A Feedback Information-Theoretic Transmission Scheme (FITTS) for Modelling Aimed Movements*

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Abstract—We build on the variability of human movements by focusing on how the stochastic variance of the limb position varies over time. This implies analyzing a whole set of trajectories rather than a single one. We show, using real data previously acquired by two independent studies, that two distinct phases appear. The first phase, where the positional variance increases steadily, is followed by a second phase where it decreases toward zero. During the second phase, the problem of aiming can be reduced to a Shannon-like communication problem where information is transmitted from a “source” (determined by the position at the end of the first phase), to a “destination” (the movement’s endpoint) over a “channel” perturbed by Gaussian noise, with the presence of a feedback link. We take advantage of a scheme by Elias, which provides a simple yet optimal solution to this problem. We further work out the Elias scheme to show that the rate of decrease of variance during the second component is a best exponential. This result is observed on real data and used to derive the well-known Fitts’ law.

*The mathematical framework developed in Section V was previously presented in [1].

I. INTRODUCTION

It has long been observed that people routinely adapt their speed so as to reliably perform pointing or grasping movements. This so-called speed-accuracy tradeoff has been studied for more than a century by different communities, such as experimental psychology [2]–[7], human-computer interaction [8]–[10], cybernetics [11], [12], and neuroscience [13], [14].

Despite the complexity and the diversity of human movements, recurrent characteristics are known and described in the literature:

a) Human movement is inherently variable: If one is repeatedly given the same pointing task, his trajectories and performance measures may be strikingly different between trials. Variability affects practically all kinematic markers—e.g., position, speed, acceleration, or jerk. One cause is the fact that a single pointing movement can involve many joints—finger, wrists, elbows, shoulders and even the back [15]. Thus, the body provides many redundant degrees of freedom to execute the same task [16]. Another source of variability is the presence of noise throughout the system, from neurons all the way to the muscles [17]. Variability can also originate from the failure to take into account the different complexities of the system [18]; it is then introduced somewhat deliberately to facilitate modeling. All this prompts the use of stochastic mathematical models to describe human movement.

b) Human movement relies on feedback: Humans cannot function properly without feedback, see e.g., Wiener’s account [19, p.95] on two patients suffering from a lack of voluntary muscle movement coordination due to lack of feedback information at the brain. Movement generation often relies on visual feedback mechanisms; this is also observed for most animals (e.g., on kitten [20]). Various experiments on occlusion and removal of light [21], [22] or removal of cursor [23] show an effect of visual feedback on virtually all kinematic properties, including accuracy and movement time, on movements as short as 100 ms [22], [24], [25].

c) Human movement can be decomposed into submovements, but also appears as resulting from continuous control: (i) Movements can be segmented into two or more submovements [5], [6], [24]. Woodworth [2] was the first to suggest that aiming was comprised of a first distance-covering phase followed by a second homing-in phase. Later it was observed that the velocity vanished multiple times before the end of the movement, marking the transition between one submovement and another. Thus many theoretic models [5], [6], [25] are derived by segmenting a single movement into several submovements, indicating intermittent control.

(ii) Nonetheless, control and feedback effects can appear continuous [27]–[29] where no visible changes occur in the kinematic profiles, yet the overall performance is improved due to the presence of feedback. The transition between intermittent and continuous control is probably the result of training and learning [2], [24], [30].

The most influential result regarding human movements is known as Fitts’ law, which predicts the time MT needed to reach a target of width W located at distance D through the index of difficulty ID = log2(1 + D/W) expressed in bits:

\[ MT = a + b \log_2(1 + D/W) = a + b \text{ID} \tag{1} \]

where \( a \) (the intercept) and \( b \) (the slope) have to be estimated from empirical data. Fitts’ law was first stated by Fitts [3] as an analogy with the Shannon capacity formula [32] and was recently formalized using the tools of Shannon’s information theory [33] (but without visual or kinesthetic feedback). The law has been found extremely useful in measuring or predicting device performance [8], [10] and human performance [34]–[37]. However, its theoretic basis was often questioned [4]–[6], [9], [10], [38], [40]. While Fitts’ law can be re-derived from many human movement models, none of them successfully incorporates the aforementioned characteristics a)–c).

The goal of this paper is to propose a model for human movement which accounts for a)–c). Our focus is on the evolution of the stochastic variance of the limb position over time. This implies analyzing a whole set of trajectories (acquired under the same experimental condition) at once,
as opposed to the common practice of considering a single trajectory.

The remainder of this paper is organized as follows. Section II reviews existing models of aimed movements. Section III describes positional variance profiles, that we expect to be unimodal. Their unimodality is validated and the effect of task parameters on them are observed from real data in Section IV. Section V proposes an information-theoretic model from which Fitts’ law is derived in Section VI. Section VII concludes.

II. PREVIOUS MODELS

We review models for human aimed movements which share similarities with our work. The interested reader is referred to [15], [24], [31], [41] for more details.

A. Early Descriptions: A Two-Component Movement

The earliest descriptions of aimed movements go back to the nineteenth century [2], [42]. It was observed that when speed increases, accuracy diminishes “when the eyes are used” [2]; and that under constant speed, increasing the distance to be covered decreases the accuracy. Based on these findings, Woodworth [2] hypothesized that an aimed movement is constituted of two serial components: an initial adjustment, whose main purpose is to cover distance, followed by a homing-in phase that relies on vision to ensure accuracy.

B. Crossman-Goodeve’s Deterministic Iterative Corrections

Crossman and Goodeve [5] proposed a discrete-time model where the change of position from one sampling instant to the next is proportional to the current error, implying that the distance to the target is reduced exponentially. Assuming a fixed time step between each control impulse, they show that the total movement time follows Fitts’ law. Unfortunately, the model is deterministic and does not account for the variability of human motor movement; its predictions are not confirmed by empirical studies [41]. p. 194.

C. Schmidt’s Law

Using a time-constrained rather than a width-constrained task, Schmidt et al. [43] observed a so-called linear speed-accuracy tradeoff for rapid movements:

$$MT = a' + b'D/\sigma,$$

(2)

where MT and D are defined as in [1], and \(\sigma\) is the standard deviation of endpoints. The discrepancy between Schmidt’s and Fitts’ law is most likely due to the method used by Schmidt et al. to determine the end of the movement [24]. Schmidt’s law is often seen as an operational formula for “ballistic” (open loop) movements of short duration, whereas Fitts’ law is seen as an operational formula for aimed (closed loop) movements of longer duration.

D. Meyer et al.’s Stochastic Optimized Submovements (SOS)

Woodworth’s two-component movement, Schmidt’s linear speed-accuracy tradeoff for ballistic movements and Crossman and Goodeve’s model are combined in the SOS model [6], [41]. Each movement is assumed to begin with a ballistic primary submovement, whose role is to cover most of the distance separating the initial point from the target and modeled with Schmidt et al.’s linear tradeoff [2]. If the obtained endpoint is outside the target, corrective submovements are needed. As a result of optimal allocation of time between each submovement, the SOS model then predicts that movement time is given by [6]

$$MT = a + b\sqrt{D/W}.$$

(3)

yet a logarithmic law can also be derived [44]. As it effectively combines elements of previous successful models into a comprehensive description, the SOS is one of the leading explanatory models for Fitts’ law [15], [24]. However, the SOS model suffers from several well-documented deficiencies, including some theoretical predictions on submovement that do not match empirical findings [6], [45]. More importantly, the model cannot account for continuous feedback because it cannot dispense with submovements.

E. Bullock-Grossberg’s Vector-Integration-To-Endpoint (VITE)

A neural network model called VITE was proposed [16] on the basis that complex synergies between muscle groups exclude a pre-planning (ballistic) strategy. Although VITE explains many aspects of human aimed movement using a feedback scheme based on the difference between current and target position, the time-continuous differential equations describing the model are deterministic, so that it fails to account for variability as well as intermittent control.

F. Elliott et al. Two-Component Description

Building on extensive experimental accounts, Elliott et al. [24], [45] proposed a description of aimed movement inspired by the two component model of Woodworth [2].

1) a first \textit{planned} component that gets the limb close to the target area. This component is based on internal representations but, contrary to Schmidt’s description, is not entirely ballistic. Instead, it is associated to a velocity regulation through kinesthetic/propioreceptive feedback.

2) when time permits, a second corrective portion is engaged to reduce spatial discrepancy between limb and target. This process, highly dependent on foveal (central) vision, involves computing the difference between limb and target position and issuing discrete corrections. With practice, these corrections appear smooth and continuous.

At present, we know no quantitative model for this two-component description.

\[\text{Here, by ballistic, we mean that there is no feedback involved at all. A ballistic movement is thus fully determined at the onset by a control impulse.}\]
G. Conclusion

None of the models above capture all of the characteristics a)–c) of human aimed movement described in Section II. Deterministic models, such as Crossman-Goodeve’s and VITE, fail to account for the variability of human movements (a) and the derivation of Fitts’ law in [33] does not include ways to integrate feedback (b). VITE does not account for intermittent control (c-i), while the celebrated SOS model fails to consider continuous corrections (c-ii) for skilled operators [2], [27].

In the sequel, we derive a two-phase model which exploits the difference between target and current position of the limb, obtained via feedback. This is consistent with empirical observations [24] and in line with the feedback schemes presented above, which all rely on the difference between target and current positions. Variability is taken into account by considering a noisy Shannon-like information transmission scheme. Contrary to previous models which model a single trajectory, a whole set of trajectories is considered; as a result both intermittent and continuous control are accounted for. In this way, the proposed model successfully satisfies a)–c).

III. Positional Variance Profiles (PVP)

As noticed in [19], trajectory variability has “surprisingly received little attention from researchers”. Most studies investigating Fitts’ law have often been limited to the measure of the spread of endpoints [7], [10] and full trajectories are often studied from the perspective of a single movement [9]. [31]. One of the main difficulties associated with considering single movements is to segment each movement correctly. For example, applying and evaluating models such as SOS requires the identification of submovements. Because movements usually end very smoothly, it can be hard to find robust methods to determine submovements and even the end of a movement.

A. Computing the PVP

In our approach, starting times are determined but stopping times need not be; there remain long stationary periods at the end of the movement, when the limb stabilizes. The trajectories obtained this way are then synchronized by using the starting time as origin. Fig. 1 displays a set of synchronized trajectories that last at most 2 s. Notice that after about 1 s, the trajectories are all stationary, and that the trajectories display a great amount of variability, which justifies the stochastic approach.

To obtain the PVP, we compute the positional variance using all trajectories at each time instant. This method not only dispenses the complicated task of determining stopping times, but also handles in the same manner trajectories with either apparent submovements or no visible submovements, thus solving the difficulty c) raised in Sect. I.

B. PVP’s Unimodality Hypothesis

Some previous works have studied whole trajectories, e.g., [46] evaluates the temporal evolution of the entropy of trajectories from a tapping task. Entropy profiles, as well as profiles of standard deviation of position were reported unimodal (an increasing phase, followed by a decreasing phase).

Other studies [14], [47] have represented positional variance at specific kinematic markers (peak acceleration, peak velocity, peak deceleration, movement time). Extrapolation of data again suggests unimodal PVP’s.

The following simple reasoning tends to justify unimodality:

1) all movements starting from the same position, initial positional variance is null (Fig. 1 at 0 s);
2) in the early stages of the movement, positional variance necessarily increases [43], [48] (Fig. 1 from 0 to 0.2 s);
3) eventually, the target is reached and the movements ends. The fact that humans are capable of reaching a target reliably implies that the positional variance can be reduced at will [49] (Fig. 1 from 1 s to 2 s).

Fig. 2. Ideal two-phase positional variance profile. The transition between the two phases occurs at \((\tau; \sigma_0^2)\).

We therefore expect unimodal PVP’s as illustrated in Fig. 2.

A first phase, for time \(t \in [0, \tau]\), where variance increases from 0 to \(\sigma_0^2\) is followed by a second phase, where variance decreases to arbitrarily low values.

IV. Empirical Analysis of PVP’s

This section provides an analysis of PVP’s from empirical data. Since many pointing studies on Fitts’ tapping task have been made, we have chosen to re-analyze two existing datasets:

A. Datasets

The first dataset is from a study by Guiard et al. [50] (hereafter G-dataset), which uses a discrete version of the tapping task [51]. Pointing was done towards a fixed line...
located at \( D = 150 \text{ mm} \), perpendicular to the direction of movement. The participants were instructed to point towards the line while varying their strategy through five conditions: from full speed emphasis (1) to full accuracy emphasis (5). Pointing data was acquired on a high-resolution graphical tablet.

The second dataset is the Pointing Dynamics Dataset (hereafter PD-dataset) by Müller et al. \(^{[52]}\) that used the reciprocal (serial) version of the tapping task, where the cursor is moved back and forth between two targets of size \( W \) separated by a distance \( D \). The two factors were \( D \) and ID = \( \log_2(1 + D/W) \); factor \( D \) had two levels \( D = \{0.212,0.353\} \) (m) and ID has four levels ID = \{2, 4, 6, 8\} (bit), and were fully crossed.

In both datasets, position is a one-dimensional real number. The raw data from each dataset was re-sampled closest to the average sampling frequency, and individual movements were extracted and synchronized as described in Sect. III. For the G-dataset, 16 participants produced movements for each of the five conditions five times, providing 80 different PVP’s. For the PD-dataset, 12 participants produced movements for 8 \((D,W)\) pairs, providing 96 PVP’s. Outliers were not removed because this would imply relying on arbitrary heuristics for which we found no satisfying method. We were nonetheless able to find compelling evidence to support our claims.

\section*{B. Unimodality of PVP’s}

We asserted unimodality on both datasets by verifying that only one sign change occurred in the derivative of the standard deviation \( \sigma(t) \).\(^1\) We considered that the sign of the derivative changes when it crosses a threshold level of \( \pm 2.5\% \) of the maximum value of the derivative. Fig. 3 displays an empirical PVP, consistent with the theoretical profile of Fig. 2 and shows its derivative in orange. There is just one sign change, when \( \sigma(t) \) is at a maximum, which indicates unimodality.

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig3.png}
\caption{In blue: Standard deviation profile \( \sigma(t) \) for Participant X of the G-dataset, performing under condition 3 (balanced speed/accuracy). In orange: Derivative of the standard deviation profile with respect to time.}
\end{figure}

We conducted this analysis on all PVP’s from both datasets: 94\% (75 out of 80) of PVP’s were found unimodal in the G-dataset; 92\% (88 out of 96) in the PD-dataset. The Hartigan & Hartigan dip test was conducted for unimodality \(^{[53]}\) on the G-dataset, where the null hypothesis is that the distribution is unimodal. We found that 6 profiles \((p < .05)\) were significantly non-unimodal out of the 80 profiles. The average p-value is just above 0.8, implying strong evidence for unimodality.

\section*{C. Effects of Instructions, D, ID and W on the PVP’s}

According to previous models as described in Sect. III, the onset of movement is primarily affected by \( D \), while the later stages are primarily affected by \( W \). Also, Fitts’ law predicts that most of the movement time’s variability can be explained by ID levels. Therefore, we investigated the effect of \( D, \) ID and \( W \) on \( \tau \), the time of maximum variance, as well as \( D_\tau \), the distance covered at maximum variance.

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig4.png}
\caption{Effects of \( D \) and \( W \) on \( \tau \) for the PD-dataset. The bars are grouped by \( D \) condition, with the 4 bars on the left corresponding to \( D = 0.212 \) m and the 4 bars on the right corresponding to \( D = 0.353 \) m. Each bar is labeled with its corresponding level of ID. For comparison, total movement time is represented with black diamonds with the same scale.}
\end{figure}

\begin{table}[h]
\centering
\begin{tabular}{cccc}
\hline
\textbf{ID} & \textbf{D} & \textbf{F(1, 88)} & \textbf{p} & \textbf{\( \eta^2 \)} \\
\hline
\textbf{D} & F(1, 88) & 4.779 & 0.031 & 0.047 \\
ID & F(3, 88) & 2.831 & 0.043 & 0.083 \\
\textbf{D×ID} & F(3, 88) & 0.236 & 0.871 & 0.007 \\
\hline
\end{tabular}
\caption{Two-way ANOVA for effects of \( D \) and ID on \( \tau \).}
\end{table}

This is confirmed by a two-way ANOVA, see Tab. I. The effects of \( D \) and ID are statistically significant at \( \alpha = 0.05 \); however the effect size is small, indicating that less than 10\% (resp. 5\%) of the variance is explained by factor ID (resp. \( D \)). In comparison, a two-way ANOVA for effect of \( D \) and ID on total movement time results in an effect size of more than 70\% of ID on movement time. Therefore, ANOVAs confirm the visual impression that the effects of the factors on \( \tau \) are small compared to the effects on movement time.

The coefficient of determination between \( \tau \) and \( W \) was found as \( \tau = 0.327 - 0.0005 \times W \), \( r^2 = 0.0557 \), \( p = 0.021 \). Although the slope is statistically significant at \( \alpha = 0.05 \), it is almost horizontal and \( r^2 \) is very close to null, showing that \( W \) has very little effect on \( \tau \) as well.
b) Effects of D, ID and W on distance $D_T$ at time $\tau$: The effects are summarized in Fig. 5. ID levels have little or no effect on $D_T$. Higher levels of D lead to larger $D_T$:

- For $D = 0.212$, $D_T$ ranges from 0.127 (for ID = 4) to 0.147 m (for ID = 6). The average $D_T$ over all ID levels is 0.138, representing $65\%$ of $D$.
- For $D = 0.353$, $D_T$ ranges from 0.217 (for ID = 2) to 0.238 m (for ID = 8). The average $D_T$ over all ID levels is 0.228, which also represents $65\%$ of $D$.

![Fig. 5. Effects of D and W on $\tau$ for the PD-dataset. The bars are grouped by D condition, with the 4 bars on the left corresponding to $D = 0.212$ m and the 4 bars on the right corresponding to $D = 0.353$ m. Each bar is labeled with the corresponding value of ID. For comparison, the level of factor D is represented in thick black horizontal lines.](image)

A two-way ANOVA reported in Table II confirms our analysis. Only $D$ is statistically significant at $\alpha = 0.05$, with very strong effect size ($\eta^2 > 60\%$), while both ID and the interaction $\text{ID} \times D$ have no significant effect on $D_T$. Linear regression between $W$ and $D_T$ gives $D_T = 0.0001 \times W + 0.1807$, $r^2 = 0.0024$, $p = 0.633$, which shows no effect of $W$ on $D_T$.

![Fig. 6. $\tau$ against instructions. Black diamonds represent total movement time.](image)

c) Effect of instructions on $\tau$ and $D_T$: These are summarized in Figs. 6 and 7.

- Effect on $\tau$: Grand mean for $\tau = 0.201$ s; $\tau$ ranges from $\tau = 0.180$ s for the ultra fast condition to $\tau = 0.240$ s for the ultra precise condition. Instructions to emphasize precision lead to higher values of $\tau$. A one-way ANOVA on the factor instruction gives $F(4, 75) = 6.200$, $p < 0.001$, $\eta^2 = 0.249$, which shows a statistically significant, moderate effect of instruction on $\tau$. In comparison, the effect of instruction on total movement time, displayed in black diamonds in Fig. 6 is much stronger ($\eta^2 = 0.491$).

- Effect on $D_T$: grand mean for $D_T = 0.102$ m; $D_T$ ranges from $D_T = 0.120$ m for the ultra fast condition to $D_T = 0.090$ m for the ultra precise condition. Instructions to emphasize speed lead to larger values of $D_T$. A one-way ANOVA on $D_T$ with factor instruction gives $F(4, 75) = 8.579$, $p < 0.001$, $\eta^2 = 0.314$, which shows a statistically significant, moderate effect on $D_T$.

D. Discussion

To summarize:

1) The duration of the first (variance-increasing) phase of the PVP is invariant to changes in $D$, $W$, ID and, to a lesser extent, to instructions. Although the effects on these factors are sometimes statistically significant, they are always very small compared to those on total movement time (time differences $\leq 100$ ms vs. $\geq 1$ s on total movement time in the PD-dataset; $\leq 60$ ms vs. $\geq 1.5$ s in the G-dataset).

2) The distance covered during that first phase is about two-thirds of the total distance to be covered ($D_T \approx 0.65 \times D$ for both levels of $D$ in the PD-dataset; $D_T \approx 0.66 \times D$ on average on conditions (2)–(4) in the G-dataset).

Trajectories in the tapping task can thus be described by (1) a first variance-increasing phase of constant duration that lasts about 200 to 300 ms, where about 2/3 of the distance is covered, followed by (2) a second variance-decreasing phase to meet the accuracy constraints. This simple two-phase description is a good match with two-component models of Woodworth and Elliott et al. [24, 45].

Considering $\tau$ as constant greatly simplifies considerations since the effect of $D$, ID, W and instructions on $\tau$ lead to difference whose magnitude matches the within groups variability. For most movements that require precision, the second phase dominates movement time. Since we expect movement time to follow Fitts’ law, only the second phase should induces Fitts’ law. It is this second phase that should be studied further to better understand the speed-accuracy tradeoff.

V. A Model for the Variance-Decreasing Phase

In the following, we propose an information-theoretic model for the second phase from which Fitts’ law will be derived.
A. Information-Theoretic Model Description

While most previous models aim at predicting an average trajectory, our goal here is to predict how the variability of a set of trajectories evolves over time. Asymptotically (for large enough datasets) the position of the limb extremity (‘limb’ in short) is random. At the end of the first phase, we may assume that it can be well approximated by a Gaussian distribution with some standard deviation $\sigma_0$ (see Fig. 2).

Thanks to the feedback, the limb position is known at the brain level. Due to eye-hand coordination and fast eye dynamics, the eye is usually pointing towards the target long before the end of the movement [24]. Hence the position of the target is also known at the brain level. The distance from limb to target can thus easily evaluated by the brain; in fact it can be readily estimated by the eye if the limb is close enough to the target.

Let $A$ be the distance from limb to target; from the above assumptions, $A$ is modeled as a Gaussian random variable with standard deviation $\sigma_0$. Movements may equally well undershoot or overshoot the target [6] when users are instructed to perform short) is random. At the end of the first phase, we may assume that it can be well approximated by a Gaussian distribution $\mathcal{N}(0, \sigma_0^2)$. We assume zero mean: $A \sim \mathcal{N}(0, \sigma_0^2)$. To complete the movement, the brain has to send $A$ to the limb: this is the role of the second phase. Consider the following scheme:

- From $A$ the brain outputs an amplitude $X_1$ to be sent to the limb:
  \[ X_1 = f(A), \]  
  where $f$ is some function performed by the brain.
- To account for the variability of the human motor system [17], we consider a noisy transmission from brain to limb, where $X_1$ gets perturbed by additive white Gaussian noise (so-called AWGN channel). The output of the channel $Y_1$ is given by
  \[ Y_1 = X_1 + Z_1 \quad \text{where} \quad Z_1 \sim \mathcal{N}(0, N). \]
- Based on the channel output $Y_1$, distance $\hat{A}_1$ is actually covered, which is the result of some function $g$ applied by the motor organs to the received $Y_1$.
- $\hat{A}_1$ is returned to the brain via ideal (noiseless) feedback where it is compared to $A$. From such a comparison a new amplitude $X_2$ is produced by the brain.

The scheme then progresses iteratively for $i = 1, 2, \ldots$. We assume that each step $i$, from the creation of $X_i$ to the reception of $\hat{A}_i$, takes $T$ seconds. Each such step is infinitesimal and thus, the whole process is an intermittent iterative correction model that becomes continuous at the limit.

At iteration $i$, the scheme is described by following equations (see Fig. 8):

1) The brain (the ‘encoder’) produces $X_i$ from $A$ and all received feedback information $\hat{A}_1^{i-1}$: $X_i = f(A, \hat{A}_1^{i-1})$.
2) The motor organs (decoder) receive $Y_i$ contaminated by Gaussian noise: $Y_i = X_i + Z_i$.

\(^3\) This Gaussian assumption is often used [6], [43] and its justification depends on the conditions under which movements are produced [42]. We verified that in the second phase, position distributions are approximately symmetric. In [49], the empirical entropy of the trajectory was compared to that of the Gaussian distribution; the difference was never more than 0.5 bits throughout the entire trajectory.

3) The covered distance $\hat{A}_i$ is a function of all previous received amplitudes: $\hat{A}_i = g(Y_i)$.

At this stage, $f$ and $g$ are still undetermined.

In Shannon’s communication-theoretic terms, the aiming task in the second phase can thus be seen as the transmission of a real value from a “source” (distance from target at the end of the first phase) to a “destination” (limb extremity) over a noisy Gaussian channel with noiseless feedback. In human-centered terms, the second phase is the one which deals specifically with aiming—to make sure that the limb reliably reaches the target, once most of the distance has been covered. This second phase likely corresponds to the submovements of the stochastic optimized submovement model [6], or to the second component in two-component models [2], [24], [45].

B. Bounds on Transmitted Information

We now leverage information-theoretic definitions.

- $P_i = \mathbb{E}[X_i^2]$, where $\mathbb{E}$ is the mathematical expectation, is the input’s average power.
- The quadratic distortion $D_n = \mathbb{E}[(A - \hat{A}_n)^2]$ is the mean-squared error of the estimation of $A$ by $\hat{A}_n$ after $n$ iterations (channel uses); this essentially corresponds to the empirical variance.
- $I(A, \hat{A}_n)$ is Shannon’s mutual information [54] between $A$ and $\hat{A}_n$.
- Shannon’s capacity $C$ [32], [54] of the AWGN Channel under power constraint $P_i \leq P$ and noise power $N$ is
  \[ C = \frac{1}{2} \log_2(1 + P/N), \]  
  expressed in bits per channel use.

**Theorem 1.** Consider the transmission scheme of Fig. 8 with an AWGN channel of capacity $C$ and noiseless feedback. For a zero-mean Gaussian source $A$ with variance $\sigma_0^2$, we have
  \[ \frac{1}{2} \log_2 \frac{\sigma_0^2}{D_n} \leq I(A, \hat{A}_n) \leq nC. \]  

The proof is given in the Appendix. The Theorem expresses that enough information should be transmitted from the brain to the limb to reduce the positional variability from the initial variance ($\sigma_0^2$) to the variance at the end of the movement ($D_n$). However, it also expresses that the transmitted information can never exceed $nC$, where $n$ is the number of iterations. Since the rate per iteration can never exceed $C$, being more accurate requires sending larger amounts of information, which in turn requires more iterations of the scheme.

C. Achieving capacity

For a given channel and number $n$ of channel uses, maximizing accuracy is equivalent to minimizing $D_n$. Similarly, for a given accuracy (distortion), minimizing time is equivalent to minimizing $n$. Optimal aiming, which consists of achieving the best possible accuracy in the least amount of time is thus achieved when equality holds in (7):

\[ \frac{1}{2} \log_2 \frac{\sigma_0^2}{D_n} = I(A, \hat{A}_n) = nC. \]
The goal of this subsection is to find the scheme that achieves optimality \([8]\). In all what follows we use the list notation \(\ell^i = (\ell_1, \ldots, \ell_i)\).

**Lemma 1.** Optimal aiming can be achieved if, and only if, we have the following conditions:

1) all considered random variables \(A, \hat{A}, A - \hat{A}, X^i, Y^i, Z^i\) are Gaussian;
2) input powers \(P_i = \mathbb{E}[X^2_i]\) are equal (to, say, \(P\));
3) endpoints \(\hat{A}\) are mutually independent;
4) channel outputs \(Y^i\) and errors \(A - \hat{A}\) are independent;
5) \(\hat{A}_i = g(Y^i)\) is a sufficient statistic of \(Y^i\) for \(A\).

The proof is given in the Appendix. Working with Gaussian variables considerably simplifies operations, as independence between Gaussian variables is equivalent to decorrelation, and the minimum mean-squared error (MMSE) estimator—which minimizes \(D_\alpha\)—reduces to a linear function in the Gaussian setting. As seen below, this implies that both \(f\) and \(g\) are linear—which is not surprising as linear functions are known to preserve normality.

Since \(g(Y^i)\) is a sufficient statistic of \(Y_i\) for \(A\), it does not matter if the feedback comes from the endpoints \(\hat{A}\) or from the outputs of the channel \(Y^i\). In this way our model can account for feedback information prior to the motor organs (e.g. kinesthetic feedback).

By working out the conditions of Lemma 1, we can derive the structure of the optimal scheme, namely the expression of \(f\) and \(g\). We first obtain \(g\) by using the well-known orthogonality principle: if a certain quantity \(x\) is to be estimated from the observed data \(y\) by the unbiased estimator \(\hat{x}(y)\), the orthogonality principle states that the following are equivalent:

- \(\hat{x}(y) = \mathbb{E}[x|y] = \mathbb{E}[xy]^t\mathbb{E}[yy^t]^{-1}y\) is the MMSE estimator;
- \(\mathbb{E}[(x - \hat{x}(y))y^t] = 0\).

**Theorem 2.** For the optimal transmission scheme, \(g(Y^i)\) is the MMSE estimator: \(g(Y^i) = \mathbb{E}[A|Y^i]\).

**Proof.** From condition \(3\), \(\mathbb{E}[(A - g(Y^i))Y^i] = 0\). The result follows immediately from the orthogonality principle. \(\square\)

The optimal scheme thus yields an endpoint \(\hat{A}_i = g(Y^i)\) obtained as the best least-squares estimation of \(A\) from all the previous observations of channel outputs \(Y^i = (Y_1, \ldots, Y_i)\).

**Theorem 3.** For the optimal transmission scheme,

\[
X_i = f(\hat{A}^{i-1}, A) = \alpha_i(A - \hat{A}_{i-1}) = \alpha_i(A - \mathbb{E}[A|Y^{i-1}]),
\]

where \(\alpha_i\) meets the power constraint \(\mathbb{E}[X^2_i] = P\).

The proof is given in the Appendix. The signal sent to the channel is thus simply the difference between the initial message \(A\) and its current estimate \(\hat{A}_{i-1}\), rescaled to meet the power constraint.

The previous two theorems formally define the encoding function \(f\) and decoding function \(g\); both are linear functions. Incidentally, the motor system is known to be able to produce linear functions \([15]\). The functions \(f\) and \(g\) are mathematically simple and biologically feasible and the distance difference \(A - \hat{A}_{i-1}\) can be readily estimated by the eye.

The next result shows that the procedure is incremental, yet optimal at each step, allowing optimal on-line control.

**Theorem 4.** Let \(A_i = X_i/\alpha_i\) be the unscaled version of \(X_i\). We have

\[
\mathbb{E}[A|Y^{i-1}] = \sum_{j=1}^{i-1} \mathbb{E}[A|Y_j] = \sum_{j=1}^{i-1} \mathbb{E}[A_j|Y_j].
\]

Again the proof is given in the Appendix. The theorem shows that the “decoding” process is recursive: At each step, a “message” \(A_i\) that is independent from the previous ones \(A^{i-1} = (A_1, \ldots, A_{i-1})\) is formed, and is then estimated optimally by least-square minimization.

Finally, we check optimality in \(8\) on the distortion:

**Theorem 5.** The quadratic distortion \(D_i = \mathbb{E}[(A - \hat{A}_i)^2]\) decreases exponentially in \(i\):

\[
D_i = \frac{\sigma_0^2}{(1 + P/N)^i}.
\]

With this scheme, it is immediately checked that capacity \(C\) is exactly achieved and the distortion decreases geometrically (divided by \(1 + P/N\)) at each iteration. The scheme successfully matches the transmission of one real value \(A\) using feedback with \(n\) independent channel uses (transmissions).

The above equations were already obtained in an information-theoretic context by Gallager and Nakiboğlu \([55]\) who discussed an older scheme by Elias \([56]\). To our knowledge, the constructive approach of the Elias scheme given here, as well as its application to model human aimed movements is novel.

A possible implementation of this scheme in the human context is given Fig. 9. The \(A_i\)’s, given by the difference between the target position \(A\) and current position \(A_{i-1}\), represent the distance that remains to be covered. This distance can readily be evaluated by the eye; because of eye-hand coordination and fast eye dynamics, the eye is usually pointing towards the target, giving high fidelity spatial information to...
the brain [23]. The distance signal is rescaled (multiplied by $\alpha_i$) and sent through the nervous system (the "channel") where it gets corrupted by additive noise $Z_i$; this noise reflects the variability characteristic of human aimed movement.

Between the channel output and the limb, MMSE estimation is performed, which amounts to multiplying the channel output $Y_i$ by the ratio between the variance of the input signal $P$ and the variance of the output signal $P + N$. Which organs could actually perform this function is unknown.

The position of the limb extremity is then updated by adding the newly decoded output to the previous ones. This operation is represented by the loop at the bottom of Fig. 9. The loop and the delay constitute a memory used to store the current position (the limb extremity remains still if no signal is sent).

VI. THEORETICAL DERIVATION OF FITTS’ LAW

The proposed scheme uses a discrete-time formalism, where $n$ in (8) is the number of the iterations of the scheme. If the iteration time is constant and equal to $T$, total time $t$ is given by $t = nT$. When considering many trajectories, it is likely that any time interval, however small will contain at least one new correction, providing a smooth variance profile. We can thus assume an infinitely small looping time $\delta T$ so that the number of iterations $n = t/\delta T$ tends to infinity. This does not entail that each trajectory is obtained as the result of a continuous on-line control, but rather that the entire set of trajectories can be described by a model where variance decreases smoothly.

A. Local result

We can rewrite (8) in terms of the second phase’s duration

$$ t = \frac{\delta T}{C} \log_2 \frac{\sigma_0^2}{D_n} = \frac{1}{C' d}, $$

(13)

where we define $d$ as the local index of difficulty for the second phase: $d = \frac{1}{2} \log_2 \frac{\sigma_0^2}{D_n}$ and $C' = C/\delta T$ the capacity in bit per second. This is very similar to Fitts’ law [11]. For practical purposes, we have $D_n = \sigma^2(t)$; rearranging terms:

$$ i_d = \log_2 \frac{\sigma_0}{\sigma} = C' t, $$

(14)

This is a local formulation for Fitts’ law that holds between any two dates in the second phase: For arbitrary small $\Delta t \geq 0$,

$$ \log_2 \sigma(t + \Delta t) = \log_2 \sigma(t) - C' \Delta t. $$

(15)

Thus, the second phase for optimal movements is characterized by a linearly decreasing standard deviation of position. This is an important improvement to the current expression of Fitts’ law, which only applies to endpoints of the movement.

B. Empirical Evidence

a) Exponential Decrease of Standard Deviation: Assessing an exponential decrease of standard deviation for optimal movements is best achieved within a log-lin scale. Within that representation, standard deviation plotted against time is expected to grow quickly during the first phase, and then decrease linearly with a slope $C'$ during the second phase as shown in [15]. When all movements have finished, the PVP should level off to a stationary phase. Fig. 10 gives an example of the fits performed on the PVP. Standard deviation (in black) increases during the first 200 ms, then decreases quasi linearly up to about 950 ms (in blue). We estimated the slope $C'$, as well as the linearity of the second phase using linear regression. In the particular example of Fig. 10 we find $r^2 = 0.998$ and $C' = 6.937$ bits/s. In the stationary phase (in orange) the standard deviation is about 0.5 mm.

We computed goodness of fit $r^2$’s for all PVPs of the PD-dataset (resp. G-dataset) and obtained an average $r^2 = 0.963$ (resp. 0.971). These findings support the theoretically computed rate [14]. The average $C'$ found was $C' = 6.25$ (resp. 5.31) bit/s. Remarkably, such values are very close to the typical 5 bit/s throughput for a Fitts’ law experiment as reported in [10].

This indicates a good consistency between our novel approach based on PVPs and the traditional approach based on total movement times.

b) Effects of Task Parameters and Instructions: As the value of $C'$ is determined by the transmission channel only, it is expected that varying the parameters $D$ and ID leave $C'$ mostly unchanged. Similarly, instructions to emphasize speed or accuracy should not significantly alter $C'$. A two way ANOVA (Tab. III) performed on the PD-dataset shows that indeed the effects of $D$ and ID on the value of $C'$ are non significant Analysis of the G-dataset shows that the effect of instruction on the value of $C'$ is also non significant at level $\alpha = 0.05 (F(4, 75) = 1.861, p = 0.126, \eta^2 = 0.09)$. 

Fig. 9. Possible implementation of how human perform the aiming task: functions and potential organ groups who can perform these functions.

![Fig. 9. Possible implementation of how human perform the aiming task: functions and potential organ groups who can perform these functions.](image)

Fig. 10. Participant X of the G-dataset, performing in the balanced condition (3). The blue dashed line is the fit for the second phase ($r^2 = 0.998$); the orange dotted line is the fit for the stationary phase.
TABLE III
TWO-WAY ANOVA FOR EFFECTS OF D AND ID ON C’.

|     | F     | p   | η²  |
|-----|-------|-----|-----|
| D   | F(1,88) = 0.985 | 0.324 | 0.011 |
| ID  | F(3,88) = 0.744 | 0.529 | 0.024 |
| D×ID | F(3,88) = 0.779 | 0.509 | 0.025 |

TABLE IV
TWO-WAY ANOVA FOR EFFECTS OF D AND ID ON r².

|     | F     | p   | η²  |
|-----|-------|-----|-----|
| D   | F(1,88) = 1.067 | 0.305 | 0.011 |
| ID  | F(3,88) = 2.735 | 0.048 | 0.084 |
| D×ID | F(3,88) = 0.232 | 0.874 | 0.007 |

For completeness, we also looked at the effects of D, ID and instructions on the goodness of fit r² for the second phase. Analysis of the PD-dataset reveals a significant, yet small effect of ID (see Table IV) because higher values of ID lead to slightly better fits. The G-dataset reveals that instructions had a significant, moderate effect on r² (F(4,75) = 1.861, p < 0.001, η² = 0.352). Thus, incentives to emphasize accuracy, be it through explicit instructions or task parameters, lead to slightly better empirical fits with the model. A possible reason might be that higher accuracy requirements lead to longer durations for the second phase, as predicted by the model. Therefore, more points are considered in the regression, and the goodness of fit is mechanically increased.

C. Deriving the Classic Fitts’ law

The local formulation obtained above can be exploited theoretically in the context of Fitts’ paradigm and should yield Fitts’ law [1]. As determined in Sect. IV, the first phase has a constant duration, say a. Second, for very short movements, Meyer’s model may be assumed: σ₀ = kD, where k is some constant and σ₀ is the standard deviation at the end of the first phase [6], [43]. We evaluated k on the P-dataset and found k = 0.12. Since the first phase covers 2/3 of the total distance as seen in Section IV, we have that σ₀ = 2k/3D. Using Equation (14), we obtain

\[ MT = a' + 1/C' \times \log_2(D/σ), \]

where \( a' = a + \log 2k/3 \), which is Mackenzie’s version of Fitts’ law [10], except for the “+1” inside the logarithm.

It is known that participants are able to keep a relative constant error rate \( ε \) (usually, \( ε \) is about 5%) across conditions [3], [10], [51]; for a Gaussian distribution of endpoints, we have that \( σ \sim \sqrt{2erf^{-1}(1 - ε)}W \), where erf \( -1(x) \) is the inverse Gaussian Error Function (see [33, Appendix 3] for an example of the computation). This leads to

\[ MT = a' + b' \log_2(D/W), \]

with \( b' = 1/C' - \log_2(2\sqrt{2erf^{-1}(1 - ε)}) \). This is Crossman’s version of Fitts’ law—equivalent to Fitts’ \( \log_2(2D/W) \)—which is very close to the one given by eq. (1), except for small values of ID.

VII. Conclusion

This paper provides a new model for voluntary movements, based on positional variance profiles (PVP), which take into account: (a) the variability of human produced movements; (b) a feedback mechanism, essential for reliable aiming; and (c) intermittent control that becomes continuous at the limit. Empirical evidence shows that in Fitts’ paradigm, the PVP is unmodulated: a first variance-increasing phase of constant duration is followed by a variance-decreasing second phase until the appropriate accuracy level is reached. We then established that:

1) The problem of aiming, during the second component, can be reduced to that of transmitting information from a source (position at the end of the first component) to a destination (limb extremity, cursor) over a channel perturbed by Gaussian noise with the presence of an ideal feedback link.

2) Using an optimal (Elias) scheme, in the sense that the transmitted information from source to destination is maximized at each step, we show that the variance can decrease at most exponentially during the second component. This result, which we call the local version of Fitts’ law, is summarized in [14].

Our analyses conducted on existing empirical data provide support in favor of the model. Variance is indeed found to decrease exponentially, and the rate at which the variance decreases is mostly independent from task parameters or participant speed-accuracy strategy.

The well known formula for Fitts’ law can be re-derived from the local Fitts’ law with the following mechanism:

1) The variability at the end of the first phase increases with the total distance to be covered, as in Schmidt’s law and Meyer et al.’s SOS model.

2) The variability at the end of the first phase is reduced during the second phase at a fixed rate: The more precision required, the more time is needed.

Thus, speed-accuracy tradeoff is mostly due to the second phase and the limited capacity of the channel.

It is interesting to observe how the information-theoretic framework appears naturally to solve the aiming problem; this reinforces the idea that information theory is a useful tool for measurement and quantification of human performance [11], [57]. The signal never actually needs to be transformed into the “vague” notion of bits, as the mutual information is simply a tangible ratio of two spatial variances.

An important question that remains open is whether or not these findings can be extrapolated to voluntary movement in general, and not just Fitts’ task. We expect the second phase to be characteristic of the way in which humans produce movements when they cannot rely on planification, either because these are not learned enough, or because they require such precision that relying on planification only is impossible. This suggests that the overall picture found for Fitts’ paradigm is valid for overlearned tasks. Another prospect is to consider non additive noise and non ideal feedback in a similar scheme. This could prove useful to model situations where feedback is non visual and only kinesthetic.
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APPENDIX A

Proof of Theorem 3. The proof uses well known techniques and inequalities from information theory [54]. For the first inequality in [17]:

\[ I(A; \hat{A}_n) = H(A) - H(A|\hat{A}_n) \]
\[ = H(A) - H(A - \hat{A}_n|\hat{A}_n) \]
\[ \geq H(A) - H(A - \hat{A}_n) \]
\[ \geq H(A) - \frac{1}{2} \log (2\pi e[(A - \hat{A}_n)^2]) \]
\[ = \frac{1}{2} \log \frac{\sigma_0^2}{D_n} \]  \hspace{1cm} (22)

we have used the definition of mutual information [18]; the fact that conditioning reduces entropy (20); the fact that the Gaussian distribution maximizes entropy under power constraints (21); and the definition of distortion and the entropy formula for a Gaussian distribution (22). For the second inequality in (17):

\[ I(A; \hat{A}_n) \leq I(A; Y^n) \]
\[ = H(Y^n) - H(Y^n|A) \]
\[ = \sum_i [H(Y_i|Y^{i-1}) - H(Y_i|Y^{i-1}, A)] \]
\[ \leq \sum_i [H(Y_i) - H(Z_i)] \]
\[ \leq \sum_i \left[ \frac{1}{2} \log(2\pi e(P_i + N)) - \frac{1}{2} \log(2\pi eN) \right] \]
\[ \leq \sum_i \frac{1}{2} \log \left( 1 + \frac{P_i}{N} \right) \leq nC \] \hspace{1cm} (29)

where we have used the data processing inequality [53] applied to the Markov chain \( A \rightarrow Y \rightarrow g(Y) = A' \) in (23); the chain rule to both entropies (25); the feedback scheme in (26); the fact that conditioning reduces entropy and independence of \( X_i \) and \( Z_i \) in (27); that fact that the Gaussian distribution maximizes entropy in (28) (where \( P_i \) and \( N \) are the powers of respectively \( X_i \) and \( Z_i \)); the concavity of the logarithm function (29).

Proof of Lemma 3. The proof consists of finding the conditions that turn the inequalities in the proof of Theorem 1 into equalities. Equality in Eqs. (20) and (23) directly imply condition 4. Equality in (21) implies that the \( A - A' \) are Gaussian. Equality in (23) implies that \( H(A|Y') = H(A|Y', g(Y')) = H(A|g(Y')) \), so that \( Y' \rightarrow g(Y') \rightarrow A \) should form a Markov chain, implying condition 5. Equality in (22) implies condition 3. Equality in (28) implies that the \( Y_i \)'s are Gaussian. Equality in (29) implies condition 5 by concavity of the logarithm. Finally, \( X_i \) is Gaussian, as the result of the sum of \( Y_i \) and \( Z_i \), both Gaussian. Similarly, \( \hat{A}_i \) is Gaussian as both \( A \) and \( A - \hat{A}_i \) are Gaussian, which finally yields condition 1.

Proof of Theorem 2. On one hand, \( X_i = \mathbb{E}[X_i|A, \hat{A}_{i-1}] \) is a linear function of \( A \) and \( A_{i-1} \), because the conditional expectation is linear for Gaussian variables. On the other hand, condition 5 reads that \( \mathbb{E}[A_i|A_{i-1}] = \mathbb{E}[g(X_i + Z_i)|A_{i-1}] = 0 \). Because \( Z_i \) is independent from \( X_i \) and \( A_{i-1} \), and \( g \) is linear, we get \( \mathbb{E}[g(X_i)|A_{i-1}] = 0 \). Combining the two results, and because \( g \) is linear, we have \( \mathbb{E}[\alpha_i(A - f(A_{i-1})), A_{i-1}] = 0 \). The orthogonality principle clearly appears and \( f = \mathbb{E}[A_i|A_{i-1}] \) is the MMSE estimator.

Proof of Theorem 4. The goal is to evaluate \( \mathbb{E}[A|Y_i] \). We first use the operational formula for the conditional expectation \( \mathbb{E}[A|Y_i] = \mathbb{E}(AY^{-1})\mathbb{E}(Y^{-1}(AY^{-1})^{-1}Y^{-1}) \). We get, using the fact that the \( Y_j \) are independent and have power \( P + N \):

\[ \mathbb{E}[A|Y_i] = \sum_{j=1}^{i-1} (P + N)^{-1}\mathbb{E}[A|Y_j] Y_j. \]

Second, it follows from previous computations that \( A - A_i = \mathbb{E}[A|Y_i] \), which is a function of the observations \( Y_i \) and therefore independent of \( Y_i \). This leads to \( A - A_i \) being independent of \( Y_i \) and the following equality: \( \mathbb{E}[A|Y_i] = \mathbb{E}[A|Y_i] \).

Combining both results, we thus get

\[ \mathbb{E}[A|Y_i] = \sum_{j=1}^{i-1} (P + N)^{-1}\mathbb{E}[A|Y_j] Y_j. \]

Proof of Theorem 5. First notice that we can write \( D_i = \mathbb{E}[(A_i - \mathbb{E}[A_i|Y_i])^2] \) as \( A - A^t = A - A_{i-1} - \mathbb{E}[A_i|Y_j] = A_i - [A_i | Y_j] \). Next, we have that

\[ D_i = \mathbb{E}(A_i^2) + \mathbb{E}[(A_i|Y_i)]^2 - 2\mathbb{E}(A_i \mathbb{E}(A_i|Y_i)) \]

From the proof above,

\[ \mathbb{E}[A_i|Y_i] = \mathbb{E}[A_i|Y_i] (P + N)^{-1} Y_i = \frac{P}{\alpha_i} (P + N)^{-1} Y_i. \]

With some calculus and using \( \mathbb{E}(A_i^2) = D_{i-1} = P/\alpha_i^2 \), we get that

\[ D_i = \frac{D_{i-1}}{1 + P/N}. \]

The proof is finished by applying this equation recursively.

REFERENCES

[1] Gori, Julien and Rioul, Olivier, “Information-Theoretic Analysis of the Speed-Accuracy Tradeoff with Feedback.” IEEE, 2018, to appear in IEEE Proc. Systems, Man, Cybernetics (SMC 2018).
[2] R. S. Woodworth, “Accuracy of voluntary movement.” The Psychological Review: Monograph Supplements, vol. 3, no. 3, p. i, 1899.
[3] P. M. Fitts, “The information capacity of the human motor system in controlling the amplitude of movement.” Journal of experimental psychology, vol. 47, no. 6, p. 381, 1954.
[4] A. T. Welford, “The measurement of sensory-motor performance: Survey and reappraisal of twelve years’ progress,” Ergonomics, vol. 3, no. 3, pp. 189–230, 1960.
[5] E. Crossman and P. Goodeve, “Feedback control of hand-movement and fitts’ law,” The Quarterly Journal of Experimental Psychology, vol. 35, no. 2, pp. 251–278, 1983, First presented at the Oxford Meeting of the Experimental Psychology Society, July 1963.
