This Paper

- Identifies distribution of random coefficients for large class of models.
- Covers discrete choice, bundles models, consideration set models, among others.
- Key feature: need to identify average demand function.
- Get traction by exploiting envelope theorem.
Example: Discrete Choice

Discrete choice with linear random coefficients

\[ v_k = \beta'_k x_k + \varepsilon_k. \]

- \( \beta \) and \( \varepsilon \) are random

- Special case is random coefficients logit, in which \( \varepsilon \) has a known distribution up to location.
  - Studied in Fox, il Kim, Ryan, Bajari (2012, JoE).

- We differ by letting both \( \beta \) and \( \varepsilon \) have nonparametric distribution.

- Identify all moments of \( \beta = (\beta_1, \ldots, \beta_K) \).
General Model

These models (and others) can be written as perturbed utility models of the form

$$Y(X, \beta, \varepsilon) \in \arg\max_{y \in B} \sum_{k=1}^{K} (\beta_k' X_k) y_k + D(y, \varepsilon).$$

- $Y$ is quantity vector for $K$ goods.
- $X$ collects regressors.
- $\varepsilon$ can be infinite dimensional.
- $B$ is nonrandom however $D(y, \varepsilon)$ can be $-\infty$ for certain combinations.
  - Allows “consideration sets.”
This paper starts with average structural function

\[ \overline{Y}(x) = \int Y(x, \beta, \varepsilon) d\tau(\beta, \varepsilon) \]

and asks what we can learn about distribution of \( \beta \).

- When \( X \) and (\( \beta, \varepsilon \)) are independent,
  \[ \overline{Y}(x) = \mathbb{E}[Y \mid X = x]. \]

- Also identifiable with endogeneity.
  - We complement Berry and Haile (2014, ECTA), who identify \( \overline{Y}(x) \) in a demand setting with instruments.
Lemma

Let

\[ V(\beta'_1 x_1, \ldots, \beta'_K x_K) = \int \left( \max_{y \in B} \sum_{k=1}^{K} y_k (\beta'_k x_k) + D(y, \varepsilon) \right) d\mu(\varepsilon). \]

Then

\[ \int Y(x, \beta, \varepsilon) d\mu(\varepsilon) = \nabla V(\beta'_1 x_1, \ldots, \beta'_K x_K) \]

at any point of differentiability.

- Related to Williams-Daly-Zachary theorem of discrete choice.
Slope-Intercept Independence

Assumption

\( \beta \) and \( \varepsilon \) are independent so we can write

\[
\overline{Y}(x) = \int \int Y(x, \beta, \varepsilon) d\mu(\varepsilon) d\nu(\beta).
\]

Notation:

\[
\overline{Y}(x, \beta) = \int Y(x, \beta, \varepsilon) d\mu(\varepsilon)
\]

\[
\overline{Y}(x) = \int \overline{Y}(x, \beta) d\nu(\beta).
\]
Assume each $x_k$ is scalar for simplicity.

Write envelope theorem as

$$\bar{Y}_k(x, \beta) = \partial_k V(\beta_1 x_1, \ldots, \beta_K x_K).$$

Differentiate envelope theorem further to get

$$\partial_{x_j} \bar{Y}_k(x, \beta) = \partial_{j,k} V(\beta_1 x_1, \ldots, \beta_K x_K) \beta_j$$

and

$$\partial_{x_\ell} \partial_{x_j} \bar{Y}_k(x, \beta) = \partial_{\ell,j,k} V(\beta_1 x_1, \ldots, \beta_K x_K) \beta_\ell \beta_j.$$
Take expectations over $\beta$ and evaluate at $x = 0$ to get:

$$\partial_{x_\ell} \partial_{x_j} \overline{Y}_k(0) = \partial_{\ell,j,k} V(0) \int \beta_\ell \beta_j d\nu(\beta).$$

- Key feature: $\partial_{\ell,j,k} V(0)$ does not depend on $\beta$. 

Example: Identification of Second Moments

\[
\begin{align*}
\partial_{x_1} \partial_{x_1} \overline{Y}_2(0) &= \partial_{1,1,2} V(0) \int \beta_1^2 \, d\nu(\beta) \\
\partial_{x_1} \partial_{x_2} \overline{Y}_2(0) &= \partial_{1,2,2} V(0) \int \beta_2 \beta_1 \, d\nu(\beta) \\
\partial_{x_2} \partial_{x_1} \overline{Y}_1(0) &= \partial_{2,1,1} V(0) \int \beta_1 \beta_2 \, d\nu(\beta) \\
\partial_{x_2} \partial_{x_2} \overline{Y}_1(0) &= \partial_{2,2,1} V(0) \int \beta_2^2 \, d\nu(\beta)
\end{align*}
\]
Example: Identification of Second Moments

\[
\frac{\partial x_1 \partial x_1 \bar{Y}_2(0)}{\partial x_2 \partial x_1 \bar{Y}_1(0)} = \frac{\partial_{1,1,2} V(0) \int \beta_1^2 d\nu(\beta)}{\partial_{2,1,1} V(0) \int \beta_1 \beta_2 d\nu(\beta)} = \frac{\int \beta_1^2 d\nu(\beta)}{\int \beta_1 \beta_2 d\nu(\beta)}
\]

- Uses symmetry \( \partial_{1,1,2} V(0) = \partial_{2,1,1} V(0) \).

- Symmetry has been used without random coefficients in Allen and Rehbeck (2019, ECTA).
Example: Identification of Second Moments

Combining other equations identifies the ratio of any two second moments.

- Identification given a scale assumption \( \int \beta_1^2 d\nu(\beta) = 1 \).
Main Result

Theorem

Assume $\int \beta_{1,1}^M d\nu(\beta)$ is known. Under regularity conditions, each $M$-th order moment of the form

$$\int \beta_{k_1,\ell_1} \cdots \beta_{k_M,\ell_M} d\nu(\beta)$$

is identified. In addition, for each $\gamma \in \{1, \ldots, K\}^{M+1}$,

$$\partial_\gamma V(0)$$

is identified.
Can identify counterfactual/welfare objects under additional assumptions.

- Integrated indirect utility

\[
V(\beta_1'x_1, \ldots, \beta_K'x_K) = \int \left( \max_{y \in B} \sum_{k=1}^{K} y_k(\beta_k'x_k) + D(y, \varepsilon) \right) d\mu(\varepsilon).
\]
How to identify $V$?

- Main result identified partial derivatives of $V$ at 0.

- If $V$ is real analytic we can identify the function globally from these derivatives.

- (Paper presents two other techniques.)
Once $V$ is identified, counterfactuals can be identified from envelope theorem.

$$\overline{Y}_k(x, \beta) = \partial_k V(\beta'_1 x_1, \ldots, \beta'_K x_K).$$
Conclusion

- Identification of moments of linear random coefficient distribution in class of perturbed utility models.
- Covers several examples in a single framework.
- Requires only the average structural function.
- Exploits the envelope theorem.