Classical and quantum geometrodynamics of 2d vacuum dilatonic black holes

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ABSTRACT We perform a canonical analysis of the system of 2d vacuum dilatonic black holes. Our basic variables are closely tied to the spacetime geometry and we do not make the field redefinitions which have been made by other authors. We present a careful discussion of asymptotics in this canonical formalism. Canonical transformations are made to variables which (on shell) have a clear spacetime significance. We are able to deduce the location of the horizon on the spatial slice (on shell) from the vanishing of a combination of canonical data. The constraints dramatically simplify in terms of the new canonical variables and quantization is easy. The physical interpretation of the variable conjugate to the ADM mass is clarified. This work closely parallels that done by Kuchař for the vacuum Schwarzschild black holes and is a starting point for a similar analysis, now in progress, for the case of a massless scalar field conformally coupled to a 2d dilatonic black hole.
I Introduction

In this paper we clarify aspects of the canonical description of vacuum 1+1 dimensional dilatonic black holes. Our analysis closely mirrors that of [1], in which vacuum Schwarzschild black holes were studied. The motivation for this work stems from our interest in the quantization of 4-d systems corresponding to spherically symmetric collapse. More specifically, we would like to study quantum aspects of spherically symmetric collapse of a massless scalar field in general relativity in 4 spacetime dimensions. This is a difficult task because the classical field equations are not exactly solvable even though the system is effectively 2 dimensional. However, the CGHS [2] model of 2d dilaton gravity with conformally coupled scalar fields is classically exactly solvable and we hope that the model can be quantized nonperturbatively. Since the 4-d system of interest is effectively 2-d we hope to gain insights into nonperturbative quantization of the 4-d case from a study of the nonperturbative quantization of the CGHS model (Note that most quantization efforts for the case in which matter is present, with a few exceptions such as the work of [3, 4], have been perturbative in character).

The first step in such a study is to present a clear analysis of the classical and quantum theory of vacuum dilatonic black holes. Bearing in mind our motivations, we would like to cast the analysis in a framework which emphasizes the similarities of this system with the vacuum Schwarzschild black holes. Both the classical and quantum theory of the latter has been clearly analysed in [1] in a canonical framework. In this paper we show that the 2d vacuum dilatonic black holes can be handled using exactly the same approach, which worked for the Schwarzschild case in [1].

In this work, we perform a canonical transformation to new canonical pairs of variables. One of these pairs consist of the mass of the spacetime and the spatial rate of change of the Killing time. An additional canonical transformation results in the Killing time itself, being a canonical variable. The vanishing of the constraints are shown to be equivalent to, modulo some subtelities, the vanishing of two of the new canonical momenta. The true degrees of freedom are parametrised by a canonical
pair, one of which is the mass of the spacetime. Its conjugate variable has a clear spacetime interpretation. Quantization of this description is almost trivial because of the simplified form of the constraints. In particular, in contrast to [3], we can find a Hilbert space representation (with an appropriate measure) for the physical operators of the theory.

This work, as we see it, has the following merits. First, it clarifies the physical interpretation of the observables of the theory (note that the observables were also given a physical interpretation in [1]; we go a little further than [3] in that we discuss parametrisation of the times at infinity). Second, the initial choice of variables used in this work is closely related to the spacetime geometry of the black hole and the whole treatment possesses a very close similarity to that in [1] which deals with the vacuum Schwarszchild case. Third, it can be viewed as a prerequisite for a similar treatment of the more interesting conformally coupled matter case.

Along the way, we give a self-consistent treatment of asymptotics in the canonical framework (note the analysis in [3] is inconsistent due to a subtle technical reason - this will be pointed out in Section 4).

The layout of the paper is as follows. In section 2, we review the global structure and the spacetime solution in the conformal gauge, of the vacuum dilatonic black holes. In section 3, we perform the canonical analysis of the action, in analogy to the ADM analysis done in 3+1 dimensions. Our canonical variables are closely tied to the geometry of the black hole spacetime and we do not make the field redefinitions of [3]. In section 4, we give a careful treatment of asymptotics in the canonical framework and identify the total energy of the system with the generator of time translations at spatial infinity. In the rest of the paper we closely mimic the treatment in [1]. The idea in [1] was to use as canonical variables, quantities which were physically significant. One hoped that the constraints of the theory simplified when written in terms of these quantities. The latter were identified by comparing the spacetime line element written in geometrically preferred coordinates with the ADM line element. Thus, in section 5, we express the mass and spatial rate of change of Killing time in terms of canonical data. In section 6 we use these quantities as canonical variables.
and see that the constraints simplify when written in terms of these variables. In section 7, we bring the ‘times at infinity’ into the canonical framework exactly as in [1]. Finally, in section 8, we give a brief description of the quantum theory based on the classical description of section 7. In section 9 we describe the classical theory by using light cone coordinates as canonical variables and quantize this description. Section 10 contains conclusions.

We have not attempted to review the enormous amount of pertinent literature and we refer the reader to review articles such as [7].

II The spacetime solution

Disregarding boundary terms, the action we deal with is that of [7]:

$$ S_D = \frac{1}{4} \int d^2x \sqrt{-g} \left( e^{-2\phi} \left( R + 4(\nabla \phi)^2 + 4(K^2) \right) \right) $$  

(1)

where $R$ is the scalar curvature of the 2 metric $g_{ab}$, $\phi$ is the dilaton field and $K$ is the cosmological constant. In this paper we use units in which $c = \frac{2K^2G}{\pi} = \hbar = 1$ where $G$ is the gravitational constant dimensions $c$ is the speed of light and $2\pi\hbar$ is Planck’s constant. With this choice, mass has units of inverse length.

The solution to the field equations in the conformal gauge is [7]:

$$ ds^2 = -\exp(2\psi) dU dV $$  

(2)

$$ \exp(-2\psi) = \exp(-2\phi) = \frac{2M}{K} - K^2UV $$  

(3)

$$ \Rightarrow R = \frac{8MK}{\frac{2M}{K} - K^2UV} $$  

(4)

The ranges of $(U, V)$ are such that $\frac{2M}{K} - K^2UV \geq 0$, with the curvature singularity along the curves $\frac{2M}{K} - K^2UV = 0$.

Note that $(U, V)$ here are like the null Kruskal coordinates $(U_s, V_s)$ for Schwarzschild. In the latter, the curvature singularity is at $U_sV_s = 1$. Here $K$ provides an extra scale,

\[\text{Our parameter } 'M' \text{ is half the parameter } 'M' \text{ which appears in [7].}\]
which allows for the definition of dimensional Kruskal like coordinates, in contrast with the dimensionless Kruskal coordinates for Schwarzschild.

The global structure of the vacuum dilatonic black hole is identical to that of the fully extended Schwarzschild solution. We label the different parts of the spacetime as follows: Region I with $U < 0, V > 0$ (the right static region), Region II with $U > 0, V > 0$ (the future dynamical region), Region III with $U > 0, V < 0$ (the left static region) and Region IV with $U < 0, V < 0$ (the past dynamical region). The horizons are at $U = 0$ and $V = 0$. We can define $(T, \rho)$ coordinates (the analog of the Killing time and Regge-Wheeler tortoise coordinates for Schwarzschild) in regions I and III as follows:

$$KV = e^{K(T+\rho)} \quad KU = -e^{-K(T-\rho)} \quad \text{in region I} \quad (5)$$

$$KV = -e^{K(T+\rho)} \quad KU = e^{-K(T-\rho)} \quad \text{in region III} \quad (6)$$

In each of the regions I and III, the line element and the coordinate ranges are

$$ds^2 = \frac{e^{(2K\rho)}}{M} + e^{(2K\rho)}[-(dT)^2 + (d\rho)^2] \quad -\infty < T, \rho < \infty \quad (7)$$

and the horizons are at $\rho \to -\infty$ with spatial infinity at $\rho \to \infty$. The metric is manifestly asymptotically flat at left and right spatial infinities in these coordinates.

### III The canonical form of the hypersurface action

We apply the standard ADM formalism of general relativity in 3+1 dimensions to the 2-d dilatonic black hole spacetimes. The spacetime is foliated by a 1 parameter family of slices ‘$\Sigma$’, the slices being labelled by a time parameter ‘$t$’, $t \in \mathbb{R}$. Each $t =$constant slice, $\Sigma$, has the topology of $\mathbb{R}$ and is coordinatized by a parameter ‘$r$’. Further, each slice is spacelike and extends from left spatial infinity to right spatial infinity, but is otherwise arbitrary (in particular, the slices are not restricted to pass through the bifurcation point $(U, V) = (0, 0)$).

The range of $r$ is $-\infty < r < \infty$ with left and right spatial infinities being approached as $r \to -\infty$ and $r \to \infty$ respectively. The ADM form of the spacetime line element
\[ ds^2 = -(N^2 - (N^r)^2 \Lambda^2)(dt)^2 + 2\Lambda^2 N^r(dt)(dr) + \Lambda^2(dr)^2. \]  

Here, \( \Lambda^2(dr)^2 \) is the induced spatial metric on \( \Sigma \) and \( N \) and \( N^r \) are the usual lapse and shift parameters. We substitute this form of the spacetime metric into the action \( S_D \) and obtain, modulo boundary terms

\[ S_{\Sigma} = \int dt dr \left( \frac{-1}{N} [(-\dot{\Lambda} + (N^r \Lambda)^)'(-\dot{R} + N^r R')R + \Lambda(-\dot{R} + N^r R')^2] ight) \]

\[ + N \left[ \frac{-RR''}{\Lambda} + \frac{\Lambda'RR'}{\Lambda^2} + K^2 R^2 \Lambda \right] \]

where dots and primes denote derivatives with respect to \( t \) and \( r \) respectively and we have defined \( R := e^{-\phi} \) to facilitate comparison with [1]. We shall supplement the above hypersurface action with the appropriate boundary terms in section 4.

The next step in the canonical analysis is to identify the momenta conjugate to the variables \( R, \Lambda \) (\( N, N^a \) will turn out to be Lagrange multipliers). The momenta are

\[ P_\Lambda = \frac{\delta S_D}{\delta \dot{\Lambda}} = \frac{1}{N}(-\dot{\Lambda} + N^r R')R \]

\[ P_R = \frac{\delta S_D}{\delta \dot{R}} = \frac{1}{N}[(\Lambda' + (N^r \Lambda)^)'R + 2\Lambda(-\dot{R} + N^r R')] \]

The canonical form of the hypersurface action is

\[ S_{\Sigma}[\Lambda, P_\Lambda, R, P_R; N, N^r] = \int dt \int_{-\infty}^{\infty} dr (P_\Lambda \dot{\Lambda} + P_R \dot{R} - NH - N^r H_r) \]

where

\[ H = \frac{P_\Lambda^2 \Lambda}{R^2} - \frac{P_R P_\Lambda}{R} + \frac{RR''}{\Lambda} - \frac{\Lambda'RR'}{\Lambda^2} - K^2 R^2 \Lambda \]

\[ H_r = P_R R' - P_\Lambda' \Lambda \]

\( H \) and \( H_r \) are the Hamiltonian and diffeomorphism constraints of the theory. \( N, N^r \) are Lagrangian multipliers. The symplectic structure can be read off from the action and the only non vanishing Poisson brackets are

\[ \{ \Lambda(y), P_\Lambda(x) \} = \{ R(x), P_R(y) \} = \delta(x, y) \]
It is straightforward to check that with these Poisson brackets the constraints are first class. It is also easy to see that $H_r$ integrated against $N^r$ generates spatial diffeomorphisms of the canonical data.

IV Asymptotics

A satisfactory treatment of asymptotics in the context of a Hamiltonian formalism must achieve the following:

(i) The choice of asymptotic conditions on the fields must be such that canonical data corresponding to classical solutions of interest are admitted.

(ii) Under evolution or under flows generated by the constraints of the theory, the canonical data should remain in the phase space. For example, the smoothness of the functional derivatives of the generating functions for such flows must be of the same type as the data themselves. Also, the boundary conditions on the data must be preserved by such flows. The latter requirement may be violated in subtle ways and the formalism must be carefully checked to see that this does not happen.

In what follows we display our choice of boundary conditions on the phase space variables. To impose (ii), above, we need to deal with appropriate functionals on the phase space. These are obtained by integrating the local expressions for $H$ and $H_r$ against the multipliers $N$ and $N^r$ to obtain $C(N) := \int_{-\infty}^{\infty} dr N H$ and $C(N^r) := \int_{-\infty}^{\infty} dr N^r H_r$. Imposition of (ii) gives boundary conditions on the $N$ and $N^r$. Next, we identify the generator of unit time translations at spatial infinity and show that it is $M$ on shell ( $M$ is the parameter in the spacetime metric). Finally, we point out why the analysis in [6] is incomplete.

Since we use the same techniques as in the analysis for 3+1 canonical gravity, we shall not discuss them in detail.

IV.1 Boundary conditions on phase space variables

At right spatial infinity ($r \rightarrow \infty$) we impose:

$$ R = e^{Kr} + \alpha_+ e^{-Kr} + O(e^{-3Kr}) $$

(16)
\[ R^2 = e^{2Kr}[1 + 2\alpha_+ e^{-2Kr} + O(e^{-4Kr})] \] \hspace{1cm} (17)

\[ \Lambda = 1 - \beta_+ e^{-2Kr} + O(e^{-4Kr}) \] \hspace{1cm} (18)

\[ \Lambda^2 = 1 - 2\beta_+ e^{-2Kr} + O(e^{-4Kr}) \] \hspace{1cm} (19)

\[ P_R \sim e^{-Kr} \] \hspace{1cm} (20)

\[ P_\Lambda \sim O(1) \] \hspace{1cm} (21)

Here \( \alpha_+ \) and \( \beta_+ \) are arbitrary (in general time dependent) parameters independent of \( r \).

At left spatial infinity \( (r \to -\infty) \) we impose exactly the same conditions except that \( r \) is replaced by \( -r \) and the ‘right’ parameters \( \alpha_+ \) and \( \beta_+ \) are replaced by the ‘left’ parameters denoted by \( \alpha_- \) and \( \beta_- \). Note that, on solution, \( \alpha_+ = \alpha_- = \beta_+ = \beta_- = \frac{M}{K} \).

**IV.2 The diffeomorphism constraint functional**

We require that

(a) \( C(N^r) \) exists. From the boundary conditions (16)-(21), above

\[ H_r \sim O(1) \] \hspace{1cm} (22)

Thus, for \( C(N^r) \) to be finite, the asymptotic behaviour of \( N^r \) should be

\[ N^r \sim O\left(\frac{1}{r^2}\right) \] \hspace{1cm} (23)

(b) The orbits generated by \( C(N^r) \) should lie within the phase space. Consider

\[ \dot{R}(x) := \{R(x), C(N^r)\} = N^r(x) R'(x) \]

\[ \Rightarrow \dot{R} \sim N^r e^{K|r|}, |r| \to \infty \] \hspace{1cm} (24)

But for \( \dot{R}(t, x) \) to satisfy the boundary condition (16), asymptotically

\[ \dot{R} \sim O(e^{-K|r|}) \]

\[ \Rightarrow N^r \sim e^{-2K|r|} \] \hspace{1cm} (25)

This is stronger than (23). For \( N^r \) satisfying (25), it can be verified that \( C(N^r) \) generates orbits which preserve the boundary conditions on the canonical variables and
is functionally differentiable without the addition of a surface term. Since asymptotic translations of \( r \) do not preserve the the boundary conditions on \( R \), there are no generators of such translations in the theory (if there were, we would identify them with spatial momenta).

### IV.3 The Hamiltonian constraint functional and the mass

We require that \( C(N) \) exists. Using the boundary conditions on the canonical variables, we find that \( H \sim e^{-2K|r|} \) near the spatial infinities. For each of the momentum dependent terms in \( H \) this is obvious. The remaining terms, individually, do not have this behaviour; but taken together, conspire to fall off as \( e^{-2K|r|} \). So, for \( C(N) \) to exist, it is possible to admit

\[
N \rightarrow N_+, \quad r \rightarrow +\infty \tag{26}
\]
\[
N \rightarrow N_-, \quad r \rightarrow -\infty \tag{27}
\]

where \( N_+, N_- \) are independent of \( r \).

Next, we require that \( C(N) \) be functionally differentiable. The variation of \( C(N) \) is of the form:

\[
\delta C(N) = \int_{-\infty}^{\infty} dr \left[ \Xi_\Lambda \delta \Lambda(r) + \Xi_R \delta R(r) + \Xi_{P_\Lambda} \delta P_\Lambda(r) + \Xi_{P_R} \delta P_R(r) \right] + \delta C(N)_{\text{surface}} \tag{28}
\]

Here \( \Xi_R, \Xi_{P_R}, \Xi_\Lambda, \Xi_{P_\Lambda} \) are local expressions involving the canonical variables and the lapse and their derivatives. The term which obstructs functional differentiability is \( \delta C(N)_{\text{surface}} \) and is given by

\[
\delta C(N)_{\text{surface}} = s(r = \infty) - s(r = -\infty) \tag{29}
\]

where

\[
s = -\frac{NRR'}{\Lambda^2} \delta \Lambda - \frac{(RN)'}{\Lambda} \delta R - \frac{RN}{\Lambda} (\delta R)' \tag{30}
\]

\( \delta C(N)_{\text{surface}} \) vanishes if \( N \rightarrow 0 \) as \( |r| \rightarrow \infty \) and \( C(N) \) is functionally differentiable for such \( N \).
To complete the analysis for the Hamiltonian constraint functional, we must check that the boundary conditions are preserved by the flows generated by $C(N)$. We have

$$\dot{P}_\Lambda := \{P_\Lambda, C(N)\} = -\Xi_\Lambda$$

$$= \frac{NP^2_\Lambda}{R^2} + \frac{NR'^2}{\Lambda^2} + \frac{N'RR'}{\Lambda^2} - NK^2R^2$$

(31)

For the boundary conditions to be satisfied, we must require $\dot{P}_\Lambda \sim O(1)$ asymptotically. From the equation for $\dot{P}_\Lambda$ and the boundary conditions on the canonical variables, it can be verified that this requirement implies

$$N' \sim e^{-2K|r|}, \quad |r| \to \infty$$

(32)

This fixes $N \sim e^{-2K|r|}$ asymptotically. It can be checked that for such $N$, $C(N)$ exists, is functionally differentiable and generates flows which preserve the boundary conditions.

Thus, just as in canonical gravity in 3+1 dimensions, the constraints $C(N), C(N^r)$ generate motions which are trivial at infinity. We have seen that the spatial momentum vanishes because constant spatial translations at infinity do not preserve the asymptotic conditions. We now turn our attention to constant time translations at the spatial infinities and identify the generators of such motions with the ADM masses.

Hence, we must impose (26) and (27). (32) still holds since we want the boundary conditions on $P_\Lambda$ to be preserved under evolution. Thus

$$N \to N_+ + O(e^{-2Kr}), \quad r \to +\infty$$

(33)

$$N \to N_- + O(e^{2Kr}), \quad r \to -\infty$$

(34)

where $N_+, N_-$ are independent of $r$. $C(N)$ is not functionally differentiable due to the presence of $\delta C(N)_{\text{surface}}$. This term, for $N$ with the above behaviour is

$$\delta C(N)_{\text{surface}} = -K(2\delta\alpha_+ - \delta\beta_+)N_+ - K(2\delta\alpha_- - \delta\beta_-)N_-$$

(35)

To restore functional differentiability, we add an appropriate term to $C(N)$ whose variation cancels $\delta C(N)_{\text{surface}}$. We call the resultant expression, (which is non zero
on the constraint surface) the true Hamiltonian $H_T$. It is given by

$$H_T(N) := C(N) + K(2\alpha_+ - \beta_+)N_+ + K(2\alpha_- - \beta_-)N_-$$  \hspace{1cm} (36)$$

$H_T$ generates time translations which are constant at right and left spatial infinities and we can identify

$$M_+ := K(2\alpha_+ - \beta_+)$$  \hspace{1cm} (37)$$

and

$$M_- := K(2\alpha_- - \beta_-)$$  \hspace{1cm} (38)$$

with the masses at right and left infinity, respectively. (Note, that on shell these expressions both reduce to the parameter $M$ in the spacetime solution.)

Thus, the correct action to use is

$$S[\Lambda, P_\Lambda, R, P_R, N, N'] = S[\Lambda, P_\Lambda, R, P_R, N, N'] + S_{\partial \Sigma}[M_-, M_+, N_-, N_+]$$  \hspace{1cm} (39)$$

where

$$S_{\partial \Sigma}[M_-, M_+, N_-, N_+] = -\int dt (N_+M_+ + N_-M_-)$$  \hspace{1cm} (40)$$

The analysis in [6] is incomplete for the following reason. In [6] the boundary conditions are a little stronger than ours - in particular, the work there assumes $\alpha = \beta$ (we have suppressed the $+$ and $-$ subscripts). However the true Hamiltonian does not necessarily preserve $\alpha = \beta$. This can be seen by looking at the evolution of $R$ and $\Lambda$:

$$\dot{R} = \{R, H_T(N)\} = -\frac{NP_\Lambda}{R}$$  \hspace{1cm} (41)$$

$$\dot{\Lambda} = \{\Lambda, H_T(N)\} = -\frac{NP_R}{R} + \frac{2NP_\Lambda \Lambda}{R^2}$$  \hspace{1cm} (42)$$

Our boundary conditions, on the other hand, suffer no such deficiency.
V  Reconstruction of mass and Killing time rate from canonical data

We would like to construct, out of the canonical data, expressions which on shell, give the mass, $M$ and spatial rate of change of Killing time, $T'$, along $\Sigma$. We guess the correct expressions, just as in [1], by comparing the ADM form of the line element (8) with that corresponding to the spacetime solution (7). So we parameterize $T = T(t, r)$ and $R = R(t, r)$ and put them into (8). Comparison with (7) gives:

$$\Lambda^2 = \Phi^2(-T'^2 + \rho'^2)$$  \hspace{1cm} (43)

$$N_r = -\frac{\dot{T}T' + \dot{\rho}\rho'}{-T'^2 + \rho'^2}$$  \hspace{1cm} (44)

$$N = \Phi \frac{\dot{T}\rho' - \dot{\rho}T'}{\sqrt{-T'^2 + \rho'^2}}$$  \hspace{1cm} (45)

where  \hspace{1cm} $\Phi^2 := \frac{e^{(2K\rho)}}{2M_K + e^{(2K\rho)}}$  \hspace{1cm} (46)

Note that \hspace{1cm} $R^2 = \frac{2M}{K} + e^{(2K\rho)}$ on solution  \hspace{1cm} (47)

(To understand the reasons for our choice of signs for the square root in the expression for $N$ see [1]). Substituting this in the expression for $P_\Lambda$ (10) we get an expression for the Killing time rate

$$T' = -\frac{P_\Lambda A}{K(R^2 - \frac{2M}{K})}$$  \hspace{1cm} (48)

Finally, making use of equation (43) for $\Lambda^2$, we get an expression for the mass

$$\frac{2M}{K} = R^2 + \frac{P_\Lambda^2}{K^2 R^2} - \frac{R'^2}{\Lambda^2 K^2}$$  \hspace{1cm} (49)

Thus we have an expression for the mass as a function of the canonical data and we can substitute this expression for $M$ in the above expression for $T'$ to get the Killing time rate also as function of the canonical data. Note the similarity of the
expressions with those in [1]. Finally, from the boundary conditions on the canonical data (16)-(21), we can infer the asymptotic behaviour of our expression for the mass. We obtain

\[ M(r = +\infty) = K(2\alpha_+ - \beta_+) \quad M(r = -\infty) = K(2\alpha_- - \beta_-) \] (50)

This is exactly what we expect from the expressions for the generators of unit time translations at spatial infinities derived in section 4.3!

VI Using $M$ and $T'$ as new canonical variables

VI.1 The canonical transformation

It is straightforward to show that

\[ \{M(x), -T'(y)\} = \delta(x, y) \] (51)

This prompts the definition

\[ P_M(x) := -T'(x) \] (52)

Since neither $M(x)$ nor $P_M(x)$ contain $P_R$, they commute under Poisson brackets with $R$. However $(M, P_M; R, P_R)$ are not a canonical chart and we need to replace $P_R$ with an appropriate variable to have a canonical chart on phase space. We use the same trick as in [1] to guess the new momentum conjugate to $R$ (which we shall refer to as $\Pi_R$). We expect $\Pi_R$ to be a density of weight one and require the diffeomorphism constraint to generate diffeomorphisms on the new canonical variables. This prompts the definition

\[ \Pi_R := P_R - \frac{1}{R'} (M' P_M + \Lambda P'_\Lambda) \] (53)

Long and straightforward calculations show that

\[ \{R(x), \Pi_R(y)\} = \delta(x, y) \quad \{\Pi_R(x), M(y)\} = \{\Pi_R(x), P_M(y)\} = \{\Pi_R(x), \Pi_R(y)\} = 0 \] (54)

Thus, we make a canonical transformation from $(\Lambda, P_\Lambda; R, P_R)$ to $(M, P_M; R, \Pi_R)$. We can equally well express the old variables in terms of the new.

\[ \Lambda^2 = \frac{R'^2}{fK^2} - \frac{P^2_{Mf}}{R'^2} \] (55)
\[ P_{\Lambda} = \frac{K f P_M}{\left( \frac{R'^2}{f K^2} - \frac{P^2_{\Lambda}}{R^2} \right)^{\frac{1}{2}}} \]  

Here we have defined

\[ f := R^2 - \frac{2M}{K} = \frac{R'^2}{(\Lambda K)^2} - \frac{P^2_{\Lambda}}{K^2 R^2} \]  

From the spacetime solution, it is apparent that (on shell), the horizons are located at \( f = 0 \). For \( f = 0 \) the canonical transformation breaks down. This is exactly what happens in the Schwarzschild case and we refer the reader to \[1\] for a discussion of issues which arise when \( f = 0 \).

**VI.2 The constraints**

Before writing the constraints in terms of the new canonical variables, it is instructive to show that \( M(x) \) is a constant of motion. From (49) and (13,14) it is easy to check that

\[ \frac{M'}{K} = -\frac{1}{RAK^2}(R'H + \frac{P_{\Lambda}}{R}H_r) \]  

Thus, as expected, the mass function doesn’t change over the slice. It is also easy to check that

\[ \{M(x), H(y)\} = -\delta(x,y)\frac{R'}{\Lambda^3 KR}H_r \]  

as well as

\[ \{M(x), H_r(y)\} = M'(x)\delta(x,y) \]  

(59), (60) and (58) show that \( M(x) \) is indeed a constant of motion.

We now proceed to write the constraints in terms of the new canonical variables. From the expressions for \( \Pi_R \), (53), and \( M', (58) \), it can be shown that

\[ H = -\frac{M'RR'}{fK} + \frac{KfP_M\Pi_R}{R} \left( \frac{R'^2}{fK^2} - \frac{P^2_{\Lambda}}{R^2} \right)^{\frac{1}{2}} \]  

\[ H_r = \Pi_R R' + P_M M' \]
Thus, the vanishing of \(H, H_r\) is equivalent to the vanishing of \(M', \Pi_R\) modulo the vanishing of \(f\). Again, arguing the same way as in [1], we can replace, as constraints, the former with the latter everywhere on \(\Sigma\) (in [1] the argument for replacing the old constraints with the new even when \(f=0\) is essentially one of continuity). We can express the old set of constraints in terms of the new ones (61,62) in matrix notation i.e.:

\[
\begin{bmatrix}
    H \\
    H_r
\end{bmatrix} = A
\begin{bmatrix}
    M' \\
    \Pi_R
\end{bmatrix}
\] (63)

where \(A\) is a 2 \(\times\) 2 matrix whose field dependent coefficients can be read of from (61,62) above. It is curious that even though the individual elements of \(A\) maybe ill defined when \(f = 0\), its determinant is independent of \(f\). In fact

\[
\text{Det}A = RK\Lambda
\] (64)

and is non vanishing as long as \(R, \Lambda \neq 0\). Since \(\Lambda^2\) is the spatial metric, \(\Lambda\) is non zero and \(R \geq 0\) by virtue of it’s definition in terms of the dilaton field, vanishing on shell only at the singularity. So, if \(\Sigma\) is away from the singularity, the behaviour of \(\text{Det}A\) supports the replacement of the old constraints with the new.

**VI.3 The action**

Written in terms of the new canonical variables, the hypersurface action is

\[
S_{\Sigma}[M, R, P_M, \Pi_R; N, N^r] = \int dt \int_{-\infty}^{\infty} dr (P_M \dot{M} + \Pi_R \dot{R} - NH - N^r H_r) \] (65)

Replacing the old constraints with the new ones gives

\[
S_{\Sigma}[M, R, P_M, \Pi_R; N, N^r] = \int dt \int_{-\infty}^{\infty} dr (P_M \dot{M} + \Pi_R \dot{R} - N^M M' - N^R \Pi_R) \] (66)

where \(N^M\) and \(N^R\) are new Lagrange multipliers related to the old ones by

\[
N^M = -\frac{RR'}{fK\Lambda} N + P_M N^r
\] (67)

\[
N^R = -\frac{K f P_M}{RA} N + R' N^r
\] (68)
From (40), the boundary action $S_{\partial \Sigma}$ is given by

$$S_{\partial \Sigma} = - \int dt (N_+ M_+ + N_- M_-)$$

(69)

where (from (37,38) and (50))

$$M_+ = M(r = \infty) \quad M_- = M(r = -\infty)$$

(70)

Note that from (67,68) and the boundary conditions on the canonical data and the lapse and shift functions, the asymptotic values of the new multipliers are $N^R \to 0$ and

$$N^M(r = +\infty) =: N_+^M = -N_+$$

(71)

$$N^M(r = -\infty) =: N_-^M = +N_-$$

(72)

The total action is just

$$S[M,R,P_M,\Pi_R; N^M, N^R] = S_{\Sigma}[M,R,P_M,\Pi_R; N^M, N^R] + S_{\partial \Sigma}[M; N^M]$$

(73)

We now raise a subtle issue (whose analog has gone unnoticed in [1]). The old variables $\Lambda, R, P_\Lambda, P_R$ were taken to be smooth functions of $r$, subject to the boundary conditions in section 4. The new variables (specifically) $P_M$ and $\Pi_R$ are not necessarily smooth functions of $r$ for arbitrary values of the old variables. In fact, on shell, on the horizon, if $\Sigma$ does not pass through the bifurcation point, the new momenta necessarily diverge. Thus, if we are to admit arbitrary slicings, not just the ones passing through the bifurcation point, we must allow for divergent values of the new momenta (at least when $f = R^2 - \frac{M}{\kappa}$ vanishes). A similar comment holds for the new Lagrange multipliers. So, if we use the action in terms of the new constraints we cannot assume the data to be smooth functions. This raises questions about the associated quantum theory. In [1] the theory is quantized without worrying about this issue - since we do not know how to resolve this issue we shall also do the same. As a result, it may be that we are only allowing slices which pass through the bifurcation point on shell (although, it may be that since $f = 0$ only on a “set of measure zero”, we can choose to ignore the issue).
VII The parametrized action

Since the form of the action (and even the notation, with the exception of the symbol for the momentum conjugate to $R$) is identical to that in [1], one can simply follow the discussion of sections VII to IX of that paper. We will be extremely succinct in this section and quickly review the relevant part of [1].

For quantization, one can deal with the action in section 6.3 in which the lapse functions are prescribed at spatial infinities, or one can parameterize the proper times at right and left infinities (denoted by $\tau_+$ and $\tau_-$) and obtain the action

$$S[M, P_M, R, \Pi_R, N^M, N^R, \tau_+, \tau_-] := S_\Sigma[M, P_M, R, \Pi_R, N^M, N^R] + S_{\partial\Sigma}[M, \tau_+, \tau_-]$$

where

$$S_{\partial\Sigma}[M, \tau_+, \tau_-] := -\int dt (M_+ \dot{\tau}_+ - M_- \dot{\tau}_-)$$

and $S_\Sigma[M, P_M, R, \Pi_R, N^M, N^R]$ is given in section 6.3. The free variations of the parametrized action with respect to all its arguments result in equations of motion equivalent to those obtained from the action in section 6.3 which had prescribed lapses at spatial infinities.

Further analysis of the parametrized action $S[M, P_M, R, \Pi_R, N^M, N^R, \tau_+, \tau_-]$ reveals that the following definitions

$$N^T(r) = -N^M(r), \quad m = M_-, \quad p = (\tau_+ - \tau_-) + \int_{-\infty}^{\infty} dr P_M(r)$$

$$T(r) = \tau_+ - \int_{-\infty}^{r} dr' P_M(r') \quad P_T(r) = -M'(r)$$

result in (upto total derivatives with respect to $t$) the reexpression of the parameterized action as

$$S[m, p, M, P_M, R, \Pi_R, N^M, N^R] = \int dt (p \dot{m} + \int_{-\infty}^{\infty} dr (P_T(r) \dot{T}(r) + \Pi_R(r) \dot{R}(r)))$$

$$- \int dt \int_{-\infty}^{\infty} dr (N^T(r) P_T(r) + N^R(r) \Pi_R(r))$$

(77)

Thus, $m$ and $p$ are constants of motion and parametrize the reduced phase space of the theory. $m$ is the ADM mass of the spacetime and $p$ has the interpretation of...
the difference between the proper time, $\tau_-$, at left infinity and the parametrization time at left infinity with the proper time $\tau_+$ at right infinity synchronized with the parametrization time at right infinity.

VIII Quantum theory

We briefly review Dirac quantization of the classical description which follows from the parameterized action (for the quantization following from the unparameterized action we refer the reader to [1]).

Since $R(r), m \geq 0$, it is better to make a point transformation on the classical phase space before we quantize. We define

$$\xi = \ln R \quad \Pi_\xi = R \Pi_R \quad \xi = \ln m \quad p_x = mp$$

and replace the $\Pi_R = 0$ constraint by $\Pi_\xi = 0$. To pass to quantum theory, we choose a coordinate representation. Thus $\Psi = \Psi(x; T, \xi)$ (Psi denotes the wavefunction). The quantum constraints are

$$\hat{P}_T\Psi = -i \frac{\delta \Psi}{\delta T(r)} = 0 \quad (80)$$

$$\hat{\Pi}_\xi\Psi = -i \frac{\delta \Psi}{\delta \xi(r)} = 0 \quad (81)$$

$$\Rightarrow \Psi = \Psi(x) \quad (82)$$

So the nontrivial operators in the theory are constructed from $\hat{x}$ and $\hat{p}_x = \frac{\partial}{\partial x}$ and the Hilbert space consists of square integrable functions of $x$ on the real line.

IX Light cone coordinates as canonical variables

IX.1 Classical theory

We now pass to a description in terms of canonical variables which correspond to (on shell) the null ‘Kruskal’ coordinates $U, V$ of section 2. Recall, from the spacetime
solution, that
\[ R^2 - \frac{2M}{K} = -K^2 UV \quad e^{2KT} = \left| \frac{V}{U} \right| \tag{83} \]

We start with the description in terms of the parametrized action \( S[m, p, M, P_M, R, \Pi_R, N^M, N^R] \).

We define new canonical variables
\[
\bar{m} = m \quad \bar{R} = R^2 - \frac{2m}{K} \quad P_T = P_T \\
\bar{p} = p + \int_{-\infty}^{\infty} dr \frac{\Pi_R}{2R} \quad \Pi_R = \frac{\Pi}{2R} \quad \bar{T} = T \tag{84}
\]

Motivated by (83) above we make a further canonical transformation (with \( \bar{m}, \bar{p} \) unchanged) to new variables (\( U, P_U, V, P_V \)):

\[
\bar{R} = -K^2 UV \quad 2K\bar{T} = \ln |V| - \ln |U| \\
P_R = -\left( \frac{P_V}{2K^2 U} + \frac{P_U}{2K^2 V} \right) \quad P_T = KV P_V - KU P_U \tag{85}
\]

Away from \( U = 0 \) or \( V = 0 \) (which correspond to the horizon on shell), we can replace the constraints \( \Pi_R = 0 = P_T \) by

\[
P_V = 0 \quad \text{(86)} \\
P_U = 0 \quad \text{(87)}
\]

If we impose that \( P_U \) and \( P_V \) are continuous functions of \( r \) on the constraint surface, they must vanish even on the horizons. Then the parametrized action becomes

\[
S[\bar{m}, \bar{p}, U, P_U, V, P_V, N^U, N^V] = \int dt (\bar{p}\dot{\bar{m}} + \int_{-\infty}^{\infty} dr (P_U(r)\dot{U}(r) + P_V(r)\dot{V}(r)) \\
- \int dt \int_{-\infty}^{\infty} dr (N^U(r)P_U(r) + N^V(r)P_V(r)) \tag{88}
\]

where \( N^U, N^V \) are the appropriately defined new Lagrange multipliers. We have refrained from looking at exactly what happens at the horizon and to what extent it is valid to replace the old constraints with (86) and (87) because there exists a much better way of defining the null coordinates and rewriting the constraints in terms of them \( \text{[]} \). This will be presented elsewhere.
Remark: if we consider the combinations of constraints $H$ and $H_r$ which generate 2 commuting copies of the Virasoro-type algebra, namely

$$H_+ := \Lambda H + H_R \quad H_- := \Lambda H - H_r$$  \hspace{1cm} (89)$$

then on the constraint surface, modulo some subtleties on the horizons, in terms of the new canonical variables

$$H_+ = V' P_V$$  \hspace{1cm} (90)$$
$$H_- = -U' P_U$$  \hspace{1cm} (91)$$

**IX.2 Quantum theory**

We describe the results of a Dirac quantization of the theory described by the action above. The fact that $m \geq 0$, is handled by using $\bar{x} := \ln \bar{m}$ and $p_\bar{x} := \bar{m} \bar{p}$ (see section 8). We choose a configuration representation, so that the wave functions depend on $(U(r), V(r), \bar{x})$. The coordinate operators $\hat{U}, \hat{V}, \hat{\bar{x}}$ act by multiplication and the momenta operators act as follows:

$$\hat{P}_U = -i(\delta/\delta U(r))$$
$$\hat{P}_V = -i(\delta/\delta V(r))$$
$$\hat{p}_\bar{x} = -i(\partial/\partial \bar{x})$$  \hspace{1cm} (92)$$

The quantum constraints are

$$\hat{P}_U \Psi(\bar{x}; U(r), V(r)) = 0$$  \hspace{1cm} (93)$$
$$\hat{P}_V \Psi(\bar{x}; U(r), V(r)) = 0$$  \hspace{1cm} (94)$$

$$\Rightarrow \Psi = \Psi(\bar{x})$$  \hspace{1cm} (95)$$

The Hilbert space consists of square integrable functions of $\bar{x}$ and we have a quantum theory identical to that in section 8.

**X Conclusions**

In this work, we have formulated the canonical description of vacuum 2-d black holes in terms of variables which have a clear spacetime interpretation. In doing this we have
followed the recipes of [1] which dealt with vacuum Schwarzschild black holes. As in [1], the classical description simplifies to such an extent that quantization becomes very easy.

When we replace the original classical constraints with new ones, care has to be taken where the horizon intersects the spatial slice. In particular, the new constraint functions are assumed to be continuous on the spatial slice and this forces them to vanish even at the horizon.

Because the classical variables used have a clear physical meaning, it becomes easier to interpret the corresponding quantum operators. We have also constructed, explicitly, a classical description (and quantized it) using the light cone coordinates as canonical variables. In [1] the analogs of these objects were the light cone Kruskal coordinates and using them explicitly was a little involved because the Regge-Wheeler tortoise coordinate plays a key role in the transformation from curvature coordinates to Kruskal coordinates.

We are trying to apply methods similar to those used in this work and in the Schwarzschild case, to the 2d black holes with conformally coupled matter.

Ultimately, we hope to learn useful lessons from this work and [8], and efforts of other workers like [3, 4] which will help in tackling the system of spherically symmetric (4-d) gravity with a (spherically symmetric) scalar field.

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After this work was completed we learnt of a recent work by Lau[9] which also contains a classical analysis similar to that in this work.

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