Restriction, corestriction
and
the characteristic homomorphism
for
0-cycles of degree 0

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0. Introduction

The main purpose of this Note is to verify (see §5) that the characteristic homomorphism [7, p. 423] of J.-L. Colliot-Thélène and J.-J. Sansuc is compatible with restriction and corestriction maps on the Chow groups of 0-cycles of degree 0. We also derive some consequences of this compatibility (see §§3, 6); as a matter of fact, it is these applications which triggered this investigation.

A word about the genesis and the organisation of this Note is in order. In the middle of February 2003, for reasons which need not concern us here, I decided to write up for publication a letter of Colliot-Thélène [4] (cf. [9, remarque 2]) in which he had indicated how the explicit proof of [9, prop. 3] — for Châtelet surfaces — was a consequence of a proposition (see prop. 1.1) valid for all rational surfaces over a field $K$ which is a finite extensions of $\mathbb{Q}_p$ ($p$ a prime number). The process of writing led me to an equivalent formulation (see remark 4.3) — perhaps a simpler one — in terms of Ext groups in place of Brauer groups; it transpired that this formulation retains its validity over an arbitrary perfect field $K$ (see §5); it leads to a similar statement for rational surfaces having a K-point; the case of a number fields (where the existence of a K-point is not needed) can be found in §6. The first three sections (§§1–3), which treat the local case, reproduce — with his kind permission and with no significant change — the contents of [4]. The §§4–6 treat the general case, including number fields.

Je remercie Jean-Louis Colliot-Thélène pour sa lettre [4], qui est à l’origine de cette Note, et pour ses conseils, ses commentaires et sa patience. I thank Joost van Hamel for his interest in this work, and for a critical reading of the manuscript.
1. The local statement

Let us take the quickest route to the statement of the main proposition of [4]; some of the definitions will be recalled in greater detail in the course of the proof in § 2.

For a variety X over a field K, we shall denote by Br(X) the group of equivalence classes of \( \mathcal{O}_X \)-algebras which are locally isomorphic to the \( \mathcal{O}_X \)-algebra \( \text{End}_{\mathcal{O}_X} V \) of endomorphisms of a vector bundle V over X, modulo those algebras which are globally isomorphic to such an algebra of endomorphisms (the Brauer group [12]); the fibre of such an algebra at any \( x \in X \) is a central simple \( K(x) \)-algebra. When X is regular of dimension \( \leq 2 \), the natural injection \( \text{Br}(X) \to H^2(X, \mathbb{G}_m) \) is an isomorphism [12, Brauer II, cor. 2.2]. We write \( \text{Br}(K) \) instead of \( \text{Br}({\text{Spec}} K) \) and \( \text{Br}(X)_0 \) for the quotient \( \text{Br}(X)/\text{Br}(K) \) modulo the subgroup of " constant " algebras; this will be called the reduced Brauer group of X. (Note that the map \( \text{Br}(K) \to \text{Br}(X) \) need not be injective, even for curves [13]). The group of 0-cycles on X, modulo rational equivalence, will be denoted by \( \text{A}_0(X) \) and — when X is proper — \( \text{A}_0(X)_0 \) will stand for the subgroup of degree-0 0-cycles (the reduced Chow group, see for example [11]).

Let L be a finite extension of K and put \( X_L = X \times_K L \). We have the restriction maps

\[
\begin{align*}
 f^* : \text{A}_0(X)_0 & \longrightarrow \text{A}_0(X_L)_0, \\
f^* : \text{Br}(X)_0 & \longrightarrow \text{Br}(X_L)_0
\end{align*}
\]

and the corestriction maps

\[
\begin{align*}
f_* : \text{A}_0(X_L)_0 & \longrightarrow \text{A}_0(X)_0, \\
f_* : \text{Br}(X_L)_0 & \longrightarrow \text{Br}(X)_0.
\end{align*}
\]

**Proposition 1.1** (Colliot-Thélène). — Let \( L \mid K \mid \mathbb{Q}_p \) (p prime) be finite extensions, and let X be a smooth, proper, absolutely connected K-surface potentially birational to \( \mathbb{P}_2 \).

a) If the restriction \( f^* \) (1) on reduced Brauer groups is surjective (resp. 0), then the corestriction \( f_* \) (2) on reduced Chow groups is injective (resp. 0).

b) If the corestriction \( f_* \) (2) on reduced Brauer groups is surjective (resp. 0), then the restriction \( f^* \) (1) on reduced Chow groups is injective (resp. 0).

2. Generalities and proof
For a variety $X$ over a field $K$, we denote by $\mathbb{Z}_0(X)$ the free commutative group on the set of closed points of $X$ and by $\mathbb{Z}_0(X)_0$ the subgroup of 0-cycles of degree 0. Manin ([14], [15]) defined a natural bilinear pairing

$$\langle \, , \rangle : \mathbb{Z}_0(X) \times \text{Br}(X) \longrightarrow \text{Br}(K)$$

which to a closed point $x \in X$ — of residue field $K' = K(x)$ — and a class $a \in \text{Br}(X)$, associates the class $\text{Cor}_{K'|K}(a(x)) \in \text{Br}(K)$, where $a(x)$ is the class in $\text{Br}(K')$ of the “ fibre ” of $a$ at $x$ (if $a$ is represented by an $\mathcal{O}_X$-algebra $A$, then $a(x) \in \text{Br}(K')$ is represented by the fibre $A(x) = A_x \otimes_{\mathcal{O}_X,x} K'$ of $A$ at $x$, which is a central simple $K'$-algebra). When $a \in \text{Br}(X)$ comes from an element of $\text{Br}(K)$ (i.e. when it is a “ constant ” algebra), one has $\langle z, a \rangle = \text{deg}(z)a$; in particular, $\langle z, a \rangle = 0$ for every $z \in \mathbb{Z}_0(X)_0$ and for every “ constant ” class $a \in \text{Br}(X)$.

Suppose now that the $K$-variety $X$ is proper. Then $\langle z, a \rangle = 0$ for any 0-cycle $z$ rationally equivalent to 0 and for every $a \in \text{Br}(X)$ ([1, Exp. XVII, § 6], cf. [2, Appendix], [7]). We thus get a pairing

$$\langle \, , \rangle : A_0(X) \times \text{Br}(X) \longrightarrow \text{Br}(K)$$

where $A_0(X)$ is the group of 0-cycles on $X$, modulo rational equivalence. Let $A_0(X)_0$ be the kernel of the degree map $\text{deg} : A_0(X) \rightarrow \mathbb{Z}$ and recall that $\text{Br}(X)_0$ stands for $\text{Br}(X)/\text{Br}(K)$. We have an induced pairing (cf. [6, p. 153])

$$\langle \, , \rangle : A_0(X)_0 \times \text{Br}(X)_0 \longrightarrow \text{Br}(K).$$

Now let $f : \text{Spec} L \rightarrow \text{Spec} K$ be the structure map corresponding to a finite separable extension $L|K$. Put $X_L = X \times_K L$; we thus have a pairing

$$\langle \, , \rangle : A_0(X_L)_0 \times \text{Br}(X_L)_0 \longrightarrow \text{Br}(L) \xrightarrow{f^*} \text{Br}(K)$$

in which $f^*$ denotes the corestriction map on Brauer groups of fields.

As consequences of the functoriality of the definitions, one has the two projection formulas

$$\langle f_X^*(z), a \rangle = \langle f_{X_L}^*(z), a \rangle \quad \text{for } z \in \mathbb{Z}_0(X_L), \ a \in \text{Br}(X),
$$

$$\langle f_X^*(z), a \rangle = \langle f_X^*(z), a \rangle \quad \text{for } z \in \mathbb{Z}_0(X), \ a \in \text{Br}(X_L),$$

where $f_X : X_L \rightarrow X$ denotes the canonical morphism and where $f_{X_L}^*$ serves to denote the corestriction maps for 0-cycles as well as for the Brauer group and, similarly, $f_X^*$ denotes the restriction maps. To simplify notation, we write $f_*$, $f^*$ — as we did in (1), (2) — instead of $f_{X_*}$, $f_{X}^*$.
Now suppose that $K$ is a finite extension of $\mathbb{Q}_p$ ($p$ a prime number). Then both $\text{Br}(K)$ and $\text{Br}(L)$ are canonically isomorphic to $\mathbb{Q}/\mathbb{Z}$ by local class field theory, and the corestriction map $f_* : \text{Br}(L) \to \text{Br}(K)$ is just the identity map $\text{Id} : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$.

Denoting by $( )^\vee$ the functor $\text{Hom}(\ , \mathbb{Q}/\mathbb{Z})$ on the category of commutative groups, the pairings $(3)$ and $(4)$ furnish the vertical arrows in the diagrams

$$
\begin{align*}
\text{A}_0(X_L)_0 & \xrightarrow{f_*} \text{A}_0(X)_0 & \text{A}_0(X)_0 & \xrightarrow{f_*} \text{A}_0(X_L)_0 \\
\downarrow & & \downarrow & \\
\text{Br}(X_L)_0^\vee & \xrightarrow{f_*^\vee} \text{Br}(X)_0^\vee, & \text{Br}(X)_0^\vee & \xrightarrow{f_*^\vee} \text{Br}(X_L)_0^\vee,
\end{align*}
$$

and the two projection formulas $(5)$ show that they are commutative.

Completion of the proof of prop. 1.1.— Now suppose that $X$ is a rational surface, which means that it becomes birational to $\mathbb{P}_2$ over a suitable (finite) extension of $K$ — it is sufficient to assume that the vertical arrows in $(6)$ are injective. Then [3, prop. 5, cf. prop. 7b)] says that the vertical arrows in $(6)$ are injective. Now, if $f^* : \text{Br}(X)_0 \to \text{Br}(X_L)_0$ (1) is surjective, then $f_*^\vee : \text{Br}(X_L)_0^\vee \to \text{Br}(X)_0^\vee$ is injective, therefore prop. 1.1a) is an immediate consequence of [3] and the commutativity of the first square $(6)$. Similarly, if $f_* : \text{Br}(X_L)_0 \to \text{Br}(X)_0$ (2) is surjective, then $f_*^\vee : \text{Br}(X)_0^\vee \to \text{Br}(X_L)_0^\vee$ is injective, and prop. 1.1b) follows from [3] and the commutativity of the second square $(6)$.

3. The local application

To illustrate the algebraic solution provided by prop. 1.1 to the arithmetical problem of determining the restriction and corestriction maps on Chow groups (0-cycles of degree 0 modulo rational equivalence) of rational surfaces over local fields, we shall content ourselves with deriving the following proposition, which appears in [9, prop. 3] with the superfluous hypothesis " $p \neq 2$ " :

**Proposition 3.1.** — Let $L | K | \mathbb{Q}_p$ ($p$ prime) be finite extensions, and let $X$ be a smooth projective $K$-surface which is $K$-birational to

$$
y^2 - dz^2 = x(x - e_1)(x - e_2) \quad (e_1 \neq e_2 \text{ in } K^\times, \ d \in K^\times, \notin K^\times 2).
$$
Then the homomorphism of restriction $A_0(X)_0 \to A_0(X_L)_0$ is trivial if the degree $n = [L : K]$ is even; it is an isomorphism if $n$ is odd.

The proof will occupy the rest of this §. An example of such an $X$ is provided by the surface $[9, (2)]$; it is $K(\sqrt{d})$-birationally to $P_2$.

If $L$ is not linearly disjoint from $K' = K(\sqrt{d})$ — $n$ is then even —, the surface $X_L$ is $L$-birationally to $P_2$; we then have $A_0(X_L)_0 = \{0\}$, and there is nothing to prove. Assume henceforth that $L$ does not contain $\sqrt{d}$.

We shall show that the restriction $f^*$ (1) on reduced Brauer groups is always an isomorphism; the corestriction $f_*$ (2) on reduced Brauer groups is an isomorphism if $n$ is odd and 0 if $n$ is even. An application of prop. 1.1 will conclude the argument.

Let $\overline{L}$ be an algebraic closure of $L$ and put $\overline{X} = X \times_K \overline{L}$, $\Gamma_K = \text{Gal}(\overline{L}/K)$, $\Gamma_L = \text{Gal}(\overline{L}/L)$; the Picard group $\overline{\mathbb{P}} = \text{Pic} \overline{X}$ — a free $\mathbb{Z}$-module of finite rank — carries a continuous action of $\Gamma_K$ and of its subgroup (of finite index) $\Gamma_L$, so one has restriction and corestriction maps

$$
    f^* : H^1(K, \overline{\mathbb{P}}) \to H^1(L, \overline{\mathbb{P}}), \quad f_* : H^1(L, \overline{\mathbb{P}}) \to H^1(K, \overline{\mathbb{P}}).
$$

**Lemma 3.2.** — One has the following commutative diagrams in which the vertical arrows are isomorphisms

$$
\begin{array}{c}
    \text{Br}(X)_0 \xrightarrow{f^*} \text{Br}(X_L)_0 \quad \quad \text{Br}(X_L)_0 \xrightarrow{f_*} \text{Br}(X)_0 \\
    \downarrow \quad \quad \downarrow \quad \quad \quad \downarrow \\
    H^1(K, \overline{\mathbb{P}}) \xrightarrow{f^*} H^1(L, \overline{\mathbb{P}}), \quad H^1(L, \overline{\mathbb{P}}) \xrightarrow{f_*} H^1(K, \overline{\mathbb{P}}).
\end{array}
$$

**Proof:** That the map $\text{Br}(X)_0 \to H^1(K, \overline{\mathbb{P}})$ in (8) is an isomorphism is just [14, lemme 3] — applicable because $H^3(K, K^\times) = \{0\}$ — since $\text{Br}(\overline{X}) = \{0\}$ (cf. [16, p. 305]) and hence $\text{Br}(X) \to \text{Br}(\overline{X})$ is the zero map. Similarly, $\text{Br}(X_L)_0 \to H^1(L, \overline{\mathbb{P}})$ is an isomorphism. The commutativity of the two squares is a consequence of fuctoriality.

**Lemma 3.3.** — One has the following commutative diagrams in which the vertical arrows are injections and the lower horizontal ones are induced by the inclusion $K^\times \to L^\times$, resp. by the norm map $L^\times \to K^\times$ :

$$
\begin{array}{c}
    H^1(K, \overline{\mathbb{P}}) \xrightarrow{f^*} H^1(L, \overline{\mathbb{P}}) \quad \quad H^1(L, \overline{\mathbb{P}}) \xrightarrow{f_*} H^1(K, \overline{\mathbb{P}}) \\
    \downarrow \quad \quad \downarrow \quad \quad \quad \downarrow \\
    (K^\times/K^\times 2)^2 \xrightarrow{f^*} (L^\times/L^\times 2)^2, \quad (L^\times/L^\times 2)^2 \xrightarrow{f_*} (K^\times/K^\times 2)^2.
\end{array}
$$
Proof: One needs an explicit description of the structure of the discrete \( \Gamma_K \)-module \( \bar{\Gamma} \), see for example [17], cf. [5, prop. 5.1]. Recalling that \( K' = K(\sqrt{d}) \), put \( G = \text{Gal}(K'|K) \) and \( X' = X \times_K K' \); the map \( \text{Pic}X' \to \bar{\Gamma} = \text{Pic}X \) is an isomorphism and hence \( H^1(K, \bar{\Gamma}) = H^1(G, \text{Pic}X') \). Now the \( G \)-module \( \text{Pic}X' \) is isomorphic — up to addition of permutation modules [5, p. 177] — to \((\mathbb{Z}[G]/\mathbb{Z})^2\) : the map \( \mathbb{Z} \to \mathbb{Z}[G] \) sends 1 to \( 1+\sigma \), where \( \sigma \) is the generator of \( G \). Therefore \( H^1(K, \bar{\Gamma}) = (\mathbb{Z}/2\mathbb{Z})^2 \), and, with this identification, the extreme vertical arrows in (g) send \( 1 \in \mathbb{Z}/2\mathbb{Z} \) to the class of \( d \in K^\times \) modulo \( K^\times \).

As \( L \) is by assumption linearly disjoint from \( K' \), a similar result is valid for the \( \Gamma_L \)-module \( \bar{\Gamma} \), for the group \( H^1(L, \bar{\Gamma}) \) and for the middle vertical arrows in (g).

Completion of the proof of prop. 3.1. — Notice first that the restriction \( f^* : H^1(K, \bar{\Gamma}) \to H^1(L, \bar{\Gamma}) \) is always an isomorphism (lemma 3.3), since \( d \in K^\times \) does not become a square in \( L^\times \) by hypothesis. So the corestriction \( f_* : A_0(X_L)_0 \to A_0(X)_0 \) is injective (lemma 3.2, prop. 1.1a).

As for the corestriction \( f_* : H^1(L, \bar{\Gamma}) \to H^1(K, \bar{\Gamma}) \), it is induced (lemma 3.3) by the norm map \( L^\times \to K^\times \), which sends \( d \) to \( d^n \) (where \( n = [L : K] \)). Therefore the restriction \( f^* : A_0(X)_0 \to A_0(X_L)_0 \) is injective if \( n \) is odd and 0 if \( n \) is even (lemma 3.2, prop. 1.1b)). The proof is complete if \( n \) is even; if \( n \) is odd, restriction \( f^* : A_0(X)_0 \to A_0(X_L)_0 \) and corestriction \( f_* : A_0(X_L)_0 \to A_0(X)_0 \) are injective, and, as the two groups are finite, \( f^* \) and \( f_* \) are both isomorphisms.

Corollary 3.4. — With the notation of prop. 3.1, the corestriction map \( f_* : A_0(X_L)_0 \to A_0(X)_0 \) is injective for all \( n \); it is bijective if \( n \) is odd.

Proof: If \( L \) contains \( \sqrt{d} \), this is trivially true, since \( A_0(X_L)_0 = \{0\} \). Otherwise, we have seen — in the course of the proof of prop. 3.1 — that this is so.

Remark 3.5. — The conclusions of prop. 3.1 and of cor. 3.4 hold — with a similar proof — for \( X \) a smooth proper surface \( K \)-birational to

\[
y^2 - dz^2 = x(x - e_1)(x - e_2) \cdots (x - e_r) \quad (r \geq 2)
\]

where the \( e_i \) are distinct elements of \( K^\times \) and where \( d \in K^\times \) is not a square.

Remark 3.6. — The groups \( A_0(X)_0 \) and \( A_0(X_L)_0 \) have been calculated for the surfaces (7); see [8, prop. 4.7], cf. [9, prop. 1] when the extension \( K'|K \) (resp. \( LK'|L \)) is unramified; see [9, prop. 2] when these extensions are ramified. The values are expressed in [9, prop. 4, 5] in terms of the type of possible bad reduction of the surfaces in question. It seems that S. Bloch...
has expressed the hope that, more generally, for all rational surfaces over a local field, the value of the Chow group of 0-cycles of degree 0 depends only on the type of bad reduction of that surface; when there is good reduction, see [3, th. A(iii)], cf. [10, th. 3]. But even the case of surfaces (10) has not yet been treated, as far as I know.

4. An equivalent formulation

Prop. 1.1 can be reformulated in terms of the characteristic homomorphism [7, p. 423] of Colliot-Thélène and Sansuc; we shall show in the next § that this homomorphism is compatible with restriction and corestriction for more general K and X.

Let S be the K-torus whose $\Gamma_K$-module of $\mathbb{L}$-rational points is $S(\mathbb{L}) = \text{Hom}_\mathbb{Z}(\mathbb{P}, \mathbb{L}^\times)$. Then one has $\text{Ext}_K^1(\mathbb{P}, \mathbb{L}^\times) = H^1(K, S(\mathbb{L}))$, and the cup product furnishes a duality of finite commutative groups

$$H^1(K, \mathbb{P}) \times H^1(K, S(\mathbb{L})) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

[18, p. 102, th. 6(b)]. We have thus a sequence of isomorphisms (cf. §3)

$$\text{Br}(X)_0^\vee = H^1(K, \mathbb{P})^\vee = H^1(K, S(\mathbb{L})) = \text{Ext}_K^1(\mathbb{P}, \mathbb{L}^\times).$$

The same holds at the level of $\mathbb{L}$ instead of $K$. As these identifications are compatible with restriction and corestriction, they yield two commutative squares — which we coalesce for typographical reasons —

$$\begin{array}{ccc}
A_0(X_0) & \to & A_0(X) \\
\downarrow \Phi & & \downarrow \Phi \\
\text{Ext}_L^1(\mathbb{P}, \mathbb{L}^\times) & \text{	extcolor{red}{f}^*} & \text{Ext}_K^1(\mathbb{P}, \mathbb{L}^\times)
\end{array}$$

in which the vertical arrows are the characteristic homomorphisms of [7].

Remark 4.1. — The lower $f^*$ in the diagram (11) consists in considering a short exact sequence of $\Gamma_K$-modules as a short exact sequence of $\Gamma_L$-modules. The lower $f_*$ in (11) associates, to the class in $\text{Ext}_L^1(\mathbb{P}, \mathbb{L}^\times)$ of a short exact sequence $\{0\} \to \mathbb{L}^\times \to E \to \mathbb{P} \to \{0\}$ of $\Gamma_L$-modules, the class...
in $\text{Ext}_K^1(\mathfrak{P}, \mathfrak{L}^\times)$ of the last row of the following diagram of $\Gamma_K$-modules:

$$
\begin{array}{ccccccc}
\{0\} & \longrightarrow & \text{Ind} \mathfrak{L}^\times & \longrightarrow & \text{Ind} E & \longrightarrow & \text{Ind} \mathfrak{P} & \longrightarrow & \{0\} \\
\| & & \| & & \| & & \| & & \\
\{0\} & \longrightarrow & \text{Ind} \mathfrak{L}^\times & \longrightarrow & E' & \longrightarrow & \mathfrak{P} & \longrightarrow & \{0\} \\
\| & & \| & & \| & & \| & & \\
\{0\} & \longrightarrow & \mathfrak{L}^\times & \longrightarrow & E'' & \longrightarrow & \mathfrak{P} & \longrightarrow & \{0\} \\
\end{array}
$$

(defining $E'$ and $E''$), where $\text{Ind} E$ stands for the $\Gamma_K$-module induced from the $\Gamma_L$-module $E$, etc.; the maps $\text{Ind} \mathfrak{L}^\times \rightarrow \mathfrak{L}^\times$ (defining the last row) and $\mathfrak{P} \rightarrow \text{Ind} \mathfrak{P}$ (defining the middle row) are the natural ones [18, p. 13], arising from the fact that the $\Gamma_L$-modules $\mathfrak{L}^\times$, $\mathfrak{P}$ are restrictions of the $\Gamma_K$-modules $\mathfrak{L}^\times$, $\mathfrak{P}$.

**Remark 4.2.** — The projection formulas (5) express the commutativity of certain diagrams, namely (6) in the special case when $K$ is a finite extension of $\mathbb{Q}_p$. We have seen that in this special case, if $X$ is a $K$-surface potentially birational to $\mathbb{P}_2$, the two squares of (6) are formally equivalent, respectively, to the two squares (11). I do not know what the relationship between (5) and (11) is in the general case, when $K$ is any perfect field and $X$ is an arbitrary $K$-variety.

**Remark 4.3.** — One obtains a reformulation of prop. 1.1 by replacing, in the hypotheses, the surjectivity of $f^*$ (1) (resp. $f_*$ (2)) on reduced Brauer groups by the injectivity of $f_*$ (resp. $f^*$) (11) on $\text{Ext}$ groups (cf. th. 6.1).

## 5. The general case

As we have seen, the proof (§ 2) of prop. 1.1 hinges on the injectivity statements of [3] and on the commutativity — a consequence of the projection formulas (5) — of the two squares in (6). The commutativity of the analogous squares (11) is a general phenomenon, as we shall show now. We thus obtain not only an alternate proof of prop. 1.1 but also its extension — along with the extensions of prop. 3.1, cor. 3.4 and remark 3.5 which follow from it — to the number field case (§ 6).

**Theorem 5.1.** — Let $L|K$ be a finite extension, with $K$ a perfect field and let $X$ be a smooth, proper, absolutely connected $K$-variety. Then the two squares in diagram (11) are commutative.
Proof: The definition of the characteristic homomorphism — recalled below — makes the commutativity of the second square of (11) obvious; let us show it for the first one.

Let $\xi$ be a 0-cycle of degree 0 on $X_L$. We then have the 0-cycles $\eta = f_*(\xi)$ on $X$ and $\overline{\eta}$ on $\overline{X}$; in fact $\overline{\eta} = n\overline{\xi}$, where $n = [L : K]$. We have to show that $f_*(\Phi(\xi)) = \Phi(\eta)$.

Let $\overline{A}$ be the semilocal ring of $\overline{X}$ along the 0-cycle $\overline{\xi}$ and put $U = U_K \times L$, where $U \subset X$ is the open subvariety complementary to the support of $\eta$. Recall [7] that $\Phi(\xi) \in \text{Ext}_L^1(\mathcal{P}, \mathcal{L}^\times)$ is the class of the extension of $\Gamma_L$-modules deduced from the short exact sequence of $\Gamma_K$-modules

\[
\{1\} \to \overline{A}^\times/\mathcal{L}^\times \to Z^1(U) \to \mathcal{P} \to \{0\}
\]

via the $\Gamma_L$-homomorphism of evaluation $\overline{A}^\times/\mathcal{L}^\times \to \mathcal{L}^\times$ at $\overline{\xi}$; similarly, $\Phi(\eta) \in \text{Ext}_K^1(\mathcal{P}, \mathcal{L}^\times)$ is deduced from (12) via the $\Gamma_K$-homomorphism of evaluation $\overline{A}^\times/\mathcal{L}^\times \to \mathcal{L}^\times$ at $\overline{\eta}$. For $f_*(\Phi(\xi))$, see the explicit description of the map $f_*$ given in remark 4.1. To show that $f_*(\Phi(\xi)) = \Phi(\eta)$, we use the following lemma.

Lemma 5.2. — Let $P$, $Q$, $R$ be $\Gamma_K$-modules and let $h: Q \to R$ be a $\Gamma_L$-homomorphism such that $nh: Q \to R$ is $\Gamma_K$-equivariant. Then the following square is commutative:

\[
\begin{array}{ccc}
\text{Ext}_K^1(P, Q) & \xrightarrow{nh} & \text{Ext}_K^1(P, R) \\
f_* \downarrow & & f_* \uparrow \\
\text{Ext}_L^1(P, Q) & \xrightarrow{h} & \text{Ext}_L^1(P, R).
\end{array}
\]

6. The global application

With the general result (th. 5.1) at hand, the statements and proofs of §§2,3 can be extended to any situation where the characteristic homomorphisms are injective. We shall content ourselves with the following

Theorem 6.1. — Assume that $X$ is a $K$-surface potentially birational to $\mathbf{P}_2$, where $K|Q$ is either a finite extension or a finitely generated extension and, in the latter case, that $X$ has a $K$-rational 0-cycle of degree 1. Let $L|K$ be a finite extension.
a) If the corestriction \( f_* \) (11) on \( \text{Ext} \) groups is injective (resp. 0), then the corestriction \( f_* \) (2) on reduced Chow groups is injective (resp. 0).

b) If the restriction \( f^* \) (11) on \( \text{Ext} \) groups is injective (resp. 0), then the restriction \( f^* \) (1) on reduced Chow groups is injective (resp. 0).

Proof: The ingredients of the proof are the same as in the local case (§§ 2, 3), up to replacing the diagram (6) by the diagram (11).

It was shown in [3] that the vertical arrows in (11) — the characteristic homomorphisms of [7] — are injective under our hypotheses; the result follows therefrom and the commutativity of (11) (th. 5.1).

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