CONICS ON A GENERIC HYPERSURFACE

MASAO JINZENJI, IKU NAKAMURA AND YASUKI SUZUKI

Abstract. In this paper, we compute the contributions from double cover maps to genus 0 degree 2 Gromov-Witten invariants of general type projective hypersurfaces. Our results correspond to a generalization of Aspinwall-Morrison formula to general type hypersurfaces in some special cases.

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1. Introduction

In this paper, we discuss a generalization of the multiple cover formula for rational Gromov-Witten invariants of Calabi-Yau manifolds [AM], [M] to double cover maps of a line $L$ on a degree $k$ hypersurface $M^k_N$ in $\mathbb{P}^{N-1}$. Naively, for a given finite set of elements $\alpha_j \in H^*(M^k_N, \mathbb{Z})$, the rational Gromov-Witten invariant $\langle O_{\alpha_1} O_{\alpha_2} \cdots O_{\alpha_n} \rangle_{0,d}$ counts the number of degree $d$ (possibly singular and reducible) rational curves on $M^k_N$ that intersect real sub-manifolds of $M^k_N$ that are Poincaré-dual to $\alpha_j$.

Recently, the mirror computation of rational Gromov-Witten invariants of $M^k_N$ with negative first Chern class ($k-N > 0$) was established in [CG], [Iri], [J]. Using the method presented in these articles, we can compute $\langle O_{e}^{m_1} O_{e}^{m_2} \cdots O_{e}^{m_n} \rangle_{0,d}$ where $e$ is the generator of $H^{1,1}(M^k_N, \mathbb{Z})$. Briefly, mirror computation of $M^k_N$ ($k > N$) in [J] goes as follows. We start from the following ODE:

\begin{equation}
(\partial_x)^{N-1} - k \cdot \exp(x) \cdot (k \partial_x + k - 1)(k \partial_x + k - 2) \cdots (k \partial_x + 1) \cdot w(x) = 0,
\end{equation}

and construct the virtual Gauss-Manin system associated with (1):

\begin{equation}
\partial_x \tilde{\psi}_{N-2-m}(x) = \tilde{\psi}_{N-1-m}(x) + \sum_{d=1}^{\infty} \exp(dx) \cdot \tilde{L}^{N,k,d}_{m} \cdot \tilde{\psi}_{N-1-m-(N-k)d}(x),
\end{equation}

where $m$ runs through all the integers and $\tilde{L}^{N,k,d}_{m}$ is non-zero only if $0 \leq m \leq N-1+(k-N)d$. From the compatibility of (1) and (2), we can derive the recursive formulas that determine all the $\tilde{L}^{N,k,d}_{m}$'s:

\begin{align*}
\sum_{n=0}^{k-1} \tilde{L}^{N,k,1}_{n} w^n &= k \cdot \prod_{j=1}^{k-1} (jw + (k-j)), \\
\sum_{m=0}^{N-1+(k-N)d} \tilde{L}^{N,k,d}_{m} z^m &= \sum_{l=2}^{d} (-1)^l \sum_{0=i_0 < \cdots < i_l = d} \times \\
&\sum_{j_1=0}^{N-1+(k-N)d} \cdots \sum_{j_l=0}^{j_{l-1}} \prod_{n=1}^{l} \left( \frac{(i_n-1 + (d-i_{n-1})z)_{j_n-j_{n-1}}}{d} \cdot \tilde{L}^{N,k,i_n-i_{n-1}}_{j_n+(N-k)i_{n-1}} \right).
\end{align*}
With these data, we can construct the formulas that represent rational three point Gromov-Witten invariants \( \langle \mathcal{O}_e \mathcal{O}_{e N - 2 - m} \mathcal{O}_{e m - 1 - (k - N) d} \rangle_d \) in terms of \( \tilde{L}_{m}^{N,k,d} \). These three point Gromov-Witten invariants are enough for reconstruction of all the rational Gromov-Witten invariants \( \langle \mathcal{O}_{e m_1} \mathcal{O}_{e m_2} \cdots \mathcal{O}_{e m_n} \rangle_{0,d} \) [KM]. In particular, we obtain the following formula in the \( d = 2 \) case:

\[
\langle \mathcal{O}_e \mathcal{O}_{e N - 2 - m} \mathcal{O}_{e m - 1 - (k - N) 2} \rangle_2 = \\
k \cdot \left( \tilde{L}_{1}^{N,k,2} - \tilde{L}_{1+2(k-N)}^{N,k,2} - 2 \tilde{L}_{1+1+(k-N)}^{N,k,1} \left( \sum_{j=0}^{k-N} (\tilde{L}_{1-j}^{N,k,1} - \tilde{L}_{1+2(k-N)-j}^{N,k,1}) \right) \right).
\]

According to the results of this procedure, rational three point Gromov-Witten invariants can be rational numbers with large denominator if \( k > N \), in contrast to the Calabi-Yau case where rational three point Gromov-Witten invariants are always integers.

One of the reasons of this rationality (non-integrality) comes from the contributions of multiple cover maps to Gromov-Witten invariants. In the Calabi-Yau case \((N = k)\), for any divisor \( m \) of \( d \) there are some contributions from degree \( m \) multiple cover maps \( \phi \) of a rational curve \( \mathbb{P}^1 \) onto a degree \( \frac{2}{m} \) rational curve \( C \hookrightarrow M_k^1 \). The contributions from the multiple cover maps are expressed in terms of the virtual fundamental class of Gromov-Witten invariants. Let \( C \) be a general degree \( d \) rational curve in \( M_k^1 \). Its normal bundle \( N_{C/M_k^1} \) is decomposed into a direct sum of line bundles as follows:

\[
N_{C/M_k^1} \simeq O_C(-1) \oplus O_C(-1) \oplus O_C^{\oplus (k-5)}.
\]

Let \( \phi : \mathbb{P}^1 \to C \) be a holomorphic map of degree \( m \). Since the pull-back \( \phi^*(N_{C/M_k^1}) \) is given by

\[
\phi^*(N_{C/M_k^1}) \simeq O_{\mathbb{P}^1}(-m) \oplus O_{\mathbb{P}^1}(-m) \oplus O_{\mathbb{P}^1}^{\oplus k-5},
\]

we obtain \( h^1(\phi^*(N_{C/M_k^1})) = 2m - 2 \). On the other hand, let \( \overline{M}_{0,0}(M,d) \) be the moduli space of 0-pointed stable maps of degree \( d \) from genus 0 curve to \( M \). Then the moduli space of \( \phi \) is the fiber space \( \pi : \overline{M}_{0,0}(C,m) \to \overline{M}_{0,0}(M_k^1,d/m) \), whose fibre \( \overline{M}_{0,0}(C,m) \) over \( C \) (fixed) has complex dimension \( 2m - 2 \). Then the push-forward of the virtual fundamental class \( \pi_* (\text{top}(H^1(\phi^* N_{C/M_k^1}))) \) can be computed only by intersection theory on the fiber \( \overline{M}_{0,0}(C,m) \), which turns out to be equal to \( 1/d \). This depends on neither the structure of the base \( \overline{M}_{0,0}(M_k^1,d/m) \) nor the global structure of the fibration.

But when \( k < N \), the situation is more complicated than \( M_k^1 \) because of negative first Chern class. Let us concentrate on the case of \( d = 2, m = 2 \) that we discuss in this paper. In this case, \( C \) is just a line \( L \) on the hypersurface \( M_N^k \). The moduli space \( \overline{M}_{0,0}(M_N^k,1) \) is a sub-manifold of \( \overline{M}_{0,0}(\mathbb{P}^{N-1},1) \), while \( \overline{M}_{0,0}(\mathbb{P}^{N-1},1) \) is the Grassmanian \( G(2, N) \), the moduli space of rank 2 quotients of \( V = \mathbb{C}^N \). As will be shown later, for a generic line \( L, N_{L/M_N^k} \) is decomposed into

\[
N_{L/M_N^k} \simeq O_L(-1)^{\oplus k-N+2} \oplus O_L^{\oplus 2N-k-5}.
\]
By pulling back it by the degree 2 map \( \phi : \mathbb{P}^1 \to L \), we obtain,
\[
\phi^* N_{L/M_N^k} \simeq O_{\mathbb{P}^1}(-2)^{\oplus k-N+2} \oplus O_{\mathbb{P}^1}^{\oplus 2N-k-5}.
\]
Therefore, \( h^1(\phi^*(N_{L/M_N^k})) = k-N+2 \), which is strictly greater than two, the complex dimension of the fiber \( \overline{M}_{0,0}(L,2) \). Thus we need to know the global structure of the fibration \( \pi \) in order to compute the multiple cover contribution to degree 2 rational Gromov-Witten invariants of \( M_N^k \).

In order to estimate the contributions from double cover maps \( \phi : \mathbb{P}^1 \to L \) to \( \langle O_{e^a}O_{e^b}O_{e^c} \rangle_{0,2} \), we first computed the number of conics, that intersect cycles Poicaré dual to \( e^a \), \( e^b \) and \( e^c \), on \( M_N^k \) (whose normal bundle are of the same type) by using the method in [K2]. Then we found the following formula by comparing these integers with the results obtained from (3):

\[
\langle O_{e^a}O_{e^b}O_{e^c} \rangle_{0,2} = (\text{number of corresponding conics}) +
\int_{G(2,N)} \left( S^k Q \right) \wedge \left[ \frac{c(S^{k-1}Q)}{1 - \frac{1}{2}c_1(Q)} \right]_{k-N} \wedge \sigma_{a-1} \wedge \sigma_{b-1} \wedge \sigma_{c-1},
\]

where \( Q \) is the universal rank 2 quotient bundle of \( G(2,N) \), \( \sigma_a \) is a Schubert cycle defined by \( \sum_{a=0}^{\infty} \sigma_a := \frac{1}{c_1(Q)} \) and \( \langle * \rangle_{k-N} \) is the operation of picking up degree \( 2(\frac{1}{2}k-N) \) part of Chern classes.

On the other hand, we have the following formula which directly follows from the definition of the virtual fundamental class of \( \overline{M}_{0,0}(M_N^k,2) \):

\[
\langle O_{e^a}O_{e^b}O_{e^c} \rangle_{0,2} = (\text{number of corresponding conics}) +
8 \int_{G(2,N)} \left( S^k Q \right) \wedge \left[ \pi_*(c_{top}(H^1(\phi^* N_{L/M_N^k}))) \right]_{k-N} \wedge \sigma_{a-1} \wedge \sigma_{b-1} \wedge \sigma_{c-1},
\]

where \( \pi : \overline{M}_{0,0}(L,2) \to \overline{M}_{0,0}(M_N^k,1) \) is the natural projection. Here, the factor 8 comes from the divisor axiom of Gromov-Witten invariants.

In this paper, we prove the following formula

\[
\pi_*(c_{top}(H^1(\phi^* N_{L/M_N^k}))) = \frac{1}{8} \left[ \frac{c(S^{k-1}Q)}{1 - \frac{1}{2}c_1(Q)} \right]_{k-N}.
\]

By combining (5) with (6), we can derive the formula (4) immediately.

From (4), we see that \( \langle O_{e^a}O_{e^b}O_{e^c} \rangle_{0,2} \) of \( M_N^k \) is a rational number with denominator at most \( 2^{k-N} \). Therefore rationality (non-integrality) of the Gromov-Witten invariant \( \langle O_{e^a}O_{e^b}O_{e^c} \rangle_{0,2} \) is caused by the effect of multiple cover map in this case.

We note here that the total moduli space of double cover maps of lines is isomorphic to \( P(S^2 Q) \) over \( G := \overline{M}_{0,0}(M_N^k,1) \to G(2,N) \), which is an algebraic \( \mathbb{Q} \)-stack \( P(S^2 Q)_{\text{stack}} \) (in the sense of Mumford). As a consequence, the union of all \( H^1(\phi^* N_{C/M_N^k}) \) turns out to be a coherent sheaf on \( P(S^2 Q)_{\text{stack}} \) with fractional Chern class in [6], as was suggested in [BT]. See [VI, Section 9].

We also did some numerical experiments on degree 3 Gromov-Witten invariants of \( M_N^k \) by using the results of [ES]. For \( k-N > 0 \), there is a new contribution from multiple cover maps to nodal conics in \( M_N^k \) that did not appear in the Calabi-Yau case. Therefore, multiple cover map contributions are far more complicated than Calabi-Yau, and we leave general analysis on this problem to future works.
This paper is organized as follows. In Section 1, we analyze characteristics of moduli space of lines in $M^k_N$ and derive $N_{L/M}^k \simeq O_L(-1)^{2N-k-5} \oplus O_L^{2N-k-5}$. In Section 2, we study the moduli space $\overline{M}_{0,0}(\mathbb{P}^1, 2)$ from the point of view of stability and identify it with $\mathbb{P}^2$ and show that the moduli space $\overline{M}_{0,0}(\mathbb{P}^1, 2)$ is isomorphic to $\mathbb{P}(S^2 Q)$ over $G$. In section 4, we describe $H^1(\phi^* N_{L/M}^k)$ as an coherent sheaf over $\mathbb{P}(S^2 Q)^{stack}$. In section 5, we derive the main theorem [13] of this paper by using Segre classes. In Section 6, we mention some generalization to degree 3 Gromov-Witten invariants.

2. Lines on a hypersurface

Let $M$ be a generic hypersurface of degree $k$ of the projective space $\mathbb{P}^{N-1} = \mathbb{P}(V)$. We assume $2N-5 \geq k \geq N-2 \geq 2$ throughout this note. In this note we count the number of rational curves of virtual degree two, namely rational curves which doubly cover lines on $M$.

Let $P = \mathbb{P}(V)$ be the projective space parameterizing all one-dimensional quotients of $V$, which is usually denoted by $\mathbb{P}(V)$ in the standard notation in algebraic geometry. In this notation let $W$ be a subspace of $V$. Then $P(W)$ is naturally a linear subspace of $P(V)$ of dimension $\dim W - 1$.

Let $G(2, V)$ be the Grassmann variety of lines in $P(V)$, the scheme parameterizing all lines of $P = P(V)$. This is also the universal scheme parameterizing all one-dimensional quotient linear spaces of $V$. Let $W$ be a two dimensional quotient linear space, $\psi \in G(2, V)$, namely $\psi : P(W) \rightarrow P(V)$ the natural immersion and $i^*_\psi : V \rightarrow W$ the quotient homomorphism. The space $W$ is denoted by $W(\psi)$ when necessary.

There exists the universal bundle $Q_{G(2, V)}$ over $G(2, V)$ and a homomorphism $i^{univ*} : O_{G(2, V)} \otimes V \rightarrow Q_{G(2, V)}$ whose fiber $i^{univ*}_\psi : V \rightarrow Q_{G(2, V), \psi}$ is the quotient $i^*_\psi : V \rightarrow W(\psi)$ of $V$ corresponding to $\psi$.

2.1. Existence of a line on $M$. Let $L = P(W)$ be a line of $P$, equivalently $W \subset G(2, V)$. Then the condition $L \subset M$ imposes at most $k+1$ conditions on $W$, while the number of moduli of lines of $P$ equals $\dim G(2, V) = 2N - 4$. Hence we infer

Lemma 2.2. If $2N \geq k+5$, then there exists at least a line on $M$.

See also [Katz, p.152]. Let $G$ be the subscheme of $G(2, V)$ parameterizing all lines of $P(V)$ lying on $M$, $Q = (Q_{G(2, V)})_G$ the restriction of $Q_{G(2, V)}$ to $G$. By Lemma 2.2 $G$ is nonempty. Let $i^* : O_G \otimes V \rightarrow Q$ be the restriction of $i^{univ*}$ to $G$. Let $P = P(Q)$ and $\pi : P \rightarrow G$ the natural projection. Then $\pi$ is the universal line of $M$ over $G$, to be more exact, the universal family over $G$ of lines lying on $M$. In other words, the natural epimorphism $i^* : O_G \otimes V \rightarrow Q$ induces a morphism $i : P \rightarrow P_G(V) := G \times P(V)$, which is a closed immersion into $P_G(V)$, thus $P$ is a subscheme of $P_G(V)$ such that $\pi = (p_1)|_P$. Let $L_\psi = P(Q_\psi)$. Note that

$L_\psi = P_\psi := \pi^{-1}(\psi) \simeq P(Q_\psi) \subset \{\psi\} \times P(V) \simeq P(V)$. 
2.3. The normal bundle $N_{L/M}$. The argument of this section is standard and well known. Let $P = P(V), L = P(W)$ and $i_W^* : V \to W \in G$. Let us recall the following exact sequence:

$$0 \to O_P \to O_P(1) \otimes V^\vee \xrightarrow{D} T_P \to 0$$

where the homomorphism $D$ is defined by

$$D(a \otimes v^\vee) = aD(v^\vee) \quad (a \in O_P(1))$$

$$\left( D_{v^\vee}F \right)(u^\vee) = \left( \frac{d}{dt} F(u^\vee + tv^\vee) \right) |_{t=0}$$

for a homogeneous polynomial $F \in S(V)$ and $u^\vee, v^\vee \in V^\vee$. We note $H^0(O_P(1)) \otimes V^\vee = V \otimes V^\vee = \text{End}(V, V)$ and that the image of $H^0(O_P)$ in $\text{End}(V, V)$ is $C \text{id}_V$.

We also have the following exact sequences:

$$0 \to TL \to (T_P)_L \to N_{L/P} \to 0$$

$$0 \to OL \to OL(1) \otimes V^\vee \xrightarrow{DL} (T_P)_L \to 0.$$

Lemma 2.4. Let $L = P(W)$. Then

$$N_{L/P} \cong OL(1) \otimes (V^\vee/W^\vee), \quad H^0(N_{L/P}) \cong W \otimes (V^\vee/W^\vee).$$

Proof. The assertion is clear from the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
0 & \to & OL & \to & OL(1) \otimes W^\vee & \xrightarrow{(DL)_{W^\vee}} & TL & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & OL & \to & OL(1) \otimes V^\vee & \xrightarrow{DL} & (T_P)_L & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & OL(1) \otimes (V^\vee/W^\vee) & \to & N_{L/P} & \to & 0
\end{array}$$

The second assertion is clear from $H^0(E(L, OL(1))) = W$. \qed

Since $TL \cong OL(2)$, there follow exact sequences

$$0 \to H^0(TL) \to H^0((T_P)_L) \to H^0(N_{L/P}) \to 0$$

$$0 \to H^0(OL) \to H^0(OL(1)) \otimes V^\vee \xrightarrow{H^0(DL)} H^0((T_P)_L) \to 0.$$

We also note

$$H^0(TL) = \text{Lie Aut}^0(L) = \text{End}(W, W)/\text{center} = \text{End}(W, W)/C \text{id}_W.$$

Since $H^0(OL(1)) = W$, we see

$$H^0((T_P)_L) = W \otimes V^\vee / \text{Im} H^0(OL) = \text{Hom}(V, W)/C \text{id}_W.$$ 

Hence we again see

$$H^0((N_{L/P})) = (\text{Hom}(V, W)/C \text{id}_W)/(\text{Hom}(W, W)/C \text{id}_W) = W \otimes (V^\vee/W^\vee) = \text{Hom}(V/W, W).$$

For any line $L = P(W)$ of $P$ the following sequence is exact:

$$0 \to N_{L/M} \to N_{L/P} \to (N_M/P)_L(\cong OL(k)) \to 0.$$

(7)
Hence so is the following sequence as well:

\[ 0 \rightarrow H^0(N_{L/M}) \rightarrow H^0(N_{L/P}) \rightarrow H^0(O_L(k)) \rightarrow 0. \]

Hence we have

**Lemma 2.5.** The following is exact:

\[ 0 \rightarrow H^0(N_{L/M}) \rightarrow W \otimes (V^\vee/W^\vee) \rightarrow H^0(D_L) \rightarrow S^kW \rightarrow H^1(N_{L/M}) \rightarrow 0. \]

**Corollary 2.6.** \( \dim G \geq 2N - k - 5 \), equality holding if \( H^1(N_{L/M}) = 0 \).

**Proof.** As is well-known, \( \dim G \geq h^0(N_{L/M}) - h^1(N_{L/M}) \). Note \( \dim W \otimes (V^\vee/W^\vee) = 2(N-2) \) and \( \dim S^kW = k + 1 \). Hence the corollary follows from Lemma 2.5. \( \square \)

**Lemma 2.7.** For a generic line \( L \) on a generic hypersurface \( M \) of degree \( k \)

\[(i) \quad N_{L/M} \simeq O_L^a \oplus O_L(-1)^{a+b}, \text{ where } a = 2N - k - 5 \text{ and } b = k - N + 2, \]

\[(ii) \quad \text{Coker } H^0(D_L) \simeq S^kW/(V^\vee/W^\vee) \text{ where } D_L := D_L \otimes O_L(-1). \]

**Proof.** Let \( M \) be a generic hypersurface of degree \( k \) and \( L \) a generic line \( L \) on \( M \). Without loss of generality we may assume that \( W^\vee \) is generated by \( e_1^\vee \) and \( e_2^\vee \), in other words, \( \psi : L \rightarrow P \) is given by

\[ \psi : [s : t] \mapsto [x_1, \ldots, x_N] = [s, t, 0, \ldots, 0]. \]

Then \( F \), the polynomial of degree \( k \) defining \( M \), is written as

\[ F = x_3F_3 + x_4F_4 + \cdots + x_NF_N \]

for some polynomials \( F_j \) of degree \( k - 1 \). Let \( f_j = \psi^*F_j = F_j(s, t, 0, \ldots, 0) \).

Now we consider the exact sequence

\[ 0 \rightarrow H^0(N_{L/M}(-1)) \rightarrow H^0(N_{L/P}(-1)) \rightarrow H^0(O_L(-1)) \rightarrow 0. \]

where we note \( H^0(N_{L/P}(-1)) = V^\vee/W^\vee \). Hence the following is exact:

\[ 0 \rightarrow H^0(N_{L/M}(-1)) \rightarrow V^\vee/W^\vee \rightarrow S^kW \rightarrow H^1(N_{L/M}(-1)) \rightarrow 0. \]

where \( H^0(D_L) \) is given by \( H^0(D_L)(e_j^\vee) = f_j \) \( (j = 3, 4, \ldots, N) \).

A generic choice of \( F \) implies a generic choice of degree \( k - 1 \) polynomials \( f_j \) \( (j = 3, 4, \cdots, N) \) in \( s \) and \( t \). By the assumptions

\[ \dim S^kW = k \geq N - 2 = \dim V^\vee/W^\vee, \]

\[ \dim W \otimes V^\vee/W^\vee = 2(N - 2) \geq k + 1 = \dim S^kW, \]

the generic choice of \( F \) implies that we can choose \( f_j \in S^kW \) \( (j = 3, 4, \cdots, N) \) (and fix once for all) such that

\[(iii) \quad f_j \quad (j = 3, 4, \cdots, N) \quad \text{are linearly independent}, \]

\[(iv) \quad Wf_3 + Wf_4 + \cdots + Wf_N = S^kW. \]
Hence $H^0(D_L^\perp)$ is injective by (iii). It follows that $H^0(N_{L/M}(-1)) = 0$. Hence (ii) is clear. Next we consider $H^0(D_L)$. By (iv), we see

$$S^kW = W \cdot H^0(D_L^\perp)(V^\vee/W^\vee) = H^0(D_L)(W \otimes V^\vee/W^\vee),$$

whence $H^0(D_L)$ is surjective. It follows that $H^1(N_{L/M}) = 0$. Hence $N_{L/M} \simeq O_L^{\oplus 3} \oplus O_L(-1)^{\oplus b}$ for some $a$ and $b$. Since $a + b = \rank(N_{L/M}) = N - 3$ and $-b = \deg(N_{L/M}) = N - 2 - k$, we have (i). $\square$

2.8. *Lines on a quintic hypersurface in $\mathbb{P}^4$. See [Katz, Appendix A] for the subsequent examples. Let $N = 5$ and $k = 5$. Hence $M$ is a hypersurface of degree 5 in $\mathbb{P}^4$, a Calabi-Yau 3-fold. Let

$$F = x_4x_1^4 + x_5x_2^2 + x_3^5 + x_4^5 + x_5^5.$$

First we note that $M = \{F = 0\}$ is nonsingular. Let $L = \{x_3 = x_4 = x_5 = 0\} = \{s, t, 0, 0, 0\}$. In this case $f_3 = 0$, $f_4 = s^4$ and $f_5 = t^4$. In the exact sequence (1) we see $H^0(N_{L/M}(-1)) = \Ker H^0(D_L^\perp) = C\varepsilon_3^\perp$ and $H^1(N_{L/M}(-1)) = \Coker H^0(D_L^\perp)$ is 3-dimensional. Hence $N_{L/M} = O_L(1) \oplus O_L(-3)$.

We summarize the above. If $\dim \Ker H^0(D_L^\perp) = 1$ and if $M$ is nonsingular, then $N_{L/M} = O_L(1) \oplus O_L(-3)$. Hence $H^0(N_{L/M}) = \Ker H^0(D_L^\perp) = W \otimes \Ker H^0(D_L^\perp)$ is 2-dimensional. Therefore we can choose $f_3 = 0$ and a linearly independent pair $f_4$ and $f_5 \in S^4W$ so that $Wf_4 + Wf_5$ is 4-dimensional. The choice $f_4 = s^4$ and $f_5 = t^4$ satisfies the conditions. This enables us to find a nonsingular hypersurface $M$ as above. However if we choose $f_3 = 0$, $f_4 = s^4$ and $f_5 = s^3t$, then $Wf_4 + Wf_5$ is 3-dimensional. Hence $M$ is singular.

Next in the same manner we find $L$ on a nonsingular hypersurface $M$ with $N_{L/M} = O_L \oplus O_L(-2)$ or $N_{L/M} = O_L(-1)^{\oplus 2}$. Let

$$F = x_3x_1^4 + x_4x_1^3x_2 + x_5x_2^2 + x_3^5 + x_4^5 + x_5^5.$$

Then we have $f_3 = s^4$, $f_4 = s^3t$ and $f_5 = t^4$. Since $Wf_3 + Wf_4 + Wf_5$ is 5-dimensional, $H^0(N_{L/M}(-1)) \simeq \Ker H^0(D_L^\perp) = 0$, $H^0(N_{L/M}) = \Ker H^0(D_L^\perp) = C(e_3^\perp - se_3^\perp)$. We see also that $\dim H^1(N_{L/M}) = \dim \Coker H^0(D_L^\perp) = 1$ and $N_{L/M} = O_L \oplus O_L(-2)$. The hypersurface $M = \{F = 0\}$ is easily shown to be nonsingular.

If $F = x_3x_1^4 + x_4x_1^3x_2 + x_5x_2^2 + x_3^5 + x_4^5 + x_5^5$ and $M = \{F = 0\}$, then $N_{L/M} = O_L(-1)^{\oplus 2}$.

2.9. *Lines on a generic hypersurface $M^2$ of $\mathbb{P}^6$. Let $N = 7$ and $k = 8$. In view of Lemma 2.2 there exists a line $L$ on any generic hypersurface $M$ of degree 8 in $\mathbb{P}(V) = \mathbb{P}^6$. In view of Lemma 2.7 $a = 1, b = 3$ and $N_{L/M} \simeq O_L \oplus O_L(-1)^{\oplus 3}$. For example let $L : x_j = 0 \ (j \geq 3)$ and we take

$$F_3 = 8x_1^7, F_4 = 8x_1^6x_2, F_5 = 8x_1^5x_2^2, F_6 = 8x_1^4x_2^3, F_7 = 8x_1^3x_2^4, F_8 = 8x_1^2x_2^5, F_9 = 8x_1^3x_2^4, F_7 = 8x_1^3x_2^4,$$

$$F = x_3F_3 + x_4F_4 + x_5F_5 + x_6F_6 + x_7F_7 + x_8^8 + x_9^8 + x_5^8 + x_6^8 + x_7^8,$$

and let $M = M^2 : F = 0$. We see that $M$ is nonsingular near $L$ and has at most isolated singularities. However it is still unclear to us whether $M = M^2$ is
nonsingular everywhere. The space $H^0(N_{L/M})$ is spanned by $te_3^\vee - se_4^\vee$, hence an infinitesimal deformation $L_\varepsilon$ of $L$ is given by

$$[s,t] \mapsto [s,t,\varepsilon t,-\varepsilon s,0,0,0]$$

which yields $F_{L_\varepsilon} = \varepsilon^8(s^8 + t^8) \equiv 0 \mod \varepsilon^8$. Since $H^1(N_{L/M}) = 0$, this infinitesimal deformation is integrable and $L$ nonsingular and one dimensional at the point $[\lim L_\varepsilon,0]$. The closure of the orbit $\pi_{\varepsilon_8}$ of $G$ that the image of $\varepsilon_8$ is spanned by all lines of $\varepsilon_8$.

We note that $M$ also contains $8$ lines

$$L' := L'_{\varepsilon_8} : \varepsilon_8 x_1 - x_2 = x_3 + \varepsilon_8 x_4 = x_j = 0 \quad (j \geq 5),$$

with $N_{L'/M} = O_{L'}(1)^{\oplus 3} \oplus O_{L'}(-6)$ where $\varepsilon_8 = -1$.

3. Stability

**Definition 3.1.** Suppose that a reductive algebraic group $G$ acts on a vector space $V$. Let $v \in V$, $v \neq 0$.

1. the vector $v$ is said to be *semistable* if there exists a $G$-invariant homogeneous polynomial $F$ on $V$ such that $F(v) \neq 0$.

2. the vector $v$ is said to be *stable* if $p$ has a closed $G$-orbit in $X_{ss}$ and the stabilizer subgroup of $v$ in $G$ is finite.

Let $\pi : V \setminus \{0\} \to \mathbf{P}(V^\vee)$ be the natural surjection. Then $v \in V$ is semistable (resp. stable) if and only if $\pi(v)$ is semistable (resp. stable).

3.2. Grassmann variety. Let $V$ be an $N$-dimensional vector space, and $G(r,N)$ the Grassmann variety parameterizing all $r$-dimensional quotient spaces of $V$. Here is a natural way of understanding $G(r,N)$ via GIT-stability. Let $U$ be an $r$-dimensional vector space, $X = \text{Hom}(V,U)$ and $\pi : X \setminus \{0\} \to \mathbf{P}(X^\vee)$ the natural map. Then $SL(U)$ acts on $X$ from the left by:

$$(g \cdot \phi^*)(v) = g \cdot (\phi^*(v)) \quad \text{for } \phi^* \in X, \ v \in V.$$ We see that for $\phi^* \in X$

$$\phi^* \text{ is } SL(U)\text{-stable} \iff \text{rank } \phi^* = r,$$

$$\phi^* \text{ is } SL(U)\text{-semistable} \iff \phi^* \text{ is } SL(U)\text{-stable}.$$ In fact, if $\text{rank } \phi^* = r-1$, then there is a one-parameter torus $T$ of $SL(U)$ such that the closure of the orbit $T \cdot \phi$ contains the zero vector as the following simple example ($r = 2$) shows

$$\lim_{t \to 0} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \lim_{t \to 0} \begin{pmatrix} ta_{11} & ta_{12} & \cdots & ta_{1N} \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$ Let $X_\delta$ be the set of all (semi)stable points and $\mathbf{P}_s$ the image of $X_\delta$ by $\pi$. It is, as we saw above, just the set of all $\phi \in X$ with $\text{rank } \phi^* = r$. Therefore the GIT-orbit space $\mathbf{P}_s/SL(U)$ is the orbit space $\mathbf{P}_s/SL(U)$ by the free action, the Grassmann variety $G(r,N)$. 
3.3. Moduli of double coverings of \( \mathbb{P}^1 \) (1). Let \( W \) and \( U \) be a pair of two dimensional vector spaces, \( X = \text{Hom}(W, S^2U) \), and \( \pi : X \setminus \{0\} \to \mathbb{P}(X^\vee) \) the natural morphism. Note that \( \text{SL}(U) \) acts on \( S^2U \) from the left via the natural action: \( \sigma(u_1u_2) = \sigma(u_1)\sigma(u_2) \) for \( \forall u_1, u_2 \in U \). Thus \( \text{SL}(U) \) acts on \( X \) from the left in the same manner in the subsection 3.2.

**Lemma 3.4.** Let \( \phi^* \in X \).

(i) \( \phi^* \) is unstable iff \( \phi^*(w) \) has a double root for any \( w \in W \),

(ii) \( \phi^* \) is semistable iff \( \phi^*(w) \) has no double roots for some nonzero \( w \in W \),

(iii) \( \phi^* \) is stable iff \( \phi^*(W) \) is a base-point free linear subsystem of \( S^2U \) on \( \text{P}(U) \).

**Proof.** We note that \( \phi^* \) is unstable iff there is a suitable basis \( s \) and \( t \) of \( U \) such that \( \phi^*(w) = a(w)s^2 \) for any \( w \in W \) since a torus orbit \( T \cdot \phi^* \) contains the zero vector. This proves (i). This also proves (ii). Next we prove (iii). If \( \phi^*(W) \) has a base point, then it is clear that \( \phi^* \) is not stable. If \( \phi^* \) is semistable and it is not stable, then we choose a basis \( s, t \) of \( U \) and a basis \( w_1, w_2 \) of \( W \) such that \( \phi^*(w_1) = st \). If \( \phi^*(w_2) = as^2 + bst \), then \( \phi^* \) is not stable. This proves the lemma. \( \square \)

**Theorem 3.5.** Let \( X_{ss} \) be the Zariski open subset of \( X \) consisting of all semistable points of \( X \), \( \pi(X_{ss}) \) the image of \( X_{ss} \) by \( \pi \), and \( Y := \pi(X_{ss})/\text{SL}(U) \). Then \( Y \simeq \mathbb{P}^2 \).

**Proof.** First consider a simplest case. We choose a basis \( s, t \) of \( U \). Let \( w_1 \) and \( w_2 \) be a basis of \( W \). \( T \) the subgroup of \( \text{SL}(U) \) of diagonal matrices and \( X' = \{ \phi^* \in X; \phi^*(w_1) = 2st \} \). Let \( Z' = \text{SL}(U) \cdot X' \).

We note that \( Z' \) is an \( \text{SL}(U) \)-invariant subset of \( X_{ss} \). We prove \( \pi(Z')/\text{SL}(U) \simeq C^2 \). Let \( \phi^* \) and \( \psi^* \) be points of \( X' \). Let \( \phi^*(w_2) = As^2 + 2Bst + Ct^2 \) and \( \psi^*(w_2) = as^2 + 2bst + ct^2 \). Then it is easy to check

\[
g \cdot \phi^* = \psi^* \quad \text{for} \quad \exists g \in \text{SL}(U) \iff g \cdot \phi^* = \psi^* \quad \text{for} \quad \exists g \in T
\]

\[
\iff A = au^2, \quad B = b, \quad C = u^{-2}c \quad \text{for} \quad \exists u \neq 0.
\]

Therefore each equivalence class of \( \pi(Z)/\text{SL}(U) \) is represented by the pair \((AC, B)\), which proves \( \pi(Z)/\text{SL}(U) \simeq C^2 \).

Now we prove the lemma. Let \( \phi^* \in X_{ss}, \phi_j = \phi^*(w_j) \) and \( \phi_0 = -(\phi_1 + \phi_2) \). Let

\[
\phi_0 = r_1s^2 + 2r_2st + r_3t^2,
\]
\[
\phi_1 = p_1s^2 + 2p_2st + p_3t^2,
\]
\[
\phi_2 = q_1s^2 + 2q_2st + q_3t^2,
\]

and we define

\[
D_1 = p_2^2 - p_1p_3, \quad D_2 = q_2^2 - q_1q_3,
\]
\[
D_0 = r_2^2 - r_1r_3 = D_1 + D_2 + 2p_2q_2 - (p_1q_3 + p_3q_1).
\]

To show the lemma, we prove the more precise isomorphism

\[
\pi(X_{ss})/\text{SL}(U) = \text{Proj} \mathbb{C}[D_0, D_1, D_2]
\]

For this purpose we define \( Y_j = \pi(\{ \phi^* \in X_{ss}; \phi_j \) has no double roots\})/\text{SL}(U) \). It suffices to prove \( Y_1 = \text{Spec} \mathbb{C}[\frac{D_0}{D_1}, \frac{D_1}{D_2}] \) by reducing it to the first simplest case.
Let $\phi^* \in Y_1$. Let $\alpha$ and $\beta$ be the roots of $\phi_1 = 0$. By the assumption $\phi_1$ has no double roots, hence $\alpha \neq \beta$. Let
\[
  u = \frac{1}{\gamma}(s - \alpha t), \quad v = \frac{1}{\gamma}(s - \beta t), \quad g = \frac{1}{\gamma} \begin{pmatrix} 1 & -\alpha \\ 1 & -\beta \end{pmatrix}
\]
where $\gamma = \sqrt{\alpha - \beta}$. Note that $g \in SL(U)$. Hence we see
\[
(\phi_1(s, t), \phi_2(s, t)) \equiv (p_1 \gamma^* uv, A_1 u^2 + 2B_1 uv + C_1 v^2)
\]
where
\[
A_1 = q_1^2 \beta^2 + 2q_2 \beta + q_3,
-B_1 = q_1 \alpha \beta + q_2 (\alpha + \beta) + q_3,
C_1 = q_1^2 \alpha^2 + 2q_2 \alpha + q_3.
\]
Thus we see
\[
(\phi_1(s, t), \phi_2(s, t)) \equiv (2st, As^2 + 2Bst + Ct^2)
\]
where
\[
A = \frac{2A_1}{p_1 \gamma^4}, \quad B = \frac{2B_1}{p_1 \gamma^4}, \quad C = \frac{2C_1}{p_1 \gamma^4}, \quad p_1 \gamma^4 = 4D_1,
AC = B^2 - \frac{D_2}{D_1}, \quad B = \frac{D_0 - D_1 - D_2}{2D_1}.
\]
Therefore by the first half of the proof
\[
Y_1 \simeq \text{Spec } \mathbb{C}[AC, B] = \text{Spec } \mathbb{C}[\frac{D_0}{D_1}, \frac{D_2}{D_1}].
\]
This completes the proof of the lemma. \(\square\)

**Corollary 3.6.** Let $Y^s = \pi(X_s) \backslash SL(U)$. Then $Y \backslash Y^s$ is a conic of $Y$ defined by
\[
Y \backslash Y^s : D_0^2 + D_1^2 + D_2^2 - 2D_0 D_1 - 2D_1 D_2 - 2D_2 D_0 = 0.
\]

**Proof.** In view of Theorem 3.5, $Y_1 \simeq \text{Spec } \mathbb{C}[AC, B]$. The complement of $Y_s$ in $Y_1$ is then the curve defined by $AC = 0$, which is easily identified with the above conic. \(\square\)

**Corollary 3.7.** Let $X^0$ be the Zariski open subset of $X$ consisting of all semistable points $\phi^*$ of $X$ with rank $\phi^* = 2$, and let $Y^0 := \pi(X^0) \backslash SL(U)$. Then $Y^0 \simeq \pi(X^0) / SL(U) \simeq Y \simeq \mathbb{P}^2$.

**Proof.** It suffices to compare $Y_1$ and $Y^0 \cap Y_1$. As in the proof of Theorem 3.5, we let $X' = \{\phi^* \in X; \phi^*(w_1) = 2st\}$. Let $Z = SL(U) \cdot X'$ and $Z^0 = SL(U) \cdot (X' \cap X^0)$.

Then with the notation in Theorem 3.5, we recall $X' = \{\phi^* \in X; \phi^*(w_1) = 2st, \phi^*(w_2) = As^2 + 2Bst + Ct^2\}$, $\pi(Z) / SL(U) \simeq \text{Spec } \mathbb{C}[AC, B]$ where $X' \cap X^0 = \{\phi^* \in X'; A \neq 0 \text{ or } C \neq 0\}$.

In the same manner as before we see $\pi(Z^0) / SL(U) \simeq \text{Spec } \mathbb{C}[AC, B]$, whence $\pi(Z^0) / SL(U) = \pi(Z) / SL(U)$. This proves $Y^0 \cap Y_1 = Y_1$. This completes the proof of the corollary. \(\square\)
3.8. Moduli of double coverings of \( \mathbf{P}(W) \) (2). There is an alternative way of understanding \( \pi(X_{k})/\text{SL}(U) \simeq \mathbf{P}^{2} \) by using the isomorphism \( S^{2}\mathbf{P}^{1} \simeq \mathbf{P}^{2} \). We use the following convention to denote a point of \( \mathbf{P}(U) = U^{\vee} \setminus \{0\}/\mathbf{G}_{m} \): \( u : v = us^{\vee} + vt^{\vee} \in U^{\vee} \) where \( s^{\vee} \) and \( t^{\vee} \) are a basis dual to \( s \) and \( t \). In what follows we fix a basis \( w_{1} \) and \( w_{2} \) of \( W \). Let \( P := (a_{1} : a_{2}) \) and \( Q := (b_{1} : b_{2}) \) be a pair of points of \( \mathbf{P}(W) \simeq \mathbf{P}^{1} \). If \( P \neq Q \), there is a double covering \( \phi : \mathbf{P}(U) \to \mathbf{P}(W) \) ramifying at \( P \) and \( Q \), unique up to isomorphism once we fix the base \( w_{1} \) and \( w_{2} \):

\[
\frac{b_{2}w_{1} - b_{1}w_{2}}{a_{2}w_{1} - a_{1}w_{2}} = \left( \frac{t}{s} \right)^{2}.
\]

Thus \( \phi \) is given explicitly by

\[
\phi_{1} := \phi^{*}(w_{1}) = b_{1}s^{2} - a_{1}t^{2}, \quad \phi_{2} := \phi^{*}(w_{2}) = b_{2}s^{2} - a_{2}t^{2}, \quad \phi_{0} = -(\phi_{1} + \phi_{2})
\]

for which we have

\[
D_{1} = a_{1}b_{1}, \quad D_{2} = a_{2}b_{2}, \quad D_{0} = (a_{1} + a_{2})(b_{1} + b_{2}).
\]

The isomorphism \( S^{2}\mathbf{P}^{1} \simeq \mathbf{P}^{2} \) is given by \((P,Q) \mapsto (D_{0},D_{1},D_{2})\). This shows

**Corollary 3.9.** We have a natural isomorphism: \( Y \simeq \mathbf{P}(S^{2}W) \).

4. The virtual normal bundle of a double covering

4.1. The case \( N = 7 \) and \( k = 8 \) revisited. We revisit the example in the subsection 2.4. Let \( N = 7 \) and \( k = 8 \). Let \( L : x_{j} = 0 \) (\( j \geq 3 \)) and we take

\[
F_{3} = 8x_{1}^{2}, \quad F_{4} = 8x_{1}x_{2}, \quad F_{5} = 8x_{1}^{2}x_{2}, \quad F_{6} = 8x_{1}^{2}x_{2}^{2}, \quad F_{7} = 8x_{2}^{2},
\]

\[
F = x_{3}F_{3} + x_{4}F_{4} + x_{5}F_{5} + x_{6}F_{6} + x_{7}F_{7} + x_{3}^{2} + x_{4}^{2} + x_{5}^{2} + x_{6}^{2} + x_{7}^{2}.
\]

and let \( M = M_{8}^{5} : F = 0 \). We often denote \( L \) also by \( \mathbf{P}(W) \) with \( W \) a two dimensional vector space for later convenience. Since \( H^{0}(D_{L}) \) is injective and \( H^{0}(D_{L}) \) is surjective, we have \( N_{L/M} \simeq O_{L} \oplus O_{L}(-1)^{\oplus 3} \). Hence \( H^{1}(N_{L/M}(-1)) = H^{1}(O_{L}(-2)^{\oplus 3}) \) is 3-dimensional. As we see easily, this follows also from the fact that Coker \( H^{0}(D_{L}) \) is freely generated by \( x_{3}^{2}, x_{4}^{2} \) and \( x_{1}x_{2} \).

Let \( \phi^{*} = (\phi_{1}, \phi_{2}) \in X^{0} \). Then Ker \( H^{0}(\phi^{*}D_{L}) \) is generated by a single element \( \phi_{2}e_{3}^{\vee} - \phi_{1}e_{4}^{\vee} \), while Coker \( H^{0}(\phi^{*}D_{L}) \) is generated by \( S^{2}U \cdot \phi_{1}^{2} \), \( S^{2}U \cdot \phi_{1}^{2} \) and \( S^{2}U \cdot \phi_{1}^{2} \). To be more precise, we see

\[
\text{Coker } H^{0}(\phi^{*}D_{L}) = \{ \phi_{1}^{2}, \phi_{1}^{2}, \phi_{1}^{2} \} \otimes S^{2}U/\{ \phi_{1}, \phi_{2} \}.
\]

In fact, this is proved as follows: first we consider the case where \( \phi_{1} \) and \( \phi_{2} \) has no common zeroes. In this case \( \phi^{*} \) gives rise to a double covering \( \phi : \mathbf{P}(U) \to \mathbf{P}(W) (= L) \), which we denote by \( L_{\phi} \) for brevity. By pulling back by \( \phi^{*} \) the normal sequence \( 0 \to N_{L/M} \to N_{L/P} \to O_{L}(k) \to 0 \) for the line \( L \) we infer an exact sequence

\[
0 \to \phi^{*}N_{L/M} \to \phi^{*}N_{L/P} \to \phi^{*}O_{L}(k) \to 0,
\]

which yields an exact sequence

\[
\begin{align*}
0 & \longrightarrow H^{0}(\phi^{*}N_{L/M}) \longrightarrow S^{2}U \otimes (V^{\vee}/W^{\vee}) \longrightarrow H^{0}(\phi^{*}D_{L}) \longrightarrow H^{0}(O_{L}(2k)) \longrightarrow 0.
\end{align*}
\]
Let $\eta = q_3e_3^\vee + \cdots + q_se_s^\vee \in \text{Ker} \, H^0(\phi^*D_L)$, $q_j \in S^2U$. Then we have

$$\phi_1^2(q_3\phi_1^3 + q_4\phi_1^4\phi_2 + q_5\phi_1^2\phi_2^3 + q_6\phi_2^5) = -q_7\phi_2^7.$$ 

Since $\phi_1$ and $\phi_2$ are mutually prime and $q_j$ is of degree two, we have $q_7 = 0$ and

$$\phi_1^2(q_3\phi_1^3 + q_4\phi_1^2\phi_2 + q_5\phi_2^3) = -q_6\phi_2^5.$$ 

Hence $q_6 = 0$ and similarly we infer also $q_5 = 0$. Thus we have $q_3\phi_1 + q_4\phi_2 = 0$. This proves that $\text{Ker} \, H^0(\phi^*D_L)$ is generated by $\phi_2e_3^\vee - \phi_1e_4^\vee$.

Next we prove that $\text{Coker} \, H^0(\phi^*D_L)$ is generated by $\phi^*\text{Coker} \, H^0(D^-_L)$ over $S^2U$, in fact over $S^2U/\phi^*(W)$. Without loss of generality we may assume that $\phi_1 = 2st$ and $\phi_2 = \lambda s^2 + 2\nu st + t^2$ for some $\lambda \neq 0$ and $\nu \in \mathbb{C}$. Let $\phi^*W = \{\phi_1, \phi_2\}$. Then one checks $U \cdot \phi^*W = S^3U$, and hence $S^2U \cdot \phi^*W = S^4U$, $S^{2m-2}U \cdot \phi^*W = S^{2m}U$ for $m \geq 2$. It follows $S^2U \cdot \phi^*(S^{m-1}W) = S^{2m}U$ for $m \geq 1$. In fact, by the induction on $m$

$$S^2U \cdot \phi^*(S^{m}W) = S^2U \cdot \phi^*(W) \cdot \phi^*(S^{m-1}W) = S^2U \cdot \phi^*(S^{m-1}W) = S^2U \cdot (S^2U \cdot \phi^*(S^{m-1}W)) = S^2U \cdot S^{2m}U = S^{2m+2}U.$$ 

Therefore $H^0(O_{L_o}(2k)) = S^{16}U = S^2U \cdot \phi^*(S^7W)$. Hence

$$\text{Coker} \, H^0(\phi^*D_L) = S^{16}U/\text{Im} \, H^0(\phi^*D_L) = S^2U \cdot \phi^*(S^7W)/S^2U \cdot \phi^*(\text{Im} \, H^0(D^-_L)) = (S^2U/\phi^*(W)) \cdot \phi^*(S^7W/\text{Im} \, H^0(D^-_L)).$$

because $\text{Coker} \, H^0(D^-_L) = S^{7}W/\text{Im} \, H^0(D^-_L)$ and $W \cdot S^{7}W \subset W \cdot \text{Im} \, H^0(D^-_L) = S^{8}W$ by the choice of $L$. This proves that $\text{Coker} \, H^0(\phi^*D_L)$ is generated by $\phi^*\text{Coker} \, H^0(D^-_L)$ over $S^2U/\phi^*(W)$. It follows $\text{Coker} \, H^0(\phi^*D_L) = (\phi^*\text{Coker} \, H^0(D^-_L)) \otimes (S^2U/\phi^*W)$.

Finally we consider the case where $\phi_1$ and $\phi_2$ has a common zero. In this case we may assume $\phi_1 = 2st$ and $\phi_2 = 2\nu st + t^2$. In this case $L_o$ is a chain of two rational curves $C'_\phi$ and $C''_\phi$ where $C_\phi$ is the proper transform of $\text{P}(U)$, where the double covering map from $L_o$ to $\text{P}(W)$ is the union of the isomorphisms $\phi'$ and $\phi''$, say, $\phi = \phi' \cup \phi''$. Let $\psi_1 = 2s$ and $\psi_2 = 2\nu s + t$. Then $\phi'$ is induced by the homomorphism $(\phi')^* \in \text{Hom}(W, U)$ such that $(\phi')^*(w_j) = \psi_j$. On the other hand let $U''_\phi = \mathbb{C} \lambda + \mathbb{C}t$, $\psi'_1 = 2t$ and $\psi''_1 = \lambda + 2\nu t$ where we note $\psi_j''$ is the linear part of $\phi_j$ in $t$ with $s = 1$. Then $C''_\phi = \text{P}(U''_\phi)$ and $\phi''$ is induced by the homomorphism $(\phi'')^* \in \text{Hom}(W, U''_\phi)$ such that $(\phi'')^*(w_j) = \psi''_j$. Furthermore the pull back by $\phi''$ of the normal sequence for $L$

$$0 \to \phi^*N_{L/M} \to \phi^*N_{L/P} \xrightarrow{\phi^*D_L} \phi^*O_L(k) \to 0,$$
yields exact sequences with natural vertical homomorphisms:

\[
egin{array}{cccccc}
0 & \rightarrow & \phi^*N_{L/M} & \rightarrow & (\phi')^*N_{L/M} \oplus (\phi'')^*N_{L/M} & \rightarrow & C & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \phi^*N_{L/P} & \rightarrow & (\phi')^*N_{L/P} \oplus (\phi'')^*N_{L/P} & \rightarrow & V^\vee/W^\vee & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \phi^*O_L(k) & \rightarrow & O_{C_\phi}(k) \oplus O_{C_{\phi}''}(k) & \rightarrow & C & \rightarrow & 0.
\end{array}
\]

This yields the following long exact sequences:

\[
egin{array}{cccccc}
0 & \rightarrow & H^0((\phi')^*N_{L/M}) & \rightarrow & U \otimes V^\vee/W^\vee & \xrightarrow{H^0((\phi')^*D_L)} & S^kU \\
& \rightarrow & H^1((\phi')^*N_{L/M}) & \rightarrow & 0 \\
0 & \rightarrow & H^0((\phi'')^*N_{L/M}) & \rightarrow & U'' \otimes V^\vee/W^\vee & \xrightarrow{H^0((\phi'')^*D_L)} & S^kU'' \\
& \rightarrow & H^1((\phi'')^*N_{L/M}) & \rightarrow & 0
\end{array}
\]

whence \(H^1((\phi')^*N_{L/M}) = H^1((\phi'')^*N_{L/M}) = 0\), and both \(H^0((\phi')^*N_{L/M})\) and \(H^0((\phi'')^*N_{L/M})\) are one-dimensional. Let \(U'\) be the subspace of \(U\) consisting of elements vanishing at \(C_\phi' \cap C''_\phi\), namely the subspace spanned by \(t\). Then the restriction of \(H^0((\phi')^*D_L)\) to \(U' \otimes V^\vee/W^\vee\) equals \(t \cdot H^0((\phi')^*D^e_L)\). Hence

\[
\text{Coker } H^0(\phi^*D_L) \simeq t \cdot S^2U/t \cdot \text{Im } H^0((\phi')^*D^e_L) \oplus \text{Coker } H^0((\phi'')^*D_L) \\
\simeq S^2U/t \cdot \text{Im } H^0((\phi')^*D^e_L) \simeq \text{Coker } H^0((\phi')^*D^e_L).
\]

One could understand the above isomorphism as

\[
\text{Coker } H^0(\phi^*D_L) = \text{Coker } H^0(D^e_L) \otimes (S^2U/\phi^*W).
\]

Thus \(H^0(\phi^*N_{L/M})\) is one-dimensional, while \(H^1(\phi^*N_{L/M})\) is 3-dimensional. This is immediately generalized into the following

**Lemma 4.2.** For any \(\phi^* \in X^0\) we have

\[
\text{Ker } H^0(\phi^*D_L) = \phi^* \text{Ker } H^0(D_L), \\
\text{Coker } H^0(\phi^*D_L) = (\phi^* \text{Coker } H^0(D^e_L)) \otimes (S^2U/\phi^*W).
\]

**Lemma 4.3.** We define a line bundle \(L_0\) (resp. \(L_1\)) on \(Y \simeq \mathbb{P}(S^2W)\) by the assignment:

\[
X^0 \ni \phi^* \mapsto \phi^* \text{Ker } H^0(D_L) \ (\text{resp. } \phi^* \text{Coker } H^0(D^e_L)).
\]

Then \(L_k \simeq \mathcal{O}_{\mathbb{P}(S^2W)}\).

**Proof.** We know that \(\phi^* \text{Ker } H^0(D_L)\) is generated by \(\phi_2e_3^\vee - \phi_1e_1^\vee\). By the SL(2)-variable change of \(s\) and \(t\), \(\phi_j\) is transformed into a new quadratic polynomial, which is however the same as the first \(\phi_j\). This shows the generator is unchanged, whence \(L_0 \simeq \mathcal{O}_{\mathbb{P}(S^2W)}\). The proof for \(L_1\) is the same. \(\square\)
Lemma 4.4. We define a coherent sheaf $L$ on the stack $Y$ $(\simeq \mathbb{P}(S^2W))$ (See Remark below) by the assignment:

$$X^0 \ni \phi^* \mapsto S^2U/\phi^*W.$$ 

Then $L^2 \simeq O_{\mathbb{P}(S^2W)}(-1)$.

Proof. The GIT-quotient $Y^0$ is covered with the images of $X'_1$:

$$X'_1 = \{(\phi_1, \phi_2) \in X^0; \phi_1 = 2st, \phi_2 = \lambda s^2 + 2\nu st + t^2, \lambda, \nu \in \mathbb{C}\},$$

$$X'_2 = \{(\phi_1, \phi_2) \in X^0; \phi_1 = ps^2 + 2gst + t^2, \phi_2 = 2st, p, q \in \mathbb{C}\}.$$ 

It is clear that the natural image of $X'_1$ in $Y$ is $Y_j$. The map $\phi$ given by $\phi^* = (\phi_1, \phi_2) \in Y_1$ has natural $\mathbb{Z}_2$ involution generated by,

$$r : (\sqrt{\lambda}s + t, \sqrt{\lambda}s - t) \mapsto (\sqrt{\lambda}s + t, -(\sqrt{\lambda}s - t)).$$

Since

$$2st = \frac{1}{2\sqrt{\lambda}}((\sqrt{\lambda}s + t)^2 - (\sqrt{\lambda}s - t)^2),$$

$$\lambda s^2 + 2\nu st + t^2 = \frac{\nu}{2\sqrt{\lambda}}((\sqrt{\lambda}s + t)^2 - (\sqrt{\lambda}s - t)^2) + \frac{1}{2}((\sqrt{\lambda}s + t)^2 + (\sqrt{\lambda}s - t)^2),$$

it is clear that,

$$r^*(\phi_1) = \phi_1, \quad r^*(\phi_2) = \phi_2, \quad r^*(\lambda s^2 - t^2) = -(\lambda s^2 - t^2).$$

Therefore, we can decompose $S^2U$ into $\langle \lambda s^2 - t^2 \rangle C \oplus \langle \phi_1, \phi_2 \rangle C$ with respect to eigenvalue of $r^*$ and take $\lambda s^2 - t^2$ as canonical generator of $S^2U/\phi^*W$. Similarly $S^2U/\phi^*W$ is generated by $ps^2 - t^2$ on $Y_2$. The problem is therefore to write $\lambda s^2 - t^2$ as an $\Gamma(O_{Y_1}(\mathbb{C})$-multiple of $pu^2 - v^2$ when we write $\phi_2 = 2uv$ by a variable change in $GL(2)$. The following variable change $(s, t) \mapsto (u, v)$ is in $GL(2)$:

$$s = \frac{\sqrt{2\alpha}}{(\beta - \alpha)^2} (2u - \frac{(\beta - \alpha)^2}{2\alpha} v), \quad t = \frac{\sqrt{2\alpha}}{(\beta - \alpha)^2} (2\beta u - \frac{(\beta - \alpha)^2}{2} v),$$

where $\alpha$, $\beta$ are roots of the equation $\lambda s^2 + 2\nu st + t^2 = 0$. Under this coordinate change, $\phi_1$ and $\phi_2$ is rewritten as follows:

$$\phi_1 = \frac{\lambda}{(\nu^2 - \lambda)^2} u^2 + 2\frac{\nu}{\nu^2 - \lambda} uv + v^2 = pu^2 + 2quv + v^2, \quad \phi_2 = 2uv.$$ 

Then we have

$$pu^2 - v^2 = -\frac{2}{\beta - \alpha} (\lambda s^2 - t^2) = -\frac{1}{\sqrt{\nu^2 - \lambda}} (\lambda s^2 - t^2) = -\sqrt{\frac{D_1}{D_2}} (\lambda s^2 - t^2).$$

Similarly by computing the effect on $S^2U/\phi^*W$ by the variable change from $X'_1$ into $X'_0$, we see that $L^2$ is isomorphic to $O_{\mathbb{P}(S^2W)}(-1)$. This completes the proof. $\square$

Remark 4.5. We remark that the space $X$ must be regared as a $\mathbb{Q}$-stack $Y^{\text{stack}}$ as follows: First we define $\phi_0 = -\phi_1 - \phi_2$. For each atlas $X'_0$ we define an atlas $Y'_0$
(α = 0, 1, 2) by

\[ Y_0^{\text{stack}} = \{ (\phi_0, \phi_1, \phi_2, \pm \psi_0) \in X^0 \times S^2 U; \phi_0 = 2st, \phi_1 = as^2 + 2bst + t^2, \psi_0 = as^2 - t^2 a, b \in \mathbb{C} \}, \]

\[ Y_1^{\text{stack}} = \{ (\phi_0, \phi_1, \phi_2, \pm \psi_1) \in X^0 \times S^2 U; \phi_2 = \lambda s^2 + 2\nu st + t^2, \psi_1 = \lambda s^2 - t^2 \lambda, \nu \in \mathbb{C} \}, \]

\[ Y_2^{\text{stack}} = \{ (\phi_0, \phi_1, \phi_2, \pm \psi_2) \in X^0 \times S^2 U; \phi_1 = ps^2 + 2qst + t^2, \phi_2 = 2st, \psi_2 = ps^2 - t^2, p, q \in \mathbb{C} \}. \]

Since \( L^2 \cong O_{P(S^2 W)}(-1) \) we have \( c_1(L) = -\frac{1}{2}c_1(O_{P(S^2 W)}(1)) \) in the Chow ring \( A(Y^{\text{stack}}) \mathbb{Q} = A(X) \mathbb{Q} = A(P(S^2 W)) \mathbb{Q} \).

5. Proof of the main theorem

**Theorem 5.1.**

\[ \pi_* (c_{\text{top}}(H^1)) = \frac{1}{8} \left[ \frac{c(S^{k-1} Q)}{1 - \frac{1}{2}c_1(Q)} \right]_{k-N}, \]

where \( \pi \) is the natural projection from \( \check{M}_{0,0}(L,2) \) to \( G \) and \([\ast]_{k-N} \) is the operation of picking up the degree \( 2(k-N) \) part of Chern classes.

**Proof.** From now on we denote the coherent sheaf \( L \) in Lemma 4.4 by \( O_P(-\frac{1}{2}) \). In view of the results from the previous section, what remains is to evaluate the top Chern class of \( (S^{k-1} Q/((V^\vee \otimes O_G)/Q^\vee )) \otimes O_P(-\frac{1}{2}) \) on \( P(S^2 Q) \). Since double cover maps parametrized by \( P(S^2 Q) \) have natural \( \mathbb{Z}_2 \) involution \( r \) given in the previous section, we have to multiply the result of integration on \( P(S^2 Q) \) by the factor \( \frac{1}{2} \) [8], [9]. With this set-up, let \( \pi^\prime : P(S^2 Q) \to G \) be the natural projection. Then what we have to compute is \( \pi_* (c_{\text{top}}(H^1)) = \frac{1}{2} \pi_* (c_{\text{top}}(H^1)) = \frac{1}{2} \pi_* (c_{\text{top}}((S^{k-1} Q/((V^\vee \otimes O_G)/Q^\vee )) \otimes O_P(-\frac{1}{2}))) \). Let \( z \) be \( c_1(O_P(1)) \). Then we obtain,

\[ \frac{1}{2} \pi_* (c_{\text{top}}((S^{k-1} Q/((V^\vee \otimes O_G)/Q^\vee )) \otimes O_P(-\frac{1}{2}))) \]

\[ = \frac{1}{2} \sum_{j=0}^{k-N+2} c_{k-N+2-j}(S^{k-1} Q \oplus Q^\vee ) \cdot \pi_*(z^j) \cdot (-\frac{1}{2})^j \]

\[ = \frac{1}{8} \sum_{j=0}^{k-N} c_{k-N-j}(S^{k-1} Q \oplus Q^\vee ) \cdot s_j(S^2 Q) \cdot (-\frac{1}{2})^j \]

\[ = \frac{1}{8} \left[ \frac{c(S^{k-1} Q) \cdot c(Q^\vee )}{1 - \frac{1}{2}c_1(S^2 Q) + \frac{1}{4}c_2(S^2 Q) - \frac{1}{8}c_3(S^2 Q)} \right]_{k-N}, \]

where \( s_j(S^2 Q) \) is the \( j \)-th Segre class of \( S^2 Q \). But if we decompose \( c(Q) \) into \( (1 + \alpha)(1 + \beta) \), we can easily see,

\[ \frac{c(Q^\vee )}{1 - \frac{1}{2}c_1(S^2 Q) + \frac{1}{4}c_2(S^2 Q) - \frac{1}{8}c_3(S^2 Q)} = \frac{(1 - \alpha)(1 - \beta)}{(1 - \alpha)(1 - \frac{1}{2}(\alpha + \beta))(1 - \beta)} = \frac{1}{1 - \frac{1}{2}c_1(Q)}. \]
Finally, by combining the above theorem with the divisor axiom of Gromov-Witten invariants, we can prove the decomposition formula of degree 2 rational Gromov-Witten invariants of $M_N^k$ found from numerical experiments.

**Corollary 5.2.**

$$\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,2} = \langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,2} + 2 \pi_*(\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,1})$$

where $\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,2} \rightarrow \langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,1}$ is the number of conics that intersect cycles Poincaré dual to $e^a, e^b$ and $e^c$. We also denote by $\pi_*(\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,1})$ the integral:

$$\int_{G(2,V)} c_{\text{top}}(S^k Q) \wedge \pi_*(\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,1}) \wedge \sigma_{a-1} \wedge \sigma_{b-1} \wedge \sigma_{c-1}.$$  

### 6. Generalization to Twisted Cubics

In this section, we present a decomposition formula of degree 3 rational Gromov-Witten invariants found from numerical experiments using the results of [LS].

**Conjecture 6.1.** If $k - N = 1$, we have the following equality:

$$\pi_*(\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,1}) = \frac{1}{27} \left( \frac{1}{24} (27k^2 - 55k + 26)(k - 1) + \frac{2}{9} c_1(Q)^2 + \frac{7}{6}(k + 1)(k - 1) + \frac{1}{9} c_2(Q) \right).$$

where $\pi : \overline{M}_{0,0}(L,3) \rightarrow \overline{M}_{0,0}(M_N^k,1)$ is the natural projection.

In the $k - N > 1$ case, we have not found the explicit formula, because in the $d = 3$ case, we have another contribution from multiple cover maps of type $(2+1) \rightarrow (1+1)$. Here multiple cover map of type $(2+1) \rightarrow (1+1)$ is the map from nodal curve $X \Rightarrow Y$, that maps the first (resp. the second) $X$ to $Y$ (resp. $L_2$) by two to one (resp. one to one). In the $k - N = 1$ case, we have also determined the contributions from multiple cover maps of $(2+1) \rightarrow (1+1)$ to nodal conics.

**Corollary 6.2.** If $k - N = 1$, $\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,3}$ is decomposed into the following contributions:

$$\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,3} = \langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,3} + \frac{1}{k} \langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,1} \langle \mathcal{O}_{e^N-5} \rangle_{0,1}$$

$$+ \frac{3}{2} \langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,1} \langle \mathcal{O}_{e^N-a-4} \mathcal{O}_{e^c} \rangle_{0,1} + \frac{3}{2} \langle \mathcal{O}_{e^b} \mathcal{O}_{e^a} \mathcal{O}_{e^c} \rangle_{0,1} \langle \mathcal{O}_{e^N-b-4} \mathcal{O}_{e^a} \rangle_{0,1}$$

$$+ 27 \pi_*(\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,1} \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,1},$$

where $\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,3} \rightarrow \langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,1}$ is the number of twisted cubics that intersect cycles Poincaré dual to $e^a, e^b$ and $e^c$.

**Proof.** In the $k - N = 1$ case, dimension of moduli space of multiple cover maps of $(2+1) \rightarrow (1+1)$ to nodal conics is given by $N - 6 + N - 6 - (N - 4) + 2 = N - 6$, hence the rank of $H^1$ is given by $N - 6 - (N - 5 - 3) = 2$. On the other hand, dimension of moduli space of $d = 2$ multiple cover maps of $P^1 \rightarrow P^1$ is 2, the degree of the form of $\pi_*(\langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,1})$ equals to $2 - 2 = 0$, where $\pi$ is the projection map.
that projects out the fiber locally isomorphic to the moduli space of \( d = 2 \) multiple cover maps. This situation is exactly the same as the Calabi-Yau case. Therefore, we can use the well-known result by Aspinwall and Morrison, that says for \( n \)-point rational Gromov-Witten invariants for Calabi-Yau manifold, \( \tilde{\pi}_*(c_{\text{top}}(H^1)) \) for degree \( d \) multiple cover map is given by,

\[
\tilde{\pi}_*(c_{\text{top}}(H^1)) = \frac{1}{d^{3-n}}.
\]

With this formula, we add up all the combinatorial possibility of insertion of external operator \( \mathcal{O}_{e^a}, \mathcal{O}_{e^b} \) and \( \mathcal{O}_{e^c} \),

\[
\frac{1}{k} \left( \langle \tilde{\pi}_*(c_{\text{top}}(H^1)) \rangle_{\mathcal{O}_{e^a}, \mathcal{O}_{e^b}, \mathcal{O}_{e^c}} \right)_{0,1} \langle \mathcal{O}_{e_{N-5}} \rangle_{0,1} \\
+ \langle \tilde{\pi}_*(c_{\text{top}}(H^1)) \rangle_{\mathcal{O}_{e^a}, \mathcal{O}_{e^b}, \mathcal{O}_{e^{a+2}}} \langle \mathcal{O}_{e_{N-4}} \mathcal{O}_{e^c} \rangle_{0,1} \\
+ \langle \tilde{\pi}_*(c_{\text{top}}(H^1)) \rangle_{\mathcal{O}_{e^b}, \mathcal{O}_{e^a}, \mathcal{O}_{e^{a+2}}} \langle \mathcal{O}_{e_{N-4}} \mathcal{O}_{e^c} \rangle_{0,1} \\
+ \langle \tilde{\pi}_*(c_{\text{top}}(H^1)) \rangle_{\mathcal{O}_{e^c}, \mathcal{O}_{e^a}, \mathcal{O}_{e^{b+2}}} \langle \mathcal{O}_{e_{N-4}} \mathcal{O}_{e^b} \rangle_{0,1} \\
+ \langle \tilde{\pi}_*(c_{\text{top}}(H^1)) \rangle_{\mathcal{O}_{e^c}, \mathcal{O}_{e^b}, \mathcal{O}_{e^{b+2}}} \langle \mathcal{O}_{e_{N-4}} \mathcal{O}_{e^c} \rangle_{0,1} \\
+ \frac{1}{2} \langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^{c+2}} \rangle_{0,1} \langle \mathcal{O}_{e_{N-4}} \mathcal{O}_{e^c} \rangle_{0,1} \\
+ \frac{1}{2} \langle \mathcal{O}_{e^a} \mathcal{O}_{e^c} \mathcal{O}_{e^{b+2}} \rangle_{0,1} \langle \mathcal{O}_{e_{N-4}} \mathcal{O}_{e^b} \rangle_{0,1} \\
+ \frac{1}{4} \langle \mathcal{O}_{e^a} \mathcal{O}_{e^b} \mathcal{O}_{e^c} \rangle_{0,1} \langle \mathcal{O}_{e_{N-5}} \rangle_{0,1}.
\]

The last expression is nothing but the formula we want.

Masao Jinzenji∗, Iku Nakamura*, Yasuki Suzuki
Division of Mathematics, Graduate School of Science, Hokkaido University
Kita-ku, Sapporo, 060-0810, Japan
e-mail address: † jin@math.sci.hokudai.ac.jp, * nakamura@math.sci.hokudai.ac.jp

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