c=2 Rational Toroidal Conformal Field Theories via the Gauss Product

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Abstract. We find a concise relation between the moduli τ, ρ of a rational Narain lattice Γ(τ, ρ) and the corresponding momentum lattices of left and right chiral algebras via the Gauss product. As a byproduct, we find an identity which counts the cardinality of a certain double coset space defined for isometries between the discriminant forms of rank two lattices.

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(0-1) Introduction – some background – Since the 80’s, string compactifications on real d-dimensional tori $T^d$ have been a source of several important ideas, such as orbifold [DHVW], T-duality [KY], etc., which later have been successfully generalized to string compactifications on more non-trivial geometries of Calabi-Yau manifolds (see, e.g. [GSW][GY][Po] for references). One thing we learned from these important developments is that certain properties of string theory (conformal field theory) often translate into deep and interesting geometrical insights when interpreted in the language of $\sigma$-models (geometry). Gepner’s correspondence between N=2 superconformal field theory (SCFT) and $\sigma$-models on Calabi-Yau manifolds is one such well-known example. Mirror Symmetry is yet another. In this paper, we will study an important property, known as rationality, of conformal string compactifications on $T^d$. That rationality translates into interesting questions on the geometry side was brought to light in recent works of [Mo][GV]. For example it was found in [Mo] that rationality of a CFT on an elliptic curve $E_\tau$ ($\cong T^2$) implies that $E_\tau$ has non-trivial endomorphisms, i.e. $E_\tau$ is of CM (complex multiplication) type. Several other deep questions on rationality of the string compactification on general Calabi-Yau manifolds have also been raised in [Mo][GV], although their answers are still conjectural.
In this paper we will restrict our attention to CFTs on $T^2$. In this case, as shown by G. Moore [Mo], one has the following characterization of rationality: a CFT on $T^2(=E_\tau)$ is rational if and only if the parameters $\tau, \rho$ are elements of the imaginary quadratic field $\mathbb{Q}(\sqrt{D})$ for some $D < 0$, where $\tau$ is the complex structure and $\rho = B + \sqrt{-1} \text{Vol}(E_\tau)$ is the complexified Kähler modulus. This allows one, in principle, to parameterize rational toroidal CFTs by such pairs $\tau, \rho$. In this paper, we further the investigation along this line by studying precisely how this parameterization can be realized. As a result, we find a precise correspondence between a Narain lattice $\Gamma(\tau, \rho)$ and a RCFT triple $(\Gamma_l, \Gamma_r, \varphi)$, which consists of a pair of momentum lattices for chiral algebras plus a gluing map $\varphi$. In our view, this is one step toward understanding the geometry of rational CFTs on $T^2$, and more generally, on Calabi-Yau manifolds. The correspondence above is realized precisely by means of the composition law of Gauss on primitive binary quadratic forms plus an important extension to the non-primitive forms. The latter is crucial if one wishes to allow arbitrary discriminants $D$.

Our result is clearly relevant to the classification problem of rational CFTs with $c = 2$ (see for example [DW] and references therein for developments on this problem), which generalizes the old and well-known classification of $c = 1$ rational CFTs given in [Gi][Ki1]. However we restrict ourselves in this paper to rational CFTs on $T^2$, which are among the best known and well-studied examples in string theory (for which we refer the readers to [HMV][GSW][Po][Ki2] and references therein).

(0-2) Main results. Our main results are:

**Theorem 4.10, Theorem 4.11, Theorem 5.7 and Theorem 5.8,**

where we obtain a complete description of rational toroidal conformal field theories in terms of the Gauss product on the classes of binary quadratic forms. The general results are summarized and restated a little differently in **Summary 5.12.**

In section 1, we briefly summarize the classical theory of binary quadratic forms and the Gauss product. The basics of the Gauss product are summarized in Appendix A, where we extend the product to non-primitive quadratic forms. In section 2, we introduce Narain lattices and their moduli space. We also describe the so-called $T$-duality group. This section is meant for setting up notations and reviewing some well-known results (see e.g. [GSW][Po][Ki2] and references therein for the original works). In section 3, we define RCFTs on $T^2$ (rational toroidal CFT) and summarize their characterizations given in [HMV] (see also [Mo][Wa][GV] for a more recent perspective). We then state our classification problem (after Proposition 3.4), and study the first part of this problem using some results of [Ni]. These results were first used in [Mo] to study of rational toroidal CFTs and rational CFTs on singular K3 surfaces, i.e. K3 surfaces of maximal Picard number 20 [SI]. In section 4, we discuss our classification in the case when the relevant lattices are primitive (which correspond to primitive quadratic forms). In section 5, we extend it to the general non-primitive case. In subsection (5-4), we will see that the diagonal RCFTs obtained in [GV] fit into our list of RCFTs in a natural way. A summary of the classification is given in **Summary 5.12.**

In our classification, the classical theory of the binary quadratic forms of Dirichlet and Gauss will come into play in an interesting and essential way (**Lemma 4.6** and **Lemma 5.1**). The reader can get a quick glance of this in examples given in subsection (0-3) below (primitive case), and also in (5-5) (non-primitive case).
(0-3) Example \((D = -39)\). It will be helpful to see how our classification works in this example now with details given later. Here we present two tables: One is the product table of the class group \(\text{Cl}(D)\) (cf. Theorem 1.1) of discriminant \(D = -39\), and the other is the table listing the RCFT data \((\Gamma_t, \Gamma_r, \varphi)\) against the Narain lattices \(\Gamma(\tau_C, \rho_C)\).

To make the first table, let us note that the class group \(\text{Cl}(D)\) here consists of the following four \(SL_2 \mathbb{Z}\) equivalence classes of binary quadratic forms:

\[
C_1 = [Q(1, 1, 10)] , \quad C_2 = [Q(2, 1, 5)] , \quad C_3 = [Q(2, -1, 5)] , \quad C_4 = [Q(3, 3, 4)] .
\]

The notation \(Q(a, b, c)\) abbreviates the quadratic form \(f(x, y) = ax^2 + bxy + cy^2\). A quadratic form can be identified with a lattice with a chosen ordered basis in which the bilinear form is given by the matrix \(\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}\). Under this identification, an \(SL_2 \mathbb{Z}\) equivalence class \(C\) of quadratic forms is nothing but an isomorphism class of lattices equipped with orientations, while a \(GL_2 \mathbb{Z}\) equivalence class \(\bar{C}\) of quadratic forms is nothing but an isomorphism class of lattices without orientations. Now we write the product table of \(\text{Cl}(D)\):

| \(\rho_{C_1}\) | \(\rho_{C_2}\) | \(\rho_{C_3}\) | \(\rho_{C_4}\) |
|-----------------|-----------------|-----------------|-----------------|
| \((\Gamma_{\bar{C}_1}, \Gamma_{\bar{C}_1}, \text{id})\) | \((\Gamma_{\bar{C}_1}, \Gamma_{\bar{C}_1}, \text{id})\) | \((\Gamma_{\bar{C}_2}, \Gamma_{\bar{C}_2}, \text{id})\) | \((\Gamma_{\bar{C}_4}, \Gamma_{\bar{C}_4}, \text{id})\) |
| \((\Gamma_{\bar{C}_2}, \Gamma_{\bar{C}_2}, \text{id})\) | \((\Gamma_{\bar{C}_1}, \Gamma_{\bar{C}_1}, \varphi_1)\) | \((\Gamma_{\bar{C}_4}, \Gamma_{\bar{C}_4}, \varphi_1)\) | \((\Gamma_{\bar{C}_2}, \Gamma_{\bar{C}_2}, \varphi_2)\) |
| \((\Gamma_{\bar{C}_3}, \Gamma_{\bar{C}_3}, \text{id})\) | \((\Gamma_{\bar{C}_1}, \Gamma_{\bar{C}_1}, \varphi_1^{-1})\) | \((\Gamma_{\bar{C}_3}, \Gamma_{\bar{C}_3}, \varphi_1)\) | \((\Gamma_{\bar{C}_2}, \Gamma_{\bar{C}_2}, \varphi_2)\) |
| \((\Gamma_{\bar{C}_4}, \Gamma_{\bar{C}_4}, \text{id})\) | \((\Gamma_{\bar{C}_1}, \Gamma_{\bar{C}_1}, \varphi_2)\) | \((\Gamma_{\bar{C}_3}, \Gamma_{\bar{C}_3}, \varphi_2)\) | \((\Gamma_{\bar{C}_1}, \Gamma_{\bar{C}_1}, \varphi_3)\) |

Table 1. Table of Gauss product \((D = -39)\).

Table 2 lists the data for the RCFTs on \(T^2\). The data that determines an RCFT consists of its momentum lattices \(\Gamma_t, \Gamma_r\) with determinant \(-D\), together with an isometry of their discriminant groups \(\varphi\). This isometry “glues” together the left and right sector of the RCFT. Equivalently, the data \(\Gamma_t, \Gamma_r, \varphi\) can also be described in terms of a Narain lattice \(\Gamma(\tau, \rho)\) which contains \(\Gamma_t, \Gamma_r\). The correspondence between triples \((\Gamma_t, \Gamma_r, \varphi)\) and Narain lattices \(\Gamma(\tau, \rho)\), \(\tau, \rho \in \mathbb{Q}(\sqrt{D})\), are shown in Table 2. The key observation here is that we have \(\bar{C} = C_i \ast C_j\), \(\bar{C}' = C_i \ast C_j\) for the triple \((\bar{\Gamma}_C, \bar{\Gamma}_C', \varphi)\) that corresponds to a Narain Lattice \(\Gamma(\bar{\tau}_C, \rho_{C_i})\).

To describe the correspondence more precisely, let us associate to each quadratic form \(Q(a, b, c)\) the complex number \(\tau_{Q(a, b, c)} = \rho_{Q(a, b, c)} = a + \frac{b}{2} + \frac{\sqrt{D}}{2} (D = b^2 - 4ac)\). Note that since \(D < 0\), these complex numbers lie in the upper half plane \(\mathbb{H}_+\). Given an \(SL_2 \mathbb{Z}\) equivalence class of quadratic forms \(C = [Q(a, b, c)]\), let \(\bar{C} = \rho_C\) be the \(SL_2 \mathbb{Z}\) orbit of \(\tau_{Q(a, b, c)}\). Given a \(GL_2 \mathbb{Z}\) equivalence class \(\bar{C}\), we denote by \(\Gamma(\bar{C})\) a lattice in the corresponding isomorphism class \(\bar{C}\) of lattices. Then Table 2 also describes the data of the RCFTs in terms of Narain lattices \(\Gamma(\tau_C, \rho_C)\).
Table 2. Table of RCFT data. RCFT data \((\Gamma_l, \Gamma_r, \phi)\) are listed against the Narain lattices \(\Gamma(\tau_{C_i}, \rho_{C_j})\). Boxed entries define the same RCFT up to worldsheet parity involution.

The lattices \(\bar{\mathcal{C}}_i\) in Table 2 are not all inequivalent. In fact, it is easy to verify that there are only three equivalence classes \(\bar{\mathcal{C}}_1, \bar{\mathcal{C}}_2 = \bar{\mathcal{C}}_3, \bar{\mathcal{C}}_4\). For brevity, we do not describe here the gluing data \(\phi_1, \phi_2, \phi_3\) explicitly, but we will discuss their general construction later.

Note that the first column (or row) corresponds to the so-called diagonal modular invariants of RCFT, whose characterization has been obtained recently in [GV] (see section (5.4)). We observe that this fits naturally into our general classification (see Proposition 5.7).

(0-4) Average one formula. As a corollary to our Main Theorems, we obtain the following “average one” formula for definite lattices. This formula connects, in an interesting way, lattice problems arising from RCFTs to the class group of binary quadratic forms (see Corollary 4.13).

\[ \frac{1}{|\text{Sym}^2 \mathcal{L}(D)|} \sum_{(\Gamma, \Gamma') \in \text{Sym}^2 \mathcal{L}(D)} |O(\Gamma) \setminus \text{Isom}(A_{\Gamma}, A_{\Gamma'})/O(\Gamma')| = 1, \tag{0.1} \]

where \(\text{Isom}(A_{\Gamma}, A_{\Gamma'})\) is the set of isometries \(\varphi : A_{\Gamma} \rightarrow A_{\Gamma'}\) and the double quotient is defined by the natural actions \(\varphi \mapsto h \cdot \varphi \cdot g^{-1}\) of lattice isometries \(g \in O(\Gamma), h \in O(\Gamma')\).

Remark. Here \(\text{Sym}^2 \mathcal{L}(D)\) denotes the set of symmetric pairs of elements of \(\mathcal{L}(D)\). The set \(\mathcal{L}(D)\) is identified with the set \(\tilde{\text{Cl}}(D)\) of \(\text{GL}_2\mathbb{Z}\) equivalence classes of the quadratic forms. When the determinant \(-D(>2)\) is a prime, the elements of \(\tilde{\text{Cl}}(D) \cong \mathcal{L}(D)\) are all isogeneous (see e.g. [Za, §12]), and we have \(\text{Isom}(A_{\Gamma}, A_{\Gamma'}) = \{\pm 1\}\). Therefore \(|O(\Gamma) \setminus \text{Isom}(A_{\Gamma}, A_{\Gamma'})/O(\Gamma')| = 1\) for all \((\Gamma, \Gamma') \in \text{Sym}^2 \mathcal{L}(D)\). That verifies (0.1) immediately in this special case. In general \(\tilde{\text{Cl}}(D) \cong \mathcal{L}(D)\) contains more than one isogeny classes, in which case the “average one formula” is a very interesting generalization. Note that \(\text{Isom}(A_{\Gamma}, A_{\Gamma'})\) is empty if \(\Gamma\) and \(\Gamma'\) are not isogeneous (see [Ni]). The average one formula (0.1) for indefinite lattices \(\mathcal{L}(D) (D > 0)\) also follows from the counting problem of general \(V\)-rationnal Narain lattices (see [HLOY2]).

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§1. Classical results on quadratic forms

In this section we summarize some classical results on (positive definite) binary quadratic forms following [Za][Ca]. These results turn out to be crucial for our classification problem.

(1-1) Quadratic forms. Let us denote by $Q(a, b, c)$ an integral quadratic form in two variables:

$$Q(a, b, c) : f(x, y) = ax^2 + bxy + cy^2 \quad (a, b, c \in \mathbb{Z}).$$

Unless stated otherwise, a quadratic form will mean a positive definite, integral, quadratic form in two variables. A quadratic form $Q(a, b, c)$ is called primitive if $\gcd(a, b, c) = 1$. The group $SL_2\mathbb{Z}$ acts on quadratic forms $A : Q(a, b, c) \mapsto Q(a', b', c')$, $(A \in SL_2\mathbb{Z})$, by

$$t A \left(\begin{array}{cc} 2a & b \\ b & 2c \end{array} \right) A = \left(\begin{array}{cc} 2a' & b' \\ b' & 2c' \end{array} \right). \quad (1.1)$$

Since $A = \pm id$ acts trivially, only the group $PSL_2\mathbb{Z}$ acts effectively. The action leaves the discriminant $D := b^2 - 4ac < 0$ unchanged. Two quadratic forms are said to be properly equivalent if they are in the same $SL_2\mathbb{Z}$ orbit. We consider the set of properly equivalent classes and denote it by

$$Cl(D) := \{ Q(a, b, c) : \text{primitive quadratic form, } D = b^2 - 4ac < 0, \ a > 0 \}/ \sim_{SL_2\mathbb{Z}}.$$

It has been known, since Gauss, that $Cl(D)$ is a finite set. Its cardinality is called a class number, denoted by $h(D) = |Cl(D)|$. We often write the set of classes by

$$Cl(D) = \{ C_1, C_2, \ldots, C_{h(D)} \}.$$

Similarly the group $GL_2\mathbb{Z}$ acts on quadratic forms by the same formula as (1.1), and two quadratic forms are said to be improperly equivalent if they are in the same $GL_2\mathbb{Z}$ orbit. We consider the set of improperly equivalent classes and denote it by

$$\widetilde{Cl}(D) := \{ Q(a, b, c) : \text{primitive quadratic form, } D = b^2 - 4ac < 0, \ a > 0 \}/ \sim_{GL_2\mathbb{Z}}.$$

There is obviously a natural surjection $q : Cl(D) \to \widetilde{Cl}(D), C \mapsto \tilde{C}$, and $q^{-1}(\tilde{C})$ has either one or two classes.

(1-2) Even lattices of rank two. An abstract (integral) lattice $\Gamma = (\Gamma, \ast, \ast)$ is a $\mathbb{Z}$-module equipped with non-degenerate bilinear form $(\ast, \ast) : \Gamma \times \Gamma \to \mathbb{Z}$. Let $\Gamma$ be a lattice. It is called even (respectively positive definite) if $(x, x) > 0$ for $x \in \Gamma$. We denote by $\Gamma(n)$ the lattice whose bilinear form is given by $n$-times the bilinear form of $\Gamma$, i.e. $\Gamma(n) = (\Gamma, n(\ast, \ast))$. We say that $\Gamma$ is primitive if $\Gamma = \Gamma'(n)$ for some even lattice $\Gamma'$ implies that $n = \pm 1$. We denote the dual lattice $\Gamma^* := \text{Hom}(\Gamma, \mathbb{Z})$ and have $|\text{det } \Gamma| = |\Gamma^*/\Gamma|$, which is called the
determinant of the lattice $\Gamma$. We consider the following set of isomorphism classes of lattices:

\[ \mathcal{L}^p(D) := \{ \Gamma : \text{even, primitive, positive definite lattices of rank two} \] 
\[ \text{and } D = -\det\Gamma \} / \text{GL}_2\mathbb{Z} \]

In this paper, both $\Gamma, [\Gamma]$ denote the isomorphism class of a lattice $\Gamma$.

For an even lattice $\Gamma$ of determinant $-D$ and rank two, choosing a basis $u_1, u_2$, we may associate a symmetric matrix,

\[ \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} := \begin{pmatrix} (u_1, u_1) \\ (u_1, u_1) \end{pmatrix} \] 
(1.2)

with $a, b, c \in \mathbb{Z}$. If $\Gamma$ is positive definite, then $4ac - b^2 = -D > 0$ and $a > 0$. Moreover if $\Gamma$ is primitive, then $\gcd(a, b, c) = 1$. Therefore for an even, positive definite, primitive lattice $\Gamma$, we may associate a primitive quadratic form $Q(a, b, c)$ by choosing a basis of $\Gamma$. It is clear that changing the basis $u_1, u_2$ results in a quadratic form which is $GL_2\mathbb{Z}$ equivalent to $Q(a, b, c)$. Similarly, since isomorphic lattices have the same $GL_2\mathbb{Z}$ orbit of the matrix (1.2), they also correspond to the same $GL_2\mathbb{Z}$ equivalence class of quadratic forms. Conversely, it is also clear that a $GL_2\mathbb{Z}$ equivalence class $\tilde{C} = \{Q(a, b, c)\} \in \tilde{\mathcal{C}}l(D)$ defines a unique isomorphism class of lattices $\Gamma = \mathbb{Z}u_1 \oplus \mathbb{Z}u_2$ by (1.2). Therefore we can identify, and will do so hereafter, the set $\mathcal{L}^p(D)$ with the set $\tilde{\mathcal{C}}l(D)$ under the natural one to one correspondence

\[ \tilde{\mathcal{C}} = \{Q(a, b, c)\} \leftrightarrow \Gamma = [\Gamma] \text{ such that (1.2) holds.} \]

By this correspondence we often write explicitly,

\[ \tilde{\mathcal{C}}l(D) = \{\Gamma_1, \Gamma_2, \cdots, \Gamma_{h(D)}\} \quad (\tilde{h}(D) = |\tilde{\mathcal{C}}l(D)|). \]

Also it will be useful to write $\tilde{\mathcal{C}}l(D) = \{q(C) | C \in Cl(D)\}$, in terms of the natural surjective map $q : Cl(D) \to \tilde{\mathcal{C}}l(D)$.

(1-3) Class group. We summarize the following nice property of the set $Cl(D)$:

**Theorem 1.1. (Gauss)** The set $Cl(D)$ has a commutative, associative, composition law (Gauss product) which makes $Cl(D)$ an abelian group with the unit being the class represented by $Q(1, 0, -\frac{D}{4}) : x^2 - \frac{D}{4}y^2$ for $D \equiv 0 \text{mod } 4$ and $Q(1, 1, \frac{1-D}{4}) : x^2 + xy + \frac{1-D}{4}y^2$ for $D \equiv 1 \text{mod } 4$.

Note that since $D$ has the shape $b^2 - 4ac$, we have $D \equiv 0$ or $1 \text{mod } 4$. The precise definition of the Gauss product is summarized in Appendix A. As it is explained there (Proposition A.7), for any given class $\mathcal{C} \in Cl(D)$, we have

\[ q^{-1}(q(C)) = \{ C, C^{-1} \} \] 
(1.4)

Hence if $C^{-1} \neq C$, then $C^{-1}$ and $C$ are improperly equivalent, and thus we have the following relation:

\[ \tilde{h}(D) = \frac{1}{2} (h(D) + \#\{C \in Cl(D) \mid C^{-1} = C\}) \] 
(1.5)
(1-4) **Non-primitive quadratic forms.** In our classification problem, lattices which are not necessarily primitive will appear. We define the following sets of equivalence classes:

\[ \text{CL}(D) := \{ Q(a, b, c) : \text{quadratic form}, D = b^2 - 4ac < 0, \ a > 0 \} / \sim_{SL_2\mathbb{Z}}, \]

and

\[ \tilde{\text{CL}}(D) := \{ Q(a, b, c) : \text{quadratic form}, D = b^2 - 4ac < 0, \ a > 0 \} / \sim_{GL_2\mathbb{Z}}. \]

Obviously these two sets respectively include \( \text{Cl}(D) \) and \( \tilde{\text{Cl}}(D) \). If \( f(x, y) \) is a quadratic form of discriminant \( D \), then \( f(x, y) = \lambda g(x, y) \) for some positive integer \( \lambda \), and some primitive quadratic form \( g(x, y) \) of discriminant \( D/\lambda^2 \). From finiteness of \( \text{Cl}(D/\lambda^2) \), it follows that \( \text{CL}(D) \) is finite. As before, we have a natural correspondence between improper equivalence classes of quadratic forms and isomorphism classes of lattices. Under this correspondence, we identify the set \( \tilde{\text{CL}}(D) \) with

\[ \mathcal{L}(D) := \{ \Gamma : \text{even, positive definite lattice of rank two and } -D = |\Gamma^*/\Gamma| \} / \text{isom}. \]

There is also a natural map \( q: \text{CL}(D) \to \tilde{\text{CL}}(D) \), and we have

\[ q^{-1}(q(C)) = \{ C, \sigma C \}, \]

where \( \sigma \in GL_2\mathbb{Z} \) is an involution which is not in \( SL_2\mathbb{Z} \). In this paper, we fix the involution \( \sigma \) so that it acts on quadratic forms by

\[ \sigma: Q(a, b, c) \mapsto Q(a, -b, c) \quad \text{(1.6)} \]

Note that if \( C \in \text{Cl}(D) \), then \( \sigma C = C^{-1} \) (see Proposition A.7 in Appendix A). As in (1.5), we have:

\[ |\tilde{\text{CL}}(D)| = \frac{1}{2} (|\text{CL}(D)| + \#\{ C \in \text{CL}(D) | \sigma C = C \}) \quad \text{(1.7)} \]

(1-5) **Reduced forms.** Each class \( C \in CL(D) \) has a special representative, which is called **the reduced form**. To describe its construction, let us consider an element

\[ S_n = \begin{pmatrix} n & 1 \\ -1 & 0 \end{pmatrix} \in SL_2\mathbb{Z}. \]

By (1.1) we have

\[ S_0 : Q(a, b, c) \mapsto Q(c, -b, a), \quad S_nS_0 : Q(a, b, c) \mapsto Q(a, b - 2na, c - nb + n^2a). \]

Using these two operations, we see that any given class \( C \in CL(D) \) contains a quadratic form \( Q(a, b, c) \) with \( a \leq c \) and \( -a < b \leq a \), and for which we have \(-D = 4ac - b^2 \geq 4a^2 - a^2 = 3a^2\), i.e.

\[ 0 < a \leq \sqrt{-D/3}. \]

Moreover, if \( a = c \), we can make \( 0 \leq b \leq a = c \). To summarize we have
Proposition 1.2. The set $\mathcal{C}(D)$ is finite. Each class $\mathcal{C} \in \mathcal{C}(D)$ can be represented by a unique form $Q(a,b,c)$, called the reduced form, satisfying

1) $-a < b \leq a$, $0 < c \leq \sqrt{-D}/3$, $c = \frac{b^2-d}{4c} \in \mathbb{Z}$, $a < c$, or
2) $0 \leq b \leq a$, $0 < c \leq \sqrt{-D}/3$, $a = c = \frac{b^2-d}{4a}$.

Note that the uniqueness of the reduced form $Q(a,b,c) \in \mathcal{C}$ for a given class $\mathcal{C} \in \mathcal{C}(D)$ is clear from the fact that $SL_2\mathbb{Z}$ is generated by $S_0$ and $S_1$.

§2. Narain lattices and toroidal compactifications

(2-1) Narain lattice. Let $\mathbb{R}^{d,d}$ be the vector space $\mathbb{R}^{2d}$ equipped with the bilinear form $(x,x)_{\mathbb{R}^{d,d}} = x_1^2 + \cdots + x_d^2 - x_{d+1}^2 - \cdots - x_{2d}^2$, of signature $(d,d)$. Let $\mathbb{R}^{d,0} \subset \mathbb{R}^{d,d}$ be the subspace of vectors $x$ with $x_{d+1} = \cdots = x_{2d} = 0$. Likewise for $\mathbb{R}^{0,d}$. Then we have an orthogonal decomposition with respect to $(\ ,\)_{\mathbb{R}^{d,d}}$:

$$\mathbb{R}^{d,d} = \mathbb{R}^{d,0} \oplus \mathbb{R}^{0,d}. \quad (2.2)$$

Let $U$ be the rank two lattice $\mathbb{Z}e \oplus \mathbb{Z}f$ with bilinear form given by $(e,e) = (f,f) = 0$, $(e,f) = 1$.

Definition 2.2. (Narain lattice)

1) A Narain lattice is a subgroup $\Gamma \subset \mathbb{R}^{d,d}$ of rank $2d$ such that $(\ ,\)_{|\Gamma}$ is even unimodular. A Narain embedding is an isometric embedding

$$\Phi : U^{\oplus d} \hookrightarrow \mathbb{R}^{d,d}.$$ 

2) Two Narain lattices $\Gamma, \Gamma'$ are said to be equivalent if $\Gamma' = g\Gamma$ for some $g \in O(d,d;\mathbb{R})$ preserving the decomposition (2.2).

Since every abstract even unimodular lattice of signature $(d,d)$ is isomorphic to $U^{\oplus d}$, every Narain lattice is the image of a Narain embedding.

Definition 2.3. (Partition function of CFT on $T^d$) Given a Narain lattice $\Gamma$, we define its partition function

$$Z^\Gamma(q,\bar{q}) = \frac{1}{\eta(q)^d\eta(\bar{q})^d} \sum_{p=\pi+p_r \in \Gamma} q^{\frac{1}{2}|p_1|^2} |\bar{q}^{\frac{1}{2}|p_1|^2}|,$$

and its parity invariant form

$$\tilde{Z}^\Gamma(q,\bar{q}) = \frac{1}{\eta(q)^d\eta(\bar{q})^d} \sum_{p=\pi+p_r \in \Gamma} \frac{1}{2} \left( q^{\frac{1}{2}|p_1|^2} q^{\frac{1}{2}|p_1|^2} + q^{\frac{1}{2}|p_1|^2} q^{\frac{1}{2}|p_1|^2} \right),$$

where $p = \pi + p_r$ is the decomposition according to (2.2) and $\|p_r\|^2 = |\langle \pi, p_r \rangle_{\mathbb{R}^{2d}}| = |\langle p_r, \pi \rangle_{\mathbb{R}^{2d}}| = |\langle p_r, p_r \rangle_{\mathbb{R}^{2d}}| \cdot \eta(q) = q^{\frac{1}{2}} \prod_{n>0} (1 - q^n)$ ($q = e^{2\pi \sqrt{-1}t}$ with $t$ in the upper half plane) is the Dedekind eta function.

Remark. 1) If $g \in O(d,d;\mathbb{R})$ preserves the decomposition (2.2), then it is obvious that $\Gamma, g\Gamma$ have the same partition function, for any Narain lattice $\Gamma$. In other words, equivalent Narain lattices have the same partition function.
2) In [Na], the partition function is computed by means of a path integral in a sigma model with target space a flat torus \( T^d \). A Narain lattice \( \Gamma \) plays the role of the momentum-winding lattice. The partition function is modular invariant, i.e. invariant under \( t \to t + 1 \) and \( t \to -1/t \), as a result of that \( \Gamma \) is even and unimodular. See, e.g., [Po, §8.4, p.252] for a proof of modular invariance. We will return to modular invariance later.

3) We introduce the parity invariant form \( \tilde{Z}^\Gamma(q, \bar{q}) \) since this fits well in our diagram given in Theorem 4.10, and also the bijection that we will prove in Proposition 4.12. The parity invariance refers to the invariance under the worldsheet parity involution (which is equivalent to the involution \( \pi_2 : (\tau, \rho) \mapsto (\tau, -\rho) \), cf. Lemma 4.9).

### (2-2) Moduli space of Narain lattices.

Here we summarize some known facts about the space which parametrizes the equivalence classes of Narain lattices. This space is known as a Narain moduli space.

Let us define the group \( O(d, d; R) \) explicitly by

\[
O(d, d; R) = \left\{ X \in \text{Mat}(2d, R) \mid ^tX \begin{pmatrix} 1_d & 0 \\ 0 & -1_d \end{pmatrix} X = \begin{pmatrix} 1_d & 0 \\ 0 & -1_d \end{pmatrix} \right\}, \tag{2.3}
\]

where \( 1_d \) represents the unit matrix of size \( d \). Let \( O(d; R) \times O(d; R) \subset O(d, d; R) \) denote the subgroup preserving the decomposition (2.2). Put

\[
E := \frac{1}{\sqrt{2}} \begin{pmatrix} 1_d & 1_d \\ 1_d & -1_d \end{pmatrix}.
\]

Let \( \Gamma_0 \) be the \( \mathbb{Z} \)-span of the column vectors of \( E \). Then we may verify that \( \Gamma_0 \) is even, unimodular indefinite lattice of rank \( 2d \) in \( \mathbb{R}^{d,d} \), hence a Narain lattice. Let \( O'(d, d; \mathbb{Z}) \subset O(d, d; R) \) denote the subgroup preserving \( \Gamma_0 \). Note that \( E^{-1} = E \). Then, it is straightforward to determine the subgroup to be

\[
O'(d, d; \mathbb{Z}) = EO(d, d; \mathbb{Z})E,
\]

with

\[
O(d, d; \mathbb{Z}) := \left\{ Y \in \text{Mat}(2d, \mathbb{Z}) \mid ^tY \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix} Y = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix} \right\}.
\]

**Proposition 2.4. (Narain moduli space)** The equivalence classes of Narain lattices are parametrized by the coset space

\[
\mathcal{M}_{d,d} := O(d; R) \times O(d; R) \backslash O(d, d; R)/O'(d, d; \mathbb{Z}). \tag{2.4}
\]

**Proof.** Given a Narain lattice \( \Gamma \subset \mathbb{R}^{d,d} \), there exists an isomorphism \( g : \Gamma_0 \to \Gamma \), since even, unimodular indefinite lattices are unique up to isomorphisms. This extends uniquely to an isometry of \( \mathbb{R}^{d,d} \) which we denote by the same \( g \in O(d, d; R) \). We associate to \( \Gamma \), the left coset \( gO'(d, d; \mathbb{Z}) \subset O(d, d; R)/O'(d, d; \mathbb{Z}) \). If \( g' : \Gamma_0 \to \Gamma \) is another isomorphism, then \( g^{-1} \cdot g' \) is an isometry of \( \Gamma_0 \), i.e. an element of \( O'(d, d; \mathbb{Z}) \). It follows that \( g' \cdot O'(d, d; \mathbb{Z}) = gO'(d, d; \mathbb{Z}) \). This shows that the correspondence \( \Gamma \mapsto gO'(d, d; \mathbb{Z}) \) is well-defined.
This correspondence is 1-1: if \( \Gamma \mapsto gO'(d, d; \mathbb{Z}) \) and \( \Gamma' \mapsto g'O'(d, d; \mathbb{Z}) \) for some \( g, g' \in O(d, d; \mathbb{R}) \) with \( gO'(d, d; \mathbb{Z}) = g'O'(d, d; \mathbb{Z}) \), then \( g^{-1} \cdot g' \in O'(d, d; \mathbb{Z}) \), i.e. \( g^{-1} \cdot g \Gamma_0 = \Gamma_0 \). It follows that \( \Gamma' = g \Gamma_0 = g \Gamma_0 = \Gamma_0 \). The correspondence is also onto: if \( g \in O(d, d; \mathbb{R}) \), then \( \Gamma = g \Gamma_0 \) is a Narain lattice with \( \Gamma \mapsto gO'(d, d; \mathbb{Z}) \). This shows that \( O(d, d; \mathbb{R}) / O'(d, d; \mathbb{Z}) \) parameterizes all Narain lattices.

Let \( \Gamma \) be a Narain lattice, and let \( \Gamma \mapsto gO'(d, d; \mathbb{Z}) \). If \( h \in O(d, d; \mathbb{R}) \times O(d, d; \mathbb{R}) \), then \( hg : \Gamma_0 \to h \Gamma \) is an isomorphism, and so \( h \Gamma \mapsto hgO'(d, d; \mathbb{Z}) \). This shows that the correspondence \( \Gamma \mapsto gO'(d, d; \mathbb{Z}) \) is compatible with the left action of \( O(d, d; \mathbb{R}) \times O(d, d; \mathbb{R}) \). This induces a 1-1 correspondence between equivalence classes of Narain lattices and \( \mathcal{M}_{d,d} \).

**Remark.** We introduce some notations here.

1) Introduce the following conjugate of \( O(d, d; \mathbb{R}) \),
\[
O'(d, d; \mathbb{R}) := E O(d, d; \mathbb{R}) E^T .
\] (2.5)

2) Put \( e_i = (0, \ldots, 0, e, 0, \ldots, 0) \), \( f_i = (0, \ldots, 0, f, 0, \ldots, 0) \in U^\oplus d \), where \( e, f \) lies in the \( i \)-th slot. Given a Narain embedding \( \Phi : U^\oplus d \hookrightarrow \mathbb{R}^{d,d} \), we define the following matrix:
\[
W(\Phi) := \frac{1}{\sqrt{2}} (\Phi(e_1 + f_1) \cdots \Phi(e_d + f_d) \Phi(e_1 - f_1) \cdots \Phi(e_d - f_d))
\]
\[
= (\Phi(e_1) \cdots \Phi(e_d) \Phi(f_1) \cdots \Phi(f_d)) \cdot E .
\] (2.6)

It is easy to check that \( W(\Phi) \in O(d, d; \mathbb{R}) \). Conversely, for any \( X \in O(d, d; \mathbb{R}) \), there is a unique Narain embedding \( \Phi \) such that \( W(\Phi) = X \). In other words, \( O(d, d; \mathbb{R}) \) parameterizes all Narain embeddings by the identification \( \Phi \equiv W(\Phi) \).

3) Likewise, \( O(d, d; \mathbb{R}) / O'(d, d; \mathbb{Z}) \) parameterizes all Narain lattices \( \Gamma = \text{Im} \Phi \) by the identification \( \Gamma \equiv W(\Phi)O'(d, d; \mathbb{Z}) \).

**Example \((d = 1)\).** We have
\[
O'(1, 1; \mathbb{R}) = \left\{ \begin{pmatrix} R & 0 \\ 0 & 1/R \end{pmatrix} \mid R \neq 0 \right\} \sqcup \left\{ \begin{pmatrix} 0 & R \\ 1/R & 0 \end{pmatrix} \mid R \neq 0 \right\} ,
\] (2.7)
where the each factor consists of two connected components. After the conjugation by (2.5), the subgroup \( O(1; \mathbb{R}) \times O(1; \mathbb{R}) \subset O(1, 1; \mathbb{R}) \) becomes a subgroup \( G \times G \) in \( O'(1, 1; \mathbb{R}) \) consisting of the four elements: \( \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Then we get
\[
G \times G \setminus \mathcal{O}'(1, 1; \mathbb{R}) = \left\{ G \times G : \begin{pmatrix} R & 0 \\ 0 & 1/R \end{pmatrix} \mid R > 0 \right\} .
\] (2.8)

Thus the four components of \( \mathcal{O}'(1, 1; \mathbb{R}) \) are collapsed to one. The subgroup \( O(1, 1; \mathbb{Z}) \subset \mathcal{O}'(1, 1; \mathbb{R}) \) consists of four elements obtained from (2.7) by setting \( R = \pm 1 \). It is easy to check that each of them acts on (2.8) either trivially or by \( R \rightarrow 1/R \), which is a well-known duality transformation. This shows that the Narain moduli space \( \mathcal{M}_{1,1} \) has the cross section \( \{ E \begin{pmatrix} R & 0 \\ 0 & 1/R \end{pmatrix} E \mid 1 \geq R > 0 \} \subset O(1, 1; \mathbb{R}) \). By Proposition 2.4, we get all the (inequivalent) Narain lattices \( \Gamma \) by letting this set act on the fixed Narain lattice \( \Gamma_0 \). Since \( \Gamma_0 \) is spanned by the
columns of the matrix \( E \), it follows that such a \( \Gamma \) is spanned by the columns of the matrix
\[
E \begin{pmatrix} R & 0 \\ 0 & 1/R \end{pmatrix} E = \frac{1}{\sqrt{2}} \begin{pmatrix} R & 1/R \\ R & -1/R \end{pmatrix}.
\]

By (2.2), each vector \( p \in \Gamma \) decomposes over \( R \)
\[
p = \frac{1}{\sqrt{2}} \begin{pmatrix} mR + n/R \\ mR - n/R \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} mR + n/R \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ mR - n/R \end{pmatrix},
\]
where \( m, n \in \mathbb{Z} \). Finally, the partition function of \( \Gamma \) is
\[
Z^\Gamma(q, \bar{q}) = \frac{1}{\eta(q) \eta(\bar{q})} \sum_{m,n \in \mathbb{Z}} q^{\frac{1}{4}(mR + nR)^2} \bar{q}^{\frac{1}{4}(mR - nR)^2}.
\]
This is the well-known partition function of toroidal compactification on \( T^1 = S^1 \) with radius \( R \).

**Proposition 2.5.** The coset space \( G \times G \setminus O'(d,d; R) \), where \( G \times G := E(O(d,R) \times O(d,R)) \setminus E \), can be represented by;
\[
G \times G \setminus O'(d,d; R) = \left\{ G_{\text{diag}} \cdot W'(\Lambda, B) \mid \Lambda \in GL(d,R), \ B \in A(d,R) \right\}.
\]
where \( W'(\Lambda, B) := \begin{pmatrix} \Lambda^{-1} & 0 \\ 0 & \Lambda \end{pmatrix} \begin{pmatrix} 1_d & -B \\ 0 & 1_d \end{pmatrix} \),
\[
(2.9)
\]
Moreover, when \( G_{\text{diag}} \cdot W'(\Lambda, B) = G_{\text{diag}} \cdot W'(\Lambda', B') \), we have \( O(d,R) \cdot \Lambda = O(d,R) \cdot \Lambda', B = B' \). So, if we fix a “gauge” for the \( O(d,R) \) action on \( GL(d,R) \) (i.e. a cross section for the orbit space \( GL(d,R)/O(d,R) \)), then \( \Lambda \) and \( B \) are uniquely determined for each orbit in the coset space \( G \times G \setminus O'(d,d; R) \).

An elementary proof of this is given in Appendix C. Here we point out that the \( \Lambda \in GL(d,R) \) may be identified with the lattice of the target torus, i.e. \( T^d = R^d/L(\Lambda) \) with \( L(\Lambda) \) being the lattice generated by the column vectors of \( \Lambda \). Under this identification, the matrix \( \Lambda^{-1} \) defines the dual torus \( (T^d)^\vee \). Also the antisymmetric matrix \( B \) represents the so-called \( B \)-field, which generalizes the electro-magnetic field in particle theory to string theory. By Proposition 2.5, the
The above generators act on the orbits \((GL(d, \mathbb{R})/O(d, \mathbb{R})) \times A(d, \mathbb{R})\) is in one-to-one correspondence with the set \((GL(d, \mathbb{R})/O(d, \mathbb{R})) \times A(d, \mathbb{R})\).

For \(d = 2\), the parametrization (2.9) can be made even more explicit. We fix a “gauge” for \(GL(2, \mathbb{R})/O(2, \mathbb{R})\) by choosing a special cross section:

\[
\Lambda_0 = \sqrt{\frac{\rho_2}{\tau_2}}\begin{pmatrix} 1 & \tau_1 \\ 0 & \tau_2 \end{pmatrix} \in GL(2, \mathbb{R}) \quad (\rho_2, \tau_2 > 0) .
\]

Note that \(\det \Lambda_0 = \rho_2\) is the volume of the torus defined by the lattice \(L(\Lambda_0)\). Letting the \(B\)-field be \(B_{12} =: \rho_1\), we arrive at the following parametrization

\[
W'(\tau, \rho) := \frac{1}{\sqrt{\rho_2 \tau_2}} \begin{pmatrix} \tau_2 & 0 & 0 & 0 \\ -\tau_1 & 1 & 0 & 0 \\ 0 & 0 & \rho_2 & \tau_1 \rho_2 \\ 0 & 0 & 0 & \tau_2 \rho_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -\rho_1 \\ 0 & 1 & \rho_1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (2.10)
\]

where \(\tau = \tau_1 + \sqrt{-1} \tau_2, \rho = \rho_1 + \sqrt{-1} \rho_2\). It is clear that \(\tau\) describes the complex structure of the torus \(T^2 = \mathbb{R}^2/L(\Lambda_0)(=: E_\tau)\). The parameter \(\rho = \rho_1 + \sqrt{-1} \rho_2 = B_{12} + \sqrt{-1} V K(E_\tau)\) is the so-called complexified Kähler modulus of \(E_\tau\). It is also clear that both parameters \(\tau\) and \(\rho\) can take arbitrary values in the upper half plane \(H^+\), i.e. \((\tau, \rho)\) ranges over all of \(H^+ \times H^+\).

The right action \(O(2, 2; \mathbb{Z})\) on the coset space \(G \times G \setminus O'(2, 2; \mathbb{R})\) is known in the physics literatures as the \(T\)-duality group.

**Proposition 2.6. (Duality transformations)**

1) The following elements generate the group \(O(2, 2; \mathbb{Z})\):

\[
S_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

\[
S_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

2) The above generators act on the orbits \((G \times G) \cdot W'(\tau, \rho)\) from the right by

\[
S_1 : (\tau, \rho) \rightarrow (-1/\tau, \rho), \quad T_1 : (\tau, \rho) \rightarrow (\tau + 1, \rho), \quad R_1 : (\tau, \rho) \rightarrow (\rho, \tau)
\]

\[
S_2 : (\tau, \rho) \rightarrow (\tau, -1/\rho), \quad T_2 : (\tau, \rho) \rightarrow (\tau, \rho + 1), \quad R_2 : (\tau, \rho) \rightarrow (-\bar{\tau}, -\bar{\rho}).
\]

3) There is a group isomorphism

\[
O(2, 2; \mathbb{Z}) \cong P(SL_2 \mathbb{Z} \times SL_2 \mathbb{Z}) \times (\mathbb{Z}_2 \times \mathbb{Z}_2)
\]

where \(P(SL_2 \mathbb{Z} \times SL_2 \mathbb{Z})\) represents the quotient group of \(SL_2 \mathbb{Z} \times SL_2 \mathbb{Z}\) by the involution: \((g, h) \mapsto (-g, -h)\), and \(\times\) denotes the semi-direct product.

**Sketch of Proof.** We will give a proof of 1) in Appendix C. By Proposition 2.5 and the parametrization of the orbit (2.10), assertion 2) is derived by straightforward calculations. For a proof of 3), we use an explicit surjective group homomorphism, which is constructed in Appendix C, \(\phi_\mathbb{Z} : SL_2 \mathbb{Z} \times SL_2 \mathbb{Z} \rightarrow O(2, 2; \mathbb{Z}) \cap \)
2) Every Narain lattice $\Gamma_1$, respectively the unimodular Lie algebras and certain generalizations such as the setup of a CFT includes (1) two chiral algebras $\mathfrak{ch}$ and their representations. For a physical exposition see [GSW][Po]. We give a mathematical exposition of chiral algebras and their representations. For a physical exposition see [GSW][Po]. We will write down $\Phi_{\tau,\rho}$ explicitly.

\begin{equation}
W(\Phi_{\tau,\rho})E = \mathbb{E}W'(\tau,\rho) =: (e_1(\tau,\rho) \ e_2(\tau,\rho) \ e_3(\tau,\rho) \ e_4(\tau,\rho)). \tag{2.11}
\end{equation}

Then we have $\langle e_i(\tau,\rho), e_{i+2}(\tau,\rho) \rangle_{\mathbb{R}^{2,2}} = 0$ ($1 \leq i, j \leq 2$). By (2.6), the Narain embedding $\Phi_{\tau,\rho}$ is given by $\Phi_{\tau,\rho}(f_i) = e_i(\tau,\rho)$, and its image is the Narain lattice

$$\Gamma(\tau,\rho) := \mathbb{Z} e_1(\tau,\rho) \oplus \mathbb{Z} e_2(\tau,\rho) \oplus \mathbb{Z} e_3(\tau,\rho) \oplus \mathbb{Z} e_4(\tau,\rho) \subset \mathbb{R}^{2,2}. \tag{2.12}$$

Explicitly the $e_i(\tau,\rho)$ are

\begin{equation}
\begin{pmatrix}
\tau_2 \\
-\tau_1 \\
\tau_2 \\
-\tau_1
\end{pmatrix}, \begin{pmatrix}
0 \\
1 \\
0 \\
1
\end{pmatrix}, \begin{pmatrix}
\rho_2 \\
-\rho_2 \\
\rho_1 \\
-\rho_1
\end{pmatrix}, \begin{pmatrix}
-\rho_1 \tau_2 + \tau_1 \rho_2 \\
\rho_1 \tau_1 + \tau_2 \rho_2 \\
-\rho_1 \tau_2 - \tau_1 \rho_2 \\
\rho_1 \tau_1 - \tau_2 \rho_2
\end{pmatrix}. \tag{2.13}
\end{equation}

The following properties of the Narain lattice $\Gamma(\tau,\rho)$ are immediate from Propositions 2.5, 2.6:

**Proposition 2.7.**
1) $\Gamma(\tau,\rho), \Gamma'(\tau',\rho')$ are equivalent if and only if $(\tau,\rho)$ and $(\tau',\rho')$ are related by a duality transformation on $\mathbb{H}_+^2$.
2) Every Narain lattice $\Gamma \subset \mathbb{R}^{2,2}$ is equivalent to $\Gamma(\tau,\rho)$ for some $\tau,\rho \in \mathbb{H}_+$.

### §3. Rational conformal field theory

**3-1) Algebraic CFT.** Algebraically, a conformal field theory (CFT) is described by its so-called chiral algebras and their representations. For a mathematical exposition see [FLM][LZ][Kac][MN]. For a physical exposition see [GSW][Po]. We give a rough schematic description of this theory here, but will be more precise when we come to CFTs on $T^2$. Examples of chiral algebras $\mathcal{A}$ often come from infinite dimensional Lie algebras and certain generalizations such as the $W$-algebras. The basic setup of a CFT includes (1) two chiral algebras $\mathcal{A}_L, \mathcal{A}_R$, which are called respectively the left and the right chiral algebras; (2) a class of representations $\mathcal{H}_{L,j}$ ($j \in \Lambda_L$) and $\mathcal{H}_{R,k}$ ($k \in \Lambda_R$), where $\Lambda_{L,R}$ are some index sets; (3) the characters of the representations $\chi_{L,j}(q) = Tr_{\mathcal{H}_{L,j}} q^{d_L}$, $\chi_{R,j}(q) = Tr_{\mathcal{H}_{R,j}} q^{d_R}$, where $d_L, d_R$ are
the scaling operators in the chiral algebras and \( q = e^{2\pi \sqrt{-1} t}, \bar{q} = e^{-2\pi \sqrt{-1} t}; \) (partition function), a real analytic modular invariant (i.e. invariant under the transformations \( t \rightarrow t + 1, t \rightarrow -1/t \)) function of the shape
\[
Z(q, \bar{q}) = \sum_{i \in A_L, j \in A_R} N_{ij} \text{ch}_{L,i}(q)\overline{\text{ch}}_{R,j}(q),
\]
where \( N_{ij} \) are some positive integers. A CFT is called rational iff its chiral algebras have only finitely many irreducible representations, in which case the index sets \( A_L, A_R \) are the finite lists of representations.

We now consider CFTs on \( T^2 \). This classes of CFTs are parameterized by equivalence classes of Narain lattices. For a generic Narain lattice \( \Gamma \), the chiral algebras are generated by vertex operators of a Heisenberg algebra (also known as "\( U(1) \) currents" in physics). The chiral algebras also contain the \( c = 2 \) Virasoro algebra as a subalgebra. The lattice \( \Gamma \) plays the role of the momentum-winding lattice. Each \( \Gamma \) are generated by vertex operators of a Heisenberg algebra (also known as "\( U(1) \) charges" of the \( U(1) \) currents). The partition function is then given by \( Z(q, \bar{q}) \), as in Definition (2.3). When the CFT becomes rational, something very interesting happens. First, the chiral algebras become significantly enlarged. Second, the infinite list of representations (of the small chiral algebras), indexed by \( \Gamma \), reconstitute themselves, and then break up into finitely many irreducible representation of the enlarged chiral algebras.

The main task of this paper is to describe Narain lattices \( \Gamma \) which yield rational CFT in terms of the momentum lattices of the left and right handed chiral algebras, and count them.

**Equations**

**(3-2) Discriminant \( A_\Gamma \).** Let \( \Gamma \) be an even, positive definite lattice. The quotient \( \Gamma^*/\Gamma \) is a group called the discriminant group of \( \Gamma \) (see [Ni] for details). Let us fix a basis \( u_1, u_2 \) of \( \Gamma \), and denote its dual basis by \( u_1^*, u_2^* \). Then the intersection form (1.2) relates these two bases by
\[
(u_1^* u_2^*) = (u_1 u_2) \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}^{-1}.
\]

The discriminant group is the abelian group generated by \( u_1^*, u_2^* \) modulo the lattice \( \Gamma \). The integral bilinear form on \( \Gamma \) extends to a rational bilinear form \( ( , ) \) on \( \Gamma^* \). Since \( \Gamma \) is even, we have a natural quadratic form \( q_\Gamma : \Gamma^*/\Gamma \rightarrow \mathbb{Q}/2\mathbb{Z} \), called the discriminant form of \( \Gamma \),
\[
q_\Gamma(v \mod \Gamma) := (v, v) \mod 2\mathbb{Z}.
\]

If \( v = m_1 u_1^* + m_2 u_2^* \), then \((v, v)\) can evaluated using the linear relation (3.3). Associated to the quadratic form \( q_\Gamma \), we have a \( \mathbb{Q}/\mathbb{Z} \)-valued bilinear form \( (w, v)_q := \frac{1}{2}(q_\Gamma(w + v) - q_\Gamma(w) - q_\Gamma(v)) \). We denote the pair \((\Gamma^*/\Gamma, q_\Gamma)\) by \( A_\Gamma \), and call it the discriminant of \( \Gamma \).

Consider the discriminants \( A_\Gamma, A_{\Gamma'} \) of two lattices \( \Gamma, \Gamma' \). The group isomorphisms which preserve the discriminant form will be called isometries of the discriminants. Clearly an isomorphism of lattices induces an isometry of their discriminants. (But the converse is not true.) We denote the set of isometries by
\[
\text{Isom}(A_\Gamma, A_{\Gamma'}) := \{ \varphi : \Gamma^*/\Gamma \rightarrow \Gamma'^*/\Gamma' \mid q_{\Gamma'}(\varphi(v)) \equiv q_{\Gamma'}(v) \mod 2\mathbb{Z} \}.
\]
Clearly, the orthogonal group $O(\Gamma)$ acts naturally on the set $\text{Isom}(A_\Gamma, A_{\Gamma'})$ from the left, and $O(\Gamma')$ from the right.

**Remark.** The set $\text{Isom}(A_\Gamma, A_{\Gamma'})$ is nonempty if and only if the two discriminants are isomorphic. This is the case exactly when the discriminant groups are isomorphic, $\Gamma^*/\Gamma \cong \Gamma'^*/\Gamma' \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2}$, and the isomorphism preserves the discriminant forms. To spell this out explicitly, let us fix the isomorphisms; $\Gamma^* \cong \Gamma^* \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2}$ (respectively $\Gamma'^*/\Gamma' \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2}$) by choosing a basis $w_1, w_2$ (respectively $w'_1, w'_2$). With respect to these bases, we represent an isomorphism $\varphi : \Gamma^*/\Gamma \to \Gamma'^*/\Gamma'$ by a matrix: $(\varphi(w_1) \varphi(w_2)) = (w'_1 w'_2) \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right)$ where $\alpha, \beta$ (respectively $\gamma, \delta$) are integers considered mod $d_1$ (d_2). Let $(\cdot, \cdot)_q$ denote the bilinear forms of $A_\Gamma, A_{\Gamma'}$. That $\varphi$ is an isometry means that $(\varphi(w_1), \varphi(w_2))_q = (w_i, w_j)_q$, i.e.

$$
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} (w'_1, w'_2)_q' & (w'_1, w'_2)_q' \\ (w'_2, w'_2)_q' & (w'_2, w'_2)_q' \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} (w_1, w_2)_q & (w_1, w_2)_q \\ (w_2, w_1)_q & (w_2, w_2)_q \end{pmatrix}. \tag{3.4}
$$

Discriminants of even, positive definite lattices of rank two and the isometries between them are central objects in our description of rational conformal field theory on $T^2$.

**Remark.**

(3-3) $c = 2$ RCFT. If $\Gamma_\Gamma, \Gamma_r$ are even, positive definite lattices of determinant $-D$ and $\varphi : A_{\Gamma_\Gamma} \to A_{\Gamma_r}$ is an isometry, then we call $(\Gamma_\Gamma, \Gamma_r, \varphi)$ a triple of determinant $-D$. We say that the triple is primitive if $\Gamma_\Gamma, \Gamma_r$ are primitive lattices. We define an equivalence relation on triples as follows:

$$(\Gamma_\Gamma, \Gamma_r, \varphi) \sim (\Gamma'_\Gamma, \Gamma'_r, \varphi')$$

$$\iff \exists \text{ isomorphisms } g : \Gamma_\Gamma \to \Gamma'_\Gamma, \ h : \Gamma_r \to \Gamma'_r \text{ such that } \varphi' = h \cdot \varphi \cdot g^{-1}.$$

Here $g : A_{\Gamma_\Gamma} \to A_{\Gamma'_\Gamma}$ is the isometry induced by $g$ and similarly for $h$. We call an equivalence class of triples $[(\Gamma_\Gamma, \Gamma_r, \varphi)]$ an RCFT data. For our description of the RCFT data, we define the following sets of equivalence classes:

$$RCFT_D := \{ \text{triples of determinant } -D \} / \sim ,$$

$$RCFT_D^{p} := \{ \text{primitive triples of determinant } -D \} / \sim ,$$

and also

$$RCFT_D(\Gamma_\Gamma, \Gamma_r) := \{ \text{triples } (\Gamma'_\Gamma, \Gamma'_r, \varphi') \text{ of determinant } -D \}
$$

with $[\Gamma'_\Gamma] = \Gamma_\Gamma, \ [\Gamma'_r] = \Gamma_r] / \sim ,$

$$RCFT_D^{p}(\Gamma_\Gamma, \Gamma_r) := \{ \text{primitive triples } (\Gamma'_\Gamma, \Gamma'_r, \varphi') \text{ of determinant } -D \}
$$

with $[\Gamma'_\Gamma] = \Gamma_\Gamma, \ [\Gamma'_r] = \Gamma_r] / \sim .$

Note that we have the obvious disjoint union:

$$RCFT_D = \bigsqcup RCFT_D(\Gamma_\Gamma, \Gamma_r),$$

where the $\Gamma_\Gamma, \Gamma_r$ range over all classes in $\overline{CL}(D) \equiv \mathcal{L}(D)$. Likewise, we have a similar decomposition for $RCFT_D^{p}$ with $\overline{CL}(D) \equiv \mathcal{L}^{p}(D)$. Note that $RCFT_D(\Gamma_\Gamma, \Gamma_r)$ is nonempty if and only if $\Gamma_\Gamma, \Gamma_r$ are in the same genus [Ni].
Lemma 3.3. Fix lattices $\Gamma_l, \Gamma_r$ in respective classes $\Gamma_l, \Gamma_r \in \mathcal{L}(D)$. Then the following map is well-defined and bijective:

$$B_{\Gamma_l, \Gamma_r} : \text{RCFT}_D(\Gamma_l, \Gamma_r) \rightarrow O(\Gamma_l) \setminus \text{Isom}(A_{\Gamma_l}, A_{\Gamma_r})/O(\Gamma_r)$$

Proof. By definition, an arbitrary class in $\text{RCFT}_D(\Gamma_l, \Gamma_r)$ may be represented by a triple $(\Gamma_l, \Gamma_r, \varphi)$ with some $\varphi \in \text{Isom}(A_{\Gamma_l}, A_{\Gamma_r})$. The triples having this shape $(\Gamma_l, \Gamma_r, *)$ are preserved exactly by the group $O(\Gamma_l) \times O(\Gamma_r)$ acting on triples by

$$(g\Gamma_l, h\Gamma_r, \tilde{h}\varphi \tilde{g}^{-1}) = (\Gamma_l, \Gamma_r, \tilde{h}\varphi \tilde{g}^{-1}).$$

This shows that the map is well-defined. Now the bijectivity of the map is also clear.

This map will be used in the next section.

Definition 3.4. ($c = 2$ RCFT) Given a triple $(\Gamma_l, \Gamma_r, \varphi)$ of determinant $-D$, we define the partition function

$$Z^{\Gamma_l, \Gamma_r, \varphi}(q, \bar{q}) = \frac{1}{|\eta(q)|^4} \sum_{a \in \Gamma_l^*/\Gamma_l} \frac{\theta_{\Gamma l}^1(q)\overline{\theta_{\Gamma r}^1}(q)}{\theta_{\Gamma l}^1(q)\overline{\theta_{\Gamma r}^1}(q)},$$

and its parity invariant form

$$\tilde{Z}^{\Gamma_l, \Gamma_r, \varphi}(q, \bar{q}) = \frac{1}{|\eta(q)|^4} \sum_{a \in \Gamma_l^*/\Gamma_l} \frac{1}{2} \left(\theta_{\Gamma l}^1(q)\overline{\theta_{\Gamma r}^1}(q) + \overline{\theta_{\Gamma l}^1}(q)\theta_{\Gamma r}^1(q)\right),$$

where $\theta_{\Gamma l}^1(q) := \sum_{v \in \Gamma} q^{v^2}/(v+a)^2$, for $a \in \Gamma^*/\Gamma$, is a theta function of the lattice $\Gamma$.

Remark. 1) It is clear that equivalent triples have the same partition function, and thus the partition function is defined for a class $[(\Gamma_l, \Gamma_r, \varphi)] \in \text{RCFT}_D$.

2) The involution $(\Gamma_l, \Gamma_r, \varphi) \mapsto (\Gamma_r, \Gamma_l, \varphi^{-1})$ on triples is obviously compatible with the equivalence relation on triples. In particular, the involution acts on the sets $\text{RCFT}_D, \text{RCFT}_D$. The parity invariant form $\tilde{Z}^{\Gamma_l, \Gamma_r, \varphi}(q, \bar{q})$ is defined so that it is invariant under this involution.

3) Note that $Z^{\Gamma_l, \Gamma_r, \varphi}(q, \bar{q})$ is nothing but a finite linear combination of products $c_{L,a}(q)c_{R,b}(q)$, where

$$c_{L,a}(q) = \frac{\theta_{\Gamma l}^1(q)}{|\eta(q)|^2}, \quad c_{R,b}(q) = \frac{\theta_{\Gamma r}^1(q)}{|\eta(q)|^2},$$

with $a \in \Gamma_l^*/\Gamma_l, b \in \Gamma_r^*/\Gamma_r$, are characters of representations of certain chiral algebras $A_L$ and $A_R$. They can also be viewed as characters of certain unitary reducible representations of the $c = 2$ Virasoro algebra (see e.g. [KP]).

Proposition 3.5. The partition function for a triple is modular invariant.

Proof. The theta function of an even lattice $\Gamma$ ($s := \text{rk} \Gamma$) has the transformation property, $\theta_{\Gamma}^s(q)|_{t \rightarrow \pm 1} = e^{\pm \sqrt{-1}t} \theta_{\Gamma}^s(q)$. Also, by the Poisson resummation formula, we have (see e.g. [KP, Proposition 3.4]):

$$\theta_{\Gamma}^s(q)|_{t \rightarrow -1/t} = \frac{1}{|\Gamma^*/\Gamma|^2} (-\sqrt{-1}i)^{\frac{s}{2}} \sum_{b \in \Gamma^*/\Gamma} e^{-2\pi \sqrt{-1}(a,b)} \theta_{\Gamma}^s(q),$$
where \((a, b) \in \mathbb{Q}/\mathbb{Z}\) is the bilinear form on the discriminant group \(\Gamma^*/\Gamma\) (see section (3-2)). Using these relations and also \(\prod_{a \in \Gamma^*/\Gamma} e^{2\pi \sqrt{-1}(a,b-b')} = d_{bb'}\), it is straightforward to verify that \(\mathbb{Z}\Gamma_t,\Gamma_r,\varphi(q,q)\) is invariant under \(t \to t+1, t \to -1/t\).

In the calculation, we use the fact that \(\varphi\) is an isometry and thus we have \((\varphi(a), b) = (a, \varphi^{-1}(b))\) for \(a \in \Gamma_t^*/\Gamma_t, b \in \Gamma_r^*/\Gamma_r\). \(\square\)

We now state our problem precisely as follows:

**Classification Problem.** (1) Formulate a correspondence between the triples \((\Gamma_1, \Gamma_r, \varphi)\) and the Narain lattices \(\Gamma(\tau, \rho)\). (2) Describe the values \(\tau, \rho \in \mathbb{H}_+\) for the equivalence classes of Narain lattices \(\Gamma(\tau, \rho)\) which correspond to the classes of triples in \(\text{RCFT}_D\), and count them.

Prop. 1.6.1 of Nikulin [Ni] gives a correspondence between primitive embeddings in a fixed unimodular lattice and isometries of discriminant groups by means of over-lattices. An equivalent approach using gluing theory can be found in [CS, Chapter 4] and references therein. This correspondence will be needed to do (1) (see below) where we seek a correspondence between abstract triples and the parameters \(\tau, \rho\). The importance of discriminant groups in the study of toroidal RCFTs was first observed in [Mo]. In the next two sections we will carry out (2). The precise counting of both Narain lattices and triples will involve taking into account a certain natural involution, \((\tau, \rho) \mapsto (\tau, -\bar{\rho})\), which represents the worldsheet parity involution. We will find that the Gauss product on the class group \(\text{Cl}(\mathcal{D})\) and its extension to \(CL(\mathcal{D})\) play a central role.

(3-4) Over-lattices. Our aim here is to define a map from \(\text{RCFT}_D\) to the Narain moduli space \(\mathcal{M}_{2,2}\). The main idea is to look at over-lattices of \(\Gamma_t \oplus \Gamma_r(-1)\).

Given a triple \((\Gamma_1, \Gamma_r, \varphi)\), put

\[\Gamma^\varphi := \{ (x, y) \in \Gamma_t^* \oplus \Gamma_r^*(-1) \mid \varphi(x \mod \Gamma_t) = y \mod \Gamma_r(-1) \}.\]  

(3.6)

It is easy to check that this is a sublattice of \(\Gamma_t^* \oplus \Gamma_r^*(-1)\) containing \(\Gamma_t \oplus \Gamma_r(-1)\), such that

\[\Gamma^\varphi/(\Gamma_t \oplus \Gamma_r(-1)) = \{ a \oplus \varphi(a) \mid a \in \Gamma_t^*/\Gamma_t \},\]

or equivalently,

\[\Gamma^\varphi = \bigcup_{a \in \Gamma_t^*/\Gamma_t} (a \oplus \varphi(a) + \Gamma_t \oplus \Gamma_r(-1)).\]  

(3.7)

(3.8)

Fix a pair of isometric embeddings \(\iota_1 : \Gamma_t \to \mathbb{R}^{2,0}, \iota_2 : \Gamma_r(-1) \to \mathbb{R}^{0,2}\). Extend \(\iota_1, \iota_2\) over \(\mathbb{R}\), and denote their extensions by \(\iota_1, \iota_2\) also.

**Proposition 3.6.** Let \((\Gamma_1, \Gamma_r, \varphi)\) be a triple, and \(\iota_1, \iota_2\) be isometric embeddings as above. Then

1) \(\Gamma^\varphi\) is an even, unimodular, integral lattice of signature \((2,2)\). Hence the image \((\iota_1 \oplus \iota_2)\Gamma^\varphi \subset \mathbb{R}^{2,2}\) is a Narain lattice.

2) The equivalence class \([((\iota_1 \oplus \iota_2)\Gamma^\varphi)]\) of Narain lattices is independent of the choices of \(\iota_1, \iota_2\). Moreover \([((\iota_1 \oplus \iota_2)\Gamma^\varphi)]\) depends only on the equivalence class of the triple \((\Gamma_1, \Gamma_r, \varphi)\).
Definition 3.7. We call a Narain lattice \( \Gamma \) to an RCFT, i.e. is not in the image of RCFT the decomposition the image of the “over-lattice map” \( \times \in \square \) proves 2).

Proposition 3.8. Proposition 3.6 and 2.4, we have a well-defined map:

\[
F : \bigcup_{D} RCFT_D \longrightarrow \mathcal{M}_{2,2}, \quad [(\Gamma_l, \Gamma_r, \varphi)] \mapsto [(\ell_1 + \ell_2)\Gamma^\varphi].
\]

Remark. 1) In the construction of \( \Gamma^\varphi \) in the proof, if we replace \( \Gamma_l, \Gamma_r \) by their images under the chosen embeddings \( \ell_1, \ell_2 \), then the resulting \( \Gamma^\varphi \) actually sits inside \( \mathbb{R}^{2,2} \), rather than just an abstract lattice.
2) Using the relation (3.8), it is a simple exercise to show that the partition function of the Narain lattice \( \Gamma = (\ell_1 + \ell_2)\Gamma^\varphi \) (Definition 2.3)) is given by

\[
Z^{\Gamma}(q, \bar{q}) = Z^{\Gamma_l, \Gamma_r, \varphi}(q, \bar{q}).
\]

(3-5) RCFTs in \( \mathcal{M}_{2,2} \). Consider the Narain lattice \( \Gamma(\tau, \rho) \) and its equivalence class \( [\Gamma(\tau, \rho)] \in \mathcal{M}_{2,2} \). For generic \( \tau, \rho \in \mathbb{H}_+ \), the class \( [\Gamma(\tau, \rho)] \) does not correspond to an RCFT, i.e. is not in the image of \( RCFT_D \) under the map (3.9) for any \( D \).

Definition 3.7. We call a Narain lattice \( \Gamma(\tau, \rho) \) rational if its class \( [\Gamma(\tau, \rho)] \) is in the image of the “over-lattice map” \( F \) given by (3.9).

To study the rationality condition for \( \Gamma(\tau, \rho) \), we consider

\[
\Pi_l := \Gamma_l \cap \mathbb{R}^{2,0}, \quad \Pi_r := \Gamma_r \cap \mathbb{R}^{0,2}.
\]

We call them the momentum lattices of (left and right chiral algebras corresponding to) \( \Gamma(\tau, \rho) \). Note that: 1) The lattices \( \Pi_l, \Pi_r \) are sublattices of \( \Gamma(\tau, \rho) \), which are zero for generic \( \tau, \rho \). 2) The natural embedding \( \Pi_r \hookrightarrow \Gamma(\tau, \rho) \) is a primitive embedding, i.e. the natural map \( \Gamma(\tau, \rho)^* \Gamma(\tau, \rho) \rightarrow \Pi_r^* \) is surjective, where \( x \in \Gamma(\tau, \rho)^* \) is mapped to \( \langle x, \ast \rangle : \Pi_l \rightarrow \mathbb{Z} \) under the natural map. Likewise for \( \Pi_l \). The following properties (Propositions 3.8 and 3.10) are well-known in the literatures [DHMV][Mo][Wa].

Proposition 3.8. The Narain lattice \( \Gamma(\tau, \rho) \) is rational if and only if

\[
\text{rk} \Pi_l = \text{rk} \Pi_r = 2.
\]

Proof. Suppose \( \Gamma(\tau, \rho) \) is rational, i.e. \( \Gamma(\tau, \rho) = \Gamma(\tau, \rho)^* \) for some triple \( (\Gamma_l, \Gamma_r, \varphi) \) with \( \Gamma_l \subset \mathbb{R}^{2,0} \), and \( \Gamma_r(-1) \subset \mathbb{R}^{0,2} \). This means that \( \Gamma^\varphi = f \Gamma(\tau, \rho) \) for some
Proof. When the both sublattices $\Pi_l$ and $\Pi_r$ have rank 2, they are orthogonal complements of each other in $\Gamma(\tau, \rho)$ (since $\Pi_l, \Pi_r$ are primitive in $\Gamma(\tau, \rho)$). Thus, for $x \in \Gamma(\tau, \rho)$, we may consider the orthogonal decomposition $x = x_l + x_r$ ($x_l \in \Pi_l \otimes \mathbb{Q}, x_r \in \Pi_r \otimes \mathbb{Q}$) over $\mathbb{Q}$. The natural (surjective) map $\Gamma(\tau, \rho) \to \Pi_l^* \to \Pi_r^*$ may be given by $x \mapsto \langle x, * \rangle = \langle x_l, * \rangle$, and has the kernel exactly equal to $\Pi_r$. Similarly we have the natural surjective map $\Gamma(\tau, \rho) \to \Pi_l^*$ with the kernel $\Pi_r$. From this, as in [Ni], we obtain
\[\Pi_l^*/\Pi_l \xrightarrow{\sim_{\ell}^\ast} \Gamma(\tau, \rho)/\left(\Pi_l \oplus \Pi_r\right) \xrightarrow{\sim_{\ell}^\ast} \Pi_r^*/\Pi_r.\] (3.13)

We have now $\iota_R \circ \iota_{\ell}^{-1} : \Pi_l^*/\Pi_l \sim \Pi_r^*/\Pi_r$, which implies that $\Pi_l$ and $\Pi_r$ have the same determinant $|\Pi_l^*/\Pi_l| = |\Pi_r^*/\Pi_r| = -D'$. Also, over the discriminant groups $\Pi_l^*/\Pi_l$ and $\Pi_r^*/\Pi_r$, we have natural quadratic forms $q_l$ and $q_r$ given by
\[q_l(x_l \bmod \Pi_l) = \langle x_l, x_l \rangle \bmod 2\mathbb{Z}, \quad q_r(x_r \bmod \Pi_r) = \langle x_r, x_r \rangle \bmod 2\mathbb{Z},\]

taking values in $\mathbb{Q}/2\mathbb{Z}$. Since $\Gamma(\tau, \rho)$ is even, integral, we have
\[q_l(x_l) + q_r(x_r) \equiv \langle x_l, x_l \rangle + \langle x_r, x_r \rangle = \langle x_l + x_r, x_l + x_r \rangle = \langle x, x \rangle \equiv 0 \bmod 2\mathbb{Z}.

Therefore the isomorphisms in (3.13) give rise to an isometry of the quadratic forms: $\iota_R \circ \iota_{\ell}^{-1} : (\Pi_l^*/\Pi_l, q_l) \to (\Pi_r^*/\Pi_r, -q_r)$. By construction, the over-lattice determined by the triple $(\Pi_l, \Pi_r(-1), \iota_R \circ \iota_{\ell}^{-1})$ is $\Gamma(\tau, \rho)$ (see Proposition 1.6.1 in [Ni]). Therefore $[\Gamma(\tau, \rho)]$ is in the image of the map $F$, i.e. $\Gamma(\tau, \rho)$ is rational.

For a rational Narain lattice $\Gamma(\tau, \rho)$, the correspondence
\[ [\Gamma(\tau, \rho)] \to [\left(\Pi_l, \Pi_r(-1), \iota_R \circ \iota_{\ell}^{-1}\right)] \] (3.14)
defines the inverse of the map
\[ F : \bigcup D \text{RCFT}_D \to \mathcal{N} := \{[\Gamma(\tau, \rho)] | \Gamma(\tau, \rho) : \text{rational}\}. \]

**Proposition 3.9.** The “over-lattice map” given by (3.9) defines a bijection
\[ F : \bigcup D \text{RCFT}_D \to \mathcal{N}. \] (3.15)

For later use, we present equivalent characterizations given in the literatures [DHMV][Mo][Wa]:

**Proposition 3.10.** Consider (3.11). The following statements are equivalent:
1) $\text{rk} \Pi_l = 2$ or $\text{rk} \Pi_r = 2$,
2) $\text{rk} \Pi_l = \text{rk} \Pi_r = 2$,
3) $\tau, \rho \in \mathbb{Q}(\sqrt{D})$ for some integer $D < 0$.

**Proof.** 1) $\Rightarrow$ 2): By symmetry, we consider only the case $\text{rk} \Pi_l = 2$. By definition of $\Pi_l$ and $\Pi_r$, they are sublattices of the Narain lattice $\Gamma(\tau, \rho)$. Furthermore we have
$\Pi_r \subset \Pi^\perp$, where $\Pi^\perp$ is the orthogonal complement of $\Pi_r$ in $\Gamma(\tau, \rho)$. Let $x \in \Pi^\perp_r$. Then we have the unique decomposition $x = x_l + x_r$ over $\mathbb{R}$ with some $x_l \in \mathbb{R}^{2,0}$ and $x_r \in \mathbb{R}^{0,2}$. Since $\Pi_l$ has rank 2, we see that $x_l \in \Pi_l \otimes \mathbb{R} = \mathbb{R}^{2,0}$. It follows that $\langle x_l, x \rangle = \langle x_1, x_1 \rangle = 0$, so that $x_l = 0$. Therefore $x = x_r \in \Pi_r$. This shows that $\Pi_r \supset \Pi^\perp_r$, hence $\Pi_r = \Pi^\perp_r$ has rank 2.

2) $\Rightarrow$ 1) is obvious.

1) $\Leftrightarrow$ 3): Fix a basis of $\Gamma(\tau, \rho)$ as in (2.12). Then an arbitrary element of $\Gamma(\tau, \rho)$ may be represented by

$$p(m, n) = m_1 e_1(\tau, \rho) + m_2 e_2(\tau, \rho) + n_1 e_3(\tau, \rho) + n_2 e_4(\tau, \rho).$$

By (2.2) we have a unique decomposition $p(m, n) = p_l(m, n) + p_r(m, n)$ over $\mathbb{R}$ where $p_l(m, n) = 1(\ast, \ast, 0, 0)$ and $p_r(m, n) = 1(0, 0, \ast, \ast)$. We have

$$p(m, n) \in \Pi_l = \Gamma(\tau, \rho) \cap \mathbb{R}^{2,0} \Leftrightarrow p_r(m, n) = 0.$$ 

This is equivalent to

$$\begin{cases} \tau_2 m_1 - \rho_2 n_1 - (\rho_1 \tau_2 + \rho_1 \rho_2)n_2 = 0, \\ -\tau_1 m_1 + m_2 + \rho_1 n_1 + (\rho_1 \tau_1 - \tau_2 \rho_2) n_2 = 0. \end{cases}$$

The integral solutions to these linear equations determine the lattice $\Pi_l$. The lattice of integral solutions ($\cong \Pi_l$) is of maximal rank, i.e. 2, if and only if they can be solved over $\mathbb{Q}$, or in other words, the normal vectors of the hyperplanes (3.17) are in $\mathbb{Q}^4$. Therefore we have

$$\mathrm{rk} \Pi_l = 2 \Leftrightarrow \tau_1, \rho_1, \rho_1 \tau_1 - \tau_2 \rho_2, \frac{\rho_2}{\tau_2} \in \mathbb{Q} \Leftrightarrow \tau_1, \tau_2, \rho_1, \rho_2, \frac{\rho_2}{\tau_2} \in \mathbb{Q}. $$

The last condition says that $\tau = \tau_1 + \sqrt{-1} \tau_2$ satisfy a quadratic equation $a \tau^2 + b \tau + c = 0$ for some integers $a, b, c$ ($b^2 - 4ac < 0$) and that the same holds for $\rho = \rho_1 + \sqrt{-1} \rho_2$. By the condition $\frac{\rho_2}{\tau_2} \in \mathbb{Q}$, we see that $\tau, \rho$ must be in the same quadratic imaginary fields $\mathbb{Q}(\sqrt{D})$. Conversely, it is straightforward to see that for $\tau, \rho \in \mathbb{Q}(\sqrt{D})$ the last condition of (3.18) holds. Thus we obtain the equivalence of 1) and 3).

**Remark.** Condition 3) above was found by Moore [Mo] to characterize rationality of CFTs on $T^2$. It is also pointed out by him that the condition that $\tau \in \mathbb{Q}(\sqrt{D})$ is equivalent to that the elliptic curve $E_\tau$ is of CM type, namely $E_\tau$ has non-trivial endomorphisms (see e.g. [Ha,IV.4]). A generalization to higher dimensional tori $T^{2d}(d \geq 1)$ has been done in [Wa, Theorem 4.5.5], where it was shown that rationality implies that $T^{2d}$ is isogeneous to a product $E_{\tau_1} \times E_{\tau_2} \times \cdots \times E_{\tau_d}$ of elliptic curves of CM types. (More precisely, it is shown that rationality is equivalent to that $T^{2d}$ is isogeneous to a product as above with each $\tau_i, \rho_i$ in $\mathbb{Q}(\sqrt{D_i})$ for some $D_i < 0$ ($i = 1, \cdots, d$).) As shown in [Mo][Wa], these points corresponding to RCFTs are dense in the Narain moduli space. It is argued in [Mo] that a similar density property holds for string compactifications on K3 surfaces, where RCFTs correspond to $\sigma$-models on singular K3 surfaces, i.e. K3 surfaces with $\rho(X) = 20$ (see e.g. [SI]).

In case of Calabi-Yau compactifications in dimension three, the connection to CFTs is less clear except at some special points such as the Gepner points, where one
has a precise dictionary comparing CFTs and the geometry of Calabi-Yau threefolds (see e.g. [Gr]). Recently, Gukov-Vafa [GV] have proposed a criterion for rationality of sigma models on Calabi-Yau threefolds $X$. They conjecture that RCFT occurs if and only if both $X$ and its mirror manifold $X^\vee$ are of CM type. Roughly, a Calabi-Yau threefold is called of CM type if its (Weil and Griffiths) intermediate Jacobians are of CM type. See [Bo, Theorem 2.3], for more precise definitions and several other equivalent definitions of CM type Calabi-Yau threefolds.

§4. Classification of $c = 2$ RCFT — primitive case —

In this section, we classify the $c = 2$ RCFT data $[(\Gamma, \Gamma_r, \varphi)]$ of primitive triples. More precisely we describe the image of the set of $RCFT^p_2$ under the “over-lattice map” $F$ in (3.15), and determine its cardinality (up to an involution).

(4.1) Narain lattices parameterized by $Cl(D)$. Given a positive definite quadratic form $Q(a, b, c) : f(x, y) = ax^2 + bxy + cy^2$, we can write

$$f(x, y) = a|x + \tau_{Q(a, b, c)}y|^2, \quad \tau_{Q(a, b, c)} := \frac{b + \sqrt{b^2 - 4ac}}{2a},$$

where $D = b^2 - 4ac < 0$. So we have a map

$$Q(a, b, c) \mapsto \tau_{Q(a, b, c)} \in H_+ := \{x + \sqrt{-1}y \mid y > 0\}.$$  \hspace{1cm} (4.4)

It is easy to verify that the $SL_2\mathbb{Z}$ action on quadratic forms is compatible with the $PSL_2\mathbb{Z}$ actions on $H_+$ under this map. This shows that the $PSL_2\mathbb{Z}$ orbit of $\tau_{Q(a, b, c)} \in H_+$ depends only on the class $C = [Q(a, b, c)] \in Cl(D)$. We denote the orbit by $\tau_C$. We also write $\rho_{Q(a, b, c)} := \tau_{Q(a, b, c)}$, $\rho_C := \tau_C$, and put

$$[\Gamma(\tau_C, \rho_{C'})] := [\Gamma(\tau_Q, \rho_{Q'})]$$

for any $Q \in C$, $Q' \in C'$. This makes sense since the equivalence class $[\Gamma(\tau, \rho)]$ is invariant under $PSL_2\mathbb{Z} \times PSL_2\mathbb{Z}$ acting on $\tau, \rho$, by Proposition 2.7). Note that if $Q(a_1, b_1, c_1), Q(a_2, b_2, c_2)$ belong to the same class $C \in Cl(D)$, then $Q(a_1, -b_1, c_1), Q(a_2, -b_2, c_2)$ also belong to the same class. Since $-\rho_{Q(a, b, c)} = \rho_{Q(a, -b, c)}$, it follows that the involution $(\tau, \rho) \mapsto (\tau, -\rho)$ on $H_+ \times H_+$, induces an involution on the classes $[\Gamma(\tau_C, \rho_{C'})]$. If $C, C'$ are primitive, then $[\Gamma(\tau_C, \rho_{C'})] \mapsto [\Gamma(\tau_C, \rho_{C' \cdot -1})]$ under this involution, because $[Q(a, -b, c)] = [Q(a, b, c)]^{-1}$ in the group $Cl(D)$ (see Proposition A.7 in Appendix A).

Definition 4.4. For negative integers $D < 0$, we put

$$\mathcal{N}^p_D := \{[\Gamma(\tau_C, \rho_{C'})] \mid C, C' \in Cl(D)\}, \quad \tilde{\mathcal{N}}^p_D := \mathcal{N}^p_D / \mathbb{Z}_2,$$

where $\mathbb{Z}_2$ denotes the involution on $\tilde{\mathcal{N}}^p_D$ induced by $(\tau, \rho) \mapsto (\tau, -\rho)$ on $H_+ \times H_+$.
Proposition 4.5. We have a canonical bijection $\tilde{\mathcal{N}}_D^0 \cong \text{Sym}^2 \tilde{\text{Cl}}(D)$. In particular, we have
\[
|\tilde{\mathcal{N}}_D^0| = \frac{1}{2} \tilde{h}(D)(\tilde{h}(D) + 1).
\] (4.6)

Proof. Obviously, we have a surjective map
\[
\text{Cl}(D) \times \text{Cl}(D) \to \tilde{\mathcal{N}}_D^0, \quad (C, C') \mapsto [\Gamma(\tau_C, \rho_{C'})] \text{ mod } \mathbb{Z}_2.
\]
Since $\mathbb{Z}_2 : [\Gamma(\tau_C, \rho_{C'})] \mapsto [\Gamma(\tau_C, \rho_{C'}^{-1})]$, the pairs $(C, C')$ and $(C, C'^{-1})$ have the same image. Since $[\Gamma(\tau, \rho)]$ is invariant under duality transformations, we have $[\Gamma(\tau, \rho)] = [\Gamma(\rho, \tau)] = [\Gamma(-\tilde{\rho}, -\tilde{\tau})]$, by Proposition 2.6. It follows that the pairs $(C, C'), (C', C), (C'^{-1}, C^{-1})$ also have the same image under the map. Note that $\tilde{\text{Cl}}(D) = \text{Cl}(D) / \approx$, where $\approx$ is the relation $C \cong C^{-1}$. This shows that our map descends to $\text{Sym}^2 \tilde{\text{Cl}}(D) \to \tilde{\mathcal{N}}_D^0$. It remains to show that this is injective.

Let $(C_1, C_1'), (C_2, C_2') \in \text{Cl}(D) \times \text{Cl}(D)$ be two pairs having the same image. Then $[\Gamma(\tau_{Q_1}, \rho_{Q_1'})] = [\Gamma(\tau_{Q_2}, \rho_{Q_2'})]$ or $[\Gamma(\tau_{Q_2}, -\tilde{\rho}_{Q_2'})]$, where $Q_i \in C_1, Q_i' \in C_i'$. By Proposition 2.7(1), it follows that $(\tau_{Q_1}, \rho_{Q_1'})$ transforms, by a duality transformation on $\mathbb{H}_+ \times \mathbb{H}_+$, to either $(\tau_{Q_2}, \rho_{Q_2'})$ or $(-\tilde{\rho}_{Q_2'}, \tau_{Q_2})(-\tilde{\rho}_{Q_2'}, \tau_{Q_2})$, by Proposition 2.6. $(\tau_{Q_i}, \rho_{Q_i'})$ transforms, by some $g \in P(SL_2 \mathbb{Z} \times SL_2 \mathbb{Z}) \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$, to one of the following:

\[
(\tau_{Q_2}, \rho_{Q_2'}), (\rho_{Q_2'}, \tau_{Q_2}), (-\tilde{\rho}_{Q_2}, -\tilde{\tau}_{Q_2}), (\tau_{Q_2}, -\tilde{\rho}_{Q_2'}), (-\tilde{\rho}_{Q_2'}, \tau_{Q_2}), (\rho_{Q_2'}, -\tilde{\tau}_{Q_2}).
\]

It follows that $(C_1, C_1')$ is equal to one of the following:

\[
(C_2, C_2'), (C_2', C_2), (C_2'^{-1}, C_2^{-1}), (C_2, C_2'^{-1}), (C_2'^{-1}, C_2), (C_2', C_2^{-1}).
\]

Therefore $(C_1, C_1')$ and $(C_2, C_2')$ represent the same element in $\text{Sym}^2 \tilde{\text{Cl}}(D)$. This shows that the map $\text{Sym}^2 \tilde{\text{Cl}}(D) \to \tilde{\mathcal{N}}_D^0$ is injective.

\[(4-2)\] Key lemma. If $\Gamma(\tau, \rho), \Gamma(\tau', \rho')$ are equivalent Narain lattices, then their left momentum lattices (3.11) are isomorphic; likewise for their right momentum lattices. By Proposition (2.7) 1), this happens if $(\tau, \rho)$ transforms to $(\tau', \rho')$ by a duality transformation on $\mathbb{H}_+ \times \mathbb{H}_+$. In particular, the isomorphism class of the momentum lattices of $\Gamma(\tau Q, \rho_{Q'})$ depends only on the equivalence classes $[Q] = C, [Q'] = C' \in \text{Cl}(D)$. We denote the isomorphism classes of the left and right momentum lattices by

\[
\Gamma(\tau_C, \rho_{C'}) \cap \mathbb{R}^{2,0}, \quad \Gamma(\tau_C, \rho_{C'}) \cap \mathbb{R}^{0,2}.
\] (4.7)

Lemma 4.6. (Key lemma) For $C_1, C_2 \in \text{Cl}(D)$, we have

\[
\Gamma(\tau_{C_1}, \rho_{C_2}) \cap \mathbb{R}^{2,0} = q(C_1 * C_2^{-1}), \quad \Gamma(\tau_{C_1}, \rho_{C_2}) \cap \mathbb{R}^{0,2} = q(C_1 * C_2)(-1),
\] (4.8)

where $*$ is the Gauss product on $\text{Cl}(D)$ and $q$ is the natural map $\text{Cl}(D) \to \tilde{\text{Cl}}(D)$.

Proof. To evaluate the isomorphism classes of the left and right momentum lattices (4.7), we choose forms $Q(a, b, c) \in C_1, Q(a', b', c') \in C_2$ which are concordant. In fact, by Lemma A.3 we can arrange that

\[
(i) \ a a' \neq 0, \ a, a' = 1 \quad (ii) \ b = b' \quad (iii) \ \frac{b^2 - D}{4aa'} \in \mathbb{Z},
\] (4.9)
where \( D = b^2 - 4ac = b'^2 - 4a'd' \). By definition, the Gauss product \( C_1 \ast C_2 \) is

\[
[Q(a, b, c)] \ast [Q(a', b', c')] := [Q(aa', b, \frac{b^2 - D}{4aa'})] .
\]

(See Appendix A for details.) We will compute a \( \mathbb{Z} \)-basis for the right momentum lattice \( \Pi_r \) of \( \Gamma(\tau, \rho) \), \( \tau := \tau_{Q(a,b,c)}, \rho := \rho_{Q(a',b',c')} \), and compare the resulting quadratic form of \( \Pi_r \) with \( Q(aa', b, \frac{b^2 - D}{4aa'}) \). Likewise for \( \Pi_l \).

• \( \mathbb{Z} \)-basis for \( \Pi_r \). A vector \( p \in \Gamma(\tau, \rho) \) lies in \( \Pi_r \) iff \( p_1 = 0 \), where \( p = p_l + p_r \) is its decomposition under (2.2). But it is difficult to find a \( \mathbb{Z} \)-basis by solving \( p_1 = 0 \) directly. A better way is to first find a \( \mathbb{Q} \)-basis, as follows. Note that \( (\Pi_r)^\perp = \Pi_l \) in \( \Gamma(\tau, \rho) \). Since \( \Pi_l \) is a rank two sublattice in a rank four lattice, it must be defined by two independent linear integral equations whose coefficients necessarily form a \( \mathbb{Q} \)-basis of \( \Pi_l \). Thus to find a \( \mathbb{Q} \)-basis of \( \Pi_r \), we can write down defining equations for \( \Pi_l \) and read off the coefficients. As in the proof of Proposition 3.8, we denote an arbitrary vector in \( \Gamma(\tau, \rho) \) by \( p(m, n) = m_1 e_1 + m_2 e_2 + n_1 e_3 + n_2 e_4 \). Then

\[
p(m, n) \in \Pi_l \iff p_r(m, n) = 0
\]

\[
\iff \left\{ \begin{array}{l}
2a'm_1 - 2a n_1 - 2bm_2 = 0 \\
-2a'bm_1 + 4aa'm_2 + 2abn_1 + (b^2 + D)n_2 = 0 .
\end{array} \right.
\]

The last two equations read \( \langle w_1, p(m, n) \rangle = \langle w_2, p(m, n) \rangle = 0 \), where

\[
w_1 = -a e_1 - b e_2 + a' e_3 , \quad w_2 = ab e_1 + (b^2 - 2ac) e_2 - a'b e_3 + 2aa' e_4 .
\]

So \( w_1, w_2 \) form a \( \mathbb{Q} \)-basis of \( \Pi_r \otimes \mathbb{Q} \). Consider the new \( \mathbb{Q} \)-basis

\[
w'_1 = w_1 = -a e_1 - b e_2 + a' e_3 , \quad w'_2 = \frac{1}{2aa'} (w_2 + bw_1) = -\frac{c}{a'} e_2 + e_4 .
\]

Note that \( w'_1 \) is an integral primitive vector because \( (a, b) = (a', a') = 1 \) and so is \( w'_2 \) by \( \frac{c}{a'} = \frac{b^2 - D}{4aa'} \in \mathbb{Z} \). By looking at the coefficient of \( e_3 \), we may claim that they form a \( \mathbb{Z} \)-basis of \( \Pi_r \).

Computing the quadratic form, we obtain

\[
\left( \begin{array}{c}
w'_1 \\
w'_2
\end{array} \right) = \left( \begin{array}{c}
w_1 \\
w_2
\end{array} \right) - \frac{c}{a'} \left( \begin{array}{c}
w_1 \\
w_2
\end{array} \right) = \left( \begin{array}{c}
-2aa' & -b \\
-b & -\frac{2c}{a'}
\end{array} \right) ,
\]

which coincides with \( -Q(aa', b, \frac{c}{a'}) \in C_1 \ast C_2 \). It follows that the isomorphism class of the lattice \( \Pi_r(-1) \) coincides with the improper equivalence class of \( C_1 \ast C_2 \), i.e. \( [\Pi_r(-1)] = q(C_1 \ast C_2) \).

• \( \mathbb{Z} \)-basis for \( \Pi_l \). As before, first we find a \( \mathbb{Q} \)-basis of \( \Pi_l \otimes \mathbb{Q} \):

\[
u_1 = 2a e_1 + 2a' e_3 , \quad u_2 = 2ab e_1 + (b^2 - D) e_2 - 2a'b e_3 + 4aa' e_4 .
\]

Put

\[
u'_1 = \frac{1}{2} u_1 = a e_1 + a' e_3 , \quad u'_2 = \frac{1}{4a} (u_2 + bu_1) = b e_1 + c e_2 + a' e_4 .
\]
Since \((a, a') = 1\), we have integers \(k, l\) satisfying \(ka + lb = 1\). Now do a further change of basis to

\[
u'' = a v_1 + a' v_3, \quad u''_b = \frac{1}{a}(u''_1 - kbu'_1) = lb v_1 + \frac{c}{a'} v_2 - kb v_3 + v_4.
\]

As before, the coefficients of \(u''_1, u''_b\) allow us to conclude that they form an integral basis for \(\Pi_l\). Computing its quadratic form, we obtain

\[
\left(\begin{array}{cc}
u''_1 & \nu''_b \\
u''_b & \nu''_2
\end{array}\right) = \left(\begin{array}{cc}2aa' & -b + 2lb \alpha' \\
-b + 2lb \alpha' & -2klb^2 + \frac{2c}{a}
\end{array}\right),
\]

which coincides with \(Q(aa', -b + 2lb \alpha', -klb^2 + \frac{2c}{a})\).

Now note that

\[
Q(a, b, c) \sim_{\mathbb{Z}(a)} Q(a, b - 2(kb)a, c_k) = Q(a, -b + 2lb \alpha', c_k)
\]

\[
Q(a', b', c') = Q(a', b, c') \sim_{\mathbb{Z}(a')} Q(a', -b, c') \sim_{\mathbb{Z}(a')} Q(a', -b + 2lb \alpha', c'_l),
\]

where \(c_k := c - (kb)b + (kb)^2a\) and \(c'_l := c' + (lb)(-b) + (lb)^2a\). It is easy to verify that \(Q(a, -b + 2lb \alpha', *)\) and \(Q(a', -b + 2lb \alpha', *)\) are concordant forms whose Gauss product coincides with (4.11). Notice that we have used the improper equivalence once in the equivalence calculation above. So we conclude that \(|\Pi_l| = q(C_1 \ast C_2^{-1})\).

\[\Box\]

Motivated by the key lemma, we introduce

**Definition 4.7.** For classes \(\Gamma_l, \Gamma_r \in \mathcal{L}(D) \equiv \tilde{Cl}(D)\), we set

\[
\mathcal{G}_D^p(\Gamma_l, \Gamma_r) := \{(C_1, C_2) \in \text{Sym}^2 Cl(D) \mid q(C_1 \ast C_2^{-1}) = \Gamma_l, q(C_1 \ast C_2) = \Gamma_r\}
\]

\[
\mathcal{G}_D^p := \bigsqcup_{\Gamma_l, \Gamma_r \in \mathcal{L}(D)} \mathcal{G}_D^p(\Gamma_l, \Gamma_r) \quad (= \text{Sym}^2 Cl(D)).
\]

Let \(\tilde{\mathcal{G}}_D^p(\Gamma_l, \Gamma_r)\) and \(\bar{\mathcal{G}}_D^p\), respectively, be the images under the natural map \(\text{Sym}^2 Cl(D) \to \text{Sym}^2 \tilde{Cl}(D)\) induced by \(q : Cl(D) \to \tilde{Cl}(D)\).

Note that \(\mathcal{G}_D^p(\Gamma_l, \Gamma_r)\) is well-defined since \(q(C_1 \ast C_2) = q(C_2 \ast C_1)\) and also \(q(C_1 \ast C_2^{-1}) = q(C_2 \ast C_1^{-1})\). Note also that \(\tilde{\mathcal{G}}_D^p(\Gamma_l, \Gamma_r) = \bar{\mathcal{G}}_D^p(\Gamma_r, \Gamma_l)\), and that \(\bar{\mathcal{G}}_D^p(\Gamma_l, \Gamma_r) = \mathcal{G}_D^p(\Gamma'_r, \Gamma'_l)\) holds if and only if \(\{\Gamma_l, \Gamma_r\} = \{\Gamma'_r, \Gamma'_l\}\). From this, we have:

\[
\tilde{\mathcal{G}}_D^p = \bigsqcup_{(\Gamma_l, \Gamma_r) \in \text{Sym}^2 \tilde{Cl}(D)} \tilde{\mathcal{G}}_D^p(\Gamma_r, \Gamma_l) \quad (= \text{Sym}^2 \tilde{Cl}(D)).
\]

---

**Classification.** Now we complete our classification.
Proposition 4.8. The inverse of the “over-lattice map” $F^{-1} : \mathcal{N} \to \bigcup D RCFT_D$ (see (3.15)) restricts to an injective map $f : N_D^p \to RCFT_D^p$, i.e. 

$$F^{-1} : \mathcal{N} \to \bigcup D RCFT_D$$
$$f : N_D^p \to RCFT_D^p$$

Proof. By (3.14), we have $F^{-1} : [\Gamma(\tau, \rho)] \mapsto [(\Pi_l, \Pi_r(\rho), \iota_R \circ \iota_L^{-1})]$. Let $[\Gamma(\tau, \rho)] \in N_D^p$, i.e. $\tau = \tau_0$, $\rho = \rho_2$ for some forms $Q_i$ of discriminant $D$. Then $[\Pi_l] = q([Q_1], [Q_2])$. It follows immediately that $\Pi_l, \Pi_r$ are both primitive lattices of determinant $-D$. This shows that $F^{-1}(N_D^p) \subset RCFT_D^p$. □

Consider the quotient

$$\tilde{RCFT}_D^p := RCFT_D^p / (\pi_2 : [(\Gamma_l, \Gamma_r, \varphi)] \mapsto [(\Gamma_r, \Gamma_l, \varphi^{-1})]),$$

and the decomposition

$$\tilde{RCFT}_D^p = \bigcup_{(\Gamma_l, \Gamma_r) \in Sym^2(\bar{C}(D))} \tilde{RCFT}_D^p(\Gamma_l, \Gamma_r), \quad (4.14)$$

with

$$\tilde{RCFT}_D^p(\Gamma_l, \Gamma_r) := \{ [(\Gamma_l, \Gamma_r, \varphi)] mod(\pi_2) \mid [\Gamma_l] = \Gamma_l, \ [\Gamma_r] = \Gamma_r \},$$

where $[(\Gamma_l, \Gamma_r, \varphi)] mod(\pi_2)$ represents the $\mathbb{Z}_2$-orbit: $[(\Gamma_l, \Gamma_r, \varphi)] \sim [(\Gamma_r, \Gamma_l, \varphi^{-1})].$

Lemma 4.9. The involutions $\pi_1 : (\tau, \rho) \mapsto (\tau, -\rho)$ and $\pi_2 : [(\Gamma_l, \Gamma_r, \varphi)] \mapsto [(\Gamma_r, \Gamma_l, \varphi^{-1})]$ are compatible with the map $f : N_D^p \to RCFT_D^p$, i.e. we have the diagram

$$\begin{array}{ccc}
[\Gamma(\tau, \rho)] & \mapsto & [(\Pi_l, \Pi_r(-1), \iota_R \circ \iota_L^{-1})] \\
\downarrow_{\pi_1} & & \downarrow_{\pi_2} \\
[\Gamma(\tau, -\bar{\rho})] & \mapsto & [(\Pi_r(-1), \Pi_l, \iota_L \circ \iota_R^{-1})].
\end{array} \quad (4.15)$$

Proof. Consider the explicit basis $e_1(\tau, \rho), \ldots, e_4(\tau, \rho)$ of $\Gamma(\tau, \rho)$ in (2.12). The involution $\pi_2 : (\tau, \rho) \mapsto (\tau, -\bar{\rho})$ exchanges the upper two components and the lower two components of each $e_i(\tau, \rho)$ (up to signs). Let $\pi_0 : \mathbb{R}^{2,2} \to \mathbb{R}^{2,2}$ be the linear map defined by $i(1,0,0,0) \leftrightarrow i(0,1,0,0), \ i(0,0,1,0) \leftrightarrow i(0,0,0,1)$. Then we have $\pi_0 \Gamma(\tau, \rho) = \Gamma(\tau, -\bar{\rho})$. Clearly $\pi_0$ is an involutive anti-isometry of $\mathbb{R}^{2,2}$, i.e. $\langle \pi_0(x), \pi_0(x) \rangle = -(x,x)$, which exchanges the two subspaces $\mathbb{R}^{2,0}$ and $\mathbb{R}^{0,2}$. Let $\Pi_l, \Pi_r$ be the left and the right momentum lattices, and $\iota_L, \iota_R$ the isomorphisms (3.13) for $\Gamma(\tau, \rho)$. Then we have

$$\pi_0 \Pi_l = \Gamma(\tau, -\bar{\rho}) \cap \mathbb{R}^{0,2} =: \Pi'_l, \quad \pi_0 \Pi_r = \Gamma(\tau, -\bar{\rho}) \cap \mathbb{R}^{2,0} =: \Pi'_r,$$

where $\Pi'_l, \Pi'_r$ are the left and the right momentum lattices of $\Gamma(\tau, -\bar{\rho})$. This shows that $\pi_0 : \Pi_l \to \Pi'_l(-1), \pi_0 : \Pi_r(-1) \to \Pi'_r$, define lattice isomorphisms. Let $\iota'_L, \iota'_R$ be the isomorphisms (3.13) for $\Gamma(\tau, -\bar{\rho})$. Then it is easy to check that $\iota_R \circ \iota_L^{-1} = \pi_0 \circ \iota_L \circ \iota_R^{-1} \circ \pi_0$. It follows that the triples $(\Pi'_l, \Pi'_r(-1), \iota_R \circ \iota_L^{-1})$, $(\Pi'_r(-1), \Pi_l, \iota_L \circ \iota_R^{-1})$ are equivalent. This shows that $f : [\Gamma(\tau, -\bar{\rho})] \mapsto [(\Pi'_r(-1), \Pi_l, \iota_L \circ \iota_R^{-1})]$. □

Now we can state our main theorems:
Theorem 4.10. Consider the map \( g : \mathcal{G}_D^p \to \mathcal{N}_D^p \) defined by \((C_1, C_2) \mapsto [\Gamma(\tau_{C_1}, \rho_{C_2})] \), and the injective map \( f : \mathcal{N}_D^p \to \text{RCFT}_D^p \) in Proposition 4.8. Then there exist corresponding induced maps \( \tilde{g} : \tilde{\mathcal{G}}_D^p \to \tilde{\mathcal{N}}_D^p \) and \( \tilde{f} : \tilde{\mathcal{N}}_D^p \to \text{RCFT}_D^p \) such that the following diagram commutes:

\[
\begin{array}{c}
\mathcal{G}_D^p \xrightarrow{g} \mathcal{N}_D^p \xrightarrow{f} \text{RCFT}_D^p \\
\downarrow q \quad \downarrow \pi_1 \quad \downarrow \pi_2 \\
\tilde{\mathcal{G}}_D^p \xrightarrow{\tilde{g}} \tilde{\mathcal{N}}_D^p \xrightarrow{\tilde{f}} \text{RCFT}_D^p
\end{array}
\]

Moreover \( \tilde{g} \) is bijective and \( \tilde{f} \) is injective.

Proof. In Proposition 4.5, we saw that the map \( \text{Cl}(D) \times \text{Cl}(D) \to \mathcal{N}_D^p, \quad (C_1, C_2) \mapsto [\Gamma(\tau_{C_1}, \rho_{C_2})] \),

induces the bijection \( \mathcal{G}_D^p = \text{Sym}^2 \text{Cl}(D) \to \mathcal{N}_D^p \) (as well as the surjection \( g : \mathcal{G}_D^p \to \mathcal{N}_D^p \) = \text{Sym}^2 \text{Cl}(D) \to \mathcal{N}_D^p \)). This is the bijection \( \tilde{g} \) we seek. The commutative diagram (4.15) implies that \( f \) induces a map \( \tilde{f} \), as required. That \( \tilde{f} \) is injective follows immediately from the fact that \( f \) is injective.

\[\blacksquare\]

Theorem 4.11.

1) The map \( f : \mathcal{N}_D^p \to \text{RCFT}_D^p \) is bijective. Hence \( \tilde{f} \) is also bijective.

2) The composition \( \tilde{f} \circ \tilde{g} \) is a bijection with

\[\tilde{f} \circ \tilde{g} \left( \tilde{\mathcal{G}}_D^p(\Gamma_l, \Gamma_r) \right) = \text{RCFT}_D^p(\Gamma_l, \Gamma_r) \quad (4.16)\]

Proof. 1) By the preceding theorem, both \( f \) and \( \tilde{f} \) are injective. Surjectivity of \( f \) will be proved in section (5-2). Thus \( \tilde{f} \) is also surjective by (4.15).

2) That \( \tilde{f} \circ \tilde{g} \) is bijective follows from 1) and the preceding theorem. Let \((C_1, C_2) \in \mathcal{G}_D^p(\Gamma_l, \Gamma_r), \) i.e. \( q(C_1 \ast C_2) = \Gamma_l \) and \( q(C_1 \ast C_2^{-1}) = \Gamma_r \). By the key lemma (Lemma 4.6), we have \( f \circ g((C_1, C_2)) = f([\Gamma(\tau_{C_1}, \rho_{C_2})]) \in \text{RCFT}_D^p(\Gamma_l, \Gamma_r) \). This shows that

\[\tilde{f} \circ \tilde{g} \left( \tilde{\mathcal{G}}_D^p(\Gamma_l, \Gamma_r) \right) \subset \text{RCFT}_D^p(\Gamma_l, \Gamma_r)\]

The reverse inclusion follows from the fact that \( \tilde{f} \circ \tilde{g} \) is a bijection.

\[\blacksquare\]

Proposition 4.12. For primitive classes \( \Gamma_l, \Gamma_r \in \text{Cl}(D) \), the bijection \( \text{B}_{\Gamma_l, \Gamma_r} : \text{RCFT}_D(\Gamma_l, \Gamma_r) \to O(\Gamma_l) \setminus \text{Isom}(A_{\Gamma_l}, A_{\Gamma_r})/O(\Gamma_r) \) in (3.5) induces a bijection

\[\text{RCFT}_D^p(\Gamma_l, \Gamma_r) \leftrightarrow O(\Gamma_l) \setminus \text{Isom}(A_{\Gamma_l}, A_{\Gamma_r})/O(\Gamma_r)\]

Proof. We only need to show that the surjective map \( \text{RCFT}_D^p(\Gamma_l, \Gamma_r) \to \text{RCFT}_D^p(\Gamma_l, \Gamma_r), \quad ([\Gamma_l, \Gamma_r, \varphi]) \mapsto ([\Gamma_l, \Gamma_r, \varphi] \text{mod}(\tau_2)) \) is also injective. This is a restriction of the map \( \tau_2 : \text{RCFT}_D^p \to \text{RCFT}_D^p \). The preimage of a given \([\Gamma_l, \Gamma_r, \varphi] \text{mod}(\tau_2)\) consists of the two RCFT data \([\Gamma_l, \Gamma_r, \varphi], [\Gamma_r, \Gamma_l, \varphi^{-1}]\). If \( \Gamma_l \neq \Gamma_r \), then only the first RCFT data lie in \( \text{RCFT}_D^p(\Gamma_l, \Gamma_r) \). If \( \Gamma_l = \Gamma_r \), then the two RCFT data are of the shape \([\Gamma, \Gamma, \psi], [\Gamma, \Gamma, \psi^{-1}]\) for some \( \psi \in O(A_{\Gamma}) \). In Appendix B, we show that \( \psi = \psi^{-1} \). Thus, in either case, the preimage of \([\Gamma_l, \Gamma_r, \varphi] \text{mod}(\tau_2)\) in \( \text{RCFT}_D^p(\Gamma_l, \Gamma_r) \) contains just one RCFT data.

\[\blacksquare\]

As a corollary to Theorem 4.11 and Proposition 4.12, we have the following equality.
Corollary 4.13. For $\Gamma_1, \Gamma_r \in \widetilde{C}l(D)$, and for arbitrary choices of lattices $\Gamma_i \in \Gamma_l$ and $\Gamma_r \in \Gamma_r$, we have
\[
|\tilde{G}_D^p(\Gamma_i, \Gamma_r)| = |O(\Gamma_i) \setminus \text{Isom}(A_{\Gamma_l}, A_{\Gamma_r})/O(\Gamma_r)|. \quad (4.17)
\]

Remark. 1) It is interesting to note that this equality connects the double coset space, which is group theoretical and arithmetical in nature, to the Gauss product, which is algebraic.
2) The average one formula in Theorem 0.1 follows immediately from the above equality and (4.13).
3) As pointed out earlier, if $\Gamma_i$ is not isogeneous to $\Gamma_r$, then the set $\text{Isom}(A_{\Gamma_i}, A_{\Gamma_r})$ is empty (see e.g. [Ni, Corollary 1.9.4]), which implies $|\tilde{G}_D^p(\Gamma_i, \Gamma_r)| = 0$. This can also be shown directly from the properties of Gauss product (see [Ca]).

§5. Classification of $c = 2$ RCFT — non-primitive case —

We now consider the general case of even, positive definite, not necessarily primitive lattices $\Gamma_l, \Gamma_r \in \mathcal{L}(D)$ of fixed determinant $-D$. As one might expect, most of the arguments in the preceding section apply to this case, once the Gauss product is extended to $\text{CL}(D)$. This extension is done in Appendix A.

(5-1) Refinement of key lemma. Let us look more closely at the proof of the key lemma (Lemma 4.6). The proof relies only on the existence of concordant forms satisfying properties (4.9). Using those properties, we found a $\mathbb{Z}$-basis for the left and right momentum lattices $\Pi_l = \Gamma(\tau_{Q_1}, \rho_{Q_2}) \cap \mathbb{R}^{2,0}$, $\Pi_r = \Gamma(\tau_{Q_1}, \rho_{Q_2}) \cap \mathbb{R}^{0,2}$ for $Q_1 \in C_1, Q_2 \in C_2$, and identified their isomorphism classes $[\Pi_l]$ and $[\Pi_r]$ with the Gauss products $q(C_1 * C_2^{-1})$ and $q(C_1 * C_2)$, respectively. Thus, in the case of non-primitive forms, if (4.9) holds for $C_1, C_2 \in \text{CL}(D)$ and the composition $C_1 * C_2$ is defined, then we expect that the same lemma holds for $C_1, C_2$, as we now show.

We recall some elementary notions introduced in Appendix A. We define the gcd $\lambda := \gcd(a, b, c)$ of a class $[Q(a, b, c)] \in \text{CL}(D)$, and the scalar multiple of a class in an obvious manner. Clearly a class is primitive iff its gcd is 1. In particular, if $\lambda$ is the gcd of $C \in \text{CL}(D)$, then $\frac{1}{\lambda}C$ is primitive. Two classes are coprime if their gcd’s are coprime. It is shown in Appendix A that there is a composition law $C_1 * C_2$ for coprime classes $C_1, C_2 \in \text{CL}(D)$, that generalizes the Gauss product for primitive classes.

Lemma 5.1. (Refinement of the key lemma) For coprime quadratic forms $C_1, C_2 \in \text{CL}(D)$, the following relations hold;
\[
\Gamma(\tau_{C_1}, \rho_{C_2}) \cap \mathbb{R}^{2,0} = q(C_1 \ast \sigma C_2), \quad \Gamma(\tau_{C_1}, \rho_{C_2}) \cap \mathbb{R}^{0,2} = q(C_1 \ast C_2)(-1), \quad (5.1)
\]
where $\ast$ is the composition law for coprime classes in $\text{CL}(D)$, and $q$ is the natural map $q : \text{CL}(D) \rightarrow \widetilde{\text{CL}}(D) \equiv \mathcal{L}(D)$.

Proof. As before, we can choose quadratic forms $Q(\alpha, \beta, \gamma) \in C_1$, $Q(\alpha', \beta', \gamma') \in C_2$ to evaluate the left hand sides of the equations (5.1). By Lemma A.3, we can arrange that
\[
(i) \quad \alpha \alpha' \neq 0, \quad (\alpha, \alpha') = 1 \quad (ii) \quad \beta = \beta' \quad (iii) \quad \frac{\beta^2 - D}{4\alpha \alpha'} \in \mathbb{Z}. \quad (5.2)
\]
Now we see that the proof of Lemma 4.6 is valid verbatim, by respectively replacing \(a, b, c; a', b', c'\) there by \(\alpha, \beta, \gamma; \alpha', \beta', \gamma'\) here, and \(C^{-1}\) there by \(\sigma C\) here (cf. Proposition A.7, 4)).

For the composition law on \(CL(D)\), the following property will be useful (see Appendix A, Remark after Definition A.6 for a proof):

**Lemma 5.2.** For coprime classes \(C_1, C_2 \in CL(D)\), if \(C_1 \ast C_2 \in CL(D)\) then \(C_1, C_2 \in CL(D)\).

**Proof of surjectivity in Theorem 4.11.** We now proceed to proving the surjectivity part of Theorem 4.11.

**Lemma 5.3.** Let \((\Gamma_1, \Gamma_2, \varphi)\) be a (not necessarily primitive) triple of determinant \(-D\) with \(\Gamma_1 \subset R^{2,0}\) and \(\Gamma_2(\varphi) \subset R^{0,2}\), and consider the corresponding over-lattice \(\Gamma^\varphi\). There exist coprime quadratic forms \(Q_1, Q_2\) of discriminant \(D\) such that \(\Gamma^\varphi\) is equivalent to \(\Gamma(\tau_{Q_1}, \rho_{Q_2})\).

**Proof.** By construction, as in section (3-4), \(\Gamma^\varphi \subset R^{2,2}\) is a Narain lattice. By Proposition 2.7 2), \(\Gamma^\varphi = h\Gamma(\tau, \rho)\) for some \(\tau, \rho \in H^\varphi\) and \(h \in O(2, R) \times O(2, R)\).

So we have

\[
\Gamma_l = \Gamma^\varphi \cap R^{2,0} = h\Pi_l, \quad \Gamma_r(-1) = \Gamma^\varphi \cap R^{0,2} = h\Pi_r, \quad (5.3)
\]

where \(\Pi_l, \Pi_r\) are the momentum lattices of the left and right chiral algebras of \(\Gamma(\tau, \rho)\). This shows that \(\text{rk } \Pi_l = \text{rk } \Pi_r = 2\). By Proposition 3.10, we have \(\tau, \rho \in Q(\sqrt{D}')\) for some \(D' < 0\). So explicitly, \(\tau\) and \(\rho\) may be written

\[
\tau = \frac{b + c\sqrt{D'}}{2a} = \frac{bc' + \sqrt{(cc')^2D'}}{2ac'}, \quad \rho = \frac{b' + c'\sqrt{D'}}{2a'} = \frac{b'c + \sqrt{(cc')^2D'}}{2a'c'},
\]

where \((a, b, c) = (a', b', c') = 1\) and \(a, a', c, c' > 0\). Now we express these values in terms of the following quadratic forms of discriminant \((Kcc')^2D'\), i.e. we have \(\tau = \tau_{Q_1}, \rho = \rho_{Q_2}\) with

\[
Q_1 = Q(Kac, Kbc', K\frac{(bc')^2 - (cc')^2D'}{4ac'}),
\]

\[
Q_2 = Q(Ka'c, Kb'c, K\frac{(b'c')^2 - (cc')^2D'}{4a'c'}),
\]

where a positive integer \(K\) is chosen so that both \(Q_1\) and \(Q_2\) are integral. Clearly, we can choose such an integer \(K\) so that \(Q_1\) and \(Q_2\) are coprime as well. Thus the composition \([Q_1] \ast [Q_2]\) makes sense. Now by Lemma 5.1, we find that \(\det \Pi_l = \det \Pi_r = (Kcc')^2D'\). It follows that \(D = (Kcc')^2D'\), since we have \(\det \Pi_l = \det \Gamma_l = D\) by (5.3). Since \(\Gamma^\varphi = h\Gamma(\tau, \rho) = h\Gamma(\tau_{Q_1}, \rho_{Q_2})\), this completes the proof. 

**Proof of the surjectivity in Theorem 4.11.** We want to show that the map \(f : N_D^p \to RCFT^p, [\Gamma(\tau, \rho)] \mapsto [(\Pi_l, \Pi_r(\varphi)) \circ \overline{\tau_{L^1}}]\) is surjective. Recall that every element in \(RCFT^p_D\) has the shape \([(\Gamma_l, \Gamma_r, \varphi)]\), where \(\Gamma_l \subset R^{2,0}\) and \(\Gamma_r(\varphi) \subset R^{0,2}\) are of determinant \(-D\) and also have primitive bilinear forms. By the preceding lemma,
we have \([\Gamma^\nu] = [\Gamma(\tau, \rho)]\), where \(\tau = \tau_{Q_1}, \rho = \rho_{Q_2}\), for some coprime forms \(Q_1, Q_2\) of discriminant \(D\). In particular, we have
\[
\Pi_l = \Gamma(\tau, \rho) \cap \mathbb{R}^{2,0} = h\Gamma^\nu \cap \mathbb{R}^{2,0} = h\Gamma_l
\]
for some \(h \in O(2; \mathbb{R}) \times O(2; \mathbb{R})\). This shows that \(\Pi_l \cong \Gamma_l\); likewise \(\Pi_r \cong \Gamma_r(-1)\).
By Lemma 5.1, we have
\[
[\Gamma_l] = [\Pi_l] = q([Q_1] * \sigma([Q_2])) , \quad [\Gamma_r(-1)] = [\Pi_r] = q([Q_1] * [Q_2]) .
\]
Since the bilinear forms of \(\Gamma_l, \Gamma_r\) are primitive, Lemma 5.2 implies \([Q_1], [Q_2]\) are also primitive, i.e. lie in \(\text{Cl}(D)\). This shows that \([\Gamma(\tau, \rho)] \in N(D)\). Hence we have \(f : [\Gamma(\tau, \rho)] \mapsto [(\Pi_l, \Pi_r(-1), \iota_R \circ \iota_L^{-1})]\). It remains to show that \([(\Pi_l, \Pi_r(-1), \iota_R \circ \iota_L^{-1})] = [(\Gamma_l, \Gamma_r, \varphi)]\). It is straightforward to check that the isomorphism \(h : \Gamma^\nu \rightarrow \Gamma(\tau, \rho)\) induces an equivalence of triples \((\Gamma_l, \Gamma_r, \varphi) \sim (\Pi_l, \Pi_r(-1), \iota_R \circ \iota_L^{-1})\).

**5-3 Classification.** Using the refined key lemma (Lemma 5.1), it is easy to generalize Theorems 4.10, 4.11, and Proposition 4.12 to the non-primitive case, with only minor modifications. The following definitions are, respectively, extensions of Definitions 4.7 and 4.8:

**Definition 5.4.** For \(\Gamma_l, \Gamma_r \in \mathcal{L}(D) \equiv \overline{\mathcal{C}L}(D)\), we set
\[
\mathcal{G}_D(\Gamma_l, \Gamma_r) := \{ (C_1, C_2) \in \text{PSym}^2\mathcal{C}L(D) \mid q(C_1 * \sigma C_2) = \Gamma_l, \quad q(C_1 * C_2) = \Gamma_r \},
\]
\[
\mathcal{G}_D := \bigsqcup_{\Gamma_l, \Gamma_r \in \mathcal{C}L(D)} \mathcal{G}_D(\Gamma_l, \Gamma_r) = \text{PSym}^2\mathcal{C}L(D) , \quad (5.4)
\]
where \(\text{PSym}^2\mathcal{C}L(D) := \{ (C, C') \in \text{Sym}^2\mathcal{C}L(D) \mid C, C' : \text{coprime} \}\). Also we define \(\mathcal{G}_D(\Gamma_l, \Gamma_r)\) and \(\mathcal{G}_D\) to be the image under the natural map \(\text{PSym}^2\mathcal{C}L(D) \rightarrow \text{PSym}^2\overline{\mathcal{C}L}(D)\) induced by \(q : \mathcal{C}L(D) \rightarrow \overline{\mathcal{C}L}(D)\).

As before, we have the following decomposition:
\[
\mathcal{G}_D = \bigsqcup_{(\Gamma_l, \Gamma_r) \in \text{PSym}^2\mathcal{C}L(D)} \mathcal{G}_D(\Gamma_l, \Gamma_r) = \text{PSym}^2\overline{\mathcal{C}L}(D) . \quad (5.5)
\]

**Definition 5.5.** We set:
\[
\mathcal{N}_D := \{ [\Gamma(\tau_{C_1}, \rho_{C_2})] \mid C_1, C_2 \in \mathcal{C}L(D) : \text{coprime} \},
\]
and \(\mathcal{N}_D\) be the image under \(q : \mathcal{C}L(D) \rightarrow \overline{\mathcal{C}L}(D)\).

Let \(\overline{\text{RCFT}}_D\) be the quotient of \(\text{RCFT}_D\) by the involution \(\pi_2\) (see Lemma 4.9). Then we have the decomposition
\[
\overline{\text{RCFT}}_D = \bigsqcup_{(\Gamma_l, \Gamma_r) \in \text{Sym}^2\overline{\mathcal{C}L}(D)} \overline{\text{RCFT}}_D(\Gamma_l, \Gamma_r) , \quad (5.6)
\]
as in (4.14).
Proposition 5.6. The inverse of the “over-lattice map” $F^{-1} : \mathcal{N} \to \cup D \text{RCFT}_D$ (see (3.15)) restricts to an injective map $f : \mathcal{N}_D \to \text{RCFT}_D$. Moreover $f$ is compatible with the involutions $\pi_1 : (\tau, \rho) \mapsto (\tau, -\rho)$ and $\pi_2 : [(\Gamma_l, \Gamma_r, \phi)] \mapsto [(\Gamma_r, \Gamma_l, \phi^{-1})]$ (see the diagram (4.15)).

Proof. If $F^{-1} : [\Gamma(C_1, \rho C_2)] \in \mathcal{N}_D \to [(\Pi_l, \Pi_r(-1), \phi)]$, then $[\Pi_l] = q(C_1 \ast \sigma C_2), [\Pi_r] = q(C_1 \ast C_2)$ by Lemma 5.1. From these relations, we conclude $\det \Pi_l = \det \Pi_r = D$ and hence $[(\Pi_l, \Pi_r(-1), \phi)] \in \text{RCFT}_D$. The compatibility with the involutions $\pi_1, \pi_2$ follows from the same argument as in Lemma 4.9.

Now Theorem 4.10 generalizes to

Theorem 5.7. Let $g$ be the map $\mathcal{G}_D \to \mathcal{N}_D$ defined by $(C_1, C_2) \mapsto [\Gamma(C_1, \rho C_2)]$, and $f$ be the injective map $\mathcal{N}_D \to \text{RCFT}_D$ given in Proposition 5.6. Then there exist corresponding induced maps $\tilde{g} : \tilde{\mathcal{G}}_D \to \tilde{\mathcal{N}}_D$ and $\tilde{f} : \tilde{\mathcal{N}}_D \to \text{RCFT}_D$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{G}_D & \xrightarrow{g} & \mathcal{N}_D \xrightarrow{f} \text{RCFT}_D \\
\downarrow q & & \downarrow \pi_1 \downarrow \pi_2 \\
\tilde{\mathcal{G}}_D & \xrightarrow{\tilde{g}} & \tilde{\mathcal{N}}_D \xrightarrow{\tilde{f}} \text{RCFT}_D
\end{array}
$$

Moreover $\tilde{g}$ is bijective and $\tilde{f}$ is injective.

Since the proof of Theorem 4.10 carries over with straightforward modifications, we do not repeat it here. Theorem 4.11 now generalizes to

Theorem 5.8.
1) The map $f : \mathcal{N}_D \to \text{RCFT}_D$ is bijective. Hence $\tilde{f}$ is also bijective.
2) The composition $\tilde{f} \circ \tilde{g}$ is a bijection with

$$
\tilde{f} \circ \tilde{g} \left( \tilde{\mathcal{G}}_D(\Gamma_l, \Gamma_r) \right) = \text{RCFT}_D(\Gamma_l, \Gamma_r).
$$

Proof. 1) By Proposition 5.6, $f$ is injective. The proof of surjectivity of $f$ is similar to that of Theorem 4.11 in section (5-2). Since $f$ is bijective, so is $\tilde{f}$ by Proposition 5.6.

2) Again, the proof of Theorem 4.11 2) carries over to the non-primitive case. □

Proposition 4.12 and its corollary, however, do not generalize immediately to non-primitive forms. This is because the relation $\varphi^2 = \text{id}$ for $\varphi \in \text{O}(A_G)$ no longer holds in general for a non-primitive lattice $\Gamma$.

Proposition 5.9.
1) If $\Gamma_l \neq \Gamma_r$, $(\Gamma_l, \Gamma_r \in \text{CL}(D))$, the bijective map $B(\Gamma_l, \Gamma_r)$ in (3.5) defines a natural one-to-one correspondence

$$
\text{RCFT}_D(\Gamma_l, \Gamma_r) \leftrightarrow \text{O}(\Gamma_l) \setminus \text{Isom}(A_{\Gamma_l}, A_{\Gamma_r})/\text{O}(\Gamma_r).
$$

2) If $\Gamma_l = \Gamma_r$, there is a one-to-one correspondence

$$
\text{RCFT}_D(\Gamma_l, \Gamma_l) \leftrightarrow (\text{O}(\Gamma) \setminus \text{Isom}(A_{\Gamma_l}, A_{\Gamma_l})/\text{O}(\Gamma))/\sim,
$$

where $\sim$ represents the identification of $[\varphi]$ with $[\varphi^{-1}]$. The correspondence is given by mapping the class $[(\Gamma, \Gamma, \varphi)] \sim [(\Gamma, \Gamma, \varphi^{-1})]$ to the class $[\varphi] \sim [\varphi^{-1}]$.

The proof of Proposition 4.12 now carries over with some slight modifications in part 2), although we omit the details here. As a corollary, we have
Corollary 5.10. For $\Gamma_l, \Gamma_r \in \overline{CL}(D)$, and for any choice of lattices $\Gamma_l \in \Gamma_l$ and $\Gamma_r \in \Gamma_r$, the following equality holds:

\[
|g_D(\Gamma_l, \Gamma_r)| = \begin{cases} 
|O(\Gamma_l) \setminus \text{Isom}(A_{\Gamma_l}, A_{\Gamma_r})/O(\Gamma_r)|, & \Gamma_l \neq \Gamma_r \\
\frac{1}{2}|O(\Gamma) \setminus \text{Isom}(A_{\Gamma_l}, A_{\Gamma_r})/O(\Gamma)| + \frac{1}{2}n_\Gamma, & \Gamma_l = \Gamma_r =: \Gamma
\end{cases}
\]  

(5.8)

where $n_\Gamma := \#\{[\varphi] \in O(\Gamma) \setminus \text{Isom}(A_{\Gamma_l}, A_{\Gamma_r})/O(\Gamma) | [\varphi^{-1}] = [\varphi]\}.$

Example. Here for the reader’s convenience, we present an example of a non-primitive lattice $\Gamma$ which has an isometry $[\varphi] \neq [\varphi^{-1}]$. The example has determinant $-D = 236$ and bilinear form

\[
\begin{pmatrix} 12 & 2 \\ 2 & 20 \end{pmatrix}
\]

with $\Gamma = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$.

The discriminant group $\Gamma^*/\Gamma$ is isomorphic to $\mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} =: \langle u_{d_1} \rangle \oplus \langle u_{d_2} \rangle$ with $d_1 = 2$, $d_2 = 118$. The generators may be chosen explicitly as $u_{d_1} = \frac{1}{2}v_2$, $u_{d_2} = \frac{1}{118}(v_1 - 6v_2)$. The discriminant form $q_\Gamma$ may be evaluated by using these expression for $u_{d_1}, u_{d_2}$. Now let us make the following orthogonal decomposition of the discriminant form (cf. Appendix B):

\[
(\Gamma^*/\Gamma, q_\Gamma) = (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \oplus \mathbb{Z}_{59}, q_1 \oplus q_2)
\]

where $q_1$ represents the discriminant form on the component $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. If we write the generators for each factor of $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{59}$, respectively, by $u, v, v^\perp$, they are given by $u = \frac{1}{2}v_2, v = \frac{1}{2}v_1, v^\perp = \frac{1}{59}(v_1 - 6v_2)$. With respect to these generators, the discriminant form $q_\Gamma$ may be represented by

\[
q_\Gamma = q_1 \oplus q_2 = \left( \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \oplus \frac{12}{59} \right).
\]

It is rather straightforward to determine all isometries in $O(A_\Gamma)$. The results are:

\[
\begin{align*}
\left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) & \oplus (\pm 1), & \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) & \oplus (\pm 1), & \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) & \oplus (\pm 1), \\
\left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right) & \oplus (\pm 1), & \left( \begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array} \right) & \oplus (\pm 1), & \left( \begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array} \right) & \oplus (\pm 1).
\end{align*}
\]  

(5.9)

As for the group $O(\Gamma)$, we see that it is trivial, i.e. $O(\Gamma) = \{\pm id\}$. Looking at the induced action on $\Gamma^*/\Gamma$, we see that $O(\Gamma)$ acts on $O(A_\Gamma)$ by $\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$ and $\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$. Hence each of the six pairs in (5.9) represents the class $O(\Gamma) \setminus O(A_\Gamma)/O(\Gamma)$. Now for the classes $[\varphi_1] = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$, $[\varphi_2] = \{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\}$, we see $[\varphi_1^{-1}] = [\varphi_2], [\varphi_1] = [\varphi_2^{-1}]$. This should be contrasted to the fact that we have $\varphi^2 = id$ (and hence $[\varphi^{-1}] = [\varphi]$) for all $\varphi \in O(A_\Gamma)$ if $\Gamma$ is a primitive lattice (see Appendix B). \hfill \Box

(5-4) Diagonal RCFTs. In their recent work [GV], Gukov and Vafa obtained a characterization of diagonal rational conformal field theories. We summarize their result as follows:
(Diagonal RCFTs) Let $\Pi$ be a lattice with quadratic form $\lambda \left( \begin{smallmatrix} 2a & b \\ b & 2c \end{smallmatrix} \right)$ such that $\gcd(a, b, c) = 1$ and $D = \lambda^2 (b^2 - 4ac)$. Then a necessary and sufficient condition for a class of Narain lattices $[\Gamma(\tau, \rho)] \in \mathcal{N}_D$ to satisfy $f([\Gamma(\tau, \rho)]) = [(\Pi, \Pi, \text{id})]$ is that

$$\tau = \frac{b + \sqrt{D/\lambda^2}}{2a}, \quad \rho = \lambda \tau,$$  \hspace{1cm} (5.10)

where the equalities are understood to be up to $\text{PSL}_2 \mathbb{Z}$ transformations.

This characterization can be seen as a special case of our general classification scheme as follows:

**Proposition 5.11. (Diagonal triples)** Under the bijection $f : \mathcal{N}_D \to \text{RCFT}_D$, the ‘diagonal’ triples $[(\Pi, \Pi, \text{id})]$ correspond to Narain lattices of the shape $[\Gamma(\tau_C, \rho_C)]$, where $[\Pi] = q(C)$, and $C_e \in \text{CL}(D)$ is the unit class (see Proposition A.7 in Appendix A). Moreover, we have $\rho_C = \lambda \tau_C$ if $C = [\lambda Q'(a, b, c)]$.

**Proof.** Put $Q := \lambda Q'(a, b, c)$, and let $(\Pi_l, \Pi_r, \iota_R \circ \iota_L^{-1})$ be the triple corresponding to the Narain lattice $\Gamma(q_C, \rho_Q)$, where $Q_e \in C_e$ is the reduced form. Then $f([\Gamma(\tau_C, \rho_C)]) = [(\Pi_l, \Pi_r, \iota_R \circ \iota_L^{-1})]$. By Lemma 5.1, we have $[\Pi_l] = [\Pi_r] = q(C * C_e) = q(C)$. In fact, by computing $\Pi_l, \Pi_r$ explicitly, as in the proof of Lemma 4.6, one finds that $\iota_R \circ \iota_L^{-1}$ is induced by an isomorphism $\iota : \Pi_l \to \Pi_r$ (cf. [GV]). It follows that

$$f([\Gamma(q_C, \rho_C)]) = [(\Pi, \Pi, \text{id})]$$ \hspace{1cm} (5.11)

where $\Pi = \Pi_l$. Conversely, given any rank two lattice $\Pi$, if $C \in \text{CL}(D)$ with $\Pi = q(C)$, then (5.11) holds.

Note that $\rho_C$ is the $\text{PSL}_2 \mathbb{Z}$ orbit of $\sqrt{D}/2$ if $D \equiv 0 \mod 4$, or of $(1 + \sqrt{D})/2$ if $D \equiv 1 \mod 4$. Also we have $\tau_{Q'(a, b, c)} = ((\lambda b)^2 + \sqrt{D})/(2\lambda a)$, and $D \equiv (\lambda b)^2 \mod 4$. It is easy to see that $\lambda \tau_{Q'(a, b, c)} = ((\lambda b)^2 + \sqrt{D})/2$ is in the $\text{PSL}_2 \mathbb{Z}$ orbit of $\sqrt{D}/2$ if $D \equiv 0 \mod 4$, or of $(1 + \sqrt{D})/2$ if $D \equiv 1 \mod 4$. It follows that $\rho_C = \lambda \tau_C$. $\square$

(5-5) **Summary and an example** $(D = -144)$.

We summarize the main results of this paper a bit differently, and illustrate them in an example as follows.

**Summary 5.12.** Any two coprime classes of positive definite quadratic forms $A, B \in \text{CL}(D)$ yield an RCFT (i.e. an element in $\text{RCFT}_D$) with the momentum lattices of the left and right chiral algebras $q(A \ast \sigma B), q(A \ast B) \in \text{CL}(D)$. Conversely, all RCFTs arise this way. Moreover, the RCFT arising from $A, B \in \text{CL}(D)$ in this way corresponds to the class of Narain lattices $[\Gamma(\tau_A, \rho_B)]$. Two pairs $(A', B')$ and $(A, B)$ yield the same RCFT iff $[\Gamma(\tau_A, \rho_B)] = [\Gamma(\tau_{A'}, \rho_{B'})]$.

We will describe the set $\hat{\text{RCFT}}_D$ for $D = -144$, in terms of the Gauss product on quadratic forms. By Proposition 1.2, we find 8 classes $C_1 \in \text{CL}(D)$:

- $C_1 = [Q(1, 0, 36)]$, $C_2 = [Q(4, 0, 9)]$, $C_3 = [Q(5, 4, 8)]$, $C_4 = [Q(5, -4, 8)]$,
- $C_5 = [Q(2, 0, 18)]$, $C_6 = [Q(3, 0, 12)]$, $C_7 = [Q(4, 4, 10)]$, $C_8 = [Q(6, 0, 6)]$,

where $C_4 = \sigma C_3$ and $C_5$ to $C_8$ are not primitive. Their compositions are given in the following table:
where \([\Gamma(\tau_1, \rho)]\) \(\sim\) \([\Gamma(\tau, -\bar{\rho})]\). The diagonal triples correspond to the Narain lattices \([\Gamma(\tau_{C_k}, \rho_{C_k})]\) for \(k = 1, \ldots, 8\). A direct computation of the double coset spaces is not difficult, and it is a good exercise to verify the equality obtained in Corollary 5.10 in this case.
Appendix A. Gauss product on $\text{CL}(D)$

In this appendix, we extend the Gauss product on $\text{CL}(D)$ to a composition law $C_1 * C_2$ on $\text{CL}(D)$, which includes non-primitive forms, but defined only when the classes $C_1, C_2 \in \text{CL}(D)$ are coprime. The composition is commutative and associative whenever defined. As seen in Lemmas 4.6, and 5.1, this extended composition law does arise naturally in our description of RCFTs.

The observation here is that the classical composition law given in terms of primitive concordant forms is valid verbatim for non-primitive forms. We say that two forms $Q(a_1, b_1, c_1)$ and $Q(a_2, b_2, c_2)$ are **concordant** if they satisfy the following conditions;

1. $a_1 a_2 \neq 0$ ,  
2. $b_1 = b_2 (=: b)$ ,  
3. $\frac{b^2 - D}{4a_1 a_2} \in \mathbb{Z}$ . 

(A.1)

Note that in case of $D < 0$, the first condition (1) is void since $a_1 \neq 0, a_2 \neq 0$ for $D = b_1^2 - 4a_1 c_1 = b_2^2 - 4a_2 c_2 < 0$. We say that the forms are coprime if $\text{gcd}(a_1, b_1, c_1, a_2, b_2, c_2) = 1$. It is clear that if $Q(a_1, b_1, c_1)$ is coprime with one quadratic form in a class $C \in \text{CL}(D)$, then it is coprime with all quadratic forms in $C$. Thus it makes sense to speak of coprime classes. We write $Q(a,b,c)$ to denote $Q(a,b,\frac{b^2 - D}{4a})$. We call the number $\text{gcd}(a,b,c)$ the gcd of the form $Q(a,b,c)$. Since equivalent forms have the same gcd, we can speak of the gcd of a class.

The following construction parallels to that given in [Ca] in the primitive case, and is valid for both $D < 0$ and $D > 0$. For the modern definition using fractional ideals, see for example [Za].

**Lemma A.1.** Consider a primitive class $C \in \text{CL}(D)$. For an arbitrary nonzero integer $M$, there is a quadratic form $Q(a,b,c) \in C$ such that $(a,M) = 1$.

**Proof.** Take a quadratic form $Q(a',b',c') : f(x,y) = a'x^2 + b'xy + c'y^2 \in C$. Then $f$ represents an integer which is coprime to $M$. To see this, let us first consider the case where $M$ has only one prime factor, say $M = p^r$. In this case the claimed integer $f(x,y)$ may be found by considering the following four cases:

1. When $p \nmid a'$. Take $x, y$ so that $p \nmid x, p|y$, then we have $(f(x,y),p) = 1$.
2. When $p|a', p \nmid c'$. Take $x, y$ so that $p|x, p \nmid y$, then we have $(f(x,y),p) = 1$.
3. When $p|a', p|c'$. In this case $p \nmid b'$ by the condition $\text{gcd}(a',b',c') = 1$ ($Q(a',b',c')$ is primitive). If we take $p \nmid x, p \nmid y$, then we have $(f(x,y),p) = 1$.

In the general case, let $M = p_0^{e_0} p_1^{e_1} \cdots p_k^{e_k}$ be the prime factorization of $M$. Put

$$S_1 = \{p_i | p_i \nmid a' \} , \quad S_2 = \{p_i | p_i|a', p_i \nmid c' \} , \quad S_3 = \{p_i | p_i|a', p_i|c' \} ,$$

and define

$$x = (\prod_{p \in S_2} p)^{\bar{x}} , \quad y = (\prod_{q \in S_1} q)^{\bar{y}} ,$$

with some integers $\bar{x}$ and $\bar{y}$ satisfying

$$p \nmid \bar{x} \quad (p \in S_1 \cup S_3) \quad \text{and} \quad p \nmid \bar{y} \quad (p \in S_2 \cup S_3).$$

Then it is clear that for the all prime factors $p_0, \cdots, p_k$ of $M$, if $p_r \in S_r$ ($r = 1, 2, 3$), then $x$ and $y$ have the properties $(r)$, and thus we have $(f(x,y),M) = 1$. 


In this way, we find \((x, y) = (n_1, n_2)\) such that \(f(n_1, n_2)\) is coprime to \(M\). We may assume that \((x, y) = (n_1, n_2)\) is primitive in \(\mathbb{Z}^2\), since otherwise we may set \((x, y) = (\frac{m}{n}, \frac{m}{n})\) with \(m = \gcd(n_1, n_2)\), preserving the property \(f(\frac{m}{n}, \frac{m}{n}), M) = 1\). When \((n_1, n_2)\) is primitive in \(\mathbb{Z}^2\), there is an \(SL_2\mathbb{Z}\) transformation \(g: (n_1, n_2) \mapsto (1,0)\). Then the quadratic form \(Q(a, b, c) = g \cdot Q(a', b', c')\) has the desired property. 

\[\square\]

**Lemma A.2.** Assume two primitive quadratic forms are equivalent: \(Q(a_1, b_1, c_1) \sim Q(a_2, b_2, c_2)\) and \(b_2 = b = b'\). If there is an integer \(l \in \mathbb{Z}\) such that \(l|c_1, l|c_2\) and \((a_1, a_2, l) = 1\), then 

\[Q(la_1, b, c_2/l) \sim Q(la_2, b, c_2/l)\cdot\]

**Proof.** Suppose \(\begin{pmatrix} r & s \\ t & u \end{pmatrix} \in SL_2\mathbb{Z}\) transforms \(Q(a_1, b_1, c_1)\) to \(Q(a_2, b_2, c_2)\), i.e.

\[
\begin{pmatrix} r & s \\ t & u \end{pmatrix} \begin{pmatrix} 2a_1 & b \\ b & 2c_1 \end{pmatrix} \begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} 2a_2 & b \\ b & 2c_2 \end{pmatrix}.
\]

Eliminating \(r, u\) from the resulting equations, we obtain \(a_2s + c_1t = a_1s + c_2t = 0\).

From these relations and our assumptions, we see that \(l|s\). Then the matrix \(\begin{pmatrix} r & s/l \\ t & u \end{pmatrix}\)

transforms \(Q(la_1, b, c_2/l)\) to \(Q(la_2, b, c_2/l)\). 

\[\square\]

**Lemma A.3.** Let \(C_1, C_2\) be coprime classes in \(CL(D)\), and \(\lambda_1, \lambda_2\) be their respective gcd’s. For an arbitrary nonzero integer \(M\), there exist quadratic forms \(Q_1 \in C_1, Q_2 \in C_2\) such that 

\[Q_1 = Q(\lambda_1a_1, b_1, \gamma_1), \quad Q_2 = Q(\lambda_2a_2, b_2, \gamma_2),\]

with \((a_1, a_2) = (a_1, M) = (a_2, M) = 1\) and also \(\beta_1 = \beta_2 =: \beta\). Furthermore \(Q_1\) and \(Q_2\) are concordant.

**Proof.** Applying Lemma A.1 to the primitive class \(\frac{1}{\lambda_1}C_1\), we obtain a quadratic form 

\[Q(a_1, b_1, c_1) \in \frac{1}{\lambda_1}C_1\]

with \((a_1, \lambda_2M) = 1\). Likewise, we have 

\[Q(a_2, b_2, c_2) \in \frac{1}{\lambda_2}C_2\]

with \((a_2, \lambda_1M) = 1\). Let 

\[Q_1 := Q(\lambda_1a_1, b_1, \gamma_1), \quad Q_2 := Q(\lambda_2a_2, b_2, \gamma_2),\]

with \(\beta_i := \lambda_i b_i, \gamma_i := \lambda_i c_i\). Since \(\lambda_1\) and \(\lambda_2\) are coprime, we have \((\lambda_1a_1, \lambda_2a_2) = 1\), hence there exist integers \(A_1, A_2\) satisfying \(\lambda_1a_1A_1 + \lambda_2a_2A_2 = 1\). Then we have the equivalence under \(SL_2\mathbb{Z}\)

\[Q_1 \sim Q(\lambda_1a_1, b_1 - 2\lambda_1a_1\beta_1 - \beta_2 - \beta - A_1, *), \quad Q_2 \sim Q(\lambda_2a_2, b_2 - 2\lambda_2a_2\beta_2 - \beta_1 - A_2, *)\cdot\]

Note that \(\beta_1^2 - \beta_2^2 \equiv 0 \mod{4}\), so that \((\beta_1 - \beta_2)/2\) is an integer. We see that the above two quadratic forms are of the form: 

\[Q_1 \sim Q(\lambda_1a_1, B, C_1), \quad Q_2 \sim Q(\lambda_2a_2, B, C_2),\]

with \(B := \lambda_2a_2A_2\beta_1 + \lambda_1a_1A_1\beta_2\), and \(C_1 = (B^2 - D)/(4\lambda_1a_1)\) and \(C_2 = (B^2 - D)/(4\lambda_2a_2)\). Since \((\lambda_1a_1, \lambda_2a_2) = 1\), the quadratic forms \(Q(\lambda_1a_1, B, C_1)\) and \(Q(\lambda_2a_2, B, C_2)\) are concordant forms which have the asserted properties. 

\[\square\]
Definition A.4. For any concordant forms $Q_1 = Q(\alpha_1, \beta, \gamma_1)$, $Q_2 = Q(\alpha_2, \beta, \gamma_2)$ of a same discriminant $D$, we set the composition of $Q_1, Q_2$ by

$$Q_1 \ast Q_2 := Q(\alpha_1\alpha_2, \beta, \frac{\beta^2 - D}{4\alpha_1\alpha_2}) .$$  \hfill (A.3)

Proposition A.5. Assume $C_1, C_2 \in CL(D)$ are coprime. The class $C_3 \in CL(D)$ of the composition $Q_1 \ast Q_2$ is independent of the choices of concordant forms $Q_1 \in C_1$, $Q_2 \in C_2$.

Proof. Suppose that $Q'_1 = Q(\alpha'_1, \beta', \gamma'_1)$, $Q'_2 = Q(\alpha'_2, \beta', \gamma'_2)$ are concordant, that $Q''_1 = Q(\alpha''_1, \beta'', \gamma''_1)$, $Q''_2 = Q(\alpha''_2, \beta'', \gamma''_2)$ are also concordant, and that $Q'_1 \sim Q''_1$, $Q'_2 \sim Q''_2$. We will show that $Q'_1 \ast Q'_2 \sim Q''_1 \ast Q''_2$.

Let $\lambda_i$ be the gcd of $Q'_i, Q''_i$. Put $M = a'_1a''_1a'_2a''_2$; $(\alpha_i := \lambda_ia'_i, \alpha''_i := \lambda_i\alpha''_i)$. Then by Lemma A.3, there exists another pair of concordant forms,

$$Q_1 = Q(\lambda_1\alpha_1, \beta, \gamma_1) \in C_1, \quad Q_2 = Q(\lambda_2\alpha_2, \beta, \gamma_2) \in C_2,$$

satisfying $(a_1, a_2) = (a_1, M) = (a_2, M) = 1$. Since $(\lambda_1, \lambda_2) = 1$, $\lambda_1|\beta, \lambda_2|\beta$, we have $\beta = \lambda_1\lambda_2b$ for some integer $b$; likewise $\beta' = \lambda_1\lambda_2b'$, and $D = \lambda_1^2\lambda_2^2D_0$. Therefore from (A.3), we have

$$Q_1 \ast Q_2 = \lambda_1\lambda_2Q(a_1a_2, b, \frac{b^2 - D_0}{4a_1a_2}) , \quad Q'_1 \ast Q'_2 = \lambda_1\lambda_2Q(a'_1a'_2, b', \frac{b'^2 - D_0}{4a'_1a'_2}) .$$

Since $(a_1a_2, a'_1a'_2) = 1$, there exist integers such that $a'_1a'_2m'_1a_2 + a_1a_2m_1 = 1$. From this, we get

$$Q_1 \ast Q_2 = \lambda_1\lambda_2Q(a_1a_2, b, *) \sim \lambda_1\lambda_2Q(a_1a_2, B, *) , \quad Q'_1 \ast Q'_2 = \lambda_1\lambda_2Q(a'_1a'_2, b', *) \sim \lambda_1\lambda_2Q(a'_1a'_2, B, *) ,$$  \hfill (A.4)

where $B := b - 2a_1a_2\frac{b+b'}{4}m_1 = b' - 2a'_1a'_2\frac{b'+b}{4}m'_1$.

Now from the shape of $B$, we find

$$Q(a_1, \lambda_2B, c_1) \sim Q(a_1, \lambda_2b, *) \sim Q(a'_1, \lambda_2b', *) \sim Q(a'_1, \lambda_2B, c'_1) ,$$

where $c_1 = \lambda_2^2(B - D_0)/4a_1$, $c'_1 = \gamma_2^2(B^2 - D_0)/4a'_1$ with $D = \lambda_1^2\lambda_2^2D_0$. Then we see that $\lambda_2a_2|c_1$ and $\lambda_2a'_2|c'_1$ since we have $(B^2 - D_0)/4a_1a_2, (B^2 - D_0)/4a'_1a'_2 \in \mathbb{Z}$ from the equivalences in (A.4) and $(a'_1a'_2, a_1a_2) = 1$. By Lemma A.2, we have

$$\lambda_2Q(a'_1a'_2, B, c_1/\lambda_2a_2) \sim \lambda_2Q(a'_1a'_2, B, c'_1/\lambda_2a'_2) .$$

In a similar way, starting from $Q(a_2, \lambda_1B, c_2) \sim Q(a_2, \lambda_1b, *) \sim Q(a'_2, \lambda_1b', *) \sim Q(a'_2, \lambda_1B, c'_2)$ with $c_2 = \lambda_1^2(B^2 - D_0)/4a_2$ and $c'_2 = \gamma_1^2(B^2 - D_0)/2a'_2$, we obtain

$$\lambda_1Q(a_1a_2, B, c_2/\lambda_1a_2) \sim \lambda_1Q(a_1a_2, B, c'_2/\lambda_1a'_2) .$$

Combining these two, and using $c_1/\lambda_2a_2 = c'_2/\lambda_1a'_1$, we obtain

$$\lambda_1\lambda_2Q(a_1a_2, B, c_2/\lambda_1a_1) \sim \lambda_1\lambda_2Q(a'_1a'_2, B, c'_1/\lambda_2a'_2) .$$  \hfill (A.5)

Now from (A.4) and (A.5), we see that $Q_1 \ast Q_2 \sim Q'_1 \ast Q'_2$. Likewise, we get $Q_1 \ast Q_2 \sim Q''_1 \ast Q''_2$. It follows that $Q'_1 \ast Q'_2 \sim Q''_1 \ast Q''_2$. \hfill \Box

By the proposition above, the composition of coprime classes in $CL(D)$ now makes sense:
**Definition A.6.** For coprime classes \( C_1, C_2 \in \text{CL}(D) \), we define
\[
C_1 * C_2 = [Q_1 * Q_2] \in \text{CL}(D)
\]
for any choice of concordant forms \( Q_1 \in C_1, Q_2 \in C_2 \).

**Remark** 1) If both classes \( C_1, C_2 \) are primitive, i.e. in \( \text{CL}(D) \), then the composition is nothing but the Gauss product.
2) If \( C_1, C_2 \in \text{CL}(D) \) are coprime classes and \( \lambda_1, \lambda_2 \) their respective gcd’s, then we have
\[
\frac{1}{\lambda_1 \lambda_2} \times C_1 * C_2 \in \text{CL}(D/(\lambda_1 \lambda_2)^2),
\]
i.e., it is a primitive class. This follows from definition (A.3) and the fact that \( \lambda_1 \lambda_2 | \beta \) and \( (\lambda_1 \lambda_2)^2 | (\beta^2 - D) \) hold because \( (\lambda_1, \lambda_2) = 1 \).
3) It follows from 2) that if either \( C_1 \) or \( C_2 \) is not primitive, then the composition \( C_1 * C_2 \) is not primitive. This is an important fact, used in our proof of surjectivity in Theorem 4.11.

The following properties of the composition law generalize the classical properties (Theorem 1.1) of Gauss’ group law on \( (\text{CL}(D), *) \).

**Proposition A.7.** For pairwise coprime classes \( C_1, C_2, C_3 \in \text{CL}(D) \), the following properties hold:
1) \( C_1 * C_2 = C_2 * C_1 \),
2) \( (C_1 * C_2) * C_3 = C_1 * (C_2 * C_3) \)
3) Let \( C_e \) be the class containing \( Q(1, 0, -\frac{D}{4}) \) for \( D \equiv 0 \mod 4 \) or \( Q(1, 1, \frac{1-D}{4}) \) for \( D \equiv 1 \mod 4 \), then \( C_e * C = C \) for any \( C \in \text{CL}(D) \).
4) If \( C_1 \in \text{CL}(D) \), i.e. primitive class, then \( \sigma C_1 * C_1 = C_e \), where \( \sigma \) is the involution defined by \( Q(a, b, c) \mapsto Q(a, -b, c) \) as in (1.6).

**Proof.**
1) This follows from Definition A.6 based on Proposition A.5 and (A.3).
2) Let \( \lambda_i \) be the gcd of \( C_i \). By Lemma A.3, we have concordant forms \( Q_1 = Q(\lambda_1 a_1, \beta_1, \gamma_1), Q_2 = Q(\lambda_2 a_2, \beta_2, \gamma_2) \) with \( (a_1, a_2) = 1 \) and \( \beta_1 = \beta_2 = \beta \). Then by Proposition A.5, we have \( Q_1 * Q_2 = Q(\lambda_1 \lambda_2 a_1 a_2, \beta, \beta) \in C_1 * C_2 \). Since \( C_1 * C_2 \) is a multiple by \( \lambda_1 \lambda_2 \) of a primitive class (see Remark 2) and \( (\lambda_1 \lambda_2, \lambda_3) = 1 \) by assumption, we may apply Lemma A.3 to coprime classes \( C_1 * C_2 \) and \( C_3 \) with \( M = a_1 a_2 \). By this we see that there exist a quadratic form \( Q(\lambda_3 a_3, \beta_3, \gamma_3) \in C_3 \) with \( (a_1 a_2, a_3) = 1 \). Now we note the following equivalences;
\[
Q(\lambda_1 \lambda_2 a_1 a_2, \beta, \gamma) \sim Q(\lambda_1 a_2 a_1 a_2, b, \gamma) \ , \ Q(\lambda_3 a_3, \beta_3, \gamma) \sim Q(\lambda_3 a_3, b, \gamma),
\]
where \( b = \beta - 2 \lambda_1 \lambda_2 a_1 a_2 \beta - \beta_3 m_{12} = \beta_3 - 2 \lambda_3 a_3 \beta - \beta_3 m_3 \) with integers \( m_{12}, m_3 \) satisfying \( a_1 a_2 m_{12} + a_3 m_3 = 1 \). From the shape of \( b \), we find
\[
Q_1 \sim Q'_1 = Q(\lambda_1 a_1, b, \gamma), \ Q_2 \sim Q'_2 = Q(\lambda_2 a_2, b, \gamma), \ Q_3 \sim Q'_3 = Q(\lambda_3 a_3, b, \gamma),
\]
with conditions \( (a_1, a_2) = (a_1, a_3) = (a_2, a_3) = 1 \). It follows that
\[
(Q'_1 * Q'_2) * Q'_3 = Q(\lambda_1 \lambda_2 \lambda_3 a_1 a_2 a_3, b, \gamma) = Q'_1 * (Q'_2 * Q'_3).
\]

Here we have used the fact that each pair of quadratic forms being composed are concordant. This yields the associativity on classes by Proposition A.5.
3) Any class $C \in CL(D)$ is coprime to the class $C_e$, and so $C \ast C_e$ is defined. For any quadratic form $Q(\lambda a, \beta, \gamma) \in C$, we can find a quadratic form $Q(1, \beta, *) \in C_e$ (by using the equivalence $Q(1, \beta', *) \sim Q(1, \beta' - 2n, *)$). Then the composition becomes $Q(\lambda a, \beta, \gamma) \ast Q(1, \beta, *) = Q(\lambda a, \beta, \gamma)$, which implies $C \ast C_e = C$.

4) Let $Q(a, b, c)$ with $ac \neq 0$ be a quadratic form in a primitive class $C$. (It is not hard to see that we can choose $Q(a, b, c)$ with $ac \neq 0$. In fact, this is obvious if $D < 0$.) Then, since $\sigma Q(a, b, c) = Q(a, -b, c) \sim Q(c, b, a)$, we have $[\sigma Q(a, b, c) \ast Q(a, b, c)] = [Q(c, b, a) \ast Q(a, b, c)] = [Q(ac, b, 1)]$. Depending on the congruence of $D$ mod 4, $Q(ac, b, 1) \sim Q(1, -b, ac)$ is equivalent either to $Q(1, 0, -D/4)$ or to $Q(1, 1, (1 - D)/4)$. This implies $\sigma C \ast C = C_e$. \(\square\)
Appendix B. $O(A_{\Gamma})$ for a primitive lattice $\Gamma$

For a primitive lattice $\Gamma \in \mathcal{L}^p(D)$, the group $O(A_{\Gamma})$, by definition, consists of all isometries of the discriminant $A_{\Gamma}$. Here we prove the following property:

**Proposition B.1.** If $\Gamma \in \mathcal{L}^p(D)$, then any element $\varphi \in O(A_{\Gamma})$ satisfies $\varphi^2 = id$.

**Proof.** First note that, for a primitive lattice $\Gamma$, the discriminant group $\Gamma^*/\Gamma$ is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_{2d}$ with $2d = |D|/2$. In the first case, the claim holds since we have the decomposition

$$O(A_{\mathbb{Z}_2}) = O(A_{\mathbb{Z}_2}) \times O(A_{\mathbb{Z}_{2d}}),$$

and $O(A_{\mathbb{Z}_2}) = \{ \pm 1 \}$ for each prime factor $p_i$ of $D$. In the second case, we have a group decomposition

$$\Gamma^* / \Gamma \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{2d} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{2^l} \oplus \mathbb{Z}_{2s+1},$$

by writing $d = 2^{l-1}(2s + 1)$ ($l \geq 1$). More precisely, if we denote $\mathbb{Z}_2 = \langle u_2 \rangle$ and $\mathbb{Z}_{2d} = \langle u_{2d} \rangle$, then $u := u_2$, $v := (2s + 1)u_{2d}$, $v' := 2^lu_{2d}$ generate each group, $\mathbb{Z}_2$, $\mathbb{Z}_{2^l}$, $\mathbb{Z}_{2s+1}$, respectively. Since $2^l(x, y) \equiv (2s + 1)(x, y) \equiv 0 \mod \mathbb{Z}$ holds for $x \in \mathbb{Z}_2 \oplus \mathbb{Z}_{2^l}$, $y \in \mathbb{Z}_{2s+1}$ and $(2^l(2s + 1)) = 1$, we see $(x, y) \equiv 0 \mod \mathbb{Z}$ and have the following orthogonal decomposition:

$$(\Gamma^*/\Gamma, qr) \cong (\mathbb{Z}_2 \oplus \mathbb{Z}_{2^l} \oplus \mathbb{Z}_{2s+1}, q_{\mathbb{Z}_2} \oplus q_{\mathbb{Z}_{2^l}} \oplus q_{\mathbb{Z}_{2s+1}}),$$

hence $O(A_{\Gamma}) \cong O(A_{\mathbb{Z}_2}) \times O(A_{\mathbb{Z}_{2s+1}})$. Thus we only need to consider $O(A_{\mathbb{Z}_2 \oplus \mathbb{Z}_{2^l}})$.

Let $u$, $v$ be the respective generators of $\mathbb{Z}_2$ and $\mathbb{Z}_{2^l}$ as above. Then an element $\varphi \in O(A_{\mathbb{Z}_2 \oplus \mathbb{Z}_{2^l}})$ can be represented by

$$\varphi : (u, v) \mapsto (u', v') = (u, v) \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right).$$

Since $2u' \equiv 0, 2^lv' \equiv 0$, it follows that $\alpha = 0, 1$, $\beta = 0, 1$, $\gamma = 0, 2^{l-1}$. Also $\delta$ must be zero or an odd integer. First, note that if a matrix $\left( \begin{array}{cc} 1 & 0 \\ 0 & \delta \end{array} \right)$ represents an isometry $\varphi \in O(A_{\mathbb{Z}_2 \oplus \mathbb{Z}_{2^l}})$, then $\delta$ must be $\pm 1$. In the following, we consider two cases, $l \geq 2$ and $l = 1$, separately.

- **$l \geq 2$:** In this case, there are five possibilities for $\varphi$:

$$\left( \begin{array}{cc} 1 & 0 \\ 0 & \pm 1 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 2^{l-1} & k \end{array} \right), \left( \begin{array}{cc} 1 & 1 \\ 0 & k \end{array} \right), \left( \begin{array}{cc} 1 & 1 \\ 2^{l-1} & k \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ 2^{l-1} & k \end{array} \right),$$

where $k$ is odd. First of all, we may exclude the last (the fifth) isometry from our consideration. To see this, write the relations between the generators: $u' = 2^{l-1}v$, $v' = u + kv$. Then we have $2^{l-1}v' = k2^{l-1}v = 2^{l-1}v$ and also $u' = ku' = k2^{l-1}v = 2^{l-1}v$. Hence we have $u' = 2^{l-1}v'$, which cannot be for the generators. Thus the fifth isometry will never appear for $l \geq 2$. 

For the first four possible isometries, we verify that each one is an involution. This is easy for the first three cases, and proceeds as follows for the fourth case: We have
\[
\left( \frac{1}{2^{l-1}}, 1 \right) \left( \frac{k - 1}{k} \right) = \left( \frac{1 + 2^{l-1}}{2^{l-1}(k + 1)}, \frac{k + 1}{2^{l-1} + k^2} \right) = \left( \frac{1}{0}, \frac{2^{l-1} + k^2}{0} \right).
\]
Since an isometry of the form \( \left( \frac{1}{0}, \frac{m}{n} \right) \) exists only for \( m = \pm 1 \), we obtain \( 2^{l-1} + k^2 = \pm 1 \). But \( 2^{l-1} + k^2 \neq -1 \mod 2^l \) for \( k \) is odd and \( l \geq 3 \). This shows that \( \varphi^2 = id \) for \( l \geq 3 \). When \( l = 2 \), the isometry takes the following form: \( \varphi = \left( \frac{1}{2}, \frac{1}{2} \right) \). Now write the discriminant form \( q_2 = \left( \frac{2}{0}, \frac{2}{0} \right) \) with respect to the generators \( u, v \), and assume that \( q_2 \) allows the isometry \( \varphi \). Then we see that the discriminant form \( q_2 \) is restricted to be
\[
q_2 = \left( \frac{1}{2}, \frac{1}{2} \right), \quad \left( \frac{1}{4}, \frac{1}{4} \right) = 0, \frac{1}{2}, 1, \frac{3}{2}.
\]
This shows that we have for the generators: \( (u, u) \equiv 1 \mod 2Z \), \( (u, v) \equiv \frac{1}{2} \mod Z \), \( (v, v) \equiv \frac{1}{2} \mod 2Z \). Note that \( l_1 := 2u, l_2 := 4v \) are elements in \( \Gamma \), and in fact, give a basis of \( \Gamma \). (Precisely, \( l_1, l_2 \) are lattice vectors chosen, respectively, from \( 2(u + \Gamma), 4(v + \Gamma).\) Then the following symmetric matrix represents the bilinear form of the lattice \( \Gamma \):
\[
\begin{pmatrix}
(l_1, l_1) & (l_1, l_2) \\
(l_2, l_1) & (l_2, l_2)
\end{pmatrix} = \begin{pmatrix}
a_0 & 4b_0 \\
4b_0 & 8c_0
\end{pmatrix}
\]
where \( a_0, b_0, c_0 \in Z \). This shows that \( \Gamma \) is not primitive, since the corresponding quadratic form becomes \( Q(2a_0, 4b_0, 4c_0) \). Therefore \( q_2 \) does not appear from a primitive lattice \( \Gamma \), and hence \( \varphi \) may be excluded.

- \( l = 1 \): In this case, there are six possibilities for \( \varphi \):
  \[
  \begin{pmatrix}
  \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}
  \end{pmatrix} .
\]
We verify \( \varphi^2 = id \) for the first four. For the rest, \( \varphi^2 = id \) does not hold, but we may exclude these by a similar argument above done for \( l = 2 \): As above assume the following possible forms for the discriminant form \( q_{2 \times 2} \):
\[
q = \begin{pmatrix}
a & 0 \\
0 & b
\end{pmatrix} \quad (a, b = 1, 2, 3), \quad q' = \begin{pmatrix}
a & \frac{1}{2} \\
\frac{1}{2} & b
\end{pmatrix} \quad (a, b = 0, 1, 2, 3),
\]
with respect to the generators \( u \) and \( v \). Then it is straightforward to see that a non-involutive isometry (i.e. the fifth or sixth of (B.3)) is possible only for the discriminant form \( q_1 := \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \). We may claim that this discriminant form \( q_1 \) never appears from a primitive lattice \( \Gamma \) in exactly the same way as above for \( l = 2 \). This completes the proof.

Note that for non-primitive lattices, the index \( d_1 \) in \( \Gamma^*/\Gamma \cong Z_{d_1} \oplus Z_{d_2} (d_1|d_2) \) can be greater than two in general. Even if we have \( d_1 = 2 \), the above proof shows that \( \varphi \) for \( l = 2 \) case or the last two cases of (B.3) for \( l = 1 \) can be possible for non-primitive lattices. The latter case has appeared in the example presented at the end of section (5-3).
Appendix C. The coset space $O(d, \mathbb{R}) \times O(d, \mathbb{R}) \setminus O(d, d; \mathbb{R})$ and $O(d, d; \mathbb{Z})$

Here we will prove Proposition 2.5, and claim 1) of Proposition 2.6.

Proof of Proposition 2.5. Let $W' \in O'(d, d; \mathbb{R})$, and write the corresponding “half conjugated matrix” $\tilde{W}$ as

$$\tilde{W} := \frac{1}{\sqrt{2}} \begin{pmatrix} 1_d & 1_d \\ 1_d & -1_d \end{pmatrix} W' = \begin{pmatrix} X & X' \\ Y & Y' \end{pmatrix}.$$ 

Then the condition that $W' \in O'(d, d; \mathbb{R})$ becomes

$$^t X X - ^t Y Y = 0_d, \quad ^t X' X - ^t Y' Y = 1_d, \quad ^t X' X - ^t Y' Y' = 0_d,$$  \hspace{1cm} (C.1) where $0_d$ is the $d \times d$ zero matrix. Here we conclude that $\det X \neq 0$. To see this, assume that the real symmetric matrix $^t X X = ^t Y Y$ has a (real) eigenvector $v(\neq 0)$ with zero eigenvalue. Then we have $^t X v v = ^t Y v v = 0$, from which we have $|X v|^2 = |Y v|^2 = 0$, i.e. $X v = Y v = 0$. However, by the second equation of (C.1), we have $v = (^t X X - ^t Y Y) v = 0$, which is a contradiction. Therefore we conclude that all eigenvalues are not zero, which means $\det X \neq 0$, $\det Y \neq 0$.

Now consider the $O(d, \mathbb{R}) \times O(d, \mathbb{R})$-orbit of the matrix $W'$:

$$\frac{1}{2} \begin{pmatrix} 1_d & 1_d \\ 1_d & -1_d \end{pmatrix} \begin{pmatrix} g_L & 0 \\ 0 & g_R \end{pmatrix} \begin{pmatrix} 1_d & 1_d \\ 1_d & -1_d \end{pmatrix} W' = \frac{1}{\sqrt{2}} \begin{pmatrix} g_L X + g_R Y & g_L X' + g_R Y' \\ g_L X - g_R Y & g_L X' - g_R Y' \end{pmatrix},$$  \hspace{1cm} (C.2)

Since $\det X \neq 0$, $g_R Y X^{-1}$ makes sense. It’s easy to check that it lies in $O(d; \mathbb{R})$.

Thus by choosing $g_L = g_R Y X^{-1}$, we get $g_L X - g_R Y = 0_d$, and (C.2) becomes

$$\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 2Y & Y' \end{pmatrix} X^{-1} \begin{pmatrix} X' + Y' \\ 0 \end{pmatrix},$$

where $g := g_R$. Put $\Lambda := \frac{1}{\sqrt{2}} (Y X^{-1} X' - Y')$. By (C.1), we find that $\sqrt{2} Y = ^t \Lambda^{-1}$ and that $^t \Lambda \frac{1}{\sqrt{2}} (Y X^{-1} X' + Y') = -\frac{1}{2} (Y' Y X^{-1} X' - ^t X' X^{-1} Y' Y') = -B$ is antisymmetric. This shows that each orbit has the shape $G_{\text{diag}} \cdot W'(\Lambda, B)$ as claimed.

Conversely, it is easy to check that each $G_{\text{diag}} \cdot W'(\Lambda, B)$ ($\Lambda \in \text{GL}(d, \mathbb{R}), B \in A(d, \mathbb{R})$) is a $O(d, \mathbb{R}) \times O(d, \mathbb{R})$-orbit. This proves the first assertion.

The last assertion of Proposition (2.5) is straightforward. \hfill \Box

We now describe generators of the discrete group $O(2, 2; \mathbb{Z})$, as in Proposition 2.6. In general, doing the same for $O(d, d; \mathbb{Z})$ ($d \geq 3$) is much harder. However for $d = 2$, we have the following nice description of $\mathbb{R}^{2,2}$. Let $\text{Mat}_{2,2}(\mathbb{R})$ be the space of $2 \times 2$ real matrices, equipped with the quadratic form $|| * || := \det(*)$. Its associated bilinear form has the signature $(2, 2)$, and therefore $\text{Mat}_{2,2}(\mathbb{R}) \cong \mathbb{R}^{2,2}$ as quadratic spaces. Consider an embedding of the hyperbolic lattice $U^{\oplus 2}$ defined by

$$\Phi_0 : \quad U^{\oplus 2} \ni e_1, e_2, f_1, f_2 \mapsto \text{Mat}_{2,2}(\mathbb{R}), \quad E_1, E_2, F_1, F_2,$$  \hspace{1cm} (C.3)
where

\[
E_1 = \sqrt{2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \sqrt{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F_1 = \sqrt{2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad F_2 = \sqrt{2} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.
\]

The embedding is isometric since \(\Phi_0(x_1e_1 + x_2e_2 + y_1f_1 + y_2f_2) = \sqrt{2} \begin{pmatrix} x_1 \\ -y_2 \\ y_1 \\ x_2 \end{pmatrix}\) has determinant \(2x_1y_1 + 2x_2y_2\). We denote by \(\Phi^R_0 : U^{2;2} \otimes \mathbb{R} = \mathbb{R}^{2;2} \cong M_{2,2}(\mathbb{R})\) the real scalar extension of \(\Phi_0\).

**Proposition C.1.** 1) The linear action \(X \mapsto gXh^{-1}\) of \((g, h) \in SL_2\mathbb{R} \times SL_2\mathbb{R}\) on \(X \in Mat_{2,2}(\mathbb{R})\) defines a group homomorphism

\[
\phi_R : SL_2\mathbb{R} \times SL_2\mathbb{R} \to O'(2, 2; \mathbb{R}) \cong O(2, 2; \mathbb{R})\,.
\]

2) \(\phi_R\) maps surjectively onto \(O'_0(2, 2; \mathbb{R})\), the connected component of the identity, and has the kernel \(\{(1, 2), (-1, 2), (-1, -1)\}\).

**Sketch of Proof.** It is easy to verify claim 1). For 2), we observe that the differential \(d\phi_R\) gives an isomorphism of the Lie algebras, and therefore the image of \(\phi_R\) is given by the connected component of the identity.

**Proposition C.2.** The restriction of \(\phi_R\) to \(SL_2\mathbb{Z} \times SL_2\mathbb{Z}\) defines a surjective group homomorphism:

\[
\phi_Z : SL_2\mathbb{Z} \times SL_2\mathbb{Z} \to O(2, 2; \mathbb{Z}) \cap O'_0(2, 2; \mathbb{R})
\]

with \(\text{Ker}(\phi_Z) = \{(1, 2), (-1, 2), (-1, -1)\}\).

**Proof.** It is clear that the image of \(\phi_Z\) lies in \(O(2, 2; \mathbb{Z}) \cap O'_0(2, 2; \mathbb{R})\) by Proposition C.1. To prove surjectivity, let us solve for \(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, h = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL_2\mathbb{Z}\) in

\[
\phi_R(g, h) = \begin{pmatrix} ad' & -ac' & -bc' & -bd' \\ -ab' & ad & ba' & bb' \\ -cb' & ca & da' & db' \\ -cd' & cc' & de' & dd' \end{pmatrix} = S,
\]

for any given \(S \in O(2, 2; \mathbb{Z}) \cap O'_0(2, 2; \mathbb{R})\). By Proposition C.1, there are solutions \(g, h \in SL_2\mathbb{R}\). To prove surjectivity, we will show that they are integral. First note that \(a^2 = \text{det} S_{12,12}, b^2 = \text{det} S_{12,34}, c^2 = \text{det} S_{34,12}, d^2 = \text{det} S_{34,34}; a'^2 = \text{det} S_{23,23}, b'^2 = -\text{det} S_{23,14}, c'^2 = -\text{det} S_{14,23}, d'^2 = -\text{det} S_{14,14}\) represents the minor made by the \(i, j\)-th rows and the \(k, l\)-th columns. From this, we see that the square of each variable \(a, b, ...\) is integer. So we can write \(a = x_a\sqrt{r_a}, \) with some integer \(x_a\) and a square free positive integer \(r_a\). Likewise, \(b = x_b\sqrt{r_b}, c = x_c\sqrt{r_c}\) and so on. The 16 components of (C.4) are required to be integers. For example, we have \(ad' = x_a x_{d'} \sqrt{r_a r_{d'}} \in \mathbb{Z}\). The last expression becomes integer only if \(r_a = r_{d'}\). Likewise we have \(r_a = r_b = \cdots = r_{d'} =: r\). Therefore we have,

\[
a, b, c, d, a', b', c', d' \in \sqrt{r} \mathbb{Z},
\]

for some square free integer \(r\). Since \(ad - bc = a'd' - b'c' = 1\), we see that the only possibility is \(r = 1\), which means that \(g, h \in SL_2\mathbb{Z}\). Thus \(\phi_Z\) is surjective. It is easy to determine the kernel of \(\phi_Z\). \(\square\)
Recall that $SL_2\mathbb{Z}$ is generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. It is easy to see that $\phi_z$ maps $(1_2, S), (1_2, T), (S, 1_2), (T, 1_2)$, respectively, to the matrices $S_1, T_1, S_2, T_2$ given in (2) of Proposition 2.6.

The two generators $R_1$ and $R_2$ are related to the other connected components of $O'(2, 2; R)$. In fact, $O'(2, 2; R)$ consists of the following four connected components:

$O'(2, 2; R) = O'_{+, +}(2, 2; R) \sqcup O'_{+, -}(2, 2; R) \sqcup O'_{-, +}(2, 2; R) \sqcup O'_{-, -}(2, 2; R)$,

where, for example, $O'_{+, -}(2, 2; R)$ consists of those elements which reverse the orientation of a positive definite two plane and preserve that of a negative definite two plane. (Note that if $g$ reverses the orientation of a single positive definite two plane, then it does so for every positive two plane. Similarly for negative definite two planes. Likewise if $g$ preserves the orientation of a two plane.) Obviously, $O'_0(2, 2; R) = O'_{+, +}(2, 2; R)$. Now giving the orientations $(e_1 + f_1) \wedge (e_2 + f_2)$ and $(e_1 - f_1) \wedge (e_2 - f_2)$, respectively, for positive and negative definite two planes, we see that $R_1$ belongs to $O'_{-, +}(2, 2; R) \cap O(2, 2; Z)$ and $R_2$ belongs to $O'_{-, -}(2, 2; R) \cap O(2, 2; Z)$. It is now clear that we have the following decomposition of $O(2, 2; Z)$:

$O(2, 2; Z) = O_{+, +}(2, 2; Z) \sqcup O_{+, -}(2, 2; Z) \sqcup O_{-, +}(2, 2; Z) \sqcup O_{-, -}(2, 2; Z)$, \hspace{1cm} (C.5)

with $O_{+, +}(2, 2; Z) = O'_0(2, 2; R) \cap O(2, 2; Z)$ and $O_{+, -}(2, 2; Z) = O_{+, +}(2, 2; Z)R_1R_2$, $O_{-, +}(2, 2; Z) = O_{+, +}(2, 2; Z)R_1$, $O_{-, -}(2, 2; Z) = O_{+, +}(2, 2; Z)R_2$. In particular, this yields property 1) of Proposition 2.6.
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