On the origin of the correspondence between classical and quantum integrable theories

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Abstract

If we start from certain functional relations as definition of a quantum integrable theory, then we can derive from them a linear integral equation. It can be extended, by introducing dynamical variables, to become an equation with the form of Marchenko’s. Then, we derive from the latter a classical (differential) Lax pair. We exemplify our method by focusing on the massive version of the ODE/IM (Ordinary Differential Equations/Integrable Models) correspondence from Quantum sine-Gordon (sG) with many moduli/masses to the classical sinh-Gordon (shG) equation, so describing, in a particular case, some super-symmetric gauge theories and the $AdS_3$ strong coupling scattering amplitudes/Wilson loops. Yet, we present it in a way which reveals its generality of application. In fact, we give some hints on how it works for spin chains.

Keywords: Integrable Field Theories; ODE/IM correspondence; Baxter TQ relation; Marchenko equation

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1 From quantum to classical theory exactly (without approximation)

One remarkable correspondence of modern mathematical physics is the so-called ODE/IM correspondence \[^1\]. In a nutshell, it starts from the monodromies of a suitable Schrödinger equation and surprisingly derives the eigenvalues of two celebrated Baxter operators, the \( Q \) and \( T \) (functions), in the case of 2D Conformal Field Theories (CFTs). The natural evolution of this correspondence, with moduli, i.e. masses, and for the ground state, has been proposed by \[^2\] and then \[^3\]: they introduced first order differential \( 2 \times 2 \) matrix operators \( D, \bar{D} \), instead of second order differential scalar one, and studied monodromies of the solutions of the Lax linear problems

\[
DP = 0, \quad \bar{D}\bar{P} = 0, \tag{1.1}
\]

with \( D \) and \( \bar{D} \) given by

\[
D = \frac{\partial}{\partial w} + \frac{1}{2} \frac{\partial \hat{\eta}}{\partial w} \sigma^3 - e^{\theta + \hat{\eta}} \sigma^+ - e^{\theta - \hat{\eta}} \sigma^- , \quad \bar{D} = \frac{\partial}{\partial \bar{w}} - \frac{1}{2} \frac{\partial \hat{\eta}}{\partial \bar{w}} \sigma^3 - e^{-\theta + \hat{\eta}} \sigma^- - e^{-\theta - \hat{\eta}} \sigma^+ , \tag{1.2}
\]

with \( \hat{\eta}(w, \bar{w}) \) a 2D classical scalar field. It satisfies the zero curvature condition \([D, \bar{D}] = 0\), which happens to be the classical sinh-Gordon (shG) equation in this case. Crucially, the coefficients of these monodromies satisfy many functional relations: \( QQ^- \), \( TQ^- \) and then \( T^- \) and \( Y^- \)-systems \[^4\], or equivalently the Thermodynamic Bethe Ansatz (TBA) equations of some integrable Quantum Field Theory (QFT), in this case sine-Gordon (sG) and generalisations with many mass scales. Importantly, they introduced in the Lax operators a set of parameters, the moduli (\( \vec{c} \) below), which turn out, in the end, to parametrise the masses of the QFT. What was lacking in this scenario was a systematic way to understand how to derive, in the opposite direction, a classical system from a quantum one: in this letter we propose a procedure to realise this program. In other words, we want to start from a quantum integrable model and then derive in a precise way a classical model associated to it. Notice that this is possible only in presence of masses. In fact, we start from a precise definition of an integrable system (field or lattice theory) in terms of \( Q \) functions (eigenvalues of \( Q \) operators) and functional relations satisfied by them, the so-called \( QQ^- \)-system. Along the lines of \[^5\] all the other integrable structures, Baxter’s and universal \( TQ^- \)-system, TBA \[^6\] and Non Linear Integral Equations for the counting functions \[^7\], the \( T^- \) and \( Y^- \)-systems, can be derived. Yet, the main idea below is that we need only to convert the universal \( TQ^- \)-system into a linear integral equation and then, by suitable Fourier transform, to a Volterra integral equation. At this point we shall naturally introduce the dynamical space \( w, \bar{w} \) of the ODE/IM correspondence by extending it into an integral equation with the form of that of Marchenko (without bound states) \[^8\], named below Marchenko-like equation. Eventually, from this equation we derive two Schrödinger equations or two classical linear Lax problems, indeed equivalent to \(^{[1.1]} \) \(^{[1.2]} \) above. We provide the details for this specific case, describing also, in a particular case, scattering amplitudes or Wilson loops with a null polygonal boundary in \( AdS_3 \) in \( \mathcal{N} = 4 \) SYM at strong coupling. But we can ostensibly adapt the method to general cases. To corroborate this statement, we give here only hints on spin chains by developing our construction for the \( XXZ \) at \( \Delta = -1/2 \) and easily yielding Stroganov’s equations \[^9\].

In the case of scattering amplitudes/Wilson loops in \( \mathcal{N} = 4 \) SYM at strong coupling our starting
Eventually we are left with a universal hypothesis is the $QQ$-system (2.20) of [5]\(^1\)
\[
Q_+ \left( \theta + \frac{i \pi}{2N}, \vec{c} \right) Q_- \left( \theta - \frac{i \pi}{2N}, \vec{c}^R \right) - Q_+ \left( \theta - \frac{i \pi}{2N}, \vec{c}^R \right) Q_- \left( \theta + \frac{i \pi}{2N}, \vec{c} \right) = -2i \cos \pi l, \tag{1.3}
\]
which connects two eigenvalues $Q_\pm(\theta, \vec{c})$, the so-called $Q$-functions, of a $Q$ (Baxter) operator. We have that $|2l| < 1$, $2N$ is a positive integer, $\theta$ is the so-called spectral parameter and $Q_\pm(\theta, \vec{c})$ are, by assumption, entire functions of $\theta$, from which, in our perspective, the next physical quantities ensue. In fact, from (1.3) we can derive Bethe equations, the $TQ$, $T$- and the $Y$-systems as we have sketched in [5] and will summarise below. In writing (1.3) we made the hypothesis that $Q_\pm$ depend also on a vector of $2N - 1$ complex coefficients $\vec{c} = (c_0, ..., c_{2N-2})$, the moduli, and on its 'rotated' version:
\[
\vec{c} \rightarrow \vec{c}^R = (c_0, ..., c_n e^{-i \pi \frac{n}{N}}, ...). \tag{1.4}
\]
In the inverted perspective of [5] (1.3) is a part of the so called $\hat{\Omega}$-symmetry. Another important assumption is the quasi-periodicity of the functions $Q_\pm$,
\[
Q_\pm(\theta - i \tau, \vec{c}^R) = e^{\mp i \pi (l + \frac{1}{2})} Q_\pm(\theta, \vec{c}), \tag{1.5}
\]
with period $\tau = \pi + \pi/N$, as its application removes from (1.3) the appearance of the 'rotation'. So that eventually we are left with a universal form for the $QQ$-system,
\[
e^{i \pi l} Q_+(\theta, \vec{c})Q_-(\theta + i \pi, \vec{c}) + e^{-i \pi l} Q_- (\theta, \vec{c})Q_+(\theta + i \pi, \vec{c}) = -2 \cos \pi l, \tag{1.6}
\]
where also shift are fixed (and no longer depending on $N$). To continue the (anti-)comparison with [5], this (along with the above quasi-periodicity) has been obtained there by using the invariance of (1.4) under the so-called $\Omega$-symmetry.

Now we move our steps from (1.6) and derive everything, in the end the associated Lax problem (1.1,1.2). First of all, we introduce a very useful quadratic construct of $Q_\pm$, the transfer matrix eigenvalue:
\[
T(\theta, \vec{c}) = \frac{i}{2 \cos \pi l} \left[ e^{-2i \pi l} Q_+(\theta + i \pi, \vec{c})Q_-(\theta - i \pi, \vec{c}) - e^{2i \pi l} Q_+(\theta - i \pi, \vec{c})Q_-(\theta + i \pi, \vec{c}) \right]. \tag{1.7}
\]
Combining relation (1.3) with the quasi-periodicity (1.5) we arrive at the functional relation
\[
T(\theta, \vec{c})Q_\pm(\theta, \vec{c}) = Q_\pm \left( \theta + i \tau - i \pi, \vec{c}^{R-1} \right) + Q_\pm \left( \theta - i \tau + i \pi, \vec{c}^R \right), \tag{1.8}
\]
with shifts depending on $\tau$. Relation (1.8) is the usual form of the Baxter $TQ$-relation for integrable models. However, for our aims it is more convenient to combine (1.6) with (1.7) to arrive to the relation
\[
T(\theta, \vec{c})Q_\pm(\theta, \vec{c}) = e^{\pm i \pi (l + \frac{1}{2})} Q_\pm(\theta + i \pi, \vec{c}) + e^{\pm i \pi (l + \frac{1}{2})} Q_\pm(\theta - i \pi, \vec{c}), \tag{1.9}
\]
which is a new $TQ$-system in a universal form, in the sense that the moduli do not rotate (and thus is more effective in their presence) and the shifts on the spectral parameter do not depend on $N$. Relation (1.9) expresses the transfer matrix $T$ via the Baxter auxiliary functions $Q_\pm$, or in reverse yields $Q_\pm$ as

\(^1\)In relation (2.20) of [5] an extra phase factor, denoted $e^{i \Phi(\theta + \frac{i \pi l}{2N}, \vec{c})}$, with $\Phi$ different from zero only in particular cases, is also present. Inclusion of this factor will not alter the conclusions of this letter, but would render notations more unwieldy. Therefore, for clarity's sake, we decided to omit it in this brief note.
solutions of a finite difference second order equation, given the 'potential' $T$. And *en passant* we remark that (1.9) constrain it to be periodic

$$T(\theta + i\tau, \vec{c}) = T(\theta, \vec{c}^R),$$

(1.10)

so that the $Q_{\pm}$ are the Floquet solutions. In the end, it is also quite natural to assume the $Q$ and $T$ functions to be real-analytic (bar represents complex conjugation):

$$\bar{Q}_\pm(\theta, \vec{c}) = Q_\pm(\bar{\theta}, \bar{\vec{c}}), \quad \bar{T}(\theta, \vec{c}) = T(\bar{\theta}, \bar{\vec{c}}).$$

(1.11)

The two properties (1.9, 1.5) are general and to be specific they have to be equipped with the asymptotic behaviour of $Q_{\pm}$, which usually fixes the integrable model and also the state thereof. To make a definite example we choose the asymptotic behaviour typical for the ground state of massive integrable quantum field theories (though we may introduce that for an excited state and carry on similarly):

$$\lim_{\text{Re } \theta \to -\infty} \ln \left[ Q_{\pm} \left( \theta + i\frac{T}{2}, \vec{c} \right) \right] = -w^{(\sigma)}_0(\vec{c})e^{\sigma \theta} + O(1), \quad |\text{Im } \theta| < \frac{T}{2},$$

(1.12)

where the renormalisation group (RG) times $w^{(1)}_0(\vec{c}) = w_0(\vec{c})$ and $w^{(-1)}_0(\vec{c}) = \bar{w}_0(\vec{c})$ drive the asymptotics when $\sigma = \pm 1$. Now, in order to make (1.12) compatible with real-analyticity (1.11) and quasi-periodicity the intuitive condition

$$\bar{w}_0(\vec{c}) = w_0(\vec{c}^{R^{-1}})$$

(1.13)

must hold. The next step is to transform the functional equation (1.9) by use of (1.5, 1.11, 1.12, 1.13) into a handier integral equation

$$Q_{\pm} \left( \theta + i\frac{T}{2}, \vec{c} \right) = q(\theta, \vec{c}) \pm \int_{-\infty}^{+\infty} \frac{d\theta'}{4\pi} \left\{ T \left( \theta' + i\frac{T}{2}, \vec{c} \right) e^{-w_0(\vec{c})(e^{\theta'} + e^{\theta})} - w_0(\vec{c})(e^{-\theta'} + e^{-\theta}) e^{\pm(\theta' - \theta')} Q_{\pm} \left( \theta' + i\frac{T}{2}, \vec{c} \right) \right\},$$

(1.14)

valid in the strip $|\text{Im } \theta| < \pi$, where the massive field theory driving term

$$q(\theta, \vec{c}) = C_{\pm}e^{\pm i\frac{T}{2}(\theta + i\tau)}, \quad C_{\pm} \in \mathbb{R},$$

(1.15)

is a consequence of (1.12). Outside the strip $|\text{Im } \theta| < \pi$ the functions $Q_{\pm} \left( \theta + i\frac{T}{2}, \vec{c} \right)$ are continued analytically. To prove it, we can 'invert' the $\pm i\pi/2$ shift operator (in the l.h.s. of (1.9)) by applying the $\tanh x$ integral kernel by virtue the (residue) relation

$$\lim_{\epsilon \to 0^+} \left[ \tanh \left( x + i\frac{\pi}{2} - i\epsilon \right) - \tanh \left( x - i\frac{\pi}{2} + i\epsilon \right) \right] = 2\pi i\delta(x), \quad x \in \mathbb{R}.$$  

(1.16)

Then, the driving term of (1.14) is the zero mode of the shift operator which reproduces the asymptotics (1.12); the shift of half the period guarantees quasi-periodicity (1.5) provided (1.11) and (1.13) hold.

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2In physical terms, the asymptotic expansion (1.12) can be easily seen as the consequence of that of a 'dressed' relativistic momentum $Z(\theta) \sim \sinh \theta$ satisfying a system of non linear integral equations [5]. In this perspective, we have found for $w_0$ an explicit form [3], which simplifies in the case of only one module, $c_0$, into $w_0 = -r/[4 \cos \pi/(2N)]$ as consequence of a unique non linear integral equation with form $Z(\theta) = MR \sinh \theta + \ldots$ with $r = MR > 0$ with $M$ is the (lightest) mass and $R$ the circumference of the cylinder.

3For excited states we expect also $T(\theta)$ to be different with some zeroes.
A crucial fact is now the observation that the integral in equation (1.14) converges at large $|\theta|$ only in some circumstances. In fact, we can restrict ourselves to the case $N$ real and only one module, $c_0$, and prove the leading asymptotic expansions by inserting (1.12) respectively inside (1.9) (with $|\text{Im}\theta| < \pi/2N - \pi/2$) and (1.8) (with $N > 1$) 

$$T\left(\theta + \frac{i\tau}{2}\right) \simeq -2\sin\pi l e^{2w_0 e\theta + 2i\bar{w}_0 e^{-\theta}}, \ N < 1; \ T\left(\theta + \frac{i\tau}{2}\right) \simeq e^{2w_1 e\theta + 2i\bar{w}_1 e^{-\theta}}, \ N > 1.$$  

(1.17)

For $N > 1$ we had to define $w_1 = -\left(\sin\frac{\pi}{2N}\right) i e^{\frac{i\pi}{2N}} w_0$, $w_0$ real, from which Re $w_1 = \left(\sin\frac{\pi}{2N}\right)^2 w_0$ drives the convergence or the divergence according to (1.12). Instead, for $N < 1$ we witness a striking divergence. Finally, the above reasoning applies for sufficiently small moduli $\bar{c}$ as the dependence of $w_0(\bar{c})$ is continuous in $\bar{c}$ [5]. However, the form of the integral suggests a regularisation by the introduction of an auxiliary space, which will turn out to be just the independent variables of the ODE side of the correspondence; we will see this in the following.

Let us start from stripping off from $Q_\pm$ the model depending terms 

$$C_\pm X_\pm(\theta, \bar{c}) = e^{i\frac{\pi}{2}} e^{i\theta + \frac{i\pi}{2}} e^{w_0(\bar{c}) e^\theta + \bar{w}_0(\bar{c}) e^{-\theta}} Q_\pm(\theta + i\frac{\tau}{2}, \bar{c}),$$  

(1.18)

so that (1.14) becomes the 'universal integral' equation 

$$X_\pm(\theta, \bar{c}) = 1 \pm \int_{-\infty}^{+\infty} \frac{d\theta'}{4\pi} \tanh \frac{\theta - \theta'}{2} T\left(\theta' + i\frac{\tau}{2}, \bar{c}\right) E(\theta', \bar{c}) X_\pm(\theta', \bar{c}),$$  

(1.19)

where 

$$E(\theta, \bar{c}) = e^{-2w_0(\bar{c}) e^\theta - 2\bar{w}_0(\bar{c}) e^{-\theta}} \simeq \lim_{\text{Re}\theta \to \pm\infty} \frac{Q_\pm(\theta + i\frac{\pi}{2}, \bar{c})}{Q_\pm(\theta + i\frac{\pi}{2} - i\bar{\pi}, \bar{c})}$$  

(1.20)

depends on the asymptotic behaviour of $Q_\pm$. A heuristically crucial point is that it has the form of two plane waves upon identification of $-i\bar{w}_0(\bar{c})$, $i\bar{w}_0(\bar{c})$ with space and of $e^{\pm \theta}$ with momenta variables. In other words, we define the (complex) momenta $\lambda = e^\theta$, $\lambda' = e^{\theta'}$ and make explicit the dependence of $X_\pm(\theta, \bar{c}) = X_\pm(w'_0, \bar{w}'_0|\lambda)$ on them and on (Wick rotated variables) $w'_0 = -i\bar{w}_0$, $\bar{w}'_0 = i\bar{w}_0$ to allow its Fourier transform, 

$$K_\pm(w'_0, \xi; \bar{w}'_0) = \int_{-\infty - i\epsilon}^{+\infty - i\epsilon} d\lambda e^{i(\xi - w'_0)\lambda}[X_\pm(w'_0, \bar{w}'_0|\lambda) - 1],$$  

(1.21)

enter the stage. Upon plugging it into equation (1.19), we obtain a Volterra equation, valid for $\xi > w'_0$, since (1.21) is zero otherwise 

$$K_\pm(w'_0, \xi; \bar{w}'_0) \pm F(w'_0 + \xi; \bar{w}'_0) \pm \int_{w'_0}^{+\infty} d\xi' K_\pm(w'_0, \xi'; \bar{w}'_0) F(\xi' + \xi; \bar{w}'_0) = 0,$$  

(1.22)

which has almost the structure of a Marchenko-type equation [8], apart from the form of the 'scattering data' 

$$F(x; \bar{w}'_0) = i \int_{0}^{+\infty} d\lambda' e^{-ix\lambda'} \bar{w}'_0/\lambda' T(\lambda' e^{i\frac{\pi}{2}}),$$  

(1.23)
which depend on \( w'_0 = -iw_0 \) (in an intricate way) because \( T \) does. Let us now generalise it to the proper form of a Marchenko equation. In fact, we can promote the RG times \( w_0(\vec{c}) = iw'_0(\vec{c}), \bar{w}_0(\vec{c}) = -iw'_0(\vec{c}) \) in (1.22, 1.23) to independent dynamical variables

\[
w = iw', \quad \bar{w} = -iw',
\]

respectively, everywhere except that in the transfer matrix \( T \) (which contains \( w'_0 \) and \( \bar{w}'_0 \)) which is left unscathed. This amounts indeed to introducing the auxiliary space of (the derivatives of) the ODE, whilst physical intuition suggests that the dependence of \( T \) (generator of the integrals of motion) on \( r = MR \) is to stay unchanged. In other words we are suitably extending the RG parameters \( w_0(\vec{c}), \bar{w}_0(\vec{c}) \) with a two dimensional space \( w', \bar{w}' \), which eventually will constitute the configuration space for the ODE side of the correspondence. Besides, the variability of the moduli \( \vec{c} \) makes this promotion of \( w_0(\vec{c}), \bar{w}_0(\vec{c}) \), which control the 'sizes' of the model, rather well motivated, although the deep meaning and relevance of the extra space of the ODE/IM in a RG perspective are still to be better investigated. In the mathematical realm, the Volterra equation (1.22) generalises to a Marchenko-like equation \( \mathcal{K} \) for \( K_\pm \)

\[
K_\pm(w'; \xi; \bar{w}') = F(w' + \xi; \bar{w}') + \int_{w'}^{+\infty} \frac{d\xi'}{2\pi} K_\pm(w'; \xi'; \bar{w}') F(\xi' + \xi; \bar{w}') = 0, \quad \xi > w',
\]

where the known term is

\[
F(x; \bar{w}') = i \int_0^{+\infty} d\lambda' e^{-ix\lambda'+2i\bar{w}'/\lambda'} T(\lambda' e^{i\frac{\lambda'}{2}}, \vec{c}).
\]

In fact, this is a regularisation of the above mentioned divergence of (1.14) as (1.25, 1.26) can be derived by extending the \( Q \)-functions \( Q(\theta + i\tau/2, \vec{c}) \) i.e. \( X_\pm(\theta, \vec{c}) \) to 'dynamical' counterparts \( X_\pm(w', \bar{w}'|\lambda) \) satisfying the well defined extensions of (1.19)

\[
X_\pm(w', \bar{w}'|\lambda) = 1 \pm \int_0^{+\infty} \frac{d\lambda'}{4\pi \lambda' \lambda + \lambda'} T(\lambda' e^{i\frac{\lambda'}{2}}, \vec{c}) e^{-2i\bar{w}'/\lambda' + 2i\bar{w}'/\lambda'} X_\pm(w', \bar{w}'|\lambda').
\]

Then, the Marchenko-like equations (1.25) come out upon Fourier transforming with the definitions

\[
K_\pm(w'; \xi; \bar{w}') = \int_{-\infty}^{+\infty} d\lambda e^{i(\xi - w')\lambda} [X_\pm(w', \bar{w}'|\lambda) - 1].
\]

Obviously, all these manoeuvres have been conceived for real \( \xi, w', \bar{w}' \). Yet, we expect that we can analytical continue in the complete plane to the point \( (w', \bar{w}') = (-iw_0(\vec{c}), iw_0(\vec{c})) \) where the quantities \( X_\pm \) should give back the integrability functions \( Q_\pm \). However, this limiting procedure is delicate and to correctly perform it we derive (for complex \( w', \bar{w}' \)) Schrödinger equations for

\[
\psi_\pm(w', \bar{w}'|\lambda) = X_\pm(w', \bar{w}'|\lambda)e^{-iw'\lambda + i\bar{w}'\lambda^{-1}},
\]

\(^5\)Although it is not a crucial issue, we note that for \( x, w'_0 \) real the integral (1.23) converges in the case \( N > 1 \) (of interest for applications to Wilson loops/scattering amplitudes). In fact, if we start from the case with only one module, \( c_0 \), as we have seen above \( w_0 < 0 \) and then at large \( |\theta| \) the asymptotic behaviour (1.17) is exponentially small thanks to \( \text{Re} w_1 = (\sin \frac{\theta}{2})^2 w_0 < 0 \). Then, the convergence still holds if we introduce a set of sufficiently small moduli \( \vec{c} \) as the dependence of \( w_0(\vec{c}) \) is continuous in \( \vec{c} \).
appearing in \((1.28)\) and with asymptotic plane wave behaviour. Actually, the Schrödinger equation originates easily from the peculiar form of the Marchenko-like equation \((1.25)\). In fact, we need to use the inverse Fourier transform of \((1.28)\) which for \(\text{Im}\lambda < 0\) i.e. \(-\pi < \text{Im}\theta < 0\) takes the form

\[
X_{\pm}(w', \bar{w}'|\lambda) - 1 = \int_{-\infty}^{+\infty} \frac{d\xi}{2\pi} e^{-i(\xi - w')\lambda} K_{\pm}(w', \xi; \bar{w}') = \int_{w}^{+\infty} \frac{d\xi}{2\pi} e^{-i(\xi - w')\lambda} K_{\pm}(w', \xi; \bar{w}').
\]

(1.30)

Then, we differentiate twice \((1.30)\) and use \((1.25)\) to arrive at the Schrödinger equations

\[
\frac{\partial^2}{\partial w'^2} \psi_{\pm}(w', \bar{w}'|\lambda) + \lambda^2 \psi_{\pm}(w', \bar{w}'|\lambda) = u_{\pm}(w'; \bar{w}') \psi_{\pm}(w', \bar{w}'|\lambda),
\]

(1.31)

with potentials

\[
u_{\pm}(w'; \bar{w}') = -\frac{d}{dw'} \frac{K_{\pm}(w', w'; \bar{w}')}{{2\pi}},
\]

(1.32)
determined by the solution of the Marchenko-like equation \((1.25)\). Despite this specific case, the method of passing from a linear integral equation to a differential one holds more in general and thus can be applied to a variety of cases for generalising the correspondence. Now, to cover the domain \(\text{Im}\theta > 0\) i.e. \(\text{Im}\lambda > 0\) we need to move the integration straight line in \((1.28)\) slightly above the real axis and define another Fourier transform

\[
\tilde{K}_{\pm}(w', \xi; \bar{w}') = \int_{-\infty + i\epsilon}^{+\infty + i\epsilon} d\lambda e^{i(\xi - w')\lambda}[X_{\pm}(w', \bar{w}'|\lambda) - 1].
\]

(1.33)

Of course, this leads to Marchenko-like equations with different forms and same known function \(F\):

\[
\tilde{K}_{\pm}(w', \xi; \bar{w}') = F(w' + \xi; \bar{w}') + \int_{-\infty}^{w'} \frac{d\xi'}{2\pi} \tilde{K}_{\pm}(w', \xi'; \bar{w}') F'(\xi' + \xi; \bar{w}') = 0, \quad \xi < w'.
\]

(1.34)

Thanks to the presence of the same function \(F\), the solutions of \((1.25, 1.34)\) are simply linked by:

\[
K_{\pm}(w', \xi; \bar{w}') = -\tilde{K}_{\pm}(w', \xi; \bar{w}').
\]

(1.35)

The inverse of \((1.33)\) gives \(X_{\pm}(w', \bar{w}'|\lambda)\) of \((1.27)\) for \(0 < \text{Im}\theta < \pi\)

\[
X_{\pm}(w', \bar{w}'|\lambda) - 1 = \int_{-\infty}^{w'} \frac{d\xi}{2\pi} e^{-i(\xi - w')\lambda} \tilde{K}_{\pm}(w', \xi; \bar{w}')
\]

(1.36)

and the corresponding wave functions \((1.29)\) satisfy, as above, the same Schrödinger equation as \((1.31)\):

\[
\frac{\partial^2}{\partial w'^2} \psi_{\pm}(w', \bar{w}'|\lambda) + \lambda^2 \psi_{\pm}(w', \bar{w}'|\lambda) = u_{\pm}(w'; \bar{w}') \psi_{\pm}(w', \bar{w}'|\lambda), \quad |\text{Im}\theta| < \pi,
\]

(1.37)

with potentials \((1.32)\). To be precise, for deriving the latter also for \(\text{real}\ \theta\) we need to prove a ‘dynamical’ analogue of the \(TQ\) relation, which relates real \(\theta\) with imaginary ones (positive and negative). Dubbed \(T\psi\) relation, it is obtained by inverting \((1.27)\) with \((1.16)\)

\[
T \left(\theta + i\frac{\tau \cdot \vec{c}}{2}\right) \psi_{\pm}(w', \bar{w}'|\theta) = \mp i \psi_{\pm}(w', \bar{w}'|\theta + i\pi) \pm i \psi_{\pm}(w', \bar{w}'|\theta - i\pi).
\]

(1.38)

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6These will result to be just the so-called Jost solutions of the Schrödinger equation.
Equations (1.37,1.38) can be interpreted with complex $w', \bar{w}'$ analytically continued with the aim of establishing the connection between the wave functions and the $Q$-functions. In fact we can derive a partial differential equation for the potentials in (1.37). We introduce a sort of 'complex conjugate' of

$$\frac{\partial^2}{\partial w'^2} \psi^\text{bar}_\pm (w', \bar{w}'\lvert \lambda) + \lambda^{-2} \psi^\text{bar}_\pm (w', \bar{w}'\lvert \lambda) = \bar{u}_\pm (w', \bar{w'}) \psi^\text{bar}_\pm (w', \bar{w}'\lvert \lambda).$$

(1.39)

with $\psi^\text{bar}_\pm (w', \bar{w}'\lvert \lambda) = \bar{\psi}_\pm (w', \bar{w}'\lvert \lambda^{-1})$. This step allows us to reformulate the Schrödinger equations (1.37,1.39) with potentials

$$u_\pm (w', \bar{w}') = \pm \frac{\partial^2}{\partial w^2} \hat{\eta}(w, \bar{w}) - \left( \frac{\partial}{\partial w} \hat{\eta}(w, \bar{w}) \right)^2, \quad w' = -iw, \; \bar{w}' = iw$$

(1.40)

from the Lax matrix linear problem (1.11) and (1.2), whose consistency condition $[D, \bar{D}] = 0$ gives

$$\frac{\partial^2}{\partial w \partial \bar{w}} \hat{\eta} = 2 \sinh 2\hat{\eta},$$

(1.41)

the classical shG equation: this is indeed a very useful tool in asymptotic analysis (for instance around $(w, \bar{w}) = (w_0(\bar{c}), \bar{w}_0(\bar{c}))$). Importantly, the potential appearing in the Schrödinger problem depends only on the transfer matrix: in fact, it can be obtained by solving (by iterations or Laplace transform) the Marchenko-like equation (1.25,1.26) for $K_\pm (w', w'; \bar{w}')$ and eventually takes on the following form in terms of Fredholm determinants or tau functions, $\tau_\pm$,

$$u_\pm (w', \bar{w}') = -2 \frac{d}{dw} K_\pm (w', w'; \bar{w}') = -2 \frac{\partial^2}{\partial w^2} \ln \tau_\pm (w', \bar{w}')$$

(1.42)

$$\ln \tau_\pm (w', \bar{w}') = \ln \det (1 \pm \bar{V}) = \text{Tr} \ln (1 \pm \bar{V}), \quad \bar{V}(\theta, \theta') = \frac{T \left( \theta + \frac{\pi}{2}, \bar{c} \right) e^{-2iw'\theta + 2i\bar{w}'e^{-\theta}}}{4\pi \cosh \theta - \theta'}$$

(1.43)

Upon carefully inspecting (1.40) and (1.42), we also obtain an expression for the shG solution

$$\hat{\eta}(w, \bar{w}) = \ln \tau_+ (-iw, i\bar{w}) - \ln \tau_- (-iw, i\bar{w}) = \sum_{n=1}^{+\infty} \frac{2}{2n-1} \int \prod_{i=1}^{2n-1} \frac{d\theta_i}{4\pi T \left( \theta_i + \frac{i\pi}{2}, \bar{c} \right) e^{-2w\theta_i - 2\bar{w}e^{-\theta_i}} \cosh \frac{\theta - \theta'}{2}}. \tag{1.44}$$

Nevertheless, the domain of validity of the equation (1.25,1.26) and its solution (1.44) depends on the behaviour of $T$ at large $\text{Re} \; \theta$. In particular, (1.17) allow us to frame the situation at least for $N$ real and small moduli except $c_0$: if $N < 1$ the multi-integrals converge for any $\text{Re} \; w > \text{Re} \; w_0$, whereas for $N > 1$ they converge only for $\text{Re} \; w > \text{Re} \; w_1 > \text{Re} \; w_0$, the last inequality holding because – as proven above – $\text{Re} \; w_1 = \left( \sin \frac{\pi}{N} \right)^2 w_0$ and $-r \sim w_0 < 0$. In other words, when $N > 1$ the convergence stops at $w_1$ before $w = w_0$ and the equations need to be analytically continued. Yet, the shG equation (1.41) remains valid. In fact, we can derive a relevant limit value by using the results for $N < 1$ and $\text{Re} \; w > \text{Re} \; w_0$: when $w \to w_0$ we can substitute $T$ in (1.44) with its leading expansion, first of (1.17), which contains $l$. Then, we absorb the phase of $w - w_0$ by a shift in $\theta_i$ and follow a general method described in (10): in the $n$-th term we change integration variables in $\alpha_i = \theta_{i+1} - \theta_i, i = 1, \ldots, 2n - 1$, integrate on $\theta_1$ to obtain the Bessel $K_0$, expand it (at leading order) for small argument proportional to $|w - w_0| \to 0$, integrate on all the remaining $\alpha_i$ and eventually resum the series into

$$\hat{\eta}(w, \bar{w}) = 2l \ln |w - w_0| + \ldots. \tag{1.45}$$
In conclusion, (1.46) is proven and connects the solutions of the two in (1.38) to obtain an integral equation for the wave-function (1.29) (as done for the diagonal part, the tau-functions (1.43)), we invert (1.28, 1.33) and combine (1.37) – solutions of integrable partial differential equations – in terms of Fredholm determinants, via a relation (1.38); 2) the expansion (1.12) as given by large

\[ \lim_{w \to w_0} (w - w_0)^{\pm 1} \psi_{\pm}(w', \bar{w}'|\theta) = Ce^{\mp \bar{\theta} t} Q_{\pm} \left( \theta + i\frac{\tau}{2}, \bar{c} \right), \quad w' = -iw, \quad \bar{w}' = i\bar{w}, \quad \text{(1.46)} \]

with a constant C to be chosen. In fact, at the leading order (1.46) the potentials (1.40) simplify to

\[ u_\pm \simeq -l(l \pm 1)/(w - w_0)^2. \]

This entails that equation (1.37) with + has two solutions labelled by \( \alpha = -l, l + 1 \) behaving as \( f_{\pm}^{(\alpha)}(w', \bar{w}') \sim (w - w_0)^\alpha \) when \( w \to w_0 \), whilst equation with – has other two solutions labelled by \( \alpha = l, -l + 1 \) behaving as \( f_{\pm}^{(\alpha)}(w', \bar{w}') \sim (w - w_0)^\alpha \). Of course, \( \psi_{\pm}(w', \bar{w}'|\lambda) \) can be expressed linearly in terms of the \( f_{\pm}^{(\alpha)}(w', \bar{w}') \) as

\[
\psi_{\pm}(w', \bar{w}'|\lambda) = B_{\pm}(\theta) f_{\pm}^{(l+1)}(w', \bar{w}') + A_{\pm}(\theta) f_{\pm}^{(-l)}(w', \bar{w}') \\
\psi_{-}(w', \bar{w}'|\lambda) = A_{-}(\theta) f_{-}^{(l)}(w', \bar{w}') + B_{-}(\theta) f_{-}^{(-l+1)}(w', \bar{w}'). 
\]

(1.47)

Since \(-1/2 < l < 1/2\), in the limit \( w \to w_0 \) the first (+) is dominated by \( f_{\pm}^{(-l)} \sim (w - w_0)^{-l} \) while the second (−) by \( f_{\pm}^{(l)} \sim (w - w_0)^l \). Then, the relations (1.46) pick up in both cases the finite quantities \( A_{\pm}(\theta) \). Upon identifying the latter respectively with \( e^{\pm \bar{\theta} t} Q_{\pm}(\theta + i\tau/2, \bar{c}) \) up to a multiplicative constant, these \( Q_{\pm}(\theta + i\tau/2, \bar{c}) \) satisfy the relevant relations: 1) the universal TQ-system (1.39) thanks to the T\( \bar{\psi} \) relation (1.38); 2) the expansion (1.12) as given by large \( \theta \) analysis on the Schrödinger equation (1.37)\(^7\)

In conclusion, (1.46) is proven and connects the solutions of the QQ relation (1.3) to the solutions of the Schrödinger equations (1.37), constructed from the former and circumventing the divergence problem of equations (1.14) (1.19).

As an interesting consequence, our method expresses the potentials of the Schrödinger equations (1.37) – solutions of integrable partial differential equations – in terms of Fredholm determinants, via a Marchenko-like equation, and then in terms of functional equations. Actually, more in general, it links Fredholm’s to Sturm-Liouville’s theory as it gives explicit and useful forms for solutions to the latter in terms of integral equations of the former: upon finding the solutions to the Marchenko-like equations (1.28) (1.33) (as done for the diagonal part, the tau-functions (1.43)), we invert (1.28) (1.33) and combine the two in (1.38) to obtain an integral equation for the wave-function (1.29)

\[ E(\theta)X_{\pm}(\theta) = -2E(\theta) + \int_{-\infty}^{+\infty} d\theta' e^{-v(\theta)-v'(\theta')} \frac{1}{4\pi \cosh \frac{2}{\theta'}} E(\theta')X_{\pm}(\theta'), \quad |\text{Im}\theta| < \pi, \quad \text{(1.48)} \]

with the definitions \( \lambda = e^{\theta} \) and

\[ E(\theta) = \sqrt{2}e^{-v(\theta)} e^{\frac{i\theta}{2}}, \quad e^{-2v(\theta)} = e^{-2iw'\theta + 2iw'\theta} T \left( \theta + i\frac{\tau}{2}, \bar{c} \right). \]

(1.49)

It allows for a solution as a recursive series in these functions and 1/cosh kernel, which is very much complementary to ODE/IM treatment and results (11).

\(^7\) Similarly, if \( N > 1 \) we can approximate \( T \) in (1.45) with the second of (1.17) to obtain the limit \( w \to w_1 \) (Re \( w_1 \) > Re \( w_0 \)): \( \bar{\eta}(w) = -\frac{1}{2} \ln |w - w_1| + \ldots \).

\(^8\) This can be achieved either via a careful large energy expansion with attention to the divergence for \( w \to w_0 \) as in (12) or via the series solution of the Fredholm integral equation below (1.38) for the wave function (1.29).
2 The conformal limit

In the present reverse pattern with respect to [5], (1.44) reveals the Ω-symmetry: \( w \to -we^{\frac{\pi}{N}}, \bar{c} \to c^R, \theta \to \theta - \frac{i\pi}{N} \) of the solution (1.44) \( \hat{\eta}(-we^{\frac{\pi}{N}}, \bar{w}e^{-\frac{\pi}{N}}, c^R) = \hat{\eta}(w, \bar{w}, \bar{c}) \) thanks to the periodicity of \( T \) (1.10). Of course, it extends to the potentials \( w = iw', \bar{w} = -i\bar{w}' \) which maintain the same form with 'rotated' \( \psi_\pm \) (broken symmetry). This simple symmetry means that \( \hat{\eta} \) may be interpreted as depending on a natural variable \( z \) defined, for instance, by means of the polynomial \( p(z, \bar{c}) = z^{2N} + \sum_{n=0}^{2N-2} c_n z^n \) and the differential

\[
\frac{dw}{dz} = \sqrt{p(z)}, \quad \frac{d\bar{w}}{d\bar{z}} = \sqrt{p(z)}, \quad w(z = 0) = w_0 . \tag{2.1}
\]

In fact, to make manifest the Ω-symmetry in the new variable \( z \to ze^{\frac{\pi}{N}}, \bar{c} \to c^R \), the function \( p \) may be left invariant: \( p(z, \bar{c}) = p(ze^{\frac{\pi}{N}}, c^R) \), and a polynomial is the simplest choice. Let us focus on the + case of (1.37) as the − sign goes pari passu. In the new variable we obtain a gauge equivalent wave function \( \psi_+(w') = (p(z))^{1/4} \psi_+(z) \) satisfying the modified Schrödinger equation

\[
- \frac{d^2}{dz^2} \psi_+ + e^{2\eta} p(z) \psi_+ - p(z) \left( \frac{1}{(p(z))^{1/4}} \frac{d^2}{dw^2} (p(z))^{1/4} + u_+ \right) \psi_+ = 0 , \tag{2.2}
\]

whose potential is given by (1.40) in the simple form (not equally elegant for the − case)

\[
p(z) \left( \frac{1}{(p(z))^{1/4}} \frac{d^2}{dw^2} (p(z))^{1/4} + u_+ \right) = \frac{\partial^2}{\partial z^2} \eta(z, \bar{z}) - \left( \frac{\partial}{\partial z} \eta(z, \bar{z}) \right)^2 , \tag{2.3}
\]

in terms of \( \eta = \hat{\eta} + \frac{1}{16} \ln p \bar{p} \), solution to the modified shG equation

\[
\partial_z \partial_{\bar{z}} \eta = e^{2\eta} - p(z)p(z)e^{-2\eta} . \tag{2.4}
\]

At this stage we can deduce the \( z, \bar{z} \to 0 \) asymptotics, when \( p(z) \approx c_0 \) entails small deviation \( w - w_0 \approx \sqrt{c_0} z \) and then, from (1.45), the elegant outcomes

\[
\eta = \hat{\eta} + \frac{1}{16} \ln p \bar{p} = l \ln(z \bar{z}) + O(1) \Rightarrow p(z) \left( \frac{1}{(p(z))^{1/4}} \frac{d^2}{dw^2} (p(z))^{1/4} + u_+ \right) = - \frac{l(l+1)}{z^2} + O(1) . \tag{2.5}
\]

In fact, last formula is enough to deduce the 2D Conformal Field Theory (CFT) case \([1]\) as given by the static case \( \bar{z} = 0 \) along with the scaling limit \( \theta \to +\infty \) with \( x, \tilde{c}_n \) fixed and driving as \( z = xe^{-\theta/(1+N)} \) and \( c_n = \tilde{c}_n e^{-\theta(2N-n)/(1+N)} \) go to zero. Eventually, \( \psi_+(z) \) goes into the CFT limit \( \psi_{cft}^+ (x) \) and the Schrödinger equation (2.2) acquires polynomial potentials

\[
- \frac{d^2}{dx^2} \psi_{cft}^+ (x) + \left( p(x, \tilde{c}) + \frac{l(l+1)}{x^2} \right) \psi_{cft}^+ (x) = 0 , \quad p(x, \tilde{c}) = x^{2N} + \sum_{n=0}^{2N-2} \tilde{c}_n x^n . \tag{2.6}
\]

This reproduces and extends known results of the conformal case \([13]\) and, in particular, with only one modulus \( \tilde{c}_0 = -E, \tilde{c}_n = 0, n \geq 1 \), it reduces to that in \([10]\).

---

9Hence, the asymptotic (1.43) of shG equation is the reason of simplicity of the original conformal ODE/IM. Yet, this regime is static (\( \bar{z} = 0 \)) and thus miss the general scenario of Lax pairs and shG evolution.

10Then the Qs become entire functions of the energy \( E \) because of quasi-periodicity \([12]\) and analyticity at \( E = 0 \).
A relevant example

In the model $l = 0$, $N = 1/2$ ($\tau = 3\pi$) with only the real module $c_0$ (i.e. $w_0 = \bar{w}_0$) calculations of our framework can be made even more explicit. In fact, the QQ-system and the quasi-periodicity read

$$Q_+ (\theta)Q_- (\theta + i\pi) + Q_- (\theta)Q_+ (\theta + i\pi) = -2, \quad Q_\pm (\theta + 3i\pi) = \pm iQ_\pm (\theta)$$  \hspace{1cm} (3.1)

and, as a consequence, the transfer matrix

$$T(\theta) = \frac{i}{2} [Q_+ (\theta + i\pi)Q_- (\theta - i\pi) - Q_+ (\theta - i\pi)Q_- (\theta + i\pi)] = 1.$$  \hspace{1cm} (3.2)

When $T$ is a constant, the tau functions $\tau_\pm$ (1.43) enjoy radial symmetry, namely they depend only on $t = 4\sqrt{w\bar{w}}$ (and not on the phase $\phi$ of $w = \frac{T}{2}e^{i\phi}$) and read

$$\ln \tau_\pm (t) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1} (\pm 1)^n}{n^2} \int \prod_{i=1}^{n} \frac{d\theta_i}{4\pi \cosh \frac{\theta_i - \theta_{i+1}}{2}}.$$  \hspace{1cm} (3.3)

Analogous dependence holds for the solution $\hat{\eta}$ (1.44) of the shG equation

$$\hat{\eta}(t) = \sum_{n=1}^{+\infty} \frac{2}{2n-1} \int \prod_{i=1}^{2n-1} \frac{d\theta_i}{4\pi \cosh \frac{\theta_i - \theta_{i+1}}{2}},$$  \hspace{1cm} (3.4)

which then satisfies the Painlevé III$_3$ equation$^{11}$

$$\frac{1}{t} \frac{d}{dt} \left( \frac{d}{dt} \hat{\eta}(t) \right) = \frac{1}{2} \sinh 2\hat{\eta}(t).$$  \hspace{1cm} (3.5)

It is an instructive exercise to compute (3.3) at small $t$ by the method used to derive (1.45), i.e. upon shifting $\alpha_i = \theta_{i+1} - \theta_i$, $i = 1, \ldots, n$ $^{10}$: $\ln \tau_\pm (t) \simeq d_\pm \ln t$, $d_\pm = (1 \mp 6)/36$. Accordingly, the potentials (1.42) for small $w'$ expand as

$$u_\pm (w') = -2 \frac{\partial^2}{\partial w'^2} \ln \tau_\pm (t) = \frac{d_\pm}{w'^2} + \ldots,$$  \hspace{1cm} (3.6)

whose leading term is just the CFT limit. In fact, the change of variables (2.1) takes the explicit form $w = \frac{2}{3}(z + c_0)^{\frac{2}{3}}$, $\bar{w} = \frac{2}{3}(\bar{z} + c_0)^{\frac{2}{3}}$ and the limit amounts to consider

$$\bar{w}' = \frac{2}{3}e^{3/2}e^{-\theta}, \quad w' = \frac{2}{3}(x + E)^{3/2}e^{-\theta}, \quad \psi_\pm (w', \bar{w}'|\lambda) = \left( w' e^\theta \right)^{\pm 1/6} \psi_\pm^{cft}(x), \quad \theta \to +\infty,$$  \hspace{1cm} (3.7)

with $x, E = -\bar{c}_0$ fixed, namely $t = 4\sqrt{w'\bar{w}'} \to 0$. Very inspiring are the conformal versions of (1.37) which take the bispectral form

$$- \frac{d^2}{dx^2} \psi_+^{cft}(x) + (x - E)\psi_+^{cft}(x) = 0,$$  \hspace{1cm} (3.8)

($l = 0, N = 1/2$ case of (2.61) solved by the Airy function $Ai(x - E)$, and

$$- \frac{d^2}{dx^2} \psi_-^{cft}(x) + (x - E)\psi_-^{cft}(x) + \frac{1}{x - E} \frac{d}{dx} \psi_-^{cft}(x) = 0,$$  \hspace{1cm} (3.9)

solved by its derivative $Ai'(x - E)$ $^{12}$. In fact, these can be simply converted into ODE in the spectral parameter $E$ and hence for $\psi_\pm^{cft} (x = 0) = Q_\pm$ (without divergence).

$^{11}$This a particular solution depending at most on one parameter, the value of $T$.

$^{12}$Here the situation is even simpler as $x$ and $E$ can swap places.
3.1 Bispectrality

Furthermore, for $T$ constant the wave-functions $\psi_{\pm}(w', \bar{w}'|\lambda)$ actually depend only on two variables $w'\lambda$ and $\bar{w}'\lambda^{-1}$ or $t$ and $\theta + i\phi$ in polar coordinates ($\lambda = e^\theta$). This means that we can swap the differential equations in $w'$ and $\bar{w}'$ \cite{III} with two in $s = e^{\theta + i\phi}$ and $t$

$$s^2 \frac{\partial^2 \psi_{\pm}}{\partial s^2} + (1 - F_{\pm}(t,s)) \frac{\partial \psi_{\pm}}{\partial s} + \frac{t}{2} \frac{d\hat{\eta}}{dt} F_{\pm}(t,s) \psi_{\pm} + \frac{t^2}{16} \left( -s^2 - s^{-2} + 2 \cosh 2\hat{\eta} - 4 \left( \frac{d\hat{\eta}}{dt} \right)^2 \right) \psi_{\pm} = 0, \quad (3.10)$$

$$t^2 \frac{\partial^2}{\partial t^2} \left( e^{\pm \frac{\eta}{2}} \psi_{\pm} \right) + t^2 \frac{d\hat{\eta}}{dt} F_{\pm}(t,s) \frac{\partial}{\partial t} \left( e^{\pm \frac{\eta}{2}} \psi_{\pm} \right) + \frac{t^2}{16} \left[ -s^2 - s^{-2} - 2 \cosh 2\hat{\eta} \right] e^{\pm \frac{\eta}{2}} \psi_{\pm} = 0, \quad (3.11)$$

with coefficients $F_{\pm}(t,s) = (se^{\pm\hat{\eta}} + s^{-1}e^{\mp\hat{\eta}})(se^{\pm\hat{\eta}} - s^{-1}e^{\mp\hat{\eta}})^{-1}$. In other words the problem becomes bispectral (in differential form) and \cite{III} yield the off-critical versions (in the conformal limit: $st = \frac{8}{7}(E-x)^3$ with $s \to +\infty$ and $t \to 0$) of the (bispectral) ODEs in $E$ stemming from \cite{III,III} by swapping $x \leftrightarrow -E$. Moreover, we can compute \cite{III} in $(w, \bar{w}) = (w_0, \bar{w}_0)$ for $\phi = 0$ upon shifting $\theta \to \theta - 3i\pi/2$, so that they become ODEs for $\psi_{\pm}(w_0, \bar{w}_0|\lambda) = Q_{\pm}(\theta)$ (simply \cite{III} with the substitution $t \to r$). An ODE for the $Q$ is a rather desirable situation for quantum integrable systems and in this case the coefficients are in terms of a Painlevé transcendent; a similar method will be exploited in the next section on spin chains.

Furthermore, the solutions $\tau_{\pm}$ and $\hat{\eta}$ enjoy radial symmetry, but in the displaced variable $w - w_0 = \frac{t}{4}e^{i\phi}$ ($\phi$ independence), even if $N \to +\infty$ with only one non-zero module $c_0$ (at fixed $l$ and $w_0 = -r/4$)\cite{IV} and anew the problem becomes bispectral. As above the wave functions depend only on two polar coordinates $t$ and $\theta + i\phi$ and equations \cite{III,III} still hold\cite{IV} albeit now with a different definition of the time $t$ and a general solution $\hat{\eta}(t)$ of Painlevé III\textsubscript{3} equation \cite{III} depending on two parameters, $l$ and $r$. At the (movable) pole $t = r$ ($w = 0 = w_1$), the function $\hat{\eta}(t)$ expands as

$$\hat{\eta}(t) = -\ln |t - r| + \ln 2 - \frac{t - r}{2r} + \frac{7 - 16u}{24r^2}(t - r)^2 + O(t - r)^3 \quad (3.12)$$

with $u$ a constant depending on $r,l$. As a consequence, we obtain for the $Q$-functions, given by $Q_{\pm}(\theta + \frac{i\pi}{2}) = \lim_{\phi \to \pi} [se^{\pm\hat{\eta}} - s^{-1}e^{\mp\hat{\eta}}]^{-\frac{1}{2}} \psi_{\pm}(\theta)$ in this limit case, the modified Mathieu equation

$$\frac{\partial^2 Q_{\pm}(\theta)}{\partial \theta^2} + \left( \frac{r^2}{8} \cosh 2\theta - u \right) Q_{\pm}(\theta) = 0. \quad (3.13)$$

It has isomonodromic deformations given by the Painlevé III\textsubscript{3} equation above (with solution fixed by the parameters $(l, r)$ or $(r, u)$) and monodromies studied in \cite{II} in connexion with supersymmetric gauge theories. It also coincides with the $N \to +\infty$ limit of the TQ \cite{I,IV}, as can be seen upon improving the second equality \cite{II} by sub-leading term $e^{2\omega_1e^{-\theta} + 2\omega_1e^{\theta}}$ \cite{III}. So that the present development may pave the way for the paramount open problem of studying the monodromies of a finite difference equation, like the TQ-relation, and somehow identify the shG equation \cite{III} as a generalisation of the Painlevé isomonodromies. In \cite{II} a seed has been sown by finding the connexion between the Floquet and the ODE/IM bases of eigenfunctions.

\textsuperscript{13}In fact, the $\Omega$-symmetry sector reduces to a straight half-line.

\textsuperscript{14}They are linear Lax associated problems for the Painlevé III\textsubscript{3} equation.
4 Spin chains: a particular point of the XXZ

For brevity’s sake, we see here our method at work for the XXZ chain in the inspiring model case with anisotropy \( \Delta = -1/2 \), without relying on supersymmetry. In our parametrisation \( N = 1/2 \) and \( l = 0 \), so that we may use our construction with \( T = 1 \). In fact, we parametrise \( \lambda = e^{\frac{3iu}{2}} \) in the \( T\psi \)-system and scale the wave function \( \psi_\pm = e^{\mp \frac{3iu}{2}} \psi_\pm \) to obtain the functional relation (in \( u \))

\[
\tilde{\psi}_\pm(w', \bar{w}'|u + \frac{2\pi}{3}) + \tilde{\psi}_\pm(w', \bar{w}'|u - \frac{2\pi}{3}) + \tilde{\psi}_\mp(w', \bar{w}'|u) = 0 .
\] (4.1)

Although it shows an extra dependence on \( w' \) and \( \bar{w}' \), for what \( u \) concerns, it coincides with the functional relation (9) of [9] for the functions \((\sin u)^{2n+1}Q^{\text{XXZ}}_\pm(u)\). Therefore the latter are given by (1.37) in the limit \( t \rightarrow 0 \) of polar coordinates \( w' = -\frac{it}{4}e^{i\phi} \), \( \bar{w}' = \frac{it}{4}e^{-i\phi} \), namely

\[
e^{-3iu-2i\phi}\left( \frac{t^2}{4} \frac{\partial^2 \tilde{\psi}_\pm}{\partial t^2} - \frac{t}{4} \frac{\partial \tilde{\psi}_\pm}{\partial t} - \frac{1}{4} \frac{\partial^2 \tilde{\psi}_\pm}{\partial \phi^2} + i \frac{\partial \tilde{\psi}_\pm}{\partial \phi} - \frac{it}{2} \frac{\partial^2 \tilde{\psi}_\pm}{\partial t \partial \phi} - d_\pm \tilde{\psi}_\pm \right) = \frac{t^2}{16} \tilde{\psi}_\pm ,
\] (4.2)

with the help of (3.6). Actually, we need to look for solutions with leading behaviour

\[
\tilde{\psi}_\pm(t, \phi|u) = t^L f_\pm \left( u + \frac{2}{3} \phi \right) + ... ,
\] (4.3)

depending all together on \( 3u + 2\phi \). Then, for real \( u \) and Bethe roots, hence \( f_\pm \), we subtract to (4.2) its complex conjugate

\[-2\sin(3u + 2\phi) \left( \frac{L^2}{4} - \frac{L}{2} - d_\pm \right) f_\pm - \frac{1}{4} \frac{\partial^2 f_\pm}{\partial \phi^2} \] \[+ 2\cos(3u + 2\phi) \left( \frac{1}{2} - L \right) \frac{\partial f_\pm}{\partial \phi} = 0 \] (4.4)

and obtain a differential equation which does not contain \( t \) any more. Eventually, derivation on \( \phi \) can be traded for derivation on \( u \):

\[
\frac{d^2 f_\pm}{du^2} - 6n \cot(3u + 2\phi) \frac{df_\pm}{du} + (c_\pm - 9n^2)f_\pm = 0 ,
\] (4.5)

with \( c_\pm = 5/2 \mp 3/2 \) and \( L = 2n+1 \). These equations are the same as (13) of [9] with \( n \) integer yielding odd number of sites \( L = 2n + 1 \) \((f_\pm \text{ are there called } f \text{ and } g)\). In conclusion, for \( \phi = 0 \) they are solved by the forms \( f_\pm(u) = (\sin u)^{2n+1}P_\pm(\cos u) \), with \( P_\pm \) polynomials of degree \( n \) (+ case) or \( n + 1 \) (− case). On the other hand, we know from above that these polynomials must be the \( Q \)-functions \( Q^{\text{XXZ}}_\pm(u) = P_\pm(\cos u) \) for the ground state of the XXZ spin chain with anisotropy \( \Delta = -1/2 \) \((2n + 1 \text{ sites})\). Arguably this procedure can be extended to generic cases with some dedicated labour [11].

5 Final considerations and perspectives.

We have presented a general procedure to find the solutions \( Q_\pm \) of the \( QQ \) relation (1.3) or \( TQ \) relation (1.8) with suitable analytic and asymptotic properties, as from the limit (1.48) of the respective wave-functions (1.37) around the ‘origin’ \( w_0 \). In brief details, we have converted the quantum universal \( TQ \) relation (1.9) into the Marchenko-like equation (1.25), written in terms of \( T \) only. The latter involves the independent variables \( w \) and \( \bar{w} \) of the ODE/IM correspondence, crucially introduced as extension of the adimensional renormalisation group parameter \( r = MR \). In fact, they are those in
which the Schrödinger (1.37) can be derived from the Marchenko-like equation. Analogously, also another Schrödinger equation (1.39) in \( \bar{w} \) holds, giving the compatibility of the two the crucial constraint on the 'potential' (1.41). In a nutshell, we have mapped a quantum integrable model into a classical one. As a consequence, the whole procedure can be carried out only in case of massive theories, being the 2D CFTs potential – not provided with classical dynamics – recovered as massless limits.

We have elaborated the procedure in the quantum homogenous sG model (sG with many masses) and found the expected classical shG equation, agreement in the CFT limit (Section 2) and in two peculiar cases leading to the radial symmetric shG, i.e. Painlevé III\(_3\) equation (transfer matrix \( T = 1 \) and \( N = \infty \), Section 3). But our method is so general that, it is supposed to be applicable many quantum integrable models; the application to the XXZ chain (Section 4) goes in that direction along with foreseeable applications to higher rank algebras. As a byproduct, it can be applied to the ordinary scattering relations as a new way to derive the usual Marchenko equation [5]

In the end, we can obtain a variety of potentials with moduli of physical meaning [13, 2, 4, 16, 5] and give an interpretation to the ODE/IM variables (1.24): this make us think that our construction is a powerful tool for charting the space of 2d (integrable) theories. In fact, we are able to have control of the CFT regime, i.e. the ultraviolet limit of the RG flow. This spanning may reveal itself also more interesting in view of correspondences of 2d theories with higher dimensional theories, like, for instance, between the \( Q-, T- \) functions and the periods of \( \mathcal{N}=2 \) SYM [12, 15, 17, 18].

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