The role of nonmetricity in metric-affine theories of gravity

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Received 13 August 2013, revised 10 December 2013
Accepted for publication 6 January 2014
Published 22 January 2014

Abstract
The intriguing choice to treat alternative theories of gravity by means of the Palatini approach, namely elevating the affine connection to the role of independent variable, contains the seed of some interesting (usually under-explored) generalizations of General Relativity, the metric-affine theories of gravity. The peculiar aspect of these theories is to provide a natural way for matter fields to be coupled to the independent connection through the covariant derivative built from the connection itself. Adopting a procedure borrowed from the effective field theory prescriptions, we study the dynamics of metric-affine theories of increasing order, that in the complete version include invariants built from curvature, nonmetricity and torsion. We show that even including terms obtained from nonmetricity and torsion to the second order density Lagrangian, the connection lacks dynamics and acts as an auxiliary field that can be algebraically eliminated, resulting in some extra interactions between metric and matter fields.

Dedicated to the memory of Francesco Caracciolo

Keywords: metric-affine theories, torsion, nonmetricity, f(R) theories, hypermomentum
PACS numbers: 04.50.Kd, 04.20.Fy

1. Introduction

Despite the name ‘Palatini approach’, Einstein himself recognized the possibility to formally describe the theory of General Relativity assuming the independence of the affine connection from the metric and performing a separate variation with respect to these two fields [1]. The independent connection is assumed to describe the parallel transport and it is used to define the covariant derivative, while the metric tensor defines distances and lengths. In the
case of General Relativity, the field equation obtained varying with respect to the *a priori* independent connection reduces to an algebraic equation expressing the fact that the affine connection is rather the Levi-Civita connection of the metric itself. Consequently, the complete equivalence with the standard treatment of General Relativity through the metric formalism is restored. However, such equivalence relies on the crucial assumption that the matter action is independent from the affine connection, that is the covariant derivatives contained in the Lagrangian density of the matter fields are the ones built from the metric connection.

The simplest generalization that one can perform is to assume that the genuine covariant derivative is the one defined, in the most natural way, from the independent connection; then, the matter action turns out to be directly dependent from the affine connection. The resulting theory, that is the prototypical version of a metric-affine theory of gravity, is formally known as the Einstein–Cartan–Sciama–Kibble (ECSK) theory [2, 3]. In ECSK theory, it is still possible to show that the independent (and not necessarily symmetric) connection can be algebraically eliminated in favour of the metric and its derivatives, plus matter fields, losing any dynamical feature [4]. This means that the connection is not propagating in the spacetime, but it is indissolubly confined to make couplings inside matter configurations. The eventual signature of the theory can be recovered only at level of the modified matter content obtained in the right-hand side of the metric field equation (once the connection is completely eliminated therein), and in a new spin–spin contact interaction term modifying the dynamics of the matter fields.

Interestingly, nothing prevents us from adding further scalar invariants with the same dimension of the Ricci scalar in the Einstein–Cartan action. Einstein–Hilbert action is the only diffeo-invariant action leading to field equations of the second order and such that the connection is the metric one. However, the freedom obtained by introducing the independent connection as a new variable motivates a further generalization of ECSK theory. The action describing a metric-affine theory is assumed to be a suitable limit, at a certain order, of a some fundamental theory; one can then follow the standard approach of effective field theory, and consider all the operators having the dimension of the specific order of the approximation. In the present case of ECSK, these operators are all the possible second order (namely, all the operators of the same dimension of the scalar curvature) invariants that can be formulated starting from the structures (torsion and nonmetricity) induced by the non-Riemannian nature of the actual spacetime.

Generalized metric-affine theories of gravity have been recently studied from different perspectives. Apart from the huge amount of work done in Einstein–Cartan gravity (see for example [5, 6] for a bird’s-eye view on recent aspects), progress was also made for what concerns the higher order versions of these theories [7], also attracted by the tantalizing possibility of relaxing the theoretical puzzles associated with the recent discovery of accelerated expansion. It is the case, for example, of Poincaré gauge theories of gravity [8], namely models whose action accommodates terms quadratic in torsion and curvature. The issue of the metric-affine counterpart for $f(R)$ modified theories of gravity was addressed in [9] and, with a specific focus to the cosmological consequences, in [10].

Different mechanisms were also proposed to alleviate the dark matter problem. It was suggested that a matter–antimatter asymmetry generated by the Hehl–Datta cubic term [11] in the Dirac equation modified by the presence of torsion, could be at the origin of the presence of dark matter in the universe [12]. The decomposition of a metric-affine theory in scalars, vectors and tensors contribution, in a way to match the results of TeVeS phenomenology, was analysed in [13]. The authors of [14], instead, suggested to fit supernovae data replacing the dark matter content expected as from the $\Lambda$ CDM model with a mean spin density of baryonic matter.
The dynamical content of a metric-affine theory of gravity was already analysed in [15] including the set of second order corrections built with torsion tensor. In this paper we are going to extend the previous result to the most general second order metric-affine theory of gravity, including quadratic operators in nonmetricity and torsion [16]. Note that, as it will be showed on the basis of a dimensional analysis, this will not include the quadratic Ricci terms, whose presence would introduce new dynamical degrees of freedom. After a brief introduction of our notation, we study the conditions under which the independent connection is reduced to the role of an auxiliary field in this metric-affine setup. We then address the problem of the minimal requirements to make the connection dynamical. Higher order theories are taken into account, bringing particular attention to the very special case of $f(\mathcal{R})$ metric-affine theories. A final section is summarizing our results.

2. Notation and conventions

Due to some ambiguities that can be found in the literature, and due to the delicate role played by the symmetries of the tensorial objects hereafter considered, let us recall the basic definitions and conventions that we will use throughout the paper. The covariant derivative with respect to the independent connection $\Gamma^\gamma_{\alpha\beta}$ of a generic $(1,1)$-tensor is defined as the following

$$\nabla_\alpha A^\beta_\gamma = \partial_\alpha A^\beta_\gamma + \Gamma^\beta_{\alpha\delta} A^\delta_\gamma - \Gamma^\delta_{\alpha\gamma} A^\beta_\delta.$$  (1)

Note that the connection $\Gamma^\gamma_{\alpha\beta}$ is not supposed to carry any symmetry. In particular, it is not supposed to covariantly conserve the metric. Using a combination of covariant derivatives of the metric, it is quite easy to show that the connection can be decomposed as

$$\Gamma^\gamma_{\alpha\beta} \equiv \{^\gamma_{\alpha\beta}\} + \frac{1}{2} \left( -Q^\gamma_{\alpha\beta} + Q^\gamma_{\beta\alpha} - Q^\gamma_{\beta\alpha} \right) + S^\gamma_{\alpha\beta} - S^\gamma_{\beta\alpha} + S^\gamma_{\alpha\beta}.$$  (2)

where $\{^\gamma_{\alpha\beta}\}$ is the Levi-Civita connection of the metric (aka Christoffel symbols of the metric) and we have defined the nonmetricity tensor $Q_{\alpha\beta\gamma} \equiv \nabla_\alpha g^\beta_\gamma$ and the antisymmetric part of the connection, otherwise known as the Cartan torsion tensor, $S_{\alpha\beta\gamma} \equiv \Gamma^\gamma_{[\alpha\beta]}$. The curvature tensor associated with the connection $\Gamma^\gamma_{\alpha\beta}$ is defined by

$$R^\mu_{\nu\rho\sigma} = \partial_\nu \Gamma^\sigma_{\mu\rho} - \partial_\mu \Gamma^\sigma_{\nu\rho} + \Gamma^\lambda_{\mu\rho} \Gamma^\sigma_{\nu\lambda} - \Gamma^\lambda_{\nu\rho} \Gamma^\sigma_{\mu\lambda}.$$  (3)

Here, the asymmetry of the connection deprives the curvature tensor of the usual properties of symmetry encountered in the (metric) Riemann tensor, that is the curvature tensor built from the Christoffels symbols of the metric. The only symmetry kept in the previous definition of the curvature tensor is the antisymmetry with respect to the first two indices. This circumstance raises an ambiguity in the definition of the Ricci tensor of $R^\mu_{\nu\rho\sigma}$: the usual prescription of the contraction of the second and the fourth indices provides the expression

$$R_{\alpha\nu\rho\gamma} \equiv R_{\alpha\rho\nu\gamma} = \partial_\gamma \Gamma^\rho_{\alpha\nu} - \partial_\nu \Gamma^\rho_{\alpha\gamma} + \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\alpha\gamma} - \Gamma^\rho_{\gamma\lambda} \Gamma^\lambda_{\alpha\nu},$$  (4)

whose related Ricci scalar is $\mathcal{R} = g^{\alpha\rho} R_{\alpha\rho\gamma\nu}$. On the other hand, two other possible contractions can be performed; contracting the third and fourth indices we obtain $R_{\alpha\rho\gamma\nu} \equiv \tilde{R}_{\alpha\rho\gamma\nu}$, a tensor also known as the homothetic curvature, antisymmetric with respect to its two indices, that reduces to $R_{\alpha\rho\gamma\nu}$ in the case of a symmetric (but still metric-incompatible) connection. The last possibility involves the use of the metric tensor, $g^{\rho\gamma} g^{\lambda\nu} R_{\alpha\rho\gamma\nu} \equiv \tilde{R}_{\alpha\rho\gamma\nu}$. However, a further contraction with the metric of the two alternative definitions of the Ricci tensor gives $g^{\alpha\beta} \tilde{R}_{\alpha\beta} = 0$ (due to the antisymmetry of the homothetic curvature) and $\tilde{R}_{\alpha\rho\gamma\nu} = -\mathcal{R}$.

1 It is worth stressing that some of the conventions used here are different from the ones adopted in [15].
leaving the Ricci scalar uniquely determined. In our effective field theory approach, we will mainly consider corrections to Einstein–Cartan Lagrangian of the same dimension of the Ricci scalar \( R \), so that the issue of the ambiguity will not be relevant for our purposes.

3. The general action

With what has been said till now, we are ready to provide a Lagrangian formulation of a metric-affine theory of gravity. Let us underline one more time the peculiar characteristic of metric-affine theories, namely, the possibility of a direct coupling of matter with the connection \( \Gamma^\gamma_{\alpha\beta} \). The general action will be of the form

\[
\mathcal{S} = \int d^4x \sqrt{-g} \left[ \mathcal{L}_G(g_{\mu\nu}, \Gamma^\gamma_{\alpha\beta}) + \mathcal{L}_M(g_{\mu\nu}, \Gamma^\gamma_{\alpha\beta}, \psi) \right],
\]

where \( g \) is the determinant of the metric, \( \mathcal{L}_G(g_{\mu\nu}, \Gamma^\gamma_{\alpha\beta}) \) and \( \mathcal{L}_M(g_{\mu\nu}, \Gamma^\gamma_{\alpha\beta}, \psi) \) are respectively the gravitational and the matter Lagrangian density (where we have made explicit the dependence of the matter from the connection), and \( \psi \) is a convenient way to refer collectively to the matter fields included in the theory under scrutiny.

We want to study the dynamics of the most general lowest order theory associated with the gravitational Lagrangian \( \mathcal{L}_G(g_{\mu\nu}, \Gamma^\gamma_{\alpha\beta}) \). Our prescription for constructing such general action is based on power counting of the dimensions of the gravitational terms. In natural units, where \( c = \hbar = \kappa_B = 1 \), lengths and times have the same dimension; given the adimensionality of the components of the metric tensor, the connection will have the dimension of the inverse of a length and consequently the Ricci tensor will have dimension \([\text{length}]^{-2}\). Since in natural units the total action must be dimensionless, the Lagrangian density must have dimension \([\text{length}]^{-4}\).

We can think the Lagrangian density as the product of a geometrical scalar invariant times an opportune overall constant, in the form of a power of a length \( L_P \), to adjust the total dimension. For the Einstein–Cartan action, for example, the correct Lagrangian density is \( \mathcal{L}_{EC}^G = \frac{R}{(16\pi L_P^2)} \).

Discarding a cosmological constant term (that in principle can be included, but, since it is not playing any role in what follows, we are omitting here for the sake of simplicity), it is not possible to build a Lagrangian density whose geometrical factor has dimension \([\text{length}]^{-1}\). This is trivially seen for the Ricci tensor case that is already of dimension \([\text{length}]^{-2}\). Nonmetricity and torsion tensor are also of dimension \([\text{length}]^{-1}\), but it is not possible to form a scalar invariant from just one rank-three tensor saturated with the metric (rank two); for such reason they will appear only as quadratic terms, hence only as terms of order \([\text{length}]^{-2}\), that is at the same order of the Ricci scalar \( R \).

The most general gravitational action with a Lagrangian density of dimension \([\text{length}]^{-2}\) has the form [16]

\[
\mathcal{S}_G = \frac{1}{16\pi L_P^2} \int d^4x \sqrt{-g} \left( \mathcal{R} + \sum_i a_i Q^2_{(i)} + \sum_i b_i Q_{(i)} \ast S_{(i)} + \sum_i c_i S^2_{(i)} \right),
\]

where the last three terms are a symbolic representation of all the possible independent contractions that can be obtained from nonmetricity \( Q \) and torsion tensor \( S \) (the symbol \( \ast \) in (6) indicates the tensorial product between torsion and nonmetricity). Using the symmetries of these two (\( Q \) is symmetric with respect to its last two indices, while \( S \) is antisymmetric with respect to its first two indices), we find the following resulting combinations.
Pure nonmetricity terms. The way we can provide invariant quantities from the product of two nonmetricity tensors is through the saturation with three metric tensors,
\[ Q_{\mu\nu} \ast Q_{\rho\sigma} \ast g^{\lambda\nu} \ast g^{\mu\sigma} = Q_{\lambda\mu}^\nu Q_{\sigma\nu}^\mu, \tag{7} \]
where the possible different independent contractions (having in mind the symmetry with respect to \{\mu\nu\} and \{\alpha\beta\}) are the following
\[ Q_{\lambda\mu\nu} \ast Q_{\gamma\alpha\beta} \ast g^{\lambda\alpha} \ast g^{\gamma\mu} = Q_{\lambda\gamma}^\alpha Q_{\mu\beta}^\nu, \tag{8} \]
\[ Q_{\lambda\mu\nu} \ast Q_{\gamma\alpha\beta} \ast g^{\lambda\mu} \ast g^{\gamma\nu} = Q_{\lambda\gamma}^\alpha Q_{\nu\beta}^\mu, \tag{9} \]
\[ Q_{\lambda\mu\nu} \ast Q_{\gamma\alpha\beta} \ast g^{\lambda\nu} \ast g^{\gamma\mu} = Q_{\lambda\gamma}^\alpha Q_{\mu\alpha}^\nu, \tag{10} \]
\[ Q_{\lambda\mu\nu} \ast Q_{\gamma\alpha\beta} \ast g^{\lambda\gamma} \ast g^{\nu\mu} = Q_{\lambda\mu}^\gamma Q_{\gamma\nu}^\mu, \tag{11} \]
\[ Q_{\lambda\mu\nu} \ast Q_{\gamma\alpha\beta} \ast g^{\lambda\gamma} \ast g^{\nu\alpha} = Q_{\lambda\nu}^\alpha Q_{\gamma\beta}^\mu. \tag{12} \]

Mixed terms. The generic nonmetricity–torsion interaction term is written as
\[ Q_{\lambda\mu\nu} \ast S_{\rho}^\gamma \ast g^{\lambda\rho} \ast g^{\mu\nu} \ast g^{\nu\sigma} = Q_{\lambda\mu}^\nu Q_{\sigma\nu}^\mu. \tag{13} \]
The identity tensor \( \delta^{\gamma\beta} \) can be used either for contracting indices in the torsion tensor or to raise one of the indices of the nonmetricity. In the former case, exploiting the antisymmetry of torsion in \{\alpha\beta\}, we can only obtain the object \( S_{\lambda} \equiv \delta_{\alpha}^{\beta} S_{\alpha\beta} \) (or an equivalent result contracting \( \gamma \) and \( \alpha \)), from which we can construct the following combinations
\[ Q_{\lambda\mu\nu} \ast S_{\alpha} \ast g^{\lambda\alpha} \ast g^{\mu\nu}, \tag{14} \]
\[ Q_{\lambda\mu\nu} \ast S_{\alpha} \ast g^{\lambda\nu} \ast g^{\mu\alpha}, \tag{15} \]
\[ Q_{\lambda\rho} \ast S_{\alpha} \ast g^{\lambda\alpha}, \tag{16} \]
where (14) and (15) are two proportional terms, due to the symmetry of \( Q \) in \{\mu\nu\}. If \( \delta^{\gamma\beta} \) in (13) acts on the nonmetricity, we have a double choice, either \( \delta^{\nu\beta} Q_{\lambda\mu\nu} \) or \( \delta^{\nu\beta} Q_{\lambda\nu\mu} \). In the first case we have
\[ Q_{\gamma\nu\mu} S_{\alpha\beta} \ast g^{\mu\alpha} \ast g^{\nu\beta} = -Q_{\gamma\mu\nu} S_{\alpha\beta} \ast g^{\mu\beta} \ast g^{\nu\alpha} = -Q_{\gamma\mu\nu} S_{\alpha\beta} \ast g^{\mu\beta} \ast g^{\nu\alpha} = 0, \tag{17} \]
in the second situation we have
\[ Q_{\lambda\nu\mu} \ast S_{\alpha\beta} \ast g^{\mu\alpha} \ast g^{\nu\beta} = 0. \tag{18} \]
Summarizing, the three independent terms one can construct from mixed terms are (14), (16) and (18).

Pure torsion terms. In this last case, the scalar quantity is built saturating the product of two torsion tensors with two \( \delta \)'s and one metric tensor in a structure like the following
\[ S_{\mu\nu} \ast S_{\alpha\beta} \ast \delta^{\gamma\beta} \ast \delta^{\gamma\alpha} \ast g^{\gamma\gamma}, \tag{19} \]
whose possible combinations assume the form
\[ g^{\mu\nu} S_{\mu} S_{\nu}, \tag{20} \]
\[ g^{\mu\nu} S_{\mu} S_{\nu} \ast g^{\lambda\lambda} \ast g^{\lambda\lambda}, \tag{21} \]
\[ g^{\mu\nu} \ast g^{\beta\gamma} \ast S_{\mu\nu} \ast S_{\alpha\beta} \ast g^{\alpha\alpha} \ast g^{\beta\beta}. \tag{22} \]

2 In what follows, we use the symbol \( '\circ' \) instead of the explicit indices for metric tensors and identity tensor in order to denote all the possible permutations of indices needed to saturate into a scalar the product of two tensors.
Taking into account the independent terms identified above, we can rewrite the three last sums in (6)
\begin{align*}
\sum_i a_i Q^{2}(i) &= a_1 Q_{\mu}^{\lambda} Q_{\mu}^{\nu} + a_2 Q_{\mu}^{\mu} Q_{\nu}^{\nu} + a_3 Q_{\mu}^{\lambda \nu} Q_{\lambda}^{\mu} + a_4 Q_{\mu}^{\lambda \nu} Q_{\lambda}^{\mu} + a_5 Q_{\mu}^{\lambda \nu} Q_{\lambda}^{\mu}, \\
\sum_i b_i Q^{2}(i) &= b_1 Q_{\mu}^{\lambda} s_{\nu}^{\lambda \nu} + b_2 Q_{\mu}^{\lambda \nu} s_{\lambda}^{\mu} + b_3 Q_{\mu}^{\lambda \nu} s_{\lambda}^{\mu}, \\
\sum_i c_i S^{2}(i) &= c_1 s_{\mu}^{\mu \nu} s_{\mu}^{\nu \lambda} + c_2 s_{\lambda}^{\mu \nu} s_{\mu}^{\nu \lambda} + c_3 s_{\nu}^{\mu \nu} s_{\mu}^{\nu \lambda}.
\end{align*}
(23)

It is interesting to note that the full Lagrangian density is free of terms obtained by
the covariant derivative of nonmetricity and torsion. This is the reason easily understood.

The independent connection can be decomposed, as already shown in (2), in the Levi-Civita
connection of the metric plus a combination \( K \) of terms in \( Q \) and \( S \), called distortion tensor
\begin{equation}
K_{\gamma} = \frac{1}{2} (-Q_{\alpha}^{\gamma} + Q_{\mu}^{\nu} - Q_{\beta}^{\gamma} + S_{\mu}^{\gamma} + S_{\nu}^{\gamma} + S_{\alpha}^{\gamma} - S_{\beta}^{\gamma} + S_{\nu}^{\gamma} + S_{\alpha}^{\gamma} + S_{\beta}^{\gamma}).
\end{equation}
(24)

Using this last condition, one can think to write the covariant derivative with respect to the
independent \( \Gamma \) as a Riemannian part (namely a covariant derivative with respect to the metric)
plus another one encoding the non-Riemannian structures of the spacetime
\begin{align*}
\nabla_{\mu} Q_{\alpha\beta\gamma} &= \nabla_{\mu} Q_{\alpha\beta\gamma} + \text{contractions} \ K \ast \ Q, \\
\nabla_{\mu} S_{\alpha\beta\gamma} &= \nabla_{\mu} S_{\alpha\beta\gamma} + \text{contractions} \ K \ast \ S.
\end{align*}
(25)

the first term on the right-hand side of both previous equations is a total divergence resulting in
a surface term, the second terms are instead contractions of the distortion tensor with \( Q \) and \( S \),
i.e. (quadratic) combinations of torsion and nonmetricity, already included in the classification
considered above.

It should be noted that one could still in principle enrich the gravitational Lagrangian
with yet another term, proportional to the Ricci scalar built from the metric tensor. This is the
only second order term that can be built from the metric. This Ricci scalar can be expressed
in terms of the Ricci scalar of the connection plus a combination of nonmetricity and torsion
tensors [17]
\begin{equation}
\frac{1}{16 \pi L^{2}} \int d^{4}x \sqrt{-g} R = \frac{1}{16 \pi L^{2}} \int d^{4}x \sqrt{-g} \left[ R - (K_{\mu \rho \mu} K^{\rho \mu} + K_{\rho \mu \lambda} K^{\rho \mu \lambda}) \right].
\end{equation}
(26)

We have discarded total derivatives leading to surface terms and can be easily reabsorbed
in the terms already present in the Lagrangian density. It is worth noticing here that such
a procedure is valid only in the case of general second order Lagrangian. For higher order
modified gravity models, it is not true anymore, see for example the \( R + f (R) \) theory [18] that
leads to the introduction of an effective further degree of freedom.

In order to conclude this section, a comment is still due: the 11 quadratic terms in
torsion and nonmetricity in (23) should be in principle complemented with the parity-violating
terms obtained from the contraction of curvature tensor, torsion and nonmetricity (as well as
for eventual combinations of them) with the totally antisymmetric Levi-Civita tensor \( \epsilon_{\alpha \beta \gamma \delta} \).

Anyway, in this paper we are working with the simplifying (realistic) hypothesis of a parity
preserving Lagrangian (as already assumed by [19]); such a requirement has also another root:
in the same spirit of [9, 15, 20] the Lagrangian can be further required to fulfil a minimal
coupling inspired construction, i.e., the general action should include only invariants that can
be built using the metric tensor to contract indexes. This procedure is automatically selecting
the parity even terms in the action, as taken into account in (23).
4. The dynamical content of the independent connection

We are basically interested in the field equation for the connection, since it is the one that should give us the possibility to re-express it as a function of matter fields plus Christoffel symbols of the metric.

Let us derive the variation of nonmetricity and torsion. One has

$$\delta Q_{\alpha\beta\gamma} = 2 g_{\lambda\nu} \Gamma_{\alpha(\beta|\gamma)}^{\lambda} \delta g_{\mu\nu} + 2 g_{\lambda\mu} \Gamma_{\alpha(\beta|\gamma)}^{\lambda} \delta g_{\nu\gamma} - 2 g_{\lambda\nu} \delta_{\mu\gamma}^{\nu} \delta_{a} \Gamma_{a}^{\lambda}, \tag{27}$$

$$\delta S^{c}_{\alpha\beta} = \delta_{\alpha}^{[c]} \delta_{\beta}^{[c]} \delta \Gamma_{\mu\nu}^{c}. \tag{28}$$

Without loss of generality, the global structure of the Palatini field equation will be of the form

$$\frac{1}{\sqrt{-g}} \left[ \frac{\Gamma}{\sqrt{-g}} \left( \sqrt{-g} g^{\mu\nu} - \frac{\Gamma}{\sqrt{-g}} \left( \sqrt{-g} G^{\mu\nu} \right) \delta_{\alpha} \right) + \sum_{j} \alpha_{j} Q^{(j)} + \sum_{j} \beta_{j} S^{(j)} = \left( 8 \pi L_{P}^{2} \right) \Delta^{\mu\nu}_{\lambda}, \tag{29}$$

where

$$\sum_{j} \alpha_{j} Q^{(j)} = \alpha_{1} \delta^{\mu\nu}_{\lambda} Q_{a}^{\mu} + \alpha_{2} \delta^{\mu\nu}_{\lambda} Q_{a}^{\mu} + \alpha_{3} \delta^{\mu\nu}_{\lambda} Q_{a}^{\mu} + \alpha_{4} \delta^{\mu\nu}_{\lambda} Q_{a}^{\mu} + \alpha_{5} \delta^{\mu\nu}_{\lambda} Q_{a}^{\mu} + \alpha_{6} \delta^{\mu\nu}_{\lambda} Q_{a}^{\mu} + \alpha_{7} \delta^{\mu\nu}_{\lambda} Q_{a}^{\mu} + \alpha_{8} \delta^{\mu\nu}_{\lambda} Q_{a}^{\mu} + \alpha_{9} \delta^{\mu\nu}_{\lambda} Q_{a}^{\mu}, \tag{30}$$

$$\sum_{j} \beta_{j} S^{(j)} = \beta_{1} \delta^{\mu\nu}_{\lambda} S_{a}^{\mu} + \beta_{2} \delta^{\mu\nu}_{\lambda} S_{a}^{\mu} + \beta_{3} \delta^{\mu\nu}_{\lambda} S_{a}^{\mu} + \beta_{4} \delta^{\mu\nu}_{\lambda} S_{a}^{\mu} + \beta_{5} \delta^{\mu\nu}_{\lambda} S_{a}^{\mu} + \beta_{6} \delta^{\mu\nu}_{\lambda} S_{a}^{\mu}, \tag{31}$$

and $\alpha_{j}$ and $\beta_{j}$ are some linear combinations of the primitive coefficients $a_{i}, b_{i}$ and $c_{i}$ used in equation (6).

$$[\alpha_{j}] = \left\{ \begin{array}{c} 2 a_{5} + b_{2} - 1 - \frac{2}{2} ; \\
2 a_{2} ; \\
8 a_{4} - b_{3} - \frac{4 a_{4} - b_{3}}{2} ; \\
2 a_{3} - 1 - \frac{4 a_{2} + b_{1} + 2}{2} ; \\
2 a_{5} + 1 - \frac{8 a_{1} - b_{2}}{2} \\
2 a_{2} - c_{1} ; \\
2 a_{3} - c_{2} + 2 ; \\
2 a_{4} - b_{3} - c_{1} ; \\
2 a_{5} + b_{1} - c_{2} \end{array} \right\}, \tag{32}$$

$$[\beta_{j}] = \left\{ b_{1} + 2 ; \\
2 b_{2} - c_{1} ; \\
b_{1} + c_{2} - 2 ; \\
c_{2} + 2 ; \\
2 c_{3} - b_{3} ; \\
b_{3} - c_{2} \right\}. \tag{33}$$

The right-hand side of (29) is given by the so-called hypermomentum tensor $\Delta^{\mu\nu}_{\lambda} \equiv \frac{1}{\sqrt{-g}} \left[ \Gamma_{\mu
u}^{\lambda} \right]$, that is, the tensor describing the intrinsic properties of matter as spin angular momentum, shear and dilatation current [27]. Also note that

$$\frac{1}{\sqrt{-g}} \left[ \Gamma_{\mu
u}^{\lambda} \right] = \delta^{\mu\nu}_{\lambda} Q_{a}^{\mu} + \frac{1}{2} \delta^{\mu\nu}_{\lambda} Q_{a}^{\mu} - Q_{a}^{\mu} - \frac{1}{2} \delta^{\mu\nu}_{\lambda} Q_{a}^{\mu} + \frac{1}{2} \delta^{\mu\nu}_{\lambda} Q_{a}^{\mu}, \tag{34}$$

that is, we can reformulate the Palatini equation just in terms of an expression linear in the torsion tensor and in the nonmetricity tensor, that we can symbolically rewrite as

$$\sum_{j} \tilde{\alpha}_{j} Q^{(j)} + \sum_{j} \tilde{\beta}_{j} S^{(j)} = \left( 8 \pi L_{P}^{2} \right) \Delta^{\mu\nu}_{\lambda}, \tag{35}$$

with $\tilde{\alpha}_{j} = \alpha_{j} + 1, \tilde{\alpha}_{3} = \alpha_{3} - 1, \tilde{\alpha}_{8} = \alpha_{8} + \frac{1}{2}, \tilde{\alpha}_{9} = \alpha_{9} - \frac{1}{2}$ and $\tilde{\alpha}_{j} = 3, 8, 9 = \alpha_{j}$. We can contract this equation in three different independent ways: with the metric $g_{\mu\nu}$, with $\delta_{\mu\nu}^{\gamma}$ and with $\delta_{\mu\nu}^{\gamma}$. We obtain respectively

$$\left( 4 \tilde{\alpha}_{2} + \tilde{\alpha}_{3} + \tilde{\alpha}_{4} + \tilde{\alpha}_{6} + \tilde{\alpha}_{7} \right) Q_{a}^{\mu} + \left( \tilde{\alpha}_{1} + \tilde{\alpha}_{5} + 4 \tilde{\alpha}_{8} + \tilde{\alpha}_{9} \right) Q_{a}^{\mu}$$

$$+ \left( 4 \beta_{1} + \beta_{2} + \beta_{3} - \beta_{4} - \beta_{9} \right) S_{a}^{\mu} = \left( 8 \pi L_{P}^{2} \right) \Delta a_{\mu}^{\alpha}, \tag{36}$$
\[
\begin{align*}
(\tilde{a}_2 + \tilde{a}_3 + \tilde{a}_4 + 4\tilde{\gamma})Q^{\mu\nu}_a + (4\tilde{a}_1 + \tilde{a}_6 + \tilde{a}_8 + \tilde{a}_9)Q^{\nu\mu}_a \\
+ (\beta_1 + \beta_2 + 4\beta_3 + \beta_4 + \beta_5)S^{\alpha\mu}_a = (8\pi L_p^2)\Delta^{\alpha\nu}_a.
\end{align*}
\] (36)

\[
(\tilde{a}_2 + 4\tilde{a}_3 + \tilde{a}_5 + \tilde{a}_6 + \tilde{\gamma})Q^{\mu\rho}_a + (\tilde{a}_1 + \tilde{a}_4 + \tilde{a}_9 + 4\tilde{a}_9)Q^{\nu\mu}_a \\
+ (\tilde{\beta}_1 + 4\tilde{\beta}_2 + \tilde{\beta}_3 - \tilde{\beta}_5 + \tilde{\beta}_6)S^{\alpha\mu}_a = (8\pi L_p^2)\Delta^{\mu\alpha}_a.
\] (37)

This is a simple linear system whose solution can be written as follows:

\[
\begin{align*}
Q^{\mu\rho}_a &= (8\pi L_p^2)(\Lambda_1\Delta_a^{\mu\rho} + B_1\Delta^{\alpha\rho}_a + C_1\Delta^{\alpha\mu}_a), \\
Q^{\nu\mu}_a &= (8\pi L_p^2)(\Lambda_2\Delta_a^{\mu\nu} + B_2\Delta^{\alpha\nu}_a + C_2\Delta^{\alpha\mu}_a), \\
S^{\alpha\mu}_a &= (8\pi L_p^2)(\Lambda_3\Delta_a^{\alpha\mu} + B_3\Delta^{\alpha\nu}_a + C_3\Delta^{\alpha\mu}_a).
\end{align*}
\] (38)

where \(\Lambda_1, B_1, C_1\) are some elementary, but rather lengthy, expressions of the coefficients \(\tilde{a}_i\) and \(\tilde{\beta}_i\). We can now use these three equations to substitute the corresponding terms in (34); their contribution is fully determined by the matter content of the theory, so we can move them on the right-hand side, where they are collectively denoted as \([f(\text{traces of } \Delta)]^{\mu\nu}_a\). What remains is the equation

\[
\begin{align*}
\tilde{a}_2Q^{\mu\nu}_a + \tilde{a}_5Q^{\nu\mu}_a + \tilde{a}_6Q^{\mu\nu}_a + \tilde{a}_9Q^{\nu\mu}_a + 4\tilde{\gamma}Q^{\nu\mu}_a \\
+ (\beta_1 + \beta_2 + 4\beta_3 + \beta_4 + \beta_5 + \beta_6)S^{\nu\mu}_a = (8\pi L_p^2)\frac{\Delta^{\alpha\nu}_a}{\theta_1\mu\nu\lambda} + (8\pi L_p^2)\frac{\Delta^{\alpha\mu}_a}{\theta_1\nu\mu\lambda} + [f(\text{traces of } \Delta)]^{\mu\nu}_a.
\end{align*}
\] (39)

The antisymmetric part with respect to \([\mu\nu]\) pairs of indices of the previous equation gives an equation to express the torsion tensor in terms of the antisymmetric part of nonmetricity tensor plus terms in hypermomenta

\[
\beta_4S^{\nu\mu}_a + (\beta_6 - \beta_4)S^{\mu\nu}_a = (\tilde{a}_6 - \tilde{a}_4)Q^{\mu\nu}_a + \beta_4\frac{\Delta^{\alpha\nu}_a}{\theta_1\mu\nu\lambda} + \beta_6\frac{\Delta^{\alpha\mu}_a}{\theta_1\nu\mu\lambda} + [f(\text{traces of } \Delta)]^{\mu\nu}_a = \Theta^\nu\mu\lambda.
\] (40)

That can be solved considering a suitable combination of the three different permutations of the indices \([\mu\nu\lambda] = (\lambda\mu\nu) \text{ and } (\mu\nu\lambda) \rightarrow (\lambda\nu\mu)\). At the end, we get:

\[
S_{\mu\nu\lambda} = \frac{2\beta_3\Theta_{\mu\nu\lambda} - \beta_4(\Theta_{\mu\nu\lambda} - \Theta_{\nu\mu\lambda} + \Theta_{\lambda\mu\nu} + \Theta_{\lambda\nu\mu})}{(\beta_4 + 2\beta_5 - \beta_6)(\beta_4 - \beta_5 - \beta_6)}
\]

\[= \left[f(\Delta)\right]_{[\mu\nu\lambda]} - \frac{2(\tilde{a}_6 - \tilde{a}_4)}{\beta_4 + 2\beta_5 - \beta_6}Q_{[\mu\nu\lambda]}. \] (41)

Since we are not interested in the exact form of the contribution of matter to the torsion tensor, we have here defined another tensor \([\tilde{f}(\Delta)]_{\mu\nu\lambda}\) that includes all the contributions coming from the hypermomenta in \(\Theta_{\mu\nu\lambda}\). Note that the tensor \(\Theta_{\mu\nu\lambda}\), and hence the torsion tensor \(S_{\mu\nu\lambda}\), is linear in nonmetricity \(Q_{\mu\nu\lambda}\). Using this expression in equation (39) to eliminate the torsion, we can rewrite it in the form

\[
\xi_1Q_{\mu\nu\lambda} + \xi_2Q_{\nu\mu\lambda} + \xi_3Q_{\lambda\mu\nu} = \left[f(\Delta)\right]_{\mu\nu\lambda},
\] (42)

where \(\xi_1 \equiv \xi(\tilde{a}_j, \tilde{\beta}_k)\) are some coefficients determined by the equations (39) and (41) and \([\tilde{f}(\Delta)]_{\mu\nu\lambda}\) is defined, in analogy to \(f\) and \(\tilde{f}\), as the collective contribution from matter to the right-hand side of the expression; equation (42) can be now solved with respect to \(Q_{\mu\nu\lambda}\) adding and subtracting the further two equations obtained permuting the indices \((\mu\nu\lambda) \rightarrow (\lambda\mu\nu)\) and \((\mu\nu\lambda) \rightarrow (\nu\mu\lambda)\). Having expressed nonmetricity in terms of just matter fields, we can reuse it in the equation for torsion to have another expression using just the matter fields. At the end, the total connection, that can be written in terms of Christoffel symbols of the metric plus distortion (where we recall that the distortion tensor (24) is a combination of nonmetricity and torsion), is hence reduced to a (not trivial) expression of metric with its derivatives and of matter fields under the guise of the hypermomenta combination.
An important point to stress is the following. Given a matter Lagrangian that does contain at most linear terms in the covariant derivative, and hence linear terms in the connection, our demonstration shows the lack of dynamics, and the consequent reduction to an auxiliary field, of $\Gamma^\lambda_{\mu\nu}$. On the other hand, some specific and exotic forms of matter can evade this fulfilment. Anyway, equations of motion of matter fields are required to be at most of second order, which forces the matter Lagrangian to contain only linear derivatives. Therefore, even in the most convoluted case, the hypermomentum tensor will be algebraic in the connection and the latter can be still eliminated at the component level.

What is the physical consequence of the lack of dynamics of the connection? Once it has been shown that the connection can be algebraically written in terms of derivatives of the metric and matter fields, it is clear that we can substitute all the terms explicitly dependent on connection (or torsion, or nonmetricity) in the field equation obtained varying with respect to the metric. Due to the extreme length of the equation, we will omit to write it here completely. It is anyway clear that, varying the general action (6) with respect to the metric, we will basically obtain terms that are quadratic in the connection (torsion/nonmetricity), and hence that will carry extra contributions of the kind \((\text{hypermomentum})^2\) to the effective stress energy tensor. This is similar to what happens in Einstein–Cartan theory as shown in [4], with the main difference that now, because of the presence of nonmetricity terms, the field equations will contain new terms of different nature, coupling matter fields (in the form of hypermomenta) to Christoffel symbols.

5. A new role for matter fields

Which kind of matter is sensitive to a minimal coupling with the connection through the covariant derivative? Clearly, scalar fields, having no spin, result to be neither sources for torsion nor for non-metricity, as can be easily understood from the fact that the covariant derivative of a scalar field is equal to its partial derivative and henceforth does not depend on the Levi-Civita independent connection.

Gauge fields (like the electromagnetic field) also do not couple to the connection, since they can be covariantly defined in terms of exterior derivatives. As a consequence, photons are not affected by the presence of torsion and non-metricity, and the causality of the theory will be preserved and completely determined from the metric structure of the spacetime. In principle one can think to break the gauge symmetry by adding a term that is manifestly gauge non-invariant, as for the mass term in the Einstein–Proca–Maxwell action, whose consequence is the emergence of a coupling between the spin of the particles and the non-Riemannian structures.

From the physical point of view, it would be interesting to study some concrete examples of matter fields coupled to connection, namely those fields for which the hypermomentum does not vanish. A (phenomenological) candidate to source connection dynamics could be a semiclassical spinning dust matter distribution, \textit{alias} a generalization of the perfect fluid in the case of non-vanishing spin, a fluid otherwise dubbed Weyssenhoff fluid [25]; even though such kind of matter is an interesting toy model, it has an unsatisfactory theoretical formulation, since there is no unambiguous Lagrangian formulation able to describe it. Instead, one has to postulate some convective forms for the energy–momentum and spin-angular momentum tensors, plus some restrictions to the fluid spin in order to ensure the integrability conditions of the equations of motion of the particles.

A better motivated example of matter field sourcing the equation for the connection is the Dirac field or any massive vector field or tensor field (not necessarily in the domain of
the standard model, whose actions contain explicit dependence on the covariant derivative and hence lead to field equations with nonzero hypermomenta. In those cases, the fields are potentially able to induce non-metricity and/or torsion. As a pedagogical example let us consider the minimally coupled Dirac Lagrangian in the simplified context of Einstein–Cartan theory. Standard model Dirac fermions can be coupled only to the metric-compatible part of the linear connection, so the matter Lagrangian reads

\[ L_{\text{Dirac}}(\Gamma) = (\bar{\psi} \gamma^\mu \psi - \bar{\psi} \gamma^\mu (\nabla_\mu \psi) - m \bar{\psi} \psi = L_{\text{Dirac}}(\{} - \bar{\psi} \gamma^{[\mu \nu \rho] \psi} K_{\rho \nu \mu}, \quad (43) \]

where the last line expresses the splitting of the Dirac Lagrangian into a Riemannian piece plus a non-Riemannian correction due to the spin-torsion coupling of the spinors. In this case the hypermomentum tensor

\[ \Delta^{\mu \nu \rho} = \bar{\psi} \gamma^{[\mu \nu \rho]} \psi \]

is the new source appearing in the field equation for the independent connection. If the equation for the connection is algebraic, as for the ECSK theory, the spin–connection coupling encoded in the hypermomentum tensor will result (once the connection is eliminated from the metric field equation) in an effective correction to the stress energy tensor in the form of a spin-spin term.

6. Higher orders terms

It is an easy task to argue that scalar Lagrangian corrections of the order \([\text{length}]^{-2n}\), with integer \(n\), are the only ones that can be written starting from our elementary geometrical objects. This is essentially due to the fact that the only quantities carrying dimension \([\text{length}]^{-1}\) are odd-rank objects (torsion, nonmetricity and covariant derivative), and they cannot be trivially saturated with the (even-rank) metric. For such reason, the next-order invariants are all the possible terms with dimension \([\text{length}]^{-4}\). It goes beyond the scope of this paper to enumerate all the possible invariants of the fourth order (just for comparison, it should be taken into account that in a spacetime with torsion and vanishing nonmetricity there are 151 independent scalar invariants [21]). It is anyway possible to show that these terms are inevitably introducing further degrees of freedom, even assuming the simplifying hypothesis of matter fields not coupled to the connection. It is what for example occurs for the generalized Palatini theories considered in [22], whose specific choice of a Lagrangian density of the form \(R + R_{\mu \nu} R_{\kappa \lambda} (ag^{\mu \kappa} g^{\nu \lambda} + bg^{\mu \lambda} g^{\nu \kappa})\) has been shown to be equivalent to Einstein gravity plus a (dynamical) Proca vector field.

A rather peculiar case is the metric-affine version of \(f(R)\) theories of gravity [9, 26]. The Ricci scalar \(R\) is invariant under the projective transformation

\[ \Gamma^\rho_{\mu \nu} \rightarrow \Gamma^\rho_{\mu \nu} + \delta^\rho_{\mu} \xi_\nu, \quad (44) \]

where \(\xi_\nu\) is an arbitrary covariant vector field. Consequently also any function of \(\mathcal{R}\) will respect the same symmetry. While this issue is not a problem when matter does not couple to the connection, for a metric-affine theory this feature can lead to inconsistent field equations. In general the matter Lagrangian is not projective invariant, neither it is reasonable to restrict the matter content to those fields that are fulfilling this property. To circumvent the problem, it is necessary to break the projective invariance of the gravitational sector, fixing the four degrees of freedom of the transformation (related to the four components of the vector field \(\xi_\nu\)) by a Lagrange multiplier.

3 It is worth reminding that those fields that do not introduce either torsion or nonmetricity because not coupled to the connection, also will not be affected by torsion and nonmetricity even if other matter fields produce it.

4 Here the distortion tensor should be read as a combination of the sole torsion tensor, as can be found from (2) once that nonmetricity is set to zero.
Note that the projective invariance is automatically broken if distortion-squared terms and, a fortiori, higher order curvature invariants are added to the action; such theories have been shown already to carry further dynamics. The simplest example is the theory considered in [18], where the Lagrangian density is assumed to be \( \mathcal{L}_G = R + f_1(R) \) (here \( R \) is the Ricci scalar built from the (metric) Riemann tensor and \( f_1(R) \) is a function of the Ricci scalar of the curvature tensor). We can now use (26) to write \( R = R + (\text{distortion})^2 \), it is then clear that the total Lagrangian turns out to be \( \mathcal{L}_G = f_2(R) + (\text{distortion})^2 \), where \( f_2(R) = R + f_1(R) \). However, this theory has been shown to carry one more effective degree of freedom. Here, we want to remain in the realm of \( f(R) \) theories of gravity, so we will skip further discussions about that. Since the number of degrees of freedom to be fixed is 4, and since the projective transformation suggests that the goal of breaking projective invariance should be achieved by constraining the connection, it is reasonable to propose an additional term in the gravitational Lagrangian involving a contraction of either nonmetricity tensor or torsion tensor.

As already shown in [23], the term of the form \( A^\mu Q_\mu \equiv A^\mu g^{\alpha\beta} Q_{\alpha\beta} \), previously proposed in [20] is not suitable for a generic \( f(R) \) metric-affine theories, since it requires the function of the Ricci scalar to reduce to the Einstein–Cartan term and the theory results to have no solutions of the field equations whenever the \( f(R) \) is nonlinear. Interestingly, the result is still valid even if we try to fix the four degrees of freedom through the other independent contraction of the nonmetricity tensor, namely adding to the Lagrangian density the Lagrange multiplier the \( B^\mu \tilde{Q}_\mu \equiv B^\mu g^{\alpha\beta} Q_{\alpha\beta} \), as it can be easily proved: in a torsion-less theory without matter fields, the independent Levi-Civita connection can be written from (2) as

\[
\Gamma^\gamma_{\alpha\beta} = \left\{ \gamma_{\alpha\beta} \right\} + \frac{1}{2} \left( -Q_a \gamma_{\beta}^a + Q^\gamma_{\beta a} - Q_{\beta a} \gamma \right); \tag{45}
\]
on the other hand, the field equation of the connection reduces to the usual \( f(R) \)-Palatini equation, that can be solved with respect to the connection to give (see for example [24])

\[
\Gamma^\gamma_{\alpha\beta} = \left\{ \gamma_{\alpha\beta} \right\} + \frac{1}{2f(R)} (2\partial^\alpha_f(R)\delta^\gamma_{\beta} - g^{\gamma\sigma} g_{\alpha\beta} \partial_{\sigma} f(R)). \tag{46}
\]

Equating (45) and (46) gives a condition expressing the contribution to nonmetricity coming from the gravitational sector of the theory

\[
- Q_a \gamma^a + Q^\gamma_{\beta a} - Q_{\beta a} \gamma = \frac{1}{f'(R)} (2\partial^\alpha_f(R)\delta^\gamma_{\beta} - g^{\gamma\sigma} g_{\alpha\beta} \partial_{\sigma} f(R)); \tag{47}
\]

we can find two independent expressions by contracting respectively \( \alpha \) and \( \beta \) indices or \( \alpha \) and \( \gamma \) in the last equation; the resulting conditions being

\[
Q_\gamma - 2\tilde{Q}_\gamma = -\frac{2}{f'(R)} \partial_{\sigma} f(R), \tag{48}
\]

\[
- Q_\beta = \frac{4}{f'(R)} \partial_{\sigma} f(R). \tag{49}
\]

It is now clear that both the constraint \( Q_\gamma = 0 \) or \( \tilde{Q}_\mu = 0 \) lead to the same result \( Q_\mu = \tilde{Q}_\mu = \partial_{\sigma} f(R) = 0 \), that makes the theory obviously inconsistent since it forces the function \( f(R) \) to be at most linear.

A viable alternative [9, 15] is the theory obtained constraining the trace of the torsion tensor \( S_{\mu\rho} \) through the term \( \mathcal{C}^\mu S_{\mu\rho} \equiv \mathcal{C}^\mu S_{\mu\rho} \)

\[
\mathcal{L} = \frac{1}{16\pi G_5} \int d^4x \sqrt{-g} \left( f(R) + C^\mu S_{\mu\nu} \right) + \mathcal{L}_M \left( g_{\mu\nu}, \Gamma^\gamma_{\alpha\beta}, \psi \right). \tag{50}
\]

5 For the latter case see for example [7, 16].

6 The condition about the absence of matter fields implies also the vanishing of the Lagrange multipliers \( A_\mu \) and \( B_\mu \) that can be proven to be proportional to the trace of the hypermomentum tensor.
whose field equations are written as

\[ f'(\mathcal{R}) R_{\mu\nu} - \frac{1}{2} f(\mathcal{R}) g_{\mu\nu} = (8\pi L_p^2) T_{\mu\nu}, \]

\[ - \nabla_\lambda (\sqrt{-g} f'(\mathcal{R}) g^{\mu\nu}) + \nabla_\sigma (\sqrt{-g} f'(\mathcal{R}) g^{\mu\sigma}) \delta^\nu_\lambda + 2 \sqrt{-g} f'(\mathcal{R}) (g^{\mu\alpha} S_{\alpha\sigma} - g^{\mu\sigma} S_{\alpha\nu} + g^{\mu\nu} S_{\alpha\lambda}) \]

\[ = (8\pi L_p^2) \sqrt{-g} \left( \Delta^{\mu\nu}_\lambda - \frac{2}{3} \Delta^{\alpha[v}_{\lambda} \delta^{\mu\nu]}_\sigma \right), \]

(51)

\[ S_{\alpha\mu} = 0. \]

Note that this choice is fully consistent: if we require vacuum solutions, the second of equations (51) simply reduces to the two conditions \( S_{\mu\nu\rho} = 0 \) and \( \nabla_\lambda (\sqrt{-g} f'(\mathcal{R}) g^{\mu\nu}) = 0 \), that are the usual field equations found in Palatini-\( f(\mathcal{R}) \) theories of gravity. On the other side, this version of \( f(\mathcal{R}) \)-metric-affine theories has the further feature to avoid propagation of torsion waves in vacuum. In fact (modulo the dependence of the matter Lagrangian on the covariant derivative, that must be at most linear), taking the antisymmetric part of second equation in (51) with respect to \( \mu \) and \( \nu \), and adding suitable permutations of the obtained expression, we can show that torsion tensor is

\[ S_{\mu\nu} = 8\pi L_p^2 (\Delta_{[\mu\nu]} - \Delta_{[\nu\mu]}), \]

(52)

namely, the torsion tensor is algebraically defined by the antisymmetric part of the hypermomentum tensor, \( \Delta_{[\mu\nu]} \).

7. Conclusions

Metric-affine theories of gravity are among the most valuable generalizations of Einstein gravity. The most appealing characteristic of these models is to make straightforward the possibility of a coupling between the geometry of the spacetime (through the independent connection) and the internal degrees of freedom of matter fields (viz., intrinsic particle spin, dilation current and shear). While the dynamics of the torsion tensor has been extensively studied in literature starting from the simplest case of the Einstein–Cartan theory till higher order theories, the role of nonmetricity has been usually shelved.

In this paper we have explored the dynamics of metric-affine theory of gravity including both torsion and nonmetricity. We have shown that, in the most general theory obtained at the second order through dimensional analysis, the independent connection is algebraically expressed in terms of matter fields, metric and their derivatives. Torsion and nonmetricity gain dynamics only when higher order Lagrangian densities are taken into account, as can be proved including a Ricci-squared correction to the Einstein-Cartan action. We have also reanalysed the rather peculiar case of \( f(\mathcal{R}) \) theories of gravity, where the projective invariance of the gravitational Lagrangian forces the introduction of a further constraint on the connection. We showed that the attempt of gauge fixing the degrees of freedom introduced by the projective transformation via nonmetricity leads always to inevitable inconsistencies.

A possible way to bypass the problem of the projective invariance is to constrain some of the degrees of freedom of torsion by adding an appropriate Lagrange multiplier. The field equations obtained from the re-arranged theory result to be fully consistent. This theory represents the metric-affine generalization of modified theories of gravity, whose phenomenology surely deserves further investigation. In particular it would be very interesting to understand what is the behaviour of the extra degrees of freedom residing in the connection when they are excited, and how matter interactions are modified when energies are well below the dynamical regime of connection. These topics will be pursued in future publications.
Acknowledgments

The author would like to thank Roberto Percacci and Yuri Obukhov for the suggestion to explore the role of nonmetricity in the dynamics of MAG. Also, a special acknowledgement to Andrea Nerozzi for his help with the xAct package [28], used to check some of the calculations in the paper. Finally, the author is grateful to Antonino Flachi and Stefano Liberati for a critical reading of the manuscript. VV is supported by FCT—Portugal through the grant SFRH/BPD/77678/2011.

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