2+1 quantum gravity with a Barbero–Immirzi-like parameter on toric spatial foliation

Rudranil Basu and Samir K Paul

S. N. Bose National Centre for Basic Sciences, JD Block, Sector III, Salt Lake City, Kolkata 700098, India
E-mail: rudranil@bose.res.in and smr@bose.res.in

Received 7 October 2009, in final form 15 March 2010
Published 26 April 2010
Online at stacks.iop.org/CQG/27/125003

Abstract

We consider gravity in 2+1 spacetime dimensions, with a negative cosmological constant and a ‘Barbero–Immirzi’ (B-I)-like parameter, when the spacetime topology is of the form $T^2 \times \mathbb{R}$. The phase space structure, both in covariant and canonical framework, is analyzed. Full quantization of the theory in the ‘constrain first’ approach reveals a finite-dimensional physical Hilbert space. An explicit construction of wavefunctions is presented. The dimension of the Hilbert space is found to depend on the ‘Barbero–Immirzi’-like parameter in an interesting fashion. A comparative study of this parameter in light of some of the recent findings in the literature for similar theories is presented.

PACS numbers: 04.50.Kd, 04.60.−m, 11.15.Yc

1. Introduction

Quantum gravity in 2+1 dimensions has been an object of serious research for quite some time. A very special feature of this theory is that the theory of 2+1 gravity in first-order formalism with/without a cosmological constant can be written as a gauge theory [1]. Although there is no propagating mode in this theory [2], it is striking that it admits of a conformal field theory (CFT) at the boundary when the theory is considered in an asymptotically AdS spacetime [3]. On the other hand, the idea of incorporating local degrees of freedom (gravitons) is quite old [4] and recently there has been proliferations through newer avenues (new massive gravity) without [5] or (cosmological new massive gravity) with a cosmological constant [6], which we will hereafter refer collectively as topologically massive gravity (TMG). Gravity in 2+1 spacetime even without graviton modes took an interesting turn after existence of black hole solutions was ensured [7]. Subsequent important works in the context of AdS/CFT correspondence [8, 9] warrant the importance of this model.

Even if one restricts oneself with 2+1 gravity models without propagating degrees of freedom, quantization of the theory poses a non-trivial problem in its own right in the sense
that one has to study this problem by quantizing the phase space keeping in mind that topology of the spacetime would play an important role in deciding the quantum theory. If the phase space is finite dimensional, one can do away with problems regarding renormalizability even in the non-perturbative regime [1, 10]. In this work, we deal with 2+1 gravity (with a negative cosmological constant) described by vielbeins (triads) and \(SO(2, 1)\) connections on a (pseudo)Riemann–Cartan manifold which is not asymptotically AdS, and aim to compare results with asymptotically AdS calculations already available in the literature. The bulk theory of 2+1 gravity, which will not be taken as TMG in our case, can be expressed as Chern–Simons theory with non-compact gauge groups, choice of the group being strictly determined by the cosmological constant [1]. Whereas 2+1 Chern–Simons theory with a compact gauge group (in the interest of gravity) can give rise to significant generalizations of these constructions [13]. That there is only global degree of freedom can be seen both in the metric formulation and in the triad-connection formulation of general relativity. All the local degrees of freedom are frozen when the constraints, all of which in this case (namely the Gauss, the Hamiltonian and the diffeomorphism) are first class, are imposed. The global degrees of freedom turn out to be finite in number if there is no topologically non-trivial boundary in the spacetime [14].

In the case of a negative cosmological constant, the Chern–Simons action corresponding to 2+1 gravity (hereafter referred as CSG) can be written as an \(SO(2, 1) \times SO(2, 1)\) gauge theory [1], which is purely topological as opposed to TMGs. The topology of the physical phase space of the theory being the nontrivial one has to take recourse to geometric quantization [15]. This formulation of quantization in the ‘constrain first’ approach was studied for an \(SL(2,\mathbb{R})\) Chern–Simons theory with rational charges [16] where a finite-dimensional Hilbert space was constructed on the almost torus part of the physical phase space and it was argued that the Hilbert space on the total phase space would be finite dimensional. Spatial slice in this case was chosen to be a torus. General quantization procedure of Chern–Simons theories in the realm of geometric quantization was exhaustively studied in [17].

More generalized versions of CSG retaining its topological nature came into prominence through the works of Mielke et al [18, 19]. In the present work we consider a special case of such generalized CSG [20] with a negative cosmological constant and a new parameter which imitates the Barbero Immirzi parameter of 3+1 gravity [21]; the possibility of this generalization also was hinted in the pioneering work [1]. The gauge group for the corresponding CSG still remains \(SO(2, 1) \times SO(2, 1)\). We discuss the quantization in the ‘constrain first’ approach where one first solves the classical constraints and then attempts to quantize the resulting phase space of gauge-invariant variables. It was revealed in [22] that this approach fails to incorporate the ‘shift’ in the central charge of the current algebra of the Wess–Zumino–Witten conformal field theory. Nevertheless, as explained in [16] that the above difficulty is overcome as one of the inequivalent Hilbert spaces has exactly the unitary structure of the vector space of the current blocks of the Wess–Zumino–Witten theory.

We also wish to point out that in a later work [23], an explicit parameterization of the physical phase space for CSG on toric spatial foliation and negative cosmological constant was done. There, in contrast to geometric quantization, the phase space was modified to a suitable cotangent bundle by a surgery of the non-trivial phase space and trivializing its topology. Conventional procedure of canonical quantization was carried out in that modified phase space. No comment however on the dimensionality of the Hilbert space was made.

In section 2 we construct the phase space for CSG with the negative cosmological constant and the Barbero–Immirzi-like parameter \(\gamma\). We note that the original CSG action with a negative cosmological constant can be written as a sum of two \(SO(2, 1)\) Chern–Simons
actions with equal and opposite signed levels. The difference in the present case follows by adding a different Lagrangian (with arbitrary coefficient) to the original one, which also gives the same equation of motion. Due to this modification, the Chern–Simons levels (for each of the two $SO(2, 1)$ sectors) carrying the footprints of both the negative cosmological constant and the new B-I-like parameter can be tuned. In the analysis (of similar TMGs) pertaining to asymptotically AdS calculations [24, 25], this freedom is crucial, in the sense that one of the Chern–Simons levels can be tuned to zero, which corresponds to the ‘chiral limit’ of the dual CFT. In our present work, we also try to make sense of this limit. The physical phase space is the moduli space of flat gauge connections, modulo gauge transformations on our choice of spatial foliation, which is genus-1 Riemann surface. It turns out to be a torus punctured at a point (may be chosen to be the origin) with a plane also punctured at a point (also chosen to be the origin of the plane) and glued to the torus through a closed curve ($S^1$) around the origin (common puncture), the plane being $Z_2$ folded through the origin.

In section 3 we discuss the geometric quantization of the phase space [15]. A complete basis for the physical Hilbert space is constructed in terms of theta functions. Note that due to the introduction of this new parameter only, both the Chern–Simons levels can be adjusted to be positive and rational. During quantization this becomes important since the dimensionality of the Hilbert space of the quantized theory is directly related to these levels. The corresponding charge in the CSG is no longer an integer owing to the fact that Weil’s integrality condition on the Chern–Simons charge disappears as a consequence of the noncompactness of the gauge group which in our case is $SO(2, 1)$ [12, 22]. A discussion on the restrictions on physical parameters coming from the quantization is also presented and compared with those from [10].

2. Phase space of 2+1 gravity with an Immirzi-like parameter

In this section we will demonstrate the classical covariant phase space (actually a pre-symplectic manifold) [26] of 2+1 gravity with a negative cosmological constant and an Immirzi-like parameter. The canonical phase space (non-covariant), the Hamiltonian structure, the constraint analysis and gauges relevant to the theory will also be discussed.

2.1. Holst-like 2+1 gravity as a Chern–Simons theory

The action for 2+1 gravity with a negative cosmological constant $\Lambda = -\frac{1}{l^2}$ on a spacetime manifold $M$ in first-order formalism is

$$I_{GR} = \frac{1}{8\pi G} \int_M e^I \wedge \left( 2d\omega_I + \epsilon^{IJK} \omega_J \wedge \omega_K + \frac{1}{3l^2} \epsilon^{IJK} e_J \wedge e_K \right),$$

(1)

where $e^I$ are the $SO(2, 1)$ orthonormal triad frame and the $\omega^I$ are connections (or canonically projected local connection) of the frame bundle with the structure group $SO(2, 1)$. The above action is well defined and differentiable in the absence of boundaries. Although in the presence of a boundary (internal and/or asymptotic) [27], one has to add suitable boundary terms to the action in order to have a finite action with well-defined (differentiable) variation.

The equations of motion for spacetimes without a boundary are

$$F_I := 2d\omega_I + \epsilon^{IJK} \omega_J \wedge \omega_K = -\frac{1}{l^2} \epsilon^{IJK} e_J \wedge e_K$$

(2)

$$T_I := de_I + \epsilon_{IJK} e_J \wedge \omega^K = 0.$$  

(3)

Note that this theory describes gravitational interaction as long as the triad system $e^I_a$ is invertible ($a, b, \ldots$ Latin indices are abstract spacetime indices). The connection 1-forms $\omega^I_a$
qualify as the spin-connections if (3) is satisfied and the spacetime becomes (pseudo)Riemann manifold as opposed to the initial structure of a (pseudo)Riemann–Cartan manifold.

A more general model for 2+1 gravity with a negative cosmological constant was introduced by Mielke et al [18, 19] and later studied extensively in [20, 28, 29], which without matter fields read

\[ I = a I_1 + b I_2 + \alpha_3 I_3 + \alpha_4 I_4, \]

where

\[ I_1 = \int_M e^l \wedge (2d\omega_l + \epsilon^{JK}_l \omega_J \wedge \omega_K) \]
\[ I_2 = \int_M \epsilon^{JK}_l e^l \wedge e_J \wedge e_K \]
\[ I_3 = \int_M \omega^l \wedge d\omega_l + \frac{1}{3} \epsilon^{JK}_l \omega^l \wedge \omega_J \wedge \omega_K \]
\[ I_4 = \int_M e^l \wedge de_l + \epsilon^{JK}_l \omega^l \wedge e_J \wedge e_K. \]

However, this model does not reproduce the equations of motion (2) and (3) for arbitrary values of the parameters \( a, b, \alpha_3, \alpha_4 \). We choose, as a special case of the above model, those values of these parameters which give the expected equations of motion as in [1, 10, 20, 30–38]:

\[ a = \frac{1}{8 \pi G}, \quad b = \frac{1}{24 \pi G l^2}, \quad \alpha_3 = \frac{l}{8 \pi G \gamma}, \quad \alpha_4 = \frac{1}{8 \pi G \gamma l}, \]

where \( \gamma \) is introduced as a new dimensionless parameter from 2+1 gravity perspective. Effectively (5) is the equation of a three-dimensional hypersurface parameterized by \( G, l, \gamma \) in the 4D parameter space of \( a, b, \alpha_3, \alpha_4 \).

It calls for a little digression for the Chern–Simons formulation. Following [1, 20] one introduces the \( SO(2, 1) \) or equivalently \( SL(2, \mathbb{R}) \) or \( SU(1, 1) \) connections for a principal bundle over the same base space of the frame bundle:

\[ A^{(\pm)}_I := \omega^I \pm \frac{e^I}{T}. \]

It is easily verifiable that the action

\[ \tilde{I} = \frac{l}{16\pi G} (I^{(+)} - I^{(-)}) \]

is the same as (1) in the absence of boundaries, where

\[ I^{(\pm)} = \int_M \left( A^{(\pm)}_I \wedge dA^{(\pm)}_I + \frac{1}{3} \epsilon^{JK}_l A^{(\pm)}_I \wedge A^{(\pm)}_J \wedge A^{(\pm)}_K \right) \]

are two Chern–Simons actions with the gauge group \( SO(2, 1) \), the lie algebras being given by

\[ \left[ J^{(+)}_I, J^{(+)}_J \right] = \epsilon_{JK} J^{(+)}_K \]
\[ \left[ J^{(-)}_I, J^{(-)}_J \right] = \epsilon_{JK} J^{(-)}_K \]
\[ \left[ J^{(+)}_I, J^{(-)}_J \right] = 0. \]

The metric on the Lie algebra is chosen to be

\[ \langle J^{(\pm)}_I, J^{(\pm)}_J \rangle = \frac{1}{2} \eta^{IJ}, \]

where \( J^{(\pm)}_I \) span the \( SO(2, 1) \) (or \( SL(2, \mathbb{R}) \) or \( SU(1, 1) \)) Lie algebras for the two theories.
One striking feature of this formulation is that the last two terms of (4) can also be incorporated in terms of \( A(\pm) \), for \( \alpha_3 = I^d \alpha_d \) as

\[
I^{(+)} + I^{(-)} = 2 \int_M \left( \omega^I \wedge d\omega_I + \frac{1}{4\pi G} \epsilon_{IJK} \omega^I \wedge \omega^J \wedge \omega^K + \frac{1}{3} \epsilon_{IJK} \omega^I \wedge e^J \wedge e^K \right)
\]

and the same equations of motion (2) and (3) are also found from varying this action.

We thus propose the action

\[
I = \frac{1}{16\pi G} (I^{(+)} - I^{(-)}) + \frac{1}{16\pi G \gamma} (I^{(+)} + I^{(-)})
\]

with a dimensionless non-zero coupling \( \gamma \). This action (9) upon variations with respect to \( A^{(\pm)} \) and \( A^{(\mp)} \) gives equations of motion as expected from Chern–Simons theories. This implies that the connections \( A^{(\pm)} \) are flat:

\[
F^{(\pm)} = dA^{(\pm)} + \epsilon_{IJK} A^{(\pm)I} \wedge A^{(\pm)J} = 0.
\]

(10)

It is also easy to check that the above flatness conditions of these \( SO(2, 1) \) bundles (10) are equivalent to the equations of motion of general relativity (2), (3).

This is a good point to stop and probe into the physical relevance of this new parameter compared with (3+1)-dimensional gravity. In order to proceed, we note that the new action is in spirit very much like the Holst action [21] used in 3+1 gravity. In our case, the parameter \( \gamma \) can superficially be thought of as being the (2+1)-dimensional counterpart of the original Barbero–Immirzi parameter. Moreover, the part \( I^{(+)} + I^{(-)} \) of the action in this light qualifies to be at par with the topological (non-dynamical) term one adds with the usual Hilbert–Palatini action in 3+1 dimensions, since this term we added (being equal to a Chern–Simons action for spacetimes we consider) is also non-dynamical. But more importantly, the contrast is in the fact that the original action, which is dynamical in the 3+1 case, is also non-dynamical here, when one considers local degrees of freedom only.

Another striking contrast between the original B-I parameter and the present one lies in the fact that in the 3+1 scenario \( \gamma \) parameterizes canonical transformations in the phase space of general relativity. From the canonical pair of the \( SU(2) \) triad (time gauge fixed and on a spatial slice) and spin-connection one goes on finding an infinitely large set of pairs parameterized by \( \gamma \). The connection is actually affected by this canonical transformation, and this whole set of parameterized connections is popularly known as the Barbero–Immirzi connection. The fact that this parameter induces canonical transformation can be checked by seeing that the symplectic structure remains invariant under the transformation on-shell. On the other hand for the case at hand, i.e. 2+1 gravity, as we will see in the following sub-section, inclusion of finite \( \gamma \) is not a canonical transformation and it does not keep the symplectic structure invariant.

2.2. Symplectic structure on the covariant phase space

Consider a globally hyperbolic spacetime manifold endowed neither with an internal nor an asymptotic boundary and let it allow foliations \( M \equiv \Sigma \times \mathbb{R} \), with \( \Sigma \) being compact and \( \partial \Sigma = 0 \).

1. On-shell (3) implies the spacetime can be the given (pseudo) Riemannian structure. With respect to the associated metric (0, 2)-tensor and a time-like vector field \( \ell \) the manifold is assumed to be Cauchy-foliated.
In view of [26] the covariant phase space, i.e. the space of solutions of the equations of motion, the theory is $V_F^{(+)} \times V_F^{(-)}$, product of spaces of flat $SO(2,1)$ connections as discussed in the last section. We now intend to find the pre-symplectic structure. For that purpose, we start with the Lagrangian 3-form that gives the above action:

$$L = \frac{l}{16\pi G} (1/\gamma + 1) \left( A^{(s)I} \wedge dA^{(s)I} + \frac{1}{3} \epsilon_{IJK} A^{(s)I} \wedge A^{(s)J} \wedge A^{(s)K} \right) + \frac{l}{16\pi G} (1/\gamma - 1) \left( A^{(-)I} \wedge dA^{(-)I} + \frac{1}{3} \epsilon_{IJK} A^{(-)I} \wedge A^{(-)J} \wedge A^{(-)K} \right).$$ (11)

The standard variation gives on-shell

$$\delta L = \frac{d\Theta(\delta)}{\Theta_1(\delta)} = \frac{d\Theta^{(s)}(\delta)}{\Theta_1(\delta)} + \frac{d\Theta^{(-)}(\delta)}{\Theta_1(\delta)},$$

where $(16\pi G/l) / \Theta_1(\pm)(\delta) = (1/\gamma \pm 1) \delta A^{(\pm)I} \wedge A^{(\pm)I}$. The procedure of second variations [26] then gives the pre-symplectic current

$$J(\delta_1, \delta_2) = J^{(s)}(\delta_1, \delta_2) + J^{(-)}(\delta_1, \delta_2),$$

where

$$J^{(\pm)}(\delta_1, \delta_2) = 2\delta_1(\Theta^{(\pm)}(\delta_2)).$$

which is a closed 2-form ($dJ(\delta_1, \delta_2) = 0$) on-shell. The closure of $J$ and the fact that we are considering spacetime manifolds which allow closed Cauchy foliations imply that the integral $\int_{\Sigma} J(\delta_1, \delta_2)$ is actually foliation independent, i.e. independent of the choice of $\Sigma$. Hence the expression $\int_{\Sigma} J(\delta_1, \delta_2)$ is manifestly covariant and qualifies as the pre-symplectic structure on $V_F^{(+)} \times V_F^{(-)}$. We thus define

$$\Omega = \Omega^{(s)} + \Omega^{(-)},$$ (12)

where

$$\Omega^{(\pm)}(\delta_1, \delta_2) = \int_{\Sigma} J^{(\pm)}(\delta_1, \delta_2) = \frac{l}{8\pi G} (1/\gamma \pm 1) \int_{\Sigma} \delta_1 A^{(\pm)I} \wedge \delta_2 A^{(\pm)I}.$$ (13)

At this point we would like to note two important features of this symplectic structure.

- In the 3+1 case the extra contribution of the Holst term (with coefficient $1/\gamma$) in the symplectic structure can be shown to vanish on-shell. Hence it is guaranteed that in the covariant phase space, $\gamma$ has the role of inducing canonical transformations. On the other hand, in the present case, it is very much clear from the above expression that the $\gamma$-dependent term cannot vanish, as suggested in the previous subsection. So what we have at hand are infinite inequivalent theories for 2+1 gravity each having different canonical structures and parameterized by different values of $\gamma$ at the classical level itself.

- The other point worth noticing is that $\Omega$ is indeed gauge invariant and it can be checked by choosing one of the two $\delta$s to produce infinitesimal $SO(2,1)$ gauge transformations or infinitesimal diffeomorphisms and keeping the other arbitrary. In both these cases $\Omega(\delta_{\delta_{SO(2,1)}}, \delta)$ and $\Omega(\delta_{\text{diffeo}}, \delta)$ vanish on the constraint surface, recognizing these two classes of vectors in the covariant phase space as the ‘gauge’ directions.

---

2 It is being called the pre-symplectic structure since as we will point out later that only on the constraint surfaces this has the property to be gauge invariant. When we have a phase space parameterized by gauge-invariant variables, this pre-symplectic structure will induce a symplectic structure on that.
2.3. Canonical phase space

From the above covariant symplectic structure one can instantly read off the following canonical equal-time (functions designating the foliations as level surfaces) Poisson brackets:

\[
\{ A_{i}^{(\pm)}(x, t), A_{j}^{(\pm)}(y, t) \} = \frac{8\pi G/\gamma}{\gamma^2 - 1} \epsilon_{ij} \eta^{IJ} \delta^2(x, y),
\]

(14)

where \( \epsilon_{ij} \) is the usual alternating symbol on \( \Sigma \).

Interestingly in terms of the Palatini variables, the above Poisson bracket reads

\[
\{ \omega_{i}^{0}(x, t), e_{j}^{0}(y, t) \} = 4\pi G \frac{\gamma}{\gamma^2 - 1} \epsilon_{ij} \eta^{IJ} \delta^2(x, y)
\]

\[
\{ \omega_{i}^{0}(x, t), \omega_{j}^{0}(y, t) \} = -4\pi G \frac{\gamma/\gamma}{\gamma^2 - 1} \epsilon_{ij} \eta^{IJ} \delta^2(x, y)
\]

\[
\{ e_{i}^{0}(x, t), e_{j}^{0}(y, t) \} = -4\pi G \frac{\gamma/\gamma}{\gamma^2 - 1} \epsilon_{ij} \eta^{IJ} \delta^2(x, y).
\]

(15)

As expected in the limit \( \gamma \to \infty \), the Poisson brackets reduce to those of usual Palatini theory:

\[
\{ \omega_{i}^{0}(x, t), e_{j}^{0}(y, t) \} = 4\pi G \epsilon_{ij} \eta^{IJ} \delta^2(x, y)
\]

\[
\{ \omega_{i}^{0}(x, t), \omega_{j}^{0}(y, t) \} = 0
\]

\[
\{ e_{i}^{0}(x, t), e_{j}^{0}(y, t) \} = 0.
\]

(16)

Here we wish to concentrate on the Hamiltonian and the constraint structure of the theory.

In terms of the Chern–Simons gauge fields, these are the \( SO(2, 1) \) Gauss constraints as illustrated below. The Legendre transformation is done by spacetime splitting of the action \( I \) given by (9):

\[
\frac{16\pi G}{l} I = \frac{1}{\gamma + 1} \int_{\Sigma} d^{2}x \epsilon^{ij} \left( - \frac{1}{4} A_{i}^{(\pm)} \partial_{0} A_{j}^{(\pm)} + 2 A_{0}^{(\pm)} \partial_{i} A_{j}^{(\pm)} + \epsilon^{JK} A_{0i}^{(\pm)} A_{j}^{(\pm)} A_{jk}^{(\pm)} \right)
\]

\[
+ \frac{1}{\gamma - 1} \int_{\Sigma} d^{2}x \epsilon^{ij} \left( - \frac{1}{4} A_{i}^{(-)} \partial_{0} A_{j}^{(-)} + 2 A_{0}^{(-)} \partial_{i} A_{j}^{(-)} + \epsilon^{JK} A_{0i}^{(-)} A_{j}^{(-)} A_{jk}^{(-)} \right).
\]

(17)

First terms in the integrands are kinetic terms and from them one can again extract (14). The Hamiltonian is given by

\[
\mathcal{H} = \mathcal{H}^{(\pm)} + \mathcal{H}^{(-)}
\]

where

\[
\mathcal{H}^{(\pm)} = \frac{l}{16\pi G} \left( \frac{1}{\gamma + 1} \epsilon^{ij} \left( 2 A_{0i}^{(\pm)} \partial_{j} A_{j}^{(\pm)} + \epsilon^{JK} A_{0i}^{(\pm)} A_{j}^{(\pm)} A_{jk}^{(\pm)} \right) \right)
\]

The fields \( A_{0j}^{(\pm)} \) are the Lagrange multipliers and we immediately have the primary constraints

\[
\mathcal{G}_{j}^{(\pm)} = \frac{l}{16\pi G} \left( \frac{1}{\gamma + 1} \epsilon^{ij} \left( \partial_{i} A_{j}^{(\pm)} + \frac{1}{2} \epsilon_{ij}^{\alpha\beta} A_{0i}^{(\pm)} A_{j}^{(\pm)} \right) \right) \approx 0.
\]

(18)

Since \( \mathcal{H}^{(\pm)} = A_{0j}^{(\pm)} \mathcal{G}_{j}^{(\pm)} \approx 0 \) the Hamiltonian is therefore weakly zero. Again the primary constraint being proportional to the Hamiltonian, there are no more secondary constraints in the theory according to Dirac. Now consider the smeared constraint

\[
\mathcal{G}^{(\pm)}(\lambda) = \int_{\Sigma} d^{2}x \lambda^{l} \mathcal{G}_{j}^{(\pm)}
\]
for some $\lambda = \lambda^I J^I \in \mathfrak{so}(2, 1)$ in the internal space. It now follows that this smeared constraints close among themselves:

$$\{G^{(\pm)}(\lambda), G^{(\pm)}(\lambda')\} = G^{(\pm)}([\lambda, \lambda'])$$

and the $SO(2, 1)$ Lie algebra is exactly implemented on the canonical phase space. Hence clearly these are the ‘Gauss’ constraints generating $SO(2, 1)$ gauge transformations separately for the $(+)$-type and the $(-)$-type gauge fields. The closure of these constraints, on the other hand, means that these are first class and there are no second class constraints. A close look at (18) reveals that this constraint is nothing but vanishing of the gauge field curvature (10) when pulled back to $\Sigma$. The temporal component $A^0_I$ is non-dynamical, being just a Lagrange multiplier. Hence all the dynamics of the theory determined by (10) is constrained as (18). Hence there is no local physical degree of freedom in the theory, which is related to the justified recognition of Chern–Simons theories as ‘topological’. We now wish to probe into implications of this constraint structure in the gravity side, the $\gamma \to \infty$ case of which was discussed by various authors, e.g. [1].

The Legendre transformation now is carried out through the spacetime split action (17) in terms of the variables pertaining relevance to the gravity counterpart of the theory as

$$\frac{16\pi G}{l} I = -2 \int_R dt \int_\Sigma d^2x \epsilon^{ij} \left[ \frac{1}{\gamma} \left( \frac{e_I^i}{\gamma} \partial_0 \partial_0 e_I^j + \frac{1}{\gamma^2} \epsilon_I^{jk} \partial_j e_K^j \right) + \frac{2}{l} \omega_I^{ij} \partial_j e^j_I \right]$$

kinetic terms

$$+ 4 \int_R dt \int_\Sigma d^2x \epsilon^{ij} \left[ \left( \frac{e_I^i}{\gamma} + \frac{1}{\gamma^2} e_0^i \right) \left( \partial_I e_J^j + \epsilon_I^{jk} \omega_J^j \right) + \left( \frac{1}{\gamma} \omega_I^{ij} \partial_0 e^j_I + \frac{1}{2} \epsilon_I^{jk} \left( \omega_J^j \omega_K^k + \frac{1}{l} \epsilon_K^{il} \epsilon_l^j \right) \right) \right].$$

(20)

One then envisages the part save the kinetic part as the Hamiltonian in the units of $\frac{l}{16\pi G}$ and $(\omega_0^I + \frac{1}{\gamma} e_0^I), \left( \frac{1}{\gamma} \omega_0^I + \frac{1}{\gamma^2} e_0^I \right)$ as the Lagrange multipliers with the following as the constraints, after suitable rescaling:

$$P^I := \frac{1 - \gamma^2}{8\pi G^2} \epsilon^{ij} \left( \partial_I e^j_I + \epsilon_I^{jk} \omega_I^j e^K_j \right) \approx 0$$

$$S^I := \frac{l(1 - \gamma^2)}{8\pi G^2} \epsilon^{ij} \left( \partial_I \omega^j_I + \frac{1}{2} \epsilon_I^{jk} \left( \omega_J^j \omega_K^k + \frac{1}{l} \epsilon_K^{il} \epsilon_l^j \right) \right) \approx 0.$$  

(21)

Let us define their smeared versions as

$$P(\lambda) := \int_\Sigma d^2x \lambda^I P_I \quad \text{and} \quad S(\lambda) := \int_\Sigma d^2x \lambda^I S_I$$

for $\lambda \in \mathfrak{so}(2, 1)$. One can also check the expected closure of the constraint algebra of $S$ and $P$ which guarantees their first-class nature:

$$\{S(\lambda), S(\lambda')\} = \gamma^{-1} S([\lambda, \lambda']) - P([\lambda, \lambda'])$$

$$\{S(\lambda), P(\lambda')\} = -S([\lambda, \lambda']) + \gamma^{-1} P([\lambda, \lambda'])$$

$$\{P(\lambda), P(\lambda')\} = \gamma^{-1} S([\lambda, \lambda']) - P([\lambda, \lambda']).$$  

(22)

3 A rescaling with the factor $\frac{1 - \gamma^2}{\gamma^2}$ is done in order to avoid apparent divergences in the constraint algebra at the points $\gamma \to \pm 1$. 

8
Linear combinations of $\omega^I_0$ and $e^I_0$ are Lagrange multipliers and hence these fields themselves are nondynamical. We thus infer that all the dynamical informations through the equations of motion (2), (3) are encoded in the constraints (21). In the limit $\gamma \to \infty$, as e.g. in [1], $P$ generates a local Lorentz, i.e. $SO(2, 1)$ Lorentz transformations and $S$ generates diffeomorphisms for the frame variables. Since in the finite $\gamma$ case too these are first class, one should expect them to generate some gauge transformations. To see changes brought in by the modified symplectic structure we first compute the transformations induced by these constraints:

\[
\{e^I_i(x, t), P(\lambda)\} = -\frac{l}{2} \left[ \gamma^{-1} \left( \frac{\partial_i \lambda^I + \epsilon^{IJK} \omega^j_0 \omega^K_0}{\partial_i \lambda^J} \right) + \frac{l}{l} \epsilon^{IJK} \lambda_J e^K_i \right], \quad (23)
\]

\[
\{\omega^I_i(x, t), P(\lambda)\} = \frac{1}{2} \left[ D_i \lambda^I + \frac{1}{l} \epsilon^{IJK} \lambda_J e^K_i \right]. \quad (24)
\]

The infinitesimal local $SO(2, 1)$ Lorentz transformations, i.e. $e \to e + \lambda \times e$, $\omega \to \omega + \partial_\lambda + \lambda \times \omega$ are seen to be successfully generated by $P(\lambda)$ in the limit $\gamma \to \infty$. But for finite $\gamma$, the transformations are deformed in a sense that infinitesimal diffeomorphisms are also generated along with Lorentz transformations. Similarly the Lie transports generated by the diffeomorphism generator $S$ are also deformed due to the modified symplectic structure as

\[
\{e^I_i(x, t), S(\lambda)\} = \frac{l}{2} \left[ D_i \lambda^I + \frac{1}{l} \epsilon^{IJK} \lambda_J e^K_i \right], \quad (25)
\]

\[
\{\omega^I_i(x, t), S(\lambda)\} = -\frac{1}{2} \left[ \gamma^{-1} D_i \lambda^I + \frac{1}{l} \epsilon^{IJK} \lambda_J e^K_i \right]. \quad (26)
\]

In this case we also note that the usual diffeomorphism generator is generating local Lorentz transformations for finite $\gamma$. We can find suitable linear combinations of these two generators which separately and purely generate local Lorentz and diffeomorphisms.

The striking difference between roles of the original 3+1 Barbero–Immirzi parameter and the present $\gamma$ can again be envisaged in terms of the usual ADM canonical pairs: the spatial metric $h_{ij}$ and the dual momentum $\pi_{ij} = \sqrt{h} \left( K_{ij} - h_{ij} K \right)$, where $K_{ij}$ is the extrinsic curvature and $K$ is its trace. Using $h_{ij} = g_{ij} = e^I_i e^J_j$, we have

\[
[h_{ij}(x, t), h_{kl}(y, t)] = -4\pi G \frac{\gamma}{r^2 - 1} \left( \delta_{ik} h_{jl} + \delta_{ij} h_{kl} + \delta_{jk} h_{il} + \delta_{jl} h_{ik} \right) \delta^2(x, y).
\]

Similar Poisson brackets involving $\pi^{ij}$ can also be calculated, which are more cumbersome. The point we get across from this bracket is that while components of the spatial metric Poisson commute in the limit $\gamma \to \infty$, it does not do so for finite $\gamma$, unlike in the 3+1 case. That $\gamma$ does not induce canonical transformation in the ADM phase space also is clear in this context.

2.4. The singularity and its resolution at $\gamma \to 1$

As is apparent from the canonical structure, e.g. (14), (15), the canonical structure blows up at the point $\gamma \to \pm 1$. This is due to the fact that the Lagrangian (11) and the action functional (9) become independent of either of the 1-form fields $A^{(k)}$ for $\gamma \to \pm 1$. As a result the symplectic structure we have constructed (12) becomes degenerate on the space $\mathcal{V}_F^{(1)} \times \mathcal{V}_F^{(-1)}$ (leaving the gauge degeneracies apart), resulting it to be non-invertible. This is clearly the reason for blowing up of the equal time Poisson brackets (14).
In order to avoid this singularity we restrict our theory to \( \gamma \in \{ \mathbb{R}^+ - \{1\} \} \) and propose the theory (9) for gravity in 2+1 dimensions. We will see that further restriction on the range of \( \gamma \) is put by the quantum theory. The borderline case \( \gamma = 1 \) can however be dealt as follows. At the point \( \gamma = 1 \) the effective theory of 2+1 gravity, as recovered from (9) easily, is described by the single-gauge 1-form \( A_1^{(s)} \) and we consider the phase space to be only coordinatized by flat connections \( A_1^{(f)} \), i.e. \( \mathcal{Y}^{(f)}_I \) with the action functional

\[
I = \frac{l}{8\pi G} \int_M \left( A_1^{(s)I} \wedge dA_1^{(s)} + \frac{1}{3} \epsilon_{IJK} A_1^{(s)I} \wedge A_1^{(s)J} \wedge A_1^{(s)K} \right).
\]  (27)

On the space \( \mathcal{Y}^{(f)}_I \) we now have the symplectic structure

\[
\Omega(\delta_1, \delta_2) = \frac{l}{4\pi G} \int_\Sigma \delta_1 A_1^{(s)I} \wedge \delta_2 A_1^{(s)}.
\]  (28)

This gives the non-singular Poisson bracket

\[
\{A_1^{(s)I}(x, t), A_1^{(s)J}(y, t)\} = \frac{4\pi G}{l} \epsilon_{IJK} \delta_2^2(x, y).
\]  (29)

In a more generalized theory, such as cosmological topologically massive gravity dealt in the first-order formalism [39–41], one deals with the action

\[
I_{\text{CTMG}} = \frac{l}{16\pi G} \int \left[ (1/\gamma + 1) F^{(s)} + (1/\gamma - 1) F^{(-)} + \varrho^I \wedge (d\epsilon_I + \epsilon_{IJK} \varrho^J \wedge \omega^K) \right]
\]

\[
= \frac{l}{16\pi G} \int \left[ (1/\gamma + 1) F^{(s)} + (1/\gamma - 1) F^{(-)} + \frac{1}{4l} \int_M \varrho^I \wedge (dA_1^{(s)} + \epsilon_{IJK} A_1^{(s)J} \wedge A_1^{(s)K}) \right]
\]

\[
- \frac{1}{4l} \int_M \varrho^I \wedge (dA_1^{(-)} + \epsilon_{IJK} A_1^{(-)J} \wedge A_1^{(-)K}) \right],
\]  (30)

where \( \varrho \) is a new 1-form field which enhances the covariant phase space and emerges as a Lagrange multiplier. The corresponding symplectic structure is

\[
\Omega_{\text{CTMG}}(\delta_1, \delta_2) = \frac{l}{8\pi G} \int_\Sigma \left[ (1/\gamma + 1) \int_\Sigma \delta_1 A_1^{(s)I} \wedge \delta_2 A_1^{(s)} + (1/\gamma - 1) \int_\Sigma \delta_1 A_1^{(-)I} \wedge \delta_2 A_1^{(-)} - \frac{1}{2l} \int_\Sigma (\delta_1 \rho^I \wedge \delta_2 A^{(s)}) \right]
\]

\[
+ \frac{1}{2l} \int_\Sigma (\delta_1 \rho^I \wedge \delta_2 A^{(-)}) \right].
\]  (31)

In contrast to the theory we have considered, this theory does not become independent of any of the dynamical variables \( (A^{(s)}, A^{(-)}, \varrho) \) as \( \gamma \rightarrow 1 \):

\[
I_{\text{CTMG}} \bigg|_{\gamma=1} = \frac{l}{16\pi G} \int \left[ 2F^{(s)} + \frac{1}{4l} \int_M \varrho^I \wedge (dA_1^{(s)} + \epsilon_{IJK} A_1^{(s)J} \wedge A_1^{(s)K}) \right]
\]

\[
- \frac{1}{4l} \int_M \varrho^I \wedge (dA_1^{(-)} + \epsilon_{IJK} A_1^{(-)J} \wedge A_1^{(-)K}) \right],
\]  (32)

and in this limit \( \gamma \rightarrow 1 \), the symplectic structure (31) remains non-degenerate:

\[
\Omega_{\text{CTMG}} \bigg|_{\gamma=1} (\delta_1, \delta_2) = \frac{l}{8\pi G} \int_\Sigma \left[ 2 \int_\Sigma \delta_1 A_1^{(s)I} \wedge \delta_2 A_1^{(s)} - \frac{1}{2l} \int_\Sigma (\delta_1 \rho^I \wedge \delta_2 A^{(s)}) \right]
\]

\[
+ \frac{1}{2l} \int_\Sigma (\delta_1 \rho^I \wedge \delta_2 A^{(-)}) \right].
\]  (33)
On the other hand there is a price one has to pay for considering TMGs in general. The theory develops a local propagating degree of freedom (graviton) and the complete non-perturbative quantization (as we present in the next section) seems far from being a plausible aim. Progress in perturbative quantization about linearized modes in TMG although has been made and relevance of the limit $\gamma \to 1$ in this context is made clear in [24].

2.5. The physical phase space

The route we choose for quantization of the system involves eliminating the gauge redundancy inherent in the theory, i.e. finding the solution space modulo gauge transformations, in the classical level itself. For the present purpose, this approach is useful in contrast to the other one which involves quantizing all degrees of freedom and then singling out the physical state space as the solution of the equation:

$$\mathcal{M}[\Psi] = 0,$$

i.e. the kernel of the quantum version of the constraints or the master constraint (regularized suitably). For illustrations of this later path one may look up the context of quantization of diffeomorphism invariant theories of connections [42], e.g. loop quantum gravity in 3+1 dimensions [43].

The advantage of the first approach, i.e. the reduced phase space (constrained first) one is that the phase space is completely coordinatized by gauge-invariant objects; another manifestation being its finite dimensionality. Quantization of a finite-dimensional phase space may acquire non-triviality only through the topology of it, as will be illustrated in the case at hand.

Now, the physical phase space is clearly

$$(V^+ F / \sim) \times (V^- F / \sim),$$

where $\sim$ means equivalence of two flat connections which are gauge related. It is thus understood [1] that at least for the case when $\Sigma$ is compact, each of the $V^\pm F / \sim$ spaces is topologically isomorphic to the space (hom : $\pi_1(\Sigma) \rightarrow SO(2, 1) / \sim$) of homomorphisms from the first homotopy group of $\Sigma$ to the gauge group modulo gauge transformations. This isomorphism is realized (parameterized) by the holonomies of the flat connections around non-contractible loops on $\Sigma$ which serve as the homomorphism maps.

For the choice of the topology of compact $\Sigma$, one may start by choosing a general $g$-genus Riemann surface. The case $g = 0$ is trivial, and the moduli space consists of two points. For $g \geq 2$, parameterization of the phase space is highly non-trivial and topology of it is still not clear in the literature, although canonical structures on those moduli spaces have been constructed [14]. As the first non-trivial case we therefore choose the case when $\Sigma$ is a genus 1 Riemann surface $T^2$. For this torus, we know that $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$, i.e. this group is freely generated by two Abelian generators $\alpha$ and $\beta$ with the relation

$$\alpha \circ \beta = \beta \circ \alpha. \quad (34)$$

Since the connections at hand are flat, their holonomies depend only upon the homotopy class of the curve over which the holonomy is defined. For this reason, as parameterizations of the $V_F^{\pm}$, we choose the holonomies

$$h^{\pm}[\alpha] := \mathcal{P} \exp \left( \int_{\alpha} A^{\pm} \right) \quad \text{and} \quad h^{\pm}[\beta] := \mathcal{P} \exp \left( \int_{\beta} A^{\pm} \right)$$

with (34) being implemented on these $SO(2, 1)$ group-valued holonomies as

$$h^{\pm}[\alpha] h^{\pm}[\beta] = h^{\pm}[\beta] h^{\pm}[\alpha]. \quad (35)$$

Here the path-ordering $\mathcal{P}$ means ordering fields with smaller parameter of the path to the left.
As is well known, these are gauge-covariant objects although their traces, the Wilson loops, are gauge invariant. Although the classical Poisson bracket algebra of Wilson loops for arbitrary genus was exhaustively studied in [14], the phase space these loops constitute is absent. On the other hand there is another simple way of finding the gauge-invariant space especially for the case of genus 1, as outlined in [16, 23]. We will for completeness briefly give the arguments reaching the construction.

Under the gauge transformations \( A^{(\pm)} \to A^{(\pm)} = g^{-1} \left( A^{(\pm)} + d \right) g \), the holonomies transform as \( h^{(\pm)}[c] \to h^{(\pm)}[c] = \chi^{-1} h^{(\pm)}[c] \chi \) for any closed curve \( c \) and some element \( \chi \in SL(2, \mathbb{R}) \).

Again from (A.1), we know that any \( SO(2, 1) \) element is conjugate to elements in any of the Abelian subgroups: \( f_\theta \) or \( g_\xi \) or \( h_\tau \). Out of the three cases, for illustrative purposes we present the elliptic case.

Let \( h^{(\pm)}[\alpha] \) be a conjugate to an element in the elliptic class. Up to proper conjugation we can write

\[
h^{(\pm)}[\alpha] = e^{-\lambda_0 \rho^{(\pm)}},
\]

and from the discussion of appendix A with (35) we must have that

\[
h^{(\pm)}[\beta] = e^{-\lambda_0 \sigma^{(\pm)}},
\]

Hence we have \( \rho^{(\pm)} \) and \( \sigma^{(\pm)} \) with the range \((0, 2\pi)\) parameterizing a sector of the gauge-invariant phase space with topology of a torus: \( S^1 \times S^1 \simeq T^2 \).

Similarly structures of the other two sectors can also be found out. One is \((\mathbb{R}^2 \setminus \{0, 0\})/\mathbb{Z}^2\), containing an orbifold singularity and another is \( S^1 \) topologically. The total phase space is therefore the product of two identical copies of \( T^2 \cup \left( \mathbb{R}^2 \setminus \{0, 0\} \right)/\mathbb{Z}^2 \cup S^1 \). To be more precise, the total phase space can be thought of as a union of a punctured torus \( \tilde{T} \), the punctured orbifold \((\mathbb{R}^2 \setminus \{0, 0\})/\mathbb{Z}^2 \) named as \( \tilde{P} \) glued together at the respective punctures through the circle \( S^1 \), identifying the \( S^1 \) as a point.

### 2.6. Symplectic structure on the phase space

If one considers periodic coordinates \( x, y \) on \( \Sigma \simeq T^2 \) with period 1, then it follows immediately that the connections

\[
A^{(\pm)} = \lambda_0 (\rho^{(\pm)} dx + \sigma^{(\pm)} dy)
\]

give the above-written holonomies parameterizing the \( \tilde{T} \) sector.

Now using (12) and (13) we have the symplectic structure \( \omega \), whose pull back to the pre-symplectic manifold \( \Omega \) on this \( \tilde{T} \) sector of the phase space, given by

\[
\omega(\delta_1, \delta_2) = \frac{l}{4\pi G} \left[ (1/\gamma + 1)\delta_1 \rho\delta_2 \sigma + (1/\gamma - 1)\delta_1 \rho\delta_2 \sigma \right]
\]

or

\[
\omega = \frac{l}{4\pi G} \left[ (1/\gamma + 1)d\rho \wedge d\sigma + (1/\gamma - 1)d\rho \wedge d\sigma \right]
\]

\[= \frac{k(\pm)}{2\pi} d\rho \wedge d\sigma + \frac{k(\pm)}{2\pi} d\rho \wedge d\sigma,
\]

where \( k^{(\pm)} = \frac{l/(1/\gamma \pm 1)}{2\pi} \) and the \( d \) are exterior differentials on the phase and the \( \wedge \) is also on this manifold, not on spacetime. Here we introduce holomorphic coordinates on \( \tilde{T} \) corresponding to a complex structure \( \tau \) on the two-dimensional space manifold \( \Sigma \) as

\[
z^{(\pm)} = \frac{1}{\pi} (\rho^{(\pm)} + \tau \sigma^{(\pm)}).
\]
Then the symplectic structure in (38) takes the form
\[ \omega = \frac{i}{4\tau_2} dz_{(+)} \wedge \bar{d}z_{(+)} + \frac{i}{4\tau_2} dz_{(-)} \wedge \bar{d}z_{(-)}. \]  
(39)

In a similar fashion, the symplectic structure on \( \tilde{P} \) is given by
\[ \omega = \frac{i}{4\tau_2} dz_{(+)} \wedge \bar{d}z_{(+)} + \frac{i}{4\tau_2} dz_{(-)} \wedge \bar{d}z_{(-)}. \]  
(40)

where \( z_{(\pm)} = \frac{1}{\pi} (x_{(\pm)} + \tau y_{(\pm)}) \), \( x, y \) being the coordinates on \( \tilde{P} \).

3. Geometric quantization of the phase space

As explained in section 2.5, the total phase space is the product of two identical copies of \( \tilde{T} \cup \tilde{P} \), \( \tilde{T} \) and \( \tilde{P} \) being glued through a circle \( S^1 \) around the puncture at \( (0,0) \). Variables relevant to each factor of this product have been distinguished until now by \( \pm \) suffices. From now on, we will remove this distinction for notational convenience and will restore when it is necessary.

Upon quantization, the total wavefunctions (holomorphic sections of the line bundle over \( \tilde{T} \cup \tilde{P} \)) should be such that the wavefunction (holomorphic sections of the line bundle over \( \tilde{T} \cup \tilde{P} \)) on \( \tilde{T} \), say \( \psi(z) \), and the wavefunction on \( \tilde{P} \), say \( \chi(z) \), should ‘match’ on the circle. The plan of quantization is therefore simple. We will first carry out the quantization on \( \tilde{T} \). Then we will consider those functions on \( \tilde{P} \) which can be found by continuation in some sense of the wavefunctions on \( \tilde{T} \).

3.1. Quantization on \( \tilde{T} \)

While performing quantization on \( \tilde{T} \) with the symplectic structure
\[ \omega = \frac{k}{2\pi} d\rho \wedge d\sigma = \frac{i}{4\tau_2} d\xi \wedge d\bar{\xi}, \]
one must keep in mind the fact that \( \tilde{T} \) is in fact punctured as opposed to being compact\(^5\). The distinction occurs from the nontriviality of the algebra of the generators of the homotopy group. The three generators of \( \pi_1(\tilde{T}) \), denoted as \( a, b \) and \( \Delta \), respectively, correspond to the usual cycles of the compact torus and the cycle winding around the puncture. They should satisfy the following relations:
\[ ab\Delta^{-1} = \Delta \quad a\Delta a^{-1} = 1 \quad b\Delta b^{-1} = 1. \]

As explained in [16, 44] \( q \in \mathbb{Z} \) dimensional unitary representation of these relations is given as follows. The unitary finite-dimensional non-trivial representations of this algebra must have the commuting generator \( \delta \) proportional to identity. Hence we have that for some \( q \)-dimensional representation
\[ \Delta_{\alpha,\beta} = e^{2\pi i p/q} \delta_{\alpha,\beta}, \]
where \( p, q \) are positive integers, coprime to each other. The reason behind choosing the rational phase will become clear shortly when we will complete the quantization.

Again, up to arbitrary \( U(1) \) phase factors \( a, b \) are represented as
\[ a_{\alpha,\beta} = e^{-2\pi i \frac{p}{q} \delta_{\alpha,\beta}} \quad b_{\alpha,\beta} = \delta_{\alpha,\beta+1} \]

\(^5\) Had the symplectic manifold \( \tilde{T} \cup \tilde{P} \) been compact, Weil’s integrality criterion would require the Chern–Simons level \( k \) to be integer valued. At this point we keep open the possibility of \( k \) being any real number.
with $\alpha, \beta \in \mathbb{Z}_q$. It is also being expected that the space of holomorphic sections should also carry the $q$ representation of this homotopy group.

Let us now consider quantization on $\mathbb{R}^2$ endowed with a complex structure $\tau$ and the above symplectic structure. The fact that the actual phase space we wish to quantize is a punctured torus will be taken into account by the action of the discretized Heisenberg group operators on the Hilbert space of parallel sections of the line bundle over $\mathbb{R}^2$. A very similar quantization scheme for a different situation may be found in [45–47].

Start from the symplectic structure on $\mathbb{R}^2$ instead of the punctured torus $\tilde{T}$ and coordinatize it by $\rho$ and $\sigma$ and endowed with a complex structure $\tau$, such that holomorphic anti-holomorphic coordinates are chosen as before:

$$\omega = \frac{k}{2\pi} d\rho \wedge d\sigma.$$  

With definition of the holomorphic coordinate $z = \frac{1}{\tau} (\rho + \tau\sigma)$ defined through the arbitrary complex structure $\tau$, this becomes

$$\omega = \frac{ik\pi}{4\tau_2} dz \wedge d\bar{z},$$

where $\tau_2 = \Im \tau$. It is easy to check that the symplectic potential

$$\Theta = \frac{ik\pi}{8\tau_2} \left[ -(\bar{z} - 2z) \, dz + (z + \xi(\bar{z})) \, d\bar{z} \right]$$

gives the above symplectic structure, for the arbitrary anti-holomorphic function $\xi(\bar{z})$. Let us now consider the Hamiltonian vector fields corresponding to the variables $\rho$ and $\sigma$:

$$\zeta_\rho = \frac{2\pi}{k} \partial_\sigma$$

$$\zeta_\sigma = -\frac{2\pi}{k} \partial_\rho. \quad (41)$$

The corresponding pre-quantum operators to these variables are therefore

$$\hat{\rho} = -i\zeta_\rho - \Theta(\zeta_\rho) + \rho$$

$$= -\frac{2i}{k} (\tau \partial_\bar{z} + \bar{\tau} \partial_z) + \frac{i\pi}{4\tau_2} (\bar{\tau}z - \tau \bar{z} - 2\tau z - \bar{\tau}\xi(\bar{z})) \quad (43)$$

$$\hat{\sigma} = -i\zeta_\sigma - \Theta(\zeta_\sigma) + \sigma$$

$$= -\frac{2i}{k} (\partial_\bar{z} + \partial_z) + \frac{i\pi}{4\tau_2} (z + \bar{z} + \xi(\bar{z})). \quad (44)$$

Now, parallel (holomorphic) sections of the line bundle $\pi : L\tilde{T} \rightarrow \tilde{T}$ over the symplectic manifold $\tilde{T}$ are classified through the kernel of the Cauchy–Riemann operator defined via the connection $\nabla = d - i\Theta$ on $L\tilde{T}$ as (in units of $\hbar=1$)

$$\nabla_\hbar \Psi(z, \bar{z}) = 0. \quad (45)$$

The ansatz for $\Psi$ can be chosen as

$$\Psi(z, \bar{z}) = e^{-\frac{ik\pi}{4\tau_2}(z\bar{z} + \Xi(\bar{z}))} \psi(z) \quad (46)$$

with $\Xi(\bar{z})$ being the primitive of $\xi(\bar{z})$ with respect to $\bar{z}$ and $\psi(z)$ is any holomorphic function. This is how the holomorphic factor $\psi(z)$ of the function $\Psi(z)$ is being singled out by the
∇_b. To find the representations of the operators corresponding to \( \hat{\sigma} \) and \( \hat{\rho} \) on the space of the holomorphic functions, we see the actions

\[
\hat{\rho} \Psi(z, \bar{z}) = e^{-\frac{ik}{8} \pi \tau^2 (z \bar{z} + \omega)} \left[ -\frac{2i}{k} \partial_z + \frac{\pi}{k} \right] \psi(z)
\]

\[
\hat{\sigma} \Psi(z, \bar{z}) = e^{-\frac{ik}{8} \pi \tau^2 (z \bar{z} + \omega)} \left[ \frac{2i}{k} \partial_{\bar{z}} \right] \psi(z)
\]

These give the representations for \( \sigma \) and \( \rho \) on the space of holomorphic sections in terms of \( \hat{\rho}' \) and \( \hat{\sigma}' \).

At this point it is necessary to note that we aim to quantize the punctured torus instead of \( \mathbb{R}^2 \). This is done by imposing periodicity conditions (for being defined on torus) through the action of the Heisenberg group and the homotopy group (accounting for the puncture) on the space of holomorphic sections. Let us therefore define homotopy matrix-valued Heisenberg operators:

\[
U(m) := b^m e^{ikm\hat{\rho}}
\]

\[
V(m) := a^m e^{-ikm\hat{\sigma}'}.
\]

The periodicity condition that

\[
U(m)V(n)\psi(z) = \psi(z)
\]

for \( m, n \in \mathbb{Z} \) therefore reduces to

\[
\psi(z + 2m + 2n\tau) = e^{-ikn\pi \tau - i\alpha m} a^{-m} b^{-n} \psi(z) \quad \text{or}
\]

\[
\psi(z + 2m + 2n\tau) = e^{-ikn\pi \tau - i\alpha m} a^{-m} b^{-n} \psi(z)
\]

in terms of components.

Let us now as a digression concentrate upon level \( I, J \) \( SU(2) \) theta functions

\[
\vartheta_{I, J}(\tau, z) := \sum_{j \in \mathbb{Z}} e^{2\pi i J \tau (j + \frac{I}{2J})^2 + 2\pi i Jz (j + \frac{I}{2J})}
\]

and define

\[
\tilde{\vartheta}_{\alpha, N}(\tau, z) := \vartheta_{\alpha N + \alpha, \frac{pq}{2}}(\tau, z/q)
\]

for \( pq \) even [16]. After some manipulations, it is easy to check that

\[
\tilde{\vartheta}_{\alpha, N}(\tau, z + 2m + 2n\tau) = e^{-\pi i (p/q) m^2 \tau - \pi i (p/q) m z} e^{2\pi i p/q m c} \tilde{\vartheta}_{\alpha + n, N}(\tau, z),
\]

the indices \( \alpha \in \{0, 1, \ldots, q - 1\} \) and \( N \in \{0, 1, \ldots, p - 1\} \). These theta functions are known to form a complete \( p \)-dimensional set over the field of complex numbers [48].

Again comparing the transformations (51) and (52) we infer that for the value \( k = p/q \), a positive rational, we have a finite \( p \)-dimensional vector space of physical states spanned by

From another point of view it can be seen that the monodromy of wavefunctions about the puncture satisfying the above relation is measured to be \( e^{2\pi i k} \). When this is related to the measure of non-commutativity \( e^{2\pi i p/q} \) of the homotopy generators due to the puncture [16], we have the relation

\[
k = \frac{p}{q}
\]

up to additive integers.
component wavefunctions, represented by theta functions depicted as above. For instance the \(N\)th wavefunction is

\[
\psi^N(z) = \begin{pmatrix}
\tilde{\vartheta}_{0,N}(\tau, z) \\
\vdots \\
\ddots \\
\tilde{\vartheta}_{q-1,N}(\tau, z)
\end{pmatrix}.
\]

Here we have only considered the case \(pq\) even. In spirit the case \(pq\) odd [16] can also be dealt at par. Distinction of that case from the present one occurs as identification of the wavefunctions satisfying (51) has to be made with a theta function with different levels.

We are considering \(k = p/q\), a positive rational. From our earlier discussions (38), we had \(k(\pm) = l(1/\gamma \pm 1)^2 G\) (in the units of \(\hbar = 1 = c\), and in 2+1 spacetime dimensions \(G\) is of dimension that of length, hence making \(k\) dimensionless) in terms of the parameters of the classical theory. From the point of view of quantization, we are restricting only those values of classical parameters for which the combinations \(k(\pm)\) are positive rational.

### 3.2. Continuation to \(\tilde{P}\)

The wavefunction \(\chi(z)\) on \(\tilde{P}\) must be of the form

\[
\chi(z) = z^\kappa \phi(z),
\]

where \(\phi(z)\) is holomorphic and \(\kappa\) is a positive rational. The factor \(z^\kappa\) in the wavefunction is necessary since it must be allowed to pick up a non-trivial phase in going around the orbifold singularity.

Also the wavefunction on the entire phase space should be such that the two functions \(\psi\) and \(\chi\) agree on the intersection and the wavefunction \(\psi(z)\) on \(\tilde{T}\) should uniquely determine that on \(\tilde{P}\) in a neighborhood of the intersection. Hence \(\chi(z)\) must take the following form around the origin:

\[
\chi^\alpha_N(z) = e^{\frac{2\pi i}{q} z^{\frac{1}{q}}} \phi^\alpha_N(z).
\]

In the above equation we have chosen \(\kappa = \frac{1}{q}\) keeping in mind that \(\chi^\alpha_N(z)\) should have exactly \(q\) number of branches. This is necessary for agreement of \(\psi\) and \(\chi\) around the puncture. \(z^{\frac{1}{q}}\) in (53) is the principal branch of \(z^\kappa\).

Again since \(\tilde{P} \equiv (\mathbb{R}^2 \setminus \{0\})/\mathbb{Z}_2\), the wavefunctions defined on it must have definite 'parity' since this results into a constant phase factor in the wavefunction. As a result \(\phi^\alpha_N(z)\) must be even or odd. This property must hold for the wavefunctions on \(\tilde{T}\) in order that the wavefunctions agree on a circle around the origin. For example in the case \(pq\) even [16], we construct from (52) wavefunctions with definite parity through the combination

\[
\psi^{a(\pm)}_N(\tau, z) = \tilde{\vartheta}_{a,N}(\tau, z) \pm \tilde{\vartheta}_{-a, -N}(\tau, z).
\]

Now in order to match the wavefunctions, we have to do the Laurent expansion around the origin. The Laurent expansion about a point say \(P_\tau\) on a torus was studied in [49]. It was shown that there exists a basis on \(C_\eta\) which is analogous to the basis \(z^n\) on a circle, where \(C_\eta\) parameterizes a compact Riemann surface in same way as a circle can parameterizes the extended complex plane, \(\eta\) being a well-defined global parameter that labels the curve \(C_\eta = \{Q : \Re(p(Q))\} = \eta, p(Q) = \int_\Omega d\eta , d\eta\) on the other hand is a differential of the third kind on the Riemann surface with poles of the first order at the points \(P_\eta\) with residues \(\pm 1\). In the case of a torus, an exact basis \(A_\eta(z)\) on \(C_\eta\) is given in [49]. These are the Laurent basis
for curves on the torus on a special system of contours \( C_\eta \). As \( \eta \rightarrow \pm \infty \), \( C_\eta \) are small circles enveloping the point \( P_\pm \). We have to match the wavefunction on the torus around a small circle about \( P_\pm \) with that on \( \tilde{P} \). We can expand \( \psi (z) \) in terms of the basis \( A_n(z) \) while for the latter we can expand \( \chi (z) \) in terms of \( z^n \). The two expressions must be equal, when an expansion of the basis \( A_n(z) \) is performed in terms of \( z^n \). Since \( \psi_N^{(\pm)} (\tau, z) \) is holomorphic we have the Laurent expansion for \( \phi_N^{(\pm)} (\tau, z) \), (which is related to \( \chi_N^{(\pm)} (\tau, z) \) through (53)) around the origin as follows:

\[
\phi_N^{(\pm)} (\tau, z) = 1 + \left[ \frac{u^2}{2!} (\pi ipq)^{-1} \partial_\tau + \frac{u^4}{4!} (\pi ipq)^{-2} \partial_\tau^2 + \cdots \right] \times \sum_j \left( e^{\pi i pq \tau x_j^\pm} \pm e^{\pi i pq \tau x_j^-} \right) + \left[ u + \frac{u^3}{3!} (\pi ipq)^{-1} \partial_\tau + \frac{u^5}{5!} (\pi ipq)^{-2} \partial_\tau^2 + \cdots \right] \times \sum_j (x_j e^{\pi i pq \tau x_j^\pm} \pm \bar{x}_j e^{\pi i pq \tau x_j^-}),
\]

with \( u = ixpz \) and \( x_j = j + \frac{2N+pq}{p} \) and \( \bar{x}_j = j - \frac{2N+pq}{p} \). \( \phi_N^{(\pm)} \) in the wavefunction (53) should have the same form as above (55). This does not determine the exact form of the above function on the entire \( \tilde{P} \). But this asymptotic form on \( \tilde{P} \) ensures the finite number \( p \) of the wavefunctions each with \( q \) components.

Hence we have at hand the full Hilbert space of the quantized theory. The dimension of the Hilbert space is \( p^{(+)} p^{(-)} \). The extensions determined by the above asymptotic form should also be ‘square integrable’ with respect to some well-defined measure \( d\mu_p \). The unitarily invariant, polarization-independent inner product associated with this Hilbert space of wavefunctions (to be more precise ‘half-densities’) is given in terms of the Kähler potential \( P \) as

\[
\left\langle \Psi, \Psi' \right\rangle = \int \prod_a \sum \mathbf{d}x \mathbf{d}z \tau_1^{-1/2} e^{-\frac{L}{2\hbar} \sum (2\mathbf{i} + 2\mathbf{z} + 2\mathbf{\xi})} \psi_a' (\mathbf{z}) \psi_a (\mathbf{x}) + \sum \mathbf{d}\mu \mathbf{\chi}_a (\mathbf{z}) \mathbf{\chi}_a (\mathbf{\xi}).
\]

### 3.3. \( \gamma \rightarrow 1 \) limit in quantum theory

It dates back to Brown and Henneaux [3], who first showed the existence of a pair of (identical) centrally extended Virasoro algebras as the canonical realizations of the asymptotic symmetries for 2+1 Einstein gravity with a negative cosmological constant on an asymptotically AdS manifold. Later various authors [29], for example, reproduced the result with equivalent theories of (4) or topologically massive gravity (TMG) [40, 50, 51] confirming an AdS(3)/CFT(2) correspondence, although with unequal central charges. In the theory we are dealing with, these central charges come out to be \( (c^{(+)}, c^{(-)}) = \left( \frac{3}{2}, 0 \right) \). In our conventions and notations.

The chiral limit i.e. \( \gamma \rightarrow 1 \) in this direction has gained importance in the recent literature for various reasons. In view of results from [24], where second-order TMG was studied on an asymptotically AdS spacetime, we see that in order to make sense of all the graviton modes \( \gamma \) should be restricted to 1. At this limit the theory becomes chiral with \( (c^{(+)}, c^{(-)}) = \left( \frac{3}{2}, 0 \right) \). Another interesting result by Grumiller et al [25] reveals that at the quantum level the chiral limit of TMG is a good candidate as a dual to a logarithmic CFT (LCFT) with central charges \( (c^{(+)}, c^{(-)}) = \left( \frac{3}{2}, 0 \right) \). More recent works with some of the interesting ramifications of TMG ‘new massive gravity’ [5] show similar progress [52]. These results were worked out on an asymptotically AdS spacetime. In the present case, however, we have considered spatial slice to be a genus 1 compact Riemann surface, without boundary. Hence chance of a CFT living at the boundary does not arise. Even if we had worked on a asymptotically AdS manifold,
the theory would not be dual to an LCFT, because for that a propagating degree of freedom is necessary, which is absent in our case.

However, there are some interesting issues in the present discussion for the limit $\gamma \to 1$.

We have inferred in section 3.1 from (38), (51) and (52) that $k_{(\pm)} = \frac{l}{G\gamma}(1/\gamma \pm 1)$ must be positive rationals. These are related to the above-discussed central charges through $k_{(\pm)} = \pm \frac{1}{3} c_{(\pm)}$. As a result, if the ratio of the AdS radius $l$ and and Planck length $G$ (in units of $\hbar = 1 = c$) is positive, we must restrict $0 < \gamma < 1$. This is in apparent contradiction to the restriction $\gamma \geq 1$ [53] put by the CFT (living in the boundary, in the case of asymptotically AdS formulation). But this may well be resolved from the point of view that our analysis is completely performed on spacetime topology (as seen clearly in the construction of the physical phase space) whose spatial foliations are compact tori and relevant ranges of $\gamma$ should depend non-trivially on the topology of spacetime and in our case restrictions coming from suitable CFT is not clear as explained in the next paragraph.

As argued in 2.4 at the point $\gamma = 1$, we describe 2+1 gravity with a negative cosmological constant through a single $SO(2, 1)$ Chern–Simons action (27). On the other hand, for a rational $SO(2, 1)$ (or any of its covers) Chern–Simons theories on genus-1 spatial foliation, existence of a dual CFT too is still not very clear, as argued in [16]. The modular transformation ($SL(2, \mathbb{Z})$) representations acting on the physical Hilbert space (as found in sections 3.1 and 3.2) appear to be one of the two factors into which modular representations of the conformal minimal models factorize. This observation points that a 2D dual theory may not be conformal, although one may identify conformal blocks (of a CFT, if it exists) labeling our wavefunctions [16].

3.4. Results on the quantization of parameters

We have explained in section 3.1 that $k_{(\pm)} = \frac{n_{(\pm)}}{q_{(\pm)}}$ are positive rationals. In [10] it has been shown that for the gauge group being an $n$-fold diagonal cover of $SO(2, 1) \times SO(2, 1)$, one requires the couplings

$$k_{(+)} \in 8n^{-1}\mathbb{Z} \text{ for } n \text{ odd}$$

$$k_{(+)} \in 4n^{-1}\mathbb{Z} \text{ for } n \text{ even and}$$

$$k_{(+)} + k_{(-)} \in 8\mathbb{Z}$$

in our notation and convention. This is in agreement with our finding that the consistent quantization procedure reveals $k_{(\pm)} = \frac{n_{(\pm)}}{q_{(\pm)}} \in \mathbb{Q}^+$ and we are considering $q_{(\pm)}$ covers of the phase space (see section 3.2) which is constructed from the gauge group. In terms of physical parameters, we have

$$\frac{l}{G} \in \mathbb{Q}^+ \text{ and}$$

$$\frac{l}{G\gamma} \in \mathbb{Q}^+$$

$$\Rightarrow \gamma \in \mathbb{Q}^+$$

which are slightly less restrictive than the results of the analysis done in [10] $\frac{l}{G} \in \mathbb{Q}^+$ and $\frac{l}{G\gamma} \in \mathbb{N} \subset \mathbb{Q}^+$. 18
4. Conclusion

The features which come out of our analysis can be summarized as follows.

Classically it is observed that $\gamma$ fails to induce canonical transformations on the canonical variables although equations of motion do not involve $\gamma$. The role of $\gamma$ is best viewed in the constraint structure of the theory which is also studied in detail. On the other hand the ‘chiral’ limit relevant in our case is $\gamma \to 1+$ as opposed to the TMGs on asymptotically AdS space times, where it is $\gamma \to 1-$. In the canonical structure the apparent singularity can also be removed as discussed in section 2.4.

Naturally different values of $\gamma$ result in inequivalent quantizations of the theory. Dimensionless $\gamma$ and the cosmological constant $-\frac{1}{l^2}$ give the dimensionality of the physical state space in a subtle manner. Note that we had $k(\pm)k(\pm) = \frac{1}{\gamma^2} - \frac{1}{4} G^2$, $k(\pm)$ being both positive integers and prime to each other. The dimension of the Hilbert space turns out to be $p(\pm)p(\pm)$ which must be a positive integer. This requirement provides allowed values of $\gamma$, for a given $\frac{l^2}{G}$ such that $\frac{l^2}{G} \in \mathbb{Q}^+$ and $\frac{l^2}{G\gamma} \in \mathbb{Q}^+$.

Acknowledgments

The authors thank Parthasarathi Majumdar for suggesting investigations on this problem and for making numerous useful remarks and comments on the manuscript. One of the authors (RB) thank the Council for Scientific and Industrial Research (CSIR), India, for support through the SPM fellowship (SPM-07/575(0061)/2009-EMR-I).

Appendix A. Conjugacy classes of $SL(2, \mathbb{R})$

Any $SL(2, \mathbb{R})$ (which is the double cover of $SO(2, 1)^\uparrow$ element $G$ can be written in its defining representation as the product of three matrices by the Iwasawa decomposition uniquely:

$$G = \left( \begin{array}{cc} \cos(\phi/2) & \sin(\phi/2) \\ -\sin(\phi/2) & \cos(\phi/2) \end{array} \right) \left( \begin{array}{cc} e^{\xi/2} & 0 \\ 0 & e^{\xi/2} \end{array} \right) \left( \begin{array}{cc} 1 & \eta \\ 0 & 1 \end{array} \right),$$

with the range of $\phi$ being compact $(-2\pi, 2\pi)$ and those of $\xi$ and $\eta$ noncompact. Note that these three matrices fall in respectively the elliptic, hyperbolic and the null or parabolic conjugacy class of $SL(2, \mathbb{R})$, in addition to forming three Abelian subgroups themselves. Also note that

$$f_\phi = \exp(\imath\sigma_2\phi/2) = e^{-\lambda_i \phi},$$
$$g_\xi = \exp(\sigma_3 \xi/2) = e^{\imath \lambda_i \xi},$$
$$h_\eta = \exp(\imath [\sigma_2 + \sigma_1] \eta/2) = e^{-(\lambda_0 + \lambda_1) \eta},$$

where $\lambda_i \in sl(2, \mathbb{R})$ with $\{\lambda_i, \lambda_j\} = \epsilon_{ijk} \lambda^k$.

We now state an important result which is used in the text. Let $g = \exp(\imath_1 \lambda_1^I)$ and $g' = \exp(\imath_2 \lambda_2^I)$ be two $SL(2, \mathbb{R})$ elements. Then the necessary and sufficient condition for $g_1g_2 = g_2g_1$ to hold is $\lambda_i = c_i \lambda_i$ for $I = 0, 1, 2$ and any $c_i \in \mathbb{R}$. This can be seen by using the Baker Campbell Hausdorff formula.

7 Since from gravity action we got a gauge theory with a Lie algebra shared commonly by $SO(2, 1)$, $SL(2, \mathbb{R})$, $SU(1, 1)$ or any covering of them, the actual group used is quite irrelevant unless one is considering transformations between disconnected components of the group manifold.
References

[1] Witten E 1988 (2+1)-Dimensional gravity as an exactly soluble system Nucl. Phys. B 311 46

[2] Deser S, Jackiw R and ‘t Hooft G 1984 Three-dimensional Einstein gravity: dynamics of flat space Ann. Phys. (NY) 152 220

[3] Brown J D and Henneaux M 1986 Central charges in the canonical realization of asymptotic symmetries: an example from three dimensional gravity Commun. Math. Phys. 104 207

[4] Deser S, Jackiw R and Templeton S 1982 Three-dimensional massive gauge theories Phys. Rev. Lett. 48 975

[5] Bergshoeff E A, Hohm O and Townsend P K 2009 Massive gravity in three dimensions Phys. Rev. Lett. 102 201301 (arXiv:0901.1766 [hep-th])

[6] Liu Y and Sun Y w 2009 Note on new massive gravity in AdS3 J. High Energy Phys. JHEP04(2009)106 (arXiv:0903.0536 [hep-th])

[7] Banados M, Teitelboim C and Zanelli J 1992 The black hole in three-dimensional space-time Phys. Rev. Lett. 69 1849 (arXiv:hep-th/9204099)

[8] Strominger A 1998 Black hole entropy from near-horizon microstates J. High Energy Phys. JHEP02(1998)009 (arXiv:hep-th/9712251)

[9] Birmingham D, Sachs I and Sen S 1998 Entropy of three-dimensional black holes in string theory Phys. Lett. B 424 275 (arXiv:hep-th/9801019)

[10] Witten E 2007 Three-dimensional gravity revisited arXiv:0706.3359 [hep-th]

[11] Verlinde H L and Verlinde E P 1989 Conformal field theory and geometric quantization Superstrings ’89 (Lectures Trieste Spring School (April 1989), and Ecole Normale Superieure, Paris (June 1989)) Ed M Green et al QCD161:T732:1989

[12] Witten E 1989 Quantum field theory and the Jones polynomial Commun. Math. Phys. 121 351

[13] Witten E 1991 Quantization of Chern–Simons gauge theory with complex gauge group Commun. Math. Phys. 137 29

[14] Imbimbo C 1992 SL(2,R) Chern–Simons topological theories Nucl. Phys. B 384 484 (arXiv:hep-th/9208016)

[15] Axelrod S, Pietra S Della and Witten E 1991 Geometric quantization of Chern–Simons gauge theory J. Diff. Geom. 33 787

[16] Nelson E W and Baekler P 1991 Topological gauge model of gravity with torsion Phys. Lett. A 156 399

[17] Baekler P, Mielke E W and Hehl F W 1992 Dynamical symmetries in topological 3D gravity with torsion Nuovo Cimento B 107 91

[18] Cacciatori S L, Caldarelli M M, Giaconnetti A, Klemm D and Mansi D S 2006 Chern–Simons formulation of three-dimensional gravity with torsion and nonmetricity J. Geom. Phys. 56 2523 (arXiv:hep-th/0507200)

[19] Holst S 1996 Barbero’s Hamiltonian derived from a generalized Hilbert–Palatini action Phys. Rev. D 53 5966 (arXiv:gr-qc/9511026)

[20] Elitzur S, Moore G W, Schwimmer A and Seiberg N 1989 Remarks on the canonical quantization of the Chern–Simons–Witten theory Nucl. Phys. B 326 108

[21] Ezawa K 1995 Chern–Simons quantization of (2+1) anti-de Sitter gravity on a torus Class. Quantum Grav. 12 373 (arXiv:hep-th/9409074)

[22] Li W, Song W and Strominger A 2008 Chiral gravity in three dimensions J. High Energy Phys. JHEP04(2008)082 (arXiv:0801.4566 [hep-th])
[25] Grumiller D and Johansson N 2008 Instability in cosmological topologically massive gravity at the chiral point
J. High Energy Phys. JHEP07(2008)134 (arXiv:0805.2610 [hep-th])
Grumiller D and Johansson N 2010 Gravity duals for logarithmic conformal field theories arXiv:1001.0002 [hep-th]

[26] Crnkovic C and Witten E 1997 Covariant description of canonical formalism in geometrical theories
Three Hundred Years of Gravitation ed S W Hawking and W Israel (Cambridge, UK: Cambridge University Press)
Ashtekar A, Bombelli L and Reula O The covariant phase space of asymptotically flat gravitational fields
Analysis, Geometry and Mechanics: 200 Years after Lagrange ed M Francaviglia and D Holm (Amsterdam:
North-Holland)

[27] Ashtekar A, Wisniewski J and Dreyer O 2003 Isolated horizons in 2+1 gravity arXiv:gr-qc/0206024
Blagojevic M and Basile M 2003 Asymptotic dynamics in 3D gravity with torsion Phys. Rev. D 68 124007
(arXiv:gr-qc/0306070)
Blagojevic M and Basile M 2003 Asymptotic symmetries in 3D gravity with torsion Phys. Rev. D 67 084032
Banerjee R, Gangopadhyay S, Mukherjee P and Roy D 2009 Symmetries of the general topologically massive
gravity in the Hamiltonian and Lagrangian formalisms arXiv:0912.1472 [gr-qc]

[28] Ashtekar A, Wisniewski J and Dreyer O 2003 Isolated horizons in 2+1 gravity arXiv:gr-qc/0206024
Blagojevic M and Cvetkovic B 2005 Conserved charges in 3D gravity with torsion Proc. 8th Workshop on What
Comes beyond the Standard Models (Bled, Slovenia, 19–29 July 2005) (arXiv:hep-ph/0512061 pp 8–16)
Manschot J 2007 AdS$_3$ partition functions reconstructed J. High Energy Phys. JHEP07(2007)103 (arXiv:0707.1159 [hep-th])

[29] Manschot J 2007 AdS 3 partition functions reconstructed J. High Energy Phys. JHEP10(2007)103
(arXiv:0707.1159 [hep-th])

[30] Grumiller D, Jackiw R and Johansson N 2008 Canonical analysis of cosmological topologically massive gravity at the chiral point arXiv:0806.4185 [hep-th]
Blagojevic M and Cvetkovic B 2009 Canonical structure of topologically massive gravity with a cosmological
constant J. High Energy Phys. JHEP05(2009)073 (arXiv:0812.4742 [gr-qc])

[31] Blagojevic M and Cvetkovic B 2009 Canonical structure of topologically massive gravity in spacelike stretched
AdS sector J. High Energy Phys. JHEP09(2009)006 (arXiv:0907.0950 [gr-qc])
Ashtekar A, Lewandowski J, Marolf D, Mourao J and Thiemann T 1995 Quantization of diffeomorphism
invariant theories of connections with local J. Math. Phys. 36 6456 (arXiv:gr-qc/9504018)
Ashtekar A and Lewandowski J 2004 Background independent quantum gravity: a status report Class. Quantum
Grav. 21 R53 (arXiv:gr-qc/0404018)

[32] ’t Hooft G 1978 On the phase transition towards permanent quark confinement Nucl. Phys. B 138 1
Thiemann T 2007 Modern Canonical Quantum General Relativity (Cambridge: Cambridge University Press)
p 819

[33] Ashtekar A, Baez J C and Krasnov K 2000 Quantum geometry of isolated horizons and black hole entropy Adv.
Theor. Math. Phys. 4 1 (arXiv:gr-qc/0005126)
Kloster S, Brannlund J and DeBenedictis A 2008 Phase-space and black hole entropy of higher genus horizons
in loop quantum gravity Class. Quantum Grav. 25 065008 (arXiv:gr-qc/0702036)
Brannlund J, Kloster S and DeBenedictis A 2009 The evolution of A black holes in the mini-superspace approximation of loop quantum gravity Phys. Rev. D 79 084023 (arXiv:0901.0010 [gr-qc])

[34] Mumford D 1983 Tata Lectures on Theta I (Boston, MA: Birkhauser)
Kirchberger I and Novikov S P 1987 Algebras of Virasoro type, Riemann surfaces and the structure of soliton
theory Funkt. Anal. Appl. 21 126
[50] Kraus P and Larsen F 2006 Holographic gravitational anomalies J. High Energy Phys. JHEP01(2006)022
arXiv:hep-th/0508218
Kraus P 2006 Lectures on black holes and the AdS(3)/CFT(2) correspondence arXiv:hep-th/0609074.s
[51] Skenderis K, Taylor M and van Rees B C 2009 Topologically massive gravity and the AdS/CFT correspondence
J. High Energy Phys. JHEP09(2009)045 (arXiv:0906.4926 [hep-th])
[52] Grumiller D and Hohm O 2009 AdS3/LCFT2—correlators in new massive gravity arXiv:0911.4274 [hep-th]
[53] Solodukhin S N 2006 Holography with gravitational Chern–Simons term Phys. Rev. D 74 024015
arXiv:hep-th/0509148
Liu Y and Sun Y W 2009 Consistent boundary conditions for new massive gravity in AdS3 J. High Energy Phys. JHEP05(2009)039 (arXiv:0903.2933 [hep-th])