ELLIPTIC CURVES IN MODULI SPACE OF STABLE BUNDLES

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Dedicated to the memory of Eckart Viehweg

Abstract. Let $M$ be the moduli space of rank 2 stable bundles with fixed determinant of degree 1 on a smooth projective curve $C$ of genus $g \geq 2$. When $C$ is generic, we show that any elliptic curve on $M$ has degree (respect to anti-canonical divisor $-K_M$) at least 6, and we give a complete classification for elliptic curves of degree 6. Moreover, if $g > 4$, we show that any elliptic curve passing through the generic point of $M$ has degree at least 12. We also formulate a conjecture for higher rank.

1. Introduction

Let $C$ be a smooth projective curve of genus $g \geq 2$ and $\mathcal{L}$ be a line bundle of degree $d$ on $C$. Let $M := SU_C(r, \mathcal{L})^s$ be the moduli space of stable vector bundles on $C$ of rank $r$ and with fixed determinant $\mathcal{L}$, which is a smooth quasi-projective Fano variety with $\text{Pic}(M) = \mathbb{Z} \cdot \Theta$ and $-K_M = 2(r, d)\Theta$, where $\Theta$ is an ample divisor. Let $B$ be a smooth projective curve of genus $b$. The degree of a curve $\phi : B \to M$ is defined to be $\deg \phi^*(-K_M)$. It seems quite natural to ask what is the lower bound of degree and to classify the curves of lowest degree.

When $B = \mathbb{P}^1$, we have determined all $\phi : \mathbb{P}^1 \to M$ with lowest degree in [6] and all $\phi : \mathbb{P}^1 \to M$ passing through the generic point of $M$ with lowest degree in [9]. In fact, one can construct $\phi : \mathbb{P} \to M$ for various projective spaces $\mathbb{P}$ such that $\phi^*(-K_M) = \mathcal{O}_\mathbb{P}(2(r, d))$, and $\phi : \mathbb{P}^{r-1} \to M$ passing through the generic point of $M$ such that $\phi^*(-K_M) = \mathcal{O}_{\mathbb{P}^{r-1}}(2r)$. Then it was proved in [6] and [9] that the images of lines in these projective spaces exhaust all minimal rational curves on $M$ (resp. minimal rational curves passing through generic point of $M$). Some applications of the results were also pointed out in [6] and [9]. Thus it is natural to ask what are the situation when $b > 0$. This note is a start to study the case of $b = 1$. It may happen that the normalization of $\phi(B)$ is $\mathbb{P}^1$. To avoid this case, we call $\phi : B \to M$
an essential elliptic curve of $M$ if the normalization of $\phi(B)$ is an elliptic curve.

It is easy to construct essential elliptic curves of degree $6(r, d)$ on $M$, and essential elliptic curves of degree $6r$ that pass through the generic point of $M$. For example, for smooth elliptic curves $B \subset \mathbb{P}$ of degree 3, the morphism $\phi : \mathbb{P} \to M$ defines essential elliptic curves $\phi|_B : B \to M$ of degree $6(r, d)$ (See Example 3.6), which are called elliptic curves of split type. For smooth elliptic curves $B \subset \mathbb{P}^{r-1}$ of degree 3, the morphism $\phi : \mathbb{P}^{r-1} \to M$ defines essential elliptic curves $\phi|_B : B \to M$ of degree $6r$ passing through the generic point of $M$ (See Example 3.5), which are called elliptic curves of Hecke type. Are they minimal elliptic curves of $M$ (resp. minimal elliptic curves passing through generic point of $M$)? Do they exhaust all minimal essential elliptic curves on $M$ (See Conjecture 4.8 for detail)?

In this note, we consider the case that $r = 2$ and $d = 1$, then $M$ is a smooth projective fano manifold of dimension $3g - 3$. When $C$ is generic, we show that any essential elliptic curve $\phi : B \to M$ has degree at least 6, and it must be an elliptic curve of split type if it has degree 6 (See Theorem 4.6). When $g > 4$ and $C$ is generic, we show that any essential elliptic curve $\phi : B \to M$ passing through the generic point of $M$ have degree at least 12 (See Theorem 4.7). When $C$ is generic, there is no nontrivial morphism from $C$ to an elliptic curve, which implies that $\text{Pic}(C \times B) = \text{Pic}(C) \times \text{Pic}(B)$. It is the condition that we need through the whole paper.

We give a brief description of the article. In Section 2, we show a formula of degree for general case. In Section 3, we show how the general formula implies the known case $B = \mathbb{P}^1$ and construct the examples of essential elliptic curves of degree $6(r, d)$ and $6r$. In Section 4, we prove the main theorems (Theorem 4.6 and Theorem 4.7), which is the special case $r = 2$, $d = 1$ of Conjecture 4.8. Although I believe the conjecture, I leave the case of $r > 2$ to other occasion.

2. The degree formula of curves in moduli spaces

Let $C$ be a smooth projective curve of genus $g \geq 2$ and $\mathcal{L}$ a line bundle on $C$ of degree $d$. Let $M = \mathcal{SU}_C(r, \mathcal{L})^s$ be the moduli spaces of stable bundles on $C$ of rank $r$, with fixed determinant $\mathcal{L}$. It is well-known that $\text{Pic}(M) = \mathbb{Z} \cdot \Theta$, where $\Theta$ is an ample divisor.

Lemma 2.1. For any smooth projective curve $B$ of genus $b$, if

$$\phi : B \to M$$

is defined by a vector bundle $E$ on $C \times B$, then

$$\deg \phi^*(-K_M) = c_2(\mathcal{E}nd^0(E)) = 2rc_2(E) - (r-1)c_1(E)^2 := \Delta(E)$$

**Proof.** In general, there is no universal bundle on $C \times M$, but there exist vector bundle $\mathcal{E}nd^0$ and projective bundle $\mathcal{P}$ on $C \times M$ such that $\mathcal{E}nd^0|_{C \times \{[V]\}} = \mathcal{E}nd^0(V)$ and $\mathcal{P}|_{C \times \{[V]\}} = \mathbb{P}(V)$ for any $[V] \in M$. Let $\pi : C \times M \to M$ be the projection, then $T_M = R^1\pi_* (\mathcal{E}nd^0)$, which commutes with base changes since $\pi_*(\mathcal{E}nd^0) = 0$.

For any curve $\phi : B \to M$, let $X := C \times B$, $\mathcal{E} = (id \times \phi)^* \mathcal{E}nd^0$ and $\pi : X = C \times B \to B$ still denote the projection. Then $\phi^* T_M = R^1 \pi_* \mathcal{E}$. By Riemann-Roch theorem, we have

$$\deg \phi^*(-K_M) = \chi(R^1 \pi_* \mathcal{E}) + (r^2 - 1)(g-1)(b-1).$$

By using Leray spectral sequence and $\chi(E) = \deg(ch(E) \cdot td(T_X))_2$, we have $\chi(R^1 \pi_* \mathcal{E}) = -\chi(E) = c_2(E) - (r^2 - 1)(g-1)(b-1)$, hence

$$\deg \phi^*(-K_M) = c_2(E).$$

If $\phi : B \to M$ is defined by a vector bundle $E$ on $X = C \times B$, then $\mathcal{E} = \mathcal{E}nd^0(E)$ (cf. the proof of lemma 2.1 in [9]). Thus

$$\deg \phi^*(-K_M) = c_2(\mathcal{E}nd^0(E)) = 2rc_2(E) - (r-1)c_1(E)^2.$$

\[\square\]

Let $f : X \to C$ be the projection. Then, for any vector bundle $E$ on $X$, there is a relative Harder-Narasimhan filtration (cf Theorem 2.3.2, page 45 in [5])

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that $F_i = E_i/E_{i-1}$ ($i = 1, \ldots n$) are flat over $C$ and its restriction to general fiber $X_p = f^{-1}(p)$ is the Harder-Narasimhan filtration of $E|_{X_p}$. Thus $F_i$ are semi-stable of slope $\mu_i$ at generic fiber of $f : X \to B$ with $\mu_1 > \mu_2 > \cdots > \mu_n$. Then we have the following theorem

**Theorem 2.2.** For any vector bundle $E$ of rank $r$ on $X$, let

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

be the relative Harder-Narasimhan filtration over $C$ with $F_i = E_i/E_{i-1}$ and $\mu_i = \mu(F_i|_{f^{-1}(x)})$ for generic $x \in C$. Let $\mu(E)$ and $\mu(E_i)$ denote the slope of $E|_{\pi^{-1}(b)}$ and $E_i|_{\pi^{-1}(b)}$ for generic $b \in B$. Then, if

$$\text{Pic}(C \times B) = \text{Pic}(C) \times \text{Pic}(B),$$
we have the following formula

\[
\Delta(E) = 2r \left( \sum_{i=1}^{n} \left( c_2(F_i) - \frac{\text{rk}(F_i) - 1}{2 \text{rk}(F_i)} c_1(F_i)^2 \right) + \sum_{i=1}^{n-1} (\mu(E) - \mu(E_i)) \text{rk}(E_i)(\mu_i - \mu_{i+1}) \right)
\]

(2.1)

Proof. It is easy to see that

\[
2c_2(E) = 2 \sum_{i=1}^{n} c_2(F_i) + 2 \sum_{i=1}^{n} c_1(E_{i-1}) c_1(F_i)
\]

\[
= 2 \sum_{i=1}^{n} c_2(F_i) + c_1(E)^2 - \sum_{i=1}^{n} c_1(F_i)^2.
\]

Thus

\[
\Delta(E) = 2r \sum_{i=1}^{n} c_2(F_i) + c_1(E)^2 - r \sum_{i=1}^{n} c_1(F_i)^2.
\]

Let \( r_i \) be the rank of \( F_i \) and \( d_i \) be the degree of \( F_i \) on the generic fiber of \( \pi : C \times B \to B \). Then we can write

\[
c_1(F_i) = f^*\mathcal{O}_C(d_i) + \pi^*\mathcal{O}_B(r_i \mu_i)
\]

where \( \mathcal{O}_C(d_i) \) (resp. \( \mathcal{O}_B(r_i \mu_i) \)) denotes a divisor of degree \( d_i \) (resp. degree \( r_i \mu_i \)) of \( C \) (resp. \( B \)). Note that

\[
c_1(F_i)^2 = 2d_i r_i \mu_i,
\]

\[
c_1(E)^2 = 2d \sum_{i=1}^{n} r_i \mu_i
\]

we have

\[
\Delta(E) = 2r \left( \sum_{i=1}^{n} c_2(F_i) + \mu(E) \sum_{i=1}^{n} r_i \mu_i - \sum_{i=1}^{n} d_i r_i \mu_i \right)
\]

\[
= 2r \left( \sum_{i=1}^{n} (c_2(F_i) - (r_i - 1)d_i \mu_i) + \mu(E) \sum_{i=1}^{n} r_i \mu_i - \sum_{i=1}^{n} d_i \mu_i \right).
\]

Let \( \text{deg}(E_i) \) denote the degree of \( E_i \) on the generic fiber of

\( \pi : C \times B \to B \).

Using \( d_i = \text{deg}(E_i) - \text{deg}(E_{i-1}) \) and \( r_i = \text{rk}(E_i) - \text{rk}(E_{i-1}) \), we have

\[
\mu(E) \sum_{i=1}^{n} r_i \mu_i - \sum_{i=1}^{n} d_i \mu_i = \sum_{i=1}^{n-1} (\mu(E) - \mu(E_i)) \text{rk}(E_i)(\mu_i - \mu_{i+1}).
\]
Since \( d_i \mu_i = c_1(F_i)^2/2r_i \), we get the formula
\[
\Delta(E) = 2r \left( \sum_{i=1}^{n} \left( c_2(F_i) - \frac{r_i - 1}{2r_i} c_1(F_i)^2 \right) + \sum_{i=1}^{n-1} (\mu(E) - \mu(E_i)) \text{rk}(E_i)(\mu_i - \mu_{i+1}) \right).
\]

\[\square\]

**Remark 2.3.** I do not know if the formula holds without the assumption that \( \text{Pic}(C \times B) = \text{Pic}(C) \times \text{Pic}(B) \). On the other hand, the assumption holds when \( B \) is an elliptic curve and \( C \) is generic.

**Theorem 2.4.** For any torsion free sheaf \( \mathcal{F} \) on \( X = C \times B \), if its restriction to a fiber of \( f : X = C \times B \rightarrow C \) is semi-stable, then
\[
\Delta(\mathcal{F}) = 2 \text{rk}(\mathcal{F}) c_2(\mathcal{F}) - (\text{rk}(\mathcal{F}) - 1)c_1(\mathcal{F})^2 \geq 0.
\]
If the determinants \( \{\det(\mathcal{F}^*)_x\}_{x \in C} \) are isomorphic each other, then \( \Delta(\mathcal{F}) = 0 \) if and only if \( \mathcal{F} \) is locally free and satisfies
- All the bundles \( \{\mathcal{F}_x := \mathcal{F}|_{\{x\} \times B}\}_{x \in C} \) are semi-stable and s-equivalent each other.
- All the bundles \( \{\mathcal{F}_y := \mathcal{F}|_{C \times \{y\}}\}_{y \in B} \) are isomorphic each other.

**Proof.** Since \( \Delta(\mathcal{F}) \geq \Delta(\mathcal{F}^*) \), we can assume that \( \mathcal{F} \) is a vector bundle. There is a \( x \in C \) such that \( \mathcal{F}_x = \mathcal{F}|_{\{x\} \times B} \) is semi-stable, so is \( \mathcal{E}nd^0(\mathcal{F})_x = \mathcal{E}nd^0(\mathcal{F}_x) \). Thus, by a theorem of Faltings (cf. Theorem I.2. of [1]), there is a vector bundle \( V \) on \( B \) such that
\[
H^0(\mathcal{E}nd^0(\mathcal{F})_x \otimes V) = H^1(\mathcal{E}nd^0(\mathcal{F})_x \otimes V) = 0,
\]
which defines a global section \( \vartheta(V) \) of the line bundle
\[
\Theta(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V) = (\det f_!(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V))^{-1}
\]
such that \( \vartheta(V)(x) \neq 0 \). By Grothendieck-Riemann-Roch theorem,
\[
c_1(\det f_!(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V)) = f_*(\text{ch}(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V) \text{td}(\pi^*T_B))_2
\]
\[
= -c_2(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V)
\]
which means that the line bundle \( \Theta(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V) \) has degree
\[
c_2(\mathcal{E}nd^0(\mathcal{F}) \otimes \pi^*V) = \text{rk}(V) \cdot c_2(\mathcal{E}nd^0(\mathcal{F})) = \text{rk}(V) \cdot \Delta(\mathcal{F})
\]
with a nonzero global section \( \vartheta(V) \). Thus \( \Delta(\mathcal{F}) \geq 0 \).

If \( \Delta(\mathcal{F}) = 0 \), then \( \mathcal{F} = \mathcal{F}^* \) must be locally free and \( \vartheta(V)(x) \neq 0 \) for any \( x \in C \), which means that for any \( x \in C \), we have
\[
H^0(\mathcal{E}nd^0(\mathcal{F})_x \otimes V) = H^1(\mathcal{E}nd^0(\mathcal{F})_x \otimes V) = 0.
\]
Then, by the theorem of Faltings, the bundles
\[ \{ \mathcal{E}nd^0(F_x) \}_{x \in C} \]
are all semi-stable. Thus, for any \( x \in C \), the bundle \( F_x := F|_{\{x\} \times B} \) is semi-stable. The bundle \( F \) defines a morphism \( \phi_F : C \to \mathcal{U}_B \) from \( C \) to the moduli space \( \mathcal{U}_B \) of semi-stable bundles on \( B \), the line bundle \( \Theta(\mathcal{E}nd^0(F) \otimes \pi^*V) \) clearly descends to a line bundle on \( \mathcal{U}_B \). If the determinants \( \text{det}(F_x) \) \((x \in C)\) are fixed, then
\[ \deg(\Theta(\mathcal{E}nd^0(F) \otimes \pi^*V)) = 0 \]
means that all \( \{F_x\}_{x \in C} \) are s-equivalence.

By using a technique of [4] (see Step 5 in the proof of Theorem 4.2 in [4], see also the proof of Theorem I.4 in [1]), we will show
\[ F|_{C \times \{y_1\}} \sim F|_{C \times \{y_2\}}, \quad \forall \ y_1, y_2 \in B. \]
Choose a nontrivial extension \( 0 \to V \to V' \to \mathcal{O}_{y_1} \to 0 \) on \( B \), let \( \mathcal{Q} \) be the Quot-scheme of rank 0 and degree 1 quotients of \( V' \), and
\[ 0 \to \mathcal{K} \to p_B^*V' \to \mathcal{O}(1) \to 0 \]
be the tautological exact sequence on \( B \times \mathcal{Q} \). Fix a point \( x_1 \in C \), then the set \( q \in \mathcal{Q} \) such that \( H^0(F_{x_1} \otimes \mathcal{K}_q) = H^1(F_{x_1} \otimes \mathcal{K}_q) = 0 \) is an open set \( U \subset \mathcal{Q} \) and \( U \neq \emptyset \) since \( q_1 = (0 \to V \to V' \to \mathcal{O}_{y_1} \to 0) \in U \).

Let \( \Gamma \subset B \times \mathbb{P}(V') \) be the graph of \( \mathbb{P}(V') \to B \), then
\[ p_B^*V' \to p_B^*|_{\Gamma} = p^*V' \to \mathcal{O}(1) \to 0 \]
drives a quotient \( p_B^*V' \to \mathcal{O}(1) \to 0 \) on \( B \times \mathbb{P}(V') \), which defines a morphism \( \mathbb{P}(V') \to \mathcal{Q} \). It is easy to see that \( \mathbb{P}(V') \to \mathcal{Q} \) is surjective (in fact, it is an isomorphism). Thus there is an open \( B_1 \subset B \) with \( y_1 \in B_1 \) such that for any \( y \in B_1 \) there exists an exact sequence
\[ 0 \to \mathcal{K}_y \to V' \to \mathcal{O}_y \to 0 \]
(2.2) such that \( H^0(F_{x_1} \otimes \mathcal{K}_q) = H^1(F_{x_1} \otimes \mathcal{K}_q) = 0 \), which implies
\[ H^0(F_x \otimes \mathcal{K}_q) = H^1(F_x \otimes \mathcal{K}_q) = 0 \quad \forall \ x \in C \]
since \( F_x \) is s-equivalent to \( F_{x_1} \) for any \( x \in C \). Pull back the exact sequence (2.2) by \( \pi : C \times B \to B \) and tensor with \( F \), we have the exact sequence
\[ 0 \to F \otimes \mathcal{K}_q \to F \otimes \mathcal{O}_y \to F_y \to 0. \]
(2.3) Take direct images of (2.2) under \( f : C \times B \to C \), we have
\[ F_y \cong f_*(F \otimes \mathcal{O}_y), \quad \forall \ y \in B_1 \]
which implies that all \( \{F_y\}_{y \in B} \) are isomorphic each other. \( \square \)
We will need the following lemma in the later computation, whose proof are straightforward computations (see [2] for the case of rank 1).

**Lemma 2.5.** Let $X$ be a smooth projective surface and $j : D \hookrightarrow X$ be an effective divisor. Then, for any vector bundle $V$ on $D$, we have

\[
\begin{align*}
    c_1(j_* V) &= \text{rk}(V) \cdot D \\
    c_2(j_* V) &= \frac{\text{rk}(V)(\text{rk}(V) + 1)}{2}D^2 - j_* c_1(V).
\end{align*}
\]

Recall that $X_t = f^{-1}(t)$ denotes the fiber of $f : X \to C$ and for any vector bundle $\mathcal{F}$ on $X$, $\mathcal{F}_t$ denote the restrictions of $\mathcal{F}$ to $X_t$.

**Lemma 2.6.** Let $\mathcal{F}_t \to W \to 0$ be a locally free quotient and

\[
0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{X}_t W \to 0
\]

be the elementary transformation of $\mathcal{F}$ along $W$ at $X_t \subset X$. Then

\[
\Delta(\mathcal{F}) = \Delta(\mathcal{F}') + 2r(\mu(\mathcal{F}_t) - \mu(W))\text{rk}(W).
\]

3. Minimal rational curves and examples of elliptic curves on moduli spaces

When $B = \mathbb{P}^1$, the condition $\text{Pic}(C \times B) = \text{Pic}(C) \times \text{Pic}(B)$ always hold and any morphism $B \to M$ is defined by a vector bundle on $C \times B$ (cf. Lemma 2.1 of [9]).

Recall that given two nonnegative integers $k, \ell$, a vector bundle $W$ of rank $r$ and degree $d$ on $C$ is $(k, \ell)$-stable, if, for each proper subbundle $W'$ of $W$, we have

\[
\frac{\deg(W') + k}{\text{rk}(W')} < \frac{\deg(W) + k - \ell}{r}.
\]

The usual stability is equivalent to $(0, 0)$-stability. The $(k, \ell)$-stability is an open condition. The proofs of following lemmas are easy and elementary (cf. [7]).

**Lemma 3.1.** If $g \geq 3$, $M$ contains $(0, 1)$-stable and $(0, 1)$-stable bundles. $M$ contains a $(1, 1)$-stable bundle $W$ except $g = 3$, $d$, $r$ both even.

**Lemma 3.2.** Let $0 \to V \to W \to \mathcal{O}_p \to 0$ be an exact sequence, where $\mathcal{O}_p$ is the 1-dimensional skyscraper sheaf at $p \in C$. If $W$ is $(k, \ell)$-stable, then $V$ is $(k, \ell - 1)$-stable.
A curve $B \to M$ defined by $E$ on $C \times B$ passes through the generic point of $M$ implies that $E_y := E|_{C \times \{y\}}$ is $(1, 1)$-stable for generic $y \in B$. Thus in the formula (2.1) of Theorem 2.2 we have

$$(\mu(E) - \mu(E_i))\text{rk}(E_i) > 1.$$ (3.1)

On the other hand, any semi-stable bundle on $B = \mathbb{P}^1$ must have integer slope. By the formula (2.1) in Theorem 2.2, we have

$$\Delta(E) > 2r$$

if $E$ is not semi-stable on the generic fiber of $f : X = C \times \mathbb{P}^1 \to C$.

When $E$ is semi-stable on the generic fiber of $f : X \to C$, by tensor $E$ with a line bundle, we can assume that $E$ is trivial on the generic fiber of $f : X \to C$. Thus $\Delta(E) = 2rc_2(E) \geq 2r$ and there must be a fiber $X_t = f^{-1}(t)$ such that $E_t = E|_{X_t}$ is not semi-stable by Theorem 2.4. If $\Delta(E) = 2r$, by Lemma 2.6 we must have $\text{rk}(W) = 1, \mu(W) = -1$ and $\Delta(F') = 0$ in Lemma 2.6. Thus $\Delta(E) = 2r$ if and only if $E$ satisfies

$$0 \to f^*V \to E \to x, \mathcal{O}_{\mathbb{P}^1}(-1) \to 0$$

which defines a so called Hecke curve. Therefore we get the main theorem in [9].

**Theorem 3.3.** If $g \geq 3$, then any rational curve of $M$ passing through the generic point of $M$ has at least degree $2r$ with respect to $-K_M$. It has degree $2r$ if and only if it is a Hecke curve except $g = 3, r = 2$ and $(2, d) = 2$.

At the end of this section, we give some examples of elliptic curves on $M$. Let us recall the construction of Hecke curves. Let $\mathcal{U}_C(r, d - 1)$ be the moduli space of stable bundles of rank $r$ and degree $d - 1$. Let

$$\mathcal{O} \subset \mathcal{U}_C(r, d - 1)$$

be the open set of $(1, 0)$-stable bundles. Let $C \times \mathcal{O} \xrightarrow{\psi} J^d(C)$ be defined as $\psi(x, V) = \mathcal{O}_C(x) \otimes \text{det}(V)$ and

$$\mathcal{R}_C := \psi^{-1}(\mathcal{L}) \subset C \times \mathcal{O},$$

which consists of the points $(x, V)$ such that $V$ are $(1, 0)$-stable bundles on $C$ with $\text{det}(V) = \mathcal{L}(-x)$. There exists a projective bundle

$$p : \mathcal{P} \to \mathcal{R}_C$$

such that for any $(x, V) \in \mathcal{R}_C$ we have $p^{-1}(x, V) = \mathbb{P}(V_x^\vee)$. Let

$$V_x^\vee \otimes \mathcal{O}_{\mathbb{P}(V_x^\vee)} \to \mathcal{O}_{\mathbb{P}(V_x^\vee)}(1) \to 0$$

be the universal quotient, $f : C \times \mathbb{P}(V_x^\vee) \to C$ be the projection, and

$$0 \to E^\vee \to f^*V^\vee \to (x) \times \mathbb{P}(V_x^\vee) \mathcal{O}_{\mathbb{P}(V_x^\vee)}(1) \to 0$$
where $E^\vee$ is defined to the kernel of the surjection. Take dual, we have
\[ 0 \to f^*V \to E \to \{x\} \times \mathbb{P}(V^\vee) \mathcal{O}_{\mathbb{P}(V^\vee)}(-1) \to 0, \]
which, at any point $\xi = (V^\vee_x \to \Lambda \to 0) \in \mathbb{P}(V^\vee_x)$, gives exact sequence
\[ 0 \to V \xrightarrow{i} E_\xi \to O_x \to 0 \]
on $C$ such that $\ker(i_x) = \Lambda^\vee \subset V_x$. $V$ being $(1,0)$-stable implies
stability of $E_\xi$. Thus (3.2) defines
\[ \Psi(x,V): \mathbb{P}(V^\vee_x) = p^{-1}(x,V) \to M. \]

**Definition 3.4.** The images (under $\{\Psi(x,V)\}_{(x,V) \in \mathbb{R}_C}$) of lines in the fibres of $p : \mathcal{P} \to \mathcal{R}_C$ are the so called **Hecke curves** in $M$. The images (under $\{\Psi(x,V)\}_{(x,V) \in \mathbb{R}_C}$) of elliptic curves in the fibres of
\[ p : \mathcal{P} \to \mathcal{R}_C \]
are called **elliptic curves of Hecke type**.

It is known (cf. [7, Lemma 5.9]) that the morphisms in (3.3) are closed immersions. By a straightforward computation, we have
\[ \Psi^*(x,V)(-K_M) = \mathcal{O}_{\mathbb{P}(V^\vee_x)}(2r). \]
For any point $[W] \in M$ and $(W_x \to C \to 0) \in \mathbb{P}(W_x)$, where $W$ is $(1,1)$-stable, we define a $(1,0)$-stable bundle $V$ by
\[ 0 \to V \xrightarrow{\alpha} W \to xC \to 0. \]
Then the images of $p^{-1}(x,V) = \mathbb{P}(V^\vee_x)$ are projective spaces that pass through $[W] \in M$, and the images of lines $\ell \subset \mathbb{P}(V^\vee_x)$ that pass through $[\ker(\alpha_x)] \in \mathbb{P}(V^\vee_x)$ are Hecke curves passing through $[W] \in M$.

**Example 3.5.** When $g \geq 4$ and $r > 2$, for generic $[W] \in M$, the images of smooth elliptic curves $B \subset \mathbb{P}(V^\vee_x)$ with degree 3 and $[\ker(\alpha_x)] \in B$ are smooth elliptic curves on $M$ that pass through $[W] \in M$, which have degree $6r$ by (3.4).

If we do not require the curve $\phi : B \to M$ passing through generic point of $M$, we may construct rational curves and elliptic curves with smaller degree. Let us recall the Construction 2.3 from [6].

For any given $r$ and $d$, let $r_1$, $r_2$ be positive integers and $d_1$, $d_2$ be integers that satisfy the equalities $r_1 + r_2 = r$, $d_1 + d_2 = d$ and
\[ \frac{r_1}{r,d} - \frac{d_1}{(r,d)} = 1, \quad \frac{r_2}{r,d} - \frac{d_2}{(r,d)} = 1. \]
Let $\mathcal{U}_C(r_1,d_1)$ (resp. $\mathcal{U}_C(r_2,d_2)$) be the moduli space of stable vector bundles with rank $r_1$ (resp. $r_2$) and degree $d_1$ (resp. $d_2$). Then, since
(r_1, d_1) = 1 and (r_2, d_2) = 1, there are universal vector bundles \( V_1, V_2 \)
on \( C \times \mathcal{U}_C(r_1, d_1) \) and \( C \times \mathcal{U}_C(r_2, d_2) \) respectively. Consider

\[
\mathcal{U}_C(r_1, d_1) \times \mathcal{U}_C(r_2, d_2) \xrightarrow{\det(\bullet) \times \det(\bullet)} J_C^{d_1} \times J_C^{d_2} \xrightarrow{(\bullet) \otimes (\bullet)} J_C^d.
\]

let \( \mathcal{R}(r_1, d_1) \) be its fiber at \([L] \in J_C^d\). The pullback of \( V_1, V_2 \) by the projection \( C \times \mathcal{R}(r_1, d_1) \rightarrow C \times \mathcal{U}_C(r_1, d_1) \) is still denoted by \( V_1, V_2 \) respectively. Let \( p : C \times \mathcal{R}(r_1, d_1) \rightarrow \mathcal{R}(r_1, d_1) \) and

\[
\mathcal{G} = R^1 p_* (V_2^\vee \otimes V_1),
\]

which is locally free of rank \( r_1 r_2 (g - 1) + (r, d) \). Let

\[
q : P(r_1, d_1) = \mathbb{P}(\mathcal{G}) \rightarrow \mathcal{R}(r_1, d_1)
\]

be the projective bundle parametrizing 1-dimensional subspaces of \( \mathcal{G} \) (\( t \in \mathcal{R}(r_1, d_1) \)) and \( f : C \times P(r_1, d_1) \rightarrow C \), \( \pi : C \times P(r_1, d_1) \rightarrow P(r_1, d_1) \) be the projections. Then there is a universal extension

\[
0 \rightarrow (id \times q)^* V_1 \otimes \pi^* \mathcal{O}_{P(r_1, d_1)}(1) \rightarrow \mathcal{E} \rightarrow (id \times q)^* V_2 \rightarrow 0
\]

on \( C \times P(r_1, d_1) \) such that for any \( x = ([V_1], [V_2], [e]) \in P(r_1, d_1) \), where \([V_i] \in \mathcal{U}_C(r_i, d_i)\) with \( \det(V_1) \otimes \det(V_2) = L \) and \([e] \subset H^1(C, V_2^\vee \otimes V_1)\) being a line through the origin, the bundle \( \mathcal{E}_{|C \times \{x\}} \) is the isomorphic class of vector bundles \( E \) given by extensions

\[
0 \rightarrow V_1 \rightarrow V \rightarrow V_2 \rightarrow 0
\]

that defined by vectors on the line \([e] \subset H^1(C, V_2^\vee \otimes V_1)\). Then \( V \) must be stable by [6] Lemma 2.2, and the sequence (3.5) defines

\[
\Phi : P(r_1, d_1) \rightarrow SU_C(r, L)^s = M.
\]

On each fiber \( q^{-1}(\xi) = \mathbb{P}(H^1(V_2^\vee \otimes V_1)) \) at \( \xi = (V_1, V_2) \), the morphisms

\[
\Phi_{\xi} := \Phi_{|q^{-1}(\xi) : q^{-1}(\xi) = \mathbb{P}(H^1(V_2^\vee \otimes V_1)) \rightarrow M}
\]
is birational and \( \Phi_{\xi}^*(-K_M) = \mathcal{O}_{\mathbb{P}(H^1(V_2^\vee \otimes V_1))}(2(r, d)) \) by [6] Lemma 2.4.

**Example 3.6.** The images of lines \( \ell \subset \mathbb{P}(H^1(V_2^\vee \otimes V_1)) \) are rational curves of degree \( 2(r, d) \) on \( M \), which is clearly the minimal degree since \(-K_M = 2(r, d) \Theta\). For smooth elliptic curves \( B \subset \mathbb{P}(H^1(V_2^\vee \otimes V_1)) \) of degree 3, the images of \( \Phi_{\xi} : B \rightarrow M \) are of degree \( 6(r, d) \). For any smooth elliptic curve \( B \subset q^{-1}(\xi) \) (\( \forall \xi \in \mathcal{R}(r_1, d_1) \)), the images of \( \Phi_{\xi} : B \rightarrow M \) are called **elliptic curves of split type**.
4. Minimal elliptic curves on moduli spaces

In this section, we consider the moduli space $M$ of rank 2 stable bundles on $C$ with a fixed determinant $L$ of degree 1. We also assume that the curve $C$ is generic in the sense that $C$ admits no surjective morphism to an elliptic curve. With this assumption, we know that $\text{Pic}(C \times B) = \text{Pic}(C) \times \text{Pic}(B)$ for any elliptic curve $B$.

For a morphism $\phi : B \to M$, it may happen that the normalization of $\phi(B)$ is a rational curve. To avoid this case, we make the following definition

**Definition 4.1.** $\phi : B \to M$ is called an essential elliptic curve of $M$ if the normalization of $\phi(B)$ is an elliptic curve.

For any morphism $\phi : B \to M$, let $E$ be the vector bundle on $X = C \times B$ that defines $\phi$. It will be free to tensor $E$ with a pull-back of line bundles on $B$. In this section, $B$ will always denote an elliptic curve.

**Proposition 4.2.** Let $\phi : B \to M$ be an essential elliptic curve of $M$ defined by a vector bundle $E$. If $E$ is not semi-stable on the generic fiber of $f : X \to C$, then

$$\Delta(E) \geq 6.$$ 

If $g = g(C) \geq 4$ and the curve $\phi : B \to M$ passes through the generic point of $M$, then

$$\Delta(E) > 12.$$ 

**Proof.** Let $0 \to E_1 \to E \to F_2 \to 0$ be the relative Harder-Narasimhan filtration over $C$. Then we have exact sequence

$$0 \to E_1|_{X_t} \to E|_{X_t} \to F_2|_{X_t} \to 0$$

on each fiber $X_t = \{t\} \times B$ of $f : X \to C$ since $E_1$, $F_2$ are flat over $C$. Thus $E_1$ is locally free (cf. Lemma 1.27 of [8]) and

$$\Delta(E) = 4c_2(F_2) + 4(\mu(E) - \mu(E_1))(\mu_1 - \mu_2)$$

where $\mu_1 = \text{deg}(E_1|_{X_t})$, $\mu_2 = \text{deg}(F_2|_{X_t})$ for $t \in C$ (cf. Theorem 2.2).

That $0 \to E_1 \to E \to F_2 \to 0$ is the relative Harder-Narasimhan filtration over $C$ means for almost $t \in C$ the exact sequences

$$0 \to E_1|_{X_t} \to E|_{X_t} \to F_2|_{X_t} \to 0$$

are the Harder-Narasimhan filtration of $E|_{X_t}$, which in particular means that $F_2$ is locally free over $f^{-1}(C \setminus T)$ where $T \subset C$ is a finite set. Thus

$$0 \to E_1|_{C \times \{y\}} \to E|_{C \times \{y\}} \to F_2|_{C \times \{y\}} \to 0, \quad \forall \ y \in B$$

are exact sequences, which imply that $F_2$ is also $B$-flat.
If $c_2(F_2) = 0$, then $F_2$ is a line bundle and there are line bundles $V_1$, $V_2$ on $C$ such that

$$E_1 = f^*V_1 \otimes \pi^*\mathcal{O}(\mu_1), \quad F_2 = f^*V_2 \otimes \pi^*\mathcal{O}(\mu_2)$$

where $\mathcal{O}(\mu_i)$ denote line bundles on $B$ of degree $\mu_i$. Replace $E$ by $E \otimes \pi^*\mathcal{O}(-\mu_2)$, we can assume that $E$ satisfies

$$0 \to f^*V_1 \otimes \pi^*\mathcal{O}(\mu_1 - \mu_2) \to E \to f^*V_2 \to 0.$$  

Let $d_i = \deg(V_i)$ and $J = \{(L_1, L_2) \in J^d_C \times J^d_C | L_1 \otimes L_2 = \mathcal{L}\}$. Then there is a projective bundle $q : P \to J$ and an universal extension

$$0 \to (id \times q)^*V_1 \otimes \pi^*\mathcal{O}_P(1) \to \mathcal{E} \to (id \times q)^*V_2 \to 0$$

on $C \times P$ such that for any $x = ([V_1], [V_2], [e]) \in P$, where $[V_i] \in J^d_C$ with $V_1 \otimes V_2 = \mathcal{L}$ and $[e] \in H^1(C, V_2^{-1} \otimes V_1)$ being a line through the origin, the bundle $\mathcal{E}|_{C \times \{x\}}$ is the isomorphic class of vector bundles $V$ given by extensions $0 \to V_1 \to V \to V_2 \to 0$ that defined by vectors on the line $[e] \subset H^1(C, V_2^{-1} \otimes V_1)$, where $\mathcal{V}_i$ denote the pullback (under $C \times J \to C \times J^d_C$) of universal line bundles, and $\pi : C \times P \to P$ denote the projection. Thus the exact sequence (4.3) induces a morphism

$$\psi : B \to \mathbb{P}^{d_2-d_1+g-2} = q^{-1}(V_1, V_2) \subset P$$

such that $\mathcal{O}(\mu_1 - \mu_2) = \psi^*\mathcal{O}_P(1)$ and $\phi : B \to M$ factors through $\psi : B \to \psi(B) \subset \mathbb{P}^{d_2-d_1+g-2}$, which implies that the normalization of $\psi(B)$ is an elliptic curve. Hence $\mu_1 - \mu_2 \geq 3$ and $\Delta(E) \geq 6$ by (4.1). If $\phi : B \to M$ passes through the generic point, then $\mu(E) - \mu(E_1) > 1$ and $\Delta(E) > 12$.

If $c_2(F_2) \neq 0$, $F_2$ is not locally free, which implies that there is a $y_0 \in B$ such that $F_2|_{C \times \{y_0\}}$ has torsion $\tau(F_2|_{C \times \{y_0\}}) \neq 0$ since $F_2$ is $B$-flat (cf. Lemma 1.27 of [8]). Let

$$0 \to \tau(F_2|_{C \times \{y_0\}}) \to F_2|_{C \times \{y_0\}} \to F_2^0 \to 0.$$  

Then $F_2^0$ being a quotient line bundle of $E|_{C \times \{y_0\}}$ implies

$$\deg(F_2^0) > \mu(E|_{C \times \{y_0\}}) = \frac{1}{2}$$

since $E|_{C \times \{y_0\}}$ is stable. By sequences (4.2) and (4.6), we have

$$\mu(E_1) = \deg(E_1|_{C \times \{y_0\}}) = 1 - \deg(F_2^0) - \dim \tau(F_2|_{C \times \{y_0\}}) \leq -1$$

which, by the formula (4.1), implies that

$$\Delta(E) \geq 4c_2(F_2) + 4\left(\frac{1}{2} + 1\right)(\mu_1 - \mu_2) \geq 10.$$  

When $\phi : B \to M$ passes through a generic point, in order to show $\Delta(E) > 12$, we note that $c_2(F_2) \neq 0$ and $F_2$ being $C$-flat also imply
that there exists a $t_0 \in C$ such that $F_2|_{X_{t_0}}$ has torsion $\tau(F_2|_{X_{t_0}}) \neq 0$. Let $0 \to \tau(F_2|_{X_{t_0}}) \to F_2|_{X_{t_0}} \to Q \to 0$ and $E' = \ker(E \to E|_{X_{t_0}}Q)$, then

$$0 \to E' \to E \to E|_{X_{t_0}}Q \to 0$$

which, for any $y \in B$, induces exact sequence

$$(4.7) \quad 0 \to E'|_{C \times \{y\}} \to E|_{C \times \{y\}} \to (t_{0,y})Q \to 0.$$  

Thus all $E'_y := E'|_{C \times \{y\}}$ are semi-stable of degree 0. If $\phi : B \to M$ passes through a generic point, then there is a $y_0 \in B$ such that $E|_{y_0}$ is $(1,1)$-stable on $X_{y_0} = C \times \{y_0\}$, thus $E|_{y_0}$ is stable by Lemma 3.2. This implies that $\Delta(E') > 0$. Otherwise $\{E'_y\}_{y \in B}$ are s-equivalent by applying Theorem 2.4 to $E|_{y_0}$ to $\pi : X \to B$, which implies $E' = f^*V \otimes \pi^*L$ for a stable bundle $V$ on $C$ and a line bundle $L$ on $B$. Then $E_t = E'_t = L \oplus L$ for any $t \neq t_0$, which is a contradiction since $E$ is not semi-stable on the generic fiber of $f : X \to C$.

To compute $\Delta(E')$, consider the Harder-Narasimhan filtration

$$0 \to E'_t \to E' \to F'_2 \to 0$$

over $C$, let $\mu'_1 = \deg(E'_t|_{X_t})$, $\mu'_2 = \deg(F'_2|_{X_t})$ for $t \in C$, then

$$\Delta(E') = 4c_2(F'_2) + 4(\mu(E') - \mu(E'_1))(\mu'_1 - \mu'_2) \geq 8.$$  

To see it, we can assume $c_2(F'_2) = 0$, then there are line bundles $V_t'$ on $C$ and line bundles $\mathcal{O}(\mu'_1)$ on $B$ of degree $\mu'_1$ such that

$$0 \to f^*V'_1 \otimes \pi^*\mathcal{O}(\mu'_1 - \mu'_2) \to E' \otimes \pi^*\mathcal{O}(\mu'_2) \to f^*V'_2 \to 0$$

which defines a morphism $\psi : B \to \mathbb{P}$ to a projective space such that $\mathcal{O}(\mu'_1 - \mu'_2) = \psi^*\mathcal{O}(1)$. Thus $\mu'_1 - \mu'_2 \geq 2$ and $\Delta(E') \geq 8$. Then

$$\Delta(E) = \Delta(E') + 4(\mu(E|_{X_{t_0}}) - \mu(Q)) \geq \Delta(E') + 6 \geq 14.$$  

Now we consider the case that $E$ is semi-stable on the generic fiber of $f : X \to C$. We can assume $0 \leq \deg(E|_{X_t}) \leq 1$ on $X_t = f^{-1}(t)$.

**Proposition 4.3.** When $E$ is semi-stable of degree 1 on the generic fiber of $f : X \to C$, we have $\Delta(E) \geq 10$. If $g > 4$ and $\phi : B \to M$ passes through the generic point, then $\Delta(E) \geq 14$.

**Proof.** It is easy to see that there is a unique stable rank 2 vector bundle with a fixed determinant of degree 1 on an elliptic curve. Thus $\Delta(E) > 0$ if and only if there exists $t_1 \in C$ such that $E_{t_1} = E|_{X_{t_1}}$ is not semi-stable.

Let $E_{t_1} \to \mathcal{O}(\mu_1) \to 0$ be the quotient of minimal degree and

$$0 \to E^{(1)} \to E \to E|_{X_{t_1}} \mathcal{O}(\mu_1) \to 0$$
be the elementary transformation of $E$ along $\mathcal{O}(\mu_1)$ at $X_{t_1}$. If $E^{(i)}$ is defined and $\Delta(E^{(i)}) > 0$, let $t_{i+1} \in C$ such that $E^{(i)}_{t_{i+1}} = E^{(i)}|_{X_{t_{i+1}}}$ is not semi-stable and $E^{(i)}_{t_{i+1}} \to \mathcal{O}(\mu_{i+1}) \to 0$ be the quotient of minimal degree, then we define $E^{(i+1)}$ to be the elementary transformation of $E^{(i)}$ along $\mathcal{O}(\mu_{i+1})$ at $X_{t_{i+1}}$, namely $E^{(i+1)}$ satisfies the exact sequence
\begin{equation}
0 \to E^{(i+1)} \to E^{(i)} \to \mathcal{O}(\mu_{i+1}) \to 0. \tag{4.8}
\end{equation}
Let $s$ be the minimal integer such that $\Delta(E^{(s)}) = 0$. Then
\begin{equation}
\Delta(E) = 2 \cdot s - 4 \sum_{i=1}^{s} \mu_i \tag{4.9}
\end{equation}
where $\mu_i \leq 0$ ($i = 1, 2, \ldots, s$). Take direct image of (4.8), we have
\begin{equation}
0 \to f_* E^{(s)} \to f_* E^{(s-1)} \to t_s H^0(\mathcal{O}(\mu_s)) \to 0 \tag{4.10}
\end{equation}
(since $R^1 f_* E^{(s)} = 0$) and $\deg(f_* E^{(i+1)}) \leq \deg(f_* E^{(i)})$, which imply
\begin{equation}
\deg(f_* E^{(s)}) \leq \deg(f_* E) - \dim H^0(\mathcal{O}(\mu_s)). \tag{4.11}
\end{equation}
Restrict (4.8) to a fiber $X_y = \pi^{-1}(y)$, we have exact sequence
\begin{equation}
0 \to E^{(i+1)}_y \to E^{(i)}_y \to (t_{i+1}, y) C \to 0, \tag{4.12}
\end{equation}
which implies that
\begin{equation}
\deg(E^{(s)}_y) = \deg(E_y) - s = 1 - s. \tag{4.13}
\end{equation}
On the other hand, by Theorem 2.4, $\Delta(E^{(s)}) = 0$ implies that there exist a stable rank 2 vector bundle $V$ of degree 1 on $B$ and a line bundle $L$ on $C$ such that $E^{(s)} = \pi^* V \otimes f^* L$. It is easy to see
\begin{equation}
\deg(E^{(s)}_y) = 2 \deg(L) = 2 \deg(f_* E^{(s)}). \tag{4.14}
\end{equation}
Thus, combine (4.11) and (4.12), we have the inequality
\begin{equation}
s \geq 1 - 2 \deg(f_* E) + 2 \dim H^0(\mathcal{O}(\mu_s)). \tag{4.13}
\end{equation}
We claim that $\deg(f_* E) \leq -1$. To show it, consider
\begin{equation}
0 \to \mathcal{F}' := f^*(f_* E) \to E \to \mathcal{F} \to 0 \tag{4.14}
\end{equation}
where $\mathcal{F}$ is locally free on $f^{-1}(C \setminus T)$ and $T \subset C$ is a finite set such that $E_t$ ($t \in T$) is not semi-stable. Thus, for any $y \in B$, the sequence
\begin{equation}
0 \to \mathcal{F}'_y \to E_y \to \mathcal{F}_y \to 0 \tag{4.15}
\end{equation}
is still exact, which implies that $\mathcal{F}$ is $B$-flat (cf. Lemma 2.1.4 of [5]). The sequence (4.15) already implies $\deg(f_* E) = \deg(\mathcal{F}') \leq 0$ since $E_y$ is stable of degree 1. Thus $\mathcal{F}$ can not be locally free since
\[
4 \cdot c_2(\mathcal{F}) = \Delta(E) - 4 \cdot \deg(f_* E) + 2 > 0.
\]
Then there is at least a \( y_0 \in B \) such that \( \mathcal{F}_{y_0} \) has torsion, otherwise \( \mathcal{F} \) is locally free (cf. Lemma 1.27 of [8]). The stability of \( E_{y_0} \) implies that \( \mathcal{F}_{y_0}/\text{torsion} \) has degree at least 1. Thus \( \deg(\mathcal{F}_{y_0}) \geq 2 \) and
\[
\deg(f_!E) = \deg(\mathcal{F}_{y_0}') \leq -1,
\]
which means \( s \geq 3 + 2 \dim H^0(\mathcal{O}(\mu_s)) \). Therefore, if \( \mu_s < 0 \), we have \( \Delta(E) \geq 2 \cdot s + 4 \geq 10 \) by (4.9). If \( \mu_s = 0 \), by tensoring \( E \) with \( \pi^*\mathcal{O}(\mu_s)^{-1} \), we may assume \( \dim H^0(\mathcal{O}(\mu_s)) = 1 \), then \( s \geq 5 \) and \( \Delta(E) \geq 10 \).

If \( \phi : B \to M \) passes through the generic point of \( M \), we claim that \( \deg(f_!E) \leq -2 \), which implies \( \Delta(E) \geq 14 \). To prove the claim, assume \( \deg(f_!E) = -1 \), we will show that \( \phi(B) \) lies in a given divisor. Note that \( \mathcal{F}_y \) must be locally free of degree 2 for generic \( y \in B \) (if \( \mathcal{F}_y \) has nontrivial torsion, then \( E_y \) has a quotient line bundle of degree at most 1, which is impossible since \( E_y \) is \((1,1)\)-stable for generic \( y \in B \)). Thus \( E_y \) satisfies \( 0 \to \xi \to E_y \to \xi^{-1} \otimes L \to 0 \) where \( \xi \) is a line bundle of degree \(-1\) on \( C \). The locus of such bundles has dimension at most \( g + h^1(\xi \otimes L^{-1}) - 1 = 2g + 1 < \dim(M) \) when \( g > 4 \). We are done. \( \square \)

Now we consider the case that \( E \) is semi-stable of degree 0 on the generic fiber of \( f : X \to C \). If \( E \) is semi-stable on every fiber of \( f : X \to C \), then \( E \) induces a non-trivial morphism
\[
\varphi_E : C \to \mathbb{P}^1
\]
(cf. [3]) such that \( \varphi_E^*\mathcal{O}_{\mathbb{P}^1}(1) = \Theta(E) = (\det f_!E)^{-1} \), which has degree \( c_2(E) \) by Grothendieck-Riemann-Roch theorem. Thus
\[
\Delta(E) = 4 \cdot c_2(E) = 4 \cdot \deg(\varphi_E) \geq 8.
\]
(4.16)

If there is a \( t_0 \in C \) such that \( E_{t_0} = E|_{X_{t_0}} \) is not semi-stable on \( X_{t_0} = f^{-1}(t_0) \), let \( E_{t_0} \to \mathcal{O}(\mu) \to 0 \) be the quotient line bundle of minimal degree \( \mu \) and \( E' = \ker(E \to X_{t_0} \mathcal{O}(\mu) \to 0) \), then we have

**Lemma 4.4.** If \( \Delta(E') = 0 \), then there is a semi-stable vector bundle \( V \) on \( C \) and a line bundle \( L \) of degree 0 on \( B \) such that
\[
E' = f^*V \otimes \pi^*L.
\]

**Proof.** By the definition, \( \{E'_t = E'|_{C \times \{t\}}\}_{t \in C} \) and \( \{E'_y = E'|_{C \times \{y\}}\}_{y \in B} \) are families of semi-stable bundles of degree 0. Apply Theorem 2.4 to \( f : X \to C \) (resp. \( \pi : X \to B \)), then \( \Delta(E') = 0 \) implies that \( \{E'_t\}_{t \in C} \) (resp. \( \{E'_y\}_{y \in B} \) are isomorphic each other. By tensor \( E \) (thus \( E' \)) with \( \pi^*L^{-1} \) (where \( L \) is a line bundle of degree 0 on \( B \)), we can assume that \( H^0(E'_t) \neq 0 \) (\( \forall \, t \in C \)), which have dimension at most 2 since \( E'_t \) is
semi-stable of degree 0. If $H^0(E'_i)$ has dimension 2, then $E' = f^*(f_*E')$ and we are done.

If $H^0(E'_i)$ has dimension 1, we will show a contradiction. In fact, by the definition of $E'$, we have an exact sequence

\[(4.17) \quad 0 \to E' \to E \to x_{t_0} \mathcal{O}(\mu) \to 0\]

where $\mathcal{O}(\mu)$ is a line bundle on $\{t_0\} \times B \cong B$ of degree $\mu < 0$. Then

$$V_1 := f_*E = f_*E'$$

is a line bundle on $C$. Since $\{E'_i\}_{t \in C}$ are isomorphic each other and $H^0(E'_i)$ has dimension 1, we have the exact sequence

\[(4.18) \quad 0 \to f^*V_1 \to E' \to f^*V_2 \otimes \pi^*L_0 \to 0\]

for a line bundle $V_2$ on $C$ and a degree 0 line bundle $L_0$ on $B$. If $L_0 \neq \mathcal{O}_B$, then $R^i f_*(f^*(V_2^{-1} \otimes V_1) \otimes L_0) = V_2^{-1} \otimes V_1 \otimes H^i(L_0) = 0$ $(i = 0, 1)$, which implies $H^1(X, f^*(V_2^{-1} \otimes V_1) \otimes L_0) = 0$ and $(4.18)$ is splitting. This is impossible since $E'_y$ is semi-stable of degree 0 and we can show that $\deg(V_1) = \deg(f_*E) \leq -1$ in the following.

To prove that $\deg(f_*E) \leq -1$, we consider the exact sequence

\[(4.19) \quad 0 \to f^*f_*E \to E \to \mathcal{F} \to 0\]

where $\mathcal{F}|_{f^{-1}(C \setminus \{t_0\})}$ is locally free of rank 1 by $(4.18)$. But $\mathcal{F}$ is not locally free (otherwise $c_2(E) = (c_1(E) - c_1(f^*f_*E)) \cdot c_1(f^*f_*E) = 0$) and for any $y \in B$ the restriction of $(4.19)$ to $X_y = \pi^{-1}(y)$

\[(4.20) \quad 0 \to f_*E \to E_y \to \mathcal{F}_y \to 0\]

is exact, which means that $\mathcal{F}$ is $B$-flat (cf. Lemma 2.1.4 of [5]). Thus, by Lemma 1.27 of [3], there is a $y_0 \in B$ such that $\mathcal{F}_{y_0}$ has torsion $\tau \neq 0$ since $\mathcal{F}$ is not locally free. Then, since $E_{y_0}$ is stable of degree 1,

$$\deg(\mathcal{F}_{y_0}) \geq 1 + \deg(\mathcal{F}_{y_0}/\tau) > 1 + \mu(E_{y_0}) = \frac{3}{2}$$

which implies $\deg(f_*E) \leq -1$ by $(4.20)$.

We have shown that $L_0$ has to be $\mathcal{O}_B$ and $(4.18)$ has to be

\[(4.21) \quad 0 \to f^*V_1 \to E' \to f^*V_2 \to 0\]

which is determined by a class of $H^1(X, f^*(V_1 \otimes V_2^{-1}))$. However, note

$$R^1 f_*(f^*(V_1 \otimes V_2^{-1})) = V_1 \otimes V_2^{-1} \otimes H^1(\mathcal{O}_B) = V_1 \otimes V_2^{-1} \text{ and }$$

$$H^0(C, V_1 \otimes V_2^{-1}) = 0,$$

by using Leray spectral sequence, we have

$$H^1(C, V_1 \otimes V_2^{-1}) \cong H^1(X, f^*(V_1 \otimes V_2^{-1})).$$
Hence there exists an extension $0 \to V_1 \to V \to V_2 \to 0$ on $C$ such that $E' \cong f^*V$, which contradicts the assumption
\[
\dim(H^0(\{t\} \times B, E'_t)) = 1.
\]

\[\square\]

**Proposition 4.5.** When $E$ is semi-stable of degree 0 on the generic fiber of $f : X \to C$, we have $\Delta(E) \geq 8$. If $C$ is not hyper-elliptic and $\phi : B \to M$ passes through a $(1,1)$-stable bundle, assume that $E$ defines an essential elliptic curve, then $\Delta(E) \geq 12$.

**Proof.** If $E$ is semi-stable on each fiber $X_t = f^{-1}(t)$, then $E$ induces a non-trivial morphism $\varphi_E : C \to \mathbb{P}$. By (4.16), $\Delta(E) \geq 8$.

If there is a $t_0 \in C$ such that $E_{t_0}$ is not semi-stable, then we have
\[
0 \to E' \to E \to x_{t_0} \mathcal{O}(\mu) \to 0
\]
where $\mathcal{O}(\mu)$ is a line bundle of degree $\mu$ on $B$. If $\Delta(E') \neq 0$, then $\Delta(E') > 0$ by Theorem 2.4. On the other hand, $c_1(E')^2 = 0$ since $E'$ has degree 0 on the generic fiber of $X \to C$ and $\text{Pic}(C \times B) = \text{Pic}(C) \times \text{Pic}(B)$. Thus $\Delta(E') = 4 \cdot c_2(E') \geq 4$, and by Lemma 2.6
\[
\Delta(E) = \Delta(E') - 4\mu \geq 8.
\]

If $\Delta(E') = 0$, by Lemma 4.4 we can assume that $E' = f^*V$, then the sequence (4.17) induces a non-trivial morphism $\varphi : B \to \mathbb{P}(V'_{t_0})$ such that $\mathcal{O}(-\mu) = \varphi^* \mathcal{O}_{\mathbb{P}(V'_{t_0})}(1)$. Thus $\Delta(E) = -4\mu \geq 8$.

Now we assume that $C$ is not hyper-elliptic and $\phi : B \to M$ passes through a $(1,1)$-stable bundle. If $E$ is semi-stable on each fiber $X_t$, then $\Delta(E) = 4 \cdot \deg(\varphi_E) \geq 12$ by (4.16) since $C$ is not hyper-elliptic.

If there is $t_0 \in C$ such that $E_{t_0}$ is not semi-stable, we claim $\Delta(E') > 0$ since $\phi : B \to M$ passes through a $(1,1)$-stable bundle. Otherwise, $E' = f^*V$ where $V$ is a $(1,0)$-stable by Lemma 3.2, then sequence (4.17) implies that $\phi : B \to M$ factors through a Hecke curve, which implies that $\phi : B \to M$ is not an essential elliptic curve. If $E'$ is semi-stable on each fiber $X_t$, then $E'$ defines a nontrivial morphism $\varphi_{E'} : C \to \mathbb{P}^1$ such that $\varphi^* \mathcal{O}_{\mathbb{P}^1}(1) = \Theta(E') = (\det f_1E')^{-1} = c_2(E')$. Thus $\Delta(E') = 4 \cdot \deg(\varphi_{E'}) \geq 12$ and $\Delta(E) = \Delta(E') - 4\mu \geq 16$.

If there is $t_0' \in C$ such that $E'_{t_0}$ is not semi-stable, then we have
\[
(4.22) \quad 0 \to \mathcal{F} \to E' \to x_{t_0'} \mathcal{O}(\mu') \to 0
\]
where $\mathcal{F}_y = \mathcal{F}|_{C \times \{y\}}$ is stable of degree $-1$ for generic $y \in B$ since $E'_{t_0}^\prime$ is stable of degree 0. If $\Delta(\mathcal{F}) \neq 0$, it is clear that $\Delta(\mathcal{F}) = 4 \cdot c_2(\mathcal{F}) \geq 4$ and $\Delta(E) = \Delta(\mathcal{F}) - 4\mu' - 4\mu \geq 12$. If $\Delta(\mathcal{F}) = 0$, by Theorem 2.4 there is a stable vector bundle $V'$ on $C$ such that $\mathcal{F}_y \cong V'$ for all $y \in B$. 

Then we can choose $F = f^*V'$, the sequence (4.22) induces a nontrivial morphism $\varphi : B \to \mathbb{P}(V'^{\vee}_{t_0})$ such that $\mathcal{O}(-\mu') = \varphi^*\mathcal{O}_{\mathbb{P}(V'^{\vee}_{t_0})}(1)$. Thus $\Delta(E') = -4\mu' \geq 8$ and $\Delta(E) = \Delta(E') - 4\mu \geq 12$.

We have seen in Example 3.6 the existence of essential elliptic curves of degree $6(r, d)$ (which is 6 in our case). Then we have shown

**Theorem 4.6.** Let $M = SU_C(2, \mathcal{L})$ be the moduli space of rank two stable bundles on $C$ with a fixed determinant of degree 1. Then, when $C$ is generic, any essential elliptic curve $\phi : B \to M$ has degree

$$\deg \phi^*(-K_M) \geq 6$$

and $\deg \phi^*(-K_M) = 6$ if and only if $\phi : B \to M$ factors through

$$\phi : B \xrightarrow{\psi} q^{-1}(\xi) = \mathbb{P}(H^1(V_2^{\vee} \otimes V_1)) \xrightarrow{\Phi_{\xi}} M$$

for some $\xi = (V_1, V_2)$ such that $\psi^*\mathcal{O}_{\mathbb{P}(H^1(V_2^{\vee} \otimes V_1))}(1)$ has degree 3.

**Proof.** By Proposition 4.2, Proposition 4.3 and Proposition 4.5, we have $\Delta(E) \geq 6$. The possible case $\Delta(E) = 6$ occurs only in Proposition 4.2 when $c_2(F_2) = 0$. This implies that $E$ must satisfy

$$0 \to f^*V_1 \otimes \pi^*\mathcal{O}(\mu_1 - \mu_2) \to E \to f^*V_2 \to 0$$

which defines $\psi : B \to \mathbb{P}(H^1(V_2^{\vee} \otimes V_1))$ such that $\psi^*\mathcal{O}_{\mathbb{P}(H^1(V_2^{\vee} \otimes V_1))}(1)$ has degree $\mu_1 - \mu_2$. Then $\Delta(E) = 6$ and (4.1) imply $\mu_1 - \mu_2 = 3$. □

**Theorem 4.7.** When $g > 4$ and $C$ is generic, any essential elliptic curve $\phi : B \to M = SU_C(2, \mathcal{L})$ that passes through the generic point must have $\deg \phi^*(-K_M) \geq 12$.

For $r > 2$, let $M = SU_C(r, \mathcal{L})$ where $\mathcal{L}$ is a line bundle of degree $d$. What is the minimal degree of essential elliptic curves on $M$? I expect the following conjecture to be true.

**Conjecture 4.8.** Let $\phi : B \to M = SU_C(r, \mathcal{L})$ be an essential elliptic curve defined by a vector bundle $E$ on $C \times M$. Then, when $C$ is a generic curve, we have

$$\deg \phi^*(-K_M) \geq 6(r, d).$$

When $(r, d) \neq r$, then $\deg \phi^*(-K_M) = 6(r, d)$ if and only if it is an elliptic curve of split type in Example 3.6. If $\phi : B \to M$ passes through the generic point and $g > 4$, then $\deg \phi^*(-K_M) \geq 6r$. 
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