On the distribution of Ramanujan sums over number fields

Sneha Chaubey · Shivani Goel

Received: 4 November 2021 / Accepted: 18 February 2023 / Published online: 26 April 2023
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2023

Abstract
For a number field \( \mathbb{K} \), and integral ideals \( \mathcal{I} \) and \( \mathcal{J} \) in its number ring \( \mathcal{O}_K \), Nowak studied the asymptotic behavior of the average of Ramanujan sums \( C_J(I) \) over both ideals \( \mathcal{I} \) and \( \mathcal{J} \). In this article, we extend this investigation by establishing asymptotic formulas for the second moment of averages of Ramanujan sums over quadratic and cubic number fields, thereby generalizing previous works of Chen, Kumchev, Robles, and Roy on moments of averages of Ramanujan sums over rationals. Additionally, using a special property of certain integral domains, we obtain second moment results for Ramanujan sums over some other number fields.

Keywords Ramanujan sums · Number fields · Perron formulas · Dedekind zeta function

Mathematics Subject Classification 11M06 · 11N37 · 11R42

1 Introduction and main results

Ramanujan, in 1918, while studying the trigonometric series representations of normalized arithmetic functions [20], introduced a function

\[
c_n(m) := \sum_{1 \leq j \leq n} e \left( \frac{mj}{n} \right) = \sum_{d \mid n} \sum_{d \mid m} d\mu \left( \frac{n}{d} \right)
\] (1.1)

Sneha Chaubey is grateful for the support from the Science and Engineering Research Board, Department of Science and Technology, Government of India, under Grant SB/S2/RJN-053/2018.

Shivani Goel
shivanig@iitd.ac.in

Sneha Chaubey
sneha@iitd.ac.in

1 Department of Mathematics, IIIT Delhi, New Delhi 110020, India
now known as the Ramanujan sum, where \( m \) and \( n \) are positive integers, \( e(x) = e^{2\pi i x} \), and \( \mu(n) \) is the Mobius function. Understanding these sums and their distribution is an important topic of study in number theory, with profound connections to problems in arithmetic such as in the proof of Vinogradov’s theorem [17, Chapter 8], Waring type formulas [13], distribution of rational numbers in short intervals [11], equipartition modulo odd integers [2], large sieve inequality [21], as well as other areas of mathematics.

Ramanujan sums have been generalized by many mathematicians in several contexts. Some examples include Cohen–Ramanujan sums [24], Anderson–Apostol sums [1], polynomial Ramanujan sums introduced by Carlitz [3] and later generalized by Cohen [5], and Ramanujan sums in the more general context of arithmetic subgroups (see [8, 12]).

The main purpose of this note is to study the distribution of Ramanujan sums defined over number fields by examining moments of its mean values. Let \( K \) be a number field and \( \mathcal{J} \) and \( \mathcal{I} \) be non-zero integral ideal in its number ring \( \mathcal{O}_K \), then Ramanujan sums over \( K \) are defined as

\[
C_{\mathcal{J}}(\mathcal{I}) := \sum_{\mathcal{J}_1 | \mathcal{J}} \mathcal{N}(\mathcal{I}_1) \mu\left(\frac{\mathcal{J}_{\mathcal{I}_1}}{\mathcal{I}_{\mathcal{I}_1}}\right). 
\]

Here, \( \mathcal{N}(\mathcal{I}_1) \) is the norm of \( \mathcal{I}_1 \) and \( \mu(\mathcal{I}) \) is the generalization of classical Mobius function such that

\[
\mu(\mathcal{I}) = \begin{cases} 
(-1)^r & \text{if } \mathcal{I} \text{ is a product of } r \text{ distinct prime ideals,} \\
0 & \text{if there exists a prime ideal } P \text{ of } \mathcal{O}_K \text{ such that } P^2 | \mathcal{I}.
\end{cases}
\]

Note that for \( K = \mathbb{Q} \), it is the usual Ramanujan sum \( c_n(m) \) in (1.1). The question on the average order over both variables \( n \) and \( m \) of \( c_n(m) \) was first considered by Chan, and Kumchev [4] motivated by applications to problems on Diophantine approximations of reals by sums of rational numbers. In [4], using both elementary and analytic techniques, they find asymptotic formulas for

\[
\sum_{m \leq y} \left( \sum_{n \leq x} c_n(m) \right)^k
\]

for \( k = 1, 2 \). Robles and Roy [22] study the first, second, and higher moments of averages of Cohen–Ramanujan sums. Although their result for higher moments (\( k \geq 3 \)) (Proposition 1.1, [22]) is incorrect, the problem of computing asymptotic formulas for higher moments even for the usual Ramanujan sums (1.1) remains open. Concerning the number field analog of Ramanujan sums, Nowak [18] showed that if \( K \) is a fixed quadratic number field, and \( y > x^{\delta} \) where \( \delta > \frac{1973}{820} = 2.40609 \cdots \), then

\[
\sum_{0 < \mathcal{N}(\mathcal{I}) \leq y} \sum_{0 < \mathcal{N}(\mathcal{J}) \leq x} C_{\mathcal{J}}(\mathcal{I}) \sim \rho_K y. 
\]
More precisely, for \( y > x \) and arbitrary \( \epsilon > 0 \),
\[
\sum_{0 < N(I) \leq y} \sum_{0 < N(J) \leq x} C_J(I) = \rho_K y + O \left( x^{\frac{1973}{1358}} y^{\frac{260}{679} + \epsilon} \right) \\
+ O \left( x^{\frac{1234823}{737394}} y^{\frac{205}{679} + \epsilon} \right) + O \left( x^{\frac{23917}{16296}} y^{\frac{8675}{16296} + \epsilon} \right) + O \left( x^2 y^\epsilon \right),
\]
where
\[
\rho_K = \lim_{t \to \infty} \frac{1}{t} \# \{ \text{integral ideals } I \text{ in } O_K : 0 < N(I) \leq t \}.
\]
(1.4)

In this note, we estimate the sum
\[
\sum_{0 < N(J) \leq x} C_J(I)
\]
in average over ideals \( I \) such that \( N(I) \in \{1, \ldots, y\} \) via the second moment. This generalizes the results in [22] for second moments of the mean value of Cohen–Ramanujan sums. We derive the following asymptotic formula for a quadratic number field \( K \).

**Theorem 1.1** Let \( K \) be a quadratic number field, \( \epsilon > 0 \) be any arbitrary small real number, then for \( y \leq x^{11/9 - \epsilon} \)
\[
\sum_{0 < N(I) \leq y} \left( \sum_{0 < N(J) \leq x} C_J(I) \right)^2 = \frac{\rho_K^2}{4 \zeta_K(2)^2} x^4 + O_q \left( x^{47/18 + \epsilon} y^{1/2} \log^{12} x \right),
\]
for \( x^{11/9 - \epsilon} \leq y < x^{36/17 - \epsilon} \)
\[
\sum_{0 < N(I) \leq y} \left( \sum_{0 < N(J) \leq x} C_J(I) \right)^2 = \frac{\rho_K^2 y}{2 \zeta_K(2)} x^2 + \frac{\rho_K^2}{4 \zeta_K(2)^2} x^4 + O_q \left( x^{27/18 + \epsilon} y^{1/2} \log^{24} x \right)
+ O_q \left( x^{47/18 + \epsilon} y^{1/2} \log^{12} x \right),
\]
and for \( y \geq x^{36/17 - \epsilon} \)
\[
\sum_{0 < N(I) \leq y} \left( \sum_{0 < N(J) \leq x} C_J(I) \right)^2 = \frac{\rho_K^2}{2 \zeta_K(2)} y x^2 + O_q \left( y x^{5 + \epsilon} \log^7 x + x^2 y^{17/18} \log^{24} x \right).
\]

With regard to other degree two extensions, for example for the field of Gaussian integers, Nowak [19] proved (1.3) with uniform error terms with \( \delta > \frac{29}{12} = 2.416 \ldots \). His result was later improved in [26] for \( \delta > 2.3235 \ldots \).
For the cubic case, a result on the first moment is derived in [15], where the authors obtain an asymptotic formula (1.3) with condition $y > x^{11/4}$

$$\sum_{0 < N(\mathcal{I}) \leq y} \sum_{0 < N(\mathcal{J}) \leq x} C_{\mathcal{J}}(\mathcal{I}) = \rho_K y + O \left( x^{\frac{8}{5}} y^{\frac{2}{5}} + x^{\frac{11}{5}} y^{\frac{1}{5}} + \epsilon \right).$$

We obtain estimates on the second moment for a cubic number field in the following theorem.

**Theorem 1.2** Let $\mathbb{K}$ be a cubic number field, $\epsilon > 0$ be any arbitrary small real number, then for $y \leq x^{237/196-\epsilon}$

$$\sum_{0 < N(\mathcal{I}) \leq y} \left( \sum_{0 < N(\mathcal{J}) \leq x} C_{\mathcal{J}}(\mathcal{I}) \right)^2 \leq \frac{\rho_K^2}{2} \xi_{\mathbb{K}}(0) x^4 + O_q \left( x^{1021/392 + \epsilon} y^{1/2} \log^{20} x \right),$$

for $x^{237/196-\epsilon} \leq y < x^{98/45-\epsilon}$

$$\sum_{0 < N(\mathcal{I}) \leq y} \left( \sum_{0 < N(\mathcal{J}) \leq x} C_{\mathcal{J}}(\mathcal{I}) \right)^2 \leq \frac{\rho_K^2}{2} \xi_{\mathbb{K}}(0) x^4 + \frac{\rho_K^2}{4} \xi_{\mathbb{K}}(0) x^4 + O_q \left( x^2 y^{45/49 + \epsilon} \log^3 x \right) + O_q \left( x^{1021/392 + \epsilon} y^{1/2} \log^{20} x \right).$$

and for $y \geq x^{98/45-\epsilon}$

$$\sum_{0 < N(\mathcal{I}) \leq y} \left( \sum_{0 < N(\mathcal{J}) \leq x} C_{\mathcal{J}}(\mathcal{I}) \right)^2 = \frac{\rho_K^2}{2} \xi_{\mathbb{K}}(0) x^2 y x^2 + O_q \left( y x^{25/14 + \epsilon} \log^{10} x + x^2 y^{45/49 + \epsilon} \log^3 x \right).$$

For mean values of Ramanujan sums over general number fields, the only known result is due to Fujisawa [6] who proved that if $\mathbb{K}$ is any number field, then for some $c > 0$, and for any $\delta > \frac{2 - \alpha}{1 - \alpha}$ where $\alpha \in [0, 1)$, with condition $y \gg x^\delta$, and $y \to \infty$

$$\sum_{0 < N(\mathcal{I}) \leq y} \sum_{0 < N(\mathcal{J}) \leq x} C_{\mathcal{J}}(\mathcal{I}) = \rho_K y + o(y). \quad (1.5)$$

His result is a consequence of a more general theorem on moments of Ramanujan sums over Dedekind domains [6, Theorem 1]. We derive asymptotic results for the second moment of Ramanujan sums over Prüfer domains in the following theorem.

**Definition 1.1** An integral domain $R$ is called a Prüfer domain if every finitely generated non-zero ideal of $R$ is invertible.
For our computations of the second moment, we use the following ideal property of Prüfer domains: If $\mathcal{I}$ and $\mathcal{J}$ are two ideals of a Prüfer domain, then

$$(\mathcal{I} + \mathcal{J})(\mathcal{I} \cap \mathcal{J}) = \mathcal{IJ}. \quad (1.6)$$

Some examples of a Prüfer domain consist of the ring of algebraic integers, the ring of entire functions in $\mathbb{C}$. For more on multiplicative ideal theory and Prüfer domains, see [7, Chapter 4].

**Theorem 1.3** Let $K$ be a number field such that its ring of integers $\mathcal{O}_K$ is a Prüfer domain. If

$$\sum_{1 \leq \mathcal{N}(\mathcal{I}) \leq y} 1 = \rho_K y + O\left(y^\alpha \right),$$

then for $x^\lambda < y$ for some $\lambda > 1$, we have

$$\sum_{0 < \mathcal{N}(\mathcal{I}) \leq y} \left( \sum_{0 < \mathcal{N}(\mathcal{J}) \leq x} C_{\mathcal{J}}(\mathcal{I}) \right)^2 = \frac{\rho_K^2}{2\zeta(2)} x^2 y + O\left(xy \log x + x^{3-\alpha}y^\alpha \right).$$

The value of $\alpha$ was estimated by Landau [14] to be $(n - 1)/(n + 1)$, where $n$ is the degree of $K$ over $\mathbb{Q}$. This was later improved by Nowak [18] and Müller [16] for the case $n = 2$ and $n = 3$, respectively.

**Remark 1.1** Theorem 1.3 holds for any number field whose corresponding ring of integers satisfies property (1.6). For the ring of integers $\mathbb{Z}$, (1.6) reduces to the fact that the gcd times lcm of any two integers is equal to the product of the integers. This property is not valid for more than two integers, complicating the computations for higher moments ($k \geq 3$).

**Remark 1.2** Theorems 1.1 and 1.2 are special cases of Theorem 1.3 with additional main terms in certain ranges of $y$, as the ring of integers for both quadratic and cubic number field is a Prüfer domain. The constants in Theorems 1.1 and 1.2 depend on the discriminant of the corresponding number fields.

### 1.1 Organization

This article is organized as follows. Section 2 covers preliminary results required to prove Theorems 1.1, 1.2, and 1.3. Section 3 contains a key result involving the average of the product of divisor functions over number fields. Section 4 contains proofs of Theorems 1.1 and 1.2 invoking the key estimate proved in Sect. 3. Finally, Sect. 5 contains proof of Theorem 1.3.
1.2 Notations

Throughout this note, we use \( \mathbb{Z} \), \( \mathbb{Q} \), and \( \mathbb{K} \) to denote the set of integers, set of rational numbers, and a number field, respectively. We denote complex numbers \( z = \sigma + it \), \( z_1 = a_1 + ib_1 \), and \( z_2 = a_2 + ib_2 \). We use \( \phi \) to denote the Euler totient function, \( \mu \) the Mobius function, \( \zeta_K(s) \) the Dedekind zeta function corresponding to a number field \( \mathbb{K} \), and \( \zeta(s) \) the Riemann zeta function. We use the Vinogradov \( \ll \) asymptotic notation, and the big oh \( O(\cdot) \) and \( o(\cdot) \) asymptotic notation. Dependence on a parameter will be denoted by a subscript.

2 Preliminaries

In this section, we state and prove some results related to the Dirichlet series of functions appearing in the proofs of Theorems 1.1 and 1.2. We start by recalling the Dirichlet series of \( C_J(I) \).

Lemma 2.1 For a number field \( \mathbb{K} \) and for \( \Re(s) > 1 \), one has

\[
\sum_{J \subseteq \mathcal{O}_K} \frac{C_J(I)}{N(J)^s} = \frac{\sigma_{\mathbb{K},(1-s)}(I)}{\zeta_K(s)},
\]

where \( \sigma_{\mathbb{K},(1-s)}(I) = \sum_{I_1 | I} N(I_1)^{1-s} \).

**Proof** From the definition of \( C_J(I) \) in (1.2), we have

\[
\sum_{J \subseteq \mathcal{O}_K} \frac{C_J(I)}{N(J)^s} = \sum_{J \subseteq \mathcal{O}_K} \frac{1}{N(J)^s} \sum_{I_1 | J} N(I_1) \mu(I_1) I_1 = \sum_{I_1 | I} \frac{1}{N(I_1)^{s-1}} \sum_{I_2 \subseteq \mathcal{O}_K} \frac{\mu(I_2)}{N(I_2)^s}
\]

\[
= \frac{\sigma_{\mathbb{K},(1-s)}(I)}{\zeta_K(s)}.
\]

The above series is absolutely convergent for \( \Re(s) > 1 \).

Lemma 2.2 For \( z \in \mathbb{C} \), and a number field \( \mathbb{K} \),

\[
\sum_{I \subseteq \mathcal{O}_K} \frac{\sigma_{\mathbb{K},z}(I)}{N(I)^s} = \zeta_K(s) \zeta(\mathbb{K}, s-z),
\]

for \( \Re(s) > \max(1 + \Re(z), 1) \).

**Proof** Using the definition of \( \sigma_{\mathbb{K},z}(I) \), we have

\[
\sum_{I \subseteq \mathcal{O}_K} \frac{\sigma_{\mathbb{K},z}(I)}{N(I)^s} = \sum_{I \subseteq \mathcal{O}_K} \frac{1}{N(I)^s} \sum_{I_1 | I} N(I_1)^z
\]

\( \square \) Springer
which are non-negative and multiplicative and which satisfy the following conditions:

\[ \Re, \text{ and it is absolutely convergent in } \Re(s) > \max(1 + \Re(z), 1). \]

**Lemma 2.3** For \( \Re(s) > \max(1, 1 + \Re(z_1), 1 + \Re(z_2), 1 + \Re(z_1 + z_2)) \), we have

\[
\sum_{I \subseteq O_K} \frac{\sigma_{K,z_1}(I) \sigma_{K,z_2}(I)}{N(I)^s} = \frac{\zeta_K(s) \zeta_K(s - z_1) \zeta_K(s - z_2) \zeta_K(s - z_1 - z_2)}{\zeta_K(2s - z_1 - z_2)}. \quad (2.3)
\]

**Proof** Fix a prime ideal \( \mathcal{P} \), then for a positive integer \( k \),

\[ \sigma_{K,z}(P^k) = \frac{N(P)^{z(k+1)} - 1}{N(P)^z - 1}. \]

Both functions \( \sigma_{K,z}(I) \), and \( \sigma_{K,z_1}(I) \sigma_{K,z_2}(I) \) are multiplicative, and hence the infinite series has an Euler product representation given by

\[
\sum_{I \subseteq O_K} \frac{\sigma_{K,z_1}(I) \sigma_{K,z_2}(I)}{N(I)^s} = \prod_{\mathcal{P} \subseteq O_K} \left( 1 + \sum_{k=1}^{\infty} \frac{\sigma_{K,z_1}(P^k) \sigma_{K,z_2}(P^k)}{N(P)^{ks}} \right)
\]

\[
= \prod_{\mathcal{P} \subseteq O_K} \left( 1 + \sum_{k=1}^{\infty} \frac{(N(P)^{z_1(k+1)} - 1)(N(P)^{z_2(k+1)} - 1)}{N(P)^{ks}(N(P)^{z_1} - 1)(N(P)^{z_2} - 1)} \right).
\]

Let \( N(P)^{-s} = x \), \( N(P)^{z_1} = y \), and \( N(P)^{z_2} = z \), then

\[
\sum_{I \subseteq O_K} \frac{\sigma_{K,z_1}(I) \sigma_{K,z_2}(I)}{N(I)^s} = \prod_{\mathcal{P} \subseteq O_K} \left( \frac{1}{y - 1} \sum_{k=0}^{\infty} x^k (y^{k+1} - 1) (z^{k+1} - 1) \right)
\]

\[
= \prod_{\mathcal{P} \subseteq O_K} \left( \frac{1}{y - 1} \left\{ \frac{yz}{1 - xy} - \frac{z}{1 - xz} - \frac{y}{1 - xy} + \frac{1}{1 - x} \right\} \right)
\]

\[
= \prod_{\mathcal{P} \subseteq O_K} \frac{1 - x^2 yz}{(1 - x)(1 - xy)(1 - xz)(1 - xyz)}.
\]

On substituting the values of \( x, y, \) and \( z \) in the above equation, we obtain Lemma 2.3.

Next, we cite two lemmas that will be useful in the next section. The first one is a Brun–Titchmarsh theorem proved by Shiu [23]. We will use it to estimate the partial sum

\[
\sum_{I \subseteq O_K} \frac{\sigma_{K,z_1}(I) \sigma_{K,z_2}(I)}{N(I) = n}.
\]

In [23], the author derives the theorem for a larger class \( M \) of arithmetic functions \( f \) which are non-negative and multiplicative and which satisfy the following conditions:
(1) For a prime \( p \), and integer \( l \geq 1 \), there exists a positive constant \( C_1 \) such that

\[
f(p^l) \leq C_1^l.
\]

(2) For every \( \epsilon > 0 \), and for \( n \geq 1 \), there exists a positive constant \( C_2 = C_2(\epsilon) \) such that

\[
f(n) \leq C_2 n^\epsilon.
\]

**Lemma 2.4** [23, Theorem 1] Let \( f \in M \), \( 0 < \alpha, \beta < 1/2 \), and \( a, k \) be integers. If \( 0 < a < k \), and \( (a, k) = 1 \), then as \( x \to \infty \)

\[
\sum_{x - y < n \leq x \atop n \equiv a \mod q} f(n) \ll \frac{y}{\phi(q) \log x} \exp \left( \sum_{p \leq x \atop p \nmid q} \frac{f(p)}{p} \right),
\]

uniformly in \( a, q \), and \( y \) provided that \( q \leq y^{1-\alpha} \), and \( x^\beta < y \leq x \).

The second lemma is a Perron-type formula for a sequence of complex numbers.

**Lemma 2.5** [22, Lemma 2.8] Let \( 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty \) be any sequence of real numbers, and let \( \{a_n\} \) be a sequence of complex numbers. Let the Dirichlet series

\[
g(s) := \sum_{n=1}^{\infty} a_n \lambda_n^{-s}
\]

be absolutely convergent for \( \sigma_a \). If \( \sigma_0 > \max(0, \sigma_a) \) and \( x > 0 \), then

\[
\sum_{\lambda_n \leq x} a_n = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} g(s) \frac{x^s}{s} ds + R,
\]

where

\[
R \ll \sum_{\frac{x}{2} < \lambda_n < 2x \atop n \neq x} |a_n| \min \left( 1, \frac{x}{T|x - \lambda_n|} \right) + \frac{4^{\sigma_0} + x^{\sigma_0}}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{\lambda_n^{\sigma_0}}.
\]

For a quadratic number field \( \mathbb{K} \) with discriminant \( q \),

\[
\zeta_{\mathbb{K}}(s) = \zeta(s)L(s, \chi_q),
\]

(2.4)

where \( \zeta(s) \) is the Riemann zeta function, and \( L(s, \chi_q) \) is the ordinary Dirichlet L-series corresponding to \( \chi_q \) and \( \chi_q \) is the Kronecker symbol of \( q \). We use bounds of
On the distribution of Ramanujan sums over number... 821

ζ(s) [25, Chapter II.3] and derive the following bounds for ζK(s):

\[
\zeta_K(\sigma + it) \ll_q \begin{cases} 
|t|^{1-2\sigma} \log^2 |t|, & -1 \leq \sigma \leq 0, \\
|t|^{1-\frac{4\sigma}{3}} \log^4 |t|, & 0 \leq \sigma \leq 1/2, \\
|t|^{\frac{2-2\sigma}{3}} \log^4 |t|, & 1/2 \leq \sigma \leq 1, \\
\log^2 |t|, & 1 \leq \sigma \leq 2, \\
1, & \sigma \geq 2,
\end{cases}
\]

(2.5)

and

\[
\frac{1}{\zeta_K(\sigma + it)} \ll_q \log^2 |t|, \quad 1 \leq \sigma \leq 2.
\]

The next lemma expresses the Dedekind zeta function for a cubic number field with discriminant \( D = dq^2 \) (\( d \) squarefree).

**Lemma 2.6** [16, Lemma 1] Let \( K \) be a cubic number field and \( D = dq^2 \) (\( d \) squarefree) its discriminant; then

1. \( K \) is a normal extension if and only if \( D = q^2 \). In this case

\[
\zeta_K(s) = \zeta(s) L(s, \chi_1) \overline{L(s, \chi_1)},
\]

(2.7)

where \( \zeta(s) \) is the Riemann zeta function and \( L(s, \chi_1) \) is the ordinary Dirichlet series corresponding to the primitive character \( \chi_1 \) modulo \( q \).

2. If \( K \) is not a normal extension, then \( d \neq 1 \), and

\[
\zeta_K(s) = \zeta(s) L(s, \chi_2),
\]

(2.8)

where \( L(s, \chi_2) \) is the Dirichlet L-series over the quadratic number field \( \mathbb{Q}(\sqrt{d}) \): 

\[
L(s, \chi_2) = \sum \chi_2(I) N(I)^{-s}.
\]

Here summation is taken over all ideals \( I \neq 0 \) in \( \mathbb{Q}(\sqrt{d}) \).

Using the above lemma, Phragmen–Lindelöf principle for a strip [10, Theorem 5.53], and the bounds given in [9], we arrive at the following bounds in the cubic case.

\[
\zeta_K(\sigma + it) \ll_q \begin{cases} 
|t|^{3(1/2-\sigma)} \log^3 |t|, & -1 \leq \sigma \leq 0, \\
|t|^{63-85\sigma + \epsilon}, & 0 \leq \sigma \leq 1/2, \\
|t|^{2(13/84+1/3)(1-\sigma)+\epsilon} = |t|^{41(1-\sigma) + \epsilon}, & 1/2 \leq \sigma \leq 1, \\
\log^3 |t|, & 1 \leq \sigma \leq 2, \\
1, & \sigma \geq 2,
\end{cases}
\]

(2.9)

and

\[
\frac{1}{\zeta_K(\sigma + it)} \ll_q \log^3 |t|, \quad 1 \leq \sigma \leq 2.
\]

(2.10)
3 A key estimate

In this section, we shall compute the average of the product of divisor functions in a number field, analogous to [22, Theorem 1.5] for divisor functions over rationals.

Lemma 3.1 Let \( \mathbb{K} \) be a number field. Then,

\[
\sum_{\substack{0 < \mathcal{N}(\mathcal{I}) \leq y \\
\mathcal{I} \subseteq \mathcal{O}_K}} \sigma_{\mathbb{K}, z_1}(\mathcal{I}) \sigma_{\mathbb{K}, z_2}(\mathcal{I}) = R_{\mathbb{K}} + E_{\mathbb{K}}.
\]

For a quadratic number field \( \mathbb{K} \), for \(-1/13 < a_1 < 0, -1/9 < a_2 < 0, -1/13 < a_1 + a_2 < 0, \) and \( |b_1|, |b_2| \ll y^{1/3} \),

\[
R_{\mathbb{K}} = \rho_{\mathbb{K}} \frac{\zeta_{\mathbb{K}}(1 - z_1) \zeta_{\mathbb{K}}(1 - z_2) \zeta_{\mathbb{K}}(1 - z_1 - z_2)}{\zeta_{\mathbb{K}}(2 - z_1 - z_2)} y^{1 + z_1} \\
+ \rho_{\mathbb{K}} \frac{\zeta_{\mathbb{K}}(1 + z_1) \zeta_{\mathbb{K}}(1 + z_1 - z_2) \zeta_{\mathbb{K}}(1 - z_2)}{\zeta_{\mathbb{K}}(2 + z_1 - z_2)} y^{1 + z_2} \\
+ \rho_{\mathbb{K}} \frac{\zeta_{\mathbb{K}}(1 + z_2) \zeta_{\mathbb{K}}(1 + z_2 - z_1) \zeta_{\mathbb{K}}(1 - z_1)}{\zeta_{\mathbb{K}}(2 - z_1 + z_2)} y^{1 + z_1 + z_2} \\
+ \rho_{\mathbb{K}} \frac{\zeta_{\mathbb{K}}(1 + z_1 + z_2) \zeta_{\mathbb{K}}(1 + z_2) \zeta_{\mathbb{K}}(1 + z_1)}{\zeta_{\mathbb{K}}(2 + z_1 + z_2)} y^{1 + z_1 + z_2},
\]

and

\[
E_{\mathbb{K}} = O_q \left( y^{\frac{17 + 5a_1 + 9a_2}{18} \log^2 y} \right).
\]

For a cubic number field \( \mathbb{K} \), for \(-16/183 < a_1 < 0, -8/49 < a_2 < 0, -16/183 < a_1 + a_2 < 0, \) and \( |b_1|, |b_2| \ll y^{3/14} \), \( R_{\mathbb{K}} \) is same as above, and

\[
E_{\mathbb{K}} = O_q \left( y^{\frac{180 + 13a_1 + 9a_2}{190} + \epsilon \log^3 y} \right).
\]

Proof For any number field \( \mathbb{K} \), one has

\[
\sum_{\mathcal{I} \subseteq \mathcal{O}_K} \frac{\sigma_{\mathbb{K}, z_1}(\mathcal{I}) \sigma_{\mathbb{K}, z_2}(\mathcal{I})}{\mathcal{N}(\mathcal{I})^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{\mathcal{I} \subseteq \mathcal{O}_K \mathcal{N}(\mathcal{I}) = n} \sigma_{\mathbb{K}, z_1}(\mathcal{I}) \sigma_{\mathbb{K}, z_2}(\mathcal{I}).
\]

Define \( A(n, z_1, z_2) := \sum_{\mathcal{I} \subseteq \mathcal{O}_K \mathcal{N}(\mathcal{I}) = n} \sigma_{\mathbb{K}, z_1}(\mathcal{I}) \sigma_{\mathbb{K}, z_2}(\mathcal{I}) \) and

\[
f(z_1, z_2, s) := \frac{\zeta_{\mathbb{K}}(s) \zeta_{\mathbb{K}}(s - z_1) \zeta_{\mathbb{K}}(s - z_2) \zeta_{\mathbb{K}}(s - z_1 - z_2)}{\zeta_{\mathbb{K}}(2s - z_1 - z_2)}.
\]
Let $\Re(z_1) = a_1$ and $\Re(z_2) = a_2$ be such that $a_1, a_2 < 0$, and $a_1 + a_2 > -1$. Consider $\alpha = 1 + \frac{1}{\log y}$. Using Lemma 2.5, we have

$$\sum_{0 < N(I) \leq \gamma} \sigma_{\mathbb{H}, z_1}(I) \sigma_{\mathbb{H}, z_2}(I) = \sum_{n \leq \gamma} A(n, z_1, z_2) = \frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} f(z_1, z_2, s) \frac{y^s}{s} ds$$

$$+ R(y; z_1, z_2), \quad (3.1)$$

where

$$R(y; z_1, z_2) \ll \sum_{y/2 < n < 2y} |A(n, z_1, z_2)| \min \left(1, \frac{y}{T|y - n|}\right)$$

$$+ \frac{y^\alpha}{T} \sum_{n=1}^{\infty} \frac{|A(n, z_1, z_2)|}{n^\alpha}. \quad (3.2)$$

For $\lambda = (1 + a_1 + a_2)/2$, we solve integral in (3.1) by modifying the line integral into a rectangular path $C$ with vertices $\alpha \pm iT$, and $\lambda \pm iT$. The poles of the integrand inside the contour $C$ are $s_1 = 1$, $s_2 = z_1 + 1$, $s_3 = z_2 + 1$, and $s_4 = z_1 + z_2 + 1$. Hence by Cauchy's residue theorem, we have

$$\frac{1}{2\pi i} \int_{C} f(z_1, z_2, s) \frac{y^s}{s} ds = \rho_{\mathbb{K}} \frac{\zeta_{\mathbb{K}}(1 - z_1) \zeta_{\mathbb{K}}(1 - z_2) \zeta_{\mathbb{K}}(1 - z_1 - z_2)}{\zeta_{\mathbb{K}}(z_1 - z_2)} y^{1+z_1}$$

$$+ \rho_{\mathbb{K}} \frac{\zeta_{\mathbb{K}}(1 + z_1) \zeta_{\mathbb{K}}(1 + z_1 - z_2) \zeta_{\mathbb{K}}(1 - z_2)}{\zeta_{\mathbb{K}}(z_1 - z_2)} y^{1+z_1}$$

$$+ \rho_{\mathbb{K}} \frac{\zeta_{\mathbb{K}}(1 + z_2) \zeta_{\mathbb{K}}(1 + z_2 - z_1) \zeta_{\mathbb{K}}(1 - z_1)}{\zeta_{\mathbb{K}}(z_1 - z_2)} y^{1+z_2}$$

$$+ \rho_{\mathbb{K}} \frac{\zeta_{\mathbb{K}}(1 + z_1 + z_2) \zeta_{\mathbb{K}}(1 + z_2) \zeta_{\mathbb{K}}(1 + z_1)}{\zeta_{\mathbb{K}}(z_1 + z_2)} y^{1+z_1+z_2}. \quad (3.3)$$

This implies

$$\frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} f(z_1, z_2, s) \frac{y^s}{s} ds = R_0 + \sum_{i=1}^{3} J_i, \quad (3.4)$$

where $R_0$ is equal to the right side of (1.4), and $J_i$’s are the line integrals along the lines $[\lambda + iT, \alpha + iT]$, $[\lambda - iT, \lambda + iT]$, and $[\alpha - iT, \lambda - iT]$, respectively. By Holder’s inequality
\[
\left( \int_{0}^{\alpha} \int_{0}^{T_{0}/2} f(z_{1}, z_{2}, \sigma + it) \frac{\sigma^{\sigma + it}}{\sigma + it} d\sigma dt \right)^{4} \\
\ll \int_{0}^{\alpha} \int_{0}^{T_{0}/2} |\zeta_{K}(\sigma + it)|^{4} y^{\sigma} d\sigma dt \\
\times \int_{0}^{\alpha} \int_{0}^{T_{0}/2} |\zeta_{K}(2(\sigma + it) - z_{1} - z_{2})(\sigma + it)| d\sigma dt \\
\times \int_{0}^{\alpha} \int_{0}^{T_{0}/2} |\zeta_{K}(\sigma + it - z_{1})|^{4} y^{\sigma} d\sigma dt \\
\times \int_{0}^{\alpha} \int_{0}^{T_{0}/2} |\zeta_{K}(2(\sigma + it) - z_{1} - z_{2})(\sigma + it)| d\sigma dt. \tag{3.5}
\]

We estimate the above integrals and the remainder \( R(y; z_{1}, z_{2}) \) in (3.2) separately for the quadratic and cubic number fields.

### 3.1 Quadratic number field

For \( K \) a quadratic number field, the Dirichlet series for \( A(n, z_{1}, z_{2}) \) can be expressed in terms of \( \zeta(s) \) and \( L(s, \chi) \) using (2.3) and (2.4). Consequently, \( A(n, z_{1}, z_{2}) \) is written as a Dirichlet convolution of coefficients of its Dirichlet series. This exercise yields the following elementary but essential bound.

\[
|A(n, z_{1}, z_{2})| \leq \left| n^{a_{1} + a_{2}} \sum_{\mathcal{N}(I) = n} \left( \sum_{I_{1}, I_{2}} 1 \right) \right|^{2} \\
\leq \sum_{d|n} \left\{ \sum_{d'|d} \left( \sum_{d_{1}|d/d'} \sum_{d_{0}} \sigma_{0}(d') \frac{\sigma_{0}(d_{1})\sigma_{0}(d_{0})}{d_{0}} \right) \right\} \sigma_{0}(n/d). \tag{3.6}
\]

This leads to \( |A(p, z_{1}, z_{2})| \leq 9 \). Taking \( T = y^{c} \) where \( c \) is a fixed real number, and dividing the interval \( y/2 < n < 2y \) according to \( \min \left( 1, \frac{y}{T|y - n|} \right) \), we arrive at

\[
\sum_{|y - n| < y^{1-c}} |A(n, z_{1}, z_{2})| \min \left( 1, \frac{y}{T|y - n|} \right) = \sum_{|y - n| < y^{1-c}} |A(n, z_{1}, z_{2})| \ll \frac{y}{T} \log^{8} T. \tag{3.7}
\]
The last estimate follows from an application of Lemma 2.4 on the function $A(n, z_1, z_2)$. Note that (3.6) ensures that the hypothesis in the lemma is satisfied. For the interval $y + y^{1-c} < n < 2y$, we have

$$\sum_{y + y^{1-c} < n < 2y} |A(n, z_1, z_2)| \min\left(1, \frac{y}{T|y-n|}\right)$$

$$\ll \frac{y}{T} \sum_{y + y^{1-c} < n < 2y} \frac{\sum_{d|n} |\sum_{d'|d} \sigma_0(d') | \sum_{d_1|d} \sigma_0(d_1) \sigma_0(d / d_1') \sigma_0(n/d)|}{n-y}$$

$$\ll \frac{y}{T} \sum_{U < n-y < 2U} \frac{\sum_{d|n} |\sum_{d'|d} \sigma_0(d') | \sum_{d_1|d} \sigma_0(d_1) \sigma_0(d / d_1') \sigma_0(n/d)|}{n-y}$$

$$\ll \frac{y}{T} \log^8 y. \quad (3.8)$$

One obtains similar bounds for $y/2 < n < y - y^{1-c}$.

Moreover,

$$\frac{y^\alpha}{T} \sum_{n=1}^{\infty} \frac{|A(n, z_1, z_2)|}{n^\alpha} \ll \frac{y}{T} \log^8 T. \quad (3.9)$$

Finally, from (3.7), (3.8), and (3.9), we deduce that

$$R(y; z_1, z_2) \ll \frac{y}{T} \log^8 T. \quad (3.10)$$

Next, we solve the line integrals using bounds in (2.5) and (2.6), we have

$$\int_{\lambda}^{\alpha} \int_{0/2}^{T_0} \frac{|\zeta_{\mathbb{R}}'(\sigma + it)|^4 y^\sigma}{t \log^4 t} d\sigma dt$$

$$\ll q \int_{0/2}^{T_0/2} \int_{\lambda}^{1/2} t^{4-16\sigma/3} \log^{18} t y^\sigma t^{1/3} \log^8 t \sigma d\sigma dt$$

$$\ll q \int_{0/2}^{T_0/2} t^{3/2} \log^{18} t \int_{\lambda}^{1/2} \left(\frac{y}{t^{16/3}}\right)^{\sigma} d\sigma dt + \int_{T_0/2}^{T_0} t^{1/2} \int_{0/2}^{\alpha} t^{8(1-\sigma)/3} \log^{18} t y^\sigma \sigma d\sigma dt$$

for $y^{3/16} < T_0 < y^{3/8}$, we have

$$\int_{\lambda}^{\alpha} \int_{0/2}^{T_0} \frac{|\zeta_{\mathbb{R}}'(\sigma + it)|^4 y^\sigma}{t \log^4 t} d\sigma dt$$

$$\ll q (T_0^{4-16\lambda/3} y^{\lambda} + y) \log^{18} T_0.$$
If $a_1 - a_2 > 0$, then we have

\[
\begin{align*}
\int_\lambda^\alpha & \int_{T_0/2}^T \frac{|\zeta_K(\sigma + it - z_1)|^4 y^{\sigma}}{|\zeta_K(2(\sigma + it) - z_1 - z_2)(\sigma + it)|} \, d\sigma dt \\
& \ll_q \int_{T_0/2}^T \int_{\lambda}^{1/2+a_1} t^{8 - 8\sigma + 8a_1/3} \log^2 \frac{t^{y^{\sigma}}}{t} \, d\sigma dt \\
& \quad + \int_{T_0/2}^T \int_{\lambda}^{1/2+a_1} t^{8 - 8\sigma + 8a_1/3} \log^2 \frac{t^{y^{\sigma}}}{t} \, d\sigma dt \\
& \quad + \int_{T_0/2}^T \int_{\lambda}^{1/2+a_1} t^{8 - 8\sigma + 8a_1/3} \log^2 \frac{t^{y^{\sigma}}}{t} \, d\sigma dt \\
& \ll_q (T_0 + y^{1+a_1}) \log^2 T_0 + y \log^2 T_0.
\end{align*}
\]

Similarly,

\[
\begin{align*}
\int_\lambda^\alpha & \int_{T_0/2}^T \frac{|\zeta_K(\sigma + it - z_1)|^4 y^{\sigma}}{|\zeta_K(2(\sigma + it) - z_1 - z_2)(\sigma + it)|} \, d\sigma dt \\
& \ll_q \int_{T_0/2}^T \int_{\lambda}^{1/2+a_2} t^{8 - 8\sigma + 8a_2/3} \log^2 \frac{t^{y^{\sigma}}}{t} \, d\sigma dt \\
& \quad + \int_{T_0/2}^T \int_{\lambda}^{1/2+a_2} t^{8 - 8\sigma + 8a_2/3} \log^2 \frac{t^{y^{\sigma}}}{t} \, d\sigma dt \\
& \quad + \int_{T_0/2}^T \int_{\lambda}^{1/2+a_2} t^{8 - 8\sigma + 8a_2/3} \log^2 \frac{t^{y^{\sigma}}}{t} \, d\sigma dt \\
& \ll_q y^{1+a_2} \log^2 T_0 + y \log^2 T_0,
\end{align*}
\]

and

\[
\begin{align*}
\int_\lambda^\alpha & \int_{T_0/2}^T \frac{|\zeta_K(\sigma + it - z_1)|^4 y^{\sigma}}{|\zeta_K(2(\sigma + it) - z_1 - z_2)(\sigma + it)|} \, d\sigma dt \\
& \ll_q \int_{T_0/2}^T \int_{\lambda}^{1+a_1+a_2} t^{8 - 8\sigma + 8(a_1 + a_2)} \log^2 \frac{t^{y^{\sigma}}}{t} \, d\sigma dt \\
& \quad + \int_{T_0/2}^T \int_{\lambda}^{1+a_1+a_2} t^{8 - 8\sigma + 8(a_1 + a_2)} \log^2 \frac{t^{y^{\sigma}}}{t} \, d\sigma dt \\
& \ll_q y^{1+a_1+a_2} \log^2 T_0 + y \log^2 T_0.
\end{align*}
\]
Collecting all the above results and substituting in (3.5), we obtain
\[
\int_{\lambda}^{\lambda} \int_{T_{0}/2}^{T_{0}} f(z_1, z_2, \sigma + it) \frac{y^{\sigma + it}}{\sigma + it} d\sigma dt \ll_q y \log^{12} T_0.
\]

Next, we choose \(T\) such that \(T_{0}/2 < T < T_0\), which gives
\[
\int_{\lambda}^{\lambda} f(z_1, z_2, \sigma + iT) \frac{y^{\sigma + iT}}{\sigma + iT} d\sigma \ll_q \frac{y}{T} \log^{12} T.
\]

Integral along the vertical line \([\lambda - iT, \lambda + iT]\) is given by
\[
\int_{\lambda - iT}^{\lambda + iT} f(z_1, z_2, s) \frac{y^s}{s} ds \ll_q \int_{-T}^{T} t^{-\frac{2}{3}} \log^{18} t \frac{y^{\lambda}}{t} dt \ll_q T^{\frac{4-2\eta}{3}} y^{\lambda} \log^{18} T.
\]

We get the required result by putting \(T = y^{1/3}\) in the above estimates.

### 3.2 Cubic number field

If \(K\) is a cubic number field, then like the case for quadratic, we write \(A(n, z_1, z_2)\) using (2.3) and (2.7) to obtain
\[
|A(n, z_1, z_2)| \leq n^{a_1 + a_2} \left| \sum_{N(I) = n} \left( \sum_{I_1 | I} 1 \right)^2 \right|
\]
\[
\leq \sum_{d | n} \left\{ \sum_{d' | d} \left( \sum_{d_1 | d'} \sigma_0(d_1) \sum_{d_2 | d/d'} \left( \sum_{d_{21} | d_2} \sigma_0(d_{21}) \sum_{d_{22} | d_2} \sigma_0(d_{22}) \right) \right) \right\}
\]
\[
\sum_{d'' | n/d} \sigma_0(d'').
\]

For a prime \(p\), a direct computation of the right-hand side of the above inequality gives the bound
\[
|A(p, z_1, z_2)| \leq 13.
\]

Choosing \(T = y^c\) where \(c\) is a fixed real number, then using Lemma 2.4, we have
\[
\sum_{|y - n| < y^{1-c}} |A(n, z_1, z_2)| \min \left(1, \frac{y}{T|y - n|} \right) = \sum_{|y - n| < y^{1-c}} |A(n, z_1, z_2)| \ll \frac{y}{T} \log^{12} T.
\]

(3.11)
The above bounds also hold true for the intervals: \( y/2 < T < y - y^{1-c} \) and \( y + y^{1-c} < n < 2y \).

Moreover,

\[
\frac{y^\alpha}{T} \sum_{n=1}^{\infty} \frac{|A(n, z_1, z_2)|}{n^\alpha} \ll \frac{y}{T} \log^{12} T. \tag{3.12}
\]

These estimates yield

\[
R(y, z_1, z_2) \ll \frac{y}{T} \log^{12} T. \tag{3.13}
\]

In the following computations, we employ Dedekind zeta bounds (2.9) and (2.10) to obtain estimates of the line integrals in (3.5). For \( y^{3/170} \leq T \leq y^{21/82} \)

\[
\int_\lambda^\alpha \int_{T_0/2}^{T_0} \frac{|\zeta(\sigma + it)|^4 y^\sigma}{|\zeta(2(\sigma + it) - z_1 - z_2)(\sigma + it)|} \, d\sigma \, dt 
\ll q \int_{T_0/2}^{T_0} \int_{1/2}^{1/2+\alpha t} t^{2(63-85\sigma)/21+\epsilon} \log^3 \frac{y^\sigma}{t} \, d\sigma \, dt 
+ \int_{T_0/2}^{T_0} \int_{1/2}^{1/2+\alpha t} t^{82(1-\sigma)/21+\epsilon} \log^3 \frac{y^\sigma}{t} \, d\sigma \, dt 
\ll q \left( T_0^{6-(170\lambda - 170\alpha t)/21+\epsilon} y \lambda + T_0^{\epsilon} \right) \log^3 T_0.
\]

If \( a_1 - a_2 > 0 \), then we have

\[
\int_\lambda^\alpha \int_{T_0/2}^{T_0} \frac{|\zeta(\sigma + it - z_1)|^4 y^\sigma}{|\zeta(2(\sigma + it) - z_1 - z_2)(\sigma + it)|} \, d\sigma \, dt 
\ll q \int_{T_0/2}^{T_0} \int_{1/2+a_1}^{1/2+a_1} t^{2(63-85\sigma + 85a_1)/21+\epsilon} \log^3 \frac{y^\sigma}{t} \, d\sigma \, dt 
+ \int_{T_0/2}^{T_0} \int_{1/2+a_1}^{1/2+a_1} t^{82(1-\sigma+a_1)/21+\epsilon} \log^3 \frac{y^\sigma}{t} \, d\sigma \, dt 
\ll q \left( T_0^{6-(170\lambda - 170\alpha t+a_1)/21+\epsilon} y \lambda + T_0^{\epsilon} \right) \log^3 T_0 + y \log^{15} T_0.
\]

Similarly,

\[
\int_\lambda^\alpha \int_{T_0/2}^{T_0} \frac{|\zeta(\sigma + it - z_2)|^4 y^\sigma}{|\zeta(2(\sigma + it) - z_1 - z_2)(\sigma + it)|} \, d\sigma \, dt 
\ll q \int_{T_0/2}^{T_0} \int_{\lambda}^{1+a_2} t^{81(1-\sigma+a_2)/21+\epsilon} \log^3 \frac{y^\sigma}{t} \, d\sigma \, dt 
+ \int_{T_0/2}^{T_0} \int_{1+a_2}^{\alpha} \log^{15} \frac{y^\sigma}{t} \, d\sigma \, dt
\]
\[ \ll q \ T^\epsilon y^{1+a_2} \log^3 T_0 + y \log^{15} T_0, \]

and

\[ \int_{T_0/2}^{T_0} \int_{\lambda}^{\alpha} |\zeta_K(\sigma + it - z_1 - z_2)|^4 y^\sigma d\sigma dt \]
\[ \ll q \ \int_{T_0/2}^{T_0} \int_{\lambda}^{\alpha} \frac{82(1-\sigma+a_1+a_2)+\epsilon}{\zeta_K(\sigma)} \log^3 t \frac{y^\sigma}{t} d\sigma dt + \int_{T_0/2}^{T_0} \int_{1+a_1+a_2}^{\alpha} \log^{15} t \frac{y^\sigma}{t} d\sigma dt \]
\[ \ll q \ T^\epsilon y^{1+a_1+a_2} \log^3 T_0 + y \log^{15} T_0. \]

The above estimates show that the double integral in (3.5) is bounded by

\[ \int_{\lambda}^{\alpha} \int_{T_0/2}^{T_0} f(z_1, z_2, \sigma + it) \frac{y^{\sigma+it}}{\sigma + it} d\sigma dt \ll q \ y T^\epsilon \log^{12} T_0. \]

Next, we choose a \( T \) such that \( T_0/2 < T < T_0 \), which gives

\[ \int_{\lambda}^{\alpha} f(z_1, z_2, \sigma + it) \frac{y^{\sigma+it}}{\sigma + it} d\sigma \ll q \ y T^\epsilon \log^{12} T. \]

Finally, the integral along the vertical line \([\lambda - iT, \lambda + iT]\) is estimated as

\[ \int_{\lambda - iT}^{\lambda + iT} f(z_1, z_2, s) \frac{y^s}{s} ds \ll q \ \int_{-T}^{T} t^{82-85a_1+\epsilon} \log^3 t \frac{y^\lambda}{t} dt \ll q \ T^{-\epsilon} y^{82-85a_1+\epsilon} \log^3 T. \]

We get the required result by putting \( T = y^{3/14} \) in the above bounds. \( \square \)

4 Second moment

Our arguments for the asymptotics of second moments for quadratic and cubic fields follow ideas from [4, 22] with several adaptations required to extend the proof for number field.

4.1 Second moment for quadratic number field

**Proof of Theorem 1.1** Let

\[ \beta_i = 1 + \frac{i}{\log y}, \]

where \( i \in \{1, 2\} \). Using Lemma 2.5, we have

\[ \sum_{0 < N(J) \leq x} C_J(T) = \frac{1}{2\pi i} \int_{\beta_i - iT}^{\beta_i + iT} \frac{\sigma_{\zeta_K(1-s)}(T)}{\zeta_K(s)} \frac{x^s}{s} ds + O \left( \frac{x \log y}{T} \sigma_{\zeta_K,0}(T) \right). \]
Squaring the two sides yields
\[
\left( \sum_{0 < \mathcal{N}(\mathcal{J}) \leq x} C_{\mathcal{I}}(\mathcal{J}) \right)^2 = \frac{1}{(2\pi i)^2} \int_{\beta_1-iT}^{\beta_1+iT} \int_{\beta_2-iT}^{\beta_2+iT} \frac{\sigma_{\mathcal{K},(1-s_1)}(\mathcal{I}) \sigma_{\mathcal{K},(1-s_2)}(\mathcal{J})}{\zeta_{\mathcal{K}}(s_1) \zeta_{\mathcal{K}}(s_2)} x^{s_1+s_2} \frac{ds_1 ds_2}{s_1 s_2} + R(x, \mathcal{I}),
\]
where
\[
R(x, \mathcal{I}) \ll \frac{x \log y}{T} \log \frac{\log x}{\log y} \frac{x^s}{s} \left( \frac{x^2 \log^2 y}{T^2} (\sigma_{\mathcal{K},0}(\mathcal{I}))^2 \right) + \frac{x^2 \log^2 y}{T} \left( \sum_{0 < \mathcal{N}(\mathcal{I}) \leq y} (\sigma_{\mathcal{K},0}(\mathcal{I}))^2 \right).
\]
Inserting (4.2) in (4.1), and summing both sides over ideals \( \mathcal{I} \) with \( \mathcal{N}(\mathcal{I}) \leq y \), we have
\[
\sum_{0 < \mathcal{N}(\mathcal{I}) \leq y} \left( \sum_{0 < \mathcal{N}(\mathcal{J}) \leq x} C_{\mathcal{I}}(\mathcal{J}) \right)^2 = \frac{1}{(2\pi i)^2} \int_{\beta_1-iT}^{\beta_1+iT} \int_{\beta_2-iT}^{\beta_2+iT} \frac{G(s_1, s_2, y) x^{s_1+s_2} \frac{ds_1 ds_2}{s_1 s_2}}{\zeta_{\mathcal{K}}(s_1) \zeta_{\mathcal{K}}(s_2)} + \mathcal{O}_{\mathcal{I}} \left( \frac{x^2 \log^2 y \log^3 T}{T} \left( \sum_{0 < \mathcal{N}(\mathcal{I}) \leq y} (\sigma_{\mathcal{K},0}(\mathcal{I}))^2 \right) \right) = I + \mathcal{O}_{\mathcal{I}} \left( \frac{x^2 \log^2 y \log^3 T}{T} \left( \sum_{0 < \mathcal{N}(\mathcal{I}) \leq y} (\sigma_{\mathcal{K},0}(\mathcal{I}))^2 \right) \right),
\]
where \( G(s_1, s_2, y) := \sum_{0 < \mathcal{N}(\mathcal{I}) \leq y} \sigma_{\mathcal{K},(1-s_1)}(\mathcal{I}) \sigma_{\mathcal{K},(1-s_2)}(\mathcal{J}) \). We take \( a_1 = a_2 = 0 \) in (3.6), and then use Lemma 2.4 to get
\[
\sum_{0 < \mathcal{N}(\mathcal{I}) \leq y} (\sigma_{\mathcal{K},0}(\mathcal{I}))^2 \ll y \log^9 y.
\]
Using Lemma 3.1, for \( |T| \ll x^{1/3} \) the integral \( I \) in (4.3) can be written as
\[
I = I_1 + I_2 + I_3 + I_4 + \mathcal{O}_{\mathcal{I}} \left( x^2 y^{17/18} \log^{24} T \right),
\]
where \( I_1, I_2, I_3, \) and \( I_4 \) are the integrals corresponding to the four terms appearing in \( R_0 \) in Lemma 3.1. We compute each of them separately below.

4.1.1 Evaluation of \( I_1 \)

To evaluate the integral \( I_1 \)
\[
I_1 = \frac{\rho_{\mathcal{K},y}}{(2\pi i)^2} \int_{\beta_1-iT}^{\beta_1+iT} \int_{\beta_2-iT}^{\beta_2+iT} \frac{\zeta_{\mathcal{K}}(s_1+s_2-1) x^{s_1+s_2} \frac{ds_1 ds_2}{s_1 s_2}}{\zeta_{\mathcal{K}}(s_1+s_2)}
\]
we shift the line integral in $s_2$-plane to a rectangular contour containing the lines $[\beta_2 + iT, 3/2 - \beta_1 + iT], [3/2 - \beta_1 + iT, 3/2 - \beta_1 - iT], [3/2 - \beta_1 - iT, \beta_2 - iT]$, and $[\beta_2 - iT, \beta_2 + iT]$. We see that $s_2 = 2 - s_1$ is the pole of the integrand inside the contour and the residue at pole is

$$\frac{\rho_{\xi} x^2}{\xi(2)s_1(2 - s_1)}.$$  

If $L_{1,1}$, $L_{1,2}$, and $L_{1,3}$ are the integrals along the lines $[3/2 - \beta_1 + iT, \beta_2 + iT]$, $[3/2 - \beta_1 - iT, 3/2 - \beta_1 + iT]$, and $[3/2 - \beta_1 - iT, \beta_2 - iT]$, respectively, then

$$|L_{1,1}|, |L_{1,3}| \ll q \int_T 1 \left| \int_{-T}^{T} \left( \int_{3/2-\beta_1}^{\beta_2} T^{2-\beta_1-\sigma} \log^8 T \frac{x^\sigma}{T} \, d\sigma \right) \frac{1}{1 + |t|} \, dt \right|
$$

$$\ll q \int_T 1 \left| \int_{-T}^{T} \left( \int_{3/2-\beta_1}^{\beta_2} \left( \frac{x}{T^{2/3}} \right)^\sigma \, d\sigma \right) \frac{1}{1 + |t|} \, dt \right|
$$

$$\ll q \frac{x^2 \log^9 T}{T^{2/3}} + \frac{x^{3/2} \log^9 T}{T^{2/3}}. \quad (4.5)$$

Furthermore, the integral along the vertical line is given by

$$|L_{1,2}| \ll q \int_T 1 \left| \int_{-T}^{T} \left| \frac{\xi(\beta_1 - 1/2 + i(t_1 + t_2))}{\xi(\beta_1 + 1/2 + i(t_1 + t_2))} \right| \frac{1}{(1 + |t_1|)(1 + |t_2|)} \, dt_1 \, dt_2 \right|
$$

$$\ll q \int_T 1 \left| \int_{-2T}^{2T} \left| \frac{\xi(\beta_1 - 1/2 + iT)}{\xi(\beta_1 + 1/2 + iT)} \right| \frac{1}{(1 + |t_1|)(1 + |t_1|)} \, dt_1 \, dt \right|
$$

$$\ll q \int_T 1 \left| \int_{-2T}^{2T} \frac{t^{1/3} \log^6 T}{(1 + |t|)} \, dt \right|
$$

$$\ll q \int_T 1 \left| \int_{-2T}^{2T} \frac{t^{1/3} \log^7 T}{T} \, dt \right|
$$

Therefore, we have

$$I_1 = \frac{\rho_{\xi}^2 y x^2}{\xi(2)} \frac{1}{2\pi i} \int_{\beta_1 - iT}^{\beta_1 + iT} \frac{1}{s_1(2 - s_1)} \, ds_1 + O_q \left( \frac{x^2 \log^9 T}{T} + \frac{y x^{3/2} T^{1/3} \log^7 T}{T} \right)
$$

$$= \frac{\rho_{\xi}^2 y x^2}{2\xi(2)} + O_q \left( \frac{x^2 \log^9 T}{T} + y x^{3/2} T^{1/3} \log^7 T \right). \quad (4.7)$$

4.1.2 Evaluation of $I_2$

The integral $I_2$ is given by

$$I_2 = \frac{\rho_{\xi} y^2}{(2\pi i)^2} \int_{\beta_1 - iT}^{\beta_1 + iT} \int_{\beta_2 - iT}^{\beta_2 + iT} \frac{\xi(2 - s_1)\xi(1 - s_1 + s_2)}{(2 - s_1)\xi(2 - s_1 + s_2)\xi(s_1)} \, ds_1 \, ds_2.$$

$\text{ Springer}$
We move the line integral over \( s_1 \)-plane to a contour with vertices \( \beta_1 + iT, 3/2 + iT, 3/2 - iT, \) and \( \beta_1 + iT \). The integrand has a simple pole at \( s_1 = s_2 \) with residue

\[
-\rho \zeta(2 - s_2) x^{2s_2} / y^{s_2}
\]

Let \( L_{2.1}, L_{2.3} \) be the integrals along the horizontal lines of the contour, and \( L_{2.2} \) be the integral along the vertical line. Thus, we have

\[
|L_{2.1}|, |L_{2.3}| \ll y^2 \int_{-T}^{T} \left( \int_{\beta_1}^{3/2} \frac{x^{2s_2}}{y^{s_2}} (2 - \sigma - it) \zeta(1 - \sigma + \beta_2 - iT + it) x^{\sigma + \beta_2} \zeta(2 - \sigma + \beta_2 - iT + it) \zeta(\sigma + i) y^{\sigma} T^T \, d\sigma \right)
\]

\[
\frac{1}{1 + |t|} \, dt
\]

\[
\ll q \frac{y^2}{T^2} \int_{-T}^{T} \left( \int_{\beta_1}^{3/2} \frac{x^{2s_2}}{y^{s_2}} (2 - 2(1 - \sigma) / 3) T^{2 - 2(1 - \sigma + \beta_2) / 3} \log T x^{\sigma} \, d\sigma \right) \frac{1}{1 + |t|} \, dt
\]

\[
\ll q \frac{y^2 \log T}{T^{2+4/3}} \int_{-T}^{T} \left( \int_{\beta_1}^{3/2} \frac{x^{4/3}}{y^{1/3}} \, d\sigma \right) \frac{1}{1 + |t|} \, dt
\]

\[
\ll q \frac{d x^{5/2} y^{1/2} \log^{11} T}{T^{4/3}} + \frac{x^2 y \log^{11} T}{T^2},
\]

and

\[
|L_{2.2}| \ll x^{5/2} y^{1/2} \int_{-T}^{T} \left( \int_{\beta_1}^{3/2} \frac{\zeta(1/2) \zeta(-1/2 + \beta_2 + i(-t_1 + t_2))}{\zeta(3/2 + i t_1) \zeta(1/2 + \beta_2 + i(-t_1 + t_2))} \, dt_1 dt_2 \right)
\]

\[
\ll q x^{5/2} y^{1/2} \int_{-2T}^{2T} \frac{\zeta(-1/2 + \beta_2 + it_1)}{\zeta(1/2 + \beta_2 + it) \zeta(1/2 + \beta_2 + it_1)} \int_{-T}^{T} \frac{t^{1/3} \log^{6} t_1}{(1 + |t_1|)^2 (1 + |t_1|)} \, dt_1 dt_2
\]

\[
\ll q x^{5/2} y^{1/2} \log^{6} T \int_{-2T}^{2T} \frac{t^{1/3} \log^{6} t}{(1 + |t|)^2} \, dt
\]

\[
\ll q x^{5/2} y^{1/2} T^{1/3} \log^{12} T.
\]

Combining the above estimates, we have

\[
I_2 = \frac{\rho^2 y^2}{\zeta(2)(2\pi i)^2} \int_{\beta_1 - iT}^{\beta_2 + iT} \frac{\zeta(2 - s_2) x^{2s_2} / y^{s_2}}{s^2_2(2 - s_2) \zeta(s_2)} \, ds_2 + O(q \left( \frac{x^2 y \log^{11} T}{T^2} + \frac{x^5/2 y^{1/2} \log^{12} T}{T^{1/3}} \right))
\]

\[
= \frac{\rho^2 y^2 x \zeta(0)}{4 \zeta(2)(2\pi i)^2} + O(q \left( \frac{x^2 y \log^{11} T}{T^2} + \frac{x^5/2 y^{1/2} T^{1/3} \log^{12} T}{T^{1/3}} \right)).
\]

### 4.1.3 Evaluation of \( I_3 \)

The integral \( I_3 \) is given as

\[
I_3 = \frac{\rho y^2}{(2\pi i)^2} \int_{\beta_1 + iT}^{\beta_2 + iT} \frac{\zeta(2 - s_2) \zeta(1 - s_2 + s_1) x^{s_1 + s_2}}{(2 - s_2) \zeta(2 - s_2 + s_1) \zeta(s_2) y^{s_2} s_1 s_2} \, ds_1 ds_2
\]

\( \text{Springer} \)
To estimate the integral $I_3$, we modify the line integration over $s_2$ to the contour containing the vertices $\beta_2 + iT, \beta_2 - iT, 3/2 + iT$, and $3/2 - iT$, and denote the integration along the lines $[\beta_2 + iT, 3/2 + iT], [3/2 + iT, 3/2 - iT]$, and $[3/2 - iT, \beta_2 - iT]$ by $L_{3,1}, L_{3,2},$ and $L_{3,3}$, respectively. There is no pole of the integrand inside the contour. So, the integral along the horizontal lines are

\[
|L_{3,1}, L_{3,3}| \ll q^2 \int_T^{-T} \left( \int_{\beta_2}^{3/2} \frac{\zeta_2(2 - \sigma - iT)\zeta_2(1 - \sigma + \beta_1 - iT + iT)\chi_2^{\sigma + \beta_1}}{\zeta_2(2 - \sigma + \beta_1 - iT + iT)\zeta_2(\sigma + iT)} \frac{1}{1 + |t|} dt \right) \int_{-T}^{T} \left( \int_{\beta_2}^{3/2} \frac{1}{1 + |t|} dt \right) \frac{y^2}{T} \frac{1}{T^2} \frac{1}{T^2}.
\]

One can evaluate the integral along the vertical line same as (4.9). Therefore

\[
|L_{3,2}| \ll q \int_T^{-T} \left( \int_{\beta_2}^{3/2} \frac{\zeta_2(2 - \sigma - iT)\zeta_2(1 - \sigma + \beta_1 - iT + iT)\chi_2^{\sigma + \beta_1}}{\zeta_2(2 - \sigma + \beta_1 - iT + iT)\zeta_2(\sigma + iT)} \frac{1}{1 + |t|} dt \right) \int_{-T}^{T} \left( \int_{\beta_2}^{3/2} \frac{1}{1 + |t|} dt \right) \frac{y^2}{T} \frac{1}{T^2} \frac{1}{T^2}.
\]

**4.1.4 Evaluation of $I_4$**

Finally, the integral $I_4$ is given by

\[
I_4 = \frac{\rho_K}{(2\pi i)^2} \int_{\beta_1 - iT}^{\beta_1 + iT} \int_{\beta_2 - iT}^{\beta_2 + iT} \frac{\zeta_2(2 - s_1)\zeta_2(2 - s_2)\zeta_2(3 - s_1 - s_2)}{\zeta_2(3 - s_1 - s_2)\zeta_2(4 - s_1 - s_2)\zeta_2(s_1)\zeta_2(s_2)} \frac{y^{3 - s_1 - s_2} x^{s_1 + s_2}}{s_1 s_2} ds_1 ds_2.
\]

We estimate $I_4$ by shifting the integration over $s_2$ to the contour with vertices $\beta_2 + iT, \beta_2 - iT, 5/2 - \beta_1 + iT,$ and $5/2 - \beta_1 - iT$. We denote the integration along the lines $[\beta_2 + iT, 5/2 - \beta_1 + iT], [5/2 - \beta_1 + iT, 5/2 - \beta_1 - iT]$, and $[5/2 - \beta_1 - iT, \beta_2 - iT]$ are $L_{4,1}, L_{4,2},$ and $L_{4,3}$, respectively, then

\[
|L_{4,1}, L_{4,3}| \ll q^2 \int_T^{-T} \left( \int_{\beta_2}^{3/2} \frac{\zeta_2(2 - \sigma - iT)\zeta_2(2 - \sigma - iT)\zeta_2(3 - \sigma - iT - iT)}{\zeta_2(3 - \sigma + iT + iT)\zeta_2(\sigma + iT)\zeta_2(\sigma)} \frac{1}{1 + |t|} dt \right) \int_{-T}^{T} \left( \int_{\beta_2}^{3/2} \frac{1}{1 + |t|} dt \right) \frac{y^2}{T} \frac{1}{T^2} \frac{1}{T^2}.
\]

(4.12)
and

\begin{align*}
|L_{4,2}| & \ll x^{5/2} y^{1/2} \int_{-T}^{T} \int_{-T}^{T} \frac{|\zeta_K(2 - \beta_1 - it)| |\zeta_K(1/2 + \beta_1 - it)| |\zeta_K(1/2 + i(-t_1 - t_2))| \, dt_1 \, dt_2}{\zeta_K(3/2 + i(-t_1 - t_2)) |\zeta_K(\beta_1 + it)| |\zeta_K(5/2 - \beta_1 + it_2)| (1 + |t_1|)^2 (1 + |t_2|)} \\
& \ll q x^{5/2} y^{1/2} \int_{-2T}^{2T} \frac{|\zeta_K(1/2 + it)| |\zeta_K(3/2 + \beta_2 + it)| \, dt}{\zeta_K(3/2 + \beta_2 + it)} \int_{-T}^{T} \frac{t_1^{1/3} \log t_1}{(1 + |t_1|)^2 (1 + |t - t_1|)} \, dt \\
& \ll q x^{5/2} y^{1/2} \log^{12} T \int_{-2T}^{2T} t^{1/3} \log^6 t \\
& \ll q x^{5/2} y^{1/2} T^{1/3} \log^{12} T. \quad (4.15)
\end{align*}

Thus,

\begin{equation}
I_4 = O_q \left( \frac{x^2 y \log^{15} T}{T^2} + x^{5/2} y^{1/2} T^{1/3} \log^{12} T \right). \quad (4.16)
\end{equation}

Collecting the results from (4.7), (4.10), (4.13), (4.16) and inserting in (4.4), we deduce

\begin{align*}
I &= \frac{\rho_K^2 \zeta(2)}{\zeta_K(2)} + \frac{\rho_K^2 x^4 \zeta_K(0)}{4 \zeta_K(2)^2} + O_q \left( \frac{x^2 y \log^9 T}{T} + y x^{3/2} T^{1/3} \log^7 T \right) \\
&+ O_q \left( x^{5/2} y^{1/2} T^{1/3} \log^{12} T + x^2 y^{17/18} \log^{24} T \right).
\end{align*}

Taking $T = x^{1/3 - \epsilon}$, we obtain the required result. \hfill \Box

4.2 Second moment for cubic number field

**Proof of Theorem 1.2** The proof uses the same steps as in the degree two case, except for several technical changes which arise due to the difference between the bounds of the Dedekind zeta function for cubic (2.9) and quadratic number fields (2.5).

In here, the optimal choice of $T$ appearing upon truncation of the infinite line is $T = x^{3/14 - \epsilon}$ for a fixed $\epsilon > 0$. Since the arguments do not change, we will omit a detailed illustration. \hfill \Box

5 Proof of Theorem 1.3

In this section, using elementary techniques, we prove the second moment for number fields of any degree satisfying condition (1.6). As the degree of a number field increases, due to large bounds for the associated Dedekind zeta function in the required regions, the error terms originating from the line integrals in Perron’s formula dominate over the main terms. Consequently, we avoid an analytic approach for higher degree number fields at the cost of losing a second main term.
Proof of Theorem 1.3} From (1.2), we have

\[
\sum_{0 < \mathcal{N}(J) \leq y} \left( \sum_{0 < \mathcal{N}(I) \leq x} C_J(I) \right)^2 = \sum_{0 < \mathcal{N}(J) \leq x} \left( \sum_{0 < \mathcal{N}(I) \leq \mathcal{N}(J)} \sum_{I \in \mathcal{I}_1} N(I) \mu(J) \right)^2 \]

\[
= \sum_{0 < \mathcal{N}(I_1, I_2) \leq x} \sum_{0 < \mathcal{N}(I_2, I_2) \leq x} N(I_1) N(I_2) \mu(J_1) \mu(J_2) \sum_{0 < \mathcal{N}(I) \leq y} 1 \quad (5.1)
\]

From the hypothesis in Theorem 1.3, the innermost sum is given by

\[
\mathcal{I} = \{ I : I \subseteq \mathcal{O}_K : 0 < \mathcal{N}(I) \leq y, \mathcal{I}_1 \cap \mathcal{I}_2 \cap I \} = \frac{\rho K y}{\mathcal{N}(\mathcal{I}_1 \cap \mathcal{I}_2)} + O \left( \left( \frac{y}{\mathcal{N}(\mathcal{I}_1 \cap \mathcal{I}_2)} \right)^\alpha \right).
\]

Using this, the left-hand side of (5.1) equals

\[
\rho K y \sum_{0 < \mathcal{N}(I_1, I_2) \leq x} \sum_{0 < \mathcal{N}(I_2, I_2) \leq x} N(I_1 + I_2) \mu(J_1) \mu(J_2) + O \left( y^\alpha \sum_{0 < \mathcal{N}(I_1, I_2) \leq x} \sum_{0 < \mathcal{N}(I_2, I_2) \leq x} N(I_1 + I_2)^\alpha N(I_1)^{1-\alpha} N(I_2)^{1-\alpha} \right) =: I_1 + I_2 \quad (5.2)
\]

Let \( I_1 + I_2 = \mathcal{A} \), then \( I_1 = \mathcal{A} \mathcal{E}_1 \), and \( I_2 = \mathcal{A} \mathcal{E}_2 \) such that \( \mathcal{E}_1 + \mathcal{E}_2 = \mathcal{O}_K \). This yields

\[
I_1 = \rho K y \sum_{0 < \mathcal{N}(\mathcal{A} \mathcal{E}_1, \mathcal{I}_1) \leq x} \sum_{\mathcal{I}_1 + \mathcal{E}_2 = \mathcal{O}_K} N(\mathcal{A}) \mu(J_1) \mu(J_2) = \rho K y \sum_{0 < \mathcal{N}(\mathcal{A} \mathcal{E}_1, \mathcal{I}_1) \leq x} \sum_{\mathcal{I}_1 + \mathcal{E}_2 = \mathcal{O}_K} N(\mathcal{A}) \mu(J_1) \mu(J_2) \sum_{\mathcal{M} \in \mathcal{I}_1 + \mathcal{E}_2} \mu(\mathcal{M})
\]

\[
= \rho K y \sum_{0 < \mathcal{N}(\mathcal{A} \mathcal{M}) \leq x} N(\mathcal{A}) \mu(\mathcal{M}) \left( \sum_{0 < \mathcal{N}(\mathcal{E}, \mathcal{M}) \leq y} \mu(\mathcal{E}) \right)^2
\]

\[
= \rho K y \sum_{0 < \mathcal{N}(\mathcal{A} \mathcal{M}) \leq x} N(\mathcal{A}) \mu(\mathcal{M}) = \frac{\rho K^2 x^2 y}{2 \zeta_K(2)} + O (xy \log x), \quad (5.3)
\]

and

\[
I_2 \leq y^\alpha \sum_{0 < \mathcal{N}(\mathcal{A} \mathcal{E}_1, \mathcal{I}_1) \leq x} \sum_{\mathcal{I}_1 + \mathcal{E}_2 = \mathcal{O}_K} N(\mathcal{A})^\alpha N(\mathcal{E}_1 A)^{1-\alpha} N(\mathcal{E}_2 A)^{1-\alpha}
\]
\[ \leq y^\alpha \sum_{0 < N(A \mathcal{E}_1, \mathcal{E}_2) \leq x} \mathcal{N}(\mathcal{A})^{2-\alpha} \mathcal{N}(\mathcal{E}_1)^{1-\alpha} \mathcal{N}(\mathcal{E}_2)^{1-\alpha} \sum_{\mathcal{M} | \mathcal{E}_1 + \mathcal{E}_2} \mu(\mathcal{M}) \]
\[ \leq y^\alpha \sum_{0 < N(A \mathcal{M}) \leq x} \mathcal{N}(\mathcal{A})^{2-\alpha} \left( \sum_{0 < N(\mathcal{E}) \leq x} \mathcal{N}(\mathcal{E})^{1-\alpha} \right)^2 \]
\[ \leq y^\alpha \sum_{0 < N(A \mathcal{M}) \leq x} \mathcal{N}(\mathcal{A})^{2-\alpha} \frac{x^{(2-\alpha)2}}{\mathcal{N}(A \mathcal{M})^{(2-\alpha)2}} \ll y^\alpha x^{3-\alpha}. \quad (5.4) \]

On substitution of (5.3) and (5.4) in (5.2), we obtain the required result.

**Acknowledgements** We are grateful to the referees for their valuable comments which had led to significant improvements in the current version.

**References**

1. Anderson, D.R., Apostol, T.M.: The evaluation of Ramanujan’s sum and generalizations. Duke Math. J. **20**, 211–216 (1953)
2. Balandraud, É.: An application of Ramanujan sums to equirepartition modulo an odd integer. Unif. Distrib. Theory 2(2), 1–17 (2007)
3. Carlitz, L.: The singular series for sums of squares of polynomials. Duke Math. J. **14**, 1105–1120 (1947)
4. Chan, T.H., Kumchev, A.V.: On sums of Ramanujan sums. Acta Arith. **152**(1), 1–10 (2012)
5. Cohen, E.: An extension of Ramanujan’s sum. Duke Math. J. **16**, 85–90 (1949)
6. Fujisawa, Y.: On sums of generalized Ramanujan sums. Indian J. Pure Appl. Math. **46**(1), 1–10 (2015)
7. Gilmer, R.: Multiplicative ideal theory, volume 90 of Queen’s Papers in Pure and Applied Mathematics. Queen’s University, Kingston, 1992. Corrected reprint of the edition (1972)
8. Grytczuk, A.: On Ramanujan sums on arithmetical semigroups. Tsukuba J. Math. **16**(2), 315–319 (1992)
9. Guangwei, Hu., Wang, Ke.: Higher moment of coefficients of Dedekind zeta function. Front. Math. China **15**(1), 57–67 (2020)
10. Iwaniec, H., Kowalski, E.: Analytic Number Theory. American Mathematical Society, vol. 53. American Mathematical Society Colloquium Publications, Providence (2004)
11. Jutila, M.: Distribution of rational numbers in short intervals. Ramanujan J. **14**(2), 321–327 (2007)
12. Knopfmacher, J.: Abstract Analytic Number Theory. Dover Books on Advanced Mathematics, 2nd edn. Dover Publications Inc, New York (1999)
13. Konvalina, J.: A generalization of Waring’s formula. J. Combin. Theory Ser. A **75**(2), 281–294 (1996)
14. Landau, E.: Einführung in die elementare und analytische theorie der algebraischen Zahlen und der Ideale. Chelsea Publishing Company, New York (1949)
15. Ma, J., Sun, H., Zhai, W.: The average size of Ramanujan sums over cubic number fields. arXiv preprint arXiv:2105.11699 (2021)
16. Müller, W.: On the distribution of ideals in cubic number fields. Monatsh. Math. **106**(3), 211–219 (1988)
17. Nathanson, M. B.: Additive number theory, volume 165 of Graduate Texts in Mathematics. Springer-Verlag, New York. Inverse problems and the geometry of sumsets (1996)
18. Nowak, W.G.: The average size of Ramanujan sums over quadratic number fields. Arch. Math. (Basel) **99**(5), 433–442 (2012)
19. Nowak, W.G.: On Ramanujan sums over the Gaussian integers. Math. Slovaca **63**(4), 725–732 (2013)
20. Ramanujan, S.: On certain trigonometrical sums and their applications in the theory of numbers [Transactions of the Cambridge Philosophical Society **22** (1918), no. 13, 259–276]. In Collected papers of Srinivasa Ramanujan, pp. 179–199. AMS Chelsea Publ., Providence, RI, (2000)
21. Ramaré, O.: Eigenvalues in the large sieve inequality. Funct. Approx. Comment. Math. **37**(2), 399–427 (2007)
22. Robles, N., Roy, A.: Moments of averages of generalized Ramanujan sums. Monatsh. Math. 182(2), 433–461 (2017)
23. Shiu, P.: A Brun–Titchmarsh theorem for multiplicative functions. J. Reine Angew. Math. 313, 161–170 (1980)
24. Sugunamma, M.: Eckford Cohen’s generalizations of Ramanujan’s trigonometrical sum $C(n, r)$. Duke Math. J. 27, 323–330 (1960)
25. Gérald Tenenbaum. Introduction to analytic and probabilistic number theory, volume 46 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1995. Translated from the second French edition by C. B. Thomas (1995)
26. Zhai, W.: The average size of Ramanujan sums over quadratic number fields. Ramanujan J. 8, 1–17 (2021)

Publisher’s Note  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.