Source integrals for multipole moments in static and axially symmetric spacetimes

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Abstract

In this article, we derive source integrals for multipole moments in axially symmetric and static spacetimes. The multipole moments can be read off the asymptotics of the metric close to spatial infinity in a hypersurface, which is orthogonal to the timelike Killing vector. Whereas for the evaluation of the source integrals the geometry needs to be known in a compact region of this hypersurface, which encloses all source, i.e. matter as well as singularities. The source integrals can be written either as volume integrals over such a region or in quasi-local form as integrals over the surface of that region.

1 Introduction

In general relativity, there were several definitions of multipoles proposed. Since this theory is non-linear, it is, however, by no means obvious that this is at all possible. Thus, it is not surprising that in early works multipoles were only defined in approximations to general relativity that lead to linear field equations and allow a classical treatment. The most definitions in this direction and beyond were covered in Thorne’s review [1].

From the 1960s on, new definitions in the full theory of isolated bodies1 started emerging. These definitions of multipole moments can roughly be divided into two classes. In the first, the metric (or quantities derived from it) are expanded at spacelike or null-like infinity. We will call these asymptotic definitions or asymptotic multipole moments. Amongst these are the definitions of Bondi, Metzner, Sachs and van der Burgh (BMSB) 2,3, Geroch and Hansen (GH) 4,5, Simon and Beig 2 (SB) 6, Janis, Newman and Unti (JNU) 7,8.

This means that all sources (matter and black holes) are located in a sphere of finite radius and the spacetime is assumed to be asymptotically flat. A precise meaning is given in Sec. 2.2.

This approach reproduces the GH multipole moments.
Thorne [1], the ADM approach [9] and the Komar integrals [10], for reviews see [1,11]. There scope of applicability varies greatly. Whereas the GH multi-pole moments are defined only in stationary spacetimes the BMSB, JNU, Thorne and ADM definitions hold in a more general setting. The Komar expressions for the mass and the angular momentum on the other hand require stationarity and stationarity and axially symmetry, respectively. Higher order multipoles are not defined in the Komar approach. Despite their conceptual differences, Gürsel showed in [12] the equivalence of the GH and Thorne’s multipole moments in case the requirements of both definitions are met. Additionally, the mass and the angular momentum in the GH, Thorne, ADM and Komar approach can be shown to agree.

In the second class fall multipoles that are determined by the metric in a compact region. Dixon’s definition in [13] falls in this class. These multipoles are given in the form of source integrals. However, it is not yet known how they are related with the asymptotic multipole moments. A main application of these multipoles is in the theory of the motion of test bodies with internal structure. But for test bodies it is obvious that there cannot be any such relation between Dixon’s multipole moments and the asymptotic multipole moments. Furthermore, Dixon’s definition is in general not applicable if caustics of geodesics appear inside the source, i.e., if the gravitational field is too strong compared to a characteristic radius of the source. Ashtekar et al. defined in [14] multipole moments of isolated horizons. These are also source integrals and require only the knowledge of the interior geometry of the horizon. In [14], it was also shown that the so defined multipoles of the Kerr black hole deviate from the GH multipoles. This effect becomes more pronounced the greater the rotation parameter. In both cases [13,14], it is interesting to find the relation of these multipole moments to asymptotically defined GH multipole moments and their physical interpretation. Source integrals might prove useful in this respect.

Other definitions, in particular for the quasi-local mass and the quasi-local angular momentum, can be found in [15]. Here we aim at source integrals for all multipole moments in static spacetimes for arbitrary sources, i.e., we want to express asymptotic multipole moments by surface or volume integrals, where the surface envelopes or the volume covers all the sources of the gravitational field. That such source integrals can be found, is not at all obvious. This is because of the non-linear nature of the Einstein equations yielding a gravitational field, which acts again as a source. To overcome the principle difficulties, we will focus here on static and axially symmetric spacetimes. In this case, the vacuum Einstein equations can be cast in an essentially linear form. Additionally, they allow the introduction of a linear system [16–18]. The latter might seem superfluous, if the the former holds. However, we will derive our source integrals relying solely on the existence of the linear system and the applicability of the inverse scattering technique. This will allow us in future work to apply the same formalism to stationary and axially symmetric isolated systems. This is especially relevant for the description of relativistic stars, where many applications are. A generalization to spacetimes with an Einstein-Maxwell field exterior to the sources seems also feasible. In all these cases, multipole moments
of the GH type exist.

Source integrals will prove useful in many respects, in particular in the search of global solutions of Einstein equations describing figures of equilibrium or relativistic stars, for recent efforts see, e.g., [19–22] and references therein. For example, an exterior solution can be constructed for a known interior solution by employing the source integrals to calculate its multipole moments. From these the exterior solution can be completely determined. This yields an exterior solution, which does not necessarily match to the interior solution. However, if it does not, this construction shows that there is no asymptotically flat solution, which can be matched to the given interior. Conversely, possible matter sources can be analyzed for given exterior solutions and their multipole moments. The latter approach is also of astrophysical interest, since often only the asymptotics of the gravitational field and the asymptotic multipole moments are accessible to experiments. Source integrals can be employed to, e.g., restrict the equation of state of a rotating perfect fluid from observed multipole moments.

Furthermore, source integrals can be used to compare numerical solutions, analytical solutions and analytical approximations by calculating their multipole moments, see for example [24]. This can also be used to approximate the vacuum exterior of a given numerical solution by an analytical one, which exhibits the correct multipole moments up to a prescribed order, see [25]. The merit of source integrals lies in the fact that they determine the multipole moments using the matter region only, which captures the internal structure of the relativistic object and is usually determined with high accuracy. Additionally, source integrals provide the means to test the accuracy of numerical methods, which are used to determine relativistic stars, cf. [26], by calculating the multipole moments in two independent ways: Firstly, using the asymptotics and, secondly, using the source integrals. This will give also a physical interpretation to possible deviations.

The paper is organized as follows: In Sec. 2 we introduce the different concepts used later, i.e., the GH and the Weyl multipole moments as well as the inverse scattering technique. Sec. 3 is devoted to the derivation of the source integrals and includes the main results, Eq. (23)-(24). In Sec. 4 we will discuss some properties of the obtained source integrals.

2 Preliminaries

In this section, we will repeat the notions that are needed in the present paper. Note, that we use geometric units, in which \( G = c = 1 \), where \( c \) is the velocity of light and \( G \) Newton’s gravitational constant. The metric has the signature \((-1, 1, 1, 1)\). Greek indices run from 0 to 3, lower-case Latin indices run from 1 to 3 and upper-case Latin indices from 1 to 2.

\[ \text{For difficulties in extracting the multipole moments from a given numerical metric, see, e.g., [23].} \]
2.1 The line element and the field equation

We consider static and axially symmetric spacetimes admitting a timelike Killing vector $\xi^a$ and a spacelike Killing vector $\eta^a$, which commutes with $\xi^a$, has closed timelike curves and vanishes at the axis of rotation. If the orbits of the so defined isometry group admit orthogonal 2-surfaces, which is the case in vacuum, in static perfect fluids or static electromagnetic fields, see, e.g., [27], then the metric can be written in the Weyl form:

$$ds^2 = e^{2k-2U} (d\rho^2 + d\zeta^2) + W^2 e^{-2U} d\varphi^2 - e^{2U} dt^2,$$

(1)

where the functions $U$, $k$ and $W$ depend on $\rho$ and $\zeta$.

Note that the metric functions $U$ and $W$ can be expressed by the Killing vectors:

$$e^{2U} = -\xi_\alpha \xi^\alpha, \quad W^2 = -\eta_\alpha \eta^\alpha \xi_\beta \xi^\beta.$$  

(2)

The Einstein equations can be inferred from the ones given in [29]. Since we will not specify the matter, we give only a complete set of combinations of the non-vanishing components of the Ricci tensor:

$$\Delta (2) U + \frac{1}{W} (U_\rho W_\rho + U_\zeta W_\zeta) = e^{2k-4U} R_{tt},$$

$$W_{\rho\rho} - W_{\zeta\zeta} + 2 (k_\zeta W_\zeta - k_\rho W_\rho + W (U_\rho^2 - U_\zeta^2)) = W (R_{\zeta\zeta} - R_{\rho\rho}),$$

$$W_\zeta k_\rho + W_\rho k_\zeta - 2 W U_\rho U_\zeta - W_{,\rho} = W R_{\rho\zeta},$$

(3a)

$$W \Delta (2) W = e^{2k-4U} W^2 R_{tt} - e^{2k} R_{\varphi\varphi},$$

$$-2 \Delta (2) k = (R_{\rho\rho} + R_{\zeta\zeta}) - e^{2k-4U} R_{tt} - \frac{e^{2k}}{W^2} R_{\varphi\varphi}$$

where $\Delta (n) = \left( \frac{\partial^2}{\partial \rho^2} + \frac{n-2}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial \zeta^2} \right)$. The fourth equation implies that we can introduce canonical Weyl coordinates $(\tilde{\rho}, \tilde{\zeta})$ with $W = \tilde{\rho}$ via a conformal transformation in vacuum or in all domains, where $\Delta (2) W = 0$ holds, including, e.g., dust. After this coordinate system is chosen, we drop the tilde again. The remaining coordinate freedom is a shift of the origin along the symmetry axis, which is characterized by $\rho = 0$. Eqs. (3a) simplify in vacuum to the well-known equations

$$\Delta (3) U = 0,$$

$$k_\zeta = 2 \rho U_\rho U_\zeta, \quad k_\rho = \rho ((U_\rho)^2 - (U_\zeta)^2),$$

(3b)

where $\Delta (n)$ is defined analogously to $\Delta (n)$ but with canonical Weyl coordinates. The last two equations determine $k$ via a line integration once $U$ is known. This $k$ automatically satisfies the last equation in (3a). Hence, only a Laplace

\[ \Box (\rho U) = 0, \]

\[ k_\rho = \rho ((U_\rho)^2 - (U_\zeta)^2), \]

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equation for $U$ remains to be solved. Therefore, the Newtonian theory and general relativity can be treated on the same formal footing. We will do so here and highlight the difference in Sec. 4. The disadvantage of using the canonical Weyl coordinates is that they cannot necessarily be introduced in the interior of the matter, where we have to use other (non-canonical) Weyl coordinates.

The physical setting, which we want to investigate, is as follows: We have in a compact region $V$ of a hypersurface given by $t = \text{const.}$ several sources, cf. Fig. 1. The surface of $V$ will be denoted by $S = \partial V$ and is assumed to be a topological 2-sphere. Note that inside $V$ it will in general be not possible to introduce canonical Weyl coordinates. The sources can be matter distributions supported in $V_i$ with $S_i = \partial V_i$. If there are black hole sources with horizons $H_i$, we assume that there is vacuum at least in a neighborhood of $H_i$, which necessary for static black holes, see, e.g., [31]. Hence, we can define a closed surfaces $S^H_i$ in the vacuum neighborhood, which encloses only the black hole with $H_i$. For relativistic stars, we can choose $V = V_1$, in which case we assume that Israel’s junction conditions are satisfied across $S = S_1$, i.e., that there are no surface distributions. In fact we will assume this for all surfaces $S_i$ and $S^H_i$. The only other restriction on the matter is that we want to be able to introduce Weyl’s line element, see [1].

\[\text{Figure 1: The different surfaces } S \text{ and } S_i, \text{ volumes } V \text{ and } V_i \text{ are depicted for a certain matter distributions (ellipsoid and torus). The curves } B \text{ and } A^\pm \text{ are relevant in Sec. 2.3. Not that this serves only as illustration and that under a certain energy condition and separation condition static } n\text{-body solutions do not exist, for details see } [30].\]
2.2 Geroch’s multipole moments

Isolated gravitating systems are described by an asymptotically flat spacetime. The precise meaning of this is defined below and used to define Geroch’s multipole moments. Let us denote our static spacetime by \((M, g)\) with the metric \(g_{\alpha\beta}\) and by \(V\) a hypersurface orthogonal to \(\xi^\alpha\) endowed with the induced metric \(h_{\alpha\beta} = -\xi^\gamma \xi_\gamma g_{\alpha\beta} + \xi_\alpha \xi_\beta\), for which we will use subsequently Latin indices. For the definition of Geroch’s multipole moments, asymptotic flatness of \(V\) is sufficient, see \([4]\):

**Definition 1** \(V\) is asymptotically flat iff there exists a point \(\Lambda\), a manifold \(\tilde{V}\) and a conformal factor \(\Omega \in C^2(\tilde{V})\), such that

1. \(\tilde{V} = V \cup \Lambda\)
2. \(\tilde{h}_{ab} = \Omega^2 h_{ab}\) is a smooth metric of \(\tilde{V}\)
3. \(\Omega = \tilde{D}_a \Omega = 0\) and \(\tilde{D}_a \tilde{D}_b \Omega = 2 \tilde{h}_{ab}\) at \(\Lambda\), where \(\tilde{D}_a\) is the covariant derivative in \(\tilde{V}\) associated with the metric \(\tilde{h}_{ab}\).

Let us define the potential \(\tilde{\psi}\)

\[
\tilde{\psi} = 1 - \left(\xi^\alpha \xi_\alpha\right)^{\frac{1}{2}} \Omega^2,
\]

which is also a scalar in \(\tilde{V}\). If we introduce the Ricci tensor \(\tilde{R}^{(3)}_{ab}\) built from the metric \(\tilde{h}_{ab}\) then the tensors \(P_{a_1...a_n}\) can be defined recursively:

\[
P_{a_1...a_n} = \tilde{\psi} \left[ \tilde{D}_{a_1} P_{a_2...a_n} - \frac{(n-1)(2n-3)}{2} \tilde{R}^{(3)}_{a_1a_2} P_{a_3...a_n} \right],
\]

where \(C[A_{a_1...a_n}]\) denotes the symmetric and trace-free part of \(A_{a_1...a_n}\). The tensors \(P_{a_1...a_n}\) evaluated at \(\Lambda\) define Geroch’s multipole moments:

\[
M_{a_1...a_n} = P_{a_1...a_n} |_{\Lambda}.
\]

The degree of freedom in the choice of the conformal factor reflects the choice of an origin, with respect to which the multipole moments are taken, see \([4]\).

Up to now, only stationarity was required. If we choose axially symmetry the multipole structure simplifies to

\[
m_r = \frac{1}{r^l} P_{a_1...a_r} z^{a_1} \cdots z^{a_r} |_{\Lambda},
\]

where the \(z^a\) is the unit vector pointing in the direction of the symmetry axis and the scalars \(m_n\) define the multipole moments completely. Hence, we will refer to them as multipole moments, as well.
Fodor et al. demonstrated in [32] that the multipole moments can be obtained directly from the Ernst potential on the axis or in the here considered static case from $U$ at the axis. If we expand $U$ along the axis, i.e., $U(\rho = 0, \zeta) = \sum_{r=1}^{\infty} U(r) |\zeta|^{-r-1}$, we characterise the solution by the set of constants $U(r)$. The result in [32] relates now the $m_r$ with the $U(r)$, i.e., if the $U(r)$ are known, we can in principle obtain the $m_r$. Thus, we will limit ourselves to discussing mainly the $U(r)$. Although the relation $m_r(U(j))$ can be obtained in principal to any order, up to now only the $m_0, \ldots, m_{10}$ were explicitly expressed using the $U(r)$, see [32]. We will give here only the first four for illustration:

$$m_0 = -U(0), \quad m_1 = -U(1), \quad m_2 = \frac{1}{3} U(0)^2 - U(2), \quad m_3 = U(0)^2 U(1) - U(3)$$

Eq. (8) shows that the mass dipole moment, $U(1)$, can be transformed away, if the mass, $U(0)$, is not vanishing. For a general discussion, further references and expressions of the center of mass, see [33].

In [34] a method to obtain the $m_r$ was proposed, which could help to overcome the non explicit structure of $m_r(U(j))$. Also the pure $2^r$ pole solutions in [35] could proof useful in this respect.

### 2.3 The linear problem of the Laplace equation

Lastly, we shortly review the linear problem associated with the Laplace equation. Although the equations involved are fairly simple, we decided to use this technique, because it is readily generalizable to the stationary case.

In the more general case of stationarity and axially symmetry, the Einstein equation, i.e., the Ernst equation, admits a linear problem, see, e.g., [16-18] and for a recent account [36]. In the static case the linear problem reads

$$\sigma, z = (1 + \lambda) U, z \sigma, \quad \sigma, \bar{z} = \left(1 + \frac{1}{\lambda}\right) U, \bar{z} \sigma,$$

where $z = \rho + i\zeta$, the spectral parameter $\lambda = \sqrt{\frac{K-i\bar{z}}{K+i\bar{z}}}$, $K \in \mathbb{C}$ and a bar denotes complex conjugation. The function $\sigma$ depends on $z, \bar{z}$ and $\lambda$. The integrability condition of Eqs. (9) is the first equation in Eq. (3b).

Next we will repeat some known properties of $\sigma$ without proof. For details we refer the reader to [36]. There are four curves of particular interest $\mathcal{A}^\pm, \mathcal{B}$ and $\mathcal{C}$, cf. Fig. 1. The axis of symmetry is divided by $\mathcal{V}$ in an upper and lower part $\mathcal{A}^+$ and $\mathcal{A}^-$, respectively. The curve $\mathcal{B}$ generates $\mathcal{S}$ by an rotation around the axis and is given by a restriction of $\mathcal{S}$ to $\varphi = 0$. Thus, we will subsequently refer to $\mathcal{A}^\pm$ and $\mathcal{B}$ as curves in a $\rho, \zeta$-plane. Lastly, $\mathcal{C}$ describes a half circle with sufficiently large radius connecting $\mathcal{A}^+$ with $\mathcal{A}^-$.\footnote{The formulas can easily inferred from [36] by setting $g_{\varphi \varphi} = 0.$}
Along $\mathcal{A}^\pm$ and $C$ Eq. (9) can be integrated. This yields for a suitable choice of the constant of integration $(0,\zeta) \in \mathcal{A}^+$:

$$
\sigma(\lambda = +1, \rho = 0, \zeta) = F(K)e^{2U(\rho=0,\zeta)},
$$

$$
\sigma(\lambda = -1, \rho = 0, \zeta) = 1,
$$

(0,\zeta) \in \mathcal{A}^-:

$$
\sigma(\lambda = +1, \rho = 0, \zeta) = e^{2U(\rho=0,\zeta)},
$$

$$
\sigma(\lambda = -1, \rho = 0, \zeta) = F(K),
$$

(10)

The function $F : \mathbb{C} \to \mathbb{C}$ is given for $K \in \mathbb{R}$ with $(\rho = 0, \zeta = K) \in \mathcal{A}^\pm$ by

$$
F(K) = \begin{cases} 
e^{-2U(\rho=0,\zeta=K)} & (0, K) \in \mathcal{A}^+ \\ ne^{2U(\rho=0,\zeta=K)} & (0, K) \in \mathcal{A}^- \end{cases}
$$

(11)

The integration along $B$ is the crucial part for our considerations in the next section.

3 Source integrals Geroch’s multipole moments

Let us assume that the line element is written in canonical Weyl coordinates in the exterior region $V^C$, cf. Sec. 2.1. The scalars $U$ and $W$, cf. (2), are also scalars in the projection $V$. Furthermore, $U$ and $W$ are supposed to be continuously differentiable in the vacuum region including $S$ and, thus, $B$. Then we can consider the linear problem (9) also along $B$ (after a projection to the $\varphi = 0$ plane):

$$
\sigma,s = \left[ U_{,A}s^A + \frac{1}{2} \left( \left( \frac{1}{\lambda + \lambda} \right) U_{,A}s^A + i \left( \frac{1}{\lambda - \lambda} \right) U_{,A}n^A \right) \right] \sigma,
$$

(12)

where $(s^A) = (s^\rho, s^\zeta) = \left( \frac{d\rho}{ds}, \frac{d\zeta}{ds} \right)$ and $(n^A) = (n^\rho, n^\zeta) = \left( -\frac{d\zeta}{ds}, \frac{d\rho}{ds} \right)$ denote the tangential vector and the outward pointing normal vector to the curve $B : s \in [s_N, s_S] \to (\rho(s), \zeta(s))$, respectively. The parameter values $s_{N/S}$ give the “north/south” pole $p_{N/S}$, i.e., $(\rho = 0, \zeta = \zeta_{N/S})$, cf. Fig. 1. Note that the tangential and the normal vectors are not necessarily normalized allowing an arbitrary parametrization of $B$.

Eq. (12) constitutes an ordinary differential equation of first order with the boundary conditions as given in (10) assuming $(0, K) \in \mathcal{A}^+ \cup \mathcal{A}^-$. As such it is an overdetermined system, which corresponds to the integrability of Eqs. (9). However, Eq. (12) is readily integrated and the compatibility condition of the boundary conditions reads

$$
U(0, K) = \frac{1}{2} \left( U(0, \zeta_N) - U(0, \zeta_S) \right) + \\
\frac{1}{4} \int_{s_N}^{s_S} \left( (\lambda^{-1} + \lambda) U_{,A}s^A + i (\lambda^{-1} - \lambda) U_{,A}n^A \right) ds.
$$

(13)
Eq. (13) determines the axis values of $U$ from the Dirichlet data and the Neumann data along an arbitrary curve $B$, which is sufficient to obtain the entire solution $U$ in $\mathcal{V}$.

The multipole moments follow from an expansion of Eq. (13) with respect to $K^{-1}$. Let us denote by $f^{(r)}$ the expansion coefficient to order $|K|^{-r-1}$ of a function $f(K)$, which is constant at infinity, i.e., $f(K) = \sum_{r=-1}^{\infty} f^{(r)}|K|^{-r-1}$. The coefficients $N_+^{(r)} = (\lambda^{-1} + \lambda)^{(r)}$ and $N_-^{(r)} = i(\lambda^{-1} - \lambda)^{(r)}$ depend still on $(\rho, \zeta)$ and satisfy the equations

$$N_+^{(r)} - N_-^{(r)} = 0,$$

$$N_+^{(r)} + N_-^{(r)} - \frac{1}{\rho} N_-^{(r)} = 0. \quad (14)$$

Eqs. (14) follow directly from the form of the spectral parameter $\lambda$, cf. after Eq. (9). This expansion is only valid for $\rho^2 + \zeta^2 < \infty$. Furthermore, $N_\pm^{(r)}$ and their radial derivatives evaluate at the axis to

$$N_\pm^{(r)}(\rho = 0, \zeta) = 0 \quad \forall r \geq -1,$$

$$N_+^{(-1)}(\rho = 0, \zeta) = 2,$$

$$N_+^{(r)}(\rho = 0, \zeta) = 0 \quad \forall r \geq 0,$$

$$N_-^{(-1)}(\rho = 0, \zeta) = 0$$

$$N_-^{(r)}(\rho = 0, \zeta) = -2\zeta^r \quad \forall r \geq 0. \quad (15)$$

The zeroth order of Eq. (13) implies together with Eq. (15) that $U^{(-1)} = 0$. This is also required by asymptotic flatness, which is assumed in the derivation of (10). Solving the Eqs. (14) yield after a lengthy calculation $N_\pm^{(r)}$ for $r \geq 0$ everywhere:

$$N_-^{(r)} = \sum_{k=0}^{\lfloor r/2 \rfloor} \frac{2(-1)^{k+1}r!\rho^{2k+1}\zeta^{r-2k}}{4^k(k!)^2(r-2k)!},$$

$$N_+^{(r)} = \sum_{k=0}^{\lfloor r/2 \rfloor} \frac{2(-1)^{k+1}r!\rho^{2k+2}\zeta^{r-2k-1}}{4^k(k!)^2(r-2k-1)(2k+2)}. \quad (16)$$

Note that $\frac{N_-^{(r)}}{\rho}$ is well-behaved also for $\rho \to 0$ for $r \geq 0$.

For orders $r \geq 0$ Eq. (13) yields together with Eq. (16) the following line integrals defining Weyl’s multipole moments

$$U^{(r)} = \frac{1}{4} \int_{\gamma^0} \left( N_+^{(r)} U_{A^r A^t s^A} + N_-^{(r)} U_{A^r A^t n^A} \right) d\gamma, \quad (17)$$
where \( \hat{s}^A \) and \( \hat{n}^A \) are the normalized vectors \( s^A \) and \( n^A \), respectively, and \( d\gamma \) denotes the proper distance along the path \( \gamma_B \), which runs along \( B \) from the north to the south pole. The functions \( N^\pm(r) \) and \( U \) are to be read as functions of \( (\rho(s), \zeta(s)) \).

Eqs. (17) are already expressions of the kind we are searching for, since they determine the multipole moments from the metric given in a compact region, i.e., they are quasi-local. But we also can rewrite these multipole moments as volume integrals justifying the term source integrals even better. The main obstacle for doing so is that Weyl’s multipole moments are given in Eq. (17) using canonical Weyl coordinates. Hence, neither the coordinate invariance of these expressions is transparent nor is obvious how to continue \( \rho \) and \( \zeta \) to \( V \).

However, \( U \) can be expressed by the norm of the timelike Killing vector, cf. Eq. (2), which can easily be continued to the interior. Let us introduce the 1-form

\[
Z_\alpha = \epsilon_{\alpha\beta\gamma\delta} W^{\beta\gamma} W^{-1} \eta^{\gamma\delta},
\]

where \( \epsilon_{\alpha\beta\gamma\delta} \) denotes the volume form of the static spacetime. \( Z_\alpha \) is closed in \( V_C \) and hypersurface orthogonal in the entire spacetimes. Thus, we can introduce a scalar potential \( Z \) with \( Z_\alpha = XZ_\alpha \), where the scalar \( X \) equals 1 in the vacuum region. In canonical Weyl coordinates in \( V_C \), the potential \( Z \) has the trivial form \( Z = \zeta + \zeta_0 \). Because we did not fix the origin of our Weyl coordinates, e.g., the value of \( \zeta_M \), we can set the constant \( \zeta_0 = 0 \) without loss of generality. This integration constant is exactly the freedom we need to change the origin with respect to which the multipole moments are defined allowing us to change to the center of mass frame. \( W = \rho \) and \( Z = \zeta \) in \( V_C \) and they are defined everywhere. Hence, we can use these two scalars to continue \( \rho \) and \( \zeta \) into \( V \). Note that \( W_\alpha \) and \( Z_\alpha \) are orthogonal everywhere and have the same norm in the \( V_C \). The line integral (17) in the covariant form reads now

\[
U^{(r)} = \frac{1}{4} \int_{\gamma_B} N_+^{(r)} (W(s), Z(s)) U_{\alpha} \hat{s}^\alpha + N_-^{(r)} (W(s), Z(s)) U_{\alpha} \hat{n}^\alpha d\gamma.
\]

(19)

The dependence on \( W(s) \) and \( Z(s) \) will be suppressed in the following expressions.

Along \( B \) the functions \( W \) and \( Z \) satisfy

\[
W,s = Z,n, \quad W,n = -Z,s,
\]

which is a consequence of the field equation and the choice of canonical Weyl coordinates. After an partial integration, we can rewrite the line integral as surface integral using the axial symmetry and Eq. (20), which yields

\[
U^{(r)} = \frac{1}{8\pi} \int_{S} e^U \left( N_-^{(r)} U_{\alpha} \hat{n}^\alpha - N_+^{(r)} W_{\alpha} \hat{n}^\alpha + N_-^{(r)} W_{\alpha} \hat{n}^\alpha U \right) dS.
\]

(21)
We denote by $dS$ the proper surface element of $S$, $S_i$ or $S^H_i$, respectively. Since we ruled out surface distributions, Israel’s junction conditions imply that Eqs. (21) can be understood as integrals over the 2-surface $S$ as seen from the exterior or the interior, see [37].

Using Stoke’s theorem and Eqs. (3) we obtain (see Fig. 1 and the end of Sec. 2.1 for a description of the $S^H_i$ and $V_i$)

$$U(=) = \frac{1}{8\pi} \sum_i \int_{S^H_i} e^{U} \left[ - \frac{N_{(r)}(W, Z)}{W} R_{\alpha\beta} \xi^\alpha \xi^\beta + N_{(r)}^{i, Z}(W, Z) U \left( \frac{W_\alpha}{W} \right) \right] -$$

$$N_{(r)}(W, Z) U \left( \frac{Z^\alpha}{W} \right) - N_{(r)}^{i, W}(W, Z) U \left( \frac{W_\alpha}{W} - Z^\alpha Z_\alpha \right) \right] dV +$$

$$\frac{1}{8\pi} \sum_i \int_{S^H_i} e^{U} \left( N_{(r)} U_{,\hat{n}} - N_{(r)}^{i, W} Z_{,\hat{n}} U + N_{(r)}^{i, Z} W_{,\hat{n}} U \right) dS.$$  

(22)

Note that the normal vector $\hat{n}_a$ points outward at $S^H_i$ and $dV$ is the proper volume element of $V$ or $V_i$, respectively. The covariant derivative with respect to the metric $g_{\alpha\beta}$ is denoted by a semicolon. The field equations in the vacuum region (3b) imply that the integrand vanishes there. Hence, we can write Weyl’s multipole moments as the contributions of the individual sources to the total Weyl moment

$$U(=) = \frac{1}{8\pi} \sum_i \int_{V_i} e^{U} \left[ - \frac{N_{(r)}(W, Z)}{W} \left( T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} \right) \xi^\alpha \xi^\beta + \right.$$  

$$N_{(r)}^{i, Z}(W, Z) U \left( \frac{Z^\alpha}{W} \right) - N_{(r)}^{i, W}(W, Z) U \left( \frac{W_\alpha}{W} - Z^\alpha Z_\alpha \right) \right] dV +$$

$$\frac{1}{8\pi} \sum_i \int_{S^H_i} e^{U} \left( N_{(r)} U_{,\hat{n}} - N_{(r)}^{i, W} Z_{,\hat{n}} U + N_{(r)}^{i, Z} W_{,\hat{n}} U \right) dS.$$  

(23)

The integrals in (23) are the source integrals or quasi-local expressions for the asymptotically defined Weyl moments and, thus, for the asymptotically defined Geroch-Hansen multipole moments. Note that the second derivatives of $W$ in Eq. (24) can be expressed by the energy momentum tensor using Eq. (3a). How the contributions of the black holes are related to the definitions of the multipole moments of isolated horizons in [14] will be investigated in a future work as well as the relation of the source integrals of $V_i$ to those of Dixon given in, e.g., [13].
Since the transformation from Weyl’s multipole moments to Geroch’s is non-linear except for the mass and the mass dipole, cf. (8), there is no linear superposition of the multipole contributions of the individual sources to the total Geroch multipole moments. Hence a mixing of the contributions of the individual sources takes place.

Of course, Eq. (23) can again be rewritten as a sum of surface integrals:

\[ U(r) = \frac{1}{8\pi} \sum_i \int_{S_i} e^{U}(N_{+i,W}^U - N_{+i,Z}^U) dS + \]

\[ \frac{1}{8\pi} \sum_i \int_{S_i} e^{U}(N_{-i,W}^U - N_{-i,Z}^U) dS. \]

(24)

It is obvious that our choice of continuation of \(\rho, \zeta\) into \(V\) is not unique and affects greatly the form of Eqs. (21)-(24), though not the value. In fact, any \(C^1\) extension of the scalars \(W, Z\) from \(V^C\) could be chosen. Thus, depending on the applications, other choices might be more appropriate.

4 Properties of the source integrals

In this section, we will discuss the consequences of Eq. (17) in more detail in the Newtonian and the general relativistic case. In the Newtonian case, Eq. (17) comprises the well known multipole definitions as we will show in the next section.

4.1 The Newtonian case

Suppose \(U\) is a solution to the Laplace equation in \(V^C\) (cf. Fig 1), then it follows by virtue of Green’s theorem from the Dirichlet and Neumann data:

\[ U(x) = -\frac{1}{4\pi} \int_S \left( G(x, y) \frac{\partial U(y)}{\partial y^a} - U(y) \frac{\partial G(x, y)}{\partial y^a} \right) \hat{n}^a dS_y, \]

(25)

where \(G(x, y)\) denotes an arbitrary Green’s function, \(dS_y\) a surface element of the boundary \(S\) and \(x \in V^C\). The integration is over \(y \in S\) and \(\hat{n}^a\) is the inward pointing with respect to \(V^C\) unit normal to \(S\) at \(y\).

Eq. (25) is equivalent to Eq. (21) in case one restricts \(x\) to the axis and makes an expansion in \(|x|^{-1}\) for the special choice \(G(x, y) = -\frac{1}{4\pi|x-y|}\). This yields surface integrals in the form of (21) with the factor \(e^U\) set to 1, a flat space surface element and with \(W = \rho\) and \(Z = \zeta\). Thus, the multipole moments still contain all the information as Eq. (25) with the difference that the latter is not accessible for stationary, axially symmetric and isolated sources in general relativity. This is the reason, why we chose the approach using the linear system.
If Stoke’s theorem is applied for these surface integrals, we arrive at

\[ U^{(r)} = \frac{1}{8\pi} \int V \left( \frac{N^{(r)}}{\rho} \Delta^{(3)} U \right) dV = \frac{1}{2} \int V \left( \frac{N^{(r)}}{\rho} \mu \right) dV, \quad (26) \]

where we made use of the Poisson equation for the Newtonian gravitational potential, \( \Delta^{(3)} U = 4\pi \mu \), with a mass density \( \mu \). These are, of course, the usual multipole moments in source integral form up to a sign. This justifies the term \'source integrals\' also for the equivalently obtained expressions \((25)\) in curved spacetime. A comparison with the well-known formulas of Newtonian theory shows that

\[ N^{(k)} = -2\rho r^k P_k(\cos \theta), \quad \forall r \geq 0 \quad (27) \]

with polar coordinates \((r, \theta)\) defined as usual: \( \rho = r \cos \theta, \, \zeta = r \sin \theta \). \( P_k \) denote the Legendre polynomials of the first kind. In the general relativistic case, these \( N^{(k)} \) depend on the extensions of \( \rho, \zeta \) into the interior and thus become polynomials in the scalars \( W \) and \( Z \).

### 4.2 The general relativistic case

In this section, we will discuss some of the properties of the quasi-local volume integral given in Eq. \((22)\). The Einstein equations are non-linear and contain already the equations of motion (Bianchi identity). Hence, we could not expect a result like \((26)\), which depends only on the mass density. We rather find source integrals \((23)\) containing terms that are not expressed explicitly by the matter distribution (all but the first term). However, all of these terms vanish in vacuum. They also vanish in matter distributions, for which we can choose \( W = \rho \) in \( V \). In those case the source integrals have the same form as in the Newtonian case.

Of course, the first multipole moment coincides with the Geroch mass and, hence, must coincide with the well-known Komar mass. With Eq. \((8)\) and Eq. \((15)\) we have \( N^{(0)}_+ = 0 \) and \( N^{(0)}_- = -2W \) such that only the first term of the integrand in Eq. \((22)\) remains:

\[ M = \frac{1}{4\pi} \sum_i \int_{V_i} \frac{R_{ab}}{\sqrt{-g}} \frac{\xi^a \xi^b}{\xi^c \xi^c} \, dV + \frac{1}{4\pi} \sum_i \int_{S^N_i} e^U U_{;a} \, dS \]

This is, of course, exactly Komar’s integral of the mass in static spacetimes. The black hole contributions can also be cast in the standard form:

\[ M_{S^N_i} = \frac{1}{4\pi} \int_{S^N_i} e^U U_{;a} \, dS = \frac{1}{8\pi} \int_{S^N_i} \epsilon_{\alpha \beta \gamma \delta} \xi^a \xi^b \xi^c \xi^d \quad (28) \]

Although the main goal of this paper is to present the derivation and definition of the source integrals, we will give here a short application. We show that static, axially symmetric and isolated dust configurations do not exist. This is an old result\(^7\) but can easily be recovered using source integrals. This demon-

\(^7\) For more general non-existence results for dust, see also [38–40].
strates also how these quasi-local expressions can be employed. Static and axially symmetric dust configurations are characterized by the energy-momentum tensor in Weyl coordinates

\[ T_{ab} = \mu e^{2U} \delta_a^r \delta_b^s. \]  

(29)

The Bianchi identity implies \( U_{,a} = 0 \) in \( V \) and, thus, at \( S \). This yields together with the quasi-local surface integrals for the Weyl moments Eq. (19)

\[ U^{(r)} = 0. \]  

(30)

Thus, the system has no mass or any other multipole moment, which implies flat space in the vacuum region. This is clearly a contradiction to a dust source with positive mass density.

5 Conclusions

We have derived in this article source integrals or quasi-local expressions for Weyl’s multipole moments and, thus, for Geroch’s multipole moments for axially symmetric and stationary sources. These source integrals can either be written as surface integrals or volume integrals. A priori, one could not expect to find any kind of source integrals at all, because of the non-linear nature of the Einstein equations. That this is possible in the here considered setting, seems not to be due to the staticity and axially symmetry and the peculiarly simple form of the field equations. But rather a linear system must be available offering a notion of integrability of the Einstein equation. Thus, it appears feasible to find source integrals not only for stationary and axially symmetric isolated systems, which describe vacuum, but also electrovacuum close to spatial infinity. These generalizations will be investigated in future work.

It should also be clarified, how the source integrals are connected to the already known source integrals for isolated horizons [14]. In [14] it was shown that the source integrals characterize the horizon uniquely. However, they do not reproduce the GH multipole moments of a Kerr black hole. In our approach, the agreement of the source integrals and the asymptotically defined Weyl or Geroch multipole moments is given by construction. Therefore, these source integrals might prove useful for identifying the contributions to the multipole moments, which yield the discrepancies between the isolated horizon multipole moments and those of Geroch and Hansen.

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References

[1] K. S. Thorne, Rev. Mod. Phys. **52**, 299 (1980).
[2] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, Proc. Roy. Soc. London A **269**, 21 (1962).
[3] R. K. Sachs, Proc. Roy. Soc. London A **270**, 103 (1962).
[4] R. Geroch, J. Math. Phys. **11**, 2580 (1970).
[5] R. O. Hansen, J. Math. Phys. **15**, 46 (1974).
[6] W. Simon and R. Beig, J. Math. Phys. **24**, 1163 (1983).
[7] A. I. Janis and E. T. Newman, J. Math. Phys. **6**, 902 (1965).
[8] E. T. Newman and T. W. J. Unti, J. Math. Phys. **3**, 891 (1962).
[9] R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. **122**, 997 (1961).
[10] A. Komar, Phys. Rev. **113**, 934 (1959).
[11] H. Quevedo, Fortschritte der Physik. **38**, 733 (1990).
[12] Y. Gürsel, Gen. Relat. Gravit. **15**, 737 (1983).
[13] W. G. Dixon, Gen. Relat. Gravit. **4**, 199 (1973).
[14] A. Ashtekar, J. Engle, T. Pawlowski, and C. v. d. Broeck, Classical Quant. Grav. **21**, 2549 (2004).
[15] L. B. Szabados, Living Rev. Relat. **12**, 4 (2009).
[16] D. Maison, Phys. Rev. Lett. **41**, 521 (1978).
[17] V. A. Belinskii and V. E. Zakharov, J. Exp. Theor. Phys. **48**, 985 (1978).
[18] G. Neugebauer, J. Phys. A: Math. Gen. **12**, L67 (1979).
[19] K. Boshkayev, H. Quevedo, and R. Ruffini, (2012), arXiv:gr-qc/1207.3043.
[20] M. Bradley, D. Eriksson, G. Fodor, and I. Récz, Phys. Rev. D **75**, 024013 (2007).
[21] R. Meinel, M. Ansorg, A. Kleinwächter, G. Neugebauer, and D. Petroff, Relativistic Figures of Equilibrium, Cambridge University Press, (2008).
[22] J. A. Cabezas, J. Martín, A. Molina, and E. Ruiz, Gen. Relat. Gravit. **39**, 707 (2007).
[23] G. Pappas and T. A. Apostolatos, Phys. Rev. Lett. **108**, 231104 (2012).
[24] V. S. Manko and E. Ruiz, Classical Quant. Grav. **21**, 5849 (2004).
[25] C. Teichmüller, M. B. Fröb, and F. Maucher, Classical Quant. Grav. 28, 155015 (2011).

[26] N. Stergioulas, Living Rev. Relat. 6 (2003).

[27] W. Kundt and M. Trümper, Z. Phys. 192, 419 (1966).

[28] E. Ayón-Beato, C. Campuzano, and A. A. García, Phys. Rev. D 74, 024014 (2006).

[29] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, Exact solutions of Einstein’s field equations, Cambridge University Press (2003).

[30] R. Beig and R. M. Schoen, Classical Quant. Grav. 26,075014 (2009).

[31] J. M. Bardeen, Rapidly rotating stars, disks, and black holes., In Black Holes (Les Astres Occlus), edited by C. Dewitt and B. S. Dewitt, 1973.

[32] G. Fodor, C. Hoenselaers, and Z. Perjés, J. Math. Phys. 30, 2252 (1989).

[33] C. Cederbaum, The Newtonian Limit of Geometrostatics, PhD Thesis (July, 2012), arXiv:gr-qc/1201.5433.

[34] J. L. Hernández-Pastora, Classical Quant. Grav. 27, 045006 (2010).

[35] T. Bäckdahl and M. Herberthson, Classical Quant. Grav. 22, 1607 (2005).

[36] G. Neugebauer and R. Meinel, J. Math. Phys. 44, 3407 (2003).

[37] W. Israel, Nuovo Cimento B Serie 44, 1 (1966).

[38] A. Caporali, Phys. Lett. A 66, 5 (1978).

[39] N. Gürlebeck, Gen. Relat. Gravit. 41, 2687 (2009), 1105.2316.

[40] H. Pfister, Classical Quant. Grav. 27, 105016 (2010).