Relativistic Lippmann - Schwinger equation

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Abstract

The classical Lippmann-Schwinger equation plays an important role in the scattering theory (non-relativistic case, Schrödinger equation). In the present paper we consider the relativistic analogue of the Lippmann-Schwinger equation. We represent the corresponding equation in the integral form. Using this integral equation we investigate the stationary scattering problems (relativistic case, Dirac equation). We consider the dynamical scattering problems (relativistic case, Dirac equation) as well.

1 Introduction

The classical integral Lippmann-Schwinger equation plays an important role in the scattering theory (non-relativistic case, Schrödinger equation). The relativistic analogue of the Lippmann-Schwinger equation was formulated in the terms of the limit values of the corresponding resolvent. In the present paper we found the limit values of the resolvent in the explicit form. Using this result, we represent relativistic Lippmann-Schwinger equation (RLS equation) as an integral equation (Sections 2 and 3). In Section 4, we consider
the dynamical scattering problems (relativistic case, Dirac equation). In Section 5, we show that the integral RLS equation is effective by investigating the stationary scattering problems (relativistic case, Dirac equation).

It is interesting to compare the results of dynamic and stationary scattering theory. The corresponding results for the radial case were obtained in [7] and [9].

2 RLS equation in the integral form

1. Let us write the Dirac equation (see [3])

\[ i \frac{\partial}{\partial t} u(r,t) = \mathcal{L}u(r,t), \]  

(2.1)

where \( u(r,t) \) is a 4 \times 1 vector function and \( r = (r_1, r_2, r_3) \). The operators \( \mathcal{L} \) and \( \mathcal{L}_0 \) are defined by the relations

\[
\mathcal{L}u = [\mathbf{e}\nu(r)I_4 + m\beta + \alpha(p + eA(r))]u, \quad \mathcal{L}_0u = (m\beta + \alpha p)u. \tag{2.2}
\]

Here \( p = -i \text{ grad} \), \( \nu \) is a scalar potential, \( A \) is a vector potential, \((-e)\) is the electron charge. Now let us define \( \alpha = [\alpha_1, \alpha_2, \alpha_3] \). The matrices \( \alpha_k \) are the 4 \times 4 matrices of the forms

\[
\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3, \tag{2.3}
\]

where

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.4}
\]

The matrices \( \beta \) and \( I_2 \) are defined by the relations

\[
\beta = \begin{pmatrix} I_2 \\ 0 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.5}
\]

2. We consider separately the unperturbed Dirac equation (2.1), (2.2), when \( \nu(r) = 0 \) and \( A(r) = 0 \). The Fourier transform is defined by

\[
\Phi(q) = Fu(r) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{iqr} u(r)dr. \tag{2.6}
\]
The inverse Fourier transform has the form

\[ u(r) = F^{-1}\Phi(q) = (2\pi)^{-3/2} \int_{Q^3} e^{-iqr} \Phi(q)dq. \]  

(2.7)

In the momentum space the unperturbed Dirac equation takes the form (see \[1\], Ch.IV):

\[ i\frac{\partial}{\partial t}\Phi(q,t) = H_0(q)\Phi(q,t), \quad q = (q_1, q_2, q_3), \]  

(2.8)

where \( H_0(q) \) and \( \Phi(q,t) \) are matrix functions of order \( 4 \times 4 \) and \( 4 \times 1 \) respectively. Here the matrix \( H_0(q) \) is defined by the relation

\[
H_0(q) = \begin{pmatrix}
m & 0 & q_3 & q_1 - iq_2 \\
0 & m & q_1 + iq_2 & -q_3 \\
q_3 & q_1 - iq_2 & -m & 0 \\
q_1 + iq_2 & -q_3 & 0 & -m
\end{pmatrix}. \tag{2.9}
\]

The eigenvalues \( \lambda_k \) and the corresponding eigenvectors \( g_k \) of \( H_0(q) \) are important in our theory. We find them below:

\[ \lambda_{1,2} = -\sqrt{m^2 + |q|^2}, \quad \lambda_{3,4} = \sqrt{m^2 + |q|^2} \quad (|q|^2 := q_1^2 + q_2^2 + q_3^2); \]  

(2.10)

\[
g_1 = \begin{pmatrix} (-q_1 + iq_2)/(m + \lambda_3) \\
q_3/(m + \lambda_3) \\
0 \\
1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} -q_3/(m + \lambda_3) \\
(q_1 + iq_2)/(m + \lambda_3) \\
(-q_1 - iq_2)/(m + \lambda_3) \\
1 \end{pmatrix}, \tag{2.11}
\]

\[
g_3 = \begin{pmatrix} (-q_1 + iq_2)/(m - \lambda_3) \\
q_3/(m - \lambda_3) \\
0 \\
1 \end{pmatrix}, \quad g_4 = \begin{pmatrix} -q_3/(m - \lambda_3) \\
(q_1 - iq_2)/(m - \lambda_3) \\
(-q_1 - iq_2)/(m - \lambda_3) \\
1 \end{pmatrix}. \tag{2.12}
\]

It follows from (2.9) and (2.10) that

\[ H_0^2(q) = (m^2 + |q|^2)I_4. \tag{2.13} \]

Hence we obtain

\[ H_0^{-1}(q) = (m^2 + |q|^2)^{-1}H_0(q). \tag{2.14} \]

Let \( |\lambda| < |\lambda_1(|q|)| \). Using (2.13) we have

\[ (H_0(q) - \lambda)^{-1} = H_0^{-1}(q) + H_0^{-1}(q)\frac{\lambda^2}{\lambda^2(|q|) - \lambda^2} + \frac{\lambda}{\lambda^2(|q|) - \lambda^2}. \tag{2.15} \]
In view of analyticity of both parts of equality (2.15) this equality is valid at all \( \lambda \notin E \), where \( E = (-\infty, -m] \cup [m, +\infty) \).

**Corollary 2.1** The operator \( L_0 \) has no eigenvalues in the interval \((-m, m)\).

3. Now we will construct a relativistic analogue of the Lippmann-Schwinger equation (RLS integral equation).

To do it we consider the expression

\[
B_{\pm}(r, \lambda) = F^{-1}[H_0(q) - (\lambda \pm i0)]^{-1},
\]

where \( \lambda = \frac{\lambda}{|\lambda|} \), \(|\lambda| > m\). Let us write the following relation (see [4], formula 721).

\[
J_1(r) = F^{-1}[m^2 + |q|^2]^{-1} = m^{1/2}K_{1/2}(m|r|)/|r|^{1/2},
\]

where \( K_p(z) \) is the modified Bessel function. It is known that (see [2])

\[
K_{1/2}(z) = K_{-1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}.
\]

According to (2.17) and (2.18) the equality

\[
J_1(r) = \sqrt{\pi/2}(e^{-m|r|/|r|})
\]

is valid. Using (2.19) we obtain

\[
J_2(r) = F^{-1}\left(\frac{q_k}{m^2 + |q|^2}\right) = i \frac{\partial}{\partial k} J_1(r) = -\sqrt{\pi/2}e^{-m|r|/|r|} \frac{r_k}{|r|^2} (m + 1/|r|).
\]

Let us calculate the expression

\[
J_{\pm}(r, \lambda) = F^{-1}\left[\frac{1}{\lambda^2(q) - (\lambda \pm i0)^2}\right], \quad \lambda \in E.
\]

In view of (2.10) and (2.21) we have

\[
J_{\pm}(r, \lambda) = -F\frac{1}{m_1^2 - |q|^2 \mp i(sgn \lambda)0}, \quad \lambda \in E.
\]

where \( m_1^2 = \lambda^2 - m^2 \), \( m_1 > 0 \). Taking into account formulas 7.19 and 7.20 from the book [4] (table of the Fourier transformation) and relation (2.18) we obtain the equalities

\[
J_{\pm}(r, \lambda) = \sqrt{\frac{\pi}{2}} e^{\pm im_1|r|/|r|}, \quad \lambda > m,
\]
\[ J_{\pm}(r, \lambda) = \sqrt{\frac{\pi}{2}} e^{\pm im|r|/|r|}, \quad \lambda < -m. \] (2.24)

Formulas (2.14), (2.19) and (2.20) imply that
\[ Q(r) = F^{-1}H^{-1}_{0}(q) = \sqrt{\pi/2} e^{-m|r|/|r|} \left[ m\beta - (m + 1/|r|)ra/|r||/|r|. \right] \] (2.25)

Here \( r\alpha = r_{1}\alpha_{1} + r_{2}\alpha_{2} + r_{3}\alpha_{3} \), matrices \( \alpha_k \) and \( \beta \) are defined by the relations (2.3)-(2.5). Due to (2.15), (2.16), (2.20) and (2.25) we have
\[ B_{\pm}(r, \lambda) = Q(r) + (2\pi)^{3/2} \lambda^{2} Q(r) \ast J_{\pm}(r, \lambda) + \lambda J_{\pm}(r, \lambda), \] (2.26)

where \( F(r) \ast G(r) = \int_{R^3} F(r-v)G(v)dv \) is the convolution of \( F(r) \) and \( G(r) \).

Now we can write the equation
\[ \phi_{\pm}(r, k, n) = e^{ikr} g_{n}(k) - (2\pi)^{-3/2} \int_{R^3} B_{\pm}(r-s, \lambda)V(s)\phi_{\pm}(s, k, n)ds, \] (2.27)

where
\[ V(r) = -e\nu(r)I_{4} + e\alpha A(r). \] (2.28)

Here the vectors \( g_{n}(k) \) are defined by the relations (2.11) and (2.12).

Equation (2.27) (RLS equation) is relativistic analogue of the Lippmann-Schwinger equation.

We note that the Lippmann-Schwinger equation play an important role in the non-relativistic scattering theory (see [6]). Our aim is to show that the constructed RLS integral equation can be effective by solving relativistic scattering problems.

3 Properties of the RLS integral equation

1. Further we assume that the matrix \( V(r) \) is self-adjoint,
\[ V(r) = V^{\ast}(r). \] (3.1)

Hence \( V(r) \) can be represented in the form
\[ V(r) = U(r)D(r)U^{\ast}(r), \] (3.2)

where \( U(r) \) is an unitary matrix, \( D(r) \) is a diagonal matrix
\[ D(r) = \text{diag}(d_{1}(r), d_{2}(r), d_{3}(r), d_{4}(r)). \] (3.3)
Let us introduce the diagonal matrices

$$D_1(r) = \text{diag}(|d_1(r)|^{1/2}, |d_2(r)|^{1/2}, |d_3(r)|^{1/2}, |d_4(r)|^{1/2})$$

(3.4)

and

$$W(r) = \text{diag}(\text{sign}d_1(r), \text{sign}d_2(r), \text{sign}d_3(r), \text{sign}d_4(r)).$$

(3.5)

Formulas (3.2)-(3.5) imply that

$$V(r) = V_1(r)W_1(r)V_1(r),$$

(3.6)

where

$$V_1(r) = U(r)D_1(r)U^*(r), \quad W_1(r) = U(r)W(r)U^*(r)$$

(3.7)

It is easy to see that

$$\|V_1(r)\|^2 = \|V(r)\|, \quad \|W_1(r)\| = 1.$$  

(3.8)

2. Modified RLS integral equation.

If $\phi_\pm (r, k, n)$ is a solution of RLS equation, then the vector-function $\psi_\pm (r, k, n) = V_1(r)\phi_\pm (r, k, n)$ is a solution of following modified RLS integral equation:

$$\psi_\pm (r, k, n) = e^{ikr}V_1(r)g_n(k) - (2\pi)^{-3/2}B_\pm (\lambda)\psi_\pm (r, k, n),$$

(3.9)

where

$$B_\pm (\lambda)f = \int_{\mathbb{R}^3} V_1(r)B_\pm (r - s, \lambda)V_1(s)f(s)ds.$$  

(3.10)

We note that the operators $B_\pm (\lambda)$ act in the Hilbert space $L_4^2(\mathbb{R}^3)$ of $4 \times 1$ vector functions.

**Theorem 3.1** Let condition (3.1) be fulfilled, the function $\|V(r)\|$ be bounded and belong to the space $L^1(\mathbb{R}^3)$. Then the operators $B_\pm (\lambda)$ are compact.

**Proof.** We represent the operators $B_\pm (\lambda)$ in the form

$$B_\pm (\lambda) = \sum_{m=1}^{3} B_\pm (m, \lambda),$$

(3.11)

where

$$B_\pm (m, \lambda)f = \int_{\mathbb{R}^3} V_1(r)B_\pm (r - s, m, \lambda)V_1(s)f(s)ds.$$  

(3.12)
Here the $4 \times 4$ matrix functions $B_\pm(r, m, \lambda)$ are defined by the relations

\begin{align}
B_\pm(r, 1, \lambda) &= \lambda J_\pm(r, \lambda), \quad (3.13) \\
B_\pm(r, 2) &= Q(r) \quad (3.14) \\
B_\pm(r, 3, \lambda) &= (2\pi)^{3/2}\lambda^2 Q(r) * J_\pm(r, \lambda). \quad (3.15)
\end{align}

Formulas (2.23), (2.24) and (3.13) imply that

\[ \|B_\pm(r, 1, \lambda)\| \leq C(\lambda)/|r|. \quad (3.16) \]

According to condition of the theorem the function $\|V(r)\|$ belongs to the Rolnik class (see [6], i.e.

\[ \int_{R^3} \int_{R^3} \frac{\|V(r)\|\|V(s)\|}{|r-s|^2} dsdr < \infty. \quad (3.17) \]

It follows from (3.12), (3.16) and (3.17) that the operator $B_\pm(1, \lambda)$ belongs to the Hilbert-Schmidt class. Hence operator $B_\pm(1, \lambda)$ is compact.

Let us consider the operator $B_\pm(2)$. In view of (2.25) we have

\[ C_1 = \int_{R^3} \|Q(r)\| dr < \infty. \quad (3.18) \]

Hence the operator $B_\pm(2)$ is bounded (see [8], section 1.4) and

\[ \|B_\pm(2)\| \leq MC_1, \quad M = \sup\|V(r)\|. \quad (3.19) \]

We represent the kernel $B_\pm(r, 2)$ in the form $B_\pm(r, 2) = B_\pm(r, 2, 1) + B_\pm(r, 2, 2)$ where

\begin{align}
B_\pm(r, 2, 1) &= Q(r), \quad 0 < r < \epsilon, \quad B_\pm(r, 2, 1) = 0, \quad r > \epsilon, \quad (3.20) \\
B_\pm(r, 2, 2) &= 0, \quad 0 < r < \epsilon, \quad B_\pm(r, 2, 2) = Q(r), \quad r > \epsilon. \quad (3.21)
\end{align}

We introduce the operators

\[ B_\pm(2, m)f = \int_{R^3} V_1(r) B_\pm(r-s, 2, m)V_1(s)f(s)ds, \quad m = 1, 2. \quad (3.22) \]

It is easy to see, that the operator $B_\pm(2, 2)$ belongs to the Hilbert= Schmidt class and

\[ \|B_\pm(2) - B_\pm(2, 2)\| = \|B_\pm(2, 1)\| \leq M \int_{0}^{\epsilon} \|Q(r)\| dr. \quad (3.23) \]
The norm $\|B_\pm(2,1)\|$ tends to zero when $\epsilon \to 0$. Hence, it follows from (3.23) that the operator $B_\pm(2)$ is compact.

To consider the operator $B_\pm(3, \lambda)$ we use the inequality

$$\|B_\pm(r, \lambda, 3)\| \leq C(\lambda) \frac{e^{-m|r|}}{|r|} \left( \frac{1}{|r|} + \frac{1}{|r|^2} \right).$$  \hspace{1cm} (3.24)$$

It follows from (3.24) and Adams theorem (see Appendix, Examples 6.2 and 6.3), that

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \|V(r)\| \|B_\pm(r-s, 3, \lambda)\|^2 \|V(s)\| ds \, dr < \infty. \hspace{1cm} (3.25)$$

According to (3.25) the operator $B_\pm(3, \lambda)$ belongs to the Hilbert-Schmidt class. Thus, all the operators $B_\pm(m, \lambda), (m = 1, 2, 3)$ are compact. The theorem is proved.

4 Wave and scattering operators, dynamical case

We introduce the operator function

$$\Theta(t) = \exp(it\mathcal{L}) \exp(-it\mathcal{L}_0).$$  \hspace{1cm} (4.1)$$

The wave operators $W_\pm(\mathcal{L}, \mathcal{L}_0)$ are defined by the relation (see [6]).

$$W_\pm(\mathcal{L}, \mathcal{L}_0) = \lim_{t \to \pm \infty} \Theta(t) P_0.$$  \hspace{1cm} (4.2)$$

Here $P_0$ is orthogonal projector on the absolutely continuous subspace $G_0$ with respect to the operator $\mathcal{L}_0$. The limit in (4.2) supposed to be in the sense of strong convergence.

**Theorem 4.1** If $V(r) = V^*(r)$, the function $\|V(r)\|$ is bounded, belongs to the space $L^1(\mathbb{R}^3)$ and

$$\int_{-\infty}^{+\infty} \int_{|r|>|t|\epsilon} \|V(r)\|^2 \, dr \, dt < \infty, \quad \epsilon > 0,$$  \hspace{1cm} (4.3)$$

then the wave operators $W_\pm(\mathcal{L}, \mathcal{L}_0)$ exist.
Proof. We use the equality
\[
\Theta(t) - I = \int_0^t \frac{d}{dt} \Theta(t) dt. \tag{4.4}
\]
Thus, to prove the formulated theorem, it is sufficient to show that the following inequality holds
\[
\int_{-\infty}^{\infty} \| \frac{d\Theta(t)}{dt} \Psi \| dt < \infty \tag{4.5}
\]
on a set \( S \) vector functions \( \Psi \) dense in \( L^2_4(R^3) \). We consider the vector functions
\[
\tilde{\Psi}(p) = \frac{1}{(2\pi)^{3/2}} \int_{R^3} V(r) \exp(-itH_0(p)) e^{ipr} \Psi(r) dr. \tag{4.6}
\]
We take such a set \( S \) of vector functions \( \Psi(r) \) that the corresponding vector functions \( \tilde{\Psi}(p) \) belong to the class \( C^\infty \) and
\[
supp \tilde{\Psi}(p) \subseteq \{ \| p \| : 0 < c_1(\Psi) < \| p \| < c_2(\Psi) < \infty \}. \tag{4.7}
\]
Let us consider the case, when \( t > 0 \). We have
\[
\| \frac{d\Theta(t)}{dt} \Psi \| = \| J(r, t) \|. \tag{4.8}
\]
where \( \Psi(r) \) belongs to the set \( S \) and
\[
J(r, t) = (2\pi)^{-3/2} \int_{R^3} V(r) \exp(-itH_0(p)) e^{ipr} \tilde{\Psi}(p) dp. \tag{4.9}
\]
Using (4.9) we obtain
\[
\| \frac{d\Theta(t)}{dt} \Psi(r) \| \leq (\| J_1(r, t) \|)^{1/2} + (\| J_2(r, t) \|)^{1/2} \tag{4.10}
\]
where
\[
J_k(r, t) = \int_{R^3} V(r) \exp[itF_k(p, r, t)] \Psi_k(p) dp, \quad k = 1, 2. \tag{4.11}
\]
Here
\[
F_k(p, r, t) = pr/t - \mu_k, \quad \mu_1 = -\mu_2 = \sqrt{|p|^2 + m^2}. \tag{4.12}
\]
We note that $\mu_1(p)$ and $\mu_2(p)$ are the eigenvalues of the matrix $H_0(p)$, the vectors $\tilde{\Psi}_k(p)$ are the corresponding eigenvectors. The stationary-phase points $p_k(r,t)$ are the solutions of the equations

$$\frac{\partial}{\partial p_s} F_k(p,r,t) = 0, \quad s = 1, 2, 3.$$  \hfill (4.13)

Thus, we have

$$r - tp/\mu_k = 0.$$ \hfill (4.14)

When $\epsilon > 0$ is small the stationary-phase points do not belong to the region $\|r\| \leq t\epsilon$. Hence integrating by parts the right side of (4.11) we have (see [5]):

$$\int_{\|r\| \leq t\epsilon} (\|J_k(r,t)\|)^2 dr \leq ct^{-4}, \quad k = 1, 2.$$ \hfill (4.15)

We use here the relation $\|V(r)\|^2 \in L(R^3)$.

Now let us consider the case when $\|r\| \geq t\epsilon$. It follows from (4.11) that

$$\|J_k(r,t)\| \leq c\|V(r)\|.$$ \hfill (4.16)

Taking into account the condition (4.3) of the theorem we obtain

$$\int_{-\infty}^{+\infty} \left[ \int_{\|r\| \geq t\epsilon} \|J_k(r,t)\|^2 dr \right]^{1/2} dt < \infty$$ \hfill (4.17)

The relations (4.10), (4.15) and (4.17) imply inequality (4.5). The theorem is proved.

**Remark 4.2** Let condition (3.1) be fulfilled. If the function $\|V(r)\|$ is bounded and

$$\|V(r)\| \leq \frac{M}{|r|^\alpha}, \quad |r| \geq \delta > 0, \quad \alpha > 3,$$ \hfill (4.18)

then the wave operators $W_{\pm}(\mathcal{L}, \mathcal{L}_0)$ exist.

**Definition 4.3** The scattering operator $S_{\pm}(\mathcal{L}, \mathcal{L}_0)$ is defined by the relation

$$S(\mathcal{L}, \mathcal{L}_0) = W_{-}^*(\mathcal{L}, \mathcal{L}_0)W_{+}(\mathcal{L}, \mathcal{L}_0).$$ \hfill (4.19)
5 Stationary scattering problem

1. In section 4 we considered the dynamical scattering problem ($t \to \infty$) for Dirac equation (2.1), (2.2). In the present section we shall investigate the stationary scattering problem for the same equation. It means that we shall investigate asymptotic behavior of $\psi_+(r, k, n)$ when $|r| \to \infty$.

(The case $\psi_-(r, k, n)$ can be investigated similarly).

Definition 5.1 We say that $\lambda \in E$ is an exceptional value if the equation

\[ I + (2\pi)^{-3/2}B_+(\lambda)\psi = 0 \]

has nontrivial solution in the space $L^2(\mathbb{R}^3)$.

We denote by $E_+$ the set of exceptional points and we denote by $E_+^*$ the set of such points $\lambda$ that $\lambda \in E, \lambda \notin E_+$.

Using Theorem 3.1 and the Fredholm alternative we obtain

Lemma 5.2 Let conditions of Theorem 3.1 be fulfilled. If $\lambda \in E_+$, then equation (3.9) has one and only one solution $\psi_+(r, k, n)$ in $L^2(\mathbb{R}^3)$.

Corollary 5.3 Let conditions of Theorem 3.1 be fulfilled. If $\lambda \in E_+$, then equation (2.27) has one and only one solution $\phi_+(r, k, n)$ which satisfies the condition $V_1(r)\phi_+(r, k, n) \in L^2(\mathbb{R}^3)$.

It can be proved (see [6], XI, III) that

Lemma 5.4 Let conditions of Theorem 3.1 be fulfilled. The set $E_+$ is closed and has Lebesgue measure equal to zero.

2. Let us consider the case $V(r) = 0$ separately.

Taking into account (2.26) we see that $B(r, \mu)$ is defined when $\Im \mu > 0$:

\[ B_+(r, \mu) = Q(r) + (2\pi)^{3/2} \mu^2 Q(r) * J_+(r, \mu) + \lambda J_+(r, \mu), \quad (5.1) \]

where

\[ J_+(r, \mu) = \sqrt{\frac{\pi}{2}} e^{im_1|r|}/|r|, \quad m_1(\mu) = \sqrt{\mu^2 - m^2}, \quad \Im \mu > 0. \quad (5.2) \]

We assume that $\Im m_1(\mu) > 0$. We introduce the operator

\[ B_0(\mu)f = \int_{\mathbb{R}^3} B_+(r - s, \mu)f(s)ds, \quad (5.3) \]
where $f(r) \in L^2(R^3)$. It is easy to see that the operator $B_0(\mu)$ is bounded in the space $L^2(R^3)$. It follows from (5.3) that

$$F[B_0(\mu)f] = (2\pi)^{3/2}F[B_0(\mu)]F(f) \quad (5.4)$$

We represent the equality (5.4) in the form

$$B_0(\mu)f = (2\pi)^{3/2}F^{-1}\{F[B_0(\mu)]F\}F^{-1}[F(f)] \quad (5.5)$$

Relation (2.2) and (2.16) imply:

$$F^{-1}\{F[B_0(\mu)]F\} = (L_0 - \mu)^{-1}. \quad (5.6)$$

It follows from (5.4) and (5.6) that

$$(L_0 - \mu)^{-1}f = (2\pi)^{-3/2}\int_{R^3} B_+(r - s, \mu)f(s)ds, \quad \Im \mu > 0. \quad (5.7)$$

3. Now we prove the main result of this section.

**Theorem 5.5** Let the $V(r) = V^*(r)$ and the function $\|V(r)\|$ be bounded and belong to the space $L^1(R^3)$. If

$$\|V_1(r)\| = O(|r|^{-3/2}), \quad |r| \to \infty, \quad (5.8)$$

then the solution $\phi_+(r, k, n)$ of RLS equation (2.27) has the form

$$\phi_+(r, k, n) = e^{ikr}g_n(k) + \frac{e^{im_1(k)|r|}}{|r|}f(\omega, k, n) + o(1/|r|), \quad |r| \to \infty, \quad (5.9)$$

where $\lambda \in E_+, \ |k|^2 = \lambda^2 - m^2, \ \omega = r/|r|$ and

$$f(\omega, k, n) = -\frac{1}{4\pi} \int_{R^3} e^{-im_1(k)s}\omega V(s)\phi_+(s, k, n)ds. \quad (5.10)$$

**Proof.** The equation (3.9) has one and only one solution $\psi_+(r, k, n)$ in the space $L^2(R^3)$. We shall estimate the integral

$$J = \int R^3 B_+(r - s, \lambda)V_1(s)W_1(s)\psi_+(s, k, n)ds. \quad (5.11)$$

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We divide the space $\mathbb{R}^3$ with respect to $s$ into three parts:
\[ M_1 = \{|s| \leq |r|/2\}, \quad M_2 = \{|r|/2 \leq |s| \leq 3|r|/2\}, \quad M_3 = \{|s| \geq 3|r|/2\}. \]
Let us introduce the integrals
\[
J_{p,m} = \int_{M_p} B_+(r - s, \lambda, m)V_1(s)W_1(s)\psi_+(s, k, n)ds, \quad (5.12)
\]
where $1 \leq p \leq 3$, $1 \leq m \leq 3$. Taking into account relations (2.23)-(2.25) and (3.13), (3.14) we have
\[
\|B_+(r, \lambda, m)\| = O(1/|r|), \quad |r| \to \infty, \quad m = 1, 2. \quad (5.13)
\]
Then the following relations
\[
|J_{1,1}| + |J_{1,2}| = O(1/|r|), \quad |J_{3,1}| + |J_{3,2}| = o(1/|r|), \quad |r| \to \infty \quad (5.14)
\]
are valid. We have used here the inequality
\[
\int_{\mathbb{R}^3} \|\psi_+(s, k, n)\|^2 ds < \infty. \quad (5.15)
\]
Condition (5.8) of the Theorem 5.5 and relations (5.11), (5.15) imply
\[
(|J_{2,1}| + |J_{2,2}|) = o(|r|^{-3/2})(\int_{M_2} \frac{ds}{|r - s|^2})^{1/2} = o(|r|^{-1}), \quad |r| \to \infty. \quad (5.16)
\]
Using (3.15) and spaces $M_p$ we obtain
\[
\|B_+(r, \lambda, 3)\| = O(1/|r|), \quad |r| \to \infty. \quad (5.17)
\]
Now in the same way as in cases $m=1$ and $m=2$ we receive:
\[
|J_{1,3}| = O(1/|r|), \quad |J_{2,3}| + |J_{3,3}| = o(1/|r|), \quad |r| \to \infty \quad (5.18)
\]
It is easy to see that
\[
|J_{1,2}| + |J_{1,3}| = o(1/|r|), \quad |r| \to \infty \quad (5.19)
\]
So, we have proved the relation
\[
J(r, \lambda, n)| = J_{1,1}| + o(1/|r|), \quad |r| \to \infty \quad (5.20)
\]
We note that $|r - s|^2 = |r|^2 - 2rs + |s|^2$. Hence we have
\[
|r - s| \sim |r| - s\omega, \quad \omega = r/|r|. \quad (5.21)
\]
The assertion of the Theorem 5.5 follows directly from (5.20) and (5.21).
Definition 5.6 The $4 \times 1$ vector function $f(\omega, k, n)$ we name the relativistic scattering amplitude.

We note that relativistic scattering amplitude is defined by formulas (5.9) and (5.10) which are similar to the corresponding formulas for non-relativistic scattering amplitude (see [6]).

4. Now we shall investigate the connection between solutions of the equation

$$L\phi = \lambda \phi$$

and the solutions of the RLS equation (2.27).

Theorem 5.7 Let the vector function $\phi(r)$ satisfies the equation (5.22) and the following conditions are fulfilled

$$\psi = V_1 \phi \in L^2_4(R^3) \text{ and } \kappa_Q \phi \in L^2_4(R^3),$$

where $\lambda \in E$ and $\kappa_Q$ is the characteristic function of a bounded domain $Q$.

If the matrix function $V(r)$ satisfies the conditions of the Theorem 3.1, then the vector function $\psi(r)$ satisfies the equation

$$\psi(r) = -(2\pi)^{-3/2} B_+ (\lambda) \psi(r), \quad \lambda \in E. \quad (5.23)$$

Proof. We use the equality

$$z(\lambda+i\epsilon)\phi = V_1(\lambda+i\epsilon-L_0)^{-1}(\lambda+i\epsilon-L)\phi = [I+(2\pi)^{-3/2} B_+ (\lambda+i\epsilon)]\psi, \quad (5.24)$$

where $\epsilon > 0$. Taking into account (5.7) and (5.22), we obtain:

$$\kappa_Q z(\lambda+i\epsilon)\phi = i\epsilon(\kappa_Q V_1)(\lambda+i\epsilon-L_0)^{-1}\phi = i\epsilon(\kappa_Q V_1)B_0(\lambda+i\epsilon)\phi \quad (5.25)$$

According to (5.1)-(5.3) the operator $(\kappa_Q V_1)B_0(\lambda+i\epsilon)$ belongs to the Hilbert-Schmidt class with norm

$$\|(\kappa_Q V_1)B_0(\lambda+i\epsilon)\| = O(\epsilon^{-1/2}), \quad \epsilon \to 0. \quad (5.26)$$

We have

$$\kappa_Q z(\lambda+i\epsilon)\phi \to 0, \quad \epsilon \to 0. \quad (5.27)$$

The assertion of the Theorem 5.7 follows directly from (5.24) and (5.27).

Corollary 5.8 Let the conditions of the Theorem 5.7 be fulfilled. If $\lambda \in E$ is eigenvalue of the corresponding operator $L$, then $\lambda \in E$. 

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**Theorem 5.9** Let the conditions of Theorem 5.5 be fulfilled and let the function \(\phi_+(r, k, n)\) be a solution of RLS equation (2.27) such that \(V_1(r)\phi_+(r, k, n)\in L^2((R^3)\). If \(\lambda\in E_+\) then the function \(\phi_+(r, k, n)\) is the solution of the equation (5.22) in the distributive sense.

**Proof.** We consider the expression
\[
(L_0 - \lambda)(L_0 - \lambda - i\epsilon)^{-1}V\phi_+ = V\phi_+ + i\epsilon(L_0 - \lambda - i\epsilon)^{-1}V\phi_+. \tag{5.28}
\]
In the same way as (5.27) we prove that
\[
\kappa_Q\epsilon \| (L_0 - \lambda - i\epsilon)^{-1}V_1 \| \to 0, \quad \epsilon \to 0. \tag{5.29}
\]
Using relation (2.27) we have
\[
(L_0 - \lambda - i\epsilon)^{-1}V\phi_+ = -\phi_+ + o(1), \quad \epsilon \to 0. \tag{5.30}
\]
Now we introduce the class of functions \(f(r)\) such that \(f(r)\in C_\infty\) and \(f(r) = 0\) when \(r\notin Q\). Taking into account (5.29) we obtain
\[
\lim_{\epsilon \to 0}((L_0 - \lambda)(L_0 - \lambda - i\epsilon)^{-1}V\phi_+, f) = (V\phi_+, f) = (\phi_+, Vf). \tag{5.31}
\]
According to (5.30) the equality
\[
\lim_{\epsilon \to 0}((L_0 - \lambda)(L_0 - \lambda - i\epsilon)^{-1}V\phi_+, f) = -(\phi_+, (L_0 - \lambda)f) \tag{5.32}
\]
holds. It follows from (5.31) and (5.32) that
\[
(\phi_+, Lf) = 0. \tag{5.33}
\]
Relation (5.33) implies the assertion of the Theorem 5.9.

### 6 Appendix

1. In this section we will consider the integral
\[
U(r) = \int_{R^3} \frac{f(s)}{|r - s|^\lambda}ds. \tag{6.1}
\]
Let us formulate a partial case of Adams theorem (see [10]).
Theorem 6.1 We assume that $1 < p < q$, $\lambda > 3/p'$, where $1/p + 1/p' = 1$. If the condition

$$\sup_{\rho > 0} \left( \rho^{3/p'-\lambda+3/q} \right) < \infty$$

(6.2)

is fulfilled, then

$$\|U\|_{L^q} = \left[ \int_{\mathbb{R}^3} |U(r)|^q dr \right]^{1/q} \leq c \|f\|_{L^p},$$

(6.3)

where the constant $c$ is independent of $f$.

Example 6.2 We consider the case when $\lambda = 1$, $p' > 3$.

According to Theorem 6.1 we have

$$1 < p = \frac{p'}{p' - 1} < \frac{3}{2}, \quad q = \frac{3p'}{p' - 3} > 3 > p, \quad U(r) \in L^q(\mathbb{R}^3).$$

(6.4)

Example 6.3 We consider the case when $\lambda = 2$, $3 < p' < 6$.

According to Theorem 6.1 we have

$$1 < p = \frac{p'}{p' - 1} < \frac{3}{2}, \quad q = \frac{3p'}{2p' - 3} > 2 > p, \quad U(r) \in L^q(\mathbb{R}^3).$$

(6.5)

2. Using Examples 6.2 and 6.3 we will prove the inequality (3.25). To do it we introduce the functions

$$U_k(r) = \frac{e^{-m|r|}}{|r|} \ast \frac{1}{|r|^k}, \quad k = 1, 2.$$  

(6.6)

We have

$$\frac{e^{-m|r|}}{|r|} \in L^p(\mathbb{R}^3), \quad 1 < p < 3/2.$$  

(6.7)

Hence, Example 6.2 implies that

$$U_1(r) \in L^q(\mathbb{R}^3), \quad U_1^2(r) \in L^{q/2}(\mathbb{R}^3), \quad q > 3.$$  

(6.8)

It follows from the conditions of Theorem 3.1 that $\|V(r)\|$ belongs to $L^p(\mathbb{R}^3)$ for all $1 \leq p < \infty$. Then we obtain

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \|V(r)\| |U_1(r - s)|^2 \|V(s)\| ds dr < \infty.$$  

(6.9)
Now we consider the function $U_2(r)$. We have

$$e^{-m|r|}/|r|^2 \in L_p(R^3), \quad 1 < p < 3/2.$$  \hspace{0.5cm} (6.10)

Hence, Example 6.3 implies that

$$U_2(r) \in L_q(R^3), \quad U_2^2(r) \in L_{q/2}(R^3), \quad q > 2.$$ \hspace{0.5cm} (6.11)

Then we obtain

$$\int_{R^3} \int_{R^3} ||V(r)||||U_2(r - s)||^2 ||V(s)|| ds dr < \infty.$$ \hspace{0.5cm} (6.12)

The inequality (3.25) follows directly from (6.9) and (6.12).

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