Discrete Polynomial Optimization with Coherent Networks of Condensates and Complex Coupling Switching

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Gain-dissipative platforms consisting of lasers, optical parametric oscillators and nonequilibrium condensates operating at the condensation/coherence threshold have been recently proposed as efficient analog simulators of 2-local spin Hamiltonians with continuous or discrete degrees of freedom. We show that nonequilibrium condensates above the threshold arranged in an interacting network may realise k-local Hamiltonians with k > 2 and lead to nontrivial phase configurations. The principle of the operation of such a system lays the ground for physics-inspired computing and the new efficient methods for finding solutions to the higher order binary optimization problems. We show how to facilitate the search for the global solution by invoking complex couplings in the system. This approach offers a highly flexible new kind of computation based on gain-dissipative simulators with complex coupling switching.

In recent years, much effort has been devoted to the development of various technological platforms that act as quantum or classical analog simulators aimed at solving certain classes of hard classical optimization problems [1–7]. It is expected that these kinds of platforms would help to efficiently solve many tasks of significant computational complexity, ranging from modelling microscopic effects and processes like the behavior of electrons in complex materials [8, 9] and finding the ground state of spin glasses [10], to the applied combinatorial optimization problems such as traveling salesman problem [11]. Large scale computational problems of this type are hard for classical von Neumann architecture which suggests looking for fully analog or hybrid digital/analog/quantum devices that can find a solution faster or find a better solution in a fixed time.

Nonequilibrium condensates, optical parametric oscillators, lasers, memristor crossbars and other platforms have been considered as annealing-inspired accelerators and demonstrated successes in finding the ground state of spin Hamiltonians with continuous or discrete variables [2, 6, 12, 13]. In particular, the Coherent Ising Machine has been shown to significantly outperform classical simulated annealing in terms of both accuracy and computation time to efficiently solve Max-CUT problems [2] and has shown better scalability than the quantum annealers [14]. Memristor - Hopfield Neural Network with massively parallel operations performed in a dense crossbar array were shown to be able to solve NP-hard Max-CUT problems predicting over four orders of magnitude advantage over digital and optical hardware [13]. Integrated photonic circuits that use self-phase modulation in two microring resonators were shown to act as an optical coherent Ising machine [15, 16]. The lattices of exciton-polariton condensates were shown to efficiently simulate the XY Hamiltonian when operating at the condensation threshold [12, 17] and the extensions to minimizing Ising and q-state Potts models were also considered [18]. In all these systems, discrete Ising 'spins' or continuous XY 'spins' are encoded in individual phase modes of the nonlinear networks. An optimization problem of interest is mapped into the quadratic binary optimization problem (QUBO) and, therefore, into the connection matrix of the Ising network. The problem of finding the optimal solution of a QUBO problem reduces to finding the ground state of the Ising Hamiltonian, which can be related to finding the 'maximum occupancy' of the collective supermode of the underlying network, as a system specific gain mechanism is continuously increased to reach the coherence threshold [12, 19].

The development of technological platforms that promise to offer a significant time or power consumption improvements in solving hard optimization problems goes hand in hand with annealing inspired optimization, when the physical principle of the device is used to formulate the classical algorithms [20]. The recent examples of which include Simulated Bifurcation algorithms inspired by quantum adiabatic optimization using a nonlinear oscillator network [21], destabilization of local minima based on degenerate parametric oscillator networks [22], parallel tempering Monte Carlo [23], and the gain-dissipative algorithm based on operation of the polariton graph simulator at the condensation Carlo [23].

The focus of all these technological and inspired implementations of the annealer-based optimization has been on QUBO, however, there is a large class of optimization problems — the higher order polynomial optimization (HOBO) — that are more naturally encoded by the k-local Hamiltonians [24, 25]. HOBO is concerned with optimizing a (high degree) multivariate polynomial function in binary variables. Our basic model is to maximize or minimize a k-th degree polynomial function f(x) where x = (x_1,...,x_i,...,x_n), x_i ∈ {-1,+1}. The examples of HOBO are ubiquitous from Hypergraph max-covering problem to Frobenius and "market split" prob-

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lems [25]. HOBO is a fundamental problem in integer programming and is also known as Fourier support graph problem. Any HOBO can be mapped into the QUBO [26], however, the overhead in the number of nodes becomes prohibitive in an actual technological platform, so it is important to consider ways to solve HOBO directly. The purpose of this article is three-fold. First, we show that Ising machines based on nonequilibrium condensates can be used to address 4-local HOBO when operating above the threshold. Secondly, inspired by the operation of the networks of nonequilibrium condensates we propose a new optimization algorithm for solving HOBO of arbitrary degree. Finally, we show that another physics-inspired method of turning on and off the complex coupling between the nonlinear condensates greatly enhances the search for global minimum.

**Polynomial optimization with coherent networks.** The optimization problem studied in this paper is

\[
\min_{x \in \{-1, 1\}^n} - \sum_{\Omega} A^k_{i_1, i_2, \ldots, i_k} x_{i_1} x_{i_2} \ldots x_{i_k},
\]

where \( \Omega = \{i_j : 1 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n\} \) and \( A^k \) is the super-symmetric tensor of degree \( k \).

To formulate the gain-dissipative platform that reaches the ground state of HOBO by finding the ‘maximum occupancy’ collective supermode of the underlying network of nonequilibrium condensates we consider the mean-field equations that govern such a network based in the Ginzburg-Landau equation [27][28]. This is a universal driven-dissipative equation that describes the behaviour of systems in the vicinity of a symmetry-breaking instability and has been used to describe lasers, thermal convection, nematic liquid crystals, and various nonequilibrium condensates [29][30]. When derived asymptotically from a generic laser model given by Maxwell-Bloch equations it has a saturable nonlinearity and can be written as

\[
\frac{1}{i} \frac{\partial \psi}{\partial t} = -\nabla^2 \psi + \tilde{U}|\psi|^2 \psi + i \left( \frac{P(r, t)}{1 + |\psi|^2} - \gamma_c \right) \psi,
\]

where \( \psi(r, t) \) is the wavefunction of the system, \( \tilde{U} \) is the strength of the delta-function interaction potential, \( \gamma_c \) is the rate of linear losses, \( b \) parametrizes the effective strength of nonlinear losses, \( P(r, t) \) describes the gain mechanism that adds particles to the system. We shall assume that \( P(r, t) \) adds particles in \( N \) spatial locations centered at \( r_i, i = 1, ..., N \), so that \( P(r, t) = \sum_i f_i(t) p_i(r) \), where \( f_i \) is the time-dependent part of the pumping at \( r = r_i \) and \( p_i(r) = p(|r - r_i|) \) is a given spatially localised pumping profile, that creates the condensate with a wavefunction \( \phi_i(r) \equiv \phi(|r - r_i|) \) centred at \( r = r_i \) and normalized so that \( 2 \pi \int |\phi(r)|^2 r \, dr = 1 \). In writing Eq. [2] we let \( h = 1 \) and \( m = 1/2 \). If the distances between the neighboring condensates are larger than the width of \( p(r) \), we employ the tight binding approximation and write the wavefunction of the system as a linear superposition of the wavefunctions of individual coherent centers \( \psi(r, t) \approx \sum_i \phi_i(t) \phi_i(r) \), where \( \phi_i(t) \) is the time-dependent complex amplitude network [18][31]. We expand the first term in the brackets of Eq. [2] in Taylor series, substitute the expressions for \( P \) and \( \psi \), multiply by \( \phi_i^* \) for \( j = 1, ..., N \) and eliminate the spatial degrees of freedom by integrating in the entire space to obtain \( N \) equations of the form

\[
\frac{d\Psi_i}{dt} = \Psi_i \left( \gamma_i - (\sigma_i + iU)|\Psi_i|^2 \right) + \sum_{j,j\neq i} J_{ij} \Psi_j + \sum_{(j,k,l)} Q_{ijkl} \Psi_j \Psi_k \Psi_l + D \xi_i(t).
\]  

In writing Eq. [3] we used the following notations: \( \gamma_i = f_i \int p|\phi|^2 \, dr - \gamma_c, \sigma_i = f_i \int p|\phi|^4 \, dr, U = \tilde{U} \int |\phi|^4 \, dr \), \( Q_{ijkl} = -b\sum_{m \in \{i,j,k,l\}} f_m \int p_m \phi_m \phi_i \phi_j \phi_l \, dr \), and \( \{i,j,k\} \) denotes the combinations of \( \{j,k,l\} \) that exclude \( k = l = i \). Here we neglected the integrals that involve products of the wavefunctions of separated condensates in the view of their smallness in comparison with the integrals of the products of the same condensates and the integrals that involve the pumping profiles. To the leading order, therefore, we kept the terms \( J_{ij} = Re[f_i \int p_i \phi_i \phi_j^* \, dr + f_j \int p_j \phi_j \phi_i^* \, dr] \) for \( j \neq i \), and denoted \( \Psi_i = a_i \exp(it \phi^2 \nabla^2 \phi) \). In writing Eq. [3] we also introduced the Langevin noise \( \xi_i(t) \). This represents intrinsic vacuum fluctuations and classical noise with a diffusion coefficient \( D \). The rate equations on \( \Psi_i(t) \) take the form similar to what we obtained for a polaritonic networks at the condensation threshold [17], but now involve higher order terms represented by the super-symmetric tensor \( Q \). At the condensation threshold these terms can be neglected, however, above the threshold these terms allow to minimize the higher order k-local Hamiltonians.

To see this, we rewrite Eq. [3] in terms of the number densities \( \rho_i \) and phases \( \theta_i \) using the Madelung transformation \( \Psi_i = \sqrt{\rho_i} \exp(i\theta_i) \) for simplicity excluding the noise:

\[
\frac{1}{2} \dot{\theta}_i(t) = (\gamma_i - \sigma_i \rho_i) \rho_i + \sum_{j,j\neq i} J_{ij} \sqrt{\rho_j \rho_i} \cos \theta_{ij} + \sum_{(j,k,l)} Q_{ijkl} \sqrt{\rho_j \rho_k \rho_l} \cos \theta_{ijkl},
\]

\[
\dot{\theta}_i(t) = -U \rho_i - \sum_{j,j\neq i} J_{ij} \sqrt{\rho_j \rho_i} \sin \theta_{ij} - \sum_{(j,k,l)} Q_{ijkl} \sqrt{\rho_j \rho_k \rho_l} \sin \theta_{ijkl},
\]

where \( \theta_{ij} = \theta_i - \theta_j \) and \( \theta_{ijkl} = \theta_i + \theta_j - \theta_k - \theta_l \).

Equation [4] describes the evolution of the higher order Kuramoto oscillators. The higher order terms affect the states even in the simplest configuration of two identical oscillators pumped with \( \gamma_i = \gamma \) for which the occupancy...
\(\rho_0 = \rho_1 = \rho_2\) at the fixed point of Eqs. \([4, 5]\) reads \(\rho_{\text{test}} = \rho_1 = \rho_2\) unless the oscillatory network is highly symmetric (all oscillators have equal number of connections of the same strength with other oscillators) the systems breaks into subsystems characterised by different frequencies. To guarantee the full synchronisation of the network we need to choose the injection rates in such a way that all oscillators have the same occupancy \(\rho_{\text{th}}\) \([17]\). For instance, this can be achieved by adjusting the pumping rates dynamically, depending on the occupancy of the oscillator at time \(t\): \(\gamma_{i} = \epsilon(\rho_{\text{th}} - \rho_i)\), where the parameter \(\epsilon\) characterizes how fast \(\gamma_{i}\) adjusts to changes in \(\rho_i\). With such density adjustments the system of \(N\) oscillators will always synchronize and achieve the stationary minimum of the Hamiltonian \(H = -\sum_{i,j} J_{ij} \cos \theta_{ij} - \sum_{i} J_{ijkl} \cos \theta_{ijkl}\).

To replace the minimization in the space of continuous spins with binary states, we will combine nonresonant pumping with resonant at twice the frequency of the condensate which introduces the terms proportional to \(\Psi^\ast\) to the right-hand side of Eq. \([2]\) (Eq. \([3]\) \([32]\). Resonant and nonresonant excitations have been previously combined in experiments on polaritons using chemical etching across the sample allowing resonant excitation from the back side of the cavity. The resonant excitation laser was synchronized with the nonresonant one and with the condense frequency \([33]\). Such combination of resonant and nonresonant excitations leads to the realisation of the minimum of the 4-local Hamiltonians of the form \(H = -\sum_{i,j} J_{ij} x_i x_j - \sum_{i} J_{ijkl} x_i x_j x_k x_l\), where we denoted the binary spin \(x_i = \cos \theta_i \in \{-1, 1\}\).

By this analysis we have predicted that nonequilibrium gain-dissipative systems when pumped above the coherence/condensation threshold may realise minimum of the higher order Hamiltonians with nontrivial phases. It is interesting to see what possible novel states and phase transitions result and we leave this questions for the future work.

**Physics-Inspired Optimization.** The formulated principle of coherence formation at a minimum of a spin Hamiltonian formulated above inspires an efficient algorithm for finding the global minimum of HOBO. For this, we extend and simplify Eq. \([6]\) to capture the mechanism of relaxation to the minimum of the HOBO but without the necessity to capture full physics of the actual system. The minimum of HOBO for \(N\) binary variables can be found by numerical integration of \(2N\) equations

\[
\frac{d\Psi_{l}}{dt} = \Psi_{l}(\gamma_{l}(t) - |\Psi_{l}|^{2}) + \sum_{\Omega} A_{i_{1},i_{2},\ldots,i_{k}}^{l} \Psi_{i_{1}} \Psi_{i_{2}} \cdots \Psi_{i_{k}}
+ g(t)\Psi_{l},
\]

\[
\frac{d|\Psi_{l}|^{2}}{dt} = \epsilon(1 - |\Psi_{l}|^{2}),
\]

where \(\Omega = \Omega/l, \gamma_{l}(t) = -\sum_{k} |A_{i_{1},i_{2},\ldots,i_{k}}^{l}|^{2}\) and \(g(t) = G \tanh(\alpha t)\), where \(G > \max_{i} \sum_{l} |A_{i_{1},i_{2},\ldots,i_{k}}^{l}|^{2}\). At the fixed point, the total occupancy of the system is equal to \(N\) and given by \(N = \sum_{l} \gamma_{l} + \sum_{l} \sum_{\Omega} A_{i_{1},i_{2},\ldots,i_{k}}^{l} \cos(\theta_{i_{1}}) \cos(\theta_{i_{2}}) \cdots \cos(\theta_{i_{k}}) \cos(\theta_{l}) + G \sum_{l} \cos(\theta_{l})\). Again, if we set the process of raising the pumping from below that guarantees that \(\gamma_{l}\) is the smallest possible injected intensity, at the fixed point the system finds the global minimum of \(k\)-local Hamiltonian \(H = -\sum_{l=1}^{N} \sum_{\Omega} \cos(\theta_{i_{1}}) \cos(\theta_{i_{2}}) \cdots \cos(\theta_{i_{k}}) \cos(\theta_{l}) - G \sum_{i=1}^{N} \cos(\theta_{i})\).

To illustrate the behavior of the system we consider the following 3-local Hamiltonian

\[
H_{\text{test}}(x) = -x_1 x_2 x_3 - x_1 x_2 x_4 - \frac{1}{4} x_2 x_3 x_4,
\]

where \(\chi = \frac{1}{2}\), so that Eq. \([6]\) becomes \(\dot{\Psi}_{l} = \Psi_{l}(\gamma_{l} - |\Psi_{l}|^{2}) + \sum_{\Omega} A_{i_{1},i_{2},\ldots,i_{k}}^{l} \Psi_{i_{1}} \Psi_{i_{2}} \cdots \Psi_{i_{k}} + g(t)\Psi_{l},\) and \(K\) is a tensor with nonzero entries \(\frac{1}{2}, \frac{1}{2}, 1\). The Hamiltonian \(H_{\text{test}}\) has three local minima: \(H_1 = \frac{1}{4}, H_2 = -\frac{3}{4}, H_3 = -\frac{3}{2}\) and the global minimum \(H_4 = -\frac{2}{3}\). To understand the basins of attraction for these stationary points we numerically integrate Eqs. \([6]\) starting with initial conditions \(\Psi_{l}(t = 0) = \frac{1}{\sqrt{20}} \exp[\alpha t_{l}]\) where the phases are uniformly distributed in the 4-dimensional space \(\theta_{l} \in [0, 2\pi]\) \([34]\). Figure \(1a\) depicts the statistics of distribution of the stationary points reached and indicates that the local minima have a larger basin of attraction then the global minimum. To facilitate the search for the global minimum the algorithm needs to allow to explore the hyperspace until the lowest lying state is found. This can be achieved by adding a noise (typically present in physical system as well), that shifts the trajectory from its deterministic path while allowing it to stay below any local minima. This can be ascertainment by decreasing \(\epsilon\) that controls the time the system spends while raising to the condensation threshold from below. We illustrate this behaviour in Fig \(1b\) that depicts the statistics of reaching local and global minima found by numerical integration of Eqs. \([6]\) using the same initial conditions as in Fig \(1a\) but with the white noise added. On Fig. \(2\) we show one of such trajectories as it approaches the global
minimum of $H_{\text{test}}$ from below.

With the growth in the number of variables and concomitant growth of the system hyperspace any local noisy perturbation of the trajectory may not be sufficient to reach the global minimum basin of attraction or it would take prohibitively long time. Recent interest in heteroclinic networks – networks that exhibit saddle states that are dynamically linked via heteroclinic connections – proposes a way to allow for a fast switching between the states [35]. Motivated by these ideas we introduced heteroclinic orbits into our model by engineering time-dependent complex couplings into the network Eqs. (6-7). Complex couplings naturally appear in our model if the energy shift due to a noncondensed reservoir $R(r, t)$ is present in the system. This introduces $gR(r, t)\psi(r, t)$ term into the right-hand side of Eq. (2), where $g$ parametrizes the strengths of the interactions between condensed and non-condensed particles. In the tight binding approximation and in analogy with the derivation of rate equations simulating the Ising and XY Hamiltonians [36], this term gives rise to $A^r_{i_1, i_2, \ldots, i_k}$ being replaced with the complex coupling $A^r_{i_1, i_2, \ldots, i_k} + iB^k_{i_1, i_2, \ldots, i_k}$, where $B$ represents the part of the coupling that comes from the energy shift due to reservoir. For instance, in polariton lattices such coupling can be turned on and off experimentally between the individual lattice elements and with strengths varying in time and space [37]. The presence of the complex part of the coupling introduces the phase lag in the system that leads to either shift of the stationary point of Eqs. (6-7) for small values of the complex part or destabilization of it by creating a saddle point [36]. In the latter case, if the complex part of the coupling is turned on, the system trajectory quickly leaves the neighborhood of the previous stationary point along the fastest direction. Including this switching dynamics into the system facilitates the search for the true global minimum by allowing fuller exploration of the phase space.

Complex coupling switching. To implement the complex coupling switching we turn one of the real coupling coefficient into the complex one with a significant complex part as soon as the system reaches a steady state. The system trajectory leaves the basin of attraction of that state and travels to a different part of the system hypercube $[0, 2\pi]^N$. When the complex part of the coupling is turned off another steady state may be found. By varying the coupling element to be switched, the duration of the switching in time and the amplitude of the imaginary part are alternatives that state and travels to a different part of the system hypercube $[0, 2\pi]^N$. When the complex part of the coupling is turned off another steady state may be found. By varying the coupling element to be switched, the duration of the switching in time and the amplitude of the imaginary part are alternatives that state and travels to a different part of the system hypercube $[0, 2\pi]^N$. When the complex part of the coupling is turned off another steady state may be found. By varying the coupling element to be switched, the duration of the switching in time and the amplitude of the imaginary part are alternatives.

Conclusions. In this paper we showed that lattices of nonequilibrium condensates when pumped above the
threshold may realise 4-local Hamiltonians with nontrivial spin structures. Combining resonant and nonresonant drive leads to minimization of discrete spin Hamiltonians, and therefore, to finding solutions to the higher order binary optimization problems. We further exploit this idea by formulating system-inspired method of computing the optimal solution of a large range of HOBO problems. Finally, we introduced the concept of computation via the mechanism of complex coupling switching – a dynamical feature that supports switching between the local minima until the true global minimum is found. This approach offers a highly flexible new kind of computation that is inspired by and compatible with many physical network realisation. We envision it becoming a part of a hybrid platform where the states of the network are fed to the physical device for an optimal performance.

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Eqs. (6-7) were Euler integrated with $\epsilon = 0.175$, $\gamma(t = 0) = -1.75$, $g(t) = 2 \tanh(0.03t - 1.5) + 1$, $D = \max(1 - \rho, 0)$.