LOCAL EXISTENCE OF SPINOR- AND TENSOR POTENTIALS

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Abstract

We give new simple direct proofs in all spacetimes for the existence of asymmetric \((n, m + 1)\)-spinor potentials for completely symmetric \((n + 1, m)\)-spinors and for the existence of symmetric \((n, 1)\)-spinor potentials for symmetric \((n + 1, 0)\)-spinors. These proofs introduce a ‘superpotential’, i.e., a potential of the potential, which also enables us to get explicit statements of the gauge freedom of the original potentials. The main application for these results is the Lanczos potential \(L_{ABCA'}\), of the Weyl spinor and the electromagnetic vector potential \(A_{AA'}\). We also investigate the possibility of existence of a symmetric potential \(H_{ABA'B'}\) for the Lanczos potential, and prove that in all Einstein spacetimes any symmetric \((3,1)\)-spinor \(L_{ABCA'}\) possesses a symmetric potential \(H_{ABA'B'}\). Potentials of this type have been found earlier in investigations of some very special spinors in restricted classes of spacetimes. All of the new spinor results are translated into tensor notation, and where possible given also for four dimensional spaces of arbitrary signature.

1 Introduction

More than 30 years ago Lanczos [13] proposed a first order potential for the Weyl tensor. However, in 1983 Bampi and Caviglia [3] showed that Lanczos’ original proof was flawed and supplied a rigorous but complicated proof of local existence for four dimensional analytic spaces, independent of signature. Illge [11] has supplied a more conventional proof of existence (by means of a Cauchy problem) in spinor notation that, in its full generality, does not seem to generalize in an obvious manner, to arbitrary signature. Moreover, it should be emphasized that Illge’s work has highlighted the simple and natural structure of the Lanczos potential in spinor notation, and makes it clear that for work in spacetimes (\(C^\infty\) manifolds with Lorentz signature) the spinor formalism is much simpler than the tensor formalism. It should also be noted that in Lorentz signature the Lanczos potential satisfies a wave equation, and the well posedness of the
corresponding Cauchy problem enabled Illge to remove the assumption about analyticity in his proof.

It is important to note that the two existence proofs supplied by Bampi and Caviglia [3], and by Illge [11], respectively, do not directly concern the Weyl tensor/spinor \( C_{abcd}/\Psi_{ABCD} \), but are valid for any tensor/spinor \( W_{abcd}/W_{ABCD} \) having the same algebraic symmetries as the Weyl tensor/spinor. Furthermore, Illge’s work also discusses the existence of potentials for completely symmetric spinors with an arbitrary number of primed and unprimed indices; in general these potentials are not symmetric.

The results in this paper for four dimensional spacetimes are given in spinor notation following the conventions of [18] where the results are natural and the calculations comparatively simple; however, we also give the results in tensor notation for four dimensional spacetimes, and where possible in four dimensional spaces of any signature. We remark that in general the Lanczos potential does not exist in dimensions higher than four [8].

Note that a spinor \( S_{A_1\cdots A_nB'_1\cdots B'_m} \) having both primed and unprimed indices is said to be (completely) symmetric if it is symmetric over both types of indices i.e.,

\[
S_{A_1\cdots A_nB'_1\cdots B'_m} = S_{(A_1\cdots A_n)(B'_1\cdots B'_m)}
\]

In Section 2 we state Illge’s theorem for the existence and uniqueness of a symmetric spinor potential \( L_{A_1\cdots A_n} \) for the symmetric spinor \( W_{AA_1\cdots A_n} \). We also state the analogous result for a spinor potential \( L_{A_1\cdots A_nB'_1\cdots B'_m} (= L_{(A_1\cdots A_n)(B'_1\cdots B'_m)}) \) for the completely symmetric spinor \( W_{AA_1\cdots A_nB'_1\cdots B'_m} \). In addition we quote the corresponding Lanczos wave equation for \( L_{ABCA'} \) and \( L_{A_1\cdots A_nB'_1\cdots B'_m} \) showing how in the latter case an algebraic constraint arises in general, if we try to demand a completely symmetric \( L_{A_1\cdots A_nB'_1\cdots B'_m} \).

In Section 3 we give a new simple proof of local existence of a symmetric Lanczos potential \( L_{ABCA'} \) of an arbitrary symmetric spinor \( W_{ABCD} \) in an arbitrary spacetime. An interesting aspect of the proof is that it also involves a potential \( T_{ABCD} = T_{(ABC)}D \) of \( L_{ABCA'} \) which may be important in itself. This proof also generalizes to spinors with other index configurations.

Furthermore, it is straightforward to translate the existence proof for the Lanczos potential into tensor notation and adapt it into an existence proof for four dimensional analytic spaces of arbitrary signature.

In Section 4 we examine the gauge freedom of the Lanczos potential. We obtain an explicit formula for the gauge freedom involving the potential \( T_{ABCD} \), analogously to electromagnetic theory where it is known that the gauge freedom in the electromagnetic potential (after its curl and divergence are specified) is given by the gradient of a scalar field that satisfies a certain wave equation. Again, the tensor version of this result is given.

As noted above, Illge has shown the existence of (asymmetric) potentials for completely symmetric spinors with an arbitrary number of primed and unprimed indices. Thus, a Lanczos potential \( L_{ABCA'} \) of some symmetric spinor \( W_{ABCD} \)
itself has spinor potentials. One example is the spinor $T_{ABCD}$ referred to above, but there are reasons why we are more interested in having a symmetric potential of the type $H_A^{AB'A'B'} = H_{(AB)(A'B'B')}$ (see e.g., [1], [3], [5] and [20]). Although such a potential does not exist in all spacetimes we demonstrate in Section 5 that it does exist in all Einstein spacetimes. In order to obtain a unique solution to the problem we will supplement the defining equation for $H_A^{AB'A'B'}$ with certain other conditions and use a technique which is similar in structure to Illge’s proof for the existence of $L_{ABCA'}$. As a result our proof of this result will lack the simplicity of the existence proof for $L_{ABCA'}$ given in Section 3. A tensor version for spacetimes and four dimensional spaces of other signatures is also given.

In Section 6 we will look in more detail at the important application to electromagnetism in curved space. We do this in order to see how the results in the first sections relate to more familiar results on potentials such as Poincare’s lemma, and what simplifications can be achieved due to the simpler index structure of the electromagnetic spinor, and due to Maxwell’s equations.

In Section 7 we discuss how the results in this paper links up with existing results and applications.

2 Preliminaries

Let $\mathcal{M}$ be a spacetime (i.e., a real, $C^\infty$, 4-dimensional manifold with a metric of signature $(+ - - -)$). For simplicity we will restrict ourselves to tensor- and spinor fields of class $C^\infty$, but note that the results given could be generalized to tensor- and spinor fields of lesser regularity by using theorems on hyperbolic systems where the fields are only assumed to be in some Sobolev space, instead of the theorems used here. For definitions of the Levi-Civita connection, the curvature spinors etc., we will follow the conventions in [18]. Also note that all indices (both tensor- and spinor indices) occurring in this paper are abstract indices [18].

Illge has shown [11] that given any symmetric spinors $W_{ABCD} = W_{(ABCD)}$, $F_{BC} = F_{(BC)}$ there exists (locally) a symmetric spinor $L_{ABCA'} = L_{(ABC)A'}$ such that

$$W_{ABCD} = 2\nabla_{(A'A'}L_{BCD)A'} , \quad F_{BC} = \nabla^{AA'}L_{ABCA'} .$$

The first of these equations is called the Weyl-Lanczos equation and such a spinor $L_{ABCA'}$ is said to be a Lanczos (spinor) potential of $W_{ABCD}$. The spinor $F_{BC}$ is called the differential gauge of $L_{ABCA'}$. When $F_{BC} = 0$ the Lanczos potential is said to be in Lanczos differential gauge. Of particular interest is the case $W_{ABCD} = \Psi_{ABCD}$ i.e., Lanczos potentials of the Weyl curvature spinor. These Lanczos potentials are spinor analogues of the Lanczos tensor potentials, originally investigated in [13]. For an extensive account of the Lanczos potential and its properties, see [1] and [6].
One of the most remarkable results concerning Lanczos potentials is Illge’s wave equation [11]. Suppose $L_{ABCA'}$ is a Lanczos potential of $W_{ABCD}$ in the differential gauge $F_{BC}$. Then $L_{ABCA'}$ satisfies the following linear wave equation

$$\Box L_{ABCA'} + 6\Phi_A'B'(A^D L_{BC})_D B' + 6\Delta L_{ABCA'} + \nabla_A^D W_{ABCD} - \frac{3}{2} \nabla_A'(A^D F_{BC}) = 0$$

Now, if $W_{ABCD}$ is actually the Weyl spinor $\Psi_{ABCD}$, if the spacetime is vacuum and if $L_{ABCA'}$ is in Lanczos differential gauge, we obtain the remarkably simple equation

$$\Box L_{ABCA'} = 0.$$ 

By letting $L_{abc}$ be the tensor equivalent of the hermitian spinor

$$L_{abc} = L_{ABCC'}\varepsilon_{A'B'} + L_{A'B'C'C}\varepsilon_{AB},$$

the tensor $L_{abc}$ has the symmetries $L_{abc} = L_{[ab]c}$, $L_{[abc]} = 0$, $L_{ab} = 0$. This last symmetry was originally thought of as a gauge condition called the Lanczos algebraic gauge; however, because of the spinor correspondence we choose to include this symmetry in the definition of the Lanczos potential. As we shall see below, it also gives us a comparatively simple form of the tensor equation corresponding to the Weyl-Lanczos equation.

We can now define a Lanczos tensor potential of the Weyl tensor $C_{abcd}$, or indeed of any tensor $W_{abcd}$ having the same algebraic symmetries as the Weyl tensor, by translating the Weyl-Lanczos equation into tensor formalism. We obtain the Weyl-Lanczos tensor equation which reads

$$W_{abcd} = L_{ab[cd]} + L_{cd[ab]} - *L^*_{ab[cd]} - *L^*_{cd[ab]}$$

where $W_{abcd}$ has the same algebraic symmetries as the Weyl tensor. This is the original definition of the Lanczos potential given in [13]. By differentiating the Weyl-Lanczos tensor equation and using the Bianchi identities and the commutators we obtain a wave equation, similar to Illge’s spinor wave equation. It is

$$\Box L_{abc} - 2L^{def} g_{[a[C_{bde}f]} + 2L_{[a}^{de} C_{b]ede} + \frac{1}{2} L_{de}^{de} C_{deab} = 0$$

in vacuum, Lanczos differential gauge and $W_{abcd} = C_{abcd}$. It is interesting to note that it is much more difficult to calculate this tensor wave equation than the corresponding spinor one, and it also appears at first to have a much more complicated structure. It includes an expression involving products of the Weyl tensor and the Lanczos potential explicitly. However, it has been confirmed that this additional expression is actually identically zero in four, and only in four dimensions [7] giving

$$\Box L_{abc} = 0$$

in agreement with the spinor equation. This is a consequence of Lovelock’s identity [13]. Therefore the Lanczos wave equation provides a striking illustration of the power of the spinor formalism.
An interesting result regarding this wave equation was proved by Edgar and Högland [6]. They showed that in a vacuum spacetime of ‘sufficient generality’ (see [7] or [4]) a spinor $L_{ABCA'}$ in Lanczos differential gauge $\nabla^{AA'}L_{ABCA'} = 0$ is a constant multiple of a Lanczos potential of the Weyl spinor if and only if $\Box L_{ABCA'} = 0$. Hence, in this particular case Illge’s wave equation is actually a sufficient condition for $L_{ABCA'}$ to be a constant multiple of a Lanczos potential of the Weyl spinor.

Illge’s theorem in [11] is actually more general than we have quoted above. Illge proves the existence of a potential similar to the one mentioned above, for the case when the symmetric spinor $W$ has an arbitrary number of indices. For easy reference we include the complete theorem of Illge in this section, together with a generalization also mentioned in [11]:

**Theorem 2.1** Let symmetric spinor fields $W_{AA_1\ldots A_n}$, $F_{A_2\ldots A_n}$, a spacelike past-compact hypersurface $\Sigma$ of class $C^\infty$ and a symmetric spinor field $\tilde{L}_{A_1\ldots A_n A'}$ defined only on $\Sigma$ be given. Then there exists a neighbourhood of $\Sigma$ in which the equations

\[
\begin{align*}
W_{AA_1\ldots A_n} &= 2\nabla_{A A'} L_{A_1\ldots A_n A'} \\
F_{A_2\ldots A_n} &= \nabla^{A A'} L_{A_1 A_2\ldots A_n A'}.
\end{align*}
\]

have a unique symmetric solution $L_{A_1\ldots A_n A'}$ satisfying $L|_\Sigma = \tilde{L}$.

We note that a spinor $W_{ABCD}$ in general has many Lanczos potentials in each differential gauge $F_{BC}$.

Following Illge [11] we attempt to generalize this theorem to symmetric spinors with both primed and unprimed indices. Let $W_{AA_1\ldots A_n B'_1\ldots B'_m}$ and $F_{A_2\ldots A_n B'_1\ldots B'_m}$ be completely symmetric spinors. We then look for a spinor $L_{A_1\ldots A_n B'_1\ldots B'_m A'}$ so that

\[
\begin{align*}
W_{AA_1\ldots A_n B'_1\ldots B'_m} &= 2\nabla_{A A'} L_{A_1\ldots A_n B'_1\ldots B'_m A'} \\
F_{A_2\ldots A_n B'_1\ldots B'_m} &= \nabla^{A A'} L_{A_1 A_2\ldots A_n B'_1\ldots B'_m A'}.
\end{align*}
\]

From this equation we see that it is natural to require that $L$ has the symmetry

\[L_{A_1\ldots A_n B'_1\ldots B'_m A'} = L_{(A_1\ldots A_n)(B'_1\ldots B'_m)A'}\]

By combining the above two equations into one, differentiating and using the commutators we arrive at a wave equation analogous to Illge’s wave equation (these calculations will be shown in detail for some special cases in later sections)

\[
0 = \Box L_{A_1\ldots A_n B'_1\ldots B'_m A'} - 2n\Phi_{B'B''} D(A_1) F_{A_2\ldots A_n B'_1\ldots B'_m B''}
\]

\footnote{From now on, a circle above a spinor field i.e., $\tilde{L}$, will always mean that the spinor field is defined only on $\Sigma.$}
Theorem 2.2

2.1 (see [11]): constraints occur, and Illge has proven the following generalization of Theorem Ricci spinor, is not ruled out. The possibility of $W$ potential of type $L_{A1\ldots An}B_1\ldots B'_mA'$, for a symmetric potential exists. If this equation can be solved we might expect to find a potential $\Phi$ constraint on the potential itself. Thus, we obtain not only a wave equation for $L$, but also an algebraic constraint on the potential $L$. Therefore we cannot, in general, find a completely symmetric solution of this equation i.e.,

$$L_{A1\ldots An}B_1\ldots B'_mA' = L_{(A_1\ldots An)(B_1'\ldots B'_m)(A')}.$$ 

Multiplying (2) by $\varepsilon A'B'_1$ gives

$$0 = n\Phi_{A'B'D(A_1)LD_{A_2\ldots An}B'_1\ldots B'_mA'B'} + (m - 1)\Psi_{A'B'C'(B_1' L_{A_1\ldots An}B'_1\ldots B'_m)}A'B'C' + \frac{1}{2} \nabla^{AA'}W_{AA_1\ldots AnA'B'_1\ldots B'_m} - \frac{n}{n + 1} \nabla(A_1 A' F_{A_2\ldots An}B'_1\ldots B'_m)A'B'_1\ldots B'_m (3)$$

Thus, we obtain not only a wave equation for $L$, but also an algebraic constraint on the potential $L$. Therefore we cannot, in general, find a completely symmetric potential for a spinor field with both primed and unprimed indices. However, we immediately see some cases where these constraints are automatically satisfied e.g., when $n = 0$, $m = 1$ providing $\nabla^{AA'}W_{AA'} = 0$. Illge [1] proves that in this case a symmetric potential exists.

Also, if $m = 1$ and $\Phi_{ABA'B'} = 0$, then the potential vanishes from this constraint equation, and we are left with just an equation for the differential gauge $F$. If this equation can be solved we might expect to find a potential for $W$. These ideas will be explored in detail in Section 6. On the other hand, if $n = 0$ and $\Psi_{ABCD} = 0$ we also see that the above equation is no longer a constraint on the potential itself.

So, in particular we see that for $W_{AA_1\ldots AnA'}$ the possibility of having a potential of type $L_{A_1\ldots AnA'B'} = L_{(A_1\ldots An)(A'B')}$ in spacetimes with vanishing Ricci spinor, is not ruled out. The possibility of $W_{AB'_1\ldots B'_m}$ having a symmetric potential $L_{B'_1\ldots B'_mA'}$ in conformally flat spacetimes is not ruled out either.

Finally we note that if we do not require complete symmetry of $L$, then no constraints occur, and Illge has proven the following generalization of Theorem 2.1 (see [1]):

**Theorem 2.2** Let symmetric spinor fields $W_{AA_1\ldots AnA'B'_1\ldots B'_m}$, $F_{A_2\ldots AnA'B'_1\ldots B'_m}$, a spacelike past-compact hypersurface $\Sigma$ of class $C^\infty$ and a spinor field

$$\tilde{L}_{A_1\ldots AnA'B'_1\ldots B'_m}A' = \tilde{L}_{(A_1\ldots An)(A')}A'.$$
defined only on $\Sigma$ be given. Then there exists a neighbourhood of $\Sigma$ in which the equations
\begin{align*}
W_{AA_1 \cdots A_n B'_1 \cdots B'_m} &= 2\nabla (A^A' L_{A_1 \cdots A_n} B'_1 \cdots B'_m A') \\
F_{A_2 \cdots A_n B'_1 \cdots B'_m} &= \nabla A^A' L_{A_1 A_2 \cdots A_n B'_1 \cdots B'_m A'}.
\end{align*}
have a unique solution $L_{A_1 \cdots A_n B'_1 \cdots B'_m A'} = L_{(A_1 \cdots A_n) (B'_1 \cdots B'_m) A'}$ that satisfies $L|_\Sigma = 0$.

In summary, Illge has shown that any symmetric spinor (in fact the symmetry condition is not necessary, although these are usually the spinors we are interested in) has a potential (actually two different potentials using Theorem 2.2 and the complex conjugate of Theorem 2.2), but it is only for symmetric spinors which are restricted to only one type of index where we can always obtain a symmetric potential.

3 Simple existence proofs for potentials of various spinors

3.1 Introduction

In this section we will give an existence proof for the Lanczos potential and its generalization. Even though the results of this section can partly be seen as special cases of Illge’s theorem in [11], they have certain advantages compared to the results in [11].

The most important advantage is that the existence proof of this section is conceptually simpler and more direct than the proof given in [11]. This is partly because it is hard to ‘separate out’ the existence part from the proof in [11]. Also the potential $T_{ABCD}$ for the Lanczos potential (whose existence could be deduced from the complex conjugate of Theorem 2.2) turns up as an essential part of the theorem. This raises the question whether this potential is important in itself.

The obvious drawback of this existence proof is that it is just an existence proof. It does not give us any uniqueness result whatsoever.

3.2 An existence proof for Lanczos potentials

Let $W_{ABCD}$ and $F_{BC}$ be arbitrary spinor fields. Our objective is to show that locally there exists a symmetric spinor $L_{ABCA'}$ such that
\begin{align*}
W_{ABCD} &= 2\nabla (A^A' L_{BCD} A') \\
F_{BC} &= \nabla A^A' L_{ABCA'}
\end{align*}
(4)

These equations can be combined into one:
\begin{align*}
2\nabla A^A' L_{BCDA'} &= W_{ABCD} - \frac{3}{2} \epsilon_{ABCD} F_{BC}
\end{align*}
(5)
Suppose there exists a spinor $T_{ABCD} = T_{(ABC)D}$ such that

$$L_{ABCA'} = \nabla_{A'}^D T_{ABCD}$$

(6)

where $L_{ABCA'}$ is a solution of (5). Note that we do not invoke (the complex conjugate of) Theorem 2.2 to ensure the existence of such a spinor. At the moment we are merely looking at necessary conditions for its existence. Equation (6) then reads

$$2\nabla_A A' T_{BCDE} = 2\nabla_A A' L_{BCDA'} = W_{ABCD} - \frac{3}{2} \varepsilon_A (B F_{CD}).$$

(7)

On the other hand,

$$2\nabla_A A' E T_{BCDE} = -\Box T_{BCDA} + 2\nabla_A (A \nabla_A') T_{BCD}^E$$

$$= -\Box T_{BCDA} - 2(X_{ABE} G_{T_{BCD}}^E + X_{ABC} G_{T_{BCD}}^E)$$

$$+ X_{AED} G_{T_{BCG}}^E - X_{AEG} G_{T_{BCD}}^E$$

$$= -\Box T_{BCDA} + 6\Psi_{AB}^E T_{CDF}^G - 6\Lambda (\varepsilon_A B T_{CD})^E$$

$$+ T_{A \{BCD \} + T_{BCDA}}$$

where we have used the commutators [18] and the fact that $X_{ABCD} = \Psi_{ABCD} + \Lambda (\varepsilon_A \varepsilon_B + \varepsilon_A \varepsilon_B \varepsilon_B \varepsilon_C).$ Combining the last equation with (7) yields the following wave equation for $T_{BCDA}$

$$0 = \Box T_{BCDA} - 6\Psi_{AB}^E T_{CDF}^G + 6\Lambda (\varepsilon_A B T_{CD})^E + T_{A \{BCD \} + T_{BCDA}}$$

$$- \frac{3}{2} \varepsilon_A (B F_{CD}) + W_{ABCD}.$$ (8)

That this equation is satisfied is a necessary condition for the existence of a Lanczos potential of the above type. We will now show that this equation can also be used to prove the existence of a Lanczos potential of said type.

**Theorem 3.1** Given symmetric spinors $W_{ABCD}$ and $F_{CD}$ there exists a spinor $L_{ABCA'} = L_{(ABC)A'}$ satisfying equations (4) locally. Also $L_{ABCA'}$ satisfies equation (8) for some spinor $T_{ABCD} = T_{(ABC)D}$.

**Proof:** Let $p \in M$ be an arbitrary point. From a theorem in [3] there exists a causal neighbourhood $U$ of $p$. Now, consider the wave equation (8). This is a linear, diagonal second order hyperbolic system for $T_{ABCD}$. Hence, from the theory for hyperbolic equations (see e.g., [22] or [3]) we know that it has a solution $T_{ABCD}$ throughout $U$. Put $L_{ABCA'} = \nabla_{A'}^D T_{ABCD}$; then, because $T_{ABCD}$ is a solution of (8) it follows that

$$2\nabla_A A' L_{BCDA'} = 2\nabla_A A' \nabla_A^E T_{BCDE} = W_{ABCD} - \frac{3}{2} \varepsilon_A (B F_{CD})$$

and as an easy consequence, equation (8) is satisfied. \qed

8
3.3 Symmetric potentials for \((n + 1, 0)\)–spinors and asymmetric potentials for \((n + 1, m)\)–spinors

Even though the most studied case is when \(W_{ABCD}\) is the Weyl curvature spinor, there is nothing special about spinors with four indices. Thus, we immediately obtain the following generalization:

**Theorem 3.2** Given symmetric spinors \(W_{AA_1 \cdots A_n}\) and \(F_{A_2 \cdots A_n}\) there exists a spinor \(L_{A_1 \cdots A_n, A'} = L_{(A_1 \cdots A_n), A'}\) satisfying the equations

\[
W_{AA_1 \cdots A_n} = 2\nabla_{(A} A' \nabla_{A_1 \cdots A_n)} A',
\]

\[
F_{A_2 \cdots A_n} = \nabla^A A' \nabla_{A_1 \cdots A_n} A'
\]

locally. Also \(L_{A_1 \cdots A_n, A'}\) satisfies the equation

\[
L_{A_1 \cdots A_n, A'} = \nabla_A B T_{A_1 \cdots A_n B}
\]

for some spinor \(T_{A_1 \cdots A_n B} = T_{(A_1 \cdots A_n), B}\).

**Proof:** It is simply a matter of going through the same calculations as in the previous section to arrive at a similar wave equation for \(T_{A_1 \cdots A_n B}\) as equation \((\ref{eq:wave})\). By the theory for hyperbolic equations, this equation will also have a solution locally. Proceed as in the proof of the previous theorem. \(\square\)

Another possible generalization of the above theorem would be to allow the spinor \(W\) to have primed indices also, and to look for a potential having one extra primed index (of course we could reverse the role of primed and unprimed indices in this argument). Unfortunately it turns out that if we write down the equation corresponding to equation \((\ref{eq:wave})\) it will not necessarily be a linear, diagonal second order hyperbolic system, if we require our potential to be completely symmetric (see equations \((\ref{eq:wave})\) and \((\ref{eq:wave})\)). However, if we remove the requirement of symmetry over the primed indices, an analogous theorem can easily be proved in exactly the same way as for the previous theorems. Thus, to be precise:

**Theorem 3.3** Given symmetric spinors \(W_{AA_1 \cdots A_n B'_1 \cdots B'_m}\) and \(F_{A_2 \cdots A_n B'_1 \cdots B'_m}\) there exists a spinor \(L_{A_1 \cdots A_n B'_1 \cdots B'_m, B'} = L_{(A_1 \cdots A_n), (B'_1 \cdots B'_m), B'}\) satisfying the equations

\[
W_{AA_1 \cdots A_n B'_1 \cdots B'_m} = 2\nabla_{(A} B' \nabla_{A_1 \cdots A_n)} B'_1 \cdots B'_m B',
\]

\[
F_{A_2 \cdots A_n B'_1 \cdots B'_m} = \nabla^{A'} B' \nabla_{A_1 \cdots A_n} B'_1 \cdots B'_m B'
\]

locally. Also \(L_{A_1 \cdots A_n B'_1 \cdots B'_m, B'}\) satisfies the equation

\[
L_{A_1 \cdots A_n B'_1 \cdots B'_m, B'} = \nabla_{B'} A' T_{A_1 \cdots A_n AB'_1 \cdots B'_m B'}
\]

for some spinor \(T_{A_1 \cdots A_n AB'_1 \cdots B'_m} = T_{(A_1 \cdots A_n), A(B'_1 \cdots B'_m)}\).
Note that (the complex conjugate of) this last theorem actually ensures that for any symmetric spinor \( L_{ABCA'} \) there exists a spinor \( T_{ABCD} = T_{(ABC)D} \) such that \( L_{ABCA'} = \nabla_A^D T_{ABCD} \).

We remark once again that the above results only guarantee local existence of the Lanczos potential in general. There is however an important class of spacetimes for which we can guarantee global existence of the Lanczos potential. If we assume that \( M \) has a global spinor structure and is globally hyperbolic i.e., contains a Cauchy surface, then equation (8) has a global solution \( T_{ABCD} \) and if we put \( L_{ABCA'} = \nabla_A^D T_{ABCD} \) then \( L_{ABCA'} \) will be globally defined, and will of course still be a Lanczos potential. Thus, in globally hyperbolic spacetimes with a global spinor structure, the above results guarantee the existence of a global Lanczos potential.

3.4 The tensor version

A spinor with the symmetries \( T_{ABCD} = T_{(ABC)D} \) can of course be decomposed into \( U_{ABCD} = T_{(ABCD)} \) and \( V_{AB} = T_{ABC} \). The wave equation (8) then splits into

\[
0 = \Box U_{ABCD} - 6 \Psi_{(AB}^{\,\,EG} U_{CD)G} + 3 \Psi_{(ABG}^{\,\,V_D)G} + 12 \Delta U_{ABCD} + W_{ABCD} \\
0 = \Box V_{BC} - 4 \Psi^{\,\,DEF}_{(B} U_{C)DEF} + \Psi_{BC}^{\,\,DE} V_{DE} - 4 \Delta V_{BC} + 2 F_{BC}
\]

(9)

Now, \( U_{ABCD} \) corresponds to a tensor \( U_{abcd} \) having Weyl symmetry, and \( V_{AB} \) corresponds to a 2-form \( V_{ab} \). As before \( L_{abc} \) is the tensor corresponding to \( L_{ABCA'} \). The differential gauge \( F_{ab} \) is defined by \( F_{ab} = L_{abc} \). In this way all the above definitions carry over to four dimensional spaces of arbitrary signature. The above proof can also be directly translated into tensors e.g., the wave equations (8) becomes

\[
\nabla^2 U_{abcd} - \frac{1}{2} (C_{ab}^{\,\,ef} U_{cdef} + C_{cd}^{\,\,ef} U_{abef}) - 2 C_{e}^{\,\,f[a} U_{b]ef} + 2 C_{d}^{\,\,e[a} U_{b]fde} \\
- \frac{3}{2} (C_{ab[c}^{\,\,d] e} V_{d]e} + C_{cd[a}^{\,\,b]e} V_{b]e}) + \frac{R}{2} U_{abcd} + W_{abcd} = 0 \\
\nabla^2 V_{bc} + 4 C^{\,\,def}_{[b} U_{c]def} + \frac{1}{2} C_{bc}^{\,\,de} V_{de} - \frac{R}{6} V_{bc} + 2 F_{bc} = 0
\]

(10)

Note that here \( \nabla^2 = \nabla_a \nabla_a \) is not necessarily a wave operator since \( M \) is of arbitrary signature. Now, if \( M \) is (real) analytic, and both \( W_{abcd} \) and \( F_{bc} \) are (real) analytic then, by the Cauchy-Kovalevskaya theorem, this system of equations always has a local solution and by translating equation (8) into tensors we can construct a Lanczos potential of \( W_{abcd} \) in the differential gauge \( F_{bc} \) from the solution of (10). We obtain that

\[
L_{abc} = - U_{abc}^{\,\,d} :d - \frac{1}{2} (V_{abc} - V_{c[a} b]) - \frac{1}{2} g_{[a} V_{b]}^{\,\,d} :d
\]

(11)
is a Lanczos potential of $W_{abcd}$ in the differential gauge $F_{bc}$.

Hence, we have shown that Lanczos potentials exist in 4-dimensional analytic spacetimes of arbitrary signature in agreement with the result of Bampi and Caviglia. We remark that this technique seems incapable of generalization to spaces of higher dimensions than four. The reason for this is that when we plug the $n$-dimensional version of (11) into the $n$-dimensional version of the Weyl-Lanczos equation the resulting equation will not be a wave equation since other terms involving second derivatives of $U_{abcd}$ and $V_{bc}$ will fail to cancel\footnote{In four dimensions these terms cancel by virtue of a special case of Lovelock’s identity in a manner analogous to the Lanczos wave equation}. Therefore existence of solutions to these equations is not guaranteed. In fact, we have strong evidence that Lanczos potentials do not exist, in general, in dimensions greater than four \footnote{In a manner analogous to the Lanczos wave equation}.

4 The gauge freedom in the Lanczos potential

4.1 General gauge transformations

The theorems of Section 3 now enable us to characterize the remaining gauge freedom in the Lanczos potential when the differential gauge is specified. Let $W_{ABCD}$ and $F_{BC}$ be given symmetric spinors. Let $L_{ABCA'}$ and $\tilde{L}_{ABCA'}$ be two Lanczos potentials of $W_{ABCD}$ in the same differential gauge $F_{BC}$ i.e.,

$$W_{ABCD} = 2\nabla_{(A'}L_{BCD)A'} = 2\nabla_{(A'}\tilde{L}_{BCD)A'}$$

$$F_{BC} = \nabla^{AA'}L_{ABCA'} = \nabla^{AA'}\tilde{L}_{ABCA'} \quad (12)$$

Put $M_{ABCA'} = \tilde{L}_{ABCA'} - L_{ABCA'}$ so that

$$2\nabla_{(A'}M_{BCD)A'} = 0, \quad \nabla^{AA'}M_{ABCA'} = 0$$

which is equivalent to

$$\nabla_{A'}M_{BCDA'} = 0. \quad (13)$$

This equation has a formal resemblance to the equation for a spin-2-field

$$\nabla_{A'}A_{W_{ABCD}} = 0$$

which has been studied by Bell and Szekeres \footnote{In a manner analogous to the Lanczos wave equation}, who found that in vacuum spacetimes of ‘sufficient generality’ (see \footnote{In a manner analogous to the Lanczos wave equation}), the only solutions to this equation are

$$W_{ABCD} = c\Psi_{ABCD}$$

where $c$ is a complex constant. Therefore we might expect (13) to have very few solutions. However, we will see that this is not the case. One reason for this is that by taking another derivative and using the commutators, we do not
obtain any additional algebraic conditions (so called Buchdahl conditions) on $M_{ABCA'}$, unlike the very strong condition on $W_{ABCD}$.

Now, according to the complex conjugate of Theorem 3.3 there exists a spinor $T_{ABCD} = T_{(ABC)D}$ such that $M_{ABCA'} = \nabla_A D T_{ABCD}$. The same calculations as in the previous section tells us that $T_{ABCD}$ must satisfy the following wave equation:

$$0 = \square T_{BCDA} - 6\Psi_{AB(E} T_{CD)GE} + 3\Psi_{(ABC} G_{DE)G} + 12\Lambda U_{ABCD}$$

where $U_{ABCD} = T_{(ABCD)}$ and $V_{BC} = T_{BCD}$. Since these equations are coupled we see that in general we need both $U_{ABCD}$ and $V_{BC}$ non-zero to get a proper gauge transformation. An important exception is when $M$ is conformally flat (i.e., $\Psi_{ABCD} = 0$) where the equations decouple, and so we could obtain gauge transformations where e.g., one of $U_{ABCD}$ and $V_{BC}$ is zero, but not the other. See however Section 4.2.

Thus, we have shown that if $M_{ABCA'}$ constitutes a gauge transformation of a Lanczos potential $L_{ABCA'}$ that does not change the differential gauge i.e., such that

$$\tilde{L}_{ABCA'} = L_{ABCA'} + M_{ABCA'}$$

is still a Lanczos potential of $W_{ABCD}$ in the differential gauge $F_{BC}$ then we can write $M_{ABCA'} = \nabla_A D T_{ABCD}$ where $T_{ABCD}$ is a solution of (14). Conversely, suppose $T_{ABCD}$ is a solution of (14) and put $M_{ABCA'} = \nabla_A D T_{ABCD}$. Then equation (14) can be rewritten as

$$0 = 2\nabla_A A' \nabla_A E T_{BCDE},$$

from which

$$2\nabla_A A' M_{BCDA'} = 0.$$ 

Decomposing into symmetric- and trace parts gives us

$$2\nabla_A A' M_{BCD} = 0, \quad \nabla A A' M_{ABCA'} = 0$$

so that $\tilde{L}_{ABCA'} = L_{ABCA'} + M_{ABCA'}$ is also a Lanczos potential of $W_{ABCD}$ in the differential gauge $F_{BC}$.

Thus, we have completely characterized the gauge transformations of the Lanczos potential that leaves the differential gauge intact. We summarize our findings in the following theorem:
Theorem 4.1 Let \( W_{ABCD} \) and \( F_{BC} \) be given symmetric spinors. Let \( L_{ABCA'} \) be a Lanczos potential of \( W_{ABCD} \) in the differential gauge \( F_{BC} \). Then the symmetric spinor \( \tilde{L}_{ABCA'} \) is also a Lanczos potential of \( W_{ABCD} \) in the differential gauge \( F_{BC} \) if and only if \( \tilde{L}_{ABCA'} = L_{ABCA'} + M_{ABCA'} \) where

\[
M_{ABCA'} = \nabla_{A'} T_{ABCD}
\]

and \( T_{ABCD} = T_{(AB)D} \) is a solution of \((14)\).

For completeness we also give the tensor version of this result. The translation itself is tedious but straightforward.

Theorem 4.2 Let \( W_{abcd} \) be a tensor having all the algebraic symmetries of the Weyl tensor and let \( F_{bc} \) be an arbitrary 2-form i.e., \( F_{bc} = F_{[bc]} \). Let \( L_{abc} \) be a Lanczos potential of \( W_{abcd} \) in the differential gauge \( F_{bc} \) i.e., \( L_{ab}^{c;}_{;c} = F_{bc} \). Let \( \tilde{L}_{abc} \) be a spinor with the same algebraic symmetries of \( L_{abc} \). Then \( \tilde{L}_{abc} \) is also a Lanczos potential of \( W_{abcd} \) in the differential gauge \( F_{bc} \) if and only if \( \tilde{L}_{abc} = L_{abc} + M_{abc} \) where

\[
M_{abc} = -U_{abc}^{d;}_{;d} - \frac{1}{2}(V_{abc} - V_{[a;b]} - \frac{1}{2}g_{[a}V_{b]}^{d;}_{;d})
\]

and where \( U_{abcd} \) have all the algebraic symmetries of the Weyl tensor, \( V_{bc} \) is a 2-form and in addition \( U_{abcd} \) and \( V_{bc} \) satisfies

\[
\Box U_{abcd} = -\frac{1}{2}(C_{ab}^{\ e}fU_{cdef} + C_{cd}^{\ ef}U_{abef}) - 2C_{c}^{\ e[a}fU_{b]def} + 2C_{d}^{\ e[a}fU_{b]efce} - \frac{3}{2}(C_{ab}^{\ e}V_{d}^{\ ;e} + C_{cd}^{\ e}V_{b}^{\ ;e}) + \frac{R}{2}U_{abcd} = 0
\]

\[
\Box V_{bc} = 4C_{c}^{\ def}[bU_{d]}^{\ ;b} + \frac{1}{2}C_{bc}^{\ de}V_{de} - \frac{R}{6}V_{bc} = 0
\]

(16)

4.2 Gauge transformations with \( U_{ABCD} = 0 \)

In this section we will consider gauge transformations for which \( U_{ABCD} = 0 \) i.e., gauge transformations of the form

\[
\tilde{L}_{ABCA'} = L_{ABCA'} - \frac{3}{4}\nabla_{A'}(AV_{BC})
\]

As we saw earlier the chances of finding such gauge transformations, if we want to preserve the differential gauge, are rather slim, except in the case when \( M \) is conformally flat. However, if we allow gauge transformations that change the differential gauge the chances are much better. First note that allowing the differential gauge to change is the same as solving only the first of equations \((15)\). When \( U_{ABCD} = 0 \) this equation becomes

\[
\Psi_{(ABC}^{\ G}V_{D)G} = 0.
\]
Introducing components in the usual way gives us the following system of equations

\[
\begin{align*}
0 &= \Psi_1 V_0 - \Psi_0 V_1 \\
0 &= 3\Psi_2 V_0 - 2\Psi_1 V_1 - \Psi_0 V_2 \\
0 &= \Psi_3 V_0 - \Psi_1 V_2 \\
0 &= \Psi_4 V_0 + 2\Psi_3 V_1 - 3\Psi_2 V_2 \\
0 &= \Psi_4 V_1 - \Psi_3 V_2
\end{align*}
\]

This system can easily be solved for the different Petrov types of the Weyl spinor \(\Psi_{ABCD}\), using the principal spinors as dyad spinors.

The result is that gauge transformations that are allowed to change the differential gauge and have \(U_{ABCD} = 0\) exist, if and only if \(\Psi_{ABCD}\) is type D, N or 0. In type D we have \(V_{AB} = -2V_1 o_A o_B\) where \(o_A\) and \(\iota_A\) are principal spinors of \(\Psi_{ABCD}\), in type N \(V_{AB} = V_2 o_A o_B\) where \(o_A\) is the principal spinor of \(\Psi_{ABCD}\) and in type 0, \(V_{AB}\) is arbitrary. We remark that gauge transformations of this type have earlier been investigated by Torres del Castillo [20], [21] in the type D and type 0 case.

It is important to note that in the whole discussion above, \(W_{ABCD}\) is arbitrary and therefore, in particular, the results do not depend on the Petrov type of \(W_{ABCD}\). They only depend on the Petrov type of the Weyl spinor \(\Psi_{ABCD}\).

5 Potentials for symmetric (3,1)-spinors in Einstein spacetimes

5.1 Introduction

In some special cases [1], [2], [5], [20] there has been found a completely symmetric spinor \(H_{ABA'B'}\) such that the spinor

\[
L_{ABCA'} = \nabla_{(A} B' H_{BC)A'B'}
\]

is a Lanczos potential of the Weyl spinor. In this section we will prove that such a spinor \(H_{ABA'B'}\) exists in all Einstein spacetimes i.e., spacetimes such that the Ricci spinor \(\Phi_{ABA'B'} = 0\). In fact, we will prove that in such spacetimes any symmetric spinor \(L_{ABCA'}\) can be written as

\[
L_{ABCA'} = \nabla_{(A} B' H_{BC)A'B'}
\]

for some spinor \(H_{ABA'B'} = H_{(AB)(A'B')}\) and that for each choice of \(L_{ABCA'}\) there exists many such spinors \(H_{ABA'B'}\). We emphasize that this result does not follow from Theorem 2.2 or Theorem 3.3 because here we are requiring complete symmetry of \(H_{ABA'B'}\).
5.2 A preliminary result

First we need a preliminary lemma, which is of interest in its own right.

**Lemma 5.1** For any symmetric spinor field \( \varphi_{AB} \), timelike or spacelike vector field \( n^{AA'} \) and complex function \( f \) there exists a unique complex vector field \( \zeta^{AA'} \) such that \( \varphi_{BC} = n_{(B^A} \zeta_{C)A'} \) and in addition \( n^{AA'} \zeta_{AA'} = f \).

**Proof:** By rescaling, it suffices to assume that \( n^{AA'} \) is a unit timelike- or spacelike vector. We start by proving uniqueness; suppose that \( \varphi_{BC} = n_{(B^A} \zeta_{C)A'} \) and \( n^{AA'} \zeta_{AA'} = f \). For the case when \( n^{AA'} \) is timelike we obtain,

\[
2 \varphi_A^B n_{BA'} + f n_{AA'} = n_A^B \zeta_B n_{BA'} + n_{BB'} \zeta_{AB'} n_{BA'} + f n_{AA'}
\]

\[
= \frac{1}{2} \zeta_{AA'} + n_{AA'} n_B^A \zeta_B n_{BB'} + \varepsilon_{BA'C} n^{CB'} \zeta_{B'B} + f n_{AA'}
\]

\[
= \zeta_{AA'}.
\]

where we have used that \( n_{B'}^C n_{CA'} = \frac{1}{2} \varepsilon_{A'B'} \). In the spacelike case, the same calculations give

\[
2 \varphi_A^B n_{BA'} - f n_{AA'} = \zeta_{AA'}.
\]

This proves the uniqueness part so now we need only verify that the above candidate for \( \zeta_{AA'} \) actually satisfies the conclusion of the lemma. As before we start with the timelike case:

\[
n^{AA'} (2 \varphi_A^B n_{BA'} + f n_{AA'}) = 2 \varphi_A^B \cdot \frac{1}{2} \varepsilon_B^A + f = f
\]

and

\[
n_{(B^A} \zeta_{C)A'} = 2 n_{(B^A} \varphi_{C)}^D n_{DA'} + f n_{(B^A} n_{C)A'} = \varepsilon_{D(B^C} \varphi_{C)}^D = \varphi_{BC}
\]

This proves the lemma in the timelike case. The spacelike case is proved in exactly the same way. \( \square \)

5.3 Construction of the spinor potential

Let \( M \) be an Einstein spacetime i.e., \( \Phi_{ABA'B'} = 0 \) and let \( L_{ABC} \) be a symmetric spinor field on \( M \). Our objective is to show that locally there exists a spinor field \( H_{ABA'B'} = H_{(AB)(A'B')} \) such that

\[
L_{ABC} = \nabla_{(A}^B H_{BC)A'B'}.
\]

We also wish to examine the gauge freedom in the potential \( H_{ABA'B'} \).

Our strategy for proving the existence of \( H_{ABA'B'} \) will be to start by deriving a wave equation for \( H_{ABA'B'} \), along with some constraint equations. Then we
use the same theorem from [3] as in Section 3 to show that these equations have a solution; finally we prove that this solution also solves equation (19).

We begin by assuming that $H_{AB'A'B'} = H_{(AB)(A'B')}$ satisfies

$$\nabla_{(A'B'} H_{BC)A'B'} = L_{ABCA'}, \quad \nabla^{AA'} H_{AB'A'B'} = \zeta_{BB'}$$

(20)

where $\zeta_{BB'}$ is a given spinor field (complex 1-form). Note that since $\tilde{\zeta}_{BB'}$ unit normal $C$ Now, let $\Sigma$ be a $C^\infty$ spacelike past-compact hypersurface with future-directed $\epsilon$-unit normal $\nu$ so that $\tilde{\zeta}_{BB'}$ acts tangentially to $\Sigma$. Hence, the above equation is equivalent to the following

$$\nabla_A B' H_{BCA'B'} = L_{ABCA'} - \frac{2}{3} \epsilon_{A(B} \nabla^{D'E'} H_{C)D'A'D'}$$

(21)

so (20) is equivalent to

where we have put $\zeta_{AA'} = \zeta_{AA'}|\Sigma$.

Next we differentiate the LHS of (21):

$$\nabla^A C' \nabla_A B' H_{BCA'B'} = \epsilon^{B'D'} \nabla^A (C' \nabla_{D'}) A H_{BCA'B'} + \frac{1}{2} \nabla^A E' \nabla_A E' H_{BCA'C'}$$

$$= - \frac{1}{2} \Box H_{BCA'C'} + \Psi_{B'E'A'C'} H_{BC} E'B'$$

$$- 4 \Lambda H_{BCA'C'}$$

(24)
where we have used that $\Phi_{ABA'B'} = 0$ along with the symmetry of $H_{ABA'B'}$. Thus, $H_{ABA'B'}$ satisfies the following wave equation:

$$
\Box H_{BCA'C'} = 2 \bar{\Psi}_{B'E'A'C'} H_{BC} + 8 \Lambda H_{BCA'C'} \\
= -2 \nabla^A C' L_{ABC'} + \frac{4}{3} \nabla_{C'} (B \zeta)_{A'}
$$

(25)

Note that this equation is actually a special case of equation (2) of Section 2.

Since $H_{BCA'C'}$ is symmetric over $(A'C')$ it follows that (25) is equivalent to

$$
\Box H_{BCA'C'} = 2 \bar{\Psi}_{B'E'A'C'} H_{BC} - 8 \Lambda H_{BCA'C'} \\
= -2 \nabla^A C' L_{ABC'} + \frac{2}{3} \nabla_{C'} (B \zeta)_{A'} + \frac{2}{3} \nabla_{A'} (B \zeta)_{C'}
$$

(26)

The second of these equations is actually equation (3) of Section 2.

After these preliminary considerations we are ready to prove our main result.

**Theorem 5.2** Suppose $M$ is an Einstein spacetime ($\Phi_{ABA'B'} = 0$) and that $\Sigma \subset M$ is a $C^\infty$ spacelike past-compact hypersurface with future directed unit normal $n_{AA'}$. Let a spinor field $L_{ABC'} = L_{(ABC')}$ and a complex function $g$ be given. Furthermore, let a spinor field $\tilde{H}_{ABA'B'} = \tilde{H}_{(AB)(A'B')}$ and a complex function $\tilde{f}$, both defined only on $\Sigma$ be given. Then there exists a neighbourhood $U$ of $\Sigma$ such that there exists a unique spinor field $H_{ABA'B'} = H_{(AB)(A'B')}$ satisfying the equations

$$
\nabla_{(A'B'} H_{BC)A'B'} = L_{ABC'} \\
\nabla_{AA'} \nabla_{BB'} H_{ABA'B'} = g \\
H_{ABA'B'}|_{\Sigma} = \tilde{H}_{ABA'B'} \\
n_{AA'} \nabla_{BB'} H_{ABA'B'}|_{\Sigma} = \tilde{f}
$$

(27)

on all of $U$.

**Proof:** An outline of the existence part of the proof is as follows. We start by solving the second of the equations (26) for $\tilde{\zeta}_{AA'}$ so that $n_{AA'} \tilde{\zeta}_{AA'} = \tilde{f}$. Then we evolve this initial data using the second equation of (26) in such a way that $\nabla_{AA'} \tilde{\zeta}_{AA'} = g$. Next we calculate the normal derivative of $H_{ABA'B'}$ using the first equation of (26) and use the so obtained Cauchy data for $H_{ABA'B'}$ to solve the first equation of (26) for $H_{ABA'B'}$. It can then be verified that this spinor field satisfies all the conditions of the theorem.

Define the symmetric spinor

$$
\tilde{\phi}_{BC} = \frac{3}{2} n_{AA'} (L_{ABC'} - \nabla_{A'B'} \tilde{H}_{BCA'B'})
$$
By Lemma 5.1 there exists a unique spinor \( \zeta_{AA'} \) such that
\[
n^{AA'} \zeta_{AA'} = \varphi
\]
and such that the second of the equations (23) is satisfied i.e.,
\[
\varphi_{BC} = n_{(B} \zeta_{(C)}|_{AA'}.
\] (28)

Our next task will be to solve for \( \zeta_{AA'} \). We want to find \( \zeta_{AA'} \) so that the following three conditions are satisfied
\[
\nabla_{(B} \zeta_{C)}|_{AA'} = \frac{3}{2} \nabla^{AA'} L_{ABCA'}
\]
\[
\nabla^{AA'} \zeta_{AA'} = g
\]
\[
\zeta_{AA'}|_{\Sigma} = \zeta_{AA'}
\] (29)

where \( \zeta_{AA'} \) is the solution of (28) obtained above. Let \( U \) be a causal neighbourhood \([9]\) of \( \Sigma \). According to Theorem 2.1 this problem has a unique solution \( \zeta_{AA'} \) in \( U \).

Next, consider the problem
\[
\Box H_{BCA'C'} = 2 \Psi_{B'EAC'C'} H_{BC'E'B'} + 8 \lambda H_{BCA'C'}
\]
\[
= -2 \nabla A_{(C'} L_{ABC)|A')} + \frac{2}{3} \nabla C'(B \zeta_{C'})|_{AA'} + \frac{2}{3} \nabla A'(B \zeta_{C'})|_{AA'}
\]
\[
= \frac{2}{3} \epsilon_{A(B \zeta_{C})|A'})|_{\Sigma}
\]
\[
\nabla H_{BCA'C'}|_{\Sigma} = H_{BCA'C'}.
\] (30)

These are the first equation of (26), the first equation of (23) and the third condition of (27). Note that the RHS of all three equations contain only known quantities. Hence this problem is a Cauchy problem for a linear, diagonal, second order hyperbolic system. According to a theorem in \([9]\) and \([22]\) this problem has a unique solution \( H_{BCA'C'} \) in \( U \).

It now remains to prove that the \( H_{BCA'C'} \) found above satisfies the conditions
\[
\nabla_{(B} H_{BCA'B')} = L_{ABCA'}, \quad \nabla^{BB'} H_{ABCA'B'} = \zeta_{AA'}.
\]
In order to do that we define
\[
\xi_{ABCA'} = \nabla_{A} H_{BCA'B'} - L_{ABCA'} + \frac{2}{3} \epsilon_{A(B \zeta_{C})|A'}.
\]

Equation (23) now implies that \( \xi_{ABCA'}|_{\Sigma} = 0 \). Because both \( H_{BCA'C'} \) and \( \zeta_{AA'} \) are constructed so that equation (26) are satisfied we have that \( \nabla^{A'} \xi_{ABCA'} = 0 \).
0. Because $\tilde{\nabla}_{AA'}$ only consists of derivatives in directions tangential to $\Sigma$, this gives us that

$$n^A C' \nabla_n \xi_{ABCA'}|\Sigma = - \tilde{\nabla}^A C'|\xi_{ABCA'}|\Sigma = -\tilde{\nabla}^A C' (\xi_{ABCA'})|\Sigma = 0.$$ 

Thus,

$$0 = n^{DC'} n^A C' \nabla_n \xi_{ABCA'}|\Sigma = \frac{1}{2} \tilde{\nabla}^{AD} \nabla_n \xi_{ABCA'}|\Sigma = -\frac{1}{2} \tilde{\nabla}_n \xi^D_{BCA'}.$$ 

Taking another derivative gives us

$$0 = \nabla_D C' \nabla^{AC'} \xi_{ABCA'} = -\frac{1}{2} \Box \xi_{DBCA'} + 2\Psi_D (B^A F^{A'} C^{A'}) - 3 \Lambda \xi_{DBCA'} - 2 \Lambda \xi_{BCD} A A'$$

$$-2 \Lambda \varepsilon_D (B^{A'} C^{A'}) A A'$$

(31)

because we assumed that $M$ is Einstein. Hence $\xi_{ABCA'}$ is a solution of the following problem

$$\Box \xi_{DBCA'} - 4 \Psi_D (B^A F^{A'} C^{A'}) F A' + 6 \Lambda \xi_{DBCA'}$$

$$+ 4 \Lambda \xi_{BCD} A A'$$

$$= 0$$

$$\xi_{DBCA'}|\Sigma = 0$$

$$\nabla_n \xi_{DBCA'}|\Sigma = 0$$

(32)

This homogeneous problem has a unique solution in $U$ according to [9]. Therefore we must have

$$\xi_{ABCA'} = 0$$

in $U$, which implies that

$$\nabla_{(A'B'} H_{BC)A'B'} = L_{ABC'} , \quad \nabla^{B'B'} H_{ABA'B'} = \zeta_{AA'}.$$ 

This proves that $H_{BCA'C'}$ satisfies all the conditions (27), which completes the existence part of the theorem.

**Uniqueness:** Remember that $\zeta_{AA'}$ was uniquely determined by the fourth condition of (27) and the second equation of (23) and that $\zeta_{AA'}$ was uniquely determined by $\zeta_{AA'}$, the second condition of (27) and the second equation of (26). Also recall that this determined the normal derivative of $H_{BCA'C'}$ on $\Sigma$ uniquely and that this normal derivative together with the third condition of (27) and the first equation of (26) determined $H_{BCA'C'}$ uniquely. This establishes uniqueness.

$\Box$
5.4 The tensor potential

It is tedious but straightforward to translate the above result into tensors. The condition $\Phi_{A'B'B'} = 0$ translates into the vanishing of the trace-free Ricci tensor $\tilde{R}_{ab} = R_{ab} - \frac{1}{4} R g_{ab}$ and as mentioned above, $L_{ABC'A'}$ corresponds to a real tensor $L_{abc}$ such that

$$L_{abc} = L_{[ab]c}, \quad L_{[abc]} = 0, \quad L_{ab} = 0$$

We also note that a spinor field $H_{ABA'B'} = H_{(AB)(A'B')}$ corresponds to a complex, symmetric and trace-free tensor field $H_{ab}$, i.e.,

$$H_{ab} = H_{(ab)}, \quad H_a^a = 0$$

**Theorem 5.3** Suppose $M$ is an Einstein spacetime ($\tilde{R}_{ab} = 0$) and that $\Sigma \subset M$ is a $C^\infty$ spacelike hypersurface with future directed unit normal $n^a$. Let a real tensor field $L_{abc}$ having the above symmetries, and a complex function $g$ be given. Furthermore, let a complex function $\bar{f}$ and a complex tensor field $\bar{H}_{ab} = \bar{H}_{(ab)}$ such that $\bar{H}_a^a = 0$, both defined only on $\Sigma$ be given. Then there exists a neighbourhood $U$ of $\Sigma$ such that there exists a unique complex tensor field $H_{ab}$ satisfying the equations

$$H_{ab} = H_{(ab)}, \quad H^a_a = 0$$

$$L_{abc} = -\nabla_{[a} H_{b]c} - \nabla_{[a} \bar{H}_{b]c} - i \nabla^*_{[a} H_{b]c} + i \nabla^*_{[a} \bar{H}_{b]c} + \frac{1}{3} \left(g_{[c[a} \nabla^d H_{b]d} + g_{c[a} \nabla^d \bar{H}_{b]d} + ig_{c[a} \nabla^d H_{b]d} - ig_{c[a} \nabla^d \bar{H}_{b]d}\right)$$

$$\nabla^a \nabla^b H_{ab} = g$$

$$H_{ab}|_{\Sigma} = \bar{H}_{ab}$$

$$n^a \nabla^b H_{ab}|_{\Sigma} = \bar{f}$$

(33)

on all of $U$.

By writing $H_{ab} = H^1_{ab} + i H^2_{ab}$ where $H^1$ and $H^2$ are real, we can simplify the second of the above conditions somewhat

$$L_{abc} = -2 \nabla_{[a} H^1_{b]c} + 2 \nabla^*_{[a} H^2_{b]c} + \frac{2}{3} \left(g_{c[a} \nabla^d H^1_{b]d} - g^*_{c[a} \nabla^d H^2_{b]d}\right).$$

(34)

This theorem can be generalized to four dimensional **analytic** spaces of arbitrary signature, just like the theorem in Section 3.
6 Comparison with electromagnetic theory

6.1 Introduction

In this section we consider electromagnetic theory in a curved spacetime. Most of the above results are applicable here too, and we will also find that due to the simple index configuration of the electromagnetic spinor ($\varphi_{AB}$ as compared to $\Psi_{ABCD}$) and also due to Maxwell’s equations, certain simplifications will occur.

6.2 The electromagnetic field and its spinor potentials

First of all we remark that as in the rest of the paper all results in this section are local in nature unless comments are made to the contrary.

Recall that the electromagnetic tensor (Maxwell tensor) is a 2-form $F_{ab} = \left[ F_{bc} \right]$. Maxwell’s equations are

$$\nabla^a F_{ab} = J_b, \quad \nabla_{[a} F_{bc]} = 0$$

where $J_b$ is the source current. The second of these equations together with Poincare’s lemma gives us the existence of a (real) 1-form $A_a$ such that

$$F_{ab} = \nabla_{[a} A_{b]}.$$  

Now, put $\alpha = \nabla^a A_a$ (i.e., $\alpha$ is analogous to the differential gauge in the above sections). If $\alpha = 0$ the electromagnetic potential $A_a$ is said to be in Lorenz gauge.

To examine the gauge freedom in $A_a$, suppose $A_a$ and $\tilde{A}_a$ are two potentials of $F_{ab}$ in the same differential gauge and put $B_a = \tilde{A}_a - A_a$. Then

$$\nabla_{[a} B_{b]} = 0, \quad \nabla^a B_a = 0. \quad (35)$$

Thus, there exists a (real) scalar field $G$ such that $B_a = \nabla_a G$ and by the second condition then $\Box G = 0$.

Conversely, take any scalar field $G$ that satisfies $\Box G = 0$ and put $B_a = \nabla_a G$. Then $B_a$ satisfies equation (35) and therefore $\tilde{A}_a = A_a + B_a$ will be a potential of $F_{ab}$ in the same differential gauge as $A_a$. Hence, we have completely characterized the gauge transformations that preserve the differential gauge.

Next we turn to the spinor formulation. As $F_{ab}$ is antisymmetric it can be written

$$F_{ab} = \varphi_{AB} \varepsilon_{A'B'} + \nabla_{A'B'} \varepsilon_{AB}$$

for some symmetric spinor $\varphi_{AB}$. Maxwell’s equations can be shown to be

$$\nabla_{A'B'} \varphi_{AB} = J_{AA'}$$

where $J_{AA'}$ is the hermitian spinor equivalent of the current $J_a$. If we apply Ilge’s Theorem 2.2 to $\varphi_{AB}$ we obtain the existence of a complex 1-form $A_{AA'}$ such that

$$\varphi_{AB} = \nabla_{(A'} A_{B)A'} \quad (36)$$
Putting $A_{AA'} = -\frac{1}{2} A_{1}^{1}_{AA'} + \frac{i}{2} A_{2}^{2}_{AA'}$ where $A_{i}^{\pm}_{AA'}$, $i = 1, 2$ are hermitian this equation becomes (in tensors)

$$F_{ab} = \nabla_{[a} A_{b]} + * \nabla_{[a} A_{b]}$$

where * denotes the Hodge dual. It is shown in [11] that solutions of this equation, with $A_{2}^{2} = 0$ exist only if $\nabla_{[a} F_{bc]} = 0$ (which is true if and only if $J_{AA'}$ is hermitian) in agreement with Poincare’s lemma.

It is interesting to note that the existence of the potential $A_{a}$ in electromagnetic theory is usually presented as a consequence of the second of Maxwell’s equations via Poincare’s lemma. However we see that the existence of the (complex) potential $A_{AA'}$ is independent of Maxwell’s equations; it is simply a consequence of Theorem 2.2. The role of Maxwell’s equations is to ensure that this potential is hermitian.

Now, we can of course use the theorems in the earlier sections to find potentials of $A_{AA'}$. From Theorem 2.2 (or 3.3) we know that we can always find an asymmetric potential $H_{A'B'}$ (however, when $A_{AA'}$ is divergence-free i.e., $\alpha = 0$ it is shown in [11] that a symmetric potential always exists, see also below) and from the complex conjugate of Theorem 2.2 (or 3.3) we can obtain an asymmetric potential $T_{AB}$. So we have two potentials for $A_{AA'}$ satisfying

$$A_{AA'} = \nabla_{A'}^{B'} T_{AB} = \nabla_{A'}^{B'} H_{A'B'}.$$

It is easily seen that if $A_{AA'}$ is hermitian then if $T_{AB}$ is a potential of $A_{AA'}$ then $H_{A'B'} = T_{A'B'}$ is also a potential of $A_{AA'}$.

It is to be noted that if $F_{ab}$ does not satisfy Maxwell’s equations then we cannot choose the electromagnetic potential $A_{AA'}$ hermitian, and there is no simple relation between the two potentials $T_{AB}$ and $H_{A'B'}$.

As before we can also obtain a wave equation for $T_{AB}$. Decomposed into its symmetric and antisymmetric parts it becomes

$$0 = \Box T_{(AB)} - 2 \Psi_{AB} C^{C} D T_{(CD)} + 8 \Lambda T_{(AB)} + 2 \varphi_{AB}$$

$$0 = \Box T_{A}^{A} + 2 \alpha$$

highlighting a formal resemblance between $T_{AB}$ and the Hertz potential in flat space.

As in Section 4 we can express the gauge freedom of $A_{AA'}$ in terms of $T_{AB}$. The result is that $A_{AA'}$ and $A_{AA'} = A_{AA'} + B_{AA'}$ are two potentials of $\varphi_{AB}$ in the differential gauge $\alpha$ if and only if

$$B_{AA'} = \nabla_{A'}^{B} T_{AB}$$

where $T_{AB}$ is a solution of

$$0 = \Box T_{(AB)} - 2 \Psi_{AB} C^{C} D T_{(CD)} + 8 \Lambda T_{(AB)}$$

$$0 = \Box T_{A}^{A}$$

(37)
But we had already expressed the gauge freedom in terms of the scalar $G$, so we might wonder what the link between $T_{AB}$ and $G$ is. To give a partial answer to this question, let $\Sigma$ be as in Section 5 and suppose

$$B_{AA'} = \nabla_{A'}^B T_{AB} = \nabla_{AA'} G,$$

where $T_{AB}$ satisfies the first of equations (38) and $G$ is an arbitrary scalar field (so that the gauge transformation $B_{AA'}$ is allowed to change the differential gauge). It follows that

$$0 = \nabla_{A'}^B (T_{AB} + \varepsilon_{AB} G).$$

By differentiating again we obtain ($S_{AB} = T_{(AB)}$, $T = T_A^A$)

$$0 = \square S_{AB} - 2\Psi_{AB} C^D S_{CD} + 8\Lambda S_{AB}$$

and by evaluating on $\Sigma$ we get

$$\nabla_n S_{AC} |_{\Sigma} = \left( -2n_A (C \nabla^{A'B} S_{A'B}) + n_A (C \nabla^{A'A} T_A^B) \right) |_{\Sigma}$$

$$\nabla_n (T + 2G) = 2n^{AA'} \nabla_{A' B} S_{AB} |_{\Sigma}$$

(40)

It easily follows that if $T |_{\Sigma} = -2G |_{\Sigma}$ and if $n^{AA'} \nabla_{A' B} S_{AB} |_{\Sigma} = 0$ then $T = -2G$ in a neighbourhood of $\Sigma$.

Finally we will look a little closer at the case when $F_{ab}$ is a 2-form that satisfy Maxwell’s equations. Poincare’s lemma (or [11]) then tells us that there exists a (hermitian) divergence-free potential $A_a = A_{AA'}$. Now, according to Illege [11] for any complex divergence-free 1-form $A_{AA'}$ there exists a symmetric spinor $T_{AB}$ such that $A_{AA'} = \nabla_{A'B} T_{AB}$. Define the 2-form $T_{ab} = T_{AB} \varepsilon_{A'B} + T_{A'B} \varepsilon_{AB}$. The tensor equations relating $A_a$ and $T_{ab}$ are then

$$\nabla^a T_{ab} = 2Re(A_a) , \quad *\nabla^a T_{ab} = 2Im(A_a)$$

As $A_a$ was chosen hermitian we obtain

$$\nabla^a T_{ab} = 2A_a , \quad *\nabla^a T_{ab} = 0$$

The second equation of these is equivalent to $\nabla_{[a} T_{bc]}$ i.e., $T_{ab}$ is a closed 2-form just like $F_{ab}$ so $T_{ab}$ also has a hermitian, divergence-free potential and so on. Hence, we get an infinite chain of potentials alternating between hermitian, divergence-free 1-forms and closed hermitian 2-forms.

7 Discussion

The most important motivation for studying the general spinor potentials of the earlier sections has been the Lanczos potential of the Weyl curvature spinor. The
discussion of this section will therefore deal mainly with those potentials and their ‘superpotentials’ $H_{AAB'B'}$ and $T_{ABCD}$.

Due no doubt in part to the rather complicated tensor version (1) of its relationship to the Weyl tensor, and also to various mistakes in some papers, the Lanczos potential has failed to attract major attention, and there is perhaps still an air of uncertainty surrounding it.

Although Bampi and Caviglia [3] identified the flaw in Lanczos’ original attempt to prove its existence, the complicated nature of their own existence proof also helped to set the Lanczos potential apart.

Although Maher and Zund [16] had discovered the very simple and natural spinor structure of $L_{ABCA'}$ as early as 1968, this result attracted little interest, perhaps because of some mistakes and misprints in this and subsequent papers of Zund’s.

Twenty years later, Illge’s work [11] highlighted and exploited the spinor representation, and also discovered for the first time the remarkably simple wave equation for the Lanczos potential of the Weyl spinor in vacuum spacetimes and Lanczos differential gauge. (Although Lanczos had calculated a wave equation for the Lanczos potential of the Weyl tensor in tensor notation, containing complicated non-linear terms obtained by everywhere replacing $C_{abcd}$ with the appropriate expression in $L_{abc}$, it contained some mistakes, which were repeated, or only partly corrected by others; no-one had suspected that these non-linear terms were actually identically zero in four dimensions.) The relative simplicity of the Lanczos spinor wave equation in the less ideal cases of non-vacuum, arbitrary differential gauge, arbitrary $W_{ABCD}$ (in particular it is linear) enabled Illge to use the wave equation in his somewhat indirect proof of existence of the Lanczos potential. More precisely he showed the equivalence of the two solution sets of the wave equation, subject to an initial value constraint and the Weyl-Lanczos equation. On the otherhand, Illge has established uniqueness results as well as existence, in his proof.

In this paper we have given an alternative very direct proof of existence in Section 3; the essential step in our proof of existence is simply appealing to the wave equation. We hope this simple proof, and the direct link with the familiar wave equation for a Hertz-like potential will highlight unambiguously the very natural and familiar structure of the Lanczos potential for the Weyl spinor, and open up the way for deeper considerations.

Further investigation is also needed to decide whether the Hertz-like potential $T_{ABCD}$ has more significance; certainly it is useful in obtaining, for the first time, an explicit expression for the gauge freedom in $L_{ABCA'}$, in Section 4.

By applying the spinor results of Sections 3, 4, 5 to electromagnetic theory in Section 6 we emphasized, as pointed out by Illge [11] that the existence of the electromagnetic potential is not dependent on the second of Maxwell’s equations, via Poincare’s lemma, which is the way in which it is usually presented. In electromagnetic theory when the electromagnetic potential $A_{AA'}$ is hermitian, the two superpotentials $T_{AB}$ and $H_{A'B'}$ are essentially equivalent, and in fact
are seen to be a spinor version of a Hertz-like potential; of course such a direct relationship is not possible for the two superpotentials of $W_{ABCD}$. Also in electromagnetic theory, as mentioned above the potential $A_{AA'}$ is hermitian; this simplification cannot apply to $L_{ABCA'}$ either; however, such a possibility exists for the potential $H_{ABA'B'}$ of $L_{ABCA'}$ (for, at least, a significant class of spacetimes), and this is one of the questions requiring further investigations.

The existence of a potential such as $L_{ABCA'}$ for $\Psi_{ABCD}$ is of course well known and thoroughly investigated in flat space in connection with the massless field equation; and indeed a chain of Hertz-like potentials, including some analogous to $T_{ABCD}$ and $H_{ABA'B'}$, have been studied. Although Penrose \[19\] has studied these using spinor techniques, his results are strictly for (conformally) flat spaces. In $H$-spaces (complex general relativity) the complex connection plays the role of a complex Lanczos potential $L_{ABCA'}$ of one of the Weyl spinors (recall that the other Weyl spinor is zero since $H$-spaces are always left-flat), and this potential itself always permits a potential $H_{ABA'B'}$. This $H$-potential is the basis for constructing physics in $H$-spaces. This is part of our motivation for investigating, in Section 5, the existence of an $H$-potential in real curved space. It is hoped, having now shown that such a potential does exist in physically important curved spaces, that (at least part of) the successful programme associated with the complex $H$-potential can be applied to this $H$-potential in real spacetimes.

A related motivation is that in earlier investigations of Lanczos potentials, the existence of such an $H_{ABA'B'}$ was not only an important aid to calculate the Lanczos potential $L_{ABCA'}$, but the possibility of it having physical and geometrical significance has also been considered. We summarize those cases below:

- Torres del Castillo \[20\] has studied spacetimes admitting a normalized spinor dyad $(\sigma^A, \iota^A)$ in which

\[
\kappa = \sigma = 0
\]

and in which the Ricci spinor satisfies

\[
\Phi_{ABA'B'}\sigma^A\sigma^B = 0
\]

He found that in all such spaces there exists a Lanczos potential $L_{ABCA'}$ of the Weyl spinor such that $L_{ABCA'}$ can be written

\[
L_{ABCA'} = \nabla_{(A'}H_{BC)A'B'}
\]

for some completely symmetric spinor $H_{ABA'B'}$. By defining

\[
\eta_{ab} = g_{ab} - H_{ab}
\]

where $H_{ab}$ is the symmetric, trace-free tensor equivalent of $H_{ABA'B'}$. Torres del Castillo obtained a complex, conformally flat metric $\eta_{ab}$. 

25
• Bergqvist and Ludvigsen \[6\] define a flat connection in the Kerr spacetime, by

\[ \hat{\nabla}_{AA'}\xi^B = \nabla_{AA'}\xi^B + 2\Gamma_{C}^{B}AA'\xi^C \]

where

\[ \Gamma_{ABC}A' = \nabla_{(A'B')}H_{C}A'B' \] (41)

and \( H_{ABA'B'} \) is hermitian and given by

\[ H_{ABA'B'} = \frac{\bar{\rho} + \rho}{4\rho^2} \Psi_{20} o_{A}o_{B}o_{A'}o_{B'} \]. (42)

where \( o^{A} \) is a principal spinor of the Weyl spinor. Subsequently Bergqvist \[5\] has shown that \( \Gamma_{(ABC)}A' \) is a Lanczos potential in the Kerr spacetime. This connection has been used by Bergqvist and Ludvigsen \[6\] to construct quasi-local momentum in the Kerr spacetime.

• In \[2\] these results are generalized to Kerr-Schild spacetimes i.e.,

\[ g_{ab} = \eta_{ab} + fl_{a}l_{b} \]

where \( \eta_{ab} \) is a flat metric, \( l^{a} \) is null and \( f \) is a real function. It is shown that providing \( l^{a} \) is geodesic and shear-free (or if another more technical condition is fulfilled) then \( H_{ab} = fl_{a}l_{b} \) is a hermitian \( H \)-potential of a Lanczos potential of the Weyl spinor, that also defines a curvature-free connection (See also \[10\]).

• In a recent paper \[14\] López-Bonilla et. al. have found, for the Kerr spacetime, an explicit Lanczos potential of the Weyl spinor, given by a hermitian \( H \)-potential of the type discussed in this paper, for the Kerr spacetime.

• Novello and Velloso \[17\] have shown that for perfect fluid spacetimes that admit a normalized timelike vector field \( u^{a}, u_{a}u^{a} = 1 \) which is shear-free and vorticity-free, so that

\[ \nabla_{a}u_{b} = u_{a}u_{b} + \frac{1}{3} \theta h_{ab} \]

where \( \dot{u}_{a} = u^{b}\nabla_{b}u_{a}, h_{ab} = g_{ab} - u_{a}u_{b} \) and \( \theta \) is the expansion of \( u^{a} \), then the tensor

\[ L_{abc} = 2u_{[a}u_{b]}u_{c} + \frac{2}{3} g_{c[a}u_{b]} \] (43)

is a Lanczos potential of the Weyl spinor (the second term is to ensure that \( L_{abc} = 0 \)). It is easy to confirm that when

\[ H_{ab} = u_{a}u_{b} - \frac{1}{4} g_{ab} = \frac{3}{4} u_{a}u_{b} - \frac{1}{4} h_{ab} \]. (44)

is substituted for \( H^1 \) (with \( H^2 = 0 \)) into equation (43) we obtain precisely the Lanczos potential (43).
We conclude with two comments. In some of the examples quoted above an $H$-potential of a Lanczos potential of the Weyl spinor was found for some non-Einstein spacetimes; it remains an open question if such a construction is possible for a significant class of non-Einstein spacetimes. Earlier in this section we commented on the possible significance of having a hermitian $H$-potential. We note, from our examples above, that in the cases where this potential was used for constructing curvature-free connections and quasi-local momentum, it was hermitian; therefore, it would appear that if these constructions are to be possible in other spaces, we need to know if hermitian superpotentials for the Weyl spinor, can be found for other spacetimes.

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