Nonsingular Global String Compactifications

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Abstract

We consider an exotic ‘compactification’ of spacetime in which there are two infinite extra dimensions, using a global string instead of a domain wall. By having a negative cosmological constant we prove the existence of a nonsingular static solution using a dynamical systems argument. A nonsingular solution also exists in the absence of a cosmological constant with a time-dependent metric. We compare and contrast this solution with the Randall-Sundrum universe and the Cohen-Kaplan spacetime, and consider the options of using such a model as a realistic resolution of the hierarchy problem.

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There has been a great deal of excitement recently over exotic compactifications of space-time, where our four-dimensional world emerges as a defect in a higher dimensional space-time. This idea, while not new (see [1,2] for some past work on this subject), has received impetus from the unusual suggestion of Randall and Sundrum [3] that a resolution of the hierarchy problem might be forthcoming from just such a scenario. In Randall and Sundrum’s original paper, spacetime was five-dimensional, and our four-dimensional spacetime emerged as a domain wall at one end of the universe; a mirror wall at the other end of the universe plus a conformal factor dependent on the distance between the two was responsible for the suppression of interactions relative to gravity on our ‘visible sector’ domain wall. In a later paper, [4], Randall and Sundrum explicitly demonstrated how the gravitational interactions effectively localised on the ‘hidden’ domain wall, by showing that a five-dimensional universe with a domain wall had low energy spin-2 excitations which were localised on that wall. A more general calculation involving a smooth wall solution has been performed in [5].

A key feature of the Randall-Sundrum solution:
\[ ds^2 = e^{-2k|y|} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2 \]  

is that in order to have a static solution, a negative cosmological constant is required in the bulk spacetime. The value of this cosmological constant, and the value of the wall energy density are related in a precise manner to the five-dimensional Planck mass and \( k \), the constant appearing in the metric (1). Without this precision balance, nonstatic solutions would have to be considered [6], see [7] (and references therein) for a comprehensive summary of domain walls in supergravity.

Domain walls are, however, familiar from another point of view: in cosmology they are one of a family of topological defect solutions which can arise in the early universe during a symmetry breaking phase transition. Generally, global defects are problematic cosmologically, and it was realised very early on [8] that the existence of domain walls with symmetry breaking scales greater than about 1 MeV must be ruled out, because a network of such defects would rapidly evolve to dominate the energy of the Universe. However, as model defects, they are particularly amenable to analytic study, and have some fascinating properties, most notably that they cause spacetime to compactify [9] under their own self-gravity. However, domain walls are not the only global defects studied in the cosmological context, global strings and monopoles (as well as textures) have also been examined. While the global monopole has a well-defined static metric [10], the global string does not. It was thought for some time that this particular defect was singular [11], its metric having been calculated by Cohen and Kaplan [12]. However, an analysis including time-dependence [13] revealed that with the correct de-Sitter like behaviour of the intrinsic metric of the worldsheet defined by the global vortex, the spacetime could be rendered nonsingular. An event horizon is present in the spacetime, and a compactification can also take place with a mirror string at the other end of the universe [14]. The analysis in [13] however, was purely within the context of Einstein gravity in four dimensions in the absence of a cosmological constant.

The similarities between the domain wall and the global string lead naturally to the question of whether one can generalise the exotic compactification scenario of [4] to the case of two extra dimensions by using a global string, rather than a wall, as was explored recently by Cohen and Kaplan [15], who generalised their original metric [12] to allow for
arbitrary dimensional spacetime. This is distinct from the original extra-dimension scenario of Arkani-Hamed et al., who regarded the two dimensional transverse space as being compact, and Sundrum, who considered the compactification of the transverse space as a result of the conical gravitational nature of N vortices (it is also distinct from the spacetime considered by Chodos and Poppitz, who generalise Sundrum’s work to include a positive cosmological constant in these conventions). Here, there is a single, global, vortex, which asymptotically closes off the spacetime. However, the solution presented by Cohen and Kaplan differs from the original Randall and Sundrum picture in a variety of ways. The first, and most obvious, is that their solution is singular, in an analogous fashion to their four-dimensional global string metric. Secondly, they did not have a cosmological constant, which was a key feature of the Randall-Sundrum set-up in order to obtain a static solution. Finally, for solving the hierarchy problem, Randall and Sundrum had a second domain wall at the other end of the universe. In this letter we point out that it is possible to remedy at least two of these problems: by adding a negative cosmological constant one can obtain a static nonsingular solution, however, the third problem probably remains insoluble, in that it is not possible to have a negative energy vortex. Essentially the trick with the global string is the same as with the domain wall - just as a negative cosmological constant counterbalances the positive impact of the wall’s self-energy to give a static solution in the brane-world, the cosmological constant can be chosen to counterbalance the global string’s tendency for cosmological expansion yielding a static solution.

We begin by considering a general static cylindrically symmetric system, first examining the solutions in the absence of the string. Just as the wall acts as a junction between two different branches of an anti-de-Sitter solution in five dimensions, we can derive an analogous patching that we might expect our string to satisfy in six dimensions. We then analyse the behaviour of the string spacetime exterior to the core, and show that there is precisely one asymptotic solution for the global string spacetime with the required properties for the brane-world conformal or ‘warp’ factor. By integrating out from the core of the string it can be shown that this does indeed match on to the physical vortex solution. The value of the cosmological constant is, however, minutely small, \( \Lambda \approx \epsilon e^{-1/\epsilon} \), where \( \epsilon \) is the effective cosmological constant on the vortex brane-world.

A global string is a vortex solution to a field theory with a spontaneously broken continuous global U(1) symmetry, the prototypical model having a ‘mexican hat’ potential:

\[
\mathcal{L} = (\nabla_\mu \Phi)^\dagger \nabla^\mu \Phi - \frac{\lambda}{4} (\Phi^\dagger \Phi - \eta^2)^2
\]

We will (for now) assume that spacetime has \( p + 3 \) dimensions, the vortex then being a \( p + 1 \)-dimensional submanifold of spacetime - a \( p \)-brane. By writing

\[
\Phi = \eta X e^{i\chi}
\]

we can reformulate the complex scalar field into two real interacting scalar fields, one of which \( (X) \) is massive, the other \( (\chi) \) being the massless Goldstone boson. In this way, the low energy theory is seen to be equivalent in \( n(= p + 3) \) dimensions to an \( (n - 2) \)-form potential. For example in four dimensions, the \( \chi \)-field is equivalent to a Kalb-Ramond \( B_{\mu\nu} \) field, and the effective action for the motion of a global string is the bosonic part of the superstring action.
A vortex solution is characterised by the existence of closed loops in space for which the phase of \( \Phi \) winds around \( \Phi = 0 \) as a closed loop is traversed. This in turn implies that \( \Phi \) itself has a zero within that loop, and this is the core of the vortex. From now on, we shall look for a solution describing an infinitely long straight vortex with winding number 1.

It can be shown (see [13] for a general argument in four dimensions) that the metric will in general have the form

\[
ds^2 = e^{2A(r)}(dt^2 - e^{2b(t)}dz_i^2) - dr^2 - C^2(r)d\theta^2
\]

where \( i = 1...p \) are the spatial coordinates in our lower dimensional spacetime. Note the presence of \( b(t) \) in the metric. In four dimensions this is crucial to obtaining a nonsingular solution. In the absence of a cosmological constant it is also needed in arbitrary numbers of dimensions to ensure nonsingularity, however, just as Randall and Sundrum tuned the bulk cosmological constant to obtain a static solution, it will turn out that we can also obtain a nonsingular static solution by tuning the cosmological constant (albeit in a far more sensitive way).

The system of equations for the global string are

\[
\frac{C''}{C} + p \left( A'' + A'^2 + \frac{A'C'}{C} \right) + \frac{p(p-1)}{2} \left( A'^2 - b^2 e^{-2A} \right) = -\epsilon \hat{T}_0^0 - \Lambda
\]

\[
p(\ddot{b} + b^2) e^{-2A} - pA'^2 - (p + 1) \frac{A'C'}{C} - \frac{p(p-1)}{2} \left( A'^2 - b^2 e^{-2A} \right) = \epsilon \hat{T}_r^r + \Lambda
\]

\[
p(\ddot{b} + b^2) e^{-2A} - (p + 1)A'' - (2p + 1)A'^2 - \frac{p(p-1)}{2} \left( A'^2 - b^2 e^{-2A} \right) = \epsilon \hat{T}_\theta^\theta + \Lambda
\]

\[
\left[ Ce^{(p+1)A}X' \right]' = Ce^{(p+1)A} \left[ \frac{X}{C^2} + \frac{1}{2}X(X^2 - 1) \right]
\]

where we have taken \( \lambda \eta^2 = 1 \), \( \epsilon = \eta^2(M_{\text{pl}})^{-p+1} \) which can be regarded as an effective cosmological constant on the brane, and the energy momentum of the global string fields is

\[
\hat{T}_0^0 = \hat{T}_z^z = \left[ X'^2 + \frac{X^2}{C^2} + \frac{1}{4}(X^2 - 1)^2 \right]
\]

\[
\hat{T}_r^r = \left[ -X'^2 + \frac{X^2}{C^2} + \frac{1}{4}(X^2 - 1)^2 \right]
\]

\[
\hat{T}_\theta^\theta = \left[ X'^2 - \frac{X^2}{C^2} + \frac{1}{4}(X^2 - 1)^2 \right]
\]

First of all, if \( p \neq 1 \), \( \dot{b}^2 = b_0 \), a constant. (If \( p = 1 \) then we have the canonical global string in four dimensions, and the analysis of [13] largely applies.) One can then directly generalise the argument of [13] to show that if \( b_0 \leq 0 \) and \( \Lambda \geq 0 \) the spacetime is necessarily singular. If \( b_0 > 0 \), \( \Lambda = 0 \), the arguments of [13] also generalise to show that there is a nonsingular solution which has an event horizon at a finite distance from the core.

However, we are interested in finding a static solution \( b_0 = 0 \). We begin by analysing the far-field behaviour of such a solution. Outside the core of the vortex, \( X = 1 \), and we see that (5) can be rewritten as
\[
\left[ A'C^{(p+1)A} \right]' \simeq -\frac{2\Lambda}{(p+1)} C^{(p+1)A}
\]

\[
\left[ C'e^{(p+1)A} \right]' \simeq -\frac{C^{(p+1)A}}{(p+1)} \left( \frac{2\epsilon(p+1)}{C^2} + 2\Lambda \right)
\]

\[
\frac{p}{2} A'^2 + \frac{A'C'}{C} \simeq -\frac{\Lambda}{(p+1)} - \frac{\epsilon}{(p+1)C^2}
\]

Setting \( \omega^2 = -2\Lambda(p+2)/(p+1) \), it is easy to see that the anti-de-Sitter solution in these cylindrical coordinates has two branches represented by \( C^{(p+1)A} = C^{p+2} \exp[\pm \omega r] \). These can be seen to be analogous to the two branches of the plane-symmetric AdS solution \( \sqrt{-g} = e^{\pm 4kr} \) which the wall in (1) interpolates between. Here, however, we do not have two disjoint regions of spacetime, so we might expect that the global string solution (analogous to the wall in (1)) should tend to the ‘negative’ AdS branch as \( r \to \infty \), in order that the spacetime is ‘closed off’ and the volume integral converges. The core of the global string will then be interpreted as a means of smoothing off the spacetime to make it nonsingular at \( r = 0 \), just as the wall interpolates between the two branches of planar AdS space. Now let us examine whether this is indeed the case.

Rather than deal directly with the far-field equations (7), we will instead, as in (13) perform a phase plane analysis of the variables

\[
x = \frac{1}{\sqrt{-2\Lambda}} \left( pA' + \frac{C'}{C} \right), \quad y = \frac{1}{\sqrt{-2\Lambda}} \frac{C'}{C}
\]

which, defining the variable \( \rho = r \sqrt{-2\Lambda} \) gives the autonomous dynamical system

\[
x' = \frac{xy}{p} - \frac{(p+1)}{p} y^2
\]

\[
y' = \frac{(p+1)}{p} x(x - y) - y^2 - \frac{p}{(p+1)}
\]

where prime now denotes \( d/d\rho \). This phase plane is characterised by an invariant hyperboloid

\[
x^2 - y^2 = \frac{p}{(p+1)}
\]

and four critical points

\[
c_1^\pm = \left( \pm \frac{p}{(p+1)}, 0 \right) \quad \text{saddle}
\]

\[
c_2^\pm = \pm \left( \frac{\sqrt{\frac{p+1}{p+2}}}{\sqrt{(p+1)(p+2)}} \right) \quad \text{attractor, repellor}
\]

A plot of the phase plane is shown in figure [1].

The asymptotic solutions corresponding to the critical points are:

\[
c_1^\pm: \quad A \sim \pm \sqrt{-2\Lambda} \frac{r}{(p+1)} , \quad C^2 \to \frac{\epsilon(p+1)}{|\Lambda|} \quad \text{as} \ r \to \infty
\]

\[
c_2^\pm: \quad A = \ln C/C_0 \sim \pm \frac{\omega}{p+2} r \quad \text{as} \ r \to \infty
\]
FIG. 1. A plot of the (x,y) phase plane. Critical points are marked with a dot and the invariant hyperboloid by the grey line. As \( p \) varies the plot alters shape, but the qualitative features remain the same.

The latter critical points are immediately identifiable as the two AdS branches, whereas the first two are quite distinct, arising purely because of the global vortex field. We see now that the global vortex can never patch onto an asymptotic AdS solution (with convergent volume integral) as the Randall-Sundrum domain wall does, since this asymptotic solution is the critical point \( c_2 \) which is a repellor in the phase plane. Instead, there exists a single trajectory terminating on \( c_1 \), which, for small \( \epsilon \), can be shown by an analogous argument to that in [13] to match on to the core solution of the equations of motion (5), provided the cosmological constant is tuned very precisely \( \Lambda = \mathcal{O}(\epsilon e^{-1/\epsilon}) \). For \( \epsilon \) close to order unity, this argument would need to be replaced by an (numerical) analysis of the full system of equations. By referring to (12a) we see that this solution has a convergent volume integral, while the transverse \((r, \theta)\) space is in fact infinite.

We therefore see that the global vortex spacetime, like the Randall-Sundrum spacetime, is the warped product of a four-dimensional Minkowski spacetime (with an exponentially decaying conformal factor) and an infinite transverse \((r, \theta)\) space, which asymptotes a cylinder. In fact, at small and intermediate distance scales the solution is imperceptibly different from the Cohen-Kaplan solution in [15]. It is only at the very large scale that the effect of the miniscule cosmological constant makes itself felt, and smoothly rounds off spacetime to the cylinder thus avoiding a singularity. Therefore, although the solution does not asymptote the AdS branch that one might initially have expected by analogy with (1), it does have an infinite transverse space, which directly parallels the metric (1). It is also quite distinct from other compactifications involving two extra dimensions, which are either compact (e.g. [16,17]), infinite (e.g. [2]), which also has electromagnetic fields and an infinite volume) or singular, such as [17]. Since the solution is so close the Cohen-Kaplan exact metric, we
can use their results [15] to obtain an estimate for \( \epsilon \) if we wish to use this spacetime in the sense of [16] to obtain a large four-dimensional Planck mass. As Cohen and Kaplan point out, a mass scale for the global vortex \( X \)-field not too far above the electroweak scale (if we wish to set the six-dimensional planck mass to be roughly at the electroweak scale) gives a large hierarchy between \( M_{4\text{Pl}} \) and \( m_{\text{ew}} \), this corresponds to \( \epsilon \approx 1/138 \) in our notation. However, unlike the Arkani-Hamed et. al. [16] set-up, the extra dimensions here are not at the millimetre scale, but are infinite, it is the warp factor that produces the relationship between \( M_{4\text{Pl}} \) and \( M_{6\text{Pl}} \).

Finally, there is one other option we could consider for the global vortex and that is whether one can have a compact \((r, \theta)\) section with a mirror vortex at the anti-pole. The only way this can be matched onto a core solution is if we take the symmetric trajectory through \( x = y = 0 \). In this case, \( \Lambda \) will still have the same order of magnitude, however, since \( e^{2A} \) returns to unity at the core of the mirror string, this does not represent a hidden/visible universe as with the original Randall-Sundrum scenario [3] and cannot be used in that sense.

In short, while global strings are certainly interesting as an alternative to domain walls in compactification processes, the minuteness of the cosmological constant, and its sensitivity to the parameter \( \epsilon \) suggests that they would be problematic in any physically realistic scenario.

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