Low-temperature conduction and DC current noise in a quantum wire with impurity

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The nonlinear conductance for tunneling through an impurity in a Luttinger liquid and the nonequilibrium DC current noise are calculated exactly for low temperatures. We present a pedestrian pathway towards the exact solution which is based on analytic properties and on a duality between weak and strong backscattering. The prefactor of the \(T^2\) enhancement is shown to be universally expressed in terms of zero temperature properties.

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Many-body correlations are essential in 1D electron systems, where the usual Fermi liquid behavior is destroyed by the interaction. The generic features of many 1D interacting fermion systems are well described in terms of the Luttinger liquid model \[3\]. In the Luttinger model, all effects of the electron-electron interaction are captured by a single dimensionless parameter \(g\) which is in the range \(g < 1\) for repulsive interaction. A sensitive experimental probe of a Luttinger liquid state is the tunneling conductance through a point contact in a 1D quantum wire \[3\]. Of interest is also the DC nonequilibrium noise \[4\]. Tunneling of edge currents in the fractional quantum Hall (FQH) regime provides another realization of a Luttinger phase. As shown by Wen \[3\], the edge state excitations are described by a (chiral) Luttinger liquid with \(g = \nu\), where \(\nu\) is the fractional filling factor.

In this paper, we make use of analytic properties and of a duality between weak and strong backscattering, which allow us to recover the exact solution of the Luttinger liquid tunneling \[3\] and the DC current noise problem \[3\] at \(T = 0\). We then study low \(T\) and compute the prefactor of the \(T^2\) enhancement exactly.

The low-energy modes of the 1D interacting electron liquid are conveniently treated in the frame work of standard bosonization \[3\]. The creation operator for spinless fermions can equivalently be expressed in terms of boson phase fields \(\theta(x)\) and \(\phi(x)\), which obey the equal-time commutation relation \([\phi(x), \theta(x')] = -(i/2)\text{sgn}(x-x')\); \(\hbar = k_B = e = 1\). The clean Luttinger liquid is described by the Hamiltonian

\[
H_L = \frac{\nu_F}{2g} \int dx \left[ (\partial_x \theta)^2 / g + g (\partial_x \phi)^2 \right].
\]

We assume a sharp band width cutoff \(\omega_c\) for the linear dispersion relation implicit in \(H_L\).

Backscattering with \(2k_F\) at a short-ranged impurity potential of strength \(V_0\) is described by the generic form

\[
H_I = -V_0 \cos[2\sqrt{\pi} \theta(0)] + V \theta(0)/\sqrt{\pi}.
\]

We have added a term describing an applied voltage drop \(V\) at the impurity. The \(\theta\) representation is convenient for the discussion of weak backscattering.

For the opposite limit of a large barrier, the \(\phi\) representation is more appropriate. Disregarding multiple electron hops, the barrier and voltage contribution takes the form (\(\Delta\) is the overlap matrix element)

\[
H_I' = -\Delta \cos[2\sqrt{\pi} \phi(0) + V \alpha].
\]

Here, \(\phi(0)\) is the discontinuity of the \(\phi\) field at the barrier. The harmonic liquid is again described by \(H_L\).

In the following, we study the nonlinear static conductance \(G(V) = I(V)/V\) through an impurity described by \(H_I\) or \(H'_I\). Using the bosonized form of the current operator in the \(\theta\) representation, we have \(G_0(V) = \lim_{t \to \infty} \langle \dot{\theta}(0, t) \rangle / \sqrt{\pi} V\), where \(\langle \cdots \rangle\) denotes the thermal average over all modes of \(H_L\) away from \(x = 0\).

The model \(H_L + H_I\) is equivalent to the problem of a Brownian particle of mass \(m\) moving in a cosine potential \(V_W B(q) = -V_0 \cos(2\pi q / q_0)\), referred to as the WB model. The correspondence can be shown by canonical transformations and examination of the equations of motion of the coordinate and momentum autocorrelation functions. The quantity \(\theta(0)\) corresponds to \(\sqrt{\pi} q / q_0\). Ohmic damping is provided by excitation of the harmonic liquid away from the barrier, and the parameter \(g\) is related to the Ohmic viscosity \(\eta\) by \(1/g = \alpha \equiv \eta \nu_0^2 / 2\pi\). The cutoff frequency \(\omega_c\) of the liquid modes is identical with \(\eta/m\). The equivalence becomes exact when the force of inertia is negligibly small compared with the friction force. For \(T \ll V_0\) and \(V \ll V_0\), this means \(\nu^2 \gg mV_W B^2(0)\), or equivalently \(\omega_c \gg 2\pi g V_0\).

By writing down a real-time path integral for the conductance or mobility \[3\], then formally expanding the exponent of the action in powers of \(V_0^2\), and for each term integrating \(\theta(x, \tau\rangle\), the resulting expression is the analog of a statistical ensemble of interacting discrete charges. Introducing the normalized conductance \(\tilde{G}(V, T, g) = G(V, T, g)/G_0(g)\), where \(G_0(g) = g/2\pi\) is the microwave conductance of the 1D quantum wire in the zero frequency limit \[3\], we have for the \(\theta\) model \[3\]

\[
\tilde{G}_\theta(V, T, g) = 1 - (\pi/V) \text{Im} U(V, T, g) \tag{1}
\]

The function \(U\) describes the interacting charge gas,

\[
U(V, T, g) = \sum_{n=1}^{\infty} (iV_0)^{2n} \int_0^{\infty} d\tau_1 d\tau_2 \cdots d\tau_{2n-1} \tag{2}
\]
The interaction $S(\tau)$ is the boson two-point function,
\begin{equation}
S(\tau) = 2g \ln(\Omega_e/\pi T) \sinh(\pi T \tau) .
\end{equation}

The $2n - 1$ integration times $\tau_j$ in order $V_0^{2n}$ are the intermediate times between the $2n$ successive charges ($\xi_j = \pm 1$; $j = 1, \ldots, 2n$), and $\tau_{jk}$ is the distance between charges $\xi_j$ and $\xi_k$. In every order $n$, the $2n$ charges have zero total charge, $\sum_{j=1}^{2n} \xi_j = 0$. There are only arrangements of which the cumulative charge quantities $p_{2n} = \sum_{j=1}^{2n} \xi_j$ are nonzero since otherwise the corresponding phase factor $\sin(\pi g p_{2n})$ would vanish.

By proper identification of the parameters, (4) – (7) is identical with the series expansion of the normalized damping [13]. Equally, there holds an exact duality between observables of the dissipative WB and TB model as shown by mapping the Hamiltonians onto each other [11].

Similarly, the strong barrier $\phi$ model $H_L + H_f$ is identical to a dissipative periodic tight-binding (TB) model, as shown by mapping the Hamiltonians onto each other [11]. To proceed, we recollect an exact duality transformation between observables of the dissipative WB and TB model holding for Ohmic [3] and for frequency-dependent damping [11]. Equally, there holds an exact duality between observables of the $\phi$ and $\theta$ model. Importantly, the conductances $G_\phi(V; T_0; V_0, g)$ and $G_\phi(V; T; \Delta, g)$ of these models are related by
\begin{equation}
G_\phi(V; T; V_0, g) = 1 - G_\phi(V; T; V_0, 1/g) ,
\end{equation}
where we displayed the relevant parameters. For repulsive interaction, $g$ is restricted to $g < 1$. If we work on the related WB or TB model, the applicable interval is $0 < g < \infty$.

Putting $T = 0$, the series (8) is a function of only $(V/T_0)^2 g^{-2}$, where $T_0 = a_0(g) V_0^{1/(1-g)} \omega_e^{-g/(1-g)}$ is the characteristic scale introduced by the impurity. The prefactor $a_0(g)$ will be specified shortly. In terms of $T_0$, the series may be written in the form
\begin{equation}
\tilde{G}(V; g) = 1 - \sum_{n=1}^{\infty} c_n(g) \left( \frac{V}{T_0} \right)^{2g^{-1} - 1} .
\end{equation}
The coefficient $c_n(g)$ is given as a $(2n - 1)$-fold integral. The circle of convergence of (10) is $v(g) < 1$, where
\begin{equation}
v(g) = (V/T_c)^{g^{-1}} \quad \text{with} \quad T_c = f(g) T_0 .
\end{equation}
The energy scale $T_c(g)$ is a crossover parameter analogous to the Kondo temperature. It is determined below.

There follows from (10) that the conductance is a power series of $(V/T_0^2)^{1/(1-g)}$ in the complementary range $v(1/g) < 1$. Demanding that the $\phi$ and $\theta$ model are different representations of the same physical problem, the scale $T_0^2$ is identical with $T_0$. Thus we have
\begin{equation}
\tilde{G}(V, g) = \sum_{n=1}^{\infty} c_n(g) \left( \frac{V}{T_0} \right)^{2(1/g-1)n} .
\end{equation}

Importantly, (11) and (12) are different expansions of the same function with complementary regions of convergence. They apply to the $\theta$ and $\phi$ model. For $g < 1$, (11) represents the high-voltage or weak-backscattering expansion, while (12) is the expansion for low voltage or strong backscattering. Conversely, for $g > 1$, the parameter regions for the series (11) and (12) are interchanged. As a result of strict duality, the entire expansions around weak and strong backscattering are related.

Kindly, the dual expansions (11) and (12) of the nonlinear conductance impose sufficiently strong conditions that it is possible to find the analytic form of the corresponding $T = 0$ scaling function.

An appropriate integral representation which creates the dual series expansions (11) and (12) reads
\begin{equation}
\tilde{G}(V, g) = \frac{1}{2\pi} \int_C \frac{dz}{z} \Gamma(u(z)) \Gamma[w(z)] F(z) \left( \frac{V}{T_0} \right)^{2z} ,
\end{equation}
where $u(z) = 1 + z/(1 - g)$, $w(z) = 1 + z g/(g - 1)$ and $F(0) = 1$. Now assume that $F(z)$ does not depend on $g$, and is an entire function of $z$. The integrand in (9) is an analytic function over the entire complex plane save for the poles $z = z_n^{(L)} \equiv (g - 1)n$ ($n = 0, 1, 2, \ldots$) and $z = z_m^{(R)} \equiv (1/g - 1)m$ ($m = 1, 2, \ldots$) where it possesses simple poles. For $g < 1$, the contour $C$ starts at $-i\infty$ and ends at $i\infty$, and circles the origin such that the set of poles $\{z_m^{(R)}\}$ lie to the left, and the set of poles $\{z_n^{(L)}\}$ lie to the right of the integration path. For $g > 1$, the path is in reverse direction and the origin is to the left.

For $v(g) < 1$, we close the contour such that all poles to the left are circled. Taking into account the residue of the enclosed poles, we find the series (11) with
\begin{equation}
c_n(g) = (-1)^{n-1} g \Gamma(g) \Gamma([g - 1)n]] / \Gamma(n) .
\end{equation}

In the complementary range $v(1/g) < 1$, the contour $C$ is closed by circling all poles to the right of the integration path. We then get the series expansion (12) with (10).

Next, we utilize the assumption that $F(z)$ does not depend on $g$. It is then possible to determine $F(z)$ by matching the expansion (11) or (12) onto the known solution for $g = 1/2$. The mobility in analytic form of the $g = 1/2$ case has been derived first in Refs. [11] for any $V_0$ and $T$. For $T = 0$ and $g = 1/2$, we get from (11) and (12)
\begin{equation}
\tilde{G}(V, 1/2) = 1 - (T_0/2V) \arctan(2V/T_0) .
\end{equation}
Upon matching the power series of (11) for $T_0 > 2V$ onto the expansion (8) with (10) at $g = 1/2$, we find
\[
F(z) = \Gamma(3/2)/\Gamma(3/2 + z),
\]
which indeed is analytic in the entire complex plane. The same form is found if the series of (11) for $T_0 < 2V$ is matched onto (8). With (13), the coefficient (10) reads
\[
c_n(g) = (-1)^{n-1} \frac{\Gamma(gn + 1)\Gamma(3/2)}{\Gamma(n+1)\Gamma(3/2 + g - 1)n}. \tag{13}
\]

Readily, the crossover scale $T_c(g)$ delimiting the regions of convergence of the expansions (8) and (8) follows from the requirement that the integrand in (8) tends to zero faster than $1/|z|$ on the semicircle of the respective path of integration for $|z| \to \infty$. With (13), we get from (11)
\[
T_c(g) = \sqrt{1 - g/g^{2/(2\pi g)}} T_0.
\]

Interestingly, we have $T_c(g) = T_B(1/g)$. This follows from the invariance of the integrand in (8) under $g \to 1/g$.

The $T = 0$ expressions (1) and (8) with (13) agree with the corresponding ones derived by Fendley et al. (6).

They utilized a suitable basis of interacting quasiparticles in which the model is integrable, and they employed sophisticated thermodynamic Bethe-ansatz (TBA) technology to calculate the non-Fermi distribution function and the density of states of the quasiparticles, which then determine the conductance by a Boltzmann-type rate expression. Our energy scale $T_0$ agrees with their scale $T_B$.

The much simpler derivation given here sheds additional light on the underlying symmetries of the models considered.

It is important to note that the representation (11) with (13), and the respective expansions (8) and (8) are exact for $0 < g < \infty$ under assumption that the correlation function $S(\tau)$ is described by the $T = 0$ form of (8).

To relate the energy scale $T_0$ in the exact solution to the parameters of the $\theta$ model, we match the $n = 1$ term of the series expansion (8) onto the $n = 1$ term of the series (11) with (10) for $T_0$. Thus we find
\[
T_0^{2-2g} = g^{2g-22-2g}[\pi/g(1/g)]^2V_0^2\omega_c^{-2g}. \tag{14}
\]

Equation (11) with (14) is an exact integral representation of the conductance of the $\theta$ model (and the equivalent WB model) at $T = 0$ for any $V$ and any positive $g$.

Likewise, we obtain the conductance in the $\phi$ model by expressing $T_0$ in terms of the parameters of this model.

Upon matching the $n = 1$ term of the series (8) onto the term of order $\Delta^2$ of the $\phi$ model, we find
\[
T_0^{2-2g} = 2^{-2g}[\pi/g(1/g)]^2\Delta^2\omega_c^{-2g}. \tag{15}
\]

Note that (14) and (15) are symmetric under the substitution $g \to 1/g$ except for the factor $g^{2g-2}$ in (14).

It is just this extra factor which is the reason for the substitution rule $V \to V/g$ in the duality relation (8).

From (14) and (15) we find the functional dependence between $V_0$ and $\Delta$
\[
\Delta^2 = \frac{[\Gamma(1 + 1/g)]^2[\Gamma(1 + g)]^{2/\alpha}}{(\omega_c/\pi)^{2+2g}V_0^{-2/g}}.
\]

Solving this for $V_0$ gives the same functional form except that $g$ is replaced by $1/g$. By use of this relation, we can equivalently express the $n = 1$ term of (8) for the $\phi$ model in terms of the correlation strength $V_0$,
\[
G^{(1)}_\phi(V, g) = \frac{g^{2g-2\alpha}[\Gamma(1/g)]^2}{\sqrt{\pi}T(1/2 + 1/g)} \left(\frac{\omega_c^2V^{2/g}}{(2\pi V_0)^{2/g}}\right). \tag{16}
\]

While the first form is of order $\Delta^2$ and is the simple golden rule expression of the conductance in the $\phi$ model, the second form is the exact single-bounce or instanton-pair contribution in the $\theta$ model, which is nonperturbative in $V_0^2$. Until now, the single-bounce contribution in the $\theta$ model has been calculated only for $1/g = \alpha \gg 1$ (14). The expression (16) agrees with the result of Ref. 14 in this limit. The expression (16) is the generalization for all $g$, but $\omega_c \gg 2\pi V_0$.

Interestingly, the single-bounce contribution as well as all multi-bounce terms in the series (8) have been found without calculating bounce actions and tricky fluctuation determinants.

For $g = 1/2 - \varepsilon$, with $|\varepsilon| \ll 1$, the expressions (11) and (8) with (14) agree with the findings in Ref. 12, where a leading-log summation of the corrections about the $g = 1/2$ solution was made.

For $g = 1 - \varepsilon$ with $|\varepsilon| \ll 1$, the series (8) or (8) gives
\[
\tilde{G}_\theta(V, 1 - \varepsilon) = \frac{1}{1 + (\pi V_0/\omega_c)^2(2\omega_c/V)^{2\varepsilon}}. \tag{17}
\]

This form agrees with the result by Matveev, Yue, and Glazman (15), who performed a leading-log summation for weak electron-electron interaction.

The limit $\alpha = 1/g \to \infty$ represents the classical limit in the related Brownian particle model. For $g = 0$, the series (8) is summed to the form
\[
\tilde{G}_\theta(0, V) = \sqrt{1 - (2\pi V_0/V)^2}, \quad V \geq 2\pi V_0. \tag{18}
\]

This expression is indeed the solution of the associated Fokker-Planck equation in the Smoluchowski limit (14) for $T = 0$. In the region $V < 2\pi V_0$, the conductance is zero because the slope of the tilted washboard potential $H_I$ changes regularly sign, and the overdamped classical particle comes to rest at a point with zero slope.

The conductance as a function of $V/T_0$ shows a smooth transition from the square root singular behavior (15) via the kink-like shape (11) to the constant behavior (17) as $g$ is increased from 0 to 1. All curves cross the line $\tilde{G} = 1/2$ in the interval $V/T_0 = 2/\sqrt{3} \pm 0.01$. 
Consider next the leading low-temperature correction. Taking into account the \( T^2 \) contribution of the correlation function \( \langle \delta I \rangle \), the interaction factor \( \langle \delta I \rangle \) reads
\[
W_n = W_n^{(0)} \left[ 1 - g T^2 \frac{\pi^2}{3} \left( \sum_{j=1}^{n-1} p_j \alpha_j \right)^2 \right],
\]
where \( W_n^{(0)} \) is the full interaction factor at \( T = 0 \). There follows from \( \langle \delta I \rangle \) that the annoying factor \( \left( \sum_{j} p_j \alpha_j \right)^2 \) may be generated by differentiation with respect to the voltage \( V \). Thus, the prefactor of the \( T^2 \) correction is universally expressed in terms of the nonlinear conductance at \( T = 0 \). For \( T \ll gV \), we have
\[
G(V, g; T) = G(V, g) + \frac{\pi^2 T^2}{3 gV} \frac{\partial^2}{\partial V^2} [V G(V, g)].
\]

Unfortunately, \( T^4 \) and higher order corrections cannot be expressed in such simple terms.

Finally, consider the nonequilibrium DC current noise \( \delta I^2(V, T) \). Within the interacting quasiparticle picture, exact expressions for the DC noise at any \( V \) and \( T \) involving TBA integral equations have been given and evaluated numerically in Ref. \[17\]. Within the Coulomb gas representation, it is straightforward to write down the series expressions of the noise at any \( V \) and \( T \) \[11\]. For the \( \theta \) model, comparison of the series of the expression
\[
\delta I_\phi^2(V, T) = \lim_{\omega \to 0} \int dt e^{i \omega t} \langle \{ \hat{\delta}(0, t), \hat{\delta}(0, 0) \} \rangle / 2 \pi .
\]
with the series \([11]\) yields at \( T = 0 \)
\[
\delta I_\phi^2(V) = -(gV/2) V \frac{\partial}{\partial V} G_\phi(V) .
\]

For a weak barrier, only the term of order \( V_0^2 \) in \([3]\) is relevant. We then have \( \delta I_\phi^2(V) = -g V G^{(1)}_\phi(V) \). In the FQHE device, this noise is due to quasiparticle tunneling. Similarly, one finds for the \( \phi \) model, either by direct calculation or by use of \([22]\) and the duality relation \([3]\),
\[
\delta I_\phi^2(V) = (V/2) \Delta(\partial/\partial \Delta) G_\phi(V) .
\]

For a strong barrier, the leading term is of order \( \Delta^2 \), yielding \( \delta I_\phi^2(V) = V G^{(1)}_\phi(V) \). In the FQHE effect, this noise results from tunneling of uncorrelated electrons. Since \( G_\phi \) is a function of \( V_0^2 V^{2g-2} \), and \( G^{(1)}_\phi \) is a function of \( \Delta^2 V^{2/g-2} \), we may express both \([21]\) and \([22]\) as
\[
\delta I^2(V) = \left[ g/2(1-g) \right] V^2 (\partial^2/\partial V^2) G(V)
\]
These results for the noise at \( T = 0 \) agree with those of Ref. \[17\] where it was however derived quite differently.

Knowing the DC noise at \( T = 0 \) exactly, allows us to calculate the exact \( T^2 \) enhancement as well. Taking up the arguments based on \([3]\), we obtain for \( T \ll gV \) and in the regime \( v(g) < 1 \), in which \([3]\) is appropriate,
\[
\delta I^2(V, T) = \delta I^2(V) + \frac{\pi^2 T^2}{3 g} \frac{\partial^2}{\partial V^2} \delta I^2(V) + 2 G_0 T.
\]

Clearly, this form applies both for the \( \theta \) and the \( \phi \) model. Importantly, the prefactor of the \( T^2 \) enhancement of the current noise is again described by zero temperature properties. The additional term \( 2 G_0 T \) is the familiar Johnson-Nyquist noise which corresponds to the Einstein relation between diffusion coefficient and mobility in the related Brownian particle model \([10]\). In the complementary regime \( v(1/g) < 1 \), we have the same expression for \( \delta I^2(V, T) \) except that the last term is absent, because the series \([3]\) applies.

To conclude, we have computed the conductance and DC current noise in a Luttinger liquid with a source of backscattering at low \( T \). A pedestrian pathway founded on duality and analytic properties guided us to the analytic solutions at \( T = 0 \). Strict duality means that the entire expansions around weak and strong backscattering are related. In the FQHE device, the crossover from weak to strong backscattering comes with a crossover from Laughlin quasiparticle tunneling to electron tunneling. We calculated in analytic form the leading low-temperature enhancement for arbitrary strength of the impurity potential.

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