Resolvent of the parallel composition and the proximity operator of the infimal postcomposition

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Abstract

In this paper we provide the resolvent computation of the parallel composition of a maximally monotone operator by a linear operator under mild assumptions. Connections with a modification of the warped resolvent are provided. In the context of convex optimization, we obtain the proximity operator of the infimal postcomposition of a convex function by a linear operator and we relax full range conditions on the linear operator to mild qualification conditions. We also introduce a generalization of the proximity operator involving a general linear bounded operator leading to a generalization of Moreau’s decomposition for composite convex optimization.

Keywords Parallel composition · Infimal postcomposition · Monotone operator theory · Proximity operators · Qualification conditions

1 Introduction

In this paper we aim at computing the resolvent of the parallel composition of $A$ by $L$ [3], defined by

$$L \triangleright A = \left( LA^{-1} L^* \right)^{-1},$$

where $\mathcal{H}$ and $\mathcal{G}$ are real Hilbert spaces, $A : \mathcal{H} \to 2^{\mathcal{H}}$ and $L : \mathcal{H} \to \mathcal{G}$ is linear and bounded. In the case when $\mathcal{H} = H \oplus H$ for some real Hilbert space $H$, $\mathcal{G} = H$, $A : (x,y) \mapsto Bx \times Cx$ for some set-valued operators $B$ and $C$ defined in $H$, and $L : (x,y) \mapsto x + y$, we have $L \triangleright A = B \Box C$ [2, Example 25.40], where $B \Box C = (B^{-1} + C^{-1})^{-1}$ is the parallel sum of $B$ and $C$, motivating the name of the
operation. The parallel composition appears naturally in composite monotone inclusions. Indeed, if \( B : G \mapsto 2^G \), the dual inclusion associated to

\[
0 \in A^* BLx,
\]

is

\[
0 \in B^{-1} u + (-L \triangleright A)^{-1} u.
\]

When \( L^* L = \alpha \Id \) for some \( \alpha \geq 0 \) or when \( L^* \) has full range, explicit formulas for the resolvent of \( LA^{-1} L^* \) depending on the resolvent of \( A \) can be found in [2, Proposition 23.25]. In [17, 33] some variants and fixed point methods to compute the resolvent are proposed under full range condition on \( L^* \) and a similar fixed point approach is used in [25] under the maximal monotonicity of \( LA^{-1} L^* \). This computation is useful in [27] for the equivalence between the primal-dual [8, 32] and Douglas-Rachford splitting (DRS) [14, 21] algorithms.

In the particular case when \( A \) is the subdifferential of a convex function \( f : H \to [-\infty, +\infty] \) satisfying dual qualification conditions, we have that \( L \triangleright A \) is the subdifferential of the infimal postcomposition of \( f \) by \( L \), defined by

\[
L \triangleright f : G \to [-\infty, +\infty] : u \mapsto \inf_{x \in H} f(x). \quad (4)
\]

This operation appears naturally when dealing with the dual of composite optimization problems since we have that \( (L \triangleright f)^* = f^* \circ L^* \) under mild assumptions [2, Proposition 13.24(iv)]. Moreover, it is related to the parallel composition via the identities

\[
L \triangleright (\partial f) = (L(\partial f^*)L^*)^{-1} = (\partial(f^* \circ L^*))^{-1} = \partial(f^* \circ L^*)^* = \partial(L \triangleright f), \quad (5)
\]

where the second equality holds if, e.g., \( 0 \in \text{sri} (\text{dom} f^* - \text{ran} L^*) \) [2, Corollary 16.53]. Therefore, under previous assumption the resolvent of \( L \triangleright (\partial f) \) and the proximity operator of \( L \triangleright f \) coincide. Moreover, since the the alternating direction method of multipliers (ADMM) for solving \( \inf (f + g^\circ L) \) is an application of DRS to the Fenchel-Rockafellar dual \( \inf (f^* \circ (-L^*) + g^*) \) [18] (see also [5, 6, 11, 12, 28]), the computation of the proximity operator of \( L \triangleright f \) is relevant in the derivation of ADMM. In the literature, several additional hypotheses have been assumed in order to ensure that the iterates of ADMM are well defined and that it converges. In particular, in [11, Theorem 5.7], the operator \( (\partial f + L^* L)^{-1} L^* \) is assumed to be single-valued, in [28, Proposition 5.2] it is assumed to have full domain, and in [5, 12] the strong monotonicity of \( (\partial f + L^* L) \) is assumed in order obtain full domain and single-valuedness. It is worth to notice that some fixed point approaches and algorithms for computing \( \text{prox}_{f^* \circ L^*} \) are proposed in [15, 22] in the context of sparse recovery in image processing.

In this paper we derive a formula for the resolvent of the parallel composition and for the proximity operator of the infimal postcomposition in a real Hilbert space with non-standard metric under mild assumptions. This is obtained from a formula of the resolvent of \( LA^{-1} L^* \) via the non-standard metric version of Moreau’s identity in [2, Proposition 23.34(iii)]. Our computation is related to a modification of
the warped resolvent defined in [7] (see [19] for a particular case). We extend and generalize [33] for parallel compositions and [5, 11, 12, 28] for infimal postcompositions related to ADMM and the well-posedness of its iterates. We also derive a generalization of Moreau’s decomposition [24] for composite maximally monotone operators and for composite convex optimization under standard assumptions by using a generalization of the proximity operator.

2 Notation and preliminaries

Throughout this paper \( \mathcal{H} \) and \( \mathcal{G} \) are real Hilbert spaces with the scalar product \( \langle \cdot | \cdot \rangle \) and associated norm \( \| \cdot \| \). The identity operator on \( \mathcal{H} \) is denoted by \( \text{Id} \). Let \( A : \mathcal{H} \to 2^{\mathcal{H}} \) be a set-valued operator. The domain of \( A \) is \( \text{dom } A = \{ x \in \mathcal{H} \mid Ax \neq \emptyset \} \), the range of \( A \) is \( \text{ran } A = \{ u \in \mathcal{H} \mid (\exists x \in \mathcal{H}) \ u \in Ax \} \), the graph of \( A \) is \( \text{gra } A = \{ (x, u) \in \mathcal{H} \times \mathcal{H} \mid u \in Ax \} \), the set of zeros of \( A \) is \( \text{zer } A = \{ x \in \mathcal{H} \mid 0 \in Ax \} \), the inverse of \( A \) is \( A^{-1} : u \mapsto \{ x \in \mathcal{H} \mid u \in Ax \} \), and its resolvent is \( J_A = (\text{Id} + A)^{-1} \). For every \( D \subset \mathcal{H} \), \( A|_D \) is the restriction of \( A \) to \( D \), which satisfies \( \text{dom } A|_D = \text{dom } A \cap D \) and, for every \( x \in D \), \( A|_D x = Ax \). The operator \( A \) is injective on \( D \) if

\[
(\forall x \in \mathcal{H})(\forall y \in \mathcal{H}) \quad Ax \cap Ay \cap D \neq \emptyset \quad \Rightarrow \quad x = y,
\]

and \( A \) is injective if it is injective on \( \mathcal{H} \). It is clear that injectivity of \( A \) on \( D \) implies its injectivity on \( D' \) when \( D' \subset D \). Moreover, the operator \( A \) is monotone if

\[
(\forall (x, u) \in \text{gra } A)(\forall (y, v) \in \text{gra } A) \quad \langle x - y | u - v \rangle \geq 0,
\]

\( A \) is strongly monotone if there exists \( \alpha > 0 \) such that

\[
(\forall (x, u) \in \text{gra } A)(\forall (y, v) \in \text{gra } A) \quad \langle x - y | u - v \rangle \geq \alpha \| x - y \|^2,
\]

and \( A \) is maximally monotone if it is monotone and, for every \( (x, u) \in \mathcal{H} \times \mathcal{H} \),

\[
(x, u) \in \text{gra } A \quad \Leftrightarrow \quad (\forall (y, v) \in \text{gra } A) \quad \langle x - y | u - v \rangle \geq 0.
\]

For every strongly monotone self-adjoint linear bounded operator \( U : \mathcal{H} \to \mathcal{G} \), we denote \( \langle \cdot | \cdot \rangle_U = \langle \cdot | U \cdot \rangle \) and \( \| \cdot \|_U = \sqrt{\langle \cdot | \cdot \rangle_U} \), which define an inner product and the associated norm in \( \mathcal{H} \), respectively.

We denote by \( \Gamma_0(\mathcal{H}) \) the class of proper lower semicontinuous convex functions \( f : \mathcal{H} \to ]-\infty, +\infty[ \). Let \( f \in \Gamma_0(\mathcal{H}) \). The Fenchel conjugate of \( f \) is defined by \( f^* : u \mapsto \sup_{x\in\mathcal{H}}(\langle x | u \rangle - f(x)) \), \( f^* \in \Gamma_0(\mathcal{H}) \), the subdifferential of \( f \) is the maximally monotone operator

\[
\partial f : x \mapsto \{ u \in \mathcal{H} \mid (\forall y \in \mathcal{H}) \ f(x) + \langle y - x | u \rangle \leq f(y) \},
\]

\( (\partial f)^{-1} = f^* \), the set of minimizers of \( f \) is denoted by \( \text{arg min}_{x \in \mathcal{H}} f(x) \), and we have that \( \text{zer } (\partial f) = \text{arg min}_{x \in \mathcal{H}} f(x) \). Given a strongly monotone self-adjoint linear operator \( U : \mathcal{H} \to \mathcal{H} \), we denote by
\[ \text{prox}_f^U : x \mapsto \arg \min_{y \in \mathcal{H}} \left( f(y) + \frac{1}{2} \| x - y \|_U^2 \right), \]  

and by \( \text{prox}_f = \text{prox}^{\text{Id}}_f \). We have [2, Proposition 24.24] (see also [9, Section 3])

\[ \text{prox}_f^U = U^{-\frac{1}{2}} \text{prox}_{f \circ U^{-\frac{1}{2}}} \frac{1}{U_{\frac{1}{2}}} \quad \text{for } U^{-1} \]  

and it is single-valued since the objective function in (11) is strongly convex. Moreover, it follows from [2, Proposition 23.34(iii)] that

\[ J_{U_A} + U J_{U^{-1}A^{-1}U^{-1}} = \text{Id}, \]  

and, in the case of convex functions, [2, Proposition 24.24] yields

\[ \text{prox}_f^U = \text{Id} - U^{-1} \text{prox}_{f^{-1}}^U \quad U = U^{-1} (\text{Id} - \text{prox}_{f^{-1}}^U) U. \]  

Given a non-empty set \( C \subset \mathcal{H} \), we denote by \( \text{span} \, C \) the closed span of \( C \), by \( \text{cone} \, C \) its conical hull. Let \( C \) be a non-empty closed convex subset of \( \mathcal{H} \). We denote by \( \text{sri} \, C = \{ x \in C \mid \text{cone} \,(C - x) = \text{span} \,(C - x) \} \) its strong relative interior, by \( \iota_C \in \Gamma_0(\mathcal{H}) \) the indicator function of \( C \), which takes the value 0 in \( C \) and \( +\infty \) otherwise, by \( P_C^U = \text{proj}_C^U \) the projection onto \( C \) with respect to \((\mathcal{H}, \langle \cdot , \cdot \rangle_U)\), and we denote \( P_C = P_C^\text{Id} \). It follows from (12) that

\[ P_C^U = U^{-\frac{1}{2}} \text{proj}_{\iota_C \circ U^{-\frac{1}{2}}} \frac{1}{U_{\frac{1}{2}}} \quad U = U^{-\frac{1}{2}} P_C \frac{1}{U_{\frac{1}{2}}} \frac{1}{U_{\frac{1}{2}}}. \]  

Given a linear bounded operator \( L : \mathcal{H} \to \mathcal{G} \), we denote its adjoint by \( L^* : \mathcal{G} \to \mathcal{H} \), its kernel (or null space) by \( \ker L \), its range by \( \text{ran} L \), and, if \( \text{ran} L \) is closed, its Moore-Penrose inverse by

\[ L^\dagger : \mathcal{G} \to \mathcal{H} : y \mapsto P_C y \]  

where \( C_y = \{ x \in \mathcal{H} \mid L^* L x = L^* y \} \). If \( L^* L \) is invertible, we have [2, Example 3.29]

\[ L^\dagger = (L^* L)^{-1} L^*. \]  

For definitions and properties of monotone operators, nonexpansive mappings, and convex analysis, the reader is referred to [2].

We now introduce a modification of the warped resolvent introduced in [7] (see also [19] for a particular case and applications). Let \( A : \mathcal{H} \to 2^{\mathcal{H}} \) be a set-valued operator and let \( K : \mathcal{H} \to \mathcal{H} \). The warped resolvent of \( A \) with kernel \( K \) is defined by \( J_A^K = (K + A)^{-1} K \). In the case when \( K \) is linear and invertible, we have

\[ J_A^K = (K + A)^{-1} K = (K(\text{Id} + K^{-1} A))^{-1} K = J_{K^{-1}A}, \]  

which has full domain and it is single-valued if \( K^{-1} A \) is maximally monotone. The following result characterizes the full domain and single-valuedness of \( J_A^K \) in a general context.
Proposition 1 Let $A : \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued operator and let $K : \mathcal{H} \to \mathcal{H}$. Then the following holds.

(i) $\text{dom} J^K_A = \mathcal{H} \Leftrightarrow \text{ran} K \subset \text{ran} (K + A)$.
(ii) $J^K_A$ is at most single-valued $\Leftrightarrow K + A$ is injective on $\text{ran} K$.

Proof (i): For every $x \in \mathcal{H}$ we have
\[
x \in \text{dom} J^K_A \Leftrightarrow (\exists u \in \mathcal{H}) \quad u \in (K + A)^{-1} K x
\Leftrightarrow (\exists u \in \mathcal{H}) \quad K x \in (K + A) u
\Leftrightarrow K x \in \text{ran} (K + A),
\]
and the result follows. (ii): First assume that $J^K_A$ is at most single-valued. In view of (6), let $x$ and $y$ in $\mathcal{H}$ and suppose that there exists $z \in \mathcal{H}$ such that $K z \in (K x + A x) \cap (K y + A y)$. Then $\{x\} \cup \{y\} \subset J^K_A z$ and single-valuedness of $J^K_A$ implies $x = y$, which yields the injectivity on $\text{ran} K$. Conversely, let $z \in \text{dom} J^K_A$ and let $x$ and $y$ in $J^K_A z$. Then, $K z \in (K x + A x) \cap (K y + A y) \cap \text{ran} K$ and injectivity on $\text{ran} K$ implies $x = y$. \qed

In [7, Definition 1.1] it is assumed that $K + A$ is injective in the whole space in order to guarantee that $J^K_A$ is single-valued, but this is a stronger assumption in general, as the following example illustrates.

Example 1 Let $\alpha > 0$, set $\mathcal{H} = \mathbb{R}$, set $K : x \mapsto \text{med}\{-1, x, 1\}$ be the median of real values $x$, $-1$, and $1$, and set $A = \alpha K$. Note that $A$ and $K$ are maximally monotone, single-valued, and $\text{ran} K = [-1, 1] \subset [-1 - \alpha, 1 + \alpha] = \text{ran} (K + A)$. Moreover, observe that $K + A = (1 + \alpha) \text{med}\{-1, \cdot, 1\}$ is injective on $\text{ran} K$ but it is not injective on $\mathbb{R}$, since $(K + A) 1 = (K + A) 2 = 1 + \alpha$.

The warped proximity operator of $f$ with kernel $K$ is defined by
\[
\text{prox}^K_f = J^K_{\partial f} = (K + \partial f)^{-1} K
\]
and note that it coincides with (12) when $K$ is strongly monotone, self-adjoint, linear, and bounded, in view of (18).

3 Resolvent of parallel composition

The following result is a generalization of [2, Proposition 23.25] and provides an explicit computation of the resolvent of $UM^*BM$ under mild assumptions.

Theorem 1 Let $\mathcal{H}$ and $\mathcal{G}$ be real Hilbert spaces, let $B : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximally monotone operator, let $M : \mathcal{G} \to \mathcal{H}$ be a linear bounded operator such that $M^*BM$ is maximally monotone in $\mathcal{G}$, and let $U : \mathcal{G} \to \mathcal{G}$ be a $\mu$–strongly monotone
self-adjoint linear operator for some \( \mu > 0 \). Then \( UM^*BM \) is maximally monotone in \((G, \langle \cdot, \cdot \rangle_{U^{-1}})\) and the following assertions hold:

(i) \( \text{ran } M \subseteq \text{dom } (MUM^* + B^{-1})^{-1} \) and 
\[
J_{UM^*BM} = \text{Id} - UM^*(MUM^* + B^{-1})^{-1}M.
\] (21)

(ii) \( \text{ran } (MUM^*) \subseteq \text{ran } (MUM^* + B^{-1}) \).

(iii) \( (MUM^* + B^{-1})_{|_{\text{ran } M}} \) is injective.

(iv) Suppose that \( \text{ran } M \) is closed. Then
\[
J_{UM^*BM} = \text{Id} - UM^*J_{B^{-1}}^{MUM^*}(\sqrt{UM^*})^\dagger \sqrt{U^{-1}}.
\] (22)

(v) Suppose that \( \text{ran } M = H \). Then
\[
J_{UM^*BM} = \text{Id} - UM^*J_{(MUM^*)^{-1}B^{-1}}(MUM^*)^{-1}M
\] (23)
\[
= P_{ker M}^{U^{-1}} + UM^*(MUM^*)^{-1}J_{MUM^*BM}.
\] (24)

**Proof** The maximal monotonicity of \( UM^*BM \) follows from [2, Proposition 20.24].

(i): For every \( x \) and \( p \) in \( G \), we have 
\[
p = J_{UM^*BM}x \iff x - p \in UM^*BMp
\]
\[
\iff (\exists v \in H) \begin{cases} x - p = UM^*v \\
\quad v \in BMp \end{cases}
\]
\[
\iff (\exists v \in H) \begin{cases} p = x - UM^*v \\
\quad Mp \in B^{-1}v \end{cases}
\]
\[
\iff (\exists v \in H) \begin{cases} p = x - UM^*v \\
\quad Mx \in MUM^*v + B^{-1}v. \end{cases}
\]
\[
\iff (\exists v \in (MUM^* + B^{-1})^{-1}Mx) \ p = x - UM^*v,
\]
and the result follows. (ii): It follows from (i) that 
\[
\text{ran } (MUM^*) \subseteq \text{ran } M \subseteq \text{dom } (MUM^* + B^{-1})^{-1} = \text{ran } (MUM^* + B^{-1}).
\] (26)

(iii): Let \( x \) and \( y \) in \( \text{ran } M \) be such that there exists \( u \in (MUM^*x + B^{-1}x) \cap (MUM^*y + B^{-1}y) \). Then, \( u - MUM^*x \in B^{-1}x \), \( u - MUM^*y \in B^{-1}y \), and the monotonicity of \( B^{-1} \) yields 
\[
0 \leq \langle -MUM^*(x - y) \mid x - y \rangle \\
= -\langle UM^*(x - y) \mid M^*(x - y) \rangle \\
\leq -\mu \|M^*(x - y)\|^2,
\] (27)
which implies \( x - y \in \ker M^* \). Since \( x - y \in \text{ran } M \subset \text{ran } M \), it follows from [2, Fact 2.25(iv)] that \( x - y \in \ker M^* \cap \text{ran } M = \{0\} \), which yields the result.

(iv): Denote by \( G_U \) the Hilbert space \( G \) endowed with the scalar product \( \langle \cdot | \cdot \rangle_U \). Note that \( M^*U = UM^*U^{-1} \), where \( M^*U \) and \( M^* \) are the adjoints of \( M \) in \( G_U \) and \( G \), respectively. Moreover, [2, Fact 2.25(iv)] and the closedness of \( \text{ran } M \) on \( G_U \) yield \( \ker M^*U = \ker M^* = \{0\} \), which yields the result.

\begin{align*}
\mathcal{G}_U = \ker M \oplus \text{ran } (UM^*)
\end{align*}

(28)
is an orthogonal decomposition of \( \mathcal{G}_U \). Hence, we have from [2, Proposition 24.24(ii) & Proposition 3.30(iii)] that

\begin{align*}
P_{\ker M}^{U^{-1}} &= \text{prox}_{\ker M}^{U^{-1}} \\
&= \sqrt{U} \text{prox}_{\ker M}^{U^{-1}} \sqrt{U} \\
&= \sqrt{U}P_{\ker M}^{U^{-1}} \sqrt{U} \\
&= \text{Id} - UM^*(\sqrt{UM^*})^\dagger \sqrt{U}^{-1},
\end{align*}

(29)

and

\begin{align*}
P_{\text{ran } (UM^*)}^{U^{-1}} &= UM^*(\sqrt{UM^*})^\dagger \sqrt{U}^{-1},
\end{align*}

(30)

where \( (\sqrt{UM^*})^\dagger \) is the Moore-Penrose inverse of \( \sqrt{UM^*} : \mathcal{H} \to \mathcal{G} \). Therefore, (i) asserts that

\begin{align*}
J_{UM^*BM} &= \text{Id} - UM^*(B^{-1} + MUM^*)^{-1}M \\
&= \text{Id} - UM^*(B^{-1} + MUM^*)^{-1}M P_{\text{ran } (UM^*)}^{U^{-1}} \\
&= \text{Id} - UM^*(B^{-1} + MUM^*)^{-1}MUM^*(\sqrt{UM^*})^\dagger \sqrt{U}^{-1} \\
&= \text{Id} - UM^*JMUM^*(\sqrt{UM^*})^\dagger \sqrt{U}^{-1},
\end{align*}

(31)

where in the last equality \( J_{B^{-1}}^{MUM^*} \) has full domain in view of (ii) and Proposition 1(i).

(v): Since \( \text{ran } M = \mathcal{H} \) is closed and \( U \) is \( \mu \)-strongly monotone for some \( \mu > 0 \), \( MUM^* \) is strongly monotone and, thus, invertible. Indeed, for every \( v \in \mathcal{H} \), [2, Fact 2.26] implies that there exists \( \alpha > 0 \) such that

\begin{align*}
\langle MUM^*v | v \rangle = \langle UM^*v | M^*v \rangle \geq \mu \|M^*v\|^2 \geq \mu \alpha^2 \|v\|^2.
\end{align*}

(32)

Hence, since \( (\sqrt{UM^*})^*(\sqrt{UM^*}) = MUM^* \), (23) follows from (iv), (18), and (17). Moreover, since (29) and (17) yield \( P_{\ker M}^{U^{-1}} = \text{Id} - UM^*(MUM^*)^{-1}M \), (24) follows from (23) and (13).
Remark 1

(i) Note that Theorem 1(i) provides the existence of zeros of the monotone operator $MUM^* + B^{-1}$ from the maximal monotonicity of $M^*BM$, which is guaranteed, e.g., if cone \(\text{ran } M - \text{dom } B = \text{span} (\text{ran } M - \text{dom } B)\) [2, Corollary 25.6] (see [4] for a weaker assumption involving the domain of the Fitzpatrick function).

(ii) Note that, from Theorem 1(i), $M^*(MUM^* + B^{-1})^{-1}M : \mathcal{G} \to \mathcal{G}$ is single-valued, even if $(MUM^* + B^{-1})^{-1}$ can be a self-valued mapping. Indeed, for every $x \in \mathcal{G}$, let $v$ and $w$ in $(MUM^* + B^{-1})^{-1}Mx$. Then, $M(x - UM^*v) \in B^{-1}v$ and $M(x - UM^*w) \in B^{-1}w$ and the monotonicity of $B^{-1}$ yields

$$0 \leq \langle -MU(M^*v - M^*w) | v - w \rangle = -\|M^*v - M^*w\|_U^2,$$

which implies $M^*v = M^*w$. This computation is consistent with the fact that the resolvent of the monotone operator $UM^*BM$ is single-valued.

(iii) Observe that Theorem 1(ii) and Proposition 1(i) imply that dom $J_{MUM^*}^{BM}$ = $\mathcal{H}$. On the other hand, the single-valuedness of $J_{B^{-1}}$ is not guaranteed since $MUM^* + B^{-1}$ is not necessarily injective on ran $(MUM^*)$ (see Proposition 1(ii)). Indeed, suppose that ker $M^* \neq \{0\}$ and that $B^{-1} = N_C$, where $C$ is the closed ball centered at 0 with radius 1. By taking $x = 0$ and $y \in (\ker M^* \setminus \{0\}) \cap \text{int } C$, we have $\{0\} = N_{C,x} \cap N_{C,y} = (MUM^*x + B^{-1}x) \cap (MUM^*y + B^{-1}y)$, $0 \in \text{ran } MUM^*$, and $x \neq y$. Since $0 = MUM^*0$, this implies $\{x, y\} \subset J_{B^{-1}}MUM^*$ and, thus, $J_{B^{-1}}MUM^*$ is not single-valued. However, when ran $M$ is closed, it follows from Theorem 1(iii) and ran $(\sqrt{UM^*})^\dagger = \text{ran } (M \sqrt{U}) = \text{ran } M$ [2, Proposition 3.30(v)] that $J_{B^{-1}}MUM^* (\sqrt{UM^*})^\dagger$ is single-valued.

(iv) In the particular case when $U = \text{Id}$, Theorem 1(v) coincides with [2, Proposition 23.25]. On the other hand, when $M = \text{Id}$ and $\mathcal{H} = \mathcal{G}$, we recover from Theorem 1(v) the Moreau’s decomposition with non-standard metric in [2, Proposition 23.34(iii)] recalled in (13).

We conclude this section with the computation of the resolvent of the parallel composition $L \triangleright A$.

Corollary 1 Let $A : \mathcal{H} \to 2^\mathcal{H}$ be a maximally monotone operator, let $L : \mathcal{H} \to \mathcal{G}$ be a linear bounded operator such that $LA^{-1}L^*$ is maximally monotone in $\mathcal{G}$. Moreover, let $U : \mathcal{G} \to \mathcal{G}$ be a self-adjoint strongly monotone linear bounded operator. Then, $U(L \triangleright A)$ is maximally monotone in $(\mathcal{G}, \langle \cdot \mid \cdot \rangle_{U^{-1}})$ and the following holds:

(i) $J_{U(L \triangleright A)} = L(A + L^*U^{-1}L)^{-1}L^*U^{-1}$.

(ii) Suppose that ran $L$ is closed. Then,

$$J_{U(L \triangleright A)} = LJ_A^{-1}U^{-1}(\sqrt{U^{-1}}L)^\dagger \sqrt{U^{-1}}.$$

(iii) Suppose that ran $L^* = \mathcal{H}$. Then,
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Proof Since $L \triangleright A = (LA^{-1}L^*)^{-1}$, the maximal monotonicity of $U(L \triangleright A)$ follows from [2, Propositions 20.22 & 20.24]. (i) By applying Theorem 1(i) to $B = A^{-1}$ and $M = L^*$, it follows from (13) that

$$J_{U(L \triangleright A)} = L J_{(L^*U^{-1}L)^{-1}A} (L^*U^{-1}L)^{-1}L^*U^{-1}. \quad (35)$$

(ii) By applying Theorem 1(iv) to $B = A^{-1}$ and $M = L^*$, we obtain

$$J_{U(L \triangleright A)} = L (A + L^*U^{-1}L)^{-1}L^*U^{-1}. \quad (36)$$

(iii) As in the proof of Theorem 1(iv), $L^*U^{-1}L$ is strongly monotone and, hence, invertible, and the result follows from (ii), (18), and (17). \hfill \Box

4 Proximity operator of the infimal postcomposition

For every $f \in \Gamma_0(H)$, every linear bounded operator $L : H \to G$, and every strongly monotone self-adjoint linear bounded operator $U : G \to G$, define

$$\text{prox}_{f,L}^U : G \to 2^H : u \mapsto \arg\min_{x \in H} \left( f(x) + \frac{1}{2} \|Lx - u\|_U^2 \right). \quad (38)$$

Note that [2, Theorem 16.3 & Theorem 16.47(i)] yield

$$(\forall u \in G)(\forall x \in H) \quad x \in \text{prox}_{f,L}^U u \iff 0 \in \partial f(x) + L^*U(Lx - u) \iff x \in (\partial f + L^*UL)^{-1}L^*Uu. \quad (39)$$

When $L = \text{Id}$, we have $\text{prox}_{f,\text{Id}}^U = \text{prox}_{f}^U$ and it is single-valued with full domain. In [1, 26] (see also [20]) an extension of definition of the classical proximity operator is studied by considering a Bregman distance instead of $\| \cdot \|_U^2$, under the assumption of uniqueness of the solution to the optimization problem in (38). In our context, the single-valuedness of $\text{prox}_{f,L}^U$ is not needed. The following result provides some properties of $\text{prox}_{f,L}^U$ in more general contexts.

Proposition 2 Let $f \in \Gamma_0(H)$, let $L : H \to G$ be a linear bounded operator, and let $U : G \to G$ be a strongly monotone self-adjoint linear bounded operator. Then, the following hold:
(i) Let $\mu > 0$ be the strong monotonicity parameter of $U$. For every $u \in \text{dom} \, \text{prox}_f^U$, $L(\text{prox}_f^U u)$ and $P_{(\ker L)^\perp}(\text{prox}_f^U u)$ are singletons.

(ii) Suppose that $\ker L = \{0\}$. Then, for every $u \in \text{dom} \, \text{prox}_f^U$, $\text{prox}_f^U u$ is a singleton.

(iii) Suppose that $\text{ran} \, L$ is closed. Then
\[
(\forall u \in \text{dom} \, \text{prox}_f^U) \quad \text{prox}_f^U u = \text{prox}_{\sqrt{UL}}^U(\sqrt{UL})^\perp \sqrt{UL}u.
\]  

(iv) Suppose that $\text{ran} \, L^* = \mathcal{H}$. Then $\text{prox}_f^U$ is single-valued, $\text{dom} \, \text{prox}_f^U = \mathcal{G}$, and
\[
(\forall u \in \mathcal{G}) \quad \text{prox}_f^U u = \{ \text{prox}_{L^*UL}^U(L^*UL)^{-1}L^*Uu \}.
\]

**Proof** (i): Let $x_1$ and $x_2$ in $\text{prox}_f^U u$. It follows from (39) applied to $x_1$ and $x_2$, the monotonicity of $\partial f$, and strong monotonicity of $U$ that
\[
0 \leq \langle -L^*UL(x_1 - x_2) \mid x_1 - x_2 \rangle = -\langle UL(x_1 - x_2) \mid L(x_1 - x_2) \rangle \\
\leq -\mu \|L(x_1 - x_2)\|^2.
\]  

Therefore, $L(x_1 - x_2) = 0$ which leads to $x_1 - x_2 \in \ker L$ and, hence, $P_{(\ker L)^\perp}x_1 = P_{(\ker L)^\perp}x_2$.

(ii): In this case $(\ker L)^\perp = \mathcal{H}$, which yields $P_{(\ker L)^\perp} = \text{Id}$ and the result follows from (i).

(iii): Note that the orthogonal decomposition in $(\mathcal{G}, \langle \cdot \mid \cdot \rangle_{U^{-1}})$ in (28) and (30) with $M = L^*$ implies that $U = P_{\text{ran}(UL)}^{U^{-1}}U + P_{\ker L^*}^{U^{-1}}U$. Therefore, it follows from (39) that, for every $x \in \mathcal{G}$ and $u \in \mathcal{H}$,
\[
x \in \text{prox}_f^U u \iff x \in (\partial f + L^*UL)^{-1}L^*P_{\text{ran}(UL)}^{U^{-1}}Uu \\
\iff x \in (\partial f + L^*UL)^{-1}L^*(UL(\sqrt{UL})^\perp \sqrt{UL}u) \\
\iff x \in \text{prox}_f^{L^*UL}(\sqrt{UL})^\perp \sqrt{UL}u,
\]  

where the last equivalence follows from (20).

(iv): Note that $\text{ran} \, L^* = \mathcal{H}$ yields, for every $x \in \mathcal{G}$, $\langle L^*ULx \mid x \rangle \geq \mu \|Lx\|^2 \geq \mu \alpha^2 \|x\|^2$, where the existence of $\alpha > 0$ is guaranteed by [2, Fact 2.26]. Therefore, $L^*UL$ is strongly monotone and, hence, invertible. Hence, the result follows from (iii) and $(\sqrt{UL})^\perp = (L^*UL)^{-1}L^*U$ in view of (17). \hfill $\square$

The following result provides sufficient conditions ensuring full domain of $\text{prox}_f^U$. This is a consequence of Theorem 1 in the optimization context and we connect the existence result with the computation of the proximity operators of...
Proposition 3 Let \( \mathcal{H} \) and \( \mathcal{G} \) be real Hilbert spaces, let \( f \in \Gamma_0(\mathcal{H}) \), let \( L : \mathcal{H} \to \mathcal{G} \) be a linear bounded operator such that

\[
0 \in \text{sri} (\text{dom} f^* - \text{ran} L^*),
\]

and let \( U : \mathcal{G} \to \mathcal{G} \) be a strongly monotone self-adjoint linear operator. Then, the following hold:

(i) \( \text{dom} \, \text{prox}^U_{f^* L} = \mathcal{G} \).
(ii) \( \text{prox}^U_{f^* L^*} = \text{Id} - UL \text{prox}^U_{f^*} U^{-1} \).
(iii) \( L \triangleright f = (f^* o L^*)^* \in \Gamma_0(\mathcal{H}) \) and \( \text{prox}^U_{L^* f} = L \text{prox}^U_{f^* L} \).

Proof (i): Since \( 0 \in \text{sri} (\text{dom} f^* - \text{ran} L^*) \), [2, Corollary 16.53(i)] yields \( \partial (f^* o L^*) = L (\partial f^*) L^* \), which is maximally monotone in \( \mathcal{H} \) because \( f^* o L^* \in \Gamma_0(\mathcal{H}) \) [2, Theorem 20.25]. Hence, by applying Theorem 1(i) to \( B = \partial f^* \) and \( M = L^* \), it follows from (39) that

\[
(\forall x \in \mathcal{H}) \quad \emptyset \neq ((\partial f^*)^{-1} + L^* UL)^{-1} L^* U x = (\partial f + L^* UL)^{-1} L^* U x = \text{prox}^U_{f^* L} x.
\]

(ii): We deduce from (12), Theorem 1(i), and (45) that

\[
\text{prox}^U_{f^* L^*} = J_{U (\partial f^* L^*)} = J_{UL (\partial f^*) L^*} = \text{Id} - UL (\partial f^*)^{-1} - L^* UL)^{-1} L^* = \text{Id} - UL \text{prox}^U_{f^*} U^{-1}.
\]

(iii): Since \( f^* o L^* \in \Gamma_0(\mathcal{H}) \), (44) and [2, Corollary 15.28] yield

\[
L \triangleright f = (f^* o L^*)^* \in \Gamma_0(\mathcal{H}).
\]

Hence, it follows from (14) and (ii) that

\[
\text{prox}^U_{L^* f} = \text{Id} - U^{-1} \text{prox}^U_{f^* L^*} U = \text{Id} - U^{-1} (\text{Id} - UL \text{prox}^U_{f^*} U^{-1}) U = L \text{prox}^U_{f^* L}
\]

and the proof is complete. \( \square \)

Without the qualification condition (44), \( \text{prox}^U_{f^* L} u \) may be empty for some \( u \in \mathcal{G} \), as the following examples illustrate.

Example 2 Suppose that \( U = \text{Id} \), that \( \text{ran} L \) is not closed, set \( f = 0 \), and let \( u \in \text{ran} L \setminus \text{ran} L \). Then, \( \text{inf}_{u \in \mathcal{H}} \| L x - u \| = 0 \) but the infimum is not attained. Observe that, since \( f^* = \iota_{\{0\}} \), we have \( \text{dom} f^* = \{0\} \) which yields \( \text{cone} (\text{dom} f^* - \text{ran} L^*) = \text{cone} (\text{ran} L^*) = \text{ran} L^* \neq \text{ran} L^* = \text{span} \text{ran} L^* \) and, thus, \( 0 \not\in \text{sri} (\text{dom} f^* - \text{ran} L^*) \).

Example 3 Suppose that \( \mathcal{H} = \mathbb{R}^2 \), \( \mathcal{G} = \mathbb{R} \), \( f : (x, y) \mapsto \exp(y) \), and \( L : (x, y) \mapsto x \). Then \( L^* : z \mapsto (z, 0) \), \( \text{ran} L^* = \mathbb{R} \times \{0\} \), and \( f^* : (u, v) \mapsto \iota_{\{0\}}(u) + \exp(v) \), where
Then, \( \text{dom} f^* = \{0\} \times [0, +\infty[ \) and cone \((\text{dom} f^* - \text{ran} L^*)\) = \(\mathbb{R} \times [0, +\infty[ \neq \mathbb{R}^2 \)

\(= \text{span} (\text{dom} f^* - \text{ran} L^*)\), which yields \(0 \notin \text{sri} (\text{dom} f^* - \text{ran} L^*)\).

**Remark 2**

(i) In the case when \(U = \mu \text{Id}\), the existence of solutions to \((38)\) is assumed in \([28, \text{Proposition 5.2(iii)\} and its uniqueness is supposed in \([11, \text{Theorem 4.7}\]. On the other hand, the strong monotonicity of \((L^*L + df)\) is assumed in \([5, 12]\) in order to guarantee the existence and uniqueness of solutions to the optimization problem in \((38)\). Under previous assumptions, the sequences of ADMM are proved to be well defined. Proposition 3(i) provides the existence of solutions to \((38)\) under the weaker condition \((44)\). It is deduced from Theorem 3 and the maximal monotonicity of \(L(df^*)L^*\), which is obtained from the qualification condition \(0 \in \text{sri} (\text{dom} f^* - \text{ran} L^*)\) and \(f^*L^* \in \Gamma_0(\mathcal{H})\) in view of \([2, \text{Corollary 16.53(i) & Theorem 20.25}\]. Moreover, it follows from Proposition 3(iii) that \(L \text{prox}^U_{f, L} \) is single-valued, which implies that the sequences generated by ADMM are well defined. In summary, under the weaker condition \(0 \in \text{sri} (\text{dom} f^* - \text{ran} L^*)\) we guarantee the existence and uniqueness of the sequences generated by ADMM under our approach, generalizing \([5, 11, 12, 28]\). A general convergence result of ADMM in this context is provided in \([6, \text{Theorem 4.6}\].

(ii) In \([15, 22]\) fixed point approaches are used in order to compute \(\text{prox}_{f \circ L}^U\) in the context of the sparse recovery in image processing. This approach leads to sub-iterations in optimization algorithms needing to compute \(\text{prox}_{f \circ L}^U\). From Proposition 3(ii) in the case when \(U = \text{Id}\), our computation is direct once \(\text{prox}_{f^* \circ L^*}^U\) is easily computable. This is the case, for instance, when \(\mathcal{H} = \mathbb{R}^n\), \(L\) is a \(n \times m\) real matrix with \(m > n\), and \(f : x \mapsto x^\top Ax/2 - z^\top x\), where \(A\) is an symmetric positive definite \(n \times n\) real matrix and \(z \in \mathbb{R}^n\). This setting appears, e.g., in signal and image processing \([10, 13, 16]\) and statistics \([29– 31]\). In this case, the computation of \(\text{prox}_{f^* \circ L^*}^U\) needs the inversion of the \(n \times n\) real matrix \(A^{-1} + LL^\top\), while \(\text{prox}_{f \circ L}^U\) need the inversion of the \(m \times m\) real matrix \(\text{Id} + L^\top AL\), which can be more expensive numerically when \(n << m\).

(iii) Note that Proposition 3(i) yields

\[\text{prox}^U_{f^* \circ L^*} + UL \text{prox}^U_{f, L} U^{-1} = \text{Id}.\]  

\((48)\)

In the case when \(L = \text{Id}\), since \(\text{prox}^U_{f, \text{Id}} = \text{prox}^U_f\), \((48)\) reduces to \([2, \text{Proposition 24.24(ii)\} , which is a non-standard metric version of Moreau’s decomposition \([24]\) first derived for mutually polar cones \([23]\).
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Declarations

Conflict of interest The authors have no conflict of interest to declare that are relevant to the content of this article.

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