Detection of some elements in the stable homotopy groups of spheres

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Abstract In this paper we constructs a new nontrivial family in the stable homotopy groups of spheres $\pi_p^{n,q+2pq+q-3}S$ which is of order $p$ and is represented by $k_0 h_n \in Ext_A^{3,p^nq+2pq+q}(\mathbb{Z}_p,\mathbb{Z}_p)$ in the Adams spectral sequence, where $p \geq 5$ is an odd prime, $n \geq 3$ and $q = 2(p-1)$. In the course of the proof, a new family of homotopy elements in $\pi_* V(1)$ which is represented by $\beta_* i'_* i_*(h_n) \in Ext_A^{2,p^nq+(p+1)q+1}(H^* V(1), \mathbb{Z}_p)$ in the Adams sequence is detected.

Keywords Stable homotopy groups of spheres, Adams spectral sequence, May spectral sequence, Steenrod algebra.

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1 Introduction and the main results

Let $A$ be the mod $p$ Steenrod algebra and $S$ be the sphere spectrum localized at an odd prime $p$. To determine the stable homotopy groups of spheres $\pi_* S$ is one of the central problems in homotopy theory. One of the main tools to reach it is the Adams spectral sequence:

$$E_2^{s,t} = Ext^s_t A(\mathbb{Z}_p, \mathbb{Z}_p) \Rightarrow \pi_{t-s} S,$$

where the $E_2^{s,t}$-term is the cohomology of $A$. So far, not so many families of homotopy elements in $\pi_* S$ have been detected. For example, a family $s_{n-1} \in \pi_p^{n,q+q-3}S$ for $n \geq 2$ which has filtration 3 in the Adams spectral sequence and is represented by

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$h_0b_{-1} \in Ext_A^{2, p^3+q+q}(\mathbb{Z}_p, \mathbb{Z}_p)$ has been detected in reference [1], where $q = 2(p - 1)$. In this paper, we also detect a family of homotopy elements in $\pi_{p^nq+pq-3}S$ which has filtration 3 and is represented by $k_0h_n \in Ext_A^{3, p^nq+2pq+q}(\mathbb{Z}_p, \mathbb{Z}_p)$ in the Adams spectral sequence.

From reference [2], $Ext_A^{1, \ast}(\mathbb{Z}_p, \mathbb{Z}_p)$ has $\mathbb{Z}_p$-bases consisting of $a_0 \in Ext_A^{1, 1}(\mathbb{Z}_p, \mathbb{Z}_p)$, $h_i \in Ext_A^{1, p^i}(\mathbb{Z}_p, \mathbb{Z}_p)$ for all $i \geq 0$ and $Ext_A^{2, \ast}(\mathbb{Z}_p, \mathbb{Z}_p)$ has $\mathbb{Z}_p$-bases consisting of $\alpha_2$, $a_0$, $a_0h_i(i > 0)$, $g_i(i \geq 0)$, $k_i(i > 0)$, $e_i(i \geq 0)$, and $h_ih_j(j \geq i + 2, i \geq 0)$ whose internal degrees are $2q + 1, 2, p^i + 1, p^{i+1} + 2p^i, 2p^{i+1} + p^i, p^{i+1}$ and $p^i + p^i$ respectively.

Let $M$ be the Moore spectrum modulo a prime $p \geq 5$ given by the cofibration

$$S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \Sigma S.$$  \hspace{1cm} (1.1)

Let $\alpha : \Sigma^q M \rightarrow M$ be the Adams map and $K$ be its cofibre given by the cofibration

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i'} K \xrightarrow{j'} \Sigma^{q+1} M,$$  \hspace{1cm} (1.2)

where $q = 2(p - 1)$. This spectrum which we briefly write as $K$ is known to be the Toda-Smith spectrum $V(1)$. Let $V(2)$ be the cofibre of $\beta : \Sigma^{(p+1)q}K \rightarrow K$ given by the cofibration

$$\Sigma^{(p+1)q}K \xrightarrow{\beta} K \xrightarrow{i} \Sigma K \xrightarrow{j} V(2) \xrightarrow{j'} \Sigma^{(p+1)q+1}K.$$  \hspace{1cm} (1.3)

Our results can be stated as follows.

**Theorem I** Let $p \geq 5, n \geq 3$, then

$$\beta_*i_*i_*(h_n) \neq 0 \in Ext_A^{2, p^n+q+(p+1)q+1}(H^*K, \mathbb{Z}_p)$$

is a permanent cycle in the Adams spectral sequence and converges to a nontrivial element $\zeta_n \in \pi_{p^n+q+(p+1)q-1}K$, where $h_n \in Ext_A^{1, p^n}(\mathbb{Z}_p, \mathbb{Z}_p)$.

**Theorem II** Let $p \geq 5, n \geq 3$, then

$$k_0h_n \neq 0 \in Ext_A^{3, p^n+2pq+q}(\mathbb{Z}_p, \mathbb{Z}_p)$$

is a permanent cycle in the Adams spectral sequence and it converges to a nontrivial element of order $p$ in $\pi_{p^n+2pq+q-3}S$.

**Remark:** The $k_0h_n$-element obtained in Theorem II is an indecomposable element in $\pi_*S$, i.e., it is not a composition of elements of lower filtration in $\pi_*S$, because $h_n(n > 0)$ is known to die in the Adams spectral sequence.

After giving some useful propositions in Section 2, the proofs of the main theorems will be given in Section 3.

**2 Some preliminaries on low-dimensional Ext groups**
In this section, we will prove some results on Ext groups of lower dimension which will be used in the proofs of the theorems.

**Proposition 2.1** Let \( p \geq 5, n \geq 3, a_0 \in \text{Ext}_A^{1,1}(\mathbb{Z}_p, \mathbb{Z}_p), h_n \in \text{Ext}_A^{1,q}(\mathbb{Z}_p, \mathbb{Z}_p) \), \( b_n \in \text{Ext}_A^{2,p^{n+1}}(\mathbb{Z}_p, \mathbb{Z}_p) \) respectively. Then we have the following:

1. \( \text{Ext}_A^{4,p^n q+(p+2)q+2}(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p\{a_0 a_0 h_1 h_n\} \).
2. \( \text{Ext}_A^{5,p^n q+(p+2)q}(\mathbb{Z}_p, \mathbb{Z}_p) = 0 \).

**Proof.** (1) See [3, Theorem 4.1].

(2) The proof is similar to that given in the proof of [4, Proposition 1.2]. We can show that in the May spectral sequence \( E_2^{5,p^n q+(p+2)q,3} = 0 \). Then

\[
\text{Ext}_A^{5,p^n q+(p+2)q}(\mathbb{Z}_p, \mathbb{Z}_p) = 0.
\]

Here the proof is omitted. \( \square \)

The following lemma is used in the proofs of many propositions in this section.

First recall spectra \( V(k) = \{V(k)_n\} \) for \( n \geq -1 \) which are so-called Toda-Smith spectra. The spectrum \( V(n) \) given in [5] such that the \( \mathbb{Z}_p \)-cohomology

\[
H^*(V(n), \mathbb{Z}_p) \cong E(n) = E(Q_0, Q_1, \cdots, Q_n),
\]

the exterior algebra generator by Milnor basis elements \( Q_0, Q_1, \cdots, Q_n \) in \( A \). The spectra \( V(n) \) for \( n \geq -1 \) are defined inductively by \( V(-1) = S \) and the cofibration

\[
\Sigma^{2(p^n-1)} V(n-1) \xrightarrow{\alpha(n)} V(n-1) \xrightarrow{i_n} V(n) \xrightarrow{j_n} \Sigma^{2p^n-1} V(n-1).
\]  

(2.1)

When \( n = 0, 1, 2 \), the above cofibration sequence just is the cofibration sequences (1.1), (1.2) and (1.3) respectively. \( \alpha(n) \) stand for the maps \( p, \alpha, \beta \) in (1.1), (1.2) and (1.3) respectively. Here \( V(-1) = S, V(0) = M, V(1) = K, i_0 = i, i_1 = i', i_2 = i, j_0 = j, j_1 = j', j_2 = j \). The existence of \( V(n) \) is assured [5, Theorem 1.1] for \( n = 1, p \geq 3 \) and for \( n = 2, p \geq 5 \).

By the definition of Ext groups, from (2.1) we can easily have the following lemma.

**Lemma 2.1** With notations as above. We have the following two long exact sequence:

\[
(1) \cdots \rightarrow \text{Ext}_A^{s-1,t-(2p^n-1)}(H^*V(n-1), \square) \xrightarrow{\alpha(n)} \text{Ext}_A^{s,t}(H^*V(n-1), \square) \xrightarrow{(i_n)} \text{Ext}_A^{s,t}(H^*V(n), \square) \xrightarrow{(j_n)} \text{Ext}_A^{s,t+1}(H^*V(n), \square) \rightarrow \cdots.
\]

\[
(2) \cdots \rightarrow \text{Ext}_A^{s-1,t-(2p^n-1)}(\square, H^*V(n-1)) \xrightarrow{\alpha(n)^*} \text{Ext}_A^{s,t}(\square, H^*V(n-1)) \xrightarrow{(i_n)^*} \text{Ext}_A^{s,t}(\square, H^*V(n)) \rightarrow \cdots.
\]

Here \( \square \) are an arbitrary \( A \)-module. \( \square \)

**Proposition 2.2** Let \( p \geq 5, n \geq 3 \). Then \( \text{Ext}_A^{3,p^n q+(p+2)q+1}(H^*M, H^*M) \) has a unique generator \( h_n g_0 \), where \( h_n g_0 \) satisfies \( i^* j_s h_n g_0 = h_n g_0, \) the generator of \( \text{Ext}_A^{3,p^n q+(p+2)q}(\mathbb{Z}_p, \mathbb{Z}_p) \) stated in [6, Table 8.1].
Proof. First consider the exact sequence

\[ \text{Ext}_A^{3p^n q + (p+2)q + 1}(\mathbb{Z}_p, \mathbb{Z}_p) \xrightarrow{j^*} \text{Ext}_A^{3p^n q + (p+2)q}(\mathbb{Z}_p, H^*M) \xrightarrow{i^*} \text{Ext}_A^{3p^n q + (p+2)q + 1}(\mathbb{Z}_p, \mathbb{Z}_p) \]

induced by (1.1). Since we know that \( \text{Ext}_A^{3p^n q + (p+2)q + 1}(\mathbb{Z}_p, \mathbb{Z}_p) \) is zero (cf. [6, Table 8.1]) and \( \text{Ext}_A^{4p^n q + (p+2)q + 1}(\mathbb{Z}_p, \mathbb{Z}_p) \) is zero (cf. [7, Proposition 2.1]), the above \( i^* \) is an isomorphism. Then we see that \( \text{Ext}_A^{3p^n q + (p+2)q}(\mathbb{Z}_p, H^*M) \) has unique generator \( h_n g_0 \), where \( h_n g_0 \) satisfies \( j^* h_n g_0 = h_n g_0 \), the unique generator of \( \text{Ext}_A^{3p^n q + (p+2)q}(\mathbb{Z}_p, \mathbb{Z}_p) \) stated in [6, Table 8.1].

At last, look at the following exact sequence induced by (1.1)

\[ \text{Ext}_A^{3p^n q + (p+2)q + 1}(\mathbb{Z}_p, H^*M) \xrightarrow{i^*} \text{Ext}_A^{3p^n q + (p+2)q + 1}(H^*M, H^*M) \xrightarrow{j^*} \text{Ext}_A^{3p^n q + (p+2)q + 1}(\mathbb{Z}_p, H^*M) \]

Since the first group is zero by virtue of \( \text{Ext}_A^{3p^n q + (p+2)q + r}(\mathbb{Z}_p, \mathbb{Z}_p) = 0 \) for \( r = 1, 2 \) (cf. [6, Table 8.1]) and the forth group is zero by virtue of the facts that \( \text{Ext}_A^{4p^n q + (p+2)q + t}(\mathbb{Z}_p, \mathbb{Z}_p) = 0 \) for \( t = 1, 2 \) (cf. [7, Proposition 2.1]), then the above \( j^* \) is an isomorphism. Thus \( \text{Ext}_A^{3p^n q + (p+2)q + 1}(H^*M, H^*M) \) has a unique generator \( j^* h_n g_0 \), where \( j^* h_n g_0 \) satisfies \( j^* h_n g_0 = h_n g_0 \). This finishes the proof of Proposition 2.2

**Proposition 2.3** Let \( p \geq 5, n \geq 3 \), then

\[ \text{Ext}_A^{3p^n q + (p+2)q}(H^*M, H^*M) \cong \mathbb{Z}_p \{i^* j^* h_n g_0, j^* i^* h_n g_0\} \]

Proof. Consider the exact sequence

\[ \text{Ext}_A^{2p^n q + (p+2)q - 1}(\mathbb{Z}_p, \mathbb{Z}_p) \xrightarrow{p^*} \text{Ext}_A^{2p^n q + (p+2)q}(\mathbb{Z}_p, \mathbb{Z}_p) \xrightarrow{i^*} \text{Ext}_A^{2p^n q + (p+2)q - 1}(\mathbb{Z}_p, \mathbb{Z}_p) \]

induced by (1.1). Since \( \text{Ext}_A^{2p^n q + (p+2)q - 1}(\mathbb{Z}_p, \mathbb{Z}_p) = 0 = \text{Ext}_A^{3p^n q + (p+2)q - 1}(\mathbb{Z}_p, \mathbb{Z}_p) \) (cf. [6, Table 8.1]), the above \( i^* \) is an isomorphism. Moreover we also know \( \text{Ext}_A^{3p^n q + (p+2)q}(\mathbb{Z}_p, \mathbb{Z}_p) = \mathbb{Z}_p \{h_n g_0\} \) (cf. [6, Table 8.1]). Thus we can have that \( \text{Ext}_A^{3p^n q + (p+2)q - 1}(\mathbb{Z}_p, H^*M) = \mathbb{Z}_p \{j^* (h_n g_0)\} \).

Now observe the following exact sequence

\[ \text{Ext}_A^{2p^n q + (p+2)q - 1}(\mathbb{Z}_p, H^*M) \xrightarrow{p^*} \text{Ext}_A^{2p^n q + (p+2)q}(\mathbb{Z}_p, H^*M) \xrightarrow{i^*} \text{Ext}_A^{2p^n q + (p+2)q - 1}(\mathbb{Z}_p, H^*M) \]

induced by (1.1). Since \( \text{Ext}_A^{2p^n q + (p+2)q + r}(\mathbb{Z}_p, \mathbb{Z}_p) = 0 \) for \( r = -1, 0 \) (cf. [2]), we can easily get that \( \text{Ext}_A^{2p^n q + (p+2)q - 1}(\mathbb{Z}_p, H^*M) = 0 \). By virtue of the fact \( \text{Ext}_A^{3p^n q + (p+2)q - 1}(\mathbb{Z}_p, H^*M) = \mathbb{Z}_p \{j^* (h_n g_0)\} \), we have that the image of the second \( p_* \) is \( p_*(j^* h_n g_0) = j^* p_*(h_n g_0) = j^* p_*(g h_n) = 0 \).
From the facts that
\[ \text{Ext}^3_{A} p^n q^{(p+2)q}(\mathbb{Z}_p, H^* M) \cong \mathbb{Z}_p \{ h_n g_0 \} \cong \mathbb{Z}_p \{ j_* h_n g_0 \} \]
and
\[ \text{Ext}^3_{A} p^n q^{(p+2)q-1}(\mathbb{Z}_p, H^* M) \cong \mathbb{Z}_p \{ j^* (h_n g_0) \} \cong \mathbb{Z}_p \{ j^* i^* j_* h_n g_0 \} \cong \mathbb{Z}_p \{ j_* j^* i^* h_n g_0 \}, \]
we can easily get that \( \text{Ext}^3_{A} p^n q^{(p+2)q}(H^* M, H^* M) \cong \mathbb{Z}_p \{ i_* j_* h_n g_0, j^* i^* h_n g_0 \} \). This shows Proposition 2.3.

\[ \text{Proposition 2.4} \] Let \( p \geq 5, n \geq 3 \), then we have
(1) \( i^* d_2(i_* j_* (h_n g_0)) \neq 0 \).
(2) \( d_2(j^* i^* (h_n g_0)) \neq 0 \), where
\[ d_2 : \text{Ext}^3_{A} p^n q^{(p+2)q}(H^* M, H^* M) \to \text{Ext}^5_{A} p^n q^{(p+2)q+1}(H^* M, H^* M) \]
is the differential of the Adams spectral sequence.

**Proof.** (1) From [7, p.488] we know that \( d_2(i_* (h_n g_0)) \neq 0 \). By Proposition 2.2, \( d_2(i_* (h_n g_0)) = d_2(i_* j_* (h_n g_0)) = d_2(i_* i_* j_* (h_n g_0)) = i^* d_2(i_* j_* (h_n g_0)) \). The desired result follows.

(2) Consider the exact sequence
\[ \text{Ext}^4_{A} p^n q^{(p+2)q}(\mathbb{Z}_p, \mathbb{Z}_p) \xrightarrow{p^*} \text{Ext}^5_{A} p^n q^{(p+2)q+1}(\mathbb{Z}_p, \mathbb{Z}_p) \xrightarrow{j_*} \text{Ext}^5_{A} p^n q^{(p+2)q}(\mathbb{Z}_p, H^* M) \xrightarrow{i^*} \text{Ext}^5_{A} p^n q^{(p+2)q+1}(\mathbb{Z}_p, H^* M) \]
induced by (1.1). We claim that the above \( j^* \) is an isomorphism. By virtue of the fact that \( \text{Ext}^5_{A} p^n q^{(p+2)q}(\mathbb{Z}_p, \mathbb{Z}_p) = 0 \) (cf. Proposition 2.1), we see that the above \( j^* \) is an epimorphism. Note that \( \text{Ext}^4_{A} p^n q^{(p+2)q}(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p \{ g_0 b_{n-1} \} \) (cf. [7, Proposition 2.1]). Since \( p^*(g_0 b_{n-1}) = a_0 g_0 b_{n-1} = 0 \) (Note: \( a_0 g_0 = 0 \) by [6, Table 8.2]), so \( \ker j^* = \text{im} p^* = 0 \), i.e., the above \( j^* \) is a monomorphism. The proof of the claim is finished. Since \( \alpha_2 b_0 h_n \neq 0 \in \text{Ext}^5_{A} p^n q^{(p+2)q+1}(\mathbb{Z}_p, \mathbb{Z}_p) \) (cf. [7, Proposition 2.1]), so by the claim we get that \( j^*(\alpha_2 b_0 h_n) \neq 0 \in \text{Ext}^5_{A} p^n q^{(p+2)q}(\mathbb{Z}_p, H^* M) \). Note the fact that \( d_2(h_n g_0) = \alpha_2 b_0 h_n \neq 0 \), so \( j^* d_2(j_* i^* h_n g_0) = j_* d_2(j^* i^* h_n g_0) \neq 0 \) by Proposition 2.2. Thus \( d_2(j^* i^* (h_n g_0)) \neq 0 \).

\[ \text{Proposition 2.5} \] Let \( p \geq 5, n \geq 3 \), then
\[ \text{Ext}^3_{A} p^n q^{(p+1)q+2}(H^* K, H^* M) = 0. \]

**Proof.** Consider the exact sequence
\[ \text{Ext}^3_{A} p^n q^{(p+1)q+3}(H^* M, \mathbb{Z}_p) \xrightarrow{j^*} \text{Ext}^3_{A} p^n q^{(p+1)q+2}(H^* M, H^* M) \xrightarrow{i^*} \text{Ext}^3_{A} p^n q^{(p+1)q+2}(H^* M, \mathbb{Z}_p). \]
Since the first and third group are zero by the facts \( \text{Ext}^3_{A} p^n q^{(p+1)q+r}(\mathbb{Z}_p, \mathbb{Z}_p) = 0 \) for \( r = 1, 2, 3 \) (cf. [6, Table 8.1]), so the second group is zero.
Look at the exact sequence

\[ \text{Ext}_A^{3p^nq+pq+2}(\mathbb{Z}_p, \mathbb{Z}_p) \xrightarrow{i_*} \text{Ext}_A^{3p^nq+pq+2}(H^*_M, \mathbb{Z}_p) \xrightarrow{j_*} \text{Ext}_A^{3p^nq+pq+1}(\mathbb{Z}_p, \mathbb{Z}_p) \xrightarrow{p_*} \text{Ext}_A^{3p^nq+pq+2}(\mathbb{Z}_p, \mathbb{Z}_p) \]

induced by (1.1). Since we know that \( \text{Ext}_A^{3p^nq+pq+1}(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p\{a_0h_1h_n\} \) (cf. [6, Table 8.1]) and \( \text{Ext}_A^{4p^nq+pq+2}(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p\{a_0a_0h_1h_n\} \) by Proposition 2.1, so the above \( p_* \) is an isomorphism. \( imj_* = 0 \) since \( p_* \) is an isomorphism. \( imi_* = 0 \) by the fact that \( \text{Ext}_A^{3p^nq+pq+2}(\mathbb{Z}_p, \mathbb{Z}_p) = 0 \) (cf. [6, Table 8.1]). Thus we can have that \( \text{Ext}_A^{3p^nq+pq+2}(H^*_M, \mathbb{Z}_p) = 0. \)

Observe the following exact sequence induced by (1.1)

\[ \text{Ext}_A^{1p^nq+(p+1)q+2}(\mathbb{Z}_p, \mathbb{Z}_p) \xrightarrow{p_*} \text{Ext}_A^{1p^nq+(p+1)q+1}(\mathbb{Z}_p, \mathbb{Z}_p) \xrightarrow{i_*} \text{Ext}_A^{1p^nq+(p+1)q+1}(H^*_M, \mathbb{Z}_p) \xrightarrow{j_*} \text{Ext}_A^{1p^nq+(p+1)q+1}(\mathbb{Z}_p, \mathbb{Z}_p). \]

Since \( \text{Ext}_A^{1p^nq+(p+1)q}(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p\{h_1h_n\} \) and \( \text{Ext}_A^{3p^nq+pq+1}(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p\{a_0h_1h_n\} \) (cf. [6, Table 8.1]), we know that the first \( p_* \) is an isomorphism. Similarly by virtue of the facts that \( \text{Ext}_A^{2p^nq+pq}(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p\{b_0h_n, h_1b_{n-1}\} \) (cf. [6, Table 8.1]) and \( \text{Ext}_A^{4p^nq+pq+1}(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p\{a_0b_0h_n, a_0h_1b_{n-1}\} \) (cf. [7, Proposition 2.1]), we get that the second \( p_* \) is also an isomorphism. Thus \( \text{Ext}_A^{3p^nq+pq+1}(H^*_M, \mathbb{Z}_p) = 0. \)

Look at the exact sequence

\[ 0 = \text{Ext}_A^{3p^nq+pq+2}(H^*_M, \mathbb{Z}_p) \xrightarrow{j_*} \text{Ext}_A^{3p^nq+pq+1}(H^*_M, H^*_M) \xrightarrow{i_*} \text{Ext}_A^{3p^nq+pq+1}(\mathbb{Z}_p, \mathbb{Z}_p) \]

induced by (1.1). It is easy to get that the second group is zero.

At last consider the following exact sequence

\[ 0 = \text{Ext}_A^{3p^nq+(p+1)q+2}(H^*_M, H^*_M) \xrightarrow{i_*} \text{Ext}_A^{3p^nq+(p+1)q+2}(H^*_K, H^*_M) \xrightarrow{j_*} \text{Ext}_A^{3p^nq+pq+1}(H^*_M, H^*_M) \]

induced by (1.2). The desired result follows. \( \square \)

**Proposition 2.6** Let \( p \geq 5, n \geq 3, \) then

\[ \text{Ext}_A^{2p^nq+(p+1)q+1}(H^*_K, \mathbb{Z}_p) \cong \mathbb{Z}_p\{\beta_*i'_*i_*\{h_n\}\}, \]

where \( \beta_* : \text{Ext}_A^{1p^nq}(H^*_K, \mathbb{Z}_p) \rightarrow \text{Ext}_A^{2p^nq+(p+1)q+1}(H^*_K, \mathbb{Z}_p) \) is the connecting homomorphism induced by \( \beta : \Sigma^{(p+1)q}K \rightarrow K. \)

**Proof.** Look at the exact sequence

\[ \text{Ext}_A^{1p^nq+pq-1}(\mathbb{Z}_p, \mathbb{Z}_p) \xrightarrow{p_*} \text{Ext}_A^{2p^nq+pq}(\mathbb{Z}_p, \mathbb{Z}_p) \xrightarrow{i_*} \text{Ext}_A^{2p^nq+pq}(H^*_M, \mathbb{Z}_p) \xrightarrow{j_*} \text{Ext}_A^{2p^nq+pq-1}(\mathbb{Z}_p, \mathbb{Z}_p) \]

induced by (1.1). Since the first group and the fourth group are zero, so the above \( i_* \) is an isomorphism. Thus we can see that \( \text{Ext}_A^{2p^nq+pq}(H^*_M, \mathbb{Z}_p) \cong \mathbb{Z}_p\{i_*\{h_1h_n\}\} \) by the fact that \( \text{Ext}_A^{2p^nq+pq}(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p\{h_1h_n\}. \)
At last observe the following exact sequence

\[ \text{Ext}_A^{2p^n q+(p+1)q+1}(H^* M, \mathbb{Z}_p) \stackrel{i'_*}{\rightarrow} \text{Ext}_A^{2p^n q+(p+1)q+1}(H^* K, \mathbb{Z}_p) \stackrel{j'_*}{\rightarrow} \text{Ext}_A^{2p^n q+pq}(H^* M, \mathbb{Z}_p) \stackrel{p_*}{\rightarrow} \text{Ext}_A^{3p^n q+(p+1)q+1}(H^* M, \mathbb{Z}_p) \]

induced by (1.2). Since the first group is zero by \( \text{Ext}_A^{2p^n q+(p+1)q+r}(\mathbb{Z}_p, \mathbb{Z}_p) = 0 \) for \( r = 0, 1 \) (cf. [2]) and the fourth group is zero by \( \text{Ext}_A^{3p^n q+(p+1)q+t}(\mathbb{Z}_p, \mathbb{Z}_p) = 0 \) for \( t = 0, 1 \) (cf. [6, Table 8.1]), then the above \( j'_* \) is an isomorphism. Thus we can have that \( \text{Ext}_A^{2p^n q+(p+1)q+1}(H^* K, \mathbb{Z}_p) \) has a unique generator \( \Delta \), which satisfies \( j'_*(\Delta) = i_*(h_1 h_n) \). From [5, (5.4)], we have that \( j' \beta i' \in \left[ \sum_{pq-1} S, M \right] \) is represented by \( i_*(h_1) \in \text{Ext}_A^{pq}(H^* M, \mathbb{Z}_p) \) in the Adams spectral sequence. It follows that \( (j' \beta i')_n(h_n) = i_*(h_1 h_n) = j'_*(\Delta) \). Note the fact that \( j'_* \) is an isomorphism.

It is easy to get that \( \beta s i'_* i_*(h_n) = \Delta \). Therefore this completes the proof of the proposition. \( \square \)

**Proposition 2.7** Let \( p \geq 5, n \geq 3 \), then

\[ \text{Ext}_A^{2p^n q+(p+1)q+1}(H^* K, H^* M) \cong \mathbb{Z}_p \{ \beta s i'_*(\tilde{h}_n) \} \]

where \( \tilde{h}_n \in \text{Ext}_A^{pq}(H^* M, H^* M) \) is the unique generator of \( \text{Ext}_A^{pq}(H^* M, H^* M) \) and satisfies \( i^*(\tilde{h}_n) = i_*(h_n) \).

**Proof.** Consider the exact sequence

\[ \text{Ext}_A^{2p^n q+pq+1}(\mathbb{Z}_p, \mathbb{Z}_p) \stackrel{i}{\rightarrow} \text{Ext}_A^{2p^n q+pq+1}(H^* M, \mathbb{Z}_p) \stackrel{j_*}{\rightarrow} \text{Ext}_A^{2p^n q+pq}(\mathbb{Z}_p, \mathbb{Z}_p) \stackrel{p_*}{\rightarrow} \text{Ext}_A^{3p^n q+pq+1}(\mathbb{Z}_p, \mathbb{Z}_p). \]

Since \( \text{Ext}_A^{2p^n q+pq+1}(\mathbb{Z}_p, \mathbb{Z}_p) = 0 \), \( im j_* = 0 \). Since \( \text{Ext}_A^{2p^n q+pq}(\mathbb{Z}_p, \mathbb{Z}_p) = \mathbb{Z}_p \{ h_1 h_n \} \) and \( \text{Ext}_A^{3p^n q+pq+1}(\mathbb{Z}_p, \mathbb{Z}_p) = \mathbb{Z}_p \{ q h_1 h_n \} \) (cf. [6, Table 8.1]), then the above \( p_* \) is an isomorphism, and then \( im j_* = 0 \). Thus \( \text{Ext}_A^{2p^n q+pq+1}(H^* M, \mathbb{Z}_p) = 0 \).

Look at the exact sequence

\[ \text{Ext}_A^{2p^n q+(p+1)q+2}(H^* M, \mathbb{Z}_p) \stackrel{i'_*}{\rightarrow} \text{Ext}_A^{2p^n q+(p+1)q+2}(H^* K, \mathbb{Z}_p) \stackrel{j'_*}{\rightarrow} \text{Ext}_A^{2p^n q+pq+1}(H^* M, \mathbb{Z}_p) = 0 \]

induced by (1.2). Since we know that \( \text{Ext}_A^{2p^n q+(p+1)q+2}(H^* M, \mathbb{Z}_p) = 0 \) by the facts that \( \text{Ext}_A^{2p^n q+(p+1)q+r}(\mathbb{Z}_p, \mathbb{Z}_p) = 0 \) for \( r = 1, 2 \) and \( \text{Ext}_A^{2p^n q+pq+1}(H^* M, \mathbb{Z}_p) = 0 \), then \( \text{Ext}_A^{2p^n q+(p+1)q+2}(H^* K, \mathbb{Z}_p) = 0 \).

Observe the following exact sequence

\[ \text{Ext}_A^{3p^n q+(p+1)q+2}(H^* M, \mathbb{Z}_p) \stackrel{i'_*}{\rightarrow} \text{Ext}_A^{3p^n q+(p+1)q+2}(H^* K, \mathbb{Z}_p) \stackrel{j'_*}{\rightarrow} \text{Ext}_A^{3p^n q+pq+1}(H^* M, \mathbb{Z}_p) \]

induced by (1.2). From the proof of Proposition 2.5 we know that the first group and the third group are zero. So the middle group is zero.
At last, look at the following exact sequence
\[
\begin{align*}
E xt_A^{2p^nq+(p+1)q+2}(H^* K, \mathbb{Z}_p) \xrightarrow{i^*} E xt_A^{2p^nq+(p+1)q+1}(H^* K, H^* \mathcal{M}) \xrightarrow{i^*} \\
E xt_A^{2p^nq+(p+1)q+1}(H^* K, \mathbb{Z}_p) \xrightarrow{p^*} E xt_A^{2p^nq+(p+1)q+2}(H^* K, \mathbb{Z}_p)
\end{align*}
\]
induced by (1.1). Since the first group and the fourth group are zero, then the above \( i^* \) is an isomorphism. Notice that \( E xt_A^{2p^nq+(p+1)q+1}(H^* K, \mathbb{Z}_p) \cong \mathbb{Z}_p \{ \beta_s i'_s i_s(h_n) \} \).
Thus we can easily have that there exists an element \( \bar{\Delta} \in E xt_A^{2p^nq+(p+1)q+1}(H^* K, H^* \mathcal{M}) \) such that \( E xt_A^{2p^nq+(p+1)q+1}(H^* K, H^* \mathcal{M}) \cong \mathbb{Z}_p \{ \bar{\Delta} \} \) and \( i^*(\bar{\Delta}) = \beta_s i'_s i_s(h_n) \). Since \( i^*(h_n) = i_s(h_n), i^*(\bar{\Delta}) = \beta_s i'_s i_s(h_n) = \beta_s i'_s i^*(h_n) = i^* \beta_s i'_s(h_n) \). Thus we have that \( \bar{\Delta} = \beta_s i'_s(h_n) \) by the fact that \( i^* \) is an isomorphism. Thus this completes the proof of the proposition.

\[\square\]

3 Proofs of the main theorems

Let
\[
\begin{array}{cccc}
\ldots \Rightarrow & \Sigma^{-2}E_2 & \cdots & \Sigma^{-1}E_1 & \cdots & \Sigma^{-0}S & \Sigma^{-2}KG_2 & \Sigma^{-1}KG_1 & KG_0 = K\mathbb{Z}_p \\
\downarrow b_2 & \downarrow b_1 & \downarrow b_0 & & & & & & \\
\Sigma^{-2}KG_2 & \Sigma^{-1}KG_1 & KG_0 & & & & & & \\
\end{array}
\]
be the minimal Adams resolution of \( S \) satisfying the following.

(1) \( E_s \xrightarrow{b_s} KG_s \xrightarrow{c_s} E_{s+1} \xrightarrow{a_s} \Sigma E_s \) are cofibrations for all \( s \geq 0 \) which induce short exact sequences in \( \mathbb{Z}_p \)-cohomology.

(2) \( KG_s \) is a wedge sum of suspensions of Eilenberg-MacLane spectra of type \( K\mathbb{Z}_p \).

(3) \( \pi_t KG_s \) are the \( E_1^{s,t} \)-terms, \( (b_s \bar{c}_{s-1})_s : \pi_t KG_{s-1} \rightarrow \pi_t KG_s \) are the \( d_1^{s-1,t} \)-differentials of the Adams spectral sequence and \( \pi_t KG_s \cong Ext^t_A( \mathbb{Z}_p, \mathbb{Z}_p ) \) (cf. [9, p.180]). Then
\[
\begin{array}{cccc}
\ldots \Rightarrow & \Sigma^{-2}E_2 \wedge W & \cdots & \Sigma^{-1}E_1 \wedge W & \cdots & \Sigma^{-0}S \wedge W & \Sigma^{-2}KG_2 \wedge W & \Sigma^{-1}KG_1 \wedge W & KG_0 \wedge W \\
\downarrow b_2 \wedge 1_W & \downarrow b_1 \wedge 1_W & \downarrow b_0 \wedge 1_W & & & & & & \\
\Sigma^{-2}KG_2 \wedge W & \Sigma^{-1}KG_1 \wedge W & KG_0 \wedge W & & & & & & \\
\end{array}
\]
is an Adams resolution of arbitrary finite spectrum \( W \).

From [10, p.204-206], the Moore spectrum \( M \) is a commutative ring spectrum with multiplication \( m_M : M \wedge M \rightarrow M \) and there is \( \bar{m}_M : \Sigma M \rightarrow M \wedge M \) such that
\[
\begin{align*}
m_M(i \wedge 1_M) &= 1_M, & (j \wedge 1_M)\bar{m}_M &= 1_M, \\
m_Mm_M &= 0, & (i \wedge 1_M)m_M + \bar{m}_M(j \wedge 1_M) &= 1_{M \wedge M}, \\
m_MT &= -m_M, & T\bar{m}_M &= \bar{m}_M, \\
m_M(1_M \wedge i) &= -1_M, & (1_M \wedge j)\bar{m}_M &= 1_M,
\end{align*}
\]
where \( T : M \wedge M \rightarrow M \wedge M \) is the switching map.

A spectrum \( X \) is called an \( M \)-module spectrum if \( p \wedge 1_X = 0 \), and consequently, the cofibration \( X \xrightarrow{p \wedge 1_X} X \xrightarrow{i \wedge 1_X} M \wedge X \xrightarrow{j \wedge 1_X} \Sigma X \) split, i.e., there is a homotopy
equivalence $M \wedge X = X \sqrt{X}$ and there are maps $m_X : M \wedge X \to X$, $\bar{m}_X : \Sigma X \to M \wedge X$ satisfying $m_X(i \wedge 1_X) = 1_X$, $(j \wedge 1_X)\bar{m}_X = 1_X$, $m_X\bar{m}_X = 0$ and $\bar{m}_X(j \wedge 1_X) + (i \wedge 1_X)m_X = 1_{M \wedge X}$. The $M$-module actions $m_X$, $\bar{m}_X$ are called associative if $m_X(1_M \wedge m_X) = -m_X(m_X \wedge 1_X)$ and $(1_M \wedge \bar{m}_M)\bar{m}_X = (\bar{m}_M \wedge 1_M)\bar{m}_X$.

Let $X$ and $X'$ be $M$-module spectra. Then we define a homomorphism $d : [\Sigma^sX', X] \to [\Sigma^{s+1}X', X]$ by $d(f) = m_X(1_M \wedge f)\bar{m}_{X'}$ for $f \in [\Sigma^sX', X]$. This operation $d$ is called a derivation (of maps between $M$-module spectra) which has the following properties:

**Lemma 3.1** [10, Theorem 2.2] (1) $d$ is a derivation: $d(fg) = fd(g) + (-1)^{|g|}d(f)g$ for $f \in [\Sigma^sX', X]$, $g \in [\Sigma^tX'', X']$, where $X$, $X'$, $X''$ are $M$-module spectra.

(2) Let $W'$, $W$ be arbitrary spectra and $h \in [\Sigma^tW', W]$. Then

$$d(h \wedge f) = (-1)^{|h|}h \wedge d(f)$$

for $f \in [\Sigma^sX', X]$.

(3) $d^2 = 0 : [\Sigma^sX', X] \to [\Sigma^{s+2}X', X]$ for associative spectra $X'$, $X$. □

From [10, (3.4)], $K$ is an $M$-module spectrum, i.e., there are $M$-module actions $m_K : K \wedge M \to K$, $\bar{m}_K : \Sigma K \to K \wedge M$ satisfying

$$m_K(1_K \wedge i) = 1_K, (1_K \wedge j)\bar{m}_K = 1_K, m_K\bar{m}_K = 0, (1_K \wedge i)m_K + (1_K \wedge j)\bar{m}_K = 1_{K \wedge M}.$$ 

Moreover, from [10, (2.6)] and [10, (3.7)] we have that $d(ij) = -1_M$, $d(\alpha) = 0$, $d(i') = 0$, $d(j') = 0$ and $d(\beta) = 0$.

**Remark 3.1** In this paper, all the notations are the same as those of [7].

Let $L$ be the cofiber of $\alpha_1 = jai : \Sigma^qS \to S$ and $K'$ be the cofiber of $\alpha : \Sigma^qS \to M$ given by the following two cofibrations:

$$\Sigma^qS \xrightarrow{\alpha_1} S \xrightarrow{i''} L \xrightarrow{j''} \Sigma^qS(see[7, (2.3)]),$$

$$\Sigma^qS \xrightarrow{\alpha} M \xrightarrow{i} K' \xrightarrow{j} \Sigma^qS(see[7, (2.4)]).$$

Let $\alpha' = \alpha_1 \wedge 1_K$. Consider the following two commutative diagram of $3 \times 3$ in the stable homotopy category:

\[
\begin{array}{ccc}
\Sigma M & \xrightarrow{\nu} & \Sigma K' \\
\setminus (v \wedge 1_M)\bar{m}_M & \xrightarrow{1_{K'} \wedge j} & \setminus y \\
\setminus 1_{K'} \wedge i & \xrightarrow{x} & \setminus \pi & \xrightarrow{jj'} & \setminus \alpha \\
K' & \xrightarrow{\pi} & K & \xrightarrow{j\alpha'} & \Sigma^2 M
\end{array}
\]

and

\[
\begin{array}{ccc}
M & \xrightarrow{(i'' \wedge 1_K)i'} & L \wedge K \\
\setminus j' & \xrightarrow{i'' \wedge 1_K} & \setminus \bar{r} \\
\setminus \alpha' & \xrightarrow{j'} & \setminus (v \wedge 1_M)\bar{m}_M \\
\Sigma^q-1 K & \xrightarrow{(j\alpha')} & \Sigma^q+1 M & \xrightarrow{\alpha} & \Sigma M
\end{array}
\]
By the above two commutative diagram of $3 \times 3$ in the stable homotopy category, we easily have the following two lemmas.

**Lemma 3.2** There exist three cofibrations

$$
K' \xrightarrow{x} K \xrightarrow{\bar{t}'} \Sigma^q S \xrightarrow{\pi} \Sigma K',
$$

(3.5)

$$
\Sigma^{-1} K \xrightarrow{j''} \Sigma M \xrightarrow{\imath} K' \wedge M \xrightarrow{\pi} K,
$$

(3.6)

$$
M \xrightarrow{(i'' \wedge 1)'} L \wedge K \xrightarrow{\bar{r}'} \Sigma^q K' \wedge M \xrightarrow{\pi} \Sigma M.
$$

(3.7)

**Lemma 3.3** $\varepsilon(v \wedge 1_M)\bar{m}_M = \alpha$, $\bar{t}(i'' \wedge 1) = (v \wedge 1_M)\bar{m}_M j'$, $\pi\bar{t} = j'' \wedge 1_K$, $\varepsilon(1_K' \wedge i)\bar{v}' = -2j'\alpha$.

From [11, p.434], there are $\bar{\Delta} \in [\Sigma^{-1} L \wedge K, K]$ and $\bar{\Delta} \in [\Sigma^{-1} K, L \wedge K]$ satisfying $\bar{\Delta}(i'' \wedge 1_K) = (j'' \wedge 1_K)\bar{\Delta} = i'' j' \in \Sigma^{-q-1} K, K]$ and $j j' \Delta = 0$. From [6, p.484], there is $\bar{\Delta}_{K'} \in [\Sigma^{-q-1} L \wedge K, K']$ such that $\bar{\Delta}_{K'}(i'' \wedge 1_K) = \bar{v} j'' \in [\Sigma^{-q-1} K, K']$ and $\bar{\Delta}(i'' \wedge 1_K) = (j'' \wedge 1_K)\bar{\Delta} = i'' j'$.

**Lemma 3.4** $\bar{\Delta}_{K'} = (1_K' \wedge j)\bar{t}$.

**Proof.** From Lemma 3.3 we have

$$(1_{K'} \wedge j)\bar{t}(i'' \wedge 1_K) = (1_{K'} \wedge j)(v \wedge 1_M)\bar{m}_M j' = (v \wedge 1_M)(1_M \wedge j)\bar{m}_M j' = v j'' = \bar{\Delta}_{K'}(i'' \wedge 1_K),$$

which shows that $(1_{K'} \wedge j)\bar{t} = \bar{\Delta}_{K'} + \bar{g}(j'' \wedge 1_K)$ for some $\bar{g} \in [K, \Sigma K']$.

Consider the exact sequence induced by (3.4)

$$
[K, \Sigma^q S] \xrightarrow{(ai)} [K, \Sigma M] \xrightarrow{\eta} [K, \Sigma K'] \xrightarrow{\eta} [K, \Sigma^q S] \xrightarrow{(ai)} [K, \Sigma^2 M].
$$

From the proof of [7, Proposition 2.18], we know that $[K, \Sigma M] = 0$. So $\imath M_v = 0$. Since $[K, \Sigma^q S] \cong \mathbb{Z}_p\{jj'\} = (ai)_*(jj') = \alpha ij j' \neq 0$. Thus we have $\imath M_v = 0$ and $[K, \Sigma K'] = 0$. Then we have $(1_{K'} \wedge j)\bar{t} = \bar{\Delta}_{K'}$.

**Lemma 3.5**[7, lemma 3.3 and (3.4)] Let $p \geq 5, n \geq 3$, then there exists an element $\eta'_{n, 2} \in [\Sigma^{p + q + q} K, E_2 \wedge K]$ such that

$$(\bar{t}_2 \wedge 1_K)\eta'_{n, 2} = h_0 n_1 \wedge 1_K \in [\Sigma^{p + q + q} K, KG_2 \wedge K], (1_{E_2} \wedge \alpha')\eta'_{n, 2} = 0,$$

where $h_0 n_1 \in \pi^{p + q + q} KG_2 \cong \text{Ext}^2_{\Lambda}(\mathbb{Z}_p, \mathbb{Z}_p)$ and $\alpha' = j \alpha \wedge 1_K \in [\Sigma^{q-1} K, K]$. There also exists an element $f_2 \in [\Sigma^{p + (p+2)(q+3)} M, E_5 \wedge L \wedge K]$ such that

$$(1_{E_2} \wedge (i'' \wedge 1_K) j)\eta'_{n, 2} = (a_2 a_3 a_4 \wedge 1_L) f_2.$$

□

**Corollary 3.1** For $f_2 \in [\Sigma^{p + (p+2)(q+3)} M, E_5 \wedge L \wedge K]$ which is given in Lemma 3.5, we have

$$
(1_{E_4} \wedge \varepsilon(1_{K'} \wedge j)) (a_4 \wedge 1_{K' \wedge M}) (1_{E_5} \wedge r) d(f_2 ij) = 0.
$$

(3.8)
Proof. From [7], we have [7, (3.6)] that \((\bar{a}_4 \wedge 1_M)(1_{E_5} \wedge \varepsilon(1_K \wedge i)\bar{\Delta}_{K'})d(f_2ij) = 0\). Here \(f_2\) is given in [7, (3.4)].

By [10, (1.7)], we have that \((\bar{a}_4 \wedge 1_M)(1_{E_5} \wedge \varepsilon(1_K \wedge i))(1_{E_5} \wedge \bar{\Delta}_{K'})d(f_2ij) = 0\). By lemma 3.4, we have

\[
(\bar{a}_4 \wedge 1_M)(1_{E_5} \wedge \varepsilon(1_K \wedge i))(1_{E_5} \wedge (1_K \wedge j)\bar{r})d(f_2ij) = 0.
\]

By [10, (1.7)], it follows that

\[
(\bar{a}_4 \wedge 1_M)(1_{E_5} \wedge \varepsilon(1_K \wedge i))(1_{E_5} \wedge (1_K \wedge j))(1_{E_5} \wedge \bar{r})d(f_2ij) = 0.
\]

Thus

\[
(\bar{a}_4 \wedge 1_M)(1_{E_5} \wedge \varepsilon(1_K \wedge ij))(1_{E_5} \wedge \bar{r})d(f_2ij) = 0.
\]

By [10, (1.7)], the corollary follows. \(\square\)

Let \(W\) be the cofibre of \(\varepsilon(1_K \wedge ij) : \Sigma^{q-2}K' \wedge M \longrightarrow M\) given by the cofibration

\[
\Sigma^{q-2}K' \wedge M \xrightarrow{\varepsilon(1_K \wedge ij)} M \xrightarrow{w_4} W \xrightarrow{u_4} \Sigma^{q-1}K' \wedge M. \tag{3.9}
\]

Lemma 3.6 There exists \(f' \in [\Sigma^{p+q+p+q+1}M, E_4 \wedge W]\) such that \((\bar{a}_2\bar{a}_3 \wedge 1_K)(1_{E_4} \wedge \pi u_4)f' = 0\).

Proof. By (3.8) and (3.9), we have that

\[
(\bar{a}_4 \wedge 1_{K' \wedge M})(1_{E_5} \wedge \bar{r})d(f_2ij) = (1_{E_4} \wedge u_4)f'
\]

with \(f' \in [\Sigma^{p+q+p+q+1}M, E_4 \wedge W]\) and by composing \((\bar{a}_2\bar{a}_3 \wedge 1_K)(1_{E_4} \wedge \pi)\) on (3.10) we have

\[
(\bar{a}_2\bar{a}_3 \wedge 1_K)(1_{E_4} \wedge \pi u_4)f' = (\bar{a}_2\bar{a}_3\bar{a}_4 \wedge 1_K)(1_{E_5} \wedge \pi \bar{r})d(f_2ij). \tag{3.11}
\]

By composing \(ij\) on [7, (3.4)], we have

\[
(1_{E_2} \wedge (i'' \wedge 1_K)\beta)\eta'_{n,2}i'ij = (\bar{a}_2\bar{a}_3\bar{a}_4 \wedge 1_{L \wedge K})f_2ij \tag{3.12}
\]

with \(\eta'_{n,2} \in [\Sigma^{p+q+q}K, E_2 \wedge K]\).

Notice that \(d(1_K) = 0\) and \(d(\beta) = 0\). Then by applying the derivation \(d\) on (3.12) we have

\[
(1_{E_2} \wedge (i'' \wedge 1_K)\beta)d(\eta'_{n,2}i'ij) = (\bar{a}_2\bar{a}_3\bar{a}_4 \wedge 1_{L \wedge K})d(f_2ij). \tag{3.13}
\]

Notice that \(\pi \bar{r} = j'' \wedge 1_K\). By composing \((1_{E_2} \wedge \pi \bar{r})\) on (3.13) we have

\[
(\bar{a}_2\bar{a}_3\bar{a}_4 \wedge 1_K)(1_{E_5} \wedge \pi \bar{r})d(f_2ij) = 0 \tag{3.14}
\]

and by (3.11), (3.14) we get

\[
(\bar{a}_2\bar{a}_3 \wedge 1_K)(1_{E_4} \wedge \pi u_4)f' = 0. \tag{3.15}
\]
Thus the Lemma is proved. \hfill \square

Let $U$ be the cofibre of $\pi u_4 : W \to \Sigma^{q-1}K$ given by the cofibration

$$W \xrightarrow{\pi u_4} \Sigma^{q-1}K \xrightarrow{w_5} U \xrightarrow{u_2} \Sigma W.$$ \hfill (3.16)

**Lemma 3.7** $w_5$ induces zero homomorphism in $\mathbb{Z}_p$-cohomology.

*Proof.* Consider the following homomorphism induced by $w_5$:

$$w^*_5 : H^*U \to H^{*+q-1}K.$$

From the cellular structures of $U$ and $K$, we can have that

$$H^t K = \begin{cases} \mathbb{Z}_p, & t = 0, 1, q+1, q+2; \\ 0, & \text{otherwise}, \end{cases}$$

and the top cell of $U$ has degree $2q+1$. It easily follows that $w^*_5$ must be a zero homomorphism in $\mathbb{Z}_p$-cohomology. \hfill \square

**Lemma 3.8** There exist three homotopy elements $f'_2 \in [\Sigma^{p\cdot q+(p+2)q}M, E_2 \land U]$, $f'_3 \in [\Sigma^{p\cdot q+(p+2)q+1}M, E_3 \land U]$ and $g_2 \in [\Sigma^{p\cdot q+(p+2)q}M, KG_2 \land W]$ such that

$$(\bar{a}_2 \bar{a}_3 \land 1_{W}) f' = (1_{E_2} \land u_5) f'_2, f'_2 = (\bar{a}_2 \land 1_U) f'_3,$$

and

$$(1_{E_3} \land u_4)(\bar{a}_3 \land 1_{W}) f' = -(1_{E_3} \land u_4 u_5) f'_3 + (1_{E_3} \land u_4)(\bar{e}_2 \land 1_{W}) g_2.$$

*Proof.* From (3.15) and (3.16), we have that

$$(\bar{a}_2 \bar{a}_3 \land 1_{W}) f' = (1_{E_2} \land u_5) f'_2$$ \hfill (3.17)

with $f'_2 \in [\Sigma^{p\cdot q+(p+2)q}M, E_2 \land U]$.

By (3.17) and (3.2) we have $(\bar{b}_2 \land 1_{W})(1_{E_2} \land u_5) f'_2 = (\bar{b}_2 \land 1_{W})(\bar{a}_2 \bar{a}_3 \land 1_{W}) f' = 0$. Thus

$$(1_{KG_2} \land u_5)(\bar{b}_2 \land 1_U) f'_2 = 0.$$ \hfill (3.18)

By (3.18), (3.16) and the fact that $w_5$ induces zero homomorphism in $\mathbb{Z}_p$-cohomology (see lemma 3.7), we have

$$(\bar{b}_2 \land 1_{U}) f'_2 = (1_{KG_2} \land w_5) g = 0$$ \hfill (3.19)

with $g \in [\Sigma^{p\cdot q+(p+1)q+1}M, KG_2 \land K]$, so by (3.2) we obtain

$$f'_2 = (\bar{a}_2 \land 1_U) f'_3$$ \hfill (3.20)

with $f'_3 \in [\Sigma^{p\cdot q+(p+2)q+1}M, E_3 \land U]$. By [10, (1.7)], from (3.20) and (3.17) we have

$$(\bar{a}_2 \bar{a}_3 \land 1_{W}) f' = -(\bar{a}_2 \land 1_{W})(1_{E_3} \land u_5) f'_3.$$ Then we have

$$(\bar{a}_3 \land 1_{W}) f' = - (1_{E_3} \land u_5) f'_3 + (\bar{e}_2 \land 1_{W}) g_2$$ \hfill (3.21)
with \( g_2 \in [\Sigma^{p^2p+(p+2)p}M, KG_2 \wedge W] \). By composing \((1_{E_3} \wedge u_4)\) on (3.21), we have
\[
(1_{E_3} \wedge u_4)(\bar{a}_3 \wedge 1_{W})f' = -(1_{E_3} \wedge u_4u_5)f'_3 + (1_{E_4} \wedge u_4)(\bar{e}_2 \wedge 1_{W})g_2.
\]
(3.22)

We finish the proof of the lemma. \( \square \)

**Lemma 3.9** the cofibre of \( \varepsilon(1_{K'} \wedge i) : \Sigma^qM \to \Sigma M \) is \( U \) given by the cofibration
\[
\Sigma^qM \xrightarrow{\varepsilon(1_{K'} \wedge i)} \Sigma M \xrightarrow{w_6} U \xrightarrow{u_6} \Sigma^{q+1}M.
\]
(3.23)
There exist two relations that
\[
u_4u_5 = (v \wedge 1_M)\bar{m}_M u_6
\]
and
\[
\varepsilon(1_{K'} \wedge i)(v \wedge 1_M)\bar{m}_M = \varepsilon(1_{K'} \wedge i)v.
\]

**Proof.** By the three cofibrations (3.6), (3.9), and (3.16), we can get the following commutative diagram (3.24) of \( 3 \times 3 \) lemma in stable homotopy category(cf. [12, p. 292-293]).

By the commutative diagram (3.24), Lemma 3.8 follows. \( \square \)

**Lemma 3.10** With notation as above. We have
\[
(\bar{a}_3\bar{a}_4 \wedge 1_{K' \wedge M})(1_{E_5} \wedge i)f(2ij)
= (1_{E_3} \wedge (v \wedge 1_M)\bar{m}_M u_6)f'_3 - (\bar{e}_2 \wedge 1_{K' \wedge M})(1_{KG_2} \wedge u_4)g_2.
\]

**Proof.** By (3.22), [10, (1.7)] and the relation \( u_4u_5 = (v \wedge 1_M)\bar{m}_M u_6 \) (see Lemma 3.9), we have
\[
(\bar{a}_3 \wedge 1_{K' \wedge M})(1_{E_4} \wedge u_4)f' = (1_{E_3} \wedge (v \wedge 1_M)\bar{m}_M u_6)f'_3 - (\bar{e}_2 \wedge 1_{K' \wedge M})(1_{KG_2} \wedge u_4)g_2.
\]
(3.25)

By composing \( (\bar{a}_3 \wedge 1_{K' \wedge M}) \) on (3.10), we have
\[
(\bar{a}_3 \wedge 1_{K' \wedge M})(1_{E_4} \wedge u_4)f' = (\bar{a}_3\bar{a}_4 \wedge 1_{K' \wedge M})(1_{E_5} \wedge i)d(f_2i).
\]
(3.26)

Combining (3.25) and (3.26) yields
\[
(\bar{a}_3\bar{a}_4 \wedge 1_{K' \wedge M})(1_{E_5} \wedge i)d(f_2i)
= (1_{E_3} \wedge (v \wedge 1_M)\bar{m}_M u_6)f'_3 - (\bar{e}_2 \wedge 1_{K' \wedge M})(1_{KG_2} \wedge u_4)g_2.
\]
(3.27)

Thus we complete the proof of this lemma. \( \square \)
Lemma 3.11 There exist two elements $f'_4 \in [\Sigma^p q^{e_2}q^{e_1} M, E_3 \wedge K]$ and $f'_5 \in [\Sigma^p q^{e_2}q^{e_1} M, E_3 \wedge K' \wedge M]$ such that

$$(1_E \wedge u)_f f'_3 = (1_E \wedge p')f'_4$$

and

$$f'_4 = (1_E \wedge \pi)f'_5.$$

Proof. By Lemma 3.3, we have $\alpha = \varepsilon(v \wedge 1_M)\bar{m}_M$. Then

$$(1_E \wedge \alpha u_6)f'_3 = (1_E \wedge \varepsilon(v \wedge 1_M)\bar{m}_M u_6)f'_3,$$

since $u_4u_5 = (v \wedge 1_M)\bar{m}_M u_6$

$$(1_E \wedge \varepsilon u_4 u_5)f'_3$$

$$(1_E \wedge \varepsilon)(1_E \wedge u_4 u_5)f'_3, \text{ by (3.22)}$$

$$(1_E \wedge \varepsilon)(1_E \wedge u_4)(\bar{c}_2 \wedge 1_W)g_2 - (1_E \wedge u_4)(\bar{a}_3 \wedge 1_W)f'$$

$$(1_E \wedge \varepsilon u_4)(\bar{c}_2 \wedge 1_W)g_2 - (1_E \wedge \varepsilon u_4)(\bar{a}_3 \wedge 1_W)f'$$

$$(\bar{c}_2 \wedge 1_M)(1_k G_2 \wedge \varepsilon u_4)g_2 - (1_E \wedge \varepsilon u_4)(\bar{a}_3 \wedge 1_W)f'$$

$$= - (1_E \wedge \varepsilon u_4)(\bar{a}_3 \wedge 1_W) f'$$

$$(1_E \wedge \varepsilon)(\bar{a}_3 \wedge 1_{K' \wedge M})(1_E \wedge u_4)f'$$

$$(1_E \wedge \varepsilon)(\bar{a}_3 \wedge 1_{K' \wedge M})(\bar{a}_4 \wedge 1_{K' \wedge M})(1_E \wedge \bar{r})d(f_2 i j)$$

$$(\bar{a}_3 \bar{a}_4 \wedge 1_M)(1_E \wedge \varepsilon \bar{r})d(f_2 i j), \text{ by (3.7)}$$

$$= 0.$$

Hence, by (1.2) we have

$$(1_E \wedge u_6)f'_3 = (1_E \wedge \varepsilon f'_4)$$

(3.28)

with $f'_4 \in [\Sigma^p q^{e_2}q^{e_1} M, E_3 \wedge K]$. Similarly, by Lemma 3.3 we have $\varepsilon(1_k' \wedge i)ij = -2j'\alpha'$. Then we have

$$-2(1_E \wedge \alpha' f'_4)$$

$$(1_E \wedge \varepsilon(1_k' \wedge i)ij f'_4)$$

$$(1_E \wedge \varepsilon(1_k' \wedge i)ij)(1_E \wedge \varepsilon(1_k' \wedge i)ij)$$

$$(1_E \wedge \varepsilon(1_k' \wedge i)ij)(1_E \wedge \varepsilon(1_k' \wedge i)ij)$$

$$(1_E \wedge \varepsilon(1_k' \wedge i)ij)(1_E \wedge \varepsilon(1_k' \wedge i)ij)$$

$$= 0.$$ 

Thus, by (3.6) we have

$$f'_4 = (1_E \wedge \pi)f'_5$$

(3.29)

with $f'_5 \in [\Sigma^p q^{e_2}q^{e_1} M, E_3 \wedge K' \wedge M]$. This completes the proof of Lemma 3.11.

Lemma 3.12 For the above $f'_5 \in [\Sigma^p q^{e_2}q^{e_1} M, E_3 \wedge K' \wedge M]$, we have

$$(\bar{b}_3 \wedge 1_{K' \wedge M})f'_5 = 0.$$
Proof. The proof will be given later. \[\square\]

Now we give the proof of Theorem I.

**Proof of Theorem I.** From Lemma 3.12, we have

\[(b_3 \land 1_{K' \land M})f'_5 = 0. \tag{3.30}\]

By virtue of (3.2), we have

\[f'_5 = (a_3 \land 1_{K' \land M})f'_6 \tag{3.31}\]

with \(f'_6 \in [\Sigma^{p+q+3} M, E_5 \land K' \land M]\). By (3.27) and (3.2), we have

\[
\begin{align*}
& (a_2 a_3 a_4 \land 1_{K' \land M})(1_{E_5} \land \bar{r})d(f_2ij) \\
& = (\bar{a}_2 \land 1_{K' \land M})(1_{E_5} \land (u \land 1_M)\bar{m}_M u_6) f'_3 \\
& = (\bar{a}_2 \land 1_{K' \land M})(1_{E_5} \land (u \land 1_M)\bar{m}_M)(1_{E_5} \land u_6)f'_3, \quad \text{by (3.28)} \\
& = (\bar{a}_2 \land 1_{K' \land M})(1_{E_5} \land (u \land 1_M)\bar{m}_M)(1_{E_5} \land j')f'_4, \quad \text{by (3.29)} \\
& = (\bar{a}_2 \land 1_{K' \land M})(1_{E_5} \land (u \land 1_M)\bar{m}_M)(1_{E_5} \land j')(1_{E_5} \land \pi)(\bar{a}_3 \land 1_{K' \land M})f'_6 \\
& = (\bar{a}_2 a_3 \land 1_{K' \land M})(1_{E_4} \land (u \land 1_M)\bar{m}_Mj')(1_{E_4} \land \pi)f'_6.
\end{align*}
\]

That is,

\[(a_2 a_3 a_4 \land 1_{K' \land M})(1_{E_5} \land \bar{r})d(f_2ij) = (\bar{a}_2 a_3 \land 1_{K' \land M})(1_{E_4} \land (u \land 1_M)\bar{m}_Mj')(1_{E_4} \land \pi)f'_6. \tag{3.32}\]

Since \([(b_4 \land 1_K)(1_{E_4} \land \pi)f'_6] \in Ext_A^{p+q+3}(H^* K, H^* M) = 0 \quad \text{(cf. [7, Proposition 2.2])}, \text{ then by (3.1), we know that the } d_1\text{-cycle } (b_4 \land 1_K)(1_{E_4} \land \pi)f'_6 \text{ is a } d_1\text{-boundary}. \text{ It follows that } (b_4 \land 1_K)(1_{E_4} \land \pi)f'_6 = (\bar{b}_4 \land 1_K)(c_3 \land 1_K)f'_7 \text{ for some } f'_7 \in [\Sigma^{p+q+2} M, KG_3 \land K]. \text{ Thus we have}

\[(1_{E_4} \land \pi)f'_6 = (\bar{c}_3 \land 1_K)f'_7 + (\bar{a}_4 \land 1_K)f'_8 \tag{3.33}\]

with \(f'_8 \in [\Sigma^{p+q+3} M, E_5 \land K]\). Then by (3.32), (3.33) and (3.2), we have

\[(\bar{a}_2 a_3 a_4 \land 1_{K' \land M})(1_{E_5} \land \bar{r})d(f_2ij) = (\bar{a}_2 a_3 \land 1_{K' \land M})(1_{E_5} \land (u \land 1_M)\bar{m}_Mj')f'_6. \tag{3.34}\]

Moreover, by composing \((1_{E_2} \land \bar{r})\) on (3.13) it is easy to get that

\[(\bar{a}_2 a_3 a_4 \land 1_{K' \land M})(1_{E_5} \land \bar{r})d(f_2ij) = (1_{E_2} \land \bar{r}(i'' \land 1_K)\beta)d(\eta'_{n,2}ij). \tag{3.35}\]

Combining (3.34) and (3.35) yields

\[(1_{E_2} \land \bar{r}(i'' \land 1_K)\beta)d(\eta'_{n,2}ij) = (\bar{a}_2 a_3 \land 1_{K' \land M})(1_{E_5} \land (u \land 1_M)\bar{m}_Mj')f'_8. \tag{3.36}\]

Notice that \(\bar{r}(i'' \land 1_K) = (u \land 1_M)\bar{m}_Mj' \) (see Lemma 3.3). Then (3.36) can turn into

\[(1_{E_2} \land (u \land 1_M)\bar{m}_Mj')d(\eta'_{n,2}ij) = (1_{E_2} \land (u \land 1_M)\bar{m}_Mj')(\bar{a}_2 a_3 a_4 \land 1_K)f'_8. \tag{3.37}\]
By (3.37) and (3.6), we have
\[(1_{E_2} \wedge j^i \beta)d(\eta_{n,2}^i ij) = (1_{E_2} \wedge j^i) (\tilde{a}_2 \tilde{a}_3 \tilde{a}_4 \wedge 1_{K}) f'_8 + (1_{E_2} \wedge j^i \alpha') f'_9 \tag{3.38}\]
with \(f'_9 \in \{\Sigma^{p^r \oplus (p+1)q+1} M, E_2 \wedge K\}. \) From [7, p.489], we know that the left hand side of (3.38) has filtration 4. However, since the first term of the right hand side of (3.38) has filtration \(\geq 5\), the second term of (3.38) must be of filtration 4. So \(f'_9\) has filtration \(\leq 3\). Notice the facts that \(E_{xt}^3 p^q \oplus (p+1)q+2(H^*K, H^*M) = 0 \) (cf. Proposition 2.5) and \(E_{xt}^{2p^q \oplus (p+1)q+1}(H^*K, H^*M) \cong \mathbb{Z}_p\{\beta_i \beta'(\tilde{h}_n)\} \) (cf. Proposition 2.7), then we have \((\tilde{b}_2 \wedge 1_{K}) f'_9 = (1_{K^G \wedge \beta}(1_{K^G \wedge \beta'}) (h_n). \) Let \(\xi_n = (\bar{a}_0 \bar{a}_1 \wedge 1_{K}) f'_9. \) Then \(\xi_n\) is represented by \(\beta \beta'(\tilde{h}_n)\) in the Adams spectral sequence. And so \(\xi_n = \xi_n\) is represented by \(i^\beta \beta' \beta'(\tilde{h}_n) = \beta_i \beta'(\tilde{h}_n) = \beta_i \beta'(h_n) \neq 0 \in E_{xt}^{3p^q \oplus (p+1)q+1}(H^*K, \mathbb{Z}_p)\) (cf. Proposition 2.6). Thus Theorem I is proved.

**Proof of Lemma 3.12.** We first recall three cofibrations given in [7]
\[
\begin{align*}
\Sigma^{-1} K & \xrightarrow{v'} \Sigma^q K' \xrightarrow{\varphi} K_2 \xrightarrow{\rho} K & \text{(see [7, (2.5)])}, \tag{3.39} \\
\Sigma^{q-1}K_1 & \xrightarrow{e(1_{K'}(\cdot))} M \xrightarrow{u_2} X \xrightarrow{u_2} \Sigma^q K' & \text{(see [7, (3.7)])}, \tag{3.40} \\
X & \xrightarrow{\varphi u_2} K_2 \xrightarrow{u_3} K' \wedge W \xrightarrow{u_3} \Sigma X & \text{(see [7, (3.10)])} \tag{3.41}
\end{align*}
\]
with the relation \(u_2 u_3 = -v' \varphi[7, (3.11)]. \) By composing \((\bar{a}_2 \wedge 1_{K'} \wedge \bar{M})\) on (3.27), we have
\[
(\bar{a}_2 \bar{a}_3 \bar{a}_4 \wedge 1_{K'} \wedge \bar{M})(1_{E_3} \wedge \tilde{r})d(f_2 i j) = (\bar{a}_2 \wedge 1_{K'} \wedge \bar{M})(1_{E_3} \wedge (v \wedge 1_{E_3} \bar{M}))f'_5,
\]
by (3.28)
\[
(\bar{a}_2 \wedge 1_{K'} \wedge \bar{M})(1_{E_3} \wedge (v \wedge 1_{E_3} \bar{M}))f'_5,
\]
by (3.29)
\[
(\bar{a}_2 \wedge 1_{K'} \wedge \bar{M})(1_{E_3} \wedge (v \wedge 1_{E_3} \bar{M}))f'_5.
\]

That is,
\[
(\bar{a}_2 \bar{a}_3 \bar{a}_4 \wedge 1_{K'} \wedge \bar{M})(1_{E_3} \wedge \tilde{r})d(f_2 i j) = (\bar{a}_2 \wedge 1_{K'} \wedge \bar{M})(1_{E_3} \wedge (v \wedge 1_{E_3} \bar{M}))f'_5.
\]

By composing \((1_{E_2} \wedge (1_{K'} \wedge j))\) on (3.42), we have
\[
(1_{E_2} \wedge (1_{K'} \wedge j))((\bar{a}_2 \bar{a}_3 \bar{a}_4 \wedge 1_{K'} \wedge \bar{M})(1_{E_3} \wedge \tilde{r})d(f_2 i j) = (1_{E_2} \wedge (1_{K'} \wedge j))((\bar{a}_2 \wedge 1_{K'} \wedge \bar{M})(1_{E_3} \wedge (v \wedge 1_{E_3} \bar{M}))f'_5.
\]

On the one hand, for the left hand side of (3.43), we have
\[
(1_{E_2} \wedge (1_{K'} \wedge j))((\bar{a}_2 \bar{a}_3 \bar{a}_4 \wedge 1_{K'} \wedge \bar{M})(1_{E_3} \wedge \tilde{r})d(f_2 i j) = -(\bar{a}_2 \bar{a}_3 \bar{a}_4 \wedge 1_{K'} \wedge \bar{M})(1_{E_3} \wedge (v \wedge 1_{E_3} \bar{M}))f'_5.
\]

On the other hand, for the right hand side of (3.43) we have
\[
(1_{E_2} \wedge (1_{K'} \wedge j))((\bar{a}_2 \wedge 1_{K'} \wedge \bar{M})(1_{E_3} \wedge (v \wedge 1_{E_3} \bar{M}))f'_5.
\]

By Lemma 3.4
\[
-(\bar{a}_2 \wedge 1_{K'} \wedge \bar{M})(1_{E_3} \wedge (v \wedge 1_{E_3} \bar{M}))f'_5.
\]

By Lemma 3.4
\[
-(\bar{a}_2 \wedge 1_{K'} \wedge \bar{M})(1_{E_3} \wedge (v \wedge 1_{E_3} \bar{M}))f'_5, \] since \((1_{M} \wedge j) \bar{M} = 1_{M}\)
\[
-(\bar{a}_2 \wedge 1_{K'} \wedge \bar{M})(1_{E_3} \wedge (v \wedge 1_{E_3} \bar{M}))f'_5, \] since \(u_2 u_3 = -v' \varphi\)
\[
(\bar{a}_2 \wedge 1_{K'} \wedge \bar{M})(1_{E_3} \wedge \bar{M})f'_5.
\]
Thus we have

\[(\bar{a}_2\bar{a}_3\bar{a}_4 \land 1_{K'}) (1_{E_5} \land \bar{\Delta}_{K'}) d(f_2 ij) = -(\bar{a}_2 \land 1_{K'}) (1_{E_3} \land u_2 u_3) f'_5. \tag{3.44}\]

Let \( X \) be the cofibre of \( \varepsilon(1_{K'} \land i) : \Sigma^{q-1} K' \longrightarrow M \) given by the cofibration

\[\Sigma^{q-1} K' \varepsilon(1_{K'} \land i) M \xrightarrow{u_2} X \xrightarrow{u_2} \Sigma^q K' \quad \text{(see [7, (3.7)])}.\]

[7,(3.8)] follows from [7, (3.7)] and [7, (3.6)] that

\[(\bar{a}_4 \land 1_{K'}) (1_{E_5} \land \bar{\Delta}_{K'}) d(f_2 ij) = (1_{E_4} \land u_2) f_3\]

for some \( f_3 \in [\Sigma^{p+q+(p+2)q+1} M, E_4 \land X] \). By composing \((\bar{a}_2\bar{a}_3 \land 1_{K'})\) on [7, (3.8)], we have

\[(\bar{a}_2\bar{a}_3\bar{a}_4 \land 1_{K'}) (1_{E_5} \land \bar{\Delta}_{K'}) d(f_2 ij) = (\bar{a}_2\bar{a}_3 \land 1_{K'}) (1_{E_4} \land u_2) f_3. \tag{3.45}\]

Combining (3.44) and (3.45) yields

\[(\bar{a}_2 \land 1_{K'}) (1_{E_3} \land u_2 u_3) f'_5 = -(\bar{a}_2\bar{a}_3 \land 1_{K'}) (1_{E_4} \land u_2) f_3. \tag{3.46}\]

By [10, (1.7)], (3.46) can turn into

\[(1_{E_2} \land u_2)(\bar{a}_2 \land 1_{X})(1_{E_3} \land u_3) f'_5 = -(1_{E_2} \land u_2)(\bar{a}_2\bar{a}_3 \land 1_{X}) f_3. \tag{3.47}\]

From (3.47) and (3.40) we have

\[(\bar{a}_2 \land 1_{X})(1_{E_3} \land u_3) f'_5 = -(\bar{a}_2\bar{a}_3 \land 1_{X}) f_3 + (1_{E_2} \land w_2) \hat{f}_4 \tag{3.48}\]

with \( \hat{f}_4 \in [\Sigma^{p+q+(p+2)q-1} M, E_2 \land M] \).

Note that \((\bar{b}_2 \land 1_{M}) \hat{f}_4 \in [\Sigma^{p+q+(p+2)q-1} M, KG_2 \land M] = 0\) by the exact sequence

\[
\begin{array}{c}
\Sigma^{p+q+(p+2)q-1} M, KG_2 \xrightarrow{(1\land i)_*} \Sigma^{p+q+(p+2)q-1} M, KG_2 \land M \xrightarrow{(1\land j)_*} \Sigma^{p+q+(p+2)q-2} M, KG_2
\end{array}
\]

induced by (1.1), where the first and the last group are zero by the fact that \(\pi_{p+q+(p+2)q+r} KG_2 \cong Ext_A^{p+q+(p+2)q+r}(\mathbb{Z}_p, \mathbb{Z}_p) = 0\) for \(r = 0, -1, -2\) (cf. [2]). Hence, \( \hat{f}_4 = (\bar{a}_2 \land 1_{M}) \hat{f}_5 \) for some \( \hat{f}_5 \in [\Sigma^{p+q+(p+2)q} M, E_3 \land M] \). By (3.2) and (3.48), we have

\[(1_{E_3} \land u_3) f'_5 = -(\bar{a}_3 \land 1_{X}) f_3 + (1_{E_2} \land w_2) \hat{f}_5 + (\bar{c}_2 \land 1_{X}) g_6 \tag{3.49}\]

with \( g_6 \in [\Sigma^{p+q+(p+2)q} M, KG_2 \land X] \). And so we have

\[(\bar{b}_3 \land 1_{X})(1_{E_3} \land u_3) f'_5 = (\bar{b}_3 \land 1_{X})(1_{E_2} \land w_2) \hat{f}_5 + (\bar{b}_3 \bar{c}_2 \land 1_{X}) g_6. \tag{3.50}\]

Since \( Ext_A^{3(p+q+(p+2)q)}(H^* M, H^* M) \cong \mathbb{Z}_{p^i} \{i, j, \overline{h}_{n g_0}, j^* \overline{h}_{n g_0}\} \) (cf. Proposition 2.3), then we can have \((\bar{b}_3 \land 1_{M}) \hat{f}_5 = \lambda_1 \overline{h}_{n g_0} \bar{ij} + \lambda_2 (1_{KG_3} \land ij) \overline{h}_{n g_0}\) for some \(\lambda_1, \lambda_2 \in \mathbb{Z}_p\), where \(\overline{h}_{n g_0} \in [\Sigma^{p+q+(p+2)q+1} M, KG_3 \land M] \). And so

\[0 = \lambda_1 (\bar{c}_3 \land 1_{M}) \overline{h}_{n g_0} \bar{ij} + \lambda_2 (\bar{c}_3 \land 1_{M}) (1_{KG_3} \land ij) \overline{h}_{n g_0}.\]
By composing $i$ on the above equality, we get that $\lambda_2(\bar{c}_3 \wedge 1_M)(1_{KG_3} \wedge ij)\bar{h}_n g_0 i j = 0$. However, since $d_2(i^*(ij)\bar{h}_n g_0) = i^* d_2(i_{ij},\bar{h}_n g_0) \neq 0$ (see Proposition 2.4 (1)), we get that $(\bar{c}_3 \wedge 1_M)(1_{KG_3} \wedge ij)\bar{h}_n g_0 i j \neq 0$. Thus, we have that

$$\lambda_2 = 0, \lambda_1(\bar{c}_3 \wedge 1_M)\bar{h}_n g_0 i j = 0.$$ 

Note that $d_2(j^*i^*\bar{h}_n g_0) \neq 0$ by Proposition 2.4 (2), then $(\bar{c}_3 \wedge 1_M)\bar{h}_n g_0 i j \neq 0$. Thus we see that $\lambda_1 = 0$. From the above discussion, we know that $(\bar{b}_3 \wedge 1_M)\bar{f}_5 = 0$ and (3.50) can turn into

$$(\bar{b}_3 \wedge 1_X)(1_{E_3} \wedge u_3)\bar{f}_5 = (\bar{b}_3 \bar{c}_2 \wedge 1_X)g_0.$$ (3.51)

The argument of the proof from [7, (3.16)] to [7, p.491] shows that $\bar{c}$ in [7, (3.16)] implies (3.51) implies (3.30) holds.

**Proof of Theorem II** By Theorem I, we get that

$$\beta_s i_s^* i_s(h_n) \neq 0 \in Ext^2_{A}(\nu^q+(p+1)q+1,H^*K,Z_p)$$

is a permanent cycle in the Adams spectral sequence and converges to a nontrivial element $\zeta_n \in \pi_{\nu^q+(p+1)q-1}K$.

Consider the following composition of maps

$$\bar{f} : \Sigma^{\nu^q+(p+1)q-1}S \xrightarrow{\zeta_n} K \xrightarrow{j j' \beta} \Sigma^{-pq+2}S.$$ 

Since $\zeta_n$ is represented up to nonzero scalar by $\beta_s i_s^* i_s(h_n) \in Ext^2_{A}(\nu^q+(p+1)q+1,H^*K,Z_p)$ in the Adams spectral sequence, then the above $\bar{f}$ is represented up to nonzero scalar by $\bar{c} = (j j' \beta)_s i_s^* i_s(h_n)$ in the Adams spectral sequence.

Meanwhile, it is well known that the $\beta$-element $\beta_2 = j j' \beta^2 i$ is represented by $k_0 \in Ext^2_{A}(\nu^q+(p+1)q+1)(\Z_p,\Z_p)$ in the Adams spectral sequence. By the knowledge of Yoneda products we can see that $\bar{f}$ is represented (up to nonzero scalar) by

$$\bar{c} = k_0 h_n \neq 0 \in Ext^3_{A}(\nu^q+2p+1)(\Z_p,\Z_p)$$

in the Adams spectral sequence(cf. [6, Table 8.1]).

Moreover, we know that $Ext^3_{A}(\nu^q+(p+1)q+(r-1))(\Z_p,\Z_p) = 0$ for $r \geq 2$, then $k_0 h_n$ cannot be hit by any differential in the Adams spectral sequence, and so the corresponding homotopy element $\bar{f} \in \pi_s S$ is nontrivial and of order $p$. This finishes the proof of Theorem II.

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