DERIVED CLASSIFICATION OF GENTLE ALGEBRAS WITH TWO CYCLES

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Abstract. We classify gentle algebras defined by quivers with two cycles under derived equivalence in a non degenerate case, by using some combinatorial invariants constructed from the quiver with relations defining these algebras. We also present a list of normal forms; any such algebra is derived equivalent to one of the algebras in the list. The article includes an Appendix presenting a slightly modified and extended version of a technical result in the unpublished manuscript [HSZ01] by Holm, Schröer and Zimmermann, describing some essential elementary transformations over the quiver with relations defining the algebra.

1. Introduction

Let $A$ be a finite-dimensional connected $k$-algebra $A$ over an algebraically closed field $k$. Denote by $D^b(A)$ the bounded derived category of the module category of finite-dimensional left $A$-modules, $A$-mod. It is an interesting problem to classify such algebras up to derived equivalence.

In particular, the family of gentle algebras is closed under derived equivalence [SZ03]. The problem of classifying gentle algebras up to derived equivalence is well understood in the case where the associated quiver has one cycle, see [AH81], [AS87], [V01], [GP99] and [BGS04]. In this paper we focus our attention on gentle algebras with two cycles.

We use combinatorial invariants $\phi_A : \mathbb{N}^2 \to \mathbb{N}$ defined in [AG07] in order to classify them under derived equivalence. Roughly speaking $\phi_A$ is obtained as follows: Start with a maximal directed path in $Q$ which contains no relations. Then continue in opposite direction as long as possible with zero relations. Repeat this until the first path appears again, say after $n$ steps. Then we obtain a pair $(n, m)$ where $m$ is the number of arrows which appeared in a zero relation. Repeat this procedure until all maximal paths without a zero relation have been used; $\phi_A$ counts then how often each pair $(n, m) \in \mathbb{N}^2$ occurred. Recall $\phi_A$ has always a finite support. Let $\{(n_1, m_1), (n_2, m_2), \ldots, (n_k, m_k)\}$ be the support of $\phi_A$, denote $\phi_A$ by $\left[(n_1, m_1), (n_2, m_2), \ldots, (n_k, m_k)\right]$ where each $(n_j, m_j)$ is written $\phi_A(n_j, m_j)$ times and the order in which they are written is arbitrary. Define also $\#\phi_A := \sum_{1 \leq j \leq k} \phi_A(n_j, m_j)$. See [AG07, 3.5] for a precise description.

We can show by induction over the number of vertices that:

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  \item [Key words and phrases.] Gentle algebras and derived equivalence.
\end{itemize}
Theorem I. If \( A = kQ/\langle P \rangle \) is a gentle algebra, \( Q \) a quiver with two cycles, then \( \#\phi_A \in \{1, 3\} \).

In the case \( \#\phi_A = 3 \), we give the complete classification of gentle algebras with quivers of two cycles, see [AG07], under derived equivalence. This main result is also proved by induction:

Theorem II. Let \( A = kQ/\langle P \rangle \) and \( B = kQ'/\langle P' \rangle \) be gentle algebras so that \( Q \) and \( Q' \) are quivers with two cycles. If \( \#\phi_A = 3 \) then,

\( A \) and \( B \) are derived equivalent if and only if \( \phi_A = \phi_B \).

For the proof of this main result we use strongly some combinatorial elementary transformations over the quiver with relations which defines a gentle algebra, introduced by Holm, Schröer and Zimmermann in the unpublished manuscript [HSZ01]. As they are constructed by using tilting complexes, they produce derived equivalent algebras, see Appendix.

In Section 2 we present the basic definitions and notation about gentle algebras, including some previous results with respect to derived invariants introduced in [AG07]. In Section 3 we give a list of normal forms with a representative of each derived equivalence class of gentle algebras with quivers of two cycles in the non degenerate case \( \#\phi_A = 3 \). It follows from the proof of Theorem II that any gentle algebra \( A \) with two cycles, \( \#\phi_A = 3 \) and more than four vertices is not only derived equivalent to one of the algebras in the list, but also can be modified into it by applying some transformations from the ones defined in the Appendix. In Section 4 we give an alternative way to apply such transformations.

Section 5 consists of technical definitions, lemmas and propositions used for the proofs of Theorems I and II presented in Section 6.

The results of this article appeared as part of the Ph.D. thesis of the author who graduated on August 30, 2006. The complete digital version of the thesis can be found on the server http://bidi.unam.mx/ or more precisely on http://132.248.9.9:8080/tesdig/Procesados_2006/0608299/Index.html.

2. Preliminaries

Let \( Q_0 \) be a set of vertices, \( Q_1 \) a set of arrows and \( s, e: Q_1 \to Q_0 \) functions which define the start resp. end point of each arrow, the tuple \( Q = (Q_0, Q_1, s, e) \) is called a quiver. For a finite and connected quiver \( Q \) the number of cycles of \( Q \) is the least number of arrows that we have to remove from \( Q \) in order to obtain a tree. It is denoted by \( c(Q) \) and can be calculated by the expression \( c(Q) = \#Q_1 - \#Q_0 + 1 \).

A sequence of arrows \( C = \alpha_n \ldots \alpha_2 \alpha_1 \) with \( s(\alpha_{i+1}) = e(\alpha_i) \) for \( 1 \leq i < n \) is called a path of length \( n \), the length is denoted by \( l(C) := n \). Also, for each \( v \in Q_0 \) we consider a trivial path \( 1_v \) of length zero. The functions \( s \) and \( e \) are extended to paths in the obvious way. We define inverse of paths as follows: \( 1_v^{-1} := 1_v \) for \( v \in Q_0 \), \( \alpha_1^{-1} \) with \( s(\alpha^{-1}) := e(\alpha) \), \( e(\alpha^{-1}) := s(\alpha) \) and \( (\alpha^{-1})^{-1} := \alpha \) for \( \alpha \in Q_1 \) and \( C^{-1} := \alpha_1^{-1} \alpha_2^{-1} \ldots \alpha_n^{-1} \) for a path \( C = \alpha_n \ldots \alpha_2 \alpha_1 \). The composition of paths \( C_1 \)
and $C_2$ in $Q$ is the concatenation of them if $s(C_2) = e(C_1)$ or 0 if $s(C_2) \neq e(C_1)$. Let $k$ be a field and $kQ$ the path algebra, with paths of $Q$ as a basis and multiplication induced by concatenation. A relation in $Q$ is a non zero linear combination of paths of length at least two, with the same start point and end point. We work with algebras $kQ/\langle P \rangle$ where $\langle P \rangle$ is the ideal of $kQ$ generated by a set of relations $P$. A path in $Q$ is identified with its corresponding class in $kQ/\langle P \rangle$.

The path algebras we study in this work fulfill very particular conditions.

**Definition 1.** We call $kQ/\langle P \rangle$ a gentle algebra if the following five conditions hold:

1. For each $v \in Q_0$, $\{\alpha \in Q_1 | s(\alpha) = v\} \leq 2$ and $\{\alpha \in Q_1 | e(\alpha) = v\} \leq 2$.
2. For each $\beta \in Q_1$, $\{\alpha \in Q_1 | s(\beta) = e(\alpha)$ and $\beta \alpha \notin P\} \leq 1$ and $\{\gamma \in Q_1 | s(\gamma) = e(\beta)$ and $\gamma \beta \notin P\} \leq 1$.
3. For each $\beta \in Q_1$ there is a bound $n(\beta)$ such that any path $\beta_1 \ldots \beta_2 \beta_1$ with $\beta_2 \beta_1 = \beta$ contains a subpath in $P$.
4. All relations in $P$ are monomials of length 2.
5. For each $\beta \in Q_1$, $\{\alpha \in Q_1 | s(\beta) = e(\alpha)$ and $\beta \alpha \in P\} \leq 1$ and $\{\gamma \in Q_1 | s(\gamma) = e(\beta)$ and $\gamma \beta \in P\} \leq 1$.

We will make a slight abuse of notation by talking about gentle algebras referring to the quiver with relations which define those algebras.

2.1. **Threads of a gentle algebra.** Let $A$ be a gentle algebra. A permitted path of $A$ is a path $C = \alpha_n \ldots \alpha_2 \alpha_1$ with no zero relations, it is called a non trivial permitted thread of $A$ if it is of maximal length, that is, for all $\beta \in Q_1$, neither $C \beta$ nor $\beta C$ is a permitted path. If $v$ is a vertex with $\{\alpha \in Q_1 | s(\alpha) = v\} \leq 1$, $\{\alpha \in Q_1 | e(\alpha) = v\} \leq 1$ and if $\beta, \gamma \in Q_1$ are such that $s(\gamma) = v = e(\beta)$ then $\gamma \beta \notin P$, we consider $1_v$ a trivial permitted thread in $v$ and denote it by $h_v$. Similarly a forbidden path of $A$ is a sequence $\Pi = \alpha_n \ldots \alpha_2 \alpha_1$ formed by pairwise different arrows in $Q$ with $\alpha_i + 1 \alpha_i \in P$ for all $i \in \{1, 2, \ldots, n - 1\}$ and it is called a non trivial forbidden thread if for all $\beta \in Q_1$, neither $\pi \beta$ nor $\beta \pi$ is a forbidden path. If $v$ is a vertex with $\{\alpha \in Q_1 | s(\alpha) = v\} \leq 1$, $\{\alpha \in Q_1 | e(\alpha) = v\} \leq 1$ and if $\beta, \gamma \in Q_1$ are such that $s(\gamma) = v = e(\beta)$ then $\gamma \beta \notin P$, we consider $1_v$ a trivial forbidden thread in $v$ and denote it by $p_v$. The existence of non trivial permitted threads is due to point (3) in Definition 1 and the existence of non trivial forbidden threads is due to the restriction of considering pairwise different arrows.

Denote by $H_A$ the set of all permitted threads of $A$, trivial and non trivial. This set describes completely the algebra $A$. Notice that certain paths can be permitted and forbidden threads at the same time.

For a gentle algebra the relations in its quiver can be described by using two functions $\sigma, \varepsilon : Q_1 \to \{1, -1\}$ as in [BR87], defined by:

1. If $\beta_1 \neq \beta_2$ are arrows with $s(\beta_1) = s(\beta_2)$, then $\sigma(\beta_1) = -\sigma(\beta_2)$.
2. If $\gamma_1 \neq \gamma_2$ are arrows with $e(\gamma_1) = e(\gamma_2)$, then $\varepsilon(\beta_1) = -\varepsilon(\beta_2)$.
3. If $\beta, \gamma$ are arrows with $s(\gamma) = e(\beta)$ and $\gamma \beta \notin P$, then $\sigma(\gamma) = -\varepsilon(\beta)$.

They can be extended to the threads of $A$ as follows. For $H = \alpha_n \ldots \alpha_2 \alpha_1$ a non trivial thread of $A$ define $\sigma(H) := \sigma(\alpha_1)$ and $\varepsilon(H) := \varepsilon(\alpha_n)$. Consider $Q$ has at least two vertices, as $Q$ is connected, for $v \in Q_0$ there exists some $\gamma \in Q_1$ with
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\[ s(\gamma) = v \text{ or } \beta \in Q_1 \text{ with } e(\beta) = v. \] If there is a trivial permitted thread at \( v \) let
\[ \sigma(h_v) = -\varepsilon(h_v) := -\sigma(\gamma) \] in the first case and \( \sigma(h_v) = \varepsilon(h_v) := \varepsilon(\beta) \) in the second one. For a trivial forbidden thread \( p_v \) define \( \sigma(p_v) = \varepsilon(p_v) := -\sigma(\gamma) \) in the first case and \( \sigma(p_v) = \varepsilon(p_v) := -\varepsilon(\beta) \) otherwise.

The considered relations are monomials of length two, so we will indicate them in the quiver by using doted lines, joining each pair of arrows which form a relation.

Identify \( \alpha \in Q_1 \) with \( (e(\alpha)^{-\varepsilon(\alpha)}, s(\alpha)^{\sigma(\alpha)}) \). More generally, identify the non prohibited trivial thread \( \alpha_n \ldots \alpha_2 \alpha_1 \) with the vector
\[ (e(\alpha_n)^{\varepsilon(\alpha_n)}, s(\alpha_n)^{\sigma(\alpha_n)}, \ldots, s(\alpha_2)^{\sigma(\alpha_2)}, s(\alpha_1)^{\sigma(\alpha_1)}) \]
and its inverse with
\[ (s(\alpha_1)^{\sigma(\alpha_1)}, s(\alpha_2)^{\sigma(\alpha_2)}, \ldots, s(\alpha_n)^{\sigma(\alpha_n)}, e(\alpha_n)^{-\varepsilon(\alpha_n)}). \]

If there is a trivial permitted thread \( h_v \) for some \( v \in Q_0 \) we identify it with \( (v^{\varepsilon(h_v)}) \). For simplicity we write \( v^+ \) instead of \( v^{+1} \) and \( v^- \) instead of \( v^{-1} \). We can describe then \( \mathcal{H}_A \) by an array \([c_1 c_2 \ldots c_r]\) where the \( c_i \) are the inverses of different elements of \( \mathcal{H}_A \) written out as columns with the above notation and \( r = \#\mathcal{H}_A \).

**Example 2.**

\[
\begin{align*}
A: & & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} \\
\alpha_{10} & & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 & \alpha_9 & \alpha_{11}
\end{align*}
\]

Define \( \sigma(\alpha_1) = \sigma(\alpha_2) = \sigma(\alpha_3) = \sigma(\alpha_4) = \sigma(\alpha_5) = \sigma(\alpha_6) = \sigma(\alpha_8) = \sigma(\alpha_{10}) = +1, \)
\( \sigma(\alpha_7) = \sigma(\alpha_9) = \sigma(\alpha_{11}) = -1, \) \( \varepsilon(\alpha_6) = \varepsilon(\alpha_8) = \varepsilon(\alpha_9) = \varepsilon(\alpha_{10}) = \varepsilon(\alpha_{11}) = +1, \) and \( \varepsilon(\alpha_1) = \varepsilon(\alpha_2) = \varepsilon(\alpha_3) = \varepsilon(\alpha_4) = \varepsilon(\alpha_5) = \varepsilon(\alpha_7) = -1. \)

Then \( \mathcal{H}_A \) consists of \( (v_5^+, v_4^+, v_3^+, v_2^+, v_1^+) \), \( (v_8^+, v_3^-, v_7^+, v_6^+) \), \( (v_7^-, v_1^-, v_9^+) \), \( (v_5^-, v_{10}^+) \), \( (v_8^-) \), \( (v_4^-) \), \( (v_6^-) \) and can be described by the expression
\[
\begin{bmatrix}
v_1^+ & v_6^+ & v_9^+ & v_{10}^+ & v_{10}^- & v_2^- & v_6^- & v_9^- \\
v_2^+ & v_7^+ & v_1^- & v_5^- & v_8^+ & v_4^+ & v_8^- & v_5^+ \\
v_3^+ & v_5^- & v_7^- & v_8^+ & v_4^+ & v_1^+ & v_3^- & v_8^+ \\
v_4^+ & v_6^+ & v_{10}^+ & v_{10}^- & v_2^- & v_6^- & v_9^- & v_5^- \\
v_5^+ & v_8^+ & v_{10}^+ & v_{10}^- & v_2^- & v_6^- & v_9^- & v_5^- \\
\end{bmatrix}
\]

Usually trivial paths will be omitted. In this case we write
\[
\begin{bmatrix}
v_1^+ & v_6^+ & v_9^+ & v_{10}^+ & v_{10}^- & v_2^- & v_6^- & v_9^- \\
v_2^+ & v_7^+ & v_1^- & v_5^- & v_8^+ & v_4^+ & v_8^- & v_5^+ \\
v_3^+ & v_5^- & v_7^- & v_8^+ & v_4^+ & v_1^+ & v_3^- & v_8^+ \\
v_4^+ & v_6^+ & v_{10}^+ & v_{10}^- & v_2^- & v_6^- & v_9^- & v_5^- \\
v_5^+ & v_8^+ & v_{10}^+ & v_{10}^- & v_2^- & v_6^- & v_9^- & v_5^- \\
\end{bmatrix}
\]

There are several forms in which we can describe \( \mathcal{H}_A \), according to the order in which we write the elements of this set.
In forthcoming examples and results we use this way of codifying a gentle algebra. The computer calculations made for this work use another way of describing a gentle algebra. Consider \( \mathcal{P}(2n) \), the collection of partitions of \( 2n \), with \( n \) the number of vertices of \( Q \), that is, \( \mathcal{P}(2n) := \{ \lambda | \lambda \vdash 2n \} \). Denote by \( \mathcal{R}(2n) \) the collection of partitions of the set \( [2n] = \{1, 2, \ldots, 2n\} \) whose elements have cardinality 2, that is

\[
\mathcal{R}(2n) := \{ R | R \text{ is a partition of } [2n], \#p = 2 \ \forall p \in R \}
\]

We can define an equivalence relation in \( \mathcal{P}(2n) \times \mathcal{R}(2n) \) and a bijection \( \Phi : \mathcal{A}(n) \to \mathcal{P}(2n) \times \mathcal{R}(2n) / \sim \) where \( \mathcal{A}(n) \) is the collection of gentle algebras whose associated quiver is connected and has \( n \) vertices. For each \( A = kQ/\langle \mathcal{P} \rangle \in \mathcal{A}(n) \), let \( H_1, H_2, \ldots, H_r \) be the different threads of \( A \) ordered in a non growing form according to its length, define \( \lambda := (l(H_1) + 1, l(H_2) + 1, \ldots, l(H_r) + 1) \). Let \( Q_0 = \{v_1, v_2, \ldots, v_n\} \), \( \sigma \) and \( \varepsilon \) be as described before, define \( f : \{ v_i^\pm | 1 \leq i \leq n, \varepsilon \in \{+1, -1\} \} \to [2n] \) as \( f(v_i^\pm) := (l(H_1) + 1) + \ldots + (l(H_j) + 1) - m + 1 \) if \( v_i^\pm \) is the \( m \) element of vector \( H_j \). Let \( \gamma \in \mathcal{R}(2n) \) be such that \( \{c, d\} \in \gamma \) if and only if there exists \( v_i \in Q_0 \) such that \( f(v_i^c) = c \) and \( f(v_i^d) = d \). Then \( \Phi(A) \) is the equivalence class of pair \( (\lambda, \gamma) \).

**Example 3.** For \( A \) as in Example 2 \( \lambda = (5, 4, 3, 2, 1, 1, 1, 1) \). Notice \( f(v_1^+) = 1, f(v_2^+) = 2, f(v_3^+) = 3, f(v_4^+) = 4, f(v_5^+) = 5, f(v_6^+) = 6, f(v_7^+) = 7, f(v_8^-) = 8, f(v_9^+) = 9, f(v_{10}^+) = 10, f(v_{11}^-) = 11, f(v_{12}^-) = 12, f(v_{13}^+) = 13, f(v_{14}^-) = 14, f(v_{15}^-) = 15, f(v_{16}^-) = 16, f(v_{17}^-) = 17, f(v_{18}^-) = 18, f(v_{19}^-) = 19, f(v_{20}^-) = 20 \). Then

\[
\gamma = \{\{1, 11\}, \{2, 17\}, \{3, 8\}, \{4, 18\}, \{5, 14\}, \{6, 19\}, \{7, 12\}, \{9, 16\}, \{10, 20\}, \{13, 15\}\}.
\]

2.2. Previous results. For a gentle algebra \( A \) there exists a derived equivalent invariant \( \phi_A : \mathbb{N}^2 \to \mathbb{N} \), see \cite{AG07} for a precise definition, which can be determined easily if \( A \) is given as a quiver with relations \( kQ/\langle \mathcal{P} \rangle \) as mentioned in Section II. As \( \phi_A \) describes the action of the suspension functor \( \Omega \tau_A^{-1} \) for the triangulated category \( \mathcal{A}_{-mod} \) on the Auslander-Reiten components which contain string modules and Auslander-Reiten triangles, see \cite{Ha88}, of the form \( X \to Y \to \tau_A^{-1}X \to \Omega \tau_A^{-1}X \) with \( Y \) indecomposable, we have the following result, see \cite{AG07}.

**Theorem A.** Let \( A \) and \( B \) be gentle algebras. If \( A \) and \( B \) are derived equivalent then \( \phi_A = \phi_B \).

Also

**Theorem C.** Let \( A = kQ/\langle \mathcal{P} \rangle \) and \( B = kQ'/\langle \mathcal{P}' \rangle \) be gentle algebras such that \( c(Q), c(Q') \leq 1 \). Then \( A \) and \( B \) are derived equivalent if and only if \( \phi_A = \phi_B \).

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1A partition of a natural \( m \) is a sequence \( \lambda = (\lambda_1, \lambda_2, \ldots) \) of naturals such that \( \lambda_1 \geq \lambda_2 \geq \ldots \), with just a finite number of no zero terms and such that \( \sum_i \lambda_i = m \); it is denoted by \( \lambda \vdash m \).

2Let \( A \) be a set, a collection \( \{A_i\}_i \) of non empty subsets of \( A \) is a partition of \( A \) if \( A_i \cap A_j = \emptyset \) for all \( i \neq j \) and \( \bigcup_i A_i = A \).
As defined in Section 3 let $\# \phi_A := \sum_{1 \leq j \leq k} \phi_A(n_j, m_j)$ where $[(n_1, m_1), \ldots, (n_k, m_k)]$ describes $\phi_A$; each $(n_j, m_j)$ in the support of $\phi_A$ is written $\phi_A(n_j, m_j)$ times and the order in which they are written is arbitrary.

In this article we prove that if $A$ is a gentle algebra with two cycles, then $\# \phi_A \in \{1, 3\}$ and we will use these invariants to classify those kind of algebras in the case where $\# \phi_A = 3$.

3. Normal forms

3.1. Three pairs of natural numbers. If $A$ is an algebra defined by a quiver with two cycles and $\phi_A = [(a_1, a_2), (b_1, b_2), (c_1, c_2)]$, we now $a_1 + b_1 + c_1 = \# \mathcal{H}_A$ and $a_2 + b_2 + c_2 = \# Q_1$. Also, the permitted threads considered as disjoint graphs form a forest with $\# \mathcal{H}_A$ trees and $2 \# Q_0$ vertices, so $\# \mathcal{H}_A = 2 \# Q_0 - \# Q_1$. By definition of $c(Q)$, $\# Q_0 = \# Q_1 - c(Q) + 1$, so $\# \mathcal{H}_A = 2(\# Q_1 - c(Q) + 1) - \# Q_1 = \# Q_1 - 2c(Q) + 2$, that is $\# \mathcal{H}_A + 2c(Q) - 2 = \# Q_1$, equivalently $a_1 + b_1 + c_1 + 2(2) - 2 = a_2 + b_2 + c_2$ or $a_1 + b_1 + c_1 + 2 = a_2 + b_2 + c_2$. Now we prove the converse, if $(a_1, a_2), (b_1, b_2)$ and $(c_1, c_2)$ are pairs such that $a_1 + b_1 + c_1 + 2 = a_2 + b_2 + c_2$ there is a gentle algebra $A$ for which $\phi_A = [(a_1, a_2), (b_1, b_2), (c_1, c_2)]$. We present a list of normal forms of all different derived equivalence classes in this case.

**Theorem 4.** Let $[(a_1, a_2), (b_1, b_2), (c_1, c_2)] \subset \mathbb{N}^2$ be such that $a_1 + b_1 + c_1 + 2 = a_2 + b_2 + c_2$. There is a gentle algebra $A$ such that $\phi_A = [(a_1, a_2), (b_1, b_2), (c_1, c_2)]$.

**Proof:**

We construct a gentle algebra $A$ for each possible collection of invariants. First we work with pairs of natural numbers $(0, a), (a, 0)$ or $(1, 1)$.

1. A gentle algebra $A$ such that $\phi_A = [(0, a), (0, b), (a + b - 2, 0)]$ with $a, b \geq 1$ and $a + b > 2$ is

   ![Diagram A](image)

   (2) A gentle algebra $A$ such that $\phi_A = [(a, 0), (b, 0), (0, a + b + 2)]$ with $a, b \geq 1$ is

   ![Diagram B](image)
(3) A gentle algebra $A$ such that $\phi_A = [(1, 1), (b, 0), (0, b + 2)]$ is

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A :
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(4) A gentle algebra $A$ such that $\phi_A = [(1, 1), (1, 1), (0, 2)]$ is

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A :
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(5) A gentle algebra $A$ such that $\phi_A = [(1, 1), (0, 1), (0, 1)]$ is

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A :
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For the general cases some trivial permitted threads are added to the previous algebras:

(6) A gentle algebra $A$ such that $\phi_A = [(k, a + k), (q, b + q), (a + b - 2 + r, r)]$ with $a \geq b \geq 1$, $a + b > 2$, $k \geq q$ if $a = b$, is

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A :
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(7) A gentle algebra $A$ such that $\phi_A = [(a + k, k), (b + q, q), (r, a + b + 2 + r)]$ with $a \geq b \geq 1$, $k \geq q$ if $a = b$ is

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A :
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(8) A gentle algebra $A$ such that $\phi_A = [(k, k), (b + q, q), (r, b + 2 + r)]$ is

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A :
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(9) A gentle algebra $A$ such that $\phi_A = [(k, k), (q, q), (r, r + 2)]$ with $k \leq q$ is

\[ A : \]

(10) A gentle algebra $A$ such that $\phi_A = [(k, k), (q, q + 1), (r, r + 1)]$ with $q \leq r$

This covers all possible cases if $\# \phi_A = 3$:

Suppose $A$ is a gentle algebra defined by a quiver with two cycles such that $\phi_A = [(a_1, a_2), (b_1, b_2), (c_1, c_2)]$. If $a_1 = a_2$ and $b_1 = b_2$ we have (9). If $a_1 = a_2$ but $b_1 \neq b_2$ and $c_1 \neq c_2$, one possibility is $b_1 > b_2$ and $c_1 < c_2$, or $b_1 < b_2$ and $c_1 < c_2$, which correspond to (8) and (10) respectively; as $a_1 + b_1 + c_1 + 2 = a_2 + b_2 + c_2$ it is not possible that $b_1 > b_2$ and $c_1 > c_2$.

Other possibility is that $a_1 \neq a_2$, $b_1 \neq b_2$ and $c_1 \neq c_2$ with $a_1 < a_2$, $b_1 < b_2$ and $c_1 > c_2$, or $a_1 > a_2$, $b_1 < b_2$ and $c_1 < c_2$, which correspond to cases (6) and (7) presented before; as $a_1 + b_1 + c_1 + 2 = a_2 + b_2 + c_2$, it is impossible that $a_1 < a_2$, $b_1 < b_2$ and $c_1 < c_2$, or that $a_1 > a_2$, $b_1 > b_2$ and $c_1 > c_2$.

We prove that any gentle algebra $A$ with two cycles and $\# \phi_A = 3$, is derived equivalent to one of the algebras presented in this section. In order to do that we will need some combinatorial transformations over the quiver with relations which define those algebras.

4. DESCRIPTION OF THE ELEMENTARY TRANSFORMATIONS IN TERMS OF THE THREADS OF $A$

Now, we analyze each one of the combinatorial transformation which preserve derived equivalence presented in the Appendix by using the description of a gentle algebra in terms of its threads. These transformations are very easy to calculate and when described in terms of its threads we see they are all constructed with a more elementary combinatorial transformation or its corresponding dual. The main reason we focus our attention on these transformations is that for any gentle algebra $A = kQ/ \langle \mathcal{P} \rangle$, $Q$ with two cycles, $\# \phi_A = 3$ and $\#Q_0 \geq 5$, all its derived equivalence class can be obtained by using these transformations.
**Definition 5.** Let $A = kQ/\langle P \rangle$ be a gentle algebra with $Q$ connected and $u, v \in Q_0$ distinct consecutive vertices. Consider a description of $A$ in terms of its threads:

$$
\mathcal{H}_A : \begin{bmatrix}
    s(H_{u^+}) & s(H_{u^-}) & s(H_{v^-}) \\
    \vdots & \vdots & \vdots \\
    \ldots & u^+ & u^- & v^- & \ldots \\
    v^+ & \vdots & \vdots \\
    \vdots & e(H_{u^-}) & e(H_{v^-}) & e(H_{u^+}) \\
\end{bmatrix}
$$

where $H_{u^-}$ and $H_{v^-}$ are threads of $A$ involving $u$ and $v$ but not the arrow $(v^+, u^+)$, and $H_{u^+}$ the thread of $A$ which involves such arrow. Denote by $m_{u^+}(\mathcal{H}_A)$ the corresponding array obtained by removing $u^+$ of its position and putting it below $v^-$, that is

$$
\mathcal{m}_{u^+}(\mathcal{H}_A) : \begin{bmatrix}
    s(H_{u^+}) & s(H_{u^-}) & s(H_{v^-}) \\
    \vdots & \vdots & \vdots \\
    \ldots & v^+ & u^- & v^- & \ldots \\
    \vdots & \vdots & u^+ \\
    e(H_{u^-}) & e(H_{v^-}) & e(H_{u^+}) & e(H_{u^-}) \\
\end{bmatrix}
$$

We say $u^+$ is moved after $v^-$. The inverse transformation is denoted by $m_{u^-}^{-1}$, in this case we say $u^+$ is moved before $v^+$. Define $m_{v^-}$ and its inverse in a similar way.

This combinatorial transformations describe the ones presented in the Appendix as we see next.

**4.1. Transformations over a vertex.** Let $A$ as in Section 7. In all cases, the transformation over a vertex $i$ corresponds to apply $m_{i^+}$ followed by $m_{i^-}$.

1. If $s_1 \neq i \neq s_2$. The array associated to $\mathcal{H}_A$ is

   $\begin{bmatrix}
   s(H_{i^+}) & s(H_{i^-}) & s(H_{i_1^-}) & s(H_{i_2^-}) \\
   \vdots & \vdots & \vdots & \vdots \\
   \ldots & p_{i_1}^+ & p_{i_2}^- & j_{1_i}^- & j_{2_i}^- & \ldots \\
   j_{1_i}^+ & j_{2_i}^+ & \vdots & \vdots \\
   \vdots & \vdots & e(H_{j_{1_i}^-}) & e(H_{j_{2_i}^-}) \\
   e(H_{i_1^+}) & e(H_{i_2^-}) \\
\end{bmatrix}$

2. If $s_1 = i = s_2$. The array associated to $\mathcal{H}_A$ is

   $\begin{bmatrix}
   s(H_{i^+}) & s(H_{i^-}) & s(H_{i_1^-}) & s(H_{i_2^-}) \\
   \vdots & \vdots & \vdots & \vdots \\
   \ldots & p_{i_1}^+ & p_{i_2}^- & j_{1_i}^- & j_{2_i}^- & \ldots \\
   j_{2_i}^+ & j_{1_i}^+ & \vdots & \vdots \\
   \vdots & \vdots & e(H_{j_{1_i}^-}) & e(H_{j_{2_i}^-}) \\
   e(H_{i_1^+}) & e(H_{i_2^-}) \\
\end{bmatrix}$

   $\mathcal{m}_{i^+}$

   $\mathcal{m}_{i^-}$
In any case, the resulting arrays correspond to the transformation over an arrow (\( T \)).

If \( s_1 = i \) the array \( H_A \) is

(a)

\[
\begin{pmatrix}
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\end{pmatrix}
\]

or

(b)

\[
\begin{pmatrix}
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\end{pmatrix}
\]

which remains invariant after applying \( m_i^- \).

(3) If \( s_1 = s_2 = i \) the array \( H_A \) is invariant under \( m_{i^+} \) and \( m_i^- \).

In any case, the resulting arrays correspond to \( H_{V_i(A)} \).

4.2. Transformation over an arrow. Let \( A \) be as in Section 7.2. We prove that the transformation over an arrow \((j^+, i^+)\) corresponds to the application of \( m_{i^+} \).

The array \( H_A \) is

\[
\begin{pmatrix}
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\end{pmatrix}
\]

which corresponds to \( H_{F_{i^+, i^+}}(A) \).
4.3. Transformation over a loop. Let $A$ be as in Section 4.3. We prove that transformation over a loop $\left(i^-, i^+\right)$ corresponds to apply $m_i^-$ followed by $m_i^+$. The array $\mathcal{H}_A$ is

$$
\begin{bmatrix}
  s(H_{i^+}) & s(H_{j^-}) \\
  \vdots & \vdots \\
  l^+ & j^- \\
  i^+ & x^+ \\
  i^- & \vdots \\
  j^+ & e(H_{j^-}) \\
  \vdots & \vdots \\
  e(H_{i^+}) & \\
\end{bmatrix}
$$

which corresponds to $H_{L(i^-, i^+)}(A)$.

5. Reduction of an algebra to its normal form

In this section we study some reductions of gentle algebras with two cycles which use the elementary transformations mentioned before, to modify them into some of the normal forms presented in Section 3. Because of the diversity of the algebras involved, even for few vertices, we do not present an algorithm to transform them into its normal form. Else, we develop induction proofs in which, by knowing the elementary transformations used to modify a gentle algebra into a normal form, we can transform the algebra obtained by adding a vertex in a convenient way also into a normal form. In this section we present some technical results necessaries in those induction proofs.

5.1. Simplification of the branches in a quiver. In order to present induction proofs over the number of vertices of the quiver which defines the algebra we need the following:

**Definition 6.** Let $Q$ be a quiver. The **degree of a vertex** is the number of arrows beginning in $v$ plus the number of arrows ending in $v$.

**Definition 7.** Let $A = kQ/\langle \mathcal{P} \rangle$ be a gentle algebra, with $Q$ a connected quiver and $x \in Q_0$ a transition vertex (that is there is just one arrow $\alpha$ which ends in the vertex, only one arrow $\beta$ which starts in it and $\beta\alpha \notin \mathcal{P}$), or a vertex of degree one which is the start point of an arrow; graphically we have one of the following cases:

(1) $\xymatrix{ & u \ar@/^/[dr]^-{\alpha} & x \ar[r]^-{\beta} & v \ar[dl]^-{\gamma} \ar@/^/[dl]^-{\beta} \ar@/^/[ul]^-{\alpha}}$

(2) $\xymatrix{ & u \ar@/^/[dr]^-{\alpha} & x \ar[r]^-{\beta} & \ar[dl]^-{\gamma} \ar@/^/[dl]^-{\beta} \ar@/^/[ul]^-{\alpha}}$
Consider the corresponding quiver with relations obtained by removing vertex \(x\), that is, an algebra \(A\{x\} := kQ'/'\langle P'\rangle\) where \(Q'_0 = Q_0\setminus\{x\}\), \(Q'_1 = (Q_1\setminus\{\alpha, \beta\})\cup\{\alpha'\}\) with \(s(\alpha') = u\), \(e(\alpha') = v\), \(P' = P \setminus \{(\alpha\gamma|\gamma \in Q_1) \cup \{\gamma\beta|\gamma \in Q_1\}\} \cup \{\alpha'\gamma|\alpha\gamma \in P\} \cup \{\gamma\alpha'|\gamma \beta \in P\}\) for case (1); and \(Q'_0 = Q_0\setminus\{x\}\), \(Q'_1 = (Q_1\setminus\{\theta\})\) \(P' = P \setminus \{\kappa\theta\}\) for case (2). That is:

\[
\begin{align*}
\begin{array}{c}
\text{u} \xrightarrow{\alpha} \text{x} \xrightarrow{\beta} \text{v} \\
\text{u} \xrightarrow{\alpha'} \text{v}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{u} \xrightarrow{\theta} \text{w} \xrightarrow{\kappa} \text{z} \\
\text{u} \xrightarrow{w} \text{v} \xrightarrow{\kappa} \text{z}
\end{array}
\end{align*}
\]

Remark 8. If \(A = kQ/'\langle P\rangle\) is a gentle algebra, with \(Q\) a quiver with two cycles and \(x \in Q_0\) as in previous definition, \(A\setminus\{x\} = kQ'/'\langle P'\rangle\) is also a gentle algebra with \(Q'\) of two cycles, because it has exactly one vertex an one arrow less than \(Q\).

Remark 9. If

\[
\begin{align*}
\begin{array}{c}
\text{u} \xrightarrow{\delta} \text{v} \xrightarrow{x} \text{v}
\end{array}
\end{align*}
\]

after applying the arrow transformation \(F_\delta\) we get

\[
\begin{align*}
\begin{array}{c}
\text{u} \xrightarrow{x} \text{v}
\end{array}
\end{align*}
\]

which is a vertex as in Definition 7(1). Similarly in the dual case.

Definition 10. Consider \(Q = (Q_0, Q_1, s, e)\) a finite connected quiver. Let \(S\) be the collection of vertices which belong to a cycle or to a trajectory which joins two vertices belonging to cycles. We say \(Q'\) a subquiver of \(Q\) is a **rooted tree in** \(v \in Q_0\) if \(Q'\) is tree, \(v\) is the only vertex of \(Q'\) which belongs to \(S\); also \(\#Q'_1\) is maximal with respect to this property. The **depth of the rooted tree in** \(v\) is the number of arrows in a trajectory which begins in \(v\), of maximal length in the subjacent graph. We say \(Q'\) a rooted tree in \(v\) is a **branch** if all vertices have degree at most two in \(Q'\).

Let us study rooted trees of a quiver associated to a gentle algebra:

Lemma 11. Let \(Q\) be a quiver of two cycles and more than five vertices, associated to a gentle algebra. If there exist rooted trees deeper than one we have one of the
following cases:

(a) \[ \begin{array}{c}
\text{1} \\
\downarrow \\
\text{2}
\end{array} \]

(b) \[ \begin{array}{c}
\text{1} \\
\downarrow \\
\text{2}
\end{array} \]

(c) \[ \begin{array}{c}
\text{1} \\
\downarrow \\
\text{2}
\end{array} \]

(d) \[ \begin{array}{c}
\text{1} \\
\downarrow \\
\text{3}
\end{array} \]

(e) \[ \begin{array}{c}
\text{1} \\
\downarrow \\
\text{2}
\end{array} \]

(f) \[ \begin{array}{c}
\text{1} \\
\downarrow \\
\text{3}
\end{array} \]

(g) \[ \begin{array}{c}
\text{1} \\
\downarrow \\
\text{2}
\end{array} \]

or some of their dual. In the previous diagrams the numbers indicate the degree of each vertex.

Proof:
Let \( u \in Q_0 \) be of degree 1 such that the only neighbor of \( u, v \), is of minimal degree. As the algebra is gentle, the quiver connected and with at least 6 vertices, the degree of \( v \) is 2, 3 or 4. If \( v \) has degree 2, the only possible cases are \((a), (b)\) or \((c)\), or their respective dual, if it has degree 3 the possibilities are \((d), (e)\) or \((f)\) or one of their dual, and if it has degree 4, the only possibility is \((g)\) or its dual.

More specifically

Lemma 12. Let \( Q \) be a two cycle quiver with more than five vertices, associated to a gentle algebra \( A \). If there are rooted trees deeper than one, \( A \) is derived equivalent under elementary transformations to a gentle algebra \( B = kQ' / \langle P' \rangle \) such that there is an \( x \in Q'_0 \) as in Definition 7 (1).

Proof: We analyze each one of the cases presented in Lemma 11. For \((a)\) there is nothing to prove and for \((b),(d),(e),(f)\) and \((g)\) the result follows from Remark 9. For \((c)\) we have:

\[ \begin{array}{c}
\text{x} \\
\delta' \\
\downarrow \\
\text{u}
\end{array} \quad \text{and} \quad \begin{array}{c}
\text{F}_{\delta^{-1}} \\
\text{F} \\
\downarrow \\
\text{v}
\end{array} \]

and we apply the dual transformations for the dual cases.

Now we analyze what happens with rooted trees of degree one.

Lemma 13. Let \( A = kQ / \langle P \rangle \) be a gentle algebra, with \( Q \) a quiver of two cycles. If there is a rooted tree in \( Q \) of degree one, there is an \( x \in Q_0 \) such that \( A \) is derived equivalent under elementary transformations to \( B = kQ' / \langle P' \rangle \) where the corresponding \( x \) is presented in one of the following situations:
with $x$ of degree 2, or

(2)

with $w$ of degree 3 and $\alpha$ and $\beta$ in a cycle.

**Proof:**
We analyze the different cases in which the rooted tree of degree one can be presented. If the situation is as in Remark 9 there is nothing to prove, we study now what happens with branches of depth one which look like:

(1)

with $w$ of degree 3. If $\alpha$ and $\beta$ belong to a cycle we have the result. If not, we apply $F_{\beta}$ to obtain

(2)

and then we use Remark 9. In the dual case we apply the corresponding inverse transformation.

Under certain conditions we can assure the existence of a vertex $x$ for which it makes sense to consider the algebra $A \setminus \{x\}$ as in Definition 7.

**Proposition 14.** Let $A = kQ/\langle P \rangle$ be a gentle algebra, with $Q$ a connected quiver of two cycles and more than 5 vertices. Then $A$ is derived equivalent under elementary transformations to $B = kQ'/\langle P' \rangle$ such that there is an $x \in Q'_0$ of the following type:

(1)

with $x$ of degree 2
with $w$ of degree 3 and $\alpha$ and $\beta$ in a cycle.

(3) 

\[ u \xrightarrow{x} v \]

with $x$ of degree 2 and $\alpha$ and $\beta$ in a cycle.

**Proof:**

If there are rooted trees the result follows from Lemmas 12 or 13.

If there are no rooted trees all vertices have degree 2, 3 or 4. We know

\[ \sum_{v \in Q_0} gr(v) = 2\#Q_1 = 2(\#Q_0 + 1) = 2\#Q_0 + 2. \]

Then all vertices must have degree 2, except one of degree 4 or two of degree 3. By hypothesis there are at least 6 vertices, so there are at least two consecutive vertices of degree 2 denote them by $u$ and $v$. If situation (1) is not presented in any of these vertices we have one of the following situations:

\[
\begin{align*}
\text{(1)} &\quad u \xrightarrow{\delta} v \\
\text{(2)} &\quad u \xrightarrow{x} v
\end{align*}
\]

or the dual of the last one. In the first situation we apply a vertex transformation

\[
\begin{align*}
\text{(1)} &\quad u \xrightarrow{v} v \\
\text{(2)} &\quad u \xrightarrow{V_u} v
\end{align*}
\]

where the degree of vertex $u$ can change but the degree of $v$ is 2, so we have (1). For the remaining cases, if the arrow $\delta$ belongs to a cycle we have (3); if not, we can apply the transformation over that arrow:

\[
\begin{align*}
\text{(1)} &\quad F_\delta \xrightarrow{v} v \\
\text{(2)} &\quad F_\delta \xrightarrow{v} v
\end{align*}
\]

and result follows by Lemma 13 else

\[
\begin{align*}
\text{(1)} &\quad F_\delta \xrightarrow{v} v \\
\text{(2)} &\quad F_\delta \xrightarrow{v} v
\end{align*}
\]

For the dual case we apply the corresponding inverse transformation to produce one of the situations already analyzed.
Much more:

**Proposition 15.** Let $A = kQ/\langle P \rangle$ be a gentle algebra, with $Q$ a quiver of two cycles and more than 5 vertices. If $A$ is not derived equivalent to an algebra $B = kQ'/\langle P' \rangle$ with an $x \in Q'_0$ as follows:

1. $u \overset{\alpha}{\rightarrow} x \overset{\beta}{\rightarrow} v$

   $x$ of degree 2

2. $u \overset{\alpha}{\rightarrow} w \overset{\beta}{\rightarrow} v$

   $w$ of degree 3 and $\alpha$ and $\beta$ in a cycle

then $\#\phi_A = 1$ or $A$ is derived equivalent to one of the representatives (1), (2) or (3) of Section 3.

**Proof:**

Suppose $A$ is not derived equivalent to an algebra whose associated quiver has a vertex $x$ as described, we know then by Lemmas 12 and 13 that $Q$ has no branches. Also $\#Q_0 \geq 6$ and by the proof of Proposition 14 there must exist a sequence of 3 arrows of type

$$u \overset{\alpha_0}{\rightarrow} u_1 \overset{\alpha_1}{\rightarrow} u_2 \overset{\alpha_2}{\rightarrow} v$$

with $u$ and $v$ of degree 2, $\alpha_0$, $\alpha_1$ and $\alpha_2$ belonging to the same cycle; moreover, any sequence consisting of consecutive vertices of degree two $u_1, u_2, \ldots u_n$, $n \geq 2$ must be like:

$$u_1 \overset{\alpha_1}{\rightarrow} u_2 \overset{\alpha_2}{\rightarrow} u_3 \ldots u_{n-1} \overset{\alpha_n}{\rightarrow} u_n$$

and is constructed by arrows which belong to the same cycle, if not, by the proof of Proposition 14 we would be able to transform the quiver with relations into another with a vertex $x$ as described. We also know every vertex in $A$ has degree two, except one of degree 4 or two of degree 3. If there is a 4 degree vertex $v$, it can be presented in one of the following ways:

- $v \overset{\alpha_1}{\rightarrow} v_1 \overset{\alpha_2}{\rightarrow} \ldots v_n \overset{\alpha_n}{\rightarrow} v_{n+1}$
- $v \overset{\alpha_1}{\rightarrow} \ldots v_n \overset{\alpha_n}{\rightarrow} v_{n+1}$

but this last case is impossible because in one of the two cycles there should be more than three arrows and by their orientation there would appear a sequence different from the ones mentioned.
before. In the first case we have a representant of type (1) and in the second case one of type (2). We study now what happens if there are two vertices of degree 3, $a$ and $b$.

By previous analysis all arrows but one or two are part of a cycle. In any case, by the conditions of the quiver with relations, $\#\phi_A = 1$ or $A$ is derived equivalent to one of representant (1), (2) or (3) of Section 3.

Case 1.- All arrows belong to some cycle. We know there must be a sequence of three arrows at least as the one mentioned before and its end terms must be $a$ and $b$. Then one of the next situations is presented:

\begin{align*}
&i) \quad \begin{array}{c}
\text{\begin{tikzpicture}
\node (a) at (0,0) {$a$};
\node (b) at (1,1) {$b$};
\node (j) at (2,0) {$j$};
\node (u1) at (2,-1) {$u_1$};
\node (u) at (1,-1) {$u$};
\draw (a) -- (b);
\draw (b) -- (j);
\draw (j) -- (u1);
\end{tikzpicture}}
\end{array}
\quad \quad \begin{array}{c}
\text{or} \quad \begin{tikzpicture}
\node (a) at (0,0) {$a$};
\node (b) at (1,1) {$b$};
\node (j) at (2,0) {$j$};
\node (u1) at (2,-1) {$u_1$};
\node (u) at (1,-1) {$u$};
\draw (a) -- (b);
\draw (b) -- (j);
\draw (j) -- (u1);
\end{tikzpicture}
\end{array}
\end{align*}

\begin{enumerate}
\item we get a trivial permitted thread in $u$ by applying $V_b^{-1}$.
\item If $j \neq a$, we obtain a trivial permitted thread in $j$ after applying $V_b^{-1}$; also, if $j = a$ we get
\begin{align*}
&\begin{array}{c}
\text{\begin{tikzpicture}
\node (a) at (0,0) {$a$};
\node (b) at (1,1) {$b$};
\node (j) at (2,0) {$j$};
\node (u1) at (2,-1) {$u_1$};
\node (u) at (1,-1) {$u$};
\draw (a) -- (b);
\draw (b) -- (j);
\draw (j) -- (u1);
\end{tikzpicture}}
\end{array}
\quad \quad \begin{array}{c}
\text{\begin{tikzpicture}
\node (a) at (0,0) {$a$};
\node (b) at (1,1) {$b$};
\node (j) at (2,0) {$j$};
\node (u1) at (2,-1) {$u_1$};
\node (u) at (1,-1) {$u$};
\draw (a) -- (b);
\draw (b) -- (j);
\draw (j) -- (u1);
\end{tikzpicture}}
\end{array}
\end{align*}
\end{enumerate}

\begin{enumerate}
\item in the first case, we produce a trivial permitted thread in $u_1$ applying $V_a$; in the second case, by the algorithm to calculate $\phi_A$ we get $\#\phi_A = 1$.
\item If $s \neq a$, we produce a trivial permitted thread in $s$ after applying $V_b$. Also, if $s = a$ we have one of the following cases:
\begin{align*}
&\begin{array}{c}
\text{\begin{tikzpicture}
\node (a) at (0,0) {$a$};
\node (b) at (1,1) {$b$};
\node (j) at (2,0) {$j$};
\node (u1) at (2,-1) {$u_1$};
\node (u) at (1,-1) {$u$};
\draw (a) -- (b);
\draw (b) -- (j);
\draw (j) -- (u1);
\end{tikzpicture}}
\end{array}
\quad \quad \begin{array}{c}
\text{\begin{tikzpicture}
\node (a) at (0,0) {$a$};
\node (b) at (1,1) {$b$};
\node (j) at (2,0) {$j$};
\node (u1) at (2,-1) {$u_1$};
\node (u) at (1,-1) {$u$};
\draw (a) -- (b);
\draw (b) -- (j);
\draw (j) -- (u1);
\end{tikzpicture}}
\end{array}
\end{align*}
\end{enumerate}

In the first one, we obtain a trivial permitted thread in $u_1$ applying $V_a$, in the second case, if $j = a$ we have one of representatives of type (3) and if $j \neq a$ we get a trivial permitted thread in $b$ after applying $V_a^{-1}$. The last two cases correspond to algebras with $\#\phi_A = 1$.

Case 2.- All arrows but one, defined by vertices $b$ and $a$, are in a cycle. We have one of the following situations
i) If \( j \neq a \) we get a trivial permitted thread in \( j \) after applying \( V_a \). If \( j = a \), that is, if there is a loop \( \lambda \) in \( a \) there are two possibilities

\[
\begin{array}{c}
\begin{array}{c}
\text{or } \begin{array}{c}
\sum_{i} m_{ij} a_i b_j \end{array}
\end{array}
\end{array}
\]

in the first one we obtain a trivial permitted thread in \( j \) by applying \( V_b^{-1} \) and in the second one we get a trivial permitted thread in \( b \) by applying \( L\lambda \).

ii) We know there can not be a loop in \( a \) because it is a finite-dimensional algebra, and then we get a trivial permitted thread in \( b \) after applying \( V_a \).

iii) If \( j_1 \neq j_2 \), we produce a trivial permitted thread in \( j_1 \) by applying \( V_a^{-1} \). If \( j_1 = j_2 \) there are only two possibilities:

\[
\begin{array}{c}
\begin{array}{c}
\text{or } \begin{array}{c}
\sum_{i} m_{ij} a_i b_j \end{array}
\end{array}
\end{array}
\]

in \( c \) by using \( V_b^{-1} \) and in the second one, after applying \( F_{\delta'}^{-1} \) we obtain a quiver in which every arrow belongs to a cycle, so it is reduced to case 1.

Case 3.- There are exactly two arrows not belonging to a cycle. There is one of the following options:

i) \[
\begin{array}{c}
\begin{array}{c}
\text{or } \begin{array}{c}
\sum_{i} m_{ij} a_i b_j \end{array}
\end{array}
\end{array}
\]

i) We can have

\[
\begin{array}{c}
\begin{array}{c}
\text{or } \begin{array}{c}
\sum_{i} m_{ij} a_i b_j \end{array}
\end{array}
\end{array}
\]

so applying \( V_{c}^{-1} \) we get a trivial permitted thread in \( x \) or applying \( F_{\delta} \) it is reduced to case 2, respectively.

ii) Is dual to i).

iii) Applying \( F_{\delta} \) we obtain an algebra like in i) or it is reduced to case 2.

Finally, recall that trivial permitted threads correspond to quivers with relations presenting transition vertices or branches, and by the conditions of \( A \) and Lemmas 12 and 13 this is impossible. The only remaining options are \#\phi_A = 1, or \( A \) derived equivalent to one of the representatives (1), (2) or (3) of Section 3.
Lemma 16. Let $A = kQ/\langle \mathcal{P} \rangle$ be a gentle algebra, with $Q$ a quiver of two cycles and $\#Q_0 \geq 6$. Let $x \in Q_0$ as in Definition 7 and $T$ an elementary transformation which can be applied to $A \setminus \{x\}$. Then $A$ is derived equivalent under elementary transformations to an algebra $B = kQ'/\langle \mathcal{P}' \rangle$ such that the corresponding $x \in Q'_0$ is also a vertex as in Definition 7 and such that $T(A \setminus \{x\}) = B \setminus \{x\}$.

Proof:
Let us analyze each one of the possible cases for vertex $x$. Observe that any elementary transformation applied to $A \setminus \{x\}$ can be also be applied to $A$. Let $H_u$ and $H_v$ be the permitted threads in $A$ which involve $u$ and $v$ and which do not involve $\alpha$ and $\beta$, $H_u$ and $H_v$ be the permitted threads in $A$ which involve $\alpha$ and $\beta$ respectively. In the corresponding presentations of $A \setminus \{x\}$ and $A$ in terms of their permitted threads we have, in each case:

(1) The presentations of $A \setminus \{x\}$ and $A$ are

$$
\begin{bmatrix}
s(H_u) & s(H_u) & s(H_v) \\
\vdots & \vdots & \vdots \\
u^+ & u^- & v \\
\vdots & \vdots & \vdots \\
e(H_u) & e(H_v)
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
s(H_u) & s(H_u) & s(H_v) & x^- \\
\vdots & \vdots & \vdots & \vdots \\
u^+ & u^- & v^- & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
e(H_u) & e(H_v)
\end{bmatrix}
$$

When the transformation $T$ does not move $u^+$ after $v^-$, or $v^+$ before $u^-$, after applying $T$ to $A$ the trivial permitted thread $x^-$ does not disappear, $x$ is also a vertex as in the first part of Definition 7 and we obtain the result defining $B = T(A)$. Now, if $u^+$ is moved after $v^-$:

$$
\begin{bmatrix}
s(H_u) & s(H_u) & s(H_v) \\
\vdots & \vdots & \vdots \\
u^+ & u^- & v^- \\
\vdots & \vdots & \vdots \\
e(H_u) & e(H_v)
\end{bmatrix}
\quad \overset{T}{\longrightarrow} \quad
\begin{bmatrix}
s(H_u) & \vdots & s(H_v) \\
\vdots & u^- & \vdots \\
v^+ & v^- & \vdots \\
\vdots & \vdots & \vdots \\
e(H_u) & e(H_v)
\end{bmatrix}
$$

In $A$ we do the following, if $e(H_v) \neq v^-$
In any case, the final array describes an algebra \( B \) which is obtained from the corresponding array of \( T(A \setminus \{x\}) \) adding a permitted thread \( x^- \) corresponding to a vertex \( x \in Q_0 \) as the one described in Definition 7. If \( T \) involves the movement of \( v^+ \) before \( u^- \), we do something similar but using transformation \( F_{(x^+,u^+)}^{-1} \).

(2) Presentations of \( A \setminus \{x\} \) and \( A \) are
The only way which could make the permitted thread in $x$ disappear would be the movement of $w^-$ before $x^-$, but this does not happen because $w$ is the beginning of a permitted thread in $A \setminus \{x\}$. In this case $B := T(A)$ solves the problem.

Applying previous lemma several times we have:

**Corollary 17.** Let $A = kQ/\langle P \rangle$ be a gentle algebra, with $Q$ a quiver with two cycles and $\#Q_0 \geq 6$. Let $x \in Q_0$ as in Definition 7 and $T$ a composition of elementary transformations such that $T(A \setminus \{x\})$ makes sense. Then $A$ is derived equivalent under elementary transformations to an algebra $B = kQ'/\langle P' \rangle$ such that the corresponding $x \in Q'_0$ is also as in Definition 7 and $T(A \setminus \{x\}) = B \setminus \{x\}$.

### 5.2. Adding a vertex to the representatives.

We analyze now the different ways in which it is possible to add a vertex as the ones described in Definition 7 to the representatives of Section 3 and we prove that in any case there exist some elementary transformations which turn these algebras into one of the normal forms. First let us see what happens if the added vertex is a transition vertex. We need the following:

**Remark 18.**

\[
\text{ is derived equivalent to } \]

\[
\text{under transformation } F_δ.
\]

Under certain conditions, adding a vertex which defines a trivial permitted thread to a branch and removing it from another produces derived equivalent algebras.
Lemma 19. The gentle algebra defined by the next quiver with relations

\[
\begin{array}{c}
\bullet w_1 \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow 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\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow 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\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \righte
is derived equivalent to

\[ \xymatrix{ x 
\ar[r] & v_{a+k} 
\ar[r] & v_{a+1} 
\ar[d] & \ar[r] & u_{a-1} 
\ar@{..}[r] & \ar[r] & u_{a+1} 
\ar[r] & \ar@{..}[r] & u_{a+k} 
\ar[r] & \ar@{..}[r] & u_{a+1} 
\ar[r] & \ar@{..}[r] & u_{a+k} } \]

\[ \xymatrix{ w_1 
\ar[r] & w_2 
\ar@{..}[r] & \ar[r] & w_r } \]

We work now with the normal forms and we analyze all different ways in which we can add a vertex as described in Definition 7. If it is a transition vertex, in most of the cases using Remark 18 and the symmetry of the normal forms, we can transform the algebra into one of the representatives by using elementary transformations, like in the case of representative (7). However, the conditions over the parameters \( a, b, k \) and \( q \) could change, we need then the following results:

**Proposition 21.** The algebras associated to the next quiver with relations are derived equivalent.

\[ \xymatrix{ x 
\ar[r] & v_{a+k} 
\ar[r] & v_{a+1} 
\ar[d] & \ar[r] & u_{a-1} 
\ar@{..}[r] & \ar[r] & u_{a+1} 
\ar[r] & \ar@{..}[r] & u_{a+k} 
\ar[r] & \ar@{..}[r] & u_{a+1} 
\ar[r] & \ar@{..}[r] & u_{a+k} } \]

\[ \xymatrix{ w_1 
\ar[r] & w_2 
\ar@{..}[r] & \ar[r] & w_r } \]

\[ \xymatrix{ v_{a+k} 
\ar[r] & v_{a+k-1} 
\ar[r] & v_{a+1} 
\ar[d] & \ar[r] & u_{a-1} 
\ar@{..}[r] & \ar[r] & u_{a+1} 
\ar[r] & \ar@{..}[r] & u_{a+k} 
\ar[r] & \ar@{..}[r] & u_{a+1} 
\ar[r] & \ar@{..}[r] & u_{a+k} } \]

\[ \xymatrix{ w_1 
\ar[r] & w_2 
\ar@{..}[r] & \ar[r] & w_r } \]

**Remark 22.** \( \phi_A = \phi_B = [(k, a + k), (k + 1, a + k + 1), (2a - 2 + r, r)] \).

**Proof:**

Applying \( V_{v_a}^{-1} \) to \( A \) we get:

\[ \xymatrix{ x 
\ar[r] & v_{a+k} 
\ar[r] & v_{a+1} 
\ar[d] & \ar[r] & u_{a-1} 
\ar@{..}[r] & \ar[r] & u_{a+1} 
\ar[r] & \ar@{..}[r] & u_{a+k} 
\ar[r] & \ar@{..}[r] & u_{a+1} 
\ar[r] & \ar@{..}[r] & u_{a+k} } \]

\[ \xymatrix{ w_1 
\ar[r] & w_2 
\ar@{..}[r] & \ar[r] & w_r } \]

Using Lemma 19 and Remarks 18 and 20 it is derived equivalent to
Renaming the vertices, it is the quiver with relations associated to $B$. 

**Proposition 23.** The algebras associated to the following quivers with relations are derived equivalent.

\[
\begin{align*}
A: & \quad x \to v_1 \to v_2 \to \ldots \to v_{a+k} \to u_0 \to u_1 \\
& \quad w_1 \to w_2 \to \ldots \to w_r \\
B: & \quad x \to v_1 \to v_2 \to \ldots \to v_{a+k} \to u_0 \to u_1 \\
& \quad w_1 \to w_2 \to \ldots \to w_r 
\end{align*}
\]

**Remark 24.** $\phi_A = \phi_B = [(k, k), (k + 1, k + 1), (r, r + 2)]$.

**Proof:**
Using Remark 18, $A$ is derived equivalent to

\[
\begin{align*}
A: & \quad x \to v_1 \to v_2 \to \ldots \to v_{q-1} \to v_q \to u_k \\
& \quad w_1 \to w_2 \to \ldots \to w_r \\
B: & \quad x \to v_1 \to v_2 \to \ldots \to v_{q-1} \to v_q \to u_k \\
& \quad w_1 \to w_2 \to \ldots \to w_r 
\end{align*}
\]

Applying the result $r$ times and renaming vertices we obtain $B$. 

**Proposition 25.** The algebras associated to the next quiver with relations are derived equivalent.

\[
\begin{align*}
A: & \quad x \to v_1 \to v_2 \to \ldots \to v_{q-1} \to v_q \to u_k \\
& \quad w_1 \to w_2 \to \ldots \to w_r \\
B: & \quad x \to v_1 \to v_2 \to \ldots \to v_{q-1} \to v_q \to u_k \\
& \quad w_1 \to w_2 \to \ldots \to w_r 
\end{align*}
\]

**Remark 26.** $\phi_A = \phi_B = [(k, k), (q + 1, q + 2), (q, q + 1)]$. 
Proof:
Using Remark 9 several times, $A$ is derived equivalent to

$$x \to v_1 \to v_{q-1} \to v_q \to u_k \to u_1 \to u_0 \to w_{q-1} \to w_2 \to w_1$$

After applying $V_{u_0}$ and $V_{w_q}$ we have:

$$x \to v_1 \to v_{q-1} \to v_q \to u_k \to u_1 \to u_0 \to w_{q-1} \to w_2 \to w_1$$

and applying $V_{u_1}$ it is derived equivalent to

$$x \to v_1 \to v_{q-1} \to v_q \to u_k \to u_1 \to u_0 \to w_{q-1} \to w_2 \to w_1$$

after renaming the vertices we get

$$x \to v_1 \to v_{q-1} \to v_q \to u_k \to u_1 \to u_0 \to w_{q-1} \to w_2 \to w_1$$

Applying $V_{u_0}$ and $V_{w_q}$ we get

$$x \to v_1 \to v_{k-1} \to v_k \to u_1 \to u_0 \to w_{q-1} \to w_2 \to w_1$$

After $k$ steps we obtain

$$x \to v_1 \to v_{q-1} \to v_q \to u_k \to u_1 \to u_0 \to w_{q-1} \to w_2 \to w_1$$

which is derived equivalent to $B$ by Remark 9.

If the added vertex is as in Definition 7 (2), and there is in fact an arrow $\kappa$ as the one described in the mentioned definition, we can reduce the problem to the case of a transition vertex using Remark 9 which has been just analyzed. If this is not the case, by Remark 20 and the type of normal forms which we are working with, it is enough to prove the next result:

**Proposition 27.** The algebras associated to the following quivers with relations are derived equivalent.
Remark 28. $\phi_A = \phi_B = [(k, a + k), (q, b + q), (a + b - 2 + r + 1, r)]$.

Proof:
Let $\delta$ be the only arrow of $A$ such that $s(\delta) = x$ and $e(\delta) = v_{b-1}$. After applying $F_\delta$ and $V_{v_{b-1}}$ we get

\[
\begin{array}{c}
v_{b+q} \rightarrow v_{b+q-1} \rightarrow \rightarrow v_{b+1} \\
v_1 \rightarrow v_2 \rightarrow \rightarrow v_{b-1} \rightarrow \rightarrow v_b \rightarrow v_{a+1} \rightarrow \rightarrow u_{a+k} \\
\end{array}
\]

\[
\begin{array}{c}
u_{a-1} \rightarrow \rightarrow w_2 \rightarrow w_1 \rightarrow w_r \\
u \rightarrow \rightarrow v_1 \rightarrow \rightarrow v_2 \rightarrow \rightarrow x \rightarrow \rightarrow v_{b-1} \rightarrow \rightarrow v_b \rightarrow v_{a+1} \rightarrow \rightarrow u_{a+k} \\
\end{array}
\]

and by Lemma 19 and Remark 20 it is derived equivalent to

\[
\begin{array}{c}
v_{b+q} \rightarrow v_{b+q-1} \rightarrow \rightarrow v_{b+1} \\
v_1 \rightarrow v_2 \rightarrow \rightarrow x \rightarrow \rightarrow v_{b-1} \rightarrow \rightarrow v_b \rightarrow v_{a+1} \rightarrow \rightarrow u_{a+k} \\
\end{array}
\]

\[
\begin{array}{c}
u_{a-1} \rightarrow \rightarrow w_2 \rightarrow w_1 \rightarrow w_r \\
\end{array}
\]

After applying $V_{v_b}$ we get:

\[
\begin{array}{c}
v_{b+q} \rightarrow v_{b+q-1} \rightarrow \rightarrow v_{b+1} \\
v_1 \rightarrow v_2 \rightarrow \rightarrow x \rightarrow \rightarrow v_{b-1} \rightarrow \rightarrow v_b \rightarrow v_{a+1} \rightarrow \rightarrow u_{a+k} \\
\end{array}
\]

\[
\begin{array}{c}
u_{a-1} \rightarrow \rightarrow w_2 \rightarrow w_1 \rightarrow w_r \\
\end{array}
\]

and by Lemma 19 and Remark 20 it is derived equivalent to $B$. \hfill \square

6. Proof of the main result

In this section we present induction proofs of Theorems I and II, which use lemmas and propositions of Section 5. For the induction basis a computer program in Groups Algorithms and Programming (GAP) version 18-may-2000 was developed in order to calculate all derived equivalent classes of gentle algebras $A$ with two cycles, $\phi_A =$
3 and at most five vertices. The program and the calculations are presented in www.matem.unam.mx/avella

From calculations we see all derived equivalent classes under elementary transformations have distinct invariants except in case of algebras defined by quivers with 4 vertices and \( \phi_A = \langle [1, 1], [1, 1], [1, 3] \rangle \), were there are two distinct classes with the same invariant:

\[
\langle [1, 2, 3, 4], [5, 6], [7, 8], [1, 5], [2, 6], [3, 7], [4, 8], [1, 1], [1, 1], [1, 3], 1 \rangle
\]

appears in the first one and

\[
\langle [1, 2, 3, 4], [5, 6], [7, 8], [1, 5], [2, 7], [3, 8], [4, 6], [1, 1], [1, 1], [1, 3], 2 \rangle
\]

in the second one. These algebras correspond to the following quiver with relations:

\[
1: \quad \begin{array}{c}
v_4 \\ \downarrow \\ v_3 \\ \downarrow \\ v_2 \\ \downarrow \\ v_1 \\
\end{array}
\]

\[
2: \quad \begin{array}{c}
v_2 \\ \downarrow \\ v_1 \\ \downarrow \\ v_0 \\
\end{array}
\]

they are a coextension and an extension in a vertex of the algebra defined by the quiver with relations

\[
v_3 \quad \begin{array}{c}
\uparrow \\ \downarrow \\ \downarrow \\ \downarrow \\
v_2 \\ \downarrow \\ \downarrow \\ v_1 \\
\end{array}
\]

the corresponding repetitive algebras, see [Rin97], are isomorphic and they have finite global dimension so they are derived equivalent, see [HW83]. However we can not transform one into the other by using the elementary transformations.

Now we prove Theorem I by induction over the number of vertices.

Let \( A = kQ/\langle P \rangle \) be a gentle algebra, with \( Q \) a connected quiver of two cycles. For \( \#Q_0 \leq 5 \) we present the calculations of \( \phi_A \) for each such algebra, see www.matem.unam.mx/avella, in all cases \( \#\phi_A \in \{1, 3\} \). Consider now \( \#Q_0 \geq 6 \).

By Proposition 14 there exists \( x \in Q_0' \) of the following type:

(1)

\[
\begin{array}{c}
u \\
\downarrow \\
x \\
\downarrow \\
v \\
\end{array}
\]

\( x \) of degree 2

(2)

\[
\begin{array}{c}
u \\
\downarrow \\
x \\
\downarrow \\
\alpha \\
\downarrow \\
\beta \\
\downarrow \\
v \\
\end{array}
\]

with \( \alpha \) and \( \beta \) belonging to a cycle and \( w \) of degree 3
with $\alpha$ and $\beta$ belonging to a cycle and $x$ of degree 2

if there is no vertex $x$ of type (1) or (2), by Proposition 15 $\# \phi_A = 1$ or $A$ is derived equivalent to one of the representatives (1), (2) or (3) and then $\# \phi_A \in \{1, 3\}$.

Consider now the case where there is a vertex $x$ of type (1) or (2). Consider the algebra $A \setminus \{x\}$. By Remark 8 we know that $A \setminus \{x\}$ is a gentle algebra associated to a quiver of two cycles with one vertex less than the one defining $A$; then, using induction hypothesis $\# \phi_{A \setminus \{x\}} \in \{1, 3\}$, which means that the algorithm to calculate the invariants of $A \setminus \{x\}$ produces three pairs of natural numbers. Now we analyze the differences presented in the corresponding algorithm applied to $A$, see [AG07, 3].

Case 1.-

Observe that the process is different only when we go through $\alpha$ and $\beta$ backwards, that is, when those arrows are considered as part of forbidden threads. Let $\Pi_\alpha = \alpha \rho_s \ldots \rho_1$ and $\Pi_\beta = \gamma_r \ldots \gamma_1 \beta$ be the forbidden threads of $A$ involving $\alpha$ and $\beta$ resp., $\hat{H}$ be the permitted thread such that $s(\hat{H}) = s(\Pi_\alpha)$ and $\sigma(\hat{H}) = -\sigma(\Pi_\alpha)$, and let $H$ be the permitted thread such that $e(H) = e(\Pi_\beta)$ and $\eta(H) = -\eta(\Pi_\beta)$. In the algorithm for $A \setminus \{x\}$ we have:

\[
\begin{align*}
H_{i}^{A \setminus \{x\}} &= H \quad (\Pi_i^{A \setminus \{x\}})^{-1} = (\gamma_r \ldots \gamma_1 \alpha' \rho_s \ldots \rho_1)^{-1} \\
H_{i+1}^{A \setminus \{x\}} &= \hat{H}
\end{align*}
\]

for some natural $i$, where $\alpha'$ being an arrow as in Definition 7. This step of the algorithm produces a pair $(n, m)$. In the corresponding algorithm for $A$ we get

\[
\begin{align*}
H_i^A &= H \quad (\Pi_i^A)^{-1} = (\Pi_\beta)^{-1} = (\gamma_r \ldots \gamma_1 \beta)^{-1} \\
H_{i+1}^A &= 1_w \quad (\Pi_i^A)^{-1} = (\Pi_\alpha)^{-1} = (\alpha \rho_s \ldots \rho_1)^{-1} \\
H_{i+2}^A &= \hat{H}
\end{align*}
\]

and we obtain the pair $(n + 1, m + 1)$.

Case 2.-

Observe the process only changes when we go through $\alpha$ and $\beta$ in the sense of the arrows, that is, when they are considered as part of permitted threads. Let $H_\alpha = \alpha \rho_s \ldots \rho_1$ and $H_\beta = \gamma_r \ldots \gamma_1 \beta$ be the permitted threads of $A$ which involve $\alpha$ and $\beta$ resp. In the algorithm $A \setminus \{x\}$ we have:

\[
\begin{align*}
H_i^{A \setminus \{x\}} &= H_\alpha = \alpha \rho_s \ldots \rho_1 \quad (\Pi_i)^{-1} = 1_w \\
H_{i+1}^{A \setminus \{x\}} &= H_\beta = \gamma_r \ldots \gamma_1 \beta
\end{align*}
\]
for some natural $i$, and this part of the algorithm produces a pair $(n, m)$. In the corresponding algorithm for $A$ we have
\[
H_i^A = H_{\alpha} = \alpha \rho_s \ldots \rho_1 \quad (\Pi_i^A)^{-1} = \eta^{-1} \\
H_{i+1}^A = 1_x \quad (\Pi_{i+1}^A)^{-1} = 1_x \\
H_{i+2}^A = \gamma_r \ldots \gamma_1 \beta \eta
\]
and then we get the pair $(n + 1, m + 1)$.

Then, $\#\phi_A \setminus \{x\} = \#\phi_A$, in fact the algorithm to calculate $\phi_A$ produces exactly the same pairs but one, which is different from the one in $\phi_A \setminus \{x\}$ only by a summand $(1, 1)$. So the result is completed.

Now we prove Theorem II which gives the classification of gentle algebras with quivers of two cycles and three series of characteristic components under derived equivalence:

For $\#Q_0 \leq 5$ we have the complete classification of gentle algebras $A = kQ/\langle P \rangle$, with quivers of two cycles under derived equivalence, presented in www.matem.unam.mx/avella. We observe the result fulfills and in fact, for $\#Q_0 = 5$ two derived equivalent algebras of that kind can be transform one into other by using a composition of elementary transformations.

Now consider a gentle algebra $A = kQ/\langle P \rangle$, $Q$ a quiver with two cycles, $\#Q_0 \geq 6$ and $\#\phi_A = 3$. We prove it is derived equivalent to one of the representatives of the ones described in Section 3. If there is $x \in Q_0$ as in (1) or (2) of Proposition 14, consider the algebra $A \setminus \{x\}$, see Definition 7. We know $A \setminus \{x\}$ is a gentle algebra with a quiver of two cycles, one vertex less than $A$ and, by the proof of Theorem I, with $\phi_A \setminus \{x\} = 3$. We apply then the induction hypothesis to $A \setminus \{x\}$, and conclude that it is derived equivalent to one of the representatives described in Section 3, call this representative $\hat{R}_{A \setminus \{x\}}$; moreover there is a composition of elementary transformations $T$ such that $T(A \setminus \{x\}) = \hat{R}_{A \setminus \{x\}}$. By Corollary 17 we know $A$ is derived equivalent to an algebra $\hat{R}_A = kQ'/\langle P' \rangle$ such that the corresponding $x \in Q'_0$ is as the one described in Definition 7 and such that $T(A \setminus \{x\}) = \hat{R}_A \setminus \{x\}$, so $R_{A \setminus \{x\}} = \hat{R}_A \setminus \{x\}$. Then $\hat{R}_A$ is obtain from the representative $R_{A \setminus \{x\}}$ by adding a vertex as the one of Definition 7. By the results of previous section, $\hat{R}_A$ is a representative of the ones described in Section 3 or can be transform into one by using elementary transformations and this concludes the proof.

APPENDIX

7. Elementary transformations of gentle algebras (after a manuscript by T. Holm, J. Schröer, A. Zimmermann)

This Appendix is a slightly modified version of a technical result in the unpublished manuscript [HSZ01]. We show here that the combinatorial transformations which are essential for the proof of Theorem II are indeed derived equivalences.

7.1. Transformations over a vertex. Let $A = kQ/\langle P \rangle$ be a gentle algebra with $Q$ connected such that there exist $\alpha_1, \alpha_2 \in Q_1 \alpha_1 \neq \alpha_2$ and $s(\alpha_1) = s(\alpha_2)$. Denote $s(\alpha_1)$ by $i$, $j_1 := e(\alpha_1)$ and $j_2 := e(\alpha_2)$. Suppose also that $j_1 \neq i \neq j_2$ ($j_1$ and $j_2$
could be equal). The description of the general situation which can be found around the vertex \( i \) in the quiver with relations is the following:

\[
\begin{array}{c}
\begin{array}{c}
s_1 \\
\sigma_1 \\
j_1 \\
\pi_1 \\
p_1 \\
\end{array}
\begin{array}{c}
s_2 \\
\sigma_2 \\
j_2 \\
\pi_2 \\
p_2 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\alpha_1 \\
i \\
\alpha_2 \\
\end{array}
\end{array}
\]

We define a new algebra \( V_i(A) = kQ'/\langle P' \rangle \) with \( Q'_0 = Q_0, Q'_1 \) and \( P' \) according to the following cases:

1. If \( s_1 \neq i \neq s_2 \) let \( \alpha'_m, \pi'_m, \sigma'_m \) for \( m \in \{1, 2\} \) be arrows such that \( j_m = s(\alpha'_m) = e(\pi'_m), i = s(\sigma'_m) = e(\alpha'_m), p_m = s(\pi'_{3-m}) \) and \( s_m = e(\sigma'_m) \). Define
   \[
   Q'_1 := (Q_1 \setminus \{\alpha_1, \alpha_2, \pi_1, \pi_2, \sigma_1, \sigma_2\}) \cup \{\alpha'_1, \alpha'_2, \pi'_1, \pi'_2, \sigma'_1, \sigma'_2\}
   \]
   and
   \[
   P' := (P \setminus \{\sigma_1\alpha_1, \sigma_2\alpha_2, \alpha_1\pi_1, \alpha_2\pi_2\}) \cup \{\sigma'_1\alpha_2, \sigma'_2\alpha_1, \alpha'_1\pi_1, \alpha'_2\pi_2\}
   \]
   that is

2. If \( s_1 = i \) we have
   (a)

\[
\begin{array}{c}
\begin{array}{c}
s_1 \\
\sigma_1 \\
j_1 \\
\pi_1 \\
p_1 \\
\end{array}
\begin{array}{c}
s_2 \\
\sigma_2 \\
j_2 \\
\pi_2 \\
p_2 \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\alpha_1 \\
i \\
\alpha_2 \\
\end{array}
\end{array}
\]

let \( \alpha'_m, \pi'_m \) for \( m \in \{1, 2\} \) and \( \sigma'_2 \) be arrows such that \( j_m = s(\alpha'_m) = e(\pi'_m), i = s(\sigma'_2) = e(\alpha'_m), p_2 = s(\pi'_1) \) and \( s_2 = e(\sigma'_2) \). Define
   \[
   Q'_1 := (Q_1 \setminus \{\alpha_1, \alpha_2, \pi_1, \pi_2, \sigma_2\}) \cup \{\alpha'_1, \alpha'_2, \pi'_1, \pi'_2, \sigma'_2\}
   \]
   and
   \[
   P' := (P \setminus \{\pi_1\alpha_1, \sigma_2\alpha_2, \alpha_1\pi_1, \alpha_2\pi_2\}) \cup \{\pi'_2\alpha_2, \sigma'_2\alpha_1, \alpha'_1\pi_1, \alpha'_2\pi_2\}
   \]
   that is
or

(b)

and we define $\alpha'_m, \pi'_m$ for $m \in \{1, 2\}$ and $\sigma'_2$ arrows such that $j_m = s(\alpha'_m) = e(\pi'_m)$, $i = s(\sigma'_2) = s(\pi'_1) = e(\alpha'_m)$, $p_1 = s(\pi'_2)$ and $s_2 = e(\sigma'_2)$. Define

$$Q'_1 := (Q \setminus \{\alpha_1, \alpha_2, \pi_1, \pi_2, \sigma_2\}) \cup \{\alpha'_1, \alpha'_2, \pi'_1, \pi'_2, \sigma'_2\}$$

and

$$P' := (P \setminus \{\pi_2\alpha_1, \sigma_2\alpha_2, \alpha_1\pi_1, \alpha_2\pi_2\}) \cup \{\pi'_1\alpha_2, \sigma'_2\alpha_1, \alpha'_1\pi'_1, \alpha'_2\pi'_2\}$$

that is

(3) If $s_1 = s_2 = i$ the algebra remains unchanged.

Denote the inverse transformation by $V_i^{-1}$.

7.2. Transformations over an arrow. Let $A = kQ/\langle P \rangle$ be a gentle algebra with $Q$ connected and $\delta \in Q_1$ such that $i := s(\delta)$ $j := e(\delta)$, $i \neq j$. The description of the general situation which can be found around the arrow $\alpha$ in the quiver with relations is the following:
Let $\hat{Q}$ be the quiver obtained of $Q$ by deleting the arrows $\beta$ and $\gamma$, that is $\hat{Q}_0 = Q_0$, $\hat{Q}_1 = Q_1 \setminus \{\beta, \gamma\}$, and $\hat{P} = \mathcal{P} \setminus \{\delta \beta, \gamma \lambda\}$. Consider the maximal connected subquivers of $\hat{Q}$ which contain vertices $b$ and $c$ and let $Q^1$ be its the union. Define also $Q^2$ as the maximal connected subquiver of $\hat{Q}$ which contains vertex $i$. If $Q^1_0 \cap Q^2_0 = \emptyset$ define a new algebra $F_{\delta}(A) = kQ' / \langle \mathcal{P}' \rangle$ taking $Q'_0 = Q_0$, $Q'_1$ and $\mathcal{P}'$ as follows. Let $\delta', \beta', \gamma', \lambda', \xi'$ be arrows such that $j = s(\delta') = e(\lambda')$, $i = s(\gamma') = s(\xi') = e(\beta') = e(\delta')$, $b = s(\beta')$, $c = e(\gamma')$, $l = s(\lambda')$ and $x = e(\xi')$. Define
\[
Q'_1 := (Q_1 \setminus \{\delta, \beta, \gamma, \lambda\}) \cup \{\delta', \beta', \gamma', \lambda', \xi\}
\]
and
\[
\mathcal{P}' := (\mathcal{P} \setminus \{\delta \beta, \gamma \lambda, \xi \delta\}) \cup \{\gamma' \delta', \xi' \beta', \delta' \lambda'\}
\]
that is

Denote the corresponding inverse transformation by $F_{\delta}^{-1}$.

7.3. **Transformation over a loop.** Let $A = kQ / \langle \mathcal{P} \rangle$ be a gentle algebra with $Q$ connected and $\lambda \in Q_1$ be a loop of $Q$, that is, $i := s(\lambda) = e(\lambda)$. The description of the general situation which can be found around the loop in the quiver with relations is the following:

Define a new algebra $L_{\lambda}(A) = kQ' / \langle \mathcal{P}' \rangle$ taking $Q'_0 = Q_0$, $Q'_1$ y $\mathcal{P}'$ as follows. Consider $\alpha', \delta', \xi'$ arrows such that $j = s(\delta') = e(\alpha')$, $i = s(\xi') = e(\delta')$, $l = s(\alpha')$ and $x = e(\xi')$. Define
\[
Q'_1 := (Q_1 \setminus \{\alpha, \delta, \xi\}) \cup \{\alpha', \delta', \xi'\}
\]
and

\[ P' := (P \setminus \{\delta\alpha, \xi\delta\}) \cup \{\delta'\alpha', \xi'\delta'\} \]

that is

\[
\begin{array}{c}
\xymatrix{
l \ar[r]^{j} & 0 \ar[r] & P_{j_1} \ar[r]^{(\alpha_1, \alpha_2)} & P_{i} \ar[r]^{\lambda} & 0 \\
\delta \ar@/^/[r]^{j} & \xi \ar@/^/[r]^{x} & l \ar[r]^{j} & 0 \ar[r] & P_{j_1} \ar[r]^{(\alpha_1, \alpha_2)} & P_{i} \ar[r]^{\lambda} & 0 \\
\delta' \ar@/^/[r]^{j} & \xi' \ar@/^/[r]^{x} & l \ar[r]^{j} & 0 \ar[r] & P_{j_1} \ar[r]^{(\alpha_1, \alpha_2)} & P_{i} \ar[r]^{\lambda} & 0
}\end{array}
\]

Denote the corresponding inverse transformation by \( L_{\lambda}^{-1} \).

We say that \( T \) is an \textit{elementary transformation} if it is a vertex, an arrow, a loop transformation or the inverse of one of those. Two gentle algebras \( A \) and \( B \) are called \textit{derived equivalent under elementary transformations} if there exists a finite sequence of elementary transformations \( T_1, T_2, \ldots, T_r \) such that \( B = T_r \cdots T_1(A) \).

7.4. Tilting complexes associated to the transformations.

**Theorem 29.** Let \( A = kQ/\langle P \rangle \) be an algebra as defined in (1) or (2) from Section 7.1. Consider

\[ T := T_c \oplus \bigoplus_{l \in Q_0 \setminus \{i\}} P_l[1] \]

with

\[ T_c := \begin{array}{c} \cdots \ar[r] & 0 \ar[r] & P_{j_1} \oplus P_{j_2} \ar[r]^{(\alpha_1, \alpha_2)} & P_{i} \ar[r]^{\lambda} & 0 \ar[r] & \cdots \end{array} \]

Then \( T \) is a tilting complex and \( \text{End}_{D^b(A)}(T) \simeq V_i(A) \).

**Proof:**

By construction \( T \) is a tilting complex and by \[SZ03\] we know that the algebra \( \text{End}_{D^b(A)}(T) \) is gentle, so \( \text{End}_{D^b(A)}(T) = kQ''/\langle P'' \rangle \) with \( Q''_0 = Q_0' \). Denote the \( \text{End}_{D^b(A)}(T) \)-indecomposable projective modules for \( m \in Q''_0 \) by \( P'_m \), which can be identified with the indecomposable summands of \( T \). The following morphisms are irreducible:

(1)

\[
\begin{align*}
\alpha'_1 := & \begin{array}{c} \cdots \ar[r] & 0 \ar[r] & P_{j_1} \ar[r] & 0 \ar[r] & 0 \ar[r] & \cdots \\
0 \ar@/^/[r]^{(0, \text{id})} & P_{j_1} \oplus P_{j_2} \ar[r]^{(\alpha_1, \alpha_2)} & P_{i} \ar[r]^{\lambda} & 0 \ar[r] & \cdots \end{array} \\
\alpha'_2 := & \begin{array}{c} \cdots \ar[r] & 0 \ar[r] & P_{j_2} \ar[r] & 0 \ar[r] & 0 \ar[r] & \cdots \\
0 \ar@/^/[r]^{(0, \text{id})} & P_{j_1} \oplus P_{j_2} \ar[r]^{(\alpha_1, \alpha_2)} & P_{i} \ar[r]^{\lambda} & 0 \ar[r] & \cdots \end{array} \\
\pi'_1 := & \begin{array}{c} \cdots \ar[r] & 0 \ar[r] & P_{j_2} \ar[r] & 0 \ar[r] & 0 \ar[r] & \cdots \\
\alpha_1 \pi_2 \ar@/^/[u] & P_{j_1} \ar[r]^{\alpha_1 \pi_2} & 0 \ar[r] & 0 \ar[r] & \cdots \end{array}
\end{align*}
\]
because they cannot be factorized through any other indecomposable sub-
mund of $T$. The morphism $\sigma_1 : P_{s_1} \to P_{j_1}$ and $\sigma_2 : P_{s_2} \to P_{j_2}$ are irreducible in $A$, but factorize through $\alpha'_1$ and $\alpha'_2$ respectively. Also, the compositions $\sigma'_1 \alpha'_2$ and $\sigma'_2 \alpha'_1$ are zero and $\alpha'_1 \pi'_1$, $\alpha'_2 \pi'_2$ are homotopic to zero.

To see there are no further irreducible maps we use Happel’s formula, see [Ha88, III.1.3 and III.1.4]:

$$\sum_i (-1)^i \dim Hom_{K^b(A)}(Q, R[i]) = \sum_{r,s} (-1)^{r-s} \dim Hom_A(Q^r, R^s)$$

where $K^b(A)$ is the homotopy category of bounded complexes of projective $A$-modules, $[\cdot]$ is the shift operator and $Q = (Q^r)_{r \in \mathbb{Z}}$, $R = (R^s)_{s \in \mathbb{Z}}$ are objects in $K^b(A)$. For direct summands of tilting complexes $\dim Hom_{K^b(A)}(Q, R[i]) = 0$ for $i \neq 0$ so in this case

$$\dim Hom_{K^b(A)}(Q, R) = \sum_{r,s} (-1)^{r-s} \dim Hom_A(Q^r, R^s)$$

For $x, y, z \in Q_0$ denote by $(x, y)$ the set of paths starting in $y$ and ending in $x$, by $(x, y; z)$ the set of paths starting in $y$ and ending in $x$ which go through $z$ and by $(x, y; \hat{z})$ the set of paths starting in $y$ and ending in $x$ which do not go through $z$. Also for $\alpha \in Q_1$ denote by $(x, y; \alpha)$ the set of paths starting in $y$ and ending in $x$ which pass through $\alpha$ and by $(x, y; \hat{\alpha})$ the set of paths starting in $y$ and ending in $x$ which do not go through $\alpha$.

By using Happel’s formula we calculate the following:

$$\dim Hom_{K^b(A)}(T_c, P_l[1]) = \dim Hom_A(P_{j_1} \oplus P_{j_2}, P_l) - \dim Hom_A(P_i, P_l)$$

$$= \#(j_1, l) + \#(j_2, l) - \#(i, l)$$

$$= \#(j_1, l; \hat{i}) + \#(j_1, l; \hat{i}) + \#(j_2, l; i)$$

$$+ \#(j_2, l; \hat{i}) - \#(i, l; \pi_1) - \#(i, l; \pi_1)$$

$$= \#(j_1, l; \hat{i}) + \#(j_2, l; \hat{i})$$

because $\#(j_1, l; i) = \#(i, l; \pi_1)$ and $\#(j_2, l; i) = \#(i, l; \pi_1)$, but only the trivial paths in $j_1$ and $j_2$ produce irreducible morphisms, the rest factorize through these.
\[ \dim\text{Hom}_{K^b(A)}(P_1[1], T_\alpha) = \dim\text{Hom}_A(P_1, P_1 \oplus P_2) - \dim\text{Hom}_A(P_1, P_1) \\
= \#(l, j_1) + \#(l, j_2) - \#(l, i) \\
= \#(l, j_1; \sigma_1) + \#(l, j_1; \sigma_1) + \#(l, j_2; \sigma_2) \\
+ \#(l, j_2; \sigma_2) - \#(l, i; \alpha_1) - \#(l, i; \alpha_2) \\
= \#(l, j_1; \sigma_1) + \#(l, j_2; \sigma_2) \]

because \#(l, j_1; \sigma_1) = \#(l, i; \alpha_1) and \#(l, j_2; \sigma_2) = \#(l, i; \alpha_2), and the only irreducible morphisms are the induced by \( \sigma_1 \) and \( \sigma_2 \)

\[ \dim\text{Hom}_{K^b(A)}(P_1[1], P_1[1]) = \dim\text{Hom}_A(P_1, P_1) \\
= \#(j_1, l) \]

so the only irreducible morphisms in this case are the ones corresponding to \( \alpha_1 \pi_2 \) for \( l = p_2 \) or the one induced by an arrow connecting \( l \) and \( j_1 \). The calculation for \( \dim\text{Hom}_{K^b(A)}(P_1[1], P_1[1]) \) is similar.

(2) (a) Let \( \alpha'_1, \alpha'_2, \pi'_1 \) and \( \sigma'_2 \) be as in (1), and

\[ \pi'_2 := \begin{array}{c}
\cdots \\
0 \rightarrow P_{j_1} \oplus P_{j_2} \rightarrow P_1 \rightarrow 0 \\
\end{array}
\]

which is also an irreducible morphism because it cannot be factorized through any other indecomposable summand of \( T \). The morphism \( \sigma_2 : P_{s_2} \rightarrow P_{j_2} \) is irreducible in \( A \), but induces a morphism which can be factorized through \( \alpha'_2 \), while \( \alpha_2 \pi_1 \) induces a morphism which can be factorized through \( \alpha'_1 \). Also, the compositions \( \sigma'_2 \alpha'_1 \) and \( \pi'_2 \alpha'_2 \) are zero and \( \alpha'_1 \pi'_1 \) is homotopic to zero as in (1) such as \( \alpha'_2 \pi'_2 \).

(b) Let \( \alpha'_1, \alpha'_2, \pi'_1 \) and \( \sigma'_2 \) as in (1), and

\[ \pi'_1 := \begin{array}{c}
\cdots \\
0 \rightarrow P_{j_1} \oplus P_{j_2} \rightarrow P_1 \rightarrow 0 \\
\end{array}
\]

which is also an irreducible morphism because can not be factorized through any other indecomposable summand of \( T \). The morphism \( \sigma_2 : P_{s_2} \rightarrow P_{j_2} \) which is irreducible in \( A \), induces a morphism which factorizes through \( \alpha'_2 \), while \( \alpha_1 \pi_2 \) induces a morphism which factorizes through \( \alpha'_1 \). Also, the compositions \( \sigma'_2 \alpha'_1 \) and \( \pi'_1 \alpha'_2 \) are zero and \( \alpha'_1 \pi'_1 \) is homotopic to zero as in (1) such as \( \alpha'_2 \pi'_1 \).

For each case, the other irreducible morphisms in \( A - \text{mod} \) produce irreducible morphisms in \( \text{End}^{b(A)}(T) - \text{mod} \) because the corresponding projective indecomposable modules are direct summands of \( T \) of degree one. We verify there are no further irreducible maps using Happel’s formula:
(a) \[
\dim \text{Hom}_{K^b(A)}(T_c, P_l[1]) = \dim \text{Hom}_A(P_{j_1} \oplus P_{j_2}, P_l) - \dim \text{Hom}_A(P_l, P_l) \\
= \#(j_1, l) + \#(j_2, l) - \#(i, l) \\
= \#(j_1, l; i) + \#(j_1, l; \hat{i}) + \#(j_2, l; i) \\
+ \#(j_2, l; \hat{i}) - \#(i, l; \pi_1) - \#(i, l; \hat{\pi}_1) \\
= \#(j_1, l; \hat{i}) + \#(j_2, l; \hat{i})
\]

because \(\#(j_1, l; i) = \#(i, l; \pi_1)\) and \(\#(j_2, l; i) = \#(i, l; \pi_1)\), but only the trivial paths in \(j_1\) and \(j_2\) produce irreducible morphisms, the rest factorize through these.

\[
\dim \text{Hom}_{K^b(A)}(P_j[1], T_c) = \dim \text{Hom}_A(P_{j_1} \oplus P_{j_2}, P_l) - \dim \text{Hom}_A(P_l, P_l) \\
= \#(j_1, l) + \#(j_2, l) - \#(i, l) \\
= \#(l, j_1; i) + \#(l, j_1; \hat{i}) + \#(l, j_2; \sigma_2) \\
+ \#(l, j_2; \hat{\sigma}_2) - \#(l, i; j_2) - \#(l, i; \hat{j}_2) \\
= \#(l, j_1; i) + \#(l, j_2; \sigma_2)
\]

because \(\#(l, j_1; \hat{i}) = \#(l, i; \hat{j}_2)\) and \(\#(l, j_2; \sigma_2) = \#(l, i; j_2)\), and the only irreducible morphisms are the induced by \(\alpha_2 \pi_1\) (in this case \(l = j_2\)) and \(\sigma_2\) (in this case \(l = s_2\)).

\[
\dim \text{Hom}_{K^b(A)}(P_{j_1}[1], P_l[1]) = \dim \text{Hom}_A(P_{j_1}, P_l) \\
= \#(j_1, l)
\]

so this are the only irreducible morphisms in this case are the one corresponding to \(\alpha_1 \pi_2\) for \(l = p_2\) or the one induced by an arrow connecting \(l\) and \(j_1\).

The calculation for \(\dim \text{Hom}_{K^b(A)}(P_{j_2}[1], P_l[1])\) is similar.

(b) \[
\dim \text{Hom}_{K^b(A)}(T_c, P_l[1]) = \dim \text{Hom}_A(P_{j_1} \oplus P_{j_2}, P_l) - \dim \text{Hom}_A(P_l, P_l) \\
= \#(j_1, l) + \#(j_2, l) - \#(i, l) \\
= \#(j_1, l; i) + \#(j_1, l; \hat{i}) + \#(j_2, l; i) \\
+ \#(j_2, l; \hat{i}) - \#(i, l; \pi_1) - \#(i, l; \hat{\pi}_1) \\
= \#(j_1, l; \hat{i}) + \#(j_2, l; \hat{i})
\]

because \(\#(j_1, l; i) = \#(i, l; \pi_1)\) and \(\#(j_2, l; i) = \#(i, l; \pi_1)\) but only the trivial paths in \(j_1\) and \(j_2\) produce irreducible morphisms, the rest factorize through these.
\[ \dim \text{Hom}_{K^b(A)}(P_1[1], T_e) = \dim \text{Hom}_A(P_i, P_{j_1} \oplus P_{j_2}) - \dim \text{Hom}_A(P_i, P_i) \]
\[ = \#(l, j_1) + \#(l, j_2) - \#(l, i) \]
\[ = \#(l, j_1; \hat{i}) + \#(l, j_1; \hat{i}) + \#(l, j_2; \sigma_2) \]
\[ + \#(l, j_2; \sigma_2) - \#(l, i; \alpha_2) - \#(l, i; \alpha_2) \]
\[ = \#(l, j_1; i) + \#(l, j_2; \sigma_2) \]

because \( \#(l, j_1; \hat{i}) = \#(l, i; \alpha_2) \) and \( \#(l, j_2; \sigma_2) = \#(l, i; \alpha_2) \), and the only irreducible morphisms are the induced by \( \alpha_1 \pi_2 \) (in this case \( l = j_1 \)) and \( \sigma_2 \) (in this case \( l = s_2 \)).

\[ \dim \text{Hom}_{K^b(A)}(P_{j_1}[1], P_1[1]) = \dim \text{Hom}_A(P_{j_1}, P_1) \]
\[ = \#(j_1, l) \]

so this are the only irreducible morphisms in this case are the one corresponding to \( \alpha_1 \pi_2 \) for \( l = p_2 \) or the one induced by an arrow connecting \( l \) and \( j_1 \).

The calculation for \( \dim \text{Hom}_{K^b(A)}(P_{j_2}[1], P_1[1]) \) is similar.

Then \( \text{End}_{D^b(A)}(T) \) can be identified with \( V_i(A) \).

\[ \square \]

**Remark 30.** In case (3) of Section 7.1, using the previous tilting complex, \( \text{End}_{D^b(A)}(T) \) identifies with an algebra defined by the same quiver with relations which defines \( A \).

**Theorem 31.** Let \( A = kQ/\langle \mathcal{P} \rangle \) as in Section 7.2. Define

\[ T := T_e \oplus \bigoplus_{m \in Q_0^1} P_m \oplus \bigoplus_{r \in Q_0^2 \setminus \{i\}} P_r[1] \]

with

\[ T_e := P_0 \to P_1 \to P_2 \to P_3 \to \cdots \]

Then \( T \) is a tilting complex and \( \text{End}_{D^b(A)}(T) \simeq F_\delta(A) \).

**Proof:**

By construction \( T \) is a tilting complex. By [SZ03] we know the algebra \( \text{End}_{D^b(A)}(T) \) is gentle, we write then \( \text{End}_{D^b(A)}(T) = kQ''/\langle \mathcal{P}' \rangle \) with \( Q''_0 = Q_0 \). Denote the \( \text{End}_{D^b(A)}(T) \)-indecomposable projective modules for \( m \in Q_0'' \) by \( P_m'' \), which are identified with the indecomposable summands of \( T \). We analyze the irreducible morphisms in \( \text{End}_{D^b(A)}(T) - \text{mod.} \) Some of them are the following:

\[ \beta' := \begin{array}{cccccccc}
\cdots & 0 & \to & P_3 & \to & P_1 & \to & 0 & \to & \cdots \\
\cdots & 0 & \to & P_3 & \to & P_1 & \to & 0 & \to & \cdots \\
\cdots & 0 & \to & P_3 & \to & P_1 & \to & 0 & \to & \cdots \\
\end{array} \]

\[ \gamma' := \begin{array}{cccccccc}
\cdots & 0 & \to & P_3 & \to & P_1 & \to & 0 & \to & \cdots \\
\cdots & 0 & \to & P_3 & \to & P_1 & \to & 0 & \to & \cdots \\
\cdots & 0 & \to & P_3 & \to & P_1 & \to & 0 & \to & \cdots \\
\end{array} \]
they do not factorize through any other indecomposable summand of $T$ and are therefore irreducible. The morphism $\xi : P_x \to P_j$ is irreducible in $A$, but produces a morphism which factorizes through $\delta'$. Also, compositions $\xi' \beta'$ and $\gamma' \delta'$ are zero and $\delta' \lambda'$ is homotopic to zero.

The rest of the irreducible morphisms in $A - \text{mod}$ produce irreducible morphisms in $\text{End}_{D^b(A)}(T - \text{mod})$ because the corresponding indecomposable projective modules are direct summands of $T$ in the suitable degrees.

We verify there are no further irreducible maps using Happel’s formula. We calculate the following:

$$\dim \text{Hom}_{K^b(A)}(T_c, P_m) = \dim \text{Hom}_A(P_i, P_m) - \dim \text{Hom}_A(P_j, P_m)$$

$$= \#(i, m)$$

but only $\beta$ produce an irreducible morphism, for $m = b$.

$$\dim \text{Hom}_{K^b(A)}(P_r[1], T_c) = \dim \text{Hom}_A(P_r, P_j) - \dim \text{Hom}_A(P_r, P_i)$$

$$= \#(r, j) - \#(r, i) = \#(r, j; x) + \#(r, j; \hat{x}) - \#(r, i; j)$$

$$= \#(r, j; x)$$

because $\#(r, j; \hat{x}) = \#(r, i; j)$, and the only irreducible morphism is the one induced by $\xi$ for $r = x$.

$$\dim \text{Hom}_{K^b(A)}(P_m, T_c) = \dim \text{Hom}_A(P_m, P_i) - \dim \text{Hom}_A(P_m, P_j)$$

$$= \#(m, i) - \#(m, j) = \#(m, i)$$

but only $\gamma$ produce an irreducible morphism, for $m = c$.

$$\dim \text{Hom}_{K^b(A)}(T_c, P_r[1]) = \dim \text{Hom}_A(P_j, P_r) - \dim \text{Hom}_A(P_r, P_r)$$

$$= \#(j, r) - \#(i, r) = \#(j, r; i) + \#(j, r; \hat{i}) - \#(i, r)$$

$$= \#(j, r; \hat{i})$$

because $\#(j, r; i) = \#(i, r)$ but the only irreducible morphism is the one induced by the trivial path in $j$ for $r = j$, the others factorize through this one.

Then $\text{End}_{D^b(A)}(T)$ identifies with $F_\delta(A)$.

So these are the only irreducible morphisms.
Theorem 32. Let $A = kQ/\langle P \rangle$ as in Section 7.3. Define

$$T := T_c \oplus \bigoplus_{m \in Q_0 \setminus \{i\}} P_m[1]$$

with

$$T_c := \cdots \to 0 \to P_j \oplus P_j \xrightarrow{(\delta, \delta \lambda)} P_i \to 0 \to \cdots$$

Then $T$ is a tilting complex and $\text{End}_{D^b(A)}(T) \simeq L_\lambda(A)$.

Proof:

By construction $T$ is a tilting complex and by [SZ03] the algebra $\text{End}_{D^b(A)}(T)$ is gentle, then $\text{End}_{D^b(A)}(T) = kQ''/\langle P'' \rangle$ with $Q_0'' = Q_0$. Denote by $P_m''$ the $\text{End}_{D^b(A)}(T)$-indecomposable projective modules for $m \in Q_0''$, which are identified with the indecomposable summands of $T$. The following morphisms:

$$\alpha' := \cdots \to 0 \to 0 \to P_i \to 0 \to \cdots$$

$$\delta' := \cdots \to 0 \to P_j \xrightarrow{(0, \text{id})} P_j \to 0 \to \cdots$$

$$\xi' := \cdots \to 0 \to P_j \oplus P_j \xrightarrow{(\delta, \delta \lambda)} P_i \to 0 \to \cdots$$

are irreducible because they cannot be factorized through any other indecomposable summand of $T$. The morphism $\xi : P_x \to P_j$ is irreducible in $A$, but the corresponding morphism in $\text{End}_{D^b(A)}(T)$ factorizes through $\delta'$ and the morphism induced by the projection of $P_j \oplus P_j$ over the first component, factorizes through

$$\lambda' := \cdots \to 0 \to P_j \oplus P_j \xrightarrow{(\delta, \delta \lambda)} P_i \to 0 \to \cdots$$

Also, notice that the compositions $\xi' \delta'$ and $\lambda' \lambda'$ are zero and $\delta' \alpha'$ is homotopic to zero.

The other morphisms which are irreducible in $A - \text{mod}$ produce irreducible morphisms in $\text{End}_{D^b(A)}(T) - \text{mod}$ because the corresponding indecomposable projective modules are direct summands of $T$ of degree one.

By using Happel’s formula we calculate the following:

$$\dim \text{Hom}_{K^b(A)}(T_c, P_m[1]) = \dim \text{Hom}_A(P_j \oplus P_j, P_m) - \dim \text{Hom}_A(P_i, P_m)$$

$$= \#2(j, m) - \#(i, m)$$

$$= 2\#(j, m; i) + 2\#(j, m; \hat{i}) - \#(i, m) = 2\#(j, m; \hat{i})$$
because \( \#(i, m) = 2\#(j, m; i) \) but only the trivial path in \( j \) induce an irreducible morphism (applied in the second coordinate) for \( m = j \), the rest factorize through this one.

\[
\dim\text{Hom}_{\text{K}^b(A)}(P_m[1], T_c) = \dim\text{Hom}_A(P_m, P_j \oplus P_j) - \dim\text{Hom}_A(P_m, P_i) \\
= 2\#(m, j) - \#(m, i) \\
= 2\#(m, j; x) + 2\#(m, j; \hat{x}) - \#(m, i) \\
= 2\#(m, j; x)
\]

because \( 2\#(m, j; \hat{x}) = \#(m, i) \), and the only irreducible morphism is the one induced by \( \xi \) for \( m = x \).

\[
\dim\text{Hom}_{\text{K}^b(A)}(P_m[1], P_t[1]) = \dim\text{Hom}_A(P_m, P_t) = \#(m, t)
\]

but the only irreducible morphism are the ones induced by arrows starting in \( l \) and ending in \( m \) and the one induced by the path \( \delta \lambda \alpha \) for \( t = l \) and \( m = j \).

Finally

\[
\dim\text{Hom}_{\text{K}^b(A)}(T_c, T_c) = \dim\text{Hom}_A(P_j \oplus P_j, P_j \oplus P_j) - \dim\text{Hom}_A(P_j \oplus P_j, P_i) \\
- \dim\text{Hom}_A(P_i, P_j \oplus P_j) + \dim\text{Hom}_A(P_i, P_i) \\
= 4\#(j, j) - 2\#(j, i) - 2\#(i, j) + \#(i, i)
\]

If \( \#(i, j) \neq 0 \) there must be only one path \( \Pi \) from \( j \) to \( i \) not involving \( \lambda \) because the algebra is gentle. Notice the only from \( j \) to \( j \) not passing through \( i \) must be the trivial path because the algebra is finite dimensional. Then

\[
(j, j) = \{1_j, \delta \lambda \Pi\}, (j, i) = \{\delta, \delta \lambda\}, (i, j) = \{\Pi, \lambda \Pi\}, (i, i) = \{1_i, \lambda\},
\]

so

\[
\dim\text{Hom}_{\text{K}^b(A)}(T_c, T_c) = 8 - 4 - 4 + 2 = 2.
\]

If \( \#(i, j) = 0 \) for each path \( \Pi \) from \( j \) to \( j \) there are exactly two paths from \( i \) to \( j \) \( \Pi \delta \lambda \) and \( \Pi \delta \) so \( \#(j, i) = 2\#(j, j) \). Therefore \( 2\#(j, i) = 4\#(j, j) \) and

\[
\dim\text{Hom}_{\text{K}^b(A)}(T_c, T_c) = \#(i, i) = 2.
\]

In both cases there is only one irreducible morphism from \( T_c \) to itself. This must be \( \lambda' \) as it does not factorizes through any of the previous irreducible morphisms we already analyzed.

We verified that there are no further irreducible morphisms and this completes the proof.

\(\Box\)
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References

[AH81] I. Assem, D. Happel, Generalized tilted algebras of type $A_n$, Comm. Algebra 9 (1981), 2101-2125.
[AS87] I. Assem, A. Skowroński, Iterated tilted algebras type $\tilde{A}_n$, Math. Z. 195 (1987), 269-290.
[AG07] D. Avella-Alaminos, C. Geiss, Combinatorial derived invariants for gentle algebras, J. Pure Appl. Algebra 212 (2008), no. 1, 228-243.
[BGS04] G. Bobiński, C. Geiss, A. Skowroński, Classifications of discrete derived categories, Cent. Eur. J. Math. 2 (2004), no. 1, 19-49.
[BR87] M.C.R. Butler, C.M. Ringel, Auslander-Reiten sequences with few middle terms and applications to string algebras, Comm. Algebra 15 (1987), no. 1-2, 145-179.
[GP99] C. Geiss, J.A. de la Peña, Auslander-Reiten components for clans, Bol. Soc. Mat. Mexicana (3) 5 (1999), 307-326.
[Ha88] D. Happel, Triangulated categories in the representation theory of finite dimensional algebras, London Math. Soc. LN Series, 119 Cambridge University Press, 1988.
[HSZ01] T. Holm, J. Schröer, A. Zimmermann, Combinatorial derived equivalences between gentle algebras, manuscript (2001), 1-14.
[HW83] D. Hughes, J. Waschbüsch, Trivial extensions of tilted algebras, Proc. London Math. Soc (3) series 46 (1983), no.2, 347-363.
[Rin97] C. M. Ringel, The repetitive algebra of a gentle algebra, Bol. Soc. Mat. Mexicana (3) 3 (1997), no. 2, 235-253.
[SZ03] J. Schröer, A. Zimmermann, Stable endomorphism algebras of modules over special biserial algebras, Math. Z. 244 (2003), no. 3, 515-530.
[Vo01] D. Vossieck, The algebras with discret derived category, Journal of Algebra 243 (2001), 168-176.

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