

where \( Z[\Psi_D] := \mathbb{E}\exp(-\Psi_D(\mathcal{P})) \) is the normalizing constant for (2.1), also called the partition function.

The following definition is central to this paper.

**Definition 2.1** For a decreasing right-continuous function \( \psi : \mathbb{R}_+ \to [0,1] \) with \( \lim_{r \to \infty} \psi(r) = 0 \) we say that the Hamiltonian \( \Psi \) is \( \psi \)-localized iff for each \( x \in \mathbb{R}^d \), each finite \( X \) and each \( r > 0 \) the add-one potential, inheriting from \( \Psi \) its translation invariance property,

\[
\Delta(x, X) := \Psi(X \cup \{x\}) - \Psi(X),
\]

satisfies

\[
0 \leq \Delta[\psi](x, X \cap B_r(x)) \leq \Delta(x, X) \leq \Delta[^{\psi}](x, X \cap B_r(x)),
\]

where \( \Delta[\psi](\cdot, \cdot) \) and \( \Delta[^{\psi}](\cdot, \cdot) \) are certain translation invariant deterministic functionals such that uniformly in \( x \in \mathbb{R}^d \) and \( X \subset \mathbb{R}^d \) we have

\[
0 \leq \exp(-\Delta[\psi](x, X \cap B_r(x))) - \exp(-\Delta[^{\psi}](x, X \cap B_r(x))) \leq \psi(r).
\]

In other words, even though determining exactly the value of the add-one potential \( \Delta(x, X) \) may require the knowledge of the whole configuration \( X \), knowing just \( X \cap B_r(x) \) we can determine the value of \( \exp(-\Delta(x, X)) \) with accuracy at least \( \psi(r) \) which tends to 0 as \( r \to \infty \). In case where both \( \Psi(X \cup \{x\}) \) and \( \Psi(X) \) are \(+\infty\) we set by convention \( \Delta(x, X) := 0 \). We also require that \( \Delta(x, \emptyset) < +\infty \) to prevent the Gibbs process \( \mathcal{P}^\Psi_D \) from concentrating on \( \emptyset \). The functionals \( \Delta[\psi](\cdot, \cdot) \) and \( \Delta[^{\psi}](\cdot, \cdot) \) will be called lower and upper add-one potentials respectively. Note that the required non-negativity of the add-one potential is not particularly restrictive because whenever the add-one potential admits a finite lower bound, possibly negative \(-a < 0\), it can be reduced to the present setting by adding \( a|X| \) to \( \Psi \) and by replacing the underlying intensity \( \tau \) with \( \tau \exp(a) \). Imposing the presence of a lower bound for the add-one potential or other related growth conditions is a usual assumption to avoid density explosions and infinite values of the partition function in (2.1), see [28].

Every Poisson point process has a \( \psi \)-localized potential, since in this case \( \Psi \equiv 0 \) and thus \( \Delta \equiv 0 \). Less trivially, a large number of Gibbs point processes, including those in modelling problems in
statistical mechanics, communication networks, and biology have ψ-Localized potentials. This includes the Strauss process, the area interaction process, processes having finite and infinite range pair potential functions, and the hard-core and Widom-Rowlinson models; see section 5 for details.

2.2 Graphical construction of Gibbs point processes with localized potentials

For a ψ-Localized potential Ψ the resulting Gibbs point process \( P_\Psi^D \) admits a particularly convenient graphical construction in the spirit of Fernández, Ferrari and García [10]-[12]. While adding a number of new ideas, in our presentation below we follow [10]-[12] as well as the developments in [4]. Consider a stationary homogeneous free birth and death process \( (\rho^D_t)_{t \in \mathbb{R}} \) in \( D \) with the following dynamics:

- A new point \( x \in D \) is born in \( \rho^D_t \) during the time interval \([t - dt, t]\) with probability \( \tau dx dt \),
- An existing point \( x \in \rho^D_t \) dies during the time interval \([t - dt, t]\) with probability \( dt \), that is the lifetimes of points of the process are independent standard exponential.

Clearly, the unique stationary and reversible measure for this process is just the law of the Poisson point process \( P \cap D \).

Consider now the following trimming procedure performed on \( \rho^D_t \), based on the ideas developed in [10]-[12]. Choose a birth site for a point \( x \in D \) at some time \( t \in \mathbb{R} \) and draw a random number \( \eta \in \mathbb{R}_+ \) from the law given by the distribution function \( 1 - \psi(\cdot) \). Then, accept it with probability \( \exp(-\Delta_1^V(x, \rho^D_{t-} \cap B_\eta(x))) / \psi(\eta) \) and reject with the complementary probability if the acceptance/rejection statuses of all points in \( \rho^D_{t-} \cap B_\eta(x) \) are determined, otherwise proceed recursively to determine the statuses of points in \( B_\eta(x) \).

Before discussing any further properties of this procedure, we have to ensure first that it actually terminates. To this end, note that each point \( x \) with the property of having the ball \( B_\eta(x) \) devoid of points from \( \rho^D_{t-} \) at its birth time \( t \) has its acceptance status determined. More generally, the acceptance status of a point \( x \) at its birth time \( t \) only depends on the status of points in \( \rho^D_{t-} \cap B_\eta(x) \), that is to say points born before and falling into \( B_\eta(x) \). We call these points causal ancestors of \( x \) and, in general, for a subset \( A \subseteq D \) by \( \text{An}_t[A] \) we denote the set of all points in \( \rho^D_t \cap A \), their causal ancestors, the causal ancestors of their ancestors and so forth throughout all past generations. The set \( \text{An}_t[A] \) is referred to as the causal ancestor cone or causal ancestor clan of \( A \) with respect to the birth and death process \( (\rho^D_t)_{t \in \mathbb{R}} \).
It is now clear that in order for our recursive status determination procedure to terminate for all points of \( \rho_t^D \) in \( A \), it is enough to have the causal ancestor cone \( \text{An}_t[A] \) finite. This is easily checked to be a.s. the case for each \( A \subseteq D \) – indeed, since \( D \) is bounded, a.s. there exists some \( s < t \) such that \( \rho_s^D = \emptyset \) and thus no ancestor clan of a point alive at time \( t \) can go past \( s \) backwards in time.

Having defined the trimming procedure above, we recursively remove from \( \rho_t^D \) the points rejected at their birth, and we write \( (\gamma_t^D)_{t \in \mathbb{R}} \) for the resulting process. Clearly, \( \gamma_t^D \) is stationary because so was \( \rho_t^D \) and the acceptance/rejection procedure is time-invariant as well. Moreover, the process \( \gamma_t^D \) is easily seen to evolve according to the following dynamics:

- Add a new point \( x \) with intensity \( \tau \exp(-\Delta(x, \gamma_t^D)) \, dx \, dt \).
- Remove an existing point with intensity \( dt \).

These are the standard Monte-Carlo dynamics for \( \mathcal{P}^{\Psi \cap D} \) as given in (2.1) and the law of \( \mathcal{P}^{\Psi} \) is its unique invariant distribution. Consequently, in full analogy with [10]-[12] the point process \( \gamma_t^D \) coincides in law with \( \mathcal{P}^{\Psi} \) for all \( t \in \mathbb{R} \).

### 2.3 Exponentially localized potentials and infinite volume limits

Recalling the definition of \( \psi \) from Definition 2.1, we henceforth assume that there is a \( C_1 > 0 \) such that

\[
\psi(r) \leq \exp(-C_1 r) \quad \forall r > 0. \tag{2.4}
\]

It should be emphasized that we require (2.4) to hold for all \( r > 0 \) and not just for \( r \) large enough. It is known, see [10]-[12] where a proof based on subcritical branching process domination is given, that if \( C_1 \) is chosen large enough, then all causal ancestor cones are a.s. finite and, in fact, there is a \( C_2 > 0 \) such that for all \( t, R \in \mathbb{R}_+ \) and \( A \subset D \) we have the crucial bound

\[
P[\text{diamAn}_t[A] \geq R + \text{diam}(A)] \leq \text{vol}(A) \exp(-C_2 R). \tag{2.5}
\]

Moreover, the constant \( C_2 \) in (2.5) does not depend on \( D \). If (2.4) is satisfied with the constant \( C_1 \) large enough so that (2.5) holds as well, then the potential \( \Psi \) is declared exponentially localized.

Putting \( D_n := [-n, n]^d \), this puts us in a position to construct the infinite volume limit (thermodynamic limit) for \( \mathcal{P}^{\Psi[D_n]} \) as \( n \to \infty \). Indeed, consider the infinite volume version \( \rho_t^D \) of our stationary free birth and death process \( \rho_t^{D_n} \), constructed as \( \rho_t^D \) with \( D \) replaced by \( \mathbb{R}^d \). Clearly, for each \( t \in \mathbb{R} \) we have that \( \rho_t \) coincides in law with \( \mathcal{P} \). Moreover, in view of (2.5) and recalling
that there did not depend on $D$, we see that the trimming procedure as described above is also valid for the infinite volume process $\rho_t$, yielding the stationary \textit{trimmed} process $\gamma_t$. These remarks justify defining the following point process, used in all that follows.

**Definition 2.2** We define the thermodynamic limit $\mathcal{P}^\Psi := \mathcal{P}^\Psi_\tau$ to be the point process coinciding in law with $\gamma_0$ and hence with $\gamma_t$ for all $t$.

To provide some further motivation for granting to $\mathcal{P}^\Psi$ the name of \textit{thermodynamic limit} note that the process $\mathcal{P}^\Psi$ enjoys the following important property: for any bounded set $D \subseteq \mathbb{R}^d$, any locally finite point configuration $\mathcal{X} \subseteq D^c$ and any finite point configuration $\mathcal{Y} \subseteq D$ the conditional law of $\mathcal{P}^\Psi \cap D$ on the event $\mathcal{P}^\Psi \cap D^c = \mathcal{X}$ is given by

$$
\frac{dL(\mathcal{P}^\Psi \cap D | \mathcal{P}^\Psi \cap D^c = \mathcal{X})}{dL(\mathcal{P} \cap D)}(\mathcal{Y}) = \frac{\exp(-\Psi(\mathcal{Y}|\mathcal{X}))}{Z_D[\Psi|\mathcal{X}]},
$$

where

$$
Z_D[\Psi|\mathcal{X}] = \mathbb{E}\exp(-\Psi(\mathcal{P} \cap D|\mathcal{X}))
$$

whereas

$$
\Psi(\mathcal{Y}|\mathcal{X}) := \lim_{r \to \infty} \Psi(\mathcal{Y} \cup \mathcal{X} \cap B(0, r)) - \Psi(\mathcal{X} \cap B(0, r))
$$

with the existence of the limit guaranteed by the localization condition (2.2). Moreover, the so-constructed $\mathcal{P}^\Psi$ is the only point process with the above properties – to see it take $D_n := [-n,n]^d \uparrow \mathbb{R}^d$ and note that in view of the graphical construction specialized for the conditional specification (2.6), the relation (2.5) guarantees that the process in some fixed bounded $A \subseteq \mathbb{R}^d$ exhibits exponentially decaying dependencies on the external configuration in $D_n^c$ as $n \to \infty$.

### 2.4 Stabilizing functionals of Gibbs point processes

In this section we specialize to our Gibbs point process setting the notion of a stabilizing functional, see [3, 23, 24, 25] and the references therein. As in section 1, let $\xi(\cdot, \cdot)$ be a translation invariant functional defined on pairs $(x, \mathcal{X})$ where $\mathcal{X}$ is a finite point collection in $\mathbb{R}^d$ and $x \in \mathcal{X}$. Further, when $x \notin \mathcal{X}$, we abbreviate $\xi(x, \mathcal{X} \cup \{x\})$ by $\xi(x, \mathcal{X})$.

Next, suppose that a given point process $\Xi$ on $\mathbb{R}^d$ is stochastically dominated by a homogeneous Poisson point process and suppose that there exists $C_3 > 0$ such that for every ball $B_r(x)$ the conditional probability of $B_r(x)$ not being hit by $\Xi$ given the external configuration $\mathcal{E} := \Xi \setminus B_r(x)$ admits the bound

$$
P[\Xi \cap B_r(x) = \emptyset | \mathcal{E}] \leq \exp(-C_3 r^d)
$$

(2.7)
uniformly in $E$. Stochastic domination and (2.7) provide upper and lower stochastic bounds on the number of points in any ball analogous to those satisfied by a homogeneous Poisson point process and for this reason such $\Xi$ are called *Poisson-like*. The next proposition tells us that the Gibbs point processes considered here are Poisson-like.

**Proposition 2.1** Every Gibbs point process $\mathcal{P}^\Psi$ with an exponentially localized potential is a Poisson-like process.

*Proof.* Indeed, the stochastic domination by $\mathcal{P}$ comes from the obvious relation $\gamma_0 \subseteq \rho_0$ in the above graphical construction of $\mathcal{P}^\Psi$ because $\rho_0$ coincides in law with $\mathcal{P}$. The second relation (2.7) follows by the graphical construction as well. Indeed, we have $\Delta(x, \emptyset) < \infty$ and hence, by (2.2) and (2.3), in the course of the dynamics given by the graphical construction the acceptance probability for a birth attempt at some $y$ inside a ball $B_{r-s}(x)$ with no points alive in the whole $B_r(x)$ is uniformly bounded away from 0, both in the location of the point $y$ attempting to be born and in the external configuration, as soon as $s$ and $r > s$ are taken large enough. On the other hand, the ball reaches a completely empty state with intensity at most 1. Consequently, the time fraction of having the ball fully empty decays exponentially with the volume of the ball uniformly in the external configuration and hence so does the probability of having no point alive in $B_r(x)$ at the time 0 by stationarity of the graphical construction in time. \hfill $\Box$

Similarly to (1.2), say that $\xi$ is a *stabilizing functional in the wide sense* if for every Poisson-like process $\Xi$ there exists an a.s. finite stabilization radius $R := R^\xi(x, \Xi)$, such that a.s.

$$\xi(x, \Xi \cap B_R(x)) = \xi(x, [\Xi \cap B_R(x)] \cup A)$$

(2.8)

for all locally finite point collections $A \subseteq \mathbb{R}^d \setminus B_R(x)$. Stabilizing functionals in the wide sense can a.s. be extended to the whole process $\Xi$, that is to say for all $x \in \mathbb{R}^d$

$$\xi(x, \Xi) := \lim_{r \to \infty} \xi(x, \Xi \cap B_R(x))$$

is a.s. well defined.

Given $s > 0$ and a Poisson-like process $\Xi$ define the tail probability

$$\tau(s) := \tau(s, \Xi) := \max \left[ \sup_{\lambda \geq 1, x \in Q_\lambda} \mathbb{P}[R(x, \Xi \cap Q_\lambda) > s], \mathbb{P}[R(x, \Xi) > s] \right].$$

Further, we say that $\xi$ is *exponentially stabilizing in the wide sense* if for every Poisson-like process $\Xi$ we have $\lim \sup_{s \to \infty} s^{-1} \log \tau(s) < 0$. Thus, if $\xi$ is exponentially stabilizing in the wide
sense, then there exists a $C_4$ such that for all $s \in \mathbb{R}_+$ we have

$$\sup_{x \in \mathbb{R}^d} \mathbb{P}[R(x, \Xi) > s] \leq \exp(-C_4s)$$

and

$$\sup_{\lambda \geq 1, x \in Q_\lambda} \mathbb{P}[R(x, \Xi \cap Q_\lambda) > s] \leq \exp(-C_4s).$$  (2.9)$$

We stress that, unlike in the standard Poisson input setting of [3, 23, 24, 25], where Poisson points in disjoint sets are independent, the configuration $\Xi \cap B_R(x)$ will, in general, depend on the configuration in $\Xi \cap B_R(x)^c$. Thus, unlike the standard Poisson input setting, the wide sense stabilization of $\xi$ at $x$ within radius $R$ does not imply that the value of $\xi(x, \Xi)$ does not depend on the configuration outside $B_R(x)$; on the other hand this value is independent of the configuration in $B_R(x)^c$ given the configuration $\Xi \cap B_R(x)$. This weak dependence feature of wide sense stabilization, which carries additional technical considerations, allows us to establish limit theory for functionals and measures in geometric probability over point sets more general than the usual Poisson and binomial point sets.

As we will see shortly, many functionals which stabilize in the standard Poisson input setting also stabilize in the wide sense. Possibly there are some functionals which stabilize over Poisson samples but which do stabilize in the wide sense, but we are not aware of these functionals. For these reasons, when the context is clear, we will henceforth abuse terminology and use the term ‘stabilization’ to mean ‘stabilization in the wide sense’, with a similar meaning for ‘exponentially stabilizing’.

### 2.5 Functionals with bounded perturbations

The theory presented in this paper is mainly confined to translation invariant geometric functionals and its extension to non-translation invariant functionals seems to require non-trivial effort. Nevertheless, a small step towards the non-translation invariant set-up can be made with only slight modification of the existing theory. This extension is the subject of the present subsection and it deals with asymptotically negligible bounded perturbations of translation-invariant functionals. To put it in formal terms, consider the following notion. Consider a Poisson-like input point process $\Xi$. Assume that $\xi(\cdot, \cdot)$ is a translation invariant geometric functional exponentially stabilizing in the wide sense and let $\hat{\xi}(\cdot, \cdot; \lambda)$ be a family of geometric functionals indexed by the extra parameter $\lambda > 0$, not assumed to be translation invariant but enjoying the following properties:

- For each $\lambda > 0$ the functional $\hat{\xi}(\cdot, \cdot; \lambda)$ admits a representation

$$\hat{\xi}(x, \mathcal{X}; \lambda) = \xi(x, \mathcal{X}) + \delta(x, \mathcal{X}; \lambda),$$  (2.10)$$
where the correction (perturbation) $\delta(x, \mathcal{X}; \lambda)$ is not necessarily translation invariant but, for all $p > 0$ it satisfies the moment bound

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}[\delta(x, \Xi; \lambda)]^p \leq \varepsilon(\lambda, p) < \infty,$$

where $\lim_{\lambda \to \infty} \varepsilon(\lambda, p) = 0$ for each fixed $p$.

- The perturbation $\delta(\cdot, \cdot; \lambda)$ satisfies the wide sense exponential stabilization with the same stabilization radius $R_\xi(\cdot, \cdot)$ as $\xi$.

If these two conditions hold, we say that $\hat{\xi}(\cdot, \cdot; \lambda)$ is an asymptotically negligible bounded perturbation of $\xi$ on input $\Xi$,; for brevity we call it just a bounded perturbation of $\xi$ in the sequel. The message of this subsection, to be made formal below, is that the asymptotic behavior of bounded perturbations of a translation invariant functional is indistinguishable from the asymptotic properties of the functional itself. This observation brings the limit theory for stochastic quantization within the compass of stabilizing functionals; see section 7.

### 3  Weak laws of large numbers and central limit theorems

We now state our main results, which show that sums of stabilizing functionals defined on Gibbsian input (with exponentially localized potential) on large cubes satisfy weak laws of large numbers and Gaussian limits as the cube size tends to infinity. For all $\lambda > 0$, let $Q_\lambda := [-\lambda^{1/d}/2, \lambda^{1/d}/2]^d$ be the volume $\lambda$ cube centered at the origin of $\mathbb{R}^d$, and let $\mu_\lambda^\xi$ be the $\lambda$-rescaled $\xi$-empirical measure on $Q_1 := [-1/2, 1/2]^d$, that is

$$\mu_\lambda^\xi := \sum_{x \in \mathcal{P} \cap Q_\lambda} \xi(x, \mathcal{P}) \delta_x / \lambda^{1/d}. \quad (3.1)$$

Let $H_\lambda^\xi := \mu_\lambda^\xi(Q_1)$ be the total mass of $\mu_\lambda^\xi$, and for future reference, define also the non-rescaled infinite-volume measure

$$\mu^\xi := \sum_{x \in \mathcal{P}} \xi(x, \mathcal{P}) \delta_x. \quad (3.2)$$

Let $p \in [0, \infty)$. Say that $\xi$ satisfies the $p$-moment condition if

$$\sup_{\lambda} \sup_{x \in \mathcal{P} \cap Q_\lambda, \mathcal{X} \in \mathcal{C}} \mathbb{E}[|\xi(x, \mathcal{P} \cup \mathcal{X})|^p] < \infty, \quad (3.3)$$

where $\mathcal{C}$ denotes the collection of all finite point sets in $\mathbb{R}^d$. 10
Recall that $\mathcal{B}(Q_1)$ denotes the set of bounded $f : Q_1 \to \mathbb{R}$ and that $\bar{\mu}_\lambda^\xi := \mu_\lambda^\xi - \mathbb{E}[\mu_\lambda^\xi]$. Under appropriate moment conditions, our first two results establish a weak law of large numbers and variance asymptotics for $\langle f, \mu_\lambda^\xi \rangle$, $f \in \mathcal{B}(Q_1)$, as $\lambda \to \infty$. Our third result shows that the finite-dimensional distributions of $(\lambda^{-1/2}(f_1, \mu_\lambda^\xi), ..., \lambda^{-1/2}(f_m, \mu_\lambda^\xi))$, $f_1, ..., f_m \in \mathcal{B}(Q_1)$, converge to those of a multivariate normal as $\lambda \to \infty$, and, in the univariate CLT we establish a rate of convergence. Finally our last general result establishes asymptotics for bounded perturbations of a translation invariant $\xi$.

**Theorem 3.1** (WLLN) Assume that $\xi$ is stabilizing and satisfies the $p$-moment condition (3.3) for some $p > 1$. We have for each $f \in \mathcal{B}(Q_1)$

$$
\lim_{\lambda \to \infty} \lambda^{-1} \mathbb{E}[\langle f, \mu_\lambda^\xi \rangle] = \tau E(\tau) \int_{Q_1} f(x)dx
$$

where

$$
E(\tau) := E^\xi(\tau) := \mathbb{E} \left[ \xi(0, \mathcal{P}^\Psi) \exp(-\Delta(0, \mathcal{P}^\Psi)) \right].
$$

Moreover, if (3.3) is satisfied for some $p > 2$ then $\lambda^{-1/2}\langle f, \mu_\lambda^\xi \rangle$ converges to $\tau E(\tau) \int_{Q_1} f(x)dx$ in $L^2$.

Note that $E(\tau)$ depends on the underlying intensity $\tau$ via $\mathcal{P}^\Psi$ even though this parameter does not explicitly show up in the defining formula. Before stating variance asymptotics write

$$
\sigma^\xi[0] := \mathbb{E} \left[ \xi^2(0, \mathcal{P}^\Psi) \exp(-\Delta(0, \mathcal{P}^\Psi)) \right]
$$

and for all $x \in \mathbb{R}^d$ define the two point correlation functions for the functional $\xi$ over the Gibbsian input $\mathcal{P}^\Psi$ by

$$
\sigma^\xi[0, x] := \mathbb{E} \left[ \xi(0, \mathcal{P}^\Psi \cup \{x\}) \xi(x, \mathcal{P}^\Psi \cup \{0\}) \exp(-\Delta(\{0, x\}, \mathcal{P}^\Psi)) \right] - [E^\xi(\tau)]^2,
$$

where $\Delta(\{x, y\}, \mathcal{X}) := \Psi(\mathcal{X} \cup \{x, y\}) - \Psi(\mathcal{X})$.

**Theorem 3.2** (Variance asymptotics) Assume that $\xi$ is exponentially stabilizing and satisfies the $p$-moment condition (3.3) for some $p > 2$. We have for each $f \in \mathcal{B}(Q_1)$

$$
\lim_{\lambda \to \infty} \lambda^{-1} \text{Var}[\langle f, \mu_\lambda^\xi \rangle] = \tau V^\xi(\tau) \int_{Q_1} f^2(x)dx,
$$

where

$$
V^\xi(\tau) := \sigma^\xi[0] + \tau \int_{\mathbb{R}^d} \sigma^\xi[0, x]dx < \infty.
$$

Letting $N(0, \sigma^2)$ denote a mean zero normal random variable with variance $\sigma^2$, we have:

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Theorem 3.3 (CLT) Assume that $\xi$ is exponentially stabilizing and satisfies the $p$-moment condition (3.3) for some $p > 2$. We have for each $f \in B(Q_1)$

$$\lambda^{-1/2} \langle f, \bar{\mu}_\lambda \rangle \overset{D}{\to} N \left(0, \tau V^\xi(\tau) \int_{Q_1} f^2(x)dx\right),$$

and the finite-dimensional distributions $(\lambda^{-1/2} \langle f_1, \bar{\mu}_\lambda \rangle, ..., \lambda^{-1/2} \langle f_m, \bar{\mu}_\lambda \rangle)$, $f_1, ..., f_m \in B(Q_1)$, converge to those of a mean zero Gaussian field with covariance kernel

$$(f_1, f_2) \mapsto \tau V^\xi(\tau) \int_{Q_1} f_1(x)f_2(x)dx.$$  

Moreover, if (3.3) is satisfied for some $p > 3$ then for all $\lambda \geq 2$ and all $f \in B(Q_1)$ we have

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}\left[ \frac{\langle f, \bar{\mu}_\lambda \rangle}{\text{Var}[\langle f, \bar{\mu}_\lambda \rangle]} \leq t \right] - \mathbb{P}[N(0,1) \leq t] \right| \leq C(\log \lambda)^{3d-1/2}.$$  

Assuming that $\hat{\xi}(\cdot, \cdot; \lambda)$ is a bounded perturbation of a stabilizing functional $\xi$ we have:

Theorem 3.4 The analogs of Theorems 3.1, 3.2 and 3.3 hold for $\xi$ replaced by $\hat{\xi}(\cdot, \cdot; \lambda)$ under their respective stabilization and moment assumptions.

Remarks. (i) Comparison with [12]. The results of [12] establish limit theory for functionals $\xi$ of weakly dependent Gibbsian input, but essentially these results require $\xi$ to have finite range (finite range test functions). Theorems 3.1-3.4 extend [12] to cases when $\xi$ has infinite range and stabilizes.

(ii) Comparison with functionals on Poisson input. Theorems 3.1-3.4 show that the established limit theory for stabilizing functionals on homogeneous Poisson input [3, 21, 23, 24, 25] is insensitive to weak Gibbsian modifications of the input. Thus the entirety of weak laws of large numbers and central limit theorems for functionals defined on homogeneous Poisson input given previously in the literature [3], [21]-[25] extend to the corresponding analogous results for functionals on point processes whose local specification (2.1) with respect to the Poisson process is exponentially localized. If the input $P^\Psi$ is Poisson, then the term $\Delta(x, P^\Psi)$ vanishes, and hence Theorem 3.1 extends the Poisson weak law of large numbers in Theorem 2.1 of [25]. Likewise, Theorem 3.2 extends the variance asymptotics of [3] and [21], whereas Theorem 3.3 extends the central limit theory of [3], [21] and [26].
Numerical evaluation of limits. We emphasize that the point process $\mathcal{P}_\Psi$ is intrinsically algorithmic; this algorithmic scheme provides an exact (perfect) sampler [12]. It is computationally efficient and yields a numerical evaluation of the limits (3.5) and (3.7).

Extensions and generalizations. The variance convergence (3.7) and the asymptotic normality (3.9) hold under weaker stabilization assumptions such as power-law stabilization (see Penrose [21]), but the resulting additional technical details obscure the main ideas of our approach, and thus we have not tried for the weakest possible stabilization conditions. Similarly, counterparts to Theorems 3.1-3.4 should hold for functionals defined in terms of non-homogenous Gibbsian input, but we do not provide the technical details here either.

4 Proofs of main results

This section is organized as follows. First, in Subsection 4.1 we establish exponential clustering properties for stabilizing functionals of processes with exponentially localized potentials. Exponential clustering is central to our approach, as it shows that the cumulants of $\langle f, \bar{\mu}_\xi \rangle$, $f \in \mathcal{B}(Q_1)$, converge to those of a normal random variable, that is to say they vanish asymptotically upon suitable re-scaling for all orders above two. Then, in Subsections 4.2 and 4.3 we establish Theorems 3.1, 3.2 and Theorem 3.3 respectively, using either the cumulant techniques developed in [3] or the Stein techniques of [26]. Subsection 4.4 provides the proof of Theorem 3.4.

4.1 Exponential clustering lemma

Let $\mathcal{P}_\Psi$ be a point process with exponentially localized potential and assume that $\xi$ is an exponentially stabilizing functional in the wide sense. For each Poisson-like configuration $\Xi$ we denote by $\xi[\Xi]$ this point configuration marked with the values of $\xi$, that is to say each $x \in \Xi$ carries the mark $\xi(x, \Xi)$. We have then

**Lemma 4.1** For each $k \geq 1$ there exist $M > 0$ and $c := c(k) > 0$ such that for any deterministic points $x_1, \ldots, x_k \in \mathbb{R}^d$ the total variation distance between $\xi[\mathcal{P}_\Psi]$ restricted to the union $B_1(x_1) \cup \ldots \cup B_1(x_k)$ and the product of respective restrictions of $\xi[\mathcal{P}_\Psi]$ to $B_1(x_1), \ldots, B_1(x_k)$ does not exceed $Mk \exp(-c \min_{i,j} \text{dist}(x_i, x_j))$.

**Proof.** The statement of the lemma is a consequence of the graphical construction of the process $\mathcal{P}_\Psi$ and of the exponential stabilization of $\xi$. To see it, observe that the considered total variation
distance does not exceed the probability of the event that the random sets

\[ A_i := A_{0\left[ \bigcup_{x \in \mathcal{P} \cap B_1(x_i)} B_R(x) \right]} \]

are not all disjoint, where \( R := R[x, \mathcal{P}^\Psi] \) and where the causal ancestor cone \( A_{0\left[ A \right]} \) is defined in section 2.2. Indeed, if all \( A_i \)'s are disjoint then the values of \( \xi \) over all points in balls \( B_1(x_i) \) depend on disjoint and hence independent portions of the free birth-and-death process in the graphical construction. To complete the proof it suffices now to show that the probability \( \mathbb{P}[A_i \cap A_j \neq \emptyset] \) decays exponentially with the distance between \( x_i \) and \( x_j \) for each \( i \) and \( j \). Now, this follows because

- The number of points in \( \mathcal{P}^\Psi \cap B_1(x_i) \) and \( \mathcal{P}^\Psi \cap B_1(x_j) \) admits super-exponentially decaying tails in view of the Poisson domination property of the Poisson-like process \( \mathcal{P}^\Psi \),
- For each such point \( x \) the stabilization radius \( R[x, \mathcal{P}^\Psi] \) admits exponentially decaying tails by the wide sense exponential stabilization (2.9),
- Consequently, the diameter of the union \( \bigcup_{x \in \mathcal{P} \cap B_1(x_i)} B_R(x) \) of such balls has exponentially decaying tails too,
- Finally, using the exponential decay relation (2.5) for causal ancestor clan sizes in the graphical construction, we conclude that the diameter of \( A_i \) also has exponentially decaying diameter.

The proof is hence complete. \( \square \)

### 4.2 Proof of Theorems 3.1 and 3.2

There are several ways to prove limit theorems for stabilizing translation invariant functionals. To illustrate the new features arising in the setting of functionals of Gibbsian input, we will first assume that \( f \) is a.e. continuous, that \( \xi \) satisfies the moment condition (3.3) for \( p = 4 \), and appeal to cumulant methods. In this setting we may directly apply the cumulant methods developed in Section 4 of [3] (especially those methods used for proving statements (i) and (ii) of Theorem 2.1 there) and hence we only provide crucial points, referring the reader to [3] for further details. The arguments in Section 4 there show that

\[
\lim_{\lambda \to \infty} \lambda^{-1} \mathbb{E}(f, \mu_\lambda^{\xi}) = c(0) \int_{Q_1} f(u)du \tag{4.12}
\]
and
\[
\lim_{\lambda \to \infty} \lambda^{-1} \text{Var}[\langle f, \mu_\lambda^x \rangle] = \int_{\mathbb{R}^d} q(0) + \int_{\mathbb{R}^d} c(0, x) dx \int_{Q_1} f^2(u) du,
\]
(4.13)
where the correlation functions \(c(x), q(x)\) and \(c(x, y), x, y \in \mathbb{R}^d\), are the respective Radon-Nikodym derivatives given by
\[
\mathbb{E}\mu^x(dx) = c(x)dx,
\]
\[
\mathbb{E}\{(\mu^x(dx))^2\} = q(x)dx \text{ and }
\]
\[
\text{Cov}[\mu^x(dx), \mu^y(dy)] = c(x, y)dxdy, x \neq y,
\]
where \(\mu^x\) is the infinite-volume empirical measure defined in (3.2). Indeed, the main idea as briefly sketched below is to use stabilization, under the guise of the exponential clustering Lemma 4.1 here, to show that when proving our results, in the \(\lambda \to \infty\) limit we can safely replace (modulo a correction of order \(o(\lambda^{1/2})\)) the considered expression \(\langle f, \mu_\lambda^x \rangle\) by \(\langle f, \mu_\infty^x \rangle\) where \([\mu_\infty]^x := \sum_{x \in \mathbb{R}^d \cap Q_1} \xi(x, \mathcal{P}^\Psi)\delta_{x/\lambda^{1/4}}\). Now, the last expression coincides with \(\int_{Q_\lambda} f_\lambda d\mu^\xi\), where \(f_\lambda(x) := f(\lambda^{-1/4}x), x \in Q_\lambda\). Consequently, \(\mathbb{E}\langle f, \mu_\lambda^x \rangle = \langle f, \mathbb{E}\mu^x \rangle\) for large \(\lambda\) is well approximated by \(\int_{Q_\lambda} f_\lambda d\mathbb{E}\mu^\xi = \int_{Q_\lambda} f_\lambda(x)c(x)dx = \lambda c(0)\int_{Q_1} f(u)du\) by translation invariance of \(\mu^x\) and upon a variable change \(u := \lambda^{-1/4}x\). This yields (4.12) for continuous \(f\). To get (4.12) in the general set-up, that is to say when \(f \in \mathcal{B}(Q_1)\) and when \(\xi\) satisfies the bounded moment condition (3.3) for some \(p > 1\), one can follow the approach of [21]. Likewise, under exponential stabilization, \(\text{Var}[(f, \mu_\lambda^x)]\) is well approximated by \(\text{Var}[\int_{Q_\lambda} f_\lambda d\mu^\xi]\), which by Campbell’s theorem, equals
\[
\int_{Q_\lambda \times Q_\lambda} f_\lambda(x)f_\lambda(y)\text{Var}\mu^\xi(dx, dy),
\]
where \(\text{Var}\mu^\xi := \mathbb{E}[\mu^\xi \otimes \mu^\xi] - [\mathbb{E}\mu^\xi \otimes \mathbb{E}\mu^\xi]\) is the variance measure of \(\mu^\xi\); see Section 4 in [3] and references therein for more details on moment measures. Using the usual decomposition of the variance measure into the diagonal and off-diagonal component [3], we see that the last expression equals
\[
\int_{Q_\lambda} f_\lambda^2(x)q(x)dx + \int_{Q_\lambda \times Q_\lambda} f_\lambda(x)f_\lambda(y)c(x, y)dxdy.
\]
(4.14)
Using the continuity of \(f_\lambda\), the translation invariance of \(c(x, y)\) and \(q(x)\), and the exponential decay of \(c(x, y)\) in the distance between \(x\) and \(y\) as guaranteed by the exponential clustering Lemma 4.1 with \(k = 2\), we come to (4.13) as required. To show (4.13) when \(f \in \mathcal{B}(Q_1)\) and when \(\xi\) satisfies the moment condition (3.3) for some \(p > 2\), we may modify the approach of [21].

Now, to calculate the correlation functions \(c(\cdot), c(\cdot, \cdot)\) and \(q(\cdot)\) in (4.12) and (4.13), note first that, given \(\mathcal{P}^\Psi\) in \(\mathbb{R}^d \setminus dx\), the probability of observing an extra point of \(\mathcal{P}^\Psi\) at \(x\) is \(\tau \exp(-\Delta(x, \mathcal{P}^\Psi))dx\) as determined by the construction of the process in Subsection 2.2, where \(\tau dx\) corresponds to the
birth attempt intensity at \( x \) whereas \( \exp(-\Delta(x, \mathcal{P}^\Psi)) \) comes from the acceptance probability. Consequently, \( \mathbb{E} \mu^\xi(dx) = \tau \mathbb{E} \xi(x, \mathcal{P}^\Psi) \exp(-\Delta(x, \mathcal{P}^\Psi))dx \) and hence

\[
c(x) = \tau \mathbb{E} \xi(x, \mathcal{P}^\Psi) \exp(-\Delta(x, \mathcal{P}^\Psi)). \tag{4.15}
\]

Likewise,

\[
q(x) = \tau \mathbb{E} \xi^2(x, \mathcal{P}^\Psi) \exp(-\Delta(x, \mathcal{P}^\Psi)). \tag{4.16}
\]

Further, for \( x, y \in \mathbb{R}^d \), given \( \mathcal{P}^\Psi \) in \( \mathbb{R}^d \backslash (dx \cup dy) \), the probability of observing extra points of \( \mathcal{P}^\Psi \) at \( x \) and \( y \) respectively is \( \tau^2 \exp(-\Delta(\{x, y\}, \mathcal{P}^\Psi))dxdy \), where again \( \tau dx \) and \( \tau dy \) are the probabilities that the birth attempts at \( x \) and \( y \) were made whereas \( \exp(-\Delta(\{x, y\}, \mathcal{P}^\Psi)) \) is the probability that they were both accepted. Consequently,

\[
c(x, y) = \tau^2 \mathbb{E} \xi(\mathcal{P}^\Psi \cup \{y\})\xi(y, \mathcal{P}^\Psi \cup \{x\}) \exp(-\Delta(\{x, y\}, \mathcal{P}^\Psi)) - c(x)c(y). \tag{4.17}
\]

In other words, \( c(x, y) = \tau^2 \sigma^\xi[0, y - x] \) with \( \sigma^\xi[\cdot, \cdot] \) as in (3.6). The required relations (3.4) and (3.7) follow now by putting (4.12) and (4.13) together with (4.15), (4.16) and (4.17) and comparing with (3.5) and (3.7). The exponential clustering Lemma 4.1 when combined with the moment conditions imposed on \( \xi \) implies that the two-point correlation \( c(x, y) \) exhibits exponential decay in the distance between \( x \) and \( y \) whence the integral \( \int_{\mathbb{R}^d} c(0, x)dx \) is finite. This observation allows us to conclude the finiteness of \( E(\tau) \) and \( V(\tau) \), as given by (3.5) and (3.8), respectively. Consequently, the \( L^2 \)-convergence stated in Theorem 3.1 follows now by the variance convergence in Theorem 3.2 and, given (4.12) and (4.13), the proof of both of these theorems is complete.

4.3 Proof of Theorem 3.3

When \( f \) is continuous on \( Q_1 \) and when \( \xi \) satisfies the moment condition (3.3) for all \( p \), the exponential clustering Lemma 4.1 allows us to use the techniques developed in Section 5 of [3], where it replaces the clustering Lemma 5.2, to show that all cumulants of \( \langle \xi, \bar{\mu}^\xi \rangle \) are all of the volume order \( \lambda \) and hence, upon the \( \lambda^{-k/2} \)-re-scaling with \( k \) being the order of the cumulant, the cumulants of order higher than two vanish asymptotically yielding the required Gaussian limit; see [3] for details.

More generally, for \( f \in \mathcal{B}(Q_1) \) and when \( \xi \) satisfies the moment condition (3.3) for all \( p > 2 \), the rate (3.11) holds by following \textit{verbatim} the the Stein approach of [26], using wide sense stabilization and the exponential clustering Lemma 4.1 instead of stabilization. Combining (3.7) and (3.11) yields (3.9) for \( f \in \mathcal{B}(Q_1) \). This completes the proof of Theorem 3.3.
4.4 Proof of Theorem 3.4

The uniformly decaying bound on all moments of the perturbation term $\delta(\cdot,\cdot;\lambda)$ in (2.10), as stated in (2.11), combined with the exponential stabilization of the perturbation, allows us to use the Hölder and Minkowski inequalities to conclude that the addition of $\delta(\cdot,\cdot;\lambda)$ to $\xi$ does not affect the asymptotic behavior of the first and second order correlation functions. This means that the cumulant-based argument for Theorems 3.1 and 3.2 carries over also for $\hat{\xi}(\cdot,\cdot;\lambda)$ with no further modifications. This yields the bounded perturbed versions of Theorems 3.1 and 3.2 for continuous test functions and under the moment condition (3.3) with $p = 4$. To relax the moment conditions as in the respective statements and to get the results for general bounded test functions we resort again to the approach of [21], which completes the proof of these two theorems for functionals with bounded perturbations. The remaining CLT Theorem 3.3 for functionals with bounded perturbations follows now easily by the stabilization property imposed on the perturbation term $\delta(\cdot,\cdot;\lambda)$ in full analogy with the respective proof of Theorem 3.3.

5 Examples of Gibbs point processes with exponentially localized potentials

The notion of an exponentially localized potential $\Psi$ is general and includes the following non-exhaustive list of the corresponding point processes $\mathcal{P}^\Psi$. If an energy functional $\Psi$ has finite interaction range so that its add-one potential satisfies $\Delta(x,\mathcal{X}) = \Delta(x,\mathcal{X} \cap B_r(x))$ for some $r$, as would be the case in many examples considered by [12], then clearly (2.4) is satisfied and there usually are natural ways of ensuring that the constant $C_1$ is large enough so that exponential localization and (2.5) hold as well. These include decreasing the intensity $\tau$ of the underlying Poisson process $\mathcal{P}$ which corresponds to increasing $\Delta(\cdot,\cdot)$ by a positive constant (low reference intensity/density regime) as well as multiplying $\Psi$ and hence also $\Delta(\cdot,\cdot)$ by some small enough $\beta > 0$ (high temperature regime). The following list is not limited to finite range energy functionals.

(i) Strauss processes. A Strauss process involves perturbing a Poisson process according to an exponential of the number of pairs of points closer than a fixed cutoff. For such processes the add-one potential depends only on points within the cut-off range and so $\psi(r)$ vanishes when $r$ exceeds this cut-off.

(ii) Point processes with pair potential function. A large class of Gibbs point processes [30], known as pairwise interaction point processes and including the Strauss process, has Hamiltonian
\[ \Psi(\mathcal{X}) := \sum \sum_{i<j} \phi(||x_i - x_j||), \quad \mathcal{X} := \{x_i\}, \]

with \( \phi \) bounded below, usually assumed to be positive by absorbing the offending constant into the intensity of the underlying Poisson process. If the pair potential function \( \phi \) has finite range, as would be the case with the Strauss process, then the potential \( \Psi \) is localized since \( \psi \) vanishes beyond the interaction range. On the other hand, suppose the pair potential function has infinite range, but satisfies the following strengthened superstability condition: \( \phi \) decays exponentially fast and \( \phi(s) = +\infty \) for \( s \leq r_0 \), that is there is a hard-core exclusion condition forbidding the presence of two points within distance less than \( r_0 \), \[28\]. In this context then the point process \( \mathcal{P}^\Psi \) is easily verified to be exponentially localized as soon as the intensity \( \tau \) is low enough (low density regime) or \( \phi \) admits a sufficiently small upper bound on its oscillations (high temperature regime).

(iii) **Area interaction point processes.** This is a germ grain process, where the grain shape is a fixed compact convex set and where the potential at each Poisson germ is determined by a function of the intersection of the grains at that germ. As a special and simple instance, suppose that the energy functional \( \Psi_D(\mathcal{X}) \) is a scalar multiple \( \gamma \) of the volume of the union of the radius \( r \) balls centered at points \( x \in \mathcal{X} \cap D \). Then, for \( \gamma \) small enough, the resulting area interaction process (consisting of ‘ordered’ points for negative interaction parameter \( \gamma \) and ‘clustered’ points for positive interaction parameter \( \gamma \)) is exponentially localized. More general energy functionals involve an additive term representing a scalar multiple of the total number of points \[1\], which can be alternatively absorbed into the intensity. As noted in \[1\], area interaction processes plausibly model certain biological processes, including those where the realization of the process represents spatial locations of plants (or animals) consuming food within distance \( r \). The energy functional is then a scalar multiple of the area of the food supplying region. These are described more fully on p. 9 of \[12\] and in \[1\].

(iv) **Point processes defined by the continuum Widom-Rowlinson model.** Another example of the point process \( \mathcal{P}^\Psi \) is that defined in terms of the continuum Widom-Rowlinson model from statistical physics, see \[31\] as well as \[13\]. Here we have fixed radii (say radius equal to \( a \)) spheres of two types, say \( A \) and \( B \), with interpenetrating spheres of similar types but hard-core exclusion between the two types. This defines a point process whose potential is exponentially localized as soon as the reference intensity is low enough, since the function \( \psi(r) \) vanishes when \( r > 2a \). It is known, see ibidem, that the continuum Widom-Rowlinson admits an equivalent reformulation in terms of single-species gas of interpenetrating spheres which is area-interacting in the sense of point (iii) above – this is seen by integrating out the positions of \( B \) particles and keeping...
track of the locations of $A$-particles only. Likewise, upon forgetting the marks carried by the particles in the two-species representation one gets the so-called random cluster representation for the Widom-Rowlinson model, see [6] and [13], from which the law of the Widom-Rowlinson model can be recovered by assigning independent and equiprobable $A$ and $B$-labels to maximal connected clusters of particles, whence the name random cluster model. Theorems 3.1-3.4 are valid for all of these equivalent models, provided the intensity is low enough, as discussed above.

(v) **Point processes given by hard-core model.** An important and natural model falling into the framework of our theory is the so-called hard-core model with low enough reference intensity. In its basic version the hard-core model, extensively studied in statistical mechanics, arises by conditioning a Poisson point process on containing no two points within distance less than $2r$ for some $r > 0$ standing for a parameter of the model. Clearly, this process admits a Gibbsian description with $\Psi$ set to $+\infty$ if there are two points closer than $2r$ from each other and 0 otherwise. Consequently, the potential is exponentially localized if the reference intensity of the underlying Poisson point process is low enough or if $r$ is small enough, which also reduces to decreasing the reference intensity upon appropriate re-scaling (in fact rather than imposing separate conditions on $r$ and the reference intensity $\tau$ it is enough to require that $\tau r^d$ be small enough, as easily checked by re-scaling).

(vi) **Truncated Poisson process.** The hard-core gas is a particular example of a truncated Poisson process. In general, a truncated Poisson process arises by conditioning a Poisson point process on the event that a certain family of constraints is fulfilled. In this paper the constraints imposed are of the following form: we fix a certain family of bounded sets and require that none of these sets contain more than a certain given number of points. Such processes are used in modelling of communication networks [2]. In particular, if we require that no ball of radius $r$ contain more than some constant number $k$ of Poisson points, then $\psi$ vanishes beyond $r$ and the associated point process has an exponentially localized potential, possibly upon decreasing the intensity.

### 6 Applications

Below we indicate some applications of our main results. This list is not exhaustive and does not include applications to e.g. germ-grain models where the germs arise as the realization of the Gibbsian point process $P^\Psi$ with an exponentially localized potential.
6.1 RSA packing with Gibbsian input

Let $\mathcal{X} \subset \mathbb{R}^d$ be locally finite. Consider a sequence of unit volume $d$-dimensional Euclidean balls $B_1, B_2, \ldots$, with centers arriving sequentially at points in $\mathcal{X}$. The first ball $B_1$ to arrive is packed and recursively, for $i = 2, 3, \ldots$ let the $i$th ball be packed if it does not overlap any ball in $B_1, B_2, \ldots, B_{i-1}$ which has already been packed. Let $\xi(x, \mathcal{X})$ be either 0 or 1, depending on whether the ball arriving at $x$ is either packed or discarded.

When $\mathcal{X}$ is the realization of a Poisson point process on $Q_\lambda$, this packing process is known as random sequential adsorption (RSA) with Poisson input on $Q_\lambda$. When $\mathcal{X}$ is the realization of an infinite sequence of independent random $d$-vectors uniformly distributed on the cube $Q_\lambda$, then this is called the RSA process with infinite binomial input; in such cases, RSA packing terminates when it is no longer possible to pack additional balls. In dimension $d = 1$, this process is known as the Rényi car parking problem [27]. In the infinite input setting and when $d = 1$ Rényi [27] (respectively Dvoretzky and Robbins [9]) proved that the total number of parked cars satisfies a weak law of large numbers (respectively central limit theorem) as $\lambda \to \infty$; recently these results were shown to hold for all dimensions in [21] and [29].

Virtually all limit results for RSA packing assume that the input is either Poisson or a fixed number of independent identically distributed random variables. To the best of our knowledge, RSA packing problems with Gibbsian input have not been considered before in the literature. The following theorem widens the scope of the existing limit results for RSA packing. Put

$$
\mu_\lambda^\xi := \sum_{x \in \mathcal{P}_\lambda \cap Q_\lambda} \xi(x, \mathcal{P}_\lambda \cap Q_\lambda) \delta_{x/\lambda^d},
$$

so that $N(\mathcal{P}_\lambda \cap Q_\lambda) := \sum_{x \in \mathcal{P}_\lambda \cap Q_\lambda} \xi(x, \mathcal{P}_\lambda \cap Q_\lambda)$ denotes the total number of balls packed on $Q_\lambda$ from the collection of balls with centers in $\mathcal{P}_\lambda \cap Q_\lambda$.

**Theorem 6.1** Let $\mathcal{P}_\lambda$ be Gibbsian input with an exponentially localized potential. Then

$$
\lambda^{-1} N(\mathcal{P}_\lambda \cap Q_\lambda) \to \tau \mathbb{E} \left[ \xi(0, \mathcal{P}_\lambda) \exp(-\Delta(0, \mathcal{P}_\lambda)) \right] \quad \text{in } L^2
$$

and

$$
\lambda^{-1} \text{Var}[N(\mathcal{P}_\lambda \cap Q_\lambda)] \to \tau V^\xi(\tau)
$$

where $V^\xi(\tau)$ is given by (3.8). The finite-dimensional distributions $(\lambda^{-1/2} \langle f_1, \mu_\lambda^\xi \rangle, \ldots, \lambda^{-1/2} \langle f_m, \mu_\lambda^\xi \rangle)$, $f_1, \ldots, f_m \in \mathcal{B}(Q_1)$, converge to those of a mean zero Gaussian field with covariance kernel

$$
(f_1, f_2) \mapsto \tau V^\xi(\tau) \int_{Q_1} f_1(x) f_2(x) dx.
$$
**Remark.** As spelled out in [24], Theorem 6.1 also applies to related packing models, including spatial birth growth models with Gibbsian input as well as RSA models with balls replaced by particles of random size/shape/charge, and ballistic deposition models.

**Proof.** The approach used in [24] shows that the packing functional $\xi(x, \cdot)$ is exponentially stabilizing on Poisson-like sets. Indeed, any Poisson-like set $\Xi$ can be coupled on the common underlying probability space with a dominating Poisson point process $\mathcal{P}_\tau$ of finite intensity $\tau$, $\tau$ large, and containing $\Xi$ a.s. Now, the idea underlying the argument in [24] shows that the packing status of a point $x$ in a configuration $\mathcal{X}$ depends on $\mathcal{X}$ only through its algorithmically determined sub-configuration $Cl[x, \mathcal{X}]$ referred to as the *causal cone* or *causal cluster* of $x$ in the presence of $\mathcal{X}$, see [24] for details. The causal cluster $Cl[x, \mathcal{X}]$ is easily seen to be non-decreasing in $\mathcal{X}$. In particular, using that $\Xi \subseteq \mathcal{P}_\tau$ yields $Cl[x, \Xi] \subseteq Cl[x, \mathcal{P}_\tau]$ a.s. for $x \in \Xi$. However, by the arguments in section 4 of [24], the causal clusters generated by points of $\mathcal{P}_\tau$ exhibit exponential decay, and hence so do causal clusters of points in $\Xi$ showing that the packing functional $\xi(x, \cdot)$ is exponentially stabilizing on Poisson-like sets, in particular on $\mathcal{P}^\Psi$. In other words, $\xi(x, \cdot)$ is exponentially stabilizing in the wide sense. Clearly $\xi$ satisfies the bounded moment condition (3.3) and therefore Theorems 3.1-3.3 show that the $\langle f, \bar{\mu}_\lambda \rangle$, $f \in \mathcal{B}(Q_1)$, satisfy the weak law of large numbers and central limit theorem given by (6.1-6.3), respectively.

6.2 Functionals of Euclidean graphs on Gibbsian input

In many cases, showing exponential stabilization of functionals of geometric graphs over Poisson point sets [3, 23], can be reduced to upper bounding the probability that regions in $\mathbb{R}^d$ are devoid of points by a term which decays exponentially with the volume of the region. When the underlying point set is Poisson, as in [3, 23], then we obtain the desired exponential decay. When the underlying point set is Poisson-like, the desired exponential decay is an immediate consequence of condition (2.7). In this way the existing stabilization proofs for functionals over Poisson point sets carry over to stabilizing functionals on Poisson-like point sets. This extends central limit theorems for functionals of Euclidean graphs on Poisson input to the corresponding central limit theorems for functionals defined over Gibbsian input. The following applications illustrate this.

(i) *k*-nearest neighbors graph. The *k*-nearest neighbors (undirected) graph on the vertex set $\mathcal{X}$, denoted $NG(\mathcal{X})$, is defined to be the graph obtained by including $\{x, y\}$ as an edge whenever $y$ is one of the $k$ nearest neighbors of $x$ and/or $x$ is one of the $k$ nearest neighbors of $y$. The *k*-nearest neighbors (directed) graph on $\mathcal{X}$, denoted $NG'(\mathcal{X})$, is obtained by placing a directed edge between
each point and its \( k \)-nearest neighbors.

**Total edge length of \( k \)-nearest neighbors graph.** Let \( L(X) \) denote the total edge length of \( NG(X) \) and let \( \xi(x,X) \) denote one half the sum of the edge lengths of edges in \( NG(X) \) which are incident to \( x \). Put

\[
\mu_\lambda^\xi := \sum_{x \in \mathcal{P} \cap Q_\lambda} \xi(x, \mathcal{P} \cap Q_\lambda) \delta_{x/\lambda^{1/d}}.
\]

**Theorem 6.2** Let \( \mathcal{P} \) be Gibbsian input with exponentially localized potential. Then

\[
\lambda^{-1} L(\mathcal{P} \cap Q_\lambda) \to \tau \mathbb{E} [\xi(0, \mathcal{P}) \exp(-\Delta(0, \mathcal{P}))] \quad \text{in } L^2
\]

and

\[
\lambda^{-1} \text{Var}[L(\mathcal{P} \cap Q_\lambda)] \to \tau V^\xi(\tau)
\]

where \( V^\xi(\tau) \) is given by (3.8). The finite-dimensional distributions \( (\lambda^{-1/2} (f_1, \bar{\mu}_\lambda^\xi), ..., \lambda^{-1/2} (f_m, \bar{\mu}_\lambda^\xi)) \), \( f_1, ..., f_m \in \mathcal{B}(Q_1) \), converge to those of a mean zero Gaussian field with covariance kernel

\[
(f_1, f_2) \mapsto \tau V^\xi(\tau) \int_{Q_1} f_1(x) f_2(x) dx.
\]

**Remark.** Theorem 6.2 generalizes Theorem 6.1 of [23], which is restricted to nearest neighbor graphs defined on Poisson input.

**Proof.** Let \( \Xi \) be a Poisson-like point set. Considering the arguments in the proofs of Theorem 6.1 of [23] and Theorem 3.1 of [3], it is easily seen that the set of edges incident to any point \( x \) in \( NG(\Xi) \) is unaffected by the addition or removal of points outside a ball of random radius \( R \). Moreover, the radius \( R \) has exponentially decaying tails, which may be seen as follows. For simplicity we prove exponential stabilization in dimension two, but the argument is easily extended to higher dimensions by using cones instead of triangles (for \( d = 1 \) we use intervals instead of triangles). For each \( t > 0 \) construct \( 6(k+1) \) triangles \( T_j(t) \), \( 1 \leq j \leq 6(k+1) \), such that \( x \) is a vertex of each triangle and such that each triangle with edge containing \( x \) has length \( t \). Let \( R_x \) be the minimum \( t \) such that each triangle contains at least one point from \( \Xi \). In such a situation, the union of the \( 6(k+1) \) triangles \( T_j(t) \), \( 1 \leq j \leq 6(k+1) \), may be partitioned into \( 6 \) equilateral triangles with common edge length \( t \), each triangle containing at least \( k+1 \) points. Then, because \( \Xi \) is Poisson-like, it follows that \( \mathbb{P}[R_x \geq t] \leq 6(k+1) \exp(-C_3 t^d) \). Moreover, as explained in the proof of Theorem 6.1 of [23], simple geometry shows that \( 4R_x \) is a radius of stabilization for the functional \( \xi \) at \( x \). Thus \( \xi \) is exponentially stabilizing.
An easy modification of the proof of Lemma 6.2 of [23] shows that moreover satisfies the $p$-moments condition (3.3) for all $p$. Therefore Theorems 3.1-3.3 show that the $\langle f, \bar{\mu}_\lambda^\xi \rangle$, $f \in \mathcal{B}(Q_1)$, satisfy the weak law of large numbers and central limit theorem given by (6.4-6.6), respectively. \[\square\]

**Number of components in nearest neighbors graph.** Let $k = 1$. Given a locally finite point set $\mathcal{X}$, let $\xi^{[c]}(x, \mathcal{X})$ denote the reciprocal of the cardinality of the component in $\text{NG}(\mathcal{X})$ which contains $x$. Thus $H(\mathcal{X}) := \sum_{x \in \mathcal{X}} \xi^{[c]}(x, \mathcal{X})$ denotes the total number of components of $\text{NG}(\mathcal{X})$. Put

$$\mu_\lambda^\xi := \sum_{x \in \mathcal{X} \cap Q_\lambda} \xi^{[c]}(x, \mathcal{X} \cap Q_\lambda) \delta_{x/\lambda^{1/d}}.$$

**Theorem 6.3** Let $\mathcal{P}^\Psi$ be Gibbsian input with exponentially localized potential. Then

$$\lambda^{-1}H(\mathcal{P}^\Psi \cap Q_\lambda) \to \tau \mathbb{E}[\xi(0, \mathcal{P}^\Psi) \exp(-\Delta(0, \mathcal{P}^\Psi))] \text{ in } L^2 \quad (6.7)$$

and

$$\lambda^{-1}\text{Var}[H(\mathcal{P}^\Psi \cap Q_\lambda)] \to \tau V^\xi(\tau) \quad (6.8)$$

where $V^\xi(\tau)$ is given by (3.8). The finite-dimensional distributions $(\lambda^{-1/2}\langle f_1, \bar{\mu}_\lambda^\xi \rangle, ..., \lambda^{-1/2}\langle f_m, \bar{\mu}_\lambda^\xi \rangle)$, $f_1, ..., f_m \in \mathcal{B}(Q_1)$, converge to those of a mean zero Gaussian field with covariance kernel

$$(f_1, f_2) \mapsto \tau V^\xi(\tau) \int_{Q_1} f_1(x)f_2(x)dx. \quad (6.9)$$

**Proof.** We establish that $\xi^{[c]}$ is exponentially stabilizing on Poisson-like sets $\Xi$ and appeal to Theorems 3.1-3.3. When $k = 1$, the Poisson-like properties of the input process and the methods of Högström and Meester [16] and Kozakova, Meester, and Nanda [18] show there are no infinite clusters in $\text{NG}(\Xi)$. Moreover, the proof of Theorem 1.1, 1.2 and Proposition 2.2 of [18] and property (2.7) of Poisson-like processes show that the finite clusters in $\text{NG}(\Xi)$ have (super)exponentially decaying cardinalities and diameters. Now we show exponential stabilization of $\xi$ as follows. Let $R_x$ be the radius of the cluster in $\text{NG}(\Xi)$ containing $x$. Write $B_r$ for $B_r(0)$. Put

$$R := \sup_{x \in B_{R_0} \cap \Xi} R_x.$$ 

It is not hard to see that $R$ has exponentially decaying tail. Indeed, writing

$$P[R > t] = \sum_{j=1}^\infty P[\sup_{x \in B_{R_0} \cap \Xi} R_x > t, 2^{-j-1} \leq R_0 < 2^j] \leq \sum_{j=1}^\infty P[\sup_{x \in B_2 \cap \Xi} R_x > t, 2^{-j-1} \leq R_0] \leq \sum_{j=1}^\infty \exp(-C2^j)P[\sup_{x \in B_2 \cap \Xi} R_x > t]$$

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and then noting that the cardinality of $x \in B_{2^j} \cap \Xi$ decays polynomially fast in $2^j$ with overwhelming probability, we obtain the desired exponential decay of $R$. We can also show that $4R$ is a radius of stabilization for $\xi^{[c]}$ at the origin (see proof of Lemma 6.1 of [23]). Since $\xi^{[c]}$ trivially satisfies the bounded moments condition (3.3) for all $p$, the weak law and central limit theorem for $H(P^\Psi \cap Q_\lambda)$ and $\lambda^{-1/2}(f_{1,\bar{\mu}_\lambda})$ follows by Theorems 3.1- 3.3.

(ii) Voronoi tessellations. Given $\mathcal{X} \subset \mathbb{R}^d$ and $x \in \mathcal{X}$, the set of points in $\mathbb{R}^d$ closer to $x$ than to any other point of $\mathcal{X}$ is a convex polyhedral cell $C(x, \mathcal{X})$. The collection of cells $C(x, \mathcal{X}), x \in \mathcal{X}$, form a partition of $\mathbb{R}^d$ which is termed the Voronoi tessellation induced by $\mathcal{X}$.

Total edge length. Given $\mathcal{X} \subset \mathbb{R}^2$, let $L(x, \mathcal{X})$ denote one half the total edge length of the finite edges in the cell $C(x, \mathcal{X})$. It is easy to see that $L$ is exponentially stabilizing on Poisson-like sets $\Xi$. Indeed, when $d = 2$, it suffices to follow the arguments in the proof of Theorem 8.1 of [23] and to note that stabilization radius depends on finding a minimum edge length $t$ such that 12 isosceles triangles with this edge length have at least one point from $\Xi$ in them. Because $\Xi$ is Poisson-like we may follow the arguments in [23] verbatim to see that $L$ stabilizes. See section 6.3 of [21] for the case $d > 2$.

As in section 6.3 of [21], we may show that $L$ also satisfies the moment condition (3.3) for $p = 3$. It follows that the total edge length $\sum_{x \in P^\Psi \cap Q_\lambda} L(x, P^\Psi \cap Q_\lambda)$ of the Voronoi tessellation on $P^\Psi \cap Q_\lambda$ satisfies the weak law and central limit theorem as $\lambda \to \infty$. In other words, the Voronoi analog of Theorem 6.2 holds.

(iii) Other proximity graphs. There are further examples where showing exponential stabilization of functionals of geometric graphs (in the wide sense) involves upper bounding the probability that regions in $\mathbb{R}^d$ are devoid of Poisson-like points. Such estimates are available in the Poisson setting and it is not difficult to extend them to Poisson-like point sets.

In this way, by modifying the methods of [23] (sections 7 and 9) and [3] (section 3.1), we obtain weak laws of large numbers and central limit theorems for the total edge length of the sphere of influence graph, the Delaunay graph, the Gabriel graph, and the relative neighborhood graph over Gibbsian input $P^\Psi$.

6.3 Gibbsian continuum percolation

Let $\mathcal{X}$ be a locally finite point set and connect all pairs of points which are at most a unit distance apart. The resulting graph is equivalent to the basic model of continuum percolation, in which
one considers the union of the radius 1 balls centered at points of \( \mathcal{X} \), see Section 12.10 in [15]. Let \( \xi^{[c]}(x, \mathcal{X}) \) be the reciprocal of the size of the component containing \( x \), so that

\[
N(\mathcal{X}) = \sum_{x \in \mathcal{X}} \xi^{[c]}(x, \mathcal{X})
\]

counts the number of finite components in \( G \).

Section 9 of [23] discusses central limit theorems for \( N(\mathcal{P} \cap Q_\lambda) \). Using Theorem 3.3 we can generalize these results to obtain a central limit theorem for the number of components \( N(\mathcal{P}^{\Psi} \cap Q_\lambda) \) in the continuum percolation model on Gibbsian input in the subcritical regime, possibly of interest in the context of sensor networks on Gibbsian point sets. To formulate this central limit theorem assume that our Gibbs point process \( \mathcal{P}^{\Psi} \) with exponentially localized potential is such that it admits a stochastic upper bound by a homogeneous Poisson point process of some intensity \( \tau \) falling into the subcritical regime of the considered continuum percolation (see Section 12.10 in [15]). Note that due to the Poisson-like nature of the input process \( \mathcal{P}^{\Psi} \) this is always possible upon spatial re-scaling. We argue that \( \xi^{[c]} \) is exponentially stabilizing on Poisson-like sets \( \Xi \) as follows. If \( \Xi \) is Poisson-like and if \( \tau \) is subcritical, then \( \Xi \) is also subcritical by stochastic domination. Consequently, the diameter of the connected cluster emanating from a given point has exponentially decaying tails, see ibidem. This yields the required exponential stabilization upon noting that \( \xi^{[c]}(x, \cdot) \) does not depend on point configurations outside the connected cluster at \( x \). Moreover, \( \xi^{[c]} \) is bounded above by one and thus satisfies the moments condition (3.3). Hence by Theorems 3.1-3.3, \( N(\mathcal{P}^{\Psi} \cap Q_\lambda) \) satisfies the weak law of large numbers and central limit theorem, exactly as in the statement of Theorem 6.3.

### 6.4 Functionals on Gibbsian loss networks

Given the Poisson point process \( \mathcal{P} := \mathcal{P}_\tau \), consider the following Gibbs point process. Fix an integer \( m \in \mathbb{N} \). Attach to each point of \( \mathcal{P} \) a bounded convex grain \( K \) and put the potential \( \Psi \) to be infinite whenever the grain \( K \) at one point has non-empty intersection with more than \( m \) other grains. This condition prohibits overcrowding, and, for more general repulsive models, one can put \( \Psi \) large and finite whenever the grain \( K \) at one point has non-empty intersection with a large number (some number less than \( m \)) of other grains. The resulting point process, which we call \( \mathcal{P}^{\Psi} \), represents a version of spatial loss networks appearing in mobile and wireless communications. As discussed in point (vi) in Subsection 5 the so defined \( \Psi \) is exponentially localized as soon as the underlying intensity \( \tau \) is small enough.
Let $\mathcal{K}$ be an open convex cone in $\mathbb{R}^d$ (a cone is a set that is invariant under dilations) with apex at the origin. Given $x, y \in \mathcal{P}^\Psi$, we say that $y$ is connected to $x$, written $x \to y$, if there is a sequence of points $\{x_i\}_{i=1}^n \in (\mathcal{K} + x) \cap \mathcal{P}^\Psi$, $|x_i - x_{i+1}| \leq 1$, $|x_1 - x| \leq 1$ and $|y - x_{n+1}| \leq 1$. If the length of this sequence does not exceed a given $m$, we write $x \to_m y$. For all $r > 0$ let $B^\mathcal{K}_r(x) := x + (\mathcal{K} \cap B_r(0))$.

Coverage functionals. The functional $\xi(x, \mathcal{P}^\Psi) := \sup \{ r \in \mathbb{R} : x \to y \text{ for all } y \in B^\mathcal{K}_r(x) \cap \mathcal{P}^\Psi \}$ determines the maximal coverage range of the network at $x$ in the direction of the cone $\mathcal{K}$.

The coverage measure is $\mu^\Psi_\lambda := \sum_{x \in \mathcal{P}^\Psi \cap Q_\lambda} \xi(x, \mathcal{P}^\Psi \cap Q_\lambda) \delta_{x/\lambda^d}$. Confining attention to $\mathcal{P}$, where $\tau$ belongs to the subcritical regime for continuum percolation, $\mathcal{P}^\Psi$ is in turn subcritical because of Poisson domination. Since the continuum percolation clusters generated by any Poisson-like set $\Xi$ have exponentially decaying diameter, it follows that $\xi$ stabilizes in the wide sense (recall the proof for the number of components in the continuum percolation model) and that $\xi$ admits an exponential moment. By appealing to Theorems 3.1 and 3.3, we obtain a weak law of large numbers and central limit theorem for both the coverage measure $\mu^\Psi_\lambda$ and the total coverage $\sum_{x \in \mathcal{P}^\Psi \cap Q_\lambda} \xi(x, \mathcal{P}^\Psi \cap Q_\lambda)$.

Network reach functional. Say that the network has reach at least $r$ at $x$ if $x \to y$ for all $y \in B^\mathcal{K}_r(x) \cap \mathcal{P}^\Psi$. Put $\xi_r(x, \mathcal{P}^\Psi) := 1$ if the network has reach at least $r$ at $x$ and otherwise put $\xi_r(x, \mathcal{P}^\Psi) := 0$. Theorems 3.1 and 3.3 yield a weak law of large numbers and central limit theorem for the total network reach $\sum_{x \in \mathcal{P}^\Psi \cap Q_\lambda} \xi_r(x, \mathcal{P}^\Psi \cap Q_\lambda)$.

Number of customers obtaining coverage. Independently mark each point $x$ of $\mathcal{P}^\Psi$ with mark $T$ (transmitter) with probability $p > 0$ and with mark $R$ (receiver) with the complement probability. Then define the reception functional $\xi(x, \mathcal{P}^\Psi)$ to be 1 if $x$ is marked with $T$ or (when $x$ is marked with $R$) if $z \to x$ for some $z$ in the transmitter set $\{z \in \mathcal{P}^\Psi : z \text{ marked with } T\}$. Put $\xi(x, \mathcal{P}^\Psi)$ to be zero otherwise. Thus $\xi(x, \cdot)$ counts when a customer at $x$ gets coverage and the limit theory for the sum $\sum_{x \in \mathcal{P}^\Psi \cap Q_\lambda} \xi(x, \mathcal{P}^\Psi \cap Q_\lambda)$, which counts the total number of receivers (customers) obtaining network coverage, is given by Theorems 3.1 and 3.3.

Connectivity functional. Given a broadcast range $r > 0$ and the transmitter set $\{z \in \mathcal{P}^\Psi : z \text{ marked with } T\}$, let $\xi_r(x, \mathcal{P}^\Psi)$ be the minimum number, say $m$, such that every point in $y \in B^\mathcal{K}_r(x) \cap \mathcal{P}^\Psi$ can be reached from some transmitter $z \in \mathcal{P}^\Psi$ with $m$ or fewer edges or hops, that is to say there exists a transmitter $z$ such that $z \to_m y$ for all $y \in B^\mathcal{K}_r(x) \cap \mathcal{P}^\Psi$. Thus all receivers in the broadcast range $r > 0$ can be linked to a transmitter in $m$ or fewer hops. Small values of $\xi_r(x, \mathcal{P}^\Psi)$ represent high network connectivity; for each $r > 0$, Theorems 3.1 and 3.3 provide a weak law.
of large numbers and central limit theorem for the connectivity functional $\sum \xi_r(z, P^\Psi \cap Q_\lambda)$ as $\lambda \to \infty$.

7 Gibbsian quantization for non-singular probability measures

Quantization for probability measures concerns the best approximation of a $d$-dimensional probability measure $P$ by a discrete measure supported by a set $\mathcal{X}_n$ having $n$ atoms. It involves a partitioning problem of the underlying space and it arises in a variety of scientific fields, including information theory, cluster analysis, stochastic processes, and mathematical models in economics [14]. The goal is to optimally represent $P$, here assumed non-singular with density $h$, with a point set $\mathcal{X}_n$, where optimality involves minimizing the $L^r$ stochastic quantization error (or ‘random distortion error’) given by

$$I(\mathcal{X}_n) := \int_{\mathbb{R}^d} \left( \min_{x \in \mathcal{X}_n} |y - x| \right)^r dP(dy) = \sum_{x \in \mathcal{X}} \int_{C(x, \mathcal{X}_n)} |y - x|^r dP(dy).$$

Recall that for all $x$ and locally finite point sets $\mathcal{X}$, $C(x, \mathcal{X})$ denotes the Voronoi cell (‘Voronoi quantizer’) induced by the Euclidean norm around $x$ with respect to $\mathcal{X}$.

The optimal (non-random) quantization error is given by $\min_{\mathcal{X}_n} I(\mathcal{X}_n)$ and the seminal work of Bucklew and Wise [5] shows that this minimal error satisfies

$$\lim_{n \to \infty} n^{r/d} \min_{\mathcal{X}_n} I(\mathcal{X}_n) = Q_{r,d} ||h||_{d/(d+r)}$$

(7.1)

where $||h||_{d/(d+r)}$ denotes the $d/(d+r)$ norm of the density $h$ and where the so-called $r$th quantization coefficient $Q_{r,d}$ is some positive constant not known to have a closed form expression.

The first order asymptotics for the distortion error on i.i.d. points sets (that is to say letting $\mathcal{X}_n$ consist of i.i.d. random variables) was first investigated by Zador [32] and later by Graf and Luschgy [14] and Cohort [7]. Letting $X_n$ be i.i.d. random variables with common density $h^{d/(d+r)}/\int h^{d/(d+r)}$ and $\omega_d$ the volume of the unit radius $d$-dimensional ball in $\mathbb{R}^d$, Zador’s theorem shows

$$\lim_{n \to \infty} n^{r/d} I(X_n) = \omega_d^{-r/d} \Gamma(1 + r/d) ||h||_{d/(d+r)},$$

(7.2)

whence (see Prop. 9.3 in [14]) the upper bound

$$Q_{r,d} \leq \omega_d^{-r/d} \Gamma(1 + r/d).$$

(7.3)
Molchanov and Tontchev [20] have pointed out the desirability for quantization via Poisson point sets and our purpose here is to establish asymptotics of the quantization error on Gibbsian input. This is done as follows. For \( \lambda > 0 \) and a finite point configuration \( \mathcal{X} \) we abbreviate \( \mathcal{X}(\lambda) := \lambda^{-1/d} \mathcal{X} \). Moreover, we write \( \tilde{\mathcal{X}} := \mathcal{X} \cap Q_1 \) so that in particular \( \tilde{\mathcal{X}}(\lambda) := \lambda^{-1/d} \mathcal{X} \cap Q_1 \). Recall also that we write \( \mathcal{P}^\Psi \) for \( \mathcal{P}^\Psi \) as in the previous sections. Consider the random point measures induced by the distortion arising from \( \tilde{\mathcal{P}}^\Psi(\lambda) \), namely

\[
\mu^\Psi_\lambda := \sum_{x \in \tilde{\mathcal{P}}^\Psi(\lambda)} \int_{C(x, \tilde{\mathcal{P}}^\Psi(\lambda))} |y-x|^\tau P(dy)\delta_x.
\]

We will be interested in the asymptotic behavior of the random integrals \( \langle f, \mu^\Psi_\lambda \rangle \). Clearly, when \( f \equiv 1 \) then \( \langle f, \mu^\Psi_\lambda \rangle \) gives another expression for the distortion \( I(\tilde{\mathcal{P}}^\Psi(\lambda)) \). On the other hand, if \( f = 1_B \), then \( \langle f, \cdot \rangle \) measures the local distortion. This section establishes mean and variance asymptotics for \( \langle f, \mu^\Psi_\lambda \rangle \) as well as convergence of the finite-dimensional distributions of \( \langle f, \mu^\Psi_\lambda \rangle \). Since we will no longer be working with translation invariant \( \xi \), we will need to appeal to Theorem 3.4.

Put

\[
M^\Psi(\tau) := \int_{C(0, \mathcal{P}^\Psi)} |w|^\tau dw \exp(-\Delta(0, \mathcal{P}^\Psi))
\]

where, recall, \( \tau \) is the intensity of the reference process \( \mathcal{P} \). Note that \( M^\Psi(\tau) \) does depend on \( \tau \) through \( \mathcal{P}^\Psi \). Changing the order of integration we have

\[
\mathbb{E} M^\Psi(\tau) = \mathbb{E} \int_{\mathbb{R}^d} |y|^\tau \mathbb{E}[\exp(-\Delta(0, \mathcal{P}^\Psi)) \mathbb{1}_{\mathcal{P}^\Psi \cap B_1(y)} = 0] dy = \quad (7.4)
\]

\[
\int_{\mathbb{R}^d} |y|^\tau \mathbb{E}[\exp(-\Delta(0, \mathcal{P}^\Psi)) \mathbb{1}_{\mathcal{P}^\Psi \cap B_1(y)} = 0] dy.
\]

In the special case where \( \Psi \equiv 0 \) (i.e. \( \mathcal{P}^\Psi \) coincides with the reference process \( \mathcal{P} \)) and where the intensity \( \tau \) of \( \mathcal{P} \) is 1 we readily get \( \mathbb{E} M^0(1) = \Gamma(1 + \frac{\tau}{2}) \omega_d^{-\tau/d} \). More generally \( \mathbb{E} M^0(\tau) = \tau^{-\frac{1}{1+\tau/d}} \Gamma(1 + \frac{\tau}{2}) \omega_d^{-\tau/d} \). Put

\[
V^\Psi(\tau) := \mathbb{E}[M^\Psi(\tau)]^2 + \int_{\mathbb{R}^d} \left( \mathbb{E} \left[ \int_{C(0, \mathcal{P}^\Psi \cup \{y\})} |w|^\tau dw \int_{C(y, \mathcal{P}^\Psi \cup \{0\})} |w-y|^\tau dw \exp[-\Delta(\{0, y\}, \mathcal{P}^\Psi)] \right] \right) \left( \mathbb{E} M^\Psi(\tau) \right)^2 dy.
\]

For any random point measure \( \rho \), recall that \( \bar{\rho} \) denotes its centered version, that is \( \bar{\rho} := \rho - \mathbb{E} \rho \).

**Theorem 7.1** Assume that the density \( h \) of \( P \) is continuous on \( Q_1 \). We have for each \( f \in \mathcal{B}(Q_1) \)

\[
\lim_{\lambda \to \infty} \lambda^{\tau/d} \langle f, \mu^\Psi_\lambda \rangle = \tau \mathbb{E} M^\Psi(\tau) \int_{Q_1} h(x)f(x)dx \quad \text{in } L^2
\]

(7.5)
and
\[
\lim_{\lambda \to \infty} \lambda^{1+2r/d} \text{Var}(f, \mu^\Psi_\lambda) = \tau V^\Psi(\tau) \int_{Q_1} f^2(x) h^2(x) dx. \tag{7.6}
\]

The finite-dimensional distributions \(\lambda^{-1/2+r/d}(\langle f_1, \mu^\Psi_\lambda \rangle, ..., \langle f_k, \mu^\Psi_\lambda \rangle)\), \(f_1, ..., f_k \in B(Q_1)\), of the random measures \((\lambda^{-1/2+r/d} \mu^\Psi_\lambda)\) converge as \(\lambda \to \infty\) to those of a mean zero Gaussian field with covariance kernel
\[
(f_1, f_2) \mapsto \tau V^\Psi(\tau) \int_{Q_1} f_1(x) f_2(x) h^2(x) dx, \quad f_1, f_2 \in B(Q_1). \tag{7.7}
\]

Upper bounds for quantization coefficients. When \(f \equiv 1\) the right hand side of (7.5) gives
\[
\lim_{\lambda \to \infty} \lambda^{r/d}(1, \mu^\Psi_\lambda) = \lim_{\lambda \to \infty} \lambda^{r/d} I(\hat{\mu}^\Psi_\lambda) = \tau \mathbb{E}[M^\Psi(\tau)].
\]
Since the Bucklew and Wise limit (7.1) is necessarily no larger than the right hand side of the above, this shows that in addition to the bound (7.3), that the \(r\)th quantization coefficient \(Q_{r,d}\) also satisfies the upper bound
\[
Q_{r,d} \leq (||h||_{d/(d+r)})^{-1} \tau \mathbb{E}[M^\Psi(\tau)].
\]
Recall from our discussion above that when \(\Psi \equiv 0\) (i.e. \(\mathcal{P}^\Psi\) is Poisson) and when \(f \equiv 1\), then the right hand side of (7.5) equals \(\tau^{-r/d} \omega^{-r/d}_d \Gamma(1 + r/d)\) and thus
\[
Q_{r,d} \leq (||h||_{d/(d+r)})^{-1} \tau^{-r/d} \omega^{-r/d}_d \Gamma(1 + r/d).
\]
We believe, although are not yet able to provide a full proof, that whereas the distortion error (7.5) is relatively large for Poisson input, it can be made smaller if we restrict to point sets which themselves enjoy some built-in repulsivity while keeping the same mean point density. Indeed, given a fixed mean number of test points it seems more economical to spread them equidistantly over the domain of target distribution than to allow for local overfulls of test points in some regions, which only result in wasting test resources with the quantization quality improvement considerably inferior to that which would be achieved should we shift the extraneous points to regions of lower test point concentration. In other words, the right hand side of (7.5) for repulsive Gibbs point processes should be smaller than the corresponding distortion for the Poisson point process with the same point density. It should be emphasized here that in order to stay within the set-up of our asymptotic theory we have to assume that the repulsivity is weak. On the other hand, it is very likely that going to some extent beyond this requirement may lead to even smaller quantization.
errors. These seem to be natural and interesting questions, yet at present we cannot handle them
with our current techniques.

Proof of Theorem 7.1. We claim that the assertions of Theorem 7.1 can be reduced to an
application of Theorem 3.4 for functionals with bounded perturbations. We do it first assuming
that the density $h$ is bounded away from 0. To this end, consider the following parametric family
of geometric functionals:

$$
\hat{\xi}(x, \mathcal{X}; \lambda) := \int_{C(x, \mathcal{X})} |y - x|^r \frac{P(d\lambda^{-1/d}y)}{\lambda h(\lambda^{-1/d}x)}
$$

Putting

$$
\xi(x, \mathcal{X}) := \int_{C(x, \mathcal{X})} |y - x|^r dy
$$

we obtain the bounded perturbed representation (2.10) for $\hat{\xi}(\cdot, \cdot; \lambda)$ with

$$
\delta(x, \mathcal{X}; \lambda) := \int_{C(x, \mathcal{X})} |y - x|^r \frac{h(\lambda^{-1/d}y) - h(\lambda^{-1/d}x)}{h(\lambda^{-1/d}x)} dy.
$$

It is easily seen that on Poisson-like input both $\xi$ and $\delta$ as given in (7.9) and (7.10) stabilize
exponentially with common stabilization radius determined by the diameter of the Voronoi cell
around the input point; see Subsection 6.2(ii). We claim that this $\delta(\cdot, \cdot; \lambda)$ also satisfies the bounded
moments condition (2.11). To this end, use the fact that $h$ is bounded away from 0 and write

$$
|\delta(x, \mathcal{X}; \lambda)| \leq C \int_{C(x, \mathcal{X})} |y - x|^r \frac{h(\lambda^{-1/d}y) - h(\lambda^{-1/d}x)}{h(\lambda^{-1/d}x)} dy.
$$

Letting $\mathcal{X} := \mathcal{P}^\psi$ and using the exponential decay of the Voronoi cell diameter on Poisson-like
input, see Subsection 6.2, we conclude from (7.11) that, for all $p > 0$ and $\lambda > 0$

$$
\sup_x \mathbb{E} \left[ |\delta(x, \mathcal{P}^\psi; \lambda)|^p \right] \leq \sup_x \mathbb{E} \left[ \int_{C(x, \mathcal{P}^\psi)} |y - x|^r (h(\lambda^{-1/d}y) - h(\lambda^{-1/d}x)) dy \right]^p.
$$

By the translation invariance of $\mathcal{P}^\psi$ and by the uniform continuity of the density $h$, this is bounded
above by

$$
\mathbb{E} \left[ \int_{C(0, \mathcal{P}^\psi)} |y|^r \omega_h(\lambda^{-1/d}|y|) dy \right]^p := L(p, r, \lambda),
$$

where $\omega_h(\cdot)$ is the modulus of continuity of $h$. Since $\mathbb{E} \left[ \int_{C(0, \mathcal{P}^\psi)} |y|^r \omega_h(\lambda^{-1/d}|y|) dy \right]^p$ is dominated by
an integrable function of $\omega$ uniformly over $\lambda$, namely by a constant multiple of the $p(r + d)^{th}$ power
of the Voronoi cell diameter, and since $\mathbb{E} \left[ \int_{C(0, \mathcal{P}^\psi)} |y|^r \omega_h(\lambda^{-1/d}|y|) dy \right]^p$ converges to zero as $\lambda \to \infty$
for almost all $\omega$, we may use the bounded convergence theorem to conclude that $L(p, r, \lambda) \to 0$
as \( \lambda \to \infty \). This is clearly enough to get (2.11) and hence \( \hat{\xi}(\cdot, \cdot; \lambda) \) is an asymptotically negligible bounded perturbation of \( \xi(\cdot, \cdot) \). To proceed, we note that the quantization empirical measure \( \mu^\Psi_\lambda \) satisfies for each \( f \in \mathcal{B}(Q_1) \)

\[
\langle f, \mu^\Psi_\lambda \rangle = \lambda^{-1-r/d} \langle fh, \mu^{\hat{\xi}}_\lambda \rangle,
\]

(7.12)

where \( \mu^{\hat{\xi}}_\lambda \) is the standard empirical measure (3.2) for \( \hat{\xi} \) as in (7.9), that is to say

\[
\mu^{\hat{\xi}}_\lambda := \sum_{x \in \mathcal{P} \cap Q_\lambda} \hat{\xi}(x, \mathcal{P} \cap Q_\lambda; \lambda) \delta_{x/\lambda^{1/d}}.
\]

On the other hand, it is easily verified that \( \xi \) satisfies all assumptions of our limit Theorems 3.1, 3.2 and 3.3. Consequently, Theorem 3.4 can be applied for \( \hat{\xi} \), which yields Theorem 7.1 via the formula (7.12) allowing us to translate results for \( \mu^{\hat{\xi}}_\lambda \) to the corresponding results for \( \mu^\Psi_\lambda \). This completes the proof of Theorem 7.1 for \( h \) bounded away from 0.

To proceed, assume now that \( h \) fails to be bounded away from 0 and, for \( \varepsilon > 0 \) put \( h_\varepsilon := \max(h, \varepsilon) \) and let \( \mu^\Psi_{\lambda, \varepsilon} \) be the version of \( \mu^\Psi_\lambda \) with \( h \) replaced by \( h_\varepsilon \). Using the definition of \( \mu^\Psi_\lambda \), and the exponential decay of the diameter of Voronoi cells in a Poisson-like environment we easily conclude that

\[
|E[\langle f, \mu^\Psi_\lambda \rangle - \langle f, \mu^\Psi_{\lambda, \varepsilon} \rangle]| = O(\lambda^{-r/d} \varepsilon).
\]

(7.13)

Likewise, using the same we get

\[
\text{Var}[\langle f, \mu^\Psi_\lambda \rangle - \langle f, \mu^\Psi_{\lambda, \varepsilon} \rangle] = O(\lambda^{-1-2r/d} \varepsilon).
\]

(7.14)

Applying Theorem 7.1 for \( h_\varepsilon \), which is legitimate due to \( h_\varepsilon \) being bounded away from 0, and then using (7.13) and (7.14) we readily get the required expectation and variance asymptotics for \( \langle f, \mu^\Psi_\lambda \rangle \) as well as the \( L^2 \) weak law of large numbers, which follows by the variance convergence. The remaining central limit theorem statement for \( \langle f, \mu^\Psi_\lambda \rangle \) follows directly by the Stein method as in Theorem 3.3, which is not affected by \( h \) being not bounded away from 0. This completes the proof of Theorem 7.1 for general \( h \).

\[ \blacksquare \]

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\section*{References}

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