IMPROVED EXTENSIBILITY CRITERIA AND GLOBAL WELL-POSEDNESS OF A COUPLED CHEMOTAXIS-FLUID MODEL ON BOUNDED DOMAINS

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ABSTRACT. This paper is contributed to the qualitative analysis of a coupled chemotaxis-fluid model on bounded domains in multiple spatial dimensions. Based on scaling-invariant argument and energy method, several optimal extensibility criteria for local classical solutions are established. As a by-product, a global well-posedness result is obtained in the two-dimensional case for general initial data.

1. Introduction. In this paper, we consider a coupled chemotaxis-fluid model:

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla P &= \nu \Delta u - n \nabla \phi, \\
\nabla \cdot u &= 0, \\
\frac{\partial n}{\partial t} + u \cdot \nabla n &= D_n \Delta n - \nabla \cdot (n \chi(c) \nabla c), \\
\frac{\partial c}{\partial t} + u \cdot \nabla c &= D_c \Delta c - nf(c),
\end{align*}
\]

\(x \in \mathbb{R}^d (d = 2, 3), t > 0\), which originally arose from the study of the dynamics of swimming bacteria, *Bacillus subtilis*, as part of the effort to understand the interaction of hydrodynamic effects and chemotactic movement [13]. Here, \(u, P, n, c\) denote the fluid velocity field, scalar pressure, bacteria density, and oxygen concentration, respectively. The constants \(\nu, D_n, D_c\) are the kinematic viscosity coefficient, bacteria diffusion coefficient, and oxygen diffusion coefficient, respectively. The functions \(\chi(c) \geq 0\) and \(f(c) \geq f(0) = 0\) are two given smooth functions of \(c\) denoting the chemotactic sensitivity and oxygen consumption rate, respectively. In the derivation of the model, the Boussinesq approximation was applied to reflect the effect due to heavy bacteria. The function \(\phi\) denotes the potential function produced by various physical mechanisms, e.g., gravitational force \((\phi(x) = x_d)\) or centrifugal force \((\phi(x) = g(|x|)))\).

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The system (1) is closely connected with classical models in fluid mechanics and chemotaxis research. On one hand, when \( n = \theta = c = 0 \equiv 1 \), one gets the well-known Navier-Stokes/Euler system:
\[
\begin{cases}
\partial_t u + u \cdot \nabla u + \nabla P = \nu \Delta u, \quad \nu \geq 0, \\
\nabla \cdot u = 0,
\end{cases}
\]
which gave birth to mathematical fluid mechanics. On the other hand, when the hydrodynamic effect drops, the model reduces to the classic Keller-Segel model of chemotaxis:
\[
\begin{cases}
\partial_t n = D_n \Delta n - \nabla \cdot (n \chi(c) \nabla c), \\
\partial_t c = D_c \Delta c - nf(c),
\end{cases}
\]
which has prominent applications in biological modeling and has been studied intensively in the mathematics literature, see e.g. the survey papers [7, 9]. Following standard terminology, we shall call (1) the “Navier-Stokes-Keller-Segel” equations.

To the authors’ knowledge, such a model is among the first generation of PDE systems bridging classic equations in mathematical fluid mechanics and chemotaxis research. Not only does the model find interests in biosciences [13], it also brings challenging questions into the mathematical community. Next, we would like to point out the facts that motivate the current work and specify the goals of this paper.

1.1. Extensibility criteria. First of all, it is well known that the two-dimensional incompressible Navier-Stokes/Euler equations are globally well-posed in the regime of large-amplitude classical solutions. However, due to the additional nonlinearities in the chemotactic component and the couplings through potential forcing and advection, such a property is not easily enjoyed by (1) under general structural conditions on \( \chi(c) \) and \( f(c) \). Hence, as one of the mostly studied topics in analysis of partial differential equations, much attention has been paid to seek sharp extensibility criteria (sufficient conditions under which local classical solutions can be extended to be global ones) for the model under general structural conditions on \( \chi(c) \) and \( f(c) \). In this direction, we find that in [3, 4] the following extensibility criteria are established for the Cauchy problem of the model in \( \mathbb{R}^d \) when \( d = 2, 3 \):

- A local solution \((u, n, c) \in L^\infty((0, T); H^2(\mathbb{R}^d)) \cap L^2((0, T); H^3(\mathbb{R}^d))\) can be extended beyond \( T \) if
  \[
  \int_0^T \left( \|u(t)\|_{L^p(\mathbb{R}^d)}^{\frac{2d}{d-2}} + \|
abla c(t)\|_{L^\infty(\mathbb{R}^d)}^2 \right) dt < \infty, \quad \text{for some } d < p \leq \infty, \tag{2}
  \]
  provided that \( \chi(\cdot) \geq 0 \) and \( f(\cdot) \geq 0 \) are smooth functions of \( c \) [3].

- A local solution \((u, n, c) \in L^\infty((0, T); H^2(\mathbb{R}^d)) \cap L^2((0, T); H^3(\mathbb{R}^d))\) can be extended beyond \( T \) if
  \[
  \int_0^T \left( \|u(t)\|_{L^p(\mathbb{R}^d)}^{\frac{2d}{d-2}} + \|n(t)\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}^{\frac{2d}{d-2}} \right) dt < \infty, \quad \text{for some } d < p \leq \infty, \quad \frac{d}{2} < q \leq \infty, \tag{3}
  \]
  provided that \( \chi(\cdot) \geq 0 \) and \( f(\cdot) \geq 0 \) are smooth and non-decreasing functions of \( c \) [4].

We observe that (2) is not optimal from the point of view of scaling invariance. Indeed, we note that (1) is invariant under the scaling transform:
\[
\begin{align*}
u \to u_{\lambda} &:= \lambda u(\lambda^2 t, \lambda x), \\
P \to P_{\lambda} &:= \lambda^2 P(\lambda^2 t, \lambda x), \\
\phi \to \phi_{\lambda} &:= \phi(\lambda^2 t, \lambda x),
\end{align*}
\]
Hence, it is straightforward to check that the energy space
\[ L^{\frac{2\nu}{d}} \left( (0, \infty); L^{p}(\mathbb{R}^d) \right), \quad d < p \leq \infty \]
is invariant for \( u \) and \( \nabla c \), and
\[ L^{\frac{2\nu}{d}} \left( (0, \infty); L^{q}(\mathbb{R}^d) \right), \quad \frac{d}{2} < q \leq \infty \]
is invariant for \( \nabla u \) and \( n \). Our first goal of this paper is to improve (2) by taking advantage of the scaling invariance property. By using a different approach, we build a new extensibility criterion for (1), which generalizes (2) to a broader range of exponents. In particular, our new extensibility criterion replaces \( \|\nabla c(t)\|_{L^{\infty}} \) in (2) by \( \|\nabla c(t)\|_{L^{\frac{2\nu}{d}}} \) for some \( d < q \leq \infty \).

The second fact that motivates this paper is concerned with the monotonicity of \( \chi(\cdot) \) and \( f(\cdot) \) required for establishing (3). We observe that experiments in [8] suggested that \( \chi(\cdot) \) and \( f(\cdot) \) are constants at large \( c \) and rapidly approach zero below a critical threshold value \( c_* \). In [5] these functions were approximated by step functions. It is then easy to see that the monotonicity of \( \chi(\cdot) \) and \( f(\cdot) \) proposed in [4] captures the essential features of the functions designed in [5]. However, from the point of view of analysis, establishing the same criterion without appealing to the monotonicity is more desirable. In addition, removing the constraint from the extensibility criterion will make the result more biologically relevant. Hence, our second goal of this paper is to establish (3) without using the monotonicity of \( \chi(\cdot) \) and \( f(\cdot) \).

Now, we state the first result of this paper. Due to physical consideration, instead of the Cauchy problem, we consider (1) on bounded domains in \( \mathbb{R}^d \) (\( d = 2, 3 \)) with physical boundaries subject to the following initial and boundary conditions:

\[
\begin{cases}
(u, n, c)(x, 0) = (u_0, n_0, c_0)(x), & x \in \Omega, \\
u \cdot n|_{\partial \Omega} = 0, \quad \nabla n \cdot n|_{\partial \Omega} = 0 = \nabla c \cdot n|_{\partial \Omega}, & t \geq 0,
\end{cases}
\]

where \( \Omega \subset \mathbb{R}^d \) is a bounded domain with smooth boundary, and \( n \) is the unit outward normal vector to \( \partial \Omega \). The first result is stated as follows.

**Theorem 1.1.** Consider the initial-boundary value problem (1) & (4) with \( \nu > 0, D_n > 0 \) and \( D_c > 0 \). Suppose that the initial data \( (u_0, n_0, c_0) \in H^2(\Omega) \) satisfy \( \nabla \cdot u_0 = 0, n_0 \geq 0, c_0 \geq 0 \) in \( \Omega \), and the boundary conditions on \( \partial \Omega \). Suppose that \( \chi(\cdot) \geq 0, f(\cdot) \geq 0, \phi(\cdot) \) are smooth functions. Let \( (u, n, c) \in L^\infty((0, T); H^2(\Omega)) \cap L^2((0, T); H^4(\Omega)) \) be a local classical solution to (1) & (4) for some finite \( T > 0 \). Then the local solution can be extended beyond \( T \) if one of the following holds true:

\[
\begin{align*}
&\int_0^T \left( \|u(t)\|_{L^p}^{\frac{2\nu}{d}} + \|\nabla c(t)\|_{L^q}^{\frac{2\nu}{d}} \right) \, dt < \infty, & \text{for some } d < p \leq \infty, \quad d < q \leq \infty; \\
&\int_0^T \left( \|u(t)\|_{L^p}^{\frac{2\nu}{d}} + \|n(t)\|_{L^q}^{\frac{2\nu}{d}} \right) \, dt < \infty, & \text{for some } d < p \leq \infty, \quad \frac{d}{2} < q \leq \infty.
\end{align*}
\]

**Remark 1.** We would like to emphasize that the first criterion in Theorem 1.1 is much more general than (2). Indeed, (2) is a special case of the first criterion when \( q = \infty \). On the other hand, the second criterion in Theorem 1.1 is the same as (3), while it does not require the monotonicity of the functions \( \chi(\cdot) \) and \( f(\cdot) \).
Remark 2. By slightly modifying the proof of Theorem 1.1, we can establish the same extensibility criteria for the Cauchy problem of (1), which improves the results obtained in [3, 4]. In addition, by combining Theorem 1.1 and classic results for the Navier-Stokes equations (cf. [10, 12]), one can identify extensibility criteria for the Cauchy problem of (1) when the hydrodynamic component, $u$, belongs to certain Besov spaces.

Remark 3. Similar extensibility criteria can be established for the Euler-Keller-Segel system, i.e., (1) when $\nu = 0$, which is associated with fluid flows with large Reynolds numbers. For example, by combining Theorem 1.1 and classic results for the Euler equations (cf. [2, 6]), one can prove the following extensibility criterion for (1) when $\nu = 0$:

$$\int_0^T \left( \|\text{curl } u(t)\|_{L^\infty(\Omega)} + \|\nabla c(t)\|_{L^p(\Omega)}^{2p} \right) dt < \infty, \quad \text{for some } d < p \leq \infty.$$  

Furthermore, $\|\text{curl } u(t)\|_{L^\infty(\Omega)}$ in the above criterion can be replaced by certain Besov norms of $u$ by combining results for the Euler equations (cf. [11]) and Theorem 1.1. The proofs will be in the same spirit of Theorem 1.1. We shall not go through the technical details in this paper.

1.2. Global well-posedness. The previous section contains information about the local (w.r.t. time) behavior of classical solutions to (1). In this section, we study the dynamics of the model in long-time run.

As a by-product of the first extensibility criterion recorded in Theorem 1.1, we prove a global well-posedness result for (1) & (4) in two space dimensions, under the structural assumptions that $\chi(c)$ and $f(c)$ are constant multiples of each other (i.e., $\chi(c) = \mu f(c)$), and $f'(c) \geq 0$, where the former one was utilized in [5] to produce plumes in numerical simulations. The result is recorded in the following theorem.

**Theorem 1.2.** Consider the initial-boundary value problem (1) & (4) with $\nu > 0$, $D_n > 0$ and $D_c > 0$. Let $d = 2$. Suppose that the initial data $(u_0, n_0, c_0) \in H^2(\Omega)$ satisfy $\nabla \cdot u_0 = 0$, $n_0 \geq 0$, $c_0 \geq 0$ in $\Omega$, and the boundary conditions on $\partial \Omega$. Suppose that $\chi(\cdot) \geq 0$, $f(\cdot) \geq 0$, $\phi(\cdot)$ is smooth functions, there is a positive constant $\mu$ such that $\chi(\cdot) = \mu f(\cdot)$, and $f'(c) \geq 0$. Then the problem (1) & (4) has a unique solution $(u, n, c) \in L^\infty((0, T); H^2(\Omega)) \cap L^2((0, T); H^3(\Omega))$ for any $0 < T < \infty$.

Remark 4. We prove Theorem 1.2 by verifying the first extensibility criterion recorded in Theorem 1.1 through analyzing a logarithmic function associated with the bacteria density function by taking advantage of the structural assumptions. We remark that the same global well-posedness result was obtained in [3] for the Cauchy problem of (1) by using (2). However, the reader will see that the proof constructed in this paper is much shorter than the one presented in [3], due to our improved extensibility criterion.

The rest of the paper contains the proofs of Theorems 1.1-1.2, which are given in Sections 2-3, respectively. The paper finishes with concluding remarks.

2. Extensibility criteria. This section is devoted to the proof of Theorem 1.1. Since local existence results can be proved by using standard arguments (for example, iteration and contraction mapping techniques), we only deal with the a priori estimates. Since there are two extensibility criteria in Theorem 1.1, we divide the proof into two subsections, each dealing with one of them. However, before splitting
the proof, we establish some universal \textit{a priori} estimates which are valid for both cases.

First, we note that, due to $\nabla \cdot \mathbf{u} = 0$, the equation for the bacteria density can be written as

$$ \partial_t n + \nabla \cdot (\mathbf{u} n) = D_n \Delta n. $$

where $\mathbf{w} = \mathbf{u} + \chi (c) \nabla c$. It follows from the properties of advection-diffusion equations and the initial condition, $n_0 \geq 0$, that

$$ n(x,t) \geq 0, \quad \forall \ (x,t) \in \bar{\Omega} \times [0,T), \quad (5) $$

where $T$ denotes the life-span of the local classical solution. Second, since $f(c) \geq 0$, by using (5), we derive from the equation for the oxygen concentration that

$$ \partial_t c + \mathbf{u} \cdot \nabla c - D_c \Delta c = -n f(c) \leq 0. $$

It follows from the properties of advection-diffusion equations and comparison principle for parabolic equations that

$$ 0 \leq c(x,t) \leq \max_{\bar{\Omega}} c_0(x), \quad \forall \ (x,t) \in \bar{\Omega} \times [0,T), \quad (6) $$

which implies the boundedness of $\chi^{(k)}(c)$ and $f^{(k)}(c)$ for any $k \geq 0$, due to the smoothness of the functions.

Next, we establish the extensibility criteria recorded in Theorem 1.1. Before proceeding to the proof of the first criterion, we would like to mention that since the proof for the 2D case is easier than the 3D case, in what follows we shall deal with the two criteria only for the 3D case.

**Notation.** Throughout the rest part of this section, unless specified, $C$ denotes a generic constant which is independent of the unknown functions, but may depend on $\Omega$, the system parameters, initial data and time. The value of the constant may vary line by line according to the context.

2.1. **The first criterion.** Let us assume for some $3 < p \leq \infty$ and $3 < q \leq \infty$,

$$ \int_0^T \left( \|u(t)\|_{L^p}^{2p} + \|c(t)\|_{L^q}^{2q} \right) dt < C. \quad (7) $$

We break the subsequent proof into eight steps.

**Step 1.** We first derive some uniform estimates for the bacteria density function by using (7). For any $m \geq 2$, testing the third equation of (1) by $n^{m-1}$ and denoting $w := n^{\frac{2}{m}}$, we calculate

$$ \frac{1}{m} \frac{d}{dt} \|w\|^2 + 4(m-1)D_n \frac{m^2}{m^2} \|\nabla w\|^2 = (m-1) \int_\Omega w \chi (c) (\nabla c \cdot \nabla w) dx. \quad (8) $$

By using the boundedness of $\chi(c)$, we infer that

$$ \left| (m-1) \int_\Omega w \chi (c) (\nabla c \cdot \nabla w) dx \right| \leq C \|\nabla c\|_{L^q} \|w\|_{L^{\frac{2q}{2q-2}}} \|\nabla w\|, \quad (9) $$

where $q$ is the same as in (7). Applying the Gagliardo-Nirenberg interpolation inequality for a function on a bounded domain in $\mathbb{R}^3$:

$$ \|g\|_{L^{\frac{2q}{2q-2}}} \leq C \|g\|^{1-\frac{2}{q}} \|\nabla g\|^{\frac{2}{q}} + \|g\|, \quad 3 < q \leq \infty, \quad (10) $$

we can update (9) as

$$ \left| (m-1) \int_\Omega w \chi (c) (\nabla c \cdot \nabla w) dx \right| \leq C \|\nabla c\|_{L^q} \left( \|w\|^{1-\frac{2}{q}} \|\nabla w\|^{\frac{2}{q}} + \|w\| \right) \|\nabla w\| $$

$$ = C \|\nabla c\|_{L^q} \left( \|w\|^{1-\frac{2}{q}} \|\nabla w\|^{1+\frac{2}{q}} + \|w\| \|\nabla w\| \right). $$
Since $q > 3$, by using Young’s inequality, we have

$$C \| \nabla c \|_{L^q} \| w \|^{(2 - \frac{3}{q})} \| \nabla w \|^{\frac{3}{q}} \leq C \| \nabla c \|_{L^q}^2 \| w \|^2 + \frac{(m-1)D_n}{m^2} \| \nabla w \|^2,$$

and

$$C \| \nabla c \|_{L^q} \| \nabla w \| \leq C \| \nabla c \|_{L^q}^2 \| w \|^2 + \frac{(m-1)D_n}{m^2} \| \nabla w \|^2 \leq C \left( \| \nabla c \|_{L^q}^{\frac{2m}{m^2}} + 1 \right) \| w \|^2 + \frac{(m-1)D_n}{m^2} \| \nabla w \|^2.$$

So we update (8) as

$$\frac{1}{m} \frac{d}{dt} \| w \|^2 + 4 \frac{(m-1)D_n}{m^2} \| \nabla w \|^2 \leq C \left( \| \nabla c \|_{L^q}^{\frac{2m}{m^2}} + 1 \right) \| w \|^2 + \frac{2(m-1)D_n}{m^2} \| \nabla w \|^2,$$

which is equivalent to

$$\frac{d}{dt} \| w \|^2 + \frac{2(m-1)D_n}{m^2} \| \nabla w \|^2 \leq C \left( \| \nabla c \|_{L^q}^{\frac{2m}{m^2}} + 1 \right) \| w \|^2,$$

where the constant $C$ depends only on $m$ and $\Omega$. Applying the Gronwall’s inequality to (11), we find

$$\| w(t) \|^2 \leq \| w_0 \|^2 \exp \left\{ C \left( \int_0^T \| \nabla c(t) \|_{L^q}^{\frac{2m}{m^2}} dt + T \right) \right\}, \quad \forall 0 < t < T.$$

Therefore, by virtue of (7), we infer that

$$\| w(t) \|^2 \leq C(m, T), \quad \forall 0 < t < T.$$

By definition, we then have

$$\| n(t) \|_{L^m} \leq C(m, T), \quad \forall 0 < t < T, \quad \forall 2 \leq m < \infty.$$  \hspace{1cm} (12)

In addition, when $m = 2$, we can easily get

$$\int_0^t \| \nabla n(\tau) \|^2 d\tau \leq C, \quad \forall 0 < t < T.$$  \hspace{1cm} (13)

**Step 2.** Now we turn to the fluid velocity equation. Testing the first equation of (1) by $u$, we derive

$$\frac{1}{2} \frac{d}{dt} \| u \|^2 + \nu \| \nabla u \|^2 = - \int_{\Omega} n \nabla \phi \cdot u \, dx \leq \| n \| \| \nabla \phi \|_{L^\infty} \| u \| \leq C \| \nabla u \| \leq \nu \frac{J}{2} \| u \|^2 + C,$$

where we have used (12), the smoothness of $\phi$, the Poincaré’s inequality for $u$, and Cauchy-Schwarz inequality. Hence, it holds that

$$\frac{1}{2} \frac{d}{dt} \| u \|^2 + \nu \frac{J}{2} \| \nabla u \|^2 \leq C.$$

Upon integrating with respect to time, we infer that

$$\| u(t) \|^2 + \int_0^t \| \nabla u(\tau) \|^2 d\tau \leq C, \quad \forall 0 < t < T.$$  \hspace{1cm} (15)
To improve the spatial regularity of $u$, we test the fluid equation of (1) by $-\nu \Delta u + \nabla P$ to get
\[
\frac{\nu}{2} \frac{d}{dt} \|\nabla u\|^2 + \|\nabla u + \nabla P\|^2 \\
= -\int_{\Omega} (u \cdot \nabla u + n \varphi) \cdot (-\nu \Delta u + \nabla P) dx \\
\leq (\|u \cdot \nabla u\| + \|n \varphi\|) \|\nabla u + \nabla P\| \\
\leq \left( \|u\|_{L^p} \|\nabla u\|_{L^{\frac{2p}{p-2}}} + \|n\| \|\varphi\|_{L^\infty} \right) \|\nabla u + \nabla P\|,
\]
where $p$ is the same as in (7). Note that, by (10), we have
\[
\|\nabla u\|_{L^{\frac{2p}{p-2}}} \leq C \left( \|\nabla u\|^{1-\frac{3}{p}} \|\Delta u\|^\frac{3}{p} + \|\nabla u\| \right).
\]
Moreover, from the classic $H^2$-theory for linear Stokes equation we know
\[
\|u\|_{H^2} + \|P\|_{H^1} \leq C \|\nabla u + \nabla P\|,
\]
which implies
\[
\|\nabla u\|_{L^{\frac{2p}{p-2}}} \leq C \left( \|\nabla u\|^{1-\frac{3}{p}} \|\Delta u\|^\frac{3}{p} + \|\nabla u\| \right).
\]
This yields the following estimate for the first term on the RHS of (16):
\[
\|u\|_{L^p} \|\nabla u\|_{L^{\frac{2p}{p-2}}} \|\nabla u + \nabla P\| \\
\leq C \|u\|_{L^p} \left( \|\nabla u\|^{1-\frac{3}{p}} \|\Delta u\|^\frac{3}{p} + \|\nabla u\| \right) \|\nabla u + \nabla P\| \\
\leq 1 - \|\nabla u + \nabla P\|^2 + C \|u\|_{L^p} \|\nabla u\|^2 + C \|u\|_{L^p} \|\nabla u\|^2 \\
\leq 1 - \|\nabla u + \nabla P\|^2 + C \left( \|u\|_{L^p}^2 + 1 \right) \|\nabla u\|^2,
\]
where we have applied the Young and Cauchy-Schwarz inequalities at various places, and used the fact that $3 < p \leq \infty$. In addition, for the second term on the RHS of (16), similar to the derivation of (14), we can easily show that
\[
\|n\| \|\nabla \varphi\|_{L^\infty} \|\nabla u + \nabla P\| \leq \frac{1}{4} - \|\nabla u + \nabla P\|^2 + C.
\]
Plugging (18) and (19) into (16), we find
\[
\frac{\nu}{2} \frac{d}{dt} \|\nabla u\|^2 + \frac{1}{2} \|\nabla u + \nabla P\|^2 \leq C \left( \|u\|_{L^p}^2 + 1 \right) \|\nabla u\|^2 + C.
\]
Applying the Gronwall’s inequality to (20), using (7) and (17), we have
\[
\|u(t)\|_{H^1}^2 + \int_0^t \|u(\tau)\|_{H^2}^2 d\tau \leq C, \quad \forall \ 0 < t < T.
\]
**Step 3.** Now we turn to the equation of the oxygen concentration. Testing the fourth equation of (1) by $-\Delta c$, we can show that for $3 < p \leq \infty$,
\[
\frac{1}{2} \frac{d}{dt} \|\nabla c\|^2 + D_c \|\Delta c\|^2 = \int_{\Omega} [u \cdot \nabla c + n f(c)] (\Delta c) dx \\
\leq \left( \|u\|_{L^p} \|\nabla c\|_{L^{\frac{2p}{p-2}}} + \|f(c)\|_{L^\infty} \|c\| \right) \|\Delta c\|.
\]
For the first term on the RHS of (22), by using (10) and Young’s inequality, we can show that
\[
\|u\|_{L^p} \|\nabla c\|_{L^p} \|\Delta c\| \leq C \|u\|_{L^p} \left( \|\nabla c\|^{1+\frac{2}{p}} \|\Delta c\|^{\frac{2}{p}} + \|\nabla c\| \right) \|\Delta c\|
\]
\[
= C \|u\|_{L^p} \|\nabla c\|^{1+\frac{2}{p}} \|\Delta c\|^{\frac{1}{p}} + C \|u\|_{L^p} \|\nabla c\| \|\Delta c\| \leq \frac{D_c}{4} \|\Delta c\|^2 + C \left( \|u\|_{L^p}^{\frac{2}{p}} + 1 \right) \|\nabla c\|^2. \tag{23}
\]

In addition, by using (6) and the smoothness of \(f(\cdot)\), we can show that
\[
\|f(c)\|_{L^\infty} \|\Delta c\| \leq C \|\Delta c\| \leq \frac{D_c}{4} \|\Delta c\|^2 + C. \tag{24}
\]

By plugging (23) and (24) into (22), we have
\[
\frac{1}{2} \frac{d}{dt} \|\nabla c\|^2 + \frac{D_c}{2} \|\Delta c\|^2 \leq C \left( \|u\|_{L^p}^{\frac{2}{p}} + 1 \right) \|\nabla c\|^2 + C. \tag{25}
\]

Applying the Gronwall’s inequality to (25), and using (7), we have
\[
\|c(t)\|_{H^1}^2 + \int_0^t \|c(\tau)\|_{H^2}^2 d\tau \leq C, \quad \forall 0 < t < T.
\]

**Step 4.** In order to achieve higher order regularity of \(c\), we perform a bootstrap argument. First, we decompose \(c\) as \(c := c_1 + c_2\), where \(c_1\) and \(c_2\) satisfy
\[
\begin{align*}
\partial_t c_1 - D_c \Delta c_1 &= -\nabla \cdot (uc), & \text{in } \Omega \times (0, T), \\
\nabla c_1 \cdot n &= 0, & \text{on } \partial \Omega \times (0, T), \\
c_1(x, 0) &= 0, & \text{in } \Omega;
\end{align*}
\]
and
\[
\begin{align*}
\partial_t c_2 - D_c \Delta c_2 &= -nf(c), & \text{in } \Omega \times (0, T), \\
\nabla c_2 \cdot n &= 0, & \text{on } \partial \Omega \times (0, T), \\
c_2(x, 0) &= c_0(x), & \text{in } \Omega,
\end{align*}
\]
respectively. We note that, due to (6), (21) and the Sobolev embedding: \(H^1 \hookrightarrow L^6\), it holds that
\[
\int_0^t \|\nabla (uc)\|_{L^6}^6 d\tau \leq \int_0^t \|u(\tau)\|_{L^6}^6 \|c(\tau)\|_{L^\infty}^6 d\tau \leq C \int_0^t \|u(\tau)\|_{H^1}^2 d\tau \leq C, \quad \forall 0 < t < T.
\]

Applying the classical regularity theory for general parabolic equations (cf. [1]) to (26), we find
\[
\int_0^t \|\nabla c_1(\tau)\|_{L^6}^6 d\tau \leq C, \quad \forall 0 < t < T. \tag{27}
\]

In addition, due to (12), (6), the smoothness of \(f(\cdot)\) and parabolic regularity, we know that (27) is also true for \(c_2\). Hence, it holds that
\[
\int_0^t \|\nabla c(\tau)\|_{L^6}^6 d\tau \leq C, \quad \forall 0 < t < T. \tag{28}
\]

As a consequence of (28), we can show that
\[
\int_0^t \|u \cdot \nabla c(\tau)\|_{L^3}^3 d\tau \leq \int_0^t \|u(\tau)\|_{L^6}^6 \|\nabla c(\tau)\|_{L^3}^3 d\tau \leq C \int_0^t \|u(\tau)\|_{H^1}^2 \|\nabla c(\tau)\|_{L^3}^3 d\tau \leq C, \quad \forall 0 < t < T. \tag{29}
\]
Applying the parabolic regularity to the fourth equation of (1), and using (12) and (29), we have
\[ \int_0^t \| c(\tau) \|_{W^{2,3}} d\tau \leq C, \quad \forall \ 0 < t < T. \]

**Step 5.** With the higher order regularity of \( c \) in hand, we now improve the regularity of \( n \). We note that the third equation of (1) can be written as
\[ \partial_t n - D_\alpha \Delta n = -\nabla \cdot (un + n\chi(c)\nabla c), \quad (30) \]
and there holds that
\[
\int_0^t \left\| (un + n\chi(c)\nabla c)(\tau) \right\|_{L^3}^6 d\tau \\
\leq C \int_0^t \left( \left\| (un)(\tau) \right\|_{L^3}^6 + \left\| (n\chi(c)\nabla c)(\tau) \right\|_{L^3}^6 \right) d\tau \\
\leq C \int_0^t \left( \left\| u(\tau) \right\|_{L^6}^3 \left\| n(\tau) \right\|_{L^6}^3 + \left\| n(\tau) \right\|_{L^6}^3 \left\| \nabla u(\tau) \right\|_{L^6}^3 \right) d\tau \leq C, \quad \forall \ 0 < t < T,
\]
where we have used various previous estimates, especially (28). Hence, applying the parabolic regularity to (30), we find that
\[
\int_0^t \left\| \nabla n(\tau) \right\|_{L^3}^6 d\tau \leq C \int_0^t \left\| (un + n\chi(c)\nabla c)(\tau) \right\|_{L^3}^6 d\tau \leq C, \quad \forall \ 0 < t < T.
\]

Now, we observe that the RHS of (30) can be written as
\[ -\nabla \cdot (un + n\chi(c)\nabla c) = -u \cdot \nabla n - \chi(c)\nabla n \cdot \nabla c - n\chi'(c)|\nabla c|^2 - n\chi(c)\Delta c. \]
By using the previous estimates and the smoothness of \( \chi(\cdot) \), we can show that for \( 0 < t < T \),
\[ \int_0^t \left\| u \cdot \nabla n(\tau) \right\|^2 d\tau \leq \int_0^t \left\| u(\tau) \right\|_{L^6}^2 \left\| \nabla n(\tau) \right\|_{L^3}^2 d\tau \leq C, \]
\[ \int_0^t \left\| (\chi(c)\nabla n \cdot \nabla c)(\tau) \right\|^2 d\tau \leq \int_0^t \left\| \nabla n(\tau) \right\|_{L^3}^2 \left\| \nabla c(\tau) \right\|_{L^6}^2 d\tau \\
\leq \left( \int_0^t \left\| \nabla n(\tau) \right\|_{L^3}^6 d\tau \right)^{\frac{1}{3}} \left( \int_0^t \left\| \nabla c(\tau) \right\|_{L^6}^6 d\tau \right)^{\frac{1}{3}} \leq C, \]
\[ \int_0^t \left\| (n\chi'(c)|\nabla c|^2)(\tau) \right\|^2 d\tau \leq \int_0^t \left\| n(\tau) \right\|_{L^6}^2 \left\| \nabla c(\tau) \right\|_{L^6}^2 d\tau \leq C, \]
\[ \int_0^t \left\| (n\chi(c)\Delta e)(\tau) \right\|^2 d\tau \leq \int_0^t \left\| n(\tau) \right\|_{L^6}^2 \left\| \Delta e(\tau) \right\|_{L^6}^2 d\tau \leq C. \]
Hence, it holds that
\[ \text{RHS of (30) } \in L^2(0,T;L^2(\Omega)). \]
This implies, by parabolic regularity, that
\[ \| n(t) \|_{H^1}^2 + \int_0^t \left( \left\| n(\tau) \right\|_{H^2}^2 + \left\| \partial_t n(\tau) \right\|^2 \right) d\tau \leq C, \quad \forall \ 0 < t < T. \quad (31) \]

**Step 6.** In this step, we use (31) to improve the regularity of \( u \). Since the information of the spatial derivatives of \( u \) is unknown on \( \partial \Omega \), we first deal with the temporal
derivatives. Taking $\partial_t$ to the first equation of (1), then testing the resulting equation by $\partial_t u$, we have
\[
\frac{1}{2} \frac{d}{dt} \|\partial_t u\|^2 + \nu \|\nabla \partial_t u\|^2 = -\int_\Omega (\partial_t u \cdot \nabla u + \partial_t n \nabla \phi + n \nabla \partial_t \phi) \cdot \partial_t u \, dx
\]
\[
\leq C \left( \|\nabla u\|_{L^2}^2 + \|\partial_t u\|_{L^2}^2 + \|\partial_t n\| \|\partial_t u\| + \|n\| \|\partial_t u\| \right)
\]
\[
\leq C \left( \|\nabla u\|_{H^1}^2 \|\partial_t u\| \|\nabla \partial_t u\| + \|\partial_t n\| \|\partial_t u\| + \|\partial_t u\| \right)
\]
\[
\leq \frac{\nu}{2} \|\nabla \partial_t u\|^2 + C \left( \|\nabla u\|_{H^1}^2 + 1 \right) \|\partial_t u\|^2 + C \left( \|\partial_t n\|^2 + 1 \right),
\]
which is equivalent to
\[
\frac{1}{2} \frac{d}{dt} \|\partial_t u\|^2 + \frac{\nu}{2} \|\nabla \partial_t u\|^2 \leq C \left( \|\nabla u\|_{H^1}^2 + 1 \right) \|\partial_t u\|^2 + C \left( \|\partial_t n\|^2 + 1 \right).
\]
Applying the Gronwall’s inequality to (32), and using (21) and (31), we find
\[
\|\partial_t u(t)\|^2 + \int_0^t \|\partial_t u(\tau)\|_{H^1}^2 d\tau \leq C, \quad \forall \ 0 < t < T.
\]
Using (33) and previous estimates, we now improve the spatial regularity of $u$ as follows. Due to (17) and the fluid equation, we have
\[
\|u\|_{H^2} \leq C - \nu \Delta u + \nabla P
\]
\[
\leq C (\|\partial_t u\| + \|u \cdot \nabla u\| + \|n \nabla \phi\|)
\]
\[
\leq C (\|\partial_t u\| + \|u\|_{L^p} \|\nabla u\|_{L^q} + \|n\| \|\nabla \phi\|_{L^\infty})
\]
\[
\leq C \left( \|\partial_t u\| + \|u\|_{H^1} \|\nabla u\|^\frac{2}{3} \|\Delta u\|^\frac{1}{3} + 1 \right)
\]
\[
\leq C \left( \|\partial_t u\| + \|\nabla u\|^\frac{2}{3} \|\Delta u\|^\frac{1}{3} + 1 \right)
\]
\[
\leq 1\|\Delta u\| + C (\|\partial_t u\| + \|\nabla u\| + 1),
\]
which implies
\[
\|u(t)\|_{H^2} \leq C (\|\partial_t u(t)\| + \|\nabla u(t)\| + 1), \quad \forall \ 0 < t < T.
\]
In a similar fashion, by using (31) and (34), we can show that
\[
\|u(t)\|_{H^3} \leq C (\|\nabla \partial_t u(t)\| + 1).
\]
Integrating with respect to time and using (33), we find
\[
\int_0^t \|u(\tau)\|_{H^3}^2 \, d\tau \leq C, \quad \forall \ 0 < t < T.
\]
This completes the proof for the regularity of $u$.

**Step 7.** Now we improve the regularity of the oxygen concentration function. Again, we first deal with the temporal derivatives, then recover the spatial derivatives by using elliptic regularity. For this purpose, we take $\partial_t$ to the fourth equation of (1), then test the resulting equation by $\partial_t c$ to get
\[
\frac{1}{2} \frac{d}{dt} \|\partial_t c\|^2 + D_c \|\nabla \partial_t c\|^2
\]
\[
= -\int_\Omega (\partial_t u \cdot \nabla c) \partial_t c \, dx - \int_\Omega (f(c)\partial_t n + n f'(c) \partial_t c) \partial_t c \, dx
\]
\[
= \int_\Omega c (\partial_t u \cdot \nabla c) \partial_t c \, dx - \int_\Omega (f(c)\partial_t n + n f'(c) \partial_t c) \partial_t c \, dx
\]
\[
\leq D_c \|\nabla \partial_t c\|^2 + C (\|c\|_{L^\infty}^2 \|\partial_t u\|^2 + \|\partial_t n\| \|\partial_t c\| + \|n\| \|\partial_t c\|^2)
\]
\[
\leq D_c \|\nabla \partial_t c\|^2 + C (\|\partial_t n\|^2 + 1),
\]
where we have applied the Cauchy-Schwarz inequality and the estimates of $\|c\|_{L^\infty}$, $\|\partial_t u\|_{L^\infty}$ and the Sobolev embedding: $H^2 \hookrightarrow L^\infty$. Applying the Gronwall’s inequality to (35), we find

$$\|\partial_t c(t)\|^2 + \int_0^t \|\partial_t c(\tau)\|^2_{H^1} d\tau \leq C, \quad \forall 0 < t < T.$$ 

Then the elliptic regularity implies that

$$\|c(t)\|_{H^2} \leq C \|\Delta c(t)\| \leq C (\|\partial_t c(t)\| + \|u \cdot \nabla c(t)\| + \|(n f(c))(t)\|) \leq C (\|\partial_t c(t)\| + \|u(t)\|_{L^\infty} \|\nabla c(t)\| + \|n(t)\|) \leq C, \quad \forall 0 < t < T.$$ 

Moreover, it holds that

$$\|c(t)\|_{H^2} \leq C \|\Delta c(t)\|_{H^1} \leq C \|\partial_t c(t)\|_{H^1} + \|u \cdot \nabla c(t)\|_{H^1} + \|(n f(c))(t)\|_{H^1}), \quad \forall 0 < t < T,$$

which implies

$$\int_0^t \|c(\tau)\|^2_{H^3} d\tau \leq C, \quad \forall 0 < t < T.$$ 

This completes the proof for the regularity of $c$.

**Step 8.** We build up the regularity of $n$ in this step. Once again, we take $\partial_t$ to the third equation of (1), then test the resulting equation by $\partial_t n$ to get

$$\frac{1}{2} \frac{d}{dt} \|\partial_t n\|^2 + D_n \|\nabla \partial_t n\|^2 = \int_{\Omega} (n \partial_t u + \chi(c) \partial_t n \nabla c + \chi'(c) n \partial_t c \nabla c + \chi(c) n \nabla \partial_t c) \cdot \nabla \partial_t n \, dx.$$ 

We estimate the RHS of (36) term by term. First, we have

$$\int_{\Omega} n \partial_t u \cdot \nabla \partial_t n \, dx \leq \frac{D_n}{8} \|\nabla \partial_t n\|^2 + C \|n \partial_t u\|^2 \leq \frac{D_n}{8} \|\nabla \partial_t n\|^2 + C \|n\|_{W^{2,\infty}}^2 \|\partial_t u\|^2 \tag{37}$$

Second, we have

$$\int_{\Omega} \chi(c) \partial_t n \nabla c \cdot \nabla \partial_t n \, dx \leq \frac{D_n}{8} \|\nabla \partial_t n\|^2 + C \|\chi(c) \partial_t n \nabla c\|^2 \leq \frac{D_n}{8} \|\nabla \partial_t n\|^2 + C \|\nabla c\|_{L^\infty} \|\partial_t n\|^2 \tag{38}$$

Third, we have

$$\int_{\Omega} \chi'(c) n \partial_t c \nabla c \cdot \nabla \partial_t n \, dx \leq \frac{D_n}{8} \|\nabla \partial_t n\|^2 + C \|\chi'(c) n \partial_t c \nabla c\|^2 \leq \frac{D_n}{8} \|\nabla \partial_t n\|^2 + C \|n\|_{H^1} \|\partial_t c\|_{H^1}^2 \|\nabla c\|_{L^\infty} \tag{39}$$

$$\leq \frac{D_n}{8} \|\nabla \partial_t n\|^2 + C \|n\|_{H^2} \|c\|_{H^3}^2.$$
Fourth, we have
\[
\int_{\Omega} \chi(c) n \nabla \partial_t c \cdot \nabla \partial_t n \, dx \leq \frac{D_n}{8} \| \nabla \partial_t n \|^2 + C \| \chi(c)n \nabla \partial_t c \|^2 \\
\leq \frac{D_n}{8} \| \nabla \partial_t n \|^2 + C \| n \|_{L^\infty}^2 \| \nabla \partial_t c \|^2 \\
\leq \frac{D_n}{8} \| \nabla \partial_t n \|^2 + C \| n \|_{H^2}^2 \| \nabla \partial_t c \|^2.
\] (40)

In order to apply the Gronwall’s inequality, we now derive an estimate for \( \| n \|_{H^2} \) in terms of \( \| \partial_t n \| \). According to the third equation of (1), we have
\[
\| n \|_{H^2}^2 \leq C \left( \| \partial_t n \|^2 + \| u \cdot \nabla n \|^2 + \| \nabla n \cdot \nabla c \|^2 + \| n \|_{L^\infty} \| \nabla c \|^2 \right) \\
\leq C \left( \| \partial_t n \|^2 + \| u \|_{L^\infty} \| \nabla n \|^2 + \| \nabla n \|_{L^2} \| \nabla c \|_{L^2}^2 + \| n \|_{L^\infty} \| \nabla c \|_{L^2} \| \nabla c \|_{L^2} \right) \\
\leq C \left( \| \partial_t n \|^2 + 1 + \| n \|_{H^2} \right) \\
\leq \frac{1}{2} \| n \|_{H^2}^2 + C \left( \| \partial_t n \|^2 + 1 \right),
\]
where we have used various Sobolev inequalities, Morrey’s inequality: \( \| F \|_{L^\infty} \leq C \| F \|_{H^1} \| F \|_{H^2} \), and previous estimates. Hence,
\[
\| n \|_{H^2}^2 \leq C \left( \| \partial_t n \|^2 + 1 \right).
\] (41)

By using (41), we can update (39) and (40) as
\[
\int_{\Omega} \chi(c) n \partial_t c \nabla c \cdot \nabla \partial_t n \, dx \leq \frac{D_n}{8} \| \nabla \partial_t n \|^2 + C \| \partial_t n \|^2 \| c \|_{H^3}^2 + C \| c \|_{H^3}^2,
\] (42)
and
\[
\int_{\Omega} \chi(c) n \nabla \partial_t c \cdot \nabla \partial_t n \, dx \leq \frac{D_n}{8} \| \nabla \partial_t n \|^2 + C \| \partial_t n \|^2 \| \nabla \partial_t c \|^2 + C \| \nabla \partial_t c \|^2.
\] (43)

Plugging (37), (38), (42) and (43) into (36), we find
\[
\frac{1}{2} \frac{d}{dt} \| \partial_t n \|^2 + \frac{D_n}{2} \| \nabla \partial_t n \|^2 \\
\leq C \left( \| c \|_{H^3}^2 + \| \nabla \partial_t c \|^2 \right) \| \partial_t n \|^2 + C \left( \| c \|_{H^3}^2 + \| \nabla \partial_t c \|^2 + \| n \|_{H^2}^2 \right).
\] (44)

Applying the Gronwall’s inequality to (44), we find
\[
\| \partial_t n(t) \|^2 + \int_0^t \| \partial_t n(\tau) \|_{H^1}^2 \, d\tau \leq C, \quad \forall \ 0 < t < T.
\] (45)

This, together with (41), implies that
\[
\| n(t) \|_{H^2}^2 \leq C, \quad \forall \ 0 < t < T.
\] (46)

A further application of the elliptic regularity and (45)–(46) finally yields
\[
\int_0^t \| n(\tau) \|_{H^2}^2 \, d\tau \leq C, \quad \forall \ 0 < t < T.
\]

This completes the proof for the regularity of \( n \), and thus the first extensibility criterion in Theorem 1.1. □
2.2. The second criterion. We now prove the second extensibility criterion by using the first one. Let us assume for some $3 < p \leq \infty$ and $\frac{3}{2} < q \leq \infty$,

$$\int_0^T \left( \| u(t) \|_{L^p}^{2p} + \| n(t) \|_{L^q}^{2q} \right) dt < C. \quad (47)$$

We would like to emphasize that (47) was established previously in [4], under the technical assumption that $\chi(\cdot)$ and $f(\cdot)$ are monotonic non-decreasing functions. Here, we remove such a constraint and prove the same extensibility criterion. The subsequent proof is broken into four steps.

**Step 1.** First, we note that (5) and (6) are still valid. Next, due to the boundary conditions, we have

$$\int_\Omega n(x, t) dx = \int_\Omega n_0(x) dx \equiv \bar{n}. \quad (48)$$

Testing the third equation of (1) by $\ln(n)$, and noting $n \geq 0$, we infer that

$$\frac{d}{dt} \left( \int_\Omega [\eta(n) - \eta(\bar{n}) - \eta'(\bar{n})(n - \bar{n})] dx \right) + 4D_n \| \nabla \sqrt{n} \|^2 = \int_\Omega \chi(c) \nabla c \cdot \nabla n \ dx. \quad (49)$$

where $\eta(n) = n \ln(n) - n$ denotes the anti-derivative of $\ln(n)$. We estimate the RHS of (49) as follows:

$$\int_\Omega \chi(c) \nabla c \cdot \nabla n \ dx \leq C \int_\Omega |\nabla c| |\nabla \sqrt{n}| \ dx$$

$$\leq C \int_\Omega |\nabla \sqrt{n}| |\nabla \sqrt{n}| \ dx$$

$$\leq C \| \nabla \sqrt{n} \|_{L^q}^q \| \nabla c \|_{L^{2q}}^{2q}$$

$$\leq \frac{D_n}{6} \| \Delta c \|^2 + C \| n \|_{L^q} \| \nabla \sqrt{n} \|^{2q} \| \nabla c \|^2.$$  \quad (50)

Next, testing the fourth equation of (1) by $-\Delta c$, we have

$$\frac{1}{2} \frac{d}{dt} \| \nabla c \|^2 + D_n \| \Delta c \|^2 = \int_\Omega (u \cdot \nabla c) \Delta c \ dx + \int_\Omega n f(c) \Delta c \ dx. \quad (51)$$

We estimate the RHS of (51) as follows. For the first term, we have

$$\int_\Omega (u \cdot \nabla c) \Delta c \ dx \leq \| u \|_{L^p} \| \nabla c \|_{L^{2p^*}} \| \Delta c \|$$

$$\leq C \| u \|_{L^p} \| \nabla c \|^{1-\frac{3}{2}} \| \Delta c \|^{1+\frac{3}{2}}$$

$$\leq \frac{D_n}{6} \| \Delta c \|^2 + C \| n \|_{L^q} \| \nabla c \|^2.$$  \quad (52)

For the second term, we have

$$\int_\Omega n f(c) \Delta c \ dx \leq C \int_\Omega \sqrt{n} \sqrt{\nabla c} \ dx$$

$$\leq C \| \sqrt{n} \|_{L^q} \| \sqrt{n} \|_{L^{2q^*}} \| \Delta c \|$$

$$\leq C \| n \|_{L^q} \left( \| \sqrt{n} \|^{1-\frac{3}{2}} \| \nabla \sqrt{n} \|^{\frac{3}{2}} + \| \sqrt{n} \| \| \Delta c \| \right)$$

$$\leq \frac{D_n}{6} \| \Delta c \|^2 + C \| n \|_{L^q} \left( \sqrt{n}^{1-\frac{3}{2}} \| \nabla \sqrt{n} \|^{\frac{3}{2}} + \sqrt{n} \right)^2.$$
For the term involving the fluid velocity on the RHS of (56), by using (17), we have
\[
\int \nabla \cdot (n \nabla \eta) \, dx + \frac{1}{2} \|
abla c \|^2 + 2D_v \|
abla \sqrt{n} \|^2 + \frac{D_v}{2} \|
abla c \|^2 \leq \frac{D_v}{6} \|
abla c \|^2 + D_v \|
abla \sqrt{n} \|^2 + C \left( \|n\|_{L^6}^{2} + 1 \right). \tag{53}
\]
Now, plugging (50) into (49), plugging (52) and (53) into (51), then adding the results, we find
\[
\frac{d}{dt} \left( \int \left[ (\eta(n) - \eta(n_0) - n) \right] \, dx + \frac{1}{2} \|
abla c \|^2 \right) + 2D_v \|
abla \sqrt{n} \|^2 + \frac{D_v}{2} \|
abla c \|^2 \leq C \left( \|n\|_{L^6}^{2} + \|u\|_{L^p}^{2} \right) \|
abla c \|^2 + C \left( \|n\|_{L^6}^{2} + 1 \right). \tag{54}
\]
Note that
\[
\int \left[ (\eta(n) - \eta(n_0) - n) \right] \, dx \geq 0.
\]
Then, applying the Gronwall’s inequality to (54) and using (47), we find
\[
\|\nabla c(t)\|^2 + \int_0^t \left( \|\nabla \sqrt{n}(\tau)\|^2 + \|\nabla c(\tau)\|^2 \right) \, d\tau \leq C, \quad \forall \, 0 < t < T. \tag{55}
\]
**Step 2.** With (55) in hand, we now improve the regularity of $u$. Testing the first equation of (1) with $-\nu \Delta u + \nabla P$, we derive
\[
\frac{\nu}{2} \frac{d}{dt} \|\nabla u\|^2 + \| - \nu \Delta u + \nabla P\|^2
\]
\[
= - \int (u \cdot \nabla u + n \nabla \phi) \cdot (-\nu \Delta u + \nabla P) \, dx \tag{56}
\]
\[
\leq \left( \|u\|_{L^p} \|\nabla u\|_{L^{2p}_x} + \|\nabla \sqrt{n}\|_{L^{2p}_x} \|\nabla \sqrt{n}\|_{L^{2p}_x} \right) \| - \nu \Delta u + \nabla P\|^2.
\]
For the term involving the fluid velocity on the RHS of (56), by using (17), we derive
\[
\|u\|_{L^p} \|\nabla u\|_{L^{2p}_x} \| - \nu \Delta u + \nabla P\|^2
\]
\[
\leq C \|u\|_{L^p} \|\nabla u\|^{1+\frac{1}{3}} - \nu \Delta u + \nabla P\|^{1+\frac{1}{3}} \tag{57}
\]
\[
\leq \frac{1}{4} \| - \nu \Delta u + \nabla P\|^2 + C \|u\|_{L^p}^{2} \|\nabla u\|^2.
\]
For the term involving the bacteria density, by using the same arguments as in (53), we have
\[
\|
abla \sqrt{n}\|_{L^{2p}_x} \|\nabla \sqrt{n}\|_{L^{2p}_x} \| - \nu \Delta u + \nabla P\|^2
\]
\[
\leq \frac{1}{4} \| - \nu \Delta u + \nabla P\|^2 + C \left( \|
abla \sqrt{n}\|^2 + \|n\|_{L^6}^{2} + 1 \right). \tag{58}
\]
Plugging (57) and (58) into (56), we have
\[
\frac{\nu}{2} \frac{d}{dt} \|\nabla u\|^2 + \frac{1}{2} \| - \nu \Delta u + \nabla P\|^2 \leq C \|u\|_{L^p}^{2} \|\nabla u\|^2 + C \left( \|
abla \sqrt{n}\|^2 + \|n\|_{L^6}^{2} + 1 \right). \tag{59}
\]
Applying the Gronwall’s inequality to (59) and using (47) and (55), we find
\[
\|u(t)\|_{H^1} + \int_0^t \|u(\tau)\|_{H^2} \, d\tau \leq C, \quad \forall \, 0 < t < T. \tag{60}
\]
Step 3. We observe that, after (55) and (60) are established, by essentially repeating the arguments on Pages 1214 – 1215 of [4], we can show that

$$\int_0^T ||(u \cdot \nabla)c(t)||_{L^q}^{2q} dt \leq C + \varepsilon \int_0^T ||\nabla^2 c(t)||_{L^q}^{2q} dt,$$

where \( \varepsilon > 0 \) is an arbitrary constant and \( C \) depends on \( \varepsilon, T \) and the constants in (55) and (60). Then, by using the maximal regularity of the heat equation (cf. [1]), we can show that

$$\int_0^T ||\partial_t c(t)||_{L^q}^{2q} dt + \int_0^T ||\nabla^2 c(t)||_{L^q}^{2q} dt \leq \tilde{C} \int_0^T ||(u \cdot \nabla)c(t)||_{L^q}^{2q} dt + C \int_0^T ||(n f(c))(t)||_{L^q}^{2q} dt + C \int_0^T ||n(t)||_{L^q}^{2q} dt + C.$$

Hence, by choosing \( \varepsilon \) small enough, such that \( \varepsilon \tilde{C} = \frac{1}{2} \), we can see that

$$\int_0^T ||\partial_t c(t)||_{L^q}^{2q} dt + \frac{1}{2} \int_0^T ||\nabla^2 c(t)||_{L^q}^{2q} dt \leq C \int_0^T ||n(t)||_{L^q}^{2q} dt + C.$$

This, together with (47), implies

$$\int_0^T ||\partial_t c(t)||_{L^q}^{2q} dt + \int_0^T ||\nabla^2 c(t)||_{L^q}^{2q} dt \leq C. \quad (61)$$

Step 4. Using (61), we now improve the regularity of \( n \). The proof of this step is similar to Step 1 of the proof of the first extensibility criterion. However, since the control quantity is switched from \( \nabla c \) to \( n \), the proof differs significantly from the first criterion. For any \( m \geq 2 \), testing the third equation of (1) by \( n^{m-1} \), we calculate

$$\frac{1}{m} \frac{d}{dt} ||n^m||_{L^1} + \frac{4 (m-1)}{m^2} \int_\Omega \partial_t ||\nabla n^\frac{m}{2} ||^2 = - \int_\Omega (m-1) \nabla \cdot (n \nabla c) \nabla n \nabla^\frac{m}{2} = - \int_\Omega n^{m-1} \chi(c) \nabla^\frac{m}{2}.$$

$$= - \int_\Omega \chi(c) \nabla^\frac{m}{2} = - \frac{m-1}{m} \int_\Omega \chi(c) \nabla^\frac{m}{2}.$$

$$\leq C \left( ||\nabla c||_{L^q}^2 + ||\Delta c||_{L^q} \right) ||n^m||_{L^\frac{2m}{m-1}}.$$

We note that, due to the Gagliardo-Nirenberg interpolation inequality:

$$||\nabla g||_{L^2}^2 \leq C ||g||_{L^2} \leq ||\nabla^2 g||_{L^q} \leq C ||\nabla n||_{L^1},$$

it holds that

$$||\nabla c||_{L^2}^2 \leq C ||c||_{L^2} \leq ||\nabla^2 c||_{L^q} \leq C ||\Delta c||_{L^q},$$

where we have used (6) and elliptic regularity. On the other hand, we apply Hölder’s inequality to get

$$||n^m||_{L^\frac{2m}{m-1}} \leq ||n^m||_{L^1} \leq ||n^m||^{\frac{2m}{2}}_{L^q}.$$

Note that

$$||n^m||_{L^q}^\frac{2m}{m} = ||n^\frac{m}{2} ||_{L^q}^\frac{m}{2}.$$

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Hence, we update the RHS of (62) as
\[
C \left( \|\nabla c\|_{L^2}^2 + \|\Delta c\|_{L^q} \right) \|n^m\|_{L^{\frac{2q}{q-1}}} \leq C \|\Delta c\|_{L^q} \|n^m\|_{L^{\frac{2q}{q-1}}} \left\| \nabla n^m \right\|_{L^q} \frac{2q}{q-1} \leq \frac{2(m-1)D_n}{m^2} \left( \|\nabla n^m\|_{L^q}^2 \right) + C \|\Delta c\|_{L^q} \|n^m\|_{L^1}.
\]
(63)

Plugging (63) into (62), we have
\[
\frac{1}{m^2} \frac{d}{dt} \|n^m\|_{L^1} + \frac{2(m-1)D_n}{m^2} \left( \|\nabla n^m\|_{L^q}^2 \right) \leq C \|\Delta c\|_{L^q} \|n^m\|_{L^1}.
\]
(64)

Applying the Gronwall’s inequality to (64) and using (61), we find
\[
\|n(t)\|_{L^m} \leq C, \quad \forall 0 < t < T, \quad \forall 2 \leq m < \infty.
\]
(65)

Again, by repeating the arguments on Page 1215 of [4] and using the maximal regularity of the heat equation, we can show that
\[
\int_0^T \|\partial_t c(t)\|_{L^2}^2 dt + \int_0^T \|\nabla^2 c(t)\|_{L^2}^2 dt \leq \frac{1}{2} \int_0^T \|\nabla^2 c(t)\|_{L^2}^2 dt + C \int_0^T \|(nf(c)')(t)\|_{L^2} dt + C, \quad \forall 3 < m \leq 6,
\]
which, together with (65), implies
\[
\int_0^T \|\partial_t c(t)\|_{L^2}^2 dt + \int_0^T \|\nabla^2 c(t)\|_{L^2}^2 dt \leq C, \quad \forall 3 < m \leq 6.
\]
In particular, by Sobolev embedding, we have
\[
\int_0^T \|\nabla c(t)\|_{L^q}^2 dt \leq C.
\]
This, together with the first component of (47), verifies (7). This completes the proof of Theorem 1.1.

3. Global well-posedness in 2D. In this section, we prove Theorem 1.2 by using the first extensibility criterion established in Theorem 1.1. First of all, it can be readily checked that when the space dimension is 2, the first extensibility criterion in Theorem 1.1 reads
\[
\int_0^T \left( \|u(t)\|_{L^p}^{2p} + \|\nabla c(t)\|_{L^q}^{2q} \right) dt < \infty, \quad \text{for some } 2 < p, q \leq \infty.
\]
(66)

Based on (66), it then suffices to prove that
\[
(u, \nabla c) \in L^4(\Omega \times (0, T)),
\]
(67)
in order to establish the global well-posedness of (1) & (4) when \(d = 2\).

Second, for simplicity, we take \(\mu = 1\) throughout this section, and thus \(\chi(c) = f(c)\) and the system of equations reads
\[
\begin{cases}
\partial_t u + u \cdot \nabla u + \nabla P = \nu \Delta u - n \nabla \phi, \\
\nabla \cdot u = 0, \\
\partial_t n + u \cdot \nabla n = D_n \Delta n - \nabla \cdot (nf(c) \nabla c), \\
\partial_t c + u \cdot \nabla c = D_c \Delta c - nf(c),
\end{cases}
\]
(68)
where \(x \in \Omega \subset \mathbb{R}^2\) and \(t > 0\).
Third, we observe that (5), (6) and (48) are still valid. We begin with an estimate similar to (49). Indeed, after integrating the RHS of (49) by parts, we find
\[
\frac{d}{dt} \left( \int_{\Omega} [\eta(n) - \eta(\bar{n}) - \eta'(\bar{n})(n - \bar{n})] \, dx \right) + 4D_n \|\nabla \sqrt{n}\|^2 
= - \int_{\Omega} [f'(c)|\nabla c|^2 + f(c) \Delta c] \, n \, dx, \tag{69}
\]
where \(\eta(n) = n \ln(n) - n\). Testing the fourth equation of (68) by \(-\Delta c\), we have
\[
\frac{1}{2} \frac{d}{dt} \|\nabla c\|^2 + D_c \|\Delta c\|^2 = \int_{\Omega} (u \cdot \nabla c) \Delta c \, dx + \int_{\Omega} n f(c) \Delta c \, dx. \tag{70}
\]
We observe, by integrating by parts,
\[
\int_{\Omega} (u \cdot \nabla c) \Delta c \, dx = - \int_{\Omega} (\nabla c)^T \cdot \nabla u \cdot (\nabla c) \, dx - \frac{1}{2} \int_{\Omega} u \cdot \nabla (|\nabla c|^2) \, dx 
= - \int_{\Omega} (\nabla c)^T \cdot \nabla u \cdot (\nabla c) \, dx,
\]
due to the boundary condition for \(u\) and the incompressibility condition. Here, a vector is regarded as a column vector. Hence, we update (70) as
\[
\frac{1}{2} \frac{d}{dt} \|\nabla c\|^2 + D_c \|\Delta c\|^2 = - \int_{\Omega} (\nabla c)^T \cdot \nabla u \cdot (\nabla c) \, dx + \int_{\Omega} n f(c) \Delta c \, dx. \tag{71}
\]
Adding (71) to (69), we get
\[
\frac{d}{dt} \left( E(n, \bar{n}) + \frac{1}{2} \|\nabla c\|^2 \right) + 4D_n \|\nabla \sqrt{n}\|^2 + D_c \|\Delta c\|^2 
= - \int_{\Omega} n f'(c)|\nabla c|^2 \, dx - \int_{\Omega} (\nabla c)^T \cdot \nabla u \cdot (\nabla c) \, dx \tag{72}
\]
where
\[
E(n, \bar{n}) \equiv \int_{\Omega} [\eta(n) - \eta(\bar{n}) - \eta'(\bar{n})(n - \bar{n})] \, dx,
\]
and we have used the assumption \(f'(c) \geq 0\) and the fact that \(n \geq 0\). By using the Gagliardo-Nirenberg interpolation inequality:
\[
\|\nabla g\|_{L^q} \leq C \left( \|g\|_{L^\infty}^{\frac{2}{q}} \|\nabla^2 g\|^{\frac{1}{2}} + \|g\| \right),
\]
the classical elliptic theory and the uniform estimate of \(\|c\|_{L^\infty}\) due to (6), we estimate the RHS of (72) as
\[
- \int_{\Omega} (\nabla c)^T \cdot \nabla u \cdot (\nabla c) \, dx \leq \|\nabla u\| \|\nabla c\|_{L^4} \leq C \|\nabla u\| (\|c\|_{L^\infty} \|\Delta c\| + \|c\|^2) \tag{73}
\]
\[
\leq D_c \|\Delta c\|^2 + C (\|u\|^2 + 1).
\]
After plugging (73) into (72), we find
\[
\frac{d}{dt} \left( E(n, \bar{n}) + \frac{1}{2} \|\nabla c\|^2 \right) + 4D_n \|\nabla \sqrt{n}\|^2 + D_c \|\Delta c\|^2 \leq C (\|u\|^2 + 1). \tag{74}
\]
Next, testing the first equation of (68) by \(u\), we deduce
\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 + \nu \|\nabla u\|^2 = - \int_{\Omega} n \nabla \phi \cdot u \, dx \leq C \|n\| \|u\|. \tag{75}
\]
Note that since $n \geq 0$, it holds that
\[
\|n\| = \|\sqrt{n}\|_L^2 \leq C (\|\sqrt{n}\|\|\nabla \sqrt{n}\| + \|\sqrt{n}\|^2)
\]
\[
= C \left(\sqrt{n}\|\nabla \sqrt{n}\| + \sqrt{n}\right),
\]
where we have used (48). Hence, we update (75) as
\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 + \nu \|\nabla u\|^2 \leq C (\|\nabla \sqrt{n}\| + 1) \|u\|
\]
\[
\leq 2D_n \|\nabla \sqrt{n}\|^2 + C (\|u\|^2 + 1).
\]
Combining (74) and (77), we have
\[
\frac{d}{dt} \left( E(n, \bar{n}) + \frac{1}{2} \|\nabla c\|^2 + \frac{1}{2} \|u\|^2 \right) + 2D_n \|\nabla \sqrt{n}\|^2 + \frac{D_c}{2} \|\Delta c\|^2 + \nu \|\nabla u\|^2
\]
\[
\leq C (\|u\|^2 + 1).
\]
Note that $E(n, \bar{n}) \geq 0$. Then applying the Gronwall’s inequality to (78), we find in particular, for $\forall 0 < t < T$,
\[
\|\nabla c(t)\|^2 + \|u(t)\|^2 + \int_0^t (\|\nabla \sqrt{n}(\tau)\|^2 + \|\Delta c(\tau)\|^2 + \|\nabla u(\tau)\|^2) \, d\tau \leq C.
\]
Since
\[
\int_0^T \|u(\tau)\|_{L^2}^2 \, d\tau \leq C \int_0^T \|u(\tau)\|^2 \|\nabla u(\tau)\|^2 \, d\tau,
\]
and
\[
\int_0^T \|\nabla c(\tau)\|_{L^2}^2 \, d\tau \leq C \int_0^T \|\nabla c(\tau)\|^2 \|\Delta c(\tau)\|^2 \, d\tau,
\]
we see that (67) follows from (79). This completes the proof of Theorem 1.2.

4. Conclusion and looking ahead. In this paper, we studied the qualitative behavior of the coupled chemotaxis-fluid model, (1). Based on scaling invariance and using energy methods, we established several extensibility criteria for local-in-time classical solutions to the model on bounded domains with physical boundaries in $\mathbb{R}^d$ ($d = 2, 3$) when the fluid velocity is zero on the boundary and the bacteria density and oxygen concentration satisfy the homogeneous Neumann boundary condition. The first criterion improves a previous one obtained in [3] by extending the exponent of the Sobolev norm associated with a control quantity to a broader range. The second one enhances the one attained in [4] by removing a structural constraint on the chemotactic sensitivity and oxygen consumption functions. As a by-product of the first criterion, we obtained a global well-posedness result for (1) in $\mathbb{R}^2$, with a shortened proof compared to the one constructed in [3], due to our improved extensibility criterion.

On the other hand, we would like to mention that many fundamental questions concerning the qualitative behavior of the coupled model still remain open at the present time. Here we just list a few:

1. The global well-posedness of classical solutions under biologically relevant structural conditions on the chemotactic sensitivity and oxygen consumption functions, rather than the one considered in Theorem 1.2, is still unknown even in two space dimensions. We shall provide definite answers to such a question in a forthcoming paper.

2. In [5], the authors numerically produced plumes aligned with the vertical direction, which showed consistency with the experimental observation reported in [8]. Nevertheless, a rigorous mathematical explanation of such a phenomenon is still
unknown. This corresponds to the study of steady state solutions of (1) subject to appropriate boundary conditions. We suspect that the underlying mechanism responsible for the emerging of plumes is either bifurcation or spectral instability. We leave the detailed study for the future.

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