Abstract

We introduce the qubit representation by complex numbers on the set of Apollonius circles with common symmetric points at 0 and 1, related with $|0\rangle$ and $|1\rangle$ states. For one qubit states we find that the Shannon entropy as a measure of randomness is a constant along Apollonius circles. For two qubit states, the concurrence as a characteristic of entanglement is taking constant value for the states on the same Apollonius circle. Geometrical meaning of concurrence as an area and as a distance in the Apollonius representation are found. Then we generalize our results to arbitrary $n$-qubit Apollonius states and show that the fidelity between given state and the symmetric one, as reflected in an axes, is a constant along Apollonius circles. For two qubits it coincides with the concurrence. For generic two qubit states we derived Apollonius representation by three complex parameters and show that the determinant formula for concurrence is related with fidelity for symmetric states by two reflections in a vertical axis and inversion in a circle. We introduce the complex concurrence and an addition formula for Apollonius states and show that for generic two qubit states its modulus satisfies the law of cosine. Finally, we show that for two qubit Apollonius state in bipolar coordinates, the complex concurrence is decribed by static one soliton solution of the nonlinear Schrödinger equation.

Keywords: qubit, Apollonius circle, concurrence, entanglement, bipolar coordinates, soliton, Nonlinear Schrödinger equation

1 Introduction

The qubit as a unit of quantum information traditionally is represented by a point on the Bloch sphere. In the coherent state form it corresponds to to complex number $z$
in extended complex plane \( \mathbb{C} \cup \{\infty\} \), where state \(|1\rangle\) is related with \(\infty\). Here we introduce the qubit representation by complex numbers on the set of Apollonius circles with common symmetric points at 0 and 1, corresponding to \(|0\rangle\) and \(|1\rangle\) states. Apollonius circles are defined as the set of points with given ratio of distances from two fixed points. Pappus d’Alexandrie (290-350 AD) in Book VII of his Collections credits discovery of these circles to Apollonius of Perga (250-170 BC) (Figure 1) though such circles was considered before by Aristotle (384-322 BC) in Meteorologica \[1\].

In recent studies on method of images in hydrodynamics we have solved problem for concentric circles \[2\], which can be conformally mapped to arbitrary position of two cylinders in the flow as Apollonius circles. This method of images implies reflection of quantum states in constructing entangled states \[3\], \[4\], and allows us to introduce the Apollonius representation of qubit states.

![Figure 1: Apollonius of Perga (252-170 BC)](image)

### 2 One Qubit in Coherent State Representation

One qubit state

\[
|\theta, \varphi\rangle = \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i\varphi} |1\rangle
\]

(1)

is determined by a point \((\theta, \varphi)\) on the Bloch sphere. If we project the Bloch sphere to the complex plane, then we get the qubit coherent state

\[
|z\rangle = \frac{|0\rangle + z|1\rangle}{\sqrt{1 + |z|^2}},
\]

(2)

where complex number \(z = \tan \frac{\theta}{2} e^{i\varphi}\) denotes the stereographic projection (Figure 2).
In this representation $|0\rangle$ state corresponds to the origin $z = 0$, but the state $|1\rangle$ is going to infinity, which produces some disadvantage (Figure 3). However, by the Hadamard gate we can move it to a finite point in the plane. As a result we get a new representation with state $|0\rangle$ at 1 and state $|1\rangle$ at -1:

$$H|z\rangle = |b\rangle = \frac{(1 + z)|0\rangle + (1 - z)|1\rangle}{\sqrt{2}\sqrt{1 + |z|^2}}.$$  

(3)

To get ordered basis qubits $|0\rangle$ and $|1\rangle$ at positions $-1$ and $1$ correspondingly, we can apply another circuit diagram:

$$|z\rangle \xrightarrow{Y} \left| -\frac{1}{z} \right\rangle \xrightarrow{H} |c\rangle$$

where the state (Figure 4)

$$|c\rangle = \frac{(z - 1)|0\rangle + (z + 1)|1\rangle}{\sqrt{2}\sqrt{1 + |z|^2}}.$$  

(4)
To compare the relation between a qubit and a bit in a more direct way, we like to fix positions of our states $|0\rangle$ and $|1\rangle$ at points 0 and 1 correspondingly. Then we replace $z$ to $2z - 1$ (scaling and translation) and get one qubit state $|a\rangle$ in the form

$$|a\rangle = \frac{(z - 1)|0\rangle + z|1\rangle}{\sqrt{|z - 1|^2 + |z|^2}} ,$$

which we call the Apollonius qubit representation (Figure 5).
where \( p_0 + p_1 = 1 \) and the ratio of probabilities is

\[
\frac{p_0}{p_1} = \frac{|z|^2}{|z - 1|^2} \equiv r^2.
\]  

\[(8)\]

**Apollonius Circle Definition:** A circle can be defined as the set of points in plane that have specified ratio of distances from two fixed points. The ratio is \( \frac{|z-a|}{|z-b|} = r \), where \( a \) and \( b \) are common symmetric points playing role of the fixed points. In our case 0 and 1 are symmetric fixed points, and ratio of probabilities \((8)\) is constant along the Apollonius circles (Figure 6).

![Figure 6: Apollonius One Qubit State](image_url)

The Apollonius circles are determined by \( r \) completely, so that the position and the radius of the circle are respectively: \( x_0 = \frac{r^2}{1 + r^2} \) and \( r_0 = \frac{r}{1 + r^2} \).

**Apollonius States and Entropy:** For Apollonius state \( |a\rangle \) we have corresponding probabilities to measure state \( |0\rangle \) and \( |1\rangle \)

\[
p_0 = |\langle 0 |z \rangle|^2 = \frac{|z - 1|^2}{|z - 1|^2 + |z|^2} = \frac{1}{1 + r^2},
\]

\[(9)\]

\[
p_1 = |\langle 1 |z \rangle|^2 = \frac{|z|^2}{|z - 1|^2 + |z|^2} = \frac{r^2}{1 + r^2}.
\]

\[(10)\]

The level of randomness for Apollonius state \( |a\rangle \) is determined by the Shannon entropy

\[
H = -p_0 \log_2 p_0 - p_1 \log_2 p_1
\]

and give us following result

\[
H(r^2) = \log_2(1 + r^2) - \frac{r^2}{1 + r^2} \log_2 r^2.
\]
This means that the Shannon entropy or level of randomness is constant along Apollonius circles (Figure 7).

**Maximally Random States:**

![Diagram of Entropy on Apollonius circles](image)

To find maximally random state we take derivative with respect to $r^2$

$$
\frac{dH}{dr^2} = -\frac{1}{(1 + r^2)^2} \log_2 r^2 = 0 \Rightarrow r = 1
$$

(11)

The second derivative gives

$$
\frac{d^2H}{(dr^2)^2} = -\frac{2}{(1 + r^2)^3} \log_2 r^2 - \frac{1}{(1 + r^2)^2} \frac{1}{r^2 \ln 2}
$$

(12)

$$
H''|_{r=1} = -\frac{1}{4 \ln 2} < 0
$$

(13)

![Graph of Entropy and Fidelity](image)

**Figure 8:** Entropy (blue line) and fidelity (pink line) between symmetric states versus $r$
and implies that \( r = 1 \) is the local maximum. Apollonius circles are level curves of the state randomness (constant entropy \( H \) level curves). The maximally random states with \( H = 1 \) are located at vertical line \( \Re z = \frac{1}{2} \). For computational basis states we have zero entropy: \( H(0) = 0 \) and \( H(\infty) = 0 \) (Figure 7).

Another characteristic which is constant along Apollonius circles is the fidelity between symmetric states, reflected in vertical axes \( \Re z = \frac{1}{2} \). This reflection corresponds to substitution \( z \rightarrow 1 - \bar{z} \) and gives the symmetric state

\[
|a_s\rangle = \frac{-\bar{z}|0\rangle + (1 - \bar{z})|1\rangle}{\sqrt{|z - 1|^2 + |z|^2}},
\]  

(14)

with fidelity

\[
F = \langle a_s|a\rangle = \frac{2|z||z - 1|}{|z - 1|^2 + |z|^2}.
\]  

(15)

In Figure 8 we show the entropy and the fidelity versus \( r \). Both curves reach maximal value at \( r = 1 \) and vanish at origin and at infinity. Comparison of these curves show that maximally random state corresponds to maximal fidelity between symmetric states and it happens when these states belong to the line \( \Re z = \frac{1}{2} \). By increasing the geometrical distance between them we decrease the level of randomness. So that states \( |0\rangle \) and \( |1\rangle \) as maximally far symmetric states are orthogonal and have fidelity vanishing.

For the distance between symmetric states we get formula

\[
||a\rangle - |a_s\rangle|| = 2 \frac{|\Re z - \frac{1}{2}|}{\sqrt{|z - 1|^2 + |z|^2}},
\]  

(16)

showing that the distance reaches maximal value for orthogonal states at \( z = 0 \) and \( z = 1 \) and on the vertical line \( \Re z = \frac{1}{2} \) it vanishes.

We can introduce another distance characteristics in terms of fidelity

\[
d = \sqrt{1 - F^2},
\]  

(17)

so that for our Apollonius qubit state and the symmetric one we find

\[
d = \frac{||z - 1|^2 - |z|^2|}{|z - 1|^2 + |z|^2} = \frac{|1 - r^2|}{1 + r^2}.
\]  

(18)

This formula shows that distance between symmetric states depends only on Apollonius circle and is determined by its parameter \( r \). It is invariant under substitution \( r \rightarrow 1/r \), corresponding to the pair of symmetric circles as reflections in axis \( \Re z = \frac{1}{2} \). For states on the line with \( r = 1 \) we have the minimal distance \( d = 0 \). For \( r = 0 \) and \( r = \infty \), corresponding to states \( |0\rangle \) and \( |1\rangle \) respectively, which are orthogonal states, we find that the distance take maximal value \( d = 1 \). This value coincides with geometrical distance between corresponding points 0 and 1. As we can see the distance \( r \) has geometrical meaning of distance between symmetric states on Apollonius circles with values \( r \) and \( 1/r \) and is just distance between two points of intersecting Apollonius circles with real line interval \([0, 1]\).
3 Apollonius Two Qubit States

By applying the CNOT gate to the product state:

\[ |a⟩ \otimes |0⟩ \xrightarrow{\text{CNOT}} |A⟩ \]

we get the Apollonius two qubit state (Figure 9)

\[ |A⟩ = \frac{(z - 1)|00⟩ + z|11⟩}{\sqrt{|z - 1|^2 + |z|^2}}. \tag{19} \]

![Figure 9: Apollonius Two Qubit State](image)

The concurrence for this state is (Figure 10)

\[ C = \frac{2|z||z - 1|}{|z - 1|^2 + |z|^2} = \frac{2r}{1 + r^2}, \]

where \( r = \frac{|z|}{|z - 1|} \). The concurrence depends on \( r \) and \( r \) depends on Apollonius circle \( \Rightarrow \) concurrence and Apollonius circle are related \( \Rightarrow \)
Concurrence is a constant along Apollonius Circle for given $r$ (Figure 11)

If $r = 1$ ⇒ we have the line $Re(z) = \frac{1}{2}$ of quantum states with $C_{max} = 1$, while states $|00\rangle$ and $|11\rangle$ with $C_{min} = 0$, correspond to common symmetric points for Apollonius circles.

### 3.1 Geometrical Meaning of Concurrence

For two qubits Apollonius state we have simple geometrical meaning of the concurrence. Since concurrence is the same for any point on the given Apollonius circle,
we consider intersection of this circle with the orthogonal one $|z - \frac{1}{2}| = \frac{1}{4}$. In Figure 12a we can see that concurrence is determined as double area of the rectangle and in Figure 12b as a distance between intersection points.

![Figure 12: a) Concurrence as an area, b) concurrence as a distance](image)

### 3.2 Concurrence and Reflection Principle

For Apollonius two qubit state $|A\rangle$ we take reflection with respect to the line $Re(z) = \frac{1}{2}$, giving the symmetric two qubit state (Figure 13)

$$|A_s\rangle = \frac{-\bar{z}|00\rangle + (1 - \bar{z})|11\rangle}{\sqrt{|z - 1|^2 + |z|^2}},$$

(20)

![Figure 13: Symmetric qubit states](image)
and fidelity between symmetric states is just the concurrence

\[ F = |\langle A_s | A \rangle| = \frac{2|z||z - 1|}{|z - 1|^2 + |z|^2} = C. \quad (21) \]

### 3.3 Generalization to n-qubit Apollonius states

By circuit:

\[ |a \rangle \otimes |0 \rangle \cdots |0 \rangle \otimes |0 \rangle \xrightarrow{\text{CNOT} \otimes I \cdots I \otimes I} |A \rangle \]

we can generate the \( n \)-qubit Apollonius state

\[ |A \rangle = \frac{(z - 1)|00\cdots0\rangle + z|11\cdots1\rangle}{\sqrt{|z - 1|^2 + |z|^2}}. \quad (22) \]

The corresponding symmetric state is

\[ |A_s \rangle = \frac{-\bar{z}|00\cdots0\rangle + (1 - \bar{z})|11\cdots1\rangle}{\sqrt{|z - 1|^2 + |z|^2}} \quad (23) \]

and fidelity between these states

\[ F = |\langle A_s | A \rangle| = \frac{2|z||z - 1|}{|z - 1|^2 + |z|^2} = \frac{2r}{1 + r^2} \quad (24) \]

is constant on Apollonius circle with fixed \( r \).

### 4 Qubit in Bipolar Coordinates

The Apollonius circle representation suggests that the bipolar coordinates could be useful in description of qubit. These coordinates have applications in navigation problems and electro-magnetic theory, determining the electric and magnetic field of two infinitely long parallel cylindrical conductors. For given complex \( z = x + iy \) we introduce two real variables, \( \tau \) and \( \sigma \) (Figure 14)

\[ z = \frac{e^\tau}{e^\tau - e^{i\sigma}}, \quad (25) \]

where

\[ \frac{|z|}{|z - 1|} = r = e^\tau. \quad (26) \]
Figure 14: Bipolar coordinates

For Cartesian coordinates we have

\[
  x = \frac{1}{2} + \frac{1}{2} \frac{\sinh \tau}{\cosh \tau - \cos \sigma},
\]

\[
  y = \frac{1}{2} \sin \sigma \frac{\cosh \tau - \cos \sigma}{2 \cosh \tau - \cos \sigma},
\]

so that

\[
  z = x + iy = \frac{1}{2} + \frac{1}{2} \frac{\sinh \tau + i \sin \sigma}{\cosh \tau - \cos \sigma}.
\]

Then for one qubit in bipolar coordinates we have

\[
  |A\rangle = \frac{1}{2} (e^{i\sigma} - e^{-\tau})|0\rangle + (e^\tau - e^{-i\sigma})|1\rangle \sqrt{\cosh \tau (\cosh \tau - \cos \sigma)}.
\]

This state up to global phase can be rewritten as

\[
  |\tau,\sigma\rangle = \frac{e^{i\sigma}|0\rangle + e^\tau|1\rangle}{\sqrt{1 + e^{2\tau}}}. \tag{31}
\]

For Apollonius two qubit state in bipolar coordinates in a similar way we have

\[
  |\tau,\sigma\rangle = \frac{e^{i\sigma}|00\rangle + e^\tau|11\rangle}{\sqrt{1 + e^{2\tau}}}.
\]

Applying the determinant formula for concurrence of this two qubit state, we find simple expression

\[
  C = \frac{1}{\cosh \tau} = sech \tau. \tag{33}
\]

It shows that concurrence is not depending on angle \(\sigma\) and is constant along the Apollonius circle with given coordinate \(\tau\). This formula suggests to consider the transition
amplitude between symmetric states in bipolar coordinates. For $n$-qubit state it gives the complex fidelity
\[ F = \langle A_s | A \rangle = Fe^{-i\sigma} = \frac{e^{-i\sigma}}{\cosh \tau}, \]
which in the case of two qubit states describe the complex version of the concurrence
\[ C = \langle A_s | A \rangle = Ce^{-i\sigma} = \frac{e^{-i\sigma}}{\cosh \tau}, \]
such that the modulus of this complex concurrence is just the usual concurrence (33) (Figure 15):
\[ |C| = |\langle A_s | A \rangle| = C = \frac{1}{\cosh \tau}. \]

![Figure 15: Soliton shape for concurrence](image)

It is interesting to notice that complex concurrence (35) is the stationary one soliton solution of the Nonlinear Schrödinger equation
\[ iC_\sigma = C_{\tau\tau} + 2|C|^2C. \]
This equation is a nonlinear integrable system admitting arbitrary $N$-soliton solutions and it has appear in many physical applications from plasma physics to fluid mechanics. We don’t know if the above result, that complex concurrence, as a transition amplitude between symmetric Apollonius states, represents soliton of NLS equation has deep meaning, but existence of such relation is really amazing.

## 5 Apollonius representation of generic two qubit state

The Apollonius states as we have introduced above are characterized by one complex parameter. For one qubit case it is the generic state. However, for multiple generic qubit states we need to introduce more parameters. Here we derive Apollonius representation for the generic two qubit state
\[ |\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle, \]
where
\[ |c_{00}|^2 + |c_{01}|^2 + |c_{10}|^2 + |c_{11}|^2 = 1. \]
Instead of four complex variables $c_{ij}$, $i, j = 0, 1$, we can introduce another set of four complex variables $\eta, \zeta, a$ and $b$ according to formulas

\begin{align}
c_{00} &= (\eta - 1) a, \\
c_{11} &= \eta a, \\
c_{01} &= (\zeta - 1) b, \\
c_{10} &= \zeta b,
\end{align}

where complex variables $a$ and $b$ we express in terms of complex $\alpha$ and $\beta$ as:

\begin{align}
a &= \frac{\alpha}{\sqrt{|\eta - 1|^2 + |\eta|^2}}, \\
b &= \frac{\beta}{\sqrt{|\zeta - 1|^2 + |\zeta|^2}}.
\end{align}

Introducing Apollonius two qubit states

\begin{align}
|\eta\rangle &= \frac{(\eta - 1)|00\rangle + \eta|11\rangle}{\sqrt{|\eta - 1|^2 + |\eta|^2}}, \\
|\zeta\rangle &= \frac{(\zeta - 1)|01\rangle + \zeta|10\rangle}{\sqrt{|\zeta - 1|^2 + |\zeta|^2}},
\end{align}

for the generic state we get superposition

\begin{align}
|\psi\rangle &= \alpha|\eta\rangle + \beta|\zeta\rangle.
\end{align}

Parameters $\alpha$ and $\beta$ can be fixed by normalization condition. We notice that Apollonius states $|\eta\rangle$ and $|\eta\rangle$ are orthogonal and normalized:

\begin{align}
\langle \eta | \eta \rangle = \langle \zeta | \zeta \rangle = 1, \\
\langle \eta | \zeta \rangle = \langle \zeta | \eta \rangle = 0.
\end{align}

It implies $|\alpha|^2 + |\beta|^2 = 1$ and we can choose

\begin{align}
\alpha &= (\xi - 1) \lambda, \\
\beta &= \xi \lambda,
\end{align}

where $\xi$ is an arbitrary complex number and

\begin{align}
|\lambda| &= \frac{1}{\sqrt{|\xi - 1|^2 + |\xi|^2}}.
\end{align}

Then by neglecting arbitrary global phase factor we have normalized generic two qubit state in Apollonius representation, characterized by three arbitrary complex numbers $\eta, \zeta$ and $\xi$:

\begin{align}
|\psi\rangle &= \frac{(\xi - 1)|\eta\rangle + \xi|\zeta\rangle}{\sqrt{|\xi - 1|^2 + |\xi|^2}}.
\end{align}

Concurrence of this state, calculated by the determinant formula is

\begin{align}
C &= \frac{2}{\sqrt{|\xi - 1|^2 + |\xi|^2}} \left| (\xi - 1)^2 \frac{\eta(\eta - 1)}{\sqrt{|\eta - 1|^2 + |\eta|^2}} - \xi^2 \frac{\zeta(\zeta - 1)}{\sqrt{|\zeta - 1|^2 + |\zeta|^2}} \right|
\end{align}

In particular cases it reduces to previous results

\begin{align}
\xi = 0 \Rightarrow C &= \frac{2|\eta||\eta - 1|}{\sqrt{|\eta - 1|^2 + |\eta|^2}}, \\
\xi = 1 \Rightarrow C &= \frac{2|\zeta||\zeta - 1|}{\sqrt{|\zeta - 1|^2 + |\zeta|^2}}.
\end{align}
5.1 Reflected qubit and concurrence

The concurrence formula (49) can be derived from the reflection principle for Apollonius generic two qubit state as

\[ C = |\langle \psi_s | \psi \rangle|, \]  

(52)

where the symmetric qubit state \( |\psi_s \rangle \) is coming from reflection of input qubits in three steps.

1) Reflection in complex plane \( \eta \) in the vertical line \( \Re \eta = \frac{1}{2} \) (Figure 16):

\[ \eta_s \equiv \eta^* = 1 - \bar{\eta} \]

\[ \eta^* = 1 - \bar{\eta} \]

Figure 16: Symmetric qubit \( |\eta \rangle \)

2) Reflection in complex plane \( \zeta \) in the vertical line \( \Re \zeta = \frac{1}{2} \) (Figure 17):

\[ \zeta_s \equiv \zeta^* = 1 - \bar{\zeta} \]

\[ \zeta^* = 1 - \bar{\zeta} \]

Figure 17: Symmetric qubit \( |\zeta \rangle \)
3) Inversion in complex plane $\xi$ in circle $|\xi - \frac{1}{2}| = \frac{1}{4}$ Figure (18):

$$\xi_s \equiv \xi^* = \frac{1}{2} + \frac{1/4}{\xi - 1/2}$$

![Diagram showing inversion in complex plane with a circle and points labeled $\xi$, $\eta$, $\zeta$, and $\xi^*$.

Figure 18: Symmetric qubit by inversion $|\xi\rangle$

The resulting state is

$$|\psi_s\rangle = \frac{(\xi^* - 1)|\eta^*\rangle + \xi^*|\zeta^*\rangle}{\sqrt{|\xi^* - 1|^2 + |\xi^*|^2}}$$

(53)

or up to global phase

$$|\psi_s\rangle = \frac{(\bar{\xi} - 1)|\eta^*\rangle - \bar{\xi}|\zeta^*\rangle}{\sqrt{|\xi - 1|^2 + |\xi|^2}}$$

(54)

where symmetric qubit states are

$$|\eta^*\rangle = -\frac{\bar{\eta}|00\rangle + (\bar{\eta} - 1)|11\rangle}{\sqrt{|\eta - 1|^2 + |\eta|^2}}$$

(55)

$$|\zeta^*\rangle = -\frac{\bar{\zeta}|01\rangle + (\bar{\zeta} - 1)|10\rangle}{\sqrt{|\zeta - 1|^2 + |\zeta|^2}}$$

(56)

Calculating the concurrence $C = |\langle\psi_s|\psi\rangle|$ we obtain the same result as by determinant formula (49).

It is instructive to see how the phase flipping gate action [5] is related with reflection of Apollonius qubits. Applying the gate to anti-unitary transformed states

$$K|\eta\rangle = |\bar{\eta}\rangle, \quad K|\zeta\rangle = |\bar{\zeta}\rangle$$

(57)
we have reflected states
\[ Y \otimes Y |\bar{\eta}\rangle = |\eta^*\rangle, \tag{58} \]
\[ Y \otimes Y |\bar{\zeta}\rangle = -|\zeta^*\rangle, \tag{59} \]
and
\[ Y \otimes Y |\bar{\psi}\rangle = (\bar{\xi} - 1)Y \otimes Y |\bar{\eta}\rangle + \bar{\xi} Y \otimes Y |\bar{\zeta}\rangle = \frac{(\bar{\xi} - 1)|\eta^*\rangle - \bar{\xi}|\zeta^*\rangle}{\sqrt{|\xi - 1|^2 + |\xi|^2}} = |\psi_s\rangle. \tag{60} \]

### 5.2 Law of Cosines for concurrence

Transition amplitude written in the form
\[ \langle \psi_s | \psi \rangle = \frac{\xi^2}{|\xi - 1|^2 + |\xi|^2} \tag{61} \]
has interesting geometrical interpretation. If we introduce the total complex concurrence, \( C = \langle \psi_s | \psi \rangle \), and two partial complex concurrences \( C_\eta = \langle \frac{1}{\eta} |\eta\rangle \) and \( C_\zeta = \langle \frac{1}{\zeta} |\zeta\rangle \) then this relation reads as a superposition principle for complex concurrences
\[ C = \mu C_\eta + \nu C_\zeta, \tag{62} \]
where complex numbers are defined as
\[ \mu = \frac{(\xi - 1)^2}{|\xi - 1|^2 + |\xi|^2}, \quad \nu = -\frac{\xi^2}{|\xi - 1|^2 + |\xi|^2}, \tag{63} \]
and satisfy
\[ |\mu| + |\nu| = 1, \quad |\nu| \frac{|\mu|}{|\mu|} = \frac{|\xi|^2}{|\xi - 1|^2} = R^2. \tag{64} \]

For partial complex concurrences we have
\[ C_\eta = \frac{2\eta(\eta - 1)}{\sqrt{|\eta - 1|^2 + |\eta|^2}}, \tag{65} \]
\[ C_\zeta = \frac{2\zeta(\zeta - 1)}{\sqrt{|\zeta - 1|^2 + |\zeta|^2}}. \tag{66} \]

From (62) we have the Law of Cosine
\[ C^2 = (|\mu|C_\eta)^2 + (|\nu|C_\zeta)^2 - 2(|\mu|C_\eta)(|\nu|C_\zeta)\cos \Phi \tag{67} \]
6 Conclusions

For arbitrary n-qubit Apollonius states, fidelity between symmetric states is constant along Apollonius circles. In one qubit case it is related with Shannon entropy and for two qubit states it coincides with concurrence. For generic two qubit states we derived Apollonius representation by three complex parameters and show that the determinant formula for concurrence is related with fidelity for symmetric states by two reflections in a vertical axis and inversion in a circle. By introduction of complex concurrence for generic two qubit states we derived the law of cosine. For two qubit Apollonius state in bipolar coordinates, the complex concurrence is described by static one soliton solution of the nonlinear Schrödinger equation.

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References

[1] Rosenfeld B. A., Apollonius of Perga, Center of Continuous Mathematical Education Publisher, Moscow, 2004.

[2] Pashaev O.K. and Yilmaz O, Vortex images and q-elementary functions, J. Physics A:Math. Theor. 41, 135207, 2008.

[3] Pashaev O.K. and Gurkan N, Energy localization in maximally entangled two- and three-qubit phase space, New Journal of Physics, 063007, 2012.

[4] Pashaev O.K., Two-circles theorem, q-periodic functions and entangled qubit states, J of Physics: Conf Series, 482, 012033, 2014.
[5] Wootters W.K., Entanglement of Formation of an arbitrary state of two qubits, Phys Rev Lett, 80, 2245-2248, 1998.