BOUNDS STATES OF THE SCHRÖDINGER-NEWTON MODEL IN LOW DIMENSIONS

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Abstract. We prove the existence of quasi-stationary symmetric solutions with exactly \( n \geq 0 \) zeros and uniqueness for \( n = 0 \) for the Schrödinger-Newton model in one dimension and in two dimensions along with an angular momentum \( m \geq 0 \). Our result is based on an analysis of the corresponding system of second-order differential equations.

1. Introduction

We consider the Schrödinger-Newton equations

\[
\begin{aligned}
&i\psi_t + \Delta \psi - \gamma \Phi \psi = 0, \\
&\Delta \Phi = |\psi|^2
\end{aligned}
\]

on \( \mathbb{R}^d, \; d \in \{1, 2\} \), which is equivalent to the nonlinear Schrödinger equation

\[
\begin{aligned}
&i\psi_t + \Delta \psi + \gamma \left( G_d(\|x\|) * |\psi|^2 \right) \psi = 0
\end{aligned}
\]

where \( G_d(\|x\|) \) denotes the Green’s function of the Laplacian on \( \mathbb{R}^d \). Of physical interest are solutions having finite energy \( E \) and particle number (or charge) \( N \) given by

\[
\begin{aligned}
&E(\psi) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \psi(x, t)|^2 \, dx - \frac{\gamma}{4} \iint_{\mathbb{R}^d} G_d(\|x - y\|) |\psi(x, t)|^2 |\psi(y, t)|^2 \, dxdy \\
&N(\psi) = \int_{\mathbb{R}^d} |\psi(x, t)|^2 \, dx,
\end{aligned}
\]

respectively.

In the physical and mathematical literature the Schrödinger-Newton system in three space dimensions has a long standing history. With \( \gamma \) designating appropriate positive coupling constants it appeared first in 1954, then in 1976 and lastly in 1996 for describing the quantum mechanics of a Polaron at rest by S. J. Pekar [1], of an electron trapped in its own hole by Ph. Choquard [2] and of self-gravitating matter by R. Penrose [3]. In 1977, E.Lieb [2] showed the existence of a unique spherically symmetric ground state in three space dimensions by solving an appropriate minimization problem. This ground state solution \( u_\omega(x), \, \omega > 0 \) is a positive spherically symmetric strictly decreasing function. In [4], P.L. Lions proved the existence of infinitely many distinct spherically symmetric solutions and claimed a proof for the existence of anisotropic bound states in [5].

While Lieb’s existence proof can be easily extended to dimensions \( d = 4 \) and \( d = 5 \), the situation has been unclear for lower dimensions due to the lack of...
positivity of the Coulomb interaction energy term. For the one-dimensional and two-dimensional problem this difficulty has been overcome recently in [6] and [7] and the existence of a unique spherically symmetric ground state has been shown by solving a minimization problem. Finally, existence and uniqueness of spherically symmetric ground states has been shown in any dimension $d \leq 6$ in [8] by employing a shooting method for the associated system of ordinary differential equations. In addition, for $d = 1$, the existence of a unique antisymmetric ground states (which is an eigenfunction of the parity-operator) has been shown in [6] and, for $d = 2$, the existence of unique angular excitations (eigenfunction of the angular momentum operator) has been shown in [7]. However, in one and two dimensions, existence of higher bound states remained open along with an angular momentum in the last case and so far only numerical studies are available indicating the existence of excited states, see e.g. [9]. Our main result proves the existence of such solutions in the attractive case $\gamma > 0$.

1.1. The one-dimensional case. We study the existence and uniqueness of quasi stationary solutions of the form

$$\psi(t, x) = \left(\frac{x}{|x|}\right)^p \varphi(|x|) e^{-i\omega t}, \quad \lim_{|x| \to \infty} \varphi(|x|) = 0,$$

where $p = \{0, 1\}$ for an odd or even function. For solutions of the form (1.5) we have $\Phi(t, x) = v(|x|)$ and $\varphi(r), v(r)$ satisfy the following system of ordinary differential equations:

$$\varphi'' = (\gamma v - \omega)\varphi$$
$$v'' = \varphi^2, \quad r \geq 0.$$

We suppose that $v(0)$ are finite and $v'(0) = 0$. In addition $\varphi$ is subject to the initial condition

$$\varphi(0) \in \mathbb{R}_+, \quad \varphi'(0) = 0 \quad \text{if } p = 0$$
$$\varphi(0) = 0, \quad \varphi'(0) \in \mathbb{R}_+ \quad \text{if } p = 1$$

The latter equation of (1.6) implies that $v' \geq 0$, and therefore for solutions $\varphi$ vanishing at infinity we have $\omega - \gamma v(0) > 0$. Changing the variables and rescaling $u(r) = Ar^{-p} \varphi(r/\sigma)$, $V(r) = B(v(r/\sigma) - \omega/\gamma) + 1$ with

$$\sigma = \sqrt{\omega - \gamma v(0)}, \quad A = \frac{\sqrt{\gamma}}{\sigma^2}, \quad B = \frac{\gamma}{\sigma^2},$$

we obtain the following system of equations:

$$u'' + \frac{2p}{r} u' = (V - 1)u$$
$$V'' = u^2 r^{2p}$$

subject to the initial conditions

$$u(0) = u_0 \in \mathbb{R}^+, \quad u'(0) = 0, \quad V(0) = 0, V'(0) = 0.$$
1.2. The two-dimensional case. We search solutions of (1.1) uniformly rotating with an angular velocity $\Omega$. In the rotating frame the equation (1.1) becomes
\[
 i\psi_t + \Delta \psi - \gamma \Phi \psi - \Omega L\psi = 0, \quad \Delta \Phi = |\psi|^2
\]
where $L = -i (x\partial_y - y\partial_x)$, the orbital angular momentum operator.

We study the existence and uniqueness of quasi stationary solutions in the rotating frame of the form
\[
\psi(t, x) = \varphi(|x|) e^{-i\omega t + im \theta}, \quad \lim_{|x| \to \infty} \varphi(|x|) = 0.
\]
If, in addition, $\varphi(|x|) \geq 0$, we call this solution the ground states for a given orbital angular momentum $m$, the other one are called excited states. For solutions of the form (1.9) we have $\Phi(t, x) = v(|x|)$ and $\varphi(r), v(r)$ satisfy the following system of ordinary differential equations:
\[
\varphi'' + \frac{1}{r} \varphi' - \frac{m^2}{r^2} \varphi = (\gamma v - (\omega - \Omega m)) \varphi
\]
(1.10)
\[
v'' + \frac{1}{r} v' = \varphi^2, \quad r \geq 0.
\]

We suppose that $v(0), \phi(0)$ are finite and $v'(0) = \phi'(0) = 0$. The latter equation implies that $v' \geq 0$, and therefore for solutions $\phi$ vanishing at infinity we have $\omega - \Omega m - \gamma v(0) > 0$. Changing the variables and rescaling $u(r) = r^{-m} \phi(r/\sigma), V(r) = B (v(r/\sigma) - (\omega - \Omega m)/\gamma) + 1$ with
\[
\sigma = \sqrt{\omega - \Omega m - \gamma v(0)}, \quad A = \frac{\sqrt{\gamma}}{\sigma^2}, \quad B = \frac{\gamma}{\sigma^2},
\]
we obtain the following system of equations:
\[
u'' + \frac{2(m + 1)}{r} u' = (V - 1) u
\]
(1.11)
\[
V'' + \frac{2}{r} V' = u^2 r^{2m}
\]
subject to the initial conditions (1.8).

1.3. The general case. Equations (1.7), (1.11) are particular cases of the following initial value problem.
\[
 u'' + \frac{2m + d - 1}{r} u' = (V - 1) u
\]
(1.12)
\[
 V'' + \frac{d - 1}{r} V' = u^2 r^{2m}
\]
subject to the initial conditions
\[
 u(0) = u_0 \in \mathbb{R}^+, \quad u'(0) = 0, \quad V(0) = 0, V'(0) = 0.
\]
Here, \( m \geq 0 \) may be regarded as a continuous parameter. By analyzing the solutions of the above initial value problem we shall prove the following result about the existence and uniqueness of ground states and the existence of the excited states:

**Theorem 1.1.** For \( d = 1, 2 \), any \( m \geq 0 \) and \( n \geq 0 \) the system (1.12) subject to the initial conditions (1.13) admits a solution \((u_{m,n}, V_{m,n})\) such that \( u_{m,n} \) has exactly \( n \) zeros and

(1.14) \[ \lim_{r \to \infty} u_{m,n}(r) = 0. \]

Moreover, the ground states \( u_{m,0} \) are unique and satisfies \( u_{m,0}(r) > 0, u'_{m,0}(r) < 0 \) on \([0, \infty[\).

To prove the main result we use an extension of the shooting method we have used in [8]. Shooting methods have been successfully applied to existence and uniqueness of solutions in boundary value problems for second order nonlinear differential equations [11], [12], [13], [14]. Our paper is organized as follows: In Section 2 we employ a shooting method to prove the existence of ground states and by induction the existence of all excited states (theorem 2.8). In Section 3 study their decay properties to prove uniqueness by analyzing the Wronskian of solutions (theorem 3.4).

## 2. Existence of bound states

We begin our study with the discussion of some general properties of solutions of (1.12) with initial values (1.13). Standard results will guarantee local existence and uniqueness of solutions, their continuous dependence on the initial values as well as on the parameter \( m \) and their regularity. As a consequence of local existence and uniqueness solutions can only have simple zeros. We shall frequently apply these properties in the sequel as well as the following integral equations for \( u' \) and \( V' \):

\[
\begin{align*}
u'(r) &= \frac{1}{r^{2m+d-1}} \int_0^r (V(s) - 1)u(s)s^{2m+d-1} \, ds \\
V'(r) &= \frac{1}{r^{d-1}} \int_0^r u^2(s)s^{2m+d-1} \, ds.
\end{align*}
\]

Since \( u_0 > 0 \), we see from integration of the equation 2.1 that \( V(r) \) is increasing, since \( d \leq 2 \), and goes to infinity. Thus for any \( u_0 \) there exist a unique \( a > 0 \), such that \( V'(a) = 1 \). This is indeed a crucial point to restrict ourselves to dimensions \( \leq 2 \).

For the initial condition \( u_0 > 0 \) of the solution \((u, V)\) for an \( m \geq 0 \), we consider the following sets:

**Definition 2.1.**

(2.2) \[ \mathcal{N}_{m,n} := \{ u_0 \in \mathbb{R}^+ : u \text{ has exactly } n \text{ zeros} \} \]

(2.3) \[ \mathcal{N}_{m,n}^0 := \left\{ u_0 \in \mathcal{N}_{m,n} : \lim_{r \to \infty} u(r) = 0 \right\} \]

(2.4) \[ \mathcal{N}_{m,n}^{\infty} := \mathcal{N}_{m,n} \setminus \mathcal{N}_{m,n}^0 \]
Note that since $u''(0) = -u_0/(2m + d)$ all solutions start strictly decreasing. Therefore if $u$ has a first critical point $r_1 > 0$ where $u > 0$, then $V(r_1) \geq 1$ and since $V$ is strictly increasing (see (2.1)) it follows again from (2.1) that $u'(r) > 0$ for all $r > r_1$. Hence every solution $u_0 \notin N_{m,0}^\infty$ is strictly decreasing in the maximal interval $(0, R)$ where $u > 0$. Obviously the sets $N_{m,n}^\infty$, $N_{m,n}^0$ are mutually disjoint and we will show that these sets forms a partition of $\mathbb{R}_+$. First we prove that $u$ can oscillate only in a finite range and then we show that $u$ has only a finite number of zero in this range.

**Lemma 2.2.** Let $a$ be such that $V(a) = 1$, then $u$ has at most one zero in the range $r > a$.

**Proof.** If $u$ has two zeros for $r > a$, there exist a critical point $\bar{r}$ such that $u(\bar{r}) \neq 0$, $u(\bar{r}) u''(\bar{r}) \leq 0$. Moreover from equation (2.1) we have

$$u(\bar{r}) u''(\bar{r}) = (V(\bar{r}) - 1)u(\bar{r})^2 > 0$$

which is the desired contradiction. \qed

As consequence of the lemma 2.2, the number of zeros of $u$ can only increase by one in a neighborhood of $u_0$.

**Lemma 2.3.** For any $u_0 \in N_{m,n}$, there exist $\varepsilon > 0$ such that $B(u_0, \varepsilon) \in N_{m,n} \cup N_{m,n+1}$, where $B(u_0, \varepsilon)$ is the open ball centered in $u_0$ with radius $\varepsilon$.

**Proof.** We choose $R > \max(a, r_n)$ where $r_n$ is the last zero of $u$. Since $u$ has simple zeros it follows from the continuous dependence on initial values that for every $\tilde{u}_0$ in a open ball $B(u_0, \varepsilon)$, $\tilde{u}$ has $n$ zeros in the interval $[0, R]$ and $\tilde{V}(R) > 1$. Finally from lemma 2.2 $\tilde{u}$ cannot have more than $n + 1$ zeros. \qed

**Lemma 2.4.** $u$ has a finite number of zeros.

**Proof.** It is sufficient to prove this assertion in the range $r \leq a$. We compare equation (1.12) to

$$\tilde{u}''(r) + \frac{2m + d - 1}{r} \tilde{u}'(r) + \tilde{u}(r) = 0,$$

which admits the solution:

$$\tilde{u}(r) = u_0 \Gamma \left( m + \frac{d}{2} \right) \left( \frac{2}{r} \right)^{(2m+d-2)/2} J_{(2m+d-2)/2}(r),$$

where $J_{(2m+d-2)/2}$ is the Bessel function of first kind of order $(2m + d - 2)/2$. By the Sturm's comparison theorem, see e.g. [14], $u$ oscillate less than $\tilde{u}$ in the range $r \leq a$. Since $J_{(2m+d-2)/2}$ has a finite number of zeros in this range, the same properties also hold for $u$. \qed

The reason of the notation $N_{m,n}^\infty$ become from the fact that the solutions $u$ in these sets goes to infinity as we will see in the next lemma.

**Lemma 2.5.** For every $u_0 \in N_{m,n}^\infty$, there exist $r_e \in (0, \infty]$ such that $\lim_{r \to r_e} |u(r)| = \infty$. 

(2.5) $N_m := \bigcup_{n=1}^\infty N_{m,n}$
Proof. We have seen in the lemma 2.2 that, for large $r$, solutions of 1.12 have no zeros and, more precisely, are not oscillatory. Then, supposing $\lim \inf_{r \to \infty} u(r) > 0$ or $\lim \sup_{r \to \infty} u(r) < 0$ leads to a contradiction. In the case $\lim \inf_{r \to \infty} u(r) = l > 0$, choosing $\tau > a$ sufficiently large and integrating (2.1) gives:

\[
   u'(r) = \frac{1}{r^{2m+d-1}} \int_{r}^{\tau} (V(s) - 1)u(s)s^{2m+d-1} ds + \left(\frac{\tau}{r}\right)^{2m+d-1} u'\left(\frac{\tau}{r}\right) \\
   \geq \frac{l}{2(2m+d)} \left(\frac{\tau}{r} - 1\right) (r - \tau) + \left(\frac{\tau}{r}\right)^{2m+1} u'\left(\frac{\tau}{r}\right) \forall r > \tau.
\]

The case $\lim \sup_{r \to \infty} u(r) < l$ can be proved in the same manner. □

Remark 1. Since as $u$ has only simple zeros and the preceding lemma, we deduce from the continuous dependence on initial values that $N_{m,n}^\infty$, $N_m$ are open sets.

Our main result theorem 1.1 states that each $N_{m,n}^0$ contain at least on element. Obviously, the set of the ground states $N_{m,n}^0$ is nonempty if both $N_m$ and $N_{m,0}^\infty$ are nonempty. As we will show in the proof of the main-theorem, the bound states are the infima of the $N_{m,n}$ sets. Thus, a necessary condition for the existence of bound states for each $n$ is these infima are ever strictly positive. We will show these properties in the two following lemmas.

**Lemma 2.6.** The set $N_{m,0}^\infty$ is non empty.

Proof. We want to show that $u_0 \in N_{m,0}^\infty$ for $u_0$ sufficiently large. Suppose on the contrary that $u_0 \in N_{m,n} \cup N_{m,0}^0$ for all $u_0 > 0$ and denote $]0, R_0[$ the maximal interval where $u > 0$ and $u' < 0$. Let $r_0 = \sqrt{2(2m+d)}$. We consider the function

\[
   f(r) := u(r) - u_0 \left(1 - \frac{r^2}{r_0^2}\right).
\]

It satisfied the differential equation

\[
   f''(r) + \frac{2m+d-1}{r} f'(r) = V(r) u(r) + u_0 - u(r).
\]

Since $f'(0) = f(0) = 0$ and the right hand-side of (2.6) we conclude that $f(r) \geq 0$ on $]0, R_0[$. Hence

\[
   u(r) \geq u_0 \left(1 - \frac{r^2}{r_0^2}\right) \quad \text{on } ]0, R_0[.
\]

and $r_0 \leq R_0$. Using that $u$ is decreasing and from the equation (2.1) for $V'$ we obtain the bound

\[
   V'(r) \geq \frac{u(r)^2}{2m+d} r^{2m+1} \quad \text{on } ]0, R_0[.
\]

Integrating this inequality and using again that $u' < 0$ yields the following estimate for $V'$:

\[
   V(r) \geq \frac{u(r)^2}{r_0^2 (m+1)} r^{2(m+1)} \quad \text{on } ]0, R_0[.
\]
We want to show that $u'(r_0) > 0$ provided $u_0$ sufficiently large which yields the desired contradiction. Using (2.7), (2.8) in equation (2.1) leads to

\[
u'(r_0) = \frac{1}{r_0^{2m+d-1}} \int_0^{r_0} (V(r) - 1) u(r) r^{2m+d-1} dr \geq \frac{1}{r_0^{2m+d+1}} \int_0^{r_0} u(r)^3 r^{4m+d+1} dr - \frac{2u_0}{r_0} \geq \frac{u_0}{r_0} (\frac{u_0^2}{m+1})^{2(m+1)} \int_0^1 (1-t^2)^3 t^{4m+d+1} dt - 2).
\]

We conclude that $u'(r_0) > 0$ for $u_0$ sufficiently large which contradicts the assumption that $N_m \bigcup N_{m,0} = [0, \infty]$.

**Lemma 2.7.** For any $N > 0$, there exist $\tilde{u}_0 > 0$ such that for each $u_0 < \tilde{u}_0$, the corresponding solution has at least $N$ zeros i.e. $u_0 \in \bigcup_{n \geq N} N_{m,n}$.

**Proof.** First we consider equation (1.12) for $u_0 > 0$ in the range $r \in [0, b]$, where $b$ is the value at which $V(b) = 1/2$. We introduce a Lyapunov function $F(r)$ defined by

\[F(r) = u^2 (1 - V) + u'^2.
\]

Then $F(0) = u_0$ and

\[F'(r) = -\frac{2(2m + d - 1)}{r} u'(r)^2 - V'(r) u(r)^2 \leq 0.
\]

Hence $F(0) \geq F(r)$ for all $r \in [0, b]$ and, using $V(r) \leq 1$ we therefore have

\[2u_0^2 \geq u(r)^2.
\]

Then from the equation 2.1 we find

\[\frac{1}{2} = V(b) = \int_0^b \int_0^r u(s) s^{2m+d-1} t^{m+1} ds dt \leq \frac{u_0^2}{(m+1)(2m+d)} b^{2(m+1)},
\]

hence

\[b \geq \left( \frac{(m+1)(2m+d)}{2u_0^2} \right)^{\frac{1}{2m+1}}.
\]

We can now compare the main equation (1.12) to the following

\[\pi''(r) + \frac{2m + d - 1}{r} \pi(r) + \frac{1}{2} \pi(r) = 0
\]

which admit the solution:

\[\pi(r) = u_0 J_{(2m+d-2)/2} \left( \frac{r}{\sqrt{2}} \right) J_{(2m+d-2)/2} \left( \frac{2\sqrt{2}}{r} \right),
\]

where $J_{(2m+d-2)/2}$ is the Bessel function of first kind. By Sturm’s comparison theorem, [14], $u$ oscillates faster than $\pi$ and has at least the same number of zero as $J_{(2m+d-2)/2}(r/\sqrt{2})$ in the range $0 \leq r \leq \left( \frac{(m+1)(2m+d)}{2u_0^2} \right)^{\frac{1}{2m+1}}$. Therefore we can choose $u_0$ sufficiently small such the number of zero of $u$ is greater or equal than $N$. \qed
Hence we have proved by the preceding lemmas the existence of ground states (see also [7]). We are now in position to prove our main existence results.

**Theorem 2.8.** For any $m \geq 0$ and $n \in \mathbb{N}$, $\alpha_{m,n} := \inf N_{m,n} \in N_{m,n}^0$. In addition for every $m \geq 0$, the solution $u_{m,0} \in N_{m,0}^0$ satisfies $u_{m,0}(r) > 0$, $u'_{m,0}(r) < 0$ on $[0, \infty[.

**Proof.** We prove this first statement by induction with the following hypothesis: For any $n \in \mathbb{N}$, $\alpha_n := \inf N_{m,n} \in N_{m,n}^0$. We have seen in the lemma 2.7 that $\alpha_0 > 0$. Since $N_{m,0}^\infty$ and $N_m$ are open sets, $\alpha_0 \in N_{m,0}$ and this result is true for $n = 0$. By hypothesis $\alpha_n \in N_{m,n}^0$ and the lemma 2.3 applied at the point $\alpha_n$ states there exist $\varepsilon > 0$ such that $B(\alpha_n, \varepsilon) \subset N_{m,n} \cup N_{m,n+1}$. Since $\alpha_n$ is the infimum of $N_{m,n}$ we have the following result

$$N_{m,n+1} \neq \emptyset \text{ and } \alpha_{n+1} < \alpha_n$$

and it follows from the lemma 2.7 that $\alpha_{n+1} > 0$.

Now we suppose on the contrary that $\alpha_{n+1} \notin N_{m,n+1}$. Since $\alpha_{n+1} < \alpha_n$, lemma 2.3 implies there exist an open ball around $\alpha_{n+1}$ which is not in $N_{n+1}$ in contradiction with the definition of $\alpha_{n+1}$. Finally, since $N_{m,n}^\infty$ is an open set, it follows that $\alpha_{n+1} \in N_{m,n}^0$.

It remains to prove the last part of the theorem. We have seen that every solution $u \notin N_{m,0}^\infty$ is strictly decreasing in the maximal interval $(0, R)$ where $u > 0$. Since the are only simple zeros, we have in particular for the ground states: $u(r) > 0$, $u'(r) < 0$ on $[0, \infty[$. □

**Remark 2.9.** We also prove with the preceding theorem that for all $m \geq 0$ the sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ of the infima is strictly decreasing as illustrate in the following figure.

The curves represent the sets $N_{m,n}$ for $n = 0, 1, 2, 3, 4, 5$ as a function of $m$ in dimension $d = 2$. 
3. Uniqueness of ground states

In this section we prove that \( N_{m,0}^0 \) has exactly one element. First of all, we show that if \( N_{m,0}^0 \) had more than one element the corresponding solutions cannot cross. We restate the no-crossing properties of \([8]\) in the following lemma.

**Lemma 3.1.** Let \( u_2(0) > u_1(0) > 0 \) and suppose that \( u_2(r), u_1(r) \) exist on \([0, R]\) such that \( u_1(r) \geq 0 \) on \([0, R]\). Then \( u_2(r) > u_1(r) \) for all \( r \in [0, R]\).

**Proof.** We consider the Wronskian of \((3.2)\)

\[
\text{Then } w(u) \text{ such that } w(\ref{3.1}) \text{ satisfies the differential equation }
\]

\[
(\ref{3.2}) \quad w(r) = u_2'(r)u_1(r) - u_1'(r)u_2(r).
\]

Then \( w \) satisfies the differential equation

\[
(\ref{3.2}) \quad w' + \frac{2m + d - 1}{r}w = (V_2 - V_1)u_1u_2.
\]

Suppose there is \( \tilde{r} \in [0, R]\) such that \( u_2(r) > u_1(r) \) on \([0, \tilde{r}]\) and \( u_1(\tilde{r}) = u_2(\tilde{r}) \geq 0 \). Then

\[
w(\tilde{r}) = (u_2'(\tilde{r}) - u_1'(\tilde{r}))u_1(\tilde{r}) \leq 0.
\]

On the other hand we have

\[
V_2'(r) - V_1'(r) = \frac{1}{r^{d-1}} \int_0^r (u_2^2(s) - u_1^2(s))s^{2m+d-1} ds > 0
\]

on \([0, \tilde{r}]\) and therefore \( V_2(r) > V_1(r) \) on \([0, \tilde{r}]\). We conclude then from the differential equation \((\ref{3.2})\) for \( w \) that \( w(r)r^{2m+d-1} \) is strictly increasing on \([0, \tilde{r}]\) and since \( w(0) = 0 \) we must have \( w(\tilde{r}) > 0 \) which is the desired contradiction. \(\square\)

**Remark 3.2.** From the no-crossing property stated in lemma 3.1 it follows immediately that \( N_m, N_m^\infty \) are intervals. More precisely, \( N_m = ]0, a[ \cup N_m^\infty = ]b, \infty[ \) with \( 0 \leq a \leq b \leq \infty \). Uniqueness of ground states is then equivalent to \( a = b \).

The important conclusion from lemma 3.1 is that two different ground state solutions cannot intersect. From the differential equation \((\ref{3.2})\) for their Wronskian \( w(r) \) we see that \( w(r)r^{2m+d-1} \) is a nonnegative strictly increasing function. However, we shall prove in the sequel that \( w(r)r^{2m+d-1} \) vanishes at infinity which yields the desired contradiction. Therefore we have to analyze the decay properties of ground states at infinity.

**Lemma 3.3.** Let \( u_0 \in N_{m,0}^0 \). Then

\[
\lim_{r \to -\infty} -\frac{u'_rV^{-\frac{r}{2}}}{u} = 1.
\]

Consequently, for any \( \kappa \in (0, 1) \),

\[
\limsup_{r \to -\infty} u(r)e^{-\kappa \int_0^r V^{1/2}(s) ds} < \infty.
\]

**Proof.** We consider the function \( z := -\frac{u'}{u}V^{-\frac{r}{2}} \) which is well defined for all \( r > 0 \) and satisfies the differential equation

\[
z' = (z^2 - 1)V^{1/2} - z \left( \frac{2m + d - 1}{r} + \frac{V'}{2V} \right) + V^{-1/2}.
\]

We also consider \( y := 2(m + 1)V - rV' \) which satisfies the following differential equation

\[
y' = (2m + d)V' - u^2r^{2m+1}.
\]
Since $u$ is decreasing, we have from the equation (2.1)
\[ V'(r) = \frac{1}{r^{d-1}} \int_0^r u(s)^2 s^{2m+d-1} ds \geq \frac{u(r)^2}{2m+d} r^{2m+1}. \]
Hence $y$ is increasing and we have the upper bound
\[ \frac{V'}{V} \leq \frac{2(m+1)}{r}. \]
Now choose $\tilde{r}$ such that
\[ \frac{2m+d}{r} V^{-1/2} \leq \frac{1}{2} \text{ for all } r \geq \tilde{r}. \]
Consider the direction field in the $(r,z)$ plane for the preceding differential equation. In the set $r \geq \tilde{r}$, $z \geq 2$ we have
\[ z' \geq (z^2 - 1) V^{1/2} - z \frac{V^{1/2}}{2} + V^{-1/2} \]
\[ \geq \frac{1}{2} z^2 V^{1/2} + \frac{1}{2} (z+1)(z-2) V^{1/2} \]
\[ \geq \frac{1}{2} z^2 V^{1/2} (\tilde{r}) \]
It follows that, should $z(r)$ ever enter this region, it would blow up at finite time after $\tilde{r}$ which is impossible. Hence $z$ remains bounded. This also implies
\[ \lim_{r \to \infty} u'(r) V^{-1/2} = 0. \]
Therefore we may apply l’Hospital’s rule. We obtain
\[ \lim_{r \to \infty} z^2 = \lim_{r \to \infty} \left( \frac{u''}{u} \frac{1}{V} + \frac{1}{2} \frac{V'}{V^{3/2}} \right) = \lim_{r \to \infty} \left( \frac{u''}{u} + \frac{1}{2} \frac{V'}{V^{3/2}} \right) = 1. \]
This proves the first part of the lemma.
Then for any $\kappa \in (0,1)$ and $r$ sufficiently large, $-\frac{w'}{w} V^{-1/2} \geq \kappa$ and the proof is completed by integrating this inequality.

Now we are in position to prove our uniqueness result:

**Theorem 3.4.** The set $\mathcal{N}_{m,0}$ has exactly one element.

**Proof.** Let $u_1(0), u_2(0) \in \mathcal{N}_{m,0}$ such that $u_2(0) > u_1(0)$. By lemma 3.1 the corresponding solutions $u_1, u_2$ cannot intersect and we have $u_2(r) > u_1(r) > 0$ for all $r \geq 0$. From the differential equation (3.2) for their Wronskian $w(r)$ we see that $w(r)r^{2m+d-1}$ is a nonnegative strictly increasing function since
\[ (w(r)r^{2m+d-1})' = (V_2 - V_1)u_1 u_2 r^{2m+d-1} > 0 \]
and $w(0) = 0$. On the other hand, we claim that
\[ \lim_{r \to \infty} w(r) r^{2m+d-1} = 0. \]
Indeed, since $u_2$ cannot intersect $u_1$ we have $\lim_{r \to \infty} V_2(r) > V_1(r) \geq 1$. Trivially, $V_2(r) \leq \frac{u_2(0)^2 r^{2(m+1)}}{2(m+1)(2m+d)}$. From the integral equation (2.1) for $u_2'$,
\[ u_2'(r) r^{2m+d-1} = \int_0^r (V_2(s) - 1)u_2(s)s^{2m+d-1} ds \]
and the decay properties of \( u_2 \) given in lemma 3.3 it follows then that \( u'_2 r^{2m+d-1} \) and \( u_2 r^{2m+d-1} \) are uniformly bounded. Therefore

\[
|w(r)| r^{2m+d-1} \leq |u_1||u'_2 r^{2m+d-1}| + |u'_1||u_2 r^{2m+d-1}| \leq c_1 |u_1| + c_2 |u'_1|
\]

for some positive constants \( c_1, c_2 \) which concludes the proof. \( \square \)

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