A Convex Maximization Problem: Continuous Case

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Abstract. We study a specific convex maximization problem in the space of continuous functions defined on a semi-infinite interval. An unexplained connection to the discrete version of this problem is investigated.

1. Problem
Consider the class $X$ of continuous functions $f : [1, \infty) \to \mathbb{R}$ satisfying

$$f(x) \geq x \quad \text{for} \quad 1 \leq x < 2$$

and

$$(y + 1)f(y) + f(x) \geq (y + 1)x \quad \text{for} \quad x \geq 2, \quad 1 \leq y \leq x.$$ 

Prove that:

(i) $x^c \in X$ if and only if $c \geq \frac{1 + \sqrt{5}}{2}$ (thus $X$ is nonempty)

(ii) $\sup_{f \in X} \int_{1}^{\infty} \frac{1}{f(x)} \, dx < \infty$

(iii) the supremum in (ii) is attained by the function $a \in X$ defined by

$$a(x) = \begin{cases} 
  x & \text{if} \quad 1 \leq x < 2 \\
  2(x - 1) & \text{if} \quad 2 \leq x < 3 \\
  3(x - 2) & \text{if} \quad 5 \leq x < 8
\end{cases}$$

on $[1, 3) \cup [5, 8)$ and elsewhere by

$$a(x) = (y + 1)(x - a(y))$$

where $y$ satisfies $(y + 1)a'(y) + a(y) = x$ and $a'(y)$ is the derivative of $a(y)$.

Remark. Part (iii) is, in essence, the continuous analog of a certain number theoretic conjecture due to Levine and O’Sullivan [3].
2. Partial Solution

After proving part (i), our treatment will be brief and rather informal. We do not prove that the conjectured maximizing function \(a(x)\) is well-defined nor that it is feasible \((a \in X)\). Our purpose is to compute the function \(a(x)\) as far as possible (assuming it makes sense!), and to describe a link between \(a(x)\) and the discrete version \(a_n\) studied in a companion paper.

2.1. Proof of (i). Define a family of functions

\[
\Phi_y(x) = (y + 1) y^c + x^c - (y + 1) x
\]

for all \(x \geq 0\) and \(y \geq 1\). It is sufficient to show that

(a) given \(1 < c < \frac{1 + \sqrt{5}}{2}\), there exist \(x \geq 2\) and \(1 \leq y \leq x\) with \(\Phi_y(x) < 0\)

(b) given \(c = \frac{1 + \sqrt{5}}{2}\), \(\Phi_y(x) \geq 0\) for all \(x \geq 0\) and \(y \geq 1\).

Observe that, for fixed \(y \geq 1\), \(\Phi_y\) is minimized (via calculus) at the point

\[
x_y = \left(\frac{y + 1}{c}\right)^{\frac{1}{c-1}}
\]

with minimum value

\[
\Phi_y(x_y) = (y + 1) \left[y^c + x_y \left(\frac{1}{c} - 1\right)\right] < \Phi_y(0)
\]

(since \(c > 1\)). Part (a) follows because

\[
\lim_{y \to \infty} \Phi_y(x_y) = -\infty
\]

(since \(1/(c-1) > c\)). Part (b) follows by contradiction, because if \(\Phi_y(x_y) < 0\), then

\[
y^c - \left(\frac{y + 1}{c}\right)^c \frac{1}{c^2} < 0
\]

(since \(1/(c-1) = c\)), which implies

\[
\left(\frac{y + 1}{y}\right)^c > c^{c+2}
\]

that is,

\[
y < \frac{1}{c^{(c+2)(c-1)} - 1} < 1
\]

But this is contrary to the hypothesis that \(y \geq 1\). QED.
2.2. **Comment on** (ii). A proof is not presently known, although it is expected to follow in a manner similar to that in the discrete case [1, 3].

2.3. **Comments on** (iii). Again, a proof is not presently known. Calculus allows us, however, to recursively unwrap the "self-generating" nature of the function \( a(x) \) to obtain some useful formulas. For example, if \( 1 \leq y < 2 \), then \( a(y) = y \), \( a'(y) = 1 \) and \( (y + 1) + y = x \); hence \( x = 2y + 1 \) and \( y = \frac{1}{2}(x - 1) \). We deduce that 
\[
 a(x) = \left( \frac{1}{2} (x - 1) + 1 \right) \left( x - \frac{1}{2} (x - 1) \right) = \frac{1}{4} (x + 1)^2 \text{ for } 3 \leq x < 5.
\]

Likewise, if \( 2 \leq y < 3 \), then \( a(y) = 2(y - 1), a'(y) = 2 \) and \( 2(y + 1) + 2(y - 1) = x \); hence \( x = 4y \) and \( y = \frac{1}{4}x \). We deduce that 
\[
 a(x) = \left( \frac{1}{4}x + 1 \right) \left( x - 2 \left( \frac{1}{4}x - 1 \right) \right) = \frac{1}{8}(x + 4)^2 \text{ for } 8 \leq x < 12.
\]

Proceeding similarly, the following is obtained:

\[
a(x) = \begin{cases} 
  x & \text{if } 1 \leq x < 2 \\
  2(x - 1) & \text{if } 2 \leq x < 3 \\
  \frac{1}{4}(x + 1)^2 & \text{if } 3 \leq x < 5 \\
  3(x - 2) & \text{if } 5 \leq x < 8 \\
  \frac{1}{8}(x + 4)^2 & \text{if } 8 \leq x < 12 \\
  4 \left( \frac{x}{3} \right)^{3/2} & \text{if } 12 \leq x < 27 \\
  \frac{1}{12}(x + 9)^2 & \text{if } 27 \leq x < 45 \\
  \frac{1}{4} \left[ 1 - 8x + \left( \frac{8x + 3}{3} \right)^{3/2} \right] & \text{if } 45 \leq x < 84 \\
  (z^2 + 1) \left( x - \frac{4z^3}{3^{3/2}} \right) & \text{if } 84 \leq x < 276 \\
  \frac{4}{81} \left[ 64 - 108x + (9x + 16)^{3/2} \right] & \text{if } 276 \leq x < 657 \\
  \left( \frac{w^2 + 5}{8} \right) \left( x - 1 + \frac{w^2}{4} - \frac{w^4}{4 \cdot 3^{3/2}} \right) & \text{if } 657 \leq x \leq 1781
\end{cases}
\]

where the auxiliary variables \( z \) and \( w \) are defined by

\[
z = \frac{1}{20^{1/3}} \left[ \left( \frac{135x^2 + 16}{5} \right)^{1/2} + 3^{3/2}x \right]^{1/3} - \frac{20^{1/3}}{5} \left[ \left( \frac{135x^2 + 16}{5} \right)^{1/2} + 3^{3/2}x \right]^{-1/3}
\]

and

\[
w = \frac{1}{5} \left[ 10 \left( 2700x^2 + 2106x + 289 \right)^{1/2} + 3^{1/2} \left( 300x + 117 \right) \right]^{1/3} + \frac{23}{5} \left[ 10 \left( 2700x^2 + 2106x + 289 \right)^{1/2} + 3^{1/2} \left( 300x + 117 \right) \right]^{-1/3} + \frac{4 \cdot 3^{1/2}}{5}
\]
We have not attempted to determine $a(x)$ for $x > 1781$.

2.4. Alternative Expression. A more compact, but less explicit formula for $a(x)$ is as follows:

$$a(x) = \begin{cases} 
  x & \text{if } 1 \leq x < 2 \\
  \max_{1 \leq y < x} (y + 1)(x - a(y)) & \text{if } x \geq 2 
\end{cases}$$

For example, if $2 \leq x < 3$, then

$$\max_{1 \leq y \leq 2} (y + 1)(x - y) = 2(x - 1)$$

since the maximum cannot occur at $y = \frac{1}{2}(x - 1) < 1$, hence it must occur at one of the endpoints $y = 1$ or $y = 2$. Since $2(x - 1) > 3(x - 2)$ for $x < 4$, the claim is true. Suppose now that

$$a(x) = \max_{1 \leq y < x} (y + 1)(x - a(y)) > 2(x - 1)$$

This, in turn, implies that

$$2(x - 1) < \max_{2 \leq y < x} (y + 1)(x - a(y)) \leq \max_{2 \leq y \leq x} (y + 1)(x - 2(y - 1)) = 3(x - 2)$$

because the maximum cannot occur at $y = \frac{1}{4}x < 1$. But $3(x - 2) < 2(x - 1)$, which yields a contradiction. Therefore $a(x) = 2(x - 1)$ for $2 \leq x < 3$.

Likewise, if $3 \leq x < 5$, then

$$\max_{1 \leq y \leq 2} (y + 1)(x - y) = \frac{1}{4}(x + 1)^2$$

since the maximum here occurs at $y = \frac{1}{2}(x - 1)$ and $1 \leq y < 2$; and if $5 \leq x < 8$, then

$$\max_{1 \leq y \leq 2} (y + 1)(x - y) = 3(x - 2)$$

since the maximum cannot occur at $y = \frac{1}{2}(x - 1) > 2$. Similar *reductio ad absurdum* reasoning gives

$$a(x) = \begin{cases} 
  \frac{1}{4}(x + 1)^2 & \text{if } 3 \leq x < 5 \\
  3(x - 2) & \text{if } 5 \leq x < 8 
\end{cases}$$

as was to be proved.

An equivalence proof applicable for all $x \geq 8$ is not known. It will be necessary to demonstrate that subinterval maximums always occur at interior points, that is, at points where the derivative $a'(x)$ vanishes.
3. Link to Discrete Case

In a companion paper, we studied the infinite sequence \(a_1, a_2, \ldots\), defined by

\[
a_1 = 1, \quad a_2 = 2, \quad a_3 = 4
\]

and, when \(i \geq 4\),

\[
a_i = (j + 1)(i - a_j)
\]

where \(j\) satisfies \((j + 1)(a_j - a_{j-1}) + a_{j-1} \leq i \leq (j + 2)(a_{j+1} - a_j) + a_j\).

An analogous alternative expression

\[
a_i = \begin{cases} 
1 & \text{if } i = 1 \\
\max_{1 \leq j < i} (j + 1)(i - a_j) & \text{if } i \geq 2
\end{cases}
\]

applies here (although in this case a rigorous equivalence proof is known). Such structural similarity leads us to expect a vague connection between the sequence \(a_i\) and the function \(a(x)\), but the precise nature of the link is difficult to anticipate. We empirically observe that the non-analytic points \(k\) of \(a(x)\), that is, the subinterval endpoints in the definition of \(a(x)\), are evidently all integers. Further, the value of \(a_k\) apparently coincides with \(a(k)\) at all such points:

\[
\begin{align*}
a_1 &= 1 = a(1) \\
a_2 &= 2 = a(2) \\
a_3 &= 4 = a(3) \\
a_5 &= 9 = a(5) \\
a_8 &= 18 = a(8) \\
a_{a12} &= 32 = a(12) \\
a_{27} &= 108 = a(27) \\
a_{a45} &= 243 = a(45) \\
a_{a84} &= 676 = a(84) \\
a_{a276} &= 4704 = a(276) \\
a_{a657} &= 19044 = a(657) \\
a_{a1781} &= 93925 = a(1781) \\
a_{a_{12460}} &= 2148412 \\
a_{a49312} &= 19916344 \\
a_{a_{245395}} &= ?
\end{align*}
\]

It is possible that this pattern breaks down at some stage beyond our computational means. We conjecture that this is not the case: that instead \(a_k\) and \(a(k)\) are equal for infinitely many integers \(k\). This intriguing correspondence between the discrete and continuous versions is presently without explanation.
REFERENCES

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