DIFFERENTIAL OPERATORS ON $G/U$ AND THE GELFAND-GRAEV ACTION

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Abstract. Let $G$ be a complex semisimple group and $U$ its maximal unipotent subgroup. We study the algebra $\mathcal{D}(G/U)$ of algebraic differential operators on $G/U$ and also its quasi-classical counterpart: the algebra of regular functions on $T^*(G/U)$, the cotangent bundle. A long time ago, S. Gelfand and M. Graev have constructed an action of the Weyl group on $\mathcal{D}(G/U)$ by algebra automorphisms. The Gelfand-Graev construction was not algebraic, it involved analytic methods in an essential way. We give a new algebraic construction of the Gelfand-Graev action, as well as its quasi-classical counterpart. Our approach is based on Hamiltonian reduction and involves the ring of Whittaker differential operators on $G/U$, a twisted analogue of $\mathcal{D}(G/U)$.

Our main result has an interpretation, via geometric Satake, in terms of spherical perverse sheaves on the affine Grassmanian for the Langlands dual group.

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1. INTRODUCTION

1.1. Let $G$ be a complex connected semisimple group and $U$ a maximal unipotent subgroup of $G$. The ring $\mathcal{D}(G/U)$ of algebraic differential operators on $G/U$ has a rich structure which was analyzed in [BGG] and studied further more recently in [BBP], [LS], and [GR]. In an unpublished paper written in the 1960’s, S. Gelfand and M. Graev have constructed an action of the Weyl group $W$ on $\mathcal{D}(G/U)$ by algebra automorphisms. This action is somewhat mysterious due to the fact that it does not come from a $W$-action on the variety $G/U$ itself. In rank 1, the action of the nontrivial element of $W = \mathbb{Z}/2\mathbb{Z}$ is, essentially, the Fourier transform of polynomial differential operators on a 2-dimensional vector space. In the case of higher rank, the action of each individual simple reflection is defined by reducing to a rank 1 case, but it is not a priori clear that the resulting automorphisms of $\mathcal{D}(G/U)$ satisfy the Coxeter relations. For a proof (essentially, due to Gelfand and Graev) that involves analytic arguments the reader is referred to [BBP, Proposition 6], cf. also [BP], [Ka], [KL] for closely related results.

One of the goals of the present paper is to provide a different approach to the Gelfand-Graev action. Specifically, we will present the algebra $\mathcal{D}(G/U)$ as a quantum Hamiltonian reduction (see Theorem 1.2.2) in such a way that the $W$-action on the algebra becomes manifest.
To explain this, recall the general setting of quantum Hamiltonian reduction. Let $A$ be an associative ring and $I$ a left ideal of $A$ (there is also a counterpart of the construction for right ideals). Thus, $A/I$ is a left $A$-module. The quantum Hamiltonian reduction of $A$ with respect to $I$ is defined to be $(\text{End}_A A/I)^{op}$, an opposite of the associative ring of $A$-module endomorphisms of $A/I$. More explicitly, let $N(I) = \{ a \in A \mid Ia \subseteq I \}$ be the normalizer of $I$ in $A$. By construction, $N(I)$ is a subring of $A$ such that $I$ is a two-sided ideal of $N(I)$. For any $a \in N(I)$, the assignment $f_a : x \mapsto xa$ induces an endomorphism of $A/I$. Moreover, this endomorphism only depends on $a \mod I$ and one has a ring isomorphism

\[
(\text{End}_A A/I)^{op} \cong N(I)/I = \{ a \in A \mid (xa - ax) \mod I = 0 \ \forall x \in I \}/I. \tag{1.1.1}
\]

The following special cases of quantum Hamiltonian reduction will be especially important for us.

Examples 1.1.2.  
(i) Let $\mathfrak{g}$ be a Lie algebra and $\iota : \mathfrak{g} \to A$ a Lie algebra map into an associative algebra $A$, i.e., a linear map such that $\iota([x,y]) = \iota(x)y - y\iota(x), \ \forall x,y \in \mathfrak{g}$. We let $I = \mathfrak{g}I$ be a left ideal of $A$ generated by the image of $\iota$. In this case, we have $N(I)/I = (A/\mathfrak{g}I)^{\mathfrak{g}}$, the centralizer of $\iota(\mathfrak{g})$ in $A/\mathfrak{g}I$. For any left, resp. right, $A$-module $M$, the right action of $(A/\mathfrak{g}I)^{\mathfrak{g}}$ on $A/\mathfrak{g}I$ induces a left, resp. right, action of $(A/\mathfrak{g}I)^{\mathfrak{g}}$ on the space $\text{Hom}_A(A/\mathfrak{g}I, M) = M^{\mathfrak{g}}$ of $\mathfrak{g}$-invariants, resp. space $M/M^{\mathfrak{g}} = M \otimes_A (A/\mathfrak{g}I)$ of $\mathfrak{g}$-coinvariants. Similarly, one can consider a right ideal, $\mathfrak{g}A$, generated by the image of $\iota$ and the corresponding algebra $(A/\mathfrak{g}A)^{\mathfrak{g}}$.

(ii) Let $A_1, A_2, Z$, be a triple of associative rings and $\iota_i : Z \to A_i$, $i = 1, 2$, a pair of ring homomorphisms. Let $I$ be a right ideal of $A_1^{op} \otimes A_2$ generated by the elements $\iota_1(z) \otimes 1 - 1 \otimes \iota_2(z)$, $z \in Z$. Then, we have $(A_1^{op} \otimes A_2)/I = A_1 \otimes_Z A_2$. Therefore, in this case, we obtain

\[N(I)/I = \{ a_1 \otimes a_2 \in A_1 \otimes Z A_2 \mid \iota_1(z) a_1 \otimes a_2 = a_1 \otimes a_2 \iota_2(z), \ \forall z \in Z \} = (A_1 \otimes_Z A_2)^{Z}, \tag{1.1.3}\]

where for any $Z$-bimodule $M$ we put $M^Z := \{ m \in M \mid zm = mz, \ \forall z \in Z \}$. Multiplication in the ring $(1.1.3)$ reads as follows:

\[
(a_1 \otimes a_2) \cdot (a_1' \otimes a_2') = (a_1' \cdot a_1) \otimes (a_2 \cdot a_2'), \quad \forall a_1, a_1' \in A_1, \ a_2, a_2' \in A_2.
\]

By construction, the space $(A_1^{op} \otimes A_2)/I = A_1 \otimes_Z A_2$ comes equipped with the structure of a left $(A_1 \otimes_Z A_2)^{Z}$-module given by the ‘inner’ action $a_1 \otimes a_2 : (a_1' \otimes a_2') \mapsto (a_1' \cdot a_1) \otimes (a_2 \cdot a_2')$, and the structure of a right $A_1^{op} \otimes A_2$-module given by the ‘outer’ action $a_1 \otimes a_2 : (a_1' \otimes a_2') \mapsto (a_1' \otimes a_1) \otimes (a_2 . a_2')$.

Let $E$ be an $(A_1, A_2)$-bimodule, resp. $(A_2, A_1)$-bimodule. There are natural isomorphisms

\[
\text{HH}^\ast(Z, E) := \text{Ext}_{Z \otimes^{op} Z}^\ast(Z, E) \cong \text{Ext}_{A_1^{op} \otimes A_2}(A_1^{op} \otimes Z A_2, E), \ \text{resp.}
\]

\[
\text{HH}_\ast(Z, E) := \text{Tor}_{Z \otimes^{op} Z}^\ast(Z, E) \cong \text{Tor}_{A_1^{op} \otimes A_2}^\ast(A_1^{op} \otimes Z A_2, E).
\]

Here, $\text{HH}^\ast(Z, -)$, resp. $\text{HH}_\ast(Z, -)$, stands for Hochschild cohomology, resp. homology. Using the above isomorphisms and the isomorphism in $(1.1.1)$, we see that each of the groups $\text{HH}^\ast(Z, E)$, resp. $\text{HH}_\ast(Z, E)$, has the natural structure of a right, resp. left, $(A_1 \otimes_Z A_2)^{Z}$-module.

1.2. Throughout the paper, we work over $\mathbb{C}$ and use the notation Sym $\mathfrak{g}$, resp. $\mathfrak{u} \mathfrak{g}$, for the symmetric, resp. enveloping, algebra of a vector space, resp. Lie algebra, $\mathfrak{g}$. For any scheme $X$ put $\mathbb{C}[X] := \Gamma(X, \mathcal{O}_X)$. Let $T^\ast X$, resp. $\mathcal{D}_X$ and $\mathcal{D}(X)$, denote the cotangent bundle, resp. the sheaf and ring of algebraic differential operators, on a smooth algebraic variety $X$.

Let $G$ be a complex semisimple group with trivial center, and $U, \bar{U}$, a pair of opposite maximal unipotent subgroups of $G$. Let $\mathfrak{g}$, resp. $\mathfrak{u}, \bar{\mathfrak{u}}$, denote the Lie algebra of $G$, resp. $U, \bar{U}$. We fix a nondegenerate character $\psi : \bar{\mathfrak{u}} \to \mathbb{C}$, i.e., such that $\psi(f_i) \neq 0$ for every simple root vector $f_i \in \bar{\mathfrak{u}}$.

The action of $G$ on itself by right translations gives a Lie algebra map $\mathfrak{g} \to \mathcal{D}(G)$. Let $\iota$, resp. $\bar{\iota}$, denote its restriction to the subalgebra $\mathfrak{u}$, resp. $\bar{\mathfrak{u}}$. It is well known, cf. e.g. [GR §3.1], that using the notation of Example 1.1.2(i), one has $\mathcal{D}(G/U) \cong (\mathcal{D}(G)/\mathcal{D}(G)\mathfrak{u})^\psi$. Let $\bar{u}^\psi$ be the image
of the map \( \bar{u} \to \mathcal{D}(G), \ x \mapsto i(x) - \psi(x) \). The algebra of Whittaker differential operators on \( G/\bar{U} \) is defined as a quantum hamiltonian reduction \( \mathcal{D}^\psi(G/\bar{U}) := (\mathcal{D}(G)/\mathcal{D}(G)\bar{u}^\psi)\bar{u}^\psi \), cf. also §6.3. The differential of the action of \( G \) on itself by left translations induces an algebra homomorphism \( \iota : U_g \to \mathcal{D}(G/U) \), resp. \( \iota^\psi : U_g \to \mathcal{D}^\psi(G/\bar{U}) \).

Let \( T \) be the abstract maximal torus of \( G \), and \( t = \text{Lie} T \). We have an imbedding \( \iota_T : U_t \hookrightarrow \mathcal{D}(T) \) as the subalgebra of translation invariant differential operators. There is a natural \( T \)-action on \( G/U \) by right translations. The differential of this action induces an algebra homomorphism \( \iota_T : U_t \to \mathcal{D}(G/U) \).

Let \( W \) be the (abstract) Weyl group, \( Z \) the center of the algebra \( U_g \), and \( hc : Z \to (U_t)^W \) the Harish-Chandra isomorphism, where \( W \)-invariants are taken with respect to the ‘dot-action’ of \( W \) on \( U_t \). One has the following diagram of algebra homomorphisms:

\[
\begin{align*}
\mathcal{D}^\psi(G/\bar{U}) & \xrightarrow{\iota^\psi} U_g \xrightarrow{hc} (U_t)^W \xrightarrow{\iota} \mathcal{D}(T) .
\end{align*}
\]

Let \( \iota_1 : Z \to \mathcal{D}(T) \), resp. \( \iota_2 : Z \to \mathcal{D}^\psi(G/\bar{U}) \), be the composite homomorphism on the right, resp. left, of \([1.2.1]\). We apply the construction of Hamiltonian reduction in the setting of Example \([1.1.2](ii)\) for the triple \( A_1 = \mathcal{D}(T) \), \( A_2 = \mathcal{D}^\psi(G/\bar{U}) \), \( Z = Z \), and the homomorphisms \( \iota_1, \iota_2 \).

With the above notation, the main result of the paper reads as follows.

**Theorem 1.2.2.** There is a natural algebra isomorphism

\[
\mathcal{D}(G/U) \cong (\mathcal{D}(T) \otimes_{Z \bar{g}} \mathcal{D}^\psi(G/\bar{U}))^{Z \bar{g}} ,
\]

such that the map \( i_* \), resp. \( i \), corresponds via \([1.2.3]\) to the map \( u \mapsto i_T(u) \otimes 1 \), resp. \( u \mapsto 1 \otimes i^\psi(u) \).

The Weyl group acts on the RHS of \([1.2.3]\) via its natural action on \( \mathcal{D}(T) \), the first tensor factor. Thanks to the theorem, one can transport the \( W \)-action on the RHS of \([1.2.3]\) to the LHS. We obtain a \( W \)-action on \( \mathcal{D}(G/U) \) by algebra automorphisms. One can check, although we will not do that in the present paper, that the \( W \)-action thus obtained is the same as the Gelfand-Graev action (it is sufficient to check this for simple reflections, which reduces to a rank one computation).

**Remark 1.2.4.** In the main body of the paper we explain how to define the varieties \( G/U, G/\bar{U} \), as well as the ring \( \mathcal{D}^\psi(G/\bar{U}) \), in a canonical way that involves neither the choice of a pair of opposite unipotent subgroups \( U, \bar{U} \), see §2.1, nor a choice of nondegenerate character \( \psi \), see §§6.3-6.4. The isomorphism in \([1.2.3]\) then becomes canonical. Therefore, our construction of the Gelfand-Graev action yields a canonical \( W \)-action, while the original construction of Gelfand and Graev depends on the choice of a pinning on \( G \).

**Theorem 1.2.2** combined with the observation at the end of Example \([1.1.2](ii)\) and the fact that each of the algebras \( \mathcal{D}(G/U) \) and \( \mathcal{D}(T) \) is isomorphic to its opposite, gives the following result:

**Corollary 1.2.5.** For each \( i \geq 0 \), the assignment \( E \mapsto \text{HH}_i(Z \bar{g}, E) \), resp. \( E \mapsto \text{HH}^i(Z \bar{g}, E) \), gives a functor from the category of \( (\mathcal{D}^\psi(G/\bar{U}), \mathcal{D}(T)) \)-bimodules to the category of \( \mathcal{D}(G/U) \)-modules.

**Remark 1.2.6.** It is not difficult to show that each of the two algebras \( \mathcal{D}(T) \) and \( \mathcal{D}^\psi(G/\bar{U}) \) is flat as (either left or right) \( Z \bar{g} \)-module. It follows that \( \text{HH}^i(\mathcal{D}(T) \otimes_{\bar{g}} \mathcal{D}^\psi(G/\bar{U})) = 0 \) for all \( i > 0 \). It might be interesting to find Hochschild cohomology groups \( \text{HH}^i(Z \bar{g}, (\mathcal{D}(T) \otimes_{\bar{g}} \mathcal{D}^\psi(G/\bar{U}))) \) for \( i > 0 \).

1.3. **Theorem 1.2.2** has a ‘quasi-classical’ counterpart. In more detail, write \( V^\perp \subseteq g^* \) for the annihilator of a vector subspace \( V \subseteq g \) and let \( \psi + \bar{u}^\perp := \{ \phi \in g^* \mid \phi|_{\bar{u}^\perp} = \psi \} \), an affine linear subspace of \( g^* \). Let

\[
\mathcal{T}^*(G/U) = G \times_U u^\perp \quad \text{resp.} \quad \mathcal{T}^\psi(G/\bar{U}) := G \times_U (\psi + \bar{u}^\perp) .
\]

(1.3.1)
Here, the equality on the left is a standard isomorphism of $G$-equivariant vector bundles on $G/U$; the equality on the right is our definition of $T^\psi(G/U)$. The variety $T^\psi(G/U)$ is a twisted cotangent bundle on $G/U$ that may be thought of as a deformation of $T^*(G/U)$. In particular, $T^\psi(G/U)$ comes equipped with a natural symplectic structure such that the map $T^\psi(G/U) \to G/U$ is a $G$-equivariant affine bundle on $G/U$ with Lagrangian fibers. Further, the $G$-action on $T^\psi(G/U)$, resp. $T^\psi(G/U)$, is Hamiltonian with moment map $\mu$, resp. $\mu^\psi$. The $T$-action on $G/U$ by right translations induces a Hamiltonian action on $T^*(G/U)$. Let $\mu_T : T^*(G/U) \to t^*$, resp. $\mu^T : T^*T \to t^*$, be the moment map for the $T$-action on $T^*(G/U)$, resp. $T^*T$. Finally, let $\mathfrak{g}^* \to \mathfrak{g}^* / G = \text{Spec}(\mathbb{C}[\mathfrak{g}^*/G])$ be the (co)adjoint quotient morphism and identify $\mathfrak{c}$ with $t^*/W$ via the Chevalley isomorphism.

A quasi-classical counterpart of diagram \ref{diagram:1.2.1} reads as follows:

$$
\begin{array}{cccccc}
T^\psi(G/U) & \xrightarrow{\mu^\psi} & \mathfrak{g}^* & \xrightarrow{\iota} & t^*/W & \xrightarrow{\mu_T} T^*T.
\end{array}
$$

Let $p_1 : T^*T \to \mathfrak{c}$, resp. $p_2 : T^\psi(G/U) \to \mathfrak{c}$, be the composite map on the right, resp. left, of the diagram. Further, let $\mathfrak{c}_\Delta \subseteq \mathfrak{c} \times \mathfrak{c}$ denote the diagonal and $T^*T \times_{\mathfrak{c}} T^\psi(G/U) := (p_1 \times p_2)^{-1}(\mathfrak{c}_\Delta)$.

We equip $T^*T \times T^\psi(G/U)$ with the natural symplectic structure of a cartesian product, where the symplectic form on the first factor is taken with a negative sign. It turns out that $T^*T \times T^\psi(G/U)$ is a smooth coisotropic subvariety of $T^*T \times T^\psi(G/U)$; furthermore, the 0-foliation on this subvariety is generated by Hamiltonian vector fields associated with the functions $\bar{z} := p_1(z) - p_2(z)$, $z \in \mathbb{C}[\mathfrak{c}]$. It follows that there is a well defined pairing

$$
\mathbb{C}[\mathfrak{c}] \times \mathbb{C}[T^*T \times_{\mathfrak{c}} T^\psi(G/U)] \to \mathbb{C}[T^*T \times_{\mathfrak{c}} T^\psi(G/U)], \quad (f, z) \mapsto \{\bar{z}, f\},
$$

induced by the Poisson bracket. Moreover, the Poisson centralizer

$$
\mathbb{C}[T^*T \times_{\mathfrak{c}} T^\psi(G/U)]^{\mathfrak{c}[\mathfrak{c}]} := \left\{ f \in \mathbb{C}[T^*T \times_{\mathfrak{c}} T^\psi(G/U)] \mid \{\bar{z}, f\} = 0 \ \forall z \in \mathbb{C}[\mathfrak{c}] \right\}
$$

inherits the structure of a Poisson algebra, cf. also Section \ref{section:3.2}.

The Poisson algebra in \ref{equation:1.3.2} is a quasi-classical counterpart of \ref{equation:1.1.3}. The quasi-classical counterpart of Theorem \ref{theorem:1.2.2} reads as follows.

**Theorem 1.3.3.** There is a natural $G \times T$-equivariant Poisson algebra isomorphism

$$
\mathbb{C}[T^*(G/U)] \cong \mathbb{C}[T^*T \times_{\mathfrak{c}} T^\psi(G/U)]^{\mathfrak{c}[\mathfrak{c}]},
$$

such that the pull-back $\mu^* : \mathbb{C}[t^*] \to \mathbb{C}[T^*(G/U)]$, resp. $\mu^* : \mathbb{C}[\mathfrak{g}^*] \to \mathbb{C}[T^*(G/U)]$, corresponds via the isomorphism, to the map $f \mapsto \mu^*_T(f) \otimes 1$, resp. $f \mapsto 1 \otimes (\mu^\psi)^*(f)$.

The natural Weyl group action on $T^*T$ induces an action on $\mathbb{C}[T^*T \times_{\mathfrak{c}} T^\psi(G/U)]$. The subalgebra $\mathbb{C}[T^*T \times_{t^*/W} T^\psi(G/U)]^{\mathfrak{c}[\mathfrak{c}]}$ is $W$-stable. Transporting the $W$-action via the isomorphism of Theorem \ref{theorem:1.3.3} yields a $W$-action on $\mathbb{C}[T^*(G/U)]$ by Poisson algebra automorphisms. Thanks to \cite{GR}, the algebra $\mathbb{C}[T^*(G/U)]$ is finitely generated. Therefore, $T^*(G/U)_{\text{aff}} := \text{Spec} \mathbb{C}[T^*(G/U)]$, the affinization of $T^*(G/U)$, is an affine variety that comes equipped with a Poisson structure and a Hamiltonian $G \times T$-action. From Theorem \ref{theorem:1.3.3} we deduce

**Corollary 1.3.4.** There exists a $W$-action on $(T^*(G/U))_{\text{aff}}$ with the following properties:

(i) The actions of $W$ and $T$ combine together to give a $W \times T$-action on $(T^*(G/U))_{\text{aff}}$ such that the moment map $(T^*(G/U))_{\text{aff}} \to t^*$, for the $T$-action, is a $W$-equivariant morphism.

(ii) The $G$-action commutes with the $W \times T$-action.

The $W$-action on $(T^*(G/U))_{\text{aff}}$ is a quasi-classical counterpart of the Gelfand-Graev action. A different construction of the same $W$-action on $(T^*(G/U))_{\text{aff}}$ was given earlier in \cite{GR} Proposition 5.5.1].
Remarks 1.3.5. (i) The $W$-action on $(T^*(G/U))_{\text{aff}}$ does not commute with the $\mathbb{C}^*$-action that comes from the dilation action on the fibers of $T^*(G/U) \to G/U$.

(ii) Analogues of Theorems 1.2.2 and 1.3.3 are likely to hold for any connected semisimple group $G$, not necessarily of adjoint type. The case of a simply connected group will be discussed in §5.4.

The geometry of $T^*(G/U)$ is, in a way, much simpler than that of $T^*(G/U)$. Indeed, the variety $T^*(G/U)$ is affine, the $G$-action on $T^*(G/U)$ is free, and the map $\mu^\psi$ is a smooth morphism with image $g_r$, the set of regular (not necessarily semisimple) elements of $g^*$. On the other hand, the variety $T^*(G/U)$ is only quasi-affine, the $G$-action on $T^*(G/U)$ is free only generically, the map $\mu : T^*(G/U) \to g^*$ is not flat and its image is the whole of $g$.

The Poisson variety $T^*(G/U)_{\text{aff}}$ is, we believe, quite interesting and it deserves further study.

In the special case of type $A$, the variety $T^*(G/U)_{\text{aff}}$ has a quiver construction, [DKS], analogous to Nakajima’s construction of $T^*(G/B)$, [NA]. In this case, a construction of the Gelfand-Graev action which is similar to the construction, due to Lusztig and Nakajima, of Weyl group actions on quiver varieties was found in [W].

We propose the following

Conjecture 1.3.6. For any semisimple group $G$, the variety $T^*(G/U)_{\text{aff}}$ has symplectic singularities, [Be]; in particular, $T^*(G/U)_{\text{aff}}$ is a union of finitely many symplectic leaves, [Kn].

Using the quiver interpretation it was shown in [J] that Conjecture 1.3.6 holds in type $A$.

In the case $G = SL_3$ (which is not a group of adjoint type) the variety $(T^*(G/U))_{\text{aff}}$ is well understood, see [J]. Specifically, it is isomorphic to the closure, $\bar{O}$, of the minimal nilpotent orbit $O$ in $so_4$. The Gelfand-Graev action of the Weyl group $W = S_3$ on $(T^*(G/U))_{\text{aff}} \cong \bar{O}$ comes from triality, and the Poisson structure on $(T^*(G/U))_{\text{aff}}$ agrees with the Kirillov-Kostant symplectic structure on $O$. Since $\bar{O} = O \cup \{0\}$, there are two symplectic leaves: $\{0\}$ and $O$. It is known that $\bar{O}$ does not have a symplectic resolution, [Fu].

1.4. Interpretation via the affine Grassmannian. Let $K = \mathbb{C}((z))$, resp. $O = \mathbb{C}[[z]]$. Let $G'$ be the Langlands dual group of $G$ and $Gr = G'(K)/G'(O)$ the affine Grassmannian. Since $G$ is of adjoint type, the group $G'$ is simply-connected, so $Gr$ is connected. The group $G'$ comes equipped with the canonical maximal torus $T'$. Let $T = G_m \times T'$. The group $T$ acts on $Gr$, where the factor $G_m$ acts by loop rotation. Let $Gr^T$ be the $T$-fixed point set. There is a canonical bijection $Gr^T \cong Q$, where $Q$ is the root lattice of $G$. We write $pt_\lambda$ for the $T$-fixed point corresponding to $\lambda \in Q$, and $i^\lambda : \{pt_\lambda\} \hookrightarrow Gr$ for the imbedding. Let $\mathbb{C}_X$ denote a constant sheaf on a space $X$. The restriction map $i^\lambda_\ast : H^1_T(Gr) = H^1_T(Gr, \mathbb{C}_{Gr}) \to H^1_T(pt_\lambda)$ is a surjective algebra homomorphism. Let $J_\lambda$ be its kernel, an ideal of $H^1_T(Gr)$.

The equivariant cohomology $H^1_T(F) = H^1_T(Gr, F)$, of a $T$-equivariant constructible complex $F$ on $Gr$, has the natural structure of a $Z$-graded $H^1_T(Gr)$-module. For such an $F$, we put $H^1_T(F)^{J_\lambda} := \{h \in H^1_T(F) \mid jh = 0, \forall j \in J_\lambda\}$.

Let $Perv_{G'(O)}(Gr)$ be the Satake category. Any object of that category is known to be a finite direct sum of the IC-sheaves $IC_\lambda := IC(Gr_\lambda)$, where $Gr_\lambda$ denotes the closure of the $G'(O)$-orbit of $pt_\lambda$. Furthermore, it is known that objects of $Perv_{G'(O)}(Gr)$ come equipped with a canonical $G_m \ltimes G'(O)$-equivariant, in particular $T$-equivariant, structure.

From Theorem 1.3.4 of the present paper combined with some results from [BF] and [GR], we will show in section 7.5 that the following theorem is essentially equivalent, via geometric Satake, to a combination of Theorems 1.2.2 and 1.3.3.

Theorem 1.4.1. For any $F \in Perv_{G'(O)}(Gr)$ and $\lambda \in Q$, the adjunction $(i^\lambda_\ast)^\dagger : H^1_T(pt_\lambda)^{J_\lambda} \to H^1_T(F)^{J_\lambda}$ induces an isomorphism

$$H^1_T(pt_\lambda)^{J_\lambda} \cong H^1_T(F)^{J_\lambda}.$$


Remarks 1.4.2. (i) Let $\mu, \lambda \in \mathbb{Q}$ be such that $pt_\lambda \in \overline{\text{Gr}}_\mu$, let $i_{\lambda, \mu} : \{pt_\lambda\} \hookrightarrow \overline{\text{Gr}}_\mu$ be the imbedding, and put $C_{\lambda, \mu} := (i_{\lambda, \mu})_\ast \mathcal{C}_{\text{pt}_\lambda}$. Write $\text{Ext}^*_T(-, -)$ for $\text{Ext}$-groups in the $T$-equivariant constructible derived category of $\overline{\text{Gr}}_\mu$. For any $\mathcal{F}$ in that category, there is a canonical isomorphism $H^*_T(i_{\lambda, \mu}_! \mathcal{F}) = \text{Ext}^*_T(C_{\lambda, \mu}, \mathcal{F})$. On the other hand, since $H^*_T(C_{\lambda, \mu})$ and $H^*_T(\text{Gr})/T_\lambda$ are isomorphic $H^*_T(\text{Gr})$-modules, we find $H^*_T(\mathcal{F})/J_\lambda = \text{Hom}_{H^*_T(\text{Gr})}(H^*_T(\mathcal{F}), H^*_T(\mathcal{F})) = \text{Hom}_{H^*_T(\text{Gr})}(H^*_T(C_{\lambda, \mu}), H^*_T(\mathcal{F}))$. Thus, the isomorphism of the theorem amounts to the claim that, for all pairs $\lambda, \mu$, as above, the functor $H^*_T(-)$ induces an isomorphism

$$\text{Ext}^*_T(C_{\lambda, \mu}, \text{IC}_\mu) \cong \text{Hom}_{H^*_T(\text{Gr})}(H^*_T(C_{\lambda, \mu}), H^*_T(\text{IC}_\mu)).$$

(ii) A non-equivariant analogue of the above isomorphism, hence of Theorem 1.4.1, is false, in general. Indeed, the groups $H^*(C_{\lambda, \mu})$ and $H^*(C_{\nu, \mu})$ are isomorphic $H^*(\text{Gr})$-modules, for any pair of points $pt_\lambda, pt_\nu \in \overline{\text{Gr}}_\mu$. On the other hand, the $H^*(\text{Gr})$-modules $H^*(i_{\lambda, \mu}_! \text{IC}_\mu)$ and $H^*(i_{\nu, \mu}_! \text{IC}_\mu)$ are clearly not necessarily isomorphic, in general.

(iii) Isomorphism (1.4.3) is reminiscent of a result from [Gi1]. Specifically, according to [Gi1] there is an analogue of isomorphism (1.4.3) where the sheaf $C_{\lambda, \mu}$ is replaced by $\text{IC}_\lambda$. However, unlike the main result of [Gi1], which holds in a much more general setting, isomorphism (1.4.3) seems to be a special feature of the Satake category. It is unlikely that an analogue of (1.4.3) holds for IC-sheaves of Schubert varieties in an arbitrary (finite dimensional, say) partial flag variety.

(iv) In the setting of Theorem 1.4.1 there is a natural map $H^*_T(\mathcal{F})/J_\lambda \rightarrow H^*_T(i_{\lambda, \mu}_! \mathcal{F})$, induced by the adjunction $\mathcal{F} \rightarrow (i_{\lambda, \mu})_* i_{\lambda, \mu}^! \mathcal{F}$. This map is not necessarily injective, in general.

1.5. Layout of the paper. In section 2 we review various (well-known) constructions which allow, in particular, to define the $G$-varieties $G/U$ and $G/\tilde{U}$ in a way that does not involve a choice of unipotent subgroups $U, \tilde{U}$. In section 3 we introduce a certain torsor of the group scheme of regular centralizers. The same torsor has appeared in a less explicit way in the work of Donagi and Gaitsgory [DG]. In section 4 we study the Miura variety, a smooth $G$-stable Lagrangian correspondence in $T^*(G/U) \times T^*(G/\tilde{U})$. This variety comes equipped with commuting actions of the Weyl group and the group scheme of regular centralizers. The Miura variety is the main geometric ingredient of the proof of Theorem 1.3.3, given in section 5.

The goal of section 6 is to construct a map $\kappa$, cf. (6.2.5), between the algebras in the LHS and RHS of (1.2.3). Theorem 6.2.7, which is a more precise version of Theorem 1.2.2, states that the constructed map is an algebra isomorphism. A key role in the construction of $\kappa$ is played by a certain $\mathcal{D}(G/U) \otimes \mathcal{D}(T) \otimes \mathcal{D}(G/\tilde{U})$-module which we call the Miura bimodule, see (6.1.3). The Miura bimodule is a slightly refined version of the quotient $M = \mathcal{D}(G)/(u\mathcal{D}(G) + \mathcal{D}(G)u^\psi)$. The latter quotient has the natural structure of a $(\mathcal{D}(G/U), \mathcal{D}(G/\tilde{U}))$-bimodule. The bimodule $M$ may be viewed as a quantization of the Miura variety.

The proofs of our main results are completed in section 7. In §7.1 we show that Theorem 6.2.7 is equivalent to a result (Theorem 7.1.8) concerning singular vectors in the universal Verma module. This result may be of independent interest. Theorem 7.1.8 is proved in §7.3 by reduction to the commutative case, i.e. to Theorem 1.3.3, via a deformation argument. The proof of Theorem 1.4.1 is given in §7.5.

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2. Geometry of $G/U$

2.1. Let $G$ be a connected semisimple group with trivial center and $\mathfrak{g} = \text{Lie} \, G$. Let $B$ be the flag variety of Borel subgroups $B \subseteq G$, equivalently, Borel subalgebras $\mathfrak{b} \subseteq \mathfrak{g}$. The tori $B/[B,B]$ associated with various Borel subgroups $B \in B$ are canonically isomorphic. Let $T = B/[B,B]$ be this universal Cartan torus for $G$. Write $t = \text{Lie} \, T$, let $Q = \text{Hom}(T, \mathbb{C}^\times) \subseteq t^*$ be the root lattice of $\mathfrak{g}$, and $W$ the universal Weyl group. Let $I$ be the set of vertices of the Dynkin diagram of $G$.

Given a Borel subalgebra $\mathfrak{b}$, let $u(\mathfrak{b}) := \{\mathfrak{b},[\mathfrak{b},\mathfrak{b}]\}$ be the nilradical of $\mathfrak{b}$, and $\mathfrak{a}(\mathfrak{b}) := u(\mathfrak{b})/\{[\mathfrak{u}(\mathfrak{b}),\mathfrak{u}(\mathfrak{b})]\}$. The weights of the $T$-action on $\mathfrak{a}(\mathfrak{b})$ are called simple roots. Write $\alpha_i$ for the simple root associated with a vertex $i \in I$. For $\lambda, \mu \in Q$, we write $\lambda \leq \mu i f \mu - \lambda = \sum_{i \in I} n_i \alpha_i$ for some nonnegative integers $n_i$. The group $G$ being adjoint, the weights of any finite dimensional representation $V$ of $G$ are contained in $Q$. Associated with $\mathfrak{b} \in B$, there is a canonical $\mathfrak{b}$-stable filtration $\mathfrak{g}^{\geq \mu,\mathfrak{b}}$, $\mu \in Q$, on $\mathfrak{g}$, such that $\text{gr}^{\mu,\mathfrak{b}} \mathfrak{g} := \mathfrak{g}^{\geq \mu,\mathfrak{b}} / \mathfrak{g}^{> \mu,\mathfrak{b}}$ is the $\mu$-weight space for the natural action of the universal Cartan algebra $t = \text{Lie} \, T$. In particular, we have $\mathfrak{g}^{\geq 0,\mathfrak{b}} = \mathfrak{b}$, resp. $\mathfrak{g}^{0,\mathfrak{b}} = t$. For each $i \in I$, the space $\text{gr}^{\alpha_i,\mathfrak{b}} \mathfrak{g}$, resp. $\text{gr}^{\alpha_i,\mathfrak{b}} \mathfrak{g}$, is 1-dimensional and we put $\text{gr}^{i}(\mathfrak{b}) = (\text{gr}^{\alpha_i,\mathfrak{b}} \mathfrak{g}) \setminus \{0\}$, resp. $\text{gr}^{i}(\mathfrak{b}) = (\text{gr}^{\alpha_i,\mathfrak{b}} \mathfrak{g}) \setminus \{0\}$. Let $\text{gr}^{i}(\mathfrak{b}) = \Pi_i \text{gr}^{i}(\mathfrak{b})$, resp. $\text{gr}^{i}(\mathfrak{b}) = \Pi_i \text{gr}^{i}(\mathfrak{b})$. The action of $T$ makes $\text{gr}^{i}(\mathfrak{b})$, resp. $\text{gr}^{i}(\mathfrak{b})$, a $T$-torsor.

We put $\mathfrak{d}(\mathfrak{b}) = \sum_{i \in I} \mathfrak{g}^{\geq -\alpha_i,\mathfrak{b}}$. The map $(s_i)_{i \in I} \mapsto \sum_i s_i$, provides a $T$-equivariant isomorphism $\oplus_i \text{gr}^{\alpha_i,\mathfrak{b}} \mathfrak{g} \rightarrow \mathfrak{a}(\mathfrak{b})$, resp. $\oplus_i \text{gr}^{\alpha_i,\mathfrak{b}} \mathfrak{g} \rightarrow \mathfrak{d}(\mathfrak{b})/\mathfrak{b}$. This gives a canonical identification of the $T$-torsor $\text{gr}^{i}(\mathfrak{b}) = \Pi_i \text{gr}^{i}(\mathfrak{b})$, resp. $\text{gr}^{i}(\mathfrak{b}) = \Pi_i \text{gr}^{i}(\mathfrak{b})$, with a unique open dense $T$-orbit in $\mathfrak{a}(\mathfrak{b})$, resp. $\mathfrak{d}(\mathfrak{b})/\mathfrak{b}$. We will abuse notation and write $x + \mathfrak{b}$, resp. $x = \mathfrak{a}(\mathfrak{b})$, for the preimage of $x \in \mathfrak{d}(\mathfrak{b})/\mathfrak{b}$, resp. the open dense $T$-orbit in $\mathfrak{d}(\mathfrak{b})/\mathfrak{b}$, under the natural projection $\mathfrak{d}(\mathfrak{b}) \rightarrow \mathfrak{d}(\mathfrak{b})/\mathfrak{b}$.

The family of torsors $\mathfrak{O}(\mathfrak{b})$, resp. $\mathfrak{O}(\mathfrak{b})$, for varying $\mathfrak{b} \in B$ gives a variety

$$\bar{B} := \{(\mathfrak{b}, s) \mid \mathfrak{b} \in B, \ s \in \mathfrak{O}(\mathfrak{b})\}, \ \text{resp.} \ \bar{B}_{\sim} := \{(\mathfrak{b}, s) \mid \mathfrak{b} \in B, \ s \in \mathfrak{O}(\mathfrak{b})\}. \ (2.1.1)$$

The group $G$ acts on $B$, resp. $\bar{B}$ and $\bar{B}_{\sim}$, in a natural way. By construction, the first projection $(\mathfrak{b}, s) \mapsto \mathfrak{b}$ is a $G$-equivariant $T$-torsor on $B$.

2.2. We say that a pair $\mathfrak{b}, \mathfrak{b}^\prime \in B$, of Borel subalgebras, is in ‘opposite position’ if $\mathfrak{b} \cap \mathfrak{b}^\prime$ is a Cartan subalgebra of $\mathfrak{g}$. In that case, one has a triangular decomposition $\mathfrak{g} = u(\mathfrak{b}) \oplus (\mathfrak{b} \cap \mathfrak{b}^\prime) \oplus u(\mathfrak{b}^\prime)$ and a diagram

$$t = \mathfrak{b}/u(\mathfrak{b}) \leftarrow \mathfrak{b}^\prime \leftarrow \mathfrak{b} \cap \mathfrak{b} \leftarrow \mathfrak{b} \rightarrow \mathfrak{b}/u(\mathfrak{b}) = t.$$ 

It follows from definitions that the resulting isomorphism between the leftmost and rightmost copy of $t$, in the diagram, is given by the map $\mathfrak{t} \mapsto w_0(\mathfrak{t})$, where $w_0$ is the longest element of $W$. The assignment $\alpha_i \mapsto -\alpha_{i^\prime}$ gives a permutation $i \mapsto i^\prime$, of the set $I$. For every $i \in I$, one has a diagram

$$\text{gr}^{-\alpha_i,\mathfrak{b}} \mathfrak{g} \leftarrow \mathfrak{g}^{\geq -\alpha_i,\mathfrak{b}} \leftarrow \mathfrak{g}^{\geq -\alpha_i,\mathfrak{b}} \cap \mathfrak{g}^{\geq \alpha_{i^\prime},\mathfrak{b}} \leftarrow \mathfrak{g}^{\geq \alpha_{i^\prime},\mathfrak{b}} \rightarrow \text{gr}^{\alpha_{i^\prime},\mathfrak{b}} \mathfrak{g}.$$ 

The compositions on the left and on the right give canonical isomorphisms

$$\text{gr}^{-\alpha_i,\mathfrak{b}} \mathfrak{g} \cong \mathfrak{g}^{\geq \alpha_{i^\prime},\mathfrak{b}} \cap \mathfrak{g}^{\geq -\alpha_i,\mathfrak{b}} \cong \text{gr}^{\alpha_{i^\prime},\mathfrak{b}} \mathfrak{g}, \ \mathfrak{d}(\mathfrak{b})/\mathfrak{b} \overset{\kappa_{+}}{\cong} \mathfrak{d}(\mathfrak{b}) \cap u(\mathfrak{b}) \overset{\kappa_{-}}{\cong} \mathfrak{a}(\mathfrak{b}).$$

Thus, the map $\kappa_{+} \circ \kappa_{-}^{-1}$ yields an isomorphism

$$\mathfrak{d}(\mathfrak{b})/\mathfrak{b} \cong \mathfrak{a}(\mathfrak{b}), \ \text{resp.} \ \kappa_{\mathfrak{b},\mathfrak{b}^\prime} : \mathfrak{O}(\mathfrak{b}) \cong \mathfrak{O}(\mathfrak{b}), \ (2.2.1)$$

to be denoted $\kappa_{\mathfrak{b},\mathfrak{b}^\prime}$, such that

$$\kappa_{\mathfrak{b},\mathfrak{b}^\prime}(ts) = w_0(t)^{-1} \kappa_{\mathfrak{b},\mathfrak{b}^\prime}(s), \ \forall \ t \in T, \ s \in \mathfrak{O}(\mathfrak{b}). \ (2.2.2)$$

Let $G_D \subseteq G \times G$, resp. $T_D \subseteq T \times T$, be the diagonal. The set $\Omega$ formed by the pairs of Borel subalgebras in opposite position is a unique open dense $G_D$-orbit in $B \times B$. Let $\tilde{\Omega}$ be the preimage
of \( \Omega \) under the projection \( B_\times B \rightarrow B \times B \). We also consider a subvariety of \( \tilde{\Omega} \) defined as follows:

\[
\Xi := \{(\tilde{b}, \tilde{x}, b, x) \in B_\times B | (b, b) \in \Omega, \ k_{\tilde{b}, b}(\tilde{x}) = x \}. \tag{2.2.3}
\]

The group \( G \times G \), resp. \( T \times T \), acts on \( B_\times \tilde{B} \) on the left, resp. right. The variety \( \tilde{\Omega} \) is \( T \times T \times G_{\Delta} \)-stable, resp. \( \Xi \) is \( T_{\Delta} \times G_{\Delta} \)-stable.

Fix a pair \( B, \tilde{B} \) of Borel subgroups in opposite position and an element \( s \in \Omega_{\Delta}(\tilde{b}) \). The stabilizer of the point \((b, k_{\tilde{b}, b}(s)) \in \tilde{B}_\times \) is \( U \) stable, resp. \( \Xi \) is \( T_\times B_{\times} \)-stable. Thus, the \( G \)-action gives a \( G \)-equivariant isomorphism \( G/U \rightarrow \tilde{B}_\times \). With this identification, we have

\[
\Xi = \{(gU/\tilde{U}, gU/U) \in G/U \times G/U, g \in G \}. \tag{2.2.4}
\]

The action of \( T_\Delta \) on \( \Xi \) is given by the formula \( t : (gU/\tilde{U}, gU/U) \mapsto (gtU/\tilde{U}, gtU/U) \).

One has a natural map \( p_\Omega : \Xi \rightarrow \Omega \), resp. \( p_- : \Xi \rightarrow B_\times \) and \( p : \Xi \rightarrow \tilde{B}_\times \), given by \( p_\Omega(b, \tilde{x}, b, x) = (b, b) \), resp. \( p_-(b, \tilde{x}, b, x) = (b, \tilde{x}) \) and \( p(b, \tilde{x}, b, x) = (b, x) \).

**Lemma 2.2.5.** (i) The map \( T \times T \rightarrow \tilde{\Omega} \), \((t, (b, \tilde{x}, b, x)) \mapsto (b, \tilde{x}, b, xt)\), is a \( T \times G_{\Delta} \)-equivariant isomorphism. Furthermore, the variety \( \Xi \) is closed in \( B_\times \tilde{B} \).

(ii) The map \( p_\Omega : \Xi \rightarrow \Omega \) is a \( G_{\Delta} \)-equivariant \( T_\Delta \)-torsor; moreover, \( \Xi \) is a \( G_{\Delta} \)-torsor.

(iii) Each of the two maps below is a \( G_{\Delta} \)-equivariant isomorphism:

\[
B_\times B \Omega \xrightarrow{p_- \times p_\Omega} \Xi \xrightarrow{p_\Omega \times p} \Omega \times B \tilde{B}. \tag{2.2.6}
\]

**Proof.** All statements except for the second statement in (i) are immediate from (2.2.3). To prove the second statement in (i), recall that the variety \( B_\times \) is quasi-affine, thanks to the Plücker imbedding. The orbits of a unipotent group action on an affine variety are known to be closed, cf. eg. [Di], §11.2.4. It follows that any \( U \)-orbit in \( \tilde{B}_\times \) is a closed subset of the affine closure of \( B_\times \), hence this orbit is closed in \( B_\times \). The fiber of the map \( p \) over \( 1U/U \in \tilde{B}_\times \) is a single \( U \)-orbit in \( B_\times \). We deduce that the \( G_{\Delta} \)-orbit of the point \((1U/U, 1U/U)\) is closed in \( G/U \times G/U \), and we are done by (2.2.4). \( \square \)

### 3. A TORSOR ON THE SET OF REGULAR ELEMENTS

3.1. The quotient \( c(b) = b/[u(b), u(b)] \) has the natural structure of a Lie algebra such that \( a(b) = u(b)/[u(b), u(b)] \) is an abelian ideal of \( c(b) \), and we have \( c(b)/a(b) = t \). If confusion is unlikely, we will use simplified notation \( d = d(b), a = a(b), \) etc. We use the Killing form to identify \( g \) with \( g^* \). We obtain the following isomorphisms:

\[
\begin{array}{cccccc}
\mathfrak{u} & \subset & \mathfrak{b} & \subset & \mathfrak{d} & \rightarrow & \mathfrak{d}/\mathfrak{u} \rightarrow \mathfrak{d}/\mathfrak{b} \subset \mathfrak{g}/\mathfrak{b} \subset \mathfrak{g}/\mathfrak{u} \\
\mathfrak{b}^\perp & \subset & \mathfrak{u}^\perp & \subset & [\mathfrak{u}, \mathfrak{u}]^\perp & \rightarrow & \mathfrak{e}^* \rightarrow \mathfrak{a}^* \subset \mathfrak{u}^* \subset \mathfrak{b}^* \end{array} \tag{3.1.1}
\]

Recall the notation \( c = g/G := \text{Spec}(\mathbb{C}[g]^G) \). We consider the following diagram:

\[
g \xrightarrow{x \mapsto (b, x): \pi} \tilde{g} := \{(b, x) \in B \times \mathfrak{g} | x \in \mathfrak{b} \} \xrightarrow{\nu: (b, x) \mapsto x \mod u(b)} t. \tag{3.1.2}
\]

The map \( \pi \) is a projective morphism, the Grothendieck-Springer morphism. The morphism \( \nu \) is smooth and the map \( \pi \times \nu \) factors through \( g \times t \). The above maps are \( G \)-equivariant where \( G \) acts diagonally on \( \tilde{g} \) and trivially on \( t \). The first projection \((b, x) \mapsto b\) makes \( \tilde{g} \) a \( G \)-equivariant vector bundle on \( B \) with fiber \( b \).
The stabilizer of an element \((b,s) \in \tilde{B}\) is the maximal unipotent subgroup associated with the Lie algebra \(u(b)\). It follows that one has
\[
\mathcal{T}^* \tilde{B} = \left\{ (b,s,x) \mid b \in B, s \in O(b), x \in u(b)^\perp \right\}.
\]
The \(G \times T\)-action on \(\tilde{B}\) induces a Hamiltonian \(G \times T\)-action on \(\mathcal{T}^* \tilde{B}\) with moment map \(\mu_{\mathcal{T}^* \tilde{B} \times \nu_{\mathcal{T}^* \tilde{B}}} : \mathcal{T}^* \tilde{B} \to g^* \times t^*\). A choice of base point \((b,s_0) \in \tilde{B}\) gives a \(G\)-equivariant isomorphism
\[
G \times_U u(b)^\perp \cong \mathcal{T}^* \tilde{B}, \quad (g,x) \mapsto (\text{Ad} g(b), \text{Ad} g(s_0), \text{Ad} g(x)),
\] (3.1.3)
where \(B\) is the Borel subgroup with Lie algebra \(b\), resp. \(U = [B,B]\) and \(\text{Ad} g(s_0)\) is an element of \(O(\text{Ad} g(b))\). Using the identifications \((3.1.1)\), we get the following \(G\)-equivariant isomorphisms:
\[
\begin{align*}
\mathcal{T}^* \tilde{B} & \xrightarrow{\mu_{\mathcal{T}^* \tilde{B} \times \nu_{\mathcal{T}^* \tilde{B}}}} (\mathcal{T}^* \tilde{B})/T \cong g^* \times g^*/G t^* \\
\tilde{g} \times_B \tilde{B} & \xrightarrow{T\text{-torsor}} \tilde{g} \xrightarrow{\pi \times \nu} g \times_{\epsilon} t,
\end{align*}
\] (3.1.4)
where \(\tilde{g} : \tilde{g} \times_B \tilde{B} \to \tilde{g}\) is a pull-back of the \(T\)-torsor \(g : \tilde{B} \to B\) via the vector bundle map \(\tilde{g} \to B\).

From now on, we will identify the first, resp. second and third, object of the top row of diagram \((3.1.4)\) with the first, resp. second and third, object of the bottom row of the diagram. Thus, we view \(\tilde{g}\) as a \(G\)-equivariant map \(\mathcal{T}^* \tilde{B} \to (\mathcal{T}^* \tilde{B})/T = \tilde{g}\), so we have \(\mu_{\mathcal{T}^* \tilde{B}} = \pi \circ \tilde{g}\), resp. \(\nu_{\mathcal{T}^* \tilde{B}} = \nu \circ \tilde{g}\).

3.2. We write \(G_x\) for the stabilizer of an element \(x \in g\) under the \(G\)-action and let \(g_x = \text{Lie} G_x\).

We say that \(x\) is regular if \(\dim g_x = \text{rk} \ g\). Let \(g_r\) be the set of regular (not necessarily semisimple) elements of \(g\). Let \(G\) act on itself by conjugation and on \(g_r \times G\) diagonally. We define
\[
\mathfrak{Z} := \{(x,g) \in g_r \times G \mid \text{Ad} g(x) = x\}.
\]
The fiber of \(\mathfrak{Z}\) over \(x \in g_r\) equals \(G_x\), which is a connected abelian group. \([Ko1]\), Proposition 14. This makes \(\mathfrak{Z} \to g_r\) a smooth connected \(G\)-equivariant abelian subgroup scheme of the constant group scheme \(g_r \times G \to g_r\), cf. eg. \([Ngo]\), §2.1.

We introduce the following incidence variety
\[
\mathcal{X} := \{(b,x) \in B \times g \mid x \in O_-(b) + b\}.
\] (3.2.1)
We let \(G\) act on \(\mathcal{X}\) diagonally. We have the following \(G\)-equivariant maps
\[
\begin{align*}
\tilde{B}_- \xrightarrow{(b,x \mod b) \mapsto (b,x)} g_r \xrightarrow{\vartheta} \mathcal{X} \xrightarrow{\mu_{\mathcal{X}} \colon (b,x) \mapsto x} g_r \xrightarrow{\vartheta} \mathfrak{c}.
\end{align*}
\] (3.2.2)
The map \(q\) is a fibration on \(\tilde{B}_-\) with affine-linear fibers \(q^{-1}(b,s) \cong s + b\), and \(\mu_{\mathcal{X}}\) is a projective morphism.

Let \(Y\) be a \(G\)-variety and \(f : Y \to g_r\) a \(G\)-equivariant map. For any \(x \in g_r\), the fiber \(f^{-1}(x)\) is \(G_x\)-stable. The family of maps \(G_x \times f^{-1}(x) \to f^{-1}(x), x \in g_r\), yields an action \(\mathfrak{Z} \times g_r Y \to Y\); furthermore, the action map is \(G\)-equivariant.

\textbf{Lemma 3.2.3.}  
(i) We have \(\mu_{\mathcal{X}}(\mathcal{X}) = g_r\), resp. \(\vartheta(g_r) = c\), where \(\vartheta : g \to c = g_r / G\) is the adjoint quotient.

(ii) The map \(\mu_{\mathcal{X}} : \mathcal{X} \to g_r\) is a \(3\)-torsor in the étale topology, in particular, \(\mu_{\mathcal{X}}\) is a smooth morphism.

(iii) The composite \(\mathcal{X} \xrightarrow{\mu_{\mathcal{X}}} g_r \xrightarrow{\vartheta} c\) is a trivial \(G\)-torsor on \(c\).
To prove the lemma, it is convenient to choose, once and for all, a principal $\mathfrak{g}_L$-triple $(e, h, f)$. Let $\mathfrak{b}_e$, resp. $\mathfrak{b}_f$, be the unique Borel subalgebra of $\mathfrak{g}$ that contains the element $e$, resp. $f$. Let $u_\mathfrak{e} = u(\mathfrak{b}_e)$, resp. $u_\mathfrak{f} = u(\mathfrak{b}_f)$. We use similar notation for the corresponding subgroups of $G$. Since $\mathfrak{O}_-(\mathfrak{b}_f)$ is a $T$-torsor, the imbedding $e + \mathfrak{b}_f \hookrightarrow \mathfrak{O}_-(\mathfrak{b}_f) + \mathfrak{b}_f$ induces a $B_f$-equivariant isomorphism

$$B_f \times U_f (e + \mathfrak{b}_f) \xrightarrow{\sim} \mathfrak{O}_-(\mathfrak{b}_f) + \mathfrak{b}_f.$$  \hfill (3.2.4)

Let $e + \mathfrak{g}_f$ be the Kostant slice. We recall the following well-known result, see [Ko2].

**Proposition 3.2.5.** (i) The map $\vartheta : \mathfrak{g}_r \rightarrow \epsilon$ is a smooth and surjective morphism; moreover, each fiber of this map is a single $G$-orbit in $\mathfrak{g}_r$.  

(ii) We have $e + \mathfrak{g}_f \subseteq \mathfrak{g}_r$. Furthermore, the slice $e + \mathfrak{g}_f$ meets every $G$-orbit in $\mathfrak{g}_r$ transversely at a single point, in particular, one has $(e + \mathfrak{g}_f) \cap \text{Ad} G(e) = \{e\}$.  

(iii) The composition $e + \mathfrak{g}_f \hookrightarrow \mathfrak{g}_r \xrightarrow{\vartheta} \epsilon$ is an isomorphism.  

(iv) The action map $U_f \times (e + \mathfrak{g}_f) \rightarrow e + \mathfrak{b}_f$ is an isomorphism. \hfill $\square$

**Proof of Lemma 3.2.3.** We have the following chain of $G$-equivariant isomorphisms:

$$X \cong G \times_B (\mathfrak{O}_-(\mathfrak{b}_f) + \mathfrak{b}_f) \cong G \times U_f (e + \mathfrak{b}_f) \cong G \times U_f (U_f \times (e + \mathfrak{g}_f)) \cong G \times (e + \mathfrak{g}_r),$$  \hfill (3.2.6)

where the first isomorphism is immediate from \[(3.2.1), the second isomorphism follows from \[(3.2.4), and the third isomorphism follows from Proposition 3.2.5(iv). Using \[(3.2.6) and the identification $\widetilde{\mathcal{B}} = G/U_f$, the maps in \[(3.2.2) read as follows:

$$\widetilde{\mathcal{B}} = G/U_f \xrightarrow{g U_f / U_f \sim (g, x); g} X \cong G \times U_f (e + \mathfrak{b}_f) \cong G \times (e + \mathfrak{g}_r) \xrightarrow{\mu_X; (g, x) \mapsto \text{Ad} g(x)} \mathfrak{g}.$$  

This yields parts (i) and (iii) of Lemma 3.2.3. Also, it is immediate from Proposition 3.2.5(i) and the explicit description above that the differential of the map $\mu_X$ is surjective, hence this map is a smooth morphism. To prove (ii) we first show that $X \rightarrow \mathfrak{g}_r$ is a $\mathfrak{z}$-quasi-torsor, i.e. the action map $a : \mathfrak{z} \times_{\mathfrak{g}_r} X \rightarrow X \times_{\mathfrak{g}_r} X$ is an isomorphism. The varieties involved are smooth since $\mathfrak{z}$ and $X$ are smooth schemes over $\mathfrak{g}_r$. Thus, it suffices to show that $a$ is a bijection. To this end, let $x \in \mathfrak{g}_r$. It follows from (iii) that the map $\mu_X : \mu_X^{-1}(\text{Ad} G(x)) \rightarrow \text{Ad} G(x)$ can be identified with the quotient map $G \rightarrow G/G_x$. This identification respects the $G$-action. Furthermore, the fiber $\mu_X^{-1}(x) \subseteq \mu_X^{-1}(\text{Ad} G(x))$ goes, via the identification, to the subgroup $G_x \subseteq G$. This yields a $G_x$-equivariant isomorphism $\mu_X^{-1}(x) \cong G_x$, proving that $X \rightarrow \mathfrak{g}_r$ is a quasi-torsor.

It remains to show that this quasi-torsor is étale locally trivial. By $G$-equivariance, it suffices to show that for any $x \in e + \mathfrak{g}_f$ the quasi-torsor has a section on an étale neighborhood of $x$. To construct such a section, recall that the Lie algebra $\mathfrak{g}_e$ is an ad $h$-stable subspace of $\mathfrak{u}_e$ and let $\mathfrak{v}$ be an arbitrary ad $h$-stable vector space complement of $\mathfrak{g}_e$ in $\mathfrak{u}_e$. The group $U_e$ being unipotent, the image of $\mathfrak{v}$ under the exponential map $u_e \mapsto U_e$ is a closed algebraic subvariety $V \subseteq U_e$; moreover, the map $\exp : \mathfrak{v} \rightarrow V$ is an isomorphism of algebraic varieties. We define a map $f : V \times (\mathfrak{O}_-(\mathfrak{b}_f) + \mathfrak{b}_f) \rightarrow \mathfrak{g}_e$ by the assignment $(v, y) \mapsto \text{Ad} v(y)$.

To complete the proof we observe that Proposition 3.2.5(ii) implies, using that $u_e = \mathfrak{v} \oplus \mathfrak{g}_e$, that one has a direct sum decomposition $\mathfrak{g} = \mathfrak{d}(\mathfrak{b}_f) \oplus [e, \mathfrak{v}]$, see Section 2.1 for the definition of $\mathfrak{d}(\mathfrak{b}_f)$. It follows that for any $x$ in a Zariski neighborhood of $e$ in $e + \mathfrak{g}_f$ one has $\mathfrak{g} = \mathfrak{d}(\mathfrak{b}_f) \oplus [x, \mathfrak{v}]$. Using the standard $G_m$-action that contracts the slice $e + \mathfrak{g}_f$ to $e$ and the fact that the vector spaces $\mathfrak{d}(\mathfrak{b}_f)$ and $\mathfrak{v}$ are $G_m$-stable, we deduce that a similar direct sum decomposition holds for all $x \in e + \mathfrak{g}_f$. Now, the set $\mathfrak{O}_-(\mathfrak{b}_f) + \mathfrak{b}_f$ is an open subset of $\mathfrak{d}(\mathfrak{b}_f)$ and for $x \in e + \mathfrak{g}_f$, we have $(1, x) \in V \times (\mathfrak{O}_-(\mathfrak{b}_f) + \mathfrak{b}_f)$. It follows from the decomposition $\mathfrak{g} = \mathfrak{d}(\mathfrak{b}_f) \oplus [x, \mathfrak{v}]$ that the differential of the map $f$ at the point $(1, x)$ is a vector space isomorphism. Hence, there is a Zariski open neighborhood $D \subseteq V \times (\mathfrak{O}_-(\mathfrak{b}_f) + \mathfrak{b}_f)$, of $(1, x)$, such that the restriction of $f$ to $D$ is an étale morphism. Now, view (as we may) $X \times_{\mathfrak{g}_r} D$ as a subvariety of $(\mathcal{B} \times \mathfrak{g}_r) \times_{\mathfrak{g}_r} D = \mathcal{B} \times D$. Then, it is immediate from definitions that the map
\[D \to \mathcal{B} \times D, (v, y) \mapsto (\text{Ad} \, v(b), v, y),\] provides a section of the \(f^* \mathfrak{z}\)-torsor \(f^* \mathcal{X} = \mathcal{X} \times_{\mathfrak{g}_r} D \to D,\) as desired. \qed

3.3. For a Borel subgroup \(\bar{B},\) one has a chain of imbeddings \(\mathcal{O}_-(\mathfrak{b}) \hookrightarrow \mathfrak{d}(\mathfrak{b})/\overline{\mathfrak{b}} = \mathfrak{a}_+^{*}(\mathfrak{b}) \hookrightarrow \mathfrak{u}(\mathfrak{b})^{*},\) cf. \((3.1.1)\). Let \(\Psi : \mathcal{O}_-(\mathfrak{b}) \to \mathfrak{u}(\mathfrak{b})^{*}\) be the composition. A character of the Lie algebra \(\mathfrak{u}(\mathfrak{b})\) is called nondegenerate if it is contained in the image of \(\Psi.\) Thus, the elements of \(\mathcal{O}_-(\mathfrak{b})\) may (and will) be identified with nondegenerate characters. It is clear from \((3.2.1)-(3.2.2)\) that one has \(q^{-1}(\mathfrak{b}, s) = \Psi(s) + \mathfrak{u}(\mathfrak{b})^{*}\), where we have used the identifications from \((3.1.1)\).

Let \(\mathfrak{u}_B\) be a vector bundle on \(\mathcal{B}\) with fibers \(\mathfrak{u}(\mathfrak{b}), \mathfrak{b} \in \mathcal{B},\) and let \(\mathfrak{u}_{\mathcal{B}_{\overline{\mathfrak{b}}}}\) be its pull-back to \(\overline{\mathcal{B}},\) via the projection \(\overline{\mathcal{B}} \to \mathcal{B}, (\bar{b}, s) \mapsto \bar{b}.\) The assignment \((\bar{b}, s) \mapsto \Psi(s)\) gives a canonical section of the dual vector bundle \(\mathfrak{u}_{\mathcal{B}_{\overline{\mathfrak{b}}}}^{*}\). We may view \(\mathfrak{u}_{\mathcal{B}_{\overline{\mathfrak{b}}}}^{*}\) as a subbundle of the trivial vector bundle \(\mathcal{B}_{\overline{\mathfrak{b}}} \times \mathfrak{g}_r^{*} \to \overline{\mathcal{B}}\).

Then, it follows from the above discussion that there is a canonical \(G\)-equivariant isomorphism

\[
\mathcal{X} \cong \Psi + \mathfrak{u}_{\mathcal{B}_{\overline{\mathfrak{b}}}}^{*} =: T^\Psi \mathcal{B}_{\overline{\mathfrak{b}}},
\]

of schemes over \(\mathcal{B}_{\overline{\mathfrak{b}}},\) where we have used the identifications from \((3.1.1).\)

Fix a point \((\bar{b}, s) \in \mathcal{B}_{\overline{\mathfrak{b}}},\) and let \(\bar{U} = [\bar{B}, \bar{B}],\) resp. \(\bar{u} = \mathfrak{u}(\mathfrak{b})\) and \(\psi = \Psi(s).\) Using the identification \(\mathcal{B}_{\overline{\mathfrak{b}}} = G/\bar{U},\) isomorphism \((3.3.1)\) takes the form

\[
\mathcal{X} \cong G \times_{\bar{U}} (\psi + \mathfrak{u}_{\overline{\mathfrak{b}}}^{*}) =: T^\psi (G/\bar{U}).
\]

Let \(m : T^* G \to \bar{u}_T\) be the moment map associated with the \(\bar{U}\)-action on \(G\) by right translations. The variety \(G \times_{\bar{U}} (\psi + \mathfrak{u}_{\overline{\mathfrak{b}}}^{*})\) may be identified with \(m^{-1}(\psi)/\bar{U},\) a Hamiltonian reduction of \(T^* G\) with respect to \((\bar{U}, \psi).\) Therefore, \(T^\psi (G/\bar{U})\) comes equipped with a symplectic structure such that the \(G\)-action on \(T^\psi (G/\bar{U})\) is Hamiltonian and the corresponding moment map goes, via \((3.3.2),\) to the map \(\mu_{\mathcal{X}}.\) The symplectic structure on \(T^\psi (G/\bar{U})\) gives, via \((3.3.2),\) a symplectic structure on \(\mathcal{X}.

It is possible to use isomorphism \((3.3.1)\) to define the symplectic structure on \(\mathcal{X}\) in a canonical way that does not involve the choice of a point \((\bar{b}, s) \in \mathcal{B}_{\overline{\mathfrak{b}}}.\) To this end, one has to generalize the Hamiltonian reduction construction to the setting of group scheme actions. In section 6.4, we will explain a quantum counterpart of such a construction that produces a sheaf \(\mathcal{D}^\Psi_{\mathcal{B}_{\overline{\mathfrak{b}}}},\) of twisted differential operators on \(\mathcal{B}_{\overline{\mathfrak{b}}},\) that may be viewed as a quantization of \(\mathcal{X}.

4. The Miura variety

4.1. We will freely use the notation of \((3.1)\). Put \(\tilde{\mathfrak{g}}_r = \pi^{-1}(\mathfrak{g}_r),\) a \(G\)-stable Zariski open subset of \(\tilde{\mathfrak{g}}_r.\)

Part (i) of the following result is due to Kostant; part (ii) is also known, cf. e.g. \((6.2),\) Lemma 5.2.1, for a proof.

**Proposition 4.1.1.** (i) The restriction of the map \(\pi \times \nu\) to \(\tilde{\mathfrak{g}}_r\) yields an isomorphism \(\tilde{\mathfrak{g}}_r \to \mathfrak{g}_r \times \mathfrak{t}.

(ii) Let \(\mathfrak{b}, \tilde{\mathfrak{b}}\) be a pair of Borel subalgebras. Then, the set \(\mathfrak{b} \cap (\mathcal{O}_-(\mathfrak{b}) + \tilde{\mathfrak{b}})\) is nonempty iff \(\mathfrak{b}\) and \(\tilde{\mathfrak{b}}\) are in opposite position.

**Definition.** The **Miura variety** is defined as \(Z := \mathcal{X} \times_{\mathfrak{g}_r} \tilde{\mathfrak{g}}_r.\) Set theoretically, we have

\[
Z = \{(\bar{b}, \tilde{b}, x) \in \mathcal{B} \times \mathcal{B} \times \mathfrak{g} \mid \tilde{b} \in \mu_{\mathcal{X}}^{-1}(x), \bar{b} \in \pi^{-1}(x)\}
\]

\[
= \{(\tilde{b}, b, x) \in \mathcal{B} \times \mathcal{B} \times \mathfrak{g} \mid x \in \mathfrak{b} \cap [\mathcal{O}_-(\mathfrak{b}) + \tilde{\mathfrak{b}}]\}.
\]

From Proposition 4.1.1(ii), we deduce

\[
Z = \mathcal{X} \times_{\bar{B}} \tilde{\mathfrak{g}}_r \cong \mathcal{X} \times_{\mathfrak{g}_r} (\mathfrak{g}_r \times \mathfrak{t}) \cong \mathcal{X} \times \mathfrak{t}.
\]
We let $G$ act on $X \times_{\bar{g}_{r}} \bar{g}_{r}$ diagonally. Let $\pi : Z \to X$, resp. $\tilde{\mu} : Z \to \tilde{g}_{r}$, be the first, resp. second, projection. These maps are $G$-equivariant and one has a diagram with cartesian squares:

$$
\begin{array}{ccc}
Z & \xrightarrow{\tilde{\mu}} & \tilde{g}_{r} \cong g_{r} \times_{\epsilon} t \\
\pi & \square & \nu \\
X & \xrightarrow{\mu} & \tilde{g}_{r} \xrightarrow{\psi} c = t/W, \\
\end{array}
$$

where $\theta$ is the quotient map. It follows from Lemma 3.2.3(iii) that the map $\nu \circ \tilde{\mu}$ is a $G$-torsor, in particular, $Z$ is smooth.

Recall the notation of 2.2. By Proposition 4.1.1, the assignment $(\bar{b}, b, x) \mapsto (\bar{b}, b)$ gives a map $Z \to \Omega$. Let $G$ act on $(\bar{B} \times B) \times t$ via its action on $\bar{B} \times B \Omega$, the first factor.

**Proposition 4.1.5.** The following map is a $G$-equivariant isomorphism

$$q_{Z} : Z \to (\bar{B} \times B) \times t, \quad (\bar{b}, b, x) \mapsto \left(\left(\bar{b}, x \mod b\right), (\bar{b}, b)\right), \, x \mod u(b)\right).$$

*Proof.* It is immediate from the construction that the map in question is $G$-equivariant. Further, we know that $\bar{B} \times B \Omega$ is a $G$-torsor, see Lemma 2.2.5. Therefore, the map $q_{Z}$ is a morphism of $G$-torsors on $t$. Hence, it is an isomorphism. □

We now give a more explicit (though less canonical) description of the Miura variety. To this end, we identify $B \times B = G/B \times G/B$. The isotropy group of the base point equals $B_{t} \cap B_{e} = G_{h}$, a maximal torus in $G$. Hence, one has a $G$-equivariant isomorphism $G/G_{h} \cong \Omega$. Let $\emptyset := u(b_{e}) \cap \emptyset(b_{t}) + b_{t}$. This is a $G_{h}$-torsor. The fiber of the projection $pr_{\Omega} : Z \to \Omega$ over the base point equals $\emptyset + g_{h}$. For $s \in \emptyset$, $h \in g_{h}$ and $t \in G_{h}$, one has $Ad t(s + h) = Ad t(s) + h$, so the set $\emptyset + g_{h}$ is $Ad G_{h}$-stable. Further, $e \in \emptyset$ and we have

$$e + g_{h} = b_{e} \cap (e + b_{t}) \subseteq b_{e} \cap [\emptyset(b_{t}) + b_{t}] = \emptyset + (b_{e} \cap b_{t}) = \emptyset + g_{h}. \tag{4.1.6}$$

We deduce $G$-equivariant isomorphisms

$$Z \cong G \times_{G_{h}} (\emptyset + g_{h}) \cong G \times_{G_{h}} (G_{h}, e + g_{h}) \cong G \times (e + g_{h}). \tag{4.1.7}$$

In particular, for any $(\bar{b}, b, x) \in Z$, there are uniquely determined elements $h \in g_{h}$ and $g \in G$ such that one has $(\bar{b}, b, x) = Ad g(b_{e}, b_{t}, h + e)$. Further, the map $G \to \bar{B} \times B \Omega$, $g \mapsto (u_{f}/u_{t}, gG_{h}/G_{h})$, is a $G$-equivariant isomorphism. With these identifications, the isomorphism $q_{Z}$ of Proposition 4.1.5 takes the form $q_{Z} : (g, h + e) \mapsto (g, h)$.

**Remark 4.1.8.** The isomorphism of Proposition 4.1.5 can also be seen as follows. Given a Borel $\bar{b}$, let $\bar{\emptyset}(_{\bar{b}}) := (\bar{b} + \emptyset(b))/u(b)$. The natural projection $d : \bar{\emptyset}_{\bar{b}}(\bar{b}) \to \bar{\emptyset}(\bar{b})$, $y \mapsto y \mod b$, is an affine bundle, a torsor under the action of $t = b/\bar{u}(b)$ viewed as an additive group.

For any Borel $b$ which is opposite to $\bar{b}$, we have a chain of maps

$$u(b) \cap (\bar{b} + \emptyset(b)) \xrightarrow{a} b \cap (\bar{b} + \emptyset(b)) \xrightarrow{b} \bar{b} + \emptyset(b) \xrightarrow{c} (b + \emptyset(b))/u(b) \xrightarrow{d} (b + \emptyset(b))/b.$$  

It is clear that the composite map $c \circ b$, resp. $d \circ c \circ b \circ a$, is an isomorphism. Therefore, the map $(c \circ b \circ a) \circ (d \circ c \circ b \circ a)^{-1} : \emptyset_{\bar{b}}(\bar{b}) \to \bar{\emptyset}_{\bar{b}}(\bar{b})$ provides a section of the $t$-torsor $d$.

One can let the Borel subalgebra $b$ vary. Specifically, we put

$$Z_{b} := \{(\bar{b}, b, x) \in \emptyset | (\bar{b}, b, x) \in \Omega, x \in u(b) \cap (\bar{b} + \emptyset(b))\};$$

$$\bar{B}_{\bar{b}} := \{(\bar{b}, y) \in \bar{B} | \bar{b} \in b, y \in \emptyset_{\bar{b}}(\bar{b})\}.$$
Similarly to the above, one has a $t$-torsor $d : \tilde{\mathcal{B}}_- \to \mathcal{B}_-$. A counterpart of the above diagram has the form

$$Z_u \xrightarrow{a} Z \xrightarrow{b} X \times_B \Omega \xrightarrow{c} \tilde{\mathcal{B}}_- \times_B \Omega \xrightarrow{d \times \text{Id}_\Omega} \tilde{\mathcal{B}}_- \times_B \Omega. \quad (4.1.9)$$

This is a diagram of smooth schemes over $\Omega$ such that, for any $(\tilde{b}, b) \in \Omega$, the corresponding fibers in (4.1.9) form the previous diagram. It follows that the composite map $c \circ b$, resp. $(d \times \text{Id}_\Omega) \circ c \circ b \circ a$, in (4.1.9) is an isomorphism. We deduce that the map $(d \times \text{Id}_\Omega) \circ c \circ b$, in (4.1.9), is a $t$-torsor and the map $a \circ ((d \times \text{Id}_\Omega) \circ c \circ b \circ a)^{-1} : \tilde{\mathcal{B}}_- \times_B \Omega \to Z$ provides a section of that torsor. This section yields a trivialization $Z \cong (\tilde{\mathcal{B}}_- \times_B \Omega) \times t$, which is the isomorphism $q_Z$ of Proposition 4.1.3.

Using the canonical isomorphism $\tilde{\mathcal{B}}_- \times_B \Omega \cong \Xi$, see (2.2.6), we will often view $q_Z$ as an isomorphism $Z \to \Xi \times t$.

4.2. Relation to dynamical Weyl groups. We let $W$ act on $\mathfrak{g}_r \times t$, resp. $\mathcal{X} \times \epsilon t$, via the natural $W$-action on $t$, the second factor. Transporting the $W$-action via the isomorphism of Proposition 4.1.4(i), resp. Proposition 4.1.5, gives a $W$-action on $\mathfrak{g}_r$, resp. $Z$. In particular, for any $x \in \mathfrak{g}_r$, we get a well-defined $W$-action $w : \mathfrak{b} \to \mathfrak{b}^w$, on the fiber $B^x = \pi^{-1}(x)$. The maps in the top row of diagram (4.1.4) are $G \times W$-equivariant.

Let $G$ be the automorphism group of the $G$-torsor $\Xi \cong \tilde{\mathcal{B}}_- \times_B \Omega$. Thus, $G$ is noncanonically isomorphic to $G$. The action of an element $w \in W$ on $Z$ gives an automorphism of the variety $\Xi \times t$ of the form $w : (x, h) \mapsto (\phi_w(h)(x), w(h))$, where $\phi_w : t \to G$ is a certain regular map. It follows from the construction that the maps $\{\phi_w, w \in W\}$ satisfy a cocycle equation:

$$\phi_{w_1}(w_2(h)) \cdot \phi_{w_2}(h) = \phi_{w_1w_2}(h), \quad \forall w_1, w_2 \in W, h \in t. \quad (4.2.1)$$

Explicit formulas for the maps $\phi_w$ are reminiscent of the formulas appearing in the theory of classical $r$-matrices and dynamical Weyl groups, cf. [GR] for some related results.

5. The key construction

5.1. Let $\mathfrak{g}_r^*$ be the set of regular semisimple elements of $\mathfrak{g}$. Given a scheme $Y$ over $\mathfrak{g}$, we put $Y_r^* := Y \times_{\mathfrak{g}} \mathfrak{g}_r^*$. For any variety $Y$, let $T_Y$ denote a constant group scheme $T \times Y \to Y$.

The goal of this subsection is to relate, following [BK] and [DG], the group schemes $\pi^* \mathfrak{z} \to \tilde{\mathfrak{g}}_r$ and $T_{\tilde{\mathfrak{g}}_r} \to \tilde{\mathfrak{g}}_r$. To this end, we will use the following known result.

**Lemma 5.1.1.** Let $B$ be a Borel subgroup with Lie algebra $\mathfrak{b}$, and $x \in \mathfrak{b} \cap \mathfrak{g}_r$. Then, $G_x \subseteq B$. Furthermore, writing $x = h + n$ for the Jordan decomposition of $x$, we have $G_x = Z(G_h) \cdot U_x$, where $U_x$ is the unipotent radical of the group $G_x$ and $Z(G_h)$, the center of $G_h$, is a torus (in particular, it is connected).

**Proof.** It is a standard fact that the group $G_h$ is connected and, moreover, if $G$ is adjoint then the group $Z(G_h)$ is a torus. The proofs of other statements in the (much harder) case where $x$ is an element of $G$ rather than $\mathfrak{g}$ can be found in [SS], §§1.6, 1.14. \qed

Given an element $(b, x) \in \tilde{\mathfrak{g}}_r$, let $\varphi_{b,x}$ be the following composition

$$\varphi_{b,x} : G_x \xrightarrow{\pi^*} \mathfrak{z} \xrightarrow{\pi^*} B \xrightarrow{B/[B,B]} T, \quad (5.1.2)$$

where the first map is well defined thanks to the lemma. It is straightforward to upgrade the construction of the map $\varphi_{b,x}$ for each individual element $(b, x) \in \tilde{\mathfrak{g}}_r$ and obtain a morphism $\varphi : \pi^* \mathfrak{z} \to T_{\tilde{\mathfrak{g}}_r}$, of group schemes on $\tilde{\mathfrak{g}}_r$.

Let the Weyl group $W$ act on $\mathcal{T} \times \tilde{\mathfrak{g}}_r$ diagonally and act on $\mathfrak{z} \times \tilde{\mathfrak{g}}_r$, through its action on $\tilde{\mathfrak{g}}_r$, the second factor. Further, we let $G$ act on $\mathcal{T} \times \tilde{\mathfrak{g}}_r$ and on $\mathfrak{z} \times \tilde{\mathfrak{g}}_r$, through its action on $\tilde{\mathfrak{g}}_r$. The actions of $W$ and $G$ on $T_{\tilde{\mathfrak{g}}_r} = \mathcal{T} \times \tilde{\mathfrak{g}}_r$, resp. $\pi^* \mathfrak{z} = \mathfrak{z} \times \tilde{\mathfrak{g}}_r$, commute. This makes $T_{\tilde{\mathfrak{g}}_r}$, resp. $\pi^* \mathfrak{z}$, a $G \times W$-equivariant group scheme on $\tilde{\mathfrak{g}}_r$.  

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Lemma 5.1.3. The morphism \( \varpi : \pi^*Z \rightarrow T \times g_r \) is a morphism of \( G \times W \)-equivariant group schemes.

Proof. It is immediate to check that the morphism \( \pi^*Z_{rs} \rightarrow T_{g_r} \) induced from \( \varpi \) by restriction is a \( G \times W \)-equivariant isomorphism. The result follows from this since \( Z_{rs} \) is Zariski dense in \( Z \). \( \Box \)

The \( W \)-equivariance of the map \( \varpi \) says that \( \varpi_{b,x}(g) = w(\varpi_{b,x}(g)) \), for any \((b,x) \in g_r\), \( g \in G_x \), \( w \in W \).

It will be convenient in what follows to view \( g_r \) as a subset of \( g^* \) rather than \( g \). The \( G \)-equivariant group scheme \( Z \) on \( g_r \) descends to a smooth group scheme \( Z_c \) on \( c \), see [Ngo, §2.1]. Thus, we have \( Z = Z_c \times_c g_r \). From the isomorphism \( g_r \cong g_r \times_c t^* \) we get

\[
\pi^*Z = Z_c \times_c g_r \times_c t^*, \quad \text{resp.} \quad T_{g_r} = T \times (g_r \times_c t^*).
\]

It is immediate from the criterion for faithfully flat descent of morphisms that the composition

\[
\pi^*Z \xrightarrow{\varpi} T_{g_r} = T \times (g_r \times_c t^*) \xrightarrow{pr_1} T \times t^* \text{ factors through a morphism, } [BK], [DG],
\]

of group schemes on \( t^* \), so that one has \( \varpi = \varpi_c \times_{t^*} \text{Id}_{g_r} \).

Let \( Z \rightarrow g_r \), resp. \( Z_c \rightarrow c \), be the Lie algebra of \( Z \), resp. \( Z_c \). Thus, \( Z \), resp. \( Z_c \), is a vector bundle on \( g_r \), resp. \( c \), and one has a canonical isomorphism \( \vartheta^*Z_c \cong Z \). The fiber of \( Z \) over \( x \in g_r \) equals \( g_x \), hence, the fiber of \( Z_c \) over \( c \in c \) is canonically identified with \( g_x \), for any \( x \in g_r \) such that \( c = \vartheta(x) \). Although the following result is known, cf. eg. [BF], p.40, we reproduce the proof since the argument will be used later.

Lemma 5.1.5. There is a canonical isomorphism \( \vartheta \cong \vartheta^*(T^*c) \), resp. \( Z_c \cong T^*c \).

Proof. It is clear that the second isomorphism follows from the first, by descent. To prove the first isomorphism, fix \( x \in g^* \) and let \( N_x \), resp. \( N^*_x \), be the normal, resp. conormal, space at \( x \) to the \( G \)-orbit of \( x \). Thus, \( N^*_x \) is a subspace of \( (g^*)^* \) and it is immediate to check that using the identification \( (g^*)^* = g \), one has \( N_x = g_x \). Assume now that \( x \) is regular and let \( c = \vartheta(x) \). Then, Proposition 3.2.5(i) implies that the differential of the map \( \vartheta \) yields an isomorphism \( d\vartheta : N_x \cong T_c(c) \). We deduce that the dual of the map \( d\vartheta \) provides a canonical isomorphism \( T_c^*(c) \rightarrow T \rightarrow g_x \). This isomorphism sends \( (df)_c \) to \( d(\vartheta^*f)_x \), where \( \vartheta^* : C[c] \rightarrow C[g^*]^G, \ f \mapsto \vartheta^*f \) is the tautological isomorphism. \( \Box \)

Let \( Y \) be a smooth symplectic manifold equipped with a morphism \( \mu_Y : Y \rightarrow c \) and an action \( Z_c \times_c Y \rightarrow Y \). Let \( \alpha : Z_c \times_c Y \rightarrow TY \) be the differential of that action and write \( \xi_F \) for the Hamiltonian vector field on \( Y \) (a section of \( TY \rightarrow Y \)) associated with a regular function \( F \in C[Y] \).

Observe that the differential of a regular function \( f \in C[c] \) may be viewed, thanks to Lemma 5.1.5, as a section \( df \in \Gamma(c, Z_c) \). We say that the \( Z_c \)-action on \( Y \) is Hamiltonian with moment map \( \mu_Y \) if the following holds:

\[
\alpha(df) = \xi_{\mu_Y f}, \quad \forall f \in C[c].
\]

To unburden notation we will write \( C[Y]^C[c] \) for the Poisson centralizer of the algebra \( C_Y^C[c] \) in the Poisson algebra of regular functions on \( Y \).

We have a \( Z_c \)-action \( Z_c \times_c (T \times t^*) \rightarrow T \times t^*, \ (z, (t, \tau)) \mapsto (\varpi(z) t, \tau) \). The \( Z_c \)-action on \( X \) gives, via the isomorphism \( X \cong T^*B_\omega \), see (3.3.1), a \( Z \)-action on \( T^*B_\omega \).

The proof of the following result is straightforward.

Lemma 5.1.7. The \( Z_c \)-action on \( T^*T = T \times t^* \), resp. \( T^*B_\omega \), is Hamiltonian with moment map \( \vartheta \circ pr_2 \), resp. \( \vartheta \circ \mu_X \). \( \Box \)
5.2. Recall the setting of diagram (3.1.4) and let \((T^*\mathcal{B})_r := \mu_{T^*\mathcal{B}}^{-1}(\mathfrak{g}_r)\). Using the identifications of diagram (3.1.4), we obtain the following pair of \(G\)-equivariant torsors on \(\mathfrak{g}_r\):

\[
\begin{array}{ccc}
Z & \xrightarrow{\pi^*3-\text{torse}} & (T^*\mathcal{B})_r \\
\downarrow{\pi^*3-\text{action}} & & \downarrow{\pi^*3-\text{action}} \\
\mathfrak{g}_r & \xrightarrow{\mathfrak{p}^{-1}-\text{torse}} & (T^*\mathcal{B})_r
\end{array}
\]

(5.2.1)

**Proposition 5.2.2.** There is a canonical morphism \(\kappa_Z : Z \to (T^*\mathcal{B})_r\), of \(G\)-equivariant schemes over \(\mathfrak{g}_r\), such that the following diagram commutes

\[
\begin{array}{ccc}
\pi^*3 \times \mathfrak{g}_r & \xrightarrow{\kappa \times \kappa_Z} & Z \\
\downarrow{T_{\mathfrak{g}_r} \times \mathfrak{g}_r} & & \downarrow{T_{\mathfrak{g}_r} \times \mathfrak{g}_r} \\
(T^*\mathcal{B})_r & \xrightarrow{\kappa_Z} & (T^*\mathcal{B})_r
\end{array}
\]

Proof. We first define \(\kappa_Z\) pointwise. To this end, we use the map \(\kappa_{b,b}\) from (2.2.1) and let \(\kappa_Z\) be given by the following assignment:

\[
\kappa_Z : Z \to (T^*\mathcal{B})_r = \mathfrak{g}_r \times_B \mathcal{B}, \quad (b, b, x) \mapsto ((b, x), (b, \kappa_{b,b}(x \mod \mathfrak{b}))).
\]

(5.2.3)

To define this map scheme theoretically, we use the isomorphisms \(p_1 \times p\) and \(p_- \times p_0\) from (2.2.6) and put \(u := (p_1 \times p) \circ (p_- \times p_0)^{-1}\). We obtain the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{B} \times_B \Omega & \xrightarrow{\pi} & \Omega \times_B \mathcal{B} & \xrightarrow{pr} & \mathcal{B} \\
\downarrow{\kappa_Z} & & \downarrow{pr_{\mathfrak{g}_r}} & & \downarrow{\varphi} \\
(T^*\mathcal{B})_r & \xrightarrow{\mathfrak{g}_r \times_B \mathcal{B}} & \mathfrak{g}_r \times_B \mathcal{B} & \xrightarrow{\mathfrak{g}_r} & \mathfrak{g}_r
\end{array}
\]

In this diagram, the composite map \((pr_{\mathfrak{g}_r} \circ u \circ \pi) \times pr_{\mathfrak{g}_r} : Z \to \mathcal{B} \times \mathfrak{g}_r\) (along the upper and lower halves of the perimeter) factors through a dotted map \(Z \to \mathfrak{g}_r \times_B \mathcal{B}\). We let this map be the required morphism \(\kappa_Z\).

We must prove that the diagram in the statement of Proposition 5.2.2 commutes. All schemes being reduced, it is sufficient to check this pointwise. Thus, let \((b, b, x) \in Z\) and put \(\mathfrak{h} := b \cap b\). This is a Cartan subalgebra, by Proposition 3.1.1 and one has a triangular decomposition \(\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{h} \oplus \mathfrak{u}\), where \(\mathfrak{u} := u(b), \mathfrak{u} := u(b)\). Let \(B, \mathfrak{U}\) and \(H\), be the group corresponding to \(\mathfrak{b}, \mathfrak{u}\) and \(h\). Fix \(g \in G_x\). We know that \(G_x \subseteq B\), so conjugation by \(g\) produces a triangular decomposition of the form \(\mathfrak{g} = u \oplus g(h) \oplus g(u)\). Since \(x \in b \cap (\partial_- (b) + b)\), one can write \(x = h + u\), where \(h \in \mathfrak{h}\) and \(u\) is a nilpotent element contained in the open \(H\)-orbit in \(u \cap \partial(b)\). Note that since \(x\) is fixed by \(g\), we have \(x = g(h) + g(u)\), where \(g(h) \in g(h)\) and \(g(u) \in g(u \cap \partial(b)) = u \cap \partial(g(b))\). Put \(u' = [u, u]\). Thus, \(x \mod \mathfrak{b} \in \Omega_- (b)\), resp. \(u \mod u' \in \Omega (b)\). Going through the construction of the map \(\kappa_{b,b}\), one finds that \(\kappa_{b,b}(x \mod \mathfrak{b}) = u \mod u'\), resp. \(\kappa_{g(b),b}(x \mod g(b)) = g(u) \mod g(u') = g(u) \mod u'\). Now, write \(z : y \mapsto z \ast_3 y\), resp. \(t : y \mapsto t \ast_T y\), for the action of \(G_x\) on \(\pi^{-1}(x)\) and \(\pi^{-1}(x)\), resp. \(T = B/U\) on \(u/u'\). By definition, one has \(z_{b,x}(g) = g \mod U\). Writing \(\kappa_{b,x}\) for the restriction of the map \(\kappa_Z\) to

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the fiber $\tilde{\mu}^{-1}(b, x)$, cf. (4.1.4), we compute:

$$\kappa_{b, x}(g \star_3 b, g \star_3 b, g(x)) = \kappa_{b, x}(g \star_3 b, b, x) = \kappa_{g(b), b}(x \text{ mod } g(b)) = g(u) \text{ mod } u'$$

$$= (g \text{ mod } U) \star T (u \text{ mod } u') = \kappa_{b, x}(g) \star_T \kappa_{b, b}(x). \quad \Box$$

Let $T_{\tilde{b}_r} \times^{\pi^*Z} Z$ be a pushout of the $\pi^*3$-torsor $Z$ via the morphism $\kappa$. Thus, $T_{\tilde{b}_r} \times^{\pi^*Z} Z$ is a $G \times W$-equivariant $T_{\tilde{b}_r}$-torsor. It follows from Proposition 5.2.2 and the universal property of pushouts, that the morphism $\kappa_Z$ induces a morphism

$$\kappa : T_{\tilde{b}_r} \times^{\pi^*Z} Z \longrightarrow (T^*\tilde{B})_r, \quad (t, (b, b, x)) \mapsto t \cdot \kappa_Z(b, b, x), \quad (5.2.4)$$

of $G$-equivariant $T$-torsors on $\tilde{g}_r$. Since any morphism of torsors is an isomorphism, we obtain

**Proposition 5.2.5.** The map $\kappa$ is an isomorphism of $G$-equivariant $T$-torsors on $\tilde{g}_r$. \hfill $\Box$

**Proof of Theorem 7.3.3.** We view $T \times t^*$ and $X$ as schemes over $c$ via the maps $T \times t^* \to t^* \to c$ and $X \to g_r \to c$, respectively, cf. Lemma 3.2.3. We find

$$T_{\tilde{b}_r} \times_{\tilde{g}_r} Z \cong (T \times t^*) \times_{\tilde{g}_r} (X \times t^*) = (T \times t^*) \times t \cdot \kappa_Z(b, b, x). \quad (5.2.6)$$

Now, we have a diagram $T_{\tilde{b}_r} \times_{\tilde{g}_r} Z \overset{g}{\to} T_{\tilde{b}_r} \times^{\pi^*Z} Z \overset{\kappa^{-1}}{\to} (T^*\tilde{B})_r \overset{j}{\to} T^*\tilde{B} \overset{\mu \star \tilde{B}}{\to} g \overset{g}{\to} c, \quad (5.2.7)$

where $j : (T^*\tilde{B})_r \hookrightarrow T^*\tilde{B}$ is an open embedding. Let $\mu : T^*\tilde{B} \overset{\mu \star \tilde{B}}{\to} g \overset{g}{\to} c$, resp. $\kappa : T^*T \times_t X \to (T^*\tilde{B})_r$, be the composite map. We obtain the following chain of $G$-equivariant algebra maps

$$C[c] \overset{\mu^*}{\to} C[T^*\tilde{B}] \overset{j^*}{\to} C[(T^*\tilde{B})_r] \overset{\kappa^*}{\to} C[T^*T \times_t X] \cong C[T^*T] \otimes_{C[c]} C[X]. \quad (5.2.8)$$

where the isomorphism on the right holds since the varieties $T^*T, Z,$ and $c$ are affine.

Since the map $g$ is an $(h \cup g)^*3$-torsor, the map $\kappa$ is a $\mu^*3_c$-torsor. Therefore, the algebra map $\mu^*$ induces an isomorphism $C[(T^*\tilde{B})_r] \cong C[T^*T \times_t X]^{[3_c]} = C[T^*T \times_t X]^{3_c}.$

We claim next that the map $j^*$ is an isomorphism. To prove this, we use that $\tilde{g} \setminus \tilde{g}_r$ has codimension $\geq 2$ in $\tilde{g}$, see e.g. [BR, Proposition 1.9.3]. Since $T^*\tilde{B}$ is a $T$-torsor over $\tilde{g}$, we conclude that $(T^*\tilde{B})_r \setminus (T^*\tilde{B})_r$ has codimension $\geq 2$ in $T^*\tilde{B}$, and the claim follows.

Combining everything together we obtain $G$-equivariant isomorphisms of $C[c]$-algebras

$$C[T^*\tilde{B}] \overset{(j^* \mu^*)^*}{\cong} C[T^*T \times_t X]^{3_c} \cong (C[T^*T] \otimes_{C[c]} C[X])^{[3_c]}. \quad (5.2.9)$$

To complete the proof, let $Y$ be an arbitrary symplectic manifold equipped with a Hamiltonian $3_c$-action with moment map $\mu_Y$, and let $C[Y]^{[3_c]}$ be the Poisson centralizer of $\mu_Y^* (C[c])$ in $C[Y]$. Then, equation (5.1.6) and the fact that the group scheme $3_c$ is connected imply that one has $C[Y]^{[3_c]} = C[Y]^{[3_c]}$. Applying this in the case $Y = T^*T \times_t X,$ from the previous isomorphisms we deduce

$$C[T^*\tilde{B}] \cong C[T^*T \times_t X]^{[3_c]} \cong (C[T^*T] \otimes_{C[c]} C[X])^{[3_c]}.$$

The theorem follows since $T^*(G/U) = T^*\tilde{B}_\_ = X$ by definition, see (3.3.2). \hfill $\Box$

**Proof of Corollary 7.3.4.** The Weyl group acts on $T_{\tilde{b}_r}$ and on $Z$. We transport the diagonal $W$-action on $T_{\tilde{b}_r} \times^{\pi^*Z} Z$ via the isomorphism $\kappa$. The resulting $W$-action on $(T^*\tilde{B})_r$ induces a $W$-action on $C[T^*\tilde{B}] = C[(T^*\tilde{B})_r]$ by algebra automorphisms.
To prove that the above defined \( W \)-action on \((T^*\tilde{B})_{\text{aff}}\) agrees with the one constructed in Proposition 5.5.1, it is sufficient to show that the two actions agree over the locus of regular semisimple elements. This is easy to check by comparing the constructions of these actions.

**Corollary 5.2.10.** In the setting of Lemma 5.1.7 the following holds:

(i) One has \( \ker \varphi_{b,x} = U_x \), resp. \( \im \varphi_{b,x} \) equals the image of the group \( Z(G_h) \) in \( T = B/[B,B] \).

(ii) If \( s \in \mathcal{O}(b) \) is such that the fiber \( \kappa^{-1}(b,x,s) \) is nonempty, then this fiber is a \( U_x \)-torsor.

**Proof.** Part (i) is immediate from Lemma 5.1.1. Part (ii) follows from (i) using that the map (5.2.4) equals \( \Xi \). In terms of (4.1.7), we obtain the following diagram:

\[
\begin{array}{cccccc}
\mathcal{X} & \xrightarrow{q} & Z & \xrightarrow{\kappa} & T^*(G/U_e) & = G \times U_e b_e \\
\tilde{B}_- & \xrightarrow{p-} & \Xi & \xrightarrow{p} & \tilde{B} & = G/U_e
\end{array}
\]

In this diagram, the map \( \tilde{\pi} \) is induced by the inclusion \( e + g_h \hookrightarrow e + b_t \), resp. the map \( \kappa \) is induced by the inclusion \( e + g_h \hookrightarrow b_e \). Further, the maps \( \tilde{\kappa} \) and \( (q \times p_{\tilde{r}}) \circ \tilde{\kappa} \) take the following form:

\[
Z = G \times (e + g_h) \xrightarrow{(q \times p_{\tilde{r}})^{-1}(\Xi)} G \times (e + b_t) \xrightarrow{q \times p_{\tilde{r}}: (g,x,y) \rightarrow g} G.
\]

**Proposition 5.3.2.** The map \( \tilde{\kappa} \) is a \( G \)-equivariant closed imbedding that makes the first projection \( Z \rightarrow \mathcal{X} \) a finite morphism and the second projection \( Z \rightarrow T^*\tilde{B} \) a birational isomorphism. Furthermore, the map \((q \times p_{\tilde{r}}) \circ \tilde{\kappa} : Z \rightarrow \Xi\) is a fibration with affine-linear fibers.

**Proof.** We claim first that the restriction of the map \( Z \rightarrow T^*\tilde{B} \) to the regular semisimple locus is an isomorphism. To see this, consider the following maps:

\[
Z_{rs} \xrightarrow{z \mapsto (1,z)} (\pi^*3 \times \pi^*3)_{rs} \xrightarrow{(\pi^*3 \times \pi^*3)} (\pi^*3)_{rs} \xrightarrow{(T_{\tilde{r}} \times \pi^*3)} (T_{\tilde{r}} \times \pi^*3)_{rs} \xrightarrow{\kappa_{rs}} (T^*\tilde{B})_{rs}.
\]

The first map above is the tautological isomorphism. As we have mentioned in the proof of Lemma 5.1.3, the second map above is easily seen to be an isomorphism. Finally, the third map is an isomorphism by Proposition 5.2.2. We conclude that the map \( Z \rightarrow T^*\tilde{B} \) is a birational isomorphism. Furthermore, the left cartesian square in (4.1.4) implies that the first projection \( Z \rightarrow \mathcal{X} \) is a finite morphism.

The scheme \( \tilde{\Xi} := (q \times p_{\tilde{r}})^{-1}(\Xi) \) is closed in \( \mathcal{X} \times T^*\tilde{B} \) since the orbit \( \Xi \) is closed in \( \tilde{B}_- \times \tilde{B} \), by Lemma 2223. Thus, proving that \( \tilde{\kappa} \) is a closed imbedding reduces to showing that so is the map \( Z \rightarrow \tilde{\Xi} \). We identify \( \Xi = G \) and use the explicit formulas for the maps \( \tilde{\kappa} \) and \((q \times p_{\tilde{r}}) \circ \tilde{\kappa} \) given above. The fiber of the projection \( \tilde{\Xi} \rightarrow \Xi \), resp. of the map \((q \times p_{\tilde{r}}) \circ \tilde{\kappa} \), over an element \( g \in G = \Xi \) may be identified with \((e + b_t) \times b_e \), resp. \( e + g_h \). Then, the restriction, \( \tilde{\kappa}_1 \), of the map \( \tilde{\kappa} \) to the fiber becomes the diagonal imbedding \( \tilde{\kappa}_1 : e + g_h \hookrightarrow (e + b_t) \times b_e \), which is a closed imbedding. It follows that \( \tilde{\kappa} \) is a closed imbedding.

Next, we note that \((e + b_t) \times b_e \) has the natural structure of an affine-linear space, a fiber of the twisted cotangent bundle on \( \tilde{B}_- \times \tilde{B} \). The image of the map \( \tilde{\kappa}_1 \) is an affine-linear subspace of that affine-linear space. Thus, the map \((q \times p_{\tilde{r}}) \circ \tilde{\kappa} : Z \rightarrow \Xi \) is an affine-linear fibration.
5.4. The case of a simply connected group. In this subsection, we explain how to adapt the constructions of previous sections to a ‘simply connected’ setting where the adjoint group $G$ is replaced by $G^\text{sc}$, a simply connected cover of $G$. To this end, let $\omega_i, \ i \in I$, be the fundamental weights of $\mathfrak{g}$, and $X^*$ the weight lattice. For each $i \in I$, we fix an irreducible finite dimensional representation $V_i$ of $\mathfrak{g}$, with highest weight $\omega_i$ (such a representation is defined uniquely up to a noncanonical isomorphism). Associated with every Borel $\mathfrak{b}$, there is a canonical $\mathfrak{b}$-stable filtration $V_i^{\geq \mu, \mathfrak{b}}$, $\mu \in X^*$, such that $\text{gr}_{\mu, \mathfrak{b}} V_i := V_i^{\geq \mu, \mathfrak{b}} / V_i^{> \mu, \mathfrak{b}}$ is a $\mu$-weight space for the natural action of the universal Cartan algebra $\mathfrak{t} = \mathfrak{b} / [\mathfrak{b}, \mathfrak{b}]$. The highest weight space $V_i^{\geq \omega_i, \mathfrak{b}}$ is the line $V_i^{u(b)}$ formed by the vectors killed by $u(b)$. Dually, one has a line $\text{gr}_{\mu, \mathfrak{b}} V_i = V_i / u(b) V_i$, where $w_0 \in W$ is the longest element. Let $O^+(\mathfrak{b})^\text{sc} = V_i^{\geq \omega_i, \mathfrak{b}} \subset \{0\}$, resp. $O^-(\mathfrak{b})^\text{sc} = \text{gr}_{\mu, \mathfrak{b}} V_i \setminus \{0\}$. The action of $T^\text{sc}$, the abelian maximal torus of $G^\text{sc}$, makes $O^+(\mathfrak{b})^\text{sc} := \prod_i O^+(\mathfrak{b})^\text{sc}$, resp. $O^-(\mathfrak{b})^\text{sc} := \prod_i O^-(\mathfrak{b})^\text{sc}$, a $T^\text{sc}$-torus. Similarly to [2.2.1], we define the following $G^\text{sc}$-equivariant $T^\text{sc}$-torsor on $\mathfrak{b}$:

$$\mathfrak{B}^\text{sc} := \{(b, s) \mid b \in \mathfrak{b}, \ s \in O(b)^\text{sc}\}, \quad \text{resp.} \quad \mathfrak{B}^\text{sc} := \{(b, s) \mid b \in \mathfrak{b}, \ s \in O_-(b)^\text{sc}\}.$$

For every $i \in I$ and a pair $b, \bar{b} \in \mathfrak{b}$ of Borel subalgebras in opposite position, the composite map $V_i^{u(b)} \hookrightarrow V_i \rightarrow V_i / u(b) V_i$ is an isomorphism. An inverse of this map induces an isomorphism

$$\Omega_i: O^-(\mathfrak{b})^\text{sc} \cong O^+(\mathfrak{b})^\text{sc},$$

which we will denote simply by $\Omega_i$. We obtain an isomorphism $\kappa_{b, b}^\text{sc} : O^+(\mathfrak{b})^\text{sc} \cong O^+(\mathfrak{b})^\text{sc}$, such that $\kappa_{b, b}(ts) = w_0(t)^{-1} \kappa_{b, b}(s)$. We define

$$\Xi^\text{sc} := \{(b, x, b, x) \in \mathfrak{B}^\text{sc} \times \mathfrak{B}^\text{sc} \mid (b, b) \in \Omega, \ \kappa_{b, b}^\text{sc}(x) = x\}.$$

Let $p^\Omega : \Xi^\text{sc} \rightarrow \Omega, \ (b, x, b, x) \mapsto (b, b)$, resp. $p^\mathfrak{b} : \Xi^\text{sc} \rightarrow \mathfrak{B}^\text{sc}, \ (b, x, b, x) \mapsto (b, x)$, and $p^\mathfrak{b} : \Xi^\text{sc} \rightarrow \mathfrak{B}^\text{sc}, \ (b, x, b, x) \mapsto (b, x)$, be the natural projection. Mimicking Section 2.2, one shows that $\Xi^\text{sc}$ is a $G_\Delta^\text{sc}$-torsor and one has the following isomorphisms, cf. [2.2.8],

$$\mathfrak{B}^\text{sc} \times B \Omega \xrightarrow{p^\Omega \times p^\Omega} \Xi^\text{sc} \xrightarrow{p^\Omega \times p^\mathfrak{b}} \Omega \times B \mathfrak{B}^\text{sc}.$$

Let $T^\text{sc}$ act on $\mathfrak{B}^\text{sc} \times \mathfrak{B}^\text{sc}$ via the $T^\text{sc}$-action on the second factor $\mathfrak{B}^\text{sc}$ (along the fibers of $\mathfrak{B}^\text{sc} \rightarrow \mathfrak{b}$). The actions of $G_\Delta^\text{sc}$ and $T^\text{sc}$ make $\mathfrak{B}^\text{sc} \times \mathfrak{B}^\text{sc}$ a $G_\Delta^\text{sc} \times T^\text{sc}$-variety. Let $\Omega^\text{sc}$ be the preimage of $\Omega$ under the map $\mathfrak{B}^\text{sc} \times \mathfrak{B}^\text{sc} \rightarrow \mathfrak{b} \times \mathfrak{b}$. The variety $\Omega^\text{sc}$ is $G_\Delta^\text{sc} \times T^\text{sc}$-stable and it contains $\Xi^\text{sc}$. Furthermore, the map $\mathfrak{B}^\text{sc} \times T^\text{sc} \rightarrow \Omega^\text{sc}$ induced by the $T^\text{sc}$-action is a $G_\Delta^\text{sc} \times T^\text{sc}$-equivariant isomorphism, cf. Lemma 2.2.1(i).

Since any Borel subgroup $B$ of $G^\text{sc}$ contains $Z(G^\text{sc})$, the center of $G^\text{sc}$, there is a canonical embedding $Z(G^\text{sc}) \hookrightarrow B / [B, B] = T^\text{sc}$. Furthermore, we have canonical isomorphisms $G^\text{sc} / Z(G^\text{sc}) \rightarrow G$, resp. $T^\text{sc} / Z(G^\text{sc}) \rightarrow T$.

Recall the notation of 2.2.1. For every $i \in I$ inside the lattice $X^*$, we have an equation $\alpha_i = \sum_{j \in I} (\alpha_j^\vee, \alpha_i) \cdot \omega_j$, where $\alpha_j^\vee$ denotes the simple coroot associated with $j \in I$. We may (and will) choose, for some Borel subalgebra $\mathfrak{b}$ and every $i \in I$, an isomorphism:

$$\text{gr}_{\alpha_i, \mathfrak{b}} \mathfrak{g} \cong \bigotimes_{j \in I} \left( V_i^{u(b)} \right)^{\otimes (\alpha_j^\vee, \alpha_i)}.$$
of 1-dimensional $T^{sc}$-modules (of the same weight). Combining these isomorphisms together we obtain a $T^{sc}$-equivariant morphism
\[
\mathcal{O}(b)^{sc} = \prod_{i \in I} \mathcal{O}(b)^{sc} \rightarrow \mathcal{O}(b) = \prod_{i \in I} \mathcal{O}(b), \quad (s_i)_{i \in I} \mapsto \left( \bigotimes_{i} s_i^{\otimes (\alpha_i^+, \alpha_i^-)} \right)_{i \in I}.
\]

The morphism above descends to an isomorphism $\mathcal{O}(b)^{sc}/Z(G^{sc}) \rightarrow \mathcal{O}(b)$, of $T$-torsors. Furthermore, this isomorphism for the chosen Borel $b$ induces an isomorphism $\tilde{B}^{sc}/Z'(G^{sc}) \rightarrow \tilde{B}$, of $G$-equivariant $T$-torsors on $\tilde{B}$. A similar construction yields an isomorphism $\tilde{B}^{sc}/Z(G^{sc}) \rightarrow \tilde{B}$, of $G$-equivariant $T$-torsors on $\tilde{B}$.

We define $\mathcal{X}^{sc} := \mathcal{X} \times_{\tilde{B}} \tilde{B}^{sc}$, resp. $Z^{sc} := \mathcal{X}^{sc} \times_{\tilde{B}} \tilde{g}_r$. Let $\mu_X^{sc}$ be the composition $\mathcal{X} \times_{\tilde{B}} \tilde{B}^{sc} \rightarrow \mathcal{X} \rightarrow \mathfrak{g}$. With these definitions, there is an analogue of the diagram from the proof of Proposition 1.2.2 that provides the construction of a morphism
\[
\kappa^{sc}_K : Z^{sc} \rightarrow \tilde{g}_r \times_{\tilde{B}} \tilde{B}^{sc} \cong T^*(\tilde{B}^{sc}),
\]
of $G^{sc}$-equivariant schemes on $\tilde{g}_r$.

Next, one has the universal centralizer group scheme $\mathfrak{z}^{sc} := \{ (g, x) \in G^{sc} \times \mathfrak{g}_r \mid \text{Ad} g(x) = x \}$. Using the $G^{sc}$-equivariant morphism $\mu_{\mathcal{X}}^{sc} : \mathcal{X}^{sc} \rightarrow \mathfrak{g}_r$, one gets a canonical action $\mathfrak{z}^{sc} \times \tilde{g}_r, \mathcal{X}^{sc} \rightarrow \mathcal{X}^{sc}$. There is a simply-connected analogue $\pi^{sc} : \pi^* \mathfrak{z}^{sc} \rightarrow T^{sc}_r$ of the map from Lemma 1.3.3. One checks that the simply-connected counterpart of the diagram of Proposition 1.2.2 commutes. This implies an analogue of Proposition 1.2.3, i.e., an isomorphism
\[
\kappa^{sc}_T : T^{sc}_r \times \pi^* \mathfrak{z}^{sc} \cong (T^*(\tilde{B}^{sc}))^r,
\]
of $G^{sc} \times T^{sc}$-equivariant schemes on $\tilde{g}_r$.

We let $W$ act on $\mathcal{X}^{sc} \times_{\tilde{B}} \tilde{B}^{sc}$ via its action on the second factor. This gives, using the isomorphism $\mathcal{X}^{sc} \times_{\tilde{B}} \tilde{B}^{sc} \cong \mathcal{X}^{sc} \times_{\tilde{B}} \tilde{B}^{sc}$, a $W$-action on $Z^{sc}$. Thus, one can transport the diagonal $W$-action on the variety in the LHS of (5.4.1) to obtain a $W$-action on $T^*(\tilde{B}^{sc})^r$.

### 6. The Miura Bimodule

#### 6.1. Given a Lie algebra $\mathfrak{g}$ and a left, resp. right, $\mathfrak{t}$-module $E$, we write $E^{\mathfrak{t}}$ for the space of $\mathfrak{t}$-invariants and use simplified notation $E/\mathfrak{t} = E/\mathfrak{t}E$, resp. $E/\mathfrak{t} = E/\mathfrak{t}E$, for $\mathfrak{t}$-coinvariants. Similarly, given a second Lie algebra $\mathfrak{t}'$ and a $(\mathcal{U} \mathfrak{t}', \mathcal{U} \mathfrak{t})$-bimodule $E$, we write $E/\mathfrak{t}' = E/\mathfrak{t}'E$.

Below, we view $\mathcal{U} \mathfrak{g}$, resp. $\mathcal{U} \mathfrak{t}$, as the algebra of left invariant, resp. invariant, differential operators on $G$, resp. $T$. Fix a Borel subgroup $B$ with unipotent radical $U$. Let $\mathfrak{b} = \text{Lie} B, \mathfrak{u} = \text{Lie} U$. Since $\mathfrak{u}$ is a Lie ideal of $\mathfrak{b}$ and the Lie algebra $t = \mathfrak{b}/\mathfrak{u}$ is abelian, we have $\mathcal{U} \mathfrak{t} = \mathcal{U} \mathfrak{b}/\mathfrak{u} = \mathfrak{u} \mathfrak{U} \mathfrak{b} = (\mathfrak{u} \mathfrak{U} \mathfrak{b})^{\mathfrak{u}} = ((\mathfrak{u} \mathfrak{U} \mathfrak{b})^{\mathfrak{u}})^1$, where $(-)^1$ stands for invariants of the adjoint $\mathfrak{t}$-action. We obtain a chain of algebra imbeddings
\[
\mathcal{U} \mathfrak{t} = ((\mathfrak{u} \mathfrak{U} \mathfrak{b})^{\mathfrak{u}})^1 \hookrightarrow ((\mathfrak{u} \mathfrak{U} \mathfrak{g})^{\mathfrak{u}})^1 \hookrightarrow (\mathfrak{u} \mathfrak{U} \mathfrak{g})^{\mathfrak{u}} \hookrightarrow (\mathfrak{u} \mathfrak{U} (G)^{\mathfrak{u}}).
\]

We put $\mathfrak{D} := (\mathfrak{u} \mathfrak{U} (G))^{\mathfrak{u}}$, let $a : \mathcal{U} \mathfrak{t} \rightarrow \mathfrak{D}$ be the composite of the maps above, and $\text{proj} : \mathcal{U} \mathfrak{b} \rightarrow \mathcal{U} \mathfrak{t} = \mathfrak{u} \mathfrak{U} \mathfrak{b}$ the projection. Thus, one has the following diagrams of algebra maps
\[
\mathfrak{D}(G) \xrightarrow{\text{proj}} \mathcal{U} \mathfrak{b} \xrightarrow{\mathfrak{i}_T \circ \text{proj}} \mathfrak{D}(T), \quad \text{resp. } \mathfrak{i}_{T} \xrightarrow{a} \mathcal{U} \mathfrak{t} \xrightarrow{\mathfrak{i}_T} \mathfrak{D}(T).
\]

We apply the construction of Hamiltonian reduction in the setting of Example 1.1.2(ii) for the triple $A_1 = \mathfrak{D}(T), A_2 = \mathfrak{D}(G), Z = \mathcal{U} \mathfrak{b}$, resp. $A_1 = \mathfrak{D}(T), A_2 = \mathfrak{D}, Z = \mathcal{U} \mathfrak{t}$, and the maps on the left, resp. right, of the diagram above. We obtain the following isomorphisms
\[
(\mathfrak{D}(T) \otimes \mathcal{U} \mathfrak{b} \mathfrak{D}(G))^{\mathfrak{b}} \cong (\mathfrak{D}(T) \otimes \mathcal{U} \mathfrak{t} (\mathfrak{u} \mathfrak{U} (G))^{\mathfrak{u}})^1 = (\mathfrak{D}(T) \otimes \mathcal{U} \mathfrak{t} \mathfrak{D})^1,
\]
where the isomorphism on the left results from the fact that one can perform Hamiltonian reduction in stages: first with respect to the Lie algebra $\mathfrak{u}$ and then with respect to $t = \mathfrak{b}/\mathfrak{u}$. Hamiltonian
We have the Proof. The ‘inner’ action of the algebra \((\mathcal{D}(T) \otimes_{\mathcal{U}b} \mathcal{D}(G))\) has the structure of a left \((\mathcal{D}(T) \otimes_{\mathcal{U}b} \mathcal{D}(G))^{\mathcal{U}b}\)-module via an ‘inner’ action, and also of a right \(\mathcal{D}(T)^{op} \otimes \mathcal{D}(G)\)-module via an ‘outer’ action, see Example 1.1.2(ii).

Fix an opposite Borel subgroup \(\bar{B}\) with Lie algebra \(\bar{b}\). Let \(\bar{U} = [\bar{B}, \bar{B}]\), resp. \(\bar{u} = [\bar{b}, \bar{b}]\). In \(\mathcal{U}b\) we have introduced the Lie subalgebra \(\bar{u}\). The Harish-Chandra homomorphism is then defined as a composition \(\sigma : \mathcal{U} \xrightarrow{\psi} \mathcal{U}(G) \otimes_{\mathcal{U}} (\mathcal{D}(G)/\bar{u}^\psi)\). The two-sided quotient \(\mathcal{U}\bar{u}^\psi / \bar{u}^\psi = (\mathcal{U}\bar{u}^\psi) / \bar{u}^\psi = \mathcal{U}(\mathcal{D}(G)/\bar{u}^\psi)\) has the natural structure of a \((\mathcal{T}, \mathcal{D}(G)/\bar{u}^\psi)\)-bimodule.

Let the Lie algebra \(\mathcal{U}\bar{u}^\psi\) act on \(\mathcal{D}(T) \otimes \mathcal{D}(G)\) by \((uT \otimes uG) \mapsto uT \otimes (uG \cdot x)\). We define the Miura bimodule as follows:

\[
\mathcal{M} := \left( \mathcal{D}(T) \otimes_{\mathcal{U}b} \mathcal{D}(G) / \bar{u}^\psi \right) = \left( \mathcal{D}(T) \otimes_{\mathcal{U}b} (u \mathcal{D}(G)) / \bar{u}^\psi \right) \quad \text{(6.1.3)}
\]

The ‘inner’ action of the algebra \((\mathcal{D}(T) \otimes_{\mathcal{U}b} \mathcal{D}(G))^{\mathcal{U}b}\) on \(\mathcal{D}(T) \otimes_{\mathcal{U}b} \mathcal{D}(G)\) survives in \(\mathcal{M}\) and it makes \(\mathcal{M}\) a left module over any of the algebras in \([6.1.2]\). The ‘outer’ action of \(\mathcal{D}(T)^{op} \otimes \mathcal{D}(G)\) descends to a right action of the algebra \(\mathcal{D}(T)^{op} \otimes \mathcal{D}(G)\), on \(\mathcal{M}\). The right action commutes with the left action; furthermore, it makes \(\mathcal{M}\) a \((\mathcal{D}(T), \mathcal{D}(G)\)-bimodule. Let \(1_M \in \mathcal{M}\) be the image of the element \(1 \otimes 1 \in \mathcal{D}(T) \otimes \mathcal{D}(G)\) in \(\mathcal{M}\).

Recall that the Harish-Chandra homomorphism \(hc : Z\mathcal{U} \rightarrow \mathcal{U}t\), where \(Z\mathcal{U}\) denotes the center of the algebra \(\mathcal{U}g\), may be defined as follows. First, one observes that the image of \(Z\mathcal{U}\) under the projection \(pr : \mathcal{U}g \rightarrow \mathcal{U}g\) is clearly contained in the subspace \((u \mathcal{U}g)^{\mathcal{U}d}t\). On the other hand, weight considerations yield an equality \((u \mathcal{U}b)^{\mathcal{U}b}t = (u \mathcal{U}g)^{\mathcal{U}d}t\). In fact, it is known that \((u \mathcal{U}g)^{\mathcal{U}d}t = (u \mathcal{U}g)^{\mathcal{U}g}\), see Lemma 7.3.3(iii) below). We obtain an algebra isomorphism \(\sigma : \mathcal{U}t \xrightarrow{\psi} (u \mathcal{U}g)^{\mathcal{U}d}t\), cf. (6.1.1).

The Harish-Chandra homomorphism is then defined as a composition \(Z\mathcal{U} \xrightarrow{pr} (u \mathcal{U}g)^{\mathcal{U}d}t \xrightarrow{\sigma^{-1}} \mathcal{U}t\).

The imbedding \(\mathcal{U}g \xrightarrow{i} \mathcal{D}(G)\) provides an isomorphism of \(Z\mathcal{U}\) and the algebra of bi-invariant differential operators on \(G\). This isomorphism descends to an algebra map \(Z\mathcal{U} \rightarrow \mathcal{D}(G)\), \(z \mapsto z^{\psi}\). The assignment \(z_1 \otimes z_2 \mapsto i_T(hc(z_1)) \otimes z_2^{\psi}\) gives an imbedding of \(Z\mathcal{U} \otimes Z\mathcal{U}\) as a subalgebra of \(\mathcal{D}(T) \otimes \mathcal{D}(G)\). Thus, we may (and will) view \(\mathcal{M}\) as a \((Z\mathcal{U}, Z\mathcal{U})\)-bimodule using the ‘outer’ action of this subalgebra.

Let \(\mathcal{M}^{Z\mathcal{U}\text{out}} := \mathcal{M}^{Z\mathcal{U}g}\), where we follow the notation of Example 1.1.2(ii) and for any \((Z\mathcal{U}, Z\mathcal{U})\)-bimodule \(E\) write \(E^{Z\mathcal{U}} := \{ x \in E \mid zx = xz, \forall z \in Z\mathcal{U}\}\).

**Lemma 6.1.4.** We have \(1_M \in \mathcal{M}^{Z\mathcal{U}\text{out}}\).

**Proof.** Fix \(z \in Z\mathcal{U}\). It is well known (and follows from Lemma 6.2.1(i) below) that inside \(\mathcal{T}\) one has \(z^{\psi} = a(hc(z))\). Hence, inside \((\mathcal{D}(T) \otimes_{\mathcal{U}} \mathcal{D})^{\mathcal{T}}\) we find

\[
i_T(hc(z)) \otimes 1 = 1 \otimes a(hc(z)) = 1 \otimes z^{\psi}, \quad \forall z \in Z\mathcal{U}. \quad \text{(6.1.5)}
\]

It is clear that inside \(u \mathcal{D}(G)/\bar{u}^\psi\), we have \(z^{\psi} = z^{\psi}\). Hence, writing \(z_T := i_T(hc(z))\), inside \(\mathcal{M} = (\mathcal{D}(T) \otimes_{\mathcal{U}b} (u \mathcal{D}(G))/\bar{u}^\psi)\), we obtain

\[
z_1 z_T = z_T(1 \otimes 1) = 1(z_T) \otimes 1 \quad \text{[6.1.5]} \quad 1(1 \otimes z_T) = 1(1 \otimes 1) z_T = 1_M z,
\]

where we have used the notation \(b(1 \otimes 1) = \text{resp. } 1(b \otimes b')1\), for the outer, resp. inner, action on \(1_M = 1 \otimes 1\) of an element \(b \otimes b\) of \(\mathcal{T}^{op} \otimes \mathcal{D}(G)/(G/U)\), resp. \((\mathcal{D}(T) \otimes_{\mathcal{U}b} \mathcal{D})^t\).

---

6.2. Let \(E\) be a \(\mathcal{U}g\)-module. For any Lie subalgebra \(\mathfrak{k} \subseteq \mathcal{U}g\), the \(Z\mathcal{U}\)-action on \(E\) survives in \(E^\mathfrak{k}\), resp. \(\mathfrak{k}/E\). In the case \(\mathfrak{k} = \mathfrak{u}\), there is also a \(\mathcal{U}t\)-action \(a \cdot x = a \ast x\) on \(E^\mathfrak{u}\), resp. \(\mathfrak{u}/E\), induced by the \(\mathcal{U}b\)-action on \(E\).

The following is a simple consequence of results of Kostant [Ko2].
Lemma 6.2.1. (i) For any \( z \in Z_\mathcal{g} \) and \( x \in E^u \), resp. \( x \in u \backslash E \), one has \( zx = hc(z) \cdot x \). In particular, there is a well defined map \( \mathcal{U}t \otimes Z_\mathcal{g} (u \backslash E) \to u \backslash E \), given by the \( \mathcal{U}t \)-action.

(ii) If \( E \) is locally nilpotent as a \( \bar{\mathcal{u}}^\psi \)-module then the composite map

\[
\text{action} \cdot (\text{Id}_{\mathcal{U}t} \otimes f_E) : \mathcal{U}t \otimes Z_\mathcal{g} E^\bar{u}^\psi \xrightarrow{\text{Id}_{\mathcal{U}t} \otimes f_E} \mathcal{U}t \otimes Z_\mathcal{g} (u \backslash E) \xrightarrow{\text{action}} u \backslash E,
\]

where \( f_E \) is the composition \( E^\bar{u}^\psi \to E \to u \backslash E \), is an isomorphism of \( \mathcal{U}t \)-modules.

Proof. Part (i) is immediate from the construction of the Harish-Chandra homomorphism. Now, let \( E \) be as in (ii). It was shown in [Ko2, §3] that the natural map \( \mathcal{U}g/\bar{u}^\psi \otimes Z_\mathcal{g} E^\bar{u}^\psi \to E \) is an isomorphism. In the case where \( E \) is finitely genenerated as a \( \mathcal{U}g \)-module a proof of this isomorphism can also be found in [GG], Theorem 6.1. To prove the general case, choose a family \( E_\alpha \) of finitely genenerated \( \mathcal{U}g \)-submodules of \( E \), such that \( E = \lim_{\to} E_\alpha \). It is clear that we have \( E^\bar{u}^\psi = \lim_{\to} (E_\alpha)^{\bar{u}^\psi} \).

We obtain

\[
\mathcal{U}g/\bar{u}^\psi \otimes Z_\mathcal{g} E^\bar{u}^\psi = \mathcal{U}g/\bar{u}^\psi \otimes Z_\mathcal{g} \left( \lim_{\to} (E_\alpha)^{\bar{u}^\psi} \right) = \lim_{\to} (\mathcal{U}g/\bar{u}^\psi \otimes Z_\mathcal{g} (E_\alpha)^{\bar{u}^\psi}) = \lim_{\to} E_\alpha = E,
\]

where the second equality holds since the functor \( \mathcal{U}g/\bar{u}^\psi \otimes Z_\mathcal{g} (-) \) commutes with direct limits.

From the above isomorphism, we deduce

\[
u \backslash E = u \backslash (\mathcal{U}g \otimes_{\mathcal{U}g} E) = (u \backslash \mathcal{U}g) \otimes_{\mathcal{U}g} E = u \backslash \mathcal{U}g \otimes_{\mathcal{U}g} (\mathcal{U}g/\bar{u}^\psi \otimes Z_\mathcal{g} E^\bar{u}^\psi)
\]

\[= (u \backslash \mathcal{U}g/\bar{u}^\psi) \otimes_{\mathcal{U}g} E^\bar{u}^\psi = \mathcal{U}t \otimes Z_\mathcal{g} E^\bar{u}^\psi,
\]

where we have used that the composite \( \mathcal{U}t = u \backslash \mathcal{U}b \hookrightarrow u \backslash \mathcal{U}g \to u \backslash \mathcal{U}g/\bar{u}^\psi \) is an isomorphism. \( \square \)

Let \( E := \mathcal{D}(G)/\bar{u}^\psi \), so \( u \backslash E = u \backslash \mathcal{D}(G)/\bar{u}^\psi \). We have natural maps

\[
\mathcal{D}(T) \otimes_{Z_\mathcal{g}} (\mathcal{D}(G)/\bar{u}^\psi) \xrightarrow{\text{Id}_{\mathcal{D}(T)} \otimes f_E} \mathcal{D}(T) \otimes_{Z_\mathcal{g}} (u \backslash \mathcal{D}(G)/\bar{u}^\psi) \to \mathcal{D}(T) \otimes_{\mathcal{U}t} (u \backslash \mathcal{D}(G)/\bar{u}^\psi).
\]

Corollary 6.2.2. The composition of the above maps yields an isomorphism \( \mathcal{D}(T) \otimes_{Z_\mathcal{g}} \mathcal{D}(G/U) \to \mathcal{M} \), of \( (\mathcal{D}(T), \mathcal{D}(G/U)) \)-bimodules with respect to the outer action.

Proof. Observe that the action of \( \bar{u}^\psi \) on \( E = \mathcal{D}(G)/\bar{u}^\psi \) by left multiplication is locally-nilpotent. Hence, we compute

\[
\mathcal{M} = \mathcal{D}(T) \otimes_{\mathcal{U}t} (u \backslash \mathcal{D}(G)/\bar{u}^\psi) \quad \text{(by (6.1.3))}
\]

\[\cong \mathcal{D}(T) \otimes_{\mathcal{U}t} (\mathcal{U}t \otimes_{Z_\mathcal{g}} (\mathcal{D}(G)/\bar{u}^\psi)) \quad \text{(by Lemma 6.2.1)}
\]

\[\cong \mathcal{D}(T) \otimes_{Z_\mathcal{g}} (\mathcal{D}(G)/\bar{u}^\psi) = \mathcal{D}(T) \otimes_{Z_\mathcal{g}} \mathcal{D}(G/U).
\]

\( \square \)

The isomorphism of the corollary restricts to an isomorphism

\[
(\mathcal{D}(T) \otimes_{Z_\mathcal{g}} \mathcal{D}(G/U))_{Z_{\mathcal{g}, \text{out}}} \cong M_{Z_{\mathcal{g}, \text{out}}}, \tag{6.2.3}
\]

For \( \mu \in \mathbb{Q} \), let \( t^\mu \in \mathbb{C}[T] \) denote the character \( \mu \) viewed as a regular function on \( T \). The algebra \( \mathcal{D}(T) \) has a weight decomposition \( \mathcal{D}(T) = \bigoplus_{\mu \in \mathbb{Q}} t^\mu \cdot \mathcal{U}t = \bigoplus_{\mu \in \mathbb{Q}} \mathcal{U}t \cdot t^\mu \), with respect to the adjoint \( t \)-action, equivalently, \( T \)-action by translations. Similarly, the adjoint \( t \)-action on \( \mathcal{D}(T) \) is semisimple, so the algebra \( \mathcal{D}(T) \) has a weight grading:

\[
\mathcal{D} = \bigoplus_{\mu \in \mathbb{Q}} \mathcal{D}^\mu, \quad \mathcal{D}^\mu := \{ u \in \mathcal{D} \mid a(h) \cdot u = u \cdot a(h) \} = \mu(h)u, \ \forall h \in t \}.
\]

It is clear that each of the spaces \( \mathcal{D}^\mu \) is stable under the \( \mathcal{U}t \)-action \( h : u \mapsto a(h) \cdot u \), on \( \mathcal{D} \). So, one has direct sum decompositions

\[
(\mathcal{D}(T) \otimes_{\mathcal{U}t} \mathcal{D}) \mathcal{U}t = \bigoplus_{\mu \in \mathbb{Q}} (\mathcal{C}t^{-\mu} \cdot \mathcal{U}t) \otimes_{\mathcal{U}t} \mathcal{D}^\mu = \bigoplus_{\mu \in \mathbb{Q}} (\mathcal{C}t^{-\mu} \otimes \mathcal{D}^\mu).
\]
Furthermore, it is easy to check that the following map:
\[ \phi : \mathcal{D} = \bigoplus_{\mu \in Q} \mathcal{D}_\mu \longrightarrow (\mathcal{D}(T) \otimes \mathcal{U}_T \mathcal{D})^{\mathcal{H}}, \quad \sum_{\mu} u_{\mu} \mapsto \sum_{\mu} (t^{-\mu} \otimes u_{\mu}), \]  
(6.2.4)
is a Q-graded algebra isomorphism.

Recall that the left (inner) action on \( \mathcal{M} \) of the algebra \((\mathcal{D}(T) \otimes \mathcal{U}_T \mathcal{D})^{\mathcal{H}}\) and the right (outer) action of the algebra \( \mathcal{D}(T)^{\text{op}} \otimes \mathcal{D}(G/U) \) commute. Therefore, Lemma 6.1.4 implies that for any \( u \in \mathcal{D} \), one has \( \phi(u)1_{\mathcal{M}} = 1_{\mathcal{M}} \kappa_{\mathcal{D}}(u) \). We define a map \( \kappa_{\mathcal{D}} \) as a composition
\[ \kappa_{\mathcal{D}} : \mathcal{D} \xrightarrow{\phi(u)1_{\mathcal{M}}} \mathcal{M} \) \( \mathcal{M} \mathcal{Z}_{\mathcal{D}, \text{out}} \) \( \mathcal{D}(T)^{\text{op}} \otimes \mathcal{Z}_{\mathcal{D}} \mathcal{D}(G/U)^{Z_{\mathcal{D}, \text{out}}} \).
(6.2.5)
Explicitly, the element \( \kappa_{\mathcal{D}}(u) \) is uniquely determined from the equation \( \phi(u)1_{\mathcal{M}} = 1_{\mathcal{M}} \kappa_{\mathcal{D}}(u) \). Using this, for any \( u, v \in \mathcal{D} \), we find
\[
1_{\mathcal{M}} \kappa_{\mathcal{D}}(uv) = \phi(uv)1_{\mathcal{M}} = \phi(u)(\phi(v)1_{\mathcal{M}}) = \phi(u)(1_{\mathcal{M}} \kappa_{\mathcal{D}}(v)) = (\phi(u)1_{\mathcal{M}}) \kappa_{\mathcal{D}}(v) = (1_{\mathcal{M}} \kappa_{\mathcal{D}}(u)) \kappa_{\mathcal{D}}(v) = 1_{\mathcal{M}}(\kappa_{\mathcal{D}}(u) \kappa_{\mathcal{D}}(v)).
\]

We deduce that \( \kappa_{\mathcal{D}} \) is an algebra homomorphism.

A \( G \)-invariant volume form on \( G \), resp. \( B \), provides an algebra isomorphism \( \mathcal{D}(G) \rightarrow (\mathcal{D}(G)^{\text{op}}) \), resp. \( \mathcal{D}(G/U) \rightarrow \mathcal{D}(G/U)^{\text{op}} \). These isomorphisms are, in fact, independent of the choices of invariant volume forms since such a form is unique up to a nonzero constant factor. Using that \( \mathcal{D}(B) = \mathcal{D}(G/U) \cong (\mathcal{D}(G)/u)^{u} \), we obtain a chain of algebra isomorphisms
\[ \mathcal{D}(B) \cong \mathcal{D}(B)^{\text{op}} \cong ((\mathcal{D}(G)/u)^{u})^{\text{op}} \cong ((\mathcal{D}(G)^{\text{op}}/u)^{u})^{\text{op}} \cong (u/\mathcal{D}(G))^{u} = \mathcal{D}. \]  
(6.2.6)
The following result is a more precise version of Theorem 1.2.2.

**Theorem 6.2.7.** The following composite map is an algebra isomorphism:
\[ \mathcal{D}(B) \xrightarrow{\mathcal{D}(B)} \mathcal{D} \xrightarrow{\kappa_{\mathcal{D}}} (\mathcal{D}(T)^{\text{op}} \otimes \mathcal{Z}_{\mathcal{D}} \mathcal{D}(G/U))^Z_{\mathcal{D}, \text{out}}. \]

### 6.3. Review of twisted differential operators (TDO).
In this subsection we extend the formalism of TDO developed by Beilinson-Bernstein [BB] to a slightly more general setting of torsors of group schemes. Such an extension will be used in the next subsection to give an intrinsic construction of the Miura bimodule that does not depend on the choice of a pair \( B, \tilde{B}, \) of opposite Borel subgroups.

Let \( N \) be a smooth affine group scheme on a smooth variety \( X \) and \( n = \text{Lie} N \) the corresponding Lie algebra, a locally free \( O_X \)-module. Let \( p : P \rightarrow X \) be an \( N \)-torsor. The \( O_P \)-module \( p^{*}n = O_P \otimes_{p^{*}O_X} p^{*}n \) comes equipped with the natural structure of a Lie algebroid. The Lie bracket is given by the formula \( [f_1 \otimes \gamma_1, f_2 \otimes \gamma_2] = f_1 f_2 \otimes [\gamma_1, \gamma_2] + f_1 \gamma_1(f_2) \otimes \gamma_2 - f_2 \gamma_2(f_1) \otimes \gamma_1 \). There is an exact sequence of locally free sheaves
\[ 0 \rightarrow p^{*}n \rightarrow T_P \rightarrow p^{*}T_X \rightarrow 0. \]  
(6.3.1)

Let \( M = \text{Aut}_N(P) \) be the group scheme of (fiberwise) automorphisms of \( P \). The Lie algebra \( m = \text{Lie} M \) is a locally free \( O_X \)-module \((p_*O_P \otimes_{O_X} n)^N\), of sections of an associated bundle \( P \times N \) \( n \) for the adjoint representation of \( N \) on \( n \). Let \( \mathfrak{k} := (p_*T_P)^N \). The commutator of vector fields gives a Lie bracket on \( \mathfrak{k} \) which is not \( O_X \)-linear in general. Applying \((p_*(-))^N\) to (6.3.1) we get an exact sequence of sheaves of Lie algebras
\[ 0 \rightarrow m \rightarrow \mathfrak{k} \xrightarrow{\alpha} T_X \rightarrow 0. \]
The map \( a \) makes \( \mathfrak{k} \) a Lie algebroid on \( X \). Similarly, the bracket
\[
[k_1 \oplus f_1, k_2 \oplus f_2] := [k_1, k_2] \oplus (a(k_1)(f_2) - a(k_2)(f_1)),
\]
gives \( \mathfrak{k} \oplus \mathcal{O}_X \) the structure of a Lie algebroid on \( X \) with anchor map \((k \oplus f) \mapsto a(k) \). There is a canonical isomorphism \( (p_*(F_1 \mathcal{D}_P))^N \cong \mathfrak{k} \oplus \mathcal{O}_X \), of Lie algebroids, where \( F_1 \mathcal{D}_P = T_P \oplus \mathcal{O}_P \) is the sheaf of first order differential operators.

Let \( \Omega_{P/X}^1 \) be the sheaf of relative 1-forms on \( P \) and \( m^* := \mathcal{H}om_{\mathcal{O}_X}(m, \mathcal{O}_X) \). We have canonical isomorphisms
\[
m^* \cong \mathfrak{k}^*/\Omega_X^1 \cong (p_*(\mathcal{O}_{P/X}))^N \cong (p_*(\mathfrak{m} \otimes \mathfrak{n}^*))^N. \tag{6.3.2}
\]
Since \( \Gamma(X, \mathfrak{n}^*)^N \subseteq \Gamma(X, (p_*(\mathfrak{m} \otimes \mathfrak{n}^*))^N) \), any \( N \)-invariant section \( \psi \in \Gamma(X, \mathfrak{n}^*) \) gives, via \( (6.3.2) \), an associated section \( \psi_m \in \Gamma(X, m^*) \).

The Lie algebroid \( \mathfrak{k} \) acts on \( m^* \) via the Lie derivative \( Lie \). A section \( \phi \) of \( m^* \) is said to be \( \mathfrak{k} \)-invariant if \( Lie_X(\phi(m)) = \phi([k, m]) \) for all local sections \( k \in \mathfrak{k}, \ m \in m. \)

Let \( \psi \in \Gamma(X, \mathfrak{n}^*) \) be an \( N \)-invariant section. Then, for any local sections \( u_1, u_2 \) of \( \mathfrak{n}, \) resp. \( m, \) one has \( \psi([u_1, u_2]) = 0 \), resp. \( \psi_m([u_1, u_2]) = 0 \). Therefore, the map \( p^* \mathfrak{n} \to F_1 \mathcal{D}_P, \ u \mapsto u - (p^* \psi)(u), \) resp. \( m \to \mathfrak{k} \oplus \mathcal{O}_X, \ u \mapsto u - \psi_m(u), \) is a morphism of Lie algebroids. Let \( \psi^{\mathfrak{g}}_P \subseteq F_1 \mathcal{D}_P, \) resp. \( \mathfrak{m}^{\psi} \subseteq \mathfrak{k} \oplus \mathcal{O}_X = (p_*(F_1 \mathcal{D}_P))^N, \) be the image of that morphism. It is clear that \( p^*(\mathfrak{m}^{\psi}) = \psi^{\mathfrak{g}}_P. \)

Assume now that the \( N \)-invariant section \( \psi \) is such that the associated section \( \psi_m \in \Gamma(X, m^*) \) is \( \mathfrak{k} \)-invariant. Then, \( \mathfrak{m}^{\psi} \) is a Lie ideal of \( (p_*(F_1 \mathcal{D}_P))^N \). It follows that \( \mathfrak{m}^{\psi} \cdot (p_* \mathcal{D}_P)^N = (p_* \mathcal{D}_P)^N \cdot \mathfrak{m}^{\psi} \) is a two-sided ideal of \( (p_* \mathcal{D}_P)^N \).

One has a push-out diagram of Lie algebroids
\[
\begin{array}{c}
0 \rightarrow \mathfrak{m} \rightarrow \mathfrak{k} \rightarrow \mathfrak{k} \oplus \mathcal{O}_X \rightarrow 0 \\
\downarrow \psi_m \downarrow \downarrow \downarrow \downarrow
0 \rightarrow \mathcal{O}_X \rightarrow \mathfrak{k} \oplus \mathcal{O}_X / \mathfrak{m}^{\psi} \rightarrow \mathcal{O}_X \rightarrow 0
\end{array}
\]
The bottom row of the diagram is a Picard algebroid on \( X \). Let \( \mathcal{D}_X^{\psi} \) be an associated TDO on \( X \), see [BB] §2.1.4. There are canonical algebra isomorphisms
\[
\mathcal{D}_X^{\psi} \cong (p_* \mathcal{D}_P)^N / (p_* \mathcal{D}_P)^N \mathfrak{m}^{\psi} \cong (p_*(\mathcal{D}_P / \mathcal{D}_P \mathfrak{n}^{\psi}_P))^N.
\]
Here, the first isomorphism follows from the construction of \( \mathcal{D}_X^{\psi} \) given in loc cit. The second isomorphism is obtained by applying the functor \( p_*(-)^N \) to the isomorphism \( p^* \mathcal{D}_X^{\psi} \cong \mathcal{D}_P / \mathcal{D}_P \mathfrak{n}^{\psi}_P. \)

The sheaf \( p^* \mathcal{D}_X^{\psi} \) acts naturally on \( p^* \mathcal{D}_X^{\psi} \) by right multiplication. This gives \( \mathcal{D}_P / \mathcal{D}_P \mathfrak{n}^{\psi}_P \) the canonical structure of a \( (\mathcal{D}_P, p^* \mathcal{D}_X^{\psi}) \)-bimodule. Furthermore, one has
\[
\mathcal{E}nd_{\mathcal{D}_P}(\mathcal{D}_P / \mathcal{D}_P \mathfrak{n}^{\psi}_P) = (p^* \mathcal{D}_X^{\psi})^{op}, \ \text{resp.} \ \mathcal{E}nd_{p^* \mathcal{D}_X^{\psi}}(\mathcal{D}_P / \mathcal{D}_P \mathfrak{n}^{\psi}_P) = \mathcal{D}_P.
\]
For any left \( \mathcal{D}_X^{\psi} \)-module \( \mathcal{E} \), the sheaf
\[
p^* \mathcal{E} = (\mathcal{O}_P \otimes_{p_* \mathcal{D}_X^{\psi}} \mathcal{D}_X^{\psi}) \otimes_{p^* \mathcal{D}_X^{\psi}} \mathcal{E} \cong (\mathcal{D}_P / \mathcal{D}_P \mathfrak{n}^{\psi}_P) \otimes_{p^* \mathcal{D}_X^{\psi}} \mathcal{E}.
\]
has the natural structure of a left \( \mathcal{D}_P \)-module. For any left, resp. right, \( \mathcal{D}_P \)-module \( \mathcal{F} \) and \( i \geq 0, \) the sheaf \( H_i(\mathfrak{m}^{\psi}, p_* \mathcal{F}) \), resp. \( H^i(\mathfrak{m}^{\psi}, p_* \mathcal{F}) \), inherits a left, resp. right, action of \( \mathcal{D}_X^{\psi}. \)

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6.4. Fix a connected linear algebraic group $G$ and a $G$-torsor $P \to pt$. Let $^b G = \text{Aut}_G(P)$ be the group of automorphisms of $P$. Further, let $X$ be a smooth $G$-variety and $p : P \to X$ a smooth, surjective $G$-equivariant morphism. It follows that $G$ acts transitively on $X$. Let $N_x \subseteq G$ denote the stabilizer of $x \in X$ in $G$. The family $N_x$, $x \in X$, forms a smooth group scheme of stabilizers $N$, a subgroup of the constant group scheme $G \times X \to X$. The map $p : P \to X$ is a $G$-equivariant $N$-torsor. The group scheme $M = \text{Aut}_N(P)$ of automorphisms of this $N$-torsor, can be identified in a natural way with a constant subgroup scheme of the group scheme $^b G \times X \to X$, that is, we have $M = ^b N \times X \to X$, where $^b N$ is a subgroup of the group $^b G$. Moreover, $P$ is a $^b G$-torsor and the map $P \to X$ descends to a $G$-equivariant isomorphism $P/^b N \cong X$.

A choice of base point $pt \in P$ gives an isomorphism $G \cong ^b G$ and a $G$-equivariant, resp. $^b G$-equivariant, isomorphism $P \cong G$, resp. $P \cong ^b G$. Furthermore, writing $x = p(pt)$, we get a group isomorphism $^b N \cong N_x$, and a $G$-equivariant isomorphism $X \cong G/N_x$.

The goal of the rest of this subsection is to define a localized, i.e. sheaf-theoretic, counterpart of the Miura bimodule. To this end, we introduce some notation. Let $U_{\mathcal{B}}$, resp. $U_{\mathcal{B}_-}$, be the group scheme of stabilizers for the $G$-action on $\mathcal{B}$ and $\mathcal{B}_-$. Write $u_{\mathcal{B}} = \text{Lie} U_{\mathcal{B}}$, resp. $u_{\mathcal{B}_-} = \text{Lie} U_{\mathcal{B}_-}$. According to Lemma $2.2.5$ the natural map $p : \Xi \to \mathcal{B}$, resp. $p_- : \Xi \to \mathcal{B}_-$, is a $U_{\mathcal{B}}$-torsor, resp. $U_{\mathcal{B}_-}$-torsor. Furthermore, $\Xi$ is a $G$-torsor.

In section $3.3$ we have constructed a canonical $G \times T$-invariant section $\Psi \in \Gamma(\mathcal{B}_-, u_{\mathcal{B}_-}^*)$ such that the value of $\Psi$ at every point $(b, s) \in \mathcal{B}_-$ is a nondegenerate character of $u(b)$. Associated with $\Psi$, there is a canonically defined sheaf $u_{\mathcal{B}_-}^* \mathcal{P}$ on $\mathcal{B}_-$, and the corresponding TDO $\mathcal{P}_{\mathcal{B}_-}$, cf. $6.3$.

Let $\mathcal{P}_{\mathcal{B}} := (u_{\mathcal{B}} \langle p_\ast \mathcal{P}_{\Xi}\rangle)^{u_{\mathcal{B}}}$ This is a TDO on $\mathcal{B}$ and there are canonical isomorphisms
\[
p^{\ast}(\mathcal{P}_{\mathcal{B}})/p_{\ast}u_{\mathcal{B}_-}^\Psi \cong p^{\ast}u_{\mathcal{B}}\mathcal{P}_{\Xi}/p_{\ast}u_{\mathcal{B}_-}^\Psi \cong p^{\ast}u_{\mathcal{B}}\mathcal{P}_{\Xi}/p_{\ast}\mathcal{P}_{\mathcal{B}_-},
\] (6.4.1)
of sheaves on $\Xi$. Furthermore, it follows from section $6.3$ that the sheaf in (6.4.1) has a natural left action of the algebra $p^{\ast}(\mathcal{P}_{\mathcal{B}})$, as well as a right action of $p_{\ast}\mathcal{P}_{\mathcal{B}_-}$. These two actions commute.

We will need a modification of the sheaf defined in (6.4.1) that incorporates the maximal torus $T$. In more detail, let $B_\mathcal{B}$, resp. $B_{\mathcal{B}_-}$, be the group scheme of stabilizers for the $G$-action on $\mathcal{B}$, resp. $T \times G$-action on $\mathcal{B}$ by $(g, t) : x \mapsto gx^{-1}t$. Let $B_{\mathcal{B}_-}$ be a pull-back of $B_\mathcal{B}$ via the projection $\mathcal{B} \to \mathcal{B}$. Thus, we have $U_{\mathcal{B}} \cong [B_{\mathcal{B}}, B_{\mathcal{B}}]$ and $B_{\mathcal{B}}/U_{\mathcal{B}} \cong T_{\mathcal{B}}$ is a constant group scheme. It follows that the composition $B_{\mathcal{B}_-} \hookrightarrow T \times G \to \mathcal{B}$ yields a $B_\mathcal{B}$-equivariant isomorphism $\mathcal{O}_\Xi \cong T \times \Xi$; furthermore, $\psi$ is a $T \times G$-torsor. Therefore, the map $p$ is a $B_{\mathcal{B}}$-torsor, resp. $p_- : T_{\mathcal{B}_-} \times \mathcal{B}_- \to \mathcal{B}_-$. The map $p_-$ equals the composition $\tilde{\psi} \hookrightarrow \mathcal{B}_- \times \mathcal{B}$.

Next, we consider a special case of the general setting at the begining of this subsection where $G := T \times G$ and $P := T \times \Xi$. Let $p : T \times \Xi \to \mathcal{B}$, resp. $p_- : T \times \Xi \to \mathcal{B}_-$, be a map defined by the formula $p(t, x) = p(x) = p(t, x)$, resp. $p_-(t, x) = p_-(x)$. By Lemma $2.2.4$ we have a $T \times G$-equivariant isomorphism $\mathcal{O}_\Xi \cong T \times \Xi$; furthermore, $\mathcal{O}_{\mathcal{B}_-}$ is a $T \times G$-torsor. Therefore, the map $p$ is a $B_{\mathcal{B}}$-torsor, resp. $p_- : T_{\mathcal{B}_-} \times \mathcal{B}_- \to \mathcal{B}_-$. The map $p_-$ equals the composition $\tilde{\psi} \hookrightarrow \mathcal{B}_- \times \mathcal{B}$.

The sheaf $(b_{\mathcal{B}} \langle p_\ast \mathcal{P}_{\mathcal{O}_\Xi}\rangle)^{b_{\mathcal{B}}}$ is a TDO on $\mathcal{B}$; moreover, using a localized version of the map (6.2.4) one constructs a natural isomorphism $p^{\ast}((b_{\mathcal{B}} \langle p_\ast \mathcal{P}_{\mathcal{O}_\Xi}\rangle)^{b_{\mathcal{B}}}) \cong p^{\ast}(\mathcal{P}_{\mathcal{B}})$.

Below, we will abuse notation and given a sheaf $\mathcal{F}$ on $\Xi$, write $\mathcal{P}_{\Xi} \mathcal{F}$ for $\mathcal{O}_{T \times \Xi} \otimes \mathcal{O}_{\mathcal{T} \times \mathcal{O}_\Xi}(\mathcal{P}_{\Xi} \mathcal{F})$. Mimicing formula (6.4.1), we define
\[
\mathcal{M} := p^{\ast}b_{\mathcal{B}}\mathcal{P}_{\mathcal{O}_\Xi}/p_{\ast}u_{\mathcal{B}_-}^\Psi \cong t \langle (p^{\ast}u_{\mathcal{B}}\mathcal{P}_{\Xi}/p_{\ast}u_{\mathcal{B}_-}^\Psi) \rangle \cong t \langle (\mathcal{P}_{\Xi}(p^{\ast}u_{\mathcal{B}}\mathcal{P}_{\mathcal{B}_-})) \rangle,
\]
where t-coinvariants in the third, resp. fourth, term are taken with respect to the t-action induced by the natural map $t \mapsto \Gamma(\Omega, t \otimes O_{\Omega}) \to \Gamma(\Omega, p^*b_B \otimes p^*u_B)$, resp. $t \mapsto D(T) \otimes \Gamma(\Xi, p^*b_B \otimes p^*u_B)$, $h \mapsto h \otimes 1 - 1 \otimes h$. It is clear that $\mathcal{M}$ is a $G \times T$-equivariant sheaf on $\bar{\Omega}$. According to section 6.3 this sheaf comes equipped with a natural left action of the algebra $p^*((b_B \setminus p_\Omega)^\mathfrak{g}) \cong p^*\bar{D}_B$, as well as a right action of $D^o \otimes p^*\bar{D}_B^\mathfrak{g}$. The left and right actions commute, so $\mathcal{M}$ acquires the structure of a $(p^*\bar{D}_B, D^o \otimes p^*\bar{D}_B^\mathfrak{g})$-bimodule.

We now fix a base point $(\bar{B}, \bar{x}, B, \kappa_{\bar{b}}(x)) \in \Xi$. This gives an identification $\Xi = G$, resp. $\Omega = G \times T$, $\bar{B} = G/U$, and $\bar{B}_- = G/\bar{U}$. The map $\mathfrak{p}$ reads: $\mathfrak{p}(t, g) = gt^{-1}U$. Let $\psi \in \bar{u}^*$ be the value of $\Psi$ at the point $(\bar{B}, \bar{x})$. We obtain identifications $p^*u_b = u \otimes O_G$, resp. $p^*u_B^\mathfrak{g} = \bar{u}^g \otimes O_G$. Using these identifications one deduces an isomorphism $\Gamma(\Xi, p^*u_B \otimes (\bar{u}^g \otimes O_G)) = u \bar{\mathcal{D}}(G)/\bar{u}^g$, and algebra isomorphisms

\[
\begin{align*}
\Gamma(\bar{B}_-, \bar{D}_B^\mathfrak{g}) & \cong \bar{D}_G(U), \\
\Gamma(\bar{\Omega}, p^*\bar{D}_B) & = \Gamma(\bar{B}, \bar{D}_B) = \Gamma(\bar{B}, (u_B \otimes \bar{D}_B)u_B) = (u \bar{\mathcal{D}}(G))^u = \bar{\mathcal{D}}.
\end{align*}
\]

Thus, the space $\Gamma(\bar{\Omega}, \mathcal{M})$ becomes a canonical isomorphism $\mathcal{M} = \bar{\mathcal{D}} \otimes \bar{\mathcal{D}}^\mathfrak{g} (G/U)$ of opposite Borels, nor the choice of a nondegenerate character $\psi \in \bar{u}^*$. Thus, the object $\Gamma(\bar{\Omega}, \mathcal{M})$ in the LHS of (6.4.3) provides a completely canonical construction of the Miura bimodule $\mathcal{M}$. Similarly, formulas (6.4.2) provide canonical constructions of the algebras $\bar{\mathcal{D}}$ and $\mathcal{D}_\mathfrak{g}(G/U)$. With these constructions, the isomorphism of Theorem 6.2.7 hence of our main Theorem 1.2.2 becomes a canonical isomorphism of canonically defined algebras.

7. Proofs of main results

7.1. Reformulation in terms of isotypic components. Throughout this section, $V$ stands for an irreducible finite dimensional $G$-representation. We make $V \otimes \mathcal{U}G$ an $\mathcal{U}\mathfrak{g}$-bimodule by letting the left, resp. right, action of $g \in \mathfrak{g}$ be defined by $g(v \otimes u) = v \otimes (gu)$, resp. $(v \otimes u)g = -(gv) \otimes u + v \otimes (ug)$. Let $ad \ g : v \otimes u \mapsto g(v \otimes u) - (v \otimes u)g = (gv) \otimes u + v \otimes (gu - ug)$, be the corresponding adjoint $\mathfrak{g}$-action. This action is locally finite.

We fix a pair $\mathfrak{b}, \bar{\mathfrak{b}}$ of opposite Borel subalgebras and identify $t$ with $\mathfrak{b} \cap \bar{\mathfrak{b}}$. A right $\mathcal{U}\mathfrak{g}$-module

\[
M = u \bar{\mathcal{U}}G = \mathcal{U}\mathfrak{g}/u(\mathcal{U}\mathfrak{g})\]

is called the universal Verma module. It is clear from definitions that one has isomorphisms

\[
\begin{align*}
\begin{array}{c}
u \otimes \mathcal{U}\mathfrak{g})^u & \cong (V \otimes \mathcal{U}\mathfrak{g})/\bar{u}^g, \\
(u \otimes \mathcal{U}\mathfrak{g})^u & \cong (V \otimes \mathfrak{M})^u.
\end{array}
\end{align*}
\]

(7.1.1)

where $(-)^u$ stands for $u$-invariants of the right, equivalently, adjoint action. The left action of the algebra $\mathcal{U}\mathfrak{b} \subseteq \mathcal{U}\mathfrak{g}$ on $V \otimes \mathcal{U}\mathfrak{g}$ descends to a $\mathcal{U}t$-action on $V \otimes M$. The isomorphism(s) in the first, resp. second, line of (7.1.1) respect the left action of the algebra $\mathcal{U}t$ and the right action of the algebra $u(\mathcal{U}\mathfrak{g})/\bar{u}^g$, resp. $u(\mathcal{U}\mathfrak{g})^u$. Using the map $Z\mathfrak{g} \to (\mathcal{U}\mathfrak{g}/\bar{u}^g)^u$, resp. $\mathcal{U}t \to (u(\mathcal{U}\mathfrak{g})^u, cf. (6.1.1)$, one may view $(V \otimes \mathfrak{M})/\bar{u}^g$ as a $(\mathcal{U}t, Z\mathfrak{g})$-bimodule, resp. $(V \otimes \mathfrak{M})^u$ as a $(\mathcal{U}t, \mathcal{U}t)$-bimodule. There is also
an adjoint $t$-action on $(V \otimes M)^u$ induced by the adjoint action of $g$ on $V \otimes U_g$. The adjoint action is semisimple, so one has a weight space decomposition

$$(V \otimes M)^u = \bigoplus_{\mu \in \mathbb{Q}} (V \otimes M)^u, \mu, \quad (7.1.2)$$

where each $\text{ad}$-$t$-weight space $(V \otimes M)^u, \mu$ is a $(U_t, U_t)$-sub-bimodule of $(V \otimes M)^u$.

Fix $\mu \in \mathbb{Q}$ and let $\tau_\mu : U_t \to U_t$ be an algebra automorphism defined on generators $t \in t$ by the assignment $t \mapsto t - \mu(t).$ Let $J_\mu$ be an ideal of the algebra $U_t \otimes Z_g$ generated by the elements $\tau_\mu(hc(z)) \otimes 1 - 1 \otimes z, z \in Z_g.$ We claim that one has an inclusion

$$(V \otimes M)^u, \mu \subseteq ((V \otimes M)^u), \mu \subset \bigoplus_{\mu \in \mathbb{Q}}, \quad (7.1.3)$$

where for any $(U_t, Z_g)$-bimodule $E$, we write $E^{J_\mu} := \{ x \in E \mid \tau_\mu(hc(z))x = xz, \forall z \in Z_g \}$.

To prove (7.1.3), let $x \in (V \otimes M)^u$. For $t \in t$, we compute $xt = tx - \text{ad}t(x) = tx - \mu(t)x = \tau_\mu(t)x$. It follows that $za = \tau_\mu(a)x$ holds for all $a \in U_t$. We deduce that for any $z \in Z_g$, one has $xz = xhc(z) = \tau_\mu(hc(z))x$, where the first equality holds by Lemma 6.2.1(i). This proves (7.1.3).

Recall that we view $U_g$ as the algebra of left invariant differential operators on $G$. Also view $C[G]$, resp. $D(G)$, as a $G \times G$-module where the first, resp. second, copy of $G$ acts by left, resp. right, translations. Thus, we have $C[G] = \bigoplus V^* \otimes V$, resp. $D(G) = C[G] \otimes U_g = \bigoplus V^* \otimes (V \otimes U_g)$, where the sum ranges over the set of isomorphism classes of irreducible $G$-representations. Hence, using (7.1.1), we obtain a decomposition

$$u \backslash D(G)/\bar{u}^\psi = \bigoplus V^* \otimes (V \otimes M)/\bar{u}^\psi, \quad \text{resp.} \quad (u \backslash D(G))/\bar{u}^\psi = \bigoplus V^* \otimes (V \otimes M)^u, \quad (7.1.4)$$

into isotypic components with respect to the $G$-action by left translations. From (6.1.3) and the first isomorphism in (7.1.4), we get

$$M = D(T) \otimes_{U_t} (u \backslash D(G)/\bar{u}^\psi) \cong C[T] \otimes (u \backslash D(G)/\bar{u}^\psi) = \bigoplus V^* \otimes (C[T] \otimes (V \otimes M)/\bar{u}^\psi).$$

Further, write $C[T] = \bigoplus_{\mu \in \mathbb{Q}} C t^{-\mu}$. We find

$$(C[T] \otimes (V \otimes M)/\bar{u}^\psi)_{Z_g, \text{out}} = \bigoplus_{\mu} (C t^{-\mu} \otimes (V \otimes M)/\bar{u}^\psi)^{Z_g, \text{out}} = \bigoplus_{\mu} C t^{-\mu} \otimes ((V \otimes M)/\bar{u}^\psi)^{J_\mu}.$$ 

Thus, we obtain a decomposition

$$M^{Z_g, \text{out}} = \bigoplus_{\mu} V^* \otimes C t^{-\mu} \otimes ((V \otimes M)/\bar{u}^\psi)^{J_\mu}. \quad (7.1.5)$$

Similarly, from the second isomorphism in (7.1.4), we deduce

$$\overset{\dagger}{\mathcal{G}} = (u \backslash D(G))/\bar{u}^\psi = \bigoplus_{\mu} \overset{\dagger}{\mathcal{G}}^\mu = \bigoplus_{\mu} V^* \otimes (V \otimes M)^u, \mu. \quad (7.1.6)$$

The map $\kappa_{\mathcal{G}, \mu}$, in (6.2.5), commutes with the $G$-action by left translations and it also respects the $\mu$-decompositions. Therefore, we conclude that this map induces, for each $(V, \mu)$, a map $\kappa_{V, \mu} : (V \otimes M)^u, \mu \to ((V \otimes M)/\bar{u}^\psi)^{J_\mu}$, between the corresponding $(V, \mu)$-components of decompositions (7.1.6) and (7.1.3), respectively. It is easy to check that the map $\kappa_{V, \mu}$ equals the composition of the following natural inclusion and projection:

$$\kappa_{V, \mu} : (V \otimes M)^u, \mu \xrightarrow{\text{(7.1.3)}} (V \otimes M)^{J_\mu} \rightarrow ((V \otimes M)/\bar{u}^\psi)^{J_\mu}. \quad (7.1.7)$$

The $(U_t, Z_g)$-bimodule structure on $V \otimes M$ induces a $(U_t, Z_g)$-bimodule structure on each of the three objects in (7.1.7). It is clear from the above discussion that Theorem 6.2.7 follows from the theorem below, to be proved in §7.3.

**Theorem 7.1.8.** For any irreducible $G$-module $V$ and $\mu \in \mathbb{Q}$, the map

$$\kappa_{V, \mu} : (V \otimes M)^u, \mu \rightarrow ((V \otimes M)/\bar{u}^\psi)^{J_\mu},$$

is an isomorphism of $(U_t, Z_g)$-bimodules.
7.2. In this subsection we discuss a Poisson counterpart of Theorem 7.1.8.

Let \( r := e + g_h \). It is known, e.g. by Proposition 3.2.5(ii)-(iv), that this affine linear space is contained in \( g_r \). Let \( \mathfrak{z}_r \) be the restriction of the group scheme \( \mathfrak{z} \) to \( r \) and \( \mathfrak{z}_r \) the Lie algebra of \( \mathfrak{z}_r \).

We put \( g_h[r] := g_h \otimes \mathbb{C}[r] \), resp. \( \mathfrak{z}[r] := \Gamma(r, \mathfrak{z}_r) \). Thus, an element of \( g_h[r] \), resp. \( \mathfrak{z}[r] \), is a polynomial map \( r \to g_h \), resp. a polynomial map \( \mathfrak{g} \to \mathfrak{g} \) such that \( \xi(x) \in g_x \) for all \( x \in r \). By Lemma 5.1.1 for any \( x \in r \) we have \( g_x \subseteq b_x \). Therefore, there is a well-defined morphism \( \mathfrak{z}_r : \mathfrak{z}_r \to g_h \times r \) and the corresponding map \( \mathfrak{z}_r : \mathfrak{z}[r] \to g_h[r] \) of global sections, such that \( (\mathfrak{z}_r(\xi))(x) = \xi(x) \mod u_e \) for all \( x \in r \). Here and below, we use identifications \( g_h = b_e/u_e = t \).

The map \( s : r \to \tilde{g}_r \), \( r \mapsto (b_e, x) \), provides a section of the map \( \pi : \pi^{-1}(r) \to r \), \( (b, x) \mapsto x \). Hence, there is a canonical isomorphism \( \mathfrak{z}_r \equiv \mathfrak{z}[r] \). The morphism \( \mathfrak{z}_r \) defined above is induced, via the isomorphism, by the morphism \( \mathfrak{z} : \pi^{-1}(r) \to \tilde{g}_r \), see Lemma 5.1.3.

Let \( T \), resp. \( \mathfrak{z}_r \), act on \( T \times r \) by \( t_1 : (t, r) \mapsto (t_1 t, r) \), resp. \( z : (t, r) \mapsto (z(t)(z), t, r) \). By 5.3.1, 4.1.7 and 5.2.6, one has \( G \times \mathfrak{z}_r \)-equivariant isomorphisms

\[
\mathcal{T}^*T \times_r X \cong T \times Z \cong T \times (G \times r) \cong (T \times r) \times G.
\]

Here \( G \) acts in the LHS, resp. RHS, through its action on \( X \), resp. action on \( G \) by left translations. The group scheme \( \mathfrak{z}_r \) acts on \( T^*T \times_r X \) diagonally and acts on \( (T \times r) \times G \) through its action on \( T \times r \). We deduce an isomorphism

\[
\mathbb{C}[\mathcal{T}^*T \times_r X] \cong \mathbb{C}[(T \times r) \times G] = \mathbb{C}[G] \otimes \mathbb{C}[T \times r],
\]

of \( G \)-representations and also of \( \mathfrak{z}[r] \)-modules. One has the decomposition \( \mathbb{C}[G] = \bigoplus V^* \otimes V \) and the weight decomposition \( \mathbb{C}[T] = \bigoplus \mu \mathbb{C}[\mu] \). Put \( V \otimes \mathbb{C}[r] = V \otimes \mathbb{C}[r] \). Separating \( G \)-isotypic components in the above isomorphism, for every \( V \), we obtain

\[
(V \otimes \mathbb{C}[\mathcal{T}^*T \times_r X])^G \cong V \otimes \mathbb{C}[T] \otimes \mathbb{C}[r] \cong \bigoplus_{\mu \in Q} (V^* \otimes \mathbb{C}[\mu]) \otimes V[r].
\]  

(7.2.1)

Write \( \xi \mathbf{v} \) for the natural pointwise action of an element \( \xi \in \mathfrak{z}[r] \) on a polynomial map \( \mathbf{v} : r \to V \) given by \( (\xi \mathbf{v})(x) := \xi(x)(\mathbf{v}(x)) \). Then, the \( \mathfrak{z}[r] \)-action on the LHS of (7.2.1) goes, via the above isomorphism, to a \( \mathfrak{z}[r] \)-action on the RHS given by the formula

\[
\xi(t^\mu \otimes \mathbf{v}) = t^\mu \otimes ((\mu \circ \mathfrak{z}_r)(\xi)) \cdot \mathbf{v} + \xi(z) \mathbf{v}, \quad \forall \mu \in Q, \mathbf{v} \in V[r],
\]

where \( \mu \in Q \) is viewed as a linear function on \( g_h = t \), so \( \mu \circ \mathfrak{z}_r \) is the composition \( \mathfrak{z}[r] \xrightarrow{\mathfrak{z}_r} g_h[r] \xrightarrow{\mu} \mathbb{C}[r] \).

Let \( z \in \mathbb{C}[c] \). We may view the differential \( d(\mathfrak{z}_r^* z) \) of the function \( \mathfrak{z}_r^* z \in \mathbb{C}[\mathfrak{g}]^G \), as a polynomial map \( g \to \mathfrak{g} \). The map being \( G \)-equivariant, it follows that \( d(\mathfrak{z}_r^* z)(x) \in g_x \), for all \( x \in \mathfrak{g} \). Therefore, restricting this map to \( r \), one gets an element of \( \mathfrak{z}[r] \) to be denoted \( \xi(z) \). In terms of the isomorphism of Lemma 5.1.5, one can identify \( \xi(z) \) with the image of the section \( dz \) of \( T^* c \) under the composition \( \Gamma(c, T^* c) \to \Gamma(c, \mathfrak{z}_r^* \mathfrak{z}_r) \to \Gamma(c, \mathfrak{z}_r^* \mathfrak{z}_r) \equiv \mathfrak{z}[r] \), where \( \mathfrak{z}_r^* \) denotes the composite map \( r \to \mathfrak{g} \). 

Let \( \tilde{z} \) be a pull-back of \( z \) in \( \mathbb{C}[c] \) via the composite map \( T^*T \times_r X \to c \), cf. 5.2.7. The Poisson bracket with \( \tilde{z} \) gives an operator \( \{\tilde{z}, -\} \) on \( \mathbb{C}[T^*T \times_r X] \). The function \( \tilde{z} \) being \( G \)-invariant, the operator \( \{\tilde{z}, -\} \) commutes with the \( G \)-action, hence restricts to an operator on each isotypic component \( (V \otimes \mathbb{C}[\mathcal{T}^*T \times_r X])^G \). Equation 5.1.6 implies that the latter operator is given, in terms of decomposition (7.2.1), by the formula

\[
\{\tilde{z}, t^\mu \otimes \mathbf{v}\} = \xi(z)(t^\mu \otimes \mathbf{v}) = t^\mu \otimes ((\mu \circ \mathfrak{z}_r)(\xi(z)) \cdot \mathbf{v} + \xi(z) \mathbf{v}).
\]

We define

\[
V[r]^{(\mu)} := \{v \in V[r] \mid \xi(\mathbf{v}) = \mu \circ \mathfrak{z}_r(\xi) \cdot \mathbf{v}, \forall \xi \in \mathfrak{z}[r]\}. 
\]  

(7.2.2)

From (5.1.6) and (7.2.1)-(7.2.2), using that the Poisson action of \( \mathbb{C}[c] \) commutes with the \( G \)-action, we find that the Poisson centralizer of \( \mathbb{C}[c] \) in the LHS of (7.2.1) has the following decomposition

\[
((V \otimes \mathbb{C}[\mathcal{T}^*T \times_r X])^G)^{\mathbb{C}[c]} \cong (V \otimes \mathbb{C}[\mathcal{T}^*T \times_r X]^{[r]})^G 
\]

(7.2.3)
\[ \cong (V \otimes \mathbb{C}[T] \otimes \mathbb{C}[\xi])^{[r]} \cong \bigoplus_{\mu \in \mathbb{Q}} \mathbb{C}t^{-\mu} \otimes V[\mu]. \]

We are going to use Theorem \[13.3\] and compare \[7.2.3\] with an analogous decomposition for \( \mathbb{C}[T^*B] \). To this end, let \( B = B_e \), resp. \( U = U_e \), and identify \( B = G/U \). The group \( B \) acts on \( \mathfrak{b} \) via the adjoint action and it also acts on every \( G \)-representation \( V \). We let \( B \) act on \( V[\mathfrak{b}] := V \otimes \mathbb{C}[\mathfrak{b}] \) diagonally. Let \( V[\mathfrak{b}]^{U,\mu} \subseteq V[\mathfrak{b}] \) denote the \( \mu \)-weight space of the residual action of \( T = B/U \) on \( V[\mathfrak{b}]^U \). The group \( G \), resp. \( T \), acts on \( G/U \) by left, resp. right, translations. Using \( G \times T \)-equivariant isomorphisms \( T^*(G/U) \cong G \times_U U \) and the Peter-Weyl decomposition we obtain, for every \( V \), an isomorphism

\[ (V \otimes \mathbb{C}[T^*B])^G \cong (V \otimes \mathbb{C}[\mathfrak{b}])^U = V[\mathfrak{b}]^U = \bigoplus_{\mu \in \mathbb{Q}} (V[\mathfrak{b}])^{U,\mu}. \] (7.2.4)

The isomorphism \( \mathbb{C}[T^*(G/U)] \cong \mathbb{C}[T^*T \times_{\epsilon} \mathcal{X}]^{[c]} \) of Theorem \[13.3\] is \( G \)-equivariant. Hence it induces for every \( V \), an isomorphism, cf. \[7.2.3\] and \[7.2.4\],

\[ \kappa_V : (V \otimes \mathbb{C}[T^*B])^G \cong V[\mathfrak{b}]^U \xrightarrow{\cong} (V \otimes \mathbb{C}[T^*T \times_{\epsilon} \mathcal{X}]^{[c]})^G. \]

View \( f \in V[\mathfrak{b}]^{U,\mu} \) as a \( V \)-valued polynomial function on \( \mathfrak{b} \) and let \( f|_\mathfrak{r} \in V[\mathfrak{r}] \) be its restriction to \( \mathfrak{r} \subseteq \mathfrak{b} \). Then, from commutativity of the diagram of Proposition \[5.2.2\] one easily derives that \( f|_\mathfrak{r} \in V[\mathfrak{r}]^{[\mu]} \) and, moreover, one has \( \kappa_V(f) = t^{-\mu} \otimes f|_\mathfrak{r} \). It follows that the map \( \kappa_V \) respects the \( \mu \)-decompositions in \( \mathfrak{g} \) and \( \mathfrak{g}^\mathfrak{c} \), respectively. Thus, this map breaks up into a direct sum of maps

\[ \kappa_{V,\mu} : V[\mathfrak{b}]^{U,\mu} \rightarrow V[\mathfrak{r}]^{[\mu]}, \quad f \mapsto f|_\mathfrak{r}, \quad \mu \in \mathbb{Q}. \]

The map \( \kappa_V \) being an isomorphism, we obtain the following result that may be viewed as a Poisson counterpart of Theorem \[6.2.7\] cf. isomorphisms \( \mathfrak{g} \) below.

**Proposition 7.2.5.** For every irreducible \( G \)-module \( V \) and \( \mu \in \mathbb{Q} \), the map \( \kappa_{V,\mu} \) is an isomorphism.

\[ \square \]

### 7.3. A deformation construction

We are going to deduce Theorem \[7.1.8\] from Proposition \[7.2.5\] by a deformation argument. To this end, we need to recall a well-known construction of deformation to the normal bundle.

Let \( X \) be a smooth connected affine algebraic variety equipped with a smooth (i.e. flat with smooth fibers) morphism \( X \rightarrow \mathbb{C} \). Let \( I \subset \mathbb{C}[X \times \mathbb{C} X] \) be the ideal of the diagonal \( X_\Delta \subset X \times \mathbb{C} X \). Write \( h \) for the coordinate on \( \mathbb{C} \) and define a \( C[h] \)-subalgebra \( A \subset C[h, -1] \otimes_{C[h]} C[X \times \mathbb{C} X] \) by \( A := \sum_{n \geq 0} h^{-n} T^n \). By definition, one has an algebra embedding \( C[X] \otimes_{C[h]} C[X] = C[X \times \mathbb{C} X] \hookrightarrow A \). The algebra \( A \), viewed as a subalgebra of \( C[h, -1] \otimes_{C[h]} C[X] \otimes_{C[h]} C[X] \), is generated by its subalgebra \( C[X] \otimes_{C[h]} C[X] \) and the elements \( z \otimes -1 \otimes z \), \( z \in C[X] \). If \( z_1, \ldots, z_r \) is a set of generators of \( C[X] \), then the \( C[h] \)-algebra \( A \) is generated by the elements \( z_i \otimes -1 \otimes z_i \), and either \( z_i \otimes 1 \), or \( 1 \otimes z_i \), for \( i = 1, \ldots, r \). Thus, \( A \) is a finitely generated algebra and we put \( \mathfrak{n}_X := \text{Spec}(A) \). The algebra embedding \( C[X \times \mathbb{C} X] \hookrightarrow A = C[\mathfrak{n}_X] \) induces a canonical morphism \( p : \mathfrak{n}_X \rightarrow X \times \mathbb{C} X \), which is an isomorphism over \( \mathbb{C} \setminus \{0\} \). It is known, cf. eg. [GR], 5.11, that the scheme \( \mathfrak{n}_X \) is smooth and the composite \( \mathfrak{n}_X \rightarrow X \times \mathbb{C} X \rightarrow \mathbb{C} \) is a smooth morphism.

The diagonal imbedding \( \Delta : X \rightarrow X_\Delta \subset X \times \mathbb{C} X \) has a canonical lift to a closed imbedding \( \epsilon : X \hookrightarrow \mathfrak{n}_X \) such that \( p \circ \epsilon = \Delta \). The defining ideal of the image of \( \epsilon \) is an ideal \( J \subset \mathfrak{n}_X \) generated by the elements \( z \otimes -1 \otimes z \), \( z \in C[X] \), where we identify elements of \( C[X] \otimes_{C[h]} C[X] \) with their images under \( p^* \). By construction, one has a \( C[h] \)-algebra isomorphism \( C[\mathfrak{n}_X]/J \cong C[X] \).

Let \( X_0 \), resp. \( \mathfrak{n}_{X_0} \), be the fiber of \( X \rightarrow \mathbb{C} \), resp. \( \mathfrak{n}_X \rightarrow \mathbb{C} \), over \( 0 \in \mathbb{C} \). The variety \( \mathfrak{n}_{X_0} \) is known to be canonically isomorphic to \( T(X_0) \), the total space of the tangent bundle on \( X_0 \). The variety \( p_{\mathfrak{n}}^{-1}(X_\Delta) \) is a union of two irreducible components, \( \epsilon(X) \) and \( T(X_0) \). These components

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meet transversely at the zero section \(X_0 \subset T(X_0)\) of the tangent bundle. It follows that the zero section, viewed as a subvariety of \(\mathfrak{g}X\), is defined by the ideal \(J + h\mathbb{C}[\mathfrak{g}X] = J + h\mathbb{C}[X \times A_\hbar, X]\).

Given a \(\mathbb{C}[\hbar]\)-module \(M\), we write \(M\big|_{\hbar=0} := M/hM\). Let \(M\) be a \(\mathbb{C}[X \times C X]\)-module such that the \(\mathbb{C}[X \times C X]\big|_{\hbar=0}\)-module \(M\big|_{\hbar=0}\) is annihilated by the ideal of the diagonal \(X_0 \subset X \times X_0\). Thus, for any \(z \in \mathbb{C}[X]\) and \(m \in M\), one has \((z \otimes 1 - 1 \otimes z)m = hm',\) for some \(m' \in \mathbb{C}[X \times C X]\). If \(M\) is \(h\)-torsion free, then the element \(m'\) is uniquely determined by \(z\) and \(m\). We conclude that for any \(h\)-torsion free \(\mathbb{C}[X \times C X]\)-module \(M\) such that \(M\big|_{\hbar=0}\) is annihilated by the ideal of the diagonal of \(X_0 \times X_0\), the \(\mathbb{C}[X \times C X]\)-action on \(M\) has a canonical extension to a \(\mathbb{C}[\mathfrak{g}X]\)-action. Furthermore, the latter action induces a \(\mathbb{C}[T(X_0)]\)-action on \(M\big|_{\hbar=0}\).

Below we abuse terminology and given \(\mathbb{C}[\hbar]\)-algebras \(A_1\) and \(A_2\), refer to left \(A_1 \otimes \mathbb{C}[\hbar] A_2^{op}\)-modules as ‘\((A_1, A_2)\)-bimodules’. Recall that a bimodule \(M\) over a commutative algebra \(A\) is said to be symmetric if \(am = ma\) for all \(a \in A, m \in M\).

Let \(U_h\ell\) denote the asymptotic enveloping algebra of a Lie algebra \(\ell\). By definition, \(U_h\ell\) is a \(\mathbb{C}[\hbar]\)-algebra generated by the vector space \(\ell\), with relations \(xy - yx = h[x, y]\) for \(x, y \in \ell\). We have \((U_h\ell)\big|_{\hbar=0} = \text{Sym}\ell\). For any \(x \in \ell\), the map \(\ell \rightarrow \ell, y \mapsto [x, y]\), extends uniquely to a \(\mathbb{C}[\hbar]\)-linear derivation \(\text{ad}x\) of the algebra \(U_h\ell\). This gives an ‘adjoint’ action of \(\ell\) on \(U_h\ell\). The ad \(\ell\)-action is locally finite.

The assignment \(h \mapsto 1 \otimes h, x \mapsto 1 \otimes x - x \otimes h\), has a unique extension to an algebra homomorphism \(U_h\ell \rightarrow U_h^{op} \otimes U_h\ell\). Via this homomorphism, for any left \(\ell\)-module \(V\) and a right \(U_h\ell\)-module \(E\), the vector space \(V \otimes E\) acquires the structure of a right \(U_h\ell\)-module. We define a left \(U_h\ell\)-action on \(V \otimes U_h\ell\) by \(x(v \otimes u) = v \otimes xu\). This makes \(V \otimes U_h\ell\) an \((U_h\ell, U_h\ell)\)-bimodule. There is also an ‘adjoint’ action of \(\ell\) on \(V \otimes U_h\ell\) defined by \(adx(v \otimes u) := -x(v) \otimes u + v \otimes \text{ad}x(u)\). The above actions are related by the formula \(x(v \otimes u) - (v \otimes u)x = h \cdot \text{ad}x(v \otimes u)\). It follows that for any \(a \in U_h\mathfrak{g}\) and \(m \in V \otimes U_h\ell\), one has \(am = ma = h(V \otimes U_h\ell)\), hence \((V \otimes U_h\ell)\big|_{\hbar=0}\) is a symmetric \((\text{Sym}\mathfrak{g}, \text{Sym}\mathfrak{g})\)-bimodule. Furthermore, the following formula

\[
\{u\big|_{\hbar=0}, m\big|_{\hbar=0}\} := ((\frac{1}{h}(um - mu))\big|_{\hbar=0}, \quad u \in U_h\mathfrak{g}, m \in V \otimes U_h\ell, \quad (7.3.1)
\]
gives \((V \otimes U_h\ell)\big|_{\hbar=0}\) the structure of a Poisson module over the algebra \(\text{Sym}\mathfrak{g}\) equipped with the Kirillov-Kostant Poisson structure.

Various constructions of previous subsections have asymptotic analogues. In particular, one has the universal asymptotic Verma module \(M_h := u\not\mid U_h\mathfrak{g}\). Let \(V\) be a finite dimensional \(G\)-representation. By the above, the vector space \(V \otimes M_h\) has the structure of a right \(U_h\mathfrak{g}\)-module. The left action of the subalgebra \(U_h\mathfrak{b} \subset U_h\mathfrak{g}\) on \(V \otimes U_h\mathfrak{g}\) descends to a left \(U_h\mathfrak{b}\)-action on \(V \otimes M_h\). This makes \(V \otimes M_h\) an \((U_h\mathfrak{b}, U_h\mathfrak{g})\)-bimodule. The adjoint \(\mathfrak{b}\)-action on \(V \otimes U_h\mathfrak{g}\) descends to an \(\mathfrak{a}\)-\(\mathfrak{b}\)-action on \(V \otimes M_h\). We write \((V \otimes M_h)^a\) for the space of \(u\)-invariants of the right, equivalently adjoint, action. Similarly to the discussion between formulas \((7.1.1)\) and \((7.1.2)\), the space \((V \otimes M_h)^a\) has the natural structure of a \((U_h\mathfrak{b}, U_h\mathfrak{g})\)-bimodule. There is also an adjoint \(\mathfrak{a}\)-action on \((V \otimes M_h)^a\) which is related to the \(\mathfrak{a}\)-\(\mathfrak{b}\)-bimodule structure by the formula \(t(v \otimes m) - (v \otimes m)t = \text{ad}t,\) for all \(t \in \mathfrak{a}, v \otimes m \in (V \otimes M_h)^a\). The \(\mathfrak{a}\)-\(\mathfrak{b}\)-action is semisimple. Thus, one has a weight decomposition \((V \otimes M_h)^a = \bigoplus_{\mu \in \mathfrak{g}} (V \otimes M_h)^a\mu\) as a direct sum of \((U_h\mathfrak{b}, U_h\mathfrak{g})\)-subbimodules, cf. \((7.1.2)\).

Observe that \((V \otimes M_h)\big|_{\hbar=0}\) is symmetric as a \((\text{Sym} \mathfrak{b}, \text{Sym} \mathfrak{b})\)\)-bimodule since \((V \otimes U_h\mathfrak{g})\big|_{\hbar=0}\) is symmetric as a \((\text{Sym} \mathfrak{g}, \text{Sym} \mathfrak{g})\)\)-bimodule, cf. discussion above formula \((7.3.1)\). Furthermore, the action of \(\text{Sym} \mathfrak{b}\) factors through an action of \(\text{Sym} \mathfrak{t}\). The identification \(\mathfrak{g}/u = \mathfrak{b}^*,\) see \((3.1.1)\), yields an identification \(\text{Sym}(\mathfrak{g}/u) = \mathbb{C}[\mathfrak{b}]\). We deduce

\[
(V \otimes M_h)\big|_{\hbar=0} \cong V \otimes \text{Sym}(\mathfrak{g}/u) = V \otimes \mathbb{C}[\mathfrak{b}], \quad (7.3.2)
\]

The projection \(\text{Sym} \mathfrak{b} \rightarrow \text{Sym} \mathfrak{t}\) corresponds, via the identifications \(\mathbb{C}[\mathfrak{t}] = \mathbb{C}[\mathfrak{t}^*]\), to a pull-back map \(pr^*_{\mathfrak{b}} : \mathbb{C}[\mathfrak{t}] \rightarrow \mathbb{C}[\mathfrak{b}]^{\mathfrak{t}^*}\) induced by the projection \(pr_{\mathfrak{b}} : \mathfrak{b} \rightarrow \mathfrak{b}/u = \mathfrak{t}\). Therefore, the action of \(\text{Sym} \mathfrak{t}\) on \((V \otimes M_h)\big|_{\hbar=0}\) corresponds, via isomorphism \((7.3.2)\), to a \(\mathbb{C}[\mathfrak{t}]\)-action on \(V \otimes \mathbb{C}[\mathfrak{b}]\) given
by $f : (v \otimes g) \mapsto v \otimes ((pr_1^* f) \cdot g)$. We remark that the map $pr_1^* : \mathbb{C}[t] \to \mathbb{C}[b]^U$ is an algebra isomorphism. This follows from the fact that the fiber of the map $pr_1$ over any regular element of $t$ is a single $U$-orbit, cf. [CG] Lemma 1.1.14], hence every $U$-invariant polynomial on $b$ must be a pull-back of a polynomial on $t$.

**Lemma 7.3.3.** (i) The left $U_t$-action makes $(V \otimes M_b)^u$ a free $U_t$-module of finite rank.

(ii) The map $(V \otimes M_b)^u |_{h=0} \to (V \otimes \mathbb{C}[b]^U)^u$ induced by \((7.3.2)\) is an isomorphism of $\mathbb{C}[t]$-modules.

(iii) The following maps induced by the imbedding $t = b/u \hookrightarrow g/u$ are algebra isomorphisms:

$$U_t \mapsto (u \setminus U_t)^u = \text{End}_{U_t} M, \quad \text{resp.} \quad U_t \mapsto (u \setminus U_h g)^u = \text{End}_{U_h g} M_h,$$

where the equalities come from \((1.1.1)\).

**Proof.** Parts (i) and (ii) of the lemma are essentially contained in [GR]. In more detail, for any Lie algebra $\mathfrak{t}$ we view (as we may) the algebra $U_t \mathfrak{t}$ as the Rees algebra associated with the PBW filtration on $U_t$. This makes $U_t \mathfrak{t}$ a non-negatively graded algebra with homogeneous components $(U_t \mathfrak{t})^{(i)}$, where $(U_t \mathfrak{t})^{(0)} = \mathbb{C}$ and $(U_t \mathfrak{t})^{(1)} = \mathfrak{t} \oplus Ch$. This applies, in particular, to $\mathfrak{t} = 0$, resp. $\mathfrak{t} = \mathfrak{g}$ and $\mathfrak{t} = t$. Let $V \otimes M_b = \oplus_{i \geq 0} (V \otimes M_b)^{(i)}$ be a grading defined by the formula $(V \otimes M_b)^{(i)} = (V \otimes 1)(U_t \mathfrak{g})^{(i)}$. One also has a grading $\mathbb{C}[t] = \oplus_{i \geq 0} \mathbb{C}[t]^i$, resp. $V \otimes \mathbb{C}[b] = \oplus_{i \geq 0} V \otimes \mathbb{C}[b]^i$, where $\mathbb{C}[-]$ denotes the space of degree $i$ homogeneous polynomials. Then, $(V \otimes M_b)^u$, resp. $(V \otimes \mathbb{C}[b]^u)$, is a graded $U_t$-submodule of $V \otimes M_b$, resp. $\mathbb{C}[t]$-submodule of $V \otimes \mathbb{C}[b]$. Now, Lemma 3.5.2 of [GR] says that the map in the statement of part (ii) is an isomorphism of graded $\mathbb{C}[t]$-modules, proving (ii).

To prove (i), let $\mathcal{D}_h(G)$ be the Rees algebra associated with the filtration on $\mathcal{D}(G)$ by order of the differential operator. Thus, $\mathcal{D}_h(G)$ is a graded $\mathbb{C}[h]$-algebra and we put $\mathcal{D}_h := (u \setminus \mathcal{D}_h(G))^u$, an asymptotic analogue of the algebra $\mathcal{D} := (u \setminus \mathcal{D}(G))^u$. There is a graded $\mathbb{C}[h]$-algebra imbedding $U_t \mapsto \mathcal{D}_h$, resp. and an isotypic decomposition $\mathcal{D}_h = \oplus V \otimes (V \otimes M_b)^u$ as direct sum of graded $U_t$-modules, an asymptotic analogue of the imbedding $a$, cf. Section 6.1 resp. decomposition \((7.1.1)\). It was shown in [GR] Proposition 3.2 that $\mathcal{D}_h$ is free as a graded $U_t$-module. It follows that each isotypic component $(V \otimes M_b)^u$ is projective as a left $U_t$-module. This projective module is necessarily free since it is a nonnegatively graded module, proving (i).

Statement (iii) is well known but we provide a proof for completeness. To this end, we apply (the proofs of) parts (i) and (ii) in the special case where $V$ is the trivial representation. We deduce that $(u \setminus U_t g)^u$ is a free graded $U_t$-module and the map $((u \setminus U_t g)^u)|_{h=0} : \mathbb{C}[b]^U$ is an isomorphism of graded $\mathbb{C}[t]$-modules. Therefore, to prove that the second map in (iii) is an isomorphism it suffices, by the graded Nakayama lemma, to prove that the composition $\mathbb{C}[t] = U_t|_{h=0} \to (u \setminus U_t g)^u|_{h=0} : \mathbb{C}[b]^U$ is an isomorphism. This composition is the pull-back map $pr_1^* : \mathbb{C}[t] \to \mathbb{C}[b]^U$, which is an isomorphism by the remark before Lemma \((7.3.3)\). Finally, we observe that the first isomorphism of part (iii) follows from the second by the specialization at $h = 1$. \hfill \Box

Let $Z_h g$ denote the center of $U_h g$. There is an asymptotic counterpart of the Harish-Chandra homomorphism defined as a composition $h c_h : Z_h g \to \text{End}_{U_h g} M_h = (u \setminus U_t g)^u \to U_t$, where the first map comes from the $Z_h g$-action on $M_h$ and the isomorphism on the right is an inverse of the isomorphism of Lemma \((7.3.3)\) (iii). One shows, mimicking standard arguments, that the map $h c_h$ yields an algebra isomorphism $h c_h : Z_h g \to (U_t W) = \mathbb{C}[t^* \times \mathbb{k}] W$. Here, $W$-invariants are taken with respect to the ‘dot-action’ of $W$ defined on generators $t \in t$ by $w : t \mapsto w(t) - \rho(w(t)) + \rho(t)$, where $\rho$ is the half-sum of positive roots. The isomorphism $h c_h$ specializes at $h = 0$ to the Chevalley isomorphism $(\text{Sym } g)^G \cong (\text{Sym } t)^W$.

Let $f_i, i \in I$, be simple root vectors for the Lie algebra $\mathfrak{u} := u t$ and $\psi \in \mathfrak{u}^*$ a nondegenerate character. Let $\mathfrak{u}_h^\psi$ be a $\mathbb{C}[h]$-submodule of $U_h g$ generated by the vector space $[u, \mathfrak{u}] \subseteq g$ and the elements $f_i - \psi(f_i), i \in I$. For any $x, y \in \mathfrak{u}_h^\psi$, we have $xy - yx \in \mathfrak{u}_h^\psi$. The adjoint action of $\mathfrak{u}$ on
$V \otimes \mathcal{U}_h \mathfrak{g}$ descends to a locally finite $u$-action on $(V \otimes \mathcal{U}_h \mathfrak{g})/\bar{u}_h^\psi$. We write $((V \otimes \mathcal{U}_h \mathfrak{g})/\bar{u}_h^\psi)^\mathfrak{a}$ for the corresponding space of $\mathfrak{u}$-invariants.

We put $\bar{u}_h^\psi := \{x - \psi(x), \ x \in \bar{u}\}$. We view this vector space as a subspace of $\text{Sym}\, \mathfrak{g}$ and let $((u + \bar{u}_h^\psi))$ denote an ideal of the algebra $\text{Sym}\, \mathfrak{g}$ generated by the subspace $u + \bar{u}_h^\psi \subseteq \text{Sym}\, \mathfrak{g}$. We have natural identifications $\text{Sym}\, \mathfrak{g}/((u + \bar{u}_h^\psi)) = \mathbb{C}[(e + b_r) \cap b_e] = \mathbb{C}[\mathfrak{r}]$. The quotient map $\text{Sym}(\mathfrak{g}/u) \to \text{Sym}\, \mathfrak{g}/((u + \bar{u}_h^\psi))$ corresponds, via the identifications, to the restriction map $\mathbb{C}[b] \to \mathbb{C}[\mathfrak{r}]$ induced by the imbedding $\mathfrak{r} \hookrightarrow b$. Using that inside $(\mathcal{U}_h \mathfrak{g})|_{h=0} = \text{Sym}\, \mathfrak{g}$ one has $\bar{u}_h^\psi|_{h=0} = \bar{u}_h^\psi$, we deduce natural isomorphisms, cf. (7.3.1),

$$((V \otimes \mathcal{M}_h)/\bar{u}_h^\psi)|_{h=0} = (u\!(V \otimes \mathcal{U}_h \mathfrak{g})/\bar{u}_h^\psi)|_{h=0} = V \otimes \text{Sym}\,(\mathfrak{g}/((u + \bar{u}_h^\psi))) = V[\mathfrak{r}].$$

Let $\mathfrak{g} = \oplus_{m \in \mathbb{Z}} \mathfrak{g}(m)$, resp. $V = \oplus_{m \in \mathbb{Z}} V(m)$, be the weight decomposition with respect to the adjoint, resp. natural, action of the element $h \in \mathfrak{g}$. Define a $\mathbb{Z}$-grading on $\mathcal{U}_h \mathfrak{g}$, to be referred to as the Kazhdan grading, by assigning the elements of $\mathfrak{g}(m) \subseteq \mathfrak{g}$, $m \in \mathbb{Z}$, degree $2 + m$ and assigning $h$ degree 2. With this grading, the algebra $\mathcal{U}_h \mathfrak{g}$ may be identified with the Rees algebra of the algebra $\mathcal{U}\mathfrak{g}$ equipped with the Kazhdan filtration, cf. [Ko2]. From now on, we will use the Kazhdan grading as our default grading.

The Lie algebra $u := u_\mathfrak{c}$ is a graded subalgebra of $\mathfrak{g}$, so the grading on $\mathcal{U}_h \mathfrak{g}$ induces one on $M_h$. We equip $V \otimes M_h$ with the standard grading on a tensor product. The Lie algebra $u_h^\psi$ is a graded $\mathbb{C}[h]$-submodule of $\mathcal{U}_h \mathfrak{g}$ since the elements $f_i$ have degree zero. Hence, the grading on $V \otimes M_h$ induces a quotient grading on $(V \otimes M_h)/u_h^\psi$, resp. $u\!(V \otimes \mathcal{U}_h \mathfrak{g})/u_h^\psi$.

We also define a grading on $(\text{Sym}\, t)[h]$ by placing the vector space $t$ in degree 2. Using the natural identifications $\mathcal{U}_h t = \mathbb{C}[t^* \times \mathbb{A}^1] = (\text{Sym}\, t)[h]$, this gives a grading on $\mathcal{U}_h t$, resp. $\mathbb{C}[t^* \times \mathbb{A}^1]$. We equip $Z_h \mathfrak{g}$, resp. $(\text{Sym}\, t)^W[h]$, with the gradings induced from the one on $\mathcal{U}_h \mathfrak{g}$, resp. $(\text{Sym}\, t)^W[h]$, by restriction. The above defined gradings induce gradings on various other objects, e.g. $\mathcal{U}_h t \otimes \mathbb{C}[h]$, in particular, on all objects which appear in Lemma [7.3.3]. It is immediate to check that all maps in the statement of lemma, as well as the asymptotic Harish-Chandra homomorphism, respect the gradings.

Lemma 7.3.5. (i) The natural map $Z_h \mathfrak{g} \to \text{End}_{\mathcal{U}_h \mathfrak{g}}(\mathcal{U}_h \mathfrak{g}/\bar{u}_h^\psi) = ((\mathcal{U}_h \mathfrak{g})/\bar{u}_h^\psi)^\mathfrak{a}$ is a graded $\mathbb{C}[h]$-algebra isomorphism.

(ii) The left $\mathcal{U}_h t$-action makes $(V \otimes M_h)/u_h^\psi$, a free graded $\mathcal{U}_h t$-module of rank $\dim V$.

(iii) There are natural isomorphisms of graded $(\mathcal{U}_h t, Z_h \mathfrak{g})$-bimodules

$$\mathcal{U}_h t \otimes Z_h \mathfrak{g} ((V \otimes \mathcal{U}_h \mathfrak{g})/\bar{u}_h^\psi) \cong u\!(V \otimes \mathcal{U}_h \mathfrak{g})/\bar{u}_h^\psi \cong (V \otimes M_h)/u_h^\psi.$$

Proof. Part (i) is essentially due to Kostant. In more detail, Kostant considers the Kazhdan filtration on $\mathcal{U}_h g$, and equips $Z_h \mathfrak{g}$, resp. $\mathcal{U}_h g/\bar{u}_h^\psi$, with induced filtrations. The natural map $Z_h \mathfrak{g} \to ((\mathcal{U}_h g)/\bar{u}_h^\psi)^\mathfrak{a}$ respects the filtrations and the map $Z_h \mathfrak{g} \to ((\mathcal{U}_h g)/\bar{u}_h^\psi)^\mathfrak{a}$, resp. $Z_h \mathfrak{g}|_{h=0} \to ((\mathcal{U}_h g)/\bar{u}_h^\psi)^{\mathfrak{a}}|_{h=0}$, may be identified with the induced map of Rees, resp. associated graded, algebras. The map of associated graded algebras is an isomorphism, by one of the main results of [Ko2]. The Kazhdan filtration on $\mathcal{U}_h g/\bar{u}_h^\psi$ being bounded below, it follows that the map of Rees algebras, that is, the map in (i), is also an isomorphism.

To prove (ii), let $\mathcal{U}(\bar{u}_h^\psi)$ be a $\mathbb{C}[h]$-subalgebra of $\mathcal{U}_h \mathfrak{g}$ generated by $\bar{u}_h^\psi$. Then, $M_h$ is free over $\mathcal{U}_h t \otimes \mathbb{C}[h] \mathcal{U}(\bar{u}_h^\psi)$, by [Ko2 Theorem 4.6]. Following Kostant, one concludes that $(V \otimes M_h)/u_h^\psi$ is free over $\mathcal{U}_h t$. Finally, the first isomorphism in (iii) is an asymptotic analogue of Lemma [6.2.1] applied in the case of $E := (V \otimes \mathcal{U}_h \mathfrak{g})/\bar{u}_h^\psi$. The second isomorphism is an asymptotic analogue of (7.3.1). □

7.4. Proof of Theorem 7.1.8 Let $Z_\mathfrak{h} := \text{Spec}(Z_h \mathfrak{g})$ and let $Z_\mathfrak{h} \to \mathbb{C}$ be the map induced by the imbedding $\mathbb{C}[h] \hookrightarrow Z_h \mathfrak{g}$. Below, we use simplified notation $\mathfrak{R}$ for the variety obtained by the
deformation construction of Section 7.3 applied to $\mathcal{Z}_h$ viewed as a scheme over $\mathbb{C}$. From Lemma 5.1.5 and the identification $(\mathcal{Z}_h)_0 = \mathcal{F}$ provided by the specialization of the asymptotic Harish-Chandra isomorphism at $h = 0$, we deduce

$$\mathbb{C}[\mathcal{Z}]|_{h=0} = \mathbb{C}[(\mathcal{Z}_h)_0] \cong \mathbb{C}[\mathcal{F}] \cong \Gamma(\mathcal{F}, \text{Sym}\mathcal{T}_c^*) \cong \Gamma(\mathcal{F}, \text{Sym}\mathcal{T}_c).$$  \hfill \text{(7.4.1)}

We view $\mathcal{U}_h t$ as a $Z_h\mathfrak{g}$-module via the imbedding $h_{hc} : Z_h\mathfrak{g} \hookrightarrow \mathcal{U}_h t$ and view $Z_h\mathfrak{g} \otimes C[\mathfrak{g}]$ resp. $\mathbb{C}[\mathfrak{g}]$, as a $Z_h\mathfrak{g}$-module via the action of $Z_h\mathfrak{g} \otimes 1$. Thus, using the identification $\mathcal{U}_h t = \mathbb{C}[t^* \times \mathbb{A}^1]$, we have $\mathcal{U}_h t \otimes_{Z_h\mathfrak{g}} \mathbb{C}[\mathfrak{g}] = \mathbb{C}[(t^* \times \mathbb{A}^1) \times Z_h \mathfrak{g}]$. The gradings on $Z_h\mathfrak{g}$ and $\mathcal{U}_h t$ induce natural gradings on $\mathbb{C}[\mathfrak{g}]$, resp. $\mathcal{U}_h t \otimes Z_h\mathfrak{g} \otimes \mathbb{C}[\mathfrak{g}]$. Recall the canonical map $p : \mathfrak{g} \to Z_h \otimes \mathbb{A}^1 Z_h$. The following composition

$$(t^* \times \mathbb{A}^1) \times Z_h \mathfrak{g} \xrightarrow{\text{Id} \times p} (t^* \times \mathbb{A}^1) \times Z_h (Z_h \otimes \mathbb{A}^1 Z_h) = (t^* \times \mathbb{A}^1) \times \mathbb{A}^1 Z_h,$$

induces a graded algebra imbedding $\mathcal{U}_h t \otimes C[\mathfrak{g}] Z_h\mathfrak{g} \hookrightarrow \mathcal{U}_h t \otimes Z_h\mathfrak{g} \otimes \mathbb{C}[\mathfrak{g}]$.

For a $(\mathcal{U}_h t \otimes C[\mathfrak{g}] Z_h\mathfrak{g})$-module $M$ one can view $M|_{h=0}$ as a $(\mathbb{C}[c] \otimes \mathbb{C}[c])$-module via the imbedding $\theta^* \otimes \text{Id} : \mathbb{C}[c] \otimes \mathbb{C}[c] \hookrightarrow \mathbb{C}[c] \otimes \mathbb{C}[c]$. From the fourth paragraph of Section 7.3 we deduce that if $M$ is $h$-torsion free and $M|_{h=0}$ is annihilated by the elements $z \otimes 1 - 1 \otimes z$, $z \in \mathbb{C}[c]$, then the action of $\mathcal{U}_h t \otimes C[\mathfrak{g}] Z_h\mathfrak{g}$ on $M$ has a unique extension to a $(\mathcal{U}_h t \otimes Z_h\mathfrak{g} \otimes \mathbb{C}[\mathfrak{g}])$-action. Furthermore, the latter action gives an action of the abelian Lie algebra $\Gamma(t^*, \theta^* \mathcal{F})$ on $M|_{h=0}$, by (7.4.1).

Fix $\mu \in \mathbb{Q}$ and let $\tau_{h,\mu} : \mathcal{U}_h t \to \mathcal{U}_h t$ be a graded algebra automorphism defined by the assignment $t \ni t \mapsto t - h \mu(t)$. Let $J_\mu$ be an ideal of the algebra $\mathcal{U}_h t \otimes Z_h\mathfrak{g} \otimes \mathbb{C}[\mathfrak{g}]$ generated, as a $(\mathcal{U}_h t \otimes 1)$-module, by the elements $\tau_{h,\mu}(h_{hc}(z)) z \otimes 1 - 1 \otimes z$, $z \in \mathbb{Z}[c]$. From the isomorphism $\mathbb{C}[\mathfrak{g}] / J \cong \mathbb{C}[c] \otimes \mathbb{A}^1$, see Section 7.3, one obtains an isomorphism $(\mathcal{U}_h t \otimes Z_h\mathfrak{g} \otimes \mathbb{C}[\mathfrak{g}]) / J_\mu \cong \mathcal{U}_h t$, such that the composition $\mathcal{U}_h t \otimes Z_h\mathfrak{g} \otimes \mathbb{C}[\mathfrak{g}] \hookrightarrow (\mathcal{U}_h t \otimes Z_h\mathfrak{g} \otimes \mathbb{C}[\mathfrak{g}]) / J_\mu \twoheadrightarrow \mathcal{U}_h t$ sends $t \otimes z$ to $t \cdot \tau_{h,\mu}(h_{hc}(z))$.

For a graded $\mathcal{U}_h t \otimes Z_h\mathfrak{g} \otimes \mathbb{C}[\mathfrak{g}]$-module $M$ we put $J_\mu = \{m \in M \mid xm = 0 \text{ for all } x \in J_\mu\}$. The ideal $J_\mu$ is homogeneous, so $J_\mu^\mathfrak{g}$ is a graded submodule of $M$.

Recall that the space $(V \otimes M_h^\mu)^\mathfrak{g}$, resp. $(V \otimes M_h)/\mu^\mathfrak{g}$, has the structure of a (graded) $(\mathcal{U}_h t, \mathcal{U}_h t)$-bimodule, resp. $(\mathcal{U}_h t, Z_h\mathfrak{g})$-bimodule. We will view these bimodules as as $(\mathcal{U}_h t \otimes C[\mathfrak{g}] Z_h\mathfrak{g})$-modules. The specialization of each of the two $(\mathcal{U}_h t \otimes C[\mathfrak{g}] Z_h\mathfrak{g})$-modules at $h = 0$ is annihilated by the elements $z \otimes 1 - 1 \otimes z$, $z \in \mathbb{C}[c]$, since $(V \otimes \mathcal{U}_h t)^{\mathcal{F}}$ is symmetric as a $(\text{Sym} \mathfrak{g}, \text{Sym} \mathfrak{g})$-bimodule. We conclude that $(V \otimes M_h^\mu)^\mathfrak{g}$, resp. $(V \otimes M_h)/\mu^\mathfrak{g}$, has the canonical structure of a graded $\mathcal{U}_h t \otimes Z_h\mathfrak{g} \otimes \mathbb{C}[\mathfrak{g}]$-module.

**Proposition 7.4.2.** For any finite dimensional G-module $V$ and $\mu \in \mathbb{Q}$ one has an inclusion $(V \otimes M_h)^\mu \subseteq ((V \otimes M_h^\mu)^\mathfrak{g})^\mu$. Moreover, the composition

$$\kappa_{V,\mu,h} : (V \otimes M_h)^\mu \hookrightarrow ((V \otimes M_h^\mu)^\mathfrak{g})^\mu \twoheadrightarrow ((V \otimes M_h)/\mu^\mathfrak{g})^\mu$$  \hfill \text{(7.4.3)}

is an isomorphism of graded $\mathcal{U}_h t \otimes Z_h\mathfrak{g} \otimes \mathbb{C}[\mathfrak{g}]$-modules.

Observe that the canonical algebra isomorphism $\mathbb{C}[\mathfrak{g}]|_{h=1} \cong (Z_h\mathfrak{g} \otimes C[\mathfrak{g}] Z_h\mathfrak{g})|_{h=1} = Z\mathfrak{g} \otimes Z\mathfrak{g}$ yields isomorphisms $(\mathcal{U}_h t \otimes Z_h\mathfrak{g} \otimes \mathbb{C}[\mathfrak{g}])|_{h=1} \cong \mathcal{U}_t \otimes Z\mathfrak{g} \otimes Z\mathfrak{g} = \mathcal{U}_t \otimes Z\mathfrak{g}$. Thus, Theorem 7.1.8 follows from the above proposition by specializing at $h = 1$.

**Proof of Proposition 7.4.2.** The proof of the first statement of the proposition is similar to the proof of (7.1.3). Specifically, let $v \otimes m \in (V \otimes M_h)^\mu$ and $t \in t$. Using the equation $(v \otimes m)t = t(v \otimes m) - h \text{ad}(t)(v \otimes m)$ and mimicking computations in the proof of (7.1.3), one finds that for all $z \in Z_h\mathfrak{g}$, we have $(v \otimes m)z = \tau_{h,\mu}(h_{hc}(z))(v \otimes m)$. Since $(V \otimes M_h)^\mu$ is a torsion free $C[\mathfrak{g}]$-module, we deduce that this module is annihilated by the elements $\tau_{h,\mu}(h_{hc}(z)) z \otimes 1 - 1 \otimes z$, $z \in Z_h\mathfrak{g}$. These elements generate the ideal $J_\mu$, and the first statement of the proposition follows.

To prove the second statement, recall that one has graded algebra isomorphisms $Z_h\mathfrak{g} \cong \mathbb{C}[c][h] \cong \mathbb{C}[z_1, \ldots, z_r, h]$, where $r = \dim t$ and $z_i, i = 1, \ldots, r$, are homogeneous elements of positive degree.

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We identify $z_i$ with an element of $Z_h \mathfrak{g}$ and consider a diagram

\[
0 \rightarrow (V \otimes M_h)^{\mu \mu} \xrightarrow{\kappa_{V,\mu,h}} (V \otimes M_h)/\bar{u}_h^g \xrightarrow{\zeta} \bigoplus_{1 \leq i \leq r} ((V \otimes M_h)/\bar{u}_h^g)(\deg z_i - 2),
\]

where $(\deg z_i - 2)$ denotes a grading shift by $\deg z_i - 2$, and the map $\zeta$ is given by the assignment

\[
(v \otimes m) \mapsto \tau_{\mu}(h\sigma(z_i)) \otimes 1 - 1 \otimes z_i (v \otimes m) \oplus \ldots \oplus \tau_{\mu}(h\sigma(z_r)) \otimes 1 - 1 \otimes z_r (v \otimes m).
\]

We use the notation of Section 7.2. The map $\kappa_{V,\mu,h}|_{h=0}$ may be identified, via Lemma 7.3.3(ii) and isomorphism (7.3.4), with the composition $V[b]^{U,\mu} \hookrightarrow V[b] \rightarrow V[c]$, where the second map is a restriction map induced by the imbedding $r = e + \mathfrak{g}_h \hookrightarrow \mathfrak{b}$. To describe the map $\zeta|_{h=0}$, recall that thanks to Lemma 5.1.5, associated with $f \in \mathbb{C}[c]$, there is a $G$-equivariant polynomial map $d(\vartheta^*f) : \mathfrak{g}_r \rightarrow \mathfrak{g}$. Let $\xi_f \in \Gamma(r, \vartheta^*_{\mathcal{E}_k})$ be the restriction of this map to $r$. In the special case $f = z_i|_{h=0} \in \mathbb{C}[\mathfrak{H}]|_{h=0} = \mathbb{C}[c]$, we put $\xi_i := \xi_f$.

One has the following algebra homomorphsims

\[
\mathbb{C}[\mathfrak{H}] \rightarrow \mathbb{C}[\mathfrak{H}]|_{h=0} \rightarrow \Gamma(c, \operatorname{Sym}_\mathbb{C} \mathcal{H}) \rightarrow \mathbb{C}[c] \otimes \mathbb{C}[c] \Gamma(c, \operatorname{Sym}_\mathbb{C} \mathcal{H}) = \Gamma(r, \operatorname{Sym}(\vartheta^*_{\mathcal{E}_k})),
\]

where the second is an isomorphism that comes from (7.4.1) and $\vartheta_r$ denotes the composite $r \hookrightarrow \mathfrak{g}_r \rightarrow c$. Let $\beta$ be the composition of the above homomorphisms. Going through definitions, one finds that $\beta$ sends $\frac{z_i \otimes 1 - 1 \otimes z_i}{h}$ to $\xi_i \in \Gamma(r, \vartheta^*_{\mathcal{E}_k}) \subset \Gamma(r, \operatorname{Sym}(\vartheta^*_{\mathcal{E}_k}))$. It follows that the composition

\[
U_{h,t} \otimes Z_{h,t} \mathbb{C}[\mathfrak{H}] \rightarrow (U_{h,t} \otimes Z_{h,t} \mathbb{C}[\mathfrak{H}]|_{h=0} \xrightarrow{\text{Id} \times \beta} \mathbb{C}[c] \otimes \mathbb{C}[c] \Gamma(c, \operatorname{Sym}(\vartheta^*_{\mathcal{E}_k})) \xrightarrow{\xi} \Gamma(r, \operatorname{Sym}(\vartheta^*_{\mathcal{E}_k}))
\]

sends the element $\frac{\tau_{\mu}(h\sigma(z_i)) \otimes 1 - 1 \otimes z_i}{h}$ to $\xi_i(\xi_i) - \mu(\xi_i(\xi_i))$. The symbol $\text{Id}$ above stands for the identification $U_{h,t}|_{h=0} = \mathbb{C}[c] \times A^1 = \mathbb{C}[c]$, and the map $\xi_i$ was defined in Section 7.2. Combining this with formula (7.3.4), it is not difficult to show that the map $(V \otimes M_h)/\bar{u}_h^g \rightarrow (V \otimes M_h)/\bar{u}_h^g$ given by the assignment $(v \otimes m) \mapsto \frac{\tau_{\mu}(h\sigma(z_i)) \otimes 1 - 1 \otimes z_i}{h} (v \otimes m)$, specializes at $h = 0$ to the map

\[
V[r] \rightarrow V[r], \quad v \mapsto \xi_i(v) - \mu(\xi_i(\xi_i)) \cdot v.
\]

Thus, from the definition of the map $\zeta$ in (7.4.4), resp. vector space $V[r]^{|\mu|}$ in (7.2.2), we deduce

$$\ker(\zeta|_{h=0}) = \ker(\zeta|_{h=0}) \quad (1) \quad V[r]^{|\mu|} = \operatorname{Im}(\kappa_{V,\mu}) = \operatorname{Im}(\kappa_{V,\mu,|h=0|}),$$

where $(1)$, resp. $(3)$, follows from the description of the map $\zeta|_{h=0}$, resp. $\kappa_{V,\mu,|h=0|}$, given above and (2) holds by Proposition 7.2.5.

To complete the proof we observe that the maps $\kappa_{V,\mu,h}$ and $\zeta$ in (7.4.4) are morphisms of finite rank free graded $U_{h,t}$-modules, by Lemma 7.3.3 and Lemma 7.3.5. The first statement of the proposition implies that $\operatorname{Im}(\kappa_{V,\mu,h}) \subset \ker(\zeta)$. Furthermore, we have shown that the specialization of diagram (7.4.4) at $h = 0$ gives an exact sequence. Thus, the second statement of the proposition is a consequence of a standard general semicontinuity result stated in the following lemma.

**Lemma 7.4.5.** Let $0 \rightarrow E' \xrightarrow{a} E \xrightarrow{b} E''$ be a sequence of morphisms of free $\mathbb{Z}$-graded $U_{h,t}$-modules of finite rank such that $b \circ a = 0$. If the induced sequence $0 \rightarrow E'|_{h=0} \xrightarrow{a} E|_{h=0} \xrightarrow{b} E''|_{h=0}$ is exact, then the original sequence is also exact.

**Proof of Lemma.** Let $p : E \xrightarrow{|h=0|} E$ denote the projection. From definitions, we find $p^{-1}(\operatorname{Im}(a)) = \operatorname{Im}(a) + hE$, resp. $p^{-1}(\ker(b)) = b^{-1}(hE'')$. Thus, we have an equality

$$\operatorname{Im}(a) + hE = b^{-1}(hE''),$$

since $\operatorname{Im}(a) = \ker(b)$ by the assumptions of the lemma.

Assume by contradiction that $\operatorname{Im}(a) \subsetneq \ker(b)$. All the modules involved are graded and the gradings on these modules are bounded below, since the algebra $U_{h,t}$ is graded by nonnegative
integers and the modules are finitely generated. Let \( e \) be a homogeneous element of \( E \) of minimal degree such that \( e \in \text{Ker}(b) \setminus \text{Im}(a) \). Since \( e \in b^{-1}(b(e)) = b^{-1}(0) \subseteq b^{-1}(hE''') \), the displayed equation above implies that there are homogeneous elements \( e' \in E' \) and \( x \in E \) such that \( e = a(e') + hx \). We compute \( h \cdot b(x) = b(hx) = b(e - a(e')) = b(e) - b(a(e')) = 0 - 0 = 0 \). The module \( E''' \) being free, hence \( h \)-torsion free, we deduce that \( x \in \text{Ker}(b) \). Since \( \deg x = \deg e - \deg h < \deg e \), we must have \( x \in \text{Im}(a) \), by our choice of the element \( e \). Writing \( x = a(y) \), we obtain \( e = a(e') + hx = a(e') + ha(y) = a(e' + hy) \). Thus, \( e \in \text{Im}(a) \), a contradiction.\[ \square \]

### 7.5. Proof of Theorem 1.4.1

Below, we are going to apply results of [BF] and [GR]. The notation used in these papers is slightly different from ours due to a different normalization of the Harish-Chandra homomorphism. Specifically, the Harish-Chandra homomorphism used in *loc cit* is defined as a composition \( \rho \circ h_{c}^{\lambda} \). The resulting imbedding \( h_{c}^{\lambda} : Z_{d}G \hookrightarrow U_{h}t \) yields an isomorphism \( Z_{d}g \cong (U_{h}t)^{W} = \mathbb{C}[t^{*} \times A^{1}]^{W} = \mathbb{C}[c][h] \) (recall that \( \rho \) denotes the half-sum of positive roots and \( W \)-invariants are taken with respect to the dot-action of \( W \)). We view \( U_{h}t \) as a \( Z_{d}t \)-algebra via the imbedding \( h_{c}^{\lambda} \) and write \( A \) for the corresponding algebra \( U_{h}t \otimes Z_{d}g \mathbb{C}[N] \).

We use the setting of Section 1.4. Thus, we have the groups \( T = G_{m} \times T^{\vee} \subseteq G_{m} \times G^{\vee}(O) \), and there are canonical isomorphisms \( H_{d}^{*}(pt) = \mathbb{C}[t^{*} \times A^{1}] = U_{d}t \), resp. \( H_{d}^{*}(G_{m} \times G^{\vee}(O))(pt) = H_{d}^{*}(G_{m} \times G^{\vee}(pt)) = \mathbb{C}[c][h] = Z_{d}g = G_{d}[Z_{d}] \). This yields natural graded algebra maps
\[
\mathbb{C}[c] \otimes \mathbb{C}[h] \twoheadrightarrow H_{d}^{*}(pt) \otimes H_{d}^{*}(pt) \otimes H_{d}^{*}(pt) \\
\cong H_{d}^{*}(G^{\vee}(O) \mathbb{C}G^{\vee}(K)/G^{\vee}(O)) \cong H_{d}^{*}(G). \]

It was shown in [BF] Theorem 1 that the composite of the above maps has a unique extension to a graded algebra isomorphism \( \mathbb{C}[N] \cong H_{d}^{*}(G_{m} \times G^{\vee}(Gr)) \). We deduce \( U_{h}t \)-algebra isomorphisms
\[
A = U_{h}t \otimes Z_{d}g \mathbb{C}[N] \cong \mathbb{C}[t^{*} \times A^{1}] \otimes \mathbb{C}[t^{*} \times A] H_{d}^{*}(G_{m} \times G^{\vee}(Gr)) \cong H_{d}^{*}(G). \]

Below, we identify the algebras \( H_{d}^{*}(Gr) \) and \( A \).

Associated with \( \lambda \in Q \), there is a graded algebra homomorphism \( \eta_{\lambda} : A \to U_{h}t \), a \( \rho \)-shifted version of the homomorphism \( U_{h}t \otimes Z_{d}g \mathbb{C}[N] \to U_{h}t \) considered in Section 7.3 such that the composition \( U_{h}t \otimes Z_{d}g \mathbb{C}[N] \hookrightarrow A \xrightarrow{\eta_{\lambda}} U_{h}t \) sends \( t \otimes z \) to \( t \cdot h_{c}^{\lambda}(z) \). The kernel of \( \eta_{\lambda} \) is an ideal \( J_{\lambda} \) of \( A \) generated by the elements \( \eta_{\lambda}(h_{c}(z)) = z, z \in Z_{d}g \). This ideal is a \( \rho \)-shifted counterpart of the ideal \( J_{\lambda} \) generated by the elements \( h_{c}(z) \), see Section 7.4.

Recall the imbedding \( i_{\lambda} : \{pt_{\lambda}\} \hookrightarrow Gr \), where \( pt_{\lambda} \) is the \( T \)-fixed point associated with \( \lambda \). It follows from [BF] §3.2 that the restriction map \( i_{\lambda}^{*} : H_{d}^{*}(Gr) \to H_{d}^{*}(pt_{\lambda}) \) corresponds, via the isomorphisms above, to the homomorphism \( \eta_{\lambda} \). Thus, the notation \( J_{\lambda} \) agrees with the notation \( J_{\lambda} = \text{Ker}(i_{\lambda}^{*}) \) used in Section 1.4. We conclude that there are canonical graded algebra isomorphisms
\[
H_{d}^{*}(pt_{\lambda}) \cong H_{d}^{*}(Gr)/J_{\lambda} \cong A/J_{\lambda} \cong U_{h}t. \tag{7.5.1} \]

For any object \( F \) of the \( T \)-equivariant constructible derived category of \( Gr \), the cohomology \( H_{d}^{*}(Gr,F) \) has the natural structure of a graded \( H_{d}^{*}(Gr) \)-module. The \( H_{d}^{*}(Gr) \)-action on \( H_{d}^{*}(Gr,F) \) factors through the quotient \( i_{\lambda}^{*} : H_{d}^{*}(Gr) \to H_{d}^{*}(pt_{\lambda}) \), since \( \text{Supp}(i_{\lambda}^{*}F) \subseteq \{pt_{\lambda}\} \). Using (7.5.1), this translates into the statement that the image of the canonical \( A \)-module map \( (i_{\lambda})_{*} : H_{d}^{*}(i_{\lambda}^{*}F) \to H_{d}^{*}(Gr,F) \) induced by the adjunction \( (i_{\lambda})_{*}i_{\lambda}^{*}F \to F \) is annihilated by the ideal \( J_{\lambda} \subseteq A \).

Now let \( F \) be an object of the Satake category \( \mathcal{P}_{\text{ev}}(G^{\vee}(O))(Gr) \). It is known that the group \( H_{d}^{*}(i_{\lambda}^{*}F) \), resp. \( H_{d}^{*}(Gr,F) \), viewed as a graded module over the subalgebra \( U_{h}t \subseteq A \), is a free graded module of finite rank; furthermore, the localization theorem in equivariant cohomology implies that the map \( (i_{\lambda})_{*} : H_{d}^{*}(i_{\lambda}^{*}F) \to H_{d}^{*}(Gr,F) \) is injective, cf. eg. [GR] §6.1-6.2.
Combining all the above, we see that proving Theorem 1.4.1 is equivalent to showing that the injective map \((i_\lambda)_t : H^*_\ell(i_\lambda^*F) \hookrightarrow (H^*_\ell(G, F))^{J_\lambda}\) is a bijection. Thus, writing \(P(M) = \sum_{k \in \mathbb{Z}} \dim M_k \cdot t^k\) for the Poincaré series of a \(\mathbb{Z}\)-graded vector space \(M = \bigoplus_{k \in \mathbb{Z}} M_k\), we are reduced to proving the following equality of Poincaré series:

\[
P(H^*_\ell(i_\lambda^*F)) = P((H^*_\ell(G, F))^{J_\lambda}).
\]  

(7.5.2)

To prove this equality, we interpret each side in terms of geometric Satake. To this end, for each \(\mu \in \mathbb{Q}\) we define a \((U_\hbar t, U_\hbar g)\)-bimodule \(M_\hbar(\mu)\), a \(\mu\)-twisted analogue of \(M_\hbar\), as follows. The bimodule \(M_\hbar(\mu)\) has the same underlying vector space as \(M_\hbar\) and the same right \(U_\hbar g\)-action. The left action of \(t \in U_\hbar t\) on \(M_\hbar(\mu)\) is defined in terms of the left action of \(t\) on \(M_\hbar\) by the formula \(t \cdot m \mapsto (t - \hbar \mu(t))m\). Put \(\lambda := \mu - \rho\). Then, the isomorphism \(\kappa_{V, \mu, h}\) of Proposition 7.4.2 translates into an isomorphism \((V \otimes M_\hbar(\rho))^{\mu, \lambda} \cong (V \otimes M_\hbar(\rho))^{J_\lambda}\) of graded \(A\)-modules.

Let \(A_{\text{grmod}}\) be the abelian category of \(\mathbb{Z}\)-graded \(A\)-modules, \(\mathbf{Rep}(G)\) the category of finite dimensional \(G\)-representations, and \(S: \mathbf{Rep}(G) \to \mathbf{Perv}_{G^\vee(O)}(\mathcal{G})\) the geometric Satake equivalence. According to [BF] Theorem 6, there is an isomorphism

\[
((-) \otimes M_\hbar(\rho))/\bar{u}^\psi_\hbar \cong H^*_\ell(G, S((-)), \quad (7.5.3)
\]

of functors \(\mathbf{Rep}(G) \to A_{\text{grmod}}\). Hence, for \(\mathcal{F} = S(V)\) and any \(\lambda \in \mathbb{Q}\), there is an isomorphism \((V \otimes M_\hbar(\rho))/\bar{u}^\psi_\hbar)^{\mu, \lambda} = (H^*_\ell(G, F))^{J_\lambda}\). On the other hand, it follows from [GR] Theorem 2.2.4, cf. also [GR] Remark 2.2.6, that the graded \(U_\hbar t\)-module \(H^*_\ell(i_\lambda^*F)\) is isomorphic to \((V \otimes M_\hbar(\rho + \lambda))^b\), where \(b\) is the Borel that contains \(u\). It is clear that we have \((V \otimes M_\hbar(\rho + \lambda))^b = (V \otimes M_\hbar(\rho))^{\mu, \lambda}\). Thus, combining the above isomorphisms of graded modules we obtain the following chain of equalities of the corresponding Poincaré series:

\[
P(H^*_\ell(i_\lambda^*F)) \xrightarrow{\text{GR}} P((V \otimes M_\hbar(\rho))^{\mu, \lambda}) \xrightarrow{\text{Prop. 7.4.2}} P((V \otimes M_\hbar(\rho))/\bar{u}^\psi_\hbar)^{\mu, \lambda} \xrightarrow{\text{BR}} P(H^*_\ell(G, F))^{J_\lambda}.
\]

This proves (7.5.2), and Theorem 1.4.1 follows. \(\square\)

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