On Counting Subring-Subcodes of Free Linear Codes Over Finite Principal Ideal Rings

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Abstract

Let $R$ be a finite principal ideal ring and $S$ the Galois extension of $R$ of degree $m$. For $k$ and $k'$, positive integers we determine the number of free $S$-linear codes $B$ of length $\ell$ with the property $k = \text{rank}_S(B)$ and $k' = \text{rank}_R(B \cap \ell')$. This corrects a wrong result \cite{1, Theorem 6} which was given in the case of finite fields.

Keywords: Finite chain ring, Galois extension, Trace map, Linear code, Chinese remainder theorem.

AMS Subject Classification 2010: 13B05, 15A03, 16P70, 94B05, 94B25.

1. Introduction

Many codes over a finite field $\mathbb{F}$ can be seen as subfield-subcodes of codes that are defined over a some extension field of $\mathbb{F}$. Unfortunately there is no general formula for the dimension or the minimum distance of such codes. However there are good bounds for some families of codes and also some formulae have been obtained for the true dimension in the case of some alternant codes, toric codes or some families of Goppa codes (see \cite{2} and the references therein). In a more general way, given a finite principal ideal ring $R$ and a Galois extension $S$ of $R$ of degree $m$, one can define an $S$-linear code $B$ of length $\ell$ as $S$-module of $S^\ell$. The rank of a linear code $\text{rank}_S(B)$ will be the minimum number of its generators. The $R$-linear code $B \cap \ell'$ is called the subring-subcode of $B$ to $R$. The relationship between subring subcodes and trace codes over finite chain rings has been studied in \cite{3} revealing a generalization of the classical Delsarte’s Theorem. In \cite{1}, Lyle determined the number of distinct $\mathbb{F}_q^\ell$-linear codes of length $\ell$ and of dimension $k$, with a fixed dimension of their subfield subcodes over $\mathbb{F}_q$. However the result is incorrect. The following example illustrates the same.

Counterexample. We have 21 different 2-dimensional codes of length 3 over $\mathbb{F}_4$. Thus from \cite{1, Theorem 6}, if follows that the $\mathbb{F}_2$-subcode of any of them is the trivial code \{(0, 0, 0)\}. Consider now $\mathbb{F}_4 = \{0, 1, \alpha, \beta\}$

\footnotesize

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Proof.

results. Section 3 provides a formula for counting the number of free on the paper. The paper is organized as follows: In Section 2 we present some definitions and preliminary results. Section 3 covers the finite field case. The Galois invariance of a code \([3, 4]\) plays an important role on the paper. The paper is organized as follows: In Section 2 we present some definitions and preliminary results. Section 3 provides a formula for counting the number of free \(S\)-linear codes of given rank whose the rank of its subring subcode is given. Finally in Section 4 we extend the results to PIRs.

2. Preliminaries

Throughout this section \(R\) denotes a finite chain ring with invariants \((q, s)\), that is all its ideals form a chain under inclusion \(R \supseteq R\theta \supseteq \cdots \supseteq R\theta^s \supseteq \{0\}\), where \(\theta\) a generator of its maximal ideal \(m = R\theta\) and \(R/m = R_q\), a finite field with \(q = p^n\) elements, \(p\) is a prime, \(n\) a positive number. The canonical projection \(\pi : R \to R_q\) naturally extends to \(R[X]\), acting on the coefficients. We say that the ring \(S\) is an extension of \(R\) and we denote it by \(SR\) if \(R\) is a subring of \(S\) and \(1_R = 1_S\). Let \(f\) be a polynomial over \(R\). Then \((f)\) is an ideal of \(R[X]\) generated by \(f\). A polynomial \(f\) is called basic irreducible if \(\pi(f)\) is irreducible over \(R_q\).

A finite ring \(S\) is a Galois extension of \(R\) of degree \(m\) if \(S \cong R[X]/(f)\), where \(f\) is a monic basic irreducible over \(R\) of degree \(m\). By \(\text{Aut}_R(S)\) we denote the group of ring automorphisms of \(S\) which fix the elements of \(R\). It is well known that if \(S\) is the Galois extension of \(R\) of degree \(m\) then \(m_S = S\theta\) and \(\text{Aut}_R(S)\) is the cyclic group of order \(m\). The map \(\Tr_S^R := \sum_{\rho \in \text{Aut}_R(S)} \rho\) is called the trace map of the Galois extension \(SR\). When \(S = R_q^m\), we write \(T_m : R_q^m \to R_q\). Note that the trace map of \(S\) over \(R\) is an \(R\)-module homomorphism of \(S\) and \(T_m \circ \pi = \pi \circ \Tr_S^R\), where \(\pi : S \to R_q^m\) is a natural map with \(\pi_{1_R} = \pi\).

An \(R\)-linear code \(C\) of length \(\ell\) is an \(R\)-submodule of \(R\). A linear code \(\mathcal{D}\) of \(C\) is an \(R\)-submodule of \(C\). \(\mathcal{D}\) is said be proper if \(\mathcal{D} \neq C\) and \(\mathcal{D} \neq \{0\}\). The code \(C\) of length \(\ell\) and rank \(k\) is said to be free if \(C \cong R^k\) as an \(R\)-module. The following result allows us to count of all free subcodes of arbitrary free \(R\)-linear code with given rank.

**Theorem 1.** If \(C\) is a free \(R\)-linear code of length \(\ell\) and rank \(k\) then the number of free \(R\)-linear subcodes of \(C\) of rank \(k'\), is given by

\[
\begin{bmatrix} k \\ k' \end{bmatrix}_{(q, s)} := q^{(s-1)(k-k')q^k}, \quad \text{where} \quad \begin{bmatrix} k \\ k' \end{bmatrix}_q := \begin{cases} 0, & \text{if } k < k' \\
1, & \text{if } k' = 0 \\
\prod_{i=0}^{k'-1} \frac{q^{k'-i} - q^{k'-i}}{q^i - q^i}, & \text{otherwise.} \end{cases}
\]

**Proof.** Use [5, Theorem 2.4.].

Let \(\mathcal{B}\) be a linear code over \(S\). We define the trace code of \(\mathcal{B}\) over \(R\) as

\[
\Tr_S^R(\mathcal{B}) = \left\{ \Tr_S^R(c_0), \Tr_S^R(c_1), \cdots, \Tr_S^R(c_{\ell-1}) \right\} \mid (c_0, c_1, \cdots, c_{\ell-1}) \in \mathcal{B},
\]

and the subring subcode of \(\mathcal{B}\) over \(R\) as \(\text{Res}_{SR}(\mathcal{B}) = \mathcal{B} \cap R^\ell\). On the other hand, given a linear code \(C\) over \(R\), the extension code of \(C\) to \(S\) is the \(S\)-submodule formed by taking all \(S\)-linear combinations of elements in \(C\).
Let $\mathcal{B}$ be a free $S$-linear code such that $\sigma(\mathcal{B}) \neq \mathcal{B}$ (i.e., it is not Galois invariant, see [3]). We define the code $\mathcal{B}_0$ as $\mathcal{B}_0 = \text{Ext}_S(\text{Res}_S(\mathcal{B}))$. It is clear that $\mathcal{B}_0$ is free as $\mathcal{B}$ is free and from [3, Corollary 1] it follows that $\mathcal{B}$ is the non-empty largest Galois-invariant subcode (G-coresubcode) of $\mathcal{B}$. The following Lemma provides us a decomposition of a non Galois-invariant free code.

**Theorem 2.** Let $\mathcal{B}$ be a non Galois-invariant free code over $S$ and $\mathcal{B}_0 = \text{Ext}_S(\text{Res}_S(\mathcal{B}))$, then $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1$, where $\mathcal{B}_1$ has the property $\text{Res}_S(\mathcal{B}_1) = \{0\}$.

**Proof.** Let $k = \text{rank}_S(\mathcal{B})$ and $k' = \text{rank}_S(\mathcal{B}_0)$. Since $\sigma(\mathcal{B}) \neq \mathcal{B}$, $k > k'$. Also, since $\mathcal{B}_0$ is free as an $S$-module, there exists a free $S$-basis $\{x_1, x_2, \ldots, x_{k'}\}$ of $\mathcal{B}_0$. Then the set $\mathcal{B}_0 + m'$ is non-empty as $k' < k$. From [6, Lemma 3.2], for any $y_1 \in \mathcal{B}\setminus \mathcal{B}_0 + m'$, the set $\{x_1, x_2, \ldots, x_{k'}, y_1\}$ is $S$-linearly independent. If $k - k' = 1$, then we are done and $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1$, where $\mathcal{B}_1 = \langle y_1 \rangle$. Otherwise, we choose $y_2 \in \mathcal{B}\setminus \langle x_1, x_2, \ldots, x_{k'}, y_1 \rangle + m'$ and by proceeding in the same manner finally we get $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1$, where $\mathcal{B}_1 = \langle y_1, y_2, \ldots, y_{k-k'} \rangle$ is a free $S$-linear code. Now, if $c \in \text{Res}_S(\mathcal{B}_1) \subseteq \text{Res}_S(\mathcal{B}) \subseteq \mathcal{B}_0$, then $c = 0$ as $\mathcal{B}_0 \cap \mathcal{B}_1 = \{0\}$. $\square$

3. Non Galois-invariant codes

Let $\mathcal{L}_\ell(S)$ be the set of all the $S$-linear codes of length $\ell$, we introduce the following set

$$E_R(\ell, m) = \{\mathcal{B} \in \mathcal{L}_\ell(S) \mid \text{Tr}_R^S(\mathcal{B}) = R^\ell\}. \quad (1)$$

Note we have that $S^\ell \in E_R(\ell, m)$ and by Delsarte’s theorem [3], $E_R(\ell, m) = \{\mathcal{B} \in \mathcal{L}_\ell(S) \mid \text{Res}_S(\mathcal{B}^\perp) = \{0\}\}$. A criterion for checking whether $\mathcal{B}$ belongs to $E_R(\ell, m)$ or not depending on the field associated to $R$ is given in the following result.

**Proposition 1.** Let $\mathcal{B}$ be an $S$-linear code then $\mathcal{B} \in E_R(\ell, m)$ if and only if $\pi(\mathcal{B}) \in E_R(\ell, m)$, where $\pi(\mathcal{B})$ is the set obtained applying $\pi$ to each element in $\mathcal{B}$ coordinatewise.

**Proof.** We have that $\pi_{|_\ell} \circ \text{Tr}_R^S = \text{Tr}_m \circ \pi$ and therefore $\pi_{|_\ell}(\text{Tr}_R^S(\mathcal{B})) = \text{Tr}_m(\pi(\mathcal{B}))$ holds. Hence $\text{Tr}_R^S(\mathcal{B}) = R^\ell$ if and only if $\text{Tr}_m(\pi(\mathcal{B})) = F_q^\ell$. $\square$

Note that for $\mathcal{B}, D \in \mathcal{L}_\ell(S)$ such that $D \subseteq \mathcal{B}$, if $D \in E_R(\ell, m)$ then it follows that $\mathcal{B} \in E_R(\ell, m)$. This fact motivates the following definition.

**Definition 1.** An $S$-linear code $\mathcal{B}$ is said to be minimal in $E_R(\ell, m)$ if $\mathcal{B} \in E_R(\ell, m)$ and $\mathcal{B}$ has no proper $S$-linear subcode in $E_R(\ell, m)$.

Note that $S^\ell$ is the minimal $S$-linear code in $E_R(\ell, m)$ if and only if $m = 1$.

**Theorem 3.** If $\mathcal{B}$ is the minimal $S$-linear code in $E_R(\ell, m)$ then $\mathcal{B}$ is free.

**Proof.** Let $\{c_1, \ldots, c_k\}$ be a set of generators in $\mathcal{B}$ on the form of [3, Lemma 1]. Consider $D$ the subcode of $\mathcal{B}$ generated by those $c_i$ such that $\pi(c_i) \neq 0$, $i = 1, \ldots, k$. Therefore $\pi(D) = \pi(\mathcal{B}) \in E_R(\ell, m)$ by Proposition 1 and hence $D \in E_R(\ell, m)$. Thus by the minimality of $\mathcal{B}$, we have $D = \mathcal{B}$, and therefore $\mathcal{B}$ is generated by elements in $(S \setminus m)^\ell$ thus it is free. $\square$

**Theorem 4.** Let $\mathcal{B} \in E_R(\ell, m)$. Then $\mathcal{B}$ is minimal if and only if $\mathcal{B}$ is free and

$$\text{rank}_S(\mathcal{B}) = \left\lceil \frac{\ell}{m} \right\rceil, \quad (2)$$

where $\left\lceil \frac{\ell}{m} \right\rceil = \min \{i \in \mathbb{N} \mid i \leq \frac{\ell}{m} \}$.  

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Proof. Suppose that $\mathcal{B}$ is minimal. From Theorem 3 it is free. Let $\{c_1, c_2, \cdots, c_k\}$ be an $S$-basis of $\mathcal{B}$ and $\{a_0, a_1, \cdots, a_{m-1}\}$ a free $R$-basis of $S$. The set $\{\text{Tr}_R^S(\alpha_jc_i) \mid 0 \leq i < m \text{ and } 1 \leq j \leq k\}$ is a generating set of $T^S_R(\mathcal{B})$. Let $u \in \{1, 2, \cdots, k\}$, and consider $D_u$ the $S$-linear subcode of $\mathcal{B}$, generated by $\{c_1, c_2, \cdots, c_k\}\setminus\{c_u\}$. Since $\mathcal{B}$ is minimal in $E_R(\ell, m)$, $\text{Tr}_R^S(D_u) \subseteq T^S_R(\mathcal{B}) = R^\ell$ and therefore $\text{rank}_R(\text{Tr}_R^S(D_u)) \leq m(k-1) < \text{rank}_R(T^S_R(\mathcal{B})) = \ell \leq mk$. Hence $k = \left\lceil \frac{\ell}{m} \right\rceil$. The converse follows straightforward by the minimality of the rank. □

From now on we will use the following notations

$$k = \left\lceil \frac{\ell}{m} \right\rceil \text{ and } E_R(\ell, m, k) = \left\{ \mathcal{B} \in E_R(\ell, m) \mid \mathcal{B} \cong S^k \text{ (as } R\text{-module)} \right\} \text{ for some } k \geq k.$$

$$M_R(\ell, m, u) = \left\{ \sum_{\mathcal{B} \in M} \mathcal{B} \mid M \subseteq E_R(\ell, m, k) \right\} \text{ and } \{ \sum_{\mathcal{B} \in M} \mathcal{B} \cong S^u \text{ (as } R\text{-module)} \}, \text{ } u \text{ an integer number. (3)}$$

**Theorem 5.** The cardinal of the set $E_R(\ell, m, k)$ is given by

$$N_R(\ell, m, k) = \sum_{u=k}^{k} (-1)^{u-k} |M_R(\ell, m, u)| \left[ k \atop u \right]_{q^{m, s}}. \quad (4)$$

Moreover $N_R(\ell, m, k) \leq N_R(\ell, m, k).$

Proof. For every $\mathcal{B} \in E_R(\ell, m, k)$ there exists $\mathcal{D} \in E_R(\ell, m, k)$ such that $\mathcal{D} \subseteq \mathcal{B}$. Thus $E_R(\ell, m, k) = \bigcup_{D \in E_R(\ell, m, k)} |D : \rightarrow\rangle^{(k)}$, where $|D : \rightarrow\rangle^{(k)} := \left\{ \mathcal{B} \in L_E(\mathcal{B}) \mid \mathcal{B} \cong S^k \text{ (as } R\text{-module)} \right\}$. Note that $|D : \rightarrow\rangle^{(k)} = \left[ k \atop u \right]_{q^{m, s}}$, for every free $S$-linear code $\mathcal{D}$ of rank $u$. By inclusion-exclusion principle,

$$N_R(\ell, m, k) = \sum_{\mathcal{B} \in M} (-1)^{\mathcal{M}-1} |D : \rightarrow\rangle^{(k)} \text{. Since } \bigcap_{\mathcal{D} \in M} |D : \rightarrow\rangle^{(k)} = \left[ \sum_{\mathcal{D} \in M} |D : \rightarrow\rangle^{(k)} \right]^\prime, \text{ we have } N_R(\ell, m, k) = \sum_{u=k}^{k} (-1)^{u-k} \sum_{\mathcal{B} \in M} |D : \rightarrow\rangle^{(k)} \text{. Therefore (4) and the last inequality follow. □}$

**Corollary 1.** Let $C$ be a free $R$-linear code of length $\ell$ and rank $k'$. The number of distinct free $S$-linear codes $\mathcal{B}$ of length $\ell$ and rank $k$ with the property $C = \text{Res}_S(\mathcal{B})$ is given by $N_R(\ell, m, \ell - k + k').$

Proof. By Theorem 2 $\mathcal{B} = \text{Ext}_S(C) \oplus \mathcal{B}_1$, where $\mathcal{B}_1$ is a free $S$-linear subcode of $\mathcal{B}$ of rank $k - k'$. Hence $\mathcal{B}_1^* \in E_R(\ell, m, \ell - k + k')$ and the number of distinct free $S$-linear codes $\mathcal{B}$ of length $\ell$, of rank $k$, with the property $C = \text{Res}_S(\mathcal{B})$ is $N_R(\ell, m, \ell - k + k')$. □

**Remark 1.** It was proved in [1] Theorem 6] in the case of finite fields that $N_{F_q}(\ell, m, \ell - k + k') = \left[ \frac{\ell - k'}{\ell - k} \right]_{q^m}$, which is in general not true. The counterexample of the same has been presented at the beginning of the paper.
Theorem 6. Let $S$ be the Galois extension of $R$ of degree $m$ and $(k, k')$ be a pair of positive integers. Then the number of free $S$-linear code $B$ of length $\ell$ and of rank $k$ with the property $k' = \text{rank}_B(\text{Res}_R(B))$ is given by
\[
\Omega_B(\ell, m, k, k') = \mathcal{N}_B(\ell, m, \ell - k + k') \left[ \begin{array}{c} \ell \\ k' \end{array} \right]_{(q,s)}.
\]

Proof. There are $\left[ \begin{array}{c} \ell \\ k' \end{array} \right]_{(q,s)}$ free $R$-linear codes $C$ of length $\ell$ and rank $k'$. For each such $C$ there are $\mathcal{N}_B(\ell, m, \ell - k + k')$ free $S$-linear codes $B$ of length $\ell$ of rank $k$, with the property $C = \text{Res}_R(B)$ and thus the result follows.

4. Counting codes over finite PIRs

In this section, we extend our results to linear codes over finite principal ideal rings (PIRs). For a detailed treatment of the theory of finite principal ideal rings we refer the reader to \cite{2,7}. Let $R$ be a finite PIR with maximal ideals $m_1, \cdots, m_u$ and $s_1, \cdots, s_u$ their indices of stability respectively. Clearly $R_t := R/m_t^u$ is a finite chain ring with maximal ideal $m_t/m_t^u$. Then we have the ring homomorphism
\[
\Phi : R \to R_1 \times \cdots \times R_u,
\]
where $\Phi_i(a) := a + m_t$ and $\Phi_i : R \to R_t$ naturally extends to $R[X]$ acting on the coefficients. Since the maximal ideals $m_1, \cdots, m_u$ of $R$ are pairwise coprime, the ring homomorphism $\Phi$ is a ring isomorphism by the Chinese remainder theorem. Thus $R = \text{CRT}(R_1, \cdots, R_u) = \Phi^{-1}(R_1 \times \cdots \times R_u)$ and we say that $R$ is the Chinese product of rings $\{R_t\}_{t=1}^u$. We extend the ring isomorphism $\Phi$ coordinatewise to an isomorphism of $R$-modules
\[
\Phi : \prod_{t=1}^u R_t^\ell \to R^\ell.
\]
If $C$ is an $R$-linear code of length $\ell$ then $C = \text{CRT}(C_1, C_2, \cdots, C_u) = \{\Phi^{-1}(c_1, c_2, \cdots, c_u) \mid c_t \in C_t\}$, where $C_t = \Phi_t(C)$ is a $R_t$-linear codes of length $\ell$. We call $C$ the Chinese product of codes $C_1, C_2, \ldots, C_u$, and $C_t$ is called $t^{\text{th}}$ component of $C$.

Lemma 1 (\cite{2}, Theorem 2.4). Let $\{C_t\}_{t=1}^u$ be $R_t$-linear codes of length $\ell$ and $C$ the Chinese product them. Then $C$ is a free $R$-linear code if and only if each $C_t$ is a free $R_t$-linear code with the rank $\text{rank}_{R_t}(C)$.

Corollary 2 (\cite{2}, Lemma 2.3). Let $C$ be an $R$-linear code of length $\ell$ and $D$ an $R$-linear subcode of $C$ with rank $k$. Then $D = \text{CRT}(D_1, D_2, \cdots, D_u)$, where $D_t = \Phi_t(D)$ is an $R_t$-linear subcode of $\Phi_t(C)$ and $\text{max}\{\text{rank}_{R_t}(D_t)\} = k$.

The following results are the counterparts for PIR’s of the ones in the previous section.

Theorem 7. Let $R$ be a principal ideal ring such that $R = \text{CRT}(R_1, R_2, \cdots, R_u)$, where $R_t$ are chain rings with invariants $(q_t, s_t)$, and $C$ be a free $R$-linear code of rank $k$. Then the number of free $R$-linear subcode $D$ of $C$, with rank $k'$ is
\[
\left[ \begin{array}{c} k \\ k' \end{array} \right]_{R} = \prod_{t=1}^u \left[ \begin{array}{c} k \\ k' \end{array} \right]_{(q_t, s_t)}
\]
From [10, Proposition 1.2(1), pp. 80] and [11, Proposition 1.1.], a Galois extension of a finite PIR is defined as follows:

**Definition 2.** Let $R$ be the Chinese product of finite chain rings $\{R_t\}_{t=1}^n$, the ring $S$ is a Galois extension of $R$ with Galois group $\mathrm{Gal}_R(S)$ if and only if

1. $S \cong R[X]/(f)$, where $\Phi_t(f)$ is monic basic irreducible polynomial over $R_t$ of degree $|\mathrm{Gal}_R(S)|$;
2. $\mathrm{Gal}_R(S)$ is a cyclic group generated by $\sigma$, where $\sigma = \Phi_t^{-1} \circ (\sigma_1 \times \sigma_2 \times \cdots \times \sigma_n) \circ \Phi$ and $\sigma_t$ is a generator of $\mathrm{Aut}_{R_t}(S_t)$, with $S_t = \Phi_t(S)$.

The following theorem presents a generalization of a Theorem 6 to PIRS.

**Theorem 8.** Let $R$ be the Chinese product of finite chain rings $\{R_t\}_{t=1}^n$, and $S$ be the Galois extension of $R$ with Galois group $\mathrm{Gal}_R(S)$. Then the number of free $S$-linear codes $\mathcal{B}$ of length $\ell$ such that $k = \text{rank}_S(\mathcal{B})$ and $k' = \text{rank}_R(\mathcal{B} \cap R^\ell)$ is

$$\hat{\Omega}_\ell(\ell, m, k, k') := \prod_{t=1}^n \Omega_{R_t}(\ell, m, k, k'),$$

where $m = |\mathrm{Gal}_R(S)|$.

**References**

[1] B. Lyle, A linear-algebra problem from algebraic coding theory, Linear Algebra Appl. 22 (1978) 223–233.
[2] F. Hernando, K. Marshall, M. E. O’Sullivan, The dimension of subcode-subfields of shortened generalized Reed-Solomon codes, Des. Codes Cryptogr. 69 (1) (2013) 131–142.
[3] E. Martínez-Moro, A. P. Nicolás, I. F. Rúa, On trace codes and Galois invariance over finite commutative chain rings, Finite Fields Appl. 22 (2013) 114–121.
[4] A. F. Tabue, E. Martínez-Moro, C. Mouaha, Galois correspondence on linear codes over finite chain rings, Submitted to Linear Algebra Appl., CoRR Arxiv: abs/1602.01242.
[5] T. Honold, I. Landjev, Linear codes over finite chain rings, Electron. J. Combin. 7 (2000) Research Paper 11, 22.
[6] S. T. Dougherty, J.-L. Kim, H. Kulosman, MDS codes over finite principal ideal rings, Des. Codes Cryptogr. 50 (1) (2009) 77–92.
[7] B. R. McDonald, Finite rings with identity, Marcel Dekker, Inc., New York, 1974, pure and Applied Mathematics, Vol. 28.
[8] S. T. Dougherty, H. Liu, Independence of vectors in codes over rings, Des. Codes Cryptogr. 51 (1) (2009) 55–68.
[9] S. T. Dougherty, S. Han, H. Liu, Higher weights for codes over rings, Applicable Algebra in Engineering, Communication and Computing 22 (2) (2011) 113–135.
[10] F. DeMeyer, E. Ingraham, Separable algebras over commutative rings, Lecture Notes in Mathematics, Vol. 181, Springer-Verlag, Berlin-New York, 1971.
[11] E. S., W. Y., On Separable Algebras Over Commutative Rings, Osaka J. Math. (4) (1967) 233–242.