MEASURES OF WEAK COMPACTNESS AND FIXED POINT THEORY

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ABSTRACT. In this paper, we study a class of Banach spaces, called $\phi$-spaces. In a natural way, we associate a measure of weak compactness in such spaces and prove an analogue of Sadovskii fixed point theorem for weakly sequentially continuous maps. A counter-example is given to justify our requirement. As an application, we establish an existence result for a Hammerstein integral equation in a Banach space.

1. Introduction

Kuratowski measure of noncompactness $\chi(C)$ of a bounded subset $C$ of a metric space $(X, d)$ is defined to be the infimum of the set of all $\varepsilon > 0$ with the following property:

$$C \text{ can be covered by finitely many sets, each of whose diameter is } \leq \varepsilon.$$ 

Let $X$ be a Banach space and let $B(X)$ denote the collection of nonempty, bounded subsets of $X$. Measure of noncompactness appear in various contexts and are very useful tools in Nonlinear Analysis (see, e.g., [5, 6, 10, 12] and the references therein). In a general way, under some condition of condensing, it allows us to obtain fixed points results. The first result in this direction was Sadovskii’s fixed point theorem [4]. It states that if $M \in B(X)$ is closed, convex and $T : M \to M$ is a condensing mapping (that is, $T$ is continuous and $\chi(T(C)) < \chi(C)$ for all $C \in B(X)$), then $T$ has a fixed point.

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Let $\phi \in \mathcal{L}(X)$. The main purpose of this paper is to investigate what conditions will ensure that the function $\chi \cdot \phi : \mathcal{B}(X) \to [0, \infty)$ is a measure of weak compactness (in the sense that if $C \in \mathcal{B}(X)$ is weakly closed and $\chi(\phi(C)) = 0$, then $C$ is weakly compact) and then to derive from this fact an analogue of Sadovskii’s theorem for weakly sequentially continuous maps. For the sake of convenience, $X$ will be called a $\phi$-space if the existence of such a $\phi$ is verified. A theorem of such type has been obtained by O’Regan [11]. Nevertheless, the definition of our measure as well as its use in the study of fixed points for such maps, seems to be new and original.

This paper is organized as follows: In Section 2, we first formulate the notion of $\phi$-spaces and give a sufficient condition for a Banach space to be a $\phi$-space. We will see that automatically reflexive spaces are $\phi$-spaces. However, there are nonreflexive $\phi$-spaces (see, Example 2). In Section 3, we extend Sadovskii’s fixed point theorem to the weak topology setting. More precisely, we prove that every map weakly sequentially continuous and $\phi$-condensing (see, Definition 3.1) has a fixed point in nonempty closed, convex and bounded subsets of a $\phi$-space (see, Theorem 3.1). Also, we will see that $\phi$-condensing is essential for the conclusion of our main result. Finally, in Section 4, we use our fixed point theory to establish an existence result for Hammerstein integral equation in a Banach space.

2. Preliminaries

Throughout this paper, $X$ stands for a real Banach space with norm $\|\cdot\|$. In what follows, we proceed to define the notion of a $\phi$-space. Let $\mathcal{F}$ be the family of all the bounded and weakly closed subsets of $X$. If $I$ is the identity map on $X$, we shall denote by $\text{span}\{I\}$ the vector space generated by $I$.

**Definition 2.1.** Let $\phi \in \mathcal{L}(X)$ be such that $\phi \notin \text{span}\{I\}$. We will say that $X$ is a $\phi$-space if the following condition is satisfied

$(\mathcal{C})$ if $C \in \mathcal{F}$ and $\phi(C)$ is a relatively compact, then $C$ is weakly compact.

**Example 1.** Every reflexive space is a $\phi$-space for any $\phi \in \mathcal{L}(X)$ with $\phi \notin \text{span}\{I\}$.

In our next result, we give a sufficient condition for a Banach space to be a $\phi$-space.
Proposition 2.1. Let \( X \) be a Banach space. If there is a \( \phi \in \mathcal{L}(X) \) such that \( \phi \notin \text{span}\{I\} \), \( \text{Ker}(\phi) = \{0\} \) and \( \|\phi^{-1}y\| \leq c\|y\| \) for all \( y \in \phi(X) \), \( c > 0 \), then \( X \) is a \( \phi \)-space.

Proof. Take an arbitrary set \( C \in \mathcal{F} \) and suppose that \( \phi(C) \) is relatively compact. Let \( (C_i) \) be a decreasing sequence of nonempty closed, convex subsets of \( C \). Since each \( C_i \) is closed and \( \|\phi^{-1}y\| \leq c\|y\| \) for all \( y \in \phi(X) \), we have that each \( \phi(C_i) \) is closed too. Thus, \( (\phi(C_i)) \) is a decreasing sequence of nonempty closed and convex subsets of \( \overline{\phi(C)} \). By the finite intersection property of compact sets we conclude that \( \bigcap_{i=1}^{\infty} \phi(C_i) \neq \emptyset \). Now since \( \phi^{-1}(\phi(C_i)) = C_i \) we get \( \bigcap_{i=1}^{\infty} C_i \neq \emptyset \). Applying now the Šmulian nested interval principle as in [2, pp. 433], one has that \( C \) is weakly compact. \( \square \)

Remark. From the open mapping theorem we know that if \( \phi \in \mathcal{L}(X) \) is one-to-one and \( \phi(X) \) is a Banach space then there exists \( c > 0 \) with \( \|\phi^{-1}y\| \leq c\|y\| \) for \( y \in X \).

Example 2. This is an example of a \( \phi \)-space that is not reflexive. Let \( C[0,1] \) be the space of the real-valued continuous functions and fix \( m \in C[0,1] \) an increasing continuous function such that \( m(t) \geq a \), for all \( t \in [0,1] \) with \( 0 < a < 1 \). Let us define \( \phi \in \mathcal{L}(C[0,1]) \) by \( \phi(x)(t) = m(t) \cdot x(t) \). Then \( \phi \notin \text{span}\{I\} \) and \( \text{Ker}(\phi) = 0 \). Moreover, if \( (Sy)(t) = m(t)^{-1} \cdot y(t) \) for all \( y \in C[0,1] \), then \( \|Sy\|_{\infty} \leq 1/m_{\infty} \cdot \|y\|_{\infty} \) for every \( y \in C[0,1] \) and \( S_{\phi(C[0,1])} \equiv \phi^{-1} \). By Proposition 2.1 \( C[0,1] \) is a \( \phi \)-space.

We are now going to define a natural measure of weak compactness in a \( \phi \)-space. For convenience, we first summarize the main properties of \( \chi \) which will be used here.

The function \( \chi : \mathcal{B}(X) \to [0,1] \) enjoys the following properties:

1. For any \( C \in \mathcal{B}(X), 0 \leq \chi(C) \leq \delta(C) \) = diameter of \( C \),
2. \( \chi(C) = 0 \) iff \( C \) is relatively compact,
3. if \( C \subset D \), then \( \chi(C) \leq \chi(D) \) for all \( C, D \in \mathcal{B}(X) \),
4. \( \chi(C \cup D) = \max\{\chi(C), \chi(D)\} \), \( \forall C, D \in \mathcal{B}(X) \),
5. \( \chi(\overline{C}) = \chi(C) \),
6. \( \chi(\text{co}(D)) = \chi(\text{co}(C)) = \chi(C) \).

The details of \( \chi \) and their properties may be found in [1].
Definition 2.2. Let \( X \) be a \( \phi \)-space. Define the measure of weak compactness on \( X \) as being the function \( \chi_{\phi} : B(X) \to [0, \infty) \) given by

\[
\chi_{\phi}(C) = \chi(\phi(C)).
\]

Remark. Clearly, \( \chi_{\phi} \) is well defined. Also, it follows from the definition of \( \phi \)-spaces and by (2) that a set \( C \subset \mathcal{F} \) is weakly compact if \( \chi_{\phi}(C) = 0 \). In addition, we have \( \chi_{\phi}(C) \leq \|\phi\| \chi(C) \) for all \( C \in B(X) \). Thus if \( C \) is compact then \( \chi_{\phi}(C) = 0 \). Hence, in \( \phi \)-spaces every compact and weakly closed set is weakly compact.

3. Fixed Point Theory

Before going to the main result of this paper, we give a useful definition. In the proof of Sadovskii theorem, \( \chi(T(C)) < \chi(C) \) plays a crucial role. This property suggests the following definition.

Definition 3.1. Let \( M \) be a bounded subset of a \( \phi \)-space \( X \). We will say that \( T : M \to M \) is ws \( \phi \)-condensing (resp. w \( \phi \)-condensing) if \( T \) is weakly sequentially continuous (resp. weakly continuous) and \( \phi \)-condensing, i.e,

\[
\chi_{\phi}(T(C)) < \chi_{\phi}(C) \text{ for all } C \in B(M),
\]

where \( B(M) \) denote the collection of all nonempty and bounded subsets of \( M \).

Our main result is as follows.

Theorem 3.1. Let \( X \) be a \( \phi \)-space and \( M \) a closed, convex and bounded subset of \( X \). If \( T : M \to M \) is a ws \( \phi \)-condensing mapping, then there is a \( y \in M \) such that \( T(y) = y \).

Since every weakly continuous mapping is weakly sequentially continuous we immediately obtain the following corollary.

Corollary 3.1. Assume \( M \) is as in Theorem 3.1. If \( T \) is w \( \phi \)-condensing from \( M \) into itself, then \( T \) has a fixed point.

The proof of Theorem 3.1 is based in the following generalization of the Schauder-Tychonov fixed point principle which was obtained by Arino, Gautier and Penot [7].
Theorem 3.2. Let $M$ be a nonempty, convex and weakly compact subset of a Banach space $X$ and let $T : M \to M$ be a weakly sequentially continuous operator. Then $T$ has at least one fixed point in the set $M$.

Proof of Theorem 3.1. We proceed as in the proof of the Sadovskii’s Theorem. Fix a point $x \in M$ and let $\Sigma$ denote the system of all closed, convex subsets of $M$ for which $x \in K$ and $T(K) \subset K$. Now set

$$B = \cap_{K \in \Sigma} K, \quad C = \overline{\operatorname{co}} \{T(B) \cup \{x\}\}.$$ 

Then $B = C$, $T(C) \subset C$ and $\phi(C) \subset \overline{\operatorname{co}} \{\phi(T(B)) \cup \{\phi(x)\}\}$. Using (3),(4) and (6), one has

$$\chi(\phi(C)) \leq \chi(\phi(T(B))) = \chi(\phi(T(C))).$$

Since $T$ is $\phi$-condensing, it follows that $\chi_\phi(C) = 0$. Then in view of the definition of $\phi$-spaces we conclude that $C$ is weakly compact since $C \in F$. Now Theorem 3.2 gives a fixed point $y \in C$ for $T$. This ends the proof.

Evidently, if $X$ is a reflexive space the assumption $\chi_\phi(T(C)) < \chi_\phi(C)$ in the Theorem 3.1 is unnecessary. But, in a nonreflexive $\phi$-space this requirement may be essential as the next example shows.

Example 4. Consider the Banach space $X = l^1$ endowed with its usual norm. Let us define $\phi : l^1 \to l^1$ by

$$\phi \left( \sum_{n \in \mathbb{N}} \alpha_n e_n \right) = \sum_{n \in \mathbb{N}} \alpha_n e_{n+1}.$$ 

Then $\phi \in \mathcal{L}(l^1)$, $\operatorname{Ker}(\phi) = \{0\}$ and $\phi \not\in \operatorname{span}\{I\}$. Now, define $S : l^1 \to l^1$ by

$$S \left( \sum_{n \in \mathbb{N}} \alpha_n e_n \right) = \sum_{n \geq 2} \alpha_n e_{n-1}.$$ 

Then, $S_{\phi(l^1)} \equiv \phi^{-1}$ and $\|S(\eta)\| \leq \|\eta\|$ for all $\eta \in l^1$. Consequently, $\|\phi^{-1}y\| \leq \|y\|$ for all $y \in \phi(l^1)$. By Proposition 2.1, $l^1$ is a $\phi$-space. Let now $M$ be the set

$$M = \left\{ \sum_{n \in \mathbb{N}} \alpha_n e_n : \alpha_n \geq 0, \sum_{n \in \mathbb{N}} \alpha_n = 1 \right\}.$$ 

Define the map $T : M \to M$ as $T = \phi|_M$. Then, $T$ is fixed point free on $M$. Moreover, one can check that $T$ is an isometry on $M$. Also it is easy to see that $T$
is weakly sequentially continuous, and
\[ \| \phi T(\eta) - \phi T(\mu) \| = \| \phi(\eta) - \phi(\mu) \|, \]
for all \( \mu, \eta \in M \), which implies that \( \chi_\phi(TC) \geq \chi_\phi(C) \) for all bounded set \( C \) in \( M \).

**Remark.** The counterexample above was inspired by the paper [3, pp. 444].

4. **Hammerstein integral equations in Banach spaces**

Let \( E \) be a Banach space and consider the Hammerstein integral equation
\[
y(t) = h(t) + \int_0^1 k(t, s)f(s, y(s))ds, \quad t \in [0, 1],
\]
where
\[
\begin{cases}
f : [0, 1] \times E \to E & \text{is such that, } f_t = f(t, \cdot) : E \to E \text{ is weakly sequentially continuous, for each } t \in [0, 1], \\
h \in C([0, 1], E) & \text{is arbitrary},
\end{cases}
\]
and
\[
\begin{cases}
k(t, s) \in L^1([0, 1], \mathbb{R}) & \text{for each } t \in [0, 1] \text{ and} \\
\text{the map } t \mapsto k(t, s) \text{ is continuous from } [0, 1] \to L^1([0, 1], \mathbb{R})
\end{cases}
\]
hold.

When \( E \) is a reflexive space and \( f \) is weakly-weakly continuous, O'Regan [9] proved that (4.1) has a solution if the additional conditions
\[
\begin{cases}
\text{there exists a nondecreasing continuous function } \\
\Omega : [0, \infty) \to (0, \infty) \text{ with } \| f(s, x) \| \leq \Omega(\| x \|) \text{ for } t \in [0, 1]
\end{cases}
\]
and
\[
(K = \sup_{t \in [0, 1]} \int_0^1 |k(t, s)|ds) \cdot \limsup_{t \to \infty} \frac{\Omega(t)}{t} \leq 1,
\]
are satisfied. The proof in [9] depends strongly on the reflexivity of \( E \) since a weak version of the Arzela-Ascoli theorem is used to guarantee that the right-side in (4.1) defines a weakly compact operator on \( C([0, 1], E) \). However, in an arbitrary Banach space \( E \) this technique may fails due to lack of weak compactness in balls. Thus it
is natural to ask what conditions are needed if the word reflexive is removed. Here
as an application of our fixed point theory, we offer the following result as a partial
answer to this question.

**Theorem 4.1.** Let $E$ be a Banach space and assume \(4.2\)–\(4.6\). In addition suppose
that for some $\phi \in \mathcal{L}(E)$ with $\phi \notin \text{span}\{I\}$, $\text{Ker}(\phi) = \{0\}$ and $\|\phi^{-1}(y)\| \leq c\|y\|$ for
all $y \in \phi(E)$ with $c > 0$ we have

$$
\|\phi\left(\int_0^1 k(t, s)(f(s, x) - f(s, y))ds\right)\|_E \leq \alpha\|\phi(x(t) - y(t))\|_E,
$$

for all $t \in [0, 1]$, $x, y \in C([0, 1], E)$ and some $0 < \alpha < 1$. Then, \(4.1\) has a solution
$y \in C([0, 1], E)$.

**Proof.** Let $X = C([0, 1], E)$ with its usual norm $\|\cdot\|$. Let us define $\phi \in \mathcal{L}(X)$ by
$(\phi x)(t) = \phi(x(t))$. Then, $\phi \notin \text{span}\{I\}$, $\text{Ker}(\phi) = \{0\}$ and $\|\phi^{-1}y\| \leq c\|y\|$ for
all $y \in \phi(C(I, E))$. Proposition 2.1 now implies that $X$ is a $\phi$-space. Now \(4.5\), \(4.6\)
guarantees that there exists $R > 0$ such that

$$
\|h\| + K\Omega(R) \leq R,
$$

(cf. \[\text{9}\]). Let $M$ be the set

$$
M = \{x \in X : \|x\| \leq R\}.
$$

Let us define the map $T : M \rightarrow M$ by

$$(Tx)(t) = h(t) + \int_0^1 k(t, s)f(s, x(s))ds, \quad t \in [0, 1].$$

We claim now that $T$ is weakly sequentially continuous. Indeed, let $(x_n) \subseteq M
such that $x_n \rightharpoonup x$ in $X$. Then, $x_n(s) \rightharpoonup x(s)$ in $E$ for each $s \in [0, 1]$. By \[\text{1.2}\], one
has that $f(s, x_n(s)) \rightharpoonup f(s, x(s))$ in $E$ for each $s \in [0, 1]$, as $n \rightarrow \infty$. Combining now
the Hahn-Banach theorem with the Dominated convergence theorem we conclude
that $(Tx_n)(t) \rightarrow (Tx)(t)$ in $E$ for each $t \in [0, 1]$, as $n \rightarrow \infty$. Now, from \[\text{1.3}\] it is easy
to check that $\{(Tx_n) : n \in \mathbb{N}\}$ is an equicontinuous subset of $X$. Thus, by Ascoli-
Arzela theorem, we may conclude that $Tx_{n_j} \rightarrow Tx$ in $X$, for some subsequence
$(x_{n_j})$ of $(x_n)$. Hence $Tx_n \rightharpoonup Tx$ in $X$, as $n \rightarrow \infty$. If not there exists a subsequence
$(Tx_{n_k})$ of $(Tx_n)$ and a weak neighborhood $V^w(Tx)$ of $Tx$ such that $Tx_{n_k} \not\in V^w(Tx)$,
for all \( k \in \mathbb{N} \). Now arguing as before we find a subsequence \( T_{x_{n_k}} \rightharpoonup T x \), which is a contradiction. This proves the claim. Further, in view of (4.7) it follows that \( T \) is \( \phi \)-condensing. Now, in order to apply Theorem 3.1 it remains to show that \( T(M) \subset M \). But (4.8) implies that \( \|T x\| \leq \|h\| + K\Omega(R) \leq R \) whenever \( x \in M \). Also, from (4.3) and (4.4) one easily verifies that \( T x \) is norm continuous if \( x \in M \).

This completes the proof of the Theorem. \( \square \)

Remark. Notice (4.6) could be replaced by condition: there exists \( R > 0 \) with \( \|h\| + K\Omega(R) \leq R \).

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