Nonrelativistic limit of the bosonic string

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Abstract

We propose the action for the nonrelativistic string invariant under general co-ordinate transformations on the string worldsheet. The Hamiltonian formulation for the nonrelativistic string is given. Particular solutions of the Euler–Lagrange equations are found in the time gauge.

1 Introduction

String theory is one of the basic fields of research in theoretical physics over last 50 years (see, e.g., [1, 2, 3]). String theory is based on the Nambu–Goto bosonic string theory, whose action was independently proposed by several authors [4–10]. This action is invariant with respect to the Poincaré group in the target space and therefore describes relativistic string.

There is a natural question: “What happens with the Nambu–Goto action in the nonrelativistic limit?” For example, the standard action from classical Newton mechanics arises in the nonrelativistic limit for a point particle (see, e.g., [11]). The problem for the string is contained in the definition of the nonrelativistic limit. The invariant expansion parameter for a point particle is given by the ratio of a particle worldline lengths. This parameter is not suited for a string because we would like to get an action which is invariant under general coordinate transformations on the nonrelativistic string worldsheet. The answer to this question was proposed in the paper [12], where the nonrelativistic limit was defined and exact solution of the equations of motion was found in the form of the rotating straight rod. Consideration of the nonrelativistic limit for the bosonic string is important both from theoretical point of view and applications, for example, in polymer physics.

Note that quantum nonrelativistic string attracted recently much attention as an individual model [13] [14]. However, the action and definition of the limit considered in [12] and [13] differ.

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2 The action

The Nambu–Goto Lagrangian is invariant under the Poincaré group acting in the $D$-dimensional Minkowskian space $\mathbb{R}^{1,D-1}$. Therefore it describes a relativistic string. The Galilei group is obtained from the Poincaré group by the formal limit of large light velocity $c \to \infty$. Let us point the problem: we would like to find an action for the bosonic string which is invariant under general coordinate transformations on the string worldsheet and consistent with the Galilei transformations in the target space. That is invariant under translations along all $D$ string coordinates and $SO(D-1)$-rotations of the space components. We would like to get this Lagrangian in the nonrelativistic limit from the Nambu–Goto action. This limit and the model were proposed in the paper \[12\].

The problem is the following. Consider a fixed point on the string worldsheet $(X^a(\tau_0, \sigma_0))$, where $X^a$, $a = 0, 1, \ldots, D-1$, are Cartesian coordinates in the Minkowski space and $\tau_0, \sigma_0$ are coordinates of the point on the string worldsheet. Together with the string, it moves in the Minkowski space $\mathbb{R}^{1,D-1}$ along the worldline $(X^a(\tau, \sigma_0))$, $\tau \in \mathbb{R}$, where $\sigma_0$ is a fixed point on the string. Let us divide the string coordinates on specially noted time and space components:

$$(X^a) := (X^0 := T, X^i := Y^i), \quad i = 1, \ldots, D-1.$$  

From the point of view of an external observer which is located in the target space this point moves with the observed velocity

$$v^i := \frac{c}{dX^0} \frac{dY^i}{dX^0} = c \frac{\dot{Y}^i}{\dot{X}^0},$$  

where $c$ is the light velocity, and differentiation is performed along the world line of the point on the string worldsheet. It is plausible to define the nonrelativistic limit in the same way as for a point particle:

$$\frac{v^i}{c} \to 0, \quad \forall i.$$  

However this limit does not satisfy the requirement of reparameterization invariance of the string worldsheet because the limit \[1\] contains differentiation only with respect to $\tau$. The solution of this problem results in the definition of the nonrelativistic limit in terms of areas but not lengths as in Eq. \[2\].

The requirement of the invariance of the action with respect to general coordinate transformations on the string worldsheet is geometrical. Then the action describes the string worldsheet but not the coordinate system chosen on it.

We consider sufficiently smooth embedding of the string worldsheet $\mathcal{U}$ in $D$-dimensional Minkowski space $\mathbb{R}^{1,D-1}$:

$$X : \mathbb{R}^2 \supset \mathcal{U} \ni (\tau, \sigma) \mapsto (X^a(\tau, \sigma)) \in \mathbb{R}^{1,D-1},$$  

where $X^a$, $a = 0, 1, \ldots, D-1$, are Cartesian coordinates in Minkowski space with metric $\eta_{ab} := \text{diag}(+ \ldots -)$ and $\tau, \sigma$ are coordinates on the string worldsheet. We assume that the coordinate $\tau \in \mathbb{R}$ is timelike and $\sigma$ is spacelike. That is $X^2 := \dot{X}^a \dot{X}^b \eta_{ab} > 0$ and $X^2 := X'^a X'^b \eta_{ab} < 0$, where the dot and prime denote respectively differentiation on $\tau$ and $\sigma$. By assumption, the space coordinate for open and closed strings is varied in the intervals $\sigma \in [0, \pi]$ and $\sigma \in [-\pi, \pi]$, respectively.
The Nambu–Goto action is proportional to the area of a string worldsheet

\[ S_{NG} := -\rho c \int d\tau d\sigma \sqrt{h} = -\rho c \int d\tau d\sigma \sqrt{(\dot{X},X')^2 - \dot{X}'^2 \dot{X}^2}, \]  

(4)

where \( \rho = \text{const} \) is the linear mass density of a string, \( c \) is the velocity of light and parenthesis denote the scalar product in Minkowskian space.

Now we define the expansion parameter. The projection of the area element of a string worldsheet \( dv := d\tau d\sigma \sqrt{|h|} \) on spacelike hypersurface \( T = \text{const} \) has the form

\[ dv_\perp = d\tau d\sigma \sqrt{(\dot{Y}'Y')^2 - \dot{Y}'^2 Y'^2} = d\tau d\sigma \sqrt{\dot{Y}'_\perp^2 Y'^2}, \]

where

\[ \dot{Y}'_\perp := \dot{Y}' - (\dot{Y}',Y)^n Y'^n \]

is the orthogonal component of the velocity vector and, for brevity, we drop indices enumerating space coordinates of the string. To derive this formula, it is sufficient to put \( T = \text{const} \) in the determinant of the induced metric \( h \). It is important to note that the volume element \( dv_\perp \) has the correct transformation properties under coordinate changes because we have the determinant of the metric induced by the embedding of the string worldsheet in the Euclidean space \( \mathbb{R}^{D-1} \subset \mathbb{R}^{1,D-1} \) under the square root. Let us denote the projection of \( dv \) on the coordinate plane \( (T,Y^i) \) by \( dv_i \). Then

\[ dv_i := d\tau d\sigma \sqrt{\dot{T}^2 Y'^2 + T'^2 Y'^2 - 2\dot{T}\dot{T}'(Y',Y')}, \]

where summation over \( i \) is absent.

We introduce the ratio

\[ \epsilon := \frac{(dv_\perp)^2}{(dv_0)^2} = \frac{\dot{Y}'_\perp^2 Y'^2}{A^2} > 0, \]

(5)

where

\[ (dv_0)^2 := \sum_{i=1}^{D-1} (dv_i)^2 := d\sigma^2 d\tau^2 A^2, \]

\[ A^2 := (\dot{T}Y' - T'\dot{Y})^2 = \dot{T}^2 Y'^2 + T'^2 \dot{Y}'^2 - 2\dot{T}\dot{T}'(Y',Y'). \]

(6)

Here and in what follows, the summation over space components of the string is performed using the Euclidean metric:

\[ \dot{Y}^2 := \dot{Y}'^2 \delta_{ij}, \quad Y'^2 := Y'^n Y'^n \delta_{ij}, \quad (Y',Y') := \dot{Y}'^n Y'^n \delta_{ij}. \]

Since area elements \( dv_\perp \) and \( dv_0 \) have the same transformation rules, the ratio \( \epsilon(x) \) is a scalar field (function). Therefore it can be used as the invariant expansion parameter assuming \( \epsilon \ll 1 \).

Let us rewrite the Nambu–Goto action \([4]\) in new notation:

\[ S_{NG} = -\rho c \int d\tau d\sigma \sqrt{A^2 - \dot{Y}'_\perp^2}. \]
Then we obtain the Lagrangian for a nonrelativistic string in the first order in expansion in $\epsilon$:

$$L_{\text{NS}} = \rho c \sqrt{A^2} \left( \frac{\dot{Y}^2 Y''^2}{2A^2} - 1 \right). \tag{7}$$

This Lagrangian is the answer to the question raised beforehand. By construction, it is invariant with respect to general coordinate transformations of $(\tau, \sigma)$ on the string worldsheet. Moreover, the Lagrangian (7) is invariant under translations $X^\lambda \to X^\lambda + \text{const}$ and global $SO(D - 1)$-rotations acting on the space string coordinates $Y$.

We now consider the Lagrangian for a nonrelativistic string (7) by itself not paying attention on how it was derived and do not assuming smallness of the parameter $\epsilon$. There are open, $\sigma \in [0, \pi]$, and closed, $\sigma \in [-\pi, \pi]$, strings as in the relativistic case.

Lagrangian (7) takes simple and visual form in the time gauge $\tau = T$, $(\dot{Y}, Y') = 0$:

$$L_{\text{NS}} \big|_{\text{time gauge}} = \rho c \sqrt{Y'^2} \left( \frac{1}{2} \dot{Y}^2 - 1 \right), \tag{8}$$

where the first summand is the kinetic term for the transverse oscillations of the string, and the second one is equal to the potential energy which is proportional to the length of the string. The common factor $\sqrt{Y'^2}$ is due to the arbitrariness in the choice of the space coordinate $\sigma$. In this gauge, the expansion parameter (5) takes the form

$$\epsilon = \frac{v^2}{c^2},$$

that is the limit $\epsilon \to 0$ is really nonrelativistic.

From now on we put $c = 1$ for simplicity.

To derive equations of motion for the nonrelativistic string, we rewrite Lagrangian (7) through independent variables $(T, Y)$, with respect to which it is varied:

$$L_{\text{NS}} = \rho \left[ \frac{\dot{Y}^2 Y'^2}{2 \sqrt{T^2 Y'^2 + T'^2 Y^2 - 2\dot{T}\dot{T}'(\dot{Y}, Y')}} - \sqrt{T^2 Y'^2 + T'^2 Y^2 - 2\dot{T}\dot{T}'(\dot{Y}, Y')} \right]. \tag{9}$$

We introduce notation to simplify the subsequent formulae:

$$P_0^\tau := \frac{\partial L_{\text{NS}}}{\partial \dot{T}} = -\frac{\rho}{\sqrt{A^2}} \left( \frac{\dot{Y}^2 Y'^2}{2A^2} + 1 \right) (\dot{T}'Y'^2 - T'(\dot{Y}, Y')), \tag{10}$$

$$P_1^\tau := \frac{\partial L_{\text{NS}}}{\partial \dot{Y}_1} = \frac{\rho}{\sqrt{A^2}} \left[ Y'^2 \dot{Y}_1 - \frac{\dot{Y}^2 Y'^2}{2A^2} + 1 \right] (T'^2 Y_1 - \dot{T} T' Y_1'), \tag{11}$$

$$P_0^\sigma := \frac{\partial L_{\text{NS}}}{\partial \dot{T}'} = -\frac{\rho}{\sqrt{A^2}} \left( \frac{\dot{Y}^2 Y'^2}{2A^2} + 1 \right) (T'^2 \dot{Y}' - \dot{T} Y'), \tag{12}$$

$$P_1^\sigma := \frac{\partial L_{\text{NS}}}{\partial \dot{Y}_n} = \frac{\rho}{\sqrt{A^2}} \left[ \dot{Y}^2 Y'_n - (\dot{Y}, Y') Y_1 - \frac{\dot{Y}^2 Y'^2}{2A^2} + 1 \right] (\dot{T}^2 Y'_n - \dot{T} T' Y_n'). \tag{13}$$

Then the Euler–Lagrange equations are rewritten as

$$\frac{\delta S_{\text{NS}}}{\delta T} = -\frac{\partial}{\partial \tau} P_0^\tau - \frac{\partial}{\partial \sigma} P_0^\sigma = 0, \tag{14}$$

$$\frac{\delta S_{\text{NS}}}{\delta Y_1} = -\frac{\partial}{\partial \tau} P_1^\tau - \frac{\partial}{\partial \sigma} P_1^\sigma = 0. \tag{15}$$
As in the case of the relativistic string, we assume vanishing variations on the boundaries \( \tau = \tau_1, \tau_2 \), and consider arbitrary variations on the boundaries \( \sigma = 0, \pi \) (free ends). Then the variational principle implies the boundary conditions
\[
P_0^\sigma \big|_{\sigma=0,\pi} = 0, \quad P_1^\sigma \big|_{\sigma=0,\pi} = 0, \quad \forall i
\]
for an open string.

In the time gauge \( \tau = T \), \( (\dot{Y}, Y') = 0 \) the following equalities hold
\[
P_0^\sigma \equiv 0, \quad P_1^\sigma = -\rho \frac{\dot{Y}^2}{\sqrt{Y'^2}} \left( \frac{\dot{Y}^2}{2} - 1 \right) Y'_i.
\]
Therefore, the boundary conditions take the form
\[
\left( \frac{\dot{Y}^2}{2} - 1 \right) \frac{Y'_i}{\sqrt{Y'^2}} \bigg|_{\sigma=0,\pi} = 0.
\]
These imply two possibilities for space string components. The first is
\[
\dot{Y}^2 \bigg|_{\sigma=0,\pi} = 2.
\]
The second implies
\[
\frac{Y'_i}{\sqrt{Y'^2}} \bigg|_{\sigma=0,\pi} = 0, \quad \forall i.
\]
Since the vector in the left hand side of the last equality has the unit length, the continuity condition is broken in the second case, and therefore we do not consider it. Thus, the boundary conditions for a free nonrelativistic string in the time gauge has the form (17). These boundary conditions for a free nonrelativistic string follow from the least action principle and mean that the string end points move with the velocity \( \sqrt{2c} \) which is greater then the velocity of light. There is no contradiction because Newton’s mechanics admits arbitrary velocities.

Equations of motion for the nonrelativistic bosonic string (14) and (15) are not independent. Invariance of the action under the choice of coordinates on the string worldsheet, due to Noether’s theorem, implies two linear identities:
\[
\frac{\delta S_{\text{NS}}}{\delta T} \partial_\alpha T + \frac{\delta S_{\text{NS}}}{\delta Y^i} \partial_\alpha Y^i \equiv 0, \quad \alpha = 0, 1.
\]
Since Lagrangian (7) and the corresponding action are invariant with respect to global translations in the target space
\[
T \mapsto T + \epsilon^0, \quad Y^i \mapsto Y^i + \epsilon^i, \quad x^\alpha \mapsto x^\alpha,
\]
where \( (\epsilon^0, \epsilon^i) \) are constant parameters, the first Noether theorem implies the conservation of the currents
\[
\partial_\alpha J_\alpha^\alpha = 0, \quad \forall \alpha,
\]
where
\[
J_0^\tau = -\frac{\partial L_{\text{NS}}}{\partial T} = -P_0^\tau, \quad J_0^\sigma = -\frac{\partial L_{\text{NS}}}{\partial T^\sigma} = -P_0^\sigma, \\
J_i^\tau = -\frac{\partial L_{\text{NS}}}{\partial Y^i} = -P_i^\tau, \quad J_i^\sigma = -\frac{\partial L_{\text{NS}}}{\partial Y^i} = -P_i^\sigma.
\]
Formally, the Euler–Lagrange equations (14), (15) coincide with the current (19) conservation. This yields the physical interpretation to the introduced notation: up to a sign, $P^T_0$ is the linear energy density and $P^i_1$ in the linear momentum density of the nonrelativistic string. The total energy and momentum are given by the integrals

\[
\mathcal{E} =: -\int d\sigma \frac{\partial L_{\text{NS}}}{\partial \dot{T}}, \\
P_i =: -\int d\sigma \frac{\partial L_{\text{NS}}}{\partial \dot{Y}^i},
\]

(20)

where integration is performed from zero to $\pi$ or from $-\pi$ to $\pi$ for open and closed strings, respectively.

Moreover, the action for the nonrelativistic string is invariant under global $\text{SO}(D-1)$-rotations which take the following infinitesimal form

\[
T \mapsto T, \quad Y^i \mapsto Y^i - Y^j \omega_j^i, \quad x^{\alpha} \mapsto x^{\alpha},
\]

where $\omega^i_j = -\omega^j_i$ are rotational parameters. Due to the first Noether’s theorem, the invariance of the action results in the current conservation on the equations of motion

\[
\partial_{\alpha} J_{ij}^{\alpha} = 0,
\]

(22)

where

\[
J_{ij}^T = Y_i \frac{\partial L_{\text{NS}}}{\partial \dot{Y}^j} - Y_j \frac{\partial L_{\text{NS}}}{\partial \dot{Y}^i}, \\
J_{ij}^\sigma = Y_i \frac{\partial L_{\text{NS}}}{\partial Y^j} - Y_j \frac{\partial L_{\text{NS}}}{\partial Y^i}.
\]

(23)

It implies conservation of the total angular momentum of the nonrelativistic string

\[
\mathcal{M}_{ij} := \int_{0,-\pi}^\pi d\sigma \left( Y_i \frac{\partial L_{\text{NS}}}{\partial \dot{Y}^j} - Y_j \frac{\partial L_{\text{NS}}}{\partial \dot{Y}^i} \right).
\]

(24)

Note that $J_{0i}^{\alpha} \equiv 0$ for space rotations.

### 3 Canonical formulation

Momentsa conjugate to $T$ and $Y^i$ are

\[
P^T_0 =: P_0^T = \frac{\partial L_{\text{NS}}}{\partial \dot{T}} = -\frac{\rho}{\sqrt{A^2}} \left( \frac{\dot{Y}^2 \dot{Y}'^2}{2A^2} + 1 \right) (\dot{T}Y'^2 - T'(\dot{Y}, Y')), \\
P^i_1 =: P_i^T = \frac{\partial L_{\text{NS}}}{\partial \dot{Y}^i} = \frac{\rho}{\sqrt{A^2}} \left[ Y^2 \dot{Y}'^2 + \left( \frac{\dot{Y}^2 \dot{Y}'^2}{2A^2} + 1 \right) (T^2 \dot{Y}^i - \dot{T}T'Y^i) \right].
\]

(25)

(26)

As in the case of relativistic string, the canonical Hamiltonian for the nonrelativistic string is equal to zero

\[
H := P^T \dot{T} + P_i^T \dot{Y}^i - L_{\text{NS}} = 0,
\]
as the consequence of straightforward calculations. It means that the dynamics of the model is entirely defined by the constraints which are present in the theory. Since the action of the nonrelativistic string

\[ S_{\text{ns}} := \rho \int d\tau d\sigma \sqrt{A^2} \left( \frac{\dot{Y}^2 Y'^2}{2A^2} - 1 \right) \]

is invariant with respect to arbitrary coordinate transformations \( \tau, \sigma \) on the string worldsheet, we expect that the model contains two primary first class constraint. Unfortunately, the form of momenta (25), (26) is relatively complicated, and seeing these constraints is problematic. Therefore we introduce notation for next calculations

\[ \dot{T}_\perp := \dot{T} - (\dot{Y}, Y')T', \]

(27)

\[ A^2 = \dot{T}_\perp^2 Y'^2 + T'^2 \dot{Y}_\perp^2, \]

\[ \epsilon A^2 = Y^2 Y'^2, \]

(28)

where the variable \( \epsilon \) is by this time not small. Then expressions for momenta (25), (26) take the form

\[ P = -\frac{\rho}{\sqrt{A^2}} \left( \frac{\epsilon}{2} + 1 \right) \dot{T}_\perp Y'^2, \]

\[ P_i = \frac{\rho}{\sqrt{A^2}} \left[ Y'^2 \dot{Y}_\perp - \left( \frac{\epsilon}{2} + 1 \right) (T'^2 \dot{Y}_\perp - \dot{T}_\perp T'Y'_i) \right]. \]

Now we derive expressions invariant with respect to \( \mathbb{SO}(D - 1) \)-rotations which are quadratic in generalized coordinates and momenta

\[ P^2 = \rho^2 \left( \frac{\epsilon}{2} + 1 \right)^2 (Y'^2 - \epsilon T'^2), \]

(30)

\[ P_i^2 := P^i P_i = \rho^2 \left( T'^2 + \epsilon Y'^2 - \epsilon T'^2 - \frac{3}{4} \epsilon^2 T'^2 \right), \]

(31)

\[ P T' = -\frac{\rho}{\sqrt{A^2}} \left( \frac{\epsilon}{2} + 1 \right) \dot{T}_\perp T' Y'^2, \]

(32)

\[ P_i Y'^i = \frac{\rho}{\sqrt{A^2}} \left( \frac{\epsilon}{2} + 1 \right) \dot{T}_\perp T' Y'^2. \]

(33)

The last two equalities imply the primary constraint

\[ H_1 := P T' + P_i Y'^i = 0. \]

(34)

This constraint is kinematical and has the same form as for the relativistic string. The dynamical constraint \( H_0 \) is more complicated. For its deriving, one can find \( \epsilon \) from equality (31) by solving quadratic equation. Afterwards this solution is to be substituted into equality (30), and this results in dynamical constraint in the form of the polynomial in canonical variables and their space derivatives. The resulting expressions are cumbersome, and we leave them for future investigations. General considerations imply that constraints \( H_0 \) and \( H_1 \) must be of the first class, and their Poisson bracket algebra should be isomorphic to conformal algebra.
4 The time gauge I

Equations of motion for the nonrelativistic string (14), (15) are complicated, and, for their simplification, we fix the time gauge using the invariance of the model with respect to general coordinate transformations on the string worldsheet. First, we fix the conformal gauge for the induced metric

\[ \dot{\mathcal{X}}^2 + \mathcal{X}'^2 = 0 \quad \Leftrightarrow \quad \dot{T}^2 - \dot{Y}^2 + T'^2 - Y'^2 = 0, \]

(35)

Next, we impose the additional condition using the residual conformal invariance \( \tau = T \)

in the same way as was done for the relativistic string. Then the conformal gauge (35) becomes

\[ \dot{Y}^2 + Y'^2 = 1, \]

(37)

\[ (\dot{Y}, Y') = 0. \]

(38)

We call conditions (36)–(38) for the nonrelativistic string the time gauge I. In Section 2, we used partial time gauge without condition (37).

In what follows, we need consequences of conditions (37), (38) obtained by differentiation:

\[ \partial_0 (\dot{Y}^2 + Y'^2 = 1) : (\dot{Y}, \ddot{Y}) + (Y', \dot{Y}') = 0, \]

(39)

\[ \partial_1 (\dot{Y}^2 + Y'^2 = 1) : (\dot{Y}, \ddot{Y}') + (Y', \dot{Y}'') = 0, \]

(40)

\[ \partial_0 ((\dot{Y}, Y') = 0) : (\dot{Y}, \ddot{Y}') + (\dot{Y}, \dot{Y}'') = 0, \]

(41)

\[ \partial_1 ((\dot{Y}, Y') = 0) : (\ddot{Y}', Y') + (\dot{Y}, Y'') = 0. \]

(42)

Formulae (39), (42) and (40), (41) imply equalities:

\[ (\dot{Y}, \ddot{Y}) - (\dot{Y}, \dot{Y}'') = 0, \]

(43)

\[ (Y', \dot{Y}'') - (Y', \ddot{Y}) = 0. \]

(44)

In the time gauge, velocities of string points are perpendicular to the string:

\[ \dot{Y}_i^\perp = \dot{Y}^i \]

and subsidiary fields combinations (6) and (5) are

\[ A^2 = Y'^2, \quad \epsilon = \dot{Y}^2. \]

(45)

Now expressions (10)–(13) take the form

\[ P_0^\tau = -\rho \sqrt{Y'^2} \left( \frac{\dot{Y}^2}{2} + 1 \right), \]

(46)

\[ P_1^\tau = \rho \sqrt{Y'^2} \dot{Y}_1, \]

(47)

\[ P_0^\sigma = 0, \]

(48)

\[ P_1^\sigma = \frac{\rho}{\sqrt{Y'^2}} \left( \frac{\dot{Y}^2}{2} - 1 \right) Y_1'. \]

(49)
and equations of motion (14), (15) are essentially simplified

\[ Y''^2 (\dot{Y}, \ddot{Y}) + (Y', \dot{Y}')(\ddot{Y}^2 + 1) = 0, \]
\[ Y''^2 \ddot{Y} + (Y', \dot{Y})' \ddot{Y} + \left(\frac{\dot{Y}^2}{2} - 1\right) Y'' + \frac{\dot{Y}^2}{2Y'^2} (Y', Y'') Y' = 0, \]

where formulae (40) and (37) are used.

Note that equalities (46)–(49) hold for weaker conditions: it is sufficient to put \( \tau = T \) and \( (\dot{Y}, Y') = 0 \).

**Proposition 4.1.** Equation (50) is the consequence of equation (51).

**Proof.** Contract Eq. (51) with \( \dot{Y}' \) and use Eqs. (42) and (40). \( \square \)

It is easily verified that contraction of Eq. (51) with \( Y'' \) reduces to condition (37). Equations of motion (51) can be rewritten as

\[ \ddot{Y} - Y'' - \ddot{Y}^2 \ddot{Y} + \frac{\dot{Y}^2}{2} Y'' + (Y', \dot{Y}')(\ddot{Y} + \frac{\dot{Y}^2}{2Y'^2} (Y', Y'') Y') = 0, \]

where equalities (37) and (40) are taken into account. Thus, equations of motion for the nonrelativistic string reduce to the system of nonlinear equations (52) and quadratic constraints (37), (38). We see that equations of motion of the nonrelativistic string in the time gauge are more complicated than equations for the relativistic string because of the nonlinearity.

Let us present the class of solutions which is singled out by the extra condition \( \dot{Y}^2 = \text{const} \) linearizing Eqs. (52) in the time gauge. Since velocities are restricted by condition (37), this class of solutions cannot describe an open string of finite length because the boundary condition (17) cannot be satisfied. Therefore we consider an open infinite string.

Let

\[ \dot{Y}^2 = \sin^2 \gamma, \quad Y'^2 = \cos^2 \gamma, \quad \gamma = \text{const} \in (0, \pi/2). \]

It is clear that constraint (37) holds for all \( \gamma \). Differentiation of these equalities yields the following relations:

\[ (\dot{Y}, \dot{Y}') = 0, \quad (Y', Y'') = 0. \]

This implies that Eqs. (52) reduce to the free wave equation in this case

\[ v^2 \ddot{Y} - Y'' = 0, \quad 0 < v^2 := \frac{2 \cos^2 \gamma}{1 + \cos^2 \gamma} < 1. \]

A general solution of this equation is

\[ Y_i = F_i(\xi) + G_i(\eta), \]

where \( F_i(\xi) \) and \( G_i(\eta) \) are arbitrary functions of cone coordinates:

\[ \xi := v\tau + \sigma, \quad \eta := v\tau - \sigma. \]
Extra conditions (53) and (38) impose the following restrictions on arbitrary functions:

\[
\begin{align*}
F'^2 + 2(F', G') + G'^2 &= \frac{\sin^2 \gamma}{v^2}, \\
F'^2 - 2(F', G') + G'^2 &= \cos^2 \gamma, \\
F'^2 - G'^2 &= 0,
\end{align*}
\]

where primes denote differentiation on respective arguments. The obtained system of equations has the solution

\[
F'^2 - G'^2 = \frac{\sin^2 \gamma + v^2 \cos^2 \gamma}{4v^2} =: a^2 > 0,
\]

\[
(F', G') = \frac{\sin 2\gamma - v^2 \cos^2 \gamma}{4v^2}.
\]

Sure, velocity \(v\) in the right hand side can be expressed by angle \(\gamma\) using the definition (54) but it does not simplify the following formulae.

Thus, we get the class of solutions for the nonrelativistic string in the time gauge \(I\), which has the form (55) where arbitrary functions are restricted by Eqs. (57) parameterized by the angle \(\gamma \in (0, \pi/2)\). These conditions have nontrivial solutions. For example, there is the solution in four dimensions

\[
F = (a\xi, 0, 0), \\
G = (b\eta, c\cos \eta, c\sin \eta),
\]

where constants \(a, b\) and \(c\) are defined by equalities

\[
\begin{align*}
a &= \sqrt{\frac{\sin^2 \gamma + v^2 \cos^2 \gamma}{v^2}}, \\
b &= \frac{\sin 2\gamma - v^2 \cos^2 \gamma}{2v\sqrt{\sin^2 \gamma + v^2 \cos^2 \gamma}}, \\
c &= \sqrt{\frac{\sin \gamma \cos \gamma}{\sin^2 \gamma + v^2 \cos^2 \gamma}}.
\end{align*}
\]

which can be simply verified. Thus, space coordinates of the nonrelativistic string in this case are

\[
Y = (a\xi + b\eta, c\cos \eta, c\sin \eta).
\]

In this way, we derive the class of exact solutions for the nonrelativistic string in the time gauge \(I\) which is parameterized by constant \(\gamma \in (0, \pi/2)\). The initial configuration for \(\tau = 0\) is

\[
Y(0, \sigma) = ((a - b)\sigma, c\cos \sigma, -c\sin \sigma),
\]

where

\[
a - b = \frac{v \cos^2 \gamma}{\sqrt{\sin^2 \gamma + v^2 \cos^2 \gamma}} = \frac{v \cos \gamma}{\sin \gamma}c.
\]

This is the spiral depicted in Fig. I. During the evolution, the spiral moves translational with constant velocity along \(Y^1\) axis and rotates simultaneously around the same axis. If \(v \to 1\), (relativistic limit) then \(\gamma \to 0\) and the radius of the spiral goes to zero, i.e. the spiral degenerates into the line.
5 The time gauge II

The theory of the nonrelativistic string allows one to impose the conformal gauge using the reparameterization invariance in the same way like it is done for the relativistic string. This gauge was used in the previous section. The conformal gauge (35) is invariant with respect to the global action of the Poincaré group in the target space. However, since we consider now the nonrelativistic string, these conditions can be changed. Therefore we impose the gauge

\[ Y'^2 = 1, \quad (\dot{Y}, Y') = 0 \]  

instead of the conformal gauge (35). The first condition means that the length of the string (from the point of view of external observer) is chosen as parameter \( \sigma \). The second condition means that the velocity vector is perpendicular to the string. In contrast to the time gauge (35) now we do not have restrictions on the square of velocity vector \( \dot{Y}^2 \).

In the gauge (61), the metric on the string worldsheet induced by the embedding (3) takes the form

\[ h_{\alpha\beta} = \begin{pmatrix} \dot{T}^2 - \dot{Y}^2 & \dot{T}\dot{T}' \\ \dot{T}\dot{T}' & T'^2 - 1 \end{pmatrix}. \]  

If we consider the metric induced by the embedding of the string worldsheet in the Euclidean subspace \( \mathbb{R}^{D-1} \subset \mathbb{R}^{1,D-1} \) (with positive definite metric) then the Riemannian metric is obtained

\[ \hat{h}_{\alpha\beta} = \begin{pmatrix} \dot{Y}^2 & 0 \\ 0 & 1 \end{pmatrix}. \]  

It is known that such coordinate system exists locally. In addition, it is not defined uniquely. Suppose that the residual symmetry allows us to impose one more condition

\[ \tau = T. \]  

We call conditions (61) and (64) the time gauge II for the nonrelativistic string. In contrast to the relativistic string, these conditions are invariant only under global translations and \( \mathbb{SO}(D-1) \)-rotations in the target space.

Let us differentiate conditions (61) on \( \tau \) and \( \sigma \), respectively:

\[ 2(Y', \dot{Y}') = 0, \quad (\dot{Y}', Y') + (\dot{Y}, Y'') = 0. \]

It implies the equality

\[ (\dot{Y}, Y'') = 0, \]  

Figure 1: Nonrelativistic bosonic string in the form of infinite spiral.
which holds in the time gauge II.

Equalities (46)–(49) are fulfilled both in the time gauge I and II.

Now equations of motion (14), (15) are
\[
\partial_0 \dot{Y}^2 = 2(\dot{Y}, \ddot{Y}) = 0, \tag{66}
\]
\[
\ddot{Y}_i - Y''_i + \frac{1}{2} \partial_1 (\dot{Y}^2 Y'_i) = 0. \tag{67}
\]

**Proposition 5.1.** Equation (66) is the consequence of Eqs. (67).

**Proof.** Substitute \(\ddot{Y}\) from Eq. (67) into Eq. (66)
\[
(\dot{Y}, \tilde{\dot{Y}}) = (\dot{Y}, Y'') - (\dot{Y}, Y''') \frac{\dot{Y}^2}{2} - (\dot{Y}, Y')(\dot{Y}, \dot{Y}') = 0,
\]
as the consequence of Eq. (61) and (65).

In the linear approximation, Equations (67) describe transverse (due to extra conditions (61)) oscillations of the string which propagate along the string with the velocity of light \(c\).

Thus the nonrelativistic string in the time gauge II is described only by the spatial components \(Y^i(\tau, \sigma)\) which satisfy equations of motion (67) and additional conditions (61). For open and closed strings, equations of motion must be supplemented by boundary conditions (17) and the periodic conditions, respectively.

We give an example of exact solution. Consider a straight open string of length \(L\) which rotates with constant angular speed in the three dimensional Minkowskian space \(\mathbb{R}^{1,2}\). Suppose that rotation takes place in the \((Y^1, Y^2)\) plane with constant angular velocity \(\omega\)
\[
Y = (\sigma \cos \omega \tau, \sigma \sin \omega \tau), \quad \sigma \in [-L/2, L/2].
\]

Here we changed the interval for the space coordinate to make rotation happen around the center of mass. Then the following equalities hold
\[
\dot{Y} = (-\sigma \omega \sin \omega \tau, \sigma \omega \cos \omega \tau),
\]
\[
Y'' = (\cos \omega \tau, \sin \omega \tau),
\]
\[
\dot{Y}^2 = \sigma^2 \omega^2 \sin^2 \omega \tau + \sigma^2 \omega^2 \cos^2 \omega \tau = \sigma^2 \omega^2,
\]
\[
Y'^2 = \cos^2 \omega \tau + \sin^2 \omega \tau = 1,
\]
\[
(\dot{Y}, Y'') = -\sigma \omega \cos \omega \tau \sin \omega \tau + \sigma \omega \sin \omega \tau \cos \omega \tau = 0,
\]
\[
\ddot{Y} = (-\sigma \omega^2 \cos \omega \tau, -\sigma \omega^2 \sin \omega \tau),
\]
\[
Y''' = (0, 0).
\]

The constraints (61) are fulfilled. It is easily verified that equations of motion (67) are also satisfied. Boundary conditions (17) define the rotational speed: \(L \omega = 2 \sqrt{2} c\). We see that the angular velocity of rotations is inversely proportional to the length of the nonrelativistic string as that for the relativistic string.
6 Conclusion

In the paper, we define the nonrelativistic limit for the bosonic Nambu–Goto string which results into the reparameterization invariant Lagrangian describing nonrelativistic bosonic string. We present the canonical formulation of the corresponding model and obtain particular solutions of the Euler–Lagrange equations: one class of solutions of the infinite moving and rotating spiral and the other solution in the form of finite straight rod rotating with constant angular velocity.

The obtained Lagrangian can be considered by itself as the model of the string. If this model is applied in nonrelativistic physics, e.g. in polymer physics, the light velocity should be replace by the velocity of speed.

References

[1] B. M. Barbashov and V. V. Nesterenko. *Introduction to the relativistic string theory*. World Scientific, Singapore, 1990.
[2] M. B. Green, J. H. Schwarz, and E. Witten. *Superstring theory*, volume 1,2. Cambridge U.P., Cambridge, 1987.
[3] L. Brink and M. Henneaux. *Principles of String Theory*. Plenum Press, New York and London, 1988.
[4] B. M. Barbashov and N. A. Chernikov. Solution and quantization of a nonlinear two-dimensional model for a Born–Infeld type field. *Sov. Phys. JETP*, 23(5):861–868, 1966.
[5] B. M. Barbashov and N. A. Chernikov. Solution of the two plane wave scattering problem in a nonlinear scalar field theory of the Born–Infeld type. *Sov. Phys. JETP*, 24(2):437–442, 1967.
[6] Y. Numbu. Duality and hydrodynamics. In *Lectures at Copenhagen Summer Symposium*, 1970. [Unpublished].
[7] H. B. Nielsen. In *15th International Conference on High Energy Physics*, Kiev, 1970. [Unpublished].
[8] L. Susskind. Dual-symmetric theory of hadrons – i. *Nuovo Cim.*, 69A:457–496, 1970.
[9] O. Hara. On origin and physical meaning of Ward-like identity in dual-resonance model. *Prog. Theor. Phys.*, 46:1549–1559, 1971.
[10] T. Goto. Relativistic quantum mechanics of one-dimensional mechanical continuum and subsidiary condition of dual resonance model. *Prog. Theor. Phys.*, 46:1560–1969, 1971.
[11] L. D. Landau and E. M. Lifshitz. *The Classical Theory of Fields*. Pergamon, New York, second edition, 1962.
[12] M. O. Katanaev. Nonrelativistic string. *Sov. J. Nucl. Phys.*, 48(1): 296–298, 1988.
[13] J. Gomis and G. Ooguri. Nonrelativistic closed string theory. *J. Math. Phys.*, 42(1):3127, 2001. [hep-th/0009181].

[14] E. Bergshoeff, J. Gomis, J. Rosseel, C. Şimşek, and Z. Yan. String theory and string Newton–Cartan geometry. *J. Phys. A*, 53:014001, 2019. [hep-th/1907.10668].