CRystal Graphs for General Linear Lie Superalgebras
And Quasi-Symmetric Functions

JAE-HOON KWON

Abstract. We give a new representation theoretic interpretation of the ring of quasi-symmetric functions. This is obtained by showing that the super analogue of the Gessel’s fundamental quasi-symmetric function can be realized as the character of an irreducible crystal for the Lie superalgebra \(\mathfrak{gl}_{m|n}\) associated to its non-standard Borel subalgebra with a maximal number of odd isotropic simple roots. We also present an algebraic characterization of these super quasi-symmetric functions.

1. Introduction

The notion of a quasi-symmetric function is a generalization of a symmetric function introduced by Gessel [6]. The ring of quasi-symmetric functions \(QSym\) has many interesting features, for example, it is a Hopf algebra whose dual is isomorphic to the ring of non-commutative symmetric functions introduced by Gefand et al. [5]. As in the case of symmetric functions, \(QSym\) and its dual have nice representation theoretic interpretations related to the representations of degenerate Hecke algebras and quantum groups of type A [16, 17]. In [17], Krob and Thibon showed that Gessel’s fundamental quasi-symmetric functions in \(n\) variables, which form a basis of \(QSym\) in \(n\) variables, are the characters of the irreducible polynomial representations of the degenerate quantum group associated to \(\mathfrak{gl}_n\). Moreover, the action of the degenerate quantum group determines a quasi-crystal graph structure on the irreducible polynomial representation, which is a subgraph of the Kashiwara’s crystal graph of Young tableaux (see also [28] for a general review on this topic).

The purpose of the present paper is to give a new representation theoretic interpretation of \(QSym\) using the crystal base theory for contragredient Lie superalgebras developed by Benkart, Kang, and Kashiwara [1]. This work was motivated by observing that unlike finite dimensional complex simple Lie algebras and their crystals, two Borel subalgebras of a general linear Lie superalgebra \(\mathfrak{gl}_{m|n}\) are not necessarily conjugate to each other, and the corresponding \(\mathfrak{gl}_{m|n}\)-crystal structures are different in general since they may have different systems of simple roots. Among the Borel conjugacy classes of \(\mathfrak{gl}_{m|n}\), we study in detail a crystal structure when the simple roots associated to a Borel subalgebra are all odd and hence isotropic, while the crystals associated to a standard Borel subalgebra of \(\mathfrak{gl}_{m|n}\) with a single odd simple root is discussed explicitly in [1]. Then it turns out that the

\[\text{2000 Mathematics Subject Classification.} \ 17B37; 05E10.\]

This research was supported by KRF Grant 2005-070-C00004.
combinatorics of these non-standard crystals is remarkably simple and nice. Our main result is that as a $\mathfrak{gl}_{m|n}$-crystal associated to this non-standard Borel conjugacy class, the connected components in the crystal of tensor powers of the natural representation are parameterized by compositions, and each of them can be realized as the set of (super) quasi-ribbon tableaux of a given composition shape (cf. [17]). An insertion algorithm for these non-standard crystals can be derived in a standard way, and it is shown to be compatible with our crystal structure. Hence, we obtain an explicit description of the Knuth equivalence or crystal equivalence for this case. Furthermore, we show that the set of Stembridge’s enriched $P$-partitions [27] is naturally equipped with this non-standard crystal structure, and then apply its combinatorial properties to decomposition of crystals including tensor product decompositions. As a corollary, it follows immediately that the irreducible characters of these non-standard $\mathfrak{gl}_{m|n}$-crystals are equal to the super analogue of fundamental quasi-symmetric functions, which are defined by a standard method of superization [7], and they form a ring isomorphic to $\mathbb{Q}Sym$ under a suitable limit.

One may regard the notion of non-standard $\mathfrak{gl}_{m|n}$-crystals as the counterpart of the Krob and Thibon’s quasi-crystals, and hence as a quasi-analogue of standard $\mathfrak{gl}_{m|n}$-crystals. In this sense, one can define a quasi-analogue of a standard $\mathfrak{gl}_{m|n}$-crystal for arbitrary $m$ and $n$ by a crystal associated to its Borel conjugacy class having a maximal number of odd isotropic simple roots. This enables us to explain the relation between symmetric functions and quasi-symmetric functions as a special case of branching rules between $\mathfrak{gl}_{m|n}$-crystals associated to a standard and a non-standard Borel conjugacy classes. We discuss a crystal structure associated to a Borel conjugacy class having a maximal number of odd isotropic simple roots, and classify its connected components occurring in the crystal of tensor powers of the natural representation, which are parameterized by certain pairs of a partition and a composition. Then we obtain an explicit branching decomposition of standard $\mathfrak{gl}_{m|n}$-crystals into non-standard ones.

The paper is organized as follows. In Section 2, we introduce the notion of abstract crystals for general linear Lie superalgebras and recall some basic properties. In Section 3, we discuss in detail a crystal structure associated to a Borel subalgebra whose simple roots are all odd isotropic. In Section 4, we study a non-standard crystal structure on the set of enriched $P$-partitions and its applications. In Section 5, we consider in general a crystal structure associated to a Borel conjugacy class with a maximal number of odd isotropic simple roots. Finally, in Section 6, we give an algebraic characterization of irreducible characters of non-standard crystals discussed in the previous section.

2. Crystal graphs for general linear Lie superalgebras

2.1. Lie superalgebra $\mathfrak{gl}_S$ and crystal graphs. Suppose that $S$ is a $\mathbb{Z}_2$-graded set with a linear ordering $\prec$. Let $\mathbb{C}^S$ be the complex superspace with a basis $\{v_b | b \in S\}$. Let $\mathfrak{gl}_S = \mathfrak{gl}(\mathbb{C}^S)$ denote the Lie superalgebra of complex linear transformations on $\mathbb{C}^S$ that vanish on a subspace of finite codimension. We call $\mathfrak{gl}_S$ a general linear Lie superalgebra (cf. [9]). In particular, when $S = \{ k \in \mathbb{Z}^\times | -m \leq k \leq n \} = [m|n]$ $(m, n \geq 0)$ with a usual
linear ordering, \( S_0 = \{-m, \ldots, -1\} \) and \( S_1 = \{1, \ldots, n\} \). \( \mathfrak{gl}_S \) is often denoted by \( \mathfrak{gl}_{m|n} \). We may identify \( \mathfrak{gl}_S \) with the space of complex matrices generated by the elementary matrices \( e_{ab} (a, b \in S) \). Then \( \mathfrak{g} = \sum_{b \in S} C e_{bb} \) is a Cartan subalgebra. Let \( \langle , \rangle \) be the natural pairing on \( \mathfrak{g}^* \times \mathfrak{h} \). Let \( \varepsilon_a \in \mathfrak{g}^* \) be determined by \( \langle e_a, \varepsilon_b \rangle = \delta_{ab} \) for \( a, b \in S \). Let \( P = \bigoplus_{b \in S} \mathbb{Z} e_b \) be the weight lattice for \( \mathfrak{gl}_S \), which is a free abelian group generated by \( e_b \) \((b \in S)\). There is a natural symmetric \( \mathbb{Z} \)-bilinear form \( ( , ) \) on \( P \) given by \( (\varepsilon_a, \varepsilon_b) = (-1)^{|a|} \delta_{ab} \) for \( a, b \in S \), where \(|a|\) denotes the degree of \( a \).

Let \( b \) be the subalgebra of \( \mathfrak{gl}_S \) spanned by \( e_{ab} \) for \( a \preceq b \in S \), which we call a Borel subalgebra. The set of positive simple roots and the set of positive roots of \( b \) with respect to \( \mathfrak{h} \) are given by

\[
\Delta (\text{or } \Delta_S) = \{ \varepsilon_b - \varepsilon_{b'} \text{ for a successive pair } b \prec b' \in S \},
\]

\[
\Phi (\text{or } \Phi_S) = \{ \varepsilon_b - \varepsilon_{b'} \text{ for } b \prec b' \in S \}.
\]

Note that \((\alpha, \alpha) = \pm 2, 0 \) for \( \alpha \in \Phi \). We put \( \ell_\alpha = (-1)^{|\alpha|} \) for \( \alpha = \varepsilon_b - \varepsilon_{b'} \in \Delta \). Let \( Q = \bigoplus_{\alpha \in \Delta} \mathbb{Z} \alpha \) be the root lattice of \( \mathfrak{gl}_S \). A partial ordering on \( P \) with respect to \( Q \) is given by \( \lambda \geq \mu \) if and only if \( \lambda - \mu \in \sum_{\alpha \in \Delta} \mathbb{Z} \alpha \) for \( \lambda, \mu \in P \).

Now, let us introduce the notion of abstract crystal graphs for \( \mathfrak{gl}_S \). Our definition is based on the crystal base theory for the quantized enveloping algebra of a contragredient Lie superalgebra developed by Benkart, Kang, and Kashiwara [1]. Throughout the paper, \( 0 \) denotes a formal symbol.

**Definition 2.1.** A crystal graph for \( \mathfrak{gl}_S \), or simply \( \mathfrak{gl}_S \)-crystal, is a set \( B \) together with the maps \( \text{wt} : B \to P, \varepsilon_\alpha, \varphi_\alpha : B \to \mathbb{Z}_{\geq 0}, e_\alpha, f_\alpha : B \to B \cup \{0\} \) \((\alpha \in \Delta)\) such that for \( b, b' \in B \),

1. if \( \alpha \) is isotropic (that is, \((\alpha, \alpha) = 0\)), then
   \[
   \varphi_\alpha(b) + \varepsilon_\alpha(b) = \begin{cases} 
   1, & \text{if } (\text{wt}(b), \alpha) \neq 0, \\
   0, & \text{otherwise},
   \end{cases}
   \]

2. if \( \alpha \) is non-isotropic, then \( \varphi_\alpha(b) - \varepsilon_\alpha(b) = \ell_\alpha(\text{wt}(b), \alpha), \)
3. if \( e_\alpha b \in B \), then \( \varepsilon_\alpha(e_\alpha b) = \varepsilon_\alpha(b) - 1, \varphi_\alpha(e_\alpha b) = \varphi_\alpha(b) + 1, \) and \( \text{wt}(e_\alpha b) = \text{wt}(b) + \alpha, \)
4. if \( f_\alpha b \in B \), then \( \varepsilon_\alpha(f_\alpha b) = \varepsilon_\alpha(b) + 1, \varphi_\alpha(f_\alpha b) = \varphi_\alpha(b) - 1, \) and \( \text{wt}(f_\alpha b) = \text{wt}(b) - \alpha, \)
5. \( f_\alpha b = b' \) if and only if \( b = e_\alpha b' \).

A \( \mathfrak{gl}_S \)-crystal \( B \) becomes a \( \Delta \)-colored oriented graph, where \( b \preceq b' \) if and only if \( b' = f_\alpha b \) \((\alpha \in \Delta)\). We call \( e_\alpha, f_\alpha \) the Kashiwara operators.

**Remark 2.2.** (1) If \((\alpha, \alpha) = 2\) for all \( \alpha \in \Delta \), or \( S_0 = S \), then \( \Phi \) is equal to the set of positive roots of type \( A_{n-1} \) with \( n = |S| \), and \( \mathfrak{gl}_S \)-crystals are the crystal graphs for \( A_{n-1} \) (cf. [11] [12]). On the other hand, if \((\alpha, \alpha) = -2\) (equivalently, \( \alpha \) is non-isotropic and \( \ell_\alpha = -1 \)) for all \( \alpha \in \Delta \), then \( \Phi \) can be identified with the set of negative roots of type \( A_{n-1} \), and \( \mathfrak{gl}_S \)-crystals are dual crystal graphs for \( A_{n-1} \).

(2) For an isotropic simple root \( \alpha \), we have \( \varepsilon_\alpha(b) = 1 \) (resp. \( \varphi_\alpha(b) = 1 \)) if \( e_\alpha b \neq 0 \) (resp. \( f_\alpha b \neq 0 \)), and \( e_\alpha^2 b = f_\alpha^2 b = 0 \) (cf. [1]).
Suppose that $B_1$ and $B_2$ are $\mathfrak{gl}_s$-crystals. We say that $B_1$ is isomorphic to $B_2$, and write $B_1 \simeq B_2$ if there is an isomorphism of $\Delta$-colored oriented graphs which preserves $\text{wt}$, $\varepsilon_\alpha$, and $\varphi_\alpha$ ($\alpha \in \Delta$). For $b_i \in B_i$ ($i = 1, 2$), let $C(b_i)$ denote the connected component of $b_i$ as a $\Delta$-colored oriented graph. We say that $b_1$ is $\mathfrak{gl}_s$-equivalent to $b_2$, if there is an isomorphism of crystal graphs $C(b_1) \to C(b_2)$ sending $b_1$ to $b_2$, and write $b_1 \simeq b_2$ for short.

We define the tensor product $B_1 \otimes B_2 = \{ b_1 \otimes b_2 \mid b_i \in B_i \ (i = 1, 2) \}$ as follows:

$$\text{wt}(b_1 \otimes b_2) = \text{wt}(b_1) + \text{wt}(b_2),$$

(1) for an isotropic simple root $\alpha \in \Delta$,

$$\chi_\alpha(b_1 \otimes b_2) = \begin{cases} \chi_\alpha(b_1), & \text{if } (\ell_\alpha = 1, (\alpha, \text{wt}(b_1)) \neq 0) \text{ or } (\ell_\alpha = -1, (\alpha, \text{wt}(b_2)) = 0), \\ \chi_\alpha(b_2), & \text{if } (\ell_\alpha = -1, (\alpha, \text{wt}(b_2)) \neq 0) \text{ or } (\ell_\alpha = 1, (\alpha, \text{wt}(b_1)) = 0), \end{cases}$$

$$x_\alpha(b_1 \otimes b_2) = \begin{cases} x_\alpha b_1 \otimes b_2, & \text{if } (\ell_\alpha = 1, (\alpha, \text{wt}(b_1)) \neq 0) \text{ or } (\ell_\alpha = -1, (\alpha, \text{wt}(b_2)) = 0), \\ b_1 \otimes x_\alpha b_2, & \text{if } (\ell_\alpha = -1, (\alpha, \text{wt}(b_2)) \neq 0) \text{ or } (\ell_\alpha = 1, (\alpha, \text{wt}(b_1)) = 0), \end{cases}$$

where $\chi = \varepsilon, \varphi$ and $x = e, f$.

(2) for a non-isotropic simple root $\alpha \in \Delta$,

$$\chi_\alpha(b_1 \otimes b_2) = \begin{cases} \max\{\chi_\alpha(b_1), \chi_\alpha(b_2) - \ell_\alpha(\alpha, \text{wt}(b_1))\}, & \text{if } (\chi = \varepsilon, \ell_\alpha = 1) \text{ or } (\chi = \varphi, \ell_\alpha = -1), \\ \max\{\chi_\alpha(b_1) + \ell_\alpha(\alpha, \text{wt}(b_2)), \chi_\alpha(b_2)\}, & \text{if } (\chi = \varphi, \ell_\alpha = 1) \text{ or } (\chi = \varepsilon, \ell_\alpha = -1), \end{cases}$$

$$e_\alpha(b_1 \otimes b_2) = \begin{cases} e_\alpha b_1 \otimes b_2, & \text{if } (\ell_\alpha = 1, \varphi_\alpha(b_1) \geq \varepsilon_\alpha(b_2)) \text{ or } (\ell_\alpha = -1, \varphi_\alpha(b_2) < \varepsilon_\alpha(b_1)), \\ b_1 \otimes e_\alpha b_2, & \text{if } (\ell_\alpha = 1, \varphi_\alpha(b_1) < \varepsilon_\alpha(b_2)) \text{ or } (\ell_\alpha = -1, \varphi_\alpha(b_2) \geq \varepsilon_\alpha(b_1)), \end{cases}$$

$$f_\alpha(b_1 \otimes b_2) = \begin{cases} f_\alpha b_1 \otimes b_2, & \text{if } (\ell_\alpha = 1, \varphi_\alpha(b_1) > \varepsilon_\alpha(b_2)) \text{ or } (\ell_\alpha = -1, \varphi_\alpha(b_2) \leq \varepsilon_\alpha(b_1)), \\ b_1 \otimes f_\alpha b_2, & \text{if } (\ell_\alpha = 1, \varphi_\alpha(b_1) \leq \varepsilon_\alpha(b_2)) \text{ or } (\ell_\alpha = -1, \varphi_\alpha(b_2) > \varepsilon_\alpha(b_1)), \end{cases}$$

where $\chi = \varepsilon, \varphi$. We assume that $0 \otimes b_2 = b_1 \otimes 0 = 0$. It is straightforward to check that $B_1 \otimes B_2$ is a $\mathfrak{gl}_s$-crystal.

2.2. Crystals of semistandard tableaux. We may regard $S$ as a $\mathfrak{gl}_s$-crystal associated to the natural representation $\mathbb{C}^S$, where $b \preceq b'$ for a successive pair $b \prec b'$ in $S$ with $\alpha = \epsilon_b - \epsilon_{b'}$, $\text{wt}(b) = \epsilon_b$, and $\varepsilon_\beta(b)$ (resp. $\varphi_\beta(b)$) is the number of $\beta$-colored arrows coming into $b$ (resp. going out of $b$) for $\beta \in \Delta$.

Let $W = W_S$ be the set of words of finite length with alphabets in $S$. The empty word is denoted by $\emptyset$. Then $W$ is a $\mathfrak{gl}_s$-crystal since we may view each non-empty word
Proposition 2.3 (cf. [1]). Let $S$ be the set of partitions. As usual, we identify $\lambda = (\lambda_k)_{k \geq 1} \in S$ with its Young diagram. For non-negative integers $m$ and $n$, we denote by $S_{m\mid n}$ the set of all $(m, n)$-hook partitions $\lambda$, that is, $\lambda_{m+1} \leq n$.

For $\lambda \in S$, let $B(\lambda) = B_S(\lambda)$ be the set of semistandard tableaux of shape $\lambda$, that is, tableaux $T$ obtained by filling a Young diagram $\lambda$ with entries in $S$ such that (1) the entries in each row (resp. column) are weakly increasing from left to right (resp. from top to bottom), and (2) the entries in $S_0$ (resp. $S_1$) are strictly increasing in each column (resp. row) (cf. [3]). Note that $B(\lambda)$ is non-empty if and only if $\lambda \in S_{m\mid n}$, where $|S_0| = m$ and $|S_1| = n$. We assume that $S_{m\mid n} = S$ if either $m$ or $n$ is infinite.

For $T \in B(\lambda)$, $T(i,j)$ denotes the entry of $T$ located in the $i$th row from the top and the $j$th column from the left. Then an admissible reading is an embedding $\psi : B(\lambda) \to W$ given by reading the entries of $T$ in $B(\lambda)$ in such a way that $T(i,j)$ should be read before $T(i+1,j)$ and $T(i,j-1)$ for each $i, j$. Then, by similar arguments as in [1], we can check that the image of $B(\lambda)$ under $\psi$ together with $0$ is stable under $e_\alpha, f_\alpha$ ($\alpha \in \Delta$), and hence $B(\lambda)$ is a subcrystal of $W$ (cf. [12]), which does not depend on the choice of $\psi$.

### Proposition 2.3 (cf. [1]). For $\lambda \in S$, $B(\lambda)$ is a $\mathfrak{gl}_S$-crystal.

For $b \in S$ and $T \in B(\lambda)$, let us denote by $b \Rightarrow T$ the tableau in $B(\mu)$ obtained by Schensted’s column bumping insertion (cf. [3] [23]), where $\mu \in S$ is given by adding a node at $\lambda$. Now, for a given word $w = w_1 \cdots w_r \in W$, we define its $P$-tableau by

$$P(w) = w_r \Rightarrow (\cdots \Rightarrow (w_2 \Rightarrow w_1)).$$

### Proposition 2.4 (cf. [1]). For $w \in W$, we have $w \simeq P(w)$.

Let us consider the case when $S = [m\mid n]$. For $\lambda \in S_{m\mid n}$, let $H_\lambda$ be the unique element in $B_{[m\mid n]}(\lambda)$ such that $\text{wt}(H_\lambda) = \sum_{i=1}^{n} \lambda_i e_{-m+i-1} + \sum_{j=1}^{n} \mu_j e_j$, where $\mu = (\mu_j)$ is the transpose of $(\lambda_{m+1}, \lambda_{m+2}, \ldots)$. Then the following is one of the main results in [1].

### Theorem 2.5 ([1]). For $\lambda \in S_{m\mid n}$, $B_{[m\mid n]}(\lambda)$ is connected with a unique highest weight element $H_\lambda$.

### Remark 2.6. (1) One of the most important properties of the crystal graph $B_{[m\mid n]}(\lambda)$ is the existence of fake highest weight vectors. That is, $B_{[m\mid n]}(\lambda)$ can have an element $T$ such that $e_\alpha T = 0$ for all $\alpha \in \Delta_{[m\mid n]}$, but $T \neq H_\lambda$.

(2) For a general $S$, $B_S(\lambda)$ is not necessarily connected. We will see some examples in later sections.

### 3. Crystals of quasi-ribbon tableaux

In this section, we discuss in detail the crystal graphs for $\mathfrak{gl}_N$, where

$$\mathcal{N} = \frac{1}{2} \mathbb{Z}_{>0} = \left\{ \frac{1}{2} < \frac{3}{2} < \frac{5}{2} < \cdots \right\},$$
with $N_0 = \mathbb{Z}_{>0}$ and $N_1 = \frac{1}{2} + \mathbb{Z}_{>0}$. Then $\Delta_N = \{ \alpha_r = \epsilon_r - \epsilon_{r+\frac{1}{2}} \mid r \in \frac{1}{2}\mathbb{Z}_{>0} \}$. Denote by $I = \frac{1}{2}\mathbb{Z}_{>0}$ the index set for the simple roots. The associated Dynkin diagram is

\[
\begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \cdots \\
\frac{1}{2} & 1 & \frac{3}{2} & 2 & &
\end{array}
\]

Note that $\ell_{\alpha_i} = (-1)^{2i}$ for $i \in I$.

3.1. Quasi-ribbon tableaux. The $\mathfrak{g}_N$-crystal $N$ is given by

\[
\begin{array}{cccccc}
\frac{1}{2} & \frac{1}{2} & 1 & \frac{3}{2} & 2 & \cdots
\end{array}
\]

From the tensor product rule of crystals given in Section 2.1, the actions of $e_i = e_{\alpha_i}, f_i = f_{\alpha_i}$ on $w = w_1 \cdots w_r \in W = W_N \ (i \in I)$ can be described more explicitly:

1. for $i \in \mathbb{Z}_{>0}$, choose the smallest $k$ $(1 \leq k \leq r)$ such that $(\alpha_i, \text{wt}(w_k)) \neq 0$. Then $e_i w$ (resp. $f_i w$) is obtained by applying $e_i$ (resp. $f_i$) to $w_k$. If there is no such $k$, then $e_i w = \mathbf{0}$ (resp. $f_i w = \mathbf{0}$).

2. for $i \in \frac{1}{2} + \mathbb{Z}_{>0}$, choose the largest $k$ $(1 \leq k \leq r)$ such that $(\alpha_i, \text{wt}(w_k)) \neq 0$. Then $e_i w$ (resp. $f_i w$) is obtained by applying $e_i$ (resp. $f_i$) to $w_k$. If there is no such $k$, then $e_i w = \mathbf{0}$ (resp. $f_i w = \mathbf{0}$).

Note that $\varepsilon_i(w)$ (resp. $\varphi_i(w)$) is the number of $i$-colored arrows coming into $w$ (resp. going out of $w$) for $w \in W$ and $i \in I$.

A composition is a finite sequence of positive integers $\alpha = (\alpha_1, \ldots, \alpha_t)$ for some $t \geq 1$, and $\sum_{i=1}^t \alpha_i = r$. Let $S(\alpha) = \{ \alpha_1 + \cdots + \alpha_i \mid i = 1, \ldots, t-1 \}$, which is a subset of $\{1, \ldots, r-1\}$. Then the map sending $\alpha$ to $S(\alpha)$ is a bijection between the set of compositions of $r$ and the set of subsets of $\{1, \ldots, r-1\}$. For $S = \{i_1 < \cdots < i_s\} \subset \{1, \ldots, r-1\}$, the inverse image is given by $\alpha(S) = \{i_1, i_2 - i_1, i_3 - i_2, \ldots, r - i_s\}$. We denote by $\mathcal{C}$ the set of compositions.

One may identify a composition with a ribbon diagram [20]. For example, a composition $\alpha = (1, 1, 3, 4, 1, 2)$ is identified with

\[
\begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \cdots
\end{array}
\]

A tableau $T$ obtained by filling a ribbon diagram $\alpha \in \mathcal{C}$ with entries in $N$ is called a quasi-ribbon tableau of shape $\alpha$ if (1) the entries in each row (resp. column) are weakly increasing from left to right (resp. from top to bottom), and (2) the entries of positive integers (resp. half integers) are strictly increasing in each column (resp. row). We denote by $B_N(\alpha)$, or simply $B(\alpha)$ if there is no confusion, the set of quasi-ribbon tableaux of shape $\alpha$ (cf. [16]).
Suppose that $T \in \mathbf{B}(\alpha)$ is given. Let $w(T)$ be the word in $\mathbf{W}$ obtained by reading the entries in $T$ row by row from top to bottom, where in each row we read the entries from right to left. For $i \in I$ and $x = e, f$, we define $x_i T$ to be the unique tableau in $\mathbf{B}(\alpha)$ satisfying $w(x_i T) = x w(T)$, where we assume that $x_i T = \mathbf{0}$ if $x_i w(T) = \mathbf{0}$.

Let $H_\alpha$ be the element in $\mathbf{B}(\alpha)$, which is defined in the following way:

$$
\begin{array}{cccc}
\frac{1}{2} & & & \\
\frac{1}{2} & 1 & 1 & \\
\frac{3}{2} & 2 & 2 & \\
\frac{5}{2} & & & \\
\frac{3}{2} & & & 3 \\
\end{array}
$$

Then $\text{wt}(H_\alpha) \geq \text{wt}(T)$ for all $T \in \mathbf{B}(\alpha)$, where $\text{wt}(T) = \text{wt}(w(T))$.

**Theorem 3.1.** For $\alpha \in \mathcal{C}$, $\mathbf{B}(\alpha)$ is a $\mathfrak{gl}_N$-crystal and

$$
\mathbf{B}(\alpha) = \{ f_{i_1} \cdots f_{i_r} H_\alpha \mid r \geq 0, \ i_1, \ldots, i_r \in I \} \setminus \{ \mathbf{0} \}.
$$

In particular, $\mathbf{B}(\alpha)$ is connected with a unique highest weight element $H_\alpha$.

**Proof.** It is straightforward to check that $x_i T \in \mathbf{B}(\alpha) \cup \{ \mathbf{0} \}$ is well-defined for $T \in \mathbf{B}(\alpha)$, $x = e, f$, and $i \in I$. Hence, identifying $T$ with $w(T)$, $\mathbf{B}(\alpha)$ becomes a subcrystal of $\mathbf{W}$.

Next, we will show that for each $T \in \mathbf{B}(\alpha)$, if $T \neq H_\alpha$, then there exists $i \in I$ such that $e_i T \neq \mathbf{0}$. Let $x$ be a node in $\alpha$, whose entry $r$ is not equal to that of $H_\alpha$. We also assume that the other entries located to the northwest of $x$, that is, the entries whose row or column indices are no more than that of $x$, are equal to those in $H_\alpha$. If $r \in \mathbb{Z}_{>0}$, then there is no $r - \frac{1}{2}$ to the left of $x$ in the same row and no more $r$ in the rows strictly lower than that of $x$.

So we have $e_{r - \frac{1}{2}} T \neq \mathbf{0}$. Similarly, if $r \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$, then there is no more $r$ to the northwest of it and no $r - \frac{1}{2}$ to the right of $r$ in the same row, which also implies that $e_{r - \frac{1}{2}} T \neq \mathbf{0}$.

This completes the proof. \hfill \Box

### 3.2. Crystal equivalence

Let $T \in \mathbf{B}(\alpha)$ be given for some $\alpha \in \mathcal{C}$. For $k \geq 1$, let $t_k$ denote the $k$th entry of $T$, where we enumerate the nodes in $T$ from northwest to southeast. Let $T^{\leq k}$ be the sub-tableau consisting of the first $k$ nodes in $T$ and $T^{\geq k}$ the sub-tableau obtained by removing $T^{\leq k-1}$ from $T$, where $T^{\leq 0}$ is assumed to be the empty tableau.

Given $b \in \mathbb{N}$, choose the smallest $t_k$ such that $b \leq t_k$ (resp. $b < t_k$) if $b \in \mathbb{Z}_{>0}$ (resp. $b \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$). Then we define $b \rightarrow T$ to be the quasi-ribbon tableau glueing $T^{\leq k-1}$, $T^{\geq k}$ and $b$, where $b$ is placed below the last node of $T^{\leq k-1}$ and to the left of the first node of $T^{\geq k}$ (cf. [16]).

**Example 3.2.**

$$
2 \rightarrow 1 \ 1 \ \frac{5}{2} \ 3 = 1 \ 1 \ \frac{5}{2} \ , \quad \frac{5}{2} \rightarrow 1 \ 1 \ \frac{5}{2} \ 3 = 1 \ 1 \ \frac{5}{2} \ 
$$

**Lemma 3.3.** For $T \in \mathbf{B}(\alpha)$ and $b \in \mathbb{N}$, we have $(b \rightarrow T) \simeq T \otimes b$. 
Proof. It is enough to show that for $i \in I$ and $x = e, f$,

$$x_i(b \rightarrow T) = \begin{cases} x_ib \rightarrow T, & \text{if } x_i(T \otimes b) = T \otimes (x_i b), \\ b \rightarrow x_i T, & \text{if } x_i(T \otimes b) = (x_i T) \otimes b, \end{cases}$$

where we understand that $0 \rightarrow T = b \rightarrow 0 = 0$.

We will prove the case when $x = f$ and $i, b \in \mathbb{Z}_{>0}$. The other cases can be verified in a similar way. First, suppose that $f_i(T \otimes b) = T \otimes f_i b$. Then $b = i$, and neither $i$ nor $i + \frac{1}{2}$ appears as an entry of $T$. This implies that $f_i(b \rightarrow T) = f_i b \rightarrow T$. Next, suppose that $f_i(T \otimes b) = f_i T \otimes b$, and $t_k = i$, the $k$th entry of $T$ from northwest, becomes $i + \frac{1}{2}$ applying $f_i$ to $T$. If $b \neq i, i + \frac{1}{2}$, then it is clear that $f_i(b \rightarrow T) = b \rightarrow f_i T$. If $b = i$, then $b$ is located to the left of $t_k$ in the same row of $b \rightarrow T$. If $b = i + \frac{1}{2}$, then $b$ is located below $t_k$ in the same column of $b \rightarrow T$. This also implies that $f_i(b \rightarrow T) = b \rightarrow f_i T$. \qed

Figure 1. $B(1, 3)$ with elements of height $\leq 6$ and arrows colored by $\{\frac{1}{2}, 1, \frac{3}{2}, 2\}$
Now, for \( w = w_1 \ldots w_r \in \mathbf{W} \), we define its quasi \( P \)-tableau by
\[
(3.1) \quad P(w) = w_r \rightarrow (\cdots \rightarrow (w_2 \rightarrow w_1)).
\]

By Theorem 3.1 and Lemma 3.3, we have

**Corollary 3.4.** For \( w \in \mathbf{W} \), we have \( w \simeq P(w) \). In particular, each connected component in \( \mathbf{W} \) is isomorphic to \( \mathbf{B}(\alpha) \) for some \( \alpha \in \mathcal{C} \).

**Corollary 3.5.** Each \( w \in \mathbf{W} \) is \( \mathfrak{gl}_N \)-equivalent to a unique quasi-ribbon tableau.

**Proof.** Let \( T \) be a quasi-ribbon tableau which is \( \mathfrak{gl}_N \)-equivalent to \( w \). Then \( T \simeq P(w) \), and hence the highest weight vectors for the connected components of \( T \) and \( P(w) \) are equal since they have the same weight. Since \( T \) and \( P(w) \) are generated by the same highest weight vector, it follows that \( T = P(w) \).

For \( \alpha \in \mathcal{C} \) with \( \alpha \models r \), a tableau \( T \) obtained by filling a ribbon diagram \( \alpha \) with \( \{1, \ldots, r\} \) is called a standard ribbon tableau of shape \( \alpha \) if the entries in each row are decreasing from left to right, and the entries in each column are increasing from top to bottom.

For \( w = w_1 \ldots w_r \in \mathbf{W} \), we define its quasi \( Q \)-tableau \( Q(w) \) to be the standard ribbon tableau of the same shape as \( P(w) \), where we fill a node of \( Q(w) \) with \( i \) if the corresponding position in \( P(w) \) is given by \( w_i \). Then as in the classical Robinson-Schensted correspondence (cf. [15]), the map \( w \mapsto (P(w), Q(w)) \) gives a bijection from \( \mathbf{W} \) to the set of pairs of a quasi-ribbon tableau and a standard ribbon tableau of the same shape (cf. [16]). Furthermore, it is straightforward to check that \( Q(x,w) = Q(w) \) whenever \( x_i w \neq 0 \) for \( i \in I \) and \( x = e, f \) (cf. Proposition 4.17 in [10] and [18]).

**Example 3.6.** Let \( w = 1 \frac{1}{2} 1 \frac{5}{2} 2 2 \). Then
\[
P(w) = \begin{pmatrix}
\frac{1}{2} & 1 & 1 \\
2 & 2 & \frac{5}{2} \\
2 & 2 & 1
\end{pmatrix}, \quad Q(w) = \begin{pmatrix}
3 & 1 \\
6 & 5 & 4
\end{pmatrix}.
\]

Summarizing the above arguments, we have

**Theorem 3.7.** For \( \alpha \in \mathcal{C} \), let \( RT(\alpha) \) be the set of standard ribbon tableaux of shape \( \alpha \). For \( T \in RT(\alpha) \), \( \mathbf{B}(T) = \{ w \in \mathbf{W} \mid Q(w) = T \} \) is isomorphic to \( \mathbf{B}(\alpha) \). Hence, as a \( \mathfrak{gl}_N \)-crystal, we have
\[
\mathbf{W} = \bigoplus_{\alpha \in \mathcal{C}} \bigoplus_{T \in RT(\alpha)} \mathbf{B}(T).
\]

3.3. Stability of crystal graphs. For \( n \in \mathbb{Z}_{>0} \), put \( \mathbb{N}_n^\prec = \{ r \in \mathbb{N} \mid r \prec n \} \). Then \( \mathbf{B}_{\mathbb{N}_n^\prec}(2^{n-1},1) \) has \( 2^{2n-1} \) elements given by
\[
(3.2) \quad f_{1/2}^{m_1/2} f_1^{m_1} \cdots f_{n-\frac{1}{2}}^{m_{n-\frac{1}{2}}} H_{(2^{n-1},1)},
\]
where \( m_i = 0, 1 \) for \( \alpha_i \in \Delta_{\mathbb{N}_n^\prec} \). Note that each element in \( \mathbf{B}_{\mathbb{N}_n^\prec}(2^{n-1},1) \) is uniquely determined by a sequence \( (m_{\frac{1}{2}}, \ldots, m_{n-\frac{1}{2}}) \in \{0,1\}^{2n-1} \) (see Figure 2).

Let us say that a node in a ribbon diagram is a corner if it is the leftmost or rightmost node in each row having at least two nodes, or it is the last node in the diagram when
enumerrated from northwest to southeast. Note that for $\alpha \in \mathcal{C}$, $\mathcal{B}_{N \leq n}(\alpha)$ is non-empty if and only if $\alpha$ has at most $2n$ corners. Suppose that $\alpha$ has either $2n-1$ or $2n$ corners, equivalently, $(\text{wt}(H_\alpha), \epsilon_r) \neq 0$ for $r < n$, and 0 for $r > n$. Then each $T$ in $\mathcal{B}_{N \leq n}(\alpha)$ differs from $H_\alpha$ only in corners. Hence, it is not difficult to see that $\mathcal{B}_{N \leq n}(\alpha)$ consists of $2^{2n-1}$ elements given by $f_{m_1/2}^m f_{m_1}^m \cdots f_{m_{n-1}/2}^m H_\alpha$, where $m_i = 0, 1$ for $\alpha_i \in \Delta_{N \leq n}$. This implies the following fact.

**Proposition 3.8.** Let $\alpha$ be a composition whose number of corners is either $2n-1$ or $2n$. Then there exists a unique bijection

$$\theta : \mathcal{B}_{N \leq n}(2^{n-1}, 1) \longrightarrow \mathcal{B}_{N \leq n}(\alpha),$$

which maps $H_{(2^{n-1}, 1)}$ to $H_\alpha$ and commutes with $x_i$ for $x = e, f$ and $\alpha_i \in \Delta_{N \leq n}$. Here we assume that $\theta(0) = 0$. In other words, $\mathcal{B}_{N \leq n}(2^{n-1}, 1)$ and $\mathcal{B}_{N \leq n}(\alpha)$ are isomorphic as $\Delta_{N \leq n}$-colored oriented graphs.

### 4. Enriched P-partitions

#### 4.1. Crystal structure on the set of enriched P-partitions

Let us recall the notion of an enriched $(P, \gamma)$-partition introduced by Stembridge [27]. We follow the notations in [27] with a little modification. Let $P = (X, \prec)$ be a finite set $X$ with a partial ordering $\prec$. Let $\gamma : X \rightarrow \mathbb{Z}_{>0}$ be an injective map, which will be called a labeling of $X$. We call a pair $(P, \gamma)$ a labeled poset. Then an enriched $(P, \gamma)$-partition is a map $\sigma : X \rightarrow \mathbb{N}$ such that for all $x < y$ in $P$,

1. $\sigma(x) \preceq \sigma(y)$,
2. $\sigma(x) = \sigma(y)$ and $\sigma(x) \in \mathbb{Z}_{>0}$ implies $\gamma(x) < \gamma(y)$,
Hence, we obtain the following.

We denote by $E(P,\gamma)$ the set of enriched $(P,\gamma)$-partitions.

Suppose that $|X| = r$. Define an embedding $\psi : E(P,\gamma) \to W$ by $\psi(\sigma) = \sigma(x_1) \cdots \sigma(x_r)$ for $\sigma \in E(P,\gamma)$, where $x_i \in X$ ($1 \leq i \leq r$) are arranged so that $\gamma(x_1) > \gamma(x_2) > \cdots > \gamma(x_r)$.

**Theorem 4.1.** The image of $E(P,\gamma)$ under $\psi$ together with $\{0\}$ is stable under $e_i, f_i$ for $i \in I$. Hence, $E(P,\gamma)$ becomes a $\mathfrak{gl}_N$-crystal.

**Proof.** We will prove that $x_i \psi(\sigma) \in \psi(E(P,\gamma))$ for $x = e, f$ and $i \in I$, when $x_i \psi(\sigma) \neq 0$.

We assume that $x = f$ and $i \in \mathbb{Z}_{>0}$ since the other cases can be checked in the same way.

Suppose that

$$f_i \psi(\sigma) = \sigma(x_1) \cdots f_i \sigma(x_s) \cdots \sigma(x_r) \neq 0,$$

where $\sigma(x_s) = i$ and hence $f_i \sigma(x_s) = i + \frac{1}{2}$. Recall that $s$ is the smallest index such that $(\alpha_i, \text{wt}(\sigma(x_s))) \neq 0$. Let $\tau : X \to \mathbb{N}$ be defined by $\tau(x_i) = \sigma(x_i)$ for $t \neq s$ and $\tau(x_s) = i + \frac{1}{2}$.

Suppose that $x_s < x_t$ in $P$. If $\sigma(x_s) = \sigma(x_t) = i$, then $\gamma(x_s) < \gamma(x_t)$, which contradicts the minimality of $s$. If $\sigma(x_s) = i < \sigma(x_t) = i + \frac{1}{2}$, then we have $\tau(x_s) = \tau(x_t)$, and $\gamma(x_s) > \gamma(x_t)$ by the minimality of $s$. If $\sigma(x_s) = i < i + \frac{1}{2} < \sigma(x_t)$, then we have $\tau(x_s) < \tau(x_t)$. Next, suppose that $x_s < x_t$ in $P$. Then it is clear that $\tau(x_s) < \tau(x_t)$. Hence, it follows that $\tau \in E(P,\gamma)$ and $\psi(\tau) = f_i \psi(\sigma)$.

We mean a linear extension of $P$ by a total ordering $w = \{ w_1 < \cdots < w_r \}$ on $P$ preserving its partial ordering. We denote by $L(P)$ the set of linear extensions of $P$. For $w \in L(P)$, let $D(w,\gamma) = \{ i | \gamma(w_i) > \gamma(w_{i+1}) \}$ be the descent of $w$ with respect to $\gamma$. Then we have

$$E(w,\gamma) = \{ \sigma : X \to \mathbb{N} | (1) \sigma(w_1) \leq \cdots \leq \sigma(w_r),$$

$$\quad (2) \sigma(w_i) = \sigma(w_{i+1}) \in \mathbb{Z}_{>0} \Rightarrow i \notin D(w,\gamma),$$

$$\quad (3) \sigma(w_i) = \sigma(w_{i+1}) \in \frac{1}{2} + \mathbb{Z}_{>0} \Rightarrow i \in D(w,\gamma) \}.$$

**Lemma 4.2.** For $w \in L(P)$, $E(w,\gamma) \simeq \mathbf{B}(\alpha)$ as a $\mathfrak{gl}_N$-crystal, where $\alpha = \alpha(D(w,\gamma)) \in \mathfrak{c}$.

**Proof.** Let us identify $w_k$ $(1 \leq k \leq r)$ with the $k$th node in the ribbon diagram $\alpha$ from its northwest. This induces an isomorphism of $\mathfrak{gl}_N$-crystals from $E(w,\gamma)$ to $\mathbf{B}(\alpha)$. □

**Remark 4.3.** Note that the crystal graph structure on $E(w,\gamma)$ depends only on $D(w,\gamma)$.

Let us consider the decomposition of $E(P,\gamma)$ as a $\mathfrak{gl}_N$-crystal. Given $\sigma \in E(P,\gamma)$, we define a linear extension $w$ on $P$ as follows:

1. Arrage the elements of $X$ in increasing order of their values of $\sigma$,
2. If there exist elements $x$ in $X$ with the same value $\sigma(x) \in \mathbb{Z}_{>0}$ (resp. $\sigma(x) \in \frac{1}{2} + \mathbb{Z}_{>0}$), then arrange them in order of their increasing (resp. decreasing) values of $\gamma$.

By definition, we have $\sigma \in E(w,\gamma)$, and this correspondence induces a bijection $\pi : E(P,\gamma) \to \bigsqcup_{w \in L(P)} E(w,\gamma)$ (Lemma 2.1 in [27]). Furthermore, it is straightforward to see that $\pi$ commutes with $x_i$ for $x = e, f$ and $i \in I$, where we assume that $\pi(0) = 0$ and $x_i 0 = 0$.

Hence, we obtain the following.
Corollary 4.4. Given a labeled poset \((P, \gamma)\), we have
\[
\mathcal{E}(P, \gamma) \simeq \bigoplus_{w \in \mathcal{L}(P)} \mathcal{E}(w, \gamma),
\]
as a \(\mathfrak{gl}_N\)-crystal. That is, for each \(\alpha \in \mathcal{C}\), the multiplicity of \(B(\alpha)\) in \(\mathcal{E}(P, \gamma)\) is equal to the number of \(w \in \mathcal{L}(P)\) such that \(\alpha(D(w, \gamma)) = \alpha\).

4.2. Decomposition of \(\mathfrak{gl}_N\)-crystals. Consider \(B(\lambda) = B_N(\lambda)\) for \(\lambda \in \mathcal{P}\) (see Section 2.2). We assume that \(\lambda\) is equipped with a partial ordering given by \(\lambda(i, j) < \lambda(j, i + 1)\) and \(\lambda(i, j) < \lambda(i + 1, j)\), where \(\lambda(i, j)\) denotes the node in \(\lambda\) located in the \(i\)th row from the top and the \(j\)th column from the left. We consider a labeled poset \((\lambda, \gamma)\), where \(\gamma\) satisfies \(\gamma(\lambda(i, j)) < \gamma(\lambda(i, j + 1))\) and \(\gamma(\lambda(i, j)) > \gamma(\lambda(i + 1, j))\). Then \(\mathcal{E}(\lambda, \gamma)\) is equal to \(B(\lambda)\) as a set, and if we use the admissible reading on \(B(\lambda)\) given by the reverse ordering of \(\gamma\)-values, then the actions of \(e_i, f_i\) \((i \in I)\) on both sets coincide.

Now, let \(ST(\lambda)\) be the set of standard tableaux of shape \(\lambda\), that is, the set of order preserving bijections \(T : \lambda \to \{1, \ldots, r\}\), where we regard \(\lambda\) as a poset with \(r\) elements. Then \(\mathcal{L}(\lambda)\) can be identified with \(ST(\lambda)\). For \(T \in ST(\lambda)\), we denote by \(D(T)\) the descent set of \(T\) as a linear extension of \(\lambda\). Hence, we have \(k \in D(T)\) if and only if the column index of \(T^{-1}(k)\) is greater than or equal to that of \(T^{-1}(k+1)\). We put \(\alpha(T) = \alpha(D(T))\) for simplicity. Hence, by Corollary 4.4 we obtain the following decomposition of \(B(\lambda)\).

Proposition 4.5. For \(\lambda \in \mathcal{P}\), we have
\[
B(\lambda) \simeq \bigoplus_{T \in ST(\lambda)} B(\alpha(T)).
\]

Remark 4.6. Proposition 4.5 can be directly extended to the case of \(B(\lambda/\mu)\) for a skew Young diagram \(\lambda/\mu\). Moreover, for a strict partition \(\lambda\), the set of shifted \(N\)-tableaux of shape \(\lambda\) is also a \(\mathfrak{gl}_N\)-crystal, and we have a similar decomposition (cf. [27]).

Let \(\alpha \in \mathcal{C}\) be given with \(\alpha \vdash r\). Let \(x_1, \ldots, x_r\) denote the nodes in the diagram of \(\alpha\) enumerated from northwest to southeast. We assume that \(\alpha\) is equipped with a total ordering \(w_\alpha = \{x_1 < \cdots < x_r\}\). We also identify \(T \in B(\alpha)\) with a function \(T : \alpha \to \mathbb{N}\). A canonical labeling \(\gamma_\alpha\) of \(\alpha\) is defined by \(\gamma_\alpha(x_i) = r - k + 1\) if \(T(x_i)\) is read in the \(k\)th letter of \(\psi(T) \in \mathcal{W}\) for \(T \in B(\alpha)\). Then clearly we have \(B(\alpha) = \mathcal{E}(w_\alpha, \gamma_\alpha)\) and the actions of \(x_i\) for \(x = e, f\) and \(i \in I\) on both sets coincide. For \(t \geq 0\), we define \(\gamma_\alpha^{[t]}\) by \(\gamma_\alpha^{[t]}(x_i) = \gamma_\alpha(x_i) + t\) for \(1 \leq i \leq r\).

Suppose that \(\alpha, \beta \in \mathcal{C}\) are given with \(\beta \vdash s\). Consider a labeled poset \((w_\alpha \cup w_\beta, \gamma_\alpha^{[s]} \cup \gamma_\beta)\), which is a disjoint union of labeled posets. Then we can check that \(B(\alpha) \otimes B(\beta)\) is isomorphic to \(\mathcal{E}(w_\alpha \cup w_\beta, \gamma_\alpha^{[s]} \cup \gamma_\beta)\) as a \(\mathfrak{gl}_N\)-crystal. By Corollary 4.4 we obtain the following tensor product decomposition.

Proposition 4.7. For \(\alpha, \beta \in \mathcal{C}\) with \(\beta \vdash s\), we have
\[
B(\alpha) \otimes B(\beta) \simeq \bigoplus_{w \in \mathcal{L}(w_\alpha \cup w_\beta)} \mathcal{E}(w, \gamma_\alpha^{[s]} \cup \gamma_\beta).
\]

Note that a linear extension of \(w_\alpha \cup w_\beta\) is called a shuffle of \(\alpha\) and \(\beta\).
4.3. RSK correspondence. Let
\[
\Omega = \{ (i,j) \in W \times W \mid
\]
\begin{enumerate}
\item $i = i_1 \cdots i_r$ and $j = j_1 \cdots j_r$ for some $r \geq 0$,
\item $(i_1, j_1) \leq \cdots \leq (i_r, j_r)$,
\item $i_k - j_k \not\in \mathbb{Z}$ implies $(i_k, j_k) \neq (i_{k-1}, j_{k-1})$,
\end{enumerate}  
\tag{4.2}

where for $(i, j)$ and $(k, l) \in \mathbb{N} \times \mathbb{N}$, the super lexicographic ordering is given by
\[
(i, j) < (k, l) \iff \begin{cases} (j < l) & \text{or}, \\
(j = l \in \mathbb{Z}_{>0}, \text{ and } i > k) & \text{or}, \\
(j = l \in \frac{1}{k} + \mathbb{Z}_{\geq 0}, \text{ and } i < k) & .
\end{cases}
\tag{4.3}
\]

Also, let $\Omega^*$ be the set of pairs $(k, l) \in W \times W$ such that $(1, k) \in \Omega$.

Given $(i, j) \in \Omega$, we define $x_i(i, j) = (x_i, x_j)$ for $x = e, f$ and $i \in I$, where we assume that $x_i(i, j) = 0$ if $x_i = 0$, and set $\mathrm{wt}(i, j) = \mathrm{wt}(i)$. Similarly, given $(k, l) \in \Omega^*$, we define $x_i^*(k, l) = (x_i, x_j)$ for $x = e, f$ and $j \in I$, and set $\mathrm{wt}^*(k, l) = \mathrm{wt}(l)$. Then as in \cite{13}, we can check that

**Lemma 4.8 (cf. \cite{13} Lemma 3.1).** Under the above hypothesis, $\Omega$ and $\Omega^*$ are $\mathfrak{gl}_N$-crystals with respect to $x_i$ and $x_i^*$ for $x = e, f$ and $i \in I$, respectively.

Consider
\[
M = \{ A = (a_{rs})_{r,s \in \mathbb{N}} \mid
\]
\begin{enumerate}
\item $a_{rs} = 0$ for all sufficiently large $r$ and $s,$
\item $a_{rs} \in \mathbb{Z}_{\geq 0}$, and $a_{rs} \leq 1$ unless $r - s \in \mathbb{Z}$.
\end{enumerate}  
\tag{4.4}

For $(i, j) \in \Omega$, define $A(i, j) = (a_{rs})$ to be the matrix in $M$, where $a_{rs}$ is the number of $k$’s such that $(i_k, j_k) = (r, s)$ for $r, s \in \mathbb{N}$. Then, it follows that the map $(i, j) \mapsto A(i, j)$ gives a bijection from $\Omega$ to $M$, where the pair of empty words $(\emptyset, \emptyset)$ corresponds to zero matrix. Similarly, we have a bijection $(k, l) \mapsto A(k, l)$ from $\Omega^*$ to $M$. With these bijections, $M$ becomes a $\mathfrak{gl}_N$-crystal with respect to both $x_i$ and $x_i^*$ for $x = e, f$ and $i \in I$ by Lemma 4.8.

For convenience, let us say that $M$ is a $\mathfrak{gl}_N^*$-crystal when we consider its crystal structure with respect to $x_i^*$.

**Lemma 4.9 (cf. \cite{13} Lemma 3.4).** $M$ is a $(\mathfrak{gl}_N, \mathfrak{gl}_N^*)$-bicrystal, that is, $e_i, f_i$ commute with $e_j^*, f_j^*$ for $i, j \in I$, where we assume that $x_i0 = x_j^*0 = 0$ for $x = e, f$.

Given $A \in M$, suppose that $A = A(i, j) = A(k, l)$ for $(i, j) \in \Omega$ and $(k, l) \in \Omega^*$. We define
\[
\varpi(A) = (P_1(A), P_2(A)) = (P(i), P(l)).
\tag{4.5}
\]

Note that $A$ is $\mathfrak{gl}_N$ (resp. $\mathfrak{gl}_N^*$)-equivalent to $P_1(A)$ (resp. $P_2(A)$).

**Proposition 4.10.** The map $\varpi$ induces an isomorphism of $(\mathfrak{gl}_N, \mathfrak{gl}_N^*)$-bicrystals
\[
\varpi : M \rightarrow \bigsqcup_{\lambda \in \mathcal{P}} \bigsqcup_{P, Q \in ST(\lambda)} B(\alpha(P)) \times B(\alpha(Q)).
\]
Proof. Given \( A \in M \), suppose that \( A = A(i,j) = A(k,l) \) for \((i,j) \in \Omega \) and \((k,l) \in \Omega^* \). Define \( \pi(A) = (P(i), P(l)) \) (see (2.2)). One can extend the arguments for Young tableaux (see, for example, §4.2 in [4]) to the super case without difficulty to prove that \( \pi \) induces a bijection from \( M \) to \( \bigsqcup_{\lambda \in \mathcal{P}} B(\lambda) \times B(\lambda) \) (cf. [13]).

Let \( \omega \) be the isomorphism given in Proposition 4.5. Then we have a bijection \( \pi' : M \to \bigsqcup_{P,Q \in ST(\lambda)} B(\alpha(P)) \times B(\alpha(Q)) \) defined by sending \( A \) to \((\omega(P(i)), \omega(P(l))) \). Furthermore, \( \omega(P(i)) \) and \( \omega(P(l)) \) are quasi-ribbon tableaux equivalent to \( i \) and \( l \) respectively, and by Corollary 3.5 it follows that \( \omega(P(i)) = P(i), \omega(P(l)) = P(l) \) and hence \( \varpi = \pi' \).

Suppose that \( A \in M \) is given. If \( x_j^i A \neq 0 \) for some \( x = e, f \) and \( j \in I \), then \( A \) is \( \mathfrak{gl}_N \)-equivalent to \( x_j^i A \) (cf. [13] Lemma 3.5), and by Lemma 3.5 we have \( P_1(x_j^i A) = P_1(A) \). Similarly, if \( x_i A \neq 0 \) for some \( x = e, f \) and \( i \in I \), then \( P_2(x_i A) = P_2(A) \). This implies that \( \varpi \) is a morphism of \( (\mathfrak{gl}_N, \mathfrak{gl}_N^*) \)-bicrystals.

Let \( \mathcal{S}_k \) be the symmetric group on \( k \) letters. For \( \sigma \in \mathcal{S}_k \), let \( D(\sigma) = \{ i : \sigma(i) > \sigma(i+1) \} \) be the descent of \( \sigma \). Put \( \alpha(\sigma) = \alpha(D(\sigma)) \). Let \( (P, Q) \) be the pair of standard tableaux of the same shape, which corresponds to \( \sigma \) under the classical Robinson-Schensted correspondence.

Then \( D(P) = D(\sigma) \) and \( D(Q) = D(\sigma^{-1}) \) [25], that is, \( \alpha(P) = \alpha(\sigma) \) and \( \alpha(Q) = \alpha(\sigma^{-1}) \).

Combining with Proposition 4.10, we obtain another version of the Gessel’s result [6] in terms of crystal graphs.

**Theorem 4.11** (cf. [6]). Let \( \mathcal{M}^k = \{ A = (a_{rs}) \in M \mid \sum_{r,s} a_{rs} = k \} \) for \( k \in \mathbb{Z}_{>0} \). Then \( \varpi \) restricts to the following isomorphism of \( (\mathfrak{gl}_N, \mathfrak{gl}_N^*) \)-bicrystals;

\[
\varpi : \mathcal{M}^k \longrightarrow \bigsqcup_{\sigma \in \mathcal{S}_k} B(\alpha(\sigma)) \times B(\alpha(\sigma^{-1})).
\]

As a corollary, we have an interesting application to permutation enumeration.

**Corollary 4.12** (cf. [6]). Given \( S, S' \subset \{ 1, \ldots, k-1 \} \), the number of permutations \( \sigma \in \mathcal{S}_k \) satisfying \( D(\sigma) = S \) and \( D(\sigma^{-1}) = S' \) is equal to the number of matrices \( A \in \mathcal{M}^k \) satisfying \( e_i A = e_j^* A = 0 \) for all \( i, j \in I \) with \( \text{wt}(A) = \text{wt}(H_{\alpha(S)}) \) and \( \text{wt}^*(A) = \text{wt}(H_{\alpha(S')}) \).

5. Non-standard Borel subalgebras and branching rule

For \( m \in \mathbb{Z}_{\geq 0} \), let us consider the crystal graphs for \( \mathfrak{gl}_{N(m)} \), where

\[
N(m) = \left\{ -m \prec \cdots \prec -1 \prec \frac{1}{2} \prec 1 \prec \frac{3}{2} \prec 2 \prec \cdots \right\}.
\]

As usual, \( N(m)_0 = N(m) \cap \mathbb{Z} \) and \( N(m)_1 = N(m) \cap \left( \frac{1}{2} + \mathbb{Z} \right) \). We assume that \( N(0) = \mathbb{N} \).

Then \( \Delta_{N(m)} = \{ \alpha_i = \epsilon_{i-1} - \epsilon_i \mid -m+1 \leq i \leq -1 \}, \quad \alpha_0 = \epsilon_{-1} - \epsilon_{\frac{1}{2}}, \quad \alpha_r = \epsilon_r - \epsilon_{r+\frac{1}{2}} \mid r \in \frac{1}{2} \mathbb{Z}_{\geq 0} \} \). Denote by \( I(m) \) the index set for simple roots. The associated Dynkin diagram is given by

\[
\begin{array}{cccccccc}
& & \circ & \cdots & \circ & \cdots & \circ & \\
& -m+1 & \prec & \cdots & \prec & 0 & \prec & \frac{1}{2} \prec \cdots \\
\end{array}
\]

For \( n \in N(m) \), we put \( N(m)_{< n} = \{ r \in N(m) \mid r \prec n \} \).
5.1. **Highest weight crystals.** Let \( \mathcal{C}(m) \) be the set of pairs \((\lambda, \alpha)\) where \(\lambda = (\lambda_1, \ldots, \lambda_m)\) is a partition with length at most \(m\), and \(\alpha = (\alpha_1, \ldots, \alpha_r)\) is a composition such that \(\alpha\) is non-empty only if \(\lambda_m \neq 0\). We assume that \(\mathcal{C}(0) = \emptyset\). One may identify \((\lambda, \alpha) \in \mathcal{C}(m)\) with a diagram obtained by placing the first node of \(\alpha\) from northwest right below the leftmost node in the \(m\)th row of \(\lambda\). For example, when \(m = 3\), \(\lambda = (5, 4, 2)\), and \(\alpha = (1, 3, 4)\), the corresponding diagram is

![Diagram](image)

We call \(\lambda\) the *body*, and \(\alpha\) the *tail* of the diagram of \((\lambda, \alpha)\). The first node of the tail from northwest will be called the *joint* of \((\lambda, \alpha)\).

Let \(B_{N(m)}(\lambda, \alpha) = B(\lambda, \alpha)\) be the set of tableaux \(T\) obtained by filling the diagram \((\lambda, \alpha) \in \mathcal{C}(m)\) with entries in \(N(m)\) such that

1. \(T\) is semistandard in the usual sense,
2. if \(b\) is the entry of its joint and \(b \in \mathbb{Z}_{>0}\), then all the entries in the body are smaller than \(b\),
3. if \(b\) is the entry of its joint and \(b \in \frac{1}{2} + \mathbb{Z}_{\geq 0}\), then all the entries in the body are smaller than or equal to \(b\).

We call \(T \in B(\lambda, \alpha)\) a *semistandard tableaux of shape* \((\lambda, \alpha)\). We define \(H_{(\lambda, \alpha)}\) to be the tableau obtained by gluing \(H_\lambda\) and \(H_\alpha\). For \(T \in B(\lambda, \alpha)\), let \(w(T)\) be the word in \(W_{N(m)} = W\) obtained from \(T\) with respect to row reading. Then for \(i \in I(m)\) and \(x = e, f\), we define \(x_i T\) to be the tableau of shape \((\lambda, \alpha)\) corresponding to \(x_i w(T)\).

**Proposition 5.1.** For \((\lambda, \alpha) \in \mathcal{C}(m)\), \(B(\lambda, \alpha)\) is a \(\mathfrak{g}l_{N(m)}\)-crystal, and

\[
B(\lambda, \alpha) = \{ f_{i_1} \cdots f_{i_r} H_{(\lambda, \alpha)} \mid r \geq 0, \ i_1, \ldots, i_r \in I(m) \} \setminus \{0\}.
\]

In particular, \(B(\lambda, \alpha)\) is connected with a unique highest weight element \(H_{(\lambda, \alpha)}\).

**Proof.** We can check that \(x_i T \in B(\lambda, \alpha) \cup \{0\}\) for \(x = e, f\), \(i \in I(m)\), and \(T \in B(\lambda, \alpha)\). Hence \(B(\lambda, \alpha)\) is a \(\mathfrak{g}l_{N(m)}\)-subcrystal of \(W\) with respect to row reading. We leave the details to the readers.

Next we claim that for \(T \in B(\lambda, \alpha)\), if \(e_i T = 0\) for all \(i \in I(m)\), then \(T = H_{(\lambda, \alpha)}\). Let \(k\) be the number of positive entries lying in the body of \(T\). If \(k = 0\), it is clear that \(T = H_{(\lambda, \alpha)}\) since the body and the tail of \(T\) should be \(H_\lambda\) and \(H_\alpha\) by Theorem 2.5 and 3.1 respectively. Suppose that \(k > 0\). Since \(e_i T = 0\) for all \(i \geq 1\), there exists at least one \(\frac{1}{k}\) in \(T\). Also, by the condition (3), there exists at least one \(\frac{1}{k}\) in the body of \(T\). Choose the \(\frac{1}{k}\) which is located in the highest position in the body of \(T\). Note that no \(-1\) can be read before such \(\frac{1}{k}\) with respect to row reading of \(T\) since \(e_i T = 0\) for \(-m + 1 \leq i \leq -1\) and hence all \(-1\) should lie in the \(m\)th row of \(T\). Then applying \(e_0\) to \(T\) changes such \(\frac{1}{k}\) to \(-1\), which is a contradiction. This completes the proof. \(\square\)
Corollary 5.2. Each $w$ in $W$ is $\mathfrak{g}l_{N(m)}$-equivalent to a unique semistandard tableau of shape $(\lambda, \alpha) \in C(m)$.

Proof. It suffices to show that each $w$ in $W$ is $\mathfrak{g}l_{N(m)}$-equivalent to a tableau $T \in B(\lambda, \alpha)$ for some $(\lambda, \alpha) \in C(m)$. Then the uniqueness follows from Proposition 5.1 together with the same arguments as in Corollary 3.3.

First, consider $T = P(w)$ which is $\mathfrak{g}l_{N(m)}$-equivalent to $w$ (cf. (2.2)). Let $b$ be the entry of $T$ located in the first column of the $(m+1)$st row, which is obviously greater than $-1$. Let $T_2$ be the subtableau of $T$ consisting of entries $b'$ such that

1. $b < b'$,
2. $b = b'$ only when $b \in \mathbb{Z} > 0$ or $b'$ is located below $b$.

Let $T_1$ be the complement of $T_2$ in $T$. By Corollary 3.3, $T_2$ is equivalent to a unique quasi-ribbon tableau $T_2'$ as a $\mathfrak{g}l_N$-crystal element.

Let $P'(w)$ be the tableau obtained by placing the first node of $T_2'$ from northwest right below the leftmost node in the $m$th row of $T_1$. By construction, we have $P'(w) \in B(\lambda, \alpha)$ for some $(\lambda, \alpha) \in C(m)$. Clearly, $w$ is $\mathfrak{g}l_{(m|0)}$-equivalent to $P'(w)$. Let $T_1'$ be the subtableau of $T_1$ consisting of positive entries, then $w$ is $\mathfrak{g}l_N$-equivalent to $T_1' \otimes T_2'$ and hence to $P'(w)$ by Proposition 4.5. In particular, we have $P'(x_i w) = x_i P'(w)$ for $x = e, f$ and $i \neq 0$. Finally, we can check that $x_0 P'(w) \neq 0$ if and only if $x_0 P'(w) \neq 0$, where $x_0 P'(w) = P'(x_0 w)$ for $x = e, f$. Since $w$ is an arbitrary given word, we conclude that $w$ is $\mathfrak{g}l_{N(m)}$-equivalent to $P'(w)$. □

From the arguments in Corollary 5.2, we can deduce the following, which is a generalization of Proposition 4.5.

Proposition 5.3. For $\lambda \in \mathcal{P}$ and $(\mu, \alpha) \in C(m)$, the multiplicity of $B(\mu, \alpha)$ in $B_{N(m)}(\lambda)$ is equal to the number of standard tableaux $T$ of shape $\lambda/\mu$ such that $\alpha(T) = \alpha$ and the smallest entry 1 of $T$ lies in the first column of $\lambda$. In particular, $B_{N(m)}(\lambda)$ is connected if the length of $\lambda$ is at most $m$.

Let $\lambda, \mu, \nu$ be partitions with length no more than $m$. Let $LR_{\mu, \nu}^\lambda$ be the set of Littlewood-Richardson tableaux of shape $\lambda/\mu$ with content $\nu$ (cf. [4, 19]). We may also regard $LR_{\mu, \nu}^\lambda$ as the set of tableaux $S \in B_{(m|0)}(\nu)$ such that $H_{\mu} \otimes S$ is $\mathfrak{g}l_{(m|0)}$-equivalent to $H_{\lambda}$ (see [21]).

Let $(\lambda, \alpha), (\mu, \beta), (\nu, \gamma) \in C(m)$ be given. Define $LR_{(\mu, \beta)(\nu, \gamma)}^{(\lambda, \alpha)}$ to be the set of quadruple $(S, T_1, T_2, w)$ satisfying the following conditions:

1. $S \in LR_{\eta, \zeta}^{\lambda}$, for some $\eta \subset \mu$ and $\zeta \subset \nu$,
2. $T_1 \in ST(\mu/\eta)$, and $T_2 \in ST(\nu/\zeta)$,
3. $w \in L(w_{\alpha(T_1)}, \beta \cup w_{\alpha(T_2)}, \gamma)$, and the composition corresponding to the descent set of $w$ is $\alpha$ (see Section 4.1 and 4.2).
4. Let $w_*$ be the smallest element in $w$, which is a node in $\mu$ or $\nu$ and then there is at least one $-1$ preceding the entry of $w_*$ when we read the word associated to an element $T_1 \otimes T_2$ in $B(\mu, \beta) \otimes B(\nu, \gamma)$, where $\eta$ in $T_1$ and $\zeta$ in $T_2$ are filled with $H_\eta$ and $S$, respectively.
Recall that given two compositions $\sigma = (\sigma_1, \ldots, \sigma_r)$ and $\tau = (\tau_1, \ldots, \tau_s)$, we mean by $\sigma \cdot \tau$ the concatenation $\sigma \cdot \tau = (\sigma_1, \ldots, \sigma_r, \tau_1, \ldots, \tau_s)$, and given $T_1 \otimes T_2 \in B(\mu, \beta) \otimes B(\nu, \gamma)$, the associated word is given by the juxtaposition $w(T_1) \cdot w(T_2)$.

**Proposition 5.4.** Let $(\lambda, \alpha), (\mu, \beta), (\nu, \gamma) \in \mathcal{C}(m)$ be given. The multiplicity of $B(\lambda, \alpha)$ in $B(\mu, \beta) \otimes B(\nu, \gamma)$ is equal to $|LR_{(\mu, \beta)(\nu, \gamma)}^{(\lambda, \alpha)}|$.

**Proof.** Let $U_1 \otimes U_2 \in B(\mu, \alpha) \otimes B(\nu, \beta)$ be such that $e_i(U_1 \otimes U_2) = 0$ for all $i \in I(m)$. Since $U_1 \otimes U_2$ is $\mathfrak{gl}_{[m|0]}$-equivalent to $H_\lambda$ for some $\lambda \in \mathcal{P}$, the subtableau of $U_1$ (resp. $U_2$) consisting of negative entries is equal to $H_\eta$ (resp. $S \in LR_{\eta}(\lambda)$) for some $\eta \subseteq \mu$ and $\zeta \subseteq \nu$.

Next, the subtableau of $U_1$ (resp. $U_2$) consisting of positive entries is $\mathfrak{gl}_{\lambda}$-equivalent to $\eta$-quasi-ribbon tableau $U_1^+$ (resp. $U_2^+$) of shape $\alpha(T_1) \cdot \beta$ (resp. $\alpha(T_2) \cdot \gamma$), where $T_i$ ($i = 1, 2$) is the standard tableau uniquely determined by the connected component of the subtableau in the body of $U_i$ consisting of positive entries (cf. Proposition 4.3). Since $e_r(U_1 \otimes U_2) = 0$ for all $r \in \frac{1}{2}\mathbb{Z}_{>0}$, $U_1^+ \otimes U_2^+$ uniquely determines a linear extension $w$ of $w_{\alpha(T_1) \cdot \beta} \cup w_{\alpha(T_2) \cdot \gamma}$ by Proposition 4.3. Then, we can check that $w$ satisfies the condition (4) in the above definition since $e_0(U_1 \otimes U_2) = 0$. Hence, we have $(S, T_1, T_2, w) \in LR_{(\mu, \beta)(\nu, \gamma)}^{(\lambda, \alpha)}$, where $\alpha$ is the composition of the descent set of $w$.

It is not difficult to see that the correspondence from $U_1 \otimes U_2$ to $(S, T_1, T_2, w)$ is reversible. Therefore, the number of highest weight vectors in $B(\mu, \beta) \otimes B(\nu, \gamma)$, whose connected components are isomorphic to $B(\lambda, \alpha)$ for $(\lambda, \alpha) \in \mathcal{C}(m)$ is equal to $|LR_{(\mu, \beta)(\nu, \gamma)}^{(\lambda, \alpha)}|$.

### 5.2. Branching rule.

Suppose that $\mathcal{S}$ and $\mathcal{T}$ are linearly ordered $\mathbb{Z}_2$-graded sets. We say that $\mathcal{S}$ and $\mathcal{T}$ are isomorphic if there exists a bijection from $\mathcal{S}$ to $\mathcal{T}$, which preserves both orderings and $\mathbb{Z}_2$-gradings, and write $\mathcal{S} \simeq \mathcal{T}$. Let $\prec$ denote the linear ordering on $\mathcal{S}$ and $\sigma$ a permutation on $\mathcal{S}$. Then $\sigma$ induces another linear ordering $\prec_\sigma$ on $\mathcal{S}$ given by $a \prec_\sigma b$ if and only if $\sigma^{-1}(a) \prec_\sigma \sigma^{-1}(b)$ for $a, b \in \mathcal{S}$. We denote by $\mathcal{S}_\sigma$ the set $\mathcal{S}$ with this new ordering.

Suppose that $\mathcal{S} = [m|n]$ for $m, n \geq 1$. The associated Borel subalgebra of $\mathfrak{gl}_{[m|n]}$ is called standard. We assume that $m \geq n$ for convenience. Let $\sigma$ be a permutation on $[m|n]$. Note that $\sigma$ induces a natural isomorphism of Lie superalgebras from $\mathfrak{gl}_{[m|n]}$ to $\mathfrak{gl}_{[m|n]}$, but the corresponding Borel subalgebras are not conjugate in general. Moreover, $\Delta_{[m|n]}^\sigma$ may have more than one odd isotropic simple roots, while $\Delta_{[m|n]}$ has only one.

Now, let $\omega$ be a permutation on $[m|n]$ such that the number of odd isotropic simple roots is maximal. We only have to consider a shuffle of $\{-m, \ldots, -1\}$ and $\{1, \ldots, n\}$, which in fact corresponds to a composite of a sequence of simple odd reflections $\sigma_{22}$. We may choose a unique $\omega$ such that $[m|n]^{\omega} \simeq N(p)^{\leq q}$ with $p = m - n$ and $q = n$.

**Example 5.5.**

$$[4|2]^{\omega} = \{-4 \prec_\omega -3 \prec_\omega 1 \prec_\omega -2 \prec_\omega 2 \prec_\omega -1\} \simeq N(2)^{\leq 2}.$$
Proposition 5.6. For $\lambda \in \mathcal{P}_{m|n}$, there exists a $\mathfrak{gl}_{(m|n)}$-crystal structure on $B_{(m|n)}(\lambda)$. For $(\mu, \alpha) \in \mathcal{C}(m-n)$, the multiplicity of $B_{(m|n)}(\mu, \alpha)$ in $B_{(m|n)}(\lambda)$ is equal to the number of standard tableaux $T$ of shape $\lambda/\mu$ such that $\alpha(T) = \alpha$ and the smallest entry 1 of $T$ lies in the first column of $\lambda$.

**Proof.** Consider the bijection $\phi : B_{(m|n)}(\lambda) \rightarrow B_{(m|n)}(\lambda)$ given by the super-analogue of switching algorithm of Benkart, Sottile and Stroomer \[ (see also \[8, 14, 24]) \]. Given $T \in B_{(m|n)}(\lambda)$, we define $x_\alpha T = \phi^{-1}(x_\alpha \phi(T))$ for $x = e, f$ and $\alpha \in \Delta_{(m|n)}$. Hence, $B_{(m|n)}(\lambda)$ becomes a $\mathfrak{gl}_{(m|n)}$-crystal, which is isomorphic to $B_{(m|n)}(\lambda)$. We obtain the decomposition of $B_{(m|n)}(\lambda)$ by Proposition \[5.3\].

Remark 6.2. For a permutation $\sigma$ on $[m|n]$, $B_{(m|n)}(\lambda)$ has a $\mathfrak{gl}_{(m|n)}$-crystal structure by the same arguments in Theorem \[5.6\].

6. **Super quasi-symmetric functions**

6.1. **Characters of $\mathfrak{gl}_{(m|n)}$-crystals.** Let $S$ be a linearly ordered $\mathbb{Z}_2$-graded set. Let $z = z_S = \{ z_b \mid b \in S \}$ be the set of formal variables. For $\mu = \sum_{b \in S} \mu_b z_b \in P$, we set $z^\mu = \prod_{b \in S} z_b^{\mu_b}$. For a connected component $C$ in $W_S$, let $\chi_C = \sum_{w \in C} z^{\text{wt}(w)}$ be the character of C, which is a well-defined formal series in $z$. Define $R_S$ to be the $\mathbb{Z}$-span of $\{ \chi_C(w) \mid w \in W_S \}$.

**Proposition 6.1.** $R_{N(m)}$ is a commutative ring with a $\mathbb{Z}$-basis $\{ \chi_{B_{N(m)}}(\lambda, \alpha) \mid (\lambda, \alpha) \in \mathcal{C}(m) \}$ for $m \in \mathbb{Z}_{\geq 0}$.

**Proof.** It follows from Corollary \[5.2\] and Proposition \[5.4\].

Remark 6.2. (1) The structure constants for $R_{N(m)}$ are given in Proposition \[5.3\].

(2) When $m = 0$, $R_N$ is isomorphic to the ring of quasi-symmetric functions by Proposition \[4.7\] and Proposition \[6.1\]. In fact, $\chi_{B_N}(\alpha)$ for $\alpha \in \mathcal{C}$ is a super quasi-symmetric function introduced in \[7\].

For $m \geq n \geq 1$, consider $R_{[m|n]}$ the ring of super symmetric polynomials with $m+n$ variables which is spanned by hook Schur polynomials

$$
\chi_{B_{[m|n]}}(\lambda) = h_{\lambda}(z_m, \ldots, z_1; z_{1}, \ldots, z_n)
$$

for $\lambda \in \mathcal{P}_{m|n}$ (cf. \[19\]). Note that $\chi_{B_{[m|n]}}(\lambda)$ does not depend on the ordering of the variables $z_m, \ldots, z_1, z_{1}, \ldots, z_n$, but the crystal structure on $B_{[m|n]}(\lambda)$ depends on the ordering on $[m|n]$. Hence, we have a natural quasi-analogue for $R_{[m|n]}$ by Proposition \[5.6\].

**Proposition 6.3.** $R_{[m|n]}$ is a subring of $R_{[m|n]}$, where $R_{[m|n]}$ is isomorphic to $R_{N(m-n)\leq n}$.

Let $p \in \mathbb{Z}_{\geq 0}$ and $q \in \mathbb{Z}_{\geq 0}$ be given. By \[5.2\], we have

$$
\chi_{B_{N(\leq q)}}(2^{2q-1}, 1) = \prod_{i \in N(\leq q) \downarrow i} (zi + z_{i+1}).
$$

In general, we have the following factorization property of $\chi_{B_{N(p)\leq q}}(\lambda, \alpha)$ (cf. \[8\]).
Proposition 6.4. For $(\lambda, \alpha) \in \mathcal{C}(p)$, suppose that the number of corners in $\alpha$ is either $2q - 1$ or $2q$. Then
\[
\operatorname{ch} B_{N(p)}(\lambda, \alpha) = h(s(z_p) \cdots z_1; z_{\frac{1}{2}}) \operatorname{ch} B_{N}\alpha(\alpha) = z^\mu h(s(z_p) \cdots z_1; z_{\frac{1}{2}}) \prod_{i \in N^{\geq q - \frac{1}{2}}} (z_i + z_{i+\frac{1}{2}}),
\]
where $\mu = \operatorname{wt}(H(\lambda)) - \operatorname{wt}(H_{(2q-1,1)})$.

Proof. By Proposition 3.8, we observe that for $T \in B_{N(p)}(\lambda, \alpha)$, the entries in $T$, which are greater than $\frac{1}{2}$ and different from those at the same place in $H(\lambda, \alpha)$, occur only in the corners of its tail. Hence we obtain a bijection from $B_{N(p)}(\lambda, \alpha)$ to $B_{N(p)}(\lambda) \times B_{N}(\alpha)$. Since $\operatorname{ch} B_{N(p)}(\lambda)$ is a hook Schur polynomial, this establishes the above identities. \qed

6.2. Characterization of super quasi-symmetric functions. We will give an algebraic characterization of $R_{N(m)}$ or $R_{N(m)}(\alpha)$ for $m \in \mathbb{Z}_{\geq 0}$ and $\alpha \in \frac{1}{2}\mathbb{Z}_{> 0}$, which is a quasi-analogue of Stembridge’s result on super symmetric polynomials [26].

Proposition 6.5 (cf. [26]). Let $t$ be an indeterminate. For $(\lambda, \alpha) \in \mathcal{C}(m)$, we have

(1) $\operatorname{ch} B_{N(m)}(\lambda, \alpha)$ is symmetric with respect to $\{ z_i | i = -m, \ldots, -1 \}$,
(2) $\operatorname{ch} B_{N(m)}(\lambda, \alpha)|_{z_r = -z_{r+1} = t}$ is independent of $t$, for $(r, s) = (-1, \frac{1}{2})$ or $s = r + \frac{1}{2}$ with $r \in \frac{1}{2}\mathbb{Z}_{> 0}$. In particular, we have $\operatorname{ch} B_{N(m)}(\lambda, \alpha)|_{z_r = -z_{r+1} = t} = \operatorname{ch} B_{N(m)}(\lambda, \alpha)$ for $r \in \frac{1}{2}\mathbb{Z}_{> 0}$.

Proof. Note that when restricted to a $\mathfrak{gl}_{N(m)}(\frac{1}{2})$-crystal, $B_{N(m)}(\lambda, \alpha)$ is a direct sum of $B_{N(m)}(\lambda, \alpha)$’s for $\mu \in \mathcal{P}_{m+1}$ up to isomorphism. Hence the condition (1) is satisfied, and $\operatorname{ch} B_{N(m)}(\lambda, \alpha)|_{z_r = -z_{r+1} = t}$ is independent of $t$ by the characterization of hook Schur polynomial in [26] (see also [19]).

Now, suppose that $r \in \frac{1}{2}\mathbb{Z}_{> 0}$ is given. Then $B_{N(m)}(\lambda, \alpha)$ is a disjoint union of
\[
B_{N(m)}(\lambda, \alpha)^0 = \{ T | e_i T = f_i T = 0 \},
B_{N(m)}(\lambda, \alpha)^+ = \{ T | f_i T \neq 0 \},
B_{N(m)}(\lambda, \alpha)^- = \{ T | e_i T \neq 0 \}.
\]
Moreover, $f_r$ gives a bijection from $B_{N(m)}(\lambda, \alpha)^+$ to $B_{N(m)}(\lambda, \alpha)^-$. Since the number of occurrences of $r$ in $T \in B_{N(m)}(\lambda, \alpha)^+$ has the different parity from that of $f_r T \in B_{N(m)}(\lambda, \alpha)^-$. It follows that
\[
\operatorname{ch} B_{N(m)}(\lambda, \alpha)^+|_{z_r = -z_{r+1} = t} = -\operatorname{ch} B_{N(m)}(\lambda, \alpha)^-|_{z_r = -z_{r+1} = t}.
\]
This implies that $\operatorname{ch} B_{N(m)}(\lambda, \alpha)|_{z_r = -z_{r+1} = t} = \operatorname{ch} B_{N(m)}(\lambda, \alpha)^0$ which is independent of $t$. On the other hand, $B_{N(m)}(\lambda, \alpha)^0$ is the set of semistandard tableaux of shape $(\lambda, \alpha)$ with entries in $N(m) \setminus \{ r, r + \frac{1}{2} \}$. Hence, by definition, we have
\[
\operatorname{ch} B_{N(m)}(\lambda, \alpha)^0 = \operatorname{ch} B_{N(m) \setminus \{ r, r + \frac{1}{2} \}}(\lambda, \alpha).
\]
The condition (2) is satisfied. \qed
\textbf{Theorem 6.6.} Suppose that $f$ is a polynomial in $z_{N(m)\leq n}$ with integral coefficients. Then $f \in R_{N(m)\leq n}$ if and only if

1. $f$ is symmetric with respect to $\{ z_i \mid i = -m, \ldots, -1 \}$,
2. $f|_{z_i=-z_{i+1}=t}$ is independent of $t$, for $(r,s) = (-1,\frac{1}{2})$ or $s = r + \frac{1}{2}$ with $r \in \frac{1}{2}\mathbb{Z}_{>0}$.

\textbf{Proof.} Let $\mathcal{R}_{m,n}$ be the ring of polynomials $f$ in $z_{N(m)\leq n}$ with integral coefficients satisfying (1) and (2). By Proposition \[15.5\] it is enough to show that $\mathcal{R}_{m,n} \subset R_{N(m)\leq n}$. We will use induction on $n$. It is clear when $n = \frac{1}{2}$ by the characterization of super symmetric polynomials \[26\].

Suppose that $n \geq 1$ and $f \in \mathcal{R}_{m,n}$ is given. By induction hypothesis, we have

$$f|_{z_{-i}=-z_i=t} = \sum_{(\lambda,\alpha) \in \mathcal{C}(m)} c_{(\lambda,\alpha)} g_{(\lambda,\alpha)},$$

where $g_{(\lambda,\alpha)} = \text{chB}_{N(m)\leq n-\frac{1}{2}}(\lambda,\alpha)$ with $c_{(\lambda,\alpha)} \in \mathbb{Z}$. Put

$$h = f - \sum_{(\lambda,\alpha)} c_{(\lambda,\alpha)} \text{chB}_{N(m)\leq n}(\lambda,\alpha).$$

Since $h|_{z_{-i}=-z_i=t} = 0$, we have $h = (z_{-\frac{1}{2}} + z_n)h^{(1)}$ for some polynomial $h^{(1)}$. Consider

$$h|_{z_{-i}=-z_i=t} = (-t + z_{n-1}) \left( h^{(1)}|_{z_{-i}=-z_i=t} \right).$$

Since the left-hand side is independent of $t$, the right-hand side must be zero, and hence $(z_{n-1} + z_{n-\frac{1}{2}})$ divides $h^{(1)}$. Repeating this procedure, we conclude that $\prod_{i \in \mathbb{Z}_{<0}} (z_i + z_{i+\frac{1}{2}})$ divides $h$. Moreover, since $h$ is super symmetric, we have $h = \prod_{i \in \mathbb{Z}_{<0}} (z_i + z_{i+\frac{1}{2}}) \prod_{i = -m}^{-1} (z_i + z_\frac{1}{2}) k$, where $k$ is a symmetric polynomial in $\{ z_{-m}, \ldots, z_{-1} \}$ with coefficients in $\mathbb{Z}[\mathbb{Z}, \ldots, z_n]$. By Proposition \[6.3\] it follows that $h$ is an integral linear combination of $\text{chB}_{N(m)\leq n}(\lambda,\alpha)$’s and so is $f$. This completes the proof. \qed

\textbf{Remark 6.7.} (1) For each $k$, let $R^k_{N(m)\leq n}$ be the subgroup of polynomials of homogeneous degree $k$ in $R_{N(m)\leq n}$, whose inductive limit is defined in a standard way and denoted by $R^k_{N(m)}$. Then we have $R_{N(m)} = \bigoplus_{k \geq 0} R^k_{N(m)}$.

(2) Let $R^\circ_N$ be the subring of $R_N$ with a $\mathbb{Z}$-basis $\{ \text{chB}_N(\lambda) \mid \lambda \in \mathcal{P} \}$. It is well-known as the ring of super symmetric functions \[3\] \[19\]. Then using the characterization of super symmetric polynomials in \[26\], one can easily deduce that for $f \in R_N$, $f$ is super symmetric, that is, $f \in R^\circ_N$ if and only if $f$ is symmetric with respect to both $z_N$ and $z_{\frac{1}{2} + z_\geq 0}$.

\textbf{Acknowledgement} Part of this this work was done during the author’s visit at National Taiwan University 2006 Summer. He thanks Prof. S.-J. Cheng for the invitation and many helpful discussions. He also thanks Prof. S.-J. Kang and G. Benkart for their interests in this work.

\textbf{References}

[1] G. Benkart, S.-J. Kang, M. Kashiwara, \textit{Crystal bases for the quantum superalgebra $U_q(\mathfrak{gl}(m,n))$}, J. Amer. Math. Soc. \textbf{13} (2000) 295-331.
[2] G. Benkart, F. Sottile, J. Stroomer, Tableau switching: algorithms and applications, J. Combin. Theory Ser. A 76 (1996) 11–43.
[3] A. Berele, A. Regev, Hook Young diagrams with applications to combinatorics and to the representations of Lie superalgebras, Adv. Math. 64 (1987) 118–175.
[4] W. Fulton, Young tableaux, London Mathematical Society Student Texts, 35. Cambridge University Press, Cambridge, 1997.
[5] I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V.S. Retakh, J.-Y. Thibon, Noncommutative symmetric functions, Adv. Math. 132 (1997) 218–348.
[6] I.M. Gessel, Multipartite P-partitions and inner products of skew Schur functions, Contemporary Math. 34 (1985) 289-301.
[7] J. Haglund, M. Haiman, N. Loehr, J.B. Remmel, A. Ulyanov, A combinatorial formula for the character of the diagonal coinvariants, Duke Math. J. 126 (2005) 195–232.
[8] M. Haiman, On mixed insertion, symmetry, and shifted Young tableaux, J. Combin. Theory Ser. A 50 (1997) 196–225.
[9] V. G. Kac, Lie superalgebras, Adv. Math. 26 (1977) 8–96.
[10] S.-J. Kang, J.-H. Kwon, Tensor product of crystal bases for $U_q(\mathfrak{gl}(m,n))$-modules, Comm. Math. Phys. 224 (2001) 705–732.
[11] M. Kashiwara, Crystalizing the $q$-analogue of universal enveloping algebras, Comm. Math. Phys. 133 (1990) 249–260.
[12] M. Kashiwara, On crystal bases, Representations of groups, CMS Conf. Proc., 16, Amer. Math. Soc., Providence, RI, (1995) 155–197.
[13] J.-H. Kwon, Crystal graphs for Lie superalgebras and Cauchy decomposition, J. Algebraic Combin. 25 (2007) 57–100.
[14] J.-H. Kwon, Rational semistandard tableaux and character formula for the Lie superalgebra $\hat{\mathfrak{gl}}_{\infty|\infty}$, Adv. Math. to appear, arXiv:math.RT/0605005.
[15] D. Knuth, Permutations, matrices, and the generalized Young tableaux, Pacific J. Math. 34 (1970) 709–727.
[16] D. Krob, J.-Y. Thibon, Noncommutative symmetric functions. IV. Quantum linear groups and Hecke algebras at $q = 0$, J. Algebraic Combin. 6 (1997) 339–376.
[17] D. Krob, J.-Y. Thibon, Noncommutative symmetric functions. V. A degenerate version of $U_q(\mathfrak{gl}_N)$, Dedicated to the memory of Marcel-Paul Schützenberger. Internat. J. Algebra Comput. 9 (1999) 405–430.
[18] B. Leclerc, J.-Y. Thibon, The Robinson-Schensted correspondence, crystal bases, and the quantum straightening at $q = 0$, The Foata Festschrift. Electron. J. Combin. 3 (1996), Research Paper 11, approx. 24 pp. (electronic).
[19] I. G. Macdonald, Symmetric functions and Hall polynomials, Oxford University Press, 2nd ed., 1995.
[20] P. A. MacMahon, Combinatorial analysis, Cambridge University Press, Cambridge, 1915.
[21] T. Nakashima, Crystal base and a generalization of the Littlewood-Richardson rule for the classical Lie algebras, Comm. Math. Phys. 154 (1993) 215–243.
[22] I. Penkov, V. Serganova, Representations of classical Lie superalgebras of type I, Indag. Math. (N.S.) 3 (1992) 419–466.
[23] J. B. Remmel, The combinatorics of $(k,l)$-hook Schur functions, Contemp. Math. 34 (1984) 253-287.
[24] J. B. Remmel, A bijective proof of a factorization theorem for $(k,l)$-hook Schur functions, Linear and Multilinear Algebra 28 (1990) 119–154.
[25] M.P. Schützenberger, Quelques remarques sur une construction de Schensted, Math. Scand. 12 (1963) 117–128.
[26] J.R. Stembridge, A characterization of supersymmetric polynomials, J. Algebra 95 (1985) 439–444.
[27] J.R. Stembridge, Enriched P-partitions, Trans. Amer. Math. Soc. 349 (1997) 763–788.
[28] J.-Y. Thibon, *Lectures on noncommutative symmetric functions*, Interaction of combinatorics and representation theory, 39–94, MSJ Mem., 11, Math. Soc. Japan, Tokyo, 2001.

Department of Mathematics, University of Seoul, 90 Cheonnong-dong, Dongdaemun-gu, Seoul 130-743, Korea

E-mail address: jhkwon@uos.ac.kr