Topological DBI actions and nonlinear instantons

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Abstract

We consider Euclidean D4 and D6-branes filling the whole $\mathbb{R}^4$ and $\mathbb{R}^6$ space, respectively. In both cases, with a constant background B-field turned on for D4-branes, we propose actions which are the same as the DBI actions up to some constant or total derivative terms. These extra terms allow us to write the action as a square of nonlinear instanton equations. As such, the actions can easily be supersymmetrized using the methods of topological field theory.

1 Introduction

D-branes in string theory are effectively described by the Dirac-Born-Infeld action. The action has also a supersymmetric extension which consists of two parts; the Dirac-Born-Infeld (DBI) part, and the Wess-Zumino part. Both parts of the action are invariant under the rigid spacetime supersymmetry transformations, however, for a specific choice of normalization of the Wess-Zumino term, the whole action turns out to have an extra local symmetry known as $\kappa$-symmetry. The $\kappa$-symmetry allows one to remove the redundant fermionic degrees of freedom [1]. Here we are only concerned with the bosonic part of the DBI action. Later on, we will explain on an alternative way to supersymmetrize the action using the methods of topological field theory [2, 3].

When a constant background B-field is turned on, the only change in the DBI action is to replace $F$ – the field strength of the $U(1)$ gauge field – by $F + B$ everywhere in the action. Using the Seiberg-Witten map, one could also recast the action in terms of the open string variables and noncommutative filed strength $\hat{F}$ [4]. In this description, the effect of $B$ appears only through the open string metric, coupling constant, and $\hat{F}$. Considering

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a supersymmetric extension of the DBI action in a constant B-field, and looking for BPS states which preserve part of the supersymmetry leads to a deformed instanton equation as the BPS condition. These deformed equations, the so-called nonlinear instanton equations \cite{5}, are our main interest in this article.

We start our discussion with the DBI action in the absence of any background field in four dimensions. By adding to the DBI action some appropriate terms, which are either constant or total derivatives, an action is derived that has instantons as its critical points \cite{7}. The structure of these extra terms is easy to guess if one notes that the action has to vanish at infinity where $F = 0$, and the fact that it is defined up to some topological terms. However, when there is a background B-field, these terms are very nontrivial to guess. In this case we derive the action by requiring, as in the case of $B = 0$, that it vanish at infinity, and its critical points coincide with the nonlinear instantons \cite{4, 5, 6}. Actually the action will turn out to be proportional to the square of nonlinear instanton equations. This will allow us to supersymmetrize the action using the topological field theory methods. As for the D6-branes in $R^6$, we look for an action which vanishes at infinity, and has some possible extra topological terms. The proposed action then will have nonlinear instantons as its critical points.

2 Topological DBI action in a background B-field

In this section, we consider first the case of $B = 0$ which has also been discussed in \cite{7}. Next, D-branes in a constant background B-field are examined. In both cases, we propose actions which are the same as the usual DBI actions up to some constant or total derivative terms. The point for choosing such actions is that they can be written as the square of some sections which have zeros on BPS configurations, and can easily be supersymmetrized. We work out the Lagrangian in the case of $B \neq 0$, and derive a supersymmetric extension which has nonlinear instantons \cite{7} as the fixed points of the corresponding fermionic symmetry. Before doing any calculations, let us first express the result for the bosonic part of the Lagrangian explicitly

$$
L = \sqrt{\det(g + M)} - \sqrt{\det(g + B)} + \frac{(1 - \text{Pf} B)}{\sqrt{\det(g + B)}} (\text{Pf} F + \frac{1}{4} \epsilon^{ijkl} B_{ij} F_{kl}) - \frac{B_{+ij} F_{ij}}{\sqrt{\det(g + B)}}. \quad (1)
$$

As can be seen, apart from the second term which is a constant, all extra terms are total derivative. These extra terms are needed if we demand that the Lagrangian vanish on BPS configurations. And we demand this to be able to supersymmetrize the action in a topological way.

To begin with, let us get on to the case of $B = 0$. Expanding the determinant appearing in the DBI action, we will have

$$
\det(g + F) = 1 + \det F + \frac{1}{2} F^2 + 2 \text{Pf} F - 2 \text{Pf} F
$$

$$
= (1 - \text{Pf} F)^2 + F^2_+. \quad (2)
$$
where $F^2 = F_{ij}F^{ij}$, $F_{ij}^+ = \frac{1}{2}(F_{ij} + \frac{1}{2}\epsilon_{ijkl}F^{kl})$, $\text{Pf}F = \frac{1}{8}\epsilon_{ijkl}F^{ij}F^{kl}$, and we use the flat metric $g_{ij} = \delta_{ij}$ everywhere. Rewriting (2), we have

$$\left(\sqrt{\det(g + F)} + 1 - \text{Pf}F\right)\left(\sqrt{\det(g + F)} - 1 + \text{Pf}F\right) = F_+^2. \quad (3)$$

However, in the following, we will show that the term

$$f = \sqrt{\det(g + F)} + 1 - \text{Pf}F$$

is positive definite. On the other hand, a natural choice for the DBI Lagrangian which vanishes at infinity (where $F = 0$), and is different from the usual DBI Lagrangian by a constant and a total derivative is

$$\mathcal{L} = \sqrt{\det(g + F)} - 1 + \text{Pf}F. \quad (4)$$

By looking at (3), we realize that choosing $\mathcal{L}$ as above has also the advantage of being proportional to the square of a section (here $F^+$), and therefore can be supersymmetrized using topological field theory methods. This will be explained shortly. Moreover, since $f$ is positive definite, then eq. (3) implies that $\mathcal{L} = 0$ if and only if $F_+ = 0$.

To see that $f$ is positive definite, first note that it can be written as

$$f = \sqrt{\det(g + F)} + 1 - \text{Pf}F = \frac{1}{2\sqrt{\det(g + F)}} \left\{ \left(\sqrt{\det(g + F)} + 1 - \text{Pf}F\right)^2 + F_+^2 \right\}. \quad (5)$$

This equation shows that the left hand side is not negative. However, if it is zero, then $F_+$ must vanish, and this implies

$$\sqrt{\det(g + F)} + 1 - \text{Pf}F = \sqrt{\det(g + F)} + 1 + \frac{1}{4}F^2$$

which is always positive and cannot be zero. Thus eq. (4) tells us that $f$ (now call it $1/h^2$) is positive definite. As such, using (3), we can write

$$\mathcal{L} = \sqrt{\det(g + F)} - 1 + \text{Pf}F = h^2F_+^2.$$

This also proves that $\mathcal{L} = 0$ if and only if $F_+ = 0$.

To extend the above results to the case of $B \neq 0$, we consider the BPS condition for Euclidean D4-branes in a background constant B-field. Noticing that $F$ has to vanish at infinity, this condition reads

$$M_{ij}^+ = \frac{1 - \text{Pf}M + \sqrt{\det(g + M)}}{1 - \text{Pf}B + \sqrt{\det(g + B)}}B_{ij}^+ \equiv \alpha B_{ij}^+, \quad (3)$$
where $M = F + B$. As in the above case with $B = 0$, in the following, we construct an action which has the same critical points as the above BPS configurations. The result will be

$$
\mathcal{L} = \sqrt{\det(g + M)} - \sqrt{\det(g + B)} + \frac{(1 - \text{Pf}B)}{\sqrt{\det(g + B)}} (\text{Pf}F + \frac{1}{4} \epsilon^{ijkl} B_{ij} F_{kl}) - \frac{B_{ij} F_{ij}}{\sqrt{\det(g + B)}}
$$

$$
\equiv \frac{(M_+ - \alpha B_+)^2}{2\alpha \sqrt{\det(g + B)}} \equiv N_+^2.
$$

Notice that $\alpha$ is positive definite (for the same reason that the left hand side of (5) is positive definite). Therefore $\mathcal{L}$ vanishes if and only if $M^+ = \alpha B^+$, i.e., it localizes on the BPS configurations.

To prove (6), first note that

$$
\frac{M^2_+ - \alpha B^2_+}{B^2_+} = \frac{1 - \text{Pf}M + \sqrt{\det(g + M)}}{1 - \text{Pf}B + \sqrt{\det(g + B)}} \equiv \alpha \beta,
$$

using eq.(3). Therefore, we can write

$$
(M_+ - \alpha B_+)^2 = M_+^2 + \alpha^2 B_+^2 - 2\alpha B_{+ij} M_+^{ij}
$$

$$
= \alpha \beta B_+^2 + \alpha^2 B_+^2 - 2\alpha B_{+ij} F_{+ij}^{ij}
$$

$$
= \alpha B_+^2 (\alpha + \beta - 2) - 2\alpha B_{+ij} F_{+ij}^{ij}.
$$

After a little algebra, the first term in the last equality can be written as

$$
\alpha B_+^2 (\alpha + \beta - 2) = 2\alpha \sqrt{\det(g + B)} \left( \frac{M_+^2 - \alpha B_+^2}{\sqrt{\det(g + M)} - \sqrt{\det(g + B)} + \frac{(1 - \text{Pf}B)}{\sqrt{\det(g + B)}} (\text{Pf}F + \frac{1}{4} \epsilon^{ijkl} B_{ij} F_{kl})} \right).
$$

Plugging this into eq. (8), finally we arrive at (3).

Now that $\mathcal{L}$ has been written as a square of $N^+$, we employ the topological field theory methods to supersymmetrize it [8]. To do so, we first introduce a ghost one-form field $\psi^i$, the fermionic partner of $A^i$, a scalar field $\phi$, and a BRST-like operator $\delta$ with an action

$$
\delta A^i = i \epsilon \psi^i, \quad \delta \psi^i = -i \partial^i \phi \quad \delta \phi = 0,
$$

where $\epsilon$ is a constant anticommuting parameter. Further we need to introduce the anti-ghost fields; a self-dual 2-form $\chi_{ij}$ (the conjugate field to $N_{ij}^+$), a scalar $\eta$, as well as a scalar $\lambda$ with ghost number 2. These transform under $\delta$ as follows

$$
\delta \chi_{ij} = \epsilon H_{ij}, \quad \delta H_{ij} = 0
$$

$$
\delta \lambda = 2i \epsilon \eta, \quad \delta \eta = 0,
$$

$$
\delta \lambda = 2i \epsilon \eta, \quad \delta \eta = 0.
$$
here the auxiliary self-dual field $H_{ij}$ has been introduced to close the algebra off shell. Let us define the operator $Q$ by

$$\delta \Phi = -i\epsilon \{Q, \Phi\}$$

for any field $\Phi$. We would like the Lagrangian to be a BRST commutator, i.e. $L_S = i\{Q,V\}$, for some gauge fermion $V$. Since $Q^2$ acting on any field is zero up to a gauge transformation, this ensures that $L$ is invariant under the fermionic symmetry $Q$ if $V$ is chosen to be gauge invariant. A minimal choice for $V$ is

$$V = \chi^{ij}(H_{ij} - 2N_{ij}^+\epsilon) + \frac{1}{2}\psi^i\partial_i \lambda$$

We now vary $V$ to get

$$L_S = i\{Q,V\} = -H_{ij}H_{ij} + 2H_{ij}N_{ij}^+ - 2i\chi^{ij}\frac{\delta N_{ij}^+}{\delta A_k}\psi^k + \frac{1}{2}\partial_i \phi \partial^i \lambda + i\psi_i \partial^i \lambda.$$

The auxiliary field $H_{ij}$ can be integrated out using its equation of motion. Doing this, we obtain

$$\begin{align*}
L_S & = N_+^2 - 2i\chi^{ij}\frac{\delta N_{ij}^+}{\delta A_k}\psi^k + \frac{1}{2}\partial_i \phi \partial^i \lambda + i\psi_i \partial^i \lambda \\
& = \sqrt{\det(g + M)} - \sqrt{\det(g + B)} + \frac{1}{\sqrt{\det(g + B)}}(\text{Pf} F + \frac{1}{4}\epsilon^{ijkl}B_{ij}F_{kl}) - \frac{B_{ij}^F F_{ij}}{\sqrt{\det(g + B)}} \\
& = 2i\chi^{ij}\frac{\delta N_{ij}^+}{\delta A_k}\psi^k + \frac{1}{2}\partial_i \phi \partial^i \lambda + i\psi_i \partial^i \lambda. 
\end{align*}$$

(9)

3 D6-branes in $\mathbb{R}^6$

The last issue we would like to discuss is the DBI action of flat Euclidean D 6-branes in $\mathbb{R}^6$ with $B = 0$. Here one may expect that, as in the case of topological action of D4-branes which has Yang-Mills instantons as its critical points, the topological action of D6-branes will have the Kähler-Yang-Mills instantons as its critical points. But, as we will see, this is not the case. Instead, the proposed topological Lagrangian

$$L = \sqrt{\det(g + F)} - 1 + \frac{1}{16}\epsilon^{ijklmn}k_{mn}F_{ij}F_{kl},$$

(10)

localizes on the solutions of the following equations;

$$\text{Pf} F = \frac{1}{2}k_{ij}F_{ij}^{ij}, \text{ and } F^{2,0} = 0,$$

(11)

The Kähler-Yang-Mills equations

$$k_{ij}F_{ij} = 0, \quad F^{2,0} = 0,$$

naturally arise in the construction of a cohomological field theory on Calabi-Yau 3-folds. There they appear as the fixed points of the corresponding BRST symmetry.
where \(i, j, \ldots = 1, \ldots, 6\), and \(k = \frac{1}{2}k_{ij}dx^i \wedge dx^j\) is the Kähler form. These equations are also derived in [5] as the BPS conditions for D6-branes. In the following, we motivate the above definition of the topological action for D6-branes in \(\mathbb{R}^6\).

To start, we use the following identity on flat Euclidean \(\mathbb{R}^6\) [1, 10]

\[
\det(g + F) = \rho_6^2,
\]

with

\[
\rho_6 = \epsilon_{ijklmn} \left\{ \frac{1}{6!} \epsilon_{ijklmn} \tau_1 \gamma_7 - \frac{1}{2 \cdot 4!} \tau_2 \gamma_{ijkl} F_{mn} + \frac{i}{16} \tau_1 \gamma_{ij} F_{kl} F_{mn} - \frac{1}{48} \tau_2 F_{ij} F_{kl} F_{mn} \right\},
\]

where \(\tau_1, \tau_2,\) and \(\tau_3\) are the Pauli matrices. Alternatively, since \(\frac{1}{4!} \epsilon_{ijklmn} \gamma_{klmn} = i \gamma_7 \gamma^{ij}\), we can write (12) as

\[
\det(g + F) = \left( \gamma_7 + i \frac{1}{16} \epsilon_{ijklmn} \gamma_{mn} F_{ij} F_{kl} \right)^2 + \left( \frac{i}{2} F^{ij} \gamma_7 \gamma_{ij} + \text{Pf} F \right)^2.
\]

Now take \(\theta\) to be a constant commuting left-handed spinor on \(\mathbb{R}^6\). We choose the gamma matrices to be hermitian and antisymmetric, and in the complex coordinate let \(\gamma_\alpha \theta = 0\), with the normalization \(\theta^\dagger \theta = 1\). The Kähler form and the holomorphic 3-form are then defined as follows [9]

\[
k_{ij} = i \theta^\dagger \gamma_{ij} \theta
\]

and

\[
C_{ijk} = \theta^\dagger \gamma_{ijk} \theta^*.
\]

Expanding eq. (13), and multiplying it by \(\theta^\dagger\) on left and \(\theta\) on right, and using (14) results in the following identity

\[
\det(g + F) = \left( 1 - \frac{1}{16} \epsilon_{ijklmn} k_{mn} F_{ij} F_{kl} \right)^2 + \left( \text{Pf} F \right)^2 - 2 \epsilon_{ijklmn} k_{ij} F_{kl} F_{mn} + 1 \frac{\tilde{F}_{\alpha \beta} \tilde{F}_{\alpha \beta}}{2}, \tag{15}
\]

where \(\alpha, \beta, \gamma \ldots\) are the complex holomorphic tangent indices, \(\tilde{F}^{ij} = \frac{1}{4} \epsilon_{ijklmn} F_{kl} F_{mn}\), and use has been made of

\[
\frac{1}{4} (k_{ij} F^{ij})^2 = \frac{1}{8} \epsilon_{ijklmn} k_{ij} F_{kl} F_{mn} + \frac{1}{2} F^2 - 2 \epsilon_{ijklmn} k_{ij} F_{kl} F_{mn}, \tag{16}
\]

for any rank 2 antisymmetric tensor in 6 dimensions. Let us now write (13) as

\[
\left( \sqrt{\det(g + F)} - 1 + \frac{1}{16} \epsilon_{ijklmn} k_{mn} F_{ij} F_{kl} \right) \left( \sqrt{\det(g + F)} + 1 - \frac{1}{16} \epsilon_{ijklmn} k_{mn} F_{ij} F_{kl} \right) \]

\[
= \left( \text{Pf} F - \frac{1}{2} F^{ij} k_{ij} \right)^2 + 2 F_{\alpha \beta} F_{\alpha \beta} + \frac{1}{2} \tilde{F}_{\alpha \beta} \tilde{F}_{\alpha \beta}. \tag{17}
\]

As in the case of D4-branes, first we prove that the term

\[
\tilde{f} = \sqrt{\det(g + F)} + 1 - \frac{1}{16} \epsilon_{ijklmn} k_{mn} F_{ij} F_{kl}
\]
on the left hand side of eq. (17) is positive definite. To show this, notice that the left hand side of eq. (17) can be written as

\[
\left(\sqrt{\det(g + F) + 1} - \frac{1}{4}k^{ij}\tilde{F}_{ij}\right)\left(\sqrt{\det(g + F) - 1} + \frac{1}{4}k^{ij}\tilde{F}_{ij}\right)
\]

\[
= -\left(\sqrt{\det(g + F) + 1} - \frac{1}{4}k^{ij}\tilde{F}_{ij}\right)\left(\sqrt{\det(g + F) + 1} - \frac{1}{4}k^{ij}\tilde{F}_{ij} - 2\sqrt{\det(g + F)}\right)
\]

\[
= -\left(\sqrt{\det(g + F) + 1} - \frac{1}{4}k^{ij}\tilde{F}_{ij}\right)^2 + 2\sqrt{\det(g + F)}\left(\sqrt{\det(g + F) + 1} - \frac{1}{4}k^{ij}\tilde{F}_{ij}\right).
\]

So, using (17), we can write

\[
2\sqrt{\det(g + F)}\left(\sqrt{\det(g + F) - 1} + \frac{1}{16}\epsilon^{ijklmn}k_{mn}F_{ij}F_{kl}\right)
\]

\[
= \left(\sqrt{\det(g + F) + 1} - \frac{1}{16}\epsilon^{ijklmn}k_{mn}F_{ij}F_{kl}\right)^2 + \left(\text{Pf} F - \frac{1}{2}F^{ij}k_{ij}\right)^2
\]

\[
+ 2F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{2}F_{\alpha\beta}\tilde{F}^{\alpha\beta}.
\]

(18)

Therefore \(\tilde{f}\) is not negative. But if it is zero, according to the above equation we must have

\[
\text{Pf} F = \frac{1}{2}k_{ij}F^{ij}, \quad F_{\alpha\beta} = 0,
\]

(equation \(\tilde{F}_{\alpha\beta} = 0\) is automatically satisfied when \(F_{\alpha\beta} = 0\)). So eq. (14) becomes

\[
\det(g + F) = \left(1 - \frac{1}{16}\epsilon^{ijklmn}k_{mn}F_{ij}F_{kl}\right)^2,
\]

(19)

we write this as

\[
\frac{1}{8}\epsilon^{ijklmn}k_{mn}F_{ij}F_{kl} = -\det(g + F) + 1 + \left(\frac{1}{16}\epsilon^{ijklmn}k_{mn}F_{ij}F_{kl}\right)^2
\]

\[
= -1 - \det F - \frac{1}{2}F^2 - \frac{1}{8}\tilde{F}^{ij}\tilde{F}_{ij} + \frac{1}{8}\left(\frac{1}{8}\epsilon^{ijklmn}k_{mn}F_{ij}F_{kl} + 8\tilde{F}^{ij}\tilde{F}_{ij}\right),
\]

where in the last line we have expanded the determinant, and used (13) with \(\tilde{F}_{\alpha\beta} = 0\). Finally the above equation implies that

\[
(1 - \frac{1}{8^2})\frac{1}{8}\epsilon^{ijklmn}k_{mn}F_{ij}F_{kl} = -\det F - \frac{1}{2}F^2 < 0.
\]

Hence

\[
\tilde{h}^{-2} \equiv \tilde{f} = \sqrt{\det(g + F) + 1} - \frac{1}{16}\epsilon^{ijklmn}k_{mn}F_{ij}F_{kl}
\]

cannot vanish and is positive definite.

The above calculations show that a good choice for the DBI Lagrangian of D6-branes in \(\mathbf{R}^6\) is the one in (10). This is justified for the following reasons. Firstly, it vanishes at
infinity where $F = 0$, and up to a constant and a total derivative is the usual DBI Lagrangian. Secondly, eq. (17) and the positive definiteness of $\tilde{f}$ allow us to write (10) as

$$L = \sqrt{\det(g + F)} - 1 + \frac{1}{16} \epsilon^{ijklmn} k_{mn} F_{ij} F_{kl}$$

$$= \tilde{h}^2 \left\{ \left( Pf F - \frac{1}{2} F^{ij} k_{ij} \right)^2 + 2 F_{\alpha\beta} F^{\alpha\beta} + \frac{1}{2} \tilde{F}_{\alpha\beta} \tilde{F}^{\alpha\beta} \right\},$$  \hspace{1cm} (20)

which shows that $L$ is zero if and only if $Pf F = \frac{1}{2} k_{ij} F^{ij}$, $F_{\alpha\beta} = 0$.

Since the right hand side of (20) is the sum of the squares of the sections

$$s^{(1)} = \tilde{h} \left( Pf F - \frac{1}{2} F^{ij} k_{ij} \right)$$

$$s^{(2)}_{\alpha\beta} = \tilde{h} F_{\alpha\beta}$$

$$s^{(3)}_{\alpha\beta} = \tilde{h} \tilde{F}_{\alpha\beta},$$

we can employ the same method that we used in the previous section to supersymmetrize the action. For the case of Kähler-Yang-Mills equations, this has been explicitly done in [9].
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