A note on deformations and mutations of fake weighted projective planes

İrem Portakal

September 13, 2018

Abstract

It has been shown by Hacking and Prokhorov that if the projective surface $X$ with quotient singularities and self-intersection number $9$ has a smoothing to the projective plane, then $X$ is the general fiber of a $\mathbb{Q}$-Gorenstein deformation of the weighted projective plane with weights giving solutions to the Markov equation. This result has been understood and generalized by combinatorial mutations of Fano triangles by Akhtar, Coates, Galkin, and Kasprzyk. In this note, we study this result by utilizing polarized T-varieties and describe the associated deformation explicitly in terms of certain Minkowski summands of so-called divisorial polytopes.

Keywords: Toric degenerations, Markov equation, Minkowski summands, weighted projective plane, $T$-variety, $\mathbb{Q}$-Gorenstein deformation, combinatorial mutation, divisorial polytope.

1 Introduction

In their paper [8], Hacking and Prokhorov generalize Manetti’s classification of surfaces $X$ with self intersection number $K_X \geq 5$. They classify all surfaces which admit a $\mathbb{Q}$-Gorenstein smoothing. As a corollary to their theorem, they show that if $X$ is a projective surface with quotient singularities and with $K_X^2 = 9$ which admits a smoothing to the projective plane, then $X$ is a $\mathbb{Q}$-Gorenstein deformation of the weighted projective plane $\mathbb{P}(a^2, b^2, c^2)$ where $(a, b, c)$ is a solution for the Markov equation, i.e. $a^2 + b^2 + c^2 = 3abc$ (Theorem 1.2). The Markov equation also appears in the context of derived categories. The solutions of the equation are in one-to-one correspondence
with the exceptional bundles on the projective plane with slopes in the interval $[0, 1/2]$ (13).

It turns out that we can understand this smoothing in terms of deformations of $T$-varieties. A normal variety $X$ is called a complexity-$d$ $T$-variety if it admits an effective algebraic torus action $T \cong (\mathbb{C}^*)^n$ such that $\dim(X) - \dim(T) = d$. The deformations of $T$-varieties have been studied by Altmann in [3] and by Ilten and Vollmert in [10]. The key idea of the theory is to associate certain Minkowski decompositions to deformations. In this note, we consider the deformations of projective toric surfaces by interpreting them as complexity-one polarized $T$-varieties. To a polarized $T$-variety, one can associate a so-called divisorial polytope consisting of a lattice polytope and piecewise affine concave function. In [9], a technique has been introduced to deform these varieties by using admissible Minkowski summands of the associated piecewise affine concave function. The latest improvement on $Q$-Gorenstein smoothings has been done combinatorially by Akhtar, Coates, Galkin and, Kasprzyk in [1] in terms of combinatorial mutations of Fano polytopes. This construction generalizes the result of Hacking and Prokhorov to fake projective planes satisfying a certain type of Diophantine equation. In fact, for this combinatorial mutation, one can relate the combinatorial mutations to deformations of projective varieties using lattice polytopes and $T$-varieties. Indeed, we associate a one-parameter deformation to a mutation of Fano triangle by utilizing divisorial polytopes. It gives an explicit exposition of the Minkowski summands of the associated deformation. In [6], Baytrev proposed a method to construct Landau-Ginzburg model mirror dual to a smooth Fano variety $X$ as a Laurent polynomial $f$ such that the Newton polytope $\text{Newt}(f)$ is the fan polytope of the small toric degeneration of $X$. The choice of the Laurent polynomial $f$ is not unique, however one can always transform $f$ to another Laurent polynomial via mutations which is also mirror dual to $X$. Although the notion of small degenerations is restrictive, this model has been extended to any $Q$-Gorenstein degeneration of the complex projective plane in [7]. In
particular, this is the case of Prokhorov-Hacking’s deformations and it is another reason that we focus our attention into two dimensional case.

Remark that by toric degenerations we mean that the variety degenerates to a toric variety as one can guess. We say that $X$ degenerates to a toric variety $X_0$ if there exists a flat morphism over a smooth curve germ such that the general fiber is isomorphic to $X$ and the zero fiber is isomorphic to the toric variety $X_0$. This may be also expressed as the deformation of $X_0$ over a smooth curve germ.

**Definition 1.1.** Let $X$ be a normal surface with quotient singularities and Picard rank 1. We say that $X$ is a $\mathbb{Q}$-Gorenstein deformation of $X_0$ if there exists a deformation $X_t$ over a smooth curve germ such that $X_t$ is isomorphic to $X$ for all $t \neq 0$ and $K_X$ is $\mathbb{Q}$-Cartier. If $X$ is smooth, the deformation is called $\mathbb{Q}$-Gorenstein smoothing.

Prokhorov and Hacking prove the following result.

**Theorem 1.2** ([8], Corollary 1.2). Let $X$ be a projective surface with quotient singularities and $K_X^2 = 9$. If $X$ admits a smoothing to the complex projective plane, then $X$ is a $\mathbb{Q}$-Gorenstein deformation of $\mathbb{P}(a^2, b^2, c^2)$ where $a^2 + b^2 + c^2 = 3abc$.

We denote the one-parameter subgroups of an algebraic torus $T \cong (\mathbb{C}^*)^n$ by $N$ and the characters of $T$ by $M$. There is a natural bilinear pairing between these two lattices which is the usual dot product

$$\langle \cdot, \cdot \rangle : M \times N \rightarrow \mathbb{Z}.$$

We let $N_\mathbb{Q} := N \otimes \mathbb{Z}_\mathbb{Q}$ and $M_\mathbb{Q} := M \otimes \mathbb{Z}_\mathbb{Q}$ be the corresponding vector spaces to the lattices $N$ and $M$.

## 2 Mutations of Fano polytopes

The notion of mutations of lattice polytopes has been introduced by Akhtar, Galkin, Coates, and Kasprzyk in their paper [1]. This construction is motivated by certain birational transformations of Laurent polynomials which have been studied in Galkin and Usnich’s paper [7]. Let $f \in \mathbb{C}[x_1^\pm, \ldots, x_n^\pm]$ be a Laurent polynomial. The period of $f$ is defined as

$$\pi_f(t) := \left( \frac{1}{2\pi i} \right)^n \int_{|x_1|=\ldots|x_n|=1} \frac{1}{1 - tf(x_1, \ldots, x_n)} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$
where $t \in \mathbb{C}$ and $|t| \ll \infty$. A mutation of $f$, as in Definition 7 of [7], is a birational transformation $\phi \in \text{Aut}(\mathbb{C}(x_1, \ldots, x_n))$ preserving the period $\pi_f$ such that $\phi(f)$ is again a Laurent polynomial. We are in particular interested in the following mutation.

**Example 1.** Let $g \in \mathbb{C}[x_2^\pm, \ldots, x_n^\pm]$ be a Laurent polynomial. The birational transformation $\phi : (x_1, \ldots, x_n) \mapsto (x_1/g, x_2, \ldots, x_n)$ is a mutation of $f$ if and only if $f$ can be written as $\sum_{i=1}^{l} f_i x_1^i$ where $f_i \in \mathbb{C}[x_2^\pm, \ldots, x_n^\pm]$ such that for $i > 0$, $f_i/g^i \in \mathbb{C}[x_2^\pm, \ldots, x_n^\pm]$.

Mutations of Laurent polynomials induce transformations of their associated Newton polytopes, which we call combinatorial mutations. Moreover, if $0 \in \text{Newt}(f)$, then above-presented type of mutation of $f$ has been associated to the deformation of the toric variety of $\text{Newt}(f)$. Note that in this case, we require that $k < 0$ and $l > 0$. We will explain this relation in Section 4.

### 2.1 Combinatorial mutations

Let $P \subset \mathbb{N}_Q$ be a full dimensional convex lattice polytope. We say $P$ is Fano, if the origin lies in the strict interior of $P$ and the vertices of $P$ are primitive lattice points. The dual of $P$ is defined as

$$P^* = \{ u \in M_Q \mid \langle u, v \rangle \geq -1, \forall v \in P \} \subset M_Q.$$  

For the construction of combinatorial mutations, we follow [1,2].

Let $w \in M$ be a primitive lattice vector. We set

$$h_{\min} := \min\{ \langle w, v \rangle | v \in P \}, \quad h_{\max} := \max\{ \langle w, v \rangle | v \in P \}.$$  

Since $P$ is a Fano polytope, we obtain that $h_{\min} < 0$ and $h_{\max} > 0$. Note that the lattice height function $w : \mathbb{N} \to \mathbb{Z}$ naturally extends to $w_Q : \mathbb{N}_Q \to \mathbb{Q}$.

**Definition 2.1.** We say that a lattice point $v \in P$ is at height $m$ with respect to $w \in M$, if $\langle w, v \rangle = m$. For each height $h \in \mathbb{Z}$, we define a hyperplane

$$H_{w,h} := \{ x \in \mathbb{N}_Q \mid \langle w, x \rangle = h \}.$$  

We let $w_h(P) = \text{conv}(H_{w,h} \cap P \cap \mathbb{N})$.

**Definition 2.2.** A factor of $P$ with respect to $w$ is a lattice polytope $F \subset \mathbb{N}_Q$ satisfying:
1. $\langle w, v \rangle = 0$ for all $v \in F$.

2. For every $h \in \mathbb{Z}$ such that $h_{\text{min}} \leq h < 0$, there exists a (possibly empty) lattice polytope $G_h \subset N_\mathbb{Q}$ at height $h$ such that

$$H_{w,h} \cap \text{vert}(P) \subseteq G_h + (-h)F \subseteq w_h(P).$$

Remark 1. A factor does not need to exist for every $w \in M$. If it exists, then we are in a position to define a combinatorial mutation of $P$. For a given combinatorial mutation of $P$, there exists a Laurent polynomial $f$ with its Newton polytope Newt$(f)$ to be equal to $P$ and the combinatorial mutation arises from an algebraic mutation.

Definition 2.3. The combinatorial mutation of $P$ with respect to height function $w$, factor $F$, and lattice polytopes $\{G_h\}$ is defined as the convex lattice polytope

$$\text{mut}_w(P, F; \{G_h\}) := \text{conv} \left( \bigcup_{h=h_{\text{min}}}^{h_{\text{max}}} G_h \cup \bigcup_{h=0}^{h_{\text{max}}} (w_h(P) + hF) \right) \subset N_\mathbb{Q}.$$ 

This definition might appear technical at first glance. Let us construct an explicit combinatorial mutation induced by the birational transformation in Example 1.

Example 2. Let $f = \sum_{i=k}^{l} f_i x_i^2 \in \mathbb{C}[x_1^+, x_2^+]$ with $k < 0$ and $l > 0$, and let $f_i, g_i \in \mathbb{C}[x_1^+]$ be Laurent polynomials for $i \in [k, l]$. Suppose that $g_i \mid f_i$ for $i > 0$. Consider the mutation $\phi: (x_1, x_2) \to (x_1, x_2/g)$. The Laurent polynomial $\phi^* f$ is the algebraic mutation of $f$ and can be written as $\sum_{i=k}^{l} (f_i/g_i) x_i^2 \in \mathbb{C}[x_1^+, x_2^+]$. Let $P := \text{Newt}(f) \subset N_\mathbb{Q}$ and $Q := \text{Newt}(\phi^* f) \subset N_\mathbb{Q}$. We set the height function for the induced combinatorial mutation $w = [0, 1] \subset M$ and the factor with respect to $w$ as $F = \text{conv}(0, (1, 0)) \subset N_\mathbb{Q}$. First infer that, for $k \leq h \leq -1$, the lattice polytopes $G_h$ are $\text{Newt}(f, g^h x_2^2)$. Moreover, by the definition of the factor, we have the condition $\text{Newt}(f, g^h x_2^2) + [0, h] \subseteq w_h(P)$. Hence, we obtain that $g = 1 + x_1$ and observe that $F = \text{Newt}(g)$.

In fact, a combinatorial mutation $P \subset N_\mathbb{Q}$ induces a piecewise linear transformation $\varphi$ of $P^* \subset M_\mathbb{Q}$ such that $(\varphi(P^*))^* = \text{mut}_w(P, F; \{G_h\})$ and it is given by

$$u \mapsto u - u_{\text{min}}w$$

where $u_{\text{min}} = \min \{ \langle u, v_F \rangle \mid v_F \in \text{vert}(F) \}$. 5
2.2 Mutations of Fano triangles

In this section, we bring our attention to the two dimensional case, more particularly to combinatorial mutations of Fano triangles. Let \( P := \text{conv}(v_1, v_2, v_3) \subset N_Q \) be a Fano triangle. Since the origin is contained in the interior of \( P \), there exists a unique choice of coprime positive integers \((\lambda_1, \lambda_2, \lambda_3)\) such that \( \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0 \). Let \( N' \subset N \) be the sublattice generated by the lattice points \( v_i \). The associated projective toric surface \( TV(P) \) is the fake projective plane with weights \( \lambda_1, \lambda_2, \lambda_3 \), i.e. it is the quotient of \( \mathbb{P}(\lambda_1, \lambda_2, \lambda_3) \) by the action of the finite group \( N/N' \) acting free in codimension one.

Proposition 2.4 ([2], Proposition 3.3). Let \( P \subset N_Q \) be a Fano triangle and \( TV(P) \) be the fake weighted projective plane with weights \( (\lambda_1, \lambda_2, \lambda_3) \). Suppose that there exists a mutation of \( P \) as \( Q = \text{mut}_w(P, F; \{G_h\}) \) for some height function \( w \) and factor \( F \) such that \( TV(Q) \) is a fake weighted projective plane. Then, up to relabelling, \( \lambda_1 \mid (\lambda_2 + \lambda_3)^2 \) and \( TV(Q) \) has weights

\[
\left( \lambda_2, \lambda_3, \frac{(\lambda_2 + \lambda_3)^2}{\lambda_1} \right).
\]

As a consequence to this proposition, one can show that the weights of \( TV(P) \) and \( TV(Q) \) belong to the solution set of the Diophantine equation

\[
mx_1x_2x_3 = k(c_1x_1^2 + c_2x_2^2 + c_3x_3^2) \tag{2.2.1}
\]

where \( m, k, c_i \in \mathbb{Z}_{>0} \) and \( c_i \) is square free. Remark that a special case of this equation is the Markov equation which has been mentioned in the introduction.
Proposition 2.5 (\cite{2}, Proposition 3.12). Let $P$ and $Q$ be Fano triangles as in Proposition 2.4. Then the weights of $\text{TV}(P)$ and $\text{TV}(Q)$ give solutions to the same Diophantine equation (2.2.1).

In the next sections, we study how these mutations give rise to deformations and hence this previous result can be considered as a generalization of Theorem 1.1 by Hacking and Prokhorov in \cite{8}.

Example 3. Consider the combinatorial mutation $\Psi$ of the triangle $P^* \subset M_Q$ to the dashed triangle $Q^* \subset M_Q$ in Figure 1. It is induced by the mutation of the type of Example 1 with $w = (0,1) \in M$ and $F = \text{conv}(0,(1,0))$. In particular, if $\text{TV}(P)$ has the weights $(a^2, b^2, c^2)$, then $\text{TV}(Q)$ has weights $(b^2, c^2, (\frac{b^2+c^2}{a^2})^2)$. Suppose now $(a,b,c)$ is a solution to the Markov equation. Since the triple $(b,c,3bc-a)$ is also a solution to Markov equation, we obtain

$$(b^2, c^2, \frac{(b^2+c^2)^2}{a^2}) = (b^2, c^2, \frac{(3abc-a^2)^2}{a^2}) = (b^2, c^2, (3bc-a)^2),$$

and thus the weights of $\text{TV}(Q)$ also give solution to the Markov equation.

3 Deformations of T-varieties

T-varieties naturally appear during the study of deformations of toric varieties preserving the torus action of the embedded torus. In this case, the total space of a deformation of a toric variety is not always toric, but they admit a lower dimensional effective torus action. The natural question to ask is if these varieties also admit a nice combinatorial description as in the toric case. A such combinatorial construction of T-varieties has been introduced in \cite{4}. The affine T-varieties are in one-to-one correspondence with the so-called p-divisors. A p-divisor is a formal product of a set of polyhedra and divisors from the "good" quotient of the T-variety by the effective torus action. For a detailed read, one can refer to the survey on the language of T-varieties \cite{5}. In this section, we restrict our attention to the combinatorial construction for polarized complexity-one T-varieties. Next, we study one-parameter deformations of these varieties in terms of admissible Minkowski decompositions.

3.1 Polarized T-varieties and Divisorial Polytopes

The correspondence between polarized toric varieties and lattice polytopes in toric geometry has been generalized to the complexity-one T-varieties case
in terms of so-called divisorial polytopes in [12]. In this section, we present this construction and in particular we apply this construction explicitly to fake weighted projective planes.

**Definition 3.1.** A divisorial polytope \((\square, \Phi)\) consists of a lattice polytope \(\square \subset M'_Q\) and a piecewise linear concave function

\[
\Phi = \sum_{P \in \mathbb{P}^1} \Phi_P : \square \to \text{Div}_Q \mathbb{P}^1
\]

such that

1. \(\text{deg} \Phi(u) > 0\) for \(u\) in the interior of \(\square\).
2. \(\text{deg} \Phi(u) > 0\) or \(\Phi(u) \sim 0\) for \(u\) a vertex of \(\square\).
3. For all \(P \in \mathbb{P}^1\), the graph of \(\Phi_P\) has its vertices in \(M'_Q \times \mathbb{Z}\).

**Theorem 3.2** ([12], Theorem 3.2). There is a one-to-one correspondence between divisorial polytopes and complexity-one T-varieties \((X, \mathcal{L})\) where \(\mathcal{L}\) is an equivariant ample line bundle.

For more details of this construction, we refer the reader to [9,12]. One can always consider a polarized toric variety with a one-dimension lower subtorus action. Let \(\triangle \subset M_Q\) be a lattice polytope in \(M\). Then there exists the exact sequence of lattices

\[
0 \to \mathbb{Z} \xrightarrow{F} \text{deg} M' \xrightarrow{\text{deg}} M' \to 0.
\]

We choose a section \(s : M' \to M\), i.e. \(\text{deg}(s) = \text{id}_{M'}\). Consider the map \(\Phi_\triangle : \text{deg}(\triangle) \to \text{Div}_Q(\mathbb{P}^1)\) given by

\[
(\Phi_\triangle)_0(u) = \max\{x \in \mathbb{Q} \mid F_Q(x) + s(u) \in \triangle \cap \text{deg}_Q^{-1}(u)\},
\]

\[
(\Phi_\triangle)_\infty(u) = -\min\{x \in \mathbb{Q} \mid F_Q(x) + s(u) \in \triangle \cap \text{deg}_Q^{-1}(u)\}.
\]

Then, \((\Phi_\triangle, \text{deg}(\triangle))\) is a divisorial polytope with respect to the restricted effective torus action \(T_{N'}\).

**Example 4.** Consider the dashed triangle \(P^* \subset M_Q\) from Figure 1. Set \(\text{deg} : M \xrightarrow{(1,0)} M'\) to be the restricted torus action and choose the section \(s : M' \xrightarrow{(1,0)} M\). Note that \(P^*\) does not need to be a lattice polytope. Let us suppose for the moment that \(P^*\) is lattice polytope. Then we obtain the lattice polytope \(\text{deg}(P^*) = \square \subset M_Q\) to be the interval \([\alpha_1, \alpha_2]\) and the piecewise linear concave functions \(\Phi_0\) and \(\Phi_\infty\) as in Figure 2.
3.2 One-parameter deformations of polarized $T$-varieties

A deformation of a projective algebraic variety $X_0$ is a flat map $\pi: \mathcal{X} \to S$ with $0 \in S$ such that $\pi^{-1}(0) = X_0$. The variety $\mathcal{X}$ is called the total space and $S$ is called the base space of the deformation. If $S$ is a open subset of $\mathbb{A}^1$, we call $\pi$ a one-parameter deformation. Since we consider polarized $T$-varieties, we will study embedded deformations in this section.

Let $X_0 \hookrightarrow \mathbb{P}^N$ be a projectively embedded variety. An embedded deformation $\pi$ of $X_0$ consists of $X \hookrightarrow \mathbb{P}_S$ such that the projection to $S$ is a deformation of $X_0$ and the embedding of $X$ restricts to the embedding of $X_0$. Our aim is to construct certain embedded one-parameter deformations of polarized $T$-varieties in terms of Minkowski summands.

**Definition 3.3.** Let $(\square, \Phi)$ be a divisorial polytope. An admissible one-parameter Minkowski decomposition of $(\square, \Phi)$ consists of two piecewise affine functions $\Phi^0_P, \Phi^1_P: \square \to \mathbb{Q}$ for some $P \in \text{Div}_{\mathbb{Q}}\mathbb{P}^1$ such that:

1. The graph of $\Phi^i_P$ has lattice vertices for $i = 0, 1$.
2. $\Phi_P(u) = \Phi^0_P(u) + \Phi^1_P(u)$ for all $u \in \square$.
3. For any full-dimensional polytope $\triangle \subset \square$ on which $\Phi_P$ is affine, $\Phi^i_P$ has non-integral slope for at most one $i \in \{0, 1\}$.

Given a one-parameter Minkowski decomposition of $(\square, \Phi)$, for any $s \in S$, define the divisorial polytope $\Phi^s(\square) : \square \to \text{Div}_{\mathbb{Q}}\mathbb{P}^1$ by
\[ \Phi^{(s)}(u) = \sum_{P \neq P'} \Phi_P(u) \otimes V(y_{P'}) + \Phi_0^0(u) \otimes V(y_P) + \Phi_1^1(u) \otimes V(y_P - s) \]

where \( y_P \in \mathbb{C}(\mathbb{P}^1) \) is a rational function with its sole zero at \( P \). Denote \( X(\Phi) \) and \( X(\Phi^{(s)}) \) to be the associated polarized \( T \)-varieties to divisorial polytopes \( (\varnothing, \Phi) \) and \( (\varnothing, \Phi^{(s)}) \).

**Theorem 3.4** ([9], Theorem 7.3.2). There exists a flat family of \( T \)-varieties \( \{ X(\Phi^{(s)}) \} \) over \( S \). Moreover, if \( \Phi \) is very ample and gives a projectively normal embedding, this deformation can be realized as an embedded deformation of \( X(\Phi^{(0)}) \cong X(\Phi) \).

The deformations of above form are called \( T \)-deformations. These deformations admit the torus \( T_N \) of \( X(\Phi^{(0)}) \) acting on the total space \( \mathcal{X} \) and (trivially) on the base space \( S \). Moreover, the maps \( \pi: \mathcal{X} \to S \) and \( X_0 \hookrightarrow X \) are \( T_N \)-equivariant.

### 4 Comparison of mutations and deformations

We now recall the mutation of the type of Example 1. Suppose that \( \text{Newt}(f) \) contains the origin in its interior.

**Theorem 4.1** ([11], Theorem 1.3). There is a flat projective family \( \pi: \mathcal{X} \to \mathbb{P}^1 \) such that \( \pi^{-1}(0) = TV(\text{Newt}(f)) \) and \( \pi^{-1}(\infty) = TV(\text{Newt}(\phi^* f)) \).

This deformation is constructed by taking the affine cone over the projective toric varieties and applying the techniques for the deformations of affine complexity-one \( T \)-varieties. Note also that, as we have shown in Example 2, this mutation induces a combinatorial mutation. We utilize this combinatorial mutation to understand Proposition 2.5 in terms of divisorial polytopes.

**Theorem 4.2.** For the combinatorial mutation \( \Psi \), there exists an one-parameter embedded deformation of the polarized variety \( TV(P) \) such that the general fiber is isomorphic to \( TV(Q) \).

**Proof.** Let us consider the mutation \( \Psi \) from the lattice polygon \( P^* \subset M_Q \) to the lattice polygon \( Q^* \subset M_Q \). A lattice polygon in \( M_Q \) is very ample. Since \( P^* \) is not necessarily a lattice polytope, we consider a dilation of \( P^* \), i.e. the polarized toric variety \( (TV(P), \mathcal{L}_{aP^*}) \) for some \( a \in \mathbb{Z}_{>0} \). As we explained in Example 3, the height function is \( w = (0, 1) \) and the factor is the line segment \( F = \text{conv}(0, (x, 0)) \). Thus, we obtain the following piecewise linear transformation \( v \mapsto v\Psi \) where \( v = (v_1, v_2) \) and,
Figure 3: An admissible one-parameter Minkowski decomposition of $\Phi_\infty$

$$\Psi = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } v_1 > 0 \\
\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} & \text{otherwise.}
\end{cases}$$

Next, we utilize the associated divisorial polytope of $TV(P)$ from Example 4. The subtorus action is constructed by the map $M \xrightarrow{(1,0)} M'$. Without loss of generality, we assume that $(u_0, u_1, u_2)$ are lattice points of the lattice polytope $aP^* \in M_Q$. In Figure 2, we observe that the divisorial polytope with respect to the restricted subtorus action consists of the line segment $[\alpha_1, \alpha_2]$ and two piecewise linear functions $\Phi_0$ and $\Phi_\infty$.

In order to construct an embedded deformation of $TV(P)$, we examine the following admissible one-parameter Minkowski decomposition of $\Phi_\infty(u) = \Phi_0^0(u) + \Phi_\infty^1(u)$ as in Figure 3. This decomposition is indeed admissible, since we obtain an integral slope for $\Phi_\infty^1$ on interval $[\alpha_1, 0]$: 

$$\frac{\beta_1 - \beta_4}{\alpha_1} = \frac{\beta_1 - \alpha_1 x - \beta_1}{\alpha_1} = -x.$$ 

Since we would like to obtain a toric variety as the general fiber of this deformation, the associated divisorial polytope $\Phi^{(s)}$ may have at most two nontrivial coefficient. Recall that the general fiber of this deformation is

$$\Phi^{(s)}(u) = \Phi_0(u) \otimes \{0\} + \Phi_0^0(u) \otimes \mathbb{V}(y_\infty) + \Phi_\infty^1(u) \otimes \mathbb{V}(y_\infty - s).$$
We get rid of the additional coefficient by adding the slope of $\frac{\beta_2 - \beta_1}{\alpha_2 - \alpha_1}$ of $\Phi_0$ to $\Phi_1$. Thus we obtain the coefficient for the divisor $\{0\} \in \text{Div}_{\mathbb{Q}}\mathbb{P}^1$ as the piecewise linear function with slopes $\frac{\beta_1 - \beta_0}{\alpha_1}$ on $[\alpha_1, 0]$ and $\frac{\beta_2 - \beta_0}{\alpha_2}$ on $[0, \alpha_2]$. Hence, we conclude that $X(\Phi^{(s)})$ corresponds to the projective toric plane $\text{TV}(Q)$.

One can interpret these embedded deformations as deformations of so-called divisorial fans in lattice $N$. This construction can be found in [12]. The $T$-deformations of divisorial polytopes can be also seen as the $T$-deformations of affine cone over $\text{TV}(P)$ which uses a generalized method of deformations of toric varieties in [3]. In fact, this is the method which has been used to prove Theorem 4.1. Moreover, if the coefficients $\Phi_0$ and $\Phi_\infty$ form an admissible one-parameter Minkowski decomposition for some $i \in \{0, 1\}$, then we can extend the deformation constructed in Theorem 4.2 to a deformation over $\mathbb{P}^1$.

From the proof of Theorem 4.2 one deduces the following interesting fact.

**Corollary 4.3.** Let $\Phi_\infty = \Phi^0_\infty + \Phi^1_\infty$ be the admissible one-parameter Minkowski decomposition associated to the combinatorial mutation $\Psi$ as in Theorem 4.2. Then $\Phi^0_\infty$ is a linear function and $\Phi^1_\infty$ has exactly two affine pieces with integral slopes. In particular, the decomposition of the slopes appear as

$$\frac{\beta_2 - \beta_1}{\alpha_1 - \alpha_2} + \left\{0, \frac{\beta_1 - \beta_4}{\alpha_1}\right\}.$$

**Example 5.** Let $P^*$ be the dual polytope $\text{conv}((-3, 1), (3, 1), (0, -1/2))$. Then $\text{TV}(P)$ is the weighted projective plane with weights $(1, 1, 4)$. Consider the ample line bundle $\mathcal{O}_{\mathbb{P}^{(1,1,4)}}(4)$. We choose the restricted torus action again as $M \xrightarrow{(1,0)} M'$. Then the associated divisorial polytope consists of lattice polytope $[-6, 6] \subset M'_Q$ and the piecewise linear function as in Figure 4. To obtain a deformation with its general fiber isomorphic to $\mathbb{P}^2$, we set the decomposition of $\Phi_\infty$ as $\Phi^0_\infty$ with slope $-\frac{1}{2}$ and $\Phi^1_\infty$ with slopes $\{0, 1\}$. It gives us a $\mathbb{Q}$-Gorenstein smoothing of $\mathbb{P}^{(1,1,4)}$ to $\mathbb{P}^2$. Now consider another admissible decomposition of $\Phi_\infty$ as in Figure 5. Note that this decomposition has slopes $\{1/2, 0\}$ and $\{0, -1/2\}$. As in the proof of Theorem 4.2 we construct the divisorial polytope $([-6, 6], \Phi^{(s)})$. Thus, we obtain the general fiber $X(\Phi^{(s)})$ being isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, this is not a $\mathbb{Q}$-Gorenstein deformation.
Figure 4: A divisorial polytope for the weighted projective plane $\mathbb{P}(1, 1, 4)$

Figure 5: A decomposition of $\Phi_\infty$ giving a deformation to $\mathbb{P}^1 \times \mathbb{P}^1$

References

[1] Mohammad E. Akhtar, Tom Coates, Sergey Galkin and Alexander M. Kasprzyk. *Minkowski polynomials and mutations*, SIGMA Symmetry Integrability Geom. Methods Appl. 8 (2012), 094, pp. 707.

[2] Mohammad E. Akhtar. Alexander M. Kasprzyk. *Mutations of fake weighted projective planes*, Proceedings of the Edinburgh Mathematical Society, Volume 59, Issue 2, May 2016, pp. 271-285.

[3] Klaus Altmann. *Minkowski sums and homogenous deformations of toric Varieties*, Tohoku Math. J. 47 (1995), 151-184.
[4] Klaus Altmann and Jürgen Hausen. *Polyhedral divisors and algebraic torus actions*. Math. Ann., 334:557-607, 2006.

[5] Klaus Altmann, Nathan Owen Ilten, Lars Petersen and, Hendrik Süß, and Robert Vollmert. *The geometry of T-varieties*, Contributions to algebraic geometry, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012, pp. 17-69.

[6] Victor Batyrev. *Toric degenerations of Fano varieties and constructing mirror manifolds*, The Fano Conference, 109-122, Univ. Torino, Turin, 2004.

[7] Sergey Galkin and Alexandr Usnich. *Mutations of potentials*, Preprint IPMU 10-0100, 2012.

[8] Paul Hacking and Yuri Prokhorov. *Smoothable del Pezzo Surfaces with quotient Singularities*, Compos. Math. 146 (2010), no.1, 169-192.

[9] Nathan Ilten. *Deformations of rational varieties with codimension one torus action*, Doctoral Thesis, FU Berlin, 2010. [urn:nbn:de:kobv:188-fudissthes00000018440-0].

[10] Nathan Ilten and Robert Vollmert. *Deformation of rational T-varieties*, J. Algebraic Geometry 21 (2012) pp. 473-493.

[11] Nathan Ilten. *Mutations of Laurent polynomials and flat families with toric fibers*, Symmetry, Integrability and Geometry: Methods and Applications 8 (2012), 047.

[12] Nathan Ilten and Hendrik Süß. *Polarized complexity-one T-Varieties*, Michigan Mathematical Journal 60 (2011) no. 3 pp. 561-578.

[13] A. N. Rudakov. *The Markov numbers and exceptional bundles on \( \mathbb{P}^2 \)*, Izv. Akad. Nauk SSSR Ser. Mat., 1988, Volume 52, Issue 1, 100-112.

Department of Mathematics, Otto-von-Guericke-University, Magdeburg, Germany

E-mail address, İ. Portakal: irem.portakal@ovgu.de