Extension of the EGS theorem to metric and Palatini $f(R)$ gravity

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Abstract

By using the equivalence between metric and Palatini $f(R)$ (or “modified”) gravities with $\omega = 0, -3/2$ Brans-Dicke theories, it is shown that the Ehlers-Geren-Sachs theorem of general relativity is extended to modified gravity. In the case of metric $f(R)$ gravity studied before, this agrees with previous literature.
1 Introduction

Through an exceptional number of publications on the subject, Sergei Odintsov has contributed to advance alternative theories of gravity and cosmology motivated by quantum corrections to the classical Einstein-Hilbert action. Among these are scalar-tensor and $f(R)$ gravitational theories and, therefore, it seems appropriate to include in this volume dedicated to Sergei a small contribution to cosmology in these theories.

In relativistic cosmology, the identification of our universe with a Friedmann-Lemaitre-Robertson-Walker (FLRW) space relies on the observations of spatial homogeneity and isotropy around us. The strongest support for this assumption, which lies at the core of relativistic cosmology (in both Einstein’s theory of general relativity and in alternative gravitational theories) comes from the observation of the high degree of isotropy of the cosmic microwave background (CMB), supplemented by the assumption that isotropy would be observed from any spatial point in the universe (the Copernican principle — such an assumption would be hard to check). The fact that a spacetime in which a family of observers exists who see the CMB isotropic around them can be identified with a FLRW space is far from trivial and, from the mathematical point of view, constitutes a kinematical characterization of FLRW spaces known as the Ehlers-Geren-Sachs (hereafter EGS) theorem \[2\]. Usually, the vanishing of acceleration, shear, and vorticity,

$$\dot{u}^a \equiv u^c \nabla_c u^a = 0, \quad \sigma_{ab} = 0, \quad \omega_{ab} = 0,$$

(1.1)

for a congruence of “typical” observers with four-velocity $u^a$ is taken to imply that the spacetime is of the FLRW type, i.e., with line element

$$ds^2 = -dt^2 + a^2(t) \gamma_{ij}(x^k) dx^i dx^j,$$

(1.2)

where $\gamma_{ij}$ is a constant curvature 3-metric. However, this is guaranteed only if $i$) matter is described by a perfect fluid, and $ii)$ the Einstein equations are imposed. These conditions enforce the vanishing of the Weyl tensor.

In its original version, the EGS theorem states that, if a congruence of timelike, freely falling observers in a dust-dominated (i.e., with vanishing pressure $P$) spacetime sees an isotropic radiation field, then (assuming that isotropy holds about every point) the spacetime is spatially homogeneous and isotropic and, therefore, a FLRW one. The original EGS theorem was generalized to an arbitrary perfect fluid that is geodesic and barotropic and with observers that are geodesics and irrotational \[3, 4\]. Moreover, an “almost EGS theorem” has been proved: spacetimes that are close to satisfying the

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1 A streamlined version of the discussion presented here was reported in \[1\].
EGS conditions are close to FLRW spaces in an appropriate sense\textsuperscript{2}. Perhaps it is not exaggerated to regard the EGS theorem and its generalizations as a cornerstone of relativistic cosmology motivating the use of the standard Big Bang model.

Since the discovery of the EGS theorem in 1968, theoretical cosmology has expanded considerably to include the possibility that general relativity may have to be augmented by adding quantum corrections which take the form of extra terms in the Einstein-Hilbert action\textsuperscript{3}

$$S_{EH} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \ R + S^m.$$  

Here we are interested in theories described by an action of the form

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \ f(R) + S^m \ ,$$  

where \( f(R) \) is an arbitrary (but twice differentiable) function of its argument. While \( f(R) \) gravity has a long history (\textsuperscript{9} - see \textsuperscript{10} for an historical review), quadratic corrections to general relativity, \textit{i.e.}, \( f(R) = R + \alpha R^2 \), were introduced early on as semiclassical corrections or counterterms to renormalize general relativity \textsuperscript{11}: such corrections, which are motivated also by string theories \textsuperscript{12}, are obviously important at large curvatures, \textit{e.g.}, in the very early universe (in which they have been used to propel inflation in Starobinski’s scenario \textsuperscript{13}) and near black holes, in which non-linear choices of the function \( f(R) \) may cure the problem of the central singularity \textsuperscript{14}. More recently, “modified” or “\( f(R) \)” gravity has seen a new lease on life after the 1998 discovery of the acceleration of the cosmic expansion using type Ia supernovae \textsuperscript{15}. While one possibility is to explain the present acceleration of the universe by postulating a mysterious form of dark energy with exotic properties (\( P \approx -\rho \), where \( \rho \) is the energy density), it has been proposed that perhaps we are seeing the first deviations from Einstein’s gravity on very large scales. The prototypical modification of the Einstein-Hilbert action consisted of the choice \( f(R) = R - \mu^4/R \), where \( \mu \approx H_0^{-1} \) is a mass scale of the order of the present value of the Hubble parameter, \textit{i.e.}, extremely small on particle physics scales \textsuperscript{16,17}. While this particular model is in gross violation of the Solar System observational constraints on the parametrized-post-Newtonian parameter \( \gamma \) \textsuperscript{18} and is subject to a violent instability \textsuperscript{19,20,21}, choices of the function \( f(R) \) that satisfy the experimental constraints and provide the correct cosmological dynamics abound in the literature (\textsuperscript{22} — see \textsuperscript{23,24} for reviews). Three versions of modified gravity exist: the first is \textit{metric \( f(R) \) gravity}, in which the action (1.3) is varied with respect to the metric,

\textsuperscript{2}For a discussion of inhomogeneous or anisotropic cosmological models admitting an isotropic radiation field, see \textsuperscript{5}.

\textsuperscript{3}Here \( R \) is the Ricci curvature of the metric \( g_{ab} \), which has determinant \( g \) and \( \kappa \equiv 8\pi G \). \( G \) is Newton’s constant, and \( S^m \) denotes the matter part of the action. We follow the notations of Ref. \textsuperscript{8}.
and provides the fourth order field equations

\[ f'(R)R_{ab} - \frac{1}{2} f(R)g_{ab} - [\nabla_a \nabla_b - g_{ab} \Box] f'(R) = \kappa T_{ab}, \tag{1.4} \]

where, as usual, \( T_{ab} = \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{ab}} \), and a prime denotes differentiation with respect to \( R \).

The second version of the theory is Palatini \( f(R) \) gravity, in which the metric \( g_{ab} \) and the connection \( \Gamma^a_{bc} \) are considered as independent quantities (i.e., the connection is not identified with the metric connection \( \{a_{bc}\} \) of \( g_{ab} \)), the Ricci tensor \( R_{ab} \) is constructed out this non-metric connection, and \( R = g^{ab} R_{ab} \). The Palatini action is

\[ S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + S^m, \tag{1.5} \]

in which the matter part of the action does not depend explicitly on the (non-metric) connection. Variation with respect to the metric and the connection \( \Gamma^a_{bc} \) provides the second order field equations

\[ f'(R)\mathcal{R}_{(ab)} - \frac{1}{2} f(R) g_{ab} = \kappa G T_{ab}, \tag{1.6} \]

\[ \nabla_c (\sqrt{-g} f'(R) g^{ab}) = 0, \tag{1.7} \]

respectively, where \( \nabla_a \) denotes the covariant derivative operator of \( \Gamma^a_{bc} \). Little is still known about a third version (metric-affine gravity), in which the matter action is allowed to depend explicitly on the (non-metric) connection \( \Gamma^a_{bc} \), and which will not be considered here.

## 2 The EGS theorem in \( f(R) \) gravity

Since the possibility that our present universe is described by some modification of general relativity is now taken rather seriously, and \( f(R) \) gravity is at least a convenient toy model (if not a serious candidate), it is natural to ask whether the basic result that allows one to identify the observed universe with a FLRW space, the EGS theorem, survives in these theories. The first investigations \[26] [7] gave an affirmative answer for the theory

\[ S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left( R + \alpha R^2 + \beta R_{ab} R^{ab} \right) + S^m \tag{2.1} \]
in the metric formalism\footnote{Due to the fact that the Gauss-Bonnet expression $R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}$ gives rise to a topological invariant, it is unnecessary to include Riemann-squared terms in the action (2.1).} subject to the additional condition (which does not appear in general relativity) that the perfect fluid filling the universe obeys a barotropic equation of state $P = P(\rho)$ with $dP/d\rho \neq 0$, which implies that surfaces of constant $P$ and surfaces of constant $\rho$ coincide. Subsequently, the validity of the EGS theorem was extended to general metric $f(R)$ gravity in Ref. [27]. Here we extend this result to Palatini modified gravity, and we provide an independent proof also for the metric version of these theories. The result is straightforward because it builds on the results of Ref. [28] that extend the validity of the EGS theorem to scalar-tensor theories of gravity described by the action

\[ S_{ST} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[ \psi R - \frac{\omega(\psi)}{\psi} \nabla^c \psi \nabla_c \psi - V(\psi) \right] + S^m, \tag{2.2} \]

where the coupling function $\omega(\psi)$ generalizes the constant Brans-Dicke parameter [30]. Now, it is well-known that metric or Palatini $f(R)$ gravity can be seen as a Brans-Dicke theory [31, 32]. In the metric formalism, with the introduction of an extra field $\phi$, the action (1.3) can be rewritten as

\[ S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[ \psi(\phi) R - V(\phi) \right] + S^m \tag{2.3} \]

when $f''(R) \neq 0$, where $\phi$ is defined by

\[ \psi(\phi) = f'(\phi) \tag{2.4} \]

and

\[ V(\phi) = \phi f'(\phi) - f(\phi). \tag{2.5} \]

The action (2.3) reduces to (1.3) trivially if $\phi = R$ and, vice-versa, variation of (2.3) with respect to $\phi$ yields

\[ (R - \phi) f''(R) = 0, \tag{2.6} \]

which implies that $\phi = R$ if $f'' \neq 0$. The action can now be seen as a Brans-Dicke action with Brans-Dicke parameter $\omega = 0$ if the field $\psi \equiv f'(\phi) = f'(R)$ is used instead of $\phi$ as the independent Brans-Dicke-like field:

\[ S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} [\psi R - U(\psi)] + S^m, \tag{2.7} \]

where

\[ U(\psi) = V(\phi(\psi)) = \psi \phi(\psi) - f(\phi(\psi)) \tag{2.8} \]
(this is called “O’Hanlon theory” or “massive dilaton gravity” \[33\] \[32\]).

In the Palatini formalism, by introducing the metric \( h_{ab} \equiv f'(\tilde{R}) g_{ab} \) conformally related to \( g_{ab} \) and the scalar \( \phi \equiv f'(\mathcal{R}) \), and using the transformation property of the Ricci scalar \( \mathcal{R} \), one obtains

\[
\mathcal{R} = R + \frac{3}{2\phi} \nabla^c \phi \nabla_c \phi - \frac{3}{2} \Box \phi ,
\]

(2.9)

and the action is equivalent to

\[
S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} [f(\chi) + f'(\chi) (\mathcal{R} - \chi)] + S^{(m)}
\]

(2.10)

if \( f'' \neq 0 \) with \( \chi = \tilde{R} \). By redefining \( \chi \) through \( \phi = f'(\chi) \), it is

\[
f(\chi) + f'(\chi) (\mathcal{R} - \chi) = \phi \mathcal{R} - \phi \chi(\phi) + f(\chi(\phi)) = \phi \mathcal{R} + \frac{3}{2} \nabla^c \phi \nabla_c \phi - V(\phi) - 3\Box \phi
\]

(2.11)

using eq. (2.9), where

\[
V(\phi) = \phi \chi(\phi) - f(\chi(\phi)).
\]

(2.12)

Apart from a boundary term, this yields

\[
S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[ \phi \mathcal{R} + \frac{3}{2\phi} \nabla^c \phi \nabla_c \phi - V(\phi) \right] + S^{(m)}
\]

(2.13)

which describes a Brans-Dicke theory with parameter \( \omega = -3/2 \), seldom considered in the literature \[34\] until the recent attempts to model the present-day cosmic acceleration.

Now, since the EGS theorem has been proved to hold for scalar-tensor gravity \[28\], it is straightforward to conclude that its validity is extended to metric modified gravity. This quick proof is consistent with the results obtained in \[27\] for general metric \( f(R) \) gravity with a more direct approach, and with the findings of \[26\] \[7\] for quadratic \( f(R) \).

The validity of the EGS theorem is then extended to Palatini \( f(R) \) gravity, which was not considered before in the EGS context. This is not entirely trivial when one considers the different order of the field equations with respect to metric \( f(R) \) theories (second order instead of fourth), and the different physics described.

3 Conclusions

The equivalence between metric and Palatini \( f(R) \) theories and \( \omega = 0, -3/2 \) Brans-Dicke theories allows for a straightforward proof of the EGS theorem for modified gravity, which
relies on the previous work [28] extending the validity of this theorem to scalar-tensor gravity. By contrast, the direct approach of [27] appears a bit cumbersome.

In addition to providing a different approach to the EGS theorem for metric $f(R)$ gravity, we provide a straightforward proof of its validity for Palatini $f(R)$ gravity, which was not considered before in this context. Although evidence is now accumulating that Palatini modified gravity is not physically viable for various reasons (see [35, 36, 37, 38, 1] for a discussion of the various theoretical aspects involved), it may still be useful as a toy model to analyze mathematical and physical features of generalized gravity theories.

Given the fact that the EGS theorem extends to Lagrangian densities of the form $\mathcal{L} = R + \alpha R^2 + \beta R_{ab}R^{ab} + \mathcal{L}^m$ [26, 7], one wonders whether it is actually valid for more general theories of the form $\mathcal{L} = f(R, R_{ab}R^{ab}, R_{abcd}R^{abcd})$, which are motivated by low-energy string corrections to general relativity and are not equivalent to a simple scalar-tensor theory. This possibility will be examined elsewhere.

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