HAS THE BLACK HOLE EQUILIBRIUM PROBLEM BEEN SOLVED?

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Abstract. When the term “black hole” was originally coined in 1968, it was immediately conjectured that the only pure vacuum equilibrium states were those of the Kerr family. Efforts to confirm this made rapid progress during the “classical phase” from 1968 to 1975, and some gaps in the argument have been closed during more recent years. However the presently available demonstration is still subject to undesirably restrictive assumptions such as non-degeneracy of the horizon, as well as analyticity and causality in the exterior.

1 Introduction

The purpose of this report is to present a brief overview of the present state of progress on the still not completely solved problem of the classification of black hole equilibrium states within the (astrophysically motivated) context of Einstein’s pure vacuum theory and its electrovac generalisation in ordinary four dimensional spacetime.

In recent years there has been a considerable resurgence of mathematical work on black hole equilibrium states, but most of it has been concerned with more or less exotic generalisations, not restricted to four dimensions, involving speculative extensions of Einstein’s theory to allow for inclusion of various scalar and other (e.g. Yang Mills type) fields such as those occurring in low energy limits of superstring theory. The present review will not even attempt to deal with this rapidly developing and open ended area of investigation. The not quite so fashionable but – as far as the observable physical world is concerned – more soundly motivated subject of black hole
equilibrium states with surrounding rings of matter (such as would result from accretion from external sources) is also beyond the scope this article.

One reason why there has not been so much recent work on what, from an astrophysical point of view, is the most important problem in black hole equilibrium theory, namely that of the pure vacuum states in four dimensions, is the widespread belief that the problem was solved long ago, and that the solution is just what was predicted by my original 1967 conjecture [1, 2], i.e. that it is completely provided by the subset of Kerr solutions [3] for which \( a^2 \leq M^2 \) where \( a = J/M \) is the ratio of the angular momentum \( J \) to the mass \( M \). This belief rapidly gained general acceptance in astrophysical circles when – following the example of Israel’s earlier 1967 work [4] providing strong evidence that (as has since been confirmed) the only strictly static (not just stationary) solutions were given by the special Schwarzschild \((J = 0)\) case – I was able [5] in 1971 to obtain a line of argument that provided rather overwhelming, though by no means absolutely watertight, mathematical evidence to the effect that the most general solution is indeed included in the Kerr family.

Many of the interested parties, particularly observationally motivated astrophysicists, considered that the conversion of the original plausibility argument into an utterly unassailable mathematical proof was merely a physically insignificant technical formality, that could be left as an exercise for the amusement of obsessively rigoristic pure mathematicians. However such lack of interest was not the only reason why subsequent progress on the problem has been rather slow. It was soon shown by those – starting with Hawking [6, 7] – who took the question seriously after all that the mathematical work needed to deal with the various apparently small technical gaps and loose ends in the comparatively simple 1971 argument [5, 8] is harder than might have been hoped. Thus despite the very considerable efforts of many people – of whom some of the most notable after Hawking [6, 7, 9] have been Robinson [10, 11, 12], followed (for the electrovac generalisation) by Bunting [13, 14] and Mazur [15, 16], and most recently (since my last comprehensive review of the subject [17]) Wald and his collaborators [18, 20, 19, 21] – there still remains a lot that needs to be done before we shall have what could be considered a mathematically definitive solution even of the pure vacuum problem, not to mention the more formidable challenge of its electrovac generalisation.

It will be convenient to present the results in chronological order, which roughly corresponds to that of their logical development except for a few cases in which newer work has provided more elegant methods of rederiving
results that were originally obtained by more laborious means. Section 2 rapidly recalls some of the relevant results (culminating in Israel’s theorem) obtained during the prolonged period of general confusion that I refer to as the “preclassical phase”, prior to the introduction of the term “black hole” and to the general recognition that the disciples of Ginzburg and Zel’dovich had been right [22] in arguing that what it represents is a generic phenomenon – not just an unstable artefact of spherical symmetry as many people, including Israel himself [23], had speculated. Section 3 describes the rapid progress made during what I refer to as the “classical phase” – the beginning of what Israel [23] has referred to as the “age of enlightenment” – immediately following the definitive formulation of the concept of a “black hole” (in terms of the “outer past event horizon” in an asymptotically flat background) so that the corresponding equilibrium state problem could at last be posed in a mathematically well defined form. Section 4 describes the substantial though slower progress that has been made in what I refer to as the “postclassical phase”, that began when the main stream of work on black holes had been diverted to quantum aspects following the discovery of the Hawking effect [24].

The final section draws the intention of newcomers to the field, for whom this article is primarily intended, to the mathematically challenging (even if physically less important) problems that still remain to be tackled: these include not only the questions concerning the technically awkward degenerate limit case and the assumptions about spherical topology and analyticity that have been discussed by Chrusciel [25] and also, in a very extensive and up to date review, by Heusler [26], but also the largely neglected question of the assumption of causality, i.e. the absence of closed timelike curves.

A propos of the latter, is ironic that while providing a fascinating account of the mental blockages that impeded earlier workers in the theory of both black holes and time machines (i.e. regions of spacetime threaded by closed timelike curves) Thorne’s recent history of “Black holes and time warps” [22] has a blind spot of its own, in that it discusses only the kind of closed timelike curve whose presence depends on topological multiconnectedness in “wormholes” of a rather artificial kind (so that the resulting causality violation is “trivial” [27] in the sense of being in principle removable by replacing the spacetime model by its locally equivalent universal covering space). What is rather surprising is Thorne’s failure to mention the kind of time machine exemplified [3] by the Kerr solutions for $a^2 > M^2$, in which causality violation of a more “flagrant” [27] (not so easily removable) kind occurs. In the Kerr black hole case $a^2 \leq M^2$ the causality violation
is confined to the interior, but the unsolved problem is whether there exist other black hole equilibrium solutions in which such causality violation occurs outside the horizon. The formal existence of such pathological black hole solutions might of course be reasonably supposed to be irrelevant for realistic physical purposes. However the same kind of objection could be raised to Thorne’s “wormhole” time machines: if the latter are nevertheless at least of sufficient mathematical interest to be worth investigating then the same applies a fortiori to black hole time machines if they exist, a possibility that is by no means excluded by any of the work carried out so far.

2 The preclassical phase (1915-67).

What I refer to as the preclassical phase in the development of black hole theory is the period of unsystematic accumulation of more or less relevant results prior to the actual use of the term “black hole”. This period began with the discovery in 1916 by Schwarzschild [28] of his famous asymptotically flat vacuum solution, whose outer region is strictly static, with a hypersurface orthogonal timelike Killing vector whose striking feature is that its magnitude tends to zero on what was at first interpreted as a spacetime singularity, but was later recognised to be interpretable as a regular boundary admitting a smooth extension to an inner region where the Killing vector becomes spacelike. This preclassical phase culminated in Israel’s 1967 discovery [4] of a mathematical argument to the effect that the Schwarzschild solution is uniquely characterised by these particular properties, i.e. the original spherical example is the only example. The significance of this discovery was the subject of an intense debate that precipitated the transition to the “classical phase”, inaugurating what Israel [23] has termed the “age of enlightenment”, which dawned when the preceeding confusion at last gave way to a clear concensus. Thrilling eyewitness accounts of the turbulent evolution of ideas in the “golden age” of rapid progress, during the transition from the preclassical to the classical phase, have been given by Israel himself [23] and from a different point of view by Thorne [22], while a historical account of the more dilatory fumbling in the early years of the preclassical phase has been given by Eisenstaedt [24].

During most of the “unenlightenned” preclassical period, from 1915 until about 1960, nearly all the relevant work, starting with that of Schwarzschild, was in fact based on the simplifying postulate of spherical symmetry. An important consequence of this restriction was demonstrated
by Birkhoff’s 1923 theorem \cite{30}, which showed that the staticity property used in Schwarzschild’s derivation of his solution need not have been postulated independently of the spherical symmetry, since it followed as an automatic consequence of the vacuum field equations. An important step towards the concept of what would be called a “black hole” was the analysis \cite{31} of the gravitational collapse of pressure free matter by Oppenheimer and Snyder in 1939. However in the pure vacuum case, on which the present review is focussed, progress stayed remarkably slow for a very long time, and people remained confused by the special limit in the Schwarzschild solution where the circumferential radius $r$ reaches the value $2M$ (in relativistic units). Despite the construction of analytic extensions beyond this limit by many earlier workers \cite{32,33,34,35,36,37} a clear understanding was obtained only after more complete extensions were made by Fronsdal \cite{38}, Kruzkal \cite{39}, and Szekeres \cite{40}.

Due to a renaissance of interest (following the observational discovery of quasars) progress was much more rapid during what Thorne \cite{22} has referred to as the “golden age”, which began during the last half dozen years of the preclassical phase and continued through what I call the classical phase (ending when most of the easiest problems had been solved at about the time Hawking diverted attention attention to less astrophysically relevant quantum effects). It was during the late “golden” period of the pre-classical phase, roughly from 1961 to 1967, that results of importance for vacuum black hole theory began to be obtained without reliance on a presupposition of spherical symmetry. The most important of these results were of course Kerr’s 1963 discovery \cite{3} of the family of stationary asymptotically flat vacuum solutions characterised by a degenerate “type D” Weyl tensor, and the 1967 Israel theorem \cite{4} referred to above.

As well as these two specific discoveries, the most significant development during this final “golden” period of the preclassical phase was the animated debate – under the leadership of Ginzburg and Zel’dovich \cite{11} in what was then the Soviet Union, of Wheeler and later on Thorne \cite{2} in the United States, and of Sciama and Penrose \cite{13} in Britain – from which the definitive conceptual machinery and technical jargon of black hole theory finally emerged. Prior to the discovery of the Kerr solution \cite{3}, when the only example considered was that of Schwarzschild, it had not been thought necessary to distinguish what Wheeler later termed the “ergosphere” – where the Killing vector generating the stationary symmetry of the exterior ceases to be timelike – from the “outer past event horizon” bounding what Wheeler later termed the “black hole” region, from which no future timelike trajec-
tory can escape to the asymptotically flat exterior. In its original version, Israel’s 1967 theorem \(^1\) (as well as its electrovac generalisation \(^{13}\)) was effectively formulated in terms of an “infinite redshift surface” that was effectively taken to be what in strict terminology was really the “ergosphere” rather than the “outer past event horizon”: this meant that the significance of the theorem for the theory for what were to be called a “black holes” was not clear until it was understood that (as shown in my thesis \(^{1, 27}\) and pointed out independently by Vishveshwara \(^{44}\)) subject to the condition of strict staticity postulated by Israel (but not in the more general stationary case exemplified by the non-spherical Kerr \(^3\) solutions) the the “outer past event horizon” actually will coincide with the ergosphere.

One of the first to appreciate the distinction between (what would come to be known as) the horizon and the ergosphere, and to recognise the members of the relevant \((a^2 \leq M^2)\) Kerr subset as prototypes of what would come to be known as black hole solutions was Boyer \(^{46, 47, 48}\). However at the time of the first detailed geometrical investigations of the Kerr solutions \(^{49, 47}\) (the purpose for which, following a suggestion by Penrose, I originally introduced the scheme of representation by the kind of conformal projection now commonly known as a “Penrose diagram”), it was assumed by Boyer and the others involved, including myself, that we were dealing just with a particularly simple case within what might turn out to be a much more extensive category. However the publication of the 1967 Israel theorem \(^4\) – which went much of the way towards proving that the Schwarzschild \(^{28}\) solution is the only strictly static example – immediately lead to the question of whether the Kerr solutions might not be similarly unique.

The explicit formulation of this suggestion came later to be loosely referred to in the singular as the “Israel-Carter conjecture”, but there were originally not one but two distinct versions. The stronger version – suggested by the manner in which the Israel theorem was originally formulated – conjectured that the relevant Kerr subfamily might be the only stationary solutions that are well behaved outside and on a regular “infinite redshift surface” – a potentially ambiguous term that in the context of the original version \(^4\) of Israel’s theorem effectively meant what was later to be termed an “ergosurface” rather than an “event horizon”. The weaker version, first written unambiguously in my 1967 thesis \(^{1, 2}\), conjectured that the relevant Kerr subfamily might be the only stationary solutions that are well behaved all the way in to a regular black hole horizon, not just outside the ergosphere.
Work by Bardeen [50] and others on the effects of stationary orbiting matter rings (which can occur outside the horizon but inside the ergosphere of an approximately Kerr background) soon made it evident that strong version of the conjecture is definitely wrong, no matter how liberally one interprets the rather vague qualifications “regular” and “well behaved”. On the other hand the upshot of the work to be described in the following sections is to confirm the validity of my weaker version, as expressed in terms of the horizon rather than the ergosphere. It is nevertheless to be remarked that, as was pointed out by Hartle and Hawking [51], the generalisation of this conjecture from the Kerr pure vacuum solutions [3] to the Kerr-Newman electrovac solutions [52] is not valid, since the solutions due to Papapetrou [53] and Majumdar [54] provide counterexamples. It is also to be emphasised that, as will be discussed in the final section, the question still remains entirely open, even in the pure vacuum case, if the interpretation of the qualification “well behaved” is relaxed so as to permit causality violation outside the horizon of the kind that is actually observed [2] to occur in the inner regions of the Kerr examples.

3 The classical phase (1968-75).

What I refer to as the classical phase in the development of black hole theory began when the appropriate conceptual framework and the corresponding generally accepted technical terminology became available, facilitating clear formulation of the relevant mathematical problems, whose solutions could then be sought by systematic research programs, not just by haphazard approach of the preclassical period. The relevant notions had already began to become clear to a small number of specialists (notably Wheeler’s associates, including Thorne and Misner, in the United States, and Penrose’s associates, including Hawking and myself in Britain) during the period of accelerated activity at the end of what I call the preclassical phase.

However it was not just theoretical progress that precipitated the rather sudden (“first order”) transition to what I call the classical phase. Just as it was the discovery of the quasar phenomenon that stimulated the “golden age” [24] of rapid progress, so also, rather similarly, it was another observational event, namely the accidental discovery of pulsars by Bell and Hewish, that inaugurated the transition from the “preclassical phase” to the “age of enlightenment” [23] during the second half of the “golden age”. Unlike the quasar phenomenon, whose underlying mechanism is far from clear even
to day, the pulsar phenomen was rapidly elucidated: in the early months
of 1968 it was already generally generally recognised to be attributable to
neutron stars, whose likely existence had long been predicted by theoreti-
cians, but whose reality had never until then been taken very seriously by
the majority of the astronomical community. The 1968 confirmation that
neutron stars definitely exist and are directly observable immediately trans-
formed the status of theoreticiens in the eyes of the observers. (Prior to 1968
even our firmest affirmations were treated with the greatest scepticism; after
1968 even our most tentative speculations, as well as our conjectures about
“black holes”, were received as oracular pronouncements.) This meant that
the beginning of the “classical phase” was characterised not only by the es-
establishment of an “enlightenned” consensus among the previously disparate
groups of specialists working in the field, but also by the recognition for the
first time by a much wider public that a new and important field of theoret-
ical astrophysics had been born. At the beginning of 1968 the term “black
hole” was known only to a handful of participants in the seminars organised
at Princeton by Wheeler; by the end of 1968 the term had already been
widely publicised in televised science fiction so that it was already known
(if not understood) by millions of people all over the world.

At a time when the existence of black holes produced by burnt out stars
throughout our galaxy and others was already widely albeit prematurely
recognised by much of the astronomical community, it became urgent for
the theoreticians actually working in the field to settle the question of the
physical relevance of the black hole scenario, which requires that it should
occur not just as an unstable special space (which was the implication that
Israel was at first inclined to draw from his theorem [23]) but as a generic
phenomenon as Zel’dovich and his collaborators had been claiming [22].

The strongest conceivable confirmation of the general validity of the
black hole scenario is what would hold if Penrose’s 1969 cosmic censorship
conjecture [55] were valid in some form. According to this vaguely worded
conjecture, in the framework of a “realistic” theory of matter the singular-
ities resulting (according to Penrose’s earlier “preclassical” closed trapped
surface theorem [43]) from gravitational collapse should generally be hidden
within the horizon of a black hole with a regular exterior. However far from
providing a satisfactory general proof of this conjecture, subsequent work on
the question (of which there has not been as much as would be warranted)
has tended to show that can be valid only if interpreted in a rather restricted
manner. Nevertheless, despite the construction of various more or less artifi-
cial counterexamples by Eardley, Smarr, Christodoulou and others [56] to at
least the broader interpretations of this conjecture, it seems clear that there will remain an extensive range of “realistic” circumstances under which the formation of a regular black hole configuration is after all to be expected.

It remains a controversial question (and in any case one that is beyond the scope of this discussion of pure vacuum equilibrium states) just how broad a range of circumstances can lead to regular black hole formation, and whether or not “naked singularities” can sometimes be formed instead under “realistic” conditions. However that may be, all that is actually needed to establish the relevance of black hole for practical physical purposes (as a crucial test of Einstein’s theory, and assuming the result is positive, as an indispensible branch of astrophysical theory) is the demonstration of effective stability with respect to small perturbations of at least some example. This essential step was first achieved in a mathematically satisfactory manner for the special case of the original prototype black hole solution, namely the Schwarzschild solution, in a crucially important quasi-normal mode analysis [57] by Vishveshwara in 1970. Another important article [58] by Price provided a more detailed account of the rate at which the solution could be expected to tend towards the Schwarzschild form under realistic circumstances as seen from the point of view of an external observer. The work of Vishveshwara and Price put the physical relevance of the subject beyond reasonable doubt by demonstrating that this particular (spherical) example is not just stable in principle but that it will also be stable in the practical sense of tending to its stationary (in this particular case actually static) limit within a timescale that is reasonably short compared with other relevant processes: in the Schwarzschild case the relevant timescale for convergence, at a given order of magnitude of the radial distance from the hole, turns out just to be comparable with the corresponding light crossing timescale.

During the remainder of the “classical phase”, important work [59, 60, 61] by Teukolsky, Press and others made substantial progress towards the confirmation that the Kerr solutions are all similarly stable so long as the specific angular momentum parameter \( a = J/M \) is less than its maximum value, \( a = M \). However the possibility that instability might set in at some intermediate value in the range \( 0 < a < M \) was not conclusively eliminated until much later on, in the “post-classical” era, when (following a deeper study of the problem by Kay and Wald [62]) the question was settled more conclusively by the publication of a powerful new method of analysis developed by Whiting [63].

While this work on the stability question was going on, one of the main
activities characterising the “classical phase” of the subject was the systematic investigation (along lines pioneered \cite{64, 65} by Christodoulou and Ruffini) of the general mechanical laws governing the behaviour of stationary and almost stationary black hole states. Work by a number of people including Hawking, Hartle, Bardeen, and myself \cite{6, 66, 67, 68, 69} (and later, as far as the electromagnetic aspects \cite{8, 70, 71, 72} are concerned, also Znajek and Damour) revealed a strong analogy with the thermodynamical behaviour of a viscous (and electrically resistive) fluid. (Following a boldly imaginative suggestion by Bekenstein \cite{73}, the suspicion that this analogy could be interpreted in terms of a deeper statistical mechanical reality was spectacularly confirmed \cite{24} when Hawking laid the foundations of quantum black hole theory.)

It was the substantial theoretical framework built up in the way during the “classical” phase that decisively confirmed the crucial importance of the equilibrium state problem on which the present article is focussed. Returning to this more specialised topic, I would start by recalling that – as well as the provision of the first convincing demonstration that (contrary to what Israel \cite{23} had at first been incined to suspect) a black hole equilibrium state can indeed be stable – a noteworthy byproduct of Vishveshwara’s epoch making paper \cite{57} was its analysis of stationary as well as dynamical perturbations, which provided evidence favorable to my uniqueness conjecture \cite{1, 2} in the form of a restricted “no hair” theorem to the effect that the only stationary pure vacuum generalisations obtainable from a Schwarzschild black hole by infinitesimal parameter variations are those of the Kerr family (in which relevant small parameter is the angular momentum $J$).

Encouraged by Vishveshwara’s confirmation \cite{57} of the importance of the problem, I immediately undertook the first systematic attempt \cite{5, 8} at verification of my uniqueness conjecture for the $a^2 \leq M^2$ Kerr solutions as stationary black hole states. For the sake of mathematical simplicity I restricted my attention to the case characterised by spherical topology and axial symmetry, conditions that could plausibly be guessed to be mathematically necessary in any case. I also ruled out consideration of conceivable cases in which closed timelike or null lines occur outside the black hole hori-

Within this framework I was able in 1971 to make two decisive steps forward, at least for the generic case for which there is a non zero value of the decay parameter $\kappa$ which (in accordance with the “zeroth” law of
black hole thermodynamics must always be constant over the horizon in a stationary state. The first of these steps was the reduction of the four dimensional vacuum black hole equilibrium problem to a two dimensional non-linear elliptic boundary problem, for which the relevant boundary conditions involve just two free parameters: the outer boundary conditions depend just on the mass $M$ and the inner boundary conditions depend just on the horizon scale parameter, $c$ which is proportional to the product of the decay parameter $\kappa$ with the horizon area $A$. The precise specification of this parameter $c$ (originally denoted by the letter $b$, and commonly denoted in more recent literature by the alternative letter $\mu$) is given by the definition $c = \kappa A/4\pi$, and its value in the particular case of the Kerr black holes is given the formula $c = \sqrt{M^2 - a^2}$ with $a = J/M$.

The second decisive step obtained in 1971 was the demonstration that the two dimensional boundary problem provided by the first step is subject to a “no hair” (i.e. no bifurcation) theorem to the effect that within a continuously differentiable family of solutions (such as the Kerr family) variation between neighbouring members is fully determined just by the corresponding variation of the pair of boundary value parameters, i.e. the solutions belong to disjoint 2-parameter families in which the individual members are fully specified just by the relevant values of $M$ and $c$. The only known example of such a family was the Kerr solution, which of course includes the only spherical limit case, namely that of Schwarzschild. The theorem therefore implied that if, contrary to my conjecture, some other non-Kerr family of solutions existed after all, then it would have the strange property of being unable to be continuously varied to a non-rotating spherical limit. On the basis of experience with other equilibrium problems this strongly suggested that, even if other families did exist, they would be unstable and therefore physically irrelevant, unlike the Kerr solutions which, by Visveshwara’s work were already known to be stable at least in the neighbourhood of the non-rotating limit.

Having drawn the conclusion from this plausible but debateable line of argument that for practical astrophysical purposes a pure vacuum black hole equilibrium could indeed be safely presumed to be described by a Kerr solution, I turned my attention to the problem of generalising this argument from the pure vacuum to the electrovac case. In the degenerate ($\kappa = 0$) case, it had been pointed out by Hartle and Hawking that the Kerr-Newman family did not provide the most general equilibrium solution, due to the existence of counterexamples provided by the Papapetrou-Majumdar solutions, but it remains plausible to conjecture that the most general non-degenerate
solutions are indeed provided by the Kerr-Newman family (whose simple spherical limit is the Reissner-Nordstrom solution). The electrovac generalisation of the first step of my 1971 argument \cite{5} turned out to be obtainable without much difficulty \cite{8}, the only difference being that the ensuing two-dimensional non-linear boundary problem now involved two extra parameters representing electric charge and magnetic monopole. As in the pure vacuum case, an essential trick was the use of a modified Ernst \cite{74} transformation based on the axial Killing vector (instead of the usual time translation generator) so as to obtain a variational formulation for which – assuming *causality* – the action would be *positive definite*. The second step was more difficult: I did not succeed in constructing a suitable electrovac generalisation of the divergence identity – with a rather complicated but (like the action) *positive definite* right hand side – that had enabled me to establish the pure vacuum “no hair” theorem \cite{5} for axisymmetric black holes in 1971, but an electrovac identity of the required – though even more complicated – form was finally obtained by Robinson \cite{10} in 1974.

While this work on the electromagnetic generalisation was going on, a deeper investigation of the underlying assumptions was initiated by Hawking \cite{6,7,9}, who made very important progress towards confirmation of the supposition that the topology would be spherical, and that the geometry would be axisymmetric. The latter was achieved by I call the “strong rigidity” theorem, which was originally advertised \cite{9} as a demonstration that – assuming analyticity – the black hole equilibrium states would indeed have to be axisymmetric (and hence by my earlier “weak rigidity” theorem \cite{66} uniformly rotating) except in the static case, which in the absence of external matter was already known – from the recent completion \cite{75,76} of the program initiated by Israel \cite{4,45} – to be not just axisymmetric but geometrically (not just topologically) spherical.

The claim to have adequately confirmed the property of axisymmetry \cite{9} was however one of several exagerations and overstatements that were too hastily put forward during that exciting “classical” period of breathlessly rapid progress. In reality, all that was mathematically established by the “strong rigidity” theorem was just that in the non axisymmetric case the equilibrium state would have to be “non-rotating” (in the technical sense that is explained in the appendix). The argument to the effect that this implied staticity depended on Hawking’s generalisation \cite{6} of the original Lichnerowicz \cite{77} staticity theorem, which in turn assumed the absence of an “ergosphere” outside the (stationary non-rotating) horizon – a litigious supposition whose purported justification in the non-rotating case was based
on heuristic considerations that have since been recognised to be fundamentally misleading, due to the existence of counterexamples. A satisfactory demonstration that the non-rotating case must after all be static was not obtained until the comparatively recent development (described in next section) of a new and much more effective approach (along lines summarised in the appendix) that was initiated by Wald in the more serene “post classical” era.

A similarly overhasty announcement of my own during the hurry of the “classical phase” was the claim to have obtained an electromagnetic generalisation of Hawking’s Lichnerowicz type of staticity theorem using just the same litigious assumption (which fails anyway for the rotating case) of the absence of an ergosphere, i.e. strict positivity, $V > 0$, of the effective gravitational potential defined as the norm, $V = -k^\mu k_\mu$, of the stationarity generator. What was shown later by a more careful analysis was that (after correction of a sign error in the original version) an even stronger and more highly litigious inequality was in fact required – until it was made finally redundant by the more effective treatment recently developed by Wald and his associates on the lines summarised in the appendix. (In dealing with the related “circularity” theorem, on which the treatment of the stationary case depends, I was more fortunate: my electromagnetic generalisation of Papapetrou’s pure vacuum prototype has stood the test of time).

Among the other noteworthy overstatements from the hastily progressing “classical” period, a particularly relevant example is Wald’s own premature claim (based on what turns out to have been an essentially circular argument) to have gone beyond my 1971 “no hair” theorem to get a more powerful uniqueness theorem of the kind that was not genuinely obtained until Robinson’s 1975 generalisation from infinitesimal to finite differences of the divergence identity I had used. In achieving this ultimate tour de force Robinson effectively strengthened the “no hair” theorem to a complete uniqueness theorem, thereby definitively excluding the – until then conceivable – existence of a presumably unstable non-Kerr branch of topologically spherical axisymmetric causally well behaved black hole solutions.

Having already succeeded in generalising my original infinitesimal divergence identity from the pure vacuum to the electrovac case, Robinson went on to try to find an analogous generalisation of his more powerful finite difference divergence identity from the pure vacuum to the electrovac case. However this turned out to be too difficult, even for him, at least by the unsystematic, trial and error, search strategy that he and I had been
using until then. As I guessed in a subsequent review [72], there was “a deep but essentially simple reason why the identities found so far should exist” and “the generalisation required to tie up the problem completely will not be constructed until after the discovery of such an explanation, which would presumably show one how to construct the required identities directly”. It was only at a later stage, in the “post classical” period that, as Heusler [26] put it “this prediction was shown to be true” when, on the basis of a deeper understanding, such direct construction methods were indeed obtained by Mazur [15, 16] and, independently, using a different (less specialised) approach, by Bunting [13, 14].

4 The post-classical phase (since 1975).

Robinson’s 1975 discovery of the finite-difference divergence identity [1] marked the end of what I call the “classical phase”, whose focal event had been the 1972 Les Houches summer school [5, 6, 50] at which all the various aspects of black hole theory were assembled and treated together for the first and probably the last time. Since 1975 the subject has split into mutually non-interacting branches. On one hand there has been the new subject of quantum black hole theory: the discovery of the Hawking effect [24] aroused interest in the possible occurrence in the early universe of microscopic black holes for which such effects might be important, and this in turn lead to an interest into the conceivable effects (e.g. as potential contributors of black hole “hair”) of various kinds of exotic (e.g. Yang Mill or dilatonic) fields that might have been relevant then. On the other hand, eschewing such rather wild speculations in favor of what more obviously exists in the real world, astrophysicists have been mainly interested in macroscopic black holes (of stellar mass and upwards) for which the only relevant long range interaction fields are still believed to be just gravitation and electromagnetism, but for which local mechanisms such as accreting plasma can produce spectacular effects that are thought to be responsible for many observed phenomena ranging in scale and distance from quasars down to galactic X-ray sources such as Cygnus X-1.

After the development of these disconnected branches, work on the pure vacuum black hole equilibrium problem and its electrovac generalisation proceeded rather slowly. Starting with Bekenstein [79], most quantum black hole theorists were more concerned about generalising the problem to hypothetical fields of various new (e.g. dilatonic types), whereas most astro-
physical black hole theorists were concerned just with accreting matter that could be treated as a small perturbation on a pure vacuum background, whose equilibrium states they supposed to have been definitely established to consist just of the relevant \((a \leq M)\) subfamily. Only a handful of mathematically oriented theorists remained acutely aware that the definitive establishment of this naive supposition was still not complete. Another reason why progress in the theory of vacuum equilibrium states slowed down in the "post classical phase" was that the problems that had been solved in the "classical phase" had of course tended to be those that were easiest.

In so far as the equilibrium problem is concerned, the most salient developments in the earlier "post classical" years were the completion referred to above by Mazur \([13, 16]\) and Bunting \([13, 14]\) of my work \([5, 8]\) and Robinson's \([10, 11]\) on the axisymmetric case, and the completion and streamlining \([12, 22, 78]\) by Robinson, Simon, Bunting and Massood-ul-alam of the work \([4, 13, 73, 70]\) initiated by Israel on the strictly static case.

Unlike the work just cited, which built upwards from the (not always entirely reliable) basis established in the classical period \([5, 6, 8, 9, 75, 76]\), a more recent resurgence of activity \([18, 20, 19, 21, 25, 84]\) – initiated by Wald and continued most recently by Chrusciel – has been more concerned with treating the shaky elements in the foundations of that underlying basis itself. This work has successfully closed an outstanding loophole in the previous line of argument by developing a new and more powerful kind of staticity theorem \([18, 20]\) for "non-rotating" black holes: instead of the litigious assumption of a lower bound on \(V\) (which was needed in the now obsolete Hawking-Lichnerowicz approach) the new theorem depends on the justifiable \([21]\) requirement that there exists a slicing by a maximal (spacelike) hypersurface. It has thereby been possible to provide \([19, 22, 54]\) a much more satisfactorily complete demonstration of what had been rather overconfidently asserted by Hawking and Ellis \([9]\), namely that subject to the assumptions of analyticity and of connectedness and non-degeneracy \((c \neq 0)\) of the horizon, the black hole equilibrium state has to be axisymmetric or static.

In so far as most of the other essential steps referred to above are concerned, introductory presentations of the key technical details are already available elsewhere in surveys such as my 1987 review \([17]\) and the very extensive and up to date treatise that has recently been provided by Heusler \([26]\). However these surveys do not include any description of the technicalities of the new improved variety of staticity theorem \([18, 20]\), whose original presentation \([18, 20]\) was somewhat obscured by extraneous compli-
cations introduced in the (so far unfulfilled) hope of generalising the result to include Yang Mills fields. As an appendix to the present survey, I have therefore provided a brief but self contained account of the way this new kind of staticity theorem is obtained in the simplest case – namely that of a pure (Einstein) vacuum.

5 What remains for the future?

As Chrusciel has emphasised [84], although many of the (declared and hidden) assumptions involved in the work during the “classical phase” have been disposed of, the more recent work on the black hole equilibrium problem is still subject to several important technical restrictions whose treatment remains as a challenge for the future.

One whose treatment should I think be given priority at this stage is the assumption of \textit{analyticity} that has been invoked in all the work on the indispensable “strong rigidity” theorem that is needed to establish axisymmetry. It is to be remarked that if, as in my early work [5], axisymmetry is simply postulated at the outset, then analyticity will be demonstrable as an automatic consequence of the ellipticity of the differential system that is obtained as a result of the “circularity” property that is established by the generalised Papapetrou theorem [80, 5]. What I would guess is that it should be possible (and probably not more difficult than the other steps that have already been achieved) to prove the necessity of analyticity for a vacuum equilibrium state it without assuming axisymmetry.

A more delicate question that remains to be settled is the possibility of equilibrium involving \textit{several disconnected black holes}. As far as the pure vacuum problem is concerned, my conjecture is that such multi black hole solutions do not exist, but they have so far been rigorously excluded only in the strictly static case [83]. The axisymmetric case has recently been studied in some detail [84, 86] by Weinstein (who denotes the horizon scale parameter \( c = \kappa A/4\pi \) by the letter \( \mu \)) but a definitive conclusion has not yet emerged. In the electrovac case the situation is certainly more complicated, since it is known [51] that there are counterexamples in which gravitational attraction is balanced by electrostatic repulsion. It is however to be noticed that the only counterexamples discovered so far, namely those of the Papapetrou-Majumdar family [53, 54] have horizons that are degenerate (in the sense of having a vanishing decay constant \( \kappa \)). It seems reasonable to conjecture that even in the electrovac case there are no non-degenerate multi black hole...
equilibrium states.

This last point leads on to a third major problem that still needs to be dealt with, namely the general treatment of the degenerate ($\kappa = 0$) case. The maximally rotating ($J^2 = M^4$) Kerr solution is still the only known pure vacuum example, and I am still inclined to conjecture that it is unique, but the problem of proving this remains entirely unsolved. As far as the electrovac problem is concerned, the only known examples are those of the Kerr-Newman [52] and Papapetrou-Majumdar [53, 54] families. Recent progress by Heusler [88] has confirmed that the latter (whose equilibrium saturates a Bogomolny type mass limit [89]) are the only strictly static examples, but for the rotating degenerate case the problem remains wide open.

I wish to conclude by drawing attention to another deeper problem that, unlike the three referred to above, has been largely overlooked even by the experts in the field, but that seems to me just as interesting from a purely mathematical point of view, even if its physical relevance is less evident. This fourth problem, is that of solving the black hole equilibrium state problem without invoking the causality axiom on which nearly all the work described above depends (e.g. for obtaining the required positivity in the successive divergence identities [5, 10, 11, 13, 15, 16, 14] used in the axisymmetric case). As remarked in the introduction, all non static Kerr solutions contain closed timelike lines, though in the black hole subfamily with $J^2 \leq M^4$ they are entirely confined inside the horizon [2, 27]. Unlike analyticity, whose failure in shock type phenomena is physically familiar in many contexts, causality – meaning the absence of closed timelike lines – is a requirement that most physicists would be prepared to take for granted as an indispensable requirement for realism in any classical field model. However the example of sphalerons suggests that despite their unacceptability at a classical level, the mathematical existence of stationary black hole states with closed timelike lines outside the horizon might have physically relevant implications in quantum theory. The discovery of such exotic configurations would be a surprise to most of us, but would not contradict any theorem obtained so far. All that can be confidently asserted at this stage is that such configurations could not be static but would have to be of rotating type.
Appendix: the new staticity theorem.

In view of the importance of the new kind of staticity theorem developed [18, 20] by Sudarsky and Wald (superceding the Lichnerowicz kind, whose adaptation to the black hole context was inadequate in the pure vacuum case dealt with by Hawking [9], and even less satisfactory in the electromagnetic case dealt by myself [8, 17]), this appendix presents a brief but self contained summary of the essential ideas. The Sudarsky Wald approach works perfectly well for the electrovac case (though it does seem to have trouble with Yang Mills fields) but in order to display the key points in the simplest possible form the description below is limited to the case of a pure (Einstein) vacuum.

It is first to be recalled that the vacuum black hole equilibrium configurations under consideration belong to the more general category of equilibrium configurations, including those of isolated states of self gravitating bodies such as neutron star models, that are invariant under the action of a time translation group whose generator \(k^\mu\) is timelike at least at large sufficiently large distance in the asymptotically flat outer region – where it can be taken to be normalised so that its magnitude tends to unity at large distance. Any such configuration will of course have a well defined asymptotic mass, \(M\) say, given by a formula of the standard Komar form

\[
M = \frac{1}{4\pi} \int_{S_\infty} k^{\mu\nu} \, dS_{\mu\nu},
\]  
(1)

(using a semi-colon to indicate covariant differentiation) where the integral is taken over a surrounding topologically spherical spacelike 2-surface \(S_\infty\) whose choice is arbitrarily adjustable without affecting the result provided it is taken sufficiently far out to be entirely in the vacuum region where the source free Einstein equations are satisfied. In the pure vacuum black hole case with which we are concerned here, the analogous integral defining the black hole mass contribution

\[
M_H = \frac{1}{4\pi} \int_{S_H} k^{\mu\nu} \, dS_{\mu\nu},
\]  
(2)

in terms of any spacelike 2-surface \(S_H\) on the horizon will give the same result: the vanishing of the Riemann tensor \(R_{\mu\nu}\) in conjunction with the Killing equation

\[
k_{(\mu;\nu)} = 0
\]  
(3)
(using round brackets for ansymmetrisation) ensures that divergence condition \( k^{\mu \nu} ; \nu = 0 \) is satisfied all the way in to the horizon, with the implication that that \( M_H = M \).

According to Hawking’s “strong rigidity theorem” ([9]) (which depends on the not yet satisfactorily justified analyticity postulate ([25]) discussed above) the null tangent vector of the horizon will be normalisable in such a way as to coincide with a “corotating” Killing vector field \( \ell^\mu \) given by a formula of the standard form

\[
\ell^\mu = k^\mu + \Omega_H h^\mu,
\]

where \( \Omega_H \) is a uniform (“rigid”) angular velocity and \( h^\mu \) is indeterminate if \( \Omega_H = 0 \) (i.e. in the “non-rotating” case) and otherwise is a well-defined (axisymmetry) Killing vector – with circular trajectories such that the correspondingly normalised angle parameter has period \( 2\pi \). (If, instead of assuming analyticity, one assumes the existence of the axisymmetry generated by \( h^\mu \) then the formula (4) is very easily derivable by my “weak rigidity” theorem ([66]).) The scale constant \( c \) of the horizon is defined by an expression of the same form as that for the mass but with the original (asymptotically timelike) Killing vector \( k^\mu \) replaced by the new the Killing vector combination \( \ell^\mu \) that is null on the horizon, i.e. it is specified by

\[
c = \frac{1}{4\pi} \int_H \ell_{\mu \nu} dS_{\mu \nu},
\]

In terms of the acceleration parameter \( \kappa \) given by \( 2\kappa^2 = \ell_{\mu \nu} \ell_{\nu \mu} \) which (like \( \Omega_H \)) must be uniform over the horizon by the “zeroth” law ([69]), the scale constant works out locally as

\[
c = \frac{\kappa \mathcal{A}}{4\pi},
\]

where \( \mathcal{A} \) is the horizon area. By the “rigidity” formula (4) (whether obtained from the “weak theorem” ([66]) assuming axisymmetry, or from the “strong theorem” ([9]) assuming analyticity) it immediately follows that the scale parameter will be expressible in terms of globally defined quantities via the Smarr type relation

\[
c = M_H - 2\Omega_H J_H,
\]

where \( M_H \) is the black hole mass contribution as defined above and \( J_H \) is the corresponding black hole angular momentum contribution,

\[
J_H = \frac{1}{16\pi} \int_H h_{\mu \nu} dS_{\mu \nu},
\]
which will be the same as the total angular momentum

\[ J = -\frac{1}{16\pi} \oint_{\infty} h^{\mu\nu} dS_{\mu\nu}, \tag{9} \]

in the pure vacuum case considered here, i.e. we shall have \( J_H = J \).

The new idea in the relatively recent work of Wald and Sudarsky \cite{18, 20} is to compare the long well known formula (7) that has just been recapitulated, which in the pure vacuum case under consideration here is evidently equivalent to the simple global mass formula

\[ M = c + 2\Omega H J, \tag{10} \]

with a mass formula of the Arnowitt Deser Misner kind, which involves integration over a spacelike 3-surface \( \Sigma \) say. The trick used by Wald and Sudarsky was to choose the hypersurface \( \Sigma \) to be maximal – a restriction that has been confirmed to be imposable without loss of generality by Wald and Chrusciel \cite{19}. This means that its second fundamental form, as expressed (using Latin letters for internal coordinate indices on the hypersurface) by \( K_{ij} \) should be trace free, i.e. \( K_i^i = 0 \) (using the induced 3-metric. i.e. the first fundamental form, for index raising). In the vacuum case this reduces the A.D.M. formula to the simple form

\[ M = \frac{1}{4\pi} \int_{\Sigma} K^{ij}K_{ij}\lambda d\Sigma + \frac{1}{4\pi} \oint_{\mathcal{H}} \lambda_i d\mathcal{S}_i, \tag{11} \]

where the surface field \( \lambda \) is given in terms of the unit normal \( n^\mu \) to \( \Sigma \) by \( \lambda = -k^\mu n_\mu \) (which will be positive). Since the axisymmetry generator \( h^\mu \) will be tangential to a section \( \Sigma \) that is maximal, one will have \( h^\mu n_\mu = 0 \), so that the “rigidity” formula (4) will simply give \( \lambda = -\ell^\mu n_\mu \) on the horizon. The boundary 2-surface contribution from the horizon can thus be evaluated as

\[ \frac{1}{4\pi} \oint_{\mathcal{H}} \lambda_i d\mathcal{S}_i = c, \tag{12} \]

where \( c \) is the scale parameter as defined by (3).

Identifying the output of this A.D.M. type mass formula (12) with that of our older Smarr type formula (10), one obtains a relationship expressible (in the pure vacuum case under consideration here) by

\[ M - c = 2\Omega H J = \frac{1}{4\pi} \int_{\Sigma} K^{ij}K_{ij}\lambda d\Sigma. \tag{13} \]
The manifest non-negativity of the integrand on the right hand side of the Wald Sudarsky identity (13) evidently entails that in the non rotating case the second fundamental form must vanish, i.e.

\[ \Omega^H = 0 \Rightarrow K_{ij} = 0. \]  

(14)

Having thus established the extrinsic flatness of the maximal hypersurface for the case of a stationary black hole with non-rotating horizon, one can straightforwardly proceed to show that as a consequence the vector \( t^\mu \) defined by

\[ t^\mu = \lambda n^\mu \]  

(15)

will automatically satisfy a Killing equation \( t_{(\mu;\nu)} = 0 \) of the same form as the one (3) satisfied by the original time translation generator, with which it will therefore be identifiable, i.e. one obtains

\[ t^\mu = k^\mu. \]  

(16)

Since \( t^\mu \) is hypersurface orthogonal by its construction (15), the desired staticity theorem is thereby established: it has been shown that the vanishing of the black hole angular velocity \( \Omega^H \) is sufficient by itself (without the need to postulate a questionable lower limit on the magnitude \( V = -k^\mu k_\mu \) as in the older treatment) to ensure hypersurface orthogonality of the time translation symmetry generator \( k^\mu \).

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