PINNACLE SETS REVISITED

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ABSTRACT. In 2017, Davis, Nelson, Petersen, and Tenner [Discrete Math. 341 (2018), 3249–3270] initiated the combinatorics of pinnacles in permutations. We provide a simple and efficient recursion to compute \( p_n(S) \), the number of permutations of \( S_n \) with pinnacle set \( S \), and a conjectural closed formula for the related numbers \( q_n(S) \). We determine the lexicographically minimal elements of the orbits of the modified Foata-Strehl action, prove that these elements form a lower ideal of the left weak order and characterize and count the maximal elements of this ideal.

1. Introduction

Given a permutation \( \sigma = \sigma_1 \ldots \sigma_n \), a well-known and well-studied statistic is the peak set of \( \sigma \) defined as
\[
\text{Peak } \sigma = \{ i \mid \sigma_{i-1} < \sigma_i > \sigma_{i+1} \} \subseteq [2, n-1].
\]

More recently, some authors [DNPT] studied a statistic very close in its definition but rather different in its behaviour: the so-called pinnacle set defined as
\[
\text{Pin } \sigma = \{ \sigma_i \mid \sigma_{i-1} < \sigma_i > \sigma_{i+1} \} \subseteq [3, n].
\]

A set \( S \) is an admissible pinnacle set (or pinnacle set, for short) if there is a permutation \( \sigma \) such that \( \text{Pin } \sigma = S \). It is easy to show that the pinnacle sets are exactly the sets \( S = \{ s_1 < s_2 < \cdots < s_p \} \) satisfying \( s_i > 2i \) [DNPT].

One main question left unsolved is how to compute efficiently the value \( p_n(S) \) defined in [DLMSSS] (denoted there by \( p_S(n) \) but we prefer the subscript to be an integer and not a set to enhance readability) as:
\[
\text{(3) } p_n(S) = \# \{ \sigma \in S_n \mid \text{Pin } \sigma = S \}.
\]

We shall provide an efficient inductive formula to compute \( p_n(S) \) (Theorem 3.1). Indeed, its complexity is polynomial in both \( n \) and \( |S| \) (Proposition 3.2). Fang in [Fang] has another strategy to very efficiently compute \( p_n(S) \).

It has been noted in [DLHHIN] that \( p_n(S) \) is divisible by \( 2^{n-1-|S|} \) and in this reference the authors constructed a set of representatives, called minimal elements, of the orbits of a group action very close to the Foata-Strehl action, called the dual Foata-Strehl action. We shall also use this construction to provide another set of representatives, this time the lexicographically smallest elements of each orbit (Subsection 4.2), show that they form a lower ideal of the left weak order (Theorem 4.6), and characterize the maximal elements of this ideal (Theorem 4.7).

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In Section 5 we study a weighted sum \( q_n(S) \) of \( p_n(S) \) presented in [DNPT] in order to simplify the general formulas. We conjecture a closed formula for \( q_n(S) \), in the form of expressions depending on the size of \( S \). The formula looks rather surprising and even if it is possible to prove some special cases when \( |S| \) is small, we have not been able to prove it in general.

In Section 6 we consider a set which is equinumerous to the pinnacle sets. We prove a formula and conjecture another one, both quite easily expressed in this context. We expect that these objects may shed some light on other enumerative questions about pinnacle sets.

Finally, in Section 7 we prove that the componentwise comparison order on pinnacle sets is compatible to the evaluation of \( p_n \) and disprove a result that was (erroneously) stated in [DNPT].

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2. Pinnacle sets

Let \( \mathfrak{P}_n(S) \) be the set corresponding to the numbers \( p_n(S) \) defined earlier:

\[ \mathfrak{P}_n(S) = \{ \sigma \in \mathfrak{S}_n \mid \text{Pin } \sigma = S \}. \]

As observed in [DLMSSS], it is immediate that if we have \( \text{max}(S) \leq n \) then

\[ p_{n+1}(S) = 2 p_n(S). \]

Indeed, starting with an element \( \sigma \) of \( \mathfrak{P}_{n+1}(S) \), the value \( n+1 \) is not a pinnacle, so it must be either at the first or at the last position of \( \sigma \). Then, removing it provides an element of \( \mathfrak{P}_n(S) \). This map is surjective on \( \mathfrak{P}_n(S) \) since given any element of \( \mathfrak{P}_n(S) \), there are two ways of going backwards (put back \( n+1 \) either in first or last position), whence the coefficient 2.

3. An efficient recursion for \( p_n(S) \)

We shall make use of the following notations. Let \( S = \{ s_1, \ldots, s_p = n \} \) be a pinnacle set. Let \( t(S) \) (or \( t \) for short when there is no ambiguity) be the largest value smaller than \( s_p \) that does not belong to \( S \). Then write \( S \) as \( S = T \cup \{ t+1, \ldots, n \} \), with \( T = \{ s \in S \mid s < t \} \).

Theorem 3.1. Let \( S = \{ s_1, \ldots, s_p = n \} \) be a pinnacle set. Let \( t \) and \( T \) be defined as before.

We then have

\[
\begin{align*}
p_n(T \cup \{ t+1, \ldots, n \}) &= p_n(T \cup \{ t, t+2, t+3, \ldots, n \}) \\
& \quad + 2 p_{n-1}(T \cup \{ t, t+1, \ldots, n-1 \}) \\
& \quad + 2(n-t) p_{n-2}(T \cup \{ t, t+1, \ldots, n-2 \}).
\end{align*}
\]

For example,

\[
p_{12}(\{4, 8, 12\}) = p_{12}(\{4, 8, 11\}) + 2 p_{11}(\{4, 8, 11\}) + 2 p_{10}(\{4, 8\})
\]
and
\begin{equation}
(8) \quad p_{12}(\{4, 8, 11, 12\}) = p_{12}(\{4, 8, 10, 12\}) + 2 p_{11}(\{4, 8, 10, 11\}) + 4 p_{10}(\{4, 8, 10\}).
\end{equation}

Proof – Let \( S = \{s_1, \ldots, s_p = n\} \) be a pinnacle set and consider the possible positions of \( t \) among the elements of \( \mathfrak{P}_n(S) \). We shall show that \( \mathfrak{P}_n(S) \) can be partitioned into three sets, each one being in correspondence with (multiple copies of) other sets \( \mathfrak{P}_{n'}(S') \).

Let \( \sigma \) be an element of \( \mathfrak{P}_n(S) \). Since \( t \) is not a pinnacle, it is either on a side of \( \sigma \), or next to at least one value greater than itself. We partition those cases into three disjoint sets of cases: either \( t \) is not next to \( t+1 \), or \( t \) is on a side (and next to \( t+1 \)), or \( t \) has two neighbours and one is \( t+1 \). This last case splits into two subcases depending on the value of the second neighbour of \( t \).

Case 1: \( t \) is not next to \( t+1 \). Since \( t \) is not a pinnacle value but \( t+1 \) is, if one exchanges \( t \) and \( t+1 \), one gets an element of \( \mathfrak{P}_{n'}(S') \), where \( S' = T \cup \{t, t+2, t+3, \ldots, n\} \): the pinnacle value \( t+1 \) is transformed into the pinnacle value \( t \) and all other values remain (non)pinnacle as in \( \sigma \). Moreover, this is a bijection onto \( \mathfrak{P}_{n'}(S') \) since applying the same elementary transposition, one gets back from \( \mathfrak{P}_{n'}(S') \) to \( \mathfrak{P}_n(S) \). This is the first term in the formula of the statement.

Otherwise, \( t \) and \( t+1 \) are next to each other. Case 2: \( t \) has as neighbours \( t+1 \) and another one smaller than itself. Then remove \( t \) from \( \sigma \) and decrement all values from \( t+1 \) up so that we end up with a permutation. We then get an element of \( \mathfrak{P}_{n-1}(S'') \), with \( S'' = T \cup \{t, t+1, \ldots, n-1\} \). Now starting with an element of \( \mathfrak{P}_{n-1}(S'') \), there are two ways to get back to \( \mathfrak{P}_n(S) \) with an inverse transform: increment all values from \( t \) up and put \( t \) either at the left or at the right of \( t+1 \), hence the coefficient 2 in the second term of the statement.

Case 3: \( t \) has no other neighbour than \( t+1 \). It is at the first or last position of \( \sigma \). If one removes both \( t \) and \( t+1 \), one gets (up to renumbering the large values) an element of \( \mathfrak{P}_{n-2}(S'''') \), with \( S''' = T \cup \{t, t+1, \ldots, n-2\} \). There are again two ways of going backwards: start with an element \( p \) of \( \mathfrak{P}_{n-2}(S'''') \) and call \( p' \) the element obtained by incrementing twice all values from \( t \) up. Then glue either \((t, t+1)\) at the beginning of \( p' \) or \((t+1, t)\) at the end of \( p' \). Since all values greater than \( t \) are pinacles, they cannot be at an extremity of \( p' \) so we indeed get in both cases an element of \( \mathfrak{P}_n(S) \). This contributes a coefficient 2 in front of the third term of the statement.

Case 4: \( t \) is next to two elements greater than itself. Then remove both \( t \) and \( t+1 \). We then get again an element of \( \mathfrak{P}_{n''}(n-2) \) up to shifting values. We then have to put back \( t \) and \( t+1 \) and since there are \((n-t-1)\) values at least equal to \( t+2 \) and two options to put back the pair \((t, t+1)\) next to any of those (either \( t+1, t \) to their left, or \( t, t+1 \) to their right), this case contributes a coefficient \( 2(n-t-1) \) in front of the third term of the statement.

Summing everything up shows the desired formula.
3.1. Known formulas for \( p_n(S) \). Let us now recover the formulas when \( S \) has at most two elements.

Consider \( S = \{n\} \). Then \( T = \emptyset \) and \( t = n - 1 \), so that
\[
 p_n(\{n\}) = p_n(\{n - 1\}) + 2 p_{n-1}(\{n - 1\}) + 2 p_{n-2}(\{\}),
\]
which simplifies into
\[
 p_n(\{n\}) = 4 p_{n-1}(\{n - 1\}) + 2^{n-2},
\]
and we recover Formula (6) of Prop. 3.6 of [DNPT]:
\[
 p_n(\{n\}) = 2^{n-2}(2^{n-2} - 1).
\]

With \( S = \{n - 1, n\} \), we have \( T = \emptyset \) and \( t = n - 2 \), and
\[
 p_n(\{n - 1, n\}) = p_n(\{n - 2, n\}) + 2 p_{n-1}(\{n - 2, n - 1\}) + 4 p_{n-2}(\{n - 2\}),
\]
and with \( S = \{\ell, n\} \) and \( \ell < n - 1 \), one obtains an equation similar to (9):
\[
 p_n(\{\ell, n\}) = p_n(\{\ell, n - 1\}) + 2 p_{n-1}(\{\ell, n - 1\}) + 2 p_{n-2}(\{\ell\}).
\]

3.2. Evaluating \( p_n(S) \) with our formula. Note that the sum in Formula (6) only contains positive coefficients and in particular is not obtained by inclusion-exclusion.

Moreover, this formula can be applied recursively in order to compute \( p_n \): the sum of the elements in each set on the right-hand side is strictly smaller than the sum of \( S \), hence showing that the iteration of the formula always stops either on a non-pinnacle set or on the empty set. In particular, the induction relies only upon the initial values \( p_n(\emptyset) = 2^{n-1} \), all derived from \( p_1(\emptyset) = 1 \).

We shall now discuss the complexity of this algorithm. It is surprisingly low. Define \( C(S) \) as the set of all pinnacle sets that will be involved at some point when iterating Formula (6) on a set \( S \). The size of \( C(S) \) measures the complexity of the computation.

Note that in Formula (6) the first and second terms have \( |S| \) elements in their pinnacle sets whereas the third one has only \( |S| - 1 \). Note on the other hand that the second term appears at a later inductive step when computing the first one. So the number of sets of size \( |S| \) in \( C(S) \) is equal to the number of sets appearing in the simpler recursion
\[
 p_n(T \cup \{t+1, \ldots, n\}) = p_n(T \cup \{t, t+2, t+3, \ldots, n\}).
\]

But this equation is easy to analyse in terms on complexity: it requires only one new element at each step, whose sum of values is strictly smaller than the previous one: it necessarily stops after \( \sum s_i \) steps, so as for the particular sets of \( C(s) \) of size \( |S| \), their number is smaller than \( n|S| \).

Now, regarding the other sets, a quick look at Formula (6) shows that the sets of cardinality \( |S| - 1 \) are all called from the set \( p_{n-2}(T \cup \{t, t+1, \ldots, n-2\}) \). This same idea applies again to sets of smaller cardinality, hence showing that \( A(S) \) is at most \( n(\text{s)}) \).
Proposition 3.2. The total number of pinnacle sets required to compute \( p_n(S) \) recursively is at most \( n|S|^2/2 \).

The complexity of our computation is therefore a (low degree) polynomial in both \( n \) and \( |S| \).

The following Python code is very efficient:

```python
def is_pinset(S):
    S = sorted(S)
    return all([S[i]>2*(i+1) for i in range(len(S))])

pindic = {tuple([]):0}
def p(S, n):
    S = tuple(sorted(S))
    if (S,n) in pindic: return pindic[S,n]
    if not is_pinset(S): return 0
    if not S: return 2**(n-1)
    if n<2: return 0
    if n>S[-1]: return 2**(n - S[-1])*p(S,S[-1])
    m = n
    while m in S: m-=1
    sa = tuple([z for z in S if z>m])
    sb = tuple([z for z in S if z<m])
    X = sb + (sa[0]-1,) + sa[1:]
    Y = sb + tuple([z-1 for z in sa])
    Z = sb + tuple([z-1 for z in sa[:-1]])
    res = p(X, n) + 2*p(Y, n-1) + 2*(n-m)*p(Z, n-2)
    pindic[S,n] = res
    return res
```

For example,

```python
>>> p({5,17,31,42,79,88,97},100)
175144760022244699153356193204473616098046926340653078867873357075537934828864484147200
```

and the number of recursive calls can be read from the dictionary

```python
>>> len(pindic)
753
```

4. A system of subrepresentatives of classes

4.1. Grouping permutations with the same pinnacle set. First note that both \( p_n(\{\} \) and \( p_n(\{k\} \) are multiples of \( 2^{n-1-|S|} \). And Formula (15) shows that it is also the case for any \( S \): the first term has the required power of 2 by the inductive hypothesis and the second and third terms have an inductive factor of \( 2^{n-2-|S|} \), but both have an extra factor 2 which yields the required power of 2.

Let us define \( p'_n(S) \) as

\[
p'_n(S) := p_n(S)/2^{n-1-|S|}.
\]

Thanks to Formula (15), the value of \( p'_n(S) \) is independent of \( n \) if \( n \geq \max(S) \) so the notation \( p'(S) \) makes sense without \( n \).
Translating Formula (6) on $p'$s, we get the following induction:

\[
p'_n(T \cup \{k+1,\ldots,n\}) = p'_n(T \cup \{k, k+2, k+3,\ldots,n\})
\]

\[+ p'_{n-1}(T \cup \{k, k+1,\ldots,n-1\})
\]

\[+ (n-k) p'_{n-2}(T \cup \{k, k+1,\ldots,n-2\}).\]

(17)

This divisibility property was already noted in [DLHHIN] where the authors construct a set of representatives called minimal elements of the orbits of their dual Foata-Strehl action (see their Definition 3.5).

4.1.1. The (modified) Foata-Strehl action of [DLHHIN]. Let $\sigma$ be a permutation and $k$ be a letter of $\sigma$. Define the factorization of $\sigma$ according to $k$ as

\[
\sigma = \alpha_k \beta_k k \gamma_k \delta_k,
\]

where $\beta_k$ (respectively $\gamma_k$) is the longest (consecutive) sequence of letters smaller than $k$ immediately to the left (resp. right) of $k$. When there is no ambiguity, we shall write $\alpha$ instead of $\alpha_k$ and similarly for the other factors.

If one represents $\sigma$ as a path with successive heights equal to the values of $\sigma$, the sets $\beta$ and $\gamma$ represent what $k$ “sees” below itself, values higher than $k$ “blocking” its view.

Then define as in [DLHHIN] the map $\varphi_k(\sigma)$ as

\[
\varphi_k(\alpha_k \beta_k k \gamma_k \delta_k) = \alpha_k \gamma_k k \beta_k \delta_k.
\]

(19)

This collection of maps satisfy some simple properties, all direct corollaries of the original paper of Foata and Strehl [FS]:

**Proposition 4.1.** Let $\sigma$ be a permutation.

- $\varphi$ does not change the pinnacle set of $\sigma$,
- $\varphi$ does not change the vale set of $\sigma$, the set of values smaller than (both) their neighbour(s).
- if one defines $\sigma' = \varphi_\ell(\sigma)$ and, if one writes $\sigma = \alpha \beta k \gamma \delta$ and $\sigma' = \alpha' \beta' k \gamma' \delta'$ as their factorizations according to $k \neq \ell$, then $\beta$ and $\beta'$ have the same values (maybe not in the same order) and the same is true for $\gamma$ and $\gamma'$.
- all the $\varphi$s commute so that if $X = \{x_1,\ldots,x_j\}$ is a set, $\varphi_X = \varphi_{x_1} \circ \cdots \circ \varphi_{x_j}$ is independent of the order of the $x$s, hence well-defined,
- the orbit of a permutation $\sigma$ is of cardinality $2^n - |S|$ where $S$ is the pinnacle set of $\sigma$.

The last property is best understood in terms of the vales of $\sigma$. Indeed, one can check that $\varphi_x(\sigma) = \sigma$ iff $x$ is a vale of $\sigma$ and since the number of vales $|V|$ is the number of peaks plus one, the cardinality of the orbit is $2^n - |V|$.

Then, given a permutation $\sigma$, the authors of [DLHHIN] define a particular subset $W(\sigma)$ of values of $\sigma$ and show that any element $\tau$ in the same orbit as $\sigma$ satisfies that $e = \varphi_W(\tau)(\tau)$ does not depend on $\tau$, and they call $e$ the FS-minimal element of the orbit. Note that this definition might be misleading since their elements are not the lexicographically minimal elements of each orbit.
Since the orbits of the modified Foata-Strehl action have cardinality $2^{n-1-|S|}$, they split the sets $\mathcal{P}_n(S)$ into $p'_n(S)$ classes. Given that the induction on $p'$ of Equation (17) comes from the induction on $p$ given in (6) which is itself an induction on sets, we translate (17) into an induction on sets $\mathcal{P}'_n(S)$. We will show later that these sets happen to be a section of the orbits, and more precisely, the lexicographically minimal elements of each orbit.

4.2. The set $\mathcal{P}'_n(S)$ enumerated by $p'_n(S)$. Let $S$ be an admissible pinnacle set and write as before $S = T \cup \{t+1, t+2, t+3, \ldots, n\}$ where $t = t(S)$.

First, as an initialization of the induction, define $\mathcal{P}'_n(\emptyset)$ as the identity permutation $1 \ldots n$. More generally, the set $\mathcal{P}'_n(S)$ with $s > n$ is obtained from $\mathcal{P}'_{n-1}(S)$ by adding $s$ to the right of all its elements.

Finally, if $n = \max(S)$, the set $\mathcal{P}'_n(S)$ is the union of the following three sets:

- First, consider the set $A_1 = \mathcal{P}'_n(T \cup \{t, t+2, t+3, \ldots, n\})$. Then our first subset of $\mathcal{P}'_n(S)$ is obtained by exchanging $t$ and $t+1$ in each element of $A_1$.
- Second, consider the set $A_2 = \mathcal{P}'_{n-1}(T \cup \{t, t+1, \ldots, n-1\})$. Then our second subset of $\mathcal{P}'_n(S)$ is obtained by sending each element of $A_2$ to that obtained by mapping its values $x$ onto

\[
\begin{cases}
  x & \text{if } x < t, \\
  t, t+1 & \text{if } x = t, \\
  x+1 & \text{otherwise.}
\end{cases}
\]  

- Third, consider the set $A_3 = \mathcal{P}'_{n-2}(T \cup \{t, t+1, \ldots, n-2\})$. Then our third subset of $\mathcal{P}'_n(S)$ is the union of $n-t$ different sets obtained from $n-t$ different maps from $A_3$.

For each value $q$ between $t+1$ and $n-1$, send an element of $A_3$ to that obtained by mapping its values $x$ onto

\[
\begin{cases}
  x & \text{if } x < t, \\
  t+1, t, x+2 & \text{if } x = q, \\
  x+2 & \text{otherwise.}
\end{cases}
\]  

The last set is obtained by mapping each element of $A_3$ to that obtained by mapping its values $x$ onto

\[
\begin{cases}
  x & \text{if } x < t, \\
  x+2 & \text{otherwise}
\end{cases}
\]  

and gluing $t+1, t$ to the right of it.

For example, one can compute that $\mathcal{P}'_6(\{4, 6\})$ is

\[
\{124365, 134265, 142365, 142563, 143635, 143562, 143625, 156243, 162435\},
\]

that $\mathcal{P}'_5(\{4, 5\})$ is

\[
\{14253, 14352, 15243\},
\]

and that $\mathcal{P}'_4(\{4\})$ is

\[
\{1243, 1342, 1423\}.
\]
so that we get the union of the following three sets as $\mathcal{P}'_n((5, 6))$ (in that case $t = 4$):

$$A_1 := \{125364, 135264, 152364, 152463, 152634, 153462, 153624, 146253, 162534\},$$

(26) $$A_2 := \{145263, 145362, 162453\},$$

$$A_3 := \{125463, 135462, 125634, 136254, 126354, 136254, 162354\}.$$

4.3. Properties of $\mathcal{P}'_n(S)$.

4.3.1. Cardinality of $\mathcal{P}'_n(S)$. The first property of $\mathcal{P}'_n(S)$ is that it is indeed a subset of the set of permutations of $S_n$ with pinnacle set $S$.

**Lemma 4.2.** Given a set $S$ and an integer $n$, the elements of $\mathcal{P}'_n(S)$ all have $S$ as pinnacle set and are distinct. Moreover, $\#\mathcal{P}'_n(S) = p'_n(S)$.

**Proof** – The facts that the pinnacle set of any element of $\mathcal{P}'_n(S)$ is $S$ and that there are no repetitions in $\mathcal{P}'_n(S)$ are immediate by induction thanks to the argument of the proof of Theorem 3.1: given any element of $\mathcal{P}'_n(S)$, one can easily find to which case it corresponds, and compute the representative of $\mathcal{P}'_n(S')$ it comes from. Note also that all maps defining the third subset of $\mathcal{P}'_n(S)$ have disjoint images. All maps being disjoint, the cardinalities of $\mathcal{P}'$ follow from the induction on $p'$s. ■

4.3.2. Characterization of the elements $\mathcal{P}'_n(S)$.

**Proposition 4.3.** A permutation $\sigma \in S_n$ with pinnacle set $S$ belongs to $\mathcal{P}'_n(S)$ iff

- it begins with 1,
- it has no double descents (no position $i$ such that $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$),
- if $k$ is a pinnacle, $\min(\beta_k) < \min(\gamma_k)$.

In our representation of permutations as landscapes, this last condition translates as: the smallest vale that a pinnacle “sees” is to its left.

**Proof** – Our proof decomposes into three steps. First, we show that all elements of $\mathcal{P}'_n(S)$ satisfy the properties of the statement. Second, we show that if $\sigma \in \mathcal{P}_n(S)$ satisfies the conditions of the statement then its preimage $\sigma'$ in the inductive definition of $\mathcal{P}_n(S)$ also does and hence belongs to a $\mathcal{P}'_n(S')$ by induction. Third, we show that starting from $\sigma'$ back to $\mathcal{P}_n(S)$, the only way to get an element satisfying the conditions is to use the maps defining $\mathcal{P}'_n(S)$.

We shall first see that all elements of $\mathcal{P}'_n(S)$ satisfy the properties of the statement. Indeed, the elements of $\mathcal{P}'_n(S)$ all begin with 1 and have no double descents directly by their inductive definition. Regarding the property on peaks, we do not add peaks in the first or second case of the induction so the property holds. Let us now consider an element $r$ of $A_3$. If one modifies some of its values and adds $t + 1, t$ to its right, the property holds for all pinnacles of $r$: either they still view the same elements to their left and right, or they now see the extra elements $t$ and $t + 1$ to their right. The new pinnacle $t + 1$ also satisfies the condition since all vales are smaller than $t$, it sees one vale to its left (which is smaller than $t$) and it only sees $t$ to its right.

Now, if one glues $(t + 1, t)$ next to a pinnacle greater than $t + 1$ in $r$, the property still holds for the same reasons as before: either a pinnacle sees the same elements to
its left and right as in \( r \) or it sees \( t \) and \( t + 1 \) as extra elements one side or the other. But since it also sees other vales that are by definition of \( t \) smaller than \( t \), seeing \( t \) and \( t + 1 \) is irrelevant to knowing which side the minimum is.

Conversely, let us show that any element \( \sigma \) satisfying the conditions belongs to a \( \mathcal{P}_n'(S) \) by induction on its pinnacle set \( S \). Define \( t(S) \) as usual on pinnacle sets and find out which case applies to \( \sigma \). Then apply the transform required to get its preimage \( \sigma' \) (see the proof of Theorem 3.1). Then \( \sigma' \) satisfies the conditions. Indeed, in Case 1, exchanging \( t \) and \( t + 1 \) if they are not neighbours does not change the position of 1, does not create a double descent, and does not change the relative order on minima around pinnacles since \( t + 1 \) was not a vale to begin with (and \( t \) and \( t + 1 \) are consecutive). In Case 2, removing \( t \) if it is next to \( t + 1 \) on one side and to a value smaller that itself on the other again does not move 1, does not create a double descent, and since \( t \) was not a vale, the minima change nowhere. In Cases 3 and 4, removing \( t \) and \( t + 1 \) if they are next to a higher pinnacle does not move 1, does not create a double descent, and all minima stay the same. So in all cases, the pre-image element \( \sigma' \) also satisfies the conditions and hence belongs to a \( \mathcal{P}_n'(S) \) by induction.

Let us now prove that going back from \( \sigma' \) to an element of \( \mathcal{P}_n(S) \) satisfying the conditions of the statement can only be done using the maps in the definition of \( \mathcal{P}_n'(S) \). Indeed, let us use the notations of this definition and assume that \( \sigma' \) is in \( A_1 \). There is only one way to get an element of \( \mathcal{P}_n(S) \) so there is nothing to prove. Now, if \( \sigma' \) is in \( A_2 \), one gets back to \( \mathcal{P}_n(S) \) by adding \( t \) either to the left or to the right of \( t + 1 \) (Case 2 in the proof of Theorem 3.1). But gluing \( t \) to the right of \( t + 1 \) does not work since this would create a double descent. Finally, if \( \sigma' \) is in \( A_3 \), we have to put back \( t \) and \( t + 1 \) together, either at an extremity of \( \sigma' \) or next to a higher pinnacle. Putting \((t + 1, t)\) at the beginning of \( \sigma' \) does not work since in that case the pinnacle \( t + 1 \) would violate the third condition. And putting \((t, t + 1)\) to the right of a higher pinnacle would also not work for the same reason. So the only inductive steps moving back to an element of \( \mathcal{P}_n(S) \) satisfying the conditions of the statement require to start from an element of a \( \mathcal{P}_n'(S) \) and only apply the induction steps defining \( \mathcal{P}_n'(S) \), hence the result.

4.3.3. \( \mathcal{P}_n'(S) \) and the orbits of FS. We shall make use of the lexicographic order on permutations and denote it by \( \leq \).

**Lemma 4.4.** Let us consider an element \( \sigma \) in an orbit of the modified Foata-Strehl action.

If \( \sigma \) has a double descent \( \sigma_{i-1} > \sigma_i > \sigma_{i+1} \), then \( \varphi_{\sigma_i}(\sigma) < \sigma \).

If \( \sigma \) has no double descent and if there are pinnacles \( k \) in \( \sigma \) so that \( \min(\beta_k) > \min(\gamma_k) \) in their factorization, let \( \ell \) be the smallest such pinnacle. Then \( \varphi_{\ell}(\sigma) < \sigma \).

**Proof** – The case of the double descent is immediate since \( \beta \) is empty and all letters in \( \gamma \) are smaller than \( \sigma_i \) and move to its left.

Let \( \ell \) be defined as in the statement. Then starting from it and moving left, it sees a nonempty succession of vales with pinnacle smaller than itself in between until it meets a higher pinnacle. Since these intermediate pinnacles \( \ell' \) are smaller than \( \ell \),
they satisfy \( \min(\beta_{\ell'}) < \min(\gamma_{\ell'}) \) so that the vales are decreasing. Moreover, \( \beta_{\ell} \) cannot begin with a descent since either it begins with 1 or it has a pinnacle before it and we assumed that \( p \) had no double descents. So the first letter of \( \beta_{\ell} \) is its minimum. The same holds to the right of \( \ell \) in \( p \), so the first letter of \( \gamma_{\ell} \) is its minimum. Therefore \( \varphi_{\ell}(p) < p \). 

\[ \text{Theorem 4.5. The elements of the sets } \mathcal{P}'_n(S) \text{ are the lexicographically minimal elements of their orbits.} \]

\textbf{Proof –} Thanks to Lemma 4.4, we know that an element that does not satisfy the conditions of 4.3 cannot be lexicographically minimal in its orbit since one can apply \( \varphi \) to it and obtain a smaller element. So the lexicographically minimal element of its orbit satisfies the conditions and thus belongs to a \( \mathcal{P}'(S) \).

Now let \( \sigma \) be the lexicographically minimal element of an orbit and consider any other element \( \sigma' \) of this orbit. Since the orbit is connected, there is a set \( X \) such that \( \varphi_X(p) = p' \). We can assume that \( X \) is minimal, so that any element of \( X \) acts non trivially on \( \sigma \). Compute \( \sigma'' = \varphi_{x_1}(\sigma) \) with \( x_1 = \min(X) \). Then \( \sigma'' \) violates the conditions next to \( x_1 \): this \( x_1 \) cannot be a vale since \( \sigma'' \neq \sigma \) so either \( x_1 \) was in the middle of a double rise in \( \sigma \) and it is a double descent in \( \sigma'' \) or \( \varphi_{x_1} \) exchanged the non-trivial sets \( \beta_{x_1} \) and \( \gamma_{x_1} \) and hence violates the third condition. Now, with all the remaining steps from \( \sigma'' \) to \( \sigma' \), the neighbourhood of \( x_1 \) does not change (see Proposition 4.1) so \( \sigma' \) also violates the conditions on \( x_1 \). So no other element in the orbit satisfies all conditions but the lexicographically minimal one.

Note that our algorithm provides an efficient way to build all lexicographically minimal elements of the dual Foata-Strehl orbits.

\[ \text{4.3.4. } \mathcal{P}'_n \text{ and the left weak order.} \]

Let us consider the whole set \( \mathcal{P}'_n \) which is the union of all \( \mathcal{P}'_n(S) \) with \( S \) a pinnacle set with maximum at most \( n \).

Recall that the left weak order on \( S_n \) is the transitive closure of the relation on permutations given by \( \sigma < \tau \) if \( s_i.\sigma = \tau \) and \( \ell(\tau) = \ell(\sigma) + 1 \) where \( s_i \) is the transposition \((i, i+1)\) and \( \ell \) is the number of inversions of permutations. And recall that a lower ideal of a poset is a subset \( S \) of this poset such that \( s \in S \) implies that \( t \in S \) for all \( t < s \).

\[ \text{Theorem 4.6. } \mathcal{P}'_n \text{ is a lower ideal of the left weak order.} \]

\textbf{Proof –} We just have to prove that if \( \tau \in \mathcal{P}'_n \), then \( s_i.\tau = \sigma \in \mathcal{P}'_n \) too if \( \ell(\tau) = \ell(\sigma) + 1 \). Thanks to their characterization, we know that \( \tau \) begins with 1, has no double descent and that \( \beta_k < \gamma_k \) for any \( k \) in the pinnacles of \( \tau \).

The first two criteria are automatic with \( \sigma \). Concerning the minima, since the exchange of \( i \) and \( i+1 \) moves \( i+1 \) further right and since \( i \) and \( i+1 \) are consecutive values, there is no situation where a left minima goes from smaller to greater than a right minima.
When given an ideal, it is customary to consider its maximal elements. In our case, the first maximal elements of our ideal are

\begin{align*}
12 \\
132 \\
1342, 1423 \\
1432, 14352 \\
14523, 15243 \\
145362, 153462 \\
154623, 156243 \\
1562453, 162453 \\
162453, 162534 \\
1546372, 1563472 \\
1564723, 1635472 \\
1645723, 1725463 \\
1725634, 1726354
\end{align*}

Recall that the standardisation process \( \text{std} \) of a word without repetition amounts to renumbering the values with 1 up to its size in the order they were in the beginning. For example, \( \text{std}(15726) = 13524 \).

**Theorem 4.7.** The maximal elements of \( \mathfrak{P}'_n \) are the elements \( \sigma \) of \( \mathfrak{P}'_n \) satisfying the extra conditions:

- Cut \( \sigma \) as \( \sigma = u.n.v \). Then \( \text{std}(u) \) and \( \text{std}(v) \) are themselves maximal elements,
- the values of \( v \) form an interval \([2, \ell]\).

**Proof** – Let us first prove that an element \( \sigma \) of \( \mathfrak{P}'_n \) that does not satisfy one extra condition cannot be maximal. First, if there is a letter \( i > 1 \) in \( u \) such that \( i + 1 \) is in \( v \), then exchange \( i \) and \( i + 1 \). This element is still in \( \mathfrak{P}'_n \) since the only condition that could fail is the pinnacle condition on \( n \), but \( n \) still sees 1 on the left. Now, if \( \sigma \) satisfies the first extra condition but not the second, \( v \) is composed of consecutive letters and if its standardized in not maximal, the same transposition (shifted by one) applied to it shows that \( v \) is not maximal either. The same argument applies to \( u \).

Conversely, let us prove that an element that satisfies both extra conditions is indeed maximal. Since \( u \) and \( v \) are maximal, there is no transposition inside \( u \) or \( v \) that could bring another element in \( \mathfrak{P}'_n \) since violating a condition is independent from what happens on the other side of \( n \). It is also impossible to move 1 so the only allowed transposition increasing the inversion number of \( \sigma \) is the transposition \((n - 1, n)\). But this one fails since after the exchange \( n - 1 \) sees 2 on its right and does not see 1 anymore on its left.

**Corollary 4.8.** Let \( M_n \) be the set of permutations defined inductively by

- \( M_1 = \{1\} \),
- \( M_2 = \{1, 2\} \),
- \( M_n \) is obtained as the union for all \( k \) of the sets \( M'_{n,k} \) where an element of \( M'_{n,k} \) is the concatenation \( a_{n-1-k}nb_k \) where \( a_{n-1-k} \) is an element of \( M_{n-1-k} \) whose values at least 2 have been shifted by \( k \) and \( b_k \) is an element of \( M_k \) whose values have been shifted by 1.

Then \( M_n \) is the set of the maximal elements of the ideal of the minimal elements of the (modified) Foata-Strehl orbits.
Corollary 4.9. The maximal elements of $\mathcal{P}'_n$ are enumerated by Sequence A007477 of [Slo] (up to a change of indices) since they satisfy the induction formula

$$p'_n = \sum_{k=1}^{n-2} p'_k p'_{n-1-k}.$$  

Here are the first terms of A007477.

$$1, 1, 1, 2, 3, 6, 11, 22, 44, 90, 187$$

and one can check that this is consistent with the list given in (27).

Proof – Immediate from the previous characterization by summing over the different positions that $n$ can occupy.

For example, one easily gets $M_4 = \{1342, 1423\}$ and $M_5 = \{14352, 14523, 15243\}$ so that $M'_0, 4$ is obtained as all concatenations of an element of $\{18796, 18967, 19687\}$ with 10 and with an element of $\{2453, 2534\}$.

5. A conjectural formula for $q_n(S)$

In [DNPT], Question 4.5, it was conjectured that

$$q_n(S) := \sum_{I \subset S} 2^{\|I\|} p_n(I)$$

had a nice formula and indeed, we shall conjecture a general formula for $q_n(S)$.

We shall write it as a product $q'_n(S) r_n(S)$, where $q'$ takes into account (almost all) the powers of 2 that appear as factors in the overall formula, and $r$ has a more complicated formula.

Let $S = \{s_1, \ldots, s_p\}$ be a pinnacle set and $n \geq s_p$. Then let

$$q'_n(S) = 2^{n-1} s_p - 2^{s_p-1} 2^{s_p-2} 2^{s_p-3} \ldots$$

We shall describe a simple but quite surprising algorithm to compute the second factor $r_n(S)$. First, define

$$D(S) = (d_1, \ldots, d_{p-1}) := (s_1 - 1, s_2 - s_1, \ldots, s_{p-1} - s_{p-2}).$$

Then build the following abstract expressions

$$E_1 := x_{1,3} + x_{1,1},$$

and

$$E_{2k+1} := f_e(E_{2k}) \text{ and } E_{2k} := f_o(E_{2k-1}),$$

where $f_o$ substitutes all elements involving the largest index $a$ as

$$f_o(x_{a,k}) = x_{a,k} \left( x_{a+1,k+\frac{1}{2}} + x_{a+1,k-\frac{1}{2}} \right)$$

with the convention $x_{a,0} = 0$, and $f_e$ substitutes all elements involving the largest index $a$ as

$$f_e(x_{a,k}) = x_{a,k} (x_{a+1,2k+1} + x_{a+1,2k-1}).$$
For example, we get the following first expressions
\begin{align}
E_2 &= f_o(E_1) = x_{1,3}(x_{2,2} + x_{2,1}) + x_{1,1}x_{2,1}, \\
E_3 &= f_o(E_2) = x_{1,3}(x_{2,2}(x_{3,5} + x_{3,3}) + x_{2,1}(x_{3,3} + x_{3,1})) + x_{1,1}x_{2,1}(x_{3,3} + x_{3,1}), \\
E_4 &= f_o(E_3) = x_{1,3}(x_{2,2}(x_{3,5}(x_{4,3} + x_{4,2}) + x_{3,3}(x_{4,2} + x_{4,1}))) \\
&+ x_{2,1}(x_{3,3}(x_{4,2} + x_{4,1}) + x_{3,1}x_{4,1}) \\
&+ x_{1,1}x_{2,1}(x_{3,3}(x_{4,2} + x_{4,1}) + x_{3,1}x_{4,1}).
\end{align}

Then, if \( S \) has \( p \) elements so that \( D(S) \) has \( p - 1 \), define
\begin{equation}
r_n(S) := ev(E_{p-1}),
\end{equation}
where \( ev \) evaluates \( x_{i,j} \) to \( j^{d_{p-i}} \).

**Conjecture 5.1.** For all pinnacle set \( S \) and all \( n \geq \max(S) \), the value of \( q_n(S) \) is equal to the product \( q'_n(S) r_n(S) \).

For example, if one replaces the \( d \)s by their values as parts of \( S \), the first formulas read
\begin{align}
q_n(\{m\}) &= 2^{n-1}2^{m-2}, \\
q_n(\{\ell, m\}) &= 2^{n-1+m-2-\ell} (3^{\ell-1} + 1), \\
q_n(\{k, \ell, m\}) &= 2^{n-1+m-2-\ell+k-2} [3^{\ell-k}(2^{k-1} + 1) + 1], \\
q_n(\{j, k, \ell, m\}) &= 2^{n-1+m-2-\ell+k-2-j} [3^{\ell-k} (2^{k-j}(5^{j-1} + 3^{j-1}) + 3^{j-1} + 1) + 3^{j-1} + 1] \\
q_n(\{i, j, k, \ell, m\}) &= 2^{n-1+m-2-\ell+k-2-j+i-2} \\
&[3^{\ell-k} (2^{k-j}(5^{j-i}(3^{i-1} + 2^{i-1}) + 3^{i-1}(2^{i-1} + 1)) + 3^{j-i}(2^{i-1} + 1)) + 3^{j-i}(2^{i-1} + 1)].
\end{align}

5.1. **Properties of our conjectural formula for \( q_n(S) \).**

5.1.1. *The variables \( x_a \).* By definition of \( f_o \), any \( E_{2k+1} \) only contains factors \( x_{2k+1,i} \) with odd \( i \) so that applying \( f_o \) to these will only put integer-valued \( j \) in the new variables \( x_{2k+2,j} \).

5.1.2. *Structure of the formula.* Thanks to its structure, it is obvious that \( E_k \) is a sum of monomials without multiplicities. Moreover, a simple induction shows that \( E_k \) exactly has \( \left( \frac{k+1}{(k+1)/2} \right) \) terms, so that these terms are in bijection with the pinnacle sets with maximum at most \( k + 2 \).

There is a simple bijection between both sets where the various \( x_a \)s encode the size of the pinnacle set but this bijection does not seem very relevant to better understanding (or proving) the formula.

5.1.3. *Special values.* By definition, \( q_n(S) \) can be nonzero even if \( S \) is not a pinnacle set. In particular, the definition implies that
\begin{equation}
q_n(\{2\} \cup S) = q_n(S).
\end{equation}

Even if our formula does not coincide in all non-pinnacle cases, it indeed does seem to coincide in that case. It was in fact a big help in finding the first formulas with small values of \( |S| \).
5.1.4. Computing $p_n(S)$. Assuming our formula for $q_n(S)$ is correct, it provides another algorithm to compute $p_n(S)$: first, compute the expressions $E_k$ up to $k = |S| - 1$, then apply these to all subsets of $S$ and apply an inclusion-exclusion process to get $p_n(S)$.

Note that it is very possible to brute-force $p_n$ on pinnacle sets with 3 or 4 values, get a general formula, and prove it using our induction. However, we do not expect this strategy to help prove our conjectural formula for $q_n$ in general.

In terms of computation time, this algorithm is much less efficient than our algorithm using Formula (6) in general since it requires to compute $2^{|S|}$ terms when our first algorithm requires $n^{|S|}$ terms.

But we can still expect that a proof and a better understanding of our formula for $q_n(S)$ could bring ideas to get a general formula for $p_n(S)$.

6. Ordered forests of complete binary trees

Among all combinatorial objects enumerated by the central binomial sequence A001405 of [Slo] are the pinnacle sets and ordered forests of complete binary trees. The bijection between both is very simple and sheds light on some structures of pinnacle sets.

Let $S = \{s_1, \ldots, s_p\}$ be a pinnacle set with $n \geq s_p$. We shall then build a sequence of complete binary trees as follows:

- set $k = n$ and start with a sequence of one tree: the tree with one node.
- Put $k$ in the rightmost empty node. If $k \in S$ then draw both children of $k$ as empty nodes. Otherwise, do not draw those children and if there is no empty node anymore, put a new tree with one node to the left of the previous one. Set $k := k - 1$ and start again this step until $k = 0$.

For example, if $S = \{4, 8, 9, 12\}$ and $n = 13$, we get the following forest:

```
  🍇  🍇  🍇  🍇  🍇  🍇  🍇  🍇  🍇  🍇  🍇  🍇  🍇  🍇
```

or, with labels

```
    1  4  9  12  13
   /   /   /   /   /
  2   3   5   8   10
     /   /   /   /   /
    6   7   11
```

Note that by construction the elements of $S$ are exactly the labels of the internal nodes of the forest.

Conversely, starting with a forest, label the nodes in left suffix order (order the trees from the left one to the right one, and within a tree, label first the left subtree of a node, then its right subtree, then the node itself). Then read the values of the internal nodes.
This is a bijection since each step can easily be reverted between the pinnacle sets and the left-suffix labelled trees.

The simplest induction on \( p_n(S) \) of Equation (5) translates on these objects as:

\[
(49) \quad p_n(F, O) = 2p_n(F),
\]

for any forest (sequence of trees) \( F \) and where \( O \) is the one-node tree.

With the help of this notation, we found two properties on \( p_n(S) \).

**Proposition 6.1.** Let us consider a sequence of trees \( F := (T_1, T_2, \ldots, T_r) \) encoding a pinnacle set.

Then

\[
(50) \quad p_n(F) = p_n(T_1) p_n(O, T_2, \ldots, T_r).
\]

**Proof** – The values in the tree \( T_1 \) are necessarily consecutive in any element of \( p_n(S) \). So we go from \( p_n(F) \) to \( p_n(O, T_2, \ldots, T_r) \) by replacing this sequence by 1 and standardizing the result. The converse operation changes 1 into all possibilities for \( p_n(T_1) \) hence explaining the multiplicative factor.

**Conjecture 6.2.** Let us consider a sequence of trees \( F := (O, T_2, T_3, \ldots, T_r) \) encoding a pinnacle set. Then

\[
(51) \quad p_n(F) = p_n(O, T_2) p_n(O, O, T_3, \ldots, T_r).
\]

A nice proof of this conjecture would probably be a first step before being able to generalize it to longer sequences of one-node trees at the beginning.

**Question 6.3.** When the first two trees of \( F \) are the tree with one node, \( p_n \) does not factorize. However, we expect there should exist a generalization of this result as an additive formula.

### 7. Orders on pinnacle sets

In [DNPT] Question 4.3 was whether there is a nontrivial order on sequences of a given size such that \( S_1 < S_2 \) would imply \( p_n(S_1) < p_n(S_2) \).

Thanks to an argument we have already seen, it is clear that if \( S_1 \) is componentwise smaller than \( S_2 \), then we indeed have \( p_n(S_1) < p_n(S_2) \) (it seems that the authors had seen that but did not write the proof). We just need to prove the property if \( S_1 = \{s_1, \ldots, s_p\} \) and \( S_2 \) is obtained by changing \( s_i \) into \( s_i + 1 \) (if \( s_i + 1 \not\in S_1 \)).

Starting with an element of \( \mathfrak{P}_n(S_1) \), exchange the values \( s_i \) and \( s_i + 1 \). Then we get an element of \( \mathfrak{P}_n(S_2) \): indeed, since \( s_i + 1 \) was not a pinnacle (not in \( S_1 \), it was not next to \( s_i \) but was either at an extremity or next to a greater value. After the exchange, all (non) pinnacles remain (non) pinnacles except \( s_i \) that changes status with \( s_i + 1 \). Note that the converse is not true (see the explanations of our inductive formula on \( p_n(S) \)): in \( \mathfrak{P}_n(S_2) \), there are elements where \( s_i \) is next to \( s_i + 1 \) and these are not in the image of the previous map.

To get more precise comparison results on pinnacle sets, we would need to inherit from small cases to larger ones, as in e.g. Proposition 3.9 in [DNPT]. Unfortunately, the provided proof is incorrect since the induction formula used to justify it does not
hold in general (and would give a direct and easy formula for $p_n(S)$). There is no real hope to patch it since

\[(52)\quad p_9(\{3, 9\}) = 1984 > p_9(\{5, 6\}) = 1152\]

whereas

\[(53)\quad p_{12}(\{3, 9, 10, 11, 12\}) = 172800 < p_{12}(\{5, 6, 10, 11, 12\}) = 207360.\]

8. Concluding remarks

We did not address here how to put algebraic structures in the picture of pinnacle sets but the strategy is the same as in the case of top-descent values of permutations. One cannot build a Hopf algebra, not even a subalgebra of the algebra on permutations, but a quotient works as in [HNTT]. This will be addressed in a forthcoming paper.

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