The four height variables of the Abelian sandpile model

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We study the four height variables in the Abelian sandpile model. We argue that the four variables are not represented by the same operator along closed boundaries, or in the bulk. Along open boundaries, we calculate all n-point correlations, and find that there, all height variables are represented by the same operator. We introduce dissipative defect points, and show that along open boundaries they are represented by the same operator as the height variables. We show that dissipative defect points along closed boundaries, or in the bulk, have no effect on weakly allowed cluster variables.

PACS numbers: 05.65.-a,05.70.-n

The Abelian sandpile model (ASM) proposed by Bak, Tang and Wiesenfeld produces power laws without any fine-tuning of parameters, and thus potentially provides an explanation for how power laws can arise in nature. Since its introduction, the ASM has been used to analyze a diverse range of systems—see [2] for a review.

The ASM is an extraordinarily simple mathematical model (see [1] for a description). However, significant aspects of the ASM are still unknown. While the height one variable is well understood, the higher height variables (two, three and four) are not. We approach this problem in this paper primarily by looking at correlation functions along or near open and closed boundaries. We also investigate the role of dissipative defect sites.

The correlation functions are computed with an elegant method, introduced by Majumdar and Dhar, who used it to calculate unit height probabilities and correlation functions [3]. The recurrent states of the ASM can be mapped to spanning trees drawn on the sandpile lattice [4, 5]. A spanning tree is a connected, directed, acyclic graph, such that for every site there is a path leading to the root, which is “off the edge of the sandpile” (i.e. the “site” connected to all open boundaries). The site is said to be a predecessor of the site if the path from to the root goes through . The probability for a site to have height is equal to the probability that in a spanning tree, exactly of its nearest neighbors will be predecessors of (NNP’s, nearest-neighbor predecessors) [4].

Certain height configurations in the ASM correspond to local restrictions on the spanning tree. For example, the condition for a site to have height one is equivalent to a spanning tree condition that it be disconnected from three of its neighbors. Any probabilities corresponding to local restrictions on the spanning tree (known as weakly allowed cluster variables) can be calculated as finite-dimensional matrix determinants with the Majumdar-Dhar method [4, 5]. These calculations require use of the lattice Green function.

Spanning trees are described by the central charge conformal field theory ($c = -2$ LCFT), and so the question arises as to how variables in the ASM should be identified with operators in the LCFT. Mahieu and Ruelle, using the Majumdar-Dhar methods, calculated correlation functions for a number of height configurations, and showed that they agree with LCFT correlations, with appropriate field identifications [6]. However, these methods do not allow the calculation of probabilities or correlations for bare higher height variables, because they are associated with predecessor relationships, which are nonlocal; a site can be a predecessor of a neighbor by a long path which goes far from either or . Priezzhev has developed methods to find the bulk probabilities for higher height variables [6]. However, the bulk correlations for higher height variables remain unknown.

The sandpile boundaries can either be open, where grains of sand can fall off the edge during topplings, or closed, where they cannot. Ivashkevich has shown that along either boundary type, the nonlocal arrow diagrams associated with higher height probabilities and two-point correlations can be written as linear combinations of local arrow diagrams [5]. He found that all boundary two-point correlation functions, between all height variables, fall off as $1/r^4$, and thus argued that all height variables correspond to the same LCFT field. We reanalyze this calculation and results below. Mahieu and Ruelle presented other evidence that the higher height variables are identical to the unit height variable in the scaling limit; however, they also pointed out that this identification appears inconsistent with LCFT operator product expansions (OPEs) [6]. We argue here that analysis of boundary correlation functions, and heights far from the boundary, indicate that the height variables should not all receive the same field identification along closed boundaries, or in the bulk. However, along open boundaries, they are identical in all n-point correlations; additionally, dissipative defect points along open boundaries receive the same field identification as the height variables.

Closed: We define, for all n-point correlation functions along closed boundaries,

$$f_c(a_1, a_2, \ldots, a_n) = \langle (\delta h_{a_1} - p_{a_1, c}) \cdots (\delta h_{a_n} - p_{a_n, c}) \rangle_c,$$  \hspace{1cm} (1)
where “c” stands for “closed,” the $a_i$‘s are heights, $h_x$ is the height at position $x$ along the boundary, and $p_{a,c}$ is the constant probability for a site on the closed boundary to have height $a_i$. Note that along closed boundaries, sites cannot have height four. Ivashkevich found

$$f_c(1, 1) = \left( \frac{9}{\pi^2} + \frac{48}{\pi^3} - \frac{64}{\pi^4} \right) \frac{1}{(x_1 - x_2)^4} + \ldots$$  \hspace{1cm} (2)

$$f_c(2, 2) = \left( \frac{61}{4\pi^2} + \frac{96}{\pi^3} - \frac{144}{\pi^4} \right) \frac{1}{(x_1 - x_2)^4} + \ldots$$  \hspace{1cm} (3)

$$f_c(3, 3) = \left( \frac{1}{4\pi^2} + \frac{8}{\pi^3} - \frac{16}{\pi^4} \right) \frac{1}{(x_1 - x_2)^4} + \ldots$$  \hspace{1cm} (4)

While these correlations all have the same power law, the coefficient in $f_c(1, 1)$ is negative, while the coefficients of $f_c(2, 2)$ and $f_c(3, 3)$ are positive. These sign differences indicate that the three height variables are in fact not all represented by the same field operator. Furthermore, the coefficients of $f_c(a, a')$ in $\mathfrak{S}$ do not factorize (into $K_a K_{a'}$), as would be expected if all three height variables were the same up to rescaling.

As a check, we have redervied all results of $\mathfrak{S}$. (There is a misprint in the result for $f_c(3, 3)$ in $\mathfrak{S}$.) While we agree with the results of $\mathfrak{S}$, the analysis there appears to have several errors. For the two-point correlations between $r_i$ and $r_j$, it divides ASM states into sets $S_{ab}$, where the states of $S_{ab}$ are allowed with heights $h_{r_i} \geq a$ and $h_{r_j} \geq b$, but forbidden otherwise. However, not all ASM states fall into such sets. For example, there are states allowed when $(h_{r_i}, h_{r_j}) = (1, 2)$, and allowed when $(h_{r_i}, h_{r_j}) = (2, 1)$, yet forbidden when $(h_{r_i}, h_{r_j}) = (1, 1)$. However, the relationship in $\mathfrak{S}$ between the $S_{ab}$ and the spanning trees is also not quite correct, and these errors end up largely, but not entirely, cancelling. The spanning tree representation is somewhat surprising. The natural assumption (made in $\mathfrak{S}$) is that all spanning trees where $r_i$ and $r_j$ each have one NNP contribute to the 2-2 correlation in the ASM. However there are some such spanning trees that do not; for example, trees where $r_i$ and $r_j$ each have one NNP, and neither is a predecessor of the other, but a neighbor of $r_i$ is a predecessor of $r_j$, and a neighbor of $r_j$ is a predecessor of $r_i$, contribute not to the 2-2 correlation, but to the 2-3 correlation. Such cases can be written as linear combinations of closed loop diagrams, and then calculated using a generalized Kirchoff theorem. Luckily, these graphs fall off as $1/(x_1 - x_2)^3$, leaving the results of $\mathfrak{S}$ unaffected.

We have calculated all closed boundary three-point correlation functions that include at least one unit height variable. Some results are

$$f_c(1, 1, 1) = \frac{2(3\pi - 8)^3}{\pi^6(x_1 - x_2)^3(x_1 - x_3)^3(x_2 - x_3)^2} + \ldots$$ \hspace{1cm} (5)

$$f_c(1, 1, 2) = -\frac{8(\pi - 3)(3\pi - 8)^2}{\pi^6(x_1 - x_2)^3(x_1 - x_3)^2(x_2 - x_3)^2}$$

The correlation functions must satisfy identities resulting from the fact that the height probabilities at any site must sum to one. For example, $\sum_{a=1}^3 f_c(a_1, a_2, a_3) = 0$. As a check, we have verified that all other three-point correlations that include at least one unit height variable satisfy all such identities. All calculated two- and three-point correlations are consistent with bulk correlations identifying $-2(3\pi - 8)/\pi^2$ with the height one variable, $(6(\pi - 4)/\pi^2)(\partial \theta \partial \bar{\theta}) + (1/2\pi)\theta^2\bar{\theta}$ with the height two variable, and $(8/\pi^2)(\partial \theta \partial \bar{\theta}) - (1/2\pi)\theta^2\bar{\theta}$ with the height three variable. There are no $\bar{\theta}$'s, since fields near a boundary of a (L)CFT can be thought of as purely holomorphic; CFT boundary correlations behave like bulk correlations, where the antiholomorphic parts of the field act like holomorphic pieces at mirror locations across the boundary. It is also consistent to make the substitution $\theta \to \bar{\theta}$, $\bar{\theta} \to \theta$ in all field identifications above, as the $c = -2$ CFT is symmetric under this transformation. (And $\bar{\theta}$ are variables in the $c = -2$ LCFT. See $\mathfrak{L}$ for a brief description of the $c = -2$ LCFT, and references therein for a more complete treatment.)

We have not been able to calculate all terms contributing to $f_c(2, 2, 2)$, but note that if complications arising from graphs such as those (to be) discussed below Figs. II and $\mathfrak{L}$ are ignored, we obtain

$$f_c(2, 2, 2) = -\frac{4(24 - 5\pi)(-576 + 384\pi - 61\pi^2)}{4\pi^6(x_1 - x_2)^3(x_1 - x_3)^2(x_2 - x_3)^2} + \ldots$$ \hspace{1cm} (8)

which is consistent with the field identification above for the height two variable above.

We have also introduced a dissipative defect site on the closed boundary. At a normal site of a closed boundary, a site topples whenever it has more than three grains, losing three grains, and sending one grain to each of its three neighbors. On an open boundary, sites topple when they have more than four grains, and thus send an extra grain off the edge. We add a “dissipative defect point” on the closed boundary, where the site can now have up to $3 + k$ grains, and where $k$ grains fall off the edge with each toppling. Accordingly, $d$ downtoppings have been added uniformly in the bulk, where it breaks criticality $\mathfrak{L}$. (But see $\mathfrak{L}$ for dissipation along a line.) We have shown that if the defect-free closed boundary Green function is $G_c(0, r_1, r_2)$, then a defect at $\bar{d}$ changes the Green function (dropping constant terms) to

$$G_c(r_1, r_2) = G_c(0, r_1, r_2) - G_c(0, r_1, \bar{d}) - G_c(0, \bar{d}, r_2)$$ \hspace{1cm} (9)
The modified Green function is independent of the value of $k$. (This is reasonable, since regardless of the amount of dissipation, the defect provides the only possible spanning tree route to the root.) Using this new Green function, for a defect at $x = 0$ on the boundary, we find

$$f_c(1) = 0$$
$$f_c(2) = \frac{1}{2\pi x_1^2} + \ldots$$

The height one probability is unaffected, to all orders, at all points, in the bulk and the boundary, by the defect (except at the defect itself). $\sum_{a=1}^{3} f_c(a) = 0$, so $f_c(3) = -f_c(2)$. Eqs. (10-11) have been numerically confirmed.

These results indicate that a dissipative defect on a closed boundary is represented by a dimension 0 operator. (Note that Mahieu and Ruelle have identified uniform dissipation in the bulk of the ASM with the integral of a dimension 0 operator) The defect has correlations with the height two and three operators, but not with the height one operator, again pointing to different field identifications for the different height variables. Since they differ along closed boundaries, they must also differ in the bulk, since CFT boundary operators are derived from OPE’s of bulk operators.

Finally, we have found the correlation function of $n$ unit height variables, at positions $x_i \ (i = 1, 2, \ldots, n)$ along the boundary. This requires local arrow constraints at $3n$ vertices of the ASM, and thus the calculation of a $3n$-dimensional matrix determinant. The matrix is divided into 3 by 3 block submatrices, such that the diagonal blocks are all identical, and the off-diagonal blocks all have the same form. A rotation makes the matrix diagonal in 2 out of every 3 rows (and columns). The universal part of the correlation function is found to be

$$\left(\frac{3\pi - 8}{\pi^2}\right)^n \det(M),$$

where $M$ is the $n$-dimensional matrix

$$M_{ij} = \begin{cases} 0 & \text{if } i = j \\ 1/(x_i - x_j)^2 & \text{if } i \neq j \end{cases}$$

This is the same as the $n$-point correlation of $-2(3\pi - 8)/\pi^2 \ (\partial \theta \partial \delta)$ in the bulk, confirming the unit height identification below Eq. (4). In $\partial$, the unit height variable in the bulk was associated with $\partial \theta \partial \delta + \partial \delta \partial \theta$.

Open: The correlation functions are simpler along open boundaries. We define the operators

$$\phi_a(x) = \frac{\delta h_{x,a} - p_{a,\text{op}}}{K_a},$$

where $a = 1, \ldots, 4$. $p_{a,\text{op}}$ is the constant probability for a site along an open boundary to have height $a$ (already found in $\phi$), and the $K_a$ are normalization factors:

$$p_{1,\text{op}} = \frac{9}{2} - \frac{42}{\pi} + \frac{320}{3\pi^2} - \frac{512}{9\pi^2}$$
$$K_1 = -\frac{3}{\pi} + \frac{80}{3\pi^2} - \frac{512}{9\pi^2}$$
$$p_{2,\text{op}} = -\frac{33}{4} + \frac{66}{\pi} - \frac{160}{3\pi^2} + \frac{1024}{9\pi^2}$$
$$K_2 = \frac{9}{\pi} - \frac{200}{3\pi^2} + \frac{1024}{9\pi^2}$$
$$p_{3,\text{op}} = \frac{15}{4} - \frac{22}{\pi} + \frac{160}{3\pi^2} - \frac{512}{9\pi^2}$$
$$K_3 = -\frac{7}{\pi} + \frac{40}{3\pi^2} - \frac{512}{9\pi^2}$$
$$p_{4,\text{op}} = 1 - \frac{2}{\pi}$$
$$K_4 = \frac{1}{\pi}$$

Using methods similar to those used to find Eq. (12), we find that the $n$-point open boundary correlation function,

$$\langle \phi_{a_1}(x_1)\phi_{a_2}(x_2)\ldots\phi_{a_n}(x_n)\rangle$$

is equal to $\det(M)$. (These results reproduce and extend the one- and two-point functions found in $\phi$. Eq. (10) is independent of the $a_i$’s, showing that the four height variables, upon rescaling, do all receive the same field assignment $(\partial \theta \partial \delta)$ along open boundaries. Apparently the different height variables correspond to different fields in the bulk, and remain different along closed boundaries, but become the same along open boundaries.

We again add dissipative defects. At the site $d = (x,y)$, we increase the toppling condition by $k > 0$, so that $k$ grains of sand are dissipated with each toppling. We assume that all open defect sites are $y = O(1)$ from the boundary (at $y = 1$). Then the new Green function is

$$G_{\text{op}}(r_1, r_2) = G_{\text{op},0}(r_1, r_2) - \frac{k}{1 + k G_{\text{op},0}(d, d)} G_{\text{op},0}(d, r_1) G_{\text{op},0}(d, r_2)$$

$G_{\text{op},0}$ is the defect-free open boundary Green function. Eqs. (12) and (14) are different because the Green function between nearby points is $O(1)$ on an open boundary, but $O(\ln L)$ on a closed boundary, where $L$ is the distance to the nearest open boundary. We have generalized Eq. (17) for multiple open dissipative defects.

We define $\phi_d(d;k)$ as the operator corresponding to the addition of a defect of strength $k$ at $d = (x,y)$, and then the multiplication of all correlation functions containing $\phi_d(d;k)$ by $\pi(1 + k G_{\text{op},0}(d,d)) / (k y^2)$. Then the connected $n$-point correlation function in Eq. (16) is still given by the connected terms of $\det(M)$, even if some of the $a_i$ are now equal to 5, and regardless of the (different) values of the $k$’s at the various defects. So the addition of a local dissipative defect near an open boundary is, like the height variables, represented by $\partial \theta \partial \delta$.

The open boundary is much more tractable than the closed boundary for several reasons. For calculating one-point functions on any boundary, when we write the nonlocal arrow diagrams as linear combinations of local arrow diagrams, we use the equivalence of certain nonlocal...
though, in the end, the loop paths end up cancelling.

graphs, fall off as 1/x along open boundaries, the graph in Fig. 2, and other analogous correlation functions. On the other hand, along closed boundaries, the graph in Fig. 2, and all analogous graphs, turn out to fall off faster than 1/x, complicating matters (although, in the end, the 1/(x1 - x2)^4 parts of these closed loop paths end up cancelling).

This produces extra terms, such as the graph in Fig. 2.

The graph in Fig. 2 can be written as a linear combination of closed loop graphs. It, and all analogous graphs, turn out to fall off faster than 1/\(x1 - x2)^4\) for the open case, and thus make no contribution to any universal correlation functions. On the other hand, along closed boundaries, the graph in Fig. 2 and other analogous graphs, fall off as 1/\(x1 - x2)^4\), complicating matters (although, in the end, the 1/(x1 - x2)^4 parts of these closed loop paths end up cancelling).

Also, the Green function decays as 1/x^2 along open boundaries (unlike the closed and bulk cases, where it grows as \(\ln(x)\)). This allows trace formulae found by Mahieu and Ruelle for 2- and 3-point correlations to be generalized for open boundary n-point functions, simplifying matters.

**Bulk**: Analysis of the higher height probabilities well in the bulk, at distances \(y \gg 1\) from a boundary provides a further argument that the higher height variables should not all receive the same bulk field identification. Suppose that they do all have the same field identification, up to rescaling. Then we would know both that

\[ p_a(y) = p_{a,B} + \frac{c_a}{y^2} + \ldots \]  

(18)

for all \(a = 1 \ldots 4\) (based on results for \(a = 1\) in [14]), and also that all height \(a\)-height \(a\) correlations in the bulk would have to have the same sign (negative [3]). Upon rescaling the height variables to give them (negative) unit norm, the \(c_a\) are rescaled to \(\tilde{c}_a\). Based on general (L)CFT arguments, the \(\tilde{c}_a\) should be universal numbers, independent of \(a\) [10, 11, 13]. (See [13] for an explicit demonstration of this, in the context of the ASM.) Then the original (unrescaled) \(c_a\) must all have the same sign. However, the four \(c_a\)’s cannot all have the same sign, since \(\sum_{a=1}^{4} p_a(y) = 1\) for all \(y\). So, contrary to our assumption, the four height variables must have different bulk field identifications.

A dissipative defect in the bulk modifies the Green function exactly as in Eq. (13), and again does not modify unit height probabilities at any sites other than the defect. However, modest numerical simulations show that higher height probabilities are changed near the defect. This parallels the the closed case, again indicating different field identifications for the higher height variables.

More generally, we can show that no height configuration probabilities that can be calculated by the removal of a set of bonds in the ASM (i.e. the weakly allowed cluster variables, and their correlations) are affected by a dissipative defect in the bulk or closed cases. This lack of correlations suggests that the analysis of weakly allowed cluster variables in [13], while impressive, should not be expected to give a fully representative picture of the field structure of the ASM.

After the bulk of this work was completed, we were informed of independent, then unpublished, calculations of Eqs. (8, 9, 10, 11), in the massive case, by G. Piroux and P. Ruelle [16]. Discussions with G. Piroux and P. Ruelle then led us to the field identifications below Eq. (12), and they further corrected an error we had made in these same equations. This work was supported by Southern Illinois University Edwardsville. We thank V. Gurarie and E. V. Ivashkevich for useful discussions.

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