ON LOWER BOUNDS OF THE SUM OF MULTIGRADED BETTI NUMBERS OF SIMPLICIAL COMPLEXES

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ABSTRACT. We find some general lower bounds of the sum of certain families of multigraded Betti numbers of any simplicial complex with a vertex coloring.

1. INTRODUCTION

An (abstract) simplicial complex on \([m] = \{1, \cdots, m\}\) is a collection \(\mathcal{K}\) of subsets \(\sigma \subseteq [m]\) such that if \(\sigma \in \mathcal{K}\), then any subset of \(\sigma\) also belongs to \(\mathcal{K}\). Here we always assume that the empty set and every one-element subset \(t_j \cup m\) belongs to \(\mathcal{K}\) and refer to \(\sigma\) as an abstract simplex of \(\mathcal{K}\). A simplex \(\sigma \in \mathcal{K}\) of cardinality \(|\sigma| = i + 1\) has dimension \(i\) and is called an \(i\)-face of \(\mathcal{K}\). In particular, any 0-face of \(\mathcal{K}\) is called a vertex of \(\mathcal{K}\). The dimension \(\dim \mathcal{K}\) of \(\mathcal{K}\) is defined to be the maximal dimension of all the faces of \(\mathcal{K}\). For any subset \(\omega \subseteq [m]\), we call \(\mathcal{K}|_\omega = \{\sigma \in \mathcal{K} \mid \sigma \subseteq \omega\}\) the full subcomplex of \(\mathcal{K}\) corresponding to \(\omega\).

Let \(k\) denote an arbitrary field and \(k[v] = k[v_1, \cdots, v_m]\) be the polynomial ring over \(k\) in the \(m\) indeterminates \(v = v_1, \cdots, v_m\). Let \(\mathbb{N} = \{0, 1, 2, \cdots\}\) be the set of nonnegative integers. A monomial in \(k[v]\) is a product \(v^a = v_1^{a_1}v_2^{a_2}\cdots v_m^{a_m}\) for a vector \(a = (a_1, \cdots, a_m) \in \mathbb{N}^m\) of nonnegative integers.

A \(k[v]\)-module \(M\) is called \(\mathbb{N}^m\)-graded if \(M = \bigoplus_{b \in \mathbb{N}^m} M_b\) and \(v^aM_b \subseteq M_{a+b}\). Given a vector \(a \in \mathbb{N}^m\), by \(k[v](-a)\) one denotes the free \(k[v]\)-module generated by an element in degree \(a\). So \(k[v](-a)\) is isomorphic to the ideal \(\langle v^a \rangle \subset k[v]\) as \(\mathbb{N}^m\)-graded modules. Moreover, any free \(\mathbb{N}^m\)-graded module of rank \(r\) is isomorphic to the direct sum \(k[v](-a_1) \oplus \cdots \oplus k[v](-a_r)\) for some vectors \(a_1, \cdots, a_r \in \mathbb{N}^m\).

An ideal \(I \subseteq k[v]\) is called a monomial ideal if it is generated by monomials. A monomial \(v^a\) is called squarefree if every coordinate of \(a\) is 0 or 1. A monomial ideal is called squarefree if it is generated by squarefree monomials.

Clearly, all the elements in \(2^m = \{\omega \mid \omega \subseteq [m]\}\) bijectively correspond to all the vectors in \(\{0, 1\}^m \subset \mathbb{N}^m\) by mapping any \(\omega \in 2^m\) to \(a_\omega \in \{0, 1\}^m\), where the \(j\)-th
coordinate $a_j$ is 1 whenever $j \in \omega$, and is 0 otherwise. With this understanding, we write
\[ v^\omega = v^{a_\omega} = \prod_{j \in \omega} v_j, \omega \subseteq [m] \quad (\text{e.g. } v^{(j)} = v_j). \] (1)

The Stanley-Reisner ideal of a simplicial complex $K$ on $[m]$ is the squarefree monomial ideal $I_K = \langle v^\tau \mid \tau \notin K \rangle$ generated by monomials in $k[v]$ corresponding to nonfaces of $K$. The quotient ring $k(K) = k[v]/I_K$ is called the Stanley-Reisner ring of $K$. By [9, Sect.1.4], $k(K)$ is a finitely generated $\mathbb{N}^m$-graded $k[v]$-module and so $k(K)$ has an $\mathbb{N}^m$-graded minimal free resolution as follows:
\[ 0 \longrightarrow F_{-h} \xrightarrow{\phi_h} F_{-h+1} \longrightarrow \cdots F_{-1} \xrightarrow{\phi_1} F_0 \longrightarrow k(K) \longrightarrow 0 \] (2)
where each $\phi_i$ is a degree-preserving homomorphism. Here we numerate the terms of a free resolution by nonpositive integers in order to view it as a cochain complex. Since each $F_{-i}$ is an $\mathbb{N}^m$-graded free $k[v]$-module, we may write
\[ F_{-i} = \bigoplus_{a \in \mathbb{N}^m} k[v]((-a)^{\beta^{k(K)}_{i,a}}, i \in \mathbb{N}. \]

The integers $\beta^{k(K)}_{i,a}$ are called the multigraded Betti numbers of $K$ with $k$-coefficients. Since the free resolution (2) is minimal, we have $\text{Tor}^k_i(K(K), k) \cong F_{-i}$ and
\[ \beta^{k(K)}_{i,a} = \dim_k \text{Tor}^k_i(K(K), k)_a, a \in \mathbb{N}^m, i \in \mathbb{N}. \] (3)

By [9, Corollary 1.40], $\beta^{k(K)}_{i,a} = 0$ for all $a \notin \{0, 1\}^m$. For brevity, we define
\[ \beta^{k(K)}_{i,\omega} = \beta^{k(K)}_{i,a_\omega}, \omega \in 2^m. \]

In addition, the Hochster’s formula tells us that (see [8] or [9, Corollary 5.12])
\[ \beta^{k(K)}_{i,\omega} = \dim_k \tilde{H}^{\omega-i-1}(K|_\omega; k), i \in \mathbb{N}. \] (4)
where $\tilde{H}^* (K|_\omega; k)$ is the reduced singular (or simplicial) cohomology of $K|_\omega$ with $k$-coefficients. Notice that $\dim (K|_\omega) \leq |\omega| - 1$. So by the Hochster’s formula, the multigraded Betti numbers of $K$ over $k$ are nothing but the usual (reduced) Betti numbers with $k$-coefficients of all the full subcomplexes of $K$.

The multigraded Betti numbers of $K$ are intimately related to a topological space $Z_K$ called the moment-angle complex of $K$. The construction of $Z_K$ first appeared in Davis-Januszkiewicz [9]. Let $D^2 = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$ be the unit 2-disk in $\mathbb{C}$ and $S^1 = \partial D^2$ be the unit circle. By definition, $Z_K$ is a subspace of the Cartesian product of $m$ copies of 2-disks:
\[ Z_K = \bigcup_{\sigma \in K} \left( \prod_{j \in \sigma} D^2_{(j)} \times \prod_{j \notin \sigma} S^1_{(j)} \right) \subseteq \prod_{j=1}^m D^2_{(j)} \] (5)
where $D^2_{(j)}$ is a copy of $D^2$ indexed by $j \in [m]$ and $S^1_{(j)} = \partial D^2_{(j)}$. In addition, we consider the index $j \in [m]$ in (5) to be increasing from the left to the right. People also use $(D^2, S^1)^K$ to denote $\mathbb{Z}_K$ in the literature and call it the polyhedral product of $(D^2, S^1)$ corresponding to $K$ (see [3, Sec.4.2]). It is shown in [1] that

$$H^q(\mathbb{Z}_K; k) \cong \bigoplus_{\omega \in [m]} \tilde{H}^{q-|\omega|-1}(K|\omega; k), \quad q \in \mathbb{N}. \quad (6)$$

So by the Hochster’s formula (4), we have

$$\dim_k H^q(\mathbb{Z}_K; k) = \sum_{\omega \in [m]} \beta_{2|\omega|-q, \omega}^{k(K)}.$$ 

If we consider $S^1$ as a Lie group with respect to the multiplication of complex numbers, there is a canonical action of the torus $T^m = (S^1)^m$ on $\mathbb{Z}_K$ where the $j$-th $S^1$-factor of $T^m$ acts on $D^2_{(j)}$ by multiplication of complex numbers.

The following inequality on the multigraded Betti numbers of $K$ is obtained by Cao-Lü [4] and Ustinovskii [12] independently.

**Theorem 1.1** (Theorem 1.4 in [4], Theorem 3.2 in [12]). For a simplicial complex $K$ on $[m]$ and any field $k$, \( \sum_q \dim_k H^q(\mathbb{Z}_K; k) \geq 2^{m-\dim(K)-1}. \)

By (6), Theorem 1.1 is equivalent to saying that the sum of all the multigraded Betti numbers of $K$ is at least \( 2^{m-\dim(K)-1}. \)

$$\sum_{\omega \in [m]} \sum_i \beta_{i, \omega}^{k(K)} \geq 2^{m-\dim(K)-1}. \quad (7)$$

By Hochster’s formula, the inequality (7) is equivalent to saying that the sum of all the usual Betti numbers of all the full subcomplexes of $K$ is at least \( 2^{m-\dim(K)-1}. \)

$$\sum_{\omega \in [m]} \left( \sum_j \dim_k \tilde{H}^j(K|\omega; k) \right) \geq 2^{m-\dim(K)-1}. \quad (8)$$

The lower bound in (7) is sharp since the equality holds for $\partial \Delta^{n_1} * \cdots * \partial \Delta^{n_s}$ where $\partial \Delta^n$ is the boundary of an $n$-simplex $\Delta^n$ and $*$ is the join of two spaces.

Moreover, in [13] Ustinovskii shows that the inequality (7) can be strengthened to be the following.

**Theorem 1.2** (Corollary 3.6 in [13]). For a simplicial complex $K$ on $[m]$ and any field $k$,

$$\sum_{\omega \in [m]} \beta_{i, \omega}^{k(K)} \geq \binom{m-\dim(K)-1}{i} \text{ for any } i \geq 0.$$ 

The proof of this theorem in [13] uses a result in [7, Corollary 2.5] on the dimensions of the Tor modules of monomial ideals in $k[v]$.

In this paper, we obtain some lower bounds of the sum of certain families of multigraded Betti numbers of $K$, which generalizes Theorem 1.2. To state our main result, let us introduce some notations.
Figure 1.

- Let $\alpha = \{\alpha_1, \cdots, \alpha_r\}$ be a partition of $[m]$ into $r$ subsets, i.e. $\alpha_i$’s are disjoint nonempty subsets of $[m]$ with $\alpha_1 \cup \cdots \cup \alpha_r = [m]$. The partition $\alpha$ is called nondegenerate for $K$ if the two vertices of any 1-simplex of $K$ belong to different $\alpha_i$. For example the trivial partition $\{\{1\}, \cdots, \{m\}\}$ of $[m]$ is nondegenerate for any simplicial complex on $[m]$. We also refer to a nondegenerate partition $\alpha$ of $[m]$ for $K$ as a vertex coloring of $K$, meaning that we put $r$ different colors to all the vertices of $K$ so that the set of vertices with the $i$-th color is $\alpha_i$ and any adjacent vertices in $K$ are assigned different colors.

- Let $[r] = \{1, \cdots, r\}$. One should keep in mind the difference between $[r]$ and the vertex set $[m]$ of $K$. For a subset $L \subseteq [r]$, define
  \[ \omega^\alpha_L = \bigcup_{i \in L} \alpha_i \subseteq [m]. \]
  It is clear that $|\omega^\alpha_L| = \sum_{i \in L} |\alpha_i| \geq |L|$. Then the partition $\alpha$ determines $2^r$ special full subcomplexes $\{K|_{\omega^\alpha_L} | L \subseteq [r]\}$ of $K$ (see Example 1).

- For any simplex $\sigma \in K$, let
  \[ I_\alpha(\sigma) = \{i \in [r] ; \sigma \cap \alpha_i \neq \emptyset\} \subseteq [r]. \]
  In other words, $I_\alpha(\sigma)$ encodes the set of colors of all the vertices of $\sigma$ defined by $\alpha$. Note that $|I_\alpha(\sigma)| \leq |\sigma|$.

**Example 1.** Let $K$ be a simplicial complex on the vertex set $[5] = \{1, 2, 3, 4, 5\}$ with a geometrical realization given by Figure 1. Let $\alpha = \{\{1\}, \{2, 4\}, \{3, 5\}\}$ be a partition of $[5]$, which is nondegenerate for $K$. The geometrical realizations of the eight special full subcomplexes of $K$ determined by $\alpha$ are shown in Figure 2.

The main result of this paper is the following theorem.
Theorem 1.3. Let $\mathcal{K}$ be a simplicial complex on $[m]$ and $k$ be an arbitrary field. If a partition $\alpha = \{\alpha_1, \cdots, \alpha_r\}$ of $[m]$ is nondegenerate for $\mathcal{K}$, then for any $q \geq 0$,

$$
\sum_{L \subseteq [r]} \beta_{q+|\omega_L^\alpha|-|L|,\omega_L^\alpha}^{\mathcal{K}} = \sum_{L \subseteq [r]} \dim_k \widetilde{H}^{(|L|-q-1)(\mathcal{K}|_{\omega_L^\alpha}; k)} \geq \left( r - \dim(\mathcal{K}) - 1 \right). \quad (9)
$$

This implies

$$
\sum_{L \subseteq [r]} \left( \sum_{j=0}^{|L|-1} \dim_k \widetilde{H}^j(\mathcal{K}|_{\omega_L^\alpha}; k) \right) \geq 2^{r-\dim(\mathcal{K})-1}. \quad (10)
$$

Note that when $r = m$ (i.e. $\alpha$ is the trivial partition of $[m]$), Theorem 1.3 gives exactly Theorem 1.2. For nontrivial nondegenerate partitions of $[m]$ for $\mathcal{K}$, Theorem 1.3 will give us some lower bounds of the sum of certain families of multigraded Betti numbers of $\mathcal{K}$. The inequality (10) tells us that the sum of all the usual Betti numbers of the $2^r$ special full subcomplexes of $\mathcal{K}$ determined by $\alpha$ also has a general lower bound. We will prove Theorem 1.3 at the end of the paper. Note that the Hochster’s formula (11) itself sheds no light on how to obtain such type of lower bounds.

Remark 1.4. It is shown in [10] that the existence of a vertex coloring by $r$ colors on a simplicial complex $\mathcal{K}$ is equivalent to some splitting properties of a vector bundle over the Davis-Januszkiewicz space of $\mathcal{K}$.

The paper is organized as follows. In Section 2, we study a quotient space of $Z_{\mathcal{K}}$ by some torus action determined by $\alpha$, whose cohomology groups are related to the multigraded Betti numbers $\beta_{i,\omega_L^\alpha}^{\mathcal{K}}$. In Section 3, we show how the multigraded...
Betti numbers $\beta^{k(\K)}_{i,\omega_\ell^q}$ are related to the Tor module of $k(\K)$ over a polynomial ring $k[u_1, \ldots, u_r]$ where the $k[u_1, \ldots, u_r]$-module structure of $k(\K)$ is determined by the partition $\alpha$. This relation leads to a proof of Theorem 1.3 at the end.

2. A Quotient Space of $Z_K$ Determined by $\alpha$

Suppose $\K$ is a simplicial complex on $[m]$. Let $Z^m = \langle e_1, \ldots, e_m \rangle$ be the canonical unimodular basis of the Lie algebra of $T^m$. For an arbitrary partition $\alpha$ of $[m]$, let $S_\alpha$ be the toral subgroup of $T^m$ corresponding to the subgroup of $Z^m$ generated by the set $\{ e_j - e_{j'} | j, j' \text{ belong to the same $\alpha_i$ for some } 1 \leq i \leq k \}$. It is easy to see that the dimension of $S_\alpha$ is $m - r$. Let $Z_K/S_\alpha$ denote the quotient space of $Z_K$ by $S_\alpha$ through the canonical action of $T^m$. Clearly $T^m/S_\alpha$ is an $r$-dimensional torus which acts on $Z_K/S_\alpha$ through the canonical action of $T^m$.

**Lemma 2.1.** For a partition $\alpha$ of $[m]$, the canonical action of $S_\alpha$ on $Z_K$ is free if and only if $\alpha$ is nondegenerate for $\K$.

**Proof.** Let $v_j$ be the center of the 2-disk $D^2_{(j)}$ in $[3]$. For any $j < j' \in [m]$, let $C_{j,j'}$ denote the circle subgroup of $T^m$ corresponding to the subgroup of $Z^m$ generated by $e_j - e_{j'}$. The fixed point set of the canonical action of $C_{j,j'}$ on $D^2_{(1)} \times \cdots \times D^2_{(m)}$ is $D^2_{(1)} \times \cdots \times D^2_{(j-1)} \times \{ v_j \} \times D^2_{(j+1)} \times \cdots \times D^2_{(j'-1)} \times \{ v_{j'} \} \times D^2_{(j'+1)} \times \cdots \times D^2_{(m)}$. In particular, the action of $C_{j,j'}$ on the component $\prod_{k \in \sigma} D^2_{(k)} \times \prod_{k \notin \sigma} S^1_{(k)} \subseteq Z_K$ has a fixed point if and only if $j$ and $j'$ are both in $\sigma$. Then it is easy to see that the canonical $S_\alpha$-action on $Z_K$ is free if and only if for any simplex $\sigma \in \K$, $|\sigma \cap \alpha_i| \leq 1$ for all $i \in [r]$, which means that $\alpha$ is nondegenerate for $\K$. \qed

For any partition $\alpha$ of $[m]$, it is shown in [14, Theorem 1.2] that there is a group isomorphism

$$H^q(Z_K/S_\alpha; k) \cong \bigoplus_{L \in [r]} \tilde{H}^{q-L-1}(|\K|; k), \forall q \geq 0. \tag{11}$$

So by the Hochster’s formula (4), we have

$$\dim_k H^q(Z_K/S_\alpha; k) = \sum_{L \in [r]} \beta^{k(\K)}_{\omega_\ell^q + |L|-q,\omega_\ell^q}. \tag{12}$$

If the partition $\alpha$ of $[m]$ is nondegenerate for $\K$, the canonical action of $S_\alpha$ on $Z_K$ is free. By [11, Theorem 1] (also see [2, 7.37]) there is an isomorphism of graded algebras for the free quotient $Z_K/S_\alpha$:

$$H^*(Z_K/S_\alpha; k) \cong \text{Tor}^{H^*(B(T^m/S_\alpha); k)}(k(\K), k) \tag{13}$$

where $B(T^m/S_\alpha)$ is the classifying space for the $r$-dimensional torus $T^m/S_\alpha$. By combining the isomorphisms in (11) and (13), we could compute the sum of the
multigraded Betti numbers in (12) by $\text{Tor}^{H^*(B(T^m/S_\alpha);k)}(k(K), k)$. But the proof in [11] does not give us any explicit formula for the isomorphism in (13), which makes this computation a little vague. In addition, the sum of multigraded Betti numbers in (12) is a little different from the sum appearing in Theorem 1.3. In Section 3, we will show how each $\beta^k_{i,\omega}$ is related to $\text{Tor}^{H^*(B(T^m/S_\alpha);k)}(k(K), k)$ via a cell decomposition of $Z_K/S_\alpha$ defined below. This cell decomposition of $Z_K/S_\alpha$ is originally constructed in [14, Section 2].

Let $\Delta^{[m]}$ be a $(m - 1)$-dimensional geometric simplex, which is the convex hull of $m$ points $v_1, \ldots, v_m$ in general position in the Euclidean space $\mathbb{R}^m$. Then $K$ is geometrically realized as a subset of $\Delta^{[m]}$, where any simplex $\sigma = \{j_1, \ldots, j_s\} \in K$ is realized as the convex hull of $v_{j_1}, \ldots, v_{j_s}$, or equivalently the join $v_{j_1} * \cdots * v_{j_s}$. For any $\omega \subseteq [m]$, let $\Delta^\omega$ denote the face of $\Delta^{[m]}$ whose vertex set is $\omega$, i.e.

$$\Delta^\omega = \ast_{j \in \omega} v_j.$$ 

If we think of $D^2$ as $S^1 \ast v$, we can rewrite the decomposition of $Z_K$ in (15) as:

$$Z_K = \bigcup_{\sigma \in K} \left( \prod_{j \in \sigma} (S^1_{(j)} \ast v_j) \times \prod_{j \not\in \sigma} S^1_{(j)} \right). \quad (14)$$

From this decomposition of $Z_K$, we can obtain a decomposition of $Z_K/S_\alpha$ as follows (see [14, Section 2]):

$$Z_K/S_\alpha = \bigcup_{\sigma \in K} \left( \prod_{i \in \Lambda_\alpha(\sigma)} (S^1_{(i)} \ast \ast_{j \in \sigma \setminus \alpha_i} v_j) \times \prod_{i \not\in [\tau]} S^1_{(i)} \right) \subseteq \prod_{i \in [\tau]} S^1_{(i)} \ast \Delta^\alpha_i \quad (15)$$

where $S^1_{(i)} = e^1_{(i)} \cup e^0_{(i)}$ denotes a copy of $S^1$ corresponding to $i \in [\tau]$ and, we consider the index $i \in [\tau]$ to be increasing from the left to the right.

**Warning:** We use the subscript $(j)$ for $j \in [m]$ while use subscript $(i)$ for $i \in [\tau]$.

A natural cell decomposition of $S^1_{(i)} \ast \Delta^\alpha_i$ is given by

$$\{(S^1 \ast \Delta^\tau)^\circ \mid \tau \subseteq \alpha_i, \tau \neq \emptyset \} \cup \{e^1_{(i)}, e^0_{(i)}\}$$

where $(S^1 \ast \Delta^\tau)^\circ$ is the interior of $S^1 \ast \Delta^\tau$. If we consider these cells as the basis of the cellular cochain complex of $S^1_{(i)} \ast \Delta^\alpha_i$, then

- the coboundary of $(S^1 \ast \Delta^\tau)^\circ$ is $\sum_{\tau \subset \omega \subseteq \alpha_i} \pm (S^1 \ast \Delta^\omega)^\circ$, $\tau \neq \emptyset$. \quad (16)

- the coboundary of $e^1_{(i)}$ is $\sum_{j \in \alpha_i} (S^1 \ast v_j)^\circ$, and the coboundary of $e^0_{(i)}$ is zero.
In the rest of this section, we always assume that the partition \( \alpha \) of \([m]\) is nondegenerate for \( \mathcal{K} \). Then for any \( \sigma \in \mathcal{K} \), we have \( |I_\alpha(\sigma)| = |\sigma| \). In other words, \( \sigma \cap \alpha_i \) is a single vertex of \( \sigma \) for any \( i \in I_\alpha(\sigma) \). For convenience, we define

\[
\sigma_{(i)} = \sigma \cap \alpha_i \in [m], \; i \in I_\alpha(\sigma).
\]

By this notation, \( \sigma = \bigcup_{i \in I_\alpha(\sigma)} \sigma_{(i)} \subseteq [m] \). So we can rewrite (15) as

\[
Z_{\mathcal{K}/S_\alpha} = \bigcup_{\sigma \in \mathcal{K}} \left( \prod_{i \in I_\alpha(\sigma)} \left( S^1_{(i)} \ast v_{\sigma_{(i)}} \right) \times \prod_{i \in [r] \backslash I_\alpha(\sigma)} S^1_{(i)} \right).
\]

For any simplex \( \sigma \in \mathcal{K} \) and \( I \subseteq [r] \backslash I_\alpha(\sigma) \), define

\[
B_{(\sigma,I)} = \prod_{i \in I_\alpha(\sigma)} \left( S^1_{(i)} \ast v_{\sigma_{(i)}} \right) ^\circ \times \prod_{i \in I} e^1_{(i)} \times \prod_{i \in [r] \backslash (I_\alpha(\sigma) \cup I)} e^0_{(i)} \subset Z_{\mathcal{K}/S_\alpha}.
\]

Then \( B_{(\sigma,I)} \) is a cell of dimension \( 2|I_\alpha(\sigma)| + |I| = 2|\sigma| + |I| \). It is easy to see that

\[
\{ B_{(\sigma,I)} \mid \sigma \in \mathcal{K}, \; I \subseteq [r] \backslash I_\alpha(\sigma) \}
\]

is a cell decomposition of \( Z_{\mathcal{K}/S_\alpha} \). Let \( C^*(Z_{\mathcal{K}/S_\alpha}; k) \) denote the cellular cochain complex determined by this cell decomposition whose generators are denoted by

\[
\{ B^*_{(\sigma,I)} \mid \sigma \in \mathcal{K}, \; I \subseteq [r] \backslash I_\alpha(\sigma) \}.
\]

To write the coboundary map of \( C^*(Z_{\mathcal{K}/S_\alpha}; k) \), we introduce the following conventions and notations.

- We orient the cell \( B_{(\sigma,I)} \) by ordering the factors in (18) so that the index \( i \in [r] \) is increasing from the left to the right and, orienting the circle \( S^1_{(i)} \) and the cells \( e^1_{(i)} \) and \( e^0_{(i)} \) in the same way for all different \( i \in [r] \).
- For any \( i \in L \subseteq [r] \), define \( \kappa(i,L) = (-1)^{c(i,L)} \), where \( c(i,L) \) is the number of elements in \( L \) that are less than \( i \). For any \( L, L' \subseteq [r] \) with \( L \cap L' = \emptyset \), we clearly have \( \kappa(i,L \cup L') = \kappa(i,L) \cdot \kappa(i,L') \).

By the above conventions, the coboundary of \( B^*_{(\sigma,I)} \) is given by:

\[
dB^*_{(\sigma,I)} = \sum_{i \in I} \kappa(i,L) \left( \sum_{\sigma \in \omega \in \mathcal{K}, \; \omega \cap \alpha_i \subseteq \{i\} \atop |\omega| = |\sigma| + 1} B^*_{(\omega,I \setminus \{i\})} \right)
\]

where \( \omega \backslash \sigma \in [m] \) denotes the only vertex of \( \omega \in \mathcal{K} \) that is not contained in \( \sigma \). By (16), the coboundary of any factor \( (S^1_{(i)} \ast v_{\sigma_{(i)}})^\circ \) in \( B_{(\sigma,I)} \) is zero since there is no simplex \( \omega \in \mathcal{K} \) with \( \omega \cap \alpha_i \) consisting of more than one vertex. In addition, when taking the coboundary of \( B^*_{(\sigma,I)} \) via the Leibniz’ rule, passing an even dimensional factor has no contribution to the sign.

For any \( L \subseteq [r] \), denote by \( C^{*\perp}(Z_{\mathcal{K}/S_\alpha}; k) \) the \( k \)-submodule of \( C^*(Z_{\mathcal{K}/S_\alpha}; k) \) generated by the set \( \{ B^*_{(\sigma,I)} \mid \sigma \in \mathcal{K}, \; I_\alpha(\sigma) \cup I = L, \; I_\alpha(\sigma) \cap I = \emptyset \} \). By the
formula [19], it is easy to see that $C^*_*\left(Z_{\mathcal{K}}/S_{\alpha}; k\right)$ is a cochain subcomplex of $C^*\left(Z_{\mathcal{K}}/S_{\alpha}; k\right)$. So we have a decomposition of cochain complexes:

$$C^*\left(Z_{\mathcal{K}}/S_{\alpha}; k\right) = \bigoplus_{l \in [r]} C^*_{\leq l}\left(Z_{\mathcal{K}}/S_{\alpha}; k\right).$$

Next, we introduce another space $\mathcal{X}(\mathcal{K}, \alpha)$ which is homotopy equivalent to $Z_{\mathcal{K}}/S_{\alpha}$. We will see in Section 3 that $\mathcal{X}(\mathcal{K}, \alpha)$ plays an important role in relating the cohomology of $Z_{\mathcal{K}}/S_{\alpha}$ with $\text{Tor}^{H^*\left(B(\tau_{m}/S_{\alpha}); k\right)}\left(k(\mathcal{K}), k\right)$. Let $S^\infty_{(j)}$ denote a copy of infinite dimensional sphere $S^\infty$ based at $v_j$ for each $j \in [m]$. We define

$$\mathcal{X}(\mathcal{K}, \alpha) := \bigcup_{\sigma \in \mathcal{K}} \left( \prod_{i \in I_{\alpha}(\sigma)} \left(S^1_{\{i\}} \ast S^\infty_{\{i\}(\sigma)}\right) \times \prod_{i \in [r] \setminus I_{\alpha}(\sigma)} S^1_{\{i\}} \right) \subseteq \prod_{i \in [r]} \left( S^1_{\{i\}} \ast S^\infty_{\{i\}(\sigma)}\right).$$

So $Z_{\mathcal{K}}/S_{\alpha}$ is a subspace of $\mathcal{X}(\mathcal{K}, \alpha)$. By contracting each $S^\infty_{(j)}$ to $v_j$, we obtain a deformation contraction from $\mathcal{X}(\mathcal{K}, \alpha)$ to $Z_{\mathcal{K}}/S_{\alpha}$. The definition of $\mathcal{X}(\mathcal{K}, \alpha)$ is inspired by the deformation of $Z_{\mathcal{K}} = (D^2, S^1)^\mathcal{K}$ to the polyhedral product $(S^\infty, S^1)^\mathcal{K}$ introduced in [21, Sec.4.5].

Next, we construct a cell decomposition of $\mathcal{X}(\mathcal{K}, \alpha)$. There is a natural cell decomposition on $S^\infty$ with exactly one cell $\xi^n$ in each dimension $n \geq 0$, where the boundary of $\xi^{2k}$ is the closure of $\xi^{2k+1}$ and the boundary of $\xi^{2k+1}$ is zero for every $k \geq 0$. We denote the cells in $S^\infty_{(j)}$ by $\{\xi^n_{(j)}\}_{n \geq 0}$ for any $j \in [m]$. In particular, $\xi^0_{(j)} = v_j$ for any $j \in [m].$

**Definition 2.2 (Weight).** We call any function $h : [m] \to \mathbb{N}$ a weight on $[m]$. The support of $h$ is defined to be $\text{supp}(h) = \{j \in [m] \mid h(j) \neq 0\}$.

- For any $j \in [m]$, define $\delta_j : [m] \to \mathbb{N}$ by $\delta_j(j) = 0$ if $j' \neq j$ and $\delta_j(j) = 1$.
- For any $\sigma \subseteq [m]$, let $1_\sigma = \sum_{j \in \sigma} \delta_j$ whose support is $\sigma$.

For any $\sigma \in \mathcal{K}$, $I \subseteq [r]$ and any weight $h$ on $[m]$ with $\text{supp}(h) = \sigma$, define a cell $B_{(\sigma,h,I)} \subset \mathcal{X}(\mathcal{K}, \alpha)$ by

$$B_{(\sigma,h,I)} = \prod_{i \in I_{\alpha}(\sigma) \setminus I} \left(S^1_{\{i\}} \ast \xi^{2h(\sigma(i))-2}_{\{i\}(\sigma)}\right)^{\circ} \times \prod_{i \in [r] \setminus I_{\alpha}(\sigma)} S^1_{\{i\}} \times \prod_{i \in I_{\alpha}(\sigma) \setminus I} e^1_{\{i\}} \times \prod_{i \in [r] \setminus I_{\alpha}(\sigma) \cup I} e^0_{\{i\}}.$$ 

It is easy to see that $\{B_{(\sigma,h,I)} : \sigma \in \mathcal{K}, \text{supp}(h) = \sigma, I \subseteq [r]\} \subset \mathcal{X}(\mathcal{K}, \alpha)$ is a cell decomposition of $\mathcal{X}(\mathcal{K}, \alpha)$. Note that $\dim B_{(\sigma,h,I)} = |I| + \sum_{j \in \sigma} 2h(j)$. Let $\iota_{\alpha} : Z_{\mathcal{K}}/S_{\alpha} \hookrightarrow \mathcal{X}(\mathcal{K}, \alpha)$ be the inclusion map. So we have

$$\iota_{\alpha}(B_{(\sigma,I)}) = B_{(\sigma,1_\alpha,I)}, \sigma \in \mathcal{K}, I \cap I_{\alpha}(\sigma) = \emptyset.$$
Let $C^*(\mathcal{X}(\mathcal{K}, \alpha); k)$ be the cellular cochain complex determined by this cell decomposition of $\mathcal{X}(\mathcal{K}, \alpha)$, whose generators are denoted by

$$\{B^*_{(\sigma, h, l)} \mid \sigma \in \mathcal{K}, \text{supp}(h) = \sigma, I \subseteq \{r\}\}.$$ 

Similarly to $B_{(\sigma, I)}$, we orient the cell $B_{(\sigma, h, l)}$ by ordering the factors in $B_{(\sigma, h, l)}$ so that the index $i \in \{r\}$ is increasing from the left to the right and, orienting the cells in all different copies of $S^1_{(i)}$ and $S^\infty_{(i)}$ in the same way. Then it is easy to see that the coboundary of $B^*_{(\sigma, h, l)}$ is given by:

$$d B^*_{(\sigma, h, l)} = \sum_{i \in I \cap I_\sigma(\sigma)} \kappa(i, l) \cdot B^*_{(\sigma, h + \delta_{\sigma[i]}, P \setminus \{i\})} \tag{22}$$

$$+ \sum_{i \in I, I_\sigma(\sigma) \sigma \subset \omega \in \mathcal{K}, \omega \sigma \in \alpha_i, \|\omega\| = \|\sigma\| + 1} \kappa(i, l) \cdot B^*_{(\omega, h + \delta_{\omega[i]}, P \setminus \{i\})}.$$ 

For any $L \subseteq \{r\}$, denote by $C^* L_{(\mathcal{X}(\mathcal{K}, \alpha); k)$ the $k$-submodule of $C^*(\mathcal{X}(\mathcal{K}, \alpha); k)$ generated by the set $\{B^*_{(\sigma, h, l)} \mid \sigma \in \mathcal{K}, \text{supp}(h) = \sigma, I_\sigma(\sigma) \cup I = L\}$. By (22), it is easy to see that $C^* L_{(\mathcal{X}(\mathcal{K}, \alpha); k)$ is a cochain subcomplex of $C^*(\mathcal{X}(\mathcal{K}, \alpha); k)$. So we have a decomposition of cochain complexes:

$$C^*(\mathcal{X}(\mathcal{K}, \alpha); k) = \bigoplus_{L \subseteq \{r\}} C^* L_{(\mathcal{X}(\mathcal{K}, \alpha); k).}$$

3. The multigraded structure of $\text{Tor}^{H^*(B(T^m/S_\alpha); k)}(k(\mathcal{K}), k)$

Let $\alpha = \{\alpha_1, \ldots, \alpha_r\}$ be a partition of $[m]$ that is nondegenerate for $\mathcal{K}$. Let us see how to compute $\text{Tor}^{H^*(B(T^m/S_\alpha); k)}(k(\mathcal{K}), k)$. Note that $H^*(B(T^m/S_\alpha); k)$ can be identified with the polynomial ring $k[u_1, \ldots, u_r]$ with $r$ variables. According to [23 7.37] (also see [3 Sec.4.8]), the $k[u_1, \ldots, u_r]$-module structure on $k(\mathcal{K})$ is given by

$$k[u_1, \ldots, u_r] \longrightarrow k(\mathcal{K})$$

$$u_i \longrightarrow \sum_{j \in \alpha_i} v_j, \ 1 \leq i \leq r. \tag{23}$$

Let $\Lambda_k[t_1, \ldots, t_r]$ denote the exterior algebra over $k$ with $r$ generators $t_1, \ldots, t_r$. We define a differential $d_\alpha$ on the tensor product $\Lambda_k[t_1, \ldots, t_r] \otimes k(\mathcal{K})$ by:

$$d_\alpha(t_i) = \sum_{j \in \alpha_i} v_j, \ 1 \leq i \leq r, \text{ and } d_\alpha(v_j) = 0, \ 1 \leq j \leq m. \tag{24}$$
We can compute \( \text{Tor}^{k^{|u_1, \ldots, u_r|}}(k(K), k) \cong \text{Tor}^{k^{|u_1, \ldots, u_r|}}(k, k(K)) \) via the Koszul resolution of \( k \) (see [2] Sec 3.4). Recall that the Koszul resolution of \( k \) is:

\[
0 \longrightarrow \Lambda_k[t_1, \ldots, t_r] \otimes k[u_1, \ldots, u_r] \longrightarrow \cdots \\
\longrightarrow \Lambda_k[t_1, \ldots, t_r] \otimes k[u_1, \ldots, u_r] \longrightarrow k[u_1, \ldots, u_r] \longrightarrow k \longrightarrow 0,
\]

where \( d(t_i) = u_i \) and \( d(u_i) = 0 \) for any \( 1 \leq i \leq r \). Then we obtain

\[
\text{Tor}^{k^{|u_1, \ldots, u_r|}}(k, k(K)) \cong H^\*(\Lambda_k[t_1, \ldots, t_r] \otimes k[u_1, \ldots, u_r]) \otimes k(K).
\]

It is easy to see that the differential on \( \Lambda_k[t_1, \ldots, t_r] \otimes k(K) \) induced from \( \text{Tor}^{k^{|u_1, \ldots, u_r|}}(k, k(K)) \) is exactly given by \( d_\alpha \) in (24).

By the definition of \( k[K] \), any element of \( k[K] \) can be uniquely written as a linear combination of monomials of the form \( v_{j_1}^{d_1} \cdots v_{j_s}^{d_s} \) where \( \{j_1, \ldots, j_s\} = \sigma \) is a simplex in \( K \) and \( d_l \geq 1 \) for all \( l = 1, \ldots, s \). For brevity, we denote the monomial \( v_{j_1}^{d_1} \cdots v_{j_s}^{d_s} \) by \( v^\sigma \) where \( h : [m] \to \mathbb{N} \) is a weight with \( \text{supp}(h) = \sigma \) defined by:

\[
h(j) = \begin{cases} a_l, & \text{if } j = j_l \in \sigma; \\ 0, & \text{if } j \notin \sigma. \end{cases}
\]

Comparing this notation with (1), we see that \( v^\sigma = v^\sigma_{I^\sigma} \) for any \( \sigma \in K \). It is clear that \( \Lambda_k[t_1, \ldots, t_r] \otimes k(K) \) is generated, as a \( k \)-module, by the set

\[
\{ t_1 v^\sigma_{i, h} \mid \sigma \in K, \text{supp}(h) = \sigma, I \subseteq [r] \},
\]

where \( t_1 = t_{i_1} \cdots t_{i_s}, I = \{i_1, \ldots, i_s\} \subseteq [r] \) with \( i_1 < \cdots < i_s \).

In addition, we define a multigrading \( mdeg^\alpha \) on \( \Lambda_k[t_1, \ldots, t_r] \otimes k(K) \) by:

\[
\begin{align*}
\text{mdeg}^\alpha(t_i) &= (-1, \{i\}) \in (\mathbb{Z}, [r]), \quad i \in [r]; \\
\text{mdeg}^\alpha(v^\sigma) &= (0, I_\alpha(\sigma)) \in (\mathbb{Z}, [r]), \quad \sigma \in K.
\end{align*}
\]

So \( \text{mdeg}^\alpha(t_1 v^\sigma) = (-|I|, I_\alpha(\sigma) \cup I) \). By the definition of \( d_\alpha \) in (24), we have

\[
d_\alpha(t_1 v^\sigma_{I^\sigma}) = \sum_{i \in I} \kappa(i, I) \cdot t_{i_1} \left( \sum_{j \in \alpha_i} v_{j} v^\sigma_{I^\sigma} \right) = \sum_{i \in I \cap I_\alpha(\sigma)} \kappa(i, I) \cdot t_{i_1} v^\sigma_{I^\sigma} + \sum_{i \in I \cap I_\alpha(\sigma)} \sum_{\sigma \subseteq \omega \subseteq K, \omega \cap \sigma = \alpha_i \atop |\omega| = |\sigma| + 1} \kappa(i, I) \cdot t_{i_1} v^\omega_{I^\sigma}.
\]

For any subset \( L \subseteq [r] \), define \((\Lambda_k[t_1, \ldots, t_r] \otimes k(K))^L \) to be the \( k \)-submodule of \( \Lambda_k[t_1, \ldots, t_r] \otimes k(K) \) generated by \( \{t_1 v^\sigma_{I^\sigma} \mid \sigma \in K, \text{supp}(h) = \sigma, I_\alpha(\sigma) \cup I = L\} \).
By the formula (25),\( d_\alpha((\Lambda_k[t_1, \cdots, t_r] \otimes \kappa(\mathcal{K}))^L) \subseteq (\Lambda_k[t_1, \cdots, t_r] \otimes \kappa(\mathcal{K}))^L \). So we have a decomposition of \( \Lambda_k[t_1, \cdots, t_r] \otimes \kappa(\mathcal{K}) \) into differential \( k \)-modules:

\[
\Lambda_k[t_1, \cdots, t_r] \otimes \kappa(\mathcal{K}) = \bigoplus_{L \leq [r]} (\Lambda_k[t_1, \cdots, t_r] \otimes \kappa(\mathcal{K}))^L.
\]

So we can define a multigraded structure on \( \text{Tor}^{\Lambda_k[t_1, \cdots, t_r]}(k, \kappa(\mathcal{K})) \) by:

\[
\text{Tor}^{\Lambda_k[t_1, \cdots, t_r]}(k, \kappa(\mathcal{K})) = \bigoplus_{L \leq [r]} \text{Tor}^{\Lambda_k[t_1, \cdots, t_r]}(k, \kappa(\mathcal{K})), \quad (26)
\]

\[
\text{Tor}^{\Lambda_k[t_1, \cdots, t_r]}(k, \kappa(\mathcal{K}))(L) := H^{-q}((\Lambda_k[t_1, \cdots, t_r] \otimes \kappa(\mathcal{K}))^L), \quad \forall q \geq 0. \quad (27)
\]

Next, we define a \( k \)-linear map

\[
\psi_\alpha : \Lambda_k[t_1, \cdots, t_r] \otimes \kappa(\mathcal{K}) \to \text{C}^*(\mathcal{X}(\mathcal{K}, \alpha); k)
\]

\[
t_{\alpha} \psi_\alpha (\sigma, h) \mapsto B_{(\sigma, h, l)}^*
\]

Obviously, \( \psi_\alpha \) is a \( k \)-linear isomorphism. Moreover, by comparing (22) with (25), we see that \( \psi_\alpha \) is a cochain map and hence an isomorphism of cochain complexes. In addition, \( \psi_\alpha \) clearly preserves the multigradings of \( \Lambda_k[t_1, \cdots, t_r] \otimes \kappa(\mathcal{K}) \) and \( \text{C}^*(\mathcal{X}(\mathcal{K}, \alpha); k) \). So we obtain the following lemma.

**Lemma 3.1.** For any \( L \subseteq [r] \), \( \psi_\alpha : (\Lambda_k[t_1, \cdots, t_r] \otimes \kappa(\mathcal{K}))^L \to \text{C}^*\text{L}(\mathcal{X}(\mathcal{K}, \alpha); k) \) induces an isomorphism in cohomology.

So for each \( L \subseteq [r] \), we have a diagram:

\[
\begin{array}{ccc}
(\Lambda_k[t_1, \cdots, t_r] \otimes \kappa(\mathcal{K}))^L & \xrightarrow{\psi_\alpha} & \text{C}^*\text{L}(\mathcal{X}(\mathcal{K}, \alpha); k) \\
\downarrow{\iota_\alpha} & & \\
\text{C}^*\text{L}(\mathcal{Z}_K/S_\alpha; k) & & \\
\end{array}
\]

where \( \iota_\alpha \) is the cochain map induced by the inclusion \( \iota_\alpha : \mathcal{Z}_K/S_\alpha \hookrightarrow \mathcal{X}(\mathcal{K}, \alpha) \). So

\[
\iota_\alpha^*(B_{(\sigma, h, l)}^*) = \begin{cases} B_{(\sigma, h, l)}^*, & \text{if } I \cap I_\alpha(\sigma) = \emptyset \text{ and } h = 1_{\sigma}; \\
0, & \text{otherwise}, \end{cases}
\]

(30)

Since \( \mathcal{Z}_K/S_\alpha \) is a deformation retract of \( \mathcal{X}(\mathcal{K}, \alpha) \), \( \iota_\alpha^* \) induces an isomorphism in cohomology. So \( \iota_\alpha^* \circ \psi_\alpha : (\Lambda_k[t_1, \cdots, t_r] \otimes \kappa(\mathcal{K}))^L \to \text{C}^*\text{L}(\mathcal{Z}_K/S_\alpha; k) \) induces an isomorphism in cohomology. More specifically, we have

\[
\iota_\alpha^* \circ \psi_\alpha (t_{\alpha} \psi_\alpha (\sigma, h)) = \begin{cases} B_{(\sigma, h, l)}^*, & \text{if } I \cap I_\alpha(\sigma) = \emptyset \text{ and } h = 1_{\sigma}; \\
0, & \text{otherwise}, \end{cases}
\]

(31)
Notice when $I \cap I_{\alpha}(\sigma) = \emptyset$ and $I_{\alpha}(\sigma) \cup I = L$, \( \dim B^*_{(\sigma, I)} = 2|I_{\alpha}(\sigma)| + |I| = 2|L| - |I| \).
So for any $q \geq 0$, $\iota_{\alpha}^* \circ \psi_{\alpha}$ induces an isomorphism
\[
H^{-q}((\Lambda_k[t_1, \ldots , t_r] \otimes k(\mathcal{K}))^L) \xrightarrow{\cong} H^{2|L|-q}(C^*_{\alpha,L}(\mathcal{Z}_K/S_{\alpha}; k)).
\]  
(32)

**Proposition 3.2.** If $\alpha = \{\alpha_1, \ldots , \alpha_r\}$ is a partition of $[m]$ that is nondegenerate for $\mathcal{K}$, then for any $q \geq 0$ and $L \subseteq [r]$, $\text{Tor}_{q,L}^{k[u_1, \ldots , u_r]}(k, k(\mathcal{K})) \cong \check{H}^{2|L|-q}(\mathcal{K}_{\omega_{\alpha}^q}; k)$.

**Proof.** By definition (27), $\text{Tor}_{q,L}^{k[u_1, \ldots , u_r]}(k, k(\mathcal{K})) = H^{-q}((\Lambda_k[t_1, \ldots , t_r] \otimes k(\mathcal{K}))^L)$. So by (32), $\text{Tor}_{q,L}^{k[u_1, \ldots , u_r]}(k, k(\mathcal{K})) \cong H^{2|L|-q}(C^*_{\alpha,L}(\mathcal{Z}_K/S_{\alpha}; k))$. Moreover, it is shown in the proof of [14] Theorem 1.2] that
\[
H^{q}(C^*_{\alpha,L}(\mathcal{Z}_K/S_{\alpha}; k)) \cong \check{H}^{q-|L|-1}(\mathcal{K}_{\omega_{\alpha}^q}; k).
\]
So we obtain $\text{Tor}_{q,L}^{k[u_1, \ldots , u_r]}(k, k(\mathcal{K})) \cong \check{H}^{2|L|-q-1}(\mathcal{K}_{\omega_{\alpha}^q}; k)$. We want to remind the reader that in [14], the space $\mathcal{Z}_K/S_{\alpha}$ is denoted by $X(\mathcal{K}, \lambda_{\alpha})$ and the full subcomplex $\mathcal{K}_{\omega_{\alpha}^q}$ is denoted by $\mathcal{K}_{\alpha,L}$. 

Finally, let us give a proof of Theorem 1.3. Our proof will use the following theorem in [5] on the Betti numbers of any multigraded module over a polynomial ring, which generalizes [7] Corollary 2.5.

**Theorem 3.3** (Theorem 3 in [5]). Let $M$ be a multigraded $k[u_1, \ldots , u_r]$-module of Krull dimension $s$. Then $\dim_k \text{Tor}_{d}^{k[u_1, \ldots , u_r]}(k; M) \geq \binom{r-s}{i}$ for every $i \geq 0$.

The Krull dimension $Kd(R)$ of a commutative ring $R$ is the maximal number of algebraically independent elements of $R$. The **Krull dimension** $Kd_{k[u_1, \ldots , u_r]}(M)$ of a $k[u_1, \ldots , u_r]$-module $M$ is defined to be the Krull dimension of the quotient ring of $k[u_1, \ldots , u_r]$ which makes $M$ a faithful module. That is,
\[
Kd_{k[u_1, \ldots , u_r]}(M) := Kd\left( k[u_1, \ldots , u_r]/\text{Ann}(M) \right)
\]
where $\text{Ann}(M)$, the annihilator, is the kernel of the natural map from $k[u_1, \ldots , u_r]$ to the ring of $k[u_1, \ldots , u_r]$-linear endomorphisms of $M$.

Suppose a partition $\alpha = \{\alpha_1, \ldots , \alpha_r\}$ of $[m]$ is nondegenerate for a simplicial complex $\mathcal{K}$ on $[m]$. Let $\{j_0, \ldots , j_d\} \subseteq [m]$ be a maximal simplex of $\mathcal{K}$ where $d = \dim(\mathcal{K})$ and assume that $j_l \in \alpha_{j_i}$, $0 \leq l \leq d$. Then with respect to the $k[u_1, \ldots , u_r]$-module structure on $k(\mathcal{K})$ defined in (23), $u_{j_0}, \ldots , u_{j_d}$ are $d+1$ algebraically independent elements of $k[u_1, \ldots , u_r]/\text{Ann}(k(\mathcal{K}))$. So we have $Kd\left( k[u_1, \ldots , u_r]/\text{Ann}(k(\mathcal{K})) \right) \geq d+1$. Conversely, it is obvious that any monomial $u_{j_1}^{a_1} \cdots u_{j_s}^{a_s}$ in $k[u_1, \ldots , u_r]$ with $s > d+1$ acts trivially on $k(\mathcal{K})$. So $Kd\left( k[u_1, \ldots , u_r]/\text{Ann}(k(\mathcal{K})) \right) \leq d+1$. So we obtain
\[
Kd_{k[u_1, \ldots , u_r]}(k(\mathcal{K})) = \dim(\mathcal{K}) + 1.
\]  
(33)
Proof of Theorem 1.3

By Proposition 3.2 and the Hochster’s formula (4), we have:
\[ \dim_k \text{Tor}_{q_L}^k(k(k)) = \dim_k \tilde{H}^{[L]-q-1}(k; k) = \beta^{k(k)}_{q+|L|\omega^q_{\alpha}}, \quad \forall q \geq 0. \]

Then since the Krull dimension of \( k(p) \) with respect to the \( k[u_1, \ldots, u_r] \)-module structure in (23) is equal to \( \dim(k) + 1 \), we obtain the following inequality from Theorem 3.3 for any \( q \geq 0 \):
\[ \sum_{L \subseteq [r]} \beta^{k(k)}_{q+|L|\omega^q_{\alpha}} = \sum_{L \subseteq [r]} \dim_k \text{Tor}_{q_L}^k(k(k)) \geq \dim_k \tilde{H}^{[L]-q-1}(k; k) \geq \left( r - \dim(k) - 1 \right). \]

Then we have
\[ \sum_{q \geq 0} \sum_{L \subseteq [r]} \dim_k \tilde{H}^{[L]-q-1}(k; k) = \sum_{q \geq 0} \sum_{L \subseteq [r]} \beta^{k(k)}_{q+|L|\omega^q_{\alpha}} \geq 2^{r-\dim(k)-1}. \]

This finishes the proof of the theorem. \( \square \)

Remark 3.4. It is not clear whether Theorem 1.3 should hold for all partitions of \([m]\). Indeed, if a partition \( \alpha \) of \([m]\) is not nondegenerate for \( K \), there is no natural way to identify \( H^*(Z_K/S^*_\alpha; k) \) with the Tor algebra of any multigraded \( k[u_1, \ldots, u_r] \)-module. However, we have not found any counterexample to the statement of Theorem 1.3 among general partitions of \([m]\) either.

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