Superspace Geometry for Supermembrane Backgrounds

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Abstract

We construct part of the superspace vielbein and tensor gauge field in terms of the component fields of 11-dimensional on-shell supergravity. The result can be utilized to describe supermembranes and corresponding matrix models for Dirichlet particles in nontrivial supergravity backgrounds to second order in anticommuting coordinates. We exhibit the $\kappa$-invariance of the corresponding supermembrane action, which at this order holds for unrestricted supergravity backgrounds, the supersymmetry covariance and the resulting surface terms in the action.

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1 Introduction

From their very beginning supermembranes \[1\] have been studied in connection with 11-dimensional supergravity \[2\]. In 11 spacetime dimensions the supermembrane can consistently couple to a superspace background that satisfies a number of constraints which are equivalent to the supergravity equations of motion. In principle the supermembrane action exists in 4, 5, 7 and 11 dimensions, analogously to the Green-Schwarz superstring \[3\] which is classically consistent in 3, 4, 6 and 10 dimensions. But interest has focused on the 11-dimensional supermembrane in the hope of providing a quantum-mechanically consistent extension of supergravity in the highest possible spacetime dimension where local supersymmetry can exist, just as the superstring defines such an extension for 10-dimensional supergravities. In this context it was expected that the massless states of the supermembrane would correspond to those of 11-dimensional supergravity. However, unlike the superstring states, the supermembrane states turn out to have a continuous mass spectrum \[4\], which makes the possible existence of massless states much more subtle to prove or disprove \[5, 6, 7\]. These features posed an obstacle to further developments in supermembrane theory.

Interest in supermembranes was rekindled by the realization that 11-dimensional supergravity does have its role to play as the long-distance approximation to M-theory \[8, 9, 10, 11, 12\]. This theory is the conjectured framework for unifying all five superstring theories and 11-dimensional supergravity. It turns out that supermembranes, M-theory and super-matrix-models are all intricately related. This is seen, for instance, from the result that the light-cone formulation of the supermembrane in flat backgrounds leads to a supersymmetric U(\(N\)) gauge quantum-mechanical model in the large-\(N\) limit \[13\]. This model \[13\], now termed ‘matrix theory’, has been conjectured to capture all the degrees of freedom of M-theory \[14\]. Furthermore there is evidence meanwhile that the supermembrane has massless states \[15\], which will presumably correspond to the states of 11-dimensional supergravity, although proper asymptotic states do not exist. The existence of such states was foreseen on the basis of identifying the Kaluza-Klein states of M-theory compactified on \(S^1\) with the Dirichlet particles and their bound states in type-IIA string theory.

From this viewpoint it is a natural question to consider the supermembrane in curved backgrounds associated with 11-dimensional supergravity, which is the subject of this paper. Such backgrounds consist of a nontrivial metric, a three-index gauge field and a gravitino field. This provides us with an action that transforms as a scalar under the combined (local) supersymmetry transformations of the background fields and the supermembrane embedding coordinates. Here it is important to realize that the supersymmetry transformations of the embedding coordinates will themselves depend on the background. When the background is supersymmetric, then the action will be supersymmetric as well. In the light-cone formulation this model will lead to models invariant under area-
preserving diffeomorphisms, which in certain situations can be approximated by matrix models in curved backgrounds. The area-preserving diffeomorphisms are then replaced by a finite group, such as U(N), but target-space diffeomorphisms are no longer manifestly realized. Matrix models in curved space have already been studied in [16]. Recently toroidal compactifications of matrix theory were considered in which the three-form gauge field of 11-dimensional gravity plays a crucial role [17]. These compactifications exhibit interesting features in which the noncommutative torus appears as a new solution to compactified matrix theory. The bosonic coupling of the membrane to the three-form gauge field will be discussed in this paper. A summary of this part of our results was presented earlier in [18]. We should also point out that classical supermembrane solutions in nontrivial backgrounds have been discussed before, see, e.g. [19].

The approach followed in this paper for constructing the supermembrane action is in principle straightforward and starts from the superspace formulation presented in [1]. The background is then characterized by the superspace vielbein and antisymmetric tensor field. For practical calculations we like to have a formulation in terms of the on-shell supergravity fields. Therefore, we need to cast the component fields into superspace, which can be done by a method sometimes referred to as ‘gauge completion’ [20, 21, 22, 23]. For 11-dimensional supergravity the first steps of this procedure have been carried out long ago [24], but unfortunately only to first order in anticommuting coordinates θ. While this suffices to identify the on-shell formulation of supergravity in superspace (see also ref. [25]), it is not sufficient for studying supermembrane interactions with the background. In this paper we therefore extend the analysis to higher order in θ and are thus able to write down the supermembrane action in a nontrivial on-shell supergravity background up to second order in θ.

At that point there is an important consistency check, namely that the action is invariant under an additional local fermionic κ-symmetry. As alluded to above, this invariance holds provided the background fields obey the equations of motion of 11-dimensional supergravity [1]. However, at second order in θ this restriction is not yet required and our results are shown to preserve the invariance. In this paper we concentrate on the superspace features and we will be brief on the light-cone formulation of the supermembrane in the supergravity background. We intend to return to a full discussion of the latter in a forthcoming publication [26].

We have organized our paper as follows. In section 2 our supergravity notations and conventions are established and as a test case the light-cone formulation of the bosonic membrane in curved backgrounds is studied. In section 3 we set the stage for an iterative computation of the component-field content of the superfields and superparameters in θ, which is then taken to second order in section 4. All results obtained in sections 3 and 4 are collected in subsection 4.3. In section 5 we turn to the explicit form of the supermembrane action coupled to background fields up to second order in θ and prove its κ-symmetry. We also
verify the manifest covariance under supersymmetry, which is only a consistency check, and determine the surface terms that follow from \(\kappa\)-symmetry and supersymmetry. Finally, we discuss possible implications and applications of our result in section 6.

2 Preliminaries

In this paper we consider superspace backgrounds that correspond to 11-dimensional supergravity. The use of certain standard conventions for the supermembrane will force us to employ specific and somewhat unconventional normalizations for the supergravity component fields. The first subsection is therefore devoted to a brief summary of 11-dimensional supergravity and will establish our notation. In the second subsection we review the supermembrane theory in superspace and indicate the effects of a nontrivial background in its bosonic truncation.

2.1 Supergravity in 11 dimensions

Supergravity in 11 spacetime dimensions is based on an “elfbein” field \(e^r_\mu\), a Majorana gravitino field \(\psi_\mu\), and a 3-rank antisymmetric gauge field \(C_{\mu\nu\rho}\). Its Lagrangian\(^1\) can be written as follows \([2]\),

\[
\mathcal{L} = -\frac{1}{2} e R(e, \omega) - 2 e \bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu \left[ \frac{1}{2} (\omega + \hat{\omega}) \right] \psi_\rho - \frac{1}{96} e (F_{\mu\nu\rho\sigma})^2 \\
- \frac{1}{2} 12 \epsilon^{\mu_1 \cdots \mu_11} F_{\mu_1\mu_2\mu_3\mu_4} F_{\mu_5\mu_6\mu_7\mu_8} C_{\mu_9\mu_{10}\mu_{11}} \\
- \frac{1}{96} e \left( \bar{\psi}_\lambda \Gamma^{\mu\nu\rho\sigma \lambda} \psi_\tau + 12 \psi^{\mu} \Gamma^{\nu\rho\sigma} \psi^\sigma \right) (F + \hat{F})_{\mu\nu\rho\sigma} ,
\]

(2.1)

where \(e = \det e^r_\mu\), \(\omega^{rs}\) denotes the spin connection and \(F_{\mu\nu\rho\sigma}\) the field strength of the antisymmetric tensor. The caret denotes that they have been made covariant with respect to local supersymmetry. We present the corresponding definitions in a sequel. The derivative \(D_\mu(\omega)\) is covariant with respect to local Lorentz transformations,

\[
D_\mu(\omega) \epsilon = \left( \partial_\mu - \frac{1}{2} \omega_\mu^{\ rs} \Gamma_{rs} \right) \epsilon.
\]

(2.2)

The supersymmetry transformations are equal to

\[
\delta e^r_\mu = 2 \bar{\epsilon} \Gamma^r \psi_\mu ,
\]

\[
\delta \psi_\mu = \left. D_\mu(\hat{\omega}) \right| \epsilon + T^{\nu\rho\sigma} \epsilon \hat{F}_{\nu\rho\sigma} ,
\]

\[
\delta C_{\mu\nu\rho} = -6 \bar{\epsilon} \Gamma_{[\mu\nu} \psi_{\rho]} .
\]

(2.3)

\(^1\)Gamma matrices satisfy \(\{\Gamma^r, \Gamma^s\} = 2 \eta_{rs}\), where \(\eta_{rs}\) is the tangent-space metric \(\eta_{rs} = \text{diag}(-, +, \cdots, +)\). Gamma matrices with multiple indices denote antisymmetrized products with unit strength. In particular \(\Gamma^{\mu_1\mu_2\cdots\mu_{11}} = 1 \epsilon^{\mu_1\mu_2\cdots\mu_{11}}\). The Dirac conjugate is defined by \(\bar{\psi} = i\gamma^0 \Gamma^0\) for a generic spinor \(\psi\).
\[ T_{\mu}^{\nu\rho\sigma\kappa} = \frac{1}{288} \left( \Gamma_{\mu}^{\nu\rho\sigma\kappa} - 8 \delta_{\mu}^{\nu} \Gamma^{\rho\sigma\kappa} \right). \] (2.4)

Note that \( \hat{F}_{\mu\nu\rho\sigma} \) is the supercovariant field strength,
\[ \hat{F}_{\mu\nu\rho\sigma} = 4 \partial_{[\mu} C_{\nu\rho\sigma]} + 12 \bar{\psi}_{[\mu} \Gamma_{\nu\rho} \psi_{\sigma]}, \] (2.5)
and that the supercovariant spin connection \( \hat{\omega}_{\mu}^{\rho\sigma} \) is the solution of the following equation,
\[ D_{[\mu}(\hat{\omega}) e_{\nu]} - \bar{\psi}_{\mu} \Gamma^{\nu} \psi_{\nu} = 0. \] (2.6)

The left-hand side of this equation is the supercovariant torsion tensor.

The Lagrangian (2.1) is derived in the context of the so-called “1.5-order” formalism, in which the spin connection is defined as a dependent field determined by its (algebraic) equation of motion, whereas its supersymmetry variation in the action is treated as if it were an independent field \[27\]. Furthermore we note the presence of a Chern-Simons-like term \( F \wedge F \wedge C \) in the Lagrangian. Under tensor gauge transformations,
\[ \delta_{C} C_{\mu\nu\rho} = 3 \partial_{[\mu} \xi_{\nu\rho]}, \] (2.7)
the corresponding action is thus only invariant up to surface terms.

We have the following bosonic field equations and Bianchi identities,
\[ R_{\mu\nu} = \frac{1}{144} g_{\mu\nu} F_{\rho\sigma\lambda\tau} F^{\rho\sigma\lambda\tau} - \frac{1}{12} F_{\mu\rho\sigma\lambda} F^{\rho\sigma\lambda}, \]
\[ \partial_{\mu} (e F^{\mu\nu\rho\sigma}) = \frac{1}{1152} \varepsilon^{\rho\sigma\mu\nu_{1}\ldots\nu_{8}} F_{\mu_{1}\mu_{2}\mu_{3}\mu_{4}} F_{\mu_{5}\mu_{6}\mu_{7}\mu_{8}}, \]
\[ \partial_{[\mu} F_{\nu\rho\sigma\lambda]} = 0, \] (2.8)
which no longer depend explicitly on the antisymmetric gauge field. An alternative form of the second equation is \[28\]
\[ \partial_{[\mu} H_{\nu_{1}\ldots\nu_{8}]} = 0, \] (2.9)
where \( H_{\mu_{1}\ldots\mu_{8}} \) is the dual field strength,
\[ H_{\mu_{1}\ldots\mu_{7}} = \frac{1}{11} \varepsilon_{\mu_{1}\ldots\mu_{11}} F^{\mu_{8}\mu_{9}\mu_{10}\mu_{11}} - \frac{1}{12} F_{\mu_{1}\mu_{2}\mu_{3}\mu_{4}} C_{\mu_{5}\mu_{6}\mu_{7}}. \] (2.10)

When the third equation of (2.8) and (2.9) receive contributions from certain source terms on the right-hand side, then the corresponding charges can be associated with the ‘flux’-integral of \( H_{\mu_{1}\ldots\mu_{7}} \) and \( F_{\mu_{1}\mu_{2}\mu_{3}\mu_{4}} \) over the boundary of an 8- and a 5-dimensional spatial volume, respectively. This volume is transverse to a \( p = 2 \) and \( p = 5 \) brane configuration, and the corresponding charges are 2- and 5-rank Lorentz tensors. For solutions of 11-dimensional supergravity that contribute to these charges, see e.g. \[29, 30, 31, 12\].
It is straightforward to evaluate the supersymmetry algebra on these fields. The commutator of two supersymmetry transformations yields a general-coordinate transformation, a supersymmetry transformation, a local Lorentz transformation, and a gauge transformation associated with the tensor gauge field,

\[ [\delta(\epsilon_1), \delta(\epsilon_2)] = \delta_{gct}(\lambda^\mu) + \delta(\epsilon_3) + \delta_L(\lambda^{rs}) + \delta_C(\xi_{\mu\nu}). \] (2.11)

The parameters of the transformations on the right-hand side are given by

\[ \xi^\mu = 2 \tilde{\epsilon}_2 \Gamma^\mu \epsilon_1, \]
\[ \epsilon_3 = -\xi^\mu \psi_\mu, \]
\[ \lambda^{rs} = -\xi^\mu \hat{\omega}^{rs}_\mu + \frac{1}{72} \tilde{\epsilon}_2 \left[ \Gamma^{rs\mu\rho\sigma} \tilde{F}_{\mu\rho\sigma} + 24 \Gamma_{\mu\nu} \tilde{F}^{rs\mu\nu} \right] \epsilon_1, \]
\[ \xi_{\mu\nu} = -\xi^\rho C_{\rho\mu\nu} - 2 \tilde{\epsilon}_2 \Gamma_{\mu\nu} \epsilon_1. \] (2.12)

### 2.2 Membranes in background fields

The 11-dimensional supermembrane is written in terms of superspace embedding coordinates \( Z^M(\zeta) = (X^\mu(\zeta), \theta^a(\zeta)) \), which are functions of the three world-volume coordinates \( \zeta^i \) \((i = 0, 1, 2)\). It couples to the superspace geometry of 11-dimensional supergravity, encoded by the supervielbein \( E_M^A \) and the antisymmetric tensor gauge superfield \( B_{MNP} \), through the action

\[ S[Z(\zeta)] = \int d^3 \zeta \left[ -\sqrt{-g(Z(\zeta))} - \frac{1}{6} \epsilon^{ijk} \Pi_i^A \Pi_j^B \Pi_k^C B_{CBA}(Z(\zeta)) \right], \] (2.13)

where \( \Pi_i^A = \partial Z^M / \partial \zeta^i \) \( E_M^A \) is the pull-back of the supervielbein to the membrane worldvolume. Here the induced metric equals \( g_{ij} = \Pi_i^\alpha \Pi_j^\beta \eta_{\alpha\beta} \), with \( \eta_{\alpha\beta} \) being the constant Lorentz-invariant metric. This action is invariant under local fermionic \( \kappa \) transformations, given that certain constraints on the background fields hold, which are equivalent to the equations of motion of 11-dimensional supergravity (2.8).

Flat superspace is characterized by

\[
\begin{align*}
E_{\mu r}^r &= \delta_{\mu r}^r, & E_{\mu}^a &= 0, \\
E_{\alpha a}^a &= \delta_{\alpha}^a, & E_{\alpha}^r &= -\langle \bar{\theta} \Gamma^r \rangle_\alpha, \\
B_{\mu\nu\alpha} &= \langle \bar{\theta} \Gamma_{\mu\nu} \rangle_\alpha, & B_{\mu\nu\beta} &= \langle \bar{\theta} \Gamma_{\mu\nu} \rangle_\alpha \langle \bar{\theta} \Gamma^\nu \rangle_\beta, \\
B_{\alpha\beta\gamma} &= \langle \bar{\theta} \Gamma_{\mu\nu} \rangle_\alpha \langle \bar{\theta} \Gamma^\mu \rangle_\beta \langle \bar{\theta} \Gamma^\nu \rangle_\gamma, & B_{\mu\nu\rho} &= 0.
\end{align*}
\] (2.14)

These quantities receive corrections in the presence of supergravity background fields \( e^r_\mu, \psi^a_\mu \) and \( C_{\mu\nu\rho} \), and it is the aim of this paper to determine some of these corrections to second order in \( \theta \).

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2 Our notation and conventions are as follows. Tangent-space indices are \( A = (r, a) \), whereas curved indices are denoted by \( M = (\mu, \alpha) \). Here \( r, \mu \) refer to commuting and \( a, \alpha \) to anticommuting coordinates. Moreover we take \( \epsilon_{012} = -\epsilon^{012} = 1 \).
In flat superspace the supermembrane Lagrangian, written in components, reads,

\[
\mathcal{L} = -\sqrt{-g(X, \theta)} - \varepsilon^{ijk} \bar{\Gamma} \mu \nu \partial_k \theta \left[ \frac{1}{2} \partial_i X^\mu (\partial_j X^\nu + \bar{\Gamma}^\nu \partial_j \theta) + \frac{1}{6} \bar{\Gamma}^\mu \partial_i \theta \bar{\Gamma}^\nu \partial_j \theta \right],
\]

(2.15)

To elucidate the generic effects of nontrivial backgrounds for membrane theories, let us confine ourselves for the moment to the purely bosonic theory and present the light-cone formulation of the membrane in a background consisting of the metric \( G_{\mu \nu} \) and the tensor gauge field \( C_{\mu \nu \rho} \). The Lagrangian density for the bosonic membrane follows directly from (2.13),

\[
\mathcal{L} = -\sqrt{-g} - \frac{1}{6} \varepsilon^{ijk} \partial_i X^\mu \partial_j X^\nu \partial_k X^\rho C_{\rho \nu \mu},
\]

(2.16)

where \( g_{ij} = \partial_i X^\mu \partial_j X^\nu \eta_{\mu \nu} \). In the light-cone formulation, the coordinates are decomposed in the usual fashion as \((X^+, X^-, X^a)\) with \(a = 1 \ldots 9\). Furthermore we use the diffeomorphisms in the target space to bring the metric in a convenient form \(32\),

\[
G_{--} = G_{a--} = 0.
\]

(2.17)

Subsequently we identify the time coordinate of the target space with the world-volume time, by imposing the condition \(X^+ = \tau\). Moreover we denote the spacesheet coordinates of the membrane by \(\sigma^r, r = 1, 2\). Following the same steps as for the membrane in flat space \(5\), one arrives at a Hamiltonian formulation of the theory in terms of coordinates and momenta. These phase-space variables are subject to a constraint, which takes the same form as for the membrane theory in flat space, namely,

\[
\phi_r = P_a \partial_r X^a + P_- \partial_r X^- \approx 0.
\]

(2.18)

Of course, the definition of the momenta in terms of the coordinates and their derivatives does involve the background fields, but at the end all explicit dependence on the background cancels out.

The Hamiltonian now follows straightforwardly. As it turns out, the background tensor field appears in the combinations

\[
C_a = -\varepsilon^{rs} \partial_r X^- \partial_s X^b C_{-ab} + \frac{1}{2} \varepsilon^{rs} \partial_r X^b \partial_s X^c C_{abc},
\]

\[
C_{\pm} = \frac{1}{2} \varepsilon^{rs} \partial_r X^a \partial_s X^b C_{\pm ab},
\]

\[
C_{+-} = \varepsilon^{rs} \partial_r X^- \partial_s X^a C_{+-a}.
\]

(2.19)

With these definitions the total Hamiltonian takes the form

\[
H = \int d^2 \sigma \left\{ \frac{G_{++}}{P_- - C_-} \left[ \frac{1}{2} \left( P_a - C_a - \frac{P_- - C_-}{G_{++}} \right) G_{a+} \right]^2 + \frac{1}{2} (\varepsilon^{rs} \partial_r X^a \partial_s X^b)^2 \right\}
- \frac{P_- - C_-}{2 G_{++}} G_{++} - C_+ - C_{+-} + \varepsilon^r \phi_r \right\}.
\]

(2.20)
where we have included the Lagrange multiplier $c^r$ coupling to the constraint (2.18). Observe that transverse indices are contracted with the metric $G_{ab}$ or its inverse.

The gauge choice $X^+ = \tau$ still allows for $\tau$-dependent reparametrizations of the world-space coordinates $\sigma^r$, which in turn induce transformations on the Lagrange multiplier $c^r$ through the Hamilton equations of motion. In addition there remains the freedom of performing tensor gauge transformations of the target-space three-form $C_{\mu\nu\rho}$. In order to rewrite (2.20) in terms of a gauge theory of area-preserving diffeomorphisms it is desirable to obtain a Hamiltonian which is polynomial in momenta and coordinates. For this the dynamics of $P_- - C_-$ needs to become trivial, i.e. $\partial_-(P_- - C_-) = 0$, allowing us to set it equal to some space-sheet density $\sqrt{w(\sigma)}$. The residual invariance group is then constituted by the area-preserving diffeomorphisms that leave $\sqrt{w}$ invariant. The $\tau$-independence of $P_- - C_-$ can be achieved by firstly assuming that the background fields are $X^{\pm}$-independent. Secondly one uses the tensor gauge transformations to set $C_{-ab}$ equal to a constant antisymmetric matrix. One then has

$$\partial_r (P_- - C_-) \approx \partial_r \left[ -\varepsilon^{rs} \partial_s X^a C_{+a} + (P_- - C_-) c^r \right]. \quad (2.21)$$

We now choose a gauge such that the right-hand side of this equation vanishes. In that case the total Hamiltonian takes the following form,

$$H = \int d^2 \sigma \left\{ \frac{G_{++} - G_{+-}}{P_- - C_-} \left[ \frac{1}{2} \left( P_a - C_a - \frac{P_- - C_-}{G_{++}} G_{a+} \right)^2 + \frac{1}{4} \left( \varepsilon^{rs} \partial_r X^a \partial_s X^b \right)^2 \right] - \frac{P_- - C_-}{2 G_{++}} G_{++} - C_+ + \frac{1}{P_- - C_-} \left[ \varepsilon^{rs} \partial_r X^a \partial_s X^b P_a C_{+ab} + C_+ C_- \right] \right\}, \quad (2.22)$$

where $P_- - C_- \propto \sqrt{w}$ and $C_{-ab}$ constant.

At this point one can impose further gauge choices and set $G_{++} = 1$ and $C_{+a} = 0$. Taking also $C_{-ab} = 0$ the corresponding Hamiltonian was recast in Lagrangian form in [18] in terms of a gauge theory of area-preserving diffeomorphisms. With both $C_{+a}$ and $C_{-ab}$ different from zero, one can go through the same procedure. As alluded to in the first reference of [17], the Lagrangian then depends explicitly on $X^-$, a feature that we have already exhibited earlier for the winding membrane [33]. However, in the case at hand, the $X^-$-dependence is rather nontrivial and clearly this is an issue that deserves more study.

With a reformulation of the membrane in background fields as a gauge theory of area-preserving diffeomorphisms at one’s disposal, one may consider its regularization through a matrix model by truncating the mode expansion for coordinates and momenta in the standard fashion [34, 3]. This leads to a replacement of Poisson brackets by commutators, integrals by traces and products of commuting
fields by symmetrized products of the corresponding matrices. At that point the original target-space covariance is affected, as the matrix reparametrizations in terms of symmetrized products of matrices do not possess a consistent multiplication structure\(^3\); this is just one of the underlying difficulties in the construction of matrix models in curved space [16]. Recently the antisymmetric constant matrix \( C_{-ab} \) was conjectured to play a role for the matrix model compactification on a noncommutative torus [17]. It should be interesting to see what the role is of (2.22) in this context. We intend to return to these issues in more detail in a future publication [26].

3 Superspace representation

In this section we introduce the method for constructing superspace backgrounds expressed in terms of the component fields of 11-dimensional supergravity. Besides this we evaluate the quantities of interest in low orders of the anticommuting coordinates \( \theta \). The superspace coordinates \( Z^M \) are given by \( Z^M = (x^\mu, \theta^a) \). The superspace geometry is encoded in the supervielbein \( E_M{}^A \) and a spin-connection field \( \Omega_M{}^{AB} \). In what follows we will not pay much attention to the spin-connection, which is not an independent field. Furthermore we have an antisymmetric tensor gauge field \( B_{MNP} \), subject to tensor gauge transformations,

\[
\delta B_{MNP} = 3 \partial_{[M} \Xi_{NP]}.
\]

(3.1)

Unless stated otherwise the derivatives with respect to \( \theta \) are always left derivatives. We remind the reader that ‘antisymmetric’ tensors in superspace satisfy the symmetry properties induced by those of superdifferentials, i.e. \( dZ^M \wedge dZ^N = (-)^{1+MN} dZ^N \wedge dZ^M \).

Under superspace diffeomorphisms corresponding to \( Z^M \rightarrow Z^M + \Xi^M(Z) \), the supervielbein and tensor gauge field transform as

\[
\delta E^{A}_M = \Xi^N \partial_N E^{A}_M + \partial_M \Xi^N E^{A}_N,
\]

\[
\delta B_{MNP} = \Xi^Q \partial_Q B_{MNP} + 3 \partial_{[M} \Xi^Q B_{NP]}.
\]

(3.2)

Tangent-frame rotations are \( Z \)-dependent Lorentz transformations that act on the vielbein according to

\[
\delta E^{A}_M = \frac{1}{2} (\Lambda^{rs} L_{rs})^{A}_B E^{B}_M,
\]

(3.3)

where the Lorentz generators \( L_{rs} \) are defined by

\[
\frac{1}{2} (\Lambda^{rs} L_{rs})^{A}_u = \Lambda^{A}_u,
\]

\[
\frac{1}{2} (\Lambda^{rs} L_{rs})^{a}_b = \frac{1}{4} \Lambda^{rs} (\Gamma_{rs})^{a}_b.
\]

(3.4)

\(^3\)We thank J. de Boer for explaining this to us.
We will not be dealing with an unrestricted superspace but one that is subject to certain constraints and gauge conditions. Furthermore, we will not describe an off-shell situation as all superfields will be expressed entirely in terms of the three component fields of on-shell 11-dimensional supergravity, the elfbein $e^r_\mu$, the antisymmetric tensor gauge field $C_{\mu\nu\rho}$ and the gravitino field $\psi_\mu$. As a result of these restrictions the residual symmetry transformations are confined to 11-dimensional diffeomorphisms with parameters $\xi^\mu(x)$, local Lorentz transformations with parameters $\lambda^{rs}(x)$, tensor-gauge transformations with parameters $\xi_{\mu\nu}(x)$ and local supersymmetry transformations with parameters $\epsilon(x)$. The purpose of this paper is to derive how the superfields are parametrized in terms of the component fields. To do this it is necessary to also determine the form of the superspace transformation parameters, $\Xi^M$, $\Lambda^{rs}$ and $\Xi_{MN}$, that generate the supersymmetry transformations. Here it is important to realize that we are dealing with a gauge-fixed situation. For that reason the superspace parameters depend on both the $x$-dependent component parameters defined above as well as on the component fields. This has two consequences. First of all, local supersymmetry transformations reside in the superspace diffeomorphisms, the Lorentz transformations and the tensor gauge transformations, as $\Xi^M$, $\Lambda^{rs}$ and $\Xi_{MN}$ are all expected to contain $\epsilon$-dependent terms. Thus, when considering supersymmetry variations of the various fields, one must in principle include each of the three possible superspace transformations. Secondly, when considering the supersymmetry algebra, it is crucial to also take into account the variations of the component fields on which the parameters $\Xi^M$, $\Lambda^{rs}$ and $\Xi_{MN}$ will depend.

In this section we present the formalism and derive the expressions for the various superfields in terms of component fields in low orders of $\theta$. This will set the stage for the evaluation of the terms of higher order in $\theta$, which is the subject of the next section, and it will allow us to establish the precise correspondence with the flat superspace conventions used for the supermembrane action in the previous subsection. The method of casting component results into superspace has a long history and is sometimes called 'gauge completion'. For results in 4 spacetime dimensions we refer the reader to [22, 23], while results in 11 dimensions in low orders of $\theta$ were presented in [24].

There are two, somewhat complimentary, ways to obtain information on the embedding of component fields in superspace geometry. One is to consider the algebra of the supersymmetry transformations as generated by the superspace transformations and to adjust it to the supersymmetry algebra of the component fields. This determines the superspace transformation parameters. The other is to compare the transformation rules for the superfields with the known transformations of the component fields. This leads to a parametrization of both the superfields and the transformation parameters in terms of the component fields and parameters. The evaluation proceeds order-by-order in the $\theta$-coordinates, but at each level one encounters ambiguities which can be fixed by suitable higher-order coordinate redefinitions and gauge choices. The first step in this iterative
procedure is the identification at zeroth-order in $\theta$ of some of the component fields and transformation parameters with corresponding components of the superfield quantities. The underlying assumption is that this identification can always be implemented by choosing an appropriate gauge. An obvious identification is given by [20, 21, 22, 23, 24],

$$
E_{\mu}^r(x, \theta = 0) = e_{\mu}^r(x),
E_{\mu}^a(x, \theta = 0) = \psi_{\mu}^a(x),
B_{\mu\nu\rho}(x, \theta = 0) = C_{\mu\nu\rho}(x),
\Xi^\mu(x, \theta = 0) = \xi^\mu(x),
\Xi^a(x, \theta = 0) = \epsilon^a(x),
\Lambda^{rs}(x, \theta = 0) = \lambda^{rs}(x),
\Xi_{\mu\nu}(x, \theta = 0) = \xi_{\mu\nu}(x).
$$

(3.5)

As explained above, the component supersymmetry transformations with parameters $\epsilon(x)$ are generated by a linear combination of a superspace diffeomorphism, a local Lorentz and a tensor gauge transformation; their corresponding parameters will be denoted by $\Xi^M(\epsilon)$, $\Lambda^{rs}(\epsilon)$ and $\Xi_{MN}(\epsilon)$, respectively. Given the embedding of the component fields into the superfields, application of these specific superspace transformations should produce the very same transformation rules that were defined directly at the component level. The structure of the commutator algebra of unrestricted infinitesimal superspace transformations is obvious. Two diffeomorphisms yield another diffeomorphism, two Lorentz transformations yield another Lorentz transformation, according to the Lorentz group structure, while two tensor transformations commute. On the other hand, a diffeomorphism and a local Lorentz transformation yield another Lorentz transformation, and a diffeomorphism and a tensor gauge transformation yield another gauge transformation. All other combinations commute.

The algebra for the restricted superspace transformations that generate the component transformations should coincide with the algebra derived directly for the component fields. However, we must take into account here that the superspace transformation parameters themselves depend on the component fields. To show the effect of this let us restrict ourselves to the diffeomorphism component of the supersymmetry transformation and consider the transformation of a generic scalar superfield,

$$
\delta(\epsilon) \Phi = \Xi^M(\epsilon) \partial_M \Phi.
$$

(3.6)

Closure of supersymmetry now implies

$$
\delta(\epsilon_1, \epsilon_2) \Phi = [\delta(\epsilon_1), \delta(\epsilon_2)] \Phi = \Xi^M(\epsilon_1, \epsilon_2) \partial_M \Phi,
$$

(3.7)

with

$$
\Xi^M(\epsilon_1, \epsilon_2) = \Xi^N(\epsilon_2) \partial_N \Xi^M(\epsilon_1) + \delta(\epsilon_1) \Xi^M(\epsilon_2) - (1 \leftrightarrow 2).
$$

(3.8)
Here the variation $\delta(\epsilon_1, \epsilon_2)$ represents the component result (2.12) of the supersymmetry commutator and $\Xi^M(\epsilon_1, \epsilon_2)$ represents the part of the resulting component transformations that are generated by superspace diffeomorphisms. Note that we are justified in restricting ourselves to the superspace diffeomorphisms, because they are the only ones that lead to a superspace diffeomorphism upon commutation. We will consider the other superspace transformations later. At zeroth-order in $\theta$ we compare the expression for $\Xi^M(\epsilon_1, \epsilon_2)$ to the result of the component supersymmetry algebra, taking into account the conditions (3.5). As it turns out, this leads to the following result,

$$\Xi^\mu(\epsilon) = \bar{\theta} \Gamma^\mu \epsilon + \sum_{n=0,3,4} \bar{\theta} \Gamma^{r_1 \cdots r_n} \epsilon \mathcal{H}^\mu_{r_1 \cdots r_n} + \mathcal{O}(\theta^2),$$

$$\Xi^a(\epsilon) = \epsilon^a - \bar{\theta} \Gamma^a \epsilon \psi^a + \sum_{n=0,3,4} \bar{\theta} \Gamma^{r_1 \cdots r_n} \epsilon \mathcal{H}^a_{r_1 \cdots r_n} + \mathcal{O}(\theta^2),$$

where the $\mathcal{H}^M_{r_1 \cdots r_n}$ are undetermined $\theta$-independent quantities.

Subsequently one compares the supersymmetry variations at $\theta = 0$ of the supervielbein components to their variation under a diffeomorphism given by $\Xi^M(\epsilon)$, i.e.,

$$E^\mu_r = e^\mu_r + 2 \bar{\theta} \Gamma^r \psi^\mu + \mathcal{O}(\theta^2),$$

$$E^a_a = \psi^a_a - \frac{1}{4} \tilde{\omega}^{rs}_\mu (\Gamma_r \theta)^a + (T_{\nu \rho \sigma} \theta)^a \tilde{F}_{\nu \rho \sigma} + \mathcal{O}(\theta^2),$$

$$E^a_\alpha = - (\bar{\theta} \Gamma^r)_{\alpha} + \sum_{n=0,3,4} (\bar{\theta} \Gamma^{r_1 \cdots r_n})_{\alpha} \mathcal{H}^r_{r_1 \cdots r_n} + \mathcal{O}(\theta^2),$$

$$E^a_\alpha = \delta^a_\alpha + \sum_{n=0,3,4} (\bar{\theta} \Gamma^{r_1 \cdots r_n})_{\alpha} \mathcal{H}^a_{r_1 \cdots r_n} + \mathcal{O}(\theta^2).$$

(3.9)

Let us briefly discuss these results. First of all we are dealing with an ambiguity in the iterative procedure reflected in the presence of the $\theta$-independent quantities $\mathcal{H}^M_{r_1 \cdots r_n}$. However, it turns out that this ambiguity can be absorbed into the definition of the superspace coordinates, according to

$$Z^M \rightarrow Z^M + \frac{1}{2} \sum_{n=0,3,4} \bar{\theta} \Gamma^{r_1 \cdots r_n} \theta \mathcal{H}^M_{r_1 \cdots r_n}.$$

(3.11)

Hence we may set $\mathcal{H}^M_{r_1 \cdots r_n} = 0$ in what follows. In that case our results for the vielbein agree with the flat-space expressions (2.14) employed for the supermembrane in the previous section (and corresponding to $\tilde{\omega}^{rs}_\mu = \tilde{F}_{\mu \nu \rho \sigma} = \psi^\mu = 0$ and $e^\mu_r = \delta^\mu_r$).

Furthermore, the fact that $E^a_\alpha = \delta^a_\alpha$ in this order implies that the local Lorentz transformation will be accompanied henceforth by a corresponding diffeomorphism given by

$$\Xi^\alpha(\lambda) = - \frac{1}{4} \lambda^a (\Gamma_r \theta)^a.$$

(3.12)
This term ensures that the various superspace components take a covariant form with respect to the local Lorentz transformations parametrized by \( \lambda^r \). In due course (3.12) will also arise at higher orders in \( \theta \) in the gauge completion procedure, as the supersymmetry commutator contains a field-dependent Lorentz transformation. A corresponding phenomenon does not occur for the 11-dimensional diffeomorphisms and the tensor gauge transformations parametrized by \( \xi^\mu \) and \( \xi_{\mu\nu} \), which do not entangle with other superspace components. This is so because the initial conditions (3.5) are fully covariant with respect to these transformations.

Before continuing we note that the components of \( \Xi^A = \Xi^M E_M^A \) remain field-independent. One expects that these tangent-space expressions will be supercovariant, so that the gravitino or the spin-connection fields cannot appear explicitly (for a discussion of this property, see \([21, 23]\)). The field-independent values for these expressions are given by

\[
\Xi^M E_M^r = 2 \bar{\theta} \Gamma^r \epsilon(x), \quad \Xi^M E_M^a = \epsilon^a(x). \tag{3.13}
\]

The above result can be regarded to some extent as a gauge condition. To see this, one may verify that (in this order of \( \theta \)) it implies that the ambiguities encoded in \( \mathcal{H}_{r_1 \ldots r_n} \) vanish. In the next section we will confirm the validity of the first of these relations to order \( \theta^2 \). The second relation, however, will receive contributions proportional to \( \hat{F}_{rstu} \) (written with flat indices). We will refrain from calculating these terms as they are not directly relevant for the purpose of this paper.

Let us now turn to the tensor field. The supersymmetry commutator for the component fields gives rise to a field-dependent tensor gauge transformation. Such a gauge transformation can arise because the tensor field is subject to both superspace diffeomorphisms and tensor gauge transformations. The commutator of a diffeomorphism and a tensor gauge transformation gives again a tensor gauge transformation and this leads to the component result. Hence the result (3.8) is incomplete for the tensor field and there is an extra tensor transformation given by

\[
\Xi_{MN}(\epsilon_1, \epsilon_2) = \Xi^P(\epsilon_2) \partial_P \Xi_{MN}(\epsilon_1) + 2 \partial_M \Xi^P(\epsilon_2) \Xi_{[P|N]}(\epsilon_1) + \delta(\epsilon_1) \Xi_{MN}(\epsilon_2) - (1 \leftrightarrow 2). \tag{3.14}
\]

Before evaluating this equation we first note that the transformation parameters \( \Xi_{MN} \) are only defined up to terms of the form \( \partial_M \Lambda_N \). We can use this feature to set all \( \Xi_{MN}(x, \theta = 0) \) other than \( \Xi_{\mu\nu}(x, \theta = 0) \) to zero (for this one chooses the \( \Lambda_M \) linear in \( \theta \)). With this simplification we compare (3.14) at \( \theta = 0 \) to the tensor component in the supersymmetry algebra (2.3) and we find

\[
\Xi_{\mu\nu}(\epsilon) = \bar{\epsilon} (C_{\mu\rho} \Gamma^\rho + \Gamma_{\mu\nu}) \theta + \sum_{n=0,3,4} \bar{\epsilon} \Gamma^{r_1 \ldots r_n} \theta \mathcal{H}_{\mu\nu r_1 \ldots r_n} + \mathcal{O}(\theta^2),
\]
\[ \Xi_{\mu\alpha}(\epsilon) = \sum_{n=0,3,4} \bar{\epsilon} \Gamma^{r_1 \cdots r_n} \theta \mathcal{H}_{\mu\alpha r_1 \cdots r_n} + \mathcal{O}(\theta^2) , \]
\[ \Xi_{\alpha\beta}(\epsilon) = \sum_{n=0,3,4} \bar{\epsilon} \Gamma^{r_1 \cdots r_n} \theta \mathcal{H}_{\alpha\beta r_1 \cdots r_n} + \mathcal{O}(\theta^2) . \] (3.15)

Again there are undetermined terms characterized by \( \theta \)-independent quantities \( \mathcal{H}_{MN r_1 \cdots r_n} \).

With this result we consider the variations of \( B_{MNP} \) under a combined superspace diffeomorphism and tensor gauge transformation. However, first we note that we can set all the components of \( B_{MNP}(x, \theta = 0) \), with the exception of \( B_{\mu\nu\rho}(x, \theta = 0) \), to zero by suitable gauge transformations with parameters linear in \( \theta \). We then establish the following results,

\[ B_{\mu\nu\rho} = C_{\mu\nu\rho} - 6 \bar{\theta} \Gamma_{[\mu\nu} \psi_{\rho]} + \mathcal{O}(\theta^2) , \]
\[ B_{\mu\alpha\nu} = (\bar{\theta} \Gamma_{\mu\nu})_{\alpha} + \sum_{n=0,3,4} (\bar{\theta} \Gamma^{r_1 \cdots r_n})_{\alpha} \mathcal{H}_{\mu\nu r_1 \cdots r_n} + \mathcal{O}(\theta^2) , \]
\[ B_{\mu\alpha\beta} = 2 \sum_{n=0,3,4} (\bar{\theta} \Gamma^{r_1 \cdots r_n})_{(\alpha} \mathcal{H}_{\beta)\mu r_1 \cdots r_n} + \mathcal{O}(\theta^2) , \]
\[ B_{\alpha\beta\gamma} = 3 \sum_{n=0,3,4} (\bar{\theta} \Gamma^{r_1 \cdots r_n})_{(\alpha} \mathcal{H}_{\beta\gamma) r_1 \cdots r_n} + \mathcal{O}(\theta^2) . \] (3.16)

Subsequently we note that all the ambiguous terms proportional to \( \mathcal{H}_{MN r_1 \cdots r_n} \) can be removed by a gauge transformation with parameters proportional to \( \theta^2 \) and equal to

\[ \Xi_{MN} = -\frac{1}{2} \sum_{n=0,3,4} \bar{\theta} \Gamma^{r_1 \cdots r_n} \theta \mathcal{H}_{MN r_1 \cdots r_n} . \] (3.17)

Hence we drop these terms here so that also the results pertaining to the tensor field agree with the flat-space values (2.14) used in the previous section.

4 Higher-order contributions

So far our results are in agreement with those of [24]. In this section we determine the higher-order contributions and go beyond the results reported in the literature. In higher orders a number of new features enters, which did not play a role in the previous section. First of all the Lorentz transformations acting on the vielbein will now become relevant as well as the supersymmetry variation of the fields in the transformation parameters when evaluating the supersymmetry commutators. The reason why the Lorentz transformations did not enter earlier is related to the fact that we did not consider the components of the superspace spin connection. The ambiguities noted in the previous section will persist, but we will no longer exhibit their explicit form in order to keep our expressions tractable. Nevertheless, we have convinced ourselves that they can be gauged away in the same fashion as before. The presence of higher-order spinor terms unavoidably
leads to the need of Fierz reorderings, which tend to be rather cumbersome in 11 dimensions. However, in all cases we could avoid explicit reorderings by making use of the well-known identity, which holds in 4, 5, 7 and 11 spacetime dimensions,

$$\bar{\psi}_1 \Gamma^\nu \psi_2 \bar{\psi}_3 \Gamma_{\mu \nu} \psi_4 = 0.$$  \hspace{1cm} (4.1)

Below we start by deriving the higher-order expressions for the vielbein and, in a second subsection, for the tensor field. We will not always, as before, completely exploit the supersymmetry commutator, but sometimes move directly to the field variations and confront their form with that induced by a superspace diffeomorphism combined with a Lorentz or with a tensor gauge transformation. In a third subsection we present a summary of all the terms obtained.

### 4.1 The vielbein at order $$\theta^2$$

We start with (3.8) for $$M = \mu$$ at order $$\theta^2$$, where we now must take into account the transformation of the component fields appearing in the superspace parameters. Using the lower-order results obtained previously and the value for $$\epsilon_3$$ given in (2.12), one can integrate the equation and obtains

$$\Xi^\mu (\epsilon) \bigg|_{g^2} = -\bar{\theta} \Gamma^\nu \epsilon \bar{\theta} \Gamma^\mu \psi_\nu.$$  \hspace{1cm} (4.2)

This result is not unique and defined up to an expression

$$\mathcal{H}_{\alpha \beta \gamma}^\mu \epsilon^\alpha \theta^\beta \theta^\gamma,$$  \hspace{1cm} (4.3)

with $$\mathcal{H}$$ a tensor antisymmetric in $$[\alpha \beta \gamma]$$. The procedure for fixing these ambiguities is the same as the one used in the previous section.

The $$M = \alpha$$ component of (3.8) proceeds in the same way, except that now also the Lorentz transformation in (2.12) enters (not through a tangent-space rotation, but through the diffeomorphism (3.12)),

$$\Xi^\alpha (\epsilon) \bigg|_{g^2} = \bar{\theta} \Gamma^\nu \epsilon \bar{\theta} \Gamma^\mu \psi_\nu \psi_\mu + \frac{1}{4} \bar{\theta} \Gamma^\nu \epsilon \hat{\omega}_\nu^{rs} (\Gamma_{rs} \theta)^\alpha + \epsilon^\beta N_{\beta}^\alpha.$$  \hspace{1cm} (4.4)

Here we also used the condition of vanishing super-torsion (2.6). The quantity $$N_{\beta}^\alpha$$ denotes terms proportional to $$\hat{F} \theta^2$$, which are much harder to integrate. They are controlled by the equation

$$\epsilon_2^2 \partial_\beta N_{\gamma}^\alpha \epsilon_1^\gamma - \bar{\theta} \Gamma^\mu \epsilon_2 (\Gamma_{\mu \nu \rho \sigma} \epsilon_1) \hat{F}_{\nu \rho \sigma \lambda} - (1 \leftrightarrow 2) = -\frac{1}{288} (\Gamma_{rs} \theta)^\alpha \epsilon_2 (\Gamma_{\nu \rho \sigma \lambda} + 24 \delta_\nu^\rho \delta_\sigma^\lambda) \epsilon_1 \hat{F}_{\nu \rho \sigma \lambda}.$$  \hspace{1cm} (4.5)

Leaving these terms aside for the moment, we continue with the vielbein transformations. The knowledge of the $$\theta^2$$-terms in $$\Xi^\mu$$ (c.f. (4.2)) suffices to evaluate the possible contributions to $$E_{\alpha}^{\alpha^*}$$. The local Lorentz transformations do
not contribute at this order in $\theta$ and one finds that all contributions cancel. This enables one to set (up to ambiguities)

$$E^r_\alpha \big|_{g_2} = 0.$$  

(4.6)

Subsequently one considers $E^r_\mu$ and obtains

$$E^r_\mu \big|_{g_2} = \tilde{\theta} \Gamma^r \left[ -\frac{1}{4} \tilde{\omega}_s^t \Gamma_{st} + T^{\nu\rho\sigma\lambda} \hat{F}_{\nu\rho\sigma\lambda} \right] \theta.$$  

(4.7)

In order to reconcile the variations with a superspace diffeomorphism, we had to include a tangent-space transformation defined by

$$\Lambda_{rs}(\epsilon) = \tilde{\epsilon} \Gamma_{\alpha} \hat{\omega}_{rs}^\alpha + \frac{1}{576} \tilde{\theta} \left( \Gamma_{rs}^{\nu\rho\sigma\lambda} \hat{F}_{\nu\rho\sigma\lambda} + 24 \Gamma_{\mu}^{\nu} \hat{F}_{\nu}^{rs} \right) \epsilon + \mathcal{O}(\theta^2).$$  

(4.8)

At this point we can verify that $\Xi^{M} E^r_M$ remains field-independent and given by the first equation of (3.13) up to terms of order $\theta^3$. As we already mentioned the second equation of (3.13) will acquire terms proportional to $\hat{F}_{rstu}$. As it turns out the vielbein component $E^r_a$ is only modified by $\hat{F}^{\theta^2}$-terms. Denoting these by $M^a_\alpha$, they are subject to the following condition,

$$\epsilon^\beta \partial_\beta M^a_\alpha = - \left\{ \tilde{\epsilon} \Gamma^{a}(T^{\nu\rho\sigma\lambda} \theta) + \frac{1}{576} \tilde{\theta} \left( \Gamma_{rs}^{\nu\rho\sigma\lambda} + 24 \delta^{r}_{\rho} \delta^{s}_{\sigma} \Gamma_{\rho\sigma} \right) \epsilon (\Gamma^{a})^\alpha \right\} \hat{F}_{\nu\rho\sigma\lambda}.$$  

(4.9)

However, neither the explicit form of these $\hat{F}^{\theta^2}$-corrections, nor the $\theta^2$-contributions to the supervielbein $E^r_\mu$, are very relevant from the membrane point of view, as they do not appear in the supermembrane action (2.13), which depends only on $\varepsilon^{ijk} \Pi^A_i \Pi^B_j \Pi^C_k B_{CBA} = \varepsilon^{ijk} \partial_i Z^M \partial_j Z^N \partial_k Z^P B_{PNM}$ and $g_{ij} = \Pi^r_i \Pi^s_j \eta_{rs}$. Therefore we refrained from determining their explicit form at this order of $\theta$.

### 4.2 The tensor field at order $\theta^2$

A brief perusal of the algebra involving the tensor gauge transformations based on (3.14) reveals the possible presence of $(\theta^2 \epsilon)$- and $(C \theta^2 \epsilon \psi)$- and $(C \theta^2 \epsilon \psi)$-terms in $\Xi_{\mu\alpha}$ and $(\theta^2 \epsilon \psi)$-terms in $\Xi_{\mu\nu}$. On the other hand no contributions are indicated for $\Xi_{\alpha\beta\gamma}$. However, we did not attempt to work out the tensor gauge parameters from the algebra, but instead proceeded directly to the variations of the tensor fields. From the variations of $B_{\alpha\beta\gamma}$ one finds

$$\Xi_{\alpha\beta\gamma}(\epsilon) \big|_{g_2} = B_{\alpha\beta\gamma} \big|_{g_2} = 0,$$  

(4.10)

up to tensor gauge transformations.

Subsequently one considers the variations to $B_{\mu\alpha\beta}$. These lead to the gauge parameter

$$\Xi_{\mu\alpha}(\epsilon) \big|_{g_2} = \frac{1}{6} \tilde{\theta} \Gamma^\nu \epsilon (\tilde{\theta} \Gamma_{\mu\nu})_\alpha + \frac{1}{6} (\theta \Gamma^\nu)_\alpha \tilde{\theta} \Gamma_{\mu\nu} \epsilon,$$  

(4.11)
and the gauge field components
\[ B_{\mu\nu}|_{g^2} = (\bar{\theta} \Gamma_{\mu\nu})_{(\alpha}(\bar{\theta} \Gamma_{\nu})_{\beta)} . \] (4.12)

In obtaining this result we reordered the fermions by making use of (4.1). Again these results are not unique and can be changed by a subsequent tensor gauge transformation with parameters proportional to $\theta^3$. In this gauge the expression for $B_{\mu\nu\rho}$ agrees with the flat-space result (2.14).

The variations of $B_{\mu\nu\rho}$ proceed in a similar way and we find
\[
\Xi_{\mu\nu}(\epsilon)|_{g^2} = \bar{\theta} \Gamma^\rho \epsilon \bar{\theta} (C_{\mu\nu} \Gamma^\rho + \Gamma_{\mu\nu}) \psi_\mu + \frac{4}{3} \bar{\theta} \Gamma^\rho \psi_{[\mu} \bar{\theta} \Gamma_{\nu] \rho \epsilon} + \frac{4}{3} \bar{\theta} \Gamma^\rho \epsilon \bar{\theta} \Gamma_{\rho[\mu} \psi_{\nu]} ,
\]
\[
B_{\mu\nu\rho}|_{g^2} = -\frac{8}{3} \bar{\theta} \Gamma^\rho \psi_{[\mu} (\bar{\theta} \Gamma_{\nu] \rho)_{\alpha} + \frac{1}{3} (\bar{\theta} \Gamma^\rho)_{\alpha} \bar{\theta} \Gamma_{\rho[\mu} \psi_{\nu]} . \] (4.13)

Again these results are subject to change under tensor gauge transformations. We used (4.1), just as in the evaluation of the remaining component, $B_{\mu\nu\rho}$, which yields the result
\[
B_{\mu\nu\rho}|_{g^2} = -3 \bar{\theta} \Gamma_{\mu\nu} \left[ -\frac{1}{4} \hat{\omega}_{[\rho}^s \Gamma_{rs} + T_{\rho[\sigma \lambda \kappa \tau} \hat{F}_{\sigma \lambda \kappa \tau]_{\theta}} \right] \theta - 12 \bar{\theta} \Gamma_{\sigma[\mu} \psi_{\nu] \bar{\theta} \Gamma^\sigma \psi_{\rho]} . \] (4.14)

### 4.3 Summary of the results

In this subsection we summarize the combined results of this and the previous section. We first present the expressions for the vielbein and the antisymmetric tensor field. Subsequently we give the expressions for the superspace transformations in terms of the component fields and transformation parameters. As the 11-dimensional coordinate transformations act in the standard way, we only list the superspace parameters corresponding to supersymmetry and local Lorentz transformations.

At order $\theta^2$ we have not fully determined the terms contributing to $E_{\mu}^a$ and neither did we fully determine the $\hat{F}$ $\theta^2$-terms in $\Xi^\alpha$ and $E_{\alpha}^a$. Our results are in agreement with those of [24] in corresponding orders of $\theta$. While high-rank tensors are of course absent in 4 dimensions, there is a clear similarity between our results and those in 4 dimensions [23].

#### 4.3.1 Vielbein and tensor field expressions

For the supervielbein $E_M^A$ we found the following expressions,
\[
E_{\mu}^r = e_{\mu}^r + 2 \bar{\theta} \Gamma^r \psi_\mu \\
\quad + \bar{\theta} \Gamma^r \left[ -\frac{1}{4} \hat{\omega}_{[\rho}^s \Gamma_{rs} + T_{\rho[\sigma \lambda \kappa \tau} \hat{F}_{\sigma \lambda \kappa \tau]_{\theta}} \right] \theta + \mathcal{O}(\theta^3) ,
\]
\[
E_{\mu}^a = \psi_{\mu}^a - \frac{1}{4} \hat{\omega}_{[\rho}^s \Gamma_{rs} (\Gamma_{s} \theta)^a + (T_{\mu[\rho \sigma \lambda \kappa \tau} \hat{F}_{\sigma \lambda \kappa \tau]_{\theta}})^a \theta + \mathcal{O}(\theta^2) ,
\]
\[
E_{\alpha}^r = - (\bar{\theta} \Gamma^r)_{\alpha} + \mathcal{O}(\theta^3) ,
\]
\[
E_{\alpha}^a = \delta_{\alpha}^a + M_{\alpha}^a + \mathcal{O}(\theta^3) . \] (4.15)
where $M_{\alpha}^a$ characterizes the $\hat{F}\theta^2$-contributions, which we did not evaluate explicitly. Observe that we determined $E_{\mu}^a$ only up to terms of order $\theta^2$. The result for the tensor field $B_{\mu\nu\rho}$ reads as follows,

\begin{align*}
B_{\mu\nu\rho} &= C_{\mu\nu\rho} - 6 \bar{\theta} \Gamma_{[\mu\nu}\psi_{\rho]} \\
&\quad - 3 \bar{\theta} \Gamma_{[\mu\nu} \left[ - \frac{1}{4} \bar{\omega}_{[\rho} \Gamma_{\sigma] \sigma^\lambda \kappa \tau} \hat{F}^{\lambda \kappa \tau} \right] \theta - 12 \bar{\theta} \Gamma_{\sigma[\mu} \psi_{\nu]} \bar{\theta} \Gamma^\sigma \psi_{\rho]} + \mathcal{O}(\theta^3), \\
B_{\mu\alpha} &= (\bar{\theta} \Gamma_{\mu\nu})_\alpha - \frac{2}{3} \bar{\theta} \Gamma_{\rho} \psi_{[\mu} (\bar{\theta} \Gamma_{\nu]} \psi_\rho)_\alpha + \frac{2}{3} (\bar{\theta} \Gamma_{\rho})_\alpha \bar{\theta} \Gamma_{\rho[\mu} \psi_{\nu]} + \mathcal{O}(\theta^3), \\
B_{\mu\beta\gamma} &= (\bar{\theta} \Gamma_{\mu\nu})_{(\alpha} (\bar{\theta} \Gamma_{\nu})_{\beta)} + \mathcal{O}(\theta^3), \\
B_{\alpha\beta\gamma} &= (\bar{\theta} \Gamma_{\mu\nu})_{(\alpha} (\bar{\theta} \Gamma_{\nu})_{\beta)} (\bar{\theta} \Gamma_{\gamma})_{\gamma) + \mathcal{O}(\theta^3).
\end{align*}

(4.16)

For completeness we included the $\theta^3$-term in $B_{\alpha\beta\gamma}$ which is known from the flat-superspace results.

### 4.3.2 Supersymmetry transformations

The supersymmetry transformations consistent with the fields specified above, are generated by superspace diffeomorphisms, local Lorentz transformations and tensor gauge transformations. The corresponding parameters are as follows. For the superspace diffeomorphisms are expressed by

\begin{align*}
\Xi^\mu(\epsilon) &= \bar{\theta} \Gamma^\mu \epsilon - \bar{\theta} \Gamma^\nu \epsilon \bar{\theta} \Gamma^\nu \psi_{\nu} + \mathcal{O}(\theta^3), \\
\Xi^\alpha(\epsilon) &= e^\alpha - \bar{\theta} \Gamma^\nu \epsilon \psi_{\mu}^\alpha \\
&\quad + \bar{\theta} \Gamma^\nu \epsilon \bar{\theta} \Gamma^\nu \psi_{\mu}^\mu + \frac{1}{4} \bar{\theta} \Gamma^\nu \epsilon \bar{\omega}_{s} (\Gamma_{rs} \theta)^\alpha + e^\beta N_{\beta}^\alpha + \mathcal{O}(\theta^3), (4.17)
\end{align*}

where $N_{\beta}^\alpha$ encodes the terms proportional to $\hat{F}\theta^2$. The Lorentz transformation is given by

\begin{equation}
\Lambda^{rs}(\epsilon) = \bar{\epsilon} \Gamma^\mu \theta \bar{\omega}_{s} (\Gamma_{s} \theta)^\alpha + \frac{1}{144} \bar{\theta} (\Gamma^{rs\mu\nu\rho\sigma} \hat{F}_{\mu\nu\rho\sigma} + 24 \Gamma_{\mu\nu} \hat{F}^{rs\mu\nu}) \epsilon + \mathcal{O}(\theta^2). (4.18)
\end{equation}

Finally, the tensor gauge transformations are parametrized by

\begin{align*}
\Xi_{\mu\nu}(\epsilon) &= \bar{\epsilon} (C_{\mu\nu} \Gamma^\rho - \Gamma_{\mu\nu}) \theta + \bar{\theta} \Gamma^\rho \epsilon \bar{\theta} (C_{\mu\nu} \Gamma^\sigma + \Gamma_{\mu\nu}) \psi_{\rho} + \frac{1}{3} \bar{\theta} \Gamma^\rho \psi_{\mu} \bar{\theta} \Gamma_{\nu] \rho} \epsilon \\
&\quad + \frac{2}{3} \bar{\theta} \Gamma^\rho \epsilon \bar{\theta} \Gamma_{\rho[\mu} \psi_{\nu]} + \mathcal{O}(\theta^3), \\
\Xi_{\mu\alpha}(\epsilon) &= \frac{1}{6} \bar{\theta} \Gamma^\nu \epsilon (\bar{\theta} \Gamma_{\mu\nu})_{\alpha} + \frac{1}{6} (\bar{\theta} \Gamma^\nu)_{\alpha} \bar{\theta} \Gamma_{\mu\nu} \epsilon + \mathcal{O}(\theta^3), \\
\Xi_{\alpha\beta}(\epsilon) &= \mathcal{O}(\theta^3). (4.19)
\end{align*}

### 4.3.3 Local Lorentz transformations

Local Lorentz transformations are generated by a superspace local Lorentz transformation combined with a diffeomorphism. The corresponding expressions are given by

\begin{align*}
\Lambda^{rs}(\lambda) &= \lambda^{rs}, \\
\Xi^\alpha(\lambda) &= -\frac{1}{4} \lambda^{rs} (\Gamma_{rs} \theta)^\alpha. (4.20)
\end{align*}
5 The supermembrane in background fields

The initial supermembrane action (2.13) is manifestly covariant under independent superspace diffeomorphisms, tangent-space Lorentz transformations and tensor gauge transformations. For the specific superspace fields associated with 11-dimensional on-shell supergravity, this is no longer true and one has to restrict oneself to the superspace transformations corresponding to the component supersymmetry, general-coordinate, local Lorentz and tensor gauge transformations. When writing (2.13) in components, utilizing the expressions found in the previous sections, one thus obtains an action that is covariant under the restricted superspace diffeomorphisms (4.17) acting on the superspace coordinates $Z^M = (X^\mu, \theta^a)$ (including the spacetime arguments of the background fields) combined with usual transformations on the component fields (we return to this point shortly). Note that the result does not constitute an invariance. Rather it implies that the actions corresponding to two different sets of background fields that are equivalent by a component gauge transformation, are the same modulo a reparametrization of the supermembrane embedding coordinates. More precisely, if we denote the superspace coordinates by $\phi$ and the background by $G(\phi)$, then the action satisfies $S[\phi, G] = S[\phi', G']$, where $G'$ is related to $G$ by a component supersymmetry transformation. Of course, when considering a background that is invariant under (a subset of) the component transformations (so that $G = G'$), then the action will be invariant under the corresponding change of the supercoordinates.

Using the previous results we may now write down the complete action of the supermembrane coupled to background fields up to order $\theta^2$. Direct substitution leads to the following result for the supervielbein pull-back,

\[
\Pi^I_i = \partial_i X^\mu \left( e^r_\mu + 2 \bar{\theta} \Gamma^r \psi_\mu - \frac{1}{4} \bar{\theta} \Gamma^{rst} \hat{\omega}_{\mu st} + \bar{\theta} \Gamma^r T_\mu^{\nu\rho\sigma\lambda} \hat{F}_{\nu\rho\sigma\lambda} \right) \\
+ \bar{\theta} \Gamma^r \partial_i \theta + O(\theta^3),
\]

\[
\Pi^a_i = \partial_i X^\mu \left( \psi^a_\mu - \frac{1}{4} \bar{\omega}^r \rho (\Gamma_r \theta)^a + (T_\mu^{\nu\rho\sigma\lambda} \theta)^a \hat{F}_{\nu\rho\sigma\lambda} \right) \\
+ \partial_i \theta^a + O(\theta^3).
\] (5.1)

Consequently the induced metric is known up to terms of order $\theta^3$.

Furthermore the pull-back of the tensor field equals

\[
-\frac{1}{6} \varepsilon^{ijk} \Pi^A_j \Pi^B_k B_{CBA} = -\frac{1}{6} \varepsilon^{ijk} \partial_i Z^M \partial_j Z^N \partial_k Z^P B_{P_{NM}} = \\
\frac{1}{6} dX^{\mu\nu\rho} \left[ C_{\mu\rho\nu} - 6 \bar{\theta} \Gamma_{\mu\nu} \psi_\rho + \frac{1}{4} \bar{\theta} \Gamma_{\rho s} \Gamma_{\mu\nu} \theta \bar{\omega}^r \rho s \\
- 3 \bar{\theta} \Gamma_{\mu\nu} T_\rho^{\sigma\lambda\kappa\tau} \theta \hat{F}_{\sigma\lambda\kappa\tau} - 12 \bar{\theta} \Gamma_{\sigma\mu} \psi_\nu \theta \Gamma^\rho \psi_\rho \right] \\
- \varepsilon^{ijk} \bar{\theta} \Gamma_{\mu\nu} \partial_i \theta \left[ \frac{1}{2} \partial_j X^\mu (\partial_j X^\nu + \bar{\theta} \Gamma^\nu \partial_j \theta) + \frac{1}{6} \bar{\theta} \Gamma^\rho \partial_i \theta \tilde{\partial}_j \Gamma^\rho \psi_\nu \\
+ \frac{1}{6} \varepsilon^{ijk} \partial_i X^\mu \partial_j X^\nu \left[ 4 \bar{\theta} \Gamma_{\rho\nu} \partial_i \theta \tilde{\partial}_j \Gamma^\rho \psi_\nu - 2 \bar{\theta} \Gamma^\rho \partial_i \theta \tilde{\partial}_j \Gamma_{\rho\nu} \psi_\nu \right] + O(\theta^3), \quad (5.2)
\]
where we have introduced the abbreviation $dX^{\mu\nu\rho} = \varepsilon^{ijk} \partial_i X^\mu \partial_j X^\nu \partial_k X^\rho$ for the world-volume form. Observe that we included also the terms of higher-order $\theta$-terms that were determined in previous sections and listed in (4.16). We will return to these terms at the end of this section. The first formula of (5.1) and (5.2) now determine the supermembrane action (2.13) up to order $\theta^3$.

As an illustration of what we stated at the beginning of this section, we consider the effect of the superspace diffeomorphisms (4.17) on $\Pi^A_i$. We only need the variations to first order in $\theta$, so that we substitute $X^\mu \rightarrow X^\mu + \bar{\theta} \Gamma^\mu \epsilon \psi_\mu$ and $\theta \rightarrow \theta + \bar{\theta} \Gamma^{\mu} \epsilon \psi_\mu$ into (5.1). For $\Pi^r_i$ this induces a variation which can be rewritten as

$$
\delta \Pi^r_i = \partial_i X^\mu \left[ \delta e^r_\mu + 2 \bar{\theta} \Gamma^r \delta \psi_\mu \right] - \Lambda_{rs}^r (\epsilon) \Pi^s_i + O(\theta^2).
$$

(5.3)

The first term on the right-hand side represents the change of $\Pi^r_i$ under the supersymmetry variations (2.3) of the background fields. The second term represents a Lorentz transformation whose parameter is given by (4.18). For the induced metric, given by $g_{ij} = \Pi^r_i \Pi^s_j \eta_{rs}$, the Lorentz transformation drops out, so that the effect of the coordinate change of $(X^\mu, \theta^a)$ is the same as when performing a supersymmetry transformation of the background fields. This implies that the first term in the supermembrane action (2.13) has indeed the required transformation behaviour.

A similar result holds for the variation of $\Pi^a_i$ under the coordinate change, but only in zeroth-order in $\theta$, as we have not determined all the $\theta^2$-contributions. An explicit calculation gives

$$
\delta \Pi^a_i = \partial_i X^\mu \left[ \delta \psi^a_\mu - \bar{\theta} \Gamma^\mu \epsilon \hat{\psi}^a_\mu \right] + \bar{\theta} \Gamma^\mu \epsilon \left( \frac{1}{4} \hat{\omega}^a_{\mu} \Gamma_{rs} - T^a_{\mu \rho \sigma \lambda} \tilde{F}_{\nu \rho \sigma \lambda} \right) \Pi^r_i \right)^a,
$$

(5.4)

where $\hat{\psi}^a_\mu$ is the supercovariant curl of the gravitino field. As expected, the terms linear in $\theta$ do not exhibit the same systematics. But we do not need the expression for $\Pi^a_i$ for the supermembrane action, so that this issue is not of immediate relevance.

Let us now consider the variation of the second term (5.2) in the supermembrane action (2.13). Its variation takes the form

$$
\delta \left( - \frac{1}{6} \varepsilon^{ijk} \Pi^A_i \Pi^B_j \Pi^C_k B_{CBA} \right) = - \partial_i \left[ \frac{1}{2} \varepsilon^{ijk} \partial_j X^\mu \partial_k X^\nu \bar{\epsilon} (C_{\mu
u\rho} \Gamma^\rho + \Gamma_{\mu
u}) \theta \right] + \frac{1}{6} dX^{\mu
u\rho} \left[ \delta C_{\mu
u\rho} - 6 \bar{\theta} \delta (\Gamma_{\mu\nu} \psi_\rho) \right] + O(\theta^2).
$$

To show that all explicit $\psi^2$-terms cancel, we again made use of (4.1). The above results were guaranteed to hold on the basis of the procedure followed in sections 3 and 4, the next feature is independent of that and concerns
the \( \kappa \)-invariance of the action. The \( \kappa \)-symmetry transformations are defined in the unrestricted superspace and will be given below. In principle, it should be possible to derive the transformation rules in the gauge-fixed superspace situation that we are working with. However, it is not necessary to do so, because we are only interested in establishing the invariance of the action. Both the original and the gauge-fixed action should be \( \kappa \)-symmetric, so that we can just use the original superspace diffeomorphisms corresponding to \( \kappa \)-symmetry and substitute them in the gauge-fixed action. These \( \kappa \)-transformations take the form of superspace coordinate changes defined by:

\[
\delta Z^M E_M^r = 0, \quad \delta Z^M E_M^a = (1 - \Gamma)^a_b \kappa^b, \tag{5.6}
\]

where \( \kappa^a(\zeta) \) is a local fermionic parameter and the matrix \( \Gamma \) is defined by

\[
\Gamma = \frac{\varepsilon^{ijk}}{6\sqrt{-g}} \Pi^r_i \Pi^s_j \Pi^t_k \Gamma_{rst}, \tag{5.7}
\]

with \( g = \det g_{ij} \). It satisfies the following properties,

\[
\Gamma^2 = 1, \quad \Gamma \Gamma = \frac{g_{ij}}{2\sqrt{-g}} \varepsilon^{ijkl} \Pi^r_i \Pi^s_j \Gamma^t_k \Gamma^u_l \theta^v. \tag{5.8}
\]

Therefore the matrix \((1 - \Gamma)\) in (5.6) is a projection operator. As a consequence, this allows one to gauge away half of the \( \theta \) degrees of freedom.

It is advantageous to expand the \( \kappa \)-transformations (5.6) as follows,

\[
\delta X^\mu = \bar{\kappa} - \Gamma^\mu \theta - \bar{\kappa} \Gamma^\nu \theta \bar{\psi}_\nu + \mathcal{O}(\theta^3), \quad \delta \theta = \kappa_+ + \psi_+ \bar{\theta} \Gamma^\mu \kappa_- + \mathcal{O}(\theta^2), \tag{5.9}
\]

where we have introduced the chiral spinor \( \kappa_- = (1 - \Gamma)\kappa \). We stress that we are not making any approximation in \( \Gamma \), which depends on the background fields and on \( \theta \) in a complicated fashion. Note that we retain the \( \theta^2 \)-contributions to \( \delta X^\mu \) for reasons that will become clear shortly. Under the variations (5.9) we then derive the following result,

\[
\delta \Pi^r_i = 2 \bar{\kappa} - \Gamma^r \delta \theta + \Pi^s_i \left[ 2 \bar{\kappa} - \Gamma^r \psi_s - 4 \bar{\theta} \Gamma^\mu \psi_s - \frac{1}{2} \bar{\kappa} - \Gamma^r \Gamma^u \theta \bar{\omega}_s^{t u} + \bar{\kappa} - \Gamma^r \theta \bar{\omega}_s^{r u} + \bar{\kappa} - \Gamma^r \bar{\theta} \Gamma^\nu \psi_s + 4 \bar{\theta} \Gamma^\nu \psi_s \bar{\kappa} - \Gamma^r \theta \bar{\omega}_s^{r u} \right] + \mathcal{O}(\theta^2). \tag{5.10}
\]

Here we rewrote the right-hand side in terms of \( \Pi^s_i \), rather than \( \partial_i X^\mu \). This is the origin of the explicit \( \psi^2 \)-term; all other explicit \( \psi^2 \)-terms cancel. It is now straightforward to obtain the \( \kappa \)-variation of the induced metric,

\[
\delta g_{ij} = 4 \Pi^r_i \bar{\kappa} - \Gamma^r \partial_j \theta + \Pi^s_i \Pi^s_j \left[ 4 \bar{\kappa} - \Gamma^r \psi_s - 8 \bar{\theta} \Gamma^\mu \psi_s - \bar{\kappa} - \Gamma^r \psi_\mu \right] + \mathcal{O}(\theta^2). \tag{5.11}
\]
Subsequently we consider the variation of the second term of the supermembrane action \( (2.13) \), which yields

\[
\delta \left( - \frac{1}{6} \varepsilon^{ijk} \Pi_i A^i \Pi_j B^j \Pi_k C^k B_{CBA} \right) = - \varepsilon^{ijk} \Pi_i \Pi_j \Pi_k B_{CBA} = - \varepsilon^{ijk} \Pi_i \Pi_j \Pi_k \bar{\kappa} - \Gamma_{rs} \partial_k \theta \\
+ \varepsilon^{ijk} \Pi_i \Pi_j \Pi_k \bar{\kappa} - \Gamma_{rs} \partial_k \theta \\
+ \frac{1}{4} \bar{\kappa} - \Gamma_{rs} \Gamma_{uv} \theta \hat{\omega}_v^u - \bar{\kappa} - \Gamma_{rs} T_i^{\mu \rho \sigma \lambda} \theta \hat{F}_{\mu \rho \sigma \lambda} + O(\theta^2) \quad (5.12)
\]

up to a total derivative which we will discuss shortly. In deriving this result, we again used \((1.1)\) to reorder the \(\psi^2\)-terms.

At this point it is rather easy to establish the \(\kappa\)-invariance of the action. Observing that the variation of the first term in the action is equal to \(-\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g^{ij} \delta g_{ij}\), replacing \(\bar{\kappa}\) by \(-\bar{\kappa}\) \(\Gamma\) and making use of the second equation \((5.8)\), one verifies directly that the variations of the two terms in the Lagrangian \((2.13)\) vanish under \(\kappa\)-symmetry, up to a surface term.

Hence we have verified that up to first order in \(\theta\) the supermembrane action transforms as a scalar under supersymmetry and is invariant under \(\kappa\)-symmetry, up to a world-volume surface term. Let us stress that at this order no need arose to make use of the 11-dimensional supergravity field equations in verifying the \(\kappa\)-symmetry of the action. We expect that this will be necessary at higher orders as is indicated by the analysis of \([1]\). In order to check these symmetries at second order we would have to know the supermembrane action up to third order in \(\theta\), as the supersymmetry as well as the \(\kappa\)-transformation of \(\theta\) contain terms of zeroth order in \(\theta\).

Finally, let us return to the surface terms that did not receive much attention earlier when establishing the supersymmetry and the \(\kappa\)-invariance. These terms are relevant when considering the open supermembrane \([35, 36, 37, 38]\). As it turns out, we can easily determine the surface term, including some contributions of higher order in \(\theta\). For \(\kappa\)-symmetry, we observe that all variations proportional to \(\partial_i \kappa\) must be generated by the surface term. Assuming that \(\kappa\)-symmetry is valid, the surface contributions can therefore be evaluated by simply collecting all variations proportional to world-volume derivatives of \(\kappa\). Moreover, these terms can only come from the Wess-Zumino-Witten sector, because the pull-back \(\Pi_i^r\) does not generate derivatives of \(\kappa\) owing to the first equation of \((5.6)\).

For supersymmetry, \(\Pi_i^r\) and the Wess-Zumino-Witten term are separately invariant in the sense explained earlier and surface terms can only come from the latter. One can thus use the same strategy and collect the variations proportional to derivatives of the supersymmetry parameter. However, now these terms come from two sources, namely from \(\partial_i Z^M\) and from the gravitino terms. To see how this works, one may compare to the calculation leading to \((5.5)\).

Because we know the variations \(\delta X^\mu\) to second and \(\delta \theta\) to first order in \(\theta\) for both supersymmetry and \(\kappa\)-symmetry, we can determine the surface contributions to order \(dX \wedge dX \theta^2\) and \(dX \wedge d\theta \theta^2\), while, for \(\kappa\)-symmetry, we also reliably
calculate those terms of the form $d\theta \wedge d\theta^2$ which are present in the flat-superspace case. In this way we find the following results. The surface term associated with $\kappa$-symmetry is given by

$$\int_{\partial M} \left[ dX^\mu \wedge dX^\nu \left[ -\frac{1}{2} \bar{\theta} (C_{\mu\nu\rho} \Gamma^\rho + \Gamma_{\mu\nu}) \kappa_- - \frac{1}{2} \bar{\psi}_\sigma (C_{\mu\nu\rho} \Gamma^\rho + \Gamma_{\mu\nu}) \theta \bar{\theta} \Gamma^\sigma \kappa_- \right. \\
+ \frac{1}{3} \bar{\psi}_\mu (\Gamma^\rho \theta \bar{\theta} \Gamma_{\rho\nu} + \Gamma_{\rho\mu} \theta \bar{\theta} \Gamma^\nu) \kappa_- \right. \\
+ \frac{1}{6} \bar{\theta} \Gamma^\mu d\theta \wedge d\bar{\theta} \left[ \Gamma^\nu \theta \bar{\theta} \Gamma_{\nu\mu} + \Gamma_{\mu\nu} \theta \bar{\theta} \Gamma^\nu + \cdots \right] \kappa_- \right].$$ (5.13)

After taking into account the gravitino variations as described above, the surface term associated with supersymmetry can be evaluated and equals

$$\int_{\partial M} \left[ dX^\mu \wedge dX^\nu \left[ \frac{1}{2} \bar{\theta} (C_{\mu\nu\rho} \Gamma^\rho + \Gamma_{\mu\nu}) \epsilon + \frac{1}{2} \bar{\psi}_\sigma (C_{\mu\nu\rho} \Gamma^\rho + \Gamma_{\mu\nu}) \theta \bar{\theta} \Gamma^\sigma \epsilon \right. \\
- \frac{2}{3} \bar{\psi}_\mu (\Gamma^\rho \theta \bar{\theta} \Gamma_{\rho\nu} + \Gamma_{\rho\mu} \theta \bar{\theta} \Gamma^\nu) \epsilon \right] \\
- \frac{1}{6} dX^\mu \wedge d\bar{\theta} \left[ \Gamma^\nu \theta \bar{\theta} \Gamma_{\nu\mu} + \Gamma_{\mu\nu} \theta \bar{\theta} \Gamma^\nu + \cdots \right] \epsilon \right].$$ (5.14)

Here the terms proportional to $d\theta \wedge d\theta$ cannot be determined, because the corresponding gravitino terms have not been obtained to sufficiently high order of $\theta$. The background-independent terms in (5.14) coincide with those given by [37] in the flat-superspace case. This provides another nontrivial verification of the correctness of our results.

The difference with the corresponding flat-case expressions [36, 37, 38] resides in the coupling to $C_{\mu\nu\rho}$ and $\psi_\mu$. However, most of the surface terms cancel by assuming a “membrane D-$p$-brane” at the boundary and imposing the Dirichlet conditions

$$\partial_{\parallel} X^{\tilde{m}} = 0, \quad \text{for } \tilde{m} = p+1, \ldots, 10,$$ (5.15)

where $\partial_{\parallel}$ defines the world-volume derivative tangential to the boundary surface. For the fermionic quantities $\theta, \psi_m, \epsilon$ and $\kappa_-$ one can impose a projection such that the only nonzero fermionic bilinears involve a product of an odd number of gamma matrices $\Gamma^m$, where $m = 0, 1, \ldots, p$. This requires $p$ to take the values 1, 5 or 9 [37, 38]. One is thus left with terms proportional to $C_{mnq}$ living on the $p$-brane at the boundary, which can presumably be dealt with by a deformation of these fermionic conditions [37]. Of course, these terms are subject to the tensor gauge transformations of 11-dimensional supergravity. This issue can be resolved by having additional degrees of freedom at the boundary of the membrane. In this connection it is relevant to observe that the 11-dimensional supergravity action itself is also not invariant under tensor gauge transformations in the presence of a boundary. Some of this has been discussed, for instance, in [11, 36].
6 Discussion

In this paper we constructed the superspace vielbein and the tensor gauge field of 11-dimensional on-shell supergravity in terms of its component fields to higher orders in $\theta$ coordinates. This enabled us to write down the 11-dimensional supermembrane action coupled to a nontrivial supergravity component-field background to second order in $\theta$. We then displayed its transformation properties under supersymmetry and exhibited the invariance of the supermembrane action under the local fermionic $\kappa$-symmetry, yielding an independent check of our superspace results. Furthermore we obtained the leading background-dependent terms of the surface terms for open supermembranes.

Having this explicit form of the supermembrane action at one’s disposal now opens up a multitude of interesting applications. The most prominent next step is the study of the supermembrane degrees of freedom in background geometries. In analogy to the bosonic case discussed in this paper, the light-cone supermembrane turns out to be equivalent to a gauge theory of area-preserving diffeomorphisms coupled to background fields, modulo corresponding assumptions on the background geometry. This $U(\infty)$ gauge theory may then in turn be regularized by a supersymmetric $U(N)$ quantum-mechanical model in curved backgrounds. Whether or not this will shed some light on the problem of formulating matrix models in curved spacetime remains to be seen. A conceptually better posed problem concerns perhaps the membrane and the matrix models in a constant antisymmetric tensor background. Other investigations of the supermembrane will deal with specific background solutions with a certain amount of residual supersymmetry. Interesting candidates for such backgrounds are the membrane [29] and the fivebrane solution [30] of 11-dimensional supergravity, as well as solutions corresponding to the product of Anti-de-Sitter spacetimes with compact manifolds [19]. Coupling to AdS solutions appears especially appealing in view of the recent results on the duality of large-$N$ superconformal field theories and supergravity on a product of AdS space with a compact manifold [39].

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