On the probability distribution of the experimental results

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Abstract
The analysis of Tables of particle properties shows that the probability distribution of the results of physical measurements is far from the conventional Gaussian \( \rho(\xi) = \exp(-\xi^2/2) \), but is more likely to follow the simple exponential law \( \rho(\xi) = \exp(-\xi) \) (\( \xi \) is the deviation of the measured from the true value in units of the presented standard error). A gap between the expected and actual probabilities grows with \( \xi \) very rapidly, amounting to \( 10^7 \) at \( \xi \approx 6 \), and is significant even at \( \xi = 2 \). A more detailed study reveals the two-component structure of the distribution: the \( \exp(-\xi) \) law is closely fulfilled up to \( \xi = 3 \), but then, at \( \xi \) larger than that, the decrease is retarded drastically. This behaviour can be associated with the existence of two various types of systematic errors, the detected and undetected ones. Within some model, both types of errors are seen to affect the form of the distribution, one at moderate \( \xi \) and the other at large \( \xi \). The first type (detected) errors are shown in some natural-looking assumptions to yield the distribution not quite equal but close to the simple exponential.

1 Introduction
It is a usual practice in experimental physics to estimate the probability for some measured quantity having a certain value with the use of Gaussian probability distribution. Namely, if the measured value is \( A \pm \Delta \), then the probability that the true value lies in the interval \( (a, a + da) \) is commonly found by the formula (Gaussian standard law):

\[
dw(a) = \frac{da}{\sqrt{2\pi}\Delta} \exp\left(-\frac{(a-A)^2}{2\Delta^2}\right).
\]

The reason for doing so is supposedly provided by the known central limit theorem asserting the validity of Gaussian law \( \square \) for any random variable satisfying certain rather general requirements (see e.g. \( \square \)). Surely, the fulfilment of these requirements as a rule is not analysed in detail, merely postulated when there are no special reasons for doubt; but the Gaussian law, whether justified or not, is extensively used and apparently borne out experimentally in many domains of statistics.

A characteristic feature of the Gaussian distribution is very rapid fall-off of the probability with the departure from the most probable value. Defining the normalized deviation

\[
\xi = \frac{|a - A|}{\Delta}
\]
and the integrated probability from (1)

\[ w(\xi) = \sqrt{\frac{2}{\pi}} \int_{\xi}^{\infty} dt \, e^{-\frac{t^2}{2}} = 1 - \Phi \left( \frac{\xi}{\sqrt{2}} \right) \]  

(Φ is the known probability integral), we have e.g. \( w(4) \sim 10^{-4}, w(5) \sim 10^{-6}, w(6) \sim 10^{-9} \), etc. These numbers are too small to be taken seriously, and they have hardly been ever put to a severe experimental test; one can only argue to have checked the Gaussian distribution in the region of moderate deviations (\( \xi \lesssim 4 \)) where probabilities are measurable. Still Gaussian estimates are widely used also in that faraway region (\( \xi \gtrsim 6 \)), evidently for the lack of a better method.

In particle physics, however, the measurement results deviating from exact values by as much as 6 errors or more are anything but an extreme rarity, their probability has in fact nothing to do with such tiny numbers as \( 10^{-9} \) and amounts to a quantity of order \( 10^{-2} \) (as estimated from the total number of data \( \sim 10^3 \) in Tables of particle properties). This huge discrepancy (about 7 orders), taken literally, is not easy to eliminate within the Gaussian-like behaving distribution functions. A suggestion had been put forward (hardly in earnest), and gained popularity, that a true estimate of probability should be obtained with substituting for the dispersion \( \Delta \) in eq. (1) an experimental error multiplied by 3, since

\[ \exp \left( -\frac{1}{2} \cdot 6^2 \right) \approx 10^{-7} \exp \left( -\frac{1}{2} \cdot \left( \frac{6}{3} \right)^2 \right). \]

The scale factor 3 for the error looks unreasonably big, but regardless of its exact value, this primitive approach appears unsatisfactory because of overstating the significance of a few wrong or inaccurate measurements with large \( \xi \) in the bulk of much better experimental data. Large deviations are obviously due to systematic errors which plague many modern ingeniously designed complex experiments and are difficult to estimate or even to unveil. As no general method of doing this is accepted and every experimentalist uses here his own discretion, there is little reason to believe a priori in Gaussian or some other form of the distribution in question, it is moreover unclear even whether such distribution of any kind should exist at all.

In 1973 the author did the work [2] intended to learn the probability distribution of measurement results with \( \xi \) in experimental way, by closer examination of Tables of particle properties. It was found that the actual distribution (with the understanding that it does exist) showed the behaviour quite different from Gaussian, decreasing with \( \xi \) much more slowly. Remarkably, the form of the distribution appeared to agree fairly well with the simple formula

\[ w(\xi) = e^{-\xi}, \]  

or, in terms of eq.(1),

\[ dw(a) = \frac{da}{\Delta} \, e^{-\frac{|a-A|}{\Delta}}. \]  

Noteworthy is the absence of a possible number coefficient in front of \( \xi \) in the exponent, i.e. this coefficient proved to equal 1 with reasonable accuracy.

In a related work by M.Roos et al. [3] the problem was treated somewhat differently. These authors also detected large departure of the real data distribution from Gaussian but gave preference to a power-behaving (Student-like) fit involving two free parameters (see eq. (13) of Sec. 2). While looking less attractive than our eq. (1), it provided better conformity
with the actual distribution; however, the arguments to its favour are not convincing and do
not allow one to decide between this and some other fit of a similar kind. The fact is that
the form of the fitting function is not specified by a theory and can be chosen more or less
arbitrarily. The present work in its major part is aimed at finding this functional form from
some probable-looking model assumptions.

The attempts to explain the non-Gaussian behaviour of data distribution made in both
works [2, 3], though different in resulting formulae, were conceptually similar and based on
the idea of taking into account the inevitable uncertainty in the experimentally determined
errors. The ambiguity of the result is a consequence of essential dependence on the assumed
form of the unknown probability distribution for the errors. Here is the weak point of this
approach: the effect of the error uncertainty, surely existing and significant as it is, escapes
solid quantitative estimates and can provide only a crude explanation of the Gaussian law
violation, defying accurate experimental tests.

It will be seen however that there exists another, in a sense competitive effect which is
more liable to evaluation and may appear dominant at some ξ. It can be designated the
variable dispersion effect and is operative even with fixed and exactly known mean squared
error, provided the dispersion of the measuring device is not constant but depends on the
value of the measured quantity. While the first effect (the error uncertainty) is caused mainly
by the systematic errors overlooked or radically underestimated by the experimentalists, the
second effect can be attributed to the influence of those systematic errors which are detected
and taken properly into account, but nevertheless induce a distortion of Gaussian law in
default of compliance with the conditions of central limit theorem.

In this work the experimental data probability distribution is reexamined on the basis of
a later and more abundant statistical material. It turns out that the distribution exhibits
a well-marked two-component structure characteristic of the superposition of two functions
with different behaviour (see fig. 2, curve 1 to be discussed below): one fall-off regime is
changed rather abruptly by the other near the point ξ = 3. It seems natural to associate these
two regimes with two various mechanisms whose contributions have different ξ dependence
and become comparable at ξ ≈ 3. The right-hand portion of the distribution curve (ξ >
3) which was not quite clearly visible with the earlier poor statistics may well have been
produced by the effect of error uncertainty, while the left-hand portion following closely the
simple-exponential law of eq. (4) could be shaped with the variable dispersion effect which
will be seen to yield a distribution of a similar form. Of particular interest is the fact that
the unit coefficient in the exponent gets a natural explanation in this scheme.

The paper is organized as follows. Sec. 2 contains an account of the method and results
of experimental determination of the data probability distribution with the use of Tables
of particle properties. The remaining part of the paper is concerned with the theoretical
interpretation of the obtained distribution. In Sec. 3 the effect of the error uncertainty is
considered. Sec. 4 is devoted to a cursory discussion of the central limit theorem and a
possible way of its expansion. In Sec. 5 which is the main in the work, a model of the
variable dispersion effect is developed and analysed. It is shown that under some rather
general assumptions the asymptotic form of the distribution at ξ ≫ 1 must be ξ⁻²exp(-ξ).
The last Sec. 6 contains a short review of the results and some concluding remarks.
2 Experimental distribution

For experimental evaluation of the data probability distribution all which is needed is a sufficiently abundant set of measurements of some quantity with the exactly known true value. Of course, the true value is normally unknown during the measurement, but we can restrict our consideration to the old experiments for which the later, much more accurate results are now available to be taken for the exact values. As for the number of measurements, although it is hardly ever large enough when dealing with only one definite physical quantity, it can well be made statistically significant by using the hypothesis of universality of the distribution, i.e. assuming the probability \( w(\xi) \) to depend only on \( \xi \) and nothing else. The probability density will then agree with the formulae

\[
dw(a) = \frac{da}{\Delta} \rho(\xi),
\]

\[
\rho(\xi) = -w'(\xi),
\]

generalizing eqs. (4) and (5). This hypothesis, if true (which point will be returned to below), allows one to take into account all measurements with the same \( \xi \) on equal terms, lumping together the results of quite dissimilar experiments, irrespective of the quantity measured or the technique employed. Note that we suppose the distribution to be symmetric about its centre, since \( \xi \) defined by eq. (2) is an even function of \( (a - A) \), so we should avoid consideration of measurements with manifestly asymmetric errors.

The distribution function \( w(\xi) \) designating the probability for the measurement result to deviate from the true quantity by \( \xi \) or more standard errors, can thus be found from any set of data by the formula

\[
w(\xi) = \frac{1}{N} \sum_{i=1}^{N} \theta \left( \frac{|A_i - a_i|}{\Delta_i} - \xi \right),
\]

where \( N \) is the total number of data in the set (supposed to be very large), \( a_i \) is the exact value of the quantity measured in \( i \)-th experiment, \( A_i \) and \( \Delta_i \) are its measured value and the error, and \( \theta(x) \) denotes the Heavyside step function (1 for \( x > 0 \) and 0 for \( x < 0 \)). When using Tables of particle properties as a data source, \( A_i \) and \( \Delta_i \) are suggested to be taken from an earlier (‘old’) Table, \( a_i \) from the latest (‘new’) Table. So in practice the values of \( a_i \) are themselves not quite exact but approximate with the new errors \( \delta_i \), and can be treated as nearly exact only when a new error \( \delta_i \) is small compared to the corresponding old error \( \Delta_i \), which is not always the case. The influence of uncertainty of \( a_i \) can be suppressed by including in the set only the data with sufficiently large ratio \( \Delta_i/\delta_i \); in what follows a rather liberal data cutoff is used

\[
\frac{\Delta_i}{\delta_i} \geq 2.5.
\]

(In changing the cutoff from 2.5 to 4 was found to reduce the number of data \( N \) by about 10% with quite negligible effect on the distribution).

The described method of finding the function \( w(\xi) \) was applied first in 1973 \( [2] \) to the Tables of 1964 \( [4] \) and now once again to the Tables of 1978 \( [6] \) (the old data), with ‘exact’ values extracted from the new Tables of 1972 \( [5] \) and 1994 \( [7] \) respectively. Apart from the restriction \( (9) \), the procedure of data selection included yet some more criteria. The following data types were rejected:

— the results treated by the compilers as unreliable or preliminary (enclosed in parentheses);
— the data taken from review articles (marked by the label RVUE);
— the results of measurements of essentially positive (by physical meaning) quantities when the error exceeded half the measured value;
— the data with asymmetric errors of the form $A^{\pm}\Delta_1$ when $\Delta_1$ and $\Delta_2$ differed by more than 10\% (otherwise the result was accepted with the error taken to be symmetric and equal to the greater of $\Delta_1$ and $\Delta_2$);
— all data relating to wide resonances, $\Gamma > 120$ MeV. In [2], moreover, no resonance data at all were considered, only those relevant to strong-interaction stable particles.

We are not dwelling on the discussion of these constraints, aimed mainly at lowering the contribution of dubious or too uncertain results and of some temporal changes in general concepts (e.g. in the common definition of the mass and width of very wide resonances). One may argue if the variation (strengthening or loosening) of formulated criteria should really have no effect but the change of the total normalization; we shall touch on this point once more in the next section.

It must be emphasized that the total statistical sample of 933 data examined in this work includes all measurement results contained in the Table [5], irrespective of the quantity taken, except only for those rejected by the above-listed restrictions. Its inspection shows that it comprises four groups of measurements having comparable weights (classified by the quantity that was measured): measurements of particle masses (223 data, or 24\%), lifetimes or widths (196 data, or 21\%), branching ratios of various decay modes (316 data, or 34\%), and the last group (198 data, or 21\%) combines all other particle characteristics (magnetic moments, form factors, parameters of angular and energy distributions of decay products) as well as some interaction constants ($g_A/g_V$, parameters of $CP$-nonconservation etc.). No account was taken of possible correlation between various quantities obtained from one experiment, they were taken to be independent if the compilers used them in statistical averaging. The number of such many-data experiments does not seem to be very small, this effect may cause some distortion and should be kept in mind. Another analogous interfering factor could be the mutual influence and correlation between the results of various experimental groups which are ideally viewed as quite independent. The significance of these effects is difficult to consider quantitatively, but one may hope it would be true to ignore them in the first approximation.

The results of the described treatment of Particle Data Tables are shown in figs. 1 and 2 (the curves labelled 1). Fig. 1 reproduces the earlier finding [4] and is given here for convenience of comparison; the newly obtained distribution of data from [5] is presented in fig. 2. The difference is seen to be modest and can well be assigned to damping of statistical fluctuations with growth of the total statistics from 209 to 933; however, it should be noted that there is a certain systematic excess of the distribution in fig. 2 over that in fig. 1 which may appear to be evidence against the above-formulated universality. Here we shall not discuss this point in much detail, but turn now to the other, much more pronounced features of the resulting distribution.

First of all, the real distribution is apparent to differ radically from Gaussian (curve 2 in figs. 1 and 2), especially at large $\xi$, the fact already noted above. The difference grows quickly with $\xi$ and ranges up to many orders at $\xi \approx 6$, but is rather large even for moderate values of $\xi$ ($\approx 3$ times at $\xi = 2$, $\approx 20$ times at $\xi = 3$). It is clear that Gaussian estimates highly underrate the deviation probabilities nearly everywhere in $\xi$ and are entirely misleading.

An essential fact here is that the real distribution curve 1 not only passes much higher than the Gaussian curve 2, but moreover exhibits a distinctly different behaviour: the
two curves have the opposite convexity (in log scale), i.e. the real probability decay rate $d[ln w]/d\xi$ decreases with $\xi$, while by Gaussian law it should be growing $\sim \xi$. It follows that there is little point in modifying Gaussian law \( \text{[1]} \) with only renormalizing the error $\Delta$ by any constant factor, as this procedure leads to nothing more than stretching the curve along the $\xi$ axis with no effect on convexity. The curves 4 and 5 in fig. 1 show that doubling, the more so tripling the error (i.e. replacing $\xi$ in eq. \( \text{[3]} \) by $\xi/2$ or $\xi/3$) results in grossly overestimating the deviation probabilities against their experimental values in the region of moderate $\xi$. Clearly, the agreement cannot be achieved by any choice of the error renormalization factor and the Gaussian-like approach itself does not work.

It is natural to try for the experimental distribution the linear-exponential approximation \( \text{[4]} \) (straight line 3 in figs. 1 and 2) which appears to fit rather well, better than one could expect without any prerequisite. Specifically, a fact deserving consideration is the unit slope of the fitting curve, which corresponds to the above-mentioned unit coefficient in the exponent of eq. \( \text{[4]} \). Of course, it may well be a mere coincidence, but it is suggestive to see here a regular phenomenon and to search for its origin.

Closer inspection of fig. 2 shows that the linear-exponential fit \( \text{[4]} \) being very good at $\xi < 3$ breaks down at large $\xi$ where the real probability curve 1 displays essentially slower fall-off. It seems that the point $\xi = 3$ marks the region where the fall-off regime undergoes a change closely resembling that of the decay curve of a two-component radioactive substance. A limited statistics (65 measurements with $\xi > 3$ of the total 933) does not allow one to recognize the functional form of the decrease at large $\xi$, but its slowing down is seen clearly enough, in distinction to fig. 1 where the statistics was too meagre for that (9 measurements with $\xi > 3$ of the total 209). As a simple illustrative example, the function with a similar behaviour

$$w_1(\xi) = \frac{e^{-\xi} + C}{1 + C}, \quad C = 0.05$$  \hspace{1cm} (10)

is displayed in fig. 2 (curve 4). The value of the constant $C$ is chosen so that the two terms in the numerator were equal at the breakpoint $\xi = 3$, the denominator serves to ensure the normalization condition

$$w_1(0) = \int_0^\infty \rho(\xi) d\xi = 1.$$  \hspace{1cm} (11)

Of course, this function cannot represent the real distribution at all $\xi$ since it does not vanish at $\xi \to \infty$, but can be considered as a reasonable approximation for moderate $\xi$ when the ‘long-lived’ component decays very slowly and its change is inappreciable in the range of $\xi$ under study.

It should be noted that both approximations \( \text{[1]} \) and \( \text{[10]} \) are expected to fail also at small $\xi$, because they do not satisfy the condition

$$\rho'(0) = -\left. \frac{d^2 w}{d\xi^2} \right|_{\xi=0} = 0$$  \hspace{1cm} (12)

which must be fulfilled if the probability density $\rho(\xi)$ is even in $\xi$ and analytic at the point $\xi = 0$. In fact at small $\xi$ the distribution follows Gaussian rather than linear-exponential law, which can be ascertained by viewing fig. 2 at some magnification; the difference is slight but unambiguous in favour of Gaussian behaviour.

Before proceeding to the interpretation of the indicated features of the experimental data probability distribution, it will be pertinent to touch briefly on the work \( \text{[3]} \) concerned with...
Figure 1:

1—experimental distribution of 209 measurement results, ref. [2]; 2—Gaussian standard law, eq. (3); 3—simple exponential law, eq. (4); 4, 5—Gaussian distribution with doubled and tripled dispersion.
Figure 2:
(1) — new experimental distribution of 993 measurement results from ref. [7]; (2), (3) — same as in fig. 1; (4) — modified exponential distribution, eq. (10).
similar problems. Its authors had a well-defined practical aim—to develop a sound procedure for averaging together the poorly reconcilable measurement results obtained by various experimental groups, and doing this required information on the real data distribution. To get it, they used a procedure differing from ours in some respects. First, they made efforts to possibly lower the influence of systematic errors and tried to select only those measurements which would not be subject to them. Second, they had to do with only one table (the latest one) and no later tables, so in place of the true value \( a \) in eq. (2) they used a weighted mean calculated by averaging the related measurement results from the same table. Thus their distribution does not quite coincide with ours and some difference would appear natural. It is worth noting that they did not merely postulate the universality condition (7) but put it to some check, separating the total sample of 306 measurements into several groups and comparing the distributions in various groups; all of them proved to be essentially the same within the probable statistical fluctuations.

In fig. 3 curve 1 shows the total data distribution from [3] represented in the form suitable for comparison with figs. 1 and 2 (in [3] a histogram for the probability density \( \rho(\xi) \) was given; here it is recast into the integral distribution \( w(\xi) \) with some interpolation). It can be observed to bear certain similarities to the distribution in fig. 2: a remarkable excess over the Gaussian distribution (curve 2), an approximately linear-exponential decrease at intermediate \( \xi \), and bendings (decay slowing down) both at small and large \( \xi \) (the latter being not quite clearly seen probably owing to deficient statistics, only 1.5 times that in fig. 1). There are also clear-cut differences: the large \( \xi \) bending (if any) occurs later in \( \xi \), and in the linear-exponential region the slope is about 1.3 instead of unity (i.e. \( w(\xi) \sim \exp(-1.3\xi) \)), so approximation (1) (curve 3) is here not good. On the whole, the distribution looks as a kind of interpolation between the Gaussian law and something like eq. (10) (curves 2 and 4 in fig. 2).

For a fitting function to describe the obtained distribution, there was taken in [3] a slight modification of Student’s formula

\[
\rho(\xi) = K \left(1 + \frac{\xi^2}{nc^2}\right)^{-\frac{n+1}{2}}
\]  

(13)

containing two free parameters, \( n \) and \( c \) (normalization constant \( K \) is defined from eq. (11)). With \( c = 1 \) and integer \( n \), eq. (13) represents the probability density of a random quantity

\[
\xi = \sqrt{\frac{n}{n+1}} \sum_{i=1}^{n+1} x_i \left(\frac{1}{n+1} \sum_{k=1}^{n+1} x_k\right)^2,
\]

(14)

which is a function of \( (n+1) \) random variables \( x_i \) all having identical normal distributions with zero mean value (or alternatively they can be viewed as the results of \( (n+1) \) independent measurements of a random variable normally distributed about zero). A scaling parameter \( c \) is introduced to take into account a possible underestimate of the measurement error by the experimentalists just in the manner discussed in Introduction. The values of the parameters adequate to experimental distribution were found to be

\[
n = 10, \quad c = 1.11.
\]
Figure 3:
①—distribution of 306 measurement results from ref. [3]; ②,③—same as in fig. 1; ④—modified Student’s distribution, eq. (13).
The corresponding integral distribution \( w(\xi) \) is represented by curve 4 in fig. 3 which practically coincides with the experimental curve 1 nearly everywhere (except for large \( \xi \) where the statistics is scanty). Notwithstanding this coincidence, the form of eq. (13) does not look convincing; below the arguments will be adduced for another dependence, more closely resembling (but not identical to) eq. (10).

3 The effect of the error uncertainty

The most prominent feature of all distributions considered above (figs. 1–3) is their large excess over the Gaussian law predictions, and its explanation is natural to try to begin with taking into account the obvious fact that the experimental error is never measured exactly but may differ significantly from the true dispersion of the measuring device. Such approach was applied in [2] and in a less explicit form in [3]; here it will be reexamined in some detail.

Let us consider a model based on the following postulates:

1. Each measurement device or experimental setup is subject to fluctuations producing at the output in place of the measured quantity \( a \) a random quantity \( x \) spread with some probability density \( p(x, a) \) which is an individual characteristic of a given device. In other words, when measuring a quantity \( a \), one obtains the result confined between \( x \) and \( x + dx \) with the probability \( p(x, a) \, dx \). The probability density \( p(x, a) \) is assumed to have a Gaussian form

\[
p(x, a) = \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{(x-a-s)^2}{2\sigma^2}} \quad (15)
\]

with two parameters \( s \) (scale shift) and \( \sigma \) (dispersion) characterizing a given device.

2. There exists in the world a huge number \( M \) of devices distributed over the parameters \( s \) and \( \sigma \) with the density \( P(s, \sigma) \), so that the number of devices with the values of the parameters between \( (s, \sigma) \) and \( (s + ds, \sigma + d\sigma) \) is \( dM = MP(s, \sigma) \, ds \, d\sigma \).

3. The parameters \( s \) and \( \sigma \) of any given device are not known exactly but measured with a limited accuracy and found, instead of \( (s, \sigma) \), to be in the range of \( (s', \sigma') \) to \( (s' + ds', \sigma' + d\sigma') \) with the probability \( q(s, \sigma|s', \sigma') \, ds' \, d\sigma' \). The probability density \( q(s, \sigma|s', \sigma') \) is supposed to be i) a function of the difference \( (s - s') \), not of \( s \) and \( s' \) separately, and ii) a homogeneous function of its four arguments, i.e.

\[
q(s, \sigma|s', \sigma') = \frac{1}{\sigma^2} q\left(\frac{s' - s}{\sigma}, \frac{\sigma'}{\sigma}\right), \quad (16)
\]

where \( q(u, t) \) is dimensionless.

In such model the experimental distribution of measurement results with the quantity

\[
\xi = \frac{x - s' - a}{\sigma'}
\]

will be

\[
\frac{dw(\xi)}{d\xi} = \int P(s, \sigma) \, ds \, d\sigma \int q(s, \sigma|s', \sigma') \, ds' \, d\sigma' \int dx \, p(x, a) \, \delta\left(\xi - \frac{x - s' - a}{\sigma'}\right)
\]

\[
= \int P(s, \sigma) \, ds \, d\sigma \int q(s, \sigma|s', \sigma') \, ds' \, d\sigma' \frac{\sigma'}{\sqrt{2\pi\sigma}} e^{-\frac{(\xi\sigma' + s' - s)^2}{2\sigma^2}}.
\]

(18)
Note that $\xi$ as defined by eq. (17) slightly differs from that of eq. (2) and is no more a positive definite quantity; to come back to former definition in eq. (15) and analogous formulae below a symmetrization with respect to $\xi \rightarrow -\xi$ should be made.

Using eq. (17) and introducing dimensionless variables
\[ u = \frac{s' - s}{\sigma}, \quad t = \frac{\sigma'}{\sigma}, \]
we can separate the integral over $ds\,d\sigma$ and take it in view of the normalization condition
\[ \int P(s, \sigma) ds\,d\sigma = 1, \]
so, as might be expected, the distribution of devices $P(s, \sigma)$ drops out of the formula and eq. (18) becomes
\[ \rho(\xi) = \frac{1}{\sqrt{2\pi}} \int du\,dt \, q(u, t) \, e^{-\frac{1}{2}(u + t \xi)^2}. \]

An essential input on the right of eq. (21) is the function $q(u, t)$ governing the accuracy of determination of the measurement error $\sigma'$, the resulting distribution being highly dependent on the suggested properties of this unknown function. Clearly, taking
\[ q(u, t) = \delta(u) \delta(t - 1), \]
we arrive at Gaussian distribution (3). Replacing $\delta(t - 1)$ by $\delta(t - k)$ (with $k$ constant) gives the scaled Gaussian law discussed in Introduction. Any kind of smearing the $\delta$-function yields generally some departure from normal distribution, and there is nothing to prevent from getting any reasonable expression for the result by choosing an appropriate form of $q(u, t)$. In particular, putting
\[ q(u, t) = \delta(u) \frac{\alpha}{t^3} e^{-\frac{\alpha}{2t^2}}, \]
we get from eqs. (21) and (7)
\[ w(\xi) = \frac{1}{\sqrt{\alpha}} e^{-\xi \sqrt{\alpha}}, \]
which at $\alpha = 1$ is just eq. (4). It can be remarked that if $q(u, t)$ has the form $\delta(u)q(t)$ with $q(t)$ smooth and vanishing both at $t = 0$ and $t \rightarrow \infty$, then the small $\xi$ behaviour of $w(\xi)$ is determined in general by the large $\xi$ asymptotics of $q(t)$, and conversely, the asymptotics of $w(\xi)$ at $\xi \rightarrow \infty$ depends on the behaviour of $q(t)$ at $t = 0$.

Among other possible expressions for the function $q(u, t)$ of interest is that one which corresponds to the parameters $c$ and $\sigma$ of a device being determined from a certain number $n$ of independent measurements of some known quantity $a$. If the results of these calibrating measurements are $x_i$ $(i = 1, \ldots n)$, then
\[ s' = \frac{1}{n} \sum_{i=1}^{n} x_i - a, \quad \sigma'^2 = \frac{1}{n - 1} \sum_{i=1}^{n} (x_i - a - s')^2, \]
so, in view of the relation $d\sigma'^2 = 2\sigma' d\sigma$ and eq. (13),
\[ q(s, \sigma|s', \sigma') = 2\sigma' \int \frac{d^n x}{(\sqrt{2\pi\sigma})^n} e^{-\frac{1}{2\sigma^2} \sum_i (x_i - a - s)^2}. \]
\[ \delta(s' + a - \frac{1}{n} \sum_i x_i) \, \delta(\sigma'^2 - \frac{1}{n - 1} \sum_i (x_i - a - s')^2) \]
\[ \sim \frac{\sigma'^{n-2}}{\sigma^n} e^{-\frac{1}{2\sigma^2} [(n - 1)\sigma^2 + n(s' - s)^2]}. \]
where use is made of the relation

$$\int d^n x \, \delta\left(\sum_{i=1}^{n} x_i\right) \delta\left(\sum_{i=1}^{n} x_i^2 - R\right) = \frac{\pi^{n-1/2}}{\sqrt{n} \Gamma\left(\frac{n-1}{2}\right)} R^{n-3}$$  \hspace{1cm} (27)$$

and the normalization factor is dropped as inessential. It follows from eqs. 26 and 16 that in this model

$$q(u, t) \sim t^{n-2} e^{-n/2 u^2 - n-1/2 t^2}. \hspace{1cm} (28)$$

Inserting this into eq. (21), we obtain

$$\rho(\xi) \sim \left(1 + \frac{n \xi^2}{n^2 - 1}\right)^{-n/2}. \hspace{1cm} (29)$$

which is nearly the same as eq. (13). (The exact eq. (13) with \(c = 1\) and \(n\) replaced by \(n - 1\) would result if the \(u\)-dependence in eq. (24) were taken to be \(\delta(u)\).)

Eq. (29) presents a typical appearance of the departure from normal distribution owing to error uncertainty induced by statistical reasons only, i.e. by insufficiently large number \(n\) of measurements. Obviously, at \(n \to \infty\) the distribution becomes Gaussian, except for very large \(\xi \gg n^{1/4}\) where the correction \(\sim \xi^4/n\) becomes essential; at still larger \(\xi \gg \sqrt{n}\) the quadratically-exponential fall-off is in fact changed by a power. In this model systematic errors are disregarded or, maybe one can say, simulated by something looking like statistical errors, which does not seem to be correct, and that probably manifests itself in the necessity of introducing a scaling parameter \(c\) into eq. (13).

The implication of the considered examples is that no definite conclusions regarding the probability distribution \(w(\xi)\) can be drawn from eq. (21) without one or other assumption about the function \(q(u, t)\) which embodies information on the mechanism of generating the error uncertainty. However, there is yet another simple model worth discussing, based on the hypothesis that the primary source of the departure from normal distribution hides in the overlooked systematic errors. It is characterized by the function \(q(u, t)\) of the form

$$q(u, t) = (\delta(u) + \mu(u)) \delta(t - 1), \hspace{1cm} (30)$$

where \(\mu(u)\) is some smooth function (an overall normalization coefficient is omitted). Contrary to both models of \[2\] and \[3\], here the the device dispersion \(\sigma\) (the spread of the distribution \text{ref15} ) appears to be measured with a good accuracy \(\sigma' = \sigma\) due to the second \(\delta\)-function and eq. \(19\), while systematic errors show up in some indefinite shift \(s\) of the measurement scale and are reflected by the addition of \(\mu(u)\) to \(\delta(u)\) in eq. \(30\). The additive form of the first multiplier corresponds to the assumption that all measurements can be divided into two separate groups of a comparable weight — the ‘good’ measurements which are free from systematic errors, and the ‘bad’ ones which are subject to them, these two groups being represented by two terms \(\delta(u)\) and \(\mu(u)\) respectively. Since the systematic errors are randomly distributed in their values without any visible correlation to a measured error (which they may exceed many times), the function \(\mu(u)\) can be considered as nearly constant at \(u \sim 1\), with some decrease at very large \(u\) rapid enough to provide the convergence of the normalization integral. So, the dependence on \(u\) in eq. \(30\) looks as a sharp peak at \(u = 0\) rising above a smooth, almost uniformly smeared background. Clearly, the weight of the background is expected to vary with changing the criteria of the data selection, and
in this sense the above-discussed universality of the distribution \( w(\xi) \) cannot exactly hold; the function \( \mu(u) \) can be viewed to contain an uncertain numerical factor depending on the selection procedure.

Inserting eq. (30) into (21), one can perform the integration if the function \( \mu(u) \) is slowly varying \( (\mu'(u) \ll 1) \), and get

\[
\rho(\xi) \approx \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}} + \mu(-\xi),
\]

(31)

since the Gaussian exponential of eq. (21) acts as a \( \delta \)-function when integrated with \( \mu(u) \).

The obtained formula is qualitatively similar to the above eq. (10) — it represents a sum of two functions, one rapidly and the other slowly decreasing, and its plot must have a bend near the value of \( \xi \) at which the two terms become equal. One of the pronounced features of experimental distribution (curve 1 in fig. 2) thus finds rather simple and natural explanation in the influence of unrevealed systematic errors. There is however a remarkable difference in the first term between eqs. (31) and (10). In the model just considered the distribution is distorted only at large \( \xi \) and remains nearly Gaussian to the left of the bending point, while the experimental curve 1 runs essentially above the Gaussian curve 2 even at \( \xi < 3 \). Moreover, had the first term of eq. (10) Gaussian-like form, the right-hand nearly horizontal portion of the curve (at \( \xi > 3 \)) would have to go much lower than it really does, since from eq. (3) \( w(3) \approx 0.003 \) instead of actual 0.05. This disparity may look not serious, as it can be easily eliminated by an appropriate smearing of the \( \delta \)-functions in eq. (30). However, such way of doing is not quite satisfactory because of its arbitrariness and indeterminacy, it leaves too much freedom in the choice of the function \( q(u, t) \) and does not allow to say anything definite about what the distribution must be like for theoretical reasons. Nor is it of any use in clarifying the origin of the unit coefficient in the exponent of eqs. (10) and (4) which is a pure accident in this scheme. Therefore it makes sense to look for some other sources of possible distribution deformations which could be effective at \( \xi < 3 \) and maybe shed light on the latter problem.

Eq. (21) can be easily extended to the case of the probability density \( p(x, a) \) in eq. (14) having a more general than Gaussian form

\[
p(x, a) = \frac{1}{\sigma} p\left(\frac{x - a - s}{\sigma}\right),
\]

(32)

then

\[
\rho(\xi) = \int du dt q(u, t) tp(u + t\xi).
\]

(33)

If the function \( p(y) \) is rapidly decreasing at large argument, the hypothesis of eq. (30) yields a similar generalization of eq. (21):

\[
\rho(\xi) \approx p(\xi) + \mu(-\xi).
\]

(34)

Again we see that at \( \xi \) not very large where the first term is dominant (the boundary value of \( \xi \) depending on the selection criteria), the experimentally measured distribution duplicates the primary scatter caused by the device, and we come back to the problem of explaining its non-Gaussian character. So, in what follows we ignore the measurement uncertainty of the function \( p(\xi) \), taking \( q(u, t) \) to be defined by eq. (22), and concentrate on examining a possible form of \( p(\xi) \) itself, which is then the same as \( \rho(\xi) \).
Normal distribution, be it justified or not, holds a specific position in statistics and is often favoured over other distributions in default of information. So, clearing up the question why should (or shouldn’t) some probability distribution be normal, it is worthwhile to recall the central limit theorem which is in fact the only argument for that. A simple version of the central limit theorem (there are many of them, see e.g. [1]) can be formulated as follows.

Let \( x_1, x_2, \ldots x_n \) be \( n \) independent random quantities with distribution density of \( x_k \) being \( \rho_k(x_k) \). Let three lowest moments of each distribution \( \rho_k \) exist:

\[
\begin{align*}
M_{x_k} &= \mu_k, \\
M (x_k - \mu_k)^2 &= \sigma_k^2, \\
M |x_k - \mu_k|^3 &= t_k^3
\end{align*}
\] (35)

(here \( Mf \) denotes mean value of \( f \)). Construct a new variable \( y = \sum_{k=1}^{n} x_k \) having the distribution density

\[
\rho(y) = \int dx_1 \ldots dx_n \rho(x_1) \ldots \rho(x_n) \delta(y - \sum_{k=1}^{n} x_k)
\] (36)

with the moments

\[
\begin{align*}
M_y &= \sum_{k=1}^{n} \mu_k = \mu, \\
M (y - \mu)^2 &= \sum_{k=1}^{n} \sigma_k^2 = \sigma^2, \\
M \sum_{k=1}^{n} |x_k - \mu_k|^3 &= \sum_{k=1}^{n} t_k^3 = t^3
\end{align*}
\] (37)

and consider the limit \( n \to \infty \). The theorem states that if the requirement

\[
\lim_{n \to \infty} \frac{t}{\sigma} = 0
\] (38)

(Liapunov’s condition) is satisfied, then, whatever the distributions \( \rho_k(x_k) \) are, the distribution \( p(y) \) tends to Gaussian with mean value \( \mu \) and dispersion \( \sigma \):

\[
\lim_{n \to \infty} p(y) = \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}.
\] (39)

The meaning of the condition (38), roughly speaking, is that it necessitates fluctuations of all \( x_k \) to make comparable contributions into fluctuations of \( y \), preventing from only a small number of them being really effective. In a typical case with all \( \sigma_k \) of the same order, \( \sigma \sim \sigma_1 n^{1/2}, \quad t \sim \sigma_1 n^{1/3}, \quad t/\sigma \sim n^{-1/6} \to 0 \). (The overall factor \( \sigma_1 \) itself is supposed to be \( \sim n^{-1/2} \), so that \( \sigma \) tends to a constant at \( n \to \infty \)).

For logical coherency we sketch here an outline of the proof. Introducing Fourier transforms

\[
\begin{align*}
\int dx e^{ixu} \rho_k(x) &= \phi_k(u), \\
\int dy e^{iyu} p(y) &= \Phi(u)
\end{align*}
\] (40)

(41)

we have from eq. (33)

\[
\Phi(u) = \prod_{k=1}^{n} \phi_k(u), \quad \ln \Phi(u) = \sum_{k=1}^{n} \ln \phi_k(u).
\] (42)
With eq. (38) being fulfilled, each \( \rho_k \) at \( n \to \infty \) is shrinking towards its maximum at \( x = \bar{x}_k \) and the exponential in the integrand of eq. (41) can be expanded near that point to yield

\[
\phi_k(u) \approx e^{iu\bar{x}_k} \left( 1 - \frac{u^2}{2} \sigma_k^2 \right),
\]

(43)

\[
\ln \phi_k(u) \approx iu\bar{x}_k - \frac{u^2}{2} \sigma_k^2.
\]

(44)

Inserting eq. (44) into (42) and using eqs. (37), we obtain

\[
\ln \Phi(u) \approx iu\bar{y} - \frac{u^2}{2} \sigma^2.
\]

(45)

An important point is that the cubic and all higher terms in the right-hand side vanish at \( n \to \infty \) due to eq. (38). Exponentiating eq. (45) and making inverse Fourier transform of \( \Phi(u) \) according to eq. (41), we get the Gaussian expression (39) for \( p(y) \).

It may be remarked that the statement of the theorem is true also in a more general case when \( y \) is not an exactly linear function of the variables \( x_k \). It is sufficient for this function to be smooth in the vicinity of the point \( x_k = \bar{x}_k \) and exhibit essential changes only with the departure from this point at distances much greater than the dispersions \( \sigma_k \), so that it could be well approximated by its linear expansion in differences \( (x_k - \bar{x}_k) \) (the coefficients of this expansion can easily be renormalized to unity by redefinition of \( x_k \)).

Another possible way of extension of the theorem is associated with abandoning the condition of independence of random variables \( x_k \). If they are correlated, then the product \( \rho_1(x_1) \cdots \rho_n(x_n) \) in the integrand of eq. (36) which is effectively proportional to

\[
e^{-\frac{1}{2} \sum_{k=1}^{n} \frac{(x_k - \bar{x}_k)^2}{\sigma_k^2}},
\]

is expected to be replaced by

\[
e^{-\frac{1}{2} T_{kl} (x_k - \bar{x}_k) (x_l - \bar{x}_l)},
\]

(46)

where \( T_{kl} \) is some non-diagonal positive definite constant \( n \times n \) matrix (Liapunov’s condition (38) is supposed to be properly modified). It is clear that such replacement does not affect the form of the function \( \rho(y) \) which remains Gaussian.

If now a random quantity has a distribution different from Gaussian, it would be well to conceive which of the conditions of the central limit theorem is violated. There are not many possibilities for that. First one, most obvious and suggesting itself, can be that the number \( n \) of effective disturbing factors is not large or only a few of them fluctuate significantly. A typical example is when all dispersions \( \sigma_k \) are negligible in comparison with one \( d_1 \) and the sum in the second eq. (37) is dominated by one term \( \sigma_1^2 \), so that all \( x_k \) except \( x_1 \) can be considered as constants and the distribution of a quantity \( y \) coincides in essence with that of \( x_1 \) up to some shift. If \( n \) is large but finite, then the dropped cubic in \( u \) term in eq. (45) is generally of order \( n(\sigma_1 u)^3 \sim (\sigma u)^3/\sqrt{n} \), and since from eq. (11) (or its inverse)

\[
u \sim \frac{y - \bar{y}}{\sigma^2},
\]

(47)

the Gaussian law is expected to break down at

\[
\xi = \frac{y - \bar{y}}{\sigma} \gtrsim n^{1/6},
\]

(48)
where this term becomes of order or more than 1. (This condition is replaced by \( \xi \gtrsim n^{1/4} \) with the analogous estimate of a quartic instead of cubic term in the case when the latter happens to vanish). It is seen that the number \( n \) must be extremely large to provide the validity of the standard law even only for \( \xi \leq 4 \).

Another natural possibility for a quantity \( y \) to have non-Gaussian distribution consists in the nonlinear dependence of \( y \) on the disturbing factors \( x_k \). The usual way of reasoning is that if the number \( n \) of \( x_k \)'s is large and each partial dispersion \( \sigma_k \) is small \( \sim \sigma n^{-1/2} \), then any smooth function \( y(\{x_k\}) \) varying at scales of \( x_k \sim \sigma \) can be linearized for small variations of \( x_k \sim \sigma_k \). However, it is not true, which is seen from the fact that if \( y \) is normally distributed then \( z = f(y) \) is generally not. This paradox emerges from the invalidity of a linear approximation for the function \( y(\{x_k\}) \) when it changes essentially only in a small number of directions in the multidimensional space of \( \{x_k\} \); in this case largeness of \( n \) is not a sufficient condition for the nonlinear terms to be negligible, in fact their total contribution is comparable in value with the linear part of the expansion. So, the nonlinearity may matter when the measured quantities are being chosen at random, without any consideration of their dependence on the disturbing factors. We shall not discuss this point in more detail since one is not led here to any definite conclusion about the character of influence of this effect on the resulting distribution; it can only be thought to weaken with averaging over a variety of dissimilar data, while in distributions of measurements of one or a few analogous quantities it may appear more important. (This is yet another possible reason for violation of the universality discussed in Sec. 2).

There is a further way worth considering to get a distribution different from Gaussian, which will be given more attention in the next section; it is associated with the possibility that the variables \( x_k \) might be correlated without representation (46) for their combined probability to hold. It can be realized e.g. by supposing the matrix \( T \) in (46) to depend on the sum of \( x_k \)'s which can be not small in spite of smallness of \( x_k \)'s owing to large \( n \). A reason for such assumption is provided by observation that the accuracy of a measuring device is mostly not constant but different in various parts of the scale, usually best in the middle and worsening to the extremities. This approach seems to have more definite consequences as to the resulting data distribution than the previous ones, which will be seen from a model considered below.

5 The variable dispersion effect

Let the total set of variables \( x_k \) determining the measurement result \( y \) consist of two subsets of \( n \) and \( n_1 \) variables respectively, the former \( n \) being entirely independent and the latter \( n_1 \) entangled together. Both numbers \( n \) and \( n_1 \) are supposed to be large enough so that all \( \rho_k(x_k) \) could be taken in the Gaussian form and thus

\[
\rho(y) = (2\pi)^{-n+n_1/2} \int d^n x \left( \prod_{k=1}^n \frac{1}{\sigma_k} e^{-x_k^2/2\sigma_k^2} \right) \cdot \\
\int d^{n_1} x (\det T)^{1/2} e^{-\frac{1}{2} T_{kl} x_k x_l} \delta \left( y - \sum_{k=1}^{n+n_1} x_k \right),
\]

(49)

where \( T_{kl} x_k x_l \) implies summation over \( k, l \) running from \( n + 1 \) to \( n + n_1 \). The mean values \( \bar{x}_k \) are put to be zero by making proper shifts of \( x_k \)'s.
Taking the inner integral over $d^{n_1} x$, we obtain

$$
\rho(y) = (2\pi)^{-\frac{n}{2}} \int d^n x \left( \prod_{k=1}^{n} \frac{1}{\sigma_k} e^{-\frac{x_k^2}{2\sigma_k^2}} \right) g \left( y - \sum_{k=1}^{n} x_k \right), \quad (50)
$$

$$
g(z) = \frac{1}{\sqrt{2\pi} \sigma_0} e^{-\frac{z^2}{2\sigma_0^2}}, \quad (51)
$$

$$
\sigma_0^2 = \sum_{kl} (T^{-1})_{kl}. \quad (52)
$$

Eq. (51) is very similar to eq. (36) with Gaussian-like probabilities $\rho_k(x_k)$; an important difference is that $\delta$-function in the integrand has been changed by Gaussian-like function $g(z)$, eq. (52). This expression looks as a natural generalization of eq. (36) for $\rho(y)$ when $y$ is not exactly equal to the sum of $x_k$ but is a random function whose values are normally distributed about this sum with the dispersion $\sigma_0$. Eq. (51) may be postulated as a hypothesis itself, without reference to eq. (49); it seems possible for it to have more general character and to be produced by some other underlying mechanisms.

It is seen that the form of the matrix $T_{kl}$ in eq. (49) does not matter much, it might well be diagonal. A significant point in what follows is the assumption that this matrix and hence the width $\sigma_0$ of the peak of the function $g(z)$ may depend on $\left( \sum_{k=1}^{n} x_k \right)$, so that further integration in eq. (51) becomes nontrivial. This dependence is supposed to model the above-mentioned variation of the measurement dispersion within the scale of a device. It should be emphasized that the two groups of variables $x_k$ ($k = 1$ to $n$ and $k = n + 1$ to $n + n_1$) play different parts in this model even in the case of a diagonal matrix $T_{kl}$ which is taken to depend only on the first group of $x_k$’s. A possible way of interpreting the second group of $n_1$ variables is to consider them as the remnants of the corrected systematic errors. The correction that an experimenter has to introduce when he detects some systematic effect, while supposed to be a constant, may in reality depend on the measured value, and this fact can be simulated by the specified variability of $\sigma_0$.

Now, to calculate $\rho(y)$, let us make again a Fourier transform of eq. (51), as we did it earlier with eq. (34):

$$
\Phi(u) = \int dy e^{iuy} \rho(y) = (2\pi)^{-\frac{n}{2}} \int d^n x \left( \prod_{k=1}^{n} \frac{1}{\sigma_k} \right) \left( \sum_{k=1}^{n} \frac{x_k^2}{\sigma_k^2} + \frac{1}{2} \sum_{k=1}^{n} \frac{x_k^2}{\sigma_k^2} + iuh - \frac{u^2}{2} \sigma_0^2(h) \right), \quad (53)
$$

where a new integration variable $h = \sum_{k=1}^{n} x_k$ is introduced. Substituting for $\delta$-function its Fourier representation, we can integrate over $x_k$ and get

$$
\Phi(u) = \frac{1}{\sqrt{2\pi} \sigma} \int dh e^{-\frac{h^2}{2\sigma^2} + iuh - \frac{u^2}{2} \sigma_0^2(h)}, \quad (54)
$$

$$
\sigma^2 = \sum_{k=1}^{n} \sigma_k^2. \quad (55)
$$

Performing now inverse Fourier transform of eq. (54), we obtain

$$
\rho(y) = \frac{1}{2\pi \sigma} \int \frac{dh}{\sigma_0(h)} e^{-\frac{h^2}{2\sigma^2} - \frac{(h - y)^2}{2\sigma_0^2(h)}}, \quad (56)
$$
In the case of $\sigma_0(h) = \text{const}$ this formula obviously gives the Gaussian expression for $\rho(y)$ with total dispersion $\Delta = \sqrt{\sigma^2 + \sigma_0^2}$. If $\sigma_0(h)$ is a slowly varying function, then the integral is nearly the same but with $\Delta$ depending on $y$ via $\sigma_0(h)$, where $h$ is a function of $y$ determined from the saddle point equation

$$h = y \frac{\sigma^2}{\sigma^2 + \sigma_0^2}.$$  \hspace{1cm} (57)

Typically $\sigma_0(h)$ is a growing function (at $h > 0$), and since $h$ is nearly proportional to $y$ from eq. (57), $\Delta$ also grows with $y$, which means that the distribution fall-off is slower than by Gaussian law.

It makes sense to consider more closely a case of a quadratic function $\sigma_0^2(h)$:

$$\sigma_0^2(h) = \sigma_0^2 + \gamma(h - h_0)^2, \quad \gamma > 0$$  \hspace{1cm} (58)

($\gamma, \sigma_0, h_0$ are some constants), which is in fact the simplest suitable form, since a linear function is not appropriate because of its sign reversal at some point. Eq. (58) can be viewed not merely as a Taylor expansion of some smooth function at small deviations ($h - h_0$), but more generally as a reasonable approximation for any function with a similar behaviour in a bounded region, i.e. having a minimum and growing progressively in both directions from this minimum.

An important property of the quadratic function $\sigma_0^2(h)$ is that it yields the simple-exponential fall-off of the distribution density $\rho(y)$ at asymptotically large $y$. Indeed, the exponent in the integrand of eq. (56) with regard to eq. (58) at $y \to \infty$, $h \to \infty$ and $h \ll y$ (this will be seen to hold for the saddle point) turns into

$$-\frac{h^2}{2\sigma^2} - \frac{y^2}{2\gamma h^2}.$$  \hspace{1cm} (59)

This function of $h$ has a sharp maximum at $h = \gamma^{-1/4} \sqrt{\sigma y}$ and its value at this point is $-y/(\sigma \sqrt{\gamma})$, so

$$\rho(y) \sim e^{-\frac{y}{\sigma \sqrt{\gamma}}}, \quad (y \to \infty)$$  \hspace{1cm} (60)

with main exponential accuracy (without pre-exponential power factors).

Another interesting consequence of the assumption (58) for $\sigma_0^2(h)$ is the possibility to calculate explicitly from eq. (54) the Fourier transform of the distribution density:

$$\Phi(u) = \frac{1}{\sqrt{1 + \gamma \sigma^2 u^2}} e^{-\frac{u^2}{2} \left( \sigma_0^2 + \gamma h_0^2 + \sigma^2 \frac{(1 - i\gamma h_0 u)^2}{1 + \gamma \sigma^2 u^2} \right)}$$  \hspace{1cm} (61)

and hence the moments $\int \rho(y)y^m dy$, which are in fact the coefficients of the Taylor expansion of $\Phi(u)$. In particular, for the dispersion $\Delta$ we have

$$\Delta^2 = \int \rho(y) y^2 dy = -\Phi''(0) = \sigma_0^2 + \gamma h_0^2 + \sigma^2 (1 + \gamma).$$  \hspace{1cm} (62)

It should be noted that eq. (58) is not yet just the expression to be compared with the experimental data distributions in figs. 1–3, since it is supposed to refer to a certain device or setup, while experimental curves involve the results obtained on a large number of them. To
get from eq. (56) the measured distribution, we must average it over the entering parameters \( \sigma, \sigma_0, h, \gamma \) peculiar to any given device, with some weight \( P \) characterizing the relative probability of finding in the world a device (setup) with certain values of the parameters. Moreover, suppose we are dealing (as was said above) only with ‘good’ measurements, whose reported errors reflect quite adequately the dispersion of the setup. Then we can obtain the distribution density \( \rho(y, \Delta) \) for the data with a given error \( \Delta \) by integrating over a range of the parameters subject to constraint (61). Introducing slightly redefined parameters

\[
b = h_0 \sqrt{\gamma}, \quad c = \sigma \sqrt{\gamma}
\] (62)

in place of \( h_0 \) and \( \gamma \), we can write by eqs. (56) and (58)

\[
\rho(y, \Delta) = \int dS_3 P(\sigma, \sigma_0, b, c) \frac{1}{2\pi \sigma} \int \frac{dh}{\sigma_0^2 + (\frac{ch}{\sigma} - b)^2} \cdot \exp \left[ -\frac{h^2}{2\sigma^2} - \frac{(y-h)^2}{2(\sigma_0^2 + (\frac{ch}{\sigma} - b)^2)} \right],
\] (63)

where the outer integral is to be taken over three-dimensional surface of a hypersphere

\[
\sigma^2 + \sigma_0^2 + b^2 + c^2 = \Delta^2
\] (64)

in the space of the parameters \( (\sigma, \sigma_0, b, c) \) (more accurately, the integration region covers 1/8 part of the hypersphere where \( \sigma > 0, \sigma_0 > 0, c > 0 \)). To put it differently,

\[
\int dS_3 = \int d\sigma d\sigma_0 db dc \delta(\Delta - \sqrt{\sigma^2 + \sigma_0^2 + b^2 + c^2}).
\] (65)

The distribution of devices \( P(\sigma, \sigma_0, b, c) \) in eq. (63) is supposed to obey the normalization condition

\[
\int P d\sigma d\sigma_0 db dc = \int P d\Delta dS_3 = 1.
\] (66)

The experimentally studied quantity is not quite \( \rho(y, \Delta) \) but rather the distribution in \( \xi = y/\Delta \):

\[
\rho(\xi) = \int dy d\Delta \rho(y, \Delta) \delta(\xi - \frac{y}{\Delta}) = \int \Delta d\Delta \rho(\xi \Delta, \Delta).
\] (67)

Inserting here eq. (63) and scaling out the variable \( \Delta \),

\[
\sigma = z_1 \Delta, \quad \sigma_0 = z_2 \Delta, \quad b = z_3 \Delta, \quad c = z_4 \Delta,
\]

\[
h = t\sigma = t z_1 \Delta, \quad dS_3 = \Delta^3 d\Omega,
\] (68)

we obtain

\[
\rho(\xi) = \int d\Omega \mathcal{P}(z) \rho_0(\xi, z),
\] (69)

\[
\rho_0(\xi, z) = \frac{1}{2\pi} \int \frac{dt}{\sqrt{z_2^2 + (tz_4 - z_3)^2}} e^{-\frac{t^2}{2} - \frac{(\xi-tz_1)^2}{2[z_2^2 + (tz_4 - z_3)^2]}},
\] (70)

\[
\mathcal{P}(z) = \int d\Delta \Delta^3 P(z_1 \Delta, z_2 \Delta, z_3 \Delta, z_4 \Delta), \quad \int d\Omega \mathcal{P}(z) = 1,
\] (71)
where now \( z \) is a unit four-dimensional vector and \( d\Omega \) an element of the surface on a unit hypersphere, \( \sum_{i=1}^{4} z_i^2 = 1 \). The \( d\Omega \) integration in eqs. (69) and (71) goes over the region \(-1 \leq z_3 \leq 1, 0 \leq z_{1,2,4} \leq 1\).

Eq. (69) seems to be of little use without knowing the distribution of devices \( P(\sigma, \sigma_0, b, c) \) which is rather indefinite, just like it was with eq. (21) containing an unknown function \( q(u, t) \). However, there turns out to be a substantial difference between these two formulae: whereas eq. (21) is sensitive to the form of the function \( q(u, t) \) and the details of its behaviour, it is quite another matter with the function \( P(z) \) in eq. (69) where the other multiplier \( \rho_0(\xi, z) \) in the integrand is a function of \( z \) sharply peaked at the spherical pole \( z_4 = 1, z_1 = z_2 = z_3 = 0 \).

To be more precise, consider the function \( \rho_0(\xi, z) \) for \( \xi \gg 1 \). Unless \( z_4 \to 0 \), the integral in eq. (70) then has two saddle points close to \( t = \pm \sqrt{\xi/z_4} \) and, in perfect analogy with eq. (59), \( \rho_0(\xi, z) \sim e^{-\xi/z_4} \).

(72)

(The case of \( z_4 \to 0 \) can be disregarded, since the integral then becomes Gaussian and \( \rho_0(\xi, z) \) falls off rapidly as \( \exp(-\xi^2/2) \) giving a negligible contribution into the \( d\Omega \) integral of eq. (69).) We conclude that \( \rho_0(\xi, z) \) at large \( \xi \) has a sharp maximum at \( z_4 = 1 \) and decreases rapidly with the departure from this pole. It means that that if we suppose \( P(z) \) to be more or less smooth on the sphere, the integral in eq. (67) will be dominated by a small region around this pole where \( P(z) \) can be replaced by a constant, so that the details of the distribution \( P(z) \) become inessential, and we get

\[
\rho(\xi) \sim e^{-\xi}.
\]

(73)

This is the result striven for: not only the distribution density \( \rho(\xi) \) decays exponentially, but even with the unit coefficient in the exponent, just as it was experimentally found. True, the justification holds only for \( \xi \gg 1 \) and depends on some hypothesis as to the form of the distribution of devices \( P(\sigma, \sigma_0, b, c) \); but this hypothesis does not seem very restrictive, and the asymptotic form (73) of \( \rho(\xi) \) may be hoped to keep approximate validity up to \( \xi \sim 1 \) as is often the case.

It is worthwhile to refine eq. (73) with the inclusion of a pre-exponential (power) factor. With this aim in view, let us rewrite eq. (68) in the large \( \xi \) limit with more accuracy. It follows from eq. (72) that \( z_4 \) can be assumed to differ from 1 only by a quantity of order \( 1/\xi \) or less, for beyond that region \( \rho_0(\xi, z) \) vanishes more rapidly than \( \sim \exp(-\xi) \) and can be put equal to zero when integrated over \( d\Omega \) in eq. (67). So we shall take

\[
1 - z_4 \approx \frac{1}{2}(z_1^2 + z_2^2 + z_3^2) < \frac{1}{\xi},
\]

(74)

which means that \( z_1, z_2, z_3 < \xi^{-1/2} \). Putting

\[
t = \pm \sqrt{\xi/z_4} + \tau
\]

and expanding the exponent in the integrand of eq. (70) in \( \tau \), one can check that the quadratic term is of order \( \tau^2/\xi \) and so \( \tau \sim \xi^{-1/2} \). Thus, keeping the terms \( \sim 1 \), we can write eq. (70) in an approximate form:

\[
\rho_0(\xi, z) \approx \frac{1}{2\pi} \int \frac{dt}{|t|} e^{-t^2/2} - \frac{\xi^2}{2t^2z_4^2} \left( 1 - \frac{2t + z_4}{\xi} + \frac{2z_4}{t} \right)
\]

in an approximate form:
\[
\approx \frac{1}{\sqrt{2\pi}\xi} e^{-\frac{\xi}{z_4}} \cosh(\sqrt{\xi}(z_1 - z_3)).
\] 

(75)

A \cosh function arises as a result of adding together the contributions of the two saddle points, \( t = \pm \sqrt{\xi/z_4} \).

Inserting this expression into eq. (70) and supposing \( T(z) \) to be smooth and nonvanishing at the point \( z_4 = 1 \), we get

\[
\rho(\xi) \approx \frac{1}{\sqrt{2\pi}\xi} e^{-\xi} T(0, 0, 0, 1) \int d\Omega e^{-\frac{\xi}{2}(z_1^2 + z_2^2 + z_3^2)} \cosh(\sqrt{\xi}(z_1 - z_3)).
\]

(76)

For small \( z_1, z_2, z_3 \sim \xi^{-1/2} \), the sphere can be approximated by a plane and thus \( d\Omega \) replaced by \( dz_1 dz_2 dz_3 \), so the integral in eq. (76) is proportional to \( \xi^{-3/2} \) and finally

\[
\rho(\xi) \sim \frac{1}{\xi^2} e^{-\xi} (\xi \to \infty).
\]

(77)

The omitted multiplicative constant includes the uncertain quantity \( T(0, 0, 0, 1) \) with a definite numerical factor, easily computable but not too interesting.

It must be admitted that the assumption of \( T(z) \to \text{const} \) at \( z_4 \to 1 \) which is of importance here, is taken rather arbitrarily and it is difficult to say what the behaviour of the function \( T(z) \) at \( z_4 \to 1 \) should be like in reality. If, in particular, \( T(z) \sim (1 - z_4)^\nu \) or, more generally, is a homogeneous function of the power \( 2\nu \) in \( z_1, z_2, z_3 \), then \( \xi^{-2} \) in eq. (77) is changed by \( \xi^{-2-\nu} \). (In terms of the parameters entering eq. (58) this implies the distribution behaving powerwise with \( \gamma \) at \( \gamma \to \infty \)). The main point however is that the exponential \( \exp(-\xi) \) is unaffected by these variations even in the case of a more rapid than power-like vanishing of \( T(z) \) at \( z_4 = 1 \).

Of some interest may be the limiting case when \( T(z) \) is concentrated near the point \( z_4 = 1 \), i.e.

\[
T(z) = \delta(z_1) \delta(z_2) \delta(z_3),
\]

(78)

which corresponds to the situation of all devices having very large \( \gamma \) in eq. (58) and \( \xi \ll \gamma \). Then eqs. (53) and (70) give

\[
\rho(\xi) = \frac{1}{\pi} \int_0^\infty \frac{dt}{t} e^{-\frac{t^2}{2} - \frac{\xi^2}{2\pi}} = \frac{1}{\pi} K_0(\xi),
\]

(79)

where \( K_0 \) is the McDonald function (modified Bessel function of the second kind). This expression holds for \( \xi \sim 1 \); it fails at \( \xi \gg \gamma \) because then the range of variation of \( z_{1,2,3} \sim \xi^{-1/2} \) becomes comparable with (or less than) the smearing of \( \delta \)-functions in eq. (78), which by eqs. (53) and (61) has the order \( \sigma/\Delta \sim \gamma^{-1/2} \). Asymptotic form of eq. (79) at \( \xi \gg 1 \)

\[
\rho(\xi) \approx \frac{1}{\sqrt{2\pi}\xi} e^{-\xi}
\]

(80)

is similar to eq. (77) but naturally differs in pre-exponential power of \( \xi \).

Eq. (79) breaks down also at \( \xi \to 0 \) where \( K_0(\xi) \) has a logarithmic singularity; it ceases to hold when \( \xi \sim \gamma^{-1/2} \) so that again the smearing of \( \delta \)-functions becomes effective (we suppose \( z_1 \sim z_2 \sim z_3 \sim \gamma^{-1/2} \), i.e. \( \sigma \sim \sigma_0 \sim h_0\sqrt{\gamma} \)). A real distribution density is finite at \( \xi = 0 \), \( \rho(0) \sim \ln 1/\gamma \). Thus the range of validity of eq. (79) is

\[
\frac{1}{\sqrt{\gamma}} \ll \xi \ll \gamma.
\]

(81)
Here $\gamma$ represents the minimal value of this parameter among all devices, which is supposed to be $\gg 1$; this case does not look realistic but is considered to illustrate the weak dependence of $\rho(\xi)$ on the distribution of devices.

6 Conclusion

Main results of this work are somewhat different from those of the previous work [2] and can now be summed up in brief as follows.

First, estimates of probability of deviation of the measured from true value of some quantity at large $\xi$ ($\gg 4$) do not seem to make any sense at all without supplementary information, neither by Gaussian nor by any other similar formulae. There is no universality at large $\xi$; typical is the existence of a critical value of $\xi$ (dependent on the sample of data) beyond which the deviation probability almost ceases to fall down. This slowly dropping large $\xi$ tail of a distribution is natural to associate with the contribution of the measurement results perverted by the overlooked systematic errors (‘bad’ measurements).

Second, there is a conspicuous departure from the standard law also at smaller $\xi$, having a different form and hardly attributable to the same effect which is expected to be negligible at small $\xi$. Among other explanations, the effect of the error uncertainty resorted to in earlier papers [3, 4] does not seem to be an apt one, for its manifestations are too indefinite and unspecific, while the experimental distribution in this region shows an intriguing correspondence with the simple exponential $\sim \exp(-\xi)$.

Third, a new effect has been found (the variable dispersion effect) capable of inducing the distribution to have asymptotic behaviour $\rho(\xi) \sim \exp(-\xi)$ at large $\xi$, with some less definite pre-exponential (power) factor depending on the distribution of measuring devices over their parameters. The effect takes into account a possibility of variation of the device dispersion with changing value of the measured quantity and may occur as a manifestation of the detected and corrected systematic errors, even in the cases of ‘good’ measurements.

It should be noted that the approach applied here is merely a model, by no means the only possible. There can exist several other sources of deviation from the standard law—the error uncertainty or those mentioned in Sec. 4. In particular, the former is hardly believable to have no appreciable influence, since the experimenters usually do not know their measurement errors with much accuracy. It may happen that various effects compensate each other and leave alone the variable dispersion effect, but one cannot say for sure if it is really so. Maybe the question could be elucidated with more abundant statistics and more detailed study of distributions in different subsamples. It seems of interest also to study the analogous distributions of data obtained in other areas of science, not only in elementary particle physics and not only in physics at all. The questions touched on here are of evident practical importance when any estimates of probability of some events are involved; one should have a clear notion of what the degree of validity of such estimates is like.

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