EXTENDING HOMEOMORPHISMS FROM PUNCTURED SURFACES TO HANDLEBODIES

ALESSIA CATTABRIGA AND MICHELE MULAZZANI

Abstract. Let $H_g$ be a genus $g$ handlebody and $MCG_{2n}(T_g)$ be the group of the isotopy classes of orientation preserving homeomorphisms of $T_g = \partial H_g$, fixing a given set of $2n$ points. In this paper we find a finite set of generators for $E_{2n}^g$, the subgroup of $MCG_{2n}(T_g)$ consisting of the isotopy classes of homeomorphisms of $T_g$ admitting an extension to the handlebody and keeping fixed the union of $n$ disjoint properly embedded trivial arcs. This result generalizes a previous one obtained by the authors for $n = 1$. The subgroup $E_{2n}^g$ turns out to be important for the study of knots and links in closed 3-manifolds via $(g,n)$-decompositions. In fact, the links represented by the isotopy classes belonging to the same left cosets of $E_{2n}^g$ in $MCG_{2n}(T_g)$ are equivalent.

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1. Introduction and preliminaries

Let $H_g$ be an oriented handlebody of genus $g \geq 0$ and $\partial H_g = T_g$. Consider a system of $n$ disjoint properly embedded trivial arcs $\mathcal{A} = \{A_1, \ldots, A_n\}$ in $H_g$ and let $P_{i1}, P_{i2}$ be the endpoints of the arc $A_i$, for $i = 1, \ldots, n$. We denote with $MCG_{2n}(T_g)$ (resp. $MCG_n(H_g)$), the group of the isotopy classes of orientation preserving homeomorphisms of $T_g$ (resp. $H_g$) fixing the set $\mathcal{P}_n = \{P_{i1}, P_{i2} \mid i = 1, \ldots, n\}$ (resp. $A_1 \cup \cdots \cup A_n$). The group $MCG_{2n}(T_g)$ is widely studied and different finite presentations of it are known (see [10, 15]). In this article we are interested in studying the subgroup $E_{2n}^g$ of $MCG_{2n}(T_g)$, which is the image of the homomorphism $MCG_n(H_g) \to MCG_{2n}(T_g)$ induced by restriction. In other words, an element $f \in MCG_{2n}(T_g)$ belongs to

\[\{A_1, \ldots, A_n\}\] properly embedded in a handlebody $H_g$ is trivial if there exist $n$ mutually disjoint embedded discs, called trivializing discs, $D_1, \ldots, D_n \subset H_g$ such that $A_i \cap D_i = A_i \cap \partial D_i = A_i$, $A_i \cap D_j = \emptyset$ and $\partial D_i - A_i \subset \partial H_g$ for all $i, j = 1, \ldots, n$ and $i \neq j$.\]
if it admits an extension to $H_g$ fixing $A_1 \cup \cdots \cup A_n$. If $h \in \mathcal{E}_{2n}^g$ we still denote with $h$ an extension of it. The case $n = 0$ is studied in [18], while in [4] we study the case $n = 1$. Moreover, in [13], the case $g = 0$ is investigated. In this paper we find a finite set of generators for $\mathcal{E}_{2n}^g$ and describe their extension to $H_g$, for each $n \geq 1$ and $g \geq 0$.

The main motivation for studying such subgroups lies in their importance in the representation of knots and links via $(g, n)$-decompositions. The notion of $(g, n)$-decomposition for links in orientable connected 3-manifolds, given by Doll in [7], extends the one of bridge (or plat) decomposition for links in $S^3$. Roughly speaking, a $(g, n)$-decomposition of a link $L \subset M$ is the data of a Heegaard surface of genus $g$ in $M$ which cuts the link into two set of $n$ disjoint trivial arcs (the underpasses and the overpasses). It is easy to see that each link admits a $(g, n)$-decomposition, for suitable $g$ and $n$ with $g \geq g_M$ and $n \geq n_L$, where $g_M$ denotes the Heegaard genus of $M$ and $n_L$ is the number of components of $L$.

The use of $(g, 1)$-decompositions of knots is revealed to be very fruitful in order to study different topics. For example the strongly-cyclic branched coverings of knots in 3-manifolds are analyzed in [6], while the Alexander polynomial of knots in rational homology spheres is investigated in [14]. The case of knots admitting $(1, 1)$-decompositions has been widely studied (see for example [2, 3, 5, 9, 12, 17]). In [11], the Heegaard Floer homology of $(1, 1)$-knots is computed, while in [16], the geometry of such knots is determined in terms of the distance of the curve complex associated with their $(1, 1)$-decompositions.

In Section 2 we give the definition of $(g, n)$-decomposition of links and describe its connections with $\mathcal{E}_{2n}^g$. In Section 3 we discuss the genus zero case, while the general case is analyzed in Section 4.

Now we briefly recall the definition of spin of a point along a curve (see [11]). Let $P$ be a point on $T_g$ and $c$ be a simple closed oriented curve on $T_g$, containing $P$. Consider a neighborhood $N$ of $c$, parametrized by coordinates $(y, \theta) \in [-1, 1] \times S^1$, where $c$ is defined by $y = 0$ and $P$ is the point of coordinates $(0, 0)$. The spin of $P$ about $c$ is the homeomorphism $s_{P,c}$ of $T_g$ obtained extending with the identity the map defined on $N$ by $(y, \theta) \mapsto (y, \theta + \theta')$ where

$$
\theta' = \begin{cases} 
2\pi(2y + 1) & \text{if } -1 \leq y \leq -1/2 \\
0 & \text{if } -1/2 \leq y \leq 1/2 \\
-2\pi(2y - 1) & \text{if } 1/2 \leq y \leq 1
\end{cases}
$$

If we denote with $t_\delta$ the right-handed Dehn twist along the simple closed curve $\delta$, then $s_{P,c} = t_{c_1}^{-1}t_{c_2}$, where $c_1$ and $c_2$ are the curves corresponding to $y = -3/4$ and $y = 3/4$ respectively.
2. \((g, n)\)-DECOMPOSITIONS OF LINKS

In this section we recall the notion of \((g, n)\)-decompositions of links and describe the connection with \(E_{2n}^g\).

A \((g, n)\)-decomposition for a link \(L\) in an orientable closed connected 3-manifold \(M\) is the data

\[
(M, L) = (H_g, A_1 \cup \cdots \cup A_n) \cup \phi (\bar{H}_g, \bar{A}_1 \cup \cdots \cup \bar{A}_n)
\]

where \(H_g, \bar{H}_g\) are two oriented handlebodies of genus \(g\), \(A = \{A_1, \ldots, A_n\}, \bar{A} = \{\bar{A}_1, \ldots, \bar{A}_n\}\) are two systems of \(n\) properly embedded trivial arcs in \(H_g\) and \(\bar{H}_g\) respectively, and \(\phi: (\partial H_g, \partial A_1 \cup \cdots \cup \partial A_n) \to (\partial \bar{H}_g, \partial \bar{A}_1 \cup \cdots \cup \partial \bar{A}_n)\) is an attaching (orientation preserving) homeomorphism (see Figure 1).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{A \((g, n)\)-decomposition.}
\end{figure}

A link admitting a \((g, n)\)-decomposition will be called a \((g, n)\)-link. It is easy to see that \(L\) admits a \((g_M, n)\)-decomposition, for a suitable \(n\), where \(g_M\) denotes the Heegaard genus of \(M\). Indeed, given a Heegaard surface \(S\) of genus \(g_M\) for \(M\), there exists an immersion of \(L\) in \(S\) with a finite number of singular points which are double points. So, by a slight modification of this immersion near each double point, we are able to embed \(L\) in \(S\), except for a finite trivial set of arcs. Moreover, if \(n_L\) denotes the number of components of \(L\), by choosing a sufficiently large \(g\), it is always possible to find a \((g, n_L)\)-decomposition of \(L\), since if \(n > n_L\) a \((g, n)\)-decomposition determines a \((g+1, n-1)\)-decomposition (see Figure 2).

Let \(L \subset M\) be a \((g, n)\)-link with \((g, n)\)-decomposition \((M, L) = (H_g, A_1 \cup \cdots \cup A_n) \cup \phi (\bar{H}_g, \bar{A}_1 \cup \cdots \cup \bar{A}_n)\) and let \(\Upsilon: \bar{H}_g \to H_g\) be a fixed (orientation preserving) homeomorphism such that \(\Upsilon(\bar{A}_i) = A_i\), for \(i = \ldots, j\).
Figure 2. From a (2, 3)-decomposition to a (3, 2)-decomposition.

1, . . . , n, then \( \varphi = \Upsilon|_{\partial H_g} \phi \) is an orientation preserving homeomorphism of \((\partial H_g, \partial A_1 \cup \cdots \cup \partial A_n)\).

Moreover, since two isotopic attaching homeomorphisms produce equivalent \((g, n)\)-links, we have a natural surjective map from \(\text{MCG}_{2n}(T_g)\), the mapping class group of the \(2n\)-punctured surface of genus \(g\), to the class \(\mathcal{L}_{g,n}\) of all \((g, n)\)-links

\[ \Theta_{g,n} : \text{MCG}_{2n}(T_g) \to \mathcal{L}_{g,n} \]

sending \( \varphi \in \text{MCG}_{2n}(T_g) \) to the link \( L_\varphi \in \mathcal{L}_{g,n} \), associated to the attaching homeomorphism \( \Upsilon|_{\partial H_g} \). Unfortunately \( \Theta_{g,n} \) is far away from being injective. Indeed, if we denote with \( \mathcal{E}_{2n}^g \) the subgroup of \( \text{MCG}_{2n}(T_g) \) consisting of the elements that admit an extension to \( H_g \), fixing \( A_1 \cup \cdots \cup A_n \) as a set, then for each \( \varepsilon \in \mathcal{E}_{2n}^g \), the link \( L_\varepsilon \) is the \( n \)-component trivial link in the connected sum of \( g \) copies of \( S^2 \times S^1 \). Moreover, for each \( \varphi \in \text{MCG}_{2n}(T_g) \), the links \( L_{\varphi\varepsilon} \) and \( L_\varphi \) are equivalent. So we can restrict ourselves to considering the surjective map

\[ \Theta'_{g,n} : \text{MCG}_{2n}(T_g)/\mathcal{E}_{2n}^g \to \mathcal{L}_{g,n}, \]

where \( \text{MCG}_{2n}(T_g)/\mathcal{E}_{2n}^g \) denotes the set of left cosets of \( \mathcal{E}_{2n}^g \) in \( \text{MCG}_{2n}(T_g) \). So, in this contest, it is important to obtain information about \( \mathcal{E}_{2n}^g \). In this paper we find a finite set of generators for this group. In order to do this, let us describe some elements of \( \mathcal{E}_{2n}^g \).
Intervals: For $i = 1, \ldots, n$, let $D_i$ be a trivializing disc for $A_i$. Consider a tubular neighborhood $N$ of $D_i$ such that $D_j \cap N = \emptyset$ for $j \neq i$ and parametrize it by $N \cong D^2 \times [0,1]$ as in Figure 3. For $i = 1, \ldots, n$, we denote with $\iota_i$ the homeomorphism of $H_g$ obtained extending by the identity the one defined on $N$ by $(\rho, \theta, t) \mapsto (\rho, \theta + \theta', t)$ where

$$
\theta' = \begin{cases} 
\pi & \text{if } 0 \leq \rho, t \leq 1/2 \\
2\pi(1-\rho) & \text{if } 1/2 \leq \rho \leq 1, 0 \leq t \leq 1/2 \\
2\pi(1-t) & \text{if } 0 \leq \rho \leq 1/2, 1/2 \leq t \leq 1 \\
4\pi(1-t)(1-\rho) & \text{if } 1/2 \leq \rho, t \leq 1.
\end{cases}
$$

By definition, $\iota_i$ exchanges the endpoints of the arc $A_i$ and fixes pointwise the other arcs.

**Figure 3.**

Exchanging two arcs: Consider a tubular neighborhood $N$ of $D_i \cup B \cup D_{i+1}$, where $B$ is a band connecting $D_i$ and $D_{i+1}$, such that $D_j \cap N = \emptyset$ for $j \neq i, i+1$, and parametrize it by $N \cong D^2 \times [0,1]$ as in Figure 4. For $i = 1, \ldots, n - 1$, we denote with $\lambda_i$ the homeomorphism of $H_g$ defined as in (1) on $N$ and as the identity outside $N$. Set $\lambda_i = \iota_{i+1}^{-1} \iota_i^{-1} \lambda_i$ exchanges the arcs $A_i$ and $A_{i+1}$ and fixes pointwise the other arcs.

Moreover, let $\text{PMCG}_{2n}(T_g)$ be the subgroup of $\text{MCG}_{2n}(T_g)$ consisting of the isotopy classes of the homeomorphisms of $T_g$ pointwise fixing the punctures and set $\mathcal{E}_{2n}^g = \text{PMCG}_{2n}(T_g) \cap \mathcal{E}_{2n}^g$. The next proposition shows that $\mathcal{E}_{2n}^g$ is generated by $\iota_1, \lambda_i, i = 1, \ldots, n - 1$, and a set of generators for $\mathcal{E}_{2n}^g$.

**Proposition 1.** Let $\Sigma_{2n}$ be the symmetric group on $2n$ letters and denote with $\Sigma'_{2n}$ the subgroup of $\Sigma_{2n}$ generated by the transposition $(1 2)$ and the permutations $(2i-1 2i+1)(2i 2i+2)$, for $i = 1, \ldots n - 1$. 

Then the exact sequence \( 1 \to \text{PMCG}_{2n}(T_g) \to \text{MCG}_{2n}(T_g) \to \Sigma_{2n} \to 1 \) restricts to an exact sequence \( 1 \to \tilde{\mathcal{E}}^{0}_{2n} \to \mathcal{E}^{0}_{2n} \to \Sigma'_{2n} \to 1 \).

Proof. First of all, we recall that the homomorphism \( p : \text{MCG}_{2n}(T_g) \to \Sigma_{2n} \) is obtained by considering the permutation induced on the punctures by the elements of \( \text{MCG}_{2n}(T_g) \), where the puncture \( P_{ij} \) corresponds to the letter \( 2i + j - 2 \), for \( i = 1, \ldots, n \) and \( j = 1, 2 \). Let \( \varepsilon \in \mathcal{E}^{0}_{2n} \), since the extension of \( \varepsilon \) induces a permutation of the arcs \( A_1, \ldots, A_n \), it is easy to see that \( p(\varepsilon) \in \Sigma'_{2n} \). Moreover if \( p(\varepsilon) = 1 \) then an extension of it fixes the arcs, so \( \varepsilon \in \mathcal{E}^{0}_{2n} \). In order to complete the proof we only need to check the surjectivity of \( \mathcal{E}^{0}_{2n} \to \Sigma'_{2n} \). This follows by observing that \( \iota_1 \) maps on \( (1 2) \) and \( \lambda_i \) maps on \( (2i - 1 2i + 1)(2i 2i + 2) \), for \( i = 1, \ldots, n - 1 \). \[\square\]

In the next two sections we will find a finite set of generators for \( \mathcal{E}^{0}_{2n} \) and \( \mathcal{E}^{g}_{2n} \).

3. The Case of Genus Zero

We introduce the homeomorphisms whose isotopy classes generate \( \mathcal{E}^{0}_{2n} \) and describe their extensions. For each \( i = 1, \ldots, n \), we set \( A'_i = D_i \cap \partial H_g \), where \( D_i \) is a trivializing disc for \( A_i \) (see Figure 1).

**Spin of a puncture:** A simple closed oriented curve \( c \subset \partial B^3 \) containing a puncture \( P_{ij} \) is called admissible if \( c \cap A'_i = \emptyset \), \( i = 1, \ldots, n \), \( i \neq j \) and \( c \cap A'_j = P_{ij} \). A spin \( s_{P_{ij}, c} \) of the puncture \( P_{ij} \) along an admissible curve \( c \) can be extended to \( B^3 \) as follows. We can suppose that a tubular neighborhood \( N \) of \( c \) in \( \partial B^3 \) does not intersect the arcs \( A'_i \), for \( i \neq j \), and so there exists an embedded ball \( \tilde{N} = [-1, 1] \times D^2 \) in \( B^3 \) such that \( [-1, 1] \times \partial D^2 \) is \( N \), \( \tilde{N} \cap A_i = \emptyset \), for \( i \neq j \) and \( \tilde{N} \cap A_j \) is an
arc \(l\) transversal to the trivial fibration of \(\tilde{N}\) in discs. We can extend \(s_{P,j,c}\) to \(B^3\) by the identity outside \(\tilde{N}\) and by making the mapping cone from the center of each disc \(\{y\} \times D^2\) inside \(\tilde{N}\). Obviously such an extension keeps \(A_i\) fixed for \(i \neq j\). Since two trivial properly embedded arcs in a ball with the same endpoints are ambient isotopic by an isotopy fixing the boundary, then \(s_{P,j,c}\) is isotopic to a homeomorphism that keeps \(l\) and so \(A_j\) fixed. Therefore, it admits an extension that keeps all the arcs fixed and, as a consequence, \(s_{P,j,c} \in E_{2n}^0\).

We denote with \(s_{ji}\) the spin of the point \(P_i\) about the curve depicted in Figure 5, for \(i, j = 1, \ldots, n\).

**Figure 5.** The spin \(s_{ji}\).

**Remark 2.** Note that if \(c = c_1 \cdots c_k\) in \(\pi_1(S^2 - \{P_{j_1}, P_{i_1}, P_{i_2} | i = 1, \ldots, n, i \neq j\}, P_{j_2})\) then \(s_{P,j,c} = s_{P_{j_2},c_k} \cdots s_{P_{j_2},c_1}\) in \(PMCG_{2n}(S^2)\). Moreover if \(c\) is admissible, then we can take the \(c_i\)'s in \(\pi_1(S^2 - (\{P_{j_1}\} \cup_{i \neq j} A_i'), P_{j_2})\) and so \(s_{P,j,c}\) is a product of the \(s_{ji}\)'s.

**Slide of an arc:** Consider a simple closed oriented curve \(c \subset T_g - \mathcal{P}_n\) such that \(c \cap A'_j\) is a single point \(P \neq P_{j_1}, P_{j_2}\). The slide \(S_{j,c}\) of the arc \(A_j\) along the curve \(c\) is defined as the spin of the point \(P\) along the curve \(c\), where we require that the parametrization of the tubular neighborhood \(N \cong [-1, 1] \times S^1\) of \(c\) is chosen such that \(A'_j\) is parametrized by \([-1/2, 1/2] \times \{0\}\), and so is kept fixed by \(S_{j,c}\). Note that \(S_{j,c}\) is isotopic to the product of spins \(s_{P_{j_1},c_1}s_{P_{j_2},c_2}\), where \(c_1\) and \(c_2\) are the curves parametrized by \([-1/2] \times S^1\) and \([1/2] \times S^1\). In order to extend \(S_{j,c}\) to the handlebody, let us consider an embedded solid torus \(\tilde{N} = [-1, 1] \times S^1 \times [0, 1]\) in \(H_g\) such that
\( N = [-1, 1] \times S^1 \times \{0\}, \quad \widetilde{N} \cap A_i = \emptyset, \) for \( i = 1, \ldots, n, \, i \neq j, \) and \( \widetilde{N} \cap A_j \subset [-1, 1] \times S^1 \times [0, 1/2]. \) Then we can extend \( S_{j,c} \) to \( H_g \) by the identity outside \( \widetilde{N}, \) and in \( \widetilde{N} \) by the map

\[
(y, \theta, t) \mapsto \begin{cases} 
(S_{j,c}(y, \theta), t) & \text{if } 0 \leq t \leq 1/2 \\
(y, \theta + \theta', t) & \text{if } 1/2 \leq t \leq 1,
\end{cases}
\]

with

\[
\theta' = \begin{cases} 
-4\pi(2 - 2t)(y + 1) + 2\pi & \text{if } -1 \leq y \leq -1/2 \\
2\pi(2t - 1) & \text{if } -1/2 \leq y \leq 1/2 \\
4\pi(2 - 2t)(y - 1) + 2\pi & \text{if } 1/2 \leq y \leq 1.
\end{cases}
\]

It is clear that \( S_{j,c} \) fixes all the arcs and so belongs to \( \mathcal{E}_{2n}^g. \)

We denote with \( S_{ji} \) and \( S'_{ji} \) the slides of the arc \( A_j \) along the curves depicted in Figure 6 (a) and (b), respectively, for \( i, j = 1, \ldots, n, \, i \neq j. \) Moreover, \( L_{ki} \) and \( M_{ki} \) are the slides of the arc \( A_k \) along the curves \( e'_{ki}, \, g'_{ki} \) depicted in Figure 9, for \( i = 1, \ldots, g \) and \( k = 1, \ldots, n. \)

**Figure 6.** Slides \( S_{ji} \) and \( S'_{ji}. \)

**Lemma 3.** If \( g = 0 \) then the following relation holds \( s_{ji} = S_{ij}s_i^{-1} \) for each \( i, j = 1, \ldots, n \) and \( i \neq j. \)
Proof. Let \( j < i \). Referring to Figure 8 we have that \( s_{ji} = t_c^{-1}t_d = s_{ii}^{-1}t_d \). Moreover, \( S_{ij} = s_{P_{p_1,c_1}s_{P_{p_2,c_2}}} = t_e^{-1}t_f^{-1}t_d = t_d \), where the last equality holds since \( g \) and \( f \) are isotopic curves and \( t_e \) is trivial. So \( s_{ji} = s_{ii}^{-1}S_{ij} \). The proof in the case \( j > i \) is completely analogous. \( \square \)

**Proposition 4**. Consider the homomorphism \( j_{g,n,m} : \operatorname{PMCG}_n(T_g) \to \operatorname{PMCG}_m(T_g) \) induced by the inclusion, with \( 1 \leq m < n \) for \( g \geq 1 \) and \( 3 \leq m < n \) for \( g = 0 \). Then \( \ker j_{g,n,m} \cong \pi_1(F_{m,n-m}(T_g)) \),
where $F_{m,n-m}(T_g)$ denotes the configuration space of $n - m$ points in $T_g$ with $m$ punctures.

The next result describes a set of generators for the group $\mathcal{E}_{2n}^0$.

**Theorem 5.** The subgroup $\mathcal{E}_{2n}^0$ of PMCG$_{2n}(S^2)$ is generated by spins $s_{jj}$ of $P_{j2}$ along the simple closed oriented curve depicted in Figure 7 and slides $S_{ji}, S'_{ji}$ of the arc $A_j$ along the curves depicted in Figure 8 (a) and (b), respectively, for $i,j = 1,\ldots,n$, $i \neq j$.

**Proof.** Let $G_{2n}^0$ be the subgroup of $\mathcal{E}_{2n}^0$ generated by the following elements

- a) spins $s_{P_{j2},c}$ of $P_{j2}$ along admissible curves on $S^2$, $j = 1,\ldots,n$;
- b) slides $S_{j,c}$ of the arc $A_j$ along simple closed oriented curves, in $S^2 - P_n$ intersecting $A_j'$ in a single point different from $P_{j1}$ and $P_{j2}$, $j = 1,\ldots,n$.

We will prove that $G_{2n}^0 = \mathcal{E}_{2n}^0$ by induction on $n$, and that $G_{2n}^0$ is generated by $s_{ji}$ with $i,j = 1,\ldots,n$, $i \leq j$ and $S_{ji}, S'_{ji}$, with $i,j = 1,\ldots,n$, $j \neq i$.

For $n = 1$ the group $\mathcal{E}_{2}^0$ is trivial since PMCG$_2(S^2)$ is trivial, so there is nothing to prove. In order to prove the inductive step we need the following lemma.

**Lemma 6.** Given an element $h \in \mathcal{E}_{2n}^0$ there exists an element $h' \in G_{2n}^0$ such that $h'h(A'_n) = A'_n$.

**Proof.** Let $A_0 = h(A'_n)$ and $D_0 = h(D_n)$. Suppose that $A_0 \cap A'_i = \emptyset$ for $i = 1,\ldots,n - 1$. Then the closed curve $c = A_0 \cup A'_n$ based on $P_{n2}$ is homotopic rel $P_{n2}$ to the product of simple closed curves $c_1 \cdots c_k$ in $S^2 - \{A'_i \mid i = 1,\ldots,n - 1\}$ and each $c_j$ is a curve of type a). Moreover $s_{P_{n2},c_k} \cdots s_{P_{n2},c_1}(A_0) = A'_n$, so in this case $h' = s_{P_{n2},c_k} \cdots s_{P_{n2},c_1}$.

Let $j \in \{1,\ldots,n - 1\}$ such that $A_0 \cap A'_j \neq \emptyset$. Up to a small deformation we can suppose that $D_j$ and $\tilde{D}_0$ intersect transversally and so the connected components of the intersection are arcs or circles. Let us consider the circular components. By an innermost argument, it is possible to choose one of them, let us say $C$, such that the union of the discs bounded by $C$ on $D_j$ and $\tilde{D}_0$ is a sphere $S$ that intersects $D_j \cap \tilde{D}_0$ only in $C$. Obviously $S$ does not contain any of the arcs $A_i$ and then, by an isotopy, the intersection $C$ can be removed. Iterating the procedure we can remove all the circular intersections. Now, let us consider the arcs. Since $h(A_n) = A_n$, then $h(A_n) \cap A_j = \emptyset$. So the endpoints of the arcs in $D_j \cap \tilde{D}_0$ are points of $A_0 \cap A'_j$. By an innermost argument, there exists an arc $B_0$ that determines a disc both
in \( \tilde{D}_0 \) and in \( D_j \), whose union is a disc \( \tilde{D}_0 \), properly embedded in the ball, that intersects \( \tilde{D}_0 \cap D_j \) only in \( B_0 \). Let \( \Delta_0 \) be the connected component of \( B^3 - \tilde{D}_0 \) that does not contain \( A_j \). If none of the \( A_i \) for \( i = 1, \ldots, n - 1 \) is contained in \( \Delta_0 \) then, by isotopy, the intersection \( B_0 \) can be removed.

Otherwise, let \( A_i \) be contained in \( \Delta_0 \). Referring to Figure 9, choose a simple oriented closed curve \( c \) on \( S^2 \) such that \( c \cap A_i' \) is a single point and \( c \cap \tilde{A}_0 \) is a single point in \( \Delta_0 \cap S^2 \) (the dashed disc in the top part of the figure represents \( \Delta_0 \cap S^2 \)). Consider the slide \( S_{i,c} \) of the arc \( A_i \) along the curve \( c \) and let \( h_i = S_{i,c} \). The figure illustrates the image of \( \tilde{A}_0 \) after the application of the homeomorphism \( S_{i,c} \) and, subsequently, of a homeomorphism isotopic to the identity. If we set \( \tilde{A}_1 = S_{i,c}(\tilde{A}_0) \) and \( \tilde{D}_1 = S_{i,c}(\tilde{D}_0) = h_i(D_0) \), we have that \( \text{card}(\tilde{D}_1 \cap D_j) = \text{card}(\tilde{D}_0 \cap D_j) \), and \( \tilde{A}_1 \cap A_i' = \tilde{A}_0 \cap A_i' \), for each \( k = 1, \ldots, n - 1 \). Moreover, by the same argument used before, it is possible to find an arc \( B_1 \) that determines a disc both in \( \tilde{D}_1 \) and in \( D_j \), whose union is a disc \( \tilde{D}_1 \), properly embedded in the ball, that intersects \( \tilde{D}_1 \cap D_j \) only in \( B_1 \). If we denote with \( \Delta_1 \) the connected component of \( B^3 - \tilde{D}_1 \) that does not contain \( A_j \) (the dashed disc in the bottom part of the figure represents \( \Delta_1 \cap S^2 \)) then \( A \cap \Delta_1 = A \cap \Delta_0 - A_i \). So, after \( k \)-steps and composing \( h \) with the opportune slides, we obtain the case of \( \Delta_k \cap A = \emptyset \) and so as before we can remove the intersection arc \( B_k \). Since the intersections of \( \tilde{D}_i \cap D_j \) are finitely many, we can remove them all and so return to the case in which \( \tilde{A}_k \cap A_i' = \emptyset \).

Since during the process we do not increase the number of intersections of \( \tilde{A}_k \) with the other arcs \( A_i' \) by applying it to the other arcs, we return to the case \( \tilde{A}_i \cap A_k' = \emptyset \) for each \( k = 1, \ldots, n - 1 \) already considered.

\[ \square \]

Continuation of the proof of Proposition 4 By the previous lemma it is enough to consider the subgroup \( \tilde{E}_{2n}^0 \) of \( \tilde{E}_{2n}^0 \), consisting of the elements fixing \( A_n' \). Let \( h \in \tilde{E}_{2n}^0 \), by hypothesis \( h(A_n) = A_n \), so we can assume that \( h \) fixes the whole disc \( D_n \), as well as a tubular neighborhood \( N \) of it. By contracting \( N \) to the point \( P_{n2} \) we obtain a surjective map \( i_1 : \tilde{E}_{2n}^0 \to E_{2n-1}^0 \), where \( E_{2n-1}^0 \) is the group of elements of \( \text{PMCG}_{2n-1}(S^2) \) that extend to the ball \( B^3 \) fixing \( A_1, \ldots, A_{n-1} \). Moreover, the surjective homomorphism \( j_{0,2n-1,2n-2} : \text{PMCG}_{2n-1}(S^2) \to \text{PMCG}_{2n-2}(S^2) \) of Proposition 4 restricts to a surjective homomorphism \( i_2 : E_{2n-1}^0 \to \tilde{E}_{2n-2}^0 \). When \( n = 2 \), the kernel of \( i_2 \) is trivial since \( \text{PMCG}_3(S^2) = \text{PMCG}_2(S^2) = 1 \). Otherwise, when \( n > 2 \), by [11] pp.
Figure 9.
158-160] and [4], ker $i_2$ is generated by $A = \{ s_{P_{i_2}, \gamma_i} \mid i = 1, \ldots, 2n - 3 \}$, where the loops $\gamma_i$ are generators of $\pi_1(S^2 - \mathcal{P}_{n-1}, P_{n2})$. By the induction hypothesis, $\mathcal{E}_{2n-2}^0$ is generated by $B = \{ s_{P_{j2}, c} \mid c \cap A'_i = \emptyset, c \cap A'_j = P_{j2}, i, j = 1, \ldots, n-1, i \neq j \}$ and $C = \{ S_{j,c} \mid c \subset S^2 - \mathcal{P}_{n-1}, (c \cap A'_j) = 1, c \cap A'_j \neq P_{j1}, P_{j2}j = 1, \ldots, n - 1 \}$. So a complete set of generators for $\mathcal{E}_{2n}^0$ is given by the generators of ker $i_1$, the lifts of elements in $A$ via $i_1$, and the lifts of elements in $B$ and $C$ via $i_2i_1$. The kernel of $i_1$ is generated by $s_{nn}$ and so it is contained in $\mathcal{G}_{2n}^0$. A spin $s_{P_{i_2}, \gamma_i} \in A$ lifts to a slide of the arc $A_n$ along a suitable curve, and so belongs to $\mathcal{G}_{2n}^0$. An element $s_{P_{j2}, c}$ in $B$ lifts to a spin still based on $P_{j2}$ along the lifting of $c$. Moreover, since we can suppose that $c$ avoids $P_{n2}$, its lifting does not intersect $A'_n$, and so belongs to $\mathcal{G}_{2n}^0$. In the same way the lifting of each element in $C$ belongs to $\mathcal{G}_{2n}^0$.

Now we find a finite set of generators for $\mathcal{G}_{2n}^0$. From Remark [2] and Lemma, it follows that elements of type a) are generated by $s_{jj}$ for and $S_{ji}$ for $i, j = 1, \ldots, n$ $i \neq j$. Note that a slide $s$ of the arc $A_j$ fixes $A'_j$. Consider the analogue of the map $i_2$ obtained by contracting a tubular neighborhood of $A'_j$ to $P_{j2}$. The kernel of the map is $s_{jj}$, while the image of $S$ is a spin $s$ based on $P_{j2}$ along a curve in $S^2 - \{ P_{j1}, P_{j1}, P_{j2} \mid i = 1, \ldots, n, i \neq j \}$. Since $s$ decomposes into a product of spins along a set of curves whose homotopy classes generate $\pi_1(S^2 - \{ P_{j1}, P_{j1}, P_{j2} \mid i = 1, \ldots, n, i \neq j \}, P_{j2})$ and the lifting of each spin is one of the $S_{ji}$'s or the $S'_{ji}$'s, then $S$ is a product of the $S_{ji}$'s and $S'_{ji}$'s.

**Corollary 7.** The subgroup $\mathcal{E}_{2n}^0$ of $\text{MCG}_{2n}(S^2)$ is generated by $\tau_i, \lambda_k, s_{11}, S_{12}, S'_{12}$, for $k = 1, \ldots, n - 1$.

**Proof.** By Theorem [4] and Proposition [1] the group $\mathcal{E}_{2n}^0$ is generated by $\tau_i, \lambda_k, s_{jj}, S_{ji}, S'_{ji}$, for $j = 1, \ldots, n$, $i \neq j$ and $k = 1, \ldots, n - 1$. Set $\Lambda_i = \lambda_1^{-1} \cdots \lambda_i^{-1}$, then the statement follows from the relations $s_{ii} = \Lambda_i^{-1} s_{11} A_i$, $S_{ji} = (\lambda_1 A_i A_j)^{-1} S_{12} \lambda_1 A_i A_j$ if $j < i$ and $S_{ji} = (\Lambda_j A_i)^{-1} S_{12} A_j A_i$ if $j > i$. □

We note that the sets of generators obtained for $\mathcal{E}_{2n}^0$ and $\mathcal{E}_{2n}^0$ is considerably smaller than the ones obtained in [13].

4. The general case

We are now ready to analyze the general case. We introduce new homeomorphisms whose isotopy classes are generators of $\mathcal{E}_{2n}^0$ and describe their extension to $H_g$.

**Semitwist of a handle:** Let $T_i$ be the 2-cell depicted in Figure [10] for $i = 1, \ldots, g$. Then $T_i$ cuts away from $H_g$ a solid torus $K_i$,
Figure 10. A model for $H_g$.

containing the $i$-th handle $V_i$. We denote with $\omega_i$ the homeomorphism of $H_g$ which is a counterclockwise rotation of $\pi$ radians of $K_i$ along $T_i$ on $K_i$ and the identity outside a tubular neighborhood of $K_i$, for $i = 1, \ldots, g$. As usual, we still denote with $\omega_i$ its restriction to $T_g$. Note that $\omega_i^2$ is isotopic to the Dehn twist along $\partial T_i$.

**Twist of a meridian disk:** Referring to Figure 10 let $\beta_i = \partial B_i$, for $i = 1, \ldots, g$. We denote with $\tau_i$ the right-handed Dehn twist along $\beta_i$. Note that $\tau_i$ admits an extension to $H_g$, whose effect is to give a complete twist to the $i$-th meridian disk $B_i$.

**Exchanging two handles:** let $C_i$ be the properly embedded 2-cell, depicted in Figure 11 for $i = 1, \ldots, g - 1$. Then $C_i$ cuts away from $H_g$ a handlebody $K'_i$ of genus two, containing the $i$-th and the $(i+1)$-th handles. Let $\rho_i$ be the homeomorphism of $H_g$ which exchanges $V_i$ and $V_{i+1}$ by a counterclockwise rotation of $\pi$ radians along $C_i$ and is the identity outside a tubular neighborhood of $K'_i$. We set $\rho_i = \omega_i^{-1}\omega_{i+1}^{-1}\rho_i$ for $i = 1, \ldots, g - 1$. Moreover, for $i < j$ we set $\rho_{ij} = \rho_i\rho_{i+1}\cdots\rho_{j-2}\rho_{j-1}\rho_j^{-1}\cdots\rho_{i+1}^{-1}\rho_i^{-1}$. Obviously $\rho_{ij}$ exchanges $i$-th handle with the $j$-th handle and keeps fixed the other handles.

**Slides of a meridian disk:** Referring to Figure 10 let $Z_i$ and $Z'_i$ be the centers of the properly embedded meridian discs $B_i$
and \( B'_i \) in \( H_g \), respectively. Moreover, denote with \( H^i_g \) the genus \((g-1)\) handlebody obtained by removing the \( i \)-th handle \( V_i \) from \( H_g \). A simple closed oriented curve \( e \) on \( \partial H^i_g - \mathcal{P}_n \) containing \( Z_i \), with \( B'_i \cap e = \emptyset \), will be called an \( i \)-loop. If we require that in the parametrization of a tubular neighborhood \( N \cong [-1,1] \times S^1 \) of \( e \) the disk \( B_i \) is contained in \([-1/2,1/2] \times S^1 \) and \( B'_i \notin N \), then the spin of \( Z_i \) along \( e \) keeps \( B_i \) and \( B'_i \) fixed, and so can be extended to \( V_i \) by the identity and to \( H^i_g \) in the same way as the extension of a slide of an arc (see page 7). We call this homeomorphism of \( H^i_g \), as well as its restriction to \( T^i_g \), a slide of \( B_i \) along the \( i \)-th loop \( e \) and denote it with \( \sigma_{i,e} \). In a completely analogous way we can define an \( i' \)-loop \( e' \) and a slide of \( B_{i'} \) along \( e' \) and denote it with \( \sigma_{i',e'} \). Note that \( \sigma_{i',e'} = \omega^{-1}_i \sigma_{i,e} \omega_i \), where \( e = \omega_i(e') \). We set \( \theta_{ij} = \sigma_{i,e_{ij}} \), \( \eta_{ij} = \sigma_{i,g_{ij}} \), \( \xi_{ik} = \sigma_{i,f_{ik}} \), \( \zeta_{ik} = \sigma_{i,l_{ik}} \), where \( e_{ij} \), \( g_{ij} \), \( f_{ik} \), \( l_{ik} \) are the oriented curves depicted in Figure 12, with \( i, j = 1, \ldots, g, i \neq j \), and \( k = 1, \ldots, n \).

**Remark 8.** By \[18\] Lemma 3.6] if the \( i \)-loop \( e \) is homotopic to the product of \( i \)-loops \( e_1 \cdots e_n \) on \( \partial H^i_g - (\mathcal{P}_n \cup \{Z_i\}) \) rel \( Z_i \), then \( \sigma_{i,e} \) is isotopic to \( \sigma_{i,e_n} \cdots \sigma_{i,e_1} \mod \tau_i \). Since the loops \( e_{ij} \), \( g_{ij} \), \( f_{ik} \), \( l_{ik} \), for \( k = 1, \ldots, n \) and \( j = 1, \ldots, g \), with \( i \neq j \), are a free set of generators for \( \pi_1(\partial H^i_g - (\mathcal{P}_n \cup \{Z_i\}), Z_i) \), then \( \theta_{ij}, \eta_{ij}, \xi_{ik}, \zeta_{ik}, \tau_i \), with \( k = 1, \ldots, n \), \( j = 1, \ldots, g \) and \( i \neq j \), generate all the slides of the \( i \)-th meridian disc \( B_i \).

Now we are ready to describe a finite set of generators for \( \mathcal{E}_n^g \).
Figure 12. The loops $e_{ij}, g_{ij}, f_{ik}, l_{ik}$, on $H_g^i$.

**Theorem 9.** The subgroup $\tilde{E}_{2n}^g$ of $\text{PMCG}_{2n}(T_g)$ is generated by $\tau_1, \omega_1, \rho_i, \theta_{12}, \eta_{12}, \xi_{1k}, \zeta_{1k}, s_{kk}, S_{kl}, S'_{kl}$ and $L_{k1}$, with $i = 1, \ldots, g - 1$ and $k, l = 1, \ldots, n, k \neq l$.

**Proof.** Let $G_{2n}^g$ be the subgroup of $\text{PMCG}_{2n}(T_g)$ generated by $\tau_i, \omega_i, \rho_i, \rho_{im}, \theta_{ij}, \eta_{ij}, \xi_{ik}, \zeta_{ik}, \sigma_i, \sigma_i', \sigma_i, \sigma_i', s_{kh}, S_{k,c}$, where $e$ is an $i$-loop, $e'$ is an $i'$-loop and $c$ is a curve on $T_g$, with $i, j = 1, \ldots, g, i \neq j, i < m$ and $k, h = 1, \ldots, n, h \leq k$.

We will prove that $G_{2n}^g = \tilde{E}_{2n}^g$, by induction on $g$, and that $G_{2n}^g$ is generated by $\tau_1, \omega_1, \rho_i, \theta_{12}, \eta_{12}, \xi_{1k}, \zeta_{1k}, s_{kk}, S_{kl}, S'_{kl}$ and $L_{k1}$, with $i = 1, \ldots, g - 1$ and $k, l = 1, \ldots, n, k \neq l$.

The case $g = 0$ is proved in Theorem 5.

Now we prove the inductive step. Let $g > 0$ and denote with $\tilde{E}_{2n}^g$ the subgroup of $E_{2n}^g$ consisting of the isotopy classes of the homeomorphisms that are the identity on the boundary of the $g$-th handle $V_g$. By the same arguments as for the proof of [18, Lemma 4.4], we have that, for each $h \in \tilde{E}_{2n}^g$, there exists an element $h' \in G_{2n}^g$ such that $h'h$ is the identity on a meridian disk of the $g$-th handle and so, up to isotopy, on all the handle. Therefore $h'h \in \tilde{E}_{2n}^g$ and so it is enough to show that $\tilde{E}_{2n}^g \subseteq G_{2n}^g$.

Let $f : T_g \to T_g$ be a homeomorphism fixing $\partial V_g$ pointwise and whose isotopy class belongs to $E_{2n}^g$. By cutting out the $g$-th handle, and capping the resulting holes with the two disks $B_g$ and $B'_g$, we can identify $f$ with a homeomorphism $f'$ of $T_{g-1}$, such that $f'|_{B_g \cup B'_g} = \text{Id}$.
and \( f' = f \) on \( T_{g-1} - (B_g \cup B_g') \). Moreover, by shrinking \( B_g \) and \( B_g' \) to their centers \( Z_g \) and \( Z_g' \), the map \( f' \) becomes a map \( \tilde{f} \) of \( T_{g-1} \) fixing \( Z_g \) and \( Z_g' \). In order to simplify the notation, we set \( P_{2n+1} = Z_g \) and \( P_{2n+2} = Z_g' \). Obviously, \( \tilde{f} \) extends to \( H_{g-1} \) fixing \( A \) pointwise.

So we obtain a surjective map \( i_1 : \tilde{\mathcal{E}}^g_{2n} \rightarrow E^g_{2n+1} \), where \( E^g_{2n+1} \) is the subgroup of \( \text{PMCG}_{2n+2}(T_{g-1}) \) consisting of the elements which extend to the handlebody fixing \( A \) pointwise. Moreover, the surjective homomorphism \( j_{g-1,2n+2,2n} : \text{PMCG}_{2n+2}(T_{g-1}) \rightarrow \text{PMCG}_{2n}(T_{g-1}) \) of Proposition 4 restricts to a surjective homomorphism \( i_1 : \tilde{\mathcal{E}}^g_{2n} \rightarrow \mathcal{E}^g_{2n+1} \). So, a set of generators of \( \tilde{\mathcal{E}}^g_{2n} \) is given by the generators of \( \ker i_1 \), the lift of the generators of \( \ker i_2 \) via \( i_1 \) and the lift of the generators of \( \mathcal{E}^g_{2n} \) via \( i_2 i_1 \). The kernel of \( i_1 \) is generated by \( \tau_g \), and so belongs to \( G^g_{2n} \). By \( [1] \) pp. 158-160 and \( [4] \) ker \( \tilde{E}^g_{2n} \) is generated by spins of \( Z_g \) and \( Z_g' \) about appropriate loops not containing \( \mathcal{P}_n \), lifting to slides of \( B_g \) and \( B_g' \) on \( T_g \), which are elements of \( G^g_{2n} \). Moreover, by the induction hypothesis \( \mathcal{E}^g_{2n-1} = G^g_{2n-1} \). Since we can suppose that the generators of \( G^g_{2n-1} \) keep \( Z_g \) and \( Z_g' \) fixed, they lift to elements of \( G^g_{2n} \).

Now we prove that \( \tau_1, \omega_1, \rho_1, \theta_{12}, \eta_{12}, \xi_{ik}, \xi_{kk}, S_{kk}, S_{kl}, S'_{kl} \) and \( L_{kl} \), with \( i = 1, \ldots, g - 1 \) and \( k, l = 1, \ldots, n, k \neq l \) generate \( G^g_{2n} \). By Remark 8 and since, as already observed, \( \sigma^i_{e^i} = \omega_i^{-1} \sigma_i e_i \omega_i \), where \( e = \omega_i(e') \), then the elements \( \theta_{ij}, \eta_{ij}, \xi_{ik}, \xi_{kk}, \tau_i, \omega_i \) for \( k = 1, \ldots, n \) and \( i, j = 1, \ldots, g \), with \( i \neq j \), generate all the slides \( \sigma_i e_i \) and \( \sigma_{e_i} \). Moreover we have \( \theta_{ij} = \rho_{j_1} \omega_i^{-2} \theta_{ji} \omega_i^2 \rho_{ji} \) if \( i > j \) and \( \theta_{ij} = \rho_{k_1} \rho_{j_2} \theta_{k_1} \rho_{j_2} \rho_{ij}^{-1} \) if \( i < j \). The same relations hold for the other slides of a meridian disk.

Analogously, if \( c = c_1 \cdots c_j \) in \( \pi_1((T_g - \mathcal{P}_{2n}) \cup \{P_{k_1}, P_{k_2}\}, P) \), with \( P = A'_{kk} \cap c \), then \( S_{c,k} = S_{c,e} \cdots S_{c,c_1} \), up to multiplication by \( s_{kk} \). Then any slide \( S_{c,k} \) is the product of \( s_{kk} \) and the slides \( L_{kl}, M_{kk}, S_{kl}, S'_{kl} \), for \( i = 1, \ldots, g \) and \( k, l = 1, \ldots, n \) with \( l \neq k \), since the corresponding curves generate \( \pi_1((T_g - \mathcal{P}_{2n}) \cup \{P_{k_1}, P_{k_2}\}, P) \). Moreover we have \( L_{kl} = \rho_{i_1} L_{k_1} \rho_{i_1}^{-1}, M_{kk} = \rho_{i_1} M_{k_1} \rho_{i_1}^{-1} \) and \( M_{kk} = \tau_{ik}^2 \xi_{ik} s_{kk} \). To end the proof it is enough to observe that \( \tau_i = \rho_{i_1} \tau_i \rho_{i_1}^{-1}, \omega_i = \rho_{i_1} \omega_i \rho_{i_1}^{-1} \) and that, by definition, \( \rho_{ij} \) is a product of \( \rho_i \)'s.

We end the paper by describing a finite set of generators for the subgroup \( \mathcal{E}^g_{2n} \).

**Theorem 10.** The subgroup \( \mathcal{E}^g_{2n} \) of \( \text{MCG}_{2n}(T_g) \) is generated by \( \iota_1, \lambda_k, \tau_1, \omega_1, \rho_1, \theta_{12}, \eta_{12}, \xi_{11}, S_{12} \) and \( L_{11} \), with \( i = 1, \ldots, g - 1, k = 1, \ldots, n - 1 \).

**Proof.** The statement follows from Proposition 11 Corollary 7 Theorem 9 and the following relations: \( s_{11} = \iota_1^2, \xi_{1h} = \)
\[ \iota_h^{-1} \xi_{1h}, \quad \xi_{1h} = \lambda_{h-1}^{-1} \cdots \lambda_1^{-1} \xi_{11} \lambda_1 \cdots \lambda_{h-1}, \quad S'_{12} = \iota_1^4 \lambda_2^2 S_{12}^{-1}, \quad S_{11} = \lambda_{h-1}^{-1} \cdots \lambda_1^{-1} L_{11} \lambda_1 \cdots \lambda_{h-1}, \text{ for each } h = 1, \ldots, n. \]

\[ \square \]

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ALESSIA CATTABRIGA, Department of Mathematics, University of Bologna, Piazza di Porta S. Donato, 5, 40126, Bologna (Italy). E-mail: cattabri@dm.unibo.it

MICHELE MULAZZANI, Department of Mathematics, C.I.R.A.M., University of Bologna, Piazza di Porta S. Donato, 5, 40126, Bologna (Italy). E-mail: mulazza@dm.unibo.it