Learning convex polyhedra with margin

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Abstract

We present an improved algorithm for quasi-properly learning convex polyhedra in the realizable PAC setting from data with a margin. Our learning algorithm constructs a consistent polyhedron as an intersection of about $t \log t$ halfspaces with constant-size margins in time polynomial in $t$ (where $t$ is the number of halfspaces forming an optimal polyhedron).

We also identify distinct generalizations of the notion of margin from hyperplanes to polyhedra and investigate how they relate geometrically; this result may have ramifications beyond the learning setting.

Keywords: Classification, polyhedra, dimensionality reduction, margin.

1 Introduction

In the theoretical PAC learning setting [Valiant 1984], one considers an abstract instance space $X$ — which, most commonly, is either the Boolean cube $\{0, 1\}^d$ or the Euclidean space $\mathbb{R}^d$. For the former setting, an extensive literature has explored the statistical and computational aspects of learning Boolean functions [Angluin 1992; Hellerstein and Servedio 2007]. Yet for the Euclidean setting, a corresponding theory of learning geometric concepts is still being actively developed [Kwek and Pitt 1998; Jain and Kinber 2003; Anderson et al. 2013; Kane et al. 2013]. The focus of this paper is the latter setting.

The simplest nontrivial geometric concept is perhaps the halfspace. These concepts are well-known to be hard to agnostically learn [Höfling et al. 1995] or even approximate [Amaldi and Kann 1995; 1998; Ben-David et al. 2003]. Even the realizable case, while commonly described as “solved” via the Perceptron algorithm or linear programming (LP), is not straightforward: The Perceptron’s runtime is quadratic in the inverse-margin, while solving the consistent hyperplane problem in strongly polynomial time is equivalent to solving the general LP problem in strongly polynomial time [Nikolov 2018; Chvátal 2018] (we include the proof in Appendix A for completeness). The problem of solving LP in strongly polynomial time has been open for decades [Bárány and Vempala 2010]. Thus, an unconditional (i.e., infinite-precision and independent of data configuration in space) polynomial-time solution for the consistent hyperplane problem hinges on the strongly polynomial LP conjecture.

If we consider not a single halfspace, but polyhedra defined by the intersection of multiple halfspaces, the computational and generalization bounds rapidly become more pessimistic. Megiddo

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(1988) showed that the problem of deciding whether two sets of points in general space can be separated by the intersection of two hyperplanes is \( \text{NP} \)-complete, and Khot and Saket (2011) showed that “unless \( \text{NP} = \text{RP} \), it is hard to (even) weakly PAC-learn intersection of two halfspaces”, even when allowed the richer class of \( \text{O}(1) \) intersecting halfspaces. Under cryptographic assumptions, Klivans and Sherstov (2009) showed that learning an intersection of \( n^\varepsilon \) halfspaces is intractable regardless of hypothesis representation.

Since the margin assumption is what allows one to find a consistent hyperplane in provably strongly polynomial time, it is natural to seek to generalize this scheme to intersections of \( t \) halfspaces each with margin \( \gamma \); we call this the \( \gamma \)-margin of a \( t \)-polyhedron. This problem was considered by Arriaga and Vempala (2006), who showed that such a polyhedron can be learned (in dimension \( d \)) in time

\[
O(dmt) + (t \log t)^{O((t/\gamma)^2 \log(t/\gamma))}
\]

with sample complexity \( m = O\left(\left(\frac{t}{\gamma^2}\right) \log(t) \log(t/\gamma)\right) \) (where we have taken the PAC-learning parameters \( \varepsilon, \delta \) to be constants). In fact, they actually construct a candidate \( t \)-polyhedron as their learner; as such, their approach is an example of proper learning, where the hypothesis is chosen from the same concept class as the true concept. In contrast, Klivans and Servedio (2008) showed that a \( \gamma \)-margin \( t \)-polyhedron can be learned by constructing a function that approximates the polyhedron’s behavior, without actually constructing a \( t \)-polyhedron. This is an example of improper learning, where the hypothesis is selected from a broader class than that of the true concept. They achieved a runtime of

\[
\min \left\{ d(t/\gamma)^{O(t \log t \log(1/\gamma))}, d \left(\frac{\log t}{\gamma}\right)^{O(1/\gamma \log t)} \right\}
\]

and sample complexity \( m = O\left((1/\gamma)^{t \log t \log(1/\gamma)}\right) \). Very recently, Goel and Klivans (2018) improved on this latter result, constructing a function hypothesis in time \( \text{poly}(d, t^{O(1/\gamma)}) \), with sample complexity exponential in \( \gamma^{-1/2} \). Our notion of learning, termed here quasi-proper, lies somewhere between proper and improper learning in terms of stringency. Quasi-proper learning requires an infinite nested family of hypotheses of growing complexity: \( H_1 \subseteq H_2 \subseteq \ldots \). A learning algorithm is said to be quasi-proper for a class \( H_k \) if it returns an \( h \in H_{\ell} \) for \( \ell \geq k \). As just stated, the definition does not formally distinguish quasi-proper and improper learning. The former is a stricter notion because we require the \( \{H_k\} \) to all belong to the same parametric class, such as all \( t \)-facet polyhedra in \( \mathbb{R}^d \), for \( t + d \leq k \). This notion could be formalized beyond the case of polyhedra, but since this paper only deals with learning the latter, such a level of generality would be out of scope.

Our results. The central contribution of the paper is improved algorithmic runtimes and sample complexity for computing separating polyhedra (Theorem 3.3). In contrast to the algorithm of Arriaga and Vempala (2006), whose runtime is exponential in \( t/\gamma^2 \), and to that of (Goel and Klivans, 2018), whose sample complexity is exponential in \( \gamma^{-1/2} \), we give an algorithm with polynomial sample complexity \( m = \tilde{O}(t/\gamma^2) \) and runtime only \( m^{O(1/\gamma^2)} \). We accomplish this by constructing an \( O(t \log m) \)-polyhedron that correctly separates the data. This means that our hypothesis is drawn from a broader class than the \( t \)-polyhedra of Arriaga and Vempala (2006) (allowing faster runtime), but from a much narrower class than the functions of Klivans and Servedio (2008); Goel and Klivans (2018) (allowing for improved sample complexity).

Complementing our algorithm, we provide the first nearly matching hardness-of-approximation bounds, which (roughly) demonstrate that an exponential dependence on \( t \gamma^{-2} \) is unavoidable for the
computation of separating $t$-polyhedra, under standard complexity-theoretic assumptions (Theorem 3.2). This motivates our consideration of $O(t \log m)$-polyhedra instead.

Our final contribution is in introducing a new and intuitive notion of polyhedron margin: This is the $\gamma$-envelope of a convex polyhedron, defined as all points within distance $\gamma$ of the polyhedron’s boundary, as opposed to the above $\gamma$-margin of the polyhedron, defined as the intersection of the $\gamma$-margins of the hyperplanes forming the polyhedron. (See Figure 1 for an illustration, and Section 2 for precise definitions.) Note that these two objects may exhibit vastly different behaviors, particularly at a sharp intersection of two or more hyperplanes. It seems to us that the envelope of a polyhedron is a more natural structure than its margin: Indeed, taking an envelope has the effect of rounding the corners of the polyhedron, and rounded polyhedra occur in nature and have been the subject of mathematical study \cite{Onaka2005,Onaka2008,Andersson2008,BogoselBonnaillieNoel2017}. Yet we find the margin more amenable to the derivation of combinatorial dimension bounds (Theorem 2.2) and algorithms (Theorem 3.3). We demonstrate that results derived for margins can be adapted to apply to envelopes as well. We prove that when confined to the unit ball, under natural conditions the $\gamma$-envelope fully contains within it the $(\gamma^2/2)$-margin (Theorem 4.2), and this implies that statistical and algorithmic results for the latter hold for the former as well. Using the same techniques, we also derive a related result of independent interest concerning expanding polyhedra (Section 4.1). In Section 5 we present some simulation results.

Related work. When general convex bodies are considered under the uniform distribution\footnote{Since the concept class of convex sets has infinite VC-dimension, without distribution assumptions, an adversarial distribution can require an arbitrarily large sample size, even in 2 dimensions \cite{KearnsVazirani1997}.} (over the unit ball or cube), exponential (in dimension and accuracy) sample-complexity bounds were obtained by \textcite{RademacherGoyal2009}. This may motivate the consideration of convex polyhedra, and indeed a number of works have studied the problem of learning convex polyhedra, including \textcite{Hegedus1994,KwekPitt1998,Anderson2013,Kane2013,Kantchelian2014}. \textcite{Hegedus1994} examines query-based exact identification of convex polyhedra with integer vertices, with runtime polynomial in the number of vertices (note that the number of vertices can be exponential in the number of facets \cite{Matousek2002}). \textcite{KwekPitt1998} also rely on membership queries (see also references therein regarding prior results, as well as strong positive results in 2 dimensions). \textcite{Anderson2013} efficiently approximately recover an unknown simplex from uniform samples inside it. \textcite{Kane2013} learn halfspaces under the log-concave distributional assumption.

The recent work of \textcite{Kantchelian2014} bears a superficial resemblance to ours, but the two are actually not directly comparable. What they term worst case margin will indeed correspond to our margin. However, their optimization problem is non-convex, and the solution relies on heuristics without rigorous run-time guarantees. Their generalization bounds exhibit a better dependence on the number $t$ of halfspaces than our Theorem 2.2 ($O(\sqrt{t})$ vs. our $O(t \log t)$). However, the hinge loss appearing in their Rademacher-based bound could be significantly worse than the 0-1 error appearing in our VC-based bound. We stress, however, that the main contribution of our paper is algorithmic rather than statistical.

2 Preliminaries

Notation. For $x \in \mathbb{R}^d$, we denote its Euclidean norm $\|x\|_2 := \sqrt{\sum_{i=1}^{d} x(i)^2}$ by $\|x\|$, and for $n \in \mathbb{N}$, we write $[n] := \{1, \ldots, n\}$. Our instance space $\mathcal{X}$ is the unit ball in $\mathbb{R}^d$: $\mathcal{X} = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$. 

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Polyhedra. A (convex) polyhedron $P \subset \mathbb{R}^d$ is the intersection of a finite number $t$ of closed halfspaces. Each halfspace is bounded by a hyperplane $(w_i, b_i) \in \mathbb{R}^d \times \mathbb{R}$ with $\|w_i\| = 1$ for each $i$, and $P$ is given by:

$$P = \left\{ x \in \mathbb{R}^d : \min_{i \in [t]} w_i \cdot x + b_i \geq 0 \right\}.$$  \hspace{1cm} (1)

(A polyhedron is not necessarily bounded. A bounded polyhedron is called a polytope.) A hyperplane $(w, b)$ is said to classify a point $x$ as positive (resp., negative) with margin $\gamma$ if $w \cdot x + b \geq \gamma$ (resp., $\leq -\gamma$). Since $\|w\| = 1$, this means that $x$ is $\gamma$-far from the hyperplane $\{x' \in \mathbb{R}^d : w \cdot x' + b = 0\}$, in $\ell_2$ distance.

Margins and envelopes. We consider two natural ways of extending this notion to polyhedra: the $\gamma$-margin and the $\gamma$-envelope. For a polyhedron defined by $t$ hyperplanes as in (1), we say that $x \in P$ is in the inner $\gamma$-margin of $P$ if

$$0 \leq \min_{i \in [t]} w_i \cdot x + b_i \leq \gamma$$

and that $x / \in P$ is in the outer $\gamma$-margin of $P$ if

$$0 > \min_{i \in [t]} w_i \cdot x + b_i \geq -\gamma.$$  

Similarly, we say that $x$ is in the outer $\gamma$-envelope of $P$ if $x \not\in P$ and $\inf_{p \in P} \|x - p\| \leq \gamma$ and that $x$ is in the inner $\gamma$-envelope of $P$ if $x \in P$ and $\inf_{p \not\in P} \|x - p\| \leq \gamma$.

We call the union of the inner and the outer $\gamma$-margins the $\gamma$-margin, and we denote it by $\partial P^{[\gamma]}$. Similarly, we call the union of the inner and the outer $\gamma$-envelopes the $\gamma$-envelope, and we denote it by $\partial P(\gamma)$.

The two notions are illustrated in Figure 1. As we show in Section 4 below, the inner envelope coincides with the inner margin, but this is not the case for the outer objects: The outer margin always contains the outer envelope, and could be of arbitrarily larger volume.

Fat-shattering dimension. Let $\mathcal{X}$ be a set and $\mathcal{F} \subset \mathbb{R}^\mathcal{X}$. For $\theta > 0$, a set $S = \{x_1, \ldots, x_m\} \subset \mathcal{X}$ is $\theta$-shattered by $\mathcal{F}$

$$\sup_{r \in \mathbb{R}^m} \min_{y \in \{-1,1\}^m} \sup_{f \in \mathcal{F}} \min_{i \in [m]} y_i (f(x_i) - r_i) \geq \theta.$$  

The $\theta$-fat-shattering dimension, denoted by fat$_\theta(\mathcal{F})$, is the size of the largest $\theta$-shattered set (possibly $\infty$).
Fat hyperplanes and polyhedra. Binary classification requires a collection of concepts mapping the instance space (in our case, the unit ball in $\mathbb{R}^d$) to $\{-1, 1\}$. Any real-valued function class can be converted into a binary concept class by composing with the sign function. The generalization error of such a predictor can be controlled by the fat-shattering dimension of the function class:

**Theorem 2.1** [Bartlett and Shawe-Taylor (1999) Theorem 4.5]. Let $\mathcal{F}$ be a collection of real-valued functions over some set $\mathcal{X}$, define $D = \text{fat}_{1/16}(\mathcal{F})$, and let $P$ be some probability distribution on $\mathcal{X} \times \{-1, 1\}$. Suppose that $(x_i, y_i), i \in [n]$ are drawn independently according to $P$, and that some $f \in \mathcal{F}$ achieves strong separation in the sense that

$$\min_{i \in [n]} y_i f(x_i) \geq 1. \quad (2)$$

Then with probability at least $1 - \delta$,

$$\mathbb{P}\{(x, y) \in \mathcal{X} \times \{-1, 1\} : \text{sign}(f(x)) \neq y\} \leq \frac{2}{n} (D \log(34en/D) \log(578n) + \log(4/\delta)).$$

In the case of hyperplanes, the strong separation condition (2) corresponds to a margin of $\gamma = 1/\|w\|$. All of our discussion of margins, envelopes, and hyperplanes continues to hold verbatim if, but rather than normalizing $\|w\| = 1$, we normalize $\gamma = 1$ and let $\|w\|$ vary. It is well-known [Bartlett and Shawe-Taylor, 1999, Theorem 4.6] that homogeneous hyperplanes — i.e., function classes of the form $\mathcal{F} = \{x \mapsto w \cdot x; \|x\| \leq 1, \|w\| \leq R\}$ — satisfy $\text{fat}_\theta(\mathcal{F}) \leq (R/\theta)^2$. For this paper, we must necessarily use inhomogeneous hyperplanes, since a general polyhedron is characterized as an intersection of such objects. To be more precise, the hyperplane $(w, b)$ induces the function $f_{w,b} : \mathbb{R}^d \to \mathbb{R}$ given by $f_{w,b}(x) = w \cdot x + b$. Fix some $R > 0$ and define $\mathcal{F} = \{f_{w,b}; \|w\| \leq R\}$.

**Remark.** A careful reader will notice that the conference version of this paper, [Gottlieb et al. (2018)], did not invoke fat-shattering dimension. Instead, the generalization bound given there, contained in Lemmas 1 and 2, relied on [Hanneke and Kontorovich (2017), Lemma 6]. Unfortunately, the latter has a mistake, as explained in [Kantorovich and Attias (2021)]; hence we follow their approach instead.

**Theorem 2.2.** For $t \in \mathbb{N}$, let $\mathcal{P}_t$ be the collection of $t$-polytopes, i.e., functions $f : \mathbb{R}^d \to \mathbb{R}$ defined by $t$ hyperplanes $(w_1, b_1), \ldots, (w_t, b_t)$:

$$f(x) = \min_{i \in [t]} w_i \cdot x + b_i, \quad \|w_i\|, |b_i| \leq R. \quad (3)$$

Then

$$\text{fat}_\theta(\mathcal{P}_t) \leq Ct \log(t) \min \left\{ \frac{R^2}{\theta^2}, d \right\}, \quad 0 < \theta \leq R, t > 1,$$

where $C > 0$ is a universal constant.

**Proof.** Observe that $\text{fat}_\theta(\mathcal{F}) = \text{fat}_\theta(-\mathcal{F})$ and so replacing min by max in (3) will not affect $\text{fat}_\theta(\mathcal{P}_t)$. The result then follows directly from [Kantorovich and Attias (2021) Theorems 2 and 3].
**Dimension reduction.** The Johnson-Lindenstrauss (JL) transform [Johnson and Lindenstrauss, 1982] takes a set $S$ of $m$ vectors in $\mathbb{R}^d$ and projects them into $k = O(\varepsilon^{-2} \log m)$ dimensions, while preserving all inter-point distances and vector norms up to $1 + \varepsilon$ distortion. That is, if $f : \mathbb{R}^d \to \mathbb{R}^k$ is a linear embedding realizing the guarantees of the JL transform on $S$, then for every $x \in S$ we have

$$ (1 - \varepsilon)\|x\| \leq \|f(x)\| \leq (1 + \varepsilon)\|x\|, $$

and for every $x, y \in S$ we have

$$ (1 - \varepsilon)\|x - y\| \leq \|f(x - y)\| \leq (1 + \varepsilon)\|x - y\|. $$

The JL transform can be realized with probability $1 - n^{-c}$ for any constant $c \geq 1$ by a randomized linear embedding, for example a projection matrix with entries drawn from a normal distribution [Achlioptas, 2003]. This embedding is oblivious, in the sense that the matrix can be chosen without knowledge of the set $S$.

It is an easy matter to show that the JL transform can also be used to approximately preserve distances to hyperplanes, as in the following lemma.

**Lemma 2.3.** Let $S$ be a set of $d$-dimensional vectors in the unit ball, $T$ be a set of normalized vectors, and $f : \mathbb{R}^d \to \mathbb{R}^k$ a linear embedding realizing the guarantees of the JL transform for $S \cup T$. Then for any $0 < \varepsilon < 1$ and some $k = O((\log |S \cup T|)/\varepsilon^2)$, with probability $1 - |S \cup T|^{-c}$ (for any constant $c > 1$) we have for all $x \in S$ and $t \in T$ that

$$ f(t) \cdot f(x) \in t \cdot x \pm \varepsilon. $$

**Proof.** Let the constant in $k$ be chosen so that the JL transform preserves distances and norms among $S \cup T$ within a factor $1 + \varepsilon'$ for $\varepsilon' = \varepsilon/5$. By the guarantees of the JL transform for the chosen value of $k$, we have that

$$ f(t) \cdot f(x) = \frac{1}{2} \left[ \|f(t)\|^2 + \|f(x)\|^2 - \|f(t) - f(x)\|^2 \right] $$

$$ \leq \frac{1}{2} \left[ (1 + \varepsilon')^2(\|t\|^2 + \|x\|^2) - (1 - \varepsilon')^2\|t - x\|^2 \right] $$

$$ < \frac{1}{2} \left[ (1 + 3\varepsilon')(\|t\|^2 + \|x\|^2) - (1 - 2\varepsilon')\|t - x\|^2 \right] $$

$$ < \frac{1}{2} \left[ 5\varepsilon'(\|t\|^2 + \|x\|^2) + 2t \cdot x \right] $$

$$ \leq 5\varepsilon' + t \cdot x. $$

$$ = \varepsilon + t \cdot x. $$

A similar argument gives that $f(t) \cdot f(x) > -\varepsilon + t \cdot x.$

**3 Computing and learning separating polyhedra**

In this section, we present algorithms to compute and learn $\gamma$-fat $t$-polyhedra. We prove hardness results for this problem, and then present our algorithms.
3.1 Hardness

We show that computing separating polyhedra is NP-hard, and even hard to approximate. Further, we will use the Exponential Time Hypothesis (ETH) to justify near-exponential dependence on \( t \gamma^{-2} \) for exact algorithms for this problem.

A consequence of ETH is that the maximum independent set and minimum graph coloring problems cannot be solved in fewer than \( c^n \) operations, for some constant \( c \) (Cygan et al. 2015). This does not necessarily imply that approximating these problems requires \( c^n \) operations: As hardness-of-approximation results utilize polynomial-time reductions, ETH implies only that the runtime is exponential in some polynomial in \( n \). However, Bonnet et al. (2015) have shown that under ETH, neither independent set nor coloring admit an \( r \)-approximation in time \( O(2^{n^c}) \) for any constant \( r \) and \( \delta > 0 \) (see Bansal et al. (2019) for upper bounds).

We begin by considering the case of a single hyperplane, as this task is a basic tool of the algorithms of Section 3.2. The following preliminary lemma builds upon Amaldi and Kann (1995, Theorem 10).

**Lemma 3.1.** Given a labelled point set \( S = |S| \) with \( p \) negative points, let \( h^* \) be a hyperplane that places all positive points of \( S \) on its positive side, and maximizes the number of negative points on its negative side — let \( \text{opt} \) be the number of these negative points. Then

1. It is NP-hard to find a hyperplane \( \bar{h} \) consistent with all positive points, and which places at least \( \text{opt}/p^{1-o(1)} \) negative points on the negative side of \( \bar{h} \). This holds when the optimal hyperplane has margin \( \gamma = \frac{1}{4\text{opt}} \) or less with respect to the correctly classified points.

2. Under ETH, a hyperplane consistent with all positive points and with \( \text{opt} \) negative points cannot be computed in time less than \( c^{1/(16\gamma^2)} \) for some constant \( c \). A hyperplane consistent with all positive points and with at least \( r\text{opt} \) negative points cannot be computed in time \( O \left( 2^{(1/(16\gamma^2))^{1-\delta}} \right) \) for any constants \( 0 < r, \delta < 1 \).

**Proof.** We reduce from maximum independent set, which for \( p \) vertices is hard to approximate to within \( p^{1-o(1)} \) (Zuckerman 2007). Given a graph \( G = (V, E) \), for each vertex \( v_i \in V \) place a negative point on the basis vector \( e_i \). Now place a positive point at the origin, and for each edge \( (v_i, v_j) \in E \), place a positive point at \( (e_i + e_j)/2 \).

Consider a hyperplane consistent with the positive points and placing \( \text{opt} \) negative points on the negative side: These negative points must represent an independent set in \( G \), for if \( (v_i, v_j) \in E \), then by construction the midpoint of \( e_i, e_j \) is positive, and so both \( e_i, e_j \) cannot lie on the negative side of the hyperplane.

Likewise, if \( G \) contained an independent set \( V' \subset V \) of size \( \text{opt} \), then we consider the hyperplane defined by the equation \( w \cdot x + \frac{3}{4\text{opt}} = 0 \), where coordinate \( w(j) = -\frac{1}{\text{opt}} \) if \( v_j \in V' \) and \( w(j) = 0 \) otherwise. It is easily verified that the distance from the hyperplane to a correctly classified negative point (i.e. a basis vector) is \( -\frac{1}{\text{opt}} + \frac{3}{4\text{opt}} = -\frac{1}{4\text{opt}} \), to the origin is \( \frac{3}{4\text{opt}} \), and to all positive points is at least \( -\frac{1}{2\sqrt{\text{opt}}} + \frac{3}{4\text{opt}} = \frac{1}{4\text{opt}} \). The first item follows.

For the second item, as the above reduction yields margin \( \gamma = \frac{1}{4\text{opt}} \geq \frac{1}{4\sqrt{p}} \), by ETH we cannot compute an exact solution in time less than \( c^p \geq c^{1/(16\gamma^2)} \), or an approximation solution in time of the order \( 2^{p^{1-\delta}} \geq 2^{(1/(16\gamma^2))^{1-\delta}} \).

We can now extend the above results for hyperplanes to similar ones for polyhedra:

**Theorem 3.2.** Given a labelled point set \( S = |S| \) with \( p \) negative points, let \( H^* \) be a collection of \( t \) halfspaces whose intersection partitions \( S \) into positive and negative sets. Then
1. It is NP-hard to find a collection $\tilde{H}$ of size less than $tp^{1-o(1)}$ whose intersection also partitions $S$ into positive and negative sets. This holds when the polyhedron implied by $H^*$ has margin $\gamma = \frac{1}{4\sqrt{p/t}}$ or less.

2. Under ETH, a collection of $t$ halfspaces whose intersection partitions $S$ into positive and negative sets cannot be computed in time less than $c^{t/(16\gamma^2)}$ for some constant $c$, nor in time $O\left(2^{t\gamma}\right)$ for any constant $\delta > 0$.

Proof. The reduction is from minimum coloring, which is hard to approximate within a factor of $n^{1-o(1)}$ [Zuckerman, 2007]. The construction is identical to that of the proof of Lemma [3.1]. Given a graph $G = (V, E)$, for each vertex $v_i \in V$ place a negative point on the basis vector $e_i$. Now place a positive point at the origin, and for each edge $(v_i, v_j) \in E$, place a positive point at $(e_i + e_j)/2$.

As above, a hyperplane consistent with the positive points and placing $m$ negative points on the negative side corresponds to an independent set of size $m$ in $G$, for if $(v_i, v_j) \in E$, then by construction the midpoint of $e_i, e_j$ is positive, and so both $e_i, e_j$ cannot lie on the negative side of the hyperplane. Then a set of $t$ hyperplanes whose intersection partitions $S$ into positive and negative sets implies a $t$-coloring on $G$.

Similarly, and as above, for any independent set $V'$ of size $m$ in $G$, there is a hyperplane that is consistent with all positive points and with the $m$ negative points corresponding to $V'$, while also achieving margin $\frac{1}{4\sqrt{m}} \geq \frac{1}{4\sqrt{p}}$ on these points. So a $t$ coloring on $G$ implies a set of $t$ hyperplanes whose intersection (a polyhedron with margin at least $\frac{1}{4\sqrt{p}}$) partitions $S$ into positive and negative sets. To enlarge the margin further, we stipulate that no color in the solution represent more than $p/t$ vertices; if a color in the optimal $t$-coloring of $G$ covers more than $p/t$ vertices, we replace it by a minimal set of colors, each coloring no more than $p/t$ vertices. This increases the total number of colors to at most $2t$, but does not affect the hardness-of-approximation result. The first item follows.

For the second item, as the above reduction yields margin $\gamma \geq \frac{1}{4\sqrt{p}}$, by ETH we cannot compute an exact solution in time less than $c^p \geq c^{t/(16\gamma^2)}$. Likewise, as above there exists a solution polyhedron formed by at most $2t$ halfspaces and achieving margin $\gamma \geq \frac{1}{4\sqrt{p/t}}$, but by ETH, finding any solution of $rt$ halfspaces for constant $r > 1$ cannot be computed in time of the order $2^{o(1-\delta)} \geq 2^{t\gamma}$. □

Theorem 3.2 roughly justifies the exponential dependence on $t\gamma^{-2}$ in the algorithm of [Arriaga and Vempala, 2006], and implies that to avoid an exponential dependence on $t$ in the runtime, we should consider broader hypothesis classes, for example $O(t \log m)$-polyhedra. We do this in the next section.

3.2 Algorithms

Here we present algorithms for computing polyhedra, and use them to give an efficient algorithm for learning polyhedra.

In what follows, we give two algorithms inspired by the work of [Arriaga and Vempala, 2006]. Both have runtime faster than the algorithm of [Arriaga and Vempala, 2006], and the second is only polynomial in $t$. The underlying idea of our algorithms is to project the points from their high-dimensional origin space into a low-dimensional target space. We can find a good halfspace in the target space using a brute-force method, and then identify a halfspace in the origin space which is consistent with the halfspace of the target space. We identify multiple such halfspaces in the origin space, and their intersection yields the solution polyhedron. Crucially, we choose these halfspaces greedily, at each step selecting the one which maximizes the number of negative points excluded from the current polyhedron.
Theorem 3.3. Given a labelled point set $S$ ($n = |S|$) for which some $\gamma$-fat t-polyhedron correctly separates the positive and negative points (i.e., the polyhedron is consistent), we can compute the following with high probability:

1. A consistent $(\gamma/4)$-fat t-polyhedron in time $n^{O((1+\gamma^{-2})\log(1/\gamma))}$.
2. A consistent $(\gamma/4)$-fat $O(t \log n)$-polyhedron in time $n^{O(\gamma^{-2} \log(1/\gamma))}$.

Before proving Theorem 3.3 we will need a preliminary lemma:

Lemma 3.4. Given any $0 < \delta < 1$, there exists a set $V$ of unit vectors of size $|V| = \delta^{-O(d)}$ with the following property: For any unit vector $w$, there exists some $v \in V$ that satisfies $v \cdot x \in w \cdot x \pm \delta$ for all vectors $x$ with $\|x\| \leq 1$. The set $V$ can be constructed in time $\delta^{-O(d)}$ with high probability.

This implies that if a set $S$ admits a hyperplane $(w, b)$ with margin $\gamma$, then $S$ admits a hyperplane $(v, b)$ (for $v \in V$) with margin at least $\gamma - \delta$.

Proof. We take $V$ to be a $\delta$-net of the unit ball, a set satisfying that every point on the ball is within distance $\delta$ of some point in $V$. Then $|V| \leq (1 + 2/\delta)^d$ ([Vershynin 2010] Lemma 5.2). For any unit vector $w$ we have for some $v \in V$ that $\|w - v\| \leq \delta$, and so for any vector $x$ satisfying $\|x\| \leq 1$ we have

$$|w \cdot x - v \cdot x| = |(w - v) \cdot x| \leq \|w - v\| \leq \delta.$$

The net can be constructed by a randomized greedy algorithm. By coupon-collector analysis, it suffices to sample $O(|V| \log |V|)$ random unit vectors. For example, each can be chosen by sampling its coordinate from $N(0, 1)$ (the standard normal distribution), and then normalizing the vector. The resulting set contains within it a $\delta$-net.

Proof of Theorem 3.3 We first apply the Johnson-Lindenstrauss transform to reduce the dimension of the points in $S$ to $k = O(\gamma^{-2} \log(n + t)) = O(\gamma^{-2} \log n)$ while achieving the guarantees of Lemma 2.3 for the points of $S$ and the $t$ halfspaces forming the optimal $\gamma$-fat t-polyhedron, with parameter $\varepsilon = \frac{\gamma}{2k}$. This means that in the embedded space, each halfspace vector $w$ is embedded into a vector which is not necessarily a unit vector, but is within distance $\frac{\gamma}{2k}$ of some unit vector in the embedded space, and so $\|f(w)\| \in [1, \frac{\gamma}{2k}]$. Each halfspace vector $w$ also satisfies $f(w) \cdot f(x) \in w \cdot x \pm \frac{\gamma}{2k}$ for all $x \in S$.

Now in the embedded space we extract a $\delta$-net $V$ of unit vectors by applying Lemma 3.4 with parameter $\delta = \frac{\gamma}{2k}$, and since the dimension of the embedded space is $k$, we have $|V| = \delta^{-O(k)}$. It follows that each halfspace vector $w$ is embedded to a point $f(w)$ within distance $\varepsilon + \delta = \frac{\gamma}{2k} + \frac{\gamma}{2k} = \frac{\gamma}{k}$ of some unit vector $v \in V$. Further, the triangle inequality gives us that $\|v - f(x)\| \leq \|f(w) - f(x)\| + \|f(w) - v\| \leq \|f(w) - f(x)\| \leq \frac{\gamma}{k}$, and so for any $x \in S$ we have

$$v \cdot f(x) = \frac{|w| + \|f(x)\| - |v - f(x)|}{2} \leq \frac{\|f(w)\| - \|f(w) - f(x)\|}{2} = \frac{f(w) \cdot f(x)}{2} \leq \frac{\gamma}{2k}.$$

Now define the set $B$ consisting of all values of the form $\frac{\gamma i}{6}$ for integer $i = \{0, 1, \ldots, \lfloor 6/\gamma \rfloor \}$. It follows that for each $d$-dimensional halfspace $(w, b)$ forming the original $\gamma$-fat t-polyhedron, there is a $k$-dimensional halfspace $(v, b')$ with $v \in V$ and $b' \in B$ satisfying $v \cdot f(x) + b' \in w \cdot x \pm \frac{\gamma}{2k} + b \pm \frac{\gamma}{2} = w \cdot x + b \pm \frac{\gamma}{k}$ for every $x \in S$. Given $(v, b')$, we can recover an approximation to $(w, b)$ in the $d$-dimensional origin space thus: Let $S' \subset S$ include all positive points in $S$, along with only those negative points of $S$ for which $|v \cdot f(x) + b'| \geq \frac{3\gamma}{4}$, and it follows that $|w \cdot x + b| \geq \frac{3\gamma}{4} - \frac{\gamma}{4} = \frac{\gamma}{2}$. As $S'$ is a separable point set with margin $\Theta(\gamma)$, we can run the Perceptron algorithm on $S'$ in time $O(dn\gamma^{-2})$, and find a $d$-dimensional halfspace $w'$ consistent with $w$ on all points at distance $\frac{\gamma}{4}$ or more from $w$. We will refer to $w'$ as the $d$-dimensional mirror of $v$. 9
We compute the $d$-dimensional mirror of every vector in $V$ for every candidate value in $B$. We then enumerate all possible $t$-polyhedra by taking intersections of all combinations of $t$ mirror halfspaces, in total time 

$$(1/\gamma)^{O(kt)} = n^{O(t\gamma^{-2}\log(1/\gamma))},$$

and choose the best one consistent with $S$. The first part of the theorem follows.

Better, we may give a greedy algorithm with a much improved runtime: First note that as the intersection of $t$ halfspaces correctly classifies all points, the best halfspace among them correctly classifies at least a $(1/t)$-fraction of the negative points with margin $\gamma$. Hence it suffices to find the $d$-dimensional mirror which is consistent with all positive points and maximizes the number of correctly classified negative points, all with margin $\gamma/4$. We choose this halfspace, remove from $S$ the correctly classified negative points, and iteratively search for the next best halfspace. After $c t \log n$ iterations (for an appropriate constant $c$), the number of remaining points is

$$n(1 - \Omega(1/t))^{ct\log n} < ne^{-\ln n} = 1,$$

and the algorithm terminates. ■

Having given an algorithm to compute $\gamma$-fat $t$-polyhedra, we can now give an efficient algorithm to learn $\gamma$-fat $t$-polyhedra. We sample $m$ points, and use the second item of Theorem 3.3 to find a $(\gamma/4)$-fat $O(t \log m)$-polyhedron consistent with the sample. By Theorem 2.2, the class of $\gamma$-fat $t$-polyhedra has fat-shattering dimension at scale $\theta = 1/16$ of order $D = O(\gamma^{-2}t \log t)$. Choosing the size of $m$ according to Theorem 2.1, we conclude:

**Theorem 3.5.** There exists an algorithm that learns $\gamma$-fat $t$-polyhedra with sample complexity

$$m = O\left(\frac{D}{\varepsilon} \log^2 \frac{1}{\varepsilon} + \log \frac{1}{\delta}\right),$$

in time $m^{O((1/\gamma^2) \log(1/\gamma))}$, where $D = O(\gamma^{-2}t \log t)$ and $\varepsilon, \delta$ are the desired accuracy and confidence levels.

### 4 Polyhedron margin and envelope

In this section, we show that the notions of margin and envelope defined in Section 2 are, in general, quite distinct. Fortunately, when confined to the unit ball $X$, one can be used to approximate the other.

Given two sets $S_1, S_2 \subseteq \mathbb{R}^d$, their Minkowski sum is given by $S_1 + S_2 = \{p + q : p \in S_1, q \in S_2\}$, and their Minkowski difference is given by $S_1 - S_2 = \{p \in \mathbb{R}^d : \{p\} + S_2 \subseteq S_1\}$. Let $B_\gamma = \{p \in \mathbb{R}^d : \|p\| \leq \gamma\}$ be a ball of radius $\gamma$ centered at the origin.

Given a polyhedron $P \in \mathbb{R}^d$ an a real number $\gamma > 0$, let

$$P^{(+\gamma)} = P + B_\gamma, \quad P^{(-\gamma)} = P - B_\gamma.$$ 

Hence, $P^{(+\gamma)}$ and $P^{(-\gamma)}$ are the results of expanding or contracting, in a certain sense, the polyhedron $P$.

Also, let $P^{[+\gamma]}$ be the result of moving each halfspace defining a facet of $P$ outwards by distance $\gamma$, and similarly, let $P^{[-\gamma]}$ be the result of moving each such halfspace inwards by distance $\gamma$. Put

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

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and choose the best one consistent with $S$. The first part of the theorem follows.

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$$m = O\left(\frac{D}{\varepsilon} \log^2 \frac{1}{\varepsilon} + \log \frac{1}{\delta}\right),$$

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### 4 Polyhedron margin and envelope

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Given a polyhedron $P \in \mathbb{R}^d$ an a real number $\gamma > 0$, let

$$P^{(+\gamma)} = P + B_\gamma, \quad P^{(-\gamma)} = P - B_\gamma.$$ 

Hence, $P^{(+\gamma)}$ and $P^{(-\gamma)}$ are the results of expanding or contracting, in a certain sense, the polyhedron $P$.

Also, let $P^{[+\gamma]}$ be the result of moving each halfspace defining a facet of $P$ outwards by distance $\gamma$, and similarly, let $P^{[-\gamma]}$ be the result of moving each such halfspace inwards by distance $\gamma$. Put
with time, each one at its own constant speed. Let $\gamma$-envelope of $P$ is given by $\partial P(\gamma) = P^{(\gamma)} \setminus P^{(-\gamma)}$, and the $\gamma$-margin of $P$ is given by $\partial P[\gamma] = P^{[\gamma]} \setminus P^{[-\gamma]}$. See Figure 1.

**Lemma 4.1.** We have $P^{(-\gamma)} = P^{[-\gamma]}$.

**Proof.** Each point in $P^{[-\gamma]}$ is at distance at least $\gamma$ from each hyperplane containing a facet of $P$, hence, it is at distance at least $\gamma$ from the boundary of $P$, so it is in $P^{(-\gamma)}$. Now, suppose for a contradiction that there exists a point $p \in P^{(-\gamma)} \setminus P^{[-\gamma]}$. This means that, on the one hand, $p$ is at distance at least $\gamma$ from every point in the boundary or the exterior of $P$, but, on the other hand, $p$ is at distance smaller than $\gamma$ from some point $q$ in some hyperplane that contains a facet of $P$. But such a point $q$ lies in the boundary or the exterior of $P$. Contradiction.

However, in the other direction we have $P^{(\gamma)} \not\subseteq P^{[\gamma]}$. Furthermore, $P^{[\gamma]}$ might contain points arbitrarily far away from $P^{(\gamma)}$ (see Figure 2, left). Moreover, the $\gamma$-margin is not monotone under set containment: There are polyhedra $Q \subseteq P$ for which $Q^{[\gamma]} \not\subseteq P^{[\gamma]}$ (see Figure 2, right).

Since the $\gamma$-margin of $P$ is not contained in the $\gamma$-envelope of $P$, we would like to find some sufficient condition under which, for some $\gamma' < \gamma$, the $\gamma'$-margin of $P$ is contained in the $\gamma$-envelope of $P$. Our solution to this problem is given in the following theorem. Recall that $\mathcal{X}$ is the unit ball in $\mathbb{R}^d$.

**Theorem 4.2.** Let $P \subseteq \mathbb{R}^d$ be a polyhedron, and let $0 < \gamma < 1$. Suppose that $P^{[-\gamma]} \cap \mathcal{X} \neq \emptyset$. Then, within $\mathcal{X}$, the $(\gamma^2/2)$-margin of $P$ is contained in the $\gamma$-envelope of $P$; meaning, $\partial P^{[\gamma^2/2]} \cap \mathcal{X} \subseteq \partial P^{(\gamma)}$.

(Without the condition $P^{[-\gamma]} \cap \mathcal{X} \neq \emptyset$, the theorem would be false. A counterexample is given by a polygon in the plane with a very sharp vertex—one with a very acute angle—inside the unit circle. In dimensions 3 and larger, counterexamples are possible even if none of the dihedral angles of the polyhedron are too small. Consider for example, in dimension 3, a polyhedron with a very sharp vertex $v$ that has an equilateral-triangle cross-section. That is, $v$ is the meeting point of three edges with very acute angles between every two edges, even though the three facets meeting at $v$ make dihedral angles close to 60 degrees.)

The proof uses the following general observation:

**Lemma 4.3.** Let $Q = Q(t)$ be an expanding polyhedron whose defining halfspaces move outwards with time, each one at its own constant speed. Let $p = p(t)$ be a point that moves in a straight line at constant speed. Suppose $t_1 < t_2 < t_3$ are such that $p(t_1) \in Q(t_1)$ and $p(t_3) \in Q(t_3)$. Then $p(t_2) \in Q(t_2)$ as well.
Proof. Otherwise, \( p \) exits one of the halfspaces and enters it again, which is impossible. \(\square\)

**Proof of Theorem 4.2** By Lemma 4.1 it suffices to show that \( P^{[+\gamma^2/2]} \cap \mathcal{X} \subseteq P^{(+\gamma)} \). Hence, let \( p \in P^{[+\gamma^2/2]} \cap \mathcal{X} \) and \( q \in P^{(-\gamma)} \cap \mathcal{X} \). Let \( s \) be the segment pq. Let \( r \) be the point in \( s \) that is at distance \( \gamma \) from \( p \). Suppose for a contradiction that \( p \notin P^{(+\gamma)} \). Then \( r \notin P \). Consider \( P = P(t) \) as a polyhedron that expands with time, as above. Let \( z = z(t) \) be a point that moves along \( s \) at constant speed, such that \( z(-\gamma) = q \) and \( z(\gamma^2/2) = p \). Since \( ||r - q|| \leq 2 \), the speed of \( s \) is at most \( 2/\gamma \). Hence, between \( t = 0 \) and \( t = \gamma^2/2 \), \( z \) moves distance at most \( \gamma \), so \( z(0) \) is already between \( r \) and \( p \). In other words, \( z \) exits \( P \) and reenters it, contradicting Lemma 4.3. \(\square\)

It follows immediately from Theorems 2.2 and 4.2 that the \( \theta \)-fat-shattering dimension of the class of \( d \)-dimensional \( t \)-polyhedra with envelope \( \gamma \) is at most

\[
O \left( t \log(t) \min \left\{ d, \frac{1}{\gamma^4} \right\} \right).
\]

Likewise, we can approximate the optimal \( t \)-polyhedron with envelope \( \gamma \) by the algorithms of Theorem 3.3 (with parameter \( \gamma' = \gamma^2/2 \)).

### 4.1 Hausdorff speed of an expanding polyhedron

The above technique also yields a result of independent interest, regarding what we call the *Hausdorff speed* of an expanding polyhedron.

Given a set \( S \subseteq \mathbb{R}^d \) and a point \( p \in \mathbb{R}^d \), let \( \delta_p(S) = \inf_{q \in S} ||p - q|| \) be the infimum of the distances between \( p \) and the points of \( S \). If \( S \) is closed and convex then there exists a unique point in \( S \) that achieves the minimum distance \( \delta_p(S) \).

Given two sets \( S_1, S_2 \subseteq \mathbb{R}^d \), the *Hausdorff distance* between them is defined as

\[
\delta(S_1, S_2) = \max \left\{ \sup_{p \in S_1} \delta_p(S_2), \sup_{p \in S_2} \delta_p(S_1) \right\}.
\]

As above, let \( P = P(t) \) be a polyhedron that expands with time, due to its bounding halfspaces moving outwards, each at its own constant speed. We define the *Hausdorff speed* of \( P \) at time \( t \) by

\[
\lim_{\gamma \to 0} \delta(P(t), P(t + \gamma))/\gamma.
\]

It is easy to see that the Hausdorff speed of \( P \) stays constant most of the time, changing only at those time moments at which \( P \) undergoes a combinatorial change (some faces of \( P \) disappear and others appear in their stead).

**Lemma 4.4.** At each combinatorial change, the Hausdorff speed of \( P \) either stays the same or decreases.

*Proof. Suppose for a contradiction that at time \( t_1 \), the Hausdorff speed of \( P \) increases from \( v_1 \) to \( v_2 \), where \( v_1 < v_2 \). Then there exists a small enough \( \gamma > 0 \) such that every point of \( P(t_1) \) is at distance at most \( \gamma v_1 \) from \( P(t_1 - \gamma) \), and such that \( P(t_1 + \gamma) \) contains a point \( p \) at distance at least \( \gamma v_2 \) from \( P(t_1) \). Let \( q \) be the point of \( P(t_1) \) that is closest to \( p \), so \( ||q - p|| = \gamma v_2 \). Let \( r \) be a point of \( P(t_1 - \gamma) \) at distance at most \( \gamma v_1 \) from \( q \). Then \( ||r - p|| \leq \gamma(v_1 + v_2) \). Hence, a moving point \( z \) that starts at \( r \) at time \( t_1 - \gamma \) and reaches \( p \) at time \( t_1 + \gamma \), moving at a constant speed along a straight line, exits \( P \) and reenters it, contradicting Lemma 4.3.* \(\square\)


5 Experimental results

Our greedy algorithmic approach motivates a simple heuristic for the construction of a consistent polyhedron: Given the \(d\)-dimensional point set \(S\) and a run-time parameter \(M\), at each iteration, we sample \(M\) \(d\)-dimensional unit vectors by sampling each coordinate from the normal \(N(0,1)\) distribution and then normalizing. Each sampled vector \(w_i\) implies a candidate direction of a \(d\)-dimensional halfspace, and we then translate the halfspace to distance \(b_i \in [0, 1]\) to the origin for the minimum \(b_i\) which places all positive points on the inside of the halfspace (that is the side of the origin). Having enumerated \(M\) halfspaces of the form \((w_i, b_i)\), we select the halfspace which maximizes the number of negative points outside the halfspace, remove these negative points from \(S\), and continue to the next iteration. The heuristic terminates when all negative points have been removed, and then outputs the chosen halfspaces.

As a proof of concept, we tested the above heuristic on randomly generated point sets and polyhedra in dimensions \(d = 2, \ldots, 20\). For each \(d\), we first randomly sampled 1000 \(d\)-dimensional data vectors from within the unit sphere: This is done by sampling each vector \(p_i\) from the unit sphere as done above, and then dividing the vector by \(u_i^{1/d}\), where \(u_i \in [0, 1]\) is sampled randomly from the uniform distribution.\(^2\) We then sampled \(d\) halfspaces whose intersection forms the target polyhedron: We sample a random direction vector \(w_j\) uniformly from the unit sphere, and then sample a random offset value \(b_j \in [.05, .95]\) to produce the halfspace \((w_j, b_j)\). The intersection of these \(d\) halfspaces gives the target (open) polyhedron. All data points inside the polyhedron with margin 0.05 were labelled as positive, all data points outside the polyhedron with margin 0.05 were labelled as negative, and the rest were discarded. We then ran the heuristic with parameter \(M = 10000\). Results are reported in Figure 3, averaged over 100 trials per dimension.

The worst results were achieved at dimension 11, where the returned polyhedra were defined by about three times more halfspaces than optimal. Thereafter the ratio decreases, and by dimension

\(^2\)It is well-known (and easy to see) that this method indeed uniformly samples from the unit ball: the standard normal distribution is spherically symmetric, while the differential volume of an infinitesimal spherical shell of radius \(r\) scales as \(r^d\) — and hence taking the \(d\)th root achieves uniformity.
15 we actually see a decrease in the number of halfspaces. We believe that this is due to known properties of high-dimensional balls: As the dimension grows, halfspaces that are offset away from the origin cut off a progressively smaller proportion of the ball’s volume, and this results in many fewer negative points and a simpler algorithmic problem.

6 Conclusions and future directions

An interesting direction for future research is efficient learning of polyhedra defined by \( k \) extremal points (as opposed to \( k \) halfspaces). Kupavskii (2020) has recently given an exponentially improved bound for the VC-dimension of these objects, but the question of learning and fat-shattering dimension are still open.

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A Reduction of linear programming to the consistent hyperplane problem

The following argument is due to Nikolov (2018), using techniques from Chvátal (2018).

In the consistent hyperplane problem, we are given two finite sets of points \( P, Q \subset \mathbb{R}^d \), and we want to find a hyperplane that strictly separates \( P \) from \( Q \). We show that, if there exists a strongly polynomial time algorithm that solves the consistent hyperplane problem for the special case \( Q = \{0\} \), then there exists a strongly polynomial time algorithm for LP. The existence of the latter is a major open problem in complexity theory, known as Smale’s 9th problem (Smale, 2000).

Recall that the problem of finding a feasible solution to an LP is equivalent to the problem of finding an optimal solution (see e.g. Matousek and Gärtner, 2006).

**Lemma A.1.** Suppose there exists a strongly polynomial time algorithm \( Z \) that, given a linear system of strict inequalities \( Ax < b \), finds a feasible solution if one exists. Then there exists a strongly polynomial time algorithm for LP (i.e. for solving a system of the form \( Ax \leq b \)).

**Proof.** Given \( Z \), we construct an algorithm \( W \) that, given an LP \( \mathcal{L} \), either finds a feasible solution for \( \mathcal{L} \) or reduces \( \mathcal{L} \) to an equivalent LP \( \mathcal{L}' \) with fewer variables and constraints. Hence, repeated application of \( W \) will produce a solution to \( \mathcal{L} \).

Let \( \mathcal{L} = \{ Ax \leq b \} \) be the given LP, with \( A \in \mathbb{Q}^{m \times n} \). Given a set of indices \( S \subseteq \{1, \ldots, m\} \), let \( \mathcal{L}(S) = \{ Ax < b : i \in S \} \) (i.e. \( \mathcal{L}(S) \) contains \( |S| \) strict inequalities and no other constraints). We say that \( S \) is maximally feasible if \( \mathcal{L}(S) \) is feasible but no \( \mathcal{L}(T) \) is feasible for \( T \supseteq S \).

It is easy to find a maximally feasible set \( S \) by making at most \( m \) queries to \( Z \). Now we claim that, if \( S \) is maximally feasible, then every solution \( x \) to the original system \( \mathcal{L} \) must have equality
\( A_j x = b_j \) for every \( j \notin S \): Indeed, suppose for a contradiction that \( A_j x < b_j \) for some \( j \notin S \). Let \( y \) be a feasible solution to \( \mathcal{L}(S) \). Then the convex combination \( \varepsilon y + (1 - \varepsilon)x \), for small enough \( \varepsilon > 0 \), is a feasible solution to \( \mathcal{L}(S \cup \{ j \}) \).

Hence, \( W \) first finds a feasible set \( S \). If \( |S| = m \), then the solution to \( \mathcal{L}(S) \) found by \( Z \) is also a solution to \( \mathcal{L} \). Otherwise, for every \( j \notin S \) we solve for a variable in the equation \( A_j x = b_j \) and substitute it into the other constraints of \( \mathcal{L} \). This way, we produce a smaller LP \( \mathcal{L}' = \{ A'x' \leq b' \} \), a solution to which yields a solution to \( \mathcal{L} \).

**Lemma A.2.** Suppose there exists a strongly polynomial time algorithm \( Y \) that, given a set of points \( P \), finds a hyperplane that strictly separates \( P \) from \( Q = \{0\} \), if one exists. Then there exists a strongly polynomial time algorithm \( Z \) as in Lemma A.1.

**Proof.** Given a strict system \( Ax < b \) with \( A \in \mathbb{Q}^{m \times n} \), we query \( Y \) with the point set \( P = \{(-A_i, b_i) : 1 \leq i \leq m\} \cup \{(0, 1)\} \). \( Y \) will return \( w = (x', t) \) such that \( p \cdot w > 0 \) for all \( p \in P \). In particular, \( t > 0 \). Hence, \( x = x'/t \) is a solution to our system. \( \square \)