DISCRETE GRADIENT DESCENT DIFFERS QUALITATIVELY FROM GRADIENT FLOW

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Abstract. We consider gradient descent on functions of the form $L_1 = |f|$ and $L_2 = f^2$, where $f : \mathbb{R}^n \to \mathbb{R}$ is any smooth function with 0 as a regular value. We show that gradient descent implemented with a discrete step size $\tau$ behaves qualitatively differently from continuous gradient descent. In both cases, let us start at an arbitrary initial point $p_0$ and flow under gradient descent toward smaller values of $L$. Under continuous gradient descent, this occurs in one phase. Under discrete gradient descent on $L_1$, if one gets close enough to the critical manifold $M = L_1^{-1}(0)$, a second, qualitatively different phase emerges. It can be described as gradient flow along $M$ (even though the gradient of $L_1$ along $M$ is undefined!), minimizing an auxiliary function $K = |\nabla f|^2$. Thus over long time scales, continuous and discrete gradient descent on $L_1$ find different minima of $L_1$, and we can characterize the difference - the minima that tend to be found by discrete gradient descent lie in a secondary critical submanifold $M' \subset M$, the locus within $M$ where $K$ is minimized. In this paper, we explain this behavior. We also study the more subtle behavior of discrete gradient descent on $L_2$.

1. Introduction

In this paper, we consider gradient descent on functions of the form $L_1 = |f|$ and $L_2 = f^2$, where $f : \mathbb{R}^n \to \mathbb{R}$ is any smooth function with 0 as a regular value. When our discussion applies to functions both of the form $L_1$ and of the form $L_2$, we call the function $L$. We show that gradient descent implemented with a discrete step size $\tau$ behaves qualitatively differently from continuous gradient descent, also called gradient flow.

Under our assumptions on $f$, the locus of global minima of $L$ is the codimension 1 submanifold $M = L_1^{-1}(0)$. In both discrete and continuous implementations, let us start at a random initialization and use gradient descent to move toward $M$. Under continuous gradient descent, this occurs in one phase. Under discrete gradient descent on $L_1$, if one gets close enough to the critical manifold $M$, there is a second qualitatively different phase of the process. This second phase can be described as gradient flow along $M$ (even though the gradient of $L$ along $M$ is undefined!), minimizing the function $K = |\nabla f|^2$.

Thus we find that when gradient descent on $L_1$ is implemented discretely, not only is $L_1$ minimized, but if the process succeeds in reaching a global minimum, there is a second process which implicitly emerges. As a result, discrete gradient descent on $L_1$, if run on long time scales, preferentially finds global minima with low values of $K$ over those with high values of $K$. In this paper, we will explain this behavior, and derive the formula for $K$. Discrete gradient descent on $L_2$ is more subtle, but we will study that case as well.

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2. Discrete gradient descent for $L_1 = |f|$: theory

Let us begin by defining discrete gradient descent with step size $\tau$ to minimize any function $L : \mathbb{R}^n \to \mathbb{R}$ by the following iterative process.

We begin at an initial position $p_0 = (x_1, \ldots, x_n)$. Suppose that after $t$ steps we have reached the point $p_t$. Then the $t + 1$st step is

$$p_{t+1} = p_t - \tau \nabla L(p_t)$$

where

$$\nabla L = \left( \frac{\partial}{\partial x_1} L, \ldots, \frac{\partial}{\partial x_n} L \right)$$

is the gradient in the standard cartesian coordinates of $\mathbb{R}^n$.

In this section we focus on a special case of discrete gradient descent. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function for which 0 is a regular value. Let $M = f^{-1}(0)$. Since we assume 0 is a regular value of $f$, $M \subset \mathbb{R}^n$ is a smooth codimension 1 submanifold. We are interested in gradient descent on the function $L_1 = |f|$. Note that $L_1$ is not differentiable along $M$, so the gradient descent step is not well defined if $p_{t-1} \in M$. Generically though, none of the points $p_t$ will lie in $M$.

**Theorem 1.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function for which 0 is a regular value. If one implements discrete gradient descent on $L_1 = |f|$ with step size $\tau > 0$, and if for some $t$ the point $p_t$ is within approximately $\tau |\nabla f|$ of $M$, then there will be a second distinct phase. This second phase can be described effectively as gradient descent along $M$ minimizing the function $K = |\nabla f|^2$.

**Proof.** We are interested in gradient descent on $L$ in the coordinate system $(x_1, \ldots, x_n)$. However, it will be easier to analyze this process in a coordinate system more natural to the manifold $M$. In general, the gradient of a function computed in a new coordinate system will not agree with the gradient of the function computed in the original coordinate system. However, if the new coordinates are orthogonal with respect to the old coordinates, meaning that the Jacobian of the change of coordinate function is an orthogonal matrix at every point, then the gradient can be correctly computed in the new coordinate system.

We wish to use tubular coordinates around $M$. $M$ has a normal vector field. Let $n$ denote the unit normal vector in the direction $f$ is increasing. For any $p \in M$, there is a neighborhood $U \subset M$ of $p$ such that there is a diffeomorphism $\phi$ from $U \times [-\epsilon, \epsilon]$ to a tubular neighborhood of $U$. We assume that an orthogonal coordinate system $(q_1, \ldots, q_{n-1})$ on $U$ exists, which we sometimes denote more compactly as $q$. For $s \in [-\epsilon, \epsilon]$, $\phi$ sends $(q, s)$ to $q + sn$. 

In this coordinate system, we can use Taylor’s theorem to expand $f$ in the variable $s$.

(2.2) \[ f(q, s) = f(q, 0) + \frac{\partial f}{\partial s}(q, 0) \cdot s + O(s^2) \]

Since $f|_M = 0$, $f(q, 0) = 0$. So

\[ f(q, s) = \frac{\partial f}{\partial s}(q, 0) \cdot s + O(s^2) \]

Meanwhile, we can compute $|\nabla f(q, 0)|$ as follows.

\[
|\nabla f(q, 0)| = \left| \left( \frac{\partial f}{\partial q_1}(q, 0), \ldots, \frac{\partial f}{\partial q_{n-1}}(q, 0), \frac{\partial f}{\partial s}(q, 0) \right) \right|
\]

\[
= \left| \left( 0, \ldots, 0, \frac{\partial f}{\partial s}(q, 0) \right) \right|
\]

\[
= \frac{\partial f}{\partial s}(q, 0)
\]

where the second equality holds because

\[
\frac{\partial f}{\partial q_k}(q, 0) = \left( s \frac{\partial}{\partial q_k} \frac{\partial f}{\partial s} \right)(q, 0)
\]

\[
= 0
\]

and where the last equality holds because we chose the orientation of $n$ so that $\frac{\partial f}{\partial s}(q, 0)$ is positive. We conclude that

(2.3) \[ f(q, s) = |\nabla f(q, 0)| \cdot s + O(s^2). \]

Now we can express $L(q, s) = |f(q, s)|$, the function we’d like to do gradient descent on, as

(2.4) \[ L(q, s) = |\nabla f(q, 0)| \cdot |s| + O(s^2). \]

Applying $\nabla$ to both sides, we obtain

(2.5) \[ \nabla |f(q, s)| = \left( \frac{\partial}{\partial q_1} |\nabla f(q, 0)| \cdot |s|, \ldots, \frac{\partial}{\partial q_{n-1}} |\nabla f(q, 0)| \cdot |s|, \frac{\partial}{\partial s} |\nabla f(q, 0)| \cdot |s| \right) + O(s^2). \]

We compute the last coordinate as
\[
\frac{\partial}{\partial s} (|(\nabla f)(q,0)| \cdot |s|) = \frac{\partial}{\partial s} (|(\nabla f)(q,0)|) \text{sgn}(s).
\]

We denote
\[
\left( \frac{\partial}{\partial q_1} (|(\nabla f)(q,0)| \cdot |s|), \ldots, \frac{\partial}{\partial q_{n-1}} (|(\nabla f)(q,0)| \cdot |s|) \right)
\]
by
\[
\nabla_q (|(\nabla f)(q,0)| \cdot |s|).
\]

Then we have
\[
\nabla L(q,s) = \nabla |f(q,s)| = \left( \nabla_q (|(\nabla f)(q,0)| \cdot |s|), |(\nabla f)(q,0)| \text{sgn}(s) \right) + O(s^2).
\]

Now we can write approximate formulas for discrete gradient descent in these coordinates, dropping the higher order terms. Under discrete gradient descent on \(|f(q,s)|\) with time step \(\tau\)
\[
(q_{t+1}, s_{t+1}) = (q_t, s_t) - \tau \nabla L(q_t, s_t).
\]

First we consider the evolution in the \(s\)-coordinate.
\[
s_{t+1} = s_t - \tau |(\nabla f)(q_t,0)| \text{sgn}(s_t).
\]

If \(s_t\) is positive and \(s_t > \tau |(\nabla f)(q,0)| \text{sgn}(s_t)\), in this step we continue decreasing the value of \(s\) toward the manifold \(M\). If \(s_t\) is in the range
\[
\tau |(\nabla f)(q,0)| \text{sgn}(s_t) > s_t > 0,
\]
then in this step we cross to the other side of \(M\). Similarly if \(s_t\) is negative. So in the \(s_t\) coordinate, under discrete gradient descent once we get close enough to \(M\), we oscillate back and forth across \(M\).

On the other hand, both when \(s_t\) is positive and when \(s_t\) is negative,
\[
(2.6) \quad q_{t+1} = q_t - \tau \nabla_q (|(\nabla f)(q_t,0)| \cdot |s_t|)
\]

So in the \(q\) coordinate, regardless of which side of \(M\) we are on, we experience discrete gradient descent in the same direction along \(M\) because \([2.7]\) depends only on \(|s_t|\) and not \(s_t\). The precise analysis is simplest if we consider a pair of steps. Suppose we start at
\[
s_t = a \tau |(\nabla f)(q,0)|
\]
for some $0 < a < 1$. Then in the next step,

$$s_{t+1} = (1 - a)\tau|\nabla f(q, 0)|$$

and the following step returns to where we started

$$s_{t+2} = (1 - (1 - a))\tau|\nabla f(q, 0)|$$

$$= a\tau|\nabla f(q, 0)|$$

The corresponding steps then in the $q$ coordinates are

$$(2.7) \quad q_{t+1} = q_t - \tau\nabla_q(||\nabla f(q, 0)|| \cdot |s_t|)$$

and

$$q_{t+2} = q_{t+1} - \tau\nabla_q(||\nabla f(q, 0)|| \cdot |s_{t+1}|)$$

$$= (q_t - \tau\nabla_q(||\nabla f(q, 0)|| \cdot |s_t|)) - \tau\nabla_q(||\nabla f(q, 0)|| \cdot |s_{t+1}|)$$

$$= q_t - \tau\nabla_q(||\nabla f(q, 0)|| \cdot (|s_t| + |s_{t+1}|))$$

$$= q_t - \tau\nabla_q(||\nabla f(q, 0)|| \cdot (|a| + |1 - a|)\tau||\nabla f(q, 0)||)$$

But since $0 < a < 1$, $|a| + |1 - a| = 1$. So

$$q_{t+2} = q_t - \tau^2\nabla_q(||\nabla f(q, 0)||^2)$$

We conclude that secondary gradient descent can be described as discrete gradient descent within the submanifold $M$ with step size $\tau^2$ minimizing the function $K = ||\nabla f(q, 0)||^2_M$. □

3. Continuous and discrete gradient descent on $L_1 = |xy - 4|$: an example

To illustrate the phenomenon described in the previous section, and see the contrast with continuous gradient descent, in this section we make a detailed study of the behavior of both continuous and discrete gradient descent when used to minimize a simple function.

Let $f = xy - 4$. In this section we consider gradient descent on $L_1 = |xy - 4|$. We visualize continuous gradient descent by plotting flow lines of the gradient field with a differential equation plotter. Next we implement discrete gradient on a computer, show the path taken, and compare to the continuous setting. Then we mathematically analyze continuous gradient descent and apply Theorem 1 to this case of discrete gradient descent and check that our analyses match the computer simulations.

3.1. Computer simulations.
3.1.1. Continuous descent. We visualize continuous gradient descent by using Pplot to show some flow lines of the gradient field of $L_1$.

3.1.2. Discrete descent. In the second iteration of this problem, we will begin at a randomly chosen initial point $p_0 = (x_0, y_0) = (1.05, .8) \in \mathbb{R}^2$ and implement discrete gradient descent in mathematica.

The locus of global minima of $L_1 = |xy - 4|$ is the hyperbola $xy = 4$, which is 1-dimensional as expected. During the execution of discrete gradient descent on this function, we observe two phases. The first phase takes us from $(1.05, .8)$ to approximately $(2.06, 1.94)$, the point on the hyperbola that we would converge to under continuous gradient descent. In the second phase, the points $p_i$ oscillate around the hyperbola, approximately converging (but never actually converging) to the point $(2, 2)$. After a long time, the steady state is to oscillate around the point $(2, 2)$, along the line $x = y$.

3.2. Theory.
3.2.1. **Continuous descent.** Now we consider continuous gradient descent on the function \( L_1 = |xy - 4| \). Beginning at an arbitrary initial point \( p_0 = (x_0, y_0) \in \mathbb{R}^2 \), we wish to find the point \( p = (x_{\text{cont}}, y_{\text{cont}}) \) that continuous gradient descent converges to. We do so by integrating the gradient field \(-\nabla L_1\). We find that if we start at \((x_0, y_0) = (C_1 + C_2, C_1 - C_2)\) and let 
\[ u = \sqrt{4 + C_1^2 C_2^2}, \]
then under continuous gradient descent we converge to the point 
\[ \left( \sqrt{u} + 2 + \sqrt{u - 2}, \sqrt{u} + 2 - \sqrt{u - 2} \right) \]
on the hyperbola \( xy = 4 \).

3.2.2. **Discrete descent.** We apply the results of the previous section to the case of discrete gradient descent on \( L_1(x, y) = |xy - 4| \). Again, we begin at an arbitrary initial point \( p_0 = (x_0, y_0) \).

By Theorem 1, there will be two phases. The first phase brings us near \((x_{\text{cont}}, y_{\text{cont}})\). In the second phase, secondary gradient descent emerges, which can be approximated as discrete gradient descent along the hyperbola \( M \) where \( xy = 4 \), minimizing the function
\[ |\nabla f(x, y)|^2 |_M = |(y, x)|^2 |_M = x^2 + y^2 |_M. \]
along \( M \).

In this case, \( M \) is one dimensional and the locus where \( |\nabla f(x, y)|^2 \) is minimized is the zero dimensional locus \( \{ (2, 2) \cup (-2, -2) \} \). So under the secondary gradient descent, we end at one of these two points which minimizes \( |\nabla f|^2 |_M \) along \( M \), just as we observed in the computer calculation. Whether we end at \((2, 2)\) or \((-2, -2)\) depends on which branch of the hyperbola \((x_{\text{cont}}, y_{\text{cont}})\) was on.

### 4. Discrete gradient descent on \( L_2 = f^2 \): computer experiments

Though the two functions \(|f|\) and \( f^2 \) are similar, discrete gradient descent with step size \( \tau \) for \( L_1 = |f| \) and \( L_2 = f^2 \) behave quite differently near \( M \). This is because near \( M \), the norm of the gradient of \(|f|\) is approximately constant on each side of \( M \). In contrast, for \( f^2 \), the norm of the gradient goes to 0 as one approaches \( M \). It is true though that at every point \( p \in \mathbb{R}^n \setminus M \), \( \nabla |f|(p) \) points in the same direction as \( \nabla f^2(p) \).

If implemented directly, discrete gradient descent on \( L_2 \) does not exhibit secondary gradient descent. As it approaches \( M \), the norm of \( \tau |\nabla f^2| \) goes to zero, and in this case discrete gradient descent converges\(^1\) to approximately the same point that continuous gradient descent does. However, there are many modified implementations of discrete gradient descent used in practice, and secondary gradient descent does emerge in some of them.

In the remainder of this section, we continue our detailed study of gradient descent on \(|xy - 4|\) and \((xy - 4)^2\) by implementing in computer simulations several modified versions of discrete gradient descent on \( L_2 = (xy - 4)^2 \). In some we observe secondary gradient descent, in some we don’t. For each we discuss why secondary descent does or does not occur.

\(^1\)Discrete gradient descent often doesn’t actually converge, due to the discrete step size. So in this paper when we use the term converges to a point in the context of discrete gradient descent, we mean that the sequence stays within a small neighborhood of that point.
Generally, if there is some reason that gradient descent doesn’t converge to a point in \( M \) but instead bounces back and forth across it \( M \), secondary descent may occur. For \( L_1 = |f| \) this bouncing arises because the size of the gradient vectors does not vanish near \( M \). For \( L_2 = f^2 \), even though the gradient field does vanish near \( M \) there are still multiple mechanisms that can cause the process to bounce back and forth across \( M \). We will explore several such mechanisms.

### 4.1. Discrete gradient descent with fixed effective step size

The first modified implementation we consider is discrete gradient descent with normalized effective step size. We begin by establishing some terminology. In the simplest implementation of discrete gradient descent \( (2.1) \), we refer to \( \tau \) as the step size, and \( \tau |\nabla L| \) as the effective step size, as it is the distance traveled in \( \mathbb{R}^n \) at each step. In this first modification, we bound the effective step size from below.

Fix a cutoff \( c \). As before, we begin at some initial position \( p_0 \). Suppose that after \( t \) steps we have reached the point \( p_t \). Then the \( t + 1 \)st step is

\[
p_{t+1} = \begin{cases} p_t - \tau \nabla L(p_t) & \text{if } |\nabla L(p_t)| > c \\ p_t - \tau c |\nabla L(p_t)| & \text{otherwise} \end{cases}
\]

With this modification, the effective step size during gradient descent does not go to zero as the sequence approaches \( M \), and the sequence can never converge exactly to a point in \( M \). Instead, the sequence will bounce back and forth near \( M \), and secondary gradient descent has the potential to emerge. In fact, this renormalized gradient field is approximately the same as \( \nabla L_1 \) near \( M \), so the analysis of Section 2 is a good approximate analysis of the dynamics. So we expect to observe two phases of gradient descent, and during the second phase, to approximately minimize \( |\nabla f|^2 \) along \( M \). Indeed, when implemented for \( L_2 = (xy - 4)^2 \) in Matlab, we observe exactly that.

![Figure 3. Discrete gradient descent with normalized effective step size on \( L = |xy - 4| \)](image)

### 4.2. Discrete gradient descent with \( \epsilon \)-jitter

A second modification of discrete gradient descent that is sometimes implemented involves adding, at each step, a small random vector to the gradient vector. We call this modification discrete gradient descent with \( \epsilon \)-jitter.

In this case, we begin at some initial position \( p_0 \). At the \( t + 1 \)st step,

\[
p_{t+1} = p_t - \tau \nabla L(p_t) - (\epsilon_{1,t}, \ldots, \epsilon_{n,t})
\]
where $\epsilon_{1,t}, \ldots, \epsilon_{n,t}$ are drawn from a gaussian distribution with norm 0 and standard deviation $\epsilon$.

In this case, once the sequence gets near $M$ the jitter causes it to bounce back and forth across $M$ indefinitely, which is the first ingredient for secondary gradient descent. However, we do not expect secondary gradient descent along $M$ in this case. This is because each step of the process that induces secondary gradient descent is primarily perpendicular to $M$, with a very small tangential component. Secondary gradient descent arises from the cumulative effect of these small tangential components over many steps. But in this setting, the jitter is on average equal in the tangential and perpendicular directions, so it masks the tangential flow. We expect that the jitter induces a random walk along $M$, rather than directed flow to $M'$. Indeed, when implemented for $L = (xy - 4)^2$ in Matlab, we observe exactly that.

![Figure 4. Discrete gradient descent with $\epsilon$-jitter on $L = |xy - 4|$](image)

4.3. $\epsilon$-noisy gradient descent. In this section, we consider a different noisy modification of discrete gradient descent, which we will call $\epsilon$-noisy gradient descent.

We begin at some initial position $p_0$. At the $t + 1^{st}$ step, we take

$$p_{t+1} = p_t - \tau \nabla L_{\epsilon_t}(p_t)$$

where $\epsilon_t$ is drawn from a gaussian distribution with mean zero and standard deviation $\epsilon$ and

$$L_{\epsilon_t} = (f + \epsilon_t)^2.$$

In the previous section, the random component perturbed the process parallel to $M$ as much as it perturbed it perpendicular to $M$, so the jitter completely overwhelmed the secondary gradient descent and we observed a random walk around $M$. However, if the added randomness were to perturb the sequence primarily in a perpendicular direction, we may again see the phenomenon of secondary gradient descent. That is what happens here.

Consider our example $f = xy - 4$. When $\epsilon$ is small, the critical manifold of $f_\epsilon(x, y) = xy - (4 - \epsilon)$ is essentially parallel to the critical manifold of $f(x, y) = xy - 4$. So if $(u, v)$ is near $M$, the gradient field of $f_\epsilon$ will push the point away from $M$ primarily perpendicularly. The tangential component of $\nabla L_\epsilon$ points in basically the same direction as the tangential component of $\nabla L$, so at each step we expect to move somewhat parallel to $M$ toward the set $M'$ minimizing $|\nabla f|^2$. Hence we expect the qualitative behavior of discrete gradient descent in this setting to be similar to that of simple discrete gradient descent in the $L = |f|$ case, with a primary gradient flow bringing the sequence to $M$, followed by a secondary gradient flow that further pushes it to $M' \subset M$. 

We will prove this in next section, but for now we implement this process in Matlab and observe the behavior we have just described.

\[ \epsilon \text{-noisy gradient descent on } L_2 = (xy - 4)^2. \]

5. \( \epsilon \)-NOISY GRADIENT DESCENT FOR \( L_2 = f^2 \): THEORY

As discussed in the previous section, discrete gradient descent on the function \( L_2 = f^2 \) generally does not display secondary gradient descent. However, modified implementations of discrete gradient descent on \( L_2 \) do. In this section, we characterize the behavior of secondary gradient descent on \( M \) for \( \epsilon \)-noisy gradient descent, as defined in the previous section.

**Theorem 2.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a smooth function for which 0 is a regular value. Suppose one implements \( \epsilon \)-noisy gradient descent on \( L_2 = f^2 \) with step size \( \tau > 0 \) and initial point \( p_0 \), resulting in a sequence of points \( p_0, p_1, \ldots \). If for some \( t \) the point \( p_t \) is within approximately \( \tau |\nabla f| \) of \( M \), then a second phase of gradient descent will take place. This secondary phase can be effectively described as discrete gradient descent with step size \( \tau^2 \epsilon^2 \) along \( M \) minimizing the function \( K = |\nabla f|^2 |_M \).

**Proof.** We are implementing a modified form of discrete gradient descent on \( L_2 = f^2 \), with update rule

\[ p_{t+1} = p_t - \tau L_{\epsilon_t}(p_t), \]

where \( \epsilon_t \) is drawn from a gaussian distribution with mean 0 and standard deviation \( \epsilon \) and

\[ L_{\epsilon_t} = (f + \epsilon_t)^2. \]

We begin by expressing \( L_{\epsilon_t} \) in tubular coordinates, using the Taylor expansion for \( f \).

\[
L_{\epsilon_t}(q,s) = (f + \epsilon_t)^2 \\
= (|\nabla f|(q,0)s + \epsilon_t)^2 + \ldots \\
= \epsilon_t^2 + 2\epsilon_t|\nabla f|s + (|\nabla f|^2)s^2 + \ldots
\]

Next, we’d like to express the update rule in the tubular coordinates constructed in Section 2. As discussed there, the gradient can be correctly computed in those coordinates.
So
\[ \nabla L_{\epsilon t} = \left( \nabla q(\epsilon_t^2 + 2\epsilon_t |\nabla f|(q,0)s + (|\nabla f|(q,0)s)^2), \frac{\partial}{\partial s}(\epsilon_t^2 + 2\epsilon_t |\nabla f|(q,0)s + (|\nabla f|(q,0)s)^2) \right) + \ldots \]
\[ = \left( \nabla q(2\epsilon_t |\nabla f|(q,0)s) + \nabla q(|\nabla f|(q,0)s)^2), 2\epsilon_t |\nabla f|(q,0) + 2|\nabla f|^2(q,0)s \right) + \ldots \]

Thus when \( s \) is small, the update rule in the \( q \)-coordinates can be approximated as
\[ q_{t+1} = q_t - \tau (\nabla q(2\epsilon_t |\nabla f|(q,0)s)) + \nabla q \left( s_t^2 |\nabla f|(q,0)^2 \right) . \]

During the process of \( \epsilon \)-noisy gradient descent, we expect \( \epsilon_t \) to be positive approximately as often as it is negative, because \( \epsilon_t \) is drawn from a gaussian distribution with mean 0. So the expected value for the first term \( s\epsilon_t |\nabla f|(q,0) \) is approximately 0, and we can approximate the update rule in the \( q \) coordinates as
\[ q_{t+1} = q_t - \tau \nabla q \left( s_t^2 |\nabla f|^2(q,0) \right) . \]

We conclude there is a secondary phase of gradient descent during which the first term \( \nabla q \left( s_t^2 |\nabla f|^2 \right) \) is minimized. To compute the expected value of this vector, we would like to know the expected value of \( s_t^2 \). To find this, we need to analyze the dynamics in the \( s \)-coordinate. In that coordinate, the update rule is
\[ s_{t+1} = s_t - 2\tau |\nabla f|(q,0)(|\nabla f|(q,0)s + \epsilon_t) \]
\[ = s_t - 2\tau |\nabla f|^2(q,0)s - 2\tau |\nabla f|(q,0)\epsilon_t \]

This is an example of an AR(1) process, and the expected value of \( s_t^2 \) is computed as
\[ \mathbb{E}(s_t^2) = \frac{(2\tau |\nabla f|\epsilon)^2}{4\tau |\nabla f|^2} \]
where the numerator is the variance of the noise, and the denominator is twice the coefficient of the second \( s_{t-1} \) term [H94].

This simplifies as
\[ \tau \epsilon^2 \]

So we conclude that in the \( q \)-coordinate, the expected value for the update rule is
\[ q_{t+1} = q_t - \tau \nabla q \left( \tau \epsilon^2 |\nabla f|^2 \right) \]
\[ = q_t - \tau^2 \epsilon^2 \nabla q \left( |\nabla f|^2 \right) \]

This is simply the update rule for discrete gradient descent on \( M \), minimizing the function \( K = |\nabla f|^2 |M \). This concludes the proof.
6. Discussion

It is a surprising but real phenomenon that discrete gradient descent behaves qualitatively differently from continuous gradient descent. In this paper, we have seen that when considering gradient descent on functions of the form $L_1 = |f|$ or $L_2 = f^2$, not only are the two qualitatively different, but this difference is robust to multiple modifications of the discrete gradient descent algorithm.

It is striking that beginning at the same initial point, on long time scales the minima found by discrete and continuous gradient descent are different. Under continuous gradient descent, if the sequence does not get trapped in a local minimum, it will converge to a global minimum $m \in M$. Under discrete gradient descent starting at the same point, if the sequence does not get trapped in a local minimum, it will arrive near $m$, but then continue moving along $M$ toward global minima that minimize not only $L$ but also the function $K = |\nabla f|^2$.

Secondary gradient descent is subtle but it is not a small effect which causes minor changes in the trajectory of gradient descent. Rather, on long time scales, it leads to discrete gradient descent tending toward significantly different minima than continuous gradient descent does when both begin at the same initial point.

References

[H94] J.D. Hamilton. *Time Series Analysis*. Princeton University Press, 1994.