“IFF” IS NOT EXPRESSIBLE IN INDEPENDENCE-FRIENDLY LOGIC

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Abstract. Ordinary first-order logic has the property that two formulas \( \phi \) and \( \psi \) have the same meaning in a structure if and only if the formula \( \phi \leftrightarrow \psi \) is true in the structure. We prove that independence-friendly logic does not have this property.

§1. Introduction. The meaning of a first-order formula \( \phi \) in a structure \( \mathfrak{A} \) is just the set of valuations that make the formula true in \( \mathfrak{A} \). That is,

\[
\phi^{\mathfrak{A}} = \{ \vec{a} \in N^{\mathfrak{A}} | \mathfrak{A} \models \phi[\vec{a}] \},
\]

where \( A \) is the universe of \( \mathfrak{A} \), and \( N \) is the number of variables in \( \phi \). Given a structure \( \mathfrak{A} \) and any two first-order formulas \( \phi \) and \( \psi \),

\[
\phi^{\mathfrak{A}} = \psi^{\mathfrak{A}}
\]

if and only if \( \mathfrak{A} \models \phi \leftrightarrow \psi \). Thus first-order logic is able to express the concept of “if and only if.”

Independence-friendly logic (IF logic) is a conservative extension of first-order logic that has the same expressive power as existential second-order logic [8, 7]. In IF logic the truth of a sentence is defined via a game between two players, Abélard (\( \forall \)) and Eloïse (\( \exists \)). The additional expressivity is obtained by modifying the quantifiers and connectives of an ordinary first-order sentence in order to restrict the information available to the existential player, Eloïse, in the associated semantic game.

In IF logic, only the information available to Eloïse is restricted, which means existential quantifiers are not dual to universal quantifiers. To compensate, negation symbols are only allowed before atomic formulas. Generalized independence-friendly logic (IFG logic) is a variant of independence-friendly logic in which the information available to both players can be restricted, making existential quantifiers dual to universal quantifiers and allowing any formula to be negated [3].

Since there are IFG-sentences that are neither true nor false, it is unclear whether IFG logic can express the concept of “if and only if.” For instance, one can define \( \phi \leftrightarrow J \psi \) as an abbreviation for the formula

\[
(\sim \phi \vee_{J} \psi) \wedge_{J} (\phi \vee_{J} \sim \psi)
\]

(the subscripts indicate what information is unavailable to the players at each move), but does this formula assert that \( \phi \) and \( \psi \) are logically equivalent? Is \( \phi \leftrightarrow J \psi \) true in a structure exactly when \( \phi \) and \( \psi \) have the same meaning in that structure? If not, is there some other syntactical combination of \( \phi \) and \( \psi \) that is
true exactly when $\phi$ and $\psi$ have the same meaning? The answer to all of these questions is no.

§2. IFG logic.

**Definition 2.1.** Given a first-order signature $\sigma$, an *atomic IFG-formula* is a pair $\langle \phi, X \rangle$ where $\phi$ is an atomic first-order formula and $X$ is a finite set of variables that includes every variable that appears in $\phi$ (and possibly more).

**Definition 2.2.** Given a first-order signature $\sigma$, the language $\mathcal{L}^\sigma_{IFG}$ is the smallest set of formulas such that:

(a) Every atomic IFG-formula is in $\mathcal{L}^\sigma_{IFG}$.
(b) If $\langle \phi, Y \rangle$ is in $\mathcal{L}^\sigma_{IFG}$ and $Y \subseteq X$, then $\langle \phi, X \rangle$ is in $\mathcal{L}^\sigma_{IFG}$.
(c) If $\langle \phi, X \rangle$ is in $\mathcal{L}^\sigma_{IFG}$, then $\langle \sim \phi, X \rangle$ is in $\mathcal{L}^\sigma_{IFG}$.
(d) If $\langle \phi, X \rangle$ and $\langle \psi, X \rangle$ are in $\mathcal{L}^\sigma_{IFG}$, and $Y \subseteq X$, then $\langle \phi \lor_Y \psi, X \rangle$ is in $\mathcal{L}^\sigma_{IFG}$.
(e) If $\langle \phi, X \rangle$ is in $\mathcal{L}^\sigma_{IFG}$, $x \in X$, and $Y \subseteq X$, then $\langle \exists x/\psi, X \rangle$ is in $\mathcal{L}^\sigma_{IFG}$.

Above $X$ and $Y$ are finite sets of variables.

From now on we will make certain assumptions about IFG-formulas that will allow us to simplify our notation. First, we will assume that the set of variables of $\mathcal{L}^\sigma_{IFG}$ is $\{ v_n \mid n \in \omega \}$. Second, since it does not matter much which particular variables appear in a formula, we will assume that variables with smaller indices are used before variables with larger indices. More precisely, if $\langle \phi, X \rangle$ is a formula, $v_j \in X$, and $i \leq j$, then $v_i \in X$. By abuse of notation, if $\langle \phi, X \rangle$ is a formula and $|X| = N$, then we will say that $\phi$ has $N$ variables and write $\phi$ for $\langle \phi, X \rangle$. As a shorthand, we will call $\phi$ an IFG$_N$-formula. Let

$$
\mathcal{L}^\sigma_{IFG_N} = \{ \phi \in \mathcal{L}^\sigma_{IFG} \mid \phi \text{ has } N \text{ variables} \}.
$$

Third, sometimes we will write $\phi \lor_J \psi$ instead of $\phi \lor_Y \psi$ and $\exists v_{n,J} \phi$ instead of $\exists v_{n,Y} \phi$, where $J = \{ j \mid v_j \in Y \}$. Finally, we will use $\phi \land_J \psi$ to abbreviate $\sim (\sim \phi \lor_J \sim \psi)$ and $\forall v_{n,J} \phi$ to abbreviate $\sim \forall v_{n,J} \sim \phi$.

Truth and falsity for IFG-sentences are defined in terms of a two-player, win-loss game of imperfect information. Given an IFG-sentence, Eloïse’s goal is to verify the sentence, and Abélard’s goal is to falsify it. The sentence is true if Eloïse has a winning strategy, and it is false if Abélard has a winning strategy. For example, consider a structure $\mathfrak{A}$ with universe $A$ and the ordinary first-order sentence

$$\forall v_0 \exists v_1 [v_0 \neq v_1].$$

First Abélard chooses an element $a$ to be the value of the variable $v_0$, then Eloïse chooses an element $b$ to be the value of the variable $v_1$. If $a \neq b$, Eloïse wins; otherwise, Abélard wins. If $\mathfrak{A}$ has more than one element Eloïse can win every play of the game; hence the sentence is true. If $\mathfrak{A}$ has only one element, then Abélard will win every play; hence the sentence is false. Now consider the IFG$_2$-sentence

$$\forall v_0 /a \exists v_1/(v_0) [v_0 \neq v_1].$$

The subscripts indicate what information is unavailable to the players at each move. The game begins as before with Abélard choosing an $a \in A$ to be the
value of $v_0$. Next Eloïse chooses an element $b \in A$ to be the value of $v_1$, but this time she must make her choice in ignorance of the value of $v_0$. Let us assume $A$ has more than one element. On the one hand, Eloïse does not have a winning strategy because she might blindly choose the same element as Abélard. Therefore the sentence is not true. On the other hand, Abélard does not have a winning strategy either, because Eloïse might get lucky and choose a different element than the one he chose. Therefore the sentence is not false.

Now we define the game semantics for formulas with free variables. Consider the IFG$_2$-formula

$$\exists v_1/(v_0)[v_0 \neq v_1].$$

In order for us to decide who wins a given play of the semantic game, at the end of the game every variable must have a value. Since $v_0$ is free, neither player has the opportunity to choose its value. To get around this difficulty, before the game begins we will assign random values to all the free variables. In fact, instead of assigning values only to the free variables, we will assign values to all of the variables. Thus the first move of the game is to choose a valuation $\vec{a} \in N^A$. Play proceeds with the players modifying the initial valuation until an atomic formula is reached, at which point the the game ends and the final valuation is used to determine the winner. In the above example, play begins with values for $v_0$ and $v_1$ being chosen at random. Then Eloïse attempts to modify the value of $v_1$ so that it is different from the value of $v_0$. Unfortunately for her, she is not allowed to see the value of $v_0$, so her task is no easier than before. However, suppose an oracle revealed to Eloïse that the initial valuation belonged to a subset $V$ of the space of all valuations $^2A$. Eloïse might be able to use this information to devise a winning strategy. For example, suppose $A = \{0,1\}$. Then $^2A = \{00,01,10,11\}$. If the oracle tells Eloïse that the initial valuation belongs to the set $V = \{00,01\}$, then Eloïse will know to choose 1 for the value of $v_1$. Thus Eloïse has a winning strategy for the game that begins by choosing the initial valuation from $V$ instead of from $^2A$. A set of valuations, such as $V$, is called a team. We say that the formula $\exists v_1/(v_0)[v_0 \neq v_1]$ is true in $\mathfrak{A}$ relative to $V$, and that $V$ is a winning team for $\phi$ in $\mathfrak{A}$.

Disjunctions and conjunctions are moves for the players, as well. In the game corresponding to the formula $\psi_1 \lor_Y \psi_2$, Eloïse must choose which disjunct she wishes to verify without knowing the values of the variables in $Y$. Dually, in the game corresponding to $\psi_1 \land_Y \psi_2$, Abélard chooses which conjunct Eloïse must verify, but his choice is not allowed to depend on the variables in $Y$.

Negation is handled by having the players switch roles. Eloïse attempts to verify $\sim \psi$ by falsifying $\psi$, and Abélard attempts to falsify $\sim \psi$ by verifying $\psi$.

In general, if $\phi$ is an IFG$_N$-formula and $V,W \subseteq ^NA$ are teams, then $\phi$ is true in $\mathfrak{A}$ relative to $V$, denoted $\mathfrak{A} \models^+ \phi[V]$, if and only if Eloïse has a winning strategy for the semantic game, given she knows the initial valuation belongs to $V$. Dually, $\phi$ is false in $\mathfrak{A}$ relative to $W$, denoted $\mathfrak{A} \models^- \phi[W]$, if and only if Abélard has a winning strategy, given he knows the initial valuation belongs to $W$. In the first case, we say that $V$ is a winning team (or trump) for $\phi$ in $\mathfrak{A}$, and in the second case, that $W$ is a losing team (or cotrump) for $\phi$ in $\mathfrak{A}$. 
Finally, we need to connect the game semantics for IFG-sentences with the game semantics for IFG-formulas. If $\phi$ is an IFG-sentence, then the initial valuation is irrelevant because the value of every variable is modified during the course of the game. If $\phi$ has free variables, then in order to have a winning strategy, a player must be able to win no matter what the initial valuation is. Therefore we define $\mathfrak{A} \models^+ \phi$ if and only if $\mathfrak{A} \models^+ \phi[A]$ and $\mathfrak{A} \models^- \phi$ if and only if $\mathfrak{A} \models^+ \phi[N\bar{A}]$. In the future, we will abbreviate similar statements by writing $\mathfrak{A} \models^+ \phi$ if and only if $\mathfrak{A} \models^+ \phi[N\bar{A}]$.

It is worth noting that restricting the information available to the players does not affect their moves, only their strategies. Therefore, restricting the information available to one player does not help his or her opponent. Eloïse has a winning strategy if and only if she wins regardless of how Abélard plays. Withholding information from her makes it harder for her to have a winning strategy, but withholding information from Abélard does not make it easier.

We hope this summary of the game semantics for IFG logic is sufficient. A more rigorous treatment can be found in [11, Section 1.2] or [12, Section 1.3].

§3. Trump semantics. Wilfrid Hodges made an important breakthrough when he found a way to define a Tarski-style semantics for independence-friendly logic [9, 10]. We now recall the necessary details.

**Definition 3.1.** Two valuations $\vec{a}, \vec{b} \in N A$ agree outside of $J \subseteq N$, denoted $\vec{a} \approx_J \vec{b}$, if
\[
\vec{a} \upharpoonright (N \setminus J) = \vec{b} \upharpoonright (N \setminus J).
\]

**Definition 3.2.** Given any set $V$, a cover of $V$ is a collection of sets $\mathcal{W}$ such that $V = \bigcup \mathcal{W}$. A disjoint cover is a cover whose members are pairwise disjoint.

**Definition 3.3.** Let $V \subseteq N A$, and let $\mathcal{W}$ be a cover of $V$. Then $\mathcal{W}$ is called $J$-saturated if every $U \in \mathcal{W}$ is closed under $\approx_J$. That is, for every $\vec{a}, \vec{b} \in V$, if $\vec{a} \approx_J \vec{b}$ and $\vec{a} \in U \in \mathcal{W}$, then $\vec{b} \in U$.

**Definition 3.4.** Define a partial operation $\bigcup_J \mathcal{W}$ on sets of teams by declaring $\bigcup_J \mathcal{W} = \bigcup \mathcal{W}$ whenever $\mathcal{W}$ is a $J$-saturated disjoint cover of $\bigcup \mathcal{W}$ and letting $\bigcup_J \mathcal{W}$ be undefined otherwise. Thus the formula $V = \bigcup_J \mathcal{W}$ asserts that $\mathcal{W}$ is a $J$-saturated disjoint cover of $V$. We will use the notation $V_1 \cup_J V_2$ to abbreviate $\bigcup_J \{V_1, V_2\}$.

**Definition 3.5.** A function $f: V \to A$ is independent of $J$, denoted $f: V \to^J A$, if $f(\vec{a}) = f(\vec{b})$ whenever $\vec{a} \approx_J \vec{b}$.

**Definition 3.6.** Let $\vec{a} \in N A$. For every $n < N$ and $b \in A$, define $\vec{a}(n : b)$ to be the valuation that is like $\vec{a}$ except the nth value has been changed to $b$, i.e.,
\[
\vec{a}(n : b) = \vec{a} \upharpoonright (N \setminus \{n\}) \cup \{(n, b)\}.
\]
Let $V, W \subseteq N A$ and $f: V \to A$. Define
\[
\begin{align*}
V(n : f) &= \{ \vec{a}(n : f(a)) \mid \vec{a} \in V \},
W(n : A) &= \{ \vec{a}(n : b) \mid \vec{a} \in W, b \in A \}.
\end{align*}
\]
The next theorem has appeared in many different forms in the literature. Hodges’ original formulation (for IF logic) appears in [9, Theorem 7.5]. Dechene’s version for IFG logic appears in [3, Theorem 5.3.5].

**Theorem 3.7 (Hodges).** Let \( \phi \) be an IFG\(_N\)-formula, let \( \mathfrak{A} \) be a suitable structure, and let \( V, W \subseteq N.A \).

- If \( \phi \) is atomic, then
  \( \begin{align*}
  (+) & \; \mathfrak{A} \models^+ \phi[V] \text{ if and only if for every } \vec{a} \in V, \; \mathfrak{A} \models \phi[\vec{a}], \\
  (-) & \; \mathfrak{A} \models^− \phi[W] \text{ if and only if for every } \vec{b} \in W, \; \mathfrak{A} \not\models \phi[\vec{b}].
  \end{align*} \)

- If \( \phi \) is \( \sim \), then
  \( \begin{align*}
  (+) & \; \mathfrak{A} \models^+ \sim[V] \text{ if and only if } \mathfrak{A} \not\models^− \psi[V], \\
  (-) & \; \mathfrak{A} \models^− \sim[W] \text{ if and only if } \mathfrak{A} \models^+ \psi[W].
  \end{align*} \)

- If \( \phi \) is \( \psi_1 \lor_J \psi_2 \), then
  \( \begin{align*}
  (+) & \; \mathfrak{A} \models^+ \psi_1 \lor_J \psi_2[V] \text{ if and only if } \mathfrak{A} \models^+ \psi_1[V_1] \text{ and } \mathfrak{A} \models^+ \psi_2[V_2] \text{ for some } V = V_1 \lor_J V_2, \\
  (-) & \; \mathfrak{A} \models^− \psi_1 \lor_J \psi_2[W] \text{ if and only if } \mathfrak{A} \models^− \psi_1[W] \text{ and } \mathfrak{A} \models^− \psi_2[W].
  \end{align*} \)

- If \( \phi \) is \( \exists v_{n/J} \psi \), then
  \( \begin{align*}
  (+) & \; \mathfrak{A} \models^+ \exists v_{n/J} \psi[V] \text{ if and only if } \mathfrak{A} \models^+ \psi[V(n : f)] \text{ for some } f : V \rightarrow A, \\
  (-) & \; \mathfrak{A} \models^− \exists v_{n/J} \psi[W] \text{ if and only if } \mathfrak{A} \models^− \psi[W(n : A)].
  \end{align*} \)

**Proof.** By two simultaneous inductions on the subformulas of \( \phi \). A full proof using the present notation can be found in [12, Theorem 1.32]. \( \square \)

§4. **IFG-cylindric set algebras.** We introduced IFG-cylindric set algebras in [11, 12] as a way to study the algebra of IFG logic.

Recall from [6, p. 2] or from [5, Definition 4.3.4 on p. 154] that the universe of \( \mathfrak{C}_N(\mathfrak{A}) \), the \( N \)-dimensional cylindric set algebra over \( \mathfrak{A} \), consists of the meanings of all the \( N \)-variable, first-order formulas expressible in the language of \( \mathfrak{A} \), where the meaning of a formula is defined by

\[
\phi^{\mathfrak{A}} = \{ \vec{a} \in N.A \mid \mathfrak{A} \models \phi[\vec{a}] \}.
\]

Similarly, the universe of \( \mathfrak{C}_{IFG_N}(\mathfrak{A}) \), the \( N \)-dimensional IFG-cylindric set algebra over \( \mathfrak{A} \), consists of the meanings of all the IFG\(_N\)-formulas expressible in the language of \( \mathfrak{A} \), where the meaning of an IFG\(_N\)-formula is given by

\[
\|\phi\|^{\mathfrak{A}} = (\|\phi\|^{+\mathfrak{A}}, \|\phi\|^{-\mathfrak{A}}),
\]

\[
\|\phi\|^{+\mathfrak{A}} = \{ V \subseteq N.A \mid \mathfrak{A} \models^+ \phi[V] \}, \quad \|\phi\|^{-\mathfrak{A}} = \{ W \subseteq N.A \mid \mathfrak{A} \models^− \phi[W] \}.
\]

More generally, we can define IFG-cylindric set algebras without reference to a base structure \( \mathfrak{A} \).

**Definition 4.1.** An **IFG-cylindric power set algebra** is an algebra whose universe is \( \mathfrak{P}(\mathfrak{P}(N.A)) \times \mathfrak{P}(\mathfrak{P}(N.A)) \), where \( A \) is a set and \( N \) is a natural number. The set \( A \) is called the base set, and the number \( N \) is called the dimension of the algebra. Every element \( X \) of an IFG-cylindric power set algebra is an ordered pair of sets of teams. We will use the notation \( X^+ \) to refer to the first coordinate of the pair, and \( X^- \) to refer to the second coordinate. There are a finite number of operations:
\begin{itemize}
  \item the constant \(0 = \langle \{\emptyset\}, \mathcal{P}(N^A) \rangle\);
  \item the constant \(1 = \langle \mathcal{P}(N^A), \{\emptyset\} \rangle\);
  \item for all \(i, j < N\), the constant \(D_{ij}\) is defined by
    \begin{align*}
      (+) \quad D_{ij}^+ &= \mathcal{P}(\{ a \in N^A \mid a_i = a_j \}), \\
      (-) \quad D_{ij}^- &= \mathcal{P}(\{ a \in N^A \mid a_i \neq a_j \});
    \end{align*}
  \item if \(X = (X^+, X^-)\), then \(X^\circ = (X^-, X^+)\);
  \item for every \(J \subseteq N\), the binary operation \(+_J\) is defined by
    \begin{align*}
      (+) \quad V \in (X +_J Y)^+ & \quad \text{if and only if} \quad V = V_1 \cup_J V_2 \quad \text{for some} \quad V_1 \in X^+ \quad \text{and} \quad V_2 \in Y^+,
      (-) \quad (X +_J Y)^- = X^- \cap Y^-;
    \end{align*}
  \item for every \(J \subseteq N\), the binary operation \(\cdot_J\) is defined by
    \begin{align*}
      (+) \quad (X \cdot_J Y)^+ &= X^+ \cap Y^+, \\
      (-) \quad W \in (X \cdot_J Y)^- & \quad \text{if and only if} \quad W = W_1 \cup_J W_2 \quad \text{for some} \quad W_1 \in X^- \quad \text{and} \quad W_2 \in Y^-;
    \end{align*}
  \item for every \(n < N\) and \(J \subseteq N\), the unary operation \(C_{n,J}\) is defined by
    \begin{align*}
      (+) \quad V \in C_{n,J}(X)^+ & \quad \text{if and only if} \quad V(n : f) \in X^+ \quad \text{for some} \quad f : V \rightarrow A, \\
      (-) \quad W \in C_{n,J}(X)^- & \quad \text{if and only if} \quad W(n : A) \in X^-.
    \end{align*}
\end{itemize}

\textbf{Definition 4.2.} An \textit{IFG-cylindric set algebra} (or \textit{IFG-algebra}, for short) is any subalgebra of an IFG-cylindric power set algebra. An \textit{IFG}_N-\textit{cylindric set algebra} (or \textit{IFG}_N-\textit{algebra}) is an IFG-cylindric set algebra of dimension \(N\).

The operations \(+_0\) and \(+_N\) are of particular interest. Since every disjoint cover of \(V\) is \(0\)-saturated, \(V \in (X +_0 Y)^+\) if and only if there is a disjoint cover \(V = V_1 \cup V_2\) such that \(V_1 \in X^+\) and \(V_2 \in Y^+\). At the other extreme, \(V = V_1 \cup_N V_2\) if and only if \(V_1 = V\) and \(V_2 = \emptyset\) or vice versa.

Also, the element \(\Omega = \langle \{\emptyset\}, \{\emptyset\} \rangle\) is present in most, but not all, IFG-algebras.

\section*{§5. Suits and double suits.} Meanings of IFG-formulas have the property that \(\|\phi\|^+ \cap \|\phi\|^- = \{\emptyset\}\), and \(V' \subseteq V \in \|\phi\|^\pm\) implies \(V' \in \|\phi\|^\pm\). These facts inspire the following definitions.

\textbf{Definition 5.1.} A nonempty set \(X^* \subseteq \mathcal{P}(N^A)\) is called a \textit{suit} if \(V' \subseteq V \in X^*\) implies \(V' \in X^*\). A \textit{double suit} is a pair \(\langle X^+, X^- \rangle\) of suits such that \(X^+ \cap X^- = \{\emptyset\}\).

\textbf{Definition 5.2.} An IFG-algebra is \textit{suited} if all of its elements are pairs of suits. It is \textit{double-suited} if all of its elements are double suits.

\textbf{Proposition 5.3} (Proposition 2.10 in [11]). The \textit{IFG}_N-\textit{algebra} generated by a set of pairs of suits is a suited \textit{IFG}_N-\textit{algebra}.

\textbf{Proposition 5.4} (Proposition 2.11 in [11]). The \textit{IFG}_N-\textit{algebra} generated by a set of double suits is a double-suited \textit{IFG}_N-\textit{algebra}. In particular, \(\mathcal{CS}_{\text{IFG}_N}(\mathfrak{A})\) is a double-suited \textit{IFG}_N-\textit{algebra}.

For the rest of the paper we will only be concerned with double suits. The next proposition is a summary of results from [11, Section 2.5].

\textbf{Proposition 5.5.} If \(X\) and \(Y\) are double suits,
“IFF” is not expressible in Independence-Friendly Logic

(a) $X +_J 0 = X = X \cdot_J 1$,
(b) $X +_J 1 = 1$ and $X \cdot_J 0 = 0$,
(c) $X +_N Y = (X^+ \cup Y^+, X^- \cap Y^-)$ and $X \cdot_N Y = (X^+ \cap Y^+, X^- \cap Y^-)$.

Part (c) implies that $+_N$ and $\cdot_N$ are lattice operations. Thus we can define a partial order on any double-suited IFG$_N$-algebra by declaring $X \leq Y$ if and only if $X +_N Y = Y$. It follows that $X \leq Y$ if and only if $X^+ \subseteq Y^+$ and $Y^- \subseteq X^-$. 

**Proposition 5.6.** If $X$ and $Y$ are double suits, $X \leq \Omega$, and $X \leq Y$, then $X +_J Y = Y$.

**Proof.** If $X \leq \Omega$, then $(X +_J Y)^+ = Y^+$. To see why, suppose $X \leq \Omega$ and $V \in (X +_J Y)^+$. Then $V = V_1 \cup V_2$ for some $V_1 \in X^+$ and $V_2 \in Y^+$, but since $X^+ = \{\emptyset\}$ we must have $V_1 = \emptyset$ and $V_2 = V$. Hence $V \in Y^+$. Conversely, suppose $V \in Y^+$. Then $V = \emptyset \cup V$, where $\emptyset \in X^+$ and $V \in Y^+$, so $V \in (X +_J Y)^+$.

If $X \leq Y$, then $Y^- \subseteq X^-$, so $(X +_J Y)^- = X^- \cap Y^- = Y^-$.

Whereas an ordinary cylindric algebra is an expansion of a Boolean algebra, we should not expect the same to be true for IFG-algebras because of the failure of the law of excluded middle in IFG logic. Somewhat miraculously, double-suited IFG-algebras do have an underlying structure that is as close to being a Boolean algebra as possible without satisfying the complementation axioms.

**Definition 5.7.** A De Morgan algebra $\mathfrak{A} = \langle A; 0, 1, \sim, \lor, \land \rangle$ is a bounded distributive lattice with an additional unary operation $\sim$ that satisfies $\sim \sim x = x$ and $\sim(x \lor y) = \sim x \land \sim y$.

**Definition 5.8.** A Kleene algebra is a De Morgan algebra that satisfies the additional axiom $x \land \sim x \leq y \lor \sim y$.

**Theorem 5.9** (Theorem 2.31 in [11]). The reduct of a double-suited IFG$_N$-algebra to the signature $\langle 0, 1, \sim, +_N, \cdot_N \rangle$ is a Kleene algebra.

Given a set $A$, let Suit$(^N A)$ denote the set of all suits in $\mathcal{P}(\mathcal{P}^N A)$, and let DSuit$(^N A)$ denote the set of all double suits in $\mathcal{P}(\mathcal{P}^N A) \times \mathcal{P}(\mathcal{P}^N A)$. Since the meaning of every IFG-formula is a double suit, the universe of $\mathcal{E}_{\text{IFG}_N}(\mathfrak{A})$ is contained in DSuit$(^N A)$. Therefore $|\text{DSuit}({}^N A)|$ gives an upper bound for the size of $\mathcal{E}_{\text{IFG}_N}(\mathfrak{A})$. Cameron and Hodges [2] count suits and double suits in the case when $N = 1$. The results of their calculations are shown in Table 1, where $m = |A|$, $f(m) = |\text{Suit}(A)|$, and $g(m) = |\text{DSuit}(A)|$. They remark that “one can think of the ratio of $g(m)$ to $2^m$ as measuring the expressive strength of IFG logic compared with ordinary first-order logic—always bearing in mind that IFG logic may have a rather unorthodox notion of what is worth expressing” [2, p. 679]. Cameron and Hodges also prove that given any finite set $A$, there is a structure $\mathfrak{A}$ such that the universe of $\mathcal{E}_{\text{IFG}_N}(\mathfrak{A})$ is exactly DSuit$(^N A)$ [2, Corollary 3.4].

**Proposition 5.10.** Let $\mathfrak{A}$ be a finite structure with at least two elements, and in which every element is named by a constant symbol. Then the universe of $\mathcal{E}_{\text{IFG}_N}(\mathfrak{A})$ is exactly DSuit$(^N A)$.
Therefore $X \in \mathfrak{C}_{\text{IFG}_N}(\mathfrak{A})$.

At this point, it is natural to ask which double suits can be the meanings of ordinary first-order formulas (that is, IFG-formulas whose independence sets are all empty). Ordinary first-order formulas have the property that $\mathfrak{A} \models^+ \phi[V]$ if and only if $\mathfrak{A} \models^+ \phi[\{\bar{a}\}]$ for all $\bar{a} \in V$. Hence if $\mathfrak{A} \models^+ \phi[V]$ and $\mathfrak{A} \models^+ \phi[V']$, then $\mathfrak{A} \models^+ \phi[V \cup V']$. It follows that the set of winning teams for an ordinary first-order formula $\phi$ is simply the power set of the set of valuations that satisfy $\phi$. That is,

$$\|\phi\|^+_\mathfrak{A} = \mathcal{P}(\phi^\mathfrak{A}).$$
The goal of this section is to show that 
then 
then 

Ordinary first-order formulas also have the property that for every \( \vec{a} \in \mathcal{N}A \) either 
\( \mathfrak{A} \models ^+ \phi[\{\vec{a}\}] \) or \( \mathfrak{A} \models ^- \phi[\{\vec{a}\}] \). These facts inspire the following definitions.

**Definition 5.11.** A double suit \( X \) is flat if there is a \( V \subseteq \mathcal{N}A \) such that 

**Proposition 5.12.** If \( X \) and \( Y \) are double suits, \( X \leq Y \), and \( Y \) is flat, then 

**Proof.** Suppose \( X \leq Y \) and \( Y^+ = \mathcal{P}(V) \). If \( V' \in (X + J Y)^+ \), then \( V' = V_1 \cup J V_2 \) for some \( V_1 \in X^+ \) and \( V_2 \in Y^+ \). Hence \( V_1, V_2 \subseteq V \) because \( X^+ \subseteq Y^+ = \mathcal{P}(V) \). Thus \( V' \subseteq V \), which implies \( V' \in Y^+ \). Conversely, if \( V' \in Y^+ \), then \( V' = \emptyset \cup J V' \), where \( \emptyset \in X^+ \) and \( V' \in Y^+ \), so \( V' \in (X + J Y)^+ \).

Also, since \( Y^- \subseteq X^- \) we have \( (X + J Y)^- = X^- \cap Y^- = Y^- \).

**Definition 5.13.** A double suit \( X \) is perfect if there is a \( V \subseteq \mathcal{N}A \) such that 

In [13], we showed that an IFG-formula \( \phi \) is equivalent to an ordinary first-order formula in a structure \( \mathfrak{A} \) if and only if \( \|\phi\|_\mathfrak{A} \) is perfect. It is worth noting that \( \mathfrak{C}_{\mathcal{N}IFG}(\mathfrak{A}) \) is generated by its perfect elements because it is generated by the meanings of atomic formulas, which are all perfect.

**§6.** \( \mathfrak{C}_{\mathcal{N}IFG}(\mathfrak{2}) \) is hereditarily simple. Let \( \mathfrak{2} \) be the structure with universe \( \{0, 1\} \) in which both elements are named by constant symbols. Then \( \mathfrak{C}_{\mathcal{N}IFG}(\mathfrak{2}) = \mathcal{DSuit}_1(\{0, 1\}) \). The distributive lattice structure of \( \mathfrak{C}_{\mathcal{N}IFG}(\mathfrak{2}) \) is shown in Figure 1, where the join operation is \( +_{\{0\}} \) and the meet operation is \( \cdot_{\{0\}} \),

\[
\begin{align*}
A &= \langle \mathcal{P}(\{0\}) \cup \mathcal{P}(\{1\}), \{\emptyset\} \rangle, \\
B &= \langle \mathcal{P}(\{0\}), \{\emptyset\} \rangle, \\
C &= \langle \mathcal{P}(\{1\}), \{\emptyset\} \rangle, \\
\|v_0 = 0\| &= \langle \mathcal{P}(\{0\}), \mathcal{P}(\{1\}) \rangle, \\
\|v_0 = 1\| &= \langle \mathcal{P}(\{1\}), \mathcal{P}(\{0\}) \rangle.
\end{align*}
\]

The goal of this section is to show that \( \mathfrak{C}_{\mathcal{N}IFG}(\mathfrak{2}) \) is hereditarily simple.

In [11, Proposition 2.5], we proved that an IFG\(_N\)-sentence can have one of only three possible meanings: 0, \( \Omega \), and 1. Thus, if we think of \( C_{0,j_0} \cdots C_{N-1,j_{N-1}} \) as a single operation that quantifies (cylindrifies) all of the variables of an IFG\(_N\)-formula, then the range of that operation is the IFG\(_N\)-algebra \( \{0, \Omega, 1\} \).

**Proposition 6.1** (Proposition 2.51 in [11]). If \( X \) is a double suit, then

\[
C_{0,j_0} \cdots C_{N-1,j_{N-1}}(X) = \begin{cases} 
1 & \text{if } X \not\subseteq \Omega, \\
\Omega & \text{if } 0 < X \leq \Omega, \\
0 & \text{if } X = 0.
\end{cases}
\]

**Lemma 6.2.** Let \( \equiv \) be a congruence on a double-suited IFG\(_N\)-algebra. If 0, \( \Omega \) (if present), or 1 are congruent to any other element, then \( \equiv \) is the total congruence.
Proof. First we will show that if $0 \equiv 1$, then $\equiv$ is the total congruence. If $0 \equiv 1$, then for every $X$ we have $X = X +_\emptyset 0 \equiv X +_\emptyset 1 = 1$. Hence $\equiv$ is the total congruence. Next we will show that if $0 \equiv \Omega$ or $1 \equiv \Omega$, then $\equiv$ is the total congruence. If $0 \equiv \Omega$, then $\Omega = \Omega +_\emptyset 0 \equiv 1$. Similarly, if $1 \equiv \Omega$, then $\Omega = \Omega +_\emptyset 1 \equiv 1$.

Now suppose $0 \neq X \equiv 0$. Then either

$$C_{0,\emptyset} \ldots C_{N-1,\emptyset}(X) = 1 \quad \text{or} \quad C_{0,\emptyset} \ldots C_{N-1,\emptyset}(X) = \Omega.$$  

In the first case, $0 = C_{0,\emptyset} \ldots C_{N-1,\emptyset}(0) \equiv C_{0,\emptyset} \ldots C_{N-1,\emptyset}(X) = 1$. In the second case, $0 = C_{0,\emptyset} \ldots C_{N-1,\emptyset}(0) \equiv C_{0,\emptyset} \ldots C_{N-1,\emptyset}(X) = \Omega$. Either way, $\equiv$ is the total congruence. In addition, if $1 \neq X \equiv 1$, then $0 \neq X^\lor \equiv 0$, so $\equiv$ is the total congruence.

Finally, if $\Omega \neq X \equiv \Omega$, then either $X \not\leq \Omega$ or $X^\lor \not\leq \Omega$. Hence either

$$\Omega = C_{0,\emptyset} \ldots C_{N-1,\emptyset}(\Omega) \equiv C_{0,\emptyset} \ldots C_{N-1,\emptyset}(X) = 1$$

or

$$\Omega = C_{0,\emptyset} \ldots C_{N-1,\emptyset}(\Omega) \equiv C_{0,\emptyset} \ldots C_{N-1,\emptyset}(X^\lor) = 1.$$  

Thus $\equiv$ is the total congruence. \hfill \dash}

Lemma 6.3. Let $\equiv$ be a congruence on any double-suited $\text{IFG}_N$-algebra that includes $\Omega$. If $X < \Omega < Y$ and $X \equiv Y$, then $\equiv$ is the total congruence.

Proof. If $X < \Omega < Y$, and $X \equiv Y$, then $\Omega = X +_N \Omega \equiv Y +_N \Omega = Y$, so by the previous lemma $\equiv$ is the total congruence. \hfill \dash
Lemma 6.4. Let $\equiv$ be a nontrivial congruence on any double-suited $\text{IFG}_N$-algebra that includes $\Omega$. Then there exist elements $X$ and $Y$ such that $X \equiv Y$ and $\Omega \leq X < Y$.

Proof. Since $\equiv$ is non-trivial, there exist distinct elements $X''$ and $Y''$ such that $X'' \equiv Y''$. Either $(X'')^+ \neq (Y'')^+$ or $(X'')^- \neq (Y'')^-$. In the first case, let $X' = X'' + \Omega$ and $Y' = Y'' + \Omega$; in the second case, let $X' = (X'')^+ + \Omega$ and $Y' = (Y'')^+ + \Omega$. In both cases, $(X')^+ \neq (Y')^+$, $X' \equiv Y'$, and $\Omega \leq X', Y'$. Now let $X = X' + \Omega X'$ and $Y = X' + \Omega Y'$. Then $X \equiv Y$ and $\Omega \leq X < Y$. \hfill \qed

Theorem 6.5. $\mathcal{C}_{\text{IFG}_1}(2)$ is simple.

Proof. By Lemma 6.4 it suffices to consider the congruences generated by pairs of elements from the interval above $\Omega$. Using the technique of perspective edges, we can see that if $A \equiv B$, then $C \equiv \Omega$, because $A \cdot_N C = C$ and $B \cdot_N C = \Omega$. Thus $\equiv$ is the total congruence by Lemma 6.2. Similarly, if $A \equiv C$ then $B \equiv \Omega$. Finally, if $B \equiv C$ then $B \equiv A$ because $B = B +_N \parallel v_0 = 0$ and $A = C +_N \parallel v_0 = 0$. \hfill \qed

Proposition 6.6. The proper subalgebras of $\mathcal{C}_{\text{IFG}_1}(2)$ are $\{0,1\}$, $\{0,\Omega,1\}$, and those shown in Figure 2.

Proof. It is easy to check that the IFG$_1$-algebras $\{0,1\}$ and $\{0,\Omega,1\}$ are subalgebras of $\mathcal{C}_{\text{IFG}_1}(2)$. Consider the subalgebra $\langle A \rangle = \{0, A^\omega, \Omega, A, 1\}$. Recall that since $\mathcal{C}_{\text{IFG}_1}(2)$ is double-suited, $X +_N 0 = X$ and $X +_N 1 = 1$. Also, $X \leq \Omega$ and $X \leq Y$ imply $X +_N Y = Y$. Thus $A^\omega +_N A^\omega = A^\omega$, $A^\omega +_N \Omega = \Omega$, and $A^\omega +_N A = A$. To finish showing $\langle A \rangle = \{0, A^\omega, \Omega, A, 1\}$ is closed under $+_\emptyset$ and $+_\{0\}$, it suffices to perform a few calculations. It is easily checked that

$$A +_\emptyset A = 1, \quad A +_{\{0\}} A = A.$$  

For example, $\{0,1\} \in (A +_\emptyset A)^+$ because $\{0,1\} = \{0\} \cup_{\emptyset} \{1\}$, where $\{0\} \in A^+$ and $\{1\} \in A^+$. $A +_{\{0\}} A = A$ because $+_\emptyset$ is a lattice join operation. Finally by Proposition 6.1, $C_{0,j}(0) = 0$, $C_{0,j}(A^\omega) = C_{0,j}(\Omega) = \Omega$, and $C_{0,j}(A) = C_{0,j}(1) = 1$.

Now consider $\langle B \rangle = \{0, B^\omega, \Omega, B, 1\}$. Since $B$ is flat $B +_\emptyset B = B$. All the other calculations are the same as for $\langle A \rangle$. Similarly $\langle C \rangle = \{0, C^\omega, \Omega, C, 1\}$.

To show $\langle A, B \rangle = \{0, A^\omega, B^\omega, \Omega, B, A, 1\}$ observe that

$$A +_\emptyset B = 1, \quad A +_{\{0\}} B = A.$$  

Similarly $\langle A, C \rangle = \{0, A^\omega, C^\omega, \Omega, C, A, 1\}$.

To show $\langle B, C \rangle = \{0, A^\omega, B^\omega, C^\omega, \Omega, C, B, A, 1\}$, observe that

$$B +_\emptyset C = 1, \quad B +_{\{0\}} C = A,$$

$$B^\omega +_\emptyset C^\omega = \Omega, \quad B^\omega +_{\{0\}} C^\omega = \Omega.$$  

Finally, note that if $\mathcal{D}$ is a subalgebra of $\mathcal{C}_{\text{IFG}_1}(2)$ that includes $\|v_0 = 0\|$, then $\|v_0 = 1\| = \|v_0 = 0\|^\omega \in \mathcal{D}$. Thus $\mathcal{D}$ includes all of the perfect elements in $\mathcal{C}_{\text{IFG}_1}(2)$. Hence $\mathcal{D} = \mathcal{C}_{\text{IFG}_1}(2)$. Similarly, if $\|v_0 = 1\| \in \mathcal{D}$, then $\mathcal{D} = \mathcal{C}_{\text{IFG}_1}(2)$. \hfill \qed

Theorem 6.7. $\mathcal{C}_{\text{IFG}_1}(2)$ is hereditarily simple.
Figure 2. Subalgebras of $\mathcal{CS}_{\text{IFG}}(2)$

Proof. It follows from Lemma 6.2 and Lemma 6.3 that the subalgebras \{0, Ω, 1\}, \langle A \rangle, \langle B \rangle, and \langle C \rangle are all simple. To show the subalgebra \langle A, B \rangle is simple, by Lemma 6.4 it suffices to show that the congruence $Cg(A, B)$ generated by $A$ and $B$ is the total congruence. Observe that if $A \equiv B$, then $1 = A +_0 A \equiv B +_0 B = B$, so $Cg(A, B)$ is the total congruence. A similar argument shows that the subalgebra \langle A, C \rangle is simple. Finally, to prove the subalgebra \langle B, C \rangle and $\mathcal{CS}_{\text{IFG}}(2)$ are simple it suffices to show that the congruences $Cg(A, B)$ and $Cg(A, C)$ are both the total congruence. But the calculations are the same as before, so we are done. ⊣
§7. \( \mathcal{C}_{\text{IFG}}(3) \) is not hereditarily simple. Let 3 be the structure with universe \( \{0, 1, 2\} \) in which all three elements are named by constant symbols. Then \( \mathcal{C}_{\text{IFG}}(3) = \text{DSuit}_1(\{0, 1, 2\}) \), which has 55 elements. Part of the lattice structure of \( \mathcal{C}_{\text{IFG}}(3) \) is shown in Figure 3. For simplicity, we only show the interval above \( \Omega \). Furthermore, we omit the falsity coordinate and denote each truth coordinate by listing the maximal winning teams. For example, the vertex labeled \( \{0, 1\}, \{2\} \) denotes the element \( \langle \mathcal{P}(\{0, 1\}) \cup \mathcal{P}(\{2\}), \emptyset \rangle \), and the vertex labeled \( \emptyset \) denotes \( \langle \{\emptyset\}, \emptyset \rangle = \Omega \). Readers familiar with the cover of [1] will recognize that Figure 3 is isomorphic to the free distributive 1-lattice on three generators. To obtain the full lattice structure of \( \mathcal{C}_{\text{IFG}}(3) \) it is necessary to flip the figure upside-down to get the interval below \( \Omega \), then fill in the sides with every possible double suit incomparable to \( \Omega \).

The goal of this section is to show that \( \mathcal{C}_{\text{IFG}}(3) \) is simple, but not hereditarily simple. In fact, every IFG\(_A\)-algebra whose universe is the collection of all double suits over a set \( A \) is simple.

**Proposition 7.1.** \( \text{DSuit}(^A) \) is simple.

**Proof.** Suppose \( X \) and \( Y \) are distinct elements of \( \text{DSuit}(^A) \) such that \( X \equiv Y \). Without loss of generality we may assume that there exists a \( V \in Y^+ \setminus X^+ \).
Let $Z = (\mathcal{P}(N A \setminus V), \mathcal{P}(V))$. Since $V \notin X^+$ we know that for every $U \in X^+$ there is an $\bar{a} \in V \setminus U$. Hence $U \cup (N A \setminus V) \neq N A$. Thus $1 \neq X +_\emptyset Z \equiv Y +_\emptyset Z = 1$. Therefore $\equiv$ is the total congruence.

Recall that in the proof that $\mathcal{E}_{\text{IFG}_{1}}(2)$ is hereditarily simple, we used the fact that $A +_\emptyset A = 1$ but $B +_\emptyset B \neq 1$. For any element $X$ of an IFG-algebra, let $nX$ be an abbreviation for $X +_0 \cdots +_0 X$.

**Definition 7.2.** The order of an element $X$ is the least positive integer $n$ such that $nX = 1$. If no such positive integer exists then the order of $X$ is infinite.

**Lemma 7.3.** Let $\equiv$ be a congruence on a double-suited $\text{IFG}_{N}$-algebra. If any two elements of different order are congruent, then $\equiv$ is the total congruence.

**Proof.** Let $X \equiv Y$. If the order of $X$ is less than the order of $Y$, then for some positive integer $n$ we have $1 = nX \equiv nY \neq 1$.

We know by Proposition 5.10 and Proposition 7.1 that $\mathcal{E}_{\text{IFG}_{1}}(3)$ is simple, but we can verify this directly by using the lemmas and the technique of perspective edges. For example, if $\{0\}, \{1\} \equiv \{0, 1\}$, then $\{0\}, \{1\}, \{2\} \equiv \{0, 1\}, \{2\}$. But $\{0\}, \{1\}, \{2\}$ has order 2, while $\{0, 1\}, \{2\}$ has order 1, so by Lemma 7.3 we have that $\equiv$ is the total congruence.

**Theorem 7.4.** $\mathcal{E}_{\text{IFG}_{1}}(3)$ is not hereditarily simple.

**Proof.** Let $A = \langle \mathcal{P}(\{0, 1\}), \{0\} \rangle$ and $B = \langle \mathcal{P}(\{0\}) \cup \mathcal{P}(\{1\}), \{0\} \rangle$. The subalgebra $\langle B \rangle = \{0, A^\lor, B^\lor, \Omega, B, A, 1\}$ is closed under $+_\emptyset$ and $+_\{0\}$ because

\[
\begin{align*}
A +_\emptyset A & = A, & A +_{\{0\}} A & = A, \\
A +_\emptyset B & = A, & A +_{\{0\}} B & = A, \\
B +_\emptyset B & = A, & B +_{\{0\}} B & = B, \\
A +_\emptyset A^\lor & = A, & A +_{\{0\}} A^\lor & = A, \\
A +_\emptyset B^\lor & = A, & A +_{\{0\}} B^\lor & = A, \\
B +_\emptyset A^\lor & = B, & B +_{\{0\}} A^\lor & = B, \\
B +_\emptyset B^\lor & = B, & B +_{\{0\}} B^\lor & = B, \\
A^\lor +_\emptyset A^\lor & = A^\lor, & A^\lor +_{\{0\}} A^\lor & = A^\lor, \\
A^\lor +_\emptyset B^\lor & = B^\lor, & A^\lor +_{\{0\}} B^\lor & = B^\lor, \\
B^\lor +_\emptyset B^\lor & = B^\lor, & B^\lor +_{\{0\}} B^\lor & = B^\lor.
\end{align*}
\]

All of the $+_\{0\}$ calculations are easy to check by looking at the lattice. The $+_\emptyset$ calculations require some computation. First, $A +_\emptyset A = A$ because $A$ is flat. Second, $A +_\emptyset B = A$ because $\{0, 1\} = \{0\} \cup_{\emptyset} \{1\}$, where $\{0\} \in A^+$ and $\{1\} \in B^+$, while $(A +_\emptyset B)^- = A^- \cap B^- = A^-$. Similarly, $B +_\emptyset B = A$. The remaining $+_\emptyset$ calculations all follow from Proposition 5.6 or Proposition 5.12. Finally, the set is closed under $C_{0,j}$ by Proposition 6.1.

Let $\equiv$ denote the equivalence relation that makes $A \equiv B$ and $A^\lor \equiv B^\lor$, but makes no other pair of distinct elements equivalent. To verify that $\equiv$ is a congruence, observe that $\equiv$ is preserved under $^\lor$ because $A^\lor \equiv B^\lor$ and $(A^\lor)^\lor =
A ≡ B = (B^\circ)^\circ. It is preserved under \(C_{0,J}\) because \(C_{0,J}(A) = 1 = C_{0,J}(B)\) and \(C_{0,J}(A^\circ) = \Omega = C_{0,J}(B^\circ)\). Finally, the calculations above show that \(\equiv\) is preserved under \(+_\emptyset\) and \(+_{\{0\}}\). Thus \(\equiv\) is a nontrivial, non-total congruence. Therefore \(\langle B \rangle\) is not simple.

\[\equiv\]

§8. "Iff" is not expressible in IFG logic. Let \(\phi \rightarrow J \psi\) be an abbreviation for \(\sim \phi \vee J \psi\), and let \(\phi \rightarrow N \psi\) be an abbreviation for

\[(\phi \rightarrow J \psi) \wedge J (\psi \rightarrow J \phi).\]

It will be useful to know when \(\mathfrak{A} \models J \phi \rightarrow \psi[V]\) and \(\mathfrak{A} \models J \phi \leftarrow \psi[V]\). It follows immediately from the definitions that for \(\phi \rightarrow J \psi\),

\[(+) \mathfrak{A} \models J \phi \rightarrow \psi[V]\] if and only if \(\mathfrak{A} \models J \phi[V_1]\) and \(\mathfrak{A} \models J \psi[V_2]\) for some \(V = V_1 \cup J V_2\),

\[(-) \mathfrak{A} \models \phi \rightarrow \psi[W]\] if and only if \(\mathfrak{A} \models \phi[W_1]\) and \(\mathfrak{A} \models \psi[W_2]\), and \(\mathfrak{A} \models \psi[W_2]\) for some \(W = W_1 \cup J W_2\).

In particular, the semantics for \(\phi \rightarrow N \psi\) are

\[(+) \mathfrak{A} \models N \phi \rightarrow \psi[V]\] if and only if \(\mathfrak{A} \models N \phi[V]\) or \(\mathfrak{A} \models N \psi[V]\),

\[(-) \mathfrak{A} \models \phi \rightarrow \psi[W]\] if and only if \(\mathfrak{A} \models \phi[W]\) and \(\mathfrak{A} \models \psi[W]\), and for \(\phi \rightarrow N \psi\),

\[(+) \mathfrak{A} \models J \phi \rightarrow \psi[V]\] if and only if \(\mathfrak{A} \models J \phi[V]\) and \(\mathfrak{A} \models J \psi[V]\), or \(\mathfrak{A} \models J \phi[V]\) and \(\mathfrak{A} \models \psi[V]\),

\[(-) \mathfrak{A} \models \phi \rightarrow \psi[W]\] if and only if \(\mathfrak{A} \models \phi[W]\) and \(\mathfrak{A} \models \psi[W]\), or \(\mathfrak{A} \models \phi[W]\) and \(\mathfrak{A} \models \psi[W]\).

For example, \(\mathfrak{A} \models N \phi \rightarrow \psi[V]\) if and only if \(\mathfrak{A} \models \phi \rightarrow (\phi \vee_N \psi) \wedge_N (\phi \vee_N \sim \psi)[V]\) if and only if \(\mathfrak{A} \models \phi \rightarrow (\phi \vee_N \psi)[V]\) and \(\mathfrak{A} \models \phi \rightarrow (\phi \vee_N \sim \psi)[V]\) if and only if \(\mathfrak{A} \models \phi[V]\) or \(\mathfrak{A} \models \psi[V]\), and \(\mathfrak{A} \models \phi[V]\) or \(\mathfrak{A} \models \psi[V]\) if and only if \(\mathfrak{A} \models \phi[V]\) and \(\mathfrak{A} \models \psi[V]\), or \(\mathfrak{A} \models \psi[V]\) and \(\mathfrak{A} \models \phi[V]\).

**Proposition 8.1.** For any IFG\(_N\)-formulas \(\phi\) and \(\psi\), \(\mathfrak{A} \models (\phi \leftrightarrow \psi)\) if and only if \(\|\phi\|_{\mathfrak{A}} = \|\psi\|_{\mathfrak{A}}\) and both are perfect.

**Proof.** Suppose \(\mathfrak{A} \models (\phi \leftrightarrow \psi)^{[N]}\). Then there exist \(V, V' \subseteq [N]\) such that \(\mathfrak{A} \models \phi[V]\), \(\mathfrak{A} \models \sim \psi^{[N] \setminus V}\), \(\mathfrak{A} \models \phi[V]\), and \(\mathfrak{A} \models \psi^{[N] \setminus V'}\). Thus \(V \cap V' = \emptyset\) and \((N, V) \cap (N \setminus V') = \emptyset\). Therefore \(V' = N \setminus V\), and \(\|\phi\|_{\mathfrak{A}} = (\mathcal{P}(V), \mathcal{P}(N \setminus V)) = \|\psi\|_{\mathfrak{A}}\). \(\square\)
**Proposition 8.2.** For any IFG$_N$-formulas $\phi$ and $\psi$, $\mathfrak{A} \models^+ \phi \leftrightarrow N \psi$ if and only if $\|\phi\|_{\mathfrak{A}} = \|\psi\|_{\mathfrak{A}} \in \{0, 1\}$.

**Proof.** Suppose $\mathfrak{A} \models^+ (\phi \leftrightarrow N \psi)[^N]$. Then $\mathfrak{A} \models^+ \phi[^N]$ and $\mathfrak{A} \models^+ \psi[^N]$, in which case $\|\phi\|_{\mathfrak{A}} = \|\psi\|_{\mathfrak{A}} = 1$, or $\mathfrak{A} \models^+ \phi[^N]$ and $\mathfrak{A} \models^+ \psi[^N]$, in which case $\|\phi\|_{\mathfrak{A}} = \|\psi\|_{\mathfrak{A}} = 0$.

**Definition 8.3.** An IFG$_N$-schema involving $k$ formula variables is an element of the smallest set $\Xi$ satisfying the following conditions.

(a) The formula variables $\alpha_0, \ldots, \alpha_{k-1}$ belong to $\Xi$.
(b) For all $i, j < N$, the formula $v_i = v_j$ belongs to $\Xi$.
(c) If $\xi$ belongs to $\Xi$, then $\sim \xi$ belongs to $\Xi$.
(d) If $\xi_1$ and $\xi_2$ belong to $\Xi$, and $J \subseteq N$, then $\xi_1 \lor J \xi_2$ belongs to $\Xi$.
(e) If $\xi$ belongs to $\Xi$, $n < N$, and $J \subseteq N$, then $\exists \nu_{n/J} \xi$ belongs to $\Xi$.

Note that the symbols $\alpha_0, \ldots, \alpha_{k-1}$ are distinct from the usual variables $v_0, \ldots, v_{k-1}$. If $\xi$ is an IFG$_N$-schema involving $k$ formula variables, and $\phi_0, \ldots, \phi_{k-1}$ are IFG$_N$-formulas, then the IFG$_N$-formula $\xi(\phi_0, \ldots, \phi_{k-1})$ is called an instance of $\xi$.

**Definition 8.4.** Every IFG$_N$-schema $\xi$ has a corresponding term $T_\xi$ in the language of IFG$_N$-algebras. $T_\xi$ is defined recursively as follows:

(a) $T_{\alpha_i} = X_i$,
(b) $T_{v_i = v_j} = D_{ij}$,
(c) $T_{\sim \xi} = (T_\xi)^\sim$,
(d) $T_{\xi_1 \lor J \xi_2} = T_{\xi_1} + J T_{\xi_2}$,
(e) $T_{\exists \nu_{n/J} \xi} = C_{n/J}(T_\xi)$.

**Lemma 8.5.** Let $\xi$ be an IFG$_N$-schema involving $k$ formula variables, and let $T_\xi$ be its corresponding term. Then for any IFG$_N$-formulas $\phi_0, \ldots, \phi_{k-1}$ and any suitable structure $\mathfrak{A}$,

$$\|\xi(\phi_0, \ldots, \phi_{k-1})\| = T_{\xi}^{\mathfrak{A}_{IFG_N}}(\|\phi_0\|, \ldots, \|\phi_{k-1}\|).$$

**Proof.** If $\xi$ is a formula variable $\alpha_i$, then $T_\xi = X_i$, so

$$\|\xi(\phi_0, \ldots, \phi_{k-1})\| = \|\phi_i\| = T_{\xi}^{\mathfrak{A}_{IFG_N}}(\|\phi_0\|, \ldots, \|\phi_{k-1}\|).$$

If $\xi$ is $v_i = v_j$, then $T_\xi = D_{ij}$, so

$$\|\xi(\phi_0, \ldots, \phi_{k-1})\| = \|v_i = v_j\| = D_{ij}^{\mathfrak{A}_{IFG_N}}(\|\phi_0\|, \ldots, \|\phi_{k-1}\|) = T_{\xi}^{\mathfrak{A}_{IFG_N}}(\|\phi_0\|, \ldots, \|\phi_{k-1}\|).$$

Now assume that

$$\|\xi(\phi_0, \ldots, \phi_{k-1})\| = T_{\xi_1}^{\mathfrak{A}_{IFG_N}}(\|\phi_0\|, \ldots, \|\phi_{k-1}\|),$$

$$\|\xi_1(\phi_0, \ldots, \phi_{k-1})\| = T_{\xi_1}^{\mathfrak{A}_{IFG_N}}(\|\phi_0\|, \ldots, \|\phi_{k-1}\|),$$

$$\|\xi_2(\phi_0, \ldots, \phi_{k-1})\| = T_{\xi_2}^{\mathfrak{A}_{IFG_N}}(\|\phi_0\|, \ldots, \|\phi_{k-1}\|).$$

Then

$$\|\sim \xi(\phi_0, \ldots, \phi_{k-1})\| = \|\xi(\phi_0, \ldots, \phi_{k-1})\|^{\sim}.$$
IFF” IS NOT EXPRESSIBLE IN INDEPENDENCE-FRIENDLY LOGIC

$$\xi_1 \lor_j \xi_2(\phi_0, \ldots, \phi_{k-1}) = \|\xi_1(\phi_0, \ldots, \phi_{k-1})\| + j \|\xi_2(\phi_0, \ldots, \phi_{k-1})\|$$

$$= T_{(\xi_1)}^{C_{IFG}(A)}(\|\phi_0\|, \ldots, \|\phi_{k-1}\|),$$

$$\forall_{\xi_1} \xi_1(\phi_0, \ldots, \phi_{k-1})$$

$$= T_{(\xi_1)}^{C_{IFG}(A)}(\|\phi_0\|, \ldots, \|\phi_{k-1}\|),$$

$$\exists_{\xi_1} \xi_1(\phi_0, \ldots, \phi_{k-1})$$

$$= T_{(\xi_1)}^{C_{IFG}(A)}(\|\phi_0\|, \ldots, \|\phi_{k-1}\|).$$

**Proposition 8.6.** Any double-suited IFG-algebra that has a term operation \(T(X,Y)\) such that \(T(X,Y) = 1\) if and only if \(X = Y\) is hereditarily simple.

**Proof.** Suppose \(\mathcal{C}\) is a double-suited IFG-algebra that has such a term operation. Then for any \(X \neq Y\) we have \(\langle 1, Z \rangle = \langle T(X,X), T(X,Y) \rangle \in Cg(X,Y)\), where \(Z\) is some element different than 1. Hence \(Cg(X,Y)\) is the total congruence. Thus \(\mathcal{C}\) is simple. Furthermore, the sentence

$$\forall X \forall Y [T(X,Y) = 1 \iff X = Y]$$

is universal, and so must hold in every subalgebra of \(\mathcal{C}\). Hence \(\mathcal{C}\) is hereditarily simple.

**Theorem 8.7.** There is no \(IFG_1\)-schema \(\xi\) involving two formula variables such that for every pair of \(IFG_1\)-formulas \(\phi\) and \(\psi\), and every suitable structure \(\mathcal{A}\), we have

$$\mathcal{A} \models \xi(\phi, \psi) \iff \|\phi\|_{\mathcal{A}} = \|\psi\|_{\mathcal{A}}.$$
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