DESCRIBING TROPICAL CURVES VIA ALGEBRAIC GEOMETRY

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Abstract. In this paper, we give a general deformation theoretical set up for the problem of the correspondence between tropical curves and holomorphic curves. Using this formulation, the correspondence theorem for non-superabundant tropical curves is naturally solved. This formulation also gives an effective combinatorial description of the superabundant tropical curves, which will give the basis for the study of correspondence between tropical curves and holomorphic curves for superabundant cases.

1. Introduction

In this paper and the sequel, we consider the correspondence between tropical curves in real affine spaces and holomorphic curves in toric varieties. This study was initiated by G. Mikhalkin’s celebrated paper \cite{Mikhalkin2004}, in which he proved the correspondence between tropical curves of any genus in $\mathbb{R}^2$, and holomorphic curves in toric surfaces specified by the combinatorial data of the tropical curves. Subsequently, B. Siebert and the author proved the correspondence between rational tropical curves in $\mathbb{R}^n$ and rational curves in $n$-dimensional toric varieties \cite{SiebertNishinou2006}.

Our first result in this paper is the unification and the extension of the above two results. Namely, the correspondence theorem for general non-superabundant tropical curves. This result was first announced by Mikhalkin in his paper \cite[Theorem 1]{Mikhalkin2005}. The terminologies used in the statement are defined or explained in Section 2.

Theorem 1. Let $(\Gamma, h) : \Gamma \to \mathbb{R}^n$ be an immersive tropical curve of genus $g$ which is non-superabundant. Let $X$ be an $n$-dimensional toric variety associated to $(\Gamma, h)$ and $\mathcal{X} \to \mathbb{C}$ be a degeneration of $X$ defined respecting $(\Gamma, h)$. Let $X_0$ be the central fiber of the family $\mathcal{X}$. Then any

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maximally degenerate pre-log curve in $X_0$ of type $(\Gamma, h)$ can be deformed into a holomorphic curve in $X$, and the degrees of freedom to deform tropical and holomorphic curves coincide.

**Remark 2.** The holomorphic curve in $X$ in the statement of the theorem is smooth when $n \geq 3$ and has at most nodal singularity when $n = 2$.

**Remark 3.** (1) From this theorem, we can deduce enumerative results for non-superabundant curves as in [10], introducing incidence conditions, markings of the edges, various weights, etc.. We leave the precise formulation to the interested readers, because it can be performed completely similarly as in [10].

(2) Theorem 1 was proved in the genus 1 case by Tyomkin [12]. Theorem 1 was also recently proved using different methods by Cheung, Fantini, Park and Ulirsch [1].

The proof of Theorem 1 is no less important than the result itself for the study of tropical curves in superabundant cases. Aside from the problem of correspondence between tropical curves and holomorphic curves, the problem of understanding superabundant tropical curves is itself not straightforward. Given a tropical curves, an immediate problems are:

- Determine whether the tropical curve is superabundant or not.
- When the tropical curve is superabundant, calculate the number of parameters of its deformation.

Both problems are not easily solved when one looks only at the tropical curve itself. In the proof of Proposition 31 which plays the key role in the proof of Theorem 1, we develop a new combinatorial method to describe sheaf cohomology groups of holomorphic curves associated to tropical curves. Using this idea, we obtain an effective combinatorial description of superabundancy of tropical curves (Theorem 35), which allows one to solve both of the problems above.

An interesting point is that while usually tropical curves are used to combinatorially understand algebraic curves, here we use algebraic geometry to understand combinatorics of tropical curves. With this description, we can study correspondence theorems for superabundant tropical curves [8]. Since the use of tropical curves has been extended to various cases [6, 7, 8], the result in this paper can also be used to the study of holomorphic curves in these cases.

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2. Preliminary

In this section, we recall and define some notations and notions which are used in this paper.

2.1. Tropical curves. First we recall some definitions about tropical curves, see [4, 10] for more information. Let $\Gamma$ be a weighted, connected finite graph. Its sets of vertices and edges are denoted $\Gamma^0$, $\Gamma^1$, and $w_\Gamma : \Gamma^1 \to \mathbb{N} \setminus \{0\}$ is the weight function. An edge $E \in \Gamma^1$ has adjacent vertices $\partial E = \{V_1, V_2\}$. Let $\Gamma^0_\infty \subset \Gamma^0$ be the set of one-valent vertices. We write $\Gamma = \Gamma \setminus \Gamma^0_\infty$. Noncompact edges of $\Gamma$ are called unbounded edges. Let $\Gamma^1_\infty$ be the set of unbounded edges. Let $\Gamma^0, \Gamma^1, w_\Gamma$ be the sets of vertices and edges of $\Gamma$ and the weight function of $\Gamma$ (induced from $w_\Gamma$ in an obvious way), respectively. Let $N$ be a free abelian group of rank $n \geq 2$ and $N_\mathbb{R} = N \otimes \mathbb{Z} \mathbb{R}$.

Definition 4 ([4, Definition 2.2]). A parametrized tropical curve in $N_\mathbb{R}$ is a proper map $h : \Gamma \to N_\mathbb{R}$ satisfying the following conditions.

(i) For every edge, $E \subset \Gamma$ the restriction $h|_E$ is an embedding with the image $h(E)$ contained in an affine line with rational slope, or $h(E)$ is a point.

(ii) For every vertex $V \in \Gamma^0$, $h(V) \in N_\mathbb{Q}$ and the following balancing condition holds. Let $E_1, \ldots, E_m \in \Gamma^1$ be the edges adjacent to $V$ and let $u_i \in N$ be the primitive integral vector emanating from $h(V)$ in the direction of $h(E_i)$. Then

$$\sum_{j=1}^m w(E_j)u_j = 0.$$  \hfill (1)

Remark 5. In [11], $h|_E$ is assumed to be an embedding (see [11, Definition 1.1]) for every edge $E$. The reason that we adopt the above definition is that those cases appear naturally when we consider superabundant tropical curves. Since $h$ is proper, an unbounded edge is not contracted.

An isomorphism of parametrized tropical curves $h : \Gamma \to \mathbb{R}^n$ and $h' : \Gamma' \to \mathbb{R}^n$ is a homeomorphism $\Phi : \Gamma \to \Gamma'$ respecting the weights such that $h = h' \circ \Phi$. 

in this paper, too. It is a great pleasure for the author to express his gratitude to him. The author was supported by Grant-in-Aid for Young Scientists (No. 19740034).
Definition 6. A tropical curve is an isomorphism class of parametrized tropical curves. A tropical curve is trivalent if any vertex of $\Gamma$ is at most trivalent. The genus of a tropical curve is the first Betti number of $\Gamma$. The set of flags of $\Gamma$ is

$$FT = \{(V, E) \mid V \in \partial E\}.$$

By (i) of Definition 4, we have a map $u : FT \to N$ sending a flag $(V, E)$ to the primitive integral vector $u_{(V, E)} \in N$ emanating from $h(V)$ in the direction of $h(E)$ or zero.

Definition 7. The combinatorial type of a tropical curve $(\Gamma, h)$ is the graph $\Gamma$ together with the map $u : FT \to N$. We write this by the pair $(\Gamma, u)$.

Remark 8. In [4], for edges contracted by the map $h$, the map $u$ giving the combinatorial type takes the value $0 \in N$, while we allow edges with nonzero value of $u$ to be contracted by the map $h$. This remark is relevant to Corollary [5].

Definition 9. The degree of a type $(\Gamma, u)$ is a function $\Delta : N \setminus \{0\} \to N$ with finite support defined by

$$\Delta(\Gamma, u)(v) := \sharp\{(V, E) \in FT \mid E \in \Gamma[1], w(E)u_{(V, E)} = v\}.$$

Let $e = |\Delta| = \sum_{v \in N \setminus \{0\}} \Delta(v)$. This is the same as the number of unbounded edges of $\Gamma$ (not necessarily of $h(\Gamma)$).

Definition 10. We call a tropical curve $(\Gamma, h)$ immersive if $h$ is an immersion. We call $(\Gamma, h)$ embedded if $h$ is an injection.

Definition 11. (i) An edge $E \in \Gamma[1]$ is called a part of a loop of $\Gamma$ if the graph given by $\Gamma \setminus E^\circ$ has lower first Betti number than $\Gamma$. Here $E^\circ$ is the interior of $E$ (that is, $E^\circ = E \setminus \partial E$).

(ii) The loops of $\Gamma$ is the subgraph of $\Gamma$ composed by the union of parts of a loop of $\Gamma$.

(iii) A bouquet of $\Gamma$ is a connected component of the loops of $\Gamma$. If the first Betti number of a bouquet is one, it is also called a loop.

In particular, a bouquet or a loop does not contain unbounded edges.

Proposition 12 ([4 Proposition 2.13]). Let $(\Gamma, h)$ be a trivalent tropical curve which is immersive. Then the moduli space of trivalent tropical curves of given combinatorial type is an open convex polyhedral domain in a real affine $k$-dimensional space, where

$$k \geq e + (n - 3)(1 - g).$$
In particular, the moduli space in the statement of Proposition 12 is defined by a set of linear inequalities $f_i > 0$. If we allow some edges of $\Gamma$ contracted by the map $h$, some of the inequalities $f_i > 0$ become not strict: $f_i \geq 0$. In particular, for tropical curves of genus 0, we have the following (see Remark 5).

**Corollary 13.** Let $(\Gamma, h)$ be a trivalent tropical curve of genus 0. Then the moduli space of trivalent tropical curves of given combinatorial type is a closed convex polyhedral domain in a real affine $k$-dimensional space, where $$k = e + (n - 3)(1 - g).$$

**Remark 14.**
(1) ‘Closed’ in the statement of the corollary does not mean ‘compact’. Since we can always parallel transport tropical curves, the moduli space is always non-compact.

(2) The inequality in Proposition 12 becomes an equality in Corollary 13 since any genus 0 tropical curve is non-superabundant (see Definition 20).

**Example 15.** Proposition 12 fails to hold for non-immersive higher genus tropical curves. For example, consider an abstract trivalent graph $\Gamma$ which has three unbounded edges $E_1, E_2, E_3$ of weight 1. Assume that the set $\Gamma \setminus \{E_1^\circ, E_2^\circ, E_3^\circ\}$ is a bouquet. The following map $h : \Gamma \to \mathbb{R}^2$ gives a tropical curve.

- $h$ maps the ends of $E_1, E_2, E_3$ to the origin $(0, 0) \in \mathbb{R}^2$.
- $h$ maps the edges $E_1, E_2, E_3$ onto the half lines $\{(x, 0) \mid x \geq 0\}, \{(0, y) \mid y \geq 0\}, \{(x, y) \mid x = y \leq 0\}$, respectively.
- $h$ contracts the other part of $\Gamma$ to $(0, 0) \in \mathbb{R}^2$.

Then it is easy to see that there is no deformation of $(\Gamma, h)$ other than parallel transport. So if the genus of $\Gamma$ is positive and $h$ is not immersive, Proposition 12 does not always hold.

As this example shows, when the map $h$ contracts loops of $\Gamma$, it becomes difficult to give a unified treatment of tropical curves. So we introduce the following assumption. Essentially, it says that an edge which is a part of a loop is not contracted.

**Assumption A**

(i) The abstract graph $\Gamma$ is always trivalent. So $(\Gamma, h)$ is always a trivalent tropical curve in the above terminology, although the image $h(\Gamma)$ may not be trivalent.
(ii) The map $h$ may contract some of the bounded edges of $\Gamma$. However, a contracted edge is not part of a loop.

**Remark 16.** In particular, immersive trivalent tropical curves satisfy Assumption A.

When a tropical curve $(\Gamma, h)$ satisfies Assumption A, we can regard each bouquet as a generalized vertex in view of Proposition 12 and its genus 0 case (Corollary 13) applies. Thus, we have the following.

**Proposition 17.** When a tropical curve $(\Gamma, h)$ satisfies Assumption A, the moduli space of trivalent tropical curves of given combinatorial type is a convex polyhedral domain in a real affine $k$-dimensional space, where

$$k \geq e + (n - 3)(1 - g).$$

\[ \square \]

**Remark 18.** (1) The polyhedral domain in the statement of Proposition 17 is neither closed or open in general, since some of the edges can be contracted while edges in the loop cannot be contracted.

(2) In contrast to Corollary 13, the equality $k = e + (n - 3)(1 - g)$ is replaced by the inequality $k \geq e + (n - 3)(1 - g)$, since the loop parts can cause superabundancy.

Moreover, note that a general member of the moduli space of Corollary 13 is immersive for all $n \geq 2$, and an embedding for all $n \geq 3$. By the remark before Proposition 17, this also applies to a tropical curve satisfying Assumption A:

**Corollary 19.** A general member of the moduli space in the statement of Proposition 17 is immersive for all $n \geq 2$ and an embedding for all $n \geq 3$.

\[ \square \]

Now we define the superabundancy of tropical curves. In view of Example 15, it is reasonable to define it only for curves satisfying Assumption A.

**Definition 20** ([11, Definition 2.22]). A trivalent tropical curve satisfying Assumption A is called superabundant if the moduli space is of dimension larger than $e + (n - 3)(1 - g)$. Otherwise it is called non-superabundant.

By Corollary 19, to see whether a tropical curve satisfying Assumption A of given combinatorial type is superabundant or not, it is enough to check it for an immersive tropical curve obtained by deforming the...
original curve. On the other hand, we will see below that the super-abundancy of an immersive tropical curve can be effectively calculated via algebraic geometry. Now we recall some notions from algebraic geometry relevant to our purpose.

2.2. Toric varieties associated to tropical curves and pre-log curves on them.

**Definition 21.** A toric variety $X$ defined by a fan $\Sigma$ is called associated to a tropical curve $(\Gamma, h)$ if the set of the rays of $\Sigma$ contains the set of the rays spanned by the vectors in the support of the degree map $\Delta : N \setminus \{0\} \to \mathbb{N}$ of $(\Gamma, h)$.

If $E$ is an unbounded edge of $h(\Gamma)$, there is an obvious unique divisor of $X$ corresponding to it. We write it as $D_E$ and call it the divisor associated to the edge $E$.

Given a tropical curve $(\Gamma, h)$ in $\mathbb{N}_R$, we can construct a polyhedral decomposition $\mathcal{P}$ of $\mathbb{N}_R$ such that $h(\Gamma)$ is contained in the 1-skeleton of $\mathcal{P}$ ([10, Proposition 3.9]). Moreover, adding divalent vertices to $\Gamma$, we can assume that $h^{-1}(\mathcal{P}[0]) = \Gamma[0]$. Here $\mathcal{P}[0]$ is the set of the vertices of $\mathcal{P}$.

Given such $\mathcal{P}$, we construct a degenerating family $X \to \mathbb{C}$ of a toric variety $X$ associated to $(\Gamma, h)$ ([10, Section 3]). We call such a family a degeneration of $X$ defined respecting $(\Gamma, h)$. Let $X_0$ be the central fiber. It is a union $X_0 = \bigcup_{v \in \mathcal{P}[0]} X_v$ of toric varieties intersecting along toric strata.

**Definition 22 ([10, Definition 4.1]).** Let $X$ be a toric variety. A holomorphic curve $C \subset X$ is torically transverse if it is disjoint from all toric strata of codimension greater than one. A stable map $\phi : C \to X$ is torically transverse if $\phi^{-1}(\text{int}X) \subset C$ is dense and $\phi(C) \subset X$ is a torically transverse curve. Here $\text{int}X$ is the complement of the union of toric divisors.

**Definition 23.** Let $C_0$ be a prestable curve. A pre-log curve on $X_0$ is a stable map $\varphi_0 : C_0 \to X_0$ with the following properties.

(i) For any $v$, the restriction $C \times_{X_0} X_v \to X_v$ is a torically transverse stable map.

(ii) Let $P \in C_0$ be a point which maps to the singular locus of $X_0$. Then $C$ has a node at $P$, and $\varphi_0$ maps the two branches $(C_0', P), (C_0'', P)$ of $C_0$ at $P$ to different irreducible components $X_{v'}, X_{v''} \subset X_0$. Moreover, if $w'$ is the intersection index of the restriction $(C_0', P) \to (X_{v'}, D')$ with the toric divisor $D' \subset X_{v'}$, and $w''$ accordingly for $(C_0'', P) \to (X_{v''}, D'')$, then $w' = w''$. 
Let $X$ be a toric variety and $D$ be the union of toric divisors. In [10, Definition 5.2], a non-constant torically transverse map $\phi : \mathbb{P}^1 \to X$ is called a line if $\sharp \phi^{-1}(D) \leq 3$. In this case, the image of $\phi$ is contained in the closure of the orbit of at most two dimensional subtorus of acting on $X$ ([10, Lemma 5.2]).

Let $\Gamma$ be a weighted tree with only one vertex $v$ and $h : \Gamma \to \mathbb{N}_\mathbb{R}$ be an immersive tropical curve. Let $\mathcal{E}_1, \ldots, \mathcal{E}_s$ be the edges of $h(\Gamma)$. Let $X$ be a toric variety associated to $(\Gamma, h)$.

**Definition 24.** A non-constant torically transverse map $\phi : \mathbb{P}^1 \to X$ is called type $(\Gamma, h)$, or type $v$ when $h$ is clear from the context, if $\phi$ satisfies the following property:

- Let $\mathcal{E}_i$ be an edge of $h(\Gamma)$ and let $w_i$ be the weight of $\mathcal{E}_i$. Then $\phi(\mathbb{P}^1)$ intersects the divisor $D_{\mathcal{E}_i}$ at a unique point with intersection multiplicity $w_i$.

Let $(\Gamma, h)$ be a tropical curve satisfying Assumption A. Let $X$ be a toric variety associated to $(\Gamma, h)$ and $X \to \mathbb{C}$ be a degeneration of $X$ defined respecting $(\Gamma, h)$. Let $X_0$ be the central fiber.

**Definition 25.** A pre-log curve $\varphi_0 : C_0 \to X_0$ is called of type $(\Gamma, h)$ if for any $v \in \Gamma^{[0]}$, the restriction of $C_0 \times_{X_0} X_{h(v)} \to X_{h(v)}$ to the component of $C_0$ corresponding to $v$ is a rational curve of type $v$.

**Definition 26 ([10, Definition 5.2]).** We assume $n \geq 3$. A pre-log curve $\varphi_0 : C_0 \to X_0$ is maximally degenerate if it is of type $(\Gamma, h)$ for some embedded tropical curve $(\Gamma, h)$.

Recall that when $n \geq 3$, a general tropical curve satisfying Assumption A is embedded. Since we tropical curves satisfying Assumption A are trivalent, the condition that $(\Gamma, h)$ is an embedding implies that the above definition is the same as the one in [10].

**Definition 27.** A trivalent embedded tropical curve $(\Gamma, h)$ is smoothable if there is a maximally degenerate curve $\varphi_0 : C_0 \to X_0$ of type $(\Gamma, h)$ with the following property. Namely, there exists a family of stable maps over $\mathbb{C}$ (or an open disk)

$$\Phi : \mathcal{C}/\mathbb{C} \to X/\mathbb{C}$$

such that $\mathcal{C}/\mathbb{C}$ is a flat family of pre-stable curves whose fiber over 0 is isomorphic to $C_0$, and the restriction of $\Phi$ to $C_0$ is a stable map equivalent to $\varphi_0$. We also call such a pre-log curve smoothable.

**Remark 28.** The smoothability of a tropical curve does not depend on the choice of a toric variety $X$ associated to it or a degeneration of $X$ defined respecting the tropical curve.
See [10, Section 5], for more information about lines and maximally degenerate pre-log curves. Given an immersive trivalent tropical curve \((\Gamma, h)\) satisfying Assumption A, we can construct maximally degenerate pre-log curves of type \((\Gamma, h)\) ([10, Proposition 5.7]), and vice versa ([10, Construction 4.4]). The arguments there extend to not necessarily immersive trivalent tropical curves and pre-log curves of type \((\Gamma, h)\), if \((\Gamma, h)\) satisfies Assumption A.

The smoothing of maximally degenerate curves or pre-log curves of type \((\Gamma, h)\) can be studied by log-smooth deformation theory [2, 3]. For informations about log structures relevant to our situation, see [10, Section 7]. We do not repeat it here, because nothing new about log structures is required here, other than those given in [10].

3. Proof of non-superabundant correspondence theorem

The purpose of this section is to give a proof of Theorem 1.

Assumption B. In this section, we assume that a tropical curve \((\Gamma, h)\) is immersive, when \(n = 2\) and an embedding when \(n \geq 3\). We assume this because this suffices for the (generic) enumeration problem for non-superabundant tropical curves (see the paragraph before Definition 11. See also Corollary 19).

Remark 29. Let \((\Gamma, h)\) be a tropical curve and \(X\) be a toric variety associated to it. Let \(\mathfrak{X} \to \mathbb{C}\) be a degeneration of \(X\) respecting \((\Gamma, h)\). Let \(\mathcal{P}\) be a polyhedral decomposition of \(N_\mathbb{R}\) defining \(\mathfrak{X}\). It was necessary to allow divalent vertices to \(\Gamma\) to assure the property \(h^{-1}(\mathcal{P}[0]) = \Gamma[0]\). So if \(\varphi_0 : C_0 \to X_0\) is a pre-log curve of type \((\Gamma, h)\), then there are components of \(C_0\) corresponding to these divalent vertices. However, these components essentially do not play any role in the argument below, so for simplicity we neglect them and consider \(\Gamma\) as if it has only trivalent vertices.

Let \((\Gamma, h)\) be a trivalent non-superabundant tropical curve and let \(\varphi_0 : C_0 \to X_0\) be a maximally degenerate pre-log curve of type \((\Gamma, h)\). We can give it log-structures as in [10, Proposition 7.1]. There are logarithmic tangent sheaves associated to those log-structures, and the tangent space and the obstruction of the deformation of such a curve are calculated in terms of these sheaves.

Suppose that a lift \(\varphi_{k-1} : C_{k-1}/O_{k-1} \to \mathfrak{X}\) of \(\varphi_0\) is constructed. Here \(O_{k-1} = \mathbb{C}[\epsilon]/\epsilon^k\). Then as in the proof of [10, Lemma 7.2], an extension \(C_k/O_k\) of \(C_{k-1}/O_{k-1}\) exists and such extensions are parametrized by the space of extensions of appropriate sheaves.
On the other hand, the obstruction to lift the map $\varphi_{k-1}$ to $C_k$ is
given by the cohomology class $H^1(C_{k-1}, \varphi_{k-1}^* \Theta_{X/C})$, here $\Theta_{X/C}$ is the
logarithmic tangent bundle relative to the base, and we will study this
group in the following subsections. As in usual deformation theory of
smooth varieties, there is a following standard result in the log-smooth
deformation theory [3].

**Proposition 30.** If $H^1(C_0, \varphi_0^* \Theta_{X/C})$ vanishes, then the maximally
degenerate curve $\varphi_0$ is smoothable. □

### 3.1. Two dimensional case.
We begin, as a warm-up, with the two-di-
mensional case which is easier and illustrates the problem to solve. Also,
it will give a simple algebraic geometric proof of a part of Mikhalkin’s
correspondence theorem ([4, Theorem 1]).

The problem is the smoothing of maximally degenerate pre-log curves
in the central fiber $X_0$ to a family of curves in $X$.

The sheaf $\varphi_0^* \Theta_{X/C}$ fits in the exact sequence

$$0 \to \Theta_{C_0/O_0} \to \varphi_0^* \Theta_{X/C} \to \varphi_0^* \Theta_{X/C}/\Theta_{C_0/O_0} \to 0.$$  

Now $\Theta_{X/C} \cong N \otimes_{\mathbb{Z}} \mathcal{O}_X$ and the logarithmic tangent bundle $\Theta_{C_0/O_0}$ has
degree $2 - 2g - e$, where $e$ is the number of unbounded edges of the
tropical curve from which we construct the pre-log curve (so that $C_0$
has $e$ marked points aside from the nodes). So the logarithmic normal
bundle $\varphi_0^* \Theta_{X/C}/\Theta_{C_0/O_0}$ has degree $2g + e - 2$. Then by Serre duality
for nodal curves, one can easily prove

$$H^1(C_0, \varphi_0^* \Theta_{X/C}/\Theta_{C_0/O_0}) = 0$$  

(this is the point where the assumption that $X$ is two dimensional
simplifies the argument).

So we have the surjection

$$H^1(C_0, \Theta_{C_0/O_0}) \to H^1(C_0, \varphi_0^* \Theta_{X/C}).$$

However, $H^1(C_0, \Theta_{C_0/O_0})$ is just the tangent space of the moduli space
of deformations of $C_0$, so the obstruction classes in $H^1(C_0, \varphi_0^* \Theta_{X/C})$ can
be cancelled when we deform the moduli of the domain of the stable
maps. Thus, we can lift $\varphi_0$ to $\varphi_k$ for any $k$ in this situation. Once
the existence of a lift of the map is shown, the remainder of the proof
of [10] applies verbatim, so this (and the results concerning weights
and incidence conditions as Propositions 5.7, 7.1 and 7.3 of [10]) gives
another proof of Mikhalkin’s correspondence theorem for plane closed
curves. □
The principle of the proof is the same for general higher dimensional cases, in so far as the curve is non-superabundant. Namely, we can show that the obstruction classes in \( H^1(C_0, \varphi_0^* \Theta_{X/C}) \) come only from the moduli of the curve itself. We will show this in the next subsection, which gives the proof of Theorem 1.

3.2. General non-superabundant cases. Let us consider the case when the rank of \( N \) is not less than three. We use the same notations as in the previous subsection. The obstruction to lift \( \varphi_0 \) is \( H^1(C_0, \varphi_0^* \Theta_{X/C}) \).

Consider the exact sequence (2) of Subsection 3.1 and the associated cohomology exact sequence. We have

\[
0 \to H^0(C_0, \Theta_{C_0/O_0}) \to H^0(C_0, \varphi_0^* \Theta_{X/C}) \to H^0(C_0, \varphi_0^* \Theta_{X/C}/\Theta_{C_0/O_0}) \to H^1(C_0, \Theta_{C_0/O_0}) \to H^1(C_0, \varphi_0^* \Theta_{X/C}) \to H^1(C_0, \varphi_0^* \Theta_{X/C}/\Theta_{C_0/O_0}) \to 0.
\]

The logarithmic tangent bundle \( \Theta_{C_0/O_0} \) is, when it is restricted to each component of \( C_0 \), isomorphic to \( O_{C_0}(-1) \). So

\[
H^0(C_0, \Theta_{C_0/O_0}) = 0.
\]

We have \( \varphi_0^* \Theta_{X/C} \cong O_{C_0} \otimes \mathbb{Z} N \). So

\[
H^0(C_0, \varphi_0^* \Theta_{X/C}) \cong \mathbb{C} \otimes \mathbb{Z} N.
\]

The cohomology group \( H^1(C_0, \Theta_{C_0/O_0}) \) is the tangent space to the moduli space of the curve \( C_0 \) itself. By Serre duality for nodal curves, the space \( H^1(C_0, \Theta_{C_0/O_0}) \) is isomorphic to the dual of the space \( H^0(C_0, \omega_{C_0} \otimes \Theta_{C_0/O_0}^\vee) \). Here \( \omega_{C_0} \) is the dualizing sheaf, which is isomorphic to the sheaf of 1-forms with logarithmic poles at nodes. So when a component \( \ell_i \) of \( C_0 \) has \( s \) nodes,

\[
\omega_{C_0} \otimes \Theta_{C_0/O_0}^\vee|_{\ell_i} \cong O_{\ell_i}(-1 + s).
\]

To give a section of \( H^0(C_0, \omega_{C_0} \otimes \Theta_{C_0/O_0}^\vee) \), the value of the section on each component must coincide at the nodes. The rank (over \( \mathbb{C} \)) of \( H^0(C_0, \omega_{C_0} \otimes \Theta_{C_0/O_0}^\vee) \) can be easily calculated as follows.

Consider the dual graph of \( C_0 \). Every vertex is trivalent (this is just the graph \( \Gamma \) since by Assumption B, \( h \) is an embedding when rank \( N = n \geq 3 \)).

We first give every vertex three dimensional vector space \( \mathbb{C}^3 \), corresponding to \( 3 = \dim H^0(\mathbb{P}^1, O(2)) \), and consider the space

\[
\prod_{v \in \Gamma^{(0)}} \mathbb{C}^3.
\]

Then every unbounded edge (it does not contribute to \( s \)) as well as inner (in other words, bounded) edge imposes one dimensional linear
conditions to the space $\prod_{v \in \Gamma[0]} \mathbb{C}^3$. This is because an unbounded edge imposes zero on the section at the corresponding marked point, and a bounded edge imposes the matching of the values of the section at the node corresponding to the edge.

The resulting vector space has dimension

$$3v - e_{\text{tot}},$$

here $v$ is the number of the vertices and $e_{\text{tot}}$ is the number of the edges. We write $$e_{\text{tot}} = e + e_{\text{inn}},$$

here $e$ is the number of the unbounded edges and $e_{\text{inn}}$ is the number of the bounded edges.

By Euler’s equality, we have

$$1 - g = v - e_{\text{inn}}.$$ 

On the other hand, since the graph is trivalent,

$$e_{\text{tot}} = 3v - e_{\text{inn}}.$$ 

From these equalities, we have

$$3v - e_{\text{tot}} = e + 3g - 3.$$ 

So

$$\dim H^1(C_0, \Theta_{C_0/O_0}) = e + 3g - 3.$$ 

Now consider $H^1(C_0, \varphi_0^* \Theta_{X/C})$. As above,

$$H^1(C_0, \varphi_0^* \Theta_{X/C}) \simeq H^0(C_0, \omega_{C_0} \otimes N^\vee).$$ 

In this case, if a component $\ell_i$ of $C_0$ has $s$ nodes, then the restriction of $\omega_{C_0} \otimes N^\vee$ will be isomorphic to $O_{\ell_i}(-2 + s) \otimes N^\vee$.

The next is the key to this section. The proof is important as well, because it plays a central role in the description of the superabundant curves in the next section.

**Proposition 31.** $\dim H^1(C_0, \varphi_0^* \Theta_{X/C}) = ng.$

**Proof.** By Serre duality, it suffices to show $\dim H^0(C_0, \omega_{C_0}) = g$. Note that a vertex of a trivalent tropical curve corresponds to a smooth rational curve $\mathbb{P}^1$ with three marked points, which is a component of a maximally degenerate curve.

Let $z$ be an affine coordinate of $\mathbb{C} \subset \mathbb{P}^1$ and let $a, b$ and $c$ be distinct points on $\mathbb{P}^1$. Assume for simplicity that none of $a, b, c$ is $\infty$. Let $\tilde{w}$ be a
sheaf of holomorphic 1-forms allowing logarithmic poles at \( a, b, c \). Then the space of sections \( \Gamma(\tilde{\omega}) \) is a two dimensional vector space spanned by

\[
\sigma = \frac{dz}{(z-a)(z-b)}, \quad \tau = \frac{dz}{(z-a)(z-c)}.
\]

Taking

\[
\frac{dz}{z-a}, \frac{dz}{z-b}, \frac{dz}{z-c}
\]

as frames of \( \tilde{\omega} \) at \( a, b, c \) respectively, the section \( \sigma \) takes values

\[
\frac{1}{a-b}, \frac{1}{b-a}, 0
\]

respectively at \( a, b, c \). Similarly, \( \tau \) takes values

\[
\frac{1}{a-c}, 0, \frac{1}{c-a}
\]

respectively at \( a, b, c \). In other words, the space of sections \( \Gamma(\tilde{\omega}) \) is identified with the subspace of \( \mathbb{C}^3 = \{(v_a, v_b, v_c) \mid v_a, v_b, v_c \in \mathbb{C}\} \) defined by

\[
v_a + v_b + v_c = 0.
\]

Using this convention, we reduce the problem to a combinatorial one. Consider a vertex \( v \) of the corresponding tropical curve \( \Gamma \) and let \( s \) be the number of bounded edges emanating from it as above.

1. When \( s = 1 \), then the space of sections of \( \mathcal{O}_v(2-2 + s) \) is trivial, and we give the value 0 to all the edges emanating from \( v \).
2. When \( s = 2 \), then we give 0 to the unbounded edge and give values \( \pm a \in \mathbb{C} \) to the remaining two edges, respectively.
3. When \( s = 3 \), we give values \( a, b, c \) satisfying \( a + b + c = 0 \) to the edges.

Thus, we give a number to each flag of \( \Gamma \). We say that a numbering is compatible when the sum of the values of the two flags associated to a bounded edge is zero, reflecting the relation of the frames

\[
\frac{dz_1}{z_1} + \frac{dz_2}{z_2} = 0
\]

at a node, here \( z_1, z_2 \) are coordinates of the two branches at the node.

Then \( \dim H^0(C_0, \omega_{C_0}) \) is the number of linearly independent compatible numberings. So our task is reduced to the calculation of the number of linearly independent compatible numberings.

Now we prove the proposition by induction on \( g \). When \( g = 0 \), there is necessarily a component with \( s = 1 \) and so we give the value 0 to the unique bounded edge. We do this for all the vertices with \( s = 1 \). Then remove all the vertices with \( s = 1 \) and the unbounded edges emanating
from them. Now we have another tree and so again there are vertices of \( s = 1 \). Two of the edges emanating from any of them have the value 0, and so the value of the remaining edge must be 0 by the rule. By induction, we have \( \dim H^0(C_0, \omega_{C_0}) = 0 \) in this case.

Now assume that we proved \( \dim H^0(C_0, \omega_{C_0}) = g \) for \( g \leq g_0 - 1 \), with \( g_0 \geq 1 \). Consider a tropical curve \((\Gamma, h)\) of genus \( g_0 \). Since \( h \) is an embedding by Assumption B, we identify \( \Gamma \) and the image \( h(\Gamma) \).

Let \( E \) be an edge which is a part of the loops of \( \Gamma \). Cutting \( E \) at the middle and extending both ends to infinity, we obtain a curve \( \Gamma' \) with genus \( g_0 - 1 \).

By induction hypothesis, there is \( g_0 - 1 \) dimensional freedom to give numbers to the flags of \( \Gamma' \) compatibly. Let \( v, v' \) be the vertices of \( E \) and choose one of the cycles of \( \Gamma \) which contains \( E \). Because \( v, v' \) has at most \( s = 2 \) in \( \Gamma' \), the numbering around \( v, v' \) looks like Figure 1. Here \( c, d, f, g, h, i, j \in \mathbb{C} \), and when \( s = 1 \) at \( v \) or \( v' \), then \( d \) or \( c \) must be 0, respectively.

Now let us return to \( \Gamma \). First let us give a value 0 to the flags \((v, vv')\) and \((v', v'v)\) and give the same values \( \pm c, \pm d, \text{ etc.} \) as \( \Gamma' \) to the remaining flags of \( \Gamma \). This is a compatible numbering of \( \Gamma \). Then give an arbitrary value \( b \) to the flag \((v, v'v')\), and add values \(-b, b, -b, \cdots \) successively to the adjacent flags of the cycle.

These again give compatible numberings of \( \Gamma \), which have one more freedom given by the the value of \( b \), compared to the numberings of \( \Gamma' \).
So we have
\[ \dim H^0(C_0, \omega_{C_0}) \geq g_0. \]

Conversely, assume \( \dim H^0(C_0, \omega_{C_0}) \geq g_0 + 1 \). Let \( \{f_k\} \) be the set of flags of \( \Gamma \) and \( S = \sum \mathbb{C}\langle f_k \rangle \) be the linear space of complex functions on this set. We write elements of \( S \) by \( \sum a_k \langle f_k \rangle \), \( a_k \in \mathbb{C} \). Note that the space \( T \) of compatible numberings is a linear subspace of \( S \). By assumption, this subspace has dimension not less than \( g_0 + 1 \). Choose any flag \( f = f_0 = (v_0, E_0) \) which is a part of some cycle of \( \Gamma \). The hyperplane \( a_0 = 0 \) cuts \( T \) so that the intersection \( U \) is a linear subspace of dimension not less than \( g_0 \). Now cut the edge \( E_0 \) at the middle point, and extend both of the ends to infinity (as in Figure 1). Then we obtain a trivalent graph \( \Gamma' \) of genus \( g_0 - 1 \). It is easy to see that each element of \( U \) gives a compatible numbering of \( \Gamma' \). On the other hand, by induction hypothesis, the dimension of the space of compatible numberings of \( \Gamma' \) is equal to \( g_0 - 1 \). This is a contradiction. So we have \( \dim H^0(C_0, \omega_{C_0}) \leq g_0 \). This proves the proposition.

**Proof of Theorem 1.** Let the dimension of \( H^0(C_0, \varphi_0^* \Theta_{X/C}/\Theta_{C_0/O_0}) \) be \( d_1 \) and the dimension of \( H^1(C_0, \varphi_0^* \Theta_{X/C}/\Theta_{C_0/O_0}) \) be \( d_2 \). As in [10], \( d_1 \) is the same as the dimension of the moduli space of the corresponding tropical curve. By the long exact sequence, we have
\[ d_1 - d_2 = n + (3g - 3 + e) - ng, \]
which is just the expected dimension of the moduli space of the corresponding tropical curve. So the tropical curve is non-superabundant if and only if \( d_2 = 0 \). In this case, the obstruction \( H^1(C_0, \varphi_0^* \Theta_{X/C}) \) is cancelled by the freedom of the moduli of the curve \( H^1(C_0, \Theta_{C_0/O_0}) \). So we can lift \( \varphi_0 \) to \( \varphi_k \) for any \( k \). Having the existence of such a lift, we can apply the proof of [10] verbatim to show that the corresponding pre-log curves actually deform into smooth curves, and that the tropical curves and corresponding holomorphic curves have the same dimensional moduli space. This proves Theorem 1.

**4. Combinatorial description of the dual space of obstructions**

Applying the same type of combinatorics introduced in the proof of Proposition 31, we can analyze \( H^1(C_0, \varphi_0^* \Theta_{X/C}/\Theta_{C_0/O_0}) \). As we saw, if \( H^1(C_0, \varphi_0^* \Theta_{X/C}/\Theta_{C_0/O_0}) \) vanishes, we know that the pre-log curves corresponding to the tropical curve can be smoothed. In this section, we give an effective method to calculate \( H^1(C_0, \varphi_0^* \Theta_{X/C}/\Theta_{C_0/O_0}) \) when it does not vanish (Theorem 35).
4.1. Calculation of superabundancy for embedded tropical curves via algebraic geometry. Since, under Assumption A, superabundancy only occurs when the dimension $n$ of the ambient space is at least 3, we assume $n \geq 3$ hereafter. In this subsection, we assume $(\Gamma, h)$ is an embedded tropical curve. Thus, in this subsection, we identify the graph $\Gamma$ and its image $h(\Gamma)$. We also assume there is no divalent vertex for notational simplicity. However, the result in this section is straightforwardly extended to the case when $(\Gamma, h)$ satisfies Assumption A (see Subsection 4.3), because essentially only the loops of $\Gamma$ affects the calculation, and Assumption A assures that $h$ is immersive on the loops.

By Serre duality, we have

$$H^1(C_0, \varphi^*_0 \Theta_X/C/\Theta_{C_0/O_0}) \cong H^0(C_0, (\varphi^*_0 \Theta_X/C/\Theta_{C_0/O_0})^\vee \otimes \omega_C)^\vee.$$ 

Recall $\Theta_{C_0/O_0} \cong O(-1)$ and $\omega_{C_0} \cong O(-2 + s)$ on each irreducible component of $C_0$, here $s$ is the number of nodes of the component. From this, it is easy to see that when $s = 1$,

$$\Gamma((\varphi^*_0 \Theta_X/C/\Theta_{C_0/O_0})^\vee \otimes \omega_{C_0}) = 0$$

when restricted to that component.

When $s = 2$, we have

$$\Gamma((\varphi^*_0 \Theta_X/C/\Theta_{C_0/O_0})^\vee \otimes \omega_{C_0}) \cong \Gamma((\varphi^*_0 \Theta_X/C/\Theta_{C_0/O_0})^\vee)$$

on the corresponding component. Note that there is a following inclusion:

$$(\varphi^*_0 \Theta_X/C/\Theta_{C_0/O_0})^\vee \subset (\varphi^*_0 \Theta_X/C)^\vee \cong N_C^\vee \otimes O_{C_0}.$$

Let $\ell$ be a component of $C_0$ and let $v$ be the vertex of the tropical curve corresponding to $\ell$. The edges emanating from $v$ span the two dimensional subspace $V_v$ of $N_C$. Then it is clear that $\Gamma((\varphi^*_0 \Theta_X/C/\Theta_{C_0/O_0})^\vee)$ is given by the subspace $V_v^+ \subset N_C^\vee$. Namely, under the convention as in the proof of Proposition 31, it is given by:

(a) Give 0 to the flag $(v, E_0)$, where $E_0$ is the unique unbounded edge emanating from $v$.

(b) Give $\pm \alpha$, where $\alpha \in V_v^+$, to the remaining flags associated to $v$.

Let us consider the case $s = 3$. For simplicity, let us first assume $n = 2$. Let $\ell$ and $v$ be as above. In this case,

$$(\varphi^*_0 \Theta_X/C_0/\Theta_{C_0})^\vee \cong O(1)^\vee \cong O(-1).$$

So $\Gamma((\varphi^*_0 \Theta_X/C_0/\Theta_{C_0/O_0})^\vee \otimes \omega_{C_0})$ is isomorphic to $C$ on $\ell$. On the other hand,

$$\Gamma((\varphi^*_0 \Theta_X/C_0/\Theta_{C_0/O_0})^\vee \otimes \omega_{C_0}) \subset \Gamma((\varphi^*_0 \Theta_X/C)^\vee \otimes \omega_{C_0}) \cong N_C^\vee \otimes C(\sigma, \tau)$$
on this component. Here \( \sigma, \tau \) are base vectors of the space of holomorphic 1-forms on \( \mathbb{P}^1 \) allowing logarithmic poles at three marked points, which we used in the proof of Proposition 31.

Let

\[(la, lb), (mc, md), (-la - mc, -lb - md)\]

be the slopes of the edges of the tropical curve \( \Gamma \) emanating from \( v \). Here \( l, m \in \mathbb{Z}_{>0} \) are weights and \( (a, b), (c, d) \) are primitive integral vectors. Recall that these edges correspond to the intersections of the line in the toric surface (defined by the two dimensional fan given by the tropical curve with one vertex \( v \)) with the toric divisors (see [10, Definition 5.1]). We set an inhomogeneous coordinate \( z \) on the line so that \( (la, lb), (mc, md), (-la - mc, -lb - md) \) correspond to 0, 1 and \( \infty \), respectively. As in the proof of Proposition 31, we can take \( \sigma, \tau \) and local frames of the sheaf at the marked points so that \( \sigma(0) = -\sigma(\infty) = 1, \sigma(1) = 0 \) and \( \tau(0) = 0, \tau(1) = -\tau(\infty) = 1 \).

**Lemma 32.** Let \( u_1, u_2 \) be the generators of \( (\mathbb{C} \cdot (a, b))^\perp, (\mathbb{C} \cdot (c, d))^\perp \) in \( \mathbb{N}_C^\vee \) such that \( u_1((c, d)) = u_2((a, b)) = 1 \). Then the space of sections

\[\Gamma(\varphi_0^* \Theta_X/\mathcal{O}_{C_0}/\mathcal{O}_0)^\vee \otimes \omega_{C_0})\]

is given by

\[\mathbb{C} \cdot \langle lu_1 \sigma - mu_2 \tau \rangle \subset N_C^\vee \otimes (\mathbb{C})\langle \sigma, \tau \rangle.\]

**Proof.** The stalks of \( \Theta_{C_0}/\mathcal{O}_0 \) at 0, 1, \( \infty \) are spanned by \( (la, lb), (mc, md), (-la - mc, -lb - md) \), respectively, considered as subsets of \( \mathbb{N}_C \otimes \mathcal{O}_{C_0} \). Sections of \( H^0(C_0, (\varphi_0^* \Theta_X/\mathcal{O}_{C_0}/\mathcal{O}_0)^\vee \otimes \omega_{C_0}) \) must annihilate these, and this condition determines the mentioned subspace in the statement. \( \square \)

From this lemma, when we represent the sections in \( \Gamma(\varphi_0^* \Theta_X/\mathcal{O}_{C_0}/\mathcal{O}_0)^\vee \otimes \omega_{C_0}) \) by a trivalent vertex with some values given to the flags as in the proof of Proposition 31, we are forced to give the values \( f\mu_1, -f\mu_2 \) and \( f(\mu_1 + \mu_2) \) of \( \mathbb{N}_C^\vee \) to the edges corresponding to 0, 1 and \( \infty \), respectively. Here \( f \) is an element of \( \mathbb{C} \).

From this, one sees that in the general case where \( n \) is not necessarily two, \( \Gamma(\varphi_0^* \Theta_X/\mathcal{O}_{C_0}/\mathcal{O}_0)^\vee \otimes \omega_{C_0}) \) is described as follows.

Namely, fix an inhomogeneous coordinate \( z \) on the rational curve \( \ell \) with three marked points 0, 1, \( \infty \), and take sections \( \sigma, \tau \) of \( \mathcal{O}(1) \) as above. Let \( E_1, E_2 \), and \( E_3 \) be the three edges of \( \Gamma \) emanating from the corresponding vertex \( v \) of \( \Gamma \) and let \( w_1, w_2, w_3 \) be their weights. Let \( n_1, n_2, n_3 \in \mathbb{N} \) be the primitive integral generators of these edges, and \( V_1, V_2, V_3 \subset \mathbb{N}_C^\vee \) be the subspaces which are the annihilators of \( \mathbb{C} \cdot n_1, \mathbb{C} \cdot n_2 \) and \( \mathbb{C} \cdot n_3 \), respectively. Then, one sees the following.
Lemma 33. The space $H^0(\ell, (\varphi_0^*\Theta_X/C/\Theta_{C_0}/O_0)^{\vee} \otimes \omega_{C_0})$ is naturally identified with the subspace

$$\{ C(w_1v_1\sigma - w_2v_2\tau) | v_1 \in V_1, v_2 \in V_2, v_1(n_2) = v_2(n_1) = 1 \}$$

of $N_C^X \otimes C(\sigma, \tau)$.

Note that elements in

$$\{ C(w_1v_1\sigma - w_2v_2\tau) | v_1 \in V_1, v_2 \in V_2, v_1(n_2) = v_2(n_1) = 1 \}$$

automatically annihilate $C \cdot n_3$ at $z = \infty$, since $w_3n_3 = -w_1n_1 - w_2n_2$. Also note that the sum of the values of a section of $H^0(\ell, (\varphi_0^*\Theta_X/C/\Theta_{C_0}/O_0)^{\vee} \otimes \omega_{C_0})$ at $0, 1, \infty$ is $0 \in N_C^X$.

Using these results, we can combinatorially describe the space of obstructions. As in the proof of Proposition 31, we give values to the flags of $\Gamma$ and impose compatibility conditions to the bounded edges. But this time, the value is in $N_C^X$, not just a complex number. Let $L = \cup_i L_i$ be the loops of $\Gamma$ (Definition 11), where $L_i$ are connected components. This is a closed subgraph of $\Gamma$. Let $\Gamma_T = \Gamma \setminus L$. A connected component of $\Gamma_T$ is a tree. There are two types of these connected components, namely:

(U) The component contains only one flag whose vertex is contained in a loop.

(B) Otherwise.

By inductive argument, it is easy to see that all the flags in a component of type (U) must have the value zero, including the unique flag whose vertex is contained in a loop. For the type (B) too, we have the following result.

Lemma 34. All the flags of a component of type (B), including the flags whose vertices are contained in the loops, must have the value $0 \in N_C^X$.

Proof. Note that $\Gamma$ can be written in the following form (Figure 2).

Here, each shaded disk corresponds to some component $L_i$ of the loops of $\Gamma$. By definition of $\{ L_i \}$, if we regard these disks as vertices, we obtain another tree $\Gamma'$.

Recall the above remark that all the edges contained in the components of type (U) have the value zero. In Figure 2, this means that all the edges (outside the shaded disks) except the ones labeled by $a, b, c, \ldots, m$ have the value zero. We call the edges outside the colored disks as the bridges.

Now, by the fact that $\Gamma'$ is a tree, it is easy to see that there is a shaded disk such that the bridges emanating from it have the value zero except one bridge. Let us call this bridge $r$ and call the remaining bridges as
Figure 2.

\(a_1, \ldots, a_s\). By the condition that the sum of the values of the three edges emanating from each vertex (of original \(\Gamma\)) is zero, we see that the sum of the values attached to \(a_1, \ldots, a_s, r\) is zero. Since \(a_1, \ldots, a_s\) have value zero, it follows that \(r\) has also the value zero. By induction on the number of colored disks, we see that all the bridges, and so all the edges of the components of type (B) also have the value zero. □

According to this lemma, we only need to consider the flags contained in some bouquet (i.e., a connected component of \(L\)). Let \(L_i\) be a bouquet. This is a graph with bivalent and trivalent vertices. Every trivalent vertex \(v\) of \(L_i\) determines a two dimensional subspace of \(N_C\) spanned by the edges emanating from it. We write it by \(V_v\) as before.

Also, every edge \(E\) of \(L_i\) determines a one dimensional subspace of \(N_C\).

Let us describe the space \(H = H^1(C_0, \varphi_0^*(\Theta_{X/C}/\Theta_{C_0/O_0})^\vee\). Let \(\{v_i\}\) be the set of trivalent vertices of \(L\). Cutting \(L\) at each \(v_i\), we obtain a set of piecewise linear segments \(\{l_m\}\). Let \(U_m\) be the linear subspace of \(N_C\) spanned by the direction vectors of the segments of \(l_m\). The following theorem follows from the argument so far.

**Theorem 35.** Let \((\Gamma, h)\) be a trivalent embedded tropical curve. Elements of the space \(H\) are described by the following procedure.

1. Give the value zero to all the flags not contained in \(L\).
(II) Give a value \( u_m \) in \( (U_m)^\perp \subset (N_C)^\vee \) to each of the flags associated to the edges of \( l_m \).

(III) The data \( \{u_m\} \) give an element of \( H \) if and only if the following conditions are satisfied.

(a) At each vertex \( v \) of \( \Gamma \),

\[
u_1 + u_2 + u_3 = 0
\]

holds as an element of \( (N_C)^\vee \). Here \( u_1, u_2, u_3 \) are the data attached to the three flags in \( \Gamma \) which have \( v \) as the vertex.

(b) The data \( \{u_m\} \) is compatible on each edge of \( l_m \), in the sense that the sum of the values attached to the two flags of any edge of \( l_m \) is zero.

\[\square\]

Theorem 36. Let \( (\Gamma, h) \) be a trivalent embedded tropical curve. Then the number of parameters to deform it is given by

\[(n - 3)(1 - g) + e + \dim H,\]

here \( H \) is the space in Theorem \[22\].

Proof. As we saw in the proof of Theorem \[1\] the rank \( d_1 \) of the cohomology group \( H^0(C_0, \varphi_0^*\Theta_X/\Theta_{C_0/O_0}) \) is the same as the dimension of the moduli space of the corresponding tropical curve. Also, recall we have the equality (see the last part of the proof of Theorem \[1\])

\[d_1 - d_2 = (n - 3)(1 - g) + e,\]

where \( d_2 \) is the rank of the space \( H \).

\[\square\]

Remark 37. We see that it almost suffices to check the conditions only at the trivalent vertices of \( L \). The conditions (I) and (III) (a) imply that at a divalent vertex of \( L \), the values \( u, u' \) associated to the relevant two flags satisfy \( u + u' = 0 \). Together with the condition (III) (b), we see that on each \( l_m \), the values associated to the flags are unique up to sign.

The following is immediate from this, because when the genus of \( \Gamma \) is one, there is no trivalent vertex in \( L \). See also \[11\].

Corollary 38. When \( \Gamma \) is a tropical curve of genus one, then \( H \cong U^\perp \), here \( U \) is the linear subspace of \( N_C \) spanned by the direction vectors of the segments of the cycle of \( \Gamma \).
4.2. Example. Let us consider genus two immersive tropical curves $\Gamma_1$ and $\Gamma_2$ in $\mathbb{R}^3$ given in Figure 3.

The curve $\Gamma_1$ has six unbounded edges of directions

$\begin{align*}
(1,0,1), & \quad (1,0,-1), \quad (-1,-1,1), \quad (-1,-1,-1), \quad (0,-1,1), \quad (0,-1,-1).
\end{align*}$

The bounded edges are:

- Three parallel vertical edges of direction $(0,0,1)$.
- Three pairs of parallel edges of directions $(1,0,0)$, $(-1,-1,0)$, $(0,1,0)$, respectively.

The curve $\Gamma_2$ is a modification of $\Gamma_1$ at the vertices $a$ and $b$. Namely:

1. Delete the edge $ab$ (as well as the neighboring unbounded edges).
2. Add a pair of parallel unbounded edges of direction $(-1,0,0)$, and a pair of parallel bounded edges of direction $(1,-1,0)$ of the same length.
3. Connect the end points $c,d$ of the bounded edges added in (2) by a segment of direction $(0,0,1)$.
4. Add unbounded edges of direction $(1,-1,1)$, $(1,-1,-1)$ at the vertices $c,d$, respectively.

Using Theorem 35, it is easy to see that $\Gamma_1$ is superabundant, while $\Gamma_2$ is non-superabundant. Namely, the set of piecewise linear segments $\{l_m\}$ of these tropical curves are given by the following three components, respectively (Figure 4).

We write the corresponding linear subspaces of $N_C \cong \mathbb{C}^3$ by $U_{l_1}, U_{l_2}$, etc.. Then, using standard nondegenerate quadratic form on $\mathbb{C}^3$ to
identify it with its dual,
\[(U_l_1)^\perp \cong \mathbb{C} \cdot (1, 0, 0), \quad (U_l_2)^\perp \cong \mathbb{C} \cdot (0, 1, 0), \quad (U_l_3)^\perp \cong \mathbb{C} \cdot (1, 1, 0).\]

Then it is easy to see that the space $H$ for $\Gamma_1$ is a one dimensional vector space. Thus, $\Gamma_1$ is superabundant.

On the other hand, since $U_{l_1} \cong \mathbb{C}^3$, $(U_{l_1})^\perp = \{0\}$. From this, it is easy to see that the space $H$ for $\Gamma_2$ is $\{0\}$. Therefore, $\Gamma_2$ is non-superabundant.

4.3. Generalization. In this subsection, we extend Theorem 36 to tropical curves which are not necessarily embedded, but satisfy Assumption A. In fact, the modification is quite straightforward. We assume the dimension $n$ of the ambient space is at least 3 as in Subsection 4.1.

Let $(\Gamma, h)$ be a tropical curve satisfying Assumption A, with a given combinatorial type. By Corollary 19, $(\Gamma, h)$ can be deformed into an embedded tropical curve $(\Gamma, h')$. Since the moduli space of tropical curves of a given combinatorial type satisfying Assumption A is a convex polyhedral domain (Proposition 17), the number of parameters to deform $(\Gamma, h')$ is the same as those to deform $(\Gamma, h)$.

On the other hand, we saw in Theorem 36 that the number of parameters to deform $(\Gamma, h')$ can be calculated from the combinatorics of the tropical curve. Note that the calculation there only make use of the loop part of $\Gamma$, so we can do the same calculation for $(\Gamma, h)$, since the map $h$, which may not an embedding but only satisfies Assumption A, does not contract edges in the loop part of $\Gamma$. Thus, we have the following.

**Theorem 39.** Let $(\Gamma, h)$ be a tropical curve satisfying Assumption A. Then the number of parameters to deform it is given by
\[(n - 3)(1 - g) + e + \dim H,
where the dimension of the space $H$ is calculated as in Theorem 36. □
References

[1] Cheung, M-W., Fantini, L., Park, J. and Ulirsch, M. Faithful realizability of tropical curves. Preprint. [arXiv:1410.4152]

[2] Kato, F., Log smooth deformation theory. Tohoku Math. J. (2) 48 (1996), no. 3, 317–354.

[3] Kato, K., Logarithmic structures of Fontaine-Illusie. Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), 191–224, Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[4] Mikhalkin, G., Enumerative tropical algebraic geometry in $\mathbb{R}^2$. J. Amer. Math. Soc. 18 (2005), no. 2, 313–377

[5] Mikhalkin, G., Tropical geometry and its applications. Preprint.

[6] Nishinou, T., Disc counting on toric varieties via tropical curves, Amer. J. Math. 134 (2012), 1423–1472.

[7] Nishinou, T., Toric degenerations, tropical curves and Gromov-Witten invariants of Fano manifolds. To appear in Canad. J. Math.

[8] Nishinou, T., Correspondence theorems for tropical curves. Preprint.

[9] Nishinou, T., Counting curves via degeneration. Preprint.

[10] Nishinou, T. and Siebert, B., Toric degenerations of toric varieties and tropical curves. Duke Math. J. 135 (2006), no. 1, 1–51.

[11] Speyer, D., Parameterizing tropical curves, I: Curves of genus zero and one. Algebra and Number Theory 8 (2014), 963–998.

[12] Tyomkin, I Tropical geometry and correspondence theorems via toric stacks. Math. Ann., Vol 553, Issue 3 (2012), 945-995.

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