Growth of matter perturbations in non-minimal teleparallel dark energy

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We study the growth rate of matter perturbations in the context of teleparallel dark energy with flat curvature. We investigate the dynamics of different theoretical scenarios based on specific forms of the scalar field potential. Allowing for non-minimal coupling between torsion scalar and scalar field, we perform a phase-space analysis of the autonomous systems of equations through the study of critical points. We thus analyze the stability of the critical points, and discuss the cosmological implications searching for possible attractor solutions at late times. Furthermore, using the latest available growth rate data, we place observational constraints on the cosmological parameters of the models through Monte Carlo numerical method. We find that the scenario with a non-minimal coupling is favoured with respect to the standard quintessence case. Also, the matter fluctuations amplitude is consistent with the most recent findings of the Planck collaboration. Finally, we compared our results with the predictions of the concordance $\Lambda$CDM model.

I. INTRODUCTION

Understanding the universe dynamics at all cosmic scales represents a challenge for theoretical cosmology $^{1, 2}$. In particular, observations show that the universe undergoes an accelerated phase at late times $^{3}$. Baryons and cold dark matter cannot speed the universe up due to the action of gravity $^{3}$ so that one needs to include within Einstein’s equation a bizarre fluid, dubbed dark energy, whose equation of state (EoS) is negative $^{3}$. Even though this scenario fairly well describes the large-scale dynamics, it fails to be predictive on dark energy’s origin. Extended theories of gravity may represent a plausible landscape toward the determination of dark energy’s nature as general relativity (GR) breaks down $^{3}$.

Although a wide number of modified gravity theories has been already investigated, we here assess modified teleparallel gravity models, initially proposed as alternatives to inflationary phases $^{3}$. In this picture, the way in which dark energy is recovered to speed up the universe today is provided by analytical functions of the torsion $T$, namely $f(T)^{10}$. The approach of $f(T)$ candidates as a robust alternative to barotropic dark energy fluids and to other extensions of Einstein’s gravity $^{11}$. Unfortunately, the function $f(T)$ is not known a priori, so any suitable $f(T)$ models aim at describing the late-time dynamics might be either postulated or reconstructed through model-independent techniques $^{12, 14}$.

We here circumscribe our attention to particular $f(T)$ models, which have reached great attention for their capability of describing both late and early epochs of universe’s evolution $^{12}$. Specifically, we consider the teleparallel dark energy scenario with a non-minimal coupling between torsion scalar and scalar field $^{16, 18}$. We thus perform a phase-space analysis by studying the stability of critical points and search for late-times attractor solutions. In particular, if the real parts of the eigenvalues are negative, the corresponding critical point is stable and represents an attractor solution. We discuss the cosmological consequences of the theoretical models emerging from different choices of the scalar field potential. We analyze the case of null, linear and exponential potentials. Moreover, we study the growth rate of matter perturbations in terms of evolution of the density contrast. We also compare the observational constraints got from the latest available data with the standard outcomes of the $\Lambda$CDM model.

The paper is structured as follows. In Sec. $\mathrm{II}$, we review the main ingredients of teleparallel gravity and $f(T)$ cosmology. In Sec. $\mathrm{III}$ we describe the dynamical system approach to study the evolution of teleparallel dark energy with non-minimal coupling. We show how to obtain an autonomous system of equations by assuming suitable functional forms for the scalar field potential. Then, in Sec. $\mathrm{IV}$ we perform a phase-space analysis of the dynamical equations and study the stability of critical points. In Sec. $\mathrm{V}$ we analyze the growth of matter overdensities and explain how the growth factor data may be used to obtain predictions on our theoretical models. In Sec. $\mathrm{VI}$ we present numerical outcomes got from Monte Carlo analyses in terms of observational constraints over the cosmological parameters. In this section, we also compare our expectations with the concordance $\Lambda$CDM paradigm. Finally, Sec. $\mathrm{VII}$ is dedicated to the discussion of the results.

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1 This is consequence of the fact that matter is under the form of dust, i.e. provides a vanishing pressure. For alternative perspectives, see for example e.g. $^{13}$. 

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II. \( f(T) \) GRAVITY AND TELEPARALLEL DARK ENERGY

In this section, we discuss the main features of modified teleparallel cosmology \([19, 24]\). Teleparallel gravity is described in terms of tetrad fields \( e_A(x^μ) \), forming an orthonormal basis for the tangent space of the manifold at each point \( x^μ \). In this representation, the tangent space makes use of the Minkowski metric \( \eta_{AB} = \text{diag}(1, -1, -1, -1) \), whereas the metric tensor is given in terms of dual vierbeins \( e^A(x^μ) \) by:

\[
g_{μν} = \eta_{AB} e^A_μ e^B_ν. \tag{1}
\]

In lieu of Levi-Civita connections used in GR, teleparallel gravity uses the Weitzenböck connections \( \hat{Γ}^λ_{μν} \), characterized by vanishing curvature with non-zero torsion. The torsion scalar is thus \([25]\):

\[
T = S_ρ^{μν} T^ρ_{μν}, \tag{2}
\]

where

\[
S_ρ^{μν} \equiv \frac{1}{2} \left(K^{μν}_ρ + δ^{μ}_{ρ} T^α_ν - δ^{ν}_{ρ} T^α_μ\right), \tag{3}
\]

and \( T_{μν} \) and \( K^{μν}_ρ \) are the torsion tensor and the contortion tensor, respectively defined as

\[
T^λ_{μν} \equiv \hat{Γ}^λ_{μν} - \hat{Γ}^λ_{νμ} = e^A_λ (e_μ e^A_ν - e_ν e^A_μ) , \tag{4}
\]

\[
K^{μν}_ρ \equiv -\frac{1}{2} (T^{μν}_ρ - T^{νμ}_ρ - T^ρ_{μν}). \tag{5}
\]

The simplest action of teleparallel gravity can be written as

\[
S = \int d^4x \, e \frac{T}{(2κ) + \mathcal{L}_m}, \quad \mathcal{L}_m = \text{det}(e^A_μ) = \sqrt{-g}, \quad κ = 8πG \text{ and } \mathcal{L}_m \text{ is the Lagrangian density for matter} \[26, 24].\]

As one varies the above action with respect to the vierbeins, the corresponding field equations are equivalent to the Einstein ones, leading to the well-known **Teleparallel Equivalent to General Relativity** (TEGR).

A first generalization of TEGR is inspired by the \( f(\bar{R}) \) gravity theories. In fact, one can substitute \( T \) in \( S \) with a generic function of the torsion scalar, defining the so-called \( f(T) \) theories \([25, 30]\). Another landscape is the addition of a canonical scalar field, reproducing the **quintessence** effects. In this respect, one can modify the action \( S \) to include a non-minimal coupling between \( T \) and a scalar field \( ϕ \):

\[
S = \int d^4x \, e \left[ \frac{T}{2κ} + \frac{1}{2} (\partial_μ ϕ \partial^μ ϕ + ξ T ϕ^2) - V(ϕ) + \mathcal{L}_m \right], \tag{6}
\]

where \( V(ϕ) \) is the scalar field potential, and \( ξ \) is the coupling constant. The non-minimal quintessence \([2] \) in the framework of teleparallel gravity is named **teleparallel dark energy** \([16]\). Here, varying the action \( (6) \) with respect to the vierbeins provides the field equations:

\[
\left[ \frac{1}{e} \partial_μ (e^ρ_μ S_ρ^{μν}) - e^λ_μ T^{ρ}_{μλ} S_ρ^{νμ} - \frac{1}{4} e^ν_μ T \right] \left( \frac{2}{κ} + 2ξ ϕ^2 \right) + e^ν_μ \partial^ρ ϕ \partial_ρ ϕ + 4ξ e^λ_μ T^{ρ}_{μν} ϕ \partial_ν ϕ - e^ν_μ \frac{1}{2} \partial_μ ϕ \partial^μ ϕ - V(ϕ) \]  

\[= e^ρ_μ T^{(m)}(μ) ν, \tag{7}\]

where \( T^{(m)}(ρ) ν \) is the matter energy-momentum tensor.

A. Cosmology of \( f(T) \) gravity

We search for cosmological solutions by considering the spatially flat Friedmann-Lemaître-Robertson-Walker metric \( ds^2 = dt^2 - a(t)^2 δij dx^i dx^j \), where \( a(t) \) is the cosmic scale factor. In this case, the vierbein fields read \( e^A_μ = \text{diag}(1, a, a, a) \), having \( T = -6H^2 \), where \( H \equiv \dot{a}/a \) is the Hubble parameter. The modified Friedmann equations with a perfect fluid source are:

\[
H^2 = \frac{κ}{3} (ρ_m + ρ_ϕ), \tag{8}
\]

\[
\dot{H} = -\frac{κ}{2} (ρ_m + ρ_ϕ + p_ϕ), \tag{9}
\]

where we have neglected the contribution of radiation, assuming pressureless matter. Here, \( ρ_m \) is the matter energy density obeying the continuity equation \( \dot{ρ}_m + 3Hρ_m = 0 \), while \( ρ_ϕ \) and \( p_ϕ \) represent the energy density and pressure of the scalar field, respectively given by

\[
ρ_ϕ = \frac{1}{2} ϕ^2 + V(ϕ) - 3H^2 ξ ϕ^2, \tag{10}
\]

\[
p_ϕ = \frac{1}{2} ϕ^2 - V(ϕ) + 4Hξ ϕ \dot{ϕ} + (3H^2 + 2\dot{H}) ξ ϕ^2. \tag{11}
\]

The Klein-Gordon equation is obtained by varying the action \( (6) \) with respect to the scalar field \( ϕ \):

\[
\ddot{ϕ} + 3H \dot{ϕ} + V_ϕ = ξ T ϕ, \tag{12}
\]

where \( V_ϕ \equiv dV/dϕ \) and \( ξ T ϕ \) represents the source term, which provides the standard Klein-Gordon without source as \( ξ → 0 \). In this limit, teleparallel dark energy reduces to ordinary quintessence at both background and perturbation levels. Further, in such a scenario the scalar

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2 In our notation, capital Latin indices run over the tangent space, while Greek indices run over the manifold.

3 In the standard scenario, the coupling is between the Ricci scalar and the scalar field.

4 Throughout the text, we use units such that \( c = 1 \).
field plays the role of dark energy, so that Eq. (12) is equivalent to the conservation equation

$$\dot{\rho}_\phi + 3H(1 + w_\phi)\rho_\phi = 0 ,$$

(13)

where \( w_\phi = p_\phi/\rho_\phi \) coincides with the dark energy equation of state that can cross the phantom divide, as one can clearly see from Eqs. (10) and (11).

### III. DYNAMICAL SYSTEM OF EQUATIONS

A powerful and elegant way to investigate the universe dynamics is to recast the cosmological equations into a dynamical system. This approach allows to combine analytical and numerical methods to obtain quantitative information on the models under study. To that purpose, we introduce the following dimensionless variables:

$$x \equiv \frac{\kappa \dot{\phi}}{\sqrt{6}H} , \quad y \equiv \frac{\kappa \sqrt{V}}{\sqrt{3}H} ,$$

(14a)

$$u \equiv \kappa \phi , \quad v \equiv \frac{\kappa \sqrt{\rho_m}}{\sqrt{3}H} .$$

(14b)

Then, Eq. (8) can be recast under the form

$$x^2 + y^2 - \xi u^2 + v^2 = 1 .$$

(15)

Moreover, from Eqs. (9)–(11) one finds

$$s = -\frac{\ddot{H}}{H^2} = \frac{3x^2 + 2\sqrt{6} \xi xu + 3v^2/2}{1 + \xi u^2} .$$

(16)

We can thus rewrite the conservation equations for \( \rho_m \) and \( \rho_\phi \) as a dynamical system for the new variables:

$$\begin{cases}
x' = (s - 3)x - \sqrt{6} \xi u - \frac{\kappa V_\phi}{\sqrt{6}H^2} , \\
y' = sy + \frac{x}{\sqrt{2H}} \frac{V_\phi}{\sqrt{V}} , \\
u' = \sqrt{6} x , \\
v' = \left( s - \frac{3}{2} \right) v .
\end{cases}$$

(17)

The ‘prime’ here denotes the derivative with respect to the number of e-folds, \( N \equiv \ln a \). In this formalism, the energy densities of the cosmic species read

$$\Omega_m = v^2 , \quad \Omega_\phi = x^2 + y^2 - \xi u^2 .$$

(18)

Here, system (17) can be numerically solved by choosing \( \Omega_m = 0.999 , y = 10^{-6} \) and \( u = 10^{-6} \) as initial conditions at \( a \ll 1 \). The dark energy equation of state is given by

$$w_\phi = \frac{x^2 - y^2 + \xi u^2 - \frac{2}{3} \xi xu + 4\sqrt{\frac{2}{3}} \xi xu}{x^2 + y^2 - \xi u^2} .$$

(19)

Once a suitable form of \( V(\phi) \) is chosen, Eqs. (17) becomes an autonomous system and one can determine the dynamics of the cosmological equations. In the following, we consider three specific forms of scalar field potential, motivating each of them through physical reasons.

#### A. Vanishing potential

The first model makes use of a vanishing potential:

$$V(\phi) = 0 .$$

(20)

This represents the simplest scenario, characterized by a tracker behaviour at early times. In this model, the late-time acceleration is realized for \( \xi < 0 \). As Eq. (20) holds, the dynamical system simply reduces to

$$\begin{cases}
x' = (s - 3)x - \sqrt{6} \xi u , \\
u' = \sqrt{6} x , \\
v' = \left( s - \frac{3}{2} \right) v .
\end{cases}$$

(21)

This model describes a field \( \phi \), free from interactions and may be used as prototype to characterize non-bounded phions.

#### B. Linear potential

The second model is teleparallel dark energy with a linear potential:

$$V(\phi) = V_0 \kappa \phi .$$

(22)

It has been shown that a linear potential provides a possible solution to the coincidence problem. In addition, this form seems to be favoured even by the anthropic principle, i.e. galaxy formation is possible only in regions where \( V(\phi) \) is well approximated by a linear function. In view of Eq. (22), the system (17) reads

$$\begin{cases}
x' = (s - 3)x - \sqrt{6} \xi u - \sqrt{3} \frac{y^2}{2} , \\
y' = sy + \sqrt{3} \frac{xy}{2} u , \\
u' = \sqrt{6} x , \\
v' = \left( s - \frac{3}{2} \right) v .
\end{cases}$$

(23)

#### C. Exponential potential

Finally, we consider an exponential potential

$$V(\phi) = V_0 e^{-\kappa \phi} .$$

(24)
Exponential potentials have been widely used in the literature to study inflation in the early universe, structure formation and dark energy dynamics \[24, 25\]. For a scalar field potential of the form \[24\], the dynamical system becomes

\[
\begin{align*}
    x' &= (s - 3)x - \sqrt{6} \xi u - \sqrt{\frac{3}{2}} y^2, \\
    y' &= sy - \sqrt{\frac{3}{2}} xy, \\
    u' &= \sqrt{s} x, \\
    v' &= (s - \frac{3}{2}) v.
\end{align*}
\]

(25)

IV. PHASE-SPACE ANALYSIS

In this section we perform a phase-space analysis for the autonomous systems obtained above. Since in the case of null potential no critical points have been found, we focus on the non-vanishing potentials. To do so, we introduce the vectors \(X = (x, y, u, v)\) and \(X' = (x', y', u', v')\). We thus find the critical points \(X_c\) satisfying the equation \(X' = 0\). To study the stability of the critical points, we set \(X = X_c + \delta X\), where \(\delta X = (\delta x, \delta y, \delta u, \delta v)\) are linear perturbations of the dynamical variables. Then, we linearize the dynamical equations to get \(\delta X' = \mathcal{M} \delta X\), where \(\mathcal{M}\) is the coefficients matrix. Finally, the eigenvalues of \(\mathcal{M}\) evaluated at each critical point determine its stability. In particular, if the real parts of all the eigenvalues are negative, the corresponding critical point is stable and represents an attractor solution at late times.

A. The case \(V(\phi) = V_0 \kappa \phi\)

In the case of linear potential, \(X' = 0\) together with Eq. (15) and the conditions \(y \geq 0, v \geq 0\) provide the following critical points:

\[
\begin{align*}
    X_c^{(1)} &= \left(0, \sqrt{\frac{2}{3}}, \frac{-1}{\sqrt{-3\sqrt{-\xi}}} 0\right), \\
    X_c^{(2)} &= \left(0, \sqrt{\frac{2}{3}}, \frac{1}{\sqrt{-3\sqrt{-\xi}}} 0\right).
\end{align*}
\]

(26)

(27)

which exist if \(\xi < 0\). The cosmological implications are thus obtained by evaluating Eqs. (13) and (19) at points \(X_c^{(1)}\) and \(X_c^{(2)}\). In both cases, we find

\[
\begin{align*}
    w_\phi &= -1, \\
    \Omega_m &= 0, \\
    \Omega_\Lambda &= 1.
\end{align*}
\]

(28)

This set represents an empty universe completely dominated by dark energy (de Sitter solution).

To study the stability of the critical points, we use system (24) and Eq. (15) which provide three independent evolution equations for the linear perturbations:

\[
\begin{align*}
    \delta x' &= (s - 3)\delta x + x\delta s - \sqrt{6}\xi\delta u - \sqrt{\frac{3}{2}} y^2 (2\delta y - y u^{-1}\delta u), \\
    \delta y' &= s\delta y + y\delta s + \sqrt{\frac{3}{2}} u^{-1} (x\delta y + y\delta x - xyu^{-1}\delta u), \\
    \delta u' &= \sqrt{6} \delta x, \\
    \delta v' &= (s - \frac{3}{2}) \delta v.
\end{align*}
\]

(29)

(30)

(31)

where

\[
\begin{align*}
    s &= \frac{3x^2 + 2\sqrt{6}\xi xu + 3(1 - x^2 - y^2 + \xi u^2)/2}{1 + \xi u^2}, \\
    \delta s &= \frac{6x\delta x + 2\sqrt{6}\xi(x\delta u + u\delta x) - 3(3\delta x + y\delta y - \xi u\delta u)}{1 + \xi u^2}.
\end{align*}
\]

(32)

(33)

We report the coefficients matrix of the perturbation equations in Appendix A. The critical points \(26\) and \(27\) give the same eigenvalues, which are obtained by solving the equation

\[
(3 + \mu)(\mu^2 + 3\mu + 18\xi) = 0.
\]

(34)

One thus gets

\[
\begin{align*}
    \mu_1 &= -3, \\
    \mu_2 &= -\frac{3}{2} \left(1 + \sqrt{1 - 8\xi}\right), \\
    \mu_3 &= -\frac{3}{2} \left(1 - \sqrt{1 - 8\xi}\right).
\end{align*}
\]

(35)

(36)

(37)

The real part of \(\mu_3\) is negative only if \(\xi > 0\), which is against the existence condition \(\xi < 0\) for the critical points. Therefore, the critical points are unstable and no attractor solutions exist for this model.

B. The case \(V(\phi) = V_0 e^{-\kappa \phi}\)

In this case, imposing \(X' = 0\) and using Eq. (15) and the conditions \(y \geq 0, v \geq 0\) give three critical points:

\[
\begin{align*}
    X_c^{(1)} &= (0, 0, 0, 1), \\
    X_c^{(2)} &= \left(0, \sqrt{2\xi - 2\sqrt{\xi(\xi - 1)}}, 1 - \sqrt{\frac{\xi - 1}{\xi}}, 0\right), \\
    X_c^{(3)} &= \left(0, \sqrt{2\xi + 2\sqrt{\xi(\xi - 1)}}, 1 + \sqrt{\frac{\xi - 1}{\xi}}, 0\right).
\end{align*}
\]

(38)

(39)

(40)

We note that point \(X_c^{(1)}\) always exists, whereas point \(X_c^{(2)}\) exists if \(\xi \geq 1\), and \(X_c^{(3)}\) exists if \(\xi < 0\) or \(\xi \geq 1\).
The cosmological behaviours at the critical points are found from Eqs. (18) and (19). In particular, at point $X_c^{(1)}$ one has

$$\begin{align*}
\Omega_m &= 1 , \\
\Omega_\phi &= 0 .
\end{align*}$$

(41)

This set corresponds to an Einstein-de Sitter universe, completely dominated by matter. On the other hand, at both points $X_c^{(2)}$ and $X_c^{(3)}$ one finds a de Sitter solution:

$$\begin{align*}
w_\phi &= -1 , \\
\Omega_m &= 0 , \\
\Omega_\phi &= 1 .
\end{align*}$$

(42)

We study the stability of the critical points from the evolution equations for linear perturbations. In this case, we have

$$\begin{align*}
\delta x' &= (s - 3)\delta x + x\delta s - \sqrt{6}\xi\delta u + \sqrt{6}y\delta y , \\
\delta y' &= s\delta y + y\delta s + \sqrt{\frac{3}{2}}(x\delta y + y\delta x) , \\
\delta u' &= \sqrt{6}\delta x ,
\end{align*}$$

(43)

(44)

(45)

where $s$ and $\delta s$ are given as in Eqs. (32) and (33), respectively. The matrix with the coefficients of the perturbation equations is reported in Appendix B. The eigenvalues $\mu_i^{(1)}$ for point $X_c^{(1)}$ are obtained as solutions of the equation

$$(3 - 2\mu)(2\mu^2 + 3\mu + 12\xi) = 0 .$$

(46)

We thus find

$$\begin{align*}
\mu_1^{(1)} &= \frac{3}{2} , \\
\mu_2^{(1)} &= \frac{1}{4}(-3 - \sqrt{9 - 96\xi}) , \\
\mu_3^{(1)} &= \frac{1}{4}(-3 + \sqrt{9 - 96\xi}) .
\end{align*}$$

(47)

(48)

(49)

Since $\mu_1^{(1)}$ is always positive, the critical point $X_c^{(1)}$ is unstable. The eigenvalues $\mu_i^{(2)}$ corresponding to point $X_c^{(2)}$ are obtained from

$$\frac{\xi(3 + \mu) \left[ 3\mu + \mu^2 + 6 \left( -2\xi + \xi(\xi - 1) \right) - 2 \left( -3\mu - \mu^2 + 6\xi \right) \left( -\xi + \sqrt{\xi(\xi - 1)} \right) \right]}{\left( \xi - \sqrt{\xi(\xi - 1)} \right)^2} = 0 ,$$

(50)

which gives

$$\begin{align*}
\mu_1^{(2)} &= -3 , \\
\mu_2^{(2)} &= -3 - 6\xi + 6\sqrt{\xi(\xi - 1)} + \sqrt{3} \left[ 3 + 64\xi^3 + 4\sqrt{\xi(\xi - 1)} + 8\xi \left( 1 + 5\sqrt{\xi(\xi - 1)} \right) - 8\xi^2 \left( 9 + 8\sqrt{\xi(\xi - 1)} \right) \right]^{1/2} , \\
\mu_3^{(2)} &= -3 + 6\xi - 6\sqrt{\xi(\xi - 1)} + \sqrt{3} \left[ 3 + 64\xi^3 + 4\sqrt{\xi(\xi - 1)} + 8\xi \left( 1 + 5\sqrt{\xi(\xi - 1)} \right) - 8\xi^2 \left( 9 + 8\sqrt{\xi(\xi - 1)} \right) \right]^{1/2} .
\end{align*}$$

(51)

(52)

(53)

The real parts of $\mu_i^{(2)}$ are all negative for $\xi > 1$, where the critical points exist. Hence, we conclude that point $X_c^{(2)}$ is stable and it represents an attractor for the universe at late times (see Fig. II).

Finally, the eigenvalues $\mu_i^{(3)}$ of point $X_c^{(3)}$ are found by solving
\[
\frac{\xi(3 + \mu)}{\left(\xi + \sqrt{\xi(x - 1)}\right)} \left[ -3\mu - \mu^2 + 6 \left(2\xi + \sqrt{\xi(x - 1)}\right) - 2 \left(-3\mu - \mu^2 + 6\xi\left(-\xi + \sqrt{\xi(x - 1)}\right)\right) \right] = 0 ,
\]

which provides
\[
\mu_1^{(3)} = -3 ,
\]
\[
\mu_2^{(3)} = -\frac{3 + 6\xi + 6\sqrt{\xi(x - 1)} + \sqrt{3} \left[3 + 64\xi^3 - 4\sqrt{\xi(x - 1)} + 8 \xi \left(1 - 5\sqrt{\xi(x - 1)}\right) + 8\xi^2 \left(-9 + 8\sqrt{\xi(x - 1)}\right)\right]^{1/2}}{-2 + 4\xi + 4\sqrt{\xi(x - 1)}},
\]
\[
\mu_3^{(3)} = \frac{3 - 6\xi - 6\sqrt{\xi(x - 1)} + \sqrt{3} \left[3 + 64\xi^3 - 4\sqrt{\xi(x - 1)} + 8 \xi \left(1 - 5\sqrt{\xi(x - 1)}\right) + 8\xi^2 \left(-9 + 8\sqrt{\xi(x - 1)}\right)\right]^{1/2}}{-2 + 4\xi + 4\sqrt{\xi(x - 1)}} .
\]

It turns out that the real parts of \(\mu_i^{(3)}\) are not all negative for \(\xi < 0\) and \(\xi \geq 1\). This implies that point \(X_0^{(3)}\) is unstable and no attractor solution corresponds to this critical point.

**FIG. 1.** Phase-space trajectories on the \(x - y\) plane for teleparallel dark energy with \(V(\phi) = V_0 e^{-\kappa\phi}\) and \(\xi = 2\). The red dot corresponds to the critical point \(X_0^{(2)}\) (cf. Eq. [39]), which represents an attractor solution of the dynamical system.

## V. GROWTH RATE OF MATTER PERTURBATIONS

Over the past years, it has become clear that the study of the background cosmological dynamics is not sufficient to discriminate between modified theories and the standard cosmological model. The study of density perturbations over the homogeneous and isotropic background could be a possible way to test possible deviations from GR and to break the degeneracy among the alternative theories [35]. Indeed, any modification of GR would affect also the growth of cosmological perturbations. An important probe in this respect is the evolution of linear matter density contrast \(\delta_m = \delta \rho_m / \rho_m\) [36, 37]:

\[
\frac{d^2 \delta_m}{da^2} + \left(\frac{3}{a} + \frac{1}{E} \frac{dE}{da}\right) \frac{d\delta_m}{da} - \frac{3}{2} \frac{\Omega_m}{\Omega_{\phi}} G_{eff} \delta_m = 0 ,
\]

where \(E(a) = H(a)/H_0\) and the subscript ‘0’ refers to quantities evaluated at the present time. In the case of teleparallel dark energy, one can write the Hubble expansion rate as follows:

\[
E(a)^2 = \Omega_{m0} a^{-3} + \Omega_{\phi0} \exp \left\{-3 \int_a^1 \left(1 + \frac{w_{\phi}}{a'}\right) da'\right\} ,
\]

where \(\Omega_{\phi0} = 1 - \Omega_{m0}\). In the quasi static approximation and in the sub-horizon regime, the effective gravitational constant \(G_{eff}\) is given by [24]

\[
G_{eff} = \frac{G}{1 + \kappa \phi^2} \left(1 - \frac{\kappa \phi^2}{2H^2(1 + \kappa \phi^2)}\right) .
\]

We note that in the case of minimal quintessence (\(\xi = 0\)), the kinetic energy of the scalar field is subdominant with respect to the dark energy at the present time, so that one recovers the expected result \(G_{eff} \simeq G\). In terms of the dynamical variables \(x\) and \(u\), Eq. [60] becomes

\[
\frac{G_{eff}}{G} = \frac{1}{1 + \xi u^2} \left(1 - \frac{3x^2}{1 + \xi u^2}\right) .
\]

To study the growth rate of matter density perturbations, we introduce the quantity [41]

\[
f(a) = \frac{d\delta_m}{d \ln a} ,
\]
Measurements from redshift space distortion and weak lensing have been obtained in the redshift interval $0 < z < 2$ for the factor

$$f\sigma_8(z) \equiv f(z)\sigma_8(z), \quad (63)$$

where $\sigma_8(a) = \sigma_8 \delta_m(a)/\delta_m(1)$ is the rms fluctuations of the linear density field inside a radius of $8h^{-1}\text{Mpc}$, and $\sigma_8$ is its present day value. In our analysis we use the very recent compilation of 63 data given in [40], which extends a previous catalogue presented in [41]. The model-dependence of these measurements requires a rescaling by the assumed fiducial cosmology. To this end, one can define the ratio

$$\rho(z) = \frac{H(z)d_A(z)}{H_{fid}(z)d_{A,fid}(z)}, \quad (64)$$

where $d_A(z)$ is the angular diameter distance given by

$$d_A(z) = (1 + z)^{-1}\int_0^z \frac{dz'}{H(z')} . \quad (65)$$

In the present case, the fiducial model refers to the ΛCDM model, $H_{fid}(z) = H_0\sqrt{\Omega_{m0}(1 + z)^3 + \Omega_{\Lambda0}}$, where $\Omega_{\Lambda0} = 1 - \Omega_{m0}$. The difference between the observed values and the values predicted by the theoretical model is given by

$$Y = f\sigma_8^{obs}(z_i) - \rho(z_i)f\sigma_8^{th}(z_i) . \quad (66)$$

Therefore, the likelihood is given as $\mathcal{L} \propto e^{-\chi^2/2}$, where

$$\chi^2 = Y^T C^{-1} Y , \quad (67)$$

where $C_{ij}$ is the the covariance matrix constructed by taking into account the correlations between the data points (see [40] for the details).

**VI. OBSERVATIONAL CONSTRAINTS**

We tested the different teleparallel dark energy models described in Sec. [III] by performing a Markov Chain Monte Carlo (MCMC) integration implemented with the Metropolis algorithm [42]. To get tighter constraints on the cosmological parameters, we complemented the growth rate factor (GRF) data with the model-independent observational Hubble data (OHD) obtained through the differential age method [43] (see [24] with the correspondent references). Thus, the combined Likelihood function of the data reads

$$\mathcal{L}_{tot} = \mathcal{L}_{GRF} \times \mathcal{L}_{OHD} . \quad (68)$$

In our analysis we fixed $H_0 = 70 \text{ km/s/Mpc}$, so that the numerical integration was done over the following parameter space $\mathcal{P}$:

$$\mathcal{P} = \{\Omega_{m0}, \xi, \sigma_8\} . \quad (69)$$

We present in Table [1] the best-fit results with relative 1σ uncertainties for different forms of the scalar field potential. Furthermore, in Figs. [2]-[4] we show the 2-D 1σ and 2σ likelihood contours for the different models. We note the similarity of the results obtained for the vanishing and the exponential potentials. In the case of linear potential, the best-fit results are slightly different compared to the other cases, although consistent within the 1σ level. In particular, the value of the coupling constant $\xi$ obtained in the case of vanishing potential is consistent with the outcomes obtained in [44] with other data surveys. Also, our $\sigma_8$ values are consistent with the latest results of the Planck collaboration [44]. The contours displayed in Figs. [2]-[4] indicate that a non-minimal gravitational coupling scenario is definitely favoured by observations with respect to the minimal quintessence at more than 2σ evidence.

![FIG. 2. Marginalized 68% and 95% confidence levels contours resulting from the MCMC analysis on teleparallel dark energy model with $V(\phi) = 0$.](image-url)
| Potential  | $\Omega_{m0}$ | $\xi$      | $\sigma_8$ |
|-----------|---------------|------------|------------|
| $V(\phi) = 0$ | 0.253 ± 0.047 | -0.342 ± 0.014 | 0.801 ± 0.050 |
| $V(\phi) = V_0 \kappa \phi$ | 0.305 ± 0.040 | -0.359 ± 0.019 | 0.772 ± 0.023 |
| $V(\phi) = V_0 e^{-\kappa \phi}$ | 0.254 ± 0.049 | -0.342 ± 0.014 | 0.798 ± 0.050 |

TABLE I. $1\sigma$ parameter constraints results from the MCMC numerical analysis on teleparallel dark energy models with different scalar field potentials.

FIG. 3. Marginalized 68% and 95% confidence levels contours resulting from the MCMC analysis on teleparallel dark energy model with $V(\phi) = V_0 \kappa \phi$.

FIG. 4. Marginalized 68% and 95% confidence levels contours resulting from the MCMC analysis on teleparallel dark energy model with $V(\phi) = V_0 e^{-\kappa \phi}$.
Finally, we compare in (Fig. 5) the growth rate for the concordance paradigm with the indicative values of $\Omega_m = 0.3$ and $\sigma_8 = 0.8$, and for the different teleparallel models resulting from our analysis. We notice that the strength of fluctuations is larger for the ΛCDM model compared to the modified gravity scenarios. Moreover, all the choices for $V(\phi)$ lead to indistinguishable curves which degenerate among them. However, due to experimental uncertainties, the ΛCDM model cannot be preferred with respect to the others. Future experiments with more precise measurements will be definitely useful to break the degeneracy.

VII. CONCLUSIONS

In the present work, we considered modified teleparallel cosmology with a non-minimal coupling between torsion and scalar field. We showed that teleparallel dark energy exhibits a richer structure of solutions compared to the minimal quintessence. To do that, we described the cosmological evolution in terms of dimensionless variables which form a dynamical system of equations. Assuming suitable forms of the scalar field potential, we were able to solve the autonomous system of equations to find the dynamical behaviour at early and late times. In particular, we analyzed the case of vanishing, linear and exponential potentials. We thus performed a phase-space analysis of the systems and investigated the cosmological solutions in correspondence of the critical points. We also studied the stability of the critical points to search for possible attractor solutions. In the case of exponential potential, we found that there exists an attractor for the universe at late times only for coupling constant $\xi > 1$.

Then, we studied the growth rate of perturbations. In particular, we wrote the evolution of the matter density contrast, specifying the effective gravitational constant in the quasi-static approximation and in the sub-horizon regime. We thus tested the viability of our theoretical scenario using the latest available data of the growth rate factor, which can be rewritten in a model-independent way after a rescaling procedure with respect to the assumed fiducial cosmology. To obtain more stringent constraints on the cosmological parameters, we complemented the growth rate data with the most recent measurements on the Hubble parameter obtained through the model-independent differential age method. To this purpose, we implemented a MCMC numerical integration by means of Metropolis’ algorithm assuming uniform distributions for the fitting parameters. The results obtained from the combined likelihood analysis show that a non-minimal coupling is strongly favoured by cosmological observations with respect to the minimal quintessence scenario. The goodness of our approach is confirmed by the results got for the amplitude of matter fluctuations, which appear consistent with the most recent findings of the Planck collaboration. We also compared our results with the predictions of the concordance ΛCDM model. We found that the teleparallel models under study are hardly distinguishable from each other. On the other hand, the ΛCDM model is not preferred over the teleparallel dark energy, due to the large uncertainties on the experimental data.

For future developments it would be interesting to consider extensions of the present scenario by including a non-zero curvature term. Also, the constraints from the cosmic microwave background anisotropies could play an important role to break the degeneracy among theories with different potentials.
ACKNOWLEDGMENTS

We are grateful to S. Capozziello for useful discussions on the topic of $f(T)$ cosmology. We also want to thank D. Babusci and S. Mancini for their useful suggestions. O. L. thanks the National Institute for Nuclear Physics for financial support and the University of Camerino for its hospitality.

Appendix A: Coefficients matrix of the perturbation equations for $V(\phi) = V_0\kappa \phi$

Here, we report the components of the perturbation coefficients matrix in the case of $V(\phi) = V_0\kappa \phi$.

\[ M_{11} = \frac{-9x^2 + 8\sqrt{6}\xi xu - 3(1 + y^2 + \xi u^2)}{2(1 + \xi u^2)}, \]  
\[ M_{12} = \frac{\sqrt{6}y}{u} - \frac{3xy}{1 + \xi u^2}, \]  
\[ M_{13} = \frac{y^2(\sqrt{6} + 2\sqrt{6}\xi u^2 + 6\xi xu^3 + \sqrt{6}\xi^2 u^4) - 2\xi u^2 (3x^3u + 2\sqrt{6}x^2(-1 + \xi u^2) + \sqrt{6}(1 + \xi u^2)^2)}{2u^2(1 + \xi u^2)^2}, \]  
\[ M_{21} = \frac{y(\sqrt{6} + 6xu + 5\sqrt{6}\xi u^2)}{2u(1 + \xi u^2)}, \]  
\[ M_{22} = \frac{\sqrt{6}x + 3u(1 + x^2 - 3y^2) + 3\xi u^3 + 5\sqrt{6}\xi xu^2}{2u(1 + \xi u^2)}, \]  
\[ M_{23} = \frac{y(-6\xi x^3u^3 + 6\xi y^2 u^3 - \sqrt{6}x(1 - 2\xi u^2 + 5\xi^2 u^4))}{2u^2(1 + \xi u^2)^2}, \]  
\[ M_{31} = \sqrt{6}, \]  
\[ M_{32} = M_{33} = 0. \]

Appendix B: Coefficients matrix of the perturbation equations for $V(\phi) = V_0 e^{-\kappa \phi}$

Here, we report the components of the perturbation coefficients matrix in the case of $V(\phi) = V_0 e^{-\kappa \phi}$.

\[ M_{11} = \frac{9x^2 + 8\sqrt{6}\xi xu - 3(1 + y^2 + \xi u^2)}{2(1 + \xi u^2)}, \]  
\[ M_{12} = y \left( \sqrt{6} - \frac{3x}{1 + \xi u^2} \right), \]  
\[ M_{13} = -\frac{\xi (3x^3 u - 3xu^2 + 2\sqrt{6}x^2(-1 + \xi u^2) + \sqrt{6}(1 + \xi u^2)^2)}{(1 + \xi u^2)^2}, \]  
\[ M_{21} = -\frac{y(-6x + \sqrt{6}(1 - 4\xi u + \xi u^2))}{2(1 + \xi u^2)}, \]  
\[ M_{22} = \frac{3(1 + x^2 - 3y^2 + \xi u^2) - \sqrt{6}x(1 - 4\xi u + \xi u^2)}{2(1 + \xi u^2)}, \]  
\[ M_{23} = -\frac{\xi y(3u(x^2 + y^2) + 2\sqrt{6}x(-1 + \xi u^2))}{(1 + \xi u^2)^2}, \]  
\[ M_{31} = \sqrt{6}, \]  
\[ M_{32} = M_{33} = 0. \]
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