First-order Nilpotent Minimum Logics: first steps

Matteo Bianchi
Department of Mathematics “Federigo Enriques”
Università degli Studi di Milano
matteo.bianchi@unimi.it

Abstract

Following the lines of the analysis done in [BPZ07, BCF07] for first-order Gödel logics, we present an analogous investigation for Nilpotent Minimum logic NM. We study decidability and reciprocal inclusion of various sets of first-order tautologies of some subalgebras of the standard Nilpotent Minimum algebra. We establish a connection between the validity in an NM-chain of certain first-order formulas and its order type. Furthermore, we analyze axiomatizability, undecidability and the monadic fragments.

1 Introduction

Nilpotent Minimum logic (NM) is a many-valued logic firstly introduced in [EG01]: its name is due to the fact that NM is complete w.r.t. the algebraic structure $\langle [0,1], 0, 1, \ast, \Rightarrow, \min, \max, 0, 1 \rangle$, with $\ast, \Rightarrow$ being Nilpotent Minimum t-norm and its residuum. Triangular norms (t-norms) are particular types of functions, originally introduced in the context of probabilistic metric spaces (see [SS05]), that can be used to give the semantics for the conjunction connective of a many-valued logic (see [KMP00] for details about t-norms and residua): for every continuous t-norm the associated residuum is an operation (that can be argued to be) suitable to give the semantics for an implication connective.

In 1998, P. Hájek wrote the monograph [Háj02a], mainly devoted to a family of many-valued logics that is strictly connected to continuous t-norms and their residua: in this framework are also included the famous Łukasiewicz and Gödel logics (in the following we will discuss more in detail about this last one).

Nilpotent Minimum t-norm, introduced in [Fod95], was an example (probably the first one) of non-continuous but left-continuous t-norm. Left-continuous t-norms are particularly important, since a t-norm admits a residuum if and only if it is left-continuous ([BEG99]): this fact stimulated analysis like [EG01], where the Monoidal t-norm based logic (MTL) was introduced, by showing that it is the logic of all left-continuous t-norms and their residua. The logic MTL is at the basis of an ample hierarchy of logics, that includes the ones introduced in [Háj02a], as well as Nilpotent Minimum logic, that was presented in [EG01]. So the birth of NM was essentially due to technical reasons: in particular, it was presented as an exemplification of a logic (of
this framework) associated to a left-continuous t-norm. However, it is worth to point out that this logic and the corresponding algebraic semantics offer many interesting features, and during the years, NM and its corresponding variety have been studied under numerous aspects. For example:

- Combinatorial aspects and description of the free algebras [AGM05, ABM07, Bus06].
- States and connection with probability theory ([AG10]).
- Computational complexity for satisfiability and tautologicity problems ([EZLM09]).
- Connections with others non-classical logics: for example, Nelson’s constructive logic ([BC10]).
- Extensions with truth constants, in the propositional and in the first-order case ([EGN06, FGN10, EGN10, EGN09]).
- Alternative semantics. This is a (joint) work in progress, but it is possible to give a temporal like semantics (on the line of [AGM08, ABM09]) to NM logic. This could be useful to show some other aspects and characteristics of this logic.

As previously argued, NM has some relation with Nelson’s logic: however, it is also connected with a famous many-valued (and superintuitionistic) logic, namely Gödel logic.

Gödel logic (G) was introduced in [Dum59] by taking inspiration from a paper by Kurt Gödel ([Göd32]). This logic was firstly defined as a superintuitionistic logic, but it can also be axiomatized as an axiomatic extension of MTL (see [Haj02b, Haj02a]). The algebraic semantics of G is given by a particular class of MTL-algebras (that also forms a subvariety of Heyting algebras), called Gödel-algebras. As pointed out in [Pre03], at the propositional level there is only one infinite valued Gödel logic, in the sense that G is complete w.r.t. every infinite totally ordered Gödel-algebra. In the first-order case G∀, however, the situation is different and there are many infinitely-valued first-order logics: this means that there are many totally ordered Gödel-algebras whose set of first-order tautologies is different w.r.t. the one of G∀. A deep analysis about first-order Gödel logics, and a general classification has been done in [Pre03, BPZ07, BCF07] (see also the general survey [Pre10]): in particular, in [BPZ07, BCF07] the sets of first-order tautologies associated to the subalgebras of [0, 1]G (the standard Gödel algebra) have been studied and a general classification about decidability has been provided, also for the monadic fragments.

Which is the previously cited connection, about NM and G? As shown in [Bus06], every NM-chain is isomorphic to the connected or the disconnected rotation (see [Bus06, Jen03]) of a Gödel-chain: this shows how strongly related are these two varieties of algebras.

We now move back to NM, and to the aim and content of this paper. Whereas the propositional level of NM has been extensively investigated, in the first-order level the situation is more delicate. For example, a systematic analysis like the one done in [BPZ07, BCF07] for Gödel logics is missing, for NM. The aim of this paper is to lay the foundations of a study of this type.
If we consider the logic associated to a totally ordered algebra, then over NM there are only two different infinite-valued logics, at the propositional level: NM and NM (see Remark 1). At the first-order level, instead, the situation is different: there are infinite totally ordered NM-algebras with negation fixpoint whose set of first-order tautologies is different w.r.t. the one of [0, 1]_{NM}. In this paper we will study the sets of first-order tautologies of some subalgebras of [0, 1]_{NM}: in particular finite NM-chains and other four infinite NM-chains (with and without negation fixpoint). Moreover we will find a connection between the validity, in an NM-chain, of certain first-order formulas and its order type. Finally, we will analyze axiomatization, undecidability and the monadic fragments.

Our investigation has been inspired by the work done for first-order Gödel logic in [BPZ07] and [BCF07], where a complete classification of the sets of first-order tautologies associated to Gödel-chains (subalgebras of [0, 1]_G) has been given. Unfortunately, we do not provide here a complete classification for the case of Nilpotent Minimum chains (for this reason the title indicates “first steps”). The infinite NM-chains discussed in this paper have been chosen essentially for their relations with some particularly important Gödel chains (G↑ and G↓, see the following sections for the definitions): as we will see, this connection will help in the study of the undecidability results.

2 Preliminaries

2.1 Syntax

Nilpotent Minimum logic was introduced in [EG01], as an extension of the Monoidal t-norm based logic (MTL): this last one is the logic at the base (in the sense that the other logics of this family are obtained by adding axiom to it) of a framework of many-valued logics initially introduced by Hájek in [Haj02a]. MTL was introduced in [EG01]: as shown in [Nog06] this logic is algebraizable in the sense of [BP89] and its equivalent algebraic semantics forms a variety (the variety of MTL-algebras). From the results of [BP89, Nog06] it follows that also every extension of MTL (a logic obtained from it by adding other axioms) is algebraizable in this way.

The language of MTL is based over connectives {&, ∧, →, ⊥} (the first three are binary, whilst the last one is 0-ary). The notion of formula is defined in the usual way.

Useful derived connectives are the following

- (negation) \( \neg \varphi := \varphi \rightarrow \bot \)
- (disjunction) \( \varphi \lor \psi := ((\varphi \rightarrow \psi) \rightarrow \psi) \land ((\psi \rightarrow \varphi) \rightarrow \varphi) \)
- (biconditional) \( \varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi) \)
- (top) \( \top := \neg \bot \)
For reader’s convenience we list the axioms of MTL

(A1) \((\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))\)

(A2) \((\varphi \& \psi) \rightarrow \varphi\)

(A3) \((\varphi \& \psi) \rightarrow (\psi \& \varphi)\)

(A4) \((\varphi \land \psi) \rightarrow \varphi\)

(A5) \((\varphi \land \psi) \rightarrow (\psi \land \varphi)\)

(A6) \((\varphi \& (\varphi \rightarrow \psi)) \rightarrow (\psi \land \varphi)\)

(A7a) \((\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \& \psi) \rightarrow \chi)\)

(A7b) \(((\varphi \& \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))\)

(A8) \(((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)\)

(A9) \(\bot \rightarrow \varphi\)

As inference rule we have modus ponens:

\[
\frac{\varphi}{\psi}
\]

\((\text{MP})\)

Gödel logic \((G)\) is obtained from MTL by adding

\begin{align*}
\text{Gödel logic (G)} & \quad \varphi \rightarrow \varphi \& \varphi.
\end{align*}

Nilpotent Minimum Logic \((\text{NM})\), introduced in [EG01], is obtained from MTL by adding the following axioms:

\begin{align*}
\text{Nilpotent Minimum Logic (NM)} & \quad \lnot \lnot \varphi \rightarrow \varphi.
\end{align*}

\[
\text{involution}
\]

\[
\text{WNM}
\]

\[
\lnot (\varphi \& \psi) \lor ((\varphi \lor \psi) \rightarrow (\varphi \& \psi)).
\]

The notions of theory, syntactic consequence, proof are defined as usual.

\textbf{2.2 Semantics}

An MTL algebra is an algebra \(\langle A, *, \Rightarrow, \cap, \cup, 0, 1 \rangle\) such that

1. \(\langle A, \cap, \cup, 0, 1 \rangle\) is a bounded lattice with minimum 0 and maximum 1.
2. \(\langle A, * , 1 \rangle\) is a commutative monoid.
3. \(\langle *, \Rightarrow \rangle\) forms a residuated pair: \(z * x \leq y \) iff \(z \leq x \Rightarrow y\) for all \(x, y, z \in A\).
4. The following axiom hold, for all \(x, y \in A\):

\[
\text{Prelinearity} \quad (x \Rightarrow y) \cup (y \Rightarrow x) = 1
\]

A totally ordered MTL-algebra is called MTL-chain.
A G-algebra is an MTL-algebra satisfying 
\[ x = x \ast x. \]

It is well known (see for example [DM71, Pre03]) that in every G-chain the following hold:

\[ x \ast y = \min(x, y) \]

\[ x \Rightarrow y = \begin{cases} 1 \text{ if } x \leq y \\ y \text{ Otherwise.} \end{cases} \]

Some examples of G-chains are the following:

- \( G^\uparrow = \langle \{1 - \frac{1}{n}: n \in \mathbb{N}^+\} \cup \{1\}, \ast, \Rightarrow, \min, \max, 0, 1 \rangle \)
- \( G^\downarrow = \langle \{\frac{1}{n}: n \in \mathbb{N}^+\} \cup \{0\}, \ast, \Rightarrow, \min, \max, 0, 1 \rangle \)
- \( G_n = \langle \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}, \ast, \Rightarrow, \min, \max, 0, 1 \rangle \)
- \( [0, 1]_G = \langle [0, 1], \ast, \Rightarrow, \min, \max, 0, 1 \rangle \)

In particular it is easy to check (see [Pre03]) that every finite G-chain of cardinality \( n \) is isomorphic to \( G_n \).

An NM-algebra is an MTL-algebra that satisfies the following equations:

\[ \sim \sim x = x \]

\[ \sim (x \ast y) \sqcup ((x \sqcap y) \Rightarrow (x \ast y)) = 1 \]

Where \( \sim \) indicates \( x \Rightarrow 0 \).

Moreover, as noted in [Gis03], in each NM-chain the following hold:

\[ x \ast y = \begin{cases} 0 \text{ if } x \leq n(y) \\ \min(x, y) \text{ Otherwise.} \end{cases} \]

\[ x \Rightarrow y = \begin{cases} 1 \text{ if } x \leq y \\ \max(n(x), y) \text{ Otherwise.} \end{cases} \]

Where \( n \) is a strong negation function, i.e. \( n: A \rightarrow A \) is an order-reversing mapping \((x < y \text{ implies } n(x) > n(y))\) such that \( n(0) = 1 \) and \( n(n(x)) = x \), for each \( x \in A \). Observe that \( n(x) = x \Rightarrow 0 \Rightarrow \sim x \), for each \( x \in A \).

A negation fixpoint is an element \( x \in A \) such that \( n(x) = x \): note that if this element exists then it must be unique (otherwise \( n \) fails to be order-reversing). A positive element is an \( x \in A \) such that \( x > n(x) \); the definition of negative element is the dual (substitute \( > \) with \( < \)).

Concerning the finite chains, in [Gis03] it is shown that two finite NM-chains with the same cardinality are isomorphic (see the remarks after [Gis03 Proposition 2]): for this reason we will denote them with \( NM_n \), \( n \) being an integer greater that 1.

We now give some examples of infinite NM-chains that will be useful in the following: for all of them the order is given by \( \leq_{\mathbb{R}} \) and \( n(x) = 1 - x \).
In this last case, \( * \) is called Nilpotent Minimum t-norm [Fod95]. Note that the first four chains of the list and every finite NM-chain are all subalgebras of \([0,1]_{NM}\).

The notion of assignment, model, satisfiability and tautology are defined as usual: we refer to [EG01] for details.

**Theorem 1** ([Pre03, Gis03]).  
- Every infinite Gödel-chain is complete w.r.t. Gödel logic.
- Every infinite NM-chain with negation fixpoint is complete w.r.t. Nilpotent Minimum logic.

Concerning the variety of NM-algebras, we have the following result:

**Theorem 2** ([Gis03 Theorem 2]).

1. An NM-chain satisfies

\[
(S_n(x_0, \ldots, x_n)) \quad \bigwedge_{i<n} ((x_i \rightarrow x_{i+1}) \rightarrow x_{i+1}) \rightarrow \bigvee_{i<n+1} x_i
\]

if and only if it has less than \(2n + 2\) elements.

2. A nontrivial NM-chain satisfies

\[
(BP(x)) \quad \neg((\neg x)^2)^2 \leftrightarrow (\neg(\neg x)^2)^2
\]

if and only if it does not contain the negation fixpoint.

**Remark 1.** As pointed out in Theorem 1, at the propositional level there is only one infinite-valued Gödel logic, that is every infinite G-chain generates the whole variety of G-algebras.

In the case of NM this is not true: indeed in [Gis03] (see also [C106]) it is shown that the variety generated by an infinite NM-chain without negation fixpoint corresponds to the one associated to the logic \( NM^- \), i.e. \( NM + BP(x) \). This result together with Theorem 1 imply that there are (if we restrict to the logics associated to a totally ordered algebra) two different infinite-valued Nilpotent Minimum logic (\( NM \) and \( NM^- \)), at the propositional level.

---

1Since two finite NM-chains with the same cardinality are isomorphic, then we can consider \( NM_n \) as defined over the set \( \{0, \frac{1}{n}, \ldots, \frac{1}{n-1}\} \).
We now introduce a construction that allows to obtain an NM-chain starting from a Gödel chain. This construction is an application of the “connected rotation” introduced in [Jen03].

**Definition 1.** Let $A$ be a Gödel chain. We can construct an NM-chain $A_{NM}$ in the following way:

- $A_{NM} = B \cup \{f\} \cup B'$, where $\langle B, \leq_{A_{NM}} \rangle = \langle A \setminus \{0\}, \leq_A \rangle$ and $\langle B' = \{b' : b \in B\}, \leq_{A_{NM}} \rangle \simeq \langle B, \geq_{A_{NM}} \rangle$.
- For every $x \in B, y \in B'$ set $x >_{A_{NM}} f >_{A_{NM}} y$.
- Define a strong negation function $n : A_{NM} \to A_{NM}$ such that $n(f) = f, n(a) = a'$ and $n(b') = b$, for every $a \in B$ and $b' \in B'$.

It is easy to see that $A_{NM}$ has negation fixpoint $f$, $B$ is the set of positive elements and $B'$ the set of negative elements: note that $1 = \max(B)$ and $1' = \min(B')$ are the maximum and minimum of $A_{NM}$. The element $1'$ will be called 0.

**Remark 2.**
- An immediate consequence of Definition 1 is that $\langle A, \leq_A \rangle$ is order isomorphic to $\langle B \cup \{f\}, \leq_{A_{NM}} \rangle$. From this fact it is easy to check that $\mathcal{A}$ is complete if and only if $A_{NM}$ is.
- It is an exercise to check that if $\mathcal{A} = G_\uparrow$, then $\mathcal{A}_{NM} = NM_{\omega}$, and if $\mathcal{A} = G_\downarrow$, then $\mathcal{A}_{NM} = NM_{\omega'}$. Moreover $NM_{\omega}$ and $NM_{\omega'}$ are the subalgebras without negation fixpoint of, respectively, $NM_{\omega'}$ and $NM_{\omega'}$.

### 2.3 First-order Nilpotent Minimum and Gödel Logics

In this section we present the first-order versions of NM and G, called NM$\forall$ and G$\forall$: more details can be found in [EG01, Haj02a].

A first-order language (we restrict to countable languages) is a pair $\langle P, C \rangle$, where $P$ is a set of predicate symbols and $C$ a set of constants (in general we do not need function symbols: see [CH10] for a development in this sense): we have the “classical” quantifiers $\forall, \exists$. The notions of term, formula, closed formula, term substitutable in a formula are defined like in the classical case ([CH10], [Haj02a]), the connectives are those of the propositional level.

A theory is a set of closed formulas.

Let $L$ be NM or G or an axiomatic extension of them: then its first-order version, $L\forall$, is axiomatized as follows:

- The axioms resulting from the axioms of $L$ by the substitution of the propositional variables by the first-order formulas.
For each evaluation over variables is defined inductively as follows:

- \( \forall x \varphi(x) \rightarrow \varphi(x/t) \) (t substitutable for \( x \) in \( \varphi(x) \))
- \( \exists x \varphi(x) \rightarrow (\exists x)\varphi(x) \) (t substitutable for \( x \) in \( \varphi(x) \))
- \( \forall x (\varphi \rightarrow \varphi_{\forall}) \rightarrow (\forall x)\varphi \) (x not free in \( \varphi \))
- \( \exists x (\varphi \rightarrow \varphi_{\exists}) \rightarrow ((\exists x)\varphi \rightarrow \varphi_{\exists}) \) (x not free in \( \varphi \))
- \( (\forall x)(\varphi \lor \varphi_{\forall}) \rightarrow ((\forall x)\varphi \lor \varphi_{\forall}) \) (x not free in \( \varphi \))

The rules of \( \mathcal{L}/ \mathcal{V} \) are: Modus Ponens: \( \frac{\varphi}{\varphi \lor \psi} \) and Generalization: \( \frac{\varphi}{\forall x \varphi} \).

As regards to semantics, we need to restrict to \( \mathcal{L} \)-chains: given an \( \mathcal{L} \)-chain \( A \), an \( A \)-interpretation (or \( A \)-model) is a structure \( M = \langle M, \{ m_c \}, \{ r_P \} \rangle \), where

- \( M \) is a non empty set.
- for each \( c \in C \), \( m_c \in M \)
- for each \( P \in P \) of arity \( n \), \( r_P : M^n \rightarrow A \).

For each evaluation over variables \( v : \text{Var} \rightarrow M \), the truth value of a formula \( \varphi (\| \varphi \|_{M,v}) \) is defined inductively as follows:

- \( \| P(x,\ldots,c,\ldots) \|_{M,v} = r_P(v(x),\ldots,m_c,\ldots) \)
- The truth value commutes with the connectives of \( \mathcal{L}/ \mathcal{V} \), i.e.
  \[
  \| \varphi \lor \psi \|_{M,v} = \sup \{ \| \varphi \|_{M,v} : \phi' \equiv \chi, \varphi' = \psi(y) \text{ for all variables except for } x \} \\
  \| \varphi \lor \psi \|_{M,v} = \inf \{ \| \varphi \|_{M,v} : \phi' \equiv \chi, \varphi' = \psi(y) \text{ for all variables except for } x \}
  \]

if these inf and sup exist in \( A \), otherwise the truth value is undefined.

A model \( M \) is called \( A \)-safe if all inf e sup necessary to define the truth value of each formula exist in \( A \). In this case, the truth value of a formula \( \varphi \) over an \( A \)-safe model is

\[
\| \varphi \|_M = \inf \{ \| \varphi \|_{M,v} : v : \text{Var} \rightarrow M \}
\]

\(^2\)For the case of \( \text{NM} \), in [CH10] theorems 2.31 and 2.32 it is showed that the axioms (31) and (32) are redundant: to maintain the notation of [EG01] we give the full list.

\(^3\)If \( P \) has arity zero, then \( \alpha \in A \).
Note that if $A$ is a standard algebra or has a lattice-reduct that is a complete lattice, then each $A$-model is safe; obviously each finite $A$-model ($M$ finite) is safe.

Finally, the notions of completeness are defined analogously to propositional level, with the difference that, with the notation $\models_A \varphi$, we mean that $\|\varphi\|_M = 1$, for each safe $A$-interpretation $M$.

**Theorem 3** ([Haj02a, Theorem 5.3.3], [EG01, theorem 9]). Let $L \in \{NM, G\}$. For each theory $T$ and formula $\varphi$ it holds that

$$T \models_L \varphi \iff T \models_{[0,1]_L} \varphi.$$  

**Remark 3.** Henceforth we will assume that the first-order language (of the type specified in the previous section) is fixed.

### 3 First-order Nilpotent Minimum logics

In this section we study and compare the sets of (first-order) tautologies associated to four different infinite NM-chains ($NM_\infty$, $NM'_\infty$, $NM_-\infty$, $NM'_-\infty$), and to the finite ones.

The choice of the four infinite chains, as explained in Remark 2, is due to their relations with the Gödel chains $G^\uparrow$ and $G^\downarrow$: as we will see in Section 3.1 this connection will help in the analysis of decidability problems.

Let $\mathcal{A}$ be an NM-chain: with the notation $\text{TAUT}_\mathcal{A} \forall$ we will denote the first-order tautologies of $\mathcal{A}$.

**Theorem 4.** For every NM-chain $\mathcal{A}$ it holds that $\text{TAUT}_{[0,1]_{\text{NM}}} \forall \subseteq \text{TAUT}_\mathcal{A} \forall$.

**Proof.** Immediate from Theorem 3 and chain completeness theorem for $\text{NM} \forall$ (see [EG01, theorem 7]).

A general result, concerning the subalgebras of $[0,1]_{\text{NM}}$, is the following.

**Proposition 1.** Let $V, W$ be the universes of two subalgebras $\mathcal{V}, \mathcal{W}$ of $[0,1]_{\text{NM}}$ (i.e. $V, W$ are two subsets of $[0,1]$ closed w.r.t. $n(x) = 1 - x$). If $V \subseteq W$, then $\text{TAUT}_{\mathcal{W}} \forall \subseteq \text{TAUT}_{\mathcal{V}} \forall$.

**Proof.** Let $\phi$ be the identity mapping over $W$, restricted to $V$: from the way in which the operations of an NM-chain are defined, an easy check shows that $\phi$ is a complete embedding (i.e. it preserves all inf and sup) from $\mathcal{V}$ to $\mathcal{W}$. From this fact, if $\|\varphi\|^\mathcal{W}_V = \alpha < 1$, then we can easily construct a model $M'$ such that $\|\varphi\|^\mathcal{W}_{M'} = \alpha$.

Now we analyze the differences between the (first-order) tautologies of $[0,1]_{\text{NM}}$ and those of the other four infinite chains that we have introduced.

**Theorem 5.**

1. $\text{TAUT}_{NM_-\infty} \forall \subset \text{TAUT}_{NM_{-\infty}} \forall$, $\text{TAUT}_{NM_+\infty} \forall \subset \text{TAUT}_{NM_+\infty} \forall$.

2. $\text{TAUT}_{[0,1]_{\text{NM}}} \forall \subset \text{TAUT}_{NM_+\infty} \forall$, $\text{TAUT}_{[0,1]_{\text{NM}}} \forall \subset \text{TAUT}_{NM_+\infty} \forall$ and $\text{TAUT}_{NM_+\infty} \forall \neq \text{TAUT}_{NM_-\infty} \forall$. 

9
Proof. 1. Immediate from Proposition \[1\] and Theorem \[2\].

2. We show that the formula

\[(\forall x)(\varphi(x) \& \nu) \iff ((\forall x)\varphi(x) \& \nu)\]

where \(x\) does not occur freely in \(\nu\), is a tautology for \(NM_\omega\) and \(NM'_\omega\), but it fails in \(NM'_\omega\) (and hence, from Theorem \[4\], it fails in \([0,1]_{\text{NM}}\).

First of all we show that \(*\) fails in \(NM'_\omega\). Consider the formula \((\forall x)(P(x) \& p) \iff ((\forall x)P(x) \& p)\), where \(p\) is a predicate of arity zero. Construct a model \(M\) (that is necessarily safe, since \(NM'_\omega\) is complete) such that its domain \(M\) is \((\frac{1}{2}, 1] \cap NM'_\omega\), \(p\) is interpreted as \(\frac{1}{2}\) and \(r_p(m) = m\), for each \(m \in M\). An easy check shows that \(\| (\forall x)(P(x) \& p) \iff ((\forall x)P(x) \& p)\|_{\text{NM}'_\omega} = \frac{1}{2}\) and hence \(NM'_\omega \not\models \ast\).

Now we show that \(NM_\omega \models \ast\). We have to check that, for each \(W \subseteq NM_\omega\) (observe that \(NM_\omega\) is a complete lattice) and \(y \in NM_\omega\), it holds that \(\inf_{w \in W} (w * y) = \inf(W) * y\). Note that, if \(W\) has minimum \(m\), then \(\inf(W) * y = m * y = \inf_{w \in W} (w * y)\). Suppose then that \(W\) has infimum but not minimum: an easy check shows that \(\inf(W) = 0\). In this last case we have that \(\inf(W) * y = 0 = \inf_{w \in W} (w * 1) \geq \inf_{w \in W} (w * y)\).

Finally we analyze \(NM'_\omega\). We have to show that \(\inf_{w \in W} \{w * x\} = \inf(W) * x\), for each \(W \subseteq NM'_\omega\) and \(x \in NM'_\omega\), when these \(\inf\) exist. If \(W\) has a minimum, say \(m\), then \(\inf_{w \in W} \{w * x\} = m * x = \inf(W) * x\); if \(W\) does not have minimum, then it does not have \(\inf\) and we are not interested to this case.

\[\square\]

Lemma 1. Let \(\mathcal{A}\) be an NM-chain: an element \(a\) does not have predecessor \(\text{(successor)}\) if and only if \(n(a)\) does not have successor (predecessor).

Proof. Immediate from the properties of the negation. \[\square\]

Consider now the following formulas:

\[C_t \quad (\exists x)(\varphi(x) \rightarrow \forall y \varphi(y))\]

\[C_i \quad (\exists x)(\exists y \varphi(y) \rightarrow \varphi(x)).\]

Their names are due to the fact that, in the context of Gödel logics they are equivalent to ask, respectively, over a Gödel-chain that “every infimum is a minimum”, and “every supremum is a maximum”; see \[BPZ07\] for details.

Theorem 6. The formulas \(C_t\) and \(C_i\) hold in every NM-chain \(\mathcal{A}\) in which every element of \(\mathcal{A} \setminus \{0, 1\}\) has a predecessor in \(\mathcal{A}\). They both fail in any other NM-chain.

\[\text{An element } x \in A \text{ has a predecessor if there is an element } p \in A \text{ such that } p < x \text{ and there are no other elements between } p \text{ and } x \text{ (i.e. there is no } c \in A \text{ such that } p < c < x). \text{ The notion of successor is defined dually.}\]
Proof. Let $\mathcal{B}$ be an NM-chain that has an element $x \in \mathcal{B} \setminus \{0, 1\}$ without predecessor in $\mathcal{B}$.

Consider the set $W = \{w \in \mathcal{B} : w < x\}$: direct inspection shows that $\sup_{w \in W}\{\sup(W) \Rightarrow w\} = \sup_{w \in W}\{x \Rightarrow w\} = \sup_{w \in W}\{\max(n(x), w)\} < 1$. This shows that $\mathcal{B} \not\models C$.

From Lemma 1 we know that $n(x)$ does not have successor. Construct the set $W = \{w \in \mathcal{B} : w > n(x)\}$: direct inspection shows that $\sup_{w \in W}\{w \Rightarrow \inf(W)\} = \sup_{w \in W}\{w \Rightarrow n(x)\} = \sup_{w \in W}\{\max(n(w), n(x))\} < 1$. This shows that $\mathcal{B} \not\models C$.

Let $\mathcal{A}$ be any NM-chain in which every element of $\mathcal{A} \setminus \{0, 1\}$ has a predecessor in $\mathcal{A}$: it follows that every element of $\mathcal{A} \setminus \{0, 1\}$ has predecessor and successor in $\mathcal{A}$. We have to check that $\sup_{w \in W}\{w \Rightarrow \inf(W)\} = 1$ and $\sup_{w \in W}\{\sup(W) \Rightarrow w\} = 1$, for every $W$ in which these inf and sup exist. If $W$ has minimum $m$, then $\sup_{w \in W}\{w \Rightarrow \inf(W)\} = m \Rightarrow m = 1$; if $W$ has maximum $n$, then $\sup_{w \in W}\{\sup(w) \Rightarrow w\} = n \Rightarrow n = 1$. If $W$ has infimum, but not minimum, then $\inf(W) = 0$ and $\sup_{w \in W}\{w \Rightarrow \inf(W)\} = \sup_{w \in W}\{n(w)\} = 1$. Finally, if $W$ has supremum, but not maximum, then $\sup(W) = 1$ and $\sup_{w \in W}\{\sup(W) \Rightarrow w\} = \sup_{w \in W}\{1 \Rightarrow w\} = 1$. □

Corollary 1.

- $C_1$ and $C_3$ belong to $\text{TAUT}_{\text{NM}_\omega}$, $\text{TAUT}_{\text{NM}_\omega}$, $\text{TAUT}_{\text{NM}_\omega}$ and $\text{TAUT}_{\text{NM}_\omega}$, for every $1 < n < \omega$.
- $C_2$ and $C_4$ fail in $\{0, 1\}_\text{NM}$ and $\text{NM}_\omega$.

Remark 4. Continuing with the analogies with Gödel logic, it can be showed (see [BPZ07] and [BLZ96]) that $C_1$ and $C_3$ are tautologies in $G_\omega$ and in every finite Gödel chain, whilst $G_1 \not\models C_1$ and $G_1 \models C_3$. Both the formulas fail in $G\phi$ (see [BLZ96]).

We prosecute our analysis of first-order tautologies with the following

Theorem 7. Let $\varphi$ be an NM$\phi$ formula. For every integer $n > 1$ and every even integer $m > 1$ it holds that

- If $NM_n \not\models \varphi$, then $\text{NM}_m \not\models \varphi$ and $\text{NM}_m \not\models \varphi$.
- If $NM_n \not\models \varphi$, then $\text{NM}_m \not\models \varphi$ and $\text{NM}_m \not\models \varphi$.

Proof. It is enough to show that $NM_n \not\models \text{NM}_m$, $NM_n \not\models \text{NM}_m$, $NM_m \not\models \text{NM}_m$, $NM_m \not\models \text{NM}_m$ preserving all inf and sup.

We begin with the case of $NM_n$.

Let $0 = c_1 < c_2 < \cdots < c_n = 1$ be the elements of $\text{NM}_n$: consider a map $\phi$ such that

- $\phi(c_1) = 0$ and $\phi(c_n) = 1$.
- If $NM_n$ has a fixpoint $f$, then $\phi(f) = \frac{1}{2}$.
- Let $c_k$ be the least positive element: we set $\phi(c_j) = 1 - \frac{1}{2+c_j}$ for every $c_n > c_j \geq c_k$.
- Let $c_h$ be the greatest negative element: we set $\phi(c_i) = \frac{1}{2+c_i}$ for every $c_i < c_1 \leq c_h$. 

11
A direct inspection shows that $\phi$ is an embedding from the two chains. Moreover, since $NM_n$ is finite, then for each $W \subseteq NM_n$, $\phi(\inf(W)) = \phi(\min(W)) = \min(\phi(W))$; analogously for sup.

Concerning the case of $NM'_n$ we have only to modify the map $\phi$ and the proof proceeds analogously to the previous case.

Let $0 = c_1 < c_2 < \cdots < c_n = 1$ be the elements of $NM_n$: consider a map $\phi'$ such that

- $\phi'(c_1) = 0$ and $\phi'(c_n) = 1$.
- If $NM_n$ has a fixpoint $f$, then $\phi'(f) = \frac{1}{2}$.
- Let $c_k$ be the greatest positive element of $NM'_\infty \setminus \{1\}$: we set $\phi'(c_k) = \frac{1}{2} + \frac{1}{2(2k-j)}$ for every $c_n > c_k \geq c_j$.
- Let $c_h$ be the least negative element of $NM'_\infty \setminus \{0\}$: we set $\phi'(c_i) = \frac{1}{2} - \frac{1}{2(2i-h)}$ for every $c_1 < c_h \leq c_i$.

Finally the proofs for $NM'\infty$ and $NM'_{\infty}$ are identical to the previous ones, except for the absence of the negation fixpoint.

**Corollary 2.** For every integer $n > 1$ we have $TAUT_{[0,1]NM_n} \subset TAUT_{NM_n}$, $TAUT_{NM_n} \subset TAUT_{NM'_n}$, $TAUT_{NM'_n} \subset TAUT_{NM'_\infty}$. Moreover, if $n$ is even, then $TAUT_{NM'\infty} \subset TAUT_{NM'_n}$, $TAUT_{NM'_n} \subset TAUT_{NM'_\infty}$.

**Proof.** From Theorems 4 and 7 we have the non-strict inclusions. To prove the strictness, direct inspection shows that the formula $\bigvee_{0 < i < n} (p_i \rightarrow p_{i+1})$ (where each $p_i$ is a predicate of arity zero) is a first-order tautology of $NM_n$, but it fails in every infinite $NM$-chain.

Differently from the results of [BPZ07] for $G_n$, it cannot be showed that $TAUT_{NM_n} \subset TAUT_{NM'_n}$. Indeed, if $NM_n$ has negation fixpoint, then (see Theorem 2) $NM_n \models \neg(\neg p^2) \leftrightarrow (\neg(\neg p^2))^2$, where $p$ is a predicate of arity zero. However $NM_{n+1} \models \neg(\neg p^2) \leftrightarrow (\neg(\neg p^2))^2$.

Note that if both the chains are with (without) negation fixpoint, then the previous problem disappear; note also that $NM_n$ has negation fixpoint if and only if $n$ is odd.

Hence, we have the following

**Theorem 8.** For each pair of integers $m, n$ such that $1 < m < n$, if $m, n$ are both even (odd), then $TAUT_{NM_m} \subset TAUT_{NM_n}$.

**Proof.** It is enough to check that $NM_m \rightarrow NM_n$, preserving all inf and sup. Take an injective map $\phi$ from the lattice reduct of $NM_m$ to the one of $NM_n$ such that:

- $\phi(0) = 0$, $\phi(1) = 1$.
- if $NM_m, NM_n$ have negation fixpoints $f, f'$, then $\phi(f) = f'$.
• φ maps all the positive elements of \( NM_m \) into the ones of \( NM_n \), preserving the order. That is, for every \( x, y \in NM_m^+ \) with \( x < y \) it holds that \( \phi(x), \phi(y) \in NM_n^+ \) and \( \phi(x) < \phi(y) \).

• for every negative element \( x \in NM_m^- \), \( \phi(x) = 1 - \phi(1 - x) \).

An easy check shows that \( \phi \) is an embedding that preserves all \( \text{inf} \) and \( \text{sup} \). This shows that \( \text{TAUT}_{NM} \models \subseteq \text{TAUT}_{NM} \models \).

To conclude, note that \( NM_m \models \forall_{i < m} (p_i \rightarrow p_{i+1}) \), but \( NM_n \not\models \forall_{i < m} (p_i \rightarrow p_{i+1}) \); hence \( \text{TAUT}_{NM} \models \subset \text{TAUT}_{NM} \models \). □

Moreover, by inspecting the previous proof, we obtain

**Corollary 3.** For every even integer \( n > 1 \), it holds that \( \text{TAUT}_{NM} \models \forall \subset \text{TAUT}_{NM} \models \).

In [BPZ07] it is shown that the first-order tautologies of \( G \) are the first-order formulas valid in all finite Gödel-chains. We will show that, under this point of view, \( NM \) plays the same role of \( G \); that is \( \text{TAUT}_{NM} \models = \bigcap_{n \geq 2} \text{TAUT}_{NM} \models \).

We start with the following lemma, that says that if an \( NM \)-model assigns to the atomic formulas truth values between a value \( \alpha \) and its negation, then the same holds for every other formula.

**Lemma 2.** Let \( M = \langle M, \{ r_p \}_{p \in P}, \{ m_c \}_{c \in C} \rangle \) be an \( NM \)-model. For \( \alpha \in \text{NM} \), consider the \( NM \)-model \( M_\alpha = \langle M, \{ r_p \}_{p \in P}, \{ m_c \}_{c \in C} \rangle \) such that, for each atomic formula \( \psi \) and every evaluation \( v \)

\[
\| \psi \|_{M_\alpha, v} = \begin{cases} 
1 & \text{if } \| \psi \|_{M, v} > |\alpha| \\
0 & \text{if } \| \psi \|_{M, v} < n(\alpha) \\
\| \psi \|_{M, v} & \text{otherwise}
\end{cases}
\]

Where \( |\alpha| = \max(\alpha, n(\alpha)) \).

Then (m) holds for every first-order formula \( \varphi \).

**Proof.** By structural induction. Since \( M_\alpha \) and \( M_{n(\alpha)} \) define the same model we will assume, without loss of generality, that \( \alpha \geq \frac{1}{2} \) (otherwise we set \( \alpha = n(\alpha) \)).

• If \( \varphi \) is atomic or \( \bot \), then there is nothing to prove.

• \( \varphi := \psi \land \chi \) and the claim holds for \( \psi \) and \( \chi \). First of all note that \( \| \psi \land \chi \|_{M, v} = \min(\| \psi \|_{M, v}, \| \chi \|_{M, v}) \) and \( \| \psi \land \chi \|_{M_\alpha, v} = \min(|\psi|_{M_\alpha, v}, |\chi|_{M_\alpha, v}) \); from the induction hypothesis, if \( \| \psi \|_{M, v} = \| \chi \|_{M, v} \), then the lemma holds.

For the other cases, note that if \( \| \psi \|_{M, v} < \| \chi \|_{M, v} \) (\( > \)), then \( \| \psi \|_{M_\alpha, v} \leq \| \chi \|_{M_\alpha, v} \) (\( \geq \)). Suppose that \( \| \psi \|_{M_\alpha, v} < \| \chi \|_{M_\alpha, v} \). If \( \| \psi \|_{M_\alpha, v} < \| \chi \|_{M_\alpha, v} \), then, applying the induction hypothesis, we have the result. The other case is \( \| \psi \|_{M_\alpha, v} = \| \chi \|_{M_\alpha, v} \in \{0, 1\} \): clearly either \( \| \chi \|_{M, v} < n(\alpha) \) or \( \| \psi \|_{M, v} > \alpha \). Again, applying the induction hypothesis, the claim follows.

• \( \varphi := \psi \& \chi \) and the claim holds for \( \psi \) and \( \chi \). We have two cases:
– \( \|\varphi\|_{M,v} = 0 \): this happens if and only if \( \|\psi\|_{M,v} \leq n(\|\chi\|_{M,v}) \). Direct inspection shows that this implies \( \|\psi\|_{M,v} \leq n(\|\chi\|_{M,v}) \) and hence \( \|\varphi\|_{M,v} = 0 \).

– \( \|\varphi\|_{M,v} = \min(\|\psi\|_{M,v}, \|\chi\|_{M,v}) > 0 \): this happens if and only if \( \|\psi\|_{M,v} > n(\|\chi\|_{M,v}) \).

If \( \|\psi\|_{M,v} < n(\alpha) \) then \( \|\varphi\|_{M,v} < n(\alpha) \) and \( \|\psi\|_{M,v} = 0 = \|\varphi\|_{M,v} \).

If \( n(\alpha) \leq \|\psi\|_{M,v} \leq \alpha \), then \( \|\psi\|_{M,v} = \|\psi\|_{M,v} \) and \( n(\|\chi\|_{M,v}) < \alpha \): if \( n(\alpha) \leq n(\|\chi\|_{M,v}) \), then \( \|\varphi\|_{M,v} = \|\varphi\|_{M,v} \), otherwise \( n(\|\chi\|_{M,v}) < n(\alpha) \), \( \|\chi\|_{M,v} > \alpha \) and hence \( \|\varphi\|_{M,v} = \|\psi\|_{M,v} = \|\varphi\|_{M,v} \), since \( \|\chi\|_{M,v} = 1 \), due to the induction hypothesis.

Finally, suppose that \( \|\psi\|_{M,v} > \alpha \). We have that \( \|\psi\|_{M,v} = 1 \): if \( n(\|\chi\|_{M,v}) > \alpha \), then \( \|\chi\|_{M,v} < n(\alpha) \) and hence \( \|\varphi\|_{M,v} = \|\chi\|_{M,v} \), from which we have \( \|\psi\|_{M,v} = 0 = \|\varphi\|_{M,v} \). If \( n(\alpha) \leq n(\|\chi\|_{M,v}) \leq \alpha \), then the same holds for \( \|\chi\|_{M,v} \) and we have \( \|\varphi\|_{M,v} = \|\chi\|_{M,v} = \|\varphi\|_{M,v} \). If \( n(\|\chi\|_{M,v}) < n(\alpha) \), then \( \|\varphi\|_{M,v} > \alpha \) and hence \( \|\varphi\|_{M,v} = 1 = \|\varphi\|_{M,v} \).

\( \varphi := \psi \rightarrow \chi \) and the claim holds for \( \psi \) and \( \chi \). We have two cases.

– \( \|\psi\|_{M,v} \leq \|\chi\|_{M,v} \): as we have already noticed, this implies \( \|\psi\|_{M,v} \leq \|\varphi\|_{M,v} \) and we have that \( \|\varphi\|_{M,v} = 1 = \|\varphi\|_{M,v} \).

– \( \|\psi\|_{M,v} > \|\chi\|_{M,v} \): it is not difficult to check that \( \|\psi\|_{M,v} \geq \|\chi\|_{M,v} \).

If the equality holds, then \( \|\psi\|_{M,v} = \|\chi\|_{M,v} \in \{0, 1\} \) and either \( \|\varphi\|_{M,v} > \alpha \) or \( \|\psi\|_{M,v} < n(\alpha) \): in both the cases \( \|\varphi\|_{M,v} = \max(n(\|\psi\|_{M,v}), \|\chi\|_{M,v}) \).

If \( \|\chi\|_{M,v} > \alpha \), then \( n(\|\psi\|_{M,v}) < n(\alpha) \) and \( \|\varphi\|_{M,v} = \|\chi\|_{M,v} = \|\varphi\|_{M,v} \): from these facts we have \( \|\psi\|_{M,v} = \|\varphi\|_{M,v} = 1 = \|\varphi\|_{M,v} \). If \( \|\varphi\|_{M,v} < n(\alpha) \), then \( n(\|\psi\|_{M,v}) = \|\varphi\|_{M,v} > \alpha \) and from the induction hypothesis we have \( \|\psi\|_{M,v} = 0 = \|\chi\|_{M,v} \) and \( \|\varphi\|_{M,v} = 1 \).

The last case is \( \|\psi\|_{M,v} > \|\chi\|_{M,v} \): we have that \( \|\varphi\|_{M,v} = \max(n(\|\psi\|_{M,v}), \|\chi\|_{M,v}) \) and \( \|\varphi\|_{M,v} = \max(n(\|\psi\|_{M,v}), \|\chi\|_{M,v}) \).

There are two subcases.

\( n(\|\psi\|_{M,v}) > \|\chi\|_{M,v} \): clearly \( \|\varphi\|_{M,v} = n(\|\psi\|_{M,v}) \). If \( n(\alpha) \leq \|\varphi\|_{M,v} \leq \alpha \), we have that \( \|\psi\|_{M,v} = \|\psi\|_{M,v} = n(\|\psi\|_{M,v}) = n(\|\varphi\|_{M,v}) \) and \( \|\varphi\|_{M,v} = \|\varphi\|_{M,v} = n(\|\psi\|_{M,v}) \) (noting that \( \|\chi\|_{M,v} \leq \|\chi\|_{M,v} \), since \( \|\psi\|_{M,v} > \|\chi\|_{M,v} \)). If \( \|\psi\|_{M,v} > \alpha \), then \( \|\varphi\|_{M,v} = 1 = n(\|\psi\|_{M,v}) < n(\alpha) \) and \( n(\|\psi\|_{M,v}) = 0 \): from these facts and the hypothesis we obtain \( n(\alpha) > \|\varphi\|_{M,v} = n(\|\psi\|_{M,v}) > \|\chi\|_{M,v} \) and hence \( \|\varphi\|_{M,v} = 0 = n(\|\psi\|_{M,v}) = \|\chi\|_{M,v} \). The last case is \( \|\psi\|_{M,v} < n(\alpha) \): we have that \( \|\varphi\|_{M,v} = n(\|\psi\|_{M,v}) > \alpha \) and hence \( 1 = n(\|\psi\|_{M,v}) = \|\varphi\|_{M,v} \).

\( \|\chi\|_{M,v} > n(\|\psi\|_{M,v}) \): we proceed analogously with the previous case.

• \( \varphi := (\forall x)\psi(x) \) and the claim holds for \( \psi(x) \): this means that, from the induction hypothesis, for every \( v' \equiv x \) we have that \( m \) holds for \( \|\psi(x)\|_{M,v'} \) and \( \|\psi(x)\|_{M,v'} \).

We distinguish three cases.
We do not analyze the case $\varphi := (\exists x)\psi(x)$, since the two quantifiers are inter-definable, in $\mathbb{NM}^\forall$, as in classical logic (see \cite{CH10} theorem 2.31).}

\begin{remark}
It is not difficult to see that the previous lemma holds even for $[0,1]_{\mathbb{NM}}$ using the same proof.
\end{remark}

\begin{theorem}
$\text{TAUT}_{[0,1]_{\mathbb{NM}}} \forall = \bigcap_{n \geq 1} \text{TAUT}_{\mathbb{NM}_n} \forall$.
\end{theorem}

\begin{proof}
The fact that $\text{TAUT}_{\mathbb{NM}_n} \forall \subseteq \bigcap_{n \geq 2} \text{TAUT}_{\mathbb{NM}_n} \forall$ follows from Corollary 2.

Concerning the reverse inclusion, suppose that $\|\varphi\|_{\mathbb{NM}_n}^{\mathbb{M}_v} = \alpha < 1$. Take $\alpha < \beta < 1$: due to Lemma 2 it is easy to check that $\|\varphi\|_{\mathbb{NM}_\beta}^{\mathbb{M}_v} \leq \alpha$. Since $\mathbb{M}_\beta$ uses only a finite number of truth values, it is easy to construct a model $\mathbb{M}_{\beta'}$ (starting from $\mathbb{M}_\beta$ and modifying the range of the various $r_{\rho}$'s) over an appropriate $\mathbb{NM}_k$ such that $\|\varphi\|_{\mathbb{NM}_\beta'}^{\mathbb{M}_{\beta'}} = \|\varphi\|_{\mathbb{NM}_\beta}^{\mathbb{M}_v}$.

\end{proof}

We now introduce a family of $\mathbb{NM}$-chains that will be useful to give an equivalent characterization of $\text{TAUT}_{[0,1]_{\mathbb{NM}}} \forall$.

For $\alpha \in (0,1)$, let $\mathcal{M}_\alpha$ be the $\mathbb{NM}$-chain defined over the universe $[1 - |\alpha|, |\alpha|] \cup \{0,1\}$ and $\mathbb{M}(x) = 1 - x$ (recall that $|\alpha| = \max(|\alpha|, n(|\alpha|))$); observe that $\mathcal{M}_\alpha$ and $\mathcal{M}_{\rho(\alpha)}$ are isomorphic and every chain of this type forms a complete lattice.

Due to Remark 5 and Theorem 9, with a proof very similar to the one of Theorem 9, we obtain the following result: this is - mutatis mutandis - the analogous of Theorem 9 for $[0,1]_{\mathbb{NM}}$ and the family of $\mathbb{NM}$-chains previously introduced.

\begin{theorem}
$\text{TAUT}_{[0,1]_{\mathbb{NM}}} \forall = \bigcap_{\alpha \in (0,1)} \text{TAUT}_{\mathcal{M}_\alpha} \forall$.
\end{theorem}

In classical (first-order) logic every formula can be written in prenex normal form: this is because the so called “quantifiers shifting laws” hold. For Nilpotent Minimum logic the situation is different: indeed, as shown in Theorem 5, some quantifier shifting laws fail in $\mathbb{NM}^\forall$. One can ask which is the situation for the $\mathbb{NM}$-chains, about these formulas.
The following theorem shows a characterization of the validity of these shifting laws in terms of the order type of an NM-chain.

**Theorem 11.** Consider the following formulas:

(1) \((\forall x)(\varphi(x) \land \psi) \leftrightarrow ((\forall x)\varphi(x) \land \psi)\)

(2) \((\exists x)(\varphi(x) \land \psi) \leftrightarrow ((\exists x)\varphi(x) \land \psi)\)

(3) \((\forall x)(\varphi(x) \lor \psi) \leftrightarrow ((\forall x)\varphi(x) \lor \psi)\)

(4) \((\exists x)(\varphi(x) \lor \psi) \leftrightarrow ((\exists x)\varphi(x) \lor \psi)\)

(5) \((\forall x)(\varphi(x) \land \psi(x)) \leftrightarrow ((\forall x)\varphi(x) \land (\forall x)\psi(x))\)

(6) \((\exists x)(\varphi(x) \land \psi(x)) \leftrightarrow ((\exists x)\varphi(x) \land (\exists x)\psi(x))\)

(7) \((\forall x)(\varphi(x) \lor \psi(x)) \leftrightarrow ((\forall x)\varphi(x) \lor (\forall x)\psi(x))\)

(8) \((\exists x)(\varphi(x) \lor \psi(x)) \leftrightarrow ((\exists x)\varphi(x) \lor (\exists x)\psi(x))\)

(9) \((\exists x)(\varphi(x) \land \varphi(x)) \leftrightarrow ((\exists x)\varphi(x) \land (\exists x)\varphi(x))\)

(10) \((\exists x)(\varphi(x) \land \varphi(x)) \leftrightarrow ((\exists x)\varphi(x) \land (\exists x)\psi(x))\)

(11) \((\forall x)(\varphi(x) \rightarrow \psi) \leftrightarrow ((\exists x)\varphi(x) \rightarrow \psi)\)

(12) \((\forall x)(\psi \rightarrow \varphi(x)) \leftrightarrow ((\forall x)\varphi(x) \rightarrow \psi)\)

(13) \(\neg(\exists x)\varphi(x) \leftrightarrow (\forall x)\neg\varphi(x)\)

(14) \(\neg(\forall x)\varphi(x) \leftrightarrow (\exists x)\neg\varphi(x)\)

(15) \((\forall x)(\varphi(x) \land \psi) \leftrightarrow ((\forall x)\varphi(x) \land (\forall x)\psi)\)

(16) \((\forall x)(\varphi(x) \land \psi(x)) \leftrightarrow ((\forall x)\varphi(x) \land (\forall x)\psi(x))\)

(17) \((\exists x)(\varphi(x) \rightarrow \psi) \leftrightarrow ((\exists x)\varphi(x) \rightarrow \psi)\)

(18) \((\exists x)(\psi \rightarrow \varphi(x)) \leftrightarrow ((\exists x)\varphi(x) \rightarrow \psi)\)

where \(x\) does not occur freely in \(\psi\). We have that

- The formulas (1)-(14) hold in every NM-chain.
- The formulas (15)-(18) hold in every NM-chain \(\mathcal{A}\) in which every element of \(\mathcal{A} \setminus \{0,1\}\) has a predecessor in \(\mathcal{A}\), and fail to hold in any other NM-chain.

**Proof.** A direct inspection.

---

**Corollary 4.**

- The formulas (1)-(18) belong to TAUT\(_{NM_m}^\forall\), TAUT\(_{NM_m}^{\forall}\), TAUT\(_{NM_m}^{\forall}\) and TAUT\(_{NM_m}^{\forall}\), for every \(1 < n < \omega\).
- The formulas (1)-(14) belong to TAUT\(_{[0,1]_{NM_m}^{\forall}}\) and TAUT\(_{NM_m}^{\forall}\).
- The formulas (15)-(18) fail in \([0,1]_{NM_m}\) and \(NM_m\).

Finally, we summarize relationship (in terms of reciprocal inclusion) between the sets of tautologies of the NM-chains studied.

**Theorem 12.** For every integer \(n > 1\) and every even integer \(m > 1\)
1. $\text{TAUT}_{[0,1]} \supseteq \bigcap_{\alpha \in (0,1)} \text{TAUT}_{\alpha}$. 

2. $\text{TAUT}_{[0,1]} \supset \text{TAUT}_{NM} \supset \text{TAUT}_{NM_{\forall}}$. 

3. $\text{TAUT}_{NM_{\forall}} \supset T_{NM_{\forall}}$, $T_{NM_{\forall}} \supset T_{NM_{\forall}^{-}} \supset T_{NM_{\forall}}$. 

4. $\text{TAUT}_{[0,1]} \supset \text{TAUT}_{NM_{\forall}^{-}} \supset \text{TAUT}_{NM_{\forall}^{-}}$. 

5. $\text{TAUT}_{NM_{\forall}^{-}} \neq \text{TAUT}_{NM_{\forall}^{-}} = \bigcap_{n \geq 2} \text{TAUT}_{NM_{\forall}^{-}}$ and hence $\text{TAUT}_{NM_{\forall}^{-}} \subset \text{TAUT}_{NM_{\forall}^{-}}$. 

This theorem can be improved: indeed in the next section we will show that $\text{TAUT}_{NM_{\forall}^{-}}$ is not recursively enumerable. As a consequence, we have that $\text{TAUT}_{[0,1]} \supset \text{TAUT}_{NM_{\forall}^{-}}$. 

### 3.1 Axiomatizability and undecidability

In this section we study if the sets of first-order tautologies associated to the NM-chains till introduced are axiomatizable or not: that is, we investigate if, given one of the previous NM-chains, there is a logic that is complete w.r.t. it. As we will see, it will be the case only for finite NM-chains: for the other chains we will have undecidability results (the set of first-order tautologies will be not recursively axiomatizable) and one open problem.

From [Gis03, Theorem 3] we can state

**Theorem 13.** For every integer $n \geq 1$

- Let $L_{NM_{2n}}$ be the logic obtained from $NM$ with the axioms $S_n(x_0, \ldots, x_n)$ and $BP(x)$ (see Theorem 2). Then $L_{NM_{2n}}$ is complete w.r.t. $NM_{2n}$.

- Let $L_{NM_{2n+1}}$ be the logic obtained from $NM$ with the axiom $S_n(x_0, \ldots, x_n)$. Then $L_{NM_{2n+1}}$ is complete w.r.t. $NM_{2n+1}$.

As regards to the first-order version of these logics, we have

**Theorem 14.** For each integer $n > 1$ and each $NM_{\forall}$ formula $\varphi$,

$$L_{NM_{n}} \vdash \varphi \iff NM_{n} \models \varphi$$

**Proof.** The soundness follows from the chain-completeness for axiomatic extensions of MTL$_{\forall}$ (see [EGOl]).

For the completeness, note that each $L_{NM_{n}}$-chain has at most $n$ elements (this follows from the axiomatization of $L_{NM_{n}}$ and Theorem 2). Moreover, it easy to see that every $L_{NM_{n}}$-chain embeds into $NM_{n}$ preserving all inf and sup. To conclude, from chain completeness theorems and the previous results we have that if $L_{NM_{n}} \not\vdash \varphi$, then $NM_{n} \not\models \varphi$.

For the case of the infinite NM-chains, we need some other machinery.

We now introduce a translation $^*$ between first-order formulas, and we will show that, given a Gödel-chain $\mathcal{A}$ and a formula $\varphi$, $\mathcal{A} \models \varphi$ if and only if $\mathcal{A}_{NM} \models \varphi^*$. This fact will be fundamental to show some undecidability results, for some of the infinite NM-chains discussed in this paper. For one of them ($NM_{\infty}^*$), however, the decidability remains an open problem.
Definition 2. Let \( \varphi \) be a formula. We define \( \varphi^* \), inductively, as follows:

- If \( \varphi \) is atomic, then \( \varphi^* :\! = \varphi^2 \).
- If \( \varphi :\! = \bot \), then \( \varphi^* :\! = \bot \).
- If \( \varphi :\! = \psi \land \chi \), then \( \varphi^* :\! = \psi^* \land \chi^* \).
- If \( \varphi :\! = \psi \lor \chi \), then \( \varphi^* :\! = \psi^* \lor \chi^* \).
- If \( \varphi :\! = \psi \rightarrow \chi \), then \( \varphi^* :\! = (\psi^* \rightarrow \chi^*)^2 \).
- If \( \varphi :\! = (\forall x)\chi \), then \( \varphi^* :\! = ((\forall x)\chi^*)^2 \).

Lemma 3. Let \( \varphi, \mathcal{A}, M = (M, (m_c)_{c \in C}, (r_p)_{p \in P}) \) be a formula, an NM-chain (call \( \mathcal{A}^+ \) the set of its positive elements) and a safe \( \mathcal{A} \)-model. Construct an \( \mathcal{A} \)-model \( M^+ = (M, (m_c)_{c \in C}, (r_p')_{p \in P}) \) such that, for every evaluation \( v \) and atomic formula \( \psi \)

\[
\|\psi\|_{\mathcal{A}^+, v} = \begin{cases} 
\|\psi\|_{M, v} & \text{if } \|\psi\|_{M, v} \in \mathcal{A}^+ \\
0 & \text{otherwise}
\end{cases}
\]

Then \( \|\varphi^*\|_{\mathcal{A}, v} = \|\varphi^*\|_{\mathcal{A}^+, v} \) for every \( v \).

Proof. By structural induction over \( \varphi \): if \( \varphi :\! = \bot \) the claim is immediate. If \( \varphi \) is atomic, then \( \varphi^* :\! = \varphi^2 \) and the claim easily follows from the definition of \( M^+ \).

If \( \varphi :\! = \psi \circ \chi \), with \( \circ \in \{\land, \lor, \rightarrow\} \), then the claim follows from the induction hypothesis over \( \psi \) and \( \chi \).

Finally, if \( \varphi :\! = (\forall x)\chi \), then by the induction hypothesis \( \|\chi^*\|_{M, w} = \|\chi^*\|_{M^+, w} \) for every \( w \equiv_x v \) and hence \( \|\varphi^*\|_{\mathcal{A}, v} = \|\varphi^*\|_{\mathcal{A}^+, v} \).

Theorem 15. Let \( \varphi \) be a formula and \( \mathcal{A} \) be an NM-chain.

1. \( \mathcal{A} \models \varphi^* \iff \|\varphi^*\|_{\mathcal{A}^+, v} \) for every safe \( \mathcal{A} \)-model \( M \) and evaluation \( v \).

2. Let \( \mathcal{B} \) be a complete NM-chain without negation fixpoint: call \( \mathcal{B}^f \) its version with negation fixpoint \( f \). It holds that

\[
\mathcal{B} \models \varphi^* \iff \mathcal{B}^f \models \varphi^*.
\]

Proof. 1. Immediate from Lemma 3.

2. Due to (1) it is enough to check that \( \|\psi\|_{\mathcal{A}^+, v} \neq f \) for every formula \( \psi \) and every \( \mathcal{A} \)-model \( M \) and evaluation \( v \). This can be done by induction over \( \psi \).

- If \( \psi \) is atomic or \( \bot \) the claim is immediate.
- If \( \psi :\! = \theta \circ \chi \), with \( \circ \in \{\land, \lor, \rightarrow\} \), then the claim follows from the induction hypothesis over \( \theta \) and \( \chi \).
• Finally, if \( \psi := (\forall x)\chi \), then by the induction hypothesis \( \|\chi\|_{M,v}^f \neq f \), for every \( w \equiv_v \psi \) if \( \|\chi\|_{M,v}^f < f \), for some \( w \equiv_v \psi \), then \( \|\chi\|_{M,v}^f < f \).

Suppose that \( \|\chi\|_{M,v}^f > f \), for every \( w \equiv_v \psi \); moreover, by contradiction, assume that \( \|\psi\|_{M,v}^f = \inf_{w \equiv_v \psi} \{\|\chi\|_{M,v}^f\} = f \). This means that the set of positive elements of \( B \) does not have infimum, in contrast with the hypothesis that \( B \) is complete.

\[ \square \]

**Theorem 16.** Let \( \phi \) be a formula, and \( \mathcal{A} \) be a Gödel chain. Consider a safe \( \mathcal{A} \)-model \( M = \langle M, \langle m_c \rangle_{c \in C}, \langle r_p \rangle_{p \in P} \rangle \); construct an \( \mathcal{A}_{NM} \)-model \( M' = \langle M, \langle m_c \rangle_{c \in C}, \langle r_p \rangle_{p \in P} \rangle \) such that, for every evaluation \( v \) and atomic formula \( \psi \)

\[ \| \psi \|_{M,v}^f = \| \psi \|_{M',v}^f. \]

Then for every evaluation \( v \) we have

\[ \| \phi \|_{M,v}^f = \| \phi^* \|_{M',v}^f. \]

**Proof.** By structural induction over \( \phi \).

- If \( \phi \) is \( \bot \) or atomic, then the claim is immediate.

- \( \phi := \psi \circ \chi \), with \( \circ \in \{\wedge, \&\} \) and the claim holds for \( \psi \) and \( \chi \). It follows that \( \|\theta\|_{M,v}^f = \|\theta\|_{M',v}^f \), for every \( v \) and with \( \theta \in \{\psi, \chi\} \); noting that these values are 0 or idempotent elements the claim follows.

- \( \phi := \psi \rightarrow \chi \) and the claim holds for \( \psi \) and \( \chi \): it follows that \( \|\theta\|_{M,v}^f = \|\theta\|_{M',v}^f \), for every \( v \) and with \( \theta \in \{\psi, \chi\} \). As previously noted, these values are idempotent elements or 0. With \( \phi^* := (\psi^* \rightarrow \chi^*)^2 \), an easy check shows that \( \|\phi\|_{M,v}^f = \|\phi^*\|_{M',v}^f \), for every \( v \).

- \( \phi := (\forall x)\psi \) and the claim holds for \( \psi \). We have that \( \|\psi\|_{M,v}^f = \|\psi\|_{M',v}^f \), for every \( v \): if there is \( w \equiv_v \psi \) such that \( \|\psi\|_{M,w}^f = 0 \), then the claim is immediate.

Suppose that \( \|\psi\|_{M,w}^f > 0 \), for every \( w \equiv_v \psi \).

- If \( \|\forall x\psi\|_{M,v}^f > 0 \), then \( \|\forall x\psi\|_{M,v}^f = \|\forall x\psi\|_{M',v}^f = \|\forall x\psi\|_{M',v}^f = \|\forall x\psi\|_{M',v}^f = \|\forall x\psi\|_{M',v}^f = 0. \)

\[ \square \]

**Corollary 5.** Let \( \phi \) be a formula, \( \mathcal{A} \) be a Gödel chain. We have that

\[ \mathcal{A} \models \phi \iff \mathcal{A}_{NM} \models \phi^*. \]
Proof. An easy consequence of Theorems 15 and 16.

Recall that a subset of \([0, 1]\) is complete if and only if it is compact with respect to the order topology (see for example [SS95]). Now, in [BPZ07] it is showed that

**Theorem 17** ([BPZ07]). Let \(A\) be a countable topologically closed subalgebra of \([0, 1]_G\) (i.e. a countable complete subalgebra of \([0, 1]_G\)). Then \(\text{TAUT}_A\forall\) is not recursively enumerable.

In our case, we have

**Theorem 18.** Let \(A\) be a countable topologically closed subalgebra of \([0, 1]_{NM}\) (i.e. a countable complete subalgebra of \([0, 1]_{NM}\)). Then \(\text{TAUT}_A\forall\) is not recursively enumerable.

**Proof.** Let \(A\) be a countable complete subalgebra of \([0, 1]_{NM}\).

If \(A\) has negation fixpoint then, due to the observations of Remark 2, we can easily find a countable complete Gödel chain \(B\) such that \(B_{NM} \simeq A\). From Theorem 15 we have that \(\varphi \in \text{TAUT}_{B\forall}\) if and only if \(\varphi^* \in \text{TAUT}_{A\forall}\): since \(\text{TAUT}_{B\forall}\) is not recursively enumerable (Theorem 17), then the same holds for \(\text{TAUT}_{A\forall}\).

If \(A\) does not have negation fixpoint, from Theorem 15 we have that \(\varphi^* \in \text{TAUT}_{A\forall}\) if and only if \(\varphi^* \in \text{TAUT}_{A\forall}\), for every \(\varphi\). Applying the argument of the previous case to \(A\), we have the theorem.

**Corollary 6.** For \(A \in \{NM, NM', NM'\}\), \(\text{TAUT}_A\forall\) is not recursively enumerable.

**Problem 1.** Which is the arithmetical complexity of \(\text{TAUT}_{NM'}\forall\)? Is it recursively axiomatizable?

### 3.1.1 Monadic fragments

In this section, we analyze the (un)decidability status of the validity problem for the monadic fragments associated to the complete subalgebras of \([0, 1]_{NM}\), as well as the four infinite NM-chains hitherto discussed. Recall that in monadic first-order logic the language contains unary predicates and no functions or constant symbols.

Let \(A\) be a subalgebra of \([0, 1]_{NM}\) (or of \([0, 1]_G\)): with \(\text{monTAUT}_A\forall\) we indicate the monadic tautologies associated to \(A\).

In [BCF07] theorem 1 it is showed that the monadic fragment of finite Gödel chains is decidable, but as noted in the subsequent remark, the proof applies to the monadic fragments of arbitrary finite-valued logics. As a consequence we have

**Theorem 19.** Let \(A\) be a finite NM-chain: we have that \(\text{monTAUT}_A\forall\) is decidable.

However, for the infinite case the situation is more difficult; indeed

**Theorem 20** ([BCF07]). Let \(A\) be an infinite complete subalgebra of \([0, 1]_G\): with the possible exception of \(A = G\), \(\text{monTAUT}_A\forall\) is undecidable.

Moving to the NM case, we obtain
Theorem 21. Let $\mathcal{A}$ be an infinite complete subalgebra of $[0, 1]_{NM}$ with the possible exception of $\mathcal{A} \in \{NM_\infty, NM^-_\infty\}$, monTAUT$_{\mathcal{A}\forall}$ is undecidable.

Proof. Let $\mathcal{A}$ be a complete subalgebra of $[0, 1]_{NM}$, with $\mathcal{A} \notin \{NM_\infty, NM^-_\infty\}$.

If $\mathcal{A}$ has negation fixpoint then, due to the observations of Remark 2, we can easily find a complete Gödel chain $\mathcal{B}$, subalgebra of $[0, 1]_G$, such that $\mathcal{B}_{NM} \approx \mathcal{A}$: since $\mathcal{A} \neq NM_\infty$, then $\mathcal{B} \neq G_\uparrow$. From Theorem 16 we have that $\varphi \in TAUT_{\mathcal{A}\forall}$ if and only if $\varphi^* \in TAUT_{\mathcal{B}\forall}$: since TAUT$_{\mathcal{B}\forall}$ is undecidable (Theorem 20), then the same holds for TAUT$_{\mathcal{A}\forall}$.

If $\mathcal{A}$ does not have negation fixpoint, from Theorem 13 we have that $\varphi^* \in TAUT_{\mathcal{A}\forall}$ if and only if $\varphi^* \in TAUT_{\mathcal{A}_f\forall}$, for every $\varphi$. Applying the argument of the previous case to $\mathcal{A}_f$, we have the theorem. 

Corollary 7. monTAUT$_{NM^-_\forall}$ is undecidable.

Problem 2. For $\mathcal{A} \in \{NM_\infty, NM^-_\infty, NM^-_\forall\}$, is monTAUT$_{\mathcal{A}\forall}$ decidable?

4 Open Problems

Inspired by the work done in [BPZ07, BCF07] for (first-order) Gödel logics, in this paper we investigated the first-order tautologies associated with particular NM-chains: moreover, we have showed some decidability and undecidability results, for the full logics and the monadic case.

Many questions are still open. The main one is the search for a full classification, in analogy with the one done for Gödel logics in [BPZ07, BCF07, Pre03], of the (existence of) first-order logics associated to the various subalgebras of $[0, 1]_{NM}$: for the subalgebras whose set of first-order tautologies is not recursively axiomatizable, instead, it could be studied its arithmetical complexity (for Gödel logics this has been done in [BPZ07, BCF07, Pre03, Háj10a, Háj10b, Háj05]). Another theme that has not been analysed here concerns the (first-order) satisfiability problem about the subalgebras of $[0, 1]_{NM}$ (for Gödel logics this has been done in [BCP09]).

We now discuss two more technical (and specific) problems, previously introduced in this paper.

Problem 1 is particularly interesting: if TAUT$_{NM^-_\forall}$ will result recursively axiomatizable, then the next step will be the search for a first-order logic complete with respect to $NM^-_\infty$. This logic could be a relevant infinite-valued logic, because $NM^-_\infty$ satisfies the quantifiers shifting rules and hence we could work with formulas in prenex normal form.

Finally, consider Problem 2 for $NM^-_\infty$, this is a particular case of Problem 1. In the case in which $\mathcal{A} \in \{NM_\infty, NM^-_\infty\}$, instead, the solution is strictly connected with the analogous problem for Gödel logic with the chain $G_\uparrow$.

Acknowledgements The author would like to thank to Professor Stefano Aguzzoli for the suggestions, in particular concerning the translation from $\varphi$ to $\varphi^*$ introduced in Definition 3.
References

[ABM07] S. Aguzzoli, M. Busaniche, and V. Marra, Spectral Duality for Finitely Generated Nilpotent Minimum Algebras, with Applications, J. Log. Comput. 17 (2007), no. 4, 749–765, doi:10.1093/logcom/exm021 2

[ABM09] S. Aguzzoli, M. Bianchi, and V. Marra, A temporal semantics for Basic Logic, Studia Logica 92 (2009), no. 2, 147–162, doi:10.1007/s11225-009-9192-3 2

[AG10] S. Aguzzoli and B. Gerla, Probability Measures in the Logic of Nilpotent Minimum, Studia Logica 94 (2010), 151–176, doi:10.1007/s11225-010-9228-8 2

[AGM05] S. Aguzzoli, B. Gerla, and C. Manara, Poset Representation for Gödel and Nilpotent Minimum Logics, Symbolic and Quantitative Approaches to Reasoning with Uncertainty (L. Godo, ed.), Lecture Notes in Computer Science, vol. 3571, Springer Berlin / Heidelberg, 2005, doi:10.1007/10.1007/11518655_56, pp. 469–469. 2

[AGM08] S. Aguzzoli, B. Gerla, and V. Marra, Embedding Gödel propositional logic into Prior’s tense logic, Proceedings of IPMU’08 (Torremolinos (Málaga)) (L. Magdalena, M. Ojeda-Aciego, and J.L. Verdegay, eds.), June 2008, http://www.gimac.uma.es/ipmu08/proceedings/papers/132-AguzzoliEtAl.pdf, pp. 992–999. 2

[BC10] M. Busaniche and R. Cignoli, Constructive Logic with Strong Negation as a Substructural Logic, J. Log. Comput. 20 (2010), no. 4, 761–793, doi:10.1093/logcom/exn081 2

[BCF07] M. Baaz, A. Ciabattoni, and C. G. Fermüller, Monadic Fragments of Gödel Logics: Decidability and Undecidability Results, Logic for Programming, Artificial Intelligence, and Reasoning - 14th International Conference, LPAR 2007, Yerevan, Armenia, October 15-19, 2007. Proceedings (N. Dershowitz and A. Voronkov, eds.), Lecture Notes in Computer Science, vol. 4790/2007, Springer Berlin / Heidelberg, 2007, doi:10.1007/978-3-540-75560-9, pp. 77–91. 2

[BCP09] M. Baaz, A. Ciabattoni, and N. Preining, SAT in Monadic Gödel Logics: A Borderline between Decidability and Undecidability, Logic, Language, Information and Computation (H. Ono, M. Kanazawa, and R. de Queiroz, eds.), Lecture Notes in Computer Science, vol. 5514, Springer Berlin / Heidelberg, 2009, doi:10.1007/978-3-642-02261-6_10, pp. 113–123. 2

[BEG99] D. Boixader, F. Esteva, and L. Godo, On the continuity of t-norms on bounded chains, Proceedings of the 8th IFSA World Congress IFSA’99 (Taipei, Taiwan), August 1999, pp. 476–479. 2
[BLZ96] M. Baaz, A. Leitsch, and R. Zach, *Incompleteness of a first-order Gödel logic and some temporal logics of programs*, Computer Science Logic - 9th International Workshop, CSL '95 Annual Conference of the EACSL Paderborn, Germany, September 22-29, 1995 Selected Papers (H. K. Büning, ed.), Lecture Notes in Computer Science, vol. 1092/1996, Springer Berlin / Heidelberg, 1996, doi:10.1109/10.1007/3-540-61377-3_28, pp. 1–15.

[BP89] W. Blok and D. Pigozzi, *Algebraizable logics*, vol. 77, Memoirs of The American Mathematical Society, no. 396, American Mathematical Society, 1989, ISBN:0-8218-2459-7 - Available on http://orion.math.iastate.edu/dpigozzi/.

[BPZ07] M. Baaz, N. Preining, and R. Zach, *First-order Gödel logics*, Ann. Pure. Appl. Logic 147 (2007), no. 1-2, 23–47, doi:10.1016/j.apal.2007.03.001.

[Bus06] M. Busaniche, *Free nilpotent minimum algebras*, Math. Log. Q. 52 (2006), no. 3, 219–236, doi:10.1002/malq.200510027.

[CH10] P. Cintula and P. Hájek, *Triangular norm predicate fuzzy logics*, Fuzzy Sets Syst. 161 (2010), no. 3, 311–346, doi:10.1016/j.fss.2009.09.006.

[CT06] R. Cignoli and P. Torrens, *Free Algebras in Varieties of Glivenko MTL-algebras Satisfying the Equation $2(x^2) = (2x)^2$*, Studia Logica 83 (2006), no. 1-3, 157–181, doi:10.1007/s11225-006-8302-8.

[DM71] J. M. Dunn and R. K. Meyer, *Algebraic Completeness Results for Dummett’s LC and Its Extensions*, Math. Log. Q. 17 (1971), no. 1, 225–230, doi:10.1002/mlq.19710170126.

[Dum59] M. Dummett, *A propositional calculus with denumerable matrix*, J. Symb. Log. 24 (1959), no. 2, 97–106, http://www.jstor.org/stable/2964753.

[EG01] F. Esteva and L. Godo, *Monoidal t-norm based logic: Towards a logic for left-continuous t-norms*, Fuzzy Sets Syst. 124 (2001), no. 3, 271–288, doi:10.1016/S0165-0114(01)00098-7.

[EGN06] F. Esteva, L. Godo, and C. Noguera, *On rational weak nilpotent minimum logics*, J. Mult.-Valued Logic Soft Comput. 12 (2006), 9–32.

[EGN09] F. Esteva, L. Godo, and C. Noguera, *First-order t-norm based fuzzy logics with truth-constants: Distinguished semantics and completeness properties*, Ann. Pure Appl. Log. 161 (2009), no. 2, 185–202, doi:10.1016/j.apal.2009.05.014.

[EGN10] F. Esteva, L. Godo, and C. Noguera, *Expanding the propositional logic of a t-norm with truth-constants: completeness results for rational semantics*, Soft Comput. 14 (2010), 273–284, doi:10.1007/s00500-009-0402-8.
[EZLM09] M. El-Zekey, W. Lotfallah, and N. Morsi, Computationally complexities of axiomatic extensions of monoidal t-norm based logic, Soft Comput. 13 (2009), 1089–1097, doi:10.1007/s00500-008-0382-0

[FGN10] E. Francesc, L. Godo, and C. Noguera, On expansions of WNM t-norm based logics with truth-constants, Fuzzy Sets Syst. 161 (2010), no. 3, 347–368, doi:10.1016/j.fss.2009.09.002

[FJG’01] S. Feferman, J. W. Dawson Jr., W. Goldfarb, C. Parsons, and W. Sieg (eds.), Kurt gödel collected works, paperback ed., vol. 1 Publications: 1929-1936, Oxford University Press, 2001, ISBN:9780195147209

[Fod95] J. Fodor, Nilpotent minimum and related connectives for fuzzy logic, Fuzzy Systems, 1995. International Joint Conference of the Fourth IEEE International Conference on Fuzzy Systems and The Second International Fuzzy Engineering Symposium., Proceedings of 1995 IEEE International Conference on, IEEE, 1995, doi:10.1109/FUZZY.1995.409964, pp. 2077–2082.

[Göd32] K. Gödel, Zum intuitionistischen Aussagenkalkül, Anzeiger Akademie der Wissenschaften Wien 69 (1932), 65–66, Reprinted in [FJG’01].

[Gis03] J. Gispert, Axiomatic extensions of the nilpotent minimum logic, Rep. Math. Logic 37 (2003), 113–123, http://www.iphils.uj.edu.pl/rml/rml-37/7-gispert.pdf.

[Háj02a] P. Hájek, Metamathematics of fuzzy logic, paperback ed., Trends in Logic, vol. 4, Kluwer Academic Publishers, 2002, ISBN:9781402003707

[Háj02b] , Observations on the monoidal t-norm logic, Fuzzy Sets Syst. 132 (2002), no. 1, 107–112, doi:10.1016/S0165-0114(02)00057-x

[Háj05] , A non-arithmetical Gödel logic, Log. J. IGPL 13 (2005), no. 4, 435–441, doi:10.1093/jigpal/jzi033

[Háj10a] , Arithmetical complexity of fuzzy predicate logics - A survey II, Ann. Pure Appl. Log. 161 (2010), no. 2, 212–219, doi:10.1016/j.apal.2009.05.015

[Háj10b] , On witnessed models in fuzzy logic III - witnessed Gödel logics, Math. Log. Q. 56 (2010), no. 2, 171–174, doi:10.1002/malq.200810047

[Jen03] S. Jenei, On the structure of rotation-invariant semigroups, Arch. Math. Log. 42 (2003), no. 5, 489–514, doi:10.1007/s00153-002-0165-8

[KMP00] E.P. Klement, R. Mesiar, and E. Pap, Triangular norms, hardcover ed., Trends in Logic, vol. 8, Kluwer Academic Publishers, 2000, ISBN:978-0-7923-6416-0


[Nog06] C. Noguera, *Algebraic study of axiomatic extensions of triangular norm based fuzzy logics*, Ph.D. thesis, IIIA-CSIC, 2006, Available on http://www.carlesnoguera.cat/files/NogueraPhDThesis.pdf.

[Pre03] N. Preining, *Complete recursive axiomatizability of Gödel logics*, Ph.D. thesis, Vienna University of Technology, Austria, 2003, Available on http://www.logic.at/staff/preining/pubs/phd.pdf.

[Pre10] —, *Gödel Logics - A Survey*, Logic for Programming, Artificial Intelligence, and Reasoning (C. Fermüller and A. Voronkov, eds.), Lecture Notes in Computer Science, vol. 6397, Springer Berlin / Heidelberg, 2010, doi:10.1007/978-3-642-16242-8, pp. 30–51.

[SS95] J. A. Seebach and L. A. Steen, *Counterexamples in topology*, reprint of 1978 ed., Dover Publications, 1995, ISBN:048668735X.

[SS05] B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, updated reprint of 1983 ed., Dover Publications, 2005, ISBN:9780486445144.