Form factors of exponential fields
for two-parametric family of integrable models

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Abstract
A two-parametric family of integrable models (the $SS$ model) that contains as particular cases several well known integrable quantum field theories is considered. After the quantum group restriction it describes a wide class of integrable perturbed conformal field theories. Exponential fields in the $SS$ model are closely related to the primary fields in these perturbed theories. We use the bosonization approach to derive an integral representation for the form factors of the exponential fields in the $SS$ model. The same representations for the sausage model and the cosine-cosine model are obtained as limiting cases. The results are tested at the special points, where the theory contains free particles.

1. Introduction
A complete set of form factors of all local and quasilocal operators, defined as matrix elements in the basis of asymptotic states, completely determines a model of quantum field theory and provides an important tool for studying its physical properties. In the case of an integrable massive model the form factors can be found exactly [1–3], as soon as the exact scattering matrix is known, as solutions of a set of functional (difference) equations named the form factor axioms. It is assumed (though not proven) that all solutions of these equations provide form factors of all local operators in the theory.

There are several techniques for solving the form factor equations. In the case of a system of neutral particles of different masses, where the two-particle $S$ matrices are scalar functions, any form factor factorizes into a product of given functions (‘minimal form factors’) and a trigonometric polynomial. The functional equations are reduced to an infinite chain of linear equations for coefficients of these polynomials [4]. For some particular local operators these equations can be solved for the whole set of form factors [5, 6].

The situation is more complicated in the case of particles with isotopic degrees of freedom, where the $S$ matrices are matrix functions while the form factors are multivector functions. In this case the form factor axioms have the form of equations in multidimensional spaces with matrix coefficients. The historically first approach to these systems was Smirnov’s integral representation [3]. It gives form factors in terms of multiple integrals. The integrand can be represented as a particular transcendental function times a function of a given class, which depends on the local operator. In this way many integrable models have been studied. The advantage of this scheme is that it provides a general solution that corresponds to the most general local operator in the theory. The main problem of this approach is identification of sets of form factors with particular local operators in the model. It was only made for several most important operators like currents and the energy-momentum tensor.

Another approach was proposed by Lukyanov [7]. It is based on the bosonization (free field representation) technique. Bosonization was first used in conformal field theory (CFT) as a way to construct an integral representation for correlation functions in the Virasoro minimal models [8]. Lukyanov showed that solutions to the form factor axioms can be found as traces of some objects, namely, vertex and screening operators, very similar to those of CFT. These operators can be expressed in terms of free bosonic operators. This
provides integral representations for form factors different from those proposed by Smirnov. In the framework of this approach, the form factors of a wide class of local operators, namely the exponential fields, were found for the sine-Gordon model and the $A$ series of the affine Toda models [9–11]. Note, that this approach is intimately related to the bosonization schemes proposed for lattice models of statistical mechanics [12].

It is worth to mention a new approach developed by Babujian and Karowski [13], based on what the authors call the off-shell Bethe ansatz. The advantage of this method is that is has to do with the same objects as the algebraic Bethe ansatz does.

In this paper we construct an integral representation for the form factors of local fields in the two-parametric family of integrable field theories, which is also known as the $SS$ model. This notation of the theory is related with the form of the scattering matrix for the fundamental particles. It is a tensor product of two sine-Gordon soliton $S$ matrices with different coupling constants. The $SS$ model has an explicit $U(1) \otimes U(1)$ symmetry, which can be extended up to the symmetry generated by two quantum affine algebras $U_q(sl_2) \otimes U_q(sl_2)$. This theory contains as the particular cases $N = 2$ supersymmetric sine-Gordon theory, $O(4)$ and $O(3)$ non-linear sigma models and several other interesting integrable theories. The $SS$ model possesses a dual sigma model representation [15], which is useful for the short-distance renormalization group analysis of the theory. This analysis together with the conformal perturbation theory gives us a rather complete description of the short-distance properties of the model. The long-distance behavior can be derived from the form factor decomposition for correlation functions of the theory.

The study of the form factors in the $SS$ model was started by Smirnov [14], who calculated the matrix elements for a set of quantum group invariant operators as, for example, stress-energy tensor and $U(1)$ currents. An important class of operators in the $SS$ model is formed by the exponential fields. In the restricted versions of the theory they correspond to primary fields in the related perturbed CFTs. Construction of an integral representation for the form factors of the exponential fields is the main problem that we consider in this paper. For this purpose we use a modification of the bosonization techniques proposed by Konno [20] for the analysis of integrable lattice models.

The paper is organized as follows. In Sec. 2 we shortly give the necessary information about the model under consideration. In Sec. 3 we describe the bosonization procedure for the model and construct a three-parameter family of form factors. In Sec. 4 we identify the constructed form factors to those of the exponential model is formed by the exponential fields. In the restricted versions of the theory they correspond to primary fields in the related perturbed CFTs. Construction of an integral representation for the form factors of the exponential fields is the main problem that we consider in this paper. For this purpose we use a modification of the bosonization techniques proposed by Konno [20] for the analysis of integrable lattice models.

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2. The model

The two-parametric family of integrable models under consideration (the $SS$ model) possesses a Lagrangian formulation in terms of three scalar fields $\varphi_1$, $\varphi_2$, $\varphi_3$. The action has the form [15]:

$$S = \int d^2x \left(\frac{(\partial_\mu \varphi_1)^2 + (\partial_\mu \varphi_2)^2 + (\partial_\mu \varphi_3)^2}{8\pi} + \frac{\mu}{\pi} \left(\cos(\alpha_1 \varphi_1 + \alpha_2 \varphi_2)e^{\beta \varphi_3} + \cos(\alpha_1 \varphi_1 - \alpha_2 \varphi_2)e^{-\beta \varphi_3}\right)\right), \quad (2.1)$$

where the parameters $\alpha_1$, $\alpha_2$ and $\beta$ satisfy the integrability condition

$$\alpha_1^2 + \alpha_2^2 - \beta^2 = 1. \quad (2.2)$$

It is convenient to introduce the notation

$$p_1 = 2\alpha_1^2, \quad p_2 = 2\alpha_2^2, \quad p_3 = -2\beta^2, \quad p_1 + p_2 + p_3 = 2 \quad (2.3)$$

and

$$\alpha_3 = -i\beta. \quad (2.4)$$

We have a two-parametric family of integrable models of quantum field theory. As particular cases this family covers a number of integrable families, like the sausage model [17] for $p_2 = 0$, $p_1 \geq 2$, the cosine-cosine model [18] for $p_1 + p_2 = 2$, $p_1, p_2 \geq 0$, the $N = 2$ supersymmetric sine-Gordon model [19] for $p_2 = 2$ and $p_1 \geq 0$. Other models, like the parafermion sine-Gordon and an integrable perturbation of the
SU(2)_{p_1-2} \times SU(2)_{p_2-2}/SU(2)_{p_1+p_2-4} coset model, can be obtained from this family by different restriction at appropriate values of \( p_1, p_2 \) [15].

There are three essentially different regimes in the theory:

\[
\begin{align*}
& p_1, p_2 > 0, \quad p_3 < 0 \quad \text{(Regime I)}; \\
& p_1, p_2, p_3 > 0 \quad \text{(Regime II)}; \\
& p_1, p_2 < 0, \quad p_3 > 0 \quad \text{(Regime III)}.
\end{align*}
\]

The regime I is unitary. Just this regime is the subject of this paper. The particle content and scattering theory in the nonunitary regimes II and III are rather complicated. The regime II is of particular interest, as it recovers the invariance under the substitutions \( \varphi_i \leftrightarrow \varphi_j, \ p_i \leftrightarrow p_j \). The spectrum and the \( S \) matrix of the model in the regime II, which can be extracted from the free field realization described below, is given in the Appendix F. A detailed description of this regime will be given elsewhere.

In the unitary regime the model possesses the \( U(1) \times U(1) \) symmetry described by the conserved topological charges\(^1\)

\[
Q_\pm = \frac{1}{2} (Q_1 \pm Q_2), \quad Q_1 = \int dx^1 \frac{\partial}{\partial t}, \quad j_\mu^i = \frac{\alpha_i}{\pi} \varepsilon^{\mu\nu} \partial_\nu \varphi_i, \quad i = 1, 2.
\]  

Classically the charges satisfy the conditions

\[
Q_\pm \in \mathbb{Z} \quad \text{or} \quad Q_1, Q_2 \in \mathbb{Z}, \quad Q_1 + Q_2 \in 2\mathbb{Z}.
\]

In the quantum case these conditions are valid for the eigenvalues of the charges. Note that the operators \( Q_1 \) and \( Q_2 \) are elements of a wider algebra, namely the above-mentioned \( U_q(\hat{sl}_2) \otimes U_{\hat{sl}_2}(\hat{sl}_2) \) with \( q_i = e^{i\pi/p_i} \).

The spectrum of the model consists of the fundamental particles \( z_{ee'} \) (\( e, e' = \pm \equiv \pm 1 \)) and their bound states. The fundamental particles are characterized by eigenvalues of the topological charges: \( Q_1 | z_{ee'} \rangle = \varepsilon | z_{ee'} \rangle, \quad Q_2 | z_{ee'} \rangle = \varepsilon' | z_{ee'} \rangle \). The mass of the fundamental quadruplet is proportional to the parameter \( \mu \) of the Lagrangian. The exact relation between these quantities has a form [15]:

\[
m = \frac{\mu}{\pi} \frac{\Gamma \left( \frac{p_1}{2} \right) \Gamma \left( \frac{p_2}{2} \right)}{\Gamma \left( \frac{p_1 + p_2}{2} \right)}.
\]

The two-particle \( S \) matrix of the fundamental particles as a function of the rapidity difference \( \theta \) is given by

\[
S_{p_1,p_2}(\theta) = -S_{p_1}(\theta) \otimes S_{p_2}(\theta)
\]

with \( S_{p}(\theta) \) being the two-soliton \( S \) matrix of the sine-Gordon model with the coupling constant \( \beta_{SG}^2 = 8\pi \frac{p}{p+1} \) [21]:

\[
S_p(\theta)^{++} = -e^{\delta_p(\theta)}, \quad S_p(\theta)^{+-} = -e^{\delta_p(\theta)} \frac{\sinh \frac{\theta}{p}}{\sinh \frac{\pi}{p}}, \quad S_p(\theta)^{+-} = -e^{\delta_p(\theta)} \frac{i \sin \frac{\pi}{p}}{\sinh \frac{\pi}{p}}, \quad S_p(\theta)^{--} = S(\theta)^{+e_1'-e_2'} = S(\theta)^{-e_1-e_2'}, \quad \delta_p(\theta) = 2 \int_0^\infty dt \frac{\sinh \frac{\pi}{p} \sinh \frac{\pi(p-1)t}{p}}{\sinh \pi t \sinh \frac{\pi t}{p}} \sin \theta.
\]

The \( S \) matrix [24] was first considered in [14] and identified as the \( S \) matrix of the model [21] in the region I in [15].

In the case when one of the parameters \( p_1, p_2 \) is less than one, the scattering matrix [24] possesses poles in the interior of the physical strip \( 0 < \text{Im} \theta < \pi \) at the points

\[
\theta = i u_i, \quad u_i = \pi - \pi p_i, \quad i = 1, 2, 3, \ldots, \quad np_i < 1, \quad \text{if} \quad 0 < p_i < 1.
\]

\(^1\)In this paper we use the notation different from that used in [15]: the charges \( Q_+ \) and \( Q_- \) here are \( Q_1 \) and \( Q_2 \) of the reference [15].
These poles correspond to bound states. As in the unitary regime \( p_1 + p_2 \geq 2 \), only one of these two sets of bound states can be present in the spectrum. The masses of the bound states are given by

\[
M_{I,n} = 2m \sin \frac{\pi p_{1n}}{2}, \quad n = 1, 2, \ldots, \quad np_1 < 1. \tag{2.12}
\]

The bound states form quadruplets different from those of the fundamental particles. For example, for \( p_1 < 1 \) each quadruplet consists of one particle \( b^{1,n}_+ \) with \( Q_2 = 2 \), one particle \( b^{1,n}_- \) with \( Q_2 = -2 \) and two particles \( b^1_{S}, b^1_{A} \) with \( Q_2 = 0 \), while \( Q_1 = 0 \) for the whole quadruplet.

3. Bosonization and Integral Representation for Form Factors

To compute form factors of local operators we would like to apply the bosonization procedure based on algebraic methods for solving the form factor equations. It was observed by Lukyanov [7] that solutions to the form factor axioms can be found in terms of representations of the corner Hamiltonian \( H \) and the vertex operators that satisfy the commutation relations of the extended Zamolodchikov–Faddeev (ZF) algebra.

\[H, Z_I(\theta)] = i \frac{d}{d\theta} Z_I(\theta) - \Omega_I Z_I(\theta).\] \hspace{1cm} \text{(3.1c)}

If the scattering matrix for two particles \( z_I \) and \( z_J \) has a pole at the point \( \theta = iu_{I,J}^K \) corresponding to a set \( K \) of bound states of the same mass, then the vertex operators should satisfy some bootstrap conditions. Namely, the vertex operator for the particles \( z_K \) with \( K \in K \) appear in the fusion of the operators \( Z_I \) and \( Z_J \). This condition can be expressed algebraically as follows. Let \( u_+ \) and \( u_- \) be defined by the equations

\[u_+ + u_- = \frac{u_{I,J}^K}{m_I}, \quad \frac{\sin u_+}{\sin u_-} = \frac{m_I}{m_J},\]

where \( m_I, m_J \) are masses of the particles \( z_I, z_J \). Then we have the relation

\[Z_I(\theta' + iu_+)Z_J(\theta - iu_-) = \frac{i}{\theta' - \theta} \sum_{K \in K} \Gamma_{I,J}^K Z_K(\theta) + O(1), \quad \theta' \to \theta\] \hspace{1cm} \text{(3.1d)}

with some constants \( \Gamma_{I,J}^K \). These constants can be found from compatibility with the conditions \( \text{3.1a} \) and \( \text{3.1b} \) and expressed in terms of the residues of the \( S \) matrix. Therefore, an important problem is to construct the basic vertex operators for the fundamental particles. Then the vertex operators for bound state particles can be derived by fusion of the basic ones.

\[\text{3This picture is deeply related [22] to the angular quantization procedure [23] where the operator } \exp(-2\pi H) \text{ is treated as a density matrix.}\]

\[\text{3The charge conjugation matrix is related to the } S \text{ matrix by the crossing symmetry condition: } \sum_{J'} C_{J,J'} S(i\pi - \theta)_{J'}^J = \sum_{J'} C_{J,J'} S(\theta)_{J'}^J.\]
Now we are in position to discuss application of the representation theory of the extended algebra (3.1) to form factors. For any representation \( \Pi \) and any element \( X \) of the algebra we introduce the following notation
\[
\langle \langle X \rangle \rangle_\Pi = \frac{\text{Tr}_H(e^{-2\pi H}X)}{\text{Tr}_H(e^{-2\pi H})},
\]
The main observation of the Ref. [7] was that the expression of the form \( \langle \langle X_{I_n}(\theta_N) \ldots X_{I_1}(\theta_1) \rangle \rangle_\Pi \) satisfies the form factor axioms, if we assume that it is a meromorphic function of the variables \( \theta_1, \ldots, \theta_N \). For \( \theta'_1 < \ldots < \theta'_m \), \( \theta_1 < \ldots < \theta_n \) the \( (m+n) \)-particle form factor of some operator \( \mathcal{O} \) is given by the relation
\[
\langle \langle X_{I_1}, \ldots, X_{I_m} | \mathcal{O}(0)| \theta_1 I_1, \ldots, \theta_n I_n \rangle \rangle 
= N_{\mathcal{O}} \langle \langle Z^{I_I}(\theta'_1 + \frac{i\pi}{2}) \ldots Z^{I_m}(\theta'_m + \frac{i\pi}{2})Z_{I_n}(\theta_n - \frac{i\pi}{2}) \ldots Z_{I_1}(\theta_1 - \frac{i\pi}{2}) \rangle \rangle_\Pi(\mathcal{O}),
\]
where
\[
Z^{I}(\theta) = (C^{-1})^{IJ} Z_J(\theta).
\]
To obtain the form factors of a particular local operator \( \mathcal{O} \) we have to choose appropriately the representation \( \Pi(\mathcal{O}) \) of the algebra (3.1). It is important that in this formulation the form factor axioms in integrable quantum field theory are a direct consequence of the relations (3.1) and the cyclic property of the trace.

The normalization factor \( N_{\mathcal{O}} \) in Eq. (3.3) cannot be established from the form factor axioms and should be calculated by other methods. The value of the central element \( \Omega_I \) on the representation \( \Pi(\mathcal{O}) \) gives the mutual locality index \( e^{2\pi i I_J} \) of the operator \( \mathcal{O} \) and the local field \( V_I(x) \) that creates the particle \( z_I \). It means that in the Euclidean space
\[
\mathcal{O}(e^{2\pi i z}e^{-2\pi i \bar{z}}) V_I(0,0) = e^{2\pi i I_I} \mathcal{O}(z, \bar{z}) V_I(0,0), \quad z = x^1 + ix^2, \quad \bar{z} = x^1 - ix^2.
\]
If the operator \( \mathcal{O} \) has a fixed Lorentzian spin \( s \) the form factors possess an additional homogeneity property:
\[
\langle \langle Z_{I_N}(\theta_N + \vartheta) \ldots Z_{I_1}(\theta_1 + \vartheta) \rangle \rangle = e^{s\vartheta} \langle \langle Z_{I_N}(\theta_N) \ldots Z_{I_1}(\theta_1) \rangle \rangle.
\]
For spinless operators (like the exponential operators, which will be considered below) the form factors are invariant with respect to overall shifts of rapidities and we have
\[
\langle \langle \text{vac} | \mathcal{O}(0)| \theta_1 I_1, \ldots, \theta_N I_N \rangle \rangle = N_{\mathcal{O}} \langle \langle Z_{I_N}(\theta_N) \ldots Z_{I_1}(\theta_1) \rangle \rangle \quad \text{for} \quad s = 0.
\]

The second important observation of the reference [7] is that the extended ZF algebras can be realized in terms of free bosons acting on a completion of a Fock space. Below we describe such a realization for the algebra (3.1) of the model (2.1).

In the case of the model (2.1) we have to introduce the vertex operators \( Z_{z_{\epsilon^I}}(\theta) \) corresponding to the fundamental particles \( z_{\epsilon^I} \). The respective \( S \) matrix of the fundamental particles is \( S_{p_1 p_2}(\theta) \) given by Eqs. (2.12), (2.11) and the charge conjugation matrix is
\[
C_{\epsilon_{1, \epsilon_1}, \epsilon_{2, \epsilon_2}} = \delta_{\epsilon_{1,-\epsilon_2}} \delta_{\epsilon_{1,-\epsilon_2}}.
\]
For the values of the parameters, where the bound states appear, e. g. \( p_1 < 1 \), the matrix elements which contain the bound states corresponding to the poles (2.11) must be taken into account. The respective vertex operators can be expressed in terms of bilinear combinations of the operators for the fundamental particles:
\[
\begin{align*}
B_{+}^{1,n}(\theta) &= -i\Gamma_{1, n}^{-1} \text{Res}_{\theta = \theta} Z_{++}(\theta + \frac{i\pi}{2}) Z_{--}(\theta - \frac{i\pi}{2}), \\
B_{-}^{1,n}(\theta) &= -i(K_{1, n}^{+} T_{1, n}^{-})^{-1} \text{Res}_{\theta = \theta} (Z_{++}(\theta + \frac{i\pi}{2}) Z_{--}(\theta - \frac{i\pi}{2}) + Z_{-+}(\theta + \frac{i\pi}{2}) Z_{+-}(\theta - \frac{i\pi}{2})), \\
B_{+}^{1,n}(\theta) &= -i(K_{1, n}^{+} T_{1, n}^{-})^{-1} \text{Res}_{\theta = \theta} (Z_{++}(\theta + \frac{i\pi}{2}) Z_{--}(\theta - \frac{i\pi}{2}) - Z_{-+}(\theta + \frac{i\pi}{2}) Z_{+-}(\theta - \frac{i\pi}{2})), \\
B_{-}^{1,n}(\theta) &= -i\Gamma_{1, n}^{-1} \text{Res}_{\theta = \theta} Z_{-+}(\theta + \frac{i\pi}{2}) Z_{+-}(\theta - \frac{i\pi}{2}).
\end{align*}
\]
with
\[
\Gamma_{1,n}^2 = \text{Res}_{\theta = i u_{1,n}} (-S_{p_1}(\theta)\Gamma_{\pm} + S_{p_2}(\theta)\Gamma_{\mp}) = (-1)^{n+1} p_1 \sin \frac{\pi}{p_1} \cdot S_{p_1}(i u_{1,n})\Gamma_{\pm} + S_{p_2}(i u_{1,n})\Gamma_{\mp},
\]
\[
K_{S}^{1,n} = \left( 2 \frac{\sin \frac{\pi}{p_2} + \sin \frac{u_{1,n}}{p_2}}{\sin \frac{u_{1,n}}{p_2}} \right)^{1/2}, \quad K_{A}^{1,n} = \left( 2 \frac{\sin \frac{\pi}{p_2} - \sin \frac{u_{1,n}}{p_2}}{\sin \frac{u_{1,n}}{p_2}} \right)^{1/2}.
\]

The nonzero elements of the charge conjugation matrix for any quadruplet of this type read
\[
C_{+-} = C_{-+} = C_{SS} = C_{AA} = 1.
\]

In the case \( p_2 < 1 \) the vertex operators \( B_{S}^{2,n}(\theta), B_{S}^{2,n}(\theta), B_{A}^{2,n}(\theta) \) are defined similarly. From the physical point of view both cases \( p_1 < 1 \) and \( p_2 < 1 \) are equivalent, but the bosonization we develop below is not symmetric with respect to exchange of \( p_1 \) and \( p_2 \). This is why we use both cases below while considering different degenerate limiting cases. The \( S \) matrices involving bound states can be found from Eqs. (3.1) and (5.8). We do not discuss them here in general because of their complicacy. Some important particular cases will be considered in Secs. 4, 5.

Note that the vertex operators \( B_{\nu}^{2,n}(\theta) \) (\( \nu = +, S, A, - \)) given by Eq. (3.8a) is not the unique choice of vertex operators for the bound states. Linear transformations that do not affect the normalization condition (3.1b) do not affect the \( N \)-particle form factor contributions into the correlation functions. We shall use this fact to get rid of cumbersome coefficients in the vertex operators of the sausage model.

Turn our attention to the operator contents of the theory (2.1). A generic operator local with respect to the fields \( \varphi_i(x) \) is a linear combination of the monomials
\[
\partial_{\mu_1}^{k_1} \varphi_{i_1} \ldots \partial_{\mu_N}^{k_N} \varphi_{i_N} e^{i a_1 \varphi_1 + i a_2 \varphi_2 + b \varphi_3}
\]
with arbitrary nonnegative integers \( k_j \). Finding form factors of all these operators is a difficult problem. Here we address a simpler problem of finding form factors for the exponential fields:
\[
O_{a_1,a_2}^{b}(x) = e^{i a_1 \varphi_1(x) + i a_2 \varphi_2(x) + b \varphi_3(x)}.
\]

To make the notations more symmetric it is convenient to introduce an imaginary parameter
\[
a_3 = -ib.
\]

With this notation we have
\[
O_{a_1,a_2,a_3}(x) = O_{a_1,a_2}^{ia_3}(x) = e^{i a_1 \varphi_1(x) + i a_2 \varphi_2(x) + i a_3 \varphi_3(x)}.
\]

The normalization factors for these operators were found in Ref. [16]:
\[
N_{a_1,a_2,a_3} = \left( \frac{\mu}{g} \right)^{\sum a_i^2} \left( \prod_i \gamma \left( \frac{1}{2} + P_i, \gamma \left( \frac{1}{2} + P \right) \right) \right)^{1/2}
\]
\[
\times \exp \int_0^\infty dt \left( \frac{1}{\sinh t} \sum_i \sinh^2 \alpha_i \alpha_i t + \frac{2 \sinh P t \prod_i \sinh P_i t}{\sinh^2 (t/2)} - e^{-t} \sum a_i^2 \right),
\]
where
\[
P = \sum_{i=1}^3 \alpha_i a_i, \quad P_i = P - 2 \alpha_i a_i, \quad i = 1, 2, 3; \quad \gamma(z) = \frac{\Gamma(z)}{\Gamma(1 - z)}.
\]

The products and sums are assumed to be over \( i = 1, 2, 3 \).

Before describing the whole bosonization procedure, we give the resulting expressions for form factors. Consider a trace function \( \langle Z_{a_1} \varphi_1(\theta_N) \ldots Z_{a_3} \varphi_3(\theta_1) \rangle \Pi(O_{a_1,a_2,a_3}) \), which, according to Eqs. (3.9), (5.8), gives the form factors of the operator \( O_{a_1,a_2,a_3}(x) \) for \( N \) fundamental particles in the model (2.1). This function is
nonzero for even $N$ with $U(1)$ charges subject to the neutrality condition $\sum_j \varepsilon_j = \sum_j \varepsilon_j' = 0$. For $N = 2n$
we have

$$\langle Z_{\varepsilon_2,\varepsilon_2', \ldots, \varepsilon_{2n},\varepsilon_{2n}'}(\theta_{2n}) \ldots Z_{\varepsilon_1,\varepsilon_1'}(\theta_1) \rangle_{(\varepsilon_{2j-1,\varepsilon_{2j}})} = \tilde{C}^n G(k_1, k_2; \bar{\theta}) \times \prod_{a_1, \ldots, a_n = \pm 1} \int_{D_1} \frac{d\xi}{2\pi} \prod_{s=1}^{n} \int_{D_2} \frac{d\eta}{2\pi} W_j(k_1, p_1, \bar{\theta}, \xi) W_j'(k_2, p_2, \bar{\theta}, \eta) \tilde{C}^{(A, B)}_{j, j'}(k_1, k_2, k_3; \xi, \eta),$$

(3.15)

where $\bar{\theta} = (\theta_1, \ldots, \theta_{2n})$, while the bold letters $\xi = (\xi_1, \ldots, \xi_n)$ etc. are $n$-tuples. The values $k_1$ (and $\kappa_i$ which appear below) are defined by Eq. (3.3). The numbers $j_1 < \ldots < j_{2n}$ ($j_1' < \ldots < j_{2n}'$) are defined so that $\varepsilon_j = -(\varepsilon_j' = -)$ for $j \in \{j_s\}_{s=1}^{n}$ ($j \in \{j_s'\}_{s=1}^{n}$) and $\varepsilon_j = + (\varepsilon_j' = +)$ otherwise. The functions in the r. h. s. read

$$\tilde{C} = c_1 c_2 G_1 G_3^2,$$

(3.16)

$$G(k_1, k_2; \bar{\theta}) = e^{\frac{1}{2}(k_1 + k_2) \sum_{j=1}^{2n} \theta_j} \prod_{1 \leq j \leq j' \leq 2n} G_{33}(\theta_j - \theta_{j'}),$$

(3.17)

$$W_j(k, p; \bar{\theta}; \xi) = \prod_{s=1}^{n} \frac{(-1)^j \pi e^{-k_\xi_j}}{\cosh x_\xi_j - \theta_j - i\pi/2} \prod_{j=1}^{2n} W(p; \xi_j - \xi_s) \prod_{j=j_s}^{2n} W(p; \xi_s - \theta_j) \right),$$

(3.18)

$$\tilde{C}^{(A, B)}_{j, j'}(k_1, k_2, k_3; \xi, \eta) = \prod_{s=1}^{n} \frac{1}{2}(B_s-A_s) \bar{G}(B_s-A_s)(\xi_s, \eta_s; \xi_s'; \eta_s'),$$

(3.19)

Here the signs $z$ are identified with $\pm 1$. The constants $C_3, \tilde{C}$, and functions $G_{33}(\theta)$, $W(p; \theta)$, $\tilde{C}^{(A, B)}_{j, j'}(\theta)$ can be found in Appendix [1]. The constants $c_i$ are given by Eq. (3.36). The integration contours $D_1 = D_s(A, B)$ in Eq. (3.14) depend on summation term and are chosen as follows. The contour $D_s$ goes from $-\infty$ to $+\infty$ below the poles of the integrand at the points $\theta_j - \frac{i\pi}{2}(p_1 - 1 + 2Mp_1) + 2\pi iN$ ($M, N = 0, 1, 2, \ldots$) and above the poles at $\theta_j - \frac{i\pi}{2}(p_1 - 1 - 2Mp_1) - 2\pi iN$. Note that the poles at $\theta_j + \frac{i\pi}{2}(p_1 - 1 - 2Mp_1)$ are absent for $j < j_s$ and the poles at $\theta_j - \frac{i\pi}{2}(p_1 - 1 - 2Mp_1)$ are absent for $j > j_s$. The contour $D_2$ goes above the points $\xi_{s'} - i\pi - 2\pi iN$ and below the points $\xi_{s'} + i\pi + 2\pi iN$, $A_s = A_s'$. A similar rule is valid for $D_2$. Besides, the contour $D_2$ goes below the poles at $\xi_{s'} -\frac{i\pi}{2}(p_1 + p_2) + 2\pi iN$ and above the poles at $\xi_{s'} -\frac{i\pi}{2}(p_1 + p_2) - 2\pi i(N + 1)$, if $B_s = A_s' +$, and below the poles at $\xi_{s'} +\frac{i\pi}{2}(p_1 + p_2) + 2\pi iN$ and above the poles $\xi_{s'} +\frac{i\pi}{2}(p_1 + p_2) - 2\pi i(N + 1)$, if $B_s = A_s' -$.

Now we describe the construction of the form factors (3.15)–(3.19) based on application of the bosonization procedure. Up to now, there is no general way to derive bosonization directly from the model and we need to guess it. So we introduce the construction in a mathematical manner, mostly without substantiations. The definitions will be justified by the check of the relations (3.11) for the final boson vertex operators (3.35). As the construction is based on Konno’s bosonization for the $U_{q, p}(sl_2)$ algebra [20], some motivations can be found in his paper.

Consider the boson operators $a_i(t)$ ($i \in \mathbb{Z}_3$) that depend on the real parameter $t$ and satisfy the commutation relations

$$[a_i(t), a_j(t')] = t \frac{\sinh^2 \frac{\pi t}{4}}{\sinh \frac{\pi t}{2}} \delta(t + t') \delta_{ij}.$$ 

(3.20)

It is useful to introduce the fields

$$\phi_i(\theta; v) = \int_{-\infty}^{\infty} dt a_i(t) e^{i\theta + i\pi v |t|/4},$$

(3.21a)

$$\phi_i(\theta; v) = \int_{-\infty}^{\infty} dt \frac{\pi}{2} a_i(t) e^{i\theta + i\pi v |t|/4},$$

(3.21b)
\[ \phi_i^{(\pm)}(\theta; v) = 2 \int_0^\infty \frac{dt}{it} \sinh \frac{\pi pt}{2} a_i(\pm t)e^{\pm i\theta v + i\pi v t/4}. \] (3.21c)

We shall also use the notation
\[ \chi_i^{(\pm)}(\theta) = \phi_i^{(\pm)}(\theta; 2 - p_i) + \phi_{i+1}^{(\pm)}(\theta; p_{i+1} - 2) - \phi_{i+2}^{(\pm)}(\theta; p_{i+1} - p_i). \] (3.21d)

In the expressions below we encounter the integrals
\[ \int_0^\infty dt f(t) \]
with \( f(t) \) having a pole at \( t = 0 \). We shall understand this integral as [24]
\[ \int_{C_0} \frac{dt}{2\pi i} f(t) \log(-t) \]
with the contour \( C_0 \) going from \( +\infty \) to \( i0 \) above the real axis, then around zero, and then below the real axis to \( +\infty \) below \( i0 \). In fact, we shall use several simple formulas listed in the Appendix A.

The Fock space is defined by the vacuum \( |0\rangle \):
\[ a_i(t)|0\rangle = 0 \quad \text{for} \quad t \geq 0. \] (3.22)
as the space spanned by the vectors
\[ a_i(-t_1) \ldots a_{i_n}(-t_n)|0\rangle, \quad t_1, \ldots, t_n > 0, \quad n = 0, 1, 2, \ldots. \]
This defines the evident normal ordering \( \ldots \). In what follows all traces are understood as traces over this Fock space.

As building blocks for the generators of the algebra \( \mathfrak{g}_{\mathfrak{su}}(1,1) \) we introduce the operators
\[ V_i(\theta) = \exp(\imath \phi_{i+1}(\theta; p_{i+1}) + \imath \phi_{i+2}(\theta; -p_{i+2})i); \] (3.23a)
\[ I_i^{(\pm)}(\theta) = \exp(-\imath \tilde{\phi}_i(\theta; p_i) \pm \imath \tilde{\phi}_i^{(\pm)}(\theta)); \] (3.23b)

These operators satisfy the following relations:
\[ V_i(\theta')V_j(\theta) = g_{ij}(\theta - \theta')V_i(\theta)V_j(\theta'); \] (3.24a)
\[ V_i(\theta')I_j^{(\pm)}(\theta) = w_{ij}^{(\pm)}(\theta - \theta')V_i(\theta')I_j^{(\pm)}(\theta'); \] (3.24b)
\[ I_i^{(\pm)}(\theta')V_i(\theta) = w_{ij}^{(\pm)}(\theta - \theta')V_i(\theta)I_j^{(\pm)}(\theta'); \] (3.24c)
\[ I_i^{(A)}(\theta')I_j^{(B)}(\theta) = \tilde{g}_{ij}^{(A\bar{B})}(\theta - \theta')I_i^{(A)}(\theta')I_j^{(B)}(\theta); \] (3.24d)

The functions \( g_{ij}(\theta) \) are defined as follows \((i, j) \) are understood modulo 3:
\[ g_{ii}(\theta) = G^{-1}(p_{i+1}, \theta)G^{-1}(p_{i+2}, \theta), \quad G(p, \theta) = \exp \int_0^{\infty} \frac{dt}{t} \frac{\sinh^2 \frac{\pi pt}{2}}{\sinh \pi t \sinh \frac{\pi pt}{2}} e^{-i\theta t}, \] (3.25a)
\[ g_{ij}(\theta) = G^{-1}_1(p_k, \theta), \quad (i \neq j, k \neq i, j), \quad G_1(p, \theta) = \exp \int_0^{\infty} \frac{dt}{t} \frac{\sinh \frac{\pi pt}{2}}{\sinh \pi t \sinh \frac{\pi pt}{2}} e^{-i\theta t}. \] (3.25b)

The functions \( w_{ij}^{(\pm)}(\theta) \) can be expressed in terms of the gamma-functions:
\[ w_{ij}^{(\pm)}(\theta) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (\cos \theta)^k, \] (3.26a)
\[ w_{i-1,i}^{(+)}(\theta) = w(p_i, 0|\theta), \quad w_{i-1,i}^{(-)}(\theta) = w(p_i, 1|\theta), \] (3.26b)

\(^4\)As our goal is calculation of traces, there is no need to introduce bra-vectors nor to discuss unitarity of the representations.
\[ w_{i+1,i}^{(+)}(\theta) = w_{i+1,i}^{(-)}(\theta) = w(p_i, 1/2|\theta), \]

where

\[ w(p, z|\theta) = r_p^{-1} \frac{\Gamma\left(\frac{N}{p} + \frac{1}{p} + z\right)}{\Gamma\left(\frac{N}{p} + \frac{1}{p} + \frac{1}{2}\right)} \quad r_p = e^{(C_E + \log \pi_p)/p} \]

with \( C_E \) being the Euler constant. Note, that all these functions have one series of poles at the points \( \theta = -i\pi + i\pi p n \) or \( \theta = -i\pi + i\pi p (n+1/2) \) \( (n = 0, 1, 2, \ldots) \) and one series of zeros at the points \( \theta = i\pi - i\pi p n \) or \( \theta = i\pi - i\pi p (n+1/2) \).

The functions \( \tilde{g}_{ij}^{(AB)}(\theta) \) \( (A, B = \pm) \) read

\[ \tilde{g}_{ii}^{(+)}(\theta) = \tilde{g}(p_i, 0, 0|\theta), \quad \tilde{g}_{ii}^{(-)}(\theta) = \tilde{g}_{ii}^{(+)}(\theta) = \frac{\imath\theta}{\pi p_i} \tilde{g}(p_i, 0, 1|\theta), \quad \tilde{g}_{i,i+1}^{(\pm)}(\theta) = \tilde{g}(p_i, 1, 1|\theta), \]

\[ \tilde{g}_{i,i+1}^{(-)}(\theta) = \tilde{g}_{i+1,i}^{(-)}(\theta) = 1, \quad \tilde{g}_{i,i+1}^{(+)}(\theta) = \frac{\theta - i\pi (p_{i+1} - 2)/2}{\theta - i\pi p_{i+1}/2}, \quad \tilde{g}_{i+1,i}^{(AB)}(\theta) = \tilde{g}_{i,i}^{(BA)}(\theta) \]

with

\[ \tilde{g}(p, z_1, z_2|\theta) = r_p^{-2} \frac{\Gamma\left(\frac{N}{p} + \frac{1}{p} + z_1\right)}{\Gamma\left(\frac{N}{p} - \frac{1}{p} + z_2\right)}. \]

The functions \( \tilde{g}_{i,i+\pm}^{(AB)}(\theta) \) are rational and provide commutativity of the operators \( I_i^{(A)} \) and \( I_j^{(B)} \) with \( i \neq j \). It will be important later that the residues at the only poles of the products \( I_i^{(\pm)}(\theta) I_{i+1}^{(\pm)}(\theta') \) satisfy the relation

\[ \text{Res}_{\theta' = \theta - i\pi p_{i+1}} I_i^{(+)}(\theta) I_{i+1}^{(+)}(\theta') = - \text{Res}_{\theta' = \theta + i\pi p_{i+1}} I_i^{(-)}(\theta) I_{i+1}^{(-)}(\theta') \]

due to the important identity for the normal products

\[ :I_i^{(+)}(\theta) I_{i+1}^{(+)}(\theta - i\pi p_{i+1}) = I_i^{(-)}(\theta) I_{i+1}^{(-)}(\theta + i\pi p_{i+1}):. \]

There is a general rule concerning positions of poles, which appear in the operator products. Let \( U_j(\theta) \) be normal ordered exponentials, like, for example, the operators \( V_1(\theta) \) or \( I_1^{(\pm)}(\theta) \). Then the product \( U_k(\theta_k)U_j(\theta_j) \) can be reduced to the normal form

\[ U_k(\theta_k)U_j(\theta_j) = g_{U_kU_j}(\theta_j - \theta_k):U_k(\theta_k)U_j(\theta_j):. \]

The set of the poles of the function \( g_{U_kU_j}(\theta_j - \theta_k) \) in the variable \( \theta_j \) is bounded in the lower half plane, i. e. the imaginary parts of all poles are greater than some constant. On the contrary, the set of the poles in the variable \( \theta_k \) is bounded in the upper half plane. The same is true for any product \( U_N(\theta_N) \ldots U_1(\theta_1) \): the set of poles in the variable \( \theta_j \) arising due to any operator \( U_k(\theta_k) \) with \( k > j \) \( (U_k \) is to the left of \( U_j) \) is bounded in the lower half plane, while the set of poles related to an operator \( U_k(\theta_k) \) with \( k < j \) \( (U_k \) is to the right of \( U_j) \) is bounded in the upper half plane.

As a result of the relations \( 3.24-3.27 \), the commutation relations of the operators introduced above are

\[ V_i(\theta_1) V_i(\theta_2) = - S_{p_{i+1}}(\theta_1 - \theta_2) V_i(\theta_1) V_i(\theta_2), \]

\[ V_i(\theta_1) V_{i+1}(\theta) = G_1(p_{i+2}; \theta_1 - \theta_2) V_{i+1}(\theta) V_i(\theta_1), \]

\[ V_i(\theta_1) I_{i+1}^{(A)}(\theta) = \frac{\sinh \frac{\theta_2 - \theta_1 - i\pi/2}{p}}{\sinh \frac{\theta_2 - \theta_1 + i\pi/2}{p}} I_{i+1}^{(A)}(\theta) V_i(\theta_1), \]

\[ V_i(\theta_1) I_{i+1}^{(A)}(\theta) = \frac{\cosh \frac{\theta_2 - \theta_1 - i\pi/2}{p}}{\cosh \frac{\theta_2 - \theta_1 + i\pi/2}{p}} I_{i+1}^{(A)}(\theta) V_i(\theta_1), \]
Fig. 1. Integration contours in the screening operators for different products that appear in Eqs. (3.30).

\[
I_i^{(A)}(\theta_1)I_{i+1}^{(B)}(\theta_2) = \frac{\sinh \frac{\theta_1 - \theta_2 + i\pi}{p}}{\sinh \frac{\theta_1 - \theta_2 - i\pi}{p}} I_i^{(B)}(\theta_2)I_i^{(A)}(\theta_1), \quad (3.30c)
\]

\[
I_i^{(A)}(\theta_1)I_i^{(B)}(\theta_2) = I_i^{(B)}(\theta_2)I_i^{(A)}(\theta_1). \quad (3.30f)
\]

Note that the commutation relations are completely independent of the values \((\pm)\) of the superscripts.

We can see from the relation \((3.30c)\) that the operator \(V_3(\theta)\) is a good candidate for the operator \(Z_{++}(\theta)\). However, it is known that the ZF algebras corresponding to nondiagonal \(S\) matrices cannot be represented in terms of pure exponentials of free fields. It is necessary to consider expressions containing integrals of normal exponentials of free fields over the spectral variable \(\theta\). These integrals first appeared in CFT [8], where they screened the total charge in the auxiliary Coulomb gas and were called screening operators. In particular, in CFT they are necessary to provide the braiding relations for the vertex operators, which are CFT analogs of the ZF operators.

In our case the analysis of the commutation relations for the vertex operators gives us the reasons to guess the following form for the screening operator:

\[
S_i(k, \kappa|\theta) = c_i \int_{C_i} \frac{d\xi}{2\pi i} I_i^{(+)}(\xi)e^\kappa - iI_i^{(-)}(\xi)e^{-\kappa}) \frac{\pi e^{-k\xi}}{\sinh \frac{\xi - \theta - i\pi/2}{p}}, \quad (3.31)
\]

with some normalization constants \(c_i\), which will be determined later. The contour \(C_i\) in this equation goes from \(-\infty\) to \(+\infty\) above the pole at the point \(\theta + i\pi/2\). As for the poles related to other operators, the contour goes below all poles arising due to the operators standing to the left of the screening operator \(S_i\) and above the poles related to the operators standing to the right of \(S_i\). This is possible due to the remark concerning the position of the poles, which is formulated above after Eq. (3.26d).

In what follows the screening operators appear in the combinations \(V_{i-1}(\theta)S_i(k, \kappa|\theta)\) and \(V_{i+1}(\theta + i\pi p_i/2)S_i(k, \kappa|\theta)\). In the first product the contour \(C_i\) goes from \(-\infty\) to \(+\infty\) above the poles at the points \(\theta + \frac{i\pi}{2} - i\pi p_i n (n = 0, 1, 2, \ldots)\) and below those at the points \(\theta - \frac{i\pi}{2} + i\pi p_i n\). In the second product it goes above the poles at \(\theta + \frac{i\pi}{2} - i\pi p_i n\) and below those at \(\theta - \frac{i\pi}{2} + i\pi p_i (n + 1)\). It is important to note that there is no real inflection of the contours between the poles at \(\theta + i\pi/2\) and \(\theta - i\pi/2\) in the first product, because there is really only one of these two poles in each term: for \(I^{(+)}(\xi)\) a pole may appear at \(\theta - i\pi/2\) and for \(I^{(-)}(\xi)\) at \(\theta + i\pi/2\). In fact, the contour must be chosen separately for each term of the considered operator product (see Fig. 1).
The meaning of the screening operator \( S_i \) is to change the topological number \( Q_i \) by \(-2\). Each screening operator \( S_i \) depend on its own pair of parameters \( k_i \) and \( \kappa_i \). We shall see that, subject to some additional conditions, their values do not affect the commutation relations of the vertex operators.

As the \( S \) matrix factorizes into a tensor product and the screening operators change the topological numbers, different screening operators must commute:

\[
[S_i(k_i, \kappa_i|\theta_1), S_j(k_j, \kappa_j|\theta_2)] = 0, \quad i \neq j.
\]  

(3.32)

This condition imposes some relations on the parameters \( k_i, \kappa_i \).

Since the screening currents \( I_{ij}^{(\pm)}(\theta) \) commute according to Eq. (3.30), the only obstacle to commutativity of the screening operators is the poles of the functions \( g_{ij}^{(+)}(\theta) \) and \( g_{ij}^{(-)}(\theta) \). Due to these poles, we have

\[
[S_1(k_1, \kappa_1|\theta_1), S_2(k_2, \kappa_2|\theta_2)] = \int \frac{d\xi}{2\pi i} \left( e^{\kappa_1 + \kappa_2 - k_1 \xi - k_2 (\xi - \frac{\pi p_2}{2})} I_1^{(+)}(\xi) I_2^{(+)}(\xi - \frac{\pi p_2}{2}) - e^{-\kappa_1 - \kappa_2 - k_1 \xi - k_2 (\xi + \frac{\pi p_2}{2})} I_1^{(-)}(\xi) I_2^{(-)}(\xi + \frac{\pi p_2}{2}) \right) \frac{\pi^3 c_1 c_2}{\sinh \frac{\pi}{2p_1} \cosh \frac{\pi}{2p_2}}.
\]

As, according to Eq. (3.30), the normal products in the right hand side coincide, the commutativity relation (3.32) holds if

\[
e^{\kappa_1 + \kappa_2 - k_1 \xi - k_2 (\xi - \frac{\pi p_2}{2})} = e^{-\kappa_1 - \kappa_2 - k_1 \xi - k_2 (\xi + \frac{\pi p_2}{2})}.
\]

To fulfill this equation we can take

\[
k_1 + k_2 = -\frac{\pi}{2} p_2 k_2,
\]

and, by cyclic permutations,

\[
k_2 + k_3 = -\frac{\pi}{2} p_3 k_3, \quad k_1 + k_3 = -\frac{\pi}{2} p_1 k_1.
\]

Solving these equations, we get

\[
k_i = -\frac{i\pi}{4} (p_i k_i + p_{i+1} k_{i+1} - p_{i+2} k_{i+2}).
\]  

(3.33)

In what follows we assume \( \kappa_1, \kappa_2, \kappa_3 \) to be the functions of \( k_1, k_2, k_3 \), given by (3.33).

Now we are ready to express the vertex operators and the corner Hamiltonian in terms of the boson operators \( a_i(t) \). To do this we introduce an auxiliary algebra generated by two elements \( \omega \) and \( \rho \) with the relations

\[
\omega^2 = \rho^2 = 1, \quad \omega \rho = -\rho \omega, \quad \text{Tr} \rho = \text{Tr} \omega = 0.
\]  

(3.34)

Then the generators of the algebra (3.3) for the model (2.1) can be represented in the form

\[
H = \int_0^\infty dt \sum_{i=1}^3 \frac{\sin \pi t}{\sinh \frac{\pi t}{2}} a_i(-t) a_i(t),
\]

(3.35a)

\[
Z_{++}(k_1, k_2, k_3|\theta) = \omega V_3(\theta) e^{(k_1 + k_2)\theta/2},
\]

(3.35b)

\[
Z_{--}(k_1, k_2, k_3|\theta) = \omega \rho V_3(\theta) S_1(k_1, k_1|\theta) e^{(k_1 + k_2)\theta/2},
\]

(3.35c)

\[
Z_{+-}(k_1, k_2, k_3|\theta) = -\omega \rho V_3(\theta) S_2(k_2, k_2|\theta - \frac{\pi p_2}{2}) e^{(k_1 + k_2)\theta/2},
\]

(3.35d)

\[
Z_{-+}(k_1, k_2, k_3|\theta) = -\omega V_3(\theta) S_1(k_1, k_1|\theta) S_2(k_2, k_2|\theta - \frac{\pi p_2}{2}) e^{(k_1 + k_2)\theta/2}.
\]

(3.35e)

These operators satisfy the commutation relations (3.3a) and (3.3c). In addition, if the normalization constants are given by

\[
c_i = -\frac{e^{2(C_E + \log p_i)/p_i}}{\pi^{3/2}} \frac{\Gamma(1 + 1/p_i)}{\Gamma(-1/p_i)} G(p_i, -i\pi),
\]  

(3.36)

---

5The charge \( Q_i \) will appear in Sec. 4 when we will consider other regions of the parameters of the model (2.1).

6Other solutions do not lead to physically different results.
they satisfy the normalization condition \(3.1b\). The proofs of the commutation relations and normalization condition is rather standard and are presented in Appendix E.

The expressions for the operators \(\text{3.35}\) define a free field realization of the algebra \(\text{3.1}\) for our model. This realization reduces the problem of construction of the integral representation for form factors to application of the Wick theorem. For arbitrary free field exponents (e. g. \(V(\theta), I_i(\pm)(\theta)\)) of the form

\[
U_j(\theta) = e^{\phi_i(\theta)} = e^{\phi_j^+(\theta)}e^{\phi_j^-(\theta)},
\]

\[
\phi_j^+(\theta) = \sum_i \int_0^\infty dt \, A_j(\pm t) a_i(\pm t)e^{\pm i\theta t},
\]

\[
\phi_j(\theta) = \phi_j^+(\theta) + \phi_j^-(\theta),
\]

the Wick theorem gives

\[
\langle \langle U_1(\theta_1) \ldots U_n(\theta_n) \rangle \rangle = \prod_{j=1}^n C_{U_j} \prod_{j<k} G_{U_j U_k}(\theta_k - \theta_j)
\]

with

\[
\log C_{U_j} = \langle \langle \phi_j^-(0)\phi_j^+(0) \rangle \rangle,
\]

\[
\log G_{U_j U_k}(\theta) = \langle \langle \phi_j(0)\phi_k(\theta) \rangle \rangle.
\]

The traces \(\text{3.39}\) can be easily calculated. The values of \(C_{U_j}, G_{U_j U_k}(u)\) for the operators, introduced in the paper, are listed in the Appendix E. The expressions \(\text{3.1b}, \text{3.19}\) are obtained from these formulas after the substitution \(I_i(\pm)(\xi) \rightarrow I_i(\pm)(\xi + i\pi p_1/2)\). These shifts of the integration variables allow one to deform two first contours depicted in Fig. 4 to the single straight contour like the third one in Fig. 4.

We have a three-parametric family of form factors labeled by \(k_1, k_2, k_3\). Now we have to identify them with a three-parametric family of local operators in the Lagrangian formulation. It will be done in the next section.

4. Identification of form factors

Due to the formal symmetry of the action \(2.1\) with respect to the substitutions \((\varphi_i, \alpha_i) \leftrightarrow (\varphi_j, \alpha_j)\), the model is unitary in three regions:

\[
I_1: \quad p_1 < 0, \quad p_2, p_3 > 0;
\]

\[
I_2: \quad p_2 < 0, \quad p_1, p_3 > 0;
\]

\[
I_3: \quad p_3 < 0, \quad p_1, p_2 > 0.
\]

What we considered above, while considering the regime I, was the region \(I_4\).

In the region \(I_1\), the operators corresponding to the fundamental particle is defined in terms of the operator \(V_i\). Namely, we can introduce the vertex operators (recall that \(i \in \mathbb{Z}_3\))

\[
Z_{++}(k_1, k_2, k_3|\theta) = \omega V_i(\theta)e^{(k_{i+1} + k_{i+2})\theta/2},
\]

\[
Z_{+-}(k_1, k_2, k_3|\theta) = \omega p_Vi(\theta)S_{i+1}(k_{i+1}, k_{i+1} + k_{i+2}|\theta)e^{(k_{i+1} + k_{i+2})\theta/2},
\]

\[
Z_{-+}(k_1, k_2, k_3|\theta) = -\omega p_Vi(\theta)S_{i+1}(k_{i+2}, k_{i+2} + k_{i+2}|\theta - \frac{i\pi p_1}{2})e^{(k_{i+1} + k_{i+2})\theta/2},
\]

\[
Z_{--}(k_1, k_2, k_3|\theta) = -\omega V_i(\theta)S_{i+1}(k_{i+1}, k_{i+1} + k_{i+2}|\theta)S_{i+2}(k_{i+2}, k_{i+2} + k_{i+2}|\theta - \frac{i\pi p_1}{2})e^{(k_{i+1} + k_{i+2})\theta/2}.
\]

Evidently, in the previous section we constructed the generators \(Z_{\pm\pm}'(k_1, k_2, k_3|\theta)\). The vertex operators \(Z_{\pm\pm}'(\theta)\) for a given \(i\) satisfy the relations \(\text{3.1}\) with the \(S\) matrix \(-S_{p+1}(\theta) \otimes S_{p+2}(\theta)\). The topological charges in these regions are given by the same formula \(\text{2.1}\) for \(Q_i\), but without the limitation to \(i = 1, 2\).

We have a three-parametric family of form factors. We expect that they are in a one-to-one correspondence with the exponential operators \(\text{3.13}\). Our main conjecture is that the correspondence between \(a_1, a_2, a_3\) and \(k_1, k_2, k_3\) analytically depends on the parameters \(p_1, p_2, p_3\) for any values of these parameters including all three regions \(I_1, I_2, I_3\). This analyticity conjecture becomes more natural, if we consider the intermediate region of parameters, corresponding to the regime \(I_4\), where all three families of vertex operators coexist (see Appendix F).
Now we want to exploit the mutual locality indices, introduced in Eq. (3.4). For this purpose we need operators that create the fundamental particles \( z_{\varepsilon'}. \) Such operators are given by

\[
\mathcal{V}_{\varepsilon\varepsilon'}^i(x) = \mathcal{O}_{\varepsilon\varepsilon'}^i(x) \exp \left( \frac{i \varepsilon \hat{\phi}_{i+1}(x)}{4\alpha_{i+1}} + \frac{i \varepsilon' \hat{\phi}_{i+2}(x)}{4\alpha_{i+2}} \right)
\]

with \( \hat{\phi}_i(x) \) being the dual field to \( \varphi_i(x), \partial^\mu \hat{\phi}_i(x) = \varepsilon^\mu \partial^i \varphi_i(x) \) and \( \mathcal{O}_{\varepsilon\varepsilon'}^i(x) \) being an operator mutually local with the operators \( \varphi_i(x). \) Indeed, consider the commutation relations \([Q_j, \mathcal{V}_{\varepsilon\varepsilon'}^i(x)]\) in the Euclidean plane. We have

\[
[Q_j, \mathcal{V}_{\varepsilon\varepsilon'}^i(x)] = -\frac{\alpha_i}{\pi} \oint dy^\mu \partial_\mu \varphi_j(y) \mathcal{V}_{\varepsilon\varepsilon'}^i(x).
\]

The last integral can be taken over a small circle around the point \( x, \) where we can neglect the interaction term in Eq. (2.1) and consider the model as that of three free bosons. It can be easily taken using the complex variables \( z = x^1 + i x^2, \bar{z} = x^1 - i x^2 \) with the result

\[
[Q_{i+1}, \mathcal{V}_{\varepsilon\varepsilon'}^i(x)] = \varepsilon \mathcal{V}_{\varepsilon\varepsilon'}^i(x), \quad [Q_{i+2}, \mathcal{V}_{\varepsilon\varepsilon'}^i(x)] = \varepsilon' \mathcal{V}_{\varepsilon\varepsilon'}^i(x).
\]

It means that this operator generally creates from the vacuum all states with topological charges \( Q_{i+1} = \varepsilon, \) \( Q_{i+2} = \varepsilon' \) including the one-particle ones.

The mutual locality index of the operator \( O_{a_1a_2a_3}(x) \) with the operator \( \mathcal{V}_{\varepsilon\varepsilon'}^i(x) \) is equal to

\[
e^{2\pi i \Omega_{\varepsilon\varepsilon'}^i} = e^{i\pi \left( a_{i+1}/\alpha_{i+1} + \varepsilon a_{i+2}/\alpha_{i+2} \right)}.
\]

On the other hand, using Eqs. (5.16), it is possible to calculate the mutual locality index of the operator \( \mathcal{V}_{\varepsilon\varepsilon'}^i(x) \) and the operator that corresponds to the form factors defined by the generators (4.2) with given \( k_1, k_2, k_3: \)

\[
e^{2\pi i \Omega_{\varepsilon\varepsilon'}^i} = e^{i\pi \left( \varepsilon k_{i+1} + \varepsilon' k_{i+2} \right)}.
\]

Hence, we conjecture that the form factors of the operator \( O_{a_1a_2a_3}(x) \) in the region \( I_3 \) are given by the traces of the operators (3.35) with

\[
k_i = \frac{a_i}{\alpha_i}, \quad \kappa_i = -\frac{i}{4} \left( p_i \frac{a_i}{\alpha_i} + p_{i+1} \frac{a_{i+1}}{\alpha_{i+1}} - p_{i+2} \frac{a_{i+2}}{\alpha_{i+2}} \right).
\]

5. Sausage Model

In the limit \( \alpha_2 \to 0, \) the action (2.1) splits into two parts

\[
S[\varphi_1, \varphi_2, \varphi_3] = S_{SM}[\varphi_1, \varphi_3] + \int d^2x \frac{(\partial_\mu \varphi_2)^2 - (2\mu)^2 \varphi_3^2}{8\pi},
\]

where \( S_{SM} \) is the action of the sausage model in the dual representation:

\[
S_{SM}[\varphi, \chi] = \int d^2x \left( \frac{(\partial_\mu \varphi)^2 + (\partial_\mu \chi)^2}{8\pi} + \frac{2\mu}{\pi} \cos \alpha \varphi \cosh \beta \chi \right), \quad \alpha^2 - \beta^2 = 1
\]

with \( \alpha = \alpha_1. \) The mass term for the field \( \varphi_2(x) \) in Eq. (1.31) appears in the limit \( p_2 \to 0 \) due to the quantum corrections (see Ref. [15] for details).

On the level of the factorized scattering theory the limit \( p_2 \to 0 \) must be performed as follows. According to Eq. (2.8), (2.12) the mass of the fundamental particles \( m \) tends to infinity, while the masses of bound states remain finite. Hence, the fundamental particles decouple. The mass of the \( n \)th bound state is \( 2\mu n, \) which means that \( n \)th bound state is simply \( n \) the first bound state particles with the same rapidity. It means that the model in this limit is completely described by the first bound state particles.
Consider the first bound state of the particles \( z_{s_1^+} \) and \( z_{s_2^-} \). This bound state forms a quadruplet. Its scattering matrix is given by the diagram

\[
\begin{array}{c}
\theta_1 - \frac{i\pi}{p}(1-p_2) \\
\theta_1 + \frac{i\pi}{p}(1-p_2) \\
\theta_2 - \frac{i\pi}{p}(1-p_2) \\
\theta_2 + \frac{i\pi}{p}(1-p_2)
\end{array}
\]

The values of the rapidities are written near the arrows. For \( p_2 = 0 \) the well known fusion phenomenon takes place. The quadruplet \( V^2 \otimes V^2 \) splits into a triplet \( V^3 \) and a singlet \( V^1 \). The \( S \) matrix acts on the spaces \( V^1 \otimes V^1, V^1 \otimes V^3 \), and \( V^3 \otimes V^1 \) trivially and on the space \( V^3 \otimes V^3 \) as a nontrivial \( S \) matrix. It means that the singlet \( Z_0^0(\theta) \) describes a free boson particle \( \varphi_2(x) \) and can be decoupled. The remaining triplet \( Z_1(\theta) \) (where the subscript \( I = +, 0, - \) labels the particles with the charge \( Q_I = 2, 0, -2 \) respectively) corresponds to particles with the mass \( M \) related to the parameter \( \mu \) in the action as

\[ M = 2\mu. \tag{5.3} \]

Scattering of these particles is described by the \( S \) matrix of the sausage model \( SM \) [17] with

\[ \lambda = \frac{1}{p} = \frac{1}{2\alpha^2}. \tag{5.4} \]

The \( S \) matrix of the sausage model has the form [17,25]

\[
S(\theta)^{++} = \frac{\sinh \frac{2-\pi}{p} \theta}{\sinh \frac{2+\pi}{p} \theta},
\]

\[
S(\theta)^{++}_{0+} = -i \frac{\sin 2\pi_p}{\sin \frac{2\pi}{p}} S(\theta)^{++}, \quad S(\theta)^{++}_{++} = \frac{\sinh \frac{2+\pi}{p} \theta}{\sinh \frac{2-\pi}{p} \theta} S(\theta)^{++},
\]

\[
S(\theta)^{--} = - \frac{\sin \frac{\pi}{p} \sin \frac{2\pi}{p}}{\sin \frac{2\pi}{p} \sin \frac{2\pi}{p}} S(\theta)^{++}, \quad S(\theta)^{--} = \frac{\sinh \frac{2-\pi}{p} \theta}{\sinh \frac{2+\pi}{p} \theta} S(\theta)^{++}, \tag{5.5}
\]

\[
S(\theta)^{++}_{+,0} = S(\theta)^{00} = S(\theta)^{++} + S(\theta)^{--},
\]

\[
S(\theta)^{00}_{11} = S(\theta)_{I_1 I_2} = S(\theta)_{I_1} S(\theta)_{I_2}.
\]

and the charge conjugation matrix is

\[ C_{IJ} = \delta_{-I,-J}. \tag{5.6} \]

To obtain the generators of the algebra [81] for the sausage model we have to consider the first bound state operators \( B_{\nu_1}^1(\theta) (\nu = +, S, A, -) \) for \( p_2 < 1 \) and then take the limit \( p_2 \to 0 \). The operators \( B_{\nu}^n(\theta) \) can be obtained from the operators \( B_{\nu}^1(\theta) \) defined in Eq. [5.8] by the substitutions \( p_1 \leftrightarrow p_2, u_{1,n} \to u_{2,n} \) and \( Z_{\nu_2}(\theta) \to Z_{\nu_2}(\theta) \).

In the limit \( p_2 \to 0 \) we easily obtain

\[
\Gamma_{2,1} = \frac{2}{\sqrt{p_1 \sin \theta}}, \quad K_{S}^{2,1} \sim 2p_2^{-1/2} \sqrt{\frac{p_1 \sin \theta}{p_1}}, \quad K_{A}^{2,1} \to \sqrt{\frac{2\cos \theta}{p_1}}. \tag{5.7}
\]

We see that the coefficients at \( B_{++}^{2,1}(\theta), B_{-+}^{2,1}(\theta), \) and \( B_{-+}^{2,1}(\theta) \) in equations analogous to Eq. [5.8] remain finite. These three operators form the triplet. The coefficients at the remaining operator \( B_{S}^{2,1}(\theta) \) is singular. This is just the singlet operator.
We note that the contributions of the oscillators $a_1(t)$ and $a_3(t)$ to the vertex operators $B^{2,1}_S$ corresponding to the singlet particle vanish, and this operator can be written as

$$B^{2,1}_S(\theta) = Z'_0(a_2|\theta) = \int_{-\infty}^{\infty} dt \, \hat{a}(t)e^{i\theta t} + iv2\pi a_2$$

with the parameter $a_2$ defined in Eq. (3.13) and the operators $\hat{a}(t)$ related with the operators $a_2(t)$ as

$$\hat{a}(t) = -\frac{\sinh \pi t}{\sinh \pi t/2} \sqrt{2\sinh \frac{\pi at}{2}} - a_2(t), \quad [\hat{a}(t), \hat{a}(t')] = 2\sinh \pi t \delta(t + t').$$

The corresponding traces consisting of the operators $Z'_0(\theta)$ are given by

$$\langle Z'_0(\theta_m + i\pi/2) ... Z'_0(\theta_1 + i\pi/2) Z'_0(\theta_1 - i\pi/2) ... \rangle \rightarrow \left\{ (i\sqrt{2\pi} a_2)^{m+n} + (\delta\text{-functional terms)} \right\}$$

They reproduce the form factors of a free field. Indeed, for a free boson field with the action

$$S[\varphi] = \int \frac{d^2x}{8\pi} \left( (\partial^2 \varphi)^2 - M^2 \varphi^2 \right)$$

we have

$$\langle \theta_1, ..., \theta_m | e^{i\varphi(0)} | \theta_1, ..., \theta_n \rangle = \sum_{k=0}^{\min(m,n)} (i\sqrt{2\pi} a)^{m+n-2k} \prod_{1 \leq i_1 < ... < i_k \leq m} \prod_{1 \leq j_1 < ... < j_k \leq n} 2\pi \delta(\theta_{i_1} - \theta_{j_1})$$

$$= (i\sqrt{2\pi} a)^{m+n} + (\delta\text{-functional terms}).$$

It is easy to see that the $\delta$-functional terms (which correspond to nonconnected diagrams) in Eq. (5.10) are the same as in Eq. (5.12).

This provides a check for our construction. The consideration in Sec. 4 does not distinguish between primary (exponential) fields (3.13) and zero spin linear combinations of their descendants (3.10). The result (6.10), as well as other free field cases described below, provide an evidence in favor of the identification of the form factors just with the primary operators.

In the triplet operators $B^{2,1}_S(\theta)$ ($\nu = +, A, -$) the contribution of the oscillators $a_2(t)$ vanishes, and we obtain the generators of the algebra (3.1) for the sausage model in terms of two families of boson operators $a(t) = a_1(t)$ and $b(t) = a_3(t)$.

Namely, let

$$[a(t), a(t')] = t - \sinh \pi t \sinh \pi t/2 \delta(t + t'),$$

$$[b(t), b(t')] = t - \sinh \pi t \sinh \pi t/2 \delta(t + t').$$

The fields associated to the operators $a(t)$ according to the rules (3.21) will be denoted as $\phi$, while those associated to the operators $b(t)$ will be denoted as $\psi$. Now we introduce some building blocks for the explicit representation of the triplet vertex operators. Let

$$V^{(\pm)}(\theta) = \exp \left( i\hat{\phi}(\theta; p) \mp i(\phi^{(\pm)}(\theta; 2 - p) - \psi^{(\pm)}(\theta; -p)) \right),$$

$$I^{(\pm)}(\theta) = \exp \left( -i\hat{\phi}(\theta; p) \pm i(\phi^{(\pm)}(\theta; 2 - p) - \psi^{(\pm)}(\theta; -p)) \right).$$

The operator products for these operators are given by

$$V^{(A)}(\theta')V^{(B)}(\theta) = g^{(AB)}(\theta - \theta') ; V^{(A)}(\theta')V^{(B)}(\theta),$$

$$I^{(A)}(\theta')V^{(B)}(\theta) = (g^{(AB)}(\theta - \theta'))^{-1} ; I^{(A)}(\theta')V^{(B)}(\theta).$$
general rule, since the screening operator \( \theta \) defines the prescription for the integration contours in the screening operators \( V \). The contour \( \theta \) is analyzed in detail in Appendix D.

We can find from Eqs. (5.15b), (5.15c) that

\[
S\pm(k, \kappa|\theta) = \int_{C\pm} \frac{dk}{2\pi i} (I^{(+)}(\xi)e^{\kappa} - iI^{(-)}(\xi)e^{-\kappa}) \frac{\pi e^{-k\xi}}{\sinh \frac{\xi - \theta - \kappa}{\rho}}. \tag{5.17}
\]

The contour \( C_+ \) (\( C_- \)) goes below (above) the pole at the point \( \theta - i\pi \) (\( \theta + i\pi \)).

The screening operators appear in the products \( S_+(k, \kappa|\theta)V^{(\pm)}(\theta) \) and \( V^{(\pm)}(\theta)S_-(k, \kappa|\theta) \). The products \( V^{(\pm)}(\theta)I^{(\pm)}(\xi) \) possess a simple pole at the point \( \xi = \theta \). Taking this fact into account we should carefully define the prescription for the integration contours in the screening operators \( S\pm(k, \kappa|\theta) \). According to the general rule, since the screening operator \( S_+ \) \( (S_-) \) is placed to the left (right) of the operator \( V^{(\pm)}(\theta) \), the contour \( C_+ \) \( (C_-) \) goes above (below) the point \( \theta \). Besides, both of them go below the poles at the points \( \theta - i\pi + \pi p n \) \((n = 0, 1, 2, \ldots)\) and above the poles at \( \theta + i\pi - \pi p n \). Similarly to the case of the operators \( S_i \), there are no inflections in the contours, because the poles \( \xi = \theta, \theta + i\pi, \theta - i\pi \) never appear in the same product simultaneously.

We can find from Eqs. (5.15a), (5.15d) that

\[
V^{(A)}(\theta)I^{(B)}(\xi) \frac{1}{\sinh \frac{\xi - \theta - i\pi}{\rho}} = I^{(B)}(\xi)V^{(A)}(\theta) \frac{1}{\sinh \frac{\xi - \theta + i\pi}{\rho}}. \tag{5.18}
\]

Using this relation we obtain

\[
S_+(k, \kappa|\theta)V(k, \kappa|\theta) = V(k, \kappa|\theta)S_-(k, \kappa|\theta) \tag{5.19}
\]

The pole at \( \xi = \theta \) is cancelled due to the identity \( g^{(+)}(\theta) = g^{(-)}(\theta) \). But for the product of the operator \( V(k, \kappa|\theta) \) with two screening operators the pole contributions do not cancel. Namely,

\[
S_+(k, \kappa|\theta)\left(S_+(k, \kappa|\theta)V(k, \kappa|\theta) - V(k, \kappa|\theta)S_-(k, \kappa|\theta)\right) = \bar{c}e^{-k(\theta - p\pi)}I^{(+)}(\theta - p\pi),
\]

\[
\left(S_+(k, \kappa|\theta)V(k, \kappa|\theta) - V(k, \kappa|\theta)S_-(k, \kappa|\theta)\right)S_-(k, \kappa|\theta) = -i\bar{c}e^{-k(\theta + p\pi)}I^{(-)}(\theta + p\pi), \tag{5.20}
\]

\[
\bar{c} = r_{\rho}2\pi^2T^2(-1/\rho).
\]

These relations take place due to pinching the contours of two screening operators between poles. The situation is analyzed in detail in Appendix D.

Now we are in position to introduce the generators of the algebra \( \mathfrak{g} \) for the sausage model. They read

\[
B^{2,1}_+(\theta) = c_1^{-1}Z_+(k, \kappa|\theta), \quad B^{2,1}_-(\theta) = -Z_0(k, \kappa|\theta), \quad B^{2,1}_-(\theta) = c_1Z_-(k, \kappa|\theta), \tag{5.21}
\]

where the constant \( c_1 \) is defined in Eq. (5.36) and the renormalized generators are

\[
H = \int_0^\infty dt \left( \frac{\sinh \pi t \sinh \frac{\pi p}{2} a(-t)a(t)}{\sinh^2 \frac{\pi t}{2}} - \frac{\sinh \pi t \sinh \frac{\pi(2-p)t}{2} b(-t)b(t)}{\sinh^2 \frac{\pi t}{2}} \right), \tag{5.22a}
\]

\[
Z_+(k, \kappa|\theta) = cpV(k, \kappa|\theta), \tag{5.22b}
\]

\[
Z_0(k, \kappa|\theta) = -c\sqrt{2\cos \frac{p}{\rho} V(k, \kappa|\theta)S_-(k, \kappa|\theta), \tag{5.22c}
\]

\[
Z_-(k, \kappa|\theta) = -cpS_+(k, \kappa|\theta)V(k, \kappa|\theta)S_-(k, \kappa|\theta). \tag{5.22d}
\]
As we have mentioned above, the definition of the bound state vertex operators is not unique. The operators $Z_I(k, \kappa|\theta)$ ($I = +, 0, -$) can be considered as vertex operators for the sausage model, because the transformation (5.21) respects the normalization condition (3.11). The guidelines for checking the ZF algebra commutation relations in this case are given in Appendix D.

The constants $k, \kappa$ are related to the parameters of the exponential operators. For the operator

$$O_{ab}(x) = e^{i\alpha x + b\chi(x)}$$

we have

$$k = \frac{a}{\alpha}, \quad \kappa = -\frac{\pi}{4} \left( \frac{a}{\alpha} + (p-2)\frac{b}{\beta} \right).$$

(5.24)

The normalization constant $c$ in Eq. (5.22) is given by

$$c = \frac{e^{2(C_k + \log \pi p)/p}}{(\pi p)^{3/2}} \frac{\Gamma^{1/2}(1/p)}{\Gamma^{3/2}(1-1/p)}.$$  

(5.25)

Now we make the integration contours for the operators $Z_0$ and $Z_-$ more obvious. The operator $Z_0$ contains the terms of the form

$$\text{const} \times \int \frac{d\xi}{2\pi i} V^{(A)}(\theta) I^{(B)}(\xi) \frac{\pi e^{k(\theta - \xi)}}{\sinh \frac{\xi - \theta - i\pi}{p}}, \quad A, B = \pm.$$ 

(5.26)

The integrations contours for these terms labeled by values of the pair $(AB)$ are shown in Fig. 2. The operator $Z_-$ consists of the terms

$$\text{const} \times \int_{C_+} \frac{d\xi_1}{2\pi i} \int_{C_-} \frac{d\xi_2}{2\pi i} V^{(A)}(\theta) I^{(B)}(\xi_1) I^{(C)}(\xi_2) \frac{\pi e^{k(\theta - \xi_1)}}{\sinh \frac{\xi_1 - \theta - i\pi}{p}} \frac{\pi e^{-k\xi_2}}{\sinh \frac{\xi_2 - \theta - i\pi}{p}}, \quad A, B, C = \pm.$$ 

(5.27)

The integration contours for these terms labeled by values of the triples $(ABC)$ are presented in Fig. 3. Note that the contour for the integration variable $\xi_2$ is always below the contour for $\xi_1$, which makes it possible to avoid the pole at the point $\xi_2 = \xi_1 + i\pi$. It is interesting to note that for the products $S_+, S_+, V$ and $V S_- S_-$ the contour for $\xi_2$ is above the contour for $\xi_1$ in the case $B = +, C = -$ only, where there is no such pole. This observation makes these operator products, which enter Eq. (5.20), well defined.

Consider now the free particle point $p = 2$. At this point the action takes the form

$$S_{SM} = \int d^2x \left( \frac{\partial \varphi^2}{8\pi} + \frac{M}{\pi} \cos \varphi + \frac{(\partial \mu \chi)^2 - M^2 \chi^2}{8\pi} \right),$$  

(5.28)

where the term $-M^2 \chi^2/8\pi$ in the Lagrangian appears due to the quantum corrections, which make the action to be well defined. Hence, it is possible to compare the result with the known results for the sine-Gordon model at the free fermion point.

At the point $p = 2$ we can derive from the $S$ matrix (5.5) that

$$Z_\pm(\theta_1)Z_\pm(\theta_2) = -Z_\pm(\theta_2)Z_\pm(\theta_1), \quad Z_+(\theta_1)Z_-(\theta_2) = -Z_-(\theta_2)Z_+(\theta_1),$$

$$Z_0(\theta_1)Z_0(\theta_2) = Z_0(\theta_2)Z_0(\theta_1), \quad Z_\pm(\theta_1)Z_0(\theta_2) = Z_0(\theta_2)Z_\pm(\theta_1).$$ 

(5.29)
It means that the operators $Z_{\pm}(\theta)$ describe free fermions while the operator $Z_0(\theta)$ describes a free boson. Despite of simplicity of commutation relations, the limit $p \to 2$ in the free field construction is rather complicated. First of all, in this limit the operators

$$V(\pm)(\theta) = e^{i\phi(\theta \pm i\pi/2)}; \quad J(\pm)(\theta) = e^{-i\phi(\theta \pm i\pi/2)}; \quad (5.30)$$

are all mutually anticommuting. This expression determines the limit for $Z_+(\theta)$. The situation with $Z_0(\theta)$ and $Z_-(\theta)$ is more complicated.

We start with the operator

$$Z_0(k, \kappa | \theta) \simeq -\frac{e^{C_E}}{2\sqrt{\pi}} \sum_{A,B=\pm} (-i)^A_B J(AB), \quad J(AB) = \int \frac{d\xi}{2\pi i} V(A)(\theta) I(\xi)(\xi) \frac{\pi e^{(\theta - \xi) - A\kappa + B\kappa}}{\sinh \frac{\pi \xi}{\theta - i\pi}}. $$

In the limit $p \to 2$ the integration contour in $J(+-)$ is pinched between the poles at $\xi = \theta + i\pi$ and $\xi = \theta - i\pi(p-1)$. The contour for the product in $J(-+)$ is pinched between the poles at $\xi = \theta - i\pi$ and $\xi = \theta - i\pi(p-1)$. This results in the following identities

$$J(+-) \simeq -\frac{2e^{-C_E}}{p-2} e^{-i\pi k - 2\kappa}; V(+) (\theta) J(-)(\theta + i\pi); + \text{finite integral},$$

$$J(-+) \simeq 2e^{-C_E} \frac{e^{-i\pi k + 2\kappa}; V(-)(\theta) J(+)(\theta - i\pi); + \text{finite integral}}{p-2}. $$

The sum of these two terms is equal to

$$J(+-) + J(-+) \simeq -\frac{2e^{-C_E}}{p-2} \left( e^{-i\pi k - 2\kappa}; \chi e^{i\psi(\theta - i\pi)(\theta - i\pi); - \psi(\theta - i\pi)(\theta - i\pi); - i\psi(\theta - i\pi)} \right).$$

It follows from Eqs. 5.24 that $i\pi k + 2\kappa \sim (p-2)^{1/2}$ and from Eq. 5.216 that $\psi(\pm)(\theta) \sim (p-2)^{1/2}$. Hence, the sum $J(+-) + J(-+)$ is proportional to $(p-2)^{-1/2}$, which is much greater than the finite contributions of $J(+-)$ and $J(-+)$. For this reason, we introduce the operators

$$\tilde{b}(t) = i\sinh \pi t \sqrt{\frac{2\sinh^2 \frac{\pi (p-2)}{2}}{t}} b(t), \quad [\tilde{b}(t), \tilde{b}(t')] = 2\sinh \pi t \delta(t + t'). \quad (5.31)$$

Finally, we obtain that the operator $Z_0(\theta)$ depends on the parameter $b$ defined in Eq. 5.216 only and is given by

$$Z_0(b|\theta) = \int_{-\infty}^{\infty} dt \tilde{b}(t)e^{i\theta t} + \sqrt{2\pi} b. \quad (5.32)$$
Comparing this with (5.8) and (5.10) we see that this correctly reproduces the form factors of the operator $e^{\phi}\chi$.

Now we proceed with the operator $Z_-(\theta)$. The integration contours are again pinched between the poles at $\xi_i = \theta \pm i\pi$ and $\xi_i = \theta \pm i\pi(p - 1)$ ($i = 1, 2$). This pinching leads to taking out one integration. The remaining integration contour goes above or below the point $\theta = 0$ depending on the kind of the remaining screening operator, $S_+$ or $S_-$. In the limit $p \to 2$ the sum of all terms reduces to a residue at $\xi = \theta$. The answer for the fermion subsystem turns out to be

$$Z_+(a|\theta) = \rho ce^{a\theta} (e^{i\phi(\theta+i\pi/2)}; e^{\pi a/2} - i e^{i\phi(\theta-i\pi/2)}; e^{-i\pi a/2}),$$

$$Z_-(a|\theta) = -\rho c^{-1} e^{a\theta} (e^{i\phi(\theta-i\pi/2)}; e^{\pi a/2} + i e^{i\phi(\theta+i\pi/2)}; e^{-i\pi a/2})$$

(5.33)

with $c = 2^{-1/2} \pi^{-1} e^{C\rho}$ for $p = 2$.

This expression can be compared to the known results for the free fermion point of the sine-Gordon model. Though it differs from the known representation [9] for the vertex operators of the sine-Gordon model at the free fermion point, it leads to the correct form factors. Indeed, the straightforward calculation of the trace gives

$$\langle Z_+(\theta_n) \ldots Z_+(\theta_1) Z_-(\theta_n) \ldots Z_-(\theta_1) \rangle$$

$$\frac{1}{(2i)^n} \sum_{\epsilon_1, \ldots, \epsilon_n = \pm 1} \frac{1}{\sin^2 \theta_1 + \sin^2 \theta_2 + \sum_{i,j} (\theta_i - \theta_j^2) \prod_{i<j} \sin \frac{\theta_i - \theta_j}{2} \sin \frac{\theta_i + \theta_j}{2} \prod_{i,j} \sin \frac{\theta_i - \theta_j}{2} \sin \frac{\theta_i + \theta_j}{2} } \prod_{i,j} \sin \frac{\theta_i - \theta_j}{2} \sin \frac{\theta_i + \theta_j}{2}$$

(5.34)

It can be proved that this expression is equal to the known one [26, 27]

$$\langle Z_+(\theta_n) \ldots Z_+(\theta_1) Z_-(\theta_n) \ldots Z_-(\theta_1) \rangle = (-)^{n(n+1)/2} e^{\sum_{i,j} (\theta_i - \theta_j)} (i \sin \pi a)^n \prod_{i<j} \sin \frac{\theta_i - \theta_j}{2} \sin \frac{\theta_i + \theta_j}{2} \prod_{i,j} \cosh \frac{\theta_i - \theta_j}{2} \sin \frac{\theta_i + \theta_j}{2}$$

(5.35)

6. **Cosine-cosine model: $p_1 + p_2 = 2$ case**

The limit $p_1 + p_2 \to 2$ ($p_3 \to 0$) is not singular for the operators corresponding to fundamental particles. Nevertheless, the first bound state splits again into a triplet and a singlet.

Let

$$a(t) = a_2(t), \quad b(t) = a_1(t), \quad p = p_2.$$ 

Without loss of generality we shall assume that

$$1 \leq p \leq 2.$$ 

(6.1)

In the limit $p_2 \to 0$ they satisfy the commutation relations (5.13). The corner Hamiltonian is given by (5.14). As in the last section, we denote by $\phi$ and $\psi$ the fields associated with the oscillators $a(t)$ and $b(t)$ according to Eqs. (5.1) respectively. Then

$$V_3(\theta) = :e^{i\phi(\theta; -p) + i\psi(\theta; 2 - p)};$$

(6.2a)

$$I_1^{(\pm)}(\theta) = :e^{i(\psi(\theta; 2 - p) + \phi(\pm)(\theta; p))};$$

(6.2b)

$$I_2^{(\pm)}(\theta) = :e^{i(\phi(\theta; p) - \psi(\pm)(\theta; 2 - p))};$$

(6.2c)

The fundamental particles are given by Eqs. (5.35b, 5.35c) with the operators $V_3$, $I_1^{(\pm)}$ given by Eq. (6.2). According to Eq. (5.35), the mass of the fundamental particles is related with the parameter $\mu$ in the action as

$$m = \frac{\mu}{\sin \frac{\pi p}{2}} = \frac{\mu}{\sin \frac{\pi (2 - p)}{2}}.$$ 

(6.3)
Beside the fundamental particles there is a set of bound states with the masses

\[ M_n = 2\mu \frac{\sin \frac{\pi(2-p)n}{2}}{\sin \frac{\pi(2-p)}{2}}, \quad n = 1, 2, \ldots, \quad n(2-p) < 1, \quad (6.4) \]

which form quadruplets as in general case.

Consider the quadruplet of the mass \( M_1 = 2\mu \). It is instructive to consider the bound state vertex operators \( B_{\nu}^{1,1} (\nu = +, S, A, -) \) defined in Eq. (5.3) for small but nonzero \( p_3 \). In this case we have

\[ \Gamma_{1,1} \to \sqrt{p_2 \sin \frac{\pi}{p_2}}, \quad K_S^{1,1} \to \sqrt{-2 \cos \frac{\pi}{p_2}}, \quad K_A^{1,1} \sim |p_3|^{1/2} \sqrt{\frac{\pi}{p_2 \sin \frac{\pi}{p_2}}}. \quad (6.5) \]

The operators \( B_{+}^{1,1}(\theta), B_{S}^{1,1}(\theta), \) and \( B_{-}^{1,1}(\theta) \) correspond to the triplet. The coefficients at these operators in Eq. (6.3) are finite. The operator \( B_{S}^{1,1}(\theta) \) corresponds to the singlet. It describes the field \( \varphi_3(x) \) that decouples and becomes a free massive field in the limit \( p_3 \to 0 \). The operator \( B_{A}^{1,1}(\theta) \) explicitly commutes with the operators \( Z_{\nu}(\theta) \). It means that the corresponding particle does not interact with the fundamental ones. Besides, it is not a bound state of the fundamental particles, because \( K_{A}^{1,1} \to 0 \) as \( p_3 \to 0 \).

It is remarkable that the operators for the first bound state particle can be written in terms of the vertex operators used for the sausage model (see Appendix A), if we assume that

\[ a(t) = a_2(t), \quad b(t) = a_1(t), \quad \sqrt{2 \cos \frac{\pi}{p}} \to -i \sqrt{2 \cos \frac{\pi}{p}} \quad (6.6) \]

in the definition of the triplet operators (6.22) and

\[ \tilde{a}(t) = \frac{\sinh \pi t}{\sinh \frac{\pi t}{2}} \sqrt{\frac{2 \sinh \frac{\pi|p_3|t}{2}}{t}} a_3(t) \quad (6.7) \]

in the definition of the singlet operator (5.8). Namely,

\[ B_{+}^{1,1}(\theta) = ic_2^{-1} Z_{+}(k_2, \kappa_2|\theta), \quad B_{S}^{1,1}(\theta) = Z_{0}(k_2, \kappa_2|\theta), \quad B_{-}^{1,1}(\theta) = -ic_2 Z_{-}(k_2, \kappa_2|\theta), \quad B_{A}^{1,1}(\theta) = Z'_{0}(a_3|\theta), \quad (6.8) \]

where the constant \( c_2 \) is defined by Eq. (4.30). The scattering matrix of the triplet is the analytical continuation of the sausage \( S \) matrix (5.1) to the region \( p < 2 \).

The fundamental particles, the first triplet bound state and the higher \((n \geq 2)\) quadruplets form the particle contents of the cosine-cosine model with the action\(^7\)

\[ S_{CC}[\varphi_1, \varphi_2] = \int d^2x \left( \frac{(\partial_\mu \varphi_1)^2 + (\partial_\mu \varphi_2)^2}{8\pi} + \frac{2\mu}{\pi} \cos \alpha_1 \varphi_1 \cos \alpha_2 \varphi_2 \right), \quad \alpha_1^2 + \alpha_2^2 = 1. \quad (6.9) \]

Note that the form factors for the first bound state of this model can be analytically continued to the sausage model region \( p > 2 \) by taking \( \alpha_2 = \alpha, \alpha_1 = -i\beta. \) The situation is very similar to that of the sine-Gordon and sinh-Gordon model, where the form factors in the sinh-Gordon model are the analytic continuation of the form factors of the sine-Gordon model that contain the first bound state particles only.

Note, that the \( S \) matrix of the fundamental quadruplet \( Z_{\nu}(\theta) \) and the triplet \( B_{\nu}^{1,1}(\theta) (\nu = +, S, -) \) is rather simple. Namely, let

\[ Z_{cc}(\theta_1) B_{\nu}^{1,1}(\theta_2) = \sum_{\nu_1 = +, S, -} S^{(1)}(\theta_1 - \theta_2) \epsilon_{\nu_1}^{\nu} B_{\nu}^{1,1}(\theta_2) Z_{cc}(\theta_1). \quad (6.10) \]

\(^7\)This is a corrected (integrable) version of the model proposed in Ref. [18].
Then, the nonzero matrix elements of the $S^{(1)}(\theta)$ matrix are given by

$$
S^{(1)}(\theta)_{++} = S^{(1)}(\theta)_{--} = - \frac{\cosh \frac{\theta - i\pi/2}{p}}{\cosh \frac{\theta + i\pi/2}{p}}, \quad S^{(1)}(\theta)_{+-} = S^{(1)}(\theta)_{-+} = - \frac{\cosh \frac{\theta - 3i\pi/2}{p}}{\cosh \frac{\theta - i\pi/2}{p}}.
$$

(6.11)

Consider now the case $p_1 = p_2 = 1$. At this point the action can be rewritten as a sum of two sine-Gordon models at the free fermion point:

$$
S_{CC}[\varphi_1, \varphi_2] = \frac{1}{2} \sum_{i=1}^{2} \int d^2 x \left( \frac{\partial_i \chi_i}{\lambda} + \mu \cos \chi_i \right), \quad \chi_{1,2}(x) = \frac{\varphi_1(x) \pm \varphi_2(x)}{\sqrt{2}}.
$$

This free fermion point provides one more check of the construction. In the limit $p_1 \to 1$ the integration contours in $Z_{\varepsilon \varepsilon'}(\theta)$ are pinched and it is possible to take all integrals explicitly. Let

$$
a_{\pm}(t) = a_1(t)e^{\pm \pi|t|/4} \pm a_2(t)e^{\mp \pi|t|/4}, \quad \{a_{\pm}(t), a_{\pm}(t')\} = t \delta(t + t'), \quad \{a_{+}(t), a_{-}(t)\} = 0.
$$

(6.12)

Define

$$
\phi_{\pm}(\theta) = \int_{-\infty}^{\infty} \frac{dt}{t} a_{\pm}(t)e^{i\delta t}.
$$

(6.13)

Then

$$
Z_{++}(k_1, k_2) = \omega e^{k_+ \theta} : e^{i\phi_+}(\theta) :,
$$

(6.14a)

$$
Z_{-+}(k_1, k_2) = \omega e^{C_{\theta}} e^{-k_- \theta} \left( e^{i\pi k_- /2} : e^{i\phi_- (\theta - i\pi/2)} : + i e^{-i\pi k_- /2} : e^{-i\phi_- (\theta + i\pi/2)} : \right),
$$

(6.14b)

$$
Z_{+-}(k_1, k_2) = \omega e^{C_{\theta}} e^{-k_+ \theta} \left( e^{i\pi k_+ /2} : e^{i\phi_+ (\theta + i\pi)} : - i e^{-i\pi k_+ /2} : e^{-i\phi_+ (\theta - i\pi)} : \right),
$$

(6.14c)

$$
Z_{-+}(k_1, k_2) = - \omega e^{C_{\theta}} e^{-k_- \theta} \left( e^{-i\pi k_- /2} : e^{-i\phi_- (\theta - i\pi)} : + i e^{i\pi k_- /2} : e^{i\phi_- (\theta + i\pi)} : \right),
$$

(6.14d)

where

$$
k_{\pm} = \frac{k_1 \pm k_2}{2} = \frac{a_1 \pm a_2}{\sqrt{2}}.
$$

(6.15)

and the operators give the form factors of the exponential fields

$$
O_{a_1a_2}(x) = e^{i a_1 \varphi_1(x) + i a_2 \varphi_2(x)} = e^{i k_- \chi_1(x) + i k_+ \chi_2(x)}.
$$

(6.16)

The vertex operators anticommute:

$$
Z_{\varepsilon_1 \varepsilon_1'}(\theta_1) Z_{\varepsilon_2 \varepsilon_2'}(\theta_2) = - Z_{\varepsilon_2 \varepsilon_2'}(\theta_2) Z_{\varepsilon_1 \varepsilon_1'}(\theta_1), \quad \text{for } p_1 = p_2 = 1.
$$

(6.17)

The pair $Z_{++}(\theta)$, $Z_{--}(\theta)$ describes the Dirac fermion related to the field $\chi_1$, while the pair $Z_{+-}(\theta)$, $Z_{-+}(\theta)$ describes that related to $\chi_2$. The representation (6.14a), (6.14d) coincides with that described in [9] for the field $e^{i a_\varphi}$ (see Eq. (6.25) for normalizations) with $a = k_+$, while the representation (6.14b), (6.14c) gives the representation (6.38) with $a = k_-$. Both of them provide the right hand side of Eq. (6.35) with $a = k_{\pm}$ as the final result for form factors.
7. Conclusion

The form factors of the exponential operators (3.11) in the model with the action (2.1) in the unitary regime can be expressed as traces of the vertex operators, which are realized in terms of three sets of free boson operators. This bosonization procedure provides an algorithm for construction of an integral representation for form factors. For the fundamental particles a general explicit integral formula (3.15)–(3.19) for any \( N \)-particle form factor has been found.

There are two important limiting cases. The first one, \( p_2 \to 0 \), gives the sausage model in the dual representation. The free field representation for the vertex operators is constructed by the limit in the free field representation for the first bound state of two fundamental particles. The second limiting case, \( p_1 + p_2 \to 2 \) gives the known cosine-cosine model, that contains the fundamental particles as well as their bound states. The first bound state forms a triplet and the respective vertex operators can be considered as analytic continuation of the vertex operators of the sausage model after the substitution \( p_1 \to p_2, p_2 \to p_3, p_3 \to p_1 \).

There are three interesting problems, which we did not solve in this paper. The first one is to compute analytically the integrals for two-particle form factors of some special exponential operators. By the analogy to the sine-Gordon theory, it is expected to be possible to compute them for \( a_i = \frac{2}{n} \alpha_i (n \in \mathbb{Z}) \). In this paper the integral was only computed for the one-particle form factor in the sausage model (see Appendix B).

Another task is to obtain a free field realization of the vertex operators for restricted models like the parafermion sine-Gordon or the coset models, which are related the family of models under consideration by the quantum group reduction. It is expected that it can be achieved by an appropriate change of the screening operators [28] together with, probably, some Felder type resolution.

The third and the most important problem is related with the prescription for calculation of the form factors of the descendent operators (3.10) and of the nonlocal with respect to the fields \( \varphi_i(x) \) operators. These problems will be addressed in forthcoming papers.

Acknowledgments

The authors are grateful to Ya. Pugai for stimulating discussions. The work was supported, in part, by EU under the contract HPRN-CT-2002–00325, by INTAS under the grant INTAS-OPEN-00–00055, by Russian Foundation for Basic Research under the grant RFBR 02–01–01015, and by Russian Ministry of Science and Technology under the Scientific Schools grant 2044.2003.2. M. L. is indebted to P. Forgacs and M. Niedermaier for organizing his visit to Université de Tours and Université Montpellier II during the autumn of 2002, where this work was started, and to CNRS for support of this visit.

Appendix A. Regularized \( t \)-integration

The Jimbo–Konno–Miwa regularization rule, described in Sec. 3, leads to the following simple formulas for regularized integrals:

\[
\exp \int_0^\infty \frac{dt}{t} e^{-zt} = \frac{e^{-C_E}}{z},
\]

\[
\exp \int_0^\infty \frac{dt}{t} \frac{e^{-zt} - 1}{e^t - 1} = e^{C_E \Gamma(z + 1)},
\]

\[
\exp \int_0^\infty \frac{dt}{t} \frac{1}{e^t - 1} = \frac{1}{\sqrt{2\pi}} e^{C_E/2},
\]

\[
\int_0^\infty dt \, f(at) = \frac{1}{a} \int_0^\infty dt \, f(t) - \text{Res}_{t=0} f(at) \cdot \log a.
\]

Here \( C_E \) is the Euler constant.
Appendix B. List of trace functions

Here we list formulas for the constants $C_{ij}$ and functions $U_{ij}^a(\theta)$ defined in Eqs. (3.39a), (3.39b), which determine the traces according to Eq. (3.38) for the operators $V_i(\theta)$, $I_{ij}^\pm(\theta)$ defined in Eqs. (3.32a) and $V^\pm(\theta), I^{(\pm)}(\theta)$ defined in Eqs. (3.14). These calculations are performed straightforwardly by application of the formula (we recall that subscripts $i, j$ are defined modulo 3):

$$\langle a_i(t) a_j(t') \rangle = t \frac{\sinh^2 \frac{\pi t}{2}}{\sinh \pi t \sinh \frac{\pi |t|}{2}} \frac{1}{1 - e^{-2\pi t}} \delta_{ij} \delta(t + t').$$

For the constants we have

$$C_{V_i} = C_i = \exp \left( -\int_0^\infty dt \frac{e^{-\pi t} \sinh^2 \frac{\pi t}{2} \sinh \frac{\pi(p_i+1+p_i+2)\delta}{2}}{2 \sinh^2 \pi t \sinh \frac{\pi(p_i+1)\delta}{2}} \right), \quad (B.1)$$

$$C_{I_{ij}^\pm} = \tilde{C}_i = e^{-\left(C_{\pi}+\log \pi p_i\right)/p_i} \frac{\pi p_i^{-1/2}}{1 + 1/p_i}, \quad (B.2)$$

$$C_{V(\pm)} = C_{I(\pm)} = \tilde{C} = \tilde{C}_{ij}|_{p_i=p_j} \quad (B.3)$$

For the functions $G_{jk}$ the result is

$$G_{V_iV_j}(\theta) = G_{ij}(\theta), \quad (B.4)$$

$$G_{V_iV_j^\pm}(\theta) = G_{ij^\pm}(\theta) = W_{ij^\pm}(\theta), \quad (B.5)$$

$$G_{I_{ij}^{(A)}/I_{ij}^{(B)}}(\theta) = G_{ij}^{(AB)}(\theta), \quad (B.6)$$

$$G_{V_iV_j^{(A)}/V_j^{(B)}}(\theta) = G_{ij}^{(A/B)}(\theta) = G_{ij}^{(AB)}(\theta)|_{p_i=p_j} \quad (B.7)$$

$$G_{V_jV_i^{(A)}/V_i^{(B)}}(\theta) = G_{ij}^{(A/B)}(\theta) = W^{(AB)}(\theta) = (G^{(AB)}(\theta))^{-1}. \quad (B.8)$$

The functions $G_{ij}(\theta)$ are rather complicated:

$$G_{ii}(\theta) = \exp \left( -\int_0^\infty dt \frac{\sinh^2 \frac{\pi t}{2} \sinh \frac{\pi(p_i+1+p_i+2)\delta}{2}}{\sinh^2 \pi t \sinh \frac{\pi(p_i+1)\delta}{2}} \right), \quad (B.9a)$$

$$G_{ij}(\theta) = \exp \left( -\int_0^\infty dt \frac{\sinh^2 \frac{\pi t}{2} \sinh \frac{\pi(p_i+1+p_i+2)\delta}{2}}{\sinh^2 \pi t \sinh \frac{\pi(p_i+1)\delta}{2}} \right) \quad (i \neq j, k \neq i, j), \quad (B.9b)$$

but they never appear in the integrands, i.e. their arguments never contain integration variables. The functions $W_{ij^\pm}(\theta)$ can be expressed in terms of a single function $W(p; \theta)$:

$$W_{ii}^{(\pm)}(\theta) = 1, \quad (B.10a)$$

$$W_{i-1,i}(\theta) = W(p_i; \theta \pm \frac{\pi}{2} p_i), \quad (B.10b)$$

$$W_{i+1,i}(\theta) = W(p_i; \theta), \quad (B.10c)$$

$$W(p; \theta) = \exp \int_0^\infty dt \frac{\sinh \frac{\pi t}{2} \cos(\theta + i\pi)t}{\sinh \pi t \sinh \frac{\pi t}{2}}. \quad (B.10d)$$

The function $W(p; \theta)$ possesses poles at the points $\theta = \frac{\pi}{2}(p - 1 + 2M) + 2\pi iN$ ($M, N = 0, 1, 2, \ldots$) and $\theta = -\frac{\pi}{2}(p - 1 + 2M) - 2\pi i(N + 1)$, and satisfies the relations

$$W(p; \theta) = W(p; \theta + 2\pi i), \quad (B.11a)$$

$$W(p; \theta - \pi)W(p; \theta) = \frac{1}{2 \cosh \frac{\pi}{\theta + 2\pi i}}, \quad (B.11b)$$

$$W(p; \theta - i\pi/2)W(p; \theta + i\pi/2) = \tanh \left( \frac{\pi}{2} + \frac{\theta}{2} \right). \quad (B.11c)$$
The functions $\bar{G}^{(AB)}_{ij}(\theta)$ are purely trigonometric:

\begin{align}
\bar{G}^{(\mp)}_{ii}(\theta) &= \pm 2 i \sinh \frac{\theta + i \pi}{p_i}, \\
\bar{G}^{(\pm)}_{ii}(\theta) &= 2 \tanh \frac{\theta + i \pi}{2} \frac{\sinh \frac{\theta + i \pi}{p_i}}{p_i}, \\
\bar{G}^{(\mp)}_{i,i+1}(\theta) &= \bar{G}^{(\mp)}_{i+1,i}(\theta) = 1, \\
\bar{G}^{(\pm)}_{i,i+1}(\theta) &= \bar{G}^{(\pm)}_{i+1,i}(\theta) = -i \coth \left( \frac{\mp \pi p_{i+1}}{4} \pm \frac{\theta}{2} \right).
\end{align}

(B.12a) (B.12b) (B.12c) (B.12d)

It means that the form factors of the sausage model can be expressed in terms of multiple integrals of trigonometric functions with two different periods.

It is important to make the following remark about the contours. To fix the contours it is necessary to consider the product in the trace and look at the poles of this expression. There are poles that appear from the operator products (4.24a, 4.24b) and the extra factors in the screening operators. The prescription for the integration contours with respect to those poles was already described in the main part of the paper. The additional poles that arise due to the trace should be treated as follows. Namely, the contours should go below the excessive poles at the points $\theta_j + i \delta_j$ with $\delta_j > 0$ and above the excessive poles at the points $\theta_j + i \delta_j$ with $\delta_j < 0$. For example, the trace

\begin{equation}
\langle \langle V^-(\theta) I^+(\xi) \rangle \rangle = \pi \bar{C}^2 W^-(\theta - \xi) = \frac{-i \pi \bar{C}^2}{2 \sinh \frac{\theta - i \pi}{p} \sinh \frac{\xi - i \pi}{p}}
\end{equation}

possesses poles at $\xi = \theta \pm i \pi$, though the product

\begin{equation}
V^-(\theta) I^+(\xi) = : V^-(\theta) I^+(\xi) : = \pi \bar{C}^2 W^-(\theta - \xi) = \frac{i \pi \bar{C}^2}{2 \sinh \frac{\theta - i \pi}{p} \sinh \frac{\xi - i \pi}{p}}
\end{equation}

possesses a pole at $\xi = \theta - i \pi$ only. It means that the integration contour for $\xi$ must go below both poles at $\theta \pm i \pi$. In contrast, in the expression

\begin{equation}
\langle \langle V^+(\theta) I^-(\xi) \rangle \rangle = \pi \bar{C}^2 W^+(\theta - \xi) = \frac{-i \pi \bar{C}^2}{2 \sinh \frac{\theta - i \pi}{p} \sinh \frac{\xi - i \pi}{p}}
\end{equation}

the integration contour must go above both poles at $\xi = \theta \pm i \pi$, because the only pole at $\xi = \theta + i \pi$ coming from the extra factors survives after removing the trace sign.

As an example of application of the integral representation, we calculate the one-particle form factor $\langle \langle Z_0(\theta) \rangle \rangle$ in the sausage model. This form factor

\begin{equation}
\langle \langle Z_0(\theta) \rangle \rangle = \frac{\langle 0 | e^{i a \varphi + b x} | \theta, 0 \rangle}{\langle 0 | e^{i a \varphi + b x} | 0 \rangle}
\end{equation}

is just a constant. Nevertheless, it gives the first nontrivial contribution to the infrared asymptotics of correlation functions.

Using the fact that $\bar{W}^{(+)}(\theta) = \bar{W}^{(-)}(\theta)$, we obtain

\begin{equation}
\langle \langle Z_0(\theta) \rangle \rangle = -e \sqrt{2 \cos \frac{\pi}{p}} \int \frac{d \xi}{2 \pi i} \langle \langle (V^+(\theta) e^{-\kappa} \right. - i V^{(-)}(\theta) e^{\kappa}) (I^+(\theta) e^{\kappa} - i I^{(-)}(\theta) e^{-\kappa}) \rangle \rangle \frac{\pi e^{k(\theta - \xi)}}{\sinh \frac{\xi - i \pi}{p}}
\end{equation}

\begin{equation}
= i e \bar{C}^2 \sqrt{2 \cos \frac{\pi}{p}} \left( \int_{C_{++}} \frac{d \xi}{2 \pi i} \bar{W}^{(+)}(\theta - \xi) \frac{\pi e^{-2 \kappa + k(\theta - \xi)}}{\sinh \frac{\xi - i \pi}{p}} + \int_{C_{--}} \frac{d \xi}{2 \pi i} \bar{W}^{(-)}(\theta - \xi) \frac{\pi e^{2 \kappa + k(\theta - \xi)}}{\sinh \frac{\xi - i \pi}{p}} \right).
\end{equation}

The contours $C_{++}$ and $C_{--}$ are chosen according to the above mentioned rule. As we have seen just now, the contour $C_{++}$ goes above the poles at $\xi = \theta \pm i \pi$, while the contour $C_{--}$ goes below the poles at these points.
With the substitution $\xi \to \xi + \theta$ we obtain

$$\langle Z_0(\theta) \rangle = -\frac{\pi c C^2}{2} \sqrt{2 \cos \frac{\pi}{p}} \left( \int_{c_+} d\xi \frac{e^{-k\xi - 2\kappa}}{2\pi \sin \frac{\xi + \pi}{p}} - \int_{c_-} d\xi \frac{e^{-k\xi - 2\kappa}}{2\pi \sin \frac{\xi - \pi}{p}} \right).$$

The integration can be done by residues, if we suppose e.g. $\text{Im} k < 0$. With these integrands, which are periodic functions times an exponential function, it is easy to sum up the residues with the result

$$\langle Z_0(\theta) \rangle = \frac{p^{1/2} \sin \frac{\pi}{p}}{\sqrt{\sin \frac{2\pi}{p}}} \sin \pi b, \quad p = 2\alpha^2 = 2\beta^2 + 2. \quad (B.13)$$

In the limit $p \to 2$ we have $\langle Z_0(\theta) \rangle \to \sqrt{2\pi} b$ in consistency with the free boson result. Note, that the result (B.13) is invariant with respect to the reflection transformation $b \to 1/\beta - b$ (see Ref. [16]).

**Appendix C. Check of the Zamolodchikov–Faddeev algebra relations**

The relation (3.1c) for the commutator $[H, Z_{c\ell'}(\theta)]$ of the corner Hamiltonian and a vertex operator can be checked straightforwardly. Here we verify the commutation relations (3.1a) of two vertex operators and prove the expression (3.36) for the normalization constants $c_i$ by checking the relation (3.1b).

We start with the simplest commutation relation

$$Z_{++}(\theta_1)Z_{++}(\theta_2) = -S_{p_1}(\theta_1)_{++} S_{p_2}(\theta_2)_{++} Z_{++}(\theta_2)Z_{++}(\theta_1),$$

where $\theta_{12} = \theta_1 - \theta_2$ and we used the fact that the $S$ matrix $S_{p_1 p_2}(\theta)$ is a tensor product of two sine-Gordon $S$ matrices (2.10). Substituting Eq. (2.10) we obtain

$$V_3(\theta_1)V_3(\theta_2) = -S_{p_1}(\theta_1)_{++} S_{p_2}(\theta_2)_{++} V_3(\theta_2)V_3(\theta_1).$$

Reduce the l.h.s. and the r.h.s. to normal products with the help of Eq. (3.24a):

$$g_{33}(-\theta_{12}) : V_3(\theta_1)V_3(\theta_2) = -S_{p_1}(\theta_1)_{++} S_{p_2}(\theta_2)_{++} g_{33}(\theta_{12}) : V_3(\theta_1)V_3(\theta_2):$$

i.e.

$$-S_{p_1}(\theta)_{++} S_{p_2}(\theta)_{++} = \frac{g_{33}(\theta)}{g_{33}(\theta)}. \quad (C.1)$$

The last identity can be obtained using Eq. (B.10).

Consider now another commutation relation

$$Z_{++}(\theta_1)Z_{--}(\theta_2) = -S_{p_1}(\theta_1)_{++} S_{p_2}(\theta_2)_{++} Z_{--}(\theta_2)Z_{++}(\theta_1) - S_{p_1}(\theta_1)_{++} S_{p_2}(\theta_2)_{++} Z_{++}(\theta_2)Z_{--}(\theta_1).$$

Using Eqs. (3.35a, b, c, d, e, f, g, h, i, j, k) we obtain that it is equivalent to the following equality

$$\sum \int d\xi \tilde{V}_3(\theta_1)\tilde{V}_3(\theta_2)\tilde{I}_1(A) : g_{33}(\theta_{12}) w_{A}^{(A)}(\xi_1 \theta_1) w_{A}^{(A)}(\xi_2 \theta_2) \sinh \frac{\xi - \theta_2 - \pi/2}{p_1} \sinh \frac{\xi - \theta_2 + \pi/2}{p_1} \sinh \frac{\xi - \theta_2 - \pi/2}{p_1} \sinh \frac{\xi - \theta_2 + \pi/2}{p_1}$$

with

$$\tilde{V}_i(\theta) = V_i(\theta) e^{\frac{k_{1+2} \theta}{2}}, \quad \tilde{I}_i(A)(\xi) = I_i(A)(\xi) e^{\kappa_i}, \quad \tilde{I}_i(A)(\xi) = -iI_i(A)(\xi) e^{-\kappa_i}.$$
We note that operator part equal to: \( \hat{V}_3(\theta_1) \hat{V}_3(\theta_2) \hat{V}_1^{(A)}(\xi) \): is the same in all terms, therefore it is sufficient to prove the identity for the remaining functions. Here and later we need the identities

\[
\begin{align*}
\frac{w_{i-1,i}^{(A)}(-\theta)}{w_{i+1,i}^{(A)}(-\theta)} &= \sinh \frac{\theta + i\pi/2}{p_i}, \quad \frac{w_{i+1,i}^{(A)}(-\theta)}{w_{i-1,i}^{(A)}(-\theta)} = \cosh \frac{\theta + i\pi/2}{p_i}, \quad g_{ii}^{(BA)}(-\theta) = \sinh \frac{\theta - i\pi/2}{p_i}.
\end{align*}
\]  

(C.2)

Dividing the functions in the l.h.s. and r.h.s. by the numerator of the function in the l.h.s. and using Eqs. (2.9), (2.10) and (C.1) we obtain

\[
\frac{1}{\sinh(x - y_2 - z/2)} = -\frac{\sinh(y_1 - y_2)}{\sinh(z - y_1 + y_2)} \frac{1}{\sinh(x - y_1 + z/2)} + \frac{\sinh z}{\sinh(x - y_1 + y_2)} \frac{1}{\sinh(x - y_1 - z/2)}
\]

with \( x = \xi/p, y_1 = \theta_1/p, z = i\pi/p \). This identity can be checked straightforwardly.

The other commutation relations can be checked in the same way. The commutation relation

\[
Z_{-+}(\theta_1)Z_{++}(\theta_2) = -S_{p_1}(\theta_12)S_{p_2}(\theta_12)Z_{+}Z_{-}(\theta_1) - S_{p_1}(\theta_12)S_{p_2}(\theta_12)Z_{++}(\theta_2)Z_{--}(\theta_1)
\]

is valid due to the identity

\[
\frac{1}{\sinh(x - y_1 - z/2)} \frac{\sinh(y_1 - y_2)}{\sinh(x - y_2 - z/2)} = -\frac{\sinh z}{\sinh(x - y_1 + y_2)} \frac{1}{\sinh(x - y_1 - z/2)} + \frac{\sinh(y_1 - y_2)}{\sinh(x - y_1 + y_2)} \frac{1}{\sinh(x - y_1 - z/2)}.
\]

The commutation relation

\[
Z_{-+}(\theta_1)Z_{++}(\theta_2) = -S_{p_1}(\theta_12)S_{p_2}(\theta_12)Z_{+}Z_{-}(\theta_1)
\]  

(C.3)

contains a two-fold integral. There is an important fact concerning the integration contours. Each product of vertex operators in this equation splits into four terms containing different operator products \( V(\theta_1)V(\theta_2) \times \hat{V}(A)(\xi)\hat{V}(B)(\xi) \), \( A, B = \pm \). For each pair \((A, B)\) we can choose the contour \( C_1^{(A,B)} \) for \( \xi \) and the contour \( C_2^{(A,B)} \) for \( \xi \), so that it satisfy the pole avoiding rules for all three operator products in Eq. \( \text{(C.3)} \). The important fact is that \( C_1^{(A,B)} \) can be deformed into \( C_2^{(A,B)} \) without meeting poles. (For \( A = B \) it simply means that \( C_1^{(A,A)} \) and \( C_2^{(A,A)} \) are two deformations the same contour.) This means that we can symmetrize the integral in the integration variables. After this symmetrization we arrive to the identity

\[
f(x_1, x_2) + f(x_2, x_1) \frac{\sinh(x_2 - x_1 - z)}{\sinh(x_2 - x_1 + z)} = 0
\]

(C.4)

with

\[
f(x_1, x_2) = \frac{1}{\sinh(x_1 - y_1 - z/2)} \frac{1}{\sinh(x_2 - y_2 - z/2)} \frac{\sinh(x_1 - y_2 + z/2)}{\sinh(x_2 - y_1 + z/2)} \frac{\sinh(x_1 - y_2 - z/2)}{\sinh(x_2 - y_1 - z/2)}.
\]

which is checked straightforwardly.

The relations proved above contain the vertex operators \( Z_{\varepsilon'\varepsilon}(\theta) \) with \( \varepsilon' = + \) only. That is why the proofs only involved the screening operator \( S_1(\theta) \). The check of relations that contain the vertex operators with \( \varepsilon = + \) go essentially in the same line. The proofs of these relations have to do with the screening operator \( S_2(\theta) \) only, and the identities to be checked are obtained from the above ones by the substitution \( y_1 \to y_1 + i\pi/2 \). The remaining relations for the vertex operators with generic \( \varepsilon, \varepsilon' \) follow from the proved ones and commutativity of the screening operators \( S_1(\theta) \) and \( S_2(\theta) \).
At last, we check the normalization condition (3.1b). It is enough to verify it for the product

\[ Z_{++}(\theta') Z_{--}(\theta) = c_1 c_2 \sum_{\gamma, \delta} \int \frac{d\xi_1}{2\pi i} \int \frac{d\xi_2}{2\pi i} \hat{V}_3(\theta') \hat{V}_3(\theta) \hat{I}_1(\xi_1) \hat{I}_2(\xi_2); \]

\[ \times g_{33}(\theta - \theta') w^{(A)}_{31}(\xi_1 - \theta') \frac{\pi w^{(A)}_{31}(\xi_1 - \theta)}{\sinh \frac{\xi_1 - \theta - \pi i/2}{\pi p_1}} \]

\[ \times g_{33}(\xi_2 - \theta') w^{(B)}_{32}(\xi_2 - \theta') \frac{\pi w^{(B)}_{32}(\xi_2 - \theta)}{\cosh \frac{\xi_2 - \theta - \pi i/2}{\pi p_1}} \]

As a rule, the pole at \( \theta' = \theta + i\pi \) is known to be related to pinching contours between poles [7]. An elaborate analysis of the poles shows that the only pinching situation takes place at the term with \( A = B = + \). The contour \( C_1 \) goes below the pole at \( \theta' = \frac{i\pi}{2} \) coming from \( w^{(A)}_{31} \) and above the pole at \( \xi_2 + \frac{i\pi p_2}{2} \) coming from \( \hat{y}^{(+)}_{12} \). The contour \( C_2 \) for \( \xi_2 \) goes, in turn, above the pole at \( \theta + \frac{i\pi}{2} \) coming from \( \pi w^{(B)}_{32}(\xi_2 - \theta) / \cosh \frac{\xi_2 - \theta - \pi i/2}{\pi p_1} \).

We have to push the contour \( C_1 \), e.g., upward. The residue at \( \xi_1 = \theta - \frac{i\pi}{2} \) contribute the singular term, while the remaining integral along the contour that goes far from poles does not contribute it, because \( \xi_2 \) will not be pinched between the points \( \xi_1 - \frac{i\pi p_2}{2} \) and \( \theta + \frac{i\pi}{2} \). In the residue we push \( C_2 \), e.g., downward. The residue at \( \xi_2 = \theta + \frac{i\pi}{2} - \frac{i\pi p_2}{2} \) contributes the singularity.

For the operator part it is easy to get

\[ \hat{V}(\theta + i\pi) \hat{V}(\theta) \hat{I}_1^{(+)}(\theta + \frac{i\pi}{2}) \hat{I}_2^{(+)}(\theta + \frac{i\pi}{2} - \frac{i\pi p_2}{2}) = 1. \]

Finally, we obtain

\[ Z_{++}(\theta') Z_{--}(\theta) = -\frac{i}{\theta' - \theta - i\pi} \pi^3 c_1 c_2 r_p^{-2} \Gamma \left( \frac{-1}{p_1} \right) \Gamma \left( \frac{-1}{p_2} \right) \frac{\Gamma \left( \frac{1}{p_1} \right) \Gamma \left( \frac{1}{p_2} \right)}{\Gamma \left( \frac{1 + 1}{p_1} \right) \Gamma \left( \frac{1 + 1}{p_2} \right)} + O(1), \]

where \( r_p \) is defined in Eq. (3.20). Comparing with the normalization condition (3.1b) we obtain the value of the product \( c_1 c_2 \). This calculation does not fix \( c_1 \) and \( c_2 \) separately. But if we demand all the operators \( Z_{\alpha\beta}(\theta) \) to be normalized by Eq. (3.1b), we get all the products \( c_1 c_{\alpha+1} \) (which are the result of cyclic permutations of subscripts in the expression for \( c_1 c_2 \)) and obtain Eq. (3.30).

**Appendix D. Vertex operators in the sausage model**

Here the operators \( Z_+ (\theta) \) and \( Z_0 (\theta) \) will be obtained directly from the general model (2.4) by taking the limit \( p_2 \rightarrow 0 \). The operator \( Z_- (\theta) \) will be found from the commutation relations of the ZF algebra for the sausage model. The proof of the commutation relations in this case has some additional complication related to the pole at \( \xi = \theta \) in the product \( V^{(+)}(\theta) I^{(+)}(\xi) \). It will be shown how to manage this pole. At the end, the normalization constant \( c \) in Eq. (5.22) will be derived.

Consider the general model (2.4) for \( p_2 < 1 \) and calculate the products

\[ Z_{\alpha+1}(\theta' + \frac{i\pi}{2}(1 - p_2)) Z_{\epsilon -}(\theta - \frac{i\pi}{2}(1 - p_2)) \]

in the vicinity of the point \( \theta' = \theta \). The residue of this pole will give the first bound state of two fundamental particles.

We begin with the product

\[ Z_{++}(\theta' + \delta) Z_{--}(\theta - \delta) \]

\[ = -\rho c_2 \int \frac{d\xi}{2\pi i} V_4(\theta' + \delta) V_5(\theta - \delta) (I_2^{(+)} \xi)e^{\xi_2} - iI_2^{(-)}(\xi)e^{-\xi_2} \]

\[ \times g_{33}(\theta - \theta' - 2\delta) w^{(+)}_{32}(\xi - \theta' - \delta) w^{(+)\xi}(\xi - \theta + \delta) \]

\[ \times \frac{\rho c_2 \xi - \theta - \pi i/2}{\cosh \frac{\xi - \theta - \pi i/2}{p_2}}. \]
with $\delta = \frac{ip}{2}(1 - p_2)$. The integration contour is pinched between the pole at the point $\xi = \theta'$ of the function $w(p_2, 1/2|\xi - \theta' - \delta)$ and that at $\xi = \theta$ of the function $\sinh^{-1} \frac{\xi - \theta}{p_2}$. Pushing the contour downward and calculating the residue of the second pole we obtain

$$Z_{++}(\theta' + \delta)Z_{+-}(\theta - \delta) = \rho \frac{iC}{\theta' - \theta} :V_3(\theta + \delta)V_3(\theta - \delta)(I_2^{(+)}(\theta)e^{\kappa_2} - iI_2^{(-)}(\theta)e^{-\kappa_2}); e^{k_1\theta} + O(1),$$

$$C = \frac{\sqrt{\pi}}{G(p_1, -i\pi)} \frac{G(p_2, -i\pi)}{G(p_2, -2\delta)}.$$

It is straightforward to find that

$$:V_3(\theta + \delta)V_3(\theta - \delta)I_2^{(+)}(\theta): \to V^{(+)}(\theta),$$

$$:V_3(\theta + \delta)V_3(\theta - \delta)I_2^{(-)}(\theta): \to V^{(-)}(\theta)$$

as $\delta \to i\pi/2$ ($p_2 \to 0$). In this limit the constant $C$ remains finite:

$$C \to 2\pi^{-1/2}G^{-1}(p_1, -i\pi), \quad p_2 \to 0.$$ 

Here we used the identity

$$\frac{G(p; -i\pi)}{G(p; -i\pi + i\pi p)} = \frac{2}{\sqrt{\pi(1 - p)}} \frac{\Gamma(1 - \frac{\theta}{p})}{\Gamma(\frac{1 - \theta}{p})} \quad \text{(D.2)}$$

As a result, we obtain

$$Z_{++}(\theta' + \delta)Z_{+-}(\theta - \delta) \to \frac{iC}{\theta' - \theta} \rho V(k_1, \kappa_1|\theta) + O(1).$$

It is easy to check that

$$\frac{c_1C}{\kappa} = \Gamma_{2,1}, \quad p_2 = 0,$$

with the constant $c$ given by Eq. (D.26) with $p = p_1$ and $\Gamma_{2,1}$ given by Eq. (D.27). It proves that $B_{++}^{2,1}(\theta) = c_1^{-1}Z_{++}(k, \kappa|\theta)$ with $Z_{++}(k, \kappa|\theta)$ defined by Eq. (D.22).

Consider now the products $O_{++}$ for $\varepsilon' + \varepsilon = 0$. For $\varepsilon' = +, \varepsilon = -$ we have

$$Z_{++}(\theta' + \delta)Z_{--}(\theta - \delta)$$

$$= c_{1,2}V_3(\theta' + \delta)V_3(\theta - \delta) \int_{c_1} \int_{c_2} \frac{d\xi_1}{2\pi} \frac{d\xi_2}{2\pi} (I_2^{(+)}(\xi_1)e^{\kappa_2} - iI_2^{(-)}(\xi_1)e^{-\kappa_2})(I_2^{(+)}(\xi_2)e^{\kappa_2} - iI_2^{(-)}(\xi_2)e^{-\kappa_2})$$

$$\times \frac{-2e^{-2\beta_1 \beta_2(\theta' + \delta - \kappa_1 - \kappa_2)}}{\sinh \frac{\xi_1 - \theta - i\pi p_2/2}{p_1} \sinh \frac{\xi_2 - \theta}{p_2}}$$

$$= c_{1,2} \sum_{A,B=\pm} \int_{c_1} \int_{c_2} \frac{d\xi_1}{2\pi i} \frac{d\xi_2}{2\pi i} \hat{V}_3(\theta' + \delta)\hat{V}_3(\theta - \delta)\hat{I}_1^{(A)}(\xi_1)\hat{I}_1^{(B)}(\xi_2):$$

$$\times w_{13}^{(A)}(\xi_1 - \theta' - \delta)\frac{\pi w_{13}^{(A)}(\xi_1 - \theta + \delta)}{\sinh \frac{\xi_1 - \theta - i\pi p_2/2}{p_1}} w_{23}^{(B)}(\xi_2 - \theta' - \delta)\frac{\pi w_{23}^{(B)}(\xi_2 - \theta + \delta)}{\sinh \frac{\xi_2 - \theta}{p_2}} \theta_{12}^{(AB)}(\xi_2 - \xi_1).$$

There are two pinching points as $\theta' \to \theta$. The first one is the same as in the case $\varepsilon' = \varepsilon = +$: the contour $C_1$ in this case is also pinched between the poles at the points $\theta'$ and $\theta$. The second one is in the term $A = B = -$ and involves both integration variables: the contour $C_1$ goes below the point $\theta' - i\pi p_2/2$ and above the point $\xi_2 + i\pi p_2$, while the contour $C_2$ goes above the point $\theta - i\pi p_2$. We want to move the contour $C_1$ so that it would go above the points $\theta + i\pi p_2/2$ and $\theta' - i\pi p_2/2$. So we have to push it through the pole $\theta + i\pi p_2/2$ in the term with $A = +, B = -$ and through the pole $\theta' - i\pi p_2/2$ in the term with $A = B = +$. After all these operations we obtain

$$Z_{++}(\theta' + \delta)Z_{--}(\theta - \delta) = \frac{\rho c_1 C}{\theta' - \theta} (J_0 + J_1 + J_2 + J_3) + O(1)$$

28
Due to the relation $J_0 = \int \frac{d\xi}{2\pi} \cdot V(\theta + \delta)V(\theta - \delta)\hat{I}_2(\theta): \hat{I}_1(\xi) \frac{\pi}{\sinh \frac{\xi - \theta - i\pi p_2}{p_1}}$, we get

$$J_1 = -C' \frac{p_1}{p_2} \cdot V(\theta + \delta)V(\theta - \delta)\hat{I}_1^{(+)}(\theta - \frac{i\pi p_2}{2})\hat{I}_2^{(+)}(\theta);,$$

$$J_2 = -C' \frac{p_1}{\Gamma - p_2} \cdot V(\theta + \delta)V(\theta - \delta)\hat{I}_1^{(+)}(\theta - \frac{i\pi p_2}{2})\hat{I}_2^{(+)}(\theta);,$$

$$J_3 = \frac{C' p_1}{\Gamma \left(1 - \frac{p_2}{p_1}\right) p_2} \cdot V(\theta + \delta)V(\theta - \delta)\hat{I}_1^{(+)}(\theta - \frac{i\pi p_2}{2})\hat{I}_2^{(+)}(\theta - i\pi p_2);,$$

$$C' = \Gamma \frac{\pi r^{-2} p_1 \left(1 - \frac{p_2}{p_1}\right) \Gamma \left(1 + \frac{1}{p_1}\right)}{1 + \frac{1}{p_1}},$$

where the integration contour in $J_0$ goes below the point $\theta - i\pi + i\pi p_2/2$ and above the points $\theta + i\pi p_2/2$, $\theta - i\pi p_2/2$. Similarly, for $\varepsilon' = -$, $\varepsilon = +$ we have

$$Z_{-}^{+}(\theta' + \delta)Z_{-}^{+}(\theta - \delta) = \frac{i c_1 C}{\theta' - \theta} (J_0' + J_1' + J_2' + J_3') + O(1).$$

Here the operators $J_0''$ and $J_1''$ ($i = 1, 2, 3$) have the form

$$J_0'' = -\int \frac{d\xi}{2\pi} \cdot V(\theta + \delta)V(\theta - \delta)\hat{I}_2(\theta): \hat{I}_1(\xi) \frac{\pi}{\sinh \frac{\xi - \theta - i\pi(2-p_2)/2}{p_1}} \sinh \frac{\xi - \theta - i\pi p_2}{p_1},$$

$$J_i'' = J_i \frac{\sinh \frac{\pi(1-p_2)}{p_1}}{\sinh \frac{\pi}{p_1}} \quad (i = 1, 2, 3),$$

where the integration contour in $J_0''$ goes above the points $\theta' + i\pi - i\pi p_2/2$, $\theta + i\pi p_2/2$, $\theta - i\pi p_2/2$. Now we should take the limit $p_2 \to 0$. In this limit the constant $C'$ remains finite:

$$C' = \pi r^{-2} \frac{\Gamma \left(1 - \frac{1}{p_1}\right) \Gamma \left(1 + \frac{1}{p_1}\right) p_2}{1 + \frac{1}{p_1}}, \quad p_2 = 0.$$

It means that

$$J_1 + J_3 = \sqrt{\pi} C' p_1 p_2^{-1/2} Z_0' (\theta) + O(p_2^{1/2}).$$

We want to obtain an expression for $B_{A}^{2,1}(\theta)$ in the limit $p_2 \to 0$. We should consider the difference

$$Z_{+}^{+}(\theta' + \delta)Z_{-}^{+}(\theta - \delta) - Z_{-}^{+}(\theta' + \delta)Z_{+}^{+}(\theta - \delta).$$

Due to the relation

$$1 + \frac{\sinh \frac{\xi - \theta - i\pi(2-p_2)/2}{p_1}}{\sinh \frac{\xi - \theta - i\pi p_2}{p_1}} = \frac{2 \sinh \frac{\xi - \theta}{p_1} \cosh \frac{i\pi(2-p_2)}{2p_1}}{\sinh \frac{\xi - \theta - i\pi(2-p_2)/2}{p_1}} = \frac{2 \sinh \frac{\xi - \theta}{p_1} \cosh \frac{i\pi}{p_1}}{\sinh \frac{\xi - \theta - i\pi}{p_1}} + O(p_2)$$

we have

$$c_1 C (J_0 - J_0') = -\frac{c_1 C}{c} \sqrt{2 \cos \frac{\pi}{p_1} Z_0(\theta)} + O(p_2) = -\Gamma K_{A}^{2,1} Z_0(\theta) + O(p_2). \quad (D.4)$$

Similarly, from the relation

$$1 - \frac{\sinh \frac{i\pi(1-p_2)}{p_1}}{\sinh \frac{i\pi}{p_1}} = \frac{2 \sinh \frac{i\pi p_2}{2p_1} \cosh \frac{i\pi(2-p_2)}{2p_1}}{\sinh \frac{i\pi}{p_1}} = O(p_2)$$
we obtain
\[ J_1 + J_3 - J'_1 - J'_3 = O(p_2^{1/2}), \quad J_2 - J'_2 = O(p_2). \]
Hence, the only finite contribution comes from the difference \( J_0 - J'_0 \). The equations \( B^{2,1}_2(\theta) = -Z_0(k, \kappa|\theta) \) with \( Z_0(k, \kappa|\theta) \) defined in Eq. \( 5.22a \).
To obtain an expression for \( B^{2,1}_2(\theta) \) we should consider the sum
\[ Z_{++}(\theta' + \delta)Z_{--}(\theta - \delta) + Z_{++}(\theta' + \delta)Z_{+-}(\theta - \delta). \]
It is easy to check that
\[ 1 - \frac{\sinh \frac{\xi - i\theta}{p_1}}{\sinh \frac{\xi - i\pi(2-p_2)/2}{p_1}} \frac{2\cosh \frac{\xi - \theta}{p_1}}{\sinh \frac{\xi - i\pi(2-p_2)/2}{p_1}} = 2 - \frac{2\cosh \frac{\xi - \theta}{p_1}}{\sinh \frac{\xi - i\pi}{p_1}} + O(p_2) \]
and hence,
\[ J_0 + J'_0 = O(1). \]
Similarly, from the relation
\[ 1 + \frac{\sinh \frac{\xi(1-p_2)}{p_1}}{\sinh \frac{\xi}{p_1}} \frac{2\cosh \frac{i\pi p_2/2}{p_1}}{\sinh \frac{\xi}{p_1}} = 2 + O(p_2) \]
we have
\[ J_1 + J_3 + J'_1 + J'_3 = 2\sqrt{\pi}c'p_1p_2^{-1/2}Z_0'(\theta) + O(p_2^{1/2}), \quad J_2 + J'_2 = O(1). \]
It is straightforward to find that
\[ \lim_{p_2 \to 0} 2\sqrt{\pi}c'p_1 = \lim_{p_2 \to 0} p_2^{1/2} K_{2,1}^{2,1}. \]
Together with Eq. \( 4.38 \) it proves the expression \( 4.38 \) for \( B^{2,1}_2(\theta) \).
Derivation of the operator \( Z_-(\theta) \) by taking the limit of Eq. \( 4.10 \) for \( \epsilon' = \epsilon = - \) would be very complicated. In fact, we do not need it at all. It is natural to suppose that the operator \( Z_-(\theta) \) has the form of a linear combination:
\[ C_{++}S_2^2(\theta)V(\theta) + C_{+-}S_+(\theta)V(\theta)S_-(\theta) + C_{-+}S_-(\theta - 2\pi i)V(\theta)S_+(\theta + 2\pi i) + C_{--}V(\theta)S_2^2(\theta). \quad (D.5) \]
All four terms in this combination are different: though the integrands are identical, the integration contours are not the same. To see it, we consider, for example, the difference
\[ \Delta(\theta) = S_+(\theta)V(\theta)S_-(\theta) - S_2^2(\theta)V(\theta). \]
We can expand it as
\[ \Delta(\theta) = \sum_{A,B,C} \Delta^{(ABC)}(\theta), \]
where
\[ \Delta^{(ABC)}(\theta) = e^2V(A)(\theta) \int_{C_+} \frac{d\xi_1}{2\pi i} \int_{C_+} \frac{d\xi_2}{2\pi i} \hat{I}^{(B)}(\xi_1) \hat{I}^{(C)}(\xi_2) \frac{\pi}{\sinh \frac{\xi_1 - i\theta - 2\pi}{p}} \frac{\pi}{\sinh \frac{\xi_2 - i\theta - 2\pi}{p}}, \]
\[ - e^2V(A)(\theta) \int_{C_+} \frac{d\xi_1}{2\pi i} \int_{C_+} \frac{d\xi_2}{2\pi i} \hat{I}^{(B)}(\xi_1) \hat{I}^{(C)}(\xi_2) \frac{\pi}{\sinh \frac{\xi_1 - i\theta - 2\pi}{p}} \frac{\pi}{\sinh \frac{\xi_2 - i\theta - 2\pi}{p}}, \]
\[ \hat{V}^{(+)}(\theta) = iV^{(+)}(\theta)e^{-\kappa+k\theta}, \quad \hat{V}^{(-)}(\theta) = V^{(-)}(\theta)e^{\kappa+k\theta}. \]
The operator \( \Delta^{(ABC)}(\theta) \) is a difference of two terms, which are distinguished by the integration contour for the variable \( \xi_2 \). For this reason the calculation amounts to finding the residue at the point \( \xi_2 = \theta \):
\[ \Delta^{(ABC)}(\theta) = \text{Res}_{\xi_2=\theta} e^{2V(A)(\theta)} \int_{C_+} \frac{d\xi_1}{2\pi i} \hat{I}^{(B)}(\xi_1) \hat{I}^{(C)}(\xi_2) \frac{\pi}{\sinh \frac{\xi_1 - i\theta - 2\pi}{p}} \frac{\pi}{\sinh \frac{\xi_2 - i\theta - 2\pi}{p}}. \]
The only nonzero contribution comes from the terms with $A = C$. As in this case
\[
\text{Res}_{\xi=\theta} \tilde{V}^{(+)}(\theta) \tilde{I}^{(+)}(\xi) = -\text{Res}_{\xi=\theta} \tilde{V}^{(-)}(\theta) \tilde{I}^{(-)}(\xi) = D \equiv \pi pr^{-2} \Gamma(1-1/p) \Gamma(1/p),
\] (D.6)
we obtain that integrands for the operators $\Delta^{(++++)}(\theta)$ and $-\Delta^{(----)}(\theta)$ coincide. The same fact takes place for the operators $\Delta^{(--\cdot)}(\theta)$ and $-\Delta^{(\cdot++)}(\theta)$. If we consider the contour in the variable $\xi_1$ for the last pair, it is the same for the operators $\Delta^{(--\cdot)}(\theta)$ and $\Delta^{(\cdot++)}(\theta)$. It means that $\Delta^{(--\cdot)}(\theta) + \Delta^{(\cdot++)}(\theta) = 0$. We have a different situation for the operators $\Delta^{(++++)}(\theta)$ and $\Delta^{(----)}(\theta)$. In this case, both integrands possess poles at $\xi_1 = \theta - i\pi$, but the origin of these poles is not the same. In the term $\Delta^{(++++)}(\theta)$ it comes from the product $\tilde{V}^{(-)}(\theta) \tilde{I}^{(+)}(\xi_1)$ and the contour for the variable $\xi_1$ must go below it (see Fig. 3). In the term $\Delta^{(++++)}(\theta)$ the pole arises from the product $\tilde{I}^{(+)}(\xi_1) \tilde{I}^{(+)}(\theta)$ and the contour for the variable $\xi_1$ must go above it. It means that
\[
\Delta(\theta) = \Delta^{(++++)}(\theta) + \Delta^{(----)}(\theta) = c^2 \text{Res}_{\xi_1=\theta-i\pi} \text{Res}_{\xi_2=\theta} \tilde{V}^{(-)}(\theta) \tilde{I}^{(+)}(\xi_1) \tilde{I}^{(-)}(\xi_2) \frac{\pi}{\sinh \frac{\xi_1-a-i\pi}{p}} \frac{\pi}{\sinh \frac{\xi_2-a-i\pi}{p}}
\]
\[
= -c^2D\frac{\pi^2p}{\sinh \frac{\pi}{p}} \tilde{I}^{(+)}(\theta-i\pi).
\]
It proves the first line of Eq. (12.20). The second line is proved similarly.

To fix the coefficients $C_{AB}$ we have to analyze the commutation relations for $Z_I(\theta)$. It will be shown below that the only solution consistent with the commutation relations is the solution \[322\], i.e.
\[
C_{++} = \text{const}, \quad C_{-+} = C_{++} = C_{-+} = 0.
\] (D.7)

Now we can prove the commutation relations of the ZF algebra for the sausage model. This problem is more difficult than in the case of the operators $Z_{ce}(\theta)$. The proof of the relations for $Z_+(\theta_1)Z_+(\theta_2)$, $Z_+(\theta_1)Z_-(\theta_2)$, and $Z_-(\theta_2)Z_+(\theta_1)$ is in the same and reduces to the identities
\[
S(\theta)_{++} = \tilde{g}_{33}(\theta) / \tilde{g}_{33}(-\theta).
\]
\[
\frac{1}{\sinh(x-y_2-z)} = \frac{1}{\sinh(y_1-y_2)} \frac{1}{\sinh(y_1-y_2-2z)} \frac{1}{\sinh(x-y_2)(x-y_1-z)}
\]
\[
- \frac{1}{\sinh(y_1-y_2-2z)} \frac{1}{\sinh(x-y_1-z)}
\]
According to Eq. (6.15) a single contour can cross the pole at $\xi = \theta$. Hence, the contour in the l. h. s. of the commutation relation can be deformed into that in the r. h. s. The situation with
\[
Z_0(\theta_1)Z_0(\theta_2) = S(\theta_1-\theta_2)_0^{-0} Z_-(\theta_2)Z_+(\theta_1) + S(\theta_1-\theta_2)^{-0}_0 Z_+(\theta_2)Z_-(\theta_1) + S(\theta_1-\theta_2)^0_0 Z_0(\theta_2)Z_0(\theta_1)
\] (D.8)
is more complicated. First, as it was mentioned in Sec. 5, the contours in the definition of the operator $Z_-(\theta_1)$ cannot cross the pole at $\theta_2$. Second, the contour in the l. h. s. in the operator $Z_0(\theta_1)$ goes above the point $\theta_2$ and the contour in $Z_0(\theta_2)$ goes below $\theta_1$. On the other hand, in the r. h. s. a contour must go above the point $\theta_1$ if it belongs to the operator $Z_1(\theta_2)$ and below $\theta_2$ if it belongs to $Z_1(\theta_1)$.

We shall prove this identity in two steps. On the first step we shall consider all integrations so as if they were done along the same contour and show that the symmetrized integrands of the l. h. s. and the r. h. s. coincide. On the second step we shall move all contours to a single one and show that the residues due to the encountered poles cancel each other.

For any product $X$ of the operators $Z_1(\theta_1)$ and $Z_1(\theta_2)$ denote by $[X]$ this product, but with the contours in the screening operators going for example above $\theta_1$ and below $\theta_2$. We denote as
\[
\Delta[X] = X - [X].
\]

On the first step we prove that
\[
[Z_0(\theta_1)Z_0(\theta_2)] = S(\theta_1-\theta_2)_0^{-0} [Z_-(\theta_2)Z_+(\theta_1)] + S(\theta_1-\theta_2)^{-0}_0 [Z_+(\theta_2)Z_-(\theta_1)] + S(\theta_1-\theta_2)^0_0 [Z_0(\theta_2)Z_0(\theta_1)].
\] (D.9)
On the second step we prove the equation

\[ \Delta[Z_0(\theta_1)Z_0(\theta_2)] = S(\theta_1 - \theta_2)_{00}^{+0} \Delta[Z_-(\theta_2)Z_+(\theta_1)] + S(\theta_1 - \theta_2)_{00}^{+0} \Delta[Z_+(\theta_2)Z_-(\theta_1)] + S(\theta_1 - \theta_2)_{00}^{00} \Delta[Z_0(\theta_2)Z_0(\theta_1)]. \]  

(D.10)

Combining Eqs. (D.9) and (D.10) we obtain Eq. (D.8).

The first step is performed just as in Appendix C and reduces the proof of Eq. (D.9) to a check of the identity of the form (C.4) with

\[ f(x_1, x_2) = \frac{-2 \cosh z}{\sinh(x_1 - y_2 - z) \sinh(x_2 - y_2 - z) \sinh(x_1 - y_2 - z)} \sinh(y_1 - y_2) \sinh(2z) \times \frac{1}{\sinh(x_1 - y_2) \sinh(x_2 - y_2)} \times \frac{1}{\sinh(x_1 - y_2) \sinh(x_2 - y_2) \sinh(x_1 - y_2 - z) \sinh(x_2 - y_2 - z)} \sinh(x_1 - y_2 - z) \sinh(x_2 - y_2 - z) \sinh(x_1 - y_2 - z) \sinh(x_2 - y_2 - z). \]

This computation would lead to the proof for operators with the same contours. In our case, however, the contours are different and on the second step we should take this fact into account.

Now we calculate \( \Delta[Z_0(\theta_1)Z_0(\theta_2)] \). We have to proceed in the same way as in the calculation of the operator \( \Delta(\theta) \) above. We skip the complete analysis and write the contributions that come from the deformation of the contours. We have to push the contour \( C_1 \) attached to \( Z_0(\theta_1) \) down across the point \( \theta_2 \) and the contour \( C_2 \) attached to \( Z_0(\theta_2) \) up across \( \theta_1 \). The result will have the form

\[ \Delta[Z_0(\theta_1)Z_0(\theta_2)] = -2c^2 \cos \frac{\pi}{p} \times \left( \text{Res}_{\xi_1 = \theta_2, \xi_2 = \theta_2 + i\pi} \frac{\hat{V}(\theta_1)\hat{I}^{(+)}(\xi_1)\hat{V}(\theta_2)\hat{I}^{(-)}(\xi_2)}{\sinh(\frac{\pi}{p} - \theta_1 - i\pi)\sinh(\frac{\pi}{p} - \theta_2 - i\pi)} + \text{Res}_{\xi_1 = \theta_1 - i\pi, \xi_2 = \theta_1} \frac{\hat{V}(-)\hat{I}^{(+)}(\xi_1)\hat{V}(\theta_2)\hat{I}^{(-)}(\xi_2)}{\sinh(\frac{\pi}{p} - \theta_1 - i\pi)\sinh(\frac{\pi}{p} - \theta_2 - i\pi)} \right). \]

Using Eq. (D.6) and the commutation relations (D.8) we obtain

\[ \Delta[Z_0(\theta_1)Z_0(\theta_2)] = -2\pi^2 c^2 Dp \cos \frac{\pi}{p} \left( \hat{V}(-)(\theta_2 + i\pi)\hat{V}(\theta_1) - \hat{V}(\theta_2)\hat{I}^{(+)}(\theta_1 - i\pi) \right) \frac{1}{\sinh(\frac{\pi}{p} - \theta_1 - i\pi)\sinh(\frac{\pi}{p} - \theta_2 - 2\pi i)}. \]

Similarly, for the terms of the r.h.s. of Eq. (D.10) we find

\[ \Delta[Z_0(\theta_2)Z_0(\theta_1)] = 0, \]

\[ \Delta[Z_-(\theta_2)Z_+(\theta_1)] = c^2 \text{Res}_{\xi_1 = \theta_2, \xi_2 = \theta_2 + i\pi} \frac{\hat{V}^{(+)}(\theta_2)\hat{I}^{(+)}(\xi_1)\hat{V}(\theta_1)}{\sinh(\frac{\pi}{p} - \theta_1 - i\pi)\sinh(\frac{\pi}{p} - \theta_2 - i\pi)} = -2\pi^2 c^2 Dp \hat{I}^{(+)}(\theta_2) - i\pi \hat{V}(\theta_1) \frac{1}{\sinh(\frac{\pi}{p})}, \]

\[ \Delta[Z_+(\theta_2)Z_-(\theta_1)] = c^2 \text{Res}_{\xi_1 = \theta_1 - i\pi, \xi_2 = \theta_2} \frac{\hat{V}^{(-)}(\theta_2)\hat{I}^{(+)}(\xi_1)\hat{V}(\theta_1)}{\sinh(\frac{\pi}{p} - \theta_1 - i\pi)\sinh(\frac{\pi}{p} - \theta_2 - i\pi)} = \pi^2 c^2 Dp \hat{I}^{(+)}(\theta - i\pi)\hat{V}(\theta_2) \frac{1}{\sinh(\frac{\pi}{p})}. \]
for the operator $Z_-(\theta)$ defined by Eq. (5.22a), i.e. for the case Eq. (D.7). Taking into account Eq. (5.5) for the explicit form of the $S$ matrix of the sausage model, we obtain Eq. (D.10), which completes the proof of Eq. (D.8). The proof of this identity is a basic one, because it fixes the form of the operator (5.22a) from that of the operators (5.22b) and (5.22c). Indeed, if we would allow all terms in Eq. (D.5) the value $\Delta[Z_+(\theta_2)Z_-(\theta_1)]$ would contain some extra terms, which are absent in $\Delta[Z_0(\theta_1)Z_0(\theta_2)]$.

The commutation relations for $Z_+(\theta_1)Z_-(\theta_2)$ can be proved in the same way. For $Z_0(\theta_1)Z_-(\theta_2)$ and $Z_-(\theta_1)Z_0(\theta_2)$ the proof is a little more complicated: it remains one integration in products like $\Delta[Z_0(\theta_1)\times Z_-\theta_2])$. First, we need to make sure that the integrations in both sides of the equation are taken along the same contour. Then we have to check that the integrands in both sides coincide identically. One more complication appears in the commutation relation for $Z_-(\theta_1)Z_-(\theta_2)$. There is a double integration in this case and we have to symmetrize the integrands in the integration variables before comparing them. We have checked all these relations, but we omit the calculations as they are very cumbersome.

The normalization constant $c$ in Eq. (5.22) can be calculated as follows. Consider the operator product $Z_+(\theta')Z_-(\theta)$. It can be shown that the only contribution to the residue at the pole $\theta' = \theta + i\pi$ comes from the product

$$c^2 \int_{C_+} \frac{d\xi_1}{2\pi} \frac{d\xi_2}{2\pi} \hat{V}(-\theta')\hat{V}(\theta)\hat{f}^+(\xi_1)\hat{f}^-(\xi_2) \frac{\pi}{\sinh \frac{\xi_1 - \theta - i\pi}{p}} \frac{\pi}{\sinh \frac{\xi_2 - \theta - i\pi}{p}}$$

$$= c^2 \int_{C_+} \frac{d\xi_1}{2\pi} \frac{d\xi_2}{2\pi} \hat{V}(-\theta')\hat{V}(\theta)\hat{f}^+(\xi_1)\hat{f}^-(\xi_2) \frac{\pi}{\sinh \frac{\xi_2 - \theta - i\pi}{p}} \frac{\pi}{\sinh \frac{\xi_1 - \theta - i\pi}{p}}$$

There are two effects here. First, there is pinching of the contour for the variable $\xi_1$ between the points $\xi_1 = \theta$ and $\xi_1 = \theta' - i\pi$ and of the contour for $\xi_2$ between the points $\xi_2 = \theta + i\pi$ and $\xi_2 = \theta'$. This gives a double pole in $\theta' - \theta$. Second, the function $\hat{g}^{(+)}(\theta - \theta')$ has a zero at $\theta - \theta' = -i\pi$. It decreases the multiplicity of the pole by one. Finally, we have

$$Z_+(\theta')Z_-(\theta) = -c^2 \hat{g}^{(+)}(-i\pi)(\theta - \theta')^{-1} \frac{\pi^2 \hat{g}^{(+)}(\theta - \theta')}{\hat{g}^{(+)}(\xi_1 - \theta - i\pi)\hat{g}^{(+)}(\xi_2 - \theta - i\pi)\sinh \frac{\xi_1 - \theta - i\pi}{p}\sinh \frac{\xi_2 - \theta - i\pi}{p}} + O(1)$$

$$= -\frac{i}{\theta' - \theta - i\pi} c^2 p^{-4}(\pi p)^4 \frac{\Gamma^3(1 - 1/p)}{\Gamma(1/p)} + O(1).$$

Demanding that the coefficient at the pole $-i/(\theta' - \theta - i\pi)$ was equal to 1, we obtain Eq. (5.26).

Appendix E. First bound state in the cosine-cosine model

Here we calculate explicitly the vertex operator $B^{1,1}_{2}(\theta)$ corresponding to the highest component $(Q = 2)$ of the quadruplet of mass $M_1$. It appears in the product

$$Z_{++}(\theta' + \frac{i\pi}{2}(1 - p_1))Z_{--}(\theta - \frac{i\pi}{2}(1 - p_1))$$

in the limit $\theta' \rightarrow \theta$. Using Eqs. (3.24a), (3.24b), (3.26a), (3.26b), we obtain for this quantity:

$$c_1 r_{p_1}^2 q_{43}(\theta - \theta' - 2\delta) \int \frac{d\xi}{2\pi} \left( V(\theta' + \delta) V(\theta - \delta) F_1^{(+)}(\xi) e^{i(\xi - (k_1 + k_2)(\theta + \theta'))/2 - k_1\xi} \right)$$

$$\times \frac{1}{p_1} \Gamma \left( \frac{i(\xi - \theta)}{p_1} - \frac{1}{2} \right) \Gamma \left( \frac{i(\xi - \theta)}{p_1} + \frac{1}{2} \right) \Gamma \left( \frac{i(\xi - \theta)}{p_1} + \frac{1}{2} \right)$$

33
Appendix F. Regime II

In this regime the theory is completely symmetric under the permutations of the pairs \( (p_1, \varphi_1), (p_2, \varphi_2), (p_3, \varphi_3) \) and possesses the \( U(1) \times U(1) \times U(1) \) symmetry described by the topological charges

\[
Q_i = \int dx^1 j_{i,0}, \quad j_i^\mu = \frac{\alpha_i}{\pi} e^{\mu \nu} \partial_\nu \varphi_i.
\]

The values of the charges must satisfy the conditions

\[
Q_i \in \mathbb{Z}, \quad Q_i + Q_j \in 2\mathbb{Z}.
\]
We would like to study this regime by means of the free field representation technique. In Sec. 4 we introduced the vertex operators $Z_{e,e'}^{i}(\theta)$ defined in the regions I, according to Eq. 112. We assume that these vertex operators can be analytically continued to the region II. It means that the spectrum in the regime II consists of three quadruplets of fundamental particles $z_{\nu}^{i}, i \in \mathbb{Z}$ (z and z' are eigenvalues of the operators $Q_{i+1}$ and $Q_{i+2}$ respectively) with the masses

$$m_{i} = M_{0} \sin \frac{\pi p_{i}}{2}, \quad M_{0} = \frac{\mu}{\pi^{2}} \Gamma \left( \frac{p_{1}}{2} \right) \Gamma \left( \frac{p_{2}}{2} \right) \Gamma \left( \frac{p_{3}}{2} \right),$$

and their bound states. The $S$ matrices of the fundamental particles are derived from the commutation relations of the vertex operators $Z_{e,e'}^{i}(\theta)$. From the free field representation described in the present paper we can find that

$$Z_{e_{1}e_{1}'}(\theta_{1})Z_{e_{2}e_{2}'}^{i}(\theta_{2}) = - \sum_{e_{3}e_{4}'} S_{p_{i+1}}(\theta_{1} - \theta_{2}) \epsilon_{e_{1}e_{2}'}^{3} \epsilon_{e_{3}e_{4}'} \epsilon_{e_{2}e_{1}'}^{4} S_{p_{i+2}}(\theta_{1} - \theta_{2}) \epsilon_{e_{3}e_{4}'}^{i} Z_{e_{3}e_{3}'}^{i}(\theta_{2}),$$

$$Z_{e_{1}e_{1}'}^{i}(\theta_{1})Z_{e_{2}e_{2}'}^{i+1}(\theta_{2}) = \epsilon \epsilon' \sum_{e_{3}e_{4}'} \hat{S}_{p_{i+2}}(\theta_{1} - \theta_{2}) \epsilon_{e_{1}e_{2}'}^{i} \epsilon_{e_{3}e_{4}'}^{i+1} Z_{e_{3}e_{3}'}^{i}(\theta_{2})Z_{e_{3}e_{3}'}^{i}(\theta_{1}), \quad (F.5a)$$

with

$$\hat{S}_{p}(\theta)_{++} = \hat{S}_{p}(\theta)_{--} = \exp \left( -i \int_{0}^{\infty} dt \frac{\tan \frac{\pi t}{2} \sin \theta t}{\sin \frac{\pi t}{2}} \right).$$

$$= \prod_{n=0}^{\infty} \frac{\Gamma^{2} \left( \frac{\pi p + 2n + 1}{2} + \frac{1}{2} \right) \Gamma \left( \frac{\pi p + 2n}{2} + \frac{1}{2} \right) \Gamma \left( \frac{\pi p}{2} + 2n + 1 \right) \Gamma \left( \frac{\pi p}{2} + 2n + 2 \right)}{\Gamma^{2} \left( \frac{\pi p + 2n + 1}{2} + \frac{1}{2} \right) \Gamma \left( \frac{\pi p + 2n}{2} + \frac{1}{2} \right) \Gamma \left( \frac{\pi p}{2} + 2n + 1 \right) \Gamma \left( \frac{\pi p}{2} + 2n + 2 \right)}, \quad (F.6)$$

$$\hat{S}_{p}(\theta)_{+-} = \hat{S}_{p}(\theta)_{-+} = \frac{\cosh \frac{\theta}{2}}{\cosh \frac{\pi p}{2}} \hat{S}_{p}(\theta)_{++}, \quad \hat{S}_{p}(\theta)_{-+} = \hat{S}_{p}(\theta)_{+-} = - \frac{\sin \frac{\theta}{2}}{\cosh \frac{\pi p}{2}} \hat{S}_{p}(\theta)_{++}.$$

Note that the matrix $\hat{S}_{p}(\theta)$ can be expressed in terms of the sine-Gordon $S$ matrix:

$$\hat{S}_{p}(\theta) = i \tan \left( \frac{\theta}{2} + \frac{\pi p}{2} \right) S_{p} \left( \theta + \frac{\pi p}{2} \right). \quad (F.6)$$

The proof of the relations (F.5b), (F.6) is similar to that of the commutation relations for $Z^{3}(\theta)$ given in Appendix C and we omit it.

The interpretation of the result is the following. Consider the $S$ matrix of two particles $z^{i}$ with the rapidity $\theta_{1}$ and $z^{j}$ with the rapidity $\theta_{2}$. The physical scattering process takes place for $\theta > 0$. Then for $j = i$ the $S$ matrix is given by the tensor product $S_{i} = -S_{p_{i+1}}(\theta) \otimes S_{p_{i+2}}(\theta)_{-},$ where the first tensor component acts on the space related to the topological charge $Q_{i+1}$ and the second one on the space related to the topological charge $Q_{i+2}$. For $j = i + 1$ the $S$ matrix is given by $S_{i,i+1}(\theta) = Q_{i} Q_{i+1} S_{p_{i+2}}(-\theta)_{-},$ where $S_{p_{i+2}}(\theta)$ acts in the space of the topological charge $Q_{i+2}$.

Now consider the bound states. There are two types of bound states. Bound states of the first type are related to the poles of the $S$ matrix for two fundamental particles of the same kind and are completely analogous to those in the regime I, described by Eqs. 4.14, 4.15. Explicitly, for any $p_{i} < 1$ ($i = 1, 2, 3$) we have series of poles of the $S$ matrices $S_{jj}(\theta)$ with $j = i \pm 1$ at the points $\theta = \nu u_{i,n}$ with $u_{i,n}$ given by Eq. (2.11). The masses of the corresponding bound states $b^{j;i,n}, \nu = +, S, A, -$ are given by

$$M_{j;i,n} = 2m_{i} \sin \frac{\pi p_{i+1}}{2}, \quad j \neq i, \quad n = 1, 2, \ldots, \quad np_{i} < 1. \quad (F.7)$$

It is important to note that

$$M_{j;i,1} = M_{j;j,1} = 2M_{0} \sin \frac{\pi p_{i}}{2} \sin \frac{\pi p_{j}}{2}, \quad j \neq i. \quad (F.8)$$

It can be checked that the quadruplets $b^{j;i,1}$ and $b^{j;j,1}$ are the same.
Bound states of the second type correspond to the poles of the $S$ matrix $S_{ij}(\theta)$ for $i \neq j$. Let $k \neq i, j$. Then the function $\hat{S}_k(\theta)$ possesses poles at the points

$$\theta = i\pi - i\frac{\pi}{2}p_k - i\pi p_k n, \quad n = 0, 1, 2, \ldots, \quad n < \frac{1}{p_k} - \frac{1}{2}. \quad (F.9)$$

The squares of the masses of the corresponding bound states read

$$m_{k,n}^2 = M_0^2 \left( \sin^2 \frac{\pi p_i}{2} + \sin^2 \frac{\pi p_j}{2} - 2 \sin \frac{\pi p_i}{2} \sin \frac{\pi p_j}{2} \cos \left( \frac{\pi p_k + \pi p_k n}{2} \right) \right). \quad (F.10)$$

We denote these bound states as $z_{k,n}^{\varepsilon, \varepsilon'}$, $\varepsilon, \varepsilon' = \pm$. Their multiplet structure is the same as that of the fundamental particle $z^k$. Moreover, $m_{k,0} = M_0 \sin \frac{\pi p_k}{2} = m_k$, and the lightest bound states $z_{k,0}^{\varepsilon, \varepsilon'}$ coincide with the fundamental particles $z^k$. The other bound state quadruplets $z_{k,n}^{\varepsilon, \varepsilon'}$, $n \geq 1$ appear if $p_k < 2/3$. Evidently, the masses of these states are in the range

$$m_k < m_{k,n} < m_i + m_j, \quad n = 1, 2, \ldots, \quad n < \frac{1}{p_k} - \frac{1}{2}.$$
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