SINGULAR ELLIPTIC EQUATION INVOLVING THE GJMS OPERATOR ON COMPACT RIEMANNIAN MANIFOLDS

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Abstract. In this paper we consider a singular elliptic equation involving the GJMS (Graham-Jenne-Mason-Sparling) operator of order $k$ on $n$-dimensional compact Riemannian manifold with $2k < n$. Mutiplicity and nonexistence results are established.

Let $(M, g)$ be an $n$-dimensional Riemannian manifold. The $k$-th GJMS operator (Graham-Jenne-Mason-Sparling, see (5)) $P_g$ is a differential operator defined for any integer $k$ if the dimension $n$ is odd, and $2k \leq n$ otherwise. In the following, we will consider the case $2k \leq n$. $P_g$ is of the form

$$P_g = \Delta^k + \text{lot}$$

where $\Delta = -\text{div}_g(\nabla)$ is the Laplacian-Beltrami operator and \text{lot} denotes the lower terms. One of the fundamental property of $P_g$ is its behavior with respect to conformal change of metrics: for $\varphi \in C^\infty(M)$, $\varphi > 0$ and $\bar{g} = \varphi^{-\frac{2}{n-2k}}g$ a conformal metric to $g$,

$$\varphi^{\frac{n+2k}{n-2k}}P_g u = P_g(\varphi u).$$

(0.1)

$P_g$ is self-adjoint with respect to the $L^2$-scalar. To $P_g$ is associated a conformal invariant scalar function denoted $Q_g$ and is called the $Q$-curvature. For $k = 1$, the GJMS operator is ( up to a constant ) the conformal Laplacian and the corresponding $Q$-curvature function is simply the scalar curvature. For $k = 2$, the GJMS operator is the Paneitz operator introduced in (13). For $2k < n$, the $Q$-curvature is $Q_g = \frac{2}{n-2k}P_g(1)$. Many works was devoted the $Q$-curvature equation in the last two decades (see [2], [3], [4], [5], [7], [9], [13], [17]). Many authors investigated the interactions of conformal methods with mathematical physic which led them to study the Einstein-scalar fields Lichnerowicz equations (see [6], [8], [12], [14], [15], [16]). These methods have been extended to scalar fields Einstein-Licherowicz type equation involving the Paneitz operator, (see [9]). In this work we analyze an Einstein-Lichnerowicz scalar field equation containing the $k$-th order GJMS operator on a Riemannian $n$-dimensional manifold with $2k < n$; more precisely we consider the following equation

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\( P_g(u) = B(x) u^{2^* - 1} + \frac{A(x)}{u^{2^* + 1}} + \frac{C(x)}{u^p} \quad u > 0 \)

where \( 2^* = \frac{2n}{n - 2k} \) and \( p > 1 \). In all the sequel of this paper we assume that the operator \( P_g \) is coercive which allows us (see Proposition 2, [17]) to endow \( H_k^2(M) \) with the following appropriated equivalent norm

\[
\|u\| = \sqrt{\int_M u P_g(u) dv_g}.
\]

So we deduce from the coercivity of \( P_g \) and the continuity of the inclusion \( H_k^2(M) \subset L^{2^*}(M) \), the existence of a constant \( S > 0 \) such that

\[
\|u\|_{2^*} \leq S \|u\|^{2^*}
\]

where \( 2^* = \frac{2n}{n - 2k} \).

Our work is organized as follows: in a first section we show the existence of a solution to equation (0.2) obtained by means of the mountain-pass theorem: more precisely we establish the following theorem.

**Theorem 1.** Let \((M, g)\) be a compact Riemannian manifold with dimension \( n > 2k \) and \( A > 0, B > 0, C > 0 \) are smooth functions on \( M \). Suppose moreover that the operator \( P_g \) is coercive and have a positive Green function. If there exists a constant \( C(n, p, k) > 0 \) depending only on \( n, p, k \) such that

\[
\frac{\|\varphi\|^2}{2^*} \int_M \frac{A(x)}{\varphi^{2^*}} dv_g \leq C(n, p, k) \left( \max_{x \in M} B(x) \right)^{\frac{2 + 2^*}{2 - 2^*}}
\]

\[
\frac{\|\varphi\|^p - 1}{p - 1} \int_M \frac{C(x)}{\varphi^{p - 1}} dv_g \leq C(n, p, k) \left( \max_{x \in M} B(x) \right)^{\frac{p + 1}{2 - 2^*}}
\]

for some smooth function \( \varphi > 0 \), then equation (0.2) admits a smooth solution.

In the second section we prove, by means of the Ekeland’s lemma, the existence of a second solution to equation (0.2). In particular, by setting

\[
t_0 = \left( \frac{1}{\max_{x \in M} B(x)} \right)^{\frac{n - 2k}{4k}} \quad \text{and} \quad a = \frac{1}{(2(n-k))^{\frac{2^*}{2}}}
\]

we obtain the following theorem:

**Theorem 2.** Let \((M, g)\) be a compact Riemannian manifold of dimension \( n > 2k \), \((k \in \mathbb{N}^*)\). Suppose that the operator \( P_g \) is coercive; has a Green positive function and there is a constant \( C(n, p, k) > 0 \) which depends only on \( n, p, k \) such that:

\[
\frac{\|\varphi\|^2}{2^*} \int_M \frac{A(x)}{\varphi^{2^*}} dv_g \leq C(n, p, k) \left( \max_{x \in M} B(x) \right)^{\frac{2 + 2^*}{2 - 2^*}}
\]
and

\[(0.8) \quad \frac{\|\varphi\|^{p-1}}{p-1} \int_M C(x) \varphi^{p-1} dv_g \leq C(n, p, k) \left( S_{\max} B(x) \right)^{\frac{p+1}{2^*-2}} \]

for some smooth function \(\varphi > 0\) on \(M\). If moreover for every \(\varepsilon \in ]0, \lambda^*[\) where \(\lambda^*\) is a positive constant, the following conditions occur

\[0 < \left( S_{\max} B(x) \right) < \frac{a}{4}\]

\[(0.9) \quad \left( \int_M \sqrt{A(x)} dv_g \right)^2 \left( \frac{8-a}{a} \right) \frac{(n-2k)Q_3}{2t_0^2} > 2^* \frac{k}{n} t_0^2 \left( 1 - \frac{a}{8} \right)\]

and

\[(0.10) \quad \int_M Q_g dv_g \neq k(n-1)\omega_n\]

where \(\omega_n\) is the volume of the round sphere, \(2^* = \frac{2n}{n-2k}\), \(3 < p < 2^* + 1\). Then the equation \((0.2)\) admits a second smooth solution.

In the last section we give a nonexistence result of solution. Mainly we show the following result:

**Theorem 3.** Given \((M, g)\) a compact Riemannian manifold of dimension \(n > 2k\), \((k \in \mathbb{N}^*)\) and \(A, B, C\) are positive smooth functions on \(M\) and \(2 < p < 2^* + 1\). Assume that

\[(0.11) \quad C(n, p, k) \left( \int_M \sqrt{B} C dv_g \right)^{\frac{2^*}{p-1+2^*}} \int_M B dv_g > (SR)^2\]

where \(S, R\) are positive constants and

\[C(n, p, k) = \frac{2^* + p - 1}{p-1} \left( \frac{p-1}{2^*} \right)^{\frac{2^*}{2^*+p-1}}\]

Then the equation \((0.2)\) has no smooth positive solution \(u\) with energy \(\|u\|_{H^k_0(M)} \leq R\).

1. **Existence of a first solution**

In this section we show the theorem. Before starting the proof, we first give an example of manifolds where the GJMS operator \(P_g\) has a positive Green function.

**Proposition 1.** Suppose that the metric \(g\) is Einstein with positive scalar curvature of dimension \(n > 2k\), then the GJMS operator \(P_g\) admits a positive Green function.
Proof. On $n$-dimensional Einstein manifold, the GJMS operator of order $k$ is given by (see [5])

$$P_g = \prod_{l=1}^{k} (\Delta - c_l Sc)$$

where $c_l = \frac{(n+2l-2)(n-2l)}{4n(n-1)}$, $Sc$ stands for the scalar curvature. If the scalar curvature is positive it is well known that the operator $\Delta - c_l Sc$ has a positive Green function. Denote by $L_l = \Delta - c_l Sc$, $l = 1, ..., k$; by definition of the Green function of $L_l$ we know that for all $u \in C^\infty(M)$,

$$\left( L_l u \right) (x) = \int_M G_{l+1}((x, y)) (L_{l+1} L_l u)(y) dv_g(y).$$

So

$$u(x) = \int_M G_l(x, z) (L_l u)(z) dv_g(z) + \frac{1}{Vol(M)} \int_M u(x) dv_g(x)$$

$$= \int_M \left( \int_M G_l(x, z) G_{l+1}((z, y)) dv_g(z) \right) (L_{l+1} L_l u)(y) dv_g(y) + \frac{1}{Vol(M)} \int_M u(x) dv_g(x)$$

and letting

$$G_{l,l+1}(x, y) = G_l \ast G_{l+1}(x, y) = \int_M G_l(x, z) G_{l+1}((z, y)) dv_g(z).$$

By induction, we get

$$u(x) = \int_M G_1 \ast ... \ast G_k(x, y) P_g(y) dv_g(y).$$

Thus $P_g$ admits a positive Green function. □

To show the existence of solutions to equation (0.2), we follow the strategy in the proof of the paper by Hebey-Pacard-Pollack [6]. We consider the following $\varepsilon$-approximating equations ($\varepsilon > 0$)

$$P_g(u) = B(x) \left( u^+ \right)^{2^b - 1} + \frac{A(x) u^+}{(\varepsilon + (u^+)^2)^{2^b + 1}} + \frac{C(x) u^+}{(\varepsilon + (u^+)^2)^{p + 1}}$$

where $2^b = \frac{2^b}{\varepsilon}$, $p > 1$. Which gives us a sequence $(u_\varepsilon)_\varepsilon$ of solutions to (1.1). The solution of equation (0.2) is then obtained as the limiting of $(u_\varepsilon)_\varepsilon$, when $\varepsilon \to 0$. To get rid of negative exponents, we consider the energy functional associated to (1.1) defined by, for any $\varepsilon > 0$

$$I_\varepsilon(u) = I^{(1)}(u) + I^{(2)}_\varepsilon(u)$$

where $I^{(1)} : H^2_k(M) \to \mathbb{R}$ is given by

$$I^{(1)}(u) = \frac{1}{2} \int_M u P_g(u) dv_g - \frac{1}{2^b} \int_M B(x) \left( u^+ \right)^{2^b} dv_g$$
and \( I^{(2)}_{\varepsilon} : H^{2}_k (M) \to \mathbb{R} \) is

\[
I^{(2)}_{\varepsilon} (u) = \frac{1}{2^*} \int_M \frac{A(x)}{(\varepsilon + (u^+)^2)^{2^*}} \, dv_g + \frac{1}{p - 1} \int_M \frac{C(x)}{(\varepsilon + (u^+)^2)^{\frac{2}{2^*}}} \, dv_g.
\]

It is easy to check the following inequality

\[
\Phi (\|u\|) \leq I^{(1)} (u) \leq \Psi (\|u\|)
\]

with

\[
\Phi (t) = \frac{1}{2} t^2 - \frac{1}{2^*} \left( \frac{S_{\max} |B|}{M} \right) t^{2^*}
\]

and

\[
\Psi (t) = \frac{1}{2} t^2 + \frac{1}{2^*} \left( \frac{\max_{x \in M} B(x)}{t} \right) t^{2^*}.
\]

The function \( \Phi (t) \) is increasing on \([0, t_0]\) and decreasing on \([t_0, +\infty[\), where

\[
t_0 = \left( \frac{1}{S_{\max} B(x)} \right)^{\frac{n - 2k}{4k}}
\]

and

\[
\Phi (t_0) = \left( \frac{1}{2} - \frac{1}{2^*} \right) \left( \frac{S_{\max} B(x)}{x \in M} \right)^{\frac{n - 2k}{2k}} = \frac{k}{n} t^{2}_{0}.
\]

**Lemma 1.** Let \( \theta > 0 \) such that

\[
\left( \frac{a}{2} \right) \frac{\theta}{2^*} < \theta^2 < a \frac{2}{2^*}
\]

where

\[
a = \frac{1}{(2(n-k))^{\frac{2}{2^*}}}
\]

and put

\[
t_1 = \theta t_0.
\]

Then we have the following double inequality

\[
\Psi (t_1) \leq \theta^2 \frac{2}{2^*} + \frac{2}{2^* - 2} \Phi (t_0) < \frac{1}{2k} \Phi (t_0).
\]

**Proof.** In fact

\[
\Psi (t_1) = \frac{1}{2} t_1^2 + S_{\max} B(x) \frac{t_{1}^{2^*}}{2^*}
\]

\[
= \theta^2 \left( \frac{1}{2} t_{0}^2 + S_{\max} B(x) \theta^2 - 2 \frac{t_{0}^{2^*}}{2^*} \right).
\]
Since
\[(1.6) \quad t_0^{2^*} \left( S_{\max B(x)} \right) = t_0^{2^*} \]
we get
\[
\Psi(t_1) = \theta^2 \left( \frac{1}{2} t_0^{2^*} + \frac{\theta^{2^* - 2}}{2^*} t_0^{2^*} \right) = \theta^2 t_0^{2^*} \left[ \frac{1}{2} + \frac{\theta^{2^* - 2}}{2^*} \right] \leq \theta^2 t_0^{2^*} \left( \frac{1}{2} + \frac{n - 2k}{2n} \right) \leq \left( 1 - \frac{k}{n} \right) \theta^2 t_0^{2^*}.
\]
And since
\[
\frac{2^* + 2}{2^* - 2} = \left( \frac{n}{k} - 1 \right) \quad \text{and} \quad \Phi(t_0) = \frac{k}{n} t_0^{2^*} \]
we infer that
\[
\Psi(t_1) \leq \theta^2 \frac{2^* + 2}{2^* - 2} \Phi(t_0) < \frac{1}{2k} \Phi(t_0).
\]

Now we check the Mountain-Pass lemma conditions for the functional $I_\varepsilon$.

**Lemma 2.** The functional $I_\varepsilon$ satisfies the following condition: there exists an open ball $B(u_1, \rho)$ of radius $\rho > 0$ and of center some $u_1$ in $H^2_k(M)$ and there are $u_2 \notin B(u_1, \rho)$ and a real number $c_o$ such that
\[
\max (I_\varepsilon(u_1), I_\varepsilon(u_2)) < c_o \leq I_\varepsilon(u)
\]
for all $u \in \partial B(u_1, \rho)$.

**Proof.** Following the strategy of the proof in the paper by Hebey-Pacard-Pollack [6], we let $\varphi \in C^\infty(M)$, $\varphi > 0$ on $M$ and without loss of generality we may assume $\|\varphi\| = 1$. Put
\[
(1.7) \quad C(n, p, k) = (2k - 1) \frac{\theta^{2^* + p}}{4n} \leq C_1(n, k) = \frac{(2k - 1) \theta^{2^*}}{4n}.
\]
The inequality \[(0.5)\] becomes
\[
(1.8) \quad \frac{1}{2^*} \int_M A(x) (t_1 \varphi)^{2^*} dv_g \leq \frac{2k - 1}{4k} \Phi(t_0).
\]
Indeed, we have
\[
(1.9) \quad \Phi(t_o) = \frac{k}{n} t_0^{2^*}. \]
and by (1.7), we get
\[
\frac{1}{2^p} \int_M \frac{A(x)}{(t_1 \varphi)^2} dv_g \leq \frac{C_1 (n, k)}{t_0^{2p} \theta^{2p}} \left( S. \text{max} \frac{B(x)}{M} \right)^{\frac{2p-2}{2-p}}
\]
\[
= \frac{2k-1}{4k} \Phi(t_0).
\]

Analogously, by putting
\[
C_2(n, p, k) = \frac{(2k-1) \theta^{p-1}}{4n}
\]
we obtain
\[
(1.10) \quad \frac{1}{p-1} \int_M \frac{C(x)}{(t_1 \varphi)^{p-1}} dv_g \leq \frac{2k-1}{4k} \Phi(t_0).
\]

By relations (1.5), (1.6), (1.8) and (1.10), we infer that
\[
I_\epsilon(t_1 \varphi) \leq \Psi(\|t_1 \varphi\|) + \frac{1}{2^p} \int_M \frac{A(x)}{\left( \epsilon + (t_1 \varphi)^2 \right)^{2p}} dv_g + \frac{1}{p-1} \int_M \frac{C(x)}{\left( \epsilon + (t_1 \varphi)^2 \right)^{\frac{p-1}{2}}} dv_g
\]
\[
= \Psi(t_1) + \frac{1}{2^p} \int_M \frac{A(x)}{\left( \epsilon + (t_1 \varphi)^2 \right)^{2p}} dv_g + \frac{1}{p-1} \int_M \frac{C(x)}{\left( \epsilon + (t_1 \varphi)^2 \right)^{\frac{p-1}{2}}} dv_g \leq \Phi(t_0).
\]

Again from (1.5), we deduce that
\[
I_\epsilon(t_0 \varphi) \geq \Phi(t_0) + \frac{1}{2^p} \int_M \frac{A(x)}{\left( \epsilon + (t_0 \varphi)^2 \right)^{2p}} dv_g + \frac{1}{p-1} \int_M \frac{C(x)}{\left( \epsilon + (t_0 \varphi)^2 \right)^{\frac{p-1}{2}}} dv_g
\]

and since \(A\) and \(C\) are assumed with positive values, we obtain
\[
I_\epsilon(t_0 \varphi) \geq \Phi(t_0).
\]

Finally from (1.11) and (1.12), we get
\[
I_\epsilon(t_1 \varphi) < \Phi(t_0) \leq I_\epsilon(t_0 \varphi).
\]

Noting that
\[
\lim_{t \to +\infty} I_\epsilon(t \varphi) = \lim_{t \to +\infty} \left[ \frac{1}{2} \|t \varphi\|_{F^2} - \frac{1}{2^p} \int_M \left( B(x) (t \varphi)^2 dv_g - \frac{A(x)}{\left( \epsilon + (t \varphi)^2 \right)^{2p}} \right) dv_g \right]
\]
\[
= \lim_{t \to +\infty} \frac{1}{p-1} \int_M \frac{C(x)}{\left( \epsilon + (t \varphi)^2 \right)^{\frac{p-1}{2}}} dv_g
\]
\[
= \lim_{t \to +\infty} \frac{1}{2^p} \left( \frac{1}{2t^{2p-2}} - \frac{1}{2^p} \int_M B(x) \varphi^{2p} dv(g) \right)
\]
and since $\int_M B(x)\varphi^2 dv_g > 0$, we obtain
$$\lim_{t \to +\infty} I_\varepsilon(t\varphi) = -\infty.$$ Consequently there is $t_2$ such that
$$t_2 > t_0 \quad \text{and} \quad I_\varepsilon(t_2\varphi) < 0.$$ Now, to have the conditions of Lemma 2 fulfilled, we just put
$$u_1 = t_1\varphi, \quad u_2 = t_2\varphi, \quad u = t_0\varphi$$ and we take $\rho = t_0 - t_1 > 0$ and $c_0 = \Phi(t_0)$.

Lemma 1 allows us to apply the Mountain-Pass Lemma to the functional $I_\varepsilon$. Let
$$C_\varepsilon = \inf_{\gamma \in \Gamma} \max_{u \in \gamma} I_\varepsilon(u)$$ where $\Gamma$ denotes the set of paths in $H^2_k(M)$ joining the functions $u_1 = t_1\varphi$ and $u_2 = t_2\varphi$. So $C_\varepsilon$ is a critical value of $I_\varepsilon$ and moreover
$$C_\varepsilon > \Phi(t_0)$$ and by putting $\gamma(t) = t\varphi$, for $t \in [t_1, t_2]$, we see that $C_\varepsilon$ is uniformly bounded when $\varepsilon$ goes to 0, so we get
$$0 < \Phi(t_0) < C_\varepsilon \leq C$$ for $\varepsilon$ sufficiently small and $C > 0$ not depending on $\varepsilon$.

Consequently there exists a sequence $(u_m)_m$ of functions in $H^2_k(M)$ such that
$$I_\varepsilon(u_m) \to \infty \quad \text{and} \quad DI_\varepsilon(u_m) \to 0 \quad \text{as} \quad m \to +\infty.$$ By Lemma 2 the sequence $(u_m)_{m \in \mathbb{N}}$ of $H^2_k(M)$ is a Palais-Smale sequence (P-S) for the functional $I_\varepsilon$.

**Theorem 4.** The Palais-Smale sequence $(u_m)_{m \in \mathbb{N}}$ is bounded in $H^2_k(M)$ and converges weakly to nontrivial smooth solution $u_\varepsilon$ of equation (1.1).

**Proof.** By (1.13) we get for any $\varphi \in H^2_k(M)$
$$DI_\varepsilon(u_m) \varphi = o(\|\varphi\|)$$ i.e. for any $\varphi \in H^2_k(M)$ one has
$$\int_M \varphi P_g u_m dv_g = \int_M B(x) (u_m^+)^{2^* - 1} \varphi dv_g$$
$$+ \int_M A(x) (u_m^+) \varphi dv_g + \int_M C(x) (u_m^+) \varphi dv_g + o(\|u_m\|)$$
in particular, for $\varphi = u_m$ we have
\[
\int_M u_m P_g u_m dv_g - \int_M B(x) (u_m^+) = \int_M A(x) (u_m^+) \frac{2}{n+1} dv_g + \int_M \frac{C(x) (u_m^+)}{n+1} dv_g + o(||u_m||).
\]

or
\[
\frac{1}{2} \int_M u_m P_g u_m dv_g + \frac{1}{2} \int_M B(x) (u_m^+) dv_g
\]

(1.16)

\[
+ \frac{1}{2} \int_M \frac{A(x) (u_m^+)}{n+1} dv_g + \frac{1}{2} \int_M \frac{C(x) (u_m^+)}{n+1} dv_g = o(||u_m||).
\]

On the other hand it comes from (1.14) that
\[
\frac{1}{2} \int_M u_m P_g u_m dv_g - \frac{1}{2} \int_M B(x) (u_m^+) dv_g
\]

(1.17)

\[
+ \frac{1}{2} \int_M \frac{A(x) (u_m^+)}{n+1} dv_g + \frac{1}{2} \int_M \frac{C(x) (u_m^+)}{n+1} dv_g = C_0 + o(||u_m||).
\]

So by adding (1.16) and (1.17) we get
\[
\frac{k}{n} \int_M B(x) (u_m^+) dv_g + \frac{1}{2} \int_M \frac{A(x) (u_m^+)}{n+1} dv_g + \frac{1}{2} \int_M \frac{A(x)}{n+1} dv_g
\]

(1.18)

\[
+ \frac{1}{2} \int_M \frac{C(x) (u_m^+)}{n+1} dv_g + \frac{1}{2} \int_M \frac{C(x)}{n+1} dv_g = C_0 + o(||u_m||).
\]

For sufficiently large $m$ we deduce that
\[
\frac{k}{n} \int_M B(x) (u_m^+) dv_g \leq 2C_0 + o(||u_m||)
\]

or
\[
\frac{1}{2} \int_M B(x) (u_m^+) dv_g \leq \frac{n}{2C_0} C_0 + o(||u_m||)
\]

and plugging this last inequality with in (1.17) we obtain
\[
\frac{1}{2} \int_M u_m P_g u_m dv (g) \leq C_0 + \frac{n}{2C_0} C_0 + o(||u_m||)
\]

\[
\leq nC_0 + \frac{n(n-2k)}{2n} C_0 + o(||u_m||) \leq 2(n-k) C_0 + o(||u_m||).
\]
Hence for $m$ large enough
\[
\int_M u_m P_g u_m dv \leq 4nC_\varepsilon + o(1) \leq 4nC_\varepsilon + 1
\]
i.e.
(1.19) \[ \|u_m\|^2 \leq 4nC_\varepsilon + 1 \]
Thus we prove the sequence $(u_m)_m$ is bounded in $H^2_k(M)$; so we can extract a subsequence, still denoted $(u_m)_m$ which verifies:
1. $u_m \to u_\varepsilon$ weakly in $H^2_k(M)$.
2. $u_m \to u_\varepsilon$ strongly in $L^p(M)$, $\forall p < \frac{2n}{n-2k}$
3. $u_m \to u_\varepsilon$ a.e. in $M$.
4. $(u_m)^{2^* - 1} \to u_\varepsilon^{2^* - 1}$ weakly in $L^{2^* - 1}(M)$.
Furthermore, putting $g(x) = \frac{1}{q}$, where $\varepsilon > 0$ and $q > 0$, we get by Lebesgue’s dominated convergence theorem that
\[
\forall k \in \mathbb{N}: \left( (u_m^+)^2 + \varepsilon \right)^{-q} < \varepsilon^{-q} \quad \text{and} \quad \varepsilon^{-q} \in L^p(M) \quad \forall p \geq 1
\]
thus $\left( (u_m^+)^2 + \varepsilon \right)^{-q} \to \left( (u_\varepsilon^+)^2 + \varepsilon \right)^{-q}$ strongly in $L^p(M)$ $\forall p \geq 1$ and with (2) we infer that $u_\varepsilon$ is a weak solution of the equation
(1.20) \[ P_g u_\varepsilon = B(x) (u_\varepsilon^+)^{2^* - 1} + \frac{A(x) u_\varepsilon^+}{(\varepsilon + (u_\varepsilon^+)^2)^{2p+1}} + \frac{C(x) u_\varepsilon^+}{(\varepsilon + (u_\varepsilon^+)^2)^{\frac{n+1}{2}}} \]
where $2^b = \frac{2^*}{2}$ and $p > 1$.
Our solution $u_\varepsilon$ is not identically zero: indeed by (1.18), we have
\[
\frac{1}{2^*} \int_M A(x) \left( \frac{u_m^+}{(\varepsilon + (u_m^+)^2)^{2^*}}dv \right) \leq C_\varepsilon + o(\|u_m\|).
\]
Now, letting $m \to +\infty$ and taking in mind (1.13), we infer that
(1.21) \[ \frac{1}{2^*} \int_M A(x) \left( \frac{u_\varepsilon^+}{(\varepsilon + (u_\varepsilon^+)^2)^{2^*}}dv \right) \leq C \]
where $C$ is the upper bound of $C_\varepsilon$.
Now if for a sequence $\varepsilon_j \to 0$ ( with $\varepsilon_j > 0$, $\forall j \in \mathbb{N}$); $u_{\varepsilon_j}$ goes to 0, then it follows that
(1.22) \[ \frac{1}{2^* (2^* - 1) \varepsilon_j^{2^*}} \int_M A(x) dv \leq C. \]
So if $j \to +\infty$, it leads to a contradiction since by assumption $A > 0$. 
Finally, for sufficiently small $\varepsilon$, $u_\varepsilon$ is a solution not identically zero of the equation (1.1).

Now we will show the regularity of $u_\varepsilon$. First we write the equation (1.20) in the form

$$P_g u_\varepsilon = b(x, u_\varepsilon) u_\varepsilon$$

where

$$b(x, u_\varepsilon) = \frac{A(x)}{(\varepsilon + (u_\varepsilon^+)^2)^{2^* - 1}} + \frac{C(x)}{(\varepsilon + (u_\varepsilon^+)^2)^{\frac{p+1}{2}}}.$$

Since $A((\varepsilon + (u_\varepsilon^+)^2)^{2^* + 1}) + C((\varepsilon + (u_\varepsilon^+)^2)^{2^* + 1}) \in L^\infty(M)$ and $u_\varepsilon \in H^2_k(M) \subset L^{2^*}(M)$, we infer that $b \in L^\infty_2(M)$. By the work of S. Mazumdar (see the proof of the theorem 5 page 28 in [10]) we obtain that $u_\varepsilon \in L^p(M)$ for any $0 < p < +\infty$. According to [1], we obtain that $u_\varepsilon \in H^p(M)$ for all $1 < p < +\infty$. By the same arguments as in the proof of proposition 8.3 in [1] we conclude that $u_\varepsilon \in C^{2k,\alpha}(M)$ with $\alpha \in (0, 1)$.

Now we are in position to prove Theorem [11].

**Proof.** From what precedes $u_\varepsilon$ is a $C^{2k}(M)$ nontrivial solution to equation (1.1), moreover $u_\varepsilon$ is a weak limit of the sequence $(u_k)_k$ which allows us by the lower semicontinuity of the norm to write

$$\|u_\varepsilon\| \leq \lim_{m \to +\infty} \inf \|u_m\|.$$

And by the inequalities (1.13), (1.19) we deduce that the sequence $(u_\varepsilon)_\varepsilon$ of the $\varepsilon$-approximating solutions is bounded in $H^2_k(M)$ for sufficiently small $\varepsilon > 0$ i.e.

$$(1.23) \quad \|u_\varepsilon\|^2 \leq 4nC + 1$$

thus we can extract a subsequence still labelled $(u_m)_m$ satisfying:

i) $u_m \rightharpoonup u$ weakly in $H^2_k(M)$

ii) $u_m \rightharpoonup u$ strongly in $L^p(M)$ for $p < 2^*$

iii) $u_m \to u$ a.e. in $M$.

iv) $u_m^{2^*-1} \rightharpoonup u^{2^*-1}$ weakly in $L^{\frac{2^*}{2^*-1}}$.

Furthermore the sequence $(u_m)_m$ is bounded below: indeed as the functions $u_k$ are continuous, denote by $x_m$ their respective maximums on $M$ and put $x_0 = \lim x_m$ (a subsequence of $(x_m)_m$ still labelled $(x_m)_m$). Since by assumption the operator $P_g$ admits a positive Green function, then we
can write
\[ u_m(x_m) = \int_M G(x_m, y) \left( B(y) (u_m^+(y))^2 - 1 + \frac{A(y) u_m^+(y)}{(\varepsilon + (u_m^+(y))^2)^{2p+1}} + \frac{C(y) u_m^+(y)}{(\varepsilon + (u_m^+(y))^2)^{\frac{p+1}{2}}} \right) \, dv_y \]
and by Fatou’s lemma, we get
\[ \liminf_m u_m(x_m) \geq \]
\[ \int_M \liminf_m \left[ \frac{G(x_m, y) (B(y) (u_m^+(y))^2 - 1 + \frac{A(y) u_m^+(y)}{(\varepsilon + (u_m^+(y))^2)^{2p+1}} + \frac{C(y) u_m^+(y)}{(\varepsilon + (u_m^+(y))^2)^{\frac{p+1}{2}}} \right) \, dv_y \]
\[ = \int_M \liminf_m G(x_m, y) \left( B(y) (u_m^+(y))^2 - 1 + \frac{A(y) u_m^+(y)}{(\varepsilon + (u_m^+(y))^2)^{2p+1}} + \frac{C(y) u_m^+(y)}{(\varepsilon + (u_m^+(y))^2)^{\frac{p+1}{2}}} \right) \, dv_y. \]
And since the functions \( A, B, C \) are positive, then \( \liminf_m u_m(x_m) = 0 \) implies that \( u^+ = 0 \). This contradicts relation (1.22). Thus, there exists \( \delta > 0 \), such that \( u_m \geq \delta \). We can once again use Lebesgue’s dominated convergence theorem to get
\[ \frac{1}{(\varepsilon_m + u_m^2)^q} \to \frac{1}{u^{2q}} \text{ strongly in } L^p(M), \forall p \geq 1, \forall q \geq 1. \]
Since for \( m \) large enough \( u_m > 0 \) there is \( \varepsilon > 0 \) such that
\[ \frac{1}{(\varepsilon_m + u_m^2)^q} \leq \frac{1}{\varepsilon^q} \text{ with } q > 0. \]
Thus by Lebesgue’s dominated convergence theorem, we infer that
\[ \frac{1}{(\varepsilon_m + u_m^2)^q} \to \frac{1}{u^{2q}} \text{ strongly in } L^p(M), \forall p \geq 1, \forall q > 0. \]
Finally, with ii), it follows that
\[ \frac{u_k}{(\varepsilon_m + u_m^2)^{2p+1}} \to \frac{1}{u^{2p+1}} \text{ strongly in } L^2(M) \]
with \( u > 0 \). Letting \( \varepsilon_m \to 0 \) in (1.20) as \( m \to +\infty \), we get that \( u \) is a weak positive solution of equation (0.2). By the same reasoning as that of the regularity of the solution \( u_\varepsilon \) of the equation (1.20) we obtain that \( u \in C^{2k,\alpha}(M) \) with \( \alpha \in (0,1) \). Since \( u > 0 \), the right-hand-side of (0.2) has the same regularity as \( u \) and by successive iterations we obtain that \( u \in C^\infty(M) \). \( \square \)
2. Existence of a second solution

According to the previous section our functional admits a local maximum $C_{\varepsilon}$, this means the following inequalities

$$I_{\varepsilon}(t_{1}\varphi) < \Phi(t_{0}) < I_{\varepsilon}(t_{0}\varphi) \leq C_{\varepsilon}$$

where $t_{0}$, $t_{1}$ are real numbers satisfying $0 < t_{1} < t_{0}$ and $\varphi \in C^{\infty}(M)$ with $\varphi > 0$ and $\|\varphi\| = 1$.

On the other hand and as $I_{\varepsilon}(t\varphi)$ tends to $-\infty$ as $t$ goes to $+\infty$, there is $t_{2} >> t_{0}$ such that $I_{\varepsilon}(t_{2}\varphi) < 0$. Now if we let $t$ and $\varepsilon$ tend both to $0^{+}$, the functional $I_{\varepsilon}$ goes to $+\infty$. Indeed,

$$\lim_{t \to 0^{+}} \left[ \lim_{\varepsilon \to 0^{+}} I_{\varepsilon}(t, \varphi) \right] = \lim_{t \to 0^{+}} \left[ I^{(1)}(t, \varphi) + I^{(2)}_{0}(t, \varphi) \right]$$

$$= \lim_{t \to 0^{+}} \int_{M} \left[ (t, \varphi)P_{g}(t, \varphi) - \frac{2}{2^{12}} B(x)(t, \varphi)^{2^{12}} \right] dv(g)$$

$$+ \lim_{t \to 0^{+}} \left[ \frac{1}{2^{12}} \int_{M} A(x)(t, \varphi)^{2^{12}}dv(g) + \frac{1}{p - 1} \int_{M} \frac{C(x)}{(t, \varphi)^{p - 1}} dv(g) \right]$$

and it follows that for $\varepsilon$ small enough, there is near 0 a real number $0 < t' << t_{1}$ such that $I_{\varepsilon}(t'\varphi) > \Phi(t_{0}) > I_{\varepsilon}(t_{1}\varphi)$. What let us see, from this fact, that our function has a local lower bound. We will give the necessary conditions for this lower bound to exist, then we show by Ekeland’s lemma that this lower bound is reached.

We will need the following version of the Ekeland’s lemma (see [11])

**Lemma 3.** Let $V$ be a Banach space, $J$ be a $C^{1}$ lower bounded function on a closed subset $F$ of $V$ and $c = \inf_{F} J$. Let $u_{\varepsilon} \in F$ such that $c \leq J(u_{\varepsilon}) \leq c + \varepsilon$. Then there is $\overline{u}_{\varepsilon} \in F$ such that

$$\left\{ \begin{array}{l}
\quad c \leq J(\overline{u}_{\varepsilon}) \leq c + \varepsilon \\
\quad \|\overline{u}_{\varepsilon} - u_{\varepsilon}\|_{V} \leq 2\sqrt{\varepsilon} \\
\quad \forall u \in F, u \neq \overline{u}_{\varepsilon}, J(u) - J(\overline{u}_{\varepsilon}) + \sqrt{\varepsilon} \|u - \overline{u}_{\varepsilon}\|_{V} > 0.
\end{array} \right.$$ 

If moreover, $\overline{u}_{\varepsilon}$ is in the interior of $F$, then

$$\|DJ(\overline{u}_{\varepsilon})\|_{V} \leq \sqrt{\varepsilon}.$$ 

We can consider the sequence $(u_{\varepsilon})_{\varepsilon}$ in the interior of $F$. Indeed if $u_{\varepsilon}$ is on the border of $F$ then by the continuity of $J$ there is $\overline{u}_{\varepsilon} \in F$ belonging to interior of $F$ such that $|J(\overline{u}_{\varepsilon}) - J(u_{\varepsilon})| < \varepsilon$. Which gives, for $\varepsilon$ sufficiently, $c - \varepsilon < J(\overline{u}_{\varepsilon}) < c + 2\varepsilon$ and $J(u) - J(\overline{u}_{\varepsilon}) + \sqrt{\varepsilon} \|u - \overline{u}_{\varepsilon}\|_{V} = J(u) - J(u_{\varepsilon}) + J(u_{\varepsilon}) - J(\overline{u}_{\varepsilon}) + \sqrt{\varepsilon} \|u - u_{\varepsilon} + u_{\varepsilon} - \overline{u}_{\varepsilon}\|_{V}$

$$\geq J(u) - J(u_{\varepsilon}) - \varepsilon + \sqrt{\varepsilon} \|u - u_{\varepsilon}\|_{V} - \sqrt{\varepsilon} \|u_{\varepsilon} - \overline{u}_{\varepsilon}\|_{V} > J(u) - J(u_{\varepsilon}) + \sqrt{\varepsilon} \|u - u_{\varepsilon}\|_{V} - 2\varepsilon > 0.$$ 

So we can speak about the differential $DJ(u_{\varepsilon})$.

Before beginning the proof of the Theorem 2 we will establish some preliminary lemmas.
Lemma 4. Let $\theta > 0$ such that

$$\left(\frac{a}{2}\right)^{\frac{2}{3p}} < \theta^2 < a^{\frac{2}{3p}}$$

where

$$a = \frac{1}{(2(n-k))^\frac{2}{3p}}$$

and put

$$t_3 = \left(\frac{a}{8}\right)^{\frac{1}{3p}} t_0$$

then we have the following inequality

$$\Phi(t_3) > \frac{a}{8} \Phi(t_0).$$

Proof. Since $(t_1$ being defined as in lemma 1)

$$t_3 = \left(\frac{a}{8}\right)^{\frac{1}{3p}} t_0 < \theta t_0 = t_1$$

and

$$\left(\frac{a}{8}\right)^{\frac{2}{3p}} > \frac{a}{8}.$$

by (2.2), we get

$$\Phi(t_3) = \frac{1}{2} t_3^2 - \left(S. \max M B(x)\right) t_3^{2p}$$

$$= \frac{1}{2} \left[ \left(\frac{a}{8}\right)^{\frac{1}{3p}} t_0 \right]^2 - \frac{1}{2} t_0^2 \frac{a}{8}$$

$$= \frac{n}{k} \left[ \frac{1}{2} \left(\frac{a}{8}\right)^{\frac{2}{3p}} - \frac{a}{8} \left(\frac{n-2k}{2n}\right)^{\frac{1}{p}} \right] t_0^2$$

Knowing by (1.4) that

$$\frac{k t_0^2}{n} = \Phi(t_0)$$

we deduce

$$\Phi(t_3) = \left[ \frac{n}{2k} \left(\frac{a}{8}\right)^{\frac{2}{3p}} - \frac{a}{8} \right] \Phi(t_0)$$

$$> \frac{a}{8} \Phi(t_0).$$

Where we used the inequality (2.3) in the last line.  \(\square\)

Lemma 5. Given a Riemannian compact manifold $(M, g)$ of dimension $n > 2k$, $k \in \mathbb{N}^*$ and $3 < p < 2^* + 1$.

If

$$\int_M Q_g dv_g \neq k(n-1) \omega_n$$

(2.4)
where $\omega_n$ is the volume of the round sphere; then there is a constant $\lambda^* > 0$ such that: $\forall \varepsilon \in [0, \lambda^*[$ the following inequality take place

$$(2.5) \quad \int_M \frac{A(x)}{(\varepsilon + (t_3 \cdot \varphi)^2)^{\frac{\sigma}{2}}} \, dv_g \geq \frac{8 - a}{a} \left( \int_M \sqrt{A(x)} \, dv_g \right)^2 \frac{(n - 2k)Q_g}{2t_0^2},$$

where $t_3, a$ are chosen as in lemma $4$.

**Proof.** Let $\varphi \in C^\infty(M), \varphi > 0$ in $M$ with $\|\varphi\| = 1$. Put

$$(2.6) \quad \beta_1 = \left[ \left( \frac{8}{8 - a} \right)^{\frac{\sigma}{2}} - 1 \right] \frac{\Omega_1}{V(M)^{\frac{\sigma}{2}}}$$

and

$$(2.7) \quad \beta_2 = \left[ \left( \frac{1}{a} \right)^{\frac{\sigma}{2}} - 1 \right] \frac{\Omega_2}{V(M)^{\frac{\sigma}{2}}}$$

where $V(M)$ denotes the volume of $M$ and

$$\Omega_1 = \left( \frac{2a}{8(n - 2k)Q_g} \right)^{\frac{\sigma}{2}} t_0^2,$$

$$\Omega_2 = \left( \frac{2(a^{\frac{\sigma}{2}} + 1)}{(n - 2k)Q_g} \right)^{\frac{\sigma}{2}} t_0^2.$$ 

Let

$$\lambda^* = \min (\beta_1, \beta_2).$$

By Hölder’s inequality, we get:

$$(2.8) \quad \left( \int_M \sqrt{A(x)} \, dv_g \right)^2 \leq I \left[ \|\varepsilon + (t_3 \cdot \varphi)^2\|_{\frac{\sigma}{2}} \right]^{\frac{\sigma}{2}}$$

where

$$I = \int_M \frac{A(x)}{(\varepsilon + (t_3 \cdot \varphi)^2)^{\frac{\sigma}{2}}} \, dv_g$$

Independently, the Minkowski’s inequality can be written

$$\|\varepsilon + (t_3 \cdot \varphi)^2\|_{\frac{\sigma}{2}} \leq \|\varepsilon\|_{\frac{\sigma}{2}} + t_3^2 \|\varphi^2\|_{\frac{\sigma}{2}}$$

consequently

$$(2.9) \quad \left[ \|\varepsilon + (t_3 \cdot \varphi)^2\|_{\frac{\sigma}{2}} \right]^{\frac{\sigma}{2}} \leq \left( \|\varepsilon\|_{\frac{\sigma}{2}} + t_3^2 \|\varphi^2\|_{\frac{\sigma}{2}} \right)^{\frac{\sigma}{2}}$$

Notice that

$$\|\varepsilon\|_{\frac{\sigma}{2}} = \varepsilon \cdot [V(M)]^{\frac{\sigma}{2}}$$
and
\[ \| \varphi^2 \|_{\frac{2}{T}} = \| \varphi \|_{\frac{2}{T}}^2. \]

From the conformal rule (0.1) of the GJMS operator \( P_g \) we have \( P_g(\varphi) = \frac{n-2k}{2} Q_g \cdot \varphi^{2^* - 1} \) after multiplication by \( \varphi \) and integration over the manifold \( M \), we get
\[ \| \varphi \|^2 = \int_M \varphi P_g(\varphi) dv_g = \frac{n-2k}{2} \int_M Q_g \cdot \varphi^{2^*} dv_g. \]

Now, by the work done in [3] and under the condition in (2.4) we can do a conformal change of the metric \( g \) to a new metric \( \tilde{g} \) such that \( Q_{\tilde{g}} \) is a constant which we suppose positive, hence
\[ \| \varphi \|^2 = \frac{n-2k}{2} Q_{\tilde{g}} \| \varphi \|_{2^*}^2 \]

since \( \| \varphi \| = 1 \) we get
\[ (2.10) \quad \| \varphi \|_{2^*}^2 = \left( \frac{2}{(n-2k)Q_{\tilde{g}}} \right)^\frac{2}{2^*} \]

and therefore (2.9) becomes
\[ \left( \| \varepsilon + (t_3 \varphi)^2 \|_{\frac{2}{T}}^2 \right)^\frac{2^*}{2^*} \leq \left( \varepsilon [V(M)]^\frac{2}{2^*} + t_3^2 \left( \frac{2}{(n-2k)Q_{\tilde{g}}} \right)^\frac{2}{2^*} \right)^\frac{2^*}{2^*}. \]

Taking account of
\[ t_3 = \left( \frac{a}{8} \right)^\frac{1}{2^*} t_0 \]

(2.8) is written as
\[ \left( \int_M \sqrt{A(x)} dv_g \right)^2 \leq I \left( \varepsilon V(M)^\frac{2}{2^*} + \left( \frac{a}{8} \right)^\frac{2}{2^*} t_0 \left( \frac{2}{(n-2k)Q_{\tilde{g}}} \right)^\frac{2}{2^*} \right)^\frac{2^*}{2^*}. \]

And since \( 0 < \varepsilon < \lambda^* \leq \beta_1 \), we get
\[ \left( \int_M \sqrt{A(x)} dv_g \right)^2 \leq I \left( \left( \frac{8}{8-a} \right)^\frac{2}{2^*} - 1 \right) \Omega_1 + \Omega_1 \leq I \left( \frac{8}{8-a} \right)^2 \Omega_1^2 \Omega_1 \leq I \left( \frac{8}{8-a} \right) \left( \frac{2a}{8(n-2k)Q_{\tilde{g}}} \right)^\frac{2}{2^*} t_0^2 \leq I \left( \frac{8}{8-a} \right) \left( \frac{2a}{8(n-2k)Q_{\tilde{g}}} \right)^\frac{2}{2^*} t_0^2 \right)^\frac{2^*}{2^*}. \]
Finally, we deduce
\[
I \geq \left( \int_M \sqrt{A(x)} \, dv_g \right)^2 \left( \frac{8 - a}{a} \right) \frac{(n - 2k)Q_{\tilde{g}}}{2t_0^{2g}}.
\]

Now we are able to prove the existence of a second solution to equation (0.2), that is to say the proof of theorem 2.

Proof. The proof will be done in four steps.

1st step. The functional \( I_\varepsilon \) has a local lower bound.

This consists to find a strictly positive real number \( \lambda^* \) such that \( \forall \varepsilon \in ]0, \lambda^*[ \) one has the following inequality

\[
I_\varepsilon(t_3\varphi) > \Phi(t_0) \quad \forall \varphi \in C^\infty(M), \|\varphi\| = 1, \text{ with } t_3 < t_1.
\]

Indeed, according to Lemma 4 inequality (2.2) and inequality (1.2), one has

\[
I_\varepsilon(t_3\varphi) = I^{(1)}(t_3\varphi) + I^{(2)}(t_3\varphi)
\]

\[
> \frac{a}{8} \Phi(t_0) + \frac{1}{2^2} \int_M \frac{A(x)}{(\varepsilon + (t_3\varphi)^2)^{\frac{p}{2}}} dv_g
\]

\[
+ \frac{1}{p - 1} \int_M \frac{C(x)}{(\varepsilon + (t_3\varphi)^2)^{\frac{p - 1}{2}}} dv_g
\]

and as by assumption

\[
\left( \int_M \sqrt{A(x)} \, dv_g \right)^2 \left( \frac{8 - a}{a} \right) \frac{(n - 2k)Q_{\tilde{g}}}{2t_0^{2g}} > 2^2 \frac{k}{n} \frac{t_0^2}{t_0^2} (1 - \frac{a}{8})
\]

and

\[
\lambda^* = \min(\beta_1, \beta_2)
\]

knowing that

\[
\Phi(t_0) = \frac{k}{n} \frac{t_0^2}{t_0^2}
\]

it follows by Lemma 5 that, \( \forall \varepsilon \in ]0, \lambda^*[ \)

\[
\frac{1}{2^2} \int_M \frac{A(x)}{(\varepsilon + (t_3\varphi)^2)^{\frac{p}{2}}} dv_g > \left( 1 - \frac{a}{8} \right) \Phi(t_0).
\]

Finally, by combination of (2.11), (2.13) and the fact that the function \( C > 0 \), we get

\[
I_\varepsilon(t_3\varphi) > \frac{a}{8} \Phi(t_0) + \left( 1 - \frac{a}{8} \right) \Phi(t_0) > \Phi(t_0).
\]
Hence our result.

2nd step. The infimum of the functional $I_\epsilon$ is reached.

Denote by $\overline{B}(0, t_1) = \{ u \in H^2_k(M) : \|u\| \leq t_1 \}$ the closed ball centred at the origin 0 of radius $t_1$ in $H^2_k(M)$. In this section we will show that $c_\epsilon = \inf_{\overline{B}(0, t_1)} I_\epsilon$ ( $c_\epsilon < \Phi(t_0)$ ) is reached. By Ekeland’s Lemma, there exists a sequence $(u_m)_{m \in \mathbb{N}}$ in $B(0, t_1)$ such that $I_\epsilon(u) \to c_\epsilon = \inf_{\overline{B}(0, t_1)} I_\epsilon$ and $DI_\epsilon(u_m) \to 0$ strongly in the dual space of $H^2_k(M)$. That is to say $(u_m)$ is a Palais-Smale sequence, so by the same arguments as in Theorem 2 and Theorem 1 we get that equation (0.2) has a smooth positive solution $v$.

3rd step. The two solutions are distinct.

To show that the two solutions $u$ and $v$ are different, we will verify that their respective energies are different.

Put $t_4 = a\frac{t_0}{2}$, it is clear that $t_4 > t_1$ (see the assumptions of Lemma 1) and since $\|v\| \leq t_1$ then if $\|u\| \geq t_4$, $u \neq v$. So we may suppose that $\|u\| < t_4$.

Imitating the computations made in the previous section and taking into account that in this time we take $\epsilon \leq \beta_2$, it is not hard to get

$$\int_M \frac{A(x)}{\left( \epsilon + (t_4 u)^2 \right)^{2\sigma}} dv_g \geq \left( \frac{n-2k}{2 \cdot a \cdot t_0^{2\sigma+2}} \right) \left( \int_M \sqrt{A(x)} dv_g \right)^2 .$$

Now, since it is easy to see that

$$\frac{1}{2^\sigma} \int_M \frac{A(x)}{u^{2^\sigma}} dv_g \geq \frac{t_4^{2\sigma}}{2^\sigma} \int_M \frac{A(x)}{\left( \epsilon + (t_4 u)^2 \right)^{2\sigma}} dv_g$$

and by the hypothesis (0.9) of Theorem 2 we infer that

$$\frac{1}{2^\sigma} \int_M \frac{A(x)}{u^{2^\sigma}} dv_g \geq \frac{a \cdot t_0^{2^\sigma-2}}{8} \Phi(t_0)$$

where $\Phi(t_0) = \kappa \frac{t_0^2}{n}$. Now, we will estimate the energy of the solution $u$

$$I(u) = \frac{1}{2} \|u\|^2 - \frac{1}{2^\sigma} \int_M B(x) u^{2^\sigma} dv_g + \frac{1}{2^\sigma} \int_M \frac{A(x)}{u^{2^\sigma}} dv_g + \frac{1}{(p-1)} \int_M \frac{C(x)}{u^{p-1}} dv_g$$

and since

$$\|u\|^2 = \int_M B(x) u^{2^\sigma} dv_g + \int_M \frac{A(x)}{u^{2^\sigma}} dv_g + \int_M \frac{C(x)}{u^{p-1}} dv_g$$
we deduce that

\[ I(u) = \frac{k}{n} \|u\|^2 + \left( \frac{1}{2^\sigma} + \frac{1}{2}\right) \int_M \frac{A(x)}{u^{2\sigma}} dv_g + \left( \frac{1}{2^\sigma} + \frac{1}{(p-1)} \right) \int_M \frac{C(x)}{u^{p-1}} dv_g \]

\[ \geq \frac{2}{2^\sigma} \int_M \frac{A(x)}{u^{2\sigma}} dv_g \]

\[ \geq 2 \frac{a t_0^{2^\sigma - 2}}{8} \Phi(t_0). \]

and taking into account of the value \( a = \frac{1}{(2(n-k))^2} \) and the fact that

\[ 0 < \left( S_{\text{max}} B(x) \right) \left( \frac{a}{4} \right) \]

we infer

\[ I(u) > \Phi(t_0). \]

Since the energy \( I(v) \) of the solution \( v \) is less than \( \Phi(t_0) \) we conclude that \( u \neq v \).

4th-step. The conditions of the theorem intersect.

Indeed, let us rewrite the condition (0.9) of Theorem 2

\[ \frac{1}{2^\sigma} \left( \int_M \sqrt{A(x)} dv_g \right)^2 > \frac{k}{8n} t_0^{2^\sigma + 2^\sigma} \frac{2a}{(n-2k)Q_g}. \]

By Hölder inequality, we get

\[ \int_M \sqrt{A(x)} dv_g = \int_M \sqrt{\frac{A(x)}{\varphi^{2\sigma}}} \varphi^{\frac{1}{2}} dv_g \]

\[ \leq \left( \int_M \frac{A(x)}{\varphi^{2\sigma}} dv_g \right) ^{\frac{1}{2}} \left( \int_M \varphi^{\sigma} dv_g \right) ^{\frac{1}{2}} \]

\[ \leq ||\varphi||_{2^\sigma}^{\frac{1}{2}} \left( \int_M \frac{A(x)}{\varphi^{2\sigma}} dv_g \right) ^{\frac{1}{2}} \]

(2.15)

From equality (2.10) in the proof of Lemma 2.3 and the fact that \( ||\varphi|| = 1 \), it comes that

\[ \frac{1}{2^\sigma} \left( \int_M \sqrt{A(x)} dv_g \right)^2 \leq \frac{1}{2^\sigma} \left( \frac{2}{(n-2k)Q_g} \right) \int_M \frac{A(x)}{\varphi^{2\sigma}} dv_g \]

with the condition (0.7) of Theorem 2, we obtain

\[ \frac{1}{2^\sigma} \left( \int_M \sqrt{A(x)} dv_g \right)^2 \leq \left( \frac{2}{(n-2k)Q_g} \right) C(n, p, k) \left( S_{\text{max}} B(x) \right)^{\frac{2^\sigma + 2^\sigma}{2^\sigma}}. \]
Since $3 < p < 2^2 + 1$, we may take $C(n,p,k) = C_1(n,p,k)$ and we have also $	heta^{2^2} \leq \theta^{p-1}$ and therefore

\[
(2.16) \quad \frac{1}{2^2} \left( \int_M \sqrt{A(x)} dv_g \right)^2 \leq \left( \frac{2}{(n-2k)Q_g} \right) \frac{2k-1}{4n} \theta^{2^2} t_0^{2+2^2}.
\]

By combining conditions (2.14) and (2.16), we get the following double inequality

\[
k \geq \frac{a}{2} < \frac{2n(n-2k)Q_g t_0^{-2^2}}{2^2} \left( \int_M \sqrt{A(x)} dv_g \right)^2 \leq \left( \frac{2}{(n-2k)Q_g} \right) \frac{2k-1}{4n} \theta^{2^2} t_0^{2+2^2}.
\]

Which in turn is equivalent to

\[
k \geq \frac{a}{2} < \frac{2n(n-2k)Q_g t_0^{-2^2}}{2^2} \left( \int_M \sqrt{A(x)} dv_g \right)^2 \leq \left( \frac{2}{(n-2k)Q_g} \right) \frac{2k-1}{4n} \theta^{2^2} t_0^{2+2^2}.
\]

and since $\frac{a}{2} < \theta^{2^2}$ we get

\[
1 < \frac{4n(n-2k)Q_g t_0^{-2^2}}{2^2 a,k} \left( \int_M \sqrt{A(x)} dv_g \right)^2 \leq \left( 2 - \frac{1}{k} \right)
\]

with $k \geq 1$. The smooth functions $A$ that fulfill the assumptions (0.6) and (0.8) of theorem 2 are those that satisfy the following double inequality

\[
C < A \leq \left( 2 - \frac{1}{k} \right) C
\]

where $C = \frac{2^2 a,k}{4n(n-2k)Q_g t_0^{-2^2} V(M)^2}$ and $V(M)$ is the volume of $M$. \hfill \square

3. Nonexistence of solution

In this section we will be placed in a closed ball $\overline{B}(0, R)$ of $H^2_k(M)$ centered at the origin $0$ and of radius $R > 0$, we prove that under some condition (inequality 0.11 of Theorem 3) that the equation (0.2) has no solution i.e. we will give the proof of Theorem 3.

Proof. (Proof of Theorem 3) Suppose that there exists a smooth positive solution $u \in H^2_k(M)$ such that $\|u\|_{H^2_k(M)} \leq R$. By multiplying both sides of equation (1.1) by $u$ and integrating over $M$, we get

\[
\int_M u P_g(u) dv_g = \int_M \left( B(x) u_{2^2} + \frac{A(x)}{u_{2^2}} + \frac{C(x)}{u^{p-1}} \right) dv_g.
\]
And since $\|u\|_{p_g} = \sqrt{\int_M u P_g(u) dv_g}$ is a norm equivalent to $\|u\|_{H^2_k(M)}$, there exists a constant $S > 0$ such that

$$\|u\| \leq S \|u\|_{H^2_k(M)}.$$ 

Then it follows that

$$(3.1) \int_M \left( B(x) u^{2^*} + \frac{A(x)}{u^{2^*}} + \frac{C(x)}{u^{p-1}} \right) dv_g \leq (SR)^2.$$ 

Moreover Hölder’s inequality allows us to write

$$(3.2) \int_M \sqrt{B(x)C(x)} dv_g \leq \left( \int_M \frac{C(x)}{u^{p-1}} dv_g \right)^{\frac{1}{2}} \left( \int_M B(x) u^{p-1} dv_g \right)^{\frac{1}{2}}.$$

By applying again the Hölder’s inequality, one finds

$$\int_M B(x) u^{p-1} dv_g = \int_M B(x)^{1-\frac{1}{2^*}} \left( B(x)^{\frac{1}{2^*}} u \right)^{p-1} dv_g$$

$$\leq \left( \int_M B(x) dv_g \right)^{\frac{2^* - p + 1}{2^*}} \left( \int_M \left( B(x)^{\frac{1}{2^*}} u \right)^{p-1} dv_g \right)^{\frac{2^*}{p-1}} \left( \int_M B(x)^{p-1} dv_g \right)^{\frac{p-1}{2^*}}.$$ 

So

$$(3.3) \int_M \sqrt{B(x)C(x)} dv_g \leq \left( \int_M B(x) dv_g \right)^{\frac{2^* - p + 1}{2^*}} \left( \int_M B(x) u^{2^*} dv_g \right)^{\frac{p-1}{2^*}} \int_M \frac{C(x)}{u^{p-1}} dv_g.$$ 

Letting

$$D = \left( \int_M \sqrt{B(x)C(x)} dv_g \right)^{2} \left( \int_M B(x) dv_g \right)^{\frac{p-2^*-1}{2^*}}$$

it comes

$$\int_M \frac{C(x)}{u^{p-1}} dv_g \geq D \left( \int_M B(x) u^{2^*} dv_g \right)^{\frac{1-p}{2^*}}.$$ 

and therefore (3.1) becomes

$$(SR)^2 \geq \int_M B(x) u^{2^*} dv_g + \int_M \frac{A(x)}{u^{2^*}} dv_g + D \left( \int_M B(x) u^{2^*} dv_g \right)^{\frac{1-p}{2^*}}.$$ 

Since $A$ is of positive values, then

$$\int_M \frac{A(x)}{u^{2^*}} dv_g \geq 0.$$
so it comes that

\[(SR)^2 \geq \int_M B(x) u^2\,dv_g + D\left(\int_M B(x) u^2\,dv_g\right)^{\frac{1-p}{2^*}}\]

and if we set

\[t = \int_M B(x) u^2\,dv_g\]

we obtain

\[(RS)^2 \geq f(t)\]

where

\[f(t) = t + Dt^{\frac{1-p}{2^*}}.\]

\(f\) has a minimum at

\[t_0 = \left(\frac{p - 1}{2^*}D\right)^{\frac{2^*}{2^*+p-1}}\]

and consequently

\[\forall t > 0, f(t) \geq \min_{t>0} f(t) = f(t_0) = \frac{2^* + p - 1}{p - 1} \left(\frac{p - 1}{2^*}D\right)^{\frac{2^*}{2^*+p-1}}.\]

Finally, replacing \(D\) by its value, we obtain

\[(RS)^2 \geq \frac{2^* + p - 1}{p - 1} \left(\frac{p - 1}{2^*}\right)^{\frac{2^*}{2^*+p-1}} \left(\int_M \sqrt{B(x)} C(x)\,dv_g\right)^{\frac{2^*}{2^*+p-1}} \left(\int_M B(x)\,dv_g\right)^{\frac{p - 1 - 2^*}{2^*+p-1}}.\]

and if we set

\[C(n,p,k) = \frac{2^* + p - 1}{p - 1} \left(\frac{p - 1}{2^*}\right)^{\frac{2^*}{2^*+p-1}}\]

then it comes that

\[(RS)^2 \geq C(n,p,k) \left(\frac{\int_M \sqrt{B(x)} C(x)\,dv_g}{\int_M B(x)\,dv_g}\right)^{\frac{2^*}{2^*+p-1}} \int_M B(x)\,dv_g.\]

\[\square\]

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