Inelastic wedge billiards

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Abstract. Billiards are simple systems used to investigate Hamiltonian dynamics in physics. When real billiards are examined experimentally, the energy dissipated in each collision must be replaced by an external stimulus to maintain the dynamics. We focus on a specific system of a driven billiard using a wedge shaped boundary to examine nonlinear and chaotic behavior. Mathematical models such as the logistic map are simple low dimensional systems that exhibit nonlinear and chaotic behavior as a single parameter is varied. This logistic map can then be used to identify a very specific mathematical parameter known as the Lyapunov exponent, which helps in identifying chaos more clearly. In the current experiment, the dynamics of a particle free to move near a horizontally shaken vertical boundary will be examined for the presence of chaos. The goal of the research is to extract a Lyapunov exponent between any two trajectories in the system. In addition, the manner in which the dynamics evolve freely through dissipative collisions provides a testbed for measurements of the velocity dependent coefficients of restitution for the billiard. A better description of hard sphere coefficients of restitution would be beneficial to a host of experiments and numerical simulations in granular physics.

1 Introduction

Gravitational billiards, where the shape of the particle boundary governs the observed dynamical motion of the particle [1–3], continue to be a rich system for studying instability and chaos in Hamiltonian dynamics. Not surprisingly, parabolic boundaries demonstrate stable orbits whose amplitude is determined by the energy of the billiard [1], while a wedge boundary demonstrates both unstable and chaotic behavior dependent upon the value of the vertex angle of the wedge [2]. An even more interesting case is that of a hyperbolic boundary that demonstrates parabolic behavior for orbits near its origin and wedge behavior for orbits near its asymptotes [3]. The motion of a particle in a wedge has been experimentally demonstrated for an optical billiard with ultra-cold particles [4].

Previously, we have demonstrated an experimental manifestation for a billiard in a driven experiment [5] for two-dimensional parabolic, wedge, and hyperbolic geometries where the particle undergoes inelastic collisions with the confining boundaries in the presence of a gravitational potential. This prior study has led to several simulations of dissipative gravitational billiards [6-8]. In addition, the study of dissipative billiards can contribute to a deeper understanding of the dissipation in granular systems [9]. Inelastic collisions have been shown to lead to clustering [10] and inelastic collapse [11], and a prior investigation analytically and numerically investigated the behavior of a particle of variable elasticity within an effective wedge boundary to search for effects of clustering and inelastic collapse [12]. For a wedge boundary, evidence is obtained of stable, unstable, period doubling, and chaotic dynamics for the billiard. While the energy injection to the system is maintained at a fixed value, the inelasticity allows the particle to move between nearby trajectories, demonstrating a variety of orbits at fixed energy input.

2 Experiment

In both the prior investigation and this study, the experiment consists of a steel frame with an interior square of 16.5 cm × 16.5 cm into which a 3.2 mm thick wedge, parabolic, or hyperbolic aluminum insert can be placed vertically with respect to gravity to examine the motion of a 3.1 mm diameter chrome steel ball within the selected boundary. The coefficient of restitution between the chrome steel ball and the aluminum boundary is approximately 0.9 but is known to be velocity dependent [13]. One of the primary motivations of this study is to reconcile results from simulations that are at odds with this assumption [6-8].

Sheets of Plexiglas constrain the motion of the particle to a thin 2D layer and at the same time allow for observation of the particle’s motion. The vertical cell is seated on a linear bearing that constrains the motion of the cell to a horizontal line along the plane of the boundary (see Figure 1). A wheel and linking armature converts the rotational motion of the Dayton 1/8 hp dc motor into a sinusoidal horizontal motion of the cell on the linear slide. The frequency of oscillation, ν, was varied between 4–7 Hz, and the peak-to-peak amplitude of oscillation was held constant at A = 2.54 cm. The system is started from rest and the particle’s motion is soon ballistic across the cell. The energy lost in the
collisions is compensated by the horizontal driving of the cell whose peak acceleration is given by \( \Gamma = A\omega^2 \), where \( \omega = 2\pi\nu \). Thus, the energy input to the system is increased or decreased by increasing and decreasing the driving frequency. The images are obtained via a 256×256 pixel 1000 frame per second (fps) Dalsa CAD6 8-bit camera. For imaging purposes, a second identical chrome steel particle is imbedded in each of the boundary inserts beneath the vertex of each surface, so that high-speed CCD imaging can track both the motion of the cell and the free particle moving ballistically within the cell. The tracking of the particle imbedded in the boundary beneath the vertex allows for an independent measurement of the shaking frequency over the entire data set. Programs written in IDL are used to track the free particle’s trajectory and extract the particle’s velocity before and after these collisions. A mapping is then made to relate the velocity and position of one collision point to the velocity and position at the following collision point.

To examine the long term behavior of the dynamics, the height at one collision can be plotted versus the height at the following collision, producing a map of the motion. To understand the goals of this study, we reproduce the results of our prior study [5] in the left column of Figure 2. The dashed lines in the figure are the \( y_n = y_{n+1} \) return map that would denote stable period-one orbits. The right column of Fig. 2 shows the time of flight between collisions versus the time of flight from the next collision. Figure 2(a) shows the \( y \) positions for the case of the parabola driven at 5.4 Hz. The trajectory of the particle is not simply a single stable orbit, but rather accesses several nearby stable orbits, resulting in an elongation of the stable fixed point along the direction of the slope 1 line. Similarly, this leads to a slight smearing of the collision times as can be seen in Fig. 2(b). Figure 2(c) shows the map of \( y \) positions of the particle collisions within the wedge driven at 6.6 Hz. The wedge is this study had a half angle of 28.5 degrees. The motion is clearly unstable and the particle is continuously driven to the top of the wedge. In Fig. 3(c) – 3(h) we examined the \( y \) positions of the particle collisions within the hyperbolic boundary when driven at 4.5 Hz, which was the focus of the prior study.

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For this study, the decision to focus on the wedge geometry was two-fold: varying the half angle of the wedge geometry allows for both the exploration for chaotic behavior as well as to further investigate the inelasticity of the particle within the experiment. In this study, we used three different wedges (shown in Figure 3) with half angles of (a) 28.5, (b) 45 and (c) 60 degrees, respectively. The use of the same 28.5 degree wedge was seen as a manner in which to check the consistency of the results between this study and the prior investigation. While the experimental details of the driving along the horizontal linear bearing and the orientation of the cell vertically to the direction of gravity naturally lends itself to a cartesian description, the collision dynamics lend themselves more naturally to a frame rotated by the value of the half angle of the wedges as shown in Figure 3(d). Within the frame of camera, data is taken that allows the accelerated motion of the free particle in the vertical direction and the constant linear momentum in the horizontal direction to be observed as shown in Figure 4. Through a simple rotation, the dynamics may easily be resolved in the natural directions normal (n) and tangential (t) to each of the three wedge surfaces. In Figure 3(d) we show such a rotation for the right hand side of the cell for the 60
degree wedge. For the left hand side of the wedge boundary (not shown) the normal vector is of course perpendicular to the wedge boundary, but the tangential direction points away from the vertex to the left. This choice results in a tangential direction on both the left and right hand sides of the vertex that is in the direction of increasing potential energy.

Fig. 3. The wedges used in this study with half angles of (a) 28.5 (b) 45 and (c) 60 degrees respectively. In (d) the rotation of the camera frame (x,y) into the boundary frame of normal (n) and tangential (t) directions is demonstrated for the right side of the 60 degree half angle wedge. Here the cells are photographed with a white background to show the wedges. In the experiment, a black background is used to more easily track the free billiard and the particle imbedded in the boundary beneath the vertex to allow the motion of the cell to be tracked.

3 Results

In Figure 4, we demonstrate the results of tracking both the free particle and the identical chrome steel ball imbedded in the wedge boundary for the cell with a half angle of 60 degrees. In the figure, we show a subset of 4000 frames of data from more than a quarter million images taken for the cell at a shaking frequency of 5.78 Hz. In arbitrary units, we show the results for the y-position ($y_{\text{ball}}$), and the x-position ($x_{\text{ball}}$) of the billiard and the oscillating x-position of the cell ($x_{\text{cell}}$) determined from tracking the imbedded particle.

The free particle demonstrates the expected accelerated motion in the y direction, constant momentum in the x direction and horizontal oscillation of the cell. In addition, the dynamics demonstrates repeating trajectories of the particle climbing along the wedge. Two very similar trajectories in the evolution of the y-position are in Figure 4 between frames 400 and 1900 and again between frames 2200 and 2800.

One method of probing the dynamics is to examine the time of flight between collisions of the billiard with the boundary. The results for one such analysis is shown in Figure 5. Here, we plot the histogram of the time between collisions for the entire data set at 5.78 Hz. In the inset of Figure 5 we demonstrate the results of the tracking program that identifies the ballistic trajectories by locating the ends of each parabolic track. While not shown in this paper, analysis demonstrates that the collisions of the billiard occur nearly uniformly throughout the shaking cycle, showing a lack of coherence between the billiard dynamics and the driving.

Fig. 4. From top to bottom, the y position of the billiard ($y_{\text{ball}}$), the x position of the billiard ($x_{\text{ball}}$) and the horizontal motion of the driven cell ($x_{\text{cell}}$). To show all three tracks on a single plot, the results are plotted in arbitrary units.

Fig. 5. Histogram for the time of flight of the billiard in the 60 degree wedge driven at 5.78 Hz. The time of flight (determined by identifying collisions from the ballistic motion of the billiard as shown in the inset) is plotted in units of the period of oscillation.

In Figure 5, three distinct peaks are apparent in the distribution of the time of flight, potentially indicating chaotic behaviour in the dynamics of the billiard. The three peaks are approximately at 0.15, 0.30 and 0.45 of the value of the shaking period (173.5 ms). The three peaks are distinct in what would otherwise be a nearly uniform distribution of the time of flight over half the period of oscillation of the cell. However, collisions occur throughout the entire period of oscillation of the cell and even sometimes over more than one cycle of the shaking (time of flight > the oscillation period).

The conversion of the dynamics into the normal and tangential directions lends well to the analysis of the coefficients of restitution [14-16], one of which is in the normal direction to the collision and the other in the direction tangent to the collision direction (the motion in the third direction is constrained by the 2D nature of the cell). In addition to the individual coefficients of restitution (CORs) based upon the change in relative speed in normal and tangential speeds, it is also possible
to quantify the dissipation in the system by examining the total change in energy of the particle before and after the collision. Hence, we define three coefficients as follows:

\[ r(v_n) = -1 \ast \frac{v_n^+}{v_n^-} \]  

(1)

\[ r(v_t) = \frac{v_t^+}{v_t^-} \]  

(2)

\[ r(E) = \left( \frac{\Delta E}{E^-} \right)^{1/2} \]  

(3)

where the +/- velocities are the relative speeds of the billiard after and before the collision, respectively. The normal COR has a -1 prefactor to address the change in direction of the particle in the collision, however the tangential COR has no such constraint. So, the normal COR should be a value between 0 and 1 but the tangential COR can have values between ±1. As the energy is positive, it should also have a value between 0 and 1. The energy COR is the square root of the ratio of the change in energy to the energy before the collision.

The results in Figures 6 and Figure 7 demonstrate that while the COR in this experiment behaves approximately as expected, there are values obtained that are larger than unity. This indicates that the rotational degree of freedom of the billiard is an important quantity. While it is not measured in this experiment, the rotational degree of freedom acts as an additional source of energy that “leaks” momentum into the normal and tangential velocities during the collisions with the boundary [16].

Fig. 6. From top to bottom, the COR calculated for the normal component of the collision velocities after to before the collision, calculated from the tangential velocity component, and from the energy change before and after the collision.

Fig. 7. The COR based on the total translational kinetic energy of the particle versus the COR based on the normal velocities.

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