Eldan’s Stochastic Localization and the KLS Hyperplane Conjecture: An Improved Lower Bound for Expansion

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Abstract

We show that the KLS constant for \( n \)-dimensional isotropic logconcave measures is \( O(n^{1/4}) \), improving on the current best bound of \( O(n^{1/3}\sqrt{\log n}) \). As corollaries we obtain the same improved bound on the thin-shell estimate, Poincaré constant and exponential concentration constant and an alternative proof of this bound for the isotropic constant; it also follows that the ball walk for sampling from an isotropic logconcave density in \( \mathbb{R}^n \) converges in \( O^{*}(n^{2.5}) \) steps from a warm start.

1 Introduction

The isoperimetry of a subset is the ratio of the measure of the boundary of the subset to the measure of the subset or its complement, whichever is smaller. The minimum such ratio over all subsets is the Cheeger constant, also called expansion or isoperimetric coefficient. This fundamental constant appears in many settings, e.g., graphs and convex bodies and plays an essential role in many lines of study.

In the geometric setting, the KLS hyperplane conjecture \([21]\) asserts that for any distribution with a logconcave density, the minimum expansion is approximated by that of a halfspace, up to a universal constant factor. Thus, if the conjecture is true, the Cheeger constant can be essentially determined simply by examining hyperplane cuts. More precisely, here is the statement. We use \( c, C \) for absolute constants, and \( \|A\|_2 \) for the spectral/operator norm of a matrix \( A \).

Conjecture 1 \([21]\). For any logconcave density \( p \) in \( \mathbb{R}^n \) with covariance matrix \( A \),

\[
\frac{1}{\psi_p} \triangleq \inf_{S \subseteq \mathbb{R}^n} \frac{\int_{\partial S} p(x)dx}{\min \left\{ \int_S p(s)dx, \int_{\mathbb{R}^n \setminus S} p(x)dx \right\}} \geq \frac{c}{\sqrt{\|A\|_2}}
\]

For an isotropic logconcave density (all eigenvalues of its covariance matrix are equal to 1), the conjectured isoperimetric ratio is an absolute constant. Note that the isoperimetric constant or KLS constant \( \psi_p \) is the reciprocal of the minimum expansion or Cheeger constant (this will be more convenient for comparisons with other constants). This conjecture was formulated by Kannan, Lovász and Simonovits in the course of their study of the convergence of a random process (the ball walk) in a convex body and they proved the following weaker bound.

Theorem 2 \([21]\). For any logconcave density \( p \) in \( \mathbb{R}^n \) with covariance matrix \( A \), the KLS constant satisfies

\[ \psi_p \leq C \sqrt{\text{Tr}(A)}. \]

For an isotropic distribution, the theorem gives a bound of \( O(\sqrt{n}) \), while the conjecture says \( O(1) \).

The conjecture has several important consequences. It implies that the ball walk mixes in \( O^*(n^{2.5}) \) steps from a warm start in any isotropic convex body (or logconcave density) in \( \mathbb{R}^n \); this is the best possible bound, and is tight e.g., for a hypercube.

The KLS conjecture has become central to modern asymptotic convex geometry. It is equivalent to a bound on the spectral gap of isotropic logconcave functions \([25]\). Although it was formulated due to an algorithmic motivation, it implies several well-known conjectures in asymptotic convex geometry. We describe these next.

The thin-shell conjecture (also known as the variance hypothesis) \([29, 6]\) says the following.

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Conjecture 3 (Thin-shell). For a random point \( X \) from an isotropic logconcave density \( p \) in \( \mathbb{R}^n \),
\[
\sigma_p^2 \overset{\text{def}}{=} \mathbb{E}((\|X\| - \sqrt{n})^2) = O(1).
\]

It implies that a random point \( X \) from an isotropic logconcave density lies in a constant-width annulus (a thin shell) with constant probability. Noting that
\[
\sigma_p^2 = \mathbb{E}((\|X\| - \sqrt{n})^2) \leq \frac{1}{n} \var(\|X\|^2) \leq C\sigma_p^2,
\]
the conjecture is equivalent to asserting that \( \var(\|X\|^2) = O(n) \) for an isotropic logconcave density. The following connection is well-known: \( \sigma_p \leq C\psi_p \). The current best bound is \( \sigma_p \leq n^{1/3} \) by Guedon and Milman [20], improving on a line of work that started with Klartag [23, 24, 18]. Eldan [14] has shown that the reverse inequality holds approximately, in a worst-case sense, namely the worst possible KLS constant over all isotropic logconcave densities in \( \mathbb{R}^n \) is bounded by the thin-shell estimate to within roughly a logarithmic factor in the dimension. This results in the current best bound of \( \psi_p \leq n^{1/3}/\sqrt{\log n} \). A weaker inequality was shown earlier by Bobkov [5] (see also [30]).

The slicing conjecture, also called the hyperplane conjecture [8, 4] is the following.

Conjecture 4 (Slicing/Isotropic constant). Any convex body of unit volume in \( \mathbb{R}^n \) contains a hyperplane section of at least constant volume. Equivalently, for any convex body \( K \) of unit volume with covariance matrix \( L_K^2 I \), the isotropic constant \( L_K = O(1) \).

The isotropic constant of a general isotropic logconcave density \( p \) with covariance a multiple of the identity is defined as \( L_p = p(0)^{1/n} \). The best current bound is \( L_p = O(n^{1/4}) \), due to Klartag [22], improving on Bourgain’s bound of \( L_p = O(n^{1/4} \log n) \) [7]. The study of this conjecture has played an influential role in the development of convex geometry over the past several decades. It was shown by Ball that the KLS conjecture implies the slicing conjecture. More recently, Eldan and Klartag [15] showed that the thin shell conjecture implies slicing, and therefore an alternative (and stronger) proof that KLS implies slicing: \( L_p \leq C\sigma_p \).

The next conjecture is a bound on the Poincaré constant for logconcave distributions.

Conjecture 5 (Poincaré constant). For any isotropic logconcave density \( p \) in \( \mathbb{R}^n \), we have
\[
Q_p^2 \overset{\text{def}}{=} \sup_{g \text{ smooth}} \frac{\var_p(g(x))}{\mathbb{E}_p\left(\|\nabla g(x)\|^2_2\right)} = O(1).
\]

It was shown by Maz’ja [34] and Cheeger [11] that this Poincaré constant is bounded by twice the KLS constant \( (Q_p \leq 2\psi_p) \). The current best bound is the same as the KLS bound.

Finally, it is conjectured that Lipschitz functions concentrate over isotropic logconcave densities.

Conjecture 6 (Lipschitz concentration). For any \( L \)-Lipschitz function \( g \) in \( \mathbb{R}^n \), and isotropic logconcave density \( p \),
\[
\mathbb{P}_{x \sim p}\left(|g(x) - \mathbb{E}g| > t\right) \leq e^{-t/(D_p L)}
\]
where \( D_p = O(1) \).

Gromov and Milman [19] showed that \( D_p \) is also bounded by the KLS constant (see Lemma ??). For more background on these conjectures, we refer the reader to [10, 2, 3].

1.1 Results

We prove the following bound, conjectured in this form in [33].

Theorem 7. For any logconcave density \( p \) in \( \mathbb{R}^n \), with covariance matrix \( A \),
\[
\psi_p \leq C \left(\text{Tr} \left( A^2 \right)\right)^{1/4}.
\]

For isotropic \( p \), this gives a bound of \( \psi_p \leq Cn^{1/4} \), improving on the current best bound. The following corollary is immediate.

Corollary 8. For any logconcave density \( p \) in \( \mathbb{R}^n \), the isotropic (slicing) constant \( L_p \), the Poincaré constant \( Q_p \), the thin-shell constant \( \sigma_p \) and the concentration coefficient \( D_p \) are all bounded by \( O\left(n^{1/4}\right) \).

We mention an algorithmic consequence.

Corollary 9. The mixing time of the ball walk to sample from an isotropic logconcave density from a warm start is \( O^* \left(n^{2.5}\right) \).
1.2 Approach

The KLS conjecture is true for Gaussian distributions. More generally, for any distribution whose density function is the product of the Gaussian density for $N(0,\sigma^2I)$ and any logconcave function, it is known that the expansion is $\Omega(1/\sigma)$. This fact is used crucially in the Gaussian cooling algorithm of [13] for computing the volume of a convex body by starting with a standard Gaussian restricted to a convex body and gradually making the variance of the Gaussian large enough that it is effectively uniform over the convex body of interest. Our overall strategy is similar in spirit — we start with an arbitrary isotropic logconcave density and gradually introduce a Gaussian term in the density of smaller and smaller variance. The isoperimetry of the resulting distribution after sufficient time will be very good since it has a large Gaussian factor. And crucially, it can be related to the isoperimetry of initial distribution. To achieve the latter, we would like to maintain the measure of a fixed subset close to its initial value as the distribution changes. For this, our proof uses the localization approach to proving high-dimensional inequalities [27, 21], and in particular, the elegant stochastic version introduced by Eldan [14] and used in subsequent papers [17, 16].

We fix a subset $E$ of the original space with the original logconcave measure of measure $\frac{1}{2}$. This is without loss of generality due to a result of [30]. In standard localization we then bisect space using a hyperplane that preserves the volume fraction of $E$. The limit of this process is 1-dimensional logconcave measures (“needles”), for which inequalities are much easier to prove. This approach runs into major difficulties for proving the KLS conjecture. While the original measure might be isotropic, the 1-dimensional measures could, in principle, have variances roughly equal to the trace of the original covariance (i.e., long thin needles), for which only much weaker inequalities hold. Stochastic localization can be viewed as the continuous time version of this process, where at each step, we pick a random direction and multiply the current density with a linear function along the chosen direction. Over time, the distribution can be viewed as a spherical Gaussian times a logconcave function, with the Gaussian gradually reducing in variance. When the Gaussian becomes sufficiently small in variance, then the overall distribution has good isoperimetric coefficient, determined by the inverse of the Gaussian standard deviation (such an inequality can be shown using standard localization, as in [12]). An important property of the infinitesimal change at each step is balance — the density at time $t$ is a martingale and therefore the expected measure of any subset is the same as the original measure. Over time, the measure of a set $E$ is a random quantity that deviates from its original value of $\frac{1}{2}$ over time. The main question is then what direction to use at each step so that (a) the measure of $E$ remains bounded and (b) the Gaussian part of the density has small variance. We show that the simplest choice, namely a pure random direction chosen from the uniform distribution suffices. The analysis needs a potential function that grows slowly but still maintains good control over the spectral norm of the current covariance matrix. The direct choice of $\|A_t\|_2$, where $A_t$ is the covariance matrix of the distribution at time $t$, is hard to control. We use $\text{Tr}(A_t^2)$. This gives us the improved bound of $O(n^{1/4})$. In the appendix, we show that a third moment assumption implies further improvement via the same localization technique\(^1\).

2 Preliminaries

In this section, we review some basic definitions and theorems that we use.\\

2.1 Stochastic calculus

In this paper, we only consider stochastic processes given by stochastic differential equations. Given real-valued stochastic processes $x_t$ and $y_t$, the quadratic variations $[x]_t$ and $[x,y]_t$ are real-valued stochastic processes defined by

$$[x]_t = \lim_{|P| \to 0} \sum_{n=1}^{\infty} (x_{\tau_n} - x_{\tau_{n-1}})^2,$$

$$[x,y]_t = \lim_{|P| \to 0} \sum_{n=1}^{\infty} (x_{\tau_n} - x_{\tau_{n-1}})(y_{\tau_n} - y_{\tau_{n-1}}),$$

where $P = \{0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots \uparrow t\}$ is a stochastic partition of the non-negative real numbers, $|P| = \max_{n} (\tau_n - \tau_{n-1})$ is called the mesh of $P$ and the limit is defined using convergence in probability. Note that $[x]_t$\(^1\)

\(^1\)In the first arXiv version of this paper, we incorrectly claimed the assumption as a lemma
is non-decreasing with \( t \) and \( [x, y]_t \) can be defined via polarization as
\[
[x, y]_t = \frac{1}{4} ([x + y]_t - [x - y]_t).
\]

For example, if the processes \( x_t \) and \( y_t \) satisfy the SDEs
\[
\begin{align*}
dx_t &= \mu(x_t)dt + \sigma(x_t)dB_t, \\
dy_t &= \nu(y_t)dt + \eta(y_t)dB_t,
\end{align*}
\]
where \( B_t \) is a Brownian motion, we have that \([x]_t = \int_0^t \sigma^2(s)ds\) and \([x, y]_t = \int_0^t \sigma(s)\eta(s)ds\) and \( [x, y]_t = \sigma(x_t)\eta(y_t)dt\); for a vector-valued SDE
\[
dx_t = \mu(x_t)dt + \Sigma(x_t)dB_t
\]
and \( dy_t = \nu(y_t)dt + M(y_t)dB_t \), we have that
\[
[x^i, x^j]_t = \int_0^t (\Sigma(x_s)\Sigma^T(x_s))_{ij}ds
\]
and \( [x^i, y^j]_t = \int_0^t (\Sigma(x_s)M^T(y_s))_{ij}ds\).

**Lemma 10** (Itô’s formula). Let \( x \) be a semimartingale and \( f \) be twice continuously differentiable function, then
\[
df(x_t) = \sum_i \frac{df(x_t)}{dx^i} dx^i + \frac{1}{2} \sum_{i,j} \frac{d^2f(x_t)}{dx^i dx^j} d[x^i, x^j]_t.
\]

The next two lemmas are well-known facts about Wiener processes; first the reflection principle.

**Lemma 11** (Reflection principle). Given a Wiener process \( W(t) \) and \( a, t \geq 0 \), then we have that
\[
\mathbb{P}( \sup_{0 \leq s \leq t} W(s) \geq a) = 2\mathbb{P}(W(t) \geq a).
\]

Second, a decomposition lemma for continuous martingales.

**Theorem 12** (Dambis, Dubins-Schwarz theorem). Every continuous local martingale \( M_t \) is of the form
\[
M_t = M_0 + W_{[M]_t}
\]
for all \( t \geq 0 \)
where \( W_s \) is a Wiener process.

### 2.2 Logconcave functions

**Lemma 13** (Dinghas; Prékopa; Leindler). The convolution of two logconcave functions is also logconcave; in particular, any linear transformation or any marginal of a logconcave density is logconcave.

The next lemma about logconcave densities is folklore, see e.g., [28].

**Lemma 14** (Logconcave moments). Given a logconcave density \( p \) in \( \mathbb{R}^n \), and any positive integer \( k \),
\[
\mathbb{E}_{x \sim p} \|x\|^k \leq (2k)^k \left( \mathbb{E}_{x \sim p} \|x\|^2 \right)^{k/2}.
\]

The following elementary concentration lemma is also well-known (this version is from [28]).

**Lemma 15** (Logconcave concentration). For any isotropic logconcave density \( p \) in \( \mathbb{R}^n \), and any \( t > 0 \),
\[
\mathbb{P}_{x \sim p} \left( \|x\| > t\sqrt{n} \right) \leq e^{-t^2/2}.
\]

A much stronger concentration bound was shown by Paouris [32].

**Lemma 16** ([32]). For any isotropic logconcave distribution and any \( t > C \),
\[
\mathbb{P}_{x \sim p} \left( \|x\| \geq t\sqrt{n} \right) \leq e^{-ct\sqrt{n}}.
\]

The following inequality bounding the small ball probability is from [3].

**Lemma 17** ([3, Thm. 10.4.7]). For any isotropic logconcave density \( p \), and any \( \epsilon < \epsilon_0 \),
\[
\mathbb{P}_{x \sim p} \left( \|x\| \leq \epsilon\sqrt{n} \right) \leq e^{\epsilon^2\sqrt{n}}
\]
were \( \epsilon_0, c \) are absolute constants.

**Definition 18.** We define \( \psi_n \) as the supremum of the KLS constant over all isotropic logconcave distributions in \( \mathbb{R}^n \).
The next lemma follows from the fact that the Poincaré constant is bounded by the KLS constant and Lemma 13.

**Lemma 19.** For any matrix $A$, and any isotropic logconcave density $p$,

$$
\text{Var}_{x \sim p} (x^T Ax) \leq O(\psi_r^2) E_{x \sim p} \left( \|Ax\|^2 \right)
$$

where $r = \text{rank}(A + A^T)$.

To prove a lower bound on the expansion, it suffices to consider subsets of measure $1/2$. This follows from the concavity of the isoperimetric profile. We quote a theorem from [30, Thm 1.8], which applies even more generally to Riemannian manifolds under suitable convexity-type assumptions.

**Theorem 20.** The Cheeger constant of any logconcave density is achieved by a subset of measure $1/2$.

### 2.3 Matrix inequalities

For any symmetric matrix $B$, we define $|B| = \sqrt{B^2}$, namely, the matrix formed by taking absolute value of all eigenvalues of $B$. For any matrix $A$, we define $R(A)$ to be the span of the rows of $A$ and $N(A)$ to be the null space of $A$. For any vector $x$ and any positive semi-definite matrix $A$, we define $\|x\|_A^2 = x^T Ax$.

**Lemma 21** (Matrix Hölder inequality). Given a symmetric matrices $A$ and $B$ and any $s, t \geq 1$ with $s^{-1} + t^{-1} = 1$, we have

$$
\text{Tr}(AB) \leq (\text{Tr}|A|^s)^{1/s} \left( \text{Tr}|B|^t \right)^{1/t}.
$$

**Lemma 22** (Lieb-Thirring Inequality [26]). Given positive semi-definite matrices $A$ and $B$ and $r \geq 1$, we have

$$
\text{Tr}((B^{1/2} AB^{1/2})^r) \leq \text{Tr}(B^{r/2} A^{1/2} B^{r/2}).
$$

Since the following lemma is stated differently in [14, 1], we show the proof from Eldan [14] here for completeness.

**Lemma 23** ([14, 1]). Given a symmetric matrix $B$, a positive semi-definite matrix $A$ and $\alpha \in [0, 1]$, we have

$$
\text{Tr}(A^\alpha BA^{1-\alpha}B) \leq \text{Tr}(AB^2).
$$

**Proof.** Without loss of generality, we can assume $A$ is diagonal. Hence, we have that

$$
\text{Tr}(A^\alpha BA^{1-\alpha}B) = \sum_{i,j} A_{ii}^\alpha A_{jj}^{1-\alpha} B_{ij}^2 \\
\leq \sum_{i,j} (\alpha A_{ii} + (1-\alpha)A_{jj}) B_{ij}^2 \\
= \alpha \sum_{i,j} A_{ii} B_{ij}^2 + (1-\alpha) \sum_{i,j} A_{jj} B_{ij}^2 = \text{Tr}(AB^2).
$$

### 3 Eldan’s stochastic localization

In this section, we consider the stochastic localization scheme introduced in [14] in slightly more general terms. In discrete localization, the idea would be to restrict the distribution with a random halfspace and repeat this process. In stochastic localization, this discrete step is replaced by infinitesimal steps, each of which is a renormalization with a linear function in a random direction. One might view this informally as an averaging over infinitesimal needles. The discrete time equivalent would be $p_{t+1}(x) = p_t(x)(1 + \sqrt{h}(x - \mu_t)^T w)$ for a sufficiently small $h$ and random Gaussian vector $w$. Using the approximation $1 + y \sim e^{y-\frac{1}{2}y^2}$, we see that over time this process introduces a negative quadratic factor in the exponent, which will be the Gaussian factor. As time tends to $\infty$, the distribution tends to a more and more concentrated Gaussian and eventually a delta function, at which point any subset has measure either 0 or 1. The idea of the proof is to stop at a time that is large enough to have a strong Gaussian factor in the density, but small enough to ensure that the measure of a set is not changed by more than a constant.
3.1 The process and its basic properties

Given a distribution with logconcave density \( p(x) \), we start at time \( t = 0 \) with this distribution and at each time \( t > 0 \), we apply an infinitesimal change to the density. This is done by picking a random direction from a Gaussian with a certain covariance matrix \( C_t \), called the control matrix. In Section 4 and 6, we use this process with \( C_t = I \).

In Section 7, we use a varying \( C_t \) to get a bound for non-isotropic distributions.

In order to construct the stochastic process, we assume that the support of \( p \) is contained in a ball of radius \( R > n \). There is only exponentially small probability outside this ball, at most \( e^{-cR} \) by Lemma 16. Moreover, since by Theorem 20, we only need to consider subsets of measure \( 1/2 \), this truncation does not affect the KLS constant of the distribution.

**Definition 24.** Given a logconcave distribution \( p \), we define the following stochastic differential equation:

\[
\begin{align*}
c_0 &= 0, & dc_t &= C_t^{1/2}dW_t + C_t\mu_t dt, \\
B_0 &= 0, & dB_t &= C_t dt,
\end{align*}
\]

where the probability distribution \( p_t \), the mean \( \mu_t \) and the covariance \( A_t \) are defined by

\[
p_t(x) = \frac{e^{c^T x - \frac{1}{2} \|x\|^2}}{\int_{\mathbb{R}^n} e^{c^T y - \frac{1}{2} \|y\|^2} p(y) dy}, \quad \mu_t = \mathbb{E}_{x \sim p_t} x, \quad A_t = \mathbb{E}_{x \sim p_t} (x - \mu_t)(x - \mu_t)^T,
\]

and the control matrices \( C_t \) are symmetric matrices to be specified later.

In Section 4 and 6, we only consider the process with \( C_t = I \) for all \( t \geq 0 \). In this case, we have that

\[
B_t = tI, \quad p_t(x) = \frac{e^{c^T x - \frac{1}{2} \|x\|^2}}{\int_{\mathbb{R}^n} e^{c^T y - \frac{1}{2} \|y\|^2} p(y) dy}.
\]

Also, since \( \mu_t \) is a bounded function that is Lipschitz with respect to \( c \) and hence standard existence and uniqueness theorems (e.g. [31, Sec 5.2]) show the existence and uniqueness of the solution on time \([0, T]\) for any \( T > 0 \). In general, we have the following result:

**Lemma 25 (Existence and Uniqueness).** If \( p(x) \) has compact support and if \( C_t^{1/2} \) are bounded and Lipschitz functions of \( A_t \) and \( B_t \), the stochastic differential equation (3.1) has a unique solution.

We defer all proofs for statements in this section, considered standard in stochastic calculus, to Section 5. Now we proceed to analyzing the process and how its parameters evolve. Roughly speaking, the first lemma below says that the stochastic process is the same as continuously multiplying \( p_t(x) \) by a random infinitesimally small linear function.

**Lemma 26 ([14, Lem 2.1]).** We have that

\[
dp_t(x) = (x - \mu_t)^T C_t^{1/2} dW_t p_t(x)
\]

for any \( x \in \mathbb{R}^n \).

By considering the derivative \( d \log p_t(x) \), we see that applying \( dp_t(x) \) as in the lemma above results in the distribution \( p_t(x) \), with the Gaussian term in the density:

\[
d \log p_t(x) = \frac{dp_t(x)}{p_t(x)} - \frac{1}{2} \frac{d[p_t(x)]}{p_t(x)^2}
\]

\[
= (x - \mu_t)^T C_t^{1/2} dW_t - \frac{1}{2} (x - \mu_t)^T C_t (x - \mu_t) dt
\]

\[
= x^T \left( C_t^{1/2} dW_t + C_t \mu_t dt \right) - \frac{1}{2} x^T C_t x dt - (\mu_t^T C_t^{1/2} dW_t + \frac{1}{2} \mu_t^T C_t \mu_t dt)
\]

\[
= x^T dc_t - \frac{1}{2} x^T dB_t x dt + g(t)
\]

where the last term is independent of \( x \) and the first two terms explain the form of \( p_t(x) \) and the appearance of the Gaussian.

Next we analyze the change of the covariance matrix.

**Lemma 27 ([14]).** We have that

\[
dA_t = \int_{\mathbb{R}^n} (x - \mu_t)(x - \mu_t)^T \left( (x - \mu_t)^T C_t^{1/2} dW_t \right) p_t(x) dx - A_t C_t A_t dt.
\]
3.2 Bounding expansion

Our plan is to bound the expansion by the spectral norm of the covariance matrix at time $t$. First, we bound the measure of a set of initial measure $\frac{1}{2}$.

**Lemma 28.** For any set $E \subset \mathbb{R}^n$ with $\int_E p(x)dx = \frac{1}{2}$ and $t \geq 0$, we have that

$$P\left(\frac{1}{4} \leq \int_E p_t(x)dx \leq \frac{3}{4}\right) \geq \frac{9}{10} - P\left(\int_0^t \left\| C_s^{1/2}A_sC_s^{1/2} \right\|_2 ds \geq \frac{1}{64}\right).$$

**Proof.** Let $g_t = \int_E p_t(x)dx$. Then, we have that

$$dg_t = \int_E (x - \mu_t)^T C_t^{1/2} dW_t p_t(x)dx$$

$$= \left(\int_E (x - \mu_t)p_t(x)dx, C_t^{1/2} dW_t \right)$$

where the integral might not be 0 because it is over the subset $E$ and not all of $\mathbb{R}^n$. Hence, we have,

$$d[g_t] = \left\| \int_E C_t^{1/2} (x - \mu_t)p_t(x)dx \right\|_2^2 dt$$

$$= \max_{\|\zeta\|_2 \leq 1} \left(\int_E \zeta^T C_t^{1/2} (x - \mu_t)p_t(x)dx \right)^2 dt$$

$$\leq \max_{\|\zeta\|_2 \leq 1} \int_{\mathbb{R}^n} \left(\zeta^T C_t^{1/2} (x - \mu_t)\right)^2 p_t(x)dx \int_{\mathbb{R}^n} p_t(x)dx dt$$

$$= \max_{\|\zeta\|_2 \leq 1} \left(\zeta^T C_t^{1/2} A_t C_t^{1/2} \zeta \right) dt$$

$$= \left\| C_t^{1/2} A_t C_t^{1/2} \right\|_2^2 dt.$$

Hence, we have that $\frac{dg_t}{dt} \leq \left\| C_t^{1/2} A_t C_t^{1/2} \right\|_2$. By the Dambis, Dubins-Schwarz theorem, there exists a Wiener process $\tilde{W}_t$ such that $g_t - g_0$ has the same distribution as $\tilde{W}_{[g_t]}$. Using $g_0 = \frac{1}{4}$, we have that

$$P\left(\frac{1}{4} \leq g_t \leq \frac{3}{4}\right) = P\left(\frac{-1}{4} \leq \tilde{W}_{[g_t]} \leq \frac{1}{4}\right)$$

$$\geq P\left(\min_{0 \leq s \leq \frac{t}{4}} \left| \tilde{W}_s \right| \leq \frac{1}{4} \text{ and } \left| g_t \right| \leq \frac{1}{64}\right)$$

$$\geq 1 - P\left(\max_{0 \leq s \leq \frac{t}{4}} \left| \tilde{W}_s \right| \geq \frac{1}{4}\right) - P\left(\left| g_t \right| > \frac{1}{64}\right)$$

$$\geq 1 - 4P\left(\tilde{W}_{\frac{t}{4}} > \frac{1}{4}\right) - P\left(\left| g_t \right| > \frac{1}{64}\right)$$

$$\geq \frac{9}{10} - P\left(\left| g_t \right| > \frac{1}{64}\right),$$

where we used reflection principle for 1-dimensional Brownian motion in $\mathbb{R}$ and the concentration of normal distribution in $\mathbb{R}$, namely $P_{x \sim \mathcal{N}(0,1)}(x > 2) \leq 0.0228$. $\square$

**Theorem 29 (Brascamp-Lieb [9]).** Let $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}^+$ be the standard Gaussian density in $\mathbb{R}^n$. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^+$ be any logconcave function. Define the density function $h$ as follows:

$$h(x) = \frac{f(x)\gamma(x)}{\int_{\mathbb{R}^n} f(y)\gamma(y) dy}.$$

Fix a unit vector $v \in \mathbb{R}^n$, let $\mu = E_h(x)$. Then, for any $\alpha \geq 1$, $E_h(|v^T(x - \mu)|^\alpha) \leq E_\gamma(|v^Tx|^\alpha)$.

Using this we derive the following well-known isoperimetric inequality that was proved in [12] and was also used in [14].

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Theorem 30 ([12, Thm. 4.4]). Let \( h(x) = f(x)e^{-\frac{1}{2}x^T B x} / \int f(y)e^{-\frac{1}{2}y^T B y} dy \) where \( f : \mathbb{R}^n \to \mathbb{R}_+ \) is an integrable logconcave function and \( B \) is positive definite. Then \( h \) is logconcave and for any measurable subset \( S \) of \( \mathbb{R}^n \),

\[
\int_{\partial S} h(x) dx = \Omega \left( \|B^{-1}\|_{2}^{-\frac{1}{2}} \right) \min \left\{ \int_{S} h(x) dx, \int_{\mathbb{R}^n \setminus S} h(x) dx \right\}.
\]

In other words, the expansion of \( h \) is \( \Omega \left( \|B^{-1}\|_{2}^{-\frac{1}{2}} \right) \).

Proof. The proof uses the localization lemma to reduce the statement to a 1-dimensional statement about a Gaussian times a logconcave density, where the Gaussian is a projection of the Gaussian \( N (0, B^{-1}) \) (but the logconcave function might be different as the limit of localization is the original function along an interval times an exponential function). We then apply the Brascamp-Lieb inequality in one dimension (Theorem 29) to prove that for the resulting one-dimensional distribution, the variance is at most that of the Gaussian, therefore at most \( \|B^{-1}\| \). The isoperimetric constant is bounded by the inverse of the standard deviation times a constant. The complete proof, in more general terms, is carried out in [12, Thm. 4.4].

We can now prove a bound on the expansion.

Lemma 31. Given a logconcave distribution \( p \). Let \( A_t \) be defined by Definition 24 using initial distribution \( p \). Suppose that there is \( T > 0 \) such that

\[
\mathbb{P} \left( \int_0^T \|C_s^{1/2} A_s C_s^{1/2}\|_2 \, ds \leq \frac{1}{64} \text{ and } B_T \succeq u I \right) \geq \frac{3}{4}
\]

Then, we have that \( \psi_p = \Omega \left( u^{-1/2} \right) \).

Proof. By Milman’s theorem [30], it suffices to consider subsets of measure \( \frac{1}{2} \). Consider any measurable subset \( E \) of \( \mathbb{R}^n \) of initial measure \( \frac{1}{2} \). By Lemma 26, \( p_t \) is a martingale and therefore

\[
\int_{\partial E} p(x) dx = \int_{\partial E} p_0(x) dx = \mathbb{E} \left( \int_{\partial E} p_t(x) dx \right).
\]

Next, by the definition of \( p_T \) (3.1), we have that \( p_T(x) \propto e^{c_T x - \frac{1}{2} x^T B_T x} p(x) \) and Theorem 30 shows that the expansion of \( E \) is \( \Omega \left( \lambda_{\min}(B_T)^{-1/2} \right) \). Hence, we have

\[
\int_{\partial E} p(x) dx = \mathbb{E} \int_{\partial E} p_T(x) dx
\]

\[
= \Omega(u^{-1/2}) \mathbb{E} \left( 1_{B_T \succeq u I} \min \left( \int_E p_T(x) dx, \int_E p_T(x) dx \right) \right)
\]

\[
\geq \Omega(u^{-1/2}) \mathbb{P} \left( B_T \succeq u I \text{ and } \frac{1}{4} \leq \int_E p_T(x) dx \leq \frac{3}{4} \right)
\]

\[
= \Omega(u^{-1/2}) \left( \mathbb{P} \left( \frac{1}{4} \leq \int_E p_T(x) dx \leq \frac{3}{4} \right) - \mathbb{P}(B_T \succeq u I \text{ is false}) \right)
\]

\[
\geq \Omega(u^{-1/2}) \left( \frac{9}{10} - \mathbb{P} \left( \int_0^t \|C_s^{1/2} A_s C_s^{1/2}\|_2 \, ds \geq \frac{1}{64} \right) - \mathbb{P}(B_T \succeq u I \text{ is false}) \right) \quad \text{(Lem 28)}
\]

\[
= \Omega(u^{-1/2})
\]

where we used the assumption at the end. Using Theorem 20, this shows that \( \psi_p = \Omega \left( u^{-1/2} \right) \).

\[
\square
\]

4 Controlling \( A_t \) via the potential \( \text{Tr}(A_t^2) \)

In this section, we only use \( C_t = I \) for the control matrix.
4.1 Third moment bounds
Here are two key lemmas about the third-order tensor of a log-concave distribution. A special case of the first inequality was used in [14]. For our main theorem, we only the first lemma with $B = I$, but we need the general case for the proof in Section 7.

Lemma 32. Given a log-concave distribution $p$ with mean $\mu$ and covariance $A$, we have that
\[
\left\| \mathbb{E}_{x \sim p} B^{1/2}(x - \mu)^T C(x - \mu) \right\|_2 = O \left( \left\| A^{1/2} B A^{1/2} \right\|_2^{1/2} \text{Tr} \left( A^{1/2} C A^{1/2} \right) \right).
\]

Proof. We first consider the case $C = vv^T$. Taking $y = A^{-1/2}(x - \mu)$ and $w = A^{1/2}v$. Then, $y$ follows an isotropic log concave distribution $\tilde{p}$ and the statement becomes
\[
\left\| \mathbb{E}_{y \sim \tilde{p}} B^{1/2} A^{1/2} y (y^T w)^2 \right\|_2 = O \left( \left\| A^{1/2} B A^{1/2} \right\|_2^{1/2} \left\| w \right\|_2^2 \right).
\]
Then, we calculate that
\[
\left\| \mathbb{E}_{y \sim \tilde{p}} B^{1/2} A^{1/2} y (y^T w)^2 \right\|_2 = \max_{\|v\|_2 \leq 1} \mathbb{E}_{y \sim \tilde{p}} (B^{1/2} A^{1/2} y)^T \zeta (y^T w)^2
\]
\[
\leq \max_{\|v\|_2 \leq 1} \sqrt{\mathbb{E}_{y \sim \tilde{p}} ((B^{1/2} A^{1/2} y)^T \zeta)^2} \sqrt{\mathbb{E}_{y \sim \tilde{p}} (y^T w)^4}
\]
\[
= O \left( \sqrt{\left\| A^{1/2} B A^{1/2} \right\|_2} \left\| w \right\|_2 \right)
\]
where we used the fact that for a fixed $w$, $y^T w$ has a one-dimensional logconcave distribution (Lemma 13) and hence Lemma 14 shows that
\[
\mathbb{E}_{y \sim \tilde{p}} (y^T w)^4 = O(1) \left( \mathbb{E}_{y \sim \tilde{p}} (y^T w)^2 \right)^2 = O(\|w\|_2^4).
\]

For a general symmetric matrix $C$, we write $C = \sum \lambda_i v_i v_i^T$ where $\lambda_i$, $v_i$ are eigenvalues and eigenvectors of $C$. Hence, we have that
\[
\left\| \mathbb{E}_{x \sim p} B^{1/2}(x - \mu)^T C(x - \mu) \right\|_2 \leq \sum_i |\lambda_i| \left\| \mathbb{E}_{x \sim p} B^{1/2}(x - \mu)^T v_i v_i^T (x - \mu) \right\|_2
\]
\[
\leq O(1) \sum_i |\lambda_i| \left\| A^{1/2} B A^{1/2} \right\|_2^{1/2} \left\| A^{1/2} v_i \right\|^2
\]
\[
= O(1) \left\| A^{1/2} B A^{1/2} \right\|_2^{1/2} \sum_i \text{Tr} \left( A^{1/2} |\lambda_i| v_i v_i^T A^{1/2} \right)
\]
\[
= O(1) \left\| A^{1/2} B A^{1/2} \right\|_2^{1/2} \text{Tr} \left( A^{1/2} C A^{1/2} \right).
\]

\[
\square
\]

Lemma 33. Given a log-concave distribution $p$ with mean $\mu$ and covariance $A$. We have
\[
\mathbb{E}_{x,y \sim p} |\langle x, y, y - \mu \rangle|^3 = O \left( \text{Tr} \left( A^2 \right)^{3/2} \right).
\]

Proof. Without loss of generality, we assume $\mu = 0$. For a fixed $x$ and random $y$, $\langle x, y \rangle$ follows a one-dimensional logconcave distribution (Lemma 13) and hence Lemma 14 shows that
\[
\mathbb{E}_{y \sim p} |\langle x, y \rangle|^3 \leq O(1) \left( \mathbb{E}_{y \sim p} (x, y)^2 \right)^{3/2} = O \left( x^T A x \right)^{3/2}.
\]
Next, we note that $A^{1/2} x$ follows a logconcave distribution (Lemma 13) and hence Lemma 14 shows that
\[
\mathbb{E}_{x,y \sim p} |\langle x, y \rangle|^3 = O(1) \mathbb{E}_{x \sim p} \left\| A^{1/2} x \right\|^3 \leq O(1) \left( \mathbb{E}_{x \sim p} \left\| A^{1/2} x \right\|^2 \right)^{3/2} = O \left( \text{Tr} \left( A^2 \right)^{3/2} \right).
\]

\[
\square
\]
4.2 Analysis of $A_t$

Using Itô's formula and Lemma 27, one can compute the derivatives of $\text{Tr} A_t^2$. Since a similar calculation appears in Sections 4, 6 and 7, we prove a common generalization in Lemma 36.

Lemma 34. Let $A_t$ be defined by Definition 24. We have that

$$d\text{Tr} A_t^2 = 2\text{E}_{x \sim p_t}(x - \mu_t)^T A_t(x - \mu_t) (x - \mu_t)^T dW_t - 2\text{Tr}(A_t^3)dt + \text{E}_{x,y \sim p_t}((x - \mu_t)^T(y - \mu_t))^3 dt.$$ 

Lemma 35. Given a logconcave distribution $p$ with covariance matrix $A$ s.t. $\text{Tr} A^2 = n$. Let $A_t$ defined by Definition 24 using initial distribution $p$. There is a universal constant $c_1$ such that

$$\mathbb{P}(\max_{t \in [0,T]} \text{Tr} (A_t^2) \geq 8n) \leq 0.01 \text{ with } T = \frac{c_1}{\sqrt{n}}.$$ 

Proof. Let $\Phi_t = \text{Tr} A_t^2$. By Lemma 34, we have that

$$d\Phi_t = -2\text{Tr}(A_t^3)dt + \text{E}_{x,y \sim p_t}((x - \mu_t)^T(y - \mu_t))^3 dt + 2\text{E}_{x,y \sim p_t}((x - \mu_t)^T A_t(x - \mu_t)(x - \mu_t)^T dW_t$$

$$\overset{\text{def}}{=} \delta_t dt + v_t^T dW_t. \tag{4.1}$$

For the drift term $\delta_t dt$, Lemma 33 shows that

$$\delta_t \leq \text{E}_{x,y \sim p_t}((x - \mu_t)^T(y - \mu_t))^3 = O\left(\text{Tr} (A_t^2)^{3/2}\right) \leq C' \Phi_t^{3/2} \tag{4.2}$$

for some universal constant $C'$. Note that we dropped the term $-2\text{Tr}(A_t^3)$ since $A_t$ is positive semidefinite and therefore the term is negative.

For the martingale term $v_t^T dW_t$, we note that

$$\|v_t\|_2 = \|\text{E}_{x,y \sim p_t}(x - \mu_t)^T A_t (x - \mu_t)(x - \mu_t)\|_2$$

$$\leq \|A_t\|_2^{1/2} \text{Tr} |A_t^2|$$

$$\leq O(\Phi_t^{5/4}). \tag{Lem 32}$$

So the drift term grows roughly as $\Phi_t^{3/2}$ while the stochastic term grows as $\Phi_t^{5/4} \sqrt{t}$. Thus, both bounds (on the drift term and the stochastic term) suggest that for $t$ up to $O\left(\frac{1}{\sqrt{n}}\right)$, the potential $\Phi_t$ remains $O(n)$. We now formalize this, by decoupling the two terms.

Let

$$f(a) = -\frac{1}{\sqrt{a+n}}.$$ 

By (4.1) and Itô’s formula, we have that

$$df(\Phi_t) = f'(\Phi_t) d\Phi_t + \frac{1}{2} f''(\Phi_t) d[\Phi]_t$$

$$= \left(\frac{\delta_t}{2(\Phi_t+n)^{3/2}} - \frac{3}{8(\Phi_t+n)^{5/2}} \right) dt + \frac{1}{2} v_t^T dW_t$$

$$\leq C' dt + dY_t \tag{4.3}$$

where $dY_t = \frac{1}{2} \frac{v_t^T dW_t}{(\Phi_t+n)^{3/2}}$, $Y_t = 0$ and $C'$ is the universal constant in (4.2).

Note that

$$\frac{d[Y]_t}{dt} = \frac{1}{4(\Phi_t+n)^3} = O(1) \frac{\Phi_t^{5/2}}{(\Phi_t+n)^3} \leq \frac{C}{\sqrt{n}}$$

By Theorem 12, there exists a Wiener process $\tilde{W}_t$ such that $Y_t$ has the same distribution as $\tilde{W}[Y]_t$. Using the reflection principle for 1-dimensional Brownian motion, we have that

$$\mathbb{P}(\max_{t \in [0,T]} Y_t \geq \gamma) \leq \mathbb{P}(\max_{t \in [0,\frac{C}{\sqrt{n}} T]} \tilde{W}_t \geq \gamma) = 2\mathbb{P}(\tilde{W}[\gamma/\sqrt{n}]_T \geq \gamma) \leq 2 \exp(-\frac{\gamma^2 \sqrt{n}}{2CT}).$$
Since $\Phi_0 = \|A_0\|_P^2 = n$, we have that $f(\Phi_0) = -\frac{1}{\sqrt{2n}}$ and therefore (4.3) shows that
\[\mathbb{P}(\max_{t \in [0,T]} f(\Phi_t) \geq -\frac{1}{\sqrt{2n}} + C'T + \gamma) \leq 2 \exp(-\frac{\gamma^2}{36n}).\]

Putting $T = \frac{1}{256(C'T + C)\sqrt{n}}$ and $\gamma = \frac{1}{4\sqrt{n}}$, we have that
\[\mathbb{P}(\max_{t \in [0,T]} f(\Phi_t) \geq -\frac{1}{3\sqrt{n}}) \leq 2 \exp(-8)).\]

Note that $f(\Phi_t) \geq -\frac{1}{3\sqrt{n}}$ implies that $\Phi_t \geq 8n$. Hence, we have that
\[\mathbb{P}(\max_{t \in [0,T]} \Phi_t \geq 8n) \leq 0.01.\]

\[\square\]

4.3 Proof of Theorem 7

Proof of Theorem 7. By rescaling, we can assume $\|A_2\|_2 = n$. By Lemma 35, we have that
\[\mathbb{P}(\max_{s \in [0,t]} \|A_2^s\|_2 \leq 8n) \geq 0.99 \quad \text{with} \quad t = \frac{c_1}{\sqrt{n}}.\]

Since $\|A_2^s\|_2 \leq 8n$ implies that $\|A_2\|_2 \leq \sqrt{8n}$, we have that
\[\mathbb{P}(\int_0^T \|A_s\|_2 ds \leq \frac{1}{64} \text{ and } B_T \geq T \cdot I) \geq 0.99\]

where $T = \min \left\{\frac{1}{64\sqrt{8}}, c_1\right\} / \sqrt{n}$. Now the theorem follows from Lemma 31.

\[\square\]

5 Localization proofs

We begin with the proof of existence of a unique solution for the SDE.

Proof of Lemma (25). We can write the stochastic differential equation as
\[dc_t = C^{1/2}(A_t, B_t)dW_t + C(A_t, B_t)\mu(c_t, B_t)dt\]
and $dB_t = C(A_t, B_t)dt$ where
\[\mu(c, B) = \int_{\mathbb{R}^n} xq(c, B, x)dx \quad \text{and} \quad A_t(c, B) = \int_{\mathbb{R}^n} (x - \mu(c, B))(x - \mu(c, B))^Tq(c, B, x)dx\]
and
\[q(c, B, x) = \frac{e^{c^T x - \frac{1}{2} \|x\|^2} p(x)}{\int_{\mathbb{R}^n} e^{c^T y - \frac{1}{2} \|y\|^2} p(y)dy}.\]

Since $p$ has compact support, we have that $q$ is Lipschitz in $c$ and $B$ variables, so are the functions $\mu$ and $A$. Next, we note that both $\mu$ and $A$ are bounded since $p$ has compact support. Since $C^{1/2}$ is bounded and Lipschitz function in $c$ and $B$ variables, so is $C$. Therefore, we can use a standard existence and uniqueness theorem (e.g. [31, Sec 5.2]) to show the existence and uniqueness of the solution on time $[0, T]$ for any $T > 0$.

\[\square\]

Next is the proof of the infinitesimal change in the density.

Proof of Lemma (26). Let $q_t(x) = e^{c^T x - \frac{1}{2} \|x\|^2} p(x)$. By Itô’s formula, applied to $f(c, B) \equiv e^{c^T x - \frac{1}{2} \|x\|^2} p(x)$, we have that
\[dq_t(x) = \left( dc^T_t x - \frac{1}{2} (dB_t x, x) + \frac{1}{2} \left( d[c^T_t x]_t - \frac{1}{2} d[(B_t x)_t]_t \right) \right) q_t(x)\]

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Note that
\[ dc^T_t x = \left< C_t^{1/2} dW_t + C_t \mu_t dt, x \right> . \]
Hence, the quadratic variations of \( c^T_t x \) is
\[ d[c^T_t x]_t = \langle C_t x, x \rangle dt. \]
Also, \( dB_t \) is a predictable process (namely, does not have a stochastic term) and hence \( d[(B_t x, x)]_t = 0 \). Therefore, this gives
\[ dq_t(x) = \left< C_t^{1/2} dW_t + C_t \mu_t dt, x \right> q_t(x). \] (5.1)

Let \( V_t = \int_{\mathbb{R}^n} q_t(y)dy \). Then, we have
\[
\begin{align*}
  dV_t &= \int_{\mathbb{R}^n} dq_t(y)dy \\
  &= \int_{\mathbb{R}^n} \left< C_t^{1/2} dW_t + C_t \mu_t dt, y \right> q_t(y)dy \\
  &= V_t \left< C_t^{1/2} dW_t + C_t \mu_t dt, \mu_t \right> .
\end{align*}
\]

By Itô’s formula, we have that
\[
\begin{align*}
  dV_t^{-1} &= -\frac{1}{V_t^2} dV_t + \frac{1}{V_t^3} d[V]_t \\
  &= -V_t^{-1} \left< C_t^{1/2} dW_t + C_t \mu_t dt, \mu_t \right> + V_t^{-1} \langle C_t \mu_t, \mu_t \rangle dt \\
  &= -V_t^{-1} \left< C_t^{1/2} dW_t, x - \mu_t \right> .
\end{align*}
\] (5.2)

Combining (5.1) and (5.2), we have that
\[
\begin{align*}
  dp_t(x) &= d(V_t^{-1} q_t(x)) \\
  &= q_t(x) dV_t^{-1} + V_t^{-1} dq_t(x) + d[V_t^{-1}, q_t(x)]_t \\
  &= -q_t(x) V_t^{-1} \left< C_t^{1/2} dW_t, \mu_t \right> + V_t^{-1} \left< C_t^{1/2} dW_t + C_t \mu_t dt, x \right> q_t(x) - V_t^{-1} \left< C_t^{1/2} \mu_t, C_t^{1/2} x \right> q_t(x) dt \\
  &= p_t(x) \left< C_t^{1/2} dW_t, x - \mu_t \right> .
\end{align*}
\]

The next proof is for the change in the covariance matrix.

Proof of Lemma (27). Recall that
\[ A_t = \int_{\mathbb{R}^n} (x - \mu_t)(x - \mu_t)^T p_t(x)dx. \]
Viewing \( A_t = f(\mu_t, p_t) \), i.e., as a function of the variables \( \mu_t \) and \( p_t \), we apply Itô’s formula. In the derivation below, we use \([\mu_t, \mu_t^T]_t\) to denote the matrix whose \( i, j \) coordinate is \([\mu_t, i, \mu_t, j]_t\). Similarly, \([\mu_t, p_t(x)]_t\) is a column vector and \([\mu_t^T, p_t(x)]_t\) is a row vector.
\[
\begin{align*}
  dA_t &= \int_{\mathbb{R}^n} (x - \mu_t)(x - \mu_t)^T dp_t(x)dx \\
  &\quad - \int_{\mathbb{R}^n} d[\mu_t(x - \mu_t)^T] p_t(x)dx - \int_{\mathbb{R}^n} (x - \mu_t)(d[\mu_t])^T p_t(x)dx \\
  &\quad - \frac{1}{2} \cdot 2 \int_{\mathbb{R}^n} (x - \mu_t)d[\mu_t^T, p_t(x)]_t dx - \frac{1}{2} \cdot 2 \int_{\mathbb{R}^n} d[\mu_t, p_t(x)]_t (x - \mu_t)^T dx \\
  &\quad + \frac{1}{2} \cdot 2d[\mu_t, \mu_t^T]_t \int_{\mathbb{R}^n} p_t(x)dx.
\end{align*}
\]
Similarly, the third term also vanishes:

\[ \int_{\mathbb{R}^n} (x - \mu_t)(d\mu_t)^T p_t(x) \, dx = 0. \]

To compute the last 3 terms, we note that

\[
d\mu_t = d \int_{\mathbb{R}^n} xp_t(x) \, dx
= \int_{\mathbb{R}^n} x(x - \mu_t)^T C_t^{1/2} dW_t p_t(x) \, dx
= \int_{\mathbb{R}^n} (x - \mu_t)(x - \mu_t)^T C_t^{1/2} dW_t p_t(x) \, dx + \int_{\mathbb{R}^n} \mu_t(x - \mu_t)^T C_t^{1/2} dW_t p_t(x) \, dx
= A_t C_t^{1/2} dW_t.
\]

Therefore, we have for the last term

\[
(\mathbf{d}[\mu_t, \mu_t^T]|_t)_{ij} = \sum_{\ell} \left( A_t C_t^{1/2} \right)_{i\ell} \left( A_t C_t^{1/2} \right)_{\ell j} dt = (A_t C_t^{1/2} (C_t^{1/2})^T A_t^T)_{ij} dt = (A_t C_t A_t)_{ij} dt
\]

which we can simply write as \( \mathbf{d}[\mu_t, \mu_t^T]|_t = A_t C_t A_t \). Similarly, we have

\[
d[\mu_t, p_t(x)|_t = p_t(x)A_t C_t (x - \mu_t) dt.
\]

This gives the fourth term

\[
\int_{\mathbb{R}^n} (x - \mu_t) d[\mu_t^T, p_t(x)|_t \, dx = \int_{\mathbb{R}^n} (x - \mu_t)(x - \mu_t)^T C_t A_t p_t(x) dt \, dx = A_t C_t A_t dt.
\]

Similarly, we have the fifth term \( \int_{\mathbb{R}^n} d[\mu_t, p_t(x)|_t (x - \mu_t)^T \, dx = A_t C_t A_t dt \). Combining all the terms, we have that

\[
A_t = \int_{\mathbb{R}^n} (x - \mu_t)(x - \mu_t)^T d p_t(x) \, dx - A_t C_t A_t dt.
\]

Next is the proof of stochastic derivative of the potential \( \Phi_t = \text{Tr}((A_t - \gamma I)^q) \).

**Lemma 36.** Let \( A_t \) be defined by Definition 24. For any integer \( q \geq 2 \), we have that

\[
d\text{Tr}((A_t - \gamma I)^q) = qE_{x \sim p_t}(x - \mu_t)^T (A_t - \gamma I)^{q-1}(x - \mu_t)(x - \mu_t)^T C_t^{1/2} dW_t - q\text{Tr}((A_t - \gamma I)^{q-1} A_t^2 C_t) dt
+ \frac{q}{2} \sum_{\alpha + \beta = q - 2} E_{x, y \sim p_t} (x - \mu_t)^T (A_t - \gamma I)^\alpha (y - \mu_t)(x - \mu_t)^T (A_t - \gamma I)^\beta (y - \mu_t)(x - \mu_t)^T C_t (y - \mu_t) dt.
\]

**Proof.** Let \( \Phi(X) = \text{Tr}((X - \gamma I)^q) \). Then the first and second-order directional derivatives of \( \Phi \) at \( X \) is given by

\[
\frac{\partial \Phi}{\partial X} |_H = q\text{Tr}((X - \gamma I)^{q-1} H) \quad \text{and} \quad \frac{\partial^2 \Phi}{\partial X \partial X} |_{H_1, H_2} = q \sum_{k=0}^{q-2} \text{Tr}((X - \gamma I)^k H_2 (X - \gamma I)^{q-2-k} H_1).
\]

Using these and Itô’s formula, we have that

\[
d\text{Tr}((A_t - \gamma I)^q) = q\text{Tr}((A_t - \gamma I)^{q-1} dA_t) + \frac{q}{2} \sum_{\alpha + \beta = q - 2} \sum_{ijkl} \text{Tr}((A_t - \gamma I)^\alpha e_{ij}(A_t - \gamma I)^\beta e_{kl}) d[A_{ij}, A_{kl}] |_t
\]

where \( e_{ij} \) is the matrix that is 1 in the entry \( (i, j) \) and 0 otherwise, and \( A_{ij} \) is the real-valued stochastic process defined by the \((i, j)^{th}\) entry of \( A_t \).
Using Lemma 27 and Lemma 26, we have that

\[ dA_t = \mathbb{E}_{x \sim p_t}(x - \mu_t)(x - \mu_t)^T (x - \mu_t)^T C_t^{1/2} dW_t - A_t A_t dt \]

\[ = \mathbb{E}_{x \sim p_t}(x - \mu_t)(x - \mu_t)^T (x - \mu_t)^T C_t^{1/2} e_z dW_{t,z} - A_t A_t dt \]

where \( W_{t,z} \) is the \( z \)th coordinate of \( W_t \). Therefore,

\[ d[A_{ij}, A_{kl}]_t = \sum_z \left( \mathbb{E}_{x \sim p_t}(x - \mu_t)_i(x - \mu_t)_j(x - \mu_t)^T C_t^{1/2} e_z \right) \left( \mathbb{E}_{x \sim p_t}(x - \mu_t)_k(x - \mu_t)_l(x - \mu_t)^T C_t^{1/2} e_z \right) dt \]

\[ = \mathbb{E}_{x, y \sim p_t}(x - \mu_t)_i(x - \mu_t)_j(y - \mu_t)_k(y - \mu_t)_l(x - \mu_t)^T C_t(y - \mu_t) dt. \] (5.4)

Using the formula for \( dA_t \) (5.3) and \( d[A_{ij}, A_{kl}]_t \) (5.4), we have that

\[ d\text{Tr}((A_t - \gamma I)^q) \]

\[ = q\mathbb{E}_{x \sim p_t}(x - \mu_t)^T (A_t - \gamma I)^{q-1}(x - \mu_t)(x - \mu_t)^T C_t^{1/2} dW_t - q\text{Tr}((A_t - \gamma I)^{q-1} A_t^2 C_t) dt \]

\[ + \frac{q}{2} \sum_{\alpha + \beta = q-2} \sum_{ijkl} \text{Tr}((A_t - \gamma I)^{\alpha} e_{ij}(A_t - \gamma I)^{\beta} e_{kl}) \mathbb{E}_{x, y \sim p_t}(x - \mu_t)_i(x - \mu_t)_j(y - \mu_t)_k(y - \mu_t)_l(x - \mu_t)^T C_t(y - \mu_t) dt \]

\[ = q\mathbb{E}_{x \sim p_t}(x - \mu_t)^T (A_t - \gamma I)^{q-1}(x - \mu_t)(x - \mu_t)^T C_t^{1/2} dW_t - q\text{Tr}((A_t - \gamma I)^{q-1} A_t^2 C_t) dt \]

\[ + \frac{q}{2} \sum_{\alpha + \beta = q-2} \mathbb{E}_{x, y \sim p_t}(x - \mu_t)^T (A_t - \gamma I)^{\alpha}(y - \mu_t)(x - \mu_t)^T (A_t - \gamma I)^{\beta}(y - \mu_t)(x - \mu_t)^T C_t(y - \mu_t) dt. \]

\[ \square \]

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6 A reduction to a third moment assumption

In this section, we use the following assumption. In the first arXiv version of this paper, we claimed this assumption as a lemma. While it might be true, our proof was not correct.

**Assumption (Third Moment).** For an isotropic logconcave distribution \( p \in \mathbb{R}^n \), \( \mathbb{E}_{x,y \sim p} \left( \langle x, y \rangle^3 \right) = O\left(n \psi_n \right) \).

Note that \( \mathbb{E}_{x,y \sim p} \left( \langle x, y \rangle^3 \right) = \sum_{i,j,k} \mathbb{E}_x (x_i x_j x_k)^2 \). Under this assumption we prove the following.

**Theorem 37.** Under the third moment assumption above, any isotropic logconcave density \( p \) in \( \mathbb{R}^n \), the KLS constant is \( \psi_p = e^{O(\sqrt{\log n \log \log n})} \).

The proof will use the same process with \( C_t = I \), but with a more sensitive potential function \( \Phi_t = \text{Tr}((A_t - I)^q) \) for even integers \( q \).

6.1 Tensor bounds

**Definition 38 (3-Tensor).** For any isotropic logconcave distribution \( p \) in \( \mathbb{R}^n \) and any symmetric matrices \( A, B \) and \( C \), we define

\[
T_p(A, B, C) = \mathbb{E}_{x,y \sim p} (x^T Ay)(x^T By)(x^T Cy).
\]

Often, we drop the subscript \( p \) to indicate the worst case bound

\[
T(A, B, C) \overset{\text{def}}{=} \sup_{\text{isotropic logconcave } p} T_p(A, B, C).
\]

**Remark.** It is clear from the definition that \( T \) is symmetric, namely \( T(A_1, A_2, A_3) = T(A_{\sigma(1)}, A_{\sigma(2)}, A_{\sigma(3)}) \) for any permutation \( \sigma \).

We first start with some simple equalities about a 3-tensor. Here we repeatedly use the elementary facts \( \text{Tr}(AB) = \text{Tr}(BA) \), \( x^T Ay = \text{Tr}(Ayx^T) \).

**Lemma 39.** For any isotropic logconcave distribution \( p \) and any symmetric matrices \( A, B \) and \( C \), we have that

\[
T_p(A, B, I) = \sum_i \text{Tr}(A \Delta_i B \Delta_i)
\]

and

\[
T_p(A, B, I) = \sum_{i,j} A_{ij} \text{Tr}(\Delta_i B \Delta_j)
\]

where \( \Delta_i = \mathbb{E}_{x \sim p} xx^T x_i \).

**Proof.** Direct calculation shows that

\[
T_p(A, B, I) = \mathbb{E}_{x,y \sim p} x^T Ayx^T Byx^T y = \sum_i \mathbb{E}_{x,y \sim p} x^T Ayx^T Byx_i y_i = \sum_i \mathbb{E}_{x,y \sim p} \text{Tr}(Axx^T Byy^T x_i y_i) = \sum_i \text{Tr}(A \Delta_i B \Delta_i)
\]
Lemma 41. For any \( A_1, A_2, A_3 \geq 0 \), we have that

\[
T(A_1, A_2, A_3) \geq 0
\]

and for any symmetric matrices \( B_1, B_2, B_3 \), we have that

\[
T(B_1, B_2, B_3) \leq T(|B_1|, |B_2|, |B_3|).
\]

Proof. Fix any isotropic logconcave distribution \( p \). We define \( \Delta_i = E_{x \sim p} x^T A_i x \) which is well defined since \( A_3 \geq 0 \). Then, we have that

\[
T_p(A_1, A_2, A_3) = E_{x \sim p} x^T A_1 y x^T A_2 y x^T A_3 y 
\]

\[
= \sum_j \text{Tr}(A_j \Delta_i) 
\]

Since \( \Delta_i \) is symmetric and \( A_1, A_2 \geq 0 \), we have that \( A_1^{1/2} \Delta_i A_2 \Delta_i A_1^{1/2} \geq 0 \) and \( \text{Tr}(A_1 \Delta_i A_2 \Delta_i) \geq 0 \). Therefore, \( T(A_1, A_2, A_3) \geq T_p(A_1, A_2, A_3) \geq 0 \).

For the second part, we write \( B_1 = B_1^{(1)} - B_1^{(2)} \) where \( B_1^{(1)} \geq 0, B_1^{(2)} \geq 0 \) and \( |B_1| = B_1^{(1)} + B_1^{(2)} \). We define \( B_2^{(1)}, B_2^{(2)}, B_3^{(1)}, B_3^{(2)} \) similarly. Note that

\[
T(B_1, B_2, B_3) = T(B_1^{(1)}, B_2^{(1)}, B_3^{(1)}) - T(B_1^{(1)}, B_2^{(1)}, B_3^{(2)}) - T(B_1^{(1)}, B_2^{(2)}, B_3^{(1)}) + T(B_1^{(1)}, B_2^{(2)}, B_3^{(2)}) 
\]

\[
- T(B_1^{(2)}, B_2^{(1)}, B_3^{(1)}) + T(B_1^{(2)}, B_2^{(1)}, B_3^{(2)}) + T(B_1^{(2)}, B_2^{(2)}, B_3^{(1)}) - T(B_1^{(2)}, B_2^{(2)}, B_3^{(2)}).
\]

Since \( B_3^{(3)} \geq 0 \), the first part of this lemma shows that every term \( T(B_1^{(i)}, B_2^{(j)}, B_3^{(k)}) \geq 0 \). Hence, we have that

\[
T(B_1, B_2, B_3) \leq T(B_1^{(1)}, B_2^{(1)}, B_3^{(1)}) + T(B_1^{(1)}, B_2^{(1)}, B_3^{(2)}) + T(B_1^{(1)}, B_2^{(2)}, B_3^{(1)}) + T(B_1^{(1)}, B_2^{(2)}, B_3^{(2)}) 
\]

\[
+ T(B_1^{(2)}, B_2^{(1)}, B_3^{(1)}) + T(B_1^{(2)}, B_2^{(1)}, B_3^{(2)}) + T(B_1^{(2)}, B_2^{(2)}, B_3^{(1)}) + T(B_1^{(2)}, B_2^{(2)}, B_3^{(2)})
\]

\[
= T(|B_1|, |B_2|, |B_3|).
\]

\]

Lemma 41. Suppose that \( \psi_k \leq \alpha k^\beta \) for all \( k \leq n \) for some \( 0 \leq \beta \leq 1/2 \) and \( \alpha \geq 1 \). Given an isotropic logconcave distribution \( p \) and an unit vector \( v \), we define \( \Delta = E_{x \sim p} x^T A x^T v \). Then, we have that

1. For any orthogonal projection matrix \( P \) with rank \( r \), we have that

\[
\text{Tr}(\Delta P \Delta) \leq O(\psi_{\min(2r, n)}^2).
\]

2. For any symmetric matrix \( A \), we have that

\[
\text{Tr}(DA) \leq O(\alpha^2 \log n) \left( \text{Tr} |A|^{1/(2\beta)} \right)^{2\beta}.
\]

Proof. We first bound \( \text{Tr}(\Delta P \Delta) \). This part of the proof is generalized from a proof by Eldan [14]. Note that \( \text{Tr}(\Delta P \Delta) = E_{x \sim p} x^T P \Delta x^T v \). Since \( E x^T v = 0 \), we have that

\[
\text{Tr}(\Delta P \Delta) \leq \sqrt{E(x^T v)^2} \sqrt{\text{Var}(x^T P \Delta x) \text{Lem} 19} \leq O \left( \psi_{\text{rank}(P \Delta + \Delta P)} \right) \sqrt{E x \| P \Delta x \|^2} = O \left( \psi_{\text{rank}(P \Delta + \Delta P)} \right) \sqrt{\text{Tr}(\Delta P \Delta)}.
\]
This gives \( \text{Tr}(\Delta P \Delta) \leq O(\psi_{\min(2r,n)}^2) \).

Now we bound \( \text{Tr}(\Delta A \Delta) \). Since \( \text{Tr}(\Delta A \Delta) \leq \text{Tr}(\Delta |A| \Delta) \), we can assume without loss of generality that \( A \succeq 0 \). We write \( A = \sum_i A_i + B \) where each \( A_i \) has eigenvalues between \( \{||A||_2 \, 2^i/n, ||A||_2 \, 2^{i+1}/n\} \) and \( B \) has eigenvalues smaller than or equals to \( ||A||_2 / n \). Clearly, we only need at most \( \lceil \log(n) + 1 \rceil \) many such \( A_i \). Let \( P_i \) be the orthogonal projection from \( \mathbb{R}^n \) to the span of the range of \( A_i \). Using \( ||A_i||_2 P_i \succeq A_i \), we have that

\[
\text{Tr}(\Delta A_i \Delta) \leq ||A_i||_2 \| \text{Tr}(\Delta P_i \Delta) \|_2 \leq O\left( \psi_{\min(2\text{rank}(A_i),n)}^2 \right) ||A_i||_2 \leq O(\alpha^2) \sum_i \text{rank}(A_i)^{2\beta} ||A_i||_2
\]

where we used the first part of this lemma in the last inequality.

Similarly, we have that

\[
\text{Tr}(\Delta B \Delta) \leq O\left( \psi_n^2 \right) ||B||_2 \leq O(n ||B||_2) \leq O(1) ||A||_2.
\]

Combining the bounds on \( \text{Tr}(\Delta A_i \Delta) \) and \( \text{Tr}(\Delta B \Delta) \), we have that

\[
\text{Tr}(\Delta A \Delta) \leq O(\alpha^2) \sum_i \text{rank}(A_i)^{2\beta} ||A_i||_2 + O(1) ||A||_2
\]

\[
\leq O(\alpha^2) \left( \sum_i \text{rank}(A_i) ||A_i||_2^{1/(2\beta)} \right)^{2\beta} \log(n)^{1-2\beta}
\]

\[
\leq O(\alpha^2 \log n) \left( \text{Tr} |A|^{1/(2\beta)} \right)^{2\beta}.
\]

\[\square\]

In the next lemma, we collect tensor related inequalities that will be useful.

**Lemma 42.** Suppose that \( \psi_k \leq \alpha k^\beta \) for all \( k \leq n \) for some \( 0 \leq \beta \leq 1/2 \) and \( \alpha \geq 1 \). For any isotropic logconcave distribution \( p \) in \( \mathbb{R}^n \) and symmetric matrices \( A \) and \( B \), we have that

1. \( \mathbb{E}_x (x^T Ax)^2 \leq O(1) (\text{Tr} |A|)^2 \),
2. \( \mathbb{E}_{x \sim p} (x^T Ax - \text{Tr} A)^2 \leq O(\psi_n^2) \text{Tr} A^2 \),
3. \( T(A, I, I) \leq O(\psi_n) ||A||_2 n \),
4. \( T(A, I, I) \leq O(\psi_n^2) \text{Tr} |A| \),
5. \( T(A, B, I) \leq O(\psi_n^2) ||B||_2 \text{Tr} |A| \) where \( r = \min(2\text{rank}(B), n) \),
6. \( T(A, B, I) \leq O(\alpha^2 \log n) \left( \text{Tr} |B|^{1/(2\beta)} \right)^{2\beta} \text{Tr} |A| \),
7. \( T(A, B, I) \leq (T(|A|^s, I, I))^{1/s} (T(|B|^t, I, I))^{1/t} \) for any \( s, t \geq 1 \) with \( s^{-1} + t^{-1} = 1 \).

**Proof.** Without loss of generality, we can assume \( A \) is diagonal by rotating space. In particular, if we want to prove something for \( \text{Tr}(A^\alpha \Delta A^\beta \Delta) \) where \( \Delta \) are symmetric matrices, we use the spectral decomposition \( A = U \Sigma U^T \) to rewrite this as

\[
\text{Tr} \left( U \Sigma^\alpha U^T \Delta U \Sigma^\beta U^T \Delta \right) = \text{Tr} \left( \Sigma^\alpha (U^T \Delta U) \Sigma^\beta (U^T \Delta U) \right)
\]

which puts us back in the same situation, but with a diagonal matrix \( A \).

Let \( \Delta_i = \mathbb{E}_{x \sim p} xx^T x_i \). For inequality 1, we note that

\[
\mathbb{E}_x (x^T Ax)^2 = \sum_{ij} A_{ii} A_{jj} \mathbb{E}_x x_i^2 x_j^2 \leq \sum_{ij} |A_{ii}| |A_{jj}| \sqrt{\mathbb{E}_x x_i^4 \mathbb{E}_x x_j^4} \leq O(1) \left( \sum_i |A_{ii}| \right)^2.
\]

For inequality 2, we note that \( \mathbb{E}_{x \sim p} (x^T Ax - \text{Tr} A)^2 = \text{Var}(x^T Ax) \leq O(\psi_n^2) \mathbb{E}_x ||Ax||^2 = O(\psi_n^2) \text{Tr} A^2 \).
For remaining inequalities, it suffices to upper bound $T$ by upper bounding $T_p$ for any isotropic logconcave distribution $p$.

For inequality 3, we note that
\[
T_p(A, I, I) \overset{\text{Lem} 39}{=} \sum_i A_{ii} \text{Tr}(\Delta_i^2) \leq \|A\|_2 \sum_i \text{Tr}(\Delta_i^2) \overset{\text{Lem} 39}{=} \|A\|_2 \|T(I, I, I)\| \leq O(n\psi^2_n) \|A\|_2
\]
where the last inequality is from the third moment assumption.

For inequality 4, we note that
\[
T_p(A, I, I) \overset{\text{Lem} 39}{=} \sum_i A_{ii} \text{Tr}(\Delta_i^2) \leq \sum_i |A_{ii}| O(\psi^2_n) = O(\psi^2_n) \text{Tr} |A|.
\]

For inequality 5, we let $P$ be the orthogonal projection from $\mathbb{R}^n$ to the span of the range of $B$. Let $r = \text{rank}(P)$. Then, we have that
\[
T_p(A, B, I) \leq T_p(|A|, |B|, I) \overset{\text{Lem} 40}{=} \sum_i |A_{ii}| \text{Tr}(\Delta_i |B| \Delta_i) \overset{\text{Lem} 39}{=} \|B\|_2 \sum_i |A_{ii}| \text{Tr}(\Delta_i P \Delta_i) \leq O(\psi^2_n) \text{Tr} |A| \|B\|_2.
\]
where we used that $|B| \leq \|B\|_2 P$ in (1).

For inequality 6, we note that
\[
T_p(A, B, I) \overset{\text{Lem} 39}{=} \sum_i A_{ii} \text{Tr}(\Delta_i B \Delta_i) \leq O(\alpha^2 \log n) \text{Tr} |A| \left( \text{Tr} |B|^{1/(2\beta)} \right)^{2\beta}.
\]

For inequality 7, we note that
\[
T_p(A, B, I) \leq T_p(|A|, |B|, I) \overset{\text{Lem} 40}{=} \sum_i \text{Tr}(|A| \Delta_i |B| \Delta_i) \leq \sum_i \text{Tr}(|A| |\Delta_i| |B| |\Delta_i|) \leq \sum_i \text{Tr}(|\Delta_i|^{1/s} |A| |\Delta_i|^{1/s} |\Delta_i|^{1/t} |B| |\Delta_i|^{1/t}) \leq \sum_i \left( \text{Tr} \left( \left( |\Delta_i|^{1/s} |A| |\Delta_i|^{1/s} \right)^{1/s} \left( |\Delta_i|^{1/t} |B| |\Delta_i|^{1/t} \right)^{1/t} \right) \right)^{1/st} \leq \sum_i \left( \text{Tr} \left( |\Delta_i| |A|^s |\Delta_i| \right)^{1/st} \left( \text{Tr} \left( |\Delta_i| |B|^t |\Delta_i| \right)^{1/t} \right) \right) \leq \sum_i \left( \text{Tr} \left( |A|^s \Delta_i^2 \right)^{1/st} \left( \text{Tr} \left( |B|^t \Delta_i^2 \right)^{1/t} \right) \right) \leq \left( T_p(|A|^s, I, I) \right)^{1/s} \left( T_p(|B|^t, I, I) \right)^{1/t} \overset{\text{Lem} 39}{=}
\]

\[\square\]

**Lemma 43.** For any positive semi-definite matrices $A, B, C$ and any $\alpha \in [0, 1]$, then
\[
T(B^{1/2}A^\alpha B^{1/2}, B^{1/2}A^{1-\alpha} B^{1/2}, C) \leq T(B^{1/2}AB^{1/2}, B, C).
\]
Proof. Fix any isotropic logconcave distribution $p$. Let $\Delta_i = E_{x \sim p} B^{1/2} x T B^{1/2} x T C^{1/2} e_i$. Then, we have that

$$T_p(B^{1/2}A^{\alpha} B^{1/2}, B^{1/2} A^{1-\alpha} B^{1/2}, C) = E_{x,y \sim p} x T B^{1/2} A^{\alpha} B^{1/2} y T B^{1/2} A^{1-\alpha} B^{1/2} x T y T C e_i$$

$$= \sum_i E \left( (y T B^{1/2} A^{\alpha} B^{1/2} x) (x T B^{1/2} A^{1-\alpha} B^{1/2} y) x T C e_i y T C e_i \right)$$

$$= \sum_i \text{Tr} \left( A^{\alpha} B^{1/2} x T B^{1/2} A^{1-\alpha} B^{1/2} y T B^{1/2} \right) \left( x T C e_i \right) \left( y T C e_i \right)$$

$$= \sum_i \text{Tr}(A^{\alpha} \Delta_i A^{1-\alpha} \Delta_i)$$

Using Lemma 23, we have that

$$\sum_i \text{Tr}(A^{\alpha} \Delta_i A^{1-\alpha} \Delta_i) \leq \sum_i \text{Tr}(A \Delta_i^2) = E_{x,y \sim p} x T B^{1/2} A B^{1/2} y T B y T C y = T_p(B^{1/2} A B^{1/2}, B, C).$$

Taking the supremum over all isotropic logconcave distributions $p$, we get the result. \square

### 6.2 Derivatives of the potential

**Lemma 44.** Let $A_t$ be defined by Definition 24. For any integer $q \geq 2$, we have that

$$d\text{Tr}((A_t - I)^q) = q E_{x \sim p} (x - \mu_t)^T (A_t - I)^{q-1} (x - \mu_t) dW_t - q \text{Tr}((A_t - I)^{q-1} A_t^2) dt$$

$$+ \frac{q}{2} \sum_{\alpha + \beta = q-2} E_{x,y \sim p} (x - \mu_t)^T (A_t - I)^\alpha (y - \mu_t) (x - \mu_t)^T (A_t - I)^\beta (y - \mu_t)^T (x - \mu_t) dt$$

We give the proof in Section 5. The next lemma bounds the stochastic process that controls this potential function.

**Lemma 45.** Let $A_t$ and $p_t$ be defined by Definition 24. Let $\Phi_t = \text{Tr}((A_t - I)^q)$ for some even integer $q \geq 2$, then we have that $d\Phi_t = \delta_t dt + v_t^T dW_t$ with

$$\delta_t \leq \frac{1}{2} q(q - 1) T(A_t(A_t - I)^{q-2}, A_t, A_t) + q(\Phi_t^{1/4} + \Phi_t^{-1/4})$$

and

$$\|v_t\|_2 \leq q \left\| E_{x \sim p} (x - \mu_t)^T (A - I)^{q-1} (x - \mu_t) (x - \mu_t)^T \right\|_2.$$

**Proof.** By Lemma 44, we have

$$d\Phi_t = q E_{x \sim p} (x - \mu_t)^T (A_t - I)^{q-1} (x - \mu_t) dW_t - q \text{Tr}((A_t - I)^{q-1} A_t^2) dt$$

$$+ \frac{q}{2} \sum_{\alpha + \beta = q-2} E_{x,y \sim p} (x - \mu_t)^T (A_t - I)^\alpha (y - \mu_t) (x - \mu_t)^T (A_t - I)^\beta (y - \mu_t)^T (x - \mu_t) dt$$

$$= q E_{x \sim p} (x - \mu_t)^T (A - I)^{q-1} (x - \mu_t) dW_t - q \text{Tr}((A_t - I)^{q-1} A_t^2) dt$$

$$+ \frac{q}{2} \sum_{\alpha + \beta = q-2} E_{x,y \sim p} (x - \mu_t)^T A_t (A_t - I)^\alpha y^T A_t (A_t - I)^\beta y^T A_t dt$$

$$\overset{d\delta_t}{=} \delta_t dt + v_t^T dW_t.$$

where $\tilde{p}_t$ is the isotropic version of $p_t$ defined by $\tilde{p}_t(x) = p(A_t^{1/2} x + \mu_t)$, $\delta_t dt$ is the drift term in $d\Phi_t$ and $v_t^T dW_t$ is the martingale term in $d\Phi_t$.

For the drift term $\alpha_t dt$, we have

$$\delta_t \leq \frac{q}{2} \sum_{\alpha + \beta = q-2} T(A_t(A_t - I)^\alpha, A_t(A_t - I)^\beta, A_t) - q \text{Tr}((A_t - I)^{q-1} A_t^2).$$

The first term in the drift is

$$\frac{q}{2} \sum_{\alpha + \beta = q-2} T(A_t(A_t - I)^\alpha, A_t(A_t - I)^\beta, A_t) \leq \frac{q}{2} \sum_{\alpha + \beta = q-2} T(A_t | A_t - I |^\alpha, A_t | A_t - I |^\beta, A_t)$$

$$\leq \frac{q}{2} \sum_{\alpha + \beta = q-2} T(A_t | A_t - I |^{q-2}, A_t, A_t)$$

$$= \frac{q(q - 1)}{2} T(A_t(A_t - I)^{q-2}, A_t, A_t).$$

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For the second term in drift, since \( q \) is even, we have that
\[
-q \text{Tr}((A_t - I)^{q-1} A_t^2) = -q \text{Tr}((A_t - I)^{q-1}(A_t - I + I)^2)
\leq -q \text{Tr}((A_t - I)^{q+1}) - q \text{Tr}((A_t - I)^{q-1})
\leq q \Phi_1^{1 + \frac{1}{q}} + q \Phi_\eta^{1 + \frac{1}{q}} \frac{1}{n^\eta}.
\]

For the martingale term \( v_t^T dW_t \), we note that
\[
\|v_t\|_2 = q \| \mathbb{E}_{x \sim p}(x - \mu_t)^T (A - I)^{q-1} (x - \mu_t)(x - \mu_t)^T \|.
\]

\[\square\]

6.3 Analysis of \( A_t \)

We first bound the drift term from Lemma 44.

**Lemma 46.** Suppose that \( \psi_k \leq \alpha k^n \) for all \( k \leq n \) for some \( 0 \leq \beta \leq \frac{1}{q} \) and \( \alpha \geq 1 \). Let \( \Phi = \text{Tr}((A - I)^q) \) for some even integer \( q \geq 2 \) and \( A \succeq 0 \). If \( \beta q \geq 2 \), then
\[
T(A(A - I)^{q-2}, A, A) \leq O(\alpha^2) \left[ n^{2\beta - \frac{2\beta}{q} + \frac{\beta}{2} \Phi^{1 - \frac{1}{q}}} + n^{2\beta - \frac{\beta}{q} + \frac{\beta}{2} \Phi^{1 - \frac{1}{q}}} + n^{2\beta - \frac{2\beta}{q} \Phi^{1 + \frac{1}{q}} + (\log n)n^{2\beta - \frac{\beta}{q} \Phi^{1 + \frac{1}{q}}}} \right].
\]

**Proof.** We have that
\[
T(A(A - I)^{q-2}, A, A)
=T((A - I)^{q-1} + (A - I)^{q-2}, (A - I) + I, (A - I) + I)
=T((A - I)^{q-1} + (A - I, A - I) + 2T((A - I)^{q-1}, A - I, I) + T((A - I)^{q-1}, I, I)
+ T((A - I)^{q-2}, A - I, A - I) + 2T((A - I)^{q-2}, A - I, I) + T((A - I)^{q-2}, I, I)
\leq T(|A - I|^{q-1}, |A - I|, |A - I|) + 2T(|A - I|^{q-1}, |A - I|, I) + T(|A - I|^{q-1}, I, I) \quad \text{(Lem 40)}
+ T(|A - I|^{q-2}, |A - I|, |A - I|) + 2T(|A - I|^{q-2}, |A - I|, I) + T(|A - I|^{q-2}, I, I)
\leq T(|A - I|^{q-1}, |A - I|, |A - I|) + 3T(|A - I|^{q-1}, |A - I|, I)
+ 3T(|A - I|^{q-2}, I, I) + T((A - I)^{q-2}, I, I)
\]
where we used Lemma 43 at the end.

For the first term in (6.1), we have that
\[
T(|A - I|^{q-1}, |A - I|, |A - I|) \leq T(|A - I|^q, |A - I|, I)
\leq O(\alpha^2 \log n) \Phi \left( \text{Tr}|A - I|^{1/2\beta} \right)^{2\beta} \quad \text{(Lem 42.6)}
\leq O(\alpha^2 \log n) \Phi \left( \text{Tr}|A - I|^{1/(2\beta q)} n^{1 - 1/(2\beta q)} \right)^{2\beta}
\leq O(\alpha^2 \log n)n^{2\beta - \frac{1}{q} \Phi^{1 + 1/q}}
\]
where we used \( 2\beta q \geq 1 \) at the last line.

For the second term in (6.1), we write
\[
|A - I| = B_1 + B_2
\]
where \( B_1 \) consists of the eigen-components of \( |A - I| \) with eigenvalues \( \leq \eta \) and \( B_2 \) is the remaining part where we will pick \( \eta \geq 0 \) later. Then, we have that
\[
T(|A - I|^{q-1}, |A - I|, I) = T(B_1^{q-1}, B_1, I) + T(B_2^{q-1}, B_2, I) + T(B_3^{q-1}, B_1, I) + T(B_3^{q-1}, B_2, I). \quad \text{(6.2)}
\]
For the first term in (6.2), we note that
\[
T(B_1^{q-1}, B_1, I) \leq T(B_1^q, I, I)
\leq O(\psi_k n) \| B_1 \|^q \quad \text{(Lem 43)}
\leq O(n^q \psi_k n).
\]
For the second term in (6.2), we note that
\[
T(B_1^{q-1}, B_2, I) \leq T(B_1^n, I, I)^{\frac{q-1}{q}} T(B_2^n, I, I)^{\frac{1}{q}} \quad \text{(Lem 42.7)}
\]
\[
\leq O(\eta^q \psi_n n)^{\frac{q-1}{q}} O(\psi_n^2 \Phi)^{\frac{1}{q}}
\quad \text{(Lem 42.3 and Lem 42.4)}
\]
where we used \(\|B_1\|_2 \leq \eta\) and \(\text{Tr} B_2^2 \leq \text{Tr} |A - I|^q = \Phi\) at the last line. For the third term in (6.2), similarly, we have
\[
T(B_2^{q-1}, B_1, I) \leq T(B_2^n, I, I)^{\frac{q-1}{q}} T(B_1^n, I, I)^{\frac{1}{q}}
\quad \text{(Lem 42.7)}
\]
\[
\leq O(\psi_n^2 \Phi)^{\frac{q-1}{q}} O(\eta^q \psi_n n)^{\frac{1}{q}}.
\quad \text{(Lem 42.3 and Lem 42.4)}
\]

For the fourth term in (6.2), we let \(P\) be the orthogonal projection from \(\mathbb{R}^n\) to the range of \(B_2\). We have that
\[
T(B_2^{q-1}, B_2, I) = T(PB_2^{q-1} P, PB_2 P, I)
\leq T(PB_2^n P, P, I) \quad \text{(Lem 43)}
\]
\[
\leq O(\psi_n^2) \text{Tr} B_2^2 \quad \text{(Lem 42.5)}
\]
where \(r \leq 2 \text{rank}(P) \leq \frac{2\Phi}{\eta^q}\). Using \(\psi_k \leq \alpha k^{\beta}\) and combining all four terms, we have that
\[
T(|A - I|^{q-1}, |A - I|, I) \leq O(\eta^q \psi_n n) + O(\eta^q \psi_n n)^{\frac{q-1}{q}} O(\psi_n^2 \Phi)^{\frac{1}{q}} + O(\psi_n^2 \Phi)^{\frac{q-1}{q}} O(\eta^q \psi_n n)^{\frac{1}{q}} + O(\psi_n^2) \Phi
\leq O(\alpha^2) \left[ \eta^q n^{1+\beta} + \eta^q n^{q-1} \left( \frac{\alpha^{q-1}}{\alpha^{q-1} + \beta} + \frac{\beta}{2(2-q)} + \frac{1}{2(2-q)} \right) \Phi \right].
\]

Balancing the last two terms and setting \(\eta = \Phi^{\frac{1}{q}} n^{-\frac{\beta}{2} - \frac{1}{2(2-q)}}\), we get that
\[
T(|A - I|^{q-1}, |A - I|, I) \leq O(\alpha^2 \psi) \left[ n^{2\beta - \frac{3\beta}{2} - \frac{q}{2(2-q)}} + n^{2\beta - \frac{3\beta}{2} - \frac{q}{2(2-q)}} + n^{2\beta - \frac{3\beta}{2} - \frac{q}{2(2-q)}} \right]
\leq O(\alpha^2 n^{2\beta(1-\frac{\beta}{2\beta-q})})
\]
where we used \(q \geq 2\) and \(\beta \geq 0\).

For the third term in (6.1), we have that
\[
T(|A - I|^{q-1}, I, I) = T(B_1^n, I, I) + T(B_2^n, I, I)
\leq O \left( \alpha n^{q-1} \left( \frac{1+\beta}{\eta} \right) \Phi \right)
\leq O(\alpha^2 n^{1+\beta} \Phi^{1-\frac{q}{q}})
\quad \text{(Lem 42.3 and Lem 42.4)}
\]
where we set \(\psi = n^q \Phi^\frac{1}{q}\) at the last line.

For the fourth term in (6.1), we have that
\[
T(|A - I|^{q-2}, I, I) = T(B_1^{q-2}, I, I) + T(B_2^{q-2}, I, I)
\leq O(\alpha n^{q-2} n^{1+\beta} + \alpha n^{2\beta} \Phi \eta)
\leq O(\alpha^2 n^{1+\beta} \Phi^{1-\frac{q}{q}})
\quad \text{(Lem 42.3 and Lem 42.4)}
\]
where we set \(\eta = n^q \Phi^\frac{1}{q}\) at the last line.

Combining all terms, we have the result.

Next we bound the martingale term.

**Lemma 47.** Let \(p\) be a logconcave distribution with covariance matrix \(A\). Let \(\Phi = \text{Tr}((A - I)^q)\) for some even integer \(q \geq 2\). Then,
\[
\|E_{x \sim p}(x - \mu_t)^T (A - I)^{q-1} (x - \mu_t)(x - \mu_t)^T\|_2 \leq O(\Phi^{1-\frac{q}{2\beta}} n^{\frac{1}{2}} + \Phi^{1-\frac{q}{2\beta}} + n^{\frac{1}{2}}).
\]
Proof. Note that
\[
\|E_{x \sim p}(x - \mu_t)^T (A - I)^{q-1} (x - \mu_t)(x - \mu_t)^T\|_2 \leq O(1) \|A\|_2^{1/2} \text{Tr} \left| A^{1/2} (A - I)^{q-1} A^{1/2} \right|
\] (Lem 32)
\[
\leq O(1) \|A\|_2^{1/2} \text{Tr}|A - I|^{q-1} + O(1) \|A\|_2^{1/2} \text{Tr}|A - I|^q
\]
\[
\leq O(1 + \Phi^{1/2}) \Phi^{1 - \frac{q}{2}} n^{\frac{q}{2}} + O(1 + \Phi^{1/2}) \Phi^{1 - \frac{q}{2}} n^{\frac{q}{2}}.
\]

Using Lemma 46 and Lemma 47, we know that \( \Phi_t = \text{Tr}((A_t - I)^q) \) satisfies the stochastic equation \( d\Phi_t = \delta_t \, dt + v_t^T \, dW_t \) with
\[
\delta_t \leq O \left( \alpha^2 q^2 \right) \left[ n^{2\beta - \frac{2q}{q} + \frac{2}{q} \Phi_t^{1 - \frac{1}{q}}} + n^{2\beta - \frac{2q}{q} + \frac{1}{q} \Phi_t^{1 - \frac{1}{q}}} + n^{2\beta - \frac{2q}{q} + \frac{1}{q} \Phi_t^{1 - \frac{1}{q}}} + (\log n)n^{2\beta - \frac{2q}{q} + \frac{1}{q} \Phi_t^{1 - \frac{1}{q}}} \right]
\]
\[
+ q(\Phi_t^{1 + \frac{1}{q}} + \Phi_t^{1 - \frac{1}{q} n^{\frac{1}{q}}})
\)
\[
\leq O \left( \alpha^2 q^2 \right) \left[ n^{2\beta - \frac{2q}{q} + \frac{2}{q} \Phi_t^{1 - \frac{1}{q}}} + n^{2\beta - \frac{2q}{q} + \frac{1}{q} \Phi_t^{1 - \frac{1}{q}}} + n^{2\beta - \frac{2q}{q} + \frac{1}{q} \Phi_t^{1 - \frac{1}{q}}} + (\log n)n^{2\beta - \frac{2q}{q} + \frac{1}{q} \Phi_t^{1 - \frac{1}{q}}} \right]
\]
and
\[
\|v_t\|_2^2 \leq O(q^2) \left[ \Phi_t^{2 - \frac{1}{q} n^{\frac{1}{q}}} + \Phi_t^{2 + \frac{1}{q} n^{\frac{1}{q}}} \right]
\]
(6.3)
where we used \( \alpha \geq 1, 2\beta q \geq 1 \) and \( q \geq 1 \) in (6.3).

Using these, one can bound the growth of \( \Phi_t \) using a stochastic Grönwall’s inequality. For completeness, we bound \( \Phi_t \) directly below.

Lemma 48. Suppose that \( \psi_k \leq \alpha k^\beta \) for all \( k \leq n \) for some \( 0 \leq \beta \leq \frac{1}{2} \) and \( \alpha \geq 1 \). Given an isotropic logconcave distribution \( p \). Let \( A_t \) be defined by Definition 24 using initial distribution \( p \). Let \( \Phi_t = \text{Tr}((A_t - I)^q) \) for some even integer \( q \geq 2 \). If \( 3q \geq 2 \) and \( n \geq q^{qq} \) for some large constant \( q \), then there is a universal constant \( c \) such that
\[
P(\max_{t \in [0,T]} \Phi_t \geq n^{1 - \frac{\beta}{q} - \frac{q}{q} \log \frac{q}{n}}) \leq 0.01 \quad \text{with} \quad T = \frac{cn^{-2\beta + \frac{q}{q}}}{q\alpha^2 \log n}
\]
Proof. The idea is to choose a a function \( \Psi_t = f(\Phi_t, t) \) so that the resulting stochastic equation for \( \Psi_t \) effectively decouples the drift and martingale terms. We use
\[
f(a, t) = \left( a + 1 + Et^q + Ft^{\frac{q}{2}} \right)^{\frac{1}{q}}
\]
with
\[
E = q^q \alpha^q n^{1 - \beta + 2\beta q} \quad \text{and} \quad F = q^q \alpha^q n^{1 - \beta + 2\beta q}.
\]
By 45 and Itô’s formula, we have that
\[
d\Psi_t = \frac{df}{d\Phi} \, d\Phi_t + \frac{df}{dt} \, dt + \frac{d^2 f}{2d\Phi d\Phi} \, d[\Phi]_t
\]
\[
\leq \frac{df}{dt} \, dt + \frac{df}{d\Phi} \, \delta_t \, dt + \frac{df}{d\Phi} \, v_t \, dW_t
\]
where we used \( f(a, t) \) is concave in \( a \) in the last line and dropped the second derivative term. The rationale for our choice of \( f(a, t) \) is that \( 1 + Et^q + Ft^{\frac{q}{2}} \) is our guess for the solution of the SDE for \( \Phi_t \), and the power \( 1/2q \) is chosen so that \( \Phi_t \) can be eliminated from the stochastic term in the bound for \( \Psi_t \) above.

For the term \( \frac{df}{d\Phi} \delta_t \, dt \), we use (6.3) and get that
\[
\frac{df}{d\Phi} \delta_t = \frac{1}{2q} \left( \Phi_t + 1 + Et^q + Ft^{\frac{q}{2}} \right)^{1 - \frac{1}{q}} \delta_t
\]
\[
\leq O(q \alpha^2) \left[ n^{2\beta - \frac{2q}{q} + \frac{2}{q} \Phi_t^{1 - \frac{1}{q}}} + n^{2\beta - \frac{2q}{q} + \frac{1}{q} \Phi_t^{1 - \frac{1}{q}}} \right]
\]
\[
(\Phi_t + 1 + Et^q + Ft^{\frac{q}{2}})^{1 - \frac{1}{q}} \Phi_t^{1 - \frac{1}{q}}
\]
\[
+ O(q \alpha^2) \left[ n^{2\beta - \frac{2q}{q} + \frac{1}{q} \Phi_t^{1 - \frac{1}{q}}} + (\log n)n^{2\beta - \frac{2q}{q} + \frac{1}{q} \Phi_t^{1 - \frac{1}{q}}} \right] \Phi_t^{1 - \frac{1}{q}}
\]
(6.5)
For the first term in (6.5), we note that
\[
\frac{n^{2\beta-\frac{2\beta}{q}+\frac{2}{q}+\frac{1}{2}}}{\Phi_t + 1 + Et^q + Ft^q} \leq \frac{n^{2\beta-\frac{2\beta}{q}+\frac{2}{q}}}{\Phi_t + 1 + Et^q + Ft^q} \leq \frac{2\beta-\frac{2\beta}{q}+\frac{2}{q}+\frac{1}{2}}{q\alpha^2 t^{\frac{1}{4}}},
\]
where we used \( F = q^2 \alpha^2 n^{1-\beta+2\beta q} \) at the end. For the second term in (6.5), we note that
\[
\frac{n^{2\beta-\frac{2\beta}{q}+\frac{2}{q}+\frac{1}{2}}}{\Phi_t + 1 + Et^q + Ft^q} \leq \frac{E^{\frac{1}{2}}}{q\alpha^2 \sqrt{t}},
\]
where we used \( E = q^2 \alpha^2 n^{1-\beta+2\beta q} \) at the end. For the third term in (6.5), assuming \( \Phi_t \leq n^{1-\beta} \log^{-\frac{2}{q}} n \), we have that
\[
\frac{n^{2\beta-\frac{2\beta}{q}+\frac{2}{q}+\frac{1}{2}}}{\Phi_t + 1 + Et^q + Ft^q} \leq \frac{E^{\frac{1}{2}}}{q\alpha^2 \sqrt{t}}.
\]
For the fourth term in (6.5), assuming \( \Phi_t \leq n^{1-\beta} \log^{-\frac{2}{q}} n \), we have that
\[
\frac{(\log n)n^{2\beta-\frac{2\beta}{q}+\frac{1}{2}}}{\Phi_t + 1 + Et^q + Ft^q} \leq \frac{E^{\frac{1}{2}}}{q\alpha^2 \sqrt{t}}.
\]
where we used \( E = q^2 \alpha^2 n^{1-\beta+2\beta q} \) at the end. Combining all four terms in (6.5), we have that
\[
\frac{df}{d\Phi} \delta_t \leq O \left( E^{\frac{1}{2}} + F^{\frac{1}{2}} t^{\frac{1}{4}} \right).
\]
For the term \( \frac{df}{dt}dt \), we have that
\[
\frac{df}{dt} dt = \frac{1}{2q} \left( \Phi_t + 1 + Et^q + Ft^q \right)^{\frac{1}{4}} = O \left( E^{\frac{1}{2}} + F^{\frac{1}{2}} t^{\frac{1}{4}} \right).
\]
For the term \( \frac{df}{d\Phi} v_t^T dW_t \), using (6.4) and assuming \( \Phi_t \leq n^{1-\beta} \log^{-\frac{2}{q}} n \), we have that
\[
\| v_t \|^2 \leq O(q^2) \left[ \frac{\Phi_t}{n^{\frac{1}{2}}} \right] \left[ n^{\frac{1}{2}} + n^{\frac{1}{2}} \right].
\]
Hence, we have that
\[
\frac{df}{d\Phi} v_t \|_2 \leq \frac{1}{4q^2 \left( \Phi_t + 1 + Et^q + Ft^q \right)^{\frac{1}{4}}} \leq O(n^{\frac{1}{2}}).
\]
Combining the terms \( \frac{df}{d\Phi} \delta_t dt \), \( \frac{df}{d\Phi} dt \) and \( \frac{df}{d\Phi} v_t^T dW_t \), we have that, when \( \Phi_t \leq n^{1-\beta} \log^{-\frac{2}{q}} n \),
\[
d\Psi_t = \frac{df}{dt} dt + \frac{df}{d\Phi} d\Phi_t + \frac{1}{2} d^2 f \frac{df}{d\Phi} d\Phi_t dt \leq C_1 \left( E^{\frac{1}{2}} + F^{\frac{1}{2}} t^{\frac{1}{4}} \right) dt + dY_t \quad (6.6)
\]
where \( Y_t \) is a martingale with \( Y_0 = 0 \) and \( \frac{d[Y]}{dt} \leq C_2 n^{2/q} \) for some universal constant \( C_1, C_2 \geq 1 \).

By Theorem 12, there exists a Wiener process \( W_t \) such that \( Y_t \) has the same distribution as \( W_{\Psi_t} \). Using the reflection principle for 1-dimensional Brownian motion, we have that
\[
\mathbb{P} \left( \max_{t \in [0,T]} Y_t \geq \gamma \right) \leq \mathbb{P} \left( \max_{t \in [0,T]} W_t \geq \gamma \right) = 2 \mathbb{P} \left( W_{C_2 n^{2/q} T} \geq \gamma \right) \leq O( \exp \left( - \frac{\gamma^2}{2C_2 n^{2/q} T} \right)).
\]
Let \( \Psi_t = n^{\frac{1}{2} - \frac{\beta}{4}} \log^{-\frac{4}{q}} n \). As long as \( \Psi_t \leq \Psi^n \), the estimate (6.6) is valid and hence
\[
\mathbb{P} \left( \max_{t \in [0,T]} \Psi_t \geq \Psi^n \right) \leq \mathbb{P} \left( \max_{t \in [0,T]} Y_t \geq \Psi^n - 1 - \int_0^T C_1 \left( E^{\frac{1}{2}} + F^{\frac{1}{2}} t^{\frac{1}{4}} \right) dt \right)
\]
where we used that $\Psi_0 = 1$ at the last line. Note that
\[
\int_0^T C_1 \left( E \frac{1}{\sqrt{2\pi}} t^{-\frac{3}{2}} + F \frac{1}{\sqrt{2\pi}} t^{-\frac{3}{2}} \right) dt = 2C_1 E \frac{1}{\sqrt{2\pi}} \sqrt{T} + 4C_1 F \frac{1}{\sqrt{2\pi}} T^{1/4} \leq 4C_1 \left( \frac{\frac{1}{4} \alpha n^{-2\beta + \beta} \sqrt{T} + \frac{1}{4} \alpha n^{-2\beta + \beta} T^{1/4} \right).
\]
Setting $T = \frac{n^{-2\beta + \beta} q^2}{2C_1 q\alpha^2 \log n}$ and using $n \geq q^{q}$ for some large constant $q$, we have that
\[
\int_0^T C_1 \left( E \frac{1}{\sqrt{2\pi}} t^{-\frac{3}{2}} + F \frac{1}{\sqrt{2\pi}} t^{-\frac{3}{2}} \right) dt < \frac{\Psi^u}{2} - 1.
\]
Hence, we have that
\[
\mathbb{P}( \max_{t \in [0,T]} \Psi_t \geq \Psi^u) \leq \mathbb{P}( \max_{t \in [0,T]} Y_t \geq \frac{\Psi^u}{2}) = O(\exp(- \left( \frac{\Psi^u}{2} \right)^2 \frac{1}{2C_2 n^{2/3}})).
\]
Note that
\[
\left( \frac{\Psi^u}{2} \right)^2 \frac{1}{2C_2 n^{2/3}} = \frac{n^{\frac{1}{4} + \frac{\beta}{2}} \log^\frac{1}{2} n \cdot 2^{16} C_1^2 q^2 \alpha^2 \log n}{n^{2\beta + \frac{\beta}{2}}} = 2^{13} C_2^{-1} n^{\frac{1}{2} + \frac{\beta}{2}} \sqrt{\log n} \geq C_2^{-1} n^\beta
\]
where we used $q \beta \geq 2$, $0 \leq \beta \leq \frac{1}{4}$, $C_1 \geq 1$, $\alpha \geq 1$, $q \geq 1$ at \textcircled{1}. Using the fact that $\Psi_t \geq \Phi_t^\frac{1}{4}$, we have that
\[
\mathbb{P}( \max_{t \in [0,T]} \Phi_t \geq n^{1 - \frac{1}{4}} \log^{-\frac{1}{2}} n) = O(\exp(-C_2^{-1} n^\beta)) \leq 0.01
\]
where we used that $q \beta \geq 2$ and $n \geq q^{q}$ for some large constant $q$. \hfill \box

**Lemma 49.** Suppose that $\psi_k \leq \alpha k^\beta$ for all $k \leq n$ for some $0 \leq \beta \leq \frac{1}{2}$ and $\alpha \geq 1$. For any even integer $q \geq 2$, if $n \geq q^q$ for some large constant $q$, we have that
\[
\psi_n \leq C \alpha \sqrt{q \log n} n^{\beta - \frac{2}{4}}
\]
for some universal constant $C$.

**Proof.** By Lemma 48, for $t$ up to $T = \frac{n^{\frac{1}{4} + \frac{\beta}{2}}}{q \alpha^2 \log n}$, with probability 0.99,
\[
\text{Tr}((A_t - I)^q) = \Phi_t \leq n^{1 - \frac{1}{4}} \log^{-\frac{1}{2}} n \quad \text{for all } 0 \leq t \leq T.
\]
Assuming this event, we have
\[
\|A_t\|_2 \leq 1 + n^{\frac{1}{4} + \frac{\beta}{2}} \log^{-\frac{1}{2}} n \quad \text{for all } 0 \leq t \leq T.
\]
and
\[
\int_0^T \|A_t\|_2 dt \leq T \cdot (1 + n^{\frac{1}{4} + \frac{\beta}{2}} \log^{-\frac{1}{2}} n) \leq \frac{C_1 n^{1 + \frac{1}{2} + \frac{\beta}{2}}}{q \alpha^2 \log^{3/2} n}
\]
which is less than $\frac{1}{64}$ when $n$ is large enough. Also, we have that $B_T = T \cdot I$. Hence, we can apply Lemma 31 and get that
\[
\psi_p = O \left( \alpha \sqrt{q \log n} n^{\beta - \frac{2}{4}} \right).
\]
Since this argument holds for any isotropic logconcave distribution, this gives the bound for $\psi_n$. \hfill \box

**Proof of Theorem 37.** Fix a large enough $n$. We start with a known bound:
\[
\psi_k \leq \alpha_1 k^{\beta_1} \quad \text{for all } k \leq n
\]
where $\alpha_1$ is some universal constant larger than $1$ and $\beta_1 = \frac{1}{4}$. Now, we apply Lemma 49 for every $k \leq n$ with $q = 2 \left\lceil \frac{1}{\beta_1} \right\rceil$. Hence, we have that
\[
\psi_k \leq 4C \alpha_1 \frac{1}{2} \sqrt{\log nk^{\beta_1} - \frac{2}{4}} \quad \text{for all } k \leq n.
\]
Repeating this process, we have that \( \psi_k \leq \alpha \ell k^{\beta_\ell} \) for all \( k \leq n \) with

\[
\alpha_{\ell+1} = 4C \alpha \beta_\ell^{-1} \sqrt{\log n}, \\
\beta_{\ell+1} = \beta_\ell - \frac{\beta_\ell^2}{16}.
\]

By induction, we have that \( \alpha_\ell = O(\ell \log n)\ell/2 \) and \( \beta_\ell \leq \frac{16}{\ell} \). Hence, we have that

\[
\psi_n \leq O(\ell \log n)^{\ell/2} n^{\frac{16}{\ell}}
\]

for all \( \ell \geq 1 \). Setting \( \ell = \left\lceil \sqrt{\log n / \log \log n} \right\rceil \), we have that

\[
\psi_n = n^{O\left(\sqrt{\frac{\log \log n}{\log n}}\right)} = \exp\left(O\left(\sqrt{\log n \log \log n}\right)\right).
\]

\[
\square
\]

7 Adaptive localization for anisotropic distributions

In this section, we show that the same third moment assumption gives the following bound on the KLS constant for arbitrary logconcave distributions.

**Theorem 50.** Under the third moment assumption of Section 6, for any logconcave density \( p \) in \( \mathbb{R}^n \) with covariance matrix \( A \), for any integer \( q \geq 1 \), the KLS constant is bounded as

\[
\psi_p \leq C q^{O(q)} \left( \text{Tr}(A^q) \right)^{\frac{1}{n}}.
\]

7.1 Controlled stochastic localization

**Definition 51.** Given a symmetric matrix \( B \), let \( E_{<u}(B) \) be the span of all eigenvectors in \( B \) with eigenvalues less than \( u \) and \( \Lambda_{<u}(B) = \dim E_{<u}(B) \). We define \( E_{\geq u}, \Lambda_{\geq u} \), etc similarly.

For this reduction, we apply localization only in the subspace where the matrix \( B_t \) controlling the Gaussian has small eigenvalues. At time \( t \), the control matrix is chosen so that it is the inverse of the projection of the current covariance matrix \( A_t \) to the subspace of the small eigenvalue of \( B_t \). This is captured in the next definition.

**Definition 52.** Given a logconcave distribution \( p \) and a threshold \( u \), we define the following process: \( p_0 = p, \ c_0 = 0, \ B_0 = 0, \ T_0 = 0 \) and for \( k \geq 1 \),

1. \( T_k = \inf\{t > T_{k-1} \mid \Lambda_{<u}(B_t) \neq \Lambda_{<u}(B_{T_{k-1}})\} \).
2. \( p_t, c_t, B_t \) are defined by Definition 24 on \( [T_{k-1}, T_k] \) with the initial data \( c_{T_{k-1}}, B_{T_{k-1}}, p_{T_{k-1}} \) (instead of 0, 0 and \( p \) ) and with the control matrix \( C_t \) given by

\[
C_t = \psi_{2\epsilon(k)}^{-2} \lim_{s \to \infty} (I + A_t + sP^{(k)})^{-1}
\]

where \( A_t \) is the covariance matrix of \( p_t \), \( r^{(k)} = \Lambda_{<u}(B_{T_{k-1}}) \), \( P^{(k)} \) is the orthogonal projection onto \( E_{\geq u}(B_{T_{k-1}}) \) and \( \psi_{2\epsilon(k)} \) is any known bound on the KLS constant for isotropic logconcave densities. Let \( r_t = \text{rank}(C_t) \).

The following lemma gives an alternative definition of \( C_t \).

**Lemma 53.** For any \( A \succ 0 \) and an orthogonal projection matrix \( P \), we let \( T = \lim_{s \to \infty} (A + sP)^{-1} \). Then, we have that

\[
T = ((I - P)A(I - P))^{\dagger}
\]

where \( \dagger \) denotes pseudoinverse. Furthermore, we have that \( \text{R}(P) = N(T) \), i.e., the rowspace of \( P \) equals the nullspace of \( T \).
Proof. By taking $P = U^T \Sigma U$, we can see that
\[
\lim_{s \to \infty} (A + sP)^{-1} = U^T \lim_{s \to \infty} (UAU^T + s\Sigma)^{-1}U
\]
and
\[(I - P)A(I - P)^\dagger = U^T (I - \Sigma UAU^T (I - \Sigma))^\dagger U.
\]
Hence, it suffices to prove the case $P$ is a diagonal matrix whose first $r$ diagonal entries are 0 and the remaining diagonal entries are 1. Write
\[A = \begin{bmatrix} A_1 & A_2 \\ A_3 \\ A_4 \end{bmatrix}
\]
where $A_1$ is a $r \times r$ matrix. Then, we have that
\[(A + sP)^{-1} = \begin{bmatrix} A_1 & A_2 \\ A_3 + sI \end{bmatrix}^{-1} = \begin{bmatrix} A_1^{-1} + A_1^{-1}A_2FA_1^{-1} & -A_1^{-1}A_2F \\ -FA_2^T A_1^{-1} \\ 0 \\ 0 \end{bmatrix}.
\]
where $F = (A_3 + sI - A_3^TA_1^{-1}A_2)^{-1}$. As $s \to \infty$, we have that $F \to 0$ and hence
\[\lim_{s \to \infty} (A + sP)^{-1} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} = ((I - P)A(I - P))^\dagger.
\]
For any $x \in R(P)$, we have that $x$ is 0 in the first $r$ coordinates and hence
\[0 \leq x^T Tx \leq x^T (I + sP)^{-1} x = \frac{\|x\|^2}{1 + s}.
\]
Taking $s \to \infty$, we have that $x^T Tx = 0$. Using $T \geq 0$, this shows that $R(P) \subset N(T)$. On the other hand, we have that
\[A + sP \preceq \|A\|_2 (I - P) + sP
\]
and hence
\[(A + sP)^{-1} \succeq \|A\|_2^{-1} (I - P) + s^{-1}P.
\]
Taking the limit, we have that $T \succeq \|A\|_2^{-1} (I - P)$. This shows that $N(T) \subset R(P)$. Hence, we have $R(P) = N(T)$.\qed

The specific formula above is not important and the reduction of this section uses only the following properties of the control matrix $C_t$.

Lemma 54. For any $t \geq 0$, we have that

1. (Focus on small values in $A_t$) $0 \preceq C_t \preceq \psi_{2r_1}^{-2} A_t^{-1}$.

2. (Focus on small values in $B_t$) $B_t \preceq uI$ and $r_t = \nu(k) = \Lambda_{<u}(B_t)$ for $t \in [\tau_{k-1} \tau_{k})$.

3. (Large step size) $\text{Tr}C_t \geq \|t_i^{1/2}/(\psi_{2r_1}(\Phi^{1/2} + 1))
\]

Proof. For the first part, since $I + A_t + sP(k) \preceq A_t$, we have that $\psi_{2r_1}^{-2} A_t^{-1} \preceq C_t \preceq 0$.

For $B_t \preceq uI$ in the second part, we prove it by a continuous induction. Let $t' = \inf_{t \geq 0} \{x^T B_t x > u \text{ for some } \|x\|^2 = 1\}$. Suppose that $t' < +\infty$. By the definition, we have that $B_t \preceq uI$ for $t \leq t'$. Fix any $t \in [\tau_{k-1} \tau_k)$.

Since $B_t \preceq uI$, we can write $B_t = B'_t + B''_t$ where all eigenvalues of $B'_t$ are $u$ and $B''_t \preceq uI$. Since $\frac{dB_t}{dt} = C_t \geq 0$ and since the number of eigenvalues being $u$ is unchanged during $[\tau_{k-1} \tau_k)$, we know that $B'_t = B'_{\tau_{k-1}}$. By the definition of $P(k)$, we have that
\[R(P(k)) = E_{\geq u}(B'_t) = E_{\geq u}(B'_t).
\]
For any $x^T B_t x \geq u \|x\|^2$, we have that $x \in E_{\geq u}(B'_t)$ because $B_t \preceq uI$. Hence, we have that $x \in R(P(k))$ and Lemma 53 shows that $R(P(k)) = N(C_t)$ and hence $x^T C_t x = 0$. Since $\frac{dB_t}{dt} = C_t$, we have that
\[x^T dB_t \frac{dt}{dt} x = 0 \text{ for any } x^T B_t x \geq u \|x\|^2 \text{ and any } t \leq t'.
\]
This contradicts the definition of \( t' \). Therefore, we have that \( B_t \preceq uI \) for all \( t \).

For \( r_t = r^{(k)} = \Lambda_{<u}(B_t) \) in the second part, Lemma 53 shows that \( R(P^{(k)}) = N(C_t) \). Therefore, we have that

\[
\begin{align*}
    n - r_t = \dim N(C_t) = \text{rank}(P^{(k)}) = n - r^{(k)} = n - \Lambda_{<u}(B_{t_{k-1}}) = n - \Lambda_{<u}(B_t)
\end{align*}
\]

where we used that the number of eigenvalues being \( u \) is unchanged during \([\tau_{k-1}, \tau_k] \) at the end.

For the third part, we use the inequality

\[
    r \leq \left( \sum_{i=1}^{r_t} x_i^q \right)^{1+\frac{1}{q}} \left( \sum_{i=1}^{r_t} x_i^{-1} \right)^{1+\frac{1}{q}} 
\]

and the fact that \( C_t \) is a rank \( r_t \) matrix. We have that

\[
    r_t \leq \left( \text{Tr} C_t^{1/q} \right)^{1+\frac{1}{q}} \left( \text{Tr} C_t \right)^{1+\frac{1}{q}}
\]

where we use \( C_t^{1/q} \) to denote the \( q^{th} \) power of pseudo inverse of \( C_t \). By Lemma 53, we have that

\[
    C_t = \psi_{2r_t}^2(I - P^{(k)})(I + A_t)(I - P^{(k)}).
\]

Hence, we have that

\[
    \text{Tr} C_t^{1/q} \overset{\text{Lem. 53}}{\leq} \psi_{2r_t}^2 \text{Tr}(I - P^{(k)})^q(I + A_t)^q(1 - P^{(k)})^q \leq \psi_{2r_t}^2 \text{Tr}(I + A_t)^q.
\]

Putting this in (7.1) gives that

\[
    \text{Tr} C_t \geq \frac{r_t^{1+\frac{1}{q}}}{\left( \text{Tr} C_t^{1/q} \right)^{1+\frac{1}{q}}} \geq \frac{r_t^{1+\frac{1}{q}}}{\psi_{2r_t}^2 (\Phi_t^{1/q} + 1)}.
\]

\[\square\]

### 7.2 Analysis of \( A_t \)

**Lemma 55.** Let \( A_t \) be defined by Definition 52. For any integer \( q \geq 2 \), we have that

\[
    d\text{Tr}(A_t^q) = q\mathbb{E}_{x \sim p_{t_i}}(x - \mu_t)^T A_t^{q-1}(x - \mu_t)(x - \mu_t)^T C_t^{1/2} dW_t - q\text{Tr}(A_t^{q+1}C_t) dt
    + \frac{1}{2} \sum_{\alpha + \beta = q-2} \mathbb{E}_{x \sim p_{t_i}}(x - \mu_t)^T A_t^\alpha(y - \mu_t)(x - \mu_t)^T A_t^{\beta}(y - \mu_t)(x - \mu_t)^T C_t(y - \mu_t) dt.
\]

**Proof.** Note that \( A_t \) is defined by concatenating solutions of finitely many SDEs. Therefore, it suffices to prove this equality for each SDE solution and this follows from Lemma 36. \[\square\]

In Section 6 that \( A_t - I \) may not be positive semi-definite and hence we need to take \( q \) to be even integer. But in this section, we analyze the process by the potential \( \text{Tr}(A_t^q) \) and hence we do not require that \( q \) to be even.

**Lemma 56.** Let \( A_t \) be defined by Definition 52. Let \( \Phi_t = \text{Tr}(A_t^q) \) for some integer \( q \geq 2 \), then we have that \( d\Phi_t = \delta_t dt + v_t^T dW_t \) with

\[
    \delta_t \leq O(q^2) \Phi_t \quad \text{and} \quad \|v_t\|_2 \leq O(q) \Phi_t.
\]

**Proof.** By Lemma 55, we have

\[
    d\Phi_t = q\mathbb{E}_{x \sim p_{t_i}}(x - \mu_t)^T A_t^{q-1}(x - \mu_t)(x - \mu_t)^T C_t^{1/2} dW_t - q\text{Tr}(A_t^{q+1}C_t) dt
    + \frac{1}{2} \sum_{\alpha + \beta = q-2} \mathbb{E}_{x \sim p_{t_i}}(x - \mu_t)^T A_t^\alpha(y - \mu_t)(x - \mu_t)^T A_t^{\beta}(y - \mu_t)(x - \mu_t)^T C_t(y - \mu_t) dt
    = q\mathbb{E}_{x \sim p}(x - \mu_t)^T A_t^{q-1}(x - \mu_t)(x - \mu_t)^T C_t^{1/2} dW_t - q\text{Tr}(A_t^{q+1}C_t) dt
    + \frac{1}{2} \sum_{\alpha + \beta = q-2} \mathbb{E}_{x \sim p}(x - \mu_t)^T A_t^{q+1}(x - \mu_t)^T y^T A_t^{q+1}(x - \mu_t)^T C_t A_t^{1/2} y dt
    \overset{\text{def}}{=} \delta_t dt + v_t^T dW_t.
\]
where \( \tilde{p}_t \) is the isotropic version of \( p_t \) defined by \( \tilde{p}_t(x) = p(A_t^{1/2}x + \mu_t) \), \( \delta_t \) is the drift term in \( d\Phi_t \) and \( v_t^d W_t \) is the martingale term in \( d\Phi_t \).

For the drift term \( \delta_t dt \), using that \( C_t \geq 0 \), we have
\[
\delta_t \leq \frac{q}{2} \sum_{\alpha + \beta = q-2} T(A_t^{\alpha+1}, A_t^{\beta+1}, A_t^{1/2} C_t A_t^{1/2}) - q T(A_t^{q+1}, C_t)
\]
\[
\leq \frac{q}{2} \sum_{\alpha + \beta = q-2} T(A_t^{\alpha+1}, A_t^{\beta+1}, A_t^{1/2} C_t A_t^{1/2})
\]
\[
\leq \frac{q}{2} \sum_{\alpha + \beta = q-2} T(A_t^q, I, A_t^{1/2} C_t A_t^{1/2})
\]
\[
\leq q^2 T(A_t^1, A_t^{1/2} C_t A_t^{1/2}, I)
\]
\[
\leq O(q^2 v_t^2) \Phi_t \left\| A_t^{1/2} C_t A_t^{1/2} \right\|_2
\]
\[
\leq O(q^2 \Phi_t).
\]

For the martingale term \( v_t^d W_t \), we note that
\[
\|v_t\|_2 = q \left\| \mathbb{E}_{x \sim p_t} (x - \mu_t)^T A_t^{-1} (x - \mu_t)(x - \mu_t)^T C_t^{1/2} \right\|
\]
\[
\leq O(q) \left\| A_t^{1/2} C_t A_t^{1/2} \right\|^{1/2}_2 \text{Tr} \left[ A_t^{1/2} A_t^{-1} A_t^{1/2} \right]
\]
\[
\leq O(q^2 \psi_{2r_t}^2) \Phi_t \leq O(q^2 \Phi_t).
\]

where we used that \( \psi_{2r_t} = O(1) \) at the last line.

Using these, one can bound the growth of \( \Phi_t \).

**Lemma 57.** Let \( A_t \) be defined by Definition 52 using initial distribution \( p \). Let \( \Phi_t = \text{Tr}(A_t^q) \) for some integer \( q \geq 2 \). Suppose that \( \Phi_0 = n \), there is a universal constant \( c \) such that
\[
\mathbb{P}(\max_{t \in [0, T_{max}]} \Phi_t \geq 2n) \leq 0.01 \quad \text{with} \quad T_{max} = \frac{1}{64(C' + C)q^2}.
\]

**Proof.** By Lemma 56 and Itô’s formula, we have that
\[
d \log \Phi_t = \Phi_t^{-1} d\Phi_t - \frac{1}{2} \Phi_t^{-2} d\{d]\_t
\]
\[
\leq \Phi_t^{-1} (\delta_t dt + v_t^d W_t)
\]
\[
\leq O(q^2) dt + \Phi_t^{-1} v_t^d W_t
\]
\[
\leq C q^2 dt + dY_t
\]
(7.2)

where \( dY_t = \Phi_t^{-1} v_t^d W_t \), \( Y_t = 0 \) and \( C \) is some universal constant.

Note that
\[
\frac{d|Y_t|}{dt} = \Phi_t^{-2} \|v_t\|_2^2 = O(q^2) \leq C' q^2
\]
for some universal constant \( C' \). By Theorem 12, there exists a Wiener process \( \tilde{W}_t \) such that \( Y_t \) has the same distribution as \( \tilde{W}_{|Y_t|} \). Using the reflection principle for 1-dimensional Brownian motion, we have that
\[
\mathbb{P}(\max_{t \in [0, T]} Y_t \geq \gamma) \leq \mathbb{P}(\max_{t \in [0, C' q^2 T]} \tilde{W}_t \geq \gamma) = 2 \mathbb{P}(\tilde{W}_{C' q^2 T} \geq \gamma) \leq 2 \exp(-\frac{\gamma^2}{2 C' q^2 T}).
\]

Since \( \Phi_0 = n \), we have that \( \log \Phi_0 = \log n \) and therefore (7.2) shows that
\[
\mathbb{P}(\max_{t \in [0, T]} \log \Phi_t \geq \log n + C q^2 T + \gamma) \leq 2 \exp(-\frac{\gamma^2}{2 C' q^2 T}).
\]

Putting \( T = \frac{1}{64(C' + C)q^2} \) and \( \gamma = \frac{1}{4} \), we have that
\[
\mathbb{P}(\max_{t \in [0, T]} \log \Phi_t \geq \log n + \frac{2}{3}) \leq 2 \exp(-8) \leq 0.01.
\]
7.3 Proof of Theorem 50

Lemma 58. Let $A_t$ be defined by Definition 52 using initial distribution $p$. Suppose that $\text{Tr}(A_t^q) = n$. Then, we have that

$$\mathbb{P}(B_{\text{max}} = uI) \geq 0.99 \quad \text{with} \quad T_{\text{max}} = \frac{1}{c_1 q^2} \quad \text{and} \quad u = q^{-c_2 q} n^{-1/q}$$

where $c_1$ and $c_2$ are universal constants.

Proof. By Lemma 57 and Lemma 54, we have that $\Phi_t \leq 2n$ for all $0 \leq t \leq T_{\text{max}}$ with 0.99 probability, subject to this event, we have that $\text{Tr}C_t \geq c \psi_{2r_t}^{-1/4} n^{-1/q}$. Let $\Psi_t = \text{Tr}B_t$. Then this shows that

$$\frac{d\Psi_t}{dt} \geq c \psi_{2r_t}^{-1/4} n^{-1/q} \geq 0.$$

By Theorem 31, we have that $\psi_n^{-2} \geq e^{-C \sqrt{\log n \log \log n}}$ for some universal constant $C \geq 0$. Hence, we have that

$$\frac{d\Psi_t}{dt} \geq c e^{-C \sqrt{\log n \log \log n}} r_t^{1+4} n^{-1/q} \geq 0.$$

Also, Lemma 54 shows that $r_t = \Lambda_{c_0} (B_t)$. Therefore, we have $u n \geq \Psi_t \geq u (n - r_t)$. Let $T_k = \inf_{t \geq 0} \{ \Psi_t \geq u n (1 - 2^{-k}) \}$. For any $t \in [T_{k-1}, T_k)$, we have that

$$r_t \geq n - \frac{\Psi_t}{u} \geq n 2^{-k}.$$

Since $e^{-C \sqrt{\log n \log \log n}} r_t^{1+4}$ is an increasing function in $r$, for $t \in [T_{k-1}, T_k)$, we have that

$$\frac{d\Psi_t}{dt} \geq c e^{-C \sqrt{\log (n 2^{-k}) \log \log (n 2^{-k})}} (n 2^{-k})^{1+4} n^{-1/q} \geq n 2^{-k} \cdot c 2^{-k/4} e^{-C \sqrt{\log (n 2^{-k}) \log \log (n 2^{-k})}}.$$

From the definition of $T_k$, we see that

$$T_k - T_{k-1} \leq \frac{u 2^{-k}}{n 2^{-k} \cdot c 2^{-k/4} e^{-C \sqrt{\log (n 2^{-k}) \log \log (n 2^{-k})}}} = O(u) 2^{k/q} e^{-C \sqrt{\log (n 2^{-k}) \log \log (n 2^{-k})}}.$$

Therefore, we have that

$$T_{\log_2 n} \leq O(u) \sum_{k=0}^{\log_2 n} 2^{k/q} e^{-C \sqrt{\log (n 2^{-k}) \log \log (n 2^{-k})}} = u q^{O(q)} n^{1/q}$$

where the last inequality can be seen by noting that the sequence is exponentially increasing with rate $1/q$ until the maximal at $2^k = n/q^{0(q^2)}$.

Setting $u = \frac{1}{q^{0(q^2)} n^q}$, we have that $T_{\log_2 n} \leq T_{\text{max}}$ and hence $\Psi_t$ increases to $u n$ and $B_t = u I$ before time $T_{\text{max}}$. After $B_t$ has increased to $u I$, $C_t = 0$ and the localization process freezes.

Proof of Theorem 50. The case $q = 1$ is proven in Theorem 2. So, we assume $q \geq 2$. By rescaling, we can assume $\text{Tr}A^q = n$. To apply Lemma 31, we note that by Lemma 58, we have that

$$\mathbb{P}(B_{T_{\text{max}}} = u I) \geq 0.99 \quad \text{with} \quad T_{\text{max}} = \frac{1}{c_1 q^2} \quad \text{and} \quad u = q^{-c_2 q} n^{-1/q}.$$

Furthermore, Lemma 54 shows that

$$\left\| C_s^{1/2} A_s C_t^{1/2} \right\|_2 = \left\| A_t^{1/2} C_t A_t^{1/2} \right\|_2 \leq \psi_{2r_t}^{-2} = O(1).$$

Therefore,

$$\mathbb{P} \left( \int_0^{T_{\text{max}}} \left\| C_s^{1/2} A_s C_s^{1/2} \right\|_2 ds \leq \frac{1}{64} \quad \text{and} \quad \text{Tr}_{\text{max}} \geq u I \right) \geq 0.99.$$

Hence, Lemma 31 shows that

$$\psi_p = \Omega \left( u^{-1/2} \right) = q^{O(q)} n^{1/2q} = q^{O(q)} (\text{Tr}A^q)^{1/2q}.$$

\[\square\]