Harmonic analysis meets stationarity: A general framework for series expansions of special Gaussian processes

Mohamed Ndaoud
Department of Statistics
CREST (UMR CNRS 9194), ENSAE
5, av. Henry Le Chatelier, 91764 Palaiseau, FRANCE

Abstract

In this paper, we present a new approach to derive series expansions for some Gaussian processes based on harmonic analysis of their covariance function. In particular, a new simple rate-optimal series expansion is derived for fractional Brownian motion. The convergence of the latter series holds in mean square and uniformly almost surely, with a rate-optimal decay of the rest of the series. We also develop a general framework of convergent series expansion for certain classes of Gaussian processes. Finally, an application to functional quantization is described.

Keywords: fractional Brownian motion, Karhunen-Loève, Fourier series, fractional Ornstein-Uhlenbeck, functional quantization.

1. Introduction

Let \( B = (B_t)_{t \in \mathbb{R}^+} \) be a centered Gaussian process. \( B \) is called fractional Brownian motion (fBm) with Hurst exponent \( H \in (0,1) \) if it has the following covariance structure

\[
\forall t, s \in \mathbb{R}^+, \quad \text{E}B_sB_t = \frac{1}{2} \left( |t|^{2H} + |s|^{2H} - |t-s|^{2H} \right).
\]

Fractional Brownian motion is a self-similar process i.e. \( \forall c, t > 0, \ B_{ct} = c^H B_t \), that has stationary increments i.e. \( \forall s, t > 0, \ B_t - B_s = B_{t-s} \). When \( H = 1/2 \), fractional Brownian motion coincides with the standard Brownian motion. Sample paths of fBm are Hölder-continuous of any order strictly less than \( H \), and hence are almost surely everywhere continuous.

One of the main challenges with fBm is its simulation, as it is the case for Gaussian processes with a complex covariance structure in general. The circulant embedding method, described in [Dietrich and Newsam(1997)], is one of the most efficient algorithms to simulate either stationary Gaussian processes or Gaussian processes with stationary increments on a finite interval \([0, T]\) for some \( T > 0 \). In particular, the latter algorithm has an \( N \log N \) complexity, where \( N \) is the number of time steps discretizing \([0, T]\). This complexity is to be compared with linear complexity for the standard Brownian motion due to the independence of its increments. Besides, circulant embedding does not allow local refinement.

Alternative approximation methods to simulate a Gaussian process involve its Karhunen-Loève expansion. The latter expansion is explicitly known for some processes such as the
Brownian motion, the Brownian bridge [Deheuvels(2007)] and the Ornstein-Uhlenbeck process [Corlay et al.(2010)], to name a few. Unfortunately, this expansion is not explicit for fBm.

**Notation.** In the rest of this paper we use the following notation. Let X and Y be two random variables, we say that $X = Y$, when X and Y have the same distribution. For given sequences $a_n$ and $b_n$, we say that $a_n = O(b_n)$ (resp $a_n = \Omega(b_n)$) when $a_n \leq c b_n$ (resp $a_n \geq c b_n$) for some absolute constant $c > 0$. We write $a_n \prec b_n$ when $a_n = O(b_n)$ and $a_n = \Omega(b_n)$. Let $X \in \mathbb{R}^p$, we denote by $\|X\|$ the Euclidean norm of X. Finally, for $x, y \in \mathbb{R}$, we denote by $x\lor y$ the maximum value between $x$ and $y$. In particular $x\lor 0$ will be denoted by $x_+$.

1.1 Related literature

In [Ayache and Taqqu(2003)], one of the first rate-optimal series expansion of fBm based on Wavelet series approximations is presented. For the sake of brevity, we only present, in what follows, trigonometric series, since they can be compared to the framework we expose further.

Later, the first trigonometric series expansion for fBm on $[0, 1]$ was discovered in [Dzhaparidze and Van Zanten(2004)]. For $0 < H < 1$, the series $(B_t^H)_{t \in [0,1]}$ is given by

$$B_t^H = \sum_{n=1}^{\infty} \frac{\sin x_n t}{x_n} X_n + \sum_{n=1}^{\infty} \frac{1 - \cos y_n t}{y_n} Y_n, \quad t \in [0,1],$$

where $(X_n)_{n \geq 1}$ and $(Y_n)_{n \geq 1}$ are i.i.d centered Gaussian random variables, $(x_n)_{n \geq 1}$ is the sequence of positive roots of the Bessel function $J_{-H}$, and $(y_n)_{n \geq 1}$ the sequence of positive roots of the Bessel function $J_{1-H}$. The variance of the Gaussian variables involved in the series is given by

$$\forall n \geq 1, \quad Var X_n = 2c_H^2 x_n^{-2H} J_{1-H}^{-2}(x_n) \quad \text{and} \quad Var Y_n = 2c_H^2 y_n^{-2H} J_{-H}^{-2}(y_n),$$

where $c_H^2 = \pi^{-1}(1 + 2H) \sin \pi H$, and $\gamma$ is the gamma function. In their paper, authors prove rate-optimality of the above series expansion in the following sense.

**Definition 1.** Let $H \in (0, 1)$ and $B^H$ a fBm with Hurst exponent $H$ on $[0, T]$ for some $T > 0$. Assume that $B^H$ is given by the series expansion

$$\forall t \in [0,T], \quad B^H_t = \sum_{i=0}^{\infty} Z_i e_i(t),$$

where $(Z_i)_{i \in \mathbb{N}}$ is a sequence of independent Gaussian random variables and $(e_i)_{i \in \mathbb{N}}$ a sequence of continuous deterministic functions. $B^H$ is said to be uniformly rate-optimal if

$$\mathbb{E} \sup_{t \in [0,T]} \left| \sum_{i=0}^{\infty} Z_i e_i(t) \right| \asymp N^{-H} \sqrt{\log N}.$$

In Kühn and Linde(2002), the rate $N^{-H} \sqrt{\log N}$ is shown to be optimal. Rate-optimality also means that no other series expansion of fBm has a faster rate of convergence. We show later how rate-optimality implies uniform convergence of the series, almost surely.
Another rate-optimal trigonometric series expansion for fBm, in the case $1 > H > 1/2$, is derived in Iglói (2005), that is close to our representation. For $1 > H > 1/2$, this expansion takes the form

$$B_t = a_0 t X_0 + \sum_{k=1}^{\infty} a_k \left( \sin(k\pi t)X_k + (1 - \cos(k\pi t))X_{-k} \right), \quad t \in [0, 1],$$

where

$$a_0 = \sqrt{\frac{\Gamma(2-2H)}{B(H - \frac{1}{2}, \frac{3}{2} - H)(2H - 1)}},$$

$$\forall k \in \mathbb{N}^*, \quad a_k = \sqrt{\frac{\Gamma(2-2H)}{B(H - \frac{1}{2}, \frac{3}{2} - H)(2H - 1)}2\Re(i \exp^{-i\pi H} \gamma(2H - 1, ik\pi))(k\pi)^{H - \frac{1}{2}}},$$

and $(X_k)_{k \in \mathbb{Z}}$ is a sequence of independent standard Gaussian random variables. $\Gamma$, $B$, and $\gamma$ are the gamma, beta, and complementary (lower) incomplete gamma functions, respectively. Even if this representation is easier to evaluate than the previous one, it still requires special functions.

### 1.2 Main contribution

In this paper, we give a constructive representation of fBm for all $0 < H < 1$ which is only based on harmonic analysis of its covariance function. Our approach is inspired from the Karhunen-Loève expansion. The latter expansion is obtained through an interesting application of the spectral theorem for compact normal operators, in conjunction with Mercer’s theorem. We give here a sketch of its proof. Let $K_B(.,.)$ be the covariance function of the process $B$ of interest on $[0, 1]^2$. Mercer’s theorem is a series representation of $K_B$ based on the diagonalization of the following linear operator

$$T_{K_B} : L^2([0, 1]) \rightarrow L^2([0, 1])$$

$$f \rightarrow \int_0^1 K_B(s,.) f(s) ds,$$

where $L^2([0, 1])$ is the space of square-integrable real-valued functions on $[0, 1]$. In particular, it states that there is an orthonormal basis $(e_i)_{i \in \mathbb{N}}$ of $L^2([0, 1])$ consisting of eigenfunctions of $T_{K_B}$ such that the corresponding sequence of eigenvalues $(\lambda_i)_{i \in \mathbb{N}}$ is nonnegative. Moreover, $K_B$ has the following representation

$$\forall s, t \in [0, 1], \quad K(s, t) = \sum_{i=0}^{\infty} \lambda_i e_i(t)e_i(s).$$

Considering $(Z_i)_{i \in \mathbb{N}}$ a sequence of independent standard Gaussian random variables, and assuming the uniform convergence of the following series

$$\forall t \in [0, 1], \quad X_t = \sum_{i=0}^{\infty} \sqrt{\lambda_i} Z_i e_i(t),$$

one may observe that $X$ is a centered Gaussian process on $[0, 1]$ with a covariance function equal to $K_B$. Hence $X$ and $B$ have the same distribution on $[0, 1]$. Since the corresponding eigenfunction sequence $(e_i)_{i \in \mathbb{N}}$ is not explicit for fBm, we follow an alternative approach.
In the case where $B$ is a stationary process or has stationary increments, $T_KB$ becomes similar to a convolution operator. It is well known that the Fourier basis is a basis of eigenfunctions for the convolution operator, when the convolution kernel is periodic. In general, we may extend the kernel into an even periodic kernel. Since this modification applies to the covariance function $K_B$, there is no guarantee that the new symmetric function $	ilde{K}_B(.,.)$ is positive. In particular, the corresponding eigenvalues are not necessary positive. The last condition is crucial in our approach, since we need to take the square root of the Fourier coefficients, as described in the Karhunen-Loève proof.

One of the main contributions, is to exhibit a new class $\Gamma$ of functions such that the Fourier coefficients are all negative. Let $T > 0$ and $\gamma$ be a real valued function on $(0,T].$ We say that $\gamma$ satisfies property ($\star$) if

- $\gamma$ is continuously differentiable, increasing and concave.
- $x^\delta \gamma'(x) = \mathcal{O} (1)$ for some $\delta \in [0,2).$

We denote by $\Gamma$ the class of such functions. Unsurprisingly, Fourier coefficients of functions in $\Gamma$ will also have an optimal decay, since the Fourier basis is orthonormal. As a consequence, we derive a new series expansion for $fBm$ given by

$$B_t^H = \sqrt{c_0 t}Z_0 + \sum_{k=1}^{\infty} \sqrt{-\frac{c_k}{2}} \left( \sin \frac{k\pi t}{T} Z_k + \left( 1 - \cos \frac{k\pi t}{T} \right) Z_{-k} \right), \quad t \in [0,T],$$

where

$$\begin{align*}
|k| \geq 1, & \quad \left\{ \begin{array}{ll}
c_0 := 0, & H < 1/2 \\
c_0 := HT^{2H-2}, & H > 1/2,
\end{array} \right. \\
c_k := \frac{2}{T} \int_0^T t^{2H}\cos \frac{k\pi t}{T} dt, & \quad H < 1/2 \\
c_k := -\frac{4H(2H-1)T}{(k\pi)^2} \int_0^T t^{2H-2}\cos \frac{k\pi t}{T} dt, & \quad H > 1/2,
\end{align*}$$

and $(Z_k)_{k \in \mathbb{Z}}$ is a sequence of independent standard Gaussian random variables. This series is rate-optimal and its convergence holds uniformly almost surely. More generally, we also derive series expansion for a general class of Gaussian processes such that the covariance operator is linked with the class $\Gamma$.

The first section is devoted to study harmonic properties of the class $\Gamma$. In the second section, we present our series expansion for $fBm$, where we prove both uniform convergence and rate-optimality. Next, we generalize this series expansion to a large class of Gaussian processes, before applying it to functional quantization.

2. On harmonic properties of the class Gamma

The present section is devoted to describe general harmonic properties of the class $\Gamma$. Let $T > 0$ and $\gamma \in \Gamma$, we define the corresponding Fourier sequence $c(\gamma) = (c_k(\gamma))_{k \in \mathbb{N}}$ such that

$$\forall k \in \mathbb{N}, \quad c_k(\gamma) := \frac{2}{T} \int_0^T \gamma(t) \cos \frac{k\pi t}{T} dt.$$

The next Proposition states some important properties of $c(\gamma)$ when $\gamma$ satisfies ($\star$). In what follows, we denote by $c_k$ the coefficient $c_k(\gamma)$, as long as there is no ambiguity.
Proposition 2. Let $\gamma$ be a function satisfying $(\star)$ and $c(\gamma)$ the sequence defined in (3), then

- $c(\gamma)$ is well defined.
- $\forall k \in \mathbb{N}^*, c_k \leq 0$.
- $c_k = \mathcal{O}_{k \to \infty}\left(\frac{1}{k^{2-\delta}}\right)$.

The proof is deferred to Appendix. It is usually not easy to reveal the sign of the Fourier coefficients of a given function. For the class of functions satisfying $(\star)$, it turns out that the previous question can be answered, based on Proposition 2. The more general question of characterizing the class of functions with negative Fourier coefficients is beyond the scope of this paper. It is also interesting to notice that for $\gamma \in \Gamma$, the singularity around $0^+$, captures the asymptotic behaviour of $c(\gamma)$ that is not trivial in general.

For $\delta \in [0,1)$, and inspecting the proof of Proposition 2, $\gamma$ has a finite limit in $0^+$, we will use in that case the notation $\gamma(0) := \lim_{x \to 0^+} \gamma(x)$. The next lemma gives a useful Fourier expansion for functions in $\Gamma$.

Lemma 3. Let $\gamma$ be a function satisfying $(\star)$ for some $\delta \in [0,1)$, then

$$\forall t \in [-T,T], \quad \gamma(|t|) = \gamma(0) + \sum_{k=1}^{\infty} c_k \left( \cos \frac{k\pi t}{T} - 1 \right).$$

Proof. Let $g : \mathbb{R} \to \mathbb{R}$ be a function such that

$$\forall t \in [-T,T], \quad g(t) = \gamma(|t|).$$

Extending $g$ into a $2T$-periodic function, it can be defined on $\mathbb{R}$. Since $g$ is an even function, its Fourier expansion is given by

$$\forall t \in [-T,T], \quad g(t) = \sum_{k=0}^{\infty} c_k \cos \frac{k\pi t}{T}, \quad (4)$$

where $c_0 = \frac{1}{T} \int_0^T \gamma(t)dt$ and $\forall k > 0$, $c_k = \frac{2}{T} \int_0^T \gamma(t) \cos \frac{k\pi t}{T}dt$. Using Proposition 2, we have

$$c_k = \mathcal{O}_{k \to \infty}\left(\frac{1}{k^{2-\delta}}\right).$$

Since $0 \leq \delta < 1$, the Fourier expansion of $g$ converges normally and hence uniformly. Replacing $t$ by $0$ in (4) we get that

$$c_0 = \gamma(0) - \sum_{k=1}^{\infty} c_k.$$

It follows that

$$g(t) = \gamma(0) + \sum_{k=1}^{\infty} c_k \left( \cos \frac{k\pi t}{T} - 1 \right).$$

The main result follows immediately. \qed
Let $\lambda := (\lambda_k)_{k \in \mathbb{N}}$ be a sequence of real numbers and $e := (e_k)_{k \in \mathbb{N}}$ a family of uniformly bounded and continuous functions on $[0, T]$. We say that $(\lambda, e)$ satisfies $(\star \star)$ if

- $\exists H > 0$, such that $\lambda_k = O_{k \to \infty} \left( \frac{1}{k^{H+1/2}} \right)$.
- $\exists L > 0$, such that $\forall k \in \mathbb{N}, \forall s, t \in [0, T], |e_k(t) - e_k(s)| \leq L |t - s|$.

Notice that for $\gamma \in \Gamma$, and setting $e = (\cos(.), \sin(.), 1 - \cos(.))$, it is easy to check that $\sqrt{-c(\gamma)}$, $e$ satisfies $(\star \star)$ for $\delta \in [0, 1)$. Before stating the next Theorem, we precise that we endow the space of continuous functions on $[0, T]$ with the supremum metric. In the rest of the paper, all convergence results in this space are implicitly with respect to the supremum metric.

**Theorem 4.** Let $(\lambda_k)_{k \in \mathbb{N}}$ be a sequence of real numbers, $(Z_k)_{k \in \mathbb{N}}$ a sequence of centered standard Gaussian random variables, and $(e_k)_{k \in \mathbb{N}}$ a family of continuous functions on $[0, T]$. Assume that $(\lambda, e)$ satisfies $(\star \star)$ for some $H > 0$, then the series $\sum_{k=0}^{N} \lambda_k e_k \left( \frac{k \pi t}{T} \right) Z_k$ converges almost surely in the space of continuous functions on $[0, T]$. Moreover, we have

$$\mathbb{E} \sup_{t \in [0, T]} \left| \sum_{k=N}^{\infty} \lambda_k e_k \left( \frac{k \pi t}{T} \right) Z_k \right| = O_{N \to \infty} \left( N^{-H} \sqrt{\log N} \right).$$

The proof is deferred to Appendix. In what follows, we will heavily use Proposition 2 along with Theorem 4. In fact, Proposition 2 describes the asymptotic behaviour of $c(\gamma)$ for $\gamma \in \Gamma$, while Theorem 4 characterizes the rate of convergence of given series expansions based on the asymptotic behaviour of $c(\gamma)$.

### 3. Constructing the fractional Brownian motion

In this section, we present first our series expansion for fBm, then prove its convergence. The construction is based on harmonic decomposition of the auto-covariance function $\gamma$ on $[0, T]$ such that $\gamma(t) = |t|^{2H}$ for some $0 < H < 1$. As described in the introduction, the diagonalization of the operator $T_{K_X}$ is not explicit for fBm. In order to benefit from the diagonalization of the convolution operator, we need to extend the auto-covariance function into a periodic function. The resulting function $\tilde{K}(.,.)$ is not guaranteed to be a covariance function. Luckily, harmonic properties of the class $\Gamma$ will be useful to get around this drawback.

Since our approach does not hold for both cases, we give results separately for both fBm with $0 < H < 1/2$ and $1 > H > 1/2$, assuming that the series converge. We prove later the convergence and rate-optimality of these series.

#### 3.1 The series expansion

The following Theorem gives an explicit series expansion for fBm when $0 < H < 1/2$, assuming the series convergence.
Theorem 5. Let $H \in (0, \frac{1}{2})$. \forall t \in [0, T]$, the function $\gamma$ is given by $\gamma(t) = t^{2H}$, and denote by $c(\gamma)$ its corresponding Fourier sequence. Let $B$ be a stochastic process given by the series expansion

$$
\forall t \in [0, T], \quad B_t = \sum_{k=1}^{\infty} \sqrt{-\frac{c_k}{2}} \left( \sin \frac{k \pi t}{T} Z_k + \left( 1 - \cos \frac{k \pi t}{T} \right) Z_{-k} \right),
$$

where $(Z_k)_{k \in \mathbb{Z}}$ denotes a sequence of independent standard Gaussian random variables, then

$$
\forall (s, t) \in [0, T]^2, \quad \mathbb{E}B_s B_t = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right).
$$

Proof. For $H \in (0, \frac{1}{2})$, we denote by $\gamma(t) := |t|^{2H}$. It is easy to check that $\gamma$ satisfies $(\star)$. Using Proposition 2, the above series is well-defined since $\forall k \geq 1, c_k \leq 0$. Because of the independence between the Gaussian random variables $Z$, it follows immediately that

$$
\mathbb{E}B_s B_t = \sum_{k=1}^{\infty} \frac{-c_k}{2} \left( \sin \frac{k \pi s}{T} \sin \frac{k \pi t}{T} + \left( 1 - \cos \frac{k \pi s}{T} \right) \left( 1 - \cos \frac{k \pi t}{T} \right) \right)
$$

$$
= \sum_{k=1}^{\infty} \frac{-c_k}{2} \left( 1 - \cos \frac{k \pi s}{T} - \cos \frac{k \pi t}{T} + \cos \frac{k \pi (t-s)}{T} \right)
$$

$$
= \sum_{k=1}^{\infty} \frac{c_k}{2} \left( \left( \cos \frac{k \pi s}{T} - 1 \right) + \left( \cos \frac{k \pi t}{T} - 1 \right) - \left( \cos \frac{k \pi (t-s)}{T} - 1 \right) \right).
$$

We conclude using Lemma 3

The previous proof does not hold for the case $1/2 < H < 1$. In fact, the Fourier coefficients $c(\gamma)$ have alternating signs in this case i.e. they do not have a constant sign. For $H > 1/2$, $\gamma$ does not satisfy property $(\star)$ any more. In particular, the change in the sign of $c(\gamma)$ is partially due to the smoothness of $\gamma'$ around 0. One may still notice that $\gamma''$ satisfies $(\star)$. The next Lemma, gives a link between $c(\gamma)$ and $c(\gamma'')$.

Lemma 6. Let $\gamma$ be a twice differentiable function on $(0, T)$ such that $\gamma'(0) \neq \gamma'(T)$. Define $f$ such that

$$
\forall t \in [0, T], \quad f(t) = \gamma(t) - \frac{\gamma'(T) - \gamma'(0)}{2T} \left( t + \frac{T \gamma'(0)}{\gamma'(T) - \gamma'(0)} \right)^2,
$$

then $\forall k \in \mathbb{N}^*$ we have

$$
c_k(f) = \left( \frac{T}{k \pi} \right)^2 c_k(-\gamma'').
$$
Proof. The function $f$ is constructed in a way such that $f'(0) = f'(T) = 0$ holds. It follows that, $\forall k \in \mathbb{N}^*$

$$
\int_0^T f(t) \cos \left( \frac{k\pi t}{T} \right) \, dt = \frac{T}{k\pi} \left[ f(t) \sin \left( \frac{k\pi t}{T} \right) \right]_0^T - \frac{T}{k\pi} \int_0^T f'(t) \sin \left( \frac{k\pi t}{T} \right) \, dt \\
= \left( \frac{T}{k\pi} \right)^2 \left[ f'(t) \cos \left( \frac{k\pi t}{T} \right) \right]_0^T - \left( \frac{T}{k\pi} \right)^2 \int_0^T f''(t) \cos \left( \frac{k\pi t}{T} \right) \, dt \\
= \left( \frac{T}{k\pi} \right)^2 \int_0^T f''(t) \cos \left( \frac{k\pi t}{T} \right) \, dt.
$$

The last equality is a consequence of orthogonality between constants and harmonics. \(\square\)

The following Theorem gives an explicit series expansion for fBm when $1/2 < H < 1$, assuming the series convergence.

**Theorem 7.** Let $H \in \left( \frac{1}{2}, 1 \right)$. \(\forall t \in [0,T]\), the function $\gamma$ is given by $\gamma(t) = -2H(2H - 1)t^{2H-2}$, and denote by $c(\gamma)$ its corresponding Fourier sequence. Let $B$ be a stochastic process given by the series expansion

$$
\forall t \in [0,T], \quad B_t = \sqrt{HT^{2H-2}}tZ_0 + \sum_{k=1}^{\infty} \frac{T}{k\pi} \sqrt{-\frac{c_k}{2}} \left( \sin \frac{k\pi t}{T} Z_k + \left( 1 - \cos \frac{k\pi t}{T} \right) Z_{-k} \right),
$$

where $(Z_k)_{k \in \mathbb{Z}}$ denotes a sequence of independent standard Gaussian random variables, then

$$
\forall (s,t) \in [0,T]^2, \quad \mathbb{E}B_sB_t = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right).
$$

**Proof.** By considering $\gamma(t) = -2H(2H - 1)t^{2H-2}$, we notice that $\gamma$ satisfies ($\star$) for $1/2 < H < 1$. Moreover

$$
\gamma'(t) = \mathcal{O}_{t \to 0^+} \left( \frac{1}{t^{3-2H}} \right).
$$

Since $1 < 3 - 2H < 2$, we get using Proposition 2 that $\forall k \geq 1$, $c_k \leq 0$, and $c_k = \mathcal{O}_{k \to \infty} \left( \frac{1}{k^{2H+1}} \right)$. We also obtain, using Lemma 6 that

$$
\frac{2}{T} \int_0^T \left( t^{2H} - HT^{2H-2}t^2 \right) \cos \left( \frac{k\pi t}{T} \right) \, dt = \left( \frac{T}{k\pi} \right)^2 c_k.
$$

Since $\frac{c_k}{k^2} = \mathcal{O}_{k \to \infty} \left( \frac{1}{k^{2H+1}} \right)$, the Fourier series converges uniformly and we can apply Lemma 3 to get

$$
\forall t \in [-T,T], \quad |t|^{2H} = HT^{2H-2}t^2 + \sum_{k=1}^{\infty} \left( \frac{T}{k\pi} \right)^2 c_k \left( \cos \frac{k\pi t}{T} - 1 \right).
$$
Since $c_k \leq 0$ the series expansion is well defined and we have
\[
E_B t_s = HT^{2H-2} ts - \frac{1}{2} \sum_{k=1}^{\infty} \left( \frac{T}{k\pi} \right)^2 c_k \left( \sin \frac{k\pi t}{T} \sin \frac{k\pi s}{T} + \left( 1 - \cos \frac{k\pi t}{T} \right) \left( 1 - \cos \frac{k\pi s}{T} \right) \right)
= HT^{2H-2} ts - \frac{1}{2} \sum_{k=1}^{\infty} \left( \frac{T}{k\pi} \right)^2 c_k \left( 1 - \cos \frac{k\pi t}{T} - \cos \frac{k\pi s}{T} + \cos \frac{k\pi (t - s)}{T} \right)
= \frac{1}{2} \left( HT^{2H-2} (t^2 + s^2 - (t - s)^2) + \sum_{k=1}^{\infty} \left( \frac{T}{k\pi} \right)^2 c_k \left( \cos \frac{k\pi t}{T} + \cos \frac{k\pi s}{T} - \cos \frac{k\pi (t - s)}{T} - 1 \right) \right)
= \frac{1}{2} \frac{|t|^{2H} + |s|^{2H} - |t - s|^{2H}}{2}. \tag{7}
\]

\section*{3.2 Convergence and rate-optimality}

After giving a new explicit representation of $B_t$, we prove its convergence in both mean square and almost surely. We also show its uniform rate-optimality. For the rest of the section, we denote more precisely by $(c_k)_{k \geq 0}$ the following sequence
\[
\begin{cases}
  c_0 := 0, & 0 < H < 1/2 \\
  c_0 := HT^{2H-2}, & 1 > H > 1/2,
\end{cases} \tag{8}
\]
and
\[
\forall k \geq 1 \begin{cases}
  c_k := \frac{2}{T} \int_0^T t^{2H} \cos \frac{k\pi t}{T} dt, & 0 < H < 1/2 \\
  c_k := -\frac{4H(2H-1)T}{(k\pi)^2} \int_0^T t^{2H-2} \cos \frac{k\pi t}{T} dt, & 1 > H > 1/2.
\end{cases} \tag{9}
\]
One may first notice, based on previous results, that $c_k \sim O_{k \to \infty} \left( \frac{1}{k^{2H+1}} \right)$. We will now consider the series expansion constructed in this section and given by
\[
\forall t \in [0, T], \ B_t = \sqrt{c_0} t Z_0 + \sum_{k=1}^{\infty} \sqrt{-c_k} \left( \sin \frac{k\pi t}{T} Z_k + \left( 1 - \cos \frac{k\pi t}{T} \right) Z_{-k} \right). \tag{10}
\]
The following theorems show the convergence of the series in mean square and almost surely uniformly. Let $N \in \mathbb{N}^*$. In what follows, we denote by $B_t^N$ the truncated series of $B$ that is given by
\[
B_t^N = \sqrt{c_0} t Z_0 + \sum_{k=1}^{N} \sqrt{-c_k} \left( \sin \frac{k\pi t}{T} Z_k + \left( 1 - \cos \frac{k\pi t}{T} \right) Z_{-k} \right).
\]

\textbf{Theorem 8.} Let $B$ be the series expansion defined in (10). $B_t^N$ converges in mean square, and its rate of convergence is given by
\[
\sup_{t \in [0, T]} \sqrt{E(B_t - B_t^N)^2} = O_{N \to \infty} \left( N^{-H} \right).
\]
Proof. As we did previously, we will use the independence of the Gaussian random variables \((Z_k)_{k \in \mathbb{Z}}\). It is straightforward that \(\forall \ t \in [0, T] \)

\[
\mathbb{E}(B_t - B_t^N)^2 = \sum_{k>N} -\frac{c_k}{2} \left( (\sin \frac{k\pi t}{T})^2 + (1 - \cos \frac{k\pi t}{T})^2 \right) = \sum_{k>N} -c_k \left( 1 - \cos \frac{k\pi t}{T} \right).
\]

Hence

\[
\sup_{t \in [0,T]} \sqrt{\mathbb{E}(B_t - B_t^N)^2} \leq \sqrt{\sum_{k>N} -c_k}.
\]

Since \(c_k = \mathcal{O}_{k \to \infty} \left( \frac{1}{k^{2H+1}} \right)\), we also have

\[
c_k = \mathcal{O}_{k \to \infty} \left( \frac{1}{k^{2H}} - \frac{1}{(k+1)^{2H}} \right).
\]

It comes, by simple comparison of the residuals of positive convergent series, that

\[
\sqrt{\sum_{k>N} -c_k} = \mathcal{O}_{N \to \infty} \left( \frac{1}{N^{H}} \right).
\]

The previous Theorem shows the mean square convergence of \(B^N\). Since \(B_N\) is a centered Gaussian process, and using the fact that the Gaussian Hilbert space is complete, we deduce that \(B\) is a centered Gaussian process with the same covariance than fBm. It follows that \(B\) is a fractional Brownian motion on \([0, T]\). We turn now to the question of rate-optimality of the series expansion.

**Theorem 9.** Let \(B\) be the series expansion defined in (10). Almost surely, \(B_t^N\) converges uniformly, and its rate of convergence is given by

\[
\mathbb{E} \sup_{t \in [0,T]} |B_t - B_t^N| \leq N^{-H} \sqrt{\log(N)}.
\]

**Proof.** We will only need to prove that the rate of convergence of the above series is faster than \(N^{-H} \sqrt{\log(N)}\) since the latter is the optimal rate of convergence for fBm as shown in Kühn and Linde(2002). By truncating the series, we have

\[
\forall t \in [0, T], \quad B_t - B_t^N = \sum_{k=N+1}^\infty -\frac{c_k}{2} \left( \sin \frac{k\pi t}{T} Z_k + \left( 1 - \cos \frac{k\pi t}{T} \right) Z_{-k} \right).
\]

Since \(\sqrt{-\frac{c_k}{2}} = \mathcal{O}_{k \to \infty} \left( \frac{1}{k^{H+1/2}} \right)\), and using the fact that \(t \to \sin(t)\) and \(t \to 1 - \cos(t)\) are both, uniformly in \(k\), 1-Lipschitz functions we can directly use Theorem 4 to conclude the proof.\[\square\]

Finally, we have derived a new series expansion for fBm. Theorem 9 shows that this series is moreover rate-optimal.
4. Generalization to special Gaussian processes

In this section, we develop a general framework for series expansion of special classes of Gaussian processes. For all these classes, we prove almost sure uniform convergence and give the corresponding rate of convergence. The question of rate-optimality of the presented series expansions is beyond the scope of this paper. We refer the reader to Proposition 4 in Luschgy and Pagès(2009) that gives some hints on rate-optimality.

The next theorem generalizes the case of fBm. We derive a series expansion for a class of Gaussian processes with stationary increments. Let $\gamma$ be a function satisfying $(\star)$ for some $\delta \in [0,1)$, and let $X$ be a centered Gaussian process. We say that $X$ is a Gaussian process of type $(A)$, if it is characterized by the following covariance structure

$$\forall t, s \in [0, T], \quad EX_tX_s = \frac{1}{2} (\gamma(t) + \gamma(s) - \gamma(|t-s|)).$$

**Theorem 10.** Assume that $\gamma$ satisfies $(\star)$ for some $\delta \in [0,1)$, and define $c(\gamma)$ its corresponding Fourier sequence. Let $(Z_k)_{k \in \mathbb{Z}}$ be a sequence of independent standard Gaussian random variables, then the series expansion

$$X_t = \sum_{k=1}^{\infty} \sqrt{-c_k} \left( \sin \frac{k\pi t}{T} Z_k + \left( 1 - \cos \frac{k\pi t}{T} \right) Z_{-k} \right), \quad t \in [0, T],$$

is a Gaussian process of type $(A)$. $X$ converges uniformly in $[0,T]$, almost surely. Moreover its rate of convergence is given by

$$\mathbb{E} \sup_{t \in [0,T]} |X_t - X_t^N| = \mathcal{O}_{N \to \infty} \left( N^{-\frac{1-\delta}{2}} \sqrt{\log(N)} \right).$$

**Proof.** Applying Proposition 2, we obtain that $\forall k \geq 1, c_k \leq 0$, hence the series is well defined. Moreover, we also have that $c_k = \mathcal{O}_{k \to \infty} \left( \frac{1}{k^{1-\delta}} \right)$. Since $\delta \in (0,1)$ we can use Theorem 4 to get that

$$\mathbb{E} \sup_{t \in [0,T]} \left| \sum_{k=N+1}^{\infty} \sqrt{-c_k} \left( \sin \frac{k\pi t}{T} Z_k + \left( 1 - \cos \frac{k\pi t}{T} \right) Z_{-k} \right) \right| = \mathcal{O}_{k \to \infty} \left( \frac{\sqrt{\log(N)}}{N^{\frac{1-\delta}{2}}} \right),$$

and that the series converges almost surely and uniformly in $[0,T]$. It follows that $X$ is a centered Gaussian process. As for its covariance, we have

$$\forall s, t \in [0, T], \quad EX_sX_t = \sum_{k=1}^{\infty} \frac{-c_k}{2} \left( 1 - \cos \frac{k\pi t}{T} - \cos \frac{k\pi s}{T} + \cos \frac{k\pi (t-s)}{T} \right).$$

We can conclude using Lemma 3 that

$$\forall s, t \in [0, T], \quad EX_sX_t = \frac{1}{2} (\gamma(t) + \gamma(s) - \gamma(|t-s|)).$$

Hence $X$ is a Gaussian process of type $(A)$.

$\square$
The next class of interest is a subclass of stationary Gaussian processes. Let $\gamma$ be a function such that $-\gamma$ satisfies $\star$ for some $\delta \in [0, 1)$, and let $X$ be a centered Gaussian process. We say that $X$ is a Gaussian process of type $(B)$, if it is characterized by the following covariance structure
\[ \forall t, s \in [0, T], \quad EX_tX_s = \gamma(|t - s|). \]

**Theorem 11.** Assume that $\gamma$ is such that $-\gamma$ satisfies $\star$ for some $\delta \in [0, 1)$, and define $c(\gamma)$ its corresponding Fourier sequence. Let $(Z_k)_{k \in \mathbb{Z}}$ be a sequence of independent standard Gaussian random variables, if $c_0 \geq 0$, then the series expansion
\[ X_t = \sqrt{c_0}Z_0 + \sum_{k=1}^{\infty} \sqrt{c_k} \left( \sin \frac{k\pi t}{T} Z_k + \cos \frac{k\pi t}{T} Z_{-k} \right), \quad t \in [0, T], \]

is a Gaussian process of type $(B)$. $X$ converges uniformly in $[0, T]$, almost surely. Moreover the series is rate-optimal, and its rate of convergence is given by
\[ E \sup_{t \in [0, T]} |X_t - X^N_t| = O_{N \to \infty} \left( N^{-1-\delta} \sqrt{\log(N)} \right). \]

**Proof.** Using the same steps as for Theorem 10, we get that the series is well defined, and that it converges uniformly almost surely. Hence $X$ is a Gaussian process. Moreover we have
\[ \forall s, t \in [0, T], \quad EX_sX_t = \sum_{k=0}^{\infty} c_k \cos \frac{k\pi (t - s)}{T} = \gamma(|t - s|). \]

It follows that $X$ is a Gaussian process of type $(B)$. Since the basis $(e_k)_{k \in \mathbb{Z}}$ used in this expansion is orthogonal, we may apply the same argument as in Proposition 4 in Luschgy and Pagès(2009), and deduce that the series is rate-optimal. \qed

One immediate consequence is a series expansion for $X$ a stationary fractional Ornstein-Uhlenbeck with $0 < H < 1/2$, where a stationary fOU is a centered Gaussian process such that
\[ \forall s, t \in [0, T], \quad EX_sX_t = e^{-|t-s|^{2H}}. \]

The last series expansion was already derived in Luschgy and Pagès(2009).

The framework we are proposing here can also be applied to Gaussian processes that are neither stationary nor with stationary increments. As an example we apply it to another class of Gaussian processes. Let $\gamma$ be a function defined on $(0, 2T)$, such that $-\gamma$ satisfies $\star$ for some $\delta \in [0, 1)$, and let $X$ be a centered Gaussian process. We say that $X$ is a Gaussian process of type $(C)$, if it is characterized by the following covariance structure
\[ \forall t, s \in [0, T], \quad EX_sX_t = \frac{1}{2} \left( \gamma(|t - s|) - \gamma(t + s) \right). \]

**Theorem 12.** Assume that $\gamma$ is such that $-\gamma$ satisfies $\star$ on $(0, 2T)$ for some $\delta \in [0, 1)$, and define $c(\gamma)$ its corresponding Fourier sequence on the interval $(0, 2T)$. Let $(Z_k)_{k \in \mathbb{N}^*}$ be a sequence of independent standard Gaussian random variables, then the series expansion
\[ X_t = \sum_{k=1}^{\infty} \sqrt{c_k} \sin \frac{k\pi t}{2T} Z_k, \quad t \in [0, T]. \]
is a Gaussian process of type (C). $X$ converges uniformly in $[0,T]$, almost surely. Moreover its rate of convergence is given by

$$E \sup_{t \in [0,T]} |X_t - X_t^N| = \mathcal{O}_{N \to \infty} \left( N^{-1/2} \sqrt{\log(N)} \right).$$

**Proof.** Using the same steps as for Theorem 10, we get that the series is well defined, and that it converges uniformly almost surely. Hence $X$ is a Gaussian process. Moreover we have

$$\forall t, s \in [0,T], \ E X_s X_t = \sum_{k=1}^{\infty} c_k \left( \sin \frac{k \pi t}{2T} \sin \frac{k \pi s}{2T} \right) = \frac{1}{2} \sum_{k=1}^{\infty} c_k \left( \cos \frac{k \pi (t - s)}{2T} - \cos \frac{k \pi (t + s)}{2T} \right).$$

As a consequence we get that

$$\forall t, s \in [0,T], \ E X_s X_t = \frac{1}{2} \left( \gamma(|t - s|) - \gamma(t + s) \right).$$

It follows that $X$ is a Gaussian process of type (C).

**Remark 13.** One may notice that, in Theorem 12, we have considered a $4T$-periodic basis instead of $2T$ because $\forall 0 < t, s < T, \ 0 \leq s + t \leq 2T$.

In order to illustrate Theorem 12, we give below two examples of corresponding series expansions.

**Example 1.** Karhunen Loève expansion of Brownian motion.

In this example, we consider the function $\gamma(t) = -|t|$ on $[0,T]$. It is easy to check that $\gamma$ satisfies conditions of Theorem 12. One may also notice that this process is a Brownian motion on $[0,T]$ since

$$\forall t, s \in [0,T], \ \frac{1}{2} (-|t - s| + |t + s|) = \min(t,s).$$

An explicit evaluation of the sequence $(c_k)_{k \in \mathbb{N}^*}$ gives

$$\forall k \in \mathbb{N}^*, \ c_k = \frac{1}{T} \int_0^{2T} -t \cos \frac{k \pi t}{2T} dt$$

$$= \frac{2}{k \pi} \int_0^{2T} \sin \frac{k \pi t}{2T} dt$$

$$= \left( 1 - (-1)^k \right) \left( \frac{2}{k \pi} \right)^2 T. \quad (12)$$

Applying Theorem 12, it follows that

$$\forall t \in [0,T], \ X_t = \sqrt{2} \sum_{k=1}^{\infty} \sqrt{\frac{T}{k - \frac{1}{2}} \pi} \sin \frac{(k - \frac{1}{2}) \pi t}{T} Z_k,$$

is a series expansion for Brownian motion on $[0,T]$, where $(Z_k)_{k \in \mathbb{N}^*}$ is a sequence of independent standard Gaussian random variables.
Example 2. A new series expansion for the generalized Ornstein-Uhlenbeck process.

In this example we consider the non-stationary Ornstein-Uhlenbeck process \((Y_t)_{t \geq 0}\) where \(Y_0\) is a Gaussian random variable with the following distribution \(N(\mu, \sigma^2_0)\). This process is a Gaussian process characterized by

\[
\forall t \geq 0, \quad \mathbb{E}Y_t = \mu e^{-\theta t} + \alpha (1 - e^{-\theta t}),
\]

and

\[
\forall s, t \geq 0, \quad \mathbb{E}(Y_s - \mathbb{E}Y_s, Y_t - \mathbb{E}Y_t) = \sigma^2_0 e^{-\theta(t+s)} + \frac{\sigma^2}{2\theta} \left(e^{-\theta(|t-s|)} - e^{-\theta(t+s)}\right),
\]

for some \(\theta > 0\) and \(\alpha, \sigma \in \mathbb{R}\). By setting \(\gamma(t) = \frac{\sigma^2}{\theta} e^{-\theta t}\), we have that \(-\gamma\) satisfies conditions of Theorem 12. Hence, and applying Theorem 12, the following expansion

\[
\forall t \in [0, T], \quad X_t = Y_0 e^{-\theta t} + \alpha (1 - e^{-\theta t}) + \sum_{k=1}^{\infty} \sqrt{c_k} \sin\left(\frac{k\pi t}{2T}\right) Z_k,
\]

is a series expansion of the generalized Ornstein-Uhlenbeck process on \([0, T]\), where \((Z_k)_{k \geq 1}\) is a sequence of independent standard Gaussian random variables, that are also independent from \(Y_0\).

An explicit evaluation of the sequence \((c_k)_{k \in \mathbb{N}^*}\) gives

\[
\forall k \geq 1, \quad \frac{\theta}{\sigma^2} c_k = \frac{1}{T} \int_0^{2T} e^{-\theta t} \cos\left(\frac{k\pi t}{2T}\right) dt = \operatorname{Re} \left(\frac{1}{T} \int_0^{2T} e^{(-\theta + i\frac{k\pi}{2T})t} dt\right) = \operatorname{Re} \left(\frac{1}{\theta T} \left(1 - (-1)^k e^{-2\theta T}\right)\right) = \frac{1}{\theta T} \frac{1 - (-1)^k e^{-2\theta T}}{1 + \left(\frac{k\pi}{2\theta T}\right)^2}.
\] (13)

The previous expansion is easier to use compared to the one known so far that includes zeros of Bessel functions.

5. Application: Functional quantization

Quantization consists in approximating a random variable taking a continuum of values in \(\mathbb{R}\) by a discrete random variable. While vector quantization deals with finite dimensional random variables, functional quantization extends the concept to the infinite dimensional setting, as it is the case for stochastic processes. Quantization of random vectors can be considered as a discretization of the probability space, providing in some sense the best approximation to the original distribution. The quantization of a random variable \(X\) taking values in \(\mathbb{R}\) consists in approximating it by the best discrete random variable \(Y\) taking finite values in \(\mathbb{R}\). If we set \(N\) to be the maximum number of values taken by \(Y\), the problem is equivalent to minimizing the error defined below

\[
\xi_N(X) = \left\{ \mathbb{E}(X - \operatorname{Proj}_\Gamma(X))^2, \quad \Gamma \subset \mathbb{R} \text{ such that } |\Gamma| \leq N \right\}.
\] (14)

A solution of (14) is an \(L^2\)-optimal quantizer of \(X\).
For a multidimensional Gaussian random variable $X$ optimal-quantization is expensive. One way to mitigate this cost, is to consider product-quantization, that is to use a cartesian product of one-dimensional optimal-quantizers of each marginale as in [Printems et al.(2005)]
. The resulting quantizer is stationary when marginals of $X$ are independent. In [Luschgy et al.(2007)Luschgy, Pagès, et al.], it is shown that Karhunen-Loève product-quantization, while it is sub-optimal, remains rate-optimal in the case of Gaussian processes.

We consider now a continuous Gaussian process $(X_t)_{t \in [0,T]}$ such that $\int_0^T \mathbb{E}|X_t|^2 dt < \infty$, and its expansion

$$\forall t \in [0,T], \quad X_t = \sum_{i=0}^{\infty} \lambda_i e_i(t)Z_i,$$

where $(\lambda_i)_{i \in \mathbb{N}}$ is a sequence of real numbers such that $\sum_{i=0}^{\infty} \lambda_i^2 < \infty$, $(e_i)_{i \in \mathbb{N}}$ is an orthonormal sequence of continuous functions, and $(Z_i)_{i \in \mathbb{N}}$ a sequence of independent standard Gaussian random variables. Notice that the Karhunen-Loève expansion is a special case of what we are introducing. In this case the error induced by replacing the process by a rate-optimal quantizer of its truncation up to order $m$ is given by

$$\xi_N(X)^2 = \int_0^T \mathbb{E}\left(X_t - \sum_{i=0}^{m} \lambda_i e_i(t)Y_i\right)^2 dt,$$

where $\forall 0 \leq i \leq m$, $Y_i$ is an optimal quantizer of $Z_i$ taking $N_i$ values and $\prod_{i=0}^{m} N_i \leq N$. More precisely we get that

$$\xi_N(X)^2 = \sum_{i=m+1}^{\infty} \lambda_i^2 + \sum_{i=0}^{m} \xi_{N_i}(N(0,\lambda_i^2)).$$

If moreover $\lambda_N^2 \sim \frac{1}{N^\delta}$, with $1 < \delta < 3$, it is shown in [Luschgy and Pagès(2002)] that, the optimal product-quantization of level $N$ is achieved when the dimension of the quantizer $m$ is of order $\log N$ and that it satisfies

$$\xi_N(X) \sim \sqrt{\log N} \frac{1-\delta}{2-\delta}.$$ 

When the basis $(e_k)_{k \in \mathbb{N}}$ chosen in the series expansion is not orthonormal, an alternative rate-optimal quantization method is presented in [Junglen and Luschgy(2010)]. The idea consists in truncating the series up to the optimal order $m = \log N$ and to consider the finite-dimensional covariance operator $K^m$ of the truncation given by

$$\forall t, s \in [0,T], \quad K^m(t,s) = \sum_{i=0}^{m} \lambda_i^2 e_i(t)e_i(s).$$

More specifically, consider $H$ a linear subspace of $L^2$ defined by $H = \text{Span}((e_i)_{0 \leq i \leq m})$. The operator $T_{K^m}$ is given by

$$T_{K^m} : H \rightarrow H \quad f \rightarrow \int_0^T K^m(s,.)f(s) ds.$$
$T_{K^m}$ is clearly an endomorphism. Unlike the Karhunen-Loève theorem, in this case we deal with a linear and symmetric operator in finite dimension. Hence there exists $(\mu^m_i)_{0 \leq i \leq m}$ a sequence of positive real numbers and $(f^m_i)_{0 \leq i \leq m}$ an orthonormal basis of $H$ such that

$$\forall t, s \in [0, T], \quad K^m(t, s) = \sum_{i=0}^{m} \mu^m_i f^m_i(t) f^m_i(s).$$

We can then assert that there exists $(Y^m_i)_{0 \leq i \leq m}$ a sequence of independent standard Gaussian random variables such that

$$\forall t \in [0, T], \quad \sum_{i=0}^{m} \lambda_i e_i(t)Z_i = \sum_{i=0}^{m} \sqrt{\mu^m_i} f^m_i(t)Y^m_i \quad a.s.$$  

Following the same argument as in [Junglen and Luschgy(2010)], if we set $m \asymp \log N$ and replace the process by a rate-optimal quantizer of $\sum_{i=0}^{m} \sqrt{\mu^m_i} f^m_i(t)Y^m_i$, we will get the quadratic quantization error

$$\xi_N(X)^2 = \mathcal{O}_{N \to \infty} \left( \sum_{i=m+1}^{\infty} \lambda_i^2 + \sum_{i=0}^{m} \xi_N(N(0, \mu_i^2)) \right).$$

Moreover, this error is rate-optimal. As an illustration, we give a rate-optimal quantization for both fBm and generalized Ornstein-Uhlenbeck with $T = 1$ and $N = 20$.

![Figure 1](image1.png)

Figure 1: Product quantization of a centered Ornstein-Uhlenbeck process, starting from $Y_0 = 0$ (left), and a fBm (right).

6. Conclusion

This paper presents a new framework to derive series expansions for a specific class of Gaussian processes based on harmonic analysis. One of the main results is a new, simple and rate-optimal series expansion for fractional Brownian motion. One of the advantages of this expansion is that its coefficients are easily computed which can reduce the complexity of simulation, especially for the case $H < 1/2$ where no other trigonometric series expansion
is known. Our approach is general and gives series expansions for a large class of Gaussian processes, in particular to the generalized Ornstein-Uhlenbeck process. The application to quantization is interesting in particular for fBm, where the series basis is not orthonormal. In this case, we show how to deal with non-orthonormality and construct a rate-optimal quantizer.

7. Acknowledgments

I would like to express my gratitude to Bruno Dupire, for giving me the opportunity to work on this subject. I also would like to thank warmly Sylvain Corlay for mentoring me during my research.
References

[Ayache and Taqqu(2003)] Antoine Ayache and Murad S Taqqu. Rate optimality of wavelet series approximations of fractional brownian motion. *Journal of Fourier Analysis and Applications*, 9(5):451–471, 2003.

[Corlay et al.(2010)] Sylvain Corlay et al. Functional quantization-based stratified sampling methods. *arXiv preprint arXiv:1008.4441*, 2010.

[Deheuvels(2007)] Paul Deheuvels. A karhunen–loève expansion for a mean-centered brownian bridge. *Statistics & probability letters*, 77(12):1190–1200, 2007.

[Dietrich and Newsam(1997)] CR Dietrich and Garry Neil Newsam. Fast and exact simulation of stationary gaussian processes through circulant embedding of the covariance matrix. *SIAM Journal on Scientific Computing*, 18(4):1088–1107, 1997.

[Dzhaparidze and Van Zanten(2004)] Kacha Dzhaparidze and Harry Van Zanten. A series expansion of fractional brownian motion. *Probability theory and related fields*, 130(1):39–55, 2004.

[Iglói(2005)] Endre Iglói. A rate-optimal trigonometric series expansion of the fractional brownian motion. *Electron. J. Probab*, 10:1381–1397, 2005.

[Itô et al.(1968)Itô, Nisio, et al.] Kiyosi Itô, Makiko Nisio, et al. On the convergence of sums of independent banach space valued random variables. *Osaka Journal of Mathematics*, 5(1):35–48, 1968.

[Junglen and Luschgy(2010)] Stefan Junglen and Harald Luschgy. A constructive sharp approach to functional quantization of stochastic processes. *Journal of Applied Mathematics*, 2010, 2010.

[Kühn and Linde(2002)] Thomas Kühn and Werner Linde. Optimal series representation of fractional brownian sheets. *Bernoulli*, pages 669–696, 2002.

[Luschgy and Pagès(2002)] Harald Luschgy and Gilles Pagès. Functional quantization of gaussian processes. *Journal of Functional Analysis*, 196(2):486–531, 2002.

[Luschgy and Pagès(2009)] Harald Luschgy and Gilles Pagès. Expansions for gaussian processes and parseval frames. *Electron. J. Probab*, 14(42):1198–1221, 2009.

[Luschgy et al.(2007)Luschgy, Pagès, et al.] Harald Luschgy, Gilles Pagès, et al. High-resolution product quantization for gaussian processes under sup-norm distortion. *Bernoulli*, 13(3):653–671, 2007.

[Printems et al.(2005)] Jacques Printems et al. Functional quantization for numerics with an application to option pricing. *Monte Carlo Methods and Applications mcma*, 11(4):407–446, 2005.
Appendix

Proof of Proposition 2. We first prove that $\gamma$ is integrable. The function $\gamma'$ is continuous on $(0, T]$, positive and for $\delta \in (0, 2)$ we have that there exists $M > 0$ and $\epsilon > 0$ such that

$$\forall x \in (0, \epsilon), \quad 0 \leq \gamma'(x) \leq \frac{M}{x^\delta}. \quad (15)$$

By integrating (15), we get

$$\gamma(x) = O_{x \to 0^+} \left(1 + x^{1-\delta}\right).$$

The last result holds also for $\delta = 0$. Since $0 \leq \delta < 2$ and $\gamma$ is continuous, it comes out that $\gamma$ is integrable on $(0, T]$. It follows that $c(\gamma)$ is well defined.

Before showing the second part, one may first notice that $\gamma'$ is positive and decreasing since $\gamma$ is concave and increasing. By a change of variable in (3), we get

$$c_k = \frac{2}{T} \int_0^{k\pi} \gamma \left(\frac{T}{k\pi} u\right) \cos(u) du.$$

Since

$$\gamma(u) \sin(u) = O_{u \to 0^+} \left(\sin(u) + u^{1-\delta} \sin(u)\right),$$

then

$$\lim_{u \to 0^+} \gamma(u) \sin(u) = 0.$$

Using integration by parts we obtain, for all $k \in \mathbb{N}^*$,

$$c_k = -\frac{2}{T} \int_0^{k\pi} \gamma' \left(\frac{T}{k\pi} u\right) \sin(u) du$$

$$= -\frac{2}{T} \int_0^{k\pi} \sum_{n=0}^{k-1} (-1)^n \gamma' \left(\frac{T(u + n\pi)}{k\pi}\right) \sin(u) du. \quad (16)$$

For $0 \leq n < k$, we define

$$v_{k,n} := \int_0^{\pi} \gamma' \left(\frac{T(u + n\pi)}{k\pi}\right) \sin(u) du.$$

It is immediate that, $\forall k \in \mathbb{N}^*$, $(v_{k,n})_{n<k}$ is nonnegative and decreasing with respect to $n$. Regrouping each pair of elements in (16), we get

$$c_k = -\frac{2}{T} \left(\frac{T}{k\pi}\right)^2 \left(\sum_{n=0}^{\left[\frac{k}{2}\right]-1} (v_{k,2n} - v_{k,2n+1}) + \frac{1 - (-1)^k}{2} v_{k,k-1}\right).$$

It follows that $c_k \leq 0, \forall k \in \mathbb{N}^*$. For the last point, we use again the second part of (16) and get

$$c_k = -\frac{2}{T} \left(\frac{T}{k\pi}\right)^2 \sum_{n=0}^{k-1} (-1)^n v_{k,n}.$$
Since the sequence $((-1)^n v_{k,n})_{n<k}$ has alternating signs and a decreasing modulus, it turns out that

$$|c_k| \leq \frac{2T^2}{k^2 \pi^2} v_{k,0} \leq \frac{2T^2}{k^2 \pi^2} \int_0^\pi \gamma'(\frac{T u}{k \pi}) \sin(u) du.$$ 

In order to conclude, it is enough to prove that

$$\int_0^\pi \gamma'(\frac{T u}{k \pi}) \sin(u) du = O_{k \to \infty} (k^\delta).$$

Since $\gamma \in \Gamma$, we can check that $x \to x^\delta \gamma'(x)$ is uniformly bounded on $[0, T]$. Let $M$ be this uniform bound, then

$$0 \leq \int_0^\pi \gamma'(\frac{T u}{k \pi}) \sin(u) du \leq M \left( \frac{k \pi}{T} \right)^\delta \int_0^\pi \frac{\sin(u)}{u^{\delta-1}} du.$$ 

As $\delta$ belongs to $[0, 2)$, it follows that $c_k = O_{k \to \infty} (k^{\delta-2})$.

This concludes the proof.

\hfill □

**Proof of Theorem 4.** Before proving the Theorem, we remind the reader the following result on the maximum of Gaussian random variables. Let $(X_i)_{1 \leq i \leq M}$ be a finite set of Gaussian random variables, then there exists $c > 0$, such that

$$E \max_{1 \leq i \leq M} |X_i| \leq c \sqrt{\log M} \max_{1 \leq i \leq M} \sqrt{EX_i^2}. \quad (17)$$

We denote by $v_k(t) := \lambda_k e^{-\frac{k^2 t}{2}} Z_k$ for $k \in \mathbb{N}$ and $t \in [0, T]$. The proof is divided in two parts. We first show that, for some $A > 0$, we have

$$\forall n \in \mathbb{N}^*, \quad E \sup_{t \in [0, T]} \left| \sum_{k=2^n}^{2^n+1-1} v_k(t) \right| \leq A \sqrt{n} 2^{-nH}. \quad (18)$$

Let $N \in \mathbb{N}$, for all $0 \leq j \leq N-1$, we denote by $I_j = \left[ j \frac{T}{N}, (j+1) \frac{T}{N} \right]$ and $t_j$ the corresponding center i.e. $t_j = (j + 1/2) \frac{T}{N}$. Let $n \in \mathbb{N}^*$, we have

$$E \sup_{t \in [0, T]} \left| \sum_{k=2^n}^{2^n+1-1} v_k(t) \right| = E \sup_{0 \leq j < N} \sup_{t \in I_j} \left| \sum_{k=2^n}^{2^n+1-1} v_k(t) \right|$$

$$\leq E \sup_{0 \leq j < N} \left| \sum_{k=2^n}^{2^n+1-1} v_k(t_j) \right| + E \sup_{0 \leq j < N} \sup_{t \in I_j} \left| \sum_{k=2^n}^{2^n+1-1} (v_k(t) - v_k(t_j)) \right|. \quad (19)$$

20
Using (17) we get

\[
E \sup_{0 \leq j < N} \left| \sum_{k=2^n}^{2^{n+1}-1} v_k(t_j) \right| \leq c \sqrt{\log N} \sup_{0 \leq j < N} \left| E \sum_{k=2^n}^{2^{n+1}-1} v_k(t_j) \right|^2
\]

\[
\leq c \sqrt{\log N} \sup_{0 \leq j < N} \left( \sum_{k=2^n}^{2^{n+1}-1} E v_k(t_j)^2 \right)
\]

\[
\leq C' \sqrt{\log N} 2^{-nH},
\]

for some \( C' > 0 \). The last inequality comes from the fact that \( E v_k(t_j)^2 \leq \lambda_k \|e_k\|_\infty^2 \leq \frac{C}{k^{1+2H}} \), for some \( C > 0 \).

For the second part of (19), we have

\[
E \sup_{0 \leq j < N} \sup_{t \in I_j} \sum_{k=2^n}^{2^{n+1}-1} (v_k(t) - v_k(t_j)) \leq E \sup_{0 \leq j < N} \sum_{k=2^n}^{2^{n+1}-1} \sup_{t \in I_j} |v_k(t) - v_k(t_j)|.
\]

Observing that \( \forall t \in I_j \) we have \( |t - t_j| \leq \frac{T}{N} \), we get that

\[
\sup_{t \in I_j} |v_k(t) - v_k(t_j)| \leq |\lambda_k|Z_k \left| e_k \left( \frac{k\pi t}{T} \right) - e_k \left( \frac{k\pi t_j}{T} \right) \right|
\]

\[
\leq C' k^{\frac{1}{2} - H} |Z_k| \frac{\pi}{N}.
\]

Replacing in (21), it follows that

\[
E \sup_{0 \leq j < N} \sup_{t \in I_j} \sum_{k=2^n}^{2^{n+1}-1} (v_k(t) - v_k(t_j)) \leq \frac{C' N}{2^{n+1}} \sum_{k=2^n}^{2^{n+1}-1} k^{\frac{1}{2} - H}
\]

\[
\leq \frac{C*}{N} 2^{n(\frac{1}{2} - H)}.
\]

Combining (20) and (23) we deduce that

\[
E \sup_{t \in [0,T]} \left| \sum_{k=2^n}^{2^{n+1}-1} v_k(t) \right| \leq C \sqrt{\log N} 2^{-nH} + \frac{C*}{N} 2^{n(\frac{1}{2} - H)}.
\]

Replacing \( N = 2^{2n} \) in (24), we prove (18).

The previous result holds even if we replace \( \sum_{k=2^n}^{2^{n+1}-1} v_k(t) \) by \( \sum_{k=M}^{2^{n+1}-1} v_k(t) \) for some \( M \in [2^n, 2^{n+1} - 1] \). Let \( N \) be a positive integer. By taking \( m = \lfloor \log N / \log 2 \rfloor \), we get

\[
E \sup_{t \in [0,T]} \left| \sum_{k=N+1}^\infty v_k(t) \right| \leq E \sup_{t \in [0,T]} \left| \sum_{k=N+1}^{2^{n+1}-1} v_k(t) \right| + \sum_{i=m+1}^\infty E \sup_{t \in [0,T]} \left| \sum_{k=2^i}^{2^{i+1}-1} v_k(t) \right|.
\]
We can conclude, using (18), that

$$ \mathbb{E} \sup_{t \in [0,T]} \left| \sum_{k=N+1}^{\infty} v_k(t) \right| \leq A \sum_{i=m}^{\infty} \sqrt{i 2^{-iH}} \leq A' \sqrt{m 2^{-mH}}. \quad (26) $$

It suffices to observe that $2^m \leq N < 2^{m+1}$, to obtain the rate of the uniform convergence. The uniform tightness implies that $\sum_{k=0}^{N} v_k$ has a weak limit in $C[0,T]$ the space of continuous functions on $[0,T]$. We remind the reader that we endow this space with the supremum metric. By the Itô-Nisio theorem, we get, as in [Itô et al. (1968)Itô, Nisio, et al.], that the process $\sum_{k=0}^{N} v_k$ converges in $C[0,T]$ almost surely.