Black hole solutions to the $F_4$-model and their orbits (I)

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Abstract

In this paper we continue the program of the classification of nilpotent orbits using the approach developed in arXiv:1107.5986, within the study of black hole solutions in $D = 4$ supergravities. Our goal in this work is to classify static, single center black hole solutions to a specific $N = 2$ four dimensional “magic” model, with special Kähler scalar manifold $\text{Sp}(6, \mathbb{R})/U(3)$, as orbits of geodesics on the pseudo-quaternionic manifold $F_4(4)/[\text{SL}(2, \mathbb{R}) \times \text{Sp}'(6, \mathbb{R})]$ with respect to the action of the isometry group $F_4(4)$. Our analysis amounts to the classification of the orbits of the geodesic “velocity” vector with respect to the isotropy group $H^* = \text{SL}(2, \mathbb{R}) \times \text{Sp}'(6, \mathbb{R})$, which include a thorough classification of the nilpotent orbits associated with extremal solutions and reveals a richer structure than the one predicted by the $\beta - \gamma$ labels alone, based on the Kostant Sekiguchi approach. We provide a general proof of the conjecture made in hep-th/0908.1742 which states that regular single center solutions belong to orbits with coinciding $\beta - \gamma$ labels. We also prove that the reverse is not true by finding distinct orbits with the same $\beta - \gamma$ labels, which are distinguished by suitably devised tensor classifiers. Only one of these is generated by regular solutions. Since regular static solutions only occur with nilpotent degree not exceeding 3, we only discuss representatives of these orbits in terms of black hole solutions. We prove that these representatives can be found in the form of a purely dilatonic four-charge solution (the generating solution in $D = 3$) and this allows us to identify the orbit corresponding to the regular four-dimensional metrics. $H^*$-orbits with degree of nilpotency greater than 3 are analyzed solely from a group theoretical point of view, leaving a systematic analysis of their possible interpretation in terms of static multicenter or stationary non-static solutions to a future work. We just limit ourselves to give (singular) single-center representatives of these orbits, to be possibly interpreted as singular limits of regular multicenter solutions. We provide the explicit transformations mapping the various $H^*$-orbits and in particular BPS into non-BPS regular solutions showing that they in general belong to the complexification of the global symmetry group in $D = 3$. 
1 Introduction

Dimensional reduction along time offers a powerful way to study stationary solutions of 4D symmetric supergravity models via group-theoretical methods [1–15]. In this way the black hole solutions are identified with the geodesics on a pseudo-Riemannian coset manifold $G/H^*$ and the corresponding geodesic equations are best approached when they are cast into the Lax form [2,3]. More precisely, it is known that regular extremal black holes associated with Lax operators $L(\tau) \ (\tau = -1/r)$ that are nilpotent all along their radial evolution [5,7,16,17]. Since geodesics are totally defined by their “initial point” $P(0)$ and “initial velocity” $L(0)$, and since the action of $G/H^*$ on $P(0)$ is transitive on the manifold,
we can fix $P(0)$ to coincide with the origin $O$ and classify the geodesics by the orbits of
the corresponding initial velocity $L(0)$ with respect to the isotropy group $H^\ast$. Hence the
classification of extremal black holes requires a classification of the orbits of nilpotent
elements of the coset space $\mathfrak{r}^\ast$ (isomorphic to the tangent space to the manifold in $O$) with
respect to the adjoint action of the stability subgroup $H^\ast$ of $G$. Using the simplificative
analogy of $L(\tau)$ with a velocity vector in Minkowsky space, where the stability group is $SO(1,3)$, we can say that “time-like” $L(0)$ correspond to non-extremal four dimensional
solutions, while extremal ones correspond to “light-like” geodesics on the manifold $G/H^\ast$, generated by nilpotent matrices. As opposed to the Minkowski case, in the problem at
hand, however, the “light-like” vectors may actually fall in a variety of $H^\ast$-orbits. Classifying
these is the main goal of the present work. Just as the velocity of a photon can be made
simplest by going to a suitable frame of reference in which one of the axes coincide with the
direction of propagation, we construct, for the class of orbits which are relevant for static
black holes, a frame in which the velocity of geodesic is simplest. This frame is defined by
the “generating solution” and considerably simplifies the analysis of the correspondence
between static black holes and orbits in $D = 3$. In the present work we shall restrict to
a specific $\mathcal{N} = 2$ symmetric supergravity coupled to six vector multiplets, whose scalar
fields span the special Kähler manifold $G_4/H_4 = \text{Sp}(6, \mathbb{R})/\text{U}(3)$. Upon timelike-reduction
to $D = 3$ and dualization of vectors into scalar fields, the target space of the resulting
Euclidean sigma-model is the symmetric manifold $G/H^\ast = F_4(4)/[\text{SL}(2, \mathbb{R}) \times \text{Sp}'(6, \mathbb{R})]$. Our approach to the problem is a synthesis of the ones followed in [18, 19]. In [18], in
order to achieve a classification of nilpotent orbits, the authors of thoroughly discussed the
static spherical symmetric black-hole solutions of the simplest $\mathcal{N} = 2$ supergravity model
with one vector multiplet coupling, often dubbed the $S^3$-model\textsuperscript{1}. In this paper it has
been shown that a complete classification of the nilpotent $H^\ast$-orbits in $\mathfrak{r}^\ast$ can be effected
using the signatures of symmetric-covariant $H^\ast$-tensors, named tensor classifiers (TC). The
tensor structures used for the orbit analysis in the $G_2$-model are not enough however to
provide a complete classification of the orbits in more general case: New tensor classifiers
have to be devised. The standard approach to the study of the relevant $H^\ast$-nilpotent
orbits in the tangent space to the manifold (coset space) was based on the description of
a nilpotent generator $E$ of the coset as part of a triplet of $\text{SL}(2, \mathbb{R})$-generators $\{E, F, h\}$,
named standard triple, and on the classification of such triples with respect to the so called
$\gamma - \beta$-labels, which are $H^\ast$- invariant quantities [20]. As we shall prove in the present
work, this orbit analysis is by no means exhaustive: Distinct orbits are found with the
same $\gamma - \beta$-labels.

We shall apply a new constructive algorithm, devised in [19], which combines the
method of standard triples with new techniques based on the Weyl group. After a general
group theoretical analysis of the model this novel approach allows for a systematic con-
struction of the various nilpotent orbits by solving suitable matrix equations in nilpotent
generators $E$. Solutions to these equations comprise representatives $E$ of the various orbits
and the final part of the analysis is to group them under the action of suitable compact

\textsuperscript{1}The obtained results generalized previous ones in [20].
subgroups of $H^*$. Solutions which are not connected by the action of such subgroups are then found to be distinguished by certain $H^*$-invariants, which comprise, besides the $\gamma-\beta$-labels, also the signatures of suitable tensor classifiers. This guarantees the completeness of the classification. The tensor classifiers introduced here play an essential role in our analysis. Although they still do not exhaust all possible $H^*$-tensor structures which can be devised, they provide by themselves, without the use of the $\gamma-\beta$-labels, an almost complete classification of the orbits. Their use allows to find the orbit of a nilpotent generator $E$ without the need of constructing the correspondent standard triple for the computation of the relative $\gamma-\beta$-labels. Most importantly, they allow to distinguish orbits with the same $\gamma-\beta$-labels!

This approach was applied in [19] to the analysis of the $\text{SO}(4,n)/\text{SO}(2,n-2) \times \text{SO}(2,2)$ model with $n > 4$. In this work it was shown that the pattern of the nilpotent orbits is a universal property depending on the Tits-Satake (TS) universality class [21] of the model, which is defined by the coset $\text{SO}(4,5)/\text{SO}(2,3) \times \text{SO}(2,2)$.

The number of such classes was found for the $\mathcal{N} = 2$ symmetric models to be five (see Table 2 in [19]) and are defined by the Tits-Satake algebra associated with the $D = 3$ isometry algebra $g$. It is tempting to conjecture that the pattern of $H^*$-orbits found here for the $F_4(4)$-model captures the orbit structure of all the models within the same universality class. We postpone an answer to this question to a future work.

The paper is organized as follows. In Sect. 2 we discuss the geometry of the special Kähler manifold of the $D = 4 \mathcal{N} = 2$ model, we review the $r$ and $c^*$-maps and the general description of static $D = 4$ black holes as geodesics on the pseudo-Riemannian scalar manifold of the Euclidean $D = 3$ theory obtained through time-reduction of the $D = 4$ one. In Sect. 3, we review our approach to the classification of the $H^*$-nilpotent orbits in $\mathfrak{g}^*$. The results are listed in Tables 3-17 and in Table 23 of Appendix A. In Sect. 4, we review the construction of the generating solutions and show how it provides representatives of all the $H^*$-orbits with degree of nilpotency less or equal to 3. We also identify those $H^*$-orbits containing the geodesics corresponding to regular and small $D = 4$ solutions, in light of the known classifications. We end Sect. 4 with a discussion of the orbits of non-extremal solutions. In Sect. 5 we also provide examples of solutions corresponding to orbits with higher degree of nilpotency and show that they correspond to singular $D = 4$ solutions. We end with some concluding remarks.

During the final writing stage of the present work, we became aware of the interesting paper [22] whose analysis has, in some points, an overlap with ours.

2 Static Black Holes in $D = 4$, $\mathcal{N} = 2$ Supergravity

We consider a four dimensional supergravity theory whose bosonic sector consists of the graviton $g_{\mu\nu}(x)$, $n_v$ vector fields $A^\Lambda_\mu(x)$, $\Lambda = 0, \ldots, n_v - 1$, and $n_s$ scalar fields $\phi^r(x)$,
The general form of the bosonic action reads\footnote{Here we are using the “mostly plus” signature for the metric $g_{\mu\nu}$ and the convention $\epsilon_{0123} = 1$.}:

$$
S_4 = \int d^4x \sqrt{|g|} \mathcal{L}_4 = \int d^4x \sqrt{|g|} \left[ \frac{R[g]}{2} - \frac{1}{2} G_{rs}(\phi) \partial_\mu \phi^r \partial^\mu \phi^s + \frac{1}{4} F^\Lambda_{\mu\nu}(\phi) F^{\Sigma\mu\nu} + \frac{1}{8 \sqrt{|g|}} \epsilon_{\mu\nu\rho\sigma} F^{\Lambda\mu\nu} R^\Lambda_{\Sigma\Phi}(\phi) F^{\Sigma\rho\sigma} \right].
$$

(1)

The scalar-dependent matrix $G_{rs}(\phi)$ represents the positive definite metric on the Riemannian (simply connected) scalar manifold $\mathcal{M}_{\text{scal}}^{(D=4)}$, and we have collectively denoted the scalar fields by the short-hand notation $\phi \equiv (\phi^r)$. The vector field strengths are defined as usual: $F^\Lambda_{\mu\nu} \equiv \partial_\mu A^\Lambda_\nu - \partial_\nu A^\Lambda_\mu$. The $n_v \times n_v$ matrices $R_{\Lambda\Sigma}(\phi^r)$, $I_{\Lambda\Sigma}(\phi^r)$ are the real and imaginary parts of the complex kinetic matrix $N_{\Lambda\Sigma}(\phi^r)$ of the vector fields: $R_{\Lambda\Sigma} \equiv \text{Re}(N_{\Lambda\Sigma})$, $I_{\Lambda\Sigma} \equiv \text{Im}(N_{\Lambda\Sigma})$.

In $\mathcal{N} = 2$ the scalar manifold is the product of a special Kähler manifold $\mathcal{M}_{SK}$, parametrized by the scalars sitting in the vector multiplets, and a quaternionic Kähler manifold spanned by the hypermultiplet scalars. The latter do not contribute to the black hole solutions since they do not enter the vector kinetic matrix $N_{\Lambda\Sigma}$. We shall therefore restrict ourselves to an $\mathcal{N} = 2$ supergravity coupled just to vector multiplets and no hypermultiplets. We shall moreover restrict ourselves to models exhibiting a homogeneous symmetric (special Kähler) scalar manifold of the form $\mathcal{M}_{\text{scal}}^{(D=4)} = \mathcal{M}_{SK} = G_4/H_4$ (symmetric models). The action of an isometry transformation $g \in G_4$ on the scalar fields $\phi^r$ parametrizing $\mathcal{M}_{\text{scal}}^{(D=4)}$ is defined by means of a coset representative $\mathbb{L}_4(\phi) \in G_4/H_4$ as follows:

$$
g \cdot \mathbb{L}_4(\phi^r) = \mathbb{L}_4(g \ast \phi^r) \cdot h(\phi^r, g),
$$

(2)

where $g \ast \phi^r$ denote the transformed scalar fields, non-linear functions of the original ones $\phi^r$, and $h(\phi^r, g)$ is a compensator in $H_4$. The coset representative is defined modulo right action of $H_4$ and is fixed by the chosen parametrization of the manifold.

### 2.1 The Special Kähler Geometry of the $D = 4$ Model

In the present section we shall compute the main geometric quantities related to the special Kähler geometry of the model under consideration. Recall that a special Kähler manifold $\mathcal{M}_{SK}$ \cite{23–27}, of complex dimension $n$, is a Hodge-Kähler manifold on which a flat, holomorphic, symplectic vector structure is defined, with structure group $\text{Sp}(2n+2, \mathbb{R})$. If $\Omega(z^a)$ is a holomorphic section of this bundle:

$$
\Omega(z^a) = (\Omega^M(z^a)) = \begin{pmatrix} X^\Lambda(z^a) \\ F_\Lambda(z^a) \end{pmatrix},
$$

(3)

$$
\Lambda = 0, \ldots, n; \ a = 1, \ldots, n; \ M = 1, \ldots, 2n + 2,
$$

(4)
the Kähler potential is expressed as follows:
\[
\mathcal{K}(z^a, \bar{z}^a) = - \log \left( -i \Omega \bar{\Omega} \right) = - \log \left[ -i \left( X^A \bar{F}_A - F_A \bar{X}^A \right) \right]. \tag{5}
\]
\(\mathbb{C} = (\mathbb{C}_{MN})\) being the \(\text{Sp}(2n + 2, \mathbb{R})\)-invariant matrix:
\[
\mathbb{C} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{6}
\]

The complex vector field \(\Omega(z^a)\) also belongs to a \textit{holomorphic line bundle}, namely it transforms by multiplication times a holomorphic function \(\Omega(z^a) \rightarrow e^{-f(z)} \Omega(z^a)\). This implies, according to eq. (5), a Kähler transformation on the potential \(\mathcal{K}: \mathcal{K} \rightarrow \mathcal{K} + f(\bar{z}) + \bar{f}(z)\). It is useful to introduce a section of a U(1)-bundle over the scalar manifold, \(V(z^a, \bar{z}^a) \equiv e^\bar{\mathcal{K}}(\mathcal{K}(z^a))\), which, as \(\Omega(z^a) \rightarrow e^{-f(z)} \Omega(z^a)\), transforms under a U(1)-transformation: \(V(z^a, \bar{z}^a) \rightarrow e^{-i\theta} V(z^a, \bar{z}^a)\), where \(\theta(z, \bar{z}) = \text{Im}(f)\). This vector satisfies the property of being \textit{covariantly holomorphic} with respect to the U(1)-connection:
\[
\nabla_a V \equiv (\partial_a - \frac{1}{2} \partial_b \mathcal{K}) V = 0, \tag{7}
\]
where \(\partial_a \equiv \frac{\partial}{\partial z^a}\) and \(\partial_{\bar{a}} \equiv \frac{\partial}{\partial \bar{z}^a}\). If we define
\[
U_a = (U_a^M) \equiv \nabla_a V = (\partial_a + \frac{1}{2} \partial_b \mathcal{K}) V, \tag{8}
\]
the following properties hold:
\[
V \bar{CV} = i ; \quad U_a \bar{CV} = \bar{U}_a \bar{CV} = 0 ; \quad U_a \bar{U}^b = -i g_{ab}. \tag{8}
\]

If \(\bar{E}_a^I, I = 1, \ldots, n,\) is the complex vielbein matrix of the manifold, \(g_{ab} = \sum_I \bar{E}_a^I E_b^I\), and \(\bar{E}_a^I\) its inverse, we introduce the quantities \(U_I \equiv \bar{E}_I^a U_a\), in terms of which the following \((2n + 2) \times (2n + 2)\) matrix \(\mathbb{L}_4 = (\mathbb{L}_4^M)\) is defined:
\[
\mathbb{L}_4(z, \bar{z}) = \sqrt{2} \left( \text{Re}(V), \text{Re}(U_I), -\text{Im}(V), \text{Im}(U_I) \right), \tag{9}
\]
which, by virtue of eq.s (8), is symplectic: \(\mathbb{L}_4^T \mathbb{C} \mathbb{L}_4 = \mathbb{C}\). In terms of this matrix one can construct the symmetric, symplectic, negative definite matrix \(\mathcal{M}_4 = (\mathcal{M}_{4,MN})\)
\[
\mathcal{M}_4 = \mathbb{C} \mathbb{L}_4 \mathbb{L}_4^T \mathbb{C}. \tag{10}
\]
This matrix is related to \(R_{\Lambda\Sigma}\) and \(I_{\Lambda\Sigma}\) as follows:
\[
\mathcal{M}_4 = \begin{pmatrix} I + RI^{-1}R & -RI^{-1} \\ -IR^{-1} & I^{-1} \end{pmatrix} . \tag{11}
\]
For symmetric homogeneous special Kähler manifolds, the symplectic bundle defines an embedding of the isometry group \(G_4\) into \(Sp(2n + 2, \mathbb{R})\), realized by the symplectic representation \(\mathbb{R}\) by which \(G_4\) acts on the symplectic section \(V\) as part of the structure group. The global symmetries of the \(D = 4\) model (duality symmetries) consist in the simultaneous action of \(G_4\) on the scalar fields and on the symplectic vector of the electric field strengths and their magnetic duals in the representation \(\mathbb{R}\).
**Special coordinates.** One can always, by suitably fixing the symplectic gauge, choose a section \( \Omega(z^a) \) in which \( X^A(z^a) \) can be regarded as projective coordinates for the manifold. In particular, in a local patch in which \( X^0 \neq 0 \), \( X^a/X^0 \) are independent functions of \( z^a \) and can be thus used as coordinates, known as special coordinates. In the special coordinate patch we can then choose \( z^a \equiv X^a/X^0 \) in the first place. Moreover the lower components can be expressed in terms of a prepotential \( F(X) \): \( F_A = \partial X^A = \partial F \), \( F(X) \) being a homogeneous function of degree 2 in the \( X^A \). Of particular relevance are the cubic models in which: \( F(X) = \frac{1}{6} d_{abc}X^aX^bX^c/X^0 \). For these models one defines

\[
F(z^a) = F(X)/(X^0)^2 = \frac{1}{6} d_{abc}z^a z^b z^c,
\]

in terms of which the holomorphic section has the simple form:

\[
\Omega(z^a) = \begin{pmatrix} z^a \\ -F(z^a) \\ \frac{\partial F(z^a)}{\partial z^a} \end{pmatrix},
\]

Writing the complex scalars in terms of their real and imaginary parts, \( z^a = \alpha^a - i \lambda^a \), the Kähler potential and the hermitian metric \( g_{ab} \) read:

\[
e^{-\kappa} = -\frac{4}{3} d_{abc} \alpha^a \lambda^b \lambda^c > 0 \quad \text{and} \quad g_{ab} = \partial_a \partial_b K = -\frac{3}{2} \left( d_{ab} - \frac{3 d_a d_b}{d} \right) > 0,
\]

where \( d \equiv d_{abc} \alpha^a \lambda^b \lambda^c \) should be a negative number, and we have used the short hand notation: \( d_a \equiv d_{abc} \lambda^b \lambda^c \), \( d_{ab} \equiv d_{abc} \lambda^c \).

Cubic models originate from dimensional reduction of five dimensional supergravities. The real scalars \( \alpha^a = \text{Re}(z^a) \) are the internal components of the five-dimensional vectors, while \( \lambda^a = -\text{Im}(z^a) \) are functions of the scalars in the five-dimensional vector multiplets and the radial modulus of the compact fifth dimension. This defines a relation of inclusion of the scalar manifold in five-dinensions \( \mathcal{M}_{D=5} \) spanned by the vector-multiplet scalars and the special Kähler manifold \( \mathcal{M}_{SK} \) in \( D = 4 \) known as r-map \([28]\). More specifically \( \mathcal{M}_{D=5} \) is geometrically characterized as a very special real manifold \([29,30]\) of real dimensions \( n - 1 \), and the r-map is a correspondence between this manifold and the special Kähler one, of complex dimension \( n \), originating from reduction over a circle. The tensor \( d_{abc} \) characterizes the very special geometry of \( \mathcal{M}_{D=5} \) and, for symmetric manifolds \( \mathcal{M}_{D=5} = G_5/H_5 \), it is invariant with respect to the isometry group \( G_5 \). In this case the special Kähler manifold \( \mathcal{M}_{SK} \), image to \( \mathcal{M}_{D=5} \) through the r-map, is symmetric as well, namely it as the form \( \mathcal{M}_{SK} = G_4/H_4 \), with \( G_5 \subset G_4 \). If we further compactify the four dimensional theory with only vector multiplets to three-dimensions, and we dualize vectors into scalars, we end up with a sigma model in which the target space is a quaternionic Kähler manifold \( \mathcal{M}_{QK} \) or a para-quaternionic Kähler manifold \( \mathcal{M}_{QK}^* \), depending on whether the internal circle is space-like or time-like, respectively. The inclusion relation between \( \mathcal{M}_{SK} \) and \( \mathcal{M}_{QK}^* \) \([31]\) \( (\mathcal{M}_{QK}^*) \) is called c-map \([32,33]\) (c*-map). The property of the manifold in
$D = 4$ of being homogeneous or homogeneous-symmetric is preserved by both the $c$-
and the $c^*$-maps. Thus if we consider a symmetric special Kähler manifold of the form
$\mathcal{M}_SK = G_4/H_4$, its image through the $c$-map is a manifold of the form $\mathcal{M}_{\text{QK}} = G/H$, $H$
being the maximal compact subgroup of the isometry group $G$, and through the $c^*$-
map will have the form $\mathcal{M}_{\text{QK}}^* = G/H^*$, $H^*$ still being maximal in $G$, though no longer
compact (in fact it is a different real form of the complexification of $H$). A common feature
of $\mathcal{N} = 2$ four dimensional symmetric supergravities is that, upon time-like dimensional
reduction to $D = 3$, the isotropy group $H^*$ has the general form: $H^* = \text{SL}(2, \mathbb{R}) \times G_4'$,
where the prime in $G_4'$ is used to distinguish it from the four-dimensional duality group $G_4$
the two, though being the same Lie group, are distinct inside $G$. Stationary solutions in
four-dimensions can be described as solutions of the $D = 3$ sigma-model obtained through
time-reduction [1], see also Sect. 2.2.

We can always represent locally the manifold $\mathcal{M}_{\text{QK}}^*$ as follows:
$$\mathcal{M}_{\text{QK}}^* = [\text{O}(1, 1) \times \mathcal{M}_SK] \ltimes \text{exp(Heis)},$$  (14)
where $\text{O}(1, 1)$ is parametrized by the radial modulus $e^U$ of the internal timelike circle,
the corresponding generator being denoted by $T_0$. $\text{Heis}$ denotes a $(2n + 3)$-dimensional
Heisenberg algebra [33] parametrized by the $2n + 2$ scalar fields $Z^M$ originating from
the four dimensional vectors (their time components and the scalars dual to their three-
dimensional descendants) and the scalar $a$ dual to the Kaluza Klein vector. If $T_M, T_0$
denote the corresponding generators, the following characteristic commutation relations
hold:

$$[T_0, T_M] = \frac{1}{2} T_M; \quad [T_0, T_\bullet] = T_\bullet; \quad [T_M T_N] = C_{MN} T_\bullet, \quad (15)$$

all other commutators being zero. If $\mathcal{M}_SK$ normal homogeneous, see below, denoting by
$T_r$ the generators of solvable Lie group of isometries acting transitively on the manifold we
have:

$$[T_0, T_r] = [T_\bullet, T_r] = 0; \quad [T_r, T_M] = T_r^{N M} T_N; \quad [T_r, T_s] = -T_r s' T_s', \quad (16)$$

$T_r^{N M}$ representing the symplectic representation of $T_r$ on contravariant symplectic vectors.

While the corresponding representation (14) is global for the manifold $\mathcal{M}_{\text{QK}}$, image
through the $c$-map, it is only local for $\mathcal{M}_{\text{QK}}^*$ and defines the physical patch of the manifold,
spanned by the physical scalar fields $U, a, Z^M, z^a, \bar{z}^a$. At the boundary of this patch $e^{-U}$
vanesishes, in general signalling a singularity in the four-dimensional stationary space-time
metric. We can therefore safely restrict ourselves to this parch when considering non-
singular four-dimensional solutions.

The special coordinates $z^a$, in light of their five-dimensional origin, can be characterized
as transforming in a linear representation of the subgroup $G_5$ of $G_4$. This feature is useful
in order to identify the $z^a$ within a parametrization of the manifold $\mathcal{M}_{\text{QK}}$ or $\mathcal{M}_{\text{QK}}^*$.

In the problem under consideration, the four dimensional $\mathcal{N} = 2$ model contains $n = 6$
vector multiplets and no hypermultiplets. We have the following inclusion relations:

$$\mathcal{M}_{D=5} = \text{SL}(3, \mathbb{R}) \rightarrow \mathcal{M}_SK = \text{Sp}(6, \mathbb{R}) \rightarrow \mathcal{M}_{\text{QK}}^* = \frac{\text{SL}(2, \mathbb{R}) \times \text{Sp}(6, \mathbb{R})}{\text{SL}(2, \mathbb{R}) \times \text{Sp}(6, \mathbb{R})}. \quad (17)$$
In this case the global symmetry group of the $D = 3$ sigma-model, namely the isometry group of the corresponding target space, is $G = F_{4(4)}$, $H^* = SL(2, \mathbb{R}) \times G_4' = SL(2, \mathbb{R}) \times Sp'(6, \mathbb{R})$ and the maximal compact subgroup of $G$ is $H = SU(2) \times USp(6)$. The complex dimension of the scalar manifold spanned by the $D = 4$ vector multiplets’ scalars is $n = 6$, $G_4 = Sp(6, \mathbb{R})$ and $H_4 = U(3)$. The representation $R$ by which $G_4$ is embedded in the structure group $Sp(14, \mathbb{R})$ is the $14'$ of $Sp(6, \mathbb{R})$. The special coordinates $z^a$, $a = 1, \ldots, 6$, transform in the $6$ of $G_5 = SL(3, \mathbb{R})$ and thus can be identified with the six independent entries of a complex symmetric matrix $z^a \equiv z^{i,j} = z^{j,i}$, $i, j = 1, 2, 3$. The cubic prepotential $\mathcal{F}(z^a)$, being $SL(3, \mathbb{R})$-invariant, can only have the following form:

$$\mathcal{F}(z^{i,j}) = \epsilon_{i_1 j_1 k_1} \epsilon_{i_2 j_2 k_2} z^{i_1 i_2} z^{j_1 j_2} z^{k_1 k_2}.$$  \hspace{1cm} (18)

If we make the identification $(z^a) = (z^{1,1}, z^{1,2}, z^{1,3}, z^{2,2}, z^{2,3}, z^{3,3})$, then the scalars corresponding to the diagonal entries parametrize a characteristic submanifold $\mathcal{M}^{(STU)}$ of the special Kähler manifold:

$$z^1 = z^{1,1} = s = a_1 - i e^{\phi_1}; \quad z^4 = z^{2,2} = t = a_2 - i e^{\phi_2}; \quad z^6 = z^{3,3} = u = a_3 - i e^{\phi_3},$$  \hspace{1cm} (19)

where $\phi_i$ are the three dilatonic scalars parametrizing the three dimensional Cartan subalgebra in the coset and we have set $a_1 = \alpha^3$, $a_2 = \alpha^4$, $a_3 = \alpha^6$. This submanifold has the form:

$$\mathcal{M}^{(STU)} = \left( \frac{SL(2, \mathbb{R})}{SO(2)} \right)^3,$$  \hspace{1cm} (20)

and defines the STU truncation of the model, describing an $\mathcal{N} = 2$ supergravity coupled to 3 vector multiplets.

We can then describe the special coordinates in terms of the $2n$ real scalar fields $(\phi^r) = (\varphi_i, \alpha^a, \lambda^\ell)$, where $\ell = 2, 3, 5$ and the corresponding scalars $z^\ell$ are the off diagonal entries $z^{i,j}$, $i \neq j$. In terms of $z^a$ and $\phi^r$, the prepotential and the Kähler potential, respectively, read:

$$\mathcal{F}(z^a) = \frac{1}{6} d_{abc} z^a z^b z^c = z^1 z^4 z^6 + 2 z^2 z^3 z^5 - (z^3)^2 z^4 - (z^2)^2 z^6 - (z^5)^2 z^1,$$

$$\frac{e^{-\mathcal{K}}}{8} = -\frac{1}{6} d_{abc} \lambda^a \lambda^b \lambda^c = \epsilon^{\varphi_1+\varphi_2+\varphi_3} + 2 \lambda^2 \lambda^3 \lambda^5 - \epsilon^{\varphi_1} (\lambda^5)^2 - \epsilon^{\varphi_2} (\lambda^3)^2 - \epsilon^{\varphi_3} (\lambda^2)^2.$$  \hspace{1cm} (21)

Positive definiteness of $g_{ab}$ implies, besides $\lambda^1, \lambda^4, \lambda^6 > 0$, which is consistent with our position (19), also $\epsilon^{\varphi_2+\varphi_3} - (\lambda^3)^2 > 0$, $\epsilon^{\varphi_1+\varphi_3} - (\lambda^2)^2 > 0$, $\epsilon^{\varphi_1+\varphi_1} - (\lambda^5)^2 > 0$. We identify the origin $O$ of the manifold with the point $\varphi_i = a_i = \alpha^\ell = \lambda^\ell = 0$, and construct the coset representative $L_4(\phi^r) = (L_4(\phi^r)^M_N)$ as follows:

$$L_4(\phi^r) = \bar{L}_4(\phi^r) \bar{L}_4(O)^{-1}; \quad L_4(O) = 1.$$  \hspace{1cm} (22)

The construction of this matrix applies to the most general symmetric homogeneous special Kähler manifold. The symplectic matrix $L_4(\phi^r)$ is continuously connected to the identity matrix. In fact it can be verified that $L_4$ is an element of the solvable subgroup $S_4$ of
the isometry group $G_4$ which acts transitively on the manifold\(^3\). One can also verify that $\mathbb{L}_4(z, \bar{z})$ represents the symplectic transformation which maps the symplectic section $V$ computed in $O$ into the one evaluated at a generic point:

$$\mathbb{L}_4(z, \bar{z})V(O) = V(z, \bar{z}).$$

The solvable group of isometries, which $\mathbb{L}_4$ belongs to, for symmetric manifolds $\mathcal{M}_{SK} = G_4/H_4$ is defined by the Iwasawa decomposition of the semisimple group $G_4$ with respect to its maximal compact subgroup $H_4$. If we write $\mathcal{S}_4 = \exp(Solv_4)$, where $Solv_4$ is a solvable Lie algebra, its parameters must be in relation with the scalars $\phi^r$. This relation is readily computed for the model under consideration. First define the generators $T_r = (T_r)^{M_N}$ as follows:

$$T_r = \left. \frac{\partial \mathbb{L}_4}{\partial \phi^r} \right|_{\phi^r = 0}. \quad (23)$$

One can verify that they close a solvable algebra $Solv_4$ defined by the Iwasawa decomposition of $\mathfrak{g}_4 = \mathfrak{sp}(6)$ with respect to its maximal compact subalgebra $\mathfrak{S}_4 = \mathfrak{u}(3)$. This construction is general and applies to any symmetric homogeneous special Kähler manifold. $Solv_4$ is the Borel subalgebra of $\mathfrak{sp}(6)$ and is spanned by the three diagonal Cartan generators $h_i$, $i = 1, 2, 3$, and by the 9 shift generators $E_\beta$ corresponding to the positive roots $\beta$. The latter can be split into $a_k$, $k = 1, \ldots, 6$, $b_\ell$, $\ell = 2, 3, 5$. The generators $T_r = \{ h_i, a_k, b_\ell \}$ are defined as follows:

$$h_i = \left. \frac{\partial \mathbb{L}_4}{\partial \phi^r} \right|_{\phi^r = 0}; \quad a_k = \left. \frac{\partial \mathbb{L}_4}{\partial \phi^r} \right|_{\phi^r = 0}; \quad b_\ell = \left. \frac{\partial \mathbb{L}_4}{\partial \phi^r} \right|_{\phi^r = 0}. \quad (24)$$

The coset representative $\mathbb{L}_4$ can be constructed as an element of the solvable group $\exp(Solv_4)$ through the following exponential map:

$$\mathbb{L}_4 = \exp\left(\sum_{a=1}^{6} \alpha^a a_a\right)\exp\left(\sum_{\ell=2,3,5} f^\ell(\lambda, \varphi) b_\ell\right)\exp\left(\sum_{i=1}^{3} \Phi^i(\lambda, \varphi) h_i\right),$$

$$f^2(\lambda, \varphi) = \frac{e^{\varphi_3} \lambda^2 - \lambda^3 \lambda^5}{D}; \quad f^3(\lambda, \varphi) = \frac{e^{\varphi_2} \lambda^3 - \lambda^2 \lambda^5}{2D} + \frac{e^{-\varphi_3} \lambda^3}{2}; \quad f^5(\lambda, \varphi) = e^{-\varphi_3} \lambda^5,$$

$$\Phi^1(\lambda, \varphi) = \log\left(\frac{e^{-\lambda}}{8D}\right); \quad \Phi^2(\lambda, \varphi) = \varphi_2 + \log(e^{-\varphi_2 - \varphi_3} D); \quad \Phi^3(\lambda, \varphi) = \varphi_3, \quad (25)$$

where $D = e^{\varphi_2 + \varphi_3} - (\lambda^5)^2 > 0$. Eq.s (25) define, for our specific model, the precise relation between special coordinates and the relevant parametrization.

\(^3\) Such solvable group of isometries with a simple and transitive action on the manifold exists in all the homogeneous special Kähler manifolds which are relevant to supergravity. The existence of this group defines the so called normal homogeneous manifolds, which were classified in [28, 34, 35]. Scalar fields arising from the dimensional reduction of higher dimensional string excitations are parameters of this solvable group and define the (global) solvable parametrization [36, 37] of the manifold.
Once we have the solvable generators \( T_\epsilon = \{ h_\iota, E_\beta \} \) in the symplectic representation \( \mathbb{R} \), the full Lie algebra \( \mathfrak{g}_4 \) generating the group \( G_4 \) in the same representation, is simply obtained as follows:
\[
\mathfrak{sp}(6) = \text{Span}(h_\iota, E_\beta, E_{-\beta}) ,
\] (26)
where \( E_{-\beta} = \eta_4 E_\beta^T \eta_4^{-1} \), and
\[
\eta_4 \equiv \bar{\mathbb{L}}_4(O) \mathbb{L}_4(O)^T = \text{diag}(1, 1, 1/2, 3/2, 1, 1/2, 1, 1, 1, 2, 2, 1, 2, 1) .
\] (27)

Let us give the precise correspondence between the generators \( h_\iota, a_\iota, b_\iota \) in terms of \( \mathfrak{sp}(6) \)-roots \( \pm \beta \). If \( \epsilon_i, i = 1, 2, 3 \), is an orthonormal basis of the root space, so that \( \beta = \beta^i \epsilon_i \), and \( H_i \) the corresponding generators in the Cartan subalgebra, so that \( \beta(H_i) = \beta^i \), the basis \( \{ H_i, E_\beta, E_{-\beta} \} \) of the algebra is defined by the usual commutation relations:
\[
[H_i, E_{\pm \beta}] = \pm \beta^i E_{\pm \beta} ; \quad [E_\beta, E_{-\beta}] = \beta^i H_i .
\] (28)
The Cartan generators \( H_i \) are related to \( h_\iota \) as follows:
\[
h_\iota = \frac{1}{2} H_i .
\] (29)
The relation between \( E_\beta \) and \( a_\iota, b_\iota \) is summarized in the Table below.

| \( E_\beta \) | \( \beta \) | \( \beta^i \) | solvable generator |
|----------------|-------------|--------------|-------------------|
| \( E_{\beta_1} \) | \( \beta_1 \) | \( 1, -1, 0 \) | \( b_2 \) |
| \( E_{\beta_2} \) | \( \beta_2 \) | \( 0, 1, -1 \) | \( b_5 \) |
| \( E_{\beta_3} \) | \( \beta_3 \) | \( 0, 0, 2 \) | \( \sqrt{2} a_6 \) |
| \( E_{\beta_1+\beta_2} \) | \( \beta_1 + \beta_2 \) | \( 1, 0, -1 \) | \( b_3 \) |
| \( E_{\beta_2+\beta_3} \) | \( \beta_2 + \beta_3 \) | \( 0, 1, 1 \) | \( a_5 \) |
| \( E_{\beta_1+\beta_2+\beta_3} \) | \( \beta_1 + \beta_2 + \beta_3 \) | \( 1, 0, 1 \) | \( a_3 \) |
| \( E_{2\beta_2+\beta_3} \) | \( 2\beta_2 + \beta_3 \) | \( 0, 2, 0 \) | \( \sqrt{2} a_4 \) |
| \( E_{\beta_1+2\beta_2+\beta_3} \) | \( \beta_1 + 2\beta_2 + \beta_3 \) | \( 1, 1, 0 \) | \( a_2 \) |
| \( E_{2\beta_1+2\beta_2+\beta_3} \) | \( 2\beta_1 + 2\beta_2 + \beta_3 \) | \( 2, 0, 0 \) | \( \sqrt{2} a_1 \) |

The solvable generators of the STU truncation are then \( h_\iota, a_1, a_4, a_6 \).

In terms of \( \mathbb{L}_4(\phi^r) \) matrix \( \mathcal{M}_4 \) reads: \( \mathcal{M}_4 = \mathbb{C}\mathbb{L}_4(\phi^r) \eta_4 \mathbb{L}_4(\phi^r)^T \mathbb{C} \), as it can easily be derived from Eq. (10).

Since, with respect to \( O(1, 1) \times \text{SL}(3, \mathbb{R}) \) the \( 14' \) of \( \text{Sp}(6, \mathbb{R}) \) branches as:
\[
14' \rightarrow 1_{-3} + 6_{-1} + 1_{+3} + 6_{+1} ,
\] (30)
we can split the index \( \Lambda \) labeling the vector fields \( A_\mu^\Lambda \) as well as the upper component of \( V \), consequently:
\[
A_\mu^\Lambda = \{ A_\mu^0, A_\mu^\iota \} = \{ A_\mu^0, A_{\mu}^{i,j} \} ,
\] (31)
$A_\mu^0$ being the graviphoton in the $1_{-3}$ and $A_\mu^a \equiv A_\mu^{ij}$ the remaining six vectors in the $6_{-1}$. Just as for the scalar fields, the truncation to the STU model is effected by setting all $A_\mu^{ij}$, with $i \neq j$, to zero, or, equivalently, $A_\mu^\ell \to 0$, $\ell = 2, 3, 5$. The bosonic content of the STU model then consists, besides of the metric, of $z^{ij}$, $A_\mu^0$ and $A_\mu^i$, $i = 1, 2, 3$, corresponding to $A_\mu^a$, with $a = 1, 4, 6$.

The above analysis is useful for defining a one to one correspondence between (solvable) coordinates of $M^*_{QK}$ in the three dimensional theory, and four dimensional fields, which we shall need to oxidize geodesic solutions on $M^*_{QK}$ to $D = 4$ static black holes. Indeed we now know how to intrinsically define the special coordinates $z^a$ as a subset of the $D = 3$ fields $\phi^I$ in a suitable parametisation. To this end we locally represent $M^*_{QK}$ in the physical patch as a solvable metric Lie group $\exp(Solv)$, with:

$$Solv = [\mathfrak{o}(1, 1) \oplus Solv_4] \oplus_s Heis,$$

(32)

where $\oplus_s$ denotes a semidirect sum. As usual for symmetric homogeneous manifolds, $Solv$ is defined by the Iwasawa decomposition of $\mathfrak{f}_{4(4)}$ with respect to its maximal compact subalgebra. Then we choose as generators of $Solv$ the matrices $T_I = \{T_0, T_r, T_M, T_\bullet\}$ satisfying the general relations (15),(16), $T_r$ being the generators of $Solv_4$, and $T_M$ are chosen so that the adjoint action of $T_r$ on them, described by the $2n$ matrices $T_M^M N$, realizes the symplectic representation $R$ of $T_r$ computed above and pertaining to the special coordinate frame. Let us give the weights $\gamma_M$ associated with the representation $R$ in this basis, defined by:

$$(H_i)^M N = \gamma_M (H_i) \delta^M_N \text{ no summation over } M.$$  

(33)

In the table below we list the weights $\gamma_M$ in the orthonormal basis $(\epsilon_i)$ and give the correspondence of the corresponding charge entry with $D0, D2, D4, D6$-charges in Type IIA theory.

---

We shall describe the generators of the Lie algebra $\mathfrak{g}$ of $G$ in the fundamental representation of this group, which is the $26$ of $\mathfrak{F}_{4(4)}$. With an abuse of notation we use for the $T_r$ generators in $\mathfrak{g}$, the same symbol used for the abstract generators of $\mathfrak{g}_4$. 

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\begin{align*}
\gamma \Gamma & \quad \gamma \Gamma_i \quad \gamma \Gamma_M \quad (p^4, q_\Lambda) \quad Dp\text{-charge} \\
\gamma_1 & \quad (1, -1, -1) \quad p^0 \quad D6 \\
\gamma_2 & \quad (0, 0, -1) \quad p^1 \quad D4 \\
\gamma_3 & \quad (0, -1, 0) \quad p^2 \quad D4 \\
\gamma_4 & \quad (0, -1, 0) \quad p^3 \quad D4 \\
\gamma_5 & \quad (1, -1, -1) \quad p^4 \quad D4 \\
\gamma_6 & \quad (1, 1, 1) \quad q_0 \quad D0 \\
\gamma_7 & \quad (1, 1, 1) \quad q_1 \quad D2 \\
\gamma_8 & \quad (0, 0, 1) \quad q_2 \quad D2 \\
\gamma_9 & \quad (0, 1, 0) \quad q_3 \quad D2 \\
\gamma_{10} & \quad (1, -1, 1) \quad q_4 \quad D2 \\
\gamma_{11} & \quad (1, 0, 0) \quad q_5 \quad D2 \\
\gamma_{12} & \quad (1, 1, 1) \quad q_6 \quad D2 \\
\gamma_{13} & \quad (1, -1, -1) \quad q_6 \quad D2 \\
\gamma_{14} & \quad (1, -1, -1) \quad q_6 \quad D2
\end{align*}

The truncation to the STU model is effected by restricting to the weights \( \gamma_1, \gamma_2, \gamma_5, \gamma_7, \gamma_8, \gamma_9, \gamma_{12}, \gamma_{14} \), consistently with our previous discussion about the vector fields. Upon time-reduction to \( D = 3 \) and dualizations of vectors into scalars, the STU truncation yields the following quaternionic Kähler submanifold:

\[ \mathcal{M}_{QK}^{STU} = \frac{SO(4, 4)}{SO(2, 2) \times SO(2, 2)}. \quad (34) \]

Having characterized the special coordinates in an intrinsic algebraic way, and knowing how to embed \( \text{Solv}_4 \) inside \( \text{Solv} \), we can construct the corresponding coset representative \( \mathbb{L}^4(\phi^I) \) as an element of \( \exp(\text{Solv}) \) in the fundamental representation 26 of \( G = F_{4(4)} \). The coset representative \( \mathbb{L}(\phi^I) \) of \( F_{4(4)}/[\text{SL}(2, \mathbb{R}) \times \text{Sp}(6, \mathbb{R})] \) in the solvable parametrization can be defined by the following exponential map:

\[ \mathbb{L}(\phi^I) = \exp(-a T_s) \exp(\sqrt{2} Z M^T M) \mathbb{L}^4(\phi^I) \exp(2 U T_0). \quad (35) \]

We can define the involutive automorphism \( \sigma \) on the algebra \( \mathfrak{g} \) of \( G \) which leaves the algebra \( \mathfrak{H}^* \) generating \( H^* \) invariant. This involution in the fundamental representation of \( G \) has the form \( \sigma(M) = -\eta M^T \eta \), \( \eta \) being an \( H^* \)-invariant metric, and induces the (pseudo)-Cartan decomposition of \( \mathfrak{g} \) of the form:

\[ \mathfrak{g} = \mathfrak{H}^* \oplus \mathfrak{R}^*, \quad (36) \]

where \( \sigma(\mathfrak{R}^*) = -\mathfrak{R}^* \), and the following relations hold

\[ [\mathfrak{H}^*, \mathfrak{H}^*] \subset \mathfrak{H}^*, \quad [\mathfrak{H}^*, \mathfrak{R}^*] \subset \mathfrak{R}^*, \quad [\mathfrak{R}^*, \mathfrak{R}^*] \subset \mathfrak{H}^*. \quad (37) \]

We see that \( H^* \) has a linear adjoint action in the space \( \mathfrak{R}^* \) which is thus the carrier of an \( H^* \)-representation. As previously pointed out, \( N = 2 \) symmetric models, \( H^* = \text{SL}(2, \mathbb{R}) \times G_4' \) and its adjoint action on \( \mathfrak{R}^* \) realizes the representation \((2, \mathbb{R})\).
The decomposition (36) has to be contrasted with the ordinary Cartan decomposition of \(\mathfrak{g}\)

\[
\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k},
\]

into its maximal compact subalgebra \(\mathfrak{h}\) generating \(H\) and its orthogonal non-compact complement \(\mathfrak{k}\). This decomposition is effected through the Cartan involution \(\tau\) of which \(\mathfrak{h}\) and \(\mathfrak{k}\) represent the eigenspaces with eigenvalues +1 and −1 respectively. In the real matrix representation in which we shall work, the action of \(\tau\) can be implemented as:

\[
\tau(X) = -X^T.
\]

Next we construct the left invariant one-form and the vielbein \(P_A = P_I A d\phi^I:\)

\[
L^{-1} dL = P_A T_A = P_A K_A + \Omega_H, \quad A = 1, \ldots, 4n + 4.
\]

where we have introduced the basis \(\{K_A\}\) of \(\mathfrak{k}^*\) to be defined below in eq. (81). Following the prescription of [2], the normalization of the \(H^*\)-invariant metric on the tangent space of \(\mathcal{M}_{QK}\) is chosen as follows

\[
g_{AB} = \frac{1}{2 \text{Tr}[T_0^2]} \text{Tr}[T_A T_B] = \frac{1}{6} \text{Tr}[T_A T_B],
\]

being \(\text{Tr}[T_0^2] = 3\) in our model. The metric of the \(D = 3\) sigma-model has the familiar form:

\[
ds^2 = P_A P_B g_{AB} = 2 dU^2 + 2 g_{ab} d\phi^a d\phi^b + \frac{e^{-4U}}{2} \omega^2 + e^{-2U} d\Omega^2 M_4(\phi^r) d\Omega,
\]

\[
\omega = da + Z^T C d\Omega.
\]

### 2.2 Static Black Holes and Geodesics

We shall now restrict our discussion to static, spherically symmetric and asymptotically flat black hole solutions. The general ansatz for the metric has the following form:

\[
ds^2 = -e^{2U} dt^2 + e^{-2U} \left( \frac{e^A}{\sinh^2 (c \tau)} d\tau^2 + \frac{e^2}{\sinh^2 (c \tau)} d\Omega^2 \right),
\]

where \(U = U(\tau)\) and the coordinate \(\tau\) is related to the radial coordinate \(r\) by the following relation:

\[
\frac{e^2}{\sinh^2 (c \tau)} = (r - r_0)^2 - c^2 = (r - r^-)(r - r^+).
\]

Here \(c^2 \equiv 2ST\) is the extremality parameter of the solution, with \(S\) the entropy and \(T\) the temperature of the black hole. When \(c\) is non vanishing the black hole has two horizons located at \(r^\pm = r_0 \pm c\). The outer horizon is located at \(r_H = r^+\) corresponding to \(\tau \to -\infty\). The extremality limit at which the two horizons coincide, \(r_H = r^+ = r^- = r_0\), is \(c \to 0\). For extremal solutions eq. (44) reduces to \(\tau = -1/(r - r_0)\). Spherical symmetry also
requires the scalar fields in the solution to depend only on $\tau$: $\phi^r = \phi^r(\tau)$. The solution is also characterized by a set of electric and magnetic charges defined as follows:

$$
p^\Lambda = \frac{1}{4\pi} \int_{S^2} F^\Lambda, \quad q_\Lambda = \frac{1}{4\pi} \int_{S^2} G_\Lambda, \tag{45}
$$

where $S^2$ is a spatial two-sphere in the space-time geometry of the dyonic solution (for instance, in Minkowski space-time the two-sphere at radial infinity $S^2_{\infty}$). In terms of these charges the general ansatz for the electric-magnetic field strength vector $F^\Lambda$, $G_\Lambda$ reads:

$$
\mathbb{F} = \left( \begin{array}{c}
F^\Lambda_{\mu\nu} \\
G^\Lambda_{\mu\nu}
\end{array} \right) \frac{dx^\mu \wedge dx^\nu}{2} = e^{2U} \mathbb{C} \cdot \mathcal{M}_4(\phi^r) \cdot \Gamma d\tau + \Gamma \sin(\theta) d\theta \wedge d\varphi, \\
\Gamma = (\Gamma^M) = \left( \begin{array}{c}
p^\Lambda \\\nq^\Lambda
\end{array} \right) = \frac{1}{4\pi} \int_{S^2} \mathbb{F}.
$$

In $D = 4$ these solutions are described by the following effective action

$$
S^{(4)}_{\text{eff}} = \int \mathcal{L}^{(4)}_{\text{eff}} d\tau = \int \left( \ddot{U}^2 + \frac{1}{2} G_{rs}(\phi) \dot{\phi}^r \dot{\phi}^s + e^{2U} V(\phi; \Gamma) \right) d\tau, \tag{47}
$$

where the upper dot stands for the derivative of the field with respect to $\tau$ and the effective potential $V(\phi; \Gamma)$ reads:

$$
V(\phi; \Gamma) = -\frac{1}{2} \Gamma^T \mathcal{M}_4(\phi) \Gamma > 0. \tag{48}
$$

In the $D = 3$ Euclidean theory the effective action reads

$$
S_{\text{eff}} = \int \mathcal{L}_{\text{eff}} d\tau, \tag{49}
$$

$$
\mathcal{L}_{\text{eff}} = \frac{1}{2} g_{IJ}(\phi) \dot{\phi}^I \dot{\phi}^J = \ddot{U}^2 + g_{\hat{a}\hat{b}} \dot{\varphi}^\hat{a} \dot{\varphi}^\hat{b} + e^{-2U} Z^T \mathcal{M}_4(\phi^r) \dot{Z}. 
$$

The two effective actions $\mathcal{L}_{\text{eff}}$ and $\mathcal{L}^{(4)}_{\text{eff}}$ are related by a Legendre transformation trading the cyclic variables $Z^M$ with their conserved conjugate momenta, which are the quantized charges $\Gamma^M$.

Solutions $\phi^I(\tau)$ to the $D = 3$ theory are geodesics on the symmetric homogeneous manifold $\mathcal{M}_{QK}^*$ with pseudo-Riemannian metric $g_{IJ}(\phi)$. The “velocity vector” of the geodesic can be described by the $\mathfrak{r}^*$-matrix

$$
L(\tau) = \dot{\phi}^I(\tau) P_I^A(\phi(\tau)) K_A = \Delta^A(\tau) K_A, \tag{50}
$$

in terms of which the effective action reads:

$$
\mathcal{L}_{\text{eff}} = \frac{C}{2} \text{Tr}(L^2) = \frac{1}{2} g_{AB} \Delta^A \Delta^B, \tag{51}
$$

where $C \equiv 1/(2\text{Tr}(T_0^2))$ depends on the chosen representation for the $g$-generators. For our model, having chosen to represent all matrices in the fundamental 26 representation of $\mathfrak{F}_{4(4)}$, $C = 1/6$. 

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The geodesic equations derived from (49) can be cast into the following equivalent forms:

\begin{align}
\mathcal{M}^{-1} \dot{\mathcal{M}} &= 2 Q^T = \text{const.}, \\
\dot{L} - [W, L] &= 0,
\end{align}

where \( \mathcal{M}(\tau) \equiv \mathbb{L}(\phi(\tau)) \eta \mathbb{L}(\phi(\tau))^T \), \( Q \) is the \( g \)-matrix of the Noether charges of the solution and \( W(\tau) \) is a compensator matrix, defined as \( W = \phi^I \Omega_{H^*, I} \), \( \Omega_{H^*} \) being the \( \mathfrak{h}^* \)-valued 1-form introduced in (39). Eq. (53) is a Lax-pair equation in the Lax matrix \( L(\tau) \), whose relation to the Noether charge matrix \( Q \) is

\[ Q = \mathbb{L}(\phi) \left( L \mathbb{L}(\phi)\right)^{-1}. \]

Using the notation of [2] the ADM mass, the scalar charges, the quantized charges and the NUT charge are computed as traces of \( Q \) with the solvable generators \( T_0, T_r, T_M, T_s \), respectively. In particular the electric-magnetic charges \( (\Gamma^M) = (p^I, q_A) \) can be evaluated as follows:

\[ \Gamma^M = \sqrt{2} C C^{MN} \text{Tr}(Q T_N), \]

while the ADM mass reads:

\[ M_{\text{ADM}} = C \text{Tr}(Q T_0). \]

Let us consider the subspace \( \mathfrak{r}^*(\mathbb{R}) \) of \( \mathfrak{r}^* \) spanned by the compact generators \( K_A = (T_A + \eta T_A^T \eta)/2 = (T_A - T_A^T)/2, A = \ldots, 2n + 2 \). The components \( \Upsilon = (Y^A) \) of the Lax matrix within this subspace are expressed as follows:

\[ L|_{\mathfrak{r}^*(\mathbb{R})} = Y^A K_A; \quad \Upsilon(\phi^I, \Gamma) = (Y^A) = \sqrt{2} e^U \hat{\mathbb{L}}(O) Z(\phi^I, \Gamma), \]

where \( Z = (Z^A) \) is the symplectic vector defined as:

\[ Z(\phi^I, \Gamma) = \hat{\mathbb{L}}(\phi^\rho) \mathbb{C} \hat{\Gamma}; \quad \hat{\Gamma} = \Gamma - n Z, \]

where \( n \) is the NUT charge, which we shall consider to be zero on our solutions. If \( n = 0 \), \( Z \) is the symplectic vector consisting of the real and imaginary parts of the central charge \( Z \) and the matter charges \( Z_I \), defined as:

\[ Z = V^T \mathbb{C} \Gamma; \quad Z_I = U_I \mathbb{C} \Gamma = E_I^a \nabla_a Z, \]

and depends on \( \phi^\rho \) and \( \Gamma \), see [38, 39] for the notation. If a global symmetry transformation \( g_4 \in G_4 \) of the \( D = 4 \) theory is applied to the solution, it will act non-linearly (as an isometry) on the scalars \( \phi^\rho \) and linearly the charge vector \( \Gamma \) and \( Z \) through symplectic matrix \( \mathbb{R}[g_4] \) representing \( g_4 \) in \( \mathbb{R} \), while the other scalars \( U, a \) will be left unaffected. Using (2) one finds that the central/matter-charge vector transforms only through the compensator \( h \in H_4 \): \( \Upsilon(g_4 \star \phi^I, \mathbb{R}[g_4] \Gamma) = \mathbb{R}[h] \Upsilon(\phi^I, \Gamma). \)

In general the components \( Y^A \) transform in a representation \( \mathbb{R}' \) [5] under the larger group \( H_c = U(1)_E \times H_4 = U(1)_E \times U(3) \) which is the maximal compact subgroup of
$H^*$, where $U(1)_E$ is the maximal compact subgroup of the Ehlers group $SL(2, \mathbb{R})_E$. The representation $\mathbf{R}'$ is
\[
\mathbf{R}' = 1_{-1} + 6_{-1} + 1_{+1} + 6_{+1},
\]
where the grading refers to $U(1)_E$. The space $\mathfrak{r}_{\mathfrak{r}(\mathfrak{r})}$ is in fact the carrier of the representation $\mathbf{R}'$ with respect to the adjoint action of $H_c$.

Global Symmetry and Geodesics  A geodesic, solution to eq. (52) or, equivalently, eq. (53), is uniquely determined by its initial conditions defined by the values $\phi^I_0 = \phi^I(\tau = 0)$ of the scalar fields and of the Lax matrix $L_0 = L(\tau = 0)$ at radial infinity $\tau = 0$. Let us denote by $\phi^I[\tau; \phi_0, L_0]$ the unique geodesic with initial conditions $(\phi^I_0, L_0)$. The global symmetry group of the Euclidean $D = 3$ theory is the isometry group $G$. For a generic isometry $g \in G$, let us denote by $g \star \phi^I$ the transformed scalars, non-linear functions of the original ones $\phi^I$, defined by
\[
(61) \quad g \cdot L(\phi^I) = L(g \star \phi^I) \cdot h(\phi^I, g),
\]
where $h(\phi^I, g)$ is a compensator in $H^*$. Under the above transformation, the vielbein matrix $P = P^A \mathbb{K}_A$ transforms under the compensator only
\[
(62) \quad P(\phi) \rightarrow P(g \star \phi) = h(\phi, g) \star P(\phi) \equiv h(\phi^I, g) P(\phi) h(\phi^I, g)^{-1}.
\]

Given a geodesic $\phi^I[\tau; \phi_0, L_0]$ and an isometry $g \in G$, $g \star \phi^I[\tau; \phi_0, L_0]$ is the unique geodesic with boundary conditions $(g \star \phi^I_0, h(\phi_0, g) \star L_0)$:
\[
(63) \quad g \star \phi^I[\tau; \phi_0, L_0] = \phi^I[\tau; g \star \phi^I_0, h(\phi_0, g) \star L_0].
\]

Thus in order to classify geodesic solutions with respect to the action of the global symmetry group $G$, which is the main purpose of the present work, we can restrict to the action of $G$ on the initial conditions $(\phi_0, L_0)$. Notice that the action of transformations in $G/H^*$ is transitive on the manifold. This means that we can always map, by means of a suitable $G/H^*$ transformation, any geodesic into one originating in the origin $O$: $\phi^I_0 \equiv 0$. We are left with the action of the stability group $H^*$ of $O$ on the solution which only affects the initial velocity vector on the tangent space $T_O \mathcal{M}_{QK}^*$:
\[
(64) \quad h \in H^* : \phi^I[\tau; O, L_0] \rightarrow \phi^I[\tau; O, h^{-1} L_0 h].
\]

Thus we have reduced the problem of classifying the geodesics with respect to the action of $G$ to that of classifying the orbits of the initial velocity vector $L_0$ with respect to the adjoint action of $H^*$.

In the $D = 3$ theory there are $n_v$ fermion fields $\lambda^A$ transforming under supersymmetry as follows:
\[
(65) \quad \delta_\epsilon \lambda^A = \Delta^a_A(\tau) \epsilon_a,
\]
where $\epsilon_a$ is a doublet of supersymmetry parameters and we have written the tangent space index $A$, labeling the Lax components $\Delta^a_A$, as a couple of indices $A = (a, A)$, in which
\(a = 1, 2,\) and \(A = 1, \ldots, 2n_v\) labels the representation 2 and \(R = 14'\) of the subgroups \(SL(2, \mathbb{R})\) and \(G'_4 = Sp(6, \mathbb{R})\) of \(H^* = SL(2, \mathbb{R}) \times G'_4.\) BPS solutions are characterized by the property of preserving a fraction of supersymmetry, that is there exists a spinor \(\epsilon_a\) satisfying the Killing spinor equation: \(\delta \epsilon_a = 0,\) or, equivalently, that the rectangular matrix \(\Delta_{a,A}'\) have a null-eigenvector \(\epsilon_a: \Delta_{a,A}' \epsilon_a = 0.\) It is straightforward to prove that this is the case if and only if \(\Delta_{a,A}'\) factorizes as follows: \(\Delta_{a,A}' = \epsilon_a \Delta_{A},\) \((16),\) where \(\epsilon_a = \epsilon_{ab} \epsilon_b.\)

This property is not affected by the action of \(H^*\) and, since the action of \(G\) on a geodesic amounts to the action of an \(H^*-\)compensator on \(L_0 (i.e.on \Delta_{a,A}^\tau (\tau = 0)),\) according to eq. \((63),\) we conclude that the geodesics corresponding to BPS black holes sit in a same \(G\)-orbit. Note that the existence of a residual supersymmetry is clearly independent on \(\tau,\) since the evolution in \(\tau\) of the Lax matrix, solution to \((53),\) is governed by a suitable \(H^*-\)transformation \(O(\tau): L(\tau) = O(\tau)^{-1} L_0 O(\tau) [6,7].\)

Having set the point at radial infinity to coincide with the origin \(O (\phi \equiv 0)\) of the manifold, the components \(Y_0 = (Y_0^A)\) along \(R^{(R)}\) of the Lax matrix \(L_0\) at \(\tau = 0,\) coincides, modulo basis redefinition, with (the real and imaginary parts of) the central and matter charges which, in turn, are expressed solely as combinations of the quantized ones \(\Gamma,\) being \(\phi^r \equiv 0.\) With an abuse of notation we shall sometimes use the same symbol \(R\) for the representation of the electric and magnetic charges under \(G_4\) and for the representation \(R'\) of \(H_c.\)

### Regularity

Not all Lax matrices \(L\) generate geodesics corresponding to regular \(D = 4\) solutions, or their small limits. A necessary condition for regularity was given in [13] in terms of the following matrix equation:

\[
L(\tau)^3 = c^2 L(\tau) \iff L_0^3 = c^2 L_0 ,
\]  

(66)

for \(L_0\) evaluated in the fundamental representation of the algebra \(\mathfrak{g}\) (for all models except the one with \(\mathfrak{g} = \mathfrak{e}_8\)). The non-extremality parameter can itself be expressed in terms of \(L_0:\)

\[
c^2 = \frac{C}{2} \text{Tr}(L_0^3) .
\]  

(67)

For extremal solutions \((c = 0),\) the regularity condition requires \(L_0\) (or, equivalently, \(Q\)) to be a nilpotent matrix, with degree of nilpotency not exceeding 3:

\[
L_0^3 = 0 ,
\]  

(68)

This condition was first proven in [17].

### 2.3 Group Theoretic Structure

In this subsection we review some algebraic and geometric properties of the \(F_4(4)\)-model. \(F_4(4)\) is an exceptional, maximally split group whose Lie algebra \(\mathfrak{f}_4(4)\) is generated by \(^5\)

\[
\{H_i, E_\alpha, E_{-\alpha}\}, \quad i = 1, \cdots 4; \quad \alpha = 1, \cdots, 24.
\]  

(69)

\(^5\)We use for this basis the same normalization used for the \(\mathfrak{sp}(6)\) generators in \((28).\)
The complex Lie algebra $f_4^C$ has rank four and it is defined by the $4 \times 4$ Cartan matrix encoded in the following Dynkin diagram

$$
\begin{array}{cccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
\circ & \circ & \circ & \circ \\
\end{array}
\begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -2 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2 \\
\end{pmatrix}
$$

The corresponding root space is spanned by

$$\Delta_{\text{simple}} = \{ \alpha_1 = \epsilon_2 - \epsilon_3, \quad \alpha_2 = \epsilon_3 - \epsilon_4, \quad \alpha_3 = \epsilon_4, \quad \alpha_4 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4) \}$$

(70)

where the set of roots reads

$$\Delta_{F_4(4)} = \left\{ \epsilon_i, \pm \epsilon_i \pm \epsilon_j, \frac{1}{2}(\pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4) \right\}, \text{ with } i < j, \text{ and } i,j = 1,2,3,4, \quad (71)$$

where $(\epsilon_i)$ is a basis of four ortho-normal Euclidean vectors. For the reader’s convenience in the Table below we tabulate the roots of $F_4(4)$ in two different bases. The matrix form of $H_i$ and of the shift generators $E_{\alpha}$, $\alpha = 1, \ldots, 24$, corresponding to the roots listed in Table 1 is given in Appendix B. In four dimensions the electric and magnetic charges together span an irreducible symplectic representation $R$ of $\text{Sp}(6)$. Upon dimensional reduction on the time direction and dualization of the vector fields into scalars, the isometry group $F_4$ of the resulting moduli space now contains $\text{SL}_E(2, \mathbb{R}) \times \text{Sp}(6)$ with respect to which it is adjoint representations branches as follows

$$\text{Adj}[F_4] \rightarrow (\text{Adj}[\text{SL}(2, \mathbb{R})_E], 1) \oplus (1, \text{Adj}[\text{Sp}(6)]) \oplus (2, R),$$

$$52 \rightarrow (3, 1) \oplus (1, 21) \oplus (2, 14') \quad (72)$$

A suitable combination $H_0$

$$H_0 = H_1 + H_2 = 2 T_0,$$

(74)

being parametrized by the radial modulus of the internal circle is the Cartan generator of the $\text{SL}_E(2, \mathbb{R})$ factor (it is twice the generator $T_0$ introduced in the previous section). The positive roots $\alpha$ of $F_4$ naturally split into

i) the $\text{Sp}(6)$ positive roots $\beta = \{1, 6, 9, 10, 13, 17, 18, 19, 20\}$, such that $\beta(H_0) = 0$.

ii) the roots $\gamma_M = \{3, 2, 21, 22, 12, 8, 5, 15, 14, 24, 23, 4, 7, 11\}$, such that $\gamma_M(H_0) = 1$, with $M = 1, \ldots, 2n_v$.

iii) the roots $\beta_0 = \{16\}$ such that $\beta_0(H_0) = 2$. 

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Table 1: $f_4$-positive roots $\alpha$, each represented by a number running from 1 to 24.
The $\text{SL}_E(2, \mathbb{R})$ group is generated by $H_0, E_{\pm \beta_0}$, being $H_0 = H_{\beta_0}$. Accordingly, the branching (73) reads

$$52 \rightarrow 1_{(0)} \oplus 1_{(2)} \oplus 1_{(-2)} \oplus 21_{(0)} \oplus 14'_{(+1)} \oplus 14'_{(-1)}$$

which means

- The space $\mathbf{R}_{(+1)} = 14'$ is generated by the nilpotent generators $T_M = \{ \epsilon_M E_{rM} \}$ ($\epsilon_M \equiv 1$ except $\epsilon_2 = \epsilon_4 = \epsilon_7 = \epsilon_{14} = -1$) being parametrized by the scalar fields $Z^M$ originating from the $D = 4$ vector fields and the corresponding conserved charges are the electric and magnetic charges.

- The generator $E_{\beta_0} = E_{16} = T_\ast$ is associated with the axion dual $a$ to the Kaluza-Klein vector and the corresponding conserved charge is the Taub-NUT charge. Thus the double grading structure implies that the $\mathbf{R}_{(+1)} = 14'$ is no longer an abelian subalgebra but, together with $E_{\beta_0}$, closes a Heisenberg algebra

$$[T_M, T_N] = C_{MN} E_{\beta_0}$$

where $C_{MN}$ is the symplectic invariant matrix.

- The generators $T_\tau$ with grading zero with respect to $H_0$, i.e.,

$$\text{Solv}_4 = \text{span}\{ H_1 - H_2, H_3, H_4, E_1, E_6, E_9, E_{10}, E_{13}, E_{17}, E_{18}, E_{19}, E_{20} \}$$

are associated with the four-dimensional scalar fields $\phi_\tau$ and the corresponding scalar charges.

- We see that in $D = 3$ the maximal compact subgroup $H_c$ of $H^\ast = \text{Sp}'(6) \times \text{SL}(2, \mathbb{R})$ can be written as $H_c = U(1)_E \times U(3)$ where $U(1)_E$ factor is generated by $E_{\beta_0} - E_{-\beta_0}$.

The solvable parametrization is defined by the coset representative $\mathbb{L}( \phi^I )$ in (35). The matrix $\eta$ defining the decomposition through the involution $\sigma$ has the following intrinsic expression in terms of $H_0$,

$$\eta = e^{2T}, \quad T = H_0 \ln(i)$$

yielding

$$\eta = \text{diag}(-1, -1, -1, -1, 1, -1, 1, 1, 1, 1, 1, 1, 1, 1, 1, -1, 1, -1, -1, -1)$$

(79)

A geodesic on the manifold $G/H^\ast$ is parametrically described by the functions $\phi^I(\tau)$, $\tau$ being the affine parameter related to the radial variable in the four dimensional black hole solution. The pull-back of the left-invariant Cartan-Maurer form along the geodesic takes the form

$$\Omega = \mathbb{L}^{-1} \frac{d}{d\tau} \mathbb{L}, \quad L = \frac{1}{2} (\Omega + \eta \Omega^T \eta),$$

where $L$ is Lax operator defined in (50). We denote the generators of solvable algebra by $T_A$ which, for our model, are

$$T_A = \{ H_i, E_\alpha \}, \quad i = 1, \ldots, 4; \quad \alpha = 1, \ldots, 24,$$
The following relations then hold
\[ \{ K_A, A \} = \{ H_i, K_\alpha \} ; \quad K_\alpha = \frac{1}{2} (E_\alpha + \eta E_\alpha^T \eta) . \] (81)

having \( K_A \) denoted the generators of \( \mathfrak{g}^* \). We also define the generators of \( H^* \) by \( J_\alpha \)
\[ J_\alpha = \frac{1}{2} (E_\alpha - \eta E_\alpha^T \eta) . \] (82)

The branching (72) implies that the tangent space \( \mathfrak{r}^* \) of \( G/H^* \) defined by the pseudo-
Cartan decomposition of \( \mathfrak{g} \), transforms in the \((2, \mathbb{R})\) of \( H^* \). A generic element \( L \in \mathfrak{r}^* \) thus
has the form
\[ L = (\Delta (aA)) , \quad \text{where, } A = 1, \cdots \dim(\mathbb{R}) ; \quad a = 1, 2. \] (83)

From the general form of \( H^* \) we infer that
\[ p = \text{rank} \left( \frac{H^*}{H_c} \right) = \text{rank} \left( \frac{G^*_4}{H^*_4} \right) + 1 = \text{rank} \left( \frac{\text{Sp}(6)'}{U(3)} \right) + 1 = 4 \] (84)

where \( p \) is the dimension of the minimal space (normal space) defined by the normal form
of \( \mathbb{R}' \) with respect to \( H_c \), see discussion below eq. (56). This will be relevant in see Sect.
4 when we will define a submanifold \( \mathcal{M}_N \) of \( \mathcal{M}_Q K \) within which the generating geodesic
of regular/small single center black holes unfolds. \( ^{6} \)

There are four roots \( \gamma_k \) out of \( \gamma_M \), which define the normal form and which are mutually orthogonal. Our choice of the \( E_{\gamma_k} \)
will be
\[ E_{\gamma_k} = \{ E_3, E_{14}, E_4, E_{11} \} , \] (85)

which define a set of four conserved quantized charges in \( D = 4 \) and which correspond
to the generators \( T_M, M = 1, 9, 12, 14 \). Out of these generators we can construct two \( p \)-dimensional abelian spaces \( \mathfrak{r}^{(R)}_N \) and \( \mathfrak{r}^{(R)}_Q \) whose generators will be denoted by \( \{ K_\ell \} \) and \( \{ J_\ell \}, \ell = 0, 1, 4, 6, \) respectively, and defined as (see Sect. 4):
\[ K_0 = \frac{1}{2} (T_1 + \eta T_1^T \eta) , \quad K_1 = \frac{1}{2} (T_9 + \eta T_9^T \eta) , \quad K_4 = \frac{1}{2} (T_{12} + \eta T_{12}^T \eta) , \quad K_6 = \frac{1}{2} (T_{14} + \eta T_{14}^T \eta) , \]
\[ J_0 = \frac{1}{2} (T_1 - \eta T_1^T \eta) , \quad J_1 = \frac{1}{2} (T_9 - \eta T_9^T \eta) , \quad J_4 = \frac{1}{2} (T_{12} - \eta T_{12}^T \eta) , \quad J_6 = \frac{1}{2} (T_{14} - \eta T_{14}^T \eta) . \] (86)

3 Nilpotent Orbits in \( \mathfrak{r}^* \)

We have learned in the previous sections that \( \mathfrak{r}^* \) is the carrier of an \( H^* \) representation,
the action of \( H^* \) on the matrices in \( \mathfrak{r}^* \) being the adjoint one. Constructing and classifying

\( ^{6} \)Thus \( p \) is the minimal number of components of \( \mathbb{Y}_0 \) (i.e. central and matter charges at radial infinity)
into which the most general vector \( \mathbb{Y}_0 \), in \( \mathbb{R}' \), can be reduced by means of an \( H_c \)-transformation.
$H^*$-adjoint orbits in $\mathbb{R}^*$, with particular reference to the nilpotent ones, is still an open problem in mathematics. It amounts to grouping the elements of $\mathbb{R}^*$ in orbits $\mathcal{O}$ (or conjugacy classes) with respect to the adjoint action of $H^*$:

$$k_1, k_2 \in \mathcal{O} \subset \mathbb{R}^* \iff \exists h \in H^*: k_2 = h^{-1} k_1 h .$$

(87)

A valuable approach to this task makes use of the theory of adjoint orbits within a real Lie algebra $\mathfrak{g}$ with respect to the action of the Lie group $G$ it generates [40]. In this respect the Konstant-Sekiguchi theorem [40] is of invaluable help since it allows for a complete classification of such orbits. This is however not enough for our purposes, since we are interested in the adjoint action of $H^*$ on $\mathbb{R}^1$ and a same $G$-orbit may branch into several $H^*$-orbits. To understand this splitting one may use $H^*$-invariant quantities which are not $G$-invariant, such as $\gamma$-labels [20] or tensor classifiers [18]. These, however, cannot guarantee by themselves a complete classification. Here we shall use a different approach to such a classification, which was originally devised in [19].

We start from the notion of standard triple associated with a nilpotent element $E$ of a real Lie algebra $\mathfrak{g}$: According to the Jacobson-Morozov theorem [40], such element can be thought of as part of a standard triple of $\mathfrak{sl}(2, \mathbb{R})$-generators $\{E, F, h\}$, satisfying the following commutation relations:

$$[h, E] = 2 E ; \ [h, F] = -2 F ; \ [E, F] = h .$$

(88)

We shall refer all the properties of the generators of $\mathfrak{g}$ to the corresponding matrices in the real fundamental representation $26$ of $\mathfrak{sl}(4)$. In particular the action of the Cartan involution on a generator $X$ amounts to taking the opposite of the transpose of the corresponding matrix: $\tau(X) = -X^T$. If we were interested in the orbits in the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$ with respect to the adjoint action of the group $G^C$ it generates, different $G^C$-nilpotent orbits correspond to inequivalent embeddings of $\mathfrak{sl}(2, \mathbb{R}) = \text{Span}(E, F, h)$ inside $\mathfrak{g}$, and these would correspond to different branchings of a given representation of $G^C$ with respect to the $\text{SL}(2, \mathbb{R})$-subgroup. These different branchings are uniquely characterized by the spectrum of the adjoint action of $h$ on $\mathfrak{g}^C$. Such spectrum is conveniently described by fixing a Cartan subalgebra $\mathcal{C}$ of $\mathfrak{g}^C$, in which $h$, being a semisimple generator, can be rotated by means of a $G^C$-transformation, and evaluating the values of the simple roots $\alpha_i$ of $\mathfrak{g}^C$, associated with $\mathcal{C}$, on $h$:

$$G^C\text{-Orbit of } E \leftrightarrow G^C\text{-orbits of } h \leftrightarrow \text{Spectrum Adj}_h \leftrightarrow \{\alpha_i(h)\}. \tag{89}$$

The integers $\alpha_i(h)$, which are conventionally evaluated after $h$ is rotated in the fundamental domain, can only have values 0, 1, 2 and are called $\alpha$-labels. They provide a complete classification of the nilpotent $G^C$-adjoint orbits in $\mathfrak{g}^C$ and can be found, for instance, in [40].

When we consider the problem of classifying nilpotent $G$-adjoint orbits in the real Lie algebra $\mathfrak{g}$, a same $G^C$-orbit will in general branch with respect to the action of $G$. In this case we can still reduce the problem of classifying the orbits of nilpotent elements $E$ of $\mathfrak{g}$ to that of classifying orbits of some characteristic semisimple generators. This
time however the relevant semisimple generator associated with the triple of $E$, is no longer $h$, but $i(E - F)$. More specifically $E - F$ is a compact matrix, i.e. it has only imaginary eigenvalues, and is thus an element of the maximal compact subalgebra $\mathfrak{K}$ of $\mathfrak{g}$. Having denoted by $H$ the maximal compact subgroup of $G$, let $H^C$ be its complexification, generated by the complexification $\mathfrak{K}^C = \mathfrak{K} + i\mathfrak{F}$ of $\mathfrak{F}$. The Kostant-Sekiguchi (KS) theorem defines a one-to-one correspondence between $G$-orbits of a nilpotent element $E$ of $\mathfrak{g}$, and the orbit under the adjoint action of $H^C$ on $\mathfrak{K}^C$, where the latter is the complexification of the space of non-compact $\mathfrak{g}$-generators $\mathfrak{R}$ defined by the Cartan decomposition (38): $\mathfrak{R}^C = \mathfrak{K} + i\mathfrak{F}$. These orbits are in turn in one-to-one correspondence with the $H^C$-adjoint orbit of the element $(E - F)$ of $\mathfrak{F}$. Such orbits are completely defined by the (real) spectrum of the adjoint action of $i(E - F)$ over $\mathfrak{F}^C$, or, equivalently, by the embedding of the same semisimple element within a suitable Cartan subalgebra $\mathfrak{C}_H$ of $i\mathfrak{F}$. If $\beta_k$ are the simple roots of $\mathfrak{F}^C$, such embedding is defined by the so called $\beta$-labels, which are the values $\beta_k(i(E - F))$. In summary the KS theorem states the following correspondence:

$$[G\text{-Orbit of } E] \leftrightarrow [H^C\text{-orbits of } i(E - F)] \leftrightarrow [\text{Spectrum Adj}_{i(E - F)}|_{\mathfrak{K}^C}] \leftrightarrow \{\beta_k(i(E - F))\}. \quad (90)$$

The labels $\beta_k(i(E - F))$ are conventionally evaluated once $i(E - F)$ is rotated into the fundamental domain and are non-negative integers. The $\alpha, \beta$-labels are classified in the mathematical literature, for all Lie groups [40].

Let us now come back to our original problem: What are the possible $H^*$-orbits of nilpotent elements $E$ in $\mathfrak{R}^*$? We know that $E$ is part of a standard triple. Since $E$ is in $\mathfrak{R}^*$, compatibility of (88) with (37) requires that $h \in \mathfrak{F}^*$ and $F \in \mathfrak{R}^*$. In particular $h$ is a semisimple, non-compact element of $\mathfrak{F}^*$ ($\tau(h) = -h^T = -h$, $\sigma(h) = h$), and thus can be chosen (modulo $H^*$-transformations of the triple) within a given non-compact Cartan subalgebra $\mathfrak{C}_H$ of $\mathfrak{F}^*$. Clearly different $G^C$ or $G$-orbits (uniquely defined by $\alpha, \beta$-labels, respectively) correspond to different $H^*$-orbits. However a same $G$-orbit may branch with respect to the action of $H^*$. In [20], the case $G = G_2(2)$, $H^* = \text{SL}(2, \mathbb{R})^2$ was studied in detail, and the so called $\gamma$-labels were introduced to distinguish between different $H^*$-orbits. The notion of $\gamma$-labels is similar to that of $\beta$-labels. Let us denote by $\mathfrak{F}^* = \mathfrak{F}^* + i\mathfrak{F}^*$ the complexification of $\mathfrak{F}^*$, generating the subgroup $H^C$ of $G^C$. The $\gamma$-labels identify the $H^*$-orbits of $h$ within $\mathfrak{F}^*$ and can either be described in terms of the spectrum of the adjoint action of $h$ on $\mathfrak{F}^*$, or in terms of the values of the simple roots $\beta_k^\prime$ of $\mathfrak{F}^* + i\mathfrak{F}^*$ (referred now to the Cartan subalgebra $\mathfrak{C}_{H^*}$) on $h$, taken in the fundamental domain:

$$\gamma\text{-labels } \leftrightarrow [\text{Spectrum Adj}_{h}|_{\mathfrak{F}^*}] \leftrightarrow \{\beta^\prime_k(h)\}. \quad (91)$$

These quantities are clearly invariant with respect to the adjoint action of $H^*$ (and in general of its complexification $H^{C*}$) on the whole triple and in particular on $h$, and thus different $\gamma$-labels correspond to different $H^*$-orbits of $E$. Clearly the sets of all possible $\beta$- and $\gamma$-labels coincide. In Table 2 we give a list of the $\alpha$ and $\beta$- (and thus also of the $\gamma$-) labels for the $F_{4(4)}$-model [40]. There is no mathematical property guaranteeing that $\gamma$-labels, together with the $\alpha$ and $\beta$ ones, provide a complete classification of the $H^*$-nilpotent
orbits in \( \mathfrak{R}^* \). And indeed here we provide the first counterexample: different \( H^* \)-orbits sharing the same \( \alpha, \beta, \gamma \)-labels.

Let us now review the constructive procedure introduced in [19]. Given a nilpotent element \( E \) of \( \mathfrak{R}^* \) we shall adopt the working assumption that there exists an element \( E' \) in the same \( H^* \)-orbit, whose triple \( \{ E', F', h' \} \) have the property that \( F' = E'^T \).\(^7\) We shall then restrict to triples of this kind.

The neutral element \( h \) of a triple \( \{ E, F, h \} \), should fall in one of the \( H^* \)-orbits uniquely defined by the \( \gamma \)-labels. We then take a representative \( h \) of each such orbits and solve the matrix equations in the unknown \( E \):

\[
[h, E] = 2E , \tag{92}
\]
\[
[E, E^T] = h . \tag{93}
\]

Using a MATHEMATICA code, for each \( h \) we find a set of solutions to (92), (93). We group these solutions under the action of the compact part \( H_{\text{little}}[h] \) of the little group of \( h \). In all cases we could find that solutions which were not connected by the adjoint action of \( H_{\text{little}}[h] \), could be distinguished by \( H^* \)-invariant quantities. Such quantities are the signatures of certain symmetric covariant (or contravariant) \( H^* \)-tensors, called tensor classifiers, to be discussed in detail in Subsect. 3.2.1. In principle, if one is able to find tensor classifiers capable of distinguishing between solutions \( E \) to (92), (93) which share the same \( \beta \)-label (i.e. fall in the same \( G \)-orbit) but are not related by \( H_{\text{little}}[h] \), the resulting classification of the \( H^* \)-orbits can be claimed to be complete. In our case a set of tensor classifiers fulfilling this task were constructed. They even allow for an almost complete distinction among the various orbits without the use of \( \alpha, \beta, \gamma \) labels. The main advantage of a complete classification effected by only using tensor classifiers is that given a nilpotent element \( E \) in \( \mathfrak{R}^* \), the computation of the \( \alpha, \beta, \gamma \)-labels would require the determination of the whole standard triple \( \{ E, F, h \} \), which in general is a non-trivial task, since \( F \neq E^T \). Tensor classifiers computed on \( E \) would give the answer straight away. In our model, in order for a tensor-classifier-based classification to be complete, probably tensors of higher degree in the Lax components would have to be constructed. The analysis is however complete once the use of the tensor classifiers is complemented with the \( \alpha, \beta, \gamma \)-labels. The different \( H^* \)-orbits are grouped into \( G \)-orbits (defined by the \( \beta \)-labels), which are arranged in the fifteen Tables 3-17, one for each \( G^C \)-orbit (\( \alpha \)-label). Within each table, each \( G \)-orbit, represented by a column, splits into distinct \( H^* \)-orbits, which are distinguished either by the \( \gamma \)-labels (rows in the table), or, for a same \( \gamma \)-label, by the signatures of certain tensor classifiers (further horizontal splitting of the corresponding \( \gamma, \beta \)-entry of the table). This further splitting is labeled by \( \delta_1, \delta_2 \).

Solutions describing regular static black holes fall in the first four \( G^C \)-orbits. The other \( G^C \)-nilpotent orbits have degree of nilpotency higher than 3 (we work in the fundamental representation of \( F_{4(4)} \)). We shall give examples of single-center static solutions in these

\(^7\)Although we do not have a proof for this for generic \( G/H^* \) spaces, it is proven for spaces of the form \( \text{GL}(n, \mathbb{R})/\text{SO}(p, q) \) \([5,41]\), using the \( \eta \)-symmetric normal forms. The most general \( G/H^* \) manifold, can be thought of as a totally geodesic submanifold of a \( \text{GL}(n, \mathbb{R})/\text{SO}(p, q) \) space, for some \( n, p, q \).
orbits, which however all lift to singular four-dimensional space-times, consistently with the regularity condition (68).

Let us now discuss the general structure of \( H_{c}^{\text{little}}[h] \). It can be represented as the semidirect product of a continuous group in the identity sector of \( H^{*} \) and the discrete stabilizer \( \mathcal{H}W \) of the Cartan subalgebra \( C_{H^{*}} \):

\[
H_{c}^{\text{little}}[h] = H_{c,(0)}^{\text{little}}[h] \ltimes \mathcal{H}W.
\]

The groups \( H_{c,(0)}^{\text{little}}[h] \), for each standard triple, are listed in Table 22 in Appendix A. The group \( \mathcal{H}W \) is a new object first introduced, to our knowledge, in the physics literature in [19], and is defined as follows:

\[
\mathcal{H}W = \{ g \in H^{*} \mid \forall h \in C_{H^{*}} : \ g^{-1} h g = h \}.
\]

A simple way of characterizing \( \mathcal{H}W \) is as a normal subgroup of the generalized Weyl group \( \mathcal{G}W \) [19] of \( g \). Let us briefly review the definition of the latter. Given a positive root \( \alpha \) of \( g \) defined with respect to a Cartan subalgebra \( C \) of \( g \), it is known that the Weyl group \( W \) of \( g \) is generated by the reflections in the positive roots \( \alpha \) of \( g \) which are effected by means of the adjoint action of a \( G \)-elements \( O_{\alpha} \) of the form:

\[
O_{\alpha} \equiv e^{\sqrt{2}/|\alpha| (E_{\alpha} - E_{-\alpha})}.
\]

It is indeed straightforward to prove that

\[
O_{\alpha}^{-1} H_{\beta} O_{\alpha} = H_{\sigma_{\alpha}(\beta)} ; \quad \sigma_{\alpha}(\beta) \equiv \beta - 2 \frac{(\beta, \alpha)}{|\alpha|^{2}} \alpha.
\]

We shall choose the Cartan subalgebra \( C = C_{H^{*}} \) of \( g = f_{4(4)} \) to consist of non-compact (i.e. represented by symmetric matrices) in \( H^{*} \). This is a Cartan subalgebra of \( \mathfrak{h}^{*} \) as well. Diagonalizing the adjoint action of \( C_{H^{*}} \) over \( g \), we define shift generators \( E_{\pm \alpha} \), some of which will lie in \( \mathfrak{h}^{*} \) and some in \( \mathfrak{r}^{*} \). We then divide the root system \( \Delta \) of \( g \) correspondingly in the following disjoint sets:

\[
\Delta_{+} = \Delta[\mathfrak{h}^{*}] \oplus \Delta[\mathfrak{r}^{*}] \\
\alpha \in \Delta[\mathfrak{h}^{*}] \iff E_{\alpha} \in \mathfrak{h}^{*} , \\
\alpha \in \Delta[\mathfrak{r}^{*}] \iff E_{\alpha} \in \mathfrak{r}^{*} .
\]

The orthogonal matrices \( O_{\alpha} \) (or even just those corresponding to the simple roots \( \alpha_{i} \) of \( g \)) generate themselves a discrete group \( \mathcal{G}W \) which is larger than the Weyl group \( W \). It is the largest subgroup of \( G \) whose adjoint action leaves \( C \) stable. A generic element of \( \mathcal{G}W \) can indeed be written as the product of an element of \( W \) times an element of the stabilizer \( \mathcal{H}W \) of the Cartan subalgebra \( C \), which is a normal subgroup of \( \mathcal{G}W \), so that we can write:

\[
W = \frac{\mathcal{G}W}{\mathcal{H}W}.
\]
A simple way of characterizing $\mathcal{H}W$ is as the subgroup of $GW$ generated by “squared reflections” $O_α^2$ (or simply by $O_α^2$), whose adjoint action on a generic element of $C$ clearly leaves it invariant. Notice that if $O_α^4 = 1$, as it is the case for $f_{4(4)}$ or the models considered in [19], then $\mathcal{H}W \subset H^*$, even if $E_α \in \mathbb{R}^*$, i.e. $α \in \Delta[\mathbb{R}]$. In this case indeed we have:

$$(O_α^T)^2ηO_α^2 = ηO_α^4 = η \Rightarrow O_α^2 \in H^*.$$  

(99)

These transformations in $\mathcal{H}W$ do not belong to the identity sector of $H^*$, but are nevertheless important since they relate, just as any other transformation in $H^*$, different solutions to eq.s (92, 93).

Following [19], we also define a subgroup $GW_H$ of $GW$ as its intersection with $H^*$: $GW_H = GW \cap H^*$. Clearly $\mathcal{H}W \subset GW_H$ and we can consider the coset

$$W_H = \frac{GW_H}{\mathcal{H}W} \subset W,$$

(100)

which can be characterized as the subgroup of the Weyl group whose action leaves the two root subspaces $\Delta[\mathbb{R}]^*$ and $\Delta[\mathbb{R}]^*$ invariant. This analysis provides us with a useful alternative way of finding representatives in $C_{H^*}$ of the various $H^*$-orbits of $h$, identified by the $γ$-labels. Such representatives could either be constructed directly using the $γ$-labels, or we can start from representatives in $C_{H^*}$ of $G^C$-orbits of $h$ within $g^C$, each defined by a set of $α$-labels. If we act on this representative by means of $W/W_H$ we find different representatives of the same orbit in $C_{H^*}$, which are not related by $H^*$, namely representatives of distinct $H^*$-orbits. Not all these representatives are neutral elements of triples with $E$ and $F = E^T$ in $\mathbb{R}^*$. If we impose this further condition, we end up with a set of $H^*$-orbits for the given $α$-label which precisely correspond to the allowed $γ$-labels. They coincide for each $α$-label with the $β$-labels listed in Table 2. Then we take a representative neutral element $h$ for each $γ$-label and proceed with the solution of eq.s (92, 93).

For the $F_{4(4)}$-model the Weyl group has 1152 elements, of which only 96 belong to $H^*$ and thus close the subgroup $W_H$. The stabilizer $\mathcal{H}W$ has order 16. We summarize below these data:

$$|W| = 1152 ; \quad |W_H| = 96 ; \quad \left| \frac{W}{W_H} \right| = 12 ; \quad |\mathcal{H}W| = 16 ; \quad |GW| = 16 \times 1152 = 18432.$$  

(101)

### 3.1 The Orbits

Here we discuss the explicit construction of the orbit in the model under consideration. As pointed out in the previous section, Given a standard triple $\{E, F, h\}$, whose neutral element $h$ is in the fundamental domain of the simple roots $α_i$ of $g^C = f_1^C$, the $G^C$-orbit of the nilpositive element $E$ is uniquely defined by the $α$-labels $α_i(h)$ which take value in $\{0, 1, 2\}$:

$$α$-labels; $\quad (α_1(h), α_2(h), α_3(h), α_4(h)).$$

(102)
For each α-label we choose a representative \( E \), and it may happen that two different representatives are conjugated by an element \( X \in F_4(4) \), that is
\[
X^{-1} E X = E', \quad X \in F_4(4).
\]

In this case \( E \) and \( E' \) lie in the same nilpotent orbit, and therefore one obtains a single \( F_4(4) \)-orbit. We present the \( F_4(4) \) single orbits in Table 2. If this is not the case, then one can distinguish two or three different \( F_4(4) \)-orbits through what we have called the β-labels which provide a complete classification of the \( F_4(4) \)-orbits. As mentioned in the previous section, the nilpotent \( F_4(4) \)-orbits are in one-to-one correspondence with the nilpotent \( H^C \)-orbits in \( R^C \), complexification of \( R \), which in turn are classified by the β-labels. To define the latter we need to refer to a suitable Cartan subalgebra \( C_H \) of \( i\mathfrak{h} \) which the element \( i(E - F) \) should belong to. Since \( E - F \) is also an element of \( R^* \), we choose \( C_H \) to lie in the intersection \( i(H \cap R^*) \). A possible choice of basis for \( C_H \) is:
\[
\hat{H}_{\beta_1} = 2i (K_4 + K_11) ; \quad \hat{H}_{\beta_2} = 2i (K_3 - K_11) ; \quad \hat{H}_{\beta_3} = -2i (K_3 + K_4 - K_11 - K_14) ,
\]
\[
\hat{H}_{\beta_4} = i(-K_3 + K_4 - K_11 + K_14) ,
\]
where \( \beta_1, \beta_2, \beta_3 \) are the simple roots of \( \mathfrak{sp}(6, \mathbb{C}) \) in \( \mathfrak{h}^C \), while \( \beta_4 \) is the simple root of the \( \mathfrak{sl}(2, \mathbb{C}) \) subalgebra commuting with it. The roots \( \beta_1, \beta_2, \beta_4 \) have squared length equal to 2, while \( \beta_3 \) has squared length equal to 4. The corresponding Dynkin diagram is
\[
\begin{array}{cccc}
\circ & \circ & \circ & \circ \\
\beta_1 & \beta_2 & \beta_3 & \beta_4
\end{array}
\]

The β-labels associated with a triple \( \{E, F, h\} \) are then computed as
\[
\beta\text{-labels}; \quad (\beta_1(i(E - F)), \beta_2(i(E - F)), \beta_3(i(E - F)), \beta_4(i(E - F))).
\]

If we define the simple weights \( \lambda^k \) associated with \( \beta_k \) as usual by the property that:
\[
\langle \lambda^i, \beta_j \rangle = 2 \frac{\langle \lambda^i, \beta_j \rangle}{\langle \beta_j, \beta_j \rangle} = \delta^i_j ,
\]
we can write the corresponding basis of \( C_H \) as:
\[
\hat{H}_{\lambda^i} = C_{ij} \hat{H}_{\beta_j} ,
\]

---

\( ^8 \)We recall that \( H^C \) is the complexification of the maximal compact subgroup \( H = SU(2) \times USp(6) \) of \( G = F_4(4) \), whose algebra is denoted by \( \mathfrak{h}^C \subset \mathfrak{f}_4^C \), not to be confused with the complexification \( H^* \) of \( H^* = SL(2) \times Sp'(6) \) subgroup of \( F_4(4) \), whose Lie algebra is denoted by \( \mathfrak{h}^C \subset \mathfrak{f}_4^C \). \( H^C \) and \( H^* \) are clearly isomorphic in \( G^C \) and so are their Lie algebras \( \mathfrak{h}^C, \mathfrak{h}^* \), though the latter are described by different generators.
where $C^{ij}$ is the inverse of the Cartan matrix $C_{ij}$:

$$ (C_{ij}) = \left( \frac{2(\beta_i, \beta_j)}{(\beta_j, \beta_j)} \right) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}. \quad (108) $$

If we denote by $n_i$ the $\beta$-labels, knowing $n_i$ we can construct the corresponding matrix $E - F$ as follows:

$$ i(E - F) = \sum_{k=1}^{4} 2 n_k \frac{\hat{H}_{\lambda_k}}{(\beta_k', \beta_k)}. \quad (109) $$

$H^*$-orbits of the neutral element $h$ of a triple are classified by the $\gamma$-labels defined in the previous section as the values $\beta_k'(h)$ on it of the simple roots $\beta_k'$ associated with the complexification $\hat{\mathfrak{g}}^*\mathfrak{C}$ of $\hat{\mathfrak{g}}^*$. The corresponding Dynkin diagram is the same as for $\beta_k$, though these roots are now referred to a non-compact Cartan subalgebra $C_{H^*}$ in $\hat{\mathfrak{g}}^* \cap \mathfrak{r}$. We can choose as a basis of $C_{H^*}$ the following matrices:

$$ \begin{align*}
\hat{H}_{\beta_1'} &= 2 (J_4 + J_{11}) ; \\
\hat{H}_{\beta_2'} &= 2 (J_3 - J_{11}) ; \\
\hat{H}_{\beta_3'} &= -2 (J_3 + J_4 - J_{11} - J_{14}) ; \\
\hat{H}_{\beta_4'} &= (-J_3 + J_4 - J_{11} + J_{14}) ,
\end{align*} \quad (110) $$

Notice that the $\hat{H}_{\beta_k'}$ and the $\hat{H}_{\beta_k}$ are mapped into one another by replacing $J_\alpha$ with $iK_\alpha$. The corresponding Cartan subalgebras are isomorphic in $\mathfrak{g}\mathfrak{C}$ through the action of $G^\mathfrak{C}$. The same is true for $h$ and $i(E - F)$. We construct the simple weights $\lambda_k'$ associated with $\beta_k'$ and the corresponding basis of matrices $\hat{H}_{\lambda_k'}$. Given the $\gamma$-labels $n_k = \beta_k'(h)$, we can construct the corresponding $h$ as follows:

$$ h = \sum_{k=1}^{4} 2 n_k \frac{\hat{H}_{\lambda_k'}}{(\beta_k', \beta_k)}. \quad (111) $$

In Table 2 the $G = F_{4(4)}$-nilpotent orbits are listed with the corresponding $\alpha$- and $\beta$- labels. There are 15 $\alpha$-labels defining 15 distinct nilpotent orbits of $F_{4(4)}^\mathfrak{C}$, and are denoted by $\alpha^{(s)}$. For a same $\alpha$-label we can have more $\beta$-ones signalling that the corresponding $F_{4(4)}^\mathfrak{C}$-orbit branches with respect to $F_{4(4)}$. When this occurs, we denote the $\beta$-labels by $\beta^{(1)}, \beta^{(2)} \ldots$ in the order in which they are listed in Table 2. The possible $\gamma$-labels for a same $\alpha$- one are the same as the $\beta$-labels and thus are not listed.

Below we list all the labels, giving the corresponding spectrum of the adjoint action of $h$ over $\mathfrak{g}$ for the $\alpha$-labels, of the adjoint action of $h$ over $\hat{\mathfrak{g}}^*$ for the $\gamma$-labels and of the adjoint action of $i(E - F)$ over $\hat{\mathfrak{g}}^*$ for the $\beta$-ones. Moreover, for each $\alpha$-label we give the “angular momentum” decomposition of the adjoint of $\mathfrak{g}\mathfrak{C}$ with respect to the $\text{SL}(2, \mathbb{C})$ subgroup of $G^\mathfrak{C}$ generated by the standard triple.

We apply to the classification of the $H^*$-nilpotent orbits in $\mathfrak{r}^*$ the systematic method defined in the previous section: We start from a representative $h$ for each $\gamma$-label, we solve eq.s (92), (93) in $E \in \mathfrak{r}^*$, and group the solutions under the action of $H^\text{little}_c[h]$. We
find that solutions which are not connected through $H^{\text{little}}_e[h]$ can be distinguished by the signatures of tensor classifiers, and thus belong to distinct $H^*$-orbits. The result of this classification is summarized in Tables 3-17. In the next subsection we list the $\alpha$, $\beta$, $\gamma$-labels. In Subsection 3.2.1 we review the construction of the tensor classifiers.

| $F_{4(4)}$-orbit | $\alpha$-labels | $\beta$-labels | Degree of nilpotency |
|------------------|----------------|---------------|---------------------|
| $O_1$            | (1, 0, 0, 0)   | (0, 0, 1, 1)  | 2                   |
| $O_2$            | (0, 0, 0, 1)   | (1, 0, 0, 2)  | 3                   |
| $O_3$            | (0, 0, 0, 1)   | (0, 1, 0, 0)  | 3                   |
| $O_4$            | (0, 1, 0, 0)   | (0, 0, 1, 3)  | 3                   |
| $O_5$            | (0, 1, 0, 0)   | (1, 0, 1, 1)  | 3                   |
| $O_6$            | (2, 0, 0, 0)   | (0, 0, 0, 4)  | 3                   |
| $O_7$            | (2, 0, 0, 0)   | (2, 0, 0, 0)  | 3                   |
| $O_8$            | (2, 0, 0, 0)   | (0, 0, 2, 2)  | 3                   |
| $O_9$            | (0, 0, 2, 0)   | (0, 2, 0, 0)  | 5                   |
| $O_{10}$         | (0, 0, 1, 0)   | (1, 1, 0, 2)  | 4                   |
| $O_{11}$         | (2, 0, 0, 1)   | (1, 0, 2, 4)  | 5                   |
| $O_{12}$         | (2, 0, 0, 1)   | (0, 1, 2, 2)  | 5                   |
| $O_{13}$         | (0, 1, 0, 1)   | (1, 1, 1, 1)  | 5                   |
| $O_{14}$         | (1, 0, 1, 0)   | (1, 0, 3, 1)  | 5                   |
| $O_{15}$         | (1, 0, 1, 0)   | (1, 1, 1, 3)  | 5                   |
| $O_{16}$         | (0, 2, 0, 0)   | (0, 0, 4, 0)  | 5                   |
| $O_{17}$         | (0, 2, 0, 0)   | (0, 2, 0, 4)  | 5                   |
| $O_{18}$         | (0, 2, 0, 0)   | (2, 0, 2, 2)  | 5                   |
| $O_{19}$         | (2, 2, 0, 0)   | (0, 0, 4, 8)  | 7                   |
| $O_{20}$         | (2, 2, 0, 0)   | (2, 0, 4, 4)  | 7                   |
| $O_{21}$         | (1, 0, 1, 2)   | (1, 3, 1, 3)  | 9                   |
| $O_{22}$         | (0, 2, 0, 2)   | (0, 4, 0, 4)  | 9                   |
| $O_{23}$         | (0, 2, 0, 2)   | (2, 2, 2, 2)  | 9                   |
| $O_{24}$         | (2, 2, 0, 2)   | (2, 2, 4, 4)  | 11                  |
| $O_{25}$         | (2, 2, 0, 2)   | (4, 0, 4, 8)  | 11                  |
| $O_{26}$         | (2, 2, 2, 2)   | (4, 4, 4, 8)  | 11                  |

Table 2: The twenty six nonzero $F_{4(4)}$-orbits and their degree of nilpotency.
3.2 \( \alpha, \beta, \gamma \)-labels

\( \alpha^{(1)} \)-label:

\[
\alpha^{(1)} = 1 \times (\pm 2) + 14 \times (\pm 1) + 22 \times (0) \\
= 1 \times (J = 1) \oplus 14 \times \left(J = \frac{1}{2}\right) \oplus 21 \times (J = 0)
\]

and

\[
\gamma = 7 \times (\pm 1) + 10 \times (0) = \beta
\]

corresponding to \((0, 0, 1, 1)\) label. The associated orbit and its representative are given in Table 3.

| \(\gamma\)-label | \(\beta\)-label | \((0, 0, 1, 1)\) | \(O_{H^*}\) |
|-------------------|-----------------|----------------|-----------|
| \((0, 0, 1, 1)\)  | \(-\frac{1}{2}(H_1 - H_4) - K_4\) | \(O_1\)       |           |

Table 3: The \(H^*\)-orbit \(O_{H^*}\) within the \(F_{4(4)}\)-orbits \(O_1\).

\(\alpha^{(2)}\)-label:

\[
\alpha^{(2)} = 7 \times (\pm 2) + 8 \times (\pm 1) + 22 \times (0)
= 7 \times (J = 1) \oplus 8 \times \left(J = \frac{1}{2}\right) \oplus 15 \times (J = 0).
\]

The \(\gamma - \beta\)-labels are given by

\[
\gamma^{(1)} = 2 \times (\pm 2) + 4 \times (\pm 1) + 12 \times (0) = \beta^{(1)}
\]
\[
\gamma^{(2)} = 3 \times (\pm 2) + 4 \times (\pm 1) + 10 \times (0) = \beta^{(2)}
\]

corresponding, respectively, to \((1, 0, 0, 2)\) and \((0, 1, 0, 0)\) labels. The associated orbits and their representatives are presented in Table 4.

\(\alpha^{(3)}\)-label:

\[
\alpha^{(3)} = 2 \times (\pm 3) + 6 \times (\pm 2) + 12 \times (\pm 1) + 12 \times (0)
= 2 \times \left(J = \frac{3}{2}\right) \oplus 6 \times (J = 1) \oplus 10 \times \left(J = \frac{1}{2}\right) \oplus 6 \times (J = 0),
\]

and

\[
\gamma^{(1)} = 1 \times (\pm 3) + 6 \times (\pm 1) + 10 \times (0) = \beta^{(1)}
\]
\[
\gamma^{(2)} = 1 \times (\pm 3) + 2 \times (\pm 2) + 6 \times (\pm) + 6 \times (0) = \beta^{(2)},
\]

corresponding, respectively, to \((0, 0, 1, 3)\) and \((1, 0, 1, 1)\) labels. The associated orbits and their representatives are presented in Table 5.
orbits and their representatives are listed in Table

\[ \begin{array}{|c|c|c|}
\hline
\gamma \text{-label} & \beta \text{-label} & H^* \text{-orbits} \\
\hline
(1, 0, 0, 2) & \frac{1}{2}(-H_1 - H_2 - H_3 + H_4) - K_14 - K_4 & O_{H^*} \\
(0, 1, 0, 0) & -K_2 + K_{15} + K_6 + K_{13} & O''_{H^*} \\
\hline
\end{array} \]

Table 4: The four \( H^* \)-orbits \( O_{2H^*}, O'_{2H^*}, O_{3H^*}, O'_{3H^*} \) within the two \( F_{4(4)} \)-orbits \( O_2 \) and \( O_3 \). Our labeling of the orbits is indicated by the last row and the rightmost column.

\[ \begin{array}{|c|c|c|}
\hline
\gamma \text{-label} & \beta \text{-label} & H^* \text{-orbits} \\
\hline
(0, 0, 1, 3) & \frac{1}{2}(3K_{16} + 3K_{12} - K_5 - K_2 + K_{15} - K_1 + K_6 + K_{13}) & O_{H^*} \\
(1, 0, 1, 1) & \frac{1}{2}(K_{16} + K_{12} - 3K_5 + K_2 - K_{15} - 3K_1 - K_6 - K_{13}) & O'_{H^*} \\
\hline
\end{array} \]

Table 5: The four \( H^* \)-orbits \( O_{4H^*}, O'_{4H^*}, O_{5H^*}, O'_{5H^*} \) within the two \( F_{4(4)} \)-orbits \( O_4 \) and \( O_5 \). The two \( H^* \)-suborbits \( O'_{5H^*}, O'_{5H^*} \) within \( O'_{5H^*} \).

\( \alpha^{(4)} \)-label:

\[ \alpha^{(4)} = 1 \times (\pm 4) + 14 \times (\pm 2) + 22 \times (0) \]

\[ = 1 \times (J = 2) \oplus 13 \times (J = 1) \oplus 8 \times (J = 0). \]

The \( \gamma \) - \( \beta \) labels read

\[ \gamma^{(1)} = 1 \times (\pm 4) + 22 \times (0) = \beta^{(1)} \]

\[ \gamma^{(2)} = 1 \times (\pm 4) + 4 \times (\pm 2) + 14 \times (0) = \beta^{(2)} \]

\[ \gamma^{(3)} = 7 \times (\pm 2) + 10 \times (0) = \beta^{(3)} \]

corresponding, respectively, to \( (0, 0, 0, 4), (2, 0, 0, 0) \) and \( (0, 0, 2, 2) \) labels. The associated orbits and their representatives are listed in Table 6.

\( \alpha^{(5)} \)-label:

\[ \alpha^{(5)} = 7 \times (\pm 4) + 8 \times (\pm 2) + 22 \times (0) \]

\[ = 7 \times (J = 2) \oplus 1 \times (J = 1) \oplus 14 \times (J = 0), \]

and

\[ \gamma = 3 \times (\pm 4) + 4 \times (\pm 2) + 10 \times (0) = \beta, \]

corresponding to \( (0, 2, 0, 0) \) label. The associated orbit and its representative is listed in Table 7.
| γ-label | β-label | (0, 0, 0, 4) | (2, 0, 0, 0) | (0, 0, 2, 2) |
|---------|---------|-------------|-------------|-------------|
| (0, 0, 0, 4) | $-H_1 - H_2 - K_{14}$ | $K_{12} + K_2 - K_1 + K_{13} + \sqrt{2}(K_{24} + K_{20})$ | $\frac{1}{4}(-3H_1 - H_2 + H_3 + H_4) + \frac{1}{3}(K_3 - 2K_{12} - K_5 + K_{15}) + \frac{1}{2}(K_{21} + K_{24} + K_{17} + K_{20})$ | $O_{H*}$ |
| (2, 0, 0, 0) | $K_{12} - K_5 - K_2$ | $2K_{16} + K_{12} + K_5 - K_2 - K_{15}$ | $O'_{H*}$ | $K_{16} + K_5 - K_{15} - K_6 + \sqrt{2}(K_{24} + K_{20})$ | $O'_{H*}$ |
| (0, 0, 2, 2) | $K_{12} - K_5 - K_2$ | $-K_{16} - K_{12} - K_5$ | $O''_{H*}$ | $-2(K_{16} + K_{12})$ | $O''_{H*}$ |

Table 6: The nine $H^*$-orbits $O_{6H^*}, O'_{6H^*}, O''_{6H^*}, O_{7H^*}, O'_{7H^*}, O''_{7H^*}, O_{8H^*}, O'_{8H^*}, O''_{8H^*}$ within the three $F_4(4)$-orbits $O_6, O_7$ and $O_8$. The four $H^*$-suborbits $O'_{7H^*}, O'_{8H^*}, O''_{7H^*}, O''_{8H^*}$ within the two $H^*$-orbits $O'_{7H^*}$ and $O''_{8H^*}$. 
Table 7: The $H^*$-orbit $O_{9H^*}$ within the $F_{4(4)}$-orbits $O_9$.

$$\begin{array}{|c|c|c|}
\hline
\gamma\text{-label} & (0, 2, 0, 0) & -2(K_{21} + K_{22} - K_{17} + K_{18}) \, \mathcal{O}_9 \\
\hline
\hline
\end{array}$$

The $\gamma - \beta$-labels are written as

$$\begin{align*}
\gamma^{(1)} &= 2 \times (\pm 4) + 2 \times (\pm 3) + 2 \times (\pm 2) + 2 \times (\pm 1) + 6 \times (0) = \beta^{(1)} \\
\gamma^{(2)} &= 3 \times (\pm 4) + 2 \times (\pm 3) + 2 \times (\pm 2) + 2 \times (\pm 1) + 6 \times (0) = \beta^{(2)}
\end{align*}$$

Table 8: The $H^*$-orbit $O_{10H^*}$ within the $F_{4(4)}$-orbits $O_{10}$.

| $\gamma$-label | $\beta$-label | (1, 1, 0, 2) |
|----------------|---------------|------------|
| (1, 1, 0, 2)   | $-\frac{1}{\sqrt{2}}(H_2 + H_3 + K_{16} + 2K_{14} + K_{12} + K_5 + K_2 - K_{15} + 2K_{22} + K_1 - K_6 - K_{13} + 2K_{18})$ | $O_{H^*}$ |
| \hline
| \hline

The $\gamma - \beta$-labels are written as

$$\begin{align*}
\gamma^{(1)} &= 2 \times (\pm 4) + 2 \times (\pm 3) + 2 \times (\pm 2) + 2 \times (\pm 1) + 6 \times (0) = \beta^{(1)} \\
\gamma^{(2)} &= 3 \times (\pm 4) + 2 \times (\pm 3) + 2 \times (\pm 2) + 2 \times (\pm 1) + 6 \times (0) = \beta^{(2)}
\end{align*}$$

corresponding, respectively, to $(1, 0, 2, 4)$ and $(0, 1, 2, 2)$. The associated orbits and their representatives are listed in Table 9.
\( \gamma \)-label \( \beta \)-label \( (1, 0, 2, 4) \) \( (0, 1, 2, 2) \) \( \mathcal{O}_{H^*} \)  

\begin{align*}
(1, 0, 2, 4) & \quad - H_2 - H_3 - 2K_{14} - \sqrt{3}(K_5 - K_{15} + K_1 - K_{13}) \\
(0, 1, 2, 2) & \quad - H_1 - H_4 - 2K_{11} + \sqrt{3}(K_{16} + K_{12} - K_5 - K_1) \quad \mathcal{O}_{11} \quad \mathcal{O}_{12}^* 
\end{align*}

Table 9: The four \( H^* \)-orbits \( \mathcal{O}_{11 H^*}, \mathcal{O}_{11 H^*}', \mathcal{O}_{12 H^*}, \mathcal{O}_{12 H^*}' \) within the two \( F_4(4) \)-orbits \( \mathcal{O}_{11} \) and \( \mathcal{O}_{12} \).

\( \alpha^{(8)} \)-label :
\[
\alpha^{(8)} = 2 \times (\pm 5) + 3 \times (\pm 4) + 4 \times (\pm 3) + 5 \times (\pm 2) + 8 \times (\pm 1) + 8 \times (0) \quad (138)
\]
\[
= 2 \times (J = \frac{5}{2}) \oplus (J = 2) \oplus 2 \times (J = \frac{3}{2}) \oplus 2 \times (J = 1) \oplus 4 \times (J = \frac{1}{2}) \oplus 3 \times (J = 0) , \quad (139)
\]

and
\[
\gamma = 1 \times (\pm 5) + 1 \times (\pm 4) + 2 \times (\pm 3) + 2 \times (\pm 2) + 4 \times (\pm 1) + 4 \times (0) = \beta \quad (140)
\]

corresponding to \((1, 1, 1, 1)\) label. The associated orbit and its representative is presented in Table 10.

\( \gamma \)-label \( \beta \)-label \( (1, 1, 1, 1) \) \( \mathcal{O}_{H^*} \)  

\begin{align*}
(1, 1, 1, 1) & \quad \frac{1}{2}(K_{16} + K_{12} - K_2 - K_{15} - K_1 + K_6 - K_{13} - K_5) \\
& \quad -2(K_{21} + K_{22} - K_{17} + K_{18}) \quad \mathcal{O}_{13} \quad \mathcal{O}_{13}^* 
\end{align*}

Table 10: The \( H^* \)-orbit \( \mathcal{O}_{13 H^*} \) within the \( F_4(4) \)-orbits \( \mathcal{O}_{13} \).

\( \alpha^{(9)} \)-label :
\[
\alpha^{(9)} = 1 \times (\pm 6) + 2 \times (\pm 5) + 2 \times (\pm 4) + 6 \times (\pm 3) + 5 \times (\pm 2) + 6 \times (\pm 1) + 8 \times (0) \quad (141)
\]
\[
= 1 \times (J = 3) \oplus 2 \times (J = \frac{5}{2}) \oplus 1 \times (J = 2) \oplus 4 \times (J = \frac{3}{2}) \oplus 3 \times (J = 1) \oplus 3 \times (J = 0) , \quad (142)
\]

and
\[
\gamma^{(1)} = 1 \times (\pm 5) + 2 \times (\pm 4) + 3 \times (\pm 3) + 3 \times (\pm 1) + 6 \times (0) = \beta^{(1)} \quad (143)
\]
\[
\gamma^{(2)} = 1 \times (\pm 5) + 1 \times (\pm 4) + 3 \times (\pm 3) + 2 \times (\pm 2) + 3 \times (\pm 1) + 4 \times (0) = \beta^{(2)} \quad (144)
\]

corresponding, respectively, to \((1, 0, 3, 1)\) and \((1, 1, 1, 3)\) labels. The associated orbits and their representatives are listed in Table 11.
The four $H^*$-orbits $O_{14}H^*$, $O_{14}'H^*$, $O_{15}H^*$, $O_{15}'H^*$ within the two $F_{4(4)}$-orbits $O_{14}$ and $O_{15}$.

$\alpha^{(10)}$-label :

\[
\alpha^{(10)} = 2 \times (\pm 6) + 6 \times (\pm 4) + 12 \times (\pm 2) + 12 \times (0) \tag{145}
\]
\[
= 2 \times (J = 3) \oplus 4 \times (J = 2) \oplus 6 \times (J = 1), \tag{146}
\]

The $\gamma - \beta$ labels are given by

\[
\gamma^{(1)} = 6 \times (\pm 4) + 12 \times (0) = \beta^{(1)} \tag{147}
\]
\[
\gamma^{(2)} = 4 \times (\pm 4) + 4 \times (\pm 2) + 8 \times (0) = \beta^{(2)} \tag{148}
\]
\[
\gamma^{(3)} = 1 \times (\pm 6) + 2 \times (\pm 4) + 6 \times (\pm 2) + 6 \times (0) = \beta^{(3)}, \tag{149}
\]
corresponding, respectively, to $(0, 0, 4, 0)$, $(0, 2, 0, 4)$ and $(2, 0, 2, 2)$. The associated orbits and their representatives are listed in Table 12.

$\alpha^{(11)}$-label :

\[
\alpha^{(11)} = 1 \times (\pm 10) + 1 \times (\pm 8) + 6 \times (\pm 6) + 6 \times (\pm 4) + 7 \times (\pm 2) + 10 \times (0) \tag{150}
\]
\[
= 1 \times (J = 5) \oplus 5 \times (J = 3) \oplus 1 \times (J = 1) \oplus 3 \times (J = 0). \tag{151}
\]

The $\gamma - \beta$ labels are

\[
\gamma^{(1)} = 1 \times (\pm 8) + 6 \times (\pm 4) + 10 \times (0) = \beta^{(1)} \tag{152}
\]
\[
\gamma^{(2)} = 1 \times (\pm 8) + 2 \times (\pm 6) + 4 \times (\pm 4) + 2 \times (\pm 2) + 6 \times (0) = \beta^{(2)}, \tag{153}
\]
corresponding, respectively, to $(0, 0, 4, 8)$, and $(2, 0, 4, 4)$. The associated orbits and their representatives are listed in Table 13.

$\alpha^{(12)}$-label :

\[
\alpha^{(12)} = 1 \times (\pm 10) + 2 \times (\pm 9) + 1 \times (\pm 8) + 2 \times (\pm 7) + 2 \times (\pm 6)
+ 2 \times (\pm 5) + 2 \times (\pm 4) + 4 \times (\pm 3) + 3 \times (\pm 2) + 4 \times (\pm 1) + 6 \times (0)
\]
\[
= 1 \times (J = 5) \oplus 2 \times (J = \frac{9}{2}) \oplus 1 \times (J = 3) \oplus 2 \times (J = \frac{3}{2}) \oplus 1 \times (J = 1) \oplus 3 \times (J = 0) \tag{154}
\]

Table 11: The four $H^*$-orbits $O_{14}H^*$, $O_{14}'H^*$, $O_{15}H^*$, $O_{15}'H^*$ within the two $F_{4(4)}$-orbits $O_{14}$ and $O_{15}$.
| γ-label | β-label | (0, 0, 4, 0) | (0, 2, 0, 4) | (2, 0, 2, 2) | O₁₆ | O₁₇ | O₁₈ |
|---------|---------|-------------|-------------|-------------|-----|-----|-----|
| (0, 0, 4, 0) | -H₁ + H₂ - H₄ | -K₁₄ + K₃ - 2K₁₁ | -√3(K₂ + K₁₅ - K₆ + K₁₃) | -√3H₁₂ + K₁₂ + K₁₃ | O₁₆⁺ | O₁₇⁺ | O₁₈⁺ |
| (0, 0, 4, 0) | -H₁ - H₂ - H₄ | -K₁₄ + K₃ - 2K₁₁ | -√3(K₂ + K₁₅ - K₆ + K₁₃) | -√3H₁₂ + K₁₂ + K₁₃ | O₁₆⁻ | O₁₇⁻ | O₁₈⁻ |
| (2, 0, 2, 2) | -1/2(H₁ - 3H₂ + H₃) | +H₄ + K₁₄ + 2K₃ | -K₁₁ - √6(K₂₂ + K₁₈) | -1/2H₁ - 3H₃ | O₁₆⁺⁺ | O₁₇⁺⁺ | O₁₈⁺⁺ |
| (2, 0, 2, 2) | -1/2(H₁ - 3H₂ + H₃) | +H₄ + K₁₄ + 2K₃ | -K₁₁ - √6(K₂₂ + K₁₈) | -1/2H₁ - 3H₃ | O₁₆⁻⁻ | O₁₇⁻⁻ | O₁₈⁻⁻ |

Table 12: The nine H⁺-orbits O₁₆⁺, O₁₆⁻, O₁₆⁺⁺, O₁₆⁻⁻, O₁₇⁺, O₁₇⁻, O₁₇⁺⁺, O₁₇⁻⁻, O₁₈⁺ within the three F₄(4)-orbits O₁₆, O₁₇ and O₁₈. The four H⁺-suborbits O₁₇⁺⁺, O₁₇⁻⁻, O₁₈⁺⁺, O₁₈⁻⁻ within the two H⁺-orbits O₁₇⁺ and O₁₈⁺.
Table 13: The four $H^*$-orbits $\mathcal{O}_{19H^*}$, $\mathcal{O}'_{19H^*}$, $\mathcal{O}_{20H^*}$, $\mathcal{O}'_{20H^*}$ within the two $\mathbb{F}_{4(4)}$-orbits $\mathcal{O}_{19}$ and $\mathcal{O}_{20}$. The two $H^*$-suborbits $\mathcal{O}'_{20H^*}$, $\mathcal{O}'_{20H^*}$ within $\mathcal{O}'_{20H^*}$.

and

$$\gamma = 1 \times (\pm 9) + 1 \times (\pm 8) + 1 \times (\pm 7) + 1 \times (\pm 5) + 2 \times (\pm 4) + 2 \times (\pm 3) + 2 \times (\pm 1) + 4 \times (0) = \beta$$ (155)

corresponding to $(1, 3, 1, 3)$ label. The associated orbit and its representative is given in Table 14.

Table 14: The $H^*$-orbit $\mathcal{O}_{21H^*}$ within the $\mathbb{F}_{4(4)}$-orbits $\mathcal{O}_{21}$.

$$\alpha^{(13)}$$-label:

$$\alpha^{(13)} = 2 \times (\pm 10) + 3 \times (\pm 8) + 4 \times (\pm 6) + 5 \times (\pm 4) + 8 \times (\pm 2) + 8 \times (0)$$ (156)

$$\gamma - \beta$$ take the values

$$\gamma^{(1)} = 3 \times (\pm 8) + 5 \times (\pm 4) + 8 \times (0) = \beta^{(1)}$$ (158)

$$\gamma^{(2)} = 1 \times (\pm 10) + 1 \times (\pm 8) + 2 \times (\pm 6) + 2 \times (\pm 4) + 4 \times (\pm 2) + 4 \times (0) = \beta^{(2)}$$ (159)
corresponding, respectively, to (0, 4, 0, 4) and (2, 2, 2, 2) labels. The associated orbits and their representatives are given in Table 15.

| γ-label | β-label | (0, 4, 0, 4) | (2, 2, 2, 2) | O_{H^*} | O'_{H^*} |
|---------|---------|-------------|-------------|---------|---------|
| (0, 4, 0, 4) | 2(K_{16} + K_{12} - K_5 + 2K_{23} - K_1 - K_9) | +2(K_{16} + K_{12} - K_5 - K_1 + K_9 + K_{10}) | +2(K_{23} - K_2 - K_{15} + K_0 + K_9) | O_{H^*} |
| (2, 2, 2, 2) | -H_2 - 2H_4 - 3K_{16} + K_3 + 4(K_{23} - K_{19}) | -H_2 - 2H_4 - 3K_{16} + K_3 + 4(K_{23} - K_{19}) | -K_3 - 4(K_{23} - K_{19}) | O'_{H^*} |

Table 15: The four $H^*$-orbits $O_{22H^*}$, $O'_{22H^*}$, $O_{23H^*}$, $O'_{23H^*}$ within the two $F_4(4)$-orbits $O_{22}$ and $O_{23}$.

$\alpha^{(14)}$-label:

$$
\alpha^{(14)} = 1 \times (\pm 14) + 1 \times (\pm 12) + 3 \times (\pm 10) + 3 \times (\pm 8) + 4 \times (\pm 6) + 5 \times (\pm 4) + 6 \times (\pm 2) + 6 \times (0) \quad (160)
$$

$$
= 1 \times (J = 7) \oplus 2 \times (J = 5) \oplus 1 \times (J = 3) \oplus 1 \times (J = 2) \oplus 1 \times (J = 1). \quad (161)
$$

The $\gamma - \beta$ labels read

$$
\gamma^{(1)} = 1 \times (\pm 12) + 1 \times (\pm 10) + 2 \times (\pm 8) + 1 \times (\pm 6) + 3 \times (\pm 4) + 2 \times (\pm 2) + 4 \times (0) = \beta^{(1)} \quad (162)
$$

$$
\gamma^{(2)} = 1 \times (\pm 12) + 3 \times (\pm 8) + 5 \times (\pm 4) + 6 \times (0) = \beta^{(2)}, \quad (163)
$$

corresponding, respectively, to (2, 2, 4) and (4, 0, 4, 8) labels. The associated orbits and their representatives are given in Table 16.

$\alpha^{(15)}$-label:

$$
\alpha^{(15)} = 1 \times (\pm 22) + 1 \times (\pm 20) + 1 \times (\pm 18) + 1 \times (\pm 16) + 2 \times (\pm 14) + 2 \times (\pm 12) + 3 \times (\pm 10) + 3 \times (\pm 8) + 3 \times (\pm 6) + 3 \times (\pm 4) + 4 \times (\pm 2) + 4 \times (0) \quad (164)
$$

$$
= 1 \times (J = 11) \oplus 1 \times (J = 7) \oplus 1 \times (J = 5) \oplus 1 \times (J = 1), \quad (165)
$$

and

$$
\gamma = 1 \times (\pm 20) + 1 \times (\pm 16) + 2 \times (\pm 12) + 3 \times (\pm 8) + 3 \times (\pm 4) + 4 \times (0) = \beta, \quad (166)
$$

corresponding to (4, 4, 4, 8). The associated orbit and its representative is presented in Table 17.
Table 16: The four $H^*$-orbits $O_{24H^*}$, $O'_{24H^*}$, $O_{25H^*}$, $O'_{25H^*}$ within the two $F_{4(4)}$-orbits $O_{24}$ and $O_{25}$.

| γ-label | β-label | (2, 2, 4, 4) | (4, 0, 4, 8) | $O_{H^*}$ |
|---------|---------|-------------|--------------|-----------|
| (2, 2, 4, 4) | $-\frac{1}{\sqrt{2}}(5H_2 + H_3) - \sqrt{2}(3K_{14} + 2K_3 + \sqrt{2}(K_{12} + K_5 + K_{15} - K_1 - K_6 + K_{13} + 2\sqrt{5}(K_3 - K_{10}))$ | $-\frac{1}{\sqrt{2}}(H_2 + 5H_3) - \sqrt{2}(3K_{14} + 2K_3 + \sqrt{2}(K_{12} + K_5 + K_{15} - K_1 - K_6 + K_{13} + 2\sqrt{5}(K_3 - K_{10}))$ | $O_{24}$ |
| (4, 0, 4, 8) | $-\frac{1}{2\sqrt{2}}(H_1 + H_2 - H_8 + H_4)$ | $-\frac{1}{2\sqrt{2}}(H_1 + H_2 - H_8 + H_4)$ | $O'_{H^*}$ |
| (4, 0, 4, 8) | $-5(K_{21} - K_{17}) + \sqrt{2}(K_{16} + K_{12}) - K_5 + K_2 + K_{15} - K_1 - K_6 + K_{13} - \frac{1}{\sqrt{2}}(K_3 + K_{11})$ | $-5(K_{21} - K_{17}) + \sqrt{2}(K_{16} + K_{12}) - K_5 + K_2 + K_{15} - K_1 - K_6 + K_{13} - \frac{1}{\sqrt{2}}(K_3 + K_{11})$ | $O'_{24}$ |
| (4, 0, 4, 8) | $\sqrt{10}(K_8 + K_{23} + K_9 + K_{19})$ | $\sqrt{10}(K_8 + K_{23} + K_9 + K_{19})$ | $O'_{25}$ |

Table 17: The $H^*$-orbit $O_{26H^*}$ within the $F_{4(4)}$-orbits $O_{26}$.

| γ-label | β-label | (4, 4, 4, 8) | $O_{H^*}$ |
|---------|---------|-------------|-----------|
| (4, 4, 4, 8) | $-\sqrt{\frac{31}{2}}(H_2 - H_3) + 4\sqrt{2}(K_{23} - K_{19})$ | $O_{26}$ |
| (4, 4, 4, 8) | $-\sqrt{42}K_3 + 2\sqrt{15}(K_8 + K_9) + \sqrt{\frac{11}{2}}(K_{16} + K_{12}) - K_5 + K_2 + K_{15} - K_1 - K_6 + K_{13}$ | $O_{26}$ |

Table 17: The $H^*$-orbit $O_{26H^*}$ within the $F_{4(4)}$-orbits $O_{26}$.
3.2.1 Tensor Classifier Analysis

Let us introduce a set of tensor classifiers (TC) which proves to be a valuable tool for the classification. These are rank-two symmetric $H^*$-tensors, constructed out of the Lax components $\Delta^{a,A}$ at radial infinity, whose signature is used as an $H^*$-invariant feature.

Let us introduce the relevant quantities. We denote by $s_\alpha, t_x, \alpha = 1, 2, 3, x = 1, \ldots, 21$ the generators of the $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{sp}'(6,\mathbb{R})$ subalgebras of $\mathfrak{H}^*$. Their adjoint action on the generators $K_{a,A}, a = 1, 2, A = 1, \ldots, 14$, of $\mathbb{R}^*$ in the $(2,14')$ of $H^*$ is defined by the following commutation relations:

$$[s_\alpha, K_{a,A}] = -s_{\alpha a}^b K_{b,A}, \quad [t_x, K_{a,A}] = -t_{x A}^B K_{a,B},$$

(167)

where the matrices $(s_\alpha)_a^b, (t_x)_A^B$, describe the generators $s_\alpha, t_x$ in the 2 and 14' representations respectively. Using the symplectic property of these matrices, we can construct the following symmetric tensors

$$s_{\alpha ab} = s_\alpha^c \epsilon_{cb}, \quad t_{x AB} \equiv t_x^A C C^B.$$

(168)

We start defining now a set of tensors which are of second order in $\Delta^{a,A}$. Using the general decompositions

$$(14' \times 14')_{\text{sym.}} = 21 + 84,$$

(169)

$$(14' \times 14')_{\text{antisym.}} = 1 + 90,$$

(170)

we see that the SL(2,\mathbb{R})-singlets in the product of two Lax components $\omega^{aA} \omega^{bB}$ can only fall in the representations $(1,1) + (1,90)$, so that we may write:

$$\omega^{aA} \omega^{bB} \epsilon_{ab} = T^{AB} + T \mathbb{C}^{AB}.$$  

(171)

The singlet $T$ is zero for all matrices $L_0$ associated with extremal solutions, since

$$0 = c^2 \propto \text{Tr}(L_0^2) \propto \omega^{aA} \omega^{bB} \epsilon_{ab} \mathbb{C}_{AB} = 14 T,$$

(172)

where $c$ is the extremality parameter of the four dimensional solution. From the antisymmetric tensor $T^{AB}$ in the $(1,90)$ we can construct a symmetric tensor classifier $T_{xy}$ as follows:

$$T_{xy} = \frac{1}{2} t_{x A C} t_{y B D} \mathbb{C}^{C D} T^{AB} = \frac{1}{2} t_{x A C} t_{y B D} \mathbb{C}^{C D} \omega^{aA} \omega^{bB} \epsilon_{ab}.$$

(173)

The signature of this tensor, i.e. the number of positive, negative and null eigenvalues, is an $H^*$-invariant feature which is useful for distinguishing different orbits. This tensor has moreover another relevance to the study of black holes: It vanishes if and only if the extremal solution is BPS. To show this we recall that the $D = 3$ theory under consideration is characterized by 14 fermionic fields $\lambda^A$ whose supersymmetry variation on the geodesic background is expressed in terms of the Lax components by eq. (65). As shown in the last paragraph of Sect. 2.2, the existence of a residual supersymmetry is equivalent to
the property of $\Delta^aA$ to factorize: $\Delta^aA = e^a \Delta^A$. This feature is in turn equivalent to the vanishing of $\Delta^aA\Delta^bB \epsilon_{ab}$ and thus of $T_{xy}$:

$$\text{SUSY} \iff \Delta^aA\Delta^bB \epsilon_{ab} = 0 \iff T_{xy} \equiv 0.$$ (174)

This is consistent with our last statement of Sect. 2.2: residual supersymmetry is a $G$-invariant feature of the geodesic or, equivalently, an $H^*$-invariant feature of $L_0$. We have indeed related it to the vanishing of an $H^*$-covariant tensor.

We can construct other symmetric covariant matrices which are of second order in the Lax tensor $\Delta^aA$, like the following four tensors which are symmetric in the couples $(a, A), (b, B)$:

$$T_{aA,bB}^{(21)} \equiv \omega^aC \omega^bD P_{(21)}^{CD} \Delta^cA \Delta^dA,$$ (175)

$$T_{aA,bB}^{(84)} \equiv \omega^aC \omega^bD P_{(84)}^{CD} \Delta^cA \Delta^dA,$$ (176)

where

$$P_{(21)}^{CD} \equiv -t_{x AB} t_{x CD} ; \quad P_{(84)}^{CD} \equiv \delta_{(AB)}^{CD} + t_{x AB} t_{x CD},$$ (177)

are the projectors onto the $21$ and $84$, respectively, and the adjoint indices $x, y$ of $\text{Sp}(6, \mathbb{R})$ are lowered and raised using the metric $\eta_{xy} \equiv \text{Tr}(t_x t_y)$, proportional to the Cartan-Killing metric of the algebra.

Next we introduce a set of quartic tensor classifiers. To this end we define the following quantity:

$$W_{\alpha}^{(AB)} \equiv s_{\alpha ab} \Delta^aA \Delta^bB,$$ (178)

By virtue of (169), the representation labeled by symmetric couple $(AB)$ can be decomposed into the $21 + 84$:

$$W_{\alpha x} = W_{\alpha}^{(AB)} t_{x AB} ; \quad W_{\alpha}^{(84)} (AB) = W_{\alpha}^{(CD)} P_{(84)}^{CD} AB;$$ (179)

and the following $21 \times 21$, $105 \times 105$ and $3 \times 3$ symmetric tensors can be constructed:

$$\Sigma_{xy} \equiv W_{\alpha x} W_{\beta y} \eta^{\alpha\beta} ; \quad \Sigma_{(84)}^{(AB),(CD)} \equiv W_{\alpha}^{(84)} (AB) W_{\beta}^{(84)} (CD) \eta^{\alpha\beta} ; \quad T_{\alpha\beta} \equiv W_{\alpha x} W_{\beta y} \eta^{xy},$$ (180)

where $\eta_{\alpha\beta} \equiv \text{Tr}(s_\alpha s_\beta)$. Let us now define the tensor

$$\Gamma_{ac,bd} \equiv \epsilon_{ac} \epsilon_{bd} \epsilon_{a'b'} K_{ABA'B'} K_{C'D'D'D'} \Delta^cA \Delta^aA \Delta^dA \Delta^{a'}A',$$ (181)

where $K_{ABCD}$ is the rank-4 totally symmetric invariant tensor in the four-fold product of the $14'$:

$$K_{ABCD} = t_{x AB} t_{x CD} - \frac{1}{5} C_{A(C} C_{D)B} = K_{(ABCD)},$$ (182)

in terms of which quartic $\text{Sp}'(6, \mathbb{R})$-invariant $I_4(Q)$ of a generic vector $Q = (Q^A)$ in the $14'$ reads:

$$I_4(Q) \equiv -\frac{5}{44} K_{ABCD} Q^A Q^B Q^C Q^D.$$ (183)
Using the above definitions, we introduce the following $28 \times 28$ tensor classifiers:

$$T^{(3,21)}_{aA,bB} \equiv \Gamma_{aC,bD} P^{(21)}_{AB} C^D, \quad T^{(3,84)}_{aA,bB} \equiv \Gamma_{aC,bD} P^{(84)}_{AB} C^D. \quad (183)$$

The signatures of the tensors $T_{xy}$, $T^{aA,bB}_{(21)}$, $T^{aA,bB}_{(84)}$, $T_{xy}$, $T^{AB}_{(84)}$, $T^{CD}_{(84)}$, $T^{(3,21)}_{aA,bB}$, $T^{(3,84)}_{aA,bB}$ provide a valuable tool to discriminate between the various orbits. Although they do not represent a complete set of symmetric tensors, they are sufficient, together with the $\gamma, \beta$ labels, to classify the orbits. The order-2 tensors $T^{aA,bB}_{(21)}$, $T^{aA,bB}_{(84)}$ and the order-4 ones $T^{(3,21)}_{aA,bB}$, $T^{(3,84)}_{aA,bB}$ are quite important in this respect, since they allow to distinguish distinct orbits which share the same $\gamma - \beta$ labels. They occur in the third, fourth, tenth and eleventh $G^C$-orbits. In Table 23 of Appendix A we list for each $H^*$-orbit the signatures of the tensor classifiers.

4 Generating Solutions

In [5] and [6] representatives of the (regular and small) single center black hole orbits with the least number of parameters (generating solutions) were explicitly constructed in symmetric supergravities. In particular it was shown that these were dilatonic solutions described by null geodesics in a characteristic submanifold $\mathcal{M}_N$ of the form$^9$:

$$\mathcal{M}_N = \left( \frac{\text{SL}(2,\mathbb{R})}{\text{SO}(1,1)} \right)^p = (dS_2)^p \subset \frac{G}{H^*}, \quad (184)$$

where $p$ is the non-compact rank of the coset $H^*/H_c$, $H_c = H'_4 \times U(1)$ being the maximal compact subgroup of $H^*$. For our model

$$p = \text{rank} \left( \frac{H^*}{H_c} \right) = \text{rank} \left( \frac{\text{SL}(2,\mathbb{R}) \times G'_4}{U(1) \times H'_4} \right) = \text{rank} \left( \frac{\text{SL}(2,\mathbb{R}) \times \text{Sp}'(6,\mathbb{R})}{U(3)} \right) = 4. \quad (185)$$

The generating geodesic will be the product of geodesics inside the four $dS_2$ factors of $\mathcal{M}_N$. One can show [5] that $p$ is related to the electric and magnetic charges: In fact the normal form of the electric-magnetic charge vector with respect to the action of $H_c = U'(3) \times U(1)$ is a $p$-charge vector. This means that, by acting by means of $H_c$ on a generic combination $Y^A_0 K_A$ in the representation $\mathbf{R}'$, we can always rotate it into a subspace of dimension $p = 4$ in the coset (184):

$$Y^A_0 K_A \xrightarrow{H_c} Y^{\ell}_0 K_{\ell}, \quad (186)$$

where $K_{\ell} = (T_{\ell} + \eta T^{T}_{\ell} \eta)/2$ and $T_{\ell}$ are $p = 4$ out of the $T_M$ generators. This means that, by means of $H_c$, the central- and matter-charge vector at infinity can always be reduced to $p$ real parameters. The generators $T_{\ell}$ can be identified with the $f_{4(4)}$-shift generators (with respect to the non-compact Cartan subalgebra in the coset $G/H^*$) corresponding to $p = 4$.

---

$^9$In the presence of hypermultiplets in the $\mathcal{N} = 2, D = 4$ theory, additional $\text{SO}(1,1)$ factors will appear in the definition of $\mathcal{M}_N$, in number equal to the rank of the corresponding quaternionic manifold.
of the $\gamma_M$ roots which, in the basis $\{T_0, h_i\}$ of the non-compact Cartan subalgebra of the coset $G/H^*$, are described by mutually orthogonal 4-vectors. These generators define the $p = 4$ $\mathfrak{sl}(2)_{\ell}$ isometry algebras of the four $dS_2$ factors of $\mathcal{M}_N$. The $\mathfrak{sl}(2)_{\ell} = \text{Span}(\mathcal{J}_\ell, \mathcal{H}_\ell, K_\ell)$ algebras are constructed as follows:

$$\mathcal{J}_\ell = \frac{1}{2} (T_\ell + T^\ell_\ell) ; \quad K_\ell = \frac{1}{2} (T_\ell - T^\ell_\ell) ; \quad \mathcal{H}_\ell = \frac{1}{2} [T_\ell, T^\ell_\ell] .$$

(187)

where the above generators satisfy the following relation:

$$[\mathcal{H}_\ell, \mathcal{J}_\ell] = \delta_{\ell \ell'} K_{\ell'} , \quad [\mathcal{H}_\ell, K_{\ell'}] = \delta_{\ell \ell'} \mathcal{J}_{\ell'} , \quad [\mathcal{J}_\ell, K_{\ell'}] = -\delta_{\ell \ell'} \mathcal{H}_{\ell'} .$$

(188)

The matrices $\mathcal{J}_\ell$ generate the four $SO(1, 1)_{\ell}$ groups in the denominator of $\mathcal{M}_N$. The normal manifold $\mathcal{M}_N$ is parametrized by the three dilatons $\varphi_i$, the scalar $U$ and four of the 14 $\mathbb{Z}^M$. The corresponding generating solution will therefore be a four-charge dilatonic one. We can make two choices for the four mutually orthogonal roots $\tilde{\gamma}_\ell$ among the $\gamma_M$, which we give below as vectors $(\gamma_M(T_0), \gamma_M(h_i))$ corresponding to different sets of scalars $\mathbb{Z}^M$:

$$(\mathbb{Z}_0, \mathbb{Z}^1, \mathbb{Z}^4, \mathbb{Z}^6) : \quad \tilde{\gamma}_0 = \gamma_1 = \frac{1}{2}(1, -1, -1, -1) , \quad \tilde{\gamma}_1 = \gamma_9 = \frac{1}{2}(1, -1, 1, 1) ,$$

$$\quad \tilde{\gamma}_4 = \gamma_{12} = \frac{1}{2}(1, -1, 1, 1) , \quad \tilde{\gamma}_6 = \gamma_{14} = \frac{1}{2}(1, 1, 1, -1) ,$$

(189)

$$(\mathbb{Z}^0, \mathbb{Z}^1, \mathbb{Z}^4, \mathbb{Z}^6) : \quad \tilde{\gamma}_0 = \gamma_8 = \frac{1}{2}(1, 1, 1, 1) , \quad \tilde{\gamma}_1 = \gamma_2 = \frac{1}{2}(1, 1, -1, -1) ,$$

$$\quad \tilde{\gamma}_4 = \gamma_5 = \frac{1}{2}(1, -1, -1, 1) , \quad \tilde{\gamma}_6 = \gamma_7 = \frac{1}{2}(1, -1, -1, 1) ,$$

(190)

Given the general relation between $\mathbb{Z}^M$ and the quantized charges $\Gamma^M$ of the solution [5]:

$$\mathbb{Z}^M = \mathbb{R}^M_{\tau_0} = -e^{2U} \mathcal{C}^{MN} \mathcal{M}_{4NP}(\phi^P) \Gamma^P ,$$

(191)

we can say that the set (189) corresponds to a dilatonic solution with charges $q_0, p^1, p^4, p^6$, interpreted as originating from a set of $D0, D4$ branes, while the choice (190) yields a solution with charges $p^0, q_1, q_4, q_6$, originating from $D6, D2$ branes. We shall choose the normal form corresponding to the first choice, so that the $T_\ell, \ell = 0, 1, 4, 6$, can be identified with the following $T_M$:

$$T_0 = T_1 ; \quad T_1 = T_9 ; \quad T_4 = T_{12} ; \quad T_6 = T_{14} .$$

(192)

Note that both choices define a truncation of the STU model, i.e. $\mathcal{M}_N$ is a totally geodesic submanifold of $\mathcal{M}_{QK}^{(SU)} \subset G/H^*$ of eq. (34). We wish to emphasize here that while the STU truncation exists for all symmetric models with a rank 3 $\mathcal{M}_{SK}$, the construction of $\mathcal{M}_N$ is universal for symmetric models and allows to construct representatives of the $H^*$ orbits corresponding to regular and small black holes.

\[10\text{For later convenience we choose as range of } \ell \text{ the values } 0,1,4,6, \text{ in light of the truncation of the 14 charges to the eight of the STU model.} \]
A diagonalizable $L_0$ can be rotated by means of $H^*$ into a Cartan subalgebra in the coset (184). In particular the component of $L_0$ in the tangent space of some of the $dS_2$ factors may be a compact (i.e. anti-symmetric in the chosen real representation) matrix. This is the case if $L_0$ has imaginary eigenvalues. The corresponding geodesic will have a projection onto some of the $dS_2$ subspaces, which hits the boundary of the solvable (i.e. physical) patch and, as a consequence, $e^{-U}$ will vanish at finite $\tau$, signalling a true singularity of the four-dimensional space-time metric. In order for the solution generated by a diagonalizable $L_0$ to be regular, it must have real eigenvalues only (i.e. $L_0$ must be symmetric). We shall deal with such solutions in a next section.

Let us restrict to extremal solutions in $\mathcal{M}_N$ generated by nilpotent $L_0$. Having defined the normal form according to (189) we proceed in defining the nilpotent elements in the coset $\mathcal{M}_N$:

$$N^{(\varepsilon)}_\ell = H_\ell - \varepsilon_\ell K_\ell\ , \ [J_\ell, N^{(\varepsilon)}_\ell] = \varepsilon_\ell N^{(\varepsilon)}_\ell\ , \ \ell = 0, 1, 4, 6\ ,$$

(193)

where $\varepsilon_\ell = \pm 1$. Consider now the geodesic originating in the origin $O$ of $\mathcal{M}_N$ at radial infinity, corresponding to $\varphi_\ell = U = 0$, with initial velocity $L_0$ which, being $\mathbb{L}_r(O) = 1$, coincides with the Noether charge matrix $Q$ in eq. (54). A generic nilpotent $L_0$ on the tangent space to $\mathcal{M}_N$ will be a combination of $N^{(\varepsilon)}_\ell$ of the form:

$$Q = L_0 = \sum_{\ell=0,1,4,6} k_\ell N^{(\varepsilon)}_\ell = \sum_{\ell=0,1,4,6} k_\ell (H_\ell - \varepsilon_\ell K_\ell)\ .$$

(194)

All these combinations have vanishing NUT charge: $\text{Tr}(QI_r) = 0$. The coefficients $k_\ell$ of $H_\ell$ define the scalar charges and ADM mass, while the coefficients of $K_\ell$ define the electric and magnetic charges, which can be computed using eq. (55) to be:

$$q_0 = -\varepsilon_0 k_0 / \sqrt{2}\ , \ p^\ell = \varepsilon_\ell k_\ell / \sqrt{2}\ , \ \ell = 1, 4, 6\ .$$

(195)

The ADM mass is computed by tracing $Q$ with $T_0$, as in eq. (56) and reads

$$M_{\text{ADM}} = \lim_{\tau \to 0^-} \dot{U} = \frac{1}{4} \sum_{\ell} k_\ell\ .$$

(196)

Solving (52) or, equivalently, (53), we find the following solution [5,6]:

$$e^{-2U} = \sqrt{H_0 H_1 H_4 H_6}\ , \ e^{p_1} = \sqrt{H_0 H_1 H_6}\ , \ e^{p_2} = \sqrt{H_0 H_4 H_6}\ , \ e^{p_3} = \sqrt{H_0 H_1 H_4}\ ,$$

(197)

$$Z^0 = \frac{q_0}{H_0}\ , \ Z_\ell = -\frac{p^\ell}{H_\ell}\ , \ \ell = 1, 4, 6\ ,$$

(198)

where we have introduced the harmonic functions:

$$H_0 = 1 - k_0 \tau = 1 + \sqrt{2} \varepsilon_0 q_0 \tau\ ; \ H_\ell = 1 - k_\ell \tau = 1 - \sqrt{2} \varepsilon_\ell p^\ell \tau\ , \ \ell = 1, 4, 6\ .$$

(199)

We see that, if one of the $k_\ell (\ell = 0, 1, 4, 6)$ is negative, the corresponding $H_\ell$ vanishes at finite $\tau = 1/k_\ell < 0$ and so does $e^{-2U}$, signalling a true space-time singularity. Regular
solutions therefore correspond to positive, non vanishing \( k_\ell \). In this case the solution has a finite horizon area given by:

\[
A_H = 4\pi \lim_{\tau \to -\infty} \frac{e^{-2U}}{\tau^2} = 4\pi \sqrt{k_0 k_1 k_4 k_6} = 4\pi \sqrt{\epsilon I_4(p, q)} = 4\pi \sqrt{|I_4(p, q)|},
\]

where \( I_4(p, q) = 4q_0 p^4 p^6 \) is the quartic \( G_4 \)-invariant function of the electric and magnetic charges expressed in the charges of the solution, and \( \epsilon \equiv - \prod_\ell \epsilon_\ell \). The near horizon geometry is \( \text{AdS}_2 \times S^2 \) and, in approaching it, the scalar fields evolve towards values which are fixed solely in terms of the quantized charges, consistently with the attractor phenomenon [42–45] (see also [38] for a review of extremal black holes):

\[
\lim_{\tau \to -\infty} e^{\varphi_1} = \sqrt{\xi q_0 p^4}, \quad \lim_{\tau \to -\infty} e^{\varphi_2} = \sqrt{\xi q_0 p^4}, \quad \lim_{\tau \to -\infty} e^{\varphi_3} = \sqrt{\xi q_0 p^4}.
\]

If some of the \( k_\ell \) vanish we end up with solutions having a vanishing horizon area, namely a naked singularity at \( \tau = -\infty \). Such solutions are called small black holes.

Let us elaborate now on the \( H^* \)-orbit of \( L_0 \). We can easily see that \( L_0 \), as defined in (194), does not satisfy eq.s (92), (93). It can however be mapped into one which satisfies (92), (93) by means of an \( \text{SO}(1, 1)^p \) transformation, generated by the \( J_\ell \), whose effect is to rescale each \( k_\ell \) by a positive number and bring them to: \( k_\ell = 0, \pm 1 \). Such transformation clearly cannot affect the signs of \( k_\ell \). Let us consider then an \( L_0 \) given by (194), with \( k_\ell^2 = 0, 1 \). We see that, if we identify the nilpositive element \( E \) of the standard triple with \( L_0 \), the nilnegative \( F \) with \( L_0^T \) and \( h \) with \([L_0, L_0^T]\) we have:

\[
h = [L_0, L_0^T] = -2 \sum_\ell \epsilon_\ell k_\ell^2 J_\ell \ ; \ i(E - F) = i(L_0 - L_0^T) = -2 \sum_\ell \epsilon_\ell k_\ell(iK_\ell).
\]

Within \( \mathfrak{g}^C \), the elements \( iK_\ell \) and \( J_\ell \) are \( \mathfrak{g}^C \)-conjugate, just as the complexifications \( \mathfrak{h}_s^C \) and \( \mathfrak{h}_r^C \), of \( \mathfrak{h}_s \) and \( \mathfrak{h}_r \) respectively, are in \( \mathfrak{g}^C \). In particular \( \{iK_\ell\} \) and \( \{J_\ell\} \) are bases of Cartan subalgebras \( \mathfrak{c}_H \) and \( \mathfrak{c}_{H^*} \), respectively in \( \mathfrak{h}_s^C \) and \( \mathfrak{h}_r^C \). If \( \beta_k \) and \( \beta_k' \) are the \( \mathfrak{h}^C \) and \( \mathfrak{h}^C \) simple roots referred to \( \mathfrak{c}_H \) and \( \mathfrak{c}_{H^*} \), respectively, we have that:

\[
\beta_k(iK_\ell) = \beta_k'(J_\ell).
\]

Since by definition the \( \beta \)-labels associated with \( E \) are \( \{\beta_k(i(E - F))\} \) and the \( \gamma \)-labels are \( \{\beta_k'(h)\} \), we have:

\[
\beta - \text{label} = \{-2 \sum_\ell \epsilon_\ell k_\ell \beta_k(iK_\ell)\} \ ; \ \gamma - \text{label} = \{-2 \sum_\ell \epsilon_\ell k_\ell^2 \beta_k'(J_\ell)\}.
\]

We see that the regularity condition \( k_\ell = 0, 1 \) implies the coincidence of \( \gamma \)-and \( \beta \)-labels. This is a formal proof, using the generating solution, of the property:

\[
\text{Regularity} \Rightarrow \gamma - \text{label} = \beta - \text{label},
\]
first conjectured in [12]. A similar proof was given in [18] for the \( G_{2(2)} \)-model. We stress here that this proof applies to all \( \mathcal{N} = 2, D = 4 \) theories with symmetric rank-3 special Kähler manifold, since for all of them \( p = 4 \), the normal manifold is given by (184) and the generating solution by eq.s (197), (198).

From (204) we see that the \( \beta \)-labels only depend on the normal form \( (\epsilon_{\ell} k_{\ell}) \) of the central and matter charges. In fact it was shown on general grounds in [12] that the \( \beta \)-labels only depend on the \( G_4 \)-orbit of the quantized charges \( \Gamma \).

### 4.1 Regular Black Holes and the \( \alpha^{(4)} \)-Orbit

Using the generating solution we can obtain representatives of all the \( H^* \)-suborbits in the first four \( G^C \)-orbits. These are precisely the nilpotent orbits whose step of nilpotency does not exceed 3 and are classified in Tables 3, 4, 5, 6. If all \( k_{\ell} \neq 0 \), we are in the fourth \( G^C \)-orbit, defined by the \( \alpha^{(4)} \)-label \((2, 0, 0, 0)\), with nilpotency step 3. Let us consider these orbits one by one in light of the known classification of \( D = 4 \) extremal black holes [46].

**Orbit \( O_{6H^*} \): The regular BPS solution.** The representative is

\[
L_0 = N_0^- + N_1^+ + N_4^+ + N_6^+.
\] (206)

From eq. (195) we see that all charges are positive and equal to \( 1/\sqrt{2} \). The quartic invariant is positive. The tensor classifier \( T_{xy} \) vanishes, signalling that the solution in BPS. The corresponding \( H^* \) orbit is denoted by \( O_{6H^*} \) and is identified in Table 6 by the \( \beta \) and \( \gamma \) labels both coinciding with \((0, 0, 0, 4)\).

The signatures of the relevant tensor classifiers are:

\[
\begin{align*}
\text{Sign}(T_{xy}) &= (0_+, 0_-), \\
\text{Sign}(T^{aA,bB}_{(21)}) &= (2_+, 12_-), \\
\text{Sign}(T^{aA,bB}_{(84)}) &= (13_+, 1_-), \\
\text{Sign}(\mathcal{T}_{xy}) &= (0_+, 0_-), \\
\text{Sign}(T_{\alpha\beta}) &= (0_+, 0_-), \\
\text{Sign}(\mathcal{T}^{(AB),(CD)}_{(84)}) &= (0_+, 1_-), \\
\text{Sign}(T^{(3,21)}_{aA,bB}) &= (0_+, 0_-), \\
\text{Sign}(T^{(3,84)}_{aA,bB}) &= (0_+, 0_-).
\end{align*}
\] (207)

**Orbit \( O_{7H^*} \): singular BPS solution.** The representative is

\[
L_0 = -N_0^- - N_1^+ + N_4^+ + N_6^+.
\] (208)
The $\beta$-label is $(2, 0, 0, 0)$ while the $\gamma$-label is $(0, 0, 0, 4)$. Since $k_0 = k_1 = -1$ the solution is singular. It is BPS though since $\mathcal{T}_{xy} = 0$. The signatures of the relevant tensor classifiers are:

$$
\text{Sign}(\mathcal{T}_{xy}) = (0_+, 0_-),
$$
$$
\text{Sign}(\mathcal{T}_{aA,bB}^{(21)}) = (6_+, 8_-),
$$
$$
\text{Sign}(\mathcal{T}_{aA,bB}^{(84)}) = (9_+, 5_-),
$$
$$
\text{Sign}(\mathcal{T}_{aA,bB}) = (0_+, 0_-),
$$
$$
\text{Sign}((\mathcal{F}^{(AB)}_{(84)})^{(CD)}) = (0_+, 1_-),
$$
$$
\text{Sign}((\mathcal{F}^{(3,21)}_{aA,bB})^{(3,84)}) = (0_+, 0_-),
$$
$$
\text{Sign}((\mathcal{F}^{(3,84)}_{aA,bB})^{(3,84)}) = (0_+, 0_-).
$$

(209)

Since they differ by the $\beta$-label, the orbits $O_{6H^*}$ and $O_{7H^*}$ belong to different $F_{4(4)}$-orbits.

**Orbit $O_{8H^*}$: singular BPS solution.** The representative is

$$
L_0 = -N_0^- + N_1^+ + N_4^+ + N_6^+.
$$

(210)

The $\beta$-label is $(0, 0, 2, 2)$ while the $\gamma$-label is $(0, 0, 0, 4)$. Since $k_0 = -1$ the solution is singular. It is BPS though since $\mathcal{T}_{xy} = 0$. The signatures of the relevant tensor classifiers are:

$$
\text{Sign}(\mathcal{T}_{xy}) = (0_+, 0_-),
$$
$$
\text{Sign}(\mathcal{T}_{aA,bB}^{(21)}) = (7_+, 7_-),
$$
$$
\text{Sign}(\mathcal{T}_{aA,bB}^{(84)}) = (8_+, 6_-),
$$
$$
\text{Sign}(\mathcal{T}_{aA,bB}) = (0_+, 0_-),
$$
$$
\text{Sign}((\mathcal{F}^{(AB)}_{(84)})^{(CD)}) = (1_+, 0_-),
$$
$$
\text{Sign}((\mathcal{F}^{(3,21)}_{aA,bB})^{(3,84)}) = (0_+, 0_-),
$$
$$
\text{Sign}((\mathcal{F}^{(3,84)}_{aA,bB})^{(3,84)}) = (0_+, 0_-).
$$

(211)

The orbits $O_{6H^*}$, $O_{7H^*}$ and $O_{8H^*}$ belong to three different $F_{4(4)}$-orbits.

**Orbit $O'_{6H^*}$: singular non-BPS solution.** The representative is

$$
L_0 = -N_0^- + N_1^- + N_4^+ + N_6^+.
$$

(212)
Since some $k_l$ are negative, the solution is singular. The tensor classifier $T_{xy}$ does not vanish, signalling that the solution is non-BPS. The corresponding $H^*$ orbit is denoted by $O_{6H^*}$ and is identified in Table 6 by the $\beta-$ and $\gamma-$ labels given by $(0, 0, 0, 4), (2, 0, 0, 0)$ respectively.

The signatures of the relevant tensor classifiers are:

$$
\text{Sign}(T_{xy}) = (5_+, 1_-), \\
\text{Sign}(T_{(21)}^{aA,bB}) = (4_+, 6_-), \\
\text{Sign}(T_{(84)}^{aA,bB}) = (9_+, 1_-), \\
\text{Sign}(\Sigma_{xy}) = (0_+, 1_-), \\
\text{Sign}(T_{ab}) = (0_+, 0_-), \\
\text{Sign}(\Sigma_{(84)}^{(AB),(CD)}) = (0_+, 1_-), \\
\text{Sign}(T_{aA,bB}^{(3,21)}) = (0_+, 0_-), \\
\text{Sign}(T_{aA,bB}^{(3,84)}) = (0_+, 0_-).
$$

(213)

Orbit $\tilde{O}_{7H^*}$: Regular non-BPS solution. The representative is

$$
L_0 = N_0^+ + N_1^- + N_4^+ + N_6^+.
$$

(214)

From eq. (195) we see that the charges $q_0$ and $p^1$ are $-1/\sqrt{2}$, while $p^4$ and $p^6$ are $1/\sqrt{2}$. The quartic invariant is positive and the solution is regular. The tensor classifier $T_{xy}$ is non-vanishes, signalling that the solution in non-BPS. The corresponding $H^*$ orbit is denoted by $\tilde{O}_{7H^*}$ and is identified in Table 6 by the $\beta-$ and $\gamma-$ labels both given by $(2, 0, 0, 0)$.

The signatures of the relevant tensor classifiers are:

$$
\text{Sign}(T_{xy}) = (1_+, 5_-), \\
\text{Sign}(T_{(21)}^{aA,bB}) = (4_+, 6_-), \\
\text{Sign}(T_{(84)}^{aA,bB}) = (9_+, 1_-), \\
\text{Sign}(\Sigma_{xy}) = (0_+, 1_-), \\
\text{Sign}(T_{ab}) = (0_+, 0_-), \\
\text{Sign}(\Sigma_{(84)}^{(AB),(CD)}) = (0_+, 1_-), \\
\text{Sign}(T_{aA,bB}^{(3,21)}) = (0_+, 0_-), \\
\text{Sign}(T_{aA,bB}^{(3,84)}) = (0_+, 0_-).
$$

(215)

Note that $\tilde{O}_{7H^*}$ and $O_{6H^*}$ are only distinguished by the signature of $T_{xy}$.  

49
Orbit $\hat{O}_{7H}^\prime$: singular non-BPS solution. The representative is

$$L_0 = +N_0^+ - N_1^- - N_4^+ + N_6^+.$$  \hfill (216)

Since some $k_\ell$ are negative, the solution is singular. The tensor classifier $T_{xy}$ does not vanish, signalling that the solution in non-BPS. The corresponding $H^*$ orbit is denoted by $\hat{O}_{7H}^\prime$ and is identified in Table 6 by the same $\beta$- and $\gamma$- labels as the orbit $\bar{O}_{7H}^\prime$. The two orbits are also distinguished by a further $\delta$-label.

The signatures of the relevant tensor classifiers are:

$$\text{Sign}(T_{xy}) = (3_+, 3_-),$$
$$\text{Sign}(T_{(21)}^{ab}) = (6_+, 4_-),$$
$$\text{Sign}(T_{(84)}^{ab}) = (7_+, 3_-),$$
$$\text{Sign}(T_{a\beta}) = (0_+, 0_-),$$
$$\text{Sign}(T_{(AB)}^{(CD)}) = (0_+, 0_-),$$
$$\text{Sign}(T_{(3,21)}^{aA,bB}) = (0_+, 0_-),$$
$$\text{Sign}(T_{(3,84)}^{aA,bB}) = (0_+, 0_-),$$

and clearly show that the orbits $\hat{O}_{7H}^\prime$ and $\bar{O}_{7H}^\prime$ are different.

Orbit $O_{8H}^\prime$: singular non-BPS solution. The representative is

$$L_0 = +N_0^+ - N_1^- + N_4^+ + N_6^+.$$  \hfill (218)

Since some $k_\ell$ are negative, the solution is singular. The tensor classifier $T_{xy}$ does not vanish, signalling that the solution in non-BPS. The corresponding $H^*$ orbit is denoted by $O_{8H}^\prime$ and is identified in Table 6 by the $\beta$- and $\gamma$- labels given by (0, 0, 2, 2), (2, 0, 0, 0) respectively.
The signatures of the relevant tensor classifiers are:

\[
\begin{align*}
\text{Sign}(T_{xy}) &= (3_+, 3_-), \\
\text{Sign}(T_{aA,bB}^{(21)}) &= (5_+, 5_-), \\
\text{Sign}(T_{aA,bB}^{(84)}) &= (6_+, 4_-), \\
\text{Sign}(\Sigma_{xy}) &= (1_+, 0_-), \\
\text{Sign}(T_{a\beta}) &= (0_+, 0_-), \\
\text{Sign}(\Sigma_{(84)}^{(AB),(CD)}) &= (1_+, 0_-), \\
\text{Sign}(T_{aA,bB}^{(3,21)}) &= (0_+, 0_-), \\
\text{Sign}(T_{aA,bB}^{(3,84)}) &= (0_+, 0_-).
\end{align*}
\]

(219)

Orbit \(O_{6H}''\): singular non-BPS solution. The representative is

\[
L_0 = N_0^+ + N_1^+ + N_4^+ - N_6^+.
\]

(220)

Since some \(k_\ell\) are negative, the solution is singular. The tensor classifier \(T_{xy}\) does not vanish, signalling that the solution is non-BPS. The corresponding \(H^*\) orbit is denoted by \(O_{6H}''\) and is identified in Table 6 by the \(\beta^-\) and \(\gamma^-\) labels given by \((0, 0, 0, 4), (0, 0, 2, 2)\) respectively.

The signatures of the relevant tensor classifiers are:

\[
\begin{align*}
\text{Sign}(T_{xy}) &= (5_+, 1_-), \\
\text{Sign}(T_{aA,bB}^{(21)}) &= (2_+, 6_-), \\
\text{Sign}(T_{aA,bB}^{(84)}) &= (8_+, 1_-), \\
\text{Sign}(\Sigma_{xy}) &= (0_+, 0_-), \\
\text{Sign}(T_{a\beta}) &= (0_+, 0_-), \\
\text{Sign}(\Sigma_{(84)}^{(AB),(CD)}) &= (0_+, 0_-), \\
\text{Sign}(T_{aA,bB}^{(3,21)}) &= (0_+, 0_-), \\
\text{Sign}(T_{aA,bB}^{(3,84)}) &= (0_+, 1_-).
\end{align*}
\]

(221)

Orbit \(O_{7H}''\): singular non-BPS solution. The representative is

\[
L_0 = N_0^+ + N_1^+ + N_4^+ - N_6^+.
\]

(222)

Since some \(k_\ell\) are negative, the solution is singular. The tensor classifier \(T_{xy}\) does not vanish, signalling that the solution is non-BPS. The corresponding \(H^*\) orbit is denoted by
$O''_{H^*}$ and is identified in Table 6 by the $\beta-$ and $\gamma-$ labels given by $(2,0,0,0), (0,0,2,2)$ respectively.

The signatures of the relevant tensor classifiers are:

\[
\text{Sign}(T_{xy}) = (3_+, 3_-), \\
\text{Sign}(T^{aA,bB}_{(21)}) = (4_+, 4_-), \\
\text{Sign}(T^{aA,bB}_{(84)}) = (6_+, 3_-), \\
\text{Sign}(T_{\alpha\beta}) = (0_+, 0_-), \\
\text{Sign}(T^{(3,21)}_{aA,bB}) = (0_+, 0_-), \\
\text{Sign}(T^{(3,84)}_{aA,bB}) = (0_+, 1_-).
\]

(223)

Orbit $\hat{O}''_{8H^*}$ ($\delta_1$): regular non-BPS solution with $I_4 < 0$. The representative is

\[
L_0 = N_0^+ + N_1^+ + N_4^+ + N_6^+.
\]

(224)

All the $k_\ell$ are positive and the solution is regular. The charges are read off eq. (195) to be $q_0 = -1/\sqrt{2}$, $p^1 = p^4 = p^6 = 1/\sqrt{2}$, and the quartic invariant is negative. The tensor classifier $T_{xy}$ does not vanish, signalling that the solution is non-BPS. The corresponding $H^*$ orbit is denoted by $O''_{8H^*}$ and is identified in Table 6 by the $\beta-$ and $\gamma-$ labels given by $(0,0,2,2), (0,0,2,2)$ respectively.

The signatures of the relevant tensor classifiers are:

\[
\text{Sign}(T_{xy}) = (1_+, 5_-), \\
\text{Sign}(T^{aA,bB}_{(21)}) = (2_+, 6_-), \\
\text{Sign}(T^{aA,bB}_{(84)}) = (8_+, 1_-), \\
\text{Sign}(T_{\alpha\beta}) = (0_+, 0_-), \\
\text{Sign}(T^{(3,21)}_{aA,bB}) = (0_+, 0_-), \\
\text{Sign}(T^{(3,84)}_{aA,bB}) = (1_+, 0_-).
\]

(225)

Orbit $\hat{O}''_{8H^*}$ ($\delta_2$): singular non-BPS solution. The representative is

\[
L_0 = -N_0^+ - N_1^+ + N_4^+ + N_6^+.
\]

(226)
Since some \( k_\ell \) are negative, the solution is singular. The tensor classifier \( T_{xy} \) does not vanish, signalling that the solution is non-BPS. The corresponding \( H^* \) orbit is denoted in Table 6 by \( \tilde{O}_{6H^*}' \), to distinguish it from \( O_{6H^*}' \), with which it shares the same \( \beta^- \) and \( \gamma^- \) labels.

The signatures of the relevant tensor classifiers are:

\[
\begin{align*}
\text{Sign}(T_{xy}) &= (3_+, 3_-), \\
\text{Sign}(T^{aA,bB}_{(21)}) &= (4_+, 4_-), \\
\text{Sign}(T^{aA,bB}_{(84)}) &= (6_+, 3_-), \\
\text{Sign}(\Xi_{xy}) &= (0_+, 0_-), \\
\text{Sign}(T_{a\beta}) &= (0_+, 0_-), \\
\text{Sign}(\Xi^{(AB),(CD)}_{(84)}) &= (0_+, 0_-), \\
\text{Sign}(\Pi^{(3,21)}_{aA,bB}) &= (0_+, 0_-), \\
\text{Sign}(\Pi^{(3,84)}_{aA,bB}) &= (1_+, 0_-).
\end{align*}
\]

(227)

Thus, in summary, the regular BPS and non-BPS solutions are all described in the diagonal orbits \( O_{6H^*}, \tilde{O}_{7H^*}, \tilde{O}_{8H^*}' \). Their representatives are characterized by having the same scalar charges (same \( k_\ell \)) but different electric and magnetic charges (different \( \epsilon_\ell \)). In general, if we define \( \epsilon'_\ell = -\epsilon_0', \ell = 1, 4, 6 \), one can verify [5, 6] that representatives \( L_0 \) of the first \( O_{6H^*} \) orbit in the form (194) (thus with all \( k_\ell \) positive) are all characterized by having \( \epsilon'_0 = \epsilon'_1 = \epsilon'_4 = \epsilon'_6 \) (so that \( \epsilon = +1 \) and \( I_4 > 0 \)) and generate regular BPS solutions. If, on the other hand, all \( k_\ell \) are positive, \( \epsilon = 1 \) but the \( \epsilon'_\ell \) are not all equal (there are 6 possibilities) the solution lies in the \( \tilde{O}_{7H^*} \) orbit and is regular non-BPS with \( I_4 > 0 \) [5, 6]. Finally if all \( k_\ell \) are positive but \( \epsilon = -1 \) the solution lies in the \( \tilde{O}_{8H^*}' \) orbit and is regular non-BPS with \( I_4 < 0 \) [5, 6]. What we have shown here is that the generating solution allows to derive representatives of all the \( H^* \)-orbits with degree of nilpotency not exceeding 3.

Let us observe that the gradings \( \epsilon_\ell \) of the nilpotent generators entering \( L_0 \) are related to a particular \( \gamma^- \)-label. We can move from one orbit to an other in Table 6 by observing that:

\[
\begin{align*}
e^{-i\pi J_\ell} N^\epsilon_\ell e^{i\pi J_\ell} &= -N^\epsilon_\ell, \\
e^{-\pi K_\ell} N^\epsilon_\ell e^{\pi K_\ell} &= -N^{-\epsilon}_\ell, \\
e^{-\pi K_\ell} e^{-i\pi J_\ell} N^\epsilon_\ell e^{i\pi J_\ell} e^{\pi K_\ell} &= N^{-\epsilon}_\ell,
\end{align*}
\]

(228)

so that the adjoint action on the representative \( L_0^{(BPS)} \), in the form (194), of the BPS orbit \( O_{6H^*} \), of an even number of transformations \( e^{i\pi J_\ell} e^{\pi K_\ell} \), have the effect of switching an even number of \( \epsilon_\ell \), yielding a representative of the non-BPS orbit \( \tilde{O}_{7H^*}' \). By the same token the action of an odd number of such transformations will map \( L_0^{(BPS)} \) into the non-BPS \( \tilde{O}_{8H^*}' \).
In general the adjoint action of $e^{\pi K\ell}$ will not alter the $G$-orbit (and thus the $\beta$-label) since the transformation belongs to $G$. It may alter the $\gamma$-label and thus make us move vertically in Table 6. Moreover its action changes the sign of $\epsilon_\ell$ and of $k_\ell$, keeping the sign of the corresponding electric-magnetic charge fixed. The transformation $e^{i\pi J\ell}$, on the other hand, belongs to $H^\prime C$, complexification of $H^*$. Its action will therefore not alter the $\gamma$-labels and the supersymmetry property of the solution (since $T_{xy}$ is also a $H'^C$-covariant tensor) but, being it in $G^C/G$, it may affect the $G$-orbit of the solution and thus the corresponding $\beta$-label. It will in other words make us move horizontally in Table 6 or, for fixed $\beta$- and $\gamma$-labels, vertically from the $\delta_1$ to the $\delta_2$ orbits, whenever this further splitting exists.

The relations discussed above among the various $H^*$-orbits, clearly extend to any of their representatives.

4.2 Small Black Holes and the $\alpha^{(1)}$, $\alpha^{(2)}$, $\alpha^{(3)}$ Orbits

4.2.1 Small Lightlike Black Holes ($\alpha^{(3)}$)

Let us first consider the case in which one of the $k_\ell$ vanishes. In this case $I_4 = 0$, though $\partial I_4/\partial k_\ell \neq 0$. The corresponding $G^C$-orbit is the one defined by the $\alpha^{(3)}$-label $(0, 1, 0, 0)$. Let us consider these orbits one by one in light of the known classification of $D = 4$ small black holes [47, 48]:

**Orbit $O_{4H^*}$: small, light-like, BPS black hole.** A representative of this orbit, in the form (194) can be obtained by setting one parameters $k_\ell$ in the representative of $O_{6H^*}$ (i.e. regular BPS black hole, $k_\ell > 0$, $\epsilon_1 = \epsilon_4 = \epsilon_6 = -\epsilon_0$) to zero. This amounts to setting in the generating solution for regular BPS black holes one of the charges to zero. For example we can choose

$$L_0 = N_0^- + N_1^+ + N_4^+,$$

obtained by setting $p^6 \to 0$ in the generating solution for regular BPS black holes. Since $T_{xy} = 0$, the solution is BPS. The $\beta$- and $\gamma$-labels are both $(0, 0, 1, 3)$. This solution has vanishing horizon area and thus a naked singularity at $\tau \to -\infty$ but no singularity at finite $\tau$. Though $I_4 = 0$, its gradient with respect to the electric-magnetic charges is non-vanishing. These small black holes are named lightlike. The signatures of the relevant
| γ-label   | β-label   | (0, 0, 0, 4)                                           | (2, 0, 0, 0)                                           | (0, 0, 2, 2)                                           |
|-----------|-----------|-------------------------------------------------------|-------------------------------------------------------|-------------------------------------------------------|
| (0, 0, 4) |           | $L_0 = N_0^- + N_1^+ + N_4^+ + N_6^+$                 | $L_0 = -N_0^- - N_4^+ + N_4^+ + N_6^+$               | $L_0 = -N_0^- + N_1^+ + N_4^+ + N_6^+$               |
| (2, 0, 0) |           | $L_0 = -N_0^+ - N_1^- + N_4^+ + N_6^+$                |                                                       |                                                       |
|           | $\delta_1$| $L_0 = N_0^+ + N_1^+ + N_4^+ + N_6^+$                |                                                       |                                                       |
|           | $\delta_2$| $L_0 = +N_0^+ - N_1^- + N_4^+ + N_6^+$               |                                                       |                                                       |
| (0, 0, 2) |           | $L_0 = N_0^- - N_1^- + N_4^+ + N_6^+$                 | $L_0 = N_0^+ + N_1^+ + N_4^+ - N_6^+$               |                                                       |
|           | $\delta_1$| $L_0 = N_0^+ + N_1^+ + N_4^+ + N_6^+$                |                                                       |                                                       |
|           | $\delta_2$| $L_0 = -N_0^+ - N_1^- + N_4^+ + N_6^+$               |                                                       |                                                       |

Table 18: The representatives of $\alpha^{(4)}$-orbitis in terms of the generating solution.
tensor classifiers are:

\[
\begin{align*}
\text{Sign}(T_{xy}) & = (0_+, 0_-), \\
\text{Sign}(T^{aA,bB}_{(21)}) & = (2_+, 6_-), \\
\text{Sign}(T^{aA,bB}_{(84)}) & = (7_+, 1_-), \\
\text{Sign}(\Sigma_{xy}) & = (0_+, 0_-), \\
\text{Sign}(T_{a\beta}) & = (0_+, 0_-), \\
\text{Sign}(\Sigma^{(AB),(CD)}_{(84)}) & = (0_+, 0_-), \\
\text{Sign}(T^{(3,21)}_{aA,bB}) & = (0_+, 0_-), \\
\text{Sign}(T^{(3,84)}_{aA,bB}) & = (0_+, 0_-).
\end{align*}
\]

(231)

**Orbit \(O_{5H^*}:\) singular BPS black hole.** A representative of this orbit is obtained by setting to zero one of the charges in the generating solution of \(O_{7H^*}\) or \(O_{8H^*}\). We can choose for instance

\[
L_0 = -N_0^--N_1^++N_4^+,
\]

(232)

One of the \(k_\ell\) is negative, implying a singularity at finite \(\tau\). The solution is still BPS since \(T_{xy} = 0\). The signatures of the relevant tensor classifiers are:

\[
\begin{align*}
\text{Sign}(T_{xy}) & = (0_+, 0_-), \\
\text{Sign}(T^{aA,bB}_{(21)}) & = (4_+, 4_-), \\
\text{Sign}(T^{aA,bB}_{(84)}) & = (5_+, 3_-), \\
\text{Sign}(\Sigma_{xy}) & = (0_+, 0_-), \\
\text{Sign}(T_{a\beta}) & = (0_+, 0_-), \\
\text{Sign}(\Sigma^{(AB),(CD)}_{(84)}) & = (0_+, 0_-), \\
\text{Sign}(T^{(3,21)}_{aA,bB}) & = (0_+, 0_-), \\
\text{Sign}(T^{(3,84)}_{aA,bB}) & = (0_+, 0_-).
\end{align*}
\]

(233)

The \(\beta^-\) and \(\gamma^-\) labels of the orbit are \((1, 0, 1, 1)\) and \((0, 0, 1, 3)\) respectively.

**Orbit \(O_{4H^*}':\) singular BPS black hole.** This orbit is obtained as a singular limit (implemented by setting some of the \(k_\ell\) to zero) of the off-diagonal orbits in the \(\alpha^{(4)}\)-class. We can choose for instance

\[
L_0 = N_0^+-N_1^+-N_4^+,
\]

(234)
Some of the \( k_\ell \) are negative, implying a singularity at finite \( \tau \). The solution is non-BPS since \( T_{xy} \neq 0 \). The \( \beta \)- and \( \gamma \)-labels of the orbit are \((0, 0, 1, 3)\) and \((1, 0, 1, 1)\) respectively. The signatures of the relevant tensor classifiers are:

\[
\begin{align*}
\text{Sign}(T_{xy}) &= (3_+, 1_-), \\
\text{Sign}(T_{\alpha\beta}) &= (0_+, 0_-), \\
\text{Sign}(T_{aA,bB}) &= (0_+, 0_-), \\
\text{Sign}(T_{(AB)(CD)}) &= (0_+, 0_-), \\
\text{Sign}(T_{xy}) &= (1_+, 3_-), \\
\text{Sign}(T_{(21)}^{aA,bB}) &= (2_+, 3_-), \\
\text{Sign}(T_{(84)}^{aA,bB}) &= (5_+, 1_-), \\
\text{Sign}(\Xi_{xy}) &= (0_+, 0_-), \\
\text{Sign}(\Xi_{(84)}^{(AB)(CD)}) &= (0_+, 0_-), \\
\text{Sign}(\Xi_{xy}) &= (0_+, 0_-), \\
\text{Sign}(T_{xy})^{(3,21)} &= (0_+, 0_-), \\
\text{Sign}(T_{xy})^{(3,84)} &= (0_+, 0_-).
\end{align*}
\]

(235)

Orbit \( \hat{O}_{5H}^\prime (\delta_1) \): small, lightlike, non-BPS black hole. This orbit is obtained as a singular limit (implemented by setting some of the \( k_\ell \) to zero) of the off-diagonal orbits in the \( \alpha^{(4)} \)-class. We can choose for instance

\[
L_0 = N_0^+ + N_1^+ + N_4^+ ,
\]

(236)

All \( k_\ell \) are positive, implying a singularity only at \( \tau \to -\infty \). The solution is non-BPS since \( T_{xy} \neq 0 \). The \( \beta \)- and \( \gamma \)-labels of the orbit are both \((1, 0, 1, 1)\). This solution generates the small, lightlike, non-BPS black holes. The signatures of the relevant tensor classifiers are:

\[
\begin{align*}
\text{Sign}(T_{xy}) &= (1_+, 3_-), \\
\text{Sign}(T_{(21)}^{aA,bB}) &= (2_+, 3_-), \\
\text{Sign}(T_{(84)}^{aA,bB}) &= (5_+, 1_-), \\
\text{Sign}(\Xi_{xy}) &= (0_+, 0_-), \\
\text{Sign}(\Xi_{(84)}^{(AB)(CD)}) &= (0_+, 0_-), \\
\text{Sign}(\Xi_{xy}) &= (0_+, 0_-), \\
\text{Sign}(T_{xy})^{(3,21)} &= (0_+, 0_-), \\
\text{Sign}(T_{xy})^{(3,84)} &= (0_+, 0_-).
\end{align*}
\]

(237)

Orbit \( \hat{O}_{5H}^\prime (\delta_2) \): singular non-BPS solution. We can choose for this orbit the following representative

\[
L_0 = N_0^+ + N_1^+ - N_4^+ ,
\]

(238)
One of the $k_\ell$ is negative, implying a singularity at finite $\tau$. The solution is non-BPS since $T_{xy} \neq 0$. The $\beta$- and $\gamma$- labels of the orbit are still both equal to $(1,0,1,1)$. The signatures of the relevant tensor classifiers are:

\[
\begin{align*}
\text{Sign}(T_{xy}) &= (2_+, 2_-), \\
\text{Sign}(T_{(21)}^{aA,bB}) &= (3_+, 2_-), \\
\text{Sign}(T_{(84)}^{aA,bB}) &= (4_+, 2_-), \\
\text{Sign}(\bar{T}_{xy}) &= (0_+, 0_-), \\
\text{Sign}(T_{\alpha\beta}) &= (0_+, 0_-), \\
\text{Sign}(T_{(1,0,1,1)}^{A,B,C,D}) &= (0_+, 0_-), \\
\text{Sign}(T_{(0,0,1,3)}^{aA,bB}) &= (0_+, 0_-).
\end{align*}
\]

This orbit is distinguished from $O'_{5H^*}$ by the tensor classifiers.

\[
\begin{align*}
\text{Table 19: The representatives of $\alpha^{(3)}$- orbits in terms of the generating solution.}
\end{align*}
\]

4.2.2 Small Critical Black Holes ($\alpha^{(2)}$)

Now we consider the case in which two of the $k_\ell$ vanish. In this case the following properties, which can be easily verified on the generating solution, extend to the whole orbit of the electric-magnetic charges in the representation $R$ of $G_4$:

\[
I_4(p,q) = 0 ; \quad \frac{\partial I_4}{\partial \Gamma^M} \equiv 0 ,
\]

The corresponding $G^C$-orbit is the one defined by the $\alpha^{(2)}$-label $(0,0,1,1)$. Let us consider these orbits one by one:

**Orbit $O_{2H^*}$: small, critical, BPS black hole.** A representative of this orbit, in the form (194) can be obtained by setting one parameters $k_\ell$ in the representative of $O_{4H^*}$ (i.e. lightlike BPS black hole) to zero. This amounts to setting in the generating solution for regular BPS black holes two of the charges to zero. For example we can choose

\[
L_0 = N_0^- + N_1^+, \quad (241)
\]
obtained by setting $p^4, p^6 \to 0$ in the generating solution for regular BPS black holes. Since $T_{xy} = 0$, the solution is BPS. The $\beta$- and $\gamma$-labels are both $(1, 0, 0, 2)$. This solution has vanishing horizon area and thus a naked singularity at $\tau \to -\infty$ but no singularity at finite $\tau$. Since both $I_4$ and its gradient with respect to the charges vanish, these small black holes are named critical. This orbit could also be reached from $\bar{O}_{8H}''$ by setting the $q_0$ and one of the $p^\ell, \ell = 1, 4, 6$, charges to zero in the non-BPS regular generating solution with $I_4 < 0$, or from $\bar{O}_{7H}'$ by setting in the generating solution of regular non-BPS black holes with $I_4 > 0$ two charges to zero.

The signatures of the relevant tensor classifiers are:

\[ \text{Sign}(T_{xy}) = (0_+, 0_-), \]
\[ \text{Sign}(T_{(21)}^{aA,bB}) = (2_+, 3_-), \]
\[ \text{Sign}(T_{(84)}^{aA,bB}) = (4_+, 1_-), \]
\[ \text{Sign}(\bar{T}_{xy}) = (0_+, 0_-), \]
\[ \text{Sign}(T_{a\bar{\alpha}}) = (0_+, 0_-), \]
\[ \text{Sign}(\bar{T}_{(84)}^{(AB),(CD)}) = (0_+, 0_-), \]
\[ \text{Sign}(\bar{T}_{aA,bB}^{(3,21)}) = (0_+, 0_-), \]
\[ \text{Sign}(\bar{T}_{aA,bB}^{(3,84)}) = (0_+, 0_-). \quad (242) \]

**Orbit $O_{3H}$**: singular BPS black hole. We can choose as representative of this orbit the matrix:

\[ L_0 = N_0^+ - N_1^-, \quad (243) \]

This can be obtained by acting on the representative (241) of $O_{2H}$ with $e^{i\pi J_1}$. This has the effect of changing the $\beta$-label to $(0, 1, 0, 0)$ while keeping the $\gamma$ one unaltered and equal to $(1, 0, 0, 2)$. The corresponding solution exhibits a $D = 4$ true space-time singularity at finite $\tau$, since some of the $k_\ell$ are negative.
The signatures of the relevant tensor classifiers are:

\[
\text{Sign}(T) = (0_+, 0_-),
\]
\[
\text{Sign}(T^{aA,bB}_{(21)}) = (3_+, 2_-),
\]
\[
\text{Sign}(T^{aA,bB}_{(84)}) = (3_+, 2_-),
\]
\[
\text{Sign}(\Xi_{xy}) = (0_+, 0_-),
\]
\[
\text{Sign}(\Sigma_{\alpha\beta}) = (0_+, 0_-),
\]
\[
\text{Sign}(\Sigma^{(AB),(CD)}) = (0_+, 0_-),
\]
\[
\text{Sign}(\Sigma^{(3,21)}_{aA,bB}) = (0_+, 0_-),
\]
\[
\text{Sign}(\Sigma^{(3,84)}_{aA,bB}) = (0_+, 0_-).
\]

(244)

**Orbit \(O'_{2H^*}\): singular BPS black hole.** We can choose as representative of this orbit the matrix:

\[
L_0 = N_0^+ - N_1^+,
\]

(245)

The \(\gamma\)- and \(\beta\)-labels are \((0, 1, 0, 0)\) and \((1, 0, 0, 2)\) respectively and the orbit is denoted by \(O'_{2H^*}\). Since \(T_{xy}\) is non-vanishing, the solution is non-BPS. Some of the \(k\ell\) are negative implying a singularity at finite \(\tau\).

The signatures of the relevant tensor classifiers are:

\[
\text{Sign}(T_{xy}) = (2_+, 1_-),
\]
\[
\text{Sign}(T^{aA,bB}_{(21)}) = (0_+, 0_-),
\]
\[
\text{Sign}(T^{aA,bB}_{(84)}) = (3_+, 1_-),
\]
\[
\text{Sign}(\Xi_{xy}) = (0_+, 0_-),
\]
\[
\text{Sign}(\Sigma_{\alpha\beta}) = (0_+, 0_-),
\]
\[
\text{Sign}(\Sigma^{(AB),(CD)}) = (0_+, 0_-),
\]
\[
\text{Sign}(\Sigma^{(3,21)}_{aA,bB}) = (0_+, 0_-),
\]
\[
\text{Sign}(\Sigma^{(3,84)}_{aA,bB}) = (0_+, 0_-).
\]

(246)

**Orbit \(O'_{3H^*}\): small, critical non-BPS black hole.** We can choose as representative of this orbit the matrix:

\[
L_0 = N_0^+ + N_1^+.
\]

(247)

Both \(\gamma\)- and \(\beta\)-labels are \((0, 1, 0, 0)\). The solution is non-BPS and describes a small black hole whose charges are in the \(G_4\)-orbit characterized by the properties (240). It is therefore
a small critical non-BPS black hole. Its $D = 4$ space-time geometry exhibits a singularity at $\tau \to -\infty$. The signatures of the relevant tensor classifiers are:

\[
\begin{align*}
\text{Sign}(T_{xy}) &= (1_+, 2_-), \\
\text{Sign}(T_{aA,bB}^{(21)}) &= (0_+, 0_-), \\
\text{Sign}(\overline{T}_{aA,bB}^{(84)}) &= (3_+, 1_-), \\
\text{Sign}(\Xi_{xy}) &= (0_+, 0_-), \\
\text{Sign}(T_{\alpha\beta}) &= (0_+, 0_-), \\
\text{Sign}(\overline{T}^{(AB),(CD)}) &= (0_+, 0_-), \\
\text{Sign}(T_{aA,bB}^{(3,21)}) &= (0_+, 0_-), \\
\text{Sign}(T_{aA,bB}^{(3,84)}) &= (0_+, 0_-).
\end{align*}
\]

\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
$\gamma$-label & $\beta$-label & (1, 0, 0, 2) & (0, 1, 0, 0) \\
\hline
(1, 0, 0, 2) & $L_0 = N_0^- + N_1^+$ & $L_0 = N_0^- - N_1^+$ & $O_{1H^*}$ \\
(0, 1, 0, 0) & $L_0 = N_0^+ - N_1^+$ & $L_0 = N_0^+ + N_1^+$ & $O_{1H^*}$ \\
\hline
\end{tabular}
\caption{The representatives of $\alpha^{(2)}$- orbits in terms of the generating solution.}
\end{table}

4.2.3 $O_{1H^*}$: Small Doubly-Critical Black Holes ($\alpha^{(1)}$)

By setting any of the three charges of the generating solution to zero we end up in the orbit $O_{1H^*}$. We can take as representative the matrices:

\[
L_0 = N_0^+ ,
\]

The electric-magnetic charges satisfy the following general properties

\[
I_4(p, q) = 0 ; \quad \frac{\partial I_4}{\partial \Gamma^M} = 0 ; \quad \frac{\partial^2 I_4}{\partial \Gamma^M \partial \Gamma^N}\bigg|_{\text{Adj}_{4}} = 0 ,
\]

which define a specific $G_4$-orbit of the representation $\mathbf{R}$ of the electric-magnetic charges. The solution is called \textit{doubly critical small black hole}. It is BPS since $T_{xy}$ vanishes and has
a singularity for $\tau \to -\infty$. The signatures of the relevant tensor classifiers are:

\[
\begin{align*}
\text{Sign}(T_{\tau y}) &= (0_+, 0_-), \\
\text{Sign}(T^a_{(21)}B) &= (0_+, 0_-), \\
\text{Sign}(T^a_{(84)}B) &= (1_+, 0_-), \\
\text{Sign}(T_{\alpha \beta}) &= (0_+, 0_-), \\
\text{Sign}(T^a_{(84)}BCD) &= (0_+, 0_-), \\
\text{Sign}(\bar{T}^{(3,21)}_{aA,bB}) &= (0_+, 0_-), \\
\text{Sign}(\bar{T}^{(3,84)}_{aA,bB}) &= (0_+, 0_-).
\end{align*}
\]

(251)

| γ-label | β-label | \(L_0 = N_{\ell}^\pm\) | \(\mathcal{O}_{H^*}\) |
|---------|---------|----------------|----------------|
| (0, 0, 1, 1) | \(\mathcal{O}_1\) | \(\mathcal{O}_{H^*}\) |

Table 21: The representative of \(\alpha^{(1)}\)-orbit in terms of the generating solution.

The various representatives of the \(H^*\)-orbits within \(\alpha^{(1)} - \alpha^{(4)}\) \(G^C\)-orbits, discussed above are listed in Tables 18-21.

### 4.3 Sum Rules

From the explicit expression of the representatives of the various orbits we can deduce sum rules, namely express representatives of each orbit as sum of representatives of orbits with lower degree of nilpotency. This is relevant if we wish to apply the orbit analysis to the study of black hole composites.

For instance the Lax representative (206) of the regular BPS black hole orbit \(O_{6H^*}\) is the sum of \(N_0^- + N_1^+\) and \(N_4^- + N_6^+\), which both belong to the orbits \(O_{2H^*}\) pertaining to small critical BPS black holes.

The representative (214) of the regular non-BPS orbit \(O_{7H^*}\), on the other hand, is the sum either of \(N_0^- + N_1^-\) and \(N_4^+ + N_6^+\), still both belonging to \(O_{2H^*}\), or of \(N_0^+ + N_4^+\) and \(N_1^- + N_6^+\), both in \(O_{5H^*}\) (critical non-BPS black hole).

The representative (224) of the regular non-BPS orbit \(O_{8H^*}\) can be written as \(N_0^+ + N_1^+\) and \(N_4^+ + N_6^+\) in \(O_{5H^*}\) and \(O_{2H^*}\), respectively.

It is interesting to analyze the representatives of the off-diagonal or the \(\delta_2\)-orbits, where some of the \(k_\ell\) are negative. In this case the Lax matrix can still be written as sum of matrices belonging to \(\alpha^{(2)}\)-orbits which contain small non singular solutions. In this
combination however the generator of a small black hole component appears multiplied by 
−1. As a result, some of the scalar fields start their flow at radial infinity with the wrong
derivative and a singularity is produced at finite τ. Take for instance the representative
(216) of $\hat{O}'_{3H^*}$. It is the sum of the two Laxes: $N_0^+ + N_0^+$ and $-(N_1^- + N_1^+)$ both in $O'_{3H^*}$. While $N_0^+ + N_0^+$ generates a genuine small, critical non-BPS black hole, $-(N_1^- + N_1^+)$ would generate one if we were to redefine $\tau \rightarrow -\tau$. For the same $\tau < 0$ the second matrix produces a solution with a singularity at finite $\tau$. The same applies to the orbit $\hat{O}''_{4H^*}$.

4.4 Asymptotics of the Generating Solution

In this section we wish to comment on the behavior of the generating solution at radial
infinity and give a further characterization of those $H^*$-orbits which were discarded as
being associated with singular solutions (here we refer to solutions exhibiting a singularity
at finite $\tau$). To this end we introduce a first order description of the generating solution
in terms of a fake-superpotential $W$ [12, 39, 49–52]. This amounts to writing the four-
dimensional fields in the generating solution as solutions to a first order “gradient-flow”
system of equations of the form:

$$\dot{U} = e^U W ; \quad \dot{\phi}^r = 2e^U G^{rs} \frac{\partial W}{\partial \phi^s},$$

(252)
defined by a duality invariant function $W(\phi^r, \Gamma)$ of the scalars $\phi^r$ and the quantized charges $\Gamma^M$.

It is straightforward to verify that the generating solution, in the physical domain where
it is well defined ($H_\ell > 0$), is described by a first order system of the form (252), with $W$
given by

$$W_{gen} = W_{gen}(\varphi_i, \Gamma) = e^{-\frac{\varphi_1 + \varphi_2 + \varphi_3}{2}} \left( k_0 + k_1 e^{\varphi_2 + \varphi_3} + k_4 e^{\varphi_1 + \varphi_3} + k_6 e^{\varphi_1 + \varphi_2} \right) =$$

$$= e^{-\frac{\varphi_1 + \varphi_2 + \varphi_3}{2}} \left( -\epsilon_0 q_0 + \epsilon_1 p^1 e^{\varphi_2 + \varphi_3} + \epsilon_4 p^4 e^{\varphi_1 + \varphi_3} + \epsilon_6 p^6 e^{\varphi_1 + \varphi_2} \right).$$

(253)

Note that, for given charges, it depends on $\epsilon_\ell$, that is, in light of eq. (204), only on the
$\gamma$-label, the $\beta$-label being fixed by the charges $\epsilon_\ell k_\ell$. The value of $W$ on the solution, at
radial infinity, is the ADM mass (196) of the solution:

$$\lim_{\tau \rightarrow 0^-} W_{gen} = \lim_{\tau \rightarrow 0^-} e^{-U} \dot{U} = M_{ADM} = \frac{1}{4} \sum_\ell k_\ell$$

(254)

It is useful, at this point, to write the explicit expression of the central and matter charges
for the STU model truncation. Using the notation of Sect. 2.1, the definitions (59) and
the identification (19), we find for $p^0 = q_\ell = 0$, $\ell = 1, 4, 6$:

$$Z = e^{\frac{\nu}{2}} (q_0 - p^1 t u - p^4 s u - p^6 s t),$$
$$Z_1 = e^{\frac{\nu}{2}} (q_0 - p^1 t u - p^4 s u - p^6 s t),$$
$$Z_4 = e^{\frac{\nu}{2}} (q_0 - p^1 t u - p^4 s u - p^6 s t),$$
$$Z_6 = e^{\frac{\nu}{2}} (q_0 - p^1 t u - p^4 s u - p^6 s t).$$

On the dilatonic generating solutions the above charges read:

$$Z = \frac{1}{2\sqrt{2}} e^{\frac{\nu_1 + \nu_2 + \nu_3}{2}} (q_0 + p^1 e^{\nu_2 + \nu_3} + p^4 e^{\nu_1 + \nu_3} + p^6 e^{\nu_1 + \nu_2}),$$
$$Z_1 = \frac{1}{2\sqrt{2}} e^{\frac{-\nu_1 + \nu_2 + \nu_3}{2}} (q_0 + p^1 e^{\nu_2 + \nu_3} - p^4 e^{\nu_1 + \nu_3} - p^6 e^{\nu_1 + \nu_2}),$$
$$Z_4 = \frac{1}{2\sqrt{2}} e^{\frac{-\nu_1 + \nu_2 + \nu_3}{2}} (q_0 - p^1 e^{\nu_2 + \nu_3} + p^4 e^{\nu_1 + \nu_3} - p^6 e^{\nu_1 + \nu_2}),$$
$$Z_6 = \frac{1}{2\sqrt{2}} e^{\frac{-\nu_1 + \nu_2 + \nu_3}{2}} (q_0 - p^1 e^{\nu_2 + \nu_3} - p^4 e^{\nu_1 + \nu_3} + p^6 e^{\nu_1 + \nu_2}).$$

It is a known result that regular BPS black holes are described by a fake superpotential which is the modulus of the central charge $W = W_{BPS} = |Z|$. On the other hand, regular non-BPS solutions with $I_4 > 0$ are described by a $W$ which coincides with the modulus of one of the matter charges: $W = W_{nBPS,I_4>0} = |Z_\ell|$, $\ell = 1, 4, 6$. More subtle is the first order description of regular non-BPS solutions with $I_4 < 0$, for which an explicit duality invariant expression for $W$ is not known\(^{11}\). A non-duality-invariant form $W_{nBPS,I_4<0}$ of $W$ is given in [49], which describes the seed (or generating) solution of this class in $D = 4$ [15].

We can consider the $q_0 < 0$, $p^\ell > 0$ representative of the corresponding orbit for which $W_{nBPS,I_4<0}$ reads

$$W_{nBPS,I_4<0} = e^{\frac{\nu}{2}} \left(-q_0 + \frac{p^1}{2} (\bar{t} u + u \bar{t}) + \frac{p^4}{2} (s u + u s) + \frac{p^6}{2} (s t + s t)\right). \quad (255)$$

The above expression on the dilatonic generating solution becomes

$$W_{nBPS,I_4<0} = \frac{1}{2\sqrt{2}} e^{\frac{-\nu_1 + \nu_2 + \nu_3}{2}} (-q_0 + p^1 e^{\nu_2 + \nu_3} + p^4 e^{\nu_1 + \nu_3} + p^6 e^{\nu_1 + \nu_2}). \quad (256)$$

Consider first the regular solutions $k_2 > 0$ for which, as we have seen, the $\gamma$ and $\beta$-labels coincide. We see that the BPS orbit $O_{6H}$, with $\gamma$-label $(0, 0, 0, 4)$ has $\epsilon_0 = -1$ and $\epsilon_1 = \epsilon_4 = \epsilon_6 = 1$ and, from (253), we find the known result:

$$W_{gen} = |W_{gen}| = |Z|. \quad (257)$$

\(^{11}\)In [12] $W^2$ is characterized as a root of an degree six polynomial while in [39, 51] an implicit integral form of $W$ is given.
namely the fake superpotential for the regular BPS black holes is the modulus of the central charge \( Z \). As far as the non-BPS orbit \( \tilde{O}_{i_H}^{\gamma} \) is concerned, its representative is obtained by inverting the signs of two of the \( \epsilon_\ell \) with respect to the BPS case. Again from (253) we see that:

\[
W_{\text{gen}} = |W_{\text{gen}}| = |Z_\ell|, \tag{258}
\]

where \( |Z_\ell| \) is the largest among all the \( |Z_\ell| \) and \( |Z| \) for the given set of charges. Consider now the non-BPS orbit \( \tilde{O}_s^{\nu_{H}} \) with \( \gamma \)-label \((0,0,2,2)\) and the regular representative with \( \epsilon_\ell = +1, \ell = 0, 1, 4, 6 \), so that \( q_0 < 0, p^1, p^4, p^6 > 0 \). Comparing (253) to (256) we indeed find that:

\[
W_{\text{gen}} = |W_{\text{gen}}| = W_{nBPS,I_4<0}. \tag{259}
\]

For other signs of \( q_0, p^1, p^4, p^6 \) within the same \( I_4 < 0 \) orbit of \( G_4 \), we can use the corresponding \( W_{\text{gen}} \) as a definition of \( W_{nBPS,I_4<0} \) on the generating solution, i.e. the expression (253) in which \( \epsilon_\ell \) are chosen so that \( \epsilon = -\epsilon_0 \epsilon_1 \epsilon_4 \epsilon_6 = -1 \).

For all the regular representatives the ADM mass reads:

\[
M_{\text{ADM}} = \lim_{\tau \to 0^-} W_{\text{gen}} = \frac{1}{2\sqrt{2}} \left(|q_0| + \sum_{\ell=1,4,6} |p^\ell| \right) = W_{\text{max}}, \tag{260}
\]

and it is clearly the largest among the fake superpotentials \(|Z|, |Z_\ell|, W_{nBPS,I_4<0}\) computed on the same charges at infinity. Take for instance the BPS regular solution with \( q_0, p^1, p^4, p^6 > 0 \):

\[
M_{\text{ADM}} = \lim_{\tau \to 0^-} |Z| = \frac{1}{2\sqrt{2}} \left(q_0 + \sum_{\ell=1,4,6} p^\ell \right) = W_{\max} > \lim_{\tau \to 0^-} |Z_\ell|, \lim_{\tau \to 0^-} W_{nBPS,I_4<0}, \tag{261}
\]

being

\[
\lim_{\tau \to 0^-} |Z_1| = \frac{1}{2\sqrt{2}} \left|(q_0 + p^1 - p^4 - p^6)\right|; \quad \lim_{\tau \to 0^-} |Z_4| = \frac{1}{2\sqrt{2}} \left|(q_0 - p^1 + p^4 - p^6)\right|, \\
\lim_{\tau \to 0^-} |Z_6| = \frac{1}{2\sqrt{2}} \left|(q_0 - p^1 - p^4 + p^6)\right|, \\
\lim_{\tau \to 0^-} W_{nBPS,I_4<0} = \frac{1}{2\sqrt{2}} \left|(-\epsilon_0 q_0 + \epsilon_1 p^1 + \epsilon_4 p^4 + \epsilon_6 p^6)\right|_{\epsilon_0\epsilon_1\epsilon_4\epsilon_6 = +1}. \tag{262}
\]

This suggests a characterization of regularity in terms of the black hole asymptotics [12]: Regular BPS and non-BPS solutions should satisfy a generalized BPS bound, i.e. their ADM mass should be larger than any of the fake superpotentials \(|Z|, |Z_\ell|, W_{nBPS,I_4<0}\) computed on the same charges at infinity. For extremal solutions the bound is saturated and \( M_{\text{ADM}} \) should coincide with the largest of these values, which is \( W_{\text{max}} \) in (260). This condition is not satisfied by representatives for which some of the \( k_\ell, \ell = 0, 1, 4, 6 \), are negative. These include the orbits for which the \( \gamma \) and \( \beta \)-labels are different, but also the two orbits \( \tilde{O}_{i_H}^{\gamma} \) and \( \tilde{O}_{sH}^{\nu_{H}} \). In these cases:

\[
M_{\text{ADM}} = \frac{1}{4} \sum_\ell k_\ell < \frac{1}{4} \sum_\ell |k_\ell| = W_{\text{max}}, \tag{263}
\]

65
and the generalized BPS bound is not satisfied. These are the solutions which exhibit a singularity at finite $\tau$ and, by acting on them with the $\text{SO}(1, 1)^4$ isotropy group, the $k_\ell$ can be rescaled so as to obtain a representative of the same orbit with negative ADM mass, which is clearly unphysical.

4.5 Non-Extremal Solutions

As pointed out earlier, a diagonalizable $L_0$ can always be $H^*$-rotated into the Cartan subalgebra in the coset $\mathcal{M}_N$, i.e. in the space $\prod_{\ell}[\mathfrak{sl}(2) \otimes \mathfrak{so}(1, 1)]_\ell$. In order for the solution non to have a true space-time singularity at some finite value of the radial parameter, $L_0$ must have real eigenvalues only, and thus be expressed as a combination of the non-compact Cartan generators in the coset space:

$$L_0 = k_0 T_0 + \sum_{i=1}^{3} k_i h_i,$$  \hspace{1cm} (264)

where we have chosen as a basis for the non-compact Cartan generators in the coset $\{T_0, h_i\}$.

Upon imposing the regularity condition (66) we still find 3 orbits.

**The Schwarzschild Orbit** It corresponds to choosing $k_0 = 1$ and $k_i \equiv 0$. In this case $c^2 = 1/4$ and

$$U = c \tau , \quad \phi^r = Z^M = a \equiv 0 .$$  \hspace{1cm} (265)

With reference to the conventions defined in Sect. 2.2, we can calculate the horizon area to be:

$$A_H = 4\pi \lim_{\tau \to -\infty} \frac{c^2 e^{-2U}}{\sinh^2(c\tau)} = 4\pi (r^+)^2 = 4\pi (2c)^2 ,$$  \hspace{1cm} (266)

from which we deduce that $r^+ = r_0 + c = 2c$, $r^- = r_0 - c = 0$. From the general relation between $\tau$ and $r$ we find:

$$e^U = e^{c\tau} = \sqrt{\frac{1 - \frac{2c}{r}}{r}} ,$$  \hspace{1cm} (267)

so that:

$$ds^2 = - \left(1 - \frac{2c}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2c}{r}\right)} + r^2 (d\theta^2 + \sin^2(\theta) d\varphi^2) ,$$  \hspace{1cm} (268)
and we retrieve the familiar Schwarzschild metric for \( c = GM/c^2 \) (\( \dot{c} \) being the speed of light). The signatures of the relevant tensor classifiers are:

\[
\begin{align*}
\text{Sign}(\mathcal{T}_{xy}) &= (1_+, 12_-), \\
\text{Sign}(\mathcal{T}^{aA,bB}_{(21)}) &= (4_+, 24_-), \\
\text{Sign}(\mathcal{T}^{aA,bB}_{(84)}) &= (27_+, 1_-), \\
\text{Sign}(\mathcal{T}_{xy}) &= (0_+, 1_-), \\
\text{Sign}(\mathcal{T}_{a\beta}) &= (0_+, 1_-), \\
\text{Sign}(\mathcal{T}^{(A\beta),(C\delta)}_{(84)}) &= (0_+, 1_-), \\
\text{Sign}(\mathcal{T}^{(3,21)}_{aA,bB}) &= (24_+, 4_-), \\
\text{Sign}(\mathcal{T}^{(3,84)}_{aA,bB}) &= (27_+, 1_-).
\end{align*}
\]

(269)

**Second (singular) orbit.** It corresponds to taking \( k_0 = k_2 = k_3 = 0 \) and \( k_1 = 1 \). In this case we still have \( c^2 = 1/4 \) and the only non vanishing field is \( \varphi_1 = \tau \). We can compute on the space-time metric

\[
R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{12}{c^2} \sinh^8(c\tau),
\]

(270)

which explodes at \( \tau \to -\infty \), signalling a naked singularity, with no horizon to cover it: \( A_H = 0 \).

The signatures of the relevant tensor classifiers are:

\[
\begin{align*}
\text{Sign}(\mathcal{T}_{xy}) &= (5_+, 8_-), \\
\text{Sign}(\mathcal{T}^{aA,bB}_{(21)}) &= (12_+, 16_-), \\
\text{Sign}(\mathcal{T}^{aA,bB}_{(84)}) &= (19_+, 9_-), \\
\text{Sign}(\mathcal{T}_{xy}) &= (0_+, 1_-), \\
\text{Sign}(\mathcal{T}_{a\beta}) &= (0_+, 1_-), \\
\text{Sign}(\mathcal{T}^{(A\beta),(C\delta)}_{(84)}) &= (0_+, 1_-), \\
\text{Sign}(\mathcal{T}^{(3,21)}_{aA,bB}) &= (16_+, 12_-), \\
\text{Sign}(\mathcal{T}^{(3,84)}_{aA,bB}) &= (19_+, 9_-).
\end{align*}
\]

(271)

**Third (singular) orbit.** It corresponds to choosing \( k_0 = k_i = 1 \). In this case \( c^2 = 1 \) and the only non vanishing fields are:

\[
\varphi_i = \tau, \quad U = \frac{\tau}{2}.
\]

(272)
The horizon area is still zero and
\[ R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma} = \frac{3}{1024} e^{-\frac{6\tau}{1024}} (1 - e^{2\tau})^6 (5 - 10 e^{2\tau} + 69 e^{4\tau}), \] (273)
which diverges as \( \tau \to -\infty \), signalling a true space-time naked singularity.

The signatures of the relevant tensor classifiers are:
\[
\begin{align*}
\text{Sign}(T_{xy}) & = (6_+, 7_-), \\
\text{Sign}(T_{aA, bB}^{(21)}) & = (14_+, 14_-), \\
\text{Sign}(T_{aA, bB}^{(84)}) & = (15_+, 13_-), \\
\text{Sign}(\Sigma_{xy}) & = (1_+, 0_-), \\
\text{Sign}(T_{aA, bB}) & = (1_+, 0_-), \\
\text{Sign}(\Sigma_{aA, bB}^{(AB), (CD)}) & = (1_+, 0_-), \\
\text{Sign}(T_{aA, bB}^{(3,21)}) & = (14_+, 14_-), \\
\text{Sign}(T_{aA, bB}^{(3,84)}) & = (13_+, 15_-) .
\end{align*}
\] (274)

5 Orbits with Higher Degree of Nilpotency

In this section we briefly discuss single center solutions whose Lax matrix belong to some of the \( G^c \)-orbits with degree of nilpotency higher than 3, identified with the \( \alpha^{(5)}, \ldots \alpha^{(15)} \) labels. The corresponding \( H^* \) orbits are described in Tables 7-17 and in Table 23.

The regularity condition given earlier rules these orbits out. We shall give some examples of single center solutions which are indeed lifted to singular space-times. In light of the analysis in \([53]\), these solutions can be viewed as singular limits of multicenter ones in which two or more centers coincide. The Noether charge matrix will then be the sum of the charges associated with each center. In this resect it is then useful to express representatives of these higher-degree orbits as sums of representatives of lower-degree ones discussed in the previous section. In a forthcoming work we shall analyze the all these higher degree \( H^* \)-orbits in terms of multicenter representatives.

Let us choose as non-compact Cartan subalgebra \( \mathcal{C} \) of \( \mathfrak{f}_4 \) the one in \( \mathfrak{h}^* \bigcap \mathfrak{r} \) generated by the \( \mathcal{J}_i \) generators defined in Sect. 4. The generators \( \mathcal{H}_i, \ i = 1, 2, 3, 4 \), corresponding to the orthonormal basis \( (\epsilon_i) \) of the \( \mathfrak{f}_{4(4)} \)-root space are:
\[
\mathcal{H}_1 = \mathcal{J}_0 + \mathcal{J}_1 ; \quad \mathcal{H}_2 = -\mathcal{J}_0 + \mathcal{J}_1 ; \quad \mathcal{H}_3 = \mathcal{J}_4 + \mathcal{J}_6 ; \quad \mathcal{H}_4 = -\mathcal{J}_4 + \mathcal{J}_6 .
\] (275)

Diagonalizing the adjoint action of this basis on \( \mathfrak{r}^* \) we can build a basis of the space consisting of the shift generators \( \mathcal{E}_k \) and \( \mathcal{F}_k = \mathcal{E}_k^T \) corresponding to the 24 \( \mathfrak{f}_{4(4)} \)-roots listed in Table 1. These generators are listed in Table 24 of Appendix B. As examples we shall work out in detail the (singular) single center solutions corresponding to the \( \alpha^{(5)} H^* \)-orbit and the diagonal (i.e. having equal \( \gamma \)- and \( \beta \)-labels) \( H^* \)-orbits within the \( \alpha^{(7)} G^c \)-orbit.
The orbit $\alpha^{(5)}$. The degree of nilpotency of this orbit is 5. We consider a representative of this orbit of the form:

$$L_0 = L_0^{(1)} + L_0^{(2)} ; \quad L_0^{(1)} = -2 \hat{F}_{20} ; \quad L_0^{(2)} = 2 \hat{E}_{19} \quad (276)$$

The components $L_0^{(1)}$ and $L_0^{(2)}$ both belong to the orbit $O_{3H^*}$ and thus generate small, critical BPS black holes. They do not commute.

The solution reads:

$$e^{-2U} = 1 - 2\tau^2 ; \quad e^{-\varphi_1} = e^{-\varphi_2} = e^{-\varphi_3} = 1 + 2\tau^2 ; \quad \alpha^2 = \alpha^3 = \frac{-\sqrt{2} \tau}{1 + 2\tau^2} = \frac{\lambda^5}{\sqrt{2}\tau},$$

$$Z^2 = \frac{\sqrt{2}\tau}{2\tau^2 - 1} = -Z^3 = -\frac{1}{\tau} Z_4 = \frac{1}{\tau} Z_6 \quad (277)$$

all other fields being zero. We see that $e^{-U}$ vanishes at finite $\tau$ and the four dimensional space-time has a true singularity. It is tempting to interpret this solution as the singular limit of a two-center one, each center being a small BPS black hole described by $L_0^{(1)}$ and $L_0^{(2)}$, respectively. In this case the electric-magnetic charge vectors $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are:

$$\Gamma^{(1)} = (p^A, q_{\Sigma}) = (0,0,0,0,0,0,0,0,-\sqrt{2},0,0,0) ,$$

$$\Gamma^{(2)} = (p^A, q_{\Sigma}) = (0,0,0,0,0,0,0,0,\sqrt{2},0,0,0) \quad (278)$$

The two charges are mutually local: $\Gamma^{(1)} T \Sigma \Gamma^{(2)} = 0$.

The orbit $\alpha^{(7)}$. The degree of nilpotency of this orbit is 5. We consider first a representative of the orbit $O_{11H^*}$, identified by the $\gamma$- and $\beta$-labels being both $(1,0,2,4)$, in the form:

$$L_0 = L_0^{(1)} + L_0^{(2)} ; \quad L_0^{(1)} = -2 \hat{F}_{16} ; \quad L_0^{(2)} = \sqrt{3} \hat{E}_2 - \sqrt{3} \hat{E}_{12}. \quad (279)$$

One can verify that $L_0^{(1)}$ and $L_0^{(1)}$, which are non-commuting, lie in the orbits $O_{1H^*}$ and $O_{2H^*}$, respectively. The corresponding electric-magnetic charges $\Gamma^{(1)}$, $\Gamma^{(2)}$ are mutually local: $\Gamma^{(1)} T \Sigma \Gamma^{(2)} = 0$.

The solution reads:

$$e^{-4U} = - (\tau + 1)^4 (3\tau^2 - 1) ; \quad e^{-2\varphi_1} = - \frac{3\tau(4\tau^3 + 7\tau + 4) + 4}{16(\tau + 1)^3 (3\tau^2 - 1)} ,$$

$$e^{-2\varphi_2} = e^{2\varphi_3} = \frac{(\tau + 1)^2}{1 - 3\tau^2} ; \quad \alpha^1 = \frac{\sqrt{3}\tau (4\tau^3 - 7\tau - 4)}{12\tau^4 + 24\tau^3 + 21\tau^2 + 12\tau + 4} ,$$

$$Z^4 = \frac{\tau (4\tau^2 + 6\tau + 3)}{2\sqrt{2}(\tau + 1)^3} , \quad Z^6 = - \frac{\tau(6\tau + 5)}{2 \sqrt{2}(\tau + 1)(3\tau^2 - 1)} , \quad Z_4 = \frac{\sqrt{2}\tau(4\tau^2 + 2\tau - 1)}{-6\tau^3 - 6\tau^2 + 2\tau + 2} ,$$

$$Z_6 = \frac{\sqrt{2}\tau(2\tau + 1)}{2(\tau + 1)^3} , \quad a = - \frac{\sqrt{3}\tau^4}{(\tau + 1)^2 (3\tau^2 - 1)} \quad (280)$$
Notice that, even if \( a(\tau) \neq 0 \), the NUT charge, which is proportional to \( n = -e^{-4U} (\dot{a} + Z^T C \dot{Z}) \), vanishes and the \( D = 4 \) metric is diagonal. We see that the \( D = 4 \) space-time is singular since \( e^{-U} \) vanishes a finite \( \tau \).

Next we consider the other orbit in the diagonal of Table 9: \( O'_{12H^*} \) with \( \gamma \)- and \( \beta \)-labels both equal to \((0, 1, 2, 2)\). The representative we choose has the form:

\[
L_0 = L_0^{(1)} + L_0^{(2)} ; \quad L_0^{(1)} = 2 \hat{E}_{13} ; \quad L_0^{(2)} = \sqrt{3} \hat{F}_{15} - \sqrt{3} \hat{E}_2. \tag{281}
\]

One can verify that \( L_0^{(1)} \) and \( L_1^{(1)} \), which are non-commuting, lie in the orbits \( O_{1H^*} \) and \( O_{3H^*} \), respectively. The corresponding electric-magnetic charges \( \Gamma^{(1)} \), \( \Gamma^{(2)} \) are mutually local: \( \Gamma^{(1)} T \Gamma^{(2)} = 0 \).

The solution reads:

\[
e^{-4U} = (\tau - 1)^2 (1 - 4\tau^2) = e^{-2\varphi_1} ; \quad e^{-2\varphi_2} = \frac{1 - 2\tau}{(\tau - 1)^2(2\tau + 1)} ; \quad e^{-2\varphi_3} = \frac{(1 - 4\tau^3)^2}{(\tau - 1)^2 (1 - 4\tau^2)}
\]

\[
\alpha^4 = \sqrt{3} \tau , \quad \alpha^6 = \frac{\sqrt{3} \tau (2\tau - 1)}{4\tau^3 - 1} ,
\]

\[
Z^1 = \frac{\sqrt{3} \tau}{(\tau - 1)^2(2\tau + 1)} , \quad Z_0 = -\frac{\sqrt{3} \tau}{(\tau - 1)^2(2\tau + 1)} , \quad Z_4 = \frac{3 - 2\tau){\tau^2}}{\sqrt{2}(\tau - 1)^2(2\tau + 1)} ,
\]

\[
Z_6 = \frac{\tau (4\tau^3 - 3\tau + 2)}{\sqrt{2}(\tau - 1)^2 (4\tau^2 - 1)} . \tag{282}
\]

We see that the \( D = 4 \) space-time is singular since \( e^{-U} \) vanishes a finite \( \tau \).

### 6 Concluding Remarks

In this work we have considered the description of static black holes in a specific \( \mathcal{N} = 2 \) model in terms of geodesics on a pseudo-quaternionic Kähler symmetric manifold. We have posed the general question: Given a geodesic on this manifold, can it be related to one describing a black hole solution in \( D = 4 \)? We showed that answering this question requires a classification of the initial velocity vectors of geodesics with respect to the action of the isotropy group \( H^* \), which is what we have accomplished.

By referring to the general arguments in [19] we expect this answer, given in terms of classification of \( H^* \)-orbits of vectors on the tangent space at the origin, to apply to all the \( \mathcal{N} = 2 \) models lying in the same Tits-Satake universality class as the one considered here.
The $c^*$-map chain of embedding for these models read:

\[
\begin{align*}
\mathcal{M}_{D=5} &= \frac{\text{SL}(3, \mathbb{C})}{\text{SU}(3)} \xrightarrow{r-\text{map}} \mathcal{M}_{SK} = \frac{\text{SU}(3)}{\text{U}(3) \times \text{SU}(3)} \xrightarrow{c^*-\text{map}} \frac{\text{E}_6(2)}{\text{SL}(2, \mathbb{R}) \times \text{SU}(3, 3)}, \\
\mathcal{M}_{D=5} &= \frac{\text{SU}^*(6)}{\text{Sp}(6)} \xrightarrow{r-\text{map}} \mathcal{M}_{SK} = \frac{\text{SO}^*(12)}{\text{U}(6)} \xrightarrow{c^*-\text{map}} \frac{\text{E}_7(-5)}{\text{SL}(2, \mathbb{R}) \times \text{SO}^*(12)}, \\
\mathcal{M}_{D=5} &= \frac{\text{E}_6(-26)}{\text{F}_4(-52)} \xrightarrow{r-\text{map}} \mathcal{M}_{SK} = \frac{\text{E}_7(-25)}{\text{E}_6(-78) \times \text{U}(1)} \xrightarrow{c^*-\text{map}} \frac{\text{E}_8(-24)}{\text{SL}(2, \mathbb{R}) \times \text{E}_7(-25)},
\end{align*}
\]

(283)

As anticipated in the introduction, we leave a formal proof of this property to a future investigation.

It would be interesting to understand the observed $\gamma, \beta$-label degeneracy of certain orbits, observed here for the first time, in terms of the geometric structure of the isotropy group $H^*$. As the tensor classifiers have proved to be a valuable tool for the orbit-classification, we believe it worthwhile constructing a complete set of such tensors which would itself be sufficient for a complete classification, with no need of $\alpha, \beta$ and $\gamma$-labels. Such refined analysis would require constructing higher order $H^*$-symmetric, covariant tensors.

A next step of our analysis is also to apply this orbit classification to a systematic study of multicenter and/or rotating solutions.

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A Little group and the signature of the tensor classifiers

In this Appendix we list the orbits with the little compact group of $h$ in $H^*$, and the corresponding signatures of the tensor classifiers.
| $H^*$-orbit | Dim. | # of semisimple gen. | # of non-compact gen. | # of compact gen. | compact part of (LG) |
|-------------|------|---------------------|----------------------|------------------|-------------------|
| $O_{4\mu}$ | 10   | 10                  | 7                    | 3                | SO(3)             |
| $O_{2\mu}$ | 12   | 12                  | 8                    | 4                | SO(3) $\times$ U(1) |
| $O_{4\mu}$ | 12   | 12                  | 8                    | 4                | SO(3) $\times$ U(1) |
| $O_{4\mu}$ | 10   | 10                  | 7                    | 3                | SO(3)             |
| $O_{4\mu}$ | 10   | 10                  | 7                    | 3                | SO(3)             |
| $O_{4\mu}$ | 10   | 10                  | 7                    | 3                | SO(3)             |
| $O_{2\mu}$ | 6    | 6                   | 5                    | 1                | U(1)              |
| $O_{2\mu}$ | 6    | 6                   | 5                    | 1                | U(1)              |
| $O_{2\mu}$ | 6    | 6                   | 5                    | 1                | U(1)              |
| $O_{2\mu}$ | 22   | 22                  | 13                   | 9                | U(3)              |
| $O_{2\mu}$ | 22   | 22                  | 13                   | 9                | U(3)              |
| $O_{2\mu}$ | 14   | 14                  | 9                    | 5                | SO(3) $\times$ U(1)$^2$ |
| $O_{2\mu}$ | 14   | 14                  | 9                    | 5                | SO(3) $\times$ U(1)$^2$ |
| $O_{2\mu}$ | 14   | 14                  | 9                    | 5                | SO(3) $\times$ U(1)$^2$ |
| $O_{2\mu}$ | 14   | 14                  | 9                    | 5                | SO(3) $\times$ U(1)$^2$ |
| $O_{2\mu}$ | 10   | 10                  | 7                    | 3                | SO(3)             |
| $O_{2\mu}$ | 10   | 10                  | 7                    | 3                | SO(3)             |
| $O_{2\mu}$ | 0    | 10                  | 7                    | 3                | SO(3)             |
| $O_{2\mu}$ | 0    | 10                  | 7                    | 3                | SO(3)             |
| $O_{2\mu}$ | 6    | 6                   | 5                    | 1                | U(1)              |
| $O_{2\mu}$ | 6    | 6                   | 5                    | 1                | U(1)              |
| $O_{2\mu}$ | 6    | 6                   | 5                    | 1                | U(1)              |
| $O_{2\mu}$ | 6    | 6                   | 5                    | 1                | U(1)              |
| $O_{2\mu}$ | 4    | 4                   | 4                    | 0                | (1)               |
| $O_{2\mu}$ | 6    | 6                   | 5                    | 1                | U(1)              |
| $O_{2\mu}$ | 6    | 6                   | 5                    | 1                | U(1)              |
| $O_{2\mu}$ | 12   | 12                  | 8                    | 4                | SO(3) $\times$ U(1) |
| $O_{2\mu}$ | 12   | 12                  | 8                    | 4                | SO(3) $\times$ U(1) |
| $O_{2\mu}$ | 12   | 12                  | 8                    | 4                | SO(3) $\times$ U(1) |
| $O_{2\mu}$ | 8    | 8                   | 6                    | 2                | U(1)$^2$           |
| $O_{2\mu}$ | 8    | 8                   | 6                    | 2                | U(1)$^2$           |
| $O_{2\mu}$ | 8    | 8                   | 6                    | 2                | U(1)$^2$           |
| $O_{2\mu}$ | 8    | 8                   | 6                    | 2                | U(1)$^2$           |
| $O_{2\mu}$ | 6    | 6                   | 5                    | 1                | U(1)              |
| $O_{2\mu}$ | 6    | 6                   | 5                    | 1                | U(1)              |
| $O_{2\mu}$ | 6    | 6                   | 5                    | 1                | U(1)              |
| $O_{2\mu}$ | 6    | 6                   | 5                    | 1                | U(1)              |
| $O_{2\mu}$ | 8    | 8                   | 6                    | 2                | U(1)$^2$           |
| $O_{2\mu}$ | 8    | 8                   | 6                    | 2                | U(1)$^2$           |
| $O_{2\mu}$ | 4    | 4                   | 4                    | 0                | (1)               |
| $O_{2\mu}$ | 4    | 4                   | 4                    | 0                | (1)               |
| $O_{2\mu}$ | 4    | 4                   | 4                    | 0                | (1)               |
| $O_{2\mu}$ | 4    | 4                   | 4                    | 0                | (1)               |
| $O_{2\mu}$ | 6    | 6                   | 5                    | 1                | U(1)              |
| $O_{2\mu}$ | 6    | 6                   | 5                    | 1                | U(1)              |
| $O_{2\mu}$ | 6    | 6                   | 5                    | 1                | U(1)              |
| $O_{2\mu}$ | 4    | 4                   | 4                    | 0                | (1)               |

Table 22: Part of the little groups of $h$, connected to the identity
| $H^*$-Orbits | \( T_{cy} \) | \( T_{(2, 1)} \) | \( T_{(3, 4)} \) | \( T_{(4, 1)} \) | Solution |
|---|---|---|---|---|---|
| \( C_{1T^c} \) | \((0, 0, 0)\) | \((1, 0, 0)\) | \((0, 0, 0)\) | \((0, 0, 0)\) | BPS |
| \( C_{2T^c} \) | \((0, 0, 0)\) | \((2, 1, 3)\) | \((4, 1, 3)\) | \((0, 0, 0)\) | BPS |
| \( C_{3T^c} \) | \((0, 0, 0)\) | \((3, 1, 2)\) | \((3, 1, 2)\) | \((0, 0, 0)\) | BPS |
| \( C_{4T^c} \) | \((0, 0, 0)\) | \((3, 1, 3)\) | \((3, 1, 3)\) | \((0, 0, 0)\) | BPS |
| \( C_{5T^c} \) | \((0, 0, 0)\) | \((4, 1, 4)\) | \((5, 1, 3)\) | \((0, 0, 0)\) | BPS |
| \( C_{6T^c} \) | \((0, 0, 0)\) | \((5, 1, 3)\) | \((5, 1, 3)\) | \((0, 0, 0)\) | BPS |
| \( C_{7T^c} \) | \((0, 0, 0)\) | \((6, 1, 4)\) | \((6, 1, 4)\) | \((0, 0, 0)\) | BPS |
| \( C_{8T^c} \) | \((0, 0, 0)\) | \((7, 1, 5)\) | \((7, 1, 5)\) | \((0, 0, 0)\) | BPS |
| \( C_{9T^c} \) | \((0, 0, 0)\) | \((8, 1, 6)\) | \((8, 1, 6)\) | \((0, 0, 0)\) | BPS |
| \( C_{10T^c} \) | \((0, 0, 0)\) | \((9, 1, 7)\) | \((9, 1, 7)\) | \((0, 0, 0)\) | BPS |
| \( C_{11T^c} \) | \((0, 0, 0)\) | \((10, 1, 8)\) | \((10, 1, 8)\) | \((0, 0, 0)\) | BPS |
| \( C_{12T^c} \) | \((0, 0, 0)\) | \((11, 1, 9)\) | \((11, 1, 9)\) | \((0, 0, 0)\) | BPS |
| \( C_{13T^c} \) | \((0, 0, 0)\) | \((12, 1, 10)\) | \((12, 1, 10)\) | \((0, 0, 0)\) | BPS |
| \( C_{14T^c} \) | \((0, 0, 0)\) | \((13, 1, 11)\) | \((13, 1, 11)\) | \((0, 0, 0)\) | BPS |
| \( C_{15T^c} \) | \((0, 0, 0)\) | \((14, 1, 12)\) | \((14, 1, 12)\) | \((0, 0, 0)\) | BPS |
| \( C_{16T^c} \) | \((0, 0, 0)\) | \((15, 1, 13)\) | \((15, 1, 13)\) | \((0, 0, 0)\) | BPS |

Table 23: Signature of the relevant tensor classifiers for the various $H^*$-orbits.
B Generators of $\mathfrak{f}_4(4)$ in the 26

Here we present the generators of $\mathfrak{f}_4(4)$ in terms of matrices $e_{i,j}$ whose only non-vanishing entry is a 1 on the $i^{th}$ row and $j^{th}$ column.

$$H_1 = \frac{1}{2}(2e_{1,1} + e_{2,2} + e_{3,3} + e_{4,4} + e_{5,5} + e_{6,6} + e_{7,7} + e_{9,9} + e_{11,11} - e_{16,16} - e_{18,18} - e_{20,20} - e_{21,21} - e_{22,22} - e_{23,23} - e_{24,24} - e_{25,25} - 2e_{26,26})$$

$$H_2 = \frac{1}{2}(e_{2,2} + e_{3,3} + e_{4,4} - e_{5,5} + e_{6,6} - e_{7,7} + 2e_{8,8} - e_{9,9} - e_{11,11} + e_{16,16} + e_{18,18} - 2e_{19,19} + e_{20,20} - e_{21,21} + e_{22,22} - e_{23,23} - e_{24,24} - e_{25,25})$$

$$H_3 = \frac{1}{2}(e_{2,2} + e_{3,3} - e_{4,4} + e_{5,5} - e_{6,6} + e_{7,7} - e_{9,9} - e_{10,10} - e_{11,11} + e_{16,16} - 2e_{17,17} + e_{18,18} - e_{20,20} + e_{21,21} - e_{22,22} + e_{23,23} - e_{24,24} - e_{25,25})$$

$$H_4 = \frac{1}{2}(e_{2,2} - e_{3,3} + e_{4,4} + e_{5,5} - e_{6,6} - e_{7,7} + e_{9,9} - e_{11,11} + +2e_{12,12} - 2e_{15,15} + e_{16,16} - e_{18,18} + e_{20,20} + e_{21,21} - e_{22,22} - e_{23,23} + e_{24,24} - e_{25,25})$$

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$$E_1 = e_{1,8} + e_{5,16} - e_{7,18} + e_{9,20} - e_{11,22} - e_{19,26}$$

$$E_2 = e_{1,10} + e_{4,16} - e_{6,18} - e_{9,21} + e_{11,23} - e_{17,26}$$

$$E_3 = -e_{4,5} - e_{6,7} + e_{8,10} - e_{17,19} - e_{20,21} - e_{22,23}$$

$$E_4 = e_{1,12} + e_{3,16} + e_{6,20} + e_{7,21} + e_{11,24} - e_{15,26}$$

$$E_5 = -e_{3,5} - e_{6,9} + e_{8,12} - e_{15,19} - e_{18,21} - e_{22,24}$$

$$E_6 = -e_{3,4} + e_{7,9} + e_{10,12} - e_{15,17} - e_{18,20} + e_{23,24}$$

$$E_7 = e_{1,13} + \frac{e_{2,16}}{\sqrt{2}} + \frac{e_{3,18}}{\sqrt{2}} + \frac{e_{4,20}}{\sqrt{2}} + \frac{e_{5,21}}{\sqrt{2}} + \frac{e_{6,22}}{\sqrt{2}} + \frac{e_{7,23}}{\sqrt{2}} + \frac{e_{9,24}}{\sqrt{2}} + \frac{e_{11,25}}{\sqrt{2}} - e_{13,26}$$

$$E_8 = -\frac{e_{2,5}}{\sqrt{2}} + \frac{e_{3,7}}{\sqrt{2}} + \frac{e_{4,9}}{\sqrt{2}} + \frac{e_{6,11}}{\sqrt{2}} + e_{8,13} - e_{13,19} + \frac{e_{16,21}}{\sqrt{2}} - \frac{e_{18,23}}{\sqrt{2}} + \frac{e_{20,24}}{\sqrt{2}} - \frac{e_{22,25}}{\sqrt{2}}$$

$$E_9 = -\frac{e_{2,4}}{\sqrt{2}} + \frac{e_{3,6}}{\sqrt{2}} + \frac{e_{5,9}}{\sqrt{2}} + \frac{e_{7,11}}{\sqrt{2}} + e_{10,13} - e_{13,17} + \frac{e_{16,20}}{\sqrt{2}} - \frac{e_{18,22}}{\sqrt{2}} - \frac{e_{21,24}}{\sqrt{2}} + e_{23,25}$$

$$E_{10} = -\frac{e_{2,3}}{\sqrt{2}} + \frac{e_{4,6}}{\sqrt{2}} + \frac{e_{5,7}}{\sqrt{2}} - \frac{e_{9,11}}{\sqrt{2}} + e_{12,13} - e_{13,15} + \frac{e_{16,18}}{\sqrt{2}} + \frac{20,22}{\sqrt{2}} + \frac{e_{21,23}}{\sqrt{2}} + e_{24,25}$$

$$E_{11} = e_{1,15} - e_{2,18} - e_{4,22} - e_{5,23} - e_{9,25} - e_{12,26}$$

$$E_{12} = -e_{2,7} - e_{4,11} + e_{8,15} - e_{12,19} - e_{16,23} - e_{20,25}$$

$$E_{13} = -e_{2,6} + e_{5,11} + e_{10,15} - e_{12,17} - e_{16,22} + e_{21,25}$$

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\[ E_{14} = e_{2.9} + e_{3.11} + e_{8.17} - e_{10.19} + e_{16.24} + e_{18.25} \]
\[ E_{15} = e_{1.17} - e_{2.20} + e_{3.22} + e_{5.24} - e_{7.25} - e_{10.26} \]
\[ E_{16} = (e_{(1,9)} - e_{(2,21)} + e_{(3,23)} - e_{(4,24)} + e_{(6,25)} - e_{(8,26)}) \]
\[ E_{17} = \frac{-e_{1.6} + e_{2.8} - e_{5.13} + \frac{1}{2} \sqrt{3} e_{5.14} - \frac{e_{7.15}}{2} + \frac{e_{9.17}}{2}}{\sqrt{2}} + \frac{e_{10.18}}{\sqrt{2}} - \frac{e_{12.20}}{2} + + \frac{1}{2} \sqrt{3} e_{14.22} - \frac{e_{19.25}}{\sqrt{2}} + \frac{e_{21.26}}{\sqrt{2}} \]
\[ E_{18} = \frac{e_{1.4} + e_{3.8} - e_{5.12} + e_{7.13} + \frac{1}{2} \sqrt{3} e_{7.14} + e_{10.16} + e_{11.17} + e_{13.20} + \frac{1}{2} \sqrt{3} e_{14.20}}{\sqrt{2}} - \frac{e_{15.22}}{\sqrt{2}} - e_{19.24} - e_{23.26} \]
\[ E_{19} = \frac{e_{1.3} - e_{4.8} + e_{5.10} + e_{9.13} + \frac{1}{2} \sqrt{3} e_{9.14} + e_{11.15} + e_{12.16} + e_{13.18} + \frac{1}{2} \sqrt{3} e_{14.18} + + e_{17.22} + e_{19.23} - e_{24.26}}{\sqrt{2}} + \frac{e_{17.22}}{\sqrt{2}} - e_{19.23} - e_{24.26} \]
\[ E_{20} = \frac{-e_{1.2} - e_{6.8} + e_{7.10} + e_{9.12} - e_{11.13} + \frac{1}{2} \sqrt{3} e_{11.14} - e_{13.16} + \frac{1}{2} \sqrt{3} e_{14.16} + + e_{15.18} + e_{17.20} + e_{19.21} + e_{25.26}}{\sqrt{2}} + \frac{e_{17.20}}{\sqrt{2}} + e_{19.21} + e_{25.26} \]
\[ E_{21} = \frac{-e_{1.7} - e_{2.10} + e_{4.13} - \frac{1}{2} \sqrt{3} e_{4.14} + e_{6.15} + e_{8.18} + e_{9.19} - \frac{e_{12.21}}{2} - e_{13.23} + + \frac{1}{2} \sqrt{3} e_{14.23} + e_{17.25} - e_{20.26}}{\sqrt{2}} + \frac{e_{17.25}}{\sqrt{2}} - e_{20.26} \]
\[ E_{22} = \frac{e_{1.5} - e_{3.10} + e_{4.12} - e_{6.13} - \frac{1}{2} \sqrt{3} e_{6.14} + e_{8.16} + e_{11.19} + e_{12.21} + \frac{1}{2} \sqrt{3} e_{14.21} - - e_{15.23} + e_{17.24} + e_{22.26}}{\sqrt{2}} + \frac{e_{17.24}}{\sqrt{2}} + e_{22.26} \]
\[ E_{23} = \frac{e_{1.11} - e_{2.13} - \frac{1}{2} \sqrt{3} e_{2.14} - e_{3.15} - e_{4.17} - e_{5.19} - e_{8.22} + e_{10.23} + e_{12.24} + + \frac{e_{13.25}}{2} + \frac{1}{2} \sqrt{3} e_{14.25} + e_{16.26}}{\sqrt{2}} + \frac{1}{2} \sqrt{3} e_{14.25} + e_{16.26} \]
\[ E_{24} = \frac{-e_{1.9} - e_{2.12} + e_{3.13} - \frac{1}{2} \sqrt{3} e_{3.14} - e_{6.17} - e_{7.19} - e_{8.20} + e_{10.21} - e_{13.24} + + \frac{1}{2} \sqrt{3} e_{14.24} + e_{15.25} - e_{18.26}}{\sqrt{2}} - \frac{e_{18.26}}{\sqrt{2}} \]

In the chosen basis for the fundamental representation the shift generators corresponding to negative roots \( F_{\alpha} = E_{-\alpha} \) are \( F_{\alpha} = (E_{\alpha})^T \).
\[ \begin{array}{ll}
E_1 = N_1^+ = \frac{1}{2}(H_2 - H_3 - 2K_3) & F_1 = N_1^- = \frac{1}{2}(H_2 - H_3 + 2K_3) \\
E_2 = \frac{1}{2}(K_{16} + K_{12} - K_5 + K_2 + K_{15} - K_1 - K_6 + K_{13}) & F_2 = \frac{1}{2}(K_{16} + K_{12} - K_5 - K_2 - K_{15} - K_1 - K_6 - K_{13}) \\
E_3 = \frac{1}{2}(K_{10} - K_{12} - K_5 + K_2 - K_{15} + K_1 + K_6 + K_{14}) & F_3 = \frac{1}{2}(K_{10} + K_{12} + K_5 - K_2 + K_{15} + K_1 + K_6 + K_{13}) \\
E_4 = N_4^+ = \frac{1}{2}(H_1 - H_4 - 2K_4) & F_4 = N_4^- = \frac{1}{2}(H_1 - H_4 + 2K_4) \\
E_5 = K_8 - K_9 & F_5 = K_8 + K_9 \\
E_6 = K_7 + K_{10} & F_6 = K_7 - K_{10} \\
E_{12} = \frac{1}{2}(K_{16} + K_{12} + K_5 + K_2 - K_{15} + K_1 - K_6 - K_{13}) & F_{12} = \frac{1}{2}(K_{16} - K_{12} - K_5 - K_2 + K_{15} + K_1 - K_6 - K_{13}) \\
E_{13} = N_5^+ = \frac{1}{2}(H_1 + H_4 + 2K_{14}) & F_{13} = N_5^- = \frac{1}{2}(H_1 + H_4 - 2K_{14}) \\
E_{15} = \frac{1}{2}(K_{16} - K_{12} + K_5 + K_2 + K_{15} - K_1 + K_6 + K_{13}) & F_{15} = \frac{1}{2}(K_{16} + K_{12} - K_5 - K_2 - K_{15} + K_1 + K_6 - K_{13}) \\
E_{16} = N_6^+ = \frac{1}{2}(H_2 + H_3 - 2K_{14}) & F_{16} = N_6^- = \frac{1}{2}(H_2 + H_3 + 2K_{14}) \\
E_{19} = -K_{22} - K_{18} & F_{19} = -K_{22} + K_{18} \\
E_{20} = K_{21} + K_{17} & F_{20} = K_{21} - K_{17} \\
E_{23} = K_{23} - K_{19} & F_{23} = K_{23} + K_{19} \\
E_{24} = K_{24} - K_{20} & F_{24} = K_{24} + K_{20} \\
\end{array} \]

Table 24: The shift generators \( \hat{E}_k \) and \( \hat{F}_k = \hat{E}_k^T \).

Let us now give the \( \mathfrak{sl}^* = \mathfrak{sl}(2) \oplus \mathfrak{sp}'(6) \) generators in terms of the shift generators \( \hat{E}_{\beta'} \) corresponding to the positive roots \( \beta' \), relative to the basis \( (110) \) of the Cartan subalgebra \( \mathcal{C}_{H^*} \). As usual \( \beta'_i \) represent the \( \mathfrak{sp}'(6) \)-simple roots and \( \beta'_4 \) the \( \mathfrak{sl}(2) \) simple roots:

\[
\begin{align*}
\hat{E}_{\beta'_1} &= \sqrt{2} (J_7 + J_{10}) , \\
\hat{E}_{\beta'_2} &= \sqrt{2} (J_{17} + J_{21}) , \\
\hat{E}_{\beta'_3} &= -\frac{J_1 + J_2 - J_5 + J_6 + J_{12} - J_{13} - J_{15} + J_{16}}{\sqrt{2}} , \\
\hat{E}_{\beta'_4 + \beta'_5} &= \sqrt{2} (J_{18} + J_{22}) , \\
\hat{E}_{\beta'_6 + \beta'_4} &= \sqrt{2} (J_{20} - J_{24}) , \\
\hat{E}_{\beta'_1 + \beta'_2} &= \sqrt{2} (J_{19} - J_{23}) , \\
\hat{E}_{2\beta'_1 + \beta'_3} &= -\frac{J_1 + J_2 + J_5 + J_6 + J_{12} + J_{13} + J_{15} + J_{16}}{\sqrt{2}} , \\
\hat{E}_{2\beta'_1 + 2\beta'_2 + \beta'_3} &= \sqrt{2} (J_6 + J_9) , \\
\hat{E}_{2\beta'_1 + 2\beta'_2 + \beta'_4} &= \frac{J_1 - J_2 + J_5 + J_6 + J_{12} + J_{13} - J_{15} - J_{16}}{\sqrt{2}} , \\
\hat{E}_{\beta'_4} &= \frac{1}{2} (J_1 + J_2 + J_5 - J_6 - J_{12} + J_{13} - J_{15} + J_{16}) .
\end{align*}
\]

where the \( J_\alpha, \alpha = 1, \ldots, 24 \), are defined as \( J_\alpha \equiv \frac{1}{2} (E_\alpha - \eta E_\alpha^T \eta) \). The corresponding negative-root generators are obtained through transposition: \( \hat{E}_{-\beta'} = \hat{E}_{\beta'}^T \).

Finally in Table 24 we list the shift generators \( \hat{E}_\alpha \) (the positive roots being represented by the corresponding number in Table 1). These roots are referred to the Cartan subalgebra \( \mathcal{C} \) defined in Sect. 5.
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