Finite Density Effect in the Gross-Neveu Model in a Weakly Curved $R^1 \times S^2$ Spacetime

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Abstract

The three-dimensional Gross-Neveu model in $R^1 \times S^2$ spacetime is considered at finite particles number density. We evaluate an effective potential of the composite scalar field $\sigma(x)$, which is expressed in terms of a scalar curvature $R$ and nonzero chemical potential $\mu$. We then derive the critical values of $(R, \mu)$ at which the system undergoes the first order phase transition from the phase with broken chiral invariance to the symmetric phase.

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I. INTRODUCTION

Last time four-fermion field theories in (2 + 1)-dimensional Minkowski spacetime, which are known as Gross – Neveu (GN) models [1], are under extensive investigation for purely theoretical motivation and also due to their applications to planar condensed matter physics. Such theories possess many desirable properties: the renormalizability in $1/N$ expansion, dynamical breaking of chiral symmetry and generation of fermion mass for a large coupling constant as in QCD [2], the analogy to the BCS theory of superconductivity in two spatial dimensions and the possibility to describe the new phenomenon of high temperature superconductivity [3], the reduction to the $S = 1/2$ quantum antiferromagnet Heisenberg model in the continuum limit [4] and so on. Main features of these models, obtained in large $N$ expansion technique, are confirmed within the framework of other nonperturbative approaches [5]. Since there are no closed physical systems in nature, the influence of different external factors on the vacuum of the simplest GN model was considered. In [6] some critical phenomena of this theory were studied at nonzero temperature $T$ and chemical potential $\mu$. Recently, on the same foundation a new property of external (cromo-)magnetic field $H$ to promote the dynamical chiral symmetry breaking was discovered [7]. (At present it is the well known effect of dynamical chiral symmetry breaking catalyst by external magnetic field [8], which is under intensive consideration [9].) The role of $T$, $H$ as well as of $\mu$, $H$ in the formation of a ground state of the GN model was also clarified [10]. The study of dynamical symmetry breaking in spacetimes with curvature and nontrivial topology is also of great importance, since in the early universe the gravity was sufficiently strong and one should take it into account. There is a reach literature on this subject (see the review [11]). The papers in [12,13] are the first ones where the effect of curvature and nontrivial topology on the chiral symmetry breaking in four - fermion models was discussed. The curvature - induced first order phase transition from a chiral symmetric to a chiral nonsymmetric phase was shown to exist in that models in the linear curvature approximation. It turns out that in specific spacetimes such as Einstein universe [14] and maximally symmetric spacetimes
The above mentioned models may be solved exactly in the leading order of large $N$ expansion technique. Finally, dynamical symmetry breaking in the external gravitational and magnetic fields was considered. It is well-known that low dimensional four-fermion field theories, especially the (2+1) - dimensional GN model, in curved spacetimes and in the nonsimply connected spacetimes may be very useful for the investigation of physical processes in thin films and in the materials with layer structure. The matter is that external stress, applied to the planar system, may change topology and curvature of a surface. The consideration of above mentioned low dimensional models in curved spaces is an interesting subject by itself, since in these cases the quantum theories can be computed exactly. The great amount of observable physical phenomena are due to nonzero particle density (superconductivity, quantum Hall effect etc.). So in the present paper the influence of both chemical potential and curvature of space on the phase structure of (2+1) - dimensional GN model is studied. Especially, we shall consider $R^1 \times S^2$ spacetime to clarify our discussion (the effect of temperature on the vacuum of GN model in this spacetime was considered in [11]). In Section II, we evaluate the one-loop effective potential in $R^1 \times S^2$ spacetime at nonzero chemical potential. Presented in Section III is the detailed analysis of the effective potential, which shows the existence of a phase transition restoring the chiral symmetry of the system as the curvature $R$ and chemical potential $\mu$ are varied. Finally, we summarize our results in Section IV.

II. EFFECTIVE POTENTIAL IN $R^1 \times S^2$ SPACETIME AT $\mu \neq 0$

The $R^1 \times S^2$ space-time is chosen to be

$$ds^2 = dt^2 - a^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

where $a$ is its constant radius. The four-fermion model in this spacetime is described by the action

$$S = \int d^3x \sqrt{-g} \left[ i \bar{\psi}_j \gamma^\mu(x) \nabla_\mu \psi_j + \frac{\lambda^2}{2N} (\bar{\psi}_j \psi_j)^2 \right],$$
where $\nabla_\mu$ is the covariant derivative and the summation over $j$ is implied ($j = 1, 2, \ldots, N$). Here fermion fields $\psi_j$ are taken in the reducible four dimensional representation of $SL(2, C)$. For this case the algebra of the $\gamma$ - matrices is presented in [2]. This action has the discrete chiral symmetry,

$$\psi \rightarrow \gamma_5 \psi. \quad (3)$$

As a result, the chiral symmetry is maintained at any order of ordinary perturbation theory. However as can be seen in different nonperturbative approaches [1,2,5] the symmetry may be broken dynamically for large values of coupling constant $\lambda$. To see the nonperturbative features such as spontaneous symmetry breaking and dynamical mass generation in the present model, it is convenient to rewrite above action in an equivalent form [1] by introducing the auxiliary field $\sigma(x)$,

$$S = \int d^3x \sqrt{-g} \left[ i\bar{\psi}_j \gamma^\mu(x) \nabla_\mu \psi_j - \sigma \bar{\psi}_j \psi_j - \frac{N}{2\lambda^2} \sigma^2 \right]. \quad (4)$$

This expression explicitly says that the vacuum expectation value of $\sigma$ field plays the role of mass for the fermions. In order to find the effective potential in the theory with the action Eq. (2) we follow [12,13] where this quantity was considered in a weak curvature approximation. First of all let us integrate over the fermion fields in Eq. (4) and evaluate an effective action $S_{\text{eff}}(\sigma)$ describing the self-interaction of $\sigma$ field:

$$\exp(iNS_{\text{eff}}(\sigma)) = \int D\psi D\bar{\psi} \exp[iS(\psi, \bar{\psi}, \sigma)]. \quad (5)$$

Here we use the $1/N$ expansion which is the fermion-loop expansion. In the mean-field approximation, where the $\sigma(x)$ field is assumed to be constant, and to the leading order in the large $N$, one can obtain the one-loop effective potential $U(\sigma)$ from the action $S_{\text{eff}}(\sigma)$:

$$U(\sigma) = \frac{\sigma^2}{2\lambda^2} + i \text{tr} \langle x | \ln(i\gamma^\mu(x) \nabla_\mu - \sigma) | x \rangle, \quad (6)$$

where tr is over indices other than spacetime indices. Using the Green function $G_F(x, y; \sigma)$ defined by the relation
\[ G_F(x, y; \sigma) \equiv \langle x | (i \gamma^\mu \nabla_\mu - \sigma)^{-1} | y \rangle, \]  

(7)

we rewrite Eq. (6) as follows:

\[ U(\sigma) = \frac{\sigma^2}{2\lambda^2} - i \text{tr} \ln G_F(x, x; \sigma). \]  

(8)

The logarithm may be eliminated from this equation by introducing the parameter \( s \):

\[ \ln \left[ \frac{K - \sigma}{K} \right] = - \int_0^\sigma ds \frac{1}{K - s}, \]  

(9)

where an operator \( K \) is given as \( i \gamma^\mu (x) \nabla_\mu \) in the present case. Therefore, Eq. (8) is rewritten in the following form:

\[ U(\sigma) = \frac{\sigma^2}{2\lambda^2} - i \int_0^\sigma ds \int \frac{d^3k}{(2\pi)^3} G_F(k; s), \]  

(10)

where the momentum-space Green function \( G_F(k; s) \) has been used. Now one may introduce the Riemann normal coordinate [21] with origin at any point in the spacetime. In this coordinate system we use the following expression for the Green function \( G_F(k; s) \):

\[ G_F(k; s) = \frac{\gamma^a k_a + s}{k^2 - s^2} - \frac{R}{12} \frac{\gamma^a k_a + s}{(k^2 - s^2)^2} + \frac{2}{3} R_{\mu \nu} k^\mu k^\nu \frac{(\gamma^a k_a + s)}{(k^2 - s^2)^3} - \frac{1}{2} \gamma^a J^{cd} R_{\epsilon \delta a \mu} k^\mu \frac{1}{(k^2 - s^2)^2}, \]  

(11)

where \( J^{ab} = \frac{1}{4} [\gamma^a, \gamma^b] \), and the Latin and Greek indices refer to a local orthonormal frame and general coordinate system, respectively. Eq. (11) is the linear approximation for the Green function \( G_F(k; s) \) in the curvature \( R \) [11-13]. According to the well known method developed in [22], to obtain eq. (11) one should neglect any terms involving derivatives higher than that of the second order in the metric tensor expansion. Now let us consider the effect of nonzero chemical potential \( \mu \) on the system. As is well-known, the fermion-number density is directly related to the chemical potential \( \mu \). Mathematically, the presence of nonzero chemical potential is realised by shifting the energy levels \( k_0 \) in the propagator \( G_F(k; s) \) by the amount of \( \mu \) [23]. Thus, we are in order to evaluate the effective potential \( U(\sigma) \) in Eq. (10) under effects of both \( R \) and \( \mu \). Using the contour integration method
we can perform the integration over momentum $k^\mu$. Denote $I_1$ the integral of the first term in $G_F(k; s)$ over $k$ and $s$. Its calculation proceeds as follows: first, the procedure of integration over $k_0$, denoted as $I'_1$, gives the result:

$$I'_1(k, s) \equiv \text{tr} \int \frac{dk_0}{2\pi} \frac{\gamma^0 (k_0 + \mu) + \gamma^i k_i + s}{(k_0 + \mu)^2 - E_k^2}$$

$$= \frac{2}{\pi} \int_{-i\infty}^{i\infty} \frac{s dz}{z^2 - E_k^2} + \frac{2}{\pi} \oint_C \frac{s dz}{z^2 - E_k^2}$$

$$= \frac{2}{\pi} \int_{-i\infty}^{i\infty} \frac{s dz}{z^2 - E_k^2} + \frac{2is}{E_k} \theta(\mu - E_k).$$

(12)

Here, $E_k^2 \equiv k^2 + s^2$, the contour $C$ is given in Fig. 1, and the unit step function $\theta(x) = 1$ for $x > 0$, $\theta(x) = 0$ for $x < 0$ has been used. Thus we get

$$I_1 \equiv -i \int_0^\sigma ds \int \frac{d^2k}{(2\pi)^2} I'_1(k, s)$$

$$= \sigma^2 \left[ \frac{\sigma}{3\pi} - \frac{\Lambda}{\pi^2} \right] + \theta(\mu - \sigma) \left[ \frac{\mu}{2\pi} \sigma^2 - \frac{1}{3\pi} \sigma^3 \right] + \theta(\sigma - \mu) \frac{\mu^3}{6},$$

(13)

where $\Lambda$ is the cutoff parameter. Here and in the following discussions, we may confine ourselves to the $\sigma \geq 0$ region due to a reflection symmetry $\sigma \leftrightarrow -\sigma$ of the effective potential $U(\sigma)$. However, note that this symmetry is broken when the system selects one of the two ground states. In a similar way one finds the contributions $I_2, I_3, I_4$ of the remaining terms of $G_F(k; s)$ to the potential (10):

$$I_2 = \frac{R}{24\pi} \left[ -\sigma + \theta(\mu - \sigma)(\sigma - \frac{1}{2\mu} \sigma^2) + \theta(\sigma - \mu) \frac{\mu^3}{2} \right],$$

$$I_3 = \frac{1}{12\pi} \left[ R \sigma - R_{00} [\theta(\mu - \sigma)(\sigma - \frac{1}{2\mu} \sigma^2) + \theta(\sigma - \mu) \frac{\mu^3}{2}] \right],$$

$$= \frac{1}{12\pi} R \sigma,$$

$$I_4 = 0.$$

(14)

In the third line of Eq.(14), we have used the relation $R_{00} = 0$, which follows from the metric Eq.(1) of the spacetime under consideration. However, the fourth line of Eq.(14) is due to a relation $\text{tr}[\gamma^i \gamma^j \gamma^k] = 0$. At this stage it is convenient to introduce the mass parameter $M$ instead of the coupling constant $\lambda$ by the following way [3]:

$$6$$
\[
\frac{1}{\lambda^2} \equiv 4 \int^\Lambda \frac{d^3k_E}{(2\pi)^3} \frac{1}{k^2_E + M^2} = \frac{2}{\pi^2} \Lambda - \frac{1}{\pi} M.
\] 

(15)

So, we shall consider the case \(\lambda > \lambda_c\) only, where \(\lambda_c^{-2} = 4 \int^\Lambda d^3k_E (2\pi)^{-3} k^{-2}_E\). Summing up all terms \(I_i\) in Eq. (13) and (14) and inserting above equation into Eq. (10), one sees that the two \(\Lambda\)-dependent terms cancel out, and thus the finite effective potential to one-loop order is obtained. Then, the \(\mu\)- and \(R\)-dependent one-loop contributions \(U^1_{R\mu}(\sigma)\) to the potential \(U(\sigma)\) are completely separated from the Minkowski-space result:

\[
U(\sigma) = U_F(\sigma) + U^1_{R\mu}(\sigma),
\]

(16)

where \(U_F(\sigma)\) is the effective potential of the original theory in flat Minkowski spacetime. Here

\[
U_F(\sigma) = \frac{\sigma^2}{3\pi} \left[ \sigma - \frac{3}{2} M \right],
\]

\[
U^1_{R\mu}(\sigma) = \frac{R}{24\pi} \sigma + \frac{1}{\pi} \theta(\mu - \sigma) \left[ (\mu - \frac{2}{3}\sigma) \frac{\sigma^2}{2} + \frac{R}{24} (\sigma - \frac{\sigma^2}{2\mu}) \right] + \frac{1}{6\pi} \theta(\sigma - \mu) \left[ \mu(\mu^2 + \frac{R}{8}) \right].
\]

(17)

In this expression one may find the following two facts. Firstly, \(U^1_{R\mu}(\sigma)\) is finite and, as \(R, \mu \to 0\), \(U^1_{R\mu}(\sigma)\) vanishes. Thus the renormalisation procedure is identical to the case of Minkowski spacetime. Secondly, in the limit \(\mu, R \to 0\), \(U(\sigma)\) is reduced to the Minkowski-space effective potential \(U_F(\sigma)\). It is well established that there are two distinct phases in the three-dimensional GN model \([1,2,7]\). For a weak coupling phase with the coupling \(\lambda < \lambda_c\) we have \(\langle \sigma \rangle = 0\). Thus the fermions are massless and the chiral symmetry remains intact. However, for the strong coupling phase \(\lambda > \lambda_c\), \(\sigma\) field has nonzero vacuum expectation value \(\langle \sigma \rangle = M\), so the chiral symmetry Eq. (2) is dynamically broken and fermions acquire mass, which is equal to the mass parameter \(M\) from Eq. (15). For the simplicity of our analysis in the next sections, we shall introduce the following rescaled dimensionless quantities defined as \(\bar{U}(x) \equiv \pi U(\sigma)/\mu^3\), \(\bar{R} \equiv R/\mu^2\), \(\bar{\sigma} \equiv \sigma/\mu\), and \(\bar{\mu} \equiv \mu/M\). In terms of these quantities, Eq. (16) is rewritten in the much simpler form:

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\[ \tilde{U}(x) = \begin{cases} 
(1 - \frac{1}{\bar{\mu}} - \frac{\tilde{R}}{24})\frac{x^2}{2} + \frac{\tilde{R}x}{12}, & \text{for } x < 1 \\
(x - \frac{3}{2\bar{\mu}})\frac{x^2}{3} + \frac{\tilde{R}x}{24} + \frac{1}{6}(1 + \frac{\tilde{R}}{8}), & \text{for } x \geq 1, 
\end{cases} \]

where one sees that \( \tilde{U}(x) \) is a continuous function at \( x = 1 \). We also wish to find the induced fermion mass \( \langle \sigma \rangle \) as a function of curvature \( R \) and chemical potential \( \mu \). Then, the gap equation for the fermion mass can be obtained by taking the derivative of the effective potential \( \tilde{U}(x) \) with respect to \( x \), and so we obtain

\[ 0 = \begin{cases} 
(1 - \frac{1}{\bar{\mu}} - \frac{\tilde{R}}{24})x + \frac{\tilde{R}}{12}, & \text{for } x < 1 \\
x^2 - \frac{x}{\bar{\mu}} + \frac{\tilde{R}}{24}, & \text{for } x \geq 1. 
\end{cases} \]

III. RESTORATION OF CHIRAL SYMMETRY

Now we shall analyze in detail the effective potential Eq. (18) in order to investigate the phase structure of the model in the \((R, \mu)\) plane. The fermion mass \( \langle \sigma \rangle \) will be derived which depends on \( R \) and \( \mu \) and the nature of the phase transitions will be discussed. To clarify our discussion, we shall consider three distinct cases: \( \mu \neq 0 \) and \( R = 0 \), then \( R \neq 0 \) and \( \mu = 0 \), and finally \( R \neq 0 \) and \( \mu \neq 0 \).

A. The case \( \mu \neq 0 \) and \( R = 0 \).

Let us first examine the effect of nonzero chemical potential on the system. In the limit \( R \to 0 \), the effective potential Eq. (18) is reduced to a simple form:

\[ \tilde{U}(x) = \begin{cases} 
(1 - \frac{1}{\bar{\mu}})^\frac{x^2}{2}, & \text{for } x < 1 \\
(x - \frac{3}{2\bar{\mu}})^\frac{x^2}{3} + \frac{1}{6}, & \text{for } x \geq 1. 
\end{cases} \]

To see a phase transition as \( \bar{\mu} \) increases from a broken phase to a symmetric one, it is necessary to examine the behavior of \( \tilde{U}(x) \) as a function of \( \bar{\mu} \). It is possible to find the following two properties of \( \tilde{U}(x) \). For \( \bar{\mu} > 1 \), \( \tilde{U}(x) \) is a monotonically increasing function of \( x \), and so the global minimum of \( \tilde{U}(x) \) occurs at \( x = 0 \). While for \( \bar{\mu} < 1 \) \( \tilde{U}(x) \) has a global minimum at \( x = 1/\bar{\mu} \) with the value:
\begin{equation}
\tilde{U}\left(x = \frac{1}{\mu}\right) = -\frac{1}{6} \frac{(1 - \tilde{\mu}^3)}{\tilde{\mu}^3}.
\end{equation}

These facts indicate that the system undergoes a phase transition from the $\langle \sigma \rangle = M$ state to the $\langle \sigma \rangle = 0$ state at the critical value $\mu_c$ of the chemical potential, given as

\begin{equation}
\mu_c = M.
\end{equation}

Solving the gap equation for the induced fermion mass, Eq. (19) with $R = 0$, one can find that

\begin{equation}
\langle \sigma \rangle = M
\end{equation}

below $\mu_c$, and $\langle \sigma \rangle = 0$ above $\mu_c$. Except at $\mu = \mu_c$, the order parameter $\langle \sigma \rangle$ does not depend on the value of $\mu$. That is, the value of order parameter $\langle \sigma \rangle$, which minimizes the potential, jumps discontinuously from $\sigma = M$ to $\sigma = 0$ at the transition point $\mu_c$. Hence, at the point $\mu = \mu_c$ we have a first order phase transition from a massive chirally broken phase to a massless chirally invariant phase of the model.

\textbf{B. The case }$R \neq 0$ \textbf{and }$\mu = 0$

In this case only the effect of curvature on the system will be considered. In the limit $\mu \rightarrow 0$, the general effective potential Eq. (18) has the following form:

\begin{equation}
U(\sigma) = U_F(\sigma) + U_R^1(\sigma)
= \frac{\sigma^2}{3\pi} \left(\sigma - \frac{3}{2} M\right) + \frac{R}{24\pi}\sigma.
\end{equation}

This expression coincides with that obtained in [14]. From Eq. (24) one can see that in the region of small values of $\sigma$ the dominant contribution to $U(\sigma)$ comes from the $R$-dependent linear term in $\sigma$. Thus, there is a potential barrier between $\sigma = 0$ and second local minimum of $U(\sigma)$. As a result, it turns out that as the curvature $R$ increases the discontinuous phase transition occurs from a chirally broken phase to a symmetric one. The critical value of the
curvature $R_c$, at which a first order phase transition occurs, is determined by the following two conditions:

$$U'(\sigma_0) = 0 \text{ and } U(\sigma_0) = 0,$$

(25)

where $\sigma_0$ denotes second nonzero local minimum of the potential. Furthermore, one may find that only for $R > R_c$ the minimum of the potential at the symmetric point $\sigma = 0$ is lower than asymmetric local minimum at a nonzero $\sigma_0$. From the gap equation Eq. (19) with $\mu = 0$ one can evaluate the local minimum of the potential $\sigma_0$,

$$\sigma_0 = \frac{M}{2} \left( 1 + \sqrt{1 - \frac{1}{6} \frac{R}{M^2}} \right),$$

(26)

which at the same time equals to the fermion mass $\langle \sigma \rangle$, induced under the influence of curvature $R$ for $R < R_c$ only. Thus, applying the critical condition Eq. (25) to the effective potential Eq. (24), one can obtain the critical curvature

$$R_c = 4.5 \, M^2.$$  

(27)

The phase transition under the influence of $R$ is a first-order one since it occurs discontinuously.

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**C. The case $R \neq 0$ and $\mu \neq 0$**

In given Subsection we are going to explore the general case when the system is specified by the curvature and finite chemical potential. To investigate the vacuum structure of the system as $R$ and $\mu$ are varied, one must first examine the behavior of the potential $\tilde{U}(x)$ as a function of $R$ and $\mu$. It is very helpful to sketch qualitatively the effective potential $\tilde{U}(x)$ from Eq. (18). For $\bar{\mu} > 1$ ($\mu > M$) the global minimum of $\tilde{U}(x)$ occurs only at $x = 0$. While for $\bar{\mu} < 1$ ($\mu < M$), the global minimum of $\tilde{U}(x)$ lies at nonzero point certainly. Therefore, when $\bar{\mu} < 1$, it turns out that the system undergoes a phase transition from the $\langle \sigma \rangle \neq 0$ vacuum state to the $\langle \sigma \rangle = 0$ state at certain critical curvature $\tilde{R}_c$ depending on $\mu$. Using a much detailed analysis of the effective potential $\tilde{U}(x)$ in Eq. (18), one can
see that until the system approaches the critical point with the increase of curvature, the second local minimum of $\tilde{U}(x)$ occurs only in the region $x > 1$. Therefore, in the procedure of determining the critical value of the curvature $\tilde{R}_c$, the effective potential needs to be considered only in the $x > 1$ region in Eq. (18).

In this case we can obtain the critical curvature $R_c$ also using the condition given in Eq. (25), with the only change $\sigma_0 \to x_0$, where $x_0$ denotes the second local minimum of the potential. That is, in the present case the phase transition under investigation is also a first-order one. As can be easily checked from the gap equation Eq. (19), the second minimum lies at the point

$$x_0 = \frac{1}{2\bar{\mu}} \left(1 + \sqrt{1 - \frac{\bar{R}\bar{\mu}^2}{6}}\right).$$

(28)

Thus, the critical condition Eq. (25) with this value for $x_0$ leads to the self-consistent relation on the critical curvature $\tilde{R}_c$:

$$16x_0^3 - \frac{24x_0^2}{\bar{\mu}} + (2x_0 + 1)\tilde{R}_c + 8 = 0,$$

(29)

where $x_0$ has the value given in Eq. (28), with $\bar{R}$ replaced by $\tilde{R}_c$. The numerical solutions of Eq. (29) are illustrated in Fig. 2. Note that as $\mu \to 0$ the $R_c$ approaches $4.5 \, M^2$ and as $R \to 0$, the $\mu_c$ approaches $M$. These limiting cases have been already discussed in the previous Subsections. Eq. (28) says that the induced fermion mass $\langle \sigma \rangle$, with $\langle \sigma \rangle = \mu x_0$, does depend on the curvature $R$ only. That is, $\langle \sigma \rangle$ does not depend on $\mu$, and thus it has the same expression as Eq. (26). In Fig. 3, the effective potentials are given for four distinct values of $R$ at fixed $\mu = \frac{M}{2}$.

IV. SUMMARY

In the present paper we have derived the effective potential of the three-dimensional Gross-Neveu model in the curved spacetime of the form $R^1 \times S^2$ and with taking into account the chemical potential $\mu$ as well. Then, the critical curvature $R_c$ at which dynamical
symmetry breaking disappears has been determined in terms of the induced fermion mass $M$ in the limit $R, \mu \to 0$ and at nonzero chemical potential $\mu$, as given in Fig. 2. As the chemical potential $\mu$ increases, the fermion-number density increases also. In Subsections A and C, it has been shown that the high density influences the symmetry behavior of the system, and so at the critical value $\mu_c$ or at the corresponding critical number density the chiral phase transition is occurs. Then, we have observed that the order parameter $\langle \sigma \rangle$ of the phase transition, corresponding to the minimum of the potential, does not depend on the value of $\mu$, except at the critical value $\mu = \mu_c$, even though the phase transition is induced by the chemical potential. This phenomenon is connected with the fact that the composed field $\sigma \sim \bar{\psi} \psi$ is a real field and carries no charge. It was observed also in two-dimensional GN model in $R^1 \times S^1$ spacetime [20], however, in that model there is another massive phase, in which fermions mass is $\mu$-dependent quantity. In Subsection B, we have shown that $R_c = 4.5M^2$. This can be roughly seen from the following two facts. Firstly, on dimensional grounds the critical curvature $R_c$ must be proportional to the square of some quantity with the dimension of mass. Secondly, the effective potential for the composite $\sigma$ field in Eq. (24) has two parameters $R$ and $M$, and so the remaining parameter apart from $R$ in this theory is $M$. Note, that our value for $R_c$ is valid only in a weak curvature limit, and thus its more accurate value can be obtained by considering higher order corrections to scalar curvature $R$. However, in such improved schemes, it is expected that the system still show the same qualitative properties as those found in the previous Sections, including the occurrence of a first order phase transition. Finally, one may consider the case of negative curvature since the present method has the advantage of being applicable to any metric. Then, Eq. (26) indicates that under the effect of negative curvature $R$ the minimum of the potential moves farther from the origin than without the curvature effect. Therefore, in this case the symmetry restoring phase transition does not happen.

We hope that the above results may be useful for condensed matter physics as well, and for astrophysical applications, especially for the description of different phenomena in the core of neutron stars.
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Figure Captions

Fig.1. The contour C in the complex $k^0$ plane. Fig.2. The critical curvature $R_c/M^2$ as a function of nonzero chemical potential $\mu/M$. In region B, chiral symmetry is broken and fermions acquire dynamical masses, while in S, the symmetry is restored by the curvature effect, and fermions become massless. Fig.3. The effective potential $\pi U(\sigma)/M^3$ as a function of $\sigma/M$ at the fixed value of $\mu/M = 1/2$. Four interesting cases of $\bar{R}$, where $\bar{R} \equiv R/M^2$, are considered, and the critical curvature $\bar{R}_c$ is then numerically obtained: $\bar{R}_c = 2.96$. 