Information-Disturbance Theorem for Mutually Unbiased Observables

Takayuki Miyadera * and Hideki Imai * †

* Research Center for Information Security (RCIS), National Institute of Advanced Industrial Science and Technology (AIST).
Daibiru building 1102, Sotokanda, Chiyoda-ku, Tokyo, 101-0021, Japan.
(e-mail: miyadera-takayuki@aist.go.jp)

† Institute of Industrial Science, University of Tokyo.
4-6-1 Komaba, Meguro-ku, Tokyo 153-8505, Japan.

Abstract

We derive a novel version of information-disturbance theorems for mutually unbiased observables. We show that the information gain by Eve inevitably makes the outcomes by Bob in the conjugate basis not only erroneous but random.

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I. INTRODUCTION

In 1984, Bennett and Brassard [1] proposed a quantum key distribution protocol which is now called as BB84 protocol. Its unconditional security was first proved by Mayers [2] in 1996 and after his proof the various proofs [3–5] have appeared. Among them, a proof by Biham, Boyer, Boykin, Mor, and Roychowdhury [4] is based upon a so-called information disturbance theorem. According to the theorem, the information gain by Eve inevitably induces errors in outcomes obtained by Bob. This disturbance enables Alice and Bob to notice the existence of eavesdroppers. As well as its application to BB84 protocol, since it can be regarded as an information theoretic version of uncertainty relation, the theorem has attracted many authors [6–8]. Recently, Boykin and Roychowdhury [9] showed a simple proof of the theorem in an arbitrary dimension by using purification technique and trace norm inequality. We, in this paper, derive a different version of the theorem. Our information-disturbance theorem is an inequality between the information gain by Eve and the randomness (rather than error probability) of the outcomes obtained by Bob. We compare our theorem with the previous one and discuss its implication.

II. SETTING

Let us begin with a setting. Three characters, Alice, Bob and Eve play their roles. Our setting is a simplified version of BB84 quantum key distribution protocol. The following analysis, however, can be applied to the full BB84 protocol with public discussion procedures. Let us consider two pairs of orthogonal states, $b := \{|0\rangle, |1\rangle\}$ and its conjugate $\overline{b} := \{|\overline{0}\rangle, |\overline{1}\rangle\}$ in $\mathbb{C}^2$. They are assumed mutually unbiased with each other. That is,

$$\langle i | \overline{k} \rangle = \sqrt{\frac{1}{2}} (-1)^{ik}$$

holds for each pair of $i, k \in \{0, 1\}$. Alice first selects $b$ or $\overline{b}$ which is used to encode a random number. Alice next randomly generates an $N$-bits sequence $i \in \{0, 1\}^N$ with probability $p(i) = \frac{1}{2^N}$. We write $A$ a random variable representing this $N$-bits sequence. Alice encodes
this information on $N$-qubits and sends them to Bob. For instance, suppose that Alice selects $b$ and generates a sequence $i = i_1 i_2 \cdots i_N$, she sends the corresponding state $|i\rangle = |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_N\rangle \in \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 =: \mathcal{H}_A \simeq \mathcal{H}_B$ to Bob. If the conjugate basis $\overline{b}$ and a sequence $j = j_1 j_2 \cdots j_N$ are chosen, the state sent to Bob is $|j\rangle = |j_1\rangle \otimes |j_2\rangle \otimes \cdots \otimes |j_N\rangle \in \mathcal{H}_A$.

Alice, after confirming that Bob actually has received $N$-qubits, informs him of the basis she used. Bob makes a measurement with respect to the basis and obtains an outcome. Let us write $B$ the random variable representing this outcome. If there is no eavesdropper, $A = B$ naturally follows. Eve wants to obtain the information of the random variable $A$. For the purpose, Eve prepares an apparatus and makes it interact with the $N$-qubits sent to Bob by Alice. Let us denote $\mathcal{H}_E$ a Hilbert space describing Eve’s apparatus. In general, Eve’s operation is described by a unitary operator $U$,

$$U : \mathcal{H}_E \otimes \mathcal{H}_A \to \mathcal{H}_E \otimes \mathcal{H}_B$$

$$|0\rangle \otimes |i\rangle \mapsto \sum_j |E_{ij}\rangle \otimes |j\rangle,$$  \hspace{1cm} (1)

where $|0\rangle$ is a normalized vector in $\mathcal{H}_E$ and $\{|E_{ij}\rangle\} \subset \mathcal{H}_E$ satisfies unitarity condition: $\sum_{j \in \{0,1\}^N} \langle E_{ij} | E_{kj} \rangle = \delta_{ik}$. After this interaction, Eve tries to make an optimal measurement on her apparatus to extract the information of $A$.

### III. INFORMATION-DISTURBANCE THEOREM

#### A. Information v.s. Error

One can show that if Eve’s operation yields herself to gain large information, error probability in qubits sent to Bob in the conjugate basis becomes inevitably large. It has been called as information-disturbance theorem and was proved in [4,9].

The representation (1) depends upon the choice of the basis. It is useful to rewrite the same unitary operator in the conjugate basis, $\overline{b}$. Using $|\overline{l}\rangle = \sum_{l \in \{0,1\}^N} |l\rangle \langle l|_{\overline{b}}$ and $|i\rangle = \sum_{j \in \{0,1\}^N} |j\rangle \langle j|i\rangle_{\overline{b}}$, we obtain $U|0\rangle \otimes |\overline{l}\rangle = \sum_{s \in \{0,1\}^N} |E_{ls}\rangle \otimes |s\rangle$, where $|E_{ls}\rangle := \sum_{i,j \in \{0,1\}^N} |E_{ij}\rangle \langle s|j\rangle \langle i|l\rangle$. 

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When Alice chooses basis $b$ and a sequence $i \in \{0,1\}^N$, a state obtained by Eve is computed as $\rho_{Eve}^i := \sum_{j \in \{0,1\}^N} |E_{ij}\rangle \langle E_{ij}|$. Later we consider how much information Eve can extract from it. When Alice chooses another basis $\overline{b}$ and a sequence $i$, Bob obtains a state $\rho_{Bob}^i = \sum_{j,l \in \{0,1\}^N} \langle E_{il}|E_{ij}\rangle |j\rangle \langle l|$. Later we consider the error it induces to the outcome.

Let us begin with Eve’s information gain. Eve performs a measurement (POVM) $X := \{X_\alpha\}$ on her state. (POVM is a family of positive operators satisfying $\sum_\alpha X_\alpha = 1$.) We put $E[X]$ a random variable representing the outcome. Probability to obtain an outcome $\alpha$ is $p(\alpha| i, b) = \text{tr} (X_\alpha \rho_{Eve}^i)$. Information gain by Eve with respect to a POVM $X$ is calculated as,

$$I(A : E[X]|b) = H(A|b) + H(E[X]|b) - H(A , E[X]|b) = \frac{1}{2^N} \sum_\alpha \sum_i p(\alpha| i) \left( \log p(\alpha| i) - \log \sum_j p(\alpha| j) \right) + N,$$

where $H(\cdot)$ means Shannon entropy. What we are interested in is its optimal value with respect to all the possible measurements by Eve:

$$I(A : E|b) := \sup \{I(A : E[X]|b) | X = \{X_\alpha\} \text{is a POVM in } H_E \}.$$

Now we consider outcomes obtained by Bob in the conjugate basis. Remind that when Alice chooses basis $\overline{b}$, the state sent to Bob is (2). Bob makes a measurement of an observable $\sum j |j\rangle \langle j|$. We put $B$ a random variable for this outcome. The probability to obtain each outcome is expressed as $p(j| i, \overline{b}) = \langle E_{ij}|E_{ij}\rangle$. Thus probability to obtain an outcome whose difference from input is $c \in \{0,1\}^N$, is

$$p(B = A \oplus c | \overline{b}) := \sum_i \frac{1}{2^N} p(i \oplus c| i, \overline{b})$$

$$= \frac{1}{2^N} \sum_i \langle E_{i,i\oplus c}|E_{i,i\oplus c}\rangle,$$

where the symbol “$\oplus$” is a bit-wise XOR operation. By use of these quantities, the information-disturbance theorem obtained by Boykin and Roychowdhury is expressed as [10]
\[ I(A : E|b) \leq 4N \sqrt{\sum_{c \neq 0} p(B = A \oplus c|b)}, \tag{4} \]

whose right hand side is proportional to the square root of the error probability in Bob’s outcome. That is, their theorem claims that the information gain by Eve makes Bob’s outcome in conjugate basis erroneous.

**B. Information v.s. Randomness**

We next derive a new information-disturbance theorem which relates information gain by Eve with randomness in Bob’s outcome.

To estimate the information gain by Eve, we introduce a symmetrized attack as in [4]. We add \( N \) auxiliary qubits to Eve’s apparatus and thus the Eve’s Hilbert space is dilated to \( \mathcal{H}_{E'} := \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \otimes \mathcal{H}_E \). Introduce a set of new vectors \( \{|E_{ij}^s\rangle\} \) in this Hilbert space \( \mathcal{H}_{E'} \) as

\[
|E_{ij}^s\rangle := \sqrt{\frac{1}{2N}} \sum_{m \in \{0,1\}^N} (-1)^{m \cdot (i \oplus j)} |m\rangle \otimes |E_{i \oplus m j \oplus m}\rangle,
\]

where “\( \oplus \)” is again a bit-wise XOR operation and “\( \cdot \)” represents bit-wise multiplications followed by their summation. Introduce a symmetrized attack as

\[
U^s : \mathcal{H}_{E'} \otimes \mathcal{H}_A \to \mathcal{H}_{E'} \otimes \mathcal{H}_B
\]

\[
(|0\rangle \otimes |0\rangle) \otimes |i\rangle \mapsto \sum_j |E_{ij}^s\rangle \otimes |j\rangle
\]

which can be extended to satisfy unitarity condition [4]. Although this symmetrized attack is different from the original attack, it is shown below that to treat this new attack is useful.

If we employ the symmetrized attack, Eve has a state described as \( \rho_{Eve,sym}^j := \sum_{j \in \{0,1\}^N} |E_{ij}^s\rangle \langle E_{ij}^s| \). To extract the information from it, she can measure the value of the auxiliary \( N \)-qubits and then apply a POVM \( X = \{X_\alpha\} \) on the original apparatus \( \mathcal{H}_E \). It is shown that this strategy gives same amount of information with the original attack. The values obtained by the first measurement are equally distributed, that is, each value \( m \) is
obtained with probability $\frac{1}{2^N}$. After obtaining a value $m$, the reduction of wave packet forces the state into

$$\rho^i_m := \sum_j |E_{i\oplus m} j\oplus m\rangle\langle E_{i\oplus m} j\oplus m|.$$ 

The second measurement gives a probability

$$p^s(\alpha|i, m) = \sum_j \langle E_{i\oplus m} j\oplus m|X_\alpha|E_{i\oplus m} j\oplus m\rangle,$$

from which it is easy to see that $p^s(\alpha|i, m) = p(\alpha|i \oplus m)$ holds. Thus by using conditional probability $p^s(\alpha|m|i) = \frac{1}{2^N}p(\alpha|i \oplus m)$, mutual information can be computed to coincide with $I(A : E|X|b)$. Taking a supremum over all the possible POVM over the full Hilbert space $\mathcal{H}_E'$ can make it larger and therefore the following inequality holds [4],

$$I(A : E|b) \leq I(A : E|b)_{sym}, \quad (5)$$

where the right hand side is the optimal information gain by the symmetrized attack.

Now we can state our theorem.

**Theorem 1** The following inequality holds:

$$I(A : E|b) \leq H(A \oplus B|\overline{b}), \quad (6)$$

where $H(\cdot)$ is the Shannon entropy. That is, the information gain by Eve in the basis $b$ makes the outcome of measurement by Bob in the conjugate basis $\overline{b}$ random.

**Proof:** We can prove the theorem by first symmetrizing the attack and next bounding Eve’s information gain by Holevo’s inequality. Thanks to (5), it is sufficient to estimate the quantity $I(A, E|b)_{sym}$ for our purpose. Holevo’s theorem [11] bounds it from above as

$$I(A, E|b)_{sym} \leq S\left(\frac{1}{2^N} \sum_i \rho^i_{\text{Eve,sym}}\right) - \sum_i \frac{1}{2^N}S\left(\rho^i_{\text{Eve,sym}}\right)$$

$$=: \chi\left(\{\rho^i_{\text{Eve,sym}}\}\right),$$

where $S(\rho)$ is von Neumann entropy of a state $\rho$. There exists a useful representation of this quantity $\chi$. Consider another additional $N$-qubits Hilbert space $\mathcal{H}_R$ and a state over $\mathcal{H}_R \otimes \mathcal{H}_E'$,
\[ \Theta := \sum_{i} \frac{1}{2^N} |i\rangle \langle i| \otimes \rho_{Eve,sym}^i. \]

Its quantum mutual entropy between \( \mathcal{H}_R \) and \( \mathcal{H}_E \) is shown to coincide with the quantity \( \chi \left( \{ \rho_{Eve,sym}^i \} \right) \),

\[ I(\Theta) := S(\Theta |_{E'}) + S(\Theta |_R) - S(\Theta) = \chi \left( \{ \rho_{Eve,sym}^i \} \right), \]

where \( \Theta |_{E'} \) is a restricted state to \( \mathcal{H}_{E'} \) of \( \Theta \) and \( \Theta |_R \) is defined in the same manner. To estimate this quantity, we consider a purification of \( \rho_{Eve,sym}^i \). Introduce another \( N \)-qubits system \( \mathcal{H}_P \) and states over \( \mathcal{H}_{E'} \otimes \mathcal{H}_P \) \[4\], \( |\varphi_i\rangle := \sum_j |E_{ij}^s \rangle \otimes |i \oplus j\rangle \). A state \( \tilde{\Theta} \) over \( \mathcal{H}_R \otimes \mathcal{H}_{E'} \otimes \mathcal{H}_P \) defined as

\[ \tilde{\Theta} := \sum_{i} \frac{1}{2^N} |i\rangle \langle i| \otimes |\varphi_i\rangle \langle \varphi_i| \]

gives \( \Theta \) if restricted to \( \mathcal{H}_R \otimes \mathcal{H}_{E'} \). By using subadditivity for the entropy difference \[12\], the mutual entropy \( I(\tilde{\Theta}) \) between \( \mathcal{H}_R \) and \( \mathcal{H}_{E'} \otimes \mathcal{H}_P \) is shown to be larger than \( I(\Theta) \). Therefore we estimate the quantity,

\[ I(\tilde{\Theta}) := S(\tilde{\Theta} |_{E'P}) + S(\tilde{\Theta} |_R) - S(\tilde{\Theta}). \]

Now we compute the restricted states over the subsystems,

\[ \tilde{\Theta} |_R = \sum_{ij} \frac{1}{2^N} \langle E_{ij}^s | E_{ij}^s \rangle |i\rangle \langle i| = \frac{1}{2^N} 1, \]

\[ \tilde{\Theta} |_{E'P} = \sum_{i} \frac{1}{2^N} |\varphi_i\rangle \langle \varphi_i|. \]

The von Neumann entropy of \( \tilde{\Theta} |_R \) is \( N \).

To compute the von Neumann entropy of \( \tilde{\Theta} \) itself, we purify this by adding an additional \( N \)-qubits \( \mathcal{H}_T \) and define a state over \( \mathcal{H}_T \otimes \mathcal{H}_R \otimes \mathcal{H}_{E'} \otimes \mathcal{H}_P \),

\[ |\Psi\rangle := \sum_i \sqrt{\frac{1}{2^N}} |i\rangle \otimes |i\rangle \otimes |\varphi_i\rangle. \]

Taking partial trace over \( \mathcal{H}_R \otimes \mathcal{H}_{E'} \otimes \mathcal{H}_P \) leads

\[ \sum_{i} \frac{1}{2^N} \langle \varphi_i | \varphi_i \rangle |i\rangle \langle i| = \frac{1}{2^N} 1. \]
whose entropy also is $N$. Thus the mutual entropy is completely determined by $\tilde{\Theta}|_{E^p}$ as

$$I(\tilde{\Theta}) = S\left(\sum_i \frac{1}{2N} |\varphi_i\rangle\langle \varphi_i|\right).$$

Now let us calculate the von Neumann entropy of $\tilde{\Theta}|_{E^p}$. Again a purification using an additional $N$-qubits $\mathcal{H}_Z$ to $\mathcal{H}_{E'} \otimes \mathcal{H}_P$ gives a state

$$|\Phi\rangle := \sum_i \sqrt{\frac{1}{2N}} |i\rangle \otimes |\varphi_i\rangle$$
on $\mathcal{H}_Z \otimes \mathcal{H}_{E'} \otimes \mathcal{H}_P$. Its restriction to $\mathcal{H}_Z$ gives

$$\sigma := \frac{1}{2N} \sum_{ij} \sum_n \langle E^s_{j} j \oplus u | E^s_{i} i \oplus u | i \rangle \langle j |$$

whose entropy agrees with $I(\tilde{\Theta})$. Let us consider its components with respect to the basis \{\{i\}\}. Since

$$\sum_u \langle E^s_{j} j \oplus u | E^s_{i} i \oplus u | i \rangle \langle j | = \frac{1}{2N} \sum_n \sum_u \langle E^s_{j} j \oplus n \oplus u | E^s_{i} i \oplus n \oplus u | i \rangle \langle j |$$

holds, it depends upon only $i \oplus j$. We write it as $f(i \oplus j)$ to represent $\sigma$ as

$$\sigma = \frac{1}{2N} \sum_{ij} f(i \oplus j) |i\rangle \langle j|,$$

which can be diagonalized by an orthonormalized vectors

$$|\mu_i\rangle := \sqrt{\frac{1}{2N}} \sum_l (-1)^t |t\rangle$$
as $\sigma = \sum_l \lambda_l |\mu_l\rangle \langle \mu_l|$, with $\lambda_l := \frac{1}{2N} \sum_t f(t)(-1)^t$. The eigenvalue $\lambda_l$ is calculated as

$$\lambda_l := \frac{1}{2N} \sum_t f(t)(-1)^t$$

$$= \frac{1}{2N} \sum_{t,n,v} \langle E^s_{v} v \oplus n | E^s_{t} t \oplus v \oplus n | (-1)^t$$

$$= \frac{(1/2N)^2}{2N} \sum_{t,n,v} \sum_{ij} \sum_{ij'} \langle E_{ij}| E_{ij'}\rangle \langle j| v \oplus n \rangle \langle v| t \oplus v \oplus n | j' \rangle \langle j'| t \oplus v \rangle (-1)^t.$$
holds, where \( i \cdot k := \sum_{n=1}^{N} i_n k_n \). It leads

\[
\lambda_l = \left( \frac{1}{2N} \right)^4 \sum_{\{i, j, j', l\}} \delta_{i \oplus j \oplus j', 0} \delta_{j \oplus j', 0} \delta_{j' \oplus l, 0} \langle E_{ij} | E_{jj'} \rangle \\
= \frac{1}{2N} \sum_{i} \langle E_{i} | E_{i} \rangle
\]

which is nothing but \( p(B = A \oplus l| \overline{b}) \) introduced in (3). Finally we obtain the following inequality,

\[
I(A : E|b) \leq H(A \oplus B| \overline{b}).
\]

Q.E.D.

**IV. DISCUSSIONS**

Below we discuss the implication of our theorem by comparing it with the former one. Since the right hand side of our inequality is determined by \( \{ p(B = A \oplus c| \overline{b}) \} \), it can be reduced to a form which includes only the term \( \sum_{c \neq 0} p(B = A \oplus c| \overline{b}) \).

**Corollary 2** [13] The following inequality between the information gain by Eve and the error probability in Bob’s outcome holds:

\[
I(A : E|b) \leq -\delta \log \delta - (1 - \delta) \log(1 - \delta) + N\delta,
\]

where \( \delta := \sum_{c \neq 0} p(B = A \oplus c| \overline{b}) \).

**Proof:**

Under the constraint \( \delta = \sum_{c \neq 0} p(B = A \oplus c| \overline{b}) \) for fixed \( \delta \), the distribution which makes the Shannon entropy \( H(A \oplus B| \overline{b}) \) maximum is \( p(B = A| \overline{b}) = 1 - \delta \) and \( p(B = A \oplus c| \overline{b}) = \frac{\delta}{2^N - 1} \) for all \( c \neq 0 \). It gives

\[
H(A \oplus B| \overline{b}) = -\delta \log \delta - (1 - \delta) \log(2^N - 1)
\]

and ends the proof. Q.E.D.
For a fixed error probability \( \delta = \sum_{c \neq 0} p(B = A \oplus c|\bar{b}) \), for sufficiently large \( N \), the term \( N\delta \) becomes dominant in the right hand side of the above equation. Thus our inequality becomes tighter than (4) in such a case.

Finally we present a situation which shows a drastic difference between the two inequalities. Suppose that Eve employs the following “attack”: Eve does not make the qubits sent by Alice interact with any apparatus, but she just converts the each value. That is, for each qubit, Eve performs a unitary operation \( |i\rangle \mapsto (-1)^i|i\rangle \) (\( i = 0, 1 \)). One can easily see that also for the conjugate basis this operation works as conversion. In this case the error probability \( \delta \) becomes 1. Thus if we employ the inequality (4), it is impossible to rule out the possibility of Eve’s information gain. On the other hand, since the error in Bob’s outcome is deterministic, the right hand side of (6) vanishes. Thus our theorem can convince us that there is no information gain by Eve.

In this paper we showed a novel version of information-disturbance theorems. According to our theorem, one can see that the information gain by Eve induces randomness to Bob's outcome in the conjugate basis. The both sides of the inequality are expressed in terms of entropy and thus seems to be natural. For large \( N \) case, in which we are usually interested in, our inequality gives tighter bound than the previously proposed ones. Moreover, our theorem can rule out the case when Eve just turns over the qubits and gains no information. Our theorem, as previous one, also relies upon the assumptions of fair probability of the random variable \( A \) and mutually unbiasedness between \( b \) and \( \bar{b} \). It will be very interesting and crucial to generalize the theorem to more general setting [14].
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