On Recursive State Estimation for Linear State-Space Models Having Quantized Output Data

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Abstract

In this paper, we study the problem of estimating the state of a dynamic state-space system where the output is subject to quantization. We compare some classical approaches and a new development in the literature to obtain the filtering and smoothing distributions of the state conditioned to quantized data. The classical approaches include the Extended Kalman filter/smooother in which we consider an approximation of the quantizer non-linearity based on the arctan function, the quantized Kalman filter/smooother, the Unscented Kalman filter/smooother, and the Sequential Monte Carlo sampling method also called particle filter/smooother. We consider a new approach based on the Gaussian sum filter/smooother where the probability mass function of the quantized data given the state is modeled as an integral equation and approximated using Gauss-Legendre quadrature. The Particle filter is addressed considering some resampling methods used to deal with the degeneracy problem. Also, the sample impoverishment caused by the resampling method is addressed by introducing diversity in the samples set using the Markov Chain Monte Carlo method. In this paper, we discuss the implementation of the aforementioned algorithms and the Particle filter/smooother implementation is studied by using different resampling methods combined with two Markov Chain algorithms. A numerical simulation is presented to analyze the accuracy of the estimation and the computational cost.

Index Terms

recursive state estimation, state-space models, filtering, smoothing, quantized data.

I. INTRODUCTION

In the last two decades, there has been a growing number of applications for sensors, networks, and sensor networks, where common problems include dealing with loss of information in signals measured with low-resolution sensors, or storing and/or transmitting a reduced representation of signals to minimize the resource consumption in a communication channel [1]. This kind of problem encompasses a nonlinear process called Quantization, which divides the input signal space into a finite or infinite (countable) number of subspaces and represents each of them by a single output value [2]. Applications with quantized data include networked control [3], [4], fault detection [3], [5], [6], cyber-physical systems [8], [9], multitarget tracking [10], and system identification [11], [12], [13], [14], to mention a few. In these applications, a key part of the development process is the ability to estimate the state of a dynamical system conditioned to the available quantized observations. For instance, [15] deals with the problem of state estimation and control of a microgrid incorporating multiple distributed energy resources, where the state estimation and control is based on uniform quantized observations that are transmitted through a wireless channel.

In general, for nonlinear and non-Gaussian systems, it is impossible to obtain closed-form expressions of the filtering and smoothing probability density functions (PDFs) and in turn a state estimators. However, for linear and Gaussian systems, the optimal solution is given by the Kalman Filter (KF) and the Rauch-Tung-Striebel smoother (KS), respectively [16]. Many sub-optimal methods have been developed in order to obtain an approximation of the desired PDFs and state estimation, see e.g. [17], [18], [19]. The Extended Kalman filter (EKF) was studied in [20] to deal with quantized data. This approach is difficult to implement since the quantizer is a non-differentiable non-linear function and requires the computation of a Jacobian matrix. The authors in [20] proposed to approximate the quantizer by using a smooth function to compute approximately the Jacobian matrix. However, due to the highly non-linearity of the quantizer, the EKF/EKS approach produces non-accuracy estimates of the state system. The Kalman filter (KF) was modified to include the quantization effect in the computation of the filtering state, which has been referred to as quantized Kalman filter (QKF) [21], [22]. The Unscented Kalman filter (UKF) was applied by [23] to deal with quantized innovation systems in a wireless sensor network environment. The UKF is based on the Unscented transformation [18] that represents the mean and covariance of a Gaussian random variable through a reduced number of points. Then, these points are propagated through the nonlinear function to accurately capture the mean and covariance of the

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propagated Gaussian random variable. The advantage of the UKF is high estimation accuracy and convergence rate, and the simple implementation compared with EKF, avoiding the computation of the Jacobian matrix required in the EKF method.

One of the most used methods in nonlinear filtering is the Sequential Monte Carlo sampling approach called particle filter (PF)\textsuperscript{[24]}. PF uses a set of weights and samples to represent approximately the filtering and smoothing PDFs. The advantages of PF/PS are the facility of implementation and the ability to deal with highly nonlinear and non-Gaussian systems. However, particle filter suffers from some drawbacks. One of them is the degeneracy problem, which means that in every iteration of the particle filter algorithm most weights are going to zero \textsuperscript{[25]}. To solve this problem, \textsuperscript{[24]} proposed an approach called resampling, in which the heavily weighted particles are replicated sometimes, and the rest of the particles are unused. A number of resampling schemes have been developed in the literature such as multinomial, stratified, systematic, residual, branch-kill, rounding-copy, Metropolis, and local selection resampling methods, to mention a few \textsuperscript{[26]}. These resampling methods have some advantages such as unbiasedness and parallelism capacity. In addition, the performance of the particle filter depends on the used resampling method \textsuperscript{[27]}. Unfortunately, the resampling process produces a loss of diversity in the particle set since the particles with high weights are replicated. This problem is called sample impoverishment \textsuperscript{[25]}, and to mitigate it, a MCMC move is usually introduced after the resampling step in order to provide diversity to the samples so that the new particle set is still distributed according to the posterior PDF \textsuperscript{[28]}. There are mainly two MCMC methods that we can use to deal with the impoverishment problem: the Gibbs and the Metropolis-Hasting (MH) sampling. Here we have used the MH algorithm and a special MH algorithm called Random-Walk Metropolis (RWM).

On the other hand, a new algorithm to deal with quantized output data is proposed in \textsuperscript{[29], 30}. In these works, the authors defined the probability mass function of the quantized data conditioned to the system state as an integral equation depending on the quantizer regions. This integral equation is solved utilizing Gauss-Legendre quadrature, which yields a model with Gaussian sum structure. This model is used to develop Gaussian sum filter/smooth to obtain closed-form filtering and smoothing distributions for systems with quantized data.

In this paper, we discuss the implementation of EKF/EKS, QKF/QKS, UKF/UKS, GSF/GSS and PF/PS algorithms to deal with the problem of quantized observations. The PF/PS algorithm is studied by combining the resampling methods: systematic (SYS), multinomial (ML), Metropolis (MT), and local selection (LS) \textsuperscript{[26]}, with the MCMC algorithms: MH and RWM \textsuperscript{[31], 32}. The analysis is carried out in terms of the accuracy of the state estimation and the computational cost. The organization of this paper is as follows: In Section II the problem of interest is defined. In Section III we present a brief review of the filtering and smoothing methods. In Section IV we present a numerical example. Finally, the concluding remarks are given in section V.

II. STATEMENT OF THE PROBLEM

This paper consider the filtering and smoothing problem for the following discrete-time time-invariant state-space system with quantized output (See Figure 1):

\begin{equation}
    x_{t+1} = Ax_t + Bu_t + w_t,
\end{equation}

\begin{equation}
    z_t = Cx_t + Du_t + v_t,
\end{equation}

\begin{equation}
    y_t = q\{z_t\},
\end{equation}

where \(x_t \in \mathbb{R}^n\) is the state vector, \(z_t \in \mathbb{R}\) is the non-quantized output, \(y_t \in \mathbb{R}\) is the quantized output, and \(u_t \in \mathbb{R}^m\) is the input of the system. The matrix \(A \in \mathbb{R}^{n \times n}\), \(B \in \mathbb{R}^{n \times m}\), \(C \in \mathbb{R}^{1 \times n}\) and \(D \in \mathbb{R}^{1 \times m}\). The nonlinear map \(q\{\cdot\}\) is the quantizer. The state noise \(w_t \in \mathbb{R}^n\) and the output noise \(v_t \in \mathbb{R}\) are zero-mean white Gaussian noises with covariance matrix \(Q\) and \(R\), respectively. Due to the random components (i.e., the noises \(w_t\) and \(v_t\)) in (1) and (2), the state-space model can be described using the state transition PDF \(p(x_{t+1}|x_t) \sim \mathcal{N}(x_{t+1}; Ax_t + Bu_t, Q)\) and the non-quantized output PDF \(p(z_t|x_t) \sim \mathcal{N}(z_t; Cx_t + Du_t, R)\) with \(x_1 \sim \mathcal{N}(x_1; \mu_1, P_1)\), where \(\mathcal{N}(x; \mu, P)\) represents a PDF corresponding to a Gaussian distribution with mean \(\mu\) and covariance matrix \(P\) of the variable \(x\). The initial condition \(x_1\), the model noise \(w_t\), and the measurement noise \(v_t\) are statistically independent random variables.

![Fig. 1. State-space model with quantized output.](image-url)

On the other hand, we consider the quantizer \(q\{\cdot\} : \mathbb{R} \rightarrow \Psi\) where \(\mathbb{R}\) is the real line and \(\Psi \subseteq \mathbb{R}\) is the output set, then \(q\{\cdot\}\) is given by (1):

\begin{equation}
    y_t = q\{z_t\} = \{\psi_k \in \Psi, R_k \subseteq \mathbb{R}; k \in \mathcal{K}\},
\end{equation}
where \( \psi_k \) is the \( k \)th output value in the output set \( \Psi \), \( R_k \) is the \( k \)th interval mapped to the value \( \psi_k \), and the indices set \( K \) defines the number of quantization levels of the output set \( \Psi \). Here we consider two types of quantizers: (i) an infinite-level quantizer (ILQ), in which the output of the quantizer has infinite (countable) levels of quantization with
\[
K = \{ \ldots, 1, 2, \ldots, L, \ldots \},
\]
where \( R_k = \{ z_t : q_{k-1} \leq z_t < q_k \} \) are disjoint intervals, and each \( \psi_k \) is the value that the quantizer takes in the region \( R_k \), and (ii) a finite-level quantizer (FLQ), in which the output of the quantizer is limited to minimum and maximum values (saturated quantizer) similar to \([4]\) with
\[
K = \{ 1, 2, \ldots, L - 1, L \}.
\]
Notice that the FLQ quantizer is comprised with finite and semi-infinite intervals given by \( R_1 = \{ z_t : q_1 < z_t \} \), \( R_L = \{ z_t : q_{L-1} \leq z_t < q_L \} \), and \( R_k = \{ z_t : q_{k-1} \leq z_t < q_k \} \), with \( k = 2, \ldots, L - 1 \).

Thus, the problem of interest can be defined as follows: Given the available data \( u_{1:N} = \{ u_1, u_2, \ldots, u_N \} \) and \( y_{1:N} = \{ y_1, y_2, \ldots, y_N \} \), where \( N \) is the data length, obtain the filtering and smoothing PDFs of the state given the quantized measurements, \( p(x_t|y_{1:t}) \) and \( p(x_{t+1}|y_{1:t}) \), respectively, the state estimators:
\[
\hat{x}_{t|t} = \mathbb{E}\{x_t|y_{1:t}\} = \int x_t p(x_t|y_{1:t}) \text{d}x_t,
\]
\[
\hat{x}_{t|N} = \mathbb{E}\{x_t|y_{1:N}\} = \int x_t p(x_t|y_{1:N}) \text{d}x_t,
\]
and the corresponding covariance matrices of the estimation error:
\[
\Sigma_{t|t} = \mathbb{E}\{(x_t - \hat{x}_{t|t})(x_t - \hat{x}_{t|t})^T|y_{1:t}\} = \int (x_t - \hat{x}_{t|t})(x_t - \hat{x}_{t|t})^T p(x_t|y_{1:t}) \text{d}x_t,
\]
\[
\Sigma_{t|N} = \mathbb{E}\{(x_t - \hat{x}_{t|N})(x_t - \hat{x}_{t|N})^T|y_{1:N}\} = \int (x_t - \hat{x}_{t|N})(x_t - \hat{x}_{t|N})^T p(x_t|y_{1:N}) \text{d}x_t,
\]
where \( t \leq N \) and \( \mathbb{E}\{x_t|y_t\} \) denotes the conditional expectation of \( x_t \) given \( y_t \).

### III. Recursive Filtering and Smoothing Methods for Quantized Output Data

#### A. Bayesian filtering and smoothing

From Bayesian approach, the filtering distribution admits the following recursion, see e.g. \([16]\):
\[
p(x_t|y_{1:t-1}) = \int p(x_t|x_{t-1})p(x_{t-1}|y_{1:t-1}) \text{d}x_{t-1},
\]
\[
p(x_t|y_{1:t}) = \frac{p(y_t|x_t)p(x_t|y_{1:t-1})}{p(y_t|y_{1:t-1})},
\]
where \( p(x_t|y_{1:t-1}) \) and \( p(x_t|y_{1:t-1}) \) are the measurement- and time-update equations. The PDF \( p(x_t|x_{t-1}) \) is directly obtained from the model in \([1]\), and \( p(y_t|y_{1:t-1}) \) is a normalization constant. On the other hand, the Bayesian smoothing equation, see e.g. \([16]\), is defined by:
\[
p(x_{t+1}|y_{1:t}) = p(x_{t+1}|y_{1:t}) \int \frac{p(x_{t+1}|y_{1:N})p(x_t|y_{1:t})}{p(x_{t+1}|y_{1:t})} \text{d}x_{t+1}.
\]
Notice that to obtain \( p(x_{t+1}|y_{1:t}) \) in \([12]\) we need the probability function \( p(y_t|x_t) \). Since \( y_t \) is a discrete random variable, the probabilistic model of \( p(y_t|x_t) \) is a PMF. Then, the measurement-update equation in \([12]\) combines both PDFs and a PMF. Here we use Generalized Probability Density Functions, see e.g. \([33]\), which comprises both discrete and absolutely continuous distributions. In \([30]\), an integral equation for \( p(y_t|x_t) \) is defined in order to solve the filtering recursion as follows
\[
p(y_t|x_t) = \int_{a_t}^{b_t} N(v_t; 0, R) \text{d}v_t,
\]
where \( a_t \) and \( b_t \) are functions of the boundary values of each region of the quantizers defined in Table \([1]\). Notice that \( y_t|x_t \) in equation \([14]\) is a non-Gaussian random variable. This leads to obtain non-Gaussian measurement- and time-update distributions. However, the EKF, QKF, and UKF filters are developed under the assumption that the measurement- and time-update distributions are Gaussian, which yields a loss of accuracy in the estimation. On the other hand, the equation \([14]\) is used in GSF/GSS and PF/PS, where the Gaussian assumption is not considered.
TABLE I
INTEGRAL LIMITS OF EQUATION [1].

| FLQ | $y_t$ | $a_t$ | $b_t$ |
|-----|-------|-------|-------|
| $\psi_1$ | $-\infty$ | $q_1 - Cx_t - Du_t$ |
| $\psi_k$, $k = 2, \ldots, L - 1$ | $q_{k-1} - Cx_t - Du_t$ | $q_k - Cx_t - Du_t$ |
| $\psi_L$ | $q_{L-1} - Cx_t - Du_t$ | $\infty$ |

| ILQ | $\psi_k$, $k = \ldots, 1, \ldots, L, \ldots$ | $q_{k-1} - Cx_t - Du_t$ | $q_k - Cx_t - Du_t$ |

B. Extended state-space system
To implement some filtering and smoothing algorithms such as the EKF/EKS and the UKF/UKS, the state-space model in [1–3] is rewritten in a extended form as follows

$$x^e_{t+1} = Ax^e_t + Bu^e_t + w^e_t,$$
$$y_t = q \{Cx^e_t\} + \xi_t,$$  \hspace{1cm} (15)

where the extended system matrices are given by

$$A = \begin{bmatrix} A & 0 \\ AC & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B & 0 \\ CB & D \end{bmatrix}, \quad C = [0 \ 1],$$  \hspace{1cm} (17)

where $x^e_t = [x^T_t \ z_t]^T$ is the extended state, $u^e_t = [u^T_t \ u^T_{t+1}]$ is the extended input with $u_{N+1} = 0$, and the initial condition $x^e_1 \sim \mathcal{N}(x^e_1; \mu^e_1, P^e_1)$ where $\mu^e_1 = [\mu^T_1 \ 0]^T$ and $P^e_1 = \text{diag}\{P_1, 1\}$. The noise $w^e_t = [w^T_t \ (Cw_t + u_{t+1})]^T$ satisfies $w^e_t \sim \mathcal{N}(w^e_t; 0, Q)$ with

$$Q = \begin{bmatrix} Q & QC^T \\ QC^T & CQC^T + R \end{bmatrix}. $$  \hspace{1cm} (18)

The noise $\xi_t$ is added in order to implement the EKF/EKS and the UKF/UKS. We assume that $\xi_t \sim \mathcal{N}(\xi_t; 0, \varepsilon)$, where the variance $\varepsilon$ is small.

C. Extended Kalman Filtering and Smoothing
The extended Kalman filter [34, 35] is an extension of the standard Kalman filter to nonlinear state-space systems with process and measurement Gaussian noises. The idea of EKF is to build a linear approximation around a state estimation by using a Taylor series expansion. The EKF is not directly applied to the problem of interest in this paper since the quantizer is a non-differentiable nonlinear function. In [20], it was suggested that it is possible to compute the Kalman gain using a Taylor series expansion. The EKF is not directly applied to the problem of interest in this paper since the quantizer is a non-differentiable nonlinear function. In [20], it was suggested that it is possible to compute the Kalman gain using a smooth arctan-based approximation of the quantizer. Following the idea in [20] and the representation of the arctan function found in [36], the following approximation of the quantizer is proposed:

$$q \{z_t\} \approx h(z_t) = \begin{cases} z_t & \text{if } |z_t| < \Delta, \\ (\Delta/\pi) \tan ((z_t - 2.5\Delta)/\rho) + 2.5\Delta & \text{if } 2\Delta \leq z_t < 3\Delta, \\ (\Delta/\pi) \tan ((z_t - 1.5\Delta)/\rho) + 1.5\Delta & \text{if } \Delta \leq z_t < 2\Delta, \\ (\Delta/\pi) \tan ((z_t - 0.5\Delta)/\rho) + 0.5\Delta & \text{if } 0 \leq z_t < \Delta, \\ (\Delta/\pi) \tan ((z_t + 0.5\Delta)/\rho) - 0.5\Delta & \text{if } -\Delta \leq z_t < 0, \\ (\Delta/\pi) \tan ((z_t + 1.5\Delta)/\rho) - 1.5\Delta & \text{if } -2\Delta \leq z_t < -\Delta, \\ (\Delta/\pi) \tan ((z_t + 2.5\Delta)/\rho) - 2.5\Delta & \text{if } -3\Delta \leq z_t < -2\Delta, \\ \vdots & \vdots \end{cases} $$  \hspace{1cm} (19)

where $\rho$ is a user parameter that defines how the approximation fits the quantizer function in the switch point, as is shown in Figure 2. On the other hand, using the smooth approximation of the quantizer it is possible to rewrite the nonlinear system as a type of linear time-varying system as follows:

$$x^e_{t+1} = Ax^e_t + Bu^e_t + w^e_t,$$  \hspace{1cm} (20)

$$y_t = Hx^e_t + F_t + \xi_t,$$  \hspace{1cm} (21)

where $H_t$ is the Jacobian matrix of $h(Cx^e_t)$ with respect to $x^e_t$ and evaluated at $x^e_{t-1}, F_t = h(Cx^e_{t-1}) - H_t x^e_{t-1}$. Then, the equations of the EKF are summarized in Algorithm [1]. One of the difficulties in applying the EKF to deal with quantized
The unscented Kalman Filter \[18\] is a deterministic-sampling-based approach that uses samples called sigma-points to propagate the mean and covariance of the system state (assumed to be a Gaussian random variable) through the nonlinear functions of the system. These propagated points accurately capture the mean and covariance of the posterior state to the 3rd order Taylor series expansion for any nonlinear function \[37\]. The key idea of UKF is directly approximating the mean functions of the system. These propagated points accurately capture the mean and covariance of the posterior state to the

\[\text{Algorithm 1: Extended Kalman Filter}\]

1. **Input:** The distribution of the initial condition \( p(x_1^0) \), e.g. \( \hat{x}_0^0 = \mu_1^0 \) and \( \Sigma_0^0 = P_1^0 \).
2. **for** \( t = 1 \) to \( N \) **do**
   3. Compute the Kalman gain using: \( K_t = \Sigma_{t|t-1}^e H_t^T \left( \varepsilon + H_t \Sigma_{t|t-1}^e H_t^T \right)^{-1} \).
   4. **Measurement Update:**
      5. Compute the filtering state \( \hat{x}_{t|t}^e \) according to \( \hat{x}_{t|t}^e = \hat{x}_{t|t-1}^e + K_t \left( y_t - h \left( \hat{x}_{t|t-1}^e \right) \right) \).
      6. Compute the covariance matrix \( \Sigma_{t|t}^e \) according to \( \Sigma_{t|t}^e = (I - K_t H_t) \Sigma_{t|t-1}^e \).
   7. **Time Update:**
      8. Compute the predicted state \( \hat{x}_{t+1|t}^e \) according to \( \hat{x}_{t+1|t}^e = A \hat{x}_{t|t}^e + B u_t^e \).
      9. Compute the covariance matrix \( \Sigma_{t+1|t}^e \) according to \( \Sigma_{t+1|t}^e = Q + \Sigma_{t|t}^e A^T \).
10. **end**
11. **Output:** The PDF \( p(x_t^e|y_1:t) \sim \mathcal{N} \left( \hat{x}_t^e; \hat{x}_{t|t}^e, \Sigma_{t|t}^e \right) \) and the PDF \( p(x_{t+1}^e|y_1:t) \sim \mathcal{N} \left( \hat{x}_{t+1}^e; \hat{x}_{t+1|t}^e, \Sigma_{t+1|t}^e \right) \) for \( t = 1, \ldots, N \).

**D. Unscented Kalman Filtering and Smoothing**

The unscented Kalman Filter \[18\] is a deterministic-sampling-based approach that uses samples called sigma-points to propagate the mean and covariance of the system state (assumed to be a Gaussian random variable) through the nonlinear functions of the system. These propagated points accurately capture the mean and covariance of the posterior state to the 3rd order Taylor series expansion for any nonlinear function \[37\]. The key idea of UKF is directly approximating the mean
Algorithm 2: Extended Kalman Smoother

1. **Input:** The PDF \( p(x_t^e|y_{1:t}) \sim \mathcal{N}(x_t^e; \hat{x}_{t|t}^e, \Sigma_t^e) \) and the PDF \( p(x_{t+1}^e|y_{1:t}) \sim \mathcal{N}(x_{t+1}^e; \hat{x}_{t+1|t}^e, \Sigma_{t+1|t}^e) \) for \( t = 1, \ldots, N \) computed in Algorithm 1.

2. **for** \( t = N \) **to** 1 **do**

3. Compute the gain \( G_t = \Sigma_t^e \Lambda_t^{-1} \).

4. Compute the smoothing state \( \hat{x}_{t|N}^e = \hat{x}_{t|t}^e + G_t \left( \bar{x}_{t+1|T}^e - \hat{x}_{t+1|t}^e \right) \).

5. Compute the covariance matrix \( \Sigma_{t|N}^e \) according to \( \Sigma_t^e = \Sigma_t^e + G_t \left( \Sigma_{t+1|T}^e - \Sigma_{t+1|t}^e \right) G_t^T \).

6. **end**

7. **Output:** The PDF \( p(x_t^e|y_{1:N}) \sim \mathcal{N}(x_t^e; \hat{x}_{t|N}^e, \Sigma_{t|N}^e) \) for \( t = 1, \ldots, N \).

and covariance of the posterior distribution instead of approximating nonlinear functions [18]. The unscented Kalman Filter is based on the unscented transformation of the random variable \( x \) into the random variable \( y \) as follows: \( y = g(x) + \nu \) where \( g(\cdot) \) is a nonlinear function, \( x \sim \mathcal{N}(x; \bar{x}, \Gamma) \), and \( \nu \sim \mathcal{N}(\nu; 0, R) \), thus the sigma-points are defined by

\[
\psi^0 = \bar{x},
\psi^\tau = \bar{x} + \sqrt{n + \lambda} \left[ \sqrt{\Sigma} \right]_\tau,
\psi^{\tau+n} = \bar{x} - \sqrt{n + \lambda} \left[ \sqrt{\Sigma} \right]_\tau,
\]

where \( \tau = 1, \ldots, n \), the scaling parameter \( \lambda = \alpha^2(n + \kappa) - n \), the parameters \( \alpha \) and \( \kappa \) determine the propagation of the sigma-points around the mean. The notation \( [\Sigma]_\tau \) refers to the \( \tau \)th column of the matrix \( \Sigma \) and \( \sqrt{\Sigma} \) represents the square root of the matrix \( \Sigma \). The weights associated to the unscented transformation are the sets

\[
\{ \omega^0, \omega^1, \ldots, \omega^{2n} \} = \{ \lambda \zeta, 0.5 \zeta, \ldots, 0.5 \zeta \},
\{ \sigma^0, \sigma^1, \ldots, \sigma^{2n} \} = \{ \lambda \zeta + \rho, 0.5 \zeta + \rho, \ldots, 0.5 \zeta \},
\]

where \( \zeta = (n + \lambda)^{-1}, \rho = 1 - \alpha^2 + \beta \). Here \( \beta \) is an additional algorithm parameter that can be used for incorporating prior information on the (non-Gaussian) distribution of \( x \). Then, the statistics of the transformed random variable are: the mean \( \mu = \sum_{\tau=0}^{2n} \omega^\tau g(\psi^\tau) \) and the covariance matrix \( \Phi = \sum_{\tau=0}^{2n} \sigma^\tau \left( [g(\psi^\tau) - \mu] [g(\psi^\tau) - \mu]^T \right) + R \). Additionally, the cross covariance matrix between \( x \) and \( y \) is given by \( \Psi = \sum_{\tau=0}^{2n} \sigma^\tau \left( [\psi^\tau - \bar{x}] [g(\psi^\tau) - \mu]^T \right) \). The steps to implement the UKF are summarized in Algorithm 3. Notice that for the problem of interest in this paper, the process equation is a linear function. Thus, the UKS algorithm is similar to EKS but using the filtering and predictive distributions obtained with the UKF algorithm.

E. Quantized Kalman Filtering and Smoothing

The quantized Kalman Filter is an alternative version of the Kalman filter (KF) that modified the measurement update equation to include the quantization effect in the computation of the filtering distributions. This modification is performed in some manners as is shown in [21], [22]. In this work, we used the following modification of the KF:

\[
x_{t|t} = x_{t|t-1} + K_t \left( y_t - q \{ Cx_{t|t-1} - Du_t \} \right),
\]

where \( K_t \) is the Kalman gain in the KF. Notice that, with the modification in [27], the QKS algorithm is similar to the standard Kalman smoother (KS).

F. Gaussian Sum Filtering and Smoothing

The Gaussian Sum Filter [29], [30] is a novel approach to deal with quantized output data. The key idea of GSF is to approximate the integral of \( p(y_t|x_t) \) given in (14) using the Gauss–Legendre quadrature rule. This approximation produces a model with Gaussian sum structure as follows:

\[
p(y_t|x_t) \approx \sum_{\tau=1}^{K} \xi^\tau \mathcal{N}(\eta^\tau; Cx_t + Du_t + \mu^\tau, R),
\]

where \( K \) is the number of points from the Gauss–Legendre quadrature rule, \( \xi^\tau, \eta^\tau, \) and \( \mu^\tau \) are defined in Table II and \( \omega^\tau \) and \( \psi^\tau \) are weights and points defined by the quadrature rule, given in, e.g., [38].
Compute the covariance matrix $\Sigma_t$.

\begin{align*}
\Sigma_t &= \sum_{\tau=1}^{2n} \sigma_{t|t-1}^\tau (Y_t^\tau - \nu_t) (Y_t^\tau - \nu_t)^T + \varepsilon .
\end{align*}

**Measurement Update:**

Compute the filtering state $x_t^e$ according to $x_t^e = x_{t-1}^e + K_t (y_t - \nu_t)$.

Compute the covariance matrix $\Sigma_t^e$ according to $\Sigma_t^e = \Sigma_{t-1}^e - K_t S_t K_t^T$.

**Time Update:**

Compute the predicted state $\hat{x}_{t+1|t}$ according to $\hat{x}_{t+1|t} = A \hat{x}_t + \hat{B} u_t$.

Compute the covariance matrix $\Sigma_{t+1|t}$ according to $\Sigma_{t+1|t} = Q + A \Sigma_t A^T$.

Output: The PDF $p(x_t|y_{1:t}) \sim \mathcal{N} \left( x_t^e; \Sigma_t^e \right)$ and the PDF $p(x_{t+1}^e|y_{1:t}) \sim \mathcal{N} \left( x_{t+1}^e; \Sigma_{t+1|t}^e \right)$ for $t = 1, \ldots, N$.

**TABLE II**

PARAMETERS OF THE $p(y_t|x_t)$ APPROXIMATION.

| FLQ | $y_t$ | $\lambda_t^\tau$ | $\eta_t^\tau$ | $\mu_t^\tau$ |
|-----|-------|-----------------|--------------|-------------|
| $\psi_1$ | $2 \omega_\tau / (1 + \psi_\tau)^2$ | $-(1 - \psi_\tau) / (1 + \psi_\tau)$ | $-q_1$ |
| $\psi_k$, $k = 2, \ldots, L - 1$ | $\omega_\tau (q_k - q_{k-1}) / 2$ | $\psi_\tau (q_k - q_{k-1}) / 2$ | $-(q_k + q_{k-1}) / 2$ |
| $\psi_L$ | $2 \omega_\tau / (1 + \psi_\tau)^2$ | $(1 - \psi_\tau) / (1 + \psi_\tau)$ | $-q_{L-1}$ |

| ILQ | $y_t$ | $\lambda_t^\tau$ | $\eta_t^\tau$ | $\mu_t^\tau$ |
|-----|-------|-----------------|--------------|-------------|
| $\psi_k$, $k = 1, \ldots, L$ | $\omega_\tau (q_k - q_{k-1}) / 2$ | $\psi_\tau (q_k - q_{k-1}) / 2$ | $-(q_k + q_{k-1}) / 2$ |

Utilizing the approximation of $p(y_t|x_t)$ given in (28), the Gaussian sum filter iterates between the following two steps:

\begin{align*}
p(x_t|y_{1:t}) &= \sum_{k=1}^{M_{t|t}} \gamma_{t|t}^k \mathcal{N}(x_t; \hat{x}_t^k, \Sigma_t^k) ,
\end{align*}

\begin{align*}
p(x_{t+1|t+1}) &= \sum_{k=1}^{M_{t+1|t}} \gamma_{t+1|t}^k \mathcal{N}(x_{t+1}; \hat{x}_{t+1}^k, \Sigma_{t+1|t}^k) ,
\end{align*}

where all quantities in this recursion can be computed following Algorithm 4. On the other hand, to compute the smoothing distribution, equation (13) is separated in two formulas to avoid the division for a non-Gaussian distribution (39). The first
formula is the backward recursion that, utilizing the approximation of \( p(y_t|x_t) \) given in (28), is defined as follows:

\[
p(y_{t+1:N}|x_t) = \sum_{k=1}^{S_{t+1}} \epsilon_{t+1}^k \lambda_{t+1}^k \exp \left\{ -\frac{1}{2} \left( x_t^T F_{t+1}^k x_t - 2G_{t+1}^k x_t + H_{t+1}^k \right) \right\},
\]

(31)

where all quantities in this recursion can be computed following Algorithm 6. The second formula computes the smoothing distribution as follows

\[
p(x_t|y_{1:N}) = \sum_{k=1}^{S_{t+1}} \epsilon_{t+1}^k N(x_t; \tilde{x}_{t+1}^k, \Sigma_{t+1}^k),
\]

(33)

where all quantities in this equation can be computed following Algorithm 6.

**Algorithm 4: Gaussian sum filter algorithm for quantized output data.**

1. **Input:** The PDF of the initial state \( p(x_1) \), e.g., \( M_{1|0} = 1, \gamma_{1|0} = 1, \tilde{x}_{1|0} = \mu_1, \Sigma_{1|0} = P_1 \). The points of the Gauss–Legendre quadrature \( \{\omega_\tau, \psi_\tau\}_{\tau=1}^K \).

2. for \( t = 1 \) to \( N \) do

3. Compute and store \( \zeta_t^\tau, \eta_t^\tau, \) and \( \mu_t^\tau \) according to table II.

4. **Measurement Update:**

5. Set \( M_{t|t} = K M_{t|t-1} \).

6. for \( \ell = 1 \) to \( M_{t|t-1} \) do

7. Compute the index \( k = (\ell - 1)K + \tau \).

8. Compute and store the weights, means, and covariances matrices as follows:

\[
\gamma_{t|t}^k = \tilde{\gamma}_{t|t}^k \left( \sum_{s=1}^{M_{t|t}} \tilde{\gamma}_{t|t}^s \right)^{-1}, \quad \tilde{\gamma}_{t|t}^k = \tilde{\gamma}_{t|t-1}^\tau N(\kappa_t^\tau, V_t^\tau),
\]

9. \( \bar{x}_{t|t}^k = \bar{x}_{t-1|t} + K_t^\tau (\eta_t^\tau - \kappa_t^\tau), \quad K_t^\tau = \Sigma_{t|t-1} C_t^T (V_t^\tau)^{-1}, \quad \kappa_t^\tau = C_t \tilde{x}_{t|t-1} + D u_t + \mu_t^\tau, \quad V_t^\tau = R + C \Sigma_{t|t-1} C_t^T. \)

10. end

11. end

12. Perform the Gaussian sum reduction algorithm according to [40] to obtain the reduced GMM of \( p(x_t|y_{1:t}) \).

13. **Time Update**

14. Set \( M_{t+1|t} = M_{t|t} \).

15. for \( k = 1 \) to \( M_{t+1|t} \) do

16. Compute and store the weights, means, and covariances matrices as follows:

\[
\gamma_{t+1|t}^k = \gamma_{t|t}^k, \quad \bar{x}_{t+1|t}^k = A \bar{x}_{t|t}^k + B u_t, \quad \Sigma_{t+1|t}^k = Q + A \Sigma_{t|t}^k A^T.
\]

17. end

18. end

**Output:** The filtering PDFs \( p(x_t|y_{1:t}) \), the predictive PDFs \( p(x_{t+1}|y_{1:t}) \), and the set \( \{\zeta_t^\tau, \eta_t^\tau, \mu_t^\tau\} \), for \( t = 1, \ldots, N \).

**G. Particle Filtering and Smoothing**

Particle filtering [25], [24] is a Monte Carlo method that approximately represents the filtering distributions \( p(x_t|y_{1:t}) \) of the state variables conditioned to the observations \( y_{1:t} \) by using a set of weighted random samples called particles so that:

\[
p(x_t|y_{1:t}) \approx \sum_{i=1}^{M} u_{t}^{(i)} \delta \left( x_t - \tilde{x}_t^{(i)} \right),
\]

(34)

where \( \delta(\cdot) \) is the Dirac delta function, \( u_{t}^{(i)} \) denotes the \( i \)th weight, \( \tilde{x}_t^{(i)} \) denotes the \( i \)th particle sampled from the filtering distribution \( p(x_t|y_{1:t}) \), and \( M \) is the number of particles. Since the filtering distribution is unknown in the current iteration, it is difficult or impossible to sample directly from it. In this case, the particles are usually generated from a known density that is chosen (by the user) to facilitate the sampling process. This is called importance sampling and the PDF is called importance
Algorithm 5: Backward-filtering algorithm for quantized output data.

1. **Input:** The initial backward measurement $p(y_N|x_N)$ at $t = N$ with parameters: $S_{N|N} = K$ and

   \[
   \begin{align*}
   \ell_{N|N}^k &= \ell_{N|N}^k, \\
   \lambda_{N|N}^k &= (\text{det} \{2\pi R\})^{-1/2}, \\
   \theta_N^k &= \eta_N^k - Du_N - \mu_N^k,
   \end{align*}
   \]

   The set $\{\ell_t, \eta_t, \mu_t^k\}$ for $t = 1, \ldots, N$ computed in algorithm 4.

2. **for** $t = N - 1$ to 1 do
   3.  **Backward Prediction**
   4.  Set $S_{t|t+1} = S_t + 1$.
   5.  for $k = 1$ to $S_{t|t+1}$ do
   6.  Compute and store the backward prediction update quantities as follows
   \[
   \begin{align*}
   \ell_{t|t+1}^k &= \ell_{t+1|t+1}^k, \\
   \lambda_{t|t+1}^k &= (\text{det} \{Q\} \text{det} \{F_{qk}\})^{-1/2} \lambda_{t+1|t+1}^k, \\
   F_{t|t+1}^k &= A^T M_{qk} A, \\
   G_{t|t+1}^k &= G_{t+1|t+1}^k F_{qk}^{-1} Q^{-1} A - u_t^T B^T M_{qk} A, \\
   H_{t|t+1}^k &= H_{t+1|t+1}^k - G_{t|t+1}^k G_{qk}^{-1} G_{t+1|t+1}^k + u_t^T B^T M_{qk} B u_t - 2u_t^T B^T Q^{-1} F_{qk}^{-1} G_{t+1|t+1}^k.
   \end{align*}
   \]
   end

7.  **Backward Measurement Update:**
8.  Set $S_{t|t} = K S_{t|t+1}$.
9.  for $\ell = 1$ to $S_{t|t}$ do
10.  for $\tau = 1$ to $K$ do
11.  Compute the index $k = (\ell - 1)K + \tau$.
12.  Compute and store the backward measurement update quantities as follows
13.  \[
   \begin{align*}
   \ell_{t|t}^k &= \tau_{t|t+1}^\ell, \\
   \lambda_{t|t}^k &= (\text{det} \{2\pi R\})^{-1/2} \lambda_{t+1|t+1}^\ell, \\
   F_{t|t}^k &= F_{t+1|t+1}^\ell + C^T R^{-1} C, \\
   G_{t|t}^k &= G_{t+1|t+1}^\ell + \theta_R^T R^{-1} C, \\
   H_{t|t}^k &= H_{t+1|t+1}^\ell + \theta_T^T R^{-1} \theta_R^\ell.
   \end{align*}
   \]
14. end
15. end
16. Compute the GMM structure of $p(y_{t:N} | x_t)$ using Lemma A.3 in [30].
17. Perform the Gaussian sum reduction algorithm according to [40] to obtain the reduced GMM structure of $p(y_{t:N} | x_t)$, see equation (54) in [30].
18. Compute and store the backward filter form of the reduced version of $p(y_{t:N} | x_t)$ using Lemma A.3 in [30].
19. end
20. **Output:** The backward prediction $p(y_{t+1:N} | x_t)$ and the backward measurement update $p(y_{t:N} | x_t)$ for $t = N, \ldots, 1$.

density. Then, the importance weight computation can be carried out in recursive fashion (Sequential importance sampling, SIS) as follows:

\[
\begin{align*}
w^{(i)}_t &\propto w^{(i)}_{t-1} \frac{p(y_t | x_t^{(i)}) p(x_t^{(i)} | x_{t-1}^{(i)})}{h(x_t^{(i)} | x_{t-1}^{(i)}, y_t)},
\end{align*}
\]

where $h(x_t | x_{t-1}, y_t)$ is the importance density and $w^{(i)}_{t-1}$ are the importance weights of the previous iteration. On the other hand, the choice of importance distribution is critical for performing the particle filtering and smoothing. The particle filter literature shows that the importance density $p(x_t | x_{t-1}^{(i)}, y_t)$ is optimal in the sense that it minimizes the variance of the importance weights $w^{(i)}_t$ [25], [16]. However, in most cases it is difficult or impossible to draw samples for this optimal importance density, except for particular cases such as a state-space model with nonlinear process and linear output equation [25]. Many sub-optimal methods have been developed to approximate the importance density such as Markov Chain Monte Carlo [41], ensemble Kalman filter [28], local linearization of the state space model, local linearization of the optimal importance distribution [25], among others. One of the most commonly used importance densities in the literature is the state transition prior $p(x_t | x_{t-1})$, see e.g. [42], [29], [25]. This choice yields a intuitive and simple to implement algorithm with $w^{(i)}_t \propto w^{(i)}_{t-1} p(y_t | x_{t-1}^{(i)})$. This algorithm is called bootstrap filter [24].

The particle filter suffers from an impossible to avoid problem called degeneracy phenomenon. As shown in [25], the variance
of the importance weights can only increase over time. This implies that after a few iterations, most particles have negligible weights, see also [43]. A consequence of the degeneracy problem is that a large computational effort is devoted to updating particles whose contribution to the final estimate is nearly zero. To solve the degeneracy problem, the resampling approach was proposed in [24]. The resampling method eliminates the particles that have small weights and the particles with large weights are replicated, generating a new set (with replacement) of equally-weighted particles.

Additionally, the resampling method used to reduce the degeneracy effect on the particles produces other unwanted issue called particle impoverishment. This means a loss of diversity in the sample set since the resampled particles will contain many repeated points that were generated from heavily weighted particles. In the worst-case scenario, all particles can be produced by a single particle with a large weight [31]. To solve the impoverishment problem, some methods such as roughening and regularization have been suggested in the literature [26]. Markov Chain Monte Carlo (MCMC) technique is another method used after the resampling step to add variability to the resampled particles [44]. The basic idea is to apply the MCMC algorithm to each resampled particle with $p(x_i | y_{1:t-1})$ as the target distribution. That is, we need to build a Markov chain by sampling a proposal particle $x^*_t$ from the proposal density. Then, $x^*_t$ is accepted only if $u \leq \varpi(x^*_t, x_t)$, with $u \sim U[0, 1]$, where $U[a_1, a_2]$ correspond to the uniform distribution over the real numbers in the interval $[a_1, a_2]$, and $\varpi(x^*_t, x_t)$ is the acceptance ratio given by

$$\varpi(x^*_t, x_t) = \min \left\{ 1, \frac{p(y_t | x^*_t)}{p(y_t | x_t)} \right\},$$ (36)

With this process, the diversity of the new particles is greater than the resampled particles, reducing the risk of particle impoverishment. Additionally, the new particles are distributed according to $p(x_i | y_{1:t})$, as desired. In this paper, we used the MH and RWM algorithms to build the MCMC step. In Algorithm 7 we summarized the steps to implement the particle filter with the MCMC step.

Similar to particle filtering, particle smoothing is a Monte Carlo method that approximately represents the smoothing distributions $p(x_t | y_{1:N})$ of the state variables conditioned to the observations $y_{1:N}$, using random samples as follows:

$$p(x_t | y_{1:N}) \approx \sum_{i=1}^{M} w^{(i)}_t \delta \left( x_t - \hat{x}^{(i)}_t \right),$$ (37)

In the resampling algorithms MR, SR, and MTR $w_t^{(i)} = 1/M$ for $i = 1, \ldots, M$. The LS algorithm produces a new set of weights, see e.g. [26].
Evaluate \( \eta \), where \( x \) is for input \( \eta \).

where \( w_{i}^{(t)} \) denotes the \( i \)th weight, \( x_{i}^{(t)} \) denotes the \( i \)th particle sampled from the smoothing distribution \( p(x_{i} | y_{1:T}) \), and \( M \) is the number of particles. Some smoothing algorithms are based on the particles provided by the particle filtering, i.e. \( x_{i}^{(t)} \), such as Backward-simulation particle smoother \([46]\), and marginal particle smoother \([25]\). Particularly in the marginal particle smoother, the weights \( w_{i}^{(t)} \) are updated in time reverse as follows:

\[
\hat{w}_{i}^{(t)} = \sum_{j=1}^{M} \frac{w_{i}^{(j)} p(x_{i}^{(j)} | x_{i+1}^{(t)})}{\sum_{k=1}^{M} w_{i}^{(k)} p(x_{i}^{(k)} | x_{i+1}^{(t)})},
\]

where \( w_{i}^{(t)} = w_{i}^{(t)} \) for \( i = 1, \ldots, M \) and the approximation of \( f(x_{i+1}^{(t)}, x_{i}^{(t)}) \) is performed using \( \hat{x}_{i}^{(t)} = x_{i}^{(t)} \) for \( t = N, \ldots, 1 \). In this paper we use the smoothing method developed in \([46]\), which for the problem of interest in this work, admits further simplifications, see also \([47]\). This smoothing method \([46]\) require the evaluation of the function \( f(x_{i+1}^{(t)}, x_{i}^{(t)}) \) given by

\[
f(x_{i+1}^{(t)}, x_{i}^{(t)}) = \exp \left\{ -\frac{1}{2} \eta_{t}^{T} Q^{-1} \eta_{t} \right\},
\]

where \( \eta_{t} = x_{i+1}^{(t)} - A x_{i}^{(t)} - B u_{t} \). In Algorithm 8 we summarize the steps to implement the particle smoother. On the other hand, provided the weights \( w^{(t)} \) and particles \( x^{(t)} \) (from particle filter or smoother) the state estimators in \([7]\) and \([8]\) and the covariance matrices of the estimation error in \([9]\) and \([10]\) can be computed as follows:

\[
\mathbb{E} \{ g(x_{i}) | s \} \approx \sum_{i=1}^{M} w^{(i)} g(x^{(i)}),
\]

\[
\mathbb{E} \{ g(x_{i}) | s \} \approx \sum_{i=1}^{M} w^{(i)} g(x^{(i)}),
\]
where \( g(x_t) \) represents a function of \( x_t \). To compute the mean and covariance matrix of the filtering and smoothing distributions \( g(x_t) = x_t \) and \( g(x_t) = (x_t - \mathbb{E}\{x_t\})(x_t - \mathbb{E}\{x_t\})^T \), respectively. The variable \( s \) represents the observations set used, that for the filtering estimations \( s = y_{1:t} \) and for smoothing estimation \( s = y_{1:N} \).

IV. NUMERICAL EXPERIMENT

In this section, we present a numerical example to analyze the performance of KF/KS, EKF/EKS, QKF/QKS, UKF/UKS, GSF/GSS, and MCMC-based PF/PS having quantized observations. We use the discrete-time system in the state-space form given in (1)-(2) with

\[
y_t = \Delta \text{round}\left( \frac{y_t}{\Delta} \right),
\]

(41)

where \( \Delta \) is a quantization step and \( \text{round} \) is the Matlab function that computes the nearest decimal or integer. The sets \( R_k \) are computed using \( q_{k-1} = y_t - 0.5\Delta \) and \( q_k = y_t + 0.5\Delta \). We compare the performance of all filtering and smoothing algorithms considering eight variations of the PF where we use the Markov Chain Monte Carlo method MH and RWM with the following resampling methods: SYS, ML, MT, and LS. For clarity of presentation we use the bootstrap filter and we solve the integral in (14) using the cumulative distribution function computed with the Matlab function \text{mvncdf}. We consider the state-space system given in (1)-(2) with \( A = 0.9, B = 1.2, C = 2.2 \), and \( D = 0.75 \). We also consider that \( w_t \sim \mathcal{N}(w_t; 0, 1) \), \( v_t \sim \mathcal{N}(v_t; 0, 0.5) \), the input signal is drawn from \( \mathcal{N}(0, 1) \), and \( x_1 \sim \mathcal{N}(x_1; 1, 0.01) \). We consider \( \Delta = 8, K = 10, \Lambda^2 = 100, \) and \( M = \{100, 500, 1000\} \) particles to perform the particle filtering and smoothing.

In Figure 4 and 5 we show the filtering and smoothing distributions, i.e. \( p(x_t|y_{1:t}) \) and \( p(x_t|y_{1:N}) \), for a time instant. We freeze the results of KF, QKF, EKF, UKF, and GSF to observe the behavior of the PF when varying the number of used samples, and when different MCMC methods are used with different resampling algorithms. These figures show that the PDFs obtained used GSF/GSS are the ones that best fit the ground truth, followed by PF/PS. Furthermore, in Figure 6 and 7 we show the boxplot of the mean square error (MSE) between the estimated and the true state running 1000 Monte Carlo experiments. These figures show the loss of accuracy in the state estimation obtained with KF/KS, EKF/EKS, QKF/QKS, and UKF/UKS, and a better performance for GSF/GSS and PF/PS (except from PF version that uses LS resampling). Additionally we can observe that in terms of accuracy, the PF/PS implementation that gives the lower MSE is the one that uses the MCMC move RWM with SYS, ML, and MT resampling methods. On the other hand, in terms of the computational load, PS in all its versions exhibit the most high execution time, followed by the GSS, EKS, UKS and KS. In Table III we ranked all the algorithms studied in this work in terms of the mean of the MSE and the execution time. This table suggests that there is a trade off between the accuracy of the estimation and the execution time in the case of PF/PS. The GSF/GSS, on the other hand, exhibits high accuracy in the estimation and a relatively small execution time.

V. CONCLUSION

In this paper, we investigate the performance of Extended Kalman filter/smooother, Quantized Kalman filter/smooother, Unscented Kalman filter/smooother, Gaussian sum filter/smooother, and particle filter/smooother for state-space models with quantized observations. The analysis was carried out in terms of the accuracy of the estimation, using the Mean square error and the computational cost. The simulations show that the PDFs of Gaussian sum filter/smooother and particle filter/smooother with a high number of particles are the ones that best fit the ground truth PDFs. However, the particle filter/smooother exhibits a high computational load when the number of samples increases. The Extended Kalman filter/smooother, Quantized Kalman filter/smooother, and Unscented Kalman filter/smooother produce results with low accuracy although its execution time is small. From simulations, we observe that the performance of the particle filter is closely related to the choice of the resampling method and the MCMC algorithms, which addresses the degeneracy problem and mitigates the sample impoverishment. We use four different resampling schemes combined with two MCMC algorithms. We found that the implementation of the MCMC-based particle filter and smoothing that produces the lower MSE is the one that uses the Random Walk Metropolis combined with the systematic resampling technique.

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TABLE III
RANK OF THE FILTERING AND SMOOTHING RECURSIVE ALGORITHMS FOR QUANTIZED DATA. REFERENCES: KF/KS, EKF/EKS [16], QKF/QKS [22], PF/PS [24], UKF/UKS [13], GSF/GSS [29], KS. THE NOTATION XX-YY-ZZ(M) DENOTES THE FOLLOWING: XX STANDS FOR THE FILTERING OR SMOOTHING ALGORITHM, YY STANDS FOR THE MCMC ALGORITHM, ZZ STANDS FOR THE RESAMPLING METHOD, AND (M) STANDS FOR THE USED NUMBER OF PARTICLES.

| Rank | MSE | Filtering | Algorithm | MSE | Smoothing | Algorithm | Smoothing Execution Time | Algorithm | Execution Time | Algorithm |
|------|-----|-----------|-----------|-----|-----------|-----------|--------------------------|-----------|----------------|-----------|
| 1    | 0.6724 | GSF | 0.5207 | GSS | 0.0026 | KS |
| 2    | 0.6740 | PF-RWM-SYS(1000) | 0.5212 | PS-RWM-SYS(1000) | 0.0031 | QKS |
| 3    | 0.6744 | PF-RWM-ML(1000) | 0.5220 | PS-RWM-ML(1000) | 0.0111 | UKS |
| 4    | 0.6754 | PF-RWM-SYS(300) | 0.5231 | PS-RWM-SYS(300) | 0.1453 | PS-RWM-SYS(100) |
| 5    | 0.6765 | PF-RWM-ML(300) | 0.5247 | PS-RWM-ML(300) | 0.1644 | PS-RWM-ML(100) |
| 6    | 0.6880 | PF-RWM-SYS(100) | 0.5393 | PS-RWM-MT(1000) | 0.1718 | PS-RWM-ML(100) |
| 7    | 0.6948 | PF-RWM-ML(100) | 0.5415 | PS-RWM-SYS(100) | 0.2077 | PS-MH-LS(100) |
| 8    | 0.7588 | PF-RWM-ML(1000) | 0.5420 | PS-RWM-MT(500) | 0.2109 | PS-MH-SYS(100) |
| 9    | 0.7830 | PF-RWM-MT(300) | 0.5570 | PS-RWM-ML(100) | 0.2354 | PS-MH-ML(100) |
| 10   | 0.9590 | PF-MH-SYS(100) | 0.5689 | PS-RWM-MT(1000) | 0.3931 | GSS |
| 11   | 0.9593 | PF-MH-MT(1000) | 0.6078 | PS-MH-SYS(100) | 0.3984 | PS-RWM-MT(100) |
| 12   | 0.9595 | PF-MH-ML(1000) | 0.6711 | PS-MH-ML(1000) | 0.4579 | PS-MH-MT(100) |
| 13   | 0.9608 | PF-MH-ML(500) | 0.6737 | PS-MH-SYS(500) | 0.4676 | EKS |
| 14   | 0.9612 | PF-MH-MT(500) | 0.6746 | PS-MH-ML(500) | 0.6048 | PS-RWM-SYS(500) |
| 15   | 0.9612 | PF-MH-ML(500) | 0.6752 | PS-MH-MT(1000) | 0.6348 | PS-RWM-ML(500) |
| 16   | 0.9686 | PF-MH-SYS(100) | 0.6781 | PS-MH-ML(500) | 0.7469 | PS-RWM-ML(500) |
| 17   | 0.9697 | PF-MH-ML(100) | 0.6927 | PS-MH-SYS(100) | 0.9772 | PS-MH-LS(500) |
| 18   | 0.9715 | PF-MH-MT(100) | 0.6927 | PS-MH-ML(100) | 1.2054 | PS-RWM-SYS(100) |
| 19   | 1.0138 | KF | 0.6974 | PS-MH-MT(100) | 1.2274 | PS-RWM-LS(100) |
| 20   | 1.6731 | PF-RWM-MT(100) | 0.7469 | PS-MH-LS(100) | 1.2709 | PS-MH-LS(500) |
| 21   | 1.8616 | QKF | 0.7497 | PS-MH-LS(500) | 1.4192 | PS-MH-ML(500) |
| 22   | 5.0549 | UKF | 0.7667 | PS-MH-SYS(100) | 1.5197 | PS-RWM-ML(100) |
| 23   | 7.3381 | PF-RWM-LS(100) | 0.9100 | KS | 1.8362 | PS-RWM-MT(500) |
| 24   | 7.3602 | PF-RWM-LS(500) | 0.9393 | PS-RWM-LS(100) | 2.0277 | PS-MH-LS(100) |
| 25   | 7.6912 | PF-RWM-LS(100) | 1.2900 | PS-RWM-LS(500) | 2.3945 | PS-MH-MT(500) |
| 26   | 8.3846 | PF-MH-LS(100) | 1.6693 | QKS | 3.0254 | PS-MH-SYS(100) |
| 27   | 8.4079 | PF-MH-LS(500) | 0.5054 | UKS | 3.3505 | PS-MH-ML(100) |
| 28   | 8.6717 | PF-MH-LS(100) | 0.4904 | PS-RWM-LS(100) | 3.6364 | PS-RWM-MT(1000) |
| 29   | 47.7827 | EKF | 33.8842 | EKS | 5.0651 | PS-MH-MT(1000) |

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Fig. 4. Filtering PDFs for a time instant. GT stands for the ground truth. KF, EKF, QKF, UKF, GSF, and PF stand for Kalman filter, extended Kalman filter, quantized Kalman filter, unscented Kalman filter, Gaussian sum filter, and particle filter, respectively. The PDFs given by the KF, EKF, QKF, UKF were frozen in all plots to observe the behavior of the PF (with RWM moves) when the number of particles increases. SYS, ML, MT, LS stand for systematic, multinomial, metropolis, and local selection resampling algorithms, respectively.

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Fig. 5. Smoothing PDFs for a time instant. GT stands for the ground truth. KS, EKS, QKS, UKS, GSS, and PS stand for Kalman smoother, extended Kalman smoother, quantized Kalman smoother, unscented Kalman smoother, Gaussian sum smoother and particle smoother, respectively. The PDFs given by the KS, EKS, QKS, UKS were frozen in all plots to observe the behavior of the PS (with RWM moves) when the number of particles increases. SYS, ML, MT, LS stand for systematic, multinomial, metropolis, and local selection resampling algorithms, respectively.

Fig. 6. Boxplot of the MSE between the estimated and true state for 1000 Monte Carlo experiments. KF, EKF, UKF, QKF, GSF, and PF stand for Kalman filter, extended Kalman filter, unscented Kalman filter, Gaussian sum filter, and particle filter, respectively. Additionally, SYS, ML, MT, LS stand for systematic, multinomial, metropolis, and local selection resampling algorithms, respectively. RWM and MH denote Random-Walk Metropolis and Metropolis-Hasting moves.

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Fig. 7. Boxplot of the MSE between the estimated and true state for 1000 Monte Carlo experiments. KS, EKS, QKS, UKS, GSS, and PS stand for Kalman smoother, extended Kalman smoother, quantized Kalman smoother, unscented Kalman smoother, Gaussian sum smoother, and particle smoother, respectively. Additionally, SYS, ML, MT, LS stand for systematic, multinomial, metropolis, and local selection resampling algorithms, respectively. RWM and MH denote Random-Walk Metropolis and Metropolis-Hasting moves.

Fig. 8. Boxplot of the execution time for 1000 Monte Carlo experiments. KS, EKS, QKS, UKS, GSS, and PS stand for Kalman smoother, extended Kalman smoother, quantized Kalman smoother, unscented Kalman smoother, Gaussian sum smoother, and particle smoother, respectively. Additionally, SYS, ML, MT, LS stand for systematic, multinomial, metropolis, and local selection resampling algorithms, respectively. RWM and MH denote Random-Walk Metropolis and Metropolis-Hasting moves.

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