THERMODYNAMICS OF GAUSSIAN FLUCTUATIONS AND PARACONDUCTIVITY IN LAYERED SUPERCONDUCTORS

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A detailed theoretical analysis of the Gaussian fluctuations of the order parameter in layered superconductors is performed within the Ginzburg-Landau (GL) theory. The available results for the Gaussian fluctuations are systematized and a number of novel formulae for the fluctuation magnetization, nonlinear magnetic susceptibility, heat capacity and high-frequency conductivity in layered superconductors are derived. We propose several new prescriptions: how to determine the lifetime constant of fluctuation Cooper pairs $\tau_0$, the in-plane coherence length $\xi_{ab}(0)$, the energy cutoff parameter $\varepsilon_c = \hbar^2/(2m_{ab}\xi_{ab}^2(0))$, and the Ginzburg number $\epsilon_{Gi}$. It is demonstrated, for example, how the spectroscopy of the lifetime can be used to verify the existence of depairing mechanisms in layered cuprates. The ultraviolet regularization of the GL free energy is carried out by means of the well-known from the field theory $\zeta$-function method. We further show that the archetype of the latter method has its origin in the century of enlightenment and the novel result is that the fluctuation part of the thermodynamic variables of the layered superconductors can be expressed in terms of the Euler $\Gamma$-function and its derivatives. Universal scaling curves for the magnetic field dependence of the paraconductivity $\sigma_{ab}(T_c, B)$, fluctuation magnetization $M(T_c, B)$ and the heat capacity $C(T_c, B)$ are found for the quasi-2D superconductors at $T_c$ and further related to the form-factor of the Cooper pairs.

Keywords: Gaussian fluctuations, Ginzburg-Landau theory, layered superconductors

1. Introduction

The writing of this review is provoked by the progress in the study of fluctuation phenomena in high-$T_c$ superconductors. The small coherence lengths in the layered cuprates $\xi_c \ll \xi_a \simeq \xi_b$ give rise to a very high density of fluctuation degrees of freedom $\propto 1/(\xi_{ab}^2(0)\xi_c(0))$ which makes the fluctuation effects easier to be observe in the high-$T_c$- rather than in the conventional superconductors. An intriguing feature of the fluctuation effects to be pointed out is that they can be observed even in the case when the interaction between fluctuations is vanishing or can be treated in a self-consistent manner. In such a case, for high quality crystals, the
fluctuations are of Gaussian nature and their theory is very simple. A number of good experimental studies have already been performed in the Gaussian regime thus initiating the Gaussian fluctuation spectroscopy for high-\(T_c\) materials.

By spectroscopy here we imply only those experiments with trivial theory where every measurement provides an immediate information for some parameter(s) important for the material science or fundamental physics of these interesting materials. Half a century ago Landau used to speak about himself as being the greatest trivializator in the theoretical physics. At present, the Ginzburg-Landau (GL) theory (called by Ginzburg also \(\Psi\)-theory) is the adequate tool to describe the fluctuation phenomena in the superconductors. The parameters of the \(\Psi\)-theory, such as coherence lengths, relaxation time \(\tau_{0,\Psi}\) of the GL order parameter \(\Psi\), the GL parameter \(\kappa_{GL} = \lambda_{ab}(0)/\xi_{ab}(0)\) are also "meeting point" between the theory and the experiment.

From one hand, these parameters are necessary for the description of the experimental data and from another hand they can be derived form the microscopic theory using the methods of the statistical mechanics. That is why the determination of the GL parameters is an important part of the investigations of every superconductor and the Gaussian fluctuation spectroscopy is an indispensable tool in these comprehensive investigations.

The purpose of this review is to systematize the known classical results for the GL Gaussian fluctuations, to derive new ones when needed, and to finally give suitable for coding formulae necessary for the further development of the Gaussian spectroscopy. The derivation of all results is described in detail and trivialized to the level of the Landau-Lifshitz encyclopedia on theoretical physics,\(^2\) the textbooks by Abrikosov\(^3\) and Tinkham\(^4\) or the well-known reviews by Cyrot\(^5\) on the GL theory, by Bulaevskii\(^6\) concerning the layered superconductors with Josephson coupling, and by Skocpol and Tinkham\(^7\) on the fluctuation phenomena in superconductors. The present work is intended as a review on the theoretical results which can be used by the Gaussian spectroscopy of fluctuations but no historical survey of the experimental research is attempted. Therefore we do explicitly refer to only a limited number of experimental studies in this field. Instead, the reader is referred to the citations-reach conference proceedings,\(^1\) but even therein a number of good works are probably not included. We do not cite directly even the epoch-creating paper by Bednorz and Müller but its spirit can be traced to every contemporary paper on high-\(T_c\) superconductivity. Even to focus on the theoretical results related to fluctuation phenomena is a very difficult problem by itself and therefore, when referring to any result one should imply "to the best of our knowledge...". One of our goals was also to fill the gap between the textbooks and experimentalists’ needs for a compilation of theoretical formulae written in common notations, appropriate for direct use.

Of course, there is a great number of interesting physical situations especially related to vortices where the fluctuations are definitely non-Gaussian. Those problems fall beyond the scope of the present review and we include only some references
from this broad field in the physics of superconductivity.\textsuperscript{8}

The review is organized as follows: in Sec. 2 the case of weak magnetic fields is considered and the thermodynamic variables are expanded in power series in the dimensionless magnetic field $h = B_z/B_{c2}(0)$. The standard notations for the thermodynamic variables in a layered superconductor are then introduced in Subsec. 2.1, and Subsec. 2.2 is dedicated to the Euler-MacLaurin summation formula in the form appropriate for the analysis of the GL results for the free energy and its ultraviolet (UV) regularization. A systematic procedure to derive the results for a layered superconductor from the results for a two-dimensional (2D) superconductor is developed in Subsec. 2.3 and the action of the introduced "layering" operator $\hat{L}$ is illustrated on the example of the formulae for the paraconductivity. Further we consider the static paraconductivity in case of perpendicular magnetic field as well as the high frequency conductivity in zero magnetic field. The power series for the nonlinear magnetic susceptibility and the magnetic moment in the Lawrence-Doniach (LD) model are derived in Subsec. 2.4 and the $\varepsilon$-method for summation of such divergent series is described in Subsec. 2.5. For practical purposes a simple FORTRAN90 program is given in the Appendix.

Further in Subsec. 2.6 we present the power series for the differential susceptibility and general weak-magnetic-field expansion formulae for the magnetization. Section 3 is dedicated to the study of the strong magnetic fields limit. Firstly, in Subsec. 3.1 the general formula for the Gibbs free energy in perpendicular to the layers magnetic field is analyzed. The fluctuation part of the thermodynamic variables is found then by differentiation in Subsec 3.2. Subsec. 3.3 is devoted to the self-consistent mean-field treatment of the fluctuation interactions in the LD model. The important limit case of an anisotropic 3D GL model is considered in Subsec. 3.4 where we derive the Gibbs free energy and the fluctuation magnetic moment. In Sec. 4 an account is given of the fitting procedure for the GL parameters which rests on theoretical results and some recommendations for the most appropriate formulae are also given for determination of the cutoff energy $\varepsilon_{\text{pc}}$ in Subsec. 4.1, the in-plane coherence length $\xi_{ab}(0)$ in Subsec. 4.2, the Cooper pair life-time constant $\tau_0$ in Subsec 4.3, and the 2D Ginzburg number in Subsec. 4.4. All new results derived throughout this review are summarized in Sec. 5 and some perspectives for the Gaussian spectroscopy are discussed as well.

2. Weak magnetic fields

2.1. Formalism

Before embarking on a detailed analysis we shall briefly introduce all entities entering the basic for our further considerations quantity—the GL functional $G$ for the Gibbs free energy in external magnetic field $H^{(\text{ext})}$. For compliance with the
previous works we follow the standard notations in which $G$ reads

$$G[\Psi_{j,n}(x,y), A(r)] = \sum_{n=-\infty}^{+\infty} \sum_{j=1}^{N} \int dx \, dy \left\{ \sum_{l=x,y} \frac{1}{2m_{ab}} \left( \frac{\hbar}{i} \frac{\partial}{\partial x_{l}} - e^{*} A_{l} \right) \Psi_{jn} \right\}^{2} + a_{0} \epsilon |\Psi_{j,n}|^{2} + \frac{1}{2} \left| \Psi_{j,n} \right|^{4} + a_{0} \gamma_{j} \left| \Psi_{j+1,n} - \Psi_{j,n} \exp \left( \frac{ie^{*}}{\hbar} \int_{z_{j,n}}^{z_{j+1,n}} A_{z} \, dz \right) \right|^{2} + \int \frac{1}{2\mu_{0}} \left( \nabla \times A - \mu_{0} H^{(\text{ext})} \right)^{2} \, dx \, dy \, dz,$$

(1)

with $A$ being the vector potential of the magnetic field $B = \nabla \times A$.

The material parameters in this sizable expression are illustrated in Fig. 1, thus we only need to note that the GL potential $a(\epsilon) = a_{0} \epsilon$ is parameterized by $a_{0} = \hbar^{2}/2m_{ab} \xi^{2}_{ab}(0)$, and $\epsilon \equiv \ln(T/T_{c}) \approx (T - T_{c})/T_{c}$ is the reduced (dimensionless) temperature. If not otherwise stated we shall make use of the SI units, thus the magnetic permeability of vacuum $\mu_{0} = 4\pi \times 10^{-7}$.

Here we will restrict ourselves to the study of fluctuations in the Gaussian regime in the normal phase not too close to the critical line $H_{c2}(T)$. In this case the nonlin-
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ear term in $G[\Psi, A]$ is negligible, $\tilde{b}|\Psi|^4/2 \to 0$. For the normal phase the magnetization is also very small and with high accuracy $\mu_0 H^{(\text{ext})} \approx B = \mu_0 (H + M) \approx \mu_0 H$.

To begin with, consider the simplest case of zero external magnetic field $H^{(\text{ext})} = 0$. Given the above assumption for $\Psi$, the GL functional is a quadratic form and one needs to sum over all eigenvalues of the energy spectrum

$$
\varepsilon_j(p, p_z) = \frac{p^2}{2m_{ab}} + \varepsilon_{c j}(p_z), \quad \varepsilon_{c j}(p_z) = a_0 \omega_j^{(N)}(\theta),
$$

where $\varepsilon_{c j}(p_z)$ are the tight-binding energy bands describing the motion of Cooper pair in $z$ direction, $p = (p_{x}, p_{y})$ is the in-plane ($ab$-plane) momentum of the fluctuating Cooper pairs and $\theta = p_z s/\hbar \in (0, 2\pi)$ is the Josephson phase. For a single layered material, $N = 1$, this corresponds to the well known Lawrence-Doniach model,$^9$

$$
\omega_1^{(LD)}(\theta) = 2\gamma_1 (1 - \cos \theta)
$$

while the case $N = 2$ is the Maki-Thompson (MT) model,$^{10}$ proposed independently by Hikami and Larkin$^{11}$ as well,

$$
\omega_j^{(MT)}(\theta) = \gamma_1 + \gamma_2 + (-1)^j \sqrt{\gamma_1^2 + \gamma_2^2 + 2\gamma_1 \gamma_2 \cos \theta}, \quad j = 1, 2.
$$

Thus, the sum over the energy spectrum gives

$$
G[\Psi] = \sum_{p, p_z, j} \langle \varepsilon_j(p, p_z) + a \rangle |\Psi_{p, p_z, j}|^2,
$$

where $\Psi_{p, p_z, j}$ is the wave function of the superconducting condensate in momentum space. We use standard periodic boundary conditions for a bulk domain of volume $V = L_x \times L_y \times L_z$ which give

$$
\sum_p = L_x L_y \int \frac{dp_x dp_y}{(2\pi \hbar)^2}, \quad \sum_{pz} = L_z \int \frac{d\theta}{2\pi s}.
$$

In order to calculate the fluctuation part of the Gibbs free energy $G(T)$ at zero magnetic field, one usually solves for every point in the momentum space $p, p_z, j$ the Gaussian integral

$$
\exp \left[ -\frac{G}{k_B T} \right] = \int \int \frac{d\Psi' d\Psi''}{2\pi} \exp \left\{ -\frac{\varepsilon + a}{k_B T} \left[ (\Psi')^2 + (\Psi'')^2 \right] \right\}
\quad = \frac{1}{2} \int_0^\infty \exp \left[ -\frac{\varepsilon + a}{k_B T} \rho \right] d\rho = \frac{k_B T/2}{\varepsilon + a},
$$

where $\Psi = \sqrt{\rho} e^{i\varphi} \equiv \Psi' + i\Psi''$, and $\varphi \in (0, 2\pi)$. Making use of this auxiliary result the calculation of the fluctuation part of the Gibbs free energy reduces to summation over the spectrum of an effective Hamiltonian, i.e.

$$
G = -k_B T \sum_{p, p_z, j} \ln \frac{k_B T/2}{\varepsilon_j(p, p_z) + a}.
$$
or, taking into account Eqs. (2) and (6),

$$G \frac{V}{V} = k_B T \int \! \frac{d (\pi p^2)}{(2\pi)^2} \frac{1}{N} \sum_{j=1}^{N} \int \! \frac{d \theta}{2\pi} \ln \left[ \left( \frac{p^2}{2m_{ab}} + a_0 \omega_j^{(N)}(\theta) + a_0 \epsilon \frac{a_0}{\hbar k_B T} \right) \right].$$  \hspace{1cm} (9)

In view of the further calculations it is also useful to introduce a dimensionless in-plane kinetic energy

$$\tilde{x} = \frac{p^2}{2m_{ab}a_0} = \left( \frac{\xi_{ab}(0)p}{\hbar} \right)^2 \in (0, c),$$ \hspace{1cm} (10)

bound by a dimensionless cutoff parameter $c$ which we consider to be an important parameter of the GL theory when applied to copper oxide superconductors. Later in Sec 4.1 we demonstrate how the value of the dimensional cutoff energy $\epsilon_{pc}$ can be determined by fitting to the experimental data. An immediate simplification to Eq. (9) can be achieved by dropping the $\frac{1}{2}k_B T/a_0 \rightarrow \text{const}$ multiplier in the argument of the logarithm as it is irrelevant for the critical behavior of the material. Furthermore, since fluctuation observables are related to non analytical dependence of the Gibbs free energy on the reduced temperature, we can substitute $T = T_c(1 + \epsilon) \approx T_c$ and the free energy per unit volume $F(\epsilon)$ is cast in more elegant form,

$$F(\epsilon) \approx \frac{G}{L_x L_y L_z} = \frac{k_B T_c}{4\pi \xi_{ab}(0)} s \int \! d\tilde{x} \frac{1}{N} \sum_{j=1}^{N} \int \! \frac{d \theta}{2\pi} \ln \left( \tilde{x} + \omega_j^{(N)}(\theta) + \epsilon \right),$$ \hspace{1cm} (12)

that could easily include the $(1 + \epsilon)$-factor in all cases when necessity appears. The physical meaning of this important for our further considerations expression is fairly transparent: one has to integrate with respect to the Josephson phase $\theta$, which describes the motion of Cooper pairs in $c$-direction, and to take into account as many different Cooper pair energy bands as are there the different superconducting layers per unit cell. Finally, integration with respect to the in-plane Cooper pair kinetic energy is to be carried out.

Consider now the important case of an external magnetic field applied parallel to the $c$-direction, i.e. perpendicular to the CuO$_2$ planes, $B = (0, 0, B)$. In this case, the in-plane kinetic energy of the Cooper pairs acquires oscillator spectrum,$^{12}$ corresponding to the quantum mechanical problem of an electron in an external magnetic field,$^2$

$$\frac{p^2}{2m_{ab}} \rightarrow \hbar \omega_c \left( n + \frac{1}{2} \right),$$ \hspace{1cm} (13)
where \( n = 0, 1, 2, 3, \ldots \) is a non-negative integer and \( \omega_c = |e^*| B/m_{ab} \) is the cyclotron frequency. The integration over the momentum space is thus reduced to summation over oscillator energy levels

\[
\int_{|p| < p_c} \frac{d^2p}{(2\pi\hbar)^2} \rightarrow \frac{B}{\Phi_0} \sum_{n=0}^{n_c-1},
\]

where \( n_c \equiv c/2\hbar \) and \( \Phi_0 = 2\pi\hbar / |e^*| = 2.07 \, \text{fT m}^2 \) is the flux quantum. The energy cutoff is to be applied now to the oscillator levels, \( \hbar \omega_c (n_c + \frac{1}{2}) = c a_0 \). Let us recall that the equation for the upper critical field \( H_{c2}(T) \) within the GL theory is nothing but the equation for annulment of the lowest energy level, \( \frac{1}{2} \hbar \omega_c + a(\epsilon) = 0 \). Thereby introducing the upper critical field linearly extrapolated to zero temperature,

\[
\mu_0 H_{c2}(0) = B_{c2}(0) \equiv T_c \left. \frac{dB_{c2}(T)}{dT} \right|_{T_c} = \frac{\Phi_0}{2\pi\xi_{ab}(0)},
\]

and the dimensionless reduced magnetic field,

\[
h \equiv \frac{B}{B_{c2}(0)} = \frac{H}{H_{c2}(0)},
\]

we obtain a linear approximation for the critical line about \( T_c \),

\[
h_{c2}(\epsilon) = \frac{H_{c2}(T)}{H_{c2}(0)} \approx -\epsilon \ll 1.
\]

With the help of the dimensionless variables introduced so far it is easily worked out that the influence of the external magnetic field is reduced to discretization of the dimensionless in-plane kinetic energy,

\[
\tilde{x} \rightarrow h(2n + 1)
\]

and the integrals of an arbitrary function \( f \) with respect to \( \tilde{x} \) are converted to sums,

\[
\int_{0}^{c} f(\tilde{x})d\tilde{x} \rightarrow 2h \sum_{n=0}^{n_c-1} f(h(2n + 1)).
\]

In fact, Max Planck discovered the quantum statistics of the black-body radiation using the same replacement. Applying this procedure to the previously derived free energy at zero magnetic field, Eq. (12), we obtain

\[
F(\epsilon) \rightarrow F(\epsilon, h) = \frac{\Delta G}{L_x L_y L_z},
\]

\[
F(\epsilon, h) = \frac{k_B T_c}{4\pi\xi_{ab}(0)^2} \sum_{n=0}^{n_c-1} 2h \sum_{n=0}^{N} 1 \sum_{j=1}^{N} \int \frac{d\theta}{2\pi} \ln \left[ h(2n + 1) + \omega_j^{(N)}(\theta) + \epsilon \right].
\]
This expression represents the starting point for all further considerations. As a first step we address in the next section the Euler-MacLaurin method and its application to the sum over the Landau levels which appears in Eq. (21).

2.2. Euler-MacLaurin summation for the free energy

Near the critical temperature, when \( \epsilon \ll c \), one can consider formally \( c \to \infty \), and \( n_c(h) \approx c/2h \to \infty \). Within such a local approximation the previous finite sums are transformed into infinite ones,

\[
2h \sum_{n=0}^{\infty} f(\epsilon + h(2n + 1)) = \hat{\Sigma}_{\text{EM}} \int_{\epsilon}^{\infty} f(\tilde{x}) d\tilde{x},
\]

where

\[
\hat{\Sigma}_{\text{EM}} \equiv \frac{h \frac{\partial}{\partial \epsilon}}{\sinh(h \frac{\partial}{\partial \epsilon})} = \sum_{n=0}^{\infty} (-1)^n \frac{2}{\pi^{2n}} \left(1 - \frac{1}{2^{2n-1}}\right) \zeta(2n) \left(h \frac{\partial}{\partial \epsilon}\right)^{2n}
\]

\[
= 1 - \frac{1}{6} h^2 \frac{\partial^2}{\partial \epsilon^2} + \frac{7}{360} h^4 \frac{\partial^4}{\partial \epsilon^4} - \frac{31}{15120} h^6 \frac{\partial^6}{\partial \epsilon^6} + \cdots
\]

is the Euler-MacLaurin operator for summation of series, in which we employ the Riemann and Hurwitz zeta functions, respectively,

\[
\zeta(\nu) = 1 + \frac{1}{2^\nu} + \frac{1}{3^\nu} + \cdots = \zeta(\nu, 1), \quad \zeta(\nu, z) = \sum_{n=0}^{\infty} \frac{1}{(n + z)^\nu},
\]

instead of Bernoulli numbers. The \( \hat{\Sigma}_{\text{EM}} \) operator can be easily obtained exploiting the exponential representation of the standard translation operator \( \hat{T}_z \), whose action is defined as follows

\[
f(b + \epsilon) = \hat{T}_z(b) f(z) \bigg|_{z=\epsilon} = \exp\left( b \frac{\partial}{\partial z}\right) f(z) \bigg|_{z=\epsilon}.
\]

If summed up from zero to infinity the above expression would give an infinite geometric progression,

\[
\sum_{n=0}^{\infty} \left[ \hat{T}_z(b) \right]^n = \sum_{n=0}^{\infty} \left[ \exp\left( b \frac{\partial}{\partial z}\right) \right]^n = \frac{1}{1 - \exp(b \frac{\partial}{\partial z})}.
\]

Let us introduce now the fluctuational part of the heat capacity,

\[
C(\epsilon) = -\frac{1}{T_c} \frac{\partial^2}{\partial \epsilon^2} F(\epsilon).
\]

Using this physical observable and Eq. (21), one can extract the magnetic field dependent part of the free energy,

\[
F(\epsilon, h) - F(\epsilon) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2}{\pi^{2n}} \left(1 - \frac{1}{2^{2n-1}}\right) \zeta(2n) h^{2n} \frac{\partial^{2(n-1)}}{\partial \epsilon^{2(n-1)}} T_c C(\epsilon)
\]

\[
= \frac{1}{6} h^2 - \frac{7}{360} h^4 \frac{\partial^2}{\partial \epsilon^2} + \frac{31}{15120} h^6 \frac{\partial^4}{\partial \epsilon^4} - \cdots T_c C(\epsilon).
\]
It is then straightforward to calculate the magnetization $M$ and the nonlinear susceptibility defined as $\chi(\epsilon, h) = M/H$, i.e.

$$M = -\frac{\partial [F(\epsilon, h) - F(\epsilon)]}{\partial B} = \chi(\epsilon, h)H. \quad (29)$$

For the Meissner-Ochsenfeld (MO) state, for example, $\chi^{(\text{MO})} = -1$. Importantly, Eq. (29) incorporates the regularized free energy which, by virtue of Eq. (28), does not contain zero magnetic field part,

$$F_{\text{reg}}(\epsilon, h) \equiv F(\epsilon, h) - F(\epsilon) := \hat{\text{Reg}}_{\text{EM}} F(\epsilon) \quad (30)$$

where

$$\hat{\text{Reg}}_{\text{EM}} = \hat{\Sigma}_{\text{EM}} - \hat{1} = 2h \sum_{n=0}^{n_c} - \int_0^c d\tilde{x} \quad (31)$$

is the Euler-MacLaurin regularization operator. This method was applied\textsuperscript{13,14,15} for calculation of zero-field limit of the magnetic susceptibility, cf. Ref. 2. Hence, inserting the $h^2$-term of Eq. (28) into Eq. (29) one finds\textsuperscript{16} for $H \to 0$

$$\chi(\epsilon) = -\frac{\mu_0 T_c C(\epsilon)}{3B^2_c(0)} = -\frac{4\pi^2 \mu_0 \xi_{ab}(0) T_c C(\epsilon)}{3\Phi_0^2} \quad (32)$$

For illustration, let us analyze how this relation between susceptibility and heat capacity can be applied to the LD model. For arbitrary multilayered structure we can calculate the curvature of the lowest dimensionless energy band in $c$-direction

$$\omega_1(\theta) = \epsilon_{c1}(p_z)/a_0,$$

according to this definition $r$ parameterizes the effective mass in $c$-direction $m_c$ for an anisotropic GL model, $|p_z| \ll \pi \hbar / s$,

$$\epsilon_{c1} \approx a_0 \frac{\theta^2}{4} = \frac{p_z^2}{2m_c} \quad (34)$$

It is now easily realized that the identity holds true,

$$r = \left( \frac{2N \xi_{c}(0)}{s} \right)^2 \quad (35),$$

which for $N = 1$ is the LD parameter $r$ that determines the effective dimensionality of the superconductor, cf. the review by Varlamov et al.\textsuperscript{17} In many other studies, e.g. Ref. 7, the wave vector $\mathbf{k} = \mathbf{p} / \hbar$ has been used as well. In terms of the latter, for the dimensionless kinetic energy in the long-wavelength approximation we have

$$\left[ \xi_{ab}(\hbar \mathbf{k}) + \epsilon_{c1}(\hbar k_z) \right] / a_0 \approx \xi_{ab}(0) \mathbf{k}^2 + \xi_{c}(0) k_z^2 = \xi_{ab}(0) \mathbf{k}^2 + r \theta^2, \quad (36)$$
where \( \varepsilon_{ab}(\hbar \mathbf{k}) \) represents the in-plane part of the kinetic energy. Let us mention that the LD model is not only applicable to single layered cuprates with \( N = 1 \), but is it also to bi-layered cuprates \( (N = 2) \) in the limit cases \( \gamma_1 \simeq \gamma_2 \) as well as in the case \( \gamma_1 \ll \gamma_2 \) when formally \( N = 1 \). That is why we use in our formulae an effective periodicity of the LD-model

\[
\text{s}_{\text{eff}} = \frac{s}{N}.
\]

For completeness we list below without deriving some of the well-known results within the LD-model. The single energy band has the form

\[
\varepsilon_c(p_z) = a_0 \omega_1 \left( \frac{p_z s}{2 \pi \hbar} \right) = \frac{\hbar^2}{m_c(s/N)^2} (1 - \cos \theta), \quad \text{where}
\]

\[
\omega_1(\theta) = \frac{1}{2} r (1 - \cos \theta) = r \sin^2 \frac{\theta}{2},
\]

being parameterized by the Josephson coupling energy

\[
J_1 = a_0 \gamma_1 = \frac{\hbar^2}{m_c(s/N)^2}, \quad r = 4 \gamma_1 = \left( 2 N \xi_c(0)/s \right)^2.
\]

For the heat capacity one has

\[
C^{(\text{LD})}(\epsilon) = \frac{k_B}{4 \pi \xi_{ab}^2(0)} N \frac{1}{s} \frac{1}{\sqrt{\epsilon + r}},
\]

and the magnetic susceptibility for a weak magnetic field applied in \( c \)-direction, according to Tsuzuki\(^{18}\) and Yamaji,\(^{20}\) reads as

\[
-\chi^{(\text{LD})}(\epsilon) = \frac{\pi}{3} \mu_0 \frac{k_B T_c}{\Phi_0} \xi_{ab}^2(0) \frac{N}{s} \frac{1}{\sqrt{\epsilon}} \frac{1}{\sqrt{\epsilon + r}} = \frac{1}{6} \frac{M_0}{H_{c2}(0)} \frac{1}{\sqrt{\epsilon}} \frac{1}{\sqrt{\epsilon + r}},
\]

where

\[
M_0 = \frac{k_B T_c}{\Phi_0} \frac{N}{s}.
\]

Before proceeding we feel it appealing to make some technical remarks concerning the representation of the general formulæ for fluctuations in arbitrary layered superconductor. To be specific, we shall demonstrate how the expressions for the magnetic susceptibility, Eq. (42), and heat capacity, Eq. (41), within the LD model can be obtained as special cases of a general procedure described in the next subsection.

**2.3. Layering operator \( \hat{L} \) illustrated on the example of paraconductivity**

In the general formula for the density of the free energy, Eq. (12), the energies related to motion in \( c \)-direction \( \varepsilon_{c}(p_z) \) enter the final result solely via the fragment \( \epsilon+\omega_j(\theta) \). Thus, in all such cases one can first solve the corresponding 2D problem and then
for a layered superconductor the result can be derived by merely averaging the 2D result with respect to the motion of the fluctuation Cooper pairs in perpendicular to the layers direction. Formally, this method reduces to introducing a layering operator \( \hat{L} \) acting on functions of \( \epsilon \); e.g. for the conductivity one would have the relation

\[
\sigma(\epsilon) = \hat{L}\sigma^{(2D)}(\epsilon) \equiv \frac{1}{N} \sum_{j=1}^{N} \int \frac{d\theta}{2\pi} \sigma^{(2D)}(\epsilon + \omega_j^{(N)}(\theta)).
\]  

(44)

In terms of the so introduced operator \( \hat{L} \) the expression for the free energy, Eq. (21), takes the form

\[
F(\epsilon) = F_0 \int_{c}^{c} d\tilde{\epsilon} \sum_{j=1}^{N} \hat{L} \ln \left( \tilde{\epsilon} + \omega_j^{(N)}(\theta) + \epsilon \right), \quad F_0 \equiv \frac{k_B T_c}{4\pi \xi_{ab}(0)} s.
\]

(45)

Besides for thermodynamic variables this operator works also for the fluctuation in-plane conductivity. Conforming with the work of Hikami and Larkin,\textsuperscript{11} for the conductivity within the LD model we have to integrate the 2D conductivity with respect to the Josephson phase,

\[
\sigma(\epsilon) = \hat{L}^{(LD)}\sigma^{(2D)}(\epsilon) \equiv \oint \frac{d\theta}{2\pi} \sigma^{(2D)}(\epsilon + \frac{1}{2}r(1 - \cos \theta)).
\]

(46)

Given a system with independent 2D layers, having density in \( c \)-direction \( N/s \), for zero magnetic field we have to average the well-known Aslamazov-Larkin expression for the static (zero-frequency) conductivity,

\[
\sigma_{AL}(\epsilon) = \frac{e^2}{16\hbar} \frac{N 1}{\epsilon s} = \frac{\pi}{8} R_{QHE}^{-1} \frac{N 1}{\epsilon s},
\]

(47)

where \( R_{QHE} \equiv 2\pi \hbar/e^2 = 25.813 \) kΩ. A simple integration gives

\[
f_{LD}(\epsilon;r) \equiv \hat{L}^{(LD)} \frac{1}{\epsilon} = \oint \frac{d\theta}{2\pi} \frac{1}{\epsilon + \frac{1}{2}r(1 - \cos \theta)} = \frac{1}{\sqrt{\epsilon(\epsilon + r)}}.
\]

(48)

We note that this integral determines both the heat capacity and magnetic susceptibility for the LD model and is widely used for fitting to experimental data. Another important integral is

\[
\hat{L}^{(LD)} \int_{\epsilon}^{c} \ln \tilde{\epsilon} \, d\tilde{\epsilon} = \int_{\epsilon}^{c} 2 \ln \left( \frac{\sqrt{\epsilon + \tilde{\epsilon} + r}}{2} \right) d\tilde{\epsilon}
\]

\[
= \left[ (2\epsilon + r) \ln \left( \sqrt{\epsilon + \sqrt{\epsilon + r}} \right) - \sqrt{\epsilon(\epsilon + r)} - \ln(4) \right] \bigg|_{\epsilon}^{c},
\]

(49)

which is used in representing the free energy at zero magnetic field, Eq. (12), cf. also Eqs. (129) and (130) below. Further, the \( \epsilon \)-derivative of this equation,

\[
\hat{L}^{(LD)} \ln \epsilon = \oint \frac{d\theta}{2\pi} \ln \left( \epsilon + \frac{1}{2}r(1 - \cos \theta) \right) = 2 \ln \left( \frac{\sqrt{\epsilon + \sqrt{\epsilon + r}}}{2} \right),
\]

(50)
is important for the calculation of the fluctuation part of the entropy and the density of fluctuation Cooper pairs. We provide also two other integrals employed in calculating the magnetoconductivity

\begin{equation}
\hat{L}^{(LD)} \frac{1}{\epsilon^2} = - \hat{L}^{(LD)} \frac{\partial}{\partial \epsilon} \frac{1}{\epsilon} = \frac{\epsilon + \frac{1}{2} r}{[\epsilon (\epsilon + r)]^{5/2}},
\end{equation}

\begin{equation}
\hat{L}^{(LD)} \frac{1}{\epsilon^3} = \frac{1}{2} \frac{\partial^2}{\partial \epsilon^2} \hat{L}^{(LD)} \frac{1}{\epsilon} = \frac{(\epsilon + r)\epsilon + \frac{3}{8} r^2}{[\epsilon (\epsilon + r)]^{5/2}}.
\end{equation}

Analogously, in the Maki-Thompson bi-layered model\(^\text{10}\) \((N = 2)\) one summation should precede the integration,\(^\text{16}\)

\begin{equation}
\sigma(\epsilon) = \hat{L}^{(MT)} \sigma^{(2D)}(\epsilon) = \frac{1}{2} \oint \frac{d\theta}{2\pi} \left[ \sigma^{(2D)}(\epsilon + \omega_1(\theta)) + \sigma^{(2D)}(\epsilon + \omega_2(\theta)) \right],
\end{equation}

i.e. in order to calculate the conductivity\(^\text{10}\) and susceptibility\(^\text{16}\) we have to add the terms

\begin{equation}
\frac{1}{\epsilon + \omega_1} + \frac{1}{\epsilon + \omega_2} = \frac{2\epsilon + (\omega_1 + \omega_2)}{\epsilon^2 + (\omega_1 + \omega_2)\epsilon + \omega_1\omega_2}.
\end{equation}

In this expression both \(\omega_1 + \omega_2\) and \(\omega_1\omega_2\) are rational, cf. Eq. (4), and the integral (53) is reduced to the integral (48),

\begin{equation}
\hat{L}^{(MT)} \frac{1}{\epsilon} = \frac{\epsilon + \gamma_1 + \gamma_2}{\sqrt{\epsilon [\epsilon + 2(\gamma_1 + \gamma_2)] (\epsilon + 2\gamma_1) (\epsilon + 2\gamma_2)}} = \frac{\epsilon + \frac{1}{4} rw}{\sqrt{(\epsilon^2 + r\epsilon w) (\epsilon^2 + r\epsilon w + \frac{1}{4} r^2 w)}} \equiv f_{MT}(\epsilon; h; r, w),
\end{equation}

where

\[
r = 4 \frac{2}{\gamma_1 + \gamma_2}, \quad u \equiv \frac{J_{\text{max}}}{J_{\text{min}}} = \frac{\gamma_{\text{max}}}{\gamma_{\text{min}}}, \quad w \equiv \frac{1}{4} \left( 2 + \frac{\gamma_1}{\gamma_2} + \frac{\gamma_2}{\gamma_1} \right) = \frac{1}{4} \left( 2 + u + \frac{1}{u} \right).
\]

Such a form, involving the \(w\) parameter, is convenient for fitting the experimental data since for both \(w = 1\) and \(w \gg 1\) cases the LD approximation holds true, which is often found to give a satisfactory explanation of the experimental observations. For more detailed discussion the reader is referred to Ref. 21. The inverse relations for the above introduced parameters read as

\begin{equation}
u = (2w - 1) + 2\sqrt{w(w - 1)}, \quad \gamma_{\text{min}} = \frac{1}{2} \left( 1 + \frac{1}{u} \right) r, \quad \gamma_{\text{max}} = \frac{1}{2} (1 + u) r,
\end{equation}

and

\begin{equation}
J_{\text{max}} = a v \gamma_{\text{max}} = \frac{\hbar^2}{2m_{ab}c_{ab}(0)} \frac{1}{2} (1 + u) r.
\end{equation}
Importantly, at known effective mass $m_{ab}$ the last equation gives the possibility one to determine the Josephson coupling energy between double CuO$_2$ planes.

Let us now illustrate in more details the action of the $\hat{L}$ operator. Towards this end we consider the famous Aslamazov-Larkin formula for the 2D conductivity $\sigma_{ab}^{(2D)}(\epsilon)$, the results for the susceptibility $-\chi_{ab}^{(2D)}(\epsilon)$ due to V. Schmidt, A. Schmid and H. Schmidt, and Ferrell and Thouless fluctuation part of the heat capacity $C^{(2D)}(\epsilon)$. With the help of the $\hat{L}$ operator for arbitrary layered superconductor, cf. Refs. 25 and 26, these three quantities can be generally written as

$$\sigma_{ab}(\epsilon) = \frac{1}{R_{QHE}} \frac{2\tau_0 k_B T_c}{h} \frac{N}{s} \hat{L} \frac{1}{\epsilon},$$  
(58)$$

$$-\chi_{ab}(\epsilon) = \frac{\pi}{3} \mu_0 \frac{k_B T_c}{\Phi_0^2} s_{ab}(0) \frac{N}{s} \hat{L} \frac{1}{\epsilon},$$  
(59)$$

$$C(\epsilon) = \frac{k_B}{4\pi \xi_{ab}^2(0)} \frac{N}{s} \hat{L} \frac{1}{\epsilon},$$  
(60)$$

where the “$ab$”-subscript in Eq. (58) indicates that the conductivity is in the $ab$-planes, while in Eq. (59) it indicates that the vanishing magnetic field is perpendicular to the same planes. It is immediately apparent that the common for all these expressions function $\hat{L} \frac{1}{\epsilon}$ cancels when calculating the $\chi/C$, $\sigma/C$, and $\sigma/\chi$ quotients. In particular, the temperature independent ratio

$$\tau_0 = \frac{\mu_0}{3} \xi_{ab}^2(0) \frac{\sigma_{ab}(\epsilon)}{-\chi_{ab}(\epsilon)} = \text{const}$$  
(61)$$

provides the best method for probing the time constant $\tau_0$ parameterizing the life time of the fluctuation Cooper pairs with zero momentum,

$$\tau(\epsilon) = \frac{\tau_0}{\epsilon}, \quad |\psi_p=0(t)|^2 \propto \exp\left(-\frac{t}{\tau(\epsilon)}\right).$$  
(62)$$

The $\tau_0$ constant participates in time-dependent GL (TDGL) theory; see for example the reviews by Cyrot, Skocpol and Tinkham, and the textbooks by Abrikosov and Tinkham. Within the weak coupling BCS theory in the case of negligible deparing mechanisms the $\tau_0$ satisfies the relation

$$\frac{\tau_{0,\Psi}^{(BCS)} k_B T_c}{h} = \frac{\pi}{8}, \quad \frac{\tau_0^{(BCS)}}{h} = \frac{\pi}{16 k_B T_c} \frac{1}{\tau(BCS)(\epsilon)} = \frac{\tau_0^{(BCS)}}{\epsilon} = \frac{\pi}{16 k_B T_c} \frac{1}{\tau(BCS)(\epsilon)}$$  
(63)$$

where $\tau_{0,\Psi}^{(BCS)} = 2 \tau_0^{(BCS)}$ is the relaxation time constant for the order parameter being two times larger. At the present experimental accuracy this BCS value agrees well with the experimental data for the layered cuprates. Thus, the above observation led us to propose the dimensionless ratio

$$\tilde{\tau}_{rel} = \frac{\tau_0}{\tau_0^{(BCS)}} = 3 \frac{k_B T_c \tau_0}{2 \pi h} = \frac{8 k_B T_c \tau_0}{3 \pi h} = \frac{16 \mu_0}{3 \pi h} s_{ab}(0) \frac{k_B T c \sigma_{ab}(\epsilon)}{-\chi_{ab}(\epsilon)} = \text{const},$$  
(64)$$
to be used for more reliable experimental data processing; any deviation of \( \tilde{\tau}_{\text{rel}} \) from unity should be interpreted as a hint towards unconventional behavior and presence of depairing mechanisms. Notice also that the BCS value \( \pi/8 = 0.393 \) in Eqs. (47), (63), and (64) is extremely robust, being originally derived for dirty 3D superconductors, and the \( \tau_0 T_c \) product remains the same for clean 2D superconductors and is not affected by the multilaminarity. The general formula for the fluctuation conductivity of a layered superconductor in perpendicular magnetic field can be also rewritten via the layering operator and relative life-time employing the 2D results by Redi, \(^{34}\) and Abrahams, Prange and Stefen \(^{38}\) (APS), cf. also Ref. 16,

\[
\sigma_{ab}(\epsilon, h) = \tilde{\tau}_{\text{rel}} \frac{e^2}{16 h} \frac{N}{s} \hat{f}_{\text{APS}}(\epsilon, h),
\]

where, for \( \epsilon + h > 0 \),

\[
f_{\text{APS}}(\epsilon, h) \equiv \frac{1}{\epsilon} \left( \frac{\epsilon}{h} \right)^2 \left[ \psi\left( 1 + \frac{\epsilon}{2h} \right) - \psi\left( 1 + \frac{\epsilon}{2h} \right) + \frac{h}{\epsilon} \right],
\]

is an universal dimensionless function of dimensionless reduced temperature \( \epsilon \) and dimensionless magnetic field \( h \). The functions

\[
\Gamma(z) \equiv \int_0^\infty e^{-t z} t^{z-1} dt, \quad \psi(z) \equiv \frac{d}{dz} \ln \Gamma(z), \quad \psi^{(1)}(z) \equiv \frac{d}{dz} \psi(z) = \zeta(2, z)
\]

are respectively the Euler gamma, digamma, and trigamma functions. This general formula is often utilized to process the experimental data for the paraconductivity. We provide also several useful asymptotics of \( f_{\text{APS}}(\epsilon, h) \) in different physical conditions,

\[
f_{\text{APS}}(\epsilon, h) \approx \begin{cases} 
\frac{2}{h} \left[ 1 - \frac{\epsilon}{2h} \ln 2 \right], & \text{if } h \gg |\epsilon| \\
\frac{4}{\epsilon + h} = 4 \frac{T_c}{T - T_{c2}(H)}, & \epsilon + h \ll h \\
\left[ 1 - \frac{1}{2} \left( \frac{h}{\epsilon} \right)^2 \right] \frac{1}{\epsilon} = \left[ 1 - \frac{h^2}{4} \frac{\partial^2}{\partial \epsilon^2} \right] \frac{1}{\epsilon}, & \text{if } h \ll \epsilon
\end{cases}
\]

For the LD model, for example, the \((h \ll \epsilon)\)-asymptotics gives, \(^{16,35,17}\) according to Eq. (52),

\[
\sigma_{ab}(\epsilon, h) \approx \tilde{\tau}_{\text{rel}} \frac{e^2}{10 h} \frac{N}{s} \left[ \frac{1}{\sqrt{\sqrt{\epsilon} (\epsilon + r)}} - \frac{h^2 (\epsilon + r) + \sqrt{\frac{3}{8} \epsilon^2}}{2} \frac{1}{(\epsilon + r)^{5/2}} \right].
\]

Note that the classical Aslamazov-Larkin result, Eq. (47), is recovered for \( r = 0, h = 0 \), and \( \tilde{\tau}_{\text{rel}} = 1 \). In the practical application to layered cuprates, however, we need to take into account the nonlocality effects. In \( \varepsilon_{\text{pc}}\)-approximation to the GL theory we have to subtract the part of the corresponding cutoff area in the 2D momentum space. Thereby, the fluctuation conductivity is given by the difference

\[
\sigma_{ab}(\epsilon, \epsilon + c, h) = \tilde{\chi} \sigma_{ab}(\epsilon, h) \equiv \sigma_{ab}(\epsilon, h) - \sigma_{ab}(\epsilon + c, h) \approx \sigma_{ab}(\epsilon, h) - \sigma_{ab}(-\epsilon, h),
\]
where a cutoff operator $\hat{C}$ is introduced, and the approximation is valid for $\epsilon \ll c$. Similarly, for the magnetization we have the same “cutoff” expression which appears when calculating the truncated sums over the Landau levels, $\sum_{0}^{n_{c}-1} = \sum_{0}^{\infty} - \sum_{n_{c}}^{\infty}$, or integral with respect to the dimensionless in-plane kinetic energy,

$$\int_{c}^{0} d\tilde{x} = \int_{0}^{\infty} d\tilde{x} - \int_{c}^{\infty} d\tilde{x}. \quad (71)$$

As a rule the GL theory allows for ultraviolet (UV) regularization—every expression can be easily regularized in the local ($c \rightarrow \infty$)-approximation. Therefore the energy cutoff parameter $c$ is not viewed as a tool for UV regularization, it is simply an important and immanent parameter of the GL theory, being of the order $c \approx 1$. The cutoff procedure has been essentially introduced from the beginning in the GL theory.40 Unfortunately, for many superconductors systematic studies for determination of the energy cutoff parameter are still missing. Here we suggest only the simplest possible interpolation formula within the LD model for $\epsilon \ll c$,

$$\sigma_{ab}(\epsilon, h) = \frac{\pi}{8} \frac{\tilde{\tau}_{\text{rel}}}{R_{\text{QHE}}} \frac{N}{s} \tilde{C} f_{\text{LD}}(\epsilon, r) \approx \frac{\tilde{\tau}_{\text{rel}}}{16h} \frac{N}{s} \left[ \frac{1}{\sqrt{\epsilon + r}} - \frac{1}{\sqrt{c(\epsilon + r)}} \right], \quad (72)$$

which takes into account only the first nonlocal correction. This simple expression fits very well39 the experimental data for YBa$_2$Cu$_3$O$_{7-\delta}$ and Bi$_2$Sr$_2$CaCu$_2$O$_{8}$.

As a last example of the action of the layering operator $\hat{L}$ we consider the 2D frequency-dependent paraconductivity at zero magnetic field. Taking the general expression for $D$-dimensional GL model41 and performing carefully the limit $D \rightarrow 2$ (note, that there is an omitted term in the expression for the 2D conductivity in Ref. 41) we get for the in-plane complex conductivity

$$\sigma_{ab}^{*}(\omega r(\epsilon)) = \sigma_{ab}^{\prime}(\frac{\omega_{T_{0}}}{\epsilon}) + i\sigma_{ab}^{\prime\prime}(\frac{\omega_{T_{0}}}{\epsilon}) = \frac{\tilde{\tau}_{\text{rel}}}{16h} \frac{N}{s} \hat{L} \left[ \frac{1}{\epsilon} \varsigma_{1}\left(\frac{\omega_{T_{0}}}{\epsilon}\right) + i \frac{\epsilon}{\epsilon} \varsigma_{2}\left(\frac{\omega_{T_{0}}}{\epsilon}\right) \right], \quad (73)$$

or in expanded notation for single layered superconductor,

$$\sigma_{ab}^{*}(\omega) = \frac{2\tau_{0}k_{B}T_{c}}{s_{\text{eff}} R_{\text{QHE}}} \int_{0}^{\pi/2} \frac{d\phi}{\pi/2} \frac{\varsigma_{1}\left(\frac{\omega_{T_{0}}}{r + r \sin^{2} \phi}\right)}{\epsilon + r \sin^{2} \phi} + i \frac{\epsilon}{\epsilon} \varsigma_{2}\left(\frac{\omega_{T_{0}}}{r + r \sin^{2} \phi}\right), \quad (74)$$

where we have for the dimensionless real and imaginary conductivity, $\varsigma_{1}(0) = 1$,

$$\varsigma_{1}(z) \equiv \frac{2}{z^{2}} \left[ z \arctan(z) - \frac{1}{2} \ln(1 + z^{2}) \right] = \frac{2}{\pi} \mathcal{P} \int_{0}^{\infty} \frac{y \varsigma_{2}(y)}{y^{2} - z^{2}} dy, \quad (75)$$

$$\varsigma_{2}(z) \equiv \frac{2}{z^{2}} \left[ \arctan(z) - z + \frac{1}{2} \ln(1 + z^{2}) \right] = \frac{2}{\pi} \mathcal{P} \int_{0}^{\infty} \frac{\varsigma_{1}(y)}{y^{2} - z^{2}} dy. \quad (76)$$

As usual in the above Kramers-Kronig integrations $\mathcal{P}$ indicates that the principal value of the integral is taken. For computer implementation of the $\hat{L}$ operator we
have to verify that, for \( r \gg 1 \), \( \hat{L} \) is simply equivalent to an incremental operator for the spatial dimensionality,

\[
\sigma^{(D+1)}(\epsilon) \approx \hat{L} \sigma^{(D)}(\epsilon).
\] (77)

In many cases the GL results for integer dimensionality are well-known and we can derive a generalization for a layered system. For both the MT and LD models the integration in Eq. (73) can be easily programmed, so we have a useful formula for fitting of the ultra high frequency measurements of \( \sigma_{ab}^{*}(\omega) \). The original explicit expressions derived from retarded electromagnetic operator by Aslamazov and Varlamov\(^{42}\) are too cumbersome to be used by experimentalists. Hence, one may realize that the GL theory is not some phenomenological alternative to the microscopic BCS theory (this scorn, dating back to the beginning of fifties, is still living even nowadays among students). The GL theory is a tool for applying the theory of superconductivity for the important for applications, let us say “hydrodynamic”, case of low frequencies and small wave-vectors. For \( \epsilon \ll r \) the frequency dependent conductivity \( \sigma_{ab}^{*}(\omega) \), having dimension \((\Omega \text{ cm})^{-1}\), from Eq. (74) displays 3D behavior, while in the opposite case of \( \epsilon \gg r \) it shows 2D character. For thin films of layered superconductors with thickness \( d_{\text{film}} \) we have to calculate the 2D conductivity \( \sigma^{(2D)} = d_{\text{film}} \sigma \), while for single layered films of conventional superconductors, for example, we have to substitute in Eq. (74) formally \( s_{\text{eff}} = d_{\text{film}} \), and certainly \( r = 0 \).

Having analyzed in detail the action of the \( \hat{L} \) operator, we developed practically all technical tools necessary to proceed our investigation of the thermodynamics of Gaussian fluctuations and fluctuation magnetization.

2.4. Power series for the magnetic moment within the LD model

We will calculate in this subsection the nonlinear susceptibility by substituting first into the free energy, Eq. (28), the heat capacity, expressed via the susceptibility from Eq. (32). Then, the formula for the magnetization, Eq. (29), gives

\[
\chi(\epsilon, h) = 6 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{\pi^{2n}} \left( 1 - \frac{1}{2^{2n-1}} \right) \zeta(2n) h^{2(n-1)} \frac{\partial^{2(n-1)}}{\partial \epsilon^{2(n-1)}} \chi(\epsilon) \chi(\epsilon).
\] (78)

Taking the LD expression for the susceptibility at zero field Eq. (42), calculating the derivatives with respect to \( \epsilon \) by means of the relation

\[
\frac{\partial^m}{\partial \epsilon^m} \frac{1}{\sqrt{\epsilon}} = \frac{(2m-1)!!}{2^m m!} \frac{1}{\sqrt{\epsilon}},
\] (79)

and defining the relative susceptibility as

\[
\tilde{\chi}_{\text{rel}}(\epsilon, h) \equiv \frac{\chi(\epsilon, h)}{\chi(\epsilon)}
\] (80)
we obtain

\[
\tilde{\chi}_{\text{rel}}(\epsilon, h; r) = 12 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(1 - \frac{1}{2^{2n+1}}\right) \frac{(2n + 2)! \zeta(2n + 2)}{2^{2n+1} \pi^{2n+2}} \times \left(\frac{h^2}{\epsilon^2}\right) \frac{2n}{n!} \sum_{m=0}^{2n} \frac{(2m - 1)!! (4n - 2m - 1)!!}{m!(2n-m)!(1 + r/\epsilon)^{2n-m}}
\]

\[
= 1 - \frac{7}{15} \frac{\epsilon^2 + \epsilon r + 3r^2/8}{(\epsilon + r)^2} \left(\frac{h}{\epsilon}\right)^2 + \cdots.
\]  

Although these series is found to be a solution to the problem of calculating the fluctuational magnetization,

\[
M(\epsilon, h) = -\frac{k_B T_c}{\Phi_0 s_{\text{eff}}} \left\{ \tilde{\chi}_{\text{rel}}(\epsilon, h; r) - \tilde{\chi}_{\text{rel}}(c + \epsilon, h; r) \right\}.
\]  

for the physical conditions of interest, i.e. an observable effect of magnetic field on the susceptibility, one needs to extend the series summation onto arguments \(h^2/\epsilon^2\) beyond the radius of convergence. Analogous series has been already reported for the 3D paraconductivity.\(^{19}\) One of the best devices for extending the convergence of series and also for calculating slowly convergent series is the \(\epsilon\)-algorithm\(^{44,45}\) based on \(\text{Padé}\) approximants.\(^{43}\) In the next section we describe a simplified version of this algorithm suitable for computer implementation.

2.5. The epsilon algorithm

The epsilon algorithm is a method for finding the limit \(L\) of infinite series

\[
L = \lim_{N \to \infty} S_N = \lim_{N \to \infty} \sum_{i=0}^{N} a_i,
\]  

in case where only the first \(N + 1\) terms \(a_i, i = 0, \ldots, N\), are known. The algorithm operates by employing two rows. The first one, called here auxiliary \(A\)-row, is initially set to zero, i.e.

\[
A_0^{[0]} = 0, \quad A_1^{[0]} = 0, \quad A_2^{[0]} = 0, \ldots \quad A_N^{[0]} = 0.
\]  

The second one is sequential \(S\)-row loaded in zero-order approximation with the partial sums of the series

\[
S_0^{[0]} = a_0, \quad S_1^{[0]} = a_0 + a_1, \ldots \quad S_N^{[0]} = a_0 + a_1 + a_2 + \cdots + a_N.
\]  

The above assignments, as indicated by Eqs. (84) and (85), constitute the initialization phase of the \(\epsilon\)-algorithm. The essence of the latter consists of filling in the
so called ε-table

\[
\begin{pmatrix}
  A_0^{[0]} & A_1^{[0]} & A_2^{[0]} & A_3^{[0]} & \ldots \\
  S_0^{[0]} & S_1^{[0]} & S_2^{[0]} & S_3^{[0]} & \ldots \\
  A_0^{[1]} & A_1^{[1]} & A_2^{[1]} & A_3^{[1]} & \ldots \\
  S_0^{[1]} & S_1^{[1]} & S_2^{[1]} & S_3^{[1]} & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
= \begin{pmatrix}
  \varepsilon_0^{[0]} & \varepsilon_1^{[0]} & \varepsilon_2^{[0]} & \ldots \\
  \varepsilon_0^{[1]} & \varepsilon_1^{[1]} & \varepsilon_2^{[1]} & \ldots \\
  \varepsilon_0^{[2]} & \varepsilon_1^{[2]} & \varepsilon_2^{[2]} & \ldots \\
  \varepsilon_0^{[3]} & \varepsilon_1^{[3]} & \varepsilon_2^{[3]} & \ldots \\
  \vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad (86)
\]

where according to the standard notations \([j/k] = P_j(z)/P_k(z)]_{z=1}\) designates a Padé approximant having power \(j\) in the nominator and, respectively, \(k\) in the denominator.\(^{43}\)

Starting from the \(A^{[0]}\)- and \(S^{[0]}\) rows every subsequent row is derived by applying the cross rule (known also as the missing identity of Frobenius). To be specific, for calculation of the \(k\)th \(A\)-row we have to solve the cross rule equation

\[
\begin{pmatrix}
  \ldots & A_{k+1}^{[k-1]} \\
  S_k^{[k-1]} & S_{k+1}^{[k-1]} \\
  A_k^{[k]} & \ldots
\end{pmatrix}
= \begin{pmatrix}
  \ldots & \text{North} \\
  \text{West} & \text{East} \\
  \text{South} & \ldots
\end{pmatrix},
\]

\[\text{(South} - \text{North}) (\text{East} - \text{West}) = 1. \quad (87)\]

Likewise, for calculating the \(k\)-th \(S\) row we have to apply the same cross rule

\[
\begin{pmatrix}
  \ldots & S_{k+1}^{[k-1]} \\
  A_k^{[k]} & A_{k+1}^{[k]} \\
  S_k^{[k]} & \ldots
\end{pmatrix}
= \begin{pmatrix}
  \ldots & \text{North} \\
  \text{West} & \text{East} \\
  \text{South} & \ldots
\end{pmatrix},
\]

\[\text{South} = \text{North} + (\text{East} - \text{West})^{-1}. \quad (88)\]

Having applied the algorithm we get in the \(S\)-rows of the ε-table, Eq. (86), a set of different Padé approximants to the limit \(L\). The \(i\)th term of the \(k\)th \(A\)-row can be easily obtained by

\[
A_i^{[k]} = A_{i+1}^{[k-1]} + (S_{i+1}^{[k-1]} - S_i^{[k-1]})^{-1}, \quad \text{for } i = 0, 1, \ldots, N - 2k + 1, \quad (89)
\]

but for practical implementation of the algorithm, we can omit the index of the approximation and to use only one auxiliary row, updating it each time,

\[
A_i := A_{i+1} + (S_{i+1} - S_i)^{-1}, \quad \text{for } i = 0, 1, \ldots, N - 2k + 1. \quad (90)
\]
For the \(k\)-th \(S\)-row, the \(i\)-th term reads as
\[
S_{i}^{[k]} = S_{i+1}^{[k-1]} + \left(A_{i+1}^{[k]} - A_{i}^{[k]}\right)^{-1}, \quad \text{for } i = 0, 1, \ldots, N - 2k, \tag{91}
\]
and can be updated in the same manner as described for the \(A\)-row,
\[
S_{i} := S_{i+1} + (A_{i+1} - A_{i})^{-1} \quad \text{for } i = 0, 1, \ldots, N - 2k. \tag{92}
\]

In order to find an estimate for the limit \(L\) of the infinite series, two different empirical criteria can be implemented. In the first one, the \(\varepsilon\)-table is scanned for a minimal difference \(|S_{i+1}^{[k-1]} - S_{i}^{[k-1]}|\). The limit \(L\) is then given by
\[
\min_{i,k} |S_{i+1}^{[k-1]} - S_{i}^{[k-1]}| \implies L \approx S_{i}^{[k-1]} . \tag{93}
\]
This minimal difference gives also an estimate for the empirical error of the method. In the second criterion the \(\varepsilon\)-table is scanned for the maximum of the East \(A\)-row element, cf Eqs. (88) and (91),
\[
\max_{i,k} |A_{i+1}^{[k]}| \implies L \approx S_{i}^{[k]} . \tag{94}
\]
The reciprocal of the maximum auxiliary value gives in this case the estimate for the empirical error of the method. It is the second criterion that we have used in the FORTRAN90 implementation of the \(\varepsilon\)-algorithm given in Appendix A. Therein we have also made use of pseudo-inverse numbers in order to ensure provisions against division by zero in Eqs. (90) and (92),
\[
z^{-1} := \begin{cases} 
0, & \text{for } z = 0 \\
1/z, & \text{for } z \neq 0.
\end{cases} \tag{95}
\]

For an illustration, consider the first approximation. In the beginning we have for the first \(A\)-row according to Eq. (89)
\[
A_{0}^{[1]} = [(a_{0} + a_{1}) - (a_{0})]^{-1} = \frac{1}{a_{1}}, \quad A_{1}^{[1]} = \frac{1}{a_{2}}, \ldots, \quad A_{N-1}^{[1]} = \frac{1}{a_{N}} . \tag{96}
\]
The first \(S\)-row then reads
\[
S_{0}^{[1]} = a_{0} + a_{1} + \frac{1}{1/a_{2} - 1/a_{1}}, \quad S_{1}^{[1]} = a_{0} + a_{1} + a_{2} + \frac{1}{1/a_{3} - 1/a_{2}}, \ldots, \tag{97}
\]
and for the last element of the \(S^{[1]}\)-row we have
\[
S_{N-2}^{[1]} = a_{0} + a_{1} + a_{2} + \cdots + a_{N-2} + a_{N-1} + (1/a_{N} - 1/a_{N-1}) . \tag{98}
\]
The above approximation \(S_{N-2}^{[1]}\) to the limit \(L\) is nothing but the well-known Aitken’s \(\Delta^{2}\)-method, which gives an exact result for the geometric progression
\[
S_{0}^{[0]} = 1 + q + q^{2} + \cdots + q^{N}, \quad S_{0}^{[1]} = S_{1}^{[1]} = S_{2}^{[1]} = \cdots = S_{N-2}^{[1]} = \frac{1}{1 - q} . \tag{99}
\]
for an arbitrary \( q \neq 1 \). This fact can rationalize the success of the \( \varepsilon \)-algorithm when applied to weak magnetic field series expansion of susceptibility. In the Euler-MacLaurin summation, Eqs. (22) and (26), we have a hidden geometric progression of translation operators.

As a rule divergent series do not exist in physics; 99\% of the divergent series born by real physical problems can be summed up by some combination of the Euler-MacLaurin method and the \( \varepsilon \)-algorithm and the reason is lies in the analytical dependence of the coefficients on the index. In the Gaussian spectroscopy of superconductors, for example, it is necessary series related to asymptotic expansion of Euler polygamma and Hurwitz zeta functions to be summed up, but the same methods could be applied to many other physical problems. The solution often can be derived by less efforts than required to verify that a series is divergent accordingly some strict mathematical criterion. Nowadays the mathematical education in the physical departments is conquered by scholastic mathematicians. Alas, none of the students of physics knows what really happens when we press the \( \sin \) key of a calculator. On the other hand this is a commercial secret of the manufacturer. The physicists do not even lightly touch the brilliant achievements of mathematics indispensable not only for the theoretical physics but for experimentalist to fit their data as well. This is the motivation why we, following the spirit of the century of enlightenment, present in Appendix A a simple FORTRAN90 program illustrating the operation of the \( \varepsilon \)-algorithm. Certainly, \textit{physics is phun},\textsuperscript{46} being in part \textit{cosa mentale},\textsuperscript{47} and every new software cannot be foolproof, but there are methods which must be taken into account in every complicated calculation.

\subsection*{2.6. Power series for differential susceptibility}

Having calculated the relative dimensionless susceptibility by employing the \( \varepsilon \)-algorithm we can recover the usual susceptibility from the dimensionless one,

\[ \chi(\varepsilon, h) = \tilde{\chi}_{\text{rel}}(\varepsilon, h) \chi(\varepsilon). \]  

In order to take into account the effects of nonlocality the cut-off area in the momentum space should subtracted out from the susceptibility

\[ \chi_{\text{\{\( g \}\}}} (\varepsilon, h) = \tilde{\chi}(\varepsilon, h) = \tilde{\chi}_{\text{rel}}(\varepsilon, h) \chi(\varepsilon) - \tilde{\chi}_{\text{rel}}(\varepsilon + \varepsilon, h) \chi(\varepsilon + \varepsilon). \]

Then we can easily find the magnetization

\[ M(H, T) = \chi_{\text{\{\( g \}\}}} (\varepsilon, h)H. \]

The calculation of the differential susceptibility

\[ \chi^{(\text{dif})}(\varepsilon, h) = \left( \frac{\partial M}{\partial H} \right)_T, \]

where \( H = H_{c2}(0)h \), gives an alternative method to determine the magnetization.

Next we define a dimensionless relative differential susceptibility

\[ \tilde{\kappa}(\varepsilon, h, r) \equiv \chi^{(\text{dif})}(\varepsilon, h)/\chi(\varepsilon). \]
For this variable, using Eq. (81), we have the series

\[
\tilde{\kappa} = 12 \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^{2n+1}}\right) \frac{(2n+2)! \, \zeta(2n+2)}{2^{2n+1} \, \pi^{2n+2}} \left(-\frac{h^2}{\epsilon^2}\right)^n \\
\times \sum_{m=0}^{2n} \frac{(2m-1)!! \, (4n-2m-1)!!}{m! \, (2n-m)! \, (1+r/\epsilon)^{2n-m}} \\
= 1 - \frac{7}{5} \left(\frac{\epsilon^2 + r \epsilon + 3r^2/8}{(\epsilon + r)^2}\right) \left(\frac{h}{\epsilon}\right)^2 + \ldots, \tag{105}
\]

which, just as done in deriving Eq. (81), can be summed up by means of the \( \varepsilon \)-algorithm. For instance, in the local GL limit we have for the magnetization

\[
M = \int_0^H \chi(T, H) dH = \chi(\epsilon) H_{c2}(0) \int_0^h \tilde{\kappa}(\epsilon, h') dh' \\
= \chi(\epsilon) H_{c2}(0) \tilde{\chi}_{\text{rel}}(\epsilon, h). \tag{106}
\]

For the further analysis, however, it is more suitable to introduce a dimensionless magnetization

\[
\tilde{m} \equiv -\frac{M}{M_0} = -\frac{\Phi_0}{k_B T_c N} s M, \quad M_0 \equiv \frac{k_B T_c N}{\Phi_0} s \tag{107}
\]

Then, using the relation

\[
\tilde{\chi}_{\text{rel}}(\epsilon, h) = \frac{1}{h} \int_0^h \tilde{\kappa}(\epsilon, h') dh' \tag{108}
\]

the result for the dimensionless fluctuation magnetization takes the form

\[
\tilde{m}(\epsilon, h) = \frac{1}{6} \frac{1}{\sqrt{\epsilon(\epsilon + r)}} \tilde{\chi}_{\text{rel}}(\epsilon, h) h = \frac{1}{6} \frac{1}{\sqrt{\epsilon(\epsilon + r)}} \int_0^h \tilde{\kappa}(\epsilon, h') dh'. \tag{109}
\]

In this section we have calculated the magnetization by means of power series in the magnetic field assuming in the beginning \( h/\epsilon \ll 1 \). In the next section we develop another method for calculating the fluctuation magnetic moment which is appropriate for strong magnetic fields and allows for studying the high magnetic field asymptotics for large enough values of the reduced magnetic field, \( h/\epsilon \gg 1 \). The overlap between these expansions about \( h/\epsilon \simeq 1 \) would be a test for the accuracy of the calculations.

3. Strong magnetic fields

3.1. General formula for the free energy

In order to derive a general formula for the Gibbs free energy for arbitrary non-vanishing magnetic field we will start again by representing the free energy density
as a sum over the energy spectrum, Eq. (21),

\[ F(\epsilon, h) = F_0 \ 2h \ \hat{L} \sum_{n=0}^{n_c-1} \left[ \ln \left( n + \frac{1}{2} + \frac{\epsilon}{2h} \right) + \ln(2h) \right], \quad (110) \]

where

\[ F_0 \equiv \frac{k_B T_c}{4\pi \xi_{ab}(0)} \frac{N}{s} = \frac{1}{2} M_0 B_{c2}(0). \quad (111) \]

The first way to go in deriving convenient for programming formula is to calculate the action of the \( \hat{L} \) operator on the integrand, cf. Ref. 48. In this case we write down the free energy as a finite sum over the Landau levels

\[ F(\epsilon, h) = \frac{k_B T_c}{4\pi \xi_{ab}(0)} \frac{N}{s} 2h \sum_{n=0}^{n_c-1} \hat{L} \ln[\epsilon + h(2n + 1)], \quad (112) \]

where, according to Eq. (50),

\[ \hat{L}^{(LD)} \ln[\epsilon + h(2n + 1)] = 2 \ln \frac{\sqrt{\epsilon + h(2n + 1) + \sqrt{\epsilon + h(2n + 1) + r}}}{2}. \quad (113) \]

This formula is useful especially in the case of strong magnetic fields when the finite series are not too long. However, in order to have a good working expression, applicable to all cases, it is much better to solve the problem analytically. Towards this end consider the last term in the integrand of Eq. (110). The summation of this constant term and simply yields the cutoff parameter \( c \)

\[ 2h \sum_{n=0}^{n_c-1} 1 = (2h)n_c = c. \quad (114) \]

Next we introduce a dimensionless function

\[ x(\epsilon, h) \equiv \frac{1}{2} + \frac{\epsilon}{2h} = \frac{\epsilon + h}{2h} = \frac{1}{2H} (T - T_{c2}(H)) \left( -\frac{\partial H_{c2}(T)}{\partial T} \right) \bigg|_{T = T_{c2} = 0}, \quad (115) \]

which is the argument of some of the analytical functions we use in the following. Further, we have to present the sum in Eq. (110) as a difference of two appropriately regularized infinite series

\[ \sum_{n=0}^{n_c-1} \ln(n + x) = \text{Reg}\zeta \sum_{n=0}^{\infty} \ln(n + x) - \text{Reg}\zeta \sum_{n_c}^{\infty} \ln(n + x). \quad (116) \]

In fact, one does not have any other possibility except the \( \zeta \)-regularization

\[ -\text{Reg}\zeta \sum_{n=0}^{\infty} \ln(n + z) = \left. \frac{\partial}{\partial \nu} \zeta(\nu, z) \right|_{\nu = 0} = \ln \frac{\Gamma(z)}{\sqrt{2\pi}} \quad (117) \]
based on one relation between the Euler Γ-function and the Hurwitz ζ-function, and the definition of the logarithmic function

$$\ln z = \lim_{\nu \to 0} \frac{z^\nu - 1}{\nu} = \frac{\partial}{\partial \nu} z^\nu \bigg|_{\nu=0}. \quad (118)$$

According to the famous results by Riemann, the analytical continuations of the ζ-function and the factorial $n!$ are unique. Therefore the UV regularization of the partition function in the GL model in an external magnetic field is practically included in the second, Gauss definition of the Γ-function as an infinite product, see e.g. Ref. 49,

$$\int_0^\infty t^{z-1} e^{-t} dt \equiv \Gamma(z) \equiv \lim_{n_c \to \infty} \frac{n_c!}{n_c^z (z+1)(z+2)\cdots(z+n_c-1)} . \quad (119)$$

Let us recall some particular values,

$$\Gamma(n+1) = n!, \quad \Gamma(1) = 0! = 1, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad (120)$$

and the Stirling’s approximation for $n_c \gg 1$, derived by Gaussian saddle point approximation applied to the first, Euler definition of the Γ-function, Eq. (119),

$$n_c! \approx \left(\frac{n_c}{e}\right)^{n_c} \sqrt{2\pi n_c}, \quad \ln (n_c!) \approx \left(n_c + \frac{1}{2}\right) \ln n_c - n_c + \ln \sqrt{2\pi}. \quad (121)$$

For the local limit or for the case of weak magnetic fields we shall also make use of the asymptotic formulae for $z \gg 1$

$$\ln \Gamma(z) \approx \left(z - \frac{1}{2}\right) \ln z - z + \frac{1}{2} \ln(2\pi) + \frac{1}{12z}, \quad (122)$$

$$\psi^{-1}(z) \approx \left(z - \frac{1}{2}\right) \ln z - z + \frac{1}{12z^2}, \quad (123)$$

$$\psi(z) \approx \ln z - \frac{1}{2z} - \frac{1}{12z^2}, \quad \psi^{(1)}(z) \equiv \zeta(2, z) \approx \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3}. \quad (124)$$

Substituting the Stirling asymptotics in the second Gauss definition, Eq. (119), and taking a logarithm we arrive at the function $\psi^{-1}(z)$, generating the polygamma functions

$$\psi^{-1}(z) \equiv \lim_{n_c \to \infty} \left\{ -\sum_{n=0}^{n_c-1} \ln(n+z) + \left(n_c - \frac{1}{2} + z\right) \ln (n_c) - n_c \right\}$$

$$= \ln \frac{\Gamma(z)}{\sqrt{2\pi}}. \quad (125)$$

As a result, the above Gauss definition for $\ln \Gamma(z)$ solves the problem for UV regularization of the infinite sum of logarithms, Eq. (117). The first derivative of this
equation gives the well-known definition of the digamma function \(\psi(z) \equiv \psi^{(0)}(z)\),

\[
\psi^{(0)}(z) \equiv \frac{d}{dz} \psi^{(-1)}(z) = -\text{Reg}_{\zeta} \sum_{n=0}^{\infty} \frac{1}{n + z} = \lim_{n \to \infty} \left\{ -\sum_{n=0}^{n_c-1} \frac{1}{n + z} + \ln(n_c) \right\}.
\]  

(126)

In particular,

\[
-\psi(1) = C_{\text{Euler}} = \lim_{n_c \to \infty} \left\{ \sum_{n=1}^{n_c-1} \frac{1}{n} - \ln(n_c) \right\} = 0.577216\ldots.
\]  

(127)

All other polygamma functions are actually Hurwitz \(\zeta\)-functions with integer first argument \(\geq 2\) and the sums are trivially convergent,

\[
\psi^{(N)}(z) = \frac{d^N}{dz^N} \psi(z) = (-1)^N N! \zeta(N + 1, z).
\]  

(128)

To summarize, we have applied the well-known \(\zeta\)-technique\(^{50}\) for UV regularization of the partition function and revealed that the archetype of this powerful method comes from the century of enlightenment and finally we can bring the free energy, Eq. (110), to the form

\[
F(\epsilon, h) = \frac{T}{T_c} F_0 \left\{ 2h \hat{L} \left[ -\ln \Gamma\left(\frac{1}{2} + \frac{\epsilon}{2h}\right) + \ln \Gamma\left(\frac{1}{2} + \frac{\epsilon + c}{2h}\right) \right] + e \ln(2h) \right\}
\]

\[
= k_B T \frac{N}{4\pi \xi_{ab}(0)} \hat{L} \hat{C} \left[ -(2h) \ln \left( \frac{\Gamma\left(\frac{\epsilon + h}{2}\right)}{\sqrt{2\pi}} \right) - c \ln(2h) \right].
\]  

(129)

For weak magnetic field, \(h \ll \epsilon\), cf. Eqs. (49), (50), (123), and (143),

\[
-(2h) \ln \left( \frac{\Gamma\left(\frac{\epsilon + h}{2}\right)}{\sqrt{2\pi}} \right) + c \ln(2h) \approx -\epsilon [\ln(\epsilon) - 1] + \frac{1}{6} \frac{h^2}{\epsilon}.
\]  

(130)

This is our main analytical result and all thermodynamic properties now can be obtained via derivatives. However, having this analytical result it is trivially to check that it can be derived by finite sums. The latter do not require UV regularization and the Euler \(\Gamma\)-function is commonly available in many textbooks on mathematical analysis.

### 3.2. Fluctuation part of thermodynamic variables

Having an analytical result for the free energy we can easily find other thermodynamic variables by differentiating. The magnetization, for example, is given by the derivative

\[
M = -\left( \frac{\partial F}{\partial B} \right)_T = -\frac{1}{B_{c2}(0)} \left( \frac{\partial F}{\partial h} \right)_\epsilon = -M_0 \hat{m},
\]  

(131)
where a dimensionless diamagnetic moment is introduced

\[ \tilde{m}(\epsilon, h) \equiv -\frac{M}{M_0} = \frac{c}{2h} \tilde{L} \left[ \ln \Gamma \left( \frac{\epsilon + h}{2h} \right) - \ln \Gamma \left( \frac{\epsilon + c + h}{2h} \right) \right] \]

\[ + \tilde{L} \left[ \frac{\epsilon}{2h} \psi \left( \frac{\epsilon + h}{2h} \right) - \frac{\epsilon + c}{2h} \psi \left( \frac{\epsilon + h + c}{2h} \right) \right]. \] (132)

In expanded notations within the LD model this formula, according to Eqs. (35), (37), (39), (43), and (46), reads as

\[ M^{(LD)}(\epsilon, h) = -\frac{k_B T_c}{\Phi_0 s_{\text{eff}}} \left( \frac{c}{2h} + \frac{2}{\pi} \int_0^{\pi} d\phi \right) \left\{ -\ln \Gamma \left( \frac{\epsilon + r \sin^2 \phi + h}{2h} \right) \right. \]

\[ - \ln \Gamma \left( \frac{\epsilon + \epsilon + r \sin^2 \phi + h}{2h} \right) + \left[ \frac{\epsilon + r \sin^2 \phi}{2h} \psi \left( \frac{\epsilon + r \sin^2 \phi + h}{2h} \right) \right] \]

\[ - \frac{c + \epsilon + r \sin^2 \phi}{2h} \psi \left( \frac{c + \epsilon + r \sin^2 \phi + h}{2h} \right) \right\}, \] (133)

where \( \phi = \frac{1}{2} \theta \). For \(|\epsilon|, h \ll r, c\) this general expression recovers the local 3D result, Eq. (180), analyzed later in Sec. 3.4, while in the opposite case of extremely high anisotropy \( r < |\epsilon|, h \ll c \) we get the local 2D result, Eq. (142). Here we want to emphasize the existence to mention a universal magnetization law at \( T = T_c \), or \( \epsilon = 0 \), which can be observed for many high-\( T_c \) materials at strong magnetic fields \( h \gg r \)

\[ -M(T_c, B) \frac{\Phi_0 s_{\text{eff}}}{k_B T_c} = \tilde{m} = \frac{1}{2} \ln 2 U_M \left( \frac{2B}{c B_{c2}(0)} \right), \] (134)

where the universal function of the nonlocal magnetization

\[ U_M(y) = \frac{2}{\ln 2} \left\{ \ln \Gamma \left( \frac{1}{y} + \frac{1}{2} \right) - \frac{1}{2} \ln \pi + \frac{1}{y} \left[ 1 - \psi \left( \frac{1}{y} + \frac{1}{2} \right) \right] \right\} \] (135)

is normalized so that \( U_M(0) = 1, U_M(\infty) = 0, y = 2h/c \). For conventional bulk superconductors the nonlocality effects on magnetization are well understood, see for example Refs. 51–56. To the best of our knowledge, the first observation of fluctuation-induced diamagnetism for a cuprate superconductor well inside the finite-magnetic-field regime was reported by Carretta et al.\textsuperscript{57} for YBa\textsubscript{2}Cu\textsubscript{3}O\textsubscript{6+x}.

Soon after, analogous measurement was reported for La\textsubscript{1.9}Sr\textsubscript{0.1}CuO\textsubscript{4} by Carballeira et al.\textsuperscript{58} Being familiar with the preliminary version of the present review (cf. Ref. 59) Carballeira et al. have entirely based their interpretation and theoretical analysis on Eq. (133) and Eq. (142) below. Alas, we find it very disappointing and impolite that the authors of Ref. 58 do not give any credits (e.g. in the author list, acknowledgments, or references section) to the author (the first author of the present review, T. M.) of the theory they have used. We will not discuss in any detail their attitude and would instead refer to the Comment.\textsuperscript{60}
Returning now to the general expression for the magnetization, Eq. (132), we derive another expression for the relative differential susceptibility based on Eq. (109)

\[
\kappa(\epsilon, h) = 6\sqrt{\epsilon(\epsilon + r)} \left( \frac{\partial m}{\partial h} \right) = 6\sqrt{\epsilon(\epsilon + r)} \left[ \frac{c}{2h^2} - \frac{\epsilon^2}{4h^3} \psi^{(1)} \left( \frac{\epsilon + h}{2h} \right) + \frac{(\epsilon + c)^2}{4h^3} \psi^{(1)} \left( \frac{\epsilon + h + c}{2h} \right) \right].
\]

(136)

The comparison of this result with Eq. (105) is one of the best methods to check the accuracy of the programmed formulae. Analogously, differentiating the free energy with respect to the temperature \( T = (1 + \epsilon)T_c \) we derive the general formula for the most singular part of the entropy (neglecting the derivative of the \( T \)-prefactor in Eq. (129)),

\[
S \equiv -\frac{1}{T_c} \frac{\partial F}{\partial \epsilon} = \frac{k_b}{4\pi \xi_{ab}(0)} \frac{N}{s} \hat{L} \left[ \psi \left( \frac{\epsilon + h}{2h} \right) - \psi \left( \frac{\epsilon + h + c}{2h} \right) \right],
\]

(137)

and the most singular part of the heat capacity

\[
C(\epsilon, h) = \frac{\partial S}{\partial \epsilon} = -\frac{1}{T_c} \frac{\partial^2 F}{\partial \epsilon^2}
= \frac{k_b}{4\pi \xi_{ab}(0)} \frac{N}{s} \frac{1}{2h} \hat{L} \left[ \psi^{(1)} \left( \frac{\epsilon + h}{2h} \right) - \psi^{(1)} \left( \frac{\epsilon + h + c}{2h} \right) \right].
\]

(138)

This expression for \( C \) can be directly derived from the starting formulae (21) and (110). The sums for the heat capacity are convergent, cf. Ref. 40, and do not require any regularization. The simplest way to reproduce the analytical result for the free energy density, Eq. (129), is to integrate two times the result for its second derivative, i.e. that for the heat capacity, cf. Ref. 66. In general finite sums from 0 to \( n_c - 1 \) for logarithms and powers can be found in many textbooks on mathematics and all our results can thus be easily checked even by experimentalists.

The fluctuation part of the entropy \( S \) is proportional to the mean square of the order parameter \( \Psi \), i.e. the volume density of fluctuation Cooper pairs. The thermally averaged density deserves a special attention because it is the main ingredient of the self-consistent treatment of the interaction of order parameter fluctuations. This Hartree type approximation due to Ullah and Dorsey\(^{61}\) will be briefly described in the next subsection.

In the following, for completeness, we will derive the local 2D asymptotics applicable for \(|\epsilon|, h \ll c\). The substitution of the first term from Eq. (123) into the general formula for the free energy, Eq. (129), gives

\[
\tilde{f}_{2D} = \frac{F(\epsilon, h)}{F_0} = -(2h) \psi^{(-1)} \left( \frac{\epsilon + h}{2h} \right) - \epsilon \ln(2h) + A(c)\epsilon + B(c) + O(1/c),
\]

(139)
where for $\epsilon \ll c$, cf. Eq. (12),

$$f_c(\epsilon) \equiv A(c)\epsilon + B(c) \approx \int_0^{c+\epsilon} \ln \tilde{x} \, d\tilde{x} \approx \epsilon \ln c + c(\ln c - 1).$$

(140)

This irrelevant for the fluctuation phenomena linear function of $\epsilon$ gives constant additions to the free energy $F_c = F_0 B(c)$, and entropy $S_c = -F_0 A(c)/T_c$ and can be omitted hereafter. The subtraction of $F_0 f_c$ from the free energy, Eq. (129), can be considered as a cutoff procedure for UV regularization,

$$\tilde{\text{Reg}}_{\text{UV}} F(\epsilon, h) = F(\epsilon, h) - (F_c - T_c S_c \epsilon),$$

(141)

which, when applied, allows the analysis of the local GL approximation to be carried out simply as $(c \to \infty)$-limit. Now a trivial differentiation gives for the dimensionless magnetization, being a positive quantity,

$$\tilde{m}_{2D}(\epsilon, h) = -\frac{M(\epsilon, h)}{M_0} = \frac{\epsilon}{2h} \left[ \psi\left(\frac{\epsilon}{2h} + \frac{1}{2}\right) - 1 \right] - \psi^{(-1)}\left(\frac{\epsilon + h}{2h}\right)$$

$$= \frac{1}{2} \frac{\partial \tilde{f}_{2D}}{\partial h}.$$ (142)

This result is also a local $(c \to \infty)$-asymptotic of Eq. (132), which for $h \ll \epsilon$ yields

$$\tilde{m}_{2D} \approx h/6\epsilon.$$

(143)

In the general case the local approximation gives $\tilde{m} = \tilde{\text{L}} \tilde{m}_{2D}$, or for the LD model

$$\tilde{m}(\epsilon, h; r) = -\frac{M(\epsilon, h)}{M_0} = \int_0^{\pi/2} \frac{d\phi}{\pi/2} \left\{ \frac{\epsilon + r \sin^2 \phi}{2h} \left[ \psi\left(\frac{\epsilon + r \sin^2 \phi}{2h} + \frac{1}{2}\right) - 1 \right] - \ln \Gamma\left(\frac{\epsilon + r \sin^2 \phi}{2h} + \frac{1}{2}\right) + \frac{1}{2} \ln(2\pi) \right\}.$$ (144)

The next differentiation with respect to the magnetic field, using Eqs. (42), (103), gives the relative dimensionless susceptibility

$$\kappa_{2D}(\epsilon, h) = 6\epsilon \left(\frac{\partial \tilde{m}}{\partial h}\right) = 12 \left(\frac{\epsilon}{2h}\right)^2 \left[ 1 - \frac{\epsilon}{2h} \psi^{(1)}\left(\frac{\epsilon}{2h} + \frac{1}{2}\right) \right] = \frac{\chi^{(\text{dif})}(\epsilon, h)}{\chi(\epsilon)}.$$

(145)

which is also a local $c \gg h, |\epsilon|$ asymptotic of Eq. (136). For the LD model after averaging with respect to the Josephson phase, according to Eq. (39), we obtain

$$\kappa(\epsilon, h; r) = 12 \int_0^{\pi/2} \frac{d\phi}{\pi/2} \left(\frac{\epsilon + r \sin^2 \phi}{2h}\right)^2 \left[ 1 - \frac{\epsilon + r \sin^2 \phi}{2h} \varsigma\left(2, \frac{\epsilon + r \sin^2 \phi}{2h}\right) \right].$$

(146)

This final result can be directly compared to low field series expansion Eq. (105). Similar differentiations of the free energy, Eq. (139), with respect to the temperatures gives the most singular part of the entropy

$$\tilde{s}_{2D} = -\frac{\partial}{\partial \epsilon} \tilde{f}_{2D} = \left[ \psi\left(\frac{\epsilon + h}{2h}\right) + \ln(2h) \right] = T_c S(\epsilon, h)/F_0,$$ (147)
and of the heat capacity
\[ \hat{L} \tilde{c}_{2D} = -\hat{L} \frac{\partial^2}{\partial \epsilon^2} \hat{f}_{2D}(\epsilon, h) = \frac{1}{2h} \psi^{(1)} \left( \frac{\epsilon + h}{2h} \right) = T_c C(\epsilon, h)/F_0. \] (148)

Restoring the \( T \) prefactor instead of \( T_c \) in Eq. (12), as was done in Eq. (129), we arrive at a slightly different expression for the fluctuation part of the free energy
\[ F = F_0 (1 + \epsilon) \hat{f}_{2D}(\epsilon, h) \] and the heat capacity
\[ \hat{L} \tilde{c}_{2D} = -(1 + \epsilon) \hat{L} \frac{\partial^2}{\partial \epsilon^2} (1 + \epsilon) \hat{f}_{2D}(\epsilon, h) 
= (1 + \epsilon) \left[ (1 + \epsilon) \hat{L} \tilde{c}_{2D} + 2 \hat{L} \tilde{s}_{2D} \right] = \frac{T_c C(\epsilon, h)}{F_0}, \] (149)

which gives
\[ C(\epsilon, h) = \frac{(1 + \epsilon)k_B N}{4\pi \xi^2_{ab}(0)} \int \left[ \frac{(1 + \epsilon) \psi^{(1)} \left( \epsilon + h \right)}{2h} + 2\psi^{(0)} \left( \epsilon + h \right) \left( \epsilon + h \right) + 2\ln(2h) \right]. \] (150)

For zero magnetic field we have
\[ C(\epsilon, h = 0) = \frac{(1 + \epsilon)k_B N}{4\pi \xi^2_{ab}(0)} \left[ (1 + \epsilon) \hat{L} \frac{1}{\epsilon} - 2\epsilon \hat{L} \ln \frac{1}{\epsilon} \right], \] (151)

which in the LD model takes the form
\[ C(\epsilon, r) = \frac{(1 + \epsilon)k_B N}{4\pi \xi^2_{ab}(0)} \left[ \frac{(1 + \epsilon)}{\sqrt{\epsilon + r}} - 2\epsilon \frac{2\ln \frac{1}{\sqrt{\epsilon + r}}}{\sqrt{\epsilon + r}} \right]. \] (152)

These expression differs from Eqs. (41), and (60). However, the \((1 + \epsilon)^2 \approx 1 + 2\epsilon\) correction and the less singular part of the heat capacity \(2(1 + \epsilon)F_0 \hat{L} \tilde{s}_{2D}/T_c\), which appears due to differentiation of \( T \) in the numerator of Eq. (12) and Eq. (129), are difficult to be identified experimentally.

For the superconducting phase below the critical temperature, \(0 < -\epsilon \ll 1\), one has to take into account more or less space homogeneous order parameter \( \Psi \), which minimizes the nongradient part of the free energy density
\[ F = a(\epsilon)n_\epsilon + \frac{1}{2}n_\epsilon^2, \] (153)

The fluctuations around this minimum
\[ \Psi = \Psi_\epsilon + \Psi' + i\Psi'', \quad n = \Psi^2 = n_\epsilon + 2\Psi_\epsilon \Psi' + (\Psi')^2 + (\Psi'')^2 \] (154)

should be considered as a small perturbation, thus only the quadratic term in the free energy is taken into account,
\[ F(\epsilon < 0) = a(\epsilon) + \frac{1}{2}bn_\epsilon^2 \approx -\frac{1}{2b}a_0^2\epsilon^2 + a_0(-2\epsilon) \left[ 1(\Psi')^2 + 0(\Psi'')^2 \right]. \] (155)
The first term in this equation corresponds to the jump in the heat capacity \( \Delta C = a_0^2/bT_c \) at \( T_c \). The linear term \( \propto \Psi' \) simply cancels. The phase fluctuations \( \propto (\Psi'')^2 \) are coupled to the plasmons and vortexes but they are irrelevant for the thermodynamic fluctuations significantly below \( T_c \). In this way mainly fluctuations related to the modulus of the order parameter are essential for the heat capacity below \( T_c \).

Finally, the comparison of the second term \( \propto \Psi' \) in Eq. (155) with the corresponding expression above \( T_c \)

\[
F(\epsilon > 0) = a(\epsilon) + \frac{1}{2} b n^2 \approx a_0(\epsilon) \left[ (\Psi')^2 + (\Psi'')^2 \right],
\]

provides a prescription to derive the fluctuation part below \( T_c \) from the fluctuation expression for the normal phase above \( T_c \).

Applying this prescription to Eq. (152) results in the following expression

\[
C(\epsilon < 0, r) = \left( 1 + \epsilon \right) \frac{k_B}{4\pi \xi_{ab}(0)} \frac{N^2}{s} \frac{1}{2} \left[ \frac{(1 + \epsilon)}{\sqrt{(-2\epsilon)(r - 2\epsilon)}} - 2\epsilon \frac{2}{\sqrt{(-2\epsilon + \sqrt{r - 2\epsilon})}} \right].
\]

These fluctuation part as well as the the phonon heat capacity should be subtracted from the experimental data in order to extract the jump \( \Delta C \) and related to it penetration depth \( \lambda_{ab}(0) \). Such a procedure, in fact, gives a purely thermodynamic method to determine the latter quantity.

The dimensionless functions Eqs. (139), (142), (145), and (148) derived with the local approximation are just as important for the thermodynamics of the layered superconductors as is the APS function for the paraconductivity, Eq. (66). The operator \( \hat{L} \) gives the possibility to extend the 2D analytical result for layered or even isotropic 3D superconductor. Additionally the \( \hat{C} \) operator gives the energy cutoff approximation for the nonlocality effects in the conducting \( CuO_2 \) planes. Therefore the analytical 2D result plays a key role for the fluctuation phenomena in layered superconductors.

We will finish the analysis of the local \( \epsilon \gg |\epsilon|, h \) 2D approximation \( h \gg r \), i.e.

\[
\mu_0 H \gg r B_{c2}(0) = \left( \frac{2\xi_{c}(0)N}{s} \right)^2 \Phi_0 \frac{1}{2\pi \xi_{ab}(0)},
\]

with the important case of strong magnetic field \( h \gg |\epsilon| \). Under these conditions \((|\epsilon|, r \ll h \ll c)\) the layered superconductors display a magnetization corresponding to the local 2D one in strong magnetic fields. The substitution of \( \epsilon = 0 \) in Eq. (142), using Eq. (120), recovers the result by Klemm, Beasley, and Luther

\[
\hat{m}(h \gg r, \epsilon \ll h) \approx 0.3465735902799726 \ldots, \quad -M \approx \frac{\ln 2}{2} \frac{k_B T_c N}{\Phi_0 s}.
\]
In the concise review by Koshelev\textsuperscript{63} on the properties of 2D GL model the calculation of $\frac{1}{2} \ln 2 \approx 0.346$ by infinite series with three decimal digits accuracy is described in great details.

### 3.3. Self-consistent approximation for the LD model

The bulk (3D) density of the fluctuation Cooper pairs $n(\epsilon, h)$ can be calculated from the general expression for the Gibbs free energy Eqs. (8) and (12). The differentiation with respect of the "chemical potential" of Cooper pairs $\mu_{\text{CP}} = -a_0 \epsilon$, according to the relation $N_{\text{CP}} = \left(\frac{\partial G}{\partial \mu_{\text{CP}}}\right)_{T,H}$, gives

$$n(\epsilon, h) = \frac{N}{s} \left\langle |\Psi_n|^2 \right\rangle = \frac{1}{a_0} \frac{\partial}{\partial \epsilon} F(\epsilon, h) = -\frac{k_B T_c}{a_0} S(\epsilon, h; c).$$

This formula can be alternatively derived by summation of the Rayleigh-Jeans asymptotics of the energy distribution of the fluctuation Cooper pairs

$$n(\epsilon, h) = \frac{1}{V} \sum_{p, p_z} \frac{k_B T_c}{\epsilon_j(p, p_z) + a} = \frac{N_s (\text{LD})}{s} \int \frac{d(\pi p^2)}{(2\pi \hbar)^2} \frac{k_B T_c}{p^2 / 2m_{ab} + a_0 \epsilon},$$

see for example the monograph by Patashinskii and Pokrovsky.\textsuperscript{64}

Let us give an illustration for zero magnetic field. In this case for the density of fluctuation Cooper pairs, using Eq. (111) and Eq. (50), we obtain

$$n(\epsilon, 0) = \frac{F_0}{a_0} 2 \ln \frac{\sqrt{c + \epsilon} + \sqrt{c + r + \epsilon}}{\sqrt{c} + \sqrt{c + r}}.$$  

This formula sets the stage for the self-consistent treatment of the order parameter fluctuations in the LD model in which the nonlinear term is replaced by its average. The idea has its origin in the Maxwell consideration of the ring of Saturn; probably it is the first work on collective phenomena in physics. Having no possibility to consider motion of all particles in detail we must search for some approximation. Within a self-consistent picture, the motion of every particle creates an average potential in which the others are moving. From the dust of the ring of Saturn to the Cooper pairs in cuprates the idea is the same, only the mechanics slightly changes. In the self-consistent approximation the nonlinear term in GL equations gives an addendum to the linear one

$$a_{\text{ren}}(\epsilon, h) = a_0 \epsilon + b n \left( \frac{a_{\text{ren}}}{a_0}, h \right),$$

where the coefficient $b = \hat{b} N/s$ can be expressed via the jump of the heat capacity $\Delta C$ at the phase transition or, which is more convenient for the high-$T_c$ cuprates, via the extrapolated to zero temperature penetration depth $1/\lambda_{ab}^2(T) = \mu_0 n(T) e^{\pi^2} / m_{ab}$, $n(T) = -a(T) / b$,

$$b = \frac{a_0^2}{T_c \Delta C} = 2 \mu_0 \left( \frac{\pi \hbar^2 k_{\text{GL}}}{\Phi_0 m_{ab}} \right)^2, \quad T_c \Delta C = \frac{1}{8\pi^2 \mu_0} \left( \frac{\Phi_0}{\lambda_{ab}(0) \xi_{ab}(0)} \right)^2.$$
where $\kappa_{GL} \equiv \lambda_{ab}(0)/\xi_{ab}(0)$ is the GL parameter. One can easily check that Eq. (165) has the same form in Gaussian units, where $\mu_0^{(\text{Gauss})} = 4\pi$. Introducing the renormalized reduced temperature $\epsilon_{\text{ren}} > 0$ for the normal phase we have the self-consistent equation

$$
\epsilon_{\text{ren}} = \ln \frac{T}{T_c} + \frac{b}{a_0} n(\epsilon_{\text{ren}}, h), \quad (166)
$$

where $n(\epsilon, h)$ is calculated by means of Gaussian saddle point approximation.\textsuperscript{61} For the LD model this equation, by virtue of Eq. (163), takes the form

$$
\epsilon_{\text{ren}} = \ln \frac{T}{T_c} + \epsilon_{\text{Gi}} 2 \ln \sqrt{\frac{\epsilon + \epsilon_{\text{ren}}}{\sqrt{\epsilon + \epsilon_{\text{ren}}} + r}} = \epsilon + \epsilon_{\text{Gi}} \hat{L}^{(\text{LD})} \ln \frac{\epsilon + \epsilon_{\text{ren}}}{\epsilon_{\text{ren}}}, \quad (167)
$$

where the dimensionless parameter

$$
\epsilon_{\text{Gi}} = \frac{bF_0}{a_0^2} = 2\pi \mu_0 N \left( \frac{\lambda_{ab}(0)}{\Phi_0} \right)^2 k_B T_c = \frac{1}{4\pi \xi_{ab}^2(0)} \frac{N k_B}{\Delta C} \quad (168)
$$

is closely related to the Ginzburg number; cf. Eq. (60) which now reads

$$
\frac{C(\epsilon)}{\Delta C} = \epsilon_{\text{Gi}} \hat{L} \frac{1}{\epsilon}, \quad (169)
$$

and the review article by Varlamov et al.\textsuperscript{17} At $T_c$, for $\epsilon_{\text{Gi}} \ll r \ll c$, Eq. (167) gives

$$
\epsilon_{\text{ren}, c} \approx \epsilon_{\text{Gi}} \ln \frac{4c}{r} \quad (170)
$$

and the effective heating $\Delta T = T_c \epsilon_{\text{ren}, c}$ constrains the fluctuation variables at $T_c$. To provide an order estimate we take for illustration $s_{\text{eff}} = 1$ nm, $\lambda_{ab}(0) = 207$ nm, $T_c = 100$ K, $\xi_{ab}(0) = 2.07$ nm, $\kappa_{GL} = 100$, $k_B = 1.381 \times 10^{-23}$ J/K. The substitution of these values in Eq. (168) gives

$$
\epsilon_{\text{Gi}} = \frac{8\pi^2 \times 1.381}{1000} \approx 11\%, \quad \frac{\epsilon_{\text{Gi}}}{6\kappa_{GL}^2} \approx 2 \times 10^{-6}. \quad (171)
$$

In the case of nonzero magnetic field the self-consistent equation for the renormalized reduced temperature, Eq. (167), according to Eqs. (137), (161), and (166), takes the form

$$
\epsilon_{\text{ren}} = \ln \frac{T}{T_c} + \epsilon_{\text{Gi}} \hat{L} \left[ -\psi \left( \frac{\epsilon_{\text{ren}} + \frac{h}{2h}}{2h} \right) + \psi \left( \frac{\epsilon + \epsilon_{\text{ren}} + \frac{h}{2h}}{2h} \right) \right], \quad (172)
$$

or, within the LD model,

$$
\epsilon_{\text{ren}} = \ln \frac{T}{T_c} + \epsilon_{\text{Gi}} \int_0^{\pi/2} \frac{d\phi}{\pi/2} \left[ -\psi \left( \frac{\epsilon_{\text{ren}} + \frac{h + r \sin^2 \phi}{2h}}{2h} \right) + \psi \left( \frac{\epsilon + \epsilon_{\text{ren}} + \frac{h + r \sin^2 \phi}{2h}}{2h} \right) \right], \quad (173)
$$
cf. also Ref. 65. For weak magnetic fields, \( h \ll \epsilon \), using the asymptotic formula for the digamma function, Eq. (124), we recover Eq. (167). The formulae pointed out could be easily programmed for the self-consistent LD fit to the paraconductivity near to the critical temperature \( T_c \). With the foregoing discussion we finish the analysis of the thermodynamics of layered superconductors. We only note that all final formulae can be used to fit the experimental data. Before proceeding however, for reliability sake, it is necessary to check if the formulae implementation correctly reproduces the 3D limit case \( r \to \infty \).

3.4. 3D test example

Every layered superconductor near the critical point \( |\epsilon|, h \ll r \) displays 3D behavior. For high-\( T_c \) cuprates, however, \( r \ll 1 \) and 3D behavior can be observed only in crystals of extremely high quality. Due to fluctuation of the stoichiometry and of the \( T_c \) 3D regime of Gaussian fluctuations may not occur. However there are many conventional layered compounds with moderate anisotropy, \( r \gtrsim 1 \), to which the 3D behavior has broader applicability. The 3D case can be derived as \( (r \to \infty) \)-asymptotics if the parabolic band approximation \( \omega_1(\theta) \approx r \theta^2/4 \), Eq. (34), is substituted into the \( \hat{L} \) operator, Eq. (46). Using the variable \( x(\epsilon, h) \) from Eq. (115) and a new dimensionless variable \( q \), defined as

\[
q = \sqrt{\frac{r}{8h}} \frac{sp_z}{\hbar}, \quad q^2 = \frac{r}{8h} \theta^2, \quad d\theta = 2\sqrt{\frac{2h}{r}} dq,
\]

we get for the regularized sum of logarithms in Eq. (112) the local approximation

\[
\hat{L}^{(LD)_{\text{Reg}}} \sum_{n=0}^{\infty} \ln \left( n + \frac{1}{2} + \frac{\epsilon}{2h} \right) \approx 2\sqrt{\frac{2h}{r}} \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} dq \frac{2\pi}{2\pi} \ln(n + x + q^2).
\]

The UV regularization in this expression is carried out with the help of the equation

\[
\hat{L}^{(LD)_{\text{Reg}}} \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} dq \frac{2\pi}{2\pi} \ln(n + x + q^2) = \zeta \left( \frac{1}{2}, x \right),
\]

which can be easily proved using derivatives of \( \zeta \)-functions

\[
\frac{d}{dx} \zeta(\nu, x) = -\nu \zeta(\nu+1, x).
\]

The second derivative of Eq. (176) is trivially convergent; the essence of the \( \zeta \)-function regularization lies in the omission of an arbitrary linear function \( A(c)x + B(c) \), being analytical with respect to \( \epsilon \) and therefore irrelevant to the critical behavior, cf. Eq. (141). In fact \( c \approx 1 \) but having dropped \( A(c) \) and \( B(c) \) we can obtain the local approximation, \( |\epsilon|, h \ll c \), as \( c \to \infty \) even if \( A(\infty) = \infty \) and \( B(\infty) = \infty \). The substitution of this UV regularization in Eq. (112), using Eq. (35),
gives the result by Mishonov\textsuperscript{66} for the fluctuation part of the Gibbs free energy

\[ G(T, H) = V F(\epsilon, h) = \frac{\sqrt{2}}{2\pi} k_B T \frac{V}{\xi_a(0)\xi_b(0)\xi_c(0)} h^{3/2} \zeta \left( -\frac{1}{2}, \frac{1}{2} + \frac{\epsilon}{2h} \right). \] (178)

This result was confirmed by Baraduc et al.\textsuperscript{48} using the same notations, with the \( \zeta \)-function presented implicitly.

In order to bridge the 3D result with the notations introduced for layered systems we can rewrite the coefficient in Eq. (178) as

\[ \sqrt{\frac{2}{\pi}} \frac{k_B T}{\xi_a(0)\xi_b(0)\xi_c(0)} = 4 \sqrt{\frac{2}{r}} \left( \frac{1}{2} M_0 B_c(0) \right). \] (179)

Now differentiation \( F(\epsilon, h) \) with respect to the magnetic field we obtain for the dimensionless magnetization, in agreement with the result by Kurkijärvi, Ambegaokar and Eilenberger\textsuperscript{54}

\[ \tilde{m}(\epsilon, h) = 3 \left( \frac{2}{r} \right)^{1/2} \sqrt{\frac{h}{r}} \left[ \zeta \left( -\frac{1}{2}, \frac{1}{2} + \frac{\epsilon}{2h} \right) - \frac{1}{3} \zeta \left( \frac{1}{2}, \frac{1}{2} + \frac{\epsilon}{2h} \right) \right]. \] (180)

The subsequent differentiation with respect to the magnetic field gives the differential susceptibility. In the particular case of strong magnetic fields, \( \epsilon \ll h \), the local approximation to the GL model, Eq. (180), gives the well known-result by Prange\textsuperscript{67} with an anisotropy correction multiplier\textsuperscript{66} \( \xi_{ab}(0)/\xi_c(0) \)

\[ \tilde{m}(0, h) = 3\sqrt{2} \times 0.0608885 \sqrt{\frac{h}{r}}, \quad -M = 3\pi^{3/2} \zeta \left( -\frac{1}{2}, \frac{1}{2} \right) \frac{k_B T_c}{\Phi_0^{3/2} \xi_c(0)} \sqrt{\mu_0 H}, \] (181)

where for the values of the \( \zeta \)-function we have

\[ \zeta \left( -\frac{1}{2}, \frac{1}{2} \right) = \left[ -1 + \frac{1}{\sqrt{2}} \right] \zeta \left( -\frac{1}{2} \right) = \text{Zeta}\{-1/2, 1/2\} = 0.0608885 \ldots, \]

\[ \zeta \left( -\frac{1}{2} \right) = \text{Zeta}\{-1/2\} = -0.207886 \ldots. \] (182)

The syntax \texttt{Zeta[...]} is used in the commercial software \textit{Mathematica}.\textsuperscript{68} We stress, however, that these are only test mathematical asymptotics for \( c \to \infty \).

For the magnetization, as well as for every quantity exhibiting UV divergences in the local limit, the nonlocal effects are strongly pronounced simply because the contribution of high momenta is significant. That is why the local approximation could be quantitatively fairly good for fitting to the data for fluctuation conductivity and heat capacity. For the magnetization in strong magnetic field regime we have to take into account the effect of nonlocality by fitting the energy cutoff parameter \( \epsilon_{pc} \). A systematic procedure for determination of the parameters of the GL theory is developed in the next section.
4. Some remarks on the fitting of the GL parameters

4.1. Determination of the cutoff energy $\varepsilon_{\text{pc}}$

Let us start with designing a general procedure to fit some parameters of the GL theory which employs only data for the in-plane paraconductivity. Later on we shall address the advantage of investigating several variables simultaneously. The first step is to extract the fluctuation part of the conductivity from the temperature dependence of the resistivity $R(T)$. For layered cuprates the resistivity of the normal phase is to within good accuracy a linear function of temperature, $R_N(T) = A_R + B_R T$, and we can fit the coefficients $A_R$ and $B_R$ far enough from the critical temperature $T_c$, e.g. in the temperature interval $(1.5 T_c, 3 T_c)$. After that we can determine the experimental data for the fluctuation conductivity

$$\sigma_i = R(T_i)^{-1} - (A_R + B_R T_i)^{-1}$$

for all experimental points $i = 1, \ldots, N_{\text{exp}}$. For bi-layered cuprates, such as YBa$_2$Cu$_3$O$_{7-\delta}$ and Bi$_2$Sr$_2$CaCu$_2$O$_8$, one can attempt fitting the data with the formula for the bi-layered model, Eq. (55), where an arbitrary lifetime $\tilde{\tau}_{\text{rel}}$ and cutoff parameter $c$ are included in the interpolation

$$\sigma(\epsilon; \tilde{\tau}_{\text{rel}}, r, w, c) = \frac{e^2}{16 \hbar} \frac{N_s}{\varepsilon} \left[ f_{\text{MT}}(\epsilon; r, w) - f_{\text{MT}}(c + \epsilon; r, w) \right],$$

where

$$f_{\text{MT}}(\epsilon; r, w) = \frac{\epsilon + \frac{1}{2}rw}{\sqrt{(\epsilon^2 + r\epsilon) (\epsilon^2 + r\epsilon + \frac{1}{4}r^2w)}} = \hat{L}^{(\text{MT})} f_{\text{APS}}(\epsilon, h = 0) = \hat{L}^{(\text{MT})} \frac{1}{\epsilon}.$$

As a next step, if necessary, one may fit the data using logarithmic plot that generates the dimensionless deviations from the $\ln(\sigma_{ab}(T))$-values,

$$x_i(r, w, c) = \ln \left[ f_{\text{MT}}(\epsilon_i; r, w) - f_{\text{MT}}(c + \epsilon_i; r, w) \right] - \ln \left( \frac{\sigma_i}{\frac{e^2}{16 \hbar} \frac{N}{s}} \right).$$

For the $x_i$ data we can calculate the mean value, the averaged square

$$\langle x \rangle = \frac{1}{N_{\text{exp}}} \sum_{i=1}^{N_{\text{exp}}} x_i, \quad \langle x^2 \rangle = \frac{1}{N_{\text{exp}}} \sum_{i=1}^{N_{\text{exp}}} x_i^2,$$

and the dispersion

$$S(r, w, c) = \langle x^2 \rangle - \langle x \rangle^2.$$

The fitting procedure is then reduced to numerically finding the minimum of the dispersion

$$S(r_0, w_0, c_0) \leq S(r, w, c).$$
in the space of parameters \((r, w, c)\). We have to start from some acceptable set of parameters, for example, \(r_0 = \frac{1}{7}\), \(w_0 = 1\), and \(c_0 = \frac{1}{2}\), and to search for the minimal value in certain range, e.g. \(r_0 \in (0, 1)\), \(w_0 \in (1, 30)\) and \(c_0 = \frac{1}{2} \in (0, 2)\). It is possible that the parameters of the normal resistivity be correct by the same procedure for minimization of the dispersion \(S(r, w, c, A_R, B_R)\). The contemporary methods of the mathematical statistics, such as the bootstrap and Jack-knife, can tell us how reliable is the set of the fitted parameters; the simplest possible realization is to decrement sequentially \(N_{\exp}\) by one and to investigate the distribution of the fitted GL parameters at every step. For example, the \(w\) parameter is almost inaccessible since for \(w = 1\) and \(w \to \infty\) we have LD-type temperature dependence of the paraconductivity. On the other hand if we try to fit the paraconductivity far from the critical temperature, e.g. \(T \in (1.02 T_c, 1.15 T_c)\) we can easily find some estimate for the cutoff parameter \(c\). In any case a good fit would be useful because as a by-product we determine the life-time of the fluctuation Cooper pairs

\[
\tilde{\tau}_{rel} = \exp\left(-\langle x \rangle (r_0, w_0, c_0)\right). \tag{190}
\]

The same procedure can be applied to the magnetic susceptibility at vanishing magnetic field which, according to Eq. (61), is proportional to the conductivity, or for the susceptibility in the LD model which, according to Eqs. (42), (72), and (168), reads

\[
-\chi_{LD}(\epsilon) = \frac{1}{6} \frac{e_{GL}}{k_B T_c} \frac{1}{\sqrt{\epsilon(r + \epsilon)}} - \frac{1}{\sqrt{(\epsilon + c)(\epsilon + c + r)}} \tag{191}
\]

where

\[
\frac{e_{GL}}{k_B T_c} = 2\pi\mu_0 \frac{k_B T_c \epsilon_{ab}(0)}{\Phi_0^2} = \frac{M_0}{H_{c2}(0)} \tag{192}
\]

The general formula for the conductivity, Eqs. (58), (70),

\[
\sigma_{ab}(\epsilon, h) = \frac{\pi}{8} \frac{\tau_{rel}}{R_{QHE}} \hat{\langle}^{(LD)} \hat{\zeta}_{APS}(\epsilon, h) \hat{\rangle}, \tag{193}
\]

which for single layered superconductor reads, cf. Eq. (133),

\[
\sigma_{ab}(\epsilon, h; r, C) = \frac{e^2}{16\hbar s_{eff}} \frac{2}{h^2} \int_0^{\pi/2} d\phi \frac{\pi/2}{\sqrt{\epsilon(r + \epsilon)}} \left\{ \left( \psi\left( \frac{1}{2} + \frac{\epsilon + r \sin^2 \phi}{2h} \right) \right) \left\{ \left( \psi\left( \frac{1}{2} + \frac{c + \epsilon + r \sin^2 \phi}{2h} \right) \right) \right\} \right\} \tag{194}
\]
gives another possibility for determining the energy cutoff parameter $c$. As most appropriate regime we recommend that the measurements of the conductivity as a function of the magnetic field to be carried out at the critical temperature $T = T_c$. In this case for strong magnetic field, $h \gg r$, the layered superconductors with strong anisotropy $r \ll 1$ show 2D behavior. The substitution $\epsilon = 0$ in Eq. (194) gives another universal law derived within the GL theory with energy cutoff

$$\frac{1}{2} B c_2(0) \frac{\sigma_{ab}(\epsilon = 0, h)}{\tau_{rel}(\epsilon^2/16h s_{eff})} = \frac{\hat{c}}{2} \frac{h}{f_{APS}(\epsilon = 0, h)}$$

where $y = 2h/c$, cf. Eq. (135).

$$\hat{U}_\sigma(y) = \frac{2}{y} \left[ \psi\left(1 + \frac{1}{y}\right) - \psi\left(\frac{1}{2} + \frac{1}{y}\right) \right],$$

$\hat{U}_\sigma(0) = 1$, and $\hat{U}_\sigma(\infty) = 0$. At best, the universal dimensionless conductivity $U_\sigma \propto B \sigma(B)$ and magnetization $U_M \propto M$ have to be fitted simultaneously using the data for the same crystal and common dimensionless argument $\propto B$. Similar universal scaling law for the heat capacity can be derived from Eq. (138)

$$\frac{2}{\zeta(2, \frac{1}{2}) B c_2(0)} C(T_c, B) = U_C\left(\frac{2}{c} \frac{B}{B c_2(0)}\right),$$

where

$$U_C(y) = 1 - \frac{\zeta\left(2, \frac{1}{2} + \frac{1}{y}\right)}{\zeta(2, \frac{1}{2})},$$

but the accuracy of thermal measurements is probably not high enough in order for this to be experimentally confirmed.

4.2. Determination of the coherence length $\xi_{ab}(0)$

The fit of every fluctuation variable as a function of the dimensional magnetic field $h$, the conductivity

$$\sigma(\epsilon, h) = \sigma(\epsilon) + \Delta\sigma(\epsilon, h),$$

for example, provides a method for determination of $B_{c2}(0)$ and $\xi_{ab}(0)$. At weak magnetic fields, $h \ll \epsilon$, the magnetoconductivity is proportional to the square of the magnetic field $\Delta\sigma(\epsilon, h) \equiv \sigma(\epsilon, h) - \sigma(\epsilon) \propto B^2$. For this small negative quantity, $0 < -\Delta\sigma(\epsilon, h) \ll \sigma(\epsilon)$, the APS result, Eq. (68), reads

$$-\Delta\sigma(\epsilon, h) \approx \frac{\hbar^2}{4} \frac{\partial^2}{\partial \epsilon^2} \sigma(\epsilon),$$
where \( h = 2\pi \xi_{ab}^2(0)B_z/\Phi_0 \). The common multiplier \( \tau_0 \) from Eq. (65) is obviously canceled in this relation because, roughly speaking, the transport takes time even in the presence of magnetic field. We note that a multiplier \( \tilde{\tau}_{rel} \) was misintroduced by M. V. Ramallo in Ref. 36 in the right-hand-side of the above equation (see Eq. (4) in Ref. 36). Thereby the old experimental data in Ref. 36 have been apparently processed by employing erroneous expression and therefore the discussion related to Fig. 2 in Ref. 36 is physically unsound. As a consequence, the life-time constant of metastable Cooper pairs in cuprates is still waiting for its first experimental determination. Nevertheless the novel theoretical result that the life-time constant \( \tau_0 \) and the diffusion coefficient of the fluctuation Cooper pairs \( \xi_{ab}^2(0)/\tau_0 \) can be determined from the \( \sigma/\chi \)-ratio remains unchanged.

Returning to Eq. (200) we note that after two-fold integration of the relation (200) in some temperature interval, e.g. \((\epsilon_a, \epsilon_b) = (0, 0.03, 0.09)\), the “noise” in the experimental data is already irrelevant and we can rewrite Eq. (200) as

\[
\xi_{ab}(0) = l_B \left[ \int_{\epsilon_a}^{\epsilon_b} \int_{\epsilon_a}^{\epsilon_b} \left(-\Delta \sigma(\epsilon', h)\right) d\epsilon' d\epsilon'' \sigma(\epsilon_a) - \sigma(\epsilon_b) + \frac{\sigma(\epsilon_b) - \sigma(\epsilon_a)}{\epsilon_b - \epsilon_a} \right]^{1/4} = \text{const},
\]

where \( l_B \) is the magnetic length

\[
l_B = \sqrt{\frac{\Phi_0}{\pi B}} = \sqrt{\frac{\hbar}{eB}} = \frac{25.6 \text{ nm}}{\sqrt{B(T)}}.
\]

For practical application we have to take into account that far from the critical temperature, even for \( T - T_c = 15\% T_c \) the fluctuation conductivity is negligible \( \sigma(0.15) \approx \sigma(\infty) = 0 \). That is why in acceptable approximation we can take \( \epsilon_b = 0.15 \approx \infty \). For \( \epsilon > \epsilon_b \) the temperature dependence of the magnetoconductivity in the numerator of Eq. (201) can be an extrapolated LD fit.

However, due to the strong critical behavior \(-\Delta \sigma \propto h^2/\epsilon^3 \) for \( \epsilon \gg r \) the influence of the interval \((\epsilon_b, \infty)\) can be neglected. In such a way, after a partial integration, we arrive at a simpler equation for determination of the in-plane coherence length, cf. Ref. 36,

\[
\xi_{ab}(0) \approx l_B \left[ \frac{1}{\sigma(\epsilon)} \left( \int_{\epsilon}^{\infty} \epsilon' (-\Delta \sigma(\epsilon', h)) d\epsilon' - \epsilon \int_{\epsilon}^{\infty} (-\Delta \sigma(\epsilon'', h)) d\epsilon'' \right) \right]^{1/4} = \text{const},
\]

where the integrations should be performed in the whole experimentally accessible temperature range above \((1 + \epsilon)T_c\). This result of the Gaussian fluctuation theory does not depend upon the \( \tau_0 \) parameter, effective mass of Cooper pairs \( m_{ab} \), and the space dimensionality. We consider this procedure for determination of the coherence length \( \xi_{ab}(0) \) as being the best one, as it is model-free and does not depend on the multilaminarity of the superconductor, i.e. on the dispersion of Cooper pairs in \( c \)-direction \( \epsilon_{c,j}(p_z) \). Equation (203) has the same form for both strongly anisotropic...
high-$T_c$ cuprates and bulk conventional dirty alloys. Of course, methods particularly based on the proximity to the critical line $H_{c2}(T)$ can be very useful in determining $\xi_{ab}(0)$ especially in the case of strong magnetic fields. For example, Eq. (68) gives another appropriate formula

$$\sigma_{ab}(\epsilon, h) \approx \tilde{\tau}_{rel} \frac{\epsilon^2 N}{16\hbar s} \frac{4}{\sqrt{(\epsilon + h)(\epsilon + h + r)}}$$

(204)

applicable for $\epsilon_{Gi} \ll \epsilon + h \ll h$. Similar result, cf. also Eq. (142),

$$M = -M_0 \tilde{m} \approx -\frac{k_B T_c N}{\Phi_0} \frac{h}{s \sqrt{(\epsilon + h)(\epsilon + h + r)}}$$

(205)

can be derived under the same physical conditions from the formula for the fluctuation magnetic moment, Eq. (133), using the approximations for $0 < x \ll 1$,

$$\Gamma(x) \approx -\ln x - \frac{1}{2} \ln(2\pi), \quad \psi(x) \approx -\frac{1}{x}.$$  (206)

The experimental investigation of the conductivity, Eq. (204), and magnetization, Eq. (205), is probably the best way to extract the upper critical field $H_{c2}(T)$ for high-$T_c$ cuprates; the $H\sigma/M$ quotient near the critical line is $2/3$ of the $\sigma/\chi$ quotient for weak magnetic fields.

4.3. Determination of the Cooper pair life-time constant $\tau_0$

Having a reliable estimate for the coherence length, the life-time constant of the metastable Cooper pairs above $T_c$ can be determined via the $\sigma/\chi$-quotient, Eq. (61). We believe that this method will become a standard procedure in the physics of high-$T_c$ materials. Certainly the most transparent method is just the fit to the phase angle of high-frequency complex fluctuation conductivity

$$\phi_\sigma(\omega \tau(\epsilon_{ren})) = \arctan \frac{\sigma''(\omega)}{\sigma'(\omega)} = \arctan \frac{\hat{L}_2(\omega \tau_0/\epsilon_{ren})}{\hat{L}_1(\omega \tau_0/\epsilon_{ren})}.$$  (207)

The state-of-the-art electronics gives such a possibility, but unfortunately the first experiments of the type$^{57,69}$ was not performed in the Gaussian region. For the development of Gaussian spectroscopy which will give results relevant for the microscopic mechanisms of superconductivity we recommend the use of the conventional thin films and high-quality low temperature cuprate films, such as Bi$_2$Sr$_2$CaCu$_2$O$_8$.

4.4. Determination of the Ginzburg number and penetration depth $\lambda_{ab}(0)$

The applicability of the self-consistent approximation in the theory of fluctuation phenomena in superconductors is strongly limited by the quality of the samples. The fluctuation of the critical temperature $\Delta T_c$, e.g. due to the oxygen stoichiometry in cuprates, should be small enough, $\Delta T_c \ll \epsilon_{Gi} T_c$, and this has to be verified empirically. If the $\sigma/\chi$ ratio remains temperature independent for $\epsilon < 3\%$ and
both \( \sigma(\epsilon) \) and \( \chi(\epsilon) \) demonstrate weak deviation from the LD fit obtained from the range \( \epsilon \in (3\%, 9\%) \), this could be considered as a hint in favor of the self-consistent approximation. In this case \( \epsilon_{\text{ci}} \) can be fitted by substituting the solution \( \epsilon_{\text{ren}}(\epsilon) \) of Eq. (167) into the LD fit to \( \sigma^{(\text{LD})}(\epsilon_{\text{ren}}) \) and \( \chi^{(\text{LD})}(\epsilon_{\text{ren}}) \). We note that the reliability in fitting \( \epsilon_{\text{ci}} \) is determined by the condition whether the self-consistent approach and the use of \( \epsilon_{\text{ren}} \) significantly improve the accuracy of the fit to the experimental data near \( T_c \). According to Eq. (168) we can parameterize \( \epsilon_{\text{ci}} \) with the help of the penetration depth \( \lambda_{ab}(0) \). In any case, an evaluation of such type should be a part of the complete set of GL parameters of the superconductor.

Another possibility for the thermodynamic determination of the penetration depth \( \lambda_{ab}(0) \) is provided by the jump in the specific heat at the critical temperature, Eq. (165). As a rule the accuracy of the determination of the penetration depth by the thermodynamic methods cannot be high, especially for high-\( T_c \) cuprates where the phonon part strongly dominates. An acceptable value of \( \ln \kappa_{\text{GL}} \) derived from the heat capacity is necessary for the establishment of a coherent understanding of the superconductivity; there is no doubt that the direct investigation of the vortex phase of the superconductors or vortex-free high-frequency measurements constitute the best methods for determination of \( \lambda_{ab}(0) \).

5. Discussion and conclusions

In the attempts to systematize the available results we had to derive in parallel new ones too. We shall summarize the most important of them starting with remarks concerning the theory. As the ultimate result we consider the representation of the fluctuation part of the Gibbs free energy by the Euler \( \Gamma \)-function Eq. (129) in Gaussian approximation. This result trivializes the derivation of all thermodynamic variables, such as fluctuation magnetization Eq. (133), or fluctuation heat capacity Eq. (150). To our knowledge this is a novel result, but we find it strange that it remained unobserved given the great attention which the fluctuations in high-\( T_c \) superconductors have attracted. The importance of fluctuations was mentioned even in the classical work by Bednorz and Müller. Fluctuations in superconductors were among the main topics in many scientific activities; the \( \Gamma \)-function is well-known to all physicist; the mathematical physics behind the 2D statistical mechanics is well developed, polygamma functions can be found in a number of BCS papers, and finally the solution turns out to be on a textbook level. Just the same is the situation for the 3D GL model. In this case the solution for the free energy is given in terms of the Hurwitz \( \zeta \)-functions. Analogous result gave the name of one of the most powerful methods in the field theory — \( \zeta \)-function method for ultraviolet regularization, but this method was never applied to the most simple problem of a 3D GL model related to numerous experiments in the physics of superconductivity.

Another simple but useful detail is the layering operator \( \hat{L} \), Eq. (44), which allows us to extend the 2D result onto layered superconductors and even to 3D superconductors. The method can be applied not only to the thermodynamic variables but to the fluctuation part of the kinetic coefficients as well. In this way we
obtained useful formulae for the in-plane fluctuation conductivity in perpendicular magnetic field, Eq. (194), and for the high-frequency Aslamazov-Larkin conductivity in layered superconductors, Eq. (74). We proposed further convenient $r$-$w$ parameters, Eq. (55), for the bi-layered model which could be utilized for experimental data processing of the fluctuation phenomena in bi-layered cuprates, such as YBa$_2$Cu$_3$O$_{7-\delta}$ for example.

The representation of the thermodynamic variables via polygamma functions is very helpful at strong magnetic fields but due to the presence of the magnetic field in denominator these results cannot be directly applied to zero-magnetic-field limit. For small magnetic fields, on the other hand, we have to use asymptotic formulae for polygamma and $\zeta$-functions with large arguments. This is the reason why the weak-magnetic-field expansion of the magnetization and the other thermodynamic variables has so bad convergence. In order to fit the experimental data for the magnetization in weak magnetic field using the new analytical result for the LD model, Eq. (82), we arrive at the problem for summation of divergent asymptotic series. At least for experimentalists this is a nontrivial problem which led us to give a prescription for usage of series from the theoretical papers. There is no doubt that the $\varepsilon$-method is one of the brilliant achievements of the applied mathematics of XX century. However, it turns out that this method was not cast in an suitable form to be employed by users like experimentalists having no time to understand how the underlying mathematics can be derived. That is why we presented an oversimplified version of this algorithm illustrated by a simple FORTRAN90 program. The latter can be also used for calculation of the differential nonlinear susceptibility at finite magnetic field, Eq. (105), which is another novel result in the present work.

Let us now address the simple final formulae that can be directly used for experimental data processing. First of all we advocate that the relation between fluctuation conductivity and magnetoconductivity, Eq. (203), provides the best method (shortly announced in Ref. 36) for determination of the in-plane coherence length $\xi_{ab}(0)$ in layered high-$T_c$ cuprates and conventional superconductor superlattices and thin films. Having such a reliable method for determination of $\xi_{ab}(0)$, the Cooper pair life-time spectroscopy can be created$^{36}$ on the basis of determination of the life-time constant $\tau_0$ by the $\sigma/\chi$ quotient, Eq. (61).

Usually science starts with some simplicity, thus it is surprising that the temperature independence of the $\sigma/\chi$, $\chi/C$, and $\sigma/C$ quotients has not attracted any attention in physics. The question of whether the high-$T_c$ cuprates are BCS superconductors, or they have a non BCS behavior, consumed more paper and brought more information pollution than that about the sense of life, about the smile of Mona Lisa. Now we possess a perfect tool to check whether this sacramental $\frac{\pi}{8}$ BCS ratio, Eq. (63), still exists in the physics of high-$T_c$ superconductivity. A careful study of the relative life-time constant $\tau_0$ by the $\sigma/\chi$ ratio, Eq. (64), will provide a unique information on the presence of depairing impurities in the superconducting cuprates. The doping dependence of this ratio will give important information for the limits of applicability of the self-consistent BCS approximation. In principle
the same life-time spectroscopy can be applied to heavy fermions and other exotic superconductors.

The methods we proposed in this review can be initially tested by means of alternative methods for determination of $\xi_{ab}(0)$, e.g. from the slope of the upper critical field $H_{c2}(T)$ defined by the fluctuation magnetization of the normal phase near the critical line, Eq. (205), being another new result derived here, or from the fluctuation conductivity of the LD model, Eq. (204). Addressing the conductivity we consider that the fitting to high frequency experimental data with the help of formulae (73) and (207) will give a direct method for determination of the relaxation time of the superconducting order parameter. A good monocrystal of layered cuprate or high-quality thin film with as low as possible critical temperature could ensure the overlap of both the suggested methods for Cooper pair life-time spectroscopy. At present we only know\(^{36}\) that for 93 K \(\text{YBa}_2\text{Cu}_3\text{O}_{7-x}\) $\tau_{0,\Psi} = 2\tau_0 = 32$ fs. We hope, however, that several experimental methods for determination of $\xi_{ab}(0)$ and $\tau_0$ will be mutually verified in the nearest future. Thus, the investigation of the Gaussian fluctuations may become a routine procedure in the materials science of superconductors.

We also believe that the development of the Gaussian spectroscopy will lead to determination of the Ginzburg number $\epsilon_{Gi}$, the energy cutoff, i.e. the maximal kinetic energy of the Cooper pairs $\varepsilon_{Gi} = c h^2 / 2 m_{ab} \xi_{ab}^2(0)$. Up to now these parameters of the GL theory are inaccessible. We hope that our derived self-consistent equation for the reduced temperature, Eq. (173), will stimulate experimentalists to reexamine the data for high-quality crystals in the region close to $\epsilon + h \approx 3\%$ in order to extract $\epsilon_{Gi}$. Virtually all final results are presented by taking into account the energy cutoff parameter $c$. The nonlocality corrections can be extracted from almost all fluctuation variables, if $\epsilon + h > 10\%$, but we suggest special new experiments to be conducted for investigation of nonlocality effects in quasi 2D superconductors at $T_c$. Analogous investigations for fluctuation diamagnetism for classical bulk superconductors are already classics in physics of superconductivity; see for example Fig. 8.5 in the well-known textbook by Tinkham.\(^4\) The universal scaling law for the heat capacity, Eq. (198), for the magnetization, Eq. (135), and conductivity, Eq. (196), versus the reduced magnetic field $y = 2h/c$ are depicted in Fig. 2.

At least for conductivity the experimental confirmation for the quasi-2D superconductors ($h \gg r, \epsilon_{Gi}$) can be easily achieved. How different are the animals ... ? The biochemists considered that what is true for \textit{Escherichia coli} holds true for the elephant. Analogously, we consider that $B \sigma(T_c, B)$ versus $B$ will be within 20\% accuracy the same for conventional Pb layers and for strongly anisotropic underdoped \textit{Bi}_2\textit{Sr}_2\textit{CaCu}_2\textit{O}_8 in spite of the bunch of sophisticated theories of high-$T_c$ superconductivity. The GL theory gives the scaling law, the notions and notations, and in this sense the language for analysis of the fluctuation phenomena. The precisely measured deviations from the GL scaling low could give the basis for further microscopic consideration using the methods of the statistical mechanics. This is the
Fig. 2. Universal scaling curves ((a) semi-logarithmic plot; (b) linear plot) of a quasi-2D superconductor for the fluctuation conductivity $B\sigma(T_c, B) \propto U_\sigma$, magnetic moment $M(T_c, B) \propto U_M$, and heat capacity $BC(T_c, B) \propto U_C$ versus dimensionless magnetic field $y = 2h/c$. The fit of the scale in horizontal direction gives the GL cutoff parameter $c$. The scales in vertical directions are related correspondingly to diffusion constant of Cooper pairs $\xi_{ab}^2(0)/\tau_0$, effective inter-layer distance $s_{eff}$, and 2D Ginzburg number $\epsilon_G$, cf. Eqs. (196), (135), and (198).

last example how the development of the Gaussian fluctuation spectroscopy could be of importance not only for the materials science but for the fundamental physics of superconductivity as well.

Acknowledgments

Some parts of this review are based on the lectures on statistical mechanics, solid state theory, numerical methods and physics of superconductivity read by one of the authors (T.M.) at the University of Sofia. He would like to thank to many of his former students for the help and collaboration in the early stages of this work, especially to D. Damianov for the collaboration on the Boltzmann equation for the fluctuation Cooper pairs$^{72}$ and to N. Zahariev for implementing the $\varepsilon$-algorithm in C. It is a pleasure to acknowledge the cooperation of C. Carballeira in implementing the $\varepsilon$-algorithm in Mathematika® and checking the numerical equivalence for $r = 0$ of Eqs. (82) and (133) using this software. The same author appreciates the discussions with Prof. Vidal on the experiments concerning $\sigma/\chi$ and other quotients. The completion of this review would be impossible without the warm hospitality of Prof. Indekeu, his support and interest to this work.

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Appendix A
PROGRAM Test
IMPLICIT NONE

INTEGER, PARAMETER :: pr = SELECTED_REAL_KIND (30,150)
REAL (pr), PARAMETER :: zero = 0.0, one = 1.0
REAL (pr) :: S(0:137), C(0:137), x, xi, arg
REAL (pr) :: rLimes
REAL (pr) :: err
INTEGER :: N
INTEGER :: i
INTEGER :: i_Pade
INTEGER :: k_Pade
INTEGER :: is

WRITE (*, '(15X,A)') '+---------------------------------------------+'
WRITE (*, '(15X,A)') '| Test driver program for subroutine Limes |'
WRITE (*, '(15X,A)') '| Calculate Ln[x] |'
WRITE (*, '(15X,A)') '+---------------------------------------------+'
WRITE (*, '(A)') ' 
WRITE (*, '(A)', ADVANCE='NO') ' Enter argument of Ln[x], x = ', arg
READ (*,*) arg
WRITE (*, '(A)', ADVANCE='NO') ' Enter the number of known terms, N = ', N
READ (*,*) N
IF (N > 137) N = 137 ! ... we like this number ;-) 

x = arg - one
xi = one
is = 1

! Initialize S to store the first N+1 known partial sums
! S0, S1, S2,..., Sn-1, Sn
! Sn = x + x^2/2 - x^3/3! + x^4/4! - ... + (-1)^n x^n/n!
! S(0) = zero ! *** lower bound of the subscript should start at 0 ! ***
DO i=1,N
  xi = xi*x
  C(i) = xi/i
  S(i) = S(i-1) + is * C(i)
  is = -is
END DO

WRITE (*, '(A)', ADVANCE='NO') ' before call: '
WRITE (*, '(F8.4,5(X,f12.4))') S(0:5)
CALL Limes & ! call subroutine Limes to calculate Ln[x]
  ( N, & ! in
    rLimes, & ! out
    i_Pade, & ! out
    k_Pade, & ! out
    err ) ! out

! Formated output of results
WRITE (*, '(A)', ADVANCE='NO') ' after call: '
WRITE (*, '(F8.4,5(X,f12.4))') S(0:5)
WRITE (*, '(A)', ADVANCE='NO') ' 
WRITE (*, '(A)', ADVANCE='NO') ' 
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WRITE (*, '(A)', ADVANCE='NO') ' 
WRITE (*, '(A)', ADVANCE='NO') ' 
WRITE (*) ;
WRITE (*, '(X,A,X,F9.3,X,A,ES12.5)') 'Ln[', arg, '] = ', LOG (arg)
WRITE (*, '(X,A,I3,A,I3,A,ES12.5)') 'rLimes[', i_Pade, ',', k_Pade, ']
WRITE (*, '(13X, 1A, ES12.5)') 'true error = ', ABS (rLimes - LOG (one + x))

!-----------------------------------------------------------------------------
CONTAINS
!-----------------------------------------------------------------------------
!+ Finds the limit of a series
SUBROUTINE Limes &
  ( N, & ! in
    S, & ! inout
    rLimes, & ! out
    i_Pade, & ! out
    k_Pade, & ! out
    err ) ! out
! Description:
! Finds the limit of a series in the case where only
! the first N+1 terms are known.
!
! Method:
! The subroutine operates by applying the epsilon-algorithm
! to the sequence of partial sums of a series supplied on input.
! For description of the algorithm, please see:
! [1] T. Mishonov and E. Penev, Int. J. Mod. Phys. B 14, 3831 (2000)
!
! Owners: Todor Mishonov & Evgeni Penev
!
! History:
! Version Date Comment
! ========= ====== ========
! 1.0 01/04/2000 Original code. T. Mishonov & E. Penev
!
! Code Description:
! Language: Fortran 90.
! Software Standards: "European Standards for Writing and
! Documenting Exchangeable Fortran 90 Code".
!
! Declarations:
IMPLICIT NONE
!* Subroutine arguments
! Scalar arguments with intent(in):
INTEGER, INTENT (IN) :: N ! width of the epsilon-table
!
! Array arguments with intent(inout):
REAL (pr), INTENT (INOUT) :: S(0:) ! sequential row of the epsilon-table
!
! Scalar arguments with intent(out):
REAL (pr), INTENT (OUT) :: rLimes ! value of the series limit
INTEGER, INTENT (OUT) :: i_Pade ! power of the numerator
INTEGER, INTENT (OUT) :: k_Pade ! power of the denominator
REAL (pr), INTENT (OUT) :: err ! empirical error
!* End of Subroutine arguments
!
! Local parameters
! these two need no description ;-)
REAL (pr), PARAMETER :: zero = 0.0
REAL (pr), PARAMETER :: one = 1.0
!
! Local scalars
REAL (pr) :: A_max ! maximum element of A
INTEGER :: i ! index variable for columns
INTEGER :: k ! index variable for rows
!
! Local arrays
REAL (pr) :: A(0:N) ! auxiliary row of the epsilon-table
!
! End of header -----------------------------------------------
Thermodynamics of Gaussian fluctuations and paraconductivity in layered superconductors

! Parse input: the algorithm cannot employ more elements than supplied on ! input, i.e. N <= size(S)

IF ( N > SIZE (S(:,)) ) THEN
  WRITE (*, '(A)') '*** Illegal input to Limes: N > size(S)'
  STOP 1
END IF

! Algorithm not applicable for N < 2

IF ( N < 2 ) THEN
  WRITE (*, '(A)') '*** Illegal input to Limes: N < 2'
  STOP 2
END IF

! I. Initialize with natural assignments

rLimes = S(N) ! the N-th partial sum
err = ABS ( S(N) - S(N-1) ) ! error -> |S(N) - S(N-1)|
i_Pade = N
k_Pade = 0
A(:) = zero ! auxiliary row initially set to zero
A_max = zero ! max. element set to zero
k = 1 ! algorithm starts from the first row

! II. Main loop: fill in the epsilon table, check for convergence ...
! (provision against division by zero employs pseudo-inverse numbers)

DO
  IF ( N - 2 * k + 1 < 0 ) EXIT

  ! Update the auxiliary row A(i) of the epsilon-table
  ! by applying the "cross rule".

  DO i=0, N - 2 * k + 1
    IF ( S(i+1) /= S(i) ) THEN
      A(i) = A(i+1) + one/(S(i+1) - S(i))
    ELSE
      A(i) = A(i+1)
    END IF
  END DO

  IF ( N - 2 * k < 0 ) EXIT

  ! Update the sequential row S(i) of the epsilon-table
  ! by applying the "cross rule".

  DO i=0, N - 2 * k
    IF ( A(i+1) /= A(i) ) THEN
      S(i) = S(i+1) + one/(A(i+1) - A(i))
    ELSE
      S(i) = S(i+1)
    END IF
  END DO

  ! Check for convergence, based on A_max; see Ref. [1]

  IF ( ABS ( A(i) ) > A_max ) THEN
    A_max = ABS ( A(i) )
    rLimes = S(i)
    k_Pade = k
    i_Pade = i + k_Pade
    err = one/A_max
  END IF

  IF ( S(i+1) == S(i) ) RETURN
  END IF

  k = k + 1 ! increment row index
END DO

END SUBROUTINE Limes

END PROGRAM Test
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