Abstract. We investigate the PDE system resulting from even electromechanical coupling in elastomers. Assuming a periodic microstructure and a periodic distribution of micro-charges of a prescribed order, we derive the homogenized system. The results depend crucially on periodicity (or adequate randomness) and on the type of microstructure under consideration. We also offer a possible path to electric enhancement if the charges are carefully tailored to the homogenized electric field.

Keywords: Homogenization, elliptic regularity, H-convergence.

1. Introduction

Ever since the discovery of the piezoelectric behavior of several types of minerals — including quartz, tourmaline, and Rochelle salt — by Pierre and Jacques Curie in the 1880s [5, 6], deformable dielectrics have been an object of uninterrupted interest in fields ranging from materials science to mathematics. This has been reinforced since the turn of the millennium when soft organic dielectrics were “re-discovered” as a class of materials with high technological potential.

In contrast to the odd coupling between mechanical and electric field, a characteristic of the hard deformable dielectrics investigated by the Curie brothers, soft organic dielectrics typically exhibit even electromechanical coupling. From a mathematical point of view, this means that the governing equations involved exhibit nonlinearity, even in the simplest asymptotic setting of small deformations. Furthermore, space charges varying at the length scale of the microstructure may assert their presence, as is the case, for example, in porous polymer electrets [2, 10] and polymer nano-particulate composites [11, 17]. This translates into equations that contain a rapidly oscillating source term and leads to anomalous behaviors [8, 14].

Our goal in this study is to investigate the homogenization of elasto-dielectrics with even electromechanical coupling that contain space charges that vary at the length scale of their microstructure; a formal analysis of that problem was presented in [12]. In addition to ignoring dissipative effects, we restrict attention to materials with periodic microstructure, quasi-static electromechanical loading conditions, and further focus on the asymptotic setting of small deformations and moderate electric fields. The derivation of the relevant local governing equations goes as follows.

Consider an elastic dielectric that occupies a bounded domain $\Omega \subset \mathbb{R}^N$ with boundary $\partial \Omega$ in its undeformed, stress-free, and polarization-free ground state. Material points are identified by their initial position vector $x$ in $\Omega$ relative to some fixed point. Upon the application of mechanical loads and electric fields, the position vector $x$ of a material point moves to a new position specified by $v = x + u(x)$, where $u$ denotes the displacement field. The associated deformation gradient is denoted by $F(x) = I + \nabla u(x)$. In the absence of magnetic fields, free currents, and body forces, and with no time dependence (see, e.g., [7]), Maxwell’s and the momentum balance equations require that

$$\text{div} \, D = Q, \quad \text{curl} \, E = 0, \quad x \in \mathbb{R}^N$$

and

$$\text{div} \, S = 0, \quad SF^T = FS^T, \quad x \in \Omega,$$

where $D(x), E(x), S(x)$ stand for the Lagrangian electric displacement field, the Lagrangian electric field, and the “total” first Piola-Kirchhoff stress tensor, while $Q(x)$ stands for the density...
charges with density $g$ respectively denoted by $\delta$ rescaled by a small parameter $\delta$;
the “total” free energy $W(x, F, E)$ is an objective function of the deformation gradient tensor $F$ and an even objective function of the electric field $E$, namely, $W(x, F, E) = W(x, QF, E) = W(x, F, -E)$ for all $Q \in SO(N)$ and arbitrary $F$ and $E$. The objectivity of $W$ implies that the balance of angular momentum $SF^T = FS^T$ is automatically satisfied. Faraday’s law $\text{curl } E = 0$ can also be satisfied automatically by the introduction of an electric potential $\varphi(x)$ such that $E(x) = -\nabla \varphi(x)$. Thus, only Gauss’s law $\text{div } D = Q$ and balance of linear momentum $\text{div } S = 0$ remain.

Now, setting $H := F - I$, a Taylor expansion of $W$ about the ground state $F = I$, $E = 0$ yields

$W(x, F, E) = -\frac{1}{2} E : \varepsilon(x) E + \frac{1}{2} H : L(x) H + H : (M(x) (E \otimes E)) - E \otimes E : (T(x) (E \otimes E)) + \ldots$, 

where $\varepsilon(x) := -\partial^2 W(x, I, 0)/\partial E^2$ is the permittivity tensor, $L(x) := \partial^2 W(x, I, 0)/\partial F^2$ is the elasticity tensor, $M(x) := 1/2 \partial^3 W(x, I, 0)/\partial F \partial E^2$ is the electrostriction tensor, and $T(x) := -1/24 \partial^4 W(x, I, 0)/\partial E^4$ is the permittivity tensor of second order. It follows that the constitutive relations that describe the electromechanical response of the elastic dielectric specialize to

$D = \varepsilon(x) E + T(x) (E \otimes E \otimes E) + \ldots$ and $S = L(x) H + M(x) (E \otimes E) + \ldots$. 

Taking the order magnitude of the deformation measure $H$ to be of order $\zeta$, with $0 < \zeta << 1$, it follows in turn that the electric field $E$ must be of order $\zeta^{1/2}$ if the elastic dielectric is to display electromechanical coupling around its ground state. To leading order, we thus have

$D = \varepsilon(x) E$ and $S = L(x) H + M(x) (E \otimes E)$. 

This is the so-called scaling of small deformations and moderate electric fields; within this scaling, by the same token, the space charge density $Q$ must be of $O(\zeta^{1/2})$.

We now detail the governing equations for the problem under investigation in this work. Assuming periodicity of the microstructure, the permittivity, elasticity, and electrostriction tensors $(\varepsilon(y), L(y), M(y))$, respectively) that characterize the local elastic dielectric response of the material are defined on a unit cell (or, more precisely, on a unit torus $\mathcal{T}$) and they are periodically rescaled by a small parameter $\delta$ to reflect the size of the microstructure. The resulting tensors are respectively denoted by $\varepsilon^\delta(x), L^\delta(x), M^\delta(x)$. 

![Figure 1](image-url)

**Figure 1.** (a) Schematics of the elastic dielectric composite $\Omega$; the boundary layer of incomplete unit cells needed to conform with the arbitrary geometry of its boundary is marked in red. (b) Schematic of the unit cell $\mathcal{T}$ that defines the periodic microstructure of the material with the explicit illustration of the distribution of space charges characterized by the space-charge density $g(y)$.

Moreover, the material is assumed to contain a distribution of periodically distributed space charges with density $g(y)$ such that

$$\int_{\mathcal{T}} g(y) \, dy = 0$$

(1.1)
so as to preserve local charge neutrality and that is rescaled in a manner similar to that of the microstructure and modulated by a slowly varying macroscopic charge $f(x)$. These space charges can be passive or active. In the case of passive charges, the slowly varying macroscopic charge $f(x)$ is fixed from the outset. Physically, this corresponds to materials wherein space charges are “glued” to material points and remain so regardless of the applied mechanical loads and electric fields. This is the case, for instance, of porous polymer electrets for which the space charges are fixed at the walls of the pores. In the case of active charges, the slowly varying macroscopic charge $f(x)$ is identified as the resulting macroscopic field for the electric potential and hence depends on the applied electric field. Physically, this corresponds to materials wherein space charge are locally mobile. This is the case, for instance, of polymer nano-particulate composites for which the space charges are locally mobile around the interfaces between the polymer and the nano-particles. Figure 1 illustrates a schematic of the material and of its periodic microstructure and space charge content.

The relevant governing equations are

$$
\begin{align*}
\text{div} \, \varepsilon^\delta \nabla \phi^\delta &= \frac{1}{\delta} g^\delta f \\
\text{div} \left[ L^\delta \nabla u^\delta + M^\delta (\nabla \phi^\delta \otimes \nabla \phi^\delta) \right] &= 0
\end{align*}
$$

(1.2)

for the electric potential $\phi^\delta$ and the displacement field $u^\delta$. For simplicity, the boundary conditions are taken to be of Dirichlet type, that is,

$$
\phi^\delta = \Phi, \quad u^\delta = 0 \quad \text{on } \partial \Omega.
$$

Note that imposing Dirichlet boundary conditions on the electric potential amounts to considering Gauss’ law inside the domain $\Omega$, and not in $\mathbb{R}^N$, a situation which corresponds to electrodes being placed along the entire boundary of $\Omega$.

**Remark 1.1.** The heuristic justification of the presence of the term $1/\delta$ in front of the space charges $g^\delta f$ is the following. Because of charge neutrality (see (1.1)), multiplication of the source term by $\delta^q$ with $q > -1$ would result in a homogenized dielectric equation without any source term, that is an equation of the form

$$
\text{div} \, \varepsilon^h \nabla \phi = 0 \quad \text{in } \Omega,
$$

where $\varepsilon^h$ is the homogenized permittivity tensor defined later in (2.9). Thus, the lowest $\delta$-order at which microscopically distributed charges will impact the homogenized dielectric equation is $\delta^{-1}$.

Of course, one can always add lower order source terms as emphasized in Remark 3.4 below, but, their impact will disappear in the effective behavior unless charge neutrality is forsaken for those terms.

The first objective of this work is to determine the purely dielectric macroscopic behavior of the material for an arbitrary but fixed (i.e., passive) distribution of space charges in the limit when the period $\delta$ of the microstructure goes to 0. This will be achieved in Section 2. The obtained results do not rest on periodicity and would still hold true in a general homogenization framework like that of $H$-convergence [16].

The second objective is to investigate the possible enhancement of the purely dielectric macroscopic behavior afforded by active space charges, when $f(x)$ is identified as the resulting macroscopic field for the electric potential. This is the object of Section 5 which we have placed at the end of this paper because of its currently speculative nature.

Finally, we determine the homogenized equations for the coupled elastic dielectric behavior of the material. This is the object of Section 4. Doing so necessitates better convergence properties on the dielectric micro-macro analysis than those provided by Section 2. To do so we combine large-scale regularity due to homogenization to local regularity properties that hold for two-phase microstructures with smooth inclusions. The technical details are the object of Section 3.

Notationwise, we denote by $\mathbf{M}^N_{\text{sym}}$ the space of symmetric $N \times N$-matrices and by $\cdot$ the Euclidean inner product between vectors in $\mathbb{R}^N$ or the Fröbenius inner product between elements of $\mathbf{M}^N_{\text{sym}}$, that is $e \cdot e' = \text{tr} \, ee'$ wit $e, e' \in \mathbf{M}^N_{\text{sym}}$. We will denote by $B_r(x)$ the open ball of center $x$ and radius $r$. 

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We will sometimes identify the torus and its subsets with the unit cube $Y = \Pi_{i=1,\ldots,N}[0,1) \subset \mathbb{R}^N$ and the corresponding subsets (denoted with the corresponding roman character) through the canonical identification $i$ between $\mathcal{T}$ and $Y$. Also we will adopt the following convention for a function $\zeta$ defined on $\mathcal{T}$. We will say that $\zeta \in H^1_1(\mathcal{T})$ if, and only if, $z = \zeta \circ i$ is such that $z \in H^1_{\text{loc}}(\mathbb{R}^N)$ and is $Y$-periodic. Further, if $\varepsilon \in L^\infty(\mathcal{T};\mathbb{M}_N^{\text{sym}})$, we will write $\text{div}\varepsilon \nabla \zeta$ for $\text{div}\{\varepsilon \circ i \nabla z\}$ and denote by $\zeta^\delta$ the periodic $H^1_{\text{loc}}$-function $z(x/\delta)$ which we will also write as $\zeta(x/\delta)$.

The rest of the notation is standard.

2. Classical homogenization of the dielectrics

In this section, we consider the dielectric part of our problem and propose to pass to the limit as the period goes to 0. As already noted, structural assumptions such as periodicity (a random distribution with good enough mixing properties would do as well), while essential in the next section, are not necessary assumptions when handling the scalar dielectric equation. We restrict our analysis to that case for simplicity but an adaptation of the results of this Section to the general setting of $H$-convergence would be straightforward (if starting directly from (2.4), rather than from (2.1)).

So, on $\Omega$, a bounded Lipschitz domain of $\mathbb{R}^N$, we consider the equation

\[
\begin{cases}
\text{div}\varepsilon^\delta \nabla \varphi^\delta = \frac{1}{\delta} g(y) f(x) \\
\varphi^\delta = \phi \text{ on } \partial \Omega
\end{cases}
\] (2.1)

with $f \in W^{1,\infty}(\Omega)$, $g \in L^2(\mathcal{T};\mathbb{R}^N)$ and $\int_\mathcal{T} g(y) \, dy = 0$, $\phi \in H^2(\partial \Omega)$ and $\varepsilon^\delta(x) := \varepsilon(\frac{x}{\delta})$ where $\varepsilon(y) \in L^\infty(\mathcal{T};\mathbb{M}_N^{\text{sym}})$ with $\gamma|\xi|^2 \leq \varepsilon(y)\xi \cdot \xi \leq \beta|\xi|^2$ for some $0 < \gamma < \beta < \infty$.

We define $\psi$ to be the unique solution in $H^1(\mathcal{T})$ of

\[
\begin{cases}
\Delta \psi(y) = g(y) \\
\int_\mathcal{T} \psi(y) \, dy = 0
\end{cases}
\] (2.2)

and note that, by elliptic regularity, $\psi \in H^2(\mathcal{T})$. We set

$$
\tau(y) := \nabla \psi(y), \quad \tau^\delta(x) := \nabla \psi(\frac{x}{\delta}),
$$

so that (2.1) reads as

\[
\begin{cases}
\text{div}(\varepsilon^\delta \nabla \varphi^\delta - f \tau^\delta) = -\tau^\delta \cdot \nabla f \\
\varphi^\delta = \phi \text{ on } \partial \Omega
\end{cases}
\] (2.4)

From (2.4) and Poincaré’s inequality, we immediately obtain that $\varphi^\delta$ is bounded in $H^1(\Omega)$ independently of $\delta$ and, upon setting

$$
q^\delta := \varepsilon^\delta \nabla \varphi^\delta - f \tau^\delta,
$$

that $q^\delta$ is bounded in $L^2(\Omega;\mathbb{R}^N)$ independently of $\delta$.

Thus, up to a subsequence (not relabeled), we conclude that

\[
\begin{cases}
\varphi^\delta \rightharpoonup \varphi \text{ weakly in } H^1(\Omega) \\
q^\delta \rightharpoonup q \text{ weakly in } L^2(\Omega;\mathbb{R}^N).
\end{cases}
\] (2.5)

Of course,

$$
\text{div} q = 0
$$

(2.6)

since $\tau^\delta L^2(\Omega;\mathbb{R}^N) \to \int_\mathcal{T} \nabla \psi(y) \, dy = 0$. It remains to identify $q$. 


To that effect, consider the periodic corrector \( w_j \) defined as follows. Set \( \chi_j \) to be the unique solution in \( H^1(T) \) to
\[
\begin{align*}
\text{div} \varepsilon \nabla (\chi_j + y_j) &= 0 \\
\int_T \chi_j \, dy &= 0.
\end{align*}
\tag{2.7}
\]
Then
\[
w_j := \chi_j + y_j
\]
Set
\[
w_j^\delta(x) := \delta \chi_j(x + \frac{x}{\delta}) + x
\]
and note that \( \nabla w_j^\delta(x) = (\nabla w)(\frac{x}{\delta}) \). Then, on the one hand, the div-curl Lemma [15] (or integration by parts) implies that, for any \( \zeta \in C^\infty_c(\Omega) \),
\[
\int_\Omega \zeta \cdot \nabla w_j^\delta \, dx \to \int_\Omega \zeta \cdot \tilde{e}_j \, dx = \int_T \zeta_{q_j} \, dy.
\]
On the other hand define, according to classical elliptic homogenization [3, Chapter 1], the symmetric constant matrix \( \varepsilon^h \) as
\[
\varepsilon^h \tilde{e}_j := \int_T \varepsilon(y) \nabla w_j \, dy.
\tag{2.9}
\]
Since \( \varepsilon \) is symmetric, another application of the div-curl Lemma yields
\[
\int_\Omega \zeta \cdot \nabla w_j^\delta \, dx = \int_\Omega \zeta \nabla \varphi \cdot \varepsilon^h \tilde{e}_j \, dx - \int_\Omega \zeta f \cdot \nabla w_j^\delta \, dx \to \int_\Omega \zeta \nabla \varphi \cdot \varepsilon^h \tilde{e}_j \, dx - \int_\Omega \zeta f a_j \, dx,
\]
with
\[
a_j := \int_T \tau(y) \cdot \nabla w_j(y) \, dy = \int_T \nabla \psi(y) \cdot (\tilde{e}_j + \nabla \chi_j(y)) \, dy = - \int_T g(y) \chi_j(y) \, dy.
\tag{2.10}
\]
Hence,
\[
q = \varepsilon^h \nabla \varphi - a f
\tag{2.11}
\]
with \( a \in \mathbb{R}^N \) given through (2.10).

A classical result of \( H \)-convergence is that
\[
\gamma |\xi|^2 \leq \varepsilon^h \xi \cdot \xi \leq \beta |\xi|^2.
\]
Thus, in view of (2.6), we conclude that \( \varphi \) is the unique \( H^1(\Omega) \)-solution of
\[
\begin{align*}
\text{div} \varepsilon^h \nabla \varphi &= a \cdot \nabla f \\
\varphi &= \phi \text{ on } \partial \Omega,
\end{align*}
\tag{2.12}
\]
so that the entire sequence \( (\varphi^\delta, q^\delta) \) converges to \( (\varphi, q) \) weakly in \( H^1(\Omega) \times L^2(\Omega; \mathbb{R}^N) \).

We now strive to improve the weak convergence results with the help of correctors. To that effect we introduce \( \theta \in H^1(T) \) to be the unique solution to
\[
\begin{align*}
\text{div}(\varepsilon \nabla \theta - \tau) &= 0 \\
\int_T \theta(y) \, dy &= 0.
\end{align*}
\tag{2.13}
\]
We set
\[
\sigma := \varepsilon \nabla \theta - \tau,
\]
and
\[
\theta^\delta(x) := \delta \theta \left( \frac{x}{\delta} \right) \quad \sigma^\delta(x) := \sigma \left( \frac{x}{\delta} \right).
\tag{2.14}
\]
so that \( \sigma^\delta = \varepsilon^\delta \nabla \theta^\delta - \tau^\delta \) and note that, by symmetry of \( \varepsilon \) and \( \theta \) of (2.7), since \( \tau \) has zero average over \( \mathcal{F} \) and in view of (2.10),

\[
\sigma^\delta_j = b_j := \int_{\mathcal{F}} \varepsilon(y) \nabla \theta(y) \cdot \nabla y_j \ dy = -\int_{\mathcal{F}} \nabla \theta(y) \cdot \varepsilon(y) \nabla x_j(y) \ dy
\]

\[
= -\int_{\mathcal{F}} (\varepsilon(y) \nabla \theta(y) - \tau(y)) \cdot \nabla x_j(y) \ dy - \int_{\mathcal{F}} \tau(y) \cdot \nabla x_j(y) \ dy
\]

\[
= -\int_{\mathcal{F}} \tau(y) \cdot \nabla x_j(y) \ dy = -\int_{\mathcal{F}} \tau(y) \cdot \nabla w_j(y) \ dy = -a_j, \text{ weakly in } L^2(\Omega; \mathbb{R}^N). \tag{2.15}
\]

We now follow a classical computation; see e.g. [4, section 4]. Take \( \Phi \) in \( C^\infty_c(\Omega; \mathbb{R}^N) \), \( \zeta, \eta \leq 1 \) in \( C^\infty_c(\Omega) \) and compute

\[
\int_{\Omega} \eta^2 \varepsilon^\delta \left( \nabla \varphi^\delta - \int_{j=1,\ldots,N} \nabla w_j^\delta \Phi_j - \nabla \theta^\delta \zeta \right) \cdot \left( \nabla \varphi^\delta - \int_{j=1,\ldots,N} \nabla w_j^\delta \Phi_j - \nabla \theta^\delta \zeta \right) \ dx =
\]

\[
\int_{\Omega} \eta^2 \left\{ \varepsilon^\delta \nabla \varphi^\delta - \tau^\delta f \right\} - \sum_{j=1,\ldots,N} \varepsilon^\delta \nabla w_j^\delta \Phi_j - \zeta (\varepsilon^\delta \nabla \theta^\delta - \tau^\delta) \right\} \cdot \left( \nabla \varphi^\delta - \int_{j=1,\ldots,N} \nabla w_j^\delta \Phi_j - \nabla \theta^\delta \zeta \right) \ dx
\]

\[
+ \int_{\Omega} \eta^2 (f - \zeta) \tau^\delta \cdot \left( \nabla \varphi^\delta - \int_{j=1,\ldots,N} \nabla w_j^\delta \Phi_j - \nabla \theta^\delta \zeta \right) \ dx
\]

\[
= \int_{\Omega} \eta^2 \left\{ \left( q^\delta - \sum_{j=1,\ldots,N} \varepsilon^\delta \nabla w_j^\delta \Phi_j - \zeta \sigma^\delta \right) + (f - \zeta) \tau^\delta \right\} \cdot \left( \nabla \varphi^\delta - \int_{j=1,\ldots,N} \nabla w_j^\delta \Phi_j - \nabla \theta^\delta \zeta \right) \ dx.
\]

Multiple applications of the div-curl Lemma, together with (2.11), (2.15), imply that

\[
\int_{\Omega} \eta^2 \left( q^\delta - \sum_{j=1,\ldots,N} \varepsilon^\delta \nabla w_j^\delta \Phi_j - \zeta \sigma^\delta \right) \cdot \left( \nabla \varphi^\delta - \int_{j=1,\ldots,N} \nabla w_j^\delta \Phi_j - \nabla \theta^\delta \zeta \right) \ dx \overset{\delta}{\to}
\]

\[
\int_{\Omega} \eta^2 \left( q - \varepsilon^\delta \Phi + \zeta \alpha \right) \cdot (\nabla \varphi - \Phi) \ dx = \int_{\Omega} \eta^2 \left( \varepsilon^\delta (\nabla \varphi - \Phi) + \alpha (\zeta - f) \right) \cdot (\nabla \varphi - \Phi) \ dx. \tag{2.16}
\]

Further,

\[
\tau^\delta \cdot \nabla \varphi^\delta = (\tau^\delta - \varepsilon^\delta \nabla \theta^\delta) \cdot \nabla \varphi^\delta + (\varepsilon^\delta \nabla \varphi^\delta - f \tau^\delta) \cdot \nabla \theta^\delta + f \tau^\delta \cdot \nabla \theta^\delta = -\sigma^\delta \cdot \nabla \varphi^\delta + q^\delta \cdot \nabla \theta^\delta + f \tau^\delta \cdot \nabla \theta^\delta,
\]

so that, setting

\[
\kappa := \int_{\mathcal{F}} \tau(y) \cdot \nabla \theta(y) \ dy, \tag{2.17}
\]

the div-curl Lemma implies, in view of (2.15), that

\[
\tau^\delta \cdot \nabla \varphi^\delta \rightarrow (a \cdot \nabla \varphi + \kappa f), \text{ weakly-* in } M_b(\Omega). \tag{2.18}
\]

**Remark 2.1.** Note for later use that, upon multiplication of the first equation in (2.13) by \( \theta \) and integration over \( \mathcal{F} \), we get \( \kappa = \int_{\mathcal{F}} \varepsilon(y) \nabla \theta(y) \cdot \nabla \theta(y) \ dy > 0 \).

Hence, since \( f \) is in particular continuous,

\[
\int_{\Omega} \eta^2 (f - \zeta) \tau^\delta \cdot \left( \nabla \varphi^\delta - \int_{j=1,\ldots,N} \nabla w_j^\delta \Phi_j - \nabla \theta^\delta \zeta \right) \ dx \overset{\delta}{\to}
\]

\[
\int_{\Omega} \eta^2 (f - \zeta) (a \cdot \nabla \varphi + \kappa f - a \cdot \nabla \Phi - \kappa \zeta) \ dx = \int_{\Omega} \eta^2 (f - \zeta) (a \cdot (\nabla \varphi - \Phi) + \kappa(f - \zeta)) \ dx. \tag{2.19}
\]
Summing the contributions \((\ref{2.16})\) and \((\ref{2.19})\), we finally obtain
\[
\lim_{\delta \to 0} \int_{\Omega} \eta^2 \varepsilon^2 \left( \nabla \varphi^\delta - \sum_{j=1}^{N} \nabla w_j \Phi_j - \nabla \theta^\delta \zeta \right) \cdot \left( \nabla \varphi^\delta - \sum_{j=1}^{N} \nabla w_j \Phi_j - \nabla \theta^\delta \zeta \right) \, dx
\]
\[
= \int_{\Omega} \eta^2 \left\{ (\varepsilon^h(\nabla \varphi - \Phi) + a(\zeta - f)) \cdot (\nabla \varphi - \Phi) + (f - \zeta)(a \cdot (\nabla \varphi - \Phi) + \kappa(f - \zeta)) \right\} \, dx
\]
\[
= \int_{\Omega} \eta^2 (\varepsilon^h(\nabla \varphi - \Phi) \cdot (\nabla \varphi - \Phi) + \kappa(f - \zeta)(f - \zeta)) \, dx.
\]
From this and the coercivity of \(\varepsilon(y)\) we conclude that, for some \(C > 0\),
\[
\gamma \limsup_{\delta} \int_{\Omega} \eta^2 \left( \nabla \varphi^\delta - \sum_{j=1}^{N} \nabla w_j \Phi_j - \nabla \theta^\delta \zeta \right) \cdot \left( \nabla \varphi^\delta - \sum_{j=1}^{N} \nabla w_j \Phi_j - \nabla \theta^\delta \zeta \right) \, dx
\]
\[
\leq C \|\nabla \varphi - \Phi\|_{L^2(\Omega;\mathbb{R}^N)}^2 + \|f - \zeta\|_{L^2(\Omega)}^2.
\]
Now, assuming that
\[
\left\{\begin{array}{c}
\partial \Omega \in C^{2,\alpha}, \quad 0 < \alpha < 1 \\
\phi \in C^{2,\alpha}(\partial \Omega) \\
f \in C^{1,\alpha}(\Omega)
\end{array}\right.
\]  \quad (2.21)
Schauder elliptic regularity applied to \((\ref{2.12})\) yields that \(\varphi \in C^{2,\alpha}(\overline{\Omega})\). Then, for any \(\lambda > 0\) we can find \(\phi, \zeta\) such that
\[
\|\nabla \varphi - \Phi\|_{C^0(\overline{\Omega})} + \|f - \zeta\|_{C^0(\overline{\Omega})} \leq \lambda
\]
so that, because \(\nabla w_j, \nabla \theta^\delta\) are bounded in \(L^2(\Omega)\) independently of \(\delta\), \((\ref{2.20})\) together with the arbitrariness of \(\eta\) implies that
\[
\nabla \varphi^\delta - \sum_{j=1}^{N} \nabla w_j \frac{\partial \varphi}{\partial x_j} - \nabla \theta^\delta f \to 0, \quad \text{strongly in } L^2_{\text{loc}}(\Omega, \mathbb{R}^N).
\]

We have proved the following
\begin{theorem}
Under assumptions \((\ref{2.21})\), \(\varphi^\delta\), unique \(H^1(\Omega)\)-solution to \((\ref{2.1})\) is such that
\[
\nabla \varphi^\delta - \sum_{j=1}^{N} \nabla w_j \frac{\partial \varphi}{\partial x_j} - \nabla \theta^\delta f \to 0, \quad \text{strongly in } L^2_{\text{loc}}(\Omega, \mathbb{R}^N)
\]
with \(w_j^\delta\) defined in \((\ref{2.8})\) and \(\theta^\delta\) defined in \((\ref{2.14})\).
\end{theorem}

\begin{remark}
If we also assume that
\[
g \in C^{0,\alpha}(\mathcal{F})
\]  \quad (2.22)
for some \(0 < \alpha < 1\), then elliptic regularity applied to \((\ref{2.2})\) implies that \(\tau \in C^{1,\alpha}(\mathcal{F}; \mathbb{R}^N)\), and thus that, in particular, the convergence in \((\ref{2.18})\) takes place weakly in \(L^2(\Omega)\). Because of that, we do not need the compactly supported smooth test \(\eta\) in all prior computations and the result of Theorem \((\ref{2.2})\) becomes
\[
\nabla \varphi^\delta - \sum_{j=1}^{N} \nabla w_j \frac{\partial \varphi}{\partial x_j} - \nabla \theta^\delta f \to 0, \quad \text{strongly in } L^2(\Omega, \mathbb{R}^N).
\]
Note that the fact that \(\varphi^\delta\) satisfies Dirichlet boundary conditions is essential in evaluating the limit of the term \(\int_{\Omega} \varepsilon^h \nabla \varphi^\delta \cdot \nabla \varphi^\delta \, dx\) in all previous computations.
\end{remark}

\begin{remark}
In the case of a two-phase microstructure, that is whenever
\[
\varepsilon(y) := \chi_{\mathcal{M}} \varepsilon + (1 - \chi_{\mathcal{M}}) \varepsilon, \quad \text{with } \mathcal{F} := \mathcal{F} \setminus \mathcal{M}
\]
\end{remark}
where $\mathcal{M}$ is a measurable subset of $\mathcal{T}$, we can modify the definition of $\varepsilon^\delta$ so that no inclusion intersects $\partial \Omega$. We define

$$\Omega_\delta := \bigcup_{z \in \mathbb{Z}^d \text{ s.t. } \delta(z + 2Y) \subset \Omega} \delta(z + 2Y),$$

note that $|\Omega \setminus \Omega_\delta| \leq C \delta$ and set

$$\varepsilon^\delta(x) := \varepsilon(\frac{x}{\delta}) 1_{\Omega_\delta} + \varepsilon_{\mathcal{M}} 1_{\Omega \setminus \Omega_\delta}.$$

With this definition of $\varepsilon^\delta$, the results of Section 2 still hold true with trivial modifications of the proofs.

Unfortunately, as described in the introduction, this result is not sufficient, when plugged into the equations of elasticity, to ensure that $\nabla \phi^\delta$ can be replaced by $\sum_{j=1}^N \nabla w_j^\delta \frac{\partial^2}{\partial x_j^2} + \nabla \theta^\delta f$ in those, or that one can perform any kind of homogenization process on the resulting system. This is why the next Section is devoted to an improvement of Theorem 2.2. The framework required for the successful completion of such a task will be much more constrained than that in the current Section. In particular, periodicity, which was merely convenience so far, will become essential. Equally essential will be the assumption that the microstructure is bi-phasal.

3. Improved estimates and correctors result for the dielectrics

Throughout this Section, we assume that $\partial \Omega \in C^{2,\alpha}$ for some $0 < \alpha < 1$. The unit torus $\mathcal{T}$ is of the form

$$\mathcal{T} = \mathcal{M} \cup \mathcal{I}, \quad \mathcal{M} \text{ closed such that } \partial \mathcal{M} \text{ is } C^{1,\beta}, \quad 0 < \beta < 1$$

$$\mathcal{I} = \mathcal{T} \setminus \mathcal{M}$$

and $\cup_{k \in \mathbb{Z}^2} k + \mathcal{I} (\mathcal{M})$ is a connected subset of $\mathbb{R}^N$ (a matrix phase).

Further, $\varepsilon(y) := \chi_{\mathcal{M}} \varepsilon_{\mathcal{M}} + (1 - \chi_{\mathcal{M}}) \varepsilon_{\mathcal{I}}$, with $\gamma|\xi|^2 \leq \varepsilon_{\mathcal{M},\mathcal{I}} \xi \cdot \xi \leq \gamma'|\xi|^2$. (3.2)

We define $\varepsilon^\delta$ as in Remark 2.4.

For some $0 < \alpha < \frac{2}{(\beta+1)N}$, we also assume that $f \in C^{1,\alpha}(\Omega)$, that $\psi$, the solution to (2.2), satisfies

$$\psi \in C^{1,\alpha}(\mathcal{T}; Y)^N,$$ (3.3)

and that $\phi$ is the restriction to $\partial \Omega$ of a function of $C^{1,\alpha}(\Omega)$ (still denoted by $\phi$). Remark that the assumed regularity of $\psi$ will be achieved if, e.g., $g \in C^{0,\alpha}(\mathcal{T})$.

In a first step, we prove $\delta$-independent $L^q$-estimates on $\nabla \phi^\delta$ for any $1 \leq q < \infty$. This is the object of the following

**Proposition 3.1.** For all $1 \leq q < \infty$, the sequence $\nabla \phi^\delta$ is bounded in $L^q(\Omega; \mathbb{R}^N)$, independently of $\delta$.

**Proof.** Step 1. First, we apply the large-scale Calderón-Zygmund estimates of [1] Theorem 7.7] to (2.4) rewritten as

$$\begin{align*}
\text{div } \varepsilon^\delta \nabla (\phi^\delta - \phi) &= \text{div}(f \tau^\delta - \varepsilon^\delta \nabla \phi + \nabla \xi^\delta) \\
\phi^\delta - \phi &= 0 \quad \text{on } \partial \Omega,
\end{align*}$$

(3.4)

where $\xi^\delta$ is the unique $H^1_0(\Omega)$-solution to

$$\Delta \xi^\delta = \nabla f \cdot \tau^\delta.$$

We get, for all $\delta > 0$ and $q \geq 2$,

$$\int_\Omega \left( \int_{B_{\delta}(z)} \chi_\Omega(z)|\nabla(\phi^\delta - \phi)|^2(z) \, dz \right)^{\frac{q}{2}} \, dx \leq C_q \left( ||\nabla(\phi^\delta - \phi)||^q_{L^2(\Omega)} + ||f \tau^\delta - \varepsilon^\delta \nabla \phi + \nabla \xi^\delta||^q_{L^q(\Omega)} \right).$$
Note that, since the coefficients are periodic, the random variable involved in [1] Theorem 7.7 is simply a constant (and in fact δ) in the present setting.

Now, since ψ is in $C^{1,\alpha}(\mathcal{T};\mathbb{R}^N)$, $\tau^\delta$ given by (2.3) is bounded in $C^{0,\alpha}(\mathcal{T};\mathbb{R}^N)$ independently of $\delta$ and elliptic regularity implies that $N\xi^\delta$ is bounded in $C^{1,\alpha}(\Omega;\mathbb{R}^N)$ independently of $\delta$.

Since by Jensen’s inequality,
\[
\int_{\Omega} \left( \int_{B_{\delta}(x)} \chi_{\Omega} |\nabla f|^q \, dz \right)^{\frac{2}{q}} \, dx \leq \int_{\Omega^d} \int_{B_{\delta}(x)} \chi_{\Omega} |\nabla f|^q \, dz \, dx = \int_{\Omega^d} \int_{\Omega} \chi_{\Omega} |\nabla f|^q \, dx \, dx = \int_{\Omega} |\nabla f|^q \, dx,
\]
the first convergence in (2.5) and the assumed regularity of the functions $f$ and $\psi$ finally yield
\[
\int_{\Omega} \left( \int_{B_{\delta}(x)} \chi_{\Omega} |\nabla \varphi|^q \, dz \right)^{\frac{2}{q}} \, dx \leq C_q \tag{3.5}
\]
for some constant $C_q$ depending on $q$ and on $\|f\|_{C^{1,\alpha}(\Omega)}$, $\|\psi\|_{C^{1,\alpha}(\mathcal{T})}$, $\|\phi\|_{C^{1,\alpha}(\Omega)}$. Estimate (3.5) enables us to control $N\varphi^\delta$ on scales larger than $\delta$. In a second step, we will derive an estimate for small scales, that is for scales smaller than $\delta$.

**Step 2.** In this step, we crucially use the two-phase character of the microstructure, as well as the $C^{1,\beta}$-regularity of the boundary of each of those phases. Take a point $x \in \Omega$ and consider the cube $Q_{2\delta}(x)$ of side-length $2\delta$ centered at $x$. We blow up equation (2.4) so as to obtain an equation on $Q_{2}(0)$.

To that effect, we set
\[
\Phi^\delta(z) := \frac{1}{\delta} \varphi^\delta(x + \delta z).
\]
Then $\Phi^\delta$ satisfies
\[
\text{div}(\varepsilon(\frac{x}{\delta} + z) \nabla \Phi^\delta - f(x + \delta z) \tau^\delta(x + \delta z)) = -\tau^\delta(x + \delta z) \nabla f(x + \delta z) \text{ in } ((\Omega - \{x\})/\delta \cap Q_2(0)). \tag{3.6}
\]
Since $\partial \Omega$ is $C^{2,\alpha}$ and $\partial \mathcal{T}$ is $C^{1,\beta}$, an elementary geometric argument shows there is domain $\Omega'$ containing $Q_1(0)$ such that $\partial(\Omega - \{x\})/\delta \cap \Omega'$ is $C^{1,\beta}$ uniformly in $x$ and $\delta$. Now, the set $(\Omega - \{x\})/\delta \cap \Omega'$ contains a number of inclusions bounded uniformly wrt $\delta$ : if there are $p$ inclusions in $\mathcal{T}$, that is if $\mathcal{T}$ has $p$ connected components, then that set contains at most $2^Np$ inclusions. Since by assumption the boundary of those inclusions is also $C^{1,\beta}$, then, in the terminology of [13], the $C^{1,\beta}$ modulus $K$ of $(\Omega - \{x\})/\delta \cap \Omega'$ does not depend on $\delta$ and [13] Theorem 1.1 applies.

Considering, for $\eta > 0$, the set $\Omega_{\delta,\eta}' = \{z \in \Omega - \{x\}/\delta \cap Q_1(0) : \text{dist}(z, \partial(\Omega - \{x\})/\delta \cap Q_1(0)) > \eta\}$ we conclude in particular that, for any $c \in \mathbb{R}$,
\[
\|\nabla_z \Phi^\delta\|_{L^\infty(\Omega_{\delta,\eta}')} \leq C_{K,\eta} \left( \|\Phi^\delta - c\|_{L^\infty(\Omega_{\delta,\eta}/\delta)} + C' \right), \tag{3.7}
\]
where $C_{K,\eta}$ is a constant that only depends on $K, \alpha, \beta, \gamma, \gamma', \eta$ while $C'$ is a constant that only depends on $\|f\|_{C^{1,\alpha}(\Omega)}$, $\|\psi\|_{C^{1,\alpha}(\mathcal{T})}$, $\|\phi\|_{C^{1,\alpha}(\Omega)}$.

We now apply De Giorgi-Nash-Moser’s theorem [9] Theorem 8.24 to (3.6). This yields in turn the following interior estimate:
\[
\|\Phi^\delta - c\|_{L^\infty(\Omega_{\delta,\eta}/\delta)} \leq C_q(\|\Phi^\delta - c\|_{L^2((\Omega - \{x\})/\delta \cap Q_1(0))} + C''), \tag{3.8}
\]
where $C_q$ depends only on $\eta, \gamma, \gamma'$ and $C''$ depends only on $\|f\|_{C^{1,\alpha}(\Omega)}$, $\|\psi\|_{C^{1,\alpha}(\mathcal{T})}$, $\|\phi\|_{C^{1,\alpha}(\Omega)}$. Inserting (3.8) into (3.7) yields
\[
\|\nabla_z \Phi^\delta\|_{L^\infty(\Omega_{\delta,\eta}')} \leq C_{K,\eta} \left( \|\Phi^\delta - c\|_{L^2((\Omega - \{x\})/\delta \cap Q_1(0))} + C''' \right),
\]
where $C_{K,\eta}$ is a constant that only depends on $K, \alpha, \beta, \gamma, \gamma', \eta$ while $C'''$ is a constant that only depends on $\|f\|_{C^{1,\alpha}(\Omega)}$, $\|\psi\|_{C^{1,\alpha}(\mathcal{T})}$, $\|\phi\|_{C^{1,\alpha}(\Omega)}$.

Now choose $c := \int_{L^2((\Omega - \{x\})/\delta \cap Q_1(0))} \Phi^\delta(z) \, dz$ and apply Poincaré-Wirtinger’s inequality to the previous estimate. We obtain
\[
\|\nabla_z \Phi^\delta\|_{L^\infty(\Omega_{\delta,\eta}')} \leq C_{K,\eta} \left( \|\nabla \Phi^\delta\|_{L^2((\Omega - \{x\})/\delta \cap Q_1(0))} + C'''' \right). \tag{3.9}
\]
Blowing \([3.9]\) down, we conclude in particular that, for a \(d > 0\) large enough, we have, for all \(\delta > 0\),

\[
\sup_{B_{\delta/4}(x)} |\nabla \varphi^\delta| \leq C'_{K,\eta} \left( \left( \int_{\Omega \cap B_{\delta}(x)} |\nabla \varphi^\delta(y)|^2 \, dy \right)^{\frac{1}{2}} + C'' \right).
\]  

(3.10)

**Step 3.** We combine the estimates obtained in the first two steps as follows.

Remark that, for all \(x \in \Omega\) and all small enough \(\delta\)'s, the assumed regularity of \(\partial \Omega\) implies the existence of \(d > 0\) such that \(|B_{\delta}(x) \cap \Omega| \geq d|B_{\eta}(x)|\). From \((3.5)\) we then get

\[
\int_\Omega \left( \int_{\Omega \cap B_{\delta}(x)} |\nabla \varphi^\delta(y)|^2 \, dy \right)^{\frac{3}{2}} \, dx \leq d^{-\frac{3}{2}} \int_\Omega \left( \int_{B_{\delta}(x)} \chi_{\Omega}(y)|\nabla \varphi^\delta(y)|^2 \, dy \right)^{\frac{3}{2}} \, dx \leq d^{-\frac{3}{2}} C_q.
\]

In turn, from \((3.10)\), we obtain

\[
\int_\Omega |\nabla \varphi^\delta|^q \, dx \leq \int_\Omega \left( \sup_{B_{\delta/4}(x)} |\nabla \varphi^\delta|^q \right) \, dx
\]

\[
\leq 2^q(C'_{K,\eta})^q \left( \int_\Omega \left( \int_{\Omega \cap B_{\delta}(x)} |\nabla \varphi^\delta(y)|^2 \, dy \right)^{\frac{3}{2}} \, dx + |\Omega|(C'')^q \right).
\]

Combining the inequalities above, we finally conclude that

\[
\int_\Omega |\nabla \varphi^\delta|^q \, dx \leq 2^q(C'_{K,\eta})^q \left( d^{-\frac{3}{2}} C_q + |\Omega|(C'')^q \right).
\]

This completes the proof of the proposition. \(\square\)

We are now in a position to improve on the convergence result of Theorem 2.2. We obtain the following

**Theorem 3.2.** Under assumptions \((2.21), (2.22), (3.1), (3.2), \varphi^\delta, \) unique \(H^1(\Omega)\)-solution to \((2.1)\) is such that, for any \(1 \leq q < \infty\),

\[
\nabla \varphi^\delta - \sum_{j=1,..,N} \nabla w_j^\delta \frac{\partial \varphi}{\partial x_j} - \nabla \theta^\delta f \rightarrow 0, \quad \text{strongly in } L^q(\Omega, \mathbb{R}^N)
\]

with \(w_j^\delta\) defined in \((2.8)\) and \(\theta^\delta\) defined in \((2.14)\).

**Proof.** First note that the regularity assumptions on the domain, \(f\) and \(\phi\) and classical Schauder regularity imply that \(\varphi\), the solution to \((2.12)\), is in \(C^{2,\alpha}(\Omega)\). Further, the regularity assumption on \(g\) and classical Schauder regularity imply that \(\tau \in C^{1,\alpha}(\mathcal{F})\). Then another application of \([13\) Theorem 1.1\), this time on \(\mathcal{F}\) which has no boundary, implies in particular that

\[
\nabla \theta, \nabla \chi_j \in L^\infty(\mathcal{F}),
\]

(3.11)

hence \(\nabla w_j\) as well. We can thus assume that, for any \(r \geq 1\), the term

\[
\sum_{j=1,..,N} \nabla w_j^\delta \frac{\partial \varphi}{\partial x_j} + \nabla \theta^\delta f \text{ is bounded in } L^r(\Omega) \text{ independently of } \delta.
\]

Set \(\theta = 1/(q - 1)\). In view of Remark 2.3 and Proposition 3.1, the uniform bound derived above yields that, for some constant \(C\) depending on \(q\) and all the data,

\[
\|(\nabla \varphi^\delta - \sum_{j=1,..,N} \nabla w_j^\delta \frac{\partial \varphi}{\partial x_j} - \nabla \theta^\delta f)\|_{L^q(\Omega)} \leq \|\nabla \varphi^\delta - \sum_{j=1,..,N} \nabla w_j^\delta \frac{\partial \varphi}{\partial x_j} - \nabla \theta^\delta f\|_{L^2(\Omega)} \times
\]

\[
\|\nabla \varphi^\delta - \sum_{j=1,..,N} \nabla w_j^\delta \frac{\partial \varphi}{\partial x_j} - \nabla \theta^\delta f\|_{L^{2q}(\Omega)} \leq C\|\left(\nabla \varphi^\delta - \sum_{j=1,..,N} \nabla w_j^\delta \frac{\partial \varphi}{\partial x_j} - \nabla \theta^\delta f\right)\|_{L^{2q}(\Omega)}^{-\theta} \to 0.
\]

Hence the result. \(\square\)
Remark 3.3. We have assumed throughout this Section that the composite is made of two phases. Nothing would change if we considered \( n \) phases instead of 2, provided that we keep the same regularity assumptions. Theorem 3.2 would still hold true, and Theorem 4.1 below as well.

Remark 3.4. All results of Sections 2, 3 remain valid if a source term of the form \( h^\delta v \) is added to the right hand-side of (1.2) with \( h \in L^\infty(\mathcal{T}) \) and \( v \in L^\infty(\Omega) \). Then, one has to add the term \((\int_{\mathcal{T}} h(y) \, dy) v\) to the right hand-side of the homogenized equation (2.12). All other results remain unchanged.

4. Homogenization of the elasto-dielectrics

We now address the elasticity part of the problem. Recall that \( L^\delta(x) := L(\frac{x}{\delta}) \) where \( L(y) \) is a measurable, symmetric linear mapping from \( M^N_{\text{sym}} \) into itself with the properties that \( \gamma |e|^2 \leq L(y)e \cdot e \leq \gamma' |e|^2 \) for a.e. \( y \in \mathcal{T} \) and some \( 0 < \gamma < \gamma' < \infty \). Also \( M^\delta(x) := M(\frac{x}{\delta}) \) with \( M \) a bounded, measurable, linear mapping from \( M^N_{\text{sym}} \) into itself.

The equations are

\[
\begin{aligned}
\{ \text{div}(L^\delta \nabla u^\delta + M^\delta (\nabla \varphi^\delta \otimes \nabla \varphi^\delta)) &= 0 \\
u^\delta &= 0 \text{ on } \partial \Omega.
\end{aligned}
\]

(4.1)

We assume that assumptions (2.21), (2.22), (3.1), (3.2) (hence also (3.3)) hold true throughout this Section.

In particular, we can apply Proposition 3.1 and we conclude, with the help of Korn and Poincaré inequalities, that \( u^\delta \) exists and is bounded in \( H^1_0(\Omega; \mathbb{R}^N) \) independently of \( \delta \).

We can also apply Theorem 3.2 and we immediately obtain that, for any \( 0 < r < \infty \),

\[
M^\delta(\nabla \varphi^\delta \otimes \nabla \varphi^\delta) - M^\delta[(\sum_{j=1,..,N} \nabla w^\delta_j \frac{\partial \varphi}{\partial x_j} + \nabla \vartheta^\delta f) \otimes (\sum_{j=1,..,N} \nabla w^\delta_j \frac{\partial \varphi}{\partial x_j} + \nabla \vartheta^\delta f)] \overset{\delta \to 0}{\rightarrow} 0, \text{ strongly in } L^r(\Omega).
\]

(4.2)

We will only use the value \( r = 2 \) hereafter. Set

\[
Z^\delta(x) := M^\delta[(\sum_{j=1,..,N} \nabla w^\delta_j \frac{\partial \varphi}{\partial x_j} + \nabla \vartheta^\delta f) \otimes (\sum_{j=1,..,N} \nabla w^\delta_j \frac{\partial \varphi}{\partial x_j} + \nabla \vartheta^\delta f)].
\]

(4.3)

Because of convergence (4.2), if \( \tilde{u}^\delta \) is the unique \( H^1_0 \)-solution to

\[
\begin{aligned}
\{ \text{div}(L^\delta \nabla \tilde{u}^\delta + Z^\delta) &= 0 \\
\tilde{u}^\delta &= 0 \text{ on } \partial \Omega,
\end{aligned}
\]

(4.4)

then

\[
u^\delta - \tilde{u}^\delta \overset{\delta \to 0}{\rightarrow} 0, \text{ strongly in } H^1_0(\Omega; \mathbb{R}^N).
\]

(4.5)

We undertake a homogenization process for the system (4.4). To that effect, we introduce the periodic corrector \( W_{ij} \) defined as follows. Set \( X_{ij} \) to be the unique solution in \( H^1(\mathcal{T}; \mathbb{R}^N) \) to

\[
\begin{aligned}
\{ \text{div} L \nabla (X_{ij} + x_i e_j) &= 0 \\
\int_{\mathcal{T}} X_{ij} \, dy &= 0.
\end{aligned}
\]

Then set

\[
W_{ij}(y) := X_{ij} + y_i e_j,
\]

and

\[
W^\delta_j(x) := \delta X_{ij}(\frac{x}{\delta}) + x_i e_j
\]

and note that \( \nabla W^\delta_j(x) = (\nabla W)_j(\frac{x}{\delta}) \).

Then an argument near identical to that which led to (2.12) would yield that \( \tilde{u}^\delta \rightharpoonup u \) in \( H^1_0(\Omega; \mathbb{R}^N) \),

(4.6)
with \( u \), unique \( H^1_0(\Omega; \mathbb{R}^N) \)-solution to
\[
\begin{cases}
\text{div}(L^h \nabla u + Z) = 0 \\
u = 0 \text{ on } \partial \Omega,
\end{cases}
\]
with \( L^h \) defined as
\[
L^h_{ij,kh} := \int_{\mathcal{S}} L(y) \nabla W_{ij} \cdot \nabla W_{kh} \, dy
\]
and \( Z \), the \( L^2(\Omega; M_{\text{sym}}^N) \)-weak limit of \( Z^\delta \) defined in (4.3), being
\[
Z := M^h(\nabla \varphi \otimes \nabla \varphi) + 2f N^h \nabla \varphi + P^h f^2
\]
where
\[
\begin{align*}
M^h_{ij,kh} &:= \int_{\mathcal{S}} M(y)(\nabla w_k(y) \otimes \nabla w_h(y)) \cdot \nabla W_{ij}(y) \, dy \\
N^h_{ij,k} &:= \int_{\mathcal{S}} M(y)(\nabla w_k(y) \otimes \nabla \theta(y)) \cdot \nabla W_{ij}(y) \, dy \\
P^h_{ij,k} &:= \int_{\mathcal{S}} M(y)(\nabla \theta(y) \otimes \nabla \theta(y)) \cdot \nabla W_{ij}(y) \, dy.
\end{align*}
\]
Convergences (4.5) and (4.6) imply the following homogenization result

**Theorem 4.1.** Under assumptions (2.21), (2.22), (3.1), (3.2), \( u^\delta \), unique \( H^1(\Omega; \mathbb{R}^N) \)-solution to (4.1), converges weakly in \( H^1(\Omega; \mathbb{R}^N) \) to the unique \( H^1_0(\Omega; \mathbb{R}^N) \)-solution \( u \) to (4.7) with \( L^h \) defined in (4.8) and \( Z \) defined through (4.9), (4.10).

**Remark 4.2.** We could, in the spirit of the previous Sections, provide a corrector result for \( u^\delta \) but will refrain from doing so because of the notational complexity.

**Remark 4.3.** Our result should be compared to that in [18], which investigates the case \( g = 0 \). In such a case \( N^h = P^h = 0 \) and the corrector results of the previous sections are markedly simpler. Also note that the results in [18] are derived under the \textit{a priori} assumption that \( \nabla \varphi^\delta \) is bounded in \( L^4(\Omega; \mathbb{R}^N) \) independently of \( \delta \); no justification for such an estimate is offered in that work.

5. THE CASE FOR ACTIVE CHARGES

The homogenized dielectric equation obtained in (2.12) can be equivalently rewritten as
\[
\text{div}(e^h \nabla \varphi - af) = 0.
\]
Now, in practice there should be several collections of charges, that is, in lieu of a charge of the form \( g^j f \), one should envision a charge of the form \( \sum_{p=1}^{N} g_p f_p \) where each pair \((g_p, f_p)\) is endowed with the same properties, namely \( \int_{\mathcal{S}} g_p(y) \, dy = 0 \), and the necessary regularities of \( g_p \) and \( f \) that were introduced in the previous subsections.

Provided that those are met, the homogenization results remain unchanged by linearity. In particular the homogenized dielectric equation becomes
\[
\begin{cases}
\text{div}(e^h \nabla \varphi - \sum_{p=1}^{N} a_p f_p) = 0 \\
\varphi = \phi \text{ on } \partial \Omega,
\end{cases}
\]
with \( a_p \in \mathbb{R}^N \) defined as (see (2.10))
\[
(a_p)_j := a_{jp} := \int_{\mathcal{S}} \tau_p(y) \cdot \nabla w_j(y) \, dy = \int_{\mathcal{S}} \nabla \psi_p(y) \cdot (\epsilon_j + \nabla \chi_j(y)) \, dy = - \int_{\mathcal{S}} g_p(y) \chi_j(y) \, dy.
\]
In (5.2), \( \tau_p = \nabla \psi_p \) with \( \psi_p \) defined as \( \psi \) was in (2.2) upon replacing \( g \) by \( g_p \).

We can thus view \( a \) as a \( N \times N \)-matrix with \( j, p \) coefficient \( a_{jp} \). Note that that matrix is not necessarily symmetric.
Active charges, if they exist, consist in an appropriate choice of \( f_p \) so that the homogenized dielectric displays an enhancement of its permittivity. In other words, one would like to choose \( f_p = \partial \varphi / \partial x_p \) so that \( \varphi \) is the solution to (5.1). Then (5.1) reads as
\[
\begin{cases}
\text{div } \tilde{\varepsilon}^h \nabla \varphi = 0 \\
\varphi = \phi \text{ on } \partial \Omega
\end{cases}
\] (5.3)
with
\[\tilde{\varepsilon}^h := \varepsilon^h - a.\]
Furthermore, if desiring electric enhancement, and not electric degradation, one should ensure that \( \tilde{\varepsilon}^h \) admits at least one positive eigenvalue with a value greater than those of \( \varepsilon^h \).

In this two-step process, one should first ensure existence of active charges, that is of a field \( \varphi \) that satisfies (5.3). This will clearly be the case if consider small enough micro-charges. Indeed, if
\[
\varepsilon \equiv \varepsilon^h \text{ small enough ensures that } \forall \xi \in \mathbb{R}^N \neq 0, -\sum_{j,p=1,...,N} a_{jp} \xi_j \xi_p \geq 0 \text{ unless } \chi(y) \parallel \xi, \text{ a.e. on } \mathcal{F}, \text{ where } \chi(y) \text{ is the vector with components } \chi_k(y), \text{ } k = 1, ..., N. \text{ In that case, } \chi(\mathcal{F})(y) = \zeta(\mathcal{F}) \chi(y), \zeta(y) \in H^1(\mathcal{F}). \text{ Then, } \zeta \text{ satisfies}
\]
\[
\text{div } \epsilon \nabla (\zeta \xi_k + y_k) = 0 \text{ in } \mathcal{F}, \text{ } k = 1, ..., N.
\]
Recalling (3.2) and the canonical identification between \( \mathcal{F} \) and the unit cell \( Y \), we obtain that
\[
z := \zeta \circ i = z_M \chi_M + z_I (1 - \chi_M) \text{ with } z \text{ periodic satisfying}
\]
\[
\begin{cases}
\Delta z_{M,I} = 0 \text{ in } M \cap S, \text{ resp. } I \cap S \\
(\varepsilon \cdot n - \varepsilon I) \frac{\partial z_{M,I}}{\partial \nu} = 0 \text{ on } \partial M \cap \partial I \cap S, \text{ } k = 1, ..., N
\end{cases}
\] (5.4)
where \( \nu \) is the exterior normal to \( \partial M \). So, since, for every \( y \in \partial M \), \( \nu(y) \neq 0 \), we must have that, for every \( y \in \partial M \cap S \), \( \xi \parallel \nu(y) \). Hence, unless \( \nu(y) \) is independent of \( y \in \partial M \cap S \), it will always be so that
\[
a \text{ is negative definite.}
\] (5.5)
Note that, if \( \omega \equiv 1 \), the case \( \nu(y) \) independent of \( y \) corresponds to rank-one layering. Upon multiplication of \( g_p = \omega \chi_p \) by a large enough factor \( \lambda \) independent of \( p \) we conclude to the existence of large enough charges such that \( \tilde{\varepsilon}^h \) is a symmetric positive definite matrix whose eigenvalues can be arbitrarily large (in absolute value) upon choosing \( \lambda \) large enough. In such a setting, we can always solve (5.3) and obtain a large enhancement.

We have proved the following

**Proposition 5.1.** Under assumptions (2.21), (3.1), (3.2), (5.5) holds true provided that

(i) The data \( g_p \), \( p = 1, ..., N \) is small enough in \( L^\infty \)-norm, in which case the enhancement, or degradation will be small; or,
(ii) The micro-structure is not that of a rank-one layering, that is, \( \chi_M(y) \) is not of the form \( \chi(y \cdot \zeta) \) with \( \chi \) the characteristic function of a closed interval in \([0,1]\) and \( \zeta \) a fixed vector in \( \mathbb{R}^N \) and \( g_p \) is chosen to be \( \lambda \chi_p \) for \( p = 1, \ldots, N \) and \( \lambda > 0 \). Then, \( \varepsilon^h \) can have arbitrarily large positive eigenvalues with an appropriate choice of \( \lambda \); or,

(iii) The interface \( \partial \mathcal{A} \) does not identify with a hyperplane in an open set \( \mathcal{S} \subset \mathcal{I} \) and \( g_p \) is chosen to be \( \lambda \omega \chi_p \) for \( p = 1, \ldots, N \) and \( \lambda > 0 \) with \( \omega \) continuous with support on \( \mathcal{S} \). Then again, \( \varepsilon^h \) can have arbitrarily large positive eigenvalues with an appropriate choice of \( \lambda \).

**Remark 5.2.** Of course, Proposition 5.1 provides no answer to the more useful question of finding a manufacturable set of micro-charges such that enhancement can occur. At present, we do not know how to offer a systematic analysis of that question.

We conclude with an example of significant enhancement that has been proposed in [14] for an assembly of coated spheres. In the spirit of Remark 5.1 we consider the setting illustrated in Figure 2. The charges for a coated sphere centered at the origin and of radius 1 are chosen to be

\[
\sum_{p=1,2,3} g_p f_p \quad \text{with} \quad g_p(x) := q \frac{\chi_i(x)}{r_p} \frac{x_p}{|x|}
\]

where \( \chi_i \) is the characteristic function of the annulus corresponding to the \( i \)-phase, \( r_p \) is the radius of the sphere corresponding to the \( p \)-phase and \( q \) is some positive constant. It is shown in [14] that, for \( f_p = E_p \) with \( E \in \mathbb{R}^3 \) a constant vector, one can embed the coated sphere in the infinite medium \( D_\infty := \mathbb{R}^3 \setminus \bar{B}(0, 1) \) (with associated characteristic function \( \chi_{D_\infty} \)) filled by a homogeneous dielectric with a well-defined value of the constant permittivity matrix \( \varepsilon^h \) so that, upon solving the dielectric equation

\[
\text{div} \varepsilon \nabla \varphi = \sum_{p=1,\ldots,N} g_p E_p \quad \text{in} \quad \mathbb{R}^N
\]

with \( \varepsilon(x) := \chi_{B_1(0)} \sum_{q=p, i, m} \chi_q(x) \varepsilon_q + \chi_{D_\infty} \varepsilon^h \) obtain as electric field in \( D_\infty \) the field \( E \) itself, that is, \( \frac{\partial \varphi}{\partial x_p} = E_p \) on \( D_\infty \). This is the essence of what is usually called the neutral inclusion process.

In [14] the adequate permittivity tensor \( \varepsilon^h \) for that process to be successful is found to be isotropic with permittivity

\[
\left\{
\begin{aligned}
\varepsilon^h := & \varepsilon_{\text{HS}} + q \varepsilon_{\text{add}}. \\
\varepsilon_{\text{HS}} := & \varepsilon_m + \frac{3 \theta_p (\varepsilon_p - \varepsilon_m)}{(2 + \theta_p) \varepsilon_m + (1 - \theta_p) \varepsilon_p} \\
\varepsilon_{\text{add}} := & \left(3 \theta_p^{1/3} (\theta_1 + \theta_p)^{1/3} (2 \varepsilon_m - \varepsilon_p) + \theta_1 (\theta_1 + \theta_p)^{1/3} (2 \varepsilon_m + \varepsilon_p) + 3 \theta_p^{1/3} (\varepsilon_p - 2 \varepsilon_m) \right) \\
& \quad \div 4 \theta_p^{1/3} [(2 + \theta_p) \varepsilon_m + (1 - \theta_p) \varepsilon_p]
\end{aligned}
\right. \quad (5.6)
\]

In (5.6) \( \varepsilon_{\text{HS}} \) is the classical overall conductivity for coated spheres in an infinite medium, often dubbed the Hashin-Shtrikman conductivity, while \( q \varepsilon_{\text{add}} \) is the enhancement (linear in the parameter \( q \)).
It remains to prove that the filling of a unit cube by homothetics of the coated sphere, together with a periodic homogenization process, will produce an overall conductivity that matches \( \tilde{\varepsilon}^h = \varepsilon \text{Id} \), a task which has yet to be fulfilled.

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