Generalized particle model: a possible source for 
Dark Energy

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Abstract. We propose a generalized particle dynamics scenario in brane world formalisms, 
for an asymptotically flat as well as for an asymptotically anti de Sitter background. The 
generalization implies that the effective mass of the particle becomes dynamical. The present 
framework results in a new model that accounts for the late acceleration of the universe. An 
effective Dark Energy equation of state, exhibiting a phantom like behavior, is generated. The 
model is derived by embedding the physical FRW universe in an effective space-time, induced 
by the generalized particle dynamics. We corroborate our results with present day observed 
cosmological parameters.

1. Introduction

In recent times theoretical understanding of Dark Energy (DE) has become the Holy Grail of 
cosmological investigations. Existence of DE is unavoidable if one wants to explain the (present 
day) accelerated expansion of the universe. However, if the dynamical laws of motion pertaining 
to General Relativity are held sacred, it seems inevitable that a paradigm shift in the properties 
of DE is needed. This is simply because normal matter creates a positive pressure that 
decelerates the universe expansion whereas DE has to generate a negative pressure to have the 
accelerated expansion.

Observational vindication [1] of the present-day acceleration of the universe, and the 
subsequent precise measurements of observable parameters [2, 3, 4] indicate that the entity 
Dark Energy (DE) which is responsible for the recent acceleration contribute to \(\sim 70\%\) of 
cosmic energy density. This DE density-fraction of total cosmic density (\(\Omega_{\text{DE}}\)) is determined 
conclusively from several independent probes (\(\Omega_{\text{DE}} = 0.726 \pm 0.015\) at 95\% C.L. from latest 
WMAP5 data [3]). On the other hand, DE effective Equation of State (EOS) (\(w_{\text{DE}}\)) is still 
inconclusive. But the big surprise is that this agent is even more exotic in nature than imagined 
before because there is some evidence that the EOS can cross the so-called Phantom Divider 
(i.e. \(w_{\text{DE}} = -1\)). Though SNLS data show no general behavior for \(w_{\text{DE}} < -1\), the analysis of 
the most reliable SNIa Gold dataset show strong indication that \(w_{\text{DE}} < -1\) [5] (the lower bound 
being \(-1.11 < w_{\text{DE}} \text{ from WMAP5 data [3]}\)), leading to the conclusion that models with phantom divider crossing are preferred over ΛCDM (or quintessential candidates) at 2σ level. This clearly 
weakens the claims of cosmological constant Λ or dynamical models like quintessence, Kessence, 
Chaplygin gas etc. [6] as viable DE models. One can resurrect the scalar field models only at 
the cost of phantom fields (quintom models) [7], with a negative kinetic energy term but they 
bring in severe instability problems and are better avoided.

In this perspective, instead of looking for contrived and phenomenologically motivated DE 
models, it seems reasonable to explore modified gravity theories [8, 9] which do not suffer from 
any such major drawbacks. But one might still feel skeptical since more often than not explicit
forms of these modified gravity theories appear to be designer made and/or fine tuned without a deeper dynamical framework based on first principles. In our work, accelerated expansion emerges in a brane world scenario. The modified gravity provides the metric of the higher dimensional spacetime in which the brane is embedded. (This well established formalism is explained below and later as we proceed.) We emphasize that in this scheme conventional relativistic dynamical principles are maintained and an extended form of spin-orbit coupling is introduced. We also have extended our model to physical (3+1)-dimensions. We also demonstrate that the phantom-like behavior can be induced without explicitly invoking the phantom field with negative kinetic energy term. It is well-known (for details see [9, 10, 13]) that by embedding techniques one can relate cosmological surface dynamics (Friedmann equations) in lower (e.g., 3+1) dimensions with particle motion in a higher (e.g., 4+1) dimensional black-hole space-time. In fact the latter is taken as asymptotically anti de Sitter space-time, taking Schwarzschild-anti de Sitter (Sch-AdS) as a representative example. In the standard brane world scenario the AdS background induces an effective cosmological constant. However, a more interesting situation occurs in our framework. Here the AdS bulk induces a dynamical quantity which is essential in imparting the phantom behavior. We demonstrate that in the subsequent cosmological scenario, the induced effective negative pressure can result in an accelerating universe, even capable of showing phantom behavior. We further establish our model by an analysis of the equation of state and a determination of the relevant parameters describing the evolution of the observable universe.

2. The Generalized Particle Formulation

We start with a generalized relativistic point particle model in a (3 + 1) dimensional spacetime where the action is given by:

\[ S = \int L \, d\tau = m \int d\tau \left[ \left( \frac{1}{2e} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - \frac{e}{2} \right) - \lambda g_{\mu\nu} \xi^\mu \dot{x}^\nu + \frac{e\lambda^2}{2} g_{\mu\nu} \xi^\mu \xi^\nu + \frac{e\beta \lambda^2}{2} \right]. \] (1)

Here \( m \) is the mass of the particle, \( \tau \) is the worldline evolution parameter, \( x^\mu(\tau) \) are the particle co-ordinates, \( e(\tau) \) is the worldline einbein introduced to make the action reparametrization invariant and the Lagrange multiplier \( \lambda(\tau) \) is an auxiliary worldline scalar variable. Furthermore \( \beta \) is a numerical constant, whilst the metric \( g_{\mu\nu}(x) \) and the vector \( \xi^\mu(x) \) are arbitrary functions of the co-ordinates \( x^\mu \).

The action is invariant under infinitesimal co-ordinate transformations of the form \( \delta x^\mu = \alpha \xi^\mu(x) \), provided the Lie-derivative of the metric with respect to \( \xi \) vanishes:

\[ \xi^\lambda \partial_\lambda g_{\mu\nu} + g_{\nu\lambda} \partial_\mu \xi^\lambda + g_{\mu\lambda} \partial_\nu \xi^\lambda = 0. \] (2)

This shows that \( \xi^\mu(x) \) is the Killing vector associated with the symmetry of the metric.

2.1. Hamiltonian formulation

To analyze the dynamics implied by the action (1), we follow the hamiltonian analysis of constrained systems as formulated by Dirac [15]. The conjugate canonical momenta \( p_e, p_\lambda, p_\mu \) are computed and the Hamiltonian follows,

\[ H = p_e \dot{e} + p_\lambda \dot{\lambda} + p_\mu \dot{x}^\mu - L = \frac{e}{2m} [p^2 + m^2 + 2m\lambda(x, p) - \beta m^2 \lambda^2]. \] (3)
The two significant constraints are:

\[ \psi_1 \equiv e(\xi \cdot p - m\beta\lambda) \approx 0; \quad \psi_2 \equiv p^2 + m^2 + \frac{1}{\beta}(\xi \cdot p)^2 \approx 0. \tag{4} \]

The first constraint signifies a conserved quantity associated with Killing vector and the second constraint is nothing but the modified mass-shell condition. It is interesting to note that in the limit \( \beta \to \infty \), we get back the normal particle model.

3. Our model in Schwarzschild background

We apply our generalized particle model in the \((3 + 1)\)-dimensional Schwarzschild background where the metric is given by

\[ g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2. \tag{5} \]

We choose the Killing vector corresponding to rotational symmetry as: \( \xi_{(\phi)}^\mu = (0, 0, 0, 1) \). Without loss of generality, we can restrict the particle to move in the equatorial plane and our generalized action in the equatorial plane \( \theta = \pi/2 \) takes the form:

\[ S = m\int \left[ \frac{1}{2e} \left(-\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{1 - \frac{2M}{r}} + r^2\dot{\theta}^2 + r^2\sin^2\theta \dot{\phi}^2 \right) - \frac{e}{2} - \lambda r^2\dot{\phi} + \frac{e\lambda^2}{2}(r^2 + \beta) \right] d\tau. \tag{6} \]

It should be noted that for our choice of Killing vector, \( (\xi \cdot p) = p_\phi \). The Hamiltonian is given by

\[ H = \frac{e}{2m}\left[m^2 - \frac{p_t^2}{1 - \frac{2M}{r}} + \left(1 - \frac{2M}{r}\right)p_r^2 + \frac{p_\phi^2}{r^2} + 2m\lambda p_\phi - \beta m^2\lambda^2 \right]. \tag{7} \]

Performing the same constraint analysis as before whilst fixing the gauge \( e = 1 \), we find the modified mass-shell constraint and the relation,

\[ 2mH = m^2 - \frac{p_t^2}{1 - \frac{2M}{r}} + \left(1 - \frac{2M}{r}\right)p_r^2 + \left(\frac{1}{r^2} + \frac{1}{\beta}\right)p_\phi^2 = 0 \quad ; \quad p_\phi = m\beta\lambda. \tag{8} \]

Consequently the Hamiltonian equations of motion are:

\[ \dot{p}_\phi = \{p_\phi, H\} = 0 \Rightarrow p_\phi = m\beta \frac{r^2\dot{\phi}}{r^2 + \beta} = m\beta\lambda = ml, \tag{9} \]

\[ \dot{p}_t = \{p_t, H\} = 0 \Rightarrow p_t = m(1 - \frac{2M}{r})\dot{t} = m\varepsilon = E. \tag{10} \]

So we have the two conserved quantities: \( p_\phi = ml \) and \( p_t = E \), where \( l \) and \( E \) are two different constants. It is worth mentioning here that \( ml \) and \( E \) are nothing but the angular momentum and energy respectively, for the particle moving in Schwarzschild background, but now depending on the parameter \( \beta \) as well. In the large \( \beta \) limit the standard results are regained.

The modified mass-shell constraint now can be written as:

\[ p^2 + m^2 + \frac{m^2l^2}{\beta} = 0 \Rightarrow p^2 + \mu^2m^2 = 0, \tag{11} \]
where $\mu$ is a dimensionless parameter defined by the relation

$$\mu^2 = 1 + \frac{l^2}{\beta}$$

and we have taken the velocity of light $c = 1$. Thus for $\mu^2 = 1$, i.e. $\beta \to \infty$, the above equation reduces to the well-known dispersion relation for normal particle model which is

$$p^2 + m^2 = 0.$$ 

It is interesting to observe that for negative $\beta$, the effective mass of the particle is reduced. We proceed further to study this behavior in details.

The radial equation of motion is given by

$$r^2 + \frac{V_{eff}(r)}{m^2} = \varepsilon^2,$$

where the effective potential $V_{eff}(r)$ is given by

$$V_{eff} = \left(1 - \frac{2M}{r}\right) \left(\mu^2 + \frac{l^2}{r^2}\right).$$

We observe from the above equation that the radial dynamics in our generalized model is like that in normal Schwarzschild case with a modified velocity of light $c^2 \to \mu^2 c^2$. We get back normal results in the limit $\beta \to \infty$.

**Figure 1.** Plot for the variation of the effective potential $V$ with the radial coordinate $r$ for $l = 100$ with different values of $\beta$

The effect of different values of $\beta$ on the dynamics of a particle with mass $m$ can be seen in Figure 1, where we plot $V_{eff}(r)$ vs. $r$ for different $\beta$ at a fixed $l = 100$. In this figure the normal line corresponds to $\beta = 10$ whereas the dotted, thick and dashed lines represent the values $\beta = 50, 10^3,-50$ respectively. Also, we have taken $M = 1, m = 0.1$. Thus the thick line (large positive $\beta$) represents the same behavior as in normal Schwarzschild case. But for negative $\beta$ (dashed line), we can clearly see a sign-flip of the effective potential. This negative sign of the effective potential can be interpreted as if there is a repulsive force acting on the particle. We study the cosmological implications of this negative $\beta$ in the following section.
3.1. Cosmological implications of negative $\beta$

In this section we shall point out a crucial implication of our model in the cosmological context. From embedding geometry using the Gauss-Codazzi equation [16] it can be shown that a $(D-1)$-dimensional surface representing a cosmological Friedmann-Robertson-Walker (FRW) metric can be embedded consistently in a $D$-dimensional black hole spacetime in such a way that the expansion of the FRW surface is realized by the particle trajectory along the radial direction in the gravitational field of the black hole (the so called expanding bubble universe) [13]. In the present article, the particle motion induces a $(2 + 1)$-dimensional FRW metric and the effective potential contains essential information for the evolution of the embedded cosmological surface. Below we shall show how this can be realized, followed by a discussion of its cosmological implications.

As demonstrated by observations, we consider the physical matter density of the universe $\rho$ to be small compared to brane tension $\rho_0$. Here we embed $(2 + 1)$ dimensional spatially flat FRW universe with the line element

$$ds^2_3 = -dr^2 + a^2(r)[d\sigma^2 + \sigma^2 d\theta^2]$$

(14)

into the $(3 + 1)$-dimensional black hole space-time induced by our generalized particle model whose line element is given by

$$ds^2_4 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2 d\Omega^2_2,$$

(15)

where $d\Omega^2_2$ is the two-sphere. The radial function $F(r)$ is

$$F(r) = \left( 1 - \frac{2M}{r} \right) \left( \frac{\mu^2 r^2 + l^2}{r^2 + l^2} \right).$$

(16)

As stated before, we identify the radial coordinate $r(t)$ of the particle with the scale factor $a(t)$ of the universe to arrive at the Friedmann equations:

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho + \frac{2M}{a^3} - \frac{l^2}{\beta} \left( \frac{1 - 2M/a}{l^2 + a^2} \right)$$

(17)

$$\ddot{a} = -\frac{4\pi G}{3} (\rho + 3p) - \frac{M}{a^3} - \frac{l^2}{\beta(l^2 + a^2)^2} \left[ l^2 - \frac{Ml^2}{a} + Ma \right].$$

(18)

For late time universe, $a$ should be large which is also the physical region in our model as we have $a > 2M$. Thus, the right hand side of (18) becomes positive for negative $\beta$ and large $a$ implying the late accelerating behavior of our universe. Further, if we consider late time universe and hence taking only the $\beta$-dominating terms in the Friedmann equations, the deceleration parameter $q$ turns out to be:

$$q = -\frac{\ddot{a}}{\dot{a}} = -\frac{l^2}{l^2 + a^2} - \frac{M}{a - 2M}.$$

(19)

This clearly reveals that the deceleration parameter is negative for the cosmologically relevant region $a > 2M$, confirming an accelerated expansion at late time.

Also the behavior for the scale factor can be analyzed to some extent with the above considerations of $\beta$-dominance. In the large $l$ limit, an analytic expression for the scale factor $a(t)$ equations can be obtained by solving the first of the Friedmann equation:

$$a(t) \approx M + \frac{1}{2} \left[ e^{l/\sqrt{-\beta}} - M^2 e^{-l/\sqrt{-\beta}} \right].$$

(20)
In the above equation, the first term is nothing but a scaling and with $M = 1$ the second term reduces to $\sinh(t/\sqrt{-\beta})$, which interpolates between a decelerating phase at early time and an accelerated expansion at late time. Thus our model behaves pretty close to $\Lambda$CDM (which matches very well with observations), with the inverse of the $\beta$ parameter playing the role of the cosmological constant $\Lambda$. Because of this identification, it is worthwhile to emphasize that negative $\beta$ is consistent with small positive value for $\Lambda$ that is favored observationally. This makes the scenario further interesting from observational ground since now one can calculate the other observable parameters and compare them with highly accurate observational data. But it should be remembered that here we have represented a toy model of our universe as it is $(2 + 1)$-dimensional. But this can be easily generalized to the physical $(3 + 1)$-dimensional universe which we do in the following section.

4. An Application to Schwarzschild-Anti de Sitter geometry

We now study the cosmological applications of our generalized particle model in our physical universe. We start with $(4 + 1)$-dimensional Schwarzschild-Anti de Sitter (Sch-AdS) background with the line element

$$ds^2_{4+1} = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2d\Omega^2_3$$  (21)

where

$$F(r) = \left(k - \frac{2M}{r^2} + \Lambda_5 r^2\right)\left(\frac{\mu^2 r^2 + l^2}{r^2 + l^2}\right)$$

and $k = (0, \pm 1)$ is the curvature parameter whilst $\Lambda_5$ is the constant curvature of the spacetime.

We embed a $(3 + 1)$-dimensional FRW spacetime $(T, \sigma, \theta, \varphi)$ with the line element

$$ds^2_{3+1} = -dT^2 + a^2(T)\left[\frac{d\sigma^2}{1 - k\sigma^2} + \sigma^2(d\theta^2 + \sin^2 \theta \ d\varphi^2)\right],$$  (22)

into the $(4 + 1)$-dimensional effective metric (21). The Friedmann equations take the form

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_4}{3}\rho + \frac{2M}{a^4} + \alpha - \Lambda_5 + \frac{\left[\ell^2(2M/a^2 - \Lambda_5 a^2)\right]}{\beta(a^2 + l^2)}$$  (23)

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_4}{3}(\rho + 3p) - \frac{2M}{a^4} + \alpha - \Lambda_5 - \frac{\left[l^2(\Lambda_5 a^4 + 2M + 2\Lambda_5 l^2 a^2)\right]}{\beta(a^2 + l^2)^2}$$  (24)

where $\dot{a} = da/dT$, $8\pi G_4/3 = 2\rho_0 (8\pi G_5/3)^2$ and $\alpha = (8\pi G_5\rho_0/3)^2$. We express the Hubble parameter $H = \dot{a}/a$ in terms of the redshift $z$ (where $1 + z = a_0/a = 1/a$). The terms containing $M (\propto a^{-4})$ contribute to the present day radiation energy density of the universe, which is a very small proportion of total radiation energy density [13]. Hence we neglect contributions from any cosmic constituent which redshifts away at the rate of radiation or faster (i. e. the terms of the order of $(1 + z)^4$ or higher) for a late time universe. With this assumption, we can express the Friedmann equation (23) in the following convenient form:

$$H^2 = H^2_0 \left[\Omega_X (1 + b(1 + z)^2) + \Omega_M (1 + z)^3\right].$$  (25)

where $\Omega_M = \rho_M/\rho_c = 8\pi G_4\rho_M/3H^2_0$ is the density parameter for matter, $\rho_c = 3H^2_0/8\pi G$ is the critical density of our universe and $\Omega_X = (\alpha - \Lambda_5 \mu^2)/H^2_0$ is the density parameter for dark energy. The effect of our modified gravity theory appears in $\Omega_X$ and in the dimensionless parameter $b = \Lambda_5(\mu^2 - 1)^2/(\alpha - \Lambda_5 \mu^2)$. Observations fix $H_0 = 70.5 \pm 1.3$ km/s/Mpc according to the
WMAP5 data [3] and $H_0 = 74.2 \pm 3.6 \text{ km/s/Mpc}$ from HST data [4]. The above equation (25) is the major result of our work the prospects of which we describe below.

$\Omega_M$ is the sum-total of the density-fraction for luminous and dark matter which contributes $0.28 \pm 0.08$ of total cosmic density (fixed independently by the CMB [2] and large scale structure data [17]. For a valid dark energy model, $\Omega_X$ accounts for the DE density. But observations indicate a universe close to the $\Lambda$CDM ($\Omega_X \simeq 0.72, \Omega_M \simeq 0.28$). This forces $b$ to be small and $\beta$ large in magnitude. Observationally, these values will be restricted by $\chi^2$ fitting, which we discuss in our later issue.

4.1. Estimation of observable parameters

First we estimate the luminosity-redshift relation $d_L(z)$ which determines the dark energy density $\Omega_X$ from observations. For our model (25), $d_L(z)$ is given by

$$d_L(z) = (1 + z) \int_0^z \frac{dz'}{H(z')} = \frac{(1 + z)}{H_0} \int_0^z \frac{dz'}{[\Omega_X (1 + b(1+z')^2) + \Omega_M(1 + z')^3]^{1/2}}. \quad (26)$$

In Figure 2 the variation of $d_L(z)$ with redshift $z$ for different $\Omega_X$ is shown using numerical integration. The plots show that a universe with $\Omega_X \sim 0.7, \Omega_M \sim 0.3$ is favored by observation, and confirms that $\Omega_X$ accounts for the dark energy density in our model. The shaded region gives the bound for $b$ as $-0.07 \leq b < 0$. Throughout the rest of the paper, we take a representative small negative value for $b = -0.05$.

The age of the universe has the currently accepted value of $13.7 \pm 0.02 \text{ Gyr}$ [2] which is very close to the estimate of our model.

In our model, the deceleration parameter $q(z)$ takes the form:

$$q(z) = \frac{-\ddot{a}/a}{\dot{a}^2/a^2} = \frac{H'(z)}{H(z)}(1 + z) - 1 = \frac{\Omega_M(1 + z)^3 - 2\Omega_X}{2(\Omega_X(1 + b(1+z)^2) + \Omega_M(1 + z)^3)}. \quad (27)$$

In Figure 3, the deceleration parameter has been plotted against redshift for $\Omega_X \sim 0.7, \Omega_M \sim 0.3$. The plot confirms that our model indeed results in an early decelerating and late accelerating universe. Moreover, onset of the recent accelerating phase, when the universe was $\sim 60\%$ of its
Figure 3. Variation of the deceleration parameter $q$ with redshift present size ($z = 0.6$), is also confirmed by our model. The effective equation of state (EOS) of DE in our model is

$$w_X(z) = \frac{2q(z) - 1}{3[1 - \Omega_M(z)]} = -1 + \frac{2b(1 + z)^2}{3}.$$  \hspace{1cm} (28)

The expression has been obtained by a binomial expansion considering small $b$ and dropping terms of order $(1 + z)^4$ or higher, as before. Obviously, since $b$ is negative and non-zero, the effective EOS of dark energy candidate satisfies $w_X < -1$. So it shows a phantom like behaviour that is as claimed by SN1a Gold data set [5].

Figure 4. Variation of the effective EOS of dark energy with redshift

In Figure 4, the effective EOS parameter of dark energy with redshift is shown for different values of $\Omega_X$. The latest WMAP5 data constrains the lower bound of the dark energy EOS today to be $-1.11 < w_{DE}$ [3]. This sets the lower bound of $b$ at $(-.15 \leq b)$, which is way below its lower bound ($-.07 \leq b$) as predicted before from the luminosity-redshift relation. This model with $(-.07 \leq b < 0)$ will thus fit well with a more precise bound for the EOS available in the future.
The statefinder parameters \[ \{r, s\} \] can distinguish dynamical models from \( \Lambda \)CDM. In our model, \( \{r, s\} \) pair is

\[
\begin{align*}
  r &= 1 - \frac{b \Omega_X (1 + z)^2}{\Omega_X (1 + b (1 + z)^2) + \Omega_M (1 + z)^2} ;
  s &= \frac{2}{3} \frac{b (1 + z)^2}{[3 + b (1 + z)^2]}.
\end{align*}
\]

Thus in our model, as \( b \) is non-trivial, the \( \{r, s\} \) pair can distinguish our alternative gravity model from \( \Lambda \)CDM for which \( r = 1, s = 0 \).

5. Summary and Future prospects

We have constructed a specific modified gravity structure, induced by a higher dimensional non-minimal particle dynamics framework. This particular generalized relativistic particle model was introduced by us in [10]. Subsequently we consider brane models embedded in this modified 4+1-dimensional AdS-Schwarzschild spacetime. The induced particle dynamics can simulate cosmological evolution which is our primary interest. We find a late time expanding universe. The most notable feature of our formulation is the possibility of having an effective phantom dark energy model without the phantom. This is (quantitatively) manifested in the crossover of the phantom barrier. Our model is qualitatively distinct, but not quantitatively far off, from the \( \Lambda \)CDM model during recent times. Hence all the positive features of \( \Lambda \)CDM along with a phantom behavior (without the problems related to the negative kinetic term) can be accommodated in our model.

Several aspects of the proposed framework can be investigated further. The parameters used in the model can be constrained observationally by using a maximum likelihood method involving the minimization of the function \( \chi^2 = \sum_{i=1}^{N} [d_L(z)_{\text{obs}} - d_L(z)_{\text{th}}]^2 / \sigma_i^2 \), where \( d_L(z)_{\text{th}} \) contains the parameters used in a specific theory, \( N \) is the number of Supernovae (taken as 157 for the most reliable Gold dataset) and \( \sigma_i \) are the 1\( \sigma \) errors from the observational method used [6]. Observationally, this is the most accurate probe of \( \Omega_{DE} \); it will further constrain the parameters used in our model. Further, the variable EOS may be reflected in the Integrated Sachs-Wolfe (ISW) effect, which will serve as another test for the model. Studying features related to perturbations in this cosmological framework is another open issue.

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