A CONSTRAINED OPTIMIZATION PROBLEM FOR
THE FOURIER TRANSFORM: EXISTENCE

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ABSTRACT. Among functions majorized by indicator functions of sets with measure one, which functions have maximal Fourier transforms in the $L^q$ norm? We prove the existence of such functions using techniques from additive combinatorics to establish a precompactness for maximizing sequences.

1. Introduction

Define the Fourier transform as $\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x) dx$ for a function $f : \mathbb{R}^d \to \mathbb{C}$. The Fourier transform is a contraction from $L^1(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$ and is unitary on $L^2(\mathbb{R}^d)$. Interpolation gives the Hausdorff-Young inequality $\|\hat{f}\|_q \leq \|f\|_p$ where $p \in (1, 2)$, $1 = \frac{1}{p} + \frac{1}{q}$. In [2], Beckner proved the sharp Hausdorff-Young inequality

(1.1) $\|\hat{f}\|_q \leq C_q^d \|f\|_p$

where $C_q = p^{1/2p} q^{-1/2q}$. In 1990, Lieb proved that Gaussians are the only maximizers of (1.1), meaning that $\|\hat{f}\|_q/\|f\|_p = C_q^d$ if and only if $f = c \exp(-Q(x,x) + v \cdot x)$ where $Q$ is a positive definite real quadratic form, $v \in \mathbb{C}^d$ and $c \in \mathbb{C}$. In 2014, Christ established a sharpened Hausdorff-Young inequality by bounding $\|\hat{f}\|_q - C_q^d \|f\|_p$ by a negative multiple of an $L^p$ distance function of $f$ to the Gaussians.

In [13], Christ proved the existence of maximizers for the ratio $\|\hat{1}_E\|_q/|E|^{1/p}$ where $E \subset \mathbb{R}^d$ is a positive Lebesgue measure set. Building on the work of Burchard in [6], Christ identified maximizing sets to be ellipsoids for exponents $q \geq 4$ sufficiently close to even integers [13].

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Another variant of the Hausdorff-Young inequality replaces indicator functions by bounded multiples and modifies the functional as follows. For \( d \geq 1, q \in (2, \infty) \), and \( p = q' \), we consider the inequality

\[
\|f\|_q \leq B_{q,d}|E|^{1/p}
\]

and define the quantities

\[
\Psi_q(E) := \sup_{|f| \leq E} \|f\|_q / \|1_E\|_p
\]

\[
B_{q,d} := \sup_E \Psi_q(E)
\]

where \( |f| < E \) means \( |f| \leq 1_E \) and the supremum is taken over all Lebesgue measurable sets \( E \subset \mathbb{R}^d \) with positive, finite Lebesgue measures. This quantity \( B_{q,d} \) is less than \( C_p^d \) by their definitions. The supremum \( \|f\|_q \) is equal to

\[
\sup_{f \in L(p,1)} \frac{\|\hat{f}\|_q}{\|f\|_p} \quad \text{where} \quad \|f\|_p = \inf \{\|a\|_{L^1} : |f| = \sum_n a_n |E_n|^{-1/p} 1_{E_n}, a_n > 0, |E_n| < \infty\}.
\]

We prove this equivalence in Proposition 2.1 in [2] Lorentz spaces are a result of real interpolation between \( L^p \) spaces. Since the quasinorm \( \| \cdot \|_\ell \) induces the standard topology on the Lorentz space \( L(p,1) \), this is a natural quantity to study.

Christ used continuum versions of theorems of Balog-Szemerédi and Freiman from additive combinatorics to understand the underlying structure of functions with nearly optimal ratio \( \|\hat{f}\|_q / \|f\|_p \) in [12] and for sets \( E \) with nearly optimal ration \( \|1_E\|_q / \|1_E\|_p \) in [13]. We use similar techniques in this paper to prove the existence of extremizers for (1.1) via a precompactness argument for extremizing sequences, presented in the following theorem.

**Theorem 1.1.** Let \( d \geq 1, q \in (2, \infty) \), \( p = q' \). Let \((E_\nu)\) be a sequence of Lebesgue measurable subsets of \( \mathbb{R}^d \) with \( |E_\nu| \in \mathbb{R}^+ \) and let \( f_\nu \) be Lebesgue measurable functions on \( \mathbb{R}^d \) satisfying \( |f_\nu| \leq 1_{E_\nu} \). Suppose that \( \lim_{\nu \to \infty} |E_\nu|^{-1/p} \|\hat{f}_\nu\|_q = B_{q,d} \). Then there exists a subsequence of indices \( \nu_k \), a Lebesgue measurable set \( A \subset \mathbb{R}^d \), a Lebesgue measurable function \( f \) on \( \mathbb{R}^d \) satisfying \( |f| \leq 1_A \), and a sequence \( (T_k) \) of affine automorphisms of \( \mathbb{R}^d \) such that

\[
\lim_{k \to \infty} \|f_{\nu_k} \circ T_k^{-1} - f\|_p = 0 \quad \text{and} \quad \lim_{k \to \infty} |T_k(E_{\nu_k})| \Delta E = 0.
\]

The existence of maximizers is a direct consequence.

A simplified outline of the proof is as follows.

(1) Begin by proving basic principles of concentration compactness: “no slacking” and “cooperation” (see [1]).

(2) If \( |f| \leq 1_E \) with \( |E| = 1 \) satisfies \( \|\hat{f}\|_q \geq \eta \) for \( \eta > 0 \), then \( f \) satisfies a related Young’s convolution inequality: for appropriate \( \gamma, r, s \), \( \|f\|_\gamma \leq c \|f\|_r \geq \eta^s \). 

(3) By continuum analogues of theorems of Balog-Szemerédi and Freiman, \( |f| \leq 1_E \) with \( |E| = 1 \) satisfying \( \|f\|_\gamma \leq \eta \) must place a portion of its \( L^p \) mass on a continuum multiprogression of controlled rank and Lebesgue measure.

(4) Combine concentration compactness principles with the specific additive structure we have from the relation to Young’s convolution inequality to conclude that a function \( |f| \leq 1_E \) satisfying \( \|\hat{f}\|_q \geq (1 - \delta)B_{q,d}|E|^{1/p} \) for small \( \delta > 0 \) is mostly supported on a multiprogression of controlled rank and size.
(5) By precomposing a near-extremizer with an affine transformation $T$, we can change variables to guarantee that the continuum multiprogression is mostly contained in $\mathbb{Z}^d \times [-\delta, \delta]^d$. We must guarantee that the Jacobian of $T$ is bounded below since otherwise we could trivially collapse any bounded set to a small neighborhood of the origin.

(6) The Fourier transform of a function living on $\mathbb{Z}^d \times [-\delta, \delta]^d$ decomposes into a discrete and a continuous Fourier transform, and a near-extremizer for (1.2) must be a near-extremizer of each step of the decomposition. Since near-extremizers of the discrete Fourier transform must mostly be supported on a single $n \in \mathbb{Z}^d$, this gives extra structure. We prove that the only multiprogression structure which is favorable at each step of the decomposition is one mostly contained in a single convex set $r \delta, \delta s \mathbb{R}^d$.

(7) If $|f| \leq 1_E$ is a near-extremizer, then $\hat{f}^{\|q\|^q-2}$ is a near-extremizer of a related dual inequality (see §3). The above reasoning may also be carried out for this dual inequality. We conclude that a significant portion of the $L^p$ mass of $f$ and $\hat{f}$ must be localized to ellipsoids (or other convex sets) of controlled size.

(8) Via a composition with an affine transformation and modulation by a character, we can assume that $f$ and $\hat{f}$ are localized (respectively) to the unit ball $B$ and and ellipsoid $E$ centered at the origin. We prove a reversed uncertainty bound: $|E||B| \leq C$ and furthermore $E \subset C \mathbb{R}$ for an appropriate $C > 0$.

(9) It follows that for any sequence of function $|f_\nu| \leq 1_{E_\nu}$ with $|E_\nu| \in \mathbb{R}^+$ and $\|\hat{f}_\nu\|_q |E_\nu|^{-1/p} \to B_{q,d}$, after $(f_\nu, E_\nu)$ is renormalized to $(F_\nu, A_\nu)$ by appropriate symmetries of the inequality, we have weakly convergent subsequences of $F_\nu$ and $1_{A_\nu}$. Finally, we get $L^p$ convergence via a convexity argument involving the $\| \cdot \|_\mathcal{L}$ norm.

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2. Results in terms of the Lorentz space $L(p,1)$

There are many quasinorms which induce the same topology on $L(p,q)$ spaces. For the special case of $p > 1$ and $q = 1$, we will show that our extremization problem can be phrased using various quasinorms (and one norm defined by Calderón) on $L(p,1)$. Let $B_{q,d}$ be as before.

**Definition 2.1.** Let $d \geq 1$. Define $\|f\|_\mathcal{L}$ for a measurable function $f : \mathbb{R}^d \to \mathbb{R}$ by

$$\|f\|_\mathcal{L} = \inf \{ \| (a_n) \|_\ell^1 : \| f \| = \sum a_n |E_n|^{-1/p} 1_{E_n}, a_n \geq 0, |E_n| < \infty \}.$$

The following definitions $\|2\|$ and $\|2\|$ are from Chapter V, §3 in $\|1\|$.

**Definition 2.2.** Let $d \geq 1$. Define $f^*$ for $t > 0$ by

$$f^*(t) = \inf \{ r : |\{ x : |f(x)| > r \}| \leq t \}.$$

**Definition 2.3.** Let $d \geq 1$, $1 \leq p < \infty$, $q$ the conjugate of $p$. Define $\| f \|_{p^*}$ for a measurable function $f : \mathbb{R}^d \to \mathbb{R}$ by

$$\| f \|_{p^*} = \frac{1}{p} \int_0^{\infty} t^{-1/q} f^*(t) dt.$$
**Definition 2.4.** Let $d \geq 1$, $1 \leq p < \infty$, $q$ the conjugate of $p$. Define $\|f\|_{p,1}$ for a measurable function $f : \mathbb{R}^d \to \mathbb{R}$ by

$$\|f\|_{p,1} = \frac{1}{p} \int_0^\infty t^{-1/q - 1} \int_0^t f^*(u)du \, dt.$$ 

**Definition 2.5.** Let $d \geq 1$, $1 \leq p < \infty$. The space $L(p,1)$ is defined as all measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ satisfying $\|f\|_{p,1} < \infty$.

See the appendix for the relationships between $\| : \mathcal{L}, \| : \mathcal{L}^*, \text{and } \| : \mathcal{P}$, and that they generate the same topology on $L(p,1)$. In particular, it is proved that $\|f\| = \|f\|_{p,1}$ for all measurable $f : \mathbb{R}^d \to \mathbb{C}$ (where one quantity is infinite if and only if the other quantity is as well).

**Proposition 2.1.** For $d \geq 1$, $q \in (2, \infty)$, and $p$ the dual exponent to $q$,

$$B_{q,d} = \sup_{f \in \mathcal{L}^p} \| \hat{f} \|_q.$$ 

**Proof.** Let $|f| = \sum a_n |E_n|^{-1/p} 1_{E_n}$ where $a_n \geq 0$ and $|E_n| < \infty$. Then $\|f\|_p \leq \|f\|_\mathcal{L}$. By the Hausdorff-Young inequality, the constant $A_\mathcal{L}$ defined by

$$(2.1) \quad A_\mathcal{L} := \sup_{f \in \mathcal{L}^p} \| \hat{f} \|_q$$ 

is finite.

We want to show that $B_{q,d} := \sup_{|E| < \infty} \sup_{|f| < 1} \frac{\|f\|_q}{|f|_\mathcal{L}} = \sup_{f \in \mathcal{L}^p} \frac{\|f\|_q}{\|f\|_\mathcal{L}} =: A_\mathcal{L}$.

If $|f| = \sum a_n |E_n|^{-1/p} 1_{E_n}$ with $a_n \geq 0$, $|E_n| < \infty$, then

$$\|f\|_p \leq \sum |a_n| |E_n|^{-1/p} \|1_{E_n}\|_q \leq B_{q,d},$$

so $A_\mathcal{L} \leq B_{q,d}$.

For the other direction, since simple functions are dense in $L^p(\mathbb{R}^d)$, it suffices to consider $f = \sum a_n 1_{A_n}$ where $A_n$ are disjoint and $f$ is majorized by the indicator of a Lebesgue measurable set $E$ of size one. Then $\sum |a_n| |A_n|^{1/p} \leq |E|^{1/p} \sum |a_n| = |E|^{1/p} |f|_1 \leq |E|^\frac{1}{p} + 1 = 1$. Rearranged, this means

$$\|\hat{f}\|_q = \frac{\|\hat{f}\|_q}{\|f\|_1} \leq \frac{\sum |a_n| |A_n|^{1/p}}{\|f\|_1} \leq \frac{\|\hat{f}\|_q}{\|f\|_\mathcal{L}},$$

so $B_{q,d} \leq A_\mathcal{L}$.

**Lemma 2.2.** If $f \in L(p,1)$ satisfies $B_{q,d} = \|f\|_\mathcal{L}^{-1} \|\hat{f}\|_q$, then

$$f = a e^{i\varphi} 1_E$$

for some scalar $a \in \mathbb{R}^+$, Lebesgue measurable function $\varphi : \mathbb{R}^d \to \mathbb{R}$, and a Lebesgue measurable set $E$ of finite measure.

**Proof.** By Lemma 2.2 we also have that $B_{q,d} = (\|f\|_{p,1}^{-1}) \|\hat{f}\|_q$. Let $E = \{(y,s) : |f(y)| > s\}$. Let $e^{i\varphi} = f/|f|$ so we can use the layer cake representation

$$f(x) = e^{i\varphi(x)} \int_0^\infty 1_E(x,t)dt.$$
Then
\[
\| \hat{f} \|_q = \left\| \left( \int_0^\infty e^{i\varphi(x)} 1_E(x,t) dt \right) \right\|_q = \left\| \int_0^\infty e^{i\varphi} 1_E(\xi,t) dt \right\|_q \\
\leq \int_0^\infty \| e^{i\varphi} 1_E(\xi,t) \|_d dt \\
\leq \int_0^\infty B_{q,d} \left\{ \{ x : |f(x)| > t \} \right\}^{1/p} dt \\
= B_{q,d} \int_0^\infty \int_0^\infty \left\{ \{ x : |f(x)| > t \} \right\}^{1/p} u^{-1/2} du dt \\
= B_{q,d} \int_0^\infty \left\{ \{ x : |f(x)| > 0 \} \right\}^{1/p} u^{-1/2} du = B_{q,d} \| f \|_{p^*}.
\]

Since \( B_{q,d} = (\| f \|_{p^*})^{-1} \| \hat{f} \|_q \), the above sequence of inequalities are actually equalities. Equality in the Minkowski integral inequality implies that for a.e. \((\xi,t) \in \mathbb{R}^d \times \mathbb{R}^+\),
\[
e^{i\varphi} 1_E(\xi,t) = h(\xi)g(t)
\]
for some measurable functions \(h,g\). Since \(e^{i\varphi} 1_E(x,t) \in L^2\), in particular, \(h \) and \(\hat{h} \) in \(L^2\).
\[
1_E(x,t) = e^{-i\varphi(x)} \tilde{h}(x)g(t).
\]
But then for every \((x,t)\) satisfying \(|f(x)| > \), we have
\[
e^{-i\varphi(x)} \tilde{h}(x)g(t) = 1.
\]
Suppose \(|f(x)| > |f(y)| > \). Then for all \(0 < t < f(y)\),
\[
e^{-i\varphi(x)} \tilde{h}(x) = g(t)^{-1} = e^{i\varphi(y)} \hat{h}(y),
\]
which is a contradiction unless \(|f(x)|\) is constant on its support. Thus \(f\) takes the form \(ae^{i\varphi} 1_S\) where \(S = \text{supp} \ f\) and \(a \in \mathbb{R}^+\).

The existence corollary to Theorem 1.1 in terms of Lorentz norms is

**Corollary 2.3.** Let \(d \geq 1, \ p \in (1,2)\), \(q\) the conjugate exponent of \(p\). First, we have \(B_{q,d} = \sup_{g \in L(p,1)} \| g \|_{p^*}^{-1} \| \hat{f} \|_q \). Second, if \(f \in L(p,1)\) satisfies \(B_{q,d} = \| f \|_{p^*}^{-1} \| \hat{f} \|_q\), then
\[
f = ae^{i\varphi} 1_E
\]
for some scalar \(a \in \mathbb{R}^+\), Lebesgue measurable function \(\varphi : \mathbb{R}^d \to \mathbb{R}\), and a Lebesgue measurable set \(E\) of finite measure.

See §8 for the proof of the Corollary 2.3.

3. **The dual inequality**

Recall the definition of the optimal constant \(B_{q,d}\)
\[
B_{q,d} = \sup_{|E| < \infty} \sup_{|f| \leq 1} \sup_{|g| \leq 1} \left\| \hat{f} \|_q \right. \\
|E|^{1/p}.
\]

By exploiting \(L^p\) duality and Plancherel’s theorem, we also have the expressions:
\[
B_{q,d} = \sup_{|E| < \infty} \sup_{|f| \leq 1} \sup_{|g| \leq 1} \left\| \hat{f} \|_q \right. \\
|E|^{1/p} = \sup_{|E| < \infty} \sup_{|g| \leq 1} \left\| 1_E \hat{g} \right\|_q \\
|E|^{1/p},
\]
the last of which motivates the following definition.

**Definition 3.1.** Let \( d \geq 1 \) and \( q \in [1, \infty) \), and \( p \) be the conjugate exponent to \( q \). Define the norm \( \| \cdot \|_{q, *} \) of a function \( g \in L^q(\mathbb{R}^d) \) to be

\[
\|g\|_{q,*} = \sup_{|E| < \infty} \int_E |g|(|E|)^{-1/p} 
\]

where the supremum is taken over Lebesgue measurable subsets \( E \subset \mathbb{R}^d \) of finite measure.

Note that by Hölder’s inequality, if \( g \in L^q \), then \( \|g\|_{q,*} \leq \|g\|_q < \infty \). Thus for \( f \in L^p \) with \( p \in (1, 2) \) and \( q \) the conjugate exponent,

\[
\|\hat{f}\|_{q,*} \leq \|f\|_p
\]

is a corollary of the Hausdorff-Young inequality.

**Proposition 3.1.** For \( d \geq 1 \), \( q \in (2, \infty) \), and \( p \) the dual exponent to \( q \),

\[
B_{q,d} = \sup_{|f|_p \leq 1} \|\hat{f}\|_{q,*}.
\]

Furthermore, if \( |f| \leq 1_E \), \( |E| < \infty \) satisfies \( \|\hat{f}\|_q \geq (1 - \delta)B_{q,d}|E|^{1/p} \) for some \( \delta > 0 \), then

\[
\|((|\hat{f}|^q)^{2\hat{f}})^{\hat{f}}\|_{q,*} \geq (1 - \delta)^q B_{q,d} \|\hat{f}\|_p.
\]

**Proof.** Let \( f \in L^p(\mathbb{R}^d) \). Consider a Lebesgue measurable set \( E \subset \mathbb{R}^d \) of finite measure such that \( |\hat{f}| \neq 0 \) a.e. on \( E \) and write \( \tilde{f} = e^{-i\varphi}|\hat{f}| \) for a real-valued phase function \( \varphi \) equal to 0 off of the support of \( \hat{f} \). Using Plancherel’s theorem, the notation \( \tilde{f}(x) = f(-x) \), and Hölder’s inequality, we then have

\[
|E|^{-1/p} \int_E |\tilde{f}| = |E|^{-1/p} \int_1^{\infty} e^{i\varphi} \tilde{f} = |E|^{-1/p} \int e^{i\varphi} 1_E \tilde{f} 
\]

\[
\leq |E|^{-1/p} \|e^{i\varphi} 1_E\|_q \|f\|_p = |E|^{-1/p} \|e^{-i\varphi} 1_E\|_q \|f\|_p
\]

so that \( \sup_{|f|_p \leq 1} \|\hat{f}\|_{q,*} \leq B_{q,d} \).

Now suppose that for \( |f| \leq 1_E \), \( |E| < \infty \), and \( \delta > 0 \) we have \( \|\hat{f}\|_q \geq (1 - \delta)B_{q,d}|E|^{1/p} \). Then \( |\hat{f}|^{q-2} \hat{f} \in L^p \) since

\[
\|\hat{f}|^{q-2} \hat{f}\|_p = \int |\hat{f}|^{p(q-2)} |\hat{f}|^p = \int |\hat{f}|^{p(q-1)} = \int |\hat{f}|^{pq(1-1/q)} = \|\hat{f}\|_q.
\]

Thus we have

\[
\int_{-E} \left|\left(|\hat{f}|^{q-2} \hat{f}\right)\right| \geq \int |\tilde{f}| \left|\left(|\hat{f}|^{q-2} \hat{f}\right)\right| 
\]

\[
\geq \int |\tilde{f}| \left|\left(|\hat{f}|^{q-2} \hat{f}\right)\right| = \int |\tilde{f}|^{q-2} \hat{f} = \int |\hat{f}|^q 
\]

\[
\geq (1 - \delta)^q B_{q,d} \|\hat{f}\|_q.
\]
Rearranging and using that \( \|\hat{f}|^{q-2}\hat{f}\|_p = \|\hat{f}\|_q^{q/p} \leq B_{q,d}^q|E|^{q/p^2} \),

\[
|E|^{-1/p} \int_{-E} \left| \left( |\hat{f}|^{q-2}\hat{f} \right) \right| \geq (1 - \delta)^q B_{q,d}^{q-\delta/p} \|\hat{f}|^{q-2}\hat{f}\|_p = (1 - \delta)^q B_{q,d} \|\hat{f}|^{q-2}\hat{f}\|_p,
\]

proving that we can find \( g \in L^p \) such that \( \|\hat{g}\|_{q,*}g\|_p^{-1} \) is arbitrarily close to \( B_{q,d} \).

For \( q \in (2, \infty) \) and \( p \) the conjugate exponent of \( q \), the inequality

\[
(3.1) \quad \|\hat{g}\|_{q,*} \leq B_{q,d}\|g\|_p.
\]

is amenable to the same analysis as our main inequality (1.2), and each lemma we prove about (1.2) will have an analogue for this dual inequality.

4. Quasi-extremal principles

We establish the quasi-extremal principles “no slacking” and “cooperation”. No slacking guarantees that a near-extremizer is a combination of small parts which must be quasi-extremizers. Cooperation guarantees that these small parts work together in a compatible way (e.g. have nontrivial intersection of supports).

**Definition 4.1.** Let \( d \geq 1, q \in (2, \infty) \) and \( p = q' \). A nonzero \( |f| \leq 1_E \in L^p \) is a \( \delta \)-quasi-extremizer for (1.2) if

\[
\|\hat{f}\|_q \geq (1 - \delta)B_{q,d}|E|^{1/p}.
\]

By a quasi-extremizer, we mean a \( \delta \)-quasi-extremizer for some small \( \delta > 0 \).

4.1. No slacking.

**Lemma 4.1** (No slacking). For any \( p, q \in (1, \infty) \) there exist \( c, C_0 < \infty \) with the following property. Let \( \delta > 0, |E| < \infty, |f| \leq 1_E \). Suppose that

\[
\|\hat{f}\|_q \geq (1 - \delta)B_{q,d}|E|^{1/p}.
\]

Suppose that \( f = g + h \) where \( g = 1_A f, h = 1_B f \) have disjoint supports and

\[
|B|^{1/p} \geq C_0 \delta^{1/p} |E|^{1/p}.
\]

Then

\[
\|\hat{h}\|_q \geq c\delta |E|^{1/p}.
\]

**Proof.** There exists \( C < \infty \) such that for any \( G, H \in L^q \),

\[
\|G + H\|_q^q \leq \|G\|_q^q + C \|G\|_q^{-1} \|H\|_q + C \|H\|_q^q.
\]

Consequently,

\[
\|\hat{g} + \hat{h}\|_q^q \leq \|\hat{g}\|_q^q + C\|\hat{g}\|_q^{-1} \|\hat{h}\|_q + C \|\hat{h}\|_q^q
\]
\[
\leq B_{q,d}^q|A|^{q/p} + C B_{q,d}^{q-1} |A|^{q-1/p} \|\hat{h}\|_q + C \|\hat{h}\|_q^q.
\]
On the other hand, $|E| = |A| + |B|$. Without loss of generality, assume $|E| = 1$, so that $|A|, |B| \leq 1$. Thus

$$
(1 - \delta)^q \leq \frac{\|f\|_q^q}{B_{q,d}|E|^{q/p}} = \frac{\|f\|_q^q}{B_{q,d}^q} \\
\leq \frac{B_{q,d}|A|^{q/p} + CB_{q,d}^{-1}|A|^{(q-1)/p}\|\hat{h}\|_q + C\|\hat{h}\|_q^2}{B_{q,d}^q} \\
= |A|^{q/p} + CB_{q,d}^{-1}|A|^{(q-1)/p}\|\hat{h}\|_q + CB_{q,d}^{-q}\|\hat{h}\|_q \\
\leq (1 - |B|)^{q/p} + CB_{q,d}^{-1}|A|^{(q-1)/p}\|\hat{h}\|_q + CB_{q,d}^{-1}\|\hat{h}\|_q \\
\leq 1 - c_1|B| + 2CB_{q,d}^{-1}\|\hat{h}\|_q.
$$

Then we have

$$
2CB_{q,d}^{-1}\|\hat{h}\|_q \geq (1 - \delta)^q - 1 + c_2|B| \\
\geq 1 - O(\delta) - 1 + |B| \\
\geq |B| - O(\delta) \\
\geq C_0^p\delta - O(\delta) \\
\geq c_3\delta
$$

provided $C_0$ is large enough. \hfill \Box

**Lemma 4.2** (No slacking dual). For each $d \geq 1$ and $q \in (2, \infty)$ there exist $\delta_0, C_0, C_0 < \infty$ with the following property. Let $\delta \in (0, \delta_0]$ and let $f = g + h$ where $f, g, h \in L^p(\mathbb{R}^d)$ and $g, h$ are disjointly supported on $A, B$ respectively. Suppose that

$$
\|f\|_{q,*} \geq (1 - \delta)B_{q,d}\|f\|_p.
$$

and that

$$
\|h\|_p \geq C_0\delta^{1/p}\|f\|_p.
$$

Then

$$
\|\hat{h}\|_{q,*} \geq c_3|E|^{1/p}.
$$

**Proof.** Using the hypothesis that $(f, E)$ is near-extremizing, we can find a subset $E \subset \mathbb{R}^d$ such that

$$
\int_E |\hat{f}|\|E|^{-1/p} \geq (1 - \delta)B_{q,d}|E|^{1/p}.
$$

Then

$$
(1 - 2\delta)\|f\|_pB_{q,d} \leq |E|^{-1/p}\int_E |\hat{f}| = |E|^{-1/p}\int_E |\hat{g} + \hat{h}| \\
\leq |E|^{-1/p}\int_E |\hat{g}| + |E|^{-1/p}\int_E |\hat{h}| \\
\leq \|\hat{g}\|_{q,*} + \|\hat{h}\|_{q,*} \\
\leq B_{q,d}\|g\|_p + \|\hat{h}\|_{q,*} \\
\leq B_{q,d}(\|f\|_p^p - \|h\|_{p'}^{p'})^{1/p} + \|\hat{h}\|_{q,*} \\
\leq B_{q,d}(\|f\|_p^p - C_0^p\delta\|f\|_p^{p'})^{1/p} + \|\hat{h}\|_{q,*}.
$$
Rearranging the above inequality gives
\[
\left( (1 - 2\delta) - (1 - C_0^p\delta)^{1/p} \right) \sum_{q,d} f^q \leq \|h\|_{q,\infty}.
\]

Finally, we can arrange for \(|C_0^p\delta| < 1\), so
\[
1 - 2\delta - (1 - C_0^p\delta)^{1/p} = -2\delta + \frac{1}{p} C_0^p\delta + O(\delta^2)
\]
\[
= (C_0^p/p - 2)\delta + O(\delta^2).
\]

If \(C_0^p/p - 2 > 0\) and \(\delta\) is small enough, we have the result.

\(\square\)

4.2. Cooperation.

Lemma 4.3. Let \(p \in [1,2)\) and \(q \in [2,\infty)\). There exist \(c, C \in \mathbb{R}^+\) with the following property. Let \(0 \neq f \in L^p\) satisfy \(\|f\|_q \geq (1 - \delta)\|B_{q,d}\|_{E}^{1/p}\). Suppose that \(f = f^\sharp + f^\flat\) where \(\text{supp } f^\sharp = A\) and \(\text{supp } f^\flat = B\) satisfy
\[
|A| + |B| \leq |E|.
\]
\[
\min(|A|^{1/p}, |B|^{1/p}) \geq \eta |E|^{1/p}.
\]

Then
\[
\|\hat{f}^\sharp \cdot \hat{f}^\flat\|_{q/2} \geq (cp^p - C\delta) |E|^{2/p}.
\]

Proof.
\[
\|\hat{f}^\sharp\|_q^2 \leq \int (|\hat{f}^\sharp|^2 + |\hat{f}^\flat|^2)|\hat{f}|^q \leq 2 \int |\hat{f}^\sharp \cdot \hat{f}^\flat||T\hat{f}|^{q-2}
\]
\[
\leq (\|\hat{f}^\sharp\|_{q/2}^2 + \|\hat{f}^\flat\|_{q/2}^2)\|\hat{f}|^{q-2}\|_{q/(q-2)} \leq 2\|\hat{f}^\sharp \cdot \hat{f}^\flat\|_{q/2}\|\hat{f}|^{q-2}\|_{q/(q-2)}
\]
\[
= (\|\hat{f}^\sharp\|_q^2 + \|\hat{f}^\flat\|_q^2)\|\hat{f}|^{q-2}\|_{q/(q-2)} \leq 2\|\hat{f}^\sharp \cdot \hat{f}^\flat\|_{q/2}\|\hat{f}|^{q-2}\|_{q/(q-2)}
\]
\[
\leq (|A|^{2/p} + |B|^{2/p})\|B_{q,d}\|_{E}^{q/(q-2)/p} + 2\|\hat{f}^\sharp \cdot \hat{f}^\flat\|_{q/2}\|B_{q,d}\|_{E}^{q-2}\|_{E}^{(q-2)/p}.
\]

Thus as in 4.3
\[
\|\hat{f}^\sharp \cdot \hat{f}^\flat\|_{q/2} \geq (2B_{q,d}^{q-2} E_{(q-2)/p})^{-1} \left( \|\hat{f}^\sharp\|_q^2 - (|A|^{2/p} + |B|^{2/p})\|B_{q,d}\|_{E}^{(q-2)/p} \right)
\]
\[
\geq (2B_{q,d}^{q-2} E_{(q-2)/p})^{-1} \left( 1 - \delta \right)\|B_{q,d}\|_{E}^{q/2} \geq (|A|^{2/p} + |B|^{2/p})\|B_{q,d}\|_{E}^{(q-2)/p}
\]
\[
\geq 2^{-1}B_{q,d}^{q-2} \left( 1 - \delta \right)\|E|^{2/p} - |A|^{2/p} - |B|^{2/p} \right).
\]

Note that
\[
(|A|^{2/p} + |B|^{2/p})^{1/2} \leq |A| + |B|
\]

with strict inequality unless |A| or |B| is 0. Without loss of generality, suppose that |E| = 1.

We want to show there exists \(c \in \mathbb{R}^+\) such that for \(\eta\) small enough and \(\eta \leq \min(|A|^{1/p}, |B|^{1/p})\),
\[
\frac{|A|^{2/p} + |B|^{2/p}}{|A| + |B|} \leq 1 - cp^p.
\]

This is true from calculus since for \(f(x) = \frac{x^{2/p} + 1}{(x+1)^{2/p}}\), \(f'(x) \leq 0\) on \((0,1)\), so the minimum occurs at \(x = 1\), and is still positive. Thus there is a slope so that the line \(y = cx\) is less than or equal to \(f\) on \([0,1]\). Take \(x = |A|/|B|\), where without loss of generality \(|A| \leq |B|\), and note that \(f(|A|/|B|) \geq c|A|/|B| \geq c|A| \geq cp^p\).
Finally, using $|A| + |B| \leq |E| = 1$,
\[
\|\hat{f} \cdot \hat{f} \|_{q/2} \geq 2^{-1} B_{q,d}^2 \left( (1 - \delta)^2 - |A|^{2/p} - |B|^{2/p} \right) \\
\geq 2^{-1} B_{q,d}^2 \left( (1 - \delta)^2 - (1 - c\eta^p) \right) \\
\geq c\eta^p - C\delta.
\]

\[\square\]

**Lemma 4.4.** For each $d \geq 1$ and $q \in (2, \infty)$ there exist $\delta_0, c, C_0 < \infty$ with the following property. Let $\delta \in (0, \delta_0]$ and let $f = g + h$ where $f, g, h \in L^p(\mathbb{R}^d)$ and $g, h$ are disjointly supported. Suppose that the following inequalities hold for $\eta^p \geq \delta$.

\[
\|\hat{f}\|_{q,*} \geq (1 - \delta) B_{q,d} \|f\|_p,
\]

\[
\min(\|g\|_p, \|h\|_p) \geq C_0 \eta \|f\|_p.
\]

Then

\[
\|\hat{g} \cdot \hat{h}\|_{q,*} \geq c\delta \|f\|_p B_{q,d}.
\]

**Proof.** Take $E \subset \mathbb{R}^d$ with $|E| \in (0, \infty)$ satisfying

\[
|E|^{-1/p} \int_E |\hat{f}| \geq (1 - 2\delta) B_{q,d} \|f\|_p.
\]

By replacing $E$ with $E \cap \{\hat{f} \neq 0\}$, we can assume that $\hat{f}$ is nonzero on $E$. For $\lambda > 0$ a large constant to be chosen later, define $E_{\lambda,g} = \{x \in E : |\hat{g}| > \lambda |\hat{h}|\}$ and $E_{\lambda,h} = \{x \in E : |\hat{h}| > \lambda |\hat{g}|\}$. Note that

\[
\int_E |\hat{f}| = \int_{E_{\lambda,g}} |\hat{f}| + \int_{E_{\lambda,h}} |\hat{f}| + \int_{E \setminus (E_{\lambda,g} \cup E_{\lambda,h})} |\hat{f}|
\]

\[
\leq (1 + 1/\lambda) \int_{E_{\lambda,g}} |\hat{g}| + (1 + 1/\lambda) \int_{E_{\lambda,h}} |\hat{h}| + \int_{E \setminus (E_{\lambda,g} \cup E_{\lambda,h})} (|\hat{g}| + |\hat{h}|)
\]

\[
\leq (1 + 1/\lambda) \int_{E_{\lambda,g}} |\hat{g}| + (1 + 1/\lambda) \int_{E_{\lambda,h}} |\hat{h}| + \int_{E \setminus (E_{\lambda,g} \cup E_{\lambda,h})} (|\hat{g}|^{1/2} \lambda^{1/2} |\hat{h}|^{1/2} + \lambda^{1/2} |\hat{g}|^{1/2} |\hat{h}|^{1/2})
\]

\[
\leq (1 + 1/\lambda) \int_{E_{\lambda,g}} |\hat{g}| + (1 + 1/\lambda) \int_{E_{\lambda,h}} |\hat{h}| + 2\lambda^{1/2} \int_{E \setminus (E_{\lambda,g} \cup E_{\lambda,h})} |\hat{g}|^{1/2} |\hat{h}|^{1/2}.
\]

Using our main inequality, we have

\[
|E_{\lambda,g}|^{1/p} |\hat{g}|_{q,*} + |E_{\lambda,h}|^{1/p} |\hat{h}|_{q,*} \leq \left( |E_{\lambda,g}|^{1/p} \|g\|_p + |E_{\lambda,h}|^{1/p} \|h\|_p \right) B_{q,d}
\]

and by Hölder’s inequality,

\[
|E_{\lambda,g}|^{1/p} \|g\|_p + |E_{\lambda,h}|^{1/p} \|h\|_p \leq (|E_{\lambda,g}|^{p/p} + |E_{\lambda,h}|^{p/p})^{1/p} (\|g\|_p^q + \|h\|_p^q)^{1/q} \leq |E|^{1/p} (\|g\|_p^q + \|h\|_p^q)^{1/q}.
\]

Then

\[
\|g\|_p^q + \|h\|_p^q \leq \max(\|g\|_p^{q-p}, \|h\|_p^{q-p}) (\|g\|_p^p + \|h\|_p^p) = \max(\|g\|_p^{q-p}, \|h\|_p^{q-p}) \|f\|_p^p.
\]

Now we use the hypothesis that $\min(\|g\|_p, \|h\|_p) \geq C_0 \eta \|f\|_p$ to say

\[
\max(\|g\|_p^p, \|h\|_p^p) = \|f\|_p^p - \min(\|g\|_p, \|h\|_p) \leq \|f\|_p^p (1 - C_0^p \eta^p).
\]
In summary,
\[
|E_{\lambda,q}|^{1/p} \|\hat{g}\|_{q,*} + |E_{\lambda,h}|^{1/p} \|\hat{h}\|_{q,*} \leq |E|^{1/p} \left( \|f\|_p^{1/q} \|f\|_p^{(q-p)/q} (1 - C_0^p \eta^p)^{(q-p)/q} \right) B_{q,d}
\]

Putting everything together, we have
\[
(1 - 2\delta) B_{q,d} \|f\|_p \leq (1 + 1/\lambda) \|f\|_p (1 - C_0^p \eta^p)^{(q-p)/q} B_{q,d} + \lambda^{1/2} |E|^{-1/p} \int_{E \setminus (E_{\lambda,h} \cup E_{\lambda,h})} |\hat{g}|^{1/2} \lambda^{1/2} |E|^{-1/p} \int_{E \setminus (E_{\lambda,h} \cup E_{\lambda,h})} |\hat{h}|^{1/2}
\]

(1 - 2\delta - (1 + 1/\lambda)(1 - (1 - p/q)C_0^p \eta^p + O(\eta^{2p})) B_{q,d} \|f\|_p \leq \lambda^{1/2} |E|^{-1/p} \int_{E \setminus (E_{\lambda,h} \cup E_{\lambda,h})} |\hat{g}|^{1/2} \lambda^{1/2} |E|^{-1/p} \int_{E \setminus (E_{\lambda,h} \cup E_{\lambda,h})} |\hat{h}|^{1/2}

The desired inequality follows from $\lambda = \delta^{-1}$, $\eta^p \geq \delta$ and $C_0$ large enough.

\[\square\]

5. Multiprogression structure of quasi-extremizers

In this section, we relate quasi-extremizers for \((1.2)\) to quasi-extremizers for Young’s convolution inequality. Then we exploit the connection between Young’s convolution inequality and principles of additive combinatorics which imply that quasi-extremizing functions for Young’s inequality have significant support on sets with arithmetic structure. We use the following definition and notation for multiprogressions of Christ.

**Definition 5.1.** A discrete multiprogression $P$ in $\mathbb{R}^d$ of rank $r$ is a function

\[
P : \prod_{i=1}^r \{0, 1, \ldots, N_i - 1\} \to \mathbb{R}^d
\]

of the form

\[
P(n_1, \ldots, n_r) = \{a + \sum_{i=1}^r n_i v_i : 0 \leq n_i < N_i\},
\]

for some $a \in \mathbb{R}^d$, some $v_j \in \mathbb{R}^d$, and some positive integers $N_1, \ldots, N_r$. A continuum multiprogression $P$ in $\mathbb{R}^d$ of rank $r$ is a function

\[
P : \prod_{i=1}^r \{0, 1, \ldots, N_i - 1\} \times [0, 1]^d \to \mathbb{R}^d
\]

of the form

\[(n_1, \ldots, n_d; y) \mapsto a + \sum_{i=1}^r n_i v_i + s y\]

where $a, v_i \in \mathbb{R}^d$ and $s \in \mathbb{R}^+$. The size of $P$ is defined to be

\[
\sigma(P) = s^d \prod_i N_i.
\]

$P$ is said to be proper if this mapping is injective.

We will identify a multiprogression with its range, and will refer to multiprogressions as if they were sets rather than functions. If $P$ is proper then the Lebesgue measure of its range equals its size. For a discussion of properties of multiprogressions, see §5 of [12].
Lemma 5.1 (Quasi-extremizers for Young’s inequality). Let \( r \in (1, \infty) \) and suppose that the exponent \( t \) defined by \( 1 + t^{-1} = 2r^{-1} \) also belongs to \((1, \infty)\). For each \( \delta > 0 \), there exist \( C_\delta, C_\delta' \in (0, \infty) \) such that for any \(|f| \leq 1_E\) with \(|E| < \infty\) and \(|E|^{2/r}(1 - \delta) \leq \|f * f\|_t\), there exist a disjoint, measurable decomposition \( E = A \cup B \) and a proper continuum multiprogression \( P \) such that

\[
A \subset P \quad \quad |P| \leq C_\delta |A| \quad \quad \text{rank } (P) \leq C_\delta \quad \quad \|f - 1_A f\|_r \leq (1 - C_\delta) \|f\|_r.
\]

Proof. This lemma follows from the proof of Lemma 6.1 in [12] where we specialize to the case that \( f_1 = f_2 \) and use that \(|E|^{2/r} \geq A^{2d} A^{d'} \|f\|^2\).

Lemma 5.2. Let \( d \geq 1 \) and \( p \in (1, 2) \). Let \( \eta > 0 \) and suppose that for a measurable set \( E \) and \(|f| \leq 1_E \in L_p(\mathbb{R}^d), |E|^{1/p} \eta \leq \|f\|_{p'}\). If \( p \leq 4/3 \), then there exists \( \gamma = \gamma(p) \in \mathbb{R}^+ \) such that \(|E|^{3/2} \eta^\gamma \leq \|f*|f|\|_{1/t}\), where \( t^{-1} = 2r^{-1} - 1 \).

Proof. First suppose that \( p \leq \frac{4}{3} \). Then applying Plancherel’s theorem and the Hausdorff-Young inequality, we have

\[
|E|^{1/p} \eta \leq \|\hat{f}\|_{p'} = \|\hat{f}*\hat{f}\|_{p'/2}^{1/2} \leq \|f*f\|_{(p'/2)'_p}^{1/2} \leq \||f|*|f|\|_{t'}^{1/2},
\]

where \( t = \frac{p'/2 - 1}{p' - 2} = \frac{p'(p-1)}{p(p-1)-2} = (2p^{-1} - 1)^{-1} \).

Write \( f(x) = g(x)e^{i\varphi(x)} \) where \( \varphi(x) \) is real-valued and \( g \geq 0 \). Note that for \( \text{Re } z > 0 \), we can define \( f_z := g^z e^{iz\varphi} \in L_p(\text{Re } z) \).

Assume that \( \frac{6}{3} < p \). Since \( \frac{2}{3} < 1 < \frac{3p}{2} \), there exists \( \theta \in (0, 1) \) such that \( 1 = (1 - \theta)p2^{-1} + \theta3p4^{-1} \). By the Three Lines Lemma proof of the Riesz-Thorin theorem,

\[
\|f\|_p \leq \sup_{\text{Re } z = p/2} \|f_z\|_2^{1-\theta} \sup_{\text{Re } z = 3p/4} \|f_z\|_p^{\theta (3p/4)} = \|f\|_p^{(1-\theta)p2^{-1}} \sup_{\text{Re } z = 3p/4} \|f_z\|_p^{\theta (3p/4)}.
\]

Combining this with the quasi-extremal hypothesis of \( f \) gives

\[
|E|^{1/p} \eta \leq \|f\|_p^{(1-\theta)p2^{-1} \sup_{\text{Re } z = 4/3} \|f_z\|_p^{\theta (4/3)'}} \leq |E|^{(1-\theta)p2^{-1} \sup_{\text{Re } z = 4/3} \|f_z* f_z\|_2^{\theta/2}} \leq |E|^{(1-\theta)p2^{-1} \sup_{\text{Re } z = 4/3} \|f_z* f_z\|_2^{\theta/2}} \leq |E|^{(1-\theta)p2^{-1} \|f\|^{4/3}_p*|f|^{4/3}_p\|_2^{\theta/2}}.
\]

Rearranging, we can write

\[
|E|^{3\theta/4} \eta \leq \|f|^{4/3}_p*|f|^{4/3}_p\|_2^{\theta/2}
\]

so \(|E|^{3/2} \eta^\gamma \leq \|f|^{4/3}_p*|f|^{4/3}_p\|_2\) for some \( \gamma > 0 \).
Lemma 5.4. Let \( \lambda, R, d, \) only then Lemma 6.2. Let \( \Lambda \) be a compact subset of \( P \) a multiprogression \( f \) decomposition \( \mu \) continuum multiprogressions of ranks \( \delta \.Adam \eta \) be functions that satisfy \( \| \varphi \|_\infty |P|^{1/p} \leq 1 \) and \( \| \psi \|_\infty |Q|^{1/p} \leq 1 \). If

\[
\| \widehat{\varphi \psi} \|_{q/2} \geq \lambda
\]

then

\[
\max(|P|, |Q|) \leq C \min(|P|, |Q|)
\]

\[
|P + Q| \leq C \min(|P|, |Q|).
\]

Lemma 6.2. Let \( d \geq 1 \), and let \( \Lambda \subset (1,2) \) be a compact set. For any \( \epsilon > 0 \) there exist \( \delta > 0 \), \( N_\epsilon < \infty \), and \( C_\epsilon < \infty \) with the following property for all \( p \in \Lambda \). Let \( |E| < \infty \) and \( |f| \leq 1_E \) be such that \( \| \hat{f} \|_q \geq (1 - \delta)B_{q,d}|E|^{1/p} \). Then there exists a disjointly supported decomposition \( f = g + h \) where \( \text{supp} g = A \) and \( \text{supp} h = B \) are disjoint and there are continuum multiprogressions \( \{ P_i : 1 \leq i \leq N_\epsilon \} \) such that

\[
|B| \leq \epsilon |E|
\]

\[
\sum_i |P_i| \leq C_\epsilon |E|
\]

\[
A < \bigcup_{i=1}^{N_\epsilon} P_i
\]
\[ \text{rank } P_1 \leq C_\epsilon \]
\[ \| g \|_p \geq c_\delta \| f \|_p. \]

**Proof.** We define an iterative process following Theorem 7.1 from [12]. Setting \( \eta_\delta = 1 - \delta \), we may apply Proposition 5.3 to obtain a disjoint decomposition \( E = A_1 \cup B_1 \) with a multiprogression \( P_1 \) satisfying
\[ |P_1| \leq C_{\eta_\delta} |A_1|, \quad \text{rank } P_1 \leq C_{\eta_\delta}, \quad \| 1_{A_1} f \|_p \geq c_{\eta_\delta} \| f \|_p. \]

Suppose that \( |B_1| > \epsilon |E| \) (the case \( |B_1| \leq \epsilon |E| \) will be analyzed below). By taking \( \delta < \epsilon / C_0 \), we can use lemma 4.1 to say
\[ \| 1_{B_1} f \|_q \geq \frac{c}{C_0} \epsilon |E|^{1/p}, \]
where \( c, C_0 \) are as in the lemma. Define \( \eta_\epsilon = \frac{c}{C_0 \epsilon}. \) Then we apply Proposition 5.3 to \( 1_{B_1} f \) to obtain a disjoint decomposition \( B_1 = A_2 \cup B_2 \) with the corresponding conclusions.

For the \( k \)-th step in the process, we halt if \( |B_{k-1}| \leq \epsilon |E| \). If \( |B_{k-1}| > \epsilon |E| \), then by Lemma 4.1 we have \( \| 1_{B_{k-1}} f \|_q \geq \eta_k |E|^{1/p} \). Then applying Proposition 5.3, we get \( B_{k-1} = A_k \cup B_k \) with the conclusions of the proposition.

We note that this process terminates after finitely many steps since all of the \( B_i \) are disjoint and after \( m + 1 \) steps, \( |E| \geq |B_1| + \cdots + |B_m| > m \epsilon |E| \). Thus we may suppose we have obtained a disjoint decomposition
\[ E = A_1 \cup \cdots \cup A_n \cup B_n \]
where \( |B_i| > \epsilon |E| \) for \( 1 \leq i < n \) and \( |B_n| \leq \epsilon |E| \). We also have multiprogressions \( P_i \) satisfying \( |P_i| \leq C_{\eta_\delta} |A_i| \), rank \( P_i \leq C_{\eta_\delta} \) and for \( 1 < i \leq n \), \( |P_i| \leq C_{\eta_\delta} |A_i| \), rank \( P_i \leq C_{\eta_\delta} \).

Thus
\[ \sum_i |P_i| \leq C_\epsilon |E|, \]
\[ A := \cup_i A_i \subset \cup_i P_i, \quad \text{rank } P_1 \leq C_\epsilon, \quad \| 1_{A_1} f \|_p \geq \| 1_{A_1} f \|_p \geq c_\delta \| f \|_p, \]
as desired. \( \square \)

**Lemma 6.3** (More structured decomposition). Let \( d \geq 1 \), and let \( \Lambda \subset (1, 2) \) be a compact set. For any \( \epsilon > 0 \) there exist \( \delta > 0 \), \( N_\epsilon < \infty \), and \( C_\epsilon < \infty \) with the following property for all \( p \in \Lambda \). Let \( |E| \leq \infty \) and \( |f| \leq 1_E \) be such that \( \| f \|_q \geq (1 - \delta) B_{q,d} |E|^{1/p} \). Then there exists a disjointly supported decomposition \( f = g + h \) where \( \text{supp } g = A \) and \( \text{supp } h = B \) are disjoint and there is a continuum multiprogression \( P \) such that
\[ |B| \leq \epsilon |E| \]
\[ |P| \leq C_\epsilon |E| \]
\[ A < P \]
\[ \text{rank } P \leq C_\epsilon \]

**Proof.** First we define \( E_\lambda = \{ x \in E : |f(x)| \leq \lambda \} \). Note that by the Hausdorff-Young inequality,
\[ \| 1_{E_\lambda} f \|_q \leq \| 1_{E_\lambda} f \|_p \leq \lambda |E|^{1/p}. \]
Assume that \( |E_\lambda| > \epsilon |E| \). Then by Lemma 4.1,
\[ \| 1_{E_\lambda} f \|_q \geq \frac{c_\epsilon}{C} |E|^{1/p} := \eta_\epsilon |E|^{1/p}. \]
Thus if we take \( \lambda = \eta_k \epsilon \), we are guaranteed that \( |E_\lambda| < \epsilon |E| \). Now without loss of generality, assume that \( |f| \geq \eta_k \epsilon \) on \( E \).

We define an iterative process with an outer and an inner loop. For the step 1 of the outer loop, let \( \eta_0 = 1 - \delta \), apply Proposition 5.3 to get \( E = A_1 \cup B_1 \) where \( A_1 \) is contained in a multiprogression \( P_1 \) satisfying the conclusions of the proposition. At step \( N \) of the outer loop, we have a disjoint decomposition

\[
f = G_N + H_N
\]

where \( \text{supp} \, G_N = A_N \) is contained in a multiprogression \( P_N \) with \( |P_N| \leq C_\epsilon |E| \), rank \( P_N \leq C_\epsilon \), and \( G_N \leq \epsilon \) if \( |B_N| < \epsilon |E| \), then we halt. Otherwise, initiate step \((N, 1)\) of the inner loop. Since \( |B_N| \geq \epsilon |E| \), by Lemma 4.1, \( |B_N| \geq \epsilon |E| \). Thus we can decompose \( B_N \) into \( S_{N,1} \) and \( R_{N,1} \) using Proposition 5.3. The halting criterion for the \((N, j)\) th step is \( |R_{N,j}| \leq \frac{1}{2} \epsilon |E| \) or \( |G_N| \leq \epsilon |E| \). If neither is satisfied in step \((N, 1)\), then \( |R_{N,1}| > \epsilon |E| \), so repeat step \((N, 1)\) replacing \( B_N \) by \( R_{N,1} \). After \( k \) iterations of the inner loop, we note that

\[
|B_N| \geq |R_{N,1}| + \cdots + |R_{N,k-1}| \geq (k-1) \epsilon |E|,
\]

so the inner loop terminates in a maximum of \( M_\epsilon \) steps.

Suppose that the inner loop terminates at step \( k \) because \( |B_{N,k}| \leq \frac{1}{2} \epsilon |E| \) but \( G_N \leq \epsilon |E| \). Then \( |G_N| \leq \epsilon |E| \). Finally we note that \( |G_N + \varphi| \geq \| f \|_q - \| 1_{R_{N,k}} f \|_q \geq (1 - \epsilon - \epsilon^{1/p}) B_{q, d} |E|^{1/p} \). Thus, choosing \( \delta \) and \( \rho \) small enough, (6.1) is a contradiction to Lemma 4.3.

Thus the halting criterion for the inner loop yields a function \( 1_{S_{N,k}} f \) such that

\[
\| G_N 1_{S_{N,k}} f \|_q \geq \| f \|_q.
\]

The function \( 1_{S_{N,k}} f \) also satisfies

\[
|1_{S_{N,k}} f| \geq c_\epsilon |1_{R_{N,k-1}} f| \geq c_\epsilon \eta_k \epsilon |R_{N,k-1}| \geq c_\epsilon \eta_k \epsilon^{1+1/p} |E|^{1/p}.
\]

If \( Q_N \) is the multiprogression associated to \( S_{N,k} \), then Lemma 6.1 (taking \( \varphi = \frac{1}{C_\epsilon |E|} 1_{A_N} f \) and \( \psi = \frac{1}{C_\epsilon |E|} 1_{S_{N,k}} f \), which satisfies the hypotheses for small enough \( \rho \)) implies that \( |P_N + Q_N| \leq C_\epsilon \min(|P_N|, |Q_N|) \) and \( \max(|P_N|, |Q_N|) \leq C_\epsilon \min(|P_N|, |Q_N|) \). Thus we can obtain a continuum multiprogression \( P_{N+1} \) of rank \( C_\epsilon \) containing \( P_N \) and \( Q_N \) and satisfying \( |P_{N+1}| \leq C_\epsilon \).

Set \( G_{N+1} = G_N + 1_{S_{N,k}} f \). Then \( H_{N+1} := f - G_{N+1} \) has support called \( B_{N+1} \). If \( |B_{N+1}| \leq \epsilon |E| \), then we’re done. If not, proceed to inner loop \( N + 1 \). Note that for each outer loop step, we have

\[
\| G_{N+1} \|_p \geq \| G_N \|_p + \| 1_{S_{N,k}} f \|_p \geq \| G_N \|_p + c_\epsilon \eta_k \epsilon^{1+1/p} |E|^{1/p}.
\]

Thus the outer loop terminates in as many as \( N_\epsilon \) steps. (Note that since the ranks of \( P_N \) and \( Q_N \) add at each step of the outer loop, the rank of the ultimate multiprogression is controlled by \( M_\epsilon > 0 \).
Lemma 6.4. Let \( d \geq 1 \), and let \( \Lambda \subset (1,2) \) be a compact set. For any \( \varepsilon > 0 \) there exist \( \delta > 0 \), \( N_\varepsilon < \infty \), and \( C_\varepsilon < \infty \) with the following property for all \( p \in \Lambda \). Let \( |E| < \infty \) and \( |f| \leq 1_E \) be such that \( \|\hat{f}\|_{q,\varepsilon} \geq (1 - \delta) B_{q,d} \|f\|_p \). Then there exists a disjointly supported decomposition \( f = g + h \) where \( \text{supp} g = A \) and \( \text{supp} h = B \) are disjoint and there is a continuum multiprogression \( P \) such that

\[
\|h\|_p \leq \varepsilon \|f\|_p
\]

\[
\|g\|_\infty |P|^{1/p} \leq C_\varepsilon \|f\|_p
\]

\[
\text{rank } P \leq C_\varepsilon
\]

Proof. Using the hypothesis \( \|\hat{f}\|_{q,\varepsilon} \geq (1 - \delta) B_{q,d} \|f\|_p \), by Lemma 5.4 we can find a disjoint decomposition \( f = g_1 + h_1 \) where \( g_1 \) is supported on a multiprogression \( P_1 \) with rank \( P_1 \leq C_\delta \), \( g_1 \|\|_p \geq c_\delta \|f\|_p \), \( \|g_1\|_\infty |P_1|^{1/p} \leq C_\delta \|f\|_p \). If \( \|h_1\|_p < \varepsilon \|f\|_p \), then we halt.

To further refine the decomposition in the case that \( \|h_1\|_p \geq \varepsilon \|f\|_p \), we define an iterative process with input \( (g_1, h_1) \) and output \( (g_2, h_2) \) where \( f = g_2 + h_2 \) and \( g_2 \), \( h_2 \) satisfy certain properties below. Apply Lemma 4.2 to conclude that \( \|h_1\|_{q,\varepsilon} \geq \eta_\varepsilon \|f\|_p \) for \( \eta_\varepsilon > 0 \). Then apply Lemma 5.4 to get \( h_1 = u_1 + v_1 \) where \( u_1 \) is supported on a multiprogression \( Q_1 \), rank \( Q_1 \leq C_\eta \), \( \|u_1\|_\infty |Q_1|^{1/p} \leq C_\eta \|f\|_p \), and \( \|u_1\|_p \geq c_\eta \|h_1\|_p \geq c_\eta \varepsilon \|f\|_p \). We can assume without loss of generality that \( c_\delta \geq c_\varepsilon \). Since \( \min(g_1\|\|_p, u_1\|\|_p) \geq c_\varepsilon \varepsilon \|f\|_p \), by Lemma 4.4 \( \|\hat{g}_1 \cdot \hat{u}_1\|_{q,\varepsilon} \geq \rho(\varepsilon)\|f\|_p B_{q,d} \), for \( \rho(\varepsilon) > 0 \). But then Lemma 6.1 (taking \( \varphi = \frac{1}{c_\delta \|f\|_p} g_1 \) and \( \psi = \frac{1}{c_\varepsilon \|f\|_p} u_1 \)) implies that \( |P_1 + Q_1| \leq C_\varepsilon \min(|P_1|, |Q_1|) \) and \( \max(|P_1|, |Q_1|) \leq C_\varepsilon \min(|P_1|, |Q_1|) \). Thus we can obtain a continuum multiprogression \( P_2 = P_1 + Q_1 \) of rank \( \leq C_\varepsilon \delta \), containing \( P_1 \) and \( Q_1 \) and satisfying \( |P_2| \leq C_\varepsilon \delta \). Let \( g_2 = g_1 + u_1 \) and let \( h_2 = v_1 \).

If \( \|h_2\|_p < \varepsilon \|f\|_p \), then halt. If \( \|h_2\|_p \geq \varepsilon \|f\|_p \), repeat the process described above with input \( (g_2, h_2) \).

After \( n \) steps of this iteration, we have a decomposition \( f = g_n + h_n \) and a multiprogression \( P_n \) of controlled size and rank containing the support of \( g_n \) and satisfying \( \|g_n\|_\infty |P_n|^{1/p} \leq C_\varepsilon \delta \|f\|_p \),

\[
\|g_n\|_p^p = \|g_1\|_p^p + \|u_1\|_p^p + \cdots + \|u_n\|_p^p \geq c_\delta^p + n c_\delta^p \|f\|_p^p.
\]

Thus the loop terminates in as many as \( n_\varepsilon \) steps. Note that since the ranks of \( P_n \) and \( Q_n \) add at each step of the process, the rank of the ultimate multiprogression is controlled by a constant depending on \( \varepsilon \). Also, \( |P_n| \leq (C_\varepsilon')^{n+1} \min(|P_1|, |Q_1|, \ldots, |Q_n|) \).

Finally we note that

\[
\|g_n\|_\infty |P_n|^{1/p} \leq (C_\varepsilon')^{n+1} (\|g_1\|_\infty |P_1|^{1/p} + \|u_1\|_\infty |Q_1|^{1/p} + \cdots + \|u_n\|_\infty |Q_n|^{1/p}) \leq (C_\varepsilon')^{n+2} \|f\|_p.
\]

\[\square\]

7. Exploitation of \( \mathbb{Z}^\kappa \times \mathbb{R}^d \)

7.1. Analysis of the discrete Hausdorff-Young inequality. Let the torus \( \mathbb{T} \) denote the quotient group \( \mathbb{R}/\mathbb{Z} \). Extend the previous notation and define the Fourier transform \( \hat{g} : \mathbb{Z}^\kappa \times \mathbb{R}^d \rightarrow \mathbb{T}^\kappa \times \mathbb{R}^d \) by

\[
\hat{g}(\theta, \xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} e^{-2\pi i \theta \cdot f(n, x)} dx
\]

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where $\theta \in \mathbb{T}^d$. This can be decomposed as $\mathcal{F} \circ \hat{\mathcal{F}}$ where
\[
\mathcal{F}g(\theta, \xi) = \sum_{n \in \mathbb{Z}^d} g(n, \xi) e^{-2\pi in \cdot \theta}
\]
\[
\hat{\mathcal{F}}f(n, \xi) = \int_{\mathbb{R}^d} f(n, x) e^{-2\pi ix \cdot \xi} dx.
\]

If we treat the operator $\mathcal{F}$ as the partial Fourier transform with respect to the first coordinate and $\hat{\mathcal{F}}$ the corresponding transform for the second coordinate, then we can say $\mathcal{F} \circ \hat{\mathcal{F}} = \hat{\mathcal{F}} \circ \mathcal{F}$ (even though the operators on the left and right are not precisely the same).

**Lemma 7.1.** Let $d, \kappa \geq 1$, and $p \in (1, 2)$, $q = p'$. The optimal constant $A(q, d, \kappa)$ in the inequality
\[
\|f\|_q \leq A(q, d, \kappa) |E|^{1/p},
\]
where $E \subset \mathbb{Z}^\kappa \times \mathbb{R}^d$ satisfies $|E| < \infty$ and $|f| \leq 1_E$ satisfies
\[
A(q, d, \kappa) = B_{q,d}.
\]
The optimal constant $A'(q, d, \kappa)$ for
\[
\|f\|_{q,*} \leq A'(q, d, \kappa) \|f\|_p
\]
for $\mathbb{Z}^\kappa \times \mathbb{R}^d$ likewise satisfies $A'(q, d, \kappa) = B_{q,d}$.

By analyzing the mixed $L^p$ norms $L^p_h L^q\xi(\mathbb{Z}^\kappa_n \times \mathbb{R}^d_\xi)$ and $L^q\xi L^p_h(\mathbb{R}^d_\xi \times \mathbb{Z}^\kappa_n)$ given respectively by
\[
\|g\|_{L^p_h L^q\xi} = \left( \sum_n \left( \int |g(n, \xi)|^q d\xi \right)^{p/q} \right)^{1/p}
\]
and
\[
\|g\|_{L^q\xi L^p_h} = \left( \int \left( \sum_n |g(n, \xi)|^p \right)^{q/p} d\xi \right)^{1/q},
\]
we will be able to prove the previous lemma. Note that there are also corresponding norms for $L^p_h L^q\theta(\mathbb{T}^d_\theta \times \mathbb{R}^d_\xi)$ and $L^q\xi L^p_h(\mathbb{T}^d_\theta \times \mathbb{R}^d_\xi)$. Since $q \geq p$, we have by Minkowski’s integral inequality that
\[
\|g\|_{L^q\xi L^p_h(\mathbb{T}^d_\kappa \times \mathbb{R}^d_\xi)} \leq \|g\|_{L^q\theta L^p_h(\mathbb{T}^d_\kappa \times \mathbb{R}^d_\xi)}.
\]
If $\mathcal{F}$ denotes the Fourier transform from $\mathbb{Z}^\kappa$ to $\mathbb{T}^\kappa$ defined by
\[
\mathcal{F}g(\theta) = \sum_n g(n) e^{-2\pi i n \cdot \theta},
\]
then the optimal constant in the corresponding Hausdorff-Young inequality for $p \in (1, 2)$ is 1. Thus if $|f| \leq 1_E$ for $E \subset \mathbb{Z}^\kappa$ and $|E| < \infty$, we have
\[
\|\mathcal{F}f\|_q \leq \|f\|_p \leq |E|^{1/p}.
\]
This means that for $g \in L^q\xi L^p_h(\mathbb{Z}^\kappa_n \times \mathbb{R}^d_\xi)$,
\[
\|\mathcal{F}g\|_{L^q\xi L^p_h} = \left( \int \int |\mathcal{F}g(\theta, \xi)|^q d\theta d\xi \right)^{1/q}
\]
\[
\leq \left( \int \left( \sum_n |g(n, \xi)|^p \right)^{q/p} d\xi \right)^{1/q},
\]
so $\mathcal{F}$ is a contraction from $L^q_{\xi} L^p_n(\mathbb{R}_\xi \times \mathbb{R}^n)$ to $L^q_{\xi} L^p_n(\mathbb{R}_\xi \times T^n)$. For $n \in \mathbb{Z}^k$, $|f| \leq 1 \in L^p(\mathbb{Z}^k \times \mathbb{R}^d)$, define

$$E_n = \{ x \in \mathbb{R}^d : (n, x) \in E \}$$

and $f_n : \mathbb{R}^d \to \mathbb{C}$ by $f_n(x) = f(n, x)$, which is in $L^p(\mathbb{R}^d)$. Note that $|f_n| \leq 1_{E_n}$, which means

$$\| \widehat{f} f \|_{L^p_n L^q_{\xi}} = \left( \sum_{n} \left( \int |\widehat{f} f(n, \xi)|^q d\xi \right)^{p/q} \right)^{1/p}$$

$$= \left( \sum_{n} \left( \int \left| \int f_n(x) e^{-2\pi i x \cdot \xi} d\xi \right|^q d\xi \right)^{p/q} \right)^{1/p}$$

$$\leq \left( \sum_{n} B^p_{q,d} |E_n| \right)^{1/p} = B_{q,d} |E|^{1/p}.$$

(7.3)

Proof. (of Lemma 7.1) Let $|f| \leq 1 \in L^p(\mathbb{Z}^k \times \mathbb{R}^d)$. We have

$$\| \mathcal{F} f \|_{L^q(\mathbb{R}^d)} \leq \| \mathcal{F} f \|_{L^q_{\xi} L^p_n} \leq \| \mathcal{F} f \|_{L^q_{\xi} L^p_n} \leq B_{q,d} |E|^{1/p},$$

where we use (7.2) in the last inequality. Thus $A(q, d, \kappa) \leq B_{q,d}$. Now let $|f| \leq 1 \in L^p(\mathbb{R}^d)$ be given. Define $\hat{E} = E \times \{0\}$ and $\hat{f} : \mathbb{Z}^k \times \mathbb{R}^d \to \mathbb{C}$ by $\hat{f}(n, x) = 0$ for $n \neq 0$ and $\hat{f}(0, x) = f(x)$. We have for $\mathcal{F}$ the Fourier transform on $\mathbb{R}^d$ defined by $\mathcal{F}g(\xi) = \int g(x) e^{-2\pi i x \cdot \xi} d\xi$ that

$$\| \mathcal{F} \mathcal{F} f \|_{L^q(\mathbb{R}^d)} = \| \mathcal{F} \hat{f}(0, \cdot) \|_{L^q_{\xi}}$$

$$= \| \mathcal{F} f \|_{L^q_{\xi} L^p_n}$$

$$\leq A(q, d, \kappa) |\hat{E}|^{1/p} = A(q, d, \kappa) |E|^{1/p}.$$

This yields the reverse inequality $A(q, d, \kappa) \geq B_{q,d}$. □

**Proposition 7.2.** Let $d, \kappa \geq 1$ and $q \in (2, \infty)$, $p = q'$. Let $\delta > 0$ be small. Let $0 \neq f \in L^d(\mathbb{Z}^k \times \mathbb{R}^d)$, $|f| \leq 1 \in E$ where $E \subset \mathbb{Z}^k \times \mathbb{R}^d$ is Lebesgue measurable and $|E| < \infty$. If $\| \hat{f} \|_q \geq (1 - \delta) B_{q,d} |E|^{1/p}$, then there exists $m \in \mathbb{Z}^k$ such that

$$|E_m| \geq (1 - \delta (1)) |E|.$$

(7.5)

**Proposition 7.3.** Let $d, \kappa \geq 1$ and $q \in (2, \infty)$, $p = q'$. Let $\delta > 0$ be small. Let $0 \neq f \in L^d(\mathbb{Z}^k \times \mathbb{R}^d)$. If $\| \hat{f} \|_{q,*, E} \geq (1 - \delta) B_{q,d} \| f \|_p$, then there exists $m \in \mathbb{Z}^k$ such that

$$\| f_m \|_{L^p(\mathbb{R}^d)} \geq (1 - \delta (1)) \| f \|_{L^p(\mathbb{Z}^k \times \mathbb{R}^d)}.$$

(7.6)

**Proof.** Since $B_{q,d} |E|^{1/p} \geq A_{q,d} \| f \|_{L^p(\mathbb{Z}^k \times \mathbb{R}^d)}$, this is an immediate result of Proposition 10.12 in [13]. □

In the analysis of $A(q, d, \kappa)$ from Lemma 7.1 we proved a string of inequalities in (7.3). Combining these inequalities with the assumption that $(f, E)$ are $\delta$-near extremizing yields the following lemma, which requires no further proof.
Lemma 7.4. Let \( d, \kappa \geq 1 \) and \( q \in (2, \infty) \). Set \( p = q' \). Let \( \delta > 0 \) and suppose \( E \subset \mathbb{Z}^{\kappa} \times \mathbb{R}^d \) be a Lebesgue measurable set with \( |E| \in \mathbb{R}^+ \) and \( f \) is a measurable function with \( |f| \leq 1_E \). If \( \|f\|_q \geq (1 - \delta)B_{q,d}|E|^{1/p} \), then all of the following hold:

\[
\begin{align*}
(7.7) \quad & \|\mathcal{F}\tilde{f}\|_{L^q_{\mathcal{E}}L^q_{\kappa}} \geq (1 - \delta)\|\tilde{f}\|_{L^q_{\kappa}L^q_{\kappa}} \\
(7.8) \quad & \|\tilde{f}\|_{L^q_{\kappa}L^q_{\kappa}} \geq (1 - \delta)\|\tilde{f}\|_{L^q_{\kappa}L^q_{\kappa}} \\
(7.9) \quad & \|\tilde{f}\|_{L^q_{\kappa}L^q_{\kappa}} \geq (1 - \delta)B_{q,d}|E|^{1/p}
\end{align*}
\]

The inequalities listed in Lemma 7.4 will be used to establish the following weak result, which is a preliminary for showing that near extremizers of the lifted problem are mostly supported on one slice of the \( \mathbb{Z}^\kappa \) variable.

Lemma 7.5. Let \( E \subset \mathbb{Z}^{\kappa} \times \mathbb{R}^d \) and \( |f| \leq 1_E \) satisfy \( \|f\|_q \geq (1 - \delta)B_{q,d}|E|^{1/p} \). There exists a disjointly supported decomposition

\[\tilde{f} f(n, \xi) = g(n, \xi) + h(n, \xi)\]

where

\[\|h\|_{L^q_{\kappa}L^q_{\kappa}} \leq o_\delta(1)|E|^{1/p}\]

and for each \( \xi \in \mathbb{R}^d \) there exists \( n(\xi) \in \mathbb{Z}^\kappa \) such that

\[g(n, \xi) = 0 \quad \text{for all} \ n \neq n(\xi).\]

**Proof.** (of Lemma 7.5) This is completely analogous to the proof of Lemma 10.14 in [13].

Let \( \eta = \delta^{1/2} \). Let \( \varphi_\xi(n) = \tilde{f} f(n, \xi) \), which is well-defined for almost every \( \xi \). Define

\[G = \{ \xi \in \mathbb{R}^d : \varphi_\xi \neq 0, \quad \|\hat{\varphi}_\xi\|_{L^q_{\kappa}} \geq (1 - \eta)\|\varphi_\xi\|_{L^q_{\kappa}} \} \]

Here, \( \hat{\cdot} \) denotes the Fourier transform for \( \mathbb{Z}^\kappa \). Then

\[
\|\mathcal{F}\tilde{f}\|_{L^q_{\kappa}L^q_{\kappa}} = \int_{R^d \setminus G} \|\hat{\varphi}_\xi\|_{L^q_{\kappa}}^q d\xi + \int_G \|\hat{\varphi}_\xi\|_{L^q_{\kappa}}^q d\xi
\]

\[
\leq (1 - \eta)^q \int_{R^d \setminus G} \|\varphi_\xi\|_{L^q_{\kappa}}^q d\xi + \int_G \|\hat{\varphi}_\xi\|_{L^q_{\kappa}}^q d\xi
\]

\[
\leq \int_{R^d} \|\tilde{f} f\|_{L^q_{\kappa}}^q d\xi - c\eta \int_{R^d \setminus G} \|\tilde{f} f\|_{L^q_{\kappa}}^q d\xi.
\]

Combining this with (7.4), we get

\[
(1 - \delta)\|\tilde{f} f\|_{L^q_{\kappa}L^q_{\kappa}} \leq \|\mathcal{F}\tilde{f}\|_{L^q_{\kappa}L^q_{\kappa}}
\]

\[
\leq \int_{R^d} \|\tilde{f} f\|_{L^q_{\kappa}}^q d\xi - c\eta \int_{R^d \setminus G} \|\tilde{f} f\|_{L^q_{\kappa}}^q d\xi.
\]

Rearranging the above inequality, we can write

\[
(7.10) \quad \int_{R^d \setminus G} \|\tilde{f} f\|_{L^q_{\kappa}}^q d\xi \leq c' \delta^{1/2} \|\tilde{f} f\|_{L^q_{\kappa}L^q_{\kappa}}.
\]

Now since for each \( \xi \in G, \|\hat{\varphi}_\xi\|_{L^q_{\kappa}} \geq (1 - \eta)\|\varphi_\xi\|_{L^q_{\kappa}}, \) we can invoke Theorem 1.3 from [1] to get \( n = n(\xi) \in \mathbb{Z}^\kappa \) such that

\[\|\varphi_\xi\|_{L^p(\mathbb{Z}^\kappa \setminus (n(\xi)))} \leq o_\eta(1)\|\varphi_\xi\|_{L^p(\mathbb{Z}^\kappa)}.\]
Define
\[ g(n, \xi) = \begin{cases} \varphi_\xi(n, \xi) & \text{if } n = n(\xi) \\ 0 & \text{else.} \end{cases} \]

Let \( h(n, \xi) := \tilde{\mathcal{F}} f(n, \xi) - g(n, \xi) \). Note that \( g \) satisfies the conclusions of the lemma by its definition. To bound \( \|h\|_{L^q_k L^\infty_p} \), we use the definition of \( g \) as well as by [7.10] to get
\[
\|h\|_{L^q_k L^\infty_p} \leq \int_G \|\mathcal{F} f\|_{L^\infty_p} d\xi + \int_{\mathbb{R}^d \setminus G} \|\mathcal{F} f\|_{L^\infty_p} d\xi
\]
\[
\leq \int_G \left( \|\tilde{\mathcal{F}} f\|_{L^p(\mathbb{Z}^\kappa \setminus n(\xi))} + |\tilde{\mathcal{F}} f(n(\xi), \xi) - g(n(\xi), \xi)| \right)^q d\xi + c \delta^{q/2} \|\mathcal{F} f\|_{L^q_k L^\infty_p}^q
\]
\[
= o_\delta(1) \|\tilde{\mathcal{F}} f\|_{L^q_k L^\infty_p}^q.
\]

**Proof.** (of Proposition [7.2]) This one should actually be done a bit more.....! Let \( \tilde{\mathcal{F}} f = g + h \) as in Lemma 7.5. Combining \( \|h\|_{L^q_k L^\infty_p} \leq o_\delta(1)(1)|E|^{1/p} \) with [7.9] implies \( \|h\|_{L^q_k L^\infty_p} \leq o_\delta(1)\|\tilde{\mathcal{F}} f\|_{L^q_k L^\infty_p} \). Using this with [7.7] gives
\[
\|g\|_{L^q_k L^\infty_p} + \|h\|_{L^q_k L^\infty_p} \geq \|\tilde{\mathcal{F}} f\|_{L^q_k L^\infty_p} \geq (1 - \delta)\|\tilde{\mathcal{F}} f\|_{L^q_k L^\infty_p}
\]
\[
\|g\|_{L^q_k L^\infty_p} \geq (1 - o_\delta(1))\|\tilde{\mathcal{F}} f\|_{L^q_k L^\infty_p}
\]
(7.11)

Note that \( \|g\|_{L^q_k L^\infty_p} = \|g\|_{L^q_k L^\infty_p} = \|g\|_{L^q_k L^\infty_p} \). Using this in (7.11) we have
\[
\|g\|_{L^q_k L^\infty_p} \geq (1 - o_\delta(1))\|\tilde{\mathcal{F}} f\|_{L^q_k L^\infty_p}
\]

Letting \( M = \sup_n \|g(n, \cdot)\|_{L^q_k}^q \) (which is finite by (7.11)), we calculate using (7.11)
\[
M^{\frac{q-p}{p}} \left( \int |g(n(\xi), \xi)|^q d\xi \right)^{1/q} \geq (1 - o_\delta(1)) M^{\frac{q-p}{p}} \left( \sum_n \left( \int |g(n(\xi), \xi)|^q d\xi \right)^{p/q} \right)^{1/p}
\]
\[
\geq (1 - o_\delta(1)) \left( \sum_n \int |g(n(\xi), \xi)|^q d\xi \right)^{1/p}
\]
\[
\geq (1 - o_\delta(1)) \left( \int |g(n(\xi), \xi)|^q d\xi \right)^{1/p}
\]

and therefore
\[
M \geq (1 - o_\delta(1)) \left( \int |g(n(\xi), \xi)|^q d\xi \right)^{\left(\frac{1}{p} - \frac{1}{q}\right)} \left( \frac{q}{p} \right) = (1 - o_\delta(1)) \int |g(n(\xi), \xi)|^q d\xi.
\]

Thus there exists \( n \in \mathbb{Z}^\kappa \) such that
\[
\int |g(n(\xi), \xi)|^q d\xi \geq (1 - o_\delta(1))(\|\tilde{\mathcal{F}} f\|_{L^q_k L^\infty_p} - \|h\|_{L^q_k L^\infty_p})^q \geq (1 - o_\delta(1)) B_{q,d} |E|^{q/p}.
\]
Then
\[ B_{q,d} |E_n|^{q/p} \geq \int |g(n, \xi)|^q d\xi \geq (1 - o_\delta(1)) B_{q,d} |E|^{q/p}, \]
so \(|E_n|^{1/p} \geq (1 - o_\delta(1)) |E|^{1/p} \).

7.2. Lifting to \( \mathbb{Z}^k \times \mathbb{R}^d \).

**Definition 7.1.** To any function \( f : \mathbb{R}^d \to \mathbb{C} \), associate the function \( f^\dagger : \mathbb{Z}^d \times \mathbb{R}^d \to \mathbb{C} \) defined by
\[
f^\dagger(n, x) = \begin{cases} f(n + x) & \text{if } x \in \mathbb{Q}_d \\ 0 & \text{if } x \notin \mathbb{Q}_d. \end{cases}
\]

For a measurable set \( E \subset \mathbb{R}^d \), let \( E^\dagger \) be the set in \( \mathbb{Z}^d \times \mathbb{R}^d \) defined by
\[ E^\dagger = \{(n, x) : n + x \in E\}. \]

**Lemma 7.6.** Let \( d \geq 1 \) and \( q \in (2, \infty) \), \( p = q' \). Let \( \delta, \eta > 0 \) be small. Let \( E \subset \mathbb{R}^d \) be a Lebesgue measurable set with \( |E| \in \mathbb{R}^+ \). Suppose that
\[ \text{distance}(x, \mathbb{Z}^d) \leq \eta \quad \text{for all } x \in E \]
and that for \(|f| \leq 1_E\),
\[ \|\hat{f}\|_{L^q(\mathbb{R}^d)} \geq (1 - \delta) A_{q,d} |E|^{1/p}. \]
Then
\[ \|\hat{f}\|_{L^q(\mathbb{T}^d \times \mathbb{R}^d)} \geq (1 - \delta - o_\eta(1)) A_{q,d} |E^\dagger|^{1/p}. \]

**Proof.** This is a direct consequence of Lemma 9.1 of [12] combined with the fact that \( B_{q,d} |E|^{1/p} (1 - \delta) \leq \|\hat{f}\|_q \) and \(|E| = |E^\dagger|\). \qed

The following lemma is analogous to the previous lemma. Ultimately, it is necessary to establish analogous results for the norm \( \| \cdot \|_{q, \infty} \) because we will use it to translate properties of near extremizers to the the Fourier transforms of near-extremizers. (at least we hope to... fingers crossed...)

**Lemma 7.7.** Let \( d \geq 1 \) and \( q \in (2, \infty) \), \( p = q' \). Let \( \delta, \eta > 0 \) be small. Let \( 0 \neq f \in L^p(\mathbb{R}^d) \). Suppose that
\[ f \neq 0 \implies \text{distance} (x, \mathbb{Z}^d) \leq \eta \]
and that
\[ \|\hat{f}\|_{L^q(\mathbb{R}^d)} \geq (1 - \delta) B_{q,d} \|f\|_p. \]
Then
\[ \|\hat{f}\|_{L^q(\mathbb{T}^d \times \mathbb{R}^d)} \geq (1 - \delta - o_\eta(1)) B_{q,d} \|f^\dagger\|_{L^p(\mathbb{Z}^d \times \mathbb{R}^d)}. \]

**Proof.** Let \( \xi = n(\xi) + \alpha(\xi) \) where \( n(\xi) \in \mathbb{Z}^d \) and \( \alpha(\xi) \in [-1/2, 1/2]^d \). We know from the proof of Lemma 9.1 in [12] that
\[ \|\hat{f}(\theta, n(\xi) + \alpha(\xi)) - \hat{f}(n(\xi) + \theta)\|_{L^q(\xi)} \leq o_\eta(1) \|f\|_p. \]
Let \( E \subset \mathbb{R}^d \) be such that \( |E|^{-1/p} \int_E |\hat{f}(\xi)|d\xi \geq (1 - 2\delta)B_{q,d}f\|_{L^p_{\tilde{p}}} \). Define the lifted set 
\( \tilde{E} = \{(\theta, \xi) \in \mathbb{T}^d \times \mathbb{R}^d : \theta + n(\xi) \in E\} \). Using (7.12), we calculate 
\[
\int_{\tilde{E}} |\hat{f}(\theta, \xi)|d\theta d\xi = \int_{E} |\hat{f}(n(\xi) + \theta)|d\theta d\xi - \int_{E} |\hat{f}^1(n(\xi) + \alpha(\xi)) - \hat{f}(n(\xi) + \theta)|d\theta d\xi \\
\geq |\tilde{E}|^{1/p}\|f\|_{L^p_{\tilde{p}}_\xi} - |\tilde{E}|^{1/p}o_\eta(1)\|f\|_{L^p_{\tilde{p}}_\xi} \\
= |\tilde{E}|^{1/p}(1 - 2\delta - o_\eta(1))B_{q,d}\|f\|_{L^p_{\tilde{p}}_\xi}.
\]

Translating to the situation in the above’s hypothesis will be much easier with the following Proposition 5.2 from [12], stated here for the reader’s convenience.

**Proposition 7.8.** (Approximation by \( \mathbb{Z}^d \)). For each \( d \geq 1 \) and \( r \geq 0 \) there exists \( c > 0 \) with the following property. Let \( P \) be a continuum multiprogression in \( \mathbb{R}^d \) of rank \( r \), whose Lebesgue measure satisfies \( |P| = 1 \). Let \( \delta \in (0, \frac{1}{2}] \). There exists \( \mathcal{T} \in \text{Aff}(d) \) whose Jacobian determinant satisfies
\[
|J(\mathcal{T})| \geq c\delta^{dr+d^2}
\]
such that
\[
\|\mathcal{T}(x)\|_{\mathbb{R}^d/\mathbb{Z}^d} < \delta \quad \text{for all} \quad x \in P.
\]

### 7.3. Spatial localization.

**Proposition 7.9.** Let \( d \geq 1 \) and \( q \in (2, \infty) \), \( p = q' \). For every \( \epsilon > 0 \) there exists \( \delta > 0 \) with the following property. Let \( E \) be a measurable set with \( |E| \in \mathbb{R}^+ \) and \( |f| \leq 1_E \). If \( \|\hat{f}\|_{q} \geq (1 - \delta)B_{q,d}|E|^{1/p} \), then there exists an ellipsoid \( \mathcal{E} \subset \mathbb{R}^d \)
\[
\|E \setminus \mathcal{E}\| \leq \epsilon |E| \\
|\mathcal{E}| \leq C_{\epsilon}|E|.
\]

**Proof.** Assume that \( |E|^{1/p}B_{q,d}(1 - \delta) \leq \|\hat{f}\|_{q} \), where \( \delta \) is to be chosen below.

1. Using the structural lemma for near extremizers of [12], Lemma 6.3 with \( \epsilon_0 > 0 \) to be chosen later, we obtain a decomposition \( E = A \cup B \) and a multiprogression \( P \) satisfying
\[
E = A \cup B, \quad A \cap B = \emptyset, \\
|B| \leq \epsilon_0|E|, \\
|P| \leq C_{\epsilon_0}|E|, \\
A < P, \\
\text{rank} \ P \leq C_{\epsilon_0}.
\]

2. By precomposing \( f \) with an affine transformation, assume without loss of generality that \( |P| = 1 \). Then for a fixed \( \delta_0 \in (0, \frac{1}{2}] \) to be chosen below, Proposition 5.2 in [12], otherwise known as Proposition 7.8 in this paper, allows us to find a \( c = c(d, p) \) as well as \( \mathcal{T} \in \text{Aff}(d) \) such that
\[
|J(\mathcal{T})| \geq c\delta_0^{dC_\epsilon+\epsilon_0} \quad \text{and} \\
\|\mathcal{T}(A)\|_{\mathbb{R}^d/\mathbb{Z}^d} < \delta_0.
\]
(3) Now taking $\eta_0 = \delta_0$ in the hypothesis of Lemma 7.9, we are guaranteed that since
\[ \|1_A(\mathcal{T}^{-1})\|_q \geq (1 - \delta)|\mathcal{T}^{-1}E|^{1/p} - \|1_B(\mathcal{T}^{-1})\|_q \geq (1 - \delta - \alpha_0(1))|\mathcal{T}^{-1}E|^{1/p} \]
and $\|\mathcal{T}(A)\|_{\mathbb{R}^d/\mathbb{Z}^d} < \delta_0$, we have
\[ \|(1_A \mathcal{T}^{-1})\|_{L^q(\mathbb{Z}^d \times \mathbb{R}^d)} \geq (1 - \delta - \alpha_0(1) - \alpha_0(1))|\mathcal{T}(A)| \]
where $^\wedge$ here denotes the Fourier transform on $\mathbb{Z}^d \times \mathbb{R}^d$. 

(4) Then Proposition 7.2 gives the existence of $m \in \mathbb{Z}^d$ such that
\[ |\mathcal{T}(A) \cap (m + [1/2, 1/2]^d)| \geq (1 - \omega(1) - \omega(1) - \omega(1))|\mathcal{T}(A)|. \]

(5) Last, we note that taking the cube $Q := m + [1/2, 1/2]^d$ gives us a cube that satisfies
\[ |E \setminus \mathcal{T}^{-1}(Q)| \leq |A| + |B| - |A \cap \mathcal{T}^{-1}(Q)| \]
\[ \leq |A| + \epsilon_0|E| - (1 - \omega(1) - \omega(1) - \omega(1))|A| \]
\[ \leq (\epsilon_0 + \omega(1) + \omega(1) + \omega(1))|E|. \]

Note that $\epsilon_0$ and $\delta_0$ may be chosen freely, and $\delta$ may be taken small enough after fixing an $\epsilon_0$ and $\delta_0$. Thus we may choose $\epsilon_0$, $\delta_0$, and then $\delta$ small enough so that $|E \setminus \mathcal{T}^{-1}(Q)| \leq \epsilon|E|$. We also note that
\[ |\mathcal{T}^{-1}(Q)| = |J(\mathcal{T})|^{-1}|Q| \]
\[ = |J(\mathcal{T})|^{-1}|P| \]
\[ \leq (\epsilon_0^{-1}d)^{-1}C_\epsilon|E| \]
\[ = \tilde{C}_\epsilon|E|. \]

Finally, since cubes and ellipses are comparable up in size up to dimensional constants, we are done.

\[ \square \]

**Proposition 7.10.** Let $d \geq 1$ and $q \in (2, \infty)$, $p = q'$. For every $\epsilon > 0$ there exists $\delta > 0$ with the following property. Let $0 = f \in L^q_\perp(\mathbb{R}^d)$ satisfy $\|\hat{f}\|_{q, \ast} \geq (1 - \delta)\hat{f}_{q, \ast}$. There exists an ellipsoid $E \subset \mathbb{R}^d$ and a decomposition $f = \phi + \psi$ such that
\[ \|\psi\|_{q'} < \epsilon\|f\|_p \]
\[ \phi \equiv 0 \quad \text{on } \mathbb{R}^d \setminus E \]
\[ \|\phi\|_{\infty} |E|^{1/p} \leq C_\epsilon \|f\|_p. \]

**Proof.** We follow an analogous argument as that in the proof of Proposition 7.9, replacing the near extremizer structure Lemma [6.3] by the analogous structure theorem for the dual problem, Lemma [6.4]. The other necessary theorems are contained in this paper.

\[ \square \]

**7.4. Frequency localization.**

**Proposition 7.11.** Let $d \geq 1$ and $q \in (2, \infty)$, $p = q'$. For every $\epsilon > 0$ there exists $\delta > 0$ with the following property. Let $E$ be a Lebesgue measurable set with $|E| \in \mathbb{R}^+$. Suppose that $|f| \leq 1_E$ satisfies $\|\hat{f}\|_q \geq (1 - \delta)\hat{f}_{q, \ast}|E|^{1/p}$. Then there exists an ellipsoid $E' \subset \mathbb{R}^d$ and a decomposition $\hat{f} = \Phi + \Psi$ such that
\[ \|\Psi\|_{q'} < \epsilon\|\hat{f}\|_p \]
\[ \Phi \equiv 0 \quad \text{on } \mathbb{R}^d \setminus E' \]
We will bound each integral defined above one at a time. First, we use the properties of \(7.15\) and \(7.16\) by Proposition 7.9 that

\[
\|\Phi\|_\infty |\mathcal{E}'|^{1/p} \leq C_{\eta}\|f\|_p.
\]

**Proof.** In the proof of Proposition 7.11 we showed that if \((f, E)\) is a near-extremizing pair for \(L^2\) then \( \hat{f} \chi^{|E|^{q-2}} \) is a near-extremizer for \(L^1\). Thus we may apply Proposition 7.10 to obtain a decomposition \( f = \varphi + \psi \) and take \( \Phi = \varphi |(2-q)/(q-1) \) and \( \Psi = \psi |(2-q)/(q-1) \) for the desired decomposition. \(\square\)

7.5. **Compatibility of approximating ellipsoids.** We will show that \( \mathcal{E} \) and \( \mathcal{E}' \) are dual to each other, up to bounded factors and independent translations. For \( s \in \mathbb{R}^+ \) and \( E \subset \mathbb{R}^d \), we consider the dilated set \( sE = \{sx : y \in E \} \).

**Definition 7.2.** The polar set \( \mathcal{E}^* \) of an ellipsoid \( \mathcal{E} \subset \mathbb{R}^d \) centered at 0 is

\[
\mathcal{E}^* = \{ y : \langle x, y \rangle \leq 1 \text{ for every } x \in \mathcal{E} \}
\]

where \( \langle \cdot, \cdot \rangle \) denotes the Euclidean inner product.

**Lemma 7.12.** Let \( d \geq 1 \) and \( \Lambda \subset (1, 2) \) be a compact set. Let \( \eta > 0 \). Fix \( p \in \Lambda \) and let \( q = q' \). Let \( \| \hat{f} \|_q \geq (1 - o_\eta(1))B_{q,d}|E|^{1/p} \) so that there exists an ellipsoid \( \mathcal{E} + u \) satisfying the conclusions of Proposition 7.9 with \( \epsilon = \eta \) and an ellipsoid \( \mathcal{E} + v \) and disjoint decomposition \( \hat{f} = \Phi + \Psi \) satisfying the conclusions of Proposition 7.10 with \( \epsilon = \eta \), where \( \mathcal{E} \) and \( \mathcal{E}' \) are ellipsoids centered at the origin and \( u, v \in \mathbb{R}^d \). Then for \( \eta < \eta_0 \), there exists a constant \( C = C(d, \Lambda, \eta) \) such that

\[
\mathcal{E} \subset CE^* \quad \text{and} \quad \mathcal{E}' \subset C\mathcal{E}^*.
\]

**Proof.** By constants, we mean quantities which are permitted to depend on \( d, \Lambda, \eta \). By replacing \( f \) and \( 1_E \) with \( e^{2\pi i x \cdot v} f(x + u) \) and \( 1_E(x + u) \), we may assume without loss of generality that \( u, v = 0 \). By dilating \( f \) and \( E \) by \( |\mathcal{E}|^{1/d} \), we may further assume that \( |\mathcal{E}| = 1 \).

First, we will prove that \( |\hat{E}| = |\mathcal{E}||\hat{E}| < C \). We have assumed that

\[
(1 - o_\eta(1))B_{q,d}|E|^{1/p} \leq \| \hat{f} \|_q,
\]

by Proposition 7.9 that

\[
|E \setminus \mathcal{E}| \leq \eta|E|, \quad |\mathcal{E}| \leq C_\eta|E|
\]

and by Proposition 7.10 that

\[
\|\Psi\|_q \leq \eta|E|^{1/p}, \quad \Phi < \hat{E}, \quad \|\Phi\|_\infty |\hat{E}|^{1/q} \leq C_\eta|E|^{1/p}.
\]

Let \( T_E \in \text{Aff}(d) \) such that \( T_E(\mathbb{B}) = \hat{E} \), so \( |T_E| = |\hat{E}| |\mathbb{B}|^{-1} \). Let \( S_\alpha = \{ \xi : |\hat{f}(\xi)| \geq \alpha|\hat{E}|^{-1/q} \} \) and \( \lambda_\eta = \{ \xi : |\hat{f}(\xi)| \leq C_\eta|E|^{1/p}|\hat{E}|^{-1/q} \} \). We decompose the following integral as

\[
\int_{\mathbb{R}^d} |\hat{f} \circ T_E|^q d\xi = \int_{\mathbb{B}_q} |\hat{f} \circ T_E|^q d\xi + \int_{\mathbb{B}_q \cap T_E^{-1}(\lambda_\eta)^c} |\hat{f} \circ T_E|^q d\xi
\]

\[
+ \int_{\mathbb{B}_q \cap T_E^{-1}(\lambda_\eta \cap S_\alpha)} |\hat{f} \circ T_E|^q d\xi + \int_{\mathbb{B}_q \cap T_E^{-1}(\lambda_\eta \cap S_\alpha)} |\hat{f} \circ T_E|^q d\xi
\]

\[
= A + B + C + D.
\]

We will bound each integral defined above one at a time. First, we use the properties of the decomposition \( \hat{f} = \Phi + \Psi \) to note that

\[
A = \int_{\mathbb{B}_q} |\Psi \circ T_E|^q d\xi \leq |T_E|^{-1} \|\Psi\|_q^q \leq |T_E|^{-1} \eta^q |E|^{q/p}.
\]
Next, we use the property that \( |\hat{f}| \leq C_\eta |E|^{1/p} \hat{E}^{-1/q} \) a.e. on \( \hat{E} \) to control \( B \). Namely,

\[
B \leq |T_{\hat{E}}|^{-1} \int_{T_{\hat{E}}} \Psi^q \leq |T_{\hat{E}}|^{-1} \eta^q |E|^{q/p}.
\]

For \( C \), we use that \( |\hat{f} \circ T_{\hat{E}}| \leq C_\eta |E|^{1/p} \hat{E}^{-1/q} \) on \( T_{\hat{E}}^{-1}(\lambda_\eta) \) to get

\[
C \leq C_\eta |E|^{q/p} \hat{E}^{-1} |B \cap T_{\hat{E}}^{-1}(S_\alpha)|.
\]

Finally, we have for \( D \) that

\[
D \leq |B \cap T_{\hat{E}}^{-1}(\lambda_\eta)\alpha| \hat{E}^{-1} = |T_{\hat{E}}|^{-1} \lambda_\eta \alpha |\hat{E}^{-1}| = |T_{\hat{E}}|^{-1} \alpha^q.
\]

Combining the upper bounds for \( A, B, C, D \) with (7.15), we have

\[
(1 - o_\eta(1))^q B_{q,d} |E|^{q/p} |T_{\hat{E}}|^{-1} \leq \int_{T_{\hat{E}}} |f \circ T_{\hat{E}}|^q d\xi
\]

\[
= A + B + C + D
\]

\[
\leq |T_{\hat{E}}|^{-1} \eta^q |E|^{q/p} + |T_{\hat{E}}|^{-1} \eta^q |E|^{q/p}
\]

\[
+ C_\eta |E|^{q/p} |\hat{E}^{-1/2}| |T_{\hat{E}}|^{-1} |\hat{E} \cap S_\alpha| + |T_{\hat{E}}|^{-1} \alpha^q
\]

Rearranging, we get

\[
C_\eta^{-q}[(1 - o_\eta(1))^q B_{q,d} - 2\eta^q - \alpha^q |E|^{-q/p}] |\hat{E} \cap \Delta| \leq |\hat{E} \cap \Delta|.
\]

Finally, since \( |\hat{E}| = 1 \) and \( |\hat{E}| \leq C_\eta |E| \), we have

\[
C_\eta^{-q}[(1 - o_\eta(1))^q B_{q,d} - 2\eta^q - \alpha^q C_\eta^{q/p}] |\hat{E} \cap \Delta| \leq |\hat{E} \cap \Delta|.
\]

Choose \( \alpha \) small enough so that

\[
\frac{1}{2} C_\eta^{-q}[(1 - o_\eta(1))^q B_{q,d} - 2\eta^q] \leq C_\eta^{-q}[(1 - o_\eta(1))^q B_{q,d} - 2\eta^q - \alpha^q C_\eta^{q/p}],
\]

so \( \alpha \) only depends on \( \eta \). Thus for \( c' = c'(\eta) > 0 \) and \( \alpha = \alpha(\eta) \), we can write

\[
c' |\hat{E} | \leq |\hat{E} \cap \Delta|.
\]

Since \( f \) is compactly supported, it is in \( L^2 \). Since \( |\hat{E}| = 1 \), note that \( |E| = |E \cap \hat{E}| + |E \setminus \hat{E}| \leq 1 + \eta |E| \), so we can assume \( |E| \leq 2 \). Using these two observations, we have

\[
4 \geq |E| \geq \|f\|_2^2 \geq \|\hat{f}\|_2^2 \geq \int_{S_\alpha} |\hat{f}(\xi)|^2 d\xi \geq \alpha^2 |\hat{E}|^{-2/q} |S_\alpha| \geq \alpha^2 c' |\hat{E}|^{-2/q},
\]

so \( |E| |\hat{E}| = |\hat{E}| \leq C' \) for \( C' = C'(\eta) \).

Now assume via composition with an affine transformation and multiplication by a character \( e^{ix \cdot y} \) that \( \hat{E} = \mathbb{B} \) and that \( \mathcal{E} = \{ x : \sum_{j=1}^d s_j^2 x_j^2 \leq 2 \} \). We wish to show that \( \mathbb{B} \subset C \mathcal{E}^* \) and that \( \mathcal{E} \subset C \mathbb{B} \), where \( C \) is permitted to depend on \( \eta \). We know from the earlier discussion that \( |\mathcal{E}| \mathbb{B} \leq C_\eta \). Since \( |\mathcal{E}| = c_d \prod_{j=1}^d s_j \), it remains to show that the smallest \( s_i \), say \( s_1 \), is bounded below. Using the same notation as earlier, we note that

\[
\|\hat{e}_{\xi} \mathcal{E} \mathbb{f}\|_q \leq B_{q,d} \|x_{\mathcal{E}} f\|_p \leq B_{q,d} s_1 |E|^{1/p}
\]
and that
\[ \|1_{E}\hat{f}\|_{L^q(\mathbb{R})} \geq \|\hat{f}\|_{L^q(\mathbb{R})} - \|1_{E}\hat{f}\|_q \]
\[ \geq \|E\hat{f}\|_{L^q(\mathbb{R}^d)} - \|\Psi\|_{L^q(\mathbb{R}^d)} - \mathcal{B}_{q,d}|E\hat{f}|^{1/p} \]
\[ \geq (1 - o_\eta(1))\mathcal{B}_{q,d}|E|^{1/p} - \eta|E|^{1/p} - \mathcal{B}_{q,d}\eta^{1/p}|E|^{1/p} \]
\[ \geq (1 - o_\eta(1))\mathcal{B}_{q,d}|E|^{1/p}. \]

We also have
\[ \|E|^{-1/p}\hat{f}\|_q \leq \mathcal{B}_{q,d}. \]

Thus we are in the situation where there are functions \( h \) satisfying \( \|\hat{\mathcal{E}}\xi, h\|_{L^q(\mathbb{R}^d)} \leq \rho \) and \( \|h\|_{L^q(\mathbb{R})} > \varepsilon'' > 0 \). If \( \varepsilon'' \) remains fixed, then as \( \rho \to 0 \), \( \|h\|_{L^q(\mathbb{R}^d)} \to \infty \). Since we have a uniform upper bound on the \( L^p \) norms of functions \( h = |E|^{-1/p}1_{E}\hat{f} \), there must be a positive lower bound for \( s_1 \) depending on \( \eta \), which completes the proof. \( \square \)

8. Precompactness

We restate Theorem 1.1 for the reader’s convenience.

**Theorem 1.1** Let \( d \geq 1 \) and \( q \in (2, \infty) \), \( p = q' \). Let \((E_\nu)\) be a sequence of Lebesgue measurable subsets of \( \mathbb{R}^d \) with \( |E_\nu| \in \mathbb{R}^+ \) and let \( f_\nu \) be Lebesgue measurable functions on \( \mathbb{R}^d \) satisfying \( |f_\nu| \leq 1_{E_\nu} \). Suppose that \( \lim_{\nu \to \infty} |E_\nu|^{-1/p}\|\hat{f}_\nu\|_q = \mathcal{B}_{q,d} \). Then there exists a subsequence of indices \( \nu_k \), a Lebesgue measurable set \( A \subset \mathbb{R}^d \), a Lebesgue measurable function \( f \) on \( \mathbb{R}^d \) satisfying \( |f| \leq 1_A \), and a sequence \((T_k)\) of affine automorphisms of \( \mathbb{R}^d \) such that
\[ \lim_{k \to \infty} \|f_{\nu_k} \circ T_k^{-1} - f\|_p = 0 \quad \text{and} \quad \lim_{k \to \infty} |T_k(E_{\nu_k})\Delta E| = 0. \]

**Lemma 8.1.** Let \( d \geq 1 \) and \( q \in (2, \infty) \), \( p = q' \). Let \((E_\nu)\) be a sequence of Lebesgue measurable subsets of \( \mathbb{R}^d \) with \( |E_\nu| \in \mathbb{R}^+ \). Let \( f_\nu \) be Lebesgue measurable functions satisfying \( |f_\nu| \leq 1_{E_\nu} \). Suppose that \( \lim_{\nu \to \infty} |E_\nu|^{-1/p}\|\hat{f}_\nu\|_q = \mathcal{B}_{q,d} \). Then there exists a sequence of elements \( T_\nu \in \text{Aff}(d) \) such that \( T_\nu(E_\nu) \subset \mathbb{B} \) for all \( \nu \) and the sequences of functions \((f_\nu \circ T_\nu^{-1})\) is precompact in \( L^q(\mathbb{R}^d) \).

**Proof.** Without loss of generality, assume that \( \text{supp} f_\nu = E_\nu \) for all \( \nu \). Let \( |f_\nu| \leq 1_{E_\nu} \) be such that
\[ \|\hat{f}_\nu\|_q \rightarrow \mathcal{B}_{q,d} \quad \text{as} \quad \nu \rightarrow \infty. \]
Let \( \delta_\nu > 0 \) be the sequence tending to zero defined by
\[ (1 - \delta_\nu)\mathcal{B}_{q,d}|E_\nu|^{1/p} = \|\hat{f}_\nu\|_q. \]

Invoking Proposition 7.9, there exist ellipsoids \( \mathcal{E}_\nu \), positive \( \varepsilon_\nu > 0 \) tending to zero, and constants \( C_\nu > 0 \) such that
\[ |E_\nu \setminus \mathcal{E}_\nu| \leq \varepsilon_\nu|E_\nu| \quad \text{and} \quad |\mathcal{E}_\nu| \leq C_\nu|E_\nu|. \]
Let \( T_\nu \in \text{Aff}(d) \) be such that \( T_\nu^{-1}(\mathcal{E}_\nu) \subset \mathbb{B} \) and \( |T_\nu^{-1}(\mathcal{E}_\nu)| \leq 1 \). Note that since
\[ |T_\nu^{-1}(E_\nu) \setminus \mathbb{B}| \leq \varepsilon_\nu|T_\nu^{-1}(E_\nu)|, \]
The hypotheses of Lemma 7.12.

Theorem 7.12 associates to $\nu, \epsilon$ for large enough $\nu$ and, by the Hausdorff-Young inequality,

$$\|f_{\nu} \circ T_{\nu} 1_\mathbb{B}\|_q \leq \|f_{\nu} \circ T_{\nu} 1_\mathbb{B}\|_q.$$  (1)

Thus

$$\frac{\|f_{\nu} \circ T_{\nu}\|_q}{\|T_{\nu}^{-1}(E_{\nu})\|^{1/p}} \to \mathcal{B}_{q,d} \quad \text{as} \quad \nu \to \infty.$$  

In addition, proving precompactness of $f_{\nu} \circ T_{\nu} 1_\mathbb{B}$ implies precompactness of $f_{\nu} \circ T_{\nu}$ since

$$\|f_{\nu} \circ T_{\nu} - f_{\nu} \circ T_{\nu} 1_\mathbb{B}\|_q = \|f_{\nu} \circ T_{\nu} 1_\mathbb{B}\|_q \leq \mathcal{B}_{q,d}\|T_{\nu}^{-1}(E_{\nu})\|^{1/p} \leq \mathcal{B}_{q,d}\epsilon_{\nu}|T_{\nu}^{-1}(E_{\nu})|^{1/p} \leq \mathcal{B}_{q,d}\epsilon_{\nu}2\mathbb{B}$$

where we used that since $|T_{\nu}^{-1}(E_{\nu})| = |T_{\nu}^{-1}(E_{\nu})\cap\mathbb{B}| + |T_{\nu}^{-1}(E_{\nu})\setminus\mathbb{B}| \leq |\mathbb{B}| + \epsilon_{\nu}|T_{\nu}^{-1}(E_{\nu})|$, for large enough $\nu$, $|T_{\nu}^{-1}(E_{\nu})| \leq 2\mathbb{B}$.

Invoking Proposition 7.11 we have a disjoint decomposition $f_{\nu} \circ T_{\nu} = \Phi_{\nu} + \Psi_{\nu}$ and ellipsoid $\mathcal{E}_{\nu}$ such that

$$\|\Psi_{\nu}\|_q \leq \epsilon_{\nu}\|\tilde{f}_{\nu}\|_q$$

and

$$\|\Phi_{\nu}\|_q |\tilde{\mathcal{E}}_{\nu}|^{1/q} \leq C_{\epsilon_{\nu}}\|\tilde{f}_{\nu}\|_q.$$  

By Lemma 7.12 there exists $C_{\nu} > 0$ such that

$\tilde{\mathcal{E}}_{\nu} \subset C_{\nu}\mathbb{B}.$

For each $\epsilon > 0$, there is an index $N < \infty$ such that for $n \geq N$, Proposition 7.11 and Lemma 7.12 associate to $f_{\nu} \circ T_{\nu} 1_\mathbb{B}$ the ellipsoid $\tilde{F}_{\nu,\epsilon}$, a disjoint decomposition $f_{\nu} \circ T_{\nu} = \Phi_{\nu,\epsilon} + \Psi_{\nu,\epsilon}$ satisfying the conclusions of Proposition 7.11 and a constant $C_{\epsilon}$ such that

$\tilde{F}_{\nu,\epsilon} \subset C_{\epsilon}\mathbb{B}.$

(Note that here we used that $\mathbb{B}$ is related to $f_{\nu} \circ T_{\nu} 1_\mathbb{B}$ and $T_{\nu}^{-1}(E_{\nu})\cap\mathbb{B}$ as required by the hypotheses of Lemma 7.12.)

Note the uniform bounds

$$\|f_{\nu} \circ T_{\nu} 1_\mathbb{B}\|_q \leq \mathcal{B}_{q,d}|\mathbb{B}|^{1/p}$$

and, by the Hausdorff-Young inequality,

$$\|\nabla f_{\nu} \circ T_{\nu} 1_\mathbb{B}\|_q \leq \|x 1_\mathbb{B}\|_p \leq |\mathbb{B}|^{1/p}.$$  

By Rellich’s theorem, on any fixed bounded subset of $\mathbb{R}^d$, we can find an $L^q$ convergent subsequence of $f_{\nu} \circ T_{\nu} 1_\mathbb{B}$. We define an iterative process. Let $f_{1,n} \circ T_{1,n} 1_\mathbb{B}$ be a convergent
subsequence of \( f_\nu \circ T_\nu 1_B \) on \( \mathbb{B} \). For \( k > 1 \), let \( f_{k,n} \circ T_{k,n} 1_B \) be a convergent subsequence of \( f_{k-1,n} \circ T_{k-1,n} 1_B \) on \( k\mathbb{B} \).

Now we prove compactness of the diagonal sequence in \( L^q(\mathbb{R}^d) \). Let \( \epsilon > 0 \). Recall the disjoint decomposition \( f_\nu \circ T_\nu 1_B = \Phi_{\nu,\epsilon} + \Psi_{\nu,\epsilon} \) where \( \Phi_{\nu,\epsilon} \) is supported on an ellipse \( F_{\nu,\epsilon} \subset C_\epsilon \mathbb{B} \) and \( \| \Psi_{\nu,\epsilon} \|_q \leq \epsilon \| f_\nu \circ T_\nu 1_B \|_q \). Thus if \( M \geq C_\epsilon \), then if \( \nu \geq N(\epsilon) \), then

\[
\int_{\mathbb{R}^d \setminus MB} |f_\nu \circ T_\nu 1_B(\xi)|^q d\xi \leq \| \Psi_{\nu,\epsilon} \|_q^q \leq \epsilon |\mathbb{B}|^{1/p}.
\]

On \( MB \), we know that the sequence \( f_{M,n} \circ T_{M,n} 1_B \) converges, so we can take \( N(M) > 0 \) large enough to guarantee that if \( n,m \geq N(M) \),

\[
\int_{MB} |f_{n,n} \circ T_{n,n} 1_B(\xi) - f_{m,m} \circ T_{m,m} 1_B(\xi)|^q d\xi < \epsilon.
\]

Putting everything together, if \( n \geq m \geq \max(N(\epsilon),N(M)) \), the diagonal sequence \( f_{n,n} \circ T_{n,n} 1_B \) satisfies

\[
\int_{\mathbb{R}^d} |f_{n,n} \circ T_{n,n} 1_B(\xi) - f_{m,m} \circ T_{m,m} 1_B(\xi)|^q d\xi = \int_{MB} |f_{n,n} \circ T_{n,n} 1_B(\xi) - f_{m,m} \circ T_{m,m} 1_B(\xi)|^q d\xi
\]

\[
+ 2^q \int_{\mathbb{R}^d \setminus MB} |f_{n,n} \circ T_{n,n} 1_B(\xi)|^q d\xi + 2^q \int_{\mathbb{R}^d \setminus MB} |f_{m,m} \circ T_{m,m} 1_B(\xi)|^q d\xi
\]

\[
\leq \epsilon + 2^q \| \Psi_{n',\epsilon} \|_q^q + 2^q \| \Psi_{m',\epsilon} \|_q^q
\]

\[
\leq \epsilon + 2^{q+1} |\mathbb{B}|^{1/p}\epsilon
\]

where \( f_{n,n} \circ T_{n,n} 1_B = f_{n'} \circ T_{n'} 1_B \), \( f_{m,m} \circ T_{m,m} 1_B = f_{m'} \circ T_{m'} 1_B \), so \( n' \geq n, m' \geq m \).

\[\square\]

**Proof.** of Theorem 1.1

From Lemma 8.1, we can assume that \( f_\nu \) is convergent and \( E_\nu \subset \mathbb{B} \). Without loss of generality, also assume that

\[
\lim_{\nu \to \epsilon} |E_\nu| = a
\]
and $|E_\nu| \leq a$ for all $\nu$. Then
\[
\lim_{\nu \to \infty} \|\hat{f}_\nu\|_q = B_{q,d}a^{1/p}.
\]

We will prove the $L^p$ convergence of a subsequence of $f_\nu$ to a limit function that has modulus equal to 1 on its support. From this, it follows that the corresponding supports converge in $L^p$ to the support of $f$.

Since $\|f_\nu\|_2 \leq |B|^{1/2}$ for all $\nu$, by the Banach-Alaoglu theorem, there is a weak-* convergent subsequence (which we just denote $f_\nu$) to a limit $f$ in $L^2$. Note that since weak-* convergence of $f_\nu$ to $f$ implies convergence as Schwartz distributions, we must have $\hat{f}_\nu$ converge to $\hat{f}$ as Schwartz distributions. Since $\hat{f}_\nu$ is a convergent sequence in $L^q$, we must therefore be true that $\hat{f}_\nu \to \hat{f}$ strongly in $L^q$.

If $\|f\|_2 = \lim_{n \to \infty} \|f_n\|_2$, then since
\[
\|f_n - f\|_2^2 = \|f_n\|_2^2 - 2\text{Re}\langle f_\nu, f \rangle + \|f\|_2^2,
\]
$f_n \to f$ in $L^2$. Thus our goal is to prove that $\lim_{n \to \infty} \|f_n\|_2 = \|f\|_2$. Fix some notation. Let $\text{supp} f = S$. Note that $S \subset \mathbb{B}$ since $f_n \to f$ and the $f_n$ are all supported in $\mathbb{B}$. Since $\|\hat{f}\|_q = \lim_{n \to \infty} \|\hat{f}_n\|_q = B_{q,d}a^{1/p}$, we know that $a \leq |S|$.

Let $F := |\hat{f}|^{q-2}\hat{f}$. Since $||\hat{f}|^{q-2}\hat{f}|^q = |\hat{f}|^q$, $F \in L^p$, so $\hat{F} \in L^q$ by the Hausdorff-Young inequality. We claim that the following supremum is equal to $B_{q,d}a^{q/p}$.
\[
\sup_{|g| \leq 1, |\text{supp}g| \leq a} |\langle g, \hat{F} \rangle| = B_{q,d}a^{q/p}
\]
where $|g| \leq 1$ and $|\text{supp}g| \leq a$ mean that $g \in L^p$, and $\langle \cdot, \cdot \rangle$ denotes the normal $L^2$ inner product.

We obtain an upper bound for $|\langle g, \hat{F} \rangle|$ by first using Plancherel’s theorem and then applying Hölder’s inequality:
\[
(8.1) \quad |\langle g, \hat{F} \rangle| = |\langle \hat{g}, F \rangle| \\
\quad \leq \|\hat{g}\|_q \|F\|_p = \|\hat{g}\|_q |\hat{f}|^{q/p} \\
\quad \leq B_{q,d} |\text{supp}g|^{1/p} B_{q,d}a^{q/p} \leq B_{q,d}a^{q/p}.
\]
The supremum achieves this upper bound since $|f_\nu| \leq 1$, $|E_\nu| \leq a$, and
\[
|\langle f_\nu, \hat{F} \rangle| = |\langle \hat{f}_\nu, F \rangle| \to |\langle \hat{f}, |\hat{f}|^{q-2}\hat{f} \rangle| = \|\hat{f}\|_q = B_{q,d}a^{q/p} \quad \text{as} \quad \nu \to \infty.
\]
Note that $|\langle f_\nu, \hat{F} \rangle| \leq |\langle f_\nu, \hat{F} \rangle| \leq \langle 1_{E_\nu}, \hat{F} \rangle \leq B_{q,d}a^{q/p}$, so by the squeeze theorem,
\[
(8.2) \quad \lim_{\nu \to \infty} \langle 1_{E_\nu}, \hat{F} \rangle = B_{q,d}a^{1/p}.
\]
Letting $e^{i\varphi} = \hat{F}/|\hat{F}|$, this is equivalent to
\[
(8.3) \quad \lim_{\nu \to \infty} \langle e^{i\varphi}1_{E_\nu}, \hat{F} \rangle = \lim_{\nu \to \infty} |\langle e^{i\varphi}1_{E_\nu}, \hat{F} \rangle| = B_{q,d}a^{1/p}.
\]
By the sequence of inequalities beginning in (8.1), this means that $\|e^{i\varphi}1_{E_\nu}\|_q \to B_{q,d}a^{1/p}$. Similarly, we note that
\[
(8.4) \quad \lim_{\nu,\mu \to \infty} \left| \frac{1}{2} e^{i\varphi}(1_{E_\nu} + 1_{E_\mu}), \hat{F} \right| = B_{q,d}a^{1/p}.
\]
and that $\| \frac{1}{2} e^{i\varphi}(1_{E_\nu} + 1_{E_\mu}) \|_q \to B_{q,da}^{1/p}$.

Recall that

$$\left\| \frac{1}{2} e^{i\varphi}(1_{E_\nu} + 1_{E_\mu}) \right\|_q \leq B_{q,da} \left\| \frac{1}{2} e^{i\varphi}(1_{E_\nu} + 1_{E_\mu}) \right\|_L$$

from Proposition 2.1. Combining this with (8.2), we conclude that

$$(8.5) \quad \liminf_{\mu,\nu \to \infty} \| 1/2(1_{E_\nu} + 1_{E_\mu}) e^{i\varphi} \|_L = \liminf_{\mu,\nu \to \infty} 1/2 \| 1_{E_\nu} + 1_{E_\mu} \|_L \geq a^{1/p}.$$ 

In addition, recall that

$$\| 1_{E_\nu} + 1_{E_\mu} \|_L = \inf \{ \| (a_n) \|_{\ell_1} : \| 1_{E_\nu} + 1_{E_\mu} \| = \sum_{n} a_n \| a_n \|^{1/p} 1_{A_n}, \quad |A_n| < \infty \}.$$ 

Using the representation

$$(1/2)(1_{E_\nu} + 1_{E_\mu}) = (1/2)|E_\nu \cap E_\mu|^{1/p}|E_\nu \cap E_\mu|^{-1/p} 1_{E_\nu \cap E_\mu} + (1/2)|E_\nu \cup E_\mu|^{1/p}|E_\nu \cup E_\mu|^{-1/p} 1_{E_\nu \cup E_\mu},$$

we have the inequality

$$(8.6) \quad a^{1/p} \leq \liminf_{\nu,\mu \to \infty} [(1/2)(1_{E_\nu} + 1_{E_\mu})]_L \leq \liminf_{\nu,\mu \to \infty} [(1/2)|E_\nu \cap E_\mu|^{1/p} + (1/2)|E_\nu \cup E_\mu|^{1/p}].$$

Note that

$$\limsup_{\nu,\mu \to \infty} [(1/2)|E_\nu \cap E_\mu|^{1/p} + (1/2)|E_\nu \cup E_\mu|^{1/p}]$$

$$= \limsup_{\nu,\mu \to \infty} [(1/2)|E_\nu \cap E_\mu|^{1/p} + (1/2)(|E_\nu| + |E_\mu| - |E_\nu \cap E_\mu|)^{1/p}]$$

$$= (1/2)a^{1/p} \limsup_{\nu,\mu \to \infty} [|E_\nu \cap E_\mu|/a]^{1/p} + (2 - (|E_\nu \cap E_\mu|/a))^{1/p}$$

$$\leq (2) a^{1/p} \limsup_{\nu,\mu \to \infty} r(|E_\nu \cap E_\mu|/a)$$

where $r(t) = t^{1/p} + (2 - t)^{1/p}$. Calculate $r'(t) = (1/p)t^{-1/q} - (1/p)(2 - t)^{-1/q}$, so the only critical point is at $t = 1$. Since $r''(1) = -\frac{1}{p(1/q - 1)} - \frac{1}{pq(2 - 1/q - 1)} < 0$, and $r(0) = 2^{1/p} < r(1) = 2$, $r(t) \leq 1$ on $[0, 1]$, with equality only when $t = 1$.

Since $|E_\nu \cap E_\mu|/a \leq 1$, we can use this in (8.7) to get

$$(1/2)a^{1/p} \limsup_{\nu,\mu \to \infty} r(|E_\nu \cap E_\mu|/a) \leq (1/2)a^{1/p}2 = a^{1/p}.$$ 

Combined with (8.6), we conclude $a^{1/p} = \lim_{\nu,\mu \to \infty} (1/2)a^{1/p}r(|E_\nu \cap E_\mu|/a)$, and thus $\lim_{\nu,\mu \to \infty} |E_\nu \cap E_\mu| = a$.

Let $A$ denote the limit set satisfying $\lim_{\nu \to \infty} |E_\nu \Delta A| = 0$. (Clearly the $L^1$ limit of indicator functions is also an indicator function.) Note that $|A| = \lim_{\nu \to \infty} |E_\nu| = a$.

Let $B = \text{supp } \tilde{f}$ and recall that $\text{supp } f = S$. We will show that

$$(8.8) \quad S \subset A, \quad S \subset B.$$ 

To show that $S \subset A$, let $e^{-i\psi} = f/|f|$ and consider

$$\langle f_\nu, 1_{S \setminus A} e^{i\varphi} \rangle \to \langle f, 1_{S \setminus A} e^{i\varphi} \rangle = \langle |f|, 1_{S \setminus A} \rangle \quad \text{as } \nu \to \infty.$$
Since \( \lim_{\nu \to \infty} \langle f_{\nu}, 1_{S\setminus A}e^{i\varphi} \rangle = 0 \) (since \( \lim_{\nu \to \infty} |\text{supp } f_{\nu}| A | = 0 \), we must have \( \langle |f|, 1_{S\setminus A} \rangle = 0 \), so \( f \equiv 0 \) on \( S \setminus A \) and therefore \( S \subset A \). Combining this with

\[
B_{q,d}a^{1/p} = \|\hat{f}\|_q \leq B_{q,d}|S|^{1/p},
\]

we conclude \( |S| = a \).

Now we show that \( S \subset B \). We know from [8.3] that

\[
\lim_{\nu \to \infty} \|e^{i\varphi} 1_{E_{\nu}} \hat{F} \| = \|f, \hat{F}\| = B_{q,d}a^{q/p}.
\]

We know that the following sequence of inequalities involving Hölder’s inequality is actually a sequence of equalities

\[
B_{q,d}q^{q/p} = \|f 1_B, \hat{F}\| = \|\hat{1}_B, F\| \\
\leq \|\hat{1}_B\|_p \|F\|_p \leq B_{q,d} |S \cap B|^{1/p} \|\hat{F}\|_q^{q/p} \\
\leq B_{q,d}a^{1/p} B_{q,d}a^{q/p} = B_{q,d}a^{q/p}.
\]

The equality in Hölder’s inequality above implies there is a constant \( c \) such that \( c |\hat{1}_B|^q = |F|^p \). Unpacking the definition of \( F \), we have \( c |\hat{1}_B|^q = |\hat{F}|^q \).

Since \( \|\hat{1}_B\|_q = B_{q,d}a^{1/p} = \|F\|_p^{q/p} = (\int |F|^p)^{1/q} = (\int c |\hat{1}_B|^q)^{1/q} = c^{1/q}\|\hat{1}_B\|_q \), it must be that \( c = 1 \) and so \( |\hat{1}_B| = |\hat{F}| \). But then by Plancherel’s theorem,

\[
\|f 1_B\|_2 = \|\hat{1}_B\|_2 = \|\hat{F}\|_2 = \|f\|_2,
\]

which can only hold if \( S \subset B \).

Since \( |S| = a \), we have

\[
\langle |f|, |\hat{F}| \rangle = \langle f e^{i\varphi} e^{i\varphi}, \hat{F} \rangle \leq \sup_{|g| \leq 1, \text{supp } g \subset A} |\langle g, \hat{F} \rangle| = \langle f, \hat{F} \rangle,
\]

so \( \langle f, \hat{F} \rangle = \langle |f|, |\hat{F}| \rangle \). In addition,

\[
\langle 1_{S} e^{i\varphi}, \hat{F} \rangle = \langle 1_{S}, |\hat{F}| \rangle \leq \langle |f|, |\hat{F}| \rangle,
\]

so \( 0 \leq \langle |f| - 1_S, |\hat{F}| \rangle \). Since \( |\hat{F}| > 0 \) and \( |f| - 1_S \leq 0 \) on \( S \), conclude that \( |f| = 1_S \).

Consider the following equalities:

\[
a = |S| = \langle f, e^{-i\varphi} \rangle = \lim_{\nu \to \infty} \langle f_{\nu}, e^{-i\varphi} \rangle \leq \|f_{\nu}\|_2 1_{E_{\nu}} \|_2 \leq \lim_{\nu \to \infty} |E_{\nu}| = a.
\]

Finally, since \( \lim_{\nu \to \infty} \|f_{\nu}\|_2 = a^{1/2} \) and \( \|f\|_2 = |S|^{1/2} = a^{1/2} \), we have the convergence

\[
\lim_{\nu \to \infty} \|f_{\nu} - f\|_2^2 = \lim_{\nu \to \infty} (\|f_{\nu}\|_2^2 - 2Re \langle f_{\nu}, f \rangle + \|f\|_2^2) = a - 2a + a = 0.
\]

\[ \square \]

**Corollary 8.2.** Let \( d \geq 1 \) and \( q \in (2, \infty) \), \( p = q' \). There exists a measurable function \( f \) and measurable subset \( E \) of \( \mathbb{R}^d \) with \( |f| \leq 1_E \) such that

\[
B_{q,d} = \frac{\|\hat{f}\|_q}{|E|^{1/p}} = \frac{\|\hat{f}\|_q}{\|f\|_E}.
\]
Proof. (of Corollary 8.2) By the proof of Theorem 1.1, we know we can find a sequence of Lebesgue measurable subsets \((E_{ij})\) of \(\mathbb{R}^d\) that majorize functions \(f_{ij}\) such that there exist \(f, g, 1_A \in L^p(\mathbb{R}^d)\) with \(||f|| = 1_A\) and \(\lim_{\nu \to \infty} \|E_{ij}^{-1/p}\| f_{ij}^\nu = B_{q,d}, \lim_{\nu \to \infty} \|f_{ij} - f\|_p = 0,\) and \(\lim_{\nu \to \infty} |E_{ij} \Delta A| = 0.\) By the proof of Theorem 1.1 we must have \(||f||_p = |A|^{1/p}\) and so

\[
\frac{\|\hat{f}\|_q}{|A|^{1/p}} = \frac{\|\hat{f}\|_q}{\|f\|_p} = B_{q,d}.
\]

\[\square\]

Proof. (of Corollary 2.3) By the proof of Theorem 1.1, we know we can find a sequence of Lebesgue measurable subsets \((E_{ij})\) of \(\mathbb{R}^d\) that majorize functions \(g_{ij}\) such that there exist \(g, 1_A \in L^p(\mathbb{R}^d)\) with \(||g|| = 1_A\) and \(\lim_{\nu \to \infty} \|E_{ij}^{-1/p}\| g_{ij}^\nu = B_{q,d}, \lim_{\nu \to \infty} \|g_{ij} - g\|_p = 0,\) and \(\lim_{\nu \to \infty} |E_{ij} \Delta A| = 0.\) Thus

\[
\frac{\|\hat{g}\|_q}{|A|^{1/p}} = \frac{\|\hat{g}\|_q}{\|g\|_p} = B_{q,d}.
\]

We know from Lemma 2.2 that \(||f||_L \leq ||f||_{p1}\). Since \(||f||_{p1}^{-1} \|\hat{f}\|_q = B_{q,d} \leq ||f||_L^{-1} \|\hat{f}\|_q\), we also have that \(||f||_L^{-1} \|\hat{f}\|_q = B_{q,d}\). Invoke Lemma 2.2 to obtain the desired structural information about \(f\).

\[\square\]

9. Appendix

9.1. The Lorentz space \(L(p, 1)\). We relate the three quasinorms on \(L(p, 1)\) defined in §Lorentz discussion. In the following lemma, we prove a formula for \(\|s\|_L\) where \(s\) is a nonnegative simple function.

**Lemma 9.1.** Let \(d \geq 1\). Let \(s = \sum_{n=1}^N a_n 1_{A_n}\) where the \(A_n\) are pairwise disjoint and of finite Lebesgue measure and \(0 < a_1 < \cdots < a_N\). Let \(a_0 = 0\) and let \(B_n = \cup_{k=n}^N A_k\) for \(n = 1, \ldots, N\). Then

\[
\|s\|_L = \sum_{n=1}^N (a_n - a_{n-1}) |B_n|^{1/p}.
\]

**Proof.** First we prove for any \(k \geq 1\) that when \(c_0 = 0 < c_1 < c_2 < \cdots < c_k\) and \(C_j = \cup_{i=j}^k E_i\) for measurable sets \(E_i \subset \mathbb{R}^d\) of finite measure,

\[
\sum_{j=1}^k (c_j - c_{j-1}) |C_j|^{1/p} \leq \sum_{j=1}^k c_j |E_j|^{1/p}.
\]

If \(k = 1\), then clearly \(\sum_{j=1}^1 (c_j^n - c_{j-1}^n) |C_j|^{1/p} = c_1 |C_1|^{1/p} = \sum_{j=1}^1 c_j |E_j|^{1/p}\). Suppose for \(k \geq 1\) that when \(c_1 < c_2 < \cdots < c_k\) and \(C_j = \cup_{i=j}^k E_i\) for measurable sets \(E_i \subset \mathbb{R}^d\) of finite measure,

\[
\sum_{j=1}^k (c_j - c_{j-1}) |C_j|^{1/p} \leq \sum_{j=1}^k c_j |E_j|^{1/p}.
\]
Then
\[
\sum_{j=1}^{k+1} (c_j - c_{j-1})|C_j^m|^{1/p} = c_1|C_1|^{1/p} + (c_2 - c_1)|C_2|^{1/p} + \cdots + (c_{k+1} - c_k)|C_{k+1}|^{1/p}
\]

so (9.1) is proved.

Next we prove the lemma inductively, where notation is as in the statement of the lemma. If \( N = 1 \), suppose \( a_11_{A_1} = \sum_{j=1}^{\infty} b_j1_{S_j} \) where \( b_j \geq 0, |S_j| < \infty \). Then
\[
a_1 |A_1|^{1/p} = \|a_11_{A_1}\|_p = \| \sum_{j=1}^{N} b_j1_{S_j} + \sum_{j=N+1}^{\infty} b_j1_{S_j}\|_p
\]
\[
\leq \sum_{j=1}^{\infty} b_j|S_j|^{1/p} + \lim_{N \to \infty} \| \sum_{j=N+1}^{\infty} b_j1_{S_j}\|_p
\]
where \( \lim_{N \to \infty} \| \sum_{j=N+1}^{\infty} b_j1_{S_j}\|_p = 0 \) by Lebesgue’s dominated convergence theorem.

Now suppose that the lemma holds for \( N - 1 \geq 1 \). Consider
\[
\sum_{n=1}^{N} a_n1_{A_n} = \sum_{j=1}^{\infty} b_j1_{S_j}
\]
where \( b_{j-1} \geq b_j \geq 0, S_j \subset \bigcup_{n=1}^{N} A_n \), the \( S_j \) are distinct, and \( |S_j| > 0 \). From (9.1), we have for each \( M > 0 \) that
\[
\sum_{j=1}^{M} (b_j - b_{j+1})|\bigcup_{k=1}^{j} S_k|^{1/p} \leq \sum_{j=1}^{M} b_j|S_j|^{1/p}.
\]

Letting \( M \to \infty \) and noting that \( \sum_{j=1}^{\infty} b_j1_{S_j} = \sum_{j=1}^{\infty} (b_j - b_{j+1})1_{\bigcup_{k=1}^{j} S_k} \), we can assume that \( S_1 \subset S_2 \subset \cdots \) and \( b_j \geq 0 \) but are not necessarily decreasing. Since the simple function \( s \) achieves its \( L^\infty \) norm on \( A_N \), and the series takes its maximum on \( S_1 \), we must have \( S_1 = A_1 \) and
\[
a_N = \sum_{j=1}^{\infty} b_j,
\]
so \( \sum_{k=j}^{\infty} b_k \to 0 \) as \( j \to \infty \). The simple function \( s \) achieves its minimum (on a set of positive measure) in \( \bigcup_{n=1}^{N} A_n \) on \( A_1 \), but the series takes the values of \( \sum_{k=j}^{\infty} b_k \) on positive measure sets in \( \bigcup_{n=1}^{N} A_n \), so there is no minimum unless \( \sum_{k=j}^{\infty} b_k \) is zero for large enough \( j \). Thus we may write
\[
\sum_{n=1}^{N} (a_n - a_{n-1})1_{B_n} = \sum_{j=1}^{M} b_j1_{S_j}
\]
where \( S_j \subset S_{j+1} \) and \( b_j > 0 \). We note \( B_1 = S_M \) and \( a_1 = b_M \). Then invoking the inductive hypothesis, we have
\[
\sum_{n=2}^{N} (a_n - a_{n-1})|B_n|^{1/p} + a_1|B_1|^{1/p} \leq \sum_{j=1}^{M-1} b_j|S_j|^{1/p} + b_M|S_M|^{1/p},
\]
as desired.

For all \( f \in L(p, 1) \),
\[
(9.2) \quad \|f\|_{p1}^* \leq \|f\|_{p1} \leq \frac{p}{p-1}\|f\|_{p1}^*,
\]
which is proved in Chapter V, §3 in [19]. From the nonincreasing property of \( f^* \), it is clear that \( \frac{1}{t} \int_0^t f^*(u)du \geq f^*(t) \) for \( t > 0 \), which implies that \( \|f\|_{p1} \geq \|f\|_{p1}^* \). This combined with (9.2) implies that \( \|f\|_{p1}^* \) is finite if and only if \( \|f\|_{p1} \) is finite.

**Lemma 9.2.** Let \( d \geq 1 \). Let \( p > 1 \) and let \( q \) be the conjugate exponent to \( p \). For all measurable functions \( f : \mathbb{R}^d \to \mathbb{C} \) with \( \|f\|_{L} < \infty \) and \( \|f\|_{p1}^* < \infty \), \( \|f\|_{L} = \|f\|_{p1}^* \).

**Proof.** First we show the equivalence for nonnegative simple functions. Write \( s = \sum_{n=1}^{N} a_n1_{A_n} \) where the \( A_n \) are pairwise disjoint and \( 0 < a_1 < \cdots < a_N \). Let \( a_0 = 0 \) and let \( B_n = \bigcup_{k=n}^{N} A_k \) for \( n = 1, \ldots, N \), and let \( |B_{N+1}| = 0 \).

Calculate
\[
\|s\|_{p1} = \frac{1}{p} \int_0^\infty t^{-1/q}s^*(t)dt = \frac{1}{p} \int_0^\infty \sum_{n=0}^{N-1} t^{-1/q} \inf \{|x : |s(x)| > r| \leq t\}dr
\]
\[
= \frac{1}{p} \sum_{n=0}^{N-1} \int_{B_{n+1}} t^{-1/q} \inf \{|x : |s(x)| > r| \leq |B_{n+1}|\}dt
\]
\[
= \frac{1}{p} \sum_{n=0}^{N-1} a_{N-n} \int_{B_{n+1}} t^{-1/q}dt
\]
\[
= \sum_{n=0}^{N-1} a_{N-n} \left( \frac{1}{|B_{n+1}|^{1/p}} - \frac{1}{|B_{n+1}|^{1/p}} \right)
\]
\[
= \sum_{n=0}^{N-1} a_{N-n} |B_{n+1}|^{1/p} - \sum_{n=0}^{N-1} a_{N-n} |B_{n+1}|^{1/p}
\]
\[
= \sum_{n=1}^{N} a_n |B_n|^{1/p} - \sum_{n=1}^{N} a_{n-1} |B_{n-1}|^{1/p}
\]
\[
= \sum_{n=1}^{N} a_n |B_n|^{1/p} - \sum_{n=1}^{N} a_{n-1} |B_{n-1}|^{1/p} = \sum_{n=1}^{N} (a_n - a_{n-1}) |B_n|^{1/p}.
\]

Thus by Lemma 9.1 we have \( \|s\|_{L} = \|s\|_{p1}^* \) for all nonnegative simple functions.

Next, consider a \( f \in L(p, 1) \) with finite support \( A \) and \( L^\infty \) norm \( M > 0 \). From the definition of \( \| \cdot \|_{L} \) and Lemma 9.1 we can choose nonnegative simple functions \( |f| - 1/n \leq s_n \leq |f| \) such that \( \lim_{n \to \infty} s_n(x) = |f(x)| \) for a.e. \( x \in \mathbb{R}^d \) and
\[
\|f\|_{L} = \lim_{n \to \infty} \|s_n\|_{L} = \lim_{n \to \infty} \|s_n\|_{p1}^*.
\]
Note that
\[
\frac{1}{p} \int_0^\infty t^{-1/q} s_n^*(t) dt = \frac{1}{p} \int_0^{|A|} t^{-1/q} s_n^*(t) dt.
\]
Since \((|f| - 1/nA)^* \leq s_n^* \leq |f|\), we have the upper bound
\[
\frac{1}{p} \int_0^{|A|} t^{-1/q} s_n^*(t) dt \leq \frac{1}{p} \int_0^{|A|} t^{-1/q} f(t) dt
\]
and the lower bound
\[
\frac{1}{p} \int_0^{|A|} t^{-1/q} \inf \{r : |\{x : s_n(x) > r\}| \leq t\} dt \geq \frac{1}{p} \int_0^{|A|} t^{-1/q} \inf \{r : |\{x : |f(x)| > r + 1/n\}| \leq t\} dt
\]
\[
= \frac{1}{p} \int_0^{|A|} t^{-1/q} \inf \{r : |\{x : |f(x)| > r\}| \leq t\} dt - p|A|^{1/p} \frac{1}{n}.
\]
Thus by the squeeze theorem, we have that \(\lim_{n \to \infty} \|s_n\|_p = \|f\|_p\).

For general \(f \in L(p, 1)\), define \(f_n = f 1_{\{|n| \leq |f| \leq n\}}\). We argue that \(\|f\|_\mathcal{L} = \lim_{n \to \infty} \|f_n\|_\mathcal{L}\).

If \(|f_n| = \sum_m a_n 1_{A_n}, |f|1_{\{|f| > n\}} = \sum_m b_m 1_{B_m}, \) and \(|f|1_{\{|f| < n\}} = \sum_k c_k 1_{C_k}\), then \(|f| = \sum_m a_n 1_{A_n} + \sum_m b_m 1_{B_m} + \sum_k c_k 1_{C_k}\) and so
\[
\|f\|_\mathcal{L} \leq \|f_n\|_\mathcal{L} + \|f|1_{\{|f| > n\}}\|_\mathcal{L} + \|f|1_{\{|f| < n\}}\|_\mathcal{L}.
\]
Since \(|f| = \sum_m e_m 1_{E_m}, e_m > 0\) with \(\sum_m e_m \|E_m\|^{1/p} < \infty\) then \(|f_n| = \sum_m e_m 1_{E_m} \cap \{|n| \leq |f| \leq n\}\) with
\[
\sum_m e_m \|E_m \cap \{|n| \leq |f| \leq n\}\|^{1/p} \leq \sum_m e_m \|E_m\|^{1/p} < \infty,
\]
we also have \(\|f_n\|_\mathcal{L} \leq \|f\|_\mathcal{L}\). To show that \(\|f\|_\mathcal{L} = \lim_{n \to \infty} \|f_n\|_\mathcal{L}\), it suffices to show that \(\lim_{n \to \infty} \|f|1_{\{|f| > n\}}\|_\mathcal{L} = \lim_{n \to \infty} \|f|1_{\{|f| > 1/n\}}\|_\mathcal{L} = 0\).

If \(|f| = \sum_m e_m 1_{E_m}, e_m > 0\) with \(\sum_m e_m \|E_m\|^{1/p} < \infty\), then
\[
\limsup_{n \to \infty} \|f|1_{\{|f| > n\}}\|_\mathcal{L} \leq \limsup_{n \to \infty} \sum_m e_m \|E_m \cap \{|f| > n\}\|^{1/p} = 0
\]
where we used the monotone convergence theorem in the last line. Similarly, we have that
\[
\limsup_{n \to \infty} \|f|1_{\{|f| < 1/n\}}\|_\mathcal{L} \leq \limsup_{n \to \infty} \sum_m e_m \|E_m \cap \{|f| < 1/n\}\|^{1/p}.
\]
Since \(\sum_m e_m \|E_m \cap \{|f| < 1/n\}\|^{1/p}\) is a decreasing sequence in \(n\),
\[
\limsup_{n \to \infty} \sum_m e_m \|E_m \cap \{|f| < 1/n\}\|^{1/p} = \inf \sum_m e_m \|E_m \cap \{|f| < 1/n\}\|^{1/p}.
\]
We also have for each \(M > 1\)
\[
\inf \sum_m e_m \|E_m \cap \{|f| < 1/n\}\|^{1/p} \leq \inf \sum_{m \leq M} e_m \|E_m \cap \{|f| < 1/n\}\|^{1/p} + \sum_{m > M} e_m \|E_m\|^{1/p}
\]
\[
= \sum_{m \leq M} e_m \|E_m \cap \{|f| = 0\}\|^{1/p} + \sum_{m > M} e_m \|E_m\|^{1/p}
\]
\[
= \sum_{m > M} e_m \|E_m\|^{1/p}.
\]
Letting \(M\) go to infinity, we have \(\lim_{n \to \infty} \|f|1_{\{|f| < 1/n\}}\|_\mathcal{L} = 0\). Conclude that \(\|f\|_\mathcal{L} = \lim_{n \to \infty} \|f_n\|_p = \|f\|_p\).
Finally, we need to show that $\lim_{n \to \infty} \|f_n\|_{p_1} = \|f\|_{p_1}$. Since $\|f_n\|_{p_1} \leq \|f\|_{p_1}$ for each $n$ and $\lim_{M \to \infty} \int_M^\infty t^{-1/q} f_n^*(t) dt \leq \lim_{M \to \infty} \int_M^\infty t^{-1/q} f^*(t) dt = 0$, it suffices to show that for each $M > 0$,

$$\lim_{n \to \infty} \int_0^M t^{-1/q} f_n^*(t) dt \geq \int_0^M t^{-1/q} f^*(t) dt.$$ 

We note that

$$\int_0^M t^{-1/q} f_n^*(t) dt = \int_0^M t^{-1/q} \inf \{ r : |\{ x : f_n(x) > r \}| \leq t \} dt$$

$$\geq \int_0^M t^{-1/q} \inf \{ r : |\{ x : f(x)1_{\{|f| \leq n \}} > r \}| \leq t \} dt - pM^{1/p} \frac{1}{n}$$

$$= \int_0^M t^{-1/q} f_n^*(t + \{|x : f(x) > n\}|) dt - pM^{1/p} \frac{1}{n}.$$ 

Since $\lim_{n \to \infty} \{|x : f(x) > n\}| = 0$ and $f^*$ is a.e. continuous, by the Lebesgue dominated convergence theorem,

$$\lim_{n \to \infty} \int_0^M t^{-1/q} f_n^*(t) dt \geq \int_0^M t^{-1/q} f^*(t) dt.$$ 

\[\square\]

**Corollary 9.3.** Let $d \geq 1$. Let $p > 1$ and let $q$ be the conjugate exponent to $p$. For all measurable functions $f : \mathbb{R}^d \to \mathbb{C}$, $\|f\|_L < \infty$ if and only if $\|f\|_{p_1} < \infty$.

**Proof.** Suppose that $\|f\|_L < \infty$. We showed in the proof of Lemma 9.2 that $\|f\|_L = \lim_{n \to \infty} \|f_n\|_L$ where $|f_n|$ are bounded with finite support and monotonically increasing a.e. to $|f|$. We also showed that for those $f_n$, $\lim_{n \to \infty} \|f_n\|_{p_1} = \|f\|_{p_1}$, so $\|f\|_{p_1} < \infty$.

Next, suppose $\|f\|_{p_1} < \infty$. By definition of $\|f\|_L$ (regardless of whether this quantity is finite or infinite), there exist simple functions $0 \leq s_n \leq |f|$ such that $\|f\|_L = \lim_{n \to \infty} \|s_n\|_L$. But we showed in the proof of Lemma 9.2 that $\|s_n\|_L = \|s_n\|_{p_1}$ for each $n$. Since $\|s_n\|_{p_1} < \|f\|_{p_1}$ for all $n$, we must have $\|f\|_L < \infty$.

\[\square\]

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