FEJÉR-TYPE POSITIVE OPERATOR BASED ON TAKENAKA–MALMQUIST SYSTEM ON UNIT CIRCLE

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Abstract. Let \( \varphi = \{ \varphi_k \}_{k=-\infty}^{\infty} \) denote the extended Takenaka–Malmquist system on unit circle \( T \) and let \( \sigma_n, \varphi(f), f \in L^1(T) \), be the Fejér-type operator based on \( \varphi \), introduced by V. N. Rusak. We give the convergence criteria for \( \sigma_n, \varphi(f) \) in Banach space \( X(T) := L^p(T) \lor C(T), p \geq 1 \). Also we prove the Voronovskaya-type theorem for \( \sigma_n, \varphi(f) \) on class of holomorphic functions representable by Cauchy-type integrals with bounded densities.

Keywords and Phrases: Holomorphic functions, Takenaka–Malmquist system, Fejér type operator, Blaschke product, Frostman condition.

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1. Introduction

Let \( C(T) \) denote a Banach space of continuous functions on the unit circle \( T := \{ z \in \mathbb{C} : |z| = 1 \} \) equipped with the norm \( ||f||_{C(T)} = \max_{t \in T} |f(t)| \).

Let a function \( f \in C(T) \) and let

\[
 f(e^{ix}) \sim \sum_{k \in \mathbb{Z}} c_k e^{ikx}
\]

be its trigonometric Fourier series. The Cesàro \((C,1)\) means of \( f \) is defined by

\[
 (1.1) \quad \sigma_0(f) = f \quad \text{and} \quad \sigma_n(f)(e^{ix}) := \frac{1}{n} \sum_{k=1}^{n} S_k(f)(e^{ix}), \quad \text{if} \quad n = 1, 2, \ldots,
\]

where \( S_k(f)(z) := \sum_{j=-k+1}^{k-1} c_j e^{ijx} \). The trigonometric polynomials \( \sigma_n(f) \) also are called the Fejér means of \( f \).

The famous L. Fejér theorem says:

(i) for each \( n = 0, 1, \ldots \), \( \sigma_n(f) \geq 0 \) on \( T \) if \( f \geq 0 \) on \( T \);
(ii) for each \( n = 0, 1, \ldots \) and \( f \in C(T) \), \( ||\sigma_n(f)||_{C(T)} \leq ||f||_{C(T)} \);
(iii) if \( f \in C(T) \), then \( \sigma_n(f) \) converge uniformly to \( f \) on \( T \) as \( n \to \infty \).
The classical proof is based on the representation of \( \sigma_n(f) \) by convolution
\[
\sigma_n(f)(e^{ix}) = (f * K_n)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it})K_n(x-t)dt,
\]
where
\[
K_n(t) := \frac{1}{n} \left( \frac{\sin \frac{nt}{2}}{\sin \frac{t}{2}} \right)^2
\]
is the approximative identity (Fejér kernel).

If the Fourier series of \( f \) is of the power series type, i.e.
\[
f(e^{ix}) \sim \sum_{k=0}^{\infty} c_k e^{ikx},
\]
then, it is readily verified that
\[
\sigma_n(f)(z) = \sum_{k=0}^{n-1} \left( 1 - \frac{k}{n} \right) c_k z^k
\]
\[
= S_n(f)(z) - \frac{z}{n} S'_n(f)(z), \quad z \in \mathbb{D},
\]
where \( f(z) = \sum_{k=0}^{\infty} c_k z^k \) is holomorphic function in the unit disc \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \) and \( S_n(f)(z) \) its partial Taylor’s sums. In this case we have the following assertion:

(iv) \( n(f(z) - \sigma_n(f)(z))/z \) converge to \( f'(z) \) uniformly on compact sets of \( \mathbb{D} \).

The assertion (iv) is a Voronovskaja type theorem for \( \sigma_n(f) \) in holomorphic case. See [6] and [15] for further generalization.

In this paper we are interested in extending of (i)–(iv) on Fejér type means of Fourier series expansion based on Takenaka–Malmquist orthonormal system.

For more precisely formulation of our goals, we need some notations and some well known facts.

Let \( \mathcal{H} \) denote the set of all functions holomorphic in \( \mathbb{D} \) and let \( dm \) be the normalized Lebesgue measure on \( \mathbb{T} \). For \( 1 \leq p \leq \infty \), the Hardy space \( H^p \) consist of those \( f \in \mathcal{H} \) for which
\[
+\infty > \|f\|_p := \begin{cases} 
\sup_{\rho \in [0,1)} \left( \int_{\mathbb{T}} |f(\rho t)|^p dm(t) \right)^{1/p}, & 1 \leq p < \infty, \\
\sup_{t \in \mathbb{T}} |f(t)|, & p = \infty.
\end{cases}
\]

It is well known that each function \( f \) from the space \( H^p \) has non-tangential limits almost everywhere on the circle \( \mathbb{T} \) and, moreover, these limit values form a measurable function belonging to \( L^p(\mathbb{T}) \). This function we will also denote by \( f \) and note that
\[
\|f\|_{L^p(\mathbb{T})} := \left( \int_{\mathbb{T}} |f|^p dm \right)^{1/p}
\]
\[
= \|f\|_p, \quad 1 \leq p < \infty;
\]
and
\[
\|f\|_{L^\infty(\mathbb{T})} = \text{ess sup}_{t \in \mathbb{T}} |f(t)|.
\]
For a given system of points $a := \{a_k\}_{k=0}^{\infty}$, $a_k \in \mathbb{D}$ (the points $a_k$ are enumerated taking into account their multiplicity), the Takenaka–Malmquist system $\varphi = \{\varphi_k\}_{k=0}^{\infty}$ is defined by

\begin{equation}
\varphi_k(z) := \frac{\sqrt{1 - |a_k|^2}}{1 - \overline{a_k} z} B_k(z), \quad k \in \mathbb{Z}_+,
\end{equation}

where

\begin{equation}
B_k(z) := \begin{cases}
1, & \text{if } k = 0, \\
\prod_{j=0}^{k-1} \frac{z - a_j}{1 - \overline{a_j} z}, & \text{if } k \in \mathbb{N},
\end{cases}
\end{equation}

is Blaschke product for the disc $\mathbb{D}$ of degree $k$.

It is well known that $\varphi$ is an orthonormal and complete system in $H^2$ if and only if $B_n(z) \to 0$ as $n \to \infty$ uniformly on compact sets in $\mathbb{D}$. The last condition is equivalent to (see [1], p. 195)

\begin{equation}
\sum_{k=0}^{\infty} (1 - |a_k|) = +\infty.
\end{equation}

So, if (1.5) is fulfilled, then for any function $f \in H^2$,

\[
\lim_{n \to \infty} \left\| f - \sum_{k=0}^{n} \langle f, \varphi_k \rangle \varphi_k \right\|_2 = 0.
\]

Consequently, we have that for any $f \in H^2$ the Fourier series

\begin{equation}
\sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k,
\end{equation}

where

\begin{equation}
\langle f, \varphi_k \rangle := \int_{\mathbb{T}} f \overline{\varphi_k} dm,
\end{equation}

convergence to $f$ uniformly in $\mathbb{D}$ as well as in $H^2$–metric on $\mathbb{T}$, provided (1.5).

Actually, we can extend the concept of Fourier series (1.6) on the space of Cauchy transforms

\[\mathcal{K} := \left\{ f(z) = \int_{\mathbb{T}} \frac{d\mu(t)}{1 - \overline{z} t} : z \in \mathbb{D}, \mu \in M \right\},\]

where $M$ is the space of finite complex Borel measures on $\mathbb{T}$. For this one, we set

\[\langle f, \varphi_k \rangle := \int_{\mathbb{T}} \overline{\varphi_k} d\mu, \quad k = 0, 1, \ldots.
\]

In fact, we have that for $f \in \mathcal{K}$

\[f(z) = \sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \varphi_k(z), \quad z \in \mathbb{D},\]

provided $\varphi$ is complete Takenaka–Malmquist system. This follows immediately from definition of $\mathcal{K}$ and the identity [3] (an analog of the Christoffel–Darboux formula)

\begin{equation}
\sum_{k=0}^{n-1} \varphi_k(z) \overline{\varphi_k(t)} = \frac{1 - B_n(z) \overline{B_n(t)}}{1 - z \overline{t}}, \quad n \in \mathbb{N}, \ z, t \in \mathbb{C}.
\end{equation}
When a function $f$ is only defined on $\mathbb{T}$ and $f \in L^1(\mathbb{T})$, then we will define $\langle f, \varphi_k \rangle$ for every $k \in \mathbb{Z}$ in same manner as (1.7), where $\varphi_{-k}(t) = \overline{t \varphi_{k-1}(t)}$ for $k \in \mathbb{N}$. In such case we associate a function $f$ with the Fourier series

\begin{equation}
(1.9) \quad f \sim \sum_{k \in \mathbb{Z}} \langle f, \varphi_k \rangle \varphi_k,
\end{equation}

based on extended Takenaka–Malmquist system $\varphi = \{\varphi_k\}_{k \in \mathbb{Z}}$.

Through the paper, we let by $\mathcal{TMS}$ denote the set of all Takenaka–Malmquist systems (including extended).

The equations (1.1), (1.2) and (1.3) hint at three possible ways to generalize of Fejér means based on $\mathcal{TMS}$.

The first one is the Cesàro $(C, 1)$ means of the series (1.9), that is

\begin{equation}
(1.10) \quad \sigma_n(f) = \frac{1}{n} \sum_{k=1}^{n} S_k(f) = \sum_{k=-n+1}^{n-1} \left(1 - \frac{|k|}{n}\right) \langle f, \varphi_k \rangle \varphi_k, \quad n = 1, 2, \ldots,
\end{equation}

where $S_k(f) := \sum_{j=-k+1}^{k-1} \langle f, \varphi_j \rangle \varphi_j$, $k \in \mathbb{N}$, are the partial sums of the series in (1.9).

Unfortunately, in such case, the assertions (i), (ii) and (iv) are not true.

For example, let $e_0(z) := 1$ and $0 < a_k < 1$, $k = 0, 1, \ldots$. Then it follows from (1.8), that

\[ e_0(t) \sim \sum_{k=0}^{\infty} \varphi_k(0) \varphi_k(t), \quad t \in \mathbb{T}, \]

provided (1.5) is satisfied.

Consequently,

\[ S_k(e_0)(z) = 1 - \overline{B_k(0)} B_k(z), \]

and therefore, by (1.10), we get

\[ \sigma_n(e_0)(z) = 1 - \frac{1}{n} \sum_{k=1}^{n} B_k(0) B_k(z). \]

It follows that

\[ \|\sigma_n(e_0)\|_{C(\mathbb{T})} = \left\| 1 - \frac{1}{n} \sum_{k=1}^{n} B_k(0) B_k \right\|_{C(\mathbb{T})} \]

\[ = 1 + \frac{1}{n} \sum_{k=1}^{n} B_k(0) B_k(-1) \]

\[ = 1 + \frac{1}{n} \sum_{k=1}^{n} \left( \prod_{j=0}^{k-1} a_j \right) \]

\[ > \|e_0\|_{C(\mathbb{T})}. \]

So, (i) and (ii) are not true in general.
Analogously, if we consider $e_0$ as a holomorphic function in $D$, we obtain that for all $z \in (-1, 0]$ and for each natural $n$

$$|n(e_0(z) - \sigma_n(e_0)(z)) - ze_0'(z)| = \left| \sum_{k=1}^{n} B_k(0)B_k(z) \right| = \sum_{k=1}^{n} \left( \prod_{j=0}^{k-1} a_j \frac{|z| + a_j}{1 + |z|a_j} \right)$$

$$\geq \sum_{k=1}^{n} \frac{1}{2k} \left( \prod_{j=0}^{k-1} a_j \right)^2 \geq \sum_{k=1}^{n} \frac{1}{2k} \left( \prod_{j=0}^{k-1} a_j \right)^2$$

This gives that (iv) is not true for all holomorphic functions.

The assertion (iii) can be rescued at least for holomorphic functions. More precisely, it was proved in [18] that $\sigma_n(f)$ converge uniformly to the function $f$ in $D$ if the boundary function $f(e^{ix})$ is continuous on $T$, provided $a_k$ are all in a compact subset of $D$ and satisfy certain (mild) condition on the distribution.

In this paper we consider the other two cases of generalization of Fejér means. Namely, we consider the operators $\sigma_{n,\varphi}$ and $\sigma_{n,\varphi}^+$ defined on $L^1(T)$ and $K$ respectively as follows:

$$\sigma_{n,\varphi}(f)(e^{ix}) := \begin{cases} f(e^{ix}), & \text{if } n = 0, \\ \frac{1}{4\pi \gamma_n(x)} \int_{-\pi}^{\pi} f(e^{iy}) \frac{\sin^2 \left( \int_{-\pi}^{y} \gamma_n(t) dt \right)}{\sin^2 \frac{y - x}{2}} dy, & \text{if } n \in \mathbb{N}, \ x \in \mathbb{R}, \end{cases}$$

where

$$\gamma_n(t) := \frac{1}{2} \sum_{k=0}^{n-1} \frac{1 - |a_k|^2}{1 - 2|a_k| \cos(t - \arg a_k) + |a_k|^2},$$

and

$$\sigma_{n,\varphi}^+(f)(z) := \begin{cases} f(z), & \text{if } n = 0, \\ S_n(f)(z) - \frac{B_n(z)}{B_n'(z)} S_n'(f)(z), & \text{if } n \in \mathbb{N}, \ z \in \mathbb{T}, \end{cases}$$

where

$$S_n(f) = \sum_{k=0}^{n-1} \langle f, \varphi_k \rangle \varphi_k.$$
It should be remarked also that for $f \in K$ the functions $\sigma^+_n(f)$, $n \in \mathbb{Z}_+$, are holomorphic in $D$, provided that $B_n/B'_n$, $n \in \mathbb{Z}_+$, are holomorphic in $D$, and are meromorphic in $D$ otherwise. But in any case, we can consider the functions

$$\delta_n,\varphi(f) := \begin{cases} 0, & \text{if } n = 0, \\ \frac{B'_n}{B_n} (f - \sigma_n^+(f)), & \text{if } n \in \mathbb{N}, \end{cases}$$

as the error functions of weighted approximation in $D$ to $f \in H^1$ by $\sigma_n^+(f)$. In view of (1.2), it is natural to conjecture that $\delta_n,\varphi(f)(z) \to f'(z)$ uniformly on compact sets in $D$ as $n \to \infty$. We will show that this is really true if (1.5) is satisfied.

Given these remarks, it is reasonable to studies the operators $\sigma_n,\varphi(f)$ and $\sigma_n^+,\varphi(f)$ separately on the unit circle $T$ and in the unit disc $D$ respectively.

Our goals are to solve the following two main problems.

**Problem 1.1.** Let $X(T)$ be one of the space $C(T)$ and $L^p(T)$, $1 \leq p < \infty$. Find necessary and sufficient condition on $\varphi \in TMS$ in order that $\sigma_n,\varphi(f)$ converges in metric $X(T)$ to $f$ for every function $f \in X(T)$.

**Problem 1.2.** Let $K$ be a class of holomorphic functions in $D$, $\varphi \in TMS$, $n \in \mathbb{N}$ and $z \in D$. Find the quantity

$$\sup_{f \in K} |\delta_n,\varphi(f)(z) - f'(z)|.$$

2. Main results

In this section we only state our main theorems and some corollaries, and the proofs of theorems will be given in Section 3.

Our main result in part of uniform convergence of $\sigma_n,\varphi$ for $C(T)$ is connected with the well-known Frostman theorem [5] (see also [8], p. 117).

**Frostman’s theorem.** Let $B$ be a infinite Blaschke product with zero-sequence $\{a_k\}_{k=0}^\infty$, and $t \in T$. Then $B$ has an angular derivative in the sense of Carathéodory at $t$ (for definition see [8], p. 62) if and only if

$$\sum_{k=0}^\infty \frac{1 - |a_k|^2}{1 - |t a_k|^2} < \infty.$$

It is shown in [16] (p.185) that there is an infinite Blaschke product that has an angular derivative at no point of $T$ or, equivalently, there is a sequence $\{a_k\}_{k=0}^\infty$ such that

$$\sum_{k=0}^\infty (1 - |a_k|) < \infty,$$

yet

$$\sum_{k=0}^\infty \frac{1 - |a_k|^2}{1 - |t a_k|^2} = \infty \quad \text{for all } t \in T.$$

**Theorem 2.1.** Suppose that $\varphi \in TMS$ and $X(T)$ is one of the space $C(T)$ and $L^p(T)$, $1 \leq p < \infty$. In order that $\|f - \sigma_n,\varphi(f)\|_{X(T)} \to 0$ as $n \to \infty$ for every $f \in X(T)$ it is necessary and sufficient that

$$\lim_{n \to \infty} \left\| \frac{1}{B'_n} \right\|_{X(T)} = 0.$$
Corollary 2.1. Let \( \varphi \in \text{TMS} \). Then for every \( f \in C(\mathbb{T}) \), \( \sigma_{n,\varphi}(f) \) converges to \( f \) uniformly on \( \mathbb{T} \) if and only if \((2.1)\) is fulfilled.

Indeed, in case \( X(\mathbb{T}) = C(\mathbb{T}) \) the norm in the left hand side of \((2.2)\) becomes

\begin{equation}
(2.3) \quad \left\| \frac{1}{B_n} \right\|_{C(\mathbb{T})} = \frac{1}{\min_{t \in \mathbb{T}} \sum_{k=0}^{n-1} \frac{1 - |a_k|^2}{|1 - \bar{t}a_k|^2}}.
\end{equation}

Corollary 2.2. Let \( \varphi \in \text{TMS} \) be such that \((2.1)\) is fulfilled. Then for every \( f \in L^p(\mathbb{T}), 1 \leq p < \infty \), we have \( \|f - \sigma_{n,\varphi}(f)\|_p \to 0 \) as \( n \to \infty \).

Indeed, similarly to \((2.3)\), we get

\begin{equation}
(2.4) \quad \left\| \frac{1 - B_n'(0)B_n}{B_n} \right\|_{X(\mathbb{T})} \leq \frac{2}{\min_{t \in \mathbb{T}} \sum_{k=0}^{n-1} \frac{1 - |a_k|^2}{|1 - \bar{t}a_k|^2}}
\end{equation}

and result follows.

Corollary 2.3. Suppose that \( \varphi \in \text{TMS} \) and \( X(\mathbb{T}) \) is one of the space \( C(\mathbb{T}) \) and \( L^p(\mathbb{T}), 1 \leq p < \infty \). If \( \varphi \) is complete, then for every \( f \in X(\mathbb{T}) \), we have \( \|f - \sigma_{n,\varphi}(f)\|_{X(\mathbb{T})} \to 0 \) as \( n \to \infty \).

This assertion readily follows from the estimates \((2.3), (3.18)\) and \((2.4)\).

Let us consider the class \( K^+ \) consisting of those holomorphic functions in \( \mathbb{D} \), which can be represented by Cauchy type integral

\begin{equation}
(2.5) \quad f(z) = K(\mu)(z) := \int_{\mathbb{T}} \frac{\mu(t)}{1 - \bar{t}z} dm(t), \quad z \in \mathbb{D},
\end{equation}

with \( \mu \in L^\infty(\mathbb{T}), \|\mu\|_\infty := \text{ess sup}_{t \in \mathbb{T}} |\mu(t)| \leq 1 \).

The following assertion gives solution of Problem 1.2.

**Theorem 2.2.** Let \( \varphi \in \text{TMS} \). Then for every \( z \in \mathbb{D} \) and \( n \in \mathbb{Z}_+ \) we have

\begin{equation}
(2.6) \quad \max_{f \in K^+} |\delta_{n,\varphi}(f)(z) - f'(z)| = \frac{|B_n(z)|}{1 - |z|^2}.
\end{equation}

For given \( z \in \mathbb{D} \) such that \( B_n(z) \neq 0 \), and for given \( n \in \mathbb{Z}_+ \) maximum is attained only for the functions

\begin{equation}
(2.7) \quad f_\ast(t) = e^{i\theta} B_n(t) \frac{t - z}{1 - \bar{t}z}, \quad \theta \in \mathbb{R}.
\end{equation}

As immediate consequence of Theorem 2.2 we have the following assertion.

**Corollary 2.4.** Let \( f \in K^+, \varphi \in \text{TMS} \) and let \( a = \{a_j\}_{j=0}^\infty \) be as in \((1.4)\). Then for every \( n \in \mathbb{N} \) we have

\begin{equation}
(2.8) \quad \delta_{n,\varphi}(f)(a_j) = f'(a_j), \quad j = 0, 1, \ldots, n - 1.
\end{equation}

Moreover, if \((1.3)\) is satisfied, then uniformly on closed sets of \( \mathbb{D} \)

\begin{equation}
\lim_{n \to \infty} \delta_{n,\varphi}(f)(z) = f'(z),
\end{equation}

and, therefore, for given \( z \in \mathbb{D} \) the equality

\begin{equation}
\lim_{n \to \infty} \delta_{n,\varphi}(f)(z) = 0
\end{equation}
hold true if and only if $f'(z) = 0$.

Now, let us consider the behavior of $|f(z) - \sigma_{n,\varphi}^+(f)(z)|$ for $z \in \overline{D}$.

We let by $S$ denote the Schur class consisting of holomorphic functions $f$ which satisfy $\sup_{z \in \overline{D}} |f(z)| \leq 1$ and let $A(D)$ denote the disk algebra of holomorphic functions in $D$ that extend continuously to $\overline{D}$ equipped with the norm $\|f\|_{A(D)} = \max_{z \in \overline{D}} |f(z)|$.

**Theorem 2.3.** Suppose $\varphi \in TMS$, $z \in \mathbb{D}$ and $n \in \mathbb{N}$. Then we have

$$
\left| \frac{B_n(z)}{B_n'(z)} \right| \frac{1 - |B_n(z)|^2}{1 - |z|^2} \leq \sup_{f \in S} |f(z) - \sigma_{n,\varphi}^+(f)(z)|
$$

$$
\leq \frac{|B_n(z)|}{|B_n'(z)|} \frac{1 - |B_n(z)|^2}{1 - |z|^2} + |B_n(z)|^2,
$$

provided $z \in \mathbb{D}$ and $|B_n'(z)| > 0$.

Moreover, we have that for all $n \in \mathbb{N}$

$$
\sup_{f \in S} \|f - \sigma_{n,\varphi}^+(f)\|_\infty = \sup_{f \in S \cap A(D)} \|f - \sigma_{n,\varphi}^+(f)\|_{A(D)} = 2.
$$

In view of this result, it is interesting to obtain the lower estimate of $|f - \sigma_{n,\varphi}^+(f)|$ for an individual function $f$. For this goal we recall that the classical Fejér means $\sigma_n(f)$ considered as a method of approximation on space $C(T)$ are saturated and its saturation order is $O(n^{-1})$ (see for example [17], p. 79). In particular, this means that the relation $\|f - \sigma_n(f)\|_{C(T)} = o(n^{-1})$ as $n \to \infty$ holds true only if $f = \text{const}$. The next corollary gives an analog of this assertion for $\sigma_{n,\varphi}$.

**Corollary 2.5.** Suppose $f \in H^\infty$, $\varphi \in TMS$ and let $a = \{a_j\}_{j=0}^{\infty}$ be as in (1.4). Then for every $n \in \mathbb{N}$ we have

$$
\|f - \sigma_{n,\varphi}^+(f)\|_{L^\infty(T)} \geq \frac{1}{n} \max_{0 \leq j \leq n-1} \left( (1 - |a_j|^2) |f'(a_j)| \right).
$$

Moreover, if (1.2) is satisfied and if

$$
\|f - \sigma_{n,\varphi}(f)\|_{L^\infty(T)} = o\left( \frac{1}{n} \right), \quad n \to \infty,
$$

then $f = \text{const}$.

For proving we fix $n \in \mathbb{N}$ and consider the function

$$
F := \frac{f - S_n(f)}{B_n}. 
$$

Since $f(a_j) - S_n(f)(a_j) = 0$ for $j = 0, \ldots, n - 1$, we have (see [17], p. 53) that $F$ belongs to $H^\infty$ space. On the other side, $\delta_{n,\varphi}(f) = B_n'F + S_n'(f)$. Therefore $\delta_{n,\varphi}(f) \in H^\infty$ and, consequently, we can apply the following well-known inequality (see [7], p. 85)

$$
(1 - |z|^2) |\delta_{n,\varphi}(f)(z)| \leq \|\delta_{n,\varphi}(f)\|_1, \quad z \in \mathbb{D}.
$$
Hence, from Corollary 2.4 we get
\[
(1 - |a_j|^2)|f'(a_j)| = (1 - |a_j|^2)|\delta_{n,\varphi}(f)(a_j)|
\leq \|\delta_{n,\varphi}(f)\|_1
\leq \left\| \frac{B'_n}{B_n}(f - \sigma_{n,\varphi}(f)) \right\|_1
\leq \|B'_n\|_1\|f - \sigma_{n,\varphi}(f)\|_{L^\infty(T)}
= n\|f - \sigma_{n,\varphi}(f)\|_{L^\infty(T)}.
\]

Here we used the following identity
\[
\|B'_n\|_1 = \sum_{j=0}^{n-1} \int_T \frac{1 - |a_j|^2}{|1 - a_j^2|^2} dm(t)
= n.
\]

The second part of assertion follows from the inequality (as above)
\[
(1 - |z|^2)|\delta_{n,\varphi}(f)(z)| \leq n\|f - \sigma_{n,\varphi}(f)\|_1, \quad \forall z \in \mathbb{D}
\]
by Corollary 2.4.

3. Proofs

Our approach in proving the main results is based on the following lemmas concerning the some properties of operators \(\sigma_{n,\varphi}\) and \(\sigma'_{n,\varphi}\).

3.1. Auxiliary lemmas. All lemmas in this subsection are new and of independent interest.

We begin with the integral representation for \(\delta_{n,\varphi}\) from which follows that \(\delta_{n,\varphi}(f)\) is holomorphic in \(\mathbb{D}\).

Lemma 3.1. Suppose that \(\varphi \in TMS\) and \(f \in K\). Then the function \(\delta_{n,\varphi}(f)\) is holomorphic in \(\mathbb{D}\).

Moreover, if \(f\) is the Cauchy transform of \(\mu \in M\), then for all \(n \in \mathbb{Z}_+\) and \(z \in \mathbb{D}\),

\[
\delta_{n,\varphi}(f)(z) = f'(z) - B_n(z) \int_T \frac{1 - |z|^2}{|1 - tz|^2} d\mu(t)
\]

Proof. Applying (1.8) we get
\[
\frac{B'_n(z)}{B_n(z)}(f(z) - S_n(f)(z)) = B'_n(z) \int_T \frac{B_n(t)}{1 - tz} d\mu(t)
\]
and
\[
f'(z) - S'_n(f)(z) = B'_n(z) \int_T \frac{B_n(t)}{1 - tz} d\mu(t) + B_n(z) \int_T \frac{B_n(t)}{(1 - tz)^2} d\mu(t).
\]
By subtracting (3.1) from (3.3) we get
\[
\delta_{n,\varphi}(f)(z) = f'(z) - B_n(z) \int_T \frac{B_n(t)}{(1 - tz)^2} d\mu(t).
\]
This means that the function \(\delta_{n,\varphi}(f)\) is holomorphic in \(\mathbb{D}\).
In order to prove (3.1) and (3.2), it remains to note that
\[ \frac{t}{(1-tz)^2} = \frac{T - z}{1 - tz} \quad t \in \mathbb{T}. \]

\[ \square \]

**Lemma 3.2.** Let \( \varphi \in TMS \) and \( e_0(t) = 1 \) for \( t \in \mathbb{T} \). Then for all \( n \in \mathbb{Z}_+ \) we have \( \sigma_{n,\varphi}(e_0) = e_0 \).

**Proof.** By the Lemma 3.1 (see (3.2) for \( d\mu = e_0 dm \)), according to the Poisson integral formula, applying to the holomorphic function \( t \mapsto (t - z)(1 - B_n(t)B_n(z))/(1 - tz) \), we have
\[ B_n'(z) \left( \frac{e_0(z) - \sigma_{n,\varphi}(e_0)(z)}{B_n(z)} \right) = \delta_{n,\varphi}(e_0)(z) \]
\[ = \int_\mathbb{T} \frac{t - z}{1 - tz} \frac{1 - B_n(t)B_n(z)}{|1 - tz|^2} dm(t) \]
\[ = 0, \quad \forall z \in \mathbb{D}. \]

The result follows, since \( B_n' \) has precisely \( n - 1 \) zeros in \( \mathbb{D} \) (see [8], p. 41). \[ \square \]

**Lemma 3.3.** Let \( B \) be a finite Blaschke product for the disc \( \mathbb{D} \) and let \( f \in H^1 \). Then for every \( z \in \mathbb{T} \) we have
\[ \lim_{\rho \to 1^-} (1 - |B(\rho z)|^2) f(\rho z) = 0. \]

**Proof.** Since
\[ \lim_{\rho \to 1^-} \frac{1 - |B(\rho z)|^2}{1 - |\rho z|^2} = |B'(z)| > 0 \]
for every \( z \in \mathbb{T} \) (see [8], p. 45), it is enough to prove the lemma in the case \( B(z) = z \).

By Cauchy integral formula, applying to the function \( t \mapsto f(t)/(1 - \rho \bar{z} t) \), we have
\[ (1 - \rho^2) f(\rho z) = \int_\mathbb{T} f(t) \frac{1 - \rho \overline{z} t}{1 - \rho z t} dm(t). \]

Since
\[ \lim_{\rho \to 1^-} \frac{1 - \rho \overline{z} t}{1 - \rho z t} = \begin{cases} -\overline{z} & \text{if } t \neq z, \\ 1 & \text{if } t = z, \end{cases} \]
by Lebesgue’s dominated convergence theorem we get
\[ \lim_{\rho \to 1^-} (1 - \rho^2) f(\rho z) = \lim_{\rho \to 1^-} \int_\mathbb{T} f(t) \frac{1 - \rho \overline{z} t}{1 - \rho z t} dm(t) \]
\[ = -\overline{z} \int_\mathbb{T} f(t) dm(t) \]
\[ = 0. \]

\[ \square \]

For \( \varphi \in TMS \) we define the kernels \( F_{n,\varphi}(t, z), n \in \mathbb{Z}_+ \), as the functions on \( \mathbb{T}^2 \) by
\[ F_{n,\varphi}(t, z) := \begin{cases} 1, & \text{if } n = 0, \\ 2\Re \sum_{k=0}^{n-1} \left( \frac{\varphi_k(z) - B_n(z)/B_n'(z)}{\varphi_k'(z)} \right) \varphi_k(t) - 1, & \text{if } n \in \mathbb{N}. \end{cases} \]
Since $B'_n(z) \neq 0$ for $z \in \mathbb{T}$ and $n \in \mathbb{N}$ (see [8], p. 40), the kernels $F_{n,\varphi}(t, z)$ are well defined.

We call $F_{n,\varphi}$ the Fejér-type kernel of order $n$ based on the system $\varphi \in TM_S$. The reason for this is that in case $\varphi_k(t) = t^k$, $F_{n,\varphi}$ coincide with the Fejér kernel $F_n(tz) := \begin{cases} 
1, & \text{if } n = 0, \\
2\text{Re} \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) z^{n-k} - 1, & \text{if } n \in \mathbb{N}.
\end{cases}$

In addition, as we will show in the following Lemma 3.4, $F_{n,\varphi}$ is positive kernel with

\[
\int_{\mathbb{T}} F_{n,\varphi}(t, z) dm(t) = 1, \quad z \in \mathbb{T}.
\]

**Lemma 3.4.** Suppose that $\varphi \in TM_S$, $f \in H^1$ and $n \in \mathbb{N}$. Then for every $z \in \mathbb{T}$ we have

\[
\sigma_{n,\varphi}^+(f)(z) = c_n(z) \int_{\mathbb{T}} f(t) \left| \frac{B_n(t) - B_n(z)}{t - z} \right|^2 dm(t)
\]

where

\[
c_n(z) := \frac{1}{\sum_{k=0}^{n-1} |\varphi_k(z)|^2} = \frac{1}{|B'_n(z)|}.
\]

Moreover, if $(t, z) \in \mathbb{T}^2$ and $x = \arg z$, $y = \arg t$ we have

\[
F_{n,\varphi}(t, z) = c_n(z) \left| \frac{B_n(t) - B_n(z)}{t - z} \right|^2 = \frac{1}{2\gamma_n(x)} \cdot \frac{\sin^2 \left( \int_x^y \gamma_n(t) dt \right)}{\sin^2 \frac{y-x}{2}}.
\]

**Proof.** Fix $z \in \mathbb{D}$ and consider the holomorphic function $F_{f,\varphi}(t) := f(t)(1 - B_n(z)B_n(t))$. Observe that

\[
\sigma_{n,\varphi}^+(F_{f,\varphi}) = \sigma_{n,\varphi}^+(f).
\]

This follows easily from the equalities

\[
S_n(fB_n)(z) = \int_{\mathbb{T}} f(t)B_n(t) \frac{1 - B_n(z)B_n(t)}{1 - tz} dm(t)
\]

\[
= \int_{\mathbb{T}} f(t)(B_n(t) - B_n(z)) \frac{1}{1 - tz} dm(t)
\]

\[
= 0, \quad \forall \ z \in \mathbb{D}.
\]
Thus applying (3.2) to $F_{f,z}$ and taken into account (3.11), we get

$$
\frac{B'_n(z)}{B_n(z)}(1 - |B_n(z)|^2)f(z) = \int_T f(t) \left| \frac{1 - B_n(t)B_n(z)}{1 - \overline{z}t} \right|^2 \, dm(t)
$$

(3.11)

$$
\frac{B'_n(z)}{B_n(z)}(1 - |B_n(z)|^2)f(z) = \int_T f(t) \left| \frac{1 - B_n(t)B_n(z)}{1 - \overline{z}t} \right|^2 \sum_{k=0}^{n-1} \varphi_k(z)\overline{\varphi_k(t)} \, dm(t).
$$

(3.12)

Since for $(t, z) \in \mathbb{T}^2$,

$$
\lim_{\rho \to 1 - \frac{t}{z}} \frac{t - \rho z}{1 - \rho z} = \begin{cases} 
\overline{z}, & \text{if } t \neq z, \\
\overline{z}, & \text{if } t = z,
\end{cases}
$$

we can go to the limit as $|z| \to 1$ in both side of (3.11) and (3.12). As a result, by Lemma 3.3 and by Lebesgue’s dominated convergence theorem we get

$$
\frac{zB'_n(z)}{B_n(z)} \sigma_{n, \varphi}^+(f)(z) = \int_T f(t) \left| \frac{B_n(t) - B_n(z)}{t - z} \right|^2 \, dm(t)
$$

$$
= \int_T f(t) \left| \sum_{k=0}^{n-1} \varphi_k(z)\overline{\varphi_k(t)} \right|^2 \, dm(t), \quad \forall z \in \mathbb{T}.
$$

By Lemma 3.2 and the Parseval identity, the last equalities implies

$$
\frac{zB'_n(z)}{B_n(z)} = \frac{zB'_n(z)}{B_n(z)} \sigma_{n, \varphi}^+(e_0(z))
$$

$$
= \int_T \left| \frac{1 - B_n(z)B_n(t)}{1 - zt} \right|^2 \, dm(t)
$$

$$
= \int_T \left| \sum_{k=0}^{n-1} \varphi_k(z)\overline{\varphi_k(t)} \right|^2 \, dm(t)
$$

$$
= \sum_{k=0}^{n-1} |\varphi_k(z)|^2.
$$

Since $zB'_n(z)/B_n(z) = |B'_n(z)|$ (see [8], p. 40), (3.6) follows.

In order to prove (3.7) and (3.9), we fix $z \in \mathbb{T}$ and consider the holomorphic function

$$
g(t) = \sum_{k=0}^{n-1} \left( \varphi_k(z) - \frac{B_n(z)}{B_n'(z)} \varphi_k'(z) \right) \varphi_k(t), \quad t \in \mathbb{D}.
$$

Differentiating (1.8) with respect to $z$ and then taking the conjugate, we obtain the formula

$$
\sum_{k=0}^{n-1} \varphi_k'(z)\overline{\varphi_k(t)} = - \frac{B_n'(z)B_n(t)}{1 - \overline{z}t} + \frac{1 - B_n(z)B_n(t)}{(1 - \overline{z})^2} t.
$$
This formula and (1.8) together give us
\[ g(0) = 1 - B_n(z)B_n(0) + \frac{B_n(z)}{B_n'(z)}B_n'(z)B_n(0) \]
\[ = 1, \]
and consequently,
\[ \int_{T} f \cdot (g - 1) dm = f(0)(g(0) - 1) \]
\[ = 0, \quad \forall f \in H^1. \]

Therefore,
\[ \sigma_{n,\varphi}^+(f)(z) = \int_{T} f(t)g(t)dm(t) \]
\[ = \int_{T} f(t) \left( g(t) + g(t) - 1 \right) dm(t) \]
\[ = \int_{T} f(t)\mathcal{F}_{n,\varphi}(t, z)dm(t) \]

and (3.7) is proved.

Now let us consider the difference
\[ \Delta_{n,z}(t) := \mathcal{F}_{n,\varphi}(t, z) - c_n(z) \left| \sum_{k=0}^{n-1} \varphi_k(z)\overline{\varphi_k(t)} \right|^2 \]
as the function defined on \( T \). From (3.6) and (3.7) we see that \( \Delta_{n,z} \) satisfy
\[ \int_{T} f\Delta_{n,z} dm = 0 \]
for every \( f \in H^\infty \). This means that \( \Delta_{n,z} \) is the nontangential limits of some function from \( H^1 \), say \( \Delta_{n,z}(0) = 0 \) (see [1], p. 85). But \( \text{Im} \Delta_{n,z}(t) = 0 \).

Thus by Schwarz integral formula, \( \Delta_{n,z}(t) = 0 \) for every \( t \in D \).

**Lemma 3.5.** Suppose that \( X(T) \) is one of space \( L^p(T) \) and \( C(T) \) and that \( X_+(D) \) is one of space \( H^p \) and \( A(D) \). Then for any \( \varphi \in TMS \) and for \( n \in \mathbb{N} \) we have
\[ \|\sigma_{n,\varphi}\|_{X(T)\to X(T)} = \|\sigma_{n,\varphi}^+\|_{X_+(D)\to X_+(D)} = 1. \]

**Proof.** The result follows immediately from Lemma 3.2 and Lemma 3.4. \( \square \)

**Lemma 3.6.** Let \( \varphi \in TMS, \alpha \in D \) be fixed and
\[ w_\alpha(z) := \frac{z - \alpha}{1 - \overline{\alpha}z}. \]
Then for every \( z \in D \) and \( n \in \mathbb{N} \) we have
(3.13)
\[ \sigma_{n,\varphi}^+(w_\alpha)(z) = \begin{cases} 
\frac{z - B_n(z)}{B_n'(z)} \left( 1 - B_n(0)B_n(z) \right), & \text{if } \alpha = 0, \\
\frac{z - \alpha}{1 - \overline{\alpha}z} - \frac{1 - |\alpha|^2}{1 - \overline{\alpha}z} \frac{B_n(z)}{1 - \overline{\alpha}z} \left( 1 - B_n(\alpha)B_n(z) \right), & \text{if } \alpha \neq 0.
\end{cases} \]
Let us consider case \( \alpha = 0 \). Differentiating (1.8) with respect to \( t \) and then letting \( t = 0 \) we get

\[
\sum_{k=0}^{n-1} \varphi_k(z) \overline{\varphi_k(0)} = z - \left( \overline{B_n'(0)} + \overline{B_n(0)} z \right) B_n(z).
\]

Differentiating this equality with respect to \( z \) gives

\[
\sum_{k=0}^{n-1} \varphi'_k(z) \overline{\varphi_k(0)} = 1 - \overline{B_n(0)} B_n(z) - \left( \overline{B_n'(0)} + \overline{B_n(0)} z \right) B'_n(z).
\]

From (3.14) we also see that

\[
\langle w_0, \varphi_j \rangle = \int_T \left( \sum_{k=0}^{n-1} \varphi_k(t) \overline{\varphi_k(0)} + \left( \overline{B_n'(0)} + \overline{B_n(0)} t \right) B_n(t) \right) \overline{\varphi_j(t)} dm(t)
\]

\[
= \overline{\varphi_j(0)} + \int_T \left( \overline{B_n'(0)} + \overline{B_n(0)} t \right) \prod_{l=j}^{n-1} \frac{t - a_l}{1 - \overline{a_l}} \sqrt{1 - |a_l|^2} dm(t)
\]

\[
= \overline{\varphi_j(0)}, \quad j = 0, 1, \ldots, n - 1.
\]

Therefore, the sums in the left-hand side of (3.14) and (3.15) are \( S_n(w_0)(z) \) and \( S'_n(w_0)(z) \) respectively.

After some calculations, we obtain

\[
\sigma_{n, w_0}^+(z) = \sum_{k=0}^{n-1} \varphi_k(0) \overline{\varphi_k(z)} - \frac{B_n(z)}{B'_n(z)} \sum_{k=0}^{n-1} \varphi_k(0) \overline{\varphi_k(z)}
\]

\[
= z - \frac{B_n(z)}{B'_n(z)} \left( 1 - \overline{B_n(0)} B_n(z) \right).
\]

Now let us consider case \( \alpha \neq 0 \). According to (1.8) and to the identity

\[
w_\alpha(z) = -\frac{1}{\alpha t} + \frac{1 - |\alpha|^2}{\alpha} \frac{1}{1 - i\alpha t},
\]

we have, similarly to the above,

\[
S_n(w_\alpha)(z) = -\frac{1}{\alpha t} S_N(e_0)(z) + \frac{1 - |\alpha|^2}{\alpha} \frac{1}{1 - i\alpha t} S_n \left( \frac{1}{1 - \alpha t} \right)(z)
\]

\[
= \frac{1}{\alpha} \sum_{k=0}^{n-1} \left( (1 - |\alpha|^2) \varphi_k(\alpha) - \varphi_k(0) \right) \varphi_k(z)
\]

\[
= \frac{1 - |\alpha|^2}{\alpha} \frac{1 - B_n(\alpha) B_n(z)}{1 - z\alpha} - \frac{1}{\alpha} \left( 1 - \overline{B_n(0)} B_n(z) \right)
\]

and

\[
S'_n(w_\alpha)(z) = \frac{1 - |\alpha|^2}{\alpha} \left( \frac{-B_n(\alpha) B'_n(z)}{1 - z\alpha} + \frac{1 - \overline{B_n(\alpha)} B_n(z)}{(1 - z\alpha)^2} \right)
\]

\[
+ \frac{1}{\alpha} \overline{B_n(0)} B'_n(z).
\]

The result follows from the above with a little algebra. \( \square \)
Lemma 3.7. Let \( \varphi \in TMS \), \( z \in \mathbb{T} \) and \( n \in \mathbb{Z}_+ \). Then we have

\[
\max_{f \in S} \left| \delta_{n, \varphi}(f)(z) - B_n'(z)\overline{B_n(z)}f(z) \right| = \begin{cases} 
\frac{1 - |B_n(z)|^2}{1 - |z|^2}, & \text{if } z \in \mathbb{D}, \\
\frac{1 - |B_n(z)|}{|B_n'(z)|}, & \text{if } z \in \mathbb{T}.
\end{cases}
\]

For given \( z \in \mathbb{T} \) and \( n \in \mathbb{Z}_+ \) maximum is attained only for the functions

\[
f(t) = e^{i\theta} \begin{cases} 
\frac{t - z}{1 - t^2} & \text{if } z \in \mathbb{D}, \\
1, & \text{if } z \in \mathbb{T},
\end{cases} \quad \theta \in \mathbb{R}.
\]

Proof. Fix \( z \in \mathbb{D} \) and consider the presentations \( (3.11) \) and \( (3.12) \). It follows that

\[
\left| \delta_{n, \varphi}(f)(z) - B_n'(z)\overline{B_n(z)}f(z) \right| = \left| \frac{B_n'(z)}{B_n(z)} \right| \left(1 - |B_n(z)|^2\right) f(z) - \sigma_{n, \varphi}^+(f)(z) \right| 
\leq \int_{\mathbb{T}} \left| f(t) \frac{t - z}{1 - t^2} \right| \left| \frac{1 - B_n(t)B_n(z)}{1 - t^2} \right|^2 dm(t) 
\leq \int_{\mathbb{T}} \left| \frac{1 - B_n(t)B_n(z)}{1 - t^2} \right|^2 dm(t) 
= \frac{1 - |B_n(z)|^2}{1 - |z|^2}.
\]

Equalities hold if and only if

\[
f(t) = e^{i\theta} \frac{t - z}{1 - t^2}
\]

for some \( \theta \in \mathbb{R} \).

Now, let us fix \( z \in \mathbb{T} \). Then we have

\[
\left| \delta_{n, \varphi}(f)(z) - B_n'(z)\overline{B_n(z)}f(z) \right| = |B_n'(z)| \sigma_{n, \varphi}^+(f)(z),
\]

and the result follows by Lemma 3.5. \( \square \)

3.2. Proof of Theorem 2.1. First of all, note that by Lemma 3.6

\[
\|w_0 - \sigma_{n, \varphi}^+(w_0)\|_{X(T)} = \left\| \frac{1 - B_n'(0)B_n}{B_n'} \right\|_{X(T)},
\]

where \( w_0(z) = z \).

To estimate the left side of the last equation we observe that

\[
\prod_{k=0}^{n-1} |a_k|^2 \leq \left| 1 - \frac{B_n'(0)}{B_n'} B_n(z) \right| \leq 2
\]

for \( z \in \mathbb{T} \). Here we used the well known estimate \( |B_n'(0)| \leq 1 - |B_n(0)|^2 \).

Therefore

\[
\prod_{k=0}^{n-1} |a_k|^2 \left\| \frac{1}{B_n'} \right\|_{X(T)} \leq \|w_0 - \sigma_{n, \varphi}^+(w_0)\|_{X(T)} \leq 2 \left\| \frac{1}{B_n'} \right\|_{X(T)}.
\]
To prove the necessity of (2.2), suppose that \( \prod_{k=0}^{n} |a_k|^2 \to 0 \) as \( n \to \infty \), yet \( \lim_{n \to \infty} \|1/B_n^i\|_{X(T)} > 0 \). In this case, \( a_k \) must satisfies the condition \((1.5)\). But 

\[
\frac{1}{2} \sum_{k=0}^{n} (1 - |a_k|) \leq \min_{i \in T} \sum_{k=0}^{n} \frac{1 - |a|^2}{1 - |t a_k|^2} \leq |B_n^i(t)|
\]

for \( t \in T \). Therefore 

\[
\left\| \frac{1}{B_n^i} \right\|_{X(T)} \leq \frac{2}{\sum_{k=0}^{n} (1 - |a_k|)} \to 0,
\]

as \( n \to \infty \). This gives a contradiction. The necessity of (2.2) follows.

To prove the sufficiency, we require the following Curtis’s generalization of Korovkin’s theorem \(2\) (see also \(4\)).

**Curtis’s theorem** Let \( T_n \) be a uniformly bounded sequence of positive operators in \( X(T) \). Then \( \lim_{n \to \infty} \| f - T_n(f) \|_{X(T)} \) for each \( f \in X(T) \) provided that \( \lim_{n \to \infty} \| e_k - T_n(e_k) \|_{X(T)} = e_k \) for \( k = 0, 1 \), where \( e_k(t) = t^k \).

To apply this theorem, we note that an analogues of the assertions \(i)\) and \(ii)\) for the operator \( \sigma_{n, \varphi} \) in \( X(T) \) follow by the definition of \( \sigma_{n, \varphi} \) and Lemma \(3.5)\)

\(i)'\) for each \( n = 0, 1, \ldots, \sigma_{n, \varphi}(f) \geq 0 \) on \( T \) if \( f \geq 0 \) on \( T \);

\(ii)'\) for each \( n = 0, 1, \ldots, \| \sigma_{n, \varphi}(f) \|_{X(T)} \leq \| f \|_{X(T)} \).

The assertions \(i)'\) and \(ii)'\) together with Lemma \(3.2\) and Curtis’s theorem, where we take \( T_n = \sigma_{n, \varphi} \), proves the sufficiency of Theorem \(2.1\) because \( \lim_{n \to \infty} \| e_1 - \sigma_{n, \varphi}(e_1) \|_{X(T)} = 0 \) according to \((3.17)\).

### 3.3. Proof of Theorem \(2.2\)

It follows from \((3.1)\) that 

\[
|\delta_{n, \varphi}(f)(z) - f'(z)| \leq |B_n(z)| \int_{T} \left| \mu(t) \frac{T - \overline{z}}{1 - \overline{z} t} B_n(t) \right| \frac{1}{|1 - \overline{z} t|^2} dm(t) 
\]

\[
= |B_n(z)| \int_{T} |\mu(t)| \frac{dm(t)}{|1 - \overline{z} t|^2} \leq \frac{|B_n(z)|}{1 - |z|^2}
\]

for every functions \( f \in \mathcal{K}^+ \) and for every \( z \in \mathbb{D} \).

If \( B_n(z) \neq 0 \), then equalities hold throughout this chain of relation if and only if 

\[
|\mu(t)| = 1
\]

and 

\[
\arg \left( \mu(t) \frac{T - \overline{z}}{1 - \overline{z} t} B_n(t) \right) = \text{const}
\]

almost everywhere on \( T \).

But this conditions are equivalent to the condition 

\[
\mu(t) = e^{i\theta} B_n(t) \frac{t - z}{1 - \overline{z} t} \quad \text{a.e. on } T
\]

for some \( \theta \in \mathbb{R} \). So, the only the functions 

\[
f_\theta(t) = K(\mu)(t) = e^{i\theta} B_n(t) \frac{t - z}{1 - \overline{z} t}, \quad \theta \in \mathbb{R},
\]
are extremal.

3.4. **Proof of Theorem 2.3** Let us note that the assertion is trivial in case $z \in \{a_0, \ldots, a_{n-1}\}$. Indeed, in this case we have $f(a_j) - \sigma^+_{n, \varphi}(f)(a_j) = B_n(z) = 0$, therefore (2.9) becomes equality.

Fix $z \in \mathbb{D} \setminus \{a_0, \ldots, a_{n-1}\} \cup \{z \in \mathbb{D} : B'_n(z) = 0\}$. Then it follows from Lemma 3.7 that for arbitrary function $f \in \mathcal{S}$

$$
|f(z) - \sigma^+_{n, \varphi}(f)(z)| \leq |(1 - |B_n(z)|^2)f(z) - \sigma^+_{n, \varphi}(f)(z)| + |B_n^2(z)f(z)|
$$

$$
= \frac{B_n(z)}{B'_n(z)} |\delta_{n, \varphi}(f)(z) - \delta'_{n, \varphi}(z)B_n(z)f(z)| + |B_n^2(z)f(z)|
$$

$$
\leq \frac{B_n(z)}{B'_n(z)} \frac{1 - |B_n(z)|^2}{1 - |z|^2} + |B_n(z)|^2.
$$

On the other side, applying (3.11) and (3.12) to the function $f(t) = (t - z)/(1 - t\bar{z})$, we get

$$
\frac{B'_n(z)}{B_n(z)} (f(z) - \sigma^+_{n, \varphi}(f)(z)) = -\frac{B'_n(z)}{B_n(z)} \sigma^+_{n, \varphi}(f)(z)
$$

$$
= \frac{1 - |B_n(z)|^2}{1 - |z|^2}
$$

and (2.9) follows.

Now fix $z \in \mathbb{T}$. Then by Lemma 3.3 we get

$$
\sup_{f \in \mathcal{S}} |f(z) - \sigma^+_{n, \varphi}(f)(z)| \leq \sup_{f \in \mathcal{S}} |f(z)| + \sup_{f \in \mathcal{S}} |\sigma^+_{n, \varphi}(f)(z)| \leq 2.
$$

In order to prove the lower estimate, consider the sequence $\{f_\lambda\}_{0 < \lambda < 1}$ of functions

$$
f_\lambda(t) = \frac{t - \lambda z}{1 - t\bar{\lambda}z}.
$$

It is clear that $f_\lambda \in \mathcal{S}$ and $|f_\lambda(t)| = 1$ for all $t \in \mathbb{T}$. According to (1.8) and to the identity

$$
\frac{t - \omega}{1 - t\omega} = -\frac{1}{\omega} + \frac{1}{\omega}(1 - |\omega|^2)\frac{1}{1 - t\bar{\omega}}, \quad 0 < |\omega| < 1,
$$

we get

$$
S_n(f_\lambda)(t) = -\frac{1}{\lambda}S_N(e_0)(z) + \frac{1}{\lambda}(1 - \lambda^2)S_n\left(\frac{1}{1 - t\bar{\lambda}z}\right)
$$

$$
= \frac{1}{\lambda} \sum_{k=0}^{n-1} \left((1 - \lambda^2)\varphi_k(\lambda z) - \varphi_k(0)\right) \varphi_k(t)
$$

$$
= \frac{1}{\lambda}(1 - \lambda^2)\frac{1 - B_n(\lambda z)B_n(t)}{1 - t\bar{\lambda}z} - \frac{1}{\lambda}(1 - B_n(0)B_n(t))
$$

and, consequence,

$$
S'_n(f_\lambda)(t) = \frac{1}{\lambda}(1 - \lambda^2) \left(-\frac{B_n(\lambda z)B'_n(t)}{1 - t\bar{\lambda}z} + \lambda\bar{\lambda} \frac{1 - B_n(\lambda z)B_n(t)}{(1 - t\bar{\lambda}z)^2}\right) + \frac{1}{\lambda}B_n(0)B'_n(t).
$$
With a little algebra it follows that

\[ f_\lambda(z) - \sigma_{n,\varphi}^+(f_\lambda)(z) = \frac{1 + \lambda}{1 - \lambda} \frac{B_n(z)}{B'_n(z)} \left( 1 - \frac{B_n(\lambda z) B_n(z)}{B'_n(z)(1 - \lambda^2)} \right). \]

Therefore

\[ \left| f_\lambda(z) - \sigma_{n,\varphi}^+(f_\lambda)(z) \right| = \frac{1 + \lambda}{1 - \lambda} \frac{1}{|B'_n(z)|} \left| 1 - \frac{B_n(\lambda z) B_n(z)}{|B'_n(z)|} \right| \geq \frac{(1 + \lambda)^2}{1 + |B_n(\lambda z)|^2} \frac{1 - |B_n(\lambda z)|^2}{|B'_n(z)|((1 - \lambda)^2)}. \]

Since

\[ \lim_{\lambda \to 1} \frac{1 - |B_n(\lambda z)|^2}{1 - \lambda^2} = |B'_n(z)|, \]

the result follows from the above relation by letting \( \lambda \to 1 \).

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FEJÉR-TYPE POSITIVE OPERATOR BASED ON TAKENAKA–MALMQVIST SYSTEM ON UNIT CIRCLE

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