AN ITERATED RESIDUE PERSPECTIVE ON STABLE GROTHENDIECK POLYNOMIALS

JUSTIN ALLMAN AND RICHÁRD RIMÁNYI

Abstract. Grothendieck polynomials are important objects in the study of the K-theory of flag varieties. They exhibit many remarkable properties which have been studied in the context of algebraic geometry and tableaux combinatorics. We introduce a new tool, similar to generating sequences, which we call the iterated residue technique. In particular, we give new proofs that the Pieri rule for stable Grothendieck polynomials and their Schur expansions exhibit alternating signs using this method.

Introduction

Grothendieck polynomials, one for each permutation, were introduced by Lascaux and Schützenberger in their seminal work [LS82] as representatives for K-classes of Schubert varieties in flag manifolds. They play the role in K-theory of the Schubert polynomials in cohomology. Stable versions of these polynomials, again corresponding to any permutation, appeared in the works of Fomin and Kirillov [FK94] [FK96] and were shown to be supersymmetric via an analogous construction to that of the Stanley symmetric functions. Buch showed that the stable Grothendieck polynomials corresponding to Grassmannian permutations, and therefore to partitions, formed a Z-linear basis for all stable Grothendieck polynomials [Buc02b]. It is these polynomials which are the subject of this note.

Over the past decade, the stable Grothendieck polynomials corresponding to partitions have become important to many problems in the realm of algebraic combinatorics, for example in [Len00, Buc02b], and in particular have appeared in several algebro-geometric contexts:

• as representatives of Schubert varieties in the K-theory of Grassmannians, e.g. in [Bri05, Buc05],
• as the proper basis to describe K-classes of degeneracy loci for Dynkin quivers, e.g. in [All13, Buc02a, Buc08, Mil05], and
• as a basis for Thom polynomials in K-theory [RS14].

In each of these settings (and more) alternating signs have been proven or at least conjectured to appear in the relevant formulae, see [Bri02, Bri05, Buc02a, Buc02b, Buc08, FL94, Len00, Mil05, RS14] and many more. These results regarding alternating signs appear to depend on an amalgam of geometric techniques e.g. in [Bri02, FL94], inventive combinatorics e.g. in [Buc02b, Buc02a, Buc08, Len00, Mil05], or identities in the theory of symmetric functions e.g. in [FG98, Len00].

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Recently the second author and Szénés have given a new formula for the stable Grothendieck polynomials in terms of *iterated residues* \[\text{RS14}\] which we now describe. Let \(\alpha = (\alpha_1, \ldots, \alpha_M)\) be a set of commuting variables and \(z = (z_1, z_2, \ldots)\) an ordered set of complex valued indeterminates. For each positive integer \(p\) define the meromorphic differential form

\[
\kappa_p(\alpha|z) = \left( \prod_{i=1}^{p} \frac{(1 - z_i)^M}{\prod_{a \in \alpha} (1 - z_i^a)} d\log(z_i) \right) \prod_{1 \leq i < j \leq p} \left( 1 - \frac{z_j}{z_i} \right).
\]

For any integer sequence \(I = (I_1, \ldots, I_p)\) the (single) stable Grothendieck polynomial is given by the *iterated residue* formula

\[
G_I(\alpha) = \text{Res}_{z=0,\infty} \left( \prod_{j=1}^{p} (1 - z_j)^{I_j - \sum_{i<j} I_i} \kappa_p(\alpha|z) \right).
\]

where \(z = (z_1, \ldots, z_p)\) and for any rational function \(f\) in the variables \(z\) we define \(\text{Res}_{z=0,\infty}(f(z) \, dz)\) to be

\[
\left( \text{Res}_{z_p=0} + \text{Res}_{z_p=\infty} \right) \circ \cdots \circ \left( \text{Res}_{z_2=0} + \text{Res}_{z_2=\infty} \right) \circ \left( \text{Res}_{z_1=0} + \text{Res}_{z_1=\infty} \right) (f(z) \, dz).
\]

For more details see also \[\text{All13}\] and \[\text{All14}\]. It is remarkable that the iterated residue formulation of \(G_I(\alpha)\) agrees with the definition, originally due to Buch \[\text{Buc02a}\], for stable Grothendieck polynomials corresponding also to non-partition integer sequences \[\text{RS14}\]. When \(I\) is not a partition we call \(G_I(\alpha)\) a fake Grothendieck polynomial. Throughout the sequel, when a statement about Grothendieck polynomials holds independently of the variables \(\alpha\) we will write simply \(G_I\) for \(G_I(\alpha)\). We trivially obtain that \(G_\emptyset = 1\) since in this case there are no residues to compute of the constant rational function equal to unity.

Using Equation (0.1), the second author and Szénés have given new results for expansions of \(K\)-theoretic Thom polynomials in the basis of Grothendieck polynomials \[\text{RS14}\], and the first author has provided a new method for computing \(K\)-classes corresponding to degeneracy loci of Dynkin quivers \[\text{All13}\] and exhibited iterated residue formulae for the multiplication and comultiplication of stable Grothendieck polynomials \[\text{All14}\].

Now we recall several facts and notations from \[\text{All14}\]. Let \(t = (t_1, t_2, \ldots)\) denote an alphabet of ordered commuting variables. Henceforth, given a finite integer sequence \(I = (I_1, \ldots, I_p)\) we denote the monomial \(\prod_{j=1}^{n} t_{i_j}^{I_j}\) succinctly by \(t^I\) in multi-index notation. Let \(\lambda = (\lambda_1, \ldots, \lambda_p)\) be a finite integer sequence, not necessarily a partition, and define the following operation on monomials in the ring \(R = \mathbb{Z}((t))\)

\[
G_{\ell}(t^\lambda) = G_{\lambda}.
\]

Moreover if \(a\) denotes any object on which a stable Grothendieck polynomial can be evaluated, for example the variables \(\alpha\) of Equation (0.1), then define

\[
G^a_{\ell}(t^\lambda) = G_{\lambda}(a).
\]

We will also refer to the \(G^a_{\ell}\) operation as an *iterated residue operation*. Such formulae sometimes appear in the literature with the title *constant term operations*. For justification of this terminology please see \[\text{All14}\] or \[\text{Rim14}\]. Extend these operations linearly to the \(\mathbb{Z}\)-submodule of \(\mathbb{Z}((t))\) for which the result is a *finite*
$\mathbb{Z}$-linear combination of stable Grothendieck polynomials. Furthermore, we allow $G_t$ to act on rational functions in $\mathbb{Z}(t)$ provided they have a formal Laurent expansion. By convention, we will always operate on the domain $t_i << t_j$ for $i < j$ when computing such expansions. It is convenient to define the factors
\[ d_i = 1 - t_i \]
\[ d_{i,j} = 1 - t_i t_j^{-1} \]
and the multiplication kernel given by the rational function
\[ K_{p,q}(t) = \frac{d_q^p}{\prod_{j=1}^{q} d_{i,p+j}}. \]

If $I = (I_1, \ldots , I_p)$ and $J = (J_1, \ldots , J_q)$ are finite integer sequences one can form the new sequence $I, J = (I_1, \ldots , I_p, J_1, \ldots , J_q)$ and if $p = q$, also form the sequence $I + J = (I_1 + J_1, \ldots , I_p + J_q)$. These notations will appear throughout.

**Proposition 0.1** (Allman [All14], Theorem 2.14). The product of the stable Grothendieck polynomials $G_I G_J$ is given by the formula
\[ G_t^{I, J} (t K_{p,q}(t)). \]

Now by analogy to the definitions above we can define another iterated residue operation $S^x_t$ as follows. Set
\[ S_t (t^{\lambda}) = s_\lambda \]
where $s_\lambda$ denotes the Schur function [Mac95, Section I.3] for the integer sequence $\lambda$. Moreover if $a$ is any object on which a Schur function can be evaluated then we also define
\[ S^a_t (t^{\lambda}) = s_\lambda (a). \]

In particular we now define a family of variables $x$ by $x_i = 1 - \alpha_i^{-1}$ for every positive integer $i$ where, for $\alpha = (\alpha_1, \ldots , \alpha_M)$ as above, we assume that $\alpha_j = 1$ for $j > M$. Sometimes the stable Grothendieck polynomials are given in terms of these variables, for example in [Buc02b], and in this case we also write $G_I (x)$ for the corresponding polynomial. We will indicate the proper context when necessary.

Since the Grothendieck polynomial $G_I (x)$ is symmetric in the variables $x$, it has an expansion in the Schur basis.

**Proposition 0.2** (Allman [All14], Corollary 3.9). Let $\lambda = (\lambda_1, \ldots , \lambda_p)$ be a partition and $x = (x_1, \ldots , x_M)$ be a set of commuting variables with $M \geq p$. Then
\[ G_t^{x} (t^{\lambda}) = S_t^x \left( t^{\lambda} \prod_{i=1}^{M} d_i^{-1} \right). \]

Both propositions depend only on algebraic manipulation of rational functions or, in the case of Proposition 0.1, some additional analytic facts from the theory of complex integration.

In this note we give two new proofs of alternating signs in theorems, both originally due to Lenart [Len00] via combinatorics and, in the case of the first proof, separately confirmed by Fulton and Lascoux [FL94] and Brion [Bri02] using geometric methods. In Section 1 we set notation and lay out relevant background material. In Section 2 we use iterated residue methods alongside Proposition 0.1 to show that the stable Grothendieck polynomial analogue of the Pieri rule exhibits alternating signs. Finally, in Section 3 we show that the Schur expansion of any
stable Grothendieck polynomial also exhibits alternating signs using Proposition 0.2.

Several other positivity and stability results from iterated residues have recently appeared. Bérczi and Szenes established the positivity of several Thom polynomials of so-called Morin singularities [BS12]. In this vein, using iterated residue techniques from [Kaz10a] Kazarian has provided formulas for Thom polynomials of contact singularities [Kaz10b]. Kaliszewski has given several new results on cohomological quiver loci [Kal13] using an iterated residue formula of the second author [Rim14].

It is our hope that iterated residue techniques will develop to provide a formal tool in attacking open conjectures regarding positivity, especially alternating signs related to Grothendieck polynomials. We view this note as a first example of techniques in such a theory.

1. Definitions and examples

1.1. Preliminaries. A weakly decreasing integer sequence \( \lambda = (\lambda_1, \ldots, \lambda_p) \) is called a partition. The length \( \ell(\lambda) \) is the number of nonzero parts and the weight \( |\lambda| \) is the sum of the parts. Allow \( \mathbf{x} = (x_1, x_2, \ldots) \) to denote a family of commuting variables; it is possible that only finitely many of these are non-zero. The Jacobi-Trudi formula (cf. [Mac95, Equation I.3.4]) defines for every partition \( \lambda \) a Schur function according to the determinant

\[
 s_\lambda = s_\lambda(\mathbf{x}) = \det (h_{\lambda_i+j-i})_{1 \leq i, j \leq p},
\]

where the symbols \( h_d \) denote the complete homogeneous symmetric functions in the variables \( \mathbf{x} \). Alternatively, \( h_d \) is defined formally as the coefficient of \( u^d \) in the expansion of the generating function \( \prod_{x \in \mathbf{x}} (1 - xu)^{-1} \). The Schur function \( s_\emptyset = 1 \) since in this case there is no determinant to compute.

To a partition \( \lambda \) and the family \( \mathbf{x} \) one also associates the (single) stable Grothendieck polynomial \( G_\lambda(\mathbf{x}) \). See for example [LS82] [FG98] [Buc02b]. It is well-known that \( G_\lambda(\mathbf{x}) \) is symmetric in \( \mathbf{x} \) and is non-homogeneous. The lowest degree term of \( G_\lambda \) is the corresponding Schur polynomial \( s_\lambda \). It is also known that for any partitions \( \lambda \) and \( \mu \), there exist unique integers \( c^\nu_{\lambda, \mu} \), only finitely many of which are non-zero, such that

\[
 G_\lambda G_\mu = \sum_{\nu} c^\nu_{\lambda, \mu} G_\nu
\]

where the sum is over all partitions \( \nu \) with \( |\nu| \geq |\lambda| + |\mu| \). Buch has given a combinatorial formula for the integers \( c^\nu_{\lambda, \mu} \) in terms of counting set-valued tableaux [Buc02a].

For general integer sequences \( I = (I_1, I_2, \ldots) \) with only finitely many non-zero parts we define the length \( \ell(I) \) to be the largest integer \( p \) for which \( I_p \) is non-zero. Such a sequence is called finite. When \( I \) is a partition, the two notions of length coincide. The weight \( |I| = \sum_k I_k \) is also defined for finite sequences. One uses the Jacobi-Trudi formula (1.1) to define for any finite non-partition integer sequence \( I = (I_1, \ldots, I_p) \) a fake Schur function

\[
 s_I = \det (h_{I_i+j-i})_{1 \leq i, j \leq p}.
\]

Now let \( I \) and \( J \) be any finite integer sequences, \( a \) and \( b \) any integers, and \( K \) a finite integer sequence with only non-positive parts. By interchanging rows in
the determinant above, one obtains the following “straightening laws” for Schur functions

\begin{align}
I(a,b,J) &= -s_{I(b-1,a+1,J)} \\
I(a,a+1,J) &= 0 \\
I(K) &= 0.
\end{align}

These relations imply that any (possibly fake) Schur function \(s_I\) is either equal to \(\pm s_\nu\) for some partition \(\nu\) or is zero. Analogously Buch has defined stable Grothendieck polynomials \(G_I\) for any integer sequence \(I\) using a formula of Lenart [Len00, Theorem 2.4] and has proven the following “straightening laws” (see [Buc02a], Section 3)

\begin{align}
G_{I(a,b,J)} &= G_{I(a+1,b,J)} + G_{I,b(a+1),J} - G_{I,b-1,a+1,J} \\
G_{I(a,a+1,J)} &= G_{I(a+1,a+1,J)} \\
G_{I,K} &= G_I.
\end{align}

The stable Grothendieck polynomial straightening relations imply that for any integer sequence \(I\) there exist unique integers \(\delta_{I,\lambda}\) only finitely many of which are nonzero, such that \(G_I = \sum_\lambda \delta_{I,\lambda} G_\lambda\). In contrast to the Schur functions this implies that \(G_I\) is never zero. Notice that in performing these operations signs can be introduced in a complicated way, i.e. the coefficients \(\delta_{I,\lambda}\) do not necessarily alternate with the difference of weights \(|\lambda| - |I|\). The laws of Equations (1.6), (1.7), and (1.8) turn out to be simple analytical consequences of Equation (0.1), see [RS14].

1.2. Iterated residues. Whenever two rational functions \(f_1\) and \(f_2\) have the property that \(G(f_1) = G(f_2)\) we say that \(f_1\) is \(G\)-equivalent to \(f_2\) and write

\[ f_1 \sim G f_2. \]

We now prove two lemmas which we will find useful later for manipulating rational functions inside the \(G\) operation.

**Lemma 1.1.** Let \(f(t)\) be a rational function independent of both \(t_i\) and \(t_{i+1}\). For any \(a, b \in \mathbb{Z}\) one has the following “straightening law”.

\[ f \cdot t_i^a t_{i+1}^b d_i \sim G f \cdot t_i^{b-1} t_{i+1}^{a+1} d_i. \]

In particular, if \(b = a + 1\) then the expression is \(G\)-equivalent to zero.

**Proof.** The numbered equation (1.9) is just an encoding of the relation (1.6) in the language of the \(G\)-operation. Similarly, the case \(b = a + 1\) encodes equation (1.7). \(\square\)

**Lemma 1.2.** Let \(F(t_1, \ldots, t_{r-1}, t_r)\) be a rational function in \(\mathbb{Z}(t)\). In particular, \(F\) is independent of \(t_i\) for \(i > r\). Suppose that no positive power of \(t_r\) appears in its Laurent expansion about zero. Then

\[ F(t_1, \ldots, t_{r-1}, t_r) \sim G F(t_1, \ldots, t_{r-1}, 1). \]

**Proof.** This is the iterated residue version of (1.8). \(\square\)
Example 1.3. Proposition 0.1 implies that \( G_1 \cdot G_1 \) is equal to applying the \( \mathcal{G}_t \) operation to the rational function

\[
t_1 t_2 \left( \frac{1 - t_1}{1 - t_1 t_2} \right),
\]

which can be rewritten as

\[
t_1 t_2 (1 - t_1) \left( 1 + \frac{t_2}{1 - t_1} \right) = t_1 t_2 (1 - t_1) + t_1^2 \frac{1 - t_1}{1 - t_2}
\]

where the second term has only non-positive powers of \( t_2 \). By Lemma 1.2 the expression above is \( \mathcal{G} \)-equivalent to the polynomial

\[
t_1 t_2 (1 - t_1) + t_1^2 = t_1 t_2 + t_1^2 t_2,
\]

and so the result of applying the operation \( \mathcal{G}_t \) is \( G_{1,1} + G_2 - G_{2,1} \).

\[\Box\]

2. The Pieri rule for Grothendieck polynomials

We devote this section to a new proof of the following theorem, a Pieri rule for the multiplication of stable Grothendieck polynomials.

Theorem 2.1 (Lenart [Len00], Theorem 3.4). For any partition \( \lambda \) and any positive integer \( n \), the product \( G_\lambda G_n \) has the form

\[
G_\lambda G_n = \sum_\nu c^\nu_\lambda G_\nu
\]

where the sum is taken over all partitions \( \nu \) with \( |\nu| \geq |\lambda| + n \). The integer \( c^\nu_\lambda \) is non-zero for only finitely many partitions \( \nu \) and furthermore satisfies the positivity property

\[
(-1)^{|\nu| - |\lambda| - n} c^\nu_\lambda \geq 0.
\]

The statement is a special case of Equation (1.2) in which the partition \( \mu \) happens to have length one. We begin by proving two lemmas which allow us to gather many terms in the expansion of Proposition 0.1 together for cancelation.

Lemma 2.2. Let \( a < b \) be positive integers and \( f(t) \) be a rational function which does not depend on \( t_i \) and \( t_{i+1} \) and set

\[
\sigma = t_i^a t_{i+1}^b \sum_{j=0}^{b-a-1} \left( \frac{t_i}{t_{i+1}} \right)^j = t_i^a t_{i+1}^b + t_i^{a+1} t_{i+1}^{b-1} + \ldots + t_i^{b-1} t_{i+1}^{a+1} + t_i^{b-a} t_{i+1}^a.
\]

Then \( f \cdot \sigma \cdot d_i \overset{\mathcal{G}}{\sim} f \cdot t_i^a t_{i+1}^b \cdot d_i \). Equivalently, if \( \sigma' \) denotes the sum

\[
\sum_{j=0}^{b-a-1} \left( t_i^a t_{i+1}^b \right)^j = t_i^a t_{i+1}^b + t_i^{a+1} t_{i+1}^{b-1} + \ldots + t_i^{b-2} t_{i+1}^{a+2} + t_i^{b-1} t_{i+1}^{a+1},
\]

then \( f \cdot \sigma' \cdot d_i \overset{\mathcal{G}}{\sim} 0 \).

Proof. It suffices to consider the case when \( f = 1 \) and \( i = 1 \). The straightening Lemma 1.1 implies that \( t_1^a t_2^b (1 - t_1) \overset{\mathcal{G}}{\sim} -t_1^{b-1} t_2^{a+1} (1 - t_1) \), and hence the first and last terms in \( \sigma' \) cancel. One notes also that the second and second-to-last terms cancel since \( t_1^a t_2^{b-1} (1 - t_1) \overset{\mathcal{G}}{\sim} -t_1^{b-2} t_2^{a+2} (1 - t_1) \). Similarly the other terms cancel in pairs, except if \( \sigma' \) has an odd number of terms (occurring whenever \( b - a \) is odd). In this
case, the uncanceled term in the middle is necessarily of the form $t_i^{a+r}t_{i+1}^{a+r+1}(1-t_1)$ where $r = (b-a-1)/2$ and Lemma 1.1 implies this is $G$-equivalent to zero. □

With $f$ and $σ$ as above, we will refer to an expression of the form $f \cdot σ \cdot d_i$ as a canceling segment of type A. In a slight abuse of language we also call $f \cdot σ' \cdot d_i$ by the same name. On the other hand, an expression of the form $f \cdot σ \cdot d_i d_{i+1}$ will be called a canceling segment of type B. We justify the latter’s title with the following lemma.

**Lemma 2.3.** Let $a \leq b$ be integers, and let $f$ and $σ$ be as in Lemma 2.2. One has

$$f \cdot σ \cdot d_i d_{i+1} \sim f \cdot t_i^a t_{i+1}^b \cdot d_i.$$  

*Proof.* As before, it is enough to consider $f = 1$ and $i = 1$. Then

$$σd_1d_2 = σd_1 - t_2σd_1.$$  

Applying Lemma 2.2 to both terms on the righthand side above and canceling appropriate terms yields the desired result. □

Let $p$ denote the length of the partition $λ$ and write $λ = (λ_1 ≥ \cdots ≥ λ_p)$. Let $I_p^p$ denote the set of all length $p$ integer sequences $I = (I_1, \ldots, I_p)$ such that $I_j ≥ 0$ for all $j$ and $|I| = \sum_j I_j ≤ n$. For each such integer sequence $I$, define a new integer sequence with $p+1$ parts $λ_I = λ + I, n - |I|$. Now define the products:

$$(2.1) \quad A_{\lambda,n}^I = \begin{cases} t^{λ_I} \cdot \prod_{i=1}^p d_i & \text{, when } |I| < n \\ t^{λ_I} \cdot \prod_{i=1}^r d_i & \text{, when } |I| = n \text{ and } r = ℓ(I) \end{cases}$$

Note that in the second case, the exponent on $t_{p+1}$ is always zero. Moreover when $r = 1$, the product appearing in the second case $\prod_{i=1}^p d_i$ is understood to be empty, i.e. has value 1, and thus $A_{\lambda,n}^{(0,0,\ldots,0)}$ is just the monomial $t^n \cdot t^λ$.

For any $1 ≤ j ≤ p$, we say that $A_{\lambda,n}^I$ is $(j, j+1)$-good if it has the property that the $j$-th part of $λ_I$ is at least the $(j+1)$-st part and, if $d_{j+1}$ appears in the product, that the $j$-th part is strictly greater than the $(j+1)$-st part. Equivalently, the latter condition means that if $A_{\lambda,n}^I$ is $(j, j+1)$-good and $λ_j + I_j = λ_{j+1} + I_{j+1}$, then $d_{j+1}$ must not be present. We say simply that $A_{\lambda,n}^I$ is good if it is $(j, j+1)$-good for all $j$ in the range $1 ≤ j ≤ p$.

**Example 2.4.** For $λ = (4, 3, 1)$ and $n = 3$ we have

$$A_{(4,3,1),3}^{(1,2,0)} = t_1^5 t_2^5 t_3^1 d_1, \quad A_{(4,3,1),3}^{(1,0,1)} = t_1^5 t_2^3 t_3^1 d_1 d_2 d_3, \quad A_{(4,3,1),3}^{(0,1,1)} = t_1^4 t_2^4 t_3^1 d_1 d_2 d_3.$$  

The first two examples are good. The last is both $(2,3)$ and $(3,4)$-good but not $(1,2)$-good. The terminology comes from the fact that when a good $A_{\lambda,n}^I$ is expanded, the exponent in multi-index notation of each monomial is a partition. In this case applying the operation $G$, evidently produces an expansion in Grothendieck polynomials which alternates in the weight of the corresponding partitions.
For example

\[ \mathcal{G}_t \left( A_{(1,2,0)}^{(4,3,1),3} \right) = G_{5,5,1} - G_{6,5,1} \]

and

\[ \mathcal{G}_t \left( A_{(1,0,1)}^{(4,3,1),3} \right) = G_{5,3,2,1} - (G_{6,3,2,1} + G_{5,4,2,1} + G_{5,3,3,1}) \\
+ (G_{6,4,2,1} + G_{6,3,3,1} + G_{5,4,3,1}) - G_{6,4,3,1}. \]

Note moreover that the lowest weight must necessarily be equal to \(|\lambda| + n\), in this case \(4 + 3 + 1 + 3 = 11 = 5 + 5 + 1 = 5 + 3 + 2 + 1\).

Proposition 1.1 implies that

\[ G_\lambda G_n = \mathcal{G}_t \left( t^{\lambda,n} K_{p,1}(t) \right) \]

and so the following result reduces the problem of performing the multiplication \(G_\lambda G_n\) to applying the \(\mathcal{G}_t\) operation to a polynomial. Observe that this proves the finiteness assertion of Theorem 2.1; namely that the product is indeed a finite sum of stable Grothendieck polynomial corresponding to partitions, possibly after applying the straightening laws.

**Proposition 2.5.** With \(\lambda, n, \) and \(A^{\lambda,n}_t\) defined as above,

\[ (2.2) \quad t^{\lambda,n} K_{p,1}(t) = \sum_{I \in \mathcal{P}^n} A^{\lambda,n}_I. \]

**Proof.** Recall that we expand the rational function \(t^{\lambda,n} K_{p,1}(t)\) on the domain where \(t_{p+1}\) is much larger than the other variables. We begin by expanding the term \(d_{1,p+1} = 1 - t_1 t_{p+1}^{-1}\) in the denominator of

\[
t^{\lambda,n} K_{p,1}(t) = t^{\lambda,n} \frac{d_1 \cdot d_2 \cdots d_p}{d_{1,p+1} \cdot d_{2,p+1} \cdots d_{p,p+1}} \]

\[= t^{\lambda,n} \left( \sum_{k=0}^{n-1} \left( \frac{t_1}{t_{p+1}} \right)^k + \frac{t_1^{n-1} t_{p+1}}{d_{1,p+1}} \right) \frac{d_1 \cdot d_2 \cdots d_p}{d_{2,p+1} \cdots d_{p,p+1}} \]

\[= (t^{\lambda,n} + t_1 t^{\lambda,n-1} + \cdots + t_1^{n-1} t^{\lambda,1}) \frac{d_1 \cdot d_2 \cdots d_p}{d_{2,p+1} \cdots d_{p,p+1}} \]

\[+ t_1^n t^{\lambda,0} \frac{d_1 \cdot d_2 \cdots d_p}{d_{1,p+1} \cdot d_{2,p+1} \cdots d_{p,p+1}}. \]

Observe that in the final rational term \(t_{p+1}\) appears with only non-positive exponents. By Lemma 1.2 this term is \(\mathcal{G}\)-equivalent to the rational function obtained from the substitution \(t_{p+1} = 1\), that is the polynomial \(t_1 t_2 \cdots t_p\). This is the summand \(A^{\lambda,n}_{\lambda(0,\ldots,0)}\).

On each remaining rational term we inductively reproduce the same technique, expanding the rest of the denominator factors \(d_{i,p+1}\) in order of increasing \(i\). Concluding with expanding the term \(d_{p,p+1}\) in this way, the result is exactly the right-hand-side of Equation (2.2). \(\square\)

**Example 2.6.** Let \(\lambda = (2,1)\) and \(n = 3\). The product \(G_{2,1} G_3\) is given by

\[ \mathcal{G}_t \left( t_1^2 t_2^3 t_3^2 \frac{(1-t_1)(1-t_2)}{(1-t_1/t_3)(1-t_2/t_3)} \right). \]
Expanding the denominator term $1 - t_1/t_3$ yields
\[
\frac{t_1^2 t_2 t_3^3 (1 - t_1)(1 - t_2)}{(1 - t_2/t_3)} + \frac{t_1^3 t_2 t_3^2 (1 - t_1)(1 - t_2)}{(1 - t_2/t_3)} + \frac{t_1^4 t_2 t_3 (1 - t_1)(1 - t_2)}{(1 - t_2/t_3)} + \frac{t_1^5 t_2}{(1 - t_1/t_3)(1 - t_2/t_3)}
\]
and after simplifying the last term by Lemma 12 this is $G$-equivalent to
\[
\frac{t_1^2 t_2 t_3^3 (1 - t_1)(1 - t_2)}{(1 - t_2/t_3)} + \frac{t_1^3 t_2 t_3^2 (1 - t_1)(1 - t_2)}{(1 - t_2/t_3)} + \frac{t_1^4 t_2 t_3 (1 - t_1)(1 - t_2)}{(1 - t_2/t_3)} + t_1^5 t_2.
\]

Expanding the denominator term $1 - t_2/t_3$ in each of the first, second, and third rational terms and applying Lemma 12 respectively yields the polynomials
\[
\begin{align*}
t_1^4 t_2 t_3^3 d_1 d_2 + t_1^3 t_2^2 t_3^2 d_1 d_2 + t_1^2 t_2^3 t_3 d_1 d_2 + t_1^2 t_2 d_1,
&t_1^3 t_2^3 t_3^2 d_1 d_2 + t_1^3 t_2^2 t_3 d_1 d_2 + t_1^3 t_2^2 d_1,
&t_1^2 t_2^4 t_3 d_1 d_2 + t_1^2 t_2^3 d_1 d_2 + t_1^2 t_2 d_1,
\end{align*}
\]

Where one identifies, for example, $A_{(1,1),3}^{(2,1)} = t_1^4 t_2 t_3^3 d_1 d_2$ and $A_{(1,3),3}^{(2,1)} = t_1^5 t_2 d_1$.  

Given $\lambda = (\lambda_1, \ldots, \lambda_p)$ and $n \in \mathbb{Z}_{\geq 0}$ it is useful to organize the sum on the right-hand-side of Equation (2.2) geometrically. Explicitly, consider the simplex $\Delta_n^\lambda$ in $\mathbb{R}^p = \{ (X_1, \ldots, X_p) : X_i \in \mathbb{R} \}$ defined by the equations
\[
\sum_{i=1}^p X_j \leq n \quad \text{and} \quad X_i \geq 0 \quad \text{for all} \ i.
\]

The $A_{I}^{\lambda,n}$ in the summation (2.2) are in bijection with the integer points in $\Delta_n^\lambda$, that is, the set $\mathcal{A}_n^\lambda$. Observe that $\Delta_n^\lambda$ depends on the integers $\ell(\lambda)$ and $n$, but not on $\lambda$ itself.

The difficulty in proving that the signs alternate in Theorem 2.1 is in dealing with fake Grothendieck polynomials which appear in the expansion of Equation (2.2). We now provide an example of how the simplex described above is helpful in organization.

**Example 2.7.** In the case of Example 2.6 we arrange the terms $A_{I}^{(2,1),3}$ for $I \in \mathcal{I}_3^2$ inside the triangle $\Delta_3^2 \subset \mathbb{R}^2$ as follows:

| $X_2$ | 3 | $t_1^2 t_2^3 d_1$ |
|-------|---|------------------|
| $X_2$ | 2 | $t_1^2 t_2^3 t_3^3 d_1 d_2$ | $t_1^3 t_2^2 t_3 d_1$ |
| $X_2$ | 1 | $t_1^2 t_2^3 t_3^3 d_1 d_2$ | $t_1^3 t_2^2 t_3 d_1 d_2$ | $t_1^3 t_2^2 d_1$ |
| $X_2$ | 0 | $t_1^2 t_2^3 t_3^3 d_1 d_2$ | $t_1^3 t_2^2 t_3 d_1 d_2$ | $t_1^3 t_2^2 t_3 d_1 d_2$ | $t_1^4 t_2 t_3 d_1$ |
| $X_1$ | 0 | $X_1$ | 1 | $X_1$ | 2 | $X_1$ | 3 |
The picture above can be simplified in two major steps. The first constructs canceling segments beginning on the line $X_2 = 0$. The second constructs canceling segments beginning on the line $X_1 = 0$.

Consider points along the line $X_2 = 0$ (the bottom facet of $\Delta^2_3$) and note that Lemma 2.2 implies that the $A^{(2,1),3}_I$ corresponding to the integer points $(0,0)$ and $(0,1)$ cancel when added together, and moreover that $A^{(2,1),3}_I$ cancels itself. In other words, the sums through the points $\{(0,0), (0,1), (0,2)\}$ and $\{(1,0), (1,1)\}$ are both canceling segments of type A. This leaves only summands $A^{(2,1),3}_I$ which are $(2,3)$-good and for which the exponent on $t_3$ never exceeds $\lambda_2 = 1$:

| $X_2$ | $t_1^2t_2^2t_3d_1$ | $t_1^2t_2^3d_1$ | $t_1t_3^2d_1$ |
|-------|---------------------|------------------|----------------|
| $X_2 = 3$ |                   |                  |               |
| $X_2 = 2$ |                   |                  |               |
| $X_2 = 1$ |                   |                  |               |
| $X_2 = 0$ |                   |                  |               |
| $X_1 = 1$ | $t_1^2t_2^3d_1$ | $t_1t_3^2d_1$ |               |
| $X_1 = 2$ |                   |                  |               |
| $X_1 = 3$ |                   |                  |               |

Now consider points on the left-hand facet $X_1 = 0$ which are not $(1,2)$-good. We can apply Lemma 2.2 to the canceling segment of type A through the points $\{(0,3), (1,2), (2,1)\}$ along the top diagonal. Doing so leaves the picture below:

| $X_2$ | $t_1^2t_2^3d_1$ | $t_1^2t_2t_3d_1d_2$ | $t_1t_2^3d_1$ |
|-------|------------------|-----------------------|----------------|
| $X_2 = 3$ |                   |                       |               |
| $X_2 = 2$ |                   |                       |               |
| $X_2 = 1$ |                   |                       |               |
| $X_2 = 0$ |                   |                       |               |
| $X_1 = 1$ | $t_1^2t_2t_3d_1d_2$ | $t_1t_2^3d_1$ |               |
| $X_1 = 2$ |                   |                       |               |
| $X_1 = 3$ |                   |                       |               |

Finally, we apply Lemma 2.3 along the next diagonal through the canceling segment of type B corresponding to points $\{(0,2), (1,1)\}$. We obtain the diagram:

| $X_2$ | $t_1^2t_2^3d_1d_2$ | $t_1t_2^3d_1d_2$ | $t_1^2t_2^2d_1$ |
|-------|---------------------|------------------|----------------|
| $X_2 = 3$ |                   |                  |               |
| $X_2 = 2$ |                   |                  |               |
| $X_2 = 1$ |                   |                  |               |
| $X_2 = 0$ |                   |                  |               |
| $X_1 = 1$ | $t_1^2t_2^3d_1d_2$ | $t_1t_2^3d_1d_2$ | $t_1^2t_2^2d_1$ |
| $X_1 = 2$ |                   |                  |               |
| $X_1 = 3$ |                   |                  |               |

In the picture above, we have deleted a $d_2$ from the $(1,1)$ box. However, we agree to leave the expression in the same location although it is no longer of the form $A^{(2,1),3}_{(1,1)}$.

Observe that the remaining summands are all good and so applying the conclusions of Example 2.4 and Proposition 2.5 implies that the signs in the product $G_{2,1}G_3$ alternate in the sense of Theorem 2.1. Moreover, when the terms above are expanded, the exponents on $t_1$ are never less than $n = 3$ but never exceed $\lambda_1 + n = 5$, those on $t_2$ never exceed $\lambda_1 = 2$, and those on $t_3$ never exceed $\lambda_2 = 1$. Explicitly, we obtain that

$$G_{2,1}G_3 = G_{5,1} + (G_{4,1,1} - G_{5,1,1} - G_{4,2,1} + G_{5,2,1}) + (G_{4,1,2} - G_{5,2}) + (G_{3,2,1} - G_{4,2,1}) = G_{5,1} + G_{4,1,1} + G_{4,2} + G_{3,2,1} - (G_{5,1,1} + 2G_{4,2,1} + G_{5,2}) + G_{5,2,1}.$$
where in the former equality we have expanded each box from the final diagram within its own parentheses, and in the latter we have arranged the terms by weight, and therefore sign. Each partition which appears above has the property that its Young diagram contains that of the partitions \((2, 1)\) and \((3)\) but is contained in the Young diagram for \((2 + 3, 2, 1) = (5, 2, 1)\).

We now show that the process in the previous example extends to an algorithm which applies to the general product \(G_{\lambda}G_n\) for any partition \(\lambda\) and any positive integer \(n\). Recall that for a partition \(\lambda\) of length \(p\), \(n \in \mathbb{Z}_{\geq 0}\), and an integer sequence \(I = (I_1, \ldots, I_p)\) we define the sequence of length \(p + 1\) given by \(\lambda_I = \lambda + I, n - |I|\).

**Proof of alternating signs in Theorem 2.1.** The proof will be by descending induction on the integer \(r\) which is allowed to range through \(1 \leq r \leq p\). At each step we will cancel some summands \(A^{\lambda, n}_I\) and ensure that those which remain are \((r, r + 1)\)-good.

Affix the \(A^{\lambda, n}_I\) to the integer points \(\Delta^n_I\) of the simplex \(\Delta^n_p\). We consider first the case \(r = p\). Begin by locating terms which are not \((p, p + 1)\)-good in the \(X_p = 0\) facet; in other words for which \(n - |I| > \lambda_p + I_p\). We restrict to \(X_p = 0\) since for fixed \(I_1, \ldots, I_{p-1}\) the minimum value of the difference \((\lambda_p + I_p) - (n - |I|)\) occurs when \(I_p = 0\). If this value is negative then begin a canceling segment of type A at the point \((I_1, \ldots, I_{p-1}, 0) \in \Delta^n_p\) moving in the direction of increasing \(X_p\). Applying this analysis at each integer point in the hyperplane \(X_p = 0\) leaves only \(A^{\lambda, n}_I\) which are \((p, p + 1)\)-good and moreover, whose exponents on \(t_{p+1}\) are no more than \(\lambda_p\).

Now suppose that \(r\) is in the range \(1 \leq r \leq p - 1\) and make the following inductive assumptions. Each \(A^{\lambda, n}_I\) remaining uncanceled from previous steps has the properties:

(I) it is \((j, j + 1)\)-good for every \(j > r\), and

(II) the \((r + 2)\)-nd part of \(\lambda_I\) lies in the range \([\lambda_{r+2}, \lambda_{r+1}]\).

Notice that the analysis of the preceding paragraph ensures the inductive assumptions are met for the value \(r = p - 1\). The conditions above allow for the possibility that the \((r + 1)\)-st part is larger than the \(r\)-th part in \(\lambda_I\), and these are precisely the terms we wish to target.

Explicitly, in the facet \(X_p = 0\) locate points for which \(A^{\lambda, n}_I\) is not \((r, r + 1)\)-good. For such points, begin a canceling segment in the direction \(\epsilon_r - \epsilon_{r+1}\) where \(\{\epsilon_i\}_{1 \leq i \leq p}\) denotes the set of standard basis vectors for \(\mathbb{R}^p\). In other words, begin a canceling segment with \(a = \lambda_r\) and \(b = \lambda_{r+1} + I_{r+1}\) in the statements of Lemmas 2.2 and 2.3. We first restrict to \(X_r = 0\) since for fixed values of \(I_j\) with \(j \neq r, r + 1\), the \(A^{\lambda, n}_I\) with the largest difference \((\lambda_{r+1} + I_{r+1}) - (\lambda_r + I_r)\) must lie on this facet. If the initial point \((I_1, \ldots, I_{r-1}, 0, I_{r+1}, I_{r+2}, \ldots, I_p)\) additionally lies in the subspace

\[
\left\{\sum_{i=1}^{p} X_i = n\right\} \cap \{X_j = 0 \text{ for all } j > r + 1\},
\]

then this forms a canceling segment of type A. Otherwise, it forms a canceling segment of type B.

We now do the type B case explicitly; the type A case is similar. In order to apply the process of the preceding paragraph one must have that \(\lambda_r \leq \lambda_{r+1} + I_{r+1}\). Set \(\delta = I_{r+1} - (\lambda_r - \lambda_{r+1})\) to be the difference, where \(0 \leq \delta \leq I_{r+1}\) by assumption; the upper bound is given by the fact that \(\lambda\) is a partition. As in Lemma 2.3 set
$$a = \lambda_r = \lambda_{r+1} + I_{r+1} - \delta$$ and $$b = \lambda_{r+1} + I_{r+1} = \lambda_r + \delta$$. Finally set

$$f = \prod_{1 \leq j \leq \frac{n}{r+1}} t_j^{\lambda_j + I_n} \left( \prod_{1 \leq i \leq s} d_i \right)$$

for some $$s > r$$ determined by Equation (2.1). Then we are to consider the sum

$$f \left( t_1^{a_1} a_2^{a_2} \cdots t_r^{a_r} + t_1^{b_1} b_2^{b_2} + \cdots + t_r^{b_r} \right) d_r d_{r+1}$$

(2.3)

$$f \cdot t_r^{a_1} \cdot d_r = f \cdot t_r^{a_1 + I_{r+1}} d_r$$

(2.4)

In the type A case $$\delta$$ is strictly positive and the term $$d_{r+1}$$ is never present. We agree that the polynomial remaining on the right-hand-side of (2.4) is identified still to the point

$$I' = (I_1, \ldots, I_r-1, \delta, I_{r+1} - \delta, I_{r+2}, \ldots, I_p)$$

even though the original factor of $$d_{r+1}$$ in $$A_{I'}^{\lambda,n}$$ has disappeared. Each term in the canceling segment must exist since for each summand $$A_{I'}^{\lambda,n}$$ which appears in (2.3), the total $$|J|$$ is fixed. Moreover, conditions (I) and (II) ensure that no term in the canceling segment (2.3) could have appeared and been cancelled by a segment formed for a larger $$r$$ value.

Equation (2.4) guarantees, since $$b \geq a$$ or $$\delta \geq 0$$, that the remaining polynomial is now $$(r, r+1)$$-good and thus satisfies the inductive assumption (I). Furthermore, since the exponent on $$t_{r+1}$$ in (2.3) must be $$\lambda_r$$, it evidently lies in the range $$[\lambda_{r+1}, \lambda_r]$$ and we have satisfied the inductive assumption (II). If any term in the $$X_r = 0$$ hyperplane was already $$(r, r+1)$$-good, then its exponent $$e_{r+1} = \lambda_{r+1} + I_{r+1}$$ on $$t_{r+1}$$ must have satisfied $$\lambda_{r+1} \leq e_{r+1} \leq \lambda_r$$. This ensures that all remaining summands identified to integer points of $$\Delta^n$$ will satisfy (I) and (II) for $$r$$ less one.

By decreasing induction on $$r$$ we are left only with good summands and, using the observation of Example 2.3 that good polynomials produce the desired alternation in signs, this algorithm proves Theorem 2.1. \hfill \Box

3. Schur expansions of single Grothendieck polynomials

We now restate another theorem of Lenart and devote this section to proving it with iterated residue techniques.

**Theorem 3.1** (Lenart [Len00], Theorem 2.8). For any partition $$\lambda = (\lambda_1, \ldots, \lambda_p)$$ and commuting family of variables $$x = (x_1, \ldots, x_M)$$ with $$M \geq p$$, one has

$$G_\lambda(x) = \sum_{\lambda \subseteq \mu \subseteq \hat{\lambda}} (-1)^{|\mu| - |\lambda|} a_{\lambda\mu} s_\mu(x)$$

(3.1)

where the sum is taken over partitions $$\mu$$ and each coefficient $$a_{\lambda\mu}$$ is non-negative. The partition $$\hat{\lambda}$$ denotes the unique maximal partition of length $$M$$ obtained by adding at most $$j - 1$$ boxes to the $$j$$-th row of the Young diagram of $$\lambda$$.

As with the $$G$$ operation, we say that rational functions $$f_1$$ and $$f_2$$ are $$S$$-equivalent when $$S_i(f_1) = S_i(f_2)$$ and write $$f_1 \overset{S}{\sim} f_2$$. We now prove two lemmas which play a role analogous to that of Lemma 1.1 but for the $$S$$ operation.
Lemma 3.2. If \( f(t) \) is a polynomial symmetric in \( t_1 \) and \( t_{i+1} \), then

\[
P_i^q f(t) = t_i^{-q} t_{i+1}^{q+1} f(t)
\]

Proof. We give the proof in the case that \( f \) depends only on \( t_1 \) and \( t_2 \); the general case is identical. Moreover, since the monomial symmetric functions form a basis for the ring of symmetric functions, we reduce to the case that \( f(t_1, t_2) = m_{a,b}(t_1, t_2) = t_1^a t_2^b + t_1^b t_2^a \). Then for any integers \( p \) and \( q \) we have

\[
\frac{t_1^p t_2^q}{t_1^p t_2^q} = t_1^{p+q} t_2^{q+1} + t_1^{a+p} t_2^{b+q}.
\]

The relation of (1.3) implies that the polynomial above is \( \mathcal{S}_t \)-equivalent to the polynomial

\[
-t_1^{a+q-1} t_2^{b+p+1} - t_1^{b+q-1} t_2^{a+p+1} = -t_1^{q-1} t_2^{p+1} (t_1^{a+b} + t_2^{a+b+1}).
\]

Remark 3.3. Lemma 3.2 implies that inside the \( \mathcal{S}_t \) operation, one can perform the standard algorithm for expressing a fake Schur polynomial \( s_\lambda \) as \( \pm s_{\nu} \) for a partition \( \nu \), i.e. Equation (1.3) on any monomial multiplied by a factor which is symmetric in the appropriate variables.

One consequence of Lemma 3.2 is the following vanishing lemma.

Lemma 3.4. For \( f \) as in Lemma 3.2

\[
\mathcal{S}_t (t_{i+1} f(t)) = 0.
\]

Proof. The result follows from Lemma 3.2 and Remark 3.3 since \( s_r r_{i+1} = 0 \) for any integer \( r \) by equation (1.4). In particular, \( s_{0,1} = 0 \). □

Now for every finite integer sequence \( I = (I_1, \ldots, I_M) \) define the new sequences \( \epsilon_k^\pm(I) = I \pm \epsilon_k \) where as in Section 2 we let \( \{\epsilon_i\}_{1 \leq i \leq M} \) denote the set of standard basis vectors. In words, either add or subtract 1 to the \( k \)-th part of \( I \) and leave the rest of the sequence unchanged. In what follows, for any integer sequence \( J = (J_1, \ldots, J_M) \), let \( d^J \) denote the product

\[
\prod_{i=1}^{M} d_i^{J_i} = \prod_{i=1}^{M} (1 - t_i)^{J_i}.
\]

Lemma 3.5. Let \( M \geq p \). Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \) be a partition and \( I = (I_1, \ldots, I_M) \) be a weakly increasing integer sequence such that \( I_k - I_{k-1} = 1 \) for some \( k \). Observe that \( \epsilon_k^\pm(I) \) is also a weakly increasing sequence.

(i) If \( \lambda_{k-1} = \lambda_k \), then

\[
\mathcal{S}_t \mathcal{d}^I \sim \mathcal{S}_t \mathcal{d}^{I^\pm(1)}.
\]

(ii) If \( \lambda_{k-1} \geq \lambda_k \) then \( \epsilon_k^\pm(I) \) remains a partition and moreover,

\[
\mathcal{S}_t \mathcal{d}^I \sim \mathcal{S}_t \mathcal{d}^{I^\pm(1)} - \mathcal{S}_t \mathcal{d}^{I^\pm(1)} \mathcal{d}^{I^\pm(1)}.
\]

Proof. For notational simplicity we provide the proof only for the case that \( M = k = 2 \); the general proof is completely analogous. Write \( I = (p, p+1) \) and \( \lambda = (a, b) \). Thus \( \epsilon_2^+(I) = (p, p+1) \) and \( \epsilon_2^+(\lambda) = (a, b+1) \). Consider the high-school algebra identity

\[
t_1^a t_2^b (1 - t_1)^p (1 - t_2)^{p+1} = t_1^a t_2^b (1 - t_2)(1 - t_1)^p (1 - t_2)^p
\]

\[
= t_1^a t_2^b (1 - t_1)^p (1 - t_2)^p - t_1^a t_2^{b+1} (1 - t_1)^p (1 - t_2)^p.
\]
We now wish to apply the operation $S_t$ to both sides above. In the case that $a = b$, the second term of the last line above vanishes by Lemma 3.4. In the case $a > b$, this term remains. □

We are now ready to establish that the Schur expansion of a stable Grothendieck polynomial alternates as desired. Recall that for $M \geq p$, $\lambda = (\lambda_1, \ldots, \lambda_p)$, $I = (0, 1, 2, \ldots, M - 1)$, and $x = (x_1, \ldots, x_M)$, Proposition 1.2 asserts that
\[ G^\lambda \left( t^\lambda \right) = S^\lambda_t \left( t^\lambda d^I \right). \]

Proof of Theorem 3.1. First observe that the polynomial inside the $S^\lambda_t$-operation satisfies the hypothesis of Lemma 3.5, explicitly with the weakly increasing sequence $k = (0, 1, 2, \ldots, M - 1)$. For each integer $2 \leq j \leq M$ define $k_j$ to be the sequence $\left( j, j + 1, \ldots, M \right)$ and form the concatenated sequence $k = k_M k_{M-1} k_{M-2} \ldots k_2$.

If one chooses the value of $k$ in the order prescribed by $k$ (reading left to right) the resulting exponents appearing on the factors $\prod d_i^{k_i}$ for every term will also satisfy the hypothesis of Lemma 3.5—in particular no adjacent exponents will ever differ by more than one. By induction on the weight of $I$, this proves the theorem since negative signs are introduced in Lemma 3.5 if and only if the weight of $|\lambda|$ is increased by exactly 1.

The fact that every partition $\mu$ appearing on the right-hand side of Equation (3.1) is contained in the interval $\lambda \subseteq \mu \subseteq \hat{\lambda}$ follows from the fact that each monomial remaining at the end of the induction must correspond to a partition, and no integer $2 \leq j \leq M$ appears more than $j - 1$ times in the sequence $k$. □

Example 3.6. Consider the partition $\lambda = (2)$ and $M = 3$. Then the Schur expansion of $G_2$ can be computed by forming the sequence $k = (3, 2, 3)$ and transforming $t_1^2 (1 - t_2)(1 - t_3)^2$ by Lemma 3.5 according to the steps:
\begin{align*}
k &= 3 \rightarrow t_1^2 (1 - t_2)(1 - t_3) \\
k &= 2 \rightarrow (t_1^2 - t_1^2 t_2)(1 - t_3) \\
k &= 3 \rightarrow (t_1^2 - (t_1^2 t_2 - t_1^2 t_2 t_3)).
\end{align*}

Notice that each monomial above corresponds to a Schur function $s_\lambda$ for a partition $\lambda$ once we apply the operation $S^\lambda_t$. Moreover, notice that there is no cancellation among the particular monomials which have appeared. In the end, we conclude that
\[ G_2(x_1, x_2, x_3) = s_2 - s_{21} + s_{211}. \]

Notice that $(2, 1, 1)$ is the partition $\hat{\lambda}$ for $\lambda = (2)$ and $M = 3$. □

References

[All13] J. Allman, Grothendieck classes of quiver cycles as iterated residues, preprint, 2013.
[All14] J. Allman, K-classes of quiver cycles, Grothendieck polynomials, and iterated residues, Ph.D. thesis, UNC–Chapel Hill, 2014.
[Bri02] M. Brion, Positivity in the Grothendieck group of complex flag varieties, J. Algebra 258 (2002), no. 1, 137–159, Special issue in celebration of Claudio Procesi’s 60th birthday. MR 1958901 (2003m:14017)
[Bri05] M. Brion, Lectures on the geometry of flag varieties, Topics in cohomological studies of algebraic varieties, Trends Math., Birkhäuser, Basel, 2005, pp. 33–85. MR 2143072 (2006f:14058)
G. Bérczi and A. Szenes, Thom polynomials of Morin singularities, Ann. of Math. 175 (2012), no. 2, 567–629. MR 2877067

[BS12] A.S. Buch, Grothendieck classes of quiver varieties, Duke Math. J. 115 (2002), no. 1, 75–103. MR 1932326

[Buc02a] A. S. Buch, A Littlewood-Richardson rule for the K-theory of Grassmannians, Acta Math. 189 (2002), no. 1, 37–78. MR 1949917

[Buc05] A. S. Buch, Combinatorial K-theory, Topics in cohomological studies of algebraic varieties, Trends Math., Birkhäuser, Basel, 2005, pp. 87–103. MR 2143073

[Buc08] A. S. Buch, Quiver coefficients of Dynkin type, Michigan Math. J. 57 (2008), 93–120, Special volume in honor of Melvin Hochster. MR 2492443

[FG98] S. Fomin and C. Greene, Noncommutative Schur functions and their applications, Discrete Mathematics 193 (1998), 179–200.

[FK94] S. Fomin and A. N. Kirillov, Grothendieck polynomials and the Yang-Baxter equation, Proc. 6th Intern. Conf. on Formal Power Series and Algebraic Combinatorics, DIMACS, 1994, pp. 183–189. MR 2307216

[FK96] S. Fomin and A. N. Kirillov, The Yang-Baxter equation, symmetric functions, and Schubert polynomials, Discrete Math. 153 (1996), 123–143. MR 1394950 (98b:05101)

[FL94] W. Fulton and A. Lascoux, A Pieri formula in the Grothendieck ring of a flag bundle, Duke Math. J. 76 (1994), no. 3, 711–729. MR 1309327 (96j:14036)

[Kal13] R. Kaliszewski, Structure of quiver polynomials and Schur positivity, Ph.D. thesis, UNC–Chapel Hill, 2013.

[Kaz10a] M. Kazarian, Gysin homomorphism and degeneracies, unpublished, 2010.

[Kaz10b] M. Kazarian, Non-associative Hilbert scheme and Thom polynomials, unpublished, 2010.

[Len00] C. Lenart, Combinatorial aspects of the K-theory of Grassmannians, Ann. Comb. 4 (2000), no. 1, 67–82. MR 1763950

[LS82] A. Lascoux and M. P. Schützenberger, Structure de Hopf de l’anneau de cohomologie et de l’anneau de Grothendieck d’une variété de drapeaux (French. English summary), C. R. Acad. Sci. Paris Sér. I Math. 295 (1982), no. 11, 629–633. MR 0686357

[Mac95] I.G. Macdonald, Symmetric functions and Hall polynomials, 2nd ed., Oxford University Press, Oxford, 1995.

[Mil05] E. Miller, Alternating formulas for K-theoretic quiver polynomials, Duke Math. J. 128 (2005), no. 1, 1–17. MR 2137947

[Rim14] R. Rimányi, Quiver polynomials in iterated residue form, J. Algebraic Combin. 40 (2014), no. 2, 527–542. MR 3239295

[RS14] R. Rimányi and A. Szenes, K-theoretic Thom polynomials and their expansions in Grothendieck polynomials, in preparation, 2014.

Department of Mathematics, Wake Forest University, P.O. Box 7388, 127 Manchester Hall, Winston–Salem, NC 27109

E-mail address: allmanjm@wfu.edu

Department of Mathematics, UNC–Chapel Hill, Phillips Hall CB#3250, Chapel Hill, NC 27599–3250

E-mail address: rimanyi@email.unc.edu