Non-power law constant flux solutions for the Smoluchowski coagulation equation

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Abstract

It is well known that for a large class of coagulation kernels, Smoluchowski coagulation equations have particular power law solutions which yield a constant flux of mass along all scales of the system. In this paper, we prove that for some choices of the coagulation kernels there are solutions with a constant flux of mass along all scales which are not power laws. The result is proved by means of a bifurcation argument.

Keywords: Smoluchowski coagulation equations; constant flux solutions; oscillatory stationary solutions; Hopf bifurcation.

Contents

1 Introduction 1
  1.1 Motivation and general background of the problem . . . . . . . . . . . . . . . 1
  1.2 Main result of the paper . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 6
  1.3 Structure of the paper and main notation . . . . . . . . . . . . . . . . . . . . 6

2 Proof of the main result 7
  2.1 Reformulation of the problem . . . . . . . . . . . . . . . . . . . . . . . . . . 7
  2.2 Linearized problem near the constant solutions. . . . . . . . . . . . . . . . . . 10
  2.3 Bifurcation of non-power law constant flux solutions. . . . . . . . . . . . . . . 11

3 Construction of the bifurcation kernel: proof of Theorem 2.2 24

1 Introduction

1.1 Motivation and general background of the problem

In this paper, we study a particular type of stationary solutions of the classical Smoluchowski coagulation equation

$$\partial_t f(x, t) = K[f](x, t)$$  \hspace{1cm} (1.1)

where

$$K[f](x, t) := \frac{1}{2} \int_0^x K(x - y, y) f(x - y, t) f(y, t) \, dy - \int_0^\infty K(x, y) f(x, t) f(y, t) \, dy , \quad x > 0 , \quad t \geq 0. \hspace{1cm} (1.2)$$
The function \( f \) describes the concentration of particles in the space of particle volumes. The collision kernel \( K \) is non-negative and symmetric, i.e., \( K(x, y) = K(y, x) \) for each \( x, y \in \mathbb{R}_+ \). This kernel contains information about the specific mechanism responsible for the coagulation of the particles. See [11, 29] where coagulation kernels have been obtained in the context of atmospheric aerosols. For a rigorous connection between coagulation kernels and dynamics of interacting particle systems, see e.g. [12, 13, 17, 24, 28], and for dynamics of stochastic processes in graphs, see e.g. [1, 7]. Several of the collision kernels arising in the applications of the Smoluchowski coagulation equation are homogeneous, i.e., there exists \( \gamma \in \mathbb{R} \) such that

\[
K(\lambda x, \lambda y) = \lambda^\gamma K(x, y) \quad \text{for any } x, y \in \mathbb{R}_+ \text{ and } \lambda > 0. \tag{1.3}
\]

Equation (1.2) can be written in a more convenient form in order to describe the transfer of volume of the clusters from smaller to larger values. To this end, we set \( g(x, t) = xf(x, t) \). Then (1.2) can be formally rewritten as

\[
\partial_t (g(x, t)) + \partial_x (J(x; f(\cdot, t))) = 0 \tag{1.4}
\]

where

\[
J(x; f) = \int_0^x dy \int_{x-y}^{\infty} dz K(y, z) yf(y) f(z). \tag{1.5}
\]

Equation (1.4) shows that the coagulation process described by Smoluchowski equation can be reinterpreted in terms of the flux of volume in the space of cluster volumes. The total flux of volume passing from the region of volumes smaller than \( x \) to the region of volumes larger than \( x \) is given by (1.5).

A particular class of solutions of Smoluchowski coagulation equation are the particle distributions \( f \) for which the flux of particles at each particular value of \( x \) is a constant (and therefore independent of \( t \) and \( x \)). Then,

\[
J(x; f) = J_0 > 0 \quad \text{for each } x > 0. \tag{1.6}
\]

In this paper, we will consider solutions to (1.6) for the above mentioned class of homogeneous kernels. Due to the homogeneity of the kernel, \( K \) can then be written in the form

\[
K(x, y) = (x + y)^\gamma \Phi \left( \frac{x}{x + y} \right) \tag{1.7}
\]

where

\[
K(s, 1-s) = \Phi(s) = \Phi(1-s), \quad 0 < s < 1. \tag{1.8}
\]

Notice that the last identity follows from the symmetry of the kernel \( K \).

Besides the homogeneity of the kernel, the main property characterizing the coagulation mechanism associated to a given coagulation kernel is the asymptotic behaviour of the function \( \Phi(s) \) as \( s \to 0^+ \) (or, equivalently, for \( s \to 1^- \)). A large class of kernels relevant to the applications of the Smoluchowski equation allows finding \( 0 < c_1 \leq c_2 < \infty \) and \( p \in \mathbb{R} \) such that

\[
c_1 s^{-p} (1-s)^{-p} \leq \Phi(s) \leq c_2 s^{-p} (1-s)^{-p} \quad \text{for } s \in (0, 1). \tag{1.9}
\]
In addition, we will show that it is possible to assume without loss of generality that
\[ \gamma + 2p \geq 0. \tag{1.10} \]
We will assume here the above positivity, and the following upper bound
\[ \gamma + 2p < 1. \tag{1.11} \]

In order to justify that (1.10) can be assumed without loss of generality, we just notice that if \( \gamma + 2p < 0 \) then (1.8) and (1.9) hold with \( \bar{p} := -\gamma + p \). Using the fact that \( \gamma + 2\bar{p} = -(\gamma + 2p) \geq 0 \) it follows that (1.9) and (1.10) hold with \( p \) replaced by \( \bar{p} \).

The class of kernels for which the conditions (1.7)-(1.11) hold, covers all the kernels satisfying
\[ c_1(x^\alpha y^\beta + x^\beta y^\alpha) \leq K(x, y) \leq c_2(x^\alpha y^\beta + x^\beta y^\alpha) \quad \text{with} \quad 0 \leq \alpha - \beta < 1, \]
as it may be seen by choosing \( \gamma = \alpha + \beta \) and \( p = -\beta \). Note that then, indeed, \( 0 \leq \gamma + 2p < 1 \).

Due to the homogeneity of the kernel \( K \) we can expect the functional \( J(x; f) \) to be constant if
\[ f(x) = \frac{C_0}{x^{\frac{\beta}{2} + p}}, \quad C_0 > 0, \tag{1.12} \]
assuming that the integral in (1.5) is defined. It turns out that the integral in (1.5) is finite with \( f \) as in (1.12) if and only if (1.11) holds. Indeed, using (1.12) in (1.5) and the upper bound in (1.9), we have
\[ J(x; f) = \int_0^1 dy \int_{-\gamma}^\gamma dz \left( y + z \right)^\gamma \Phi \left( \frac{y}{y + z} \right) \frac{y}{y^{\frac{\beta}{2} + p} z^{\frac{\alpha}{2} + p}} \leq c_2 \int_0^1 dy \int_0^\gamma dz \left( y + z \right)^\gamma \left( \frac{y}{y + z} \right)^{-p} \left( \frac{z}{y + z} \right)^{-p} \frac{y}{y^{\frac{\beta}{2} + p} z^{\frac{\alpha}{2} + p}}. \]

We now decompose the integral above in the contributions of the regions where \( y < z \) and \( y \geq z \). We can then estimate \( J(x; f) \) by
\[ J(x; f) \leq C \int_0^1 dy \int_\frac{\gamma}{2}^\gamma dz \left( y + z \right)^\gamma \left( \frac{y}{y + z} \right)^{-p} \left( \frac{z}{y + z} \right)^{-p} \frac{y}{y^{\frac{\beta}{2} + p} z^{\frac{\alpha}{2} + p}} + C \int_\frac{\gamma}{2}^1 dy \int_1^y dz \left( y + z \right)^\gamma \left( \frac{y}{y + z} \right)^{-p} \left( \frac{z}{y + z} \right)^{-p} \frac{y}{y^{\frac{\beta}{2} + p} z^{\frac{\alpha}{2} + p}} =: J_1 + J_2. \tag{1.13} \]

A straightforward computation leads to
\[ J_1 \leq C \int_0^1 dy \int_{\frac{\gamma}{2} + \frac{p}{2}}^\gamma dz \frac{(z^{\gamma + p})^{\frac{\gamma}{2} + p}}{z^{\frac{\alpha}{2} + p}} < \infty \quad \text{if} \quad \gamma + 2p < 1, \]
\[ J_2 \leq C \int_\frac{\gamma}{2}^1 dy \int_{\frac{\gamma}{2} + \frac{p}{2}}^\gamma dz \frac{(z^{\gamma + p})^{\frac{\gamma}{2} + p}}{z^{\frac{\alpha}{2} + p}} \leq C \int_0^1 dy \int_{1-y}^1 dz \frac{(z^{\gamma + p})^{\frac{\gamma}{2} + p}}{z^{\frac{\alpha}{2} + p}} \leq C \int_0^1 \frac{dz}{z^{\frac{\alpha}{2} + p} + \gamma} < \infty \quad \text{if} \quad \gamma + 2p < 1. \]

Hence, \( J(x; f) \) is finite if \( \gamma + 2p < 1 \).

It has been seen in [10] that (1.11) is a necessary and sufficient condition for the existence of solutions of (1.6) including non-homogeneous kernels. A similar result for a slightly more restrictive class of coagulation kernels has been found in [8].
Notice that if (1.11) holds, a solution of the equation (1.6) is given by (1.12) with $C_0$ given by
\[
C_0 = b_{\Phi, \gamma} \sqrt{J_0} \text{ where } b_{\Phi, \gamma} = \left( \int_0^1 dy \int_{1-y}^{\infty} dz \frac{(y + z)^\gamma}{z^{\gamma+1}} \Phi \left( \frac{y}{y + z} \right) \right)^{-1/2}.
\] (1.14)

Notice that $b_{\Phi, \gamma} < \infty$, due to (1.11), since the integrand is bounded by $\frac{z^{\gamma-1}}{y^{\gamma+1}} \chi\{z \leq 1\} \chi\{0 \leq y \leq 1\}$ for $y \leq z$ and by $\frac{1}{z^{\gamma+1}} \chi\{\max\{1-z, 1/2\} \leq y \leq 1\}$ for $y > z$.

Our goal is to prove the existence of other solutions of (1.6) which are not power laws. These non-power law solutions will be shown to exist for a large class of homogeneous kernels with homogeneity $\gamma$.

The power law solutions (1.12) are well known and they have been extensively used in the analysis of coagulation problems (cf. [27], as well as [5] where an argument analogous to the one used in the framework of wave turbulence introduced in [31] has been applied to coagulation equations). On the other hand, it has been proved in [8] that the stationary solutions of the coagulation equation with injection, i.e., the solutions of the equation
\[
K[f](x, t) + \eta(x) = 0
\] (1.15)
where $\eta$ is compactly supported in $(0, \infty)$, behave as a constant flux solution as $x \to \infty$, i.e., as a solution of (1.5), (1.6). We remark that in the case of the constant kernel this result has been considered in [6], including the convergence to equilibrium.

It is natural to ask if the only possible solutions of (1.5), (1.6) are the power laws (1.12). In this paper, we prove that this is not the case. More precisely, we will prove that there exist kernels $K$ satisfying (1.7)–(1.11) for which, in addition to the power law solutions (1.12), there are other constant flux solutions, i.e., solutions of (1.5), (1.6) which have a form different from (1.12).
The class of non-power law solutions of (1.5), (1.6) will be obtained by means of a bifurcation argument. Specifically, we will show in this paper (cf. Theorem 1.1) that there are kernels \( K \) satisfying (1.7)–(1.9) for which there exist solutions of (1.5), (1.6) which have approximately the form
\[
f(x) \simeq \frac{C_0}{x^{\gamma + \frac{3}{2}}} \left[ 1 + \epsilon \cos (\alpha \log (x)) \right]
\] (1.16)
for some suitable \( \alpha > 0 \) and a small \( \epsilon \neq 0 \). We have illustrated one such solutions in Figure 1.

Actually, we will prove that the bifurcation of non-power law solutions to (1.5), (1.6) can be obtained for kernels \( K \) satisfying not only (1.9) but also the more restrictive condition
\[
\Phi (s) \sim as^{-p} \text{ as } s \to 0^+ \text{ with } p \in \mathbb{R}, \text{ with } a > 0.
\] (1.17)
Then, due to (1.8), we have also
\[
\Phi (s) \sim (1-s)^{-p} \text{ as } s \to 1^-.
\] (1.18)

Solutions of the Smoluchowski coagulation equation, as well as more general coagulation-fragmentation models, which exhibit oscillations in the volume variable \( x \) have been found for several choices of the coagulation kernel \( K \). In [15], it has been found that for coagulation \( K(x,y) \) kernels strongly concentrated along the line \( x \simeq y \) the self-similar solutions yielding the asymptotic behaviour of the solutions of the coagulation equation (without particle injection) develop an oscillatory behaviour. Asymptotic expansions of these oscillatory behaviours have been obtained in [19] and [23]. The papers [3, 4] provide a rigorous construction of a class of solutions of the coagulation-fragmentation equation with kernels \( K(x,y) \) concentrated near the line \( x = y \) for which the solutions converge asymptotically to a sequence of Dirac masses. In the particular case of the so-called diagonal kernels \( K(x,y) = x^{\gamma + 1} \delta (x-y) \), a full description of the long time asymptotics of the solutions of the Smoluchowski coagulation equation has been obtained in [18]. It turns out that the solutions to (1.2) with the kernel \( K(x,y) = x^{\gamma + 1} \delta (x-y) \) and \( \gamma < 1 \), exhibit oscillations for most of the initial data both in time as well as in the volume variable \( x \).

In all the examples described above, the results have been obtained for coagulation or coagulation-fragmentation equations without injection of monomers. Ref. [20] concerns a discrete coagulation equation for which injection of clusters with two very different sizes takes place, namely monomers and clusters with size \( I_s \gg 1 \). In the situation considered in [20], it would be natural to expect the solutions to behave for long times as stationary solutions which follow a power law for large cluster sizes. Indeed, this turns out to be the observed behaviour for large times. However, due to the large difference of sizes between the clusters of size \( I_s \) and the monomers, the stationary solutions oscillate in the variable \( x \) in a large interval of cluster sizes \( x \) until eventually the oscillations are damped and the stationary solution finally approaches a power law for large values of \( x \). Contrarily, in the stationary solutions that we construct here, the oscillations are present for all sizes and they result from properties of the coagulation kernel rather than from the source term.

The present work does not concern oscillations in time that have been obtained in the literature using both numerical and analytical methods (including rigorous results), cf. [21, 22, 25].
As remarked earlier, the condition \((1.11)\) is a necessary and sufficient condition for the existence of stationary solutions to \((1.15)\), i.e., with a source term. This problem has been considered in the case of the discrete coagulation equation for kernels of the form \(K (x, y) = x^{\gamma} y^{p - \gamma} + y^{\gamma} x^{p - \gamma}\) and if the source term \(\eta (x)\) is a Kronecker delta at the monomers. In \([13]\), formal asymptotic formulas for the concentration of large cluster sizes are derived using generating functions. These results indicate that solutions to \((1.15)\) exist only if the condition \((1.11)\) holds. More recently, in the case of both discrete and continuous coagulation equations and general coagulation kernels, it has been rigorously proved in \([8]\) that \((1.11)\) is a necessary and sufficient condition for the existence of solutions of \((1.15)\). In fact, the condition \((1.11)\) ensures the so-called locality property for the class of equations \((1.15)\): the most relevant collisions are those between particles with comparable sizes and not the collisions between particles with very different sizes (cf. \([16]\)). A similar property has been extensively used in the study of the class of kinetic equations arising in the theory of Wave Turbulence (cf. \([30]\)).

### 1.2 Main result of the paper

In order to formulate the main result of this paper, it is convenient to introduce some notation to characterize the class of admissible kernels. We will denote as \(\mathcal{K}_{\gamma, p}\) the class of continuous functions \(K \in C ((0, \infty)^2)\) of the form \(\Phi\) \((\gamma, \Phi)\) where \(\Phi \in C (0, 1)\) satisfies \(\Phi (s) > 0\) for \(s \in (0, 1)\) and the limit \(a := \lim_{s \to 0^+} [s^p \Phi (s)]\) exists and is strictly positive. We endow \(\mathcal{K}_{\gamma, p}\) with a structure of metric space by means of the metric

\[
\text{dist} (\Phi_1, \Phi_2) = \sup \{s^p |\Phi_1 (s) - \Phi_2 (s)| : s \in (0, 1)\} \quad (1.19)
\]

We note that the metric space \(\mathcal{K}_{\gamma, \lambda}\) is not complete because the strict positivity of the kernels can be lost taking limits.

The following theorem presents the main result of this paper.

**Theorem 1.1** Let \(J_0 > 0\). For each \(\gamma, p \in \mathbb{R}\) satisfying \((1.10)\) and \((1.11)\), there exists a one-parameter family of kernels \(K : I \to \mathcal{K}_{\gamma, p}\), with \(I = (-\delta, \delta)\), for some \(\delta > 0\). The mapping \(K\) is continuous if \(\mathcal{K}_{\gamma, p}\) is endowed with the topology generated by the metric \((1.19)\). Moreover, for each \(\delta_0 \in I \setminus \{0\}\) there are at least two different solutions of \((1.5), (1.6)\). One of the solutions is given by \((1.12)\) with \(C_0\) as in \((1.12)\). The second solution \(f_{\delta_0}\) has the property that there exists \(Q > 1\) (independent of \(\delta_0\)) such that \(f_{\delta_0} (Qx) = Q^{-\frac{\gamma + p}{2}} f_{\delta_0} (x)\) and the function \(f_{\delta_0}\) is not a power law of the form \((1.12)\) in the interval \((1, Q)\).

**Remark 1.2** The kernels obtained in the proof of the Theorem are of the form \((1.7), (1.8)\) with \(\Phi \in C^\infty (0, 1)\). As an example with \(\gamma = 0\), we have plotted a representation of the non-power law solution constructed in the proof in Figure 7.

### 1.3 Structure of the paper and main notation

To quantify asymptotic properties of functions, we rely here on the following fairly standard notations. We write “\(f \sim g\) as \(x \to \infty\)” to denote \(\frac{f(x)}{g(x)} \to 1\), as \(x \to \infty\). Moreover, given two functions \(f, g\), we write “\(f \approx g\) in an interval \(I \subset \mathbb{R}\)” if \(\frac{g(x)}{2} \leq f(x) \leq 2g(x)\) for \(x \in I \subset \mathbb{R}\), and the notation “\(f \ll g\)” is used if the quotient \(\frac{g(x)}{f(x)}\) can be made arbitrarily large for \(x\) sufficiently large.
The complex conjugate of $a \in \mathbb{C}$ is denoted by $\bar{a}$. The indicator function of any set $A \subset \mathbb{R}$ will be denoted by $\chi_A$.

The plan of the paper is the following. In Section 2.1, we reformulate the problem (1.5), (1.6) using a more convenient set of variables in such a way that power law solutions (1.12) become a constant solution. Then, in Section 2.2, we formulate the main properties of the linearized version of the problem around this power law solution which are proved later in Section 3. In Section 2.3, we study the full nonlinear problem using a Hopf bifurcation type of argument, concluding the proof of Theorem 1.1.

2 Proof of the main result

In order to prove Theorem 1.1, it is convenient to reformulate the problem (1.5)–(1.6) using a different set of variables in which the solutions (1.12) become constant. We will discuss later in this section how to linearize the problem (1.5) around the power law solution or, equivalently, the reformulated problem around the constant solution. The information obtained from the linearized problem will be used later to prove a bifurcation result for the full nonlinear problem that will imply Theorem 1.1.

2.1 Reformulation of the problem

Here, we reformulate the problem (1.5)–(1.6) so that the solutions (1.12) become constant. Notice that (1.5)–(1.6) is invariant under a rescaling group. In the new set of variables that we introduce in this section, the rescaling group becomes the group of translations. This will be convenient in order to bifurcate the non-constant flux solutions that we study in this paper.

We define a function $H : \mathbb{R} \to \mathbb{R}$ such that

$$H(X) = x^{\frac{\gamma}{2}} f(x), \quad x = e^X. \quad (2.1)$$

Then, setting $y = e^Y$, $z = e^Z$ we obtain

$$K(y, z) y f(y) f(z) = \frac{1}{y^{1+2} z^{1+2}} W(Y - Z) H(Y) H(Z)$$

where

$$W(Y) = \frac{1}{2} \left( e^{\frac{Y}{2}} + e^{-\frac{Y}{2}} \right) \gamma \left[ \Phi \left( \frac{1}{1 + e^Y} \right) + \Phi \left( \frac{1}{1 + e^{-Y}} \right) \right].$$

Using (1.8), we find that $\Phi \left( \frac{1}{1 + e^Y} \right) = \Phi \left( \frac{1}{1 + e^{-Y}} \right)$, and thus

$$W(Y) = \left( e^{\frac{Y}{2}} + e^{-\frac{Y}{2}} \right) \gamma \Phi \left( \frac{1}{1 + e^Y} \right) = \left( e^{\frac{Y}{2}} + e^{-\frac{Y}{2}} \right) \gamma \Phi \left( \frac{1}{1 + e^{-Y}} \right), \quad Y \in \mathbb{R}. \quad (2.2)$$

Therefore, $W$ is symmetric, $W(-Y) = W(Y)$. Moreover, (1.17), (1.18) imply

$$W(Y) \sim e^{(\frac{\gamma}{2} + p) |Y|} \text{ as } |Y| \to \infty. \quad (2.3)$$

We further observe that, given any function $W$ satisfying $W(Y) = W(-Y)$ and (2.3), we can obtain a function $\Phi : (0, 1) \to \mathbb{R}$ such that (2.2) holds. Indeed, using the change of
Figure 2: Representation of the shape of a typical function $\Phi > 0$ (cf. (2.4)) associated to the coagulation kernel (cf. (1.7)) considered in this paper, with $\gamma = 0$ and $p = 0$.

variable $\frac{1}{1+e^x} = s$ in (2.2), we obtain

$$
\Phi (s) = \frac{1}{\left(\sqrt{\frac{1-s}{s}} + \sqrt{\frac{s}{1-s}}\right)^\gamma} W \left( \log \left( \frac{1-s}{s} \right) \right), \ s \in (0, 1). \tag{2.4}
$$

A representation of a typical function $\Phi$ appearing later in the proof has been illustrated in Figure 2.

We can now reformulate the problem (1.5)–(1.6) as

$$
B (H_1, H_2; W) (X) = J_0, \ X \in \mathbb{R}, \tag{2.5}
$$

where

$$
B (H_1, H_2; W) (X) := \int_{-\infty}^{X} dY \int_{\log(1-e^{Y-X})}^{\infty} dZ \left[ e^{\frac{1}{2}(Y-Z)} W (Y-Z) \right] H_1 (Y) H_2 (Z). \tag{2.6}
$$

We will use repeatedly the following property of the operator $B$.

**Proposition 2.1** For each $W \in L^\infty_{\text{loc}} (\mathbb{R})$ satisfying (2.3) the operator $B$ defined by means of (2.6) defines a bilinear operator from $L^\infty (\mathbb{R}) \times L^\infty (\mathbb{R})$ to $\mathbb{R}$ which satisfies

$$
\| B (H_1, H_2; W) (\cdot) \|_{L^\infty (\mathbb{R})} \leq C_W \| H_1 (\cdot) \|_{L^\infty (\mathbb{R})} \| H_2 (\cdot) \|_{L^\infty (\mathbb{R})} \tag{2.7}
$$

where

$$
C_W = \int_{-\infty}^{0} dY \int_{\log(1-e^{Y})}^{\infty} dZ \left[ e^{\frac{1}{2}(Y-Z)} |W (Y-Z)| \right] < \infty. \tag{2.8}
$$
Note that the integral on the right-hand side of this formula is independent of $X$, hence

$$\sup_{X \in \mathbb{R}} \left( \int_{-\infty}^{X} dY \int_{X+\log(1-e^{-Y})}^{\infty} dZ \left[ e^{\frac{1}{2}(Y-Z)} |W(Y-Z)| \right] \right).$$

(2.9)

Proof: From the definition of the bilinear operator $B$ (cf. (2.6)) we immediately obtain the estimate

$$\|B(H_1, H_2; W)(\cdot)\|_{L^\infty(\mathbb{R})} \leq \|H_1(\cdot)\|_{L^\infty(\mathbb{R})} \|H_2(\cdot)\|_{L^\infty(\mathbb{R})}$$

$$= \sup_{X \in \mathbb{R}} \left( \int_{-\infty}^{X} dY \int_{X+\log(1-e^{-Y})}^{\infty} dZ \left[ e^{\frac{1}{2}(Y-Z)} |W(Y-Z)| \right] \right).$$

(2.9)

Note that the integral on the right-hand side of this formula is independent of $X$, hence

$$\int_{-\infty}^{0} dY \int_{\log(1-e^{-Y})}^{\infty} dZ \left[ e^{\frac{1}{2}(Y-Z)} |W(Y-Z)| \right] \leq C \int_{-\infty}^{0} dY \int_{\log(1-e^{-Y})}^{\infty} dZ \left[ e^{\frac{1}{2}(Y-Z)} e^{\left(\frac{1}{2}+p\right)Y-Z} \right]$$

where we used (2.3) in the last inequality. We now show that the integral on the right hand side of the equation above is finite. Indeed,

$$\int_{-\infty}^{0} dY \int_{\log(1-e^{-Y})}^{\infty} dZ \left[ e^{\frac{1}{2}(Y-Z)} e^{\left(\frac{1}{2}+p\right)Y-Z} \right]$$

$$= \int_{-\infty}^{0} dY \int_{\max\{\log(1-e^{-Y}), Y\}}^{\infty} dZ \left[ e^{\frac{1}{2}(Y-Z)} e^{\left(\frac{1}{2}+p\right)(Y-Z)} \right]$$

$$+ \int_{-\log(2)}^{0} dY \int_{\log(1-e^{-Y})}^{\max\{\log(1-e^{-Y}), Y\}} dZ \left[ e^{\frac{1}{2}(Y-Z)} e^{\left(\frac{1}{2}+p\right)(Y-Z)} \right]$$

$$\leq \int_{-\infty}^{0} dY \int_{\max\{1, -\log(2)\}}^{\infty} dZ \left[ e^{\left(\frac{1}{2}+p \cdot \frac{1}{2}\right)(Y-Z)} \right] + \int_{-\log(2)}^{0} dY \int_{\log(1-e^{-Y})}^{\max\{\log(1-e^{-Y}), -\log(2)\}} dZ \left[ e^{\left(\frac{1}{2}+\frac{1}{2}+p\right)(Y-Z)} \right]$$

$$\leq C_1(\gamma, p) + C_2(\gamma, p) \leq C_1(\gamma, p) + C_2(\gamma, p)$$

where $C_1(\gamma, p)$, $C_2(\gamma, p)$ are finite due to (1.11). Combining this with (2.9) we obtain (2.7) with $C_W$ as in (2.8).

In the variables (2.1), the constant flux solution (1.12), (1.14) becomes the following solution of (2.5)

$$H_s(X) = b_{\upphi, \gamma} \sqrt{J_0}$$

(2.10)

with $b_{\upphi, \gamma}$ as in (1.14). Notice for further reference that

$$B(H_s, H_s; W)(X) = J_0.$$  

(2.11)

Our goal is to prove the existence of solutions to (2.5) different from (2.10) using a bifurcation argument. More precisely, we will obtain solutions of (2.5) which are different from (2.10) but are close to constant for some particular choices of kernel. To this end we first consider a linearized version of (2.5).
2.2 Linearized problem near the constant solutions.

We first study the linearized problem obtained from (2.5) for small perturbations of (2.10). To this end, we write

\[ H(X) = H_s(X) [1 + \varphi(X)] = b_{q,\gamma} \sqrt{J_0} [1 + \varphi(X)]. \]  

(2.12)

Our goal is to obtain a class of kernels \( W_0 \) for which the linearized problem has non trivial solutions. Plugging (2.12) into (2.5), assuming that the kernel \( W = W_0 \), and neglecting quadratic terms in \( \varphi \), we obtain the linearized problem

\[ L(\varphi; W_0) \equiv B(1, \varphi; W_0) + B(\varphi, 1; W_0) = 0. \]  

(2.13)

Using (2.6) we can rewrite (2.13) as

\[ L(\varphi; W_0)(X) = \int_{-\infty}^{X} dY \int_{X + \log(1 - e^{Y - X})}^{\infty} dZ \left[ e^{\frac{1}{2}(Y - Z)} W_0(Y - Z) \right] (\varphi(Y) + \varphi(Z)) = 0. \]  

(2.14)

Since we want to obtain solutions of the nonlinear problem (2.5) by means of a perturbative argument, it is natural to look for bounded, non trivial solutions of (2.14). Moreover, due to the fact that the operator \( L(\cdot; W_0) \) commutes with the group of translations, we look for solutions of (2.14) with the form \( \varphi(X) = e^{ikX} \) for some \( k \in \mathbb{R} \). In order to have nonconstant solutions, we need \( k \in \mathbb{R} \setminus \{0\} \). We later prove the following result.

**Theorem 2.2** For each \( \gamma, p \) satisfying (1.10) and (1.11), there exists a function \( W_0 = W_0(X), W_0 \in H^\infty(D_\beta) \) where \( H^\infty(D_\beta) \) denotes the Hardy space (cf. [20]) of bounded, analytic functions in the open domain

\[ D_\beta = \{ X \in \mathbb{C} : |Im(X)| < (1 + \beta |Re(X)|) \} \subset \mathbb{C} \]  

(2.15)

for some \( \beta > 0 \) (depending on \( \gamma, p \)). Moreover, \( W_0 \) satisfies

(i) \( W_0(X) = W_0(-X) \) for \( X \in D_\beta \)

(ii) \( W_0(X) > 0 \) for \( X \in \mathbb{R} \)

(iii) the following asymptotics (cf. (2.3))

\[ W_0(X) \sim \exp \left( \left( \frac{\gamma}{2} + p \right) \sqrt{X^2 + 1} \right) \left[ 1 + O \left( e^{-\kappa|X|} \right) \right] \]  

(2.16)

as \( |X| = \sqrt{(Re(X))^2 + (Im(X))^2} \to \infty \), \( X \in D_\beta \) with \( \kappa > 0 \).

Moreover, there exists a function \( \Psi(k; W_0) \) analytic in \( D_\beta \) such that

\[ L(e^{ik}; W_0) = \Psi(k; W_0) e^{ik}, \quad k \in \mathbb{R}. \]  

(2.17)

The function \( \Psi(k; W_0) \) satisfies \( \Psi(-k; W_0) = \overline{\Psi(k; W_0)} \) and there exists \( k_s \in \mathbb{R}, \ k_s > 0 \) such that

\[ \Psi(k_s; W_0) = \Psi(-k_s; W_0) = 0 \]  

(2.18)
and with the property that \( \Psi(k;W_0) \neq 0 \) for any \( k \in \mathbb{R} \) such that \(|k| > |k_s|\). The asymptotic behavior of the function \( \Psi(k;W_0) \) as \(|k| \to \infty\) is

\[
\Psi(k;W_0) \sim a \operatorname{sgn}(k)e^{\frac{\pi \operatorname{sgn}(k)}{2}(\frac{\gamma}{2}+p-\frac{1}{2})|k|^\frac{1}{2}+(\frac{\gamma}{2}+p)} \quad \text{as} \quad |k| \to \infty
\]  

(2.19)

where

\[
a = \frac{2i}{1+\gamma+2p} \Gamma \left( \frac{1}{2} - \left( \frac{\gamma}{2} + p \right) \right).
\]

**Remark 2.3** Theorem 2.2 implies that for \( W_0 \) as in that Theorem we have

\[
\mathcal{L} \left( e^{ik_0};W_0 \right) (X) \equiv B \left( 1,e^{ik_0};W_0 \right) (X) + B \left( e^{ik_0},1;W_0 \right) (X) = 0 \quad \text{for} \quad X \in \mathbb{R}.
\]

Given that \( W_0 \) is real in the real line this implies, taking real and imaginary parts

\[
\mathcal{L} \left( \cos(k_0 \cdot);W_0 \right) (X) \equiv B \left( 1,\cos(k_0 \cdot);W_0 \right) (X) + B \left( \cos(k_0 \cdot),1;W_0 \right) (X) = 0 \quad \text{for} \quad X \in \mathbb{R}
\]

\[
\mathcal{L} \left( \sin(k_0 \cdot);W_0 \right) (X) \equiv B \left( 1,\sin(k_0 \cdot);W_0 \right) (X) + B \left( \sin(k_0 \cdot),1;W_0 \right) (X) = 0 \quad \text{for} \quad X \in \mathbb{R}.
\]

Theorem 2.2 will be proved in Section 3.

**Remark 2.4** It would be possible to prove the results of this paper with functions \( W_0 \) satisfying just some differentiability conditions, instead of the analyticity condition formulated in Theorem 2.2. The main reason to use analytic functions in a wedge extending toward infinity is because this allows to obtain automatically estimates for the derivatives of \( W \). In particular, the validity of the asymptotic formula (2.19) in a wedge implies the validity of asymptotic formulas for the derivatives along the real line. These formulas are obtained by just formally differentiating both sides of the asymptotic formula (2.19); the results is a consequence of the classical Cauchy estimates for analytic functions.

Under the analyticity assumption the solutions that we obtain are very regular. In particular, the singularities of the function \( \Phi(s) \) as \( s \to 0^+ \) or \( s \to 1^- \) are given by some power law.

### 2.3 Bifurcation of non-power law constant flux solutions.

We recall that our goal is to obtain solutions of (2.5) with \( B \) as in (2.6). Due to the invariance of the problem under rescaling of \( H \), we can assume that \( b_{k_s} \sqrt{\mathcal{J}_0} = 1 \). Then \( H_s(X) = 1 \). Furthermore, we choose \( k_s \) as in Theorem 2.2 and then set \( T = \frac{2\pi}{k_s} \). Our goal is to construct kernels \( W \), close in some suitable sense to \( W_0 \) (cf. the metric (1.19)), and nonconstant functions \( H \) having a period \( T \) such that

\[
B(H,H;W)(X) = J_0 = \frac{1}{(b_{k_s} \gamma)^2}, \quad X \in \mathbb{R}.
\]

(2.20)

Our plan is to obtain \( H \) which behaves approximately as \( 1 + s \cos(kX) \) where \( s \) is a small real number. Due to the invariance of the problem under translations, we could equally well obtain functions \( H \) behaving approximately as \( 1 + s \cos(k_s(X-X_0)) \), with \( X_0 \in \mathbb{R} \) and \( s \) small.

We describe now in detail the functional spaces in which the operator \( B \) acts. We will assume that the operator acts on spaces of real functions.
We will denote as $H^1_{\text{per}}([0, T])$ the Hilbert space obtained as the closure of the functions $f : [0, T] \rightarrow [0, T]$ satisfying

$$f \in C^\infty([0, T]) \quad , \quad \partial_X^\ell f (0) = \partial_X^\ell f (T) \quad , \quad \text{for any } \ell = 0, 1, 2, 3, \ldots$$  \tag{2.21}

with the norm

$$\|f\| = \left( \int_0^T \left[ |f(X)|^2 + |\partial_X f (X)|^2 \right] \, dX \right)^{\frac{1}{2}}. \quad \tag{2.22}$$

Notice that we can identify the elements of the space $H^1_{\text{per}}([0, T])$ with the functions $f : \mathbb{R} \rightarrow \mathbb{R}$ which satisfy $f (X) = f (X + T)$ for each $X \in \mathbb{R}$ as well as $\|f\| < \infty$ with $\|\cdot\|$ as in (2.22). With this identification, we have

$$\|f\| = \left( \int_a^{a+T} \left[ |f(X)|^2 + |\partial_X f (X)|^2 \right] \, dX \right)^{\frac{1}{2}} \quad \text{for each } a \in \mathbb{R}.$$  

Morrey’s inequality implies that $f$ is Hölder continuous in the whole real line as well as the estimate $\|f\|_{L^{\infty}(\mathbb{R})} \leq C \|f\|$ for some constant $C$ depending on $T$.

We will use also the spaces $H^s_{\text{per}}([0, T])$ with $s > 0$. These spaces are the closure of (2.21) with the norm

$$\|f\|_{H^s_{\text{per}}} = \left( \sum_{n \in \mathbb{Z}} \left( 1 + |n|^{2s} \right) |a_n|^2 \right)^{\frac{1}{2}} \quad \text{with} \quad a_n = \frac{1}{T} \int_0^T f (X) e^{-\frac{2\pi inX}{T}} \, dX.$$  \tag{2.23}

Suppose that $k_\ast$ is as in Theorem 2.2. We now introduce two subspaces of $H^1_{\text{per}}([0, T])$ as follows

$$V_1 = \text{span} \left\{ \cos (k_\ast \cdot), \sin (k_\ast \cdot) \right\}, \quad V_2 = \text{span} \left\{ \cos (nk_\ast \cdot), \sin (nk_\ast \cdot) : n \neq 0, -1, 1, n \in \mathbb{Z} \right\}$$  \tag{2.24}

where span denotes finite linear combinations and the closure is with respect to the topology of $H^1_{\text{per}}([0, T])$. We define subspaces

$$Z_0 = \text{span} \left\{ 1 \right\},$$

$$Z_1 = \text{span} \left\{ \cos (k_\ast \cdot), \sin (k_\ast \cdot) \right\},$$

$$Z_2 = \text{span} \left\{ \cos (nk_\ast \cdot), \sin (nk_\ast \cdot) : n \neq 0, -1, 1, n \in \mathbb{Z} \right\}$$  \tag{2.25}

where the closure here is understood with respect to the topology of $L^2([0, T])$. We will denote as $P_0, P_1, P_2$ the orthogonal projection of $L^2([0, T])$ into $Z_0$, $Z_1, Z_2$ respectively. These projections are given by

$$P_0 f = \int_0^1 f(sT) \, ds, \quad P_1 f = 2 \cos (k_\ast x) \int_0^1 \cos (2\pi s) f(sT) \, ds + 2 \sin (k_\ast x) \int_0^1 \sin (2\pi s) f(sT) \, ds,$$

$$P_2 = I - P_0 - P_1.$$

We now remark the following. Suppose that $W$ is a locally bounded function that satisfies $W (Y) = W (-Y)$ and (2.5). Then, using that $\|f\|_{L^{\infty}(\mathbb{R})} \leq C \|f\|$ we readily obtain that

$$\|B(H, H; W)\|_{L^2} \leq C_1 \|B(H, H; W)\|_{L^{\infty}} \leq C_2 \|H\|^2.$$
where $C_1, C_2$ depend on $T$ and $W$. Our goal is to obtain nonconstant solutions of \ref{eq:2.20}. We now indicate the strategy that we will follow to prove the existence of such solutions. Notice that due to the homogeneity of $B(H;H;W)$ in $H$, if we obtain a solution of the problem

$$B \left( \tilde{H}, \tilde{H}; W \right) = K_0$$

\tag{2.26}

for some $K_0 > 0$, we would then obtain a solution of \ref{eq:2.20} by means of $H = \sqrt{K_0} \tilde{H}$. Applying the operators $P_1, P_2$ into the equation \ref{eq:2.26} we obtain

$$P_1 B \left( \tilde{H}, \tilde{H}; W \right)(X) = P_2 B \left( \tilde{H}, \tilde{H}; W \right)(X) = 0.$$  

\tag{2.27}

Our plan is to prove the existence of solutions of \ref{eq:2.27} for some function $\tilde{H}$ close to 1. Using that $B(1,1;W) > 0$ it will then follow using a continuity argument that \ref{eq:2.26} holds for some $K_0 > 0$. More precisely, we look for a solution of the equations \ref{eq:2.27} with the form

$$\tilde{H}(X) = 1 + U(X), \quad U(X) = s\varphi(X) + \psi(X,s)$$

\tag{2.28}

where

$$\varphi \in V_1, \quad \psi \in V_2$$

\tag{2.29}

and $|s|$ is small and where $W$ is close to $W_0$ in the uniform convergence norm. The solution that we will obtain will satisfy $||\psi|| \leq Cs^2$. Under these assumptions we will obtain

$$P_0 B \left( \tilde{H}, \tilde{H}; W \right) = P_0 B(1,1;W) + P_0 B(1,U;W) + P_0 B(U,1;W) + P_0 B(U,U;W)$$

and using that $P_0 B(1,U;W) = P_0 B(U,1;W) = 0$ as well as the fact that $B(1,1;W)$ is constant, we obtain

$$P_0 B \left( \tilde{H}, \tilde{H}; W \right) = B(1,1;W) + P_0 B(U,U;W) \equiv K_0.$$  

Since $W$ is close to $W_0$ in the uniform convergence norm, we obtain that $B(1,1;W) = \frac{1}{(b_{w_0})^2} > 0$. Therefore, assuming that $|s|$ is sufficiently small and using also \ref{eq:2.27}, we obtain that $\tilde{H}$ solves \ref{eq:2.26} for some $K_0 > 0$.

We have then reduced the problem \ref{eq:2.20} to finding a solution of \ref{eq:2.27}--\ref{eq:2.29} for $|s|$ small and with $\psi$ satisfying $||\psi|| \leq Cs^2$. Due to the invariance of the problem \ref{eq:2.20} under translations in the variable $X$ we can assume without loss of generality that $\varphi(X) = \cos(kX)$.

We now introduce an auxiliary linear operator $\mathcal{L}$. For each $W \in L^\infty(\mathbb{R})$ satisfying $W(Y) = W(-Y)$, $Y \in \mathbb{R}$ and the asymptotics \ref{eq:2.3}, we define (cf. \ref{eq:2.13})

$$\mathcal{L}(\varphi; W) \equiv B(1,\varphi; W) + B(\varphi,1;W).$$  

\tag{2.30}

We have that for each $W$ satisfying the previous assumptions the operator $\mathcal{L}(\cdot; W)$ is well defined in $H^1_{per}([0,T])$. Moreover,

$$\mathcal{L}(V_1; W) \subset Z_1, \quad \mathcal{L}(V_2; W) \subset Z_2$$

\tag{2.31}

with $V_1$, $V_2$ as in \ref{eq:2.24} and $Z_1$, $Z_2$ are as in \ref{eq:2.25}. Choosing $W_0$ as in Theorem \ref{thm:2.2} we have also $\mathcal{L}(V_1; W_0) = \{0\}$ and in particular

$$\mathcal{L}(\cos(k_s\cdot); W_0) = 0.$$  

\tag{2.32}
We now make precise the choice of $W$. We will choose $W$ with the form
\begin{equation}
W = W_0 + W_1
\end{equation}
where $W_1$ is the linear combination of two functions $W_{1,1}$, $W_{1,2}$ which are analytic in the domain $D_\beta \subset \mathbb{C}$, $\beta > 0$, introduced in the statement of Theorem 2.2 (cf. (2.15)). Moreover, $W_{1,1}$, $W_{1,2}$ tend to zero as $|Y| \to \infty$ and satisfy
\begin{equation}
W_{1,1}(Y) = W_{1,1}(-Y) \quad \text{and} \quad W_{1,2}(Y) = W_{1,2}(-Y) \quad \text{for} \quad Y \in D_\beta.
\end{equation}

We have the following result.

**Lemma 2.5** Let $k_* > 0$ be as in Theorem 2.2. There exist two functions $W_{1,1}$, $W_{1,2}$ analytic in $D_\beta$ for some $\beta > 0$, real in the real axis, satisfying (2.34) as well as the estimate
\begin{equation}
|W_{1,1}(Y)| + |W_{1,2}(Y)| \leq \exp \left( -a|Y|^2 \right) \quad \text{for} \quad Y \in D_\beta
\end{equation}
for some $a > 0$, such that
\begin{equation}
\text{span}\{ \mathcal{L}(\varphi; W_{1,1}), \mathcal{L}(\varphi; W_{1,2}) \} = Z_1
\end{equation}
with $\varphi_0(X) = \cos(k_*X)$.

**Proof:** Suppose that $W_{1,1}$, $W_{1,2}$ are two functions analytic in $D_\beta$, satisfying (2.34), as well as $W_{1,j}(Y) \in \mathbb{R}$ for $Y \in \mathbb{R}$, and that decay sufficiently fast to ensure that $|\Psi(k_*; W_{1,1})|$, $|\Psi(k_*; W_{1,2})|$ defined by means of (3.1), (3.16) are finite. Using (2.17) we can write
\begin{equation}
\mathcal{L}(\varphi; W_{1,j}) = \frac{1}{2} \left[ \Psi(k_*; W_{1,j}) e^{ik_*} + \Psi(-k_*; W_{1,j}) e^{-ik_*} \right], \quad j = 1, 2
\end{equation}
where $\Psi(k_*; W_{1,j})$ is as in (3.16), (3.1). Using that $\Psi(-k_*; W_{1,j}) = \overline{\Psi(k_*; W_{1,j})}$ we can rewrite (2.37) as
\begin{equation}
\mathcal{L}(\varphi; W_{1,j}) = \text{Re} \left( \Psi(k_*; W_{1,j}) e^{ik_*} \right), \quad j = 1, 2.
\end{equation}

Writing $\Psi(k_*; W_{1,j})$ in polar form $|\Psi(k_*; W_{1,j})| e^{i\arg(\Psi(k_*; W_{1,j}))}$ we readily see that the functions $\{ \mathcal{L}(\varphi; W_{1,j}) \}_{j=1,2}$ are linearly independent in $L^2(\mathbb{R})$ iff the vectors
\begin{align*}
\left( \begin{array}{c}
\text{Re} \left( \Psi(k_*; W_{1,1}) \right) \\
\text{Im} \left( \Psi(k_*; W_{1,1}) \right)
\end{array} \right), \quad \left( \begin{array}{c}
\text{Re} \left( \Psi(k_*; W_{1,2}) \right) \\
\text{Im} \left( \Psi(k_*; W_{1,2}) \right)
\end{array} \right)
\end{align*}
considered as elements of $\mathbb{R}^2$ are linearly independent. Using (3.16), (3.1) we easily see that in the case of $W_{1,j}(z) = \delta(z - z_j), \quad j = 1, 2$, for some $z_1, z_2 \in \mathbb{R}$ we would have
\begin{equation}
\Psi(k_*; W_{1,j}) = \frac{G(z_j, k_*)}{ik_*}, \quad j = 1, 2,
\end{equation}

Since the function $z \to \arg(G(z, k_*))$ for a fixed value of $k_* > 0$ is not constant, we obtain the existence of two values $z_1, z_2$ yielding the desired linear independence conditions. In order to obtain $W_{1,1}$, $W_{1,2}$ with the desired symmetry, analyticity conditions and decay at infinity we argue as in the proof of Theorem 2.2. More precisely, we replace the Dirac masses $\delta(z - z_j), \quad j = 1, 2$ by the functions $\zeta_\epsilon(z - z_1), \quad \zeta_\epsilon(z - z_2)$ where the function $\zeta_\epsilon$ is as in (3.19) and $\epsilon > 0$. Using the continuity of the functions $W_{1,j} \mapsto \Psi(k_*; W_{1,j}), \quad j = 1, 2$, in the weak topology, we obtain functions $W_{1,1}$, $W_{1,2}$ with the properties stated in the Lemma. \(\Box\)
We can now continue with the analysis of the nonlinear bifurcation problem which has been reduced to the analysis of (2.27)-(2.29) with \(|s|\) small. We will prove now the following result.

**Theorem 2.6** Suppose that \(W_0, k, \) are as in Theorem 2.3 and \(W_{1,1}, W_{1,2} \) are as in Lemma 2.5. Let \(\varphi_0 (X) = \cos (k, X)\). Then, there exists \(s_0 > 0\) and a constant \(C > 0\) depending only on \(W_0, W_{1,1}, W_{1,2} \) such that, for any \(s \in \mathbb{R} \) satisfying \(|s| \leq s_0\) there exist \(\alpha_1, \alpha_2 \in \mathbb{R} \) and \(\psi \in V_2\) satisfying \(\|\psi(\cdot, s)\| \leq C |s|^2\) such that the function \(H\) defined by means of (2.28) solves (2.27) with

\[
W = W_0 + \alpha_1 W_{1,1} + \alpha_2 W_{1,2}
\]

(2.38)

where

\[
W > 0.
\]

(2.39)

**Remark 2.7** The value of \(s_0\) depends on the function \(W_0\) in Theorem 2.2. As a matter of fact, the construction of the function \(W_0\) that will be made in Section 3 will depend on several parameters \(z_a, z_b, a, b, \varepsilon\), which are assumed to be fixed. Then, the parameter \(s_0\) depends on all these parameters.

**Proof of Theorem 2.6** First part. We first reformulate the problem (2.27) as a fixed point problem for a suitable operator. Using (2.6) as well as the definition of the operator \(L\) in (2.30) we obtain

\[
B\left(1 + U, 1 + W; W\right) = B\left(1, 1; W\right) + L\left(U; W\right) + B\left(U, U; W\right).
\]

Taking now the operators \(P_1\) and \(P_2\) of this expression and using that \(P_j B\left(1, 1; W\right) = 0\) for \(j = 1, 2\) we can rewrite (2.27) as

\[
P_1 L\left(U; W\right) + P_1 B\left(U, U; W\right) = 0
\]

\[
P_2 L\left(U; W\right) + P_2 B\left(U, U; W\right) = 0.
\]

Using now (2.31) and the formula of \(U\) in (2.28) we obtain

\[
s P_1 L\left(\varphi; W\right) + P_1 B\left(U, U; W\right) = 0, \quad P_2 L\left(\psi; W\right) + P_2 B\left(U, U; W\right) = 0.
\]

(2.40)

We write \(W_1 = \alpha_1 W_{1,1} + \alpha_2 W_{1,2}\). Then, assuming that \(W\) has the form (2.38) we obtain

\[
s P_1 L\left(\varphi_0; W_0\right) + s P_1 L\left(\varphi_0; W_1\right) + P_1 B\left(U, U; W\right) = 0.
\]

Using then (2.32) we have \(L\left(\varphi_0; W_0\right) = 0\). On the other hand, we have that \(L\left(\varphi_0; W_1\right) \in Z_1\). Then

\[
s L\left(\varphi_0; W_1\right) + P_1 B\left(U, U; W\right) = 0
\]

or equivalently

\[
\alpha_1 L\left(\varphi_0; W_{1,1}\right) + \alpha_2 L\left(\varphi_0; W_{1,2}\right) = -\frac{1}{s} P_1 B\left(U, U; W_0 + W_1\right).
\]

(2.41)

Due to Lemma 2.5 we have that the vectors \(\{L\left(\varphi_0; W_{1,1}\right), L\left(\varphi_0; W_{1,2}\right)\}\) span \(Z_1\). Therefore, there exist two linear forms \(\ell_j : Z_1 \rightarrow \mathbb{R}, j = 1, 2\), such that for any \(w \in Z_1\) we have

\[
w = \ell_1 \left(w\right) L\left(\varphi_0; W_{1,1}\right) + \ell_2 \left(w\right) L\left(\varphi_0; W_{1,2}\right).
\]
Notice that $|\ell_1(w)| + |\ell_2(w)| \leq C \|w\|_{L^1(\mathbb{R})}$. The constant $C$ depends on the functions $W_{1,1}, W_{1,2}$ but these will be assumed to be assumed to be fixed in all the remaining argument.

Then, due to (2.41), it follows that the constants $\alpha_1, \alpha_2$ in (2.38) must be chosen as

$$\alpha_1 = -\frac{1}{s} \ell_1(P_1 B (U, U; W_0 + W_1)), \quad \alpha_2 = -\frac{1}{s} \ell_2(P_1 B (U, U; W_0 + W_1)).$$

(2.42)

This choice of $\alpha_1, \alpha_2$ implies the first equation in (2.40). Notice that (2.42) is an equation for the coefficients $\alpha_1, \alpha_2$ because $W_1$ depends on both of them. Nevertheless, it has a suitable form for a fixed point argument. It only remains to reformulate the second equation in (2.40). This requires to examine the invertibility properties of the operator $P_2 \mathcal{L}$. Indeed, this follows from the fact that $\Psi (nk_s; W) \neq 0$ for any $n \neq 0, \pm 1$. Indeed, this follows from the fact that $\Psi (k; W_0) \neq 0$ for $|k| > k_s$ as well as the fact that the functions $\Psi (k; W)$ depend continuously on the function $W$ in the uniform topology of measures. Moreover, the asymptotics of $\Psi (k; W_0)$ as $|k| \to \infty$ is given by (2.19) and this asymptotic behaviour is not modified adding to $W_0$ the function $\alpha_1 W_{1,1} + \alpha_2 W_{1,2}$. Therefore, the claim follows if $s_0$ is small enough.

The operator $P_2 \mathcal{L} (\cdot; W)$ acts in Fourier as follows. A function $f \in Z_2$ can be represented by means of a Fourier series with the form

$$f (X) = \sum_{n \in \mathbb{Z} \backslash \{0 \pm 1\}} a_n e^{ink_X},$$

where in addition we have $a_n = a_{-n}$. Then, using (2.17) we would obtain

$$A_W (f) (X) := P_2 \mathcal{L} (f; W) (X) = \sum_{n \in \mathbb{Z} \backslash \{0 \pm 1\}} \Psi (nk_s; W) a_n e^{ink_X}. \tag{2.43}$$

The expression (2.43) defines an operator acting on $f \in H^1_{\text{per}} ([0, T]) \cap Z_2$ for each function $W$ as in (2.38).

Using (2.19), it then follows that $A_W (f) \in H^{1-\frac{1}{4} p} (\mathbb{R}) \cap \{a_0 = a_1 = 1, a_{-1} = 0\} = H^{1-\frac{1}{4} p} ([0, T]) \cap Z_2$ where the spaces $H^{1-\frac{1}{4} p} (\mathbb{R})$ are endowed with the norm (2.23). Therefore, the inverse $(A_W)^{-1}$ is defined in the space $H^{1-\frac{1}{4} p} ([0, T]) \cap Z_2$ and it transforms this space in a subspace of $H^1_{\text{per}} ([0, T])$. Moreover, we have

$$\left\| (A_W)^{-1} (\varphi) \right\|_{H^{1-\frac{1}{4} p} (\mathbb{R})} \leq C \left\| \varphi \right\|_{H^{1-\frac{1}{4} p} ([0, T])} \tag{2.44}$$

for each $\varphi \in H^{1-\frac{1}{4} p} ([0, T]) \cap Z_2$.

We can now rewrite the second equation in (2.40) in a more convenient form in order to reformulate (2.40) as a fixed point problem for $(\alpha_1, \alpha_2, \psi)$. More precisely, suppose that we have $B (U, U; W) \in H^{1-\frac{1}{4} p} ([0, T])$ for any $U \in H^1_{\text{per}} ([0, T])$. We can then write

$$P_2 \mathcal{L} (\psi; W_0) + P_2 \mathcal{L} (\psi; W_1) = -P_2 B (U, U; W),$$

$$\psi = - (A_W)^{-1} (P_2 \mathcal{L} (\psi; W_1)) - (A_W)^{-1} (P_2 B (U, U; W_0 + W_1)). \tag{2.45}$$
Notice that (2.42), (2.45) yield a reformulation of the problem (2.40) as a fixed point problem for \((\alpha_1, \alpha_2, \psi)\), assuming that \(W_1 = \alpha_1 W_{1,1} + \alpha_2 W_{1,2}, \; U = s\varphi + \psi\). More precisely, if we define the mapping

\[
T \left( \begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\psi
\end{array} \right) = \left( \begin{array}{c}
-\frac{1}{2}\ell_1 (P_1 B (U; U; W_0 + W_1)) \\
-\frac{1}{2}\ell_2 (P_1 B (U; U; W_0 + W_1)) \\
-(\mathcal{A} W)^{-1} (P_2 L (\psi; W_1)) - (\mathcal{A} W)^{-1} (P_2 B (U; U; W_0 + W_1))
\end{array} \right)
\]

we have that (2.42), (2.45) can be rewritten as

\[
\left( \begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\psi
\end{array} \right) = T \left( \begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\psi
\end{array} \right).
\]

In order to conclude the proof of Theorem 2.6 we must check that the operator in (2.46) is well defined for \(U \in H^1_{\text{per}} \left([0, T]\right)\). Specifically we need to prove that \(B (U; U; W)\) transforms \(U \in H^1_{\text{per}} \left([0, T]\right)\) into \(H^{1-\frac{1}{2} + p}_{\text{per}} \left([0, T]\right)\). To this end we prove the following Lemma.

**Lemma 2.8** Let \(k_*\) be as in Theorem 2.2. Suppose that \(W\) is analytic in a domain \(D_\beta\) for some \(\beta > 0\). Suppose that (2.16) holds, with \(\gamma + 2p < 1\). Let \(B\) be as in (2.6). Then, the following estimate holds

\[
\|B (U_1, U_2; W)\|_{H^{1-\frac{1}{2} + p}_{\text{per}} \left([0, T]\right)} \leq C\|U_1\| \|U_2\|, \; U_1, U_2 \in H^1_{\text{per}} \left([0, T]\right)
\]

where \(C\) depends only on \(\gamma, p, \kappa, k_*\), the multiplicative constant in the term \(O(e^{-\kappa |X|})\) and the supremum of \(W\) in \(D_\beta\).

**Remark 2.9** Notice that \(W_0\) satisfies also (2.16) due to (2.38) and Lemma 2.3.

**Proof:** Suppose that \(U_1, U_2 \in H^1_{\text{per}} \left([0, T]\right)\). We can represent them as

\[
U_1 (X) = \sum_{n \in \mathbb{Z}} a_n e^{inkX}, \quad U_2 (X) = \sum_{n \in \mathbb{Z}} b_n e^{inkX}, \quad a_n = a_{-n}, \quad b_n = b_{-n},
\]

where we recall that \(k_* = \frac{2\pi}{\lambda}\). Notice that the series defining \(U_1\) converges absolutely since

\[
\sum_{n \in \mathbb{Z}} |a_n| \leq \left(\sum_{n \in \mathbb{Z}} n^2 |a_n|^2\right)^{\frac{1}{2}} \left(\sum_{n \in \mathbb{Z}} \frac{1}{n^2}\right)^{\frac{1}{2}} \leq C \|U_1\|. \quad \text{The same argument applies to } U_2.
\]

Then, using the definition of the operator \(B\) we obtain

\[
B (U_1, U_2; W) (X) = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} a_m b_n \int_{-\infty}^{X} dY \int_{X + \log(1 - e^{-Y})}^{\infty} dZ \left[ e^{\frac{1}{2}(Y-Z)W(Y-Z)} \right] e^{imkX} e^{inkZ}.
\]

\[
= \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} J_{m,n} a_m b_n e^{i(m+n)k_*X} \sum_{\ell \in \mathbb{Z}} \left[ \sum_{m+n = \ell} J_{m,n} a_m b_n \right] e^{ik_*X}
\]

where

\[
J_{m,n} = \int_{-\infty}^{0} dY \int_{\log(1 - e^{-Y})}^{\infty} dZ \left[ e^{\frac{1}{2}(Y-Z)W(Y-Z)} \right] e^{imk_*X} e^{ink_*Z}.
\]

17
Notice that the coefficients $J_{m,n}$ are well defined due to (2.16). We can compute them, using the change of variables $\xi = Z - Y$, $d\xi = dZ$ and applying Fubini's Theorem as follows

$$J_{m,n} = \int_{-\infty}^{0} e^{ink_{s}Y} dY \int_{\log(1-\epsilon^{\eta})}^{\infty} dZ \left[ e^{-\frac{\xi}{2}(Z-Y)} W(Z-Y) \right] e^{ink_{s}Z}$$

$$= \int_{-\infty}^{0} e^{i(m+n)k_{s}Y} dY \int_{-Y+\log(1-\epsilon^{\eta})}^{\infty} \left[ e^{-\frac{\xi}{2}e^{ink_{s}\xi}} W(\xi) \right] d\xi$$

$$= \int_{-\infty}^{\infty} e^{-\frac{\xi}{2}e^{ink_{s}\xi}} W(\xi) d\xi \int_{-\log(\epsilon^{\eta}+1)}^{0} e^{i(m+n)k_{s}Y} dY$$

$$= \frac{1}{i(m+n)k_{s}} \int_{-\infty}^{\infty} e^{-\frac{\xi}{2}e^{ink_{s}\xi}} W(\xi) \left[ 1 - \frac{1}{(e^{\eta}+1)^{i(m+n)k_{s}}} \right] d\xi \quad (2.50)$$

if $m + n \neq 0$, and

$$J_{m,n} = \int_{-\infty}^{\infty} e^{-\frac{\xi}{2}e^{ink_{s}\xi}} W(\xi) \log \left( e^{\xi} + 1 \right) d\xi \quad \text{if } m + n = 0. \quad (2.51)$$

Due to (2.49), in order to estimate the Fourier coefficients of $B(U_{1},U_{2};W)$ we need to derive bounds for the sums

$$\sum_{m+n=\ell} J_{m,n} a_{m}b_{n}. \quad (2.52)$$

Using (2.50) we obtain that the coefficients $J_{m,n}$ with $m + n = \ell \neq 0$ are given by

$$J_{m,n} = \frac{1}{i\ell k_{s}} \int_{-\infty}^{\infty} e^{-\frac{\xi}{2}e^{ink_{s}\xi}} W(\xi) \left[ 1 - \frac{1}{(e^{\eta}+1)^{i(m+n)k_{s}}} \right] d\xi, \quad m + n = \ell. \quad (2.53)$$

In order to estimate these coefficients we write

$$J_{m,n} = J_{m,n}^{(1)} + J_{m,n}^{(2)} \quad (2.54)$$

where

$$J_{m,n}^{(1)} = \frac{1}{i\ell k_{s}} \int_{-\infty}^{0} e^{-\frac{\xi}{2}e^{ink_{s}\xi}} W(\xi) \left[ 1 - \frac{1}{(e^{\eta}+1)^{i(m+n)k_{s}}} \right] d\xi$$

$$J_{m,n}^{(2)} = \frac{1}{i\ell k_{s}} \int_{0}^{\infty} e^{-\frac{\xi}{2}e^{ink_{s}\xi}} W(\xi) \left[ 1 - \frac{1}{(e^{\eta}+1)^{i(m+n)k_{s}}} \right] d\xi.$$

Using (2.16) and that $k_{s} > 0$, we obtain

$$\left| J_{m,n}^{(2)} \right| \leq \frac{C}{|\ell| k_{s}}, \quad \ell \neq 0. \quad (2.55)$$

In order to estimate $J_{m,n}^{(1)}$ we use the change of variables $\eta = \log \left( 1 + e^{\xi} \right)$, whence $\xi = \log \left( e^{\eta} - 1 \right)$ and $d\xi = \frac{e^{\eta}}{e^{\eta}-1} d\eta$. Then

$$J_{m,n}^{(1)} = \frac{1}{i\ell k_{s}} \int_{0}^{\log(2)} \frac{W \left( \log \left( e^{\eta} - 1 \right) \right)}{(e^{\eta}-1)^{\frac{1}{2}}} \left( e^{\eta} - 1 \right)^{ink_{s}} \left[ 1 - e^{-i\ell k_{s}\eta} \right] \frac{e^{\eta}}{e^{\eta}-1} d\eta$$

$$= \frac{1}{i\ell k_{s}} \int_{0}^{\log(2)} \frac{e^{\eta} W \left( \log \left( e^{\eta} - 1 \right) \right)}{(e^{\eta}-1)^{\frac{1}{2}}^{ink_{s}}} \left( 1 - e^{-i\ell k_{s}\eta} \right) d\eta.$$
The analyticity properties of $W$ imply that the function $Q(\eta) = e^\eta W(\log(e^\eta - 1))$ is analytic in a domain containing the interval $(0, \log(2)]$. The domain contains a whole neighbourhood of the point $\eta = \log(2)$. Concerning the neighbourhood of $\eta = 0$ we can see, using Taylor series for $e^\eta$ that we have analyticity of $Q$ if $|\eta|$ is small enough and $\arg \log(\eta) \in (-\pi - \beta, -\pi + \beta)$ with $\beta > 0$, i.e. the region of analyticity of $Q$ covers the whole set $|\eta| \leq \delta$ with $\delta$ small, $\arg(\delta) \in (-\pi, \pi)$. Moreover, the function $Q$ can be extended analytically along the negative real axis in a small neighbourhood of the origin to yield a multivalued function.

The asymptotic behaviour of $Q$ near the origin can be obtained using (2.16). We have

$$Q(\eta) = (\eta)^{-\left(\frac{2}{3} + \rho\right)} \left[1 + O(|\eta|^\kappa)\right] \text{ as } |\eta| \to 0.$$ 

Therefore

$$|Q(\eta)| \leq C |\eta|^{-\left(\frac{2}{3} + \rho\right)} \text{ for } 0 < \eta \leq \log(2).$$

We then write

$$J_{m,n}^{(1)} = \frac{1}{i\ell k_*} \int_0^{\log(2)} \frac{Q(\eta)}{(e^\eta - 1)^{\frac{s}{2} - ink_*}} \left(1 - e^{-i\ell k_* \eta}\right) \, d\eta = \frac{1}{i\ell k_*} \int_0^{\log(2)} \frac{Q(\eta)}{(e^\eta - 1)^{\frac{s}{2}} (e^\eta - 1)^{-ink_*}} \, d\eta.$$

Using that $|e^\eta - 1| = 1$ and $|1 - e^{-i\ell k_* \eta}| \leq C \ell |\eta|$ for $\eta \in (0, \log(2))$ and $\ell \in \mathbb{Z}$ we obtain

$$|J_{m,n}^{(1)}| \leq C \frac{\ell}{|\ell|} \int_0^{\log(2)} \frac{|Q(\eta)| \ell |\eta|}{(e^\eta - 1)^{\frac{s}{2}}} \, d\eta \leq C \int_0^{\log(2)} \frac{d\eta}{|\eta|^k} \leq C.$$ 

Combining this with (2.55) we obtain

$$|J_{m,n}| \leq C \text{ for } m + n = 0 \neq 0.$$ 

On the other hand, (2.51) combined with (2.55) implies

$$|J_{m,n}| \leq C \text{ for } m + n = 0.$$ 

Therefore

$$|J_{m,n}| \leq C$$

(2.56)

for $m$ and $n$ arbitrary integers.

We can now estimate the $\ell$–th Fourier coefficient of $B(U_1, U_2; W)$ which is given in (2.52) as

$$\left| \sum_{m+n=\ell} J_{m,n} a_m b_n \right| \leq C \sum_{m+n=\ell} |a_m| |b_n|.$$ 

(2.57)

We have

$$\|B(U_1, U_2; W)\|_{H^{\frac{2}{3}+\rho}(-\frac{2}{3}+\rho)}^2 = \sum_{\ell \in \mathbb{Z}} \left(1 + 1/|\ell|\right)^{(1-\gamma-2\rho)^2} \sum_{m+n=\ell} |J_{m,n} a_m b_n|^2$$

19
and using (2.57) we obtain
\[
\|B(U_1, U_2; W)\|_{H^1_{\text{per}}([\pi, \pi])} \leq C \sum_{\ell \in \mathbb{Z}} (1 + |\ell|) \left( \sum_{n \in \mathbb{Z}} |a_{\ell - n}| |b_n| \right)^2. \tag{2.58}
\]

In order to estimate the right-hand side of (2.58), we define the following periodic functions in \(S^1\):
\[
\psi_1(x) = \sum_{n \in \mathbb{Z}} |a_n| e^{inx}, \quad \psi_2(x) = \sum_{n \in \mathbb{Z}} |b_n| e^{inx}, \quad x \in [-\pi, \pi].
\]

Using Plancherel formula as well as the fact that \(U_1, U_2\) are in \(H^1_{\text{per}}(\mathbb{R})\) we can estimate the right-hand side of (2.58) as
\[
C \left[ \|\psi_1 \psi_2\|_{L^2(-\pi, \pi)}^2 + \|\partial_x (\psi_1 \psi_2)\|_{L^2(-\pi, \pi)}^2 \right]. \tag{2.59}
\]

Moreover,
\[
\|\psi_1\|_{L^2(-\pi, \pi)}^2 + \|\partial_x (\psi_1)\|_{L^2(-\pi, \pi)}^2 \leq C \|U_1\|^2,
\]
\[
\|\psi_2\|_{L^2(-\pi, \pi)}^2 + \|\partial_x (\psi_2)\|_{L^2(-\pi, \pi)}^2 \leq C \|U_2\|^2
\]
and
\[
\|\psi_j\|_{L^\infty(-\pi, \pi)} \leq C \|U_j\|, \quad j = 1, 2.
\]

Then, the term in (2.59) can be estimated by \(C \|U_1\|^2 \|U_2\|^2\). Therefore, using also (2.58) we obtain
\[
\|B(U_1, U_2; W)\|_{H^1_{\text{per}}([\pi, \pi])} \leq C \|U_1\|^2 \|U_2\|^2
\]
whence the Lemma follows. \(\square\)

We can estimate easily the dependence of the bilinear operator in \(W_1\).

**Lemma 2.10** Let \(\alpha \in \mathbb{R}^2\) with \(\alpha = (\alpha_1, \alpha_2)\) and \(W_1 = \alpha_1 W_{1,1} + \alpha_2 W_{1,2}\) with \(W_{1,1}, W_{1,2}\) as in Lemma 2.8. Then, the following estimate holds
\[
\|B(U_1, U_2; W_1)\|_{H^1\text{per}([\pi, \pi])} \leq C |\alpha| \|U_1\| \|U_2\|, \quad U_1, U_2 \in H^1\text{per}([0, T]), \tag{2.60}
\]

**Proof:** Suppose that \(U_1, U_2 \in H^1\text{per}([0, T])\). We represent then using a Fourier series as in the Proof of Lemma 2.8
\[
U_1(X) = \sum_{n \in \mathbb{Z}} a_n e^{ink_x} X, \quad U_2(X) = \sum_{n \in \mathbb{Z}} b_n e^{ink_x} X, \quad a_n = a_{-n}, \quad b_n = b_{-n}.
\]

Then, arguing as in the Proof of Lemma 2.8 we obtain
\[
B(U_1, U_2; W_1)(X) = \sum_{\ell \in \mathbb{Z}} \left[ \sum_{m+n=n} J_{m,n} a_m b_n \right] e^{i\ell k_x} X
\]
where \(J_{m,n}\) is given in (2.50)-(2.51).
We have $|J_{m,n}| \leq C$ if $\ell = m + n = 0$. To estimate $J_{m,n}$ for $\ell \neq 0$ we write

$$J_{m,n} = J_{m,n}^{(1)} + J_{m,n}^{(2)}$$

(2.61)

where

$$J_{m,n}^{(1)} = \frac{1}{i\ell k_*} \int_{-\infty}^{0} e^{-\xi} e^{i\ell k_* \xi} W_1(\xi) \left[ 1 - \frac{1}{(e^{\xi} + 1)^{itk_*}} \right] d\xi$$

$$J_{m,n}^{(2)} = \frac{1}{i\ell k_*} \int_{0}^{\infty} e^{-\xi} e^{i\ell k_* \xi} W_1(\xi) \left[ 1 - \frac{1}{(e^{\xi} + 1)^{itk_*}} \right] d\xi.$$

Arguing as in the Proof of Lemma 2.8 we obtain

$$\left| J_{m,n}^{(2)} \right| \leq C \left| \ell \right| k_* |\alpha|, \ell \neq 0.$$

Moreover, due to the fast decay of $W_1(\xi)$ as $\xi \to -\infty$ (cf. (2.35)) we can obtain a similar estimate for $J_{m,n}^{(1)}$. Therefore

$$\left| J_{m,n}^{(1)} \right| \leq C \left| \ell \right| k_* |\alpha|, \ell \neq 0$$

whence, combining all the estimates we obtain

$$|J_{m,n}| \leq C \frac{1}{1 + |\ell|} |\alpha|, \ell \neq 0 \text{ for } m + n = \ell$$

(2.62)

where $C$ depends on $k_*$ but not on $m, n$. We then obtain the estimate arguing as in the Proof of Lemma 2.8. Indeed, the proof of that Lemma just relies on the boundedness of $|J_{m,n}|$. The estimate (2.62) implies that $|J_{m,n}| \leq C |\alpha|$. Using this, a simple adaptation of the argument in the Proof of Lemma 2.8 yields (2.60) whence the result follows. \qed

Lemma 2.10 yields also estimates for the linear operator $\mathcal{L}(U; W_1)$.

**Lemma 2.11** Let $k_*$ be as in Theorem 2.2 and $\alpha, W_1$ as in Lemma 2.8. Let $\mathcal{L}(\varphi; W_1)$ be as in (2.38) with $W = W_1$ with $\varphi \in H^1_{per}([0, T])$. Then

$$\| \mathcal{L}(\varphi; W_1) \|_{H^1_{per}([0, T])} \leq C |\alpha| \| \varphi \|, \varphi \in H^1_{per}([0, T])$$

where $C$ depends on $k_*$ but it is independent of $\alpha$ and $\varphi$.

**Proof:** It is just a consequence of the fact that $\mathcal{L}(\varphi; W_1) \equiv B(1, \varphi; W_1) + B(\varphi, 1; W_1)$ and (2.60) in Lemma 2.10 \qed

We can conclude now the Proof of Theorem 2.6.

**End of the Proof of Theorem 2.6** The problem can be reformulated as a fixed point. To this end we introduce a Hilbert space

$$\mathcal{H} = \{ (\alpha_1, \alpha_2, \psi) : \alpha_1, \alpha_2 \in \mathbb{R}, \| \psi \| < \infty \}$$
as well as a mapping \( T : \mathcal{H} \to \mathcal{H} \)

\[
T(\alpha_1, \alpha_2, \psi) = (T_1, T_2, T_3)(\alpha_1, \alpha_2, \psi) \in \mathcal{H}
\]  

(2.63)

where

\[
T_1(\alpha_1, \alpha_2, \psi) = -\frac{1}{s} \ell_1 (P_1 B (U, U; W_0 + W_1))
\]

(2.64)

\[
T_2(\alpha_1, \alpha_2, \psi) = -\frac{1}{s} \ell_2 (P_1 B (U, U; W_0 + W_1))
\]

\[
T_3(\alpha_1, \alpha_2, \psi) = -(A_W)^{-1} (P_2 \mathcal{L} (\psi; W_1)) - (A_W)^{-1} (P_2 B (U, U; W_0 + W_1))
\]

Then, the problem (2.42), (2.45) is equivalent to the fixed point problem

\[
(\alpha_1, \alpha_2, \psi) = T(\alpha_1, \alpha_2, \psi).
\]

(2.65)

We now define the set \( K_{M,s_0} = \{(\alpha_1, \alpha_2, \psi) : |\alpha_1| + |\alpha_2| \leq M |s|, \, \|\psi\| \leq |s|, \, |s| \leq s_0\} \).

We introduce in \( K_{M,s_0} \) the metric

\[
\text{dist} \left( (\alpha_1^{(1)}, \alpha_2^{(1)}, \psi^{(1)}), (\alpha_1^{(2)}, \alpha_2^{(2)}, \psi^{(2)}) \right) = |\alpha_1^{(1)} - \alpha_1^{(2)}| + |\alpha_2^{(1)} - \alpha_2^{(2)}| + M \|\psi^{(1)} - \psi^{(2)}\|.
\]

(2.66)

We will show that if \( M \) is chosen sufficiently large and \( s_0 \) sufficiently small the operator \( T \) defined by means of (2.63), (2.64) is contractive. We emphasize here that the whole argument is made assuming that the function \( W_0 \) in (2.38) is fixed. The function \( W_0 \) depends on \( \varepsilon \) and then all the constants \( C \) in the following might depend on \( \varepsilon \). However, we will choose the constants \( C \) independent of \( M \) and \( s \). In the following argument, we need to assume that \( M \) is sufficiently large (depending on \( \varepsilon \)) and then \( s_0 \) sufficiently small (depending on \( M \)).

We first prove that \( T \) transforms \( K_{M,s_0} \) into itself. To this end we derive estimates for \( T_1(\alpha_1, \alpha_2, \psi), \, T_2(\alpha_1, \alpha_2, \psi) \) for each \( (\alpha_1, \alpha_2, \psi) \in K_{M,s_0} \). Assuming that \( M |s| \) is sufficiently small we can apply Lemma 2.8 to estimate \( \|B (U, U; W_0 + W_1)\|_{H_{2r}^{-\frac{1}{2}+p} ([0, T])} \leq C |s|^2 \) with \( C \) independent of \( M \). Then \( |T_1(\alpha_1, \alpha_2, \psi)| + |T_2(\alpha_1, \alpha_2, \psi)| \leq C_0 |s| \) with \( C_0 \) independent of \( M \). Therefore, choosing \( M > C_0 \) we obtain that the two components \( T_1(\alpha_1, \alpha_2, \psi) \), \( T_2(\alpha_1, \alpha_2, \psi) \) satisfy the two inequalities required in the definition of \( K_{M,s_0} \). On the other hand, using Lemma 2.11 and the fact that \( P_2 \) is a projection operator, we can estimate \( P_2 \mathcal{L} (\psi; W_1) \) as \( \|P_2 \mathcal{L} (\psi; W_1)\|_{H_{2r}^{-\frac{1}{2}+p} ([0, T])} \leq C M |s|^2 \) with \( C \) independent of \( M \). On the other hand, Lemma 2.8 implies \( \|B (U, U; W_0 + W_1)\|_{H_{2r}^{-\frac{1}{2}+p} ([0, T])} \leq C |s|^2 \) assuming that \( M |s| \) is sufficiently small. Using (2.44) we obtain

\[
\| (A_W)^{-1} (P_2 \mathcal{L} (\psi; W_1)) \| \leq C M |s|^2
\]

(2.67)

\[
\| (A_W)^{-1} (P_2 B (U, U; W_0 + W_1)) \| \leq C |s|^2.
\]

(2.68)

Therefore, if we choose \( |s| \) small enough we obtain \( \|T_3(\alpha_1, \alpha_2, \psi)\| \leq |s| \). This estimate combined with the estimates for \( T_1(\alpha_1, \alpha_2, \psi), \, T_2(\alpha_1, \alpha_2, \psi) \) obtained above imply that for each \( (\alpha_1, \alpha_2, \psi) \in K_{M,s_0} \) we have \( T(\alpha_1, \alpha_2, \psi) \in K_{M,s_0} \).
It remains to show that the operator $T$ is contractive. Given \( \left( \alpha_1^{(j)}, \alpha_2^{(j)}, \psi^{(j)} \right) \in K_{M,s_0}, j = 1, 2 \), we write

\[
U^{(j)} = s\varphi + \psi^{(j)}, \quad W_1 = \alpha_1^{(j)}W_{1,1} + \alpha_2^{(j)}W_{1,2}, \quad \alpha^{(j)} = \left( \alpha_1^{(j)}, \alpha_2^{(j)} \right), \quad j = 1, 2.
\]

Then, using that $B(U,U;W)$ is multilinear in its arguments, and using also Lemmas \ref{lem:continuous} and \ref{lem:contractive} we obtain

\[
\left\| B\left(U^{(1)}, U^{(1)}; W_0 + W_1^{(1)}\right) - B\left(U^{(2)}, U^{(2)}; W_0 + W_1^{(2)}\right) \right\|_{H^2_{\text{per}}(\mathbb{R})^{\varphi+\psi}} \leq C |s| \left\| \psi^{(1)} - \psi^{(2)} \right\| + C |s|^2 \left| \alpha^{(1)} - \alpha^{(2)} \right|
\]

where $C$ is independent of $M$. Then, using the first two equations of (2.64) as well as (2.66) we obtain

\[
\left| T_k \left( \alpha_1^{(1)}, \alpha_2^{(1)}, \psi^{(1)} \right) - T_k \left( \alpha_1^{(2)}, \alpha_2^{(2)}, \psi^{(2)} \right) \right| \leq C \left\| \psi^{(1)} - \psi^{(2)} \right\| + C |s| \left| \alpha^{(1)} - \alpha^{(2)} \right|
\]

\[
\leq C \left( \frac{1}{M} + |s| \right) \text{dist} \left( \left( \alpha_1^{(1)}, \alpha_2^{(1)}, \psi^{(1)} \right), \left( \alpha_1^{(2)}, \alpha_2^{(2)}, \psi^{(2)} \right) \right)
\]

for $k = 1, 2$. On the other hand, using the last equation in (2.64) combined with Lemmas \ref{lem:continuous} and \ref{lem:contractive} we obtain

\[
\left\| T_3 \left( \alpha_1^{(1)}, \alpha_2^{(1)}, \psi^{(1)} \right) - T_3 \left( \alpha_1^{(2)}, \alpha_2^{(2)}, \psi^{(2)} \right) \right\| \leq CM |s| \left\| \psi^{(1)} - \psi^{(2)} \right\| + C |s| \left| \alpha^{(1)} - \alpha^{(2)} \right| + C |s|^2 \left| \alpha^{(1)} - \alpha^{(2)} \right|
\]

Multiplying (2.70) by $M$, adding the result to (2.69) and using (2.66) we obtain

\[
\text{dist} \left( T \left( \alpha_1^{(1)}, \alpha_2^{(1)}, \psi^{(1)} \right), T \left( \alpha_1^{(2)}, \alpha_2^{(2)}, \psi^{(2)} \right) \right) \leq C \left( \frac{1}{M} + M |s| \right) \text{dist} \left( \left( \alpha_1^{(1)}, \alpha_2^{(1)}, \psi^{(1)} \right), \left( \alpha_1^{(2)}, \alpha_2^{(2)}, \psi^{(2)} \right) \right)
\]

Then, choosing $M$ sufficiently large and then $|s| \leq s_0$ we obtain that the operator $T$ is contractive, whence the result follows. Notice that if $s_0$ is sufficiently small we have that $W > 0$. The quadratic estimate of $\psi$ follows from the estimates (2.67)-(2.68) and from the definition of $T$ (2.65). \hfill \Box

**Remark 2.12** Notice that Theorem \ref{thm:existence} implies the existence of nonconstant solutions of (2.20) for each $J_0 > 0$ and kernels $W = W_0 + W_1$ with $|s| \leq s_0$. Indeed, we have already seen that the value of $J_0$ can be assumed to be any positive number by means of a rescaling argument. Then, given that $B(1,1;W_0) = a > 0$, we obtain that the solutions $\hat{H}$ of (2.27) obtained in Theorem 2.6 satisfy $B\left( \hat{H}, \hat{H}; W_0 \right) = J_0$ with $J_0 > 0$ if $s_0$ is small enough due to the continuous dependence of $B\left( \hat{H}, \hat{H}; W_0 \right)$ on $s$. 

23
We can now reformulate Theorem 2.6 in terms of the original set of variables (cf. (1.5), (1.6)) in order to prove Theorem 1.1.

Proof of Theorem 1.1. It is just a consequence of Theorem 2.6 and Remark 2.12 using the change of variables (2.1). Notice that the function $X \to e^{ \frac{(\gamma + 3)x}{2} } f(e^{x}) = \tilde{H}(X)$ is periodic with period $T = 2\pi / k_*$, with $k_*$ as in Theorem 2.2, whence
\[
e^{ \frac{(\gamma + 3)x}{2} } f(e^{x}) = e^{ \frac{(\gamma + 3)x}{2} } e^{ \frac{(\gamma + 3)x}{2} } f(e^{T}e^{x}) .
\]

Therefore
\[
f(Qx) = \frac{f(x)}{Q^{(\gamma + 3)x}} \text{ for each } x > 0 \text{ with } Q = e^{T}.
\]

We have $f(x) = \tilde{H}(\log(x))x^{-(\gamma + 3)/2}$. Notice that the function $f$ is not a power law, since $\tilde{H}$ is not constant for $s \neq 0$. Finally, the continuity of the family of kernels $K$ in the topology induced by the metric (1.19) follows from (2.4), the asymptotics of $W_0$ in (2.16), the estimate (2.35) and equation (2.38) as well as the continuity in $s$ of the functions $\alpha_1$, $\alpha_2$ in (2.38).

Then the result follows. \hfill \Box

## 3 Construction of the bifurcation kernel: proof of Theorem 2.2

In order to prove Theorem 2.2 we first need an auxiliary result which allows to study the properties of some complex-valued functions.

**Lemma 3.1** We define a function $G : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, $G = G(z,k)$, by means of
\[
G(z,k) = e^{-\frac{z}{2}} \left(1 + e^{ikz}\right) \left(1 - \frac{1}{(e^z + 1)^{ik}}\right) + e^{\frac{z}{2}} \left(1 + e^{-ikz}\right) \left(1 - \frac{1}{(e^{-z} + 1)^{ik}}\right) . \tag{3.1}
\]

There exists $\delta_0 > 0$ sufficiently small such that, for any $\delta \in (0,\delta_0)$ there exists $z_0 = z_0(\delta) > 0$ sufficiently large such that, if $z_b \geq z_0$ and $z_n = (1 + \delta)z_b$ there exist infinitely many values $k_n \in \mathbb{R}$, with $n \in \mathbb{N}$ such that
\[
\frac{G(z_a,k_n)}{G(z_b,k_n)} \in \mathbb{R} \quad \text{with} \quad \frac{G(z_a,k_n)}{G(z_b,k_n)} < 0 . \tag{3.2}
\]

Moreover, for each $\delta$ and $z_b$ there exist a number $\sigma > 0$ and sequences of positive numbers $\{\varepsilon_{n,1}\}_{n \in \mathbb{N}}$, $\{\varepsilon_{n,2}\}_{n \in \mathbb{N}}$ such that
\[
\arg(G(z_b,k_n - \varepsilon_{n,1})) - \arg(G(z_a,k_n - \varepsilon_{n,1})) = \pi - \sigma \tag{3.3}
\]
\[
\arg(G(z_b,k_n + \varepsilon_{n,2})) - \arg(G(z_a,k_n + \varepsilon_{n,2})) = \pi + \sigma \tag{3.4}
\]
and also
\[
0 < R_1 e^{\frac{z_a}{2}} \leq |G(z_a,k)| \leq R_2 e^{\frac{z_a}{2}} < \infty \quad , \quad 0 < R_1 e^{\frac{z_b}{2}} \leq |G(z_a,k)| \leq R_2 e^{\frac{z_b}{2}} < \infty \tag{3.5}
\]
\[
0 < R_1 e^{\frac{z_b}{2}} \leq |G(z_b,k)| \leq R_2 e^{\frac{z_b}{2}} < \infty \quad , \quad 0 < R_1 e^{\frac{z_b}{2}} \leq |G(z_b,k)| \leq R_2 e^{\frac{z_b}{2}} < \infty \tag{3.6}
\]
for $k \in [k_n - \varepsilon_{n,1},k_n + \varepsilon_{n,2}]$ for some $R_1$, $R_2$ independent of $z_a$, $z_b$. \hfill \Box
Proof of Lemma 3.1. We define \( G_1 (z, k) = e^{\frac{z}{i}} (1 + e^{-ikz}) \left( 1 - \frac{1}{(e^{z+1})^2} \right) \). Using that \( |e^{ikz}| = |(e^z + 1)^{ik}| = 1 \) we obtain

\[
|G (z, k) - G_1 (z, k)| \leq 4e^{-\frac{2 \pi}{3}} \quad \text{for all} \quad (z, k) \in \mathbb{R} \times \mathbb{R}, \quad z \geq z_0.
\]  

(3.7)

The function \( G_1 (z, k) \) can be written as

\[
G_1 (z, k) = e^{\frac{z}{i}} \left( 1 + e^{-ikz} \right) \left( 1 - \exp \left( -ik\omega (z) \right) \right)
\]  

(3.8)

where

\[
\omega (z) = \log \left( 1 + e^{-z} \right).
\]  

(3.9)

Notice that (3.8) implies that \( G_1 (z, k) = e^{\frac{z}{i}} \zeta_1 \zeta_2 \) where \( \zeta_1, \zeta_2 \in \{ \zeta \in \mathbb{C} : |\zeta - 1| = 1 \} \). As \( k \) varies from \(-\infty\) to \(+\infty\) we have that \( \zeta_1 \) and \( \zeta_2 \) rotate in a circle of radius 1 centered at the point \( \zeta = 1 \) in the counterclockwise sense with angular velocity \( z \) and \( \omega (z) \) respectively. We will denote

\[
(\zeta_1, a) = \left( 1 + e^{-ikza} \right), \quad (\zeta_1, b) = \left( 1 + e^{-ikzb} \right), \quad (\zeta_2, a) = \left( 1 + e^{-ik\omega (za)} \right), \quad (\zeta_2, b) = \left( 1 + e^{-ik\omega (zb)} \right).
\]

Notice that all points \((\zeta_1, a - 1), (\zeta_1, b - 1), (\zeta_2, a - 1), (\zeta_2, b - 1), (\frac{\zeta_1 - 1}{\zeta_1, b - 1}), (\frac{\zeta_2 - 1}{\zeta_2, a - 1})\) are in the unit circle \( \{ \zeta \in \mathbb{C} : |\zeta| = 1 \} \) for any \( k \in \mathbb{R} \). The six points rotate around the origin in the clockwise sense as \( k \) increases with constant angular velocity \( z_a, z_b, \omega (z_a), \omega (z_b), (z_a - z_b) = \delta z_b, (\omega (z_b) - \omega (z_a)) \) respectively. This implies that the six points \((\zeta_1, a - 1), (\zeta_1, b - 1), (\zeta_2, a - 1), (\zeta_2, b - 1), (\frac{\zeta_1 - 1}{\zeta_1, b - 1}), (\frac{\zeta_2 - 1}{\zeta_2, a - 1})\) reach arbitrary positions in the unit circle in periods

\[
\frac{2\pi}{z_a}, \frac{2\pi}{z_b}, \frac{2\pi}{\omega (z_a)}, \frac{2\pi}{\omega (z_b)}, \frac{2\pi}{\delta z_b}, \frac{2\pi}{\omega (z_b) - \omega (z_a)}
\]

respectively. In particular choosing \( \delta \) sufficiently small and \( z_0 \) sufficiently large (depending on \( \delta \)) we would obtain, using (3.9) that

\[
\frac{2\pi}{z_a} \approx \frac{2\pi}{z_b} \ll \frac{2\pi}{\delta z_b} \ll \frac{2\pi}{\omega (z_b) - \omega (z_a)} \approx \frac{2\pi}{\omega (z_b)} \ll \frac{2\pi}{\omega (z_a)}
\]  

(3.10)

for \( z \in (z_b, (1 + \delta)z_b) \) with \( z_b \geq z_0 \) and \( \delta \) sufficiently small and \( z_0 \) sufficiently large. The inequalities (3.10) imply that for sufficiently large values of \( k \) the points \((\zeta_1, a - 1), (\zeta_1, b - 1)\) make an arbitrary angle. On the other hand they rotate with a much faster angular speed \( z_a \approx z_b \). Therefore, for any prescribed angle \( \alpha \in (0, 2\pi) \) and any \( \varepsilon_0 > 0 \), if we choose \( \delta \) sufficiently small and \( z_0 \) large there are infinitely many values \( k \) for which the points \((\zeta_1, a - 1)\) can be at any position of the unit circle with the angle between \((\zeta_1, a - 1)\) and \((\zeta_1, b - 1)\) is contained in \((\alpha - \varepsilon_0, \alpha + \varepsilon_0)\). The estimates (3.10) imply also that for \( k \) sufficiently large we can make a similar claim about the points \((\zeta_2, a - 1), (\zeta_2, b - 1)\). Moreover, (3.10) has also the consequence that for small \( \delta \) and large \( z_0 \) the points \((\zeta_2, a - 1), (\zeta_2, b - 1)\) rotate much more slowly as \( k \) increases as the points \((\zeta_1, a - 1), (\zeta_1, b - 1)\) and \((\zeta_2, a - 1), (\zeta_2, b - 1)\).

Therefore, for any \( \alpha, \beta, \gamma \in (0, 2\pi) \) and any \( \varepsilon_0 > 0 \) we can choose \( \delta \) sufficiently small and \( z_0 \) large such there are infinitely many values \( k \) such that \((\zeta_2, a - 1) \in (\gamma - \varepsilon_0, \gamma + \varepsilon_0)\), \((\zeta_1, a - 1)\) is at any position of the circle, the angle between \((\zeta_1, a - 1)\) and \((\zeta_1, b - 1)\) is contained in the interval \((\alpha - \varepsilon_0, \alpha + \varepsilon_0)\) and the angle between \((\zeta_2, a - 1)\) and \((\zeta_2, b - 1)\) is in \((\beta - \varepsilon_0, \beta + \varepsilon_0)\).

We denote as \( \xi_1, \xi_2 \) two points of the circle \( \{ \zeta \in \mathbb{C} : |\zeta - 1| = 1 \} \) such that \( \xi_1 = \rho_1 e^{i\theta_1}, \xi_2 = \rho_2 e^{i\theta_2} \) with \( 0 < \theta_1 < \frac{\pi}{2} \), \( -\frac{\pi}{2} < \theta_2 < 0 \) and such that \( 2\theta_1 - 2\theta_2 = \pi - s \) where \( s > 0 \) is a
small number. Notice that \( \rho_1 > 0, \rho_2 > 0 \). Notice that \( \xi_1 = 1 + e^{i\varphi_1}, \xi_2 = 1 + e^{i\varphi_2} \) with \( 0 < \varphi_1 < \pi \) and \( -\pi < \varphi_2 < 0 \). We have \( \theta_j = \frac{\varphi_j}{2}, j = 1, 2 \) as it can be seen noticing that the vector connecting the points 0 and \( 1 + \exp(i\varphi_j) \) is a diagonal of the parallelogram consisting of the points \( \{0, 1, \exp(i\varphi_1), 1 + \exp(i\varphi_1)\} \). Notice that \( \rho_1 \) and \( \rho_2 \) are independent of \( z_a, z_b \) and \( k \).

We choose \( \varepsilon_0 > 0 \) small and we select \( k = k_0 \) such that \( \zeta_{1,b} = \xi_1, |\arg(\zeta_{2,b} - \xi_1)| < \varepsilon_0, |\arg(\zeta_{1,a} - \xi_2)| < \varepsilon_0, |\arg(\zeta_{2,a} - \xi_2)| < \varepsilon_0 \). Notice that we need to take \( b \) sufficiently small and \( z_0 \) sufficiently large to ensure that infinitely many of such a values \( k_0 \) exist and that these values diverge to \( +\infty \). Therefore, using that \( G_1(z, k) = e^{i\frac{\pi}{2}}\xi_1\zeta_2 \) it follows that, assuming that \( \varepsilon_0 \) has been chosen sufficiently small, we have that \( \arg(G_1(z, k_0)) = \arg(G_1(z, k_0)) \)

\[
\pi - 2s < \arg(G_1(z, k_0)) - \arg(G_1(z, k_0)) < \pi - \frac{s}{2}.
\]

Using (3.7) it then follows that, if \( z_0 \) is chosen sufficiently large (depending only on \( s, \theta_1, \theta_2 \)) we have

\[
\pi - 3s < \arg(G(z, k_0)) - \arg(G(z, k_0)) < \pi - \frac{s}{4}.
\] (3.11)

We now examine the values of \( G(z, k), G(z, k) \) for \( k > k_0 \). Notice that as \( k \) increases we obtain that \( \frac{\zeta_{1,b}}{\zeta_{1,a}} = e^{-i\delta_2} = e^{i\delta_2} \). Then, in times of order \( \frac{s}{\delta_2} \), we obtain that \( \arg(\zeta_{1,b}) - \arg(\zeta_{1,k}) \) increases an amount of order \( s \). Then, using the fact that \( \zeta_{2,a}, \zeta_{2,b} \) change much more slowly than \( \frac{\zeta_{1,b}}{\zeta_{1,a}} \) (cf. (3.10)), we obtain that there exists \( k_1 \in \{k_0, k_0 + \frac{5}{s\delta_2}\} \) such that

\[
\pi + \frac{s}{2} < \arg(G_1(z, k_1)) - \arg(G_1(z, k_1)) < \pi + 2s
\]

and in addition \( \frac{1}{2} |G_1(z, k)| \leq |G_1(z, k_0)| \leq 2 |G_1(z, k)| \) and \( \frac{1}{2} |G_1(z, k)| \leq |G_1(z, k_0)| \leq 2 |G_1(z, k)| \) for all \( k \in [k_0, k_0 + 1] \). Then, using (3.7) it follows that if \( z_0 \) is sufficiently large we have

\[
\pi + \frac{s}{4} < \arg(G(z, k_1)) - \arg(G(z, k_1)) < \pi + 3s
\] (3.12)

and also

\[
\frac{1}{3} |G(z, k)| \leq |G(z, k_0)| \leq 3 |G(z, k)| \quad \text{and} \quad \frac{1}{3} |G(z, k)| \leq |G(z, k_0)| \leq 3 |G(z, k)|
\] (3.13)

for all \( k \in [k_0, k_0 + 1] \). Moreover, since \( G_1(z, k) = e^{i\frac{\pi}{2}}\zeta_1\zeta_2, \zeta_{1,a} \) and \( \zeta_{2,a} \) are close to \( \zeta_2 \) and \( \zeta_{1,b} \) and \( \zeta_{2,b} \) are close to \( \zeta_1 \) and \( |\xi_1| \) and \( |\xi_2| \) are strictly positive (and independent of \( k, z_a, z_b \)), it then follows that \( |G_1(z, k)| \geq C_0e^{i\frac{2\pi}{3}}, |G_1(z, k)| \geq C_0e^{i\frac{2\pi}{3}} \) if \( k \in [k_0, k_0 + 1] \), for some \( C_0 > 0 \) (independent of \( k, z_a, z_b \)). Then, combining (3.11) and (3.12) it follows that, if \( z_0 \) is chosen sufficiently large, there exists at least one value \( k_* \in (k_0, k_0 + 1) \) such that

\[
\arg(G(z, k_*)) = \arg(G(z, k_*)) + \pi.
\]

The existence of the sequences \( \{\varepsilon_{n,1}\}_{n \in \mathbb{N}}, \{\varepsilon_{n,2}\}_{n \in \mathbb{N}} \) satisfying (3.3), (3.4) with, say \( \sigma = \frac{\pi}{3} \) follows from a similar argument. Estimates (3.5), (3.6) follow from (3.7), (3.8) and the way in which \( \zeta_{1,a}, \zeta_{1,b}, \zeta_{2,a}, \zeta_{2,b} \) have been chosen. This concludes the proof of the Lemma. □

In Figure 3 we illustrate how the alignment between vectors \( G(z_a, k_0) \) and \( G(z_b, k_0) \) takes place.

**Remark 3.2** We notice that Lemma 3.1 expresses the fact that for large values of \( z \) the function \( G(z, k) \) can be approximated as \( G_1(z, k) \). This function is proportional to the product
Figure 3: Left: representation of the vectors $G$ normalized rotating counter-clockwise for several values of $k \in [19.31, 19.53]$. Right: representation of the functions $k \to \text{Re} \left( \overline{G(z_b,k)}G(z_a,k) \right)$ and $k \to \text{Im} \left( \overline{G(z_b,k)}G(z_a,k) \right)$ with $z_a = 2$ and $z_b = 1$.

of to complex numbers which are obtained rotating at different angular speeds around a circle with radius one around the point 1 of the complex plane. Actually one of the numbers (namely $\zeta_2$) rotates much more slowly than the other. Therefore, after placing $\zeta_2, a, \zeta_2, b$ at convenient positions, the whole problem reduces to placing $\zeta_1, a, \zeta_1, b$ also at the correct places in order to obtain that $G(z_a,k)$ and $G(z_b,k)$ point in opposite directions, something that is possible due to the fact that the points $\zeta_1, a, \zeta_1, b$ rotate around 1 at different angular speeds. A key point of the argument is to choose $\zeta_1, a, \zeta_1, b$ and $\zeta_2, b$ in such a way that their modulus is bounded from below. This fact, combined with (3.7) allows to treat $G(z_a,k)$ and $G(z_b,k)$ as perturbations of $G_1(z_a,k)$ and $G_1(z_b,k)$ respectively.

In Figure 3 (right) we plot the functions

$$k \to \text{Re} \left( \overline{G(z_b,k)}G(z_a,k) \right), \quad k \to \text{Im} \left( \overline{G(z_b,k)}G(z_a,k) \right)$$

for the values $z_b = 1, \; z_a = 2$. These functions allow to identify the alignment of the vectors $G(z_a,k)$ and $G(z_b,k)$. Indeed, given two complex numbers $Z_1, Z_2 \in \mathbb{C}$ we can identify them with vectors $V_1, V_2 \in \mathbb{R}^2$ given by $V_j = (\text{Re}(Z_j), \text{Im}(Z_j)), \; j = 1,2$. We have the following identity

$$\bar{Z}_1Z_2 = V_1 \cdot V_2 + i \det(V_1, V_2).$$

Therefore, the two vectors associated to the complex numbers $Z_1, \; Z_2$ are parallel if $\det(V_1, V_2) = \text{Im} (\bar{Z}_1Z_2) = 0$. Moreover, they point in opposite directions if in addition $V_1 \cdot V_2 = \text{Re} (\bar{Z}_1Z_2) < 0$.

Figure 3 (left) shows the existence in the range $k \in [19, 20]$ of one value of $k$ such that $G(z_b,k) = -cG(z_a,k)$ for some $c < 0, \; c \in \mathbb{R}$.

We now come back to the proof of Theorem 2.2.
Proof of Theorem 2.2. \( \Psi (k; W_0) \) defined by means of (2.17), can be written, using (2.14), as

\[
\Psi (k; W_0) = e^{-ikX} \int_{-\infty}^{X} dY \int_{1-\log(1-e^{-Y})}^{\infty} dZ \left[ e^{\frac{i}{2}(Y-Z)} W_0 (Y-Z) \right] \left( e^{ikY} + e^{ikZ} \right)
\]

\[
= \int_{-\infty}^{0} dY \int_{\log(1-e^{-Y})}^{\infty} dZ \left[ e^{\frac{i}{2}(Y-Z)} W_0 (Y-Z) \right] \left( e^{ikY} + e^{ikZ} \right).
\]

It is readily seen that this function is well defined for \( k \in \mathbb{R} \) if \( W_0 \) satisfies (2.16) and (1.11) holds. The fact that \( \Psi (-k; W_0) = \Psi (k; W_0) \) follows immediately from the fact that \( W_0 \) is real in \( \mathbb{R} \).

In order to simplify the formula for \( \Psi (k; W_0) \) we use the change of variables \( Z - Y = z, \, dZ = dz \). Then, using also that \( W_0 (X) = W_0 (-X) \) we obtain

\[
\Psi (k; W_0) = \int_{-\infty}^{0} e^{ikY} dY \int_{\log(1-e^{-Y})}^{\infty} e^{-\frac{i}{2}W_0 (z)} \left( 1 + e^{ikz} \right) dz.
\]

On the other hand, applying Fubini’s Theorem and using that if \( z \geq \log (1 - e^{-Y}) - Y \) if and only if \( Y \geq - \log (e^z + 1) \), we obtain

\[
\Psi (k; W_0) = \int_{-\infty}^{\infty} e^{-\frac{i}{2}W_0 (z)} \left( 1 + e^{ikz} \right) dz \int_{-\log(e^z + 1)}^{0} e^{ikY} dY.
\] (3.14)

Thus, computing the integral in \( Y \) we obtain

\[
\Psi (k; W_0) = \frac{1}{ik} \int_{-\infty}^{\infty} e^{-\frac{i}{2}W_0 (z)} \left( 1 + e^{ikz} \right) \left[ 1 - \frac{1}{(e^z + 1)^{ik}} \right] dz.
\] (3.15)

Using the symmetry \( W_0 (z) = W_0 (-z) \) we can rewrite \( \Psi (k; W_0) \) as

\[
\Psi (k; W_0) = \frac{1}{ik} \int_{0}^{\infty} W_0 (z) G(z, k) dz
\] (3.16)

where \( G(z, k) \) is as in (3.1). It is readily seen that for each fixed \( k \in \mathbb{R} \) we have \( G(z, k) = O \left( e^{-\frac{|z|}{2}} \right) \) as \( |z| \to \infty \), \( z \in \mathbb{R} \). Then, if \( W_0 \) satisfies (2.16) we obtain that the integral in (3.16) is well defined due to (1.11).

Due to (2.17) the problem has been reduced to finding a function \( W_0 \), with the properties (i), (ii), (iii) stated in the Theorem, such that the corresponding function \( \Psi (k; W_0) \) has a zero \( k \in \mathbb{R} \setminus \{0\} \).

In order to show that such a function \( W_0 \) exists we will use the continuity properties of the function \( \Psi (k; W_0) \) with respect to \( W_0 \) if \( W_0 \) experiences small changes in the weak topology of measures. We will take \( W_0 \) as an even perturbation of the following combination of Dirac measures

\[
W_{0,s} (z) = a\delta (z - z_a) + b\delta (z - z_b), \quad z > 0,
\] (3.17)

where \( a > 0, b > 0, z_a > 0, \, z_b > 0 \). Using (3.16) we obtain

\[
\Psi (k; W_{0,s}) = \frac{1}{ik} \left[ aG(z_a, k) + bG(z_b, k) \right].
\] (3.18)
In order to obtain a function $W_0$ with the regularity and the asymptotic behaviour stated in the Theorem, we introduce some auxiliary functions

$$\zeta(z) = e^{-z^2}, \quad \zeta_\varepsilon(z) = \frac{1}{\varepsilon} e^{\left(\frac{z}{\varepsilon}\right)}, \quad \varepsilon > 0, \quad z \in \mathbb{R}. \quad (3.19)$$

We define

$$W_{0,1}^\varepsilon(z) = a\zeta_\varepsilon(z - z_0) + b\zeta_\varepsilon(z - z_b), \quad z > 0 \quad (3.20)$$

where $a, b, z_a, z_b$ are as in (3.17). Notice that $W_{0,1}^\varepsilon \rightharpoonup W_{0,1}$ as $\varepsilon \to 0$ in the weak topology of $\mathcal{M}_+ (\mathbb{R})$. We define $\Psi (k; W_{0,1}^\varepsilon)$ using (3.16). It readily follows that $\lim_{\varepsilon \to 0} \Psi (k; W_{0,1}^\varepsilon) = \Psi (k; W_{0,1})$ uniformly in compact sets of $k$.

On the other hand we define

$$W_{0,2}^\varepsilon(z) = [1 - \zeta(\varepsilon z)] \exp\left( \left(\frac{\gamma}{2} + p\right) \sqrt{z^2 + 1}\right), \quad z > 0. \quad (3.21)$$

Note that $W_{0,1}^\varepsilon$ and $W_{0,2}^\varepsilon$ are extended to the whole real line by means of

$$W_{0,1}^\varepsilon(z) = W_{0,1}^\varepsilon(-z), \quad W_{0,2}^\varepsilon(z) = W_{0,2}^\varepsilon(-z). \quad (3.22)$$

Notice that we can define $\Psi (k; W_{0}^\varepsilon)$ using (3.16) since the integral there is convergent for the function $W_{0,2}^\varepsilon$ for each $k \in \mathbb{R}$. Notice that since $|G(z, k)| \leq C_L e^{-\frac{|z|^2}{2}}$ for $|k| \leq L$ we have that $\Psi (k; W_{0}^\varepsilon)$ converges uniformly to zero as $\varepsilon \to 0$ uniformly in compact sets of $k$ due to (1.11). We now define

$$W_0^\varepsilon = W_{0,1}^\varepsilon + W_{0,2}^\varepsilon. \quad (3.23)$$

Then $\Psi (k; W_0^\varepsilon) = \Psi (k; W_{0,1}^\varepsilon) + \Psi (k; W_{0,2}^\varepsilon)$. We then have

$$\lim_{\varepsilon \to 0} \Psi (k; W_0^\varepsilon) = \Psi (k; W_{0,1}). \quad (3.24)$$

uniformly in compact sets of $k$. Moreover, we can write

$$\Psi (k; W_{0,1}^\varepsilon) = a\Psi (k; \zeta_\varepsilon(\cdot - z_a)) + b\Psi (k; \zeta_\varepsilon(\cdot - z_b)) \quad (3.25)$$

and we have also

$$\lim_{\varepsilon \to 0} \Psi (k; \zeta_\varepsilon(\cdot - z_a)) = G(z_a, k), \quad \lim_{\varepsilon \to 0} \Psi (k; \zeta_\varepsilon(\cdot - z_b)) = G(z_b, k) \quad (3.26)$$

uniformly in compact sets of $k$.

We claim that choosing $\varepsilon > 0$ we can find $k_\varepsilon \in \mathbb{R}$ as well as $a > 0$ and $b > 0$ such that $\Psi (k_\varepsilon; W_0^\varepsilon) = 0$. Notice that the function $W_0^\varepsilon$ depends on $a, b$ although we do not write this dependence explicitly. We write $\Psi (k; \zeta_\varepsilon(\cdot - z_a)), \Psi (k; \zeta_\varepsilon(\cdot - z_b))$ in polar coordinates

$$\Psi (k; \zeta_\varepsilon(\cdot - z_a)) = R_a e^{i\theta_a}, \quad \Psi (k; \zeta_\varepsilon(\cdot - z_b)) = R_b e^{i\theta_b} \quad (3.27)$$

where $R_a > 0$, $R_b > 0$ as well as $\theta_a, \theta_b$ are functions of $k$. Moreover (3.25) combined with Lemma 3.1 (in particular (3.3)-(3.6)) imply that for $\varepsilon > 0$ sufficiently small there exist at least one value of $k_0 \in \mathbb{R}$, $k_0 > 0$ and some $\delta_1 > 0$, $\sigma > 0$ independent of $\varepsilon$ such that

$$\pi - 2\sigma \leq \theta_b - \theta_a \leq \pi - 2\sigma \quad (3.28)$$

if $k = k_0 - \delta_1$, \[ \begin{cases} 
\pi + \sigma \leq \theta_b - \theta_a \leq \pi + 2\sigma & \text{if } k = k_0 + \delta_1,
\end{cases} \]
\[
\frac{1}{L} \leq R_a \leq L, \quad \frac{1}{L} \leq R_b \leq L \quad \text{if} \quad k \in [k_0 - \delta_1, k_0 + \delta_1]
\]  
(3.29)

for some \( L > 0 \) independent of \( \varepsilon \).

We now select \( a, b \) in (3.24) as follows

\[
a = \frac{1}{R_a}, \quad b = \frac{\sigma}{R_b},
\]

where \( \sigma > 0 \) is a numerical constant to be determined. Notice that \( R_a, R_b \) are functions of \( k \) and therefore \( a, b \) are also functions. Due to (3.29) we have that \( \frac{1}{L} \leq a \leq L, \quad \frac{2}{L} \leq a \leq \sigma L \).

Using (3.24), (3.26) and our choice of \( a, b \) we obtain

\[
\Psi (k; W_0^0) = e^{i \theta a} + \sigma e^{i \theta b} + \Psi (k; W_0^{\ast, 2}).
\]

We assume that \( |\sigma - 1| \leq \frac{1}{L} \). Using the fact that \( \Psi (k; W_0^{\ast, 2}) \) tends to zero as \( \varepsilon \to 0 \) we obtain, combining continuity argument with (3.27), (3.28) that if \( \varepsilon \) is sufficiently small there exists \( k_* \in (k_0 - \delta_1, k_0 + \delta_1) \) such that \( (e^{i \theta a} + \Psi (k_*; W_0^{\ast})) = -\lambda_0 e^{i \theta b} \) for some \( \lambda_0 > 0 \). Notice that \( \lambda_0 \) is close to 1 if \( \varepsilon \) is sufficiently small. Choosing then \( \sigma = \lambda_0 \) we obtain that \( \Psi (k_*; W_0^0) = 0 \). Using also \( \Psi (-k; W_0) = \Psi (k; W_0) \) we obtain that (2.18) holds with \( W_0 = W_0^\ast \). Notice that \( W_0 \) is analytic in the domain \( D_\beta \).

It only remains to prove the asymptotic formula (2.19). Since this will imply that \( \Psi (k; W_0) \) will be different from zero for large \( |k| \). In particular, this implies that we can define \( k_* \) as the positive root of \( \Psi (k; W_0) \) in the real line with the largest value of \( |k| \). Using (3.15) we obtain

\[
\Psi (k; W_0) = \frac{1}{ik} [G_1 (k) + G_2 (k)]
\]

where

\[
G_1 (k) = \int_{-\infty}^{\infty} e^{-\frac{z}{2}} W_0 (z) \left[ 1 - \frac{1}{(e^z + 1)^{ik}} \right] dz, \quad G_2 (k) = \int_{-\infty}^{\infty} e^{-\frac{z}{2}} W_0 (z) e^{iz} \left[ 1 - \frac{1}{(e^z + 1)^{ik}} \right] dz.
\]

We now write

\[
G_1 (k) = G_{1,1} (k) + G_{1,2} (k)
\]

\[
G_{1,1} (k) = \int_{-\infty}^{0} e^{-\frac{z}{2}} W_0 (z) \left[ 1 - \frac{1}{(e^z + 1)^{ik}} \right] dz, \quad G_{1,2} (k) = \int_{0}^{\infty} e^{-\frac{z}{2}} W_0 (z) \left[ 1 - \frac{1}{(e^z + 1)^{ik}} \right] dz
\]

and

\[
G_2 (k) = G_{2,1} (k) + G_{2,2} (k)
\]

\[
G_{2,1} (k) = \int_{-\infty}^{0} e^{-\frac{z}{2}} W_0 (z) e^{iz} \left[ 1 - \frac{1}{(e^z + 1)^{ik}} \right] dz, \quad G_{2,2} (k) = \int_{0}^{\infty} e^{-\frac{z}{2}} W_0 (z) e^{iz} \left[ 1 - \frac{1}{(e^z + 1)^{ik}} \right] dz.
\]

Using the change of variables \( \xi = \log (e^z + 1) \) we readily obtain, using Riemann-Lebesgue Lemma, as well as (2.16), that

\[
\int_{0}^{\infty} e^{-\frac{z}{2}} W_0 (z) \frac{dz}{(e^z + 1)^{ik}} \to 0 \quad \text{as} \quad |k| \to \infty.
\]
Therefore
\[ G_{1,2} (k) \to \int_0^\infty e^{-z} W_0 (z) \, dz \text{ as } |k| \to \infty . \]  
(3.30)

Similarly, we can write \( G_{2,2} (k) \) as
\begin{align*}
G_{2,2} (k) &= \int_0^\infty e^{-\frac{z}{2}} W_0 (z) e^{ikz} \, dz - \int_0^\infty e^{-\frac{z}{2}} W_0 (z) \left( \frac{e^z}{e^z + 1} \right)^k \, dz .
\end{align*}

The first integral on the right converges to zero as \(|k| \to \infty\) due to Riemann-Lebesgue. On the other hand, using the change of variables \( \xi = \log \left( \frac{e^z e^z + 1}{1} \right) \) in the second integral on the right-hand side we obtain also that the resulting integral converges to zero as \(|k| \to \infty\), using again Riemann-Lebesgue. Thus
\[ G_{2,2} (k) \to 0 \text{ as } |k| \to \infty . \]  
(3.31)

It remains to study the asymptotics of \( G_{1,1} (k) \) and \( G_{2,1} (k) \) as \(|k| \to \infty\). Using the change of variables \( z \to -z \), we can rewrite \( G_{1,1}, G_{2,1} \) as
\begin{align*}
G_{1,1} (k) &= \int_0^\infty e^{\frac{z}{2}} W_0 (z) \left[ 1 - \frac{1}{(1 + e^{-z})^k} \right] \, dz \\
G_{2,1} (k) &= \int_0^\infty e^{\frac{z}{2}} W_0 (z) e^{-ikz} \left[ 1 - \frac{1}{(1 + e^{-z})^k} \right] \, dz
\end{align*}
where we used also that \( W_0 (z) = W_0 (-z) \). Setting now
\[ H (z; k) = \int_0^z e^{\frac{\xi}{2}} W_0 (\xi) e^{-ik\xi} \, d\xi \]  
(3.32)

we can rewrite \( G_{1,1}, G_{2,1} \) as
\begin{align*}
G_{1,1} (k) &= \int_0^\infty \frac{\partial H (z; 0)}{\partial z} \left[ 1 - \frac{1}{(1 + e^{-z})^k} \right] \, dz \\
G_{2,1} (k) &= \int_0^\infty \frac{\partial H (z; k)}{\partial z} \left[ 1 - \frac{1}{(1 + e^{-z})^k} \right] \, dz.
\end{align*}
(3.33)

Integrating by parts in (3.33), i.e. the formula of \( G_{1,1} (k) \), we obtain
\[ G_{1,1} (k) = ik \int_0^\infty \left[ H (z; 0) \frac{e^{-z}}{1 + e^{-z}} \right] \frac{dz}{(1 + e^{-z})^k} . \]

Using (2.16) we obtain that, since \( 1 + \gamma + 2p \geq 1 > 0 \), the following asymptotics holds
\[ H (z; 0) \frac{e^{-z}}{1 + e^{-z}} \sim \frac{2}{1 + \gamma + 2p} e^{(\frac{1}{2} + p - \frac{1}{2})z} \left[ 1 + O \left( e^{-\kappa |z|} \right) \right] \text{ as } \text{Re} (z) \to \infty , 
\]  
\( z \in D_\beta \)
with \( \kappa > 0 \). Using the change of variables \( X = \log (1 + e^{-z}) \) (and hence \( z = \log \left( \frac{1}{\sqrt{1 - e^X}} \right) \), 
\( dz = -\frac{e^X}{e^X - 1} \, dX \)) we can then write
\[ G_{1,1} (k) = ik \int_0^{\log(2)} \frac{e^X F (X)}{e^X - 1} e^{-ikX} \, dX \]  
(3.35)
where \( F (\log (1 + e^{-z})) = \frac{e^{-z} H(z;0)}{1 + e^{-z}} \). The function \( F \) is analytic in a wedge around the interval \([0, \log (2)]\) and it satisfies

\[
F (X) = \frac{2}{1 + \gamma + 2p} (X)^{\frac{1}{2} - \left( \frac{1}{2} + p \right)} [1 + O (|X|^\epsilon)] \quad \text{as} \quad |X| \to 0 \quad \text{if} \quad 1 + \gamma + 2p > 0 \quad (3.36)
\]

with \( X \) contained in the portion of a cone \( \{|\text{Im} (X)| \leq \theta |\text{Re} (X)| : |X| \leq \delta_0\} \) with \( \theta > 0 \) and \( \delta_0 > 0 \) small. Similar asymptotic formulas for the derivatives of \( F (X) \) can be obtained differentiating formally both sides of (3.35). Using this approximation for \( F (X) \) in (3.35) and using the change of variables \( kX \to \gamma \), we have

\[
\int X e^{-ikX} dX = \cdots = |k| \left( \frac{1}{2} + \frac{1}{2} + p \right) e^{-i\frac{\pi}{2}} |k|^{\frac{1}{2} + \left( \frac{1}{2} + p \right)} \Gamma \left( 1 + \frac{1}{2} \right) (X)^{\frac{1}{2} - \left( \frac{1}{2} + p \right)} e^{-i\frac{\pi}{2}} |k|^{\frac{1}{2} + \left( \frac{1}{2} + p \right)}
\]

Indeed, from (3.35), using (3.36) we have

\[
\frac{2}{1 + \gamma + 2p} (ik) \int_0^\infty (X)^{\frac{1}{2} - \left( \frac{1}{2} + p \right)} e^{-i\frac{\pi}{2}} |k|^{\frac{1}{2} + \left( \frac{1}{2} + p \right)} \Gamma \left( 1 + \frac{1}{2} \right) (X)^{\frac{1}{2} - \left( \frac{1}{2} + p \right)} e^{-i\frac{\pi}{2}} |k|^{\frac{1}{2} + \left( \frac{1}{2} + p \right)} = a \text{sgn} (k) e^{\frac{i\pi}{2} - \left( \frac{1}{2} + p \right)} |k|^{\frac{1}{2} + \left( \frac{1}{2} + p \right)}
\]

with \( a \) as in (3.38).

It now remains to estimate the contribution of \( G_{2,1} (k) \). To this end we integrate by parts in (3.34) to obtain

\[
G_{2,1} (k) = \int_0^\infty \frac{\partial H (z; k)}{\partial z} \left[ 1 - \frac{1}{(1 + e^{-z})^{ik}} \right] dz = ik \int_0^\infty H (z; k) \frac{e^{-z}}{1 + e^{-z}} \frac{1}{(1 + e^{-z})^{ik}} dz.
\]

We can estimate \( H (z; k) \) using again integration by parts. Then

\[
H (z; k) = \int_{[0, z]} e^{\xi} W_0 (\xi) e^{-ik \xi} d\xi = \frac{1}{ik} \left[ W_0 (0) - e^{iz} W_0 (z) e^{-ikz} \right] + \frac{1}{ik} \int_{[0, z]} d\xi \left( e^{i\xi} W_0 (\xi) \right) e^{-ik \xi} d\xi \quad (3.39)
\]

\[
(3.40)
\]

32
hence
\[ |H(z;k)| \leq \frac{C}{|k|} e^{\left(\frac{\gamma + p}{2} + \frac{1}{2}\right)z}. \]

Then, using that \( \gamma + 2p < 1 \) we obtain
\[ |G_{2,1}(k)| \leq C \text{ for } |k| \geq 1. \tag{3.41} \]

Combining now (3.30), (3.31), (3.37), (3.41) we obtain
\[
\Psi(k;W_0) = \frac{1}{ik} [G_1(k) + G_2(k)] \sim \text{sgn}(k) e^{i \frac{\text{sgn}(k)}{2} \left(\frac{\gamma + p}{2} \right) |k|^{\frac{1}{2} + \left(\frac{\gamma + p}{2}\right)}} \text{ as } |k| \to \infty
\]
whence (2.19) follows.

\[\Box\]

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