PETERSSON PRODUCTS OF BASES OF SPACES OF CUSP FORMS AND ESTIMATES FOR FOURIER COEFFICIENTS

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Abstract. We prove a bound for the Fourier coefficients of a cusp form of integral weight which is not a newform by computing an explicit orthogonal basis for the space of cusp forms of given integral weight and level.

1. Introduction.

The Fourier coefficients $a(F, n)$ of a cusp form $F$ of integral weight $k$ for the group $\Gamma_0(M)$ are bounded from above by $\sigma_0(n)n^{k-\frac{1}{2}}$ if $F$ is a primitive form (also called normalized newform) by the famous Ramanujan-Petersson-Deligne bound. For applications one often needs bounds for an arbitrary cusp form which is a linear combination of old and new forms. Such bounds have first been given in special cases in [5, 6]. The first step for this is the construction of an explicit orthogonal basis for the space $S_k(\Gamma_0(M), \chi)$. Starting from the usual basis of translates of primitive forms and using the well known fact that translates of different primitive forms are pairwise orthogonal, one is left with the task to orthogonalize the translates of the same primitive form, in particular, one has to compute their Petersson scalar products. Choie and Kohnen in [4] and Iwaniec, Luo, Sarnak in [9] cover arbitrary integral weights, square free level and trivial character, using Rankin $L$-functions for the computation of the Petersson products of translates of a primitive form. By the same method, Rouymi [14] treated prime power level and trivial character. His approach was generalized to arbitrary levels and trivial character by Ng Ming Ho in his unpublished master thesis [12]. Blomer and Miličić in [3] treat Maassforms and holomorphic modular forms for arbitrary level and trivial character by the same method.

In this note we investigate the case of arbitrary level and arbitrary character with a rather elementary approach. In order to compute the Petersson product of two translates of the same primitive form we use the trace operator sending a form of level $M$ to a form of level $N$ dividing $M$. Together with the well known fact that the $p$-th Hecke operator on forms of level $N$ can be obtained by first translating the argument by a factor $p$ and then applying the trace operator from level $Np$ down to level $N$ this allows us to express the scalar products quite easily in terms of Hecke eigenvalues of the underlying primitive form. The formulas we get and the relations between the Hecke eigenvalues $\lambda_f(1, p^j)$ of a primitive form $f$ for varying $j$ imply then that each element of the orthogonal basis obtained by the Gram-Schmidt procedure involves only very few of the translates of its underlying primitive form. For forms of half integral weight our approach works in essentially the same way as far as the computation of the Petersson product of a Hecke eigenform with its translates is concerned. Since the theory of newforms is in this case completely...
known only for the Kohnen plus space in square free level, it is however not clear how large the part of the space of all cusp forms of a given arbitrary level is that is covered by our result.

We then use in the integral weight case the orthogonal basis to obtain an explicit bound for the Fourier coefficient \(a(F, n)\) of an arbitrary cusp form \(F\) in terms of the Petersson norm \(⟨F, F⟩\) and the level \(M\).

In applications to the theory of integral quadratic forms it is usually possible to compute or at least bound \(⟨F, F⟩\) for the cusp form \(F\) at hand (the difference between a genus theta series and a theta series), so that our result is directly applicable to such problems; this will be worked out separately.

An estimate for the Fourier coefficients in the half integral weight case could in principle be obtained in the same way as in the integral weight case discussed above as long as one has an explicit bound for the Fourier coefficients with square free index of a Hecke eigenform. Unfortunately most of the known estimates (see [2, Appendix 2] involve constants which are not explicitly known, and we prefer not to discuss this possibility in detail in the present paper.

This article is an extension of work from the master thesis of the second named author at Universität des Saarlandes, 2014.

After the first version of this article was posted in the matharxiv Ng Ming Ho sent us his master thesis, from which we also learnt of the previous work of Iwaniec, Luo and Sarnak and of Rouymi. We thank Ng Ming Ho for providing this information to us.

### 2. Trace operator and scalar products

Let \(N \mid M\) be integers and let \(χ\) be a Dirichlet character modulo \(N\); we denote the Dirichlet character modulo \(M\) induced by it by \(χ\) as well. We have induced characters on the groups \(Γ_0(N), Γ_0(M)\) given by

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto χ(d)
\]

as usual and denote these again by \(χ\).

For an integer \(k\) we denote by \(M_k(Γ_0(N), χ), S_k(Γ_0(N), χ)\) the spaces of modular forms respectively cusp forms of weight \(k\) and character \(χ\) for the group \(Γ_0(N)\). On \(S_k(Γ_0(N), χ)\) we consider the Petersson inner product given by

\[
⟨f, g⟩ := \langle f, g⟩_{Γ_0(M)} := \frac{1}{(SL_2(\mathbb{Z}) : Γ_0(N))} \int_{\mathcal{F}} f(x + iy)g(x + iy)y^{-k-2}dxdy,
\]

where \(\mathcal{F}\) is a fundamental domain for the action of \(Γ_0(N)\) on the upper half plane \(H \subseteq \mathbb{C}\) by fractional linear transformations. The normalization chosen implies that for \(N \mid M\) and \(f, g \in S_k(Γ_0(N), χ) \subseteq S_k(Γ_0(M), χ)\) we have \(⟨f, g⟩_{Γ_0(N)} = ⟨f, g⟩_{Γ_0(M)}\).

For \(γ = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})\) with \(\text{det}(γ) > 0\) we write as usual \(f|kγ(z) = \text{det}(γ)^{k/2}(cz + d)^{-k}f(\frac{az + b}{cz + d})\).

We define trace operators as in [10, 1]:
Definition 1. For $N | M$ and $\chi$ as above we put for $f \in M_k(\Gamma_0(M), \chi)$:

$$f|_{k \text{tr} M}^N = \frac{1}{(\Gamma_0(N) : \Gamma_0(M))} \sum_i \chi(\alpha_i) f|_{k \alpha_i},$$

where $\Gamma_0(N) = \bigcup \Gamma_0(M) \alpha_i$ is a disjoint coset decomposition.

Lemma 2. The definition above is independent of the choice of coset representatives. One has $f|_{k \text{tr} M}^N \in M_k(\Gamma_0(N), \chi)$ and $f|_{k \text{tr} M}^N \in S_k(\Gamma_0(N), \chi)$ if $f$ is cuspidal.

Proof. This is a routine calculation, see e.g. [1, Prop. 2.1]. □

Lemma 3. With notations as above one has for $f \in S_k(\Gamma_0(N), \chi)$, $g \in S_k(\Gamma_0(M), \chi)$:

$$\langle f, g \rangle = \langle f, g|_{k \text{tr} M}^N \rangle,$$

where the Petersson product on the left hand side is with respect to $\Gamma_0(M)$ and that on the right hand side is with respect to $\Gamma_0(N)$.

Proof. One has for $\alpha_i \in \Gamma_0(N)$:

$$\langle f, \chi(\alpha_i) g|_{k \alpha_i} \rangle = \langle \chi(\alpha_i) f|_{k \alpha_i^{-1}} \rangle, g \rangle = \langle f, g \rangle,$$

which implies the assertion. □

Definition 4. Let $\gcd(\ell, N) = 1$

a) With $\delta_\ell := \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \in \text{GL}_2^+(\mathbb{Q})$ we put

$$f|_{k \text{tr} M}^{V(\ell)} := f(\ell z) = \ell^{-k/2} f|_{k \delta_\ell(z)}$$

for $f \in M_k(\Gamma_0(N), \chi)$.

b) For $\ell | m$ we denote by $T_N(\ell, m)$ the Hecke operator given by the double coset

$$\Gamma_0(N) \begin{pmatrix} \ell & 0 \\ 0 & m \end{pmatrix} \Gamma_0(N).$$

c) For $\ell | m$ we denote by $T_N^*(m, \ell)$ the Hecke operator given by the double coset

$$\Gamma_0(N) \begin{pmatrix} m & 0 \\ 0 & \ell \end{pmatrix} \Gamma_0(N).$$

Remark. a) It is well-known (see [11, §4.5]) that on spaces of cusp forms of level $N$ the operator $T^*(m, \ell)$ is adjoint to $T(m, \ell)$ with respect to the Petersson inner product.

b) As usual we write

$$T_N(n) = \sum_{\ell m = n} T_N(\ell, m), \quad T_N^*(n) = \sum_{\ell m = n} T_N^*(m, \ell).$$

Lemma 5. Let $f \in S_k(\Gamma_0(N), \chi)$ and $d \in \mathbb{N}$. Then

$$(\Gamma_0(N) : \Gamma_0(Nd))(f|_{k \text{tr} M}^N) |_{k \text{tr} N^d}^d = \frac{1}{d^{k-1}} f|_{k \text{tr} N}(d, 1).$$
Proof. Putting $\Gamma_0(N) = \bigcup_{i} \Gamma_0(Nd) \alpha_i$ we have (using $\delta_d^{-1} \Gamma_0(N) \delta_d \cap \Gamma_0(N) = \Gamma_0(Nd)$ and the proof of Prop. 3.1 of [15]):

\[
\begin{align*}
\langle f|kT_{N}\rangle (d, 1) &= d^{k-1} \sum_{i} \chi(\alpha_i)[d^{-k/2}f|k\delta_d]|k\alpha_i \\
&= d^{k-1} \sum_{i} \chi(\alpha_i)(f|kV_d)|k\alpha_i \\
&= (\Gamma_0(N) : \Gamma_0(Nd))d^{k-1}(f|kV_d)|ktr_{N}^d.
\end{align*}
\]

\[\square\]

**Theorem 6.** Let $f \in S_k(\Gamma_0(N), \chi)$ be a primitive form, let $m, n \in \mathbb{N}$ with $\gcd(m, n) = d$. Then

\[
\langle f|kV_m, f|kV_n \rangle = \frac{\lambda(1, \frac{m}{d}) \lambda(1, \frac{n}{d})}{(\frac{m}{d})^k \prod_{\substack{p | mn \ \text{prime} \ \text{odd}}} (1 + \frac{1}{p})} \langle f, f \rangle,
\]

where we denote by $\lambda(1, \frac{m}{d})$ the $T(1, \frac{m}{d})$-eigenvalue of $f$ (and analogously for $\lambda(1, \frac{n}{d})$).

**Proof.** Since we have

\[
\langle f|kV_m, f|kV_n \rangle = \langle f|kV_m/dkV_d, f|kV_n/dkV_d \rangle
\]

\[
= d^{-k} \langle f|kV_m/dk\delta_d, f|kV_n/dk\delta_d \rangle
\]

\[
= d^{-k} \langle f|kV_m/d, f|kV_n/d \rangle,
\]

we can restrict attention the case

\[
d = \gcd(m, n) = 1.
\]

In that case we have

\[
\langle f|kV_m, f|kV_n \rangle = \langle f|kV_m, f|kV_n \mid tr_{mn}^{mN} \rangle
\]

\[
= \frac{1}{(\Gamma_0(mN) : \Gamma_0(mnN))} \frac{1}{n^{k-1}} \langle f|kV_m, f|kT_{mN}^*(n, 1) \rangle,
\]

where we used Lemma 3 and Lemma 4. We split $n$ as $n = \tilde{n}n'$ with $\gcd(\tilde{n}, N) = 1$ and $n'|N^\infty$ (i.e., $n'$ is divisible only by primes dividing $N$) and have

\[
T_{mN}^*(n, 1) = T_{mN}^*(\tilde{n}, 1)T_{mN}^*(n', 1)
\]

\[
f|kT_{mN}^*(\tilde{n}, 1) = f|kT_{N}^*(\tilde{n}, 1)
\]

\[
= \lambda(1, \tilde{n})f
\]

since $T_{N}^*(\tilde{n}, 1)$ is adjoint to $T_{N}^*(1, \tilde{n})$; in the same way we see

\[
f|kT_{mN}^*(n', 1) = f|kT_{N}^*(n', 1)
\]

\[
= \lambda(1, n')f.
\]

This gives us

\[
\langle f|kV_m, f|kV_n \rangle = \frac{1}{n^{k-1}(\Gamma_0(mN) : \Gamma_0(mnN))} \cdot \lambda(1, \tilde{n})\lambda(1, n') \langle f|kV_m, f \rangle
\]

\[
= \frac{\lambda(1, n)}{n^{k-1}(\Gamma_0(mN) : \Gamma_0(mnN))} \langle f, f|kV_m \rangle.
\]
In particular, we get

\[ \langle f, f \mid k V_m \rangle = \frac{\lambda(1, m)}{(\Gamma_0(N) : \Gamma_0(mN))^{m^{k-1}}} (f, f), \]

and thus (computing the group index in the denominator)

\[ \langle f \mid k V_m, f \mid k V_n \rangle = \frac{\lambda(1, n)\lambda(1, m)}{(mn)^{k-1}} \langle f, f \rangle, \]

\[ = \frac{\lambda(1, n)\lambda(1, m)}{(mn)^k} \prod_{\nu | mn} (1 + \frac{1}{\nu}) \langle f, f \rangle, \]

as asserted.

3. ORTHOGONAL BASES FOR SPACES OF CUSP FORMS

The formulas for the Petersson products derived in the previous section allow to construct an orthogonal basis by Gram Schmidt orthogonalization. As we learnt from Ng Ming Ho after version one of this article was posted, this has been done for trivial character in [14] for prime power level and in [12] for general level. For the sake of completeness and since [12] is at present not published we give here our version of it.

We recall first the well-known fact (see e.g. [11, Lemma 4.6.9] that the space \( S_k(\Gamma_0(M), \chi) \) has a basis consisting of the \( f \mid V_\ell \), where \( f \) runs over the primitive (normalized Hecke eigenforms) of levels \( N \mid M \) where \( N \) is divisible by the conductor of \( \chi \), and where \( \ell \) is a positive integer such that \( \ell N \) divides \( M \). We will call this basis the basis of translates of newforms.

**Lemma 7.** Let \( f \in S_k(\Gamma_0(N, \chi)) \) be a primitive form, let \( m_1, m'_1, m_2, m'_2 \) be positive integers with \( \gcd(m_1m'_1, m_2m'_2) = 1 \) put \( \tilde{f} = \frac{f}{\sqrt{\langle f, f \rangle}} \). Then

\[ \langle \tilde{f} \mid k V_{m_1}, \tilde{f} \mid k V_{m'_1} \rangle \cdot \langle \tilde{f} \mid k V_{m_2}, \tilde{f} \mid k V_{m'_2} \rangle = \langle \tilde{f} \mid k V_{m_1}, \tilde{f} \mid k V_{m'_2} \rangle \cdot \langle \tilde{f} \mid k V_{m_2}, \tilde{f} \mid k V_{m'_1} \rangle. \]

**Proof.** This follows directly from the theorem above. \( \square \)

It is well-known that for primitive forms \( f \neq g \) all translates of \( f \) by some \( V_{m'} \) are orthogonal to all translates of \( g \) by some \( V_{m'} \). Our lemma above shows that for a primitive form \( f \in S_k(\Gamma_0(N), \chi) \) for some \( N \mid M \) the space of translates of \( f \) in \( S_k(\Gamma_0(M), \chi) \) is isometric (with respect to Petersson norms) to the tensor products of the spaces \( W_{p_i}^{(f)} \) for the \( p_i \mid \frac{M}{N} \) consisting of \( p_i \)-power-translates of \( f \). An isometry is given by the unique linear map sending

\[ \tilde{f} \mid V_{p_1}^{r_1} \otimes \cdots \otimes \tilde{f} \mid V_{p_2}^{r_2} \text{ to } \tilde{f} \mid V_{p_1}^{r_1} \cdots p_2^{r_2}, \]

where \( \tilde{f} = \frac{f}{\sqrt{\langle f, f \rangle}}. \)

To construct an orthogonal basis for \( S_k(\Gamma_0(M), \chi) \) it suffices therefore to do that for each space \( W_{p_i} \).
Theorem 8. Let $f \in S_k(\Gamma_0(N), \chi)$ be a primitive form, put $\tilde{f} = \frac{f}{\sqrt{\langle f, f \rangle}}$, let $p$ be a prime number, $r \in \mathbb{N}$, let $W_p(f)$ be the space generated by $f, f|V_p, \ldots, f|V_{pr}$.

a) If $p \mid N$ the space $W_p(f)$ has an orthogonal basis consisting of

$$g_0 = \tilde{f}, \quad g_j = p^{j/2}(\tilde{f}|_{V_p} - \frac{\lambda(1, p)}{p^k}\tilde{f}|_{V_{p^j-1}}) \text{ for } 1 \leq j \leq r$$

with

$$\langle g_0, g_0 \rangle = 1, \quad \langle g_j, g_j \rangle = 1 - \frac{|\lambda(1, p)|^2}{p^k} \text{ for } 1 \leq j \leq r.$$

b) If $p \nmid N$ the space $W_p(f)$ has an orthogonal basis consisting of

$$g_0 = \tilde{f}, \quad g_1 = p^{k/2}(\tilde{f}|_{V_p} - \frac{\lambda(1, p)}{p^k(1 + \frac{1}{p})^2}\tilde{f}),$$

$$g_j = p^{j/2}(\tilde{f}|_{V_p} - \frac{\lambda(1, p)}{p^k}\tilde{f}|_{V_{p^j-1}} + \frac{\chi(p)}{p^{k+1}}\tilde{f}|_{V_{p^j-2}}) \text{ for } 2 \leq j \leq r$$

for $2 \leq j \leq r$, with

$$\langle g_0, g_0 \rangle = 1, \quad \langle g_1, g_1 \rangle = 1 - \frac{|\lambda(1, p)|^2}{p^k(1 + \frac{1}{p})^2},$$

$$\langle g_j, g_j \rangle = (1 - \frac{1}{p^2})(1 - \frac{|\lambda(1, p)|^2}{p^k(1 + \frac{1}{p})^2}) \text{ for } 2 \leq j \leq r.$$

Proof. a) In the case $p|N$ we have by Theorem 6 for $0 \leq i \leq j \leq r$:

$$\langle \tilde{f}|_{V_p}, \tilde{f}|_{V_{p^i}} \rangle = p^{-ik}\langle \tilde{f}, \tilde{f}|_{V_{p^{i-1}}} \rangle$$

$$= p^{-ik}\lambda(1, p^{i-1})$$

$$= p^{-ik}\lambda(1, p^j)$$

This gives for $1 \leq j \leq r$

$$\langle g_0, g_j \rangle = p^{j/2}\langle \tilde{f}, \tilde{f}|_{V_p} - \frac{\lambda(1, p)}{p^k}\tilde{f}|_{V_{p^j-1}} \rangle$$

$$= p^{j/2}\left(\frac{\lambda(1, p)}{p^k}\frac{\lambda(1, p^{j-1})}{p^{k+1}(j-1)}\right)$$

$$= 0,$$

because of $\lambda(1, p)\lambda(1, p^{j-1}) = \lambda(1, p^j)$ for $p|M$. 
Similarly, we see for $1 \leq i < j \leq r$

$$\langle g_i, g_j \rangle = p^{(i+j)/2} \langle \hat{f}_i | k V_{p^i}, \hat{f}_j | k V_{p^j} \rangle - \frac{\lambda(1, p)}{p^k} \langle \hat{f}_i | k V_{p^{i+1}}, \hat{f}_j | k V_{p^{j-1}} \rangle - \frac{\lambda(1, p)}{p^k} \langle \hat{f}_i | k V_{p^{i+1}}, \hat{f}_j | k V_{p^{j-1}} \rangle$$

$$= p^{(i+j)/2} (p^{-j} k \lambda(1, p)^{i-j}) + \frac{\lambda(1, p)^2}{p^{2k}} |p^{-j} \lambda(1, p)|^2 - \frac{\lambda(1, p)}{p^k} p^{-j} k \lambda(1, p)^{i-j}$$

$$= 0$$

because of $\lambda(1, p) \lambda(1, p^{i-j}) = \lambda(1, p)(p^{i-j})$ and $\lambda(1, p) \lambda(1, p^{i-j+1}) = \lambda(1, p) \lambda(1, p^{i-j}) = |\lambda(1, p)|^2 \lambda(1, p^{i-j})$.

Finally we have for $1 \leq j \leq r$

$$\langle g_j, g_j \rangle = p^k \langle \hat{f}_i | k V_{p^i}, \hat{f}_j | k V_{p^j} \rangle - \frac{\lambda(1, q)}{p^k} \langle \hat{f}_i | k V_{p^{i+1}}, \hat{f}_j | k V_{p^{j-1}} \rangle$$

$$= p^k (p^{-j} k \lambda(1, p)^{i-j}) + \frac{\lambda(1, p)^2}{p^{2k}} |p^{-j} \lambda(1, p)|^2 - \frac{\lambda(1, p)}{p^k} p^{-j} k \lambda(1, p)^{i-j}$$

$$= (1 - \frac{|\lambda(1, p)|^2}{p^k})$$

b) Consider now the case $p \nmid N$. From Theorem 6 we have for $0 \leq i < j \leq r$

$$\langle \hat{f}_i | k V_{p^i}, \hat{f}_j | k V_{p^j} \rangle = \frac{\lambda(1, p^{i-j})}{p^k(1 + \frac{1}{p})}$$

and

$$\langle \hat{f}_i | k V_{p^i}, \hat{f}_j | k V_{p^j} \rangle = \frac{1}{p^k}.$$ 

From Lemma 4.5.7 we have $\lambda(1, p^2) = \lambda(1, p)^2 - (p+1)p^{k-2} \chi(p)$ and $\lambda(1, p^i) = \lambda(1, p) \lambda(1, p^{i-1}) - p^{k-1} \chi(p) T(1, p^{j-2})$ for $j \geq 3$.

This gives us first

$$\langle g_0, g_1 \rangle = p^{k/2} \langle \hat{f}_i | k V_{p^i}, \hat{f}_j | k V_{p^j} \rangle$$

$$= 0.$$ 

For $i \geq 1$ we get

$$p^{-(i+1)/2} \langle \hat{f}_i | k V_{p^i}, g_{i+1} \rangle = \langle \hat{f}_i | k V_{p^i}, \hat{f}_i | k V_{p^i+1} \rangle - \frac{-\lambda(1, p)}{(p^{(i+1)/2} (1 + \frac{1}{p})}$$

$$+ \frac{\chi(p)}{p^{k+1}} \langle \hat{f}_i | k V_{p^i}, \hat{f}_j | k V_{p^{j-1}} \rangle$$

$$= \frac{\lambda(1, p)}{p^{(i+1)/2} (1 + \frac{1}{p})} - \frac{\lambda(1, p)}{p^{k+1}} + \frac{\lambda(1, p)}{p^{k+1}}$$

$$= 0$$

(using $\chi(p) \lambda(1, p) = \lambda(1, p)$, see [11] Theorem 4.5.4]).
For $0 \leq i < j \leq r$ with $j \geq 2 + i$ we obtain
\[
p^{-jk/2}(\tilde{f}|_{k}V_{p^{i}}, g_{j}) = (\tilde{f}|_{k}V_{p^{i}}, \tilde{f}|_{k}V_{p^{j}}) - \frac{\lambda(1,p)}{p^{k}}(\tilde{f}|_{k}V_{p^{i}}, \tilde{f}|_{k}V_{p^{j-1}})
\]
\[
+ \frac{\chi(p)}{p^{k+1}}(\tilde{f}|_{k}V_{p^{i}}, \tilde{f}|_{k}V_{p^{j-2}})
\]
\[
- \frac{\lambda(1,p^{j-i})}{p^{jk}(1 + \frac{2}{p})} - \frac{\lambda(1,p)\lambda(1,p^{j-i-1})}{p^{k}p^{(j-1)k}(1 + \frac{2}{p})} + \frac{\chi(p)\lambda(1,p^{j-i-2})}{p^{(j-2)k}(1 + \frac{2}{p})}
\]
\[
=0.
\]

Taken together we see that the $g_{i}$ form an orthogonal basis, it remains to compute the $\langle g_{i}, g_{i} \rangle$.
For this, $\langle g_{0}, g_{0} \rangle = 1$ is clear.

Next, we have
\[
\langle g_{1}, g_{1} \rangle = \langle g_{1}, p^{k/2} \tilde{f}|_{k}V_{p} \rangle
\]
\[
= p^{k}(\tilde{f}|_{k}V_{p}, \tilde{f}|_{k}V_{p}) - p^{k/2} \cdot \frac{\lambda(1,p)}{p^{k/2}(1 + \frac{2}{p})}(\tilde{f}, \tilde{f}|_{k}V_{p})
\]
\[
= 1 - \frac{\lambda(1,p)}{(1 + \frac{2}{p})} \cdot \frac{\lambda(1,p)}{p^{k}}
\]
\[
= 1 - \frac{\lambda(1,p)^{2}}{p^{k}(1 + \frac{2}{p})}.
\]

For $j \geq 2$ we see
\[
\langle g_{j}, g_{j} \rangle = \langle g_{j}, p^{jk/2} \tilde{f}|_{k}V_{p^{i}} \rangle
\]
\[
= p^{jk}(\tilde{f}|_{k}V_{p^{i}}, \tilde{f}|_{k}V_{p^{i}}) - \frac{\lambda(1,p)}{p^{k}}p^{jk}(\tilde{f}|_{k}V_{p^{i-1}}, \tilde{f}|_{k}V_{p^{i}})
\]
\[
+ \frac{\chi(p)p^{jk}}{(\tilde{f}|_{k}V_{p^{i-2}}, \tilde{f}|_{k}V_{p^{i}})}
\]
\[
= 1 - \frac{\lambda(1,p)^{2}}{p^{k}(1 + \frac{2}{p})} + \frac{\chi(p)\lambda(1,p^{2})}{p^{k+1}(1 + \frac{2}{p})}
\]

Using again $\lambda(1,p^{2}) = \lambda(1,p)^{2} - (p+1)p^{k-2}\chi(p)$ and $\chi(p)\lambda(1,p) = \lambda(1,p)$ we obtain the assertion. \hfill \square

4. Half integral weights

For positive integers $\kappa, N$ we denote by $M_{k}(4N, \chi)$ the space of holomorphic modular forms of weight $k = \kappa + \frac{1}{2}$ and character $\chi$ for the group $\Gamma_{0}(4N)$. For the relevant definitions and notations see [15]. In particular, we denote by $\mathfrak{G}$ the covering group of $GL_{2}(\mathbb{R})$ defined there and by $\gamma \mapsto \gamma^{*}$ the embedding of $\Gamma_{0}(4)$ into $\mathfrak{G}$ with image $\Delta_{0}(4)$. We can extend this embedding by putting $\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}^{*} = \begin{pmatrix} 1 & 0 \\ 0 & m^{2} \end{pmatrix}$ and $\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}^{*} = \begin{pmatrix} m^{2} & 0 \\ 0 & 1 \end{pmatrix}$ and $\gamma_{1}\alpha\gamma_{2}^{*} = \gamma_{1}^{*}\alpha^{*}\gamma_{2}^{*}$ for $\gamma_{1}, \gamma_{2} \in \Gamma_{0}(4)$ and $\alpha$ one of the above matrices of determinant $m^{2}$. In the sequel we will omit the superscript $\ast$ if this can cause no confusion.
We also use the action of double cosets of integral matrices of non zero square determinant on half integral weight modular forms of level $4N$ as defined there. In particular we have associated to the double coset with respect to $\Delta_0(4N)$ of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & m^2 \end{smallmatrix}\right), \ m \neq 1$ the Hecke operators $T_{4N}(1, m^2)$ which for $m \mid 4N$ coincide with the Hecke operators $U(m^2)$ sending $\sum_n a_f(n) e(nz)$ to $\sum_n a_f(nm^2) e(nz)$. By considering a modular form of level $4N$ as a form of level $\text{lcm}(m, 4N)$ we can let $U(m^2)$ act on forms of any level divisible by $4$. The operator $T_{4N}(m^2, 1)$ associated to the double coset of $\left(\begin{smallmatrix} m^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with respect to $\Gamma_0(4N)$ is adjoint to $T_{4N}(1, m^2)$ with respect to the Petersson product and coincides with it if one has $\gcd(m, 4N) = 1$, we write then as usual $T_{4N}(m^2)$. For $N$ dividing $M$ we have as in the integral weight case a trace operator $\text{tr}_M^N$ from $M_k(4M, \chi)$ to $M_k(4N, \chi)$ sending cusp forms to cusp forms and satisfying for cusp forms $f, g$

$$\langle f, g \rangle = \langle f, g \mid_k \text{tr}_M^N \rangle,$$

where the Petersson product on the left hand side is with respect to $\Gamma_0(M)$ and that on the right hand side is with respect to $\Gamma_0(N)$.

In the theory of half integral weight modular forms there are two different methods used for the definition of oldforms, namely using the operator $V_d$ as in the integral weight case (but with square determinant), raising the level by a factor $d^2$, and using the operator $U(p^2)$ for a prime not dividing the level, raising the level by a factor $p$. We start with the first method.

**Proposition 9.** Let $k = \kappa + \frac{1}{2}$ be half integral, let $f \in S_k(\Gamma_0(N), \chi)$ and $d \in \mathbb{N}$. Then

$$(\Gamma_0(N) : \Gamma_0(ND^2))(f |_k \text{V}_d) |_k \text{tr}_{N}^{N_d} = \frac{1}{d^{2(k-1)}} f |_k T_{N}(d^2, 1).$$

In particular, if $p$ is a prime with $p \nmid 4N$ and $f$ is an eigenform of the Hecke operator $T(p^2)$ with eigenvalue $\lambda_p$, we have

$$\langle p^2 + p \rangle (f |_k \text{V}_d) |_k \text{tr}_{N}^{N_p} = \frac{\lambda_p}{p^2(k-1)} f$$

and

$$\langle f, f \rangle |_k \text{V}_d = \frac{\lambda_p}{(p^2 + p)p^{2(k-1)}} \langle f, f \rangle.$$

**Proof.** This is proven in the same way as Lemma 5. Notice that in the case of half integral weight we can only use shift operators $V_d$ and Hecke operators $T_N^d(2, 1)$ with squares $d^2$.

**Proposition 10.** Let $k = \kappa + \frac{1}{2}$ be half integral, let $f \in S_k(\Gamma_0(N), \chi)$ and $p \nmid 4N$ be a prime. Then

$$f \mid_{k} U(p^2) |_{k} \text{tr}_{N}^{N_p} = p^2 f |_{k} T(p^2).$$

In particular, if $f$ is an eigenform of the Hecke operator $T(p^2)$ with eigenvalue $\lambda_p$, we have

$$\langle f, f \rangle |_{k} U(p^2) = p^2 \lambda_p \langle f, f \rangle.$$

**Proof.** With $\alpha_b = \left(\begin{smallmatrix} 1 & b \\ 0 & p \end{smallmatrix}\right)$ we have (see [10])

$$f |_k U(p^2) = f |_k \Gamma_0(4N) \alpha_0 \Gamma_0(4Np) = (p^2)^{\frac{p-1}{2}} \sum_{b=0}^{p^2-1} f |_k \alpha_b.$$
Moreover, we have $\Gamma_0(4N)\alpha_0\Gamma_0(4Np) = \cup_b\Gamma_0(4N)\alpha_b$, and by Section 3.1 of [15],
$\Gamma_0(4N)\alpha_0\Gamma_0(4Np)\Gamma_0(4Np)\Gamma_0(4N) = (p + 1)p^2\Gamma_0(4N)\alpha_0\Gamma_0(4N)$. From this the first assertion follows, and the second one follows in the same way as in the integral weight case, using Lemma 8 which is valid for half integral weight too. $\square$

As mentioned in the introduction, because of the lack of a satisfactory theory of oldforms and newforms in the half integral weight case we finish the investigation of this case here without trying to find good orthogonal bases for the space of all cusp forms.

5. Fourier coefficients of cusp forms

For the rest of this paper we concentrate again on the case of modular forms of integral weight $k$.

Theorem 11. The space $S_k(\Gamma_0(M), \chi)$ has an orthonormal basis $(h_1, \ldots, h_d)$, where each $h_i$ is an eigenform of all Hecke operators $T(p)$ for $p \nmid M$ and where the Fourier coefficients $a(h_i, n)$ satisfy

$$|a(h_i, n)| \leq 2\sqrt{\pi}e^{2\pi}\sigma_0(n)n^{\frac{k+1}{2}} \cdot M^{\frac{k+1}{2}} \prod_{p | M} (1 + \frac{1}{p})^3 \frac{1}{1 - \frac{1}{p^2}}.$$

Proof. We write $g_j = \phi_{p,j}(\tilde{f})$ for the basis vectors $g_j \in W_p(f)$ constructed in Theorem 8 and view $\phi_{p,j}$ as an operator transforming a modular form $g$ into the expression on the right hand side (with $g$ in place of $\tilde{f}$) of the definition of $g_j$. Obviously, these operators commute. As noticed after Lemma 4 the space $S_k(\Gamma_0(M), \chi)$ has then an orthogonal basis consisting of the $(\prod_{p \mid M} \phi_{p,j_p})(\tilde{f})$, where $f$ runs over the primitive forms of levels $N_f \mid M$ in $S_k(\Gamma_0(M), \chi)$ and $j_p \geq 0$ over the integers satisfying $N_f p^{j_p} \mid M$.

Examining the Proof of Theorem 8 we see that the Petersson norm of $(\prod_{i} \phi_{p_i,j_{p_i}})(\tilde{f})$ is equal to the product over $i$ of the norms of the $\phi_{p_i,j_{p_i}}(\tilde{f})$, which were computed in that theorem.

Analogously, we can decompose the computation of a bound for the Fourier coefficients of $(\prod_{i} \phi_{p_i,j_{p_i}})(\tilde{f})$ into the computation of such a bound for each $\phi_{p_i,j_{p_i}}(\tilde{f})$. Looking at the $g_j$ again, we have for $p | N_f$ (using $|a(f, n)| \leq \sigma_0(n)n^{\frac{k+1}{2}}$ and $|\lambda(1, p)| \leq \frac{1}{p^{\frac{k+1}{2}}}$ for primitive forms $f$ and $p \mid N_f$)

$$\langle f, f \rangle^\frac{k}{2} |a(g_0, n)| \leq \sigma_0(n)n^{\frac{k+1}{2}}$$

and

$$\langle f, f \rangle^\frac{k}{2} |a(g_j, n)| \leq \frac{n^{\frac{k}{2}}}{p^{j}} \sigma_0\left(\frac{n}{p^j}\right) \frac{n^{\frac{k+1}{2}}}{p^{j+1}}$$

$$+ p^{j+1} \sqrt{p} \sigma_0\left(\frac{n}{p^{j+1}}\right) \frac{n^{\frac{k+1}{2}}}{p^{j+1}} \frac{1}{p^{j+1}}$$

for $j \geq 1$, where the terms involving $\frac{n}{p^{j+1}}$, $\frac{n}{p^{j}}$ appear only if the respective quotient is integral. This gives $\langle f, f \rangle^\frac{k}{2} |a(g_j, n)| \leq \sigma_0(n)n^{\frac{k+1}{2}} p^j (1 + \frac{1}{p})$ for $j \geq 1$, and we see that this estimate holds indeed for all $j$. 

For $p \nmid N$ we obtain (with $|\lambda(1,p)| \leq 2p^{k-1}$ for $p \nmid N_f$):

$$
\langle f, f \rangle^{\frac{1}{2}} |a(g_0, n)| \leq \sigma_0(n)n^{k-1}
$$

$$
\langle f, f \rangle^{\frac{1}{2}} |a(g_1, n)| \leq p^{\frac{k}{2}} \sigma_0(\frac{n}{p})(\frac{n}{p})^{k-1}
$$

$$
+ 2\sigma_0(n)n^{k-1} \cdot \frac{p^{k-1}}{p^{\frac{k}{2}}(1 + \frac{1}{p})}
$$

$$
\leq \sigma_0(n)n^{k-1}p^{\frac{k}{2}}(1 + \frac{2}{p(1 + \frac{1}{p})})
$$

and for $j \geq 2$

$$
\langle f, f \rangle^{\frac{1}{2}} |a(g_j, n)| \leq p^{\frac{k}{2}}(\sigma_0(\frac{n}{p^j})(\frac{n}{p^j})^{k-1} + 2 \cdot \frac{p^{k-1}}{p^j} \cdot \sigma_0(\frac{n}{p^j-1})(\frac{n}{p^j-1})^{k-1})
$$

$$
+ \frac{1}{p^{k+1}}\sigma_0(\frac{n}{p^{j-2}})(\frac{n}{p^{j-2}})^{k-1})
$$

$$
\leq \sigma_0(n)n^{k-1}p^{\frac{k}{2}}(1 + \frac{1}{p})^2,
$$

and we see that the latter bound holds for all $j$.

Finally, to estimate $\langle f, f \rangle$ for the primitive form $f$ from below we choose the fundamental domain $\mathcal{F}$ so that it contains $\{x + iy \in \mathbb{H} \mid |x| < \frac{1}{2}, y > 1\}$, use $a(f, 1) = 1$ and get as in [5]

$$
\langle f, f \rangle \geq (4\pi e^{4\pi N_f} \cdot \prod_{p \nmid N_f} (1 + \frac{1}{p}))^{-1}
$$

from the trivial bound $\int_{\mathcal{F}} |f(x + iy)|^2y^{k-2}dxdy \geq \int_1^\infty \exp(-4\pi y)dy$.

Improvements on this are possible by [7] [8] but have been made effective so far only in few cases, see [13]. At least if the conductor $M_\chi$ of the character $\chi$ is small compared to $M$ these don’t give much for our present purpose because of the additional factors coming from oldforms which we computed above.

Putting things together and comparing the bounds in the cases $p | N$ and $p \nmid N$, we arrive for $h$ equal to the quotient of one of the $\prod_{p \mid M} a_{\phi_{p,jp}}(f)$ by its Petersson norm at the common bound

$$
|a(h, n)| \leq 2\sqrt{\pi}e^{2\pi} \sigma_0(n)n^{k-1}M^{\frac{k}{2}} \prod_{p \mid M} \frac{(1 + \frac{1}{p})^3}{\sqrt{1 - \frac{1}{p^2}}}
$$

for both cases as asserted.

**Theorem 12.** Let $F \in S_k(\Gamma_0(M), \chi)$. Then the Fourier coefficients $a(F, n)$ satisfy

$$
|a(F, n)| \leq 2\sqrt{\pi}e^{2\pi} \sqrt{\langle F, F \rangle} \cdot (\dim S_k(\Gamma_0(M), \chi))^{\frac{1}{2}} \cdot \sigma_0(n)n^{k-1}M^{\frac{k}{2}} \cdot \prod_{p \mid M} \frac{(1 + \frac{1}{p})^3}{\sqrt{1 - \frac{1}{p^2}}}
$$

**Proof.** This follows immediately from the previous theorem, using the Cauchy-Schwarz inequality.
Remark. a) As indicated above it should be possible to improve on the factor $M^{\frac{1}{2}}$ in the bound for $a(F,n)$ if the conductor $M_\chi$ of the character $\chi$ is equal to $M$ or at least relatively large compared to $M$ by using an effective version of the bound for the Petersson norm of a primitive form from [7, 8].

b) For $\gcd(n, M) = 1$ we obtain the better estimate

$$|a(h_i, n)| \leq 2\sqrt{\pi e^{2\pi}} \sigma_0(n)^{\frac{k-1}{2}} \cdot M^{\frac{1}{2}} \prod_{p|M} \frac{(1 + \frac{1}{p})}{\sqrt{1 - \frac{1}{p^2}}} \cdot \langle F, F \rangle \cdot (\dim S_k(\Gamma_0(M), \chi))^{\frac{1}{2}} \cdot \sigma_0(n)^{\frac{k-1}{2}} \cdot M^{\frac{1}{2}} \cdot \prod_{p|M} \frac{(1 + \frac{1}{p})}{\sqrt{1 - \frac{1}{p^2}}}.$$ 

in Theorem 12 and hence

$$|a(F, n)| \leq 2\sqrt{\pi e^{2\pi}} \sqrt{\langle F, F \rangle} \cdot (\dim S_k(\Gamma_0(M), \chi))^{\frac{1}{2}} \cdot \sigma_0(n)^{\frac{k-1}{2}} \cdot M^{\frac{1}{2}} \cdot \prod_{p|M} \frac{(1 + \frac{1}{p})}{\sqrt{1 - \frac{1}{p^2}}}.$$ 

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