LOCAL VERSIONS OF THE WIENER–LÉVY THEOREM

S.YU.FAVOROV

Abstract. Let \( h \) be a real-analytic function in the neighborhood of some compact set \( K \) on the plane. We show that for any complex measure on the Euclidean space of a finite total variation without singular components with the Fourier–Stieltjes transform \( f(y) \) there exists another measure of a finite total variation with the Fourier transform \( g(y) \) with the property \( g(y) = h(f(y)) \) for each \( y \) such that \( f(y) \) belongs to \( K \).

AMS Mathematics Subject Classification: 42B10, 42B05

Keywords: Wiener–Lévy Theorem, Fourier transform, absolute convergent Diriclet series, pure point measure, real-analytic function

1. Introduction. It is well known that for each absolutely convergent Fourier series \( F(t) \) such that \( F(t) \neq 0 \) for all \( t \) the function \( 1/F(t) \) also has an absolutely convergent Fourier-series expansion (the Wiener Theorem). Its natural generalization is known as the Wiener–Lévy Theorem (see, for example, [9], Ch.VI):

Theorem 1. Let

\[
F(t) = \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n t}
\]

be an absolutely convergent Fourier series, and \( h(z) \) be a holomorphic function on a neighborhood of the closure of the range of \( F \). Then the function \( h(F(t)) \) admits an absolutely convergent Fourier series expansion as well.

Clearly, for \( h(z) = 1/z \) we get the Wiener Theorem.

The next variant of this theorem for functions on \( \mathbb{R} \) is also known as the Wiener–Lévy Theorem (see for example [1], Ch.III or [6], Ch.I)

Theorem 2. Let \( \hat{f}(y) \) be the Fourier transform of some function \( f \in L^1(\mathbb{R}) \), and \( h(z) \) be a holomorphic function on a neighborhood of the closure of the range of \( \hat{f} \) such that \( h(0) = 0 \). Then there is a function \( g \in L^1(\mathbb{R}) \) such that its Fourier transform \( \hat{g} \) coincides with \( h(\hat{f}(y)) \).

This theorem admits a generalization to functions from \( L^1(G) \), where \( G \) is an arbitrary locally compact abelian group, and one can replace a holomorphic function \( h \) by any real-analytic function. Then if we replace here the absolutely continuous measure \( f(x)dx \) by a pure point measure \( \sum_n a_n \delta_{s_n} \) (\( \delta_s \), as usual, means the unit mass at the point \( s \)) with \( \sum_n |a_n| < \infty \), we obtain the Wiener–Lévy Theorem for Diriclet series. On the other hand, for each locally compact abelian non discrete group \( G \) there exists a measure \( \mu \) with a finite total variation such that values of its Fourier transform \( \hat{\mu} \) is bounded away from zero on the dual group \( \hat{G} \) (hence the function \( 1/z \) is holomorphic on the closure of the set \( \hat{\mu}(\hat{G}) \)), but there is no measure \( \nu \) on \( G \) such that its Fourier transform \( \hat{\nu} = 1/\hat{\mu} \).

Also, the requirement of the real-analyticity is necessary for most groups \( G \) (in particular for \( G = \mathbb{R}^d \)) to fulfill the Wiener–Lévy Theorem (see [7], Ch.5, 6).

Also note that the Fourier transforms of pure point measures were considered earlier in [5] in connection with the problem of complete reconstruction of band-limited functions.
The local form of the Wiener–Lévy Theorem is of greatest interest for our study (see [6], Ch.6):

**Theorem 3.** Let $G$ be a locally compact abelian group, let $K$ be a compact subset of the dual group $\hat{G}$, let $f \in L^1(G)$, and let $h(z)$ be a holomorphic function on a neighborhood of the closure of the set $\hat{f}(K)$. Then there is a function $g \in L^1(G)$ such that its Fourier transform $\hat{g}$ coincides with $h(f(y))$ for all $y \in K$.

In our article we consider the case when $K$ is a compact subset of the complex plane $\mathbb{C}$ and the function $h(z)$ is analytic (or real-analytic) on a neighborhood of $K$. Of course if $\hat{G}$ is a compact abelian group then $\hat{f}^{-1}(K)$ is a compact subset of $\hat{G}$, hence we are in the conditions of Theorem 3. But for $G = \mathbb{R}^d$ we obtain a result stronger than Theorem 3. Theorems of this type were used by us to study Poisson measures in [2] – [4].

2. **Notations and Preliminaries.** To formulate our results we have to recall some definitions.

Denote by $M(G)$ the set of complex measures on the locally compact group $G$ with a finite total variation $\|\mu\|$, by $M_d(G)$ the set of pure point measures from $M(G)$, and by $M_ad$ the set of measures from $M(G)$ containing only pure point and absolutely continuous (with respect to the Haar measure) components, i.e., without singular components. The Fourier transform of $\mu \in M(G)$ is defined by the equality

$$\hat{\mu}(y) = \int_G (-x, y) \mu(dx), \quad y \in \hat{G},$$

where $\hat{G}$ is the group of characters on $G$, and $(x, y)$ means the action of the character $y$ on $x \in G$. In particular, in the case $G = \mathbb{R}^d$ we have

$$\hat{\mu}(y) = \int_{\mathbb{R}^d} e^{-2\pi i (x, y)} \mu(dx), \quad y \in \mathbb{R}^d,$$

where $(x, y)$ means the scalar product of $x$ and $y$. If the measure $\mu$ is absolute continuous and $\mu(dx) = f(x)dx$, we will also write

$$\hat{f}(y) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i (x, y)} dx.$$

Furthermore, a complex-valued function $h$, defined on an open set $V \subset \mathbb{C}$ is said to be real-analytic on $V$ if to every point $z \in V$ there corresponds the expansion

$$h(\xi + i\eta) = \sum_{k, n=0}^{\infty} c_{k,n}(\xi - \Re z)^k(\eta - \Im z)^n, \quad \xi, \eta \in \mathbb{R}, \quad c_{k,n} \in \mathbb{C},$$

which converges in some disc

$$D(z) = \{(\xi, \eta) \in \mathbb{R}^2 : |\xi - \Re z|^2 + |\eta - \Im z|^2 < r_z^2\}.$$

Note that series (1) converges also in the ball

$$B(z) = \{(\xi, \eta) \in \mathbb{C}^2 : |\xi - \Re z|^2 + |\eta - \Im z|^2 < r_z^2\},$$

and for any intersecting balls $B(z_1)$ and $B(z_2)$ the set

$$B(z_1) \cap B(z_2) \cap \mathbb{R}^2 = D(z_1) \cap D(z_2)$$

is the set of uniqueness for analytic functions of two variables. Therefore, if the function $h(z)$ is real-analytic in a neighborhood of some compact set $K \subset \mathbb{C}$, then it has a continuation to the neighborhood $\bigcup_{z \in K} B(z) \subset \mathbb{C}^2$ of $K$ as an analytic function of two complex variables $\xi$, $\eta$. 

2
3. Main results.

Theorem 4. Let $\mu$ be a measure from $M_{ad}(\mathbb{R}^d)$, let $h(z)$ be a real-analytic function on a neighborhood of some compact set $K \subset \mathbb{C}$. Then there is a measure $\nu \in M_{ad}(\mathbb{R}^d)$ such that, for every $y \in \mathbb{R}^d$ for which $\hat{\mu}(y) \in K$, we have $\hat{\nu}(y) = h(\hat{\mu}(y))$.

In particular, if $h(z) = 1/z$ or $h(z) = 1/|z|^\alpha$ for $|z| \geq \varepsilon$ and $h(z) = 0$ for $|z| \leq \varepsilon/2$, we obtain the following result:

Corollary. For any $\mu \in M_{ad}(\mathbb{R}^d)$ and $\varepsilon > 0$, $\alpha > 0$ there are measures $\nu_\varepsilon, \nu_{\alpha,\varepsilon} \in M_{ad}(\mathbb{R}^d)$ such that in the case $|\hat{\mu}(y)| \geq \varepsilon$ we have $\hat{\nu}_\varepsilon(y) = 1/\hat{\mu}(y)$, $\hat{\nu}_{\alpha,\varepsilon}(y) = 1/|\hat{\mu}(y)|^\alpha$, and in the case $|\hat{\mu}(y)| \leq \varepsilon/2$ we have $\hat{\nu}_\varepsilon(y) = 0$.

The reasoning in the proof of Theorem 4 also provides the following statement:

Theorem 5. Let $\mu$ be a measure from $M_d(G)$ for a locally compact abelian group $G$, and let $h(z)$ be a real-analytic function on a neighborhood of some compact set $K \subset \mathbb{C}$. Then there is a measure $\nu \in M_d(G)$ such that, for every $y \in G$ for which $\hat{\mu}(y) \in K$, we have $\hat{\nu}(y) = h(\hat{\mu}(y))$. The support of the measure $\nu$ lies in $\text{Lin}_\varepsilon \text{supp} \mu$.

Here supp $\mu$ means the set $\{x \in G : \mu(\{x\}) \neq 0\}$ if $\mu \in M_d(G)$.

For the case $G = \mathbb{R}^d$ and holomorphic $h$ on a neighborhood $V \subset \mathbb{C}$ of $K$ Theorem 5 was proved in [2]. Its application allowed us to obtain in [2] some strengthening of one of the theorems in the theory of Fourier quasicrystals. Another application of Theorem 5 to Kahane’s property of discrete sets see [3].

4. Auxiliary lemmas and their proofs. We will use Schwartz’ space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing $C^\infty$-functions on $\mathbb{R}^d$ with the topology defined by a countable number of norms

$$N_n(\varphi) = \sup_{x \in \mathbb{R}^d} \left\{ (1 + |x|)^n \max_{k_1 + \ldots + k_d \leq n} \left| \frac{\partial^{k_1} \varphi}{\partial x_1^{k_1}} \ldots \frac{\partial^{k_d} \varphi}{\partial x_d^{k_d}}(x) \right| \right\}, \quad n = 0, 1, 2, \ldots$$

The Fourier transform is a continuous linear one-to-one mapping of $\mathcal{S}(\mathbb{R}^d)$ onto $\mathcal{S}(\mathbb{R}^d)$, and the set of $C^\infty$-functions with bounded support is dense in $\mathcal{S}(\mathbb{R}^d)$ (see [8]).

Lemma 1. For every $f \in L^1(\mathbb{R}^d)$ and every $\varepsilon > 0$ there is $\varphi \in \mathcal{S}(\mathbb{R}^d)$ such that $\|f - \varphi\|_{L^1} < \varepsilon$ and $\hat{\varphi}$ has a compact support.

Proof. Take $f_1 \in L^1(\mathbb{R}^d)$ such that $\|f - f_1\|_{L^1} < \varepsilon/3$ and $f_1$ has a compact support. The convolution $f_2 = f_1 * \varphi$ with a suitable $C^\infty$-function $\varphi(x) \geq 0$ with support in a small ball such that $\int_{\mathbb{R}^d} \varphi(x) dx = 1$ has the properties $\|f_2 - f_1\|_{L^1} < \varepsilon/3$ and $f_2 \in \mathcal{S}(\mathbb{R}^d)$.

Therefore, $\hat{f}_2 \in \mathcal{S}(\mathbb{R}^d)$ as well, and there is a sequence of $C^\infty$-functions with compact supports that converges to $\hat{f}_2$ in the space $\mathcal{S}(\mathbb{R}^d)$. Let $\{v_n\}$ be the images the functions from this sequence under the inverse Fourier transform. Clearly, $v_n \to f_2$ in the space $\mathcal{S}(\mathbb{R}^d)$, therefore,

$$\|f_2 - v_n\|_{L^1} \leq \sup_{\mathbb{R}^d} (1 + |x|)^{d+1} |f_2(x) - v_n(x)| \int_{\mathbb{R}^d} (1 + |x|)^{-d-1} dx \leq C(d) N_{d+1}(f_2 - v_n) \to 0$$

as $n \to \infty$. Hence, $\|f_2 - v_n\|_{L^1} < \varepsilon/3$ for a suitable $v_n$, and $\|f - v_n\|_{L^1} < \varepsilon$. 

Lemma 2. There is a constant $C = C(r, d)$ such that for every $v \in \mathcal{S}(\mathbb{R}^d)$ with the property supp $\hat{v} \subset B(0, r)$ we get

$$\|v\|_{L^1} \leq C(r, d) \left( \|\hat{v}\|_{\infty} + \sum_{j=1}^{d} \|((\partial^j / \partial y_j^j))\hat{v}\|_{\infty} \right),$$

with $l = d + 1$ for odd $d$ and $l = d + 2$ for even $d$. 

3
Proof. We have
\[
\hat{v}(y) = \int_{\mathbb{R}^d} v(x)e^{-2\pi i (x,y)}dx, \quad (\partial^j / \partial y^j)\hat{v}(y) = (-2\pi i)^j \int_{\mathbb{R}^d} x^j v(x)e^{-2\pi i (x,y)}dx, \quad j = 1, \ldots, d.
\]
Hence, \(v(x)(1 + \sum_{j=1}^d |x_j|^l)\) is the inverse Fourier transform of the function
\[
\hat{v}(y) + \left(\frac{i}{2\pi}\right)^l \sum_{j=1}^d (\partial^j / \partial y^j)\hat{v}(y).
\]
Since \(\text{supp} \hat{v} \subseteq B(0,r)\), we get
\[
\|v\|_{L^1} \leq C \left(\|\hat{v}\|_\infty + \sum_{j=1}^d \| (\partial^j / \partial y^j)\hat{v}\|_\infty \right) \int_{\mathbb{R}^d} \frac{dx}{1 + |x_1|^l + |x_2|^l + \cdots + |x_d|^l}
\]
with a constant \(C\) depending on \(d\) and \(r\). \(\Box\)

Lemma 3. Let \(T(\Theta, \tau)\) be \(C^\infty\)-function in variables \(\Theta = (\theta_1, \ldots, \theta_N) \in \mathbb{R}^N\) and \(\tau \in [0,1]^2\), and let \(T(\Theta, \tau)\) be periodic with periods 1 in each coordinate \(\theta_1, \ldots, \theta_N\). Then its Fourier series
\[
T(\Theta, \tau) = \sum_{k \in \mathbb{Z}^N} b_k(\tau)e^{2\pi i (k,\Theta)}, \quad k = (k_1, \ldots, k_N),
\]
is absolutely convergent and \(\sum_k |b_k(\tau)| < C\) uniformly in \(\tau\).

Proof. We have
\[
b_k(\tau) = \int_{[0,1]^N} T(\Theta, \tau)e^{-2\pi i (k,\Theta)}d\Theta.
\]
Integrating this equality twice in parts over each variable \(\theta_j\) such that \(j \in J(k) = \{j : k_j \neq 0\}\), we get
\[
|b_k(\tau)| \leq \sup_{(\Theta, \tau) \in [0,1]^{N+2}} \left| \left( \prod_{j \in J(k)} (-4\pi^2 k_j^2)^{-1} \partial^2 / \partial \theta_j^2 \right) T(\Theta, \tau) \right|.
\]
Taking into account that every derivative of \(T(\Theta, \tau)\) is uniformly bounded in \(\tau \in [0,1]^2\), we get the estimate
\[
|b_k(\tau)| \leq C \min\{1, k_1^{-2}\} \cdots \min\{1, k_N^{-2}\},
\]
where \(C\) depends on neither \(\tau\) nor \(k\). This estimate implies the assertion of the lemma. \(\Box\)

5. Proofs of the main results.

Proof of Theorem 4. Let \(U\) be an open set in \(\mathbb{C}^2\) such that \(h\) has a holomorphic continuation to \(U\) as a function of two complex variables. Set \(\varepsilon < (1/13)\text{dist}(K, \partial U)\). Let \(\varphi(|\zeta|)\) be \(C^\infty\)-differentiable nonnegative function with a support in \(B(0, \varepsilon) \subseteq \mathbb{C}^2\) such that \(\int_{\mathbb{C}^2} \varphi(|\zeta|)m(d\zeta) = 1\) (here \(m(d\zeta)\) means the Lebesgue measure in \(\mathbb{C}^2\)). Consider \(C^\infty\)-function
\[
H(z) = \int_{\text{dist}(\zeta, K) < 9\varepsilon} h(\zeta)\varphi(|z - \zeta|)m(d\zeta).
\]
If \(\text{dist}(z, K) < 8\varepsilon\), we get
\[
H(z) = \int_{|\zeta| \leq \varepsilon} h(z - \zeta)\varphi(|\zeta|)m(d\zeta).
\]
Since an average over any sphere of a holomorphic function of many variables equals the meaning of the function in the center of the sphere, we obtain that \(H(z) = h(z)\) on the set \(\{z : \text{dist}(z, K) < 7\varepsilon\}\) and \(H(z) = 0\) on the set \(\{z : \text{dist}(z, K) > 10\varepsilon\}\).
Let 
\[ \mu = f dx + \sum_{n} a_n \delta_{\gamma_n}, \quad f \in L^1(\mathbb{R}^d), \quad \sum_{n} |a_n| < \infty. \]

Using Lemma 1, take a function \( v \in S(\mathbb{R}^d) \) such that \( \|f - v\|_{L^1} < \varepsilon \) and supp \( \hat{v} \) is a compact set. Pick \( N < \infty \) such that \( \sum_{n>N} |a_n| < \varepsilon \), and define the measure \( s = \sum_{n=1}^{N} a_n \delta_{\gamma_n} \). Note that
\[ \hat{s}(y) = \sum_{n \leq N} a_n e^{-2\pi i \langle \gamma_n, y \rangle}. \]

Since \( \|\mu - v dx - s\| < 2\varepsilon \), we see that \( \|\hat{\mu}(y) - \hat{v}(y) - \hat{s}(y)\|_{\infty} < 2\varepsilon \). Put
\[ \alpha(y) = \Re(\hat{v}(y) + \hat{s}(y)), \quad \beta(y) = \Im(\hat{v}(y) + \hat{s}(y)). \]

Consider the function
\[ F(y) = \frac{1}{(2\pi i)^2} \int_{|\alpha(y) - \zeta_1| = 3\varepsilon} \int_{|\beta(y) - \zeta_2| = 3\varepsilon} \frac{H(\zeta_1 + i\zeta_2) d\zeta_1 d\zeta_2}{(\zeta_1 - \Re(\hat{\mu}(y)))(\zeta_2 - \Im(\hat{\mu}(y)))}. \]

If \( \hat{\mu}(y) \in K \), then \( \text{dist}(\alpha(y) + i\beta(y), K) < 2\varepsilon \). Therefore,
\[ \mathcal{E} = \{ (\zeta_1, \zeta_2) : |\zeta_1 - \alpha(y)| \leq 3\varepsilon, |\zeta_2 - \beta(y)| \leq 3\varepsilon \} \subset \{ z : \text{dist}(z, K) < 7\varepsilon \}, \]
and \( H(z) = h(z) \) in a neighborhood of \( \mathcal{E} \). Using the Cauchy integral formula for the polydisk \( \mathcal{E} \), we obtain
\[ F(y) = h(\Re(\hat{\mu}(y) + i\Im(\hat{\mu}(y))) = h(\hat{\mu}(y)). \]

Furthermore, we have for all \( y \in \mathbb{R}^d \)
\[ F(y) = \int_0^1 \int_0^1 H(\alpha(y) + 3\varepsilon e^{2\pi i r_1} i \beta(y) + 3\varepsilon e^{2\pi i r_1} i 3\varepsilon e^{2\pi i r_2} 9\varepsilon^2 e^{2\pi i (r_1 + r_2)} d\tau_1 d\tau_2. \]

Since
\[ \left| \frac{\Re(\hat{\mu}(y) - \alpha(y))}{3\varepsilon e^{2\pi i r_1}} \right| < 2/3, \quad \left| \frac{\Im(\hat{\mu}(y) - \beta(y))}{3\varepsilon e^{2\pi i r_2}} \right| < 2/3, \]
we get
\[ \frac{1}{(1 - \frac{\Re(\hat{\mu}(y) - \alpha(y))}{3\varepsilon e^{2\pi i r_1}})(1 - \frac{\Im(\hat{\mu}(y) - \beta(y))}{3\varepsilon e^{2\pi i r_2}})} = \sum_{p,q=0}^{\infty} \left( \frac{\Re(\hat{\mu}(y) - \alpha(y))}{3\varepsilon e^{2\pi i r_1}} \right)^p \left( \frac{\Im(\hat{\mu}(y) - \beta(y))}{3\varepsilon e^{2\pi i r_2}} \right)^q, \]
and
\[ F(y) = \sum_{p,q=0}^{\infty} \left( \frac{\Re(\hat{\mu}(y) - \alpha(y))}{3\varepsilon} \right)^p \left( \frac{\Im(\hat{\mu}(y) - \beta(y))}{3\varepsilon} \right)^q \int_0^1 \int_0^1 A(y, \tau_1, \tau_2) + D(y, \tau_1, \tau_2) d\tau_1 d\tau_2, \]
with
\[ A(y, \tau_1, \tau_2) = H(\hat{v}(y) + \hat{s}(y) + 3\varepsilon e^{2\pi i r_1} i 3\varepsilon e^{2\pi i r_2}) - H(\hat{s}(y) + 3\varepsilon e^{2\pi i r_1} + i3\varepsilon e^{2\pi i r_2}), \]
\[ D(y, \tau_1, \tau_2) = H(\hat{s}(y) + 3\varepsilon e^{2\pi i r_1} + i3\varepsilon e^{2\pi i r_2}). \]

Define two measures on \( \mathbb{R}^d \)
\[ \lambda_R(x) = 1/2 \left( \mu(x) - v(x) dx - s(x) + \mu(-x) - v(-x) dx - s(-x) \right), \]
\[ \lambda_I(x) = (1/2i) \left( \mu(x) - v(x) dx - s(x) - \mu(-x) - v(-x) dx - s(-x) \right). \]

It is easily seen that \( \|\lambda_R\| < 2\varepsilon, \|\lambda_I\| < 2\varepsilon \), and
\[ \hat{\lambda}_R(y) = \Re(\hat{\mu}(y) - \alpha(y)), \quad \hat{\lambda}_I(y) = \Im(\hat{\mu}(y) - \beta(y)). \]

Since the Fourier transform of convolution of measures equals the product of the Fourier
transform of the measures, we get
\begin{equation}
[(\lambda_R/3\varepsilon)^p * (\lambda_I/3\varepsilon)^q] \overset{\sim}{\longrightarrow} \left(\frac{\Re \hat{\mu}(y) - \alpha(y)}{3\varepsilon}\right)^p \left(\frac{3\hat{\mu}(y) - \beta(y)}{3\varepsilon}\right)^q.
\end{equation}

Also, the variation of convolution of measures does not exceed the product of variations
of the measures, hence
\begin{equation}
\|[(\lambda_R/3\varepsilon)^p * (\lambda_I/3\varepsilon)^q]\| < (2/3)^{p+q}.
\end{equation}

On the other hand, since $\text{supp } A(y, \tau_1, \tau_2) \subset \text{supp } v$ and $A(y, \tau_1, \tau_2) \in C^\infty$, we see that
$A(y, \tau_1, \tau_2) \in \mathcal{S}(\mathbb{R}^d)$. Therefore there exists $u_{\tau_1, \tau_2}(x) \in \mathcal{S}(\mathbb{R}^d)$ such that
$\hat{u}_{\tau_1, \tau_2}(y) = \hat{A}(y, \tau_1, \tau_2)$ for every fixed $\tau_1, \tau_2$. Then the function $A(y, \tau_1, \tau_2)$ and all its
derivatives of order at most $d + 2$ are bounded uniformly in $\tau_1, \tau_2 \in [0, 1]^2$. By Lemma 2,
$\|u_{\tau_1, \tau_2}\|_{L^1}$ is uniformly bounded too. Set
\begin{equation}
k_{p,q}(x) = \int_0^1 \int_0^1 u_{\tau_1, \tau_2}(x) e^{2\pi i (p\tau_1 + q\tau_2)} d\tau_1 d\tau_2 \in L^1(\mathbb{R}^d).
\end{equation}

By Fubini’s Theorem,
\begin{equation}
\sup_{p,q} \|k_{p,q}\|_{L^1} < \infty,
\end{equation}
and
\begin{equation}
\hat{k}_{p,q}(g) = \int_0^1 \int_0^1 A(y, \tau_1, \tau_2) e^{2\pi i (p\tau_1 + q\tau_2)} d\tau_1 d\tau_2.
\end{equation}

Next, apply Lemma 3 to the function $H \left(\sum_{n \leq N} a_n e^{2\pi i \theta_n} + 3\varepsilon e^{2\pi i r_1} + i3\varepsilon e^{2\pi i r_2}\right)$. We get
\begin{equation}
H \left(\sum_{n \leq N} a_n e^{2\pi i \theta_n} + 3\varepsilon e^{2\pi i r_1} + i3\varepsilon e^{2\pi i r_2}\right) = \sum_{k \in \mathbb{Z}^N} b_k(\tau_1, \tau_2) e^{2\pi i (k, \theta)},
\end{equation}
with the condition
\begin{equation}
\sup_{\tau_1, \tau_2} \sum_k |b_k(\tau_1, \tau_2)| < \infty.
\end{equation}

If we replace in (6) $\theta_n$ by $-\langle \gamma_n, y \rangle$ for each $n$, we get the function\footnote{Note that some $\rho_k$ may coincide}
\begin{equation}
\sum_{k \in \mathbb{Z}} b_k(\tau_1, \tau_2) e^{2\pi i (\rho_k, y)} = p_k \in \text{Lin}_\mathbb{Z} \{\gamma_n\}_{n=1}^\infty.
\end{equation}

This function is the Fourier transform of the measure $\sum_{k \in \mathbb{Z}} b_k(\tau_1, \tau_2) \delta_{-\rho_k}$. Set
\begin{equation}
\nu_{p,q} = \sum_{k \in \mathbb{Z}} c_k(p, q) \delta_{-\rho_k} \quad \text{with} \quad c_k(p, q) = \int_0^1 \int_0^1 b_k(\tau_1, \tau_2) e^{2\pi i (p\tau_1 + q\tau_2)} d\tau_1 d\tau_2.
\end{equation}

It follows from (6), (7) and Fubini’s Theorem that
\begin{equation}
\sup_{p,q} \|\nu_{p,q}\| < \infty,
\end{equation}
and
\begin{equation}
\hat{\nu}_{p,q}(y) = \int_0^1 \int_0^1 D(y, \tau_1, \tau_2) e^{2\pi i (p\tau_1 + q\tau_2)} d\tau_1 d\tau_2.
\end{equation}

Finally put
\begin{equation}
\nu = \sum_{p,q=0}^\infty (\lambda_R/3\varepsilon)^p * (\lambda_I/3\varepsilon)^q * (\hat{k}_{p,q} dx + \nu_{p,q}).
\end{equation}
We have
\[ \|\nu\| \leq \sum_{p,q=0}^{\infty} \|\lambda_R/3\varepsilon\|p \|\lambda_I/3\varepsilon\|q (\|\kappa_{p,q}\|_{L^1} + \|\nu_{p,q}\|). \]

It follows from (3), (4), and (8), that \( \nu \) has a finite total variation, and, by (2), (5), and (9), that \( \hat{\nu}(y) = F(y) \).

**Proof of Theorem 5.** Let \( \mu = \sum_n a_n\delta_{\gamma_n} \) with \( \gamma_n \in G \) and \( \sum_n |a_n| < \infty \). Then
\[ \hat{\mu}(y) = \sum_n a_n(-\gamma_n,y), \quad y \in \hat{G}. \]

Replace \( e^{-2\pi i(y,\gamma_n)} \) by \( (-\gamma_n,y) \) in the previous proof. Further, we do not use Lemmas 1 and 2, but put \( A(y,\tau_1,\tau_2) \equiv 0 \), \( u(x,\tau_1,\tau_2) \equiv 0 \), and \( k_{p,q}(x) \equiv 0 \forall p,q \). Then, repeating the reasoning in the proof of Theorem 4, we obtain the assertion of Theorem 5.

**Remark.** If the function \( h \) is holomorphic in a neighborhood of the compact set \( K \), then there is an alternative proof of Theorem 5. Indeed, let \( \Gamma \) denote the group \( G \) with respect to the discrete topology. Clearly, every pure point measure \( \mu \in M(G) \) is the function \( f \in L^1(\Gamma) \) at the same time. Therefore \( \hat{\mu} \) extends to the continuous function \( \hat{f} \) on the compact group \( \hat{\Gamma} \). Then \( \hat{f}^{-1}(K) \) is a compact subset of \( \hat{\Gamma} \), and we may apply Theorem 3. In order to obtain a statement about the support of the measure \( \nu \), one has to replace the group \( \Gamma \) by the group \( \operatorname{Lin}_Z \operatorname{supp} \mu \).

I am very grateful to Professor Hans Georg Feichtinger for pointing me the papers, which contain results close to those in my article, and for his attention to my work.

**References**

[1] Akhiezer, N.I.: Theory of Approximation. F.Ungar Pub. (1956)
[2] Favorov, S.Yu.: Large Fourier quasicryals and Wiener’s Theorem. Journal of Fourier Analysis and Applications. 25(2), 377-392 (2019)
[3] Favorov, S.Yu.: Local Wiener’s Theorem and Coherent Sets of Frequencies. Analysis Math., 46 (4) (2020), 737–746
[4] Favorov, S.Yu.:Temperate distributions with locally finite support and spectrum on Euclidean spaces. arXiv.org/abs/2106.07067 [math.FA] 15 June 2021
[5] Feichtinger, H.G.: Discretization of convolution and reconstruction of band-limited functions from irregular sampling. in: Progress in approximation theory. Academic Press, Boston, M. A., 333-345 (1991).
[6] Reiter, H., Stegeman J.D.: Classical Harmonic Analysis and Locally Compact Groups. Oxford University Press, Oxford, (2000)
[7] Rudin, W.: Fourier Analysis on Groups. Interscience Publications, a Division of John Wiley and Sons, New York (1962)
[8] Rudin, W.: Functional Analysis. McGraw -Hill Book Company, New York (1973)
[9] Zygmund, A.: Trigonometric Series. Cambridge University Press, Cambridge (2002)

Serhii Favorov, Karazin’s Kharkiv National University Svobody sq., 4, 61022, Kharkiv, Ukraine

Email address: sfavorov@gmail.com