A Foundation of Lazy Streaming Graphs

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Abstract A streaming graph system continuously processes a stream of operations over a large graph. In the big data processing ecosystem, this performance-critical data processing paradigm is emerging with increasing relevance. Lazy processing is a collection of important optimization techniques for streaming graphs, but designing correct, expressive, and efficient lazy streaming graphs is challenging. In this paper, we lay a foundation for lazy streaming graph processing. The resulting DG Calculus features fine-grained in-data lazy processing, endowed with expressive optimizations such as batching, fusion, and splicing. We establish the soundness of DG Calculus through bisimulation with a system for eager graph processing. To the best of our knowledge, DG Calculus is the first foundational calculus for streaming graphs.

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1 Introduction

Streaming graph systems \cite{5,12,13,27,29,32} are emerging as an important family of software systems in large-scale data processing. Such systems continuously process a stream of operations, the operation stream, consisting of both queries from the graph and updates to it. With their real-world deployments such as Twitter and Facebook, streaming graphs provide the bedrock for numerous applications running on cloud computing platforms and data centers.

Despite the popularity and relevance of streaming graph systems, they have so far received little attention in rigorous studies. Indeed, streaming graphs are often deployed in performance-critical scenarios, so a focus on experimental system design in earlier work is understandable. As we look forward however, streaming graph systems start to become a core part of software infrastructure, and a lack of rigorous studies may make them the most vulnerable link in building high-assurance systems. As it turns out, building correct, expressive, and efficient streaming graph systems is a challenging task where the three goals often compete with each other. In a nutshell, even though a naive streaming graph system implementation may indeed be as simple as elementary graph data structure manipulation, the demand on the efficiency of a realistic streaming graph system immediately calls for the expressiveness in their optimizations, so much so that optimization design is arguably the design of streaming graph systems itself; expressive optimizations however introduce non-trivial challenges to correctness.

Lazy Graph Processing The case in point — and the focus of this paper — is lazy processing. Building on the “haste makes waste” philosophy, lazy data processing may delay some operations applied to the graph. The most basic form of lazy processing, batching, is implemented in virtually all real-world streaming graphs. Here, multiple operations from the operation stream may be combined in a “batch,” so that only one graph traversal is needed for all batched operations.

Earlier systems perform batching only at the level of the top-level graph: when a batch of operations is ready to be processed, the traversal of the graph will be a monolithic step, \textit{i.e.}, the graph processing engine will traverse the entire graph until all operations in the batch are processed. This primitive form of batching is indeed superior in performance than the baseline of no batching at all, but its monolithic nature of traversal is undesirable due to prolonged latency. Refining the granularity of lazy processing is an important dimension of batching optimization design \cite{22,24}. For example, a recent streaming graph system, DeltaGraph \cite{11}, allows for delayed and potentially batched operations to be propagated through the graph via a sequence of local steps, each of which only involves moving the delayed operation(s) from a graph node to one of its neighbor nodes. Experiments show this flavor of fine-grained lazy data processing can significantly improve performance, and composes well with opportunistic computing \cite{8}.

Looking ahead, a significant challenge against wider adoption of finer-grained lazy data processing is correctness. For monolithic batching, correctness may be self-evident. As we shall see however, the correctness of fine-grained lazy data processing is more
subtle. Fundamentally, if a graph processing operation — be it a query or an update — is no longer performed in one monolithic step, will the same result be returned at the end? Indeed, the overall challenge is well-known in language design: just as with great power comes great responsibility, with great expressiveness comes great need for correctness.

**DG Calculus** In this paper, we introduce DG Calculus, the first formal system for streaming graphs. Building on a minimal core set of graph processing operations, DG Calculus defines the operational semantics of processing a continuously evolving graph in the presence of operation streams consisting of queries and updates. We first formalize the conventional *eager data processing* — the processing style where each operation is immediately processed through a graph traversal — in the DG\(^E\) Calculus. The centerpiece of our calculus is DG\(^L\) Calculus, a reduction system that captures the essence of fine-grained lazy data processing.

Minimalistic as DG Calculus is, the calculus is intentionally expressive for features that may intersect with lazy data processing. In addition to batching, we further support two important forms of optimizations enabled by lazy processing. Known as *fusion*, the first form of optimization allows multiple operations to be combined through algebraic laws into one during the lazy propagation. The second optimization further allows *dependent* operations to be propagated separately, with the dependency resolved through an in-graph substitution we call *splicing*.

A key design goal of DG Calculus is to establish the correctness of lazy data processing of streaming graphs. Recall that the propagation of delayed operation(s) consists of local steps. Taking the bird’s eye view of the entire graph, this naturally entails a design where multiple operations (or multiple batches of operations) may be undergoing propagation at different neighborhoods of the graph in a *decentralized* manner, and the selection of which operation (or batch) to take the next local step is *non-deterministic*. The local, decentralized, and non-deterministic nature of propagation — together with optimizations of batching, fusion and splicing — calls for the non-trivial question of whether such a graph processing design can produce the same results as in eager processing. DG Calculus affirmatively answers this question by establishing *observable equivalence* between DG\(^L\) Calculus and DG\(^E\) Calculus through bisimulation. Intuitively, it says processing with fine-grained lazy processing produces the same results as eager data processing, modulo permutation of the order of the results due to non-determinism.

**Contributions** Designing correct, expressive, and efficient streaming graph systems with querying, updating, and optimizations unified under one system is a challenging problem in data management. The foundational route we take is less explored in an exciting domain rich in experimental research. DG Calculus complements existing efficiency-oriented research by rigorously establishing correctness while promoting expressiveness. Overall, this paper makes the following contributions:

- It lays a foundation for streaming graphs via DG\(^E\) Calculus, a calculus to intuitively capture the essence of (eager) streaming graph processing.
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- It elucidates lazy processing of streaming graphs via \( \text{DG}^L \) Calculus, a calculus with expressive support for lazy propagation (decentralization, locality, and nondeterminism) and in-data optimization (batching, fusion, and splicing).
- It presents a bisimulation result to establish the soundness of lazy data processing through observable equivalence between the two modes of data processing.

2 Lazy Data Processing in Examples

In this section we highlight key features of a streaming graph that \( \text{DG} \) Calculus formally captures. To connect with the rest of the paper, the running example in Fig. 1a is written in the \( \text{DG} \) Calculus grammar extended with standard programming features.

Example 1 (A Streaming Server and Client). In Fig. 1a a simple server function maintains a streaming graph \( g \). The operation stream is represented by \( \text{ostream} \), whose post-processing results — forming a result stream — is represented by \( \text{rstream} \). Each operation \( \text{op} \), when applied to the graph \( g \), will compute the post-processing graph \( g' \) and a result \( r \). In this example, we choose to represent \( g \) mutably to highlight the persistence nature of data maintained by the server. This choice is orthogonal to \( \text{DG} \) Calculus, with a purely functional core. The (logical) graph after all operations are processed is shown in Fig. 1b.

DG Calculus-specific expressions are highlighted in red. Expression unit represents the smallest graph whose key and value are both a pre-defined \( \text{kr} \). In DG Calculus, operations are first-class citizens, with some examples we now introduce.

Example 2 (Data-Processing Operations). First, operation \( \text{add} v k \) adds a new node, together with a new edge from the node with key \( k \) to the newly added node. The newly added node has a freshly generated key — to be returned to the client — and its associated value is \( v \). Together we call them the key–value pair. Second, operation \( \text{adde} k' k \) adds a new edge from the node of key \( k \) to the node of key \( k' \). Third, operation \( \text{map} e \) applies the \( e \) function to every node in the streaming graph.

DG Calculus supports dependencies between operations. To express the dependencies — such as the result of one operation may be used as an argument of another — we further accompany each operation with a label \( \ell \).

Example 3 (Labeled and Dependent Operations). We write \( \ell_1 \rightarrow \text{add} v_1 \text{kr} \) to mean an operation whose generated key of the newly added node can be referenced by \( \ell_1 \), and thus another dependent operation \( \ell_3 \rightarrow \text{add} v_3 \ell_1 \) will add a new node to be the child of the node referenced by \( \ell_1 \).

At runtime we maintain the graph through a standard representation where the graph is represented through a sequence of nodes, each of which holds the keys of nodes it forms out-edges with. This mundane notation is in essence identical to reference graphs, a prevalent representation for representing data structures.
(* Server *)
let g = ref unit in
let server ostream =
case ostream of
  |⇒ ⇒
  |op :: ostream'⇒
    let (g'; r) = op g
    in g := g';
r :: server ostream'

(* Client *)
let v1, v2, v3, v4, v5 = ... in
let rstream = server [(ℓ1 7→ add v1 kr),
                        (ℓ3 7→ add v3 ℓ1),
                        (ℓ2 7→ add v2 ℓ1),
                        (ℓ4 7→ add ℓ3 ℓ2),
                        (ℓ5 7→ add v4 kr),
                        (ℓ6 7→ add v5 ℓ5),
                        (ℓ7 7→ add ℓ5 ℓ2)]

(a) pseudocode
(b) a (logical) graph illustration

Figure 1  A Streaming Graph Example

(a) (b) (c)

Legend: defines O store as

Figure 2  Lazy Graph Processing

For simplicity, the sub-graph consisting of nodes kr, v1, v2, and v3 — corresponding with the first four operations applied to server in Fig. 1 — will be used in later examples.

The client code in Fig. 1 applies an operation stream to server, for which we also informally say the operation stream is issued. With fine-grained lazy processing, an operation does not need to immediately realize, i.e., lead to a monolithic traversal of the graph until the node(s) for which the operation is intended for is reached. Instead, the operation may be memoized inside a graph node, in a structure we call an operation store, or O store for short.

Example 4 (Fine-Grained Lazy Processing). In Fig. 2 three example runtime representations of the streaming data structure are presented. No operations has been realized in Fig. 2(a); the first two operations have been realized in Fig. 2(b); all are realized in Fig. 2(c).

The O store is decentralized: it may be associated with any node. The delayed operations propagate through nodes in a local manner: an operation is only removed from the O store of a node and placed to the O store of an adjacent node in the structure of the graph. At any given moment, how many steps of propagation have happened in the graph and which adjacent nodes are involved in the propagation are both non-deterministic. As a result, all 3 runtime representations are possible in DG Calculus.
Figure 3  Optimizations in Lazy Processing

With fine-grained lazy processing, important graph processing optimizations can be aptly defined through refined support of in-data propagation. From now on, let us take the fully realized tree in Fig. 2(c) as the runtime representation of \( g \) managed by the server function.

Example 5 (Batching). As in Fig. 3(a), the two add operations may be propagated together. If the two add operations had to propagate separately, two graph traversals would be needed.

Example 6 (Fusion). When operations in the stream are related, it is possible to compose them — following predefined fusion rules — and only propagate the composed one. In Fig. 3(b), the two map operations are “fused” into one. Such fusion can happen in the \( O \) store of any node.

Example 7 (Splicing). In Fig. 3(c), the two operations are dependent. If the add operation had to pessimistically wait until the second add operation is completed, lazy processing would be de facto disabled. In DG\( ^L \) Calculus, the first add operation does not need to be realized before the second add operation is issued. Instead, the two operations may be propagated in-data at the same time, and their dependency is resolved through substitution when the post-realization result of the add operation encounters the first add operation during their propagations in-data.

3  Streaming Graphs

In this section, we introduce basic definitions for constructing a streaming graph runtime. We define the graph representation in § 3.1, graph processing operations in § 3.2 and a high-level semantic account over a streaming graph runtime in § 3.3 and § 3.4.

Throughout this paper we use some common notations. We use notation \( \bar{\tau} \) to refer to a sequence of \( \tau_1, \tau_2, \ldots, \tau_n \) for some \( n \geq 0 \). When \( n = 0 \), we also write it as \( \emptyset \) to refer to the empty sequence. Given sequence \( \bar{\tau} \), we call \( \tau_1 \) the head and \( \tau_n \) the end. We denote \( |\bar{\tau}| \) as \( n \). Given two sequences \( \bar{\nu} \) and \( \bar{\sigma} \), notation \( \bar{\nu}, \bar{\sigma} \) represents their concatenation. We further define \( \bar{\nu}, \emptyset \) and \( \emptyset, \bar{\nu} \) as \( \bar{\nu} \). When the sequence does not contain duplicates, and the order among its elements does not matter, we liberally
treat it as a set, and apply common operators such as \( \in \). We call a sequence in the form of \( \iota_1 \mapsto \iota'_1, \ldots, \iota_n \mapsto \iota'_n \) a mapping, defined only when \( \iota_1, \iota_2, \ldots, \iota_n \) are distinct. Given mapping \( \mu \), we denote \( \mu(\iota_i) \) for \( \iota'_i \) for any \( i = 1..n \), and \( \text{dom}(\mu) \) as \( \{\iota_1, \ldots, \iota_n\} \) and \( \text{ran}(\mu) \) as \( \{\iota'_1, \ldots, \iota'_n\} \). We define the zero arity tuple as \( \emptyset \). Symbols referred to in this paragraph — \( \iota, \varphi, \mu \) — may represent any meta-level element in the rest of the paper.

### 3.1 Runtime Graph Representation

As shown in Fig. 4, a graph in our calculus \( G \) consists of a sequence of graph nodes (g), each of which includes a node payload (n) containing a key (k), a value (v), and an out-edge set (k). This representation is mundane, except that we further associate each node with an O store and an R store. We have discussed the motivation behind O store in §2. The R store, or result store, is designed for recording the results from (post-processed) operations as they propagate back. The symmetric design of the O store and the R store captures the bi-directional flow of propagation: as operations propagate through the graph in its traversal, results propagate back in the opposite direction. Observe that both stores are defined per-node, illustrating the decentralized nature of our lazy processing design. For eager graph processing, both stores are \( \emptyset \) for all nodes. For convenience, we define helper function \( \epsilon(n) \) to expand \( n \) to a graph node with both stores set to \( \emptyset \):
As graph traversal plays a central role in graph processing, we introduce two special nodes for the first and the last node of traversal. A root node is defined with built-in key $kr$ and value of the same, and a final node is defined with built-in key $kf$ and value of the same. We define unit as $\varepsilon((kr;kr);\emptyset)$, $\varepsilon((kf;kf);\emptyset)$.

Values are minimally defined as keys; in other words, keys are not only indices for nodes, but may also be used as primitive data, e.g., as values of the key–value pair.

Example (Runtime Graph Representation). The graph in Fig. 2c is encoded in the syntax of DG Calculus as follows:

$$
\begin{align*}
\varepsilon((kr;kr);\{k_1\}), \\
\varepsilon((k_1;v_1;\{k_2,k_3\})), \\
\varepsilon((k_2;v_2;\{k_3\})), \\
\varepsilon((k_3;v_3;\emptyset)), \\
\varepsilon((kf;kf;\emptyset))
\end{align*}
$$

where $k_1, k_2,$ and $k_3$, are the generated keys corresponding to the values $v_1, v_2,$ and $v_3$.

3.2 Graph Operations

In this minimal calculus, we choose to support six common operations for graph processing. In addition to the three operations we explained in Example 2 — the add, adde, and map operations — we support (1) $\text{find } e$ to query the node with key $e$ from the graph; (2) $\text{nbO } e$ to query the set of keys that the node with key $e$ has edges to; (3) $\text{nbI } e \bar{k}$ to query the set of keys that have edges to the node with key $e$. The $\bar{k}$ argument serves as an accumulator in the formal system, which can be set as $\emptyset$ for end users.

For operations, we make the semantic distinction between singular operations ($\hat{\omega}$) and bulk operations ($\hat{\omega}$). During a graph traversal, a singular operation can be realized over a single node (or a small number of adjacent nodes on the traversal sequence). In contrast, the realization of a bulk operation may involve a large number of nodes, or even all nodes of the graph. As both kinds of operations are frequently supported in streaming graphs, they represent distinct dimensions of expressiveness that DG Calculus chooses to support.

Example 9 (Singular Operation). The $\text{find } e$ operation is only realized over the node that contains the key $e$.

Example 10 (Bulk Operation). The $\text{map}$ operation performs per-node processing for the entire graph. Another example is the $\text{nbI } e \bar{k}$ expression for in-edge processing. Here, the nodes which has an outgoing edge to node $e$ may scatter through the entire traversal sequence.

1 This requirement slightly differs from the informal illustration we used in Fig. 2 where the final node, only serving a formal purpose, is omitted.
Beyond singular vs. bulk operations, our choice of the operations aims at representing distinctive design dimensions of graph processing. First, the add and adde operations enable topology change, allowing us to study fine-grained laziness in the more challenging use scenario of dynamic graphs, whereas nbI, nbO, and find are queries in nature. Second, the pair of nbO and nbI operations are topology-dependent, whereas the others are not, known as vertex-centric operations.

### 3.3 Runtime Configuration

We represent a program runtime configuration $P$ as a triple, consisting of the graph $G$, the operation stream $O$ that forms a sequence of operations to process the graph, and the result stream $R$ that keeps the post-processing results. This triple captures the intuition that a streaming graph program runtime issues a stream of operations to the graph, processes each operation over it, and produces a stream of results.

The reader may note that the operation stream and the result stream defined here share the same structure as the $O$ store and $R$ store carried within graph nodes, respectively. This correspondence is not a coincidence. Indeed, one may think of each $O$ store or each $R$ store — within each graph node — forms a stream of its own. In figurative terms, each such stream is a per-node “streamlet”, and while taking the graph as a whole, all “streamlets” in combination form the flow of operations and results within the graph. The operation stream and the result stream in the program runtime configuration are the streams at the boundary between the graph and its processing client: the former can be viewed as the operations before they flow into any of the “streamlet” in the graph, whereas the latter can be viewed as the results after they flow out from the last “streamlet” in the graph.

Each operation in the operation stream is labeled, i.e., each operation $\omega$ is associated with a distinct label $\ell \in \text{LABEL}$. As we discussed in §2, labels provide a simple name binding mechanism for dependent operations. In our calculus, the binding between the label and its corresponding operation does not change. For convenience we introduce function $\text{ops}(\ell)$ to compute the operation $\ell$ labels. Similarly, each result in the result stream is also labeled, indicating the operation with said label has now produced a result. We also allow bulk operations themselves to be results, a feature we will elaborate in §4.1.

### 3.4 A High-Level Semantic Account for Streaming Graphs

The reduction relation $P \stackrel{m}{\longrightarrow} P'$ says that configuration $P$ one-step reduces to configuration $P'$ under processing mode parameter $m$. DG Calculus is a parameterized calculus; the processing mode parameter $m$ allows us to study both eager data processing ($m = E$) and lazy data processing ($m = L$) under one system. The relation is defined by the single rule below:

\[
G_1,O' \xrightarrow{m} G_2,R' \\
G_1,(O',O),R \xrightarrow{m} G_2,[(R'),(R',R)]
\]
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| Relation | Description                      | See   |
|----------|----------------------------------|-------|
| $m\rightarrow$ | Program Configuration Reduction | §3.3  |
| $E\rightarrow$ | Eager Graph Reduction           | §4    |
| $L\rightarrow$ | Lazy Graph Reduction             | §5    |
| $\Rightarrow$ | Hotspot Reduction                | §4.1  |
| $\rightarrow$ | Fusion Reduction                 | §5.1  |

Table 1 A Summary of Key Reduction Relations

and we use $m\rightarrow^*$ to represent the reflexive and transitive closure of $m\rightarrow$. The rule is primarily conditioned by a relation $G_1, O \rightarrow m^* G_2, R$ which intuitively says that the graph $G_1$ one-step reduces to $G_2$ given the operation stream $O$ while producing the result stream $R$. The concrete definitions of this relation under two processing modes are the core of $DG$ Calculus that we will elaborate in §4 and §5. The substitution $O[R]$ where $R = \ell \rightarrow \rho$ and $|R| = n$ is defined as identical to $O$ except every occurrence of $\ell_i$ is replaced with $\rho_i$ where $i = 1\ldots n$.

This reduction relation is interesting in two ways. First, it allows multiple operations in the operation stream to be processed at a time, which can be viewed as a primitive form of batching that happens at the boundary between the graph server and the client. Second, once a result labeled $\ell$ is returned from (lazy or eager) graph processing, its value is used to substitute label $\ell$ that may appear in the remaining operations.

Finally, we define function $\text{init}(O)$ to compute the initial configuration of program $O$. Specifically, $\text{init}(O) \triangleq (\text{unit}; O; \emptyset)$. We informally say that program $O$ reduces to the graph $G$ and result stream $R$ iff $\text{init}(O) \rightarrow m^* (G; \emptyset; R)$.

To help readers navigate through the paper, Table 1 presents a bird’s eye view of key reduction relations introduced in this paper.

4 $DG^E$ Calculus: Eager Processing in Streaming Graphs

In this section, we define the standard form of streaming graph processing, which we term as eager processing. This is achieved through defining the $m\rightarrow$ relation where $m = E$. Read independently, the $DG^E$ Calculus provides a minimal core on streaming graphs.

Rather than defining eager processing in a monolithic manner, we modularize our semantics in two parts: the $E\rightarrow$ relation is devoted to illustrate the eager aspect of propagation and processing; and the hotspot reduction relation is devoted to illustrating how an operation is realized. A practical benefit of this modularized definition is that lazy processing can share the same definition of the hotspot reduction.
AddN \( (k; v; \overline{k}) \), add \( v' \) \( k \) \( \Rightarrow (\langle k; v; \overline{k}, k \rangle, \langle k; v'; \emptyset \rangle, k) \) if \( k \neq k_f, k \) fresh

AddE \( (k; v; \overline{k}) \), adde \( v' \) \( k \) \( \Rightarrow (\langle k; v; \overline{k}, k' \rangle, \langle k; v'; k'' \rangle) \), \( k \)

Find \( (k; v; \overline{k}) \), find \( k \) \( \Rightarrow (\langle k; v; \overline{k}, k \rangle, \langle k; v' \rangle) \)

NbOut \( (k; v; \overline{k}) \), nbO \( k \) \( \Rightarrow (\langle k; v; k' \rangle, k' \) \( \overline{k} \) \( ) \)

NbIn \( (k'; v; \overline{k}) \), nbI \( k \) \( \overline{k}_1 \) \( \Rightarrow (\langle k'; v; k'' \rangle, \langle k' \rangle, k_1 \) \( ) \) if \( k \in k'' \)

Map \( (k; v; \overline{k}) \), map \( e \) \( \Rightarrow (\langle k; v'; \overline{k}, k \rangle, \langle k; v' \rangle) \)

\( (\lambda x.e) \) \( v \rightarrow e[v/x] \)

\begin{align*}
\frac{e \rightarrow e''}{e \rightarrow e''} & \quad \frac{e' \rightarrow e''}{(\lambda x.e) \ e' \rightarrow (\lambda x.e) \ e''}
\end{align*}

**Figure 5** Hotspot Reduction: \( n, \omega \Rightarrow \overline{n}, \rho \)

**Figure 6** Call-by-Value Reduction: \( e \rightarrow e \)

### 4.1 The Hotspot Reduction

Let us first introduce a reduction system to capture the realization of an operation over a node payload in the graph. We informally call this node payload in the graph a hotspot. The hotspot reduction relation \( n, \omega \Rightarrow \overline{n}, \rho \), which we define in Fig. 5, says that hotspot \( n \) one-step reduces to a sequence of nodes \( \overline{n} \) when an operation \( \omega \) is realized, and it produces a result \( \rho \).

**Example 11** (Dynamic Graph). In Fig. 5, AddN illuminates the support of dynamic streaming graphs of DG Calculus. The newly added node is inserted after the node it is attached to in the hotspot. Thanks to dynamic graph support, the first post-reduction meta-element of the hotspot relation is a node sequence, not a node.

Map involves function application. We use \( \rightarrow \) to represent the standard call-by-value reduction, whose standard semantics is shown in Fig. 6, and \( \rightarrow^* \) as the reflexive and transitive closure of \( \rightarrow \). Note that we are only interested in reductions that strongly normalize to a value. This is consistent with large-scale graph processing where the operation applied to a node must terminate.

While NbOut is a singular operation to compute the hotspot node’s out-edge set, NbIn must be defined as a bulk operation where each hotspot is defined as one whose out-edge set includes the queried node.

**Example 12** (Neighbors In and Neighbors Out). Consider the operation \( \text{nbI} \ k_3 \emptyset \), when issued to the graph in Example 5 (illustrated in Fig. 2) the result \( \{k_1, k_2\} \) will be computed. When issued to the same graph, the operation \( \text{nbO} \ k_1 \) will result in \( \{k_2, k_3\} \) being computed.

Beyond graph-specific details, the hotspot reduction is most interesting for the philosophy of defining the semantics of operations over hotspots: regardless of singular operations or bulk operations, realizing an operation strongly exhibits local behavior.

**Example 13** (Local Behavior for Singular Operations). According to NbOut, the realization of this operation — finding the set of keys which the current node has edges to — can be simply defined over a single node in the graph.
A bulk operation does process a large number of nodes, but processing a bulk operation can still be divided into local steps of realization, each of which only considers a single hotspot.

Example 14 (Local Behavior for Bulk Operations). Map is only defined over a single hotspot and does not immediately produce a value. For the operation map e, the post-reduction result is map e itself, capturing the intuition that the operation has been realized up to the hotspot so far, but it should be further realized in the rest of the graph.

4.2 Eager Semantics for Graph Processing

Eager graph processing is defined by relation \( G, O \xrightarrow{E} G', R \) as in Fig 7. It says that the graph \( G \) one-step reduces to \( G' \) when being processed with an operation stream \( O \) in the eager processing mode, and it produces the result stream \( R \). In eager processing, the issuance of an operation must be one by one. To be consistent with lazy data processing however (so that we can formally relate the two systems), we choose to define the relation over the stream instead of one operation/result. \( E \rightarrow \) serves as the bridge between this general multi-issuance form and all other rules, which require one operation/result to be processed at a time.

The essence of eager processing is that processing each operation entails the traversal of the potentially large graph before another operation can be processed. The traversal either ends with the operation being realized in a hotspot, as shown in \( E \rightarrow \) for singular operations and \( E \rightarrow \omega \) for bulk operations, or the traversal reaches the very last node of the graph. For the latter (see E-Final), an operation-specific value should be returned. We define a convenience function \( \text{final} \) as follows, with the only interesting case being the \( \text{nbI} \) operation produces a result kept in the accumulator.

\[
\text{final}(\omega) = \begin{cases} 
  kf & \text{if } \omega \in \{\text{add e e, adde e e, find e, map e, nb0 e}\} \\
  k & \text{if } \omega = \text{nbI e k}
\end{cases}
\]
Example 15 (Eager Processing). Consider the following configuration based on the program from Fig. 3(c): \( e(\langle k; r; \{ k_1 \} \rangle), G; \ell_8 \rightarrow add v_6 k_1, \ell_9 \rightarrow add v_7 \ell_1 \) being reduced is as follows:

\[
\begin{align*}
\langle k_1; v_1; \{ k_2, k_3 \} \rangle, &\quad add v_6 k_1 \Rightarrow (\langle k_1; v_1; \{ k_2, k_3 \} \rangle, \langle k_6; v_6; \emptyset \rangle), k_6 &\quad \text{ADDN} \\
(e(\langle k_1; v_1; \{ k_2, k_3 \} \rangle), G), &\quad \ell_8 \rightarrow add v_6 k_1 \rightarrow G', \ell_8 \rightarrow k_6 &\quad \text{E-ω} \\
(e(\langle k; r; \{ k_1 \} \rangle), G), &\quad \ell_8 \rightarrow add v_6 k_1 \rightarrow e(\langle k; r; \{ k_1 \} \rangle), G', \ell_8 \rightarrow k_6 &\quad \text{E-↓}
\end{align*}
\]

Subsequently, the reduction derivation which leads to the \( \ell_9 \rightarrow add v_7 k_6 \) being realized is as follows:

\[
\begin{align*}
\langle k_6; v_6; \emptyset \rangle, &\quad add v_7 k_6 \Rightarrow (\langle k_6; v_6; \{ k_7 \} \rangle, \langle k_7; v_7; \emptyset \rangle), k_7 &\quad \text{ADDN} \\
(e(\langle k_6; v_6; \emptyset \rangle), G_1), &\quad \ell_9 \rightarrow add v_7 k_6 \rightarrow e(\langle k_6; v_6; \{ k_7 \} \rangle), (k_7; v_7; \emptyset), G_1, &\quad \text{E-ω} \\
G', &\quad \ell_9 \rightarrow k_7 \rightarrow G'', &\quad \text{E-↓}
\end{align*}
\]

\[
\begin{align*}
&\quad (e(\langle k; r; \{ k_1 \} \rangle), G'), \quad \ell_9 \rightarrow add v_7 k_6 \rightarrow e(\langle k; r; \{ k_1 \} \rangle), G'', &\quad \text{E-↓} \\
&\quad (e(\langle k; r; \{ k_1 \} \rangle), G'), \quad \ell_9 \rightarrow k_7 \rightarrow \ell_9 \rightarrow k_7 \rightarrow k_6
\end{align*}
\]

where \( G = e(\langle \{ k_1; v_1; \{ k_2, k_3 \} \} \rangle), G_1 \) and \( G' = e(\langle k_1; v_1; \{ k_2, k_3, k_6 \} \rangle), e(\langle k_6; v_6; \emptyset \rangle), G_1 \) and \( G'' = e(\langle k_1; v_1; \{ k_2, k_3, k_6 \} \rangle), e(\langle k_6; v_6; \{ k_7 \} \rangle), e(\langle k_7; v_7; \emptyset \rangle), G_1 \) and where \( k_1, k_6, k_7 \) are the generated keys corresponding to the values \( v_1, v_6, \text{and} v_7 \).

As the example suggests, one step of program configuration reduction can only be established through a derivation that requires potentially multiple E-↓ instances, intuitively representing the traversal of the graph. Indeed, the only leaf nodes in the derivation in the eager semantic system must be instances of E-ω, E-→, or E-Final. E-↓ is inductively defined and cannot be a leaf node. As we shall see, constructing a derivation of lazy processing does not require a graph traversal.

5 DG\(^I\) Calculus: Lazy Processing in Streaming Graphs

In this section, we define lazy processing in streaming graphs. This is achieved through defining the \( \rightarrow_m^o \) relation where \( m = I \). Lazy graph processing is defined by relation \( G, O \xrightarrow{m} G', R \) as defined in Fig. 8. It says that the graph \( G \) one-step reduces to \( G' \) when being processed with an operation stream \( O \) in the lazy processing mode, and it produces the result stream \( R \).
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\[
\text{L-LAZY } (\langle n; O'; R \rangle, G), O \xrightarrow{L} (\langle n; (O', O); R \rangle, G), \emptyset
\]

\[
\text{L-CLAIM } \quad \text{fv}(O) \cap \text{dom}(R) = \emptyset \\
\quad (\langle n; O; (R', R) \rangle, G), \emptyset \xrightarrow{L} (\langle n; O; R', G \rangle, R)
\]

\[
\text{L-}\omega \\
\quad n, \omega \Rightarrow (n', \bar{n}' \rho), \rho \\
\quad (G_1, \langle n; (\ell \rightarrow \omega; O); R \rangle, G_2), \emptyset \xrightarrow{L} (G_1, \langle n'; O'; (\ell \rightarrow \rho; R') \rangle, \bar{e}(n', G_2), \emptyset)
\]

\[
\text{L-}\omega \\
\quad n, \bar{\omega} \Rightarrow (n', \bar{n}' \rho), G_2, \ell \rightarrow \rho \xrightarrow{L} G_2', \emptyset \\
\quad (G_1, \langle n; (\ell \rightarrow \omega; O); R \rangle, G_2), \emptyset \xrightarrow{L} (G_1, \langle n'; O; \bar{e}(n', G_2'), \emptyset
\]

\[
\text{L-FINAL } (G, \langle n; (\ell \rightarrow \omega; O); R \rangle), \emptyset \xrightarrow{L} (G, \langle n; O; \ell \rightarrow \text{final}(\omega, R) \rangle, \emptyset)
\]

\[
\text{fv}(O) \cap \text{dom}(R) = \emptyset \\
\forall \omega \in \text{ran}(O). \exists n', \bar{n}' \rho, n \Rightarrow n', \rho \\
\quad \text{latent}(O) \\
\quad (G_1, \langle n; (O', O); R \rangle, G_2), \emptyset \xrightarrow{L} (G_1, \langle n; O'; R \rangle, G_2') , \emptyset
\]

\[
\text{L-FUSE } (G_1, \langle n; O; R \rangle, G_2), \emptyset \xrightarrow{L} (G_1, \langle n; O; R; R_1, G_2 \rangle, \emptyset)
\]

\[
\text{O \Rightarrow O', R_1} \\
\quad (G_1, \langle n; O_1, O_2; R \rangle, G_2), \emptyset \xrightarrow{L} (G_1, \langle n; O_1, O_2; R_1, R_2 \rangle, G_2), \emptyset
\]

Figure 8 DG\textsuperscript{L} Calculus Operational Semantics (Lazy System): \( G, O \xrightarrow{L} G, R \)

Among the rules in Fig. 8 the first two rules — L-Lazy and L-Claim — capture the behavior at the boundary between a graph and its client. In all other rules, the reductions solely happen in data: none depends on the client-side operation stream or result stream. In essence, they show how fine-grained lazy processing has decoupled the data from its client, so that the most interesting graph processing happens within the graph.

Before we proceed, let us introduce the function \( \odot \) whose domain is the set of unlabeled operations. The function \( \odot(\omega) \) computes the target of the operation \( \omega \), i.e., where the hotspot is, as defined by the key of the hotspot node. For bulk operations, where there is no one single target, we use kr as the target. We define \( \odot(\omega) \) as follows:

\[
\odot(\omega) \equiv \begin{cases} 
\text{e} & \text{if } \omega \in \{\text{add } e', \text{adde } e', \text{find } e, \text{nb } 0 \ e\} \\
\text{kr} & \text{otherwise}
\end{cases}
\]
5.1 Operation Issuance and Result Claim

As we discussed in §3, the O and R stores are essential for lazy processing, revealing the decentralized nature of lazy data processing: each node has its local view on what operations are delayed (O) and what results have been computed so far (R). Just as the order of operations in the configuration’s operation stream reflects the order of graph processing, the operation stream in each O store must also preserve the chronological order of operations. Informally, for two operations we say the one on the left is older while the one on the right is newer.

As L-Lazy reveals, an operation stream can be issued — appended to the end of the operation stream of the root node — in constant time, demonstrating the key difference between DGÉ Calculus and DGŁ Calculus. Issuing an operation in DGŁ Calculus does not require the same operation be realized.

Example 16 (Lazy Processing). Consider the configuration presented in Example 15.

One possible reduction derivation of P using lazy processing is as follows:

\[
\text{L-Lazy}
\]

\[
L^\rightarrow \begin{array}{l}
((\langle\langle k_r; k_r; \{k_1\}\rangle; \emptyset; \emptyset), G), \\
\ell_8 \mapsto \text{add } v_6 \ k_1
\end{array}
\]

\[
\begin{array}{l}
((\langle\langle k_r; k_r; \{k_1\}\rangle; \emptyset; \emptyset), G), \\
\ell_8 \mapsto \text{add } v_6 \ k_1, \ell_9 \mapsto \text{add } v_6 \ \ell_8; \\
\emptyset
\end{array}
\]

This reduction sequence consists of one constant-time step. Notably, the second operation labeled with \( \ell_9 \) could now be removed from the configuration’s operation stream before the first operation labeled with \( \ell_8 \) is realized.

This example, albeit simple, reveals the key difference between DGÉ Calculus and DGŁ Calculus. Here, issuing an operation does not require the same operation be realized. In contrast, recall that DGÉ Calculus would require the operation labeled \( \ell_8 \) be realized before \( \ell_9 \) can be issued.

L-Claim is the dual of L-Lazy. When the result stream associated with the focused node of the graph is not empty, it is ready to be claimed: the sub-sequence including the end of the focused node’s result stream may be “lifted” as the result of the reduction. Since the operation stream may contain operations which depend on the results to be claimed, we require that the labels associated with the claimed result sub-sequence do not appear in the operation stream. If they do, reduction may still happen, but via instances of L-Splice, which we will detail in §5.1. We define \( \rho(O) \) as the set of labels appearing in any \( \omega \) such that \( \omega \in \text{ran}(O) \). Both L-Lazy and L-Claim demonstrate batching at work. The issued operations and the claimed results are in the form of streams, not single operations or results.

Realization When an operation encounters its hotspot, the operation is realized. Concretely, a singular operation (L-\( \omega \)) at the head of the operation stream of any node in the graph can produce a result upon reaching its hotspot, with its result placed to
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the head of the result stream belonging to the hotspot node. A bulk operation (L-ω) however will continue the lazy propagation despite reaching a hotspot already. L-Final is the counterpart of E-Final in lazy processing.

The definitions for L-ω and L-ω here both depend on the ⇒ relation, which we defined in §4.1. This helps explain why we decoupled the hotspot reduction from the rest of the rules when eager data processing was defined. Indeed, eager or lazy, how an operation is realized remains the same.

Lazy Propagation One important feature of DG Calculus is in-data propagation. Intuitively, an operation is propagated “down” the graph (i.e., following the traversal order) until it reaches the hotspot. From that point on, a result is created and propagated “up” the graph (i.e., against the traversal order). In our reduction system, the essence of “down” propagation is captured by L↓ and the “up” propagation is captured by L↑.

A key observation is that each propagation step — be it “down” or “up” — is local, i.e., it only involves two nodes, adjacent to each other during traversal. This leads to a decentralized propagation design where different operations or results may be propagating in different neighborhoods of the graph in a non-deterministic manner as long as the reduction system allows.

Example (Lazy Propagation). The following is one possible snapshot of the graph — with ongoing propagation — from Fig.3(a):

Both rules illustrate batching at propagation time: O and R are used in the propagation instead of a single operation or result.

The first predicate in the prequent of L↓ is needed for a similar reason as in L-Claim, except that we now propagate an operation stream “down” the graph, instead of moving a result stream “up.” The second predicate is shared with L↓, meaning the current node cannot lead to a hotspot reduction for the operations intended to be propagated. The third predicate relies on the definition below:

Definition 1 (Latent Dependency). An operation stream O has latent dependencies, written as latent(O), iff there exists any ω ∈ ran(O), ⊗(ω) = ℓ and ops(ℓ) = find e for some e.

The motivation behind this definition is that it is not always possible to know whether a lazily processed operation has propagated to its target hotspot. When the key of the target is not yet known — i.e., in the form of a label — the conservative assumption is it may indeed compute to the key of the current node. That should forbid propagation, because the hotspot otherwise would be missed. To see why the find e operation may introduce latent dependencies, recall that this operation computes to the value of node e, which in turn could be a key. If such a key happens
In Data Fusion: \( O \to O, R \)

Splicing and Fusion  
Rule L-Splice captures the notion that the result stream may carry the values of labels which operations in the operation stream may depend on. Simply put, this rule fulfills data dependency between the operation propagating “down” and the result propagating “up”.

Example 18 (Splicing). In Example 15 if the root node has an O store that contains \( \ell_{15} \to \text{add} v_9 \ell_{14} \), and an R store that contains \( \ell_{14} \to k_1 \), L-Splice will transform the same operation in the O store to \( \ell_{15} \to \text{add} v_9 k_1 \). Note that the latent dependency is gone, and operation \( \ell_{15} \to \text{add} v_9 k_1 \) can now propagate.

In-data fusion is captured by rule L-Fuse, using reduction relation \( O \to O', R \), defined in Fig. 9 which says that \( O \) one-step reduces to \( O' \) producing labeled result \( R \). Operations within the graph may be rearranged or combined before they are realized, therefore reducing the overall number of computation steps involved in the round trip including the “down” operation propagation and the “up” result propagation. Note that this relation is unique for lazy graph processing, as eagerness prevents such an optimization.

Fusion in lazy graph processing is defined by five rules: F-MapMap, two map operations can be fused into one which composes the two mapping functions; F-MapAdd, a map operation can propagate past an add operation by applying the mapping function to the new node’s value; F-FindAdd, a find operation for the value of a delayed add operation can immediately return; F-NborsInAddN, the edges in to a new node is the currently calculated neighbors plus the node key for the attachment node; F-NborsInAddE, a nbI operation can propagate past an associated add operation by accounting for the new edge. They can be illustrated by the following examples.

\begin{align*}
F-\text{MapMap} & : \ell \mapsto \text{map } e, \ell' \mapsto \text{map } e' \mapsto (\lambda x. e'(x)), \ell \mapsto kf \\
F-\text{MapAdd} & : \ell \mapsto \text{add } v k, \ell' \mapsto \text{map } e \mapsto (\ell' \mapsto \text{map } e, \ell \mapsto \text{add } (e v) k), \emptyset \\
F-\text{FindAdd} & : \ell \mapsto \text{add } e e', \ell' \mapsto \text{find } \ell \mapsto \ell \mapsto \text{add } e e', \ell' \mapsto e \\
F-\text{NborsInAddN} & : \ell \mapsto \text{add } e k, \ell' \mapsto \text{nbI } \ell \mapsto \ell \mapsto \text{add } e k, \ell' \mapsto (\ell, k) \\
F-\text{NborsInAddE} & : \ell \mapsto \text{add } d k k, \ell' \mapsto \text{nbI } \ell \mapsto (\ell \mapsto \text{nbI } k(k_0, k), \ell \mapsto \text{add } d k k), \emptyset \\
\end{align*}
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Example 20 (Map Composition). Imagine we have two mapping operations \( \ell \rightarrow \text{map}(\lambda x.x + 1) \) and then \( \ell' \rightarrow \text{map}(\lambda x.x * 2) \). According to F-MapMap they may be composed into \( \ell' \rightarrow \text{map}(\lambda x.((\lambda x.x + 1) x)) \) while result \( \ell \rightarrow \text{ kf can be returned.} \)

Example 21 (Map in the Presence of Node Change). Imagine we have an add-node operation \( \ell \rightarrow \text{add v k} \) and then a mapping operation \( \ell' \rightarrow \text{map}(\lambda x.x + 1) \). According to F-MapAdd, the mapping operation can skip over the add operation, transforming the add operation into \( \ell \rightarrow \text{add (v + 1 k}. \)

Example 22 (Find in the Presence of Node Change). Imagine we have an add-node operation \( \ell \rightarrow \text{add v k} \) and then a find operation \( \ell' \rightarrow \text{find} \). According to F-FindAdd, the find operation can immediately produce the result \( \ell' \rightarrow v \).

Example 23 (In-Edge Computation in the Presence of Node Change). Imagine we have an add-node operation \( \ell \rightarrow \text{add v k} \) and then a neighbors-in operation \( \ell' \rightarrow \text{nbI} \ell \bar{k} \). According to F-NbInAddN, the neighbors-in operation can immediately produce the result \( \ell' \rightarrow (\bar{k}, k) \).

Example 24 (In-Edge Computation in the Presence of Edge Change). Imagine we have an add-edge operation \( \ell \rightarrow \text{addEdge} k k' \) and then a neighbors-in operation \( \ell' \rightarrow \text{nbI} k \bar{k} \). According to F-NbInAddE, the neighbors-in operation can skip over the add operation, and transform itself into \( \ell' \rightarrow \text{nbI} k(\bar{k}, k') \).

Together, these fusion rules are examples that share the spirit of “short-circuiting” the operation(s) before they are fully realized, leading to reduced need for propagation.

6 Meta-Theory

We now study the properties of DG Calculus, with a focus on establishing the relationship between DG\(^{C}\) Calculus and DG\(^{E}\) Calculus. First let us introduce a simple notion of a well-formed configuration:

Definition 2 (Well-Formed Configuration). Predicate \( \text{wf}(\langle G; O; R \rangle) \) holds iff

1. (Root Node) there exists \( \langle n; O'; R' \rangle \in G \) such that \( n = \langle kr; kr; \bar{k} \rangle \) for some \( \bar{k} \);
2. (Final Node) there exists \( \langle n'; O'; R' \rangle \in G \); such that \( n = \langle kf; kf; \emptyset \rangle \);
3. (Key Uniqueness) if \( G = G_1, \langle k_1; v_1; \bar{k}_1 \rangle; O_1; R_1 \rangle, G_2, \langle (k_2; v_2; \bar{k}_2) ; O_2; R_2 \rangle, G_3 \) for some \( G_1, G_2, \) and \( G_3 \), then \( k_1 \neq k_2 \);
4. (Label Uniqueness) If \( G = (n_1; O_1; R_1), \ldots, (n_m; O_m; R_m) \) then \( \text{dom}(O_1), \ldots, \text{dom}(O_m), \text{dom}(R_1), \ldots, \text{dom}(R_m) \) are all distinct;
5. (Top-Level Dependent Result Resolution) \( Fv(O) \cap \text{dom}(R) = \emptyset \).

Lemma 1 (Well-Formedness Preservation). If \( \text{wf}(P_1) \), and \( P_1 \xrightarrow{m} P_2 \), then \( \text{wf}(P_2) \).

Proof Sketch. Case analysis on E- rules and L- rules.
The two systems clearly demonstrate different traits in determinism:

**Lemma 2** (DG^E Calculus Deterministic Reduction). If \( P = \langle G; O; R \rangle \) and \( O \neq \emptyset \) and \( wf(P) \) then there exists a unique \( P' \) such that \( P \xrightarrow{E} P' \).

*Proof Sketch.* The determinism of \( \xrightarrow{E} \) trivially follows the determinism of \( \xrightarrow{} \). The latter can be proved by induction, and case analysis on the last step of derivation over the \( \xrightarrow{} \) rules. \qed

**Lemma 3** (DG^L Calculus Non-Deterministic Reduction). If \( P = \langle G; O; R \rangle \) and \( wf(P) \) and at least two of the following are non empty: (i) \( O \); (ii) any \( O' \) for \( \langle n; O'; R' \rangle \in G \); (iii) any \( R' \) for \( \langle n; O'; R' \rangle \in G \), then there exists \( P \xrightarrow{L} P' \) and \( P \xrightarrow{L} P'' \) where \( P' \neq P'' \).

*Proof Sketch.* Observe that both \( L\downarrow \) and \( L\uparrow \) may be non-deterministically applied on any node in the graph. \qed

Given the non-deterministic nature of lazy propagation, it is natural to ask whether different reduction sequences from the same configuration may produce the same result (confluence), and more importantly, if they all indeed produce the same result, how is that result related to what is produced by DG^E Calculus (observable equivalence). We answer these questions next.

Intuitively, configurations \( P \) in DG^L Calculus and \( P' \) in DG^E Calculus “correspond” to each other when there is

- **Data Correspondence:** the two must finally compute the same graph.
- **Result Correspondence:** for every operation computed in the two configurations \( P \) and \( P' \), they must yield the same result.

To capture the notion of “finally”, we introduce configuration summary \( s \), defined as \( \langle \pi; R \rangle \), which intuitively says that stream processing produces “final” results \( R \) and the “final” logical graph consists of \( \pi \) nodes. Configuration summary supports a simple update operator we define now:

**Definition 3** (Summary Update). Operator \( s \leftarrow \langle O; R \rangle \) computes a configuration summary from \( s \) given \( O \) and \( R \), defined as:

\[
\langle \pi; R_0 \rangle \leftarrow \langle O; R \rangle \triangleq \langle \pi'; R_0, R' \rangle \quad \text{if} \quad \langle \epsilon(n); O[R]; R \rangle \xrightarrow{E}^* \langle \epsilon(\pi'); \emptyset; R' \rangle
\]

On the high level, this definition says that we rely on the standard eager data processing to formally define the notion of “final” above. Fortunately, Lemma 2 shows the reduction system is deterministic, so the operator is well-defined. Intuitively, this update operator provides an accumulator for computing the configuration summary. Now we can easily define configuration summary for DG^E Calculus and DG^L Calculus as follows:

**Definition 4** (DG^E Calculus Configuration Summary). Operator \( \triangleleft^E \) computes the summary of a DG^E Calculus configuration. \( \triangleleft^E(G; O; R) \triangleq \langle \pi; \emptyset \rangle \leftarrow \langle O; R \rangle \) where \( G = \langle \pi; O; R \rangle \).

**Definition 5** (DG^L Calculus Configuration Summary). Operator \( \triangleleft^L \) computes the summary of a DG^L Calculus configuration:
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\[ \xi^L(G; O; R) \triangleq (\pi; \emptyset) \leftarrow (O_u; R_u) \ldots \leftarrow (O_1; R_1) \leftarrow (O; R) \]

where \(|G| = u\) and \(G = (n; O; R)\).

**Definition 6** (Endo-Bisimilarity). We say configuration \(P\) and configuration \(P'\) are \(m\)-bisimilar, denoted as \(P \sim^m P'\), if \(\xi^m P = (\pi; R)\) and \(\xi^m P' = (\pi'; R')\) and \(\pi = \pi'\) and \(R = R'\) is a permutation of \(R\) where \(m = E\) or \(m = L\).

First, since the summary update definition uses eager reduction, we can now show that eager graph processing preserves endo-bisimilarity.

**Lemma 4** (Endo-bisimilarity Over Eager Reduction). For any \(P_1\) and \(P_2\) where \(w (P_1)\) and \(P_1 \xrightarrow{E} P_2\) then \(P_1 \sim P_2\).

**Proof Sketch.** Let \(P_i = (e(n_i); O_i; R_i)\) for \(i = 1, 2\). We wish to establish \(n_i \sim n_j\) and \(R_i \sim R_j\) where \((\pi_i; R_i) = (\pi_j; R_j)\sim (\pi_i; \emptyset) \leftarrow (O_i; R_i)\) for \(i = 1, 2\). The latter according to Definition 5 implies \(e(n_i); O_i; R_i; R_i) \xrightarrow{E} (e(n_j); \emptyset; R_j)\) for \(i = 1, 2\). With \(w (P_1)\) and Lemma 1, we know \(w (P_2)\). The reduction thus is \((e(n_i); O_i; R_i) \xrightarrow{E} (e(n_j); \emptyset; R_j)\). By assumption, \((e(n_1); O_1; R_1) \xrightarrow{E} (e(n_2); O_2; R_2)\) holds. The conclusion follows the basic definition of transitivity and determinism of eager data processing (Lemma 2). \(\blacksquare\)

We now show that lazy graph processing also preserves endo-bisimilarity. The key lemma for establishing this is in-graph data processing does not alter the result of summary computation according to Definition 5.

**Lemma 5** (Endo-Bisimilarity Over In-Graph Lazy Reduction). If \(G_1, \emptyset \xrightarrow{L}. G_2, \emptyset\) then \((G_1; \emptyset; \emptyset) \sim (G_2; \emptyset; \emptyset)\).

**Proof Sketch.** Case analysis over the \(L\)- rules. Here we only sketch the case for \(L-\downarrow\). Not to lose generality, let us consider the rule is applied to a propagation of \(O\) from node \(n_a\) to \(n_b\). where

\[
G_1 = (n_1; O_1; R_1), \ldots, (n_a; O; O_a; R_a), (n_b; O_b; R_b), \ldots, (n_m; O_m; R_m)
\]

\[
G_2 = (n_1; O_1; R_1), \ldots, (n_a; O_a; R_a), (n_b; O_b; O; R_b), \ldots, (n_m; O_m; R_m)
\]

The conclusion follows the following equation and Definition 3:

\[
(\pi; \emptyset) \leftarrow (O_a; R_a) \ldots \leftarrow (O; O_a; R_a) \leftarrow (O_b; R_b) \ldots \leftarrow (O_1; R_1)
\]

\[
= (\pi; \emptyset) \leftarrow (O_m; R_m) \ldots \leftarrow (O_a; R_a) \leftarrow (O_b, O; R_b) \ldots \leftarrow (O_1; R_1)
\]

\(\blacksquare\)

**Lemma 6** (Endo-bisimilarity Over Lazy Reduction). For any \(P_1\) and \(P_2\) where \(w (P_1)\) and and \(P_1 \xrightarrow{L} P_2\) then \(P_1 \sim P_2\).

**Proof Sketch.** Let \(P_i = (G_i; O_i; R_i)\) for \(i = 1, 2\). The assumption entails \((G_1; (O', O); R_1) \xrightarrow{L} (G_2; O[R']; R'(R')); (R_1)\) where \(G_1, O' \xrightarrow{L} G_2, R'\) where \(O_1 = O', O\) and \(O_2 = O[R']\) and \(R_2 = R', R_1\). Case analysis over \(L\)- rules. The most important rules involve in-graph data processing, which is established by Lemma 5. \(\blacksquare\)
We are now ready to relate configurations from DG\textsuperscript{E} Calculus and DG\textsuperscript{L} Calculus through bisimilarity.

**Definition 7 (E-L Bisimilarity).** We say well-formed configurations \( P \) and \( P' \) are E-L bisimilar, denoted as \( P \Leftrightarrow E \sim L P' \), iff \( \Diamond^E P = \langle n; R \rangle \) and \( \Diamond^L P' = \langle n'; R' \rangle \) and \( n = n' \) and \( R \) is a permutation of \( R' \).

The key theorem we prove is bisimulation between DG\textsuperscript{E} Calculus and DG\textsuperscript{L} Calculus which we state in the two theorems below.

**Theorem 1 (Simulating Eager Processing with Lazy Processing).** For any well-formed \( P \) and \( P \rightarrow E P' \) there exists some derivation for \( P \rightarrow L P' \).

*Proof Sketch.* The insight is that for every derivation that leads to \( P \rightarrow E P' \), the last step can always be simulated with a pair of L-Lazy and L-Claim, and every instance of E-\( \downarrow \) can be simulated with a pair of L-\( \uparrow \) and L-\( \downarrow \).

The other way of simulation — from lazy processing to eager processing — requires some care. In particular, note that the \( O \) store and the \( R \) store for graph nodes may be non-empty in lazy processing, but eager processing is not defined over such configurations. Simulation however can still be established:

**Theorem 2 (Simulating Lazy Processing with Eager Processing).** For any well-formed \( P_1 \) and \( P_1 \rightarrow L P'_1 \) there exists some \( P_0 \) and \( P'_0 \) where \( P_0 \rightarrow E P' \) and \( P_0 \Leftrightarrow E \sim L P_1 \), and \( P'_0 \Leftrightarrow E \sim L P'_1 \).

*Proof Sketch.* Case analysis over \( \rightarrow L \). There are three subcases. For all rules that involve in-graph data processing, \( P_0 \) and \( P'_0 \) can both be constructed as identical, with its graph component consisting of the (same) final nodes in the configuration summary of \( P_1 \), as well as \( \emptyset \) for the \( O \) store and the \( R \) store for each node. \( \rightarrow E^* \) is a reflexive instance of \( \rightarrow E \). For L-Lazy and L-Claim, apply induction.

We are now ready to establish some high-level properties about our calculus. First, even though DG\textsuperscript{L} Calculus is non-deterministic (Lemma 8), all reduction sequences from the same configuration produce the same result:

**Lemma 7 (DG\textsuperscript{L} Calculus Confluence).** If \( P \rightarrow L^* \langle G; \emptyset; R \rangle \) and \( P \rightarrow L^* \langle G'; \emptyset; R' \rangle \) and where \( \text{wf}(P) \) and \( P = \langle G_0; O_0; R_0 \rangle \) and \( |R| = |R'| = |R_0| + |O_0| \) then \( R \) is a permutation of \( R' \).

*Proof Sketch.* Follows from Lemma 8. A permutation is possible when fusion might re-order the order in which results are returned.

Both DG\textsuperscript{E} Calculus and DG\textsuperscript{L} Calculus always progress:

**Theorem 3 (Progress).** For any program \( O \) and \( \text{init}(O) \rightarrow m^* P \) then either \( P = \langle G; \emptyset; R \rangle \) for some \( G \) and \( R \) or there exists some \( P' \neq P \) and \( P \rightarrow m P' \).
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Proof Sketch. Every operation in DG\(^E\) Calculus either realizes or reaches the final node. More rigorously, induction over \(\rightarrow\) and case analysis on the last step of derivation. For DG\(^L\) Calculus, the theorem follows a simple case analysis over \(\rightarrow\) rules.

We can now establish that DG\(^L\) Calculus and DG\(^E\) Calculus produce the same results up to permutation.

**Theorem 4** (Observable Equivalence). For any program \(O\), if \(\text{init}(O) \rightarrow^* \langle G; \emptyset; R \rangle\) and \(\text{init}(O) \rightarrow^* \langle G'; \emptyset; R' \rangle\) and \(|R| = |R'| = |O|\), then \(R\) is a permutation of \(R'\).

Proof Sketch. Follows from Lemma\(\text{[6]}\), Theorem\(\text{[1]}\), Lemma\(\text{[7]}\) and Lemma\(\text{[3]}\).

This important property justifies the correctness of lazy data processing. This is a not a trivial result considering the decentralized, local, and non-deterministic nature of propagation adopted by lazy processing, and the expressive optimizations of batching, fusion, and splicing unified in the same system. DG\(^E\) Calculus and DG\(^L\) Calculus share the same programming model (see Fig.\(\text{[1]}\)), and their difference is entirely semantic. With observable equivalence, the choice of eager/lazy data processing can (largely) be viewed as a black box. This is good news for upgrading streaming graph systems from eager data processing to lazy data processing.

## 7 Related Work

We are unaware of theoretical frameworks modeling the behavior of streaming graphs, nor lazy data processing over them. In this section, we place streaming graphs in context — a fast-developing field primarily experimental in nature — with a focus on features that overlap with our interest. In the long run, DG\(^L\) Calculus may serve as a foundation to rigorously model some of the systems/features we describe below.

“Haste makes waste” is a recurring theme that motivates numerous latency-oriented optimizations in streaming graphs. Top-level “all or nothing”-style of batching is a basic technique used in the majority of streaming graph systems \([16]\). Pregel \([24]\) and GraphLab \([22]\) allow low-level MPI messages across graph partitions on different clusters to be batched, which can be viewed as a lower-level semantics-oblivious implementation of decentralized batching. Through its timely dataflow design, Naiad \([15]\) tolerates delay in streaming graph processing with timestamps. Kickstarter \([32]\) reduces latency through incrementally correcting the error in the approximation result. Reflective consistency \([10]\) is a system where deferred synchronization leads to relaxed consistency. Our calculus is motivated by DeltaGraph \([11]\), a Haskell library that allows graph-manipulating operations to be cached, propagated, and fused within the graph. Their experiments demonstrated the performance benefits of fine-grained lazy processing.

Beyond streaming graphs, lazy data processing is also an important research direction in streaming databases. Query prefetching \([26]\) resorts to static analysis and transformation to combine queries and reduce the number of round-trips to access.
data. Query synthesis \[7\] addresses inefficiencies in object-relational mapping libraries by synthesizing efficient SQL queries. Sloth \[6\] adopts a notion of extended lazy evaluation to dynamically batch SQL queries. Query results may also be cached, such as through Sqlcache \[28\]. Query consolidation \[30\] allows multiple user-defined functions to be merged. Together, these systems motivate the need for latency-oriented optimization through being “lazy,” which DG\(^L\) Calculus lays a foundation for. In design, their primary focus is on the optimization at the client-data boundary, analogous to a DG\(^L\) Calculus setting where \(O\) is only allowed at the root level of the graph.

In DG Calculus, the results produced in the result stream may not follow the same order as their corresponding operations in the operation stream. This is aligned with the typical use scenarios of streaming graphs, where applications are latency- and throughput-sensitive, and the order of results generally does not matter \[23\]. From the view of the client, this is a flavor of asynchronous data processing \[3, 44, 54\], common in large-scale data processing. Asynchronous language support is also widely known \[18, 21\]. Unlike existing work where asynchrony serves as an enabling technique for concurrency, asynchrony in DG Calculus is a natural effect of lazy data processing.

Streaming graphs should not be confused with data streaming systems, despite their close resemblance in naming. With precursors in data-flow languages such as Lucid \[2\] and Lustre \[4\], and examples in stream languages such as StreamIt \[31\], data streaming systems support data themselves as a stream, passed through structured operations composed via dataflow combinators. This is nearly a dual view of ours, where operations form a stream passed through structured data.

Lazy data processing is broadly related to lazy evaluation, a classic semantic feature in programming languages. In the call-by-name \(\lambda\) calculus, the argument of a function may not be evaluated upon function application. Haller et al. \[17\] describes a programming model for Spark \[35\]-inspired distributed data processing, where deferred evaluation is supported at the boundary of nodes. Incremental computing systems \[1, 13, 20, 25\] maintain the propagation latent in a function’s control/data flow, and efficiently perform recomputation along the propagation path only when necessary. If one viewed a graph-processing operation as a function and the graph it operates on as the argument, DG\(^L\) Calculus at its essence calls for a dual propagation design: the function propagates within the argument.

The general idea of combining and re-ordering expressions is a basic idea in compiler optimization, such as deforestation \[33\] and Stream Fusion \[9\].

## Conclusion

This paper introduces DG Calculus, a foundation for lazy streaming graph processing. DG Calculus features fine-grained in-data lazy processing which allows for optimizations such as batching, fusion, and splicing. The soundness of DG Calculus is established through a bisimulation to eager streaming graph processing.
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