On the AF-algebra of a Hecke eigenform

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Abstract

An AF-algebra is assigned to each cusp form $f$ of weight two; we study properties of this operator algebra, when $f$ is a Hecke eigenform.

Key words and phrases: cusp forms, AF-algebras

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1 Introduction

A. The Modularity Theorem asserts that:

All rational elliptic curves arise from modular forms.

This result is tremendously important, since it leads to a spectacular proof of Fermat’s Last Theorem. The reader of the excellent book [4] on the Modularity Theorem will find on page (x) of the introduction the following interesting object. (We shall modify the original text to match our notation, which can be found in Section 2.) Denote by $f_N \in S_2(\Gamma_0(N))$ a Hecke eigenform and by $f_N^\sigma$ all its conjugates; consider a lattice $\Lambda_{f_N}$ generated by the complex periods of holomorphic forms $\omega_N^\sigma = f_N^\sigma dz$ on the Riemann surface $X_0(N) = \mathbb{H}^*/\Gamma_0(N)$. If $|\sigma|$ is the number of conjugates, the abelian variety $A_{f_N} := \mathbb{C}^{[\sigma]}/\Lambda_{f_N}$ is said to be associated to the eigenform $f_N$; it has the following remarkable property (the Modularity Theorem):

There exists a homomorphism of $A_{f_N}$ onto a rational elliptic curve.

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Let $\phi_N = \Re (\omega_N)$ be the real part of $\omega_N$; it is a closed form on the surface $X_0(N)$. (Alternatively, one can take for $\phi_N$ the imaginary part of $\omega_N$.) Clearly, $\omega_N$ defines a unique form $\phi_N$; the converse follows from the Hubbard-Masur Theorem \cite{6}. Since $\omega_N$ and $\phi_N$ define each other, what object will replace the associated variety $A_f$ in the case of $\phi_N$? Roughly speaking, it is shown in this paper that such a replacement is given by an operator algebra $A_f$ coming from the real periods of the form $\phi_N$; we study the basic properties of such an algebra (Theorem 1).

### B. The AF-algebra $A_f$.

Let $f \in S_2(\Gamma_0(N))$ be a cusp form and $\omega = f dz$ the corresponding holomorphic differential on $X_0(N)$. We shall denote by $\phi = \Re (\omega)$ a closed form on $X_0(N)$ and consider its periods $\lambda_i = \int_{\gamma_i} \phi$ against a basis $\gamma_1$ in the (relative) homology group $H_1(X_0(N), Z(\phi); \mathbb{Z})$, where $Z(\phi)$ is the set of zeros of $\phi$. Assume $\lambda_i > 0$ and consider the vector $\theta = (\theta_1, \ldots, \theta_{n-1})$ with $\theta_i = \lambda_{i+1}/\lambda_1$. The Jacobi-Perron continued fraction of $\theta$ \cite{2} is given by the formula:

\[
\left( \begin{array}{c} 1 \\ \theta \end{array} \right) = \lim_{i \to \infty} \left( \begin{array}{cc} 0 & 1 \\ I & b_i \end{array} \right) \cdots \left( \begin{array}{cc} 0 & 1 \\ I & b_i \end{array} \right) \left( \begin{array}{c} 0 \\ 1 \end{array} \right) = \lim_{i \to \infty} B_i \left( \begin{array}{c} 0 \\ 1 \end{array} \right),
\]

where $b_i = (b_1^{(i)}, \ldots, b_{n-1}^{(i)})^T$ is a vector of non-negative integers, $I$ is the unit matrix and $\mathbb{I} = (0, \ldots, 0, 1)^T$. By $A_f$ we shall understand the AF-algebra given its Bratteli diagram with partial multiplicity matrices $B_i$. Recall that an AF-algebra is called stationary if $B_i = B = \text{Const}$ \cite{5}. When two non-similar matrices $B$ and $B'$ have the same characteristic polynomial, the corresponding stationary AF-algebras will be called companion AF-algebras. Denote by $A_{f_N}$ an AF-algebra, such that $f_N \in S_2(\Gamma_0(N))$ is a Hecke eigenform. Our main result can be stated as follows.

**Theorem 1** The AF-algebra $A_{f_N}$ is stationary unless $f_N$ is a rational eigenform, in which case $A_{f_N} \cong \mathbb{C}$; moreover, $A_{f_N}$ and $A_{f_N}$ are companion AF-algebras.

The paper is organized as follows. The minimal preliminary results are expounded in Section 2, where we introduce the Hecke eigenforms, the AF-algebras and the Jacobi-Perron continued fractions. Theorem \cite{1} is proved in Section 3.

## 2 Preliminaries

### A. The Hecke eigenforms.

Let $N > 1$ be a natural number and consider
Let $H$ be a (finite index) subgroup of the modular group given by the formula:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid c \equiv 0 \mod N \right\}.$$ 

Let $\mathbb{H} = \{ z = x + iy \in \mathbb{C} \mid y > 0 \}$ be the upper half-plane and let $\Gamma_0(N)$ act on $\mathbb{H}$ by the linear fractional transformations; consider an orbifold $\mathbb{H}/\Gamma_0(N)$. To compactify the orbifold at the cusps, one adds a boundary to $\mathbb{H}$, so that $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{ \infty \}$ and the compact Riemann surface $X_0(N) = \mathbb{H}^*/\Gamma_0(N)$ is called a modular curve. The meromorphic functions $f(z)$ on $\mathbb{H}$ that vanish at the cusps and such that

$$f \left( \frac{az + b}{cz + d} \right) = \frac{1}{(cz + d)^2} f(z), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),$$

are called cusp forms of weight two; the (complex linear) space of such forms will be denoted by $S_2(\Gamma_0(N))$. The formula $f(z) \mapsto \omega = f(z)dz$ defines an isomorphism $S_2(\Gamma_0(N)) \cong \Omega_{hol}(X_0(N))$, where $\Omega_{hol}(X_0(N))$ is the space of holomorphic differentials on the Riemann surface $X_0(N)$. Note that $\dim_{\mathbb{C}}(S_2(\Gamma_0(N))) = \dim_{\mathbb{C}}(\Omega_{hol}(X_0(N))) = g$, where $g = g(N)$ is the genus of the surface $X_0(N)$. A Hecke operator, $T_n$, acts on $S_2(\Gamma_0(N))$ by the formula $T_n f = \sum_{\gamma \in \Gamma} \gamma f$, where $\gamma(m) = \sum_{a} |GCD(m, n)| a c m / a^2$ and $f(z) = \sum_{m \in \mathbb{Z}} c(m) q^m$ is the Fourier series of the cusp form $f$ at $q = e^{2 \pi i z}$. Further, $T_n$ is a self-adjoint linear operator on the vector space $S_2(\Gamma_0(N))$ endowed with the Petersson inner product; the algebra $\mathbb{T}_N := \mathbb{Z}[T_1, T_2, \ldots]$ is a commutative algebra. Any cusp form $f_N \in S_2(\Gamma_0(N))$ that is an eigenvector for one (and hence all) of $T_n$, is referred to as a Hecke eigenform; such an eigenform is called rational whenever its Fourier coefficients $c(m) \in \mathbb{Z}$. The Fourier coefficients $c(m)$ of $f_N$ are algebraic integers, and we denote by $\mathbb{K}_{f_N} = \mathbb{Q}(c(m))$ an extension of the field $\mathbb{Q}$ by the Fourier coefficients of $f_N$. Then $\mathbb{K}_{f_N}$ is a real algebraic number field of degree $1 \leq \deg (\mathbb{K}_{f_N}/\mathbb{Q}) \leq g$, where $g$ is the genus of the surface $X_0(N)$ [4], Proposition 6.6.4. Any embedding $\sigma : \mathbb{K}_{f_N} \to \mathbb{C}$ conjugates $f_N$ by acting on its coefficients; we write the corresponding Hecke eigenform $f_N^\sigma(z) := \sum_{m \in \mathbb{Z}} \sigma(c(m)) q^m$.

**B. The AF-algebras.** A $C^*$-algebra is an algebra $A$ over $\mathbb{C}$ with a norm $a \mapsto ||a||$ and an involution $a \mapsto a^*$ such that it is complete with respect to the norm and $||ab|| \leq ||a|| \cdot ||b||$ and $||a^*a|| = ||a^2||$ for all $a, b \in A$. Any commutative $C^*$-algebra is isomorphic to the algebra $C_0(X)$ of continuous
complex-valued functions on some locally compact Hausdorff space $X$; otherwise, $A$ represents a noncommutative topological space. The $C^*$-algebras $A$ and $A'$ are said to be stably isomorphic (Morita equivalent) if $A \otimes \mathcal{K} \cong A' \otimes \mathcal{K}$, where $\mathcal{K}$ is the $C^*$-algebra of compact operators; roughly speaking, stable isomorphism means that $A$ and $A'$ are homeomorphic as noncommutative topological spaces.

An AF-algebra (Approximately Finite $C^*$-algebra) is defined to be the norm closure of an ascending sequence of finite dimensional $C^*$-algebras $M_n$, where $M_n$ is the $C^*$-algebra of the $n \times n$ matrices with entries in $\mathbb{C}$. Here the index $n = (n_1, \ldots, n_k)$ represents the semi-simple matrix algebra $M_{n_1} \oplus \ldots \oplus M_{n_k}$. The ascending sequence mentioned above can be written as $M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \ldots$, where $M_i$ are the finite dimensional $C^*$-algebras and $\varphi_i$ the homomorphisms between such algebras. The homomorphisms $\varphi_i$ can be arranged into a graph as follows. Let $M_i = M_{i_1} \oplus \ldots \oplus M_{i_k}$ and $M_{i'} = M_{i_1'} \oplus \ldots \oplus M_{i_k'}$ be the semi-simple $C^*$-algebras and $\varphi_i : M_i \rightarrow M_{i'}$ the homomorphism. One has two sets of vertices $V_{i_1}, \ldots, V_{i_k}$ and $V_{i_1'}, \ldots, V_{i_k'}$, joined by $b_{r_{i_s}}$ edges whenever the summand $M_{i_{t_s}}$ contains $b_{r_{i_s}}$ copies of the summand $M_{i_{t_s}'}$ under the embedding $\varphi_i$. As $i$ varies, one obtains an infinite graph called the Bratteli diagram of the AF-algebra. The matrix $B = (b_{r_{i_s}})$ is known as a partial multiplicity matrix; an infinite sequence of $B_i$ defines a unique AF-algebra.

For a unital $C^*$-algebra $A$, let $V(A)$ be the union (over $n$) of projections in the $n \times n$ matrix $C^*$-algebra with entries in $A$; projections $p, q \in V(A)$ are equivalent if there exists a partial isometry $u$ such that $p = u^*u$ and $q = uu^*$. The equivalence class of projection $p$ is denoted by $[p]$; the equivalence classes of orthogonal projections can be made to a semigroup by putting $[p] + [q] = [p + q]$. The Grothendieck completion of this semigroup to an abelian group is called the $K_0$-group of the algebra $A$. The functor $A \rightarrow K_0(A)$ maps the category of unital $C^*$-algebras into the category of abelian groups, so that projections in the algebra $A$ correspond to a positive cone $K_0^+ \subset K_0(A)$ and the unit element $1 \in A$ corresponds to an order unit $u \in K_0(A)$. The ordered abelian group $(K_0, K_0^+, u)$ with an order unit is called a dimension group; an order-isomorphism class of the latter we denote by $(G, G^+)$.  

C. The Jacobi-Perron fractions. Let $a_1, a_2 \in \mathbb{N}$ such that $a_2 \leq a_1$. Recall that the greatest common divisor of $a_1, a_2$, $\text{GCD}(a_1, a_2)$, can be determined
from the Euclidean algorithm:

\[
\begin{align*}
    a_1 &= a_2 b_1 + r_3 \\
    a_2 &= r_3 b_2 + r_4 \\
    r_3 &= r_4 b_3 + r_5 \\
    &\vdots \\
    r_{k-3} &= r_{k-2} b_{k-1} + r_{k-1} \\
    r_{k-2} &= r_{k-1} b_k,
\end{align*}
\]

where \( b_i \in \mathbb{N} \) and \( \gcd(a_1, a_2) = r_{k-1} \). The Euclidean algorithm can be written as the regular continued fraction

\[
\theta = \frac{a_1}{a_2} = b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots + \cfrac{1}{b_k}}} = [b_1, \ldots, b_k].
\]

If \( a_1, a_2 \) are non-commensurable in the sense that \( \theta \in \mathbb{R} - \mathbb{Q} \), then the Euclidean algorithm never stops, and \( \theta = [b_1, b_2, \ldots] \). Note that the regular continued fraction can be written in matrix form

\[
\begin{pmatrix} 1 \\ \theta \end{pmatrix} = \lim_{k \to \infty} \begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & b_k \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

The Jacobi-Perron algorithm and connected (multidimensional) continued fraction generalizes the Euclidean algorithm to the case \( \gcd(a_1, \ldots, a_n) \) when \( n \geq 2 \). Namely, let \( \lambda = (\lambda_1, \ldots, \lambda_n) \), \( \lambda_i \in \mathbb{R} - \mathbb{Q} \) and \( \theta_{i-1} = \frac{\lambda_i}{\lambda_{i-1}} \), where \( 1 \leq i \leq n \). The continued fraction

\[
\begin{pmatrix} 1 \\ \theta_1 \\ \vdots \\ \theta_{n-1} \end{pmatrix} = \lim_{k \to \infty} \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & b_1^{(1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_{n-1}^{(1)} \end{pmatrix} \cdots \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & b_1^{(k)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_{n-1}^{(k)} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

where \( b_i^{(j)} \in \mathbb{N} \cup \{0\} \), is called the Jacobi-Perron algorithm (JPA). Unlike the regular continued fraction algorithm, the JPA may diverge for certain vectors \( \lambda \in \mathbb{R}^n \). However, for points of a generic subset of \( \mathbb{R}^n \), the JPA converges [1]; in particular, the JPA for periodic fractions is always convergent.
3 Proof of theorem

A standard dictionary ([5]) between AF-algebras and their dimension groups is adopted. Instead of dealing with $\mathfrak{A}_f$, we work with its dimension group $G_{\mathfrak{A}_f} = (G, G^+)$, where $G \cong \mathbb{Z}^n$ is the lattice and $G^+ = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid \theta_1 x_1 + \ldots + \theta_{n-1} x_{n-1} + x_n \geq 0\}$ is a positive cone. Recall, that $G_{\mathfrak{A}_f}$ is abelian group with an order, which defines the AF-algebra $\mathfrak{A}_f$, up to a stable isomorphism. We arrange the proof in a series of lemmas. First, let us show, that $\mathfrak{A}_f$ is a correctly-defined AF-algebra.

**Lemma 1** The $\mathfrak{A}_f$ does not depend, up to a stable isomorphism, on a basis in $H_1(X_0(N), Z(\phi); \mathbb{Z})$.

**Proof.** Denote by $m := \mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n$ a $\mathbb{Z}$-module in the real line $\mathbb{R}$. Let $\{\gamma_j'\}$ be a new basis in $H_1(X_0(N), Z(\phi); \mathbb{Z})$, such that $\gamma_j' = \sum_{j=1}^n a_{ij} \gamma_j$ for matrix $A = (a_{ij}) \in \text{GL}_n(\mathbb{Z})$. Using the integration rules, one gets: $\lambda_i' = \int_{\gamma_i'} \phi = \int_{\sum_{j=1}^n a_{ij} \gamma_j} \phi = \sum_{j=1}^n \int_{\gamma_j} \phi = \sum_{j=1}^n a_{ij} \lambda_j$. Thus, $m' = m$ and a change of basis in the homology group $H_1(X_0(N), Z(\phi); \mathbb{Z})$ amounts to a change of basis in the module $m$. It is an easy exercise to show that there exists a linear transformation of $\mathbb{Z}^n$ sending the positive cone $G^+$ of $G_{\mathfrak{A}_f}$ to the positive cone $(G^+)'$ of $G_{\mathfrak{A}_f}'$. In other words, $\mathfrak{A}_f'$ and $\mathfrak{A}_f$ are stably isomorphic. □

**Lemma 2** The (scaled) periods $\lambda_i$ belong to the field $\mathbb{K}_{f_N}$.

**Proof.** Let $m = \mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n$ be a $\mathbb{Z}$-module generated by $\lambda_i$; we seek the effect of the Hecke operators $T_m$ on $m$. By the definition of a Hecke eigenform, $T_m f_N = c(m) f_N$ for all $T_m \in T_N$. In view of the isomorphism $S_2(\Gamma_0(N)) \cong \Omega_{hol}(X_0(N))$, one gets $T_m \omega_N = c(m) \omega_N$, where $\omega_N = f_N dz$. Then $\Re (T_m \omega_N) = T_m (\Re (\omega_N)) = \Re (c(m) \omega_N) = c(m) \Re (\omega_N)$. Therefore, $T_m \phi_N = c(m) \phi_N$, where $\theta_N = \Re (\omega_N)$. The action of $T_m$ on $\mathbb{Z}$-module $m$ can be written as $T_m (m) = \int_{H_1} T_m \phi_N = \int_{H_1} c(m) \phi_N = c(m) m$, where $H_1 := H_1(X_0(N), Z(\phi_N); \mathbb{Z})$. Thus, the Hecke operator $T_m$ acts on the module $m$ as multiplication by an algebraic integer $c(m) \in \mathbb{K}_{f_N}$.

The action of $T_m$ on $m = \mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n$ can be written as $T_m \lambda = c(m) \lambda$, where $\lambda = \lambda_1, \ldots, \lambda_n$; thus, $T_m$ is a linear operator (on the space $\mathbb{R}^n$), whose eigenvector $\lambda$ corresponds to the eigenvalue $c(m)$. It is an easy exercise in linear algebra that $\lambda$ can be scaled so that all $\lambda_i$ lie in the same field as $c(m)$; lemma 2 follows. □
Case I. Let \( f_N \) be not a rational eigenform; then \( n = \deg (\mathbb{K}_{f_N}/\mathbb{Q}) \geq 2 \). Note, that \( m = \mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n \) is a full (i.e. the maximal rank) \( \mathbb{Z} \)-module in the number field \( \mathbb{K}_{f_N} \). Indeed, \( \text{rank} \ (m) \) cannot exceed \( n \), since \( m \subset \mathbb{K}_{f_N} \) and \( \mathbb{K}_{f_N} \) is a vector space (over \( \mathbb{Q} \)) of dimension \( n \). On the other hand, \( (\lambda_1, \ldots, \lambda_n) \) is a basis of the field \( \mathbb{K}_{f_N} \) and, as such, \( \text{rank} \ (m) \) cannot be less than \( n \); thus, \( \text{rank} \ (m) = n \).

**Lemma 3** The vector \( (\lambda_1, \ldots, \lambda_n) \) has a periodic (Jacobi-Perron) continued fraction.

**Proof.** Since \( m \subset \mathbb{K}_{f_N} \) is a full \( \mathbb{Z} \)-module, its endomorphism ring \( \text{End} \ (m) = \{ \alpha \in \mathbb{K}_{f_N} : \alpha m \subseteq m \} \) is an order (a subring of the ring of integers) of the number field \( \mathbb{K}_{f_N} \); let \( u \) be a unit of the order \( \mathbb{K}_{f_N} \), p 112. The action of \( u \) on \( m \) can be written in a matrix form \( A\lambda = u\lambda \), where \( A \in \text{GL}_n(\mathbb{Z}) \); with no loss of generality, one can assume the matrix \( A \) to be non-negative in a proper basis of \( m \).

According to [1], Prop.3, the matrix \( A \) can be uniquely factorized as \( A = \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \ldots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix} \), where vectors \( b_i = (b_{i1}, \ldots, b_{in-1})^T \) have non-negative integer entries. By [7], Satz XII, the periodic continued fraction

\[
\begin{pmatrix} 1 \\ \theta' \end{pmatrix} = \text{Per} \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \ldots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix}
\]

converges to a vector \( \lambda' = (\lambda'_1, \ldots, \lambda'_n) \) which satisfies the equation \( A\lambda' = u\lambda' \). Since \( A\lambda = u\lambda \), the vectors \( \lambda \) and \( \lambda' \) are collinear; but collinear vectors have the same continued fractions [2]. \( \square \)

The first case of Theorem [1] follows from lemma [3], since \( \mathfrak{A}_{f_N} \) is a stationary AF-algebra, whose period is given by the matrix \( A \).

Case II. Let \( f_N \) be a rational eigenform; in this case \( n = 1 \) and \( \mathbb{K}_{f_N} = \mathbb{Q} \). The Bratteli diagram of \( \mathfrak{A}_{f_N} \) is finite and one-dimensional; therefore, \( \mathfrak{A}_{f_N} \cong M_1(\mathbb{C}) = \mathbb{C} \). This argument finishes the proof of the first part of Theorem [1].

To prove the second part, let on the contrary \( A \neq A' \) be similar matrices. To find \( S \) such that \( A' = S^{-1}AS \), notice that \( m'^\sigma = \lambda'^\sigma_1\mathbb{Z} + \ldots + \lambda'^\sigma_n\mathbb{Z} \). Since \( m'^\sigma = m \), \( \lambda'^\sigma_j = \sum s_{ij} \lambda_i \), where \( S = (s_{ij}) \); but \( s^k \) = Id for some integer \( k \) and thus \( S^k = I \). Therefore, \( (A')^k = (S^{-1}AS)^k = A^k \) and \( A' = A \), which contradicts our assumption. On the other hand, \( \lambda'^\sigma_j \in \mathbb{K}_{f_N} \) implies that the
characteristic polynomials $\text{char} (A) = \text{char} (A')$; therefore, $\mathfrak{A}_{f_N}$ and $\mathfrak{A}_{f_N^\sigma}$ are companion AF-algebras. □

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