A combinatorial approach to discrete geometry

L. Bombelli* and Miguel Lorente†

*University of Mississipi. USA
†Universidad de Oviedo, Spain

Abstract. We present a paralell approach to discrete geometry: the first one introduces Voronoi cell complexes from statistical tessellations in order to know the mean scalar curvature in term of the mean number of edges of a cell [1]. The second one gives the restriction of a graph from a regular tessellation in order to calculate the curvature from pure combinatorial properties of the graph [2].

Our proposal is based in some epistemological pressupositions: the macroscopic continuous geometry is only a fiction, very usefull for describing phenomena at certain scales, but it is only an approximation to the true geometry. In the discrete geometry one starts from a set of elements and the relation among them without presuposing space and time as a background.

1. Introduction. In recent years some approaches to quantum gravity have suggested the hypothesis of a discrete space time [3] as a consequence of the combinatorial properties of spin networks underlying the structure of space [4] and implemented with the hypothesis of causal sets [5][6].

In our model we have to choose the discrete quantities in such a way that in the continuous limit they become the classical ones. The direct way consists on calculating some continuous quantity in a 2d-manifold. By the inverse way, we can start directly from the graph and find some embedding where the corresponding quantities become analog, like the genus of some graph [7], or the curvature in a triangulated manifold.

According to Bombelli one can scatter points in a Lorentzian manifold, and then keep the statistical distribution of points from which some discrete quantities can be defined, such as curvature, from combinatorial properties of the set of relations [8].

2. Reflection groups and tessellations. Let $P$ a finite sided n-dimensional convex polyhedron in a metric space $X$, all of whose dihedral angles are submultiple of $\pi$. Then the group generated by the reflection of $X$ in the sides of $P$, $\{S_i\}$ is a discrete reflection group $\Gamma$ with respect to the polyhedron $P$.

Let $\Delta$ be an n-simplex in $X$ all of whose dihedral angles are submultiple of $\pi$. The group $\Gamma$ generated by the reflections of $X$ in the sides of $\Delta$ is an n-simplex discrete reflection group. Notice that $X$ can be $S^n$, $E^n$ or $H^n$. The classification of all the irreducible n-simplex (spherical, euclidean and hyperbolic) reflection groups is complete [9].

Assume that $n = 2$. Then $\Delta$ is a triangle in $X$, whose angles $\frac{\pi}{l}$, $\frac{\pi}{m}$, $\frac{\pi}{n}$ are submultiple of $\pi$. If we call $T(l,m,n)$ the group $\Gamma$ generated by the reflections in the sides of $\Delta$, $T(l,m,n)$ is call a triangle reflection group.

If $X = S^2$ the only spherical triangle reflection groups are:

$$T(2,2,2), \quad T(2,2,n) \quad n > 2, \quad T(2,3,3), \quad T(2,3,4), \quad T(2,3,5)$$
If $X = E^2$ we have the euclidean triangle reflection groups:

$$T(3, 3, 3), \quad T(2, 4, 4), \quad T(2, 3, 6)$$

If $X = H^2$ we have the hyperbolic triangle reflection groups:

$$T(2, m, n) m \geq n \geq 3, \quad T(l, m, n) l \geq m \geq n \geq 3$$

Geometrically a reflection can be represented by a linear transformation which fixes an hyperplane pointwise and sends some non zero vector to its negative. In the metric space $X$ we construct vectors $\{\alpha_i\}$ in one to one correspondence to the sides $\{S_i\}$ defined before, in such a way that the angle between $\alpha_i$ and $\alpha_j$ will be compatible with the values of $k_{ij}$, namely, $\vartheta (\alpha_i, \alpha_j) = \frac{\pi}{k_{ij}}$, ($k_{ij}$, positive integer o infinite, $k_{ii} = 1$).

In order to construct a reflection with respect to these vectors $\{\alpha_i\}$ we define a non-degenerate symmetric bilinear form on $X$ by the formulas

$$\langle \alpha_i, \alpha_j \rangle = -\cos \frac{\pi}{k_{ij}}$$

This expression is interpreted to be $-1$ for $k_{ij} = \infty$. Obviously $\langle \alpha_i, \alpha_i \rangle = 1$, and $\langle \alpha_i, \alpha_j \rangle \leq 0$ for $i \neq j$. For each vector $\alpha_i$ we can define a reflection $S_i$ on $X$:

$$S_i \beta = \beta - 2 \langle \alpha_i, \beta \rangle \alpha_i \quad , \quad \beta \in X$$

clearly $S_i \alpha_i = -\alpha_i$ and all $\gamma$ satisfying $\langle \alpha_i, \gamma \rangle = 0$ belong to a plane invariant under $S_i$.

It can be proved that the collection of the polyhedra obtained by the reflections on the side of $\Delta$ is a tessellation of $X$ all the n-simplex (compact or non-compact) reflection groups lead to regular tessellation of $X$.

In Figure 1 we give now some examples of tessellations in $H^2$ generating by reflecting in the sides of the hyperbolic triangle $T(2, 3, 8)$.

3. Gauss curvature of continuous tessellations. Two dimensional tessellations in $X$ ($= S^2, E^2$ or $H^2$) are generated by 2-simplex (triangle) reflection group.

In $S^2$ the geodesic triangle are spherical.

The excess of the interior angles of a spherical triangle is:

$$\varepsilon = \alpha + \beta + \gamma - \pi = \frac{\pi}{l} + \frac{\pi}{m} + \frac{\pi}{n} - \pi$$

It can be proved that this excess is always positive [9]

The area of the triangle $T(x, y, z)$ is

$$\text{Area} \{T(x, y, z)\} = \alpha + \beta + \gamma - \pi = \varepsilon$$

The excess of the interior angles of an euclidean triangle is

$$\varepsilon = \alpha + \beta + \gamma - \pi = 0$$

In $H^2$ the geodesic triangles are hyperbolic.
The excess of the interior angles of an hyperbolic triangle is
\[ \varepsilon = \alpha + \beta + \gamma - \pi = \frac{\pi}{l} + \frac{\pi}{m} + \frac{\pi}{n} - \pi \]

It can be proved that this excess is always negative [9].

The area of an hyperbolic triangle \( T(x,y,z) \) is
\[ \text{Area}(T) = \pi - (\alpha + \beta + \gamma) \]

We can now apply these results to the curvature of the surfaces corresponding to the 2-dimensional regular tessellations (spherical, euclidean or hyperbolic). According to Gauss-Bonet theorem the excess angle of some geodesic triangle \( T \) is equal to the integral of the gaussian curvature \( K \) over \( T \)
\[ \varepsilon = \alpha + \beta + \gamma - \pi = \int \int_{T} K d\sigma \]

where \( d\sigma \) is the area element. If \( K=\text{const.} \)
\[ K = \frac{\varepsilon}{A} \]

Applying this formula to the above results, we have:
- \( K = 1 \) for spherical geodesic triangles
- \( K = 0 \) for euclidean triangles
- \( K = -1 \) for hyperbolic geodesic triangle

4. Curvature on planar graphs. A graph is a par \( G = \{V,E\} \) where \( V \) is a non-empty set of vertices and \( E \) an unordered 2-set of vertices, called edges, in such a way that two vertices are incident to an edge.

A graph can be defined in an abstract way using only combinatorial properties of vertices and edges, or can be obtained from geometrical objects. For instance, given a particular tessellation described in section 2, we keep the edges and vertices of all the triangles and eliminate the embedding manifold in such a way that we are left with the points (vertices) and relations among those (edges). In Figure 2, we have drawn the graphs that we have derived by this method from the tessellations given in Figure 1, where the vertices are represented by points and the edges by arrows. In a graph one can define such elements as path, circuit, length, distance. For instance, in a given graph one may travel from one vertex to another using several edges. the set of the vertices visited in that journey is called a path. These definitions coincide with the standard ones when the graph is embedded in some continuous manifold.

One can define the excess of this triad of vertices as the quantity, in analogy with (1),
\[ \delta = \frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1 \]

where \( 2l, 2m, 2n \) are the number of edges incident in each of the three vertices, which correspond to \( 2l \)--valued, \( 2m \)--valued or \( 2n \)--valued vertices, respectively.
If we define the spherical, euclidean or hyperbolic graph, that is obtained from a spherical, euclidean or hyperbolic tessellation respectively, we can check

\[ \delta > 0, \text{ for a spherical graph} \]

\[ \delta = 0, \text{ for an euclidean graph} \]

\[ \delta < 0, \text{ for an hyperbolic graph} \]

We define the area and the area of the triad \( T(l, m, n) \) in an hyperbolic graph

\[ \sigma(T) = 1 - \left( \frac{1}{l} + \frac{1}{m} + \frac{1}{n} \right) \]

Similarly, we define the curvature of a triad \( T(l, m, n) \)

\[ K(T) = \frac{\delta}{\sigma} = \begin{cases} 
1, & \text{for a spherical graph} \\
0, & \text{for an euclidean graph} \\
-1, & \text{for an hyperbolic graph} 
\end{cases} \]

an expression that can be considered the discrete version of the Gauss-Bonet theorem.

5. **Statistical approach to Gaussian curvature.** In a random distribution of points we can apply the combinatorial approach to calculate the curvature as in the regular case. We can consider a cell decomposition of a manifold by the embedding of a cell complex \( \Omega \), formed by \( N_0 \) vertices, \( N_1 \) edges, \( N_2 \) faces, \( N_3 \) tetrahedra and so on, satisfying

\[ \sum_{k=0}^{D} (-1)^k N_k(\Omega) = (-1)^D \chi(\Omega) \]

One of the most useful cell decomposition is a triangulation in which all the cell complex are simplicial, satisfying \( N_1(\Omega) = \frac{1}{2}(D + 1)N_0(\Omega) \). Given a random distribution of points \( p_i \) we construct the Voronoi complex as the set of points \( p_j \) that are closer to \( p_i \) than other point \( p_j \). The Delaunay complex is the dual to Voronoi, and it is formed by all the points \( p_i \) and the edges joining them (see Figure 3). A Voronoi complex gives rise to a non-regular tessellation. If we substract the embedding manifold we are left with
FIGURE 3. Example of 2D Voronoi (thin lines) and Delaunay (thick lines) complexes; the points they are based on are the Delaunay vertices.

a graph, where we can calculate curvature from combinatorial properties of Voronoi complex. Suppose we have a 2-dimensional Voronoi graph $\Omega$ with elements $N_0, N_1, N_2$ satisfying

$$N_1 = \frac{1}{2} (2 + 1) N_0, \quad N_0 - N_1 + N_2 = \chi(\Omega), \quad \chi \quad \text{Euler number.}$$

In order to calculate the mean curvature, we use the Gauss-Bonet formula:

$$\bar{N}_1 = 2 \frac{N_1(\Omega)}{N_2(\Omega)} = G \left( 1 - \frac{\chi(\Omega)}{N_2} \right) = G \left( 1 - \int K d\sigma \right) = G \left( 1 - \frac{K}{4\pi \rho} \right)$$

whence $K = 4\pi \rho \left( 1 - \frac{1}{G} \bar{N}_1 \right)$, where $\rho$ is the density of the cell complex in $\Omega$.

This work was partially supported by Ministerio Educación y Ciencia, grant BFM 2003-00313/FIS

REFERENCES

1. L. Bombelli et al., “Semiclassical quantum gravity: Statistics of Combinatorial Riemannian Geometries”, arXiv: gr-qc/0409006.
2. M. Lorente, “A discrete curvature on a planar graph”, arXiv: gr-qc/0412094.
3. T. Regge, R.M. Williams, “Discrete Structure in Gravity”, J. Math Phys., 44 (2000) 3964-3984.
4. R. Penrose, “Angular momentum: an approach to combinatorial space time” in Quantum Theory and Beyond (ed. T. Bastin), Cambridge U. Press, Cambridge, 1971, pp. 151-181.
5. L. Bombelli, J. Lee, D. Meyer, R.D. Sorkin, “Space time as a causal set”, Phys. Rev. Lett. 59 (1987) 521-524. F. Markopoulou, L. Smolin, “Causal evolution of spin networks”, Nucl. Phys. B 508 (1997) 409-430.
6. M. Lorente, “Causal spin networks and the Structure of Space and Time”, 12th Int. Congress on Logic, Methodology and Philosophy of Science, Universidad de Oviedo, 2003.
7. P. Kramer, M. Lorente “Surface embedding, topology and dualization for spin networks”, J. Phys. A: Math. Gen. 35 (2002) 8563-8574.
8. L. Bombelli, “Statistical Lorentzian geometry”, arXiv: gr-qc/0002053.
9. J.G. Ratcliffe, Foundation of Hyperbolic Manifolds, Springer, New York 1994, pp. 293-296.