CONSTRUCTIVE NONCOMMUTATIVE INVARIANT THEORY

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Abstract. A constructive Hilbert-Nagata Theorem is proved for the algebra of invariants in a Lie nilpotent relatively free associative algebra endowed with an action induced by a representation of a reductive group. More precisely, the problem of finding generators of the subalgebra of invariants in the relatively free algebra is reduced to finding generators in the commutative algebra of polynomial invariants for a representation associated canonically to the original representation.

1. Introduction

Let $K$ be a field of characteristic zero. In this paper by ‘algebra’ we mean an associative $K$-algebra with unity, unless explicitly stated otherwise. For a finite dimensional $K$-vector space $V$ denote by $T(V)$ the tensor algebra of $V$; that is, $T(V)$ is the free algebra generated by a basis of $V$. Given a variety $R$ of algebras we write $F(R, V)$ for the relatively free algebra in $R$ generated by a basis of $V$. Recall that $R$ consists of all algebras that satisfy a given set of polynomial identities, so $F(R, V)$ is the factor algebra of $T(V)$ modulo a T-ideal (an ideal stable under all algebra endomorphisms of $T(V)$). Since we deal with algebras with unity and $K$ is infinite, $R$ necessarily contains the variety of commutative algebras, and we have the canonical surjections

$$T(V) \rightarrow F(R, V) \rightarrow S(V),$$

where $S(V)$ is the symmetric tensor algebra of $V$ (i.e., the commutative polynomial algebra generated by a basis of $V$). The above algebra surjections are $GL(V)$-equivariant, and they are homomorphisms of graded algebras, where $T(V)$ is endowed with the standard grading, $V \subset T(V)$ being the degree 1 homogeneous component in $T(V)$. Note that when $R$ is the variety of all algebras we have that $F(R, V) = T(V)$, and when $R$ is the variety of commutative algebras we have $F(R, V) = S(V)$.

Throughout the paper $G$ will denote a linear algebraic group. Suppose that $V$ is a rational $G$-module; that is, we are given a group homomorphism $\rho : G \rightarrow GL(V)$ that is a morphism of affine algebraic varieties (from now on we shall simply say that $V$ is a $G$-module). The action of $G$ on $V$ induces an action on $T(V)$, $F(R, V)$,
and $S(V)$ via automorphisms of graded algebras, and the above surjections are $G$-equivariant. We are interested in the subalgebra

$$F(R, V)^G = \{ f \in F(R, V) \mid g \cdot f = f \text{ for all } g \in G \}$$

of $G$-invariants. We refer to [7] and [9] for surveys on results concerning subalgebras of invariants in relatively free algebras.

For an integer $p \geq 1$ denote by $N_p$ the variety of Lie nilpotent algebras of Lie nilpotency index less or equal to $p$. In other words, $N_p$ is the variety of algebras satisfying the polynomial identity $[x_1, \ldots, x_{p+1}] = 0$. Here $[x_1, x_2] = x_1x_2 - x_2x_1$, and for $i \geq 3$ we have $[x_1, \ldots, x_i] = [[x_1, \ldots, x_{i-1}], x_i]$.

**Remark 1.1.** For a (nonunitary) associative algebra nilpotence of index $\leq p$ means that the algebra satisfies the polynomial identity $x_1 \cdots x_p = 0$. By analogy with group theory a Lie algebra is nilpotent of index $\leq p$, if it satisfies the polynomial identity $[x_1, \ldots, x_{p+1}] = 0$.

Our starting point is the following non-commutative generalization of the Hilbert-Nagata theorem (see for example [10, Theorem A, p.3]):

**Theorem 1.2.** ([5, Theorem 3.1]) Suppose that $G$ is reductive, and $R \subseteq N_p$ for some $p \geq 1$. Then $F(R, V)^G$ is a finitely generated algebra for any $G$-module $V$.

**Remark 1.3.** The assumption $R \subseteq N_p$ for some $p \geq 1$ above is necessary, as the following converse of Theorem 1.2 is also shown in [5]: If $\dim_K(V) \geq 2$ and $F(R, V)^G$ is finitely generated for all reductive subgroups of $GL(V)$, then $R \subseteq N_p$ for some $p \geq 1$.

The proof of Theorem 1.2 in [5] is non-constructive, since it uses the Noetherian property of $F(R, V)$ for $R \subset N_p$, similarly to the fundamental paper of Hilbert [12] on the commutative case $p = 1$, where the Hilbert Basis Theorem was proved. Hilbert gave a more constructive proof of the commutative case in [13] (explicit degree bounds for the generators were first proved by Popov [19], [20], and stronger bounds more recently by Derksen [2]). For constructive (commutative) invariant theory of reductive groups see also [1], [3].

In the present paper we make Theorem 1.2 constructive. In algorithms for computing generators of algebras of invariants a crucial role is played by degree bounds. For example, an a priori degree bound for the generators of the algebra $F(R, V)^G$ implies an algorithm to find explicit generators (by solving systems of linear equations). From a different perspective, sometimes there is a qualitatively known process that yields generators of the algebra of invariants, and the aim is to derive from it a degree bound for a minimal generating system, to have some quantitative information on the algebra of invariants.

In order to discuss degree bounds we need to introduce some notation. Given a graded algebra $A = \bigoplus_{d=0}^{\infty} A_d$ we denote by $\beta(A)$ the minimal non-negative integer $d$ such that $A$ is generated by homogeneous elements of degree at most $d$ (and write $\beta(A) = \infty$ if there is no such $d$). Set

$$\beta(G) := \sup\{\beta(S(V)^G) \mid V \text{ is a } G\text{-module}\}.$$ 

It is a classical theorem of E. Noether [18] that for $G$ finite we have $\beta(G) \leq |G|$. On the other hand Derksen and Kemper [3, Theorem 2.1] proved that $\beta(G) = \infty$.
for any infinite group $G$. So for an infinite group $G$, finite degree bounds may hold only for restricted classes of $G$-modules.

Given a set $\tau$ of isomorphism classes of simple $G$-modules, denote by $\text{add}(\tau)$ the class of all $G$-modules that are finite direct sums of simple modules whose isomorphism class belongs to $\tau$. This definition is particularly natural when the group $G$ is reductive, because in that case any $G$-module decomposes as a direct sum of simple $G$-modules. By slight abuse of notation we write $V \in \tau$ if the isomorphism class of the $G$-module $V$ belongs to $\tau$. Moreover, write $\tau^\otimes p$ for the isomorphism classes of simple summands of all $G$-modules $V_1 \otimes \cdots \otimes V_q$, where $V_i \in \tau$ and $q \leq p$. Set

$$\beta_{\tau} := \sup \{ \beta(S(V)^G) \mid V \in \text{add}(\tau) \}.$$ 

Weyl’s theorem [23] on polarizations implies the following:

**Proposition 1.4.** Let $G$ be a reductive group and let $\tau$ be a finite set of isomorphism classes of simple $G$-modules. Then the number $\beta_{\tau}$ is finite.

**Proof.** Let $V_1, \ldots, V_q$ be simple $G$-modules representing the isomorphism classes in $\tau$. Let $V$ be an arbitrary $G$-module in $\text{add}(\tau)$. Then $V \cong \sum_{i=1}^q m_i V_i$, where the $m_i$ are non-negative integers and $m_i V_i$ stands for the direct sum of $m_i$ copies of $V_i$. Weyl’s theorem on polarizations (the special case $h = 1$ of Theorem 2.1 below) implies that

$$\beta(S(V)^G) \leq \beta(\sum_{i=1}^q \min \{m_i, \dim(V_i)\} V_i^G) \leq \beta(\sum_{i=1}^q \dim(V_i) V_i^G) \leq \beta(S(\sum_{i=1}^q \dim(V_i) V_i^G)).$$

(The second inequality above follows from the fact that if $A$ is a submodule of the $G$-module $B$, then there is a $G$-equivariant graded $K$-algebra surjection $S(B) \to S(A)$ mapping some of the variables to zero). Since (1) holds for any $V \in \text{add}(\tau)$, we conclude the equality

$$\beta_{\tau} = \beta(S(\sum_{i=1}^q \dim(V_i) V_i^G)).$$

The assumption on $G$ guarantees that the number on the right hand side above is finite. $\square$

Turning to a variety $\mathcal{R}$ of associative algebras, we write

$$\beta_{\tau, \mathcal{R}} := \sup \{ \beta(F(\mathcal{R}, V)^G) \mid V \in \text{add}(\tau) \},$$

where $\beta(F(\mathcal{R}, V)^G)$ is the supremum of the degrees of the elements in a minimal homogeneous generating system of the algebra $F(\mathcal{R}, V)^G$. Having established this notation we are in position to state the following corollary of the results of the present paper:

**Theorem 1.5.** Suppose that the group $G$ is reductive and the variety $\mathcal{R}$ is contained in $\mathcal{R}_p$ for some $p \geq 1$. Let $\tau$ be a finite set of isomorphism classes of simple $G$-modules. Then we have the inequality

$$\beta_{\tau, \mathcal{R}} \leq p \beta_{\tau^\otimes p}.$$
In fact the results of this paper give more: under the assumptions of Theorem 1.5 the construction of an explicit generating system of $F(\mathfrak{R}, V)^G$ is reduced to finding a generating system of a commutative algebra of $G$-invariants $S(W)^G$ for a $G$-module $W$ associated canonically to $V$. We need some preparations to give the precise statement.

Fix an integer $p \geq 1$, a variety $\mathfrak{R}$ contained in $\mathfrak{R}_p$, and a $G$-module $V$ (where $G$ is an arbitrary linear algebraic group). Set $F := F(\mathfrak{R}, V)$. Consider the subspaces in $T(V)$ given by

$$V^{[\leq p]} := \text{Span}_K \{ [v_1, \ldots, v_d] | v_1, \ldots, v_d \in V \}$$

for $d = 1, 2, \ldots$, and $V^{[1]} := V$. As a consequence of the Jacobi identity we have

$$[V^{[d]}, V^{[e]}] \subseteq V^{[d+e]}.$$ The subspace $V^{[d]}$ is a $GL(V)$-submodule in the $d^{\text{th}}$ tensor power $V^{\otimes d}$ of $V$. Set

$$V^{[\leq p]} := \bigoplus_{d=1}^{p} V^{[d]},$$

and let $T := T(V^{[\leq p]})$ be the tensor algebra generated by $V^{[\leq p]}$. Denote by $e_i$ the $i^{\text{th}}$ standard basis vector $e_i := (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}_0^p$ (the coordinate 1 is in the $i^{\text{th}}$ position). The algebra $T$ has an $\mathbb{N}_0^p$-grading

$$T = \bigoplus_{\alpha \in \mathbb{N}_0^p} T_\alpha, \quad T_{e_i} = V^{[i]} \subseteq V^{[\leq p]} \text{ for } i = 1, \ldots, p.$$ Note that $GL(V)$ acts on $T$ via $\mathbb{N}_0^p$-graded algebra automorphisms. The symmetric tensor algebra $S := S(V^{[\leq p]})$ is endowed with the analogous $\mathbb{N}_0^p$-grading, so the natural surjection $\pi_S : T \to S$ is a $GL(V)$-equivariant homomorphism of $\mathbb{N}_0^p$-graded algebras.

The tautological linear embedding of $V^{[\leq p]}$ into $T(V)$ extends to a $GL(V)$-equivariant algebra surjection $\pi : T \to T(V)$. Composing this with the natural surjection $\nu : T(V) \to F(\mathfrak{R}, V)$ we obtain

$$\pi_F := \nu \circ \pi : T \to F.$$ For $\alpha \in \mathbb{N}_0^p$ write $|\alpha| := \sum_{i=1}^{p} \alpha_i$. Denote by $\iota_\alpha : S_\alpha \to T_\alpha$ the linear map given by

$$\iota_\alpha(v_1 \cdots v_{|\alpha|}) = \frac{1}{|\alpha|!} \sum_{\sigma \in \text{Sym}_{|\alpha|}} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(|\alpha|)}$$

(where Sym$_d$ stands for the symmetric group on $\{1, \ldots, d\}$ and $v_1, \ldots, v_{|\alpha|} \in V^{[\leq p]}$). Obviously $\iota_\alpha$ is $GL(V)$-equivariant (as it is even $GL(V^{[\leq p]})$-equivariant), and hence $\iota_\alpha$ is $G$-equivariant (just like $\pi_S$). Thus $\iota_\alpha(S_G^\alpha) \subseteq T_G^\alpha$. Moreover, $\pi_S \circ \iota_\alpha$ is the identity map on $S_\alpha$, consequently $\pi_S(\iota_\alpha(S_G^\alpha)) = S_G^\alpha$. In particular, $\pi_S(T_G^\alpha) = S_G^\alpha$.

**Theorem 1.6.** Let $\mathfrak{R}$ be a variety contained in $\mathfrak{R}_p$ for some $p \geq 1$. Let $V$ be a $G$-module, and $S$, $T$, $F$ the algebras defined above. Take a multihomogeneous (with respect to the $\mathbb{N}_0^p$-grading) $K$-algebra generating system $\{f_\lambda | \lambda \in \Lambda\}$ of $S_G^\alpha$, and set

$$\tilde{f}_\lambda := \iota_{\alpha_\lambda}(f_\lambda) \in T_G^\alpha, \quad \lambda \in \Lambda.$$
where \( \alpha_\lambda \) stands for the multidegree of \( f_\lambda \). Then the \( K \)-algebra \( F^G \) is generated by \( \{ \pi_F(f_\lambda) \mid \lambda \in \Lambda \} \).

**Remark 1.7.** The algebra \( S^G \) is known to be finitely generated (in addition to the case when \( G \) is reductive) also when \( G \) is a maximal unipotent subgroup of a reductive group (see [11] or [10, Theorem 9.4]), and consequently when \( G \) is a Borel subgroup of a reductive group. In these cases Theorem 1.6 reduces the construction of a finite generating system of \( F^G \) to the construction of a finite generating system in the commutative algebra of invariants \( S^G \).

The paper is organized as follows. Since the technique of polarization in commutative invariant theory is fundamental for Proposition 1.4, first in Section 2 we investigate to what extent it works in our noncommutative setup. Theorem 2.1 is a noncommutative generalization of Weyl’s theorem on polarization, however, it applies for a class of varieties different from those appearing in Theorem 1.2. In Section 3 we prove Theorem 1.6, and deduce Theorem 1.5 from it.

2. A noncommutative generalization of Weyl’s theorem

In this section we assume that the variety \( \mathfrak{R} \) is generated (in the sense of universal algebra) by a finitely generated algebra. Kemer [14] proved that then \( \mathfrak{R} \) satisfies a Capelli identity. This means that there exists a positive integer \( h = h(\mathfrak{R}) \) such that all algebras in \( \mathfrak{R} \) satisfy the polynomial identity

\[
\sum_{\pi \in S_{h+1}} (-1)^\pi x_{\pi(1)}y_1 x_{\pi(2)}y_2 \cdots y_h x_{\pi(h+1)} = 0.
\]

(5)

Our focus is on the case when \( V = U + mW \), where

\[ mW = W + \cdots + W \]

is the direct sum of \( m \) copies of a \( G \)-module \( W \), and \( U \) is a \( G \)-module. We shall identify \( mW \) with \( W \otimes K^m \), which is naturally a \( GL(W) \times GL(K^m) \)-module (here \( K^m \) stands for the space of column vectors of length \( m \) over \( K \)). Also \( V = U + mW \) is naturally a \( GL(U) \times GL(W) \times GL(K^m) \)-module. Moreover, for \( l \leq m \) we shall identify \( K^l \) with the subspace of \( K^m \) consisting of column vectors whose last \( m - l \) coordinates are zero. Accordingly \( lW \) is identified with the subspace of \( mW \) consisting of \( m \)-tuples of elements of \( W \) with the zero vector as the last \( m - l \) component. In this way \( F(\mathfrak{R}, U + lW) \) becomes a subalgebra of \( F(\mathfrak{R}, U + mW) \).

**Theorem 2.1.** Let \( G \) be a group, \( U \) and \( W \) \( G \)-modules, \( n = \dim(W) \), and let \( B \) be a set of generators of the algebra \( F(\mathfrak{R}, U + nhW)^G \) where \( h = h(\mathfrak{R}) \) as above. Then for any \( m \geq nh \) the algebra \( F(\mathfrak{R}, U + mW)^G \) is generated by

\[ \{ g \cdot f \mid g \in GL(K^m), f \in B \}. \]

Recall that the relatively free algebra \( F(\mathfrak{R}, V) \) is graded such that the subspace \( V \) is the degree 1 homogeneous component.

**Corollary 2.2.** In the scenario of Theorem 2.1 for all positive integers \( m \) we have the inequality

\[ \beta(F(\mathfrak{R}, U + mW)^G) \leq \beta(F(\mathfrak{R}, U + nhW)^G). \]

**Remark 2.3.** In the special case when \( \mathfrak{R} \) is the variety of commutative algebras we have \( h(\mathfrak{R}) = 1 \), and hence Theorem 2.1 in this special case recovers Weyl’s theorem on polarization, which is a main theme in [23].
In order to prove Theorem 2.1 we need to recall some facts about polynomial representations of the general linear group. A partition \( \lambda \) of \( d \) (we write \( \lambda + d \)) is a finite non-increasing sequence \( \lambda = (\lambda_1, \lambda_2, \ldots) \) of non-negative integers (with the convention that zeros can be freely appended to or removed from the end) such that \( \sum \lambda_i = d \). Write \( \mathrm{ht}(\lambda) \) for the number of non-zero components of \( \lambda \). Denote by \( S^\lambda(\cdot) \) the Schur functor (see for example [21]), so for a finite dimensional \( K \)-vector space \( E \) and a partition \( \lambda \) with \( \mathrm{ht}(\lambda) \leq \dim(E) \), we have that \( S^\lambda(E) \) is a simple polynomial \( GL(E) \)-module.

**Lemma 2.4.** If the simple \( GL(E) \)-module \( S^\mu(E) \) occurs as a direct summand in \( S^\lambda(U + nE) \), where \( n \) is a positive integer and \( GL(E) \) acts trivially on the finite dimensional \( K \)-vector space \( U \), then \( \mathrm{ht}(\mu) \leq n \cdot \mathrm{ht}(\lambda) \).

*Proof.* Suppose that \( S^\mu(E) \) occurs as a direct summand in \( S^\lambda(V) \), where \( V = U + nE \). It follows from Pieri’s rules (see [17]) that

\[
S^\lambda(V) \text{ is a direct summand in } \bigotimes_{i=1}^{h(\lambda)} S^{(\lambda_i)}(V),
\]

where for the partition \((d)\) with only one non-zero part, \( S^{(d)}(V) \) is the \( d \)th symmetric tensor power of \( V \). Therefore \( S^\mu(E) \) is a direct summand in \( \bigotimes_{i=1}^{h(\lambda)} S^{(\lambda_i)}(U + nE) \).

Any simple \( GL(E) \)-module summand of \( S^{(\lambda_i)}(U + nE) \) is isomorphic to \( S^{(\mu^i)}(E) \) for some partition \( \mu^i = (\nu^i_1, \ldots, \nu^i_n) \) with \( \mathrm{ht}(\nu_i) \leq n \) by the Cauchy identity (see [21]). Therefore again by (6), \( S^\mu(E) \) is a direct summand in \( S^{\nu^1}(E) \otimes \cdots \otimes S^{\nu^h(\lambda)}(E) \), hence \( S^\mu(E) \) is a direct summand in

\[
\bigotimes_{i=1}^{h(\lambda)} \bigotimes_{j=1}^{n} S^{(\nu^i_j)}(E).
\]

We conclude by Pieri’s rules that \( \mathrm{ht}(\mu) \) is bounded by the number \( nh(\lambda) \) of tensor factors in the above expression. \( \square \)

*Proof of Theorem 2.1.* Set \( V = U + mW = U + W \otimes K^m \). Since \( F(\mathcal{R}, V) \) satisfies the Capelli identity (5), by a theorem of Regev [22] (see also [8, Theorem 2.3.4]) no \( S^\lambda(V) \) with \( \mathrm{ht}(\lambda) > h = h(\mathcal{R}) \) occurs as a summand of \( F(\mathcal{R}, V) \). So the degree \( d \) homogeneous component of \( F(\mathcal{R}, V) \) has the \( GL(V) \)-module decomposition

\[
F(\mathcal{R}, V)_d \cong \sum_{\lambda \vdash d \atop \mathrm{ht}(\lambda) \leq h} m^\lambda S^\lambda(V)
\]

with some non-negative integers \( m^\lambda \). Setting \( E = K^m \) and \( n = \dim(W) \), Lemma 2.4 shows that as a \( GL(E) \)-module, \( F(\mathcal{R}, U + W \otimes E)_d \) decomposes as

\[
F(\mathcal{R}, U + nE)_d \cong \sum_{\mu \vdash d \atop \mathrm{ht}(\mu) \leq nh} r^\mu S^\mu(E)
\]

with some non-negative integers \( r^\mu \). Since the actions of \( G \) and \( GL(E) \) commute, the algebra \( F(\mathcal{R}, U + W \otimes E)^G \) is a \( GL(E) \)-submodule in \( F(\mathcal{R}, U + W \otimes E) \). Thus each simple \( GL(E) \)-module direct summand of \( F(\mathcal{R}, U + W \otimes E)^G \) is isomorphic to \( S^\mu(E) \) for some partition \( \mu \) with \( \mathrm{ht}(\mu) \leq nh \). Such a summand is
generated by a highest weight vector \( w \), having the property that for an element 
\( g = \text{diag}(z_1, \ldots, z_m) \in GL(K^m) = GL(E) \) we have \( g \cdot w = (z_1^{\mu_1} \cdots z_m^{\mu_m})w \). Since 
\( \mu_{nh+1} = \cdots = \mu_m = 0 \), we conclude that \( w \) belongs to the subalgebra \( F(\mathfrak{g}, U + W \otimes K^{nh}) = F(\mathfrak{g}, U + nhW) \) of \( F(\mathfrak{g}, U + W \otimes E) \). Thus \( F(\mathfrak{g}, U + W \otimes E)^G \) is contained in the \( GL(E) \)-submodule of \( F(\mathfrak{g}, U + W \otimes E) \) generated by \( F(\mathfrak{g}, U + W \otimes K^{nh})^G \).

Now \( \{ g : f \mid g \in GL(K^m), f \in B \} \) spans a \( GL(E) \)-submodule, hence it generates a 
\( GL(E) \)-stable subalgebra of \( F(\mathfrak{g}, U + W \otimes E)^G \). By construction this subalgebra contains \( F(\mathfrak{g}, U + W \otimes K^{nh})^G \), so it contains \( F(\mathfrak{g}, U + W \otimes E)^G \). \( \square \)

3. Lie nilpotent relatively free algebras

Throughout this section \( p \geq 1 \) is a fixed integer, \( \mathfrak{g} \) is a variety contained in 
\( \mathfrak{n}_p \), \( V \) is a \( G \)-module, \( F = F(\mathfrak{g}, V), T = T(V^{\leq p}), S = S(V^{\leq p}) \) as defined in 
Section 1.

The \( \mathbb{N}_0^p \)-grading \( \boxed{3} \) on \( T = \bigoplus_{\alpha \in \mathbb{N}_0^p} T_\alpha \) induces an \( \mathbb{N}_0 \)-grading \( T = \bigoplus_{d=0}^{\infty} T_d \) where

\[
T_d := \bigoplus_{\alpha : \sum_{i=1}^{p} (i-1)\alpha_i = d} T_\alpha.
\]

In particular, for \( d = 1, \ldots, p \) we have \( V^{[d]} \subset T_{d-1} \). Similarly \( S = \bigoplus_{d=0}^{\infty} S_d \),

\[
S_d := \bigoplus_{\alpha : \sum_{i=1}^{p} (i-1)\alpha_i = d} S_\alpha.
\]

Since the natural surjection \( \pi_S : T \to S \) preserves the \( \mathbb{N}_0^p \)-grading \( \boxed{3} \), it preserves also the \( \mathbb{N}_0 \)-grading \( \boxed{3} \). The \( \mathbb{N}_0 \)-gradings on \( T \) and \( S \) determine a descending filtration by ideals

\[
T = T(0) \supseteq T(1) \supseteq T(2) \supseteq \cdots \ \ \ S = S(0) \supseteq S(1) \supseteq S(2) \supseteq \cdots,
\]

where \( T(d) := \bigoplus_{j \geq d} T_j \) and \( S(d) := \bigoplus_{j \geq d} S_j \). We shall identify the factor \( T(d)/T(d+1) \)
(respectively \( S(d)/S(d+1) \)) with \( T_d \) (respectively \( S_d \)) in the obvious way.

The tautological linear embedding of \( V^{[\leq p]} \) into \( T(V) \) extends to an algebra surjection \( \pi : T \to T(V) \). This is \( GL(V) \)-equivariant, but does not preserve the grading (recall that \( T(V) \) is endowed with the standard grading where \( V \subset T(V) \) is the degree 1 homogeneous component). A homogeneous element in \( T \) may not be mapped to a homogeneous element of \( T(V) \). However, the multihomogeneous component \( T_\alpha \subset T \) is mapped under \( \pi \) into the homogeneous component of \( T(V) \) of degree \( \sum_{i=1}^{p} i\alpha_i \).

Define \( F(d) := \pi_T(T(d)) \), then

\[
F = F(0) \supseteq F(1) \supseteq F(2) \supseteq \cdots
\]
is a descending filtration of \( F \) by ideals. However, unlike the filtrations of \( T \) and \( S \) above, this descends to the zero space in finitely many steps by a result of Latyshev \( \boxed{16} \), asserting in particular that if a finitely generated algebra satisfies a non-matrix identity (i.e., a polynomial identity that does not hold for the algebra
of $2 \times 2$ matrices over $K$), then its commutator ideal is nilpotent. The natural surjection $\pi_S : T \to S$ and the homomorphism $\pi_F$ defined in (4) induce surjective linear maps between the factors of the filtrations as follows:

$$
T_d = T(d)/T(d+1) \xrightarrow{\pi_F} F(d)/F(d+1)
$$

$$
S_d = S(d)/S(d+1)
$$

Lemma 3.1. For all $d = 0, 1, \ldots$ we have $\ker(\pi_{S,d}) \subseteq \ker(\pi_{F,d})$.

Proof. Clearly $\ker(\pi_S)$ is generated as a two-sided ideal by elements of the form $[v, w]$, where $v, w$ range over a vector space spanning set of $V^{[\leq p]} \subset T$. Therefore the degree $d$ homogeneous component of $\ker(\pi_S)$ is spanned by elements of the form $v_1 \cdots v_{k+1} v_{k+2} \cdots v_i$, where $v_j \in V^{[d_j]}$ for $j = 1, \ldots, l$, $d_j \in \{1, \ldots, p\}$, and

$$
\sum_{i=1}^l (d_i - 1) = d. \text{ If } d_{k+1} + d_{k+2} \geq p + 1, \text{ then } \pi([v_{k+1}, v_{k+2}]) \in V^{[p+1]} \subset T(V) \text{ by (2), whence } \pi_F([v_{k+1}, v_{k+2}]) = \nu(\pi([v_{k+1}, v_{k+2}])) \in \nu(V^{[p+1]}) = \{0\}. \text{ Otherwise there exists a } u \in V^{[d_{k+1} + d_{k+2}]} \subset T \text{ such that } \pi(u) = \pi([v_{k+1}, v_{k+2}]). \text{ Now the degree of } v_1 \cdots v_{k+1} v_{k+2} \cdots v_i \text{ is }
$$

$$(d_1 - 1) + \cdots + (d_k - 1) + (d_{k+1} + d_{k+2} - 1) + (d_{k+3} - 1) + \cdots + (d_l - 1) = 1 + \sum_{i=1}^l (d_i - 1) = d + 1,$$

so $v_1 \cdots v_{k+1} v_{k+2} \cdots v_l \in T_{d+1}$ and hence

$$
\pi_F(v_1 \cdots v_{k+1} v_{k+2} v_{k+3} \cdots v_l) = \pi_F(v_1 \cdots v_k u v_{k+3} \cdots v_l) \subseteq F(d+1).
$$

It follows that $v_1 \cdots v_k v_{k+1} v_{k+2} v_{k+3} \cdots v_l \in \ker(\pi_{F,d})$. □

Corollary 3.2. For $d = 0, 1, 2, \ldots$ there exists a unique linear map

$$
\gamma_d : S_d \to F(d)/F(d+1) \text{ such that } \pi_{F,d} = \gamma_d \circ \pi_{S,d}.
$$

Moreover, the map $\gamma_d$ is necessarily surjective onto $F(d)/F(d+1)$ and is $GL(V)$-equivariant.

Proof. The existence of $\gamma_d$ is an immediate consequence of Lemma 3.1. The equality $\pi_{F,d} = \gamma_d \circ \pi_{S,d}$ and surjectivity of $\pi_{S,d}$ force its uniqueness. Moreover, since $\pi_{F,d}$ is surjective onto $F(d)/F(d+1)$, the map $\gamma_d$ must be also surjective. As both $\pi_{S,d}$ and $\pi_{F,d}$ are $GL(V)$-equivariant, the map $\gamma_d$ is $GL(V)$-equivariant as well. □

Lemma 3.3. Let $A, B$ be algebras with descending chains of ideals $A = A(0) \supseteq A(1) \supseteq A(2) \supseteq \ldots$, $B = B(0) \supseteq B(1) \supseteq B(2) \supseteq \ldots$, such that $B(N) = 0$ for some positive integer $N$. Let $\rho : A \to B$ be an algebra homomorphism such that $\rho(A(d)) \subseteq B(d)$ for all $d$. Assume that the induced linear maps

$$
\rho_d : A(d)/A(d+1) \to B(d)/B(d+1) \text{ are surjective for all } d.
$$

Then $\rho$ is surjective onto $B$.

Proof. It is sufficient to show that $\rho(A) \supseteq B(d)$ for all non-negative integers $d$. This trivially holds for $d \geq N$, as $B(N) = 0$. Now by a descending induction on $d$ we show that this holds also for $d < N$. Indeed, suppose that $\rho(A) \supseteq B(d + 1)$. Take any $b \in B(d)$. By (9) there exists an $a \in A(d)$ such that $\rho(a) - b \in B(d + 1)$. Now
B(d + 1) ⊆ \text{Im}(\rho) and \rho(a) \in \text{Im}(\rho) imply b \in \text{Im}(\rho). So B(d) ⊆ \text{Im}(\rho), finishing the proof of the inductive step.

Proof of Theorem 1.6. Recall that \( \gamma_d : S_d \to F(d)/F(d+1) \) is surjective by Corollary 3.2. Moreover, \( \gamma_d \) is \( \text{GL}(V) \)-equivariant, and \( S_d \) is a completely reducible \( \text{GL}(V) \)-module. Therefore there exists a \( \text{GL}(V) \)-module homomorphism \( \mu : F(d)/F(d+1) \to S_d \) such that \( \gamma_d \circ \mu \) is the identity map of \( F(d)/F(d+1) \). Since \( \mu \) is \( \text{GL}(V) \)-equivariant, it is also \( G \)-equivariant, hence \( \mu ((F(d)/(d+1)^G) \subseteq S_d^G \), and consequently \( \gamma_d(S_d^G) \supseteq \gamma_d((F(d)/(d+1)^G)) = (F(d)/(d+1)^G) \). Thus \( \gamma_d \) restricts to a surjective linear map

\[
S_d^G \to (F(d)/(d+1)^G).
\]

Denote by \( A \) the subalgebra of \( T \) generated by \{ \( f_\lambda \mid \lambda \in \Lambda \} \). By construction, \( \pi_S(A) \) contains all generators \( \pi_S(f_\lambda) = f_\lambda \) of \( S^G \), hence \( \pi_S(A) = S^G \). Furthermore, since \( A \) is generated by homogeneous elements (with respect to the grading \( \mathbf{S} \)), it is a graded subalgebra \( A = \bigoplus_{d=0}^{\infty} A_d \) of \( T \), and \( \pi_S(A_d) = S^G_d \) for all \( d \). Hence we have

\[
(10) \quad \pi_{F,d}(A_d) = (\gamma_d \circ \pi_{S,d})(A_d) = \gamma_d(S_d^G) = (F(d)/F(d+1))^G = (F(d)^G)/(d+1)^G.
\]

The last equality above follows from the complete reducibility of \( F(d) \) as a \( \text{GL}(V) \)-module; indeed, this implies that \( F(d)/F(d+1) \) embeds as a \( \text{GL}(V) \)-module (hence \( G \)-module) direct summand into \( F(d) \), therefore the natural surjection \( F(d) \to F(d)/F(d+1) \) restricts to a surjective linear map \( F(d)^G \to (F(d)/F(d+1))^G \), implying in turn that \( (F(d)/F(d+1))^G = (F(d)^G)/(d+1)^G \).

Now apply Lemma 3.3 for the algebra \( A \) with \( A(d) := \bigoplus_{k \geq d} A_k \) and \( B := F^G \), \( B(d) := F(d)^G \), and \( \rho := \pi_F|_A \). We identify \( A_d \) and \( A(d)/A(d+1) \) in the obvious way. Condition (9) holds by (10) and \( B(N) = \{0\} \) for \( N \) sufficiently large by (10), so we conclude that \( \pi_F(A) = F^G \). Thus \( \pi_F \) maps the generating system \{ \( f_\lambda \mid \lambda \in \Lambda \} \) of \( A \) to a generating system of \( F^G \).

Proof of Theorem 1.6. Let \( V \) be a \( G \)-module in \( \operatorname{add}(\tau) \). Now \( G \) is reductive, so we may take a finite minimal multihomogeneous (with respect to the \( \mathbb{N}_0^G \)-grading \( \mathbf{S} \)) generating system \{ \( f_\lambda \mid \lambda \in \Lambda \} \) of \( S(V^{[\leq \rho]})^G \). Consider the generating system \{ \( \pi_F(f_\lambda) \mid \lambda \in \Lambda \} \) of \( F(\mathfrak{R}, V)^G \) given by Theorem 1.6. Note that \{ \( f_\lambda \mid \lambda \in \Lambda \} \) is a minimal homogeneous generating system of \( S^G \) with respect to the standard grading, where \( V^{[\leq \rho]} \) is the degree 1 homogeneous component of \( S \). Since \( G \) is reductive, all \( G \)-modules are completely reducible. Moreover, \( V^{[\rho]} \) is a \( \text{GL}(V) \)-module (hence \( G \)-module) direct summand in \( V^{\otimes \rho} \). It follows that \( V^{[\leq \rho]} \) belongs to \( \operatorname{add}(\tau^{\otimes \rho}) \), implying that \( \deg(f_\lambda) = \sum_{i=1}^{p} \alpha_{\lambda,i} \leq \beta_{\tau^{\otimes \rho}} \) for each \( \lambda \in \Lambda \) (recall that the multidegree of \( f_\lambda \) is \( \alpha_{\lambda} = (\alpha_{\lambda,1}, \ldots, \alpha_{\lambda,p}) \)). So we have

\[
\deg(\pi_F(f_\lambda)) = \sum_{i=1}^{p} i\alpha_{\lambda,i} \leq p \sum_{i=1}^{p} \alpha_{\lambda,i} = p \deg(f_\lambda) \leq p\beta_{\tau^{\otimes \rho}}.
\]

In other words, \( F(\mathfrak{R}, V)^G \) is generated by elements of degree at most \( p\beta_{\tau^{\otimes \rho}} \). This holds for any \( G \)-module \( V \), therefore the desired inequality \( \beta_{\tau^{\otimes \rho}} \leq p\beta_{\tau^{\otimes \rho}} \) holds. \( \square \)
Remark 3.4. In the proof of Theorem 1.5 we used that all simple $G$-submodules of $V^q$ are contained in $V^q$, because $V^q$ is a $GL(V)$-submodule of $V^q$. We mention that it was proved by Klyachko\cite{Klyachko} that for $q \neq 4,6$, all simple $GL(V)$-submodules of $V^q$ different from the $q$th exterior or symmetric powers of $V$ are contained in $V^q$.

Following \cite{DomokosDrensky6} for a finite group $G$ we set
$$\beta(G, \mathcal{R}) = \sup \{ \beta(F(\mathcal{R}, V)^G | V \text{ is a } G\text{-module}) \}.$$  

Corollary 3.5. For a finite group $G$ we have
$$\beta(G, \mathcal{R}_p) \leq p\beta(G).$$

Remark 3.6. The results of \cite{DomokosDrensky6} provide an upper bound for $\beta(G, \mathcal{R}_p)$ of different nature.

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