Balancedly splittable Hadamard matrices

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Abstract

Balancedly splittable Hadamard matrices are introduced and studied. A connection is made to the Hadamard diagonalizable strongly regular graphs, maximal equiangular lines set, and unbiased Hadamard matrices. Several construction methods are presented. As an application, commutative association schemes of 4, 5, and 6 classes are constructed.

1 Introduction

An $n \times n$ matrix $H$ is a Hadamard matrix of order $n$ if its entries are $1, -1$ and it satisfies $HH^\top = I_n$, where $I_n$ denotes the identity matrix of order $n$. Hadamard matrices $H$ are shown to be related to other combinatorial objects such as combinatorial designs, distance regular graphs of diameter 4, and the following under some regularity conditions:

- strongly regular graphs if $H$ is symmetric with constant diagonal [8],
- doubly regular tournaments if $H$ is skew-symmetric [22],
- symmetric or non-symmetric association schemes with 3 classes if $H$ is of symmetric or skew-symmetric Bush-type [9].

A Hadamard matrix $H$ of order $n$ is said to be balancedly splittable if there is an $\ell \times n$ submatrix $H_1$ of $H$ such that inner products for any two distinct column vectors of $H_1$ take at most two values. More precisely, there exist integers $a, b$ and the adjacency matrix $A$ of a graph such that $H_1^\top H_1 = \ell I_n + aA + b(J_n - A - I_n)$, where $J_n$ denotes the all-ones matrix of order $n$.

We will show that the matrix $A$ is (switching equivalent) to the adjacency matrix of a strongly regular graph, and the graph is Hadamard diagonalizable in the sense of Barik, Fallat, and Kirkland [2]. The case $b = -a$ corresponds to a maximal equiangular lines set, and to the unbiased Hadamard matrices. We propose various constructions and provide non-existence results for balancedly splittable Hadamard matrices. A construction provided in Section 3.6 is related to recent work on complex Hadamard matrices [7] and the submatrix $H_1$ of a balancedly Hadamard matrix is essentially the same as the quasi-symmetric design constructed in [12]. It turns out that the property of balancedly splittable leads to a new relation between Hadamard matrices and various combinatorial objects.

As a further application we will make a connection to association schemes and provide schemes with 4, 5, 6-classes by using balancedly splittable Hadamard matrices and Latin squares. We will demonstrate that our approach relates to different concepts presented in [11, 13, 15, 16, 18], particularly in constructing unbiased Hadamard matrices and association schemes.

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2 Balancedly splittable Hadamard matrices

Definition 2.1. A Hadamard matrix $H$ of order $n$ is balancedly splittable if by suitably permuting its rows it can be transformed to $H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$ such that the matrix $H_1^\top H_1$ has at most two distinct off-diagonal entries. In this case we say that $H$ is balancedly splittable with respect to $H_1$.

Let $H_1$ be an $\ell \times n$ matrix. Then there exist integers $a, b$ and a $(0,1)$-matrix $A$ such that $a \geq b$ and

$$H_1^\top H_1 = \ell I_n + aA + b(J_n - A - I_n). \quad (2.1)$$

The tuple of values $(n, \ell, a, b)$ is said to be the parameters of a balancedly splittable Hadamard matrix of order $n$ with respect to $H_1$.

By the equation $H_1^\top H_1 + H_2^\top H_2 = nI_n$, we have the following lemma.

Lemma 2.2. Let $H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$ be a Hadamard matrix of order $n$ with $\ell \times n$ matrix $H_1$, $1 \leq \ell < n$. Then $H$ is balancedly splittable with the parameters $(n, \ell, a, b)$ with respect to $H_1$ if and only if $H$ is balancedly splittable with the parameters $(n, n - \ell, -b, -a)$ with respect to $H_2$.

The following are examples with $a = b$.

Example 2.3. (1) Any Hadamard matrix is balancedly splittable with respect to itself with the parameters $(n, n, 0, 0)$.

(2) A Hadamard matrix $H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$ of order $n$ with $H_1$ the all-ones vector is balancedly splittable with respect to $H_1$ with the parameters $(n, 1, 1, 1)$.

Conversely, it is easy to characterize a balancedly splittable Hadamard matrix to have $H_1^\top H_1$ with the only one distinct off-diagonal entry, as shown below.

Proposition 2.4. If a Hadamard matrix $H$ is balancedly splittable with $H_1^\top H_1 = \ell I_n + a(J_n - I_n)$, then $(\ell, a) \in \{(1, 1), (n - 1, -1), (n, 0)\}$.

Proof. Squaring the equation $H_1^\top H_1 = \ell I_n + a(J_n - I_n)$ yields that $n(\ell - a)I_n + naJ_n = (\ell - a)^2 I_n + (2(\ell - a)a + a^2 n)J_n$. Comparing coefficients with $1 \leq \ell \leq n$, we have that $(\ell, a) \in \{(1, 1), (n - 1, -1), (n, 0)\}$. \qed

In the rest of the paper, we focus on balancedly splittable Hadamard matrices $H$ such that $H_1^\top H_1$ has exactly two distinct values off diagonal. The following is an obvious example.

Example 2.5. Let $H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$ be a Hadamard matrix of order $n$ with $1 \times n$ matrix $H_1$. If $H_1$ is not equal to the all-ones vector, then $H$ is balancedly splittable with respect to $H_1$ with the parameters $(n, 1, 1, -1)$.

Throughout the rest of the paper, we assume that $1 < \ell < n - 1$ in order to avoid the trivial cases.

A strongly regular graph with parameters $(v, k, \lambda, \mu)$ is a regular graph with $v$ vertices and degree $k$ such that every two adjacent (non-adjacent resp.) vertices have $\lambda$ ($\mu$ resp.) common neighbors. The Seidel matrix of a graph with adjacency matrix $A$ is $S = J_v - I_v - 2A$. A strong
graph is such that its Seidel matrix $S$ satisfies the property that $S^2$ is a linear combination of $S, I_v, J_v$. It is known that a graph is strongly regular if and only if it is regular and strong, see for [21] and [3] Chapter 10.

Balancedly splittable Hadamard matrices are related to strong graphs and strongly regular graphs, as shown in the following proposition.

**Proposition 2.6.** Let $H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$ be a balancedly splittable Hadamard matrix of order $n$ with $H_1 H_1^\top = \ell I_n + a A + b (J_n - A - I_n)$ where $A$ is an $n \times n$ $(0,1)$-matrix, and $1 < \ell < n-1, b < a$.

(1) If $b = -a$, $S = J_n - I_n - 2A$ is the Seidel matrix of a strong graph satisfying $S^2 = \frac{n-2\ell}{n} S + \frac{\ell(n-\ell)}{n} I_n$ and $n = \frac{n^2 - 2}{\ell - a}$. The strong graph is switching equivalent to a strongly regular graph with the parameters $(n,k,\lambda,\mu) where k,\lambda,\mu$ are the following:

$$k = \frac{(a-1)\ell(a+\ell)}{2a(\ell-a^2)}, \quad \lambda = \frac{(a+\ell)(3a^2 + a\ell - a - 3\ell)}{4a(\ell-a^2)}, \quad \mu = \frac{(a-1)(\ell^2 - a^2)}{4a(\ell-a^2)}. \quad (2.2)$$

(2) If $b \neq -a$, then $A$ is the adjacency matrix of a strongly regular graph with parameters $(n,k,\lambda,\mu)$, where $b, k, \mu, \nu$ are either one of the following:

(2.4)

Furthermore if (a) occurs, then each row of $H_1$ is orthogonal to the all-ones vector.

**Proof.** Squaring (2.1) with the fact that $H_1 H_1^\top = n I_\ell$, we have that

$$(\ell I_n + a A + b (J_n - A - I_n))^2 = n(\ell I_n + a A + b (J_n - A - I_n)). \quad (2.3)$$

Simplifying (2.3) by $b \neq a$ yields that

$$A^2 = \frac{1}{(a-b)^2} \left( (a-b)(n-2\ell+2b)A + (\ell-b)(n-\ell+b)I_n + b(n-nb-2\ell+2b)J_n - (a-b)A J_n + J_n A \right). \quad (2.4)$$

For $x \in \{1, \ldots, n\}$, let $k_x$ denote the degree of $x$ in the graph whose adjacency matrix is $A$. Then comparing the $(x,x)$-entry in (2.3) shows that

$$\ell^2 + a^2 k_x + b^2 (n-1-k_x) = n \ell. \quad (2.5)$$

(1): For the case $b = -a$, by $H_1 H_1^\top = \ell I_n + a S$, (2.3) is reduced to $S^2 = \frac{n-2\ell}{a} S + \frac{\ell(n-\ell)}{n} I_n$, and (2.3) shows that $\ell^2 + a^2 (n-1) = n \ell$. Since $\ell \neq 1$, we have that $\ell \neq a^2$. Thus $n = \frac{n^2 - 2}{\ell - a^2}$.

Normalize the Hadamard matrix $H$ so that the last row of $H$ equals to the all-ones vector. Then multiplying $J_n$ by $H_1 H_1 = (\ell + a) I_n + 2a A - a J_n$, we have $2a A J_n = (an - \ell - a) J_n$. Since $a \neq 0$, the graph is regular with valency $k = \frac{an - \ell - a}{2a}$. The strong graph with the Seidel matrix $S$ is regular, and thus it is strongly regular. Let $(n,k,\lambda,\mu)$ be its parameters. The parameters are determined as in (2.2) by substituting $b = -a$ and $A J_n = J_n A = \frac{an - \ell - a}{2a} J_n$ into (2.4) with use of $n = \frac{n^2 - 2}{\ell - a^2}$.
(2): By the assumption that \( b \neq \pm a \), \( \text{[2.5]} \) shows that \( k_x = \frac{n\ell - \ell^2 - b^2(n-1)}{a^2 - b^2} \), which is independent of the particular choice of \( x \). Thus \( A \) is the adjacency matrix of a regular graph of degree \( k \) given as

\[
k := \frac{n\ell - \ell^2 - b^2(n-1)}{a^2 - b^2}. \tag{2.6}
\]

To use the fact that \( AJ_n = J_nA = kJ_n \), \( \text{[2.4]} \) shows that the matrix \( A \) is the adjacency matrix of a strongly regular graph with parameters \((n, k, \lambda, \mu)\) where \( \lambda, \mu \) are determined as follows:

\[
\lambda = \frac{n(a^2 - a(b-1)b + b^3 - 2b\ell) + 2(b - \ell)(a^2 + ab - b(b + \ell))}{(a-b)^2(a+b)}, \tag{2.7}
\]

\[
\mu = \frac{bn(-ab + a + b^2 + b - 2\ell) + 2b(a - \ell)(b - \ell)}{(a-b)^2(a+b)}. \tag{2.8}
\]

Substituting \( \text{[2.7]}, \text{[2.8]} \) into the well-known formula \( k(k - \lambda - 1) = (n-k-1)\mu \) and simplifying it, we have

\[
((a(n-1) + \ell)b - (-a + \ell - n))(a(n-1) + \ell - n)b - (a - \ell)(\ell - n)) = 0. \tag{2.9}
\]

Since \( a(n-1) + \ell > 0 \) and \( a(n-1) + \ell - n > 0 \), \( \text{[2.9]} \) implies that \( b = \frac{\ell(-a+\ell-n)}{a(n-1)+\ell} \) or \( b = \frac{\ell(a-\ell-\ell-n)}{a(n-1)+\ell-n} \).

Thus by \( \text{[2.6]}, \text{[2.7]}, \text{[2.8]} \) we obtain the desired formula for \( k, \lambda \) and \( \mu \).

For \( (a) \), pre-multiplying the all-ones column vector \( \mathbf{1} \) and post-multiplying its transpose by \( \text{[2.1]} \) shows that

\[
(H_1\mathbf{1})^\top(H_1\mathbf{1}) = \mathbf{1}^\top(\ell I_n + aA + b(J_n - A - I_n)) = (\ell + ak + b(n-1-k))n = 0,
\]

where we used \( b = \frac{\ell(-a+\ell-n)}{a(n-1)+\ell}, k = \frac{\ell(n-\ell-1)}{n(a^2+\ell)-(a-\ell)^2} \) in the last equality. Thus \( H_1\mathbf{1} = \mathbf{0} \) holds where \( \mathbf{0} \) is the zero vector.

**Remark 2.7.** It is routinely checked that the parameters of the strongly regular graphs in \( (2) \) \( (a) \) are valid for the case \( (1) \) to use \( \frac{\ell(n-\ell)}{a^2} = n-1 \).

**Remark 2.8.** Let \( H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \) be a Hadamard matrix of order \( n \).

(1) It is easy to check that \( H \) is balancedly splittable with the parameters \((n, \ell, a, b)\) with respect to \( H_1 \) fitting into Proposition \( \text{[2.6]} \)(2)(a) if and only if \( H \) is balancedly splittable with the parameters \((n, n-\ell, -b, -a)\) with respect to \( H_2 \) fitting into Proposition \( \text{[2.6]} \)(2)(b).

(2) Assume that \( H \) is a balancedly splittable Hadamard matrix of order \( n \) with the parameters \((n, \ell, a, -a)\) with respect to \( H_1 \) and the last row being the all-ones vector. Then the Hadamard matrix \( H \) is a balancedly splittable Hadamard matrix of order \( n \) with the parameters \((n, n-\ell-1, a-1, -a-1)\) with respect to the submatrix of \( H \) obtained by deleting \( H_1 \) and the last row from \( H \).

**Remark 2.9.** A strongly regular graph is said to be *imprimitive* if either the graph or its complement is disconnected. This is equivalent to \( k = \lambda + 1 \) or \( k = \mu \). The former occurs in \( (2)(a) \) if and only if \( a = \ell \). The latter occurs in \( (2)(a) \) if and only if \( a = 0 \).

A graph is said to be *Hadamard diagonalizable* if its Laplacian matrix \( L \) is diagonalized by a Hadamard matrix, that is, there exists a Hadamard matrix \( H \) such that \( HLH^\top \) is a diagonal matrix \([2]\). It turns out that a Hadamard diagonalizable graph is regular \([2, \text{Theorem 5}]\). Therefore a graph is Hadamard diagonalizable if and only if its adjacency matrix is diagonalized by a Hadamard matrix.
Corollary 2.10.  

(1) If a Hadamard matrix \( H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \) of order \( n \) is balancedly splittable with respect to \( H_1 \) with parameters \((n, \ell, a, b)\) such that \( H_2 \) has the all-ones row vector, then the strongly regular graph constructed in Theorem 2.6 is Hadamard diagonalizable by \( H \).

(2) Conversely, if a strongly regular graph on \( n \) vertices which is Hadamard diagonalizable by a normalized Hadamard matrix \( H \), then \( H \) is balancedly splittable with parameters \((n, \ell, a, b)\) where

Proof. Assume (1) to be true. It holds that

\[ HH_1^\top H_1 H_2^\top = \text{diag}(n^2, \ldots, n^2, 0, \ldots, 0). \]

Without loss of generality, we may assume that the last row of \( H_2 \) is the all-ones vector. Then we have that \( HJ_n H_1^\top = \text{diag}(0, \ldots, 0, n^2) \). Pre-multiplying \( H \) and post-multiplying \( H_1^\top \) by (2.1) and simplifying it yields that

\[ HAH^\top = \frac{n}{a-b} \text{diag}(-\ell+b+n, \ldots, -\ell+b+n, -\ell+b, \ldots, -\ell+b, -\ell-b(n-1)). \]

Therefore \( A \) is Hadamard diagonalizable by \( H \).

Conversely assume (2), namely let \( A \) be a strongly regular graph which is diagonalized by a normalized Hadamard matrix \( H \) of order \( n \). Without loss of generality \( H \) has the all-ones vector as the last row. Then it holds that by a suitable rearranging rows and columns of \( A \),

\[ HAH^\top = \text{diag}(\theta, \ldots, \theta, \tau, \ldots, \tau, k) \]

where \( k \) is the valency of \( A \), and \( \theta, \tau \) are distinct eigenvalues of \( A \) (one of which may be equal to \( k \)), and \( \ell \) is the multiplicity of \( \theta \). Write \( H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} H_1' \\ 1^\top \end{pmatrix} \) where \( H_1 \) is the \( \ell \times n \) matrix and \( H_2 \) is the \((n-\ell) \times n \) matrix and \( H_2' \) is the \((n-\ell-1) \times n \) submatrix of \( H_2 \). Pre-multiplying \( H^\top \) and post-multiplying \( H \) by (2.10) provides

\[ n^2 A = \theta H_1^\top H_1 + \tau (H_2')^\top H_2' + kJ_n. \]

(2.11)

On the other hand, by \( H^\top H = nI \), we have

\[ nI_n = H_1^\top H_1 + (H_2')^\top H_2' + J_n. \]

(2.12)

Therefore by (2.11) and (2.12) we have that

\[ H_1^\top H_1 = \frac{1}{\theta - \tau} (n^2 A - \tau nI_n - (k - \tau)J_n). \]

Thus a Hadamard matrix \( H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \) is balancedly splittable with respect to the \( \ell \times n \) matrix \( H_1 \) such that \( H_2 \) has the all-ones row vector. \( \square \)

We list the feasible parameters in Table 1 for (1) with \( n \leq 1024, \ell \leq n/2 \) and those in Table 2 for (2)(a) with \( n \leq 64 \) and \( 0 < a < \ell \). In the tables, E stands for “exists” and NE stands for “does not exist”.

The following upper bound is due to Delsarte, Goethals and Seidel [5]. The finite set \( \mathcal{X} \) of \( \mathbb{R}^m \) satisfying the assumption in Proposition 2.11 is referred to as an equiangular lines set.
### Table 1: $n \leq 1024, \ell \leq n/2$

| $n$ | $\ell$ | $a$ | $b$ | $k$ | $\lambda$ | $\mu$ | Notes |
|-----|--------|-----|-----|-----|-----------|-------|-------|
| 16  | 6      | 2   |     |     |           |       | E     |
| 36  | 15     | 3   |     |     |           |       | NE, Prop 2.16 |
| 64  | 28     | 4   |     |     |           |       | E     |
| 100 | 45     | 5   |     |     |           |       | NE, Prop 2.16 |
| 120 | 35     | 5   |     |     |           |       | NE, Prop 2.17 |
| 144 | 66     | 6   |     |     |           |       | NE, Prop 2.16 |
| 196 | 91     | 7   |     |     |           |       | NE, Prop 2.16 |
| 256 | 120    | 8   |     |     |           |       | E     |
| 280 | 63     | 7   |     |     |           |       | NE, Prop 2.16 |
| 288 | 42     | 6   |     |     |           |       | NE, Prop 2.16 |
| 320 | 88     | 8   |     |     |           |       | NE, Prop 2.16 |
| 324 | 153    | 9   |     |     |           |       | NE, Prop 2.16 |
| 400 | 190    | 10  |     |     |           |       | NE, Prop 2.16 |
| 484 | 231    | 11  |     |     |           |       | NE, Prop 2.16 |
| 528 | 187    | 11  |     |     |           |       | NE, Prop 2.16 |
| 540 | 99     | 9   |     |     |           |       | NE, Prop 2.17 |
| 560 | 130    | 10  |     |     |           |       | E     |
| 576 | 276    | 12  |     |     |           |       | E     |
| 616 | 165    | 11  |     |     |           |       | NE, Prop 2.17 |
| 640 | 72     | 8   |     |     |           |       | NE, Prop 2.16 |
| 676 | 325    | 13  |     |     |           |       | NE, Prop 2.16 |
| 780 | 247    | 13  |     |     |           |       | NE, Prop 2.17 |
| 784 | 378    | 14  |     |     |           |       | NE, Prop 2.16 |
| 900 | 435    | 15  |     |     |           |       | NE, Prop 2.16 |
| 924 | 143    | 11  |     |     |           |       | NE, Prop 2.16 |
| 936 | 221    | 13  |     |     |           |       | NE, Prop 2.16 |
| 1008 | 266 | 14  |     |     |           |       | NE, Prop 2.16 |
| 1024| 496    | 16  |     |     |           |       | E     |

### Table 2: $n \leq 64$ and $0 < a < \ell$

| $n$ | $\ell$ | $a$ | $b$ | $k$ | $\lambda$ | $\mu$ | Notes |
|-----|--------|-----|-----|-----|-----------|-------|-------|
| 16  | 5      | 1   | -3  | 10  | 6         | 6     | E, Remark 2.8(2) |
| 16  | 9      | 1   | -3  | 9   | 4         | 6     | E, Remark 2.8(2) |
| 36  | 10     | 4   | -2  | 10  | 4         | 2     | NE, Prop 2.21(1) |
| 36  | 14     | 2   | -4  | 21  | 12        | 12    | NE, Prop 2.21(3) |
| 36  | 20     | 2   | -4  | 20  | 10        | 12    | NE, Prop 2.21(4) |
| 36  | 25     | 1   | -5  | 25  | 16        | 20    | NE, Prop 2.21(2) |
| 64  | 14     | 6   | -2  | 14  | 6         | 2     | E, Theorem 3.1   |
| 64  | 18     | 2   | -6  | 45  | 32        | 30    |               |
| 64  | 21     | 5   | -3  | 21  | 8         | 6     |               |
| 64  | 27     | 3   | -5  | 36  | 20        | 20    | E, Remark 2.8(2) |
| 64  | 35     | 3   | -5  | 35  | 18        | 20    | E, Remark 2.8(2) |
| 64  | 42     | 2   | -6  | 42  | 26        | 30    |               |
| 64  | 45     | 5   | -3  | 18  | 2         | 6     |               |
| 64  | 49     | 1   | -7  | 49  | 36        | 42    | E, Theorem 3.1   |


Proposition 2.11. Let $X \subset \mathbb{R}^m$ be a set of unit vectors such that $|\langle v, w \rangle| = \alpha$ for all $v, w \in X, v \neq w$. If $m < \frac{1}{\alpha^2}$, then

$$|X| \leq \frac{n(1-\alpha^2)}{1-m\alpha^2}. \quad (2.13)$$

Proposition 2.12. If there exists a balancedly splittable Hadamard matrix with the parameters $(n, \ell, a, -a)$, then there exists an equiangular lines set in $\mathbb{R}^\ell$ with inner product $\frac{\sqrt{n-\ell}}{\sqrt{\ell(n-1)}}$ attaining equality in (2.13).

Proof. Let $H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$ be a balancedly Hadamard matrix with respect to an $\ell \times n$ matrix $H_1$. Let $X$ be the set of column vectors of $H_2$ normalized by dividing by $\sqrt{\ell}$, so $a^2\ell^2 = \frac{n-\ell}{1-\ell}$. Then $X$ is a subset of $n$ unit vectors of $\mathbb{R}^\ell$ such that $|\langle v, w \rangle| = \frac{\sqrt{n-\ell}}{\sqrt{\ell(n-1)}}$ for all $v, w \in X, v \neq w$. It can be seen that $m < \frac{1}{\alpha^2}$ for $(m, \alpha) = (\ell, \frac{\sqrt{n-\ell}}{1-\ell})$ and that the right hand side in (2.13) in this case equals to $n$. Thus our equiangular lines set attains the bound in (2.13). \qed

Two Hadamard matrices $H$ and $K$ of order $n$ are said to be unbiased if $\frac{1}{\sqrt{n}}HK^\top$ is a Hadamard matrix of order $n$.

Proposition 2.13. Let $H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$ be a balancedly splittable Hadamard matrix of order $n$ with $H_1^\top H_1 = \ell I_n + aS$ where $a \neq 0$ and $S$ is an $n \times n$ $(0, 1, -1)$-matrix. Then the following are equivalent.

1. $K := \frac{1}{2a}(H_1^\top H_1 - H_2^\top H_2)$ is a Hadamard matrix.
2. $(\ell, a) = ((n \pm \sqrt{n})/2, \sqrt{n}/2)$.

In this case, $n = 4k^2$ for some integer $k$ and the Hadamard matrices $H$ and $K$ are unbiased.

Proof. Since $H_1 H_2^\top$ and $H_2^\top H_1$ are zero matrices and $H_1 H_1^\top = nI_\ell, H_2 H_2^\top = nI_{n-\ell}$,

$$KK^\top = K^2 = \frac{1}{4a^2}(H_1^\top H_1 - H_2^\top H_2)^2 = \frac{1}{4a^2}(H_1^\top(H_1 H_1^\top)H_1 + H_2^\top(H_2 H_2^\top)H_2)$$

$$= \frac{n}{4a^2}(H_1^\top H_1 + H_2^\top H_2) = \frac{n^2}{4a^2}I_n.$$

Therefore $K$ is a Hadamard matrix if and only if $K$ is a $(1, -1)$-matrix and $a = \sqrt{n}/2$. Since $K = \frac{1}{2a}(H_1^\top H_1 - H_2^\top H_2) = \frac{1}{2a}((2\ell-n)I_n - 2aS)$, $K$ is a $(1, -1)$-matrix if and only if $(2\ell-n)/(2a) = \pm 1$. By Proposition 2.14(1) the latter is equivalent to $\ell = (n \pm \sqrt{n})/2$. Therefore (1) is equivalent to (2).

If (1) and (2) hold, then $a = \sqrt{n}/2$ is an integer. Therefore $n$ must be a square of an even integer. And we have that $HK^\top = \sqrt{n} \begin{pmatrix} H_1 \\ -H_2 \end{pmatrix}$. Thus $H$ and $K$ are unbiased. \qed

A Hadamard matrix of order $n$ is said to be regular if $1^\top H = \sqrt{n}1^\top$. In this case $n$ must be square.

Proposition 2.14. Any balanced splittable Hadamard matrix of order $4n^2$ with the parameters $(\ell, a, b) = (2n^2 - n, n, -n)$ is equivalent to a regular Hadamard matrix.
Proof. Let $H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$ where $H_1$ is an $\ell \times n$ matrix. Since multiplying a signed permutation matrix by $H$ from the left keeps the property of balancedly splittable, we may assume that $H_1$ has the all-ones first column and $H_2$ has the negative all-ones first column. By the assumption $b = -a$, multiplying a signed permutation matrix by $H$ from the right also keeps the property of balancedly splittable. This implies that $1^\top H_1 = (-2n^2 + n, n, \ldots , n)$ and $1^\top H_2 = (2n^2 + n, n, \ldots , n)$. Therefore $1^\top H = 1^\top H_1 + 1^\top H_2 = (2n, 2n, \ldots , 2n)$, which proves that $H$ is equivalent to a regular Hadamard matrix. \hfill $\square$

Remark 2.15. The Hadamard matrices of order 16 with Hall’s classes IV or V are not equivalent to regular Hadamard matrices, and thus are not balancedly splittable with the parameters $(\ell, a, -b) = (6, 2, -2)$. See [20] for Hall’s classes of Hadamard matrices.

We now present two non-existence results.

**Proposition 2.16.** There is no balancedly splittable Hadamard matrix with the parameters $(n, \ell, a, -a)$, $\ell + a \not\equiv 0 \pmod{4}$, $1 < \ell < n - 1$.

**Proof.** Assume that there exists such a balancedly Hadamard matrix $H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$ where $H_1$ is an $\ell \times n$ matrix. By multiplying $H$ on both sides by signed permutation matrices, we may assume that $H_1^\top H_1$ has $a$ as its entries in the first row and the first column except $(1,1)$-entry. Now we claim that $H_1^\top H_1 = \ell I + a(J - I)$. Indeed, suppose for the contrary that there exist two columns, say $i$-th and $j$-th columns, distinct from the first column such that their inner product equals to $-a$. Let $x, y, x, w$ be non-negative integers such that

- the first column = $(+ \cdots + + \cdots + + \cdots +)^\top$,
- the $i$-th column = $(+ \cdots + + \cdots + - \cdots - - \cdots -)^\top$,
- the $j$-th column = $(+ \cdots + - \cdots - + \cdots + - \cdots -)^\top$.

Then it follows that

$$\begin{cases}
x + y + z + w = \ell, \\
x + y - z - w = a, \\
x - y + z - w = a, \\
x - y - z + w = -a.
\end{cases}$$

Solving these equations yields $(x, y, z, w) = (\frac{\ell + a}{4}, \frac{\ell + a}{4}, \frac{\ell + 3a}{4}, \frac{\ell - 3a}{4})$, which is impossible because $\ell + a \not\equiv 0 \pmod{4}$. Therefore we have $H_1^\top H_1 = \ell I + a(J - I)$.

However, $H_1^\top H_1 = \ell I + a(J - I)$ contradicts Proposition 2.14 by $1 < \ell < n - 1$. Therefore we conclude that such a balancedly splittable Hadamard matrix does not exist. \hfill $\square$

In a similar way the following is proved.

**Proposition 2.17.** There is no balancedly splittable Hadamard matrix with the parameters $(n, \ell, a, -a)$, $\ell \not\equiv a \pmod{4}$ and $a > 1$.

**Proof.** In the same way as in Proposition 2.16 we may assume that there exists such a balancedly Hadamard matrix $H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$ where $H_1$ is an $\ell \times n$ matrix and $H_1^\top H_1$ has $a$ as its entries in the first row and the first column except $(1,1)$-entry. Now we claim that

$$H_1^\top H_1 = \begin{pmatrix} \ell & a1^\top \\ a1 & \ell I_{n-1} - a(J_{n-1} - I_{n-1}) \end{pmatrix}.$$
Indeed, suppose to the contrary that there exist two columns, say $i$-th and $j$-th columns, distinct from the first column such that their inner product equals to $a$. Let $x, y, x, w$ be non-negative integers such that

- the first column $= (\cdot \cdot \cdot + \cdot \cdot \cdot + \cdot \cdot \cdot + \cdot \cdot \cdot)^\top$,
- the $i$-th column $= (\cdot \cdot \cdot + \cdot \cdot \cdot - \cdot \cdot \cdot - \cdot \cdot \cdot)^\top$,
- the $j$-th column $= (\cdot \cdot \cdot - \cdot \cdot \cdot + \cdot \cdot \cdot - \cdot \cdot \cdot)^\top$.

Then it is seen that

$$
\begin{cases}
  x + y + z + w = \ell, \\
  x + y - z - w = a, \\
  x - y + z - w = a, \\
  x - y - z + w = a.
\end{cases}
$$

Solving these equations yields $(x, y, z, w) = \left( \frac{l - 3a}{4}, \frac{l - a}{4}, \frac{l - a}{4}, \frac{l - a}{4} \right)$, which is impossible because $l \not\equiv a \pmod{4}$. Therefore we have $H_1^\top H_1 = \begin{pmatrix} \ell & a1^\top \\ a1 & \ell I_{n-1} - a(J_{n-1} - I_{n-1}) \end{pmatrix}$. It can be seen that $\ell - (n-1)a$ is one of the eigenvalues of $\begin{pmatrix} \ell & a1^\top \\ a1 & \ell I_{n-1} - a(J_{n-1} - I_{n-1}) \end{pmatrix}$ However, $\ell - (n-1)a < 0$ for $a > 1$, which contradicts the fact that all the singular values of $H_1$ are nonnegative. Therefore we conclude that such a balancedly splittable Hadamard matrix does not exist.

There are exactly three inequivalent Hadamard matrices of order 16 with maximal excess 64. Those are contained in the Hall’s classes I, II, and III. The corresponding strongly regular graphs are $K_4 \times K_4$ and the Shrikhande graph [23]. These three Hadamard matrices are balancedly splittable. The following are three examples of order 16.

**Example 2.18.** The Hadamard matrix of order 16 of Hall’s class I, that is, the Sylvester Hadamard matrix is balancedly splittable with parameters $(16, 6, 2, -2)$. The corresponding strongly regular graph is $K_4 \times K_4$.

**Example 2.19.** The Hadamard matrix of order 16 of Hall’s class II is balancedly splittable with parameters $(16, 6, 2, -2)$. The corresponding strongly regular graph is $K_4 \times K_4$.

**Example 2.20.** The Hadamard matrix of order 16 of Hall’s class III is balancedly splittable with parameters $(16, 6, 2, -2)$. The corresponding strongly regular graph is the Shrikhande graph [23].

Though the classification of Hadamard matrices of order 36 has not been finished yet, we have the non-existence results for balancedly splittable Hadamard matrices of order 36 by dealing with the eigenspaces of the attached strongly regular graphs.

**Proposition 2.21.**

1. There is no balancedly splittable Hadamard matrix of order 36 with the parameters $(36, 10, 4, -2)$.
2. There is no balancedly splittable Hadamard matrix of order 36 with the parameters $(36, 25, 1, -5)$.
3. There is no balancedly splittable Hadamard matrix of order 36 with the parameters $(36, 14, 2, -4)$.
4. There is no balancedly splittable Hadamard matrix of order 36 with the parameters $(36, 20, 2, -4)$.
Proof. (1): If there would exist such a Hadamard matrix $H$, then $H$ must come from the unique strongly regular graph with the parameters $(36, 10, 4, 2)$ having the adjacency matrix $A$. The matrix $B := 12I + 6A - 2J$ is written as $12I + 6A - 2J = H_1^TH_1$ where $H_1$ is a $10 \times 36$ $(1, -1)$-matrix. Then the eigenvectors of $B$ with eigenvalue $36$ are the row vectors of the matrix $(I_{10}, X)$ where $X$ is
\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
-1 & 1 & 1 & 1 \\
-1 & 0 & 0 & 0 \\
-0 & 0 & 0 & 0 \\
-0 & 1 & 0 & 0 \\
-0 & 0 & 1 & 0 \\
-0 & 0 & 0 & 1
\end{pmatrix},
\]
and $-$ stands for $-1$. By computer search, there are no mutually orthogonal 10 eigenvectors of $B$ with eigenvalue 36 and entries $1, -1$. Therefore there is no Hadamard matrix $H$ of order 36 such that any $10 \times 36$ submatrix $H_1$ of $H$ satisfies that $12I + 6A - 2J = H_1^TH_1$.

The proofs for (2), (3), and (4) are the same as that of (1).

Note that the strongly regular graph for (2) is the complement of that for (1). There exist 180 strongly regular graphs with the parameters $(36, 21, 12, 12)$ which correspond to the case (3), and there exist 32548 strongly regular graphs with the parameters $(36, 20, 10, 12)$ which correspond to the case (4).

3 Constructions

In this section, we construct several balancedly splittable Hadamard matrices.

3.1 Construction for $(n, \ell, a, b) = (m^2, (m - 1)^2, 1, -m + 1), (m^2, 2m - 2, m - 2, -2)$, $m$ an order for a Hadamard matrix

Theorem 3.1. Let $m$ be the order for a Hadamard matrix. Then there exists a balancedly splittable Hadamard matrix of order $m^2$ with the parameters $(m^2, (m - 1)^2, 1, -m + 1)$ and $(m^2, 2m - 2, m - 2, -2)$.

Proof. Let $H$ be a Hadamard matrix of order $m$. Normalize $H$ so that $H = \begin{pmatrix} 1^T \\ H_1 \end{pmatrix}$. Then $H \otimes H$ is a Hadamard matrix and has $H_1 \otimes H_1$ as a submatrix of $H \otimes H$. Then using the property that $H_1^TH_1 = mI_m - J_m$, we have
\[
(H_1 \otimes H_1)^TH_1 \otimes H_1 = H_1^TH_1 \otimes H_1^TH_1 = (mI_m - J_m) \otimes (mI_m - J_m),
\]
which has only two distinct entries off diagonal. Therefore $H \otimes H$ is a balancedly splittable Hadamard matrix of order $m^2$ with the parameters $(m^2, (m - 1)^2, 1, -m + 1)$. Note that $H \otimes H$ is normalized and the all-ones row vector is not a row vector of $H_1 \otimes H_1$. Then we use the fact in Remark 2.8 (2) to show that $H \otimes H$ is also a balancedly splittable Hadamard matrix of order $m^2$ with the parameters $(m^2, 2m - 2, m - 2, -2)$.

3.2 Construction for $(n, \ell, a, b) = (m^2, m, m, 0)$, $m$ an order for a Hadamard matrix

Theorem 3.2. Let $m$ be the order for a Hadamard matrix. Then there exists a balancedly splittable Hadamard matrix of order $m^2$ with the parameters $(m^2, m, m, 0)$. 

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Proof. Let \( H \) be a Hadamard matrix of order \( m \). Let \( r_i \) be the \( i \)-th row of \( H \) for \( i \in \{1, \ldots, m\} \) and normalize \( H \) so that \( r_1 \) is the all-ones vector. Define an \( m^2 \times m^2 \) matrix \( M \) by \( M = (r_i^T r_j)_{i,j=1}^m \). Then \( M \) is a Hadamard matrix of order \( m^2 \). Let \( M_1 = (r_j^T r_1)_{j=1}^m \) be a submatrix of \( M \). By \( r_i r_j^T = m \delta_{i,j} \) and \( r_1 = 1_m^T \), we have

\[
M_1^T M_1 = \begin{pmatrix} r_1^T r_1 \\ r_1^T r_2 \\ \vdots \\ r_1^T r_m \end{pmatrix} \begin{pmatrix} r_1^T r_1 & r_1^T r_2 & \ldots & r_1^T r_m \end{pmatrix} = \begin{pmatrix} mJ_m & O & \ldots & O \\ O & mJ_m & \ldots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \ldots & mJ_m \end{pmatrix},
\]

which has only two distinct entries off diagonal. Therefore \( M \) is a balancedly splittable Hadamard matrix of order \( m^2 \) with the parameters \((m^2, m, m, 0)\). \(\square\)

### 3.3 Construction for \((n, \ell, a, b) = (km, k(m-1), 0, -k)\), \(k, m\) orders for Hadamard matrices

**Theorem 3.3.** Let \( k, m \) be the orders for Hadamard matrices. Then there exists a balancedly splittable Hadamard matrix of order \( km \) with the parameters \( (km, k(m-1), 0, -k) \).

**Proof.** Let \( H, K \) be Hadamard matrices of order \( k, m \) respectively. Normalize \( K \) so that \( K = \begin{pmatrix} 1^T \\ K_1 \end{pmatrix} \). Then \( H \otimes K \) is a Hadamard matrix and has \( H \otimes K_1 \) as a submatrix. Then using the property that \( K_1^T K_1 = mI_m - J_m \), we have

\[
(H \otimes K_1)^T (H \otimes K_1) = H^T H \otimes K_1^T K_1 = kI_k \otimes (mI_m - J_m),
\]

which has only two distinct entries off diagonal. Therefore \( H \otimes K \) is a balancedly splittable Hadamard matrix of order \( km \) with the parameters \((km, k(m-1), 0, -k)\). \(\square\)

### 3.4 Construction for \((n, \ell, a, b) = (n, n - 2, 0, -2)\), \(n\) an order for a Hadamard matrix

The following result is the same as [2, Observation 2].

**Theorem 3.4.** Let \( n \) be the order for a Hadamard matrix. Then there exists a balancedly splittable Hadamard matrix of order \( n \) with the parameters \((n, n - 2, 0, -2)\).

**Proof.** Let \( H \) be a Hadamard matrix of order \( n \). Normalize the first two rows of \( H \) so that

\[
H = \begin{pmatrix} 1_{n/2}^T & 1_{n/2}^T \\ 1_{n/2}^T & -1_{n/2}^T \\ H_1 \end{pmatrix}.
\]

Then \( H_1^T H_1 = \begin{pmatrix} nI_{n/2} - 2J_{n/2} & O_{n/2} \\ O_{n/2} & nI_{n/2} - 2J_{n/2} \end{pmatrix} \), which has only two distinct entries off diagonal, where \( O_{n/2} \) denotes the zero matrix of order \( n/2 \). It follows that \( H \) is a balancedly splittable Hadamard matrix of order \( n^2 \) with the parameters \((n, n - 2, 0, -2)\). \(\square\)

### 3.5 Construction for \((n, \ell, a, b) = (4^m, 2^m, 2^m, 0), (4^m, 2^{m-1}(2^m - 1), 2^{m-1}, -2^{m-1})\), \(m\) a positive integer

Let \( H_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \) and define \( H_m = H_{m-1} \otimes H_1 \) recursively for \( m \geq 2 \). Then \( H_m \) is a Hadamard matrix of order \( 2^m \), which is called *Sylvester-type.*
**Lemma 3.5.** If there exist a balancedly splittable Hadamard matrix of order $n_i^2$ with $(\ell_i, a_i, b_i) = (n_i, n_i, 0)$ for $i = 1, 2$, then there exists a balancedly splittable Hadamard matrix $H$ of order $n_1^2 n_2^2$ with $(\ell, a, b) = (n_1 n_2, n_1 n_2, 0)$.

**Proof.** Let $H_i = \begin{pmatrix} H_i,1 \\ H_i,2 \end{pmatrix}$ be balancedly splittable Hadamard matrices of order $n_i^2$ with $(\ell_i, a_i, b_i) = (n_i, n_i, 0)$ with respect to $H_i,1$ for $i = 1, 2$. Then, by Remark 2.9, $H_i^\top H_i,1 = n_i I_{n_i} \otimes J_{n_i}$ for $i = 1, 2$. A Hadamard matrix $H_1 \otimes H_2$ has a submatrix $H_{1,1} \otimes H_{2,1}$. Then,

$$(H_{1,1} \otimes H_{2,1})^\top (H_{1,1} \otimes H_{2,1}) = H_{1,1}^\top H_{1,1} \otimes H_{2,1}^\top H_{2,1} = n_1 n_2 I_{n_1} \otimes J_{n_1} \otimes I_{n_2} \otimes J_{n_2},$$

which is permutationally equal to $n_1 n_2 I_{n_1 n_2} \otimes J_{n_1 n_2}$. This proves that $H_1 \otimes H_2$ is balancedly splittable. \[\square\]

**Lemma 3.6.** If there exist a balancedly splittable Hadamard matrix of order $n_i$ with $(\ell_i, a_i, b_i) = ((n_i + \sqrt{n_i^2}/2, \sqrt{n_i^2}/2, -\sqrt{n_i^2}/2)$ for $i = 1, 2$, then there exists a balancedly splittable Hadamard matrix $H$ of order $n_1 n_2$ with $(\ell, a, b) = ((n_1 n_2 + \sqrt{n_1 n_2^2}/2, \sqrt{n_1 n_2^2}/2, -\sqrt{n_1 n_2^2}/2)$.

**Proof.** Let $H_i = \begin{pmatrix} H_i,1 \\ H_i,2 \end{pmatrix}$ be balancedly splittable Hadamard matrices of order $n_i$ with $(\ell_i, a_i, b_i) = ((n_i + \sqrt{n_i^2}/2, \sqrt{n_i^2}/2, -\sqrt{n_i^2}/2)$ for $i = 1, 2$. Then $H_i^\top H_i,1 = \ell_i I_{n_i} + a_i S_i$ for some Seidel matrices $S_i$, $i = 1, 2$. We consider a Hadamard matrix $H_1 \otimes H_2$, and it has $K := \begin{pmatrix} H_{1,1} \otimes H_{2,1} \\ H_{1,2} \otimes H_{2,2} \end{pmatrix}$ as a submatrix. Then

$$K^\top K = (H_{1,1} \otimes H_{2,1})^\top (H_{1,1} \otimes H_{2,1}) + (H_{1,2} \otimes H_{2,2})^\top (H_{1,2} \otimes H_{2,2})$$

$$= H_{1,1}^\top H_{1,1} \otimes H_{2,1}^\top H_{2,1} + H_{1,2}^\top H_{1,2} \otimes H_{2,2}^\top H_{2,2}$$

$$= (\ell_1 I_{n_1} + a_1 S_1) \otimes (\ell_2 I_{n_2} + a_2 S_2) + ((n_1 - \ell_1) I_{n_1} - a_1 S_1) \otimes ((n_2 - \ell_2) - a_2 S_2)$$

$$= (\ell_1 \ell_2 + (n_1 - \ell_1)(n_2 - \ell_2)) I_{n_1 n_2} + (2 \ell_1 n_1 a_2 I_{n_1} \otimes S_2 + a_1 (2 \ell_2 - n_2) S_1 \otimes I_{n_2} + 2 a_1 a_2 S_1 \otimes S_2$$

$$= \frac{n_1 n_2 + \sqrt{n_1 n_2^2}}{2} I_{n_1 n_2} + \frac{\sqrt{n_1 n_2^2}}{2} I_{n_1 \otimes S_2} + \frac{\sqrt{n_1 n_2^2}}{2} I_{n_2 \otimes S_1} + \frac{\sqrt{n_1 n_2^2}}{2} S_1 \otimes S_2,$$

which has only two distinct off-diagonal entries. Thus $H_1 \otimes H_2$ is balancedly splittable. \[\square\]

A balancedly splittable Hadamard matrix $H$ of order $n^2$ is said to be twin if $H = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix}$ satisfies that $H$ is balancedly splittable with parameters $(n^2, n, n, 0)$ with respect to $H_1$ and with parameters $(n^2, n(n-1)/2, n/2, -n/2)$ with respect to $H_2$ and $H_3$.

**Theorem 3.7.** The Sylvester-type Hadamard matrix of order $4^m$ is twin balancedly splittable.

**Proof.** Let $H$ be the Sylvester-type Hadamard matrix of order $4$:

$$H = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

where $H_1$ is a $2 \times 4$ matrix and $H_2, H_3$ are both $1 \times 4$ matrices. The result follows from Lemma 3.5, Lemma 3.6, and the above Hadamard matrix of order $4$. \[\square\]
3.6 Construction for \((n, \ell, a, b) = (q(q + 1), q, q, -1)\), where \(q\) an order of a skew-symmetric Hadamard matrix

In [27], it is shown that the existence of a skew-symmetric Hadamard matrix of order \(q + 1\) implies that the existence of a Hadamard matrix of order \((q - 1)q\). We review the construction and its generalization.

In [7], the following matrices \(J_m^{(q)}\) and \(A_m^{(q)}\) are used in order to construct a quaternary complex Hadamard matrix. Let \(q + 1\) be the order of a skew type Hadamard matrix \(H\). Multiply some rows and columns of \(H\) by \(-1\), if necessary, we may assume that

\[
H = \begin{pmatrix}
1 & 1^T \\
-1 & I + Q
\end{pmatrix}.
\]

The \((0, \pm 1)\)-matrix \(Q = (q_{ij})_{i,j=1}^q\), called the skew symmetric core of the skew type Hadamard matrix, is skew symmetric and satisfies that \(J_qQ = QJ_q = O_q\), and \(QQ^\top = qI_q - J_q\). For any odd prime power \(q\), see [21] for Paley’s construction.

Let \(q\) be the order of a skew symmetric core \(Q\). Define the following matrices recursively for each nonnegative integer \(m\):

\[
J_m^{(q)} = \begin{cases}
J_1 & \text{if } m = 0, \\
J_q \otimes A_{m-1}^{(q)} & \text{otherwise},
\end{cases}
\]

\[
A_m^{(q)} = \begin{cases}
J_1 & \text{if } m = 0, \\
I_q \otimes J_{m-1}^{(q)} + Q \otimes A_{m-1}^{(q)} & \text{otherwise}.
\end{cases}
\]

For a normalized Hadamard matrix of \(H\) of order \(q + 1\) with skew symmetric core \(Q\), the matrix \(C = H - I\) is a conference matrix, that is, \(CC^\top = qI\). We define \(M = -I_{q+1} \otimes J_1^{(q)} + C \otimes A_1^{(q)}\).

**Theorem 3.8.** The matrix \(M\) is a balancedly splittable Hadamard matrix of order \(q(q + 1)\) with \((n, \ell, a, b) = (q(q + 1), q, q, -1)\).

**Proof.** To use the properties that \(J_1^{(q)} (J_1^{(q)})^\top + qA_1^{(q)} (A_1^{(q)})^\top = q(q + 1)I_q\) and \(J_1^{(q)} (A_1^{(q)})^\top = A_1^{(q)} (J_1^{(q)})^\top\), we have

\[
MM^\top = (-I_{q+1} \otimes J_1^{(q)} + C \otimes A_1^{(q)}) (-I_{q+1} \otimes (J_1^{(q)})^\top + C^\top \otimes (A_1^{(q)})^\top)
\]

\[
= I_{q+1} \otimes (J_1^{(q)} (J_1^{(q)})^\top - C \otimes A_1^{(q)} (J_1^{(q)})^\top - C^\top \otimes J_1^{(q)} (A_1^{(q)})^\top + CC^\top \otimes A_1^{(q)} (A_1^{(q)})^\top)
\]

\[
= I_{q+1} \otimes (J_1^{(q)} (J_1^{(q)})^\top + qA_1^{(q)} (A_1^{(q)})^\top) - C \otimes (A_1^{(q)} (J_1^{(q)})^\top - J_1^{(q)} (A_1^{(q)})^\top)
\]

\[
= q(q + 1)I_{q+1} \otimes I_q.
\]

Therefore \(M\) is a Hadamard matrix. Next we show that \(M\) is balancedly splittable with respect to \(M_1\) obtained from \(M\) by restricting rows to the first \(q\) rows.

Since \(M_1 = \left( -J_1^{(q)} \ A_1^{(q)} \ \cdots \ A_1^{(q)} \right) \) and

\[
(J_1^{(q)})^\top J_1^{(q)} = (J_q)^2 = qJ_q,
\]

\[
(J_1^{(q)})^\top A_1^{(q)} = J_q (I_q + Q) = J_q,
\]

\[
(A_1^{(q)})^\top A_1^{(q)} = (I_q + Q^\top) (I_q + Q) = I_q + Q^\top Q = (q + 1)I_q - J_q,
\]

we have

\[
M_1^\top M_1 = \begin{pmatrix}
qJ_q & -J_q & \cdots & -J_q \\
-J_q & (q + 1)I_q - J_q & \cdots & (q + 1)I_q - J_q \\
\vdots & \vdots & \ddots & \vdots \\
-J_q & (q + 1)I_q - J_q & \cdots & (q + 1)I_q - J_q
\end{pmatrix}.
\]

Thus \(M\) is balancedly splittable. 

\[\square\]
4 Commutative association schemes

In this section we define commutative association schemes.

A d-class commutative association scheme, see [1], with a finite vertex set \( X \), is a set of non-zero \((0,1)\)-matrices \( A_0, A_1, \ldots, A_d \) with rows and columns indexed by \( X \), such that

1. \( A_0 = I_{|X|} \),
2. \( \sum_{i=0}^{d} A_i = J_{|X|} \),
3. \( A_i^T \in \{ A_1, \ldots, A_d \} \) for any \( i \in \{ 1, \ldots, d \} \),
4. for all \( i, j \), \( A_i A_j = \sum_{k=0}^{d} p_{i,j}^k A_k \) for some non-negative integers \( p_{i,j}^k \),
5. for all \( i, j \), \( A_i A_j = A_j A_i \).

The association scheme is said to be symmetric if \( A_i^T = A_i \) for any \( i \), and non-symmetric otherwise. Note that if symmetric matrices \( A_i \) \((0 \leq i \leq d)\) satisfy (4), then (5) must follow. The vector space spanned by \( A_i \)'s over the real number field forms a commutative algebra, denoted by \( A \) and is called the Bose-Mesner algebra. Then there exists a basis of \( A \) consisting of primitive idempotents, say \( E_0 = (1/|X|)J_{|X|}, E_1, \ldots, E_d \). Since \( A_0, A_1, \ldots, A_d \) and \( \{ E_0, E_1, \ldots, E_d \} \) are two bases of \( A \), there exist the change-of-basis matrices \( P = (P_{i,j})_{i,j=0}^{d} \), \( Q = (Q_{i,j})_{i,j=0}^{d} \) so that

\[
A_i = \sum_{j=0}^{d} P_{j,i} E_j, \quad E_j = \frac{1}{|X|} \sum_{i=0}^{d} Q_{i,j} A_i.
\]

The matrices \( P, Q \) are said to be the first and second eigenmatrices respectively.

**Example 4.1.** Construction in Subsection [1] is closely related to the binary Hamming schemes \( H(n,2) \). Let \( X = \mathbb{Z}_2^n \) and \( R_i = \{(x,y) \mid x,y \in X, d(x,y) = i\} \) for \( i = 0,1,\ldots,n \), where \( d(x,y) \) is the Hamming distance between \( x \) and \( y \). Define \( A_i \) to be the adjacency matrix of a graph \((X,R_i)\) for \( i = 0,1,\ldots,n \). Then the matrices \( A_0, A_1, \ldots, A_n \) is a symmetric association scheme, which is called the binary Hamming association scheme, denoted by \( H(n,2) \).

We denote adjacency matrices of the Hamming scheme \( H(n,2) \) by \( A_i^{(n)} \).

The Hamming scheme \( H(n+1,2) \) is described as a fusion scheme of the product of schemes \( H(n,2) \) and \( H(1,2) \) as follows, see also [3]. For association schemes \( \{ A_0^i, A_1^i, \ldots, A_d^i \} \) and \( \{ A_0'^i, A_1'^i, \ldots, A_d'^i \} \), the product of these two is an association schemes with non-zero matrices \((0,1)\) \( A_i^i \otimes A_j^j \) where \( 0 \leq i \leq d_1, 0 \leq j \leq d_2 \). Take two association schemes as the Hamming schemes \( H(n,2) \) and \( H(1,2) \) respectively, then we have

\[
A_i^{(n+1)} = \sum_{j+k=i} A_j^{(n)} \otimes A_k^{(1)} = A_i^{(n)} \otimes A_0^{(1)} + A_{i-1}^{(n)} \otimes A_1^{(1)}
\]

for \( i \in \{ 1, \ldots, n + 1 \} \). It follows now that the adjacency matrices of the binary Hamming scheme are diagonalizable by the Sylvester Hadamard matrices. For \( n = 1 \), the adjacency matrices \( A_0 = I_2, A_1 = J_2 - I_2 \) are diagonalizable by \( H_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \). For \( n \geq 2 \), it follows from the recurrence above that \( H_n = H_1^\otimes n \) diagonalizes the adjacency matrices of \( H(n,2) \).

A subscheme or fusion scheme of the association scheme \((X,\{R_i\}_{i=0}^{d})\) is an association scheme \((X,\{\cup_{j \in \Lambda_i} R_j\}_{i=0}^{c})\) for some decomposition \( \{ \Lambda_0, \Lambda_1, \ldots, \Lambda_c \} \) of \( \{ 0,1,\ldots,d \} \) such that \( \Lambda_0 = \{ 0 \} \).

Muzychuk [20] classified the subschemes of \( H(n,2) \) for \( n \geq 9 \). By [20] Theorem 2.1, there are exactly two cases for the subschemes to be primitive strongly regular graphs:
• \( n \) is even and \( \Lambda_1 = \{ k \mid 1 \leq k \leq n, k \equiv 0, 1 \pmod{4} \} \), \( \Lambda_2 = \{ k \mid 1 \leq k \leq n, k \equiv 2, 3 \pmod{4} \} \).

• \( n \) is even and \( \Lambda_1 = \{ k \mid 1 \leq k \leq n, k \equiv 0, 3 \pmod{4} \} \), \( \Lambda_2 = \{ k \mid 1 \leq k \leq n, k \equiv 1, 2 \pmod{4} \} \).

The parameters of these strongly regular graphs are
\[
(n, k, \lambda, \mu) = (4^m, 2^{m-1}(2^m \pm 1), 2^{m-1}(2^m \pm 1), 2^{m-1}(2^m \pm 1))
\]
and their complements.

The Doob schemes are the association schemes with the same parameters as the binary Hamming schemes [9]. By Example 2.19, the Doob schemes are Hadamard diagonalizable, and this scheme has the fusion schemes which yield strongly regular graphs.

5 Construction of commutative association schemes

Let \( H \) be a Hadamard matrix of order \( n \) with rows \( r_1, \ldots, r_n \). For \( i \in \{1, \ldots, n\} \), let \( C_i = r_i^\top r_i \). We call \( C_1, \ldots, C_n \) the auxiliary matrices of \( H \). The auxiliary matrices play an important role to construct association schemes. The following is basic properties for the auxiliary matrices.

**Lemma 5.1.** [14]

1. \( \sum_{i=1}^n C_i = nI_n \).
2. For any \( i \in \{1, \ldots, n\} \), \( C_i^2 = nC_i \).
3. For any distinct \( i, j \in \{1, \ldots, n\} \), \( C_i C_j = 0_n \).

Note that for a Hadamard matrix \( H \), letting \( H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \) where \( H_1 \) is an \( \ell \times n \) matrix, it holds that \( \sum_{i=1}^\ell C_i = H_1^\top H_1 \).

Some combinatorial objects and association schemes are obtained from a balancedly split-table Hadamard matrix of order \( n \) such that \( \sum_{i=1}^\ell C_i \) has exactly one off-diagonal entries and some Latin squares as follows:

• symmetric or skew-symmetric Bush type Hadamard matrices and 3-class association schemes from the case \((\ell, a) = (n, 0)\) with \( C_1 = J_n \) and a symmetric Latin squares with constant diagonal, as described in [25], [9].

• symmetric or skew-symmetric regular \((0, \frac{1}{n-1})\) biangular matrices and 4-class association schemes from the case \((\ell, a) = (n-1, 1)\) with \( C_1 = J_n \) and a symmetric Latin square with constant diagonal, refer to [16].

• unbiased Hadamard matrices and 4-class association schemes from the case \((\ell, a) = (n, 0)\) and mutually unique fixed symbol (UFS) Latin squares, see [11] [19].

• unbiased Bush-type Hadamard matrices and 5-class association schemes from the case \((\ell, a) = (n, 0)\) with \( C_1 = J_n \) and mutually UFS Latin squares as defined in [15].

• unbiased biangular vectors (more generally linked systems of symmetric group divisible designs) and 5-class association schemes from the case \((\ell, a) = (n-1, 1)\) with \( C_1 = J_n \) and mutually UFS Latin squares, as shown in [13], and [17].
In the following subsections, we construct symmetric or non-symmetric association schemes with 4, 5 or 6-classes from a balancerly splittable Hadamard matrix such that $\sum_{i=1}^{\ell} C_i$ has exactly two distinct off diagonal entries and some Latin squares. Throughout the following subsections, we assume that $H$ is a balancedly splittable Hadamard matrix of order $n$ with auxiliary matrices $C_1, \ldots, C_n$ satisfying $\sum_{i=1}^{\ell} C_i = \ell I_n + aA + b(J_n - A - I_n)$ where $A$ is an $n \times n (0, 1)$-matrix, $a \neq b$, and $C_i J_n = O_n$ for $i \in \{1, \ldots, \ell\}$. According to Proposition 2.6 $b = \left(\frac{\ell(a+b-n)}{a(n-1)+\ell}\right)$ and the matrix $A$ is the adjacency matrix of a strongly regular graph with the parameters $(n, k, \lambda, \mu)$ given as:

\[
k = \frac{\ell n(n-\ell-1)}{n(a^2 + \ell) - (a-\ell)^2},
\]

\[
\lambda = \frac{n(n^2(a^3 + \ell^2) - 2(\ell+1)n(a^3 + \ell^2) + 2a\ell + a + \ell(\ell+2))(a-\ell)^2)}{(a-\ell)^2 - n(a^2 + \ell)^2},
\]

\[
\mu = \frac{\ell n(a-\ell)(\ell-n+1)(a-\ell+n)}{(a-\ell)^2 - n(a^2 + \ell)^2}.
\]

We use $C_0 := O_n$ and a Latin square $L = (L_{i,j})_{i,j \in S}$ on the symbol set $S$ where $S$ equals to $\{1, \ldots, \ell\}$ or $\{0, 1, \ldots, \ell\}$, and denote $\tilde{L}$ to be $\tilde{L} = (C_{L_{i,j}})_{i,j \in S}$.

For the remaining part of the paper, we use a variant of Mutually Orthogonal Latin Squares (MOLS) which we call UFS (Unique Fix Symbol) suitable for the way we apply it. Two Latin squares $L_1$ and $L_2$ of size $n$ on the same symbol set are called to be UFS Latin squares, if every superimposition of each row of $L_1$ on each row of $L_2$ results in only one element of the form $(a, a)$. In effect, each permutation of symbols between the rows of the two Latin squares has a Unique Fixed Symbol. A set of Latin squares in which every distinct pair of Latin squares are UFS Latin square is called mutually UFS Latin squares. Note that UFS Latin squares are called suitable Latin squares in [11] and elsewhere. See [11] for the equivalenee of existence between mutually UFS Latin squares and mutually orthogonal Latin squares.

**Lemma 5.2.** Let $L_1, L_2$ be UFS Latin squares on the symbol set $\{1, \ldots, n\}$ with the $(i,j)$-entry equal to $l(i,j), l'(i,j)$ respectively. An $n \times n$ array with the $(i,j)$-entry equal to $b$ determined by $b = l(i,a) = l'(j,a)$ for the unique $a \in \{1, \ldots, n\}$, is a Latin square.

The following lemma will be used in Subsections 5.1, 5.2, 5.3. We omit its easy proof.

**Lemma 5.3.** Let $H$ be a balancedly splittable Hadamard matrix of order $n$. If $C_i J_n = J_n C_i = O_n$, then $AC_i = C_i A = (n-\ell + b)C_i$.

**Lemma 5.4.** Let $H$ be a balancedly splittable Hadamard matrix of order $n$ with $\sum_{i=1}^{\ell} C_i = \ell I_n + aA + b(J_n - A - I_n)$ and $C_i J_n = O_n$ for $i \in \{1, \ldots, \ell\}$. Let $L$ be a Latin square on the symbol set $S$ where $S$ equals to $\{1, \ldots, \ell\}$ or $\{0, 1, \ldots, \ell\}$. Then $\tilde{L} L^T = nI_{|S|} \otimes (|S| I_n + aA + b(J_n - A - I_n))$.

**Lemma 5.5.** Let $H$ be a balancedly splittable Hadamard matrix of order $n$ with $\sum_{i=1}^{\ell} C_i = \ell I_n + aA + b(J_n - A - I_n)$ and $C_i J_n = O_n$ for $i \in \{1, \ldots, \ell\}$. Let $L_1, \ldots, L_f$ be mutually UFS Latin squares on the symbol set $S$ where $S$ equals to $\{1, \ldots, \ell\}$ or $\{0, 1, \ldots, \ell\}$. For distinct $i, j \in \{1, \ldots, f\}$, $\tilde{L}_i^\prime \tilde{L}_j^\prime = n \tilde{L}_{i,j}$, where $\tilde{L}_{i,j}$ is the Latin square determined from $L_1, L_2$ by Lemma 5.2.

Then the following holds: for any distinct $i, j, k \in \{1, \ldots, f\}$, $L_{i,k}$ and $L_{j,k}$ are UFS and the Latin square obtained from $L_{i,k}$ and $L_{j,k}$ in this ordering via Lemma 5.2 coincides with $L_{i,j}$.
5.1 Symmetric and non-symmetric association schemes with 4-classes

In this subsection, we will use a symmetric Latin square with constant diagonal, which is known to exist for order \( v \) a positive even integer, see [13]. Assume that \( \ell \) is an odd integer and let \( L \) be a symmetric Latin square of order \( \ell + 1 \) on the symbol set \( \{0, 1, \ldots, \ell\} \) with constant diagonal 0.

We define disjoint \((0,1)\)-matrices \( A_i \) \((i \in \{0, 1, \ldots, 4\})\) as

\[
A_0 = I_{(\ell+1)n}, \quad A_1 = I_{\ell+1} \otimes A, \quad A_2 = I_{\ell+1} \otimes (J_n - A - I_n), \quad \bar{L} = A_3 - A_4.
\]

**Theorem 5.6.** The set of matrices \( \{A_0, A_1, \ldots, A_4\} \) is a symmetric association scheme with 4-classes.

**Proof.** It is routine to see that \( A_0 = I_{(\ell+1)n} \) s are disjoint symmetric \((0,1)\)-matrices such that \( \sum_{i=0}^{4} A_i = J_{(\ell+1)n} \) and each \( A_i \) is symmetric. Let \( \mathcal{A} = \text{span}_\mathbb{R} \{A_0, A_1, \ldots, A_4\} \). We will check (4) in the definition of the association scheme for each case.

(i): For \( i, j \in \{1, 2\} \), (AS4) follows from the fact that \( A \) is the adjacency matrix of a strongly regular graph.

(ii): It follows from Lemma 5.3 that \( A_i(A_3 - A_4), (A_3 - A_4)A_i \in \mathcal{A} \) for \( i = 1, 2 \). Since \( A_3 + A_4 = I_{\ell+1} \otimes (J_n - I_n) \) and \( A \) is in particular the adjacency matrix of a regular graph, \( A_i(A_3 + A_4), (A_3 + A_4)A_i \in \mathcal{A} \) for \( i = 1, 2 \). Thus (AS4) holds for \( (i, j) \in \{(1, 2) \times \{3, 4\} \cup \{(3, 4) \times \{1, 2\}\} \).

(iii): For \( i, j \in \{3, 4\} \), \( (A_3 - A_4)^2 = (\bar{L})^2 \in \mathcal{A} \) by Lemma 5.4. By \( A_3 + A_4 = (J_{\ell+1} - I_{\ell+1}) \otimes J_n \) and \( C_i J_n = O_n \) for \( i \in \{1, \ldots, \ell\} \), \( (A_3 + A_4)(A_3 - A_4) = (A_3 - A_4)(A_3 + A_4) = O_n \in \mathcal{A} \) and \( (A_3 + A_4)^2 \in \mathcal{A} \). These implies that each component of \( (A_3^2, A_3 A_4, A_4 A_3, A_4^2) \) belongs to \( \mathcal{A} \) where \( H \) is a Hadamard matrix. Since \( H \) is invertible, each of \( A_3^2, A_3 A_4, A_4 A_3, A_4^2 \) belongs to \( \mathcal{A} \).

This completes the proof. \( \square \)

Then the eigenmatrices \( P \) and \( Q \) are as follows:

\[
P = \begin{pmatrix}
1 & (\ell+1)n & \ell n & \ell n & \ell n \\
(\ell+1)n & (\ell+1)n & \ell n & \ell n & \ell n \\
(\ell+1)n & \ell n & (\ell+1)n & \ell n & \ell n \\
(\ell+1)n & \ell n & \ell n & (\ell+1)n & \ell n \\
(\ell+1)n & \ell n & \ell n & \ell n & (\ell+1)n
\end{pmatrix}
\]

\[
Q = \begin{pmatrix}
1 & \ell & \ell & \ell & \ell \\
1 & \ell & \ell & \ell & \ell \\
1 & \ell & \ell & \ell & \ell \\
1 & \ell & \ell & \ell & \ell \\
1 & \ell & \ell & \ell & \ell \\
\end{pmatrix}
\]

By a slight modification, we obtain non-symmetric association schemes with 4-classes. Under the same setting on \( L, \ell \), and \( C_i \) as above, we define \( \bar{L} = (\epsilon_{i,j} C_{L_{i,j}})_{(\ell+1)n} \), where \( \epsilon_{i,j} = 1 \) if \( i \leq j \) and -1 if \( i > j \). We define disjoint \((0,1)\)-matrices \( A_i \) \((i \in \{0, 1, \ldots, 4\})\) as

\[
A_0 = I_{(\ell+1)n}, \quad A_1 = I_{\ell+1} \otimes A, \quad A_2 = I_{\ell+1} \otimes (J_n - A - I_n), \quad \bar{L} = A_3 - A_4.
\]

Note that \( A_3^2 = A_4 \).

**Theorem 5.7.** The set of matrices \( \{A_0, A_1, \ldots, A_4\} \) is a non-symmetric association scheme with 4-classes.
The eigenmatrices $\tilde{P}$ and $\tilde{Q}$ are obtained from $P, Q$ by changing $\tilde{P}_{i,j} = \sqrt{-1} P_{i,j}$ for $i \in \{2, 3\}, j \in \{3, 4\}$ and $\tilde{Q}_{i,j} = \sqrt{-1} Q_{i,j}$ for $i \in \{3, 4\}, j \in \{2, 3\}$.

### 5.2 Symmetric association schemes with 5-classes

Let $L_1, \ldots, L_f$ be mutually UFS Latin squares on $\{1, \ldots, \ell\}$. We now construct a symmetric association scheme with 5-classes from a balancedly splittable Hadamard matrix and mutually UFS Latin squares. Consider the Gram matrix $G$ of the row vectors of $L_i$ ($i \in \{1, \ldots, f\}$) defined by

$$G = \left( \begin{array}{cccc} \tilde{L_1} \tilde{L}_1^\top & \tilde{L}_1 \tilde{L}_2^\top & \cdots & \tilde{L}_1 \tilde{L}_f^\top \\ \tilde{L}_2 \tilde{L}_1^\top & \tilde{L}_2 \tilde{L}_2^\top & \cdots & \tilde{L}_2 \tilde{L}_f^\top \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{L}_f \tilde{L}_1^\top & \tilde{L}_f \tilde{L}_2^\top & \cdots & \tilde{L}_f \tilde{L}_f^\top \end{array} \right).$$

The entries of $G$ are $\{n\ell, na, nb, \pm n, 0\}$. Decompose the matrix $G$ into its entries as

$$G = n\ell A_0 + n(aA_1 + bA_2) + n(3-A_4).$$

Then the disjoint $(0, 1)$-matrices $A_i$ ($i \in \{0, 1, \ldots, 4\}$) satisfy $\sum_{i=0}^4 A_i = J_{f\ell n} - J_f \otimes (J_{\ell} - I_{\ell}) \otimes J_n$. We now define

$$A_5 = I_f \otimes (J_{\ell} - I_{\ell}) \otimes J_n.$$

Note that $A_1 = I_f \otimes I_{\ell} \otimes A$, $A_2 = I_f \otimes I_{\ell} \otimes (J_n - A - I_n)$ and

$$A_3 - A_4 = \frac{1}{n} \begin{pmatrix} O_{\ell n} & \tilde{L}_1 \tilde{L}_2^\top & \cdots & \tilde{L}_1 \tilde{L}_f^\top \\ \tilde{L}_2 \tilde{L}_1^\top & O_{\ell n} & \cdots & \tilde{L}_2 \tilde{L}_f^\top \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{L}_f \tilde{L}_1^\top & \tilde{L}_f \tilde{L}_2^\top & \cdots & O_{\ell n} \end{pmatrix},$$

$$A_3 + A_4 = (J_f - I_f) \otimes J_{\ell} \otimes J_n.$$

The following is the main result of this subsection.

**Theorem 5.8.** Let $H$ be a balancedly splittable Hadamard matrix of order $n$ with $\sum_{i=1}^f C_i = \ell I_n + aA + b(J_n - A - I_n)$ where $A$ is the adjacency matrix of a regular graph, and $L_1, \ldots, L_f$ be mutually UFS Latin square on $\{1, \ldots, \ell\}$. Then the set of matrices $\{A_0, A_1, \ldots, A_5\}$ defined above is a symmetric association scheme with 5-classes.

**Proof.** It is easy to see that $A_0 = I_{f\ell n}$, $A_i$'s are disjoint symmetric $(0, 1)$-matrices such that $\sum_{i=0}^5 A_i = J_{f\ell n}$, and each $A_i$ is symmetric. Let $A = \text{span}_R \{A_0, A_1, \ldots, A_5\}$.

First it can be shown that $\text{span}_R \{A_0, A_1, A_2, A_3 + A_4, A_5\}$ is closed under the matrix multiplication. Next we show that the products $A_i(A_3 - A_4)$ for $i \in \{1, 2, 5\}$ and $(A_3 - A_4)^2$ are linear combinations of $A_0, A_1, \ldots, A_5$, from which (4) in the definition of the association scheme follows. The equation $A_5(A_3 - A_4) = O_{f\ell n}$ can be shown, and the cases for $A_i(A_3 - A_4)$ for $i \in \{1, 2\}$ follow from the following.

Since $(I_{\ell} \otimes J_n) \tilde{L}_j = O_{\ell n}$ for each $j$, we have that $(A_0 + A_1 + A_2)(A_3 - A_4) = O_{f\ell n}$. Therefore $(A_1 + A_2)(A_3 - A_4) = -A_3 + A_4$.

Since $(\sum_{i=1}^f I_{\ell} \otimes C_i) \tilde{L}_j = \tilde{L}_j$ for each $j$, we have that $(aA_1 + bA_2)(A_3 - A_4) = (n-\ell)(A_3 - A_4)$.\]
Finally from Lemmas 5.2, 5.3 it follows that
\[(A_3 - A_4)^2 = n(f - 1)(\ell A_0 + aA_1 + bA_2) + n(f - 2)(A_3 - A_4).\]
This completes the proof.

Then the eigenmatrices \(P\) and \(Q\) are as follows:

\[
P = \begin{pmatrix}
\ell(n-\ell-1) & (\ell+\alpha(n-1))^2 & \frac{1}{2}(f-1)\ell n & \frac{1}{2}(f-1)\ell n & (\ell - 1)n \\
\frac{(n-\ell-1)^2}{(n-\ell+1)n} & \frac{1}{2}(f-1)n & \frac{1}{2}(n-f)n & 0 \\
\frac{1}{2}(f-1)n & \frac{1}{2}(n-f)n & 0 & 0 & 0 \\
0 & 0 & -n & n & 0 \\
0 & 0 & 0 & -n & (\ell - 1)n \\
\end{pmatrix}
\]

\[
Q = \begin{pmatrix}
1 & f(n-\ell-1) & f(\ell-1) & (f-\ell)^2 & f - 1 \\
\frac{1}{\ell-a(n-1)} & \frac{1}{\ell+a(n-1)} & \frac{1}{\ell+1} & \frac{1}{\ell-a(n-1)} & f - 1 \\
0 & 0 & -f & 0 & f - 1 \\
0 & 0 & 0 & \ell & -1 \\
0 & 0 & 0 & 0 & f - 1 \\
\end{pmatrix}
\]

5.3 Symmetric association schemes with 6-classes

Let \(L_1, \ldots, L_f\) be mutually UFS Latin squares on \(\{0, 1, \ldots, \ell\}\) with constant diagonal 0. We now construct a symmetric association scheme with 6-classes from a balancedly splittable Hadamard matrix and mutually UFS Latin squares. Consider the Gram matrix \(G\) of the row vectors of \(\hat{L}_i\) \((i \in \{1, \ldots, f\})\) defined by \(G = (\hat{L}_i\hat{L}_j^\top)_i,j=1\).

The entries of \(G\) are \(\{n\ell, na, nb, \pm n, 0\}\). Decompose the matrix \(G\) into its entries as
\[G = n\ell A_0 + n(aA_1 + bA_2) + n(A_3 - A_4).\]
Then the disjoint \((0, 1)\)-matrices \(A_i\) \((i \in \{0, 1, \ldots, 4\})\) satisfy \(\sum_{i=0}^4 A_i = J_f(\ell+1)n - (J_f \otimes (J_{\ell+1} - I_{\ell+1}) \otimes J_n + (J_f - I_f) \otimes I_{\ell+1} \otimes J_n).\) We now define
\[A_5 = J_f \otimes (J_{\ell+1} - I_{\ell+1}) \otimes J_n, \quad A_6 = (J_f - I_f) \otimes I_{\ell+1} \otimes J_n.\]

Note that \(A_1 = I_f \otimes I_{\ell+1} \otimes A, A_2 = I_f \otimes I_{\ell+1} \otimes (J_n - A - I_n)\) and
\[
A_3 - A_4 = \frac{1}{n} \begin{pmatrix}
O_{(\ell+1)n} & \hat{L}_1\hat{L}_2^\top & \cdots & \hat{L}_1\hat{L}_f^\top \\
\hat{L}_2\hat{L}_1^\top & O_{(\ell+1)n} & \cdots & \hat{L}_2\hat{L}_f^\top \\
\vdots & \vdots & \ddots & \vdots \\
\hat{L}_f\hat{L}_1^\top & \hat{L}_f\hat{L}_2^\top & \cdots & O_{(\ell+1)n} \\
\end{pmatrix},
\]

\[
A_3 + A_4 = (J_f - I_f) \otimes (J_{\ell+1} - I_{\ell+1}) \otimes J_n.
\]

The following is the main result of this subsection.

**Theorem 5.9.** Let \(H\) be a balancedly splittable Hadamard matrix of order \(n\) with \(\sum_{i=1}^\ell C_i = \ell I_n + aA + b(J_n - A - I_n)\) where \(A\) is the adjacency matrix of a regular graph, and \(L_1, \ldots, L_f\) be mutually UFS Latin square on \(\{0, 1, \ldots, \ell\}\) with constant diagonal 0. Then the set of matrices \(\{A_0, A_1, \ldots, A_6\}\) defined above is a symmetric association scheme with 6-classes.
Proof. The proof is similar to that of Theorem 5.8.

Then the eigenmatrices $P$ and $Q$ are as follows:

$$
Q = \begin{pmatrix}
1 & \ell & \ell + 1 & 0 & 1 & 0 \\
1 & \ell & \ell + 1 & 0 & 1 & 0 \\
1 & \ell & \ell + 1 & 0 & 1 & 0 \\
1 & -\ell - 1 & 0 & 0 & 0 & 0 \\
1 & -\ell - 1 & 0 & 0 & 0 & 0 \\
1 & -\ell - 1 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$P = \begin{pmatrix}
1 & \frac{\ell(\ell-n+1)}{n} & \frac{(\ell+a(n-1))^2}{(n-1)a^2+2a+\ell(n-\ell)} & \ell(\ell+1)n & \frac{1}{2}(f-1)n & \frac{1}{2}(f-1)n \\
1 & \frac{-\ell(n-2a+\ell(n-\ell))}{n} & \frac{-(\ell+a(n-1))^2}{(n-1)a^2+2a+\ell(n-\ell)} & \ell(\ell+1)n & \frac{1}{2}(f-1)n & \frac{1}{2}(f-1)n \\
1 & \frac{-\ell(n-2a+\ell(n-\ell))}{n} & \frac{-(\ell+a(n-1))^2}{(n-1)a^2+2a+\ell(n-\ell)} & \ell(\ell+1)n & \frac{1}{2}(f-1)n & \frac{1}{2}(f-1)n \\
1 & -\ell + 1 & 0 & 0 & 0 & 0 \\
1 & -\ell + 1 & 0 & 0 & 0 & 0 \\
1 & -\ell + 1 & 0 & 0 & 0 & 0
\end{pmatrix}
$
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