On Liouville type theorems in the stationary non-Newtonian fluids

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Abstract

In this paper we prove a Liouville type theorem for the stationary equations of a non-Newtonian fluid in $\mathbb{R}^3$ with the viscous part of the stress tensor $A_p(u) = \text{div}(|D(u)|^{p-2}D(u))$, where $D(u) = \frac{1}{2}(\nabla u + (\nabla u)^\top)$ and $\frac{9}{5} < p < 3$. We consider a weak solution $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^3)$ and its potential function $V = (V_{ij}) \in W^{2,p}_{\text{loc}}(\mathbb{R}^3)$, i.e. $\nabla \cdot V = u$. We show that there exists a constant $s_0 = s_0(p)$ such that if the $L^s$ mean oscillation of $V$ for $s > s_0$ satisfies a certain growth condition at infinity, then the velocity field vanishes. Our result includes the previous results [5, 6] as particular cases.

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1 Introduction

We consider a power law model of non-Newtonian fluid in $\mathbb{R}^3$

\[
\begin{aligned}
&-A_p(u) + (u \cdot \nabla)u = -\nabla \pi \quad \text{in} \quad \mathbb{R}^3, \\
&\text{div } u = 0,
\end{aligned}
\]

where $u = (u_1(x), u_2(x), u_3(x))$ is the velocity field, $\pi = \pi(x)$ is the pressure field. The diffusion term is represented by

$A_p(u) = \text{div}(|D(u)|^{p-2}D(u))$, \quad $1 < p < +\infty$

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and the deviatoric stress tensor is interpreted as $|D|^{p-2}D = \sigma(D)$, where $D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is the symmetric gradient. If $2 < p < +\infty$, the equations describe shear thickening fluids, of which viscosity increases along with shear rate $|D(u)|$. If $1 < p < 2$, shear thinning fluids satisfy them. In the case of $p = 2$, (1.1) corresponds to the usual stationary Navier-Stokes equations which represent Newtonian fluids. We refer to Wilkinson [1] for continuum mechanical background of the above system.

The Liouville problem for the stationary Navier-Stokes equations (Galdi [4], Remark X. 9.4, pp. 729) has attracted considerable attention in the mathematical fluid mechanics. Though it is still open, there are positive answers under additional conditions (see [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]). And as a generalization, Liouville type theorems for non-Newtonian fluids have been investigated (see [18, 5]).

Let $u \in L^1_{loc}(\mathbb{R}^3)$ be a vector field, and let $V = (V_{ij}) \in L^1_{loc}(\mathbb{R}^3; \mathbb{R}^3 \times \mathbb{R}^3)$ be a matrix valued function satisfying $\text{div } V = u$ in the distributional sense. In [6] Chae and Wolf proved Liouville type theorem for the stationary Navier-Stokes equations when the following is assumed

\[
\left( \int_{B(r)} |V - V_{B(r)}|^s dx \right)^\frac{1}{s} \leq C r^{\min\{\frac{3}{2} - \frac{3}{s}, \frac{3}{p} - \frac{3}{s}\}} \quad \forall 1 < r < +\infty
\]

for some $3 < s < +\infty$. They also considered it in (1.1) when $\frac{5}{3} < p < 3$ but only for $s = \frac{3p}{2p-3}$ in [5]. We generalize these results.

As is well known, weak solutions are actually smooth for $p = 2$. Otherwise there is only partial regularity of weak solutions [2, 3]. In this paper, we consider weak solutions, which is defined as follows:

**Definition 1.1.** Let $\frac{5}{3} < p < 3$. A function $u \in W^{1,p}_{loc}(\mathbb{R}^3)$ is called a weak solution to (1.1) if

\[
\int_{\mathbb{R}^3} (|D(u)|^{p-2}D(u) - u \otimes u) : D(\varphi) dx = 0
\]

is fulfilled for all vector fields $\varphi \in C^\infty_c(\mathbb{R}^3)$ with $\text{div } \varphi = 0$.

**Remark 1.2.** Let $u$ be a weak solution. Then, there exists $\pi \in L^{\frac{p}{p-1}}_{loc}(\mathbb{R}^3)$ satisfying

\[
\int_{B(r)} |\pi - \pi_{B(r)}|^s dx \leq C \int_{B(r)} ||D(u)|^{p-2}D(u) - u \otimes u|^s dx, \quad \forall 0 < r < +\infty
\]

for all $\frac{3}{2} \leq s \leq \frac{p}{p-1}$. And $(u, \pi)$ holds

\[
\int_{\mathbb{R}^3} (|D(u)|^{p-2}D(u) - u \otimes u) : D(\varphi) dx = \int_{\mathbb{R}^3} \pi \text{div } \varphi dx
\]

for any vector field $\varphi \in W^{1,p}(\mathbb{R}^3)$ with compact support. Hence, (1.4) replaces (1.2). We refer to [2] for a brief explanation.
Remark 1.3. Let \( (u, \pi) \) be a weak solution and \( \phi \in C^\infty_c(\mathbb{R}^3) \). If we take \( \varphi = u\phi \), then \((1.4)\) with \( \varphi \in W^{1, p}(\mathbb{R}^3) \) yields the local energy equality
\[
\int_{\mathbb{R}^3} |D(u)|^p \phi dx = - \int_{\mathbb{R}^3} |D(u)|^{p-2} D(u) : u \otimes \nabla \phi dx + \int_{\mathbb{R}^3} \left( \frac{1}{2} |u|^2 + \pi \right) u \cdot \nabla \phi dx.
\]

(1.5)

Theorem 1.4. Let \( \frac{9}{5} < p < 3 \), and \( \frac{3}{2} < s < +\infty \) satisfy
\[
s \geq \frac{3p}{2(2p-3)}, \quad s > \frac{9-3p}{2p-3}.
\]

Let \( (u, \pi) \in W^{1, p}_{\text{loc}}(\mathbb{R}^3) \times L^\infty_{\text{loc}}(\mathbb{R}^3) \) be a weak solution to \((1.1)\). We set
\[
\alpha(p, s) := \min \left\{ \frac{1}{3} - \frac{5p-9}{s(2p-3)}, \frac{3}{p} - \frac{4}{3} \right\}.
\]

If we assume there exists a potential \( V \in W^{2, p}_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^{3 \times 3}) \) such that
\[
\left( \int_{B(r)} |V - V_{B(r)}|^s dx \right)^{\frac{1}{s}} \leq CR^{\alpha(p, s)} \quad \forall 1 < r < +\infty,
\]
then \( u \equiv 0 \).

Remark 1.5. If \( s = \frac{3p}{2p-3} \), then we obtain \( \alpha = \frac{9-4p}{3p} \), which consistent with Theorem 1.3 (ii) in \([5]\).

Remark 1.6. In the case of \( p = 2 \), we have \( 3 < s < +\infty \) and \( \alpha = \min \{ \frac{s}{3} - 1, \frac{s}{6} \} \), corresponding to Theorem 1.1 in \([7]\).

We denote by \( C(p, s) = C \) a generic constant that may vary from line to line.

2 Caccioppoli type inequalities

To prove Theorem 1.4 we need the following two lemmas.

Lemma 2.1. Let \( \frac{9}{5} < p < 3 \). Let \( u \in W^{1, p}_{\text{loc}}(\mathbb{R}^3) \) and \( 1 \leq R < +\infty \). For \( 0 < \rho < R \), we let \( \psi \in C^\infty_c(B(R)) \) satisfy \( 0 \leq \psi \leq 1 \) and \( |\nabla \psi| \leq C(R - \rho)^{-1} \). If we assume there exists \( V \in W^{2, p}_{\text{loc}}(\mathbb{R}^3) \) such that \( \text{div} \, V = u \) and \((1.6)\) for some \( \frac{3p}{2(2p-3)} \leq s \leq \frac{3p}{2p-3} \), then for \( s \geq p \)
\[
\int_{B(R)} |\psi^p u|^p dx \leq CR^{\frac{3}{2} + \frac{6}{p} - \frac{p(5p-9)}{2(2p-3)}} \|u\|_{L^p}^p + C(R - \rho)^{-p} R^{3 + \frac{6}{3} - \frac{p(5p-9)}{2(2p-3)}},
\]
and for \( s < p \)
\[
\int_{B(R)} |\psi^p u|^p dx \leq CR^{\frac{p^2}{2p-3}} \|\nabla u\|_{L^p}^{p(3p-3s+3p)} + C(R - \rho)^{-p} R^{\frac{3}{2} + \frac{3}{2} - \frac{5}{2}(2p-3)(p+3s-3p)} \|\nabla u\|_{L^p}^p + C(R - \rho)^{-p} R^{\frac{3}{2} + \frac{3p^2}{2(2p-3)}} \|\nabla u\|_{L^p}^{p(3p-3s+3p)} + C(R - \rho)^{-p} R^{\frac{3}{2} + \frac{3p^2}{2(2p-3)}}.
\]
Proof. We first let $\frac{9}{5} < p < 2$. Because $s \geq \frac{3p}{2(2p-3)} \geq p$, we show (2.7) in this case. Hölder’s inequality implies that

$$
\int_{B(R)} |\psi^p u|^p \, dx \leq |B(R)|^{1 - \frac{p}{2}} \left( \int_{B(R)} |\psi^p u|^2 \, dx \right)^{\frac{p}{2}} = CR^{\frac{3p}{2}} \left( \int_{B(R)} |u|^2 \psi^{2p} \, dx \right)^{\frac{p}{2}}.
$$

(2.9)

Recalling $u = \text{div} \, V$ and using integration by parts, we have

$$
\int_{B(R)} |u|^2 \psi^{2p} \, dx = \int_{B(R)} \partial_i \left( V_{ij} - (V_{ij})_{B(R)} \right) u_j \psi^{2p} \, dx
$$

$$
= - \int_{B(R)} \left( V_{ij} - (V_{ij})_{B(R)} \right) \partial_i u_j \psi^{2p} \, dx - 2p \int_{B(R)} \left( V_{ij} - (V_{ij})_{B(R)} \right) u_j \psi^{2p-1} \partial_i \psi \, dx
$$

$$
\leq \int_{B(R)} |V - V_{B(R)}| \|\psi \nabla u|\psi^{2p-1} \, dx + 2p \int_{B(R)} |V - V_{B(R)}| \|\psi^p u|\nabla \psi|\psi^{p-1} \, dx
$$

$$
\leq \int_{B(R)} |V - V_{B(R)}| \|\psi \nabla u| \, dx + C(R - \rho)^{-1} \int_{B(R)} |V - V_{B(R)}| \|\psi^p u| \, dx.
$$

Note that $\frac{3p}{2(2p-3)} \leq s \leq \frac{3p}{2p-3}$ implies

$$
\frac{1}{s} + \frac{1}{p} < 1
$$

and

$$
\alpha = \frac{1}{3} - \frac{5p - 9}{s(2p - 3)}.
$$

Using Hölder’s inequality and (1.6), we obtain

$$
\int_{B(R)} |u|^2 \psi^{2p} \, dx \leq \left( \int_{B(R)} |V - V_{B(R)}|^s \, dx \right)^{\frac{1}{s}} \|\psi \nabla u\|_{L^p} \|B(R)|^{1 - \frac{1}{p}}
$$

$$
+ C(R - \rho)^{-1} \left( \int_{B(R)} |V - V_{B(R)}|^s \, dx \right)^{\frac{1}{s}} \|\psi^p u\|_{L^p} |B(R)|^{1 - \frac{1}{p}}
$$

$$
\leq CR^{\frac{10}{s} - \frac{5p - 9}{s(2p - 3)} - \frac{p}{2}} \|\psi \nabla u\|_{L^p} + C(R - \rho)^{-1} R^{\frac{10}{s} - \frac{5p - 9}{s(2p - 3)} - \frac{p}{2}} \|\psi^p u\|_{L^p}.
$$

Inserting this inequality into (2.9) and applying Young’s inequality, we have

$$
\int_{B(R)} |\psi^p u|^p \, dx \leq CR^{\frac{3}{2} + \frac{9}{6} - \frac{p(5p - 9)}{2(2p - 3)}} \|\psi \nabla u\|_{L^p}^{\frac{p}{2}} + C(R - \rho)^{-p} R^{\frac{3}{2} + \frac{9}{6} - \frac{p(5p - 9)}{2(2p - 3)}} \|\psi^p u\|_{L^p}^{\frac{p}{2}}
$$

$$
\leq CR^{\frac{3}{2} + \frac{9}{6} - \frac{p(5p - 9)}{2(2p - 3)}} \|\psi \nabla u\|_{L^p}^{\frac{p}{2}} + C(R - \rho)^{-p} R^{\frac{3}{2} + \frac{9}{6} - \frac{p(5p - 9)}{2(2p - 3)}} + \frac{1}{2} \int_{B(R)} |\psi^p u|^p \, dx,
$$

which implies (2.7).
Now, we let $2 \leq p < 3$. Using $u = \text{div} \mathbf{v}$, integration by parts and Hölder’s inequality, we find
\[
\int_{B(R)} |\psi^p u|^p \, dx = \int_{B(R)} \partial_i (\mathbf{V}_{ij} - (\mathbf{V}_{ij})_{B(R)}) u_j |u|^{p-2} \psi^2 \, dx \\
= - \int_{B(R)} (\mathbf{V}_{ij} - (\mathbf{V}_{ij})_{B(R)}) \left( \partial_i u_j |u|^{p-2} + (p-2) u_j u_k \partial_i u_k |u|^{p-4} \right) \psi^2 \, dx \\
- p^2 \int_{B(R)} (\mathbf{V}_{ij} - (\mathbf{V}_{ij})_{B(R)}) u_j |u|^{p-2} \psi^{2-1} \partial_i \psi \, dx \\
\leq C \int_{B(R)} |\mathbf{V} - \mathbf{V}_{B(R)}| |\nabla \psi u| |\psi^{p+1} u|^{p-2} \, dx \\
+ C(R - \rho)^{-1} \int_{B(R)} |\mathbf{V} - \mathbf{V}_{B(R)}| |\psi^{p+1} u| \, dx \\
\leq C \|\mathbf{V} - \mathbf{V}_{B(R)}\|_{L^s(B(R))} \|\nabla \psi u\|_{L^p} \|\psi^{p+1} u\|_{L^{p-2}}^{p-2} \frac{p sp}{s p - s - p} \\
+ C(R - \rho)^{-1} \|\mathbf{V} - \mathbf{V}_{B(R)}\|_{L^s(B(R))} \|\psi^{p+1} u\|_{L^{p-1}} \frac{1}{2} = I + II.
\]
We assume $s \geq p$ first. Since
\[
1 \leq \frac{sp}{sp - s - p} \leq \frac{p}{p - 2}
\]
by applying Hölder’s inequality to $I$ we have
\[
I \leq C \|\mathbf{V} - \mathbf{V}_{B(R)}\|_{L^s(B(R))} \|\nabla \psi u\|_{L^p} \|\psi^{p+1} u\|_{L^{p-2}}^{p-2} |B(R)|^{\frac{1}{2} - \frac{1}{5}}.
\]
Subsequently, we use (1.6) and Young’s inequality. This yields
\[
I \leq CR^\frac{3}{p} \left( \int_{B(R)} |\mathbf{V} - \mathbf{V}_{B(R)}| \, dx \right)^\frac{1}{2} \|\nabla \psi u\|_{L^p} \|\psi^{p+1} u\|_{L^{p-2}}^{p-2} \\
\leq CR^\frac{3}{p} \frac{1}{2} \frac{sp - s - p}{2s(2p - 3)} \|\nabla \psi u\|_{L^p} \|\psi^{p+1} u\|_{L^{p-2}}^{p-2} \\
\leq CR^\frac{3}{p} \frac{p}{6} \frac{p(5p - 9)}{2s(2p - 3)} \|\nabla \psi u\|_{L^p} \|\psi^{p+1} u\|_{L^{p-1}} \frac{1}{4} \int |\psi^p u|^p \, dx.
\] (2.10)
Arguing similarly to the above, having
\[
1 < \frac{s}{s - 1} \leq \frac{p}{p - 1},
\]
we can calculate $II$ as follows
\[
II \leq C(R - \rho)^{-1} \|\mathbf{V} - \mathbf{V}_{B(R)}\|_{L^s(B(R))} \|\psi^{p+1} u\|_{L^{p-1}} |B(R)|^{\frac{1}{2} - \frac{1}{5}} \\
\leq C(R - \rho)^{-1} R^\frac{3}{p} \left( \int_{B(R)} |\mathbf{V} - \mathbf{V}_{B(R)}| \, dx \right)^\frac{1}{2} \|\psi^{p+1} u\|_{L^{p-1}}
\]

\[ \leq C(R - \rho)^{-1} R_p^{\frac{3}{2} + \frac{1}{3} - \frac{5p - 9}{2(2p - 3)}} \|\psi^p u\|_{L_p}^{p - 1} \]
\[ \leq C(R - \rho)^{-p} R_p^{3 + \frac{3}{2} - \frac{5p(5p - 9)}{2(2p - 3)}} + \frac{1}{4} \int |\psi^p u|^p dx. \quad (2.11) \]

(2.10) and (2.11) show (2.7).

Now we assume \( s < p \). Since it holds
\[ \frac{p}{p - 2} < \frac{sp}{sp - s - p} < \frac{3p}{3 - p} < \frac{3p}{(3 - p)(p - 2)}, \]
the standard interpolation inequality implies that
\[ \|\psi^{p+1} u\|^{p-2}_{L^{sp-3p+sp}} \leq \|\psi^{p+1} u\|^{p-2}_{L^{sp-3p}} \leq \|\psi^{p+1} u\|^{p-2}_{L^{sp}} \]
\[ = \|\psi^{p+1} u\|^{sp}_{L^{sp}} \leq C \left( \|\psi^{p+1} u\|_{L^p} + (R - \rho)^{-1} \|\psi^p u\|_{L^p} \right) \]
\[ \leq C \|\psi^p u\|_{L^p}^{sp(p-2) - 3p + 3s} \|\psi^{p+1} u\|_{L^p}^{sp(p-2) - 3p + 3s} + C(R - \rho)^{-3p - 3s} \|\psi^p u\|_{L^p}^{p-2}. \]

Thus, by using Sobolev inequality here, we arrive at
\[ \|\psi^{p+1} u\|^{p-2}_{L^{sp-3p+sp}} \leq C \left( \|\psi^{p+1} u\|_{L^p} + (R - \rho)^{-1} \|\psi^p u\|_{L^p} \right) \]
\[ \leq C \|\psi^p u\|_{L^p}^{sp(p-2) - 3p + 3s} \|\psi^{p+1} u\|_{L^p}^{sp(p-2) - 3p + 3s} + C(R - \rho)^{-3p - 3s} \|\psi^p u\|_{L^p}^{p-2}. \]

Inserting the inequality we just obtained into \( I \) and applying (1.6) along with Young’s inequality, we find
\[ I \leq CR_{\frac{3}{2}} \left( \int_{B(R)} |V - V_{B(R)}|^s dx \right)^{\frac{1}{s}} \|\psi^p u\|_{L^p}^{sp + 3p - 3s} + \|\psi^{p+1} u\|_{L^p}^{sp(p-2) - 3p + 3s} \]
\[ + C(R - \rho)^{3p - 3s} R_{\frac{3}{2}} \left( \int_{B(R)} |V - V_{B(R)}|^s dx \right)^{\frac{1}{s}} \|\psi^p u\|_{L^p}^{sp(p-2) - 3p + 3s} \]
\[ \leq CR_{\frac{3}{2}} \left( \int_{B(R)} |V - V_{B(R)}|^s dx \right)^{\frac{1}{s}} \|\psi^p u\|_{L^p}^{sp + 3p - 3s} + \|\psi^{p+1} u\|_{L^p}^{sp(p-2) - 3p + 3s} \]
\[ + C(R - \rho)^{3p - 3s} \|\psi^p u\|_{L^p}^{sp(p-2) - 3p + 3s} \|\psi^p u\|_{L^p}^{p-2} \]
\[ \leq CR_{\frac{3}{2}} \|\psi^p u\|_{L^p}^{p(p+1)-3p + 3s} + C(R - \rho)^{\frac{3p^2}{2} + \frac{3p^2}{2} + \left( \frac{5p - 9}{2(2p - 3)} \right) \|\psi^p u\|_{L^p}^2 \]
\[ + \frac{1}{4} \int |\psi^p u|^p dx. \quad (2.12) \]

We note that \( s < p \) implies
\[ \frac{p}{p - 1} < \frac{s}{s - 1} < \frac{3p}{2(3 - p)} < \frac{3p}{(3 - p)(p - 1)}. \]

Thus, using Interpolation inequality and Sobolev inequality, we infer
\[ \|\psi^{p+1} u\|^{p-1}_{L^{sp-3p+sp}} \leq \|\psi^{p+1} u\|^{p-1}_{L^{sp-3p}} \|\psi^{p+1} u\|^{p-1}_{L^{sp}} \]
\[ + \frac{1}{4} \int |\psi^p u|^p dx. \]

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\[
\psi \in W^{1,p}(\mathbb{R}^n) \quad \text{for} \quad V \in W^{2,p}(\mathbb{R}^n) \quad \text{such that} \quad \rho \leq \psi \leq R \quad \text{and} \quad |\nabla \psi| \leq C(R - \rho)^{-1}.
\]

Inserting this inequality into II and using (1.6) and Young's inequality, it follows

\[
II \leq C(R - \rho)^{-1} R^\frac{2}{3} \left( \int_{B(R)} |V - V_{B(R)}|^s \, dx \right)^\frac{1}{s} \left\| \psi \nabla u \right\|_{L^p}^{\frac{3p - 3s}{sp}} \left\| \psi u \right\|_{L^p}^{\frac{sp(1 - 3s)}{sp}}
+ C(R - \rho)^{-1} \left( \int_{B(R)} |V - V_{B(R)}|^s \, dx \right)^\frac{1}{s} \left\| \psi \nabla u \right\|_{L^p}^{\frac{3p - 3s}{sp}} \left\| \psi u \right\|_{L^p}^{\frac{sp(1 - 3s)}{sp}}
\]

By (2.12) and (2.13) we complete the proof. \qed

Lemma 2.2. Let \( \frac{2}{3} < p < 3 \). Let \( u \in W^{1,p}_{loc}(\mathbb{R}^3) \) and \( 1 \leq R < +\infty \). For \( 0 < \rho < R \), we let \( \psi \in C^\infty_c(B(R)) \) satisfy \( 0 \leq \psi \leq 1 \) and \( |\nabla \psi| \leq C(R - \rho)^{-1} \). If we assume there exists \( V \in W^{2,p}_{loc}(\mathbb{R}^3) \) such that \( \text{div} \ V = u \) and (1.6) for some \( \frac{3p}{2(2p - 3)} \leq s \leq \frac{3p}{2p - 3} \), then for \( s \geq 3 \)

\[
\frac{9p}{2p + 3p - 3s} 2p + 3p - 3s \frac{9p}{2p + 3p - 3s} + C(R - \rho)^{-1} \left\| \psi \nabla u \right\|_{L^p}^{\frac{9p}{2p + 3p - 3s}} + C(R - \rho)^{-1} \left\| \psi \nabla u \right\|_{L^p}^{\frac{9p}{2p + 3p - 3s}}
\]

and for \( s < 3 \)

\[
\frac{9p}{2p + 3p - 3s} 3s(2p - 3) 2p + 3p - 3s \frac{9p}{2p + 3p - 3s} + C(R - \rho)^{-1} \left\| \psi \nabla u \right\|_{L^p}^{\frac{9p}{2p + 3p - 3s}} + C(R - \rho)^{-1} \left\| \psi \nabla u \right\|_{L^p}^{\frac{9p}{2p + 3p - 3s}}
\]

and for \( s < 3 \)

\[
\frac{9p}{2p + 3p - 3s} 3s(2p - 3) 2p + 3p - 3s \frac{9p}{2p + 3p - 3s} + C(R - \rho)^{-1} \left\| \psi \nabla u \right\|_{L^p}^{\frac{9p}{2p + 3p - 3s}} + C(R - \rho)^{-1} \left\| \psi \nabla u \right\|_{L^p}^{\frac{9p}{2p + 3p - 3s}}
\]

\[
\left\| \psi \nabla u \right\|_{L^p}^{\frac{9p}{2p + 3p - 3s}} + C(R - \rho)^{-1} \left\| \psi \nabla u \right\|_{L^p}^{\frac{9p}{2p + 3p - 3s}}
\]

(2.14)
Proof. Arguing similarly to the proof of Lemma 2.1, we get

\[
\int_{B(R)} |\psi^3 u|^3 \, dx \leq C \|V - V_{B(R)}\|_{L^s(B(R))} \|\psi \nabla u\|_{L^p} \|\psi^4 u\|_{L^{sp-s-p}}^{sp} + C(R - \rho)^{-1} \|V - V_{B(R)}\|_{L^s(B(R))} \|\psi^4 u\|_{L^{\frac{sp-s-p}{3}}}^2 = I + II.
\]

Firstly, we estimate \(I\). Since

\[
3 \leq \frac{sp}{sp - s - p} \leq \frac{3p}{3 - p},
\]

Gagliardo–Nirenberg interpolation inequality and Hölder’s inequality imply that

\[
\|\psi^4 u\|_{L^{sp-s-p}} \leq C \left( \|\psi^4 \nabla u\|_{L^p} + (R - \rho)^{-1} \|\psi^3 u\|_{L^p} \right)^{\frac{3p}{3p - 2sp}} \|\psi^4 u\|_{L^3}^{\frac{4sp - 3p - 6s}{3(2p - s)}} \leq C \|\psi \nabla u\|_{L^p}^{\frac{3p}{3p - 2sp}} \|\psi^3 u\|_{L^3}^{\frac{4sp - 3p - 6s}{3(2p - s)}} + C \left( (R - \rho)^{-1} \|\psi^p u\|_{L^p} \right)^{\frac{3p}{3p - 2sp} \|\psi^3 u\|_{L^3}^{\frac{4sp - 3p - 6s}{3(2p - s)}}}.
\]

We insert it into \(I\) and use (1.6), and then we apply Young’s inequality twice. This yields

\[
I \leq CR^\frac{3}{2} + \frac{\frac{3p}{3p - 2sp}}{\frac{3p}{3p - 2sp} + \frac{3p}{3p - 2sp}} \|\psi \nabla u\|_{L^p}^{\frac{3p}{3p - 2sp}} \|\psi^3 u\|_{L^3}^{\frac{4sp - 3p - 6s}{3(2p - s)}} + C \left( (R - \rho)^{-1} \|\psi^p u\|_{L^p} \right)^{\frac{3p}{3p - 2sp} \|\psi^3 u\|_{L^3}^{\frac{4sp - 3p - 6s}{3(2p - s)}}}
\]

\[
\leq CR\|\psi \nabla u\|_{L^p}^{\frac{9p}{2sp + 3p - 4s}} + C R \|\psi \nabla u\|_{L^p}^{\frac{3p}{2sp + 3p - 4s}} \left( (R - \rho)^{-1} \|\psi^p u\|_{L^p} \right)^{\frac{9p}{2sp + 3p - 4s}} + \frac{1}{4} \int |\psi^3 u|^3 \, dx.
\] (2.16)

Secondly, we estimate \(II\). First, let \(s \geq 3\). If we apply Hölder’s inequality to \(II\) and use (1.6) along with Young’s inequality, it follows

\[
II \leq C(R - \rho)^{-1} R \left( \int_{B(R)} |V - V_{B(R)}|^s \, dx \right)^{\frac{1}{s}} \|\psi^3 u\|_{L^3}^2 \leq C(R - \rho)^{-3} R^{\frac{3s}{2sp - 3p} - \frac{3(2p - s)}{3(2p - s)}} + \frac{1}{4} \int |\psi^3 u|^3 \, dx.
\] (2.17)

(2.16) and (2.17) shows (2.14). Now we let \(s < 3\). Since

\[
\frac{3}{2} < \frac{s}{s - 1} \leq \frac{3p}{6 - p} < \frac{3p}{2(3 - p)},
\]

the interpolation inequality and Sobolev inequality imply that

\[
\|\psi^4 u\|_{L^{\frac{sp-s-p}{3}}} \leq \|\psi^4 u\|_{L^{\frac{sp-s-p}{3}}}^{\frac{p(3-s)}{2(2p-3)}} \|\psi^4 u\|_{L^{\frac{sp-s-p}{3}}}^{\frac{5sp-3p-6s}{2(2p-3)}} \|\psi^4 u\|_{L^2}^4.
\]
By (2.16) and (2.18) we complete the proof.

Similarly as above, we obtain

\[ II \leq C(R - \rho)^{-1} R^{\frac{3}{2} + \frac{3}{2p - 3} - \frac{5p - 9}{2p - 3}} \|\psi\nabla u\|_{L^p} \leq 3 (R - \rho)^{-1} \|\psi^p u\|_{L^p} + \int \frac{1}{4} \left| \psi^3 u \right|^3 dx. \]  

(2.18)

By (2.16) and (2.18) we complete the proof.

3 Proof of Theorem 1.4

We assume all conditions for Theorem 1.4 are fulfilled. Note that in the case of $\frac{3p}{2p - 3} < s < +\infty$, we have by Jensen’s inequality and (1.6)

\[
\left( \frac{1}{|B(r)|} \int_{B(r)} |V - V_{B(r)}|^{\frac{3p}{2p - 3}} dx \right)^{\frac{2p - 3}{3p}} \leq \left( \frac{1}{|B(r)|} \int_{B(r)} |V - V_{B(r)}|^s dx \right)^{\frac{1}{s}} \leq C R^{\frac{3}{2p} - \frac{4}{3}}.
\]

This shows that we can reduce (1.6) to that case of $s = \frac{3p}{2p - 3}$. Hence, in general we may restrict the range of $s$ to

\[
\frac{3p}{2p - 3} \leq s \leq \frac{3p}{2p - 3}, \quad s > \frac{9 - 3p}{2p - 3}.
\]

Let $1 < r < +\infty$ be arbitrarily chosen. We set $r \leq \rho < R \leq 4r$ and $\overline{R} = \frac{R + \rho}{2}$.

The first claim is that

\[
\int_{\mathbb{R}^3} |\nabla u|^p \, dx < +\infty.
\]  

(3.19)

Let $\zeta \in C_c^\infty(B(\overline{R}))$ be a radially non-increasing function such that $0 \leq \zeta \leq 1$, $\zeta = 1$ on $B(\rho)$ and $|\nabla \zeta| \leq C(R - \rho)^{-1}$ for some $C > 0$. If we insert $\phi = \zeta^p$ into (1.25), then we have

\[
\int_{B(\overline{R})} |D(u)|^p \zeta^p \, dx = - \int_{B(\overline{R})} |D(u)|^{p-2} D(u) : u \otimes \nabla \zeta^p \, dx
\]

\[
+ \frac{1}{2} \int_{B(\overline{R})} |u|^2 u \cdot \nabla \zeta^p \, dx + \int_{B(\overline{R})} (\pi - \pi_B(\overline{R})) u \cdot \nabla \zeta^p \, dx.
\]
Hölder’s inequality and Young’s inequality imply that
\[ \int_{B(R)} |\nabla u|^p \gamma^p dx \leq C \int_{B(R)} |u|^p |\nabla \gamma|^p dx + C \int_{B(R)} |u|^3 |\nabla \gamma|^p dx. \]

Employing Calderón-Zygmund’s inequality, we obtain
\[ \int |\nabla (u \gamma)|^p dx \leq C \int |D(u)|^p \gamma^p dx + C \int |u|^p |\nabla \gamma|^p dx. \] (3.20)

Using (3.20) along with (1.3), it follows
\[ \int_{B(\rho)} |\nabla u|^p dx \leq C (R - \rho)^{-p} \int_{B(\rho)} |u|^p dx + C (R - \rho)^{-1} \int_{B(\rho)} |u|^3 dx \]
\[ + C (R - \rho)^{-1} R^{\frac{2}{p} - 1} \left( \int_{B(\rho)} |\nabla u|^p dx \right)^{\frac{2}{p} (1 - \frac{1}{p})}. \]

We consider \( \psi \in C^\infty_c (B(R)) \) a radially non-increasing function satisfying \( 0 \leq \psi \leq 1 \), \( \psi = 1 \) on \( B(R) \) and \( |\nabla \psi| \leq C (R - \rho)^{-1} \) for some \( C > 0 \). By the properties of \( \psi \) we have that
\[ \int_{B(\rho)} |\nabla u|^p dx \leq C (R - \rho)^{-p} \int_{B(\rho)} |\psi u|^p dx + C (R - \rho)^{-1} \int_{B(\rho)} |\psi u|^3 dx \]
\[ + C (R - \rho)^{-1} R^{\frac{2}{p} - 1} \left( \int_{B(\rho)} |\nabla u|^p dx \right)^{\frac{2}{p} (1 - \frac{1}{p})} = I + II + III. \] (3.21)

Let \( \epsilon > 0 \) be an arbitrary real number. Before calculating \( II \), we note that \( \psi \) satisfies the assumptions for Lemma 2.1 and Lemma 2.2. Observing that for \( s > \frac{9 - 3p}{2p - 3} \)
\[ \frac{9p}{2sp + 3p - 3s} < p, \]
we may apply Young’s inequality to (2.14) for \( s \geq 3 \). This yields
\[ II \leq C(\epsilon) (R - \rho)^{- \frac{3(5p - 9)}{s(2p - 3)}} R^{\frac{2p + 3p - 3s}{2p - 3}} + I \]
\[ + C (R - \rho)^{-4} R^{\frac{3(5p - 9)}{s(2p - 3)}} + \epsilon \int_{B(R)} |\psi |^p dx. \]

We continue estimating \( I \). Since
\[ 4 - \frac{3(5p - 9)}{s(2p - 3)} < 4, \]
we get for $R > 1$

$$II \leq C(\epsilon)(R - \rho)^{-\frac{2sp + 3p - 3s}{2sp - 3s + 3p - 9}} R^\frac{2sp + 3p - 3s}{2sp - 3s + 3p - 9} + C(R - \rho)^{-4} R^4 + \epsilon \int_{B(R)} |\psi \nabla u|^p dx.$$ 

In the case of $s < 3$, we see that for $s > \frac{9 - 3p}{2p - 3}$, it holds

$$\frac{3p(3 - s)}{sp - 3s + 3p} < \frac{3p(3 - s)}{s(2p - 3) - 3s + 3p} < p.$$ 

Thus (2.15) and Young’s inequality give

$$II \leq C(\epsilon)(R - \rho)^{-\frac{2sp + 3p - 3s}{2sp - 3s + 3p - 9}} R^\frac{2sp + 3p - 3s}{2sp - 3s + 3p - 9} + I + C(\epsilon)(R - \rho)^{-\frac{7sp + 3p - 12s}{sp + 3p - 9}} R^\frac{2sp + 3p - 3s}{sp + 3p - 9} + \epsilon \int_{B(R)} |\psi \nabla u|^p dx.$$ 

Since

$$\frac{2sp + 3p - 3s}{sp + 3p - 9} < \frac{7sp + 3p - 12s}{sp + 3p - 9},$$

$R > 1$ shows that

$$II \leq C(\epsilon)(R - \rho)^{-\frac{2sp + 3p - 3s}{2sp - 3s + 3p - 9}} R^\frac{2sp + 3p - 3s}{2sp - 3s + 3p - 9} + C(\epsilon)(R - \rho)^{-\frac{7sp + 3p - 12s}{spa + 3p - 9}} R^\frac{7sp + 3p - 12s}{spa + 3p - 9} + \epsilon \int_{B(R)} |\psi \nabla u|^p dx.$$ 

Hence, in both cases we obtain the following estimate

$$II \leq C(\epsilon)(R - \rho)^{-\frac{2sp + 3p - 3s}{2sp - 3s + 3p - 9}} R^\frac{2sp + 3p - 3s}{2sp - 3s + 3p - 9} + C(\epsilon)(R - \rho)^{-\frac{7sp + 3p - 12s}{sp + 3p - 9}} R^\frac{7sp + 3p - 12s}{sp + 3p - 9} + \epsilon \int_{B(R)} |\psi \nabla u|^p dx. \quad (3.22)$$

Now we estimate $I$. If $s \geq p$, using (2.7) and Young’s inequality, we see that

$$I \leq C(\epsilon)(R - \rho)^{-2p} R^{3 + \frac{p(5p - 9)}{s(2p - 3)}} + \epsilon \int_{B(R)} |\psi \nabla u|^p dx.$$ 

Since

$$3 + \frac{p}{3} - \frac{p(5p - 9)}{s(2p - 3)} \leq 6 - \frac{4p}{3},$$

for $s \leq \frac{3p}{2p - 3}$, $R > 1$ implies

$$I \leq C(\epsilon)(R - \rho)^{-2p} R^{6 - \frac{4p}{3}} + \epsilon \int_{B(R)} |\psi \nabla u|^p dx \quad (3.23)$$

$$\leq C(\epsilon)(R - \rho)^{-2p} R^{2p} + \epsilon \int_{B(R)} |\psi \nabla u|^p dx.$$
Notice that for $s < p$,

$$0 < \frac{p(sp - 3s + 3p)}{2sp - 3s + 3p} < p$$

and

$$0 < \frac{p(3p - 3s)}{sp - 3s + 3p} < p.$$  

By applying Young’s inequality to (2.8) we get

$$I \leq C(\epsilon)(R - \rho)^{-2p + 3 - \frac{2p}{3p - 3}} R^\frac{p^2}{3p - 3} + \epsilon \int_{B(R)} |\psi \nabla u|^p dx. \quad (3.24)$$

Since

$$\frac{p}{3} + \frac{p^2}{s(2p - 3)} < 2p - 3 + \frac{3p}{s} \leq 6p - 9 \quad (3.25)$$

for $s \geq \frac{3p}{2(2p - 3)}$, $R > 1$ and $R(R - \rho)^{-1} > 1$ imply that

$$I \leq C(\epsilon)(R - \rho)^{-(6p - 9)} R^{6p - 9} + \epsilon \int_{B(R)} |\psi \nabla u|^p dx. \quad (3.26)$$

Therefore, in each case we obtain

$$I \leq C(\epsilon)(R - \rho)^{-2p} R^{2p} + C(\epsilon)(R - \rho)^{-6p - 9} + \epsilon \int_{B(R)} |\psi \nabla u|^p dx. \quad (3.26)$$

Applying Young’s inequality to $III$, we find

$$III \leq C(\epsilon)(R - \rho)^{-2p} R^{2p} + \epsilon \int_{B(R)} |\psi \nabla u|^p dx. \quad (3.27)$$

By $3 < \frac{2p}{3-p}$ and $R > 1$, it follows that

$$III \leq C(\epsilon)(R - \rho)^{-2p} R^{2p} + \epsilon \int_{B(R)} |\psi \nabla u|^p dx. \quad (3.27)$$

We define

$$\gamma := \max \left\{ \frac{7sp + 3p - 12s}{sp + 3p - 9}, \frac{2sp + 3p - 3s}{2sp - 3s + 3p - 9}, \frac{2p, 6p - 9}{2p, 3 - p} \right\}.$$ 

From (3.22), (3.26), (3.27) and $R(R - \rho)^{-1} > 1$ we deduce that

$$I + II + III \leq C(\epsilon)(R - \rho)^{-\gamma R^\gamma} + \epsilon \int_{B(R)} |\nabla u|^p dx.$$ 

Inserting this estimate into (3.21) and applying the iteration Lemma in [19, Lemma 3.1] for sufficiently small $\epsilon > 0$, we are led to

$$\int_{B(R)} |\nabla u|^p dx \leq C(R - \rho)^{-\gamma R^\gamma}.$$
By taking $R = 2r$, $\rho = r$ and passing $r \to +\infty$, we obtain (3.19).

Secondly we claim that

$$r^{-p} \int_{B(2r) \setminus B(r)} |u|^p \, dx = o(1) \quad \text{as} \quad r \to +\infty. \quad (3.28)$$

We consider a cut-off function $\psi \in C_c^\infty(B(4r) \setminus B(\frac{r}{2}))$ satisfying $0 \leq \psi \leq 1$, $\psi = 1$ on $B(2r) \setminus B(r)$ and $|\nabla \psi| \leq Cr^{-1}$. Then $\psi$ satisfies the assumptions for Lemma 2.1 when $R = 4r$ and $\rho = r$. Hence, in the case of $s \geq p$ we use (3.23) to obtain

$$r^{-p} \int_{B(4r)} |\psi^p u|^p \, dx \leq C r^{\frac{10p-10}{3}} + C \int_{B(4r)} |\psi \nabla u|^p \, dx \leq C r^{\frac{10p-10}{3}} + C \int_{B(4r) \setminus B(\frac{r}{2})} |\nabla u|^p \, dx.$$ 

If $s < p$, using (3.24), we have that

$$r^{-p} \int_{B(4r)} |\psi^p u|^p \, dx \leq C r^{\frac{2sp}{3} + \frac{sp}{2} + \frac{3p}{2} - \frac{3p}{3} + \frac{3p}{3}} + C \int_{B(4r) \setminus B(\frac{r}{2})} |\nabla u|^p \, dx.$$ 

Thus, observing (3.25) and (3.19), we obtain

$$r^{-p} \int_{B(4r)} |\psi^p u|^p \, dx = o(1) \quad \text{as} \quad r \to +\infty \quad (3.29)$$

which implies (3.28).

Next, we claim

$$r^{-1} \int_{B(2r) \setminus B(r)} |u|^3 \, dx = o(1) \quad \text{as} \quad r \to +\infty \quad (3.30)$$

and

$$r^{-1} \int_{B(r)} |u|^3 \, dx = O(1) \quad \text{as} \quad r \to +\infty. \quad (3.31)$$

We set the same function $\psi \in C_c^\infty(B(4r) \setminus B(\frac{r}{2}))$ with $R = 4r$ and $\rho = r$. For $s \geq 3$ we can use (2.14) to infer

$$r^{-1} \int_{B(4r)} |\psi^3 u|^3 \, dx \leq C \left( \int_{B(4r) \setminus B(\frac{r}{2})} |\nabla u|^p \, dx \right)^{\frac{9}{2sp+4p-3s}} + C \left( r^{-p} \int_{B(4r)} |\psi^p u|^p \, dx \right)^{\frac{9}{2sp+4p-3s}} + Cr^{-3(5p-9)/(5p-3s)}.$$ 

In case $s < 3$, (2.15) gives that

$$r^{-1} \int_{B(4r)} |\psi^3 u|^3 \, dx$$
In each case (3.19) and (3.29) imply (3.30).

Let

\[ \text{Proof of Theorem 1.4} \]

We observe (1.5) with \( \psi \in W^{1,p}(B(2r)) \), and apply Hölder’s inequality and Calderón-Zygmund’s inequality to get

\[ \int_{B(r)} |\nabla u|^p \, dx \leq C \int_{B(r)} |u|^p |\nabla \phi|^p \, dx + C \int_{B(r)} |u|^2 |\nabla \phi| \, dx \]

\[ + C \int_{B(r)} |\pi - \pi_{B(2r)}| |u| |\nabla \phi| \, dx \]

\[ \leq Cr^{-p} \int_{B(2r) \setminus B(r)} |u|^p \, dx + Cr^{-1} \int_{B(2r) \setminus B(r)} |u|^3 \, dx \]

\[ + Cr^{-1} \int_{B(2r) \setminus B(r)} |\pi - \pi_{B(2r)}| |u| \, dx = IV + V + VI. \]

The properties (3.28) and (3.30) directly shows that

\[ IV + V \to 0 \quad \text{as} \quad r \to +\infty. \]

To estimate VI we use Hölder’s inequality and (1.3) when \( s = \frac{3}{2} \). This yields

\[ VI \leq C \left( r^{-1} \int_{B(2r)} |\pi - \pi_{B(2r)}|^2 \, dx \right)^{\frac{3}{4}} \left( r^{-1} \int_{B(2r) \setminus B(r)} |u|^3 \, dx \right)^{\frac{1}{4}} \]
\[ \leq C \left( r^{-1} \int_{B(2r)} |\nabla u|^\frac{3(p-1)}{2} \, dx + r^{-1} \int_{B(2r)} |u|^3 \, dx \right) \frac{4}{3} \left( r^{-1} \int_{B(2r) \setminus B(r)} |u|^3 \, dx \right) \frac{1}{3}. \]

According to (3.31), it is sufficient to show that

\[ r^{-1} \int_{B(2r)} |\nabla u|^\frac{3(p-1)}{2} \, dx = O(1) \quad \text{as} \quad r \to +\infty. \]

On the other hand, Hölder’s inequality with \( \frac{3(p-1)}{2} < p \) implies that

\[ r^{-1} \int_{B(2r)} |\nabla u|^\frac{3(p-1)}{2} \, dx \leq r^{-1} |B(2r)|^\frac{3}{2p} \left( \int_{B(2r)} |u|^p \, dx \right) \frac{3(p-1)}{2p} \]

\[ = r^{\frac{3}{2p} - \frac{3}{2}} \left( \int_{B(2r)} |u|^p \, dx \right) \frac{3(p-1)}{2p}. \]

This implies

\[ VI \to 0 \quad \text{as} \quad r \to +\infty \]

and

\[ \int_{B(r)} |\nabla u|^p = o(1) \quad \text{as} \quad r \to +\infty. \]

Accordingly, \( u \equiv \text{const} \) and by means of (3.30), we have that \( u \equiv 0 \).  

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