In this work, considering the background dynamics of flat Friedmann-Lemaitre-Robertson-Walker (FLRW) model of the universe, we investigate a non-canonical scalar field model as dark energy candidate which interacting with the pressureless dust as dark matter in view of dynamical systems analysis. Two interactions from phenomenological point of view are chosen: one is depending on Hubble parameter $H$, another is local, independent of Hubble parameter. In Interaction model 1, an inverse square form of potential as well as coupling function associated with scalar field is chosen and a two dimensional autonomous system is obtained. From the 2D autonomous system, we obtain scalar field dominated solutions representing late time accelerated evolution of the universe. Late time scaling solutions are also realized by the accelerated evolution of the universe attracted in quintessence era. Center Manifold Theory can provide the sufficient conditions on model parameters such that the de Sitter like solutions can be stable attractor at late time in this model. In the Interaction model 2, potential as well as coupling function are considered to be evolved exponentially on scalar field and as a result of which a four dimensional autonomous system is achieved. From the analysis of 4D system, we obtain non-hyperbolic sets of critical points which are analyzed by the Center Manifold Theory. In this model, de Sitter like solutions represent the transient evolution of the universe.

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1. INTRODUCTION

Theoretical cosmology does not agree with the recent observational evidences [1, 2] which predict that currently the Universe is enduring an accelerated expansion. So, cosmologists have been trying to explain these observational predictions for more than one decade separated in two groups. One group of cosmologists has been trying to explain the observational facts by introducing an exotic quantity commonly known as dark energy (DE) with large negative pressure, while the other group is focused to explain the observational evidences theoretically by modifying the Einstein Gravity theory.

In the literature, there are various choices of DE but cosmological constant is observationally more preferable as well as the simplest one. But this choice also consists of two severe conceptual problems, (1) cosmological constant problem and (2) coincidence problem. On the other hand, there is no well-accepted modification of gravity theory as an alternative choice to the cosmologists. Cosmological constant problem can be overcome by introducing new type of time varying DE model in which the canonical scalar field model: Quintessence is the popular one where equation of state parameter $\omega_\phi$ for Quintessence takes any value in the range $-1 < \omega_\phi < -\frac{1}{3}$. After that various DE models based on scalar field such as K-essence, phantom, quintom etc. have been studied in the literature. The scalar field model, where energy density and pressure having a non-canonical form of kinetic term, i.e., kinetic term appears with a coupling function (non-canonical term) depending on scalar field $\phi$, is the non-canonical scalar field model [3–7] which has also been studied extensively to solve several cosmological issues [8]. On the other hand, coincidence problem can be alleviated by introducing an interaction term in between the dark energy and the dark matter. Since at present the evolution of the universe is assumed to be dominated by the dark energy and the dark matter, the possibility of having interaction between them cannot be ignored. Also, an appropriate form of interaction can provide a possible mechanism to alleviate the coincidence problem. However, due to unknown nature of DE, one can choose the interaction term phenomenologically which can be a function of energy density of either DE or DM

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or both [9]. Several interacting DE models where interaction terms are taken in phenomenological background have been investigated in the literature (see Refs. [10–25]).

Motivated from the above fact, in the present manuscript, cosmological implications of a non-canonical scalar field model have been studied in the background of homogeneous and isotropic flat FLRW space-time. The system of cosmological equations are nonlinear differential equations. There is no well known method to find the exact solution of the system. Dynamical system analysis is an elegant tool to study nonlinear systems. Dynamical systems analysis allows us to gain a qualitative understanding of any cosmological model. Apparently many recent research works are going on this approach to understand cosmological models both geometrical and physical point of view. To survey some of the current literatures we cite papers [26–30] where authors studied cosmological evolutions of the Universe by dynamical system analysis. By suitable change of variables Einstein field equations are transformed to an autonomous set of equations in terms of variables in a 2D and 4D phase space for various choices of interactions. So the question arises that whether the cosmological solution of the non-canonical model is stable when subject to perturbation in the phase space. The fluctuation, near a hyperbolic critical point [31, 32], can not change much about the stability but a small perturbation near a non-hyperbolic critical point appears to be different stability criteria. So it is important to study non-hyperbolic critical points. As Hartman-Grobman theorem is not appropriate to characterize non-hyperbolic critical points, we thoroughly study Center Manifold theory and apply it to characterize the non-hyperbolic critical points. We study the physical significance or novelty of the corresponding results.

Motivated from scalar-tensor theory, and for mathematical simplicity we choose two interactions: one is depending directly on Hubble parameter $H$ and energy density of matter $\rho_m$, another is chosen locally, depending only on energy density $\rho$ and independent of $H$. For the first interaction model, the potential and the coupling function (non-canonical term) of the scalar field are considered as inverse square form. As a result, autonomous system of dynamical system is reduced to a two dimensional. On the other hand, for interaction model 2, the potential and the coupling function are considered to be depended exponentially on scalar field $\phi$. This leads to a four dimensional autonomous system. In 2D system, for interaction model 1, we obtain some interesting critical points including hyperbolic or non-hyperbolic type. Linear stability theory shows that accelerated scalar field dominated solutions are the late time attractor. This study also reveals that the accelerated scaling solutions are attractors in quintessence era. Moreover, the Center Manifold Theorem provides the sufficient conditions for critical points to be the late time de Sitter attractors. Further, interaction model 2 with exponential potential by the Center Manifold Theorem exhibits that the non-hyperbolic sets of critical points being the saddle-like solutions show transient nature of evolution whether they are scalar field dominated or scaling like solutions.

We organize our work in the following manner: In the next section 2, we start with non-canonical scalar field model, by choosing phase space variables we construct the autonomous system of ordinary differential equations. Phase space analysis of the autonomous system by considering two interaction models with different potential and coupling have been presented in the section 3. After that Center Manifold theorem has been studied for non-hyperbolic type critical points in 4. We present cosmological implications in a section in 5. Finally, We conclude with a brief discussion in 6.

## 2. NON-CANONICAL SCALAR FIELD AND FORMATION OF AUTONOMOUS SYSTEM

The action for a non-canonical scalar field minimally coupled to gravity can be written as

$$A = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa} - \frac{1}{2} \lambda(\phi) g^{\mu\nu} (\nabla_\mu \phi)(\nabla_\nu \phi) - V(\phi) \right] + A_m. \quad (1)$$

Here $A_m$ is the action corresponding to the matter content in the universe [36]. In the present case the matter is chosen as cold dark matter (CDM) in the form of dust. $\lambda(\phi)$, the coupling function is chosen as arbitrary function of the scalar field $\phi$, $V(\phi)$ is the potential of the scalar field and $\kappa^2 = 8\pi G$ is the Gravitational Coupling parameter. The present cosmological model is considered in the background of homogeneous and isotropic flat Friedmann-Lemaître-Robertson-Walker (FLRW) space-time manifold having line element

$$ds^2 = -dt^2 + a^2(t) \left[ d\theta^2 + r^2 \left( d\phi^2 + \sin^2 \theta d\phi^2 \right) \right]. \quad (2)$$

Friedmann equation and acceleration equation for non-canonical scalar field in presence of pressureless dust in the background of FLRW metric (using natural units ($\kappa^2 = 8\pi G = c = 1$)) are:

$$H^2 = \frac{1}{3} (\rho_m + \rho) \quad (3)$$
and
\[ 2\dot{H} = -(\rho_m + \rho_\phi + p_\phi), \]  
(4)

where \( \dot{H} = \frac{\dot{a}}{a} \) is Hubble parameter and \( \rho_m \) is the energy density for dark matter (DM) taken as pressureless \( (p_m = 0) \) dust with equation of state parameter \( \omega_m = 0 \). Non-canonical scalar field governed by its energy density \( \rho_\phi \) and pressure \( p_\phi \) which are combined with potential and kinetic terms as follows:

\[ \rho_\phi = \frac{1}{2} \lambda(\phi) \dot{\phi}^2 + V(\phi) \]  
(5)

and

\[ p_\phi = \frac{1}{2} \lambda(\phi) \dot{\phi}^2 - V(\phi), \]  
(6)

where \( V(\phi) \) is the potential function of the scalar field and \( \lambda(\phi) \) describes the coupling function of scalar field \( \phi \) connected with the kinetic term of the scalar field. This scalar field reduces to the canonical (quintessence) scalar field when the non-canonical term \( \lambda(\phi) = 1 \), and it describes the phantom for \( \lambda(\phi) = -1 \). The equation of state parameter for scalar field is \( \omega_\phi = \frac{p_\phi}{\rho_\phi} \). Total (effective) energy density and pressure for the model \( \rho_{\text{eff}} = \rho_m + \rho_\phi \) and \( p_{\text{eff}} = p_\phi \) constitute the total (effective) equation of state parameter in the form:

\[ \omega_{\text{eff}} = \frac{p_{\text{eff}}}{\rho_{\text{eff}}} = \frac{p_\phi}{\rho_m + \rho_\phi}. \]  
(7)

In an interacting scenario when DE interacts with DM through an interaction term \( Q \), the individual component (DE and DM) conserves separately and hence the conservation equations for scalar field and matter (with the term \( Q \)) take the following form:

\[ \dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = -Q \]  
(8)

and

\[ \dot{\rho}_m + 3H\rho_m = Q, \]  
(9)

where the sign of interaction term \( Q \) specifies the direction of energy flow in the dark sector. For example, \( Q > 0 \) indicates the flow of energy transfers from DE to DM while for \( Q < 0 \) the flow of energy transfers from DM to DE. Now, evolution equation for the scalar field (from Eqn. (8), using Eqn. (5) and Eqn. (6)) with non-canonical term takes the form

\[ \frac{1}{2} \lambda' \dot{\phi}^2 + \lambda \ddot{\phi} + V' + 3H\lambda \dot{\phi} = -\frac{Q}{\phi}, \]  
(10)

where the symbol ‘prime’ denotes the derivative with respect to scalar field \( (\prime \equiv \frac{d}{d\phi}) \). Since the evolution equations are very much complicated in the interacting scenario it is quite impossible to obtain an exact analytical solution from the Friedmann equation, acceleration equation and conservation. We try to obtain a qualitative picture of this cosmological model, we adopt dynamical system approach and for that choose the following dynamical variables

\[ x = \frac{\dot{\phi}}{\sqrt{2}}, \quad y = \frac{\sqrt{V}}{\sqrt{3}H}, \quad z = \frac{\sqrt{\lambda}}{\sqrt{3}H}. \]  
(11)

Using the variables (in equation (11)) the evolution equations reduce (after some algebra) to the following autonomous system of ordinary differential equations:

\[ \frac{dx}{dN} = -3x - \sqrt{\frac{3}{2}} \frac{\lambda'}{\lambda \sqrt{\lambda}} x^2 z - \sqrt{\frac{3}{2}} \frac{V'}{\sqrt{V}} y^3 - \frac{Q}{6xz^2 H^3}, \]  
\[ \frac{dy}{dN} = y \left[ \sqrt{\frac{3}{2}} \frac{V'}{\sqrt{V}} xy + \frac{3}{2} \left(1 + x^2 z^2 - y^2 \right) \right], \]  
\[ \frac{dz}{dN} = z \left[ \sqrt{\frac{3}{2}} \frac{\lambda'}{\lambda \sqrt{\lambda}} xz + \frac{3}{2} \left(1 + x^2 z^2 - y^2 \right) \right]. \]  
(12)
Here, \( N = \ln a \) is chosen as the independent variable. Now, the system (12) cannot be a complete autonomous dynamical system of ODEs for the dynamical variables \( x(N), y(N) \) and \( z(N) \) due to involvement of the terms containing \( \frac{V'}{\sqrt{V}} \) and \( \frac{\lambda}{\sqrt{\lambda}} \) which are explicit functions of scalar field \( \phi \). However, to close the system, one has to convert the terms into dynamical variables (by setting \( s(\phi) = \frac{V'}{\sqrt{V}} \) and \( u(\phi) = \frac{\lambda}{\sqrt{\lambda}} \)). The time evolution equations of \( s(\phi) = \frac{V'}{\sqrt{V}} \) and \( u(\phi) = \frac{\lambda}{\sqrt{\lambda}} \) are as follows:

\[
\frac{ds}{dN} = \sqrt{6} x y s^2 \left( \Gamma_s - \frac{3}{2} \right), \tag{13}
\]

\[
\frac{du}{dN} = \sqrt{6} x y u^2 \left( \Gamma_u - \frac{3}{2} \right), \tag{14}
\]

where \( \Gamma_s = \frac{V''}{\sqrt{V}} \) and \( \Gamma_u = \frac{\lambda \phi''}{\sqrt{\lambda}} \). Note that the system (12) with the equations (13) and (14) forms a complete five dimensional autonomous system of ODEs of variables \( x(N), y(N), z(N), s(N) \) and \( u(N) \). It is very much complicated to handle 5D autonomous system rather than 3D system and for inverse square potential \( V \propto \phi^{-2} \), the terms \( s^2(\Gamma_s - \frac{3}{2}) \) vanishes which implies \( \frac{ds}{dN} = 0 \), i.e., when \( s = \text{constant} \). Similarly, for coupling function \( \lambda(\phi) \propto \phi^{-2} \), we get \( u = \text{constant} \) which implies that \( \frac{du}{dN} = 0 \). Thus, the resulting autonomous system will be of three dimensional system when \( s \) and \( u \) are non-zero constant parameters (for inverse square potential and coupling function) which we shall discuss in the next section.

Now, the cosmological parameters can be written in terms of dynamical variables as follows:

Density parameter for non-canonical scalar field (DE) is

\[
\Omega_\phi = x^2 z^2 + y^2, \tag{15}
\]

the equation of state parameter for non-canonical scalar field (DE) is

\[
\omega_\phi = \frac{x^2 z^2 - y^2}{x^2 z^2 + y^2}, \tag{16}
\]

the effective equation of state parameter for the model is

\[
\omega_{eff} = x^2 z^2 - y^2, \tag{17}
\]

and the deceleration parameter is

\[
q = -1 + \frac{3}{2}(1 + \omega_{eff}) \tag{18}
\]

One can obtain the conditions for acceleration by setting either \( q < 0 \), or \( \omega_{eff} < -\frac{1}{3} \) and for deceleration: \( q > 0 \) i.e. \( \omega_{eff} > -\frac{1}{3} \). Friedmann equation (3) gives the constraint equation in terms of density parameter for DM is

\[
\Omega_m = 1 - x^2 z^2 - y^2 \tag{19}
\]

by which due to the energy condition \( 0 \leq \Omega_m \leq 1 \), we obtain the constraints for dynamical variables in the region:

\[
0 \leq x^2 z^2 + y^2 \leq 1. \tag{20}
\]

In the following section, we shall explicitly analyze the autonomous system (12) for two different choices of the interaction terms, namely, (i) \( Q_1 \propto H \rho_m \) and (ii) \( Q_2 \propto \rho_m \) with the potential \( V(\phi) \) and the coupling function \( \lambda(\phi) \) of the scalar field are considered as inverse square and exponential form respectively.

### 3. Phase Space Analysis of Autonomous System (12) with Different Choices of Interaction Terms and Potentials:

We shall now present a detailed phase space analysis of the general autonomous system (12), where first analyze the interaction \( Q_1 \propto H \rho_m \) [33, 34] with inverse square potential and coupling and then we analyze \( Q_2 \propto \rho_m \) [34, 35] with the exponential form of potential and coupling function of the scalar field. Hyperbolic type critical points are analyzed by using the Hartman-Grobman theorem (linear stability theory) and the Center Manifold Theory is applied for checking the stability of non-hyperbolic critical points.
3.1. Interaction Model 1: \( Q_1 : Q \propto H \rho_m \)

First, we consider the interaction term

\[
Q = 6\eta H \rho_m,
\]

where \( \eta \) is the coupling parameter measures the strength of the interaction. Positivity of the coupling parameter indicates the energy transfers from DE to DM which is required to alleviate the coincidence problem and is compatible with second law of thermodynamics. Using the term of (21) in the autonomous system (12) with inverse square form of potential and coupling function of scalar field, i.e., \( V(\phi) = V_0 \phi^{-2} \) and \( \lambda(\phi) = \lambda_0 \phi^{-2} \) such that \( s \) and \( u \) become constant parameters which leading to a 3D autonomous system as \( \frac{dx}{dN} = 0 \) in equation (13) and \( \frac{du}{dN} = 0 \) in equation (14). Note that for such inverse square form of potential and of non-canonical coupling term with \( V_0 = \lambda_0 \), the autonomous system leads to a two dimensional system because the dynamical behavior of two variables \( y \) and \( z \) in Eqn.(11) are same. As a result, in the study of this model with inverse square potential term for both the potential \( (V(\phi)) \) and coupling \( (\lambda(\phi)) \), we shall study only 2D dynamical system by replacing the variable \( z \) with \( y \), and the constant parameter \( u \) with \( s \). Now, 2D autonomous system in \( x - y \) takes the following form:

\[
\frac{dx}{dN} = -3x - \sqrt{\frac{3}{2}} s x^2 y - \sqrt{\frac{3}{2}} s y - \frac{3\eta}{x y^2}(1 - x^2 y^2 - y^2),
\]

\[
\frac{dy}{dN} = y \left[ \sqrt{\frac{3}{2}} s x y + \frac{3}{2} \left(1 + x^2 y^2 - y^2\right) \right].
\]

The system (22) has singularities at \( x=0 \) and \( y=0 \) respectively. To remove these, the right hand side (r.h.s) of each equation of the system (22) is multiplied by \( xy^2 \). Then the system (22) reduces to the form:

\[
\frac{dx}{dN} = -3x^2 y^2 - \sqrt{\frac{3}{2}} s x^3 y^3 - \sqrt{\frac{3}{2}} s x y^3 - 3\eta(1 - x^2 y^2 - y^2),
\]

\[
\frac{dy}{dN} = x y^3 \left[ \sqrt{\frac{3}{2}} s x y + \frac{3}{2} \left(1 + x^2 y^2 - y^2\right) \right],
\]

The critical points extracted from the system (23) are the following:

- **I. Critical Point**: \( A_1 = (0,1) \)
- **II. Critical Point**: \( B_1 = (0,-1) \)
- **III. Critical Point**: \( C_1 = \left(-\frac{s}{s-\sqrt{s}}, \sqrt{1-\frac{x^2}{s}}\right) \)
- **IV. Critical Point**: \( D_1 = \left(\frac{s}{\sqrt{s-\sqrt{s}}}, -\sqrt{1-\frac{x^2}{s}}\right) \)
- **V. Critical Point**: \( E_1 = \left(\frac{\sqrt{3}(2s-1)}{\sqrt{4s^2+3(2s-1)^2}}, \frac{\sqrt{4s^2+3(2s-1)^2}}{\sqrt{2s}}\right) \)
- **VI. Critical Point**: \( F_1 = \left(-\frac{\sqrt{3}(2s-1)}{\sqrt{4s^2+3(2s-1)^2}}, -\frac{\sqrt{4s^2+3(2s-1)^2}}{\sqrt{2s}}\right) \)

3.1.1. Phase space analysis of interaction 1

We shall now discuss the phase space analysis of critical points extracted from the 2D autonomous system (23). Critical points and their corresponding physical parameters are shown in the table (I).

- Critical points \( A_1 \) and \( B_1 \) exist for all parameter values \( s \) and \( \eta \) in the phase plane \( x-y \). The points are completely dominated by the potential energy of the scalar field (DE) which behaves as cosmological constant (\( \omega_0 = -1 \)). Acceleration of the universe is always possible near the points (since \( \omega_{eff} = -1, q = -1 \)). Eigenvalues of the
perturbed matrix for the critical point $A_1$: $\{\mu_1 = 0$ and $\mu_2 = -\frac{\sqrt{3}}{2}s\}$. The point $A_1$ is non-hyperbolic type critical point and linear stability theory fails to describe the nature of the point. However, the point $A_1$ has a one dimensional stable manifold for $s > 0$. On the other hand, eigenvalues for the point $B_1$: $\{\mu_1 = 0$ and $\mu_2 = \frac{\sqrt{3}}{2}s\}$ showing that the point is also a non-hyperbolic type critical point and has one dimensional stable manifold for $s < 0$. Since the critical points namely, $A_1$ and $B_1$ may be physically interesting as these points describe the de Sitter expansion of the universe (see Table 1). To obtain the conditions for stability of these points we employ the Center Manifold Theory which is discussed in the next section 4.

- Critical points $C_1$ and $D_1$ exist for $-\sqrt{6} < s < \sqrt{6}$ and for all values of $\eta$. The critical points are completely dominated by scalar field which behaves as any perfect fluid model (since $\omega_\phi = \frac{s^2}{3} - 1$). Therefore, DE can behave as quintessence, cosmological constant or any other exotic type fluid depending on parameter $s$. In particular, for $s^2 < 2$ ($s \neq 0$), the scalar field behaves as quintessence like fluid. For this restriction on parameter, the points represent accelerating universe (as $\omega_{eff}$ and $q$ satisfy $\omega_{eff} < -\frac{1}{3}$, $q < 0$ for this case). For $s = 0$, scalar field behaves as cosmological constant and for that case evolution of the points are characterised by de Sitter expansion ($\omega_\phi = -1$, $\omega_{eff} = -1$, $q = -1$). Eigenvalues of linearized Jacobian matrix for the point $C_1$:

$$\{\mu_1 = \frac{1}{12}s \left(6 - s^2\right)^{3/2} \text{ and } \mu_2 = -\frac{1}{6}s \left(6\eta + s^2 - 3\right)\sqrt{6 - s^2}\}$$

Critical point $C_1$ is hyperbolic in nature and it is stable attractor either for: $(\eta \leq -\frac{1}{2}$ and $-\sqrt{6} < s < 0)$ or for $(-\frac{1}{2} < \eta < \frac{1}{2}$ and $-\sqrt{3 - 6\eta} < s < 0)$ while the point represents unstable source either for $(\eta \leq -\frac{1}{2}$ and $0 < s < \sqrt{6})$ or for $(-\frac{1}{2} < \eta < \frac{1}{2}$ and $0 < s < \sqrt{3 - 6\eta})$. Therefore, the critical point $C_1$ represents the accelerated past attractor in $\{(\eta \leq \frac{1}{6}$ and $0 < s < \sqrt{2}) \text{ or } (\frac{1}{6} < \eta < \frac{1}{2}$ and $0 < s < \sqrt{3 - 6\eta})\}$. For this parameter restrictions, the point can describe the early accelerated scalar field dominated phase of the universe.

The evolution of the critical point $C_1$ is attracted only in quintessence era either for $(\eta \leq \frac{1}{6}$ and $-\sqrt{2} < s < 0)$ or for $(\frac{1}{6} < \eta < \frac{1}{2}$ and $-\sqrt{3 - 6\eta} < s < 0)$.

Eigenvalues of the critical point $D_1$ are:

$$\{\mu_1 = -\frac{1}{12}s \left(6 - s^2\right)^{3/2} \text{ and } \mu_2 = \frac{1}{6}s \left(6\eta + s^2 - 3\right)\sqrt{6 - s^2}\}$$

The point is also hyperbolic and represents stable attractor for: either $(\eta \leq -\frac{1}{2}$ and $0 < s < \sqrt{6})$ or for $(-\frac{1}{2} < \eta < \frac{1}{2}$ and $0 < s < \sqrt{3 - 6\eta})$. It represents unstable source for: $(\eta \leq -\frac{1}{2}$ and $-\sqrt{6} < s < 0)$ or $(-\frac{1}{2} < \eta < \frac{1}{2}$ and $-\sqrt{3 - 6\eta} < s < 0)$. Therefore, the point $D_1$ describes the accelerated past attractor for: $(\eta \leq \frac{1}{6}$ and $-\sqrt{2} < s < 0)$ or $(\frac{1}{6} < \eta < \frac{1}{2}$ and $-\sqrt{3 - 6\eta} < s < 0)$.

The point $D_1$ corresponds to a stable attractor only in quintessence era for the parameter restrictions: $(\eta \leq \frac{1}{6}$ and $0 < s < \sqrt{2})$ or $(\frac{1}{6} < \eta < \frac{1}{2}$ and $0 < s < \sqrt{3 - 6\eta})$. Therefore, one can conclude that when the point $C_1$ corresponds to a scalar field dominated accelerated attractor at late times, the point $D_1$ shows its early accelerated past attractor in phase plane and vice-versa respectively.

- Critical points $E_1$ and $F_1$ describe scaling solutions in phase plane and exist for

(i) $-\frac{1}{2} < \eta < 0$ and $\left(-\frac{1}{2}\sqrt{3} \left(\frac{4\eta - 4\eta^2 - 1}{\eta}\right) s \leq -\sqrt{3 - 6\eta}$ or $\sqrt{3 - 6\eta} \leq s < \frac{1}{2}\sqrt{3} \left(\frac{4\eta - 4\eta^2 - 1}{\eta}\right) \right)$, or

(ii) $0 \leq \eta < \frac{1}{2}$ and $\left(s \leq -\sqrt{3 - 6\eta}$ or $s \geq \sqrt{3 - 6\eta}\right)$, or

(iii) $\eta = \frac{1}{2}$ and $(s < 0 \text{ or } s > 0)$.

The ratio of the energy densities of DE and DM for both the points is $\frac{\Omega_\phi}{\Omega_m} = \frac{2\sqrt{s^2 + 3(2s - 1)^2}}{(2s - 1)(3 - s^2 - 6\eta)}$. Non-canonical scalar field behaves as any perfect fluid (as $\omega_\phi = -\frac{2s^2}{2\sqrt{s^2 + 3(2s - 1)^2}}$). There exists acceleration for both the points if $\eta > \frac{1}{6}$. For $\eta = \frac{1}{6}$ irrespective of $s$, the scalar field DE behaves as cosmological constant, and accelerated de Sitter solution exists for these critical points (since for this case: $\omega_{eff} = q = -1, \Omega_\phi = 1, \Omega_m = 0$). On the other hand, for uncoupled case $\eta = 0$, scalar field DE behaves as dust ($\omega_{\phi} = 0$) and dominated decelerating scaling solutions obtained ($\omega_{eff} = 0, q = \frac{1}{2}, \Omega_m = 1 - \frac{3}{4}, \Omega_\phi = \frac{3}{4}$).

Eigenvalues of the linearized Jacobian matrix for the critical point $E_1$:

$$\{\mu_1 = \frac{\sqrt{3}(-s^2(4\eta(3\eta + s^2 - 3) + 3)(4\eta(9\eta + s^2 - 3) - 3) - \Delta E_1)}{8s^2(4\eta(3\eta + s^2 - 3) + 3)^{3/2}}, \mu_2 = \frac{\sqrt{3}(-s^2(4\eta(3\eta + s^2 - 3) + 3)(4\eta(9\eta + s^2 - 3) - 3) + \Delta E_1)}{8s^2(4\eta(3\eta + s^2 - 3) + 3)^{3/2}}\}$$

where,
The Critical Points and the corresponding physical parameters for the interaction model $Q = 6\eta H\rho_m$ for power law potential are presented.

| Critical Points | $\Omega_m$ | $\Omega_\phi$ | $\omega_\phi$ | $\omega_{eff}$ | $q$ |
|----------------|------------|---------------|---------------|---------------|----|
| $A_1$          | 0          | 1             | $-1$          | $-1$          | $-1$ |
| $B_1$          | 0          | 1             | $-1$          | $-1$          | $-1$ |
| $C_1$          | 0          | 1             | $\frac{2}{3}^2 - 1$ | $\frac{2}{3}^2 - 1$ | $\frac{2}{3}^2 - 1$ |
| $D_1$          | 0          | 1             | $\frac{2}{3} - 1$ | $\frac{2}{3} - 1$ | $\frac{2}{3} - 1$ |
| $E_1$          | $\frac{1 - 2\eta(s^2 + 6\eta - 3)}{s^2}$ | $\frac{2\eta^2 + 3(2\eta - 1)^2}{2s^2}$ | $-\frac{2\eta^2 + 3(2\eta - 1)^2}{2s^2}$ | $-2\eta$ | $\frac{1}{2} - 3\eta$ |
| $F_1$          | $\frac{1 - 2\eta(s^2 + 6\eta - 3)}{s^2}$ | $\frac{2\eta^2 + 3(2\eta - 1)^2}{2s^2}$ | $-\frac{2\eta^2 + 3(2\eta - 1)^2}{2s^2}$ | $-2\eta$ | $\frac{1}{2} - 3\eta$ |

$\Delta_{E1} = \sqrt{s^4(4\eta(3\eta + s^2 - 3) + 3)^4(-216(2\eta - 1)^3 + 16\eta^2 s^6 - 24\eta(4(\eta - 3)\eta + 5)s^4 - 9(1 - 2\eta)^2 (60\eta^2 + 76\eta + 7)s^2)}$. The point $E_1$ is hyperbolic type critical point and represents stable solution for $0 < \eta < \frac{1}{2}$ and $(s < -\frac{1}{2} \sqrt{3} \sqrt{\frac{4\eta - 12\eta^2 + 1}{\eta}} \text{ or } s > \frac{1}{2} \sqrt{3} \sqrt{\frac{4\eta - 12\eta^2 + 1}{\eta}})$.

Eigenvalues of the point $F_1$:

$$\mu_1 = \sqrt{3} s \left(4\eta(3\eta + s^2 - 3) + 3 \sqrt{(4\eta(9\eta + s^2 - 3) - 3) - \Delta_{F1}} \right) \text{ and } \mu_2 = \sqrt{3} s \left(4\eta(3\eta + s^2 - 3) + 3 \sqrt{(4\eta(9\eta + s^2 - 3) - 3) + \Delta_{F1}} \right),$$

where,

$\Delta_{F1} = \sqrt{s^4(4\eta(3\eta + s^2 - 3) + 3)^4(-216(2\eta - 1)^3 + 16\eta^2 s^6 - 24\eta(4(\eta - 3)\eta + 5)s^4 - 9(1 - 2\eta)^2 (60\eta^2 + 76\eta + 7)s^2)}$. The point $F_1$ represents stable attractor for:

(i) $-\frac{1}{2} < \eta < 0$ and $\left(-\frac{1}{2} \sqrt{3} \sqrt{\frac{4\eta - 12\eta^2 - 1}{\eta}} < s < -\sqrt{3 - 6\eta} \text{ or } \sqrt{3 - 6\eta} < s < \frac{1}{2} \sqrt{3} \sqrt{\frac{4\eta - 12\eta^2 - 1}{\eta}} \right)$, or

(ii) $\eta = 0$ and $(s < -\sqrt{3} \text{ or } s > \sqrt{3})$, or

(iii) $0 < \eta < \frac{1}{2}$ and $\left(-\frac{1}{2} \sqrt{3} \sqrt{\frac{4\eta - 12\eta^2 - 1}{\eta}} < s < -\sqrt{3 - 6\eta} \text{ or } \sqrt{3 - 6\eta} < s < \frac{1}{2} \sqrt{3} \sqrt{\frac{4\eta - 12\eta^2 + 1}{\eta}} \right)$.

Therefore, in the interacting scenario, the point $F_1$ represents the scaling attractor at late times when DE behaves as any perfect fluid, while for uncoupled case ($\eta = 0$), DE behaves as dust and the point $F_1$ represents dust dominated decelerated stable solution (for $\eta = 0$, $s^2 > 3$). So, uncoupled model can predict the future decelerated scaling solution by the point $F_1$ which is shown in Fig.(3).

3.2. Interaction Model 2 : $Q \propto \rho_m$

Now, in this sub-section we shall discuss another interaction model in which the term $Q$ is chosen locally, i.e., not depending directly on Hubble parameter $H$ as [34, 35]

$$Q = \Gamma \rho_m,$$

where $\Gamma$ stands for a constant parameter which for $\Gamma > 0$ indicating the flow of energy from DE to DM and reverse for $\Gamma < 0$. Now, letting $\gamma = \frac{\Gamma}{\Omega_0}$ and introducing a new dynamical variable

$$r = \frac{H_0}{H + H_0}, \text{ where } H_0 \text{ is constant } (25)$$
FIG. 1: The figure shows stability of critical points in $x - y$ plane. In panel (a) trajectories shows that late time stable solution $C_1$ is attracted in quintessence era and $D_1$ is a past attractor for $\eta = 0.01$ and $s = -1$. On the other hand In (b) trajectories shows that late time stable solution $D_1$ is attracted in quintessence era and $C_1$ represents the past attractor for $\eta = 0.01$ and $s = 1$. Evolution of cosmological parameters for the critical points are in panel (c) for $\eta = 0.01$ and $s = -1$ showing that the late time accelerated solutions attracted in quintessence era.

the autonomous system (12) with $V(\phi) = V_0 e^{(\beta H_0 \phi)}$ and $\lambda(\phi) = \lambda_0 e^{(\alpha H_0 \phi)}$ leads to the following 4D autonomous system:

\[
\begin{align*}
\frac{dx}{dN} &= -3x - \frac{\alpha}{\sqrt{2}} \frac{x^2 r}{(1 - r)} - \frac{\beta}{\sqrt{2}} \frac{y^2 r}{z^2 (1 - r)} - \frac{\gamma}{2} \frac{r}{(1 - r)} \frac{(1 - x^2 z^2 - y^2)}{x z^2}, \\
\frac{dy}{dN} &= y \left[ \frac{\beta}{\sqrt{2}} \frac{r}{(1 - r)} x + \frac{3}{2} \frac{1 + x^2 z^2 - y^2}{1 - r} \right] , \\
\frac{dz}{dN} &= z \left[ \frac{\alpha}{\sqrt{2}} \frac{r}{(1 - r)} x + \frac{3}{2} \frac{1 + x^2 z^2 - y^2}{1 - r} \right] , \\
\frac{dr}{dN} &= \frac{3}{2} \frac{r}{(1 - r)} (1 + x^2 z^2 - y^2) ,
\end{align*}
\]

where $\frac{\lambda'}{\lambda H_0} = \alpha = \text{constant}$ and $\frac{V'}{V H_0} = \beta = \text{constant}$. Now, the system (26) has singularities at $x = 0, z = 0$ and $r = 1$ respectively. So, we multiply the right hand side (r.h.s) of (26) by $x z^2 (1 - r)$. Then the system (26) reduces to
FIG. 2: The figure shows stability of critical points in $x-y$ plane. In panel (a) trajectories shows that late time scaling solution $E_1$ is attracted in quintessence era for $\eta = 0.33, s = 2.8$. On the other hand In (b) trajectories shows that late time scaling solution $F_1$ is attracted in quintessence era for $\eta = 0.2, s = -1.89$. Evolution of cosmological parameters for the critical points are in panel (c) for $\eta = 0.33, s = 2.8$ showing that the late time accelerated solutions attracted in quintessence era.

FIG. 3: Trajectories shows that late time decelerated scaling solution $F_1$ is attracted in quintessence era for $\eta = 0, s = -1.89$. 
TABLE II: The existence of Critical Points and their corresponding physical parameters for the interaction model $Q^2 = \Gamma \rho_m$ for exponential potential.

| Critical Points | $(x, y, z, r)$ | $\Omega_m$ | $\Omega_\phi$ | $\omega_\phi$ | $\omega_{eff}$ | $q$ |
|-----------------|----------------|----------|-------------|-------------|-------------|-----|
| $A_2$           | $(0, y_c, z_c, 0)$ | $1 - y^2_c$ | $y^2_c$ | $-1$ | $-y^2_c$ | $1 - \frac{3\gamma^2}{2}$ |
| $B_2$           | $(0, 1, z_c, r_c)$ | $0$ | $1$ | $-1$ | $-1$ | $-1$ |
| $C_2$           | $(0, -1, z_c, r_c)$ | $0$ | $1$ | $-1$ | $-1$ | $-1$ |
| $D_2$           | $(x_c, y_c, 0, 0)$ | $1 - y^2_c$ | $y^2_c$ | $-1$ | $-y^2_c$ | $1 - \frac{3\gamma^2}{2}$ |
| $E_2$           | $(x_c, \frac{\gamma}{\sqrt{1 - \sqrt{2}x_c}}, 0, r_c)$ | $\frac{\gamma}{\sqrt{2}x_c - \gamma}$ | $\frac{\gamma}{\sqrt{2}x_c + \gamma}$ | $-1$ | $\frac{\gamma}{\sqrt{2}x_c - \gamma}$ | $\frac{\gamma}{\sqrt{2}x_c + 2\gamma}$ |
| $F_2$           | $(x_c, -\frac{\gamma}{\sqrt{1 - \sqrt{2}x_c}}, 0, r_c)$ | $\frac{\gamma}{\sqrt{2}x_c - \gamma}$ | $\frac{\gamma}{\sqrt{2}x_c + \gamma}$ | $-1$ | $\frac{\gamma}{\sqrt{2}x_c - \gamma}$ | $\frac{\gamma}{\sqrt{2}x_c + 2\gamma}$ |

We present the critical points and the corresponding physical parameters in the table (II). The following are the critical points for the system (27):

- I. Set of Critical Points: $A_2 = (0, y_c, z_c, 0)$
- II. Set of Critical Point: $B_2 = (0, 1, z_c, r_c)$
- III. Set of Critical Points: $C_2 = (0, -1, z_c, r_c)$
- IV. Set of Critical Points: $D_2 = (x_c, y_c, 0, 0)$
- V. Set of Critical: $E_2 = \left(x_c, \frac{\gamma}{\sqrt{1 - \sqrt{2}x_c}}, 0, r_c\right)$
- VI. Set of Critical Points: $F_2 = \left(x_c, -\frac{\gamma}{\sqrt{1 - \sqrt{2}x_c}}, 0, r_c\right)$

3.2.1. Phase space analysis for Interaction 2

We shall discuss the phase space analysis of sets of critical points arise from the 4D autonomous system.

- Set of critical points $A_2$ exists for all parameters $\alpha, \beta, \gamma$. The set corresponds to a scaling solution in phase space. Dark energy associated with the set behaves as cosmological constant ($\omega_\phi = -1$). Accelerated evolution of the universe ($\omega_{eff} \rightarrow -1$) is achieved by the set of points when it becomes DE dominated for $y_c \rightarrow 1$. On the other hand, for $y_c \rightarrow 0$ the set describe DM dominated decelerated universe. Evolution of the cosmological parameters are shown in Fig.(4) for the parameter values $\alpha = \beta = \gamma = 1$. Eigenvalues of perturbation matrix are $\{\mu_1 = 0, \mu_2 = 0, \mu_3 = 0, \mu_4 = 0\}$. As it is not appropriate to apply Hartman-Grobman theorem or Center Manifold Theory for the case of no non-zero eigenvalue, so stability analysis of the set of points $A_2$ is found to be complicated.

- Set of critical points $B_2$ and $C_2$ are same in all respects in the cosmological point of view. These sets represent completely potential energy of scalar field dominated solutions where DE behaves as cosmological constant ($\omega_\phi = -1$). There exists accelerating universe always near the set of critical points ($\omega_{eff} = -1, q = -1$).
FIG. 4: Evolution of energy density parameters for DE and DM, when initial conditions are chosen near the set of points $A_2$ for the parameter values $\alpha = \beta = \gamma = 1$.

Eigenvalues for the sets $B_2$ and $C_2$ are: \( \{ \mu_1 = -\frac{1}{\sqrt{2}}\beta r_c, \mu_2 = 0, \mu_3 = 0, \mu_4 = 0 \} \). These sets are non-hyperbolic and not normally hyperbolic sets in nature since there are three zero-eigenvalues at each point on the sets. To study the nature of the sets $B_2$ and $C_2$, we employ Center Manifold Theory which is presented in the next section 4.

- Set of critical points $D_2$ corresponds to a scaling solution behaves similarly to the set of points $A_2$ in the phase space. It is also a non-hyperbolic type set since the eigenvalues at the set of points are: \( \{ \mu_1 = 0, \mu_2 = 0, \mu_3 = 0, \mu_4 = 0 \} \). As we mentioned earlier that it is complicated to characterize the stability of these points since all associated eigenvalues are zero.

- Sets of critical points $E_2$ and $F_2$ are same in all respects. They correspond to DE-DM scaling solutions in the phase space. Depending on some parameter restrictions, the sets show the accelerating universe. Eigenvalues of perturbation matrix at the set of points $E_2$ and $F_2$ are: \( \{ \mu_1 = \frac{\sqrt{2}\beta r_c}{2(\sqrt{2}\beta r_c - \gamma)}, \mu_2 = 0, \mu_3 = 0, \mu_4 = 0 \} \) which indicates the sets are non-hyperbolic in nature and to characterize these points linear stability theory or Hartman-Grobman theorem is not appropriate to apply. So we apply Center Manifold Theory in the next section (4) to analyze non-hyperbolic critical points.
4. CENTER MANIFOLD THEORY

Mathematical background

We follow Perko [37] & [38], for discussing the mathematical background of the center manifold theory. When Jacobian matrix corresponding to the given autonomous system at the critical point has zero eigenvalue(s), linear theory fails to provide information on the stability of that critical point. In this case, use of centre manifold is interested because it reduce the dimension of the system near that critical point so that stability of the reduced system can be investigated. There always exists an invariant local centre manifold $W^c$ passing through the fixed point to which the system could be restricted to study its behaviour in the neighbourhood of the fixed point. The stability of the reduced system determines the stability of the system at that point.

Let $u \in \mathbb{R}^c$ and $v \in \mathbb{R}^s$. An arbitrary dynamical system with zero eigenvalues in the Jacobian matrix can always be written in the following form

\[
\begin{align*}
\dot{u} &= Au + f(u, v), \\
\dot{v} &= Bu + g(u, v),
\end{align*}
\]

where

\[
\begin{align*}
f(0, 0) &= 0, \quad Df(0, 0) = 0, \\
g(0, 0) &= 0, \quad Dg(0, 0) = 0.
\end{align*}
\]

For $\delta$ sufficiently small, the centre manifold for the system (28) is defined by the space

\[ W^c(0) = \{(u, v) \in \mathbb{R}^c \times \mathbb{R}^s | v = h(u), |u| < \delta, h(0) = 0, Dh(0) = 0\} \]

for all $u \in \mathbb{R}^c$ with $|u| < \delta$.

Next, we need to construct this centre manifold explicitly. By differentiating the defining equation $v = h(u)$ with respect to the independent variable, we get $\dot{v} = Dh(u)\dot{u}$ where we used the chain rule. Eliminating $\dot{u}$ and $\dot{v}$ via (28), one arrives and the following quasilinear partial differential equation which $h$ has to satisfy

\[
N(h(u)) = Dh(u)[Au + f(u, h(u))] - Bh(u) - g(u, h(u)) = 0,
\]

and the flow on the center manifold $W^c(0)$ is defined by the system of differential equations

\[
\dot{u} = Cu + f(u, h(u))
\]

for all $u \in \mathbb{R}^c$ with $|u| < \delta$.

4.1. Stability Analysis for non-hyperbolic critical points of Interaction model 1

4.1.1. Critical point $A_1$

The Jacobian matrix at the critical point $A_1$ corresponding to the autonomous system (23) can be put as

\[
J(A_1) = \begin{bmatrix}
-\sqrt{\frac{s}{2}} & s \\
0 & 0
\end{bmatrix}.
\]
The eigenvalues of $J(A_1)$ are $-\sqrt{\frac{2}{s}}$ and 0 and the corresponding eigenvectors are $[1,0]^T$ and $[2\sqrt{\frac{2}{s}},1]^T$ respectively. Since the critical point $A_1$ is non-hyperbolic in nature, so we use center manifold theory for analyzing the stability of this critical point. First we take the shifting transformation $x = X, y = Y + 1$ so that the critical point $A_1$ moves to the origin. From the entries of the Jacobian matrix we can see that there is a linear term of $y$ corresponding to the first equation of the autonomous system (23). But the eigen value 0 of the Jacobian matrix (33) is corresponding to second equation of the autonomous system (23). So we have to introduce another coordinate system $(x_t, y_t)$ in terms of $(X, Y)$. By using the eigenvectors of the Jacobian matrix (33), we introduce the following coordinate system

$$
\begin{bmatrix}
x_t \\
y_t
\end{bmatrix} = \begin{bmatrix} 1 & -2\sqrt{\frac{2}{s}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}
$$

(34)

and in these new coordinates the equations in the autonomous system (23) are transformed into

$$
\begin{bmatrix}
x_t' \\
y_t'
\end{bmatrix} = \begin{bmatrix} -\sqrt{\frac{2}{s}}s & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} \text{non linear terms} \\
0 
\end{bmatrix}.
$$

(35)

By center manifold theory there exists a continuously differentiable function $h: \mathbb{R} \to \mathbb{R}$ such that

$$x_t = h(y_t) = a_1 y_t^2 + a_2 y_t^3 + \text{higher order terms}
$$

(36)

Differentiating both side of (36) with respect to $N$, then we get

$$\frac{dx_t}{dN} = (2a_1 y_t + 3a_2 y_t^2) \frac{dy_t}{dN}
$$

(37)

By inserting the expression of $y_t'$ from (35) and then also by inserting (36) and comparing the coefficient of lowest order terms, we have $a_1 = -\sqrt{\frac{2}{s}} \left( 5 + 24 \frac{s^2}{s^3} \right)$. We only concern about the non-zero coefficients of the lowest power terms in center manifold theory as we analyze arbitrary small neighbourhood of the origin, so it is not needed to determine the another coefficients. Hence, the expression of the center manifold can be written as

$$x_t = -\sqrt{\frac{2}{s}} \left( 5 + 24 \frac{s^2}{s^3} \right) y_t^2 + \mathcal{O}(y_t^3)
$$

(38)

This implies that one dimensional center manifold lies on the $x_t y_t$ plane and tangent to the center subspace $y_t$ axis at the origin. The flow on the center manifold near the origin is determined by

$$\frac{dy_t}{dN} = \begin{cases} 
6\sqrt{\frac{2}{s}}(2\eta-1)y_t^2 + \mathcal{O}(y_t^3), & \text{for } \eta \neq \frac{1}{2} \\
\frac{6\sqrt{2}}{s^2}(1-\frac{7}{s^2}) y_t^3 + \mathcal{O}(y_t^4), & \text{for } \eta = \frac{1}{2}
\end{cases}
$$

(39)

For $\eta \neq \frac{1}{2}$, we see that the degree of the lowest order term is even which implies that for all possible choices of $s$ and $\eta$ except $\eta = \frac{1}{2}$ the vector field near the origin is unstable [39] due to saddle node nature of the origin (see figure 5, 6, 7 and 8).

For $\eta = \frac{1}{2}$, the stability of the vector field depends on the value of $s$. For $0 < s < 2\sqrt{\frac{2}{7}}$ and $-2\sqrt{\frac{7}{2}} < s < 0$ the vector field near the origin is unstable due to saddle node nature of the origin (see figure 9(a) and 10(b)), the origin is an unstable node while $s < -2\sqrt{\frac{7}{2}}$ (see figure 9(a)) and for $s > 2\sqrt{\frac{7}{2}}$ the vector field near the origin is stable due to stable node nature of the origin (see figure 10(a)). As the new coordinate system $(x_t, y_t)$ is topologically equivalent to the original coordinate system $(x, y)$, so we conclude that in the original coordinate system the behaviour of the critical point $A_1$ is same as the behaviour of the origin in the new coordinate system.

4.1.2. Critical point $B_1$

The Jacobian matrix at the critical point $B_1$ corresponding to the autonomous system (23) can be put as

$$J(B_1) = \begin{bmatrix} \sqrt{\frac{2}{s}} & -6\eta \\ 0 & 0 \end{bmatrix}.
$$

(40)
The eigenvalues of $J(A_1)$ are $\sqrt{\frac{2}{3}} s$ and 0 and the corresponding eigenvectors are $[1, 0]^T$ and $[2\sqrt{\frac{2}{3}}, 1]^T$ respectively. Similar as $A_1$ first we take the shifting transformation $x = X, y = Y - 1$ such that the critical point $A_2$ moves to the origin. Then we introduce the matrix transformation (34) and due to this transformation the Jacobian matrix $J(B_1)$ converts into its diagonal form. Proceeding in similar way as above, after putting the corresponding arguments, the center manifold can be expressed as

$$x_t = \sqrt{\frac{6}{s}} \left( 5 + 24 \frac{n^2}{s^2} \right) y_t^2 + \mathcal{O}(y_t^3)$$

and the flow on the center manifold is determined by

$$\frac{dy_t}{dN} = \begin{cases} 
6\sqrt{\frac{6}{s}} (2\eta - 1) y_t^2 + \mathcal{O}(y_t^3), & \text{for } \eta \neq \frac{1}{2} \\
\frac{6\sqrt{\frac{6}{s}}}{s} y_t^2 + \mathcal{O}(y_t^3), & \text{for } \eta = \frac{1}{2}
\end{cases}$$

Similarly, due to present of even degree lowest order term for $\eta \neq \frac{1}{2}$, we conclude that the origin is a saddle node for all possible choices $\eta$ and $s$. For $\eta > \frac{1}{2}, s > 0$ the phase plot is same as FIG.8(a), for $\eta < 0, s < 0$ the phase plot is same as FIG.6(b) and for $0 < \eta < \frac{1}{2}, s > 0$ the phase plot is same as FIG.7. Again, for $0 < \eta < \frac{1}{2}, s < 0$ the phase
FIG. 7: Vector field near the origin for the critical point $A_1$. The phase plot is drawn for $\eta > \frac{1}{2}$ and $s < 0$.

FIG. 8: Vector field near the origin for the critical point $A_1$. The phase plot (a) is for $0 < \eta < \frac{1}{2}, s < 0$ and (b) is for $\eta < 0, s < 0$.

plot is same as FIG.5, for $\eta < 0, s > 0$ the phase plot is same as FIG.8(b) and for $\eta < 0, s > 0$ the phase plot is same as FIG.6(a). For $\eta = \frac{1}{2}$, the origin is an unstable node (see figure 9(b)) for $s > 0$ and for $s < 0$ origin is a stable node (see figure 10(a)).
FIG. 9: Vector field near the origin for the critical point $A_1$. The phase plot (a) is for $\eta = 1/2, 0 < s < 2\sqrt{\frac{3}{7}}$ and (b) is for $\eta = 1/2, s < -2\sqrt{\frac{3}{7}}$.

FIG. 10: Vector field near the origin for the critical point $A_1$. The phase plot (a) is for $\eta = 1/2, s > 2\sqrt{\frac{3}{7}}$ and (b) is for $\eta = 1/2, -2\sqrt{\frac{3}{7}} < s < 0$.

4.2. Stability analysis for non-hyperbolic sets of Interaction model 2

4.2.1. Set of critical points $B_2$

The Jacobian matrix at the critical point $B_2$ corresponding to the autonomous system (27) can be put as

$$J(B_2) = \begin{bmatrix} -\frac{1}{\sqrt{2}} r_c \beta & \gamma r_c & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (43)$$
The eigenvalues of \( J(B_2) \) are \(-\frac{1}{\sqrt{2}}r_c\beta, 0, 0 \) and 0. \([1, 0, 0, 0]^T\) be the eigenvector corresponding to the eigenvalue \(-\frac{1}{\sqrt{2}}r_c\beta \) and \([\frac{\sqrt{2}}{\beta}, 1, 0, 0]^T\), \([0, 0, 1, 0]^T\) and \([0, 0, 0, 1]^T\) are the eigenvectors corresponding to the eigenvalue 0. For a fixed \( z_c \) and \( r_c \), first we take the shifting transformation \( x = X, y = Y + 1, z = Z + z_c, r = R + r_c \) so that the critical point \( B_2 \) moves to the origin. Now we introduce another coordinate system \((x_t, y_t, z_t, r_t)\) in terms of \((X, Y, Z, R)\) so that \( J(B_2) \) modifies to its diagonal form. By using the eigenvectors of \( J(B_2) \), we introduce the following coordinate system

\[
\begin{bmatrix}
 x_t \\
 y_t \\
 z_t \\
 r_t
\end{bmatrix} = \begin{bmatrix}
 1 - \frac{\sqrt{2}}{\beta} & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
 X \\
 Y \\
 Z \\
 R
\end{bmatrix}
\] (44)

and in this new coordinate system the autonomous system (27) is transformed into

\[
\begin{bmatrix}
 x'_t \\
 y'_t \\
 z'_t \\
 r'_t
\end{bmatrix} = \begin{bmatrix}
 -\frac{1}{\sqrt{2}}r_c\beta & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
 x_t \\
 y_t \\
 z_t \\
 r_t
\end{bmatrix} + \begin{bmatrix}
 \text{non} \\
 \text{lin} \\
 \text{ear} \\
 \text{terms}
\end{bmatrix}
\] (45)

By center manifold theory there exists a continuously differentiable function \( h: \mathbb{R}^3 \rightarrow \mathbb{R} \) such that

\[
x_t = h(y_t, z_t, r_t) = a_1 y_t^2 + a_2 z_t^2 + a_3 r_t^2 + a_4 y_t z_t + a_5 y_t r_t + a_6 z_t r_t + \text{higher order terms}
\] (46)

Differentiating both side of (46) with respect to \( N \), then we get

\[
\frac{dx_t}{dN} = \begin{bmatrix}
 2a_1 y_t + a_4 z_t + a_5 r_t \\
 2a_2 z_t + a_4 y_t + a_6 r_t \\
 2a_3 r_t + a_5 y_t + a_6 z_t
\end{bmatrix} \begin{bmatrix}
 y_t \\
 z_t \\
 r_t
\end{bmatrix} + \begin{bmatrix}
 \frac{dy_t}{dN} \\
 \frac{dz_t}{dN} \\
 \frac{dr_t}{dN}
\end{bmatrix}
\] (47)

By inserting the expression of \( y'_t, z'_t \) and \( r'_t \) from (45) and then also by inserting (46) and comparing the coefficient of lowest order terms, we have \( a_1 = -\frac{\sqrt{2}}{\beta} \left( \frac{3}{2} \right) \gamma + \frac{\gamma^2}{\beta^2} \), \( a_2 = a_3 = a_4 = a_5 = a_6 = 0 \). Hence, the center manifold (up to second order) can be defined as

\[
x_t = -\frac{\sqrt{2}}{\beta} \left( \frac{3}{2} \gamma + \frac{\gamma^2}{\beta^2} \right) y_t^2 + \text{higher order terms}
\] (48)

and the flow on the center manifold (up to second order) is obtained by

\[
y'_t = -\frac{3\sqrt{2}\gamma z_c}{\beta} \left( 1 - r_c \left( 1 + \frac{\gamma}{3} \right) \right) y_t^2 + \text{higher order terms},
\] (49)

\[
z'_t = -\frac{3\sqrt{2}\gamma z_c^3}{\beta} \left( 1 - r_c \left( 1 + \frac{\alpha \gamma}{3\beta} \right) \right) y_t^2 + \text{higher order terms},
\] (50)

\[
r'_t = -\frac{3\sqrt{2}\gamma r_c z^2}{\beta} \left( 1 - r_c \left( 2 - \frac{r_c}{\beta} \right) \right) y_t^2 + \text{higher order terms}.
\] (51)

To determine the direction of the vector field near the origin we only consider the flow equation (49) as the lowest order term of the expression of center manifold depends on \( y_t \) and the lowest order term of the R.H.S. of Eqs. (50) and (51) depends only on \( y_t \). Since, for \( r_c \neq \frac{1}{1+\frac{\gamma}{3}} \) the degree of the lowest order term of the R.H.S. of (49) is even, so we can conclude that the origin is a saddle node and unstable in nature while \( r_c \neq \frac{1}{1+\frac{\gamma}{3}} \). If the coefficient of \( y_t^2 \) in the R.H.S. of (49) is positive then the vector field near the origin is same as FIG.6(b) for \( r_c\beta > 0 \) and the vector field near the origin is same as FIG.11(b) for \( r_c\beta < 0 \). Again if the coefficient of \( y_t^2 \) in the R.H.S. of (49) is negative then the vector field near the origin is same as FIG.11(a) for \( r_c\beta > 0 \) and the vector field near the origin is same as FIG.8(b) for \( r_c\beta < 0 \). Hence, we conclude that in old coordinate system \((x, y, z, r)\), the critical point \( B_2 \) is unstable due to its saddle node nature. Otherwise, that means for \( r_c = \frac{1}{1+\frac{\gamma}{3}} \) the critical point \( B_2 \) will be stable or unstable which depends on the sign of the coefficients of the lowest order term of (49).
4.2.2. Set of critical points $C_2$

The Jacobian matrix at the critical point $C_2$ corresponding to the autonomous system (27) can be put as

$$
J(C_2) = \begin{bmatrix}
-\frac{1}{\sqrt{2}}r_c\beta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
$$  \hfill (52)

The eigenvalues of $J(C_2)$ are $-\frac{1}{\sqrt{2}}r_c\beta$, 0, 0 and 0. $[1, 0, 0, 0]^T$ be the eigenvector corresponding to the eigenvalue $-\frac{1}{\sqrt{2}}r_c\beta$ and $\left[\frac{-\sqrt{2}\gamma}{\beta}, 1, 0, 0\right]^T$, $[0, 0, 1, 0]^T$ and $[0, 0, 0, 1]^T$ are the eigenvectors corresponding to the eigenvalue 0. For a fixed $z_c$ and $r_c$, first we take the shifting transformation $x = X, y = Y - 1, z = Z + z_c, r = R + r_c$ so that the critical point $C_2$ moves to the origin. Now we introduce another coordinate system $(x_t, y_t, z_t, r_t)$ in terms of $(X, Y, Z, R)$ so that $J(C_2)$ modifies to its diagonal form. By using the eigenvectors of $J(C_2)$, we introduce the following coordinate system

$$
\begin{bmatrix}
x_t \\
y_t \\
z_t \\
r_t
\end{bmatrix} = 
\begin{bmatrix}
1 & \frac{-\sqrt{2}\gamma}{\beta} & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} 
\begin{bmatrix}
X \\
Y \\
Z \\
R
\end{bmatrix}.
$$  \hfill (53)

and in these new coordinates the autonomous system (27) is transformed into

$$
\begin{bmatrix}
x_t' \\
y_t' \\
z_t' \\
r_t'
\end{bmatrix} = 
\begin{bmatrix}
-\frac{1}{\sqrt{2}}r_c\beta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} 
\begin{bmatrix}
x_t \\
y_t \\
z_t \\
r_t
\end{bmatrix} + 
\begin{bmatrix}
\text{non} \\
\text{lin} \\
\text{ear} \\
\text{terms}
\end{bmatrix}.
$$  \hfill (54)
By using similar arguments which we have already mentioned in the stability analysis of $B_2$, the center manifold (up to second order) can be defined as

$$x_t = -\frac{\sqrt{2}}{\beta} \left( \frac{3}{2} \gamma + \frac{\gamma^2 z^2}{\beta^2} \right) y_t^2 + \text{higher order terms}$$  \hspace{1cm} (55)$$

and the flow on the center manifold (up to second order) is obtained by

$$y_t' = \frac{3\sqrt{2} \gamma z^2}{\beta} \left( 1 - r_c \left( 1 + \frac{\gamma}{3} \right) \right) y_t^2 + \text{higher order terms},$$  \hspace{1cm} (56)$$

$$z_t' = -\frac{3\sqrt{2} \gamma z^3}{\beta} \left( 1 - r_c \left( 1 + \frac{\alpha \gamma}{3 \beta} \right) \right) y_t^2 + \text{higher order terms},$$  \hspace{1cm} (57)$$

$$r_t' = -\frac{3\sqrt{2} \gamma z c^2}{\beta} \left( 1 - r_c \left( 2 - \frac{r_c}{\beta} \right) \right) y_t^2 + \text{higher order terms}.$$  \hspace{1cm} (58)$$

In this case also we see that for $r_c = \frac{1}{1 + \frac{\beta}{\alpha}}$, the degree of the lowest order term in the R.H.S. of (56) is even, it follows that for $r_c = \frac{1}{1 + \frac{\beta}{\alpha}}$ the origin in the new coordinate system, that means., the critical point $C_2$ in the original coordinate system is a saddle node and unstable in nature. Otherwise, the critical point $C_2$ will be stable or unstable which depends on the sign of the coefficients of the lowest order term of (56). For this critical point, the dynamics on the vector field near the origin will be same as above drawn figures.

### 4.2.3. Set of critical points $E_2$

The Jacobian matrix at the critical point $E_2$ corresponding to the autonomous system (27) can be put as

$$J(E_2) = \begin{bmatrix} \frac{\sqrt{2} \beta \gamma r_c}{2(\sqrt{2} \beta x_c - \gamma)} & r_c \sqrt{\gamma} - \sqrt{2} \beta x_c & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \hspace{1cm} (59)$$

The eigenvalues of $J(E_2)$ are \[ \frac{\sqrt{2} \beta \gamma r_c}{2(\sqrt{2} \beta x_c - \gamma)}, 0, 0 \text{ and } 0. \] $[1, 0, 0, 0]^T$ be the eigenvector corresponding to the eigenvalue \[ \frac{\sqrt{2} \beta \gamma r_c}{2(\sqrt{2} \beta x_c - \gamma)} \] and \[ \frac{\sqrt{2} (\gamma - \sqrt{2} \beta x_c)^2}{\beta \sqrt{\gamma}}, 1, 0, 0]^T \] and $[0, 0, 0, 1]^T$ are the eigenvectors corresponding to the eigenvalue 0. For a fixed $x_c$ and $r_c$, first we take the shifting transformation $x = X + x_c, y = Y + \frac{\sqrt{2} \beta \gamma r_c}{2(\sqrt{2} \beta x_c - \gamma)}, z = Z, r = R + r_c$ so that the critical point $E_2$ moves to the origin. Now we introduce another coordinate system $(x_t, y_t, z_t, r_t)$ in terms of $(X, Y, Z, R)$ so that $J(E_2)$ modifies to its diagonal form. By using the eigenvectors of $J(E_2)$, we introduce the following coordinate system

$$\begin{bmatrix} x_t \\ y_t \\ z_t \\ r_t \end{bmatrix} = \begin{bmatrix} 1 - \frac{\sqrt{2} (\gamma - \sqrt{2} \beta x_c)^2}{\beta \sqrt{\gamma}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ R \end{bmatrix} \hspace{1cm} (60)$$

and corresponding to these coordinate system the autonomous system (27) is transformed into

$$\begin{bmatrix} x_t' \\ y_t' \\ z_t' \\ r_t' \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2} \beta \gamma r_c}{2(\sqrt{2} \beta x_c - \gamma)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \\ r_t \end{bmatrix} + \begin{bmatrix} \text{non} \text{lin} \text{ear} \text{ terms} \end{bmatrix} \hspace{1cm} (61)$$
By center manifold theory there exists a continuously differentiable function \( h: \mathbb{R}^3 \to \mathbb{R} \) such that
\[
x_t = h(y_t, z_t, r_t) = a_1 y_t^2 + a_2 z_t^2 + a_3 r_t^2 + a_4 y_t z_t + a_5 y_t r_t + a_6 z_t r_t + \text{higher order terms}
\] (62)

Differentiating both side of (62) with respect to \( N \), we get
\[
\frac{dx_t}{dN} = \begin{bmatrix} \frac{dy_t}{dN} \\ \frac{dz_t}{dN} \\ \frac{dr_t}{dN} \end{bmatrix} = \begin{bmatrix} 2a_1 y_t + a_4 z_t + a_5 r_t \\ 2a_2 z_t + a_4 y_t + a_6 r_t \\ 2a_3 r_t + a_5 y_t + a_6 z_t \end{bmatrix}
\] (63)

By inserting the expression of \( y_t', z_t' \) and \( r_t' \) from (45) and then also by inserting (61) and comparing the coefficient of lowest order terms, we have
\[
a_1 = -\frac{3}{\sqrt{2\beta\gamma}}(\gamma - \sqrt{2}\beta x_c)^2, a_2 = \frac{x_c^2}{\sqrt{2\beta\gamma}}(\gamma + \sqrt{2}(\alpha - 2\beta)x_c)(\sqrt{2}\beta x_c - \gamma), a_3 = a_4 = a_6 = 0, a_5 = \frac{4x_c}{r_c\sqrt{\gamma}} \sqrt{\gamma - \sqrt{2}\beta x_c}.
\]

Hence, the center manifold (up to second order) can be written as
\[
x_t = a_1 y_t^2 + a_2 z_t^2 + a_5 y_t r_t + \text{higher order terms}
\] (64)

and the flow on the center manifold (up to second order) is obtained by
\[
y_t' = \left\{ (1 - \gamma)(1 - r_c) \frac{3\sqrt{\gamma x_c}}{2(\gamma - \sqrt{2}\beta x_c)^{3/2}} \right\} z_t^2 + \text{higher order terms},
\] (65)
\[
z_t' = \left\{ \frac{\alpha r_c x_c^2}{\sqrt{2}} + \frac{3\beta x_c^2}{\sqrt{2}(\gamma - \sqrt{2}\beta x_c)} (r_c - 1) \right\} z_t^3 + \text{higher order terms},
\] (66)
\[
r_t' = -\frac{3r_c x_c^2 \beta}{\sqrt{2}} (1 - r_c)^2 z_t^2 + \text{higher order terms}.
\] (67)

As the R.H.S. of the expression of the center manifold depends on three coordinates \( y_t, z_t \) and \( r_t \), so here we can only determine the dynamics on the center manifold near the origin by using the flow equations (65), (66) and (67). For a meaningful choices of \( x_c, \beta, \gamma, r_c, \alpha, r \), the dynamics in the 3D coordinate system near the origin is shown as in FIG.12.

![FIG. 12: Dynamics on the center manifold is given locally (up to second order) by (65), (66) and (67). For existence the domain of critical point and to satisfy the condition (20) we have drawn this 3-dimensional phase plot for \( x_c = -1, \beta = 1, \gamma = 2, r_c = 2, \alpha = 1 \), which conclude that the origin origin is unstable in nature.](image-url)
4.2.4. Set of critical points $F_2$

The Jacobian matrix at the critical point $F_2$ corresponding to the autonomous system (27) can be put as

$$J(F_2) = \begin{bmatrix} \frac{\sqrt{\gamma} \beta r_c}{2(\sqrt{\gamma} \beta x_c - \gamma)} - r_c \sqrt{\gamma} \sqrt{\gamma} - \sqrt{2} \beta x_c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  \hspace{1cm} (68)

The eigenvalues of $J(F_2)$ are $\frac{\sqrt{\gamma} \beta r_c}{2(\sqrt{\gamma} \beta x_c - \gamma)}$, 0, 0 and 0. $[1, 0, 0, 0]^T$ be the eigenvector corresponding to the eigenvalue $\frac{\sqrt{\gamma} \beta r_c}{2(\sqrt{\gamma} \beta x_c - \gamma)}$ and $[\frac{1}{\sqrt{2} \beta x_c}, 1, 0, 0]^T$, $[0, 0, 1, 0]^T$ and $[0, 0, 0, 1]^T$ are the eigenvectors corresponding to the eigenvalue 0. For a fixed $x_c$ and $r_c$, first we take the shifting transformation $x = X + x_c, y = Y - \frac{\sqrt{\gamma} \beta r_c}{2(\sqrt{\gamma} \beta x_c - \gamma)}$, $z = Z, r = R + r_c$ so that the critical point $E_2$ moves to the origin. Now we introduce another coordinate system $(x_t, y_t, z_t, r_t)$ in terms of $(X, Y, Z, R)$ so that $J(F_2)$ converts to its diagonal form. By using the eigenvectors of $J(F_2)$, we introduce the following coordinate system

$$\begin{bmatrix} x_t \\ y_t \\ z_t \\ r_t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ R \end{bmatrix}$$  \hspace{1cm} (69)

and corresponding to these coordinate system the autonomous system (27) is transformed into

$$\begin{bmatrix} x_t' \\ y_t' \\ z_t' \\ r_t' \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2} \beta x_c}{2(\sqrt{\gamma} \beta x_c - \gamma)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \\ z_t \\ r_t \end{bmatrix} + \begin{bmatrix} \text{non} \\ \text{lin} \\ \text{ear} \\ \text{terms} \end{bmatrix}.$$  \hspace{1cm} (70)

By center manifold theory there exists a continuously differentiable function $h: \mathbb{R}^3 \rightarrow \mathbb{R}$ such that

$$x_t = h(y_t, z_t, r_t) = a_1 y_t^2 + a_2 z_t^2 + a_3 r_t^2 + a_4 y_t z_t + a_5 y_t r_t + a_6 z_t r_t + \text{higher order terms}.$$  \hspace{1cm} (71)

Differentiating both side of (71) with respect to $N$, we get

$$\frac{dx_t}{dN} = \begin{bmatrix} 2a_1 y_t + a_4 z_t + a_5 r_t \\ 2a_2 z_t + a_4 y_t + a_6 r_t \\ 2a_3 r_t + a_5 y_t + a_6 z_t \end{bmatrix}$$  \hspace{1cm} (72)

By inserting the expression of $y_t'$, $z_t'$ and $r_t'$ from (69) and then also by inserting (71) and comparing the coefficient of lowest order terms, we have $a_1 = \frac{3}{\sqrt{2} \beta x_c} (\gamma - \sqrt{2} \beta x_c)^2$, $a_2 = -\frac{3}{\sqrt{2} \beta x_c} (\gamma + \sqrt{2} (\alpha - 2 \beta) x_c)(\sqrt{2} \gamma - \sqrt{2} \beta x_c)$, $a_3 = a_4 = a_5 = a_6 = 0$. Hence, the center manifold (up to second order) can be written as

$$x_t = a_1 y_t^2 + a_2 z_t^2 + \text{higher order terms}$$  \hspace{1cm} (73)

and the flow on the center manifold (up to second order) is obtained by

$$y_t' = -\left\{ (1 - \gamma)(1 - c) - \frac{3 \sqrt{\gamma} x_c}{2(\gamma - \sqrt{2} \beta x_c)^2} - \frac{\beta \gamma r_c x_c^2}{2 \sqrt{\gamma} - \sqrt{2} \beta x_c} \right\} z_t^2 + \text{higher order terms},$$  \hspace{1cm} (74)

$$z_t' = \left\{ \frac{\alpha r_c x_c^2}{\sqrt{2}} + \frac{3 \beta x_c^2}{\sqrt{2} (\gamma - \sqrt{2} \beta x_c)} (r_c - 1) \right\} z_t^3 + \text{higher order terms},$$  \hspace{1cm} (75)

$$r_t' = -\frac{3 \beta x_c^2}{\sqrt{2} (1 - r_c)^2} z_t^2 + \text{higher order terms}.$$  \hspace{1cm} (76)
The plot of the center manifolds for \( r_t = 0 \), that means., in \( x_t y_t z_t \) coordinate system are shown as in FIG. 13 and FIG. 14 for several choices of \( x_c, \beta, \gamma, r_c \) and \( \alpha \). The projection of the vector field in arbitrary small neighbourhood of the origin on \( y_t z_t \) plane corresponding to the same choices of \( x_c, \beta, \gamma, r_c \) and \( \alpha \) are also shown as in FIG. 13 and FIG. 14.

FIG. 13: Left hand side figure is the plot of the center manifold corresponding to the critical point \( F_2 \) and the projection of the vector field in arbitrary small neighbourhood of the origin on \( y_t z_t \) plane is shown as at right hand side. For existence the domain of critical point and to satisfy the condition (20) we have drawn this 3-dimensional phase plot for \( x_c = -1, \beta = 1, \gamma = 2, r_c = 2, \alpha = 1 \). From the plot of the vector field we conclude that the vector field is unstable about the \( y_t \) axis.

FIG. 14: Left hand side figure is the plot of the center manifold corresponding to the critical point \( F_2 \) and the projection of the vector field in arbitrary small neighbourhood of the origin on \( y_t z_t \) plane is shown as at right hand side. For existence the domain of critical point and to satisfy the condition (20) we have drawn the center manifold and the phase plot for \( x_c = 1, \beta = -1, \gamma = 2, r_c = 2, \alpha = 1 \). From the plot of the vector field we conclude that for this case also the vector field is unstable about \( y_t \) axis.
5. **COSMOLOGICAL IMPLICATIONS**

5.1. Interaction Model 1

The critical points $A_1$ and $B_1$ represent the universe dominated by potential energy of the scalar field (DM is absent here since $\Omega_m = 0$), where scalar field described by cosmological constant like fluid ($\omega_{eff} = -1$) and effective equation of state is $\omega_{eff} = -1$. The critical points with one vanishing eigenvalue are non-hyperbolic type. Linear stability theory fails to show the exact nature of the system near the points. However, there exists a one dimensional stable manifold for the point $A_1$ when $s > 0$. We employ Center Manifold Theory (CMT) to understand the dynamical behavior after deriving the evolution equations on the center manifold which is shown in the last section. The flow of the Center manifold regarding the point $A_1$ is clearly shown in various figures. Thus, the accelerated de Sitter like solution represented by $A_1$ (with $\omega_{eff} = -1$, $\Omega_m = 0$, $\Omega_\phi = 1$) is a source (past attractor) for $\eta = \frac{1}{2}$, $s > -2\sqrt{\frac{3}{7}}$ (see in Fig. 9(b)) and is a stable attractor for $\eta = \frac{1}{2}$, $s > 2\sqrt{\frac{3}{7}}$ (see in Fig. 10(a)), otherwise it represents the transient era (saddle like solution) in the evolution of the universe (section 4 is being referred for detail analysis). In a similar manner we find the condition for which the point $B_1$ has one dimensional stable manifold is that $s < 0$, and the points describe the de Sitter expansion of the universe.

Scalar field dominated solutions $C_1$ and $D_1$ correspond to the late time stable attractors in quintessence era with $\Omega_\phi = 1$, so coincidence problem cannot be alleviated by these points. The evolution of late time accelerated solution $C_1$ is attracted in quintessence era either for ($\eta \leq \frac{1}{6}$ and $-\sqrt{2} < s < 0$) or for ($\frac{1}{6} < \eta < \frac{1}{2}$ and $-\sqrt{3 - 6\eta} < s < 0$), while in this parameter region, the point $D_1$ represents the past attractor corresponding to early accelerated phase of the universe. On the other hand, the critical point $D_1$ corresponds to a late time accelerated stable attractor either for ($\eta \leq \frac{1}{6}$ and $0 < s < \sqrt{2}$) or ($\frac{1}{6} < \eta < \frac{1}{2}$ and $0 < s < \sqrt{3 - 6\eta}$), but in the parameter region the point $C_1$ is a past attractor corresponding to an early accelerated universe. Therefore, the late time accelerated evolutions of the universe attracted in quintessence era (where points are future attractors) as well as early time accelerated era of the universe (where the points are past attractor) are achieved by the points $C_1$ and $D_1$. In Fig.1(a), for $\eta = 0.01, s = -1$, the point $C_1$ shows accelerated late time stable attractor in quintessence era whereas the point $D_1$ represents the accelerated past attractor. In Fig.1(b)), the point $D_1$ represents late time accelerated stable attractor, while the point $C_1$ is an accelerated past attractor for $\eta = 0.01, s = 1$. Fig.1(c) refers to the evolution of cosmological parameters for $\eta = 0.01$ and $s = -1$ showing that the evolution of the accelerated universe is attracted at late times in the quintessence era.

Critical points $E_1$ and $F_1$ represent DM-DE scaling solutions with $0 < \Omega_\phi < 1$ in the phase plane therefore, the points can alleviate the coincidence problem. The points are hyperbolic type, so linear stability theory is sufficient to describe the nature of the points. For $\eta = \frac{1}{2}$, the points $E_1$ and $F_1$ describe the de Sitter expansions of the universe ($\Omega_\phi = 1, \Omega_m = 0, \omega_{eff} = -1$) but both the points are unstable here. For $\eta = 0$, the point $F_1$ can predict the future decelerated dust dominated universe as for $\eta = 0$, the physical parameters are: $\omega_{eff} = 0$, and DE behaves as dust $\omega_\phi = 0$ and the point is stable for $s^2 > 3$. Fig. (3) shows that the dust dominated decelerated scaling attractor is represented by the point $F_1$ for $\eta = 0, s = -1.89$. It is worth mentioning that the points $E_1$ and $F_1$ show physically interesting nature when coupling of interaction constrained as $\eta < \frac{1}{2}$. Despite this, depending on other parameter restrictions, both the points correspond to late time accelerated scaling attractors with ratio of energy densities of DE and DM being unity which can alleviate the coincidence problem. Fig.(2(a)) shows for the parameters $\eta = 0.33, s = 2.8$ that the point $E_1$ represents accelerated scaling attractor with $\Omega_\phi \approx 0.7, \Omega_m \approx 0.3, \omega_{eff} \approx -0.66$ and $q \approx -0.49$ which agrees the present observed accelerated universe. Also, Fig.2(c) shows that for the same parameters values that the late time accelerated evolution of the universe is attracted in quintessence era. On the other hand, the Fig.(2(b)) shows that the point $F_1$ is a scaling attractor which is accelerating in nature for $\eta = 0.2, s = -1.89$. Here, the cosmological parameters: $\Omega_\phi \approx 0.70, \Omega_m \approx 0.30, \omega_{eff} \approx -0.4$ and $q \approx -0.1$. For both the cases, coincidence problem can be alleviated successfully as late time accelerated attractor is achieved satisfying $\frac{\Omega_\phi}{\Omega_m} = O(1)$.

5.2. Interaction model 2

For the Interaction model 2, we have obtained several number of non-isolated set of critical points in 4D phase-space. DE behaves as cosmological constant for all the sets. The Center Manifold Theory reveals that the accelerated de Sitter like solutions namely, the sets $B_2$ and $C_2$ (with $\Omega_\phi = 1, \Omega_m = 0, \omega_{eff} = -1, q = -1$) represent the transient era in the evolution of the universe dominated by potential of the scalar field (see Fig. (11), where the sets $B_2$ and $C_2$ are
saddle like unstable solutions in the phase space). The matter-scalar field scaling solutions are described by the sets of critical points namely $E_2$ and $F_2$ which are non-hyperbolic in nature and by Center Manifold Theory, the sets also represent the transient era in the evolution of the universe (see Figs. (12), (13) and (14)). Thus, the interaction model 2 produces non-hyperbolic sets of critical points either dominated by scalar field or representing scaling solutions have the transient nature of evolution since by the analysis of CMT all the points behave as saddle like nature.

6. CONCLUDING REMARKS

We have studied the dynamics of coupled non-canonical scalar field model taken as DE candidate with pressureless dust as DM in the background of spatially flat FLRW metric. Two interaction models with inverse square form and exponential form of the potential $V(\phi)$ and the coupling term $\lambda(\phi)$ (non-canonical term associated with scalar field) have been investigated. Evolution equations are complicated in nature so we have performed dynamical systems analysis to achieve the qualitative behavior of the cosmological model considered here. We have obtained hyperbolic as well as non-hyperbolic type critical points. Linear stability theory is sufficient to provide the nature of hyperbolic critical points. For non-hyperbolic type critical points, we have performed the Center Manifold Theory in detail to obtain exact dynamical nature of the points by finding center manifold for each point.

In the interaction model 1, where $Q \propto H \rho_m$, we choose potential $V(\phi)$ as well as coupling function $\lambda(\phi)$ of the scalar field to be evolved in inverse square form $V(\phi) = \lambda(\phi) = V_0 \phi^{-2}$, where $V_0$ is a constant. As a result of which, the dynamical variables $y$ and $z$ are not independent at all. Eventually the autonomous system reduces to a 2D system since $y = z$. For the same reason the constant parameter $s$ will have the same value as $u$. In this 2D system, we have obtained hyperbolic as well as non-hyperbolic type critical points. For the non-hyperbolic type points $A_1$ and $B_1$, we study the Center Manifold Theorem which is presented in separate section in 4. We have obtained parameter restrictions $\eta = \frac{1}{2}$, $s > 2\sqrt{\frac{2}{3}}$ for $A_1$ to be (see in Fig. 10(a)) the late time de Sitter solution of the universe.

Hyperbolic type critical points such as $C_1$ and $D_1$, by the linear stability theory, are very much interesting from the cosmological point of view. It is worthy to note that whenever $C_1$ describes the late time accelerated scalar field dominated stable attractor solution in some parameter region in the phase plane $x - y$, the point $D_1$ represents the accelerated past attractor (unstable source) there. This scenario confirms that our late time accelerated universe is attracted in quintessence era described by $C_1$ while early accelerated phase of the universe described by $D_1$. Points $E_1$ and $F_1$ are also hyperbolic type and linear stability theory helps to find the nature of the points. Although the points describe the accelerated de Sitter solution for $\eta = \frac{1}{2}$, but they are unstable there. Further, the point $F_1$ describes late time decelerated dust dominated universe for $\eta = 0$. On the other hand, the points $E_1$ and $F_1$ are physically interesting at late times only for $\eta < \frac{1}{2}$. For some restrictions on parameters, the points represent the late time accelerated scaling attractors satisfying the order of energy density of DE and DM to be unity and alleviating the coincidence problem.

We have studied another interacting model 2, where the interaction term $Q \propto \rho_m$ is chosen locally i.e., independent of Hubble parameter $H$ and the potential of scalar field and the coupling function of scalar field are chosen as exponential type in such a way that the dynamical variables $y$ and $z$ are independent. As a result, the autonomous system will be of 4D where Hubble parameter $H$ is considered as a dynamical variable. In this interacting scenario, we have obtained a collection of non-isolated set of critical points, all of which are non-hyperbolic in nature but not normally hyperbolic sets. So, the Center Manifold Theorem is employed to realize the dynamics of evolution near the sets. We have obtained de Sitter like solutions dominated by potential of scalar field having transient nature of evolution. At the same time, we have also obtained matter-scalar field scaling solutions having the transient nature in the phase space.

Finally, one can conclude that the phase space analysis of the interaction model 1 can only produce late time attractor solutions whereas the interaction model 2 cannot give attractor solutions.

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