THE 2-PARAMETER GREEN FUNCTIONS
FOR 8-DIMENSIONAL SPIN GROUPS

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Abstract. The 2-parameter Green functions occur as a crucial ingredient in the character formula for Lusztig induction in finite reductive groups. Still, very little is known about these functions, in particular in the case of groups arising from algebraic groups with disconnected centre. We collect some basic properties and then apply these, together with some explicit computations, to determine all 2-parameter Green functions for 8-dimensional spin groups in odd characteristic, whose centre is disconnected of order 4.

1. Introduction

The 2-parameter Green functions are a central ingredient in the character formula for Lusztig induction and restriction in finite groups of Lie type. Despite of their importance, little seems to be known about their values. We collect some basic properties and compute these functions for several families of groups of small rank, with connected and non-connected centre.

Our main result is the complete determination of the 2-parameter Green functions for the family of spin groups Spin_8^+(q) for odd q, see Theorem 3.4 which we give in the form of tables at the end of this paper.

Our results also lead us to formulate two conjectures on the values of 2-parameter Green functions, the first (Conjecture 2.3) for arbitrary finite reductive groups, the second (Conjecture 2.9) particular to groups of type A.

2. 2-parameter Green functions

2.1. Definition and first properties. Let G be connected reductive with a Steinberg map F : G → G. For an F-stable Levi subgroup L of a parabolic subgroup P of G with Levi decomposition P = UL,

\[ Q_L^G : G_{uni}^F \times L_{uni}^F \to \mathbb{Q}, \quad (u, v) \mapsto \frac{1}{|L^F|} \sum_{i \geq 0} (-1)^i \text{Tr}(\text{Res}(u, v) | H_i^\ell(L^{-1}(U))), \]

is called the associated 2-parameter Green function. Here \( L : G \to G, \quad g \mapsto g^{-1}F(g) \), denotes the Lang map, \( G_{uni} \) is the set of unipotent elements of G, and \( H_i^\ell \) is \( \ell \)-adic cohomology with compact support, for a prime \( \ell \) different from the characteristic of G.
By an abuse of notation we omit the parabolic subgroup $P$ from our notation. This is justified as by the Mackey formula for Lusztig induction, the 2-parameter Green function is independent of $P$ in all situations that we consider here. It follows in particular that $Q_L^G(u, v_1) = Q_L^G(u, v_2)$ whenever $v_1, v_2 \in L^F$ are conjugate in $N_G(L)^F$. Not much else seems to be known about the $Q_L^G$.

The relevance of 2-parameter Green functions comes from the fact that they occur in the character formula for Lusztig induction (see [2, Prop. 12.2]):

**Theorem 2.1.** Let $L$ be an $F$-stable Levi subgroup of $G$. Then

$$R_L^G(\psi)(g) = \frac{1}{|L^F|} \sum_{h \in G^F} |C_{G}^{\psi}(s)F| \sum_{v \in C_{G}^{\psi}(s)_{\text{uni}}} Q_{C_{G}^{\psi}(s)}^G(u, v^{-1}) h\psi(sv),$$

for $\psi \in \text{Irr}(L^F)$ and $g \in G^F$ with Jordan decomposition $g = su$.

If $L = T$ is a maximal torus, so $P$ is a Borel subgroup of $G$, then $L^F_{\text{uni}} = \{1\}$ and the defining formula shows that $Q_L^G(u, 1) = R_L^G(1)(u)$ ($u \in G^F_{\text{uni}}$), which is the usual (1-parameter) Green function. The values of $Q_L^G(u, v)$ at $u = 1$ are known for any $L$, see [2, p. 174]:

$$Q_L^G(1, v) = \begin{cases} 1_{G^F} : L^F \mid u^F & \text{if } v = 1, \\ 0 & \text{otherwise.} \end{cases}$$

If $L$ is a split Levi subgroup of $G$, that is, if we can choose $P$ and hence its unipotent radical $U$ to be $F$-stable, then

$$Q_L^G(u, v) = \frac{1}{|L^F|} \{|gU^F | g \in G^F, u^g \in vU^F\}|$$

can be computed in an elementary way. This shows:

**Proposition 2.2.** Assume that $P$ is an $F$-stable parabolic subgroup of $G$ with Levi decomposition $P = U.L$. Then for $v \in L^F_{\text{uni}}, u \in G^F_{\text{uni}}$ we have:

(a) $|vL^F| Q_L^G(u, v) \in \mathbb{Z}_{\geq 0}$.

(b) If $Q_L^G(u, v^{-1}) \neq 0$ then $v^G \subseteq u^G \subseteq \text{Ind}^G_{L}(vL)$.

(c) If $u$ is regular unipotent, then there is a unique $L^F$-class $C$ of regular unipotent elements of $L^F$ such that

$$Q_L^G(u, v) = \begin{cases} |vL^F|^{-1} & \text{if } v \in C, \\ 0 & \text{otherwise.} \end{cases}$$

Here, $\overline{C}$ denotes the closure of a class $C$ and $\text{Ind}^G_{L}(C)$ is the induced class in the sense of Lusztig–Spaltenstein [7].

**Proof.** Observe that if $g \in G^F$ is such that $u^g \in vU^F$ then for any $c \in C_L(v)^F$, the element $gc$ has the same property, so $\{|gU^F | u^g \in vU^F\}|$ is divisible by $|C_L(v)^F|$, whence [2] shows that $|vL^F| Q_L^G(u, v)$ is an integer.

If $Q_L^G(u, v^{-1}) \neq 0$ there is $g \in G^F$ with $u^g \in vU^F$, so up to replacing $u$ by a conjugate, we have $u \in vU^F$. Now by definition the induced class $C := \text{Ind}^G_{L}(vL)$ has the property that $C \cap vL^F$ is dense in $vL^F$. Hence, $u \in \overline{C}$. Moreover, we have $u = vx$ for some
The above formula yields a linear system of equations for \( \tilde{R} \) to consider the case when \( L \leq T \). It follows that only the regular unipotent class of \( L \) induces to the regular unipotent class of \( G \), and thus \( Q^G_L(u,v) = 0 \) unless \( v \) is regular. Now assume that \( u \in vU^F \), and so \( Q^G_L(u,v) \neq 0 \). Since \( u \) is regular, it lies in a unique Borel subgroup \( B \leq P \) of \( L \). Thus, if \( g \in G^F \) with \( g^{-1}ug \in P \) then \( g \in P^F \). In particular, \( g^{-1}ug \in v^F \) for some \( v' \in L^F_\text{uni} \) implies that \( v,v' \) are \( L^F \)-conjugate. It is clear that there are exactly \( |C_L(v)^F| \) many cosets \( gU^F \) with \( g^{-1}ug \in vU^F \).

All of our examples indicate that the previous result continues to hold in the general case (see also Corollary 2.6 Proposition 2.7 and the tables in Section 3):

**Conjecture 2.3.** The conclusions of Proposition 2.2 continue to hold for arbitrary \( F \)-stable Levi subgroups of \( G \).

We write \( \tilde{Q}^G_L \) for the matrix \( \left( |v|^{L^F} Q^G_L(u,v^{-1}) \right)_{v,u} \), with rows and columns indexed by the unipotent conjugacy classes of \( L^F, G^F \) respectively.

### 2.2. A linear system

On unipotent elements \( u \in G^F_\text{uni} \) the character formula from Theorem 2.1 reads

\[
R^G_L(\psi)(u) = \sum_{v \in L^F_\text{uni}} Q^G_L(u,v^{-1}) \psi(v)
\]

for \( \psi \in \text{Irr}(L^F) \).

**Lemma 2.4.** Let \( M \leq G \) be an \( F \)-stable Levi subgroup containing \( L \). Then

\[
\tilde{Q}^G_L = \tilde{Q}^G_M \cdot \tilde{Q}^G_M.
\]

**Proof.** By transitivity of Lusztig induction [2 Prop. 9.1.8] we have \( R^G_L = R^G_M \circ R^M_L \), so

\[
R^G_L(\psi)(u) = \sum_{v \in M^F_\text{uni}} Q^M_M(u,v^{-1}) \sum_{x \in L^F_\text{uni}} Q^M_L(v,x^{-1}) \psi(x)
\]

for all \( u \in G^F_\text{uni} \) and all class functions \( \psi \) on \( L^F \). The claim follows.

Thus, for an inductive determination of the 2-parameter Green functions it is sufficient to consider the case when \( L \leq G \) is maximal among \( F \)-stable Levi subgroups. Now let \( T \leq L \) be an \( F \)-stable maximal torus, then Lemma 2.4 gives \( \tilde{R}^G_T = \tilde{R}^L_T \cdot \tilde{Q}^G_L \), where we have set \( \tilde{R}^G_T := (R^G_T(1)(u))_u \). The \( L^F \)-conjugacy classes of \( F \)-stable maximal tori of \( L \) are parametrised by \( F \)-conjugacy classes in the Weyl group \( W_L \) of \( L \) (see [2 4.2.22]). Thus the above formula yields a linear system of equations

\[
R^G_T(w)(1)(u) = \sum_{v \in L^F_\text{uni}} R^L_T(w)(1)(v) Q^G_L(u,v^{-1}) \quad (w \in W_L)
\]

for \( \tilde{Q}^G_L \) with coefficient matrix \( (R^L_T(w)(1)(v))_{w,v} \).
Let’s rewrite this by passing to almost characters. For \( \varphi \in \text{Irr}(W_L)^F \) choose and fix an extension to \( W_L \langle F \rangle \) and let \( \tilde{\varphi} \) denote its restriction to the coset \( W_L \langle F \rangle \). Then

\[
R_{\tilde{\varphi}}^L := \frac{1}{|W_L|} \sum_{w \in W_L} \tilde{\varphi}(wF) R_{T_w}^L(1)
\]

is the corresponding almost character, so we have

\[
R_{T_w}^L(1) = \sum_{\varphi \in \text{Irr}(W_L)^F} \tilde{\varphi}(wF) R_{\varphi}^L.
\]

Averaging the above equation (*) over all \( w \in W_L \) we find

\[
\frac{1}{|W_L|} \sum_{w \in W_L} \tilde{\varphi}(wF) \sum_{v \in L_{\text{uni}}} Q_G^G(u, v^{-1}) R_{T_w}^L(v) = \sum_{v \in L_{\text{uni}}} Q_G^G(u, v^{-1}) R_{\varphi}^L(v)
\]

on the right hand side, and on the left

\[
\frac{1}{|W_L|} \sum_{w \in W_L} \tilde{\varphi}(wF) R_{T_w}^G(1)(u) = \frac{1}{|W_L|} \sum_{w} \tilde{\varphi}(wF) \sum_{\psi \in \text{Irr}(W)^F} \tilde{\psi}(wF) R_{\psi}^G(u)

= \sum_{\psi \in \text{Irr}(W)^F} R_{\psi}^G(u) \frac{1}{|W_L|} \sum_{w} \tilde{\varphi}(wF) \tilde{\psi}(wF)

= \sum_{\psi \in \text{Irr}(W)^F} R_{\psi}^G(u) \langle \tilde{\varphi}, \tilde{\psi} \rangle_{W_L,F} = R_{\text{Ind}_{W_L \langle F \rangle}^G}^G(\tilde{\varphi})(u)
\]

by Frobenius reciprocity. With \( u \) running over the unipotent classes in \( G_{\text{uni}}^F \) the equations

\[
R_{\text{Ind}_{W_L \langle F \rangle}^G}^G(\tilde{\varphi})(u) = \sum_{v \in L_{\text{uni}}} Q_G^G(u, v^{-1}) R_{\varphi}^L(v) \quad \text{(for } \varphi \in \text{Irr}(W_L)^F \text{)}
\]

also give a linear system of equations for \( \tilde{Q}_G^L \), with coefficient matrix \( (R_{\varphi}^L(v))_{\varphi,v} \).

**Proposition 2.5.** Assume that the number of unipotent classes of \( L^F \) equals \( |\text{Irr}(W_L)^F| \). Then the 2-parameter Green functions \( Q_G^L \) are uniquely determined by the ordinary Green functions of \( G \) and of \( L \).

**Proof.** This follows from the above considerations and the fact that the square (by assumption) matrix \( (R_{\varphi}^L(v))_{\varphi,v} \) of values of almost characters on unipotent classes of \( L^F \) is known to be invertible. \( \square \)

The assumption of Proposition 2.5 is satisfied, for example, if \( L \) has connected centre and only components of type \( A \). Some examples will be given in Section 2.3. For Levi subgroups of other types, the above just gives restrictions on the values of the 2-parameter Green functions. See also Section 3 below.

We then get the following values on regular unipotent elements of \( G \):

**Corollary 2.6.** In the situation of Proposition 2.5 if \( u \) is regular unipotent in \( G^F \) then

\[
\tilde{Q}_L^G(u, v) = \begin{cases} 
1 & \text{for } v \text{ regular unipotent}, \\
0 & \text{otherwise}.
\end{cases}
\]
Proof. It is known that the ordinary Green functions take constant value 1 on regular unipotent elements. Clearly, the claimed values yield a solution to the system of equations (*) for $\hat{Q}_G^L$.

The conclusion needs no longer be true when $Z(L)$ is disconnected, see e.g. Proposition 3.1.

2.3. Some examples in $\text{GL}_n$. We apply Proposition 2.5 to compute some explicit values. For this recall that the conjugacy classes of unipotent elements of $\text{GL}_n$ and of $\text{GL}_n(q)$ are parametrised by partitions of $n$ via their Jordan canonical form, and the same parametrisation holds for the irreducible characters of its Weyl group, the symmetric group $S_n$. We write $u_\lambda$ for a unipotent element in $\text{GL}_n(q)$ with Jordan form of shape $\lambda \vdash n$, and $\varphi_\lambda \in \text{Irr}(S_n)$ for the character parametrised by $\lambda$.

In $\text{GL}_n$, the order relation on unipotent classes is given by the dominance order on partitions. We can now deduce the following vanishing result; for this, let us order the irreducible characters of $W$ and of $W_L$ by some partial order compatible with the dominance order, and the unipotent classes in $G$ and $L$ by the same order.

Proposition 2.7. Let $G = \text{GL}_n$ with a split Frobenius map and $L$ an $F$-stable Levi subgroup. Let $u_\lambda \in G_{\text{uni}}$ and $v_\mu \in L_{\text{uni}}$. Then $\hat{Q}_G^L(u,v) = 0$ unless $\lambda \preceq \kappa$ for some constituent $\varphi_\kappa$ of $\text{Ind}_{W_L,F}(\tilde{\varphi}_\mu)$.

Proof. It is known that $R_G^L(v_\mu) = 0$ unless $\mu \preceq \nu$. In particular, $(R_G^L(v))_{\varphi,\varphi'}$ is upper triangular. Fix $u = u_\lambda$. Then clearly the (unique) solution to the linear system satisfied by the Green functions will have $\hat{Q}_G^L(u,v_\mu) = 0$ whenever the left hand side entries $R_{\text{Ind}_{W_L,F}(\tilde{\varphi}_\nu)}^L(u)$ vanish for all $\nu \preceq \mu$. By what we said before this happens unless $\lambda \preceq \kappa$ for some constituent $\varphi_\kappa$ of $\text{Ind}_{W_L,F}(\tilde{\varphi}_\mu)$.

For example, let $L$ be of type $\text{GL}_{n-1}$ in $G$ of type $\text{GL}_n$. Let $u_\lambda \in G_{\text{uni}}$ and $v_\mu \in L_{\text{uni}}$ be parametrised by $\lambda \vdash n$, $\mu \vdash n - 1$ respectively. Then $\hat{Q}_G^L(u,v) = 0$ unless $\mu$ is obtained from $\lambda$ by removing one node. Indeed, the induced class is labelled by the partition obtained by increasing the first part by 1, whence the claim follows from Proposition 2.7.

Example 2.8. (a) We compute the (modified) 2-parameter Green functions for $G = \text{GL}_3$ with the standard Frobenius map. Via the Jordan normal form, the unipotent conjugacy classes of $\text{GL}_3$ are parametrised by partitions of 3, which we order $1^3, 21, 3$. For $L$ the only non-trivial standard Levi subgroup, of type $A_1$, we find

$$\hat{Q}_L^G = \begin{pmatrix} \Phi_3 & 1 & . \\ . & q & 1 \end{pmatrix}$$

where the columns are labelled by the unipotent classes of $G$, and the rows by those of $L$. Here, as in subsequent tables, $\Phi_d$ denotes the $d$th cyclotomic polynomial evaluated at $q$, and "." stands for “0”.

(b) Next let $G = \text{GL}_4$, with the unipotent classes ordered as $1^4, 21^2, 2^2, 31, 4$. First let $L_1$ be a standard Levi subgroup of type $A_2$. Its unipotent classes are labelled by the partitions $1^3, 21, 3$ of 3, and we obtain the matrix of (modified) 2-parameter Green functions.
functions
\[ \overline{Q}_L = \begin{pmatrix} 
\Phi_2 \Phi_4 & 1 & \ldots & \\
q \Phi_2 & \Phi_2 & 1 & \\
\ldots & \ldots & q & 1 
\end{pmatrix}. \]

Next, let \( L_2 \) be the standard Levi subgroup of type \( A_1^2 \). The resulting matrix is
\[ \overline{Q}_{L_2} = \begin{pmatrix} 
\Phi_3 \Phi_4 & \Phi_2 & 1 & \ldots \\
q^2 & 1 & \ldots & \\
\ldots & q^2 & 1 & \\
\ldots & \ldots & q \Phi_2 & \Phi_1 & 1 
\end{pmatrix}. \]

where the rows are labelled by the unipotent conjugacy classes of \( L_2 \) parametrised by the pairs \((1^2, 1^2), (2, 1^2), (1^2, 2), (2, 2)\) of partitions of 2. Note that the second and third row agree, as the second and third unipotent class of \( L_2^F \) are conjugate in \( N_G(L)^F \). For the twisted Levi subgroup \( L_3 \) of type \( A_1(q^2). (q^2 - 1) \) we find
\[ \overline{Q}_{L_3} = \begin{pmatrix} 
\Phi_3^2 \Phi_3 & -\Phi_1 & 1 & \ldots \\
q \Phi_1 - \Phi_1 & 1 & \ldots & \\
\ldots & \ldots & q \Phi_2 & \Phi_1 & 1 
\end{pmatrix}. \]

Finally, the split Levi subgroup of type \( A_1 \) is not maximal and we may apply Lemma 2.4, while for its twisted version \( L_4 \) of type \( A_1(q). (q^2 - 1)(q - 1) \) we obtain
\[ \overline{Q}_{L_4} = \begin{pmatrix} 
-\Phi_1 \Phi_3 \Phi_4 & 1 & -\Phi_1 & 1 \\
\ldots & -q^2 \Phi_1 & q \Phi_2 & 1 
\end{pmatrix}. \]

(c) For \( G = GL_3 \) and a standard Levi subgroup \( L_1 \) of type \( A_3 \) we obtain
\[ \overline{Q}_{L_1} = \begin{pmatrix} 
\Phi_5 & 1 & \ldots & \\
q \Phi_3 & \Phi_2 & 1 & \\
\ldots & \ldots & q^2 & 1 \\
\ldots & \ldots & q \Phi_2 & q & 1 \\
\ldots & \ldots & \ldots & q & 1 
\end{pmatrix}. \]

(compare with Proposition 2.7), while for a standard Levi subgroup \( L_2 \) of type \( A_2 A_1 \) we find
\[ \overline{Q}_{L_2} = \begin{pmatrix} 
\Phi_4 \Phi_5 & \Phi_3 & 1 & \ldots & \\
q^3 & \Phi_3 & q \Phi_2 & \Phi_2 & 1 \\
\ldots & \ldots & q \Phi_2 & q \Phi_2 & q \\
\ldots & \ldots & \ldots & q^2 & 1 \\
\ldots & \ldots & \ldots & \ldots & q \Phi_1 & 1 
\end{pmatrix}. \]

where the classes of \( L \) are ordered as \((1^3, 1^2), (1^3, 2), (21, 1^2), (21, 2), (3, 1^2), (3, 2)\). Note that here two distinct unipotent classes of \( L \) fuse into the same \( G \)-class (corresponding to rows 2 and 3) but the Green functions differ: the classes are not \( N_G(L)^F \)-conjugate.

The above examples seem to indicate the following conjecture:

**Conjecture 2.9.** Let \( G \) be of type \( A_1 \), \( L \leq G \) an \( F \)-stable Levi subgroup, \( v \in L_{uni}^F \) and \( u \) is the induced class of \( v^L \) in the sense of Lusztig–Spaltenstein. Then \( \overline{Q}_L(u,v) = 1 \).

For example, if \( L = GL_m \cdot GL_{n-m} < G = GL_n \) and \( v \) is parametrised by \((\lambda_1, \lambda_2)\) then we should have \( \overline{Q}_L(u,v) = 1 \) for \( u \) parametrised by \( \lambda_1 + \lambda_2 \).
2.4. Some examples in classical groups. In this section we consider some further cases for $G$ with connected centre and $F$ a Frobenius endomorphism with respect to an $\mathbb{F}_q$-structure, where $q$ is odd. In this situation, the assumptions of Proposition 2.5 are satisfied for groups of adjoint types $B_2, B_3, C_3, D_4$, for example.

Example 2.10. (a) Let $G = SO_5$ and $L$ a Levi subgroup of type $A_1$ containing long root subgroups. Then we find

$$
\begin{pmatrix}
\Phi_2 \Phi_4 & 1 & 2 & \cdots & . \\
. & q \Phi_2 & \Phi_1 & \Phi_2 & . \\
. & . & 1 & . & 2 \\
. & . & . & . & . \\
. & . & . & . & . \\
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
-\Phi_1 \Phi_4 & 1 & 2 & \cdots & . \\
. & q \Phi_1 & -\Phi_1 & -\Phi_2 & 1 \\
. & . & 1 & . & 2 \\
. & . & . & . & . \\
. & . & . & . & . \\
\end{pmatrix}
$$

for the split and twisted version, respectively, while for the short root $A_1$ we get

$$
\begin{pmatrix}
\Phi_2 \Phi_4 & \Phi_2 & \Phi_1 & 1 & . \\
. & . & 2q & . & 1 \\
. & . & . & . & . \\
. & . & . & . & . \\
. & . & . & . & . \\
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
-\Phi_1 \Phi_4 & -\Phi_1 & -\Phi_1 & 1 & 1 \\
. & . & . & . & . \\
. & . & . & . & . \\
. & . & . & . & . \\
. & . & . & . & . \\
\end{pmatrix}
$$

(b) Let $G = SO_7$ and $L$ be a split Levi subgroup of type $B_2$. Here, $L$ has again as many classes as its Weyl group has irreducible characters, so the system has a unique solution

$$
\begin{pmatrix}
\Phi_2 \Phi_4 \Phi_6 & \Phi_2 & 1 & 1 & . & . & . \\
. & q^2 \Phi_2 & . & 1 & . & . & . \\
. & . & q^2 \Phi_2 & q \Phi_2 & \frac{1}{2} q \Phi_1 & \frac{1}{2} q \Phi_2 & 1 & . \\
. & . & . & q \Phi_4 & \frac{1}{2} q \Phi_1 & \frac{1}{2} q \Phi_2 & 1 & . \\
. & . & . & . & . & . & 2q & 1 \\
\end{pmatrix}
$$

(c) Let $G$ be of adjoint type $C_3$ and $L$ split of type $C_2$. Then we find

$$
\begin{pmatrix}
\Phi_2 \Phi_3 \Phi_6 & 1 & 2 & . & . & . \\
. & q \Phi_2 \Phi_4 & \Phi_2 & \Phi_2 & . & 1 & . \\
. & . & q^2 \Phi_2 & \frac{1}{2} q \Phi_2 & \Phi_2 & . & 1 & . \\
. & . & . & q \Phi_4 & \frac{1}{2} q \Phi_1 & \frac{1}{2} q \Phi_2 & 1 & . \\
. & . & . & . & . & . & q \Phi_2 & q & q & 1 \\
\end{pmatrix}
$$

(d) Let $G$ be of type $D_4$ with connected centre, $F$ a split Frobenius and $L$ a split Levi subgroup of type $A_3$. Ordering the 13 unipotent classes of $G^F$ by their Jordan normal forms

$$
1^8, 2^21^4, 2^4+, 2^4-, 31^5, 32^21, 321^2 \text{ (two classes), } 42^2+, 42^2-, 51^3, 53, 71
$$

we find

$$
\begin{pmatrix}
\Phi_2 \Phi_4 \Phi_6 & \Phi_2 & . & . & . & . & 1 & . & . & . & . & . & . & . \\
. & q^2 \Phi_2 \Phi_4 & \Phi_2 \Phi_4 & \Phi_2 \Phi_4 & . & 1 & 2 & . & . & . & . & . & . & . \\
. & . & . & . & . & 2q^2 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & q \Phi_2 \Phi_4 & q \Phi_2 & q \Phi_1 & q \Phi_2 & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
\end{pmatrix}
$$

for one of the three possible embeddings, and a suitable permutation of the columns $(3, 4, 5)(9, 10, 11)$ for the other two. The Green functions for a twisted Levi subgroup of type $^2A_3(q).(q + 1)$ are related to those of $A_3(q).(q - 1)$ as follows: The ordinary Green functions of $^2A_3(q)$ are obtained from those of $A_3(q)$ by replacing $q$ by $-q$ by Emmola duality [5]. This also entails a permutation of maximal tori in the linear system ($\ast$). Therefore, the Green functions of $^2A_3(q).(q + 1)$ are obtained from those of $A_3(q).(q - 1)$
by replacing $q$ by $-q$ and swapping the classes with Jordan normal form $3^21^2$ (which have centraliser orders $q^8(q \pm 1)^2$).

Next consider a split Levi subgroup $L$ of type $A_1^3$. Here, $\tilde{Q}_L^G$ equals

$$
\begin{pmatrix}
\Phi_2\Phi_3\Phi_6 \Phi_2 \Phi_4 \Phi_2 \Phi_4 \Phi_2 \Phi_4 \Phi_2 1 1 & . & . & . & . \\
q^2\Phi_2 \Phi_2 \Phi_4 & q^2\Phi_4 2q & . & 2q & . & 1 & . & . \\
q^4\Phi_2 & q^4\Phi_4 & q^2\Phi_4 & . & 2q & 1 & . & . \\
. & q^2\Phi_2\Phi_4 & . & . & . & . & . & 2q\Phi_1 - \Phi_2 & 1 & . & . \\
. & . & q^2\Phi_2\Phi_4 & . & q^2\Phi_4 & 2q\Phi_1 & . & . & . & . & 1 & . \\
. & . & . & q^2\Phi_2\Phi_4 & q^2\Phi_4 & 2q\Phi_1 & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & q\Phi_2 q 1 & 1 & . & . \\
\end{pmatrix}
$$

where $* = q^4 + 3q^3 + 3q^2 + q + 1$. The symmetry from triality, cyclically permuting the classes 3,4,5 and 9,10,11 of $G^F$, as well as the classes 2,3,4 and 5,6,7 of $L^F$, is clearly visible. Again, the Green functions for a twisted Levi subgroup of type $A_1(q^3).(q+1)$ are obtained by replacing $q$ by $-q$ and interchanging the two classes with Jordan normal form $3^21^2$. For a split Levi subgroup of type $A_1(q^3).(q+1)$ we find

$$
\begin{pmatrix}
-q^2\Phi_1\Phi_2\Phi_6 & \Phi_1^2\Phi_2^2 & \Phi_1^2\Phi_2^2 & -\Phi_1\Phi_2 & -\Phi_1\Phi_2 & -\Phi_1\Phi_2 & -\Phi_1\Phi_2 & -\Phi_1\Phi_2 & 1 & . & . \\
. & q^2\Phi_1\Phi_2\Phi_6 & . & q\Phi_2\Phi_4 & q\Phi_2 \Phi_2 & \Phi_1 & 1 & . & . \\
. & . & . & 2q^2 & . & 1 & . & . & . & . & . & . & . & q\Phi_2 q 1 & 1 & . & . \\
\end{pmatrix}
$$

(e) Let $G$ be of type $D_4$ with twisted Frobenius such that $G^F = 2D_4(q)$ and $L$ an $F$-stable Levi subgroup of twisted type $2A_3(q).(q-1)$. Then we get

$$\tilde{Q}_L^G = \begin{pmatrix}
\Phi_3\Phi_8 & \Phi_2 & 1 & . & . & . & . \\
. & q^2\Phi_4 & . & 1 & 2 & . & . \\
. & q\Phi_2\Phi_4 & q\Phi_2 \Phi_2 & \Phi_1 & 1 & . & . \\
. & . & . & 2q^2 & . & 1 & . \\
. & . & . & . & . & q\Phi_2 q 1 & 1 & . & . \\
\end{pmatrix}
$$

(f) Let $G$ be of type $D_4$ with triality Frobenius and $L$ an $F$-stable Levi subgroup of type $A_1(q^3).(q-1)$. Then we obtain

$$\tilde{Q}_L^G = \begin{pmatrix}
\Phi_2\Phi_3\Phi_6\Phi_{12} & q^4+q+1 & \Phi_2 & 1 & 1 & . & . \\
. & q^3\Phi_2 & q\Phi_3 & q\Phi_6 & \Phi_2 & 1 & . & . \\
\end{pmatrix},
$$

while for a Levi subgroup of type $A_1(q).(q^3-1)$ one has

$$\tilde{Q}_L^G = \begin{pmatrix}
\Phi_2\Phi_3\Phi_6\Phi_{12} & \Phi_2\Phi_6 & q^3+q^2+1 & \Phi_3 & \Phi_6 & 1 & . & . \\
. & q^4\Phi_2\Phi_6 & q^2\Phi_1\Phi_2 & q(q^2-q-1) & q\Phi_6 & 1 & . \\
\end{pmatrix}.
$$

### 3. The Green functions for $\text{Spin}^+_8(q)$

In this section we determine all 2-parameter Green functions from maximal $F$-stable Levi subgroups of the simply connected groups of type $D_4$ with a split Frobenius, that is, for the groups $\text{Spin}^+_8(q)$ with $q$ an odd prime power. For this, we first study the relation to the Green functions for groups with connected centre.
3.1. Groups with non-connected centre. For groups $G$ with non-connected centre, the number of unipotent classes is bigger than the number of irreducible characters of the Weyl group, so that Proposition 2.5 cannot be applied. Still, additional considerations sometimes lead to a solution.

For this, let $G \hookrightarrow \tilde{G}$ be an $F$-equivariant regular embedding, so that $\tilde{G} = GZ(\tilde{G})$ has connected centre and derived subgroup $[\tilde{G}, \tilde{G}] = G$. Then for any Levi subgroup $L \leq G$, $\tilde{L} := LZ(\tilde{G})$ is a Levi subgroup of $\tilde{G}$, $F$-stable if $L$ is. As in our case, when $G$ is of type $D_4$, all proper Levi subgroups of $G$, and hence of $\tilde{G}$, are of type $A$, all 2-parameter Green functions for $\tilde{G}$ can be computed with Proposition 2.5. Then we just need to descend to the simply connected group $G$.

Let us write $L_G$ for the Lang map on $G$. Now while $L^{-1}_G(U)$ is not necessarily invariant under the action of $G \times \tilde{L}$, it is so under the diagonally embedded subgroup $\Delta(\tilde{L}) \cong \tilde{L}$ of $G \times \tilde{L}$ (with $\Delta(l) = (l, l^{-1})$ for $l \in \tilde{L}$). So $\Delta(\tilde{L})$ acts on the $\ell$-adic cohomology groups $H^i_c(L^{-1}_G(U))$. In particular, the 2-parameter Green functions $Q^G_L$ are invariant under the diagonal action of $\tilde{L}$:

$$Q^G_L(u, v) = Q^L_I(u^l, v^l) \quad \text{for all } l \in \tilde{L}.$$ 

Furthermore, the 2-parameter Green functions $Q^G_L$ are induced from those of $L$ inside $G$ (see [1, Prop. 2.2.1(b)]). This implies relations between the 2-parameter Green functions of $\tilde{G}$ and $G$. Suppose that the unipotent class of $u \in G^F$ splits into $n$ classes of $G^F$ and that the unipotent class of $v \in \tilde{L}^F$ splits into $m$ classes of $L^F$, then by [1, Prop. 2.2.1(b)]

$$Q^G_L(u, v) = \sum_{i=1}^m \dot{Q}^G_L(u, v_i) = \frac{m}{n} \sum_{i=1}^n \dot{Q}^G_L(u_i, v)$$

where $u_i \in G^F$ are the representatives of the $G^F$-classes in $[u]_{G^F}$, and $v_i \in L^F$ are the representatives of the $L^F$-classes in $[u]_{L^F}$.

In the examples that we consider below, five cases appear:

- If $n = 1$ then (3) directly gives
  $$\dot{Q}^G_L(u, v_i) = \frac{1}{m} \dot{Q}^G_L(u, v) \quad \text{for } i = 1, \ldots, m.$$ 

- If $m = 1$ then (3) directly gives
  $$\dot{Q}^G_L(u_i, v) = \dot{Q}^G_L(u, v) \quad \text{for } i = 1, \ldots, n.$$ 

- If $n = m = 2$ then the value $Q := \dot{Q}^G_L(u, v)$ is replaced in the matrix of $\dot{Q}^G_L$ by the submatrix

$$\begin{array}{cc}
v_1 & u_1 \\
v_2 & Q - a \\
u_2 & a \\
u_1 & Q - a \\
\end{array}$$

where $a := \dot{Q}^G_L(u_1, v_1)$. 
• If $n = 4$, $m = 2$ then the value $Q := \tilde{Q}_L^G(u, v)$ is replaced in the matrix of $\tilde{Q}_L^G$ by the submatrix

|   | $u_1$ | $u_2$ | $u_3$ | $u_4$ |
|---|-------|-------|-------|-------|
| $v_1$ | $a_1$ | $a_2$ | $a_3$ | $2Q - a_1 - a_2 - a_3$ |
| $v_2$ | $Q - a_1$ | $Q - a_2$ | $Q - a_3$ | $a_1 + a_2 + a_3 - Q$ |

where $a_i := \tilde{Q}_L^G(u_i, v_1)$ for $i = 1, 2, 3$.

• If $n = 2$, $m = 4$ then the value $Q := \tilde{Q}_L^G(u, v)$ is replaced in the matrix of $\tilde{Q}_L^G$ by the submatrix

|   | $u_1$ | $u_2$ |
|---|-------|-------|
| $v_1$ | $a_1$ | $Q/2 - a_1$ |
| $v_2$ | $a_2$ | $Q/2 - a_2$ |
| $v_3$ | $a_3$ | $Q/2 - a_3$ |
| $v_4$ | $Q - a_1 - a_2 - a_3$ | $a_1 + a_2 + a_3 - Q/2$ |

where $a_i := \tilde{Q}_L^G(u_i, v_i)$ for $i = 1, 2, 3$.

So, every pair of splitting classes introduces some unknown entries in $\tilde{Q}_L^G$. The number of unknowns can be reduced thanks to the diagonal action of $L^F$.

We get the explicit action of $L^F$ by following the construction of Bonnafé in [1] around Proposition 2.2.2. There, he translates the action of an element $l \in L^F/L^F$ (on the unipotent elements of $G^F$ and $L^F$) to the action (by conjugation) of a representative $l_z \in L$ for $z \in H^1(F, Z(L))$ such that $l_z^{-1}F(l_z) = z$. Since there is a canonical surjection $H^1(F, Z) \to H^1(F, Z(L))$ we choose, instead, for all $z \in H^1(F, Z) = Z$ an element $g \in G$ such that $g^{-1}F(g) = z$, where $Z = Z(G)$. This gives us the desired action for all split Levi subgroups. For the non-split ones we replace $F$ by the associated twisted Frobenius map $F'$. So for all $z \in H^1(F, Z) = Z$ we have $g, g' \in G$ such that $g^{-1}F(g) = z = g'^{-1}F'(g')$. Then the action is given by $(u, v) \mapsto (u^g, v^{g'})$ for $u \in G^F$ and $v \in L^{F'}$.

We will see how to find the remaining unknowns in the following examples.

3.2. The Green functions for $\text{SL}_4(q)$ and $\text{SU}_4(q)$. It will be convenient (in view of Lemma 2.1) to first compute the Green functions for the Levi subgroups of types $A_3$ and $2A_3$.

**Proposition 3.1.** Let $q$ be an odd prime power, $G = \text{SL}_4$ when $q \equiv 3 \pmod{4}$, or $G = \text{SL}_4/\langle \pm 1 \rangle$ when $q \equiv 1 \pmod{4}$, and $F$ a split Frobenius with respect to an $\mathbb{F}_q$-structure on $G$. Then

\[
\tilde{Q}_{L_1}^G = \begin{pmatrix}
\Phi_3 \Phi_4 & \Phi_2 & 1 & 1 & \ldots \\
q \Phi_2 & \ldots & 1 & \ldots \\
q^2 \Phi_2 & \ldots & 1 & \ldots \\
\ldots & q \Phi_2 & \ldots & q \Phi_2 & 1 \\
\ldots & \ldots & q \Phi_2 & \ldots & q \Phi_2 & 1 \\
\ldots & \ldots & \ldots & q \Phi_2 & \ldots & q \Phi_2 & 1 \\
\end{pmatrix}
\]

for a split Levi subgroup $L_1$ with $L_1^F = A_1(q)^2.(q - 1)$, and

\[
\tilde{Q}_{L_2}^G = \begin{pmatrix}
\Phi_4^2 \Phi_3 - \Phi_1 & 1 & 1 & \ldots \\
\Phi_4^2 \Phi_3 - \Phi_1 & \ldots & q \Phi_1 & -q \Phi_1 & 1 \\
\ldots & q \Phi_1 & \ldots & q \Phi_1 & -q \Phi_1 & 1 \\
\ldots & \ldots & q \Phi_1 & \ldots & q \Phi_1 & -q \Phi_1 & 1 \\
\ldots & \ldots & \ldots & q \Phi_1 & \ldots & q \Phi_1 & -q \Phi_1 & 1 \\
\ldots & \ldots & \ldots & \ldots & q \Phi_1 & \ldots & q \Phi_1 & -q \Phi_1 & 1 \\
\end{pmatrix}
\]
for a non-split Levi subgroup $L_2$ with $L_2^F = A_1(q^2).(q + 1)$.

Proof. In both cases, $G = G^F$ has centre of order 2 and seven unipotent classes with representatives $u_1, \ldots , u_7$, since the classes of $G$ with Jordan normal forms $2^2$ and 4 both split into two classes in the finite group. Moreover, as $L_1$ is split of type $A_1^2$, the Levi subgroup $L_1^F$ has five unipotent classes with representatives $v_1, \ldots , v_5$ of Jordan types $(1^2,1^2), (1^2,2), (2,1^2)$ and two of type $(2,2)$. Now by the known Green functions for $L_1$ in $G$ (see Example 2.8(b)) and our considerations above, we only need to worry about $u$ and $v$ having two blocks of size 2 or $u$ being regular unipotent. From the explicit realisation in the split case as value of a permutation character (see (2)) one computes that $Q_{L_1}^G(u,v^{-1}) = 0$ for $u,v$ with two Jordan blocks, but not conjugate in $G^F$. Also, $Q_{L_1}^G(u,v^{-1}) = 0$ for $v$ with two Jordan blocks, and $u$ one out of the two regular unipotent classes. So, with respect to a suitable ordering of the splitting classes, we find the stated result.

Now assume that $L_2$ is of non-split type $A_1^2$, so $L_2^F = A_1(q^2).(q + 1)$. The class of regular unipotent elements of $L_2$ splits into two $L_2^F$-classes, with representatives $v_2, v_3$ say. Again, by the known Green functions for $L_2$ in $G$ (see Example 2.8(b)) and our considerations above there are two unknown values in $\tilde{Q}_{L_2}^G$, denoted $a_1 := \tilde{Q}_{L_2}^G(u_3, v_2)$ and $a_2 := \tilde{Q}_{L_2}^G(u_6, v_2)$.

Consider the class function $\psi$ on $L_2^F$ that takes values 1 and $-1$ on $v_2, v_3$ respectively, and zero everywhere else. All $R_{L_2^F}(\theta)$ have equal values on $v_2, v_3$, so $\psi$ is orthogonal to all Deligne–Lusztig characters and hence cuspidal. By the Mackey formula this means, since $|N_G(L_2)^F : L_2^F| = 2$, that $\langle R_{L_2}^G(\psi), R_{L_2}^G(\psi) \rangle = N \langle \psi, \psi \rangle$, with $N$ being equal 1 or 2 (depending on whether $v_2$ and $v_3$ fuse in $G^F$). On the other hand, the norm of $R_{L_2}^G(\psi)$ can be computed from the character formula using the (unknown) values of $\tilde{Q}_{L_2}^G(u,v)$ on the pairs of splitting classes. Then the norm equation above gives that

$$(2a_1 - q\Phi_1)^2 + q^2\Phi_1(2a_2 - 1)^2 = Nq^3\Phi_1.$$ 

Next, let $\xi$ be the class function on $L_1^F$ that takes values 1 resp. $-1$ on the two regular unipotent classes, respectively. Since $\psi$ is cuspidal, the Mackey formula shows that $\langle R_{L_1}^G(\xi), R_{L_2}^G(\psi) \rangle = 0$. Using the known values of $\tilde{Q}_{L_1}^G$, this translates to $a_1 = q\Phi_1(1 - a_2)$. This system of equations has rational solutions only for $N = 2$ (meaning that the classes of $v_2$ and $v_3$ don’t fuse in $G^F$). The solutions of this system are $(a_1, a_2) \in \{(q\Phi_1, 0), (0,1)\}$.

Both correspond to the a matrix as in the statement, but in one case the second and third lines are interchanged. This was to be expected, since we did not choose explicit representatives for the classes of $v_2$ and $v_3$. \hfill \Box

To fix the table for a given choice of representative, we can proceed as follows. Both solutions verify Conjecture 5.2 of [3] which in turn proves Conjecture 5.2’ of the same article in our case. This states that the Lusztig restriction of Gel’fand–Graev characters of $G^F$ are Gel’fand–Graev characters of $L_2^F$ (up to a possible sign, easily chosen by imposing the positivity of the degree). By explicit computation of the Gel’fand–Graev characters this eliminates one of the solutions and thus leads to a unique result.

A similar argument also shows the following:
Proposition 3.2. Let $G = SL_4$ and $G^F = SL_4(q)$ with $q \equiv 1 \pmod{4}$. Then

$$\tilde{Q}^G_{L_1} = \begin{pmatrix} \Phi_3 \Phi_4 & \Phi_2 & 1 & 1 & \cdots & \cdots \\ q^2 & \cdots & \cdots & 1 & \cdots & \cdots \\ q^2 & \cdots & \cdots & 1 & \cdots & \cdots \\ \cdots & q \Phi_2 & \frac{1}{2} \Phi_1 & 1 & 1 & \cdots \\ \cdots & \cdots & q \Phi_2 & \frac{1}{2} \Phi_1 & 1 & 1 \end{pmatrix}$$

for a split Levi subgroup $L_1$ of type $A_1^2$, and

$$\tilde{Q}^G_{L_2} = \begin{pmatrix} \Phi_2 \Phi_3 & -\Phi_1 & 1 & 1 & \cdots & \cdots \\ \cdots & q \Phi_1 & -\frac{1}{2} \Phi_1 & 1 & 1 & \cdots \\ \cdots & \cdots & q \Phi_1 & -\frac{1}{2} \Phi_1 & 1 & 1 \end{pmatrix}$$

for a non-split Levi subgroup $L_2$ of type $A_1^2$.

Proof. This directly follows from Proposition 3.1 and [1, Cor. 2.2.3]. We have checked the condition for the latter explicitly (by replacing the Frobenius morphism with the twisted Frobenius associated with the twisted Levi).

Here is the counter-part of Proposition 3.1 for $SU_4(q)$:

Proposition 3.3. Let $q$ be an odd prime power and $G = SL_4$ when $q \equiv 1 \pmod{4}$, or $G = SL_4/(\pm 1)$ when $q \equiv 3 \pmod{4}$, and $F$ a twisted Frobenius with respect to an $\mathbb{F}_q$-structure on $G$. Then

$$\tilde{Q}^G_{L_1} = \begin{pmatrix} \Phi_6 \Phi_3 & \Phi_6 & 1 & 1 & \cdots & \cdots \\ q^2 \Phi_2 & \cdots & \cdots & 1 & \cdots & \cdots \\ \cdots & q \Phi_2 & \frac{1}{2} \Phi_1 & 1 & 1 & \cdots \\ \cdots & \cdots & q \Phi_2 & \frac{1}{2} \Phi_1 & 1 & 1 \end{pmatrix}$$

for a split Levi subgroup $L_1$ with $L_1^F = A_1^2(q^2).q - 1)$, and

$$\tilde{Q}^G_{L_2} = \begin{pmatrix} \Phi_6 \Phi_3 & -\Phi_1 & 1 & 1 & \cdots & \cdots \\ \cdots & q^2 & \cdots & \cdots & 1 & \cdots \\ \cdots & \cdots & q \Phi_1 & -\frac{1}{2} \Phi_2 & 1 & \cdots \\ \cdots & \cdots & \cdots & q \Phi_1 & -\frac{1}{2} \Phi_2 & 1 \end{pmatrix}$$

for a non-split Levi subgroup $L_2$ with $L_2^F = A_1^2(q^2).q + 1)$.

Proof. Instead of doing the same computation twice for the stated groups and congruences we did the computation once for the Levi subgroups $A_1(q^2).(q^2 - 1)$ and $A_1(q^2).(q + 1)^2$ inside the Levi $2A_3(q)(q + 1)$ of Spin$_8^\pm(q)$. Then, [1, Cor. 2.2.3] assures that the solution is what we claim. The argument is analogous to the proof of Proposition 3.1. By explicit calculations one finds the result for the split case. Then, thanks to scalar products of Lusztig-induced cuspidal class functions and the norm equation one can fix the unknowns for the non-split case. Again the Lusztig restriction of Gel’fand–Graev characters fixes the position of the values. 

□
3.3. The Green functions for Spin$^+_8(q)$. Let $G = \text{Spin}_8$ with a split Frobenius $F$, such that $G^F = \text{Spin}^+_8(q)$ with $q$ odd. Moreover, let $G \rightarrow \tilde{G}$ be a regular embedding. Of the 13 unipotent classes of $G$ six split into two and three split into four classes in $G^F$. By the general discussion at the beginning of the section this means that every splitting unipotent class of a Levi subgroup $L$ introduces at least 9 unknowns in the matrix $\tilde{Q}^G_{L_i}$. It turns out that this number can be reduced to 5 thanks to the diagonal action of $\tilde{L}^F$. So we are missing 5 equations per Levi subgroup and splitting class of that Levi.

The types of maximal Levi subgroups with disconnected centre of Spin$^+_8(q)$ are $M_1 = A_3(q)(q - 1)$, $M_2 = A_1(q)^3(q - 1)$, $M_3 = 2A_3(q)(q + 1)$, $M_4 = A_1(q)^3(q + 1)$ and $L_5 = A_1(q^2)(q^2 + 1)$. It turns out that to compute the Green functions for these Levi we need also to consider minimal Levi subgroups with disconnected centre $L_1 = A_1(q)^2(q - 1)^2$, $L_2 = A_1(q^2)(q^2 - 1)$, $L_3 = A_1(q^2)(q^2 - 1)$ and $L_4 = A_1(q^2)(q + 1)^2$. Figure 1 summarizes the situation by showing the diagram of containments for these Levi subgroups. Each of these has one splitting unipotent class, and we denote by $f_i$ the (cuspidal) class function of $L_i$ that has values 1, -1 on the splitting classes and 0 elsewhere.

**Figure 1.** Subgroup lattice of Levi subgroups with disconnected centre of Spin$^+_8(q)$, up to triality. The lines represent inclusions: a single line corresponds to Levi already treated in Propositions 3.1 and 3.3, a double line indicates a split Levi and, therefore, that the Green function can be computed explicitly using (2), dashed lines indicate non-split Levi for which the Green functions must be computed with methods similar to those used in the proof of Proposition 3.1.

The reason why we need $L_1$ to $L_4$ is because we want to emulate the proof of Proposition 3.1. While

$$\langle R^G_{L_i} f_i, R^G_{L_j} f_j \rangle = 0 \quad \text{for } i, j = 1, \ldots, 5, \ i \neq j$$

by the Mackey formula, an analogous formula for $M_i, i = 1, \ldots, 4$, need not hold since a class function $f$ of $M_i$ taking 1, -1 on splitting unipotent classes is not cuspidal. However,

$$\langle R^G_{M_i} f, R^G_{L_j} f_j \rangle = 0 \quad \text{for all } i, j \text{ such that } L_j \not\subset M_i$$

again, by the Mackey formula.
In conclusion, the plan to get the 2-parameter Green functions of Spin\(_8^+(q)\) is the following. First, we determine \(\tilde{Q}_G^L\) for the split Levi subgroups \(L\) by explicit computations with (2). Second, we compute \(\tilde{Q}_G^L\) for \(i = 2, \ldots, 5\) thanks to Propositions 3.1 and 3.3 and Lemma 2.4. Then using this knowledge and Lemma 2.4 we get \(\tilde{Q}_G^L\) for the maximal Levi subgroups \(L\).

Unfortunately, for the non-split Levis only 4 of the 5 equations (one of which is the norm equation) needed to find the unknowns can be obtained by the procedure described above. However, Digne, Lehrer and Michel prove in [4, Thm. 3.7] and [3, Conj. 5.2 and 5.2'] that, when \(q\) is "large enough" (\(q > q_0\) for \(q_0\) a constant depending only on the Dynkin diagram), for all regular unipotent elements \(u\) of \(G^F\), \(\tilde{Q}_G^L(u, v) = 1\) for exactly one regular unipotent class \((v)^L\) and \(\tilde{Q}_G^L(u, v') = 0\) for all other regular unipotent elements \(V'\) of \(L^F\).

This is gives the last equation.

**Theorem 3.4.** The 2-parameter Green functions for \(G = \text{Spin}_8^+(q)\), \(q\) odd, satisfy:

(a) For the maximal \(F\)-stable split Levi subgroups \(L\) of \(G\), \(\tilde{Q}_G^L\) is given in Tables 1–2,

(b) For the maximal \(F\)-stable non-split Levi subgroups \(L\) of \(G\), \(\tilde{Q}_G^L\) is given in Tables 3–5 for large enough \(q\).

Representatives for all the unipotent classes are given in the Steinberg presentation in Tables 6–9, the numbering is chosen such that the centre node of the Dynkin diagram is labelled 3.

The tables for Spin\(_8^+(q)\) and for the Levi subgroups of type \(A_3^1\) also show the action of the triality automorphism on the unipotent classes.

**Proof.** Part (a) follows by explicit computation of the fusion of unipotent classes of \(G^F\) and an application of (2). This gives the 2-parameter Green functions for \(M_1\), \(M_2\) and \(L_1\). Thanks to Lemma 2.4 and Proposition 3.1 we also obtain \(\tilde{Q}_G^L\).

Analogously to Proposition 3.1 we can compute \(\tilde{Q}_G^{M_2}\) (explicitly) and \(\tilde{Q}_G^{L_3}\) (by scalar products). Again, Lemma 2.4 now gives \(\tilde{Q}_G^L\).

The situation is now graphically summarized in Figure 1. The Levi subgroups in solid boxes are those with known Green functions (which are valid for all odd \(q\)), while those in dashed boxes have to be found and, unfortunately, will be shown to be valid only for large enough \(q\). The reason is the following. Both \(\tilde{Q}_G^{L_4}\) and \(\tilde{Q}_G^{L_5}\) have 5 unknown entries and in total we get 7 scalar products of the form

\[\langle R_{L_1}^G f_i, R_{L_2}^G f_j \rangle = 0 \quad \text{for } i \neq j,\]

plus two norm equations. Since (as can be seen from Figure 1) equations of the form

\[\langle R_{M_i}^G f, R_{L_j}^G f_j \rangle = 0 \quad \text{for } L_j \nsubseteq M_i\]

do not provide new information, the only course of action that we see, at present, is to use the above mentioned result of Digne, Lehrer and Michel.

Setting \(\tilde{Q}_G^L(u, v)\) to 1 or 0 for pairs of regular unipotent elements allows us to solve the system of equations. Then, the Lusztig restriction of explicitly computed Gel’fand–Graev characters allows us to select the only correct solution. We mention, however, that the norm equation now yields two rational solutions. For \(q > 3\), only one of them
satisfies $|C_L(v)|\tilde{Q}_{GL}(u,v) \in \mathbb{Z}$ (which has to hold by the definition of the modified Green functions.

Thanks to the knowledge of $\tilde{Q}_{GL}$ for $i = 2, 3, 4$ we can repeatedly use Lemma 2.4 to determine the remaining Green functions.

The unipotent conjugacy classes are computed explicitly by fusing the classes of a maximal unipotent subgroup under elements of a maximally split torus and elements of the Weyl group. The precise action of the triality automorphism follows from the knowledge of the fusion of the unipotent classes. □

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Table 1. $\tilde{Q}_L^G(u, v)$ for the split Levi subgroup $A_3(q)(q - 1)$ of Spin$_8^+(q)$.

|   | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ | $u_7$ | $u_8$ | $u_9$ | $u_{10}$ | $u_{11}$ | $u_{12}$ |
|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|----------|
| $v_1$ | $\Phi_2 \Phi_4 \Phi_6$ | $\Phi_2$ | . | . | . | 1 | 1 | . | . | . | . | . |
| $v_2$ | . | $q^2 \Phi_2$ | $\Phi_2 \Phi_4$ | $\Phi_2 \Phi_4$ | $\Phi_2 \Phi_4$ | . | . | 1 | 1 | 1 | 1 |
| $v_3$ | . | . | . | . | . | $q \Phi_2 \Phi_4$ | $q \Phi_2$ | $q \Phi_2$ | . |
| $v_4$ | . | . | . | . | . | . | $q \Phi_2 \Phi_4$ | $q \Phi_2$ | $q \Phi_2$ |
| $v_5$ | . | . | . | . | . | . | . | . | . | . | . | . |
| $v_6$ | . | . | . | . | . | . | . | . | . | . | . | . |
| $v_7$ | . | . | . | . | . | . | . | . | . | . | . | . |

|   | $u_{13}$ | $u_{14}$ | $u_{15}$ | $u_{16}$ | $u_{17}$ | $u_{18}$ | $u_{19}$ | $u_{20}$ | $u_{21}$ | $u_{22}$ | $u_{23}$ | $u_{24}$ | $u_{25}$ | $u_{26}$ | $u_{27}$ | $u_{28}$ |
|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $v_1$ | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $v_2$ | . | 2 | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $v_3$ | $\frac{2}{\sqrt{2}} \Phi_2$ | $\frac{2}{\sqrt{2}} \Phi_2$ | . | . | . | 1 | . | . | . | . | . | . | . | . | . |
| $v_4$ | $\frac{2}{\sqrt{2}} \Phi_2$ | $\frac{2}{\sqrt{2}} \Phi_2$ | . | . | . | 1 | . | . | . | . | . | . | . | . | . |
| $v_5$ | $2q^2$ | $\Phi_2$ | $\Phi_2$ | $\Phi_2$ | $\Phi_2$ | . | 1 | 1 | 1 | 1 | . | . | . | . | . |
| $v_6$ | . | . | . | . | . | $q \Phi_2$ | . | 1 | 1 | 1 | . | . | . | . | . |
| $v_7$ | . | . | . | . | . | $q \Phi_2$ | . | 1 | 1 | 1 | . | . | . | . | . |

$q \equiv 1 \text{ (mod } 4)$

|   | $u_{21}$ | $u_{22}$ | $u_{23}$ | $u_{24}$ |
|---|-------|-------|-------|-------|
| $v_6$ | $q$ | . | $q$ | . |
| $v_7$ | $q$ | . | $q$ | . |

$q \equiv 3 \text{ (mod } 4)$

|   | $u_{21}$ | $u_{22}$ | $u_{23}$ | $u_{24}$ |
|---|-------|-------|-------|-------|
| $v_6$ | . | $q$ | . | $q$ |
| $v_7$ | $q$ | . | $q$ | . |
**Table 2.** $\hat{Q}_L^G(u,v)$ for the split Levi subgroup $A_1(q)^3(q-1)$ of $\text{Spin}^+_8(q)$.

| $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ | $u_7$ | $u_8$ | $u_9$ | $u_{10}$ | $u_{11}$ | $u_{12}$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|----------|----------|
| $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ |
| $\Phi_2$ | $\Phi_2$ | $\Phi_2$ | $\Phi_2$ | $\Phi_2$ | $\Phi_2$ | $\Phi_2$ | $\Phi_2$ | $\Phi_2$ | $\Phi_2$ | $\Phi_2$ | $\Phi_2$ |
| $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ |
| $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ | $\Phi_2\Phi_3\Phi_4\Phi_5$ |

**Table 2.** $\hat{Q}_L^G(u,v)$ for the split Levi subgroup $A_1(q)^3(q-1)$ of $\text{Spin}^+_8(q)$.

| $u_{13}$ | $u_{14}$ | $u_{15}$ | $u_{16}$ | $u_{17}$ | $u_{18}$ | $u_{19}$ | $u_{20}$ | $u_{21}$ | $u_{22}$ | $u_{23}$ | $u_{24}$ |
|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 1        | 1        | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        |
| 2        | 2$\Phi_2$ | .        | .        | 1        | 1        | .        | .        | .        | .        | .        | .        |
| 3        | 2$\Phi_2$ | .        | 1        | 1        | .        | .        | .        | .        | .        | .        | .        |
| 4        | 2$\Phi_2$ | .        | .        | .        | 1        | 1        | .        | .        | .        | .        | .        |
| 5        | $\Phi_2$ | .        | .        | .        | .        | 1        | 1        | .        | .        | .        | .        |
| 6        | $\Phi_2$ | .        | .        | .        | .        | .        | 1        | 1        | .        | .        | .        |
| 7        | $\Phi_2$ | .        | .        | .        | .        | .        | .        | 1        | 1        | .        | .        |
| 8        | $\Phi_2$ | .        | .        | .        | .        | .        | .        | .        | 1        | 1        | .        |
| 9        | $\Phi_2$ | .        | .        | .        | .        | .        | .        | .        | .        | 1        | 1        |
| 10       | $\Phi_2$ | .        | .        | .        | .        | .        | .        | .        | .        | .        | 1        |
| 11       | $\Phi_2\Phi_3\Phi_4\Phi_5$ | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        |
| 12       | $\Phi_2\Phi_3\Phi_4\Phi_5$ | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        |
| 13       | $\Phi_2\Phi_3\Phi_4\Phi_5$ | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        |
| 14       | $\Phi_2\Phi_3\Phi_4\Phi_5$ | .        | .        | .        | .        | .        | .        | .        | .        | .        | .        |

*$s = q^4 + 3q^3 + 3q^2 + q + 1$
Table 3. (Modified) 2-parameter Green functions for $A_3(q)(q + 1)$ for $q$ large.

| $v_1$ | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ | $u_7$ | $u_8$ | $u_9$ | $u_{10}$ | $u_{11}$ | $u_{12}$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-----------|-----------|-----------|
| $v_2$ | $q^3\Phi_4 - \Phi_4 - \Phi_3\Phi_4 - \Phi_1\Phi_4 - \Phi_1\Phi_4$ | . | . | . | . | 1 | . | . | . | . | . | . |
| $v_3$ | $q\Phi_3\Phi_4 - q\Phi_1 - q\Phi_1$ | . | . | . | . | . | . | . | . | . | . | . |
| $v_4$ | $q\Phi_3\Phi_4 - q\Phi_1 - q\Phi_1$ | . | . | . | . | . | . | . | . | . | . | . |
| $v_5$ | $q\Phi_3\Phi_4 - q\Phi_1 - q\Phi_1$ | . | . | . | . | . | . | . | . | . | . | . |
| $v_6$ | $q\Phi_3\Phi_4 - q\Phi_1 - q\Phi_1$ | . | . | . | . | . | . | . | . | . | . | . |
| $v_7$ | $q\Phi_3\Phi_4 - q\Phi_1 - q\Phi_1$ | . | . | . | . | . | . | . | . | . | . | . |

$q \equiv 1 \pmod{4}$

| $v_6$ | $-q - q$ | $-q$ | $-q$ | $-q$ |
| $v_7$ | $-q - q - q$ | $-q - q$ |

$q \equiv 3 \pmod{4}$

| $v_6$ | $-q - q - q$ | $-q - q$ | $-q - q$ |
| $v_7$ | $-q - q - q$ | $-q - q$ | $-q - q$ |
Table 4. (Modified) 2-parameter Green functions for $A_1(q)^3(q + 1)$ for $q$ large.

|       | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ | $u_7$ | $u_8$ |
|-------|------|------|------|------|------|------|------|------|
| $v_1$ | $-\Phi_1\Phi_1\Phi_1\Phi_6$ | $*\Phi_1\Phi_1\Phi_1\Phi_4$ | $-\Phi_1\Phi_4$ | $-\Phi_1\Phi_4$ | $-\Phi_1\Phi_4$ | $-\Phi_1\Phi_4$ | $-\Phi_1\Phi_4$ | $-\Phi_1\Phi_4$ |
| $v_2$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ |
| $v_3$ | $-q^3\Phi_1$ | $-q^3\Phi_1$ | $-q^3\Phi_1$ | $-q^3\Phi_1$ | $-q^3\Phi_1$ | $-q^3\Phi_1$ | $-q^3\Phi_1$ | $-q^3\Phi_1$ |
| $v_4$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ |
| $v_5$ | $-q^3\Phi_1$ | $-q^3\Phi_1$ | $-q^3\Phi_1$ | $-q^3\Phi_1$ | $-q^3\Phi_1$ | $-q^3\Phi_1$ | $-q^3\Phi_1$ | $-q^3\Phi_1$ |
| $v_6$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ |
| $v_7$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ |
| $v_8$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ |
| $v_9$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ |
| $v_{10}$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ |
| $v_{11}$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ |
| $v_{12}$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ |
| $v_{13}$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ |
| $v_{14}$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ | $q^3\Phi_1$ |

$q \equiv 1 \pmod{4}$

|       | $u_{21}$ | $u_{22}$ | $u_{23}$ | $u_{24}$ | $u_{25}$ | $u_{26}$ | $u_{27}$ | $u_{28}$ |
|-------|------|------|------|------|------|------|------|------|
| $v_1$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| $v_2$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| $v_3$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| $v_4$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| $v_5$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| $v_6$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| $v_7$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| $v_8$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| $v_9$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| $v_{10}$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| $v_{11}$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| $v_{12}$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| $v_{13}$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |
| $v_{14}$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ | $*$ |

$q \equiv 3 \pmod{4}$

|       | $u_{21}$ | $u_{22}$ | $u_{23}$ | $u_{24}$ |
|-------|------|------|------|------|
| $v_1$ | $*$ | $*$ | $*$ | $*$ |
| $v_2$ | $*$ | $*$ | $*$ | $*$ |
| $v_3$ | $*$ | $*$ | $*$ | $*$ |
| $v_4$ | $*$ | $*$ | $*$ | $*$ |
| $v_5$ | $*$ | $*$ | $*$ | $*$ |
| $v_6$ | $*$ | $*$ | $*$ | $*$ |
| $v_7$ | $*$ | $*$ | $*$ | $*$ |
| $v_8$ | $*$ | $*$ | $*$ | $*$ |
| $v_9$ | $*$ | $*$ | $*$ | $*$ |
| $v_{10}$ | $*$ | $*$ | $*$ | $*$ |
| $v_{11}$ | $*$ | $*$ | $*$ | $*$ |
| $v_{12}$ | $*$ | $*$ | $*$ | $*$ |
| $v_{13}$ | $*$ | $*$ | $*$ | $*$ |
| $v_{14}$ | $*$ | $*$ | $*$ | $*$ |

$* = q^4 - 3q^3 + 3q^2 - q + 1$
Table 5. (Modified) 2-parameter Green functions for $A_1(q^2)(q^2 + 1)$ for $q$ large.

\[
\begin{array}{cccccccccc}
& u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \\
\hline
v_1 & -\Phi_1 \Phi_2 \Phi_3 \Phi_6 & \Phi_1^2 \Phi_5^2 & \Phi_1 \Phi_2^2 & \Phi_1^2 \Phi_5^2 & \Phi_1 \Phi_2^2 & \Phi_1^2 \Phi_5^2 & \Phi_1 \Phi_2 & \Phi_2 \\
v_2 & - & - & - & - & - & - & - & - \\
v_3 & - & - & - & - & - & - & - & - \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
& u_9 & u_{10} & u_{11} & u_{12} & u_{13} & u_{14} & u_{15} & u_{16} & u_{17} & u_{18} & u_{19} & u_{20} \\
\hline
v_1 & -\Phi_1 \Phi_2 & -\Phi_1 \Phi_2 & -\Phi_1 \Phi_2 & -\Phi_1 \Phi_2 & -\Phi_1 \Phi_2 & -\Phi_1 \Phi_2 & -\Phi_1 \Phi_2 & -\Phi_2 \\
v_2 & q^2 \Phi_1 \Phi_2 & q^2 \Phi_1 \Phi_2 & q^2 \Phi_1 \Phi_2 & q^2 \Phi_1 \Phi_2 & q^2 \Phi_1 \Phi_2 & q^2 \Phi_1 \Phi_2 & q^2 \Phi_1 \Phi_2 & q^2 \Phi_2 \\
v_3 & q^2 \Phi_1 \Phi_2 & q^2 \Phi_1 \Phi_2 & q^2 \Phi_1 \Phi_2 & q^2 \Phi_1 \Phi_2 & q^2 \Phi_1 \Phi_2 & q^2 \Phi_1 \Phi_2 & q^2 \Phi_1 \Phi_2 & q^2 \Phi_2 \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
& u_{21} & u_{22} & u_{23} & u_{24} & u_{25} & u_{26} & u_{27} & u_{28} \\
\hline
v_1 & - & - & - & - & - & - & - & - \\
v_2 & - & - & - & - & - & - & - & - \\
v_3 & - & - & - & - & - & - & - & - \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
& u_{21} & u_{22} & u_{23} & u_{24} \\
\hline
v_2 & \frac{q}{2} & -\frac{q}{2} & \frac{q}{2} & -\frac{q}{2} \\
v_3 & -\frac{q}{2} & \frac{q}{2} & -\frac{q}{2} & \frac{q}{2} \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
& u_{21} & u_{22} & u_{23} & u_{24} \\
\hline
v_2 & -\frac{q}{2} & \frac{q}{2} & -\frac{q}{2} & \frac{q}{2} \\
v_3 & -\frac{q}{2} & \frac{q}{2} & -\frac{q}{2} & \frac{q}{2} \\
\end{array}
\]

$q \equiv 1 \pmod{4}$

$q \equiv 3 \pmod{4}$
Table 6. Unipotent classes of Spin$_q^+(q)$. The table shows the Jordan normal form, the numbering of the class, a representative in Steinberg presentation ($\langle \mu \rangle = F_\rho^\times$) and the image under the triality automorphism. On the classes with Jordan form 53 the triality automorphism acts differently depending on the congruence of $q$. The action for $q \equiv 1 \text{ (4)}$ is given first.

\[
\begin{array}{|c|c|c|}
\hline
1^8 & u_1 & 1 \\
\hline
2^4^+ & u_2 & u_1(1) \\
\hline
2^4^- & u_3 & u_1(1)u_2(1) \\
     & u_4 & u_1(\mu)u_2(1) \\
\hline
3^5 & u_5 & u_1(1)u_4(1) \\
     & u_6 & u_1(\mu)u_4(1) \\
\hline
32^1 & u_7 & u_2(1)u_4(1) \\
     & u_8 & u_2(\mu)u_4(1) \\
\hline
32^1 & u_9 & u_1(1)u_2(1)u_4(1) \\
     & u_{10} & u_1(1)u_2(\mu)u_4(1) \\
     & u_{11} & u_1(\mu)u_2(1)u_4(1) \\
     & u_{12} & u_1(\mu)u_2(\mu)u_4(1) \\
\hline
32^1 & u_{13} & u_1(1)u_3(1) \\
\hline
\hline
\end{array}
\]

Table 7. Representatives of the unipotent classes of the Levi subgroups $A_3(q)(q - 1)$ and $2A_3(q)(q + 1)$, where $\langle \rho \rangle = F_\rho^\times$ and $\mu = \rho^{q+1}$.

\[
\begin{array}{|c|c|}
\hline
v_1 & 1 \\
\hline
v_3 & u_2(1) \\
\hline
v_4 & u_2(\mu) \\
\hline
v_5 & u_2(1)u_3(1) \\
\hline
v_6 & u_2(1)u_3(1)u_4(1) \\
\hline
v_7 & u_2(\mu)u_3(1)u_4(1) \\
\hline
v_2 & u_3(1) \\
\hline
v_3 & u_2(1)u_4(1) \\
\hline
v_4 & u_2(\rho)u_4(\rho^\theta) \\
\hline
v_5 & u_2(\rho)u_4(\rho^\theta)u_6(1)u_7(1) \\
\hline
v_6 & u_2(1)u_4(1)u_3(1) \\
\hline
v_7 & u_2(\rho)u_4(\rho^\theta)u_3(1) \\
\hline
\end{array}
\]
Table 8. Representatives of the unipotent classes of the Levi subgroups $A_1(q)^3(q-1)$ and $A_1(q)^3(q+1)$ with the action of triality, where $\langle \mu \rangle = \mathbb{F}_q^\times$.

| $v_1$ | 1 | $v_1$ |
|-------|---|-------|
| $v_2$ | $u_1(1)$ | $v_4$ |
| $v_3$ | $u_2(1)$ | $v_2$ |
| $v_4$ | $u_4(1)$ | $v_3$ |
| $v_5$ | $u_1(1)u_2(1)$ | $v_9$ |
| $v_6$ | $u_1(\mu)u_2(1)$ | $v_{10}$ |
| $v_7$ | $u_3(1)u_4(1)$ | $v_5$ |
| $v_8$ | $u_4(\mu)u_3(1)$ | $v_6$ |
| $v_9$ | $u_2(1)u_4(1)$ | $v_7$ |
| $v_{10}$ | $u_2(\mu)u_4(1)$ | $v_8$ |
| $v_{11}$ | $u_3(1)u_2(1)u_4(1)$ | $v_{11}$ |
| $v_{12}$ | $u_1(1)u_2(\mu)u_4(1)$ | $v_{14}$ |
| $v_{13}$ | $u_1(\mu)u_2(1)u_4(1)$ | $v_{12}$ |
| $v_{14}$ | $u_1(\mu)u_2(\mu)u_4(1)$ | $v_{13}$ |

Table 9. Representatives of the unipotent classes of a non-split Levi subgroup of type $A_1(q^2)(q^2+1)$, where $\langle \rho \rangle = \mathbb{F}_{q^2}^\times$.

| $v_1$ | 1 |
|-------|---|
| $v_2$ | $u_2(1)u_4(1)$ |
| $v_3$ | $u_2(\rho)u_4(\rho^q)$ |