Integrals of Motion for Discrete-Time Optimal Control Problems

Delfim F. M. Torres
delfim@mat.ua.pt
Department of Mathematics
University of Aveiro
3810-193 Aveiro, Portugal
http://www.mat.ua.pt/delfim

Abstract

We obtain a discrete time analog of E. Noether's theorem in Optimal Control, asserting that integrals of motion associated to the discrete time Pontryagin Maximum Principle can be computed from the quasi-invariance properties of the discrete time Lagrangian and discrete time control system. As corollaries, results for first-order and higher-order discrete problems of the calculus of variations are obtained.

Keywords: discrete time optimal control, discrete time calculus of variations, discrete time mechanics, discrete time Pontryagin extremals, quasi-invariance up to difference gauge terms, discrete version of Noether's theorem.

Mathematics Subject Classification 2000: 49-99, 39A12.

1 Introduction

Most physical systems encountered in nature exhibit symmetries: there exists appropriate infinitesimal-parameter family of transformations which keep the system invariant. From the well-known theorem of Emmy Noether [26, 27], one can discover the integrals of motion from those invariance transformations. Noether’s theorem plays a fundamental role in modern physics, and is usually formulated in the context of the calculus of variations: from the invariance properties of the variational integrals, the integrals of motion of the respective Euler-Lagrange differential equations, that is, expressions which are preserved...
along the extremals, are obtained. The result is, however, much more than a theorem. It is an universal principle, which can be formalized in a precise statement, as a theorem, on very different contexts and, for each such context, under very different assumptions. Let us consider, for example, classical mechanics or, more generally, the calculus of variations. Typically, Noether transformations are considered to be point-transformations (they are considered to be functions of coordinates and time), but one can consider more general transformations depending also on velocities and higher derivatives \[10\] or within the broader context of dynamical symmetries \[20\]. For an example of an integral of motion which comes from an invariance transformation depending on velocities, see \[29\]. In most formulations of Noether’s principle, the Noether transformations keep the integral functional invariant (cf. e.g. \[15\] §1.5)). It is possible, however, to consider transformations of the problem up to an exact differential (cf. e.g. \[31\] p. 73)), called a gauge-term \[33\]. Once strictly-invariance of the integral functional is no more imposed, one can think considering additional terms in the variation of the Lagrangian – see the quasi-invariance and semi-invariance notions introduced by the author respectively in \[38\] and \[39\]. Formulations of Noether’s principle are possible for problems of the calculus of variations: on Euclidean spaces (cf. e.g. \[18\]) or on manifolds (cf. e.g. \[19\]); with single or multiple integrals (cf. e.g. \[5\]); with higher-order derivatives (cf. e.g. \[11\]); with holonomic or nonholonomic constraints (cf. e.g. \[33\] Ch. 7), \[35\]); and so on. Other contexts for which Noether’s theorems are available include supermechanics \[8\], control systems \[42, 25\], and optimal control (see e.g. \[9, 6, 37, 40\]). For a survey see \[36, 41\]. Here we are interested in providing a formulation of the Noether’s principle in the discrete time setting. For a description of discrete time mechanics, discrete time calculus of variations, and discrete optimal control see, e.g., \[13, 14, 28, 29\], \[7\], and \[12\]. Illustrative examples of real-life problems which can be modeled in such framework can be found in \[34\] Ch. 8. Versions of the Noether’s principle for the discrete calculus of variations, and applicable to discrete analogues of classical mechanics, appeared earlier in \[16, 17, 21, 3, 11, 44\], motivated by the advances of numerical and computational methods. There, the discrete analog of Noether’s theorem is obtained from the discrete analog of the Euler-Lagrange equations. To the best of our knowledge, no Noether type theorem is available for the discrete time optimal control setting. One such formulation is our concern here. The result is obtained from the discrete time version of the Pontryagin maximum principle. As corollaries, we obtain generalizations of the previous results for first-order and higher-order discrete problems of the calculus of variations which are quasi-invariant and not necessarily invariant.

2 Discrete-Time Optimal Control

Without loss of generality (cf. \[24\] §2), we consider the discrete optimal control problem in Lagrange form. The time \(k\) is a discrete variable: \(k \in \mathbb{Z}\). The horizon consists of \(N\) periods, \(k = M, M + 1, \ldots, M + N - 1\), where \(M\) and \(N\) are fixed
integers, instead of a continuous interval. We look for a finite control sequence $u(k) \in \mathbb{R}^r$, $k = M, \ldots, M + N - 1$, and the corresponding state sequence $x(k) \in \mathbb{R}^n$, $k = M, \ldots, M + N$, which minimizes or maximizes the sum

$$ J [x(\cdot), u(\cdot)] = \sum_{k=M}^{M+N-1} L(k, x(k), u(k)) , $$

subject to the discrete time control system

$$ x(k + 1) = \varphi(k, x(k), u(k)) , \quad k = M, \ldots, M + N - 1 , $$

the boundary conditions

$$ x(M) = x_M , \quad x(M + N) = x_{M+N} , $$

and the control constraint

$$ u(k) \in \Omega \subseteq \mathbb{R}^r , \quad k = M, \ldots, M + N - 1 . $$

A sequence-pair $(x(k), u(k))$, $k = M, \ldots, M + N - 1$, satisfying the recurrence relation (1) and conditions (2), is said to be admissible: $x(k)$ is an admissible state sequence and $u(k)$ an admissible control sequence. Functions $L(k, x, u) : \{M, \ldots, M + N - 1\} \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$ and $\varphi(k, x, u) : \{M, \ldots, M + N - 1\} \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ are assumed to be continuously differentiable with respect to $x$ and $u$ for all fixed $k = M, \ldots, M + N - 1$, and convex in $u$ for any fixed $k$ and $x$. They are in general nonlinear. The control constraint set $\Omega$ is assumed to be convex. The problem is denoted by $(P)$.

**Remark 2.1.** For continuous optimal control problems, the convexity assumptions we are imposing are not needed in order to derive the Pontryagin maximum principle [30]. This differs from the discrete time optimal control setting. Our hypothesis can be, however, weakened to directional convexity or even more weak conditions (see [24], [34, §8.3] and references in [32, Ch. 6] and [24]).

**Remark 2.2.** It is possible to formulate problem $(P)$ with the first-order difference equations (1) in terms of the forward or backward difference operators $\Delta$ or $\nabla$, defined by $\Delta x(k) = x(k + 1) - x(k)$, $\nabla x(k) = x(k) - x(k-1)$. The results of the paper are written in those terms in a straightforward way.

The following theorem provide a first-order necessary optimality condition (cf. e.g. [4, §3.3.3], [22, §6.2 Ch. 6]) in the form of Pontryagin’s maximum principle [30]. For a good survey on the history of the development of maximum principle to the optimization of discrete time systems, we refer the reader to [24].

**Theorem 2.1 (Discrete-Time Maximum Principle).** If $(x(k), u(k))$ is a minimizer or a maximizer of the problem $(P)$, then there exists a nonzero sequence-pair $(\psi_0, \psi(k))$, $k = M + 1, \ldots, M + N$, where $\psi_0$ is a constant less or equal than zero and $\psi(k)$ is in $\mathbb{R}^n$, such that the sequence-quadruple

$$(x(k), u(k), \psi_0, \psi(k + 1)) , \quad k = M, \ldots, M + N - 1 ,$$

satisfies:
(i) the Hamiltonian system

\[
\begin{aligned}
  x(k+1) &= \frac{\partial H}{\partial \psi}(k, x(k), u(k), \psi_0, \psi(k+1)), \quad k = M, \ldots, M + N - 1, \\
  \psi(k) &= \frac{\partial H}{\partial x}(k, x(k), u(k), \psi_0, \psi(k+1)), \quad k = M + 1, \ldots, M + N - 1;
\end{aligned}
\]  

(ii) the maximality condition

\[
H(k, x(k), u(k), \psi_0, \psi(k+1)) = \max_{u \in \Omega} H(k, x(k), u, \psi_0, \psi(k+1)),
\]  

with the Hamiltonian

\[
H(k, x, u, \psi_0, \psi) = \psi_0 L(k, x, u) + \psi \cdot \varphi(k, x, u).
\]

Remark 2.3. The first equation in the Hamiltonian system is just the control system \(1\). The second equation in the Hamiltonian system is known as the adjoint system. The multipliers \(\psi(\cdot)\) are called adjoint multipliers or co-state variables.

Remark 2.4. In the absence of the initial conditions \(x(M) = x_M\) and/or terminal conditions \(x(M + N) = x_{M+N}\), there corresponds additional conditions in the Discrete-Time Maximum Principle called transversality conditions. Our version of Noether’s theorem only require the use of the adjoint system and maximality condition. Therefore, the result is valid under all types of boundary conditions under consideration.

Definition 2.1. A sequence-quadruple \((x(k), u(k), \psi_0, \psi(k+1)), k = M, \ldots, M + N - 1, \psi_0 \leq 0\), satisfying the Hamiltonian system and the maximality condition, is called an extremal for problem \((P)\). An extremal is said to be normal if \(\psi_0 \neq 0\) and abnormal if \(\psi_0 = 0\).

Remark 2.5. As we will see on Section\(5\), there are no abnormal extremals both for first-order and higher-order discrete problems of the calculus of variations. In particular, there are no abnormal extremals for problems of discrete time mechanics. For our general problem \((P)\), however, abnormal extremals do exist. In fact, they happen to occur frequently. For a throughout study of abnormal extremals see \[2\].

3 Integrals of Motion

We obtain a systematic procedure to establish integrals of motion, i.e., to establish expressions which are preserved on the extremals of the discrete optimal control problem \((P)\), from the (quasi-)invariance properties of the discrete Lagrangian \(L(k, x(k), u(k))\) and discrete control system \(x(k+1) = \varphi(k, x(k), u(k))\).
Definition 3.1. Let $X : \{ M, \ldots, M + N - 1 \} \times \mathbb{R}^n \times \Omega \times B(0; \varepsilon) \to \mathbb{R}^n$, $\varepsilon > 0$, $B(0; \varepsilon) = \left\{ s = (s_1, \ldots, s_\rho) \mid \|s\| = \sqrt{\sum_{i=1}^\rho (s_i)^2} < \varepsilon \right\}$, be an infinitesimal $\rho$-parameter transformation such that for each $k, k = M, \ldots, M + N - 1$, $X(k, \cdot, \cdot, \cdot)$ is continuously differentiable with respect to all arguments, and such that $X(k, x, u, 0) = x$ for all $k = M, \ldots, M + N - 1$, $x \in \mathbb{R}^n$, and $u \in \Omega$. If there exists a real function $\Phi(k, x, u, s)$ and for all $s \in B(0; \varepsilon)$ and admissible $(x(k), u(k))$ there exists a control sequence $u(k, s), u(k, 0) = u(k)$, such that:

$$L(k, x(k), u(k)) + \Delta \Phi(k, x(k), u(k), s) + \delta(k, x(k), u(k), s) = L(k, X(k, x(k), u(k), s), u(k, s)),$$  \hspace{1cm} (5)

$$X(k + 1, x(k + 1), u(k + 1), s) + \delta(k, x(k), u(k), s) = \varphi(k, X(k, x(k), u(k), s), u(k, s)),$$ \hspace{1cm} (6)

for each $k = M, \ldots, M + N - 1$ and where $\delta(k, x, u, s)$ is an arbitrary function satisfying

$$\frac{\partial \delta(k, x, u, s)}{\partial s_i} \bigg|_{s=0} = 0, \hspace{0.5cm} i = 1, \ldots, \rho,$$ \hspace{1cm} (7)

for each $k, x, u$, then the problem $(P)$ is said to be quasi-invariant with respect to the transformation $X(k, x, u, s)$ up to the difference gauge term $\Phi(k, x(k), u(k), s)$.

Remark 3.1. In the relation (5), $\Delta$ is the forward difference operator:

$$\Delta \Phi(k, x(k), u(k), s) = \Phi(k + 1, x(k + 1), u(k + 1), s) - \Phi(k, x(k), u(k), s).$$

Remark 3.2. When $\delta \equiv 0$ and $\Phi \equiv 0$, we have (strict-)invariance. The term quasi-invariant refers to the possibility of $\delta$ to be different from zero.

Theorem 3.1 (Discrete-Time Noether Theorem). If $(P)$ is quasi-invariant with respect to the $\rho$-parameter transformation $X$ up to the difference gauge term $\Phi$, in the sense of Definition 3.1, then all its extremals $(x(k), u(k), \psi_0, \psi(k))$, $k = M, \ldots, M + N - 1$, satisfy the following $\rho$ expressions ($i = 1, \ldots, \rho$):

$$\psi_0 \frac{\partial \Phi(k, x(k), u(k), s)}{\partial s_i} \bigg|_{s=0} + \psi(k) \cdot \frac{\partial X(k, x(k), u(k), s)}{\partial s_i} \bigg|_{s=0} = \text{constant}.$$  \hspace{1cm} (8)

Remark 3.3. The integrals of motion obtained by Theorem 3.1 are “momentum” integrals. Due to the fact that time $k$ is discrete, one can not vary $k$ continuously and, for that reason, one can not obtain the “energy” integrals as in the continuous optimal control case (cf. [37, 40]). To address the problem another method needs to be developed. This will be addressed in a forthcoming paper.

Remark 3.4. Together with the continuous results in [37, 40], Theorem 3.1 provides a framework to obtain a generalization of Noether’s theorem for hybrid-systems. This and related questions are under study and will be addressed elsewhere.
Proof. Let \((x(k), u(k), \psi_0, \psi(k))\) be an extremal for problem \((P)\). Differentiating \((6)\) and \((7)\) with respect to the parameter \(s_i, \ i = 1, \ldots, \rho\), and setting \(s = (s_1, \ldots, s_\rho) = 0\), we get (recall \((5)\) and that \(X(k, x(k), u(k), 0) = x(k), u(k, 0) = u(k)\)):

\[
\Delta \frac{\partial}{\partial s_i} \Phi (k, x(k), u(k), s) \bigg|_{s=0} = \frac{\partial L}{\partial x} (k, x(k), u(k)) \cdot \frac{\partial}{\partial s_i} X (k, x(k), u(k), s) \bigg|_{s=0} + \frac{\partial L}{\partial u} (k, x(k), u(k)) \cdot \frac{\partial}{\partial s_i} u (k, s) \bigg|_{s=0}, \tag{8}
\]

\[
\frac{\partial}{\partial s_i} X (k + 1, x(k + 1), u(k + 1), s) \bigg|_{s=0} = \frac{\partial \varphi}{\partial x} (k, x(k), u(k)) \cdot \frac{\partial}{\partial s_i} X (k, x(k), u(k), s) \bigg|_{s=0} + \frac{\partial \varphi}{\partial u} (k, x(k), u(k)) \cdot \frac{\partial}{\partial s_i} u (k, s) \bigg|_{s=0}. \tag{9}
\]

From the adjoint system \(\psi(k) = \frac{\partial \mu}{\partial x} (k, x(k), u(k), \psi_0, \psi(k + 1))\), we know that

\[
- \psi_0 \frac{\partial L}{\partial x} (k, x(k), u(k)) = \psi(k + 1) \cdot \frac{\partial \varphi}{\partial x} (k, x(k), u(k)) - \psi(k),
\]

and multiplying \((8)\) by \(-\psi_0\) one obtains:

\[
\psi_0 \left( \Delta \frac{\partial}{\partial s_i} \Phi (k, x(k), u(k), s) \bigg|_{s=0} - \frac{\partial L}{\partial u} (k, x(k), u(k)) \cdot \frac{\partial}{\partial s_i} u (k, s) \bigg|_{s=0} \right)
+ \left( \psi(k + 1) \cdot \frac{\partial \varphi}{\partial x} (k, x(k), u(k)) - \psi(k) \right) \cdot \frac{\partial}{\partial s_i} X (k, x(k), u(k), s) \bigg|_{s=0} = 0. \tag{10}
\]

As far as \(u(k, 0) = u(k)\), according to the maximality condition of the Discrete-Time Maximum Principle the function

\[
s \mapsto \psi_0 L (k, x(k), u(k, s)) + \psi(k + 1) \cdot \varphi (k, x(k), u(k, s))
\]

attains its maximum for \(s = 0\). Therefore,

\[
\frac{\partial}{\partial s_i} \left( \psi_0 L (k, x(k), u(k, s)) + \psi(k + 1) \cdot \varphi (k, x(k), u(k, s)) \right) \bigg|_{s=0} = 0,
\]

that is,

\[
\psi_0 \frac{\partial L}{\partial u} (k, x(k), u(k)) \cdot \frac{\partial}{\partial s_i} u (k, s) \bigg|_{s=0} + \psi(k + 1) \cdot \frac{\partial \varphi}{\partial u} (k, x(k), u(k)) \cdot \frac{\partial}{\partial s_i} u (k, s) \bigg|_{s=0} = 0. \tag{11}
\]
From (10) and (11) it comes
\[ \psi_0 \Delta \frac{\partial}{\partial s_i} \Phi (k, x(k), u(k), s) \bigg|_{s=0} + \left( \psi(k+1) \cdot \frac{\partial \varphi}{\partial x}(k, x(k), u(k)) - \psi(k) \right) \cdot \frac{\partial}{\partial s_i} X(k, x(k), u(k), s) \bigg|_{s=0} + \psi(k+1) \cdot \frac{\partial \varphi}{\partial u}(k, x(k), u(k)) \cdot \frac{\partial}{\partial s_i} u(k, s) \bigg|_{s=0} = 0. \]

Using (9), this last equality is equivalent to
\[ \Delta \left( \psi_0 \frac{\partial}{\partial s_i} \Phi (k, x(k), u(k), s) \bigg|_{s=0} + \psi(k) \cdot \frac{\partial}{\partial s_i} X(k, x(k), u(k), s) \bigg|_{s=0} \right) = 0. \]

The proof is complete. \( \square \)

4 An Example

We now illustrate the use of Theorem 3.1 by the following example \((n = 3, r = 2, \Omega = \mathbb{R}^2)\):
\[ \sum_k \left( (u_1(k))^2 - (u_2(k))^2 \right) \rightarrow \text{extr}, \]
\[ \begin{cases} x_1(k+1) = x_2(k) + u_1(k), \\ x_2(k+1) = x_1(k) + u_2(k), \\ x_3(k+1) = x_2(k)u_1(k), \end{cases} \]
subject to fixed endpoints. In this case the Hamiltonian is given by
\[ H(x_1, x_2, u_1, u_2, \psi_0, \psi_1, \psi_2, \psi_3) = \psi_0 L(u_1, u_2) + \psi_1 \varphi_1(x_2, u_1) + \psi_2 \varphi_2(x_1, u_2) + \psi_3 \varphi_3(x_2, u_1), \]
with \( L(u_1, u_2) = u_1^2 - u_2^2, \ \varphi_1(x_2, u_1) = x_2 + u_1, \ \varphi_2(x_1, u_2) = x_1 + u_2, \) and \( \varphi_3(x_2, u_1) = x_2u_1. \) From the adjoint system we get the evolution equations
\[ \begin{align*}
\psi_1(k) &= \psi_2(k + 1), \\
\psi_2(k) &= \psi_1(k + 1) + \psi_3(k + 1)u_1(k), \\
\psi_3(k) &= 0,
\end{align*} \]
while from the maximality conditions we get \((\psi_3 = 0)\)
\[ \begin{align*}
\psi_1(k + 1) &= -2\psi_0 u_1(k), \\
\psi_2(k + 1) &= 2\psi_0 u_2(k).
\end{align*} \]
There are no abnormal extremals for the problem, and one can fix \(\psi_0 = -\frac{1}{2}\).

The extremals are obtained solving five difference-equations of order one,

\[
\begin{align*}
    x_1(k+1) &= x_2(k) + \psi_1(k+1), \\
    x_2(k+1) &= x_1(k) - \psi_2(k+1), \\
    x_3(k+1) &= x_2(k)\psi_1(k+1), \\
    \psi_1(k+1) &= \psi_2(k), \\
    \psi_2(k+1) &= \psi_1(k),
\end{align*}
\]

together with the boundary conditions (or the transversality conditions), by standard techniques. On the other hand, the problem is quasi-invariant with respect to the one-parameter \((\rho = 1)\) transformations

\[
\begin{align*}
    X_1(x_1(k), s) &= x_1(k) + 2s, \\
    X_2(x_2(k), s) &= x_2(k) + s, \\
    X_3(x_1(k), x_3(k), s) &= x_3(k) + sx_1(k),
\end{align*}
\]

up to the difference gauge term \(\Phi(x_1(k), x_2(k), s) = 2(x_1(k) + x_2(k))\ s\). To see that we choose

\[
u_1(k, s) = u_1(k) + s, \quad u_2(k, s) = u_2(k) - s,
\]

in the Definition \[5.2.1\]. We notice that \(X_1(x_1(k), 0) = x_1(k), X_2(x_2(k), 0) = x_2(k), X_3(x_1(k), x_3(k), 0) = x_3(k), u_1(k, 0) = u_1(k),\) and \(u_2(k, 0) = u_2(k)\). Direct verifications show that the quasi-invariance conditions are satisfied:

\[
\begin{align*}
    L(u_1(k, s), u_2(k, s)) &= (u_1(k))^2 - (u_2(k))^2 + 2(u_1(k) + u_2(k))s \\
    &= L(u_1(k), u_2(k)) + 2(x_1(k+1) - x_2(k) + x_2(k+1) - x_1(k))s \\
    &= L(u_1(k), u_2(k)) + \Delta \Phi(x_1(k), x_2(k), s),
\end{align*}
\]

\[
\begin{align*}
    \varphi_1(X_2(x_2(k), s), u_1(k, s)) &= x_2(k) + u_1(k) + 2s = x_1(k+1) + 2s \\
    &= X_1(x_1(k+1), s), \\
    \varphi_2(X_1(x_1(k), s), u_2(k, s)) &= x_1(k) + u_2(k) + s = x_2(k+1) + s \\
    &= X_2(x_2(k+1), s), \\
    \varphi_3(X_2(x_2(k), s), u_1(k, s)) &= (x_2(k) + s)(u_1(k) + s) \\
    &= x_2(k)u_1(k) + s(x_2(k) + u_1(k)) + s^2 \\
    &= x_3(k+1) + sx_1(k+1) + \delta(s) = X_3(k+1) + \delta(s).
\end{align*}
\]

By Theorem \[5.2.1\] we obtain the following conservation law for the problem:

\[
2\psi_0(x_1(k) + x_2(k)) + 2\psi_1(k) + \psi_2(k) + \psi_3(k)x_1(k) = \text{constant}. \quad (12)
\]

Using the information from the discrete time maximum principle, condition \[12\] is equivalent to

\[
(x_1(k) + x_2(k)) + 2u_2(k) - u_1(k) = \text{constant}. \quad (13)
\]
The conservation law (13) is a necessary optimality condition. It is trivially satisfied choosing the control variables according to:

\[ u_1(k) = x_1(k), \]
\[ u_2(k) = -\frac{1}{2} x_2(k). \]

An extremal is then obtained with the co-state variables given by

\[ \psi_1(k) = -u_2(k) = \frac{1}{2} x_2(k), \]
\[ \psi_2(k) = u_1(k) = x_1(k), \]
\[ \psi_3(k) = 0. \]

5 Discrete Calculus of Variations

We now obtain a discrete Noether’s theorem for the problems of the discrete time calculus of variations which are quasi-invariant with respect to infinitesimal transformations having \( \rho \) parameters, \( \rho \geq 1 \), up to a difference gauge term.

5.1 The fundamental Problem

The fundamental problem in the discrete calculus of variations is a special case of our problem \( (P) \): \( r = n \); no restrictions on the controls \( (\Omega = \mathbb{R}^n) \); \( \varphi(k, x, u) = u \). The problem is then to determine a finite sequence \( x(k) \in \mathbb{R}^n, k = M, \ldots, M + N, x(M) = x_M, x(M + N) = x_{M+N}, \) for which the discrete cost function

\[ J[x(\cdot)] = \sum_{k=M}^{M+N-1} L(k, x(k), x(k+1)) \]

is extremized. The maximality condition in the Theorem 2.1 implies in this case the conditions

\[ \frac{\partial H}{\partial u}(k, x(k), u(k), \psi_0, \psi(k+1)) = 0, \quad k = M, \ldots, M + N - 1, \]

that is,

\[ \psi(k+1) = -\psi_0 \frac{\partial L}{\partial u}(k, x(k), x(k+1)) , \quad (14) \]

while from the adjoint system one gets

\[ \psi(k) = \psi_0 \frac{\partial L}{\partial x}(k, x(k), x(k+1)) . \quad (15) \]

We note that no abnormal extremals exist for the fundamental problem of the discrete calculus of variations: \( \psi_0 = 0 \) implies that \( \psi(k+1) \) is zero for all \( k = M, \ldots, M + N - 1 \), a possibility excluded by the discrete time maximum
principle. So it must be the case that $\psi_0 \neq 0$. From (14) and (15), a necessary condition to have an extremum is that $x(k), k = M, \ldots, M + N - 2$, must satisfy the second-order difference equation

$$\frac{\partial L}{\partial x}(k + 1, x(k + 1), x(k + 2)) + \frac{\partial L}{\partial u}(k, x(k), x(k + 1)) = 0.$$  \hspace{1cm} (16)

Equations (16) share resemblances with the continuous Euler-Lagrange equations, and are called the discrete Euler-Lagrange equations.

**Definition 5.1.** The discrete Lagrangian $L(k, x(k), x(k + 1))$ is said to be quasi-invariant with respect to the infinitesimal $\rho$-parameter transformation $X(k, x, u, s), s = (s_1, \ldots, s_\rho), \|s\| < \varepsilon, X(k, x, u, 0) = x$ for all $k = M, \ldots, M + N - 1, x, u \in \mathbb{R}^n$, up to the difference gauge term $\Phi(k, x(k), x(k + 1), s)$, if for each $k = M, \ldots, M + N - 2$

$$L(k, x(k), x(k + 1)) + \Delta\Phi(k, x(k), x(k + 1), s) + \delta(k, x(k), x(k + 1), s)$$

$$= L(k, X(k, x(k), x(k + 1), s), X(k + 1, x(k + 1), x(k + 2), s)),$$ \hspace{1cm} (17)

where $\delta(\cdot, \cdot, \cdot, \cdot)$ is a function satisfying (7).

**Corollary 5.1.** If $L(k, x(k), x(k + 1))$ is quasi-invariant with respect to the $\rho$-parameter transformation $X$ up to the difference gauge term $\Phi$, in the sense of Definition 5.1, then all solutions $x(k), k = M, \ldots, M + N - 2$, of the discrete Euler-Lagrange difference equation (16) satisfy

$$\frac{\partial L}{\partial u}(k, x(k), x(k + 1)) \cdot \frac{\partial}{\partial s_i}X(k, x(k), x(k + 1), s) \bigg|_{s=0} - \frac{\partial}{\partial s_i}\Phi(k, x(k), x(k + 1), s) \bigg|_{s=0} = \text{constant},$$

$i = 1, \ldots, \rho$.

### 5.2 Higher Order Discrete Problems

Let us now consider the problem of optimizing

$$\sum_k L(k, x(k), x(k + 1), \ldots, x(k + m)),$$ \hspace{1cm} (18)

where $L(k, x^0, x^1, \ldots, x^n)$ is continuously differentiable with respect to all variables. This problem is analogous to the continuous problems of the calculus of variations for which the Lagrangian $L$ depends on higher-order derivatives. It
is easily written in the optimal control form $(P)$. Introducing the notation
\[ x^0(k) = x(k), \]
\[ x^1(k) = x(k + 1), \]
\[ \vdots \]
\[ x^{m-1}(k) = x(k + m - 1), \]
\[ u(k) = x(k + m), \]
one gets:
\[ \sum_k L \left( k, x^0(k), \ldots, x^{m-1}(k), u(k) \right) \rightarrow \text{extr}, \]
\[ \begin{cases} x^0(k + 1) = x^1(k), \\ x^1(k + 1) = x^2(k), \\ \vdots \\ x^{m-2}(k + 1) = x^{m-1}(k), \\ x^{m-1}(k + 1) = u(k). \end{cases} \]

The Hamiltonian is given by
\[ H = \psi_0 L \left( k, x^0(k), \ldots, x^{m-1}(k), u(k) \right) + \left( \sum_{j=0}^{m-2} \psi^j \cdot x^{j+1} \right) + \psi^{m-1} u. \]

From the maximality condition
\[ \psi^{m-1}(k + 1) = -\psi_0 \frac{\partial L}{\partial x^m} \left( k, x^0(k), \ldots, x^{m-1}(k), u(k) \right), \quad (19) \]
while from the adjoint system
\[ \psi^j(k) = \psi_0 \frac{\partial L}{\partial x^j} \left( k, x^0(k), \ldots, x^{m-1}(k), u(k) \right) + \psi^{j-1}(k + 1), \quad (20) \]

\[ j = 1, \ldots, m - 1. \] From (19), (20), and (21), we conclude that: similarly to the fundamental problem of the calculus of variations, no abnormal extremals exist in the higher order case; the equation
\[ \sum_{j=0}^{m} \frac{\partial L}{\partial x^j} \left( k + m - j, x^0(k + m - j), \ldots, x^{m-1}(k + m - j), u(k + m - j) \right) = 0 \]
holds. Going back to the initial notation, (22) is nothing more than the discrete Euler-Poisson equation of order $2m$ for the $m$-th order discrete problem of the
The calculus of variations:\(^{13,15}\):
\[
\sum_{j=0}^{m} \frac{\partial L}{\partial x^j} (k + m - j, x(k + m - j), \ldots, x(k + 2m - 1 - j), x(k + 2m - j)) = 0.
\]

(23)

**Definition 5.2.** The discrete Lagrangian \( L(k, x(k), \ldots, x(k + m)) \) is said to be quasi-invariant with respect to the infinitesimal \( \rho \)-parameter transformation \( X(k, x^0, \ldots, x^m, s) \), \( s = (s_1, \ldots, s_\rho) \), \( \|s\| < \varepsilon \), \( X(k, x^0, \ldots, x^m, 0) = x^0 \) for all \( k \), and \( x^l, j = 0, \ldots, m \), up to the difference gauge term \( \Phi(k, x(k), \ldots, x(k + m), s) \), if for each \( k \)
\[
L(k, x(k), \ldots, x(k + m)) + \Delta \Phi(k, x(k), \ldots, x(k + m), s) + \delta(k, x(k), \ldots, x(k + m), s)
= L(k, X(k, x(k), \ldots, x(k + m), s), \ldots, X(k + m, x(k + m), \ldots, x(k + 2m), s))
\]

(24)

where \( \frac{\partial \delta}{\partial s_i} = 0, i = 1, \ldots, \rho \).

**Corollary 5.2.** If \( L(k, x(k), \ldots, x(k + m)) \) is quasi-invariant with respect to the \( \rho \)-parameter transformation \( X \) up to the difference gauge term \( \Phi \), in the sense of Definition 5.2, then all solutions \( x(k) \) of the discrete Euler-Poisson difference equation \((23)\) satisfy
\[
\frac{\partial}{\partial s_i} \Phi(k, x(k), \ldots, x(k + m), s) \bigg|_{s=0}
+ \sum_{j=0}^{m-1} \sum_{l=0}^{j} \frac{\partial L}{\partial x^l} (k + j - l, x(k + j - l), \ldots, x(k + j - l + m))
\cdot \frac{\partial}{\partial s_i} X(k + j, x(k + j), \ldots, x(k + j + m), s) \bigg|_{s=0}
= \text{constant},
\]

(25)
i = 1, \ldots, \rho.

In the case \( m = 1 \) the discrete Euler-Poisson equation \((23)\) reduces to the discrete Euler-Lagrange equation \((16)\), and the conservation law \((25)\) reduces to
\[
\frac{\partial}{\partial s_i} \Phi(k, x(k), x(k + 1), s) \bigg|_{s=0}
+ \frac{\partial L}{\partial x^0} (k, x(k), x(k + 1)) \cdot \frac{\partial}{\partial s_i} X(k, x(k), x(k + 1), s) \bigg|_{s=0}
= \text{constant},
\]
or, which is the same,
\[
\frac{\partial}{\partial s_i} \Phi(k, x(k), x(k + 1), s) \bigg|_{s=0}
- \frac{\partial L}{\partial x^1} (k - 1, x(k - 1), x(k)) \cdot \frac{\partial}{\partial s_i} X(k, x(k), x(k + 1), s) \bigg|_{s=0}
= \text{constant}.
\]

This is precisely the conservation law given by Corollary 5.1.
References

[1] D. Anderson. Noether’s theorem in generalized mechanics. *J. Phys. A*, 6:299–305, 1973. Zbl 0256.49049 MR 54:6786

[2] A. V. Arutyunov. *Optimality conditions*. Kluwer Academic Publishers, Dordrecht, 2000. Zbl pre01657516 MR 1845332

[3] J. C. Baez and J. W. Gilliam. An algebraic approach to discrete mechanics. *Lett. Math. Phys.*, 31(3):205–212, 1994. Zbl 0805.58031 MR 95i:58098

[4] D. P. Bertsekas. *Dynamic programming and optimal control. I*. Athena Scientific, Belmont, MA, 2 edition, 2000.

[5] P. Blanchard and E. Brüning. *Variational methods in mathematical physics*. Springer-Verlag, Berlin, 1992. Zbl 0756.49023 MR 95b:58049

[6] G. Blankenstein and A. van der Schaft. Optimal control and implicit Hamiltonian systems. In *Nonlinear control in the year 2000, Vol. 1 (Paris)*, pages 185–205. Springer, London, 2001. MR 1806135

[7] J. A. Cadzow. Discrete calculus of variations. *Int. J. Control, I. Ser.*, 11:393–407, 1970. Zbl 0193.07601

[8] J. F. Cariñena and H. Figueroa. A geometrical version of Noether’s theorem in supermechanics. *Rep. Math. Phys.*, 34(3):277–303, 1994. Zbl 0846.58008 MR 96g:58011

[9] D. S. Djukic. Noether’s theorem for optimum control systems. *Internat. J. Control (1)*, 18:667–672, 1973. Zbl 0281.49009 MR 49:5979

[10] I. M. Gelfand and S. V. Fomin. *Calculus of variations*. Dover Publications, Mineola, NY, 2000. Zbl 0964.49001

[11] J. W. Gilliam. *Lagrangian and Symplectic Techniques in Discrete Mechanics*. Ph.D. dissertation, University of California Riverside, August 1996.

[12] R. Hilscher and V. Zeidan. Discrete optimal control: the accessory problem and necessary optimality conditions. *J. Math. Anal. Appl.*, 243(2):429–452, 2000. Zbl pre01439372 MR 2001k:49058

[13] G. Jaroszkiewicz and K. Norton. Principles of discrete time mechanics. I. Particle systems. *J. Phys. A*, 30(9):3115–3144, 1997. Zbl 0949.70014 MR 98k:81076

[14] G. Jaroszkiewicz and K. Norton. Principles of discrete time mechanics. II. Classical field theory. *J. Phys. A*, 30(9):3145–3163, 1997. Zbl 0949.70015 MR 98k:81077

[15] J. Jost and X. Li-Jost. *Calculus of variations*. Cambridge University Press, Cambridge, 1998. Zbl 0913.49001 MR 2000m:49002
[16] J. D. Logan. First integrals in the discrete variational calculus. *Aequationes Math.*, 9:210–220, 1973. Zbl 0268.49022 MR 48:6739

[17] J. D. Logan. Higher dimensional problems in the discrete calculus of variations. *Internat. J. Control (1)*, 17:315–320, 1973. Zbl 0251.49007 MR 48:6741

[18] J. D. Logan. *Invariant variational principles*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1977. MR 58:18024

[19] D. Lovelock and H. Rund. *Tensors, differential forms, and variational principles*. Wiley-Interscience [John Wiley & Sons], New York, 1975. MR 57:13703

[20] M. Lutzky. Dynamical symmetries and conserved quantities. *J. Phys. A: Math. Gen.*, 12(7):973–981, 1979. Zbl 0413.70005 MR 81c:58054

[21] S. Maeda. Extension of discrete Noether theorem. *Math. Japon.*, 26(1):85–90, 1981. Zbl 0458.70002 MR 82g:70041

[22] P. Michel. Programmes mathématiques mixtes. Application au principe du maximum en temps discret dans le cas déterministe et dans le cas stochastique. *RAIRO Rech. Opér.*, 14(1):1–19, 1980. Zbl 0438.49023 MR 81i:90157

[23] S. Moyo and P. G. L. Leach. A note on the construction of the Ermakov-Lewis invariant. *J. Phys. A: Math. Gen.*, 35:5333–5345, 2002.

[24] Z. Nahorski, H. F. Ravn, and R. V. V. Vidal. The discrete-time maximum principle: a survey and some new results. *Internat. J. Control*, 40(3):533–554, 1984. Zbl 0549.49021 MR 86c:49028

[25] H. Nijmeijer and A. van der Schaft. Controlled invariance for nonlinear systems. *IEEE Trans. Automat. Control*, 27(4):904–914, 1982. Zbl 0492.93035 MR 83k:93011

[26] E. Noether. Invariante variationsprobleme. *Gött. Nachr.*, pages 235–257, 1918. JFM 46.0770.01

[27] E. Noether. Invariant variation problems. *Transport Theory Statist. Phys.*, 1(3):186–207, 1971. English translation of the original paper [26]. Zbl 0292.49008 MR 53:10538

[28] K. Norton and G. Jaroszkiewicz. Principles of discrete time mechanics. III. Quantum field theory. *J. Phys. A*, 31(3):977–1000, 1998. Zbl 0961.81125 MR 99m:81275a

[29] K. Norton and G. Jaroszkiewicz. Principles of discrete time mechanics. IV. The Dirac equation, particles and oscillons. *J. Phys. A*, 31(3):1001–1023, 1998. Zbl 0961.81126 MR 99m:81275b
[30] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko. The mathematical theory of optimal processes. Interscience Publishers John Wiley & Sons, Inc. New York-London, 1962. Zbl 0882.01027 MR 29:3316b

[31] H. Rund. The Hamilton-Jacobi theory in the calculus of variations: Its role in mathematics and physics. D. Van Nostrand Co., Ltd., London-Toronto, Ont.-New York, 1966. Zbl 0141.10602 MR 37:5752

[32] A. P. Sage and C. C. I. White. Optimum systems control. Prentice-Hall Inc., Englewood Cliffs, N.J., 2nd edition edition, 1977. Zbl 0388.49002

[33] W. Sarlet and F. Cantrijn. Generalizations of Noether’s theorem in classical mechanics. SIAM Rev., 23(4):467–494, 1981. Zbl 0474.70014 MR 83c:70020

[34] S. P. Sethi and G. L. Thompson. Optimal control theory. Kluwer Academic Publishers, Boston, MA, second edition, 2000. Applications to management science and economics. Zbl pre01734433 MR 1883960

[35] J. Sniatycki. Nonholonomic Noether theorem and reduction of symmetries. Rep. Math. Phys., 42(1-2):5–23, 1998. Zbl 0947.70013 MR 2000a:37060

[36] H. J. Sussmann. Symmetries and integrals of motion in optimal control. In Geometry in nonlinear control and differential inclusions (Warsaw, 1993), pages 379–393. Polish Acad. Sci., Warsaw, 1995. Zbl 0891.49011 MR 96i:49007

[37] D. F. M. Torres. Conservation laws in optimal control. In Dynamics, Bifurcations and Control, volume 273 of Lecture Notes in Control and Information Sciences, pages 287–296. Springer-Verlag, Berlin, Heidelberg, 2002. Zbl pre01819752 MR 1901565

[38] D. F. M. Torres. Conserved quantities along the Pontryagin extremals of quasi-invariant optimal control problems. In Proc. 10th Mediterranean Conference on Control and Automation, MED2002, Lisbon, Portugal, 10 pp. (electronic), 2002.

[39] D. F. M. Torres. On optimal control problems which admit an infinite continuous group of transformations. In Proc. 5th Portuguese Conference on Automatic Control, Controlo 2002, Aveiro, Portugal, pages 247–251, 2002.

[40] D. F. M. Torres. On the Noether theorem for optimal control. European Journal of Control, 8(1):56–63, 2002.

[41] D. F. M. Torres. Carathéodory-equivalence, Noether theorems, and Tonelli full-regularity in the calculus of variations and optimal control. Special Issue of the J. of Mathematical Sciences, Submitted for publication.
[42] A. van der Schaft. Symmetries and conservation laws for Hamiltonian systems with inputs and outputs: a generalization of Noether’s theorem. *Systems Control Lett.*, 1(2):108–115, 1981/82. Zbl 0482.93038 MR 83k:49054

[43] F. Y. M. Wan. *Introduction to the calculus of variations and its applications*. Chapman & Hall, New York, 1995. Zbl 0843.49001 MR 97a:49001

[44] J. M. Wendlandt and J. E. Marsden. Mechanical integrators derived from a discrete variational principle. *Phys. D*, 106(3-4):223–246, 1997. Zbl 0963.70507 MR 99d:70004