Transposition of a local-density-dependent pion-nucleus potential to an effective density-linear potential - generalized Seki-Masutani relations

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We have shown that a local-density-dependent term, $F[\rho(r)\rho(r)]$, of the pion-nucleus potential with a nuclear density, $\rho(r)$, can be transposed into a conventional density-linear term, $F(\rho_c)\rho(r)$, with an effective nuclear density, $\rho_c$, which is found to be close to $\sim 0.6\rho(0)$ for most $\pi^-$ bound states. The presently found relations, generalized Seki-Masutani relations, for the density-quadratic term, the medium-modified isovector term and the double-scattering isoscalar term of the $s$-wave pion-nucleus interaction assure that the constant parameters in the conventional pion-nucleus potential are interpreted as being effective ones in the light of density-dependent effects.

1. Introduction

The so-called “anomalous $s$-wave repulsion” in the pion-nucleus interaction has been a long-standing problem in pion physics (see, for instance, [1]). Many authors (see, for instance, [1,2]) have claimed the existence of a large extra repulsion arising from the density quadratic term, $ReB_0\rho(r)^2$, from global fits of pionic atom data to the Ericson-Ericson potential [3,4]:

\begin{align}
V(r) &= U_s(r) + U_p(r), \\
U_s(r) &= -(2\pi/m_\pi) \\
&\quad \times \left[ \epsilon_1(b_0\rho(r) + b_1\Delta\rho(r)) + \epsilon_2B_0\rho(r)^2 \right], \\
\rho(r) &= \rho_p(r) + \rho_n(r), \\
\Delta\rho(r) &= \rho_n(r) - \rho_p(r),
\end{align}

where $U_s(r)$ and $U_p(r)$ are for $s$-wave and $p$-wave parts, respectively, and $\rho_p(r)$ and $\rho_n(r)$ are the proton and neutron density distributions, respectively, and $\epsilon_1 = 1 + m_\pi/M_N$ and $\epsilon_2 = 1 + m_\pi/2M_N$ (hereafter, we adopt units of $m^{-1}_n$, $m^{-1}_p$ and $m^{-1}_\pi$ for $b_0$, $b_1$ and $ReB_0$, respectively). New light has been shed on this problem, namely, the possibility of an anomalous repulsion in the isovector part ($b_1$) as a unique signature of chiral symmetry restoration was pointed out [5,6]. In this respect, the deeply bound $\pi^-$ states which were produced in heavy nuclei (in $^{207}$Pb [7–10] and in $^{205}$Pb [11,12]) are of particular importance, since the binding energies and widths of $1s\pi^-$ states depend nearly entirely on the $s$-wave potential ($U_s$) and, thus, only the $1s$ states in heavy nuclei ($N > Z$) provide unique information on the isovector part of the $s$-wave potential. In fact, the above conjecture has been proved by using new information from the $1s\pi^-$ state in $^{205}$Pb. The experiment indicates an enhanced isovector parameter, which is related to a decreased order parameter of chiral symmetry breaking in the nuclear medium, $b_1^\text{free}/b_1 = f_\pi^2/f_\pi^2 = 0.78^{+0.13}_{-0.09} [12]$. Recent global fits of pionic atom data by Friedman [13,14] have shown consistency with the above view.

Since the potential parameter $b_1$ is nuclear-density dependent in the theoretical context of Weise [5], a question can be raised as to the mutual relation in adopting the conventional Ericson-Ericson potential with constant coefficients (model C) and a modified local-density-dependent (model LDD) potential. In the present note we show that both methods are equally valid and are connected to each other, since a density-dependent term can be effectively linearized. Our procedure gives an answer to the following question: where do the bound pions probe the nuclear potential? This problem is closely related to the so-called Seki-Masutani (SM) relation [16], which emphasized a strong correlation between the density-linear isoscalar parameter ($b_0$) and the density-quadratic parameter ($ReB_0$). In a sense, we generalize the SM relation.

2. Where does a bound $\pi^-$ probe the nuclear potential?

First, we examine which part ($r$) of the nuclear density, $\rho(r)$, is probed by a bound $\pi^-$. Two typical nuclei, $^{16}$O and $^{208}$Pb, are considered. The former possesses only shallow bound states, whereas the latter accommodates deeply bound states. The nuclear matter densities are assumed to take a 2-parameter Fermi distribution with a half-density radius ($r_p$ and $r_n$) and a diffuseness parameter ($z_p$ and $z_n$). The parameter values were chosen as follows: $r_p = 6.652$ fm, $r_n = 6.892$ fm, and $a_p = a_n = 0.5234$ fm for $^{208}$Pb, and $c_p = c_n = 2.482$ fm and $a_p = a_n = 0.5234$ fm for $^{16}$O. The central nucleon densities are: $\rho_0 = 0.150$ for $^{208}$Pb and 0.172 for $^{16}$O. We take known potential parameters for the $p$-wave parts [1] and a set of $s$-wave parameters ($b_0 = -0.028$, $b_1 = -0.12$, $ReB_0 = 0$ and $ImB_0 = 0.055$).

Figure 1 shows the $\pi^-$ densities ($R_{\text{nl}}(r)^2$), the nuclear densities ($\rho(r)$) and the overlapping densities (namely, the nuclear densities probed by $\pi^-$), defined as

$$S(r) = \rho(r)|R_{\text{nl}}(r)^2|^2r^2,$$

for $^{16}$O (1s and 2p) and $^{208}$Pb (1s, 2p and 3d). From these figures we notice that the overlapping density is
peaked at a radius slightly less than the half-density radius, nearly independent of the nucleus and the π− quantum numbers. This means that the bound π− effectively probes a fraction of the full nuclear density ($\rho_0 = \rho(0)$): 

$$\rho_e \sim 0.60 \rho_0.$$  

(5)

This is a key to intuitively understanding the following results of numerical calculations.

3. Seki-Masutani relation

Seki and Masutani [16] emphasized the presence of a strong correlation between $b_0$ and Re$B_0$ in describing pionic-atom binding energies and widths, which we call the Seki-Masutani relation. Toki et al. found the same correlation also exists for the deeply bound 1s and 2p states in $^{208}$Pb [17,18]. Recently, this correlation has been revisited, and the following common relation has been established both theoretically and empirically [12]:

$$b_0^e \equiv b_0 + 0.215 \text{Re}B_0 = \text{constant.}$$  

(6)

This means that the binding energies (and widths) are nearly unchanged by varying either $b_0$ or Re$B_0$ as long as these parameters are moved together to fulfill the above SM relation. In other words, none of $b_0$ and Re$B_0$ can be uniquely determined from given binding energies; only their combination, $b_0^e$, can be experimentally determined.

Actually, various global fits gave widely distributed values with large errors for these parameters. For instance, the best-fit values of $\{b_0, \text{Re}B_0\}$ are $\{0.024(15), -0.26(3)\}$ in KLTK90 [2] and $\{0.000(15), -0.14(7)\}$ in BFG97 [1]. With a different definition of $b_0$, namely, $b_0^F = b_0 - \Delta b_0^{\text{DS}}$, in which the double-scattering term ($\Delta b_0^{\text{DS}}$, to be discussed later) is subtracted, Friedman obtained three different sets for $\{b_0^F, \text{Re}B_0\}$: $\{0.018(19), -0.14(4)\}$ (F02a) [13], $\{0.030(10), -0.21(4)\}$ (F02b) [14] and $\{0.020(10), -0.15(4)\}$ [14]. These widely split values are shown by large open circles with both vertical and horizontal error bars in Fig. 2. They appear to be almost meaningless, contrary to their claim that the parameter Re$B_0$ is well determined. On the other hands, a $\chi^2$ fit with a gridding Re$B_0$ yields a series of points, as shown by closed squares (FG98 [15]) and by small open circles (F02a [13]), both of which indicate linear relations parallel to the SM correlation.

Recently, the value of $b_0$ with a gridding Re$B_0$ was precisely determined by $\chi^2$ fits of the 1s $\pi^-$ binding energies in six light symmetric nuclei ($^{12}$C, $^{14}$N, $^{16}$O, $^{20}$Ne, $^{24}$Mg and $^{28}$Si) [12]. The obtained data sets, shown by closed circles in Fig. 2 (their sizes correspond to the error bars), prove the above SM relation (6) perfectly, yielding a precise value of

$$b_0^e = -0.028 \pm 0.001.$$  

(7)

It is to be noted that the $b_0^e$ composed of the widely distributed values of $b_0$ and Re$B_0$ in each of KLTK90, BFG97, F02a and F02b would yield a value, close to $-0.030 \sim -0.034$ (after subtraction of the double scattering term in the case of F02a and F02b), which is not so much distributed as the individual values of $b_0$ and Re$B_0$. These authors seemed to be puzzled by the large-ness of their [Re$B_0$] values compared with Im$B_0 = 0.055$. On the other hand, they never took seriously the other discrepancy that their $b_0$'s (after subtraction of the double scattering term) take large positive values, meaning a strongly attractive $b_0$ in contrast to the nearly vanishing free $\pi$N value [19,20]. Such odd consequences from the global fits must be artifacts, arising from regarding $b_0$ and Re$B_0$ as independent uncorrelated parameters.

4. Seki-Masutani Ansatz and its generalization

Seki and Masutani [16] showed that the correlation between $b_0$ and Re$B_0$ can be understood by imposing an effective replacement of the expectation value of the density-quadratic term as,

$$< \rho(r)^2 > \rightarrow \rho_e < \rho(r),$$  

(8)

which is expected to hold for any pionic atom state with a common value of $\rho_e$ (Seki-Masutani Ansatz). They claimed an effective nuclear density $\rho_e \sim 0.50 \rho_0$. We understand that the localization of the overlapping densities, $S(r)$, near $r \sim r_e$, as shown in Fig. 1, is the key to justify the SM Ansatz.

We now examine how the Seki-Masutani Ansatz applies to various functional forms of the potential, $U_s(\rho(r))$. Let us add the following form to the real part of the s-wave pion-nucleus potential:

$$\Delta U_s(r) = F[\rho(r)] \rho(r),$$  

(9)

where $F[\rho(r)]$ is a local-density-dependent (LDD) functional coefficient, representing an additional density-non-linear potential involving a local density, $\rho(r)$. We solve the Klein-Gordon equation with LDD potentials and compare the numerical results with those without invoking LDD assumptions, thereby examining how the effects of the LDD potentials are expressed. Following the spirit of SM, we study how this effect is transposed into an effective coefficient,

$$F[\rho(r)] \rho(r) \rightarrow \mathcal{F}(\rho_e) \rho(r),$$  

(10)

where $\mathcal{F}(\rho_e)$ is an effective constant parameter involving an effective nuclear density, $\rho_e$. Namely,

$$\mathcal{F}(\rho_e) = \frac{< F[\rho(r)] \rho(r) >}{< \rho(r) >}.$$  

(11)

5. Polynomial case

Let us first consider the case of a density-polynomial term, which can be expressed as
The effective density of the above LDD potential can be replaced by a conventional potential, and the second relation is a new relation for the $D\rho(r)^3$ term.

The first relation is the well-known Seki-Masutani relation for the $B\rho(r)^2$ term, and the second relation is a new relation for the $D\rho(r)^3$ term.

The effective density does not depend much on the individual states, as we have found in the cases of $^{16}\text{O}$ and $^{208}\text{Pb}$. We are therefore tempted to expect that the concept of the effective density applies to any functional form of $U(\rho(r))$.

**6. Density-dependent $b_1$ parameter**

Next, let us consider the case of an isovector s-wave potential which is expected to be enhanced due to a partial restoration of the chiral symmetry in the nuclear medium [5,6]. We assume the following potential according to Weise [5]:

$$b_1(r)\Delta\rho(r) = b_1^{\text{free}}\frac{\Delta\rho(r)}{1 - \alpha \rho(r)},$$

where $\alpha$ is a constant parameter representing the density-dependent effect. We examine whether or not the effect of the above LDD potential can be replaced by a conventional potential with a constant parameter involving an effective density $\rho_e$, as in

$$\rho_e = \frac{\Delta\rho(r)}{1 - \alpha \rho(r)}.$$

Numerical calculations of the binding energies yielded the dependence of $B_{1s}$ on the parameter $\alpha$, as shown in Fig. 3. On the other hand, the standard procedure assuming $b_1$ to be a constant gave another relation between $B_{1s}$ and $b_1$. The two relations are found to be nearly identical, if we take $b_1 = \frac{\rho_e}{1 - \alpha \rho_e}$ with

$$\rho_e \sim 0.090 \sim 0.60 \rho_0.$$
eq.(20) with $\alpha$ as a running parameter was adopted. The results were compared with those calculated with $b_1$ as a constant, as shown in Table I. The effective density, $\rho_c/\rho_0 \sim 0.60$, for $^{16}\text{O}$ and $^{208}\text{Pb}$ agrees well with that found for $b_1(\rho)$. This means that the $\tilde{b}_1$ effectively converted from the isovector LDD potential term has the same effect as the density-dependent $b_1(\rho)$ in the double-scattering formula, eq.(26).

If we assert that the isoscalar potential strength $b_0^\rho$, eq.(7), is composed of the free $b_0$ (essentially zero), the double scattering term and $\text{Re}B_0$, we can deduce $\text{Re}B_0$ empirically as [12]

$$\text{Re}B_0 = \frac{b_0^\rho - \Delta U_{\text{LDD}}}{0.215} \sim -0.038 \pm 0.025.$$  (27)

This value looks reasonable, when compared with $\text{Im}B_0 = 0.055$.

8. RIA correction

The present procedure can be applied to any kind of additional potential term. Friedman [13,14] takes into account the following correction

$$\Delta U_{\text{RIA}} = -\frac{2\pi}{m} \frac{3m}{m_\pi} d_0 \frac{(1 + \alpha \rho)^2 - 1}{2} k_F(\rho)^2 \rho(r),$$  (28)

where $d_0 = -0.19$ $m_\pi^{-3}$ and $\alpha = 2.7$ fm$^{-3}$. The effect of adding $\Delta U_{\text{RIA}}$ was calculated and was found to be equivalent to invoking an additional isoscalar term

$$\Delta b_0^\text{RIA} \sim -0.021.$$  (29)

This correction can be obtained also by setting $\rho(r) \rightarrow \rho_c \sim 0.5 \rho_0$.

9. Angle transformation term

Finally, we make some comments on the so called angle transformation term as an additional correction to the EE potential,

$$\Delta U_{\text{AT}} = -(2\pi/m_\pi)[(m_\pi/2M)\epsilon_1^{-1}\nabla^2]c_0\rho + c_1 \Delta \rho]$$
$$+ (m_\pi/M)\epsilon_2^{-1}\nabla^2(C_0\rho_c\rho A),$$  (30)

which has been taken into account in the global fits of BFG97, FG98, F02a and F02b, but not in the present paper nor in Ref. [12]. This correction behaves like an s-wave potential, though it originates from the p-wave parameters. We examined how this correction is expressed in terms of the s-wave parameters. We found that the inclusion of this correction brings the following changes:

$$\Delta b_0^{\text{AT}} \approx +0.003,$$  (31)

for both $^{16}\text{O}$ and $^{208}\text{Pb}$, and

$$\Delta \text{Im}B_0^{\text{AT}} \approx -0.008 \text{ for } ^{16}\text{O},$$
$$\approx -0.014 \text{ for } ^{208}\text{Pb}.$$  (32)

Namely, this correction brings an attractive effect to the real part and a narrowing effect to the imaginary part. Thus, the best-fit parameters ($b_0^{\rho}$) with and without this correction have differences, as given above. A small difference in $b_0^{\rho}$ between the above global fits and the present analysis is attributed to this correction. On the other hand, there is no sizable effect on $b_1$: $\Delta b_1^{\text{AT}} \approx +0.003$.

10. Conclusion

We have shown that the Seki-Masutani relation between $b_0$ and $\text{Re}B_0$ holds for the 1s binding energies of light and heavy pionic atoms, both theoretically and empirically. If we admit this fact, the parameter for the isoscalar s-wave interaction is well represented by another parameter, $b_0^{\rho}$, as given in eq.(6,7), which takes care of the strong correlation between $b_0$ and $\text{Re}B_0$. The reason for this relation is understood from the fact that the overlapping density of any bound $\pi^-$ with the nuclear density is peaked at the radius which is slightly smaller than the half-density radius; the nuclear density effectively probed by a bound $\pi^-$ is around $0.6 \rho_0$.

We have shown that the Seki-Masutani Ansatz can be generalized for any functional form of the density-dependent potential: a LDD term, $F(\rho(r))\rho(r)$, can be transposed into a corresponding term $F(\rho_c)\rho(r)$ with a constant parameter, where $\rho_c$ is an effective parameter. Some important cases are summarized in Table II. In other words, we can say that the conventional potential form persists to be valid, because it can represent a LDD potential; the constant parameters in the conventional potential are regarded as being density-dependent parameters at $\rho \sim \rho_c \sim 0.60 \rho_0$. This also means that the observed enhancement of $b_1$ (interpreted as a reduction of the chiral order parameter) in $^{205}\text{Pb}$ [12], $R = b_1^{\text{free}}/b_1 = f_\pi^2/f_\pi^2 = 0.78^{+0.13}_{-0.09}$, is for the effective nuclear density of $\rho_c \sim 0.60 \rho_0$, and thus, the reduction that would occur for the full nuclear density is expected to be $R \sim 0.63$.

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TABLE I. Relation between the coefficient α representing the density-dependent $b_1(\rho(r)) = b_1^{free}/[1 - \alpha \rho(r)]$ and the equivalent isovector parameter $b_1$ in giving the same 1s binding energies.

| Nucleus | α  | $1 - b_1^{free}/b_1$ | $\rho_e$ | $\rho_e/\rho_0$ |
|---------|----|----------------------|---------|-----------------|
| $^{16}$O | 2  | 0.215                | 0.1075  | 0.63            |
|         | 3  | 0.323                | 0.1077  | 0.63            |
|         | 4  | 0.428                | 0.1070  | 0.63            |
| $^{208}$Pb | 2 | 0.185                | 0.0925  | 0.62            |
|         | 3  | 0.277                | 0.0923  | 0.62            |
|         | 4  | 0.370                | 0.0925  | 0.62            |

TABLE II. Summary of the effective densities, $\rho_e/\rho_0$, obtained for various types of local-density-dependent functions, $F[\rho(r)]\rho(r)$.

| $F[\rho]$ | $1s^{16}$O | $1s^{208}$Pb | Remark                     |
|-----------|------------|---------------|----------------------------|
| $\rho$    | 0.42       | 0.59          | SM relation                |
| $1/(1 - \alpha \rho)$ | 0.60       | density-dependent $b_1$ |
| $\rho^{1/3}$ | 0.40       | 0.54          | effective Fermi momentum   |
| $\rho^{1/3}/(1 - \alpha \rho)^2$ | 0.63       | 0.62          | double-scattering term      |
FIG. 1. Overlapping probabilities (lower frame) of the $\pi^-$ densities (upper frame) with the nucleon densities (middle frame) in typical pionic bound states: (Left) $^{16}\text{O}$. (Right) $^{208}\text{Pb}$. The vertical broken lines show the half-density proton radii and the vertical dash-dotted line is for the half-density neutron radius in $^{208}\text{Pb}$. 
FIG. 2. Seki-Masutani relation between $b_0$ and Re$B_0$. Best-fit values of $b_0$ versus Re$B_0$ as a gridding variable, obtained in $\chi^2$ minimization using the 1s pionic atom data in $6N=Z$ nuclei, are shown by closed circles, whose sizes are equal to the fitting uncertainties. They lie on the SM lines: $b_0^* = b_0 + 0.215\cdot\text{Re}B_0 = -0.0280 \pm 0.0010$. The $\{b_0, \text{Re}B_0\}$ parameters obtained from global fits of KLTK90 [2], BFG97 [1], F02a [13] and one of F02b [14] are shown by large open circles with both vertical and horizontal error bars. Also shown are $\{b_0, \text{Re}B_0\}$ sets from FG98 [15] (closed squares) and F02a [13] (small open circles), without conversion from $b_0$ to $b_0^*$ in their convention.

FIG. 3. The 1s binding energy calculated in the LDD potential, eq.(20), as a function of $\alpha$, compared with that in a conventional potential with $b_1$ as a constant parameter.