Stabilizing unstable periodic orbit of unknown fractional-order systems via adaptive delayed feedback control

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Abstract
This article presents an adaptive nonlinear delayed feedback control scheme for stabilizing the unstable periodic orbit of unknown fractional-order chaotic systems. The proposed control framework uses the Lyapunov approach and sliding mode control technique to guarantee that the closed-loop system is asymptotically stable on a periodic trajectory sufficiently close to the unstable periodic orbit of the system. The proposed method has two significant advantages. First, it employs a direct adaptive control method, making it easy to implement this method on systems with unknown parameters. Second, the framework requires only the period of the unstable periodic orbit. The robustness of the closed-loop system against system uncertainties and external disturbances with unknown bounds is guaranteed. Simulations on fractional-order duffing and gyro systems are used to illustrate the effectiveness of the theoretical results. The simulation results demonstrate that our approach outperforms the previously developed linear feedback control method for stabilizing unstable periodic orbits in fractional-order chaotic systems, particularly in reducing steady-state error and achieving faster convergence of tracking error.

Keywords
Fractional-order systems, adaptive control, chaos, delayed feedback, unstable periodic orbit

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Introduction
Fractional Calculus is the generalization of integrals and derivatives to noninteger orders. It has been used in several fields, for example, soft robotics,1 vibration,3 electromagnetic problems,5 power grid,6 piezoelectric actuators,6 and drug delivery systems.7 There are several reasons that fractional-order (FO) systems have been employed in control theory and dynamical systems, for instance, their accuracy in modeling dynamical systems, larger stability areas, and capacity to satisfy more control criteria.8,9 During the last few years, numerous control schemes have been developed for FO systems. Shahvali et al.10 studied the distributed adaptive dynamic event-based consensus control for nonlinear uncertain multi-agent systems. Zamani et al.11 addressed the fixed-time formation problem in FO multi-agent systems. Lori et al.12 developed a sliding mode control method to control the position of a quadrotor in the presence of external disturbances.13 Ayten et al.14 presented the speed and direction angle control of a wheeled mobile robot based on an FO adaptive model-based PID-type sliding mode control technique. Machado et al.15 addressed the modeling and dynamical analysis of soccer teams using two modeling perspectives, which are adopted based on the concepts of fractional calculus. Shahvali et al.16 proposed an adaptive fault compensation control for nonlinear uncertain FO systems. Huong et al.17 studied the

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problem of global asymptotic stability analysis, and mixed \( H_\alpha \) and passive control for a class of control FO nonlinear systems using the Lyapunov direct method. Essa et al.\(^{18}\) represented the application of FO controllers on an experimental and simulation model of a hydraulic servo system. Wang et al.\(^{19}\) investigated an adaptive FO nonsingular terminal sliding mode control scheme based on time delay estimation. Yaghooti et al.\(^{20}\) proposed an adaptive FO PID control for FO nonlinear systems. Furthermore, all practical control systems are subject to operational constraints, including restricted dimensions and limited control capacity. Solutions to these issues have been proposed in the form of Model Predictive Control and Explicit Reference Governor.\(^{21–24}\) Similar to integer-order systems, FO systems can also be subject to these operational constraints. Consequently, these constrained control schemes have been developed and adapted for use with FO systems.\(^7\)

The research area of chaotic dynamics and control of FO dynamical systems has gained significant attention and popularity in recent years. Chaos is an intriguing phenomenon of FO systems that can endanger many systems. Nonlinear FO systems of orders lower than three are able to generate chaotic attractors.\(^{25,26}\) Moreover, it is demonstrated that UPO found in these systems can serve as a generalization of the integer-order case and potentially provide more accurate system modeling for a number of applications in robotics\(^{27,28}\) and optimal motion planning.\(^{29,30}\) Some control techniques have been applied for stabilizing the UPO of FO chaotic systems such as linear feedback control. Rahimi et al.\(^{31}\) represented techniques for finding unstable periodic orbits in chaotic FO systems and for the stabilization of founded UPOs. Sadeghian et al.\(^{32}\) illustrated a method to apply feedback of measured states using the period of fixed points (in discrete systems) and periodic orbits (in continuous systems), in which there is no need for information for fixed point and periodic orbits. Naceri et al.\(^{33}\) introduced a control algorithm based on the delayed feedback control method to stabilize a FO system on an unstable fixed point. Zheng et al.\(^{34}\) presented the stabilization of a FO chaotic system on its original equilibrium point using the Takagi–Sugeno (T–S) fuzzy models and prediction-based feedback controls. Shahvali et al.\(^{35}\) created an adaptive neural network backstepping control method for the nonlinear double-integrator FO systems. Rabah et al.\(^{36}\) studied the behavior of the fractional Li system and chaos suppression using a FO proportional integral derivative controller. Danca et al.\(^{37}\) investigated a class of nonlinear impulsive Caputo differential equations of fractional order, which models chaotic systems. Layeghi et al.\(^{38}\) designed a fuzzy adaptive sliding mode scheme to stabilize the UPOs of chaotic systems by using a combination of fuzzy identification and sliding mode control.

The control schemes previously discussed exhibit two significant limitations. First, they are developed based on systems whose dynamics are fully known. However, in many cases, it is challenging to analytically determine system parameters due to uncertainties and external disturbances. Second, these algorithms necessitate complete knowledge of the UPO’s trajectory, which may not always be feasible to obtain.

This article introduces a new algorithm for stabilizing UPOs in unknown FO chaotic systems. The proposed algorithm addresses the limitations of previous techniques by utilizing a direct adaptive control method that can be applied to systems with unknown parameters without the need for prior knowledge of system dynamics. In addition, the proposed framework only requires knowledge of the UPO’s period, eliminating the need for complete information about the UPO’s trajectory as required by previously developed methods in the literature. It is assumed that the system is subject to uncertainties and external disturbances with unknown bounds. Therefore, an adaptive framework is designed such that the upper bounds of uncertainties and disturbances are estimated while stabilizing the system’s UPO. To do so, this method utilizes an adaptive nonlinear delayed feedback control along with a sliding mode control technique. In order to cope with the external disturbances and system uncertainties, an appropriate sliding surface is utilized to derive adaptation laws, resulting in a stable closed-loop system.

The rest of the article is organized as follows: Section “Preliminaries” illustrates a succinct review of the fundamental principles of fractional calculus. Section “Problem statement” presents the problem formulation. In section “Stability analysis and control scheme design,” an adaptive sliding mode control method is proposed. The parameters are updated using an adaptation mechanism. In section “Simulation results,” the developed control method is implemented to stabilize UPOs of FO chaotic gyro and duffing systems. Numerical simulation results support the efficacy of the suggested framework. The final section presents a succinct conclusion.

**Preliminaries**

The theory of noninteger-order derivatives was first introduced by Leibnitz and subsequently, several definitions were proposed. In this article, we use Caputo’s definition. The purpose of this section is to give a brief overview of key concepts and stability theorems that are fundamental to the study of FO systems.

**Definition 1.** The fractional integral of order \( \alpha \in \mathbb{R} \) is defined as\(^{39}\)

\[
\frac{\partial^\alpha}{\partial t^\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau
\]

where \( \Gamma(\cdot) \) stands for the standard Gamma function, such that \( \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \).
Definition 2. The Caputo fractional derivative of order \( \alpha \in \mathbb{R} \) is defined as\(^{39}\)

\[
\mathcal{C}_0^\alpha D_t^\alpha f(t) = \begin{cases} 
\frac{1}{\Gamma(p-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-p+1}} \, d\tau, & p-1 < \alpha < p \\
\frac{d}{dt} f(t), & \alpha = p
\end{cases}
\]

(2)

where \( p \) is an integer number such that \( p-1 < \alpha < p \).

**Notation.** For the sake of conciseness, we will use \( D_t^\alpha \) and \( F_t^\alpha \) to denote the operators \( \mathcal{C}_0^\alpha \) and \( \mathcal{D}_0^\alpha \).

**Theorem 1.** Let us define a commensurate certain FO linear system as\(^{40}\)

\[
D_t^\alpha x = Ax
\]

(3)

where \( x \in \mathbb{R}^n \) represents the state vector of the system; \( A \in \mathbb{R}^{n \times n} \) is the state matrix; and \( \alpha \in \mathbb{R} \) belongs to an interval \( (0, 2) \) and stands for the order of fractional derivatives. The asymptotic stability of the system represented by equation (3) can be determined by evaluating the satisfaction of the following criteria

\[
|\arg(\lambda_i)| > \alpha \frac{\pi}{2}, \quad 1 \leq i \leq n
\]

(4)

where \( \lambda_i \), \( 1 \leq i \leq n \), denotes the eigenvalues of matrix \( A \).

By using the Lyapunov direct method, the asymptotic stability of FO nonlinear systems can be studied. In the following theorem, the Lyapunov direct method is extended to the case of FO systems, which leads to the Mittag–Leffler stability.\(^{41}\)

**Theorem 2.** By using the Caputo derivative, a FO nonlinear system is defined as\(^{41}\)

\[
D_t^\alpha x = f(t, x)
\]

(5)

where \( x \in \mathbb{R}^n \) presents the state vector of the system; \( \alpha \in \mathbb{R} \) is the order of fractional derivatives of the system. Let \( x = 0 \) represent the system (5) equilibrium point, and \( V(t, x(t)) \) be a continuously differentiable function and a candidate for the Lyapunov function, and also \( \xi_i \) \( (i = 1, 2, 3) \) be class-\( \kappa \) functions such that

\[
\xi_1(||x||) \leq V(t, x(t)) \leq \xi_2(||x||)
\]

(6)

\[
D_t^\alpha V(t, x(t)) \leq -\xi_3(||x||)
\]

(7)

where \( \nu \in (0, 1) \). So, the FO system of equation (5) is asymptotically stable.

In what follows, we introduce some lemmas which will be used to prove the stability of our proposed algorithm.

**Lemma 1.** Let \( x(t) \in \mathbb{R}^n \) represent a vector of continuously differentiable functions. Afterward, for all of \( t \geq 0 \) it continues to hold\(^{42}\)

\[
D_t^\alpha (x^T P x) = (D_t^\alpha x)^T P x + x^T P D_t^\alpha x; \quad \forall \alpha \in (0, 1)
\]

(8)

where \( P \in \mathbb{R}^{n \times n} \) states a symmetric positive definite matrix.

Barbalat’s Lemma is developed for FO nonlinear systems as follows.\(^{43}\)

**Lemma 2.** Let \( \psi : \mathbb{R} \rightarrow \mathbb{R} \) represent a uniformly continuous function on \([0, \infty)\). Suppose \( q \) and \( N \) are two positive constants such that \( F_t^\alpha |\psi|^q \leq N \) for all \( t > 0 \) with \( \alpha \in (0, 1) \). Then

\[
\lim_{t \to \infty} \psi(t) = 0
\]

(9)

**Problem statement**

Consider the following control-affine FO chaotic system

\[
\begin{cases}
D_t^\alpha x_i = x_{i+1}, & i = 1, \ldots, n-1 \\
D_t^\alpha x_n = f(t, x) + F^\alpha(t, x) \theta + d(t) + g(t, x)u(t)
\end{cases}
\]

(10)

where \( 0 < \alpha < 1 \) represents the fractional derivative order of the system, \( x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n \) is the system state vector, \( f(\cdot, \cdot) \) and \( g(\cdot, \cdot) \) are nonlinear and continuously differentiable functions. It is assumed that \( g(\cdot, \cdot) \neq 0 \), and that \( F^\alpha(t, x) \theta \) represents the system uncertainties, where \( F(\cdot, \cdot) \in \mathbb{R}^m \) is known and \( \theta \in \mathbb{R}^m \) is a vector of unknown parameters. \( d(\cdot) \in \mathbb{R} \) represents the external disturbance, and it is assumed that the disturbance is bounded \((|d(\cdot)| \leq k < \infty)\), but the value of boundary \((k > 0)\) is unknown. The system input is represented by \( u(t) \in \mathbb{R} \) and it is assumed that the system exhibits chaotic behavior for \( u = 0 \). The function \( f(t, x) + F^\alpha(t, x) \theta \) is \( T \)-periodic with period \( T \), and without any control or disturbance, the system (10) exhibits an unstable periodic response with a known period of \( T \).

The primary objective of this study is to design and implement a robust adaptive control framework that effectively makes the chaotic system track a periodic orbit that is approximate to one of the system’s UPOs. To achieve this, an adaptive control approach is employed, as the exact plant function is not fully known. The proposed adaptive laws are designed to ensure the stability of the closed-loop control system, while the use of sliding mode techniques provides robustness against uncertainties and external disturbances of unknown magnitudes. This research aims to provide a novel solution to the challenging problem of
chaotic systems and contribute to the field of control engineering.

**Stability analysis and control scheme design**

This section presents the design of a nonlinear delayed feedback control framework for the FO system defined by equation (10). A delayed state is defined as

\[ \bar{u}(t) = u(t - T), \quad \bar{d}(t) = d(t - T), \quad \text{and} \quad \theta = \theta(t - T). \]

To obtain the closed-loop system’s error dynamics, one can subtract equation (11) from equation (10)

\[
\begin{align*}
D^p_\bar{t}x_i - D^p_\bar{t}\bar{x}_i &= x_{i+1} - \bar{x}_{i+1}; \quad i = 1, \ldots, n - 1 \\
D^p_\bar{t}\bar{x}_n - D^p_\bar{t}x_n &= f(t, x) - f(t - T, \bar{x}) \\
&+ F^\top(t, x)\theta - F^\top(t - T, \bar{x})\theta \\
&+ \bar{d}(t) + g(t, x)\bar{u}(t) - g(t - T, \bar{x})\bar{u}(t)
\end{align*}
\]

where \( \bar{u}(t) = u(t - T), \quad \bar{d}(t) = d(t - T), \) and \( \theta = \theta(t - T). \) To obtain the closed-loop system’s error dynamics, one can subtract equation (11) from equation (10)

\[
\begin{align*}
D^p_\bar{t}x_i - D^p_\bar{t}\bar{x}_i &= x_{i+1} - \bar{x}_{i+1}; \quad i = 1, \ldots, n - 1 \\
D^p_\bar{t}\bar{x}_n - D^p_\bar{t}x_n &= f(t, x) - f(t - T, \bar{x}) \\
&+ F^\top(t, x)\theta - F^\top(t - T, \bar{x})\theta \\
&+ \bar{d} - d \\
&+ g(t, x)\bar{u}(t) - g(t - T, \bar{x})\bar{u}(t)
\end{align*}
\]

Let us use \( e = x - \bar{x} \) to define the error vector. The equation (12) can therefore be rewritten as

\[
\begin{align*}
D^p_\bar{t}e_i &= e_{i+1}; \quad 1 \leq i \leq n - 1 \\
D^p_\bar{t}e_n &= f(t, x) - f(t - T, \bar{x}) \\
&+ F^\top(t, x)\theta - F^\top(t - T, \bar{x})\theta \\
&+ \bar{d} - d \\
&+ g(t, x)\bar{u}(t) - g(t - T, \bar{x})\bar{u}(t)
\end{align*}
\]

To stabilize a periodic orbit of the system, we should obtain the system input \( \bar{u}(t) \) such that the following requirements are fulfilled

\[
\lim_{t \to \infty} ||e(t)|| = 0 \quad \lim_{t \to \infty} ||x(t) - x(t - T)|| = 0
\]

**Theorem 3.** Let \( \bar{x}(t) = x(t - T) \) and \( \bar{u}(t) = u(t - T) \), where \( T \) is the period of the UPOs of the chaotic system (10). If the following control input and adaptation laws are applied to the system (10), the chaotic behavior of the system is substituted by a regular periodic one

\[ u = u_{eq} + u_{ad} + u_s \]

where

\[
u_{eq} = -\sum_{i=1}^{n} \eta_i e_i + f(t, x) - f(t - T, \bar{x}) - g(t - T, \bar{x})\bar{u}
\]

\[
u_{ad} = -\frac{F^\top(t, x)\theta(t) - F^\top(t - T, \bar{x})\theta(t) + 2\hat{k}\text{sgn}(S)}{g(t, x)}
\]

\[
u_s = -(M + \mu)\text{sgn}(S)
\]

\[
D^p_\bar{t}\bar{u} = 2\gamma_k |S|
\]

where \( \theta_i \in \mathbb{R} \) and \( \hat{k} \in \mathbb{R} \) are estimates of \( \theta_i \in \mathbb{R} \) and \( k \in \mathbb{R} \), respectively; \( \gamma_k \in \mathbb{R} \) and \( \gamma_i \in \mathbb{R} \) are elements of the vectors \( \theta \in \mathbb{R}^m \) and \( F \in \mathbb{R}^{m^2} \), respectively; \( \mu \in \mathbb{R} \) is an arbitrary positive number; \( S \in \mathbb{R} \) is a sliding surface defined by

\[
S = e_n + \sum_{i=1}^{n} \eta_i \bar{P}^i e_i
\]

where \( \eta_i, i = 1, \ldots, n \), are positive real constant numbers chosen such that the dynamic of the sliding surface is stabilized asymptotically, \( S = 0 \), is provided.

**Remark 1.** All the system state variables and the sliding surface have to be continuously differentiable and measurable since this is a simplifying requirement that is frequently seen in controlling FO systems. This presumption is typical, particularly when a FO system is being attempted to be controlled.

**Remark 2.** For the existence of a sliding mode, it is necessary that the dynamics obtained by \( S = 0 \) be asymptotically stable. By applying the time derivative of order \( \alpha \) to equation (21), we obtain

\[
D^p_\bar{t} \bar{S} = D^p_\bar{t} \bar{e}_n + \sum_{i=1}^{n} \eta_i \bar{e}_i
\]

Therefore, the sliding surface dynamics can be obtained as follows

\[
D^p_\bar{t} \bar{e}_n = -\sum_{i=1}^{n} \eta_i \bar{e}_i
\]
where \( \eta_1, \ldots, \eta_n \) are chosen such that the roots of equation \( s^m + \sum_{i=1}^{m} \eta_i s^{i-1} = 0 \) satisfy the condition (10) and the sliding surface is asymptotically stable. Using Theorem 1, this condition can be expressed as

\[
|\arg(s)| > \alpha \frac{\pi}{2}
\]  

(24)

where \( s_i, i = 1, \ldots, n \), denote the roots of the above-mentioned equation.

**Proof.** Consider the following Lyapunov function

\[
V = \frac{1}{2} S^2 + \sum_{i=1}^{m} \frac{1}{2 \gamma_i} (\theta_i - \hat{\theta}_i)^2 + \frac{1}{2 \gamma_k} (k - \hat{k})^2
\]  

(25)

By applying the time derivative of order \( \alpha \) to the Lyapunov function (25), and using Lemma 1 along with some algebraic manipulations, we can obtain the following inequality for the derivative of the Lyapunov function

\[
D^\alpha_t V \leq SD^\alpha_t S - \sum_{i=1}^{m} \frac{1}{\gamma_i} (\theta_i - \hat{\theta}_i) D^\alpha_t \theta_i - \frac{1}{\gamma_k} (k - \hat{k}) D^\alpha_t \hat{k}
\]

\[
\leq S \left( D^\alpha_t c_s + \sum_{i=1}^{n} \eta_i \epsilon_i \right) - \sum_{i=1}^{m} \frac{1}{\gamma_i} (\theta_i - \hat{\theta}_i) D^\alpha_t \hat{\theta}_i
\]

\[
- \frac{1}{\gamma_k} (k - \hat{k}) D^\alpha_t \hat{k}
\]  

(26)

Substituting equation (13) into equation (26) yields

\[
D^\alpha_t V \leq S \left( \sum_{i=1}^{n} \eta_i \epsilon_i + f(t, x) - f(t - T, \hat{x})
\right.

\[
- F^\top(t - T, \hat{x}) \hat{\theta} + d - \hat{d}
\]

\[
+ g(t, x)u - g(t - T, \hat{x})\hat{u}
\]

\[
- \sum_{i=1}^{m} \frac{1}{\gamma_i} (\theta_i - \hat{\theta}_i) D^\alpha_t \hat{\theta}_i
\]

\[
- \frac{1}{\gamma_k} (k - \hat{k}) D^\alpha_t \hat{k}
\]  

(27)

Since it is assumed that the disturbance is bounded with an unknown upper bound, \( |d(t)| \leq k \), one may easily check that \( |d - \hat{d}| \leq 2k \), and substituting this upper bound into equation (27) yields

\[
D^\alpha_t V \leq S \left( \sum_{i=1}^{n} \eta_i \epsilon_i + f(t, x) - f(t - T, \hat{x})
\right.

\[
+ F^\top(t, x)\theta - F^\top(t - T, \hat{x})\hat{\theta}
\]

\[
+ g(t, x)u - g(t - T, \hat{x})\hat{u}
\]

\[
- \sum_{i=1}^{m} \frac{1}{\gamma_i} (\theta_i - \hat{\theta}_i) D^\alpha_t \hat{\theta}_i
\]

\[
- \frac{1}{\gamma_k} (k - \hat{k}) D^\alpha_t \hat{k}
\]  

Using control inputs equations (15)–(17) in inequality equation (28) yields

\[
D^\alpha_t V \leq S(F^\top(t, x)\theta - F^\top(t - T, \hat{x})\hat{\theta}
\]

\[
+ F^\top(t - T, \hat{x})\hat{\theta} + g(t, x)u
\]

\[
+ 2k|\gamma| - 2k|\gamma| - \sum_{i=1}^{m} \frac{1}{\gamma_i} (\theta_i - \hat{\theta}_i) D^\alpha_t \hat{\theta}_i
\]

\[
- \frac{1}{\gamma_k} (k - \hat{k}) D^\alpha_t \hat{k}
\]  

(29)

Adding the term \( S(-F^\top(t - T, \hat{x})\theta + F^\top(t - T, \hat{x})\hat{\theta}) \) to the right-hand side of the inequality equation (29) results in

\[
D^\alpha_t V \leq S(F^\top(t, x)\theta - F^\top(t - T, \hat{x})\theta
\]

\[
+ F^\top(t - T, \hat{x})\hat{\theta} - F^\top(t - T, \hat{x})\hat{\theta}
\]

\[
- F^\top(t, x)\hat{\theta} + F^\top(t - T, \hat{x})\hat{\theta} + g(t, x)u
\]

\[
+ 2k|\gamma| - 2k|\gamma| - \sum_{i=1}^{m} \frac{1}{\gamma_i} (\theta_i - \hat{\theta}_i) D^\alpha_t \hat{\theta}_i
\]

\[
- \frac{1}{\gamma_k} (k - \hat{k}) D^\alpha_t \hat{k}
\]  

(30)

After some mathematical manipulations, inequality equation (30) can be written as

\[
D^\alpha_t V \leq S(F^\top(t - T, \hat{x})(\theta - \hat{\theta}) + g(t, x)u
\]

\[
+ (k - \hat{k}) \left( 2k|\gamma| - \frac{1}{\gamma_k} D^\alpha_t \hat{k} \right)
\]

\[
+ \sum_{i=1}^{m} \left((\theta_i - \hat{\theta}_i)(S(F_i(t, x) - F_i(t - T, \hat{x})))\right)
\]

\[
- \sum_{i=1}^{m} \left( \frac{1}{\gamma_i} (\theta_i - \hat{\theta}_i) D^\alpha_t \hat{\theta}_i \right)
\]  

(31)

Substituting adaptation laws (19) and (20) into inequality equation (31) yields

\[
D^\alpha_t V \leq S(F^\top(t - T, \hat{x})(\theta(t) - \theta(t) - T)) + g(t, x)u
\]  

(32)
Remark 3. As a special case, if $\theta(t) = \theta(t - T)$ or $\theta$ is constant, then $u_c$ can be selected as

$$u_c = -\frac{\mu \text{sgn}(S)}{g(t, x)}$$

Proof. As we know $\theta(t) = \theta(t - T)$, so equation (32) will be simplified as follows

$$D^\mu_t V \leq S g(t, x) u_c$$

By substituting equation (37) into equation (36), one can obtain equation (33).

Remark 4. The implementation of control input may clatter as a result of the sign function’s application in the equations of $u_{ad}$ and $u_c$. Saturation or sigmoid functions can take the place of the sign function to avoid this problem. These functions are defined as follows\(^{45}\)

$$\text{sat}(\frac{S}{\delta}) = \begin{cases} \frac{S}{\delta} & \frac{S}{\delta} < 1 \\ \text{sgn}(\frac{S}{\delta}) & \frac{S}{\delta} \geq 1 \end{cases}$$

where $\delta$ is a small positive real number

$$\text{sigmoid}(S) = \frac{1}{1 + e^{-S}}$$

Simulation results

The proposed scheme has been evaluated on FO duffing and gyro systems to demonstrate its effectiveness in controlling chaotic behavior.

FO duffing system

In this simulation, the well-known FO duffing system is considered as follows

$$\begin{cases} D^\mu_t x_1 = x_2 \\ D^\mu_t x_2 = -\theta_1 x_1 - \theta_2 x_1^3 - \theta_3 x_2 + \theta_4 \cos(\omega t) \end{cases}$$

By setting $\theta_1 = -1$, $\theta_2 = 1$, $\theta_3 = 0.15$, $\theta_4 = 0.3$, $\omega = 1$, and $\alpha = 0.96$, the FO duffing equation shows chaotic behavior. Moreover, the existence of UPOs in the FO duffing system with these parameter values is proven in the literature.\(^{25}\)

Adding the external disturbance $d(t)$ and control input $u(t)$ to the system dynamics gives

$$\begin{cases} D^\mu_t x_1 = x_2 \\ D^\mu_t x_2 = -\theta_1 x_1 - \theta_2 x_1^3 - \theta_3 x_2 + \theta_4 \cos(\omega t) + d(t) + g(t, x) u(t) \end{cases}$$

In this case, $d(t)$ is chosen as $0.2 \cos(2t)$ which is bounded by $|d(t)| < k = 0.2$. The parameter $k$ is assumed to be unknown and should be updated by the adaptation law (20). To use the proposed method, we consider $f(t, x) = 0$ and $g(t, x) = 1 + x_1^2$. In addition, $F(t, x)$ and $\theta$ are defined as follows

$$F(t, x) = \begin{bmatrix} -x_1 & -x_1^3 & -x_2 & \cos(\omega t) \end{bmatrix}^T$$

$$\theta = [\theta_1 \theta_2 \theta_3 \theta_4]^T$$

Therefore, equation (41) can be rewritten as follows

$$\begin{cases} D^\mu_t x_1 = x_2 \\ D^\mu_t x_2 = f(t, x) + F^T(t, x) \theta + d(t) + g(t, x) u(t) \end{cases}$$

The initial conditions are considered as $\dot{k}(0) = 0.1$, $\theta(0) = [-1.5 1.5 0.2 0.5]^T$, and $x(0) = [0.15 0.1]^T$. The adaptation coefficients are established to be $\gamma_k = 1$, and $\gamma = [5 5 5 5]^T$. The Predictive-Evaluate-Correct-Evaluate method\(^{46}\) is used to solve the fractional differential equations, and the time step size is set to 0.005s. To resolve the fractional differential equations, the Predictive-Evaluate-Correct-Evaluate algorithm\(^{46}\) is used with a time step size of 0.005s.

The proposed adaptive delayed feedback control algorithm is applied to a well-known fractional duffing
system to stabilize its UPO with period $T = 2\pi$ in Figures 1–2. Figure 1 (solid black line) illustrates the time history of the state variables and the main UPO of the system by the dashed blue lines. It is worth mentioning that the controller is applied at $t = 4T$. As these figures demonstrate, the UPO of the FO duffing system is completely stabilized after almost 10 seconds of applying the controller. So the chaotic behavior of the system is suppressed, and the state trajectories of the system track the main UPO. Although we assumed that $f = 0$ and all the system’s parameters are unknown, the controller can stabilize the system, and this shows the adaptivity of the controller. In addition, the controller is able to stabilize the system’s UPO regardless of the presence of uncertainty and external disturbance with unknown upper bounds, illustrating the proposed algorithm’s robustness against system uncertainty and disturbance.

Figure 2 illustrates the trajectory of the sliding surface and control input. As the system approaches the unstable periodic orbit (UPO), a lower control signal is needed. The trajectory sets on the UPO with zero control signal when the chaotic system trajectory reaches the exact system UPO. However, due to unknown parameters and disturbances, the primary UPO may not be fully stabilized, and the controller action may not completely converge to zero. Consequently, the system trajectories converge to the close vicinity of the main UPO.

The time history for the estimated value of the external disturbance’s upper bound in the FO duffing system is illustrated in Figure 3. It can be observed that, similar to the system’s UPO, the estimated value represented by $\hat{k}$, converges to its final value after almost 30 seconds. Since the estimated value is the upper bound of the disturbance, even though it converges to the final value slower than other parameters, the UPO is stabilized before $\hat{k}$ converges to the final value.

A linear feedback control method is developed for stabilizing the UPO of FO chaotic systems by Rahimi et al.\textsuperscript{31} We have applied the linear feedback control method to the duffing system and compared the results to our proposed algorithm. As Figure 1 illustrates, our method converges to the main UPO of the system faster than the linear feedback method. Furthermore, Figure 4 compares the error of these two methods with the main system UPO. From the comparison, it can be concluded that the adaptive delayed feedback control scheme exhibits a lower steady-state error compared to the linear feedback control method.

**FO gyro system**

In this section, we consider the FO chaotic gyro system in the following form
By setting $\alpha = 0.98$, $\psi_1 = 10$, $\psi_2 = 0.5$, $\psi_3 = 0.05$, $\beta = 1$, $\omega = 25$, and $f = 35.5$, the FO gyro system exhibits chaos. Furthermore, the existence of UPOs in this system is shown in.25

Adding the external disturbance $d(t)$ and control input $u(t)$ to the system dynamics gives

$$
\begin{align*}
D_t^\alpha x_1 &= x_2 \\
D_t^\beta x_2 &= -\psi_1^2 \frac{(1-\cos x_1)^2}{\sin^2 x_1} - \psi_2 x_2 - \psi_3 x_2^2 \\
&+ \beta \sin x_1 + f \sin \omega t \sin x_1 \\
&+ d(t) + g(t, x) u(t)
\end{align*}
$$

The disturbance is considered as $0.1 \sin(t)$ whose upper bound is $k = 0.1$. It is presumed that the disturbance upper bound is not known and ought to be estimated by the adaptation law (20). Similar to the previous example, we consider $f(t, x) = f \sin \omega t \sin x_1$ and $g(t, x) = 1 + x_1^2$. Also, $F(t, x)$ and $\theta$ are defined as follows

$$
F(t, x) = \left[ -\frac{(1-\cos x_1)^2}{\sin^2 x_1} - x_2 - x_2^3 \sin x_1 \right]^T
$$

$$
\theta = [\psi_1^2 \psi_2 \psi_3 \beta]^T
$$

The initial conditions are considered as $\dot{x}(0) = 0.1$, $\theta(0) = [-1.5 1.5 0.2 0.5]^T$, and $x(0) = [0.15 0.1]^T$. The adaptation parameters are has been chosen as $\gamma = [2 2 2 2]^T$ and $\gamma_k = 2$.

The simulation results are presented for stabilization of the periodic orbit of the gyro system (46) with period $T = 4\pi$ in Figures 5–6. Figure 5 (solid black line) shows the time history of the state variables. In this figure, the main UPO of the system is demonstrated by the dashed blue lines. After 5 s of applying the control input, the UPO is stabilized. Again similar to the previous part, to assess the robustness of the proposed control algorithm, system uncertainty and disturbance are added to the closed-loop system. Moreover, the adaptivity of the proposed algorithm is illustrated by assuming that all the system’s parameters are unknown. Figure 6 demonstrates the time history of the control input and sliding surface. Since we have considered system uncertainty and disturbance in the numerical simulations, the trajectories of the system converge to close vicinity of the main UPO, and the control input, $u$, has not converged to zero.
Figure 7 illustrates time history for the estimated value of the external disturbance’s upper bound in system (46). After almost 16 s, \( k \) is convergent to its final value, much like the system’s UPO. The upper bound of the disturbance may take longer to converge to its final value as compared to other parameters. However, despite this slower convergence rate, the system is able to stabilize on the UPO even before the estimated value reaches its final value.

We have applied the linear feedback control method to the duffing system and compared the results to our proposed algorithm. As Figure 1 illustrates, our method converges to the main UPO of the system faster than the linear feedback method. Furthermore, Figure 4 compares the error of these two methods with the main system UPO. One can conclude that the steady-state error in the linear feedback control method is greater than the adaptive delayed feedback control scheme.

Similar to duffing system, our presented method and the linear feedback control method have been applied to the fractional gyro system. As depicted in Figure 5 our algorithm demonstrates a higher convergence rate to the main system’s UPO compared to the linear feedback control. Moreover, Figure 8 illustrates a comparison of the UPO tracking error between these two methods. Based on this comparison, it can be inferred that the adaptive delayed feedback control scheme exhibits a lower steady-state error than the linear feedback control technique.

**Conclusion**

This article develops a robust adaptive nonlinear delayed feedback control strategy for a class of uncertain FO chaotic systems. The control framework employs the FO sliding mode control approach which guarantees the robustness of the closed-loop system against system uncertainties and external disturbances. The control input and adaptation mechanism are constructed from a proper sliding surface, using the Lyapunov approach. A key advantage of the proposed method is its independence from the UPO’s trajectory, requiring only the period of an unstable periodic orbit for controller design. In the proposed framework, the stabilized orbit may not precisely represent the main UPO of the chaotic system due to external disturbances and system uncertainties. However, the stabilized orbit closely approximates the main UPO of the chaotic system. Finally, the proposed adaptive delayed feedback control scheme is applied to FO duffing and gyro systems, and numerical simulations are included to illustrate our findings and validate our theoretical results. These results show that the performance of our method is better than the linear feedback control method which is previously developed for stabilizing UPOs in FO chaotic systems in terms of steady-state error and convergence rate of tracking error.

Future research will focus on extending the proposed technique to systems with more general structures than system (10) and evaluating the method in real-world systems through experimental setups. In addition, future work will address the consideration of constraints on control inputs and state variables and the extension of the Explicit Reference Governor scheme to FO nonlinear systems.

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