Research Article

On the Computation of Some Topological Descriptors to Find Closed Formulas for Certain Chemical Graphs

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In this research paper, we will compute the topological indices (degree based) such as the ordinary generalized geometric-arithmetic (OGA) index, first and second Gourava indices, first and second hyper-Gourava indices, general Randić index $R_c(G)$, for $c = \pm 1, \pm 1/2$, harmonic index, general version of the harmonic index, atom-bond connectivity (ABC) index, SK, SK1, and SK2 indices, sum-connectivity index, general sum-connectivity index, and first general Zagreb and forgotten topological indices for various types of chemical networks such as the subdivided polythiophene network, subdivided hexagonal network, subdivided backbone DNA network, and subdivided honeycomb network. The discussion on the aforementioned networks will give us very remarkable results by using the aforementioned topological indices.

1. Introduction

The branch of mathematics that is related to the study of implementation of chemistry and graph theory together is called chemical graph theory. This theory is used to model the molecules of a chemical compound mathematically. This theory helps us to understand the physical properties of that chemical/molecular compound. In this theory, we construct the structure of a chemical compound in the form of a graph. In chemical graph theory, atoms are used as nodes, and bonds between the atoms are utilized as edges. A topological index is a numerical parameter of a graph that explains its topology. The topological index is also called a molecular descriptor and a connectivity index. It is obtained by transforming the chemical information into a numerical quantity. Topological indices are used as molecular descriptors in the construction of quantitative structure-activity relationships and quantitative structure-property relationships as well. The theoretical models such as quantitative structure-activity relationships (QSARs) relate mathematically physical/chemical properties to the structure of a molecule. Topological indices such as ordinary generalized geometric-arithmetic (OGA) index, first and second Gourava indices, first and second hyper-Gourava indices, general Randić index $R_{\gamma}(G)$, for $\gamma = \pm 1, \pm 1/2$, harmonic index, general version of harmonic index [1, 2], atom-bond connectivity (ABC) index [3, 4], SK, SK1, and SK2 indices, sum-connectivity index, general sum-connectivity index, and first general Zagreb [5] and forgotten topological indices have very significant roles in QSAR and QSPR studies and are used to discuss the bioactivity of molecular structures.

In 2009, D. Vukičević and B. Furtula established the first GA index in [6–11]. The first geometric-arithmetic (GA) index of a graph $\xi$ was calculated by

$$GA(\xi) = \sum_{g \in E(\xi)} \frac{d_g d_h}{d_g + d_h}$$

(1)

An ordinary geometric-arithmetic index of $\xi$ was produced in 2011 in [12] and formulated by, for each positive real number $k$,
were formulated by second Gourava and hyper-Gourava indices in [13, 14]. The first and second Gourava and hyper-Gourava indices of a graph \( G \) are defined as:

\[
GO_1(\xi) = \sum_{\{d_g, d_h\}} \left[ (d_g + d_h) + (d_g d_h) \right],
\]

\[
GO_2(\xi) = \sum_{\{d_g, d_h\}} \left[ (d_g + d_h) + (d_g d_h)^2 \right],
\]

\[
HGO_1(\xi) = \sum_{\{d_g, d_h\}} \left[ (d_g + d_h) + (d_g d_h) \right]^2,
\]

\[
HGO_2(\xi) = \sum_{\{d_g, d_h\}} \left[ (d_g + d_h) + (d_g d_h)^2 \right].
\]

In 1975, Randić index [15–17] was introduced by Milan Randić. It is often used in chemoinformatics to investigate the compounds of chemicals. The Randić index is also known as the general Randić index [22, 23] and is given by

\[
R_{(1/2)}(\xi) = \sum_{uv \in E(\xi)} \frac{1}{\sqrt{(d_u + d_v)}},
\]

where \( d_u \) and \( d_v \) are the degrees of the nodes.

In 2008, Ernesto Estrada et al. [25, 26] introduced a new topological index, named atom-bond connectivity (ABC) index, calculated by

\[
ABC(G) = \sum_{gh \in E(G)} \left[ \frac{d_g + d_h - 2}{d_g d_h} \right].
\]

The ABC index is an excellent valuable index in the formation of heat in alkanes [25, 26].

**Definition 1.** For a graph \( \xi \), the SK index [27] can be computed by

\[
SK(\xi) = \sum_{gh \in E(\xi)} \left[ \frac{d_g + d_h}{2} \right].
\]

Let \( d_g \) and \( d_h \) be the degrees of nodes \( g \) and \( h \) in \( \xi \), respectively.

**Definition 2.** For a graph \( \xi \), the SK1 index can be computed by

\[
SK_1(\xi) = \sum_{gh \in E(\xi)} \left[ \frac{d_g d_h}{2} \right].
\]

Let \( d_g \) and \( d_h \) be the degrees of nodes \( g \) and \( h \) in \( \xi \), respectively.

**Definition 3.** For a graph \( \xi \), the SK2 index can be computed by

\[
SK_2(\xi) = \sum_{gh \in E(\xi)} \left[ \frac{d_g + d_h}{2} \right]^2.
\]

Let \( d_g \) and \( d_h \) be the degrees of nodes \( g \) and \( h \) in \( \xi \), respectively.

In 2009, B. Lučić proposed the sum-connectivity index \( \chi_c \) in [28] calculated by

\[
\chi_{c-(1/2)}(G) = \sum_{gh \in E(G)} \left[ \frac{d_g + d_h}{2} \right]^{-1/2}.
\]

In 2010, B. Zhou and Trinajstić furnished an index named general sum-connectivity index in [24, 29] and formulated as follows:

\[
\chi_h(\xi) = \sum_{gh \in E(\xi)} \left[ \frac{d_g + d_h}{2} \right]^k.
\]

2. **Topological Indices on Certain Chemical Graphs**

In this part of the research paper, we will compute the topological indices (degree based) such as ordinary generalized geometric-arithmetic (OGA) index, first and second Gourava indices, first and second hyper-Gourava indices, general Randić index \( R_{(1/2)}(\xi) \), for \( \gamma = [-1, \pm 1/2] \), harmonic index, general version of harmonic index, atom-bond connectivity (ABC) index, SK, SK1, and SK2 indices, sum-connectivity index, general sum-connectivity index, first general Zagreb index, and forgotten topological indices for various types of chemical networks such as subdivided...
polythiophene network, subdivided hexagonal network, subdivided backbone DNA network, and subdivided honeycomb network.

2.1. Results for the Subdivided Polythiophene Network. Polythiophenes are rings with five elements having one heteroatom together with their benzo and other carbonyl. Polythiophene is used in electronic devices such as water purification devices, biosensors, and light-emitting diodes and in hydrogen storage [39]. In a subdivided polythiophene network, shown in Figure 1, we insert another vertex (degree 2) in every edge of ξ. In this way, we get a subdivided polythiophene network. In this section, we compute the subdivided polythiophene network using the above-defined topological indices. In the subdivided polythiophene network SPLYₙ, we have the number of nodes 11n - 1 and edges 12n - 2. A subdivided polythiophene network for n = 5 is shown in Figure 1. We get two kinds of edges (degree based) that are (2, 2) and (2, 3). Table 1 gives us two types of edges. A subdivided polythiophene network SPLY₅ is displayed in Figure 1.

**Theorem 1. For the subdivided polythiophene network, SPLYₙ, the ordinary generalized geometric-arithmetic index is calculated by**

\[ OGA_1(\xi) = 6n \left( 1 + \left[ \frac{\sqrt{24}}{5} \right]^k \right) + \left[ 4 - 6 \left[ \frac{\sqrt{24}}{5} \right]^k \right], \]  

(15)

**Proof.** By letting ξ as a subdivided polythiophene network SPLYₙ, from Table 1, we know

\[ OGA_1(\xi) = \sum_{gh \in E(\xi)} \left[ \left( \frac{\sqrt{4d_g d_h}}{2} \right)^k \left[ \frac{\sqrt{16}}{2} + (6n - 6) \left[ \frac{\sqrt{24}}{5} \right]^k \right], \]  

(16)

\[ OGA_k(\xi) = (6n + 4) \left[ \frac{\sqrt{16}}{2} + (6n - 6) \left[ \frac{\sqrt{24}}{5} \right]^k \right], \]

and by doing some calculations, we get

\[ OGA_k(\xi) = 6n \left[ 1 + \left( \frac{\sqrt{24}}{5} \right)^k \right] + \left[ 4 - 6 \left( \frac{\sqrt{24}}{5} \right)^k \right]. \]  

(17)

**Theorem 2. For the subdivided polythiophene network, SPLYₙ, the first and second Gourava indices are calculated by**

\[ GO_1(\xi) = \sum_{gh \in E(\xi)} [(d_g + d_h) + (d_g d_h)]^2, \]  

\[ GO_2(\xi) = \sum_{gh \in E(\xi)} [(d_g + d_h) + (d_g d_h)]^2, \]  

and by doing some calculations, we get

\[ GO_1(\xi) = (6n + 4) \left[ (44 + (44) + (6n - 6) [5 + (6)] = 114n - 34, \]  

\[ GO_2(\xi) = (6n + 4) [16] + (6n - 6) [30] = 276n - 116. \]  

(19)

**Theorem 3. For the subdivided polythiophene network, SPLYₙ, the first and second hyper-Gourava indices are calculated by**

\[ HGO_1(\xi) = \sum_{gh \in E(\xi)} \left( \frac{\sqrt{4d_g d_h}}{2} \right)^k \left( \frac{\sqrt{16}}{2} + (6n - 6) \left( \frac{\sqrt{24}}{5} \right)^k \right), \]

\[ HGO_2(\xi) = \sum_{gh \in E(\xi)} \left( \frac{\sqrt{4d_g d_h}}{2} \right)^k \left( \frac{\sqrt{16}}{2} + (6n - 6) \left( \frac{\sqrt{24}}{5} \right)^k \right), \]

and by doing some calculations, we get

\[ HGO_1(\xi) = (6n + 4) [64] + (6n - 6) [121] = 1110n - 470, \]  

\[ HGO_2(\xi) = (6n + 4) [256] + (6n - 6) [900] = 6936n - 4376. \]  

(21)

**Theorem 4. For the subdivided polythiophene network, SPLYₙ, the general Randić’ index is calculated by**

\[ R_{\gamma}(SPLY_n) = \left\{ \begin{array}{ll} \frac{5}{2} n, & \text{for } \gamma = -1, \\ n \left( \frac{109}{20} \right) + \left( \frac{8}{5} \right), & \text{for } \gamma = -\frac{1}{2}, \\ n \left( \frac{267}{10} \right) - \left( \frac{67}{10} \right), & \text{for } \gamma = \frac{1}{2}, \\ 60n - 20, & \text{for } \gamma = 1. \end{array} \right. \]  

(22)

**Proof.** By letting ξ as a subdivided polythiophene network SPLYₙ of n dimensions, we have the number of nodes and
edges in $SPLY_n$ as $|V(SPLY_n)| = 11n - 1$ and $|E(SPLY_n)| = 12n - 2$, respectively.

We know that

$$R_\gamma (\xi) = \sum_{gh \in E(\xi)} (d_g, d_h)^\gamma,$$  \hspace{1cm} (23)

for $\gamma = \{-1, 1, -1/2, 1/2\}$.

Case 1: If $\gamma = -1$, the application of Randic index $R_\gamma (\xi)$

$$R_{-1} (\xi) = \sum_{gh \in E(\xi)} \frac{1}{d_g, d_h},$$  \hspace{1cm} (24)

using (23). From Table 1, we know $R_{-1} (\xi) = (6n + 4)(4)^{-1} + (6n - 6)(6)^{-1}$. By doing some calculations, we get $R_{-1} (\xi) = (5/2)n$.

Case 2: If $\gamma = (1/2)$, the application of Randic index $R_\gamma (\xi)$

$$R_{(1/2)} (\xi) = \sum_{gh \in E(\xi)} \sqrt{d_g, d_h},$$  \hspace{1cm} (25)

using (23),

$$R_{(1/2)} (\xi) = (6n + 4) \frac{1}{\sqrt{4}} + (6n - 6) \frac{1}{\sqrt{6}},$$  \hspace{1cm} (26)

and by doing some calculations, we get $R_{(1/2)} (\xi) = n(109/20) + (8/5)$.

Case 3: If $\gamma = (1/2)$, the application of Randic index $R_\gamma (\xi)$

$$R_{(1/2)} (\xi) = \sum_{gh \in E(\xi)} \frac{1}{d_g, d_h},$$  \hspace{1cm} (27)

using (23),

$$R_{(1/2)} (\xi) = (6n + 4) \sqrt{4} + (6n - 6) \sqrt{6},$$  \hspace{1cm} (28)

and by doing some calculations, we get $R_{(1/2)} (\xi) = n(267/10) - (67/10)$.

Case 4: If $\gamma = 1$, the application of Randic index $R_\gamma (\xi)$

$$R_1 (\xi) = \sum_{gh \in E(\xi)} (d_g, d_h)^1,$$  \hspace{1cm} (29)

using (23),

$$R_1 (\xi) = (6n + 4)(4) + (6n - 6)(6),$$  \hspace{1cm} (30)

and by doing some calculations, we get $R_1 (\xi) = 60n - 20$.

**Theorem 5.** For the subdivided polythiophene network $SPLY_n$, the harmonic index is calculated by

$$HI(\xi) = \sum_{gh \in E(\xi)} \frac{2}{d_g + d_h},$$  \hspace{1cm} (31)

and by doing some calculations, we get

$$HI(\xi) = (6n + 4) \left( \frac{1}{2} + (6n - 6) \frac{2}{5} \right) = \frac{2}{5} (n - 1).$$  \hspace{1cm} (32)

**Theorem 6.** For the subdivided polythiophene network $SPLY_n$, the general version of the harmonic index is calculated by

$$H_kI(\xi) = \frac{6n}{k} \left( \frac{5^k + 4^k}{6^{k+1}} + 4 \left( \frac{5^k}{6^{k+1}} - 6 \right) \right),$$  \hspace{1cm} (33)

and by doing some calculations, we get

$$H_kI(\xi) = (6n + 4) \left( \frac{2}{2 + 2} \right)^k + (6n - 6) \left( \frac{2}{2 + 3} \right)^k,$$

$$= \frac{6n \left( 5^k + 4^k \right) + 4 \left( 5^k - 6 \right) \left( 4^k \right)}{(10)^k}.$$  \hspace{1cm} (34)

**Theorem 7.** For the subdivided polythiophene network $SPLY_n$, the atom-bond connectivity index is calculated by

$$ABC(\xi) = 6\sqrt{2n} - \sqrt{2}.$$  \hspace{1cm} (35)

**Theorem 8.** For the subdivided polythiophene network $SPLY_n$, $SK$, $SK_1$, and $SK_2$ indices are calculated by $SK(\xi) = 27n - 7$, $SK_1(\xi) = 30n - 10$, and $SK_2(\xi) = (1/2) \left( 123n - 43 \right)$, respectively.

**Proof.** By letting $\xi$ as a subdivided polythiophene network $SPLY_n$, from Table 1, we know

$$HI(\xi) = \frac{2n - 2}{5},$$  \hspace{1cm} (36)

and by doing some calculations, we get

$$HI(\xi) = (6n + 4) \left( \frac{1}{2} + (6n - 6) \frac{2}{5} \right) = \frac{2}{5} (n - 1).$$  \hspace{1cm} (37)

**Proof.** By letting $\xi$ as a subdivided polythiophene network $SPLY_n$, from Table 1, we know

$$ABC(\xi) = 6\sqrt{2n} - \sqrt{2}.$$  \hspace{1cm} (38)

and by doing some calculations, we get

$$ABC(\xi) = \frac{2n - 2}{5} \frac{2}{4} + (6n - 6) \frac{2}{6} = \sqrt{2} (6n - 1).$$  \hspace{1cm} (39)
For the subdivided polythiophene network, the sum-connectivity index is calculated by
\[ SPLY_n = \sum_{gh \in E(\xi)} \left[ \frac{d_g + d_h}{2} \right], \]
and by doing some calculations, we get
\[ SPLY_n = (6n + 4)(2) + (6n - 6) \left( \frac{5}{2} \right) = 27n - 7, \]
\[ SPLY_n = (6n + 4)(2) + (6n - 6)(3) = 30n - 10, \]
\[ SPLY_n = (6n + 4)(4) + (6n - 6) \left( \frac{25}{4} \right) = \frac{1}{2} (123n - 43). \]

**Theorem 9.** For the subdivided polythiophene network, \( SPLY_n \) the sum-connectivity index is calculated by
\[ \chi_{-1(1/2)}(\xi) = n \left\{ 3 + \frac{6}{\sqrt{5}} \right\} + \left\{ 2 - \frac{6}{\sqrt{5}} \right\}. \]

*Proof.* By letting \( \xi \) as a subdivided polythiophene network \( SPLY_n \) from Table 1, we know
\[ \chi_{-1(1/2)}(\xi) = \sum_{gh \in E(\xi)} \left[ d_g + d_h \right]^{-1(1/2)}, \]
\[ \chi_{-1(1/2)}(\xi) = (6n + 4) \left( \frac{1}{2} \right) + (6n - 6) \left( \frac{1}{\sqrt{5}} \right), \]
and by doing some calculations, we get
\[ \chi_{-1(1/2)}(\xi) = n \left\{ 3 + \frac{6}{\sqrt{5}} \right\} + \left\{ 2 - \frac{6}{\sqrt{5}} \right\}. \]

**Theorem 10.** For the subdivided polythiophene network, \( SPLY_n \) the general sum-connectivity index is calculated by
\[ \chi_k(\xi) = 6n \left[ 5^k + 4^k \right] + 4 \left[ 4^k \right] - 6 \left[ 5^k \right]. \]

*Proof.* By letting \( \xi \) as a subdivided polythiophene network \( SPLY_n \) from Table 1, we know
\[ \chi_k(\xi) = \sum_{gh \in E(\xi)} \left[ d_g + d_h \right]^k, \]
\[ \chi_k(\xi) = (6n + 4) \left( 4^k \right) + (6n - 6) \left( 5^k \right), \]
and by doing some calculations, we get
\[ \chi_k(\xi) = 6n \left( 5^k + 4^k \right) + 4 \left( 4^k \right) - 6 \left( 5^k \right). \]
Figure 2: SHXn.

| (dg, dh) for gh ∈ E(ξ) | Number of E(ξ) |
|------------------------|-----------------|
| (2, 3)                 | 18              |
| (2, 4)                 | 24(n - 2)       |
| (2, 6)                 | 6(3n^2 - 9n + 7) |

\[
\text{OGA}_k(ξ) = 18n^2 \left( \frac{\sqrt{48}}{8} \right)^k + n \left[ 24 \left( \frac{\sqrt{32}}{6} \right)^k - 54 \left( \frac{\sqrt{48}}{8} \right)^k \right] + \left[ 42 \left( \frac{\sqrt{48}}{8} \right)^k - 48 \left( \frac{\sqrt{32}}{6} \right)^k \right].
\]  

**Theorem 14.** For the subdivided hexagonal network, SHXn, the first and second Gourava indices are calculated by
\[
\text{GO}_1(ξ) = 360n^2 - 744n + 366, \\
\text{GO}_2(ξ) = 1728n^2 - 4032n + 2268.
\]

**Proof.** By letting ξ as a subdivided hexagonal network SHXn, from Table 2, we know
\[
\text{GO}_1(ξ) = \sum_{gh \in E(ξ)} \left[ (d_g + d_h) + (d_g d_h) \right], \\
\text{GO}_2(ξ) = \sum_{gh \in E(ξ)} \left[ (d_g + d_h) + (d_g d_h) \right],
\]
and by doing some calculations, we get
\[
\text{GO}_1(ξ) = (18)[5] + (6) + 24(n - 2)[6] + (8) + 6[3n^2 - 9n + 7] \times (12) = 360n^2 - 744n + 366, \\
\text{GO}_2(ξ) = (18)[30] + 24(n - 2)[48] + 6[3n^2 - 9n + 7] \times (96), \\
= 1728n^2 - 4032n + 2268.
\]

**Theorem 15.** For the subdivided hexagonal network, SHXn, the first and second hyper-Gourava indices are calculated by \( \text{HGO}_1(ξ) = 7200n^2 - 16896n + 9570 \) and \( \text{HGO}_2(ξ) = 165888n^2 - 442368n + 292680 \).

**Proof.** By letting ξ as a subdivided hexagonal network SHXn, from Table 2, we know
\[
\text{HGO}_1(ξ) = \sum_{gh \in E(ξ)} \left[ (d_g + d_h) + (d_g d_h) \right]^2, \\
\text{HGO}_2(ξ) = \sum_{gh \in E(ξ)} \left[ (d_g + d_h) + (d_g d_h) \right]^2,
\]
and by doing some calculations, we get
\[
\text{HGO}_1(ξ) = (18)[121] + 24(n - 2)[196] + 6[3n^2 - 9n + 7] \times (400) = 7200n^2 - 16896n + 9570, \\
\text{HGO}_2(ξ) = (18)[900] + 24(n - 2)[2304] + 6[3n^2 - 9n + 7] \times (9216), \\
= 165888n^2 - 442368n + 292680.
\]
Theorem 16. For the subdivided hexagonal network, $SHX_n$, $n > 1$, the general Randić index is calculated by

$$R_y(\text{SHX}_n) = \begin{cases} 
\frac{3n^2 - 3n + 1}{2}, & \text{for } y = -1, \\
3n^2\sqrt{3} + n\{\sqrt{72} - 9\sqrt{3}\} + 2.501, & \text{for } y = -\frac{1}{2},
\end{cases}$$

$$62.35n^2 - 119.17n + 53.82, \quad \text{for } y = \frac{1}{2},$$

$$216n^2 - 456n + 228, \quad \text{for } y = 1.$$  (62)

Proof. By letting $\xi$ as a subdivided hexagonal network $SHX_n$ of $n$ dimensions, we have the number of nodes and edges in $SHX_n$ as $|V(SHX_n)| = 12n^2 - 18N + 7$ for $n > 1$ and $|E(SHX_n)| = 18n^2 - 30n + 12$ for $n > 1$, respectively. We know that

$$R_y(\xi) = \sum_{gh \in E(\xi)} (d_g, d_h)^y,$$  (63)

for $y = \{-1, 1, -1/2, 1/2\}$.

Case 1: if $y = -1$, the application of Randić index $R_y(\xi)$

$$R_{-1}(\xi) = \sum_{gh \in E(\xi)} \frac{1}{d_g d_h}.$$  (64)

using (63). From Table 2, we know

$$R_{-1}(\xi) = (18)(6)^{-1} + 24(n - 2)(8)^{-1} + 6\left(3n^2 - 9n + 7\right)(12)^{-1}.$$  (65)

By doing some calculations, we get

$$R_{-1}(\xi) = (3n^2 - 3n + 1/2).$$

Case 2: if $y = -1/2$, the application of Randić index $R_y(\xi)$

$$R_{-1/2}(\xi) = \sum_{gh \in E(\xi)} \frac{1}{\sqrt{(d_g, d_h)}},$$  (66)

using (63),

$$R_{-1/2}(\xi) = \frac{1}{\sqrt{6}} + 24(n - 2)\frac{1}{\sqrt{8}} + 6\left(3n^2 - 9n + 7\right)\frac{1}{\sqrt{12}}.$$  (67)

By doing some calculations, we get

$$R_{-1/2}(\xi) = 3n^2\sqrt{3} + n\{\sqrt{72} - 9\sqrt{3}\} + 2.501.$$  (68)

Case 3: if $y = 1/2$, the application of Randić index $R_y(\xi)$

$$R_{1/2}(\xi) = \sum_{gh \in E(\xi)} \sqrt{d_g, d_h},$$  (69)

using (63),

$$R_{1/2}(\xi) = (18)\sqrt{6} + 24(n - 2)\sqrt{8} + 6\left(3n^2 - 9n + 7\right)\sqrt{12}.$$  (70)

By doing some calculations, we get

$$R_{1/2}(\xi) = 62.35n^2 - 119.17n + 53.82.$$  (71)

Case 4: if $y = 1$, the application of Randić index $R_y(\xi)$

$$R_{1}(\xi) = \sum_{gh \in E(\xi)} (d_g, d_h)^1,$$  (72)

using (63),

$$R_{1}(\xi) = (18)(6) + 24(n - 2)(8) + 6\left(3n^2 - 9n + 7\right)(12).$$  (73)

By doing some calculations, we get

$$R_{1}(\xi) = 216n^2 - 456n + 228.$$  (74)

Theorem 17. For the subdivided hexagonal network, $SHX_n$, the harmonic index is calculated by

$$HI(\xi) = 4.5n^2 - 5.5n + 1.7.$$  (75)

Proof. By letting $\xi$ as a subdivided hexagonal network $SHX_n$, from Table 2, we know

$$HI(\xi) = \sum_{gh \in E(\xi)} \frac{2}{d_g + d_h}.$$  (76)

By doing some calculations, we get

$$HI(\xi) = \left(18 \left[\frac{2}{5}\right] + 24(n - 2)\left[\frac{2}{6}\right] + 6\left(3n^2 - 9n + 7\right)\left[\frac{2}{8}\right]\right),$$

$$= \frac{1}{10} \left(45n^2 - 55n + 17\right) = 4.5n^2 - 5.5n + 1.7.$$  (77)

Theorem 18. For the subdivided hexagonal network, $SHX_n$, the general version of the harmonic index is calculated by

$$H_k(\xi) = 18n^2 \left[4^{-k}\right] + 2n\left[12\left[3^{-k}\right] - 27\left[4^{-k}\right]\right] + \left[18\left[2^k\left[5^{-k}\right]\right] - 48\left[3^{-k}\right] + 42\left[4^{-k}\right]\right].$$  (78)

Proof. By letting $\xi$ as a subdivided hexagonal network $SHX_n$, from Table 2, we know

$$H_k(\xi) = \sum_{gh \in E(\xi)} \left[\frac{2}{d_g + d_h}\right]^k.$$  (79)

By doing some calculations, we get
from Table 2, we know the atom-bond connectivity index is calculated by the subdivided hexagonal network, \( \text{SHX}_n \), Theorem 19.

\[
ABC(\xi) = 9\sqrt{2n^2} - 15\sqrt{2n} + 6\sqrt{2}.
\]  

(81)

Proof. By letting \( \xi \) as a subdivided hexagonal network \( \text{SHX}_n \), from Table 2, we know

\[
ABC(\xi) = \sum_{g \in E(\xi)} \sqrt{\frac{d_g + d_h - 2}{d_g d_h}},
\]  

(82)

and by doing some calculations, we get

\[
ABC(\xi) = (18)\sqrt{\frac{2 + 3 - 2}{6}} + 24(n - 2)\sqrt{\frac{2 + 4 - 2}{8}} + 6(3n^2 - 9n + 7)\sqrt{\frac{2 + 6 - 2}{12}}
= 9\sqrt{2n^2} - 15\sqrt{2n} + 6\sqrt{2}.
\]  

(83)

Theorem 20. For the subdivided hexagonal network, \( \text{SHX}_n \), \( SK, SK_1, \) and \( SK_2 \) indices are calculated by \( SK(\xi) = 72n^2 - 144n + 69 \), \( SK_1(\xi) = 108n^2 - 228n + 498 \), and \( SK_2(\xi) = 288n^2 - 648n + 7052 \), respectively.

Proof. By letting \( \xi \) as a subdivided hexagonal network \( \text{SHX}_n \), from Table 2, we know

\[
SK(\xi) = \sum_{g \in E(\xi)} \left[ \frac{d_g + d_h}{2} \right],
\]  

(84)

\[
SK_1(\xi) = \sum_{g \in E(\xi)} \frac{d_g d_h}{2},
\]  

(85)

and \( SK_2(\xi) = \sum_{g \in E(\xi)} \left[ \frac{d_g + d_h}{2} \right]^2 \),

and by doing some calculations, we get

\[
SK(\xi) = (18)\left( \frac{5}{2} + 24(n - 2)\left( \frac{6}{2} \right) + 6\left( 3n^2 - 9n + 7 \right)\left( \frac{8}{2} \right) \right)
= 72n^2 - 144n + 69.
\]  

\[
SK_1(\xi) = 18(3) + 24(n - 2)\left( \frac{4}{3} \right) + 36\left( 3n^2 - 9n + 7 \right)
= 108n^2 - 228n + 498.
\]

\[
SK_2(\xi) = (18)\left( \frac{25}{4} \right) + 24(n - 2)\left( \frac{9}{4} \right) + 6\left( 3n^2 - 9n + 7 \right)\left( \frac{16}{4} \right)
= 288n^2 - 648n + 7052.
\]  

(86)

Theorem 21. For the subdivided hexagonal network, \( \text{SHX}_n \), the sum-connectivity index is calculated by

\[
\chi_{-(1/2)}(\xi) = 18n^2 \left[ \frac{1}{\sqrt{8}} \right] + n \left( \frac{24}{\sqrt{6}} - \frac{54}{\sqrt{8}} \right) + \left( \frac{18}{\sqrt{5}} - \frac{48}{\sqrt{6}} + \frac{42}{\sqrt{8}} \right).
\]

(87)

Proof. By letting \( \xi \) as a subdivided hexagonal network \( \text{SHX}_n \), from Table 2, we know

\[
\chi_{-(1/2)}(\xi) = \sum_{g \in E(\xi)} \left[ \frac{d_g + d_h}{2} \right]^{-1/2},
\]

and by doing some calculations, we get

\[
\chi_{-(1/2)}(\xi) = (18)\left( \frac{1}{\sqrt{6}} \right) + 24(n - 2)\left( \frac{1}{\sqrt{8}} \right)
+ 6\left( 3n^2 - 9n + 7 \right)\left( \frac{1}{\sqrt{12}} \right),
\]

(88)

Theorem 22. For the subdivided hexagonal network, \( \text{SHX}_n \), the general sum-connectivity index is calculated by

\[
\chi_k(\xi) = 18n^2 \left[ 8^k \right] + n \left[ 24(6^k) - 54(8^k) \right] + \left( 18(5^k) - 48(6^k) + 42(8^k) \right).
\]

(89)

Proof. By letting \( \xi \) as a subdivided hexagonal network \( \text{SHX}_n \), from Table 2, we know
\[
\chi_k(\xi) = \sum_{gh \in E(\xi)} [d^k_g + d^k_h],
\]
\[
\chi_k(\xi) = (18\{5^k\} + 24(n-2)(6^k) + 6(3n^2 - 9n + 7)(8^k),
\]
and by doing some calculations, we get
\[
\chi_k(\xi) = = 18n^2\{8^k\} + n[24(6^k) - 54(8^k)] + 18(5^k) - 48(6^k) + 42(8^k).\]

**Theorem 23.** For the subdivided hexagonal network, SHXₙ, the first general Zagreb index is calculated by
\[
kM_1(\xi) = n^2\{\frac{3}{2}6^k\} + 9\{2^k\} + n[6\{4^k\} - 15\{2^k\} - 9\{6^k\}] + 6\{2^k\} + 6\{3^k\} - 12\{4^k\} + 7\{6^k\}.\]

**Proof.** By letting \(\xi\) as a subdivided hexagonal network SHXₙ, from Table 2, we know
\[
kM_1(\xi) = \sum_{gh \in E(\xi)} [d^{k-1}_g + d^{k-1}_h], \quad k > 1,
\]
\[
kM_1(\xi) = (18\{\frac{2}{2} + \frac{3}{3}\} + 24n - 48\{\frac{2}{2} + \frac{4}{4}\}) + 6(3n^2 - 9n + 7)(2^{k-1} + 6^{k-1}).
\]
and by doing some calculations, we get
\[
kM_1(\xi) = n^2\{36^k + 92^k\} + n64^k - 152^k - 96^k + 62^k + 63^k - 124^k + 76^k.\]

**Theorem 24.** For the subdivided hexagonal network, SHXₙ, the forgotten index is calculated by
\[
F(\xi) = 2\{360n^2 - 840n + 477\}.
\]

**Proof.** By letting \(\xi\) as a subdivided hexagonal network SHXₙ, from Table 2, we know
\[
F(\xi) = \sum_{gh \in E(\xi)} [d^2_g + d^2_h],
\]
\[
F(\xi) = (18)(13) + 24(n-2)(20) + 6(3n^2 - 9n + 7)(40),
\]
and by doing some calculations, we get
\[
F(\xi) = 2\{360n^2 - 840n + 477\}.
\]

2.3. Results for the Subdivided Backbone DNA Network.

The structure of DNA is called a double helix as it is made of two strands that wind around each other that looks like a staircase [40]. Each strand has a backbone made of deoxyribose, sugar, and a phosphate group. These sugar and phosphates make up the backbone, while the nitrogen bases are found in the centre and hold the two strands together. There are 4 bases attached to each sugar which are adenine, cytosine, guanine, and thymine. Both ends of DNA have a number, i.e., one end is ‘5’ and the other is ‘3’. In a subdivided backbone DNA network, shown in Figure 3, we insert another node (degree 2) in each edge of \(\xi\). In this way, we get a subdivided backbone DNA network of \(n\) dimensions. A subdivided backbone DNA network for \(n = 4\) is displayed in Figure 3. A subdivided backbone DNA network is symbolized as SBBDNA\((n)\). The order and size of SBBDNA\((n)\) are \(15n - 5\) and \(16n - 6\), respectively. We obtain two types of edges (degree based) that are (2,2) and (2,3). Table 3 gives us two kinds of edges. A subdivided backbone DNA network SBBDNA\((4)\) is shown in Figure 3.

**Theorem 25.** For the subdivided backbone DNA network, SBBDNA\((n)\), the ordinary generalized geometric-arithmetic index is calculated by
\[
OGA_k(\xi) = 2\left\{n + 3\left\{\sqrt{\frac{24}{5}}\right\} - 3\left\{\sqrt{\frac{24}{5}}\right\}^k\right\}.
\]

**Proof.** By letting \(\xi\) as a subdivided backbone DNA network SBBDNA\((n)\), from Table 3, we know
\[
OGA_k(\xi) = \sum_{gh \in E(\xi)} \left\{\sqrt{4d_gd_h}\right\}^k,
\]
\[
OGA_k(\xi) = 10n\left\{\sqrt{\frac{16}{2}} + 2 + (6n - 6)\right\}\left\{\sqrt{\frac{24}{5}}\right\}^k,
\]
and by doing some calculations, we get
\[
OGA_k(\xi) = 2\left\{n + 3\left\{\sqrt{\frac{24}{5}}\right\}^k\right\} - 3\left\{\sqrt{\frac{24}{5}}\right\}^k\right\}.
\]

**Theorem 26.** For the subdivided backbone DNA network, SBBDNA\((n)\), the first and second Gourava indices are calculated by \(GO_1(\xi) = 146n - 66\) and \(GO_2(\xi) = 340n - 180\).

**Proof.** By letting \(\xi\) as a subdivided backbone DNA network SBBDNA\((n)\), from Table 3, we know
\[
GO_1(\xi) = \sum_{gh \in E(\xi)} \left\{d_g + d_h + d_gd_h\right\},
\]
\[
GO_2(\xi) = \sum_{gh \in E(\xi)} \left\{d_g + d_h + d_gd_h\right\},
\]
and by doing some calculations, we get
\[
GO_1(\xi) = \sum_{gh \in E(\xi)} \left\{d_g + d_h + d_gd_h\right\},
\]
\[
GO_2(\xi) = \sum_{gh \in E(\xi)} \left\{d_g + d_h + d_gd_h\right\}.
\]
SBBDNA(n), the general Randić index is calculated by

\[ R_{\gamma}(\xi) = \sum_{gh \in E(\xi)} \left( d_g \cdot d_h \right)^{\gamma}, \]  

(107)

for \( \gamma = \{-1, 1/2, 1/2\} \).

Case 1: if \( \gamma = -1 \), the application of Randić’ index \( R_{-1}(\xi) \)

\[ R_{-1}(\xi) = \sum_{gh \in E(\xi)} \frac{1}{d_g \cdot d_h}, \]  

(108)

using (107). From Table 3, we know \( R_{-1}(\xi) = 10n(4)^{-1} + (6n - 6)(6)^{-1} \). By doing some calculations, we get \( R_{-1}(\xi) = (7/2)n - 1 \).

Case 2: if \( \gamma = -(1/2) \), the application of Randić’ index \( R_{-1}(\xi) \)

\[ R_{-(1/2)}(\xi) = \sum_{gh \in E(\xi)} \left( \frac{1}{\sqrt{d_g \cdot d_h}} \right), \]  

(109)

and by doing some calculations, we get \( R_{-(1/2)}(\xi) = (n(1+1/2) - (49/20)). \)

Case 3: if \( \gamma = 1/2 \), the application of Randić’ index \( R_{1}(\xi) \)

\[ R_{1/2}(\xi) = \sum_{gh \in E(\xi)} \sqrt{d_g \cdot d_h}, \]  

(110)

and by doing some calculations, we get \( R_{1/2}(\xi) = n(1/2) - (147/10). \)

Case 4: if \( \gamma = 1 \), the application of Randić’ index \( R_{1}(\xi) \)

\[ R_{1}(\xi) = \sum_{gh \in E(\xi)} \left( d_g \cdot d_h \right)^{1}, \]  

(111)

using (107),

\[ R_{1/2}(\xi) = 10n\sqrt{4} + (6n - 6)\sqrt{6}, \]  

(112)

and by doing some calculations, we get \( R_{1/2}(\xi) = n(347/10) - (147/10). \)

We know that

\[ H = \frac{37n - 12}{5}, \]  

(115)

Proof. By letting \( \xi \) as a subdivided backbone DNA network SBBDNA(n), from Table 3, we know
\[
H_I(\xi) = \sum_{g \in E(\xi)} \frac{2}{d_g + d_h}
\]

and by doing some calculations, we get
\[
\frac{H_I(\xi)}{10n} = 1 + (6n - 6) \left(\frac{d_g + d_h}{2}\right) \left(\frac{2}{5}\right) = \frac{37n - 12}{5}.
\]

**Theorem 30.** For the subdivided backbone DNA network, \(SBBDNA(n)\), the general version of the harmonic index is calculated by
\[
H_k I(\xi) = 2n \left(\frac{5}{2k^2} + \frac{3}{5k^2}\right) - \frac{6}{5^2} k^2.
\]

**Proof.** By letting \(\xi\) as a subdivided backbone DNA network \(SBBDNA(n)\), from Table 3, we know
\[
H_k I(\xi) = \sum_{g \in E(\xi)} \left(\frac{2}{d_g + d_h}\right)^k,
\]

and by some calculations, we get
\[
H_k I(\xi) = 10n \left(\frac{2}{2^2 + 2^2}\right)^k + (6n - 6) \left(\frac{2}{2 + 3}\right)^k = 2n \left(\frac{5}{2k^2} + \frac{3}{5k^2}\right) - \frac{6}{5^2} k^2.
\]

**Theorem 31.** For the subdivided backbone DNA network, \(SBBDNA(n)\), the atom-bond connectivity index is calculated by
\[
ABC(\xi) = \frac{2(8n - 3)}{\sqrt{2}}.
\]

**Proof.** By letting \(\xi\) as a subdivided backbone DNA network \(SBBDNA(n)\), from Table 3, we know
\[
ABC(\xi) = \sum_{g \in E(\xi)} \sqrt{\frac{d_g + d_h - 2}{d_g d_h}}.
\]

By doing some calculations, we get
\[
ABC(\xi) = 10n \sqrt{\frac{2^2 + 2^2}{4} + (6n - 6) \left(\frac{2 + 3}{6}\right)^2} = \frac{2(8n - 3)}{\sqrt{2}}.
\]

**Theorem 32.** For the subdivided backbone DNA network, \(SBBDNA(n)\), \(SK\), \(SK_1\), and \(SK_2\) indices are calculated by \(SK(\xi) = 35n - 15\), \(SK_1(\xi) = 38n - 18\), and \(SK_2(\xi) = (1/2) (155n - 75)\) respectively.

**Proof.** By letting \(\xi\) as a subdivided backbone DNA network \(SBBDNA(n)\), from Table 3, we know
\[
SK(\xi) = \sum_{g \in E(\xi)} \left[\frac{d_g + d_h}{2}\right],
\]

\[
SK_1(\xi) = \sum_{g \in E(\xi)} \left[\frac{d_g d_h}{2}\right],
\]

\[
SK_2(\xi) = \sum_{g \in E(\xi)} \left[\frac{d_g + d_h}{2}\right]^2.
\]

By doing some calculations, we get
\[
SK(\xi) = 10n (2 + (6n - 6) \left(\frac{5}{2}\right)) = 35n - 15,
\]

\[
SK_1(\xi) = 10n (2 + (6n - 6) (3)) = 38n - 18,
\]

\[
SK_2(\xi) = 10n (4 + (6n - 6) \left(\frac{25}{4}\right)) = \frac{1}{2} (155n - 75).
\]

**Theorem 33.** For the subdivided backbone DNA network, \(SBBDNA(n)\), the sum-connectivity index is calculated by
\[
\chi_{-(1/2)}(\xi) = n \left(\frac{5 + 6}{\sqrt{5}}\right) - \frac{6}{\sqrt{5}}.
\]

**Proof.** By letting \(\xi\) as a subdivided backbone DNA network \(SBBDNA(n)\), from Table 3, we know
\[
\chi_{-(1/2)}(\xi) = \sum_{g \in E(\xi)} \left[\frac{d_g + d_h}{2}\right]^{-1/2},
\]

\[
\chi_{-(1/2)}(\xi) = 10n \left(\frac{1}{2}\right) + (6n - 6) \left(\frac{1}{\sqrt{5}}\right).
\]

By doing some calculations, we get
\[
\chi_{-(1/2)}(\xi) = n \left(\frac{5 + 6}{\sqrt{5}}\right) - \frac{6}{\sqrt{5}}.
\]

**Theorem 34.** For the subdivided backbone DNA network, \(SBBDNA(n)\), the general sum-connectivity index is calculated by
\[
\chi_k(\xi) = n \left[10 \left\{4^k\right\} + 6 \left\{5^k\right\}\right] - \left\{5^k\right\}.
\]

**Proof.** By letting \(\xi\) as a subdivided backbone DNA network \(SBBDNA(n)\), from Table 3, we know
\[
\chi_k(\xi) = \sum_{g \in E(\xi)} \left[\frac{d_g + d_h}{2}\right]^k,
\]

\[
\chi_k(\xi) = 10n \left(4^k\right) + (6n - 6) \left(5^k\right).
\]

By doing some calculations, we get
\[
\chi_k(\xi) = n \left[10 \left\{4^k\right\} + 6 \left\{5^k\right\}\right] - \left\{5^k\right\}.
\]
2.4. Results for the Subdivided Honeycomb Network. The honeycomb network is a hexagon. It can be made in different methods. The first honeycomb network is symbolized by \(HC(1)\). The next honeycomb network is produced by attaching more hexagons to each of its edges. This newly formed honeycomb network is symbolized by \(HC(2)\); similarly, the next honeycomb network is produced by attaching more hexagons to each of its edges. In this way, the newly formed honeycomb network is denoted by \(HC(n)\). By repeating this process, we finally obtain a honeycomb network of \(n\) dimensions and denote by \(HC(n)\). The honeycomb network is being used in computer graphics, image processing, and cellular phone base stations; moreover, it is used in chemistry for the representation of benzenoid hydrocarbons. To get the subdivided honeycomb network shown in Figure 4, we insert a new node on each of its edges. The \(n\)-dimensional subdivided honeycomb network is symbolized by \(SHC_n\). A subdivided honeycomb network for \(n = 4\) is displayed in Figure 4. The number of nodes and edges in the subdivided honeycomb networks are \(15n^2 - 3n\) and \(18n^2 - 6n\), respectively. We have obtained two different types of edges in \(SHC_4\) shown in Table 4, whereas Figure 4 shows \(SHC_4\).

**Theorem 35.** For the subdivided backbone DNA network, \(SBBDNA(n)\), the first general Zagreb index is calculated by

\[
M_1(\xi) = n \left[ 10 \left\{ 2^k \right\} + 3 \left\{ 2^k \right\} + 2 \left\{ 3^k \right\} \right] - \left\{ 3 \left\{ 2^k \right\} + 2 \left\{ 3^k \right\} \right\}
\]

Proof. By letting \(G\) as a subdivided backbone DNA network, \(SBBDNA(n)\), from Table 3, we know

\[
M_1(\xi) = \sum_{gh \in E(\xi)} \left[ d_g^{k-1} + d_h^{k-1} \right], \quad k > 1,
\]

and by doing some calculations, we get

\[
M_1(\xi) = 10n \left( 2^k \right) + (6n - 6) \left( 2^{k-1} + 3^{k-1} \right),
\]

and by some calculations, we get

\[
M_1(\xi) = n \left[ 10 \left\{ 2^k \right\} + 3 \left\{ 2^k \right\} + 2 \left\{ 3^k \right\} \right] - \left\{ 3 \left\{ 2^k \right\} + 2 \left\{ 3^k \right\} \right\}
\]

**Theorem 36.** For the subdivided backbone DNA network, \(SBBDNA(n)\), the forgotten index is calculated by

\[
F(\xi) = 2[79n - 39].
\]

Proof. By letting \(G\) as a subdivided backbone DNA network, \(SBBDNA(n)\), from Table 3, we know

\[
F(\xi) = \sum_{gh \in E(\xi)} \left[ d_g^2 + d_h^2 \right],
\]

and by some calculations, we get

\[
F(\xi) = 2[79n - 39].
\]

| \(d_g, d_h\) for \(gh \in E(\xi)\) | Number of \(E(\xi)\) |
|---|---|
| (2, 2) | 12n |
| (2, 3) | 18n(n - 1) |

**Theorem 37.** For the subdivided honeycomb network, \(SHC_n\), the ordinary generalized geometric-arithmetic index is calculated by

\[
OGA_k(\xi) = 6 \left[ \frac{\sqrt{24}}{5} \right]^k + n \left[ 2 - 3 \left( \frac{\sqrt{24}}{5} \right) \right].
\]

Proof. By letting \(G\) as a subdivided honeycomb network, \(SHC_n\), from Table 4, we know

\[
OGA_k(\xi) = \sum_{gh \in E(\xi)} \left[ \frac{4d_gd_h}{d_g + d_h} \right]^k,
\]

and by some calculations, we get

\[
OGA_k(\xi) = 6 \left[ \frac{\sqrt{24}}{5} \right]^k + n \left[ 2 - 3 \left( \frac{\sqrt{24}}{5} \right) \right].
\]

**Theorem 38.** For the subdivided honeycomb network, \(SHC_n\), the first and second Gourava indices are calculated by

\[
GO_1(\xi) = 198n^2 - 102n \quad \text{and} \quad GO_2(\xi) = 540n^2 - 492n.
\]
Proof. By letting \( \xi \) as a subdivided honeycomb network \( \text{SHC}_n \), from Table 4, we know

\[
\begin{align*}
\text{GO}_1 (\xi) &= \sum_{gh \in E(\xi)} \left( (d_g + d_h) + (d_g d_h) \right), \\
\text{GO}_2 (\xi) &= \sum_{gh \in E(\xi)} \left( (d_g + d_h) + (d_g d_h) \right),
\end{align*}
\]

and by doing some calculations, we get

\[
\begin{align*}
\text{GO}_1 (\xi) &= (12n)[(4) + (4)] + 18n(n-1)[(5) + (6)] \\
&= 198n^2 - 102n, \\
\text{GO}_2 (\xi) &= (12n)[16] + 18n(n-1)[30] = 540n^2 - 492n.
\end{align*}
\]

Theorem 39. For the subdivided honeycomb network, \( \text{SHC}_n \), the first and second hyper-Gourava indices are calculated by \( \text{HGO}_1 (\xi) = 2178n^2 - 1410n \) and \( \text{HGO}_2 (\xi) = 16200n^2 - 13128n. \)

Proof. By letting \( \xi \) as the subdivided honeycomb network \( \text{SHC}_n \), from Table 4, we know

\[
\begin{align*}
\text{HGO}_1 (\xi) &= \sum_{gh \in E(\xi)} \left( (d_g + d_h) + (d_g d_h) \right)^2, \\
\text{HGO}_2 (\xi) &= \sum_{gh \in E(\xi)} \left( (d_g + d_h) + (d_g d_h) \right)^2,
\end{align*}
\]

and by doing some calculations, we get

\[
\begin{align*}
\text{HGO}_1 (\xi) &= (12n)[64] + 18n(n-1)[121] \\
&= 2178n^2 - 1410n, \\
\text{HGO}_2 (\xi) &= (12n)[256] + 18n(n-1)[900] \\
&= 16200n^2 - 13128n.
\end{align*}
\]

Theorem 40. For the subdivided honeycomb network, \( \text{SHC}_n \), the general Randic’ index is calculated by

\[
R_\gamma (\text{SHC}_n) = \begin{cases} 
3n^2, & \text{for } \gamma = -1, \\
3\left( n^2 \sqrt{6} + n\sqrt{2} - 3\sqrt{6} \right), & \text{for } \gamma = -\frac{1}{2}, \\
6\left( 3n^2 \sqrt{6} + n\left( 4 - 3\sqrt{6} \right) \right), & \text{for } \gamma = \frac{1}{2}, \\
108n^2 - 60n, & \text{for } \gamma = 1.
\end{cases}
\]

Proof. By letting \( \xi \) as a subdivided honeycomb network \( \text{SHC}_n \) of \( n \) dimensions, we have the order and size of \( \xi \) in \( \text{SHC}_n \) as \( |V(\text{SHC}_n)| = 15n^2 - 3n \) and \( |E(\text{SHC}_n)| = 18n^2 - 6n \), respectively. We know that

\[
R_\gamma (\xi) = \sum_{gh \in E(\xi)} (d_g \cdot d_h)^\gamma, \quad (146)
\]

for \( \gamma = \{-1, -1/2, 1/2\} \).

Case 1: if \( \gamma = -1 \), the application of Randic’ index \( R_\gamma (\xi) \)

\[
R_{-1} (\xi) = \sum_{gh \in E(\xi)} \frac{1}{d_g d_h}, \quad (147)
\]

using (146). From Table 4, we obtain \( R_{-1} (\xi) = 12n(4)^{-1} + 18n(n-1)(6)^{-1}. \)

By doing some calculations, we obtain \( R_{-1} (\xi) = 3n^2. \)

Case 2: if \( \gamma = -(1/2) \), the application of Randic’ index \( R_\gamma (\xi) \)

\[
R_{-(1/2)} (\xi) = \sum_{gh \in E(\xi)} \left( \frac{1}{d_g d_h} \right)^{1/2}, \quad (148)
\]

using (146). From Table 4, we obtain \( R_{-(1/2)} (\xi) = 12n(1/\sqrt{4}) + 18n(n-1)(1/\sqrt{6}). \)

By doing some calculations, we obtain \( R_{-(1/2)} (\xi) = 3n^2 \sqrt{6} + n\left( 2 - \sqrt{6} \right) \).

Case 3: if \( \gamma = 1/2 \), the application of Randic index \( R_\gamma (\xi) \)

\[
R_{1/2} (\xi) = \sum_{gh \in E(\xi)} \sqrt{d_g d_h}, \quad (149)
\]

using (146). From Table 4, we obtain \( R_{1/2} (\xi) = 12n\sqrt{4} + 18n(n-1)\sqrt{6}. \)

By doing some calculations, we obtain \( R_{1/2} (\xi) = 6(3n^2 \sqrt{6} + n\left( 4 - 3\sqrt{6} \right) \).

Case 4: if \( \gamma = 1 \), the application of Randic’ index \( R_\gamma (\xi) \)

\[
R_1 (\xi) = \sum_{gh \in E(\xi)} (d_g \cdot d_h), \quad (150)
\]

using (146). From Table 4, we obtain \( R_1 (\xi) = 12n(4) + 18n(n-1)(6). \)

By doing some calculations, we obtain \( R_1 (\xi) = 108n^2 - 60n. \)

Theorem 41. For the subdivided honeycomb network, \( \text{SHC}_n \), the harmonic index is calculated by

\[
\text{HI} (\xi) = \frac{36n^2 - 6n}{5}, \quad (151)
\]

Proof. By letting \( \xi \) as a subdivided honeycomb network \( \text{SHC}_n \), from Table 4, we know

\[
\text{HI} (\xi) = \sum_{gh \in E(\xi)} \frac{2}{d_g + d_h}, \quad (152)
\]

and by doing some calculations, we get
and by doing some calculations, we get
\begin{equation}
HI(\xi) = (12n)\left[\frac{1}{2}\right] + 18n(n-1)\left[\frac{2}{5}\right] = \frac{36n^2 - 6n}{5},
\end{equation}
(153)

**Theorem 42.** For the subdivided honeycomb network, SHC\(_n\), the general version of the harmonic index is calculated by
\begin{equation}
H_kI(\xi) = 6\left[3n^2\left[\frac{2}{5}\right]^k + n\left[\frac{2}{2-2} + 3\left[\frac{2}{5}\right]^k\right]\right]
\end{equation}
(154)

**Proof.** By letting \(\xi\) as a subdivided honeycomb network SHC\(_n\), from Table 4, we know
\begin{equation}
H_kI(\xi) = \sum_{gh\in E(\xi)}\left[\frac{2}{d_g + d_h}\right]^k,
\end{equation}
(155)
and by doing some calculations, we get
\begin{equation}
H_kI(\xi) = (12n)\left[\frac{2}{2+2}\right]^k + 18n(n-1)\left[\frac{2}{2+3}\right]^k,
\end{equation}
(156)

\begin{equation}
H_kI(\xi) = 6\left[3n^2\left[\frac{2}{5}\right]^k + n\left[\frac{2}{2-2} + 3\left[\frac{2}{5}\right]^k\right]\right].
\end{equation}

**Theorem 43.** For the subdivided honeycomb network, SHC\(_n\), the atom-bond connectivity index is calculated by
\begin{equation}
ABC(\xi) = n\sqrt{2}|9n - 3|.
\end{equation}
(157)

**Proof.** By letting \(\xi\) as a subdivided honeycomb network SHC\(_n\), from Table 4, we know
\begin{equation}
ABC(\xi) = \sum_{gh\in E(\xi)}\sqrt{d_g + d_h - 2}/d_gd_h,
\end{equation}
(158)
and by doing some calculations, we get
\begin{equation}
ABC(\xi) = 12n\sqrt{\frac{2+2-2}{4} + 18n(n-1)\sqrt{\frac{2+3-2}{6}}},
\end{equation}
(159)
\begin{equation}
= n\sqrt{2}|9n - 3|.
\end{equation}

**Theorem 44.** For the subdivided honeycomb network, SHC\(_n\), SK, SK\(_1\), and SK\(_2\) indices are calculated by \(SK(\xi) = 45n^2 - 21n\), \(SK_1(\xi) = 54n^2 - 30n\), and \(SK_2(\xi) = (1/2)(225n^2 - 129n)\), respectively.

**Proof.** By letting \(\xi\) as a subdivided honeycomb network SHC\(_n\), from Table 4, we know
\begin{equation}
SK(\xi) = \sum_{gh\in E(\xi)}\left[\frac{d_g + d_h}{2}\right],
\end{equation}
\begin{equation}
SK_1(\xi) = \sum_{gh\in E(\xi)}\left[\frac{d_g d_h}{2}\right],
\end{equation}
(160)
\begin{equation}
SK_2(\xi) = \sum_{gh\in E(\xi)}\left[\frac{d_g + d_h}{2}\right]^2,
\end{equation}
(160)
and by doing some calculations, we get
\(SK(\xi) = (12n)(2) + 18n(n-1)(\frac{5}{2}) = 45n^2 - 21n\),
\(SK_1(\xi) = (12n)(2) + 18n(n-1)(3) = 54n^2 - 30n\),
\(SK_2(\xi) = (12n)(4) + 18n(n-1)(\frac{25}{4}) = \frac{1}{2}(225n^2 - 129n)\).
(161)

**Theorem 45.** For the subdivided honeycomb network, SHC\(_n\), the sum-connectivity index is calculated by
\begin{equation}
X_{(1/2)}(\xi) = \frac{1}{\sqrt{5}}\left[18n^2 + n[6\sqrt{5} - 18]\right].
\end{equation}
(162)

**Proof.** By letting \(\xi\) as a subdivided honeycomb network SHC\(_n\), from Table 4, we know
\begin{equation}
X_{(1/2)}(\xi) = \sum_{gh\in E(\xi)}\left[d_g + d_h\right]^{-1/2},
\end{equation}
(163)
\begin{equation}
X_{(1/2)}(\xi) = (12n)\left[\frac{1}{2}\right] + 18n(n-1)\left[\frac{1}{\sqrt{5}}\right],
\end{equation}
and by doing some calculations, we get
\begin{equation}
X_{(1/2)}(\xi) = \frac{1}{\sqrt{5}}\left[18n^2 + n[6\sqrt{5} - 18]\right].
\end{equation}
(164)

**Theorem 46.** For the subdivided honeycomb network, SHC\(_n\), the general sum-connectivity index is calculated by
\begin{equation}
\chi_k(\xi) = 18\left[5^k\right]n^2 + 6n\left[2\left[4^k\right] - 3\left[5^k\right]\right].
\end{equation}
(165)

**Proof.** By letting \(\xi\) as a subdivided honeycomb network SHC\(_n\), from Table 4, we know
\begin{equation}
\chi_k(\xi) = \sum_{gh\in E(\xi)}\left[d_g + d_h\right]^k,
\end{equation}
(166)
\begin{equation}
\chi_k(\xi) = (12n)\left(4^k\right) + 18n(n-1)(5^k),
\end{equation}
and by doing some calculations, we get
\[ \chi_k(\xi) = 18\{5^k\}n^2 + 6n\{2\{4^k\} - 3\{5^k\}\}. \] (167)

**Theorem 47.** For the subdivided honeycomb network, \( SHC_m \), the first general Zagreb index is calculated by
\[ kM_1(\xi) = 18n^2\{2k^{-1} + 3k^{-1}\} + 6n\{2\{2^k\} - 3n\{2k^{-1} + 3k^{-1}\}\}. \] (168)

**Proof.** By letting \( \xi \) as a subdivided honeycomb network \( SHC_m \), from Table 4, we know
\[ kM_1(\xi) = \sum_{gh\in E(\xi)} [d_g^{k-1} + d_h^{k-1}], \quad k > 1, \] (169)
and by doing some calculations, we get
\[ kM_1(\xi) = 18n^2\{2k^{-1} + 3k^{-1}\} + 6n\{2\{2^k\} - 3n\{2k^{-1} + 3k^{-1}\}\}. \] (170)

**Theorem 48.** For the subdivided honeycomb network, \( SHC_m \), the forgotten index is calculated by
\[ F(\xi) = 2n[117n - 69]. \] (171)

**Proof.** By letting \( \xi \) as a subdivided honeycomb network \( SHC_m \), from Table 4, we know
\[ F(\xi) = \sum_{gh\in E(\xi)} [d_g^{2} + d_h^{2}], \] (172)
and by doing some calculations, we get \( F(\xi) = 2n[117n - 69] \).

### 3. Conclusions

In this paper, we have computed the topological indices (degree based) such as ordinary generalized geometric-arithmetic (OGA) index, first and second Gourava indices, first and second hyper-Gourava indices, general Randić' index \( R_y(\xi) \), for \( y = \{ \pm 1, \pm (1/2) \} \), harmonic index, general version of the harmonic index, atom-bond connectivity (ABC) index, SK, SK1, and SK2 indices, sum-connectivity index, general sum-connectivity index, and first general Zagreb and forgotten topological indices for different kinds of chemical networks such as the subdivided polythiophene network, subdivided hexagonal network, subdivided backbone DNA network, and subdivided honeycomb network. The above computed topological indices are used as molecular descriptors in the construction of "quantitative structure-activity relationships and quantitative structure-property relationships." These indices give us results that can be correlated with the molecular structures to understand their chemical and physical properties.

For the next research papers, our goal is to compute more topological indices for some new graphs to know their topologies.

### Data Availability

No data were used to support this study.

### Disclosure

This paper has not been published elsewhere, and it will not be submitted anywhere else for publication.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

### Authors’ Contributions

All authors contributed equally to this work.

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