THE EFFECTIVE BOSONIC HAMILTONIAN FOR EXCITONS RECONSIDERED

M. Combescot and O. Betbeder-Matibet

Groupe de Physique des Solides, Université Denis Diderot et Université Pierre et Marie Curie,
CNRS, Tour 23, 2 place Jussieu, 75251 Paris Cedex 05, France

Abstract

The effective bosonic hamiltonian for excitons, extensively quoted up to now, cannot be correct because it is (surprisingly) non-hermitian. The oversight physically originates from the intrinsic difficulty of properly defining electron-hole interactions between excitons when dealing with exchange terms. By using our commutation technique, we show that the fermionic character of the excitons cannot be forced into a dressed Coulomb interaction only: The effective bosonic hamiltonian must contain purely fermionic terms of the same order as the Coulomb terms. They are necessary to ensure hermiticity, and they do not reduce to a two-body interaction, Pauli exclusion being N-body by essence.
In the very low density and temperature regime, the electron-hole (e-h) pairs of a semiconductor are bound into hydrogen-like states known as excitons. In this regime, it is widely accepted [1–6] that the semiconductor can be represented by an effective hamiltonian,

\[ H_{\text{eff}} = \sum_i E_i B^+_i B_i + \frac{1}{2} \sum_{ijmn} E_{mnij} B^+_m B^+_n B_i B_j, \]

in which the excitons are assumed to be real bosons: \( B^+_i \) is the \( i \) boson-exciton creation operator, \([B_i, B^+_j] = \delta_{ij}\), \( i \) standing for \((\nu_i, q_i)\), where \( \nu_i \) characterizes the exciton level and \( q_i \) its center of mass momentum, the exciton energy being \( E_i = \varepsilon_{\nu_i} + E_{q_i} \). The second term of Eq. (1) comes from interactions between excitons. The \( X-X \) scattering \( E_{mnij} \), reported up to now, is a sum of a direct and an exchange Coulomb term. This exchange term results from the composite nature of the excitons. There are in fact two ways of forming two excitons with two electrons \((e_1, e_2)\) and two holes \((h_1, h_2)\), depending on how the e and h are coupled. This quite harmless evidence generates major difficulties when dealing with e-h Coulomb interactions. Indeed, \( V(\mathbf{r}_{e_1} - \mathbf{r}_{h_2}) = V_{e_1h_2} \) is an interaction between two excitons if they are formed with \((e_1, h_1)\) and \((e_2, h_2)\), while it is an interaction inside one of them if they are formed with \((e_1, h_2)\) and \((e_2, h_1)\). The concept of e-h interaction between excitons is thus ambiguous for exchange processes. One of the problems of the effective hamiltonian used up to now comes from this difficulty.

In this letter, we first reconsider the widely accepted exciton effective hamiltonian, and point out its (surprising) non hermiticity. In a second part, we recall the commutation technique introduced in a previous problem [7,8] dealing with interacting excitons, namely the exciton optical Stark shift, and we use it to derive a new \( X-X \) scattering. It contains a purely fermionic contribution, necessary to ensure hermiticity, which is conceptually quite different from a Coulomb interaction dressed by fermionic effects. We also find that for \( N > 2 \) excitons, the effective bosonic hamiltonian must contain \((3,...,N)\)-body scatterings of the same order as the Coulomb terms. They have a purely fermionic origin: While Coulomb interaction is basically 2-body, Pauli exclusion between the electrons of the \( N \) excitons is intrinsically \( N \)-body, so that one cannot get rid of it by 2-body operators only.
1. The former result. The former \( E_{mnij} \) reads [1]

\[
E_{mnij} = \int de_1 de_2 dh_1 dh_2 \phi_m^*(e_1, h_1) \phi_n^*(e_2, h_2) (V_{e_1e_2} + V_{h_1h_2} - V_{e_1h_2} - V_{e_2h_1}) \\
\times [\phi_i(e_1, h_1) \phi_j(e_2, h_2) - \phi_i(e_1, h_2) \phi_j(e_2, h_1)],
\]

(2)

where \( e \) stands for \( r \) and \( \phi_i(e, h) \) is the wave function of the \( i \) exciton. The first term of the bracket gives the direct part \( E_{mnij}^{\text{dir}} \) of the \( X-X \) interaction, while the usual exchange part \( E_{mnij}^{\text{exc}} \) comes from the second term. Although widely quoted, this \( X-X \) scattering cannot be correct: Indeed, as \( E_{mnij} \neq E_{ijmn}^{\ast} \), the corresponding effective hamiltonian is non-hermitian. Strangely enough, this alarming point seems to have stayed unnoticed up to now. The problem comes from the e-h interactions in the exchange part. These e-h interactions a priori contain four terms, \( V_{e_1h_1} + V_{e_2h_2} + V_{e_1h_2} + V_{e_2h_1} \). \( V_{e_1h_1} \) (resp. \( V_{e_2h_2} \)) has clearly to be dropped because it is a Coulomb interaction inside the \( m \) (resp. \( n \)) exciton. However, on this basis, \( V_{e_1h_2} \) (resp. \( V_{e_2h_1} \)) should be dropped from the exchange term because it is a Coulomb interaction inside the \( i \) (resp. \( j \)) “exchange” exciton. The correct \( X-X \) exchange scattering should thus contain either no e-h interaction at all, or possibly all them four:

There is no reason to keep two of them only [9].

2. The commutation technique. The e-h interaction between excitons is in fact quite subtle, as it is impossible to split the e-h terms of the bare Coulomb interaction operator into a part which binds the exciton (through repeated e-h interactions) and a part which makes the excitons to interact.

In one of our previous works, we already faced such a difficulty. We overcame it by introducing \([7,8]\) the operator \( V_i^+ \) defined as \([H_{sc}, B_i^+] = E_i B_i^+ + V_i^+\), where \( H_{sc} \) is the exact semiconductor hamiltonian, and \( B_i^+ \) the exact exciton creation operator given in the appendix (Eq. (A1)). If the excitons were non-interacting, we would have \([H_{sc}, B_i^+] = E_i B_i^+\) only, so that \( V_i^+ \) does come from interaction between excitons. From its explicit value given in Eq. (A2), we get

\[
[[H_{sc}, B_i^+], B_j^+] = [V_i^+, B_j^+] = \sum_{mn} E_{mnij}^{\text{dir}} B_m^+ B_n^+,
\]

(3)
where \( \xi_{mnij}^{dir} = (\xi_{ijmn}^{dir})^* \) is just \( \mathcal{E}_{mnij}^{dir} \) properly symmetrized (see Eq. (A8)).

Besides \( V_i^{+} \), there is another “interaction” between excitons which plays a crucial part. It comes from the fermionic character of the exciton which appears through the boson-departure operator \( D_{ij} = \delta_{ij} - [B_i, B_j^{+}] \). When acting on \( B_j^{+} \), this operator gives

\[
[D_{mi}, B_j^{+}] = -[[B_m, B_i^{+}], B_j^{+}] = 2 \sum_n \lambda_{mnij} B_n^{+},
\]

(4)

with \( \lambda_{mnij} = \lambda_{ijmn}^{*} \) given in Eq. (A8). \( \lambda_{mnij} \) corresponds to cross the e (or the h) of two excitons, so that it relates two excitons with e and h bound in a different way:

\[
B_i^{+} B_j^{+} = -\sum_{mn} \lambda_{mnij} B_m^{+} B_n^{+}.
\]

(5)

Equations (3, 4) are the key equations of our commutation technique. They allow to calculate any quantity dealing with interacting excitons.

3. The two exchange terms. From them we get

\[
H_{sc} B_i^{+} B_j^{+}|0\rangle = (E_i + E_j) B_i^{+} B_j^{+}|0\rangle + \sum_{mn} \xi_{mnij}^{dir} B_m^{+} B_n^{+}|0\rangle,
\]

(6)

\[
\langle 0|B_mB_n B_i^{+} B_j^{+}|0\rangle = \delta_{mi} \delta_{nj} + \delta_{mj} \delta_{ni} - 2 \lambda_{mnij}.
\]

(7)

This gives for the matrix elements of the exact hamiltonian \( H_{sc} \), calculated with \( H_{sc} \) acting on the right or on the left,

\[
\langle 0|B_mB_n H_{sc} B_i^{+} B_j^{+}|0\rangle = (E_i + E_j) (\delta_{mi} \delta_{nj} + \delta_{mj} \delta_{ni} - 2 \lambda_{mnij}) + 2 (\xi_{mnij}^{dir} - \sum_{rs} \lambda_{mnrs} \xi_{rsij}^{dir})
\]

(8)

\[
= (E_m + E_n) (\delta_{mi} \delta_{nj} + \delta_{mj} \delta_{ni} - 2 \lambda_{mnij}) + 2 (\xi_{mnij}^{dir} - \sum_{rs} \xi_{mnrs}^{dir} \lambda_{rsij}).
\]

(9)

Two Coulomb exchange terms thus appear,

\[
\xi_{mnij}^{left} = \sum_{rs} \lambda_{mnrs} \xi_{rsij}^{dir}, \quad \xi_{mnij}^{right} = \sum_{rs} \xi_{mnrs}^{dir} \lambda_{rsij} = (\xi_{ijmn}^{left})^{*}.
\]

(10)

Due to Eqs. (8, 9), they verify

\[
(E_m + E_n) \lambda_{mnij} + \xi_{mnij}^{right} = (E_i + E_j) \lambda_{mnij} + \xi_{mnij}^{left}.
\]

(11)
They are thus equal for $E_m + E_n = E_i + E_j$ only, i.e. diagonal processes, $(mn) = (ij)$, or possibly scattering between excitons staying inside the same exciton level and having an infinite total mass. Using the expression of $\xi_{mnij}^{\text{right}}$ given in the appendix, we find that $\xi_{mnij}^{\text{right}} = E_{\text{exc}}^{mnij}$ (properly symmetrized), with its e-h interactions “inside” the $i$ and $j$ excitons. The other exchange term $\xi_{mnij}^{\text{left}}$, with its e-h interactions “inside” the $m$ and $n$ excitons, does not appear in $E_{mnij}$.

From Eqs. (A4,A7), we find that the $\lambda$’s as well as the direct, left and right $\xi$’s write as a sum over one free $k$ of a product of four $\langle k|x_{\nu_i}\rangle$. For bound states, each $\langle k|x_{\nu_i}\rangle$ induces a $(a_x^3/V)^{1/2}$ factor (where $a_x$ is the exciton Bohr radius and $V$ the sample volume), so that the $\lambda$’s and the various $\xi$’s are all of the order of $a_x^3/V$ (even if $\xi_{mnij}^{\text{right}}$ appears as a sum of $\xi_{mnij}^{\text{dir}}$).

4. The new $H_{\text{eff}}$. We can think of identifying $H_{\text{eff}} B_i^+ B_j^+ |0\rangle$ with Eq. (6). This would lead to $E_{mnij} = \xi_{mnij}^{\text{dir}}$. (Note that due to Eq. (5), we could replace the prefactor of the $B_m^+ B_n^+$ term in Eq. (6) by $(a \xi_{mnij}^{\text{dir}} - b \xi_{mnij}^{\text{left}})/(a + b)$, with arbitrary $(a, b)$; the hermiticity of $H_{\text{eff}}$ however forces $b = 0$). As this $X$-$X$ scattering, without exchange terms, completely misses the fermionic character of the excitons, it looks reasonable to reject it.

A better way to determine an exciton effective bosonic hamiltonian is to force its matrix elements to be the same as the ones of the exact hamiltonian. If we introduce the (normalized) $N$ exact-exciton state,

$$|\psi_{i_1,\ldots,i_N}^{(N)}\rangle = B_{i_1}^+ \cdots B_{i_N}^+ |0\rangle / \langle 0|B_{i_1} \cdots B_{i_N} B_{i_1}^+ \cdots B_{i_N}^+ |0\rangle^{1/2}, \quad (12)$$

and the $N$ boson-exciton state $|\widetilde{\psi}_{i_1,\ldots,i_N}^{(N)}\rangle$ with $B_{i_1}^+$ replaced by $\bar{B}_{i_1}^+$, this condition reads

$$\langle \widetilde{\psi}_{i_1,\ldots,i_N}^{(N)}|H_{\text{eff}}|\psi_{i_1,\ldots,i_N}^{(N)}\rangle = \langle \psi_{i_1,\ldots,i_N}^{(N)}|H_{\text{sc}}|\psi_{i_1,\ldots,i_N}^{(N)}\rangle. \quad (13)$$

i) In the one-exciton subspace, Eq. (13) immediately gives $E_{ij} = E_i \delta_{ij}$, for the one-body part $H_{\text{eff}}$ written as $\sum_{ij} E_{ij} B_i^+ \bar{B}_j^+$.

ii) In the two-exciton subspace, Eqs. (10,11,13) lead to

$$E_{mnij}^{\text{new}} = \rho_{mnij} \left[ \xi_{mnij}^{\text{dir}} - \xi_{mnij}^{\text{right}} - (1 - \delta_{mnij}) (E_m + E_n) \lambda_{mnij} \right] \quad (14)$$
\[ \rho_{mnij} = \left[ \frac{(1 + \delta_{mn})(1 + \delta_{ij})}{(1 + \delta_{mn} - 2\lambda_{mnmn})(1 + \delta_{ij} - 2\lambda_{ijij})} \right]^{1/2}, \quad (15) \]

where \( \delta_{mnij} = 1 \) for \((mn) = (ij)\) and 0 otherwise. Eq. (14) shows that the non-diagonal scatterings have a purely fermionic contribution in \( \lambda_{mnij} \), which is necessary to ensure the hermiticity of \( H_{\text{eff}} \) (see Eqs. (10, 11)).

This \( \mathcal{E}_{mnij}^{\text{new}} \) can be rewritten in a more symmetrical way by introducing

\[ \xi_{\text{exc}}^{\text{mnij}} = \frac{1}{2} (\xi_{\text{mnij}}^{\text{right}} + \xi_{\text{mnij}}^{\text{left}}) = \frac{1}{2} \int de_1 de_2 dh_1 dh_2 \phi_m^*(e_1, h_1) \phi_n^*(e_2, h_2) \]
\[ \times [V_{e_1e_2} + V_{h_1h_2} - \frac{1}{2} (V_{e_1h_1} + V_{e_2h_2} + V_{e_1h_2} + V_{e_2h_1})] \phi_i(e_1, h_1) \phi_j(e_2, h_2) + (m \leftrightarrow n), \quad (16) \]

which is such that \( \xi_{\text{exc}}^{\text{mnij}} = (\xi_{\text{ijmn}}^{\text{exc}})^* \). From Eqs. (11, 14, 16) we then get

\[ \mathcal{E}_{mnij}^{\text{new}} = \xi_{\text{mnij}}^{\text{dir}} - \xi_{\text{mnij}}^{\text{exc}} - \eta_{mnij} + O((a_3^3/V)^2), \quad (17) \]
\[ \eta_{mnij} = \frac{1}{2} (1 - \delta_{mnij}) (E_m + E_n + E_i + E_j) \lambda_{mnij}, \quad (18) \]

as, for bound states, the \( \lambda \) and the \( \xi \)'s are all in \( a_3^3/V \), while \( \rho_{mnij} = 1 + O(a_3^3/V) \).

Equation (17) shows that the \( a_3^3/V \) leading term of the two-body part of \( H_{\text{eff}} \) writes as three (hermitian) operators. The first one, associated to \( \xi_{\text{mnij}}^{\text{dir}} \), comes from direct Coulomb terms in which all interactions are unambiguously interactions between excitons. The second operator, associated to \( \xi_{\text{mnij}}^{\text{exc}} \), comes from exchange Coulomb terms in which the concept of interactions between excitons is rather ambiguous, since the e-h contributions are interactions inside one of the four \((m, n, i, j)\) excitons. The third operator, associated to \( \eta_{mnij} \), is purely fermionic. It appears in all non-diagonal scatterings. Let us stress that, when properly symmetrized, the former \( \mathcal{E}_{mnij} \) is equal to \( \xi_{\text{mnij}}^{\text{dir}} - \xi_{\text{mnij}}^{\text{right}} \) so that it is correct for \((mn) = (ij)\) only [10].

iii) In the three-exciton subspace, we find that, except for diagonal processes, Eq. (13) cannot be fulfilled with two-body scatterings only: Besides the terms found for \( N = 2 \), the effective hamiltonian must contain a purely fermionic 3-body operator,

\[ H_{\text{eff}}^{(3)} = \frac{1}{3!} \sum_{lmn,ijk} \left[ -\eta_{lmn,ijk} + O((a_3^3/V)^2) \right] B^+_l B^+_m B^+_n B^-_i B^-_j B^-_k, \quad (19) \]
\[ \eta_{lmn,ijk} = \frac{1}{3} (1 - \delta_{lmnijk}) \left\{ [E_l \delta_{li} \lambda_{mnjk} + (i \leftrightarrow j) + (i \leftrightarrow k)] + (l \leftrightarrow m) + (l \leftrightarrow n) \right\}, \]  

where \( \delta_{lmnijk} = 1 \) if \( (lmn) = (ijk) \) and 0 otherwise.

In a similar way, Eq. (13) written in the \( N \)-exciton subspace, shows that \( H_{\text{eff}} \) must contain a set of \( (3, \ldots, N) \)-body operators of the order of \( a_x^3/V \) as the \( X-X \) part. This is after all not surprising: Pauli exclusion being \( N \)-body by essence since all the electrons of the \( N \) excitons must be different, the fermionic character of the exciton has to appear through \( N \)-body scatterings. Pauli exclusion between close-to-boson particles in fact generates a new "many-body" effect which is conceptually quite different from the usual one. Indeed, Coulomb interaction being a 2-body interaction, the usual many-body effects it induces are due to \( (2 \times 2) \) correlations between couples of electrons or holes. Here, Pauli exclusion is \( N \)-body in itself, so that the many-body effects it induces are already in the bare operators.

The consequences of this work as well as its extension to excitons with angular momentum variables (easy to include along reference \([11]\)) will be presented in an extended paper.
APPENDIX A:

- The exact exciton creation operator \( B^+_i \) is related to the creation operators of free e-h pairs with same total momentum \( q \) through

\[
B^+_i = \sum_{k_i} \langle k_i | x_{\nu_i} \rangle a^{+}_k b^{+}_h, \quad a^{+}_k b^{+}_h = \sum_{\nu_i} \langle x_{\nu_i} | k_i \rangle B^+_i, \quad (A1)
\]

where \( x_{\nu_i} \) being the exciton relative motion eigenstate, and \( K_i^e = k_i + \alpha_e q_i, K_i^h = -k_i + \alpha_h q_i \) the e and h momenta of the \((k_i, q_i)\) free pair, with \( \alpha_{e,h} = m_{e,h}/(m_e + m_h) \).

- Using these relations, we can write \( V^+_i \) as

\[
V^+_i = \sum_{m,n,q \neq 0} V_q \gamma_{mi}(q) B^+_m \sum_p (a^+_p a_p - b^+_p b_p), \quad (A2)
\]

where \( V_q = 4\pi e^2/V q^2 \) in 3D. \( \gamma_{mi}(q) = \delta_{q_m,q_i+q} \langle x_{\nu_m} | e^{i\alpha_n q}r - e^{-i\alpha_e q}r | x_{\nu_i} \rangle \) characterizes the scattering of an \( i \) exciton into a \( m \) state under a \( q \) excitation. Inserting Eq. \((A2)\) into Eq. \((3)\), we find

\[
\xi^\text{dir}_{mnij} = \frac{1}{2} \sum_{q \neq 0} V_q \gamma_{mi}(q) \gamma_{nj}(-q) + (m \leftrightarrow n), \quad (A3)
\]

in which we have symmetrized \( \xi^\text{dir}_{mnij} \) in order to have it invariant under \((m \leftrightarrow n)\) or \((i \leftrightarrow j)\). It will be useful to note that \( \xi^\text{dir}_{mnij} \) also reads

\[
\xi^\text{dir}_{mnij} = \sum_{k_i,k_j,k_m,k_n} \langle x_{\nu_m} | k_m \rangle \langle x_{\nu_i} | k_i \rangle \langle k_i | x_{\nu_j} \rangle \langle k_j | x_{\nu_j} \rangle \left[ \sum_{q \neq 0} V_q \delta^\text{dir}_{mnij}(q) \right], \quad (A4)
\]

\[
\delta^\text{dir}_{mnij}(q) = \frac{1}{2} \left( \delta_{K_i^e,K_j^e+q} \delta_{K_i^h,K_j^h} - (e \leftrightarrow h) \right) \left( \delta_{K_i^h,K_j^h-q} \delta_{K_i^e,K_j^e} - (e \leftrightarrow h) \right) + (m \leftrightarrow n). \quad (A5)
\]

\( \delta^\text{dir}_{mnij}(q) \) corresponds to the set of momentum conservations of all direct interactions between two excitons having a \( q \) momentum transfer. Since its four \( \delta \)'s impose \( q_i + q_j = q_m + q_n \), there are three conditions only between the four \( k \)'s so that the sum of Eq. \((A4)\) has only one free \( k \). We can easily check that Eq. \((A4)\) reads, in real space,

\[
\xi^\text{dir}_{mnij} = \frac{1}{2} \int de_1 de_2 dh_1 dh_2 \phi^*_m(e_1, h_1) \phi^*_n(e_2, h_2) (V_{e_1e_2} + V_{h_1h_2} - V_{e_1h_2} - V_{e_2h_1})
\]

\[
\times \phi_i(e_1, h_1) \phi_j(e_2, h_2) + (m \leftrightarrow n). \quad (A6)
\]
Using Eq. (A1), we find that $\lambda_{mnij}$, defined in Eq. (4), reads as the expression (A4) of $\xi_{mnij}^{\text{dir}}$, except for the last bracket which is replaced by

$$\Delta_{mnij} = \frac{1}{2} \delta_{K^e_m, K^e_i} \delta_{K^h_n, K^h_i} \delta_{K^e_j, K^e_j} \delta_{K^h_k, K^h_i} + (m \leftrightarrow n)$$

(A7)

These $\delta$'s correspond to cross the $e$ or the $h$ of the two excitons. In real space, $\lambda_{mnij}$ reads

$$\lambda_{mnij} = \frac{1}{2} \int de_1 de_2 dh_1 dh_2 \phi^*_m(e_1, h_1) \phi^*_n(e_2, h_2) \phi_i(e_1, h_2) \phi_j(e_2, h_1) + (m \leftrightarrow n). \quad (A8)$$

Eqs. (A4), (A5), (A7) show that $\xi_{mnij}^{\text{right}}$, defined in Eq. (9), reads as $\xi_{mnij}^{\text{dir}}$ (Eq. (A4)) with $\delta_{mnij}^{\text{dir}}(q)$ replaced by $\delta_{mnij}^{\text{right}}(q)$ deduced from Eq. (A5) by $(K^h_i \leftrightarrow K^h_j)$. The set of $\delta$’s of this $\delta_{mnij}^{\text{right}}(q)$ corresponds to the momentum conservations for exchange processes.
REFERENCES

[1] E. HANAMURA and H. HAUG, Phys. Report 33, 209 (1977)

[2] H. HAUG and S. SCHMITT-RINK, Progr. Quantum Electron. 9, 3 (1984)

[3] A. IVANOV, H. HAUG and L. KELDYSH, Phys. Report 296, 237 (1998)

[4] V. AXT and S. MUKAMEL, Rev. Mod. Phys. 70, 145 (1998)

[5] C. CIUTI, V. SAVONA, C. PIERMAROCCHI, A. QUATTROPANI and P. SCHWENDIMANN, Phys. Rev. B 58, 7926 (1998). They seem to have 3 “exchange” terms: One is just the \((m \leftrightarrow n)\) term of \(\xi^\text{dir}_{mnij}\), while the two others are the two parts of \(\xi^\text{right}_{mnij}\). In all the papers we have seen, \(\xi^\text{left}_{mnij}\) is missing.

[6] J. INOUE, T. BRANDES and A. SHIMIZU, Phys. Rev. B 61, 2863 (2000)

[7] M. COMBESCOT and R. COMBESCOT, Phys. Rev. Lett. 61, 117 (1988)

[8] M. COMBESCOT, Phys. Report 221, 168 (1992) and references therein.

[9] See Eqs. (5.7-9) of Ref. [2]. By using this “unit-operator projection” technique in a consistent way, it is however possible to produce the two missing terms \((V_{e_1h_1} + V_{e_2h_2})\): One just has to form the \((m, n)\) excitons with \((e_1, h_2)\) and \((e_2, h_1)\) and the \((i, j)\) excitons with \((e_1, h_1)\) and \((e_2, h_2)\). Let us stress that within this technique, based on Coulomb interaction only, there is no way to produce the purely fermionic term \(\eta_{mnij}\) in Eq. (17).

[10] While a correct diagonal term is enough to calculate \(\langle H_{sc} \rangle\), non-diagonal terms are necessary for effects like the ones considered in Ref. [3].

[11] M. COMBESCOT, Phys. Rev. B 41, 3517 (1990)

[12] In Ref. [4][5][6], \(V_i^+\) was written in terms of free e-h pairs instead of \(B_i^+\)’s, which was convenient enough for the purpose of these past works.