An inverse Sanov theorem for curved exponential families∗

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Abstract

We prove the large deviation principle (LDP) for posterior distributions arising from curved exponential families in a parametric setting, allowing misspecification of the model. Moreover, motivated by the so called inverse Sanov Theorem, obtained in a nonparametric setting by Ganesh and O’Connell (1999 and 2000), we study the relationship between the rate function for the LDP studied in this paper, and the one for the LDP for the corresponding maximum likelihood estimators. In our setting, even in the non misspecified case, it is not true in general that the rate functions for posterior distributions and for maximum likelihood estimators are Kullback-Leibler divergences with exchanged arguments. Finally, the results of the paper has some further interest for the case of exponential families with a dual one (see Letac (2021+)).

Keywords: Bayesian consistency, Large deviations, Curved exponential families, Kullback-Leibler divergence, Information geometry, Misspecified statistical models.

MSC2010 classification: 60F10, 62F12, 62F15.

1 Introduction

The interest for Bayesian consistency has grown in the last decades, especially in the nonparametric framework, see the survey papers of Ghosal, Ghosh and Ramamoorthi (1998) and Wasserman (1998). Some more recent developments have addressed the issue of misspecification, too, see Kleijn and Van der Vaart (2006). Most of the literature concerns sufficient conditions for the prior distribution to ensure consistency. The work by Ganesh and O’Connell (2000) has been a source of inspiration for this paper. Under a Dirichlet process prior on a compact state space $K$, they prove a Large Deviation Principle (LDP, see [5] for a definition) on $\mathcal{P}(K)$, the family of probability measures on $K$, for the family of posterior distributions, as the simple size grows. When the empirical distribution converges weakly to some law $P_0$ (i.e. the true law), such LDP is governed by the following rate function (evaluated at $P \in \mathcal{P}(K)$)

$$D(P_0||P) = \begin{cases} \int_K \log\left(\frac{dP_0}{dP}\right) dP_0, & \text{if } P_0 \text{ is absolutely continuous w.r.t. } P, \\ +\infty, & \text{otherwise} \end{cases}$$

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which is the celebrated Kullback-Leibler divergence of $P_0$ with respect to $P$. Consistency is a consequence of this result, since $D(P_0||P) \geq 0$, and the equality holds if and only if $P = P_0$. Now observe that the LDP provided by the celebrated Sanov theorem for the empirical distribution of i.i.d. samples drawn from $P_0$ is governed by the rate function $D(P||P_0)$, the Kullback-Leibler divergence of $P$ with respect to $P_0$; so, for this reason, the authors called their result the inverse Sanov theorem because the rate functions in the two LDP’s are obtained one from the other by exchanging the arguments in $D(\cdot||\cdot)$. Notice also that, even if in a rather formal way, the empirical distribution can be regarded as a non-parametric maximum likelihood estimator of the true distribution, giving a statistical flavour to the Sanov theorem.

The role of the prior distribution for consistency in parametric problems is rather clearcut. Indeed, in general what matters about the prior is only its support, as it entails the choice of a specific statistical model for the observations. As a matter of fact, in an earlier paper by Ganesh and O’Connell (1999) the inverse Sanov theorem is proved for a finite sample space, without any restriction on the prior, except that its support must include the limit assumed for the empirical distribution, which is nothing but the assumption that the model is not misspecified.

Motivated by previous works by the first author (Macci and Petrella (2009) and Macci (2014)), in the present paper we focus our attention on the analysis of parametric problems, establishing a kind of parametric inverse Sanov theorem. By this we mean a LDP for the sequence of posterior distributions, with the rate function of the form (1), but restricted to the parametric family assumed for the data. In addition our derivation covers also the misspecified case. This parametric family is assumed to be a curved exponential family, which in this context means a general subfamily of a full exponential family, called in the sequel the saturated model. The saturated model is generated by some positive $\sigma$-finite Borel measure $\lambda$ on $\mathbb{R}^d$, with cumulant generating function
\[
\kappa(\theta) = \log \int_{\mathbb{R}^d} e^{\theta \cdot x} \lambda(dx), \quad \theta \in \mathbb{R}^d,
\]
which is regular, that is is not concentrated on a proper affine submanifold of $\mathbb{R}^d$ and with open essential domain (domain of finiteness), denoted by $\text{dom}(\kappa)$. A more general situation will be discussed in Section 4. The full exponential family generated by $\lambda$ is defined through the densities
\[
\frac{dP_\theta}{d\lambda}(x) = e^{\theta \cdot x - \kappa(\theta)}, \quad \theta \in \text{dom}(\kappa).
\]
It is well known that the function $\kappa$ is smooth in $\text{dom}(\kappa)$ and
\[
\nabla \kappa(\theta) = \int x P_\theta(dx), \quad \theta \in \text{dom}(\kappa).
\]
The normalized log-likelihood function, evaluated on the empirical mean $\bar{x}_n = \frac{1}{n} \sum_{i=1}^n x_i$ of an observed sample $x_1, x_2, \ldots, x_n$ of points in $\mathbb{R}^d$, is defined by
\[
l(\theta; \bar{x}_n) = \theta \cdot \bar{x}_n - \kappa(\theta), \quad \theta \in \mathbb{R}^d.
\]
It is understood that the log-likelihood is set equal to $-\infty$ outside $\text{dom}(\kappa)$.

Within the Bayesian approach, one needs to specify also a prior distribution on the parameter $\theta$, that is a probability distribution $\nu$ on $\mathbb{R}^d$ supported by $\text{dom}(\kappa)$. Any subfamily of (2) of the form $\{P_\theta, \theta \in T\}$, where $T \subset \text{dom}(\kappa)$ is a Borel set with $\nu(T) = 1$ (called a measurable support of $\nu$ in the sequel), is a statistical model compatible with the choice of $\nu$. In order to avoid a separate treatment for cases which are not practically relevant, we will always assume that $\nu$ is atomless. Moreover we assume $T$ to be contained in the (topological) support $S(\nu)$ of $\nu$, that is the complement of the largest open set with $\nu$-probability 0. Clearly $\nu(S(\nu) \setminus T) = 0$. The definition of
support applies also to the \( \sigma \)-finite measure \( \lambda \) as well: its convex hull \( C(\lambda) \) will play a fundamental role in the sequel.

By Bayes’ formula the posterior distribution, conditional to \( x_1, x_2, \ldots, x_n \), is given by

\[
\pi_n(A|x_n) = \frac{\int_A \exp(n l(\theta; x_n)) \nu(d\theta)}{\int_T \exp(n l(\theta; x_n))\nu(d\theta)},
\]

where \( A \) is any Borel subset of \( T \). In view of the main theorem, we recall that a sequence of probability measures \( \{\phi_n, n \in \mathbb{N}\} \) on some topological space \( T \), satisfies a LDP with a rate function \( I: T \to [0, +\infty) \) if \( I \) is a lower semi-continuous function and

\[
- \inf_{\theta \in \text{int}(A)} I(\theta) \leq \liminf_{n \to \infty} \frac{1}{n} \ln \phi_n(A) \leq \limsup_{n \to \infty} \frac{1}{n} \ln \phi_n(A) \leq - \inf_{\theta \in \bar{A}} I(\theta),
\]

where \( \text{int}(A) \) and \( \bar{A} \) are the interior and the closure of \( A \), respectively.

Now we are ready to state the main result of the paper.

**Theorem 1.** If the sequence \( \{\bar{x}_n, n \in \mathbb{N}\} \) converges to some \( \mu_0 \) belonging to the interior of \( C(\lambda) \), the sequence of probability measures \( \{\pi_n(\bar{x}_n), n \in \mathbb{N}\} \) on \( T \) defined in (4), satisfies a LDP with rate function

\[
I(\theta) = l(\theta_\nu; \mu_0) - l(\theta; \mu_0), \quad \theta \in T,
\]

where \( \theta_\nu \in \text{dom}(\kappa) \) is any maximizer of \( l(\cdot; \mu_0) \) in \( S(\nu) \).

The assumption that \( \bar{x}_n \) converges to \( \mu_0 \in \text{int}(C(\lambda)) \) is quite natural. By the Strong Law of the Large Numbers, if we consider i.i.d. samples drawn from a distribution with mean \( \mu_0 \in \text{int}(C(\lambda)) \), such assumption holds almost surely. The regularity of \( \kappa \) implies that \( l(\cdot; \mu_0) \) is an upper semi-continuous function with compact superlevel sets (see e.g. Barndorff-Nielsen (1978), page 150), from which the existence (but not the uniqueness) of \( \theta_\nu \in S(\nu) \) is guaranteed. Note that, by (3), \( \theta_\nu \) can be interpreted as a limiting Maximum Likelihood Estimate of \( \theta \) in \( T \), because \( \mu_0 \) is supposed to be the limit of the sample means \( \bar{x}_n \) (as \( n \to \infty \)).

In addition, by regularity, there exists \( \theta_0 \in \text{dom}(\kappa) \) with \( \nabla \kappa(\theta_0) = \mu_0 \) and it is unique, since \( \kappa \) is strictly convex in \( \text{dom}(\kappa) \). By differentiation, one checks that \( \theta_0 \) is also the unique minimum point of \( l(\cdot; \mu_0) \) in the whole \( \text{dom}(\kappa) \): it is the limiting unrestricted MLE. Indeed, for \( \theta \in \text{dom}(\kappa) \) we have

\[
l(\theta_0; \mu_0) - l(\theta; \mu_0) = (\theta_0 - \theta) \cdot \mu_0 - \kappa(\theta_0) + \kappa(\theta) = \int \log \frac{dP_{\theta_0}}{dP_\theta} dP_{\theta_0} = D(P_{\theta_0}||P_\theta),
\]

which is positive except when \( \theta = \theta_0 \). Now either \( \theta_0 \in S(\nu) \), which is the "inverse Sanov regime", in which case \( \theta_\nu = \theta_0 \) and \( I(\theta) \) is given by (7) (extended to \( +\infty \) out of \( \text{dom}(\kappa) \)), or \( \theta_0 \notin S(\nu) \), which means that the model is misspecified. Also when \( \theta_\nu \neq \theta_0 \) the rate function can be rewritten as an "excess of divergence over the minimum", in the form

\[
I(\theta) = \{l(\theta_0; \mu_0) - l(\theta; \mu_0)\} - \{l(\theta_0; \mu_0) - l(\theta_\nu; \mu_0)\} = D(P_{\theta_0}||P_\theta) - D(P_{\theta_0}||P_{\theta_\nu}),
\]

for \( \theta \in T \subset \text{dom}(\kappa) \), and \( +\infty \) elsewhere. In addition, the function \( I \) is clearly a lower semi-continuous function.

It is worth to mention that also in the misspecified case the rate function can be written itself as a divergence by means of the Pythagorean identity for linear subfamilies stated below, first proved by Simon (1973). In Section [3] will give an example to illustrate the failure of the property for genuinely curved subfamilies.
Proposition 1. Let $\mu_0 \in \text{int}(C(\lambda))$ and let $T = L \cap \text{dom}(\kappa)$ be a measurable support of $\nu$, where $L$ is an affine submanifold of $\mathbb{R}^d$. Then $\theta_\nu$ is the only vector in $T$ such that the difference $\nabla \kappa(\theta_\nu) - \mu_0$ is orthogonal to $L$. Moreover for any $\theta \in T$ it holds

$$D(P_{\theta_0} \parallel P_\theta) = D(P_{\theta_0} \parallel P_{\theta_\nu}) + D(P_{\theta_\nu} \parallel P_\theta). \quad (8)$$

Proof. The first statement can be found in Brown (1986), Theorem 5.8 and Construction 5.9. The second statement follows from the first since $\theta - \theta_\nu \in L$ and writing down (8) using (7) the former is reduced to

$$(\theta - \theta_\nu) \cdot (\nabla \kappa(\theta_\nu) - \mu_0) = 0.$$  

From the previous result it follows that when the statistical model entailed by the prior is a linear subfamily of the saturated model, the rate function $g$ governing the LDP for the sequence of posteriors (4), is the same for a misspecified case $\theta_0 \notin T$ and a correctly specified one in which this is replaced by $\theta_\nu \in T$, as long as $\nabla \kappa(\theta_0) = \mu_0$ and $\theta_\nu$ is obtained from $\mu_0$ as in Theorem 1 (indeed notice $S(\nu) \setminus T \subset \partial \text{dom}(\kappa)$).

Finally observe also that the various choices of $T$ in the rate function (8) allow to consider different statistical models which are embedded in the saturated model, with a different relative topology in which the LDP holds.

The proof of Theorem 1 will be given in Section 2; it relies on some general facts about convex conjugate functions. In Section 3 we will discuss its frequentist counterpart, namely the LDP for the Maximum Likelihood Estimator (again denoted by MLE). The analysis of large deviations for consistent estimators in classical statistics dates back to the results of Bahadur (see Bahadur et al. (1980)). The application to the MLE in exponential families was discussed by Kester and Kallenberg (1986) and Arcones (2006), who observed that the parametric analogue of the Sanov theorem holds only for linear subfamilies. This is due to the failure of the Pythagorean identity for genuinely curved families, which will be illustrated through an example. Section 4 is devoted to examine an extension of Theorem 1 for non regular families. Since this is more cumbersome to state, we have decided to put it in a separate section. The last section deals with exponential families generated by dual measures, a concept which arise quite naturally from the subject of the paper (see Letac, 2021+).

2 Proof of the main theorem

Before giving the proof of Theorem 1 we need to recall some general facts about natural exponential families, that can be found in the books of Barndorff-Nielsen (1978) and Brown (1986). As anticipated in the introduction we assume that $\lambda$ is a regular $\sigma$-finite Borel measure on $\mathbb{R}^d$. Then the cumulant generating function $\kappa$ is a convex (and lower semi-continuous) function on $\mathbb{R}^d$, which is strictly convex (and continuous) in $\text{dom}(\kappa)$ (see e.g. Barndorff-Nielsen (1978), Theorem 7.1). Moreover $\kappa$ is differentiable in $\text{dom}(\kappa)$, and $\nabla \kappa$ maps $\text{dom}(\kappa)$ diffeomorphically onto the interior of $C(\lambda)$ (see e.g. Barndorff-Nielsen (1978), page 121). Throughout the paper, we set

$$l(\theta, t) = \theta \cdot t - \kappa(\theta),$$

for $\theta, t \in \mathbb{R}^d$, and we consider the conjugate function of $\kappa$ defined by

$$\kappa^*(t) = \sup_{\theta \in \mathbb{R}^d} l(\theta; t) = \sup_{\theta \in \text{dom}(\kappa)} l(\theta; t). \quad (9)$$
It is a lower semi-continuous convex function and differentiable in the interior of its essential domain, which coincides with \( \text{int}(C(\lambda)) \), being

\[
\text{int}(C(\lambda)) \subset \text{dom}(\kappa^*) \subset C(\lambda)
\]

(see e.g. Barndorff-Nielsen (1978), Theorems 9.1, 9.2 and 9.13). The gradient \( \nabla \kappa^* \) is the inverse mapping to \( \nabla \kappa \), thus it is defined in \( \text{int}(C(\lambda)) \) onto \( \text{dom}(\kappa) \). Moreover

\[
\kappa^*(t) + \kappa(\theta) \geq \theta \cdot t
\]

whereas and the equality holds if and only if \( t = \nabla \kappa(\theta) \), with \( \theta \in \text{dom}(\kappa) \). As a consequence

\[
\kappa^*(t) = \nabla \kappa^*(t) \cdot t - \kappa(\nabla \kappa^*(t)) = l(\nabla \kappa^*(t); t) \ t \in \text{int}(C(\lambda)). \tag{10}
\]

Thus, for \( \bar{x}_n \in \text{int}(C(\lambda)) \), \( \nabla \kappa^*(\bar{x}_n) \) is the MLE for the parameter \( \theta \) in the saturated model. Moreover \( \theta_0 = \nabla \kappa^*(\mu_0) \) in \( \text{int}(\lambda) \) (since \( \mu_0 = \nabla \kappa(\theta_0) \)), and

\[
D(P_{\theta_0}||P_\theta) = l(\theta_0; \nabla \kappa(\theta_0)) - l(\theta; \nabla \kappa(\theta_0)) = \kappa^*(\nabla \kappa(\theta_0)) - l(\theta; \nabla \kappa(\theta_0)), \tag{11}
\]

for \( \theta_0, \theta \in \text{dom}(\kappa) \).

In order to discuss also the LDP’s for MLE’s it is worth observing that, once defined

\[
l^*(t; \theta) = t \cdot \theta - \kappa^*(t),
\]

we can also write (similarly to \( \text{(11)} \))

\[
\kappa(\theta) = \nabla \kappa(\theta) \cdot \theta - \kappa^*(\nabla \kappa(\theta)) = l^*(\nabla \kappa(\theta); \theta)) = \sup_{t \in \mathbb{R}^d} l^*(t; \theta), \ \theta \in \text{dom}(\kappa),
\]

and from \( \text{(11)} \) we get

\[
D(P_{\theta_0}||P_\theta) = \kappa^*(\nabla \kappa(\theta_0)) - l(\theta; \nabla \kappa(\theta_0)) = \kappa(\theta) + \kappa^*(\nabla \kappa(\theta_0)) - \theta \cdot \nabla \kappa(\theta_0) = \kappa^*(\nabla \kappa(\theta_0); \theta), \ \theta_0, \theta \in \text{dom}(\kappa). \tag{12}
\]

Finally for an arbitrary set \( B \subset \mathbb{R}^d \) define

\[
\kappa_B^*(t) = \sup_{\theta \in B} l(\theta; t) = \sup_{\theta \in \mathbb{R}^d} \{ \theta \cdot t - \kappa(\theta) - \delta(\theta|B) \},
\]

where \( \delta(\theta|B) = 0 \) if \( \theta \in B \) and \( \delta(\theta|B) = +\infty \) if \( \theta \notin B \). The function \( \kappa_B^*(t) \) is again a lower semi-continuous convex function, being a supremum of affine functions. Given that \( \kappa_B^*(t) \leq \kappa^*(t) \) (the former being a supremum constrained to a smaller domain), it is \( \text{dom}(\kappa^*) \subset \text{dom}(\kappa_B^*) \), and therefore \( \text{int}(C(\lambda)) \subset \text{int}(\text{dom}(\kappa_B^*)) \). Since a convex function is continuous in the interior of its effective domain (see e.g. Roberts and Varberg (1973), Theorem D, page 93), the function \( \kappa_B^* \) is always continuous in \( \text{int}(C(\lambda)) \), whatever is the choice of the set \( B \).

For proving Theorem \( \text{[III]} \) first we need the following lemma.

**Lemma 2.** Under the assumptions of Theorem \( \text{[III]} \)

\[
\lim_{n \to \infty} \frac{1}{n} \ln \int_{S(\nu)} \exp(nl(\theta; \bar{x}_n)) \nu(d\theta) = \kappa^*_S(\nu)(\mu_0) = l(\theta_\nu; \mu_0).
\]
Proof. First of all, recall that by assumption

$$\kappa_{S(\nu)}^*(\mu_0) = l(\theta_\nu; \mu_0),$$

where $\theta_\nu \in S(\nu)$. Now, replacing the integrand with its supremum over the support $S(\nu)$, we immediately have

$$\frac{1}{n} \ln \int_{S(\nu)} \exp(nl(\theta; \bar{x}_n)) \nu(d\theta) \leq \sup_{\theta \in S(\nu)} l(\theta; \bar{x}_n) = \kappa_{S(\nu)}^*(\bar{x}_n).$$

Hence if $\bar{x}_n$ tends to $\mu_0 \in \text{int}(C(\lambda))$ as $n$ tends to $+\infty$, then $\bar{x}_n$ is eventually in $\text{int}(C(\lambda))$. By continuity of $\kappa_{S(\nu)}^*$ within this set, $\kappa_{S(\nu)}^*(\bar{x}_n)$ tends to $\kappa_{S(\nu)}^*(\mu_0)$ and

$$\lim_{n \to \infty} \frac{1}{n} \ln \int_{S(\nu)} \exp(nl(\theta; \bar{x}_n)) \nu(d\theta) \leq \kappa_{S(\nu)}^*(\mu_0).$$

For the reverse inequality let $B(\theta_\nu, \epsilon)$ be the ball of radius $\epsilon$ and center $\theta_\nu$ and observe that for any $\epsilon > 0$, $\nu(B(\theta_\nu, \epsilon)) > 0$, from which

$$\inf_{\theta \in S(\nu) \cap B(\theta_\nu, \epsilon)} l(\theta; \bar{x}_n) + \frac{1}{n} \ln \nu(B(\theta_\nu, \epsilon)) \leq \frac{1}{n} \ln \int_{S(\nu)} \exp(nl(\theta; \bar{x}_n)) \nu(d\theta).$$

Sending $n$ to $+\infty$ first, and then $\epsilon$ to $0$, one gets

$$\sup_{\epsilon > 0} \lim_{n \to \infty} \inf_{\theta \in S(\nu) \cap B(\theta_\nu, \epsilon)} l(\theta; \bar{x}_n) \leq \lim_{n \to \infty} \frac{1}{n} \ln \int_{S(\nu)} \exp(nl(\theta; \bar{x}_n)) \nu(d\theta). \quad (13)$$

Finally we prove that the left hand side in display (13) cannot be smaller than $\kappa_{S(\nu)}^*(\mu_0) = l(\theta_\nu; \mu_0)$. Reasoning by contradiction, suppose that for some $\delta > 0$

$$\sup_{\epsilon > 0} \lim_{n \to \infty} \inf_{\theta \in S(\nu) \cap B(\theta_\nu, \epsilon)} l(\theta; \bar{x}_n) < l(\theta_\nu; \mu_0) - \delta.$$

Then for any positive integer $m$ there exist $\theta_m \in S(\nu) \cap B(\theta_\nu, m^{-1})$ and an integer $n_m$ such that

$$\theta_m \cdot \bar{x}_{n_m} - \kappa(\theta_m) < \theta_\nu \cdot \mu_0 - \kappa(\theta_\nu) - \delta. \quad (14)$$

Now $\theta_m$ converges to $\theta_\nu$, $\kappa(\theta_m)$ converges to $\kappa(\theta_\nu)$, and $n_m$ can be chosen to be increasing with $m$. As $m \to \infty$ we get the convergence of the left hand side of (13) to $l(\theta_\nu; \mu_0) - \kappa(\theta_\nu)$, which is impossible since we assumed $\delta > 0$. So we have proved that

$$\lim_{n \to \infty} \frac{1}{n} \ln \int \exp(nl(\theta; \bar{x}_n)) \nu(d\theta) \geq l(\theta_\nu; \mu_0) = \kappa_{S(\nu)}^*(\mu_0), \quad (15)$$

ending the proof. \hfill \Box

**Proof of Theorem 1** The proof of the upper bound consists in estimating the numerator of the Bayes’ formula (1). Choose $A = B \cap T$, where $B$ is a Borel set of $\mathbb{R}^d$ and $T$ is a measurable support of $\nu$. Then, with exactly the same argument of the previous lemma

$$\limsup_{n \to \infty} \frac{1}{n} \ln \int_{B \cap T} \exp(nl(\theta; \bar{x}_n)) \nu(d\theta) \leq \lim_{n \to \infty} \kappa_{B \cap T}^*(\bar{x}_n) = \kappa_{B \cap T}^*(\mu_0), \quad (16)$$

which, together with Lemma 2, implies the rightmost inequality in (5), with the rate function $I$ defined in (3). Indeed the supremum in (16) is increased once it is taken in the closure of $B \cap T$ (in the relative topology of $T$).
As far as the lower bound is concerned, let \( O \cap T \) be the interior of the measurable set \( B \cap T \) in the relative topology of \( T \). Thus \( O \) is an open set, and \( O \cap T \subset B \cap T \). Repeating the argument of the previous proof with any \( \theta_* \in O \cap T \) replacing \( \theta_\nu \) and \( T \) replacing \( S(\nu) \), one arrives at

\[
\liminf_{n \to \infty} \frac{1}{n} \ln \int_{B \cap T} \exp(nl(\theta; \bar{x}_n))\nu(d\theta) \geq l(\theta_*; \mu_0).
\]

As a consequence the leftmost inequality in (5) is readily obtained.

**Example 3.** The Hardy-Weinberg family of distributions, in its simplest form with two alleles (see e.g. Barndorff-Nielsen (1978), Example 8.10), is a subfamily of the family of all distributions over 3 outcomes, coded with the 3 vectors in the plane \( O = (0,0), e_1 = (1,0), e_2 = (0,1) \). By choosing

\[
\lambda = \frac{1}{2} \delta_0 + \frac{1}{4} \delta_{e_1} + \frac{1}{4} \delta_{e_2},
\]

this family is represented as the natural exponential family generated by \( \lambda \), with the natural parameter \( \theta = (\theta_1, \theta_2) \in \mathbb{R}^2 \) and

\[
\kappa(\theta) = \log \int \exp \{ \theta \cdot t \} \lambda(dt) = \log(2 + e^{\theta_1} + e^{\theta_2}) - 2 \log 2.
\]

The probabilities of these outcomes are

\[
P_\theta(e_i) = \frac{\partial \kappa}{\partial \theta_i} = \frac{e^{\theta_i}}{2 + e^{\theta_1} + e^{\theta_2}}, \quad i = 1, 2, \quad P_\theta(0) = \frac{2}{2 + e^{\theta_1} + e^{\theta_2}}.
\]

The Hardy-Weinberg subfamily assumes that these probabilities arise from a binomial distribution with 2 trials, where \( O \) corresponds to one success and one failure, hence they are subject to the constraints

\[
P_\theta(0) = 2\sqrt{P_\theta(e_1)P_\theta(e_2)}
\]

which in term of the natural parameters becomes

\[
HW = \{\theta_1 + \theta_2 = 0\},
\]

taken to be the support of the prior distribution \( \nu \). Let \( t = (x, y) \) be any vector with positive components and \( x + y < 1 \), which means that \( (x,y) \) belongs to the interior of \( C(\lambda) \). With simple computations, the maximizer of the likelihood function \( l(\theta; t) \) with \( \theta \in HW \) is given by

\[
\theta_{\nu,1} = \log(1 + x - y) - \log(1 - x + y), \theta_{\nu,2} = -\theta_{\nu,1}.
\]

Then, by Theorem 4, if the sequence \( \{\bar{x}_n, n \in \mathbb{N}\} \) converges to \( t = (x, y) \) as \( n \to \infty \), the sequence of probability measures \( \{\pi_n(\cdot | \bar{x}_n), n \in \mathbb{N}\} \) on \( T = HW \) satisfies a LDP with rate function This is better visualized in terms of the parameter \( p(\theta_1) = \frac{e^{\theta_1/2}}{\sqrt{2 + e^{\theta_1} + e^{\theta_2}}} \), which is the success probability of the underlying binomial distribution. Since \( p_0 = \frac{1+x-y}{2} \) is the limiting MLE of this parameter, with simple computations one gets

\[
I_1(\theta_1) = 2 \left\{ p_0^2 \log \frac{p_0}{p(\theta_1)} + p_0(1 - p_0) \log \frac{p_0(1 - p_0)}{p(\theta_1)(1 - p(\theta_1))} + (1 - p_0)^2 \log \frac{1 - p_0}{1 - p(\theta_1)} \right\},
\]

the Kullback-Leibler divergence between two binomials with 2 trials and probability of success \( p_0 \) and \( p(\theta_1) \), respectively, in agreement with Proposition 4.
3 Large deviation principles for the MLE

This section is devoted to review what can be considered as the frequentist counterpart of Theorem 3, namely the LDP for a MLE in a curved exponential family. Let $x_1, x_2, \ldots, x_n$ be an i.i.d. sample drawn from $P_{\theta_0}$ belonging to the family $\{\}$, where $\theta_0 \in \text{dom}(\kappa)$. A Maximum Likelihood Estimator constrained to a measurable parameter set $T \subset \text{dom}(\kappa)$ is a measurable mapping $\varphi : \mathbb{R}^d \to T$ such that

$$l(\varphi(\bar{x}_n); \bar{x}_n) \geq l(\theta; \bar{x}_n), \ \theta \in T,$$

almost surely with respect to $P_{\theta_0}$. In order to relate this terminology with that of the Introduction, notice that we are allowed to say that the values of a maximum likelihood estimator are maximum likelihood estimates (see (3)). When $\bar{x}_n \in \text{int}(C(\lambda))$, subtracting the maximum value of the unconstrained likelihood function $\kappa^*(\bar{x}_n) = l(\nabla \kappa^*(\bar{x}_n); \bar{x}_n)$ from both sides (see (10) with $t = \bar{x}_n$), the above inequality is equivalent formulated as

$$D(P_{\nabla \kappa^*(\bar{x}_n)}||P_{\varphi(\bar{x}_n)}) \leq D(P_{\nabla \kappa^*(\bar{x}_n)}||P_{\theta}), \ \theta \in T.$$

Under suitable assumptions, the following result can be derived using rather general results of the theory of large deviations.

**Theorem 4.** Suppose that $x_1, \ldots, x_n$ is an i.i.d. sample drawn from $P_{\theta_0}$ and the sample mean $\bar{x}_n$ takes values in $\text{int}(C(\lambda))$ a.s. Moreover suppose that there exists a continuous function $\varphi : \mathbb{R}^d \to T$ which is a MLE constrained to $T$. Then the sequence $\{\varphi(\bar{x}_n), n \in \mathbb{N}\}$ satisfies a LDP with a rate function

$$\hat{I}(\theta) = \inf \{D(P_{\theta'}||P_{\theta_0}) : \theta' \in \text{dom}(\kappa), \varphi(\nabla \kappa(\theta')) = \theta, \ \theta \in T. \ (18)$$

**Proof.** By Cramér’s theorem (see e.g. Theorem 2.2.30 in Dembo and Zeitouni (1998)), the sample mean of i.i.d. random variables drawn from $P_{\theta_0}$ satisfies a LDP with rate function

$$\iota(t) = \sup_{\theta} \left\{ t \cdot \theta - \log \int e^{(\theta_0 + \theta) \cdot x - \kappa(\theta_0)} \lambda(dx) \right\} = \sup_{\theta} \{ t \cdot \theta - \kappa(\theta + \theta_0) + \kappa(\theta_0) \},$$

with the supremum on the whole space $\mathbb{R}^d$. Therefore, by (11) and a straightforward change of variable, for $t \in \text{int}(C(\lambda))$ we have

$$\iota(t) = \kappa^*(t) - l(\theta_0; t) = l(\nabla \kappa^*(t); t) - l(\theta_0; t) = D(P_{\nabla \kappa^*(t)}||P_{\theta_0}).$$

Finally, by the continuity of $\varphi$, we use the contraction principle (see e.g. Dembo and Zeitouni (1998), Section 4.1.4) and we get the LDP of $\{\varphi(\bar{x}_n), n \in \mathbb{N}\}$ with rate function $\hat{I}$ defined by

$$\hat{I}(\theta) = \inf \{\iota(t) : t \in \text{int}(C(\lambda)), \varphi(t) = \theta \} = \inf \{\iota(\nabla \kappa(\theta')) : \theta' \in \text{dom}(\kappa), \varphi(\nabla \kappa(\theta')) = \theta \},$$

which is easily seen to coincide with (18) since $\nabla \kappa$ and $\nabla \kappa^*$ are inverse to each other, between $\text{int}(C(\lambda))$ and $\text{dom}(\kappa)$.

As a consequence of the previous result, under the conditions stated therein, the rate function $\hat{I}(\theta)$, for any $\theta \in T$, is computed by solving the following geometrical problem: find the parameter vector $\theta'$ "closest" to $\theta_0$ within the "surface of constant MLE"

$$M_{\theta} = \{ \theta' \in \text{dom}(\kappa) : \varphi(\nabla \kappa(\theta')) = \theta \}$$

in the sense of minimizing $D(P_{\theta'}||P_{\theta_0})$. If

$$D(P_{\theta}||P_{\theta_0}) \leq D(P_{\theta'}||P_{\theta_0}), \forall \theta' \in M_{\theta} \ (19)$$
then the quantity \( \bar{T}(\theta) \) is equal to \( D(P_\theta || P_{\theta_0}) \); therefore when \( \theta_0 \in T \), we can say that the "parametric" Sanov theorem holds.

The property \( \mathcal{I} \) holds for the full exponential family, i.e. when \( T = \text{dom}(\kappa) \). Indeed the MLE estimator \( \varphi = \varphi_T = \nabla \kappa^* \) is injective, hence the set \( M_\theta \) reduces to a point. More generally it holds under the assumptions of the following proposition, which is immediately obtained from Proposition \( \mathcal{I} \).

**Proposition 2.** When \( T = L \cap \text{dom}(\kappa) \), \( L \) being an affine submanifold of \( \mathbb{R}^d \), if under i.i.d. sampling from \( P_{\theta_0} \), with \( \theta_0 \in T \), the sample mean \( \bar{x}_n \) takes values a.s. in \( \text{int}(C(\lambda)) \), then there is a uniquely defined MLE \( \varphi_T \) and

\[
D(P_{\nabla \kappa^*(t)} || P_{\theta_0}) = D(P_{\nabla \kappa^*(t)} || P_{\varphi_T(t)}) + D(P_{\varphi_T(t)} || P_{\theta_0}).
\]

for any \( t \in \text{int}(C(\lambda)) \). As a consequence, as long as \( \varphi(T(t) = \theta) \), it holds

\[
D(P_{\theta} || P_{\theta_0}) \leq D(P_{\nabla \kappa^*(t)} || P_{\theta_0}).
\]

When \( T \) has not the form prescribed by the previous result the above displayed property fails, as illustrated by the following example.

**Example 5.** The family of Gaussian distributions with mean equal to the standard deviation form a one-parameter curved subfamily of the two-parameter Gaussian exponential family. Recall that the family of distributions in the cartesian plane that are images of Gaussian laws \( N(\mu, \sigma^2) \) on the real line under the mapping \( x \rightarrow (x, q(x)) \), where \( q(x) = x^2 \), is a natural exponential family, with a generating measure \( \lambda \) that can be chosen equal to the image of the Lebesgue measure under the above mapping. The natural parameters \( (\theta_1, \theta_2) \) are then

\[
\theta_1 = \frac{\mu}{\sigma^2} \in \mathbb{R}, \quad \theta_2 = -\frac{1}{2\sigma^2} < 0,
\]

and the cumulant generating function is

\[
\kappa(\theta_1, \theta_2) = -\frac{1}{2} \left( 2 \log 2 + \log \pi + \log (\theta_2) + \frac{\theta^2}{2\theta_2} \right),
\]

whose gradient is given by

\[
\frac{\partial \kappa}{\partial \theta_1} = \frac{\theta_1}{2\theta_2}, \quad \frac{\partial \kappa}{\partial \theta_2} = \frac{\theta_1^2}{4\theta_2^2} - \frac{1}{2\theta_2}.
\]

Since \( \lambda \) is supported by the graph of the function \( q \), the set \( \text{int}(C(\lambda)) \) is the subset of the plane above this graph. It is clear that the mean of a sample of size \( n \) drawn from any law of this exponential family will lie on this set unless all the elements of the sample are equal, which clearly happens with probability zero.

The subfamily of laws with mean equal to the standard deviation corresponds to the following curve in the natural parameter space

\[
T = \left\{ (\theta_1, \theta_2) : \theta_2 = -\frac{1}{2} \theta_1^2 = -\frac{1}{2} q(\theta_1), \quad \theta_1 > 0 \right\},
\]

whose image under the mapping \( \nabla \kappa \) is readily checked to be the graph of the function \( 2q \), restricted to the first quadrant of the plane. The first order condition for the maximization of the likelihood in \( \theta_1 \), with \( T \) parametrized as in \( \mathcal{I} \), when \( x = \bar{x}_n \) and \( y = (x^2)_n = \sum_{i=1}^n x_i^2 / n \), gives the following equation

\[
\left( x - \frac{\partial \kappa}{\partial \theta_1}(\theta_1, -\frac{1}{2} \theta_1^2) \right) - \theta_1 \left( y - \frac{\partial \kappa}{\partial \theta_2}(\theta_1, -\frac{1}{2} \theta_1^2) \right) = \left( x - \frac{1}{\theta_1} \right) - \theta_1 \left( y - \frac{2}{\theta_1^2} \right) = 0,
\]
whose unique positive solution is

\[ \theta_1 = \frac{x + \sqrt{x^2 + 4y}}{2y} = \varphi(x, y), \]  

with \( \varphi \) continuous in \( \text{int}(C(\lambda)) \). The conditions of Theorem 4 are thus satisfied: the sequence of MLE's satisfies a LDP in \( T \) with rate function of the form \( 18 \), i.e.

\[ \tilde{I}(\theta_1, \theta_2) = \tilde{I}_1(\theta_1), \quad \theta_2 = -\frac{1}{2} \theta_1^2, \quad \theta_1 > 0, \]

where \( \tilde{I}_1 \) has to be determined. In order to this observe that, by means of \( 12 \), the minimization problem appearing in \( 18 \) can be rephrased as the maximization of \( l^*((x, y); \theta_0) \) constrained to \( \varphi(x, y) = \theta_1 \). Now observe that the set of \( (x, y) \) such that \( 22 \) is satisfied can be described as the graph of the function

\[ y = g(x) = \frac{1}{\theta_1^2} + \frac{1}{\theta_1} x \]

so, if we set \( \theta_0 = (\theta_{0,1}, \theta_{0,2}) \), the first order condition for such a maximization problem in \( x \) is

\[ \theta_{0,1} + \frac{\theta_{0,2}}{\theta_1} - \frac{\partial \kappa^*}{\partial x}(x, g(x)) \left( \frac{\partial \kappa^*}{\partial y}(x, g(x)) \frac{1}{\theta_1} \right) = 0. \]

By recalling that \( \nabla \kappa \) and \( \nabla \kappa^* \) are inverse to each other and taking \( 20 \) into account, this gives the quadratic equation in \( z = \theta_1 x \)

\[ 2(\theta_1 \theta_{0,1} + \theta_{0,2}) z^2 - 2 \left( \theta_1 \theta_{0,1} + \theta_{0,2} - \theta_1^2 \right) z - \left[ 2(\theta_1 \theta_{0,1} + \theta_{0,2}) + \theta_1^2 \right] = 0. \]

Finally assume that \( \theta_{0,2} = -\frac{1}{2} g(\theta_{0,1}) = -\frac{1}{2} \theta_{0,1}^2, \) for \( \theta_{0,1} > 0, \) that is \( (\theta_{0,1}, \theta_{0,2}) \) belongs to \( T \). The above equation has the solution \( z = 1 \), corresponding to \( x = \frac{1}{\theta_1} \), provided \( \tau = \frac{\theta_{0,1}}{\theta_1} \) satisfies \( \tau(2 - \tau) = 1 \), which is equivalent to \( \tau = 1 \). As a consequence

\[ \tilde{I}_1(\theta_1) \leq D(P_{\theta_1, \theta_0} \| P_{\theta_{0,1}, \theta_{0,2} - \frac{1}{2} g(\theta_{0,1})}), \quad \theta_1 > 0, \]

and the equality holds if and only if \( \theta_1 = \theta_{0,1} \). So the ”parametric” Sanov theorem fails because, for all values \( \theta_1 \neq \theta_{0,1} \), we have the strict inequality.

4 LDP’s when the set \( \text{dom}(\kappa) \) is not open

The aim of this section is to explain under which circumstances the LDP stated as Theorem 11 continues to hold when the essential domain of the cumulant generating function \( \kappa \) (of the reference measure \( \lambda \)) is not open. In this case \( \kappa \) remains continuous in the interior of \( \text{dom}(\kappa) \), but this is not necessarily true at boundary points of \( \text{dom}(\kappa) \). The basic assumption remains unchanged: the sequence \( \{\bar{x}_n, n \in \mathbb{N}\} \) converges to \( \mu_0 \in \text{int}(C(\lambda)) \), which ensures that there exists \( \theta_\nu \in S(\nu) \cap \text{dom}(\kappa) \) such that

\[ \kappa^*_S(\nu)(\mu_0) = l(\theta_\nu; \mu_0). \]

The existence of a maximizer \( \theta_\nu \in S(\nu) \cap \text{dom}(\kappa) \) is guaranteed by the fact that \( l(\cdot; \mu_0) \) is upper semi-continuous with bounded superlevel sets, and \( S(\nu) \) is a closed set.

Now it is convenient to introduce the following notion.

**Definition 6.** Let \( B \subset \text{dom}(\kappa) \) and \( \theta \in B \cap \partial \text{dom}(\kappa) \). We say that \( \theta \) is a continuity point for \( \kappa \) on \( B \) if for any sequence \( \{\theta_\ell\} \subset B \cap \text{int}(\text{dom}(\kappa)) \) such that \( \theta_\ell \to \theta \) it happens that \( \kappa(\theta_\ell) \to \kappa(\theta) \) as \( \ell \to \infty \).
Theorem 7. Let $\lambda$ be any positive $\sigma$-finite Borel measure on $\mathbb{R}^d$ which is not concentrated on a proper affine submanifold and whose cumulant generating function $\kappa$ has an essential domain $\text{dom}(\kappa)$ with non empty interior. Moreover let $\nu$ be a probability measure on $\text{dom}(\kappa)$ and $T \subset S(\nu)$ be a measurable support of it.

Suppose the following conditions (A), (B) and (C) hold:

(A) The sequence $\{x_n, n \in \mathbb{N}\}$ converges to $\mu_0 \in \text{int}(C(\lambda))$;
(B) either $\theta_\nu \in \text{int}(\text{dom}(\kappa))$, or it is a continuity point for $\kappa$ on $S(\nu)$;
(C) any $\theta \in T \cap \partial \text{dom}(\kappa)$ is a continuity point for $\kappa$ on $T$.

Then the sequence of probability measures $\{\pi_n(\cdot | x_n), n \in \mathbb{N}\}$ on $T$, defined in (6), has a LDP with rate function (6).

Proof. As far as the upper bounds in Lemma 2 and Theorem 1 are concerned the assumption about $\mu_0$ continue to work as in the regular case. Indeed, for any $B \subset \mathbb{R}^d$, the convex function $\kappa_B^\nu$ is continuous in the interior of its essential domain, which necessarily includes the interior of $C(\lambda)$.

As far as the lower bounds in Lemma 2 and Theorem 1 are concerned, we have to take into account that $\theta_\nu$ and $\theta_\ast$ may lie in the boundary of $\text{dom}(\kappa)$. However, assumption (B) and assumption (C) ensures that for any $\ell \in \mathbb{N}$, one can replace $\theta_\nu$ and $\theta_\ast$ with $\theta_\nu^\ell, \theta_\ast^\ell \in S(\nu)$ and $\theta_\ast^\ell \in T$ with the property

$$l(\theta_\nu^\ell; \mu_0) > l(\theta_\nu; \mu_0) - \delta, \quad l(\theta_\ast^\ell; \mu_0) > l(\theta_\ast; \mu_0) - \delta,$$

respectively. Now one repeats the proof of (15) and (17) to prove these lower bounds with the right hand side equal to $l(\theta_\nu^\ell; \mu_0) - \delta$ and $l(\theta_\ast^\ell; \mu_0) - \delta$, respectively. Since they hold irrespectively of $\delta$, they continue to hold with $\delta$ replaced by zero.

Assumptions (B) and (C) always hold for one-dimensional exponential families with non-atomic priors, by lower semi-continuity of the function $\kappa$.

The following classical example (see e.g. Barndorff-Nielsen (1978), Example 7.3, page 104), serves as an illustration of the previous result.

Example 8. Consider the probability measure $\mu$ on $\mathbb{R}^2$ given by

$$\lambda(dx_1, dx_2) = \frac{1}{2\sqrt{\pi(1 + x_1^2)^3/2}} \exp \left\{ - \left( \frac{x_1^2 + x_2^2}{4(1 + x_1^2)} \right) \right\} dx_1 dx_2.$$

The essential domain of its cumulant generating function $\kappa$ is given by

$$\text{dom}(\kappa) = \{ (\theta_1, \theta_2) \in \mathbb{R}^2 : \theta_2 \in (-1, 1) \} \cup \{ (0, \pm 1) \}.$$

The peculiarity of this example is that on the boundary points $(0, \pm 1)$ of $\text{dom}(\kappa)$ the function $\kappa$ is finite but it is not continuous. Indeed on the curve

$$\theta_2 = \sqrt{1 - \theta_1^2}, \quad \theta_1 \in (0, 1]$$

the function $\kappa$ tends to $+\infty$ as $\theta_1 \to 0$. Now define $\eta(z) = (z, \sqrt{1 - z^2})$ for $z \in (0, 1)$, let $\gamma$ be the uniform prior on $M = (0, 1)$ and let $\nu = \gamma \circ \eta^{-1}$ be its image measure on $\text{dom}(\kappa)$. Clearly $T = \eta(M)$ is a measurable support of $\nu$, whereas its topological support contains also the limit point $(0, 1)$. We are going to apply Theorem 7 to this example. Since $\eta$ is a homeomorphism of $M$ onto $T$, the LDP can be immediately transferred to $M$, as it will be done in the following.

The measure $\lambda$ has the whole $\mathbb{R}^2$ as support, so assuming that $x_n$ converges to some $\mu_0 = (\mu_0_1, \mu_0_2) \in \mathbb{R}^2$ guarantees assumption (A) of the theorem. As a consequence the function $l(\eta(z); \mu_0)$ defined for $z \in M$, which is continuous in any interval $[\varepsilon, 1]$ with $\varepsilon > 0$ and tends to $-\infty$ as $z \to 0$,
has a maximizer in \( z \), say \( \tilde{z} \), which depends on \( \mu_0 \), whose image under \( \eta(\tilde{z}) \) is an interior point of \( \text{dom}(\kappa) \). So assumption (B) is automatic. As far as assumption (C) is concerned nothing has to be verified since \( M \) does not contain any boundary point of \( \text{dom}(\kappa) \). As a consequence Theorem 7 can be applied, with the rate function given by

\[
\iota_M(z) = l(\eta(\tilde{z}); \mu_0) - l(\eta(z); \mu_0)
= (\tilde{z} - z)\mu_{0,1} + (\sqrt{1 - \tilde{z}^3} - \sqrt{1 - z^3})\mu_{0,2} - \kappa(\tilde{z}, \sqrt{1 - \tilde{z}^3}) + \kappa(z, \sqrt{1 - z^3}),
\]

where \( z \in M \). In particular the rate function tends to \(+\infty\) as \( z \to 0 \): indeed the rate function does not see at all that the cumulant generating function is finite at the boundary point \((0,1)\), since this is not contained in the image of \( M \) under \( \eta \).

Now suppose that \( T = \eta(M) \), hence \( z = 0 \) is added and mapped into the limit point \((0,1)\). Then Theorem 7 cannot be applied because of the failure of assumption (C). As a matter of fact the LDP fails on spheres of radius \( \varepsilon > 0 \) suitably small around \((0,1)\), whose posterior probability approaches zero with an exponential rate that can be made arbitrarily large by chosing \( \varepsilon \) small enough, which is not compatible with the finite value \( \iota_M(0) = l((0,1); \mu_0) \).

Finally notice that if the curve \((23)\) is replaced by

\[
\theta_2 = 1 - \theta_1, \quad \theta_1 \in (0,1)
\]

again with a uniform prior on \( \theta_1 \), the above phenomenon disappears, due to the continuity of \( \kappa \) on the curve, up to the boundary point \((0,1)\), inherited by lower semi-continuity on the whole of \( \mathbb{R}^2 \).

## 5 Dual families and inverse LDP’s

The final section is devoted to a more specific topic, for which we need to revise our notation: the function \( \kappa \) will be denoted by \( \kappa_\lambda \) in the following. We also indicate by \( \psi_\lambda \) the inverse function of \( \nabla \kappa_\lambda \). We assume that \( \nabla \kappa_\lambda \) is a diffeomorphism between of \( \text{dom}(\kappa_\lambda) \) onto \( \text{int}(C(\lambda)) \) and \( \psi_\lambda \) goes the other way round. This is guaranteed when \( \kappa_\lambda \) is regular, but more generally if \( \kappa_\lambda \) is steep. This means that \( \lambda \) is not concentrated on a proper affine submanifold of \( \mathbb{R}^d \); its essential domain has a non-empty interior, and \( \|\nabla \kappa_\lambda(\theta_n)\| \to +\infty \) whenever \( \{\theta_n\} \) is a sequence of points in \( \text{int}(\text{dom}(\kappa_\lambda)) \) such that \( \theta_n \to \theta \) for some \( \theta \in \partial \text{dom}(\kappa_\lambda) \). This is the appropriate setting for the following definition, which is inspired by Barndoff-Nielsen (1978, Section 9.1, page 142); here, for our purposes, we give a slight modification of the definition in Letac (2021+), Section 3.1.

**Definition 9.** Let \( \lambda \) be a \( \sigma \)-finite Borel measure on \( \mathbb{R}^d \) with a steep cumulant generating function \( \kappa_\lambda \). A \( \sigma \)-finite Borel measure \( \lambda^* \) on \( \mathbb{R}^d \), with a steep cumulant generating function \( \kappa_{\lambda^*} \) as well, is called the dual measure of \( \lambda \) when

\[
\kappa_{\lambda^*} = \kappa_\lambda^*,
\]

where \( \kappa_\lambda^* \) is the conjugate function of \( \kappa_\lambda \), defined in (9).

We warn the reader that not all \( \sigma \)-finite Borel measures on \( \mathbb{R}^d \) have dual measures (see Letac (2021+) for specific examples). The notion of dual measure has immediate consequences that we collect here:

i) the dual measure \( \lambda^* \) has the measure \( \lambda \) as its dual;
ii) the measure \( \lambda^* \) has the properties

\[
\text{int}(\text{dom}(\kappa_{\lambda^*})) = \text{int}(C(\lambda)), \quad \text{int}(C(\lambda^*)) = \text{int}(\text{dom}(\kappa_\lambda));
\]

iii) \( \nabla \kappa_{\lambda^*} = \psi_\lambda \), the inverse of \( \nabla \kappa_\lambda \), is defined on \( \text{int}(C(\lambda)) \) onto \( \text{dom}(\kappa_\lambda) \).
When \( \lambda \) has a dual measure we have an interesting connection between LDP’s concerning the full exponential families generated by \( \lambda \) and \( \lambda^* \) (see Corollary 1). This is a consequence of the following proposition in which we illustrate the relation between Kullback-Leibler divergences computed within the exponential families \( \{ P_\theta, \theta \in \text{dom}(\kappa_\lambda) \} \) and \( \{ Q_\mu, \mu \in \text{dom}(\kappa_{\lambda^*}) \} \) defined by

\[
\frac{dP_\theta}{d\lambda}(x) = \exp\{\theta \cdot x - \kappa_\lambda(\theta)\}, \quad \frac{dQ_\mu}{d\lambda^*}(x) = \exp\{\mu \cdot x - \kappa_{\lambda^*}(\mu)\}. \tag{24}
\]

**Proposition 3.** Let \( \lambda \) and \( \lambda^* \) be \( \sigma \)-finite Borel measures on \( \mathbb{R}^d \) which are dual. For \( \theta, \theta_0 \in \text{int}(\text{dom}(\kappa_\lambda)), \) define

\[
\mu = \nabla \kappa_\lambda(\theta), \quad \mu_0 = \nabla \kappa_\lambda(\theta_0) \in \text{int}(\text{dom}(\kappa_{\lambda^*})).
\]

Then

\[
D(P_{\theta_0} \| P_\theta) = D(Q_{\mu} \| Q_{\mu_0}).
\]

**Proof.** The desired equality can be checked as follows, by taking into account that \( \theta = \nabla \kappa_{\lambda^*}(\mu) \):

\[
D(P_{\theta_0} \| P_\theta) = (\kappa_\lambda)^*(\mu_0) - l(\theta; \mu_0)
\]

\[
= \kappa_{\lambda^*}(\mu_0) - \theta \cdot \mu_0 + \kappa_\lambda(\theta) = (\mu - \mu_0) \cdot \theta - \kappa_{\lambda^*}(\mu) + \kappa_{\lambda^*}(\mu_0) = D(Q_{\mu} \| Q_{\mu_0}).
\]

\( \square \)

From the above proposition the following corollary can be easily obtained. In order to avoid confusion, we consider posterior distributions and MLE’s for the natural parameters of both the dual exponential families defined in [24].

**Corollary 1.** Suppose \( \lambda \) is a regular \( \sigma \)-finite measure on \( \mathbb{R}^d \). Let \( \nu \) be a probability measure on \( T = \text{dom}(\kappa_\lambda) \), supported by its closure. Let \( \{ \bar{x}_n \} \) be a sequence in \( \mathbb{R}^d \) converging to \( \mu_0 \in \text{int}(C(\lambda)) \), and consider the sequence \( \{ \pi_n(\cdot|\bar{x}_n) \} \) of posterior distributions on \( T \) defined in [1]. Finally let \( \theta_0 \) be such that \( \nabla \kappa_\lambda(\theta_0) = \mu_0 \).

Next suppose that \( \lambda \) has a dual measure \( \lambda^* \). Consider i.i.d. random variables \( x_1^*, \ldots, x_n^* \) drawn from \( Q_{\mu_0} \), defined in [24], and suppose that their sample mean \( \bar{x}_n^* \) takes values in \( C(\lambda^*) = \text{dom}(\kappa_{\lambda^*}) \) with probability 1.

Then the sequence of posterior laws \( \{ \pi_n(\cdot|\bar{x}_n) \} \) (for the family \( \{ P_\theta \} \)) and the sequence of laws of the MLE \( \{ \nabla \kappa_\lambda(\cdot|\bar{x}_n^*) \} \) of the natural parameter (for the family \( \{ Q_\mu \} \)), have LDP’s with rate functions \( J \) and \( \tilde{J} \) on \( \text{dom}(\kappa_\lambda) \) and \( \text{dom}(\kappa_{\lambda^*}) \), respectively, related by

\[
J(\theta) = D(P_{\theta_0} \| P_\theta) = D(Q_{\mu} \| Q_{\mu_0}) = \tilde{J}(\mu),
\]

when \( \mu = \nabla \kappa_\lambda(\theta) \).

Differently with respect to the presentation of the previous result, in the following classical example of dual measures the change of variable in the posterior distribution allows to obtain the identity of the rate functions for the two LDP’s.

**Example 10.** The probability measure \( \lambda \) equal to the Poisson law with mean 1 generates the Poisson exponential family, that is clearly regular with \( \text{dom}(\kappa) = \mathbb{R} \). Its cumulant generating function is given by \( \kappa_\lambda(\theta) = e^\theta - 1 \) and once a prior distribution \( \omega \) is placed on the mean value parameter \( \mu = e^\theta \) the sequence of distributions conditional to \( \bar{x}_n \) on the same parameter is given by

\[
\omega_n(B|\bar{x}_n) = \frac{\int_B e^{\mu \log \mu - \mu} \omega(d\mu)}{\int_{-\infty}^{\infty} e^{\mu \log \mu - \mu} \omega(d\mu)}.
\]
where \( B \) is any Borel subset of the positive real line. The function \( \kappa_\lambda \) has the convex conjugate

\[
\kappa_\lambda^*(\mu) = \begin{cases} 
\mu \log(\mu) - \mu + 1 & \text{for } \mu \geq 0 \\
+\infty & \text{otherwise,}
\end{cases}
\]

which is, up to the additive constant 1, the cumulant generating function of the probability measure \( \lambda^* \) with density

\[
f(y) = \frac{\pi}{4} \int_0^{+\infty} e^{\pi v/4} \cos(-v - vy + v \log v) dv, \quad y \in \mathbb{R}
\]

with respect to the Lebesgue measure on the real line. This is the density of \(-X - 1\), where \( X \) has the Landau distribution (see Landau (1967); see also Eaton et al. (1971) for some more information about this law).

The family \( \{Q_\mu, \mu > 0\} \), appearing in the previous corollary, is the exponential family generated by \( \lambda^* \). The normalized log-likelihood function, as a function of the natural parameter \( \mu \), is given by

\[
\mu \cdot \bar{x}_n^* - \kappa_\lambda^*(\mu),
\]

which is maximized by

\[
\mu = \nabla \kappa_\lambda(\bar{x}_n^*) = e^{\bar{x}_n^*}.
\]

As a result, for \( x_1^*, \ldots, x_n^* \) i.i.d. from \( Q_{\mu_0} \), as \( n \to \infty \) the sequence of laws of the statistics \( e^{\bar{x}_n^*} \) obeys the same LDP as the sequence of posterior laws \( \{\omega_n(\cdot | \bar{x}_n)\} \) on the positive real line, when \( \{\bar{x}_n\} \) is a sequence converging to \( \mu_0 > 0 \), the rate function being in both cases

\[
\bar{J}(\mu) = D(Q_\mu || Q_{\mu_0}) = \mu_0 \log(\mu_0/\mu) + \mu - \mu_0, \quad \mu > 0.
\]

(25)

Indeed the rate function for the LDP for posterior laws is obtained by computing

\[
D(P_{\theta_0} || P_\theta) = (\theta_0 - \theta)e^{\theta_0} - e^{\theta_0} + e^\theta
\]

and substituting \( e^{\theta_0} = \mu_0 \) and \( e^\theta = \mu \), by Theorem 1 and a trivial application of the contraction principle (the function \( \theta \to r^\theta \) is a homeomorphism of the real line onto its positive side) it is immediately checked that one arrives to (25).

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