GLOBAL EXISTENCE AND POINTWISE DECAY FOR THE NULL CONDITION

SHI-ZHUO LOOI AND MIHAI TOHANEANU

Abstract. We study the nonlinear wave equation with the null condition and small initial data on nonstationary spacetimes. Under the assumption that the solution to the linear equation satisfies local energy bounds, we prove global existence of solutions, and obtain sharp pointwise bounds.

1. Introduction

In this paper, we study the wave equation with the null condition on a variety of spacetimes. The goal is twofold: to prove global existence, and to obtain sharp pointwise decay.

The paper is structured as follows. Section 1 introduces our result, and some of the history of the problem. Section 2 contains some notation, a discussion of local energy estimates, and the rigorous statement of our main theorem. Sections 3 and 4 contain the proof of global existence. Sections 5, 6 and 7 are dedicated to the sharp pointwise bounds.

1.1. Statement of the result. We consider the operator

\[ P := \partial_\alpha g^{\alpha\beta}(t,x)\partial_\beta + g^{\omega}(t,x)\Delta_\omega + B^\alpha(t,x)\partial_\alpha + V(t,x). \]  

(1.1)

Here \( \Delta_\omega \) denotes the Laplace operator on the unit sphere, and \( \alpha, \beta \) range across \( 0, \ldots, 3 \). The main assumptions on \( P \) are that it is hyperbolic, asymptotically flat, and that the linear evolution satisfies strong local energy decay (and thus the Hamiltonian flow must be nontrapping); the precise conditions on the potential \( V \), the coefficients \( B, g^\omega \) and the Lorentzian metric \( g \) are given in the main result, Theorem 2.2. On the other hand, we allow time-dependent coefficients, as well as large perturbations of \( \Box \).

We study the nonlinear Cauchy problem

\[ P\phi = S^{\alpha\beta}\partial_\alpha \phi \partial_\beta \phi, \quad (\phi(0), \partial_t \phi(0)) = (\phi_0, \phi_1), \]  

(1.2)

where \( S^{\alpha\beta} \in \mathbb{R} \) are constants such that \( S^{\alpha\beta} = S^{\beta\alpha} \), and

\[ S^{\alpha\beta} \xi_\alpha \xi_\beta = 0 \]  

(1.3)

for all \( \xi \) such that \( \xi_0^2 = \sum_{j=1}^3 \xi_j^2 \). We will also write \( S^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi = Q(\partial_\phi, \partial_\phi) \).

Our main theorem states, informally, that if the solution to the linear wave equation \( P\phi = F \) satisfies strong local energy bounds, then (1.2) with small initial data admits a unique global solution. Moreover, we prove global pointwise decay rates of \( (t-r)^{-1} (t+r)^{-1} \) for the solution and vector fields applied to it. The rate of decay coincides with the one obtained by Christodolou [6] in the case \( P = \Box \) by using the conformal method; we believe this rate to be sharp. See Theorem 2.2 for the precise statement.

1.2. History. The semilinear wave equation in \( \mathbb{R}^{1+3} \)

\[ \Box \phi = Q(\partial_\phi, \partial_\phi), \quad \phi|_{t=0} = \phi_0, \quad \partial_t \phi|_{t=0} = \phi_1 \]  

(1.4)

for small initial data has been studied extensively. It is known that the solution blows up in finite time if \( Q(\partial_\phi, \partial_\phi) = (\partial_\phi)^2 \), see [10]. On the other hand, if the nonlinearity satisfies the null condition (1.3), first identified by Klainerman [13], it was shown independently in [6] and [14] that the solution exists globally. This result was extended to quasilinear systems with multiple speeds, as well as the case of exterior domains; see, for instance, [26], [27], [28], [8], [29].
There have also been many works for small data in the variable coefficient case. Almost global existence for nontrapping metrics was shown in [5], [38]. Global existence for stationary, small perturbations of Minkowski was shown in [39], and for nonstationary, compactly supported perturbations in [46]. In the context of black holes, global existence was shown in [24] for Kerr space-times with small angular momentum, and in [4] for the Reissner-Nordström backgrounds. See also the upcoming [25] for sharp pointwise bounds and asymptotics, given certain assumptions, for a variety of nonlinearities.

The results of this paper can also be extended to the quasilinear problem for nontrapping metrics, as well as to the semilinear problem satisfying the null condition as long as weak local energy decay holds, such as on Schwarzschild and subextremal Kerr.

2. Notation and statement of the main theorem

2.1. Notation for implicit constants. We write \( X \lesssim Y \) to denote \( |X| \leq CY \) for an implicit constant \( C \) which may vary by line. Similarly, \( X \ll Y \) will denote \( |X| \leq cY \) for a sufficiently small constant \( c > 0 \). In Sections 3 and 4, all implicit constants are allowed to depend only on the coefficients of \( P \). In Sections 5-7 the constants may also depend on the initial data \( \phi(0) \).

2.2. Vector fields. In \( \mathbb{R}^{1+3} \), we consider the three (ordered) sets

\[
\partial := (\partial_t, \partial_1, \partial_2, \partial_3), \quad \Omega := (x^i\partial_j - x^j\partial_i), \quad S := t\partial_t + \sum_{i=1}^3 x^i\partial_i,
\]

which are, respectively, the generators of translations, rotations and scaling. We also denote by \( \vartheta \) the angular derivatives, and

\[
\partial r := (\partial_t + \partial_r, \vartheta)
\]

We set

\[
Z := (\partial, \Omega, S)
\]

and we define the function class

\[
S^Z(f)
\]

to be the collection of functions \( g : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R} \) such that

\[
|Z^J g(t, x)| \lesssim_J |f|
\]

whenever \( J \) is a multiindex. We will frequently use \( f = \langle r \rangle^k \) for some real \( k \). We also define

\[
S^Z_{radial}(f) := \{ g \in S^Z(f) : g \text{ is spherically symmetric} \}
\]

We denote

\[
\phi_J := Z^J \phi := \partial^i \Omega^j S^k u,
\]

\[
\phi_{\leq m} := (\phi_J)_{|J| \leq m}
\]

Let \( \| \cdot \| \) be any norm used in this paper. Given any nonnegative integer \( N \geq 0 \), we write \( \| g_{\leq N} \| \) to denote \( \sum_{|J| \leq N} \| g_J \| \).

2.3. Local energy norms. We consider a partition of \( \mathbb{R}^3 \) into the dyadic sets \( A_R = \{ R \leq \langle r \rangle \leq 2R \} \) for \( R \geq 1 \).

Define the local energy norm \( LE \)

\[
\| u \|_{LE} = \sup_R \| \langle r \rangle^{-\frac{1}{2}} u \|_{L^2([0, \infty) \times A_R)}
\]

\[
\| u \|_{LE[0, \epsilon]} = \sup_R \| \langle r \rangle^{-\frac{1}{2}} u \|_{L^2([0, \epsilon] \times A_R)},
\]

(2.2)
its $H^1$ counterpart
\[
\|u\|_{LE^1} = \|\nabla u\|_{LE} + \|\langle r \rangle^{-1} u\|_{LE}
\]
(2.3)
and as well the dual norm
\[
\begin{align*}
\|f\|_{LE^*} &= \sum_R \|\langle r \rangle^{1/2} f\|_{L^2([0,\infty) \times A_R)} \\
\|f\|_{LE^*[t_0,t_1]} &= \sum_R \|\langle r \rangle^{1/2} f\|_{L^2([t_0,t_1] \times A_R)}.
\end{align*}
\]
(2.4)

Then we have the following scale invariant local energy estimate on Minkowski backgrounds:
\[
\|\nabla u\|_{L^\infty L^2_x} + \|u\|_{LE^1} \lesssim \|\nabla u(0)\|_{L^2} + \|\Box u\|_{LE^*+L^1_t L^2_x}
\]
(2.5)
and a similar estimate involving the $LE^1[t_0,t_1]$ and $LE^*[t_0,t_1]$ norms.

The first estimate of this kind was obtained by Morawetz for the Klein-Gordon equation [33]. There are many similar results obtained in the case of small perturbations of the Minkowski space-time; see, for example, [11], [12], [37], [41], [42], [2], [29] and [31]. Even for large perturbations, in the absence of trapping, (2.5) still sometimes holds, see for instance [5], [30]. In the presence of trapping, (2.5) is known to fail, see [34], [35]. We will assume that a similar estimate holds, with no loss of derivatives, for our operator $P$ after commuting with vector fields in $Z$.

**Definition 2.1** (Local energy decay). We say that $P$ has the strong local energy decay (SLED) property if the following estimate holds for all $m \geq 0$, and $0 \leq T_0 < T_1 \leq \infty$:
\[
\|\phi_{\leq m}\|_{LE^1([T_0,T_1] \times \mathbb{R}^3)} \lesssim_m \|\partial \phi_{\leq m}(T_0)\|_{L^2(\mathbb{R}^3)} + \|(P\phi)_{\leq m}\|_{(L^1 L^2+LE^*)([T_0,T_1] \times \mathbb{R}^3)}.
\]
(2.6)

Here the implicit constant may depend on $m$, but not $T_0$ and $T_1$.

For example, **Definition 2.1** holds if the operator $P$ is a small perturbation of $\Box$, see for instance [31]. More generally, strong local energy estimates (for $m = 0$) were shown in [30], provided there are no negative eigenvalues or real resonances. Provided that $P$ is stationary, one can extend the result to (2.6) by commuting with the vectors fields in $Z$.

**2.4. Statement of the main theorem.** Let $h = g - m$, where $m$ denotes the Minkowski metric. We make the following assumptions on the coefficients of $P$: Let $\sigma \in (0,\infty)$ be real.
\[
\begin{align*}
h^{\alpha\beta} B^\alpha &\in S^Z((r)^{-1-\sigma}) \\
\partial_\alpha B^\alpha, V &\in S^Z((r)^{-2-\sigma}) \\
g^{\alpha\beta} &\in S^Z_{\text{radial}}((r)^{-2-\sigma})
\end{align*}
\]
(2.7)

**Theorem 2.2** (Main theorem). Assume that $P$ has the SLED property (**Definition 2.1**), and that the coefficients of $P$ satisfy (2.7).

i) Assume that $(\phi(0), \partial_t \phi(0)) \in H^{13}(\mathbb{R}^3) \times H^{12}(\mathbb{R}^3)$. Then there is $\epsilon_0 > 0$ so that, if
\[
\|\partial \phi_{\leq 12}(0)\|_{L^2(\mathbb{R}^3)} \leq \epsilon_0,
\]
then (1.2) has a unique global solution.

ii) Fix $m \in \mathbb{N}$. Then there is an integer $N \gg m$ so that, if we assume in addition that
\[
\|\langle r \rangle^{-1/2} \partial \phi_{\leq N}(0)\|_{L^2(\mathbb{R}^3)} < \infty
\]
them the solution satisfies
\[
\phi_{\leq m}(t,x) \lesssim \langle v \rangle^{-1} \langle u \rangle^{-1}
\]
where $v := t + r, u := t - r$.

We expect the rate of decay to be sharp even for the Minkowski metric.
2.5. **Notation for dyadic numbers and conical subregions.** We work only with dyadic numbers that are at least 1. We denote dyadic numbers by capital letters for that variable; for instance, dyadic numbers that form the ranges for radial (resp. temporal and distance from the cone \( \{ |x| = t \} \)) variables will be denoted by \( R \) (resp. \( T \) and \( U \)); thus
\[
R, T, U \geq 1.
\]
We choose dyadic integers for \( T \) and a power \( a \) for \( R, U \)—thus \( R = a^k \) for \( k \geq 1 \)—different from 2 but not much larger than 2, for instance in the interval \((2, 5]\), such that for every \( j \in \mathbb{N} \), there exists \( j' \in \mathbb{N} \) with
\[
a^{j'} = \frac{3}{8} 2^j.
\]

2.5.1. **Dyadic decomposition.** We decompose the region \( \{ r \leq t \} \) based on either distance from the cone \( \{ r = t \} \) or distance from the origin \( \{ r = 0 \} \). We fix a dyadic number \( T \).

\[
C_T := \begin{cases} 
\{ (t, x) \in [0, \infty) \times \mathbb{R}^3 : T \leq t \leq 2T, \quad r \leq t \} & T > 1 \\
\{ (t, x) \in [0, \infty) \times \mathbb{R}^3 : 0 < t < 2, \quad r \leq t \} & T = 1
\end{cases}
\]

\[
C_T^R := \begin{cases} 
C_T \cap \{ R < r < 2R \} & R > 1 \\
C_T \cap \{ 0 < r < 2 \} & R = 1
\end{cases}
\]

\[
C_T^U := \begin{cases} 
\{ (t, x) \in [0, \infty) \times \mathbb{R}^3 : T \leq t \leq 2T \} \cap \{ U < |t - r| < 2U \} & U > 1 \\
\{ (t, x) \in [0, \infty) \times \mathbb{R}^3 : T \leq t \leq 2T \} \cap \{ 0 < |t - r| < 2 \} & U = 1
\end{cases}
\]

If a need arises to distinguish between the \( R = 1 \) and \( U = 1 \) cases, we shall write \( C_T^{R=1} \) and \( C_T^{U=1} \) respectively. We define
\[
C_T^{<3T/4} := \bigcup_{R < 3T/8} C_T^R.
\]

Now letting \( R > T \), we define
\[
C_R^T := \{ (t, x) \in [0, \infty) \times \mathbb{R}^3 : r \geq t, T \leq t \leq 2T, R \leq r \leq 2R, R \leq |r - t| \leq 2R \}
\]
\[
C_R^R := \{ (t, x) \in [0, \infty) \times \mathbb{R}^3 : T \leq t \leq R, R \leq r \leq 2R, R/2 \leq |r - t| \leq 2R \}
\]

\( C_R^T, C_R^U, C_R^R \) and \( C_R^T \) are where we shall apply Sobolev embedding, which allows us to obtain pointwise bounds from \( L^2 \) bounds. Given any subset of these conical regions, a tilde atop the symbol \( C \) will denote a slight enlargement of that subset on their respective scales; for example, \( \tilde{C}_R^T \) denotes a slightly larger set containing \( C_R^T \).

2.6. **Notation for the symbols \( n \) and \( N \).** Throughout the paper the integer \( N \) will denote a fixed and sufficiently large positive number, signifying the highest total number of vector fields that will ever be applied to the solution \( \phi \) to \((1.2)\) in the paper.

We use the convention that the value of \( n \) may vary by line.

2.7. **The use of the tilde symbol.** If \( \Sigma \) is a set, we shall use \( \tilde{\Sigma} \) to indicate a slight enlargement of \( \Sigma \), and we only perform a finite number of slight enlargements in this paper to dyadic subregions. The symbol \( \tilde{\Sigma} \) may vary by line.

If \( f \) is a function, we shall typically use \( \tilde{f} \) to denote commuting vector fields applied to \( f \).

2.8. **Intervals.** Given a number \( T > 0 \), let \( I_T := [T, 2T] \).
2.9. Other notation. If \( x = (x^1, x^2, x^3) \in \mathbb{R}^3 \), we write
\[
r := \left( \sum_{i=1}^{3} (x^i)^2 \right)^{1/2}, \quad u := t - r, \quad v := t + r.
\]
We write \( \Box := -\partial_t^2 + \Delta \). Let \( \langle a \rangle := (1 + |a|^2)^{1/2} \).

Let
\[
\mathcal{R}_1 := \{ R : R < |u|/8 \}, \quad \mathcal{R}_2 := \{ R : |u|/8 < R < t \}.
\]
This notation will be used only when \( r < t \).

**Definition 2.3.** Let \( \mathbb{R}_+ := [0, \infty) \).

- Let \( D_{tr} \) denote
\[
D_{tr} := \{ (\rho, s) \in \mathbb{R}^2_+ : -(t + r) \leq s - \rho \leq t - r, \ |t - r| \leq s + \rho \leq t + r \}.
\]
When we work with \( D_{tr} \) we shall use \( (\rho, s) \) as variables, and \( D_{tr}^R \) is short for \( D_{tr}^{\rho < R} \).

- For \( R > 1 \), let
\[
D_{tr}^R := D_{tr} \cap \{ (\rho, s) : R < \rho < 2R \}
\]
and let
\[
D_{tr}^{R=1} := D_{tr} \cap \{ (\rho, s) : \rho < 2 \}.
\]

We write \( dA := dsdp \).

3. Poinwise bounds from local energy

In this section we will show that local energy bounds imply certain weak pointwise bounds, see Proposition 3.4 and Proposition 3.5. Nevertheless, these bounds are sufficient to prove global existence in Section 4.

We start with the following Klainerman-Sideris type estimate for the second derivative.

**Lemma 3.1.** Assume \( \phi \) is sufficiently regular. We then have for all \( r \gg 1 \)
\[
|\partial^2 \phi| \lesssim \left( \frac{1}{r} + \frac{1}{\langle u \rangle} \right) |\partial \phi| + \left( 1 + \frac{t}{\langle u \rangle} \right) \langle r \rangle |(P\phi)|.
\]

**Proof.** Note first that
\[
|\partial^2 \phi| \lesssim \left( \frac{1}{r} + \frac{1}{\langle u \rangle} \right) |\partial \phi| + \left( 1 + \frac{t}{\langle u \rangle} \right) |(\Box \phi)|.
\]

The case \( |J| = 0 \) is an immediate consequence of Lemma 2.3 from [16]. The general case follows after commuting with vector fields.

It is thus enough to estimate the difference \( P - \Box \). We write by (1.1)
\[
P - \Box = h^{\alpha \beta} \partial_\alpha \partial_\beta + (\partial_\alpha h^{\alpha \beta}) \partial_\beta + g^\alpha(t,x) \Delta_\alpha + B^a(t,x) \partial_a + V(t,x)
\]
Using the assumptions on the coefficients in subsection 2.7 we have
\[
(P - \Box)\phi \in S^Z(\langle r \rangle^{-1-\sigma})(\partial^2 \phi + \partial \phi) + S^Z(\langle r \rangle^{-2-\sigma})\Omega^2 \phi
\]
After applying vector fields we thus obtain
\[
\left| (P - \Box)\phi \right| \lesssim S^Z(\langle r \rangle^{-1-\sigma})|\partial^2 \phi| + S^Z(\langle r \rangle^{-2-\sigma})|\partial \phi| + S^Z(\langle r \rangle^{-2-\sigma})|\phi|.
\]

The conclusion now follows from (3.2) and (3.3), since the first term on the RHS of (3.3) can be absorbed in the LHS of (3.2) for \( r \gg 1 \).

The main tool for turning local energy estimates into pointwise bounds is the following lemma

**Lemma 3.2** (Dyadically localised bounds). Let \( w \in C^4 \).
• For all \( T \geq 1 \) and \( 1 \leq U \leq 3T/8 \), we have
\[
\|w\|_{L^\infty(C_T^U)} \lesssim \sum_{i \leq 1, j \leq 2} \frac{1}{(T^3U)^{1/2}} \|S^i \Omega^j w\|_{L^2(C_T^U)} + \left( \frac{U}{T^3} \right) \frac{1}{2} \|\partial_t S^i \Omega^j w\|_{L^2(C_T^U)}.
\] (3.4)

• For all \( T \geq 1 \) and \( R > T \), we have
\[
\|w\|_{L^\infty(C_T^R)} \lesssim \sum_{i \leq 1, j \leq 2} \frac{1}{(R^3T)^{1/2}} \|S^i \Omega^j w\|_{L^2(C_T^R)} + \frac{1}{(RT)^{1/2}} \|\partial_t S^i \Omega^j w\|_{L^2(C_T^R)}.
\] (3.5)

• For all \( T \geq 1 \) and \( 1 \leq R \leq 3T/8 \), we have
\[
\|w\|_{L^\infty(C_T^R)} \lesssim \sum_{i \leq 1, j \leq 2} \frac{1}{(R^3T)^{1/2}} \|S^i \Omega^j w\|_{L^2(C_T^R)} + \frac{1}{R} \|\partial_t S^i \Omega^j w\|_{L^2(C_T^R)}.
\] (3.6)

• For all \( T \geq 1 \) and \( R > T \), we have
\[
\|w\|_{L^\infty(C_T^R)} \lesssim \sum_{i \leq 1, j \leq 2} \frac{1}{R^2} \|S^i \Omega^j w\|_{L^2(C_T^R)} + \frac{1}{R} \|\partial_t S^i \Omega^j w\|_{L^2(C_T^R)}.
\] (3.7)

The proof of this lemma can be found in [22] (see also [32] and [23]). For (3.7), note that \(|C_T^R|^{1/2} \sim R^2\), which explains the \(1/R^2\) factor.

We will also use the following lemma near the cone, which is a slight extension of Lemma 9.1 in [19].

**Lemma 3.3.** If \( f \in C^1 \), then
\[
\int_{t/2}^{3t/2} \langle t - r \rangle^{-2-\gamma} f(t, x)^2 dx \lesssim \int_{t/4}^{7t/4} |\partial_r f(t, x)|^2 dx + \frac{1}{t^2} \left( \int_{t/4}^{t/2} f(t, x)^2 dx + \int_{3t/2}^{7t/4} f(t, x)^2 dx \right)
\] (3.8)

**Proof.** Let \( \chi : [0, \infty) \to [0, 1] \) be a cutoff such that \( \chi(s) = 1 \) for \( 1/2 \leq s \leq 3/2 \) and 0 when \( s \leq 1/4 \) and \( s \geq 7/4 \). We will show that, if \( \gamma > -1/2 \), and \( \gamma \neq 1/2 \), then
\[
\int \langle t - r \rangle^{-2-2\gamma} \chi(r/t) f(r, \omega) r^2 dr \lesssim \int \langle t - r \rangle^{-2-2\gamma} |\partial_r f(r, \omega) \chi(r/t)|^2 r^2 dr + \frac{1}{t^2} \int \langle t - r \rangle^{-2\gamma} |f(r, \omega) \chi'(r/t)|^2 r^2 dr.
\]
The conclusion follows if we take \( \gamma = 0 \) and integrate over \( \omega \).

We have
\[
f(r, \omega) \chi^2(r/t) - f(7t/4, \omega)^2 \chi((7t/4)/t) = \frac{1}{2} \int_r^{7t/4} f(\rho, \omega) \chi(\rho/t) \cdot \partial_r (f(\rho, \omega) \chi(\rho/t)) d\rho.
\]
Hence
\[
f(r, \omega)^2 \chi(r/t) r^2 \lesssim f(7t/4, \omega)^2 (3t/2)^2 + 2 \int_r^{7t/4} |f(\rho, \omega) \chi(\rho/t) | \cdot \partial_r (f(\rho, \omega) \chi(\rho/t)) | d\rho
\]
Recall that \( \chi(7t/4) = 0 \). We multiply by \( \langle t - r \rangle^{-2-2\gamma} \) and integrate \( r \) from \( t/4 \) to \( 7t/4 \). This yields
\[
\int_{t/4}^{7t/4} \langle t - r \rangle^{-2-2\gamma} \chi(r/t) f(r, \omega)^2 r^2 dr \lesssim \int_{t/4}^{7t/4} \langle t - r \rangle^{-1-2\gamma} |f(r, \omega) \chi(r/t) | \partial_r (f(r, \omega) \chi(r/t)) | r^2 dr
\]
By the chain rule, \( \partial_r (\chi(r/t)) \lesssim \chi'(r/t) \cdot \frac{1}{t} \). Thus by Cauchy-Schwarz and the chain rule
\[
\int_{t/4}^{7t/4} \langle t - r \rangle^{-2-2\gamma} f(r, \omega)^2 r^2 dr \lesssim \int_{t/4}^{7t/4} \langle t - r \rangle^{-2\gamma} |\partial_r f(r, \omega) \chi(r/t) |^2 r^2 dr
\]
\[+ \frac{1}{t^2} \int_{t/4}^{7t/4} \langle t - r \rangle^{-2\gamma} |f(r, \omega) \chi'(r/t) |^2 r^2 dr.
\]
This concludes the proof of the lemma.

Our next proposition yields global pointwise bounds for \( \phi_t \) under the assumption that the local energy norms are finite. These estimates are sharp from that point of view, but can be improved for solutions to \((1.2)\), see Sections 5-7.

**Proposition 3.4.** Let \( T \) be fixed and \( \phi \) be any sufficiently regular function. There is a fixed positive integer \( k \), such that for any multi-index \( J \) with \( |J| \leq N - k \), we have:
\[
|\phi_J| \leq C_{|J|}^k \| \phi_{[J]k} \|_{LE^1[T,2T]} \|u\|^{1/2}(u)^{-1}.
\] (3.9)

**Proof.** Away from the cone, (3.9) is a straightforward consequence of (3.5) and (3.6). For the \( C^k_T \) region, one uses (3.4) in conjunction with Lemma 3.3:
\[
\|\phi_J\|_{L^\infty(C^k_T)} \lesssim T^{-\frac{1}{2}} U^\frac{1}{2} \left(\|U^{-1}\phi_{[J]3}\|_{L^2(C^k_T)} + \|\partial_r \phi_{[J]3}\|_{L^2(C^k_T)}\right)
\]
\[
\lesssim T^{-\frac{1}{2}} U^\frac{1}{2} \left(\|\partial_r \phi_{[J]3}\|_{L^2(C^k_T)} + \frac{1}{T} \|\phi_{[J]3}\|_{L^2[T,2T]}L^2(r\asymp T)\right)
\]
\[
\lesssim \frac{U^\frac{1}{2}}{T} \|\phi_{[J]3}\|_{LE^1[T,2T]}.
\]

□

We now obtain an improved bound on the derivatives. Note that here it is crucial that \( \phi \) is a solution to \((1.2)\), since we need to use Lemma 3.1. Let
\[
\mu := \mu(t,r) := \min((t),\langle u\rangle)^{1/2}.
\]

**Proposition 3.5.** Let \( T \) be fixed and \( \phi \) solve \((1.2)\) for the times \( t \in I_T \). Then for any dyadic region \( C \in \{C^R_T, C^T_R, C^k_T\} \) and \( m \geq 0 \) we have
\[
\|\partial \phi_{\leq m}\|_{L^\infty(C)} \leq \tilde{C} m \frac{1}{\mu} \left(\frac{1}{R} + \|\partial \phi_{\leq m+3}\|_{L^\infty(C)}\right) \|\phi_{\leq m+5}\|_{LE^1[T,2T]} \] (3.10)

**Proof.** Note first that if \( r \lesssim 1 \), the bound follows immediately from Lemma 3.2. Subsequently, we assume that \( r \gg 1 \).

Note first that
\[
(P\phi)_{\leq m+3} \lesssim \left|\left(\partial \phi_{\leq m+3}\right) \right|
\]
\[
(P\phi)_{\leq m+3} \lesssim \left|\left(\partial \phi_{\leq m+3}\right) \right|
\]
\[
(P\phi)_{\leq m+3} \lesssim \left|\left(\partial \phi_{\leq m+3}\right) \right|
\]

We also have

1. In \( C^T_R \) and \( C^R_T \), the bound in (3.1) is
\[
\partial^2 \phi_{\leq m+3} \lesssim R^{-1}|\partial \phi_{\leq m+4}| + R^{-2}|\phi_{\leq m+5}| + |(P\phi)_{\leq m+3}|. \] (3.11)

2. In \( C^k_T \), the bound in (3.1) is
\[
\partial^2 \phi_{\leq m+3} \lesssim U^{-1}|\phi_{\leq m+4}| + \frac{1}{TU}|\phi_{\leq m+5}| + TU^{-1}|(P\phi)_{\leq m+3}|. \] (3.12)

We now apply Lemma 3.2 in our region \( C \). When \( C = C^k_T \) we obtain, using (3.6) and (3.11):
\[
\|\partial \phi_{\leq m}\|_{L^\infty(C^k_T)} \lesssim \frac{1}{(RT)^{1/2}} \|\partial \phi_{\leq m+3}\|_{L^2(C^k_T)} + \frac{1}{(RT)^{1/2}} \|\partial^2 \phi_{\leq m+3}\|_{L^2(C^k_T)}
\]
\[
\lesssim \frac{1}{RT^{1/2}} \|\phi_{\leq m+5}\|_{LE^1[T,2T]} + \frac{1}{(RT)^{1/2}} \|\phi_{\leq m+3}\|_{L^2(C^k_T)}
\]
\[
\lesssim \left(\frac{1}{RT^{1/2}} + T^{-1/2} \|\phi_{\leq m+3}\|_{L^\infty(C^k_T)}\right) \|\phi_{\leq m+5}\|_{LE^1[T,2T]}
\]

Similar computations yield, using (3.5) and (3.11):
\[
\|\partial \phi_{\leq m}\|_{L^\infty(C^k_R)} \lesssim \left(\frac{1}{RT^{1/2}} + T^{-1/2} \|\phi_{\leq m+3}\|_{L^\infty(C^k_R)}\right) \|\phi_{\leq m+5}\|_{LE^1[T,2T]}
\]
When \( C = C^m_T \) we obtain by (3.4) and (3.12):

\[
\| \partial \phi \leq m \|_{L^\infty(C^m_T)} \lesssim \left( \frac{1}{T U^{1/2}} + U^{-1/2} \| \partial \phi \leq m+1 \|_{L^\infty(C^m_T)} \right) \| \phi \leq m+5 \|_{LE^1[T,2T]}
\]

This completes the proof of Proposition 3.5. \( \square \)

Remark 3.6. We only need to use the following estimate (3.10):

\[
\| \partial \phi \leq m \|_{L^\infty(C)} \leq C_m \left( \frac{1}{r_\mu} + \| \partial \phi \leq m+1 \|_{L^\infty(C)} \right) \| \phi \leq m+5 \|_{LE^1[T,2T]}
\]

(3.13)

Thus in \( r \leq 3t/2 \) we have

\[
\partial \phi \leq m \lesssim (r)^{-1} \langle u \rangle^{-1/2} \| \phi \leq m+5 \|_{LE^1[T,2T]}
\]

4. THE PROOF OF SMALL DATA GLOBAL EXISTENCE

We are now ready to prove our first theorem. For any \( N \in \mathbb{N} \), define

\[
\mathcal{E}_N(t) = \| \partial \phi \leq N \|_{L^\infty[0,t]L^2} + \| \phi \leq N \|_{LE^1[0,t]}
\]

We also define \( \bar{N} = N - 1 \), and \( N_1 = N/2 \).

Theorem 4.1. Assume that \( \phi \) solves (1.2). Then there exists a global classical solution to (1.2), provided that the initial data is smooth and satisfies, for some sufficiently small \( \epsilon_0 \ll 1 \),

\[
\mathcal{E}_N(0) \leq \epsilon_0
\]

for any natural number \( N \geq 12 \).

Moreover, write \( \mathcal{E}_N(0) = \nu_N \epsilon \) where \( \nu_N \) is a small constant to be determined later (see (4.14)). Then for any \( \delta > 0 \), there is some \( \bar{C} > 0 \) so that

\[
\mathcal{E}_N(t) \leq \bar{C} \langle t \rangle^\delta \nu_N \epsilon,
\]

(4.1)

\[
|\phi \leq N_1| \leq \epsilon \langle u \rangle^{1/2} \langle v \rangle^{-1}, \quad |\partial \phi \leq N_1| \leq \frac{\epsilon}{(r)_\mu^{1/2}}.
\]

(4.2)

Proof. The proof will be by a bootstrap argument. Clearly (4.1) and (4.2) hold for small times. Assuming now that (4.1) and (4.2) hold for \( 0 \leq t \leq T \), we improve the constants by a factor of \( 1/2 \). Thus by continuity the solution exists for all time.

The proof that (4.1) holds with a better constant uses, crucially, the smallness of \( \epsilon \). For instance, as the reader can verify below, in the region \( r \leq t/2 \) we manage to absorb the nonlinearity to the left-hand side by way of this smallness. We note that in the course of proving (4.1) we can actually assume \( \bar{C} \) to be as big as desired.

The proof that (4.2) holds with a better constant makes use of Proposition 3.4 and Proposition 3.5 and the previous paragraph. By using the fact, now already proved, that \( \mathcal{E}_N(T) = O(\nu_N \epsilon) \) (see previous paragraph) where \( \nu_N \) is a constant independent of \( T \) and the other constants involved in the proof, we can choose \( \nu_N \) small enough so that (4.2) indeed holds with a better constant. This concludes our overview of the proof.

Note that

1. By Sobolev embeddings and the smallness of the initial data, the estimates in (4.2) hold for time 0. By local existence theory and the continuity in time of the functions in (4.2), the estimates in (4.2) hold for all sufficiently small times.

2. The estimates in (4.1) hold for time 0 by assumption that the initial data satisfies \( \mathcal{E}_N(0) \leq \nu_N \epsilon \). The estimates in (4.1) hold for all sufficiently small times by continuity in time of the norms involved in \( \mathcal{E}_N(t), k \leq N \)—where we choose \( \bar{C} \) to be big enough.

Note also that, since \( Q \) satisfies the null condition, we have that

\[
Q(\partial \phi, \partial \phi) \in S^2(1) \partial \phi \overline{\partial \phi}
\]
Moreover,
\[ \partial \phi \in S^2 \left( \frac{(u)}{r} \right) \partial \phi + S^2 \left( \frac{1}{r} \right) Z \phi, \quad \overline{\partial} \phi \in S^2 \left( \frac{(u)}{t} \right) \overline{\partial} \phi + S^2 \left( \frac{1}{t} \right) Z \phi, \]  
(4.3)

which combined with (4.2) yields
\[ \overline{\partial} \phi \leq N_1(t, x) \mid_{\frac{1}{2} \leq r \leq 1} \leq \epsilon \left( \frac{u}{r} \right)^{1/2}. \]  
(4.4)

Assume that (4.1) and (4.2) hold for all \( 0 \leq t \leq T \). Let
\[ \mathcal{N} := L^1 L^2 + LE^* \]
be the space in which we place the nonlinearity.

We start with the bound for \( \mathcal{E}_N \). We will use \( L^1 L^2 \) near the cone, and \( LE^* \) away from it.

Given the assumption of local energy decay (2.6), we have
\[ \mathcal{E}_N(t) \leq C_N \left( \mathcal{E}_N(0) + \|Q_{\leq N}\|_{\mathcal{N}[0,t]} \right). \]  
(4.5)

We define
\[ S_1 := \{(s, x) : 0 \leq s \leq t, |x| \leq s/2 \} \]
\[ S_2 = \{(s, x) : 0 \leq s \leq t, s/2 \leq |x| \leq 3s/2 \} \]
\[ S_3 := \{(s, x) : 0 \leq s \leq t, |x| \geq 3s/2 \} \]

In \( S_1 \) we have by (4.2)
\[ \|Q_{\leq N}\|_{LE^*(S_1)} \lesssim \| \partial \phi \|_{LE^*(S_1)} \lesssim \epsilon \| \phi \|_{LE^*[0,t]} \lesssim \epsilon \| \phi \|_{LE^*[0,t]} \]  
(4.6)

On the other hand, in \( S_2 \cup S_3 \) (4.2) implies that \( |\partial \phi| \lesssim \epsilon \), \( 1 \). We thus obtain
\[ \|Q_{\leq N}\|_{L^1 L^2(S_2 \cup S_3)} \lesssim \epsilon \| \phi \|_{L^1 L^2(S_2 \cup S_3)} \lesssim \epsilon \int_0^t \langle s \rangle^{-1} \mathcal{E}_N(s)ds \]  
(4.7)

We thus obtain, by (4.5), (4.6), and (4.7):
\[ \mathcal{E}_N(t) \leq C_N \left( \mathcal{E}_N(0) + \epsilon \int_0^t \langle s \rangle^{-1} \mathcal{E}_N(s)ds \right) \]

By Gronwall’s inequality,
\[ \mathcal{E}_N(t) \leq C_N \mathcal{E}_N(0) \exp \left( C_N \int_0^t \langle s \rangle^{-1}ds \right) \leq C_N \mathcal{E}_N(0) \langle t \rangle^{C_N \epsilon}. \]  
(4.8)

We now choose \( \epsilon_0 \) small enough so that \( C_N \epsilon_0 \leq \delta \), and \( \tilde{C} = 2C_N \).

We will now show that \( \mathcal{E}_N \) is bounded, where \( N := N - 1 \). More precisely,
\[ \mathcal{E}_N(t) \leq C_N \mu_N \epsilon \]  
(4.9)

We first see, similarly to (4.6), that
\[ \|Q_{\leq N}\|_{LE^*(S_1)} \lesssim \epsilon \| \phi \|_{LE^*[0,t]} \]  
(4.10)

Since (4.2) implies that \( |\partial \phi| \lesssim \epsilon(t)^{-3/2} \) in \( S_3 \), we obtain
\[ \|Q_{\leq N}\|_{L^1 L^2(S_3)} \lesssim \epsilon \int_0^t \langle s \rangle^{-3/2} \mathcal{E}_N(s)ds \lesssim \epsilon \int_0^t \langle s \rangle^{-3/2} \mathcal{E}_N(s)ds \lesssim \epsilon \mathcal{E}_N(0) \]  
(4.11)

where the last bound holds by (4.8).

For the bound in \( S_2 \), however, we proceed differently. Let \( |\alpha| \leq N/2 \) and let \( |\alpha + \beta| = N \).

We have
\[ \|Q_{\leq N}\|_{L^1 L^2(S_2)} \lesssim \sum_{\alpha, \beta} \| \partial \phi_\alpha \partial \phi_\beta \|_{L^1 L^2(S_2)} + \| \partial \phi_\alpha \partial \phi_\beta \|_{L^1 L^2(S_2)}. \]
We begin with the second term. By (4.4) we have
\[
\|\partial\phi_\alpha \partial\phi_\beta \|_{L^1 L^2(S_2)} \lesssim \epsilon \| \langle s \rangle^{-3/2} \partial\phi \lesssim N \| L^1 L^2(S_2) \| \lesssim \epsilon \int_0^t \langle s \rangle^{-3/2} \mathcal{E}_N(s) \, ds \lesssim \epsilon \mathcal{E}_N(0)
\] (4.12)
where the last bound holds by (4.8).

For the first term, we use (4.3), (4.2), Lemma 3.3 and (4.1):
\[
\|\partial\phi_\alpha \partial\phi_\beta \|_{L^1 L^2(S_2)} \lesssim \|\partial\phi_\alpha \langle s - r \rangle \langle r \rangle^{-1} \partial\phi_\beta \|_{L^1 L^2(S_2)} + \|\partial\phi_\alpha \langle r \rangle^{-1} S\phi_\beta \|_{L^1 L^2(S_2)} \\
\lesssim \epsilon \int_0^t \langle s \rangle^{-3/2} \mathcal{E}_N(s) \, ds + \int_0^t \epsilon \| \langle s - r \rangle^{-1/2} \langle r \rangle^{-2} S\phi_\beta(s,) \|_{L^2(s/2 \leq |x| \leq 3s/2)} \, ds \\
\lesssim \epsilon \mathcal{E}_N(t) + \int_0^t \epsilon \| \langle s - r \rangle^{-1} \langle r \rangle^{-3/2} S\phi_\beta(s,) \|_{L^2(s/2 \leq |x| \leq 3s/2)} \, ds \\
\lesssim \epsilon \mathcal{E}_N(t) + \int_0^t \epsilon \langle s \rangle^{-3/2} \| \langle s - r \rangle^{-1} S\phi_\beta(s,) \|_{L^2(s/2 \leq |x| \leq 3s/2)} \, ds \\
\lesssim \epsilon \mathcal{E}_N(t) + \int_0^t \epsilon \langle s \rangle^{-3/2} (\| \partial_s S\phi_\beta(s,) \|_{L^2} + \| r^{-1} S\phi_\beta(s,) \|_{L^2}) \, ds \\
\lesssim \epsilon \mathcal{E}_N(t) + \int_0^t \epsilon \langle s \rangle^{-3/2} \mathcal{E}_{N+1}(s) \, ds \lesssim \epsilon \mathcal{E}_N(t) + \epsilon \mathcal{E}_N(0)
\] (4.10), (4.11) and (4.12) imply
\[
\mathcal{E}_N(t) \lesssim \epsilon \mathcal{E}_N(t) + \epsilon \mathcal{E}_N(0)
\]
and (4.9) follows for small enough \( \epsilon \).

We now improve the constants in (4.2). We pick \( \nu_N \) so that
\[
2 \nu_N \bar{C}_N \max(C_N, C_N) \leq \frac{1}{2}
\] (4.14)

We have by Proposition 3.4, (4.9) and (4.14):
\[
|\phi_{\lesssim N}| \leq \bar{C}_N \langle u \rangle^{1/2} \langle v \rangle^{-1} \mathcal{E}_N(T) \leq \bar{C}_N \nu_N \epsilon \langle u \rangle^{1/2} \langle v \rangle^{-1} \leq \frac{1}{2} \epsilon \langle u \rangle^{1/2} \langle v \rangle^{-1}
\]

To improve the constant for the derivative, we note that, by Proposition 3.5’s (3.10) and the fact that \( \frac{N+1}{2} \leq N \) and \( N + 5 \leq \tilde{N} \):
\[
\| \partial \phi_{\lesssim N} \|_{L^\infty(C)} \leq \bar{C}_N \left( \frac{1}{(r)} \frac{1}{\mu} + \| \partial \phi_{\lesssim N} \|_{L^\infty(C)} \right) \mathcal{E}_N(T)
\]

Using (4.9) we see that we can absorb the second term on the right to the left as long as (4.14) holds. We now obtain
\[
|\partial \phi_{\lesssim N}| \leq 2 \bar{C}_N \frac{1}{(r)} \frac{1}{\mu} \mathcal{E}_N(T) \leq \epsilon \frac{1}{2} \frac{1}{(r)} \frac{1}{\mu}
\]

We have thus improved the constants in (4.2) by 1/2. This concludes the continuity argument and the proof of small data global existence.

\[ \Box \]

5. Preliminaries to the Iteration

Remark 5.1 (The initial data). Let \( w := S(t, 0) \phi[0] \) denote the solution to the free wave equation with initial data \( \phi[0] \) at time 0. Then for any multiindex \( J \) with \( |J| = O_N(1) \),
\[
w_J(t, x) = \frac{1}{|\partial B(x, t)|} \int_{\partial B(x, t)} (\phi_0)_{J}(y) + \nabla_y(\phi_0)_{J}(y) \cdot (y - x) + t(\phi_1)_{J}(y) \, dS(y).
\] (5.1)

By (5.1) and the assumptions \( (\phi(0), \partial_t \phi(0)) \in L^{13}(\mathbb{R}^3) \times L^{12}(\mathbb{R}^3) \),
\[
\| \langle r \rangle^{1/2} \partial \phi_{\lesssim N}(0) \|_{L^2} < \infty,
\]
we have \[ w_J \lesssim \langle u \rangle^{-1} \langle w \rangle^{-1}. \]

5.1. Overview of the iteration. The iteration proceeds as follows. First we note that by Remark 5.1, we may assume zero initial data in the following iteration. Second, note that (4.1) is already optimal when \( t - 1 < r < t + 1 \). Third, we distinguish the nonlinearity and the coefficients of \( P - \square \), and for both of these, we apply the fundamental solution and iterate. That is, we decompose \( \phi \leq m \) into \( \phi \leq m = f_L + f_N \) where \( f_L \) solves the linear part of (5.4). The iterations for \( f_L \) and \( f_N \) proceed in lockstep with one another.

Since the iteration for \( \{ r < t - 1 \} \) would depend on the values of the solution and its vector fields in the region where \( r > t \), we shall first complete the iteration in \( \{ r > t + 1 \} \). For the iteration in \( \{ r < t - 1 \} \), we note that the decay rates obtained from the fundamental solution are insufficient in the region \( \{ r < t/2 \} \). To remedy this, we prove Proposition 7.2. With the new decay rates obtained from Proposition 7.2, we are then able to obtain new decay rates for the solution and its vector fields. At every step of the iteration, Lemma 5.2 is used to turn the decay gained at previous steps into new decay rates.

A little more precisely: The fundamental solution gives us an improvement for \( r \phi \leq m \), say \( r \phi \leq m \lesssim \langle u \rangle^{-\alpha} \) for some real \( \alpha \), and then by Proposition 7.2, we obtain \( v \phi \leq m \lesssim \langle u \rangle^{-\alpha} \). Then, using this improvement for \( \phi \leq m \), we improve the decay rate for the derivatives \( \partial \phi \leq m \). This improvement for the derivatives \( \partial \phi \leq m \) is then used to improve the decay rates for \( r \phi \leq m \) using the fundamental solution. We then again apply Proposition 7.2, and this cyclical iteration between these two improvements continues until we reach the final decay rate \( v \phi \leq m \lesssim \langle u \rangle^{-1} \).

To simplify the iteration—in particular, to avoid the appearance of logarithms—we shall reduce the value of \( \sigma \) if necessary to be equal to some positive irrational number less than the original value of \( \sigma \). We also take \( 0 < \sigma \ll 1 \).

5.2. Setting up the problem. We rewrite (1.2) as
\[ \square \phi = (\square - P) \phi + Q = -\partial_t (h^{\alpha \beta} \partial_\beta \phi + B^\alpha \phi) - g^\omega \Delta_\omega \phi - (V - \partial_\alpha B^\alpha) \phi + Q \]

Using the assumptions (2.7), we can thus write
\[ \square \phi \in \partial (S^Z(r^{-1-\sigma}) \phi \leq 1) + S^Z(r^{-2-\sigma}) \phi \leq 2 + Q \]

Pick any multiindex \( |J| \leq N_1 - 2 \). We have after commuting
\[ \square \phi_J \in \partial (S^Z(r^{-1-\sigma}) \phi \leq m+1) + S^Z(r^{-2-\sigma}) \phi \leq m+2 + Q \leq m \] (5.2)

When we commute vector fields with the null form in (1.2), we obtain more than one null form, but for the purposes of pointwise decay iteration we may treat all of these null forms as a single null form, which by a slight abuse of notation we also denote by \( Q \).

When \( r \leq t/2 \) or \( r \geq 3t/2 \) we will gain a factor of \( 1/r \) for the derivative. On the other hand, we only gain a factor of \( 1/\langle u \rangle \) for the derivative in the region \( t/2 \leq r \leq 3t/2 \), which causes an additional issue. To deal with it, we remark that, for any function \( w \), we have
\[ \partial w \in S^Z(r^{-1}) w \leq 1 + S^Z(1) \partial_t w, \quad r \geq t/2 \] (5.3)

This is obvious for \( \partial_t \) and \( \partial \), whereas for \( \partial_r \), we write
\[ \partial_r = \frac{S}{r} - \frac{t}{r} \partial_t. \]

Let \( \chi \text{cone} \) be a cutoff subordinated to the region \( t/2 \leq r \leq 3t/2 \). We now rewrite (5.2) as
\[ \square \phi_J \in S^Z(r^{-2-\sigma}) \phi \leq m+2 + (1 - \chi \text{cone}) (S^Z(r^{-1-\sigma}) \partial \phi \leq m+1) + \partial_t (\chi \text{cone} S^Z(r^{-1-\sigma}) \phi \leq m+1) + Q \leq m \] (5.4)

We now write \( \phi_J = \sum_{j=1}^3 \phi_j \) where
\[ \square \phi_1 = G_1, \quad G_1 \in S^Z(r^{-2-\sigma}) \phi \leq m+2 + (1 - \chi \text{cone}) (S^Z(r^{-1-\sigma}) \partial \phi \leq m+1) \]
\[ \square \phi_2 = \partial_t G_2, \quad G_2 \in \chi \text{cone} S^Z(r^{-1-\sigma}) \phi \leq m+1 \]
\[ \square \phi_3 = Q \leq m = G_3 \] (5.5)
Finally, from now on $n$ will represent a large constant, which does not depend on $m$, but may increase from one estimate to the next. We will not track the exact value of $n$ needed.

### 5.3. Estimates for the fundamental solution.

We have the following result, which is similar to previous classical results, see for instance [9], [3], [43], [44].

**Lemma 5.2.** Let $m \geq 0$ be an integer and suppose that $\psi : [0, \infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ solves

\[
\Box \psi(t, x) = g(t, x), \quad \psi(0) = 0, \quad \partial_t \psi(0) = 0.
\]

Define

\[
h(t, r) = \sum_{\alpha = 0}^{2} \| \Omega^\alpha g(t, r) \|_{L^2(S^2)}
\]

Assume that

\[
h(t, r) \lesssim \frac{1}{\langle r \rangle^\alpha \langle v \rangle^\beta \langle u \rangle^\eta}, \quad 2 < \alpha < 3, \quad \beta \geq 0, \quad \eta \geq -1/2.
\]

Define

\[
\tilde{\eta} = \begin{cases} 
\eta - 2, & \eta < 1 \\
-1, & \eta \geq 1
\end{cases}.
\]

We then have in both the interior region $\{ r < t - 1 \}$ (without additional restrictions on $\alpha + \beta + \eta$), and in the exterior region $\{ r > t + 1 \}$ in the case $\alpha + \beta + \eta > 3$:

\[
\psi(t, x) \lesssim \frac{1}{\langle r \rangle^\alpha \langle v \rangle^\beta \langle u \rangle^{\tilde{\eta} + 1}}. 
\]

On the other hand, if $\alpha + \beta + \eta < 3$ and $r \geq t + 1$, we have

\[
\psi(t, x) \lesssim r^{2 - (\alpha + \beta + \eta)}. 
\]

**Proof.** A detailed proof of (5.7) can be found in Lemma 5.5 of [22] (see also Lemma 6.1 in [45]). The idea is to use Sobolev embedding and the positivity of the fundamental solution of $\Box$ to show that

\[
r \psi \lesssim \int_{D_{tr}} \rho h(s, \rho) ds d\rho,
\]

where $D_{tr}$ is the backwards light cone with vertex $(r, t)$, and use the pointwise bounds on $h$.

Let us now prove (5.8). In this case $D_{tr} \subset \{ r - t \leq u' \leq r + t, \quad r - t \leq \rho \leq r + t \}$ and we obtain, using that $\langle u' \rangle \lesssim \rho$ and $\rho > t$ in $D_{tr}$:

\[
r \psi \lesssim \int_{r-t}^{r+t} \int_{r-t}^{r+t} \rho^{1 - \alpha - \beta} \langle u' \rangle^{-\eta} d\rho du' \lesssim \int_{r-t}^{r+t} \langle u' \rangle^{2 - (\alpha + \beta + \eta)} du' \lesssim (t + r)^{3 - (\alpha + \beta + \eta)}
\]

which finishes the proof. $\Box$

In view of (5.4), we will also need the following result for an inhomogeneity of the form $\partial_t g$ supported near the cone. The result is similar to Lemma 5.2, except that we gain an extra factor of $\langle u \rangle$ in the estimate.

**Lemma 5.3.** Let $\psi$ solve

\[
\Box \psi = \partial_t g, \quad \psi(0) = 0, \quad \partial_t \psi(0) = 0,
\]

where $g$ is supported in $\{ t/2 \leq |x| \leq 3t/2 \}$. Let $h$ be as in (5.6), and assume that

\[
|h| + |Sh| + |\Omega h| + (t - r)|\partial h| \lesssim \frac{1}{\langle r \rangle^\alpha \langle v \rangle^\beta \langle u \rangle^\eta}, \quad 2 < \alpha < 3, \quad \eta \geq -1/2
\]

We then have in the interior region $\{ r < t - 1 \}$, as well as in the exterior region $\{ r > t + 1 \}$ in the case $\alpha + \eta > 3$:

\[
\psi(t, x) \lesssim \frac{1}{\langle r \rangle^\alpha \langle v \rangle^\beta \langle u \rangle^{\eta + 1}}.
\]

(5.10)
Proof. Let \( \tilde{\psi} \) be the solution to

\[
\Box \tilde{\psi} = g, \quad \tilde{\psi}[0] = 0.
\]

Clearly \( \psi = \partial_t \tilde{\psi} \). We also note that in the support of \( g \) we have

\[
(t\partial_t + x_i\partial_t)h \lesssim |Sh| + |\Omega h| + (t-r)|\partial_t h|.
\]

We now apply Lemma 5.2 (with \( \beta = 0 \)) to \( \nabla \tilde{\psi}, \Omega \tilde{\psi}, S\tilde{\psi} \), and use the fact that

\[
\langle u \rangle \partial_t \tilde{\psi} \lesssim |\nabla \tilde{\psi}| + |S\tilde{\psi}| + |\Omega \tilde{\psi}| + \sum_i |(t\partial_t + x_i\partial_t)\tilde{\psi}|.
\]

\[\square\]

5.4. Derivative bounds. We will now derive better bounds for the derivatives; roughly speaking, the derivative gains \( 1/\langle r \rangle \) away from the cone, and \( 1/\langle u \rangle \) near the cone. The idea is that we can use Morawetz type estimates in the various dyadic regions defined in Section 2.

Proposition 5.4. Let \( \phi \) solve (1.2), and assume that

\[
\phi_{m+n} \lesssim \langle r \rangle^{-\alpha} \langle t \rangle^{-\beta} \langle u \rangle^{-\eta}, \quad (5.11)
\]

for some sufficiently large \( n \). We then have

\[
\partial \phi_{m} \lesssim \langle r \rangle^{-\alpha} \langle t \rangle^{-\beta} \langle u \rangle^{-\eta} \nu^{-1}, \quad \nu := \min(\langle r \rangle, \langle u \rangle)
\]

Proof. It is enough to show that, if \( \mathcal{R} \in \{C^R, C^R_T\} \), we have

\[
\| \partial \phi_{m} \|_{L^\infty(\mathcal{R})} \lesssim \frac{1}{\mathcal{R}} \| \phi_{m+n} \|_{L^\infty(\mathcal{R})} + \mathcal{R}^{-1/2} \| \partial \phi_{m+n} \|_{L^\infty(\mathcal{R})} \quad (5.12)
\]

and if \( \mathcal{R} = C^U_T \), then

\[
\| \partial \phi_{m} \|_{L^\infty(\mathcal{R})} \lesssim \frac{1}{\mathcal{R}} \| \phi_{m+n} \|_{L^\infty(\mathcal{R})} + U^{-1/2} \| \partial \phi_{m+n} \|_{L^\infty(\mathcal{R})} \quad (5.13)
\]

Indeed, plugging (5.11) into the bounds (5.12) and (5.13) yields for a large enough value of \( n \):

\[
\| \partial \phi_{m+n} \|_{L^\infty(\mathcal{R})} \lesssim \langle r \rangle^{-\alpha} \langle t \rangle^{-\beta} \langle u \rangle^{-\eta} \nu^{-1/2}
\]

and now plugging the above estimate in (5.12) and (5.13) finishes the proof.

We now prove (5.12) and (5.13). Given a function \( w \), we have

\[
\| \partial w \|_{L^2(\mathcal{R})} \lesssim \frac{\| w \|_{L^2(\mathcal{R})}}{\nu} + \| \langle r \rangle \langle Pw \rangle \|_{L^2(\mathcal{R})}. \quad (5.14)
\]

(We refer the reader to [22] or [32] for a proof.)

We remark that the second part of (4.2) can now be improved to

\[
\| \partial \phi_{N} \|_{L^2(\mathcal{R})} \lesssim \frac{\epsilon}{\langle r \rangle \nu^{1/2}}. \quad (5.15)
\]

when \( r > t \). Indeed, this follows by the same arguments from Proposition 3.5 in the region \( C^R_T \).

Note now that, by (4.2) and (5.15), we have that

\[
(P\phi)_{m+n} \lesssim \| \partial \phi \|_{L^2(\mathcal{R})} \| \partial \phi \|_{L^2(\mathcal{R})} \lesssim \frac{1}{\nu^{1/2}} \| \partial \phi \|_{L^2(\mathcal{R})} \quad (5.16)
\]

and thus (5.14) and (5.16) imply

\[
\| \partial \phi_{m+n} \|_{L^2(\mathcal{R})} \lesssim \frac{\| \phi_{m+n} \|_{L^2(\mathcal{R})}}{\nu} + \nu^{-1/2} \| \partial \phi_{m+n} \|_{L^2(\mathcal{R})}. \quad (5.17)
\]

We now return to Lemma 3.2, using Lemma 3.1 and (5.16) to bound the second-order derivatives pointwise and (5.17) to bound the first-order derivatives in \( L^2 \). We find

\[
\| \partial \phi \|_{L^\infty(\mathcal{R})} \lesssim |\mathcal{R}|^{-\frac{1}{2}} \left( \frac{\| \phi_{m+n} \|_{L^2(\mathcal{R})}}{\nu} + \| \nu^{-1/2} \partial \phi_{m} \|_{L^2(\mathcal{R})} \right) + |\mathcal{R}|^{-\frac{1}{2}} \| \langle r \rangle \langle P\phi \rangle \|_{L^2(\mathcal{R})} \right) \lesssim \frac{\| \partial \phi_{m+n} \|_{L^\infty(\mathcal{R})}}{\nu^{1/2}} + \frac{\| \partial \phi_{m+n+1} \|_{L^\infty(\mathcal{R})}}{\nu^{1/2}} \quad (5.18)
\]

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where the second line follows by (5.16). This finishes the proof of (5.12) and (5.13). \qed

6. The iteration in \( \{ r > t + 1 \} \)

In this section we prove the optimal pointwise bounds in the region \( r > t + 1 \).

**Theorem 6.1.** If \( r > t + 1 \), then

\[
\phi \lesssim \langle r \rangle^{-1} \langle u \rangle^{-1}.
\]

**Proof.** We begin with the bounds (4.1) and (5.15), combined with (4.3), which in the outside region translate to

\[
|\phi| \lesssim \frac{1}{\langle r \rangle^{1/2}}, \quad |\partial \phi| \lesssim \frac{1}{\langle r \rangle^{1/2}}, \quad \overline{\partial \phi} \lesssim \frac{1}{\langle r \rangle^{3/2}}.
\]

Since \( \langle u \rangle \leq \langle r \rangle \), this can be weakened to

\[
|\phi| \lesssim \frac{1}{\langle r \rangle^{1/2}}, \quad |\partial \phi| \lesssim \frac{1}{\langle r \rangle^{1/2}}, \quad \overline{\partial \phi} \lesssim \frac{1}{\langle r \rangle^{3/2}}.
\]

Recall the decomposition (5.5), and let

\[
H_i = \sum_{k=0}^{2} \|\Omega^k(G_i)\|_{L^2(S^2)}.
\]

We thus have, using (6.2) and (6.1) for \( H_3 \):

\[
H_1 \lesssim \frac{1}{\langle r \rangle^{3/2 + \sigma}}, \quad \partial_1 H_2 \lesssim \frac{1}{\langle r \rangle^{3/2 + \sigma}}, \quad H_3 \lesssim \frac{1}{\langle r \rangle^{2 + \lambda}}, \quad \lambda \in (0, 1)
\]

By (5.8) with \( \alpha = 5/2 + \sigma, \beta = 0, \) and \( \eta = 0 \), we obtain

\[
(\phi_1)_{m+n} \lesssim r^{-1/2 - \sigma}
\]

which gains a factor of \( \langle r \rangle^{-\sigma} \) compared to (6.2). Similarly (5.8) with \( \alpha = 3/2 + \sigma, \beta = 0, \) and \( \eta = 1 \) yields

\[
(\phi_2)_{m+n} \lesssim r^{-1/2 - \sigma}
\]

Finally, (5.8) with \( \alpha = 2 + \sigma, \beta = 0, \) and \( \eta = 1/2 \) yields

\[
(\phi_3)_{m+n} \lesssim r^{-1/2 - \sigma}
\]

The three inequalities above, combined with Proposition 5.4 and (4.3), give the following improved bounds (by a factor of \( \langle r \rangle^{-\sigma} \))

\[
|\phi| \lesssim \frac{1}{\langle r \rangle^{1 + 1/2 + \sigma}}, \quad |\partial \phi| \lesssim \frac{1}{\langle r \rangle^{1 + 1/2 + \sigma}}, \quad \overline{\partial \phi} \lesssim \frac{1}{\langle r \rangle^{3/2 + \sigma}}.
\]

We now repeat the iteration, replacing \( \alpha \) by \( \alpha + \sigma \) and applying (5.8). The process stops after \( \lceil \frac{1}{2\sigma} \rceil \) steps, when (5.8), combined with Proposition 5.4 and (4.3), yield

\[
|\phi| \lesssim \frac{1}{\langle r \rangle^{3 + \sigma}}, \quad |\partial \phi| \lesssim \frac{1}{\langle r \rangle^{3 + \sigma}}, \quad \overline{\partial \phi} \lesssim \frac{1}{\langle r \rangle^{3 + \sigma}}.
\]

We now switch to using (5.7) for \( \phi_1 \) and \( \phi_3 \), and (5.10) for \( \phi_2 \). Note that (6.4) implies

\[
H_1 \lesssim \frac{1}{\langle r \rangle^{3 + \sigma}}, \quad H_2 \lesssim \frac{1}{\langle r \rangle^{2 + \sigma}}, \quad H_3 \lesssim \frac{1}{\langle r \rangle^{3 + \sigma}}.
\]

By (5.7) with \( \alpha = 2 + \sigma, \beta = 1, \) and \( \eta = 0 \), we obtain

\[
(\phi_1)_{m+n} \lesssim r^{-1} \langle u \rangle^{-\sigma}
\]

Similarly (5.10) with \( \alpha = 2 + \sigma, \) and \( \eta = 0 \) yields

\[
(\phi_2)_{m+n} \lesssim r^{-1} \langle u \rangle^{-\sigma}
\]
Finally, (5.7) with $\alpha = 5/2$, $\beta = 1/2$, and $\eta = \sigma$ yields

\[(\phi_3)_{m+n} \lesssim r^{-1}(u)^{-\sigma}\]

The three inequalities above, combined with Proposition 5.4 and (4.3), give the following improved bounds (by a factor of $\langle u \rangle^{-\sigma}$)

\[|\phi_{m+n}| \lesssim \frac{1}{\langle r \rangle \langle u \rangle^{1+\sigma}}, \quad \partial \phi_{m+n} \lesssim \frac{1}{\langle r \rangle \langle u \rangle^{1+\sigma}}, \quad \mathcal{D} \phi_{m+n} \lesssim \frac{1}{\langle r \rangle^2 \langle u \rangle^{\sigma}}. \tag{6.5}\]

We now repeat the iteration, but we carefully observe that this is the last step where we can improve decay for $\phi_3$. Indeed, we have

\[H_3 \lesssim \frac{1}{\langle r \rangle^3 \langle u \rangle^{1+2\sigma}}\]

and (5.7) now yields

\[(\phi_3)_{m+n} \lesssim r^{-1}(u)^{-1}\]

which does not improve as we gain powers of $\langle u \rangle$. On the other hand, we can continue improving the decay rates of $\phi_1$ and $\phi_2$ all the way to

\[(\phi_1)_{m}, (\phi_2)_{m} \lesssim r^{-1}(u)^{-1}\]

which finishes the proof. \qed

7. The iteration in \{r < t - 1\}

In the interior region the iteration is similarly based on Lemma 5.2, with an additional twist. It turns out that plugging in a bound of $\langle r \rangle^{-1}(u)^{\eta}$ on the right hand side is not enough to gain decay. Instead, we first need to turn the $r^{-1}$ factor into a $t^{-1}$ factor.

7.1. Converting $r$ decay to $t$ decay. We start with the following lemma:

**Lemma 7.1.** Assume that $\phi$ solves (1.2). We then have

\[\|\phi_{m}\|_{L^1_t(C^\leq 3T/4)} \lesssim T^{-1}\|\langle r \rangle \phi_{m+n}\|_{L^1_t(C^\leq 3T/4)} + \|Q_{m+n}\|_{L^1_t(C^\leq 3T/4)}, \tag{7.1}\]

**Proof.** One begins with (2.6), and we may assume that $\phi$ is supported in $C^\leq 3T/4$ because the commutator $[P, \chi_{C^\leq 3T/4}]$ can be controlled (here, $\chi_{C^\leq 3T/4}$ is a smooth cutoff). Thus

\[\|\phi_{m}\|_{L^1_t(C^\leq 3T/4)} \lesssim \|\partial \phi_m(T)\|_{L^2_t} + \|Q_m\|_{L^1_t(C^\leq 3T/4)}\]

and from here transitions the spatial norm $L^2_t$ to the $L^2_{t,x}$ norm in the usual way (averaging in time using the scaling vector field $S$) and then transitions to the $L^1_t$ norm. For details, we refer the reader to [22] or [32]. \qed

The next proposition uses the previous lemma to turn $r$-decay in \{r < t/2\} into $t$-decay.

**Proposition 7.2.** Let $\phi$ solve (1.2). Assume that

\[\phi_{m+n} \lesssim \langle r \rangle^{-1}(u)^{-q}, \quad \partial \phi_{m+n} \lesssim \langle r \rangle^{-1}(u)^{-q+\sigma}, \tag{7.2}\]

for some $q \geq -1/2$. We then have

\[\|\phi_{m}\|_{L^\infty_t(C^\leq 3T/4)} \lesssim \langle t \rangle^{-1}(u)^{-q}. \tag{7.3}\]

**Proof.** We estimate the right hand side of (7.1). Using (7.2) we compute

\[T^{-1}\|\langle r \rangle \phi_{m+n}\|_{L^1_t(C^\leq 3T/4)} \lesssim T^{-1}T^{1/2-q} = T^{-1/2-q}\]

\[\|Q_{m+n}\|_{L^1_t(C^\leq 3T/4)} \lesssim \sum R \|\langle r \rangle^{-3/2} t^{-2q+2\sigma}\|_{L^2_t(2T \times A_R)} \lesssim T^{-3/2-2q+2\sigma} \ln T \lesssim T^{-1/2-q}\]
where the last inequality holds for all $q > -1 + 2 \sigma$. Therefore Lemma 7.1 implies

$$\|\phi_{\leq m}\|_{L^1(\mathbb{C}_{T}^{< \frac{3T}{4}})} \lesssim T^{-1/2-q},$$

and the conclusion follows by (3.6).

7.2. The iteration. Finally, we are ready to improve the bounds in the interior region.

**Theorem 7.3.** If $r < t - 1$, then

$$\phi_{\leq m} \lesssim \langle r \rangle^{-1} \langle u \rangle^{-1}.\]$$

**Proof.** As before, we begin with the bounds (4.1) and (4.2), combined with (4.3), which in the inside region translate to

$$|\phi_{\leq m+n}| \lesssim \frac{\langle u \rangle^{1/2}}{\langle t \rangle}, \quad |\partial_1 \phi_{\leq m+n}| \lesssim \frac{1}{\langle r \rangle \langle u \rangle^{1/2}}, \quad |\partial_2 \phi_{\leq m+n}| \lesssim \frac{\langle u \rangle^{1/2}}{\langle r \rangle \langle t \rangle}. \quad (7.4)$$

We thus have, using (7.4):

$$H_1 \lesssim \frac{\langle u \rangle^{1/2}}{\langle r \rangle^{2+\sigma} \langle t \rangle}, \quad \partial_1 H_2 \lesssim \frac{1}{\langle r \rangle^{1+\sigma} \langle u \rangle \langle t \rangle^{1/2}}, \quad H_3 \lesssim \frac{1}{\langle r \rangle^2 \langle t \rangle}.$$  

By (5.7) with $\alpha = 2 + \sigma$, $\beta = 1$, and $\eta = -1/2$, we obtain

$$\phi_{1} \lesssim \langle r \rangle^{-1} \langle u \rangle^{1/2-\sigma}$$

Similarly (5.7) with $\alpha = 2 + \sigma$, $\beta = 0$, and $\eta = 1/2$ yields

$$\phi_{2} \lesssim \langle r \rangle^{-1} \langle u \rangle^{1/2-\sigma}$$

Finally, (5.7) with $\alpha = 2 + \sigma$, $\beta = 1 - \sigma$, and $\eta = 0$ yields

$$\phi_{3} \lesssim \langle r \rangle^{-1} \langle u \rangle^{1/2-\sigma}$$

The three inequalities above give

$$\phi_{\leq m+n} \lesssim \langle r \rangle^{-1} \langle u \rangle^{1/2-\sigma}$$

Now recall that by (7.4) we also have that

$$|\partial_\phi_{\leq m+n}| \lesssim \frac{1}{\langle r \rangle \langle u \rangle^{1/2}}$$

We can now apply Proposition 7.2 (with $q = -1/2 + \eta$), in conjunction with Proposition 5.4 and (4.3), to obtain the following improved bounds (by a factor of $\langle u \rangle^{-\sigma}$):

$$|\phi_{\leq m+n}| \lesssim \frac{\langle u \rangle^{1/2-\sigma}}{\langle t \rangle}, \quad |\partial_\phi_{\leq m+n}| \lesssim \frac{1}{\langle r \rangle \langle u \rangle^{1/2+\sigma}}, \quad |\overline{\partial_\phi_{\leq m+n}}| \lesssim \frac{\langle u \rangle^{1/2-\sigma}}{\langle r \rangle \langle t \rangle}. \quad (7.5)$$

We now repeat the iteration, replacing $\eta$ by $\eta + \sigma$, applying (5.8) and then turning the $r$ decay into $t$ decay by Proposition 7.2. The process stops after $\frac{1}{2\sigma}$ steps, when (5.7), combined with Proposition 7.2, Proposition 5.4 and (4.3), yield

$$|\phi_{\leq m+n}| \lesssim \frac{1}{\langle t \rangle}, \quad |\partial_\phi_{\leq m+n}| \lesssim \frac{1}{\langle r \rangle \langle u \rangle}, \quad |\overline{\partial_\phi_{\leq m+n}}| \lesssim \frac{1}{\langle r \rangle \langle t \rangle}. \quad (7.6)$$

At this point we switch to using (5.10) for $\phi_2$, and the iteration process follows the same pattern as in Section 6, with the extra use of Proposition 7.2 to turn factors of $\langle r \rangle^{-1}$ into factors of $\langle t \rangle^{-1}$. \qed
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Department of Mathematics, University of Kentucky, Lexington, KY 40506
Email address: Shizhuo.Looi@uky.edu