Quantum bound to chaos and the semiclassical limit

Jorge Kurchan

1Laboratoire de Physique Statistique, École Normale Supérieure, PSL Research University; Université Paris Diderot Sorbonne Paris-Cité; Sorbonne Universités UPMC Univ Paris 06; CNRS; 24 rue Lhomond, 75005 Paris, France.

We discuss the quantum bound on chaos in the context of the free propagation of a particle in an arbitrarily curved surface at low temperatures. The semiclassical calculation of the Lyapunov exponent can be performed in much the same way as the corresponding one for the ‘Loschmidt echo’. The bound appears here as the impossibility to scatter a wave, by effect of the curvature, over characteristic lengths smaller than the de Broglie wavelength.

In a recent paper a bound to chaos, imposed by quantum dynamics, was derived. Although the original context of the discussion was that of Black Holes and String Theory, the result itself is a general elementary property of quantum mechanics, and the proof only relies on Schrödinger’s equation plus reasonable ‘clustering’ properties of operators [1] (see also [2, 3]). In this note, I discuss the appearance of this result in a setting in which it is an intuitive property of a wave dynamics in a scattering medium.

I. FIDELITY

Consider a system evolving with a Hamiltonian $H$ for a time $t$, and then backwards with a slightly different hamiltonian $-H + \delta H$: a ‘Loschmidt echo’ [4,5] experiment (see, in the present context, [1]). A particular case is when at time $t$, the system is subjected to a rapid ‘kick’ with Hamiltonian $\delta H(t') = B\delta(t' - t)$, and subsequently is evolved back for a time $t$ with $-H$. In classical mechanics, the returning trajectory is the time-reversed of the initial trajectory, except that its beginning has been perturbed. It is clear then that if the system is chaotic, and the time $t$ is long enough, the return point differs from the initial point by a magnitude that does not depend on the nature of the kick (for $t\lambda \gg 1$), and $C$ is a constant that does.

An analogous result holds for a quantum system if one prepares it in a localized state, and evolves it with a Hamiltonian and back with a slightly perturbed Hamiltonian. If one measures appropriately the overlap between the initial and the returning wavefunction, there is a regime where this difference scales exponentially in time, with the (quantum Lyapunov) exponent independent of the perturbation between the outgoing and ingoing Hamiltonians [5]. Several years ago, Jalabert and Pastawski [8] showed that for a semiclassical system, and for a given range of perturbation and times, this quantum exponent corresponds to the classical one (although an ‘annealed’ version thereof, see discussion in [1]).

Consider a Hermitian operator $A$, which we shall assume is concentrated in some region of phase space, as seen, for example, in its coherent-state or Wigner representation. We introduce a measure of the ‘fidelity’ $F = \text{Tr}[A_{\text{tran}} A]$, comparing the operator $A$ with the one $A_{\text{tran}}$ transformed by Loschmidt echo trajectory. For a ‘kick’ perturbation, we have:

$$F = \text{Tr}[A_{\text{tran}} A] = \text{Tr}\left\{e^{i\frac{\delta H}{\hbar}t} e^{i\frac{\delta B}{\hbar}e^{-i\frac{\delta H}{\hbar}}} A e^{i\frac{\delta H}{\hbar}t} e^{i\frac{\delta B}{\hbar}e^{-i\frac{\delta H}{\hbar}}} \right\} = \text{Tr}\left\{A e^{i\frac{\delta H}{\hbar}t} e^{i\frac{\delta B}{\hbar}e^{-i\frac{\delta H}{\hbar}}} A e^{i\frac{\delta H}{\hbar}t} e^{i\frac{\delta B}{\hbar}e^{-i\frac{\delta H}{\hbar}}} \right\}$$

(1)

If $\delta = 0$ we clearly get $F = \text{Tr}[A^2]$. Putting

$$B(t) = e^{i\frac{\delta H}{\hbar}t} B(t) e^{-i\frac{\delta H}{\hbar}}$$

and developing in powers of $\delta$ (linear response) $e^{i\frac{\delta B}{\hbar}B(t)} = 1 + i\frac{\delta}{\hbar} B(t) - \frac{1}{2}\frac{\delta^2}{\hbar^2} B^2(t) + ...$ we get:

$$\text{Tr}[A^2] - \text{Tr}[A_{\text{tran}} A] = -\frac{\delta^2}{\hbar^2} \text{Tr}\{B(t)AB(t)A\} + \frac{\delta^2}{\hbar^2} \text{Tr}\{B^2(t)A^2\} = -\frac{\delta^2}{2\hbar^2} \text{Tr}\{[B(t),A]^2\}$$

(3)

The measure we are seeking is thus

$$F_1 = -\text{Tr}\{[B(t),A]^2\}$$

(4)
which is positive since it is minus the trace of an anti-Hermitean operator squared. We need the operator $A$ to be concentrated, in order to localize our measure in a region of phase space, and to make the trace converge. Two choices have been proposed. A traditional one is to consider a localized wavefunction $|\psi\rangle$ and:

$$A = |\psi\rangle\langle\psi| \quad \Rightarrow \quad F = |\langle\psi| e^{it\mathcal{H}} e^{-i\mathcal{H}\beta} |\psi\rangle|^2$$

(here in the ‘kick’ version, although more commonly with a continuous perturbation $\delta H$). A more recent one that lends itself better for the study of an extensive system at temperature $T = \frac{1}{\beta}$, is obtained by choosing a smooth (possibly unlocalized) Hermitean operator $W$ and to render it concentrated on an energy shell, at least in the thermodynamic limit, by putting in $[1]$:  

$$A \propto e^{-\frac{i}{\hbar}\mathcal{H}^2} W e^{-\frac{\beta}{2}H} \Rightarrow \quad F_1 \propto -\text{Tr} \left\{ B(t) e^{-\frac{i}{\hbar}\mathcal{H}^2} W e^{-\frac{\beta}{2}H} B(t) e^{-\frac{i}{\hbar}\mathcal{H}^2} W e^{-\frac{\beta}{2}H} \right\} + \text{Tr} \left\{ B^2(t) e^{-\frac{i}{\hbar}\mathcal{H}^2} W e^{-\frac{\beta}{2}H} W e^{-\frac{\beta}{2}H} \right\}$$

The symbol $\propto$ means that $A$ is normalized by $\text{Tr} e^{-\beta H/2}$. The second term only contributes a two-time function and its precise form is not relevant beyond canceling constants, we keep it for convenience here. (The regularization strategy of splitting the equilibrium density operator in factors – here four of them – is standard in Chemical Physics, see [22].) For this form, in some cases there is a Lyapunov regime:

$$F_1(t) \sim f_1 e^{\lambda t}$$

In this context, it has been shown that there is a bound to chaos [1] of purely quantum nature:

$$\beta \hbar \lambda \leq 2\pi$$

### II. MODEL

In the semiclassical case, where there is no apparent limitation, the question immediately arises of how does quantum mechanics intervene [23]. We start by considering a Hamiltonian system of the form

$$H_o = \sum_{i=1}^{N'} \frac{\dot{q}_i^2}{2m} + V(q)$$

in the semiclassical limit $\hbar \to 0$. For the bound to be effective, we need the product $\beta \hbar$ to be finite, and this may only be achieved in the semiclassical limit for very low temperatures, corresponding to the lowest classical energies of the system. It is clear that if the classical ground state does not have dimension larger than two, there will be no chaos: the system will sit on the classical ground state and perform vibrations and quantum fluctuations around it. In order to render the system interesting from our point of view, we consider a potential such that the minimum of $V(q)$ is a manifold of dimension $N < N'$. Thus, even at the lowest energies, the system will have long-distance motion at the bottom of the well’. Since we are interested in such a motion, we may concentrate on it by working with the quantum mechanics in the curved space of minima of the potential (a ‘quantum wire’ [13] or ‘quantum conducting surface’ problem). The Hamiltonian restricted to that manifold is constructed as follows: we set up a system of $N < N'$ curvilinear coordinates $q_i$ of the manifold of minima, and the associated momenta $p_i$, $i = 1, \ldots, N$ [17]. Let the metric tensor be $g_{ij}(q)$ so that length element $\delta l^2 = g_{ij} dq_i dq_j$. The Hamiltonian is proportional to the Laplace Beltrami operator [17]

$$H = -\frac{\hbar^2}{2m} g^{-\frac{1}{2}} \partial_i g^{ij} g^{\frac{1}{2}} \partial_j = \frac{\hbar^2}{2m} g^{-\frac{1}{2}} Q_i^{\dagger} g^{ij} Q_j g^{\frac{1}{2}}$$

where $g = \text{det} \{g_{ij}\}(q)$ and $Q_i = g^{\frac{1}{2}} \partial_i g^{-\frac{1}{2}}$. The r.h.s. shows that the spectrum is positive semidefinite, including zero if and only if the system is bounded. The imaginary-time evolution with Hamiltonian [10] is just diffusion on the constrained surface. The lowest eigenvalues of [10] indeed correspond to the inverse thermalization times associated with this diffusion process, the eigenvalue zero to the flat invariant measure. Equation [6] can be viewed as an expectation of various observables (correlations and responses) of a process with four diffusion stretches lasting $\beta/4$.

A two-dimensional version of the constant negative surface has been extensively studied as the simplest, and one of the few solvable chaotic systems, as was discussed in detail by Balazs and Voros [10],

$$H = -\frac{\hbar^2}{8m R^2} (1 - q_{1}^2 - q_{2}^2)^2 \left[ \frac{\partial^2}{\partial q_{1}^2} + \frac{\partial^2}{\partial q_{2}^2} \right]$$
where $R$ is the ‘radius’ of the hyperboloid. The classical Lyapunov exponent is $\lambda = \sqrt{2E/mR^2}$. Expectation values may be computed analytically, because eigenbases are known. We shall not do this interesting calculation here, but keep the example in mind throughout.

**Classical Limit**

In the classical limit, the Hamiltonian is given by the kinetic energy of the constrained motion:

$$H = \frac{1}{2m} p_i g^{ij} p_j$$  \hspace{1cm} (12)

and the Lagrangian:

$$L(q, \dot{q}) = m \frac{1}{2} \dot{q}_i g_{ij} \dot{q}_j$$  \hspace{1cm} (13)

where $g_{ij}$ is the inverse of $g^{ij}$. The geodesic lengths are then given by the extrema of:

$$\text{Length} = \hat{\int} dt L^{1/2}$$  \hspace{1cm} (14)

Because $L$ is a constant of motion, the Lagrange equations for geodesics and for the action lead to proportional equations. We hence conclude that the classical trajectories are geodesics in the coordinate space of the $q_i$, traversed with a velocity that is proportional to the square root of the (purely kinetic) energy.

If the system is bounded, we may compute the equilibrium quantities via:

$$Z = \int dq dp \ e^{-\beta H} \propto (mT)^{\frac{N}{2}} \int dq \ g^{1/2} \Rightarrow \left( \frac{E}{N} \right)_{\text{class. equil.}} = \left( \mathcal{E} \right)_{\text{class. equil.}} = \frac{T}{2}$$  \hspace{1cm} (15)

$\mathcal{E}$ is the energy per degree of freedom.

It is interesting to observe that in going from (9) to its low-energy version (12), a kind of reparametrization invariance appears. Given two points in configuration space and two times $t_1$ and $t_2$, there is a trajectory in space that will join the points in time $t_1$ and another in $t_2$ just by running over the same path at two different speeds. Note that this property was absent in the original system (9). Time-reparametrization plays an important role in quantum glass models that saturate the bound $[8, 14, 15, 18]$.

**Classical Lyapunov exponent and Geodesic Deviation**

Let us now assume that the ground state manifold is such that the geodesics are chaotic. As mentioned above, an example is the constant negative curvature surface in two dimensions $[16]$. Two nearby trajectories in phase space $(q_i, p_i)$ and $(q_i + \delta q_i, p_i + \delta p_i)$ will separate exponentially

$$\sqrt{\sum_i \left[ (\delta q_i)^2 + (\delta p_i)^2 \right]}(t) \propto e^{\lambda t}$$  \hspace{1cm} (16)
It is easy to write a differential equation for the evolution of the phase-space Lyapunov vector \((\delta q_i, \delta p_i)\), which will be as usual first order in time. In this particular case, one may also easily eliminate \(\delta p_i\) and obtain a second order differential equation for the \(\delta q_i\) (the ‘Geodesic Deviation’) exclusively. We need here only to remark that it is of the form \[20\] \(\delta \ddot{q}_i = G(R, \dot{q}, \delta q_i)\), with \(G\) quadratic in \(\dot{q}_i\) and linear in both the Riemann curvature tensor \(R\) and \(\delta q_i\). In a chaotic system, the purely spatial separation is expected to scale exponentially with the geodesic length \(\ell_G\) along the trajectory (Fig. 1)

\[
\Delta(\ell_G) \sim \Delta(0) e^{\ell_G/\ell_N}
\]  
(17)

where \(\ell_N\), the Lyapunov length, is the typical distance travelled for the separation of two nearby trajectories to be multiplied by \(e\). We expect that in a system with a good thermodynamic limit, \(\ell_N\) scales with the size as

\[
\ell_N = \ell_o \sqrt{N}
\]  
(18)

with \(\ell_o\) a length that has a finite thermodynamic limit. We may understand this scaling as follows: if we consider the Cartesian product of two independent chaotic systems with the same Lyapunov length, and consider pairs of trajectories for both systems starting from separations \(\Delta_0\). After a length \(\ell_o\) travelled by each, their separations will achieve \(\Delta(t)\) for both. But this happens in a combined length travelled that is now \(\sqrt{2}\ell_o\), while the total separation has increased only by \(e\).

Consider now the usual Lyapunov exponent associated with time. It is clear that the same pair of nearby geodesics traversed at twice the speed will diverge twice as fast. Because the energy is quadratic in the speed, we conclude then that:

\[
\lambda = \frac{\sqrt{2E/m}}{\ell_N} \quad \Rightarrow \quad \lambda = \frac{\sqrt{2E/m}}{\ell_o}
\]  
(19)

The right hand side confirms that \(\ell_o\) has a good thermodynamic limit if \(\lambda\) also does. (The largest Lyapunov value is expected to be \(O(1)\) in \(N\), although there have been some debate about how general this is, see \[19\].) A quantum system will have a thermal de Broglie length defined as \[24\]:

\[
\ell = \left(\frac{4\pi^2\hbar^2}{T m}\right)^{1/2}
\]  
(20)

This is a quantity per degree of freedom. The wavelength along a trajectory in full phase-space will be \(\sqrt{N}\) times that, i.e. \(\propto \sqrt{N}\ell\). Let us compare Lyapunov and deBroglie lengths:

\[
\ell_N \div \sqrt{N} \ell \quad \ell_0 \div \ell = \left(\frac{4\pi^2\hbar^2}{T m}\right)^{1/2}
\]

\[
\frac{\sqrt{2E/mN}}{\lambda} \div \left(\frac{4\pi^2\hbar^2}{T m}\right)^{1/2}
\]

\[
\frac{\sqrt{T/m}}{\lambda} \div \left(\frac{4\pi^2\hbar^2}{T m}\right)^{1/2}
\]

Rearranging, we find that both lengths are comparable if:

\[
\frac{T}{\hbar} \sim \lambda
\]  
(22)

where we omitted the numerical factors in the last, dimensional comparison.

The bound \[8\] has hence the meaning that when the thermal deBroglie length is of the order or larger than the characteristic ‘chaos’ length \(l_o\), the semiclassical estimate breaks down and chaoticity is bounded, see Figure 1. It is hard to scatter a wavefront along a distance shorter than its wavelength.

### III. Calculation

Let us make this argument more precise. In order to do this, we make a semiclassical calculation in a way that closely resembles the one of Jalabert and Pastawski \[8\] \[10\]. We sketch the calculation here and leave the details for
the Appendix. We have to compute (23), the first term of which is

\[ T = \text{Tr} \left\{ e^{i\frac{H}{\hbar}} B e^{i\frac{\Phi}{\hbar}} e^{-\frac{q}{2} H} W e^{-\frac{q}{2} H} e^{i\frac{H}{\hbar}} B e^{-i\frac{\Phi}{\hbar}} e^{-\frac{q}{2} H} W e^{-\frac{q}{2} H} \right\} \]  

(23)

We rescale quantities using:

\[ t_o = \beta \hbar \quad ; \quad \ell = \left( \frac{4\pi^2 \hbar^2}{m} \right)^{1/2} \quad \hat{t} = \frac{\ell}{4\pi^2 t_o} = \frac{t}{2\pi} \sqrt{\frac{T}{m}} \sim \text{distance}/2\pi \]  

(24)

The value of $2\hat{t}$ is the typical distance traveled by a classical particle at temperature $T$, it is much longer than the thermal length. To recap, we have three quantities, $\ell_o$, $\ell$ and $\hat{t}$: ‘spatial divergence’, thermal deBroglie and trajectory lengths, respectively.

We shall now write the real and imaginary time propagations as:

\[ \text{real time} = e^{-i\frac{\Phi}{\hbar}} = e^{-i\frac{1}{2} [-\frac{\hbar}{2} \hat{p} + \frac{\hbar}{2} \hat{q}]} = e^{-\frac{q}{2} \hat{H}} \quad \text{; \ imag. time} = e^{-\frac{\Phi}{\hbar}} = e^{i\frac{1}{2} \frac{\hbar}{2} \hat{p} - i\frac{1}{2} \frac{\hbar}{2} \hat{q}} = e^{-D} \]  

which implicitly define $\hat{H}$ and $D$. We recognize the real time propagations as semiclassical, with $\ell$ playing the role usually played by $\hbar$. As to the imaginary propagations, it is diffusion in curved space with ‘noise’ of small amplitude $\propto \ell$. Let us introduce a basis:

\[ T = \int dq_1...dq_n W(q_2) B(q_4) W(q_6) B(q_8) \langle q_1 | e^{\pm i\frac{\Phi}{\hbar}} | q_8 \rangle \langle q_8 | e^{-i\frac{\Phi}{\hbar}} | q_7 \rangle \langle q_7 | e^{-D} | q_9 \rangle \langle q_9 | e^{i\frac{\Phi}{\hbar}} | q_4 \rangle \langle q_4 | e^{-\frac{q}{2} \hat{H}} | q_3 \rangle \langle q_3 | e^{-D} | q_2 \rangle \langle q_2 | e^{-D} | q_1 \rangle \]  

(26)

We now use the fact that the real time propagations are semiclassical, and that the diffsusions are over very small $O(\ell)$ lengths (Fig. 2). We first write the short diffusion equations introducing $O(1)$ lengths $y_1, y_2$:

\[ R(q_5, y_1) = \langle q_5 + \ell \frac{y_1}{2} | e^{-D} W e^{-D} | q_5 - \ell \frac{y_1}{2} \rangle \quad \text{and} \quad R(q_1, y_2) = \langle q_1 + \ell \frac{y_2}{2} | e^{-D} W e^{-D} | q_1 - \ell \frac{y_2}{2} \rangle , \]  

(27)

and compute (Appendix), to leading order in $\ell$:

\[ R(q_5, y_1) \propto e^{-8\pi^2 L(q_5, y_1)} \]  

(28)

and similarly for $R(q_1, y_2)$ (one may omit constant prefactors, as they appear in the denominator too). Note that the $y_i$ play the role of velocities in the Lagrangian $L$. For the integral, we have:

\[ \int dq_1 dq_2 dq_3 dq_8 dq_9 dq_{12} B(q_4) R(q_5, y_1) R(q_1, y_2) \langle q_1 - \ell \frac{y_1}{2} | e^{\pm i\frac{\Phi}{\hbar}} | q_8 \rangle \langle q_8 | e^{-i\frac{\Phi}{\hbar}} | q_7 \rangle \langle q_7 | e^{-D} | q_9 \rangle \langle q_9 | e^{i\frac{\Phi}{\hbar}} | q_4 \rangle \langle q_4 | e^{-\frac{q}{2} \hat{H}} | q_1 + \ell \frac{y_2}{2} \rangle \]  

(29)

Next, we introduce the Van Vleck formula for the semiclassical propagator:

\[ \langle q' | e^{\pm i\frac{\Phi}{\hbar}} | q \rangle \sim A^{\pm}(q', q) e^{\pm \frac{i}{\hbar} S^c(q', q)} \]  

(30)

We have to take into account the shifts of $y_1$ and $y_2$ in the brackets. Using the fact that the derivative of the classical action $S^c$ with respect to the coordinate gives the momentum, we may put:

\[ S^c(q' + \ell z, q) \sim S^c(q', q) + \ell p' \quad ; \quad S^c(q', q + \ell z) \sim S^c(q', q) + \ell z p \]  

(31)

where $p, p'$ denote the classical momentum at the time corresponding to $q$ and $q'$, respectively. The shifts $y_1, y_2$ in the $A^{\pm}$ are negligible, since they are of $O(\ell)$. We then have:

\[ \int dq_1 dq_2 dq_3 dq_8 dq_9 dq_{12} B(q_4) \langle q_4 | e^{i\Phi_3 y_1 + i\Phi_2 y_2 - 8\pi^2 L(q_5, y_1) - 8\pi^2 L(q_1, y_2)} \rangle \langle q_1 | e^{-\frac{q}{2} \hat{H}} | q_8 \rangle \langle q_8 | e^{-\frac{q}{2} \hat{H}} | q_9 \rangle \langle q_9 | e^{i\Phi_3 y_1} A^+ (q_1, q_9) \rangle \langle q_1 | e^{-\frac{q}{2} \hat{H}} | q_4 \rangle \langle q_4 | e^{-\frac{q}{2} \hat{H}} | q_3 \rangle \langle q_3 | e^{i\Phi_2 y_2} A^- (q_4, q_3) \rangle \langle q_3 | e^{-\frac{q}{2} \hat{H}} | q_2 \rangle \langle q_2 | e^{-\frac{q}{2} \hat{H}} | q_1 \rangle \]  

(32)

We next calculate the integrals over $q_4, q_5, q_8$ by saddle point evaluation. Derivatives of the action $S^c$ with respect to $q_4, q_5, q_8$ give the momenta at the end and at the beginning of the subsequent classical path, in such a way that the classical trajectory ‘bounces back’ three times, retracing its steps, see Fig. 2. Note that the saddle point does not
fix the endpoint of the trajectory (or, equivalently, the momentum), which is a zero-mode at the classical level. On the other hand, we are not yet integrating over the initial position. Denoting \(q_8^\ell = q_F + \ell^2 \hat{q}_8\), \(q_0^\ell = q_I + \ell^2 \hat{q}_8\), and \(q_8 = q_F + \ell^2 \hat{q}_8\), we compute the Gaussian integral around the saddle point, taking care of zero-modes and of Jacobians associated with changes of variables. After a few steps (see the Appendix), we get

\[
T \propto \int dp dq dq_5 \ B(q_F + \ell^2 \hat{q}_4) B(q_F) W(q_I + \ell^2 \hat{q}_5) W(q_I) \ e^{-\frac{1}{16\pi^2} H(p_I, q_I)} |\det \Lambda^{-1}| \ \exp \left\{ i\hat{t} \left[ \Lambda^{-1}_{ab} q_4 a q_5 b \right] \right\}
\]  

(33)

where we have defined:

\[
\frac{\partial^2 S^c}{\partial q_{iF} \partial q_{jF}} = \frac{\partial p_{iF}}{\partial q_{jF}} = \frac{\partial p_{Fj}}{\partial q_{Fi}} = [\Lambda^{-1}]_{ij}(q_I, q_F, \hat{t})
\]

(34)

The leading term is cancelled by the second term in (6). The expectations with two \(\hat{q}\) contribute only to connected two-time correlations that have decayed in the time regime that we are studying because chaos leads to cancellations (see the Appendix). We are left with the expectation containing four \(\hat{q}\), which, after rearrangement, may be expressed in terms of classical Poisson brackets. Assembling all the pieces together, we get the final, purely classical, result:

\[
F_1 \propto \ell^2 \int dp dq \ {B_F, W_I}^2(q_I, p_I, 2\pi\hat{t}) e^{-H_o(p_I, q_I)} \int dp dq_5 \ e^{-H_o(p_I, q_I)}
\]

(35)

where we have defined the classical adimensional Hamiltonian

\[
H_o = \frac{1}{2} p_i p_j g^{ij}
\]

(36)

The meaning of this formula is clear. We are summing over trajectories of different (rescaled) speeds, starting at all points, and weighted with the Gibbs measure. The time is the rescaled time \(\hat{t}\), which accounts for the absence of the temperature in the exponent, so that this expression is, for given \(\hat{t}\), purely geometrical. This is as far as we can go for a system with few degrees of freedom in the canonical framework.

Canonical vs. micro-canonical

For systems with few degrees of freedom, quantities have thermal fluctuations, and there is no single velocity associated with all trajectories in \(\{\hat{q}\}\). Let us consider for simplicity the case in which the measure is concentrated on a single energy, as in the thermodynamic limit. Introducing a delta function and rescaling with the energy density \(E_o \to \sqrt{2E_o} \hat{p}\) we have:

\[
\int dp dq \ {B_F, W_I}^2(q, p) e^{-NH_o(p, q, 4\pi\hat{t})} = \int dE_o e^{-N E_o} \int dqdp \ \delta(H_o(q, p) - E_o) \ {B_F, W_I}^2(q, p)
\]

\[
\int dE_o e^{-NE_o + \frac{N-1}{2} \ln 2E_o} \int dq dp \ \delta \left(H_o(q, p) - \frac{1}{2}\right) \ {B_F, W_I}^2(q, \sqrt{2E_o}p, 2\pi\hat{t})
\]

(37)

(we have dropped the subindex ‘I’ in the arguments \(q, p\)). Putting all together we may write:

\[
F_1 \propto \ell \int dE_o e^{-N E_o + \frac{N-1}{2} \ln 2E_o} \frac{\langle \{B_F, W_I\}^2 \rangle(2\pi\hat{t})}{\langle \{B_F, W_I\}^2 \rangle(2\pi\hat{t})}
\]

(38)

where

\[
S = \ln \frac{1}{N} \int dp dq \ \delta(E_o(q, p) - 1/2) = \frac{1}{N} \ln \int dp dq \ g^{ij} + \text{const}
\]

(39)

is a purely geometric contribution to the entropy, and

\[
\langle \{B_F, W_I(q)\}^2 \rangle(2\pi\hat{t}) \equiv \int dq dp \ \delta(H_o(q, \sqrt{2E_o}p, 2\pi\hat{t}), W_I(q))^2 \ \delta(H_o(q, p) - \frac{1}{2})
\]

(40)
FIG. 2: The trajectories: full and dashed lines are real-time propagation and diffusion, respectively.

FIG. 3: The bound acts at the point where the thermal wavelength become of the same order as the Lyapunov length. At longer thermal wavelengths (the shaded region) quantum dynamics dominates of [8].

The delta function imposes unit speed. It is now clear that if the number of degrees of degrees of freedom is large, and the system has a thermodynamic limit, a single energy density $E_o = \frac{1}{2}$ will dominate and the argument of the Poisson bracket in (10) will simply be $(q,p)$. Here the limit of large $N$ has to be made before the limit of large $t$, otherwise the Poisson bracket term will distort the thermodynamic measure. For this value, we define the reference Lyapunov exponent $\lambda_0$, a purely geometric quantity

$$\langle\langle\{B_F, W_I\}\rangle\rangle(2\pi t) \sim e^{\lambda_0 t} \sim e^{t\lambda_0}\sqrt{\frac{T}{\pi}}$$

which implies that $\lambda_0 = \frac{1}{\ell_o}$, the spatial geodesic separation we defined above, and

$$\lambda = \frac{1}{\ell_o} \sqrt{\frac{T}{m}}$$

which is the classical estimate we made at the beginning, i.e. the expected result that semiclassical and classical Lyapunov exponents coincide.

(41)
Quantum length $\ell$ and characteristic length(s) of the geometry

The situation we have is sketched in Fig 3. For short de Broglie wavelengths, the adimensional Lyapunov exponent $\beta\hbar\lambda$ is proportional to $\ell_o$. If the system has a single characteristic length $R$, as for example the case of the pseudosphere, whose only parameter is its ‘radius’ $R$, then the classical Lyapunov exponent scales with this length, and there is only the question of how does the thermal deBroglie length compare with the geometric one. As we consider longer quantum wavelengths, the semiclassical approximation becomes less good, and in the regime where the wavelength becomes large with respect to all the geometrical features completely breaks down. It is not even clear that a Lyapunov regime – exponentially dependent on $t$ and independent of $B$ and $W$ – exists for the Loschmidt echo, and one would expect that chaoticity becomes smaller.

In the example of motion on the pseudosphere, the Hamiltonian has a continuous spectrum starting from a lowest eigenvalue \[ E_0 = \frac{\hbar^2}{m\ell_o^2} = \frac{\hbar^2}{mR^2} \] (44)

The same system bounded to a large region, will have a similar spectrum (this time not continuous), plus a ‘flat’ state of eigenvalue zero $|0\rangle$. Let us compare the thermal energy with the lowest nonzero eigenvector:

$$
\beta E_0 = \frac{\hbar^2}{mT\ell_o^2} = \left( \frac{\ell}{\ell_o} \right)^2
$$

We see what happens when $\ell$ becomes larger than $\ell_o$: the factor in the exponent of $e^{-\beta H}$ projects onto the lowest, non-propagating mode. We may think of the evolution as diffusion over a long time $\beta$, larger than the ergodic time of the diffusive system. We may estimate that the result of $F_1$ for $\beta E_o \gg 0$ is, up to exponentially small $(\ell/\ell_o)^2$:

$$
F_1 = \langle B^2 \rangle_0 \langle W^2 \rangle_0 - \langle B \rangle_0^2 \langle W \rangle_0^2
$$

where $\langle \bullet \rangle_0$ denotes flat averages over the manifold. Again, it seems doubtful that there is chaos in this regime $\ell \ll \ell_o$ (the gray region), and it is not even clear whether a Lyapunov exponent independent of $W, B$, governing an exponential relaxation, still exists.

Is it possible to have chaos all the way down to zero temperature? It is easy to construct a (very artificial) model that does. For this we have to use the fact, already mentioned above, that the Lyapunov exponent of the Cartesian product of two weakly or non-interacting systems is just the largest of the exponents of the individual systems. Consider then a $2N$ dimensional system composed of $N$ hyperbolic systems like (11), each with typical ‘radius’ $R_1 < R_2 < ... < R_N$. The total Lyapunov of the combined system will be composed of the envelope of the highest of the curves of Fig. 3, each one with a characteristic $\ell_{o1} < \ell_{o2} < ... < \ell_{oN}$. In such a case, one may imagine that as one moves into the deeper quantum regime (longer thermal lengths), the wave propagation ignores the shorter geometric lengths but is scattered by the longest. In a system with regions with all levels of curvature, the quantum Lyapunov would then be governed by the smaller curvature of the order of the inverse deBroglie length, and this is compatible with $\beta\hbar\lambda$ stabilizing as the thermal length becomes longer and longer. The example of critical opalescence comes to mind: at exactly the critical point there are fluctuations of density – and hence of refractive index – at all scales, and the fluid looks white because it scatters all wavelengths. It is tempting to compare the situation with systems like SYK that saturate the bound at exactly zero temperature and have a critical, gapless point there.

IV. CONCLUSIONS

The example of a particle moving on a curved surface, or of a ‘bosonic’ system with a classical ground-state manifold of high dimension, is a context where one may picture the action of the quantum bounds on chaos in an intuitive way. It would be interesting to discuss the effect of ‘driving’ such a system, and computing transport and dissipation coefficients, and the limits quantum mechanics places upon them.

Another interesting exercise would to obtain a bound that applies to the Loschmidt echo in the more conventional setting of a wave packet with typical energy $E$, rather than in the canonical ensemble. It seems clear that some such bound should exist, and would be useful for systems with few degrees of freedom.
Appendix

Let us first compute the diffusion intervals.

\[ R(q_5, y_1) = \langle q_5 + \ell \frac{y_1}{2} | e^{-D W} e^{-D} | q_5 - \ell \frac{y_1}{2} \rangle \]

\[ \propto \int dq' e^{-\left( \frac{i \pi^2}{2} \right) q_{ij} \left( q_5 + \ell \frac{y_1}{q} - q' \right)} W(q') e^{-\left( \frac{i \pi^2}{2} \right) q_{ij} \left( q_5 - \ell \frac{y_1}{q} - q' \right)} J(q_{ij}) \]

\[ \sim W(q_5) e^{-4 \pi^2 q_{ij} (q_5 y_1 y_1) - 2 \pi^2 L_o (q_5 y_1)/2} \]

and similarly for \( R(q_1, y_2) \). Note that the \( y_i \) play the role of velocities in the adimensionalized Lagrangian \( L_o \).

For the full expression we then have, using the semiclassical propagators:

\[ T = \int dq_1 dq_2 dq_3 dq_4 dq_5 dq_6 dq_7 dq_8 B(q_4) B(q_8) e^{i p S^c \pi_1 + i p_1 y_2 - 8 \pi^2 L_o (q_5 y_1) - 8 \pi^2 L_o (q_1 y_1)} A^+ \left( q_1 - \ell \frac{y_2}{2}, q_8 \right) e^{i S^c (q_1, q_8)} \]

\[ A^- \left( q_5 + \ell \frac{y_1}{2}, q_4 \right) A^+ \left( q_5 - \ell \frac{y_1}{2}, q_4 \right) e^{i S^c (q_5, q_4)} A^- \left( q_4, q_1 + \ell \frac{y_2}{2} \right) e^{i S^c (q_4, q_1)} \]

We next calculate the integrals over \( q_4, q_5, q_8 \) by saddle point evaluation. We get:

\[ q_i^c = q_i^o = q_i \quad \text{and} \quad q_i^c = q_i^o = q_F \]

As mentioned above, derivatives of the action \( S^c \) with respect to \( q_4, q_5, q_8 \) give the momenta at the end and at the beginning of the subsequent classical path, in such a way that the classical trajectory ‘bounces back’ three times, retracing its steps see Fig. 2. The speed is the same in modulus in the four trajectories, but is as yet undetermined. Or, in other words, the saddle point does not fix the endpoint of the trajectory, which is then a zero-mode at the classical level.

Denoting \( q_i^c = q_F + \ell \frac{y_1}{2} q_i, q_o^c = q_i + \ell \frac{y_1}{2} q_i \), and \( q_5 = q_F + \ell \frac{y_1}{2} q_5 \), we compute the integral around the saddle point. Rather than expressing it as a path integral, we write an expression for the Gaussian fluctuations around each trajectory. For this we expand

\[ S^c (q_i + \ell \frac{y_1}{2} q_i, q_F + \ell \frac{y_1}{2} q_F) = S^c (q_i, q_F) + \ldots + \ell \frac{y_1}{2} \left[ \frac{\partial^2 S^c}{\partial q_i \partial q_j} \hat{q}_i \hat{q}_j + \frac{\partial^2 S^c}{\partial q_i \partial q_j} \hat{q}_i \hat{q}_j + \frac{\partial^2 S^c}{\partial q_i \partial q_j} \hat{q}_i \hat{q}_j \right] \]

\[ A^+ \left( q_1 - \ell \frac{y_2}{2}, q_8 \right) \approx A^+ (q_1, q_F) = A^- (q_F, q_1) \]

where, as usual, linear terms will cancel at the saddle point, and second derivatives are evaluated on the saddle-point values \( q_1, q_F \). We have kept the leading order in the \( A^\pm \). Note that

\[ \det [A]^-1 = |A(q_1, q_F)|^2 \]

We now develop the classical action up to second order in \( \hat{q}_4, \hat{q}_5, \) and \( \hat{q}_5 \) in the exponent. We also need to integrate over the multiple saddle points, each with a given \( q_F \). The Gaussian fluctuations naturally contain zero-modes associated to the multiplicity of saddles, we must suppress them with a factor \( \delta (\hat{q}_i) \), otherwise we would be double-counting the integral over \( q_F \). All in all, we get:

\[ \int dq_F \, dq_1 dq_2 dq_3 dq_4 dq_5 dq_6 dq_7 dq_8 B(q_F + \ell \frac{y_1}{2} q_4) B(q_F) W(q_1 + \ell \frac{y_1}{2} q_5) W(q_1) e^{ip S^c \pi_1 + ip_1 y_2 - 8 \pi^2 L_o (q_5 y_1) - 8 \pi^2 L_o (q_1 y_1)} \]

\[ |A(q_1, q_F)|^4 \exp \left\{ i t [\Lambda_{ab} \hat{q}_4 \hat{q}_5] \right\} \]

Note the cancellations of terms with second derivatives with respect to a single vector \( \frac{\partial^2 S^c}{\partial q_{ij} \partial q_{ij}} \), etc., and also the absence of terms with \( \hat{q}_5 \) due to the \( \delta (\hat{q}_5) \). We now integrate over \( y_1 \) and \( y_2 \), which transforms \( L_o \) into the adimensional Hamiltonian \( \{36\} \) in the exponent, to obtain:

\[ \int dq_F \, dq_1 dq_2 dq_3 dq_4 dq_5 dq_6 dq_7 dq_8 B(q_F + \ell \frac{y_1}{2} q_4) B(q_F) W(q_1 + \ell \frac{y_1}{2} q_5) W(q_1) e^{-\frac{1}{2} \Lambda_{ab} \hat{q}_{4a} \hat{q}_{5b}} |A(q_1, q_F)|^4 \exp \left\{ i t [\Lambda_{ab}^{-1} \hat{q}_{4a} \hat{q}_{5b}] \right\} \]

(53)
The final step is to change integration variables from \((q_I, q_F)\) to \((q_I, p)\):

\[
dp = |A(q_I, q_F)|^2 \dq_F
\]

so that:

\[
T = \int \dp \dq_I \dq_F \dq_5 \dq_6 \ B(q_F + \ell^2 \hat{q}_5) B(q_F) W(q_I + \ell^2 \hat{q}_5) W(q_I) \ e^{-\frac{1}{4\pi^2} H_{\hat{q}_5} (p_I, q_I)} \ \text{det}[\Lambda^{-1}] \ \exp \left\{ i \hat{t} \left[ \Lambda^{-1}_{ab} \hat{q}_a \hat{q}_b \right] \right\}
\]

Note that the normalization is a constant independent of \(\Lambda\).

Let us expand

\[
W(q_c + \ell^2 \hat{q}_5) = W(q_I + \ell^2 \frac{\partial W}{\partial q_j} \hat{q}_{5j} + \ell \frac{\partial^2 W}{\partial q_j \partial q_k} \hat{q}_{5j} \hat{q}_{5k} + \ldots) \quad \text{and} \quad B(q_F + \ell^2 \hat{q}_{4j}) = B(q_c) + \ell^2 \frac{\partial B}{\partial q_j} \hat{q}_{4j} + \ell \frac{\partial^2 B}{\partial q_j \partial q_k} \hat{q}_{4j} \hat{q}_{4k} + \ldots
\]

Substituting these developments in (55) there are several terms that contribute:

- The leading term proportional to \(B^2(q_F) W^2(q_I)\), which is cancelled by the corresponding one coming from the second term in (54).
- Expectations of terms of the form \(B(q_F) W(q_I) W(q_J) B(q_F) \langle q_5, q_4 \rangle \) (comma denotes derivative) are proportional to \(\Lambda_{ij}\). They are zero at long times for classical reasons: the basis that diagonalizes \(\Lambda_{ij}\) varies wildly from trajectory to trajectory in a chaotic system. This is how the two-time connected correlation function vanishes at the times we are interested in. Similar two-time contributions come from the second term in (54), and are negligible for the same reason.
- Finally, the relevant contribution, of \(O(\ell)\) comes from:

\[
B(q_F) W(q_I) W_{ij} (q_I) B_{kl}(q_F) \langle q_5, q_3, q_4 k, q_4 l \rangle = -B(q_F) W(q_I) W_{ij} (q_I) B_{kl}(q_F) \Lambda_{ik} \Lambda_{jl} \]

Up to a total derivative in \(q_I\) and \(q_F\) this yields, neglecting rapidly oscillating terms

\[
B_{ij}(q_F) W_{ij}(q_I) W_{kl}(q_I) \Lambda_{ik} \Lambda_{jl} = \{ B(q_F), W(q_I) \}^2
\]

where we have used

\[
B_{ij} \Lambda_{jl} W_{ij} = \frac{\partial B(q_F)}{\partial q_{Fj}} \frac{\partial W(q_I)}{\partial q_{ij}} = \frac{\partial B(q_F)}{\partial p_{Fj}} \frac{\partial W(q_I)}{\partial q_{ij}} = -\{ B(q_F), W(q_I) \}
\]

Putting all the pieces together, we get

\[
F_1 \propto \ell \frac{\int \dp \dq_I \{ B_{F}, W_I \}^2 (q_I, p_I, \hat{t}) \ e^{-\frac{1}{4\pi^2} H_{p_I} (p_I, q_I)}}{\int \dp \dq_I \ e^{-\frac{1}{4\pi^2} H_{p_I} (p_I, q_I)}}
\]

rescaling \(p \to 2\pi p\) and then \(\hat{t} \to 2\pi \hat{t}\), we get rid of the prefactor in the exponent and we obtain the final result (59).

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This was discussed, for the case of billiards in [1], and also in [12] in relation to the entanglement rate.

Note that a factor is not the conventional one often used.