Theta Functions and Adiabatic Curvature on a Torus

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Abstract

Let $M$ be a complex torus, $L_{\hat{\mu}} \to M$ be positive line bundles parametrized by $\hat{\mu} \in \text{Pic}^0(M)$, and $E \to \text{Pic}^0(M)$ be a vector bundle with $E|_{\hat{\mu}} \cong H^0(M, L_{\hat{\mu}})$. We endow the total family $\{L_{\hat{\mu}}\}_{\hat{\mu}}$ with a Hermitian metric that induces the $L^2$-metric on $H^0(M, L_{\hat{\mu}})$ hence on $E$. By using theta functions $\{\theta_m\}_m$ on $M \times M$ as a family of functions on the first factor $M$ with parameters in the second factor $M$, our computation of the full curvature tensor $\Theta_E$ of $E$ with respect to this $L^2$-metric shows that $\Theta_E$ is essentially an identity matrix multiplied by a constant 2-form, which yields in particular the adiabatic curvature $c_1(E)$. After a natural base change $M \to \hat{M}$ so that $E \times \hat{M} M \coloneqq E'$, we also obtain that $E'$ splits holomorphically into a direct sum of line bundles each of which is isomorphic to $L_{\hat{\mu}=0}^*$. Physically, the spaces $H^0(M, L_{\hat{\mu}})$ correspond to the lowest eigenvalue with respect to certain family of Hamiltonian operators on $M$ parametrized by $\hat{\mu}$ or in physical notation, by wave vectors $k$.

Keywords: Theta functions, Complex torus, Picard variety, Poincaré line bundle, Connection, Curvature, Characters, Holonomy, Fourier-Mukai transform.

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1. Introduction

Let $M$ be a complex torus. To consider the set of all positive line bundles $L \to M$ with the same first Chern classes, one may first pick any positive line bundle $L_0 \to M$ with the required $c_1(L_0) = [\omega]$ for some closed $(1, 1)$ form $\omega$ which is integral, positive and of constant coefficients. Write $\delta$ for the degree of $L_0$. For any holomorphic automorphism $T : M \to M$, $c_1(T^*L_0) = [\omega]$ and it is well-known that all line bundles on $M$ with the same $c_1$ can arise in this way. In fact, it is known that $T$ is a translation $T_\mu : M \to M$ on $M$ for some fixed $\mu \in M$. We denote $T^*L_0$ by $L_\mu$.

This can be placed in another context by means of Poincaré line bundle $P : M \times \hat{M}$ where $\hat{M} = \text{Pic}^0(M)$. Let $\pi_1$, $\pi_2$ be the two projections of $M \times \hat{M}$ to $M$, $\hat{M}$ respectively. Write $\bar{E} = \pi_1^*L_0 \otimes P$. Thinking of $\bar{E}|_{M \times \{\hat{\mu}\}}$ on $M \times \{\hat{\mu}\}$ as a family of line bundles $L_{\hat{\mu}}$ on $M \cong M \times \{\hat{\mu}\}$, one has the associated family of vector spaces $H^0(M, L_{\hat{\mu}})$ varying with $\hat{\mu}$. 

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It forms a holomorphic vector bundle $E$ on $\hat{M}$. Similarly, we have a holomorphic vector bundle $E'$ on $M$ with $E'_{|\mu} = H^0(M, L_{\mu})$. This type of construction is closely related to the Fourier-Mukai transform. See [13]. There is a map $\varphi_{L_0} : M \to \hat{M}$ sending $\mu \in M$ to $T_{\mu}^* L_0 \otimes L_0^* \in \hat{M}$. For precise notations and details, we refer to later appropriate sections.

A natural question of interest in this paper is to ask for the full curvature of $E$. We have:

Theorem 1.1. (= Theorem 8.5.) In the notations as above, there exists a Hermitian metric $h_{\hat{E}}$ on $\hat{E}$ such that the induced $L^2$-metric on $E$, denoted by $h_E$, has the curvature

$$\Theta(E, h_E) = (2\pi i)\omega(\text{Id})_{\delta \times \delta}$$

where $(\text{Id})_{\delta \times \delta}$ denotes the $\delta \times \delta$ identity matrix. Therefore $c_1(E, h_E) = -\delta \omega$ (at the level of differential forms).

Our study into this question was influenced by a related work of C. T. Prieto [13] where he studied similar questions on compact Riemann surfaces but restricted to $c_1$. Among other things, he placed his computations in the framework of local family index theorems, and derived the $c_1$ from the theorem of Bismut-Gillet-Souleé [6] in this regard. To invoke these theorems, the Quillen metric need be introduced as an extra ingredient. By contrast, we use theta functions for explicit computations and achieve the full curvature $\Theta$ of $E$.

In fact, the above $\Theta$ is obtained via the following result of independent interest, which appears to be of algebraic geometry in nature.

Theorem 1.2. (See (8.13).) We have $\varphi_{L_0}^* E = E'$ on $M$. Moreover, $E'$ splits holomorphically into a direct sum of holomorphic line bundles each of which is isomorphic to $L_0^*$, the dual of $L_0$.

There are rich connections between these problems and physics, for which we refer mathematically minded readers to the nice presentation by Prieto in [13, Introduction], including the term "adiabatic curvature". For physical interest, it is desirable to compute the adiabatic curvature of spectral bundles (cf. [1]), where our space of holomorphic sections corresponds to the lowest eigenvalue under suitable interpretation. Some interesting results in this direction (for higher eigenvalues) have been obtained by Prieto in [12] and [13]. Put in this perspective, our present work is far from being complete. Another immediate question is to ask for the higher dimensional generalization of Theorem 1.1 say, on an Abelian variety. Further, our present approach is transcendental in nature, and from the purely algebraic point of view, it is not altogether clear how Theorem 1.2 can be proved in an algebraic manner. A third question of interest appears to be a study into all of these problems under deformation of complex structures on $M$. We hope to come back to (some of) these questions in future publications.

We remark that the theoretical and experimental aspects of the role played by the first Chern class $c_1$ have long been noticed by physicists under study of, among others, "geometric phases in quantum systems" in general and the quantum Hall effect in particular (cf. [7], [10], [14]). In these settings the adiabatic curvature usually refers to the $c_1$ (or $\frac{2\pi}{\hbar} c_1$) of spectral bundles associated with certain Hamiltonian operators depending on parameters such as wave vectors (cf. [7], (13.26) in p. 314]). While the theoretical/abstract
formula for the (full) curvature is already available, some physical approaches to the actual computation are carried out using, for instance, "magnetic translation operators" (cf. \cite{2} and references therein) and even noncommutative geometry methods (cf. \cite{4}). To the best of our understanding, these studies and explicit results focus only on $c_1$ rather than the full curvature tensor as done here.

The full curvature in related contexts has been of interest in the mathematical literature. Indeed, it appears in disguise of the Chern character of the index bundle (see \cite{5}) and more recently, it also plays an important role in the work of B. Berndtsson for vector bundles associated to holomorphic fibrations (see \cite{3}).

To outline our approach, some difficulties are in order. It is natural to consider metrics $h_{\hat{\mu}}$ on $L_{\hat{\mu}}$ for $\mu \in \hat{M}$ which are of constant curvature $\frac{2\pi i}{\omega}$. As this curvature condition determines $h_{\hat{\mu}}$ only up to multiplicative constants, one is required not only to make a choice but also, more importantly, to do it in a consistent manner with respect to $\hat{\mu}$ globally. By this, among others, we are led to the Poincaré line bundle $P \rightarrow M \times \hat{M}$. But we found it much less illuminating if we fell into the description of $P$ in terms of complex algebraic geometry as usually given in the literature. Fortunately, the needed differential geometric aspects on the Poincaré line bundle $P$ have been developed in part by \cite{8} from the gauge theory perspective (cf. Section 6). This is precisely what we resort to here, and by proving an identification theorem, we can endow $P$ with certain metric geometry data (cf. Section 7).

Next, from the physical point of view it is natural to use the $L^2$-metric of the system for the curvature computation. For this purpose, the explicit theta functions as global sections are expected to deserve a try. However, as far as the Theorem 1.2 is concerned, our difficulty lies in that the choice of these functions \textit{a priori} depends on $\hat{\mu}$ although the curvature computation only makes use of a local basis of theta functions valid around $\hat{\mu}$, for $L_{\hat{\mu}} \rightarrow M \times \{\hat{\mu}\}$. We are therefore led to exploit a \textit{global} property of these ($\hat{\mu}$-dependent) theta functions (cf. Section 2 and Section 3). For the formulation it turns out to get most simplified if we shift the viewpoint about parameters from $\hat{\mu} \in \hat{M}$ to $\mu \in M$ via the map $\varphi_{L_0} : M \rightarrow \hat{M}$ as given precedingly (cf. Section 4). We thus form the theta functions on $M \times M$ as a family of functions defined on the first $M$ as well as parametrized by the second $M$ (cf. Section 5). In this way we can eventually accomplish a holomorphic splitting of the vector bundle in the sense of Theorem 1.2.

In retrospect, it remains somewhat unexpected why the $L^2$-metric property of these global theta functions so formed, behave nicely to suit our (computational) need. Indeed, it is only after the explicit computation that we find this neat fact. See the main technical Lemma 5.2 for details. Nevertheless, we are prompted to perceive Theorem 1.2 as a conceptual picture in support of the computational result Theorem 1.1 (cf. Remark 5.4 and ii) of Remark 8.6).

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2. Holomorphic line bundles over the compact Riemann surface \( M = V/\Lambda \)

The principal aim of this section is to collect the background materials and to fix the notations for later use. Basic references are, for instance, [9] and [11]. Let \( V \) be a complex vector space of dimension 1 and \( \Lambda = \mathbb{Z} \{ \lambda_1, \lambda_2 \} \subseteq V \) be a discrete lattice where \( \text{Im} \frac{\lambda_2}{\lambda_1} > 0 \).

The compact Riemann surface \( M = V/\Lambda \) is a complex torus. Let \( L \) be a holomorphic line bundle over \( M \). The first Chern class \( c_1(L) \) of \( L \) is a complete invariant of \( L \) as a \( C^\infty \) line bundle. The Picard group \( \text{Pic}(M) \) are the isomorphic classes of holomorphic line bundles over \( M \). The connected component \( \text{Pic}^0(M) \) of \( \text{Pic}(M) \) represents all the equivalent classes of degree 0 holomorphic line bundles over \( M \).

We let \( \{dx_1, dx_2\} \) be the 1-forms on \( V \) dual to \( \{\lambda_1, \lambda_2\} \), that is, \( \int_\lambda dx_j = \delta_{ij} \). In terms of this basis, any positive holomorphic line bundle \( L = \omega \) over \( M \) has a Hodge form \( \omega = \delta \, dx_1 \wedge dx_2 \) on \( M \) satisfying \( c_1(L) = [\omega] \), \( \delta \in \mathbb{N} \).

To fix the complex coordinates, choose a \( \delta \in \mathbb{N} \) and let \( e_1 = \lambda_1/\delta \), \( \tau = \tau_1 + i\tau_2 = \frac{\lambda_2}{\lambda_1} \). We write \( \lambda_1 = \delta e_1, \lambda_2 = \tau e_1 \) with \( \tau_2 > 0 \). Let \( z = z_1 + iz_2 \) with \( z_1, z_2 \in \mathbb{R} \), be the complex coordinate on \( V \) (and on \( M \)) such that \( dz \) is dual to \( e_1 \).

We denote \( ze_1 \in V \) by \( z \) whenever there is no danger of confusion. One has

\[
\begin{align*}
\text{d}z &= \delta \, dx_1 + \tau \, dx_2, \\
\text{d}z &= \delta \, dx_1 + \tau \, dx_2. \\
\end{align*}
\]

We define \( L_0 \) to be the holomorphic line bundle over \( M \) given by multipliers

\[
\begin{align*}
e_{\lambda_1}(z) &\equiv 1 \\
e_{\lambda_2}(z) &= e^{-2\pi iz - \pi i\tau}. \\
\end{align*}
\]

Notice that any system of multipliers \( \{e_{\lambda} \in \mathcal{O}^*(V)\}_{\lambda \in \Lambda} \) for a holomorphic line bundle \( L \) on \( M = V/\Lambda \) has to satisfy the compatibility relations:

\[
e_{\lambda'}(z + \lambda) e_{\lambda}(z) = e_{\lambda}(z + \lambda') e_{\lambda'}(z) = e_{\lambda + \lambda'}(z), \quad \forall \lambda, \lambda' \in \Lambda.
\]

It is known that \( c_1(L_0) = [\omega] \), \( \omega = \delta \, dx_1 \wedge dx_2 \).

This description helps to give an explicit basis of global sections. More precisely, write \( \pi : V \to M = V/\Lambda \) for the projection. There is a trivialization \( \phi : \pi^*L_0 \to V \times \mathbb{C} \) of \( \pi^*L_0 \) such that for any global holomorphic section \( \theta \) of \( L_0 \to M \), the function \( \theta := (\phi^{-1})^*(\pi^*\theta) \) is a quasi-periodic entire function on \( V \) satisfying

\[
\begin{align*}
\theta(z + \lambda_1) &= \theta(z) \\
\theta(z + \lambda_2) &= e^{-2\pi iz - \pi i\tau} \theta(z), \quad \forall z \in V.
\end{align*}
\]

By the same token, a Hermitian metric \( h_{L_0}(z) > 0 \) on \( L_0 \) where

\[
||\tilde{\theta}(\pi(z))||_{h_{L_0}}^2 := h_{L_0}(z) |\theta(z)|^2,
\]
is also characterized by the quasi-periodic property:
\[
\begin{align*}
    h_{L_0}(z + \lambda_1) &= h_{L_0}(z), \\
    h_{L_0}(z + \lambda_2) &= e^{-4\pi z_2 - 2\pi \tau_2} h_{L_0}(z), \quad \forall z \in V.
\end{align*}
\tag{2.5}
\]

**Lemma 2.1.** For the holomorphic line bundle \( L_0 \to M \), one can use the quasi-periodic entire functions on \( V \)
\[
\theta_m(z) := \sum_{k \in \mathbb{Z}} e^{\pi ik^2 \tau} e^{2\pi i m k} e^{2\pi i (\frac{k^2 + m^2}{2} \tau)} e^{2\pi i m \delta z}, \quad m = 0, 1, \ldots, \delta - 1,
\tag{2.6}
\]
as a basis of global holomorphic sections of \( L_0 \), and
\[
h_{L_0}(z) := e^{-2\tau (z)^2}
\tag{2.7}
\]
as a metric on \( L_0 \).

**Proof.** For the special case \( \delta = 1, m = 0 \)
\[
\theta_0(z) = \sum_{k \in \mathbb{Z}} e^{\pi ik^2 \tau} e^{2\pi iz} = \vartheta(z, \tau) \quad \forall z \in V
\tag{2.8}
\]
is the Riemann theta function. For general \( \delta \in \mathbb{N}, m = 0, \ldots, \delta - 1, \)
\[
\theta_m(z) = e^{2\pi i m \tau} \vartheta(z + \frac{m}{\delta} \tau, \tau) \quad \forall z \in V
\tag{2.9}
\]
is a translate of \( \vartheta(z, \tau) \) multiplied by the exponential factor \( e^{2\pi i m \tau} \). The lemma follows easily from (2.5), (2.6) and the quasi-periodic property of the Riemann theta function.

For any \( \mu e_1 \in V, \mu = \mu_1 + i \mu_2 \), we have a map
\[
\mathcal{T}_\mu : M \mapsto M
\]
defined by the translation by \([\mu] \in M\). Let \( L_\mu := \mathcal{T}_\mu^* L_0 \to M \). Then \( L_\mu \) can be given by multipliers
\[
\begin{align*}
    e_{\lambda_1}(z) &\equiv 1 \\
    e_{\lambda_2}(z) &= e^{-2\pi iz - 2\pi i \mu - \pi i \tau}.
\end{align*}
\tag{2.10}
\]
In the same vein as before, any global holomorphic sections \( \tilde{\theta} \) of \( L_\mu \to M \) can be described via quasi-periodic entire functions \( \theta \) on \( V \) satisfying
\[
\begin{align*}
    \theta(z + \lambda_1) &= \theta(z) \\
    \theta(z + \lambda_2) &= e^{-2\pi i (z + \mu) - \pi i \tau} \theta(z), \quad \forall z \in V.
\end{align*}
\tag{2.11}
\]
and the metric $h_{L\mu}(z)$ on $L_\mu \to M$:

$$
\begin{align*}
    h_{L\mu}(z + \lambda_1) &= h_{L\mu}(z) \\
    h_{L\mu}(z + \lambda_2) &= e^{-4\pi(z_2 + \mu_2) - 2\pi \tau_2} h_{L\mu}(z), \quad \forall z \in V.
\end{align*} \tag{2.12}
$$

It is well known that all the holomorphic line bundles on $M$ having the same first Chern class as $L_0$ can be represented as a translate of $L_0$. As a consequence, by Lemma 2.11 and (2.12), one has:

**Lemma 2.2.** Fix a $\mu \in V$. For the holomorphic line bundle $L_\mu \to M$ as defined above, one can use the quasi-periodic entire functions on $V$:

$$
\theta_m(z, \mu) = \sum_{k \in \mathbb{Z}} e^{\pi i k^2 \tau} e^{2\pi i v_m k} e^{2\pi i (k\delta + \mu)} \delta(z + \mu), \quad m = 0, 1, ..., \delta - 1 \tag{2.13}
$$

as a basis of global holomorphic sections of $L_\mu$, and

$$
h_{L\mu}(z) = h_L(z + \mu) = e^{-2\pi \tau_2 (z_2 + \mu_2)^2} \tag{2.14}
$$
as a metric on $L_\mu$.

3. The dual torus $\hat{M} = \text{Pic}^0(M)$ of $M$

The notational convention here follows that of [9, p. 307-317] unless specified otherwise. We have a natural identification for the set $\text{Pic}^0(M)$:

$$
\text{Pic}^0(M) \cong \frac{H^1(M, \mathcal{O})}{H^1(M, \mathbb{Z})} \cong \frac{H^{0,1}_\sigma(M)}{H^1(M, \mathbb{Z})} \tag{3.1}
$$

via the long exact cohomology sequence associated with the exponential sheaf sequence for the first isomorphism, and the Dolbeault isomorphism for the second, where the map $H^1(M, \mathbb{Z}) \to H^{0,1}_\sigma(M)$ is given by

$$
\omega \mapsto \omega^{0,1}.
$$

The image of $H^1(M, \mathbb{Z})$ in $\overline{V}^* = H^{0,1}_\sigma(M)$ is the lattice $\overline{\Lambda}^* = \mathbb{Z} \{ dx_1^*, dx_2^* \}$ which consists exactly of conjugate linear functionals on $V$ whose real part is half-integral on $\Lambda \subseteq V$. See below. $\text{Pic}^0(M) = \overline{V}^*/\overline{\Lambda}^*$ is often called the dual torus of $M$, and denoted as $\hat{M}$.

To be precise, we write the conjugate linear part of $dx_1$, $dx_2$ as

$$
\begin{align*}
    dx_1^* &= \Pi_{11} d\overline{\tau} = \frac{1}{2\tau_2} (1 - i \frac{\tau_1}{\tau_2}) d\overline{\tau} \\
    dx_2^* &= \Pi_{21} d\overline{\tau} = \frac{i}{2\tau_2} d\overline{\tau}
\end{align*} \tag{3.2}
$$

from (cf. (2.1))
\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{dx_1}{\Pi_{11}} + \frac{1}{2\pi i} (1 + i\frac{\mu}{\tau_2}) \, dz + \frac{1}{2\pi i} (1 - i\frac{\mu}{\tau_2}) \, d\bar{z}
\frac{dx_2}{\Pi_{21}} + \frac{1}{2\tau_2} \, dz + \frac{i}{2\tau_2} \, d\bar{z}.
\end{array} \right.
\end{align*}
\]

(3.3)

Re-ordering \{dx_1^*, dx_2^*\} we set
\[
\begin{align*}
dy_1^* &= -dx_2^*, \\
dy_2^* &= dx_1^* = \frac{\tau}{\delta} \, dy_1^*.
\end{align*}
\]

Setting \(e_1^* := \frac{dy_1^*}{\delta}\), we have the lattice
\[
\Lambda^* = \mathbb{Z}\{dy_1^*, dy_2^*\} = \mathbb{Z}\{e_1^*, \frac{\tau}{\delta} e_1^*\}.
\]

One has the map \(\varphi_{L_0}: M \to \text{Pic}^0(M)\) defined, via the translation \(T\mu: M \to M\) with \([\mu] \in M\), by
\[
\varphi_{L_0}([\mu]) = T^*\mu L_0 \otimes L_0^*, \quad \forall \mu \in V,
\]
and the natural lifting map \(\widetilde{\varphi_{L_0}}: V \to \overline{V^*}\) of \(\varphi_{L_0}\). In general, \(\varphi_{L_0}\) is not an isomorphism unless \(\delta = 1\).

The following property is well-known:

**Property 1.** \(\widetilde{\varphi_{L_0}}: V \to \overline{V^*}\) is a complex linear transformation such that
\[
\widetilde{\varphi_{L_0}}(e_1) = e_1^*. \tag{3.4}
\]

**Proof.** Let us go back to the map
\[
H^{0,1}_\delta(M) \xrightarrow{\delta} H^1(M, \mathcal{O}) \xrightarrow{p} \frac{H^1(M, \mathcal{O})}{H^1(M, \mathbb{Z})} \cong \text{Pic}^0(M) \tag{3.5}
\]
where \(\delta\) is the Dolbeault isomorphism and \(p\) is the projection. For any \(\alpha = \sigma \, d\bar{z} \in H^{0,1}_\delta(M)\), \(p \circ \delta\) sends \(\alpha\) to the line bundle given by the multipliers
\[
\begin{align*}
e^{2\pi i \sigma \delta} && e^{2\pi i \sigma \tau_2}.
\end{align*}
\]

Note that this choice of line bundles is dual to the one given in [9, p. 315-316].

Multiplying the trivializations by the function \(f(z) = e^{-2\pi i \sigma z}\) yields the normalized multipliers
\[
\begin{align*}
e^{2\pi i \sigma \delta} && e^{2\pi i \sigma \tau_2}.
\end{align*}
\]

On the other hand, the multipliers of \(T^*\mu L_0 \otimes L_0^*\) are, via (2.2) and (2.10),
\[
\begin{align*}
e^{2\pi i \mu} \equiv 1 \\
e^{4\pi i \sigma \tau_2}.
\end{align*}
\]

(3.7)
Plugging $\mu = 1 (\mu e_1 = e_1)$ into (3.8) and setting $\alpha = e_1^* = \frac{-i}{2\tau_2} d\tau = \sigma d\tau$ in (3.7), one obtains

\[ (3.7) = (3.8), \]

hence (3.4). We omit the proof that $\tilde{\varphi}_{L_0}$ is complex linear.

We should also recall the **Poincaré line bundle**. Let $\hat{\mu} = \hat{\mu}_1 + i \hat{\mu}_2$ be the complex coordinate on $\nabla^*$ (and on $\hat{M}$) such that $d\hat{\mu}$ is dual to $e_1^*$. As previously, an element $\hat{\mu} e_1^* \in \nabla^*$ is interchangeably written as $\hat{\mu} \in \nabla^*$. We denote the line bundle corresponding to $[\hat{\mu}] \in \hat{M} = \text{Pic}^0(M)$ in (3.1) by $P_{[\hat{\mu}]}$ or $P_{\hat{\mu}}$ if there is no danger of confusion. By Property 1 above, we can also write

\[ P_{\hat{\mu}} = P_{\varphi_{L_0}([\mu])} \cong T_{\mu}^* L_0 \otimes L_0^*, \quad \forall \mu \in V, \tag{3.9} \]

where $\mu$ and $\hat{\mu}$ are related by $\tilde{\varphi}_{L_0} (\mu e_1) = \hat{\mu} e_1^*$. The following lemma is standard.

**Lemma 3.1.** There is a unique holomorphic line bundle $P \to M \times \hat{M}$ called the Poincaré line bundle satisfying:

1. $P|_{M \times (\hat{\mu})} \cong P_{\mu}$,
2. $P|_{\{\mu\} \times \hat{M}}$ is a holomorphically trivial line bundle.

4. A holomorphic line bundle $\tilde{K} \to M_1 \times M_2 \cong M \times M$

As explained in the second half of Introduction, we would like to "accomodate" the $\mu$-dependent theta functions $\theta_{m}(z, \mu)$ of previous sections. For this need, we introduce an intermediate line bundle $\tilde{K}$ in this section. Let $M_1 \cong M_2 \cong M$ and $\pi: V \times V \to M_1 \times M_2 = V/\Lambda \times V/\Lambda$ with projections $\pi_i: M_1 \times M_2 \to M_i$, $i = 1, 2$. We denote by $\lambda_{10} = \lambda_1$ and $\lambda_{20} = \lambda_2$ for the first lattice and $\lambda_{01} = \lambda_1$ and $\lambda_{02} = \lambda_2$ for the second one. Recall the map $\varphi_{L_0}: M \to \hat{M}$ in the preceding section, and form $\text{Id} \times \varphi_{L_0}: M \times M \to M \times \hat{M}$.

**Definition 4.1.** We define the holomorphic line bundle $\tilde{K} := \pi_1^* L_0 \otimes (\text{Id} \times \varphi_{L_0})^* P \otimes \pi_2^* L_0 \to M_1 \times M_2$ where $P \to M \times \hat{M}$ is the Poincaré line bundle.

**Proposition 4.2.** In notations as above, a system of multipliers of $\tilde{K}$ can be

\[
\begin{align*}
&\{ e_{\lambda_{10}}(z, \mu) \equiv 1, \quad e_{\lambda_{01}}(z, \mu) \equiv 1 \\
& e_{\lambda_{20}}(z, \mu) = e^{-2\pi i z - 2\pi i \mu - \pi i \tau}, \quad e_{\lambda_{02}}(z, \mu) = e^{-2\pi i z - 2\pi i \mu - \pi i \tau}. \tag{4.1}
\end{align*}
\]

**Proof.** Recall that a holomorphic line bundle on $M \times M = V/\Lambda \times V/\Lambda$ is essentially described by a set of data: a system of multipliers $\{ e_{\lambda_{10}}, e_{\lambda_{20}}, e_{\lambda_{01}}, e_{\lambda_{02}} \in \mathcal{O}^*(V \times V) \}$ satisfying the **compatibility relations** (cf. (2.3)) : for $\{ i, j \} = \{ 1, 2 \}$

\[
\begin{align*}
&\{ e_{\lambda_{10}}(z + \lambda_j, \mu) e_{\lambda_{0j}}(z, \mu) = e_{\lambda_{0j}}(z + \lambda_i, \mu) e_{\lambda_{10}}(z, \mu) \\
& e_{\lambda_{01}}(z, \mu + \lambda_j) e_{\lambda_{0j}}(z, \mu) = e_{\lambda_{0j}}(z + \lambda_i, \mu) e_{\lambda_{01}}(z, \mu) \\
& e_{\lambda_{02}}(z, \mu + \lambda_j) e_{\lambda_{0j}}(z, \mu) = e_{\lambda_{0j}}(z, \mu + \lambda_i) e_{\lambda_{02}}(z, \mu). \tag{4.2}
\end{align*}
\]

Proof.
To break things down, the multipliers of $\pi_1^* L_0$ can be

$$
\begin{align*}
\varepsilon_{\lambda_{10}}(z, \mu) &\equiv 1, \quad \varepsilon_{\lambda_{20}}(z, \mu) = e^{-2\pi iz - \pi i\tau} \\
\varepsilon_{\lambda_{01}}(z, \mu) &\equiv 1, \quad \varepsilon_{\lambda_{02}}(z, \mu) \equiv 1
\end{align*}
$$

(4.3)

and the multipliers of $\pi_2^* L_0$ can be similarly expressed. As we will see soon, a system of multipliers of $(\text{Id} \times \varphi_{L_0})^* P$ can be chosen to be

$$
\begin{align*}
\varepsilon_{\lambda_{10}}(z, \mu) &\equiv 1, \quad \varepsilon_{\lambda_{20}}(z, \mu) = e^{-2\pi i\mu} \\
\varepsilon_{\lambda_{01}}(z, \mu) &\equiv 1, \quad \varepsilon_{\lambda_{02}}(z, \mu) = e^{-2\pi iz}.
\end{align*}
$$

(4.4)

Obviously all these multipliers satisfy (4.2). So (4.4) does define a holomorphic line bundle, tentatively denoted by $J$, on $M_1 \times M_2$.

To see the above claim (4.4), note first that a system of multipliers of $(\text{Id} \times \varphi_{L_0})^* P|_{M \times \{\mu\}} \cong \mathcal{T}_\mu^* L_0 \otimes L_0^* \cong P_{\varphi_{L_0}(\mu)} \to M_1 \times M_2$

can be

$$
\begin{align*}
\varepsilon_{\lambda_1}(z) &\equiv 1 \\
\varepsilon_{\lambda_2}(z) = e^{-2\pi i\mu}
\end{align*}
$$

(4.5)

and that of the trivial line bundle $(\text{Id} \times \varphi_{L_0})^* P|_{\{0\} \times M}$

e_{\lambda_1}(\mu) = e_{\lambda_2}(\mu) \equiv 1.

(4.6)

One observes that (4.4) or $J$ satisfies (4.5) and (4.6). The claim that

$$
J \cong (\text{Id} \times \varphi_{L_0})^* P
$$

follows from the same type of arguments of [9, p. 329] for the proof of the uniqueness of Poincaré line bundle. Our claim (4.4) is proved.

Finally, (4.1) follows from (4.3) (for $\pi_1^* L_0$ and similarly for $\pi_2^* L_0$) and (4.4).

By Proposition 4.2, any global holomorphic sections $\tilde{\theta}$ of $\tilde{K} \to M_1 \times M_2$ can be represented by quasi-periodic holomorphic functions on $V \times V$ satisfying, for all $z, \mu \in V$,

$$
\begin{align*}
\theta(z + \lambda_1, \mu) &= \theta(z, \mu) = \theta(z, \mu + \lambda_1) \\
\theta(z + \lambda_2, \mu) &= e^{-2\pi iz - 2\pi i\mu - \pi i\tau} \theta(z, \mu) = \theta(z, \mu + \lambda_2)
\end{align*}
$$

(4.7)

and any Hermitian metric $h(z, \mu)$ on $\tilde{K} \to M_1 \times M_2$:

$$
\begin{align*}
h(z + \lambda_1, \mu) &= h(z, \mu) = h(z, \mu + \lambda_1) \\
h(z + \lambda_2, \mu) &= e^{-4\pi iz - 4\pi \mu - 2\pi \tau_2} h(z, \mu) = h(z, \mu + \lambda_2).
\end{align*}
$$

(4.8)
An application of Proposition 4.2 is to exploit those $\mu$-dependent theta functions $\theta_m(z, \mu)$. Recall that in Lemma 2.2, \{\theta_m(z, \mu)\}_m represents a basis of the global holomorphic sections of $L_\mu$ for each individual $\mu \in V$. As $\mu$ varies, it seems tempting to think that \{\theta_m(z, \mu)\}_m naturally extends the sections \{\theta_m(z, 0)\}_m of $L_0$ via the Poincaré line bundle along the $\mu$-direction. This is not quite the case, however.

Indeed, a global property that this family of functions \{\theta_m(z, \mu)\}_m possess is the following.

**Theorem 4.3.** For the holomorphic line bundle $\tilde{K} = \pi_1^*L_0 \otimes (\text{Id} \times \varphi_L)^*P \otimes \pi_2^*L_0 \to M_1 \times M_2$, one has the quasi-periodic holomorphic functions on $V \times V$

$$\theta_m(z, \mu) = \sum_{k \in \mathbb{Z}} e^{\pi ik^2 \tau} e^{2\pi i \tau \frac{m}{\delta} (k+\frac{m}{\delta})} e^{2\pi i \delta \frac{m}{\delta} (z+\mu)}, \quad m = 0, 1, \ldots, \delta - 1,$$

(4.9)
as a basis of global holomorphic sections of $\tilde{K}$, and

$$h(z, \mu) = e^{\frac{\pi}{\tau_2} (z^2 + \mu^2)}$$

(4.10)
as a metric on $\tilde{K}$, which on the restriction $\tilde{K}|_{M \times \{\mu\}}$ induces the metric $h_{L_\mu}$ in (2.14).

**Proof.** Let $\omega := z + \mu$. By using the quasi-periodic property of $\theta_m(\omega)$ and $h_{L_0}(\omega)$ in (2.11) and (2.12), we see that the functions (4.9) and (4.10) satisfy (4.7) and (4.8). The theorem follows. \qed

We can now equip the line bundle

$$(\text{Id} \times \varphi_{L_0})^*P \to M_1 \times M_2$$

with a metric. Since $\tilde{K} = \pi_1^*L_0 \otimes (\text{Id} \times \varphi_L)^*P \otimes \pi_2^*L_0 \to M_1 \times M_2$, by the metric $h(z, \mu)$ on $\tilde{K}$ (cf. (4.10)) and the metric $h_{L_0}(z)$ (cf. (2.7)), one finds the induced metric

$$h_{(\text{Id} \times \varphi_{L_0})^*P}(z, \mu) = e^{-\frac{4\pi}{\tau_2} z^2 \mu^2}$$

(4.11)
on $(\text{Id} \times \varphi_{L_0})^*P$. Let's now calculate the curvature of this metric.

**Theorem 4.4.** The curvature of the metric in (4.11) is

$$\Theta_{(\text{Id} \times \varphi_{L_0})^*P}(z, \mu) = \frac{\pi}{\tau_2} (dz \wedge d\mu + d\mu \wedge dz).$$

(4.12)

**Proof.** The curvature $\Theta_{\tilde{K}}$ of $(\tilde{K}, h(z, \mu))$ is

$$\Theta_{\tilde{K}}(z, \mu) = - \partial \overline{\partial} \log(h(z, \mu))$$

$$= \frac{\pi}{\tau_2} \left( dz \wedge d\bar{z} + d\bar{z} \wedge dz + d\mu \wedge d\bar{z} + d\bar{z} \wedge d\mu \wedge dz \right)$$

(4.13)
and the curvature $\Theta_{L_0}(z)$ of $(L_0, h_{L_0}(z))$ is

$$\Theta_{L_0}(z) = - \partial \overline{\partial} \log(h_{L_0}(z)) = \frac{\pi}{\tau_2} dz \wedge d\overline{z}. \quad (4.14)$$

Now (4.12) follows from (4.13) and (4.14). \qed

5. The holomorphic vector bundle $K \to M_2 \cong M$

To facilitate the curvature computation later on, we shall now discuss the direct image bundle $K$ of $\tilde{K}$ in the preceding section. Recalling the line bundle $\tilde{K} \to M_1 \times M_2$ (cf. Definition 4.11), we form the push-forward $K := \pi_2^* \tilde{K}$ which is a holomorphic vector bundle on $M_2$. One sees that $K = \pi_2^*((\pi_1^* L_0 \otimes (Id \times \varphi_L)^* P) \otimes L_0)$ on $M_2$ by the standard projection formula.

**Definition 5.1.** Define a metric $(\cdot, \cdot)_{h_{L_\mu}}$ on $K$ by the $L^2$ inner product using $(\cdot, \cdot)_{h_{L_\mu}}$ on $K_{|\mu} = H^0(M, \tilde{K}_{|M \times \{\mu\}})$ (cf. the last statement in Theorem 4.3):

$$(\theta(z), \theta'(z))_{h_{L_\mu}} := \int_M h_{L_\mu}(z) \theta(z) \overline{\theta'(z)} \left( \frac{i}{2} dz \wedge d\overline{z} \right) \quad (5.1)$$

where $\theta, \theta'$ are global holomorphic sections of $L_\mu$.

The main lemma for our computations is as follows.

**Lemma 5.2.** With the inner product $(\cdot, \cdot)_{h_{L_\mu}}$, the holomorphic sections $\theta_m(z, \mu) = \sum_{k \in \mathbb{Z}} e^{\pi i k^2 \tau} e^{2\pi i r \frac{m}{\delta}} e^{2\pi i (\frac{k + m}{\delta}) (z + \mu)}$, $m = 0, 1, \ldots, \delta - 1$, constitute an orthogonal basis of $H^0(M, L_\mu)$, where

$$(\theta_m(z, \mu), \theta_m(z, \mu))_{h_{L_\mu}} = \sqrt{\frac{\tau_2}{2}} \delta e^{2m^2 \tau_2}, \quad m = 0, 1, \ldots, \delta - 1. \quad (5.3)$$

**Proof.** By (5.1), we have

$$(\theta_m(z, \mu), \theta_{m'}(z, \mu))_{h_{L_\mu}} = \int_M h_{L_\mu}(z) \theta_m(z, \mu) \overline{\theta_{m'}(z, \mu)} \left( \frac{i}{2} dz \wedge d\overline{z} \right)$$

$$= \int_0^{\tau_2} \int_0^\delta \sum_{k,j \in \mathbb{Z}} e^{\frac{-2\pi}{\tau_2} (z_2 + \mu_2)^2} \left( e^{\pi i k^2 \tau} e^{2\pi i r \frac{m}{\delta}} e^{2\pi i (\frac{k + m}{\delta}) (z + \mu)} \right) \left( e^{-\pi i j^2 \tau} e^{-2\pi i r \frac{m'}{\delta}} e^{-2\pi i (\frac{j + m'}{\delta}) (\overline{z} + \overline{\mu})} \right) d\overline{z_1} d\overline{z_2} \quad (5.4)$$
where \( z = z_1 + iz_2, z_1, z_2 \in \mathbb{R} \). The terms in (5.4) related to \( z_1 \) are

\[
\int_0^\delta e^{2\pi i z_1 (k-j + \frac{m-m'}{8})} \, dz_1
\]

which survive only when \( k = j \) and \( m = m' \). The lemma follows by straightforward calculations in the following aspects:

i) change of variable \( t := \frac{1}{\tau_2} (z_2 + \mu_2) \),

ii) the union of the domains of definite integrals

\[
\sum_{k \in \mathbb{Z}} \int_{\frac{\mu_2}{\tau_2}}^{1+\frac{\mu_2}{\tau_2}} e^{-2\pi \tau_2 (t+k+\frac{\mu}{\tau_2})^2} \, dt = \int_{-\infty}^{\infty} e^{-2\pi \tau_2 t^2} \, dt,
\]

iii) the Gaussian integral (where we use \( A = 2\pi \tau_2 \))

\[
\int_{-\infty}^{\infty} e^{-At^2} \, dt = \frac{\sqrt{\pi}}{\sqrt{A}}, \quad A > 0.
\]

By this lemma, the value of \( (\theta_m(z, \mu), \theta_m(z, \mu))_{k,\mu} \) in Definition 5.1 is independent of \( \mu \). We obtain the first statement of the following theorem.

**Theorem 5.3.** (1) On \( K \), the curvature tensor of the metric \( (\ , \ )_h \) defined in Definition 5.1 is identically zero.

(2) \( K \) splits holomorphically into a direct sum of holomorphically trivial line bundles

\[
K = \bigoplus_{m=0}^{\delta} K_m \text{where each } K_m \text{ has the canonical section identified as } \theta_m \text{ of Lemma 5.2}
\]

**Proof.** The first statement is observed precedingly; the second statement follows from Theorem 4.3 Lemma 5.2 and the first statement.

**Remark 5.4.** For the above second statement, there is an argument without using metric. Since \( M \) is of dimension one, each \( \theta_m \) of Lemma 5.2 generates a holomorphic line subbundle of \( K \to M_2 \cong M \), still denoted by \( K_m \to M_2 \). It is not difficult to see that \( \theta_m \) is actually nowhere vanishing on \( M_2 \) by using the fact that by construction, it arises from translates of the ordinary theta functions. Hence \( K_m \) is holomorphically trivial. By similar arguments, \( \{\theta_m\}_m \) is also independent everywhere on \( M \) and hence a global basis for \( K \to M \).

**6. Connection on the line bundle \( \mathcal{P} \to M \times M^* \)**

The vector bundle to be computed is going to live on \( \hat{M} \). For this reason and others as explained earlier in Introduction, we are led to differential geometric aspects of the Poincaré line bundle in this section and the next one. Here, we view the Riemann surface

\[12\]
as a real 2-dimensional smooth manifold and introduce a differential geometric description of the Poincaré line bundle with a connection on it. We follow closely the treatment in [8, Subsections 3.2.1 and 3.2.2], but use a suitable sign convention more adapted to our purpose.

To begin with, we write \( V \cong \mathbb{R}^2 \), and \( M = V / \Lambda \) where \( \Lambda = \{ \lambda_1, \lambda_2 \} = \{ (\delta, 0), (\tau_1, \tau_2) \}, \delta \in \mathbb{N}, \tau_2 > 0 \). Let \( \Lambda^* = \{ dx_1, dx_2 \} \) be the dual basis of \( \Lambda \); that is, \( \int_{\lambda_i} dx_j = \delta_{ij} \). Let \( V^* := \text{Hom}(V, \mathbb{R}) \) be the dual space of \( V \). Any \( \xi \in V^* \) is a 1-form with constant real coefficients. That is, \( \xi = \xi_1 dx_1 + \xi_2 dx_2 \) with \( \xi_1, \xi_2 \in \mathbb{R} \). We define \( M^* := V^*/2\pi \Lambda^* \), (6.1)

and write \([\xi]\) as the equivalent class of \( \xi \) in \( M^* \).

Let \( \underline{C}_V : V \times \mathbb{C} \to V \) be the trivial complex line bundle over \( V \). An element \( \xi \in V^* \) gives rise to a character \( \chi_\xi : \Lambda \to U(1) \) by

\[
\chi_\xi(\lambda) := e^{-i <\xi, \lambda>}
\]

where \( <\xi, \lambda> = \xi(\lambda) \in \mathbb{R} \). The set \( \Lambda \) acts on \( \underline{C}_V \) by

\[
\lambda \circ (x, \sigma) := (x + \lambda, \chi_\xi(\lambda) \sigma) \tag{6.3}
\]

This action preserves the horizontal foliation in \( \underline{C}_V \) which thus descends to a flat connection, denoted by \( d \), on the quotient bundle over \( M \). For \( \xi = \xi_1 dx_1 + \xi_2 dx_2 \in V^* \), one can define a flat \( U(1) \) connection on the complex line bundle \( \underline{C}_M : M \times \mathbb{C} \to M \) by

\[
\nabla^\xi := d + i\xi \tag{6.4}
\]

It is a simple fact that the gauge equivalence classes of flat line bundles on \( M \) are parametrized by \( M^* := V^*/2\pi \Lambda^* \). We write

\[
\underline{T}_[\xi] := \left( \underline{C}_V / \Lambda, \nabla^\xi \right) \tag{6.5}
\]

for the flat line bundle on \( M \) corresponding to the connection \( \nabla^\xi \), \( \xi \in V^* \). With the connection \( \nabla^\xi \), it is seen that the parallel transport along the loops is given by \( \chi_\xi \).

Remark that in (6.4) the sign convention is actually consistent with that in [8] as far as \( \underline{T}_[\xi] \) is concerned, because by [8, proof of Proposition 2.2.3] as remarked in [8, p. 83], their \( L_\xi \) is seen to be the same as \( \underline{T}_[\xi] \) above; see also [8, proof of Lemma 3.2.14, p. 86].

Dually, for any given \( x \in V \) we define a character \( \chi_x : 2\pi \Lambda^* \to U(1) \) by

\[
\chi_x(2\pi \nu) := e^{-2\pi i <\nu, x>} \tag{6.6}
\]

So we get flat line bundles \( \underline{T}_[x] \) over \( M^* \) with parallel transport \( \chi_x \).

The above picture paves the way for the following lemma.

**Lemma 6.1.** There is a complex line bundle \( \mathcal{P} \) over \( M \times M^* \) with a unitary connection, such that the restriction of \( \mathcal{P} \) to each \( M_{[\xi]} := M \times \{ [\xi] \} \) is isomorphic (as a line bundle with connection) to \( \underline{T}_[\xi] \) and the restriction to each \( M^*_{[x]} := \{ [x] \} \times M^* \) is isomorphic to
To be more precise, we consider the connection 1-form $A = i \xi$, $\xi \in V^*$ on the trivial line bundle $\mathbb{C}_{|M \times V^*} : (M \times V^*) \times \mathbb{C} \to M \times V^*$. We can lift the actions of $2\pi \Lambda^*$ on $M \times V^*$ to $\mathbb{C}_{|M \times V^*}$ by

$$2\pi \nu \circ (x, \xi, \sigma) := (x, \xi + 2\pi \nu, e^{-2\pi i <\nu, x>} \sigma), \quad \forall \nu \in \Lambda^*.$$  \hspace{1cm} (6.7)

This action preserves the connection $d + A$ and hence induces a connection on the line bundle

$$\mathcal{P} := \mathbb{C}_{|M \times V^*} / 2\pi \Lambda^* \to M \times M^*,$$  \hspace{1cm} (6.8)

denoted as $\nabla^\mathcal{P}$. It is worthwhile mentioning that although the connection is flat on each slice $\mathcal{P}_{|M \times \{\xi\}} \cong \mathcal{T}[\xi]$, it is not flat on the entire $\mathcal{P}$. Indeed the curvature of $\nabla^\mathcal{P} = d + A$ is

$$dA + A \wedge A = i (d\xi_1 \wedge dx_1 + d\xi_2 \wedge dx_2).$$  \hspace{1cm} (6.9)

Similarly, if we define a metric $h_{\mathbb{C}_{|M \times V^*}}(x, \xi) \equiv 1$ on the trivial line bundle $\mathbb{C}_{|M \times V^*}$, or equivalently,

$$< (x, \xi, \sigma_1), (x, \xi, \sigma_2) >_{\mathbb{C}_{|M \times V^*}} := \sigma_1 \cdot \sigma_2,$$  \hspace{1cm} (6.10)

then the metric (6.10) is preserved by the action of $2\pi \Lambda^*$ in (6.8). Thus it induces a metric on $\mathcal{P}$, denoted as $h_\mathcal{P}$.

One sees that the connection $\nabla^\mathcal{P}$ and the metric $h_\mathcal{P}$ just defined are compatible on $\mathcal{P}$, that is, the connection is unitary with respect to the metric as required in Lemma 6.1.

The holomorphic structure on the line bundle $\mathcal{P}$ is discussed in the next section.

7. Identify $\mathcal{P}$ with the Poincaré line bundle $\mathcal{P}$

The following lemma is almost immediate. It is included to make the transformation in coordinates more transparent.

**Lemma 7.1.** One has

$$Iso : \hat{\mathcal{M}} \sim \to M^*.$$  \hspace{1cm} (7.1)

**Proof.** Recall that $\hat{\mathcal{M}} = \text{Pic}^0(M) \cong H^0_{\mathbb{Q}}(M) / H^1(M, \mathbb{Z})$ with the image of $H^1(M, \mathbb{Z})$ in $H^0_{\mathbb{Q}}(M)$ as $\overline{\Lambda}^* = \{ n_1 dx_1^* + n_2 dx_2^* \mid n_1, n_2 \in \mathbb{Z} \}$ in the notations of Section 8. We write

$$\text{Pic}^0(M) = \hat{\mathcal{M}} = \left\{ \begin{array}{l} c_1 dx_1^* + c_2 dx_2^* \mid c_1, c_2 \in \mathbb{R} \\ n_1 dx_1^* + n_2 dx_2^* \mid n_1, n_2 \in \mathbb{Z} \\ \hat{\mu} e_1^* \mid \hat{\mu} \in \mathbb{C} \end{array} \right\}$$  \hspace{1cm} (7.1)

where $\hat{\mu} = \mu_1 + i\mu_2$. Similarly, from (6.1),

$$M^* = \left\{ \begin{array}{l} \xi_1 dx_1 + \xi_2 dx_2 \mid \xi_1, \xi_2 \in \mathbb{R} \\ 2\pi k_1 dx_1 + 2\pi k_2 dx_2 \mid k_1, k_2 \in \mathbb{Z} \end{array} \right\}$$  \hspace{1cm} (7.2)
We have the group isomorphism $Iso: \hat{M} \to M^*$ by sending $dx_1^*$ to $2\pi dx_1$ and $dx_2^*$ to $2\pi dx_2$ with
\[
\begin{cases}
\xi_1 = \frac{2\pi}{\tau_2} \hat{\mu}_2 \\
\xi_2 = \frac{2\pi}{\tau_2} (\tau_1 \hat{\mu}_2 - \tau_2 \hat{\mu}_1)
\end{cases}
\] equivalently
\[
\begin{cases}
\hat{\mu}_2 = \frac{1}{2\pi} \xi_1 + \frac{1}{2\pi} \frac{\tau_2}{\tau_1} \xi_1 \\
\hat{\mu}_1 = \frac{1}{2\pi} \frac{\tau_2}{\tau_1} \xi_1.
\end{cases}
\] (7.3)
In particular, $Iso(\Lambda^*) = 2\pi \Lambda^*$.

Recall the line bundle $P \to M \times \hat{M}$ of Lemma 3.1. By the above lemma, $M^*$ admits a complex structure inherited from that of $\hat{M}$. To compare $P$ and $P$, we first note that the global connection $\nabla^P$ in the preceding section on the line bundle $P \to M \times M^*$ of Lemma 6.1 gives a holomorphic structure on $P$ (where the $M$ has been identified with the previous $M$ automatically as a complex torus).

To see this, define $\tilde{Iso} := (Id, Iso): M \times \hat{M} \to M \times M^*$ with $Iso: \hat{M} \to M^*$ in Lemma 7.1. Let's form the pull-back bundle $\tilde{Iso}^* P$ equipped with the pull-back metric $\tilde{Iso}^* h_P$ and the pull-back connection $\tilde{\nabla} := \tilde{Iso}^* \nabla^P$. By $\nabla^P = d + i\xi$, the connection is seen to be
\[
\tilde{\nabla} = d + \frac{\pi}{\tau_2} (-\hat{\mu} dz + \hat{\mu} d\bar{z})
\] and the curvature $\Theta_{\tilde{\nabla}}$ of $\tilde{\nabla}$ is
\[
dA + A \wedge A = \frac{\pi}{\tau_2} (dz \wedge d\bar{\mu} + d\hat{\mu} \wedge d\bar{z}).
\] (7.4)
Remark that the calculation to derive (7.4) is merely to plug (7.3) and (3.3) into (6.9). Now that the curvature of $\tilde{\nabla}$ is of type $(1,1)$, it is well-known that $\tilde{\nabla}$ gives rise to a holomorphic structure on $\tilde{Iso}^* P$. This implies the above claim.

We shall now identify $P$ and $P$.

**Theorem 7.2.** In the notations as above, let $P \to M \times \hat{M}$ be the Poincaré line bundle of Lemma 3.1, and $P \to M \times M^*$ of Lemma 6.1 be equipped with the holomorphic structure as given precedingly. Then
\[
P \cong \tilde{Iso}^* P.
\] (7.5)
**Proof.** By Lemma 3.1, $P$ is the unique holomorphic line bundle on $M \times \hat{M}$ satisfying
\[
(1) P|_{M \times \{\hat{\mu}\}} \cong P_{\hat{\mu}}.
(2) P|_{\{0\} \times \hat{M}} \text{ is holomorphically trivial on } \{0\} \times \hat{M}.
\]
To show that $P \cong \tilde{Iso}^* P$ where $\tilde{Iso} = (Id, Iso)$ as defined prior to Theorem 7.2, it therefore suffices to prove the following for $P \to M \times M^*$:
(1') for any $[\xi] \in M^*$, the line bundle $L_{[\xi]} \simeq P_{|M \times\{[\xi]\}}$ is holomorphically isomorphic to $P_{Iso^{-1}}(\{[\xi]\}) = P_{\hat{\mu}}$.

(2') $P_{|\{0\} \times M^*}$ is holomorphically trivial on $\{0\} \times M^*$.

To prove (1'), from the action in (6.3) that
$$\lambda \circ (x, \sigma) = (x + \lambda, \chi_\xi(\lambda) \sigma) = (x + \lambda, e^{-i<\xi,\lambda>} \sigma),$$

the holonomy transforms the basis $\lambda$ by $\chi_\xi(\lambda)$ as remarked earlier. Accordingly, the multipliers of $L_{[\xi]}$ which transforms inversely, are

$$\begin{cases}
    e_{\lambda_1}(z) = e^{i\xi_1} \\
    e_{\lambda_2}(z) = e^{i\xi_2}.
\end{cases} \quad (7.6)$$

Recall that the multipliers of $P_{\hat{\mu}}$ are (cf. (3.9), (3.8) and the complex linearity of (3.4))

$$\begin{cases}
    e_{\lambda_1}(z) = 1 \\
    e_{\lambda_2}(z) = e^{-2\pi i \hat{\mu}}.
\end{cases} \quad (7.7)$$

To match the above two sets of multipliers (7.6) and (7.7), define a line bundle $L_{\Delta, \xi} \rightarrow M$ with the (constant) multipliers

$$\begin{cases}
    e_{\lambda_1}(z) = e^{i\alpha \delta} = e^{i\xi_1} \\
    e_{\lambda_2}(z) = e^{i\alpha \tau} = e^{i\xi_1}\xi_1
\end{cases} \quad (7.8)$$

where $\alpha = \frac{\xi_1}{\delta} \in \mathbb{R}$. The function
$$\Phi_\xi(z) = e^{i\alpha z} \quad (7.9)$$

satisfying the quasi-periodic property with respect to (7.8) (see Section 2 and (2.1)) is then a global, nowhere vanishing section of $L_{\Delta, \xi}$. Therefore $L_{\Delta, \xi}$ is holomorphically trivial on $M$.

Via (7.7) and (7.8), the multipliers of the line bundle $P_{\hat{\mu}} \otimes L_{\Delta, \xi}$ become

$$\begin{cases}
    e_{\lambda_1}(z) = 1 \cdot e^{i\xi_1} = e^{i\xi_1} \\
    e_{\lambda_2}(z) = e^{-2\pi i \hat{\mu}} \cdot e^{i\xi_1} \xi_1 = e^{i\xi_2},
\end{cases} \quad (7.10)$$

where the second multiplier uses (7.3). Therefore, $L_{[\xi]} \simeq P_{\hat{\mu}}$ holomorphically, proving (1').

It remains to prove (2'). Recall that the action in (6.7)

$$2\pi \nu \circ (x, \xi, \sigma) := (x, \xi + 2\pi \nu, e^{-2\pi i <\nu, x>} \sigma), \quad \forall \nu \in \Lambda^*.$$  

At $x = 0$, this becomes

$$2\pi \nu \circ (0, \xi, \sigma) = (0, \xi + 2\pi \nu, \sigma) \quad \forall \nu \in \Lambda^*.$$  

Since $\sigma$ is unchanged, it follows that $P_{|\{0\} \times M^*}$ has trivial multipliers and hence a holomor-
phically trivial line bundle on $M^*$, proving (2').

8. Main Results

We shall now organize our preceding results and prove our main results here. By Theorem 7.2 that $P \cong \tilde{\text{Iso}}^* P$, we can pull back the metric $h_P$ and the connection $\nabla^P = d + i\xi$ on $P$ via the map $\tilde{\text{Iso}}$, and get a metric and a compatible connection on $P$

$$h_P := \tilde{\text{Iso}}^* h_P, \quad \nabla^P := \tilde{\text{Iso}}^* \nabla^P.$$

Write $\Theta_P$ for the curvature of $\nabla^P$. If we combine (7.4) with Theorem 4.4 in Section 4 (see also (3.4)), we have the first part of the following theorem.

**Theorem 8.1.** Recalling that $h_{(\text{Id} \times \varphi_{L_0})^* P}$ and $\Theta_{(\text{Id} \times \varphi_{L_0})^* P}$ on $(\text{Id} \times \varphi_{L_0})^* P \to M \times M$ (see (4.11) and (4.12)), one has the following. On $M \times M$,

1. $(\text{Id} \times \varphi_{L_0})^* \Theta_P = \Theta_{(\text{Id} \times \varphi_{L_0})^* P}$.
2. $(\text{Id} \times \varphi_{L_0})^* h_P = h_{(\text{Id} \times \varphi_{L_0})^* P}$.

**Proof.** The first part of the theorem is just noted. In turn, it yields that the two metrics in the second statement differ at most by a multiplicative constant $c$. If one restricts both metrics to $\{0\} \times M$, one sees that $c = 1$. 

To proceed further, we form some vector bundles as follows.

**Definition 8.2.** Define the line bundles

$$\tilde{E} := \pi_1^* L_0 \otimes P \to M \times \hat{M}, \quad \tilde{E}' := \pi_1^* L_0 \otimes (\text{Id} \times \varphi_{L_0})^* P \to M_1 \times M_2$$

where $M_1 \cong M_2 \cong M$, and the vector bundles

$$E := (\pi_2)_* \tilde{E} \to \hat{M}, \quad E' := (\pi_2)_* \tilde{E}' \to M_2.$$

The transformation from $L_0 \to M$ to $E \to \hat{M}$ (or $E' \to M$) can be placed in the context of the so-called Fourier-Mukai transform, but we shall not go into it here. We refer to [13, Section 5] for more details.

In what follows, we shall interchangeably use the identification $P \cong \tilde{\text{Iso}}^* P$ obtained in Theorem 7.2. First equip $\tilde{E}, \tilde{E}'$ with metrics

$$h_{\tilde{E}} = \pi_1^* h_{L_0} \otimes h_P \quad \text{where} \quad h_P = \tilde{\text{Iso}}^* h_P,$$

(cf. (2.7) for $h_{L_0}$ and (4.11))

$$h_{\tilde{E}'} = \pi_1^* h_{L_0} \otimes h_{(\text{Id} \times \varphi_{L_0})^* P}$$

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respectively. By (2) of Theorem 8.1, one has
\[ h_{\tilde{E}'} = (\text{Id} \times \varphi_{L_0})^* h_{\tilde{E}}. \] (8.6)

We shall now equip the vector bundle \( E \) with a metric given by the \( L^2 \)-metric on \( E|_\mu = H^0(M, L_\mu) \) using \( h_{\tilde{E}} \), and similarly the \( L^2 \)-metric on \( E'|_\mu = H^0(M, L_\mu) \) using \( h_{\tilde{E}'} \). These \( L^2 \)-metrics on \( E \) and \( E' \) are denoted by \( h_E \) and \( h_{E'} \) respectively.

Recall that \( \tilde{K} = \pi_1^* L_0 \otimes (\text{Id} \times \varphi_{L_0})^* P \otimes \pi_2^* L_0 \). By the explicit expressions (4.11) and (2.7), one sees that
\[ h_{\tilde{E}'} = e^{-\frac{2\pi}{\tau_2}(z_2^2 + 2z_2 \mu_2)}. \] (8.7)

We summarize the above in the following.

**Proposition 8.3.**
\[ (\tilde{E}', h_{\tilde{E}'}) = (\text{Id} \times \varphi_{L_0})^* (\tilde{E}, h_{\tilde{E}}) \] (8.8)
where \( h_{\tilde{E}} \) and \( h_{\tilde{E}'} \) are defined as in (8.4) and (8.5). As a consequence,
\[ (E', h_{E'}) = \varphi_{L_0}^*(E, h_E) \] (8.9)
with the curvatures
\[ \Theta(E', h_{E'}) = \varphi_{L_0}^* \Theta(E, h_E). \] (8.10)

Recall that \( K \rightarrow M_2 \) is the vector bundle \( K|_\mu = H^0(M, \tilde{K}|_{M \times \{\mu\}}) \) of Section 5. As vector bundles
\[ K = E' \otimes L_0, \quad E' = K \otimes L_0^* \] (8.11)
where \( L_0^* \) is the dual of \( L_0 \).

By Theorem 5.3 that \( K \) splits into line bundles (each of which is holomorphically trivial)
\[ K = \bigoplus_{m=0}^{\delta-1} K_m, \] (8.12)
it follows that
\[ E' = K \otimes L_0^* = \bigoplus_{m=0}^{\delta-1} (K_m \otimes L_0^*) = \bigoplus_{m=0}^{\delta-1} L_0^*. \] (8.13)

By Theorem 5.3 (8.13), and (4.14), the curvature of \( E' \) is immediately computed as follows.

**Theorem 8.4.** Let’s denote by \((\text{Id})_{\delta \times \delta}\) the \( \delta \times \delta \) identity matrix. Then we have
\[ \Theta(E', h_{E'}) = -\Theta_{L_0}(\mu)(\text{Id})_{\delta \times \delta} = \frac{-\pi}{\tau_2} d\mu \wedge d\bar{\mu} (\text{Id})_{\delta \times \delta}. \] (8.14)
Combining Theorem 8.4 and Proposition 8.3 (see also (3.4)), we have
Theorem 8.5. (1) $\Theta(E, h_E) = -\frac{i}{\tau_2} d\bar{\mu} \wedge d\mu \ (Id)^{\delta \times \delta}$.

(2) As a consequence of (1), the first Chern class of $E$ is

$$c_1(E, h_E) = \frac{-i\delta}{2\tau_2} d\bar{\mu} \wedge d\mu$$

(at the level of differential forms).

Remark 8.6. i) Our computational result of $c_1(E)$ agrees with that of the torus case in [13, Theorem 12] of Prieto, in view of his Remark 10 and various notations in p. 388, p. 381 and p. 386.

ii) It is unclear to us whether Theorem 8.5 can be proved independently of Theorem 8.4, mainly due to the fact that our description of $(\mu$-dependent) theta functions is most conveniently given on $M \times M$ rather than on $M \times \hat{M}$, as remarked earlier in Introduction.

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