Initial-boundary value problems for linear diffusion equation with multiple time-fractional derivatives

Zhiyuan Li$^1$, Masahiro Yamamoto$^2$

$^{1,2}$Graduate School of Mathematical Sciences, The University of Tokyo
E-mail: $^1$zyli@ms.u-tokyo.ac.jp, $^2$myama@ms.u-tokyo.ac.jp

Abstract

In this paper, we discuss initial-boundary value problems for linear diffusion equation with multiple time-fractional derivatives. By means of the Mittag-Leffler function and the eigenfunction expansion, we reduce the problem to an integral equation for a solution, and we apply the fixed-point theorem to prove the unique existence and the Hölder regularity of solution. For the case of the homogeneous equation, the solution can be analytically extended to a sector $\{ z \in \mathbb{C} ; z \neq 0, |\arg z| < \frac{\pi}{2} \}$. In the case where all the coefficients of the time-fractional derivatives are positive constants, by the Laplace transform and the analyticity, we can prove that if a function satisfies the fractional diffusion equation and the homogeneous Neumann boundary condition on arbitrary subboundary as well as the homogeneous Dirichlet boundary condition on the whole boundary, then it vanishes identically.

Keywords: fractional diffusion equation, multiple time-fractional derivatives, initial-boundary value problem, eigenfunction expansion, fixed point, unique existence of solution

1 Introduction

We assume $\Omega$ to be a bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary $\partial \Omega$. We consider an initial-boundary value problem for a diffusion equation with multiple fractional time derivatives:

\[
\begin{align*}
\partial_t^{\alpha_1} u(t) + \sum_{j=2}^{\ell} q_j \partial_t^{\alpha_j} u(t) &= -A u(t) + B(x) \cdot \nabla u(t) + F(x,t), \quad t > 0, \\
u(x,0) &= a, \quad x \in \Omega, \\
u(x,t) &= 0, \quad x \in \partial \Omega, \quad t \in (0,T).
\end{align*}
\]

Here $0 < \alpha_\ell < \cdots < \alpha_2 < \alpha_1 < 1$. For $\alpha \in (0,1)$, by $\partial_t^{\alpha}$ we denote the Caputo fractional derivative with respect to $t$:

\[
\partial_t^{\alpha} g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{d}{d\tau} g(\tau) d\tau
\]

and $\Gamma$ is the Gamma function. See, e.g., Podlubny [31] and Kilbas et al [18] for the definition and properties of the Caputo derivative.
The operator $A$ denotes a second-order partial differential operator in the following form

$$(-Au)(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial}{\partial x_j} u(x) \right) + b(x)u(x), \quad x \in \Omega,$$

for $u \in H^2(\Omega) \cap H^1_0(\Omega)$, and we assume that $a_{ij} = a_{ji} \in C^1(\Omega)$, $1 \leq i, j \leq d$, $b \in C(\Omega)$, $b(x) \leq 0$ for $x \in \Omega$ and that there exists a constant $\nu > 0$ such that

$$\nu \sum_{j=1}^{d} \xi_j^2 \leq \sum_{j,k=1}^{d} a_{jk}(x)\xi_j \xi_k, \quad x \in \Omega, \quad \xi \in \mathbb{R}^d.$$

The classical diffusion equation with integer-order derivative has played important roles in modelling contaminants diffusion processes. However, in recent two decades, more experimental data in some diffusion processes in highly heterogeneous media, show that the classical model may be inadequate in order to interpret experimental data. For example, Adams and Gelhar [2] points out that field data in a saturated zone of a highly heterogeneous aquifer indicate a long-tailed profile in the spatial distribution of densities as the time passes, which is difficult to be interpreted by the classical diffusion equation. The above phenomenon of long-tailed profile has been investigated by many researchers, and see Berkowitz et al [3], Giona et al [11], Y. Hatano and N. Hatano [14], and the references therein. For better model equation, an equation where the first-order time derivative is replaced by a derivative of fractional order $\alpha \in (0, 1)$, has been proposed. As defined below, the fractional derivative possesses the memory effect and leads to realization of slow diffusion. This modified model is presented as a useful approach for the description of transport dynamics in complex system that are governed by anomalous diffusion and non-exponential relaxation patterns, and attracted great attention from different areas. For numerical calculation, see Beson et al [5], Meerschaert et al [29], Diethelm and Luchko [9] and the references therein. For theoretical aspects, see Gorenflo et al [12], Hanyaga [13], Luchko and Gorenflo [21], Luchko [22, 23, 24], Sakamoto and Yamamoto [34], Xu et al [37], etc. From a viewpoint of the stochastic analysis, one can regard the time-fractional diffusion equation as a macroscopic model derived from the continuous-time random walk. Metzler and Klafter [28] demonstrated that a fractional diffusion equation describes a non-Markovian diffusion process with a memory. Roman and Alemany [32] investigated continuous-time random walks on fractals and showed that the average probability density of random walks on fractals obeys a diffusion equation with a fractional time derivative asymptotically.

In some recent publications such as e.g. [6, 7, 35], the time-fractional diffusion equations of distributed order is investigated. A distributed order derivative is an integral of fractional derivatives with respect to continuously changing orders. Differential equations of the distributed order and some of their applications were considered e.g., in [19, 27, 36]. One important particular case of the time-fractional diffusion equation of distributed order is that the weight function is taken in form of a finite linear combination of the Dirac $\delta$-functions with the positive weight coefficients (see e.g., [22, 23]). This yields the so-called diffusion equation with multiple time-fractional derivatives, which is studied in our paper. As for diffusion equations with multiple fractional time derivatives, see also Jiang et al [17], Gejji et al [10], and the the references therein. The article [17] discusses the case where the spatial dimension is one, the coefficients are constant and the spatial fractional derivative is considered, and establishes the formula of the solution. In the paper [10], a solution to an initial-boundary value problem is formally represented by Fourier series and the multivariate Mittag-Leffler function. As for multivariate Mittag-Leffler functions, see e.g., [21]. However no proofs for the convergence of the series and for the uniqueness of the solution are given in [10]. A proof of the convergence of the series defining the solution of the
more general distributed order fractional Cauchy problems in bounded domains can be found in the paper [20]. The paper [24] proves the unique existence of the solution, the maximum principle and related properties in the case where the coefficients of the time derivatives are positive and depend on x, and the arguments are based on the Fourier method, that is, the separation of the variables.

These papers mainly discuss the case where the spatial differential operators is a symmetric elliptic operator.

The paper [3] proves the uniqueness of solution to an initial-boundary problem for a symmetric fractional diffusion equation with two terms of time-fractional derivatives, by assuming the existence of the solution. The method is similar to [21] and [34].

In this article, following [25] and [3], we deal with the initial-boundary value problems for linear diffusion equation with multiple time-derivatives. The difference is that here we investigate the linear non-symmetric diffusion equation with the variable coefficients of fractional time derivatives not necessary constant or positive variable. Such kind of equation simulates the advection and so can be regarded as more feasible model equation than symmetric fractional diffusion equations in modelling diffusion in porous media.

The rest of this article is organized as follows:

Section 2: The fixed point method is applied to prove unique existence as well as regularity of solution to (1).

Section 3: Based on the existence result in section 2, we prove the regularity of Hölder of the solution step by step from the continuous regularity to the Hölder regularity with index under the assumption that initial condition \(a = 0\) and the source term \(F \in C^{\theta}([0,T];L^2(\Omega))\) with the compatibility condition \(F(0) = 0\).

Section 4: We prove that the solution can be analytically extended to a sector \(\{z \in \mathbb{C}; z \neq 0, |\arg z| < \frac{\pi}{2}\}\) when \(F = 0\) and \(a \in L^2(\Omega)\).

Section 5: The analyticity of the solution and the Laplace transform are applied to show that if a function satisfies the fractional diffusion equation and the homogeneous Neumann boundary condition on arbitrary subboundary as well as the homogeneous Dirichlet boundary condition on the whole boundary, then it vanishes identically. This is a weak type of unique continuation.

### 2 Existence, uniqueness and regularity of the solution

Let \(L^2(\Omega)\) be a usual \(L^2\)-space with the inner product \((\cdot, \cdot)\), and \(H^k(\Omega), H^s_0(\Omega)\) denote Sobolev spaces (e.g., Adams [1]). We set \(\|a\|_{L^2(\Omega)} = (a,a)^{\frac{1}{2}}\).

We define an operator \(A\) in \(L^2(\Omega)\) by

\[(Au)(x) = (Au)(x), \quad x \in \Omega, \quad D(A) = H^2(\Omega) \cap H^1_0(\Omega).\]

Then the fractional power \(A^\gamma\) is defined for \(\gamma \in \mathbb{R}\) (e.g., [30]), and \(D(A^\gamma) \subset H^{2\gamma}(\Omega), D(A^{\frac{1}{2}}) = H^1_0(\Omega)\) for example. We note that \(\|u\|_{D(A^\gamma)} := \|A^\gamma u\|_{L^2(\Omega)}\) is stronger than \(\|u\|_{L^2(\Omega)}\) for \(\gamma > 0\).

Since \(A\) is a symmetric uniformly elliptic operator, the spectrum of \(A\) is entirely composed of eigenvalues and counting according to the multiplicities, we can set \(0 < \lambda_1 \leq \lambda_2 \leq \cdots\). By \(\phi_n \in D(A)\), we denote the orthonormal eigenfunction corresponding to \(\lambda_n\): \(A\phi_n = \lambda_n \phi_n\). Then the sequence \(\{\phi_n\}_{n \in \mathbb{N}}\) is orthonormal basis in \(L^2(\Omega)\). Then we see that

\[D(A^\gamma) = \left\{ \psi \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^{2\gamma} |(\psi, \phi_n)|^2 < \infty \right\}\]
and that $D(A^\gamma)$ is a Hilbert space with the norm

$$
\|\psi\|_{D(A^\gamma)} = \left( \sum_{n=1}^{\infty} \lambda_n^{2\gamma} |(\psi, \phi_n)|^2 \right)^{\frac{1}{2}}.
$$

Moreover we define the Mittag-Leffler function by

$$
E_{\alpha, \beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C},
$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are arbitrary constants. By the power series, we can directly verify that $E_{\alpha, \beta}(z)$ is an entire function of $z \in \mathbb{C}$.

Now we define an operator $S(t) : L^2(\Omega) \to L^2(\Omega)$ for $t > 0$ by

$$
S(t)a := \sum_{n=1}^{\infty} (a, \phi_n)E_{\alpha_1, 1}(-\lambda_n t^{\alpha_1})\phi_n \text{ in } L^2(\Omega)
$$

for $a \in L^2(\Omega)$. Moreover the term-wise differentiations are possible and give

$$
S'(t)a := -\sum_{n=1}^{\infty} \lambda_n (a, \phi_n)t^{\alpha_1-1}E_{\alpha_1, 1}(-\lambda_n t^{\alpha_1})\phi_n \text{ in } L^2(\Omega)
$$

$$
S''(t)a := -\sum_{n=1}^{\infty} \lambda_n (a, \phi_n)t^{\alpha_1-2}E_{\alpha_1, 1-1}(-\lambda_n t^{\alpha_1})\phi_n \text{ in } L^2(\Omega)
$$

for $a \in L^2(\Omega)$, $t > 0$. Moreover, it is known (See, e.g., Theorem 1.6 in [31]) that there exists constant $C > 0$ such that

$$
\|A^{-1}S'(t)\| \leq Ct^{\alpha_1-1-\alpha_1}\gamma, \quad 0 < t \leq T, \quad 0 \leq \gamma \leq 1.
$$

(3)

$$
\|A^{-1}S''(t)\| \leq Ct^{\alpha_1-2-\alpha_1}\gamma, \quad 0 < t \leq T, \quad 0 \leq \gamma \leq 1,
$$

(4)

where $\| \cdot \|$ denotes the operator norm from $L^2(\Omega)$ to $L^2(\Omega)$.

Now we are ready to state our first main result.

**Theorem 2.1** (a priori estimate). Suppose that $a \in L^2(\Omega)$, $F \in L^\infty(0, T; L^2(\Omega))$, $B(x) := (B_1(x), \ldots, B_d(x))$, $B_i \in W^{2, \infty}(\Omega)$, $1 \leq i \leq d$, $q \in W^{2, \infty}(\Omega)$. Let $0 < \alpha_1 < \cdots < \alpha_1 < 1$ and $u(t) \in D(A^\gamma)$, $0 < t \leq T$ satisfy (7). Then

$$
\|u(t)\|_{H^{2\gamma}(\Omega)} \leq C \left( t^{-\alpha_1}\gamma \|a\|_{L^2(\Omega)} + \|F\|_{L^\infty(0, T; L^2(\Omega))} \right) + 0 < t \leq T,
$$

where $\gamma \in \left( \frac{1}{2}, 1 \right)$ and $C > 0$ is a constant which is independent of $a$, $F$ in (7), but may depend on $T$, $d$, $\{\alpha_j\}_{j=1}^{d}$, $\gamma$, $\Omega$ and the coefficients of the operator $A$, $\{q_i\}_{i=2}^{d}$.

In [9] a similar fractional diffusion equation is discussed for $F = 0$ and $B = 0$ and a similar regularity is proved. However [3] assumes an extra condition $\alpha_1 + \alpha_1 > 1$, and our main result needs not such an assumption.
Proof. Since \( u \) is the solution of our initial-boundary value problem (??), by an argument similar to the proof of Theorem 1 in [3], we find

\[
\begin{aligned}
\hspace{2mm} & u(t) = -\int_0^t A^{-1} S'(t-\tau)(B \cdot \nabla u + F) d\tau + \sum_{i=2}^\ell \frac{1}{\Gamma(1-\alpha_i)} \int_0^t A^{-1} S'(t-\tau)(t-\tau)^{-\alpha_i}(q_i u(\tau)) d\tau \\
& + \sum_{i=2}^\ell \frac{1}{\Gamma(1-\alpha_i)} \int_0^t \int_0^{t-\tau} A^{-1} S''(t-\eta-\tau)(\eta^{-\alpha_i} - (t-\tau)^{-\alpha_i}) \eta q_i u(\tau) d\tau d\eta \\
& - \sum_{i=2}^\ell \frac{1}{\Gamma(1-\alpha_i)} \int_0^t A^{-1} S'(t-\tau)^{-(\alpha_i+1)} q_i d\tau + S(t)a := \sum_{k=1}^5 I_k(t), \hspace{2mm} 0 < t \leq T. 
\end{aligned}
\]

(5)

Now let us turn to the evaluation of the integral equation (??). For this purpose, taking the fact \( D(A^\gamma) \subseteq D(A^\beta)(\forall \alpha \geq \beta \geq 0) \), noting that \( D(A^\gamma) = H^\gamma_0(\Omega) \), it follows that

\[ \| B \cdot \nabla u \|_{L^2(\Omega)} \leq C \| u \|_{H^1(\Omega)} \leq C_1 \| u \|_{D(A_4)} \leq C_2 \| u \|_{D(A^\gamma)}, \hspace{2mm} \forall \gamma \in [\frac{1}{2}, 1). \]

Moreover, since \( q_j \in W^{2,\infty}(\Omega), \hspace{2mm} j = 2, \cdots, \ell \) by the interpolation theory (see, e.g., [20]), we have \( \| A^\gamma(q_j u) \|_{L^2(\Omega)} \leq C \| A^\gamma u \|_{L^2(\Omega)} \) for any \( \gamma \in [0, 1] \), any \( u \in D(A^\gamma) \). Therefore for \( I(t) := I_1(t) + I_2(t) + I_4(t) + I_5(t), \hspace{2mm} \forall 0 < t \leq T \), using (3) and (4), we have the following estimate

\[
I(t) \leq C_{\gamma} \| A^\gamma u(t) \|_{L^2(\Omega)} + C \sum_{i=2}^\ell \int_0^t (t-\tau)^{\alpha_i+1} \| A^\gamma u(\tau) \|_{L^2(\Omega)} d\tau + C \int_0^t (t-\tau)^{\alpha_i-1} \| A^\gamma u(\tau) \|_{L^2(\Omega)} d\tau.
\]

In order to evaluate \( I_3(t) \) we need more technical treatment to the integral in \( I_3(t) \). In fact, after the change of variable \( \tilde{\tau} = t - \tau \), and letting \( \tilde{\eta} = \frac{\eta}{\tilde{\tau}} \), we have

\[
\| I_3(t) \|_{L^2(\Omega)} \leq C \sum_{i=2}^\ell \int_0^t \left[ \int_0^1 (\tilde{\tau} - \tilde{\eta})^{\alpha_i+2} (\tilde{\tau}^{\alpha_i} - \tilde{\eta}^{\alpha_i}) \tilde{\tau} d\tilde{\eta} \right] \| A^\gamma u(t - \tilde{\tau}) \|_{L^2(\Omega)} d\tilde{\tau}. 
\]

Finally, we are to prove \( J_i := \int_0^1 (1-\eta)^{\alpha_i-2} (\eta^{-\alpha_i} - 1) d\eta < \infty, \hspace{2mm} j = 2, \cdots, \ell \). In fact, we represent \( J_i \) as follows:

\[
J_i = \int_0^\frac{1}{2} (1-\eta)^{\alpha_i-2} (\eta^{-\alpha_i} - 1) d\eta + \int_\frac{1}{2}^1 (1-\eta)^{\alpha_i-2} (\eta^{-\alpha_i} - 1) d\eta := J_i' + J_i''.
\]

For \( J_i' \). Using the inequality \( (1-\eta)^{\alpha_i-2} \leq \frac{1}{2^{\alpha_i-2}}, \hspace{2mm} \forall \eta \in [0, \frac{1}{2}] \), we derive

\[
J_i' \leq C \int_0^\frac{1}{2} (\eta^{-\alpha_i} - 1) d\eta < \infty.
\]
For $J''_i$. Using the inequality $\eta^{-\alpha_i} - 1 \leq C(1 - \eta)\eta^{-\alpha_i - 1}$, $\forall \eta \in (0, 1)$, $j = 2, \ldots, \ell$. This inequality is proved, for example, by means of the mean value theorem. Then we can deduce

$$J''_i \leq C \int_\frac{1}{p}^1 (1 - \eta)\alpha_i - 2 (1 - \eta)\eta^{-\alpha_i - 1} d\eta = C \int_\frac{1}{p}^1 (1 - \eta)\alpha_i - 1 \eta^{-\alpha_i - 1} d\eta < \infty.$$ 

Collecting the estimates for $J'_i$ and $J''_i$, we have

$$\|I_3(t)\|_{L^2(\Omega)} \leq C \sum_{i=2}^\ell \int_0^t \tau^\alpha_i - \alpha_i - 1 \|A^\gamma u(t - \tau)\|_{L^2(\Omega)} d\tau = C \sum_{i=2}^\ell \int_0^t (t - \tau)\alpha_i - \alpha_i - 1 \|A^\gamma u(\tau)\|_{L^2(\Omega)} d\tau.$$ 

Finally $A^\gamma u$ can be estimated as follows: for $0 < t \leq T$, $\gamma \in [\frac{1}{2}, 1)$,

$$\|A^\gamma u(t)\|_{L^2(\Omega)} \leq C \int_0^t (t - \tau)^\alpha \|A^\gamma u(\tau)\|_{L^2(\Omega)} d\tau + C(\|F\|_{L^\infty(0, T; L^2(\Omega))} + \|a\|_{L^2(\Omega) t^{-\alpha\gamma}}),$$

where $\alpha = \min(\alpha_1 - \alpha_1, \alpha_1 - \alpha_2)) = \alpha_1 - \max(\alpha_1, \alpha_2)$. Moreover, from the general Gronwall inequality (e.g., Lemma 7.1.1 in [14]), we see that

$$\|A^\gamma u(t)\|_{L^2(\Omega)} \leq C (t^{-\alpha\gamma}) \|a\|_{L^2(\Omega)} + \|F\|_{L^\infty(0, T; L^2(\Omega))}), 0 < t \leq T,$$

where the constant $C > 0$ only depend on $d$, $\{\alpha_j\}_{j=1}^\ell$, $\gamma$, $T$, $B$, $\Omega$ and the coefficients of $A$, $\{q_j\}_{j=2}^\ell$.

Next we consider the unique existence of our fractional diffusion equation. On the basis of the fact that the solution $u$ to the problem (11) satisfies the integral equation (12), we call the function $u$ which satisfies (15), $u \in C([0, T]; L^2(\Omega))$ and $u(t) \in H_0^1(\Omega)$, a.e. $t \in [0, T]$ as the mild solution of the initial-boundary problem (11).

For any fixed $T > 0$, we define the operator $K$ as follows:

$$K u(t) = \sum_{k=1}^5 I_k(t),$$

where $I_k(t)$, $k = 1, \ldots, 5$ are defined in (15). The homogeneous equation is firstly investigated. Local existence result for mild solution was established via Banach's Fixed Point Theorem. Namely, the following theorem holds.

**Theorem 2.2** (Local existence). Suppose that $0 < \alpha_\ell \leq \cdots \leq \alpha_1 < 1$, $a \in L^2(\Omega)$, $B(x) := (B_1(x), \ldots, B_d(x))$, $B_i \in W^{2, \infty}(\Omega)(i = 1, \ldots, d)$, $q_j \in W^{2, \infty}(\Omega)(2 \leq j \leq \ell)$, and $F = 0$. Then there exists a mild solution to (12) in the space $L^p(0, \delta; H_0^1(\Omega)) \cap C([0, \delta]; L^2(\Omega))$ with $\delta \in [\frac{1}{2}, 1)$, where $\delta > 0$ is small enough, and $p \in (\frac{1}{\alpha_1}, \frac{2}{\alpha_1})$.

In order to prove Theorem 2.2 we give some estimates, which are organized in the following lemmas.

**Lemma 2.1.** Under the assumptions of Theorem 2.2. Then the following estimate

$$\|K u(t)\|_{L^2(\Omega)} \leq C T^\alpha_1 - \frac{1}{T} \|u\|_{L^p(0, T; H_0^1(\Omega))} + C \sum_{i=2}^\ell T^\alpha_i - \alpha_1 \|u\|_{C([0, T]; L^2(\Omega))} + C \|a\|_{L^2(\Omega)}, \quad t \in [0, T]$$

holds for each function $u \in L^p(0, T; H_0^1(\Omega)) \cap C([0, T]; L^2(\Omega))$, $p \in (\frac{1}{\alpha_1}, \frac{2}{\alpha_1})$. 

6
Here and henceforth in this section, $C > 0$ denotes constants only depend on $d$, $(\alpha_j)_{j=1}^{\ell}$, $\gamma$, $\Omega$ and the coefficients of the operator $A$, $(q_i)_{i=2}^{\ell}$. Moreover, $C_T > 0$ denotes constants only depending on $T$, $d$, $(\alpha_j)_{j=1}^{\ell}$, $\gamma$, $\Omega$ and the coefficients of the operator $A$, $(q_i)_{i=2}^{\ell}$.

**Proof.** Using the assumption $u \in L^p(0, T; H^1_0(\Omega)) \cap C([0, T]; L^2(\Omega))$, we apply the same argument to Theorem 2.1 and conclude that, for any $t \in [0, T]$,

$$\|Ku\|_{L^2(\Omega)} \leq C \int_0^t (t - \tau)^{\alpha_1 - 1}\|u(\tau)\|_{H^1(\Omega)} d\tau + C \sum_{i=2}^{\ell} \int_0^t (t - \tau)^{\alpha_1 - 1 - \alpha_i} d\tau \|u\|_{C([0, T]; L^2(\Omega))} + C\|a\|_{L^2(\Omega)}.$$ 

Moreover, let $p' > 0$ be the conjugate number of $p$, that is, $\frac{1}{p} + \frac{1}{p'} = 1$. Since $p \in \left(\frac{1}{\alpha_1}, \frac{2}{\alpha_1}\right)$ implies

$$(\alpha_1 - 1)p' + 1 = p'(\alpha_1 - 1 + \frac{1}{p'}) = p'\left(\alpha_1 - 1 - \frac{1}{p}\right) > 0,$$

it follows from Hölder’s inequality that, for $0 \leq t \leq T$,

$$\|Ku(t)\|_{L^2(\Omega)} \leq C \left(\int_0^t (t - \tau)^{\alpha_1 - 1 - \frac{1}{p'}} d\tau\right)^{\frac{1}{p'}} \left(\int_0^t \|u(\tau)\|_{H^1(\Omega)}^{p'} d\tau\right)^{\frac{1}{p'}} + C \sum_{i=2}^{\ell} (t - \tau)^{\alpha_1 - \alpha_i} \|u\|_{C([0, T]; L^2(\Omega))} + C\|a\|_{L^2(\Omega)}$$

$$\leq CT^{\alpha_1 - \frac{1}{p'}} \|u\|_{L^p(0, T; H^1_0(\Omega))} + C \sum_{i=2}^{\ell} T^{\alpha_1 - \alpha_i} \|u\|_{C([0, T]; L^2(\Omega))} + C\|a\|_{L^2(\Omega)}.$$

\[\square\]

**Lemma 2.2.** Under the assumptions of Theorem 2.1. Then for $\forall u \in L^p(0, T; H^1_0(\Omega))$ the following estimate holds:

$$\|Ku\|_{L^p(0, T; H^1_0(\Omega))} \leq C \cdot (\sum_{i=2}^{\ell} T^{\alpha_1 - \alpha_i} + T^{\frac{\alpha_1}{p'}})\|u\|_{L^p(0, T; H^1_0(\Omega))} + CT^{\frac{\alpha_1}{p'} - \frac{\alpha_i}{2}}\|a\|_{L^2(\Omega)}.$$ 

**Proof.** Similar to the calculation in Theorem 2.1, we find

$$\|A^\frac{1}{2}Ku(t)\|_{L^2(\Omega)} \leq C \int_0^t ((t - \tau)^{\frac{\alpha_1}{2} - 1} + \sum_{i=2}^{\ell} (t - \tau)^{\alpha_1 - \alpha_i - 1})\|A^\frac{1}{2}u(\tau)\|_{L^2(\Omega)} d\tau$$

$$+ C(t^{\frac{\alpha_1}{2}} + \sum_{i=2}^{\ell} t^{\frac{\alpha_1}{2} - \alpha_i})\|a\|_{L^2(\Omega)}.$$ 

Therefore, since $|a + b|^p \leq 2^p(|a|^p + |b|^p)$, for all $a, b \in \mathbb{R}$, we have

$$\int_0^T \|Ku(t)\|_{D(A^\frac{1}{2})}^p dt \leq C 2^p \int_0^T \left(\int_0^t ((t - \tau)^{\frac{\alpha_1}{2} - 1} + \sum_{i=2}^{\ell} (t - \tau)^{\alpha_1 - \alpha_i - 1})\|A^\frac{1}{2}u(\tau)\|_{L^2(\Omega)} d\tau\right)^p dt$$

$$+ C 2^p \int_0^T (t^{\frac{\alpha_1}{2}} + \sum_{i=2}^{\ell} t^{\frac{\alpha_1}{2} - \alpha_i})^p\|a\|_{L^2(\Omega)}^p dt.$$
By Young’s inequality for the convolution, noting \( p \in (\frac{1}{\alpha_1}, \frac{2}{\alpha_1}) \) implies \( \frac{2}{p} > 1 \), so that

\[
\int_0^T \| Ku(t) \|_{D(\frac{1}{2})}^p \, dt \leq C \left( \int_0^T (\tau^{\frac{1}{2}} + \sum_{i=2}^{\ell} \tau^{\alpha_1-1-\alpha_i}) \, d\tau \right)^{\frac{2}{p}} \int_0^T \| u(\tau) \|_{D(\frac{1}{2})}^p \, d\tau \\
+ CT^{1-\frac{2}{p}} \| a \|_{L^2(\Omega)}^p + C \sum_{i=2}^{\ell} T^{\alpha_1-\alpha_i} \| a \|_{L^2(\Omega)}^{p+1}
\]

Finally, we obtain

\[
\| Ku \|_{L^p(0,T;D(\frac{1}{2}))} \leq C \left( T^{\frac{\alpha_1}{2}} + \sum_{i=2}^{\ell} T^{\alpha_1-\alpha_i} \right) \| u \|_{L^p(0,T;H_{\alpha_1}^1(\Omega))} \\
+ CT^{\frac{\alpha_1}{2}} \| a \|_{L^2(\Omega)} + C \sum_{i=2}^{\ell} T^{\alpha_1-\alpha_i+1} \| a \|_{L^2(\Omega)}
\]

\[\square\]

**The proof of Theorem 2.3** We set \( X_T := L^p(0,T;D(\frac{1}{2})) \cap C([0,T];L^2(\Omega)) \) and \( \| \cdot \|_T := \| \cdot \|_{L^p(0,T;D(\frac{1}{2}))} + \| \cdot \|_{C([0,T];L^2(\Omega))} \). It is easy to see that \( \| \cdot \|_T \) is a norm of \( X_T \), and \( (X_T, \| \cdot \|_T) \) is a Banach space.

Assuming \( u_1, u_2 \in X_T \), by an argument similar to the proof of Lemma 2.1 and Lemma 2.2, we derive that there exists a constant \( C > 0 \) such that the following estimates hold:

\[
\| Ku_1 - Ku_2 \|_{C([0,T];L^2(\Omega))} \leq C T^{\alpha_1-\frac{1}{p}} \| u_1 - u_2 \|_{L^p(0,T;H_{\alpha_1}^1(\Omega))} + C \sum_{i=2}^{\ell} T^{\alpha_1-\alpha_i} \| u_1 - u_2 \|_{C([0,T];L^2(\Omega))}
\]

\[
\| Ku_1 - Ku_2 \|_{L^p(0,T;H_{\alpha_1}^1(\Omega))} \leq C \left( T^{\frac{\alpha_1}{2}} + \sum_{i=2}^{\ell} T^{\alpha_1-\alpha_i} \right) \| u_1 - u_2 \|_{L^p(0,T;H_{\alpha_1}^1(\Omega))}
\]

Setting \( T = \delta \), the above calculations leads to

\[\text{(7)}\]

\[
\| Ku_1 - Ku_2 \|_{C([0,\delta];L^2(\Omega))} \leq C \left( \delta^{\alpha_1-\frac{1}{p}} + \sum_{i=2}^{\ell} \delta^{\alpha_1-\alpha_i} \right) \| u_1 - u_2 \|_{X_\delta},
\]

\[\text{(8)}\]

\[
\| Ku_1 - Ku_2 \|_{L^p(0,\delta;H_{\alpha_1}^1(\Omega))} \leq C \left( \delta^{\frac{\alpha_1}{2}} + \sum_{i=2}^{\ell} \delta^{\alpha_1-\alpha_i} \right) \| u_1 - u_2 \|_{L^p(0,\delta;H_{\alpha_1}^1(\Omega))}.
\]

From the above two estimates \((7)\) and \((8)\), it follows that the operator \( K \) is a contracted operator from \( (X_\delta, \| \cdot \|_\delta) \) into itself when \( \delta > 0 \) is small enough. Consequently, there exists a unique fixed point \( u \in X \) such that \( Ku(t) = u(t) \), \( \forall t \in [0, \delta] \).

On the basis of Theorem 2.2, we can further prove the global existence of the mild solution to the initial-boundary value problem \((1)\).

**Theorem 2.3** (Global existence). Under the assumption of Theorem 2.2. Then for any \( T > 0 \) being fixed, there exists a mild solution to \((1)\) in \( L^p(0,T;H_{\alpha_1}^1(\Omega)) \cap C([0,T];L^2(\Omega)) \), \( p \in (\frac{1}{\alpha_1}, \frac{2}{\alpha_1}) \).
Proof. For any fixed $T > 0$, without loss of generality, we assume that $T > \delta$, where $\delta$ is defined in Theorem 2.2. On the interval $[0, \delta]$, we see that $u \in X_\delta := L^p(0, \delta; H^1_0(\Omega)) \cap C([0, \delta]; L^2(\Omega))$ satisfies (5). Here for convenience, we set $\delta = T > \delta$ in Theorem 2.2. On the interval $[0, \delta]$, we have the integral equation of $U_t = \bar{u}$ also satisfies the above inequality. Therefore by the general Gronwall’s inequality (e.g. Theorem 7.1.1 in [15]), we have

$$\|U(t)\|_{L^2(\Omega)} \leq C_T \int_0^t (t - \tau)^{\alpha - 1} \|U(\tau)\|_{L^2(\Omega)} d\tau + C_T \|a\|_{L^2(\Omega)} t^{-\frac{\alpha}{2}}, \quad 0 < t \leq \delta < T, \quad (9)$$

where $\alpha = \min(\alpha_1 - \alpha_2, \alpha_1 - \alpha_2)$, and the constant $C > 0$ is independent of $\delta$.

We set $U(t) = \|U^\sharp(t)\|_{L^2(\Omega)}$ in $[0, \delta]$, and $U = 0$ in the interval $(\delta, T]$. It is easy to see that $U(t)$ also satisfies the above inequality. Therefore by the general Gronwall’s inequality (e.g. Theorem 7.1.1 in [15]), we have

$$U(t) \leq C_T t^{-\frac{\alpha}{2}} \|a\|_{L^2(\Omega)}, \quad t \in [0, T].$$

Therefore

$$\|u(t)\|_{D(A^\sharp)} \leq C_T t^{-\frac{\alpha}{2}} \|a\|_{L^2(\Omega)}, \quad t \in (0, \delta). \quad (9)$$

Hence by the integral equation of $u$ and the above inequality, similarly to Theorem 2.1 we have

$$\|u(t)\|_{L^2(\Omega)} \leq C_T (t^{-\frac{\alpha}{2}} + 1) \|a\|_{L^2(\Omega)} + C_T \int_0^t \sum_{i=2}^\ell \frac{(t - \tau)^{\alpha_1 - \alpha_2}}{1 - \tau} \|u(\tau)\|_{L^2(\Omega)} d\tau, \quad t \in [0, \delta].$$

Again from the general Gronwall’s inequality, and similarly to (9), we can prove that

$$\|u(t)\|_{L^2(\Omega)} \leq C_T \|a\|_{L^2(\Omega)}, \quad t \in [0, \delta], \quad (10)$$

where $C_T > 0$ depending only on $T$, $\{q_i\}_{i=2}^\ell$, $d$, $\{\alpha_j\}_{j=1}^\ell$, $B$, $\Omega$ and the coefficients of the operator $A$.

Next we study our initial-boundary problem on the interval $[\frac{\delta}{2}, \delta]$. Denoting $t_0 := \frac{\delta}{2}$, then $[\frac{\delta}{2}, \delta]$ can be rewritten as $[t_0, 2t_0]$.

It is easy to see that the representation of the solution

$$u(t) = \sum_{i=2}^\ell \int_0^t A^{-1} S'(t - \tau) q_i(x) \partial_t^{\alpha_1} u(\tau) d\tau - \int_0^t A^{-1} S'(t - \tau) B(x) \cdot \nabla u(\tau) d\tau + S(t) a.$$ 

still holds on interval $[t_0, 2t_0]$. For $t \in [t_0, 2t_0]$, we break up the integral into two parts

$$u(t) = \sum_{i=2}^\ell \int_0^t A^{-1} S'(t - \tau) q_i(x) \partial_t^{\alpha_1} u(\tau) d\tau - \int_0^t A^{-1} S'(t - \tau) B(x) \cdot \nabla u(\tau) d\tau$$

$$+ \sum_{i=2}^\ell \int_0^{t_0} A^{-1} S'(t - \tau) q_i(x) \partial_t^{\alpha_1} u(\tau) d\tau - \int_0^{t_0} A^{-1} S'(t - \tau) B(x) \cdot \nabla u(\tau) d\tau + S(t) a.$$
We define a new operator $E$ on interval $[t_0, 3t_0]$ by

$$E v(t) = \sum_{i=2}^{\ell} \int_{t_0}^{t} A^{-1} S(t - \tau) q_i(x) \partial_i^\alpha v(\tau) d\tau - \int_{t_0}^{t} A^{-1} S(t - \tau) B(x) \cdot \nabla v(\tau) d\tau$$

$$+ \sum_{i=2}^{\ell} \int_{t_0}^{t} A^{-1} S(t - \tau) q_i(x) \partial_i^\alpha u(\tau) d\tau - \int_{t_0}^{t} A^{-1} S(t - \tau) B(x) \cdot \nabla u(\tau) d\tau + S(t)a$$

$$= \sum_{i=2}^{\ell} \int_{t_0}^{t} A^{-1} S(t - \tau) q_i(x) \partial_i^\alpha v(\tau) d\tau - \int_{t_0}^{t} A^{-1} S(t - \tau) B(x) \cdot \nabla v(\tau) d\tau + I(t).$$

Now let us turn to the evaluation of $I(t)$, $t \in [t_0, 3t_0]$. The use of (9) and (10) leads to

$$\|I(t)\|_{L^2(\Omega)} \leq C_T \left[ \int_{t_0}^{t} (t - \tau)^{\alpha_1 - 1} \tau^{-\frac{2}{\alpha_2}} d\tau + \int_{t_0}^{t} (t - \tau)^{\alpha_1 - \alpha_1 - 1} d\tau \right] \|a\|_{L^2(\Omega)} + C_T \|a\|_{L^2(\Omega)}.$$

Therefore $\|I(t)\|_{L^2(\Omega)} \leq C T \|a\|_{L^2(\Omega)}$, $t \in [t_0, 3t_0]$. By an argument similar to the proof of Lemma 2.1 and Lemma 2.2 we obtain that the operator $E$ maps $X_1 := C([t_0, 3t_0]; L^2(\Omega)) \cap L^p(t_0, 3t_0; H^1_0(\Omega))$ into itself, where $p \in \left( \frac{1}{\alpha_2}, \frac{2}{\alpha_2} \right)$.

Let $v_1, v_2 \in X_1$. By the definition of the operator $E$, similarly to Theorem 2.2 we have

$$\|E v_1(t) - E v_2(t)\|_{L^2(\Omega)}$$

$$\leq C \int_{t_0}^{t} (t - \tau)^{\alpha_1 - 1} \|v_1(\tau) - v_2(\tau)\|_{D(A^{\frac{1}{2}})} d\tau + C \int_{t_0}^{t} (t - \tau)^{\alpha_1 - \alpha_1 - 1} \|v_1(\tau) - v_2(\tau)\|_{C([t_0, 3t_0]; L^2(\Omega))} d\tau.$$

Consequently, the use of Hölder’s inequality and $D(A^{\frac{1}{2}}) = H^1_0(\Omega)$ yields that, for any $t \in [t_0, 3t_0]$, the following estimate

$$\|E v_1(t) - E v_2(t)\|_{L^2(\Omega)} \leq C \left( (2t_0)^{\alpha_1 - \frac{1}{p}} + \sum_{i=2}^{\ell} (2t_0)^{\alpha_1 - \alpha_1} \right) \|v_1 - v_2\|_{X_1}, \; p \in \left( \frac{1}{\alpha_2}, \frac{2}{\alpha_2} \right).$$

Moreover, similarly to the argument in Theorem 2.2 we can prove that

$$\|E v_1(t) - E v_2(t)\|_{D(A^{\frac{1}{2}})}$$

$$\leq C \int_{t_0}^{t} (t - \tau)^{\alpha_1 - 1} \|v_1(\tau) - v_2(\tau)\|_{D(A^{\frac{1}{2}})} d\tau + C \int_{t_0}^{t} (t - \tau)^{\alpha_1 - \alpha_1 - 1} \|v_1(\tau) - v_2(\tau)\|_{D(A^{\frac{1}{2}})} d\tau.$$

Then by Young’s inequality, for $p \in \left( \frac{1}{\alpha_2}, \frac{2}{\alpha_2} \right)$ we have

$$\int_{t_0}^{3t_0} \|E v_1(t) - E v_2(t)\|_{D(A^{\frac{1}{2}})}^p dt \leq C \left( (2t_0)^{\frac{1}{p}} + \sum_{i=2}^{\ell} (2t_0)^{\alpha_1 - \alpha_1} \right) \|v_1 - v_2\|_{L^p(t_0, 3t_0; H^1_0(\Omega))}^p.$$

By the notation $t_0 = \frac{\delta}{2}$, we see that

$$\|E v_1(t) - E v_2(t)\|_{L^2(\Omega)} \leq C \left( \delta^{\alpha_1 - \frac{1}{p}} + \sum_{i=2}^{\ell} \delta^{\alpha_1 - \alpha_1} \right) \|v_1 - v_2\|_{X_1}, \; t \in [t_0, 3t_0] \quad (11)$$
\[ \| \mathcal{E}v_1 - \mathcal{E}v_2 \|_{L^p(t_0, 3t_0; H^1_0(\Omega))} \leq C \left( \sum_{i=2}^\ell \delta^{\alpha_i - \alpha_1} + \delta^{\alpha_1} \right) \| v_1 - v_2 \|_{L^p(t_0, 3t_0; H^1_0(\Omega))}. \]  

(12)

From that the constant \( C \) in (11) and (12) are same to the constant in (7) and (8) and the choice of \( \delta \) in Theorem 2.2, we can deduce that the operator \( \mathcal{E} \) is strictly contracted operator from \( X_1, \| \cdot \| \) into itself, where the norm \( \| \cdot \| \) defined by

\[ \| u \| := \| u \|_{L^p(t_0, 3t_0; H^1_0(\Omega))} + \| u \|_{C([t_0, 3t_0]; L^2(\Omega))}. \]

Therefore, there exists a unique fixed point \( v \in X_1 \) such that \( \mathcal{E}v(t) = v(t) \) in \( [t_0, 3t_0] \), that is,

\[ v(t) = \sum_{i=2}^\ell \int_{t_0}^t A^{-1}S'(t - \tau)q_i(x)\partial_t^{\alpha_i}v(\tau)d\tau - \int_{t_0}^t A^{-1}S'(t - \tau)B(x) \cdot \nabla v(\tau)d\tau + I(t), \quad t_0 \leq t \leq 3t_0. \]

Additionally, from the uniqueness argument we can see that \( u(t) = v(t) \) in \( [t_0, 2t_0] = [\frac{t}{2}, \delta] \), then we define a new function \( \tilde{u} \) by

\[ \tilde{u} = \begin{cases} 
 u, & t \in [0, 2t_0]; \\
 v, & t \in [2t_0, 3t_0]. 
\end{cases} \]

Repeating the above argument to the interval pair \( ([0, 3t_0], [2t_0, 4t_0]) \), we can obtain that the mild solution exists on the larger interval \( [0, 4t_0] \), and go on. Finally the existence interval of the mild solution to our problem (9-10) can be extended to the interval \( [0, T] \), where \( T > 0 \) is constant chosen at the very beginning.

The discussion for the non-homogeneous equation with initial condition being zero is similar to the homogeneous equation. We list the results of homogeneous and non-homogeneous equations in the following theorem.

**Theorem 2.4.** Let \( \{\alpha_j\}_{j=1}^\ell \) satisfy \( 0 < \alpha_1 < \cdots < \alpha_1 < 1 \), and \( \gamma \in \left[ \frac{1}{\alpha_1}, 1 \right) \).

1. Let \( a \in L^2(\Omega), \ F = 0 \), then the initial-boundary value problem (9-10) admits a unique mild solution \( u \in C([0, T]; D(A^{\gamma})) \cap C([0, T]; L^2(\Omega)) \). Moreover, there exists a constant \( C > 0 \) such that

\[ \| u \|_{C([0, T]; L^2(\Omega))} \leq C_T \| a \|_{L^2(\Omega)}, \]

and

\[ \| u(t) \|_{D(A^{\gamma})} \leq C_T t^{-\alpha_1 \gamma} \| a \|_{L^2(\Omega)}, \quad 0 < t \leq T. \]

2. Let \( F \in L^\infty(0, T; L^2(\Omega)), \ a = 0 \), then the initial-boundary problem (9-10) admits a unique solution \( u \in C([0, T]; D(A^{\gamma})) \cap L^2(0, T; H^2(\Omega)) \). Moreover the following estimate holds:

\[ \| u \|_{C([0, T]; D(A^{\gamma}))} + \| u \|_{L^2(0, T; H^2(\Omega))} \leq C_T \| F \|_{L^\infty(0, T; L^2(\Omega))}. \]

**Proof.** (1). From Theorem 2.3, we see that there exists a unique mild solution in \( L^p(0, T; H^1_0(\Omega)) \cap C([0, T]; L^2(\Omega)), \ p \in (\frac{1}{\alpha_1}, \frac{2}{\alpha_1}), \) such that

\[ u(t) = \sum_{i=2}^\ell \int_0^t A^{-1}S'(t - \tau)q_i(x)\partial_t^{\alpha_i}u(\tau)d\tau - \int_0^t A^{-1}S'(t - \tau)B(x) \cdot \nabla u(\tau)d\tau + S(t)a. \]

11
By the density argument, we deduce that $u \in C$ which yields that

Consequently, step by step, we obtain that for any $\gamma$ Repeating the above argument, we deduce $u \in \bar{C}$ Therefore, by a similarly argument to Theorem 2.1, we can prove that

where $\bar{0} = \alpha_1 - \max(\alpha_1 \gamma, \alpha_2)$. By general Gronwall’s inequality, the following estimate holds:

By the density argument, we deduce that $u \in C((0, T]; H_0^1(\Omega))$. For any arbitrary fixed small $0 < \varepsilon < \frac{1}{2}$, taking $A^{\frac{1}{2}} + \varepsilon$ on the both sides of the above integral equation of $u$, by (6) and (13), similarly to the proof of Theorem 2.1, we derive that

which yields that $u \in C((0, T]; D(A^{\frac{1}{2}} + \varepsilon))$ and

Repeating the above argument, we deduce $u \in C((0, T]; D(A^{\frac{1}{2}} + 2\varepsilon))$ and

Consequently, step by step, we obtain that for any $\gamma \in \left[\frac{1}{2}, 1\right)$ there exists a constant $C > 0$ depending only on $d, \{q_i\}_{i=2}^\ell, \{\alpha_j\}_{j=1}^\ell, \gamma, B, \Omega, T$ and the coefficients of the operator $A$, such that $u \in C((0, T]; D(A^\gamma))$ and $\|A^\gamma u(t)\|_{L^2(\Omega)} \leq Ct^{-\alpha_1} \|u\|_{L^2(\Omega)}, t \in (0, T]$.

(2) Firstly, we show the existence of the mild solution of initial-boundary value problem (\ref{eq:1}). It is easy to show that the operator $K$ defined by (6) maps the space $X_2$ into itself, where

with the norm $\|\cdot\|_{C([0, T]; D(A^\gamma))}$. Moreover, by induction, we can obtain that for any $u, v$ in $X_2$, the following estimation

$$
\|K^n u(t) - K^n v(t)\|_{D(A^\gamma)} \leq \frac{M_{1+n}^n}{(\bar{\alpha} + 1)} \|u - v\|_{C([0, T]; D(A^\gamma))}, 0 < t \leq T
$$
holds, where \( \tilde{\alpha} = \alpha_1 - \max(\alpha_1 \gamma, \alpha_2) \). In fact, by the definition of the operator \( K \) and the induction assumption, we have

\[
\| K^{n+1}u(t) - K^{n+1}v(t) \|_{D(\mathcal{A}^\gamma)} \leq C \int_0^t (t - \tau)^{\tilde{\alpha} - 1} \frac{M_1^{n+1} \tilde{\alpha} n}{\Gamma(\tilde{\alpha} n + 1)} \| u - v \|_{C([0, T]; D(\mathcal{A}^\gamma)))} d\tau, \quad 0 \leq t \leq T.
\]

Using \( B(\alpha, \beta) := \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} = \int_0^1 s^\alpha - 1 (1 - s)^{\beta - 1} ds, \alpha, \beta > 0 \), setting \( M_1 := C\Gamma(\tilde{\alpha}) \), we see that

\[
\| K^{n+1}u(t) - K^{n+1}v(t) \|_{D(\mathcal{A}^\gamma)} \leq \frac{CM_1^n}{\Gamma(\tilde{\alpha} n + 1)} B(\tilde{\alpha}, \tilde{\alpha} n + 1) t^{\tilde{\alpha}(n+1)} \| u - v \|_{C([0, T]; D(\mathcal{A}^\gamma)))} = \frac{M_1^{n+1} \tilde{\alpha} n}{\Gamma(\tilde{\alpha} n + 1 + 1)} \| u - v \|_{C([0, T]; D(\mathcal{A}^\gamma)))}, \quad 0 \leq t \leq T.
\]

Consequently, for \( n \in \mathbb{N} \) big enough, we can see that \( K \) is a strictly contracted operator from \( X_2 \) to \( X_2 \). Then there exists a unique fixed point \( \tilde{u} \in X_2 \) such that \( K^n \tilde{u} = \tilde{u} \). It is easy to show that \( \tilde{u} \) is also the fixed point of the operator \( K \), that is, \( K\tilde{u} = \tilde{u} \), and therefore the fixed point of \( K : X_2 \to X_2 \) is also unique.

Similar to the proof of Theorem 2.1, we can prove \( u \in C([0, T]; D(\mathcal{A}^\gamma)) \) and the estimation

\[
\| u \|_{C([0, T]; D(\mathcal{A}^\gamma)))} \leq C\| F \|_{L^\infty(0, T; L^2(\Omega))}.
\]

The use of \( \| \nabla u \|_{L^2(\Omega)} \leq C\| F \|_{L^\infty(0, T; L^2(\Omega))} \) and Theorem 2.1 in [34] leads to

\[
I_1(t) := \int_0^t A^{-1} S'(t - \tau)(B \cdot \nabla u + F)d\tau \in L^2(0, T; H^2(\Omega)),
\]

and

\[
\| I_1 \|_{L^2(0, T; H^2(\Omega))] \leq C\| F \|_{L^\infty(0, T; L^2(\Omega))}.
\]

Next we show that \( \sum_{i=2}^\ell \int_0^t A^{-1} S'(t - \tau)q_i \partial_\tau^i u d\tau := I_2(t) + I_3(t) \in C([0, T]; H^2(\Omega)) \), where \( I_2(t), I_3(t) \) are defined in (53).

For any \( \gamma \in [\frac{3}{2}, 1) \), \( \varepsilon_0 > 0 \) small enough such that \( \alpha_1 - \alpha_i - \alpha_1 \varepsilon_0 > 0 \) \( (i = 2, \cdots, \ell) \), from (14), similarly to Theorem 2.1, we see that

\[
\| A^\gamma + \varepsilon_0 (I_2 + I_3)(t) \|_{L^2(\Omega)} \leq C \int_0^t \sum_{i=2}^\ell (t - \tau)^{\alpha_1 - \alpha_i - \alpha_1 \varepsilon_0 - 1} \| A^\gamma u \|_{L^2(\Omega)} d\tau \leq C \int_0^t \sum_{i=2}^\ell (t - \tau)^{\alpha_1 - \alpha_i - \alpha_1 \varepsilon_0 - 1} d\tau \| F \|_{L^\infty(0, T; L^2(\Omega))} \leq C\| F \|_{L^\infty(0, T; L^2(\Omega))}.
\]

Since \( \gamma \in [\frac{3}{2}, 1) \), we can choose \( \gamma \) such that \( \gamma + \varepsilon_0 = 1 \), then we have

\[
\| A(I_2 + I_3)(t) \|_{L^2(\Omega)} \leq C\| F \|_{L^\infty(0, T; L^2(\Omega))}, \quad 0 \leq t \leq T.
\]

Hence \( I_2 + I_3 \in C([0, T]; H^2(\Omega)) \). Collecting the above estimates (15) and (16), we derive \( u \in L^2(0, T; H^2(\Omega)) \), and

\[
\| u \|_{L^2(0, T; H^2(\Omega))} \leq C\| F \|_{L^\infty(0, T; L^2(\Omega))}.
\]

\[ \square \]
3 Hölder regularity

Here we assume that $a = 0$, $F \in C^\theta([0,T];L^2(\Omega))$ with $\theta \in (0,1)$ and $F(0) = 0$. We expect to obtain some Hölder estimate for the mild solution to the equation (??).

We set $X_\theta := \{Au \in C^\theta([0,T];L^2(\Omega)), u(0) = 0\}$, with Hölder norm

$$\|Au\|_\theta := \|Au\|_{C^\theta([0,T];L^2(\Omega))} + \sup_{t_1 \neq t_2 \in [0,T]} \frac{\|Au(t_1) - Au(t_2)\|_{L^2(\Omega)}}{|t_1 - t_2|^\theta}.$$

From the arguments in the above section, we can formally obtain that $u(t) = K(t)$, $t \in [0,T]$, where the operator $K$ is defined by (9). Firstly, we want to prove that the operator $K$ can improve the Hölder’s regularity. Namely, the following lemma holds.

Lemma 3.1. There exist constant $\varepsilon > 0$ small enough, and constant $C > 0$, for any $\theta' \in [0,\theta)$, the following estimate

$$\|Ku(t + h) - Ku(t)\|_{L^2(\Omega)} \leq C(h^\theta \|F\|_\theta + h^{\theta'}\|Au\|_\theta')$$

holds any $u \in X_{\theta'}$.

Proof. We recall the definition of $I_j$, $j = 1,\cdots,5$ in (3). Let $h > 0$, and $t,t + h \in [0,T]$. Representing $A_{11}$ as

$$A_{11}(t) = \int_0^t S'(\tau)F(t - \tau)d\tau + \int_0^t S'(\tau)B \cdot \nabla u(t - \tau)d\tau =: I_{11}(t) + I_{12}(t).$$

By an argument similar to the proof of Theorem 2.4 in [34], we can prove that

$$\|I_{11}(t + h) - I_{11}(t)\|_{L^2(\Omega)} \leq C h^\theta \|F\|_\theta.$$

For $0 \leq t < t + h \leq T$, $h < 1$, we have

$$I_{12}(t + h) - I_{12}(t) = \int_{t-h}^t S'(\tau + h)(B \cdot \nabla u(t - \tau))d\tau - \int_0^t S'(\tau)(B \cdot \nabla u(t - \tau))d\tau$$

$$= \int_0^t A^{-\xi} S'(\tau + h)A^\xi(B \cdot \nabla u(t - \tau) - B \cdot \nabla u(t))d\tau + (S(t + h) - S(t)) (B \cdot \nabla u(t))$$

$$+ \int_0^t A^{-\xi}(S'(\tau + h) - S'(\tau))A^\xi(B \cdot \nabla u(t - \tau) - B \cdot \nabla u(t))d\tau =: J_1^h(t) + J_2^h(t) + J_3^h(t).$$

Using the estimate (3) and (4), we find

$$\|J_1^h(t)\|_{L^2(\Omega)} \leq C \int_0^t (\tau + h)^\frac{\alpha}{2} - (\tau + h)^{\theta'} \|Au\|_{\theta'} \leq C h^{\frac{\alpha}{2} + \theta'} \|Au\|_{\theta'}.$$

and noting that $\int_0^{\tau + h} S''(\xi)d\xi = S'(t + h) - S'(t)$, and again using (3), we can prove that

$$\|J_2^h(t)\|_{L^2(\Omega)} \leq C h^{\frac{\alpha}{2} + \theta'} \int_0^t (\tau + h)^{\frac{\alpha}{2} - 1} d\tau \|Au\|_{\theta'}.$$

(17)

In the case when $0 \leq t \leq h$, according to (17), we derive the estimate

$$\|J_3^h(t)\|_{L^2(\Omega)} \leq C h^{\frac{\alpha}{2} + \theta'} \int_0^1 (\tau + h)^{\frac{\alpha}{2} - 1} d\tau \|Au\|_{\theta'} \leq C h^{\frac{\alpha}{2} + \theta'} \|Au\|_{\theta'}.$$
As for the case $t > h$, we represent $\mathcal{J}_3^h(t)$ as

$$
\|\mathcal{J}_3^h(t)\|_{L^2(\Omega)} \leq C h^{\alpha_i + \theta'} \left( \int_0^1 + \int_0^h \right) (\tau^{\alpha_i - 1} - (\tau + 1)^{\alpha_i - 1}) \tau^{\theta'} d\tau \|Au\|_{\partial\Omega},
$$

Therefore, since \((\ref{eq:18})\) and the inequality $\tau^{\alpha_i - 1} - (\tau + 1)^{\alpha_i - 1} \leq C \tau^{\alpha_i - 2}, \tau > 1$, it follows that

$$
\|\mathcal{J}_3^h(t)\|_{L^2(\Omega)} \leq C h^{\alpha_i + \theta'} \|Au\|_{\partial\Omega} + C h^{\alpha_i + \theta'} \int_1^\infty \tau^{\alpha_i - 2} \tau^{\theta'} d\tau \|Au\|_{\partial\Omega} \leq C h^{\alpha_i + \theta'} \|Au\|_{\partial\Omega}.
$$

We estimate $\mathcal{J}_3^h(t)$. Since $u(0) = 0$, it follows that $\|A^{\frac{1}{2}} B \cdot \nabla u(t)\|_{L^2(\Omega)} \leq C t^\theta' \|Au\|_{\partial\Omega}$, we can prove that

$$
\|\mathcal{J}_3^h(t)\|_{L^2(\Omega)} = \left\| \int_0^{t+h} A^{-\frac{1}{2}} S'(\eta)d\eta A^{\frac{1}{2}} B \cdot \nabla u(t) d\eta \right\|_{L^2(\Omega)} \leq C h^{\alpha_i + \theta'} \|Au\|_{\partial\Omega}.
$$

Therefore, we proved that $\|I_{12}(t + h) - I_{12}(t)\|_{L^2(\Omega)} \leq C h^{\theta'} \|Au\|_{\partial\Omega}, 0 \leq t < t + h \leq T$. We are to estimate $AI_2(t)$. Similarly to the calculation of $AI_1(t)$, for any $0 \leq t < t + h \leq T$, we have

$$
\begin{align*}
(AI_2(t + h) - AI_2(t)) &= \sum_{i=2}^\ell \frac{1}{\Gamma(1 - \alpha_i)} \int_0^0 A^{-1} S'(\tau + h)(h + h)^{-\alpha_i} A(q_i u(t - \tau) - q_i u(t)) d\tau \\
&+ \sum_{i=2}^\ell \frac{1}{\Gamma(1 - \alpha_i)} \int_0^0 A^{-1} S'(\tau + h)(h + h)^{-\alpha_i} - S'(\tau)^{-\alpha_i}) A(q_i u(t - \tau) - q_i u(t)) d\tau \\
&+ \sum_{i=2}^\ell \frac{1}{\Gamma(1 - \alpha_i)} \int_0^{t+h} A^{-1} S'(\tau)^{-\alpha_i} A(q_i u(t) d\tau = I_{21}^h(t) + I_{22}^h(t) + I_{23}^h(t).
\end{align*}
$$

Therefore, in terms of estimate $\|A^{-1} S'(t)\| \leq C t^{\alpha_i - 1}, \forall t > 0 \text{ and that } u \in X_{\theta''},$ we obtain

$$
\|I_{21}^h(t)\|_{L^2(\Omega)} \leq C \int_0^0 \left( \sum_{i=2}^\ell (\tau + h)^{\alpha_i - 1} \eta^{\theta'} \|Au\|_{\partial\Omega} \leq C \sum_{i=2}^\ell h^{\alpha_i - 1} \eta^{\alpha_i + \theta'} \|Au\|_{\partial\Omega}, 0 \leq t < t + h \leq T,
$$

and noting that $\int_0^{t+h} dS'(\eta^{\alpha_i - 1}) = S'(\tau + h)(h + h)^{-\alpha_i} - S'(\tau)^{-\alpha_i}, i = 2, \ldots, \ell$. Therefore for $0 \leq t < t + h \leq T.$

If $0 \leq t \leq h$, then we have

$$
\|I_{23}^h(t)\|_{L^2(\Omega)} \leq C \sum_{i=2}^\ell h^{\alpha_i - 1} \eta^{\alpha_i + \theta'} \|Au\|_{\partial\Omega}, 0 \leq t < t + h \leq T.
$$
If $t > h$, we choose $\varepsilon > 0$, $\theta'' \in [0, \theta']$ such that $\alpha_1 - \alpha_\ell + \theta'' < 1$, and $\alpha_1 - \alpha_2 + \theta'' = \theta + \varepsilon$. In fact, we can choose $\theta'' := c_0 \min\{1 - \alpha_1 + \alpha_\ell, \theta'\}$, $\varepsilon := \alpha_1 - \alpha_2 + \theta'' - \theta'$, here $0 < c_0 < 1$ is sufficiently close to 1. Using the fact that $C^\alpha([0, T]; L^2(\Omega)) \subset C^\beta([0, T]; L^2(\Omega))$, where $0 \leq \beta \leq \alpha \leq 1$, we obtain that for $0 < h < 1$ the following estimate

$$\|I_{22}^h(t)\|_{L^2(\Omega)} \leq C \sum_{i=2}^{\ell} h^{\alpha_1 - \alpha_i + \theta''} \left(1 + \sum_{i=2}^{\ell} \int_1^\infty (\tau^{\alpha_1 - \alpha_i - 1} - (\tau + 1)^{\alpha_1 - \alpha_i - 1}) \tau^{\theta''} d\tau\right) \|Au\|_{\theta''}$$

$$\leq C \sum_{i=2}^{\ell} h^{\alpha_1 - \alpha_i + \theta''} \|Au\|_{\theta''} + C \sum_{i=2}^{\ell} h^{\alpha_1 - \alpha_i + \theta''} \int_1^\infty \tau^{\alpha_1 - \alpha_i - 1} \tau^{\theta''} d\tau \|Au\|_{\theta''} \leq C h^{\theta'' + \varepsilon} \|Au\|_{\theta''},$$

holds. Choosing $\varepsilon > 0$, $\theta'' \in [0, \theta']$ as in the calculation of $I_{22}^h(t)$, then we have

$$\|I_{23}^h(t)\|_{L^2(\Omega)} \leq C \int_t^{t+h} \sum_{i=2}^{\ell} \tau^{\alpha_1 - \alpha_i - 1} t^{\theta''} d\tau \|Au\|_{\theta''} \leq C \int_t^{t+h} \sum_{i=2}^{\ell} \tau^{\alpha_1 - \alpha_i - 1} t^{\theta''} d\tau \|Au\|_{\theta''}.$$ 

Consequently, we have

$$\|AI_2(t+h) - AI_2(t)\|_{L^2(\Omega)} \leq Ch^{\theta'' + \varepsilon} \|Au\|_{\theta''}, 0 \leq t < t+h \leq T, 0 < h < 1.$$ 

It remains to estimate $\|AI_3(t+h) - AI_3(t)\|_{L^2(\Omega)}$. For brevity, for $i = 2, \ldots, \ell$ we set

$$g_i(\tau) := \frac{1}{\Gamma(1-\alpha_i)} \int_0^\tau A^{-1} S'_{\theta''}((\tau - \eta)(\eta^{-\alpha_i} - \tau^{-\alpha_i})) d\eta.$$ 

For $0 \leq t < t+h \leq T$, similarly to the above calculation, we have

$$AI_3(t+h) - AI_3(t)$$

$$= \sum_{i=2}^{\ell} \int_0^t g_i(\tau+h)A(q_i u(t-\tau) - q_i u(t))d\tau + \sum_{i=2}^{\ell} \int_0^t (g_i(\tau+h) - g_i(\tau))A(q_i u(t-\tau) - q_i u(t))d\tau$$

$$+ \sum_{i=2}^{\ell} \int_t^{t+h} g_i(\tau)A(q_i u(t))d\tau =: I_{31}^h(t) + I_{32}^h(t) + I_{33}^h(t), \quad 0 \leq t < t+h \leq T.$$ 

Again applying the estimate (3) and (4), we obtain, for any $0 \leq t < t+h \leq T$

$$\|I_{31}^h(t)\|_{L^2(\Omega)} \leq C \sum_{i=2}^{\ell} \int_0^t \int_h^{\tau+h} (\tau + h - \eta)\alpha_i - 2(\eta^{-\alpha_i} - (\tau + h)^{-\alpha_i}) d\eta |\tau|^{\theta''} d\tau \|Au\|_{\theta''}.$$ 

By changing variable $\tilde{\eta} = \frac{\eta}{\tau+h}$, we find, for any $0 \leq t < t+h \leq T, 0 < h < 1$

$$\|I_{31}^h(t)\|_{L^2(\Omega)} \leq C \sum_{i=2}^{\ell} \int_0^h \int_0^{1} (1 - \eta)\alpha_i - 2(\eta^{-\alpha_i} - 1) d\eta |\tau|^{\theta''} (\tau + h)^{\alpha_i - 1} d\tau \|Au\|_{\theta''}$$

$$\leq C h^{\alpha_1 - \alpha_2 + \theta''} \|Au\|_{\theta''} = C h^{\theta'' + \varepsilon} \|Au\|_{\theta''}.$$ 

Choosing $\varepsilon > 0$, $\theta'' \in [0, \theta']$ as in the calculation of $I_{22}^h(t)$, we have for $0 \leq t < t+h \leq T$

$$\|I_{33}^h(t)\|_{L^2(\Omega)} \leq C \sum_{i=2}^{\ell} \int_t^{t+h} \int_0^r (\tau - \eta)\alpha_i - 2(\eta^{-\alpha_i} - \tau^{-\alpha_i}) d\eta d\tau \|Au\|_{\theta''}.$$
\[
\Gamma(1 - \alpha_i)(g_i(\tau + h) - g_i(\tau)) = \int_{-h}^{\tau} A^{-1} S^\tau(\tau - \eta)((\eta + h)^{-\alpha_i} - (\tau + h)^{-\alpha_i})d\eta
\]
\[
+ \int_{\tau}^{0} A^{-1} S^\tau(\tau - \eta)((\eta + h)^{-\alpha_i} - (\tau + h)^{-\alpha_i} - \eta^{-\alpha_i} + \tau^{-\alpha_i})d\eta := J^h_i(\tau) + J^h_{i2}(\tau).
\]

The case \(t \leq h\) is considered firstly. To estimate \(J^h_4(\tau)\) \(0 \leq \tau \leq t \leq h\), we represent it in the form
\[
\|J^h_4(\tau)\| \leq C \left( \int_{-h}^{-\frac{\tau}{2}} + \int_{\frac{\tau}{2}}^{0} \right) (\tau - \eta)^{\alpha_i - 2}((\eta + h)^{-\alpha_i} - (\tau + h)^{-\alpha_i})d\eta := J^h_{41} + J^h_{42}.
\]

For \(J^h_{41}\). Using the inequality \((\tau - \eta)^{\alpha_i - 2} \leq \left(\frac{\tau}{2}\right)^{\alpha_i - 2}, \forall \tau \in [0, h], \forall \eta \in [-h, -\frac{h}{2}]\), we have
\[
\|J^h_{41}(\tau)\| \leq C \int_{-h}^{-\frac{\tau}{2}} \eta^{\alpha_i - 2}((\eta + h)^{-\alpha_i} - (\tau + h)^{-\alpha_i})d\eta \leq C\eta^{\alpha_i - 1}.
\]

For \(J^h_{42}\). Noting that \((\eta + h)^{-\alpha_i} - (\tau + h)^{-\alpha_i} \leq C(\tau - \eta)(\eta + h)^{-\alpha_i - 1}, \forall \tau \in [0, h], \forall \eta \in [-\frac{h}{2}, 0]\), we have
\[
\|J^h_{42}(\tau)\| \leq C \int_{-\frac{\tau}{2}}^{0} (\tau - \eta)^{\alpha_i - 1}(\eta + h)^{-\alpha_i - 1}d\eta \leq C\tau^{\alpha_i - 1}h^{-\alpha_i}.
\]

Finally we deduce
\[
\|J^h_4(\tau)\| \leq CH^{\alpha_i - 1} + C\eta^{\alpha_i - 1}, \forall 0 < \tau \leq h.
\]

It remains to estimate \(J^h_5(\tau)(0 \leq \tau \leq h)\). From (10), we see that
\[
\|J^h_5(\tau)\| \leq C \int_0^\tau (\tau - \eta)^{\alpha_i + 2}((\eta + h)^{-\alpha_i} - (\tau + h)^{-\alpha_i} - \eta^{-\alpha_i} + \tau^{-\alpha_i})d\eta
\]
\[
\leq C \int_0^\frac{\tau}{2} (\tau - \eta)^{\alpha_i + 1}((\eta + h)^{-\alpha_i} - (\tau + h)^{-\alpha_i} + \eta^{-\alpha_i} + \tau^{-\alpha_i})d\eta
\]
\[
+ C \int_\frac{\tau}{2}^\tau (\tau - \eta)^{\alpha_i - 2}((\eta + h)^{-\alpha_i} - (\tau + h)^{-\alpha_i} - \eta^{-\alpha_i} + \tau^{-\alpha_i})d\eta := J^h_{51} + J^h_{52}, 0 < \tau \leq h.
\]

For \(J^h_{51}\). Using the inequality \((\tau - \eta)^{\alpha_i + 1} \leq (\frac{\tau}{2})^{\alpha_i + 1}, \forall \tau \geq 0, \forall \eta \in [0, \frac{\tau}{2}]\), we have
\[
\|J^h_{51}(\tau)\| \leq C \int_0^\tau \tau^{\alpha_i - 1}(\eta + h)^{-\alpha_i - 1}d\eta \leq C\tau^{\alpha_i - 1}(\eta + h)^{-\alpha_i - 1}, 0 < \tau \leq h.
\]

For \(J^h_{52}\). Using the inequality
\[
(\eta + h)^{-\alpha_i} - (\tau + h)^{-\alpha_i} \leq C(\tau - \eta)(\eta + h)^{-\alpha_i - 1}, 0 \leq \eta \leq \tau,
\]
we have
\[ \|J^h_{22}(\tau)\| \leq C \int_{\tau}^{t} (\tau - \eta)^{\alpha_1} \eta^{-\alpha_1} d\eta \leq C\tau^{\alpha_1 - \alpha_i - 1}, \ 0 < \tau \leq h.\]

Collecting the estimates for \( J^h_{21}(\tau) \) and \( J^h_{22}(\tau) \) on \( 0 < \tau \leq t \leq h \) we obtain
\[ \|g_i(\tau + h) - g_i(\tau)\| \leq C\tau^{\alpha_1 - \alpha_i - 1}, \ 0 < \tau \leq t \leq h, i = 2, \ldots, \ell. \] (19)

We choose \( \varepsilon > 0, \theta'' \in [0, \theta'] \) as in the calculation of \( I^h_{22}(t) \). Then (19) yields the estimate
\[ \|I^h_{22}(t)\|_{L^2(\Omega)} \leq C \sum_{i=2}^{\ell} \int_{0}^{t} \|g_i(\tau + h) - g_i(\tau)\|_{\theta''} \|Au\|_{\theta''} \leq Ch^{\theta' + \varepsilon} \|Au\|_{\theta''}, 0 \leq t \leq h < 1. \]

In the case when \( t > h \), from the above calculation, we have the following estimate
\[ \|I^h_{22}(t)\|_{L^2(\Omega)} \leq Ch^{\theta' + \varepsilon} \|Au\|_{\theta''} + \sum_{i=2}^{\ell} \int_{h}^{t} \|g_i(\tau + h) - g_i(\tau)\|_{\theta''} (Au(\tau) - Au(t))\|_{L^2(\Omega)} \].

(20)

After the change of variables, we represent \( g_i(\tau + h) - g_i(\tau) \) in the form:
\[ \Gamma(1 - \alpha_i)(g_i(\tau + h) - g_i(\tau)) = \left((\tau + h)^{1-\alpha_i} - \tau^{1-\alpha_i}\right) \int_{0}^{1} A^{-1} S''\left((1 - \eta)(\tau + h)\right)(\eta^{-\alpha_i} - 1) d\eta \]
\[ - \tau^{1-\alpha_i} \int_{0}^{1} A^{-1} \int_{\tau - \eta \tau}^{(1-\eta)(\tau + h)} S''(\xi)(\xi^{-\alpha_i} - 1) d\xi. \]

Further, we shall need the inequalities:
\[ (\tau + h)^{1-\alpha_i} - \tau^{1-\alpha_i} \leq Ch^{\tau - \alpha_i}, \forall \eta \in [h, t], \]
\[ \|A^{-1} S''(\xi)\| \leq C\xi^{\alpha_1 - 3}, \forall \xi > 0. \]

Then we have
\[ \Gamma(1 - \alpha_i)\|g_i(\tau + h) - g_i(\tau)\| \]
\[ \leq Ch^{\tau - \alpha_i} \int_{0}^{1} \left((1 - \eta)(\tau + h)\right)^{1-\alpha_i - 2} (\eta^{-\alpha_i} - 1) d\eta + C\tau^{1-\alpha_i} \int_{0}^{1} \int_{\tau - \eta \tau}^{(1-\eta)(\tau + h)} \xi^{\alpha_1 - 3} d\xi (\eta^{-\alpha_i} - 1) d\eta \]
\[ \leq Ch^{\tau - \alpha_i - 2} \int_{0}^{1} (1 - \eta)^{\alpha_1 - 2} (\eta^{-\alpha_i} - 1) d\eta \leq Ch^{\tau - \alpha_i - 2}, \ h \leq \tau \leq t. \]

Consequently \( \|g_i(\tau + h) - g_i(\tau)\| \leq Ch^{\tau - \alpha_i - 2}, \forall \tau \geq h. \) Applying this estimate to (20), for \( t > h, 0 < h < 1 \), we deduce that
\[ \|I^h_{22}(t)\|_{L^2(\Omega)} \leq Ch^{\theta' + \varepsilon} \|Au\|_{\theta''} + C \sum_{i=2}^{\ell} \int_{h}^{t} \int_{0}^{\infty} h\tau^{\alpha_1 - \alpha_i + \theta'' - 2} d\tau \|Au\|_{\theta''} \leq Ch^{\theta' + \varepsilon} \|Au\|_{\theta''}. \]

Collecting the above estimates, we have
\[ \|AKu(t + h) - AKu(t)\|_{L^2(\Omega)} \leq Ch^{\theta} \|F\|_{\theta} + Ch^{\theta' + \varepsilon} \|Au\|_{\theta''}, \forall \theta' \in [0, \theta). \]

(21)

This complete the proof of the lemma. \( \square \)
Theorem 3.1 (Hölder estimate). Suppose that \( a = 0 \) and \( F \in X_\theta \) with \( \theta \in (0, 1) \). Then the initial-boundary value problem (1) admits a unique mild solution \( u \in X_\theta \). Moreover there exists a constant \( C > 0 \) such that
\[
\|Au\|_\theta \leq C\|F\|_\theta.
\]

Proof. Since \( C^0([0, T]; L^2(\Omega)) \subset L^\infty(0, T; L^2(\Omega)) \), and applying the Theorem 2.4, we find a unique \( u \in C([0, T]; D(A)) \) for each \( \gamma \in (0, 1) \) such that \( u = Ku \), and for any \( \gamma \in (0, 1) \), there exists constant \( C > 0 \) such that
\[
\|u\|_{C([0, T]; D(A^\gamma))} \leq C\|F\|_{C([0, T]; L^2(\Omega))}.
\]
Moreover, we can obtain \( u \in C([0, T]; H^2(\Omega)) \) and the following estimate \( \|u\|_{C([0, T]; H^2(\Omega))} \leq C\|F\|_\theta \), that is \( \|Au\|_{C([0, T]; L^2(\Omega))} \leq C\|F\|_\theta \).

By Lemma 3.1 ([23]), we can see that for \( 0 \leq t < t + h \leq T, 0 < h < 1 \),
\[
\|Au(t + h) - Au(t)\|_{L^2(\Omega)} \leq Ch^\theta\|F\|_\theta + Ch^{\theta + \varepsilon}\|Au\|_\theta, \forall 0 \leq \theta \in [0, \theta).
\]
Let \( \theta = 0 \) in (23). We have \( \|Au\|_\varepsilon \leq C\|F\|_\theta + \|Au\|_0 \leq C\|F\|_\theta \). Next, let \( \theta = \varepsilon \). Repeating the argument in the above, we have \( \|Au\|_{2\varepsilon} \leq C\|F\|_\theta + \|Au\|_\varepsilon \leq C\|F\|_\theta \). Step by step, it is clear from the above argument that \( \|Au\|_\theta \leq C\|F\|_\theta \). Then theorem is thus proved. \( \square \)

4 Analyticity

We set \( F = 0 \) in initial-boundary value problem (1):

\[
\begin{aligned}
\frac{\partial u}{\partial t} + \sum_{j=2}^{\ell} q_j(x) \frac{\partial^{\alpha_j} u}{\partial t^{\alpha_j}} &= -Au(t) + B(x) \cdot \nabla u(t), \ 0 < t \leq T, \\
0 &= u(x, 0) = a, \ x \in \Omega, \\
0 &= u(x, t) = 0, \ x \in \partial \Omega, \ t \in (0, T).
\end{aligned}
\]

According to the results in Theorem 2.4, we see that the solution to (24) satisfies the integral equation (3). After the change of variables, we have
\[
\begin{aligned}
u(t) &= S(t)a - \sum_{i=2}^{\ell} \frac{t^{1-\alpha_i}}{\Gamma(1-\alpha_i)} \int_0^1 A^{-1}S'((1-\tau)t)\tau^{-\alpha_i} q_i \, \! d\tau t \int_0^1 A^{-1}S'(\tau t)B \cdot \nabla u((1-\tau)t) \! \! d\tau \\
&\quad + \sum_{i=2}^{\ell} \frac{t^{1-\alpha_i}}{\Gamma(1-\alpha_i)} \int_0^1 A^{-1}S'((1-\tau)t)\tau^{-\alpha_i} q_i u((1-\tau)t) \! \! d\tau \\
&\quad + \sum_{i=2}^{\ell} \frac{t^{2-\alpha_i}}{\Gamma(1-\alpha_i)} \int_0^1 A^{-1}S''((1-\eta)) (\eta^{-\alpha_i} - 1)q_i u((1-\tau)t) \! \! d\eta d\tau.
\end{aligned}
\]

Moreover, we extend the variable \( t \) in (25) from \((0, T)\) to the sector \( S := \{ z \neq 0; |\arg z| < \frac{\pi}{2} \} \), and setting \( u_0 = 0 \), we define \( u_{n+1}(z)(n = 0, 1, \cdots) \), \( z \in S \) as follows:
\[
\begin{aligned}
u_{n+1}(z) &= \sum_{i=2}^{\ell} \frac{z^{1-\alpha_i}}{\Gamma(1-\alpha_i)} \int_0^1 A^{-1}S'((1-\tau)z)\tau^{-\alpha_i} q_i \, \! d\tau z \int_0^1 A^{-1}S'(\tau z)B \cdot \nabla u_n((1-\tau)z) \! \! d\tau \\
&\quad + S(z)a + \sum_{i=2}^{\ell} \frac{z^{1-\alpha_i}}{\Gamma(1-\alpha_i)} \int_0^1 A^{-1}S'(\tau z)\tau^{-\alpha_i} q_i u_n((1-\tau)z) \! \! d\tau \\
&\quad + S(z)u_0 + \sum_{i=2}^{\ell} \frac{z^{2-\alpha_i}}{\Gamma(1-\alpha_i)} \int_0^1 A^{-1}S''((1-\eta)) (\eta^{-\alpha_i} - 1)q_i u_n((1-\tau)z) \! \! d\eta d\tau.
\end{aligned}
\]
Lemma 4.1. For any constants $0 < \theta < \frac{\pi}{2}$ and $T > 0$, there exist constants $M > 0$ and $M_1 > 0$ such that the following estimate

$$
\| A^T u_{n+1}(z) - A^T u_n(z) \|_{L^2(\Omega)} \leq M_1 M_n \left| z \right|^{\alpha n - \frac{\alpha_i}{2}} \| a \|_{L^2(\Omega)}, \quad n \in \mathbb{N}
$$

holds for $\forall z \in \mathcal{S}_T^\theta := \{ z \in \mathcal{S} : |\Re z| \leq T, |\arg z| \leq \theta \}$, where $\alpha := \min_{i=2}^\ell \{ \alpha_i - \alpha, \frac{\alpha_i}{2} \}$.

Proof. Firstly, for $n = 0$, by (3) and (4), noting that $|\tau|^{\alpha - \alpha_i} \leq C |\tau|^{- \alpha_i}$, $z \in \mathcal{S}_T^\theta$, we have

$$
\| A^T (u_1(z) - u_0(z)) \|_{L^2(\Omega)} \leq C |\tau|^{- \alpha_i} \| a \|_{L^2(\Omega)} + C \sum_{i=2}^\ell \int_0^1 (1 - \tau) \tau^{\alpha_i - 1 - \alpha_i} d\tau \| a \|_{L^2(\Omega)}
$$

$$
\leq C |\tau|^{- \alpha_i} \| a \|_{L^2(\Omega)}.
$$

Next, for any $n \in \mathbb{N}$, taking the operator $A^T$ on the both side of (26), by the induction assumption and the (3) and (4) for the $z \in \mathcal{S}$, similarly to the argument in the Theorem 2.1, we can prove that

$$
\| u_{n+1}(z) - u_n(z) \|_{D(A^T)} \leq CM_1 M_n \left| z \right|^{\alpha n - \frac{\alpha_i}{2}} \int_0^1 \tau^{\alpha_i - 1} (1 - \tau)^{\alpha(n-1) - \frac{\alpha_i}{2}} d\tau \| a \|_{L^2(\Omega)}
$$

$$
= CM_1 M_n \left| z \right|^{\alpha n - \frac{\alpha_i}{2}} B(\alpha, \alpha(n-1) + 1 - \frac{\alpha_i}{2}) \| a \|_{L^2(\Omega)}
$$

$$
= CM_1 \Gamma(\alpha) M_n \left| z \right|^{\alpha n - \frac{\alpha_i}{2}} \| a \|_{L^2(\Omega)} = M_1 M_n \left| z \right|^{\alpha n - \frac{\alpha_i}{2}} \| a \|_{L^2(\Omega)},
$$

where we set $M := CT(\alpha)$, $M_1 := CT(1 - \frac{\alpha_i}{2})$. \hfill \Box

Theorem 4.1. Suppose that $u \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H^1_0(\Omega))$ is the mild solution to (21), then $u : (0,T) \to H^1_0(\Omega)$ can be analytically extended to a sector $S = \{ z \neq 0 ; |\arg z| < \frac{\pi}{2} \}$.

Proof. For any $\delta > 0$, we denote $\mathcal{S}_T^\theta_{\delta} := \{ z \in \mathcal{S}_T^\theta : |z| \geq \delta \}$. From the definition of $u_n(z)$ in (26), it is easy to show that $A^T u_n(z) : \mathcal{S}_T^\theta_{\delta} \to L^2(\Omega)$ is analytic. Hence according to Lemma 4.1 there exists $\tilde{u}(z) \in L^2(\Omega)$ such that $\| A^T u_n(z) - A^T \tilde{u}(z) \|_{L^2(\Omega)}$ uniformly tends to 0, $z \in \mathcal{S}_T^\theta_{\delta}$, as $n \to \infty$. Therefore $A^T \tilde{u}(z)$ is analytic in $\mathcal{S}_T^\theta_{\delta}$. Moreover, since $\delta$, $T$, $\theta$ are arbitrarily chosen, then we deduce $A^T \tilde{u}(z)$ is actually analytic in the sector $S$. 

\[\begin{aligned} + \ell \int_0^1 \int_0^1 A^{-1} S''((1 - \eta)\tau z)(\eta^{-\alpha_i} - 1)q_i u_n((1 - \tau)z) \tau^{1 - \alpha_i} \eta d\eta d\tau. \end{aligned}\]
Finally, we show that $\tilde{u}(z)$ is just the mild solution $u$ to (24) when the variable $z$ is restricted on the interval $(0, T)$. In fact, denoting the imaginary part of $\tilde{u}(t)$, $\forall t \in (0, T)$ as $\Im u(t)$, we see that $\Im u(t)$ is the mild solution to the following initial-boundary problem:

$$
\begin{cases}
\partial_t^{\alpha_j} u(t) + \sum_{j=2}^{\ell} q_j(x) \partial_t^{\alpha_j} u(t) = -Au(t) + B(x) \cdot \nabla u(t), & t > 0, \\
u(x, 0) = 0, & x \in \Omega, \\
u(x, t) = 0, & x \in \partial\Omega, \ t \in (0, T).
\end{cases}
$$

Using the uniqueness result of the above problem, we have $\Im u(t) = 0, \forall t \in (0, T)$. So that again by the uniqueness argument we see that $\tilde{u}(t) = u(t), \forall t \in (0, T)$. This completes the proof of the theorem. □

5 The weak unique continuation

For the parabolic equation, there are a well known principle called unique continuation, generally speaking, any solution of a parabolic equation that is defined on a domain $D$ must vanish in all of $D$ if it vanishes on an open set in $D$ (see, e.g., [3]). Here for the fractional diffusion equation, we are to establish similar result to the parabolic equation.

**Theorem 5.1 (Weak unique continuation).** Assuming that $0 < \alpha_\ell < \cdots < \alpha_2 < \alpha_1 < 1, a \in L^2(\Omega).$ Let $q_j$ be constant for $j = 2, \cdots, \ell$. Suppose that $u \in C([0, T]) \cap C((0, T) \cap H^2(\Omega) \cap H^1_0(\Omega))$ is the mild solution to the following initial-boundary value problem:

$$
\begin{cases}
\partial_t^{\alpha_j} u(x, t) + \sum_{j=2}^{\ell} q_j(x) \partial_t^{\alpha_j} u(x, t) = -Au(x, t), & 0 < t \leq T, \\
u(x, 0) = a(x), & x \in \Omega, \\
u(x, t) = 0, & x \in \partial\Omega, \ t \in (0, T].
\end{cases}
$$

Let $\omega \subset \Omega$ be an arbitrarily chosen subdomain and let $T > 0$. Then $u(x, t) = 0, x \in \omega, 0 < t < T$, implies $u = 0$ in $\Omega \times (0, T)$.

**Proof.** According to Theorem [21], we can uniquely extend the existence interval of $u$ into $[0, \infty)$. Therefore, we can describe the solution $u(t)$ to (28) by the Laplace transform:

$$
L(u)(x, s) = \sum_{n=1}^{\infty} h_n(s) (\alpha_n(x) \phi_n(x), \ x \in \Omega, \Re s > M.
$$

Here $M > 0$ is a constant such that the Laplace transform of $u$ converges for $\Re s > M$.

$$
h_n(s) = \frac{s^{\alpha_1} + \sum_{j=2}^{\ell} q_j s^{\alpha_j}}{s^{\alpha_1} + \sum_{j=2}^{\ell} q_j s^{\alpha_j} + \lambda_n}.
$$

Let $\omega \subset \Omega$ be an arbitrarily fixed sub domain. According to the Theorem [14,11] we have $u : (0, \infty) \rightarrow L^2(\Omega)$ can be analytically extended to a sector $S = \{z \neq 0; |\arg z| < \frac{\pi}{2}\}$. From the definition for the analytic of vector-valued function, we see that

$$
\lim_{\Delta z \rightarrow 0} \frac{u(z + \Delta z) - u(z)}{\Delta z}
$$

exists in the topology of $L^2(\Omega)$, for any $z \in S$. Then $u : S \rightarrow L^2(\omega)$ is also analytic. Therefore, $u : S \rightarrow L^2(\omega)$ is also weakly analytic (see, e.g., Theorem 3.31 on p.82 in [3]). That is, for any
\( \varphi \in L^2(\omega) \), the function \((u(z), \varphi)_{L^2(\omega)}\) is analytic in the ordinary sense. \( u|_{\omega \times (0,T)} = 0 \) derives \((u(t), \varphi)_{L^2(\omega)} = 0 \) for any \( t \in (0,T) \). By analyticity, we recognize that \((u(t), \varphi)_{L^2(\omega)} = 0 \) for any \( t \in (0,\infty) \). Since \( \varphi \) is chosen arbitrarily, so \( u = 0 \) in \( \omega \times (0,\infty) \). Therefore

\[
L(u)(s) = \sum_{n=1}^{\infty} \frac{s^{\alpha_1} + \sum_{j=2}^{f} q_j s^{\alpha_j}}{s^{\alpha_1} + \sum_{j=2}^{f} q_j s^{\alpha_j} + \lambda_n} (a, \phi_n)\phi_n = 0 \text{ in } \omega, \, \Re s > M.
\]

Setting \( \eta = s^{\alpha_1} + \sum_{j=2}^{f} q_j s^{\alpha_j} \), we see that \( \eta \) varies over some domain \( E \subset \mathbb{C} \) as \( s \) varies over \( \Re s > M \). Therefore

\[
\sum_{n=1}^{\infty} \frac{1}{\eta + \lambda_n} (a, \phi_n)\phi_n(x) = 0, \text{ in } \omega, \, \eta \in E.
\]

We set \( \sigma(A) = \{\mu_k\}_{k \in \mathbb{N}} \) and we denote by \( \{\varphi_{kj}\}_{1 \leq j \leq m_k} \) an orthonormal basis of \( \ker(\mu - A) \). Note that we regards \( \sigma(A) \) as set, not as sequence with multiplicities. Therefore we can rewrite (29) by

\[
J(\eta) := \sum_{k=1}^{m} \sum_{j=1}^{m_k} \frac{1}{\eta + \mu_k} (a, \phi_n)\varphi_{kj} = 0 \text{ in } \omega, \, \forall \eta \in E.
\]

It is easy to see that \( J : E \to L^2(\Omega) \) is analytic. Moreover, we can derive that \( J(\eta) \) defined in (30) holds for \( \eta \in \mathbb{C} \setminus \{-\mu_k\}_{k \in \mathbb{N}} \). We can take a suitable disk which includes \(-\mu_k\) and does not include \(-\mu_i \neq \mu_k \). Integrating (30) in this disk, we have

\[
u_k = \sum_{j=1}^{m_k} (a, \varphi_{kj})\varphi_{kj} = 0 \text{ in } \omega.
\]

Since \((A - \mu_k)\nu_k = 0 \) and \( \nu_k = 0 \) in \( \omega \), the unique continuation (e.g., Isakov [16]) implies \( \nu_k = 0 \) in \( \Omega \) for each \( k \in \mathbb{N} \). Since \( \{\varphi_{kj}\}_{1 \leq j \leq m_k} \) is linearly independent in \( \Omega \), we see that \((a, \varphi_{kj}) = 0 \) for \( 1 \leq j \leq m_k, \, k \in \mathbb{N} \). Therefore \( u = 0 \) in \( \Omega \times (0,T) \). This completes the proof of the theorem. \( \square \)

**Corollary 5.1.** Let \( \Gamma \) be an open subset of \( \partial \Omega \), and \( u \in C([0,T]; L^2(\Omega)) \cap C((0,T); H^{2}(\Omega) \cap H^1_0(\Omega)) \) satisfy the problem (28) and \( \partial_{\nu} u|_{\Gamma \times (0,T)} = 0 \). Then \( u = 0 \) in \( \Omega \times (0,T) \).

**Proof.** Since the boundary \( \partial \Omega \) of \( \Omega \) is smooth enough, we can choose an open set \( \omega \) such that \( \omega \cap \Omega = \Gamma \) and the boundary of the new domain \( \tilde{\Omega} := \Omega \cup \omega \) is smooth enough. We set

\[
\tilde{u}(x,t) := \begin{cases} u(x,t), & \forall (t,x) \in \Omega \times (0,T), \\ 0, & \forall (t,x) \in \omega \times (0,T). \end{cases}
\]

According to the condition \( u|_{\partial \Omega \times (0,T)} = \partial_{\nu} u|_{\Gamma \times (0,T)} = 0 \), it is easy to see that the new function \( \tilde{u} \) belongs to \( C([0,T]; L^2(\tilde{\Omega})) \cap C((0,T); H^{2}(\tilde{\Omega}) \cap H^1_0(\tilde{\Omega})) \).

On the other hand, the proof of the weak unique continuation is divided into the following steps: Step 1. Extending existence interval of the solution to \((0,\infty)\) and taking the fractional integral operator \( J^{\alpha_1} \) on the both side of the equation, that is,

\[
\begin{cases} u - a + \sum_{j=2}^{f} q_j J^{\alpha_j - \alpha_1} (u - a) = -AJ^{\alpha_1}u, & x \in \Omega, \, t > 0, \\ u = 0, & x \in \partial \Omega, \, t > 0. \end{cases}
\]

Step 2. Taking Laplace transform on the both side of the above equation, we can obtain the weak unique continuation by some results in the complex analysis.
Using the fact that \( u \) is also the solution of problem (32), then \( \tilde{u} \) is the solution of the following problem

\[
\begin{aligned}
\tilde{u} - \tilde{a} + \sum_{j=2}^{t} q_j J^{\alpha_1 - \alpha_j} (\tilde{u} - \tilde{a}) &= -\mathcal{A} J^{\alpha_1} \tilde{u}, \quad t > 0, \\
\tilde{u} &= 0, \quad x \in \partial \tilde{\Omega}, \quad t > 0,
\end{aligned}
\]

(33)

where

\[
\tilde{a}(x) := \begin{cases} a(x), & \forall \ x \in \Omega, \\
0, & \forall \ x \in \bar{\omega}. \end{cases}
\]

(34)

From Theorem 5.1 we deduce \( \tilde{u} = 0 \) in \( \tilde{\Omega} \times (0, T) \), that is, \( u = 0 \) in \( \Omega \times (0, T) \). This completes the proof of corollary.

6 Conclusions

Initial-boundary value problem for the linear diffusion equation with multiple time-fractional derivatives was investigated. We proved unique existence of the solution by the eigenfunction expansion and Laplace transform as well as the Hölder regularity and related properties.

(i) In the case where the equation with single fractional time-derivative, the solution to the initial-boundary value problem in [34] can easily achieve the \( C((0, T]; H^2(\Omega)) \) regularity, while we only proved that the solution is in \( C((0, T]; H^{2\gamma}(\Omega)) \), \( \gamma \in (0, 1) \) (see Theorem 2.4), and we do not know whether the solution can achieve the regularity as in [34].

(ii) We only proved the Hölder’s regularity with the initial condition being zero. We guess that there may be a Hölder regularity similar to Theorem 2.4 in [34].

(iii) For the classical diffusion equation, we have that the unique continuation principle holds without \( u|_{\partial \Omega \times (0, T)} = 0 \) (e.g., [16]). However for our case, we do not know whether the uniqueness holds without such kind of assumption.

References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1999.

[2] E. E. Adams, L. W. Gelhar, Field study of dispersion in a heterogeneous aquifer 2: Spatial moments analysis, Water Resources Research 28(1992) 3293-3307.

[3] S. Beckers and M. Yamamoto, Regularity and unique existence of solution to linear diffusion equation with multiple time-fractional derivatives. *Control and Optimization with PDE Constraints (2013)* ed K. Bredies, C. Clason, K. Kunisch and G. von Winckel (Basel: Birkhäuser).

[4] B. Berkowitz, H. Scher and S. E. Silliman, Anomalous transport in laboratory-scale heterogeneous porous media. Water Resource Research 36(2000) 149-158.

[5] D. A. Benson, R. Schumer, M. M. Meerschaert and S. W. Wheatcraft, Fractional dispersion, levy motion, and the MADE tracer tests. Transport in Porous Media 42(2001) 211-240.
[6] A.V. Chechkin, R. Gorenflo, I.M. Sokolov, Retarding subdiffusion and accelerating superdiffusion governed by distributed order fractional diffusion equations, Phys. Rev. E 66(2002) 1-7.

[7] A.V. Chechkin, R. Gorenflo, I.M. Sokolov, V.Yu. Gonchar, Distributed order time fractional diffusion equation, Fract. Calc. Appl. Anal. 6(2003) 259-279.

[8] M. Choulli, Une Introduction aux Problems Inverses Elliptiques et Paraboliques. Springer-Verlag, 2009.

[9] K. Diethelm and Y. Luchko, Numerical solution of linear multi-term initial value problems of fractional order. J. Comput. Anal. Appl. 6 (2004), 243-263.

[10] V. Daftardar-Gejji, S. Bhalekar, Boundary value problems for multi-term fractional differential equations, J. Math. Anal. Appl. 345 (2008) 754-765.

[11] M. Giona, S. Gerbelli and H. E. Roman, Fractional diffusion equation and relaxation in complex viscoelastic materials. Physica A 191(1992) 449-453.

[12] R. Gorenflo, Y. Luchko and P. P. Zabrejko, On solvability of linear fractional differential equations in Banach spaces. Frac. Calc. Appl. Anal. 2(1999) 163-176.

[13] A. Hanyaga, Multidimensional solutions of time-fractional diffusion-wave equations. Proc. R. Soc. Lond. (Ser. A) 458(2002) 933-957.

[14] Y. Hatno, N. Hatano, Dispersive transport of ions in column experiments: an explanation of long-tailed profiles, Water Resources Res. 34(1980) 1027-1033.

[15] D. Henry, Geometric Theory of Semilinear Parabolic Equations. Springer-Verlag, Berlin, 1981.

[16] V. Isakov, Inverse Problems for Partial Differential Equations. Springer-Verlag, New York, 1998.

[17] H. Jiang, F. Liu, I. Turner and K. Burrage, Analytical solutions for the multi-term time-space Caputo-Riesz fractional advection-diffusion equations on a finite domain. J. Math. Anal. Appl. 389 (2012), 1117-1127.

[18] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Amsterdam: Elsevier, 2006.

[19] A.N. Kochubei, Distributed order calculus and equations of ultraslow diffusion, J. Math. Anal. Appl. 340(2008) 252-281.

[20] J. L. Lions; E. Magenes, Non-homogeneous Boundary Value Problems and Applications. Springer-Verlag, 1972.

[21] Y. Luchko, R. Gorenflo, An operational method for solving fractional differential equations with the Caputo derivatives, Acta Math. Vietnam. 24(1999) 207-233.

[22] Y. Luchko, Boundary value problems for the generalized time-fractional diffusion equation of distributed order, Fract. Calc. Appl. Anal. 12(2009) 409-422.

[23] Y. Luchko, Maximum principle for the generalized time-fractional diffusion equation. J. Math. Anal. Appl. 351 (2009), 218-223.
[24] Y. Luchko, Some uniqueness and existence results for the initial-boundary-value problems for the generalized time-fractional diffusion equation. Computers and Mathematics with Applications 59 (2010), 1766-1772.

[25] Y. Luchko, Initial-boundary-value problems for the generalized multi-term time-fractional diffusion equation. J. Math. Anal. Appl. 374 (2011), 538-548.

[26] M.M. Meerschaert, E. Nane, P. Vellaisamy, Distributed-order fractional Cauchy problems on bounded domains, arXiv:0912.2521v1.

[27] M. M. Meerschaert, E. Nane, H.P. Scheffler, Stochastic model for ultraslow diffusion, Stochastic Process. Appl 116(2006)1215-1235.

[28] R. Metzler and J. Klafter, Boundary value problems for fractional diffusion equations. Physica A 278(2000) 107-125.

[29] M. M. Meerschaert and C. Tadjeran, Finite difference approximations for fractional advection-dispersion flow equations. Journals of Computational and Applied Mathematics 172(2004) 65-77.

[30] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag, Berlin, 1992.

[31] I. Podlubny, *Fractional Differential Equations*. Academic Press, San Diego, 1999.

[32] H. E. Roman, P. A. Alemany, Continuous-time random walks and the fractional diffusion equation. J. Phys. A 27(1994) 3407-3410.

[33] Water Rudin, *Functional Analysis 2nd ed.*, McGraw-Hill, 1991.

[34] K. Sakamoto; M. Yamamoto, Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. J. Math. Anal. Appl. 382(2011), 426-447.

[35] L. M. Sokolov, A.V. Chechkin, J. Klafter, Distributed-order fractional kinetics, Acta Phys. Polon. B 35(2004) 1323-1341.

[36] S. Umarov, R. Gorenflo, Cauchy and nonlocal multi-point problems for distributed order pseudo-differential equations, Z. Anal. Anwend. 24(2005) 449-466.

[37] X. Xu, J. Cheng and M. Yamamoto, Carleman estimate for a fractional diffusion equation with half order and application. Applicable Analysis 90(2011) 1355-1371.