The work of J. A. Wheeler in the mid 1960’s showed that for smooth equations of state no stable stellar configurations with central densities above that corresponding to the limiting mass of “neutron stars” (in the generic sense) were stable against acoustical vibrational modes. A perturbation would cause any such star to collapse to a black hole or explode. Accordingly, there has been no reason to expect that a stable degenerate family of stars with higher density than the known white dwarfs and neutron stars might exist. We have found a class of exceptions corresponding to certain equations of state that describe a first order phase transition. We discuss how such a higher density family of stars could be formed in nature, and how the promising new exploration of oscillations in the X-ray brightness of accreting neutron stars might provide a means of identifying them. Our proof of the possible existence of a third family of degenerate stars is one of principle and rests on general principles like causality, microstability of matter and General Relativity.

Since the early work by Wheeler and collaborators [1], there have been two reasons to doubt that there are any families of stable degenerate stars above the density of the known white dwarf and neutron star families. The first reason is a physical one and can be understood as follows. Concerning the class of degenerate stars, white dwarfs are stabilized by degenerate electron pressure which fails at such density that electron capture reduces their effectiveness. Stability is reestablished at densities about five orders of magnitude higher when the baryon Fermi pressure (and ultimately the short-range nuclear repulsive interaction) supports neutron stars. There is no evident mechanism for stabilizing a denser family. At higher density the Fermi pressure of nucleons and hyperons is replaced—not supplemented—by the pressure of their quark constituents if the principle of asymptotic freedom [2] is correct. Indeed a phase transition will generally reduce the pressure at a given energy density, and ultimately bring about the termination of stability to gravitational collapse. Moreover, at ever higher density the Fermi pressure of quarks must be shared among a greater number of flavors according to the Pauli principle. A mechanism for stabilizing a third family is therefore not apparent from the point of view of Fermi pressure.

The second reason for the general belief that no stable stellar configurations exist with central densities higher than those of neutron stars is due to an analysis performed by Wheeler et al. Those authors showed analytically for polytropic equations of state that for general relativistic stars there is a denumerable infinity of turning points of the mass as a function of central density and therefore an infinity of sequences for which the mass has positive slope. However, all configurations with densities greater than the first mass limit for neutron stars are unstable to acoustical radial vibrations, and end either by exploding or imploding to a black hole. It seemed plausible that this result was not peculiar to the polytropes, but would hold for any (at least reasonably) smooth equation of state and there are numerical examples in the litera-
ture that demonstrate that the theorem holds more generally. Moreover, and this is important, the demonstration of instability of polytropes above the neutron star family proves: (1) that positive slope of stellar mass with respect to central density \(dM/d\rho_c > 0\)—is not a sufficient condition for stability. (2) Therefore, the burden of proof that any proposed third family of degenerate stars is stable requires a demonstration that the normal modes of radial vibration of stars in the family are stable rather than leading to collapse or explosion.

We have found exceptions to the expectation, corresponding to equations of state that are physical in the sense that they are causal and microscopically stable (Le Chatelier’s principle) but insufficiently smooth to obey the quoted theorem. Under certain combinations of parameters defining the nuclear and quark deconfined equations of state, there exists a sequence of high density stellar configurations above the neutron stars. The “neutron star” sequence is terminated by the softening in the equation of state by the mixed phase when a substantial core of mixed phase is attained. A new sequence at higher density is stabilized by replacement of the mixed phase by a pure quark phase core. In this case there are stars of the same mass but radically different quark content and also of size.

![Diagram](https://example.com/diagram.png)

**FIG. 2.** Stellar sequence of neutron stars and their twins as a function of central density. Both segments with positive slope correspond to stable configurations since their normal modes of vibration are found to be stable.

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We hasten to add however that what determines the stability of stars is the equation of state. Any other phase transition that has similar effects on the equation of state as the class we have found could likewise give rise to a third stable family. In any case we refer to the denser sequence as non-identical twins of neutron stars because in both cases it is the Fermi pressure of particles carrying baryon number that supports the star against gravity in addition to repulsion at short distance between any nucleons that may be present. We demonstrate that in principle it is permitted by the laws of nature that there can exist two families of relativistic stars with partially overlapping regions of mass, for which two ‘neutron’ stars of the same mass exist, but in which baryon number is distributed differently among the various species of baryons or quarks.

An example of non-identical twins is shown in Fig. 3 where we compare the density profiles. The low-density star lies on the first segment of Fig. 3 with positive slope and the high-density star lies on the second. Details of the models of hadronic and quark matter can be found in ref. [4]. The nuclear matter properties that are used to determine the hadronic equation of state (beside the well established saturation density, binding and symmetry energy coefficient) are the compression modulus and effective nucleon mass at saturation, \(K = 290\) MeV and \(m^*/m = 0.66\), respectively. The bag constant for the quark equation of state is \(B^{1/4} = 180\) MeV and the quark model equation of state is given in Ref. [5] with \(\alpha_s = 0\).

Stars on both segments of the stellar sequence shown in Fig. 2 having positive slope are stable. However, positive slope \(dM/d\rho_c > 0\) is a necessary but not a sufficient condition for stability. Usually, and in all cases tested heretofore, the second and all others are unstable to radial vibrations [1,2]. Stability can be hinted at from the behavior of the mass-radius relation but a definitive test involves an analysis of the radial modes of oscillation [3]. The squared frequency \(\omega^2\) of the fundamental mode is plotted in Fig. 3. Positive values indicate stability and correspond to the segments with positive slope in Fig. 2. The analysis shows that the fundamental (nodeless \(n = 0\)) oscillation becomes unstable at the first maximum, as usual, but unusually, stability of this mode is regained at the following minimum, to be lost again at the next maximum. The usual pattern is that at the maximum in the neutron star family the fundamental mode becomes unstable, and at each higher minimum and maximum a higher mode in order \(n = 1, 2, 3 \cdots\) becomes unstable.

We used the equilibrium equation of state shown in Fig. 3 for the calculation of the oscillation modes. We do not know the time-scale for equilibration of the transition between quark matter and nuclear matter and how it compares with typical periods of pressure oscillations. This does not matter. The turning points in the stellar sequence between stability and instability occur only at the zeros of \(\omega^2\), and because the stellar oscillations are infinitely slow at the turning points their location can be determined exactly. Thus, independent of the timescales, the stability for all members of the stellar sequence can be determined.
FIG. 3. The square of the frequency of the fundamental radial vibration. Perturbations grow as $\exp(\omega t)$ when $\omega^2 < 0$. The segments with $\omega^2 > 0$ are stable since the time dependence of a radial perturbation is $\exp(i\omega t)$.

The features of the stellar sequence shown in Fig. 2 can be identified with features in the equation of state shown in Fig. 4. The limiting mass star at $\approx 1.418 M_\odot$ has a central density $\approx 900 \text{ MeV/fm}^3$ falling very near the upper end of the mixed phase where all quarks become deconfined. One can see from the equation of state near the end of the mixed phase that $dp/d\varepsilon$ becomes small and therefore also the adiabatic index, $\Gamma = \frac{d \ln p}{d \ln \rho} = \frac{p + \varepsilon}{p} \cdot \frac{dp}{d\varepsilon}$ (where $p$, $\varepsilon$, and $\rho$ denote pressure, energy density and baryon density). In this upper region of the mixed phase, the pressure exerted by matter is insufficient to prevent collapse. The adiabatic index is discontinuous at the density boundary between the mixed and pure phases, being smaller in the mixed than in the pure phases. This is a characteristic of phase equilibrium. The larger adiabatic index in the pure quark phase restores stability over a short range of central densities between $\sim 1150 \text{ MeV/fm}^3$ and $\sim 1550 \text{ MeV/fm}^3$, above which the mass becomes too large to be supported by the Fermi pressure of quark matter and the second sequence of stable stars terminates.

Twins, found here for the deconfinement transition, are likely to be a more general phenomenon for any first order phase transition so long as the low-density sequence of stars terminates at central densities that fall close to to the end of the mixed phase which in neutron star matter will have the form shown in Fig. 4. (Generally, the pressure increases monotonically on an isotherm as in the figure for a first order phase transition in a substance having more than one conserved charge $\mathcal{Q}$.)

From the above proof of stability we have shown that in principle a third family of stable degenerate stars could exist that lies in density above the white dwarfs and neutron stars. In such a case, stars in the third family could exist whose masses match a small range of normal neutron stars. Each star on the high density branch has a non-identical twin on the low-density branch, having the same mass but different composition and radius.
some white dwarfs may collapse by accretion to a neutron star.

Can such twins be identified? One possible avenue is through observations on the so-called quasi-periodic oscillations in the X-ray brightness of accreting neutron stars. According to theory, mass and radius determinations may be possible. If twins exist, then the mass-radius curve will exhibit two segments of stable stars instead of one, and observed stars will fall on one or the other of the two distinct segments. The discovery of only about two stars on each branch with a radius resolution of a kilometer in our example would suffice to establish the existence of twin branches.

Are there even higher density families of relativistic compact stars? It is not ruled out by virtue of the proof given here that a third family is allowed by reason of structure introduced into the equation of state by a phase transition. There could be a series of phase transitions following each other at a succession of densities. But the quark deconfinement phase transition may be the ultimate attainable in compact stars that are stable to collapse to black holes. At such densities that matter is in the pure quark phase, the equation of state is likely to be smooth like a polytrope. From that density on we are assured of the denumerable infinity of turning points in the stellar sequence having an increasing number of unstable normal modes such as was found by Wheeler et. al.

Our proof that a third family of degenerate stars is allowed by the laws of nature is not a proof of their actual existence, quite aside from questions concerning the mode of formation. From the discussion in the introduction, it is not a question of new degrees of freedom being engaged. It is a question of the behavior of the equation of state of superdense matter. We have no knowledge from experiment of a single point on the equation of state above nuclear density. But we do have expectations of phase transitions, and asymptotic freedom of quarks would appear to elevate one of them to a law of nature. A phase transition can produce the requisite structure discussed above into the equation of state so as to restore stability for a finite density range after stability has been lost by the canonical neutron stars. The possible existence of a third family of compact stars hinges on such details that we may never determine in laboratory experiments. The actual discovery of members of the third family therefore would reveal, however imperfectly, a non-smooth behavior of the equation of state possibly caused by a change in phase of dense matter—that we may never know by laboratory experimentation.

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