THE YANG-MILLS FLOW NEAR THE BOUNDARY
OF TEICHMÜLLER SPACE

GEORGIOS D. DASKALOPOULOS AND RICHARD A. WENTWORTH

Abstract. We study the behavior of the Yang-Mills flow for unitary connections on compact
and non-compact oriented surfaces with varying metrics. The flow can be used to define a one
dimensional foliation on the space of SU(2) representations of a once punctured surface. This
foliation universalizes over Teichmüller space and is equivariant with respect to the action of the
mapping class group. It is shown how to extend the foliation as a singular foliation over the Strebel
boundary of Teichmüller space, and continuity of this extension is the main result of the paper.

1. Introduction

The Morse theory of the Yang-Mills functional on the space of gauge equivalence classes of
unitary connections on a hermitian vector bundle over a Riemann surface was introduced in the
seminal paper of Atiyah and Bott [AB]. Further properties of the gradient flow of the functional
were obtained in [D] and [R]. The minimal critical set can be identified on the one hand with
conjugacy classes of (projective) unitary representations of the fundamental group of the surface
via the holonomy map, and on the other hand with the moduli space of semistable holomorphic
vector bundles via the theorem of Narasimhan and Seshadri (cf. [NS, Do1]), and the analysis of
[AB] shows that the Yang-Mills flow can be used effectively to study the topology of this space.

In [DW2] we studied the behavior of the moduli space of vector bundles as the conformal structure
on the Riemann surface degenerates. A natural question to pose is whether the Yang-Mills flow
itself behaves in a reasonable fashion under similar degenerations. In the case where the metric
degeneration is to a cone metric, one may regard such a description as a non-linear version of the
convergence of eigenvalues and eigenfunctions of the Laplace operator obtained in [IWJ].

The specific problem we consider is the following: let X denote a compact surface of genus g ≥ 2,
with a prescribed point p, and set X* = X \ {p}. We define
\begin{equation}
R_\alpha = \text{Hom}_\alpha (\pi_1(X^\ast), SU(2))/SU(2)
\end{equation}
where the subscript \(\alpha\) indicates that the holonomy of the representation around \(p\) is conjugate
to the diagonal matrix with entries \(e^{\pm 2\pi i \alpha}\). We take \(0 \leq \alpha \leq 1/2\). Thus, \(R_0\) is identified with
conjugacy classes of \(SU(2)\) representations of the fundamental group of the closed surface \(X\).
Fixing a conformal structure \([\sigma^\ast]\) on \((X, \{p\})\) we can define a smooth, surjective map (the “Hecke
Correspondence”; see [DDW] and below) \( \pi_{\alpha}^{[\sigma^*]} : \mathcal{R}_\alpha \to \mathcal{R}_0 \) for \( \alpha \neq 0, 1/2 \), which is an \( S^2 \)-fibration over the irreducibles. Roughly speaking, the map is defined as follows: for a flat \( SU(2) \) connection \( A \) with holonomy \( \alpha \) about \( p \), we act on \( A \) by a singular complex gauge transformation \( g \) to bring the connection \( A_0 = g(A) \) into the standard, trivial form \( d \) on a neighborhood of \( p \). Thus, \( A_0 \) may be regarded as a connection on the closed surface \( X \) which is, however, no longer flat. The map is then defined by using the Yang-Mills flow to obtain from \( A_0 \) a flat connection \( \pi_{\alpha}^{[\sigma^*]}([A]) \). This definition clearly involves the choice of conformal structure in an important way, and understanding this behavior is the motivation for this paper.

The map \( \pi_{\alpha}^{[\sigma^*]} \) can be generalized to a map \( \pi_{\alpha}^{[\sigma^*]} : \mathcal{R}_\alpha \to \mathcal{R}_\beta \) for any \( 0 \leq \beta \leq \alpha < 1/2 \), and this will be a homeomorphism for \( \beta \neq 0 \) (see Sec. 2.3). We obtain from this a 1-dimensional foliation \( \mathcal{F}^{[\sigma^*]} \) of \( \mathcal{R} = \bigcup_{0 < \alpha < 1/2} \mathcal{R}_\alpha \). A similar question to the one posed above is the dependence of this foliation on \( [\sigma^*] \). For example, while it may be intuitively clear that \( \mathcal{F}^{[\sigma^*]} \) varies continuously with \( [\sigma^*] \), a differentiable structure is less obvious. Our first result is thus an explicit determination of variational formulas governing \( \mathcal{F}^{[\sigma^*]} \). As a consequence, we prove that the foliation is \( C^1 \). The second result is a description of the behavior when the conformal structure degenerates. Here we show that for a dense set of boundary points on Teichmüller space, the foliations actually converge away from some singularities.

More precisely, let \( \mathcal{T}_{\text{aug.}}(g, 1) \) denote the augmented Teichmüller space (cf. [A]) obtained by adding nodal Riemann surfaces. These are obtained by collapsing a collection \( \Phi \) of disjoint simple closed boundary incompressible curves on \( X^* \). In the topology of \( \mathcal{T}_{\text{aug.}}(g, 1) \), nodal surfaces may be approached by the “pinching” degeneration familiar from the Deligne-Mumford compactification of the moduli of curves. Thus, there is a family of conformal structures \( \sigma^*(\ell) \) on \( X^* \) degenerating as \( \ell \to 0 \) to a conformal structure \( \sigma^*(0) \) on \( X^* \setminus \Phi \), where a tubular neighborhood of each curve in \( c \in \Phi \) is conformal to the coordinate axes in \( \mathbb{C}^2 \), minus the origin. The collapsed curve \( c \), corresponding to the origin, is called the node (see Sec. 4.2 for more details). To see the effect of pinching on the representation variety, we first make the following:

**Definition 1.1.** Given a system \( \Phi \) of disjoint simple closed boundary incompressible curves on \( X^* \) and an irreducible representation \( \rho : \pi_1(X^*) \to SU(2) \), we say that \( \rho \) is accidentally reducible with respect to \( \Phi \) if the restriction of \( \rho \) to the fundamental group of any component of \( X^* \setminus \Phi \) is reducible. We denote by \( \mathcal{R}^{\Phi} \subset \mathcal{R} \) the closed subspace of conjugacy classes of accidentally reducible representations.

In Sec. 3 we prove the:

**Main Theorem.** For each equivalence class \( \sigma^*(0) \) of nodal conformal structures on \( X^* \setminus \Phi \) there exists a smooth 1-dimensional foliation \( \mathcal{F}^{[\sigma^*(0)]} \subset \mathcal{R} \setminus \mathcal{R}^{\Phi} \) such that for all paths \( [\sigma^*(\ell)] \to [\sigma^*(0)] \) in \( \mathcal{T}_{\text{aug.}}(g, 1) \) as above, \( \mathcal{F}^{[\sigma^*(\ell)]} \to \mathcal{F}^{[\sigma^*(0)]} \) on \( \mathcal{R} \setminus \mathcal{R}^{\Phi} \) in the Hausdorff sense.

This result is in contrast to the algebraic situation: for each conformal structure \( [\sigma^*] \) one can identify the space \( \mathcal{R}_\alpha \) with the moduli space of parabolic stable bundles \( \mathcal{M}_\alpha^{[\sigma^*]} \) (cf. [MS]). This
universalizes over $\mathcal{T}(g,1)$ to define a holomorphic fibration $\widetilde{M}_\alpha$. Furthermore, the Mehta-Seshadri Theorem defines an identification of $\widetilde{M}_\alpha$ with the trivial fibration $\tilde{R}_\alpha = \mathcal{T}(g,1) \times R_\alpha$. In this setting, the map $\pi_\alpha^{(\sigma^*)} : M_\alpha^{(\sigma^*)} \rightarrow M$ is the elementary transformation at $p$; for $0 < \beta < \alpha < 1/2$, $\pi_\alpha^{(\sigma^*)}$ is simply the identity. Using algebro-geometric methods, it is possible to compactify $\tilde{M}_\alpha$ and $\tilde{M}$ over the Deligne-Mumford compactification of $M(g,1)$ by adding the appropriate moduli space of torsion-free sheaves on nodal curves. Furthermore, one can show that the maps $\pi_\alpha^{(\sigma^*)}$ extend holomorphically over the compactification. This is similar in spirit to the degeneration used in [DW2]. But the map defined in the Main Theorem differs significantly from this algebraic compactification in the sense that there does not appear to be a version of the Narasimhan-Seshadri theorem relating the moduli space of torsion-free sheaves on a nodal curve to the subspace of $R_\alpha$ mentioned above.

We make two further comments on the proof: (i) In the case where the Yang-Mills flow starting from a connection $[A]$ converges to an irreducible flat connection $[B]$, one can find a complex gauge transformation $g$ such that $[B] = g[A]$. This is not true, however, if $[B]$ is reducible. As $[B]$ moves closer to the reducibles one expects to lose control of the $C^0$ bound on $g$. Reducibility of $[B]$ can be detected via the existence of a kernel for the associated Laplace operator $\Delta_B$ acting on the traceless, skew-hermitian endomorphisms of $E$. As $[B]$ approaches the reducibles the first eigenvalue of this operator goes to zero, and one might therefore look for an explicit $C^0$ bound on $g$ which depends on $[B]$ only through the first eigenvalue of $\Delta_B$. Such an estimate exists and will be used in Sec. 5. As the surface degenerates to a cone metric we have convergence of eigenvalues. The $C^0$ bound on $g$ mentioned in the previous paragraph will persist, provided that the limiting eigenvalues are non-zero. This is the motivation for the definition of accidental reducibles (Def. 1.1).

(ii) By a theorem of Wolpert [Wo] the Strebel boundary is at finite Weil-Petersson distance from the base point $[\sigma_0]$ in Teichmüller space. From the structure of the first variation formula and the bounds mentioned above, one expects $\pi_\alpha^{(\sigma^*)(\ell)}[A]$ to have finite length, and hence to converge, for paths $[\sigma^*(\ell)] \rightarrow [\sigma^*(0)]$. For other points in the boundary we have no such conclusion, as they are at infinite Weil-Petersson distance. Though our proof will be along more direct lines, this is the basic explanation for the convergence in the Main Theorem, and for why an extension of $\pi_\alpha^{(\sigma^*)}$ to the full Thurston boundary remains a difficult question. An interesting technical point in the proof below, however, is a restriction on the cone angle of the degenerate metric required to prove the convergence.

This paper is organized as follows: in Sec. 2 we review Donaldson’s approach to the Yang-Mills flow and Råde’s version of L. Simon’s estimate. We discuss Simpson’s flow for singular metrics, and we show that the flow at infinity of a singular connection preserves the conjugacy class of the holonomy about the singularity (see Cor. 2.4). We also interpret this construction by passing to branched covers. All this permits a definition of the foliation $\mathcal{F}^{(\sigma^*)}$. Sec. 3 contains the proof of the first variational formula Thm. 3.1 for the action of the complex gauge group. In particular, we show that the first variation for a path $[g_\epsilon A_\epsilon]$ of gauge equivalence classes of flat connections
is independent of the derivative $\dot{g}$. This may be regarded as a kind of analogue of Ahlfors’ result for the first variation of the hyperbolic area element for quasi-conformal maps. In Sec. 4 we prove estimates for the eigenvalue problems for sections of vector bundles and degenerations to cone metrics. Finally, the Main Theorem is proved in Sec. 5.

A word concerning notation: if $\sigma$ is a Riemannian metric on a surface $X$, then integrals over $X$, unless otherwise specified, will be assumed to be taken with respect to $\sigma$. If $\sigma$ is a conformal metric and $z$ a conformal coordinate, we sometimes write $\sigma(z)|dz|^2$ for the area form. If $f$ is a function on $X$, or more generally a section of a vector bundle $V$ with a fiber metric $H$, and $\alpha \geq 1$, we set

$$\|f\|_\alpha = \left\{ \int_X |f|^\alpha \right\}^{1/\alpha}$$

In the above, if $f$ is a section, then $|f| = |f|_H$ involves the fiber metric $H$. When we want to emphasize the choice of metrics, we will write $\|f\|_{\alpha;\sigma}$, or $\|f\|_{\alpha;\sigma,H}$, or perhaps even $\|f\|_{\alpha;H}$ when there is no risk of confusion of the two metrics. Other Sobolev norms will be denoted with an explicit subscript, e.g. $\|f\|_{L^2_\sigma}$, with $\|f\|_\infty$ (resp. $\|f\|_{\infty,H}$) for the $L^\infty$ norm of $|f|$ (resp. $|f|_H$).

Finally, we also use the following abbreviation for $2 \times 2$ diagonal matrices:

$$\text{diag}(\lambda_1, \lambda_2) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$ 

Acknowledgements. We would like to thank Peter Li and Karen Uhlenbeck for discussions, and the Max-Planck Institute in Bonn and the MSRI for their generous hospitality. Thanks also to G. Berger for suggesting Lemma 2.7.

2. Gauge Theory

2.1. Review of the Yang-Mills Flow. Let $E$ be a rank 2 hermitian vector bundle on a closed, compact surface $X$. The Yang-Mills flow is given by the equation:

$$\frac{dA(t)}{dt} + D_{A(t)}^* F_{A(t)} = 0,$$

where $A(t)$ is a time dependent connection, $F_A$ is the curvature, and $D_A^* = - \ast D_A \ast$ is the $L^2$ formal adjoint of covariant differentiation $D_A$ with respect to the connection $A$. Eq. (2.1) depends on a choice of a Riemannian metric on $X$ through the Hodge $\ast$, and this dependence will be the main theme of the paper.

Let $\mathcal{A}$ denote the space of unitary connections on $E$ and $\mathcal{G}$ the group of unitary gauge transformations. Donaldson proved in [Do2] that the Yang-Mills flow equations (2.1) with initial condition:

$$A(0) = A_0,$$

have a unique solution for all time in $\mathcal{A}/\mathcal{G}$. Subsequently, Råde was able to prove that the initial value problem (2.1), (2.2) has a unique solution for all time in $\mathcal{A}$ before we mod out by the gauge group. We will not need Råde’s existence result in this paper, although some of his estimates on the asymptotics of the flow as $t \to \infty$ will be important for our arguments.
Eq. (2.1) is the $L^2$-gradient flow for the Yang-Mills functional $YM(A) = \|F_A\|_2^2$, and for a solution $A(t)$ of (2.1),

$$
\frac{d}{dt}YM(A(t)) = -\nabla YM(A(t)) = -\|D_A^*F_A\|_2^2.
$$

(2.3)

The critical points of $YM(A)$ are solutions of the Yang-Mills equations $D_A^*F_A = 0$, and the minima are given by the Hermitian-Yang-Mills (or projectively flat) equation $*F_A = \mu I$ for a constant $\mu$.

Donaldson’s approach to solving eq.’s (2.1)-(2.2) up to gauge is to solve instead the non-linear heat equation:

$$
H^{-1}(t)\frac{dH(t)}{dt} = -\sqrt{-1} * F_{\bar{\partial}E,H(t)},
$$

(2.4)

for a family of metrics $H(t)$ with initial conditions:

$$
H(0) = H_0.
$$

(2.5)

In the above, $F_{\bar{\partial}E,H(t)}$ denotes the curvature of the unique connection compatible with holomorphic structure $\bar{\partial}E$ on $E$ and unitary with respect to the metric $H(t)$.

The two systems yield the same solution in $A/G$, for if $A(t) = g(t)A_0$ is a solution of (2.1), then $H(t) = g(t)g(t)^*H_0$ is a solution to (2.4); conversely, if $H(t) = h(t)H_0$ is a solution to (2.4), then $A(t) = h^{1/2}(t)A_0$ is real gauge equivalent to a solution of (2.1). See [Do2] for more details. Also notice that it is easy to factor out the trace part of the connection and gauge transformations in eq.’s (2.1)-(2.2) and (2.4)-(2.5). Therefore, in the following we shall assume that the solutions to the above equations all preserve determinants.

We now review the analogue of L. Simon’s result, due to Råde, concerning the asymptotics of the solutions to (2.1)-(2.2):

**Proposition 2.1** ([R, Prop. 7.2]). Let $A_\infty$ be an irreducible Yang-Mills connection. There exist constants $\varepsilon_1, c > 0$ such that for any $A$ satisfying $\|A - A_\infty\|_{L^4} \leq \varepsilon_1$, we have:

$$
\|D_A^*F_A\|_2 \geq c |YM(A) - YM(A_\infty)|^{1/2}.
$$

(2.6)

Let $\text{ad} E$ denote the bundle of skew-hermitian endomorphisms of $E$, and let $(\text{ad} E)_0$ denote the subbundle of traceless ones. Then the constants $\varepsilon_1, c$ depend only on the first eigenvalue of the Laplacian associated to $A_\infty$ acting on sections of $(\text{ad} E)_0$ and on the constant governing the inclusion $L^2 \hookrightarrow L^4$ (notice that by Kato’s inequality this is essentially the Sobolev constant $s_1$ for functions – see (4.2) below).

The initial value problem (2.4)-(2.5) has solutions over non-compact surfaces as well. More precisely, let $X$ be a compact Riemann surface as before, choose “punctures” $p_1, \ldots, p_k \in X$, and let $X' = X \setminus \{p_1, \ldots, p_k\}$. Fix a metric on $X'$ whose expression in terms of a conformal coordinate $z$ on $X$ centered at any one of the $p_i$’s has the form $ds^2 = \sigma(z)|dz|^2$, with:

$$
\int_X |\sigma(z)|^p|dz|^2 < \infty,
$$

(2.7)
for some $p > 1$ (cf. [Si1], Prop. 2.4). Let $E$ be a holomorphic bundle on $X$. Given a hermitian metric $H_0$ on $E$ with $\| * F_{\varphi_{E,H_0}} \|_\infty < \infty$, we define:

$$\text{deg}(E,H_0) = \sqrt{-1} \int_X \text{Tr}(*F_{\varphi_{E,H_0}}).$$

Then $E$ is said to be $H_0$-stable (resp. semistable) if for any proper holomorphic subbundle $F$ of $E$ we have:

$$\frac{\text{deg}(F,H_0)}{\text{rk} F} < \frac{\text{deg}(E,H_0)}{\text{rk} E} \quad (\text{resp.} \leq).$$

Simpson proved the following:

**Theorem 2.2** (cf. [Si1], Prop. 6.6 and the proof of Thm. 1). (i) Given a holomorphic vector bundle $E$ on $X'$ with hermitian metric $H_0$ such that $\| * F_{\varphi_{E,H_0}} \|_{\infty,H_0} < \infty$, there exists a unique solution $H(t) = h(t)H_0$ to (2.4)-(2.5) with constant determinant and having the property that for any finite $T > 0$,

$$(2.8) \quad \sup_{T \geq 0} \| h(t) \|_{\infty,H_0} < \infty.$$

(ii) If, in addition, $E$ is $H_0$-stable, then (2.8) holds uniformly for all $T$. Furthermore, $H(t) = h(t)H_0$ converges weakly in $L^p_p$ to a solution $H(\infty) = h(\infty)H_0$ of the Hermitian-Yang-Mills equation with $\| h(\infty) \|_{\infty,H_0} < \infty$.

This result has important consequences which we will need in the proof of our main theorem: given a holomorphic bundle $E \to X^*$ as above and a puncture $p \in X \backslash X^*$, choose a local holomorphic coordinate $z : U \to \Delta$ centered at $p$ and defined on a neighborhood $U$, and a local holomorphic frame $\{ f_1, f_2 \}$ over $U$ such that $H_0$ is in the standard form $\text{diag}(|z|^{2\alpha}, |z|^{-2\alpha})$. The identification is chosen so that a unitary frame $\{ e_1, e_2 \}$ is given by $e_1 = |z|^{-\alpha} f_1$, $e_2 = |z|^{\alpha} f_2$, and the hermitian connection $D_0$ with respect to this frame is in the form $D_0 = d + \text{diag}(ia, -ia) d\theta$.

**Proposition 2.3.** The gauge transformation $h(\infty)$ from Thm. 2.2, Part (ii), is independent of the choice of conformal metric satisfying (2.7). Furthermore, if $\alpha < 1/2$ then $h(\infty)$ extends continuously at $p$, and $h(\infty)(p)$ is diagonal with respect to the frame $\{ e_1, e_2 \}$.

**Proof.** For the first statement, note that the action of the complex gauge group $G^C$ on $A$ is independent of the conformal factor. Therefore, the $C^0$ bound from Thm. 2.2, (ii), and the argument in [Do1] prove uniqueness. To show that $h(\infty)$ extends if $\alpha < 1/2$, denote by $D_0$ the hermitian connection on $E$ associated to $H_0$. Let $g_1$ be a singular gauge transformation of the form $g_1 = \text{diag}(|z|^{-\alpha}, |z|^{\alpha})$ near $p$. Then $g_1(D_0) = d$ near $p$. Furthermore, there exists $0 \leq \beta \leq 1/2$ and a real gauge transformation $\ell$ with det $\ell = 1$ such that if $g_2 = \text{diag}(|z|^{-\beta}, |z|^{\beta})$ near $p$, then $g_2 \ell h^{1/2}(\infty)(D_0) = d$ (cf. [DW1], Lemma 2.7). It follows that $g_2 \ell h^{1/2}(\infty)g_1^{-1}(d) = d$, hence $g_2 \ell h^{1/2}(\infty)g_1^{-1}$ is holomorphic on the punctured disk $\Delta^*$. On the other hand, since $|\alpha + \beta| < 1$, this matrix cannot have a pole at $p$, and it therefore extends continuously. Thus, we may write:

$$g_2 \ell h^{1/2}(\infty)g_1^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$
where \(a, b, c, d\) are holomorphic in \(\Delta\). Then:

\[
\ell h^{1/2}(\infty) = g_2^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} g_1 = \begin{pmatrix} |z|^{\beta - \alpha} a & |z|^\alpha b \\ |z|^{-\alpha} c & |z|^{-\beta} d \end{pmatrix}.
\]

Suppose that \(\alpha < \beta\). Since the entries of the last matrix are bounded, we must have \(c(0) = 0\) and \(d(0) = 0\), which contradicts \(\det h^{1/2}(\infty) = 1\) (as mentioned above we always fix the determinant by projecting away the trace part of the connection; see \([\text{Si1}]\)). A similar argument holds for \(\alpha > \beta\).

Hence, \(\alpha = \beta\). Finally, since \(b\) and \(c\) are holomorphic, \(c(0) = 0\), and \(\ell h^{1/2}(\infty)\) is diagonal at \(p\). In particular, \((\ell h^{1/2}(\infty))^*(\ell h^{1/2}(\infty))(p) = h(\infty)(p)\) is diagonal at \(p\).

**Corollary 2.4.** If \(E\) is a holomorphic bundle on \(X^*\) which is \(H_0\)-stable, then the Yang-Mills flow at infinity preserves the conjugacy class of the holonomy around the punctures.

**Proof.** The case \(0 < \alpha < 1/2\) follows from the proof of Prop. 2.3 above. For \(\alpha = 0\), the metric \(H_0\) is smooth at \(p\) and the flow (2.4)-(2.5) extends smoothly on \(X\). It follows that the holonomy of the limit remains trivial. The case \(\alpha = 1/2\) also follows from the proof of Prop. 2.3, for if \(\beta < 1/2\) then we again have \(|\alpha + \beta| < 1\), and the same argument gives a contradiction.

**Remark 2.5.** It follows by the arguments in \([\text{Si1}]\) that if \(H_0\) is of the form \(\text{diag}(|z|^{2\alpha}, |z|^{-2\alpha})\) with respect to a holomorphic frame \(\{f_1, f_2\}\) near \(p\), then \(H_0\)-stability coincides with Seshadri’s parabolic stability with respect to the weights \(\{\alpha, -\alpha\}\). For more details we refer to \([\text{DW1}, \text{DW2}]\).

### 2.2. Representation Varieties and Branched Covers.

Let \(X\) be a compact surface, \(p \in X\), and \(X^* = X \setminus \{p\}\). We denote by \(\mathcal{R}(X^*)\) the space of conjugacy classes of \(SU(2)\) representations of the free group \(\pi_1(X^*)\). We may also identify \(\mathcal{R}(X^*)\) with the space of gauge equivalence classes of flat connections on a trivial rank two hermitian bundle \(E\) on \(X^*\). Given a real number \(0 \leq \alpha \leq 1/2\), we denote by \(\mathcal{R}_\alpha\) the subspace of flat connections on \(E\) with holonomy matrix conjugate to \(\exp \{2\pi i \text{diag}(\alpha, -\alpha)\}\) around the puncture \(p\). Then \(\mathcal{R}_0\) is naturally identified with the space \(\mathcal{R}(X)\) of equivalence classes of flat connections on the trivial bundle over the compact surface \(X\). Notice that for \(\alpha \neq 0\), \(\mathcal{R}_\alpha\) consists entirely of irreducible representations. For future reference, we let \(\mathcal{R}(X)_{\text{irr}} \subset \mathcal{R}(X)\) denote the open set of irreducible representations of \(\pi_1(X)\).

In this section, we will sketch how to reduce certain analytical questions for the Yang-Mills flow on \(E\) over \(X^*\) to the flow on a bundle over a branched cover of the compact surface \(X\). Let \(\beta = k/n\) where \(k, n\) are positive coprime integers. Consider a regular, cyclic, \(n\)-fold, holomorphic branched cover \(\hat{X}\) of \(X\) with \(p\) in the ramification divisor \(B\). Let \(q : \hat{X} \to X\) be the covering map, \(\hat{p} = q^{-1}(p)\), \(\hat{B} = q^{-1}(B)\), \(\hat{X}^* = \hat{X} \setminus \{\hat{p}\}\), \(\hat{U} = q^{-1}(U)\), and \(\hat{U}^* = \hat{U} \cap \hat{X}^*\). Let \(E = X^* \times \mathbb{C}^2\) be the trivial rank 2 vector bundle on \(X^*\). We construct a bundle \(\hat{E}\) over \(\hat{X}\) by gluing \(q^*(E)\) on \(\hat{X} \setminus \hat{U}\) with \(\hat{U} \times \mathbb{C}^2\) via the gauge transformation:

\[
\hat{s} : \hat{U}^* \to SU(2) : \hat{s}(w) = \hat{s}(\hat{\varphi}, \hat{\theta}) = \begin{pmatrix} e^{-ik\theta} & 0 \\
0 & e^{ik\theta} \end{pmatrix}.
\]
Since \( \det \hat{s} = 1 \), it follows that \( \deg \hat{E} = 0 \); hence, \( \hat{E} \) is isomorphic to the trivial bundle. Although there is no natural global trivialization, there is a trivialization of \( \hat{E}|_{\hat{X}^*} = q^*E|_{\hat{X}^*} \) induced by the one on \( E \).

On \( E \) we fix the trivial hermitian metric \( H_0 \). Also, since \( \hat{s} \) is unitary, the trivial metrics on \( q^*E \) and \( U \times \mathbb{C}^2 \) glue together to define a hermitian metric \( \hat{H}_0 \) on \( \hat{E} \). Let \( \mathcal{A}_\beta \) denote the space of unitary connections of \( E \) on \( X^* \), flat in a neighborhood of \( p \) with holonomy conjugate to \( \exp \{ 2\pi i \text{diag}(\beta, -\beta) \} \). Given \( A \in \mathcal{A}_\beta \), choose a real gauge transformation \( g \) such that:

\[
g(A)|_{U^i} = d + \begin{pmatrix} i\beta & 0 \\ 0 & -i\beta \end{pmatrix} d\theta.
\]

It follows that:

\[
q^*g(A)|_{U^*} = d + \begin{pmatrix} ik & 0 \\ 0 & -ik \end{pmatrix} d\hat{\theta}.
\]

By gluing \( q^*g(A) \) with the trivial connection \( d\hat{t} \) via \( \hat{s} \), we obtain a unitary connection \( \hat{A} \) on \( \hat{E} \). Let \( \hat{A} \) denote the space of unitary connections on \( \hat{E} \). Define:

\[
\hat{q}: \mathcal{A}_\beta \to \hat{A}, \quad \hat{q}(A) = q^*(g^{-1})\hat{A},
\]

where in the above, \( q^*(g) = g \circ q \). It is easily checked that \( \hat{q} \) is well-defined and real gauge equivariant. In particular, it induces a map \( \hat{q}: \mathcal{R}_\beta \to \hat{\mathcal{R}} = \mathcal{R}(\hat{X}) = \hat{A}_{flat}/\hat{S} \), where \( \hat{A}_{flat} \) are the flat connections on \( \hat{E} \) and \( \hat{S} \) is the real gauge group. We note the following:

**Proposition 2.6.** If \( \beta = k/n \) with \( n \) odd, then \( \hat{q}(\mathcal{R}_\beta) \subset \hat{\mathcal{R}}_{\text{irr}} \). Furthermore, given a collection \( \Phi \) of simple closed boundary incompressible curves in \( X^* \), mutually disjoint and disjoint from \( B \), \( \hat{\Phi} = q^{-1}(\Phi) \), and \([A] \in \mathcal{R}_\beta \) which is not accidentally reducible with respect to \( \Phi \) (see Def. [1]), then \( \hat{q}([A]) \) is not accidentally reducible with respect to \( \hat{\Phi} \).

**Proof.** We first prove:

**Lemma 2.7.** Consider an exact sequence of groups \( 1 \to H \to G \to Q \to 1 \), and suppose that \( Q \) is abelian with no index 2 subgroup. Then the restriction to \( H \) of any irreducible \( SU(2) \) representation of \( G \) is also irreducible.

**Proof.** Suppose, to the contrary, that there is an irreducible \( \rho: G \to SU(2) \) which is reducible on \( H \). Then there is a maximal torus \( T \) in \( SU(2) \) such that the image of \( H \) lies in \( T \). Since \( Q \) is abelian and \( \rho \) is irreducible, the image of \( H \) cannot be contained entirely in the center of \( SU(2) \). This implies that the image of \( G \) lies in \( N(T) \), the normalizer of \( T \). Now we have the exact sequence for the Weyl group \( 1 \to T \to N(T) \to \mathbb{Z}/2\mathbb{Z} \to 1 \). Since the image of \( G \) is not contained in \( T \), we must have a surjection \( Q \to \mathbb{Z}/2\mathbb{Z} \to 1 \); contradiction.

Continuing with the proof of Prop. 2.4, as remarked above, a point \([A] \in \mathcal{R}_\beta \) gives a conjugacy class of irreducible representations of \( \pi_1(X^*) \) which, in turn, induce irreducible representations of \( \pi_1(X \setminus B) \). Now \( \hat{q}[A] \) is irreducible as a representation of \( \pi_1(\hat{X}) \) if and only if it induces an irreducible representation of \( \pi_1(\hat{X} \setminus \hat{B}) \). The first statement then follows from Lemma 2.7 by
setting \( H = \pi_1(\hat{X} \setminus \hat{B}) \), \( G = \pi_1(X \setminus B) \), and \( Q = \mathbb{Z}/n\mathbb{Z} \). For the second statement, consider a connected component \( Y \) of \( X \setminus \Phi \). If \( Y \cap B \neq \emptyset \), then \( \hat{Y} = q^{-1}(Y) \) is a connected component of \( \hat{X} \setminus \hat{\Phi} \), and the argument is as above. If \( Y \cap B = \emptyset \), then for each connected component \( \hat{Y}_i \) of \( q^{-1}(Y) \), \( \pi_1(\hat{Y}_i) \simeq \pi_1(Y) \), so \( \hat{q}[A] \) is irreducible there as well.

Now consider the effect of eq.’s (2.4)-(2.5) under \( \hat{q} \). First we choose a conformal metric \( \sigma \) on \( X \setminus B \) with the following property: for every \( b \in B \), express the map \( q \) locally as \( z = w^\alpha \) for coordinates \( z \) on centered at \( b \) and \( w \) centered at \( \hat{b} \). Then we assume that \( \sigma(z) = 1/\alpha^2|z|^{2(1-1/n)} \). Notice that such a metric satisfies condition (2.7). This is an example of a cone metric (see (4.1)). Let \( \hat{\sigma} \) be the pull-back metric on \( \hat{X} \setminus \hat{B} \). Then the condition on \( \sigma \) implies that \( \hat{\sigma} \) extends to a smooth conformal metric on \( \hat{X} \).

Next, fix a connection \( A_0 \) on \( E \) over \( X^* \) and let \( \hat{A}_0 = \hat{q}(A_0) \). Let \( H(t) = \hat{h}(t)H_0 \) be a solution of (2.4)-(2.5) on \( E \) over \( X \setminus B \), where the holomorphic structure on \( E \) is defined by \( A_0^{0,1} \). Then \( \hat{H}(t) = q^*H(t) = \hat{h}(t)\hat{H}_0 \) is a solution of the same equations on \( \hat{X} \setminus \hat{B} \) with respect to the holomorphic structure \( q^*A_0^{0,1} \). Since \( q \) is smooth and \( h(t) \) satisfies the estimate (2.8), the same is true for \( \hat{h}(t) \). Hence, by the uniqueness properties of the flow, we obtain

**Lemma 2.8.** The restriction of the flow \( \hat{H}(t) \) to \( \hat{X} \setminus \hat{B} \) coincides with \( q^*H(t) \).

This allows us to reduce estimates of \( h(t) \) over \( X^* \) to estimates of \( \hat{h}(t) \) over \( \hat{X} \). For example:

**Lemma 2.9.** Let \( \Omega \) be an open set with compact closure in \( X \setminus B \). Then there exists a constant \( C = C(\Omega) \) such that:

\[
\|h(t)\|_{L^k_\alpha(\Omega)} \leq C \|\hat{h}(t)\|_{L^k_\alpha(\hat{\Omega})}.
\]

In the above, \( h(t) \) is the solution to (2.4)-(2.5) on \( X^* \), and \( \hat{h}(t) \) is the solution over \( \hat{X} \).

### 2.3. Definition of the Foliation.

Let \( X, X^* \) be as above. Given \( 0 \leq \beta < \alpha < 1/2, \) or \( 0 < \alpha < \beta \leq 1/2, \) and \( [\sigma^*] \in T(g, 1) \), we will now give a rigorous definition of the twist maps \( \pi_{\alpha\beta}^{[\sigma^*]} : \mathcal{R}_\alpha \to \mathcal{R}_\beta \). The conformal structure on \( X^* \) extends to \( X \). Let \( \sigma \) denote a choice of smooth metric on \( X \) compatible with this conformal structure. Let \( z : U \to \Delta \) be a holomorphic coordinate centered at \( p \), and let \( \varphi_1 \) be a smooth cut-off function supported in \( \Delta_{1/3} \) and identically equal to 1 on \( \Delta_{1/6} \). We define a singular complex gauge transformation of \( E \) by setting:

\[
(2.9) \quad g_{\alpha\beta}(x) = \exp \left\{ \varphi_1(z(x)) \begin{pmatrix} (\alpha - \beta) \log |z(x)| & 0 \\ 0 & (\beta - \alpha) \log |z(x)| \end{pmatrix} \right\},
\]

for \( x \in U \) and extending by the identity elsewhere. We now come to the definition of \( \pi_{\alpha\beta}^{[\sigma^*]} : \)

**Definition 2.10.** Given \( [A] \in \mathcal{R}_\alpha \), choose a representative \( A \) in the standard form \( \text{diag}(i\alpha, -i\alpha)d\theta \) when pulled back to \( \Delta \) via \( z \). Then \( \tilde{A} = g_{\alpha\beta}(A) \) will be of the form \( \text{diag}(i\beta, -i\beta)d\theta \) over \( \Delta_{1/6} \). Furthermore, it is easy to check (cf. [MS, DW]) that the holomorphic structure induced by \( \tilde{A} \) is parabolic stable, and hence by Remark 2.3 we can flow the connection \( \tilde{A} \) with respect to the metric.
σ at infinite time. Thm. 2.2 guarantees that $\tilde{A}$ flows to a flat connection $\tilde{A}(\infty)$ which, by Cor. 2.4 has holonomy conjugate to $\exp\{2\pi i \text{diag}(\beta, -\beta)\}$. We set:

$$\pi_{\alpha\beta}^{[\sigma^*]}[A] = [\tilde{A}(\infty)],$$

the class of $\tilde{A}(\infty)$ in $R_\beta$.

It is straightforward to show that the definition of $\pi_{\alpha\beta}^{[\sigma^*]}$ is independent of the choices made, i.e. the coordinate $z$, the cutoff function $\varphi_1$, the lift of the conformal structure, and the lift of $[A]$. In other words, the only dependence of $\pi_{\alpha\beta}^{[\sigma^*]}$ is through the class $[\sigma^*] \in T(g, 1)$. Set $R = \bigcup_{0 < \alpha < 1/2} R_\alpha \subset R(X^*)$. The next two lemmas are left to the reader:

**Lemma 2.11.** For $0 \leq \gamma < \beta < \alpha < 1/2$ or $0 < \alpha < \beta < \gamma \leq 1/2$ and $[\sigma^*] \in T(g, 1)$, $\pi_{\alpha\gamma}^{[\sigma^*]} = \pi_{\beta\gamma}^{[\sigma^*]} \circ \pi_{\alpha\beta}^{[\sigma^*]}$.

**Lemma 2.12.** Given $[\sigma^*] \in T(g, 1)$, $\alpha \in (0, 1/2)$, and $[A] \in R_\alpha$, the map $[0, 1/2] \rightarrow R : \beta \mapsto \pi_{\alpha\beta}^{[\sigma^*]}[A]$ is continuous. Furthermore, its restriction to $(0, 1/2)$ is smooth.

For convenience, for $\beta = \alpha$ we define $\pi_{\alpha\beta}^{[\sigma^*]}$ to be the identity.

**Definition 2.13.** Fix $[\sigma^*] \in T(g, 1)$, $[A] \in R_\alpha$, $0 < \alpha < 1/2$, and let:

$$\mathcal{F}^{[\sigma^*]} = \bigcup_{0 < \beta < 1/2} \pi_{\alpha\beta}^{[\sigma^*]}[A], \quad \mathcal{F}^{[\sigma^*]} = \bigcup_{[A] \in R^*} \mathcal{F}^{[\sigma^*]}_{[A]}.$$

It follows from the above discussion that $\mathcal{F}^{[\sigma^*]}$ is a smooth 1-dimensional foliation of $R^* = R \setminus \text{hol}^{-1}\{0, 1/2\}$.

We now turn to the definition of the limiting foliation $\mathcal{F}^{[\sigma^*(0)]}$ discussed in the Introduction. Let $\Phi$ denote a collection of disjoint simple closed curves on $X^*$, and let $\sigma^*(0)$ be a conformal structure on the pinched surface $X_0^* = X^* \setminus \Phi$. Let us elaborate on this (see Sec. 4.2 for more details). By definition, the conformal structure on the pinched surface $X_0^*$ arises from a conformal structure $\sigma^*$ on $X^*$ as follows: for each $c \in \Phi$ there is a tubular, cylindrical neighborhood $C$ of $c$ which is conformally equivalent, with respect to $\sigma^*$, to the intersection in $\mathbb{C}^2$ of a neighborhood of the origin with the annulus $zw = \varepsilon$ for some non-zero complex number $\varepsilon$. In these coordinates, $c$ is the set where $|z| = |w| = |\varepsilon|^{1/2}$. The conformal structure $\sigma^*(0)$ then replaces $C$ with the pinched annulus $zw = 0$.

Let $R^\Phi \subset R$ be the set of $\Phi$-accidental reducibles. Given $[A] \in R_\alpha \setminus R^\Phi$, let $A$ be a lift of $[A]$ to a connection which has the standard form $d + \text{diag}(i\gamma, -i\gamma)d\theta$ in a coordinate neighborhood $z$ of each $c \in \Phi$ as described above. Of course, the holonomy $\gamma$ depends on the component $c$. By assumption, the restriction of $A$ to any component of $X_0^*$ is irreducible. Hence, for $\varepsilon([A]) > 0$ sufficiently small and $|\beta - \alpha| < \varepsilon([A])$, the holomorphic structure associated to the twists $g_{\alpha\beta}(A)$ on $X_0^*$ is parabolic stable for the choice of weights $\beta$ and $\{\gamma_c\}_{c \in \Phi}$. 


The Yang-Mills Flow

**Definition 2.14.** Set $A(0, \infty)$ to be the flow of $g_{\alpha\beta}(A)$ (at infinite time) with respect to a metric on $X_0^*$ compatible with the conformal structure and satisfying the condition (2.7). As before, the real gauge equivalence class $[A(0, \infty)]$ of $A(0, \infty)$ is independent of all the choices made.

**Theorem 2.15.** There is a well-defined lift of $[A(0, \infty)]$ to an element in $\mathcal{R}_\beta$.

**Proof.** Let $U_\pm$ be coordinate neighborhoods $z$ and $w$ corresponding to a curve $c \in \Phi$ as before, and suppose the holonomy is $\gamma$. Thus we have:

$$A|_{U_\pm} = d \pm \begin{pmatrix} i\gamma & 0 \\ 0 & -i\gamma \end{pmatrix} \, d\theta_\pm,$$

with respect to unitary frames $e_1^\pm, e_2^\pm$ of $E|_{U_\pm}$. By Cor. 2.4 there are real gauge transformations $g_\pm$ of $E|_{U_\pm}$ such that:

$$(2.11) \quad g_\pm (A(0, \infty))|_{U_\pm} = d \pm \begin{pmatrix} i\gamma & 0 \\ 0 & -i\gamma \end{pmatrix} \, d\theta_\pm.$$

The $g_\pm$ may be extended to a global real gauge transformation $g$ of $E|_{X_0^*}$ which is the identity away from a small neighborhood of $U_\pm$. By using the identification of the frame $\{e_1^\pm, e_2^\pm\}$ with $\{e_1^-, e_2^\pm\}$, (2.11) implies that the pull-back of $g(A(0, \infty))$ to $X^* \setminus c$ extends smoothly over $c$. By repeating the above for every curve $c \in \Phi$ we obtain a flat connection on $X^* \setminus \Phi$ with the correct holonomy. The resulting conjugacy class is unique, because a conjugacy class of connections on $E|_{X_0^*}$ together with gluing data determine a unique conjugacy class of connections on $E|_{X^*}$.

The construction above may be summarized as follows: the connection $A$ on $X^*$ determines a flat connection on $X^* \setminus \Phi$, along with “gluing parameters” across the curves $c \in \Phi$. The twisted connection $g_{\alpha\beta}(A)$ has a small amount of curvature in the component $Y$ of $X^* \setminus \Phi$ containing $\{p\}$, and it agrees with $A$ on all the other components. The flow $A(0, \infty)$ then runs the Simpson flow on $Y$, applied to $g_{\alpha\beta}(A)$, and leaves the connection on the other components fixed. Since the holonomies around the curves $c \in \Phi$ bounding $Y$ remain unchanged for the flow at infinite time (Cor. 2.4), we may use the original gluing parameters to reconstruct a flat connection on $X^*$.

**Definition 2.16.** For each $[A] \in \mathcal{R}_\alpha$ and $\beta$ satisfying $|\alpha - \beta| < \varepsilon([A])$ as above, define $\pi_{\alpha\beta}^{[\sigma^*(0)]}[A]$ to be the lift of $[A(0, \infty)]$ described in Thm. 2.14. Furthermore, we set:

$$\mathcal{F}^{[\sigma^*(0)]}_{[A]} = \bigcup_{\alpha-\varepsilon([A]) < \beta < \alpha + \varepsilon([A])} \pi_{\alpha\beta}^\Phi[A].$$

It is straightforward to check a composition rule as in Lemma 2.14. We may therefore define:

$$\mathcal{F}^{[\sigma^*(0)]} = \bigcup_{[A] \in \mathcal{R}} \mathcal{F}^{[\sigma^*(0)]}_{[A]}.$$

Then $\mathcal{F}^{[\sigma^*(0)]}$ is a smooth foliation of $\mathcal{R} \setminus \Phi_{\Phi}$. 

3. Differentiability of the Foliation

3.1. First Order Variational Formula. In describing the behavior of the Yang-Mills flow applied to a connection $A$ as the conformal structure $\sigma$ on a closed surface varies, there are two considerations: first, the complex gauge transformation $g$ describing the flow, i.e. such that $g(A)$ is flat, will depend in a complicated way on $\sigma$. Second, while the real gauge group acts on the space of unitary connections in a manner independent of the conformal structure, the complex gauge group does not.

Assume that we have fixed a conformal structure on $X$. Let $\mu_\epsilon$ be a differentiable family of Beltrami differentials on $X$ with $\mu_0 = 0$, $\mu_\epsilon|_{\epsilon=0} = \nu$. Let $g_\epsilon$ be a differentiable family of complex gauge transformations on $E$ with $g_0 = g$, $g_\epsilon|_{\epsilon=0} = \dot{g}$. Finally, let $A_\epsilon$ be differentiable family of unitary connections on $E$ with $A_0 = A$, $A_\epsilon|_{\epsilon=0} = \dot{A}$. Set $\gamma_\epsilon = g_\epsilon A_\epsilon$, and $\dot{\gamma} = \gamma_\epsilon|_{\epsilon=0}$. We emphasize that the action of the complex gauge group is with respect to the $\mu_\epsilon$-deformed conformal structure. Also, we regard Beltrami differentials $\nu$ as endomorphisms $\Omega^{1,0}(X) \to \Omega^{0,1}(X)$, which extend to endomorphisms on forms with values in $E$.

**Theorem 3.1.** Let $\mu_\epsilon, g_\epsilon, A_\epsilon$, and $\gamma_\epsilon$ be as above. Then:

$$\dot{\gamma}^{0,1} = \bar{\partial}_{g(A)}(g^{-1}\dot{g}) - \nu \left[ (\partial_{A}g^*)(g^*)^{-1} + (g^*)^{-1}\partial_{A}g^* \right] + g^{-1}\bar{A}^{0,1}g,$$

where the $(0,1)$ part is taken with respect to the fixed conformal structure on $X$, and $g^*$ denotes the fiberwise adjoint of $g$ with respect to the background metric $H_0$.

**Remark 3.2.** If $\gamma_\epsilon$ is a path of flat connections, then in $\mathcal{A}/\mathcal{E}$ a representative for the tangent vector $[\dot{\gamma}]^{0,1}$ may be taken to be harmonic. Projecting to the harmonics, we see that the first term in the expression above vanishes. Hence, we conclude that the first variation of the Yang-Mills flow is independent of $\dot{g}$.

**Proof of Thm. 3.1.** We first note how the Cauchy-Riemann operators deform: fix a unitary connection $A$ on $E$ and a Beltrami differential $\mu$.

**Lemma 3.3.** Let $\partial_{A,\mu} : \Omega^0(E) \to \Omega^{0,1}(X_\mu, E) \subset \Omega^1(E)$ be the $\bar{\partial}$-operator on $X_\mu$ associated to $A$. Then $\partial_{A,\mu}$ is given by

$$\partial_{A,\mu}s = \frac{1}{1 - |\mu|^2} \left( \bar{\partial}s - \mu \partial s + \bar{\mu} \bar{\partial}s - |\mu|^2 \partial s \right),$$

for smooth sections $s \in \Omega^0(E)$.

**Proof.** By choosing local frames, the lemma follows from the corresponding statement for $\partial$ acting on functions. Thus, let $f$ be a function, $z$ a local conformal coordinate on $X$, and $w = w_\epsilon$ solutions to the Beltrami equation $w_\epsilon = \mu_\epsilon z w$. Here we have expressed $\mu = \mu_\epsilon z \partial \bar{z} \otimes (\partial/\partial z)$. Write:

$$df = f_z dz + f_{\bar{z}} d\bar{z} = f_w dw + f_{\bar{w}} d\bar{w},$$
and use \( dw = w_z(dz + \mu \tilde{\omega} d\bar{z}) \) to obtain:
\[
\begin{align*}
    f_z &= f_z w_z + f_{\tilde{\omega}} \bar{z} \tilde{\mu}_z \ , \quad f_{\bar{z}} = f_{\tilde{\omega}} w_z \mu_z \bar{z} + f_{\tilde{\omega}} \bar{z} \\
\end{align*}
\]

Multiplying the first equation by \( \mu \) and subtracting the second, we have:
\[
    f_{\tilde{\omega}} \bar{z} (1 - |\mu|^2) = f_{\bar{z}} - f_z \mu_z \bar{z} .
\]

Now multiply through by \( d\bar{z} + \tilde{\mu}_z dz \) to obtain:
\[
    \bar{\partial} \mu = f_{\tilde{\omega}} d\tilde{\omega} = \frac{1}{1 - |\mu|^2} \left( f_z d\bar{z} - \mu \tilde{\omega} d\bar{z} f_z + \tilde{\mu}_z dz f_z - |\mu|^2 f_z dz \right) .
\]

The result follows by observing that:
\[
    \mu \tilde{\omega} d\bar{z} f_z = \mu \tilde{\omega} d\bar{z} \otimes (\partial/\partial z)(f_z dz) = \mu \partial f , \quad \tilde{\mu}_z dz f_z = \tilde{\mu}_z dz \otimes (\partial/\partial z)(f_z dz) = \tilde{\mu} \partial f .
\]

Continuing with the proof of Thm. 3.1, the action of the complex gauge group is given by:
\[
    g(D_A) = d + lg^{-1}A^{0,1}g + g^*A^{1,0}(g^*)^{-1} + g^{-1} \bar{\partial}_\mu g - (\partial_\mu g^*)(g^*)^{-1} ,
\]
where
\[
    A^{0,1} = A_{\tilde{\omega}} d\tilde{\omega} = A_z d\bar{z} - \epsilon \nu(A_z dz) + \nu(A_z d\bar{z}) + O(\epsilon^2) ,
\]
\[
    A^{1,0} = A_{\tilde{\omega}} dw = A_z dz - \nu(A_z d\bar{z}) + \nu(A_z dz) + O(\epsilon^2) .
\]

From Lemma 3.3 we have:
\[
    \partial_\mu g = \tilde{\partial} g + \epsilon \left( \tilde{\partial} \bar{g} - \nu(\partial g) + \nu(\tilde{\partial} g) \right) + O(\epsilon^2) ,
\]
\[
    \partial_\mu g^* = \partial g^* + \epsilon \left( \partial \bar{g}^* - \nu(\tilde{\partial} g^*) + \nu(\partial g^*) \right) + O(\epsilon^2) .
\]

Applying this and (3.2) to (3.1) yields the result.

3.2. Differentiability of the Twisted Connection. The aim of this section is to prove the following:

**Proposition 3.4.** Let \( \sigma^*_\varepsilon \) be a continuously differentiable family of the metrics representing a path \( [\sigma^*_\varepsilon] \subset \mathcal{T}(g, 1) \). Then there is a family of singular gauge transformations \( g^*_\alpha_\beta \) such that \( g^*_\alpha_\beta(A) \) is a continuously differentiable path of connections.

**Proof.** We differentiate at \( \varepsilon = 0 \). Let \( \mu \varepsilon \) be the differentiable path of Beltrami differentials on \( X \) with \( \mu_0 = 0 \) and \( \mu_\varepsilon|_{\varepsilon=0} = \nu \), associated to \( \sigma^*_\varepsilon \). Let \( z : U \to \Delta \) be a local coordinate on a neighborhood \( U \) of \( p \), conformal with respect to the conformal structure determined by \( \sigma^*_0 \). Fix \( \varphi_0 \) a smooth cut-off function supported in \( \Delta \) and identically 1 on \( \Delta_{2/3} \). We obtain Beltrami differentials \( \tilde{\mu} \) on \( C \) by extending \( \varphi_0 \mu \) by zero. For \( \varepsilon \geq 0 \) small, we consider the solution \( w_\varepsilon \) to the Beltrami equation on \( C \):
\[
    w_z = \tilde{\mu} \tilde{\omega} w_z ,
\]
normalized such that:
\[
(3.4) \quad w_\varepsilon(0) = 0, \quad w_\varepsilon(1) = 1, \quad w_\varepsilon(\infty) = \infty.
\]
Let \( \dot{w} \) denote \( \partial w_\varepsilon / \partial \varepsilon \) at \( \varepsilon = 0 \). We also set \( \dot{\nu} = \dot{\mu} \).

To compute the derivative, we must take into account the change of frame. The problem is local, so suppose we have a fixed (trivial) hermitian rank 2 vector bundle over the complex plane with global unitary frame \( e_\pm \) and singular connection \( A_\alpha \) such that:
\[
(3.5) \quad D_\varepsilon A_\pm = \pm i\alpha \theta \otimes e_\pm.
\]
Let \( \theta_\varepsilon \) denote the theta coordinate of \( w_\varepsilon \).

**Lemma 3.5.** Define a frame \( e_\varepsilon^\pm = \exp \{ \pm i\nu_\alpha_\varepsilon \} e_\varepsilon^+ \) such that \( D_\varepsilon A_\varepsilon^\pm = \pm i\alpha \theta \otimes e_\pm \). Normalize by setting \( u_\alpha^\varepsilon(1) = 0, u_\alpha^\varepsilon(0) \equiv 0 \), and set \( \bar{u}_\alpha = \bar{u}_\alpha^\varepsilon |_{\varepsilon = 0} \). Then
\[
\dot{u}_\alpha(z) = \frac{\alpha}{2} \left( \frac{\dot{w}}{z} - \frac{\bar{w}}{z} \right).
\]

**Proof.** Differentiate to obtain:
\[
D_\varepsilon A_\varepsilon^+ = i\dot{u}_\alpha \otimes e_\varepsilon^+ + i\alpha \theta \otimes e_\varepsilon^+ \implies d\dot{u}_\alpha = \alpha \theta - \alpha \theta.
\]
Hence, \( d\dot{u}_\alpha = \alpha \theta \). Now:
\[
d\theta_\varepsilon = \frac{1}{2i} \left( \frac{d\dot{w}}{w} - \frac{d\bar{w}}{\bar{w}} \right) \implies d\theta = \frac{1}{2i} \left( d\frac{\dot{w}}{z} - d\frac{\bar{w}}{\bar{z}} \right).
\]
Since \( \dot{w}(1) = 0 \) by the normalization (3.4), the result follows upon integration. \( \square \)

We now choose one more smooth cut-off function in addition to \( \varphi_0 \) and \( \varphi_1 \): let \( \varphi_2 \) be smooth, supported on \( \Delta \) and is identically 1 on \( \Delta_{1/3} \). Set \( u_\alpha^\varepsilon = u_\alpha^\varepsilon - u_\alpha^\varepsilon \), and define the complex gauge transformation:
\[
g_\alpha^\varepsilon = \begin{pmatrix} \exp \left(i\varphi_2 u_\alpha^\varepsilon + \varphi_1 (\alpha - \beta) \log |w_\varepsilon| \right) & 0 \\ 0 & \exp \left(-i\varphi_2 u_\alpha^\varepsilon - \varphi_1 (\alpha - \beta) \log |w_\varepsilon| \right) \end{pmatrix}.
\]
Now consider the family of singular connections \( \gamma_\varepsilon = g_\varepsilon A_\alpha \), where the action of the complex gauge transformation is with respect to the complex structure \( w_\varepsilon \). The derivative in the \( e^+ \) direction, for example, is given by (see Thm. 3.1):
\[
(\dot{\gamma}_{0,1})_{e^+} = \frac{\partial}{\partial z} \left[ \varphi_2 (\alpha - \beta) \frac{1}{2} \left( \frac{\dot{w}}{z} + \frac{\bar{w}}{\bar{z}} \right) - 2(\alpha - \beta) \nu \frac{\partial}{\partial \bar{z}} (\varphi_1 \log |z|^2) \right].
\]
By Lemma 3.5 this is:
\[
(\dot{\gamma}_{0,1})_{e^+} = \frac{\partial}{\partial \bar{z}} \left[ \varphi_2 (\alpha - \beta) \frac{1}{2} \left( \frac{\dot{w}}{z} + \frac{\bar{w}}{\bar{z}} \right) + \varphi_1 \left( \frac{w}{2} + \frac{\bar{w}}{\bar{z}} \right) \right] - 2(\alpha - \beta) \nu \frac{\partial}{\partial z} (\varphi_1 \log |z|^2) .
\]
By the choice of cut-off functions it is easily verified that the support of \( \gamma^{0,1}_{e^+} \) lies in the annulus \( 1/6 \leq |z| \leq 2/3 \). The continuity follows from this expression. \( \square \)
3.3. Differentiability. We combine Thm. 3.1 and Prop. 3.4 to prove:

**Theorem 3.6.** Let \( \sigma^*_\varepsilon \) be a continuously differentiable family of metrics representing a path \([\sigma^*_\varepsilon] \in T(y,1)\). Then for each \([A] \in \mathcal{R}_\alpha \), \( \pi^*_{\alpha\beta}[A] \) is a continuously differentiable path in \( \mathcal{R}_\beta \).

As an immediate consequence:

**Corollary 3.7.** Given \( 0 < \beta < \alpha < 1/2 \), the universal Hecke correspondence \( \tilde{\pi}_{\alpha\beta} : \tilde{\mathcal{R}}_\alpha \to \tilde{\mathcal{R}}_\beta \) is continuously differentiable. The same is true for \( \beta = 0 \) on the preimage of the irreducible representations.

**Proof of Thm. 3.6.** First, notice that by choosing a rational number \( \beta < k/n < \alpha \) and using Lemma 2.11 it suffices to prove the result for rational holonomies. Second, by Def. 2.10 and Prop. 3.4 it suffices to show that given a continuously differentiable path of connections \( A_\varepsilon \in A_\beta \), the path \([A_\varepsilon(\infty)] \in \mathcal{R}_\beta \) is also continuously differentiable. Finally, by passing to a branched cover as in Sec. 2.2, it suffices to prove the result for closed surfaces. We continue with the notation as in Sec. 2.2.

Let \( \hat{A}_\varepsilon \in \hat{A} \) be a continuously differentiable path. Note that by Prop. 2.4 and Lemma 2.8 we may assume the \( \hat{A}_\varepsilon \) are stable. Since \( \hat{A}_\varepsilon(\infty) = g_\varepsilon \hat{A}_\varepsilon \), where \( g_\varepsilon \) is a complex gauge transformation, it suffices by the first variational formula (Thm. 3.1) to show that \( h_\varepsilon \) is a smooth family of complex gauge transformations. This can be achieved by the implicit function theorem as follows: consider the map

\[
f : \Omega^0(\sqrt{-1} \text{ad}_0 \hat{E}) \times \hat{A}_{\text{stable}} \times \text{Met}_{-1} \longrightarrow \Omega^0(\sqrt{-1} \text{ad}_0 \hat{E})
\]

\[
f(u, \hat{A}, \hat{\sigma}) = \sqrt{-1} * \hat{\sigma} \partial^\hat{A} (e^{-u} \partial^\hat{A} e^u).
\]

It is easily verified that the map \( f \) is smooth, and:

\[(\delta_u f)_{(u, \hat{A}, \hat{\sigma})} (\delta u) = \Delta^\hat{\sigma}_{e^u \hat{A}} (\delta u).
\]

By the implicit function theorem, the solution \( u = u(\hat{A}, \hat{\sigma}) \) depends smoothly on \((\hat{A}, \hat{\sigma})\), and this completes the proof.

4. Eigenvalue and Eigenfunction Estimates

4.1. Estimates. We shall need eigenfunction and eigenvalue estimates on sections of a vector bundle \( V \) equipped with a Riemannian metric, a metric connection \( A \), and Laplace operator \( \Delta_A \).

In the following, we shall assume \( X \) is a compact surface, possibly with boundary, with a smooth metric of area \( a \). We define (cf. [Li, eq. (0.4)]) the Sobolev constant \( s_1 \) to be the supremum over all constants \( s \) satisfying: \( \inf_{a \in \mathbb{R}} \|f - a\|_2^2 \leq \|df\|_2^2 \), for all smooth functions \( f \). In case \( \partial X \neq \emptyset \), we define another constant associated to the Dirichlet problem: let \( s_2 \) be the supremum over all constants \( s \) satisfying: \( s \|f\|_2^2 \leq \|df\|_2^2 \), for all smooth compactly supported functions \( f \). In this section we prove the following:
Theorem 4.1. There is a universal constant $C$ with the following property: let $\varphi$ be an eigensection of $V$ with eigenvalue $\lambda$. Then:

1. If $\partial X = \emptyset$ or if $\partial X \neq \emptyset$ and $\varphi$ satisfies Neumann boundary conditions, then:

$$\|\varphi\|_\infty^2 \leq C \left( a^{-1} + \left( \frac{4\lambda}{a} \right)^2 a \right) \|\varphi\|_2^2 .$$

2. If $\partial X \neq \emptyset$ and $\varphi$ satisfies Dirichlet boundary conditions, then:

$$\|\varphi\|_\infty^2 \leq C \left( \frac{4\lambda}{a} \right)^2 a \|\varphi\|_2^2 .$$

We also need a lower bound on the growth of eigenvalues. While heat kernel estimates as in [CL] might perhaps give more precise estimates, we shall only need the following:

Theorem 4.2. There is a universal constant $C$ with the following property:

1. If $\lambda_k$ denotes the $k$-th eigenvalue for sections of $V$ for the closed or Neumann boundary problem, then:

$$k \leq C \text{rk} V \left( 1 + \frac{4\lambda_k a}{s_1} \right)^3 .$$

2. If $\mu_k$ denotes the $k$-th eigenvalue for sections of $V$ for the Dirichlet boundary problem, then:

$$k \leq C \text{rk} V \left( \frac{4\mu_k a}{s_2} \right)^3 .$$

Thm.’s 4.1 and 4.2 stated above are generalizations of the results of P. Li [Li] (see also [CL]) on eigenvalues and eigenfunctions of the Laplacian on forms. For the sake of completeness, however, we shall sketch the important steps involved in the proofs of the results for sections. We point out that the main difference in the estimates is the extra term $a^{-1}$ in Thm. 4.1, Part (1), and the 1 in Thm. 4.2, Part (1). These are basically due to the lack of an à priori lower bound on the eigenvalues in terms of Sobolev constants. The key estimate is the following (cf. [Li], eq. (2.7) and (3.1)):

Lemma 4.3. Let $\varphi$ be a smooth section of $V$ and $A$ a smooth metric connection on $V$. If $\partial X \neq \emptyset$, we assume that $\varphi$ satisfies either Dirichlet or Neumann boundary conditions. Then for any $\alpha > 1$,

$$\int_X |\varphi|^{2\alpha - 2} \langle \varphi, \Delta A \varphi \rangle \geq \frac{2\alpha - 1}{\alpha^2} \int_X |d|\varphi|^\alpha|^2 .$$

Proof. Using $2\langle \varphi, D_A \varphi \rangle = |d|\varphi|^2$, we have for $\alpha > 1$:

$$\langle D_A (|\varphi|^{2\alpha - 2} \varphi), D_A \varphi \rangle = \frac{\alpha - 1}{2} |\varphi|^{2\alpha - 4} (d|\varphi|^2, d|\varphi|^2) + |\varphi|^{2\alpha - 2} |D_A \varphi|^2 .$$

By Kato’s inequality, the right hand side above is

$$\geq \frac{\alpha - 1}{2} |\varphi|^{2\alpha - 4} (d|\varphi|^2, d|\varphi|^2) + |\varphi|^{2\alpha - 2} |d|\varphi|^2 = \frac{2\alpha - 1}{\alpha^2} |d|\varphi|^\alpha|^2 .$$

The lemma then follows from integration by parts. \qed
Proof of Thm. 4.1. We shall only prove Part (1), the proofs of the other statements being similar. Assume $\Delta_A \varphi = \lambda \varphi$. Applying (4.2) to $f = |\varphi|^{\infty}$ and using Lemma 4.3, we obtain:

$$
\lambda \int_X |\varphi|^{2\alpha} \geq \frac{2\alpha - 1}{\alpha^2} \left( \frac{a}{s_1} \right)^{1/2} \left( \int_X |\varphi|^{4\alpha} \right)^{1/2} - \frac{a}{s_1} \int_X |\varphi|^{2\alpha},
$$

or,

$$
\left( a^{-1/2} + \frac{\alpha^2}{2\alpha - 1} \frac{4\lambda a^{1/2}}{s_1} \right)^{1/2} \|\varphi\| \geq \|\varphi\|_{4\alpha}.
$$

Set $\alpha = 2^k$ to obtain, by iteration,

$$
\|\varphi\|_2 \prod_{j=0}^k \left( a^{-1/2} + \frac{2^{2j}}{2^{2j+1} - 1} \frac{4\lambda a^{1/2}}{s_1} \right)^{1/2j+1} \geq \|\varphi\|_{2^{k+2}}.
$$

Letting $k \to \infty$ yields $\|\varphi\|_\infty \leq \|\varphi\|_2 \prod_{j=0}^\infty \left( 1 + \frac{2^{2j}}{2^{2j+1} - 1} \frac{4\lambda a}{s_1} \right)^{1/2j}$. The result then follows from the following simple:

Lemma 4.4. Fix $\gamma > 0$. Then for $\beta > 1$ there is a constant $C(\beta)$ independent of $\gamma$ such that:

$$
\prod_{j=0}^\infty \left( 1 + \frac{\gamma^{2j} \lambda a}{\beta - 1} \right)^{1/\beta} \leq C(\beta) \left( 1 + \gamma^{\beta/\beta-1} \right).
$$

Proof of Thm. 4.4. Again, we shall concentrate on Part (1). Let the first $k$ eigenvalues and eigensections be denoted $0 \leq \lambda_1 \leq \ldots \leq \lambda_k$, and $\varphi_1, \ldots, \varphi_k$, respectively.

Lemma 4.5. For any $\psi \in \text{span} \{\varphi_1, \ldots, \varphi_k\}$ and any $l \geq 1$,

$$
\frac{\|\psi\|_{2l+2}}{a^{1/2l+2}} \leq \left( 1 + \frac{4\lambda_k a}{s_1} \frac{2^{2l+1}}{2^{2l+1} - 1} \right)^{1/2l+1} \frac{\|\psi\|_{2l+1}}{a^{1/2l+1}}.
$$

Proof. By Lemma 4.3 for $\alpha = 2^l$ and (4.2) we have:

$$
\frac{2^{2l+1} - 1}{2^{2l}} \frac{s_1}{4} \left\{ a^{-1/2} \|\psi\|_{2l+1} - a^{-1} \|\psi\|_{2l+2} \right\} \leq \int_X \|\psi\|_{2l+1} \langle \psi, \Delta_A \psi \rangle - \|\psi\|_{2l+2} \langle \psi, \Delta_A \psi \rangle
$$

$$
\leq \left\{ \int_X \|\psi\|_{2l+1} \right\}^{2l+1} \left\{ \int_X |\Delta_A \psi|_{2l+1}^{2l+1} \right\}^{1/2l+1}.
$$

Write $\psi = \sum_{j=1}^k c_j \varphi_j$. Now it follows as in [4, Lemma 17] that there is a subset $J \subset \{1, \ldots, k\}$ such that:

$$
\left\{ \int_X |\Delta_A \psi|_{2l+1}^{2l+1} \right\}^{1/2l+1} \leq \left\{ \int_X |\Delta_A \psi|_{2l+2}^{2l+2} \right\}^{1/2l+2} \leq \left\{ \int_X \sum_{j=1}^k \lambda_j c_j \varphi_j \right\}_{2l+2}^{2l+2} \leq \lambda_k \left\{ \int_X \sum_{j \in J} \lambda_j c_j \varphi_j \right\}_{2l+2}^{2l+2} \leq \lambda_k \left\{ \int_X \sum_{j \in J} \varphi_j \right\}_{2l+2}^{2l+2}.
$$
Therefore,
\[ a^{-1/2} \left( \frac{\|\psi\|_{2^l+2}}{\|\psi\|_{2^l+1}} \right)^{2^l+1-1} \leq \frac{4\lambda_k}{s_1^2} \frac{2^l}{2^l+1} - 1 a^{1/2^l+2} + a^{-1} \frac{\|\psi\|_{2^l+1}}{\|\psi\|_{2^l+2}}. \]

Using the fact that \( \|\psi\|_{2^l+1} \leq a^{1/2^l+2} \|\psi\|_{2^l+2} \), we have:
\[ \left( \frac{\|\psi\|_{2^l+2}}{\|\psi\|_{2^l+1}} \right)^{2^l+1} \leq \left( a^{-1/2} + \frac{4\lambda_k}{s_1} \frac{2^l}{2^l+1} - 1 a^{1/2^l+2} \right) a^{1/2^l+2}, \]
from which the lemma follows.

For each \( l \geq 1 \), choose \( \psi_l \in \text{span}\{\varphi_1, \ldots, \varphi_k\} \) so that \( \|\varphi\|_{2^l+2} \|\psi_l\|_2 \leq \|\psi_l\|_{2^l+2} \|\varphi\|_2 \) for all \( \varphi \in \text{span}\{\varphi_1, \ldots, \varphi_k\} \). Then repeated application of Lemma 4.3 gives:
\[ \frac{\|\varphi\|_\infty}{\|\varphi\|_2} \leq \frac{\|\psi_0\|_4}{\|\varphi\|_2} \prod_{l=1}^{\infty} \left( 1 + \frac{4\lambda_k a}{s_1 \left( 2^l+1 \right)} \right)^{1/2^l+1-1}. \]

From Lemma 4.3 and 4.2, we also have:
\[ \frac{\|\psi_0\|_8}{\|\varphi\|_2} \leq a^{-1/2} \left( 1 + \frac{4\lambda_k a}{s_1} \right). \]

Using Lemma 4.4 and the fact that \( 1/(2^l+1-1) \leq 1/2^l \) we conclude that for all \( \varphi \in \text{span}\{\varphi_1, \ldots, \varphi_k\} \)
\[ \frac{\|\varphi\|_\infty}{\|\varphi\|_2} \leq C a^{-1} \left( 1 + \frac{4\lambda_k a}{s_1} \right)^3. \]

Part (1) of Thm. 4.2 now follows from the following:

**Lemma 4.6** ([4, Lemma 11]). Let \( W \) be a finite dimensional subspace of \( L^2(V) \cap C^0 \). Then there is \( \varphi \in W \) such that \( \dim W \|\varphi\|_2^2 \leq (\text{rk} V) a\|\varphi\|_\infty^2 \).

4.2. Conic Degeneration. Recall the notion of a cone metric on a manifold \( C(Y) = (0,1) \times Y \) (cf. [Ch1], [Ch2], [JW1]): this is a metric of the form \( ds^2 = dr^2 + r^2 \sigma \), where \( \sigma \) is a (smooth) metric on \( Y \). An \( n \)-dimensional manifold \( X \) with metric \( \sigma \) on \( X \setminus \{p\} \) is said to have a cone metric if for some choice of \((Y, \sigma)\) and some neighborhood \( U \) of \( p \) in \( X \), \( U \setminus \{p\} \) is isometric to \( C(Y) \); in this case we call \( p \) a cone point. We generalize this notion to that of a cone double point, by which we mean locally the union of two copies of \( C(Y) \) with the singularity identified. For example, for surfaces it is natural to view such a singularity as arising from the following family of degenerating metrics:
on the cylinder \( C = \{(x,y) : -1 \leq x \leq 1, 0 \leq y \leq 2\pi\} / \{(x,0) \sim (x,2\pi)\} \), define, for \( 0 \leq \ell \leq 1 \), the metrics \( ds^2 = dx^2 + (\ell + (1-\ell)x^2) \kappa^2 dy^2 \). The parameter \( 0 < \kappa \leq 1 \), introduced here for convenience, is a fixed cone angle. The metric for \( \ell = 0 \) has a cone double point as described above. We shall refer to a family of metrics \( \sigma(\ell) \) on a compact, connected surface \( X \) as a conic degenerating family if \( X \) contains a finite collection of disjoint cylinders \( C_1, \ldots, C_k \), all of which are isometric to \( C(\ell) = (C, ds^2) \), and \( \sigma(\ell) \) converges smoothly to a Riemannian metric on \( X \setminus C_1 \cup \cdots \cup C_k \).

Eigenvalue problems are still well-behaved on compact manifolds with cone metrics. We shall be considering the following situation: let \( A \) be a connection on a (real) vector bundle \( V \to X \) with Riemannian metric. If \( X \) has cone points, then we allow \( A \) to be singular at those points. It
will suffice to assume further that with respect to some choice of orthonormal frame and conformal coordinates \((r, \theta)\) near the cone point, \(\mathbf{A}\) has the form \(b \otimes d\theta\), where \(b\) is a constant diagonal matrix.

Consider the operator \(\Delta_\mathbf{A} = D_\mathbf{A}^* D_\mathbf{A}\) acting on sections \(\varphi\) of \(V\) satisfying \(\varphi, D_\mathbf{A}\varphi, \Delta_\mathbf{A}\varphi\) in \(L^2\). Then by a generalization of the result in [Ch1], Stokes theorem holds for sections of \(V\) and the cone metric. Hence, the \(L^2\) extension of \(\Delta_\mathbf{A}\) is self-adjoint. Moreover, the spectrum of \(\Delta_\mathbf{A}\) is discrete with finite multiplicity, and the eigenfunctions are smooth away from the singularities (cf. [Ch2]).

We will be using three important properties of conic degenerating families. The first follows from a direct computation:

**Proposition 4.7.** Let \((X, \sigma(\ell))\) be a conic degenerating family of surfaces. Then the eigenvalues of the Ricci curvature tensor are uniformly bounded by a constant times \(1/\ell\).

**Proof.** We briefly sketch the computation: it clearly suffices to compute the Ricci tensor \(R^i_k\) for \(C(\ell) = (C, ds_\ell^2)\). We use coordinates \(x_1 = x, x_2 = y\), so that:

\[
\sigma_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & (\ell + (1 - \ell)x^2) \kappa^2 \end{pmatrix}.
\]

Then one finds for the Christoffel symbols: \(\Gamma^1_{11} = \Gamma^2_{11} = \Gamma^1_{12} = \Gamma^2_{22} = 0\), and

\[
\Gamma^1_{22} = -\frac{1}{2} \frac{\partial \sigma_{22}}{\partial x} = -(1 - \ell)x\kappa^2, \quad \Gamma^2_{12} = \frac{1}{2} \sigma_{22} \frac{\partial \sigma_{22}}{\partial x} = \frac{(1 - \ell)x}{\ell + (1 - \ell)x^2}.
\]

The operator

\[
R^i_k = \sigma^{lj} \left( \frac{\partial \Gamma^j_l}{\partial x^k} - \frac{\partial \Gamma^j_k}{\partial x^l} + \Gamma^j_l \Gamma^i_k - \Gamma^i_j \Gamma^k_l \right).
\]

From this, one verifies \(R^1_1 = R^2_2 = 0\), and

\[
R^1_1 = \sigma_{22} \left( \frac{\partial \Gamma^2_{22}}{\partial x} - \Gamma^2_{12} \partial_2 \Gamma^1_{22} \right) = \frac{-\ell(1 - \ell)}{(\ell + (1 - \ell)x^2)^2},
\]

\[
R^2_2 = -\frac{\partial \Gamma^2_{12}}{\partial x} - \Gamma^2_{12} \partial_2 \Gamma^2_{12} = \frac{-\ell(1 - \ell)}{(\ell + (1 - \ell)x^2)^2}.
\]

The result follows.

The second fact is a comparison between conic degeneration and the well-known “plumbing” construction used to study holomorphic degenerating families of Riemann surfaces (cf. [DW2]). Consider the annuli given by: \(\Upsilon_\varepsilon = \{(z, w) \in \mathbb{C}^2 : |z|, |w| \leq 1, zw = \varepsilon\}\). This is a holomorphic family for \(\varepsilon\) in a punctured disk, but we will take \(\varepsilon\) to be real. The central fiber \(\varepsilon = 0\) is a “pinched” annulus – two disks with coordinates \(z\) and \(w\) identified at \(z = w = 0\). We shall be interested in metrics on \(\Upsilon_\varepsilon\) which are conformal with respect to the complex structure induced from \(\mathbb{C}^2\) and which degenerate to cone metrics \(ds_0^2\) on \(\Upsilon_0\) proportional to metrics of the following form: for some \(0 < \kappa \leq 1\) and in the coordinate \(z\),

\[
ds_0^2 = \kappa^2 |z|^{2(\kappa - 1)} |dz|^2
\]
These can always be constructed:

**Proposition 4.8.** Fix $\kappa$, $0 < \kappa \leq 1$. Then there is a function $\varepsilon = \varepsilon(\ell)$ depending on $\kappa$ and a conformal conic degenerating family of metrics on $\Upsilon_{\varepsilon(\ell)}$ converging to the metric $\left(4.4\right)$. Furthermore, the parameters $\varepsilon$ and $\ell$ are related by the following bound:

$$\frac{\ell^{2/\kappa}}{4^{1/\kappa}} \leq \varepsilon(\ell) \leq \ell^{1/\kappa},$$

for $\ell \leq 3/4$.

**Proof.** We solve for conformal coordinates $z = re^{i\theta}$ on the portion of $\Upsilon_{\varepsilon(\ell)}$ where $0 \leq x \leq 1$. Take $\theta = \theta(y) = y$, and $r = f(x, \ell)$ for $f$ increasing, $f(1, \ell) = 1$. For such a solution, we may write $ds^2_\ell = \sigma(z, \ell) dz^2$, or:

$$\frac{dr^2}{(f')^2(x, \ell)} + (\ell + (1 - \ell)x^2)\kappa^2 d\theta^2 = \sigma(z, \ell)(dr^2 + r^2 d\theta^2).$$

where $f'(x, \ell)$ is the partial derivative with respect to $x$. This implies:

$$\frac{\partial}{\partial x} \log f(x, \ell) = \frac{1}{\kappa(\ell + (1 - \ell)x^2)^{1/2}}, \quad \sigma(z, \ell) = \frac{1}{(f')^2(x, \ell)}.$$

Applying the initial condition, we find the solution

$$f(x, \ell) = \left[\frac{(1 - \ell)^{1/2}x}{1 + (1 - \ell)^{1/2}} + \frac{(1 - \ell)^{1/2}}{1 + (1 - \ell)^{1/2}} \sqrt{x^2 + \frac{\ell}{1 - \ell}}\right]^{1/\kappa(1 - \ell)^{1/2}},$$

$$\varepsilon(\ell) = f'^2(0, \ell) = \left[\frac{\ell}{(1 + (1 - \ell)^{1/2})^2}\right]^{1/\kappa(1 - \ell)^{1/2}}.$$

Notice that the relationship between $\varepsilon$ and $f$ in the second line follows from the fact that the middle of the cylinder corresponds to $|z| = |w| = \sqrt{\varepsilon}$. The convergence and bounds on $\varepsilon(\ell)$ easily follow.

Lastly, we comment on the behavior of the Sobolev constants under conic degeneration. We quote the following result:

**Proposition 4.9 (JW1, Thm. 2.4 and Prop. 2.6).** For the conic degenerating cylinder $C(\ell)$ in Sec. 4.2, there is a constant $c$ such that $s_2(C(\ell)) \geq c > 0$ for all $\ell > 0$.

This result bears on the inclusion $L^2_1 \hookrightarrow L^4$. Recall that for a smooth function $f$ the definition of the Sobolev constant implies (see [4], Lemma 4):

$$\|df\|_2^2 \geq \frac{s_1}{4} \left(a^{-1/2}\|f\|_4^2 - a^{-1}\|f\|_2^2\right),$$

where $a$ denotes the area of $X$. For a conic degenerating family, $s_1 \to 0$ if and only if there is a separating pinching cylinder (see [JW1, Cor. 2.9]). However, by the proposition above, the Sobolev constants $s_2$ of the component regions, i.e. the degenerating cylinders and their complements, all remain bounded away from zero. Thus, applying a cut-off function to (4.2), one easily proves
Proposition 4.10. For a conic degenerating family \((X, \sigma(\ell))\) there is a constant \(c > 0\) independent of \(\ell\) such that for all smooth functions \(f\) on \(X\) and all \(\ell > 0\),
\[
\|f\|_{4, \sigma(\ell)}^2 \leq c \left( \|df\|_{2^*, \sigma(\ell)}^2 + \|f\|_{2^*, \sigma(\ell)}^2 \right).
\]

4.3. Uniform Bounds. If \((X, \sigma(\ell))\) is a conic degeneration, then a vector bundle \(V \to X\) with connection \(A\) such that \(A\) has the standard form \(b \otimes d\theta\), \(b\) constant diagonal, on the cylinder \(C\), naturally defines a family of operators \(\Delta_A\) on \((X, \sigma(\ell))\). By the comments at the end of Sec. 4.2 it is reasonable to ask whether the eigenvalues and eigenfunctions of \(\Delta_A\) on \((X, \sigma(\ell))\) converge as \(\ell \to 0\) to those of the limiting cone metric. This is what we call spectral convergence. The estimates from Thms. 4.1 and 4.2 are the key elements needed to prove spectral convergence for this degenerating family (cf. [JW1, JW2]). The first step is uniform \(C^0\) bounds on eigenfunctions and uniform growth of eigenvalues:

Corollary 4.11. If \((X, \sigma(\ell))\) is a conic degenerating family with vector bundle \(V\) and connection \(A\), then there are constants \(C_1\), \(C_2\) independent of \(\ell\) such that if \(\varphi(\ell)\) is a normalized eigensection of \(V\) with eigenvalue \(\lambda(\ell)\) then \(\|\varphi(\ell)\|_\infty \leq C_1 + C_2 \lambda^5(\ell)\) for all \(\ell\).

Corollary 4.12. If \((X, \sigma(\ell))\) is a conic degenerating family, then there is a constant \(C\) and an integer \(N\), both independent of \(\ell\), such that if \(\lambda_k(\ell)\) denotes the \(k\)-th eigenvalue for the closed problem for sections of \(V\), then \(\lambda_k(\ell) \geq C k^{1/3}\) for all \(\ell\) and all \(k \geq N\).

The arguments in the references mentioned above then apply to give:

Theorem 4.13. For a conic degenerating family we have spectral convergence for sections of \(V\).

Applying this to the particular case of a unitary connection \(A\) on \(E\) and the eigenvalue \(\lambda_1\) of \(\Delta_A\) acting on sections of \((\text{ad} E)_0\), we have:

Corollary 4.14. Let \(A\) be an irreducible connection on \(X\). Then for a conic degenerating family \((X, \sigma(\ell))\), \(\lambda_1(\ell) \to 0\) if and only if \(A\) is accidentally reducible.

Proof of Cor. 4.14. In the non-separating case, it follows from [JW1, Cor. 2.9 and Thm. 2.4] that the Sobolev constant \(\mathfrak{s}_1\) is bounded away from zero. The result then follows from Part (1) of Thm. 4.1. In the separating case, following [JW2] we divide the degenerating surface into three regions \(X^\pm\) and \(C\) with the induced metrics from \(\sigma(\ell)\) (denoted also by \(\sigma(\ell)\)). Let \(\hat{C} \subset C\) be a fixed subcylinder and \(\hat{X}^\pm\) the corresponding complementary regions. By definition, the degenerating family \((\hat{C}, \sigma(\ell))\) is isometric to the standard degenerating cylinder; therefore, by Prop. 4.9 it follows that the Sobolev constant \(\mathfrak{s}_2\) for \((\hat{C}, \sigma(\ell))\) is bounded away from zero. The same is true for \((\hat{X}^\pm, \sigma(\ell))\), since these form a smooth family of metrics on the surfaces with boundary for \(\ell \geq 0\).

Choose a smooth cut-off function \(\eta\) on \(X\) such that \(\eta \equiv 1\) on \(\hat{C}\) and \(\eta \equiv 0\) on \(X^\pm\). Suppose \(\varphi(\ell)\) is a normalized eigensection on \((X, \sigma(\ell))\) with eigenvalue \(\lambda(\ell)\). We denote by \(((1 - \eta)\varphi(\ell))^\pm\) the restriction of \(((1 - \eta)\varphi(\ell))\) to \(\hat{X}^\pm\). It clearly suffices to find bounds on \(((1 - \eta)\varphi(\ell))^\pm\) and \(\eta \varphi\).
separately. Notice that $\Delta^m_A((1 - \eta)\varphi(\ell))^\pm$ and $\Delta^m_A\eta\varphi$ have $L^2$ bounds depending on $m$ and $\lambda(\ell)$, but otherwise independent of $\ell$. More precisely, the bound may be taken of the form $C_1 + C_2\lambda^m(\ell)$.

This is because $d\eta$ and $d(1 - \eta)$ are supported in $\tilde{X}^\pm$ where the higher derivatives of $\varphi$ may be uniformly bounded by an application of the elliptic estimate.

Let $\psi^\pm_k(\ell)$ and $\psi^\pm_k(\ell)$ denote normalized eigensections with eigenvalues $\mu^\pm_k(\ell)$ and $\mu^C_k(\ell)$ for the Dirichlet problems on $\tilde{X}^\pm$ and $\tilde{C}$, respectively. Consider the Fourier expansions:

\[(1 - \eta)\varphi(\ell))^\pm = \sum_{k=1}^\infty a_k^\pm(\ell)\psi^\pm_k(\ell) \quad \eta\varphi(\ell) = \sum_{k=1}^\infty b_k(\ell)\psi^C_k(\ell) .\]

Since the Sobolev constants for the individual pieces are bounded away from zero, it follows from Thm. 4.1 that we have uniform bounds:

\[\|\psi^\pm_k(\ell)\|_\infty \leq C\mu^\pm_k(\ell) \quad \|\psi^C_k(\ell)\|_\infty \leq C\mu^C_k(\ell) .\]

Here the constant $C$ may be chosen independent of $k$ and $\ell$. Hence, it suffices to show that there is some $B$ of the required form satisfying:

\[\sum_{k=1}^\infty |a_k^\pm(\ell)|\mu^\pm_k(\ell) \leq B ,\]

\[\sum_{k=1}^\infty |b_k(\ell)|\mu^C_k(\ell) \leq B ,\]

for all $\ell$. Consider (4.4). By definition, we have:

\[b_k(\ell) = \int_C \langle \eta\varphi(\ell) , \psi^C_k(\ell) \rangle d\sigma(\ell) = \frac{1}{(\mu^C_k(\ell))^m} \int_C \langle \eta\varphi(\ell) , \Delta^m_A\psi^C_k(\ell) \rangle d\sigma(\ell) = \frac{1}{(\mu^C_k(\ell))^m} \int_C \langle \Delta^m_A(\eta\varphi(\ell)) , \psi^C_k(\ell) \rangle d\sigma(\ell) .\]

Since we assume uniform $L^2$ bounds on $\Delta^m_A(\eta\varphi(\ell))$, we obtain a uniform bound

\[\mu^C_k(\ell) |b_k(\ell)| \leq (C_1 + C_2\lambda^m(\ell))(\mu^C_k(\ell))^{1-m} ,\]

where the constants $C_1, C_2$ may be chosen independent of $\ell$. By Part (2) of Thm. 4.2 above we have $\mu^C_k(\ell) \geq C'k^{1/3}$ for some constant $C'$ independent of $k$ and $\ell$. Hence,

\[\mu^C_k(\ell) |b_k(\ell)| \leq (C'_1 + C'_2\lambda^m(\ell))k^{(1-m)/3} ,\]

where the constants $C'_1, C'_2$ may be chosen independent of $k$ and $\ell$. Since the sum over $k$ of the terms on the right hand side converges for $m \geq 5$, the desired bound in (4.4) is obtained. The proof for (4.3) is similar.

Proof of Cor. 4.12. In the non-separating case, it follows from [JW1, Cor. 2.9 and Thm. 2.4] that the Sobolev constant $c_1$ is bounded away from zero. The result then follows from Part (1) of Thm. 4.2. In the separating case, consider the three regions $X^\pm$, $C$ as in the proof of Cor. 4.12 above. By domain monotonicity, it suffices to prove the result for the Neumann spectra for $(X^\pm, \sigma(\ell))$ and $(C, \sigma(\ell))$. Since $(X^\pm, \sigma(\ell))$ form a smooth family of metrics on the surfaces with boundary,
it follows that the Sobolev constants $s_1$ for these regions are uniformly bounded away from zero; hence, by Part (1) of Thm. 4.2 the Neumann spectra of these surfaces has uniform growth as in the statement of the corollary. As above, the Sobolev constant $s_2$ for $(C, \sigma(\ell))$ is also uniformly bounded away from zero. Hence, by Part (2) of Thm. 4.2 the Dirichlet spectra of these surfaces has uniform growth as in the statement of the corollary. On the other hand, because of the form of the connection on $C$, the Neumann spectrum may be bounded below by the Dirichlet spectrum after shifting the index by two as in [W]. This completes the proof.  

4.4. Heat Kernel Estimates. It is well-known that eigenvalue and eigenfunction estimates produce estimates on solutions to the linear heat equation. This will also be useful in the non-linear Yang-Mills flow. The result we need is the following:

**Theorem 4.15.** Let $(X, \sigma(\ell))$ is a conic degenerating family and fix $T > 0$. There is a constant $C$ depending on $T$ but independent of $\ell$ such that if $v(t, x)$ is the solution to the heat equation with initial conditions $v(0, x) = v_0$, then $\sup_x |v(t, x)| \leq C\|v_0\|_2,\sigma(\ell)$ for all $t \geq T$.

**Proof.** The solution $v(t, x)$ may be written:

$$v(t, x) = \sum_{i=0}^{\infty} e^{-t \lambda_i} a_i \varphi_i(x),$$

where $|a_i| \leq \|v_0\|_2,\sigma(\ell)$. By Cor. 4.11 for the ordinary Laplacian on functions, there are constants $C_1$ and $C_2$ independent of $\ell$ such that:

$$\sup_x |v(t, x)| \leq \|v_0\|_2,\sigma(\ell) \sum_{i=0}^{\infty} e^{-t \lambda_i(\ell)} (C_1 + C_2 \lambda^5_i(\ell)).$$

Thus, we must show that for fixed $T_0 > 0$ there is a constant $C$ depending only on $T_0$, $C_1$, and $C_2$ such that for all $t \geq T_0$ and all $\ell > 0$,

$$\sum_{i=0}^{\infty} e^{-t \lambda_i} (C_1 + C_2 \lambda^5_i(\ell)) \leq \tilde{C}.$$

First choose $\Lambda_0$ such that for all $\Lambda \geq \Lambda_0$ we have $\Lambda^6(C_1 + C_2 \lambda^5_i)e^{-\Lambda T} \leq 1$. Now by Cor. 4.12 we can find a constant $C_3$ such that $\lambda_i \geq C_3 i^{1/3}$, for $i$ sufficiently large, say $i \geq N$, independent of $\ell$. We further prescribe $N$ such that for $i \geq N$, $\lambda_i \geq \Lambda_0$. Finally, let:

$$C_4 = \sup_{\Lambda > 0} (C_1 + C_2 \lambda^5_i)e^{-\Lambda T}.$$

Then for $t \geq T$,

$$\sum_{i=0}^{\infty} e^{-t \lambda_i} (C_1 + C_2 \lambda^5_i) = \sum_{i=0}^{N-1} e^{-t \lambda_i} (C_1 + C_2 \lambda^5_i) + \sum_{i=N}^{\infty} e^{-t \lambda_i} (C_1 + C_2 \lambda^5_i) \leq NC_4 + \frac{1}{C_3} \sum_{i=N}^{\infty} \frac{1}{i^2}.$$  

This proves the result.  

\[ \square \]
5. Proof of the Main Theorem

5.1. Outline of the Proof. Let \([\sigma^*(\ell)]\) be a degeneration in \(T_{aug.}(g, 1)\) to a nodal Riemann surface with conformal structure \(\sigma^*(0)\) associated to a collection \(\Phi\) of simple closed curves. Recall that \(\mathcal{R}^\Phi \subset \mathcal{R}\) denotes the \(\Phi\)-accidentally reducible representations. In this section we are going to show that for a given \([A] \in \mathcal{R}_\alpha \setminus \mathcal{R}^\Phi\), and \(|\beta - \alpha| < \varepsilon_0\), where \(\varepsilon_0\) is sufficiently small as in Sec. 2.3.

\[
\lim_{\tau \to \infty} \pi_{\alpha\beta}[\sigma^*(\ell)] [A] = \pi_{\alpha\beta}[\sigma^*(0)] [A].
\]

By standard compactness arguments, this will suffice to prove the Main Theorem. As representatives for the degenerating conformal structure we may choose lifts (5.1) standard form around the point \(p\) will follow from Thm. 5.1 and the following: the form \(\Xi\) in Thm. 5.1 and transverse arcs \(\Gamma_i\), we may choose lifts the pinching cylinders return to the initial holonomies of \(A\) across the cylinder (cf. the discussion preceding Def. 2.16). So the second part of the proof is to show that the holonomies \(\alpha\) converge for curves supported away from the pinching cylinders. This does not suffice, however. Indeed, this statement, combined with the results on Simpson’s flow from Sec. 3 show only that the limiting holonomies of \(A(\ell, \infty)\) around the pinching cylinders return to the initial holonomies of \(A_0\) as \(\ell \to 0\). This would still allow for the possibility of a change of framing, or gluing parameters, across the cylinder (cf. the discussion preceding Def. 2.16). So the second part of the proof is to show that the holonomies across the pinching cylinders, as measured with respect to the framing coming from \(A_0\), are very nearly trivial.

We present these two results as Thm.’s 5.1 and 5.2 below:

**Theorem 5.1.** For any set \(\{\Xi_j\}_{j=1}^N\) of closed curves supported in \(\pi_1(X_0^*)\) we have

\[
\{\text{hol} \xi_j A(\ell, \infty)\}_{j=1}^N \longrightarrow \{\text{hol} \xi_j A(0, \infty)\}_{j=1}^N,
\]

modulo overall conjugation by \(SU(2)\).

Next, recall from Thm. 2.13 that the manner by which a connection \([A(0, \infty)]\) produces a point in \(\mathcal{R}_\beta\) is to use the initial framings. Consider a cylinder \((C, ds^2)\) in \(X^*\) on which the twisted initial connection \(A_0\) is flat. We may choose a unitary frame \(\{e_1, e_2\}\) such that in terms of the coordinates \((x, y)\) in Sec. 4.2, \(A_0\) has the form \(dA_0 = d + \text{diag}(i\gamma, -i\gamma)dy\). We fix this frame once and for all throughout the degeneration. Now for a small transverse arc \(\Gamma_\varepsilon = \{(x, y) \in C : -\varepsilon \leq x \leq \varepsilon, y = y_0\}\) and flat connection \(A(\ell, \infty)\), we measure the holonomy \(\text{hol}_{\Gamma_\varepsilon} A(\ell, \infty)\) by parallel translating the frame \(\{e_1, e_2\}\) along \(\Gamma_\varepsilon\). For example, notice that by our choice of lift \([A(0, \infty)]\) in Thm. 2.13, \(\text{hol}_{\Gamma_\varepsilon} A(0, \infty) = \text{I}\) for any choice of \(\varepsilon\).

Since any closed curve on \(X^*\) may be written, up to homotopy, as a concatenation of curves of the form \(\Xi\) in Thm. 5.1 and transverse arcs \(\Gamma_i\), one for each component \(c_i \in \Phi\), we see that (5.1) will follow from Thm. 5.1 and the following:
**Theorem 5.2.** For any $\delta > 0$ there is $\varepsilon_0 > 0$ and $\ell_0$ such that for all $\Gamma_{\varepsilon_0}^i$ and all $\ell \geq \ell_0$,

$$\left| \text{hol}_{\Gamma_{\varepsilon_0}^i} (A(\ell, \infty)) - I \right| < \delta .$$

The proof of Thm. 5.2 will occupy the next two subsections, and the proof of Thm. 5.2 will be given in subsection 5.4. One of the key ingredients in the proofs is the $C^0$ estimate for the metric and the curvature found in Cor. 5.10 below.

5.2. **Proof of Theorem 5.1**. Throughout this section, $\Omega$ is an open set with compact closure in $X^* \setminus \Phi$. As before, we fix $[A] \in \mathcal{R}_\alpha$. Choose a representative $A$, and let $A_0 = g_{0\alpha}(A)$ be the twist of $A$ at $p$. We assume that $\beta = k/n$, where $k, n$ are positive coprime integers, and $n$ is odd.

We denote by $A(\ell, t)$ the solution of the Yang-Mills flow (2.1)-(2.2), by $h(\ell, t)$ the solution to the non-linear heat equation (2.4)-(2.5) on $(X, \sigma^*(\ell))$, and by $h(0, t)$ the solution to the same equations on $X^*_0 = X^* \setminus \Phi$ with the degenerate metric $\sigma^*(0)$, the holomorphic structure on the bundle being determined by $A_0$.

As a first step, we show that as we degenerate the metric on $X^*$, $h(\ell, t)$ converges to $h(0, t)$ uniformly on compact sets. In the following, $p$ is any number strictly greater than 1. Also, $*_{\ell}$ and $*_{\omega}$ will denote the Hodge *'s on $X^*$ and $X^* \setminus \Phi$ with respect to the metrics $\sigma^*(\ell)$ and $\sigma^*(0)$, respectively.

**Proposition 5.3.** Given $T > 0$, $\log h(\ell, T) \to \log h(0, T)$ weakly in $L^p_{2,\text{loc}}$. In particular, the convergence is strong in $C^1(\Omega)$.

**Proof.** In Sec. 5.3, we will obtain $C^0$ bounds for $h(\ell, t)$ and $*_{\ell} F_{A(\ell, t)}$ independent of $\ell$ (see Cor. 5.10). Assuming these results, since

$$h^{-1/2}(\ell, t) F_{A(\ell, t)} h^{1/2}(\ell, t) = F_{A_0} + \partial_{A_0} \left( h^{-1}(\ell, t) \partial_{A_0} h(\ell, t) \right) ,$$

standard elliptic estimates imply $L^p_{2,\text{loc}}$ estimates for $h(\ell, t)$ uniform in $\ell$ and $0 \leq t \leq T$. By eq. (2.4), this implies an $L^p_{2,1,\text{loc}}$ estimate on $X^* \times [0, T]$, uniform in $\ell$, where the 1 refers to the time derivative. It follows that $h(\ell, t)$ converges to some $\tilde{h}(0, t)$ weakly in $L^p_{2,\text{loc}}$. The uniform $C^0$ bounds imply that $\tilde{h}(0, t)$ and $*_{\omega} F^{\partial_{A_0}}_{A_0} \tilde{h}(0, t) H_0$ are also bounded uniformly for $t \in [0, T]$. The uniqueness part of Thm. 2.2 shows that $\tilde{h}(0, T) = h(0, T)$ as desired. \qed

**Corollary 5.4.** Given $\varepsilon > 0$ there exists $T > 0$ and $\ell_0 = \ell_0(\varepsilon, T) > 0$ such that for $\ell \geq \ell_0$,

$$\| \log h(\ell, T) - \log h(0, \infty) \|_{C^1(\Omega)} < \varepsilon ,$$

and similarly,

$$\| h^{1/2}(\ell, T) A_0 - h^{1/2}(0, \infty) A_0 \|_{C^0(\Omega)} < \varepsilon .$$

**Proof.** By Thm. 2.2, $h(0, t) \to h(0, \infty)$ uniformly in $C^1(\Omega)$, so we can choose $T_1$ such that for all $t \geq T_1$,

$$\| \log h(0, t) - \log h(0, \infty) \|_{C^1(\Omega)} < \varepsilon /2 .$$
Take $T \geq T_1$. By Prop. 5.3, $\exists \ell_0 = \ell_0(\varepsilon, T)$ such that for all $\ell \geq \ell_0$, 
$$
\| \log h(\ell, t) - \log h(0, T) \|_{C^1(\Omega)} < \varepsilon/2 .
$$
The result follows. \hfill \Box 

Now let $A(\ell, t)$ be the solution of the Yang-Mills flow on $X^*$ as before. Write 
(5.2) 
$$
A(\ell, \infty) = \tilde{g}(\ell, t)A(\ell, t) , \quad \tilde{h}(\ell, t) = \tilde{g}^*(\ell, t)\tilde{g}(\ell, t) .
$$

Proposition 5.5. Given $\varepsilon > 0$, there is a $T_0 > 0$ independent of $\ell$ such that for all $t \geq T_0$, 
$$
| \log \tilde{h}(\ell, t) |_{C^1(\Omega)} < \varepsilon .
$$
We shall also prove this result in the following subsection. Here we show how Prop. 5.5 implies Thm. 5.1. Let $T = \max(T_0, T_1)$, where $T_0$ and $T_1$ are as in Prop. 5.5 and Prop. 5.3, respectively. Let $\ell_0$ be chosen as in Cor. 5.4, and choose $\ell \geq \ell_0$. Notice that both $h^{1/2}(\ell, T)A_0$ and $\tilde{h}^{1/2}(\ell, T)A(\ell, \infty)$ are real gauge equivalent to $A(\ell, T)$. We therefore can write: 
$$
\tilde{h}(\ell, T)A(\ell, \infty) = k(\ell, T)h^{1/2}(\ell, T)A_0 ,
$$
where $k(\ell, T)$ is a real gauge transformation. By Cor. 5.3, 
$$
\left\| \tilde{h}^{1/2}(\ell, T)A(\ell, \infty) - k(\ell, T)h^{1/2}(0, \infty)A_0 \right\|_{C^0(\Omega)} \leq c\varepsilon ,
$$
for $c$ depending only on the Sobolev embedding $L^p_1(\Omega) \hookrightarrow L^\infty(\Omega)$, and may be taken independent of $\ell$. On the other hand, by Prop. 5.3, 
$$
\left\| \tilde{h}^{1/2}(\ell, T)A(\ell, \infty) - A(\ell, \infty) \right\|_{L^\infty(\Omega)} \leq \varepsilon .
$$
It follows that: 
$$
\left\| A(\ell, \infty) - k(\ell, T)h^{1/2}(0, \infty)A_0 \right\|_{L^\infty(\Omega)} \leq (c + 1)\varepsilon .
$$
Since $A(\ell, \infty)$ and $k(\ell, T)h^{1/2}(0, \infty)A_0$ are $C^0$-close, their holonomies around the $\Xi_j$ are also close. Finally, since $k(\ell, T)h^{1/2}(0, \infty)A_0$ and $A(0, \infty)$ are real gauge equivalent, the theorem follows.

5.3. Proof of Proposition 5.5. We begin with some preliminary results:

Proposition 5.6. Let $[A] \in \mathcal{R}_\alpha$ be as before. Then given $\varepsilon_0 > 0$ there is $\delta > 0$ such that for $|\beta - \alpha| < \delta$ there exists a twist $A_0 = g_{\alpha \beta}(A)$ and a flat connection $A_\infty$ with the standard form of holonomy $\beta$ around $p$, and such that $\| A_0 - A_\infty \|_{C^0(\Sigma)} < \varepsilon_0$. 

Proof. Let $C$ denote a component of the pinching region and $A_0$ the twist of $A$ at $p \notin C$. Let $h$ denote the flow at $\infty$ for eq.’s (2.4)-(2.5) associated to $\tilde{A}_0$, the initial hermitian structure $H_0$, and the degenerate metric $\sigma(0)$ on $X_0^*$. Let $\tilde{A}_\infty$ denote the hermitian connection associated to $\tilde{A}_0$ and $hH_0$. We may assume that $A_0$ is in the standard form $d + \text{diag}(i\gamma, -i\gamma)d\theta$ on a slightly larger cylinder $C_1 \supset \overline{C}$. Let $C_2$ be an open cylinder such that $\overline{C} \subset C_2 \subset \overline{C}_2 \subset C_1$. Since the initial curvature $\| *F_{A_0} \|_{C^0(\Sigma)}$ can be made arbitrarily small for $|\beta - \alpha|$ small, and since $A_0$ is stable on $X_0^*$, it follows (cf. [11]) that $\| \log h \|_{C^1(V)}$, and hence also $\| A_0 - \tilde{A}_\infty \|_{C^0(V)}$, can be made arbitrarily small.
for a relatively compact set \( V \subset X_0^* \). By [DW1, Lemma 2.7] there is a real gauge transformation \( g \) such that \( g(\tilde{A}_\infty) = A_0 \) on \( C_2 \). Moreover, it is clear from the proof of that lemma that by taking \( V \) so that \( X \setminus V \subset C \) and using the fact that \( \|A_0 - \tilde{A}_\infty\|_{C^0(V)} \) is small, we may conclude that \( \| \log g \|_{C^0(C_1 \setminus C_2)} \) is small. By bootstrapping we find that \( \| \log g \|_{L^1_t(C_1 \setminus C_2)} \) is small; hence, we can extend \( g \) to a real gauge transformation of \( E \) over \( X^* \) with \( g \equiv I \) on \( X \setminus C_1 \) and \( \|g - I\|_{L^1_t(X \setminus C_2)} \) small. Set \( A_\infty = g(\tilde{A}_\infty) \). Then \( A_\infty \) extends to a connection over \( X^* \), and the desired estimate holds.

In order to get our estimate for the metric in \( X^* \), we need again to pass to branched covers. Assume that \( p \in X^* \setminus X \) is outside the pinching region, and let \( q : \tilde{X} \to X \) be a regular cyclic branched cover of degree \( n \), chosen so that all branch points lie outside the pinching region. We choose metrics \( \tilde{\sigma}(\ell) \) on \( \tilde{X} \) so that \( \tilde{\sigma}(\ell) \to \tilde{\sigma}(0) \) is a conic degeneration, and with respect to the induced conformal structures from \( \tilde{\sigma}(\ell) \) and \( \sigma(\ell) \), the map \( q \) is holomorphic. Recall the map \( \hat{q} : \mathcal{A}_{k/n} \to \tilde{A} : A \mapsto \tilde{A} \).

**Lemma 5.7.** Let \( A \in \mathcal{A}_{k/n} \) be a flat connection which lies outside the \( \Phi \)-accidental reducibles. Then there is a \( \lambda \) such that \( \lambda_1(\Delta_{\tilde{A}}^{\tilde{\sigma}(\ell)}) \geq \lambda > 0 \) for all \( \ell \).

**Proof.** According to Cor. 4.14, \( \lambda_1(\Delta_{\tilde{A}}^{\tilde{\sigma}(\ell)}) \to 0 \) if and only if \( \tilde{A} \) is accidentally reducible on \( \tilde{X}_0 \). But this is ruled out by the assumptions and Prop. 2.6.

The next step is to reduce to the case of a closed surface. First, notice that it suffices to get \( C^0 \) estimates for \( \tilde{h}(\ell,t) \) and \( *_\ell F_{\tilde{A}(\ell,t)} \). Also, by Lemma 2.3, and shrinking \( \Omega \) slightly to avoid branch points, it suffices to show:

\[
\| \log \tilde{h}(\ell,t) \|_{C^1(\tilde{\Omega})} < \varepsilon.
\]

Note also that by Prop. 5.6 we have a flat connection \( \tilde{A}_\infty \) such that \( \| \tilde{A}_\infty - \tilde{A}_0 \|_4 < \varepsilon_0 \), where \( \varepsilon_0 \) can be taken arbitrarily small. Furthermore, by Lemma 5.7 we may assume that \( \lambda_1(\Delta_{\tilde{A}}^{\tilde{\sigma}(\ell)}) \geq \lambda > 0 \) uniformly in \( \ell \). It follows by the fundamental estimate [R, Prop. 7.2 – second method] that if \( \varepsilon_0 \) is less than a universal constant \( \varepsilon_1 \), depending only on \( \lambda \), then

\[
\left\| D_{\tilde{A}_0}^{*_\ell} F_{\tilde{A}_0} \right\|_{2,\tilde{\sigma}(\ell)} \geq c \left\| *_\ell F_{\tilde{A}_0} \right\|_{2,\tilde{\sigma}(\ell)}.
\]

Since from now on we will work on a closed surface \( \tilde{X} \), we henceforth omit the hats from the notation. Again, \( \sigma(\ell) \to \sigma(0) \) is a conic degeneration of a closed surface \( X \). We begin with the following:

**Proposition 5.8.** Let \( A(\ell,t) \) be a solution of the Yang-Mills flow with initial condition \( A_0 \), \( \| *_\ell F_{A_0} \|_\infty < B \), and such that the fundamental estimate

\[
\left\| D_{A(\ell,t)}^{*_\ell} F_{A(\ell,t)} \right\|_{2,\sigma(\ell)} \geq c \left\| *_\ell F_{A(\ell,t)} \right\|_{2,\sigma(\ell)};
\]

\[
\left\| D_{A_0}^{*_\ell} F_{A_0} \right\|_{2,\tilde{\sigma}(\ell)} \geq c \left\| *_\ell F_{A_0} \right\|_{2,\tilde{\sigma}(\ell)}
\]
holds for a constant $c$ independent of $\ell$ and for $0 \leq t < \tilde{T} \leq \infty$. Then there are constants $c_1, c_2$ depending on $c$ and $B$, but independent of $\ell$ and $\tilde{T}$, such that for all $0 \leq t \leq \tilde{T}$:

1. $\|A(\ell,t)\|_{\infty} \leq c_1 e^{-c t/2}$; and,
2. $\|h(\ell,t)\|_{\infty} \leq c_2$.

**Proof.** For (1), first note that from (2.3):

$$\frac{d}{dt} \|A(\ell,t)\|_{2,\sigma(t)} = -\frac{c}{2} \left\| D_{A(\ell,t)}^* F_A(\ell,t) \right\|_{2,\sigma(t)}^2 \leq -\frac{c}{2} \left\| D_{A(\ell,t)}^* F_A(\ell,t) \right\|_{2,\sigma(t)}^2,$$

from which we obtain $\|A(\ell,t)\|_{2,\sigma(t)} \leq c_1 e^{-c t/2}$. Let $u(t,x) = \|A(\ell,t)\|_{2,\sigma(t)}$ be the pointwise norm. By [Do2, Prop. 16] it follows that $\dot{u} + \Delta_\sigma u \leq 0$, where $\Delta_\sigma$ is the ordinary Laplacian with respect to the metric $\sigma$. By the maximum principle, we can first choose $c_2$ so that (1) holds for $0 \leq t \leq 1$. Now we rechoose $c_2$ according to the heat kernel bounds of Thm. 4.13 and again apply the maximum principle.

For (2) consider the metric flow (2.4):

$$h^{-1} \frac{dh}{dt} = -\sqrt{-1} \ast \overline{\ast} F H_0, \quad h(0) = I.$$

Multiplying the equation through by $h$ and taking traces, we find:

$$\frac{d}{dt} \log \text{Tr} h \leq \tilde{C} \|\ast \overline{\ast} F \|_{\infty},$$

for some numerical constant $\tilde{C}$. By (1), the right hand side is uniformly integrable on $0 \leq t \leq \tilde{T}$. Since $\det h(t) = 1$, the result follows. \qed

**Proposition 5.9.** Let $A_0$ be as before and $\|A(\ell,t)\|_{\infty} < \varepsilon_2$. Then there exists a constant $c$ independent of $\ell$ such that:

$$\left\| D_{A(\ell,t)}^* F_A(\ell,t) \right\|_{2,\sigma(t)} \geq c \|\ast \ast F_A(\ell,t)\|_{2,\sigma(t)},$$

for all $t \geq 0$, where $A(\ell,t)$ is the solution of the Yang-Mills flow (2.4) with initial condition $A(\ell,0) = A_0$, and with respect to the metric $\sigma(\ell)$.

**Proof.** By (5.4), there is $c > 0$ such that:

$$\left\| D_{A(\ell,0)}^* F_A(\ell,0) \right\|_{2,\sigma(t)} \geq c \|\ast \ast F_A(\ell,0)\|_{2,\sigma(t)};$$

uniformly in $\ell$. Let $J = \{t \in [0,\infty) : \text{the estimate (5.6) holds on } [0,t]\}$. Then $J \neq \emptyset$. Let $T = \sup J$. We assume $T < \infty$, and derive a contradiction.

**Claim 1.** For the constant $c$ appearing in (5.6), $\|A(\ell,t) - A(\ell,0)\|_{2,\sigma(t)} < c^{-1} \varepsilon_2$.

By (5.5),

$$\frac{d}{dt} \|\ast \ast F_A(\ell,t)\|_{2,\sigma(t)} \leq -c \left\| D_{A(\ell,t)}^* F_A(\ell,t) \right\|_{2,\sigma(t)} = -c \left\| \frac{\partial A}{\partial t}(\ell,t) \right\|_{2,\sigma(t)},$$

for all $t \geq 0$. By (5.6), $\|\ast \ast F_A(\ell,t)\|_{2,\sigma(t)} \leq c \|\ast \ast F_A(\ell,0)\|_{2,\sigma(t)}$.
from which the estimate in Claim 1 follows by integration.

**Claim 2.** Write \( A(\ell, T) = g(\ell)A_0, h(\ell) = g(\ell)g^*(\ell). \) Then \( \|g(\ell)\|_\infty \) and \( \|g^{-1}(\ell)\|_\infty \) are uniformly bounded independent of \( \ell. \)

This is immediate from Prop. 5.8.

**Claim 3.** There exists a real gauge transformation \( u(\ell) \) such that
\[
\|u(\ell)A(\ell, T) - A_0\|_{L_2^2(A_0); \sigma(\ell)} < c\varepsilon_2,
\]
where \( c \) is independent of \( \ell. \)

Write \( A(\ell, T) = g(\ell)A_0 \) as before. By Claim 1, \( \|g^{-1}(\ell)\partial_{A_0}g(\ell)\|_{2;\sigma(\ell)} \leq c\varepsilon_2, \) and by Claim 2, \( \|\partial_{A_0}g(\ell)\|_{2;\sigma(\ell)} \leq c\varepsilon_2 \) (in this as in the following, \( c \) will denote a generic constant independent of \( \ell). \)

By elliptic regularity and the fact that \( \lambda_1 \left( \Delta^\sigma_{A_0} \right) \geq \lambda > 0 \) (see Prop. 2.4) it follows that
\[
\|\partial_{A_0}g(\ell)\|_{2;\sigma(\ell)} \leq \|d_{A_0}g(\ell)\|_{2;\sigma(\ell)} \leq \|g(\ell)\|_{L_2^2(A_0); \sigma(\ell)} \leq \lambda^{-1}\|\partial_{A_0}g(\ell)\|_{2;\sigma(\ell)} \leq c\varepsilon_2,
\]
where \( g(\ell) \) is the \( L^2 \)-orthogonal projection to the perp space of \( \text{ker}d_{A_0}. \) This result, combined with Claim 2, implies that \( \|h^{-1}(\ell)\partial_{A_0}h(\ell)\|_{2;\sigma(\ell)} \leq c\varepsilon_2. \) On the other hand,
\[
g^{-1}(\ell)F_{A(\ell, T)}g(\ell) = F_{A_0} + \partial_{A_0} (h^{-1}(\ell)\partial_{A_0}h(\ell)).
\]

Hence, by again applying elliptic regularity, \( \|h^{-1}(\ell)\partial_{A_0}h(\ell)\|_{L_2^2(A_0); \sigma(\ell)} \leq c\varepsilon_2. \) Since the connection defined from \( \partial_{A_0} \) and \( h(\ell)H_0 \) is real gauge equivalent to \( A(\ell, T), \) the proof of Claim 3 is complete.

For notational simplicity, set \( A(\ell, 0) = u(\ell)A(\ell, T). \) Then:
\[
(5.7) \quad \|A(\ell, 0) - A_0\|_{L_2^2(A_0); \sigma(\ell)} < c\varepsilon_2.
\]

**Claim 4.** Let \( \tilde{A}(\ell, t) \) denote the Yang-Mills flow with initial condition \( A(\ell, 0) \) and with respect to the metric \( \sigma(\ell). \) Then there is \( \delta > 0 \) and a real gauge transformation \( v(\ell, t) \) such that:
\[
\left\|v(\ell, t)A(\ell, t) - A_0\right\|_{4;\sigma(\ell)} < \varepsilon_1/2,
\]
for \( 0 < t < \delta \) and all \( \ell. \) Here, \( \varepsilon_1 \) is the universal constant so that Råde’s estimate (2.6) holds.

The proof of Claim 4 will be accomplished in three stages:

(i) Given \( \eta > 0, \exists \delta_1 > 0 \) such that for \( 0 \leq t \leq \delta_1 \) and all \( \ell, \|\log h(\ell, t)\|_{\infty} < \eta, \) where \( h(\ell, t) \) is the solution of (2.4)-(2.5) with respect to the holomorphic structure defined by \( \partial_{A(\ell, 0)} \). This follows by Prop. 5.8 as in Claim 2.
(ii) Given \( \eta > 0 \), \( \exists \, \delta_2 > 0 \) such that for \( 0 \leq t \leq \delta_2 \) and all \( \ell \), \( \| A(\ell, t) - A(\ell, 0) \|_{2, \sigma(\ell)} < \eta \). This obtains from the following inequalities:

\[
\| A(\ell, t) - A(\ell, 0) \|_{2, \sigma(\ell)} = \left\| \int_0^t \frac{d}{dt} A(\ell, t) dt \right\|_{2, \sigma(\ell)} \leq \int_0^t \left\| \frac{d}{dt} A(\ell, t) \right\| dt \\
= \int_0^t \left\| D^{\ast}_A(\ell, t) F_{A(\ell, t)} \right\|_{2, \sigma(\ell)} dt \leq t^{1/2} \left( \int_0^t \left\| D^{\ast}_A(\ell, t) F_{A(\ell, t)} \right\|^2_{2, \sigma(\ell)} dt \right)^{1/2} \\
= t^{1/2} \left( \| \ast \ell F_{A(\ell, 0)} \|_{2, \sigma(\ell)}^2 - \| \ast \ell F_{A(\ell, t)} \|_{2, \sigma(\ell)}^2 \right) \leq c t^{1/2} \varepsilon .
\]

(iii) Let \( A(\ell, t) = g(\ell, t) A(\ell, 0), \) \( h(\ell, t) = g^{\ast}(\ell, t) g(\ell, t), \) and \( \delta = \min(\delta_1, \delta_2) \). As in Claim 3, we first obtain \( \| \partial_{A(\ell, 0)} g(\ell, t) \|_{2, \sigma(\ell)} \leq c \eta \) for \( 0 \leq t \leq \delta \). On the other hand, by choosing \( \varepsilon_2 \) sufficiently small, it follows by (5.7) that we may assume that \( \lambda_1(\Delta(\ell, 0)) \geq \lambda/2 > 0 \). As in Claim 3, we obtain for \( 0 \leq t \leq \delta \),

\[
\| h^{-1}(\ell, t) \partial_{A(\ell, 0)} g(\ell, t) \|_{L^2_t(A_0, \sigma(\ell))} \leq c(\varepsilon_2 + \eta) .
\]

By taking \( \varepsilon_2 \) and \( \eta \) sufficiently small with respect to \( \varepsilon_1 \), and applying this result together with Kato’s inequality and the uniform embedding \( L^2_t \hookrightarrow L^4 \) for functions (see Prop. 4.10), we obtain Claim 4.

Now we are ready to complete the proof of the proposition. By Claim 4, and by taking \( \varepsilon_0 \) in Prop. 5.6 sufficiently small, it follows that \( \| v(\ell, t) A(\ell, t) - A_\infty \|_{4, \sigma(\ell)} < \varepsilon_1 \); hence,

\[
\left\| D^{\ast}_A(\ell, t) F_{A(\ell, t)} \right\|_{2, \sigma(\ell)} \geq c \| \ast \ell F_{A(\ell, t)} \|_{2, \sigma(\ell)} ,
\]

for \( 0 \leq t \leq T + \delta \), contradicting the assumption that \( T = \sup \mathcal{J} \).

The following corollary is an immediate consequence of Prop.’s 5.8 and 5.9, and completes the proof of the \( C^0 \) estimate:

**Corollary 5.10.** Let \( A_0 \) be as in Prop. 5.7. Then there are constants \( c_1, \, c_2, \) and \( c_3 \) independent of \( \ell \) and \( t \) such that

1. \( \| \ast \ell F_{A(\ell, t)} \|_{\infty} \leq c_1 e^{-c_2 t} \) for \( 0 \leq t \leq \tilde{T} \);
2. \( \| h(\ell, t) \|_{\infty} \leq c_3 \).

We now proceed with the proof of (5.3):

**Corollary 5.11.** Let \( \Omega \subset X \) be as before. Write \( A(\ell, t) = \tilde{g}(\ell, t) A(\ell, \infty), \) \( \tilde{h}(\ell, t) = \tilde{g}^{\ast}(\ell, t) \tilde{g}(\ell, t) \). Given \( \varepsilon > 0 \), \( \exists \, T_0 > 0 \) independent of \( \ell \) such that for all \( t \geq T_0 \), \( \| \log \tilde{h}(\ell, t) \|_{C^1(\Omega)} < \varepsilon \).

**Proof.** By Prop.’s 5.9 and 5.8, we have \( \| \ast \ell F_{A(\ell, t)} \|_{\infty} \to 0 \) as \( t \to \infty \) uniformly in \( \ell \), and:

\[
\int_0^\infty \| \ast \ell F_{A(\ell, t)} \|_{\infty} dt < \infty ,
\]
uniformly in $\ell$. From this we deduce as in Claims 1 and 2 that:

$$\lim_{t \to \infty} \| A(\ell, t) - A(\ell, \infty) \|_{2,\sigma(\ell)} = 0,$$

and that $\| \tilde{g}(\ell, t) \|_{2,\sigma(\ell)}$ and $\| \tilde{g}^{-1}(\ell, t) \|_{2,\sigma(\ell)}$ are bounded, all uniformly in $\ell$. Hence,

$$\| \tilde{\partial}_A(\ell, \infty) \tilde{g}(\ell, t) \|_{2,\sigma(\ell)},$$

can be made arbitrarily small independent of $\ell$. Since $\lambda_1(\Delta_{A(\ell, \infty)}) \geq \lambda > 0$ uniformly in $\ell$, it follows that $\| g(\ell, t) \|_{L^2_{1,\sigma(\ell)}}$ can be made arbitrarily small uniformly in $\ell$. Finally, by using the curvature estimate:

$$\tilde{g}^{-1}(\ell, t) F_{A(\ell, t)} \tilde{g}(\ell, t) = \tilde{\partial}_A(\ell, \infty) \left( \tilde{h}^{-1}(\ell, t) \partial_{A(\ell, \infty)} \tilde{h}(\ell, t) \right),$$

we obtain the corollary by bootstrapping in $\Omega$. This proves (5.3), and thus completes the proof of Prop. 5.5.

5.4. Proof of Theorem 5.2. Let $h(\ell, \infty)$ denote the limit at infinite time of the solution to eq.’s (2.4)-(2.5). Then up to real gauge, $A(\ell, \infty)$ is $h^{-1/2}(\ell, \infty) \cdot A_0$. It suffices to obtain an estimate of the form:

$$\int_{\sqrt{\bar{\varepsilon}(\ell)}}^{\varepsilon_0} |_{\partial/\partial r} \left\{ h^{-1/2}(\ell, \infty) \partial_{A_0} h^{-1/2}(\ell, \infty) - c.c. \right\} \, dr < \delta,$$

(5.8)

for sufficiently small $\varepsilon_0$ and $\ell$. In the above, the coordinate $r$ is $|z|$ for the conformal coordinates on $(C, ds^2)$ from Prop. 4.8, and $|_{\partial/\partial r}$ denotes contraction of the form in the $r$ direction. The notation $c.c.$ means hermitian conjugation of the previous term. Also, note that we have taken the square root $\sqrt{\bar{\varepsilon}}$, since with respect to the conformal coordinates in Sec. 4.2 this corresponds to a point in the middle of the cylinder, i.e. $x = 0$. The estimate for the whole arc $\Gamma^{\varepsilon_0}$ follows from the estimates in the $z$ and $w$ coordinates separately.

The square root on the gauge transformation $h$ is difficult to work with directly, so we will eliminate it by using a $C^0$ bound. Namely, to prove (5.8), is suffices to prove:

1. There is $B$ such that $\| h(\ell, \infty) \| \leq B$ for all $\ell$;
2. $\int_{\sqrt{\bar{\varepsilon}(\ell)}}^{\varepsilon_0} |_{\partial/\partial r} \left\{ h^{-1} \partial_{A_0} h - c.c. \right\} \, dr < \delta$ for $\varepsilon_0$ sufficiently small and $\sqrt{\bar{\varepsilon}(\ell)} \leq \varepsilon$.

We further reduce this with the following:

Proposition 5.12. Suppose that there is a constant $B$ such that (1) holds. Suppose in addition that $\| h^{-1} \partial_{A_0} h - c.c. \|_{2,\sigma(\ell)} \leq B$ for all $\ell$. Then (2) also holds.

Proof. Set $w = h^{-1} \partial_{A_0} h - c.c.$ The first step of the proof is to show that the hypotheses imply an estimate of the form:

$$\| w \|_{4,\sigma(\ell)} \leq B/\ell^2,$$

(5.9)
where $B$ is independent of $\ell$. Consider the function $|w|$. By Prop. 4.10, it suffices to estimate $d|w|$. By Kato’s inequality, \( |d|w| |^2 \leq |\nabla A_0 w|^2 \). So it suffices to estimate the right hand side. We now apply the Weitzenböck formula for a 1-form $\omega$ with values in the self-adjoint bundle:

\[
\Delta_{A_0} \omega = -\nabla^*_{A_0} \nabla_{A_0} \omega + \{ R, \omega \} + \{ F_{A_0}, \omega \} .
\]

Now $F_{A_0}$ is uniformly bounded in any norm, say $C^1$, and we have $\Delta_{A_0} h^{-1} \partial_{A_0} h = -\bar{\partial}_{A_0} F_{A_0}$; so $\Delta_{A_0} w$ is bounded as well. Therefore, the desired estimate is obtained, provided we can estimate the term $\{ R, w \}$. The explicit formula can be found in [Wu], and it involves the operator $R_k^i = \sigma^{ij} R_{kij}^i$ (see [Wu, p. 953]). Then (5.9) follows from Prop. 4.7.

We now consider the problem on a cylinder $(C, ds^2)$. Choose a local holomorphic frame $\{ f_i \}$ for $A_0$ adapted to $\{ e_i \}$ as in Sec. 2.1. In conformal coordinates, we may write:

\[
h^{-1} \partial_{A_0} h = \sum_{n \in \mathbb{Z}} c_n^i z^n f_i \otimes f_j^* \otimes dz ,
\]

since $\bar{\partial}_{A_0} (h^{-1} \partial_{A_0} h) = 0$. The coefficients $c_n^i$ depend on $\ell$, and we wish to estimate them. For convenience, we set $|c_{-1}|^2 = |c_{11}^1|^2 + |c_{22}^2|^2$. We assume the frame has been chosen such that $|f_i \otimes f_j^*| = |z|^{\gamma_i - \gamma_j}$, where $\gamma_1 = \gamma$, $\gamma_2 = -\gamma$. We will assume that $\gamma \neq 0$, the argument in the case $\gamma = 0$ being similar. Also, since we assume that $\sigma(\ell)$ converges to a cone metric of the type (4.1), the integrals below will be carried out with respect to this metric. A simple computation shows that these estimates are valid. We have:

\[
|h^{-1} \partial_{A_0} h|^2 = \sum_{m,n \in \mathbb{Z}} c_m^i c_n^j z^m \bar{z}^n |z|^{\gamma_i - \gamma_j} dz^2 .
\]

The $L^2$ bound on $w$ implies one on $|h^{-1} \partial_{A_0} h|$, which in turn implies that there is a constant $B$ independent of $\ell$ such that:

\[
|c_{m}^i|^2 \int_{|z| \leq 1} |z|^{2m + \gamma_i - \gamma_j} |dz|^2 \leq B .
\]

Therefore, there is another constant independent of $\ell$, which we also denote by $B$, such that:

(i) For $m \geq 0$, $|c_{m}^i| \leq B$. Also, $|c_{-1}^1| \leq B$;

(ii) For $m \leq -2$, or $(m, i, j) = (-1, 2, 1)$, $|c_{m}^i| \leq B|z|^{-m - \frac{1}{2}(\gamma_i - \gamma_j) - 1}$;

(iii) $|c_{-1}|^2 \log(1/\varepsilon(\ell)) \leq B$.

We now apply this to:

\[
\int_{\varepsilon^{1/2}}^{\varepsilon_0} |\partial_{A_0} h| d\varepsilon \leq \sum_{(m, i, j) \neq (-1, i, i)} |c_{m}^i| \int_{\varepsilon^{1/2}}^{\varepsilon_0} r^{m + \frac{1}{2}(\gamma_i - \gamma_j)} dr + 2|c_{-1}| \log(1/\varepsilon) .
\]

By the estimates (i) and (ii) above, the first term on the right hand side may be made arbitrarily small for small $\varepsilon_0$, independent of $\ell$. To estimate the second term, we use the $L^1$ bound (5.4). Because of the log term, it suffices to show that $|c_{-1}|$ vanishes as some power of $\varepsilon$. The estimate (iii) is not sufficient.
To use the $L^4$ bound, we first isolate the $m = -1$ terms. Again, (5.9) implies a similar bound on the $L^4$ norm of $h^{-1}\partial A_0 h$. Write:

$$|h^{-1}\partial A_0 h|^2 = \frac{|c_{-1}|^2 + |c_{-1}^2||z|^{2\gamma} + |c_{-1}^4||z|^{-2\gamma}}{|z|^2} + g,$$

Integrating over a subcylinder $C_a$, we have:

$$\int_{C_a} \frac{|c_{-1}|^4}{|z|^4} ds_0^2 \leq B \int_{C_a} \left( |h^{-1}\partial A_0 h|^4 + g^2 + \frac{|c_{-1}^2|^4|z|^{4\gamma}}{|z|^4} + \frac{|c_{-1}^4|^4|z|^{-4\gamma}}{|z|^4} \right) ds_0^2. \tag{5.10}$$

The subcylinder is given by $C_a = \{ z \in C : \varepsilon a \leq |z| \leq \varepsilon_0 \}$, where $a$ will be chosen as follows:

**Claim.** For:

$$\frac{1 + \kappa}{3 - \kappa} > a > \frac{\kappa}{2 - \kappa},$$

there is a constant $B$ independent of $\ell$ so that:

$$\int_{C_a} g^2 ds_0^2 \leq \frac{B}{\varepsilon^{2\kappa}}.$$

Assuming this claim, we complete the proof. By (5.9) and Prop. 1.8 applied to (5.10), we have:

$$\int_{C_a} \frac{|c_{-1}|^4}{|z|^4} ds_0^2 \leq \frac{B}{\varepsilon^{2\kappa}} + \int_{C_a} \left( \frac{|c_{-1}^2|^4|z|^{4\gamma}}{|z|^4} + \frac{|c_{-1}^4|^4|z|^{-4\gamma}}{|z|^4} \right) ds_0^2.$$

Carrying out the integrals on both sides, this implies $|c_{-1}|^4 \leq B (\varepsilon^{(4-2\kappa)a-2\kappa} + \varepsilon^{4\gamma(1-a)})$. By the choice of $a$ in (5.11), $(4 - 2\kappa)a - 2\kappa > 0$, so we are finished.

It remains to prove the claim. First, by the expression for $|h^{-1}\partial A_0 h|^2$, note that the terms in the series for $g$ involving the coefficients $c_n$ and $c_0$ are bounded uniformly by $\varepsilon_0^{(|m|+|n|)b}$ for some $b > 0$ and $|m|, |n|$ large, depending upon $a$. Therefore, to bound the integral of $g^2$, it suffices to bound the squares of the individual terms. Of the terms which appear, there are three types which need to be estimated:

I. $|c_{ij}|^2 |c_{m}^j| |c_{n}^i| |z|^{2(m+n)+2(\gamma_i-\gamma_j)}$, where $m, n \leq -2$;

II. $|c_{-1}^2|^2 |c_{-1}^4||z|^{2(n-1)-2\gamma}$ and $|c_{-1}^2|^2 |c_{-1}^4||z|^{2(n-1)+2\gamma}$, where $n \leq -2$;

III. $|c_{-1}^2|^2 |c_{-1}^4|^2 |z|^{2(n-1)}$, where $n \leq -2$.

Terms of type I give integrals of the form (using (i) above):

$$\int_{\varepsilon a}^{\varepsilon a} |c_{m}^j|^2 |c_{n}^i|^2 |z|^{2(m+n)+2(\gamma_i-\gamma_j)-1+2\kappa} dr \leq B|c_{m}^j|^2 |c_{n}^i|^2 |z|^{2(m+n)+2(\gamma_i-\gamma_j)+2\kappa}a$$

$$\leq B \varepsilon^{-2(m+n)(1-a)-2(\gamma_i-\gamma_j)(1-a)-2(2-\kappa a)}.$$

But:

$$-2(m+n)(1-a) - 2(\gamma_i - \gamma_j)(1-a) - 2(2 - \kappa a) \geq -2(m+n)(1-a) - 2(1-a) - 2(2 - \kappa a)$$

$$\geq -2(m+n+3)(1-a),$$
so these terms are, in fact, bounded. For type III, we apply the same estimate to get:

$$
\int_{\epsilon_0}^{\epsilon_a} |c_{-1}|^2 |c_{n+1}^i|^2 r^{2(n-1)-1+2\kappa} dr \leq B |c_{-1}|^2 \epsilon^{-2(n-1)(1-a)-2(2-\kappa a)}.
$$

Since $n \leq -2$, the exponent of $\epsilon$ is $\geq -2\kappa$ by the assumption on $a$ in (5.11). Type II is similar to these two computations. 

Since the hypotheses of Prop. 5.12 are satisfied by Cor. 5.10 (2), and the uniform $L^2$ bound on $|h^{-1/2} \partial h^{1/2}|$, this completes the proof of (5.1) and also of the Main Theorem.

References

[A] W. Abikoff, “The Real Analytic Theory of Teichmüller Space,” Lect. Notes in Math. 820, Springer-Verlag, Berlin, 1980.

[AB] M.F. Atiyah and R. Bott, The Yang-Mills equations over Riemann surfaces, Phil. Trans. R. Soc. Lond. A 308 (1982), 523-615.

[Ch1] J. Cheeger, On the spectral geometry of spaces with cone-like singularities, Proc. Nat. Acad. Sci. USA 76 (1979), 2103-2106.

[Ch2] J. Cheeger, On the Hodge theory of Riemannian manifolds, in “Geometry of the Laplace Operator,” Proc. Sympos. Pure Math. 36, Amer. Math. Soc., Providence, 1980, 91-146.

[CL] S.-Y. Cheng and P. Li, Heat kernel estimates and lower bound of eigenvalues, Comment. Math. Helvetici 56 (1981), 327-338.

[D] G. Daskalopoulos, The topology of the space of stable bundles on a Riemann surface, J. Diff. Geom. 36 (1992), 699-746.

[DDW] G. Daskalopoulos, S. Dostoglou, and R. Wentworth, Families of $SU(2)$ representations for mapping cylinders of periodic monodromy, Proc. Edinburgh Math. Soc. 40 (1997), 383-392.

[DW1] G. Daskalopoulos and R. Wentworth, Geometric quantization for the moduli space of vector bundles with parabolic structure, in geometry, topology and physics, proceedings of the first Brazil-USA workshop, Campinas, Brazil, June 30- July 7, 1996. B.N. Apanasov, S.B. Bradlow, W.A. Rodrigues, Jr., K.K. Uhlenbeck (editors), Walter de Gruyter Publishers, 1997, 119-155.

[DW2] G. Daskalopoulos and R. Wentworth, Local degeneration of the moduli space of vector bundles and factorization of rank two theta functions I, Math. Ann. 297 (1993), 417-466.

[Do1] S.K. Donaldson, A new proof of a theorem of Narasimhan and Seshadri, J. Diff. Geom. 18 (1983), 269-277.

[Do2] S.K. Donaldson, Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles, Proc. London Math. Soc. 50 (1985), 1-26.

[G] F. Gardiner, “Teichmüller Theory and Quadratic Differentials,” Wiley Interscience, New York, 1987.

[JW1] L. Ji and R. Wentworth, Spectral convergence on degenerating surfaces, Duke Math. J. 66 (1992), 469-501.

[JW2] L. Ji and R. Wentworth, Correction to ”Spectral convergence on degenerating surfaces”, Duke Math. J. 90 (1997), 205-207.

[Li] P. Li, On the Sobolev constant and the $p$-spectrum of a compact Riemannian manifold, Ann. Scient. Éc. Norm. Sup. (4) 13 (1980), 451-469.

[MS] V. Mehta and C. Seshadri, Moduli of vector bundles on curves with parabolic structures, Math. Ann. 248 (1980), 205-239.

[NS] M.S. Narasimhan and C. Seshadri, Stable and unitary vector bundles on a compact Riemann surface, Ann. of Math. 82 (1965), 540-567.

[R] J. Råde, On the Yang-Mills heat equation in two and three dimensions, J. Reine. Angew. Math. 431 (1992), 123-163.

[S1] C. Simpson, Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization, J. Amer. Math. Soc. 1 (1988), 867-918.

[S2] C. Simpson, Harmonic bundles on noncompact curves, J. Amer. Math. Soc. 3 (1991), 713-770.

[W] H. Weinberger, “Variational Methods for Eigenvalue Approximation,” SIAM, Philadelphia, 1974.

[Wo] S. Wolpert, Noncompleteness of the Weil-Petersson metric for Teichmüller space, Pacific J. Math. 61 (1975), 573-577.
[Wu] H. Wu, *The Bochner technique*, in Proceedings of the 1980 Beijing Symposium on Differential Geometry and Differential Equations, Vol. 1, 2, 3 (Beijing, 1980), Science Press, Beijing, 1982, 929-1071.

Department of Mathematics, Brown University, Providence, RI 02912  
E-mail address: daskal@gauss.math.brown.edu

Department of Mathematics, University of California, Irvine, CA 92697  
E-mail address: raw@math.uci.edu