Exponential confidence interval based on the recursive Wolverton - Wagner density estimation.

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Abstract.

We derive the exponential non improvable Grand Lebesgue Space norm decreasing estimations for tail of distribution for exact normed deviation for the famous recursive Wolverton - Wagner multivariate statistical density estimation.

We consider pointwise as well as Lebesgue - Riesz norm error of statistical density of measurement.

Key words and phrases. Probability, random variable and vector (r.v.), density of distribution, Hölder’s and other functional class of functions, Tchernov’s inequality, Young - Fenchel transform, weight, regression problem, mixes and ordinary Lebesgue - Riesz and Grand Lebesgue Space norm and spaces, kernel, bandwidth, condition of orthogonality, bias and variation, convergence, uniform norm, convergence almost everywhere, consistence, recursive Wolverton - Wagner multivariate statistical density estimation, optimization.
1 Statement of problem. Notations and definitions. Previous results.

Let \( (\Omega, M, P) \) be probability space with expectation \( E \) and variance \( \text{Var} \). Let also \( \{\xi_k\}, k = 1, 2, \ldots, n \) be a sequence of independent, identical distributed (i, i.d.) random vectors (r.v.) taking the values in the ordinary Euclidean space \( \mathbb{R}^d \), \( d = 1, 2, \ldots \) and having certain non-known density of a distribution \( f = f(x), x \in \mathbb{R}^d \). C.Wolverton and T.J.Wagner in [31] offered the following famous statistical estimation \( f_{WW}^n(x) = f_n(x) \) for \( f(\cdot) \).

Let \( \{h_k\}, k = 1, 2, \ldots \) be some positive sequence of real numbers such that \( \lim_{k \to \infty} h_k = 0 \).

Let also \( K = K(x), x \in \mathbb{R}^d \) be certain kernel, i.e. measurable even function for which

\[
\int_{\mathbb{R}^d} K(x) \, dx = 1. \tag{1}
\]

Then by definition

\[
f_{WW}^n(x) = f_n(x) \overset{\text{def}}{=} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{h_k^d} K \left( \frac{x - \xi_k}{h_k} \right). \tag{2}
\]

Recall that the classical kernel, or Parzen - Rosenblatt’s estimate \( f_{PR}^n(x) \) has a form

\[
f_{PR}^n(x) := \frac{1}{nh_n^d} \sum_{k=1}^{n} K \left( \frac{x - \xi_k}{h_n} \right), \tag{3}
\]

see [24], [25].

Note that the Wolverton - Wagner estimate obeys a very important recursion property:

\[
f_{WW}^n(x) = \frac{n - 1}{n} f_{WW}^{n-1}(x) + \frac{1}{nh_n^d} K \left( \frac{x - \xi_n}{h_n} \right).
\]

"The recurrent definition of probability density estimates" \( f_{WW}^n(x) \) has two obvious advantages: 1) there is no need to memorize data, i.e. if the estimate \( f_{WW}^{n-1}(x) \) is known, then \( f_{WW}^n(x) \) can be calculated by means of the last observation \( \xi_n \) only, without using the sampling \( \xi_1, \xi_2, \ldots, \xi_{n-1} \); 2) the asymptotic dispersion of the estimate \( f_{WW}^n(x) \) does not exceed the dispersion of the estimate" \( f_{PR}^n(x) \), see [22].

Our aim in this report is to deduce the exact exponential decreasing estimate for the tail of deviation probability

\[
P_{WW}^n(u) \overset{\text{def}}{=} \sup_{x \in \mathbb{R}^d} \mathbb{P}(B_n | f_{WW}^n(x) - f(x) > u), \quad u \geq 1, \tag{4}
\]
i.e. under exact optimal deterministic numerical sequence $B_n$, such that $\lim_{n \to \infty} B_n = \infty$.

For the Parzen - Rosenblatt estimate $f_{n}^{PR}(x)$ these estimates was obtained e.g. in [23], chapter 5, sections 5.2 - 5.6.

We will use some facts from the theory of the so - called Grand Lebesgue Spaces (GLS), devoted in particular the Banach spaces of random variables having exponential decreasing tails of distributions, see e.g. [2], [7], [8], [17], [18], [20], [23] etc.

Note that the distribution of the normed deviation $B_n(f_{n}^{WW} - f)$ in different Lebesgue - Riesz spaces $L_p(R^d \otimes \Omega)$:

$$\Delta_{n,p} := E \int_{R^d} B_n^p \left| f_{n}^{WW}(x) - f(x) \right|^p dx$$

was investigated in many works, see e.g. [10], [22], [24], [25], [27], [28], [29], [30], [31] etc. The optimal choose of $\{h_k\}$ and the kernel $K(x)$ are devoted the following works [9], [16], [19], [21]. The case when the r.v. - s. are dependent is investigated in [19], [26].

**Let us reproduce some used for us notations and conditions from this theory.**

Let $(\beta, L)$ be certain positive numbers. Denote by $l = l(\beta) = [\beta]$ an integer part for $\beta$, i.e. a maximal integer number less than $\beta$:

$$l(\beta) = [\beta] = \max \{ j = 0, 1, 2, \ldots : j \leq \beta \}.$$ 

For instance, $l(0.3) = 0$, $l(\pi) = 3$. Correspondingly, the fractional part $\{\beta\}$ for $\beta$ is equal $\{\beta\} := \beta - [\beta]$.

Introduce as ordinary the functional class $\Sigma(\beta, L)$ as follows

$$\Sigma(\beta, L) = \{ f : R^d \to R; \forall m = \vec{m} : |m| \leq [\beta] \Rightarrow \frac{\partial^m f}{\partial x^m} \in H(\{\beta\}, L) \}, \quad (5)$$

where $H(\alpha, L)$ denotes the Hölder class of the functions

$$H(\alpha, L) = \{ g : R^d \to R, |g(x) - g(y)| \leq L \cdot |x - y|^{\alpha} \}, \quad \alpha \in (0, 1]$$

As usually

$$|z| = \sqrt{(z, z)}, \ m = \vec{m} = \{m_1, m_2, \ldots, m_d\}, \ |m| = \sum_{i=1}^{d} m_i.$$ 

In the case when $\beta$ is integer number, the derivative in (5) is assumed to be continuous and bounded:
∀\bar{m} = \bar{m} : |\bar{m}| = \beta = [\beta] \Rightarrow \sup_{x \in \mathbb{R}^d} \left| \frac{\partial \bar{m} f}{\partial \bar{m}} \right| \leq L.

We suppose henceforth that the density function belongs to some set $\Sigma(\beta, L)$ for some non-trivial value $\beta \in (0, \infty)$:

$$f(\cdot) \in \Sigma(\beta) \overset{\text{def}}{=} \bigcup_{L \in (0, \infty)} \Sigma(\beta, L), \; \beta > 0.$$  

(6)

As for the kernel $K$. We impose on $K$ the following conditions

$$K(-x) = K(x); \; \int_{\mathbb{R}^d} K(x) \; dx = 1; \; \int_{\mathbb{R}^d} K^2(x) dx < \infty;$$  

(7)

$$K(\cdot) \in C(\mathbb{R}^d), \; \int_{\mathbb{R}^d} |K(x)| dx < \infty.$$  

(8)

The following conditions may be named as conditions of orthogonality:

$$\forall \bar{m} : |\bar{m}| \leq [\beta] \Rightarrow \int_{\mathbb{R}^d} \bar{m}(x) V(x) \; dx = 0.$$  

(9)

The last conditions (9) may be used only for the investigation of bias $\delta_n(x)$ of these statistics

$$\delta_n = \delta_n(x) \overset{\text{def}}{=} \mathbb{E} f_{n}^{WW}(x) - f(x).$$  

(10)

In detail, as long as $f \in \Sigma(\beta)$ and by virtue of (9)

$$\delta^{(k)} \overset{\text{def}}{=} h_k^{-d} \int_{\mathbb{R}^d} K\left( \frac{x - y}{h_k} \right) f(y) \; dy - f(x) \sim h_k^\beta, \; h_k \to 0+,$$

therefore

$$|\delta_n| \sim C_1(\beta, L) \; n^{-1} \left[ \sum_{k=1}^{n} h_k^\beta \right],$$  

(11)

see [22], [10], [19].

2 Main result.

Let us investigate now the Variance of the considered Wolver- Wagner $f_{n}^{WW}(x)$ statistic, of course under formulated above restrictions. We have

$$\text{Var} \left\{ f_{n}^{WW}(x) \right\} = \frac{1}{n^2} \sum_{k=1}^{n} h_k^{-d} \text{Var} \left\{ K\left( \frac{x - \xi_k}{h_k} \right) \right\} \approx \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{h_k^d}.$$  

Let’s form the classical target functional
\[ Z = Z_n(h_1, h_2, \ldots, h_n) \overset{\text{def}}{=} \mathbb{E} \left[ f_n^W(x) - f(x) \right]^2; \]

then

\[ Z_n(h_1, h_2, \ldots, h_n) \geq \frac{1}{n^2} \left\{ \sum_{k=1}^n \frac{1}{h_k^2} + \left[ \sum_{k=1}^n h_k^2 \right]^2 \right\}. \tag{12} \]

The (asymptotic) minimal value of the functional \( Z_n(h_1, h_2, \ldots, h_n) \) relative the variables \( \{h_k\} \) subject to our limitations is attained on the values

\[ h_k \sim C_2(\beta, d, L) \ k^{-1/(2\beta+d)} \tag{13} \]

and wherein

\[ \min Z_n = n^{-2\beta/(2\beta+d)}. \tag{14} \]

So, the speed of convergence \( f_n^W(x) \to f(x) \) as \( n \to \infty \) is equal to \( n^{-\beta/(2\beta+d)} \):

\[ \left[ \mathbb{E}(f_n^W(x) - f(x))^2 \right]^{1/2} \sim n^{-\beta/(2\beta+d)}, \tag{15} \]

alike ones for the Parzen - Rosenblatt estimates.

On the other words, the value \( B_n \) in (4) must be chose as follows:

\[ B_n = n^{\beta/(2\beta+d)}. \tag{16} \]

Note that the one - dimensional case \( d = 1 \) was considered in [9].

We suppose henceforth that the values \( \{h_k\}, B_n \) are chose optimally in accordance with (13) and (16).

Define the following tail probability

\[ Q_n^W(u) \overset{\text{def}}{=} \sup_{x \in \mathbb{R}^d} \mathbb{P}(B_n| f_n^W(x) - \mathbb{E}f_n^W(x) | > u ), \ u \geq 1. \tag{17} \]

Note that the following value is bounded:

\[ \sup_x \sup_n B_n|f(x) - \mathbb{E}f_n^W(x)| = C_3 = C_3(d, \beta, L) < \infty. \]

Therefore

\[ \mathbb{P}_n^W(u) \leq Q_n^W(u - C_3). \]

Evidently, the r.v. \( f_n^W(x) - \mathbb{E}f_n^W(x) \) is centered (mean zero).

**Theorem 2.1.** We propose under formulated above conditions
\[
\sup_n Q_n^{WW}(u) \leq 2 \exp \left[ -C_4(d, \beta, L) u^{\frac{2\beta+d}{2d+4}} \right], \quad u \geq 1. \tag{18}
\]

**Proof.** First of all we need for applying the theory of Grand Lebesgue Spaces (GLS) the estimate of an exponential moment

\[
E_n[Q, \lambda, \beta] \overset{\text{def}}{=} \mathbf{E} \exp \left[ \lambda B_n(f_n^{WW}(x) - \mathbf{E}f_n^{WW}(x)) \right], \quad \lambda \in \mathbb{R}. \tag{19}
\]

Denote for this purpose

\[
\Theta_n = B_n(f_n^{WW}(x) - \mathbf{E}f_n^{WW}(x)) = n^{-\frac{d+d}{2d+4}} \sum_{k=1}^{n} h_k^{-d} K^\circ \left( \frac{x - \xi_k}{h_k} \right) = \sum_{k=1}^{n} \theta_{k,n}, \quad \theta_{k,n} := n^{-\frac{\beta+d}{2d+4}} h_k^{-d} K^\circ \left( \frac{x - \xi_k}{h_k} \right),
\]

where as ordinary for arbitrary r.v. \( \eta \Rightarrow \eta^o \overset{\text{def}}{=} \eta - \mathbf{E}\eta. \) We have

\[
E_n[Q, \lambda, \beta] = \mathbf{E} e^{\lambda \Theta_n} = \prod_{k=1}^{n} \mathbf{E} e^{\lambda \theta_{k,n}}.
\]

Let us consider two possibilities.

**A.** \( 0 < \lambda \leq n^{-\frac{d+d}{2d+4}} h_n^{-d}/(2 \sup_x K(x)) \leq 1, \)

or equally

\[
\lambda \in \left( 0, \, C_5 n^{\frac{\beta}{2d+4}} \right).
\]

We use the following elementary inequality

\[
y \in (0, 1) \Rightarrow e^y \leq 1 + y + y^2.
\]

Therefore

\[
E_{k,n}(\lambda) := \mathbf{E} \exp \left( \lambda \theta_{k,n} \right) \leq 1 + C \lambda^2 \text{Var}(\theta_{k,n}) \leq \exp \left( C \lambda^2 \text{Var}(\theta_{k,n}) \right);
\]

\[
E_n[Q, \lambda, \beta] \leq \exp \left( C \lambda^2 \sum_{k=1}^{n} \text{Var}(\theta_{k,n}) \right) \leq \exp(C \lambda^2).
\]

**B.** Let us investigate an opposite possibility

\[
\lambda \geq C_5 n^{\frac{\beta}{2d+4}}.
\]

But then
\[ n \leq C \lambda^{\frac{2\beta + d}{\beta}}, \]

and we deduce

\[ \lambda \Theta_n = \lambda n^{-\frac{\beta + d}{2\beta + d}} \sum_{k=1}^{n} h_k^{-d} K^o \left( \frac{x - \xi_k}{h_k} \right), \]

and following

\[ \lambda |\Theta_n| \leq C \lambda n^{\frac{\beta + d}{2\beta + d}} \leq C \lambda^{\frac{2\beta + d}{\beta}}, \]

\[ \mathbb{E} \exp(\lambda \Theta_n) \leq \exp \left( C \lambda^{\frac{2\beta + d}{\beta}} \right). \]

The case when \( \lambda < 0 \) is considered quite analogously.

Denote \( m = m(n) = n^{\beta/(2\beta + d)} \). Let us introduce the following function

\[ \phi(\lambda) = \phi_m(\lambda) = \phi_{\beta,d,n}(\lambda) = \lambda^2 I(|\lambda| \leq m) + |\lambda|^{(2\beta + d)/\beta} I(|\lambda| > m), \]

where as ordinary \( I(A) \) denotes the indicator function of the set \( A \).

We obtained actually

\[ \ln \mathbb{E} (\lambda \Theta_n) \leq \phi_m(C \lambda), \lambda \in \mathbb{R}. \]  (20)

It follows from the theory of Grand Lebesgue Spaces (GLS), see e.g. [7], [8], [15], [20] that

\[ \mathbb{P}(|\Theta_n| > Cu) \leq \exp \left( -\phi_m^*(u) \right), u \geq 0, \]

modified Tchernov’s inequality. Here as usually \( \phi^*(\cdot) \) denotes the classical Young-Fenchel transform

\[ \phi^*(u) \overset{def}{=} \sup_{\lambda \in \mathbb{R}} (\lambda u - \phi(\lambda)). \]

We deduce after simple calculations

\[ \mathbb{P}(|\Theta_n| > Cu) \leq \exp \left( -u^2 \right), u \in \left( 0, n^{\beta/(2\beta + d)} \right); \]  (21)

\[ \mathbb{P}(|\Theta_n| > Cu) \leq \exp \left( -u^{(2\beta + d)/(\beta + d)} \right), u \geq n^{\beta/(2\beta + d)}. \]  (22)

The announced result (18) follows immediately from (21) and (22); it is easily to verify that obtained estimate for \( \mathbb{P}(|\Theta_n| > Cu) \) reaches its maximum relative the variable \( n \) only for the value \( n = 1 \).

**Remark 2.1.** Note that the inequality of the form (18) of Theorem 2.1 is true also for the classical Parzen-Rosenblatt estimation, see [23], chapter 5, sections 1-2.
Remark 2.2. It is known, see [23], chapter 5, section 3 that the result (18) is essentially non-improvable. Indeed, there holds the following lower estimate under our conditions for arbitrary density statistics $\hat{f}$:

$$\sup_n \sup_x P \left( B_n |\hat{f}(x) - f(x)| > u \right) \geq 2 \exp \left[ -C_{14}(d, \beta, L) \frac{u^{2d+4}}{d+4} \right], \quad u \geq 1. \quad (23)$$

3 Error estimate in Lebesgue-Riesz norms.

Let $\mu$ be arbitrary Borelian finite: $\mu(R^d) = 1$ measure on the whole space $X := R^d$. For instance,

$$\mu(A) = \frac{\nu(A \cap D)}{\nu(D)},$$

where $\nu$ is ordinary Lebesgue measure and $D$ is fixed measurable non-trivial set: $0 < \nu(D) < \infty$ or $\mu = \delta_{x_0}$—unit delta Dirac measure concentrated at the point $x_0 \in X$.

Introduce as ordinary the classical Lebesgue-Riesz space $L_p = L_p(R^d, \mu)$ as a set of all the (measurable) functions $g: R^d \to R$ having a finite norm

$$||g||_p := \left[ \int_{R^d} |g(x)|^p \mu(dx) \right]^{1/p}, \quad p \in [1, \infty).$$

We intent in this section to evaluate the error estimation of the Wolverton-Wagner statistics in the $L_p$ norm:

$$R_{n,p}(u) \overset{df}{=} P( B_n ||f_{n}^{WW} - f||_p > u ), \quad u \geq 1. \quad (24)$$

Note that case $p = 2$ (Hilbert space) was considered in many works, e.g. [9], [10], [22] at all. The $L_1$ approach was investigated in the monograph [11].

Theorem 3.1. We propose again under formulated above conditions

$$\sup_n R_{n,p}(u) \leq \exp \left[ -C_5(d, \beta, L, p) \left( u - C_3 \right)^{\frac{2d+4}{d+4}} \right], \quad u \geq C_3. \quad (25)$$

Proof. We need for this purpose to apply the theory of the so-called mixed Lebesgue-Riesz spaces, e.g. [3], [4]. Indeed, let us introduce the following two mixed Lebesgue-Riesz spaces containing on all the bi-measurable numerical valued random processes (fields) $\eta = \eta(x, \omega), \quad x \in R^d, \quad \omega \in \Omega$ having a finite norm

$$||\eta||_{p, x, \omega, \Omega} \overset{df}{=} || \eta ||_{p, x} \quad (26)$$
\[
\left\{ \mathbb{E} \left[ \int_X |\eta(x, \omega)|^p \mu(dx) \right]^{r/p} \right\}^{1/r}, \quad X = \mathbb{R}^d, \ p, r \geq 1.
\]
and correspondingly
\[
||\eta||_{r, \Omega, p, X} \overset{\text{def}}{=} ||\eta||_{r, \Omega} ||_{p, X} =
\]
\[
\left\{ \int_X \left[ \mathbb{E}|\eta(x)|^r \right]^{p/r} \mu(dx) \right\}^{1/p}, \quad X = \mathbb{R}^d, \ p, r \geq 1.
\]
Evidently, in general case
\[
||\eta||_{p, X, r, \Omega} \neq ||\eta||_{r, \Omega, p, X},
\]
but always
\[
||\eta||_{p, X, pr, \Omega} \leq ||\eta||_{pr, \Omega, p, X}.
\]
It follows from the theory of Grand Lebesgue Spaces, [20], [23], chapter 1, section 1.5 that if the r.v. \( \zeta \) satisfies the inequality
\[
P(|\zeta| > u) \leq \exp \left( -C u^{(2\beta + d)/(\beta + d)} \right),
\]
then
\[
\sup_{r \geq 1} \left[ r^{-\beta/(2\beta + d)} ||\eta||_{r, \Omega} \right] < \infty,
\]
and inverse proposition is also true. Therefore it follows from the estimation (22) that uniformly relative the parameter \( n \)
\[
||\Theta_n||_{r, \Omega} \leq C_9(\beta, L, d) r^{\beta/(2\beta + d)}, \ r \geq 1..
\]
We obtain from the relation (30) denoting \( \kappa = ||\Theta_n||_{p, X} : \)
\[
||\kappa||_{pr, \Omega} \leq C_{10}(pr) \beta/(2\beta + d),
\]
or equally taking into account the boundedness of the measure \( \mu \) and applying Lyapunov’s inequality
\[
||\kappa||_s \leq C_{11}(\beta, d, L; p) s^{\beta/(2\beta + d)}, \ s \geq 1;
\]
which is completely equal the the assertion of theorem 3.1.

**Corollary 3.1.** It follows from theorem 3.1
\[
P(||f_n^{WW} - f||_p > C(v + C_3)) \leq \Delta_n(v),
\]
where
\[
\Delta_n(v) := \exp \left( -n^{\beta/(\beta + d)} \times v^{(2\beta + d)/(\beta + d)} \right), \ v = \text{const} \geq 1.
\]
∀ v > 1 \Rightarrow \sum_{n=1}^{\infty} \Delta_n(v) < \infty,

we conclude that for all the values \( p \in [1, \infty) \) the Wolverton - Wagner’s statistics converges in the \( L_p \) norm with probability one:

\[
P(\|f^W - f\|_p \to 0) = 1.
\] (33)

**Corollary 3.2.** Moreover, define for each the value \( p \in [1, \infty) \) the variables

\[
\tau_n \overset{\text{def}}{=} ||f_n^W - f||_p, \quad \tau \overset{\text{def}}{=} \sup_n ||f_n^W - f||_p = \sup_n \tau_n.
\]

As long as

\[
P(\tau > v) \leq \sum_{n=1}^{\infty} P(\tau_n > v), \quad v \geq 1,
\]

we get after some calculations

\[
P(\tau > C_{12}(v + C_3)) \leq \sum_{n=1}^{\infty} \exp \left( -v^{(2\beta+d)/(\beta+d)} \cdot n^{\beta/(\beta+d)} \right) \leq C_{14} v^{-(2\beta+d)/\beta}, \quad v \geq 1.
\]

4 Concluding remarks.

**A.** It is interest by our opinion to deduce the optimal density estimation as well as confidence region in the uniform norm \( L_\infty \):

\[
||g||_\infty := \sup_{x \in \mathbb{R}^d} |g(x)|.
\]

Perhaps, one can set for this purpose

\[
h_k := \frac{(\ln k)^\gamma}{k^{1/(2\beta+d)}}.
\]

**B.** Offered here method may be generalized on the so called regression problem, i.e. when

\[
\eta_i = f(x_i) + \epsilon_i, \quad i = 1, 2, \ldots, n.
\]

**C.** For the practical using it may be interest to investigate the weight approximate for density, e.g.
\[ \Gamma_{n,w}[f_n^{WW}, f] := \sup_n \sup_x \mathbb{B}_n \| f_n^{WW}(\cdot) - f(\cdot) \|, \]

where \( w = w(x) \) is certain weight, i.e. non-negative numerical valued measurable function.

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