Dynamic crossover in the persistence probability of manifolds at criticality

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Abstract. We investigate the persistence properties of critical \(d\)-dimensional systems relaxing from an initial state with non-vanishing order parameter (e.g., the magnetization in the Ising model), focusing on the dynamics of the global order parameter of a \(d'\)-dimensional manifold. The persistence probability \(P_c(t)\) shows three distinct long-time decays depending on the value of the parameter \(\zeta = (D-2+\eta)/z\) which also controls the relaxation of the persistence probability in the case of a disordered initial state (vanishing order parameter) as a function of the codimension \(D = d-d'\) and of the critical exponents \(\eta\) and \(z\). We find that the asymptotic behavior of \(P_c(t)\) is exponential for \(\zeta > 1\), stretched exponential for \(0 \leq \zeta \leq 1\), and algebraic for \(\zeta < 0\). Whereas the exponential and stretched exponential relaxations are not affected by the initial value of the order parameter, we predict and observe a crossover between two different power-law decays when the algebraic relaxation occurs, as in the case \(d' = d\) of the global order parameter. We confirm via Monte Carlo simulations our analytical predictions by studying the magnetization of a line and of a plane of the two- and three-dimensional Ising models, respectively, with Glauber dynamics. The measured exponents of the ultimate algebraic decays are in a rather good agreement with our analytical predictions for the Ising universality class. In spite of this agreement, the expected scaling behavior of the persistence probability as a function of time and of the initial value of the order parameter remains problematic. In this context, the non-equilibrium dynamics of the O(\(n\)) model in the limit \(n \to \infty\) and its subtle connection with the spherical model are also discussed in detail.

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we show that the correlation functions of the components of the order parameter which are respectively parallel and transverse to its average value within the O\((n \to \infty)\) model correspond to the correlation functions of the local and global order parameters of the spherical model.

**Keywords:** classical phase transitions (theory), persistence (theory)

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1. Introduction

After decades of research, understanding the statistics of first-passage times for non-Markovian stochastic processes remains a challenging issue. Of particular interest in this context is the persistence probability \(P(t)\), which, for a stochastic process \(X(t) \geq 0 \in \mathbb{R}\) of zero mean \(\langle X(t) \rangle = 0\), is defined as the probability that \(X\) does not change sign within the time interval \([0,t]\). In terms of \(P(t)\), the probability density of the first time at which the process crosses \(X = 0\) (zero crossing) is \(-dP(t)/dt\). Such first-passage problems have been widely studied by mathematicians since the early 1960s [1]–[3], often inspired by engineering applications. During the last fifteen years these problems have received considerable attention in the context of non-equilibrium statistical mechanics of spatially extended systems, both theoretically [4,5] and experimentally [6]. In various relevant physical situations, ranging from coarsening dynamics to fluctuating interfaces and polymer chains, \(P(t)\) turns out to decay algebraically at large times, \(P(t) \sim t^{-\theta_c}\), where \(\theta_c\) is a non-trivial exponent, the prediction of which becomes particularly challenging for non-Markovian processes [5].
In statistical physics, persistence properties were first studied for the coarsening dynamics of ferromagnetic spin models evolving at zero temperature $T = 0$ from random initial conditions \[4\]. In this case the local magnetization, i.e., the value $\pm 1$ of a single spin, is the physically relevant stochastic process $X(t)$ and the corresponding persistence probability turns out to decay algebraically at long times. By contrast, at any non-vanishing temperature $T > 0$, the spins fluctuate very rapidly in time due to the coupling to the thermal bath and therefore the persistence probability of an individual spin decays exponentially in time. However, it was shown in \[7\] that the global magnetization, i.e., the spatial average of the local magnetization over the entire (large) sample, is characterized by a persistence probability $P_c(t)$—referred to as global persistence—which decays algebraically in time $P_c(t) \sim t^{-\theta_g}$ at temperatures $T$ below the critical temperature $T_c$ of the model. The case corresponding to $T = T_c$, which we shall focus on here, is particularly interesting because $\theta_g$ turns out to be a new universal exponent associated with the critical behavior of these systems. The global persistence for critical dynamics has since been studied in a variety of instances \[8\]–\[10\](s e e a l s o \[12\] for a recent study of the global persistence below $T_c$).

In order to understand how the long-time behavior of the persistence probability at fixed $T = T_c$ crosses over from the exponential form of the local magnetization to the algebraic form of the global one, Majumdar and Bray \[13\] introduced and studied the persistence of the total magnetization of a $d'$-dimensional submanifold of a $d$-dimensional system, with $0 \leq d' \leq d$ (a similar idea was put forward in \[10\] and further studied in \[11\]). The two limiting cases $d' = 0$ and $d' = d$ correspond, respectively, to the local and the global magnetization and therefore a crossover from an exponential ($d' = 0$) to an algebraic ($d' = d$) decay is expected in the persistence probability as $d'$ is varied from 0 to $d$. Interestingly, it turns out that as a function of $d'$ the persistence probability $P_c(t)$ of the manifold displays three qualitatively different long-time behaviors, depending on the value of a single parameter $\zeta = (D - 2 + \eta)/z$, where $D = d - d'$ is the codimension of the manifold, $z$ the dynamical critical exponent and $\eta$ the Fisher exponent which characterizes the anomalous algebraic decay of the static two-point spatial correlation function of the spins. One finds as a function of $\zeta$ \[13\]

$$P_c(t) \sim \begin{cases} t^{-\theta_0(d,d')} & \text{for } \zeta < 0, \\ \exp(-a_1 t^\zeta) & \text{for } 0 \leq \zeta \leq 1, \\ \exp(-b_1 t) & \text{for } \zeta > 1, \end{cases} \tag{1}$$

where the exponent $\theta_0(d,d')$ is a new universal exponent which depends on both $d'$ and $d$, whereas $a_1$ and $b_1$ are non-universal constants. Interestingly, from the mathematical point of view, the somewhat unexpected stretched exponential behavior for $0 \leq \zeta \leq 1$ emerges as a consequence of a theorem due to Newell and Rosenblatt \[2\] which connects the long-time decay of the persistence probability of a stationary process to the long-time decay of its two-time correlation function. In passing, we mention that a behavior similar to equation (1) was also predicted and observed numerically for the non-equilibrium correlation function and the global persistence of spins close to a free surface of a semi-infinite three-dimensional Ising model with Glauber dynamics \[14\].

The results mentioned above—including the crossover in equation (1)—were obtained for the critical coarsening of a system which is initially prepared in a completely disordered state characterized by a vanishing initial value $m_0 = 0$ of the magnetization or, in general
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Table 1. Time dependence of the persistence probability $P_c(t)$ of a $d'$-dimensional submanifold of a $d$-dimensional system relaxing from an initial state with non-vanishing order parameter $m_0$, as a function of the parameter $\zeta = (d-d' - 2 + \eta)/z$. (See the main text for details.)

| Condition | $P_c(t)$ for $t_{\text{micr}} \ll t \ll \tau_m$ | $P_c(t)$ for $\tau_m \ll t$ |
|-----------|---------------------------------|----------------|
| $\zeta < 0$ | $P_c(t) \sim t^{-\theta_{0}(d,d')}$ | $P_c(t) \sim t^{-\theta_{\infty}(d,d')}$ |
| $0 \leq \zeta \leq 1$ | $P_c(t) \sim \exp(-a_1 t^\zeta)$ | $P_c(t) \sim \exp(-b_1 t)$ |
| $\zeta \geq 1$ | $P_c(t) \sim \exp(-c_1 t)$ | |

terms, of the order parameter. Due to the absence of symmetry-breaking fields, this initial condition implies that the average value $m(t)$ of the fluctuating magnetization at time $t$ over the possible realizations of the process vanishes. For $m_0 \neq 0$, instead, after a non-universal transient, the average magnetization $m(t)$ grows in time as $m(t) \propto m_0 t^{\theta'_0}$ for $t \ll \tau_m \propto m_0^{-1/\kappa}$ whereas, for $t \gg \tau_m$, $m(t)$ decays algebraically to zero as $m(t) \propto t^{-\beta/(\nu z)}$. These different time dependences are characterized by the universal exponents $\theta'_0 > 0$ (the so-called initial-slip exponent [15]) and

$$\kappa = \theta'_0 + \beta/(\nu z),$$

where $\beta$, $\nu$ and $z$ are the usual static and dynamic (equilibrium) critical exponents, respectively. In [16], we have demonstrated that a non-vanishing value of $m_0$ results in a temporal crossover in the persistence probability $P_c(t)$ of the thermal fluctuations $\delta m(t)$ of the fluctuating magnetization around its average value $m(t)$, such that

$$P_c(t) \sim \begin{cases} t^{-\theta_0} & \text{for } t_{\text{micr}} \ll t \ll \tau_m, \\ t^{-\theta_{\infty}} & \text{for } t \gg \tau_m, \end{cases}$$

where $t_{\text{micr}}$ is a microscopic time scale. On the basis of a renormalization-group analysis of the dynamics of the order parameter up to the first order in the dimensional expansion and of Monte Carlo simulations of the two-dimensional Ising model with Glauber dynamics, we concluded that the two exponents $\theta_0$ and $\theta_{\infty}$ are indeed different, with $\theta_0 < \theta_{\infty}$.

In the present study, we investigate both analytically and numerically the interplay between the crossovers described by equations (1) and (3), focusing primarily on the case of the Ising model universality class with relaxational dynamics. We argue that for $\zeta > 0$ the qualitative behavior of the persistence probability in equation (1) is not affected by a non-vanishing initial magnetization. For $\zeta < 0$, instead, we predict a crossover between two distinct algebraic decays characterized by different exponents $\theta_0(d, d')$ and $\theta_{\infty}(d, d')$, as in equation (3). The conclusions of our analysis of the time dependence of the persistence probability are summarized in table 1.

The rest of the paper is organized as follows. In section 2, we describe the continuous model that we shall study and we present a scaling analysis which yields the behaviors mentioned above for the persistence probability $P_c(t \gg \tau_m)$ at criticality; see table 1. In section 3, we present an analytic approach to the calculation of $\theta_{\infty}(d, d')$ for the Ising universality class with relaxational dynamics. In particular, in section 3.1, we focus on the Gaussian approximation, whereas in section 3.2, we discuss the effects of non-Gaussian fluctuations. In section 4, we present the results of our numerical simulations of the Ising...
model with Glauber dynamics and we compare them with the analytical predictions of section 3. In section 5, we study the persistence of manifolds for the $O(n)$ model in the limit $n \to \infty$ and, in passing, we discuss the connection between the non-equilibrium dynamics of this model and that of the spherical model. Our conclusions and perspectives are then presented in section 6.

2. Model and scaling analysis

Here we focus primarily on the Ising model on a $d$-dimensional hypercubic lattice, evolving with Glauber dynamics at its critical point and we study the persistence properties of the associated order parameter both analytically and numerically. The universal aspects of the relaxation of this model are captured by the case $n = 1$ of so-called Model A [17] for the $n$-component fluctuating local order parameter $\varphi_{\alpha}(x, t)$, $\alpha = 1, \ldots, n$ (e.g., the coarse-grained density of spins at point $x \equiv (x_1, \ldots, x_d)$ in the Ising model):

$$\partial_t \varphi(x, t) = -\frac{\delta \mathcal{H}[\varphi]}{\delta \varphi(x, t)} + \eta(x, t),$$

where $\eta(x, t)$ is a Gaussian white noise with $\langle \eta_\alpha(x, t) \rangle = 0$ and $\langle \eta_\alpha(x, t) \eta_\beta(x', t') \rangle = 2T \delta_{\alpha\beta} \delta^d(x - x') \delta(t - t')$ and $T$ is the temperature of the thermal bath. In what follows we consider the critical case $T = T_c$. In equation (4), the friction coefficient has been set to 1 and $\mathcal{H}$ is the $O(n)$-symmetric Landau–Ginzburg functional:

$$\mathcal{H}[\varphi] = \int d^d x \left[ \frac{1}{2} (\nabla \varphi)^2 + \frac{1}{2} r_0 \varphi^2 + \frac{g_0}{4!} (\varphi^2)^2 \right],$$

where $\int d^d x \equiv \int \prod_{i=1}^d dx_i$, $r_0$ is a parameter which has to be tuned to a critical value $r_{0,c}$ in order to approach the critical point at $T = T_c$ (here $r_{0,c} = 0$), and $g_0 > 0$ is the bare coupling constant.

At the initial time $t = 0$, the system is assumed to be prepared in a random spin configuration with a non-vanishing average order parameter $M_0$ along the direction in the internal space which will be denoted by 1, $[\varphi_{\alpha}(x, 0)]_0 = M_0 \delta_{\alpha 1}$ ($\alpha = 1, \ldots, n$), and short-range correlations $[\varphi_{\alpha}(x, 0) \varphi_{\beta}(x', 0)]_0 - M_0^2 \delta_{\alpha 1} \delta_{\beta 1} = \tau_0^{-1} \delta_{\alpha\beta} \delta^d(x - x')$, where $[\cdot \cdot \cdot]_0$ stands for the average over the distribution of the initial configuration. It turns out that $\tau_0^{-1}$ is irrelevant in determining the leading scaling properties [15] and therefore we set $\tau_0^{-1} = 0$. Here we first focus on the case $n = 1$ of the Ising universality class, whereas in section 5 we shall extend our analysis to $n > 1$. The stochastic process that we are interested in is the dynamics of the fluctuations of the magnetization around its average value, i.e.,

$$\psi(x, t) = \varphi(x, t) - M(t), \quad \text{where } M(t) = \langle \varphi(x, t) \rangle$$

is the average magnetization and $\langle \ldots \rangle$ stands for the average over the possible realizations of the thermal noise $\eta$. Assuming the total system to be on a hypercubic lattice of large volume $L^d$, the total (fluctuating) magnetization of the $d'$-dimensional submanifold is given by

$$\bar{M}(x_{d'+1}, \ldots, x_d, t) = \frac{1}{L^{d'}} \int \prod_{i=1}^{d'} dx_i \psi(x_1, \ldots, x_{d'}, x_{d'+1}, \ldots, x_d, t),$$

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and it depends on the remaining $D = d - d'$ spatial coordinates, which will be denoted by the vector $\mathbf{r} = (r_1, \ldots, r_D) \equiv (x_{d+1}, \ldots, x_d)$.

The analytical calculation of the persistence probability $P_c(t)$ of the process $\tilde{M}(\mathbf{r}, t)$ relies on the observation that the field $\tilde{M}(\mathbf{r}, t)$ is a Gaussian random variable for $d' > 0$. Indeed, for a $d'$-dimensional manifold of linear size $L$, $\tilde{M}(\mathbf{r}, t)$ is the sum of $L^{d'}$ ($d' > 0$) local fluctuating degrees of freedom, e.g., spins in a ferromagnet, which are correlated in space only across a finite, time-dependent and growing correlation length $\xi(t)$. In the thermodynamic limit $L \gg \xi(t)$ the number $\sim [L/\xi(t)]^{d'}$ of effectively independent variables contributing to the sum becomes large and the central limit theorem implies that $\tilde{M}(\mathbf{r}, t)$ is a Gaussian process, for which powerful tools have been developed in order to determine the persistence exponent $[5, 18]$. In particular, the Gaussian nature of the process implies that $P_c(t)$ is solely determined by the two-time correlation function $C_{\tilde{M}}(t, t') \equiv \langle \tilde{M}(\mathbf{r}, t)\tilde{M}(\mathbf{r}, t') \rangle$. For $t' < t < \tau_m$, the correlation function $C_{\tilde{M}}(t, t')$ coincides with the one for the case with vanishing initial magnetization $\tau_m \to \infty$, which was studied in [13]. The corresponding persistence probability $P_c(t)$ displays the behavior summarized in table 1 for $t < \tau_m$. However, a non-vanishing average $M_0$ of the initial order parameter affects the behavior of $C_{\tilde{M}}(t, t')$ as soon as $t', t \sim \tau_m$. In the long-time regime $t > t' \gg \tau_m$, one can take advantage of the results presented in [19] for the scaling behavior of the two-time correlation function of the Fourier transform $\tilde{\psi}(\mathbf{Q}, t) = \int d^d \mathbf{x} e^{i \mathbf{Q} \cdot \mathbf{x}} \psi(\mathbf{x}, t)$ of the local fluctuation $\psi(\mathbf{x}, t)$ of the magnetization (see equation (6)). Here and in what follows, the Fourier transform of $\psi(\mathbf{x}, t)$ will be simply denoted by using $\mathbf{Q} \equiv (Q_1, \ldots, Q_d)$ as an argument of $\psi$ and we will also assume $t > t'$. This yields the following scaling form for the Fourier transform $\tilde{M}(\mathbf{q}, t) \propto \psi(\mathbf{Q} = (0, \ldots, 0, q_1, \ldots, q_D), t)$ of $\tilde{M}(\mathbf{r}, t)$:

$$\langle \tilde{M}(\mathbf{q}, t)\tilde{M}(-\mathbf{q}, t') \rangle = \frac{1}{q^{2-\eta}} \left( \frac{t}{t'} \right)^{\theta-1} f_C \left( q^2 (t-t'), \frac{t}{t'} \right),$$

where $\mathbf{q} = (q_1, \ldots, q_D)$, $q = |\mathbf{q}|$, and $\theta = -\beta \delta / (\nu z) = -(d + 2 - \eta) / (2z)$ [19]. With the prefactor $(t/t')^{\theta-1}$ explicitly indicated, the function $f_C$ is such that $f_C(v, u \to \infty) = f_{C,\infty}(v)$ for fixed $v$. The correlation function $C_{\tilde{M}}(t, t')$ which determines the persistence probability is then obtained from the correlation function in equation (8) by integration over momenta:

$$C_{\tilde{M}}(t, t') \equiv \langle \tilde{M}(\mathbf{r}, t)\tilde{M}(\mathbf{r}, t') \rangle = \int \prod_{i=1}^D \frac{dq_i}{2\pi} \langle \tilde{M}(\mathbf{q}, t)\tilde{M}(-\mathbf{q}, t') \rangle. \quad (9)$$

The scaling form (8) has the same structure as the one discussed in [13] for $m_0 = 0$, the only differences being in the specific value the exponent $\theta$ and in the form of the scaling function $f_C(v, u)$. Accordingly, the argument presented in [13]—which is actually independent of these differences—leads to the conclusion that the persistence probability for $t \gg \tau_m$ behaves as

$$P_c(t) \sim \begin{cases} t^{-\theta_\infty(d, d')} & \text{for } \zeta < 0 \\ \exp(-a_1 t^\zeta) & \text{for } 0 \leq \zeta \leq 1, \\ \exp(-b_1 t) & \text{for } \zeta > 1, \end{cases} \quad (10)$$

where the non-universal constants $a_1$ and $b_1$ are the same as in equation (1). Indeed, in the case $\zeta \propto D - 2 + \eta \geq 0$, the integral in equation (9) for the equal-time correlation function

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In the case that we are currently interested in, $D - 2 + \eta \approx \zeta < 0$ and the integral in equation (9)—necessarily defined with a large wavevector cutoff $\Lambda$—is convergent for $\Lambda \to \infty$, so $C_M(t, t) = \langle \tilde{M}^2(r, t) \rangle$ (and therefore $C_M(t, t')$) has a well-defined $\Lambda$-independent limit which can be taken from the outset, provided that the expressions below are understood to be valid for times much larger than the time scale $t_\Lambda$ (= $t_{\text{mic}}$ of equations (1)) and (3)) associated with that cutoff.

In order to study the persistence properties it is convenient to focus on the normalized process $X(r, t) = \tilde{M}(r, t)/\langle \tilde{M}^2(r, t) \rangle^{1/2}$ associated with $\tilde{M}(r, t)$ (see, for instance, [7]), which is characterized by a unit variance. The scaling form (8), together with equation (9), implies that the correlation function of $X(r, t)$ has the following scaling form for $t > t' \gg \tau_m$:

$$\langle X(r, t)X(r, t') \rangle = \left(\frac{t}{t'}^\nu\right)^{-\mu_\infty(d, d')} F_\infty(t/t'),$$

where $F_\infty(x)$ is a non-constant function, with the asymptotic behavior $F_\infty(x \to \infty) \sim \text{const} \neq 0$ and

$$\mu_\infty(d, d') = - (\theta - 1) + \frac{\zeta}{2}$$

and

$$\tau_m = 1 + \frac{d + D}{2z}. \quad (13)$$

Even though the Gaussian process described by equation (11) is not stationary, it becomes so when the logarithmic time $T = \log t$ is introduced. In fact, this yields $\langle X(T)X(T') \rangle = e^{-\mu_\infty(T-T')} F_\infty(e^{T-T'}) = C_{st}(T - T')$ with $C_{st}(\Delta T) \sim \exp[-\mu_\infty(d, d')\Delta T]$ for large $\Delta T$, which implies [3] that the persistence probability $P_\infty(t) \sim \exp[-\theta_\infty(d, d') T]$, i.e., algebraically $P_\infty(t) \sim t^{-\theta_\infty(d, d')}$ in terms of the original time variable. In addition, if $F_\infty(x)$ is actually independent of $x,$
i.e., if it is a constant, the process can be mapped onto a Markovian process for which \( \theta_\infty(d, d') = \mu_\infty(d, d') \). In a sense, \( \mu_\infty \) provides a sort of ‘Markovian approximation’ of the exponent \( \theta_\infty \). In the generic case, however, the exponent \( \theta_\infty(d, d') \) depends in a non-trivial fashion on both the exponent \( \mu_\infty(d, d') \) (see equation (11)) and the full scaling function \( F_\infty(x) \), which, for the present case \( D \neq 0 \), is known only within the Gaussian approximation of equations (4) and (5) [19] (whereas for \( D = 0 \), an expression for the first correction in the \( \epsilon \)-expansion about the spatial dimension \( d = 4 - \epsilon \) is available [19]). It actually turns out (see further below) that the function \( F_\infty(x) \) is non-trivial already within the Gaussian approximation, a fact that makes the calculation of the exponent \( \theta_\infty(d, d') \) a rather difficult task. Here we present a perturbative expansion of this exponent which is valid for small codimension \( D \ll 1 \). Indeed, one notes that \( F_\infty(x) \) reduces to a constant for \( D = 0 \) and therefore the persistence exponent \( \theta_\infty(d, d') \) is given by its Markovian approximation \( \theta_\infty(d, d') = \mu_\infty \) [5, 16]. For small \( D \) one can take advantage of the perturbative formula derived in [7, 9] in order to expand the non-Markovian process for \( D \neq 0 \) and its persistence exponent \( \theta_\infty \) around the Markovian one corresponding to \( D = 0 \).

3.1. The Gaussian approximation

The Gaussian approximation discussed below is exact in dimension \( d > 4 \) and it is obtained by neglecting nonlinear contributions in \( \psi \) to the Langevin equation (4) once it has been expressed in terms of \( \psi(x, t) \) and \( m^2 \equiv g_0 M^2 / 2 \) (see, e.g., [19]). This yields the evolution equation

\[
\partial_t \psi(x, t) = \nabla^2 - m^2(t)\psi(x, t) + \eta(x, t) \quad \text{where} \quad \partial_t m(t) + \frac{1}{3} m^3(t) = 0, \tag{14}
\]

with the initial condition \( m(t = 0) = m_0 \). In order to determine the persistence exponent, we need to calculate the correlation function \( C_M \) of the order parameter of the manifold (see equations (7) and (9)). In turn, \( C_M \) can be inferred from the two-time correlation function of the Fourier components \( \psi(Q, t) = \int d^d x e^{iQ \cdot x} \psi(x, t) \), where \( Q \equiv (Q_1, \ldots, q_d) \), which was calculated in [19] (see equation (59) therein):

\[
C_Q(t, t') \equiv \langle \psi(Q, t)\psi(-Q, t) \rangle = \frac{2e^{-Q^2(t+t')}}{[(t+\tau_m)(t'+\tau_m)]^{3/2}} \int_0^{t_<} dt_1(t_1+\tau_m)\sum_{i=1}^{3} e^{2Q^2i}, \tag{15}
\]

where \( t_< = \min\{t, t'\} \) and \( \tau_m = 3/(2m_0^3) \). The correlation function \( C_M(t, t') = \langle \tilde{M}(r, t)\tilde{M}(r, t') \rangle \) in space follows from the integration (9) of the correlation of \( \tilde{M}(q, t) \sim \psi(Q = (0, \ldots, 0, q_1, \ldots, q_d), t) \) (see equation (7)):

\[
C_M(t, t') = \langle \tilde{M}(r, t)\tilde{M}(r, t') \rangle = \int \prod_{i=1}^{D} \frac{d t_i}{2\pi} C_{Q = (0, q)}(t, t')
= \frac{2c_D^D1^{D-2}}{[(t+\tau_m)(t'+\tau_m)]^{3/2}} \int_0^{\tilde{t}_1^{D/2}} d \tilde{t}_1(\tilde{t}_1+1)^3(\tilde{t}_1+\tilde{t}-2\tilde{t}_1)^{-D/2}, \tag{16}
\]

where, in the first line, we used the notation \( Q = (0, q) \equiv (0, \ldots, 0, q_1, q_2, \ldots, q_d) \) and, in the second, we introduced \( c_D = (4\pi)^{-D/2} \) and the dimensionless time variables \( \tilde{t} \equiv t/\tau_m \) and \( \tilde{t}' \equiv t'/\tau_m \), assuming \( t' < t \). The correlation function \( \langle X(r, t > t')X(r, t') \rangle \) of the
normalized process $X(\mathbf{r}, t) = \tilde{M}(\mathbf{r}, t)/\langle \tilde{M}^2(\mathbf{r}, t) \rangle^{1/2}$ is therefore given by

$$\langle X(\mathbf{r}, t)X(\mathbf{r}, t') \rangle = C_{\tilde{M}}(t, t')/\sqrt{C_{\tilde{M}}(t, t)C_{\tilde{M}}(t', t')} = \left( \frac{t'}{t} \right)^{-(D+2)/4} \frac{I_{D/2}(t'/t, \tilde{v})}{[I_{D/2}(1, t)I_{D/2}(1, t')]^{1/2}},$$

where

$$I_a(x, u) = \int_0^1 dv (1 + uv)^3[1 + x(1 - 2v)]^{-a}. \quad \text{(18)}$$

Note that in order for $C_{\tilde{M}}(t, t)$ to be defined, $I_{D/2}(1, u) \sim \int_1^1 dv (1 - v)^{-D/2}$ has to be convergent, which requires $D < 2$, consistently with the assumption $\zeta = (D - 2)/2 < 0$ ($\eta = 0$ and $z = 2$ within the present approximation). In contrast to the case $D = 0$ for the global persistence, discussed in [16] (see equation (6) therein), there is no suitable choice of a function $L(t)$ such that the correlation function (17) for $D \neq 0$ takes the form of a ratio $L(t')/L(t)$, which corresponds to a Markovian process. Accordingly, the Gaussian process $\{X(t)\}_{t \geq 0}$ for $D \neq 0$ displays significant non-Markovian features even within the Gaussian approximation.

In the asymptotic regime $\tilde{t} > \tilde{t}' \gg 1$ we are interested in $I_n(x, u \gg 1) \simeq u^3 \int_0^1 dv v^3[1 + x(1 - 2v)]^{-a}$ and therefore the correlation function (17) takes the form

$$\langle X(\mathbf{r}, t)X(\mathbf{r}, t') \rangle \simeq (t'/t)^{(2+D/4)}\mathcal{A}(t'/t), \quad \text{for } t > t' \gg \tau_m, \quad \text{(19)}$$

where

$$\mathcal{A}(x) = A_D \int_0^1 dv v^3[1 + x(1 - 2v)]^{-D/2}, \quad \text{(20)}$$

and $A_D = [2^{-D/2} \times 6\Gamma(1 - D/2)\Gamma(5 - D/2)]^{-1} = (8 - D)(6 - D)(4 - D)(2 - D)/(3 \times 25 - D/2)$, with finite $\mathcal{A}(0) = A_D/4$ and, by definition, $\mathcal{A}(1) = 1$. The correlation function (19) has the form (11) with $d = 4$, $z = 2$, i.e., $\mu_\infty = 2 + D/4$ and $F_\infty(t'/t') = \mathcal{A}(t'/t)$. The scaling function $\mathcal{A}(t'/t)$ is indeed a non-trivial function of its argument and it reduces to a constant only for $D = 0$, i.e., for vanishing codimension. As anticipated, one can take advantage of this fact in order to determine $\theta_\infty(d, d')$ perturbatively, by expanding around the Markovian Gaussian process for $D = 0$ according to [9]. First, one introduces the logarithmic time $T = \log t$ and expands $\langle X(\mathbf{r}, T)X(\mathbf{r}, T') \rangle$ for small $D$:

$$\langle X(\mathbf{r}, T)X(\mathbf{r}, T') \rangle = e^{-\bar{\mu}_\infty(T-T')}\bar{\mathcal{A}}(e^{\bar{T}_-T')}, \quad \text{(21)}$$

with $\bar{\mu}_\infty = 2$,

$$\bar{\mathcal{A}}(x) \equiv x^{D/4}\mathcal{A}(x) = 1 + D\bar{A}_1(x) + \mathcal{O}(D^2), \quad \text{(22)}$$

and

$$\bar{A}_1(x) = \frac{1}{2} \left[ -\frac{11}{6} + \frac{y}{3} + \frac{y^2}{2} + y^3 - \frac{1}{2} \log (2y - 1) + \frac{y^4 - 1}{2} \log (y - 1) - y^4 \log y \right], \quad \text{(23)}$$

where $y = (x^{-1} + 1)/2$.

The function $\bar{A}_1(x)$ is responsible, at the lowest order in $D$, for the non-Markovian corrections to $\theta_\infty \equiv \bar{R}\bar{\mu}_\infty$ which can be calculated using the perturbation theory of [9]:

$$\bar{R} \equiv \frac{\theta_\infty}{\bar{\mu}_\infty} = 1 - D\frac{2\bar{\mu}_\infty}{\pi} \int_0^1 dx x^{\bar{\mu}_\infty-1}\frac{\bar{A}_1(x)}{(1 - x^{2\bar{\mu}_\infty})^{3/2}} + \mathcal{O}(D^2). \quad \text{(24)}$$

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Alternatively, a different (yet equivalent) numerical estimate of $\theta_\infty$ which, as expected, has the same series expansion as $\theta_\infty$ can be obtained by expanding up to first order in $D$ only $\mathcal{A}(x) = 1 + D \tilde{A}_1(x) + \mathcal{O}(D^2)$ on the rhs of equation (19) while keeping the full $D$-dependence of the value of $\mu_\infty = 2 + D/4$, which corresponds to the Markovian approximation. Accordingly, one has $\theta_\infty(d, d') = \mu_\infty \mathcal{R} = (2 + D/4)\mathcal{R}$ where the ratio $\mathcal{R}$ is given here by equation (24) in which $\tilde{\mu}_\infty \rightarrow \mu_\infty = 2 + O(D)$ and $\tilde{A}_1(x) \mapsto A_1(x) \equiv \tilde{A}_1(x) - (\ln x)/4$.

With these substitutions one finds $\mathcal{R} = \mathcal{R} - D/8 + O(D^2)$ and therefore
\[
\theta_\infty^{[2]}(d, d' = d - D) = 2(1 + D/8)[1 + D \times 0.723 \ldots + O(D^2)],
\]
which, as expected, has the same series expansion as $\theta_\infty^{[1]}(d, d' = d - D)$ up to the first order in $D$ and provides the numerical estimate
\[
\theta_\infty^{[2]}(d, d' = d - D) = 3.87 \ldots
\]
for $D = 1$.

These expressions for the persistence exponent $\theta_\infty(d, d - D)$ have been derived accounting only for the effects of Gaussian fluctuations of the order parameter and therefore they become increasingly accurate as the non-Gaussian fluctuations become less relevant, i.e., as the spatial dimensionality $d$ of the model approaches and exceeds 4. Accordingly, the numerical estimates $\theta_\infty^{[1,2]}(d, d - D)$ in equations (26) and (28) are expected to become increasingly accurate as $d$ increases for a fixed small codimension $D$.

In section 3.2 we shall compare these analytical predictions, extrapolated to $D = 1$ and provided by equations (27) and (29), to the results of Monte Carlo simulations of the Ising model with Glauber dynamics in spatial dimensionality $d = 2$ and 3, discussed in section 4. Figure 1(a) summarizes the available estimates of $\theta_\infty(d, d - D)$ as a function of the codimension $D$ and of the space dimensionality $d$ of the model. The solid and the dashed lines correspond to the estimates $\theta_\infty^{[1]}(d, d - D)$ (equation (26)) and $\theta_\infty^{[2]}(d, d - D)$ (equation (28)), respectively, derived within the Gaussian model. The vertical bars for $D = 0$ and $D = 1$ indicate the corresponding Monte Carlo estimates (see [16] and section 4 below, respectively) for the Ising model with Glauber dynamics for $d = 2$ (gray) and $d = 3$ (black). As expected, the Gaussian approximation provides a rather accurate estimate of the actual value of $\theta_\infty$ for $d = 3$. For comparison we report here also the codimension expansion of $\theta_0(d, d')$—corresponding to the case $m_0 = 0$—which can be determined as detailed above for $\theta_\infty$ on the basis of equations (17) and (18) in the limit $t' < t \ll 1$. In this case, $I_a(x, u \ll 1) \approx \int_0^1 dv [1 + x(1 - 2v)]^{-a} = [(1 + x)^{1-a} - (1 - x)^{1-a}]/[2(1 - a)x]$ and therefore
\[
\langle X(r, t)X(r, t') \rangle \sim (t/t')^{-(2 + D)/4} B(t'/t), \quad \text{for } \tau_m \gg t > t',
\]

\[\text{doi:10.1088/1742-5468/2010/12/P12029} \]
Figure 1. Persistence exponents $\theta_{\infty,0}(d, d-D)$ of the global order parameter of a manifold of codimension $D$ as a function of $D$, in the case of (a) non-vanishing and (b) vanishing initial value of the order parameter. The solid and the dashed curves in the two panels correspond to the estimates $\theta_{\infty,0}^1$ (equations (26) and (33)) and $\theta_{\infty,0}^2$ (equations (28) and (34)), respectively, derived within the Gaussian model universality class with relaxational dynamics (equation (4)). For $D \geq 2$, i.e., $\zeta \geq 0$ (shaded areas) the asymptotic decay of the persistence probability of this model is no longer algebraic. The vertical bars for $D = 0$ (see [16] and [20, 21] for (a) and (b), respectively) and $D = 1$ (present work) indicate the corresponding Monte Carlo estimates of the persistence exponents for the Ising model universality class with Glauber dynamics for $d = 3$ (black) and $d = 2$ (gray). In panel (b), the black and gray circles for $D = 1$ indicate the preliminary Monte Carlo estimates of [13].

where

$$B(x) = \frac{(1 + x)^{1-D/2} - (1 - x)^{1-D/2}}{2^{-1-D/2}x},$$

with finite $B(0) = 2^{D/2}(1 - D/2)$ and, by definition, $B(1) = 1$. The Markovian approximation to $\theta_0$ is given by $\mu_0 \equiv 1/2 + D/4$ and corresponds to $B(x) \equiv 1$ in equation (30), i.e., to having $D = 0$. This expression for $\mu_0$ agrees with the hyperscaling relation [7]

$$\mu_0 = -\theta_{\infty} - 1 + \zeta/2,$$

which can be derived by the same arguments as those leading to equation (12), taking into account that in this case $m_0 = 0$, the correlation function of the order parameter fluctuations scales as in equation (8) with $\theta$ replaced by the initial-slip exponent $\theta_{\infty} = \theta'_{\infty} - (2 - \eta - z)/z$ [15]. For the present Gaussian model $\theta_{\infty} = 0$ and $\eta = 0$. (Note that with the identification $\lambda/(z) \leftrightarrow -\theta + \zeta$, expression (32) yields the expression for the persistence exponent $\mu_0$ which is reported after equation (8) in [13] in terms of $\lambda$.)

Taking advantage of equation (24) and of the series expansion of $B$ for small $D$ one finds

$$\theta_{0}^1(d, d-D) = \frac{1}{2}[1 + D \times (\sqrt{2} - 1/2) + O(D^2)],$$

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which coincides with the case $d = 4$ of the expression derived in [13] for the $O(n \to \infty)$ model. An equivalent estimate of $\theta_0$ is obtained as discussed above for $\theta_\infty$:

$$\theta_0^{[2]}(d, d - D) = \frac{1}{2}(1 + D/2)[1 + D \times (\sqrt{2} - 1) + \mathcal{O}(D^2)].$$ (34)

Figure 1(b) summarizes the available estimates of $\theta_0(d, d - D)$ as a function of the codimension $D$ and of the space dimensionality $d$ of the model. As in panel (a) of the same figure, the solid and the dashed lines correspond to the estimates $\theta_0^{[1]}(d, d - D)$ (equation (33)) and $\theta_0^{[2]}(d, d - D)$ (equation (34)), respectively, derived within the Gaussian model. The vertical bars for $D = 0$ and $D = 1$ indicate the corresponding Monte Carlo estimates (see [20, 21] and section 4 below, respectively) for the Ising model with Glauber dynamics for $d = 2$ (gray) and $d = 3$ (black). In panel (b), the gray and black circles for $D = 1$ indicate the corresponding preliminary Monte Carlo estimates reported in [13], which turn out to be, respectively, marginally compatible and significantly different from the ones for $d = 2$ (gray) and $d = 3$ (black) presented below in section 4.

### 3.2. Beyond the Gaussian approximation (perturbatively)

According to the analytical predictions within the Gaussian model indicated by the solid and dashed lines in figure 1, the persistence exponents $\theta_\infty^{[1,2]}(d, d - D)$ increase upon increasing the codimension $D \geq 0$, until $D$ reaches the value $D = 2$, above which (shaded areas in figure 1) the relaxation of the persistence probability is no longer algebraic. The contributions of non-Gaussian fluctuations to the global persistence exponents $\theta_\infty$ and $\theta_0$ within the $O(n)$ universality class have been calculated analytically and perturbatively in [16] and [7, 9], respectively, only for the case $D = 0$. (For $D > 0$ the persistence exponent $\theta_0$ is studied in [13] only in the limit $n \to \infty$; see section 5 below.) For the Ising universality class ($n = 1$) these corrections decrease $\theta_\infty,0$ compared to its Gaussian value, as the spatial dimensionality $d$ of the model decreases below the upper critical dimensionality $d = 4$. On the basis of these behaviors of $\theta_\infty,0(d, d - D)$ for $d > 4$ (Gaussian model) as a function of (small) $D > 0$ and for $D = 0$ as a function of (small) $4 - d$, it is difficult to predict analytically the qualitative dependence on $D$ of $\theta_\infty,0(d, d - D)$ at fixed $d$ resulting from the combined effect of these trends as a function of $D$ and $d$. The same problem arises in the more general case of the $O(n)$ universality class with generic $n$. In this respect, it would be desirable to extend to the case $D \neq 0$ the analysis of the contribution of non-Gaussian fluctuations to the persistence exponent of the $O(n)$ universality class, beyond the $n \to \infty$ limit, following [16] and [7, 9] for the cases $m_0 \neq 0$ and $m_0 = 0$, respectively. This requires the knowledge of the analytic expression (e.g., in a dimensional expansion around the upper critical dimensionality $d = 4$) of the wavevector dependence of the correlation function of the order parameter $C_Q(t, t')$ beyond the Gaussian approximation (equation (15)). At present, however, such an analytic expression within the $O(n)$ universality class with $m_0 \neq 0$ is available only for $Q = 0$ (see [19, 22]) and, in the limit $n \to \infty$, for transverse fluctuations (see [22]; in section 5.2 we shall comment on the relation between the $O(n \to \infty)$ model and the spherical model investigated, in this context, in [23, 24]). For a vanishing initial value of the order parameter $m_0 = 0$, instead, the expression of $C_Q(t, t')$ beyond the Gaussian approximation and for finite $n$ is known for generic $Q$ only up to the first order in the dimensional expansion around $d = 4$ [25] and up to the second order for $Q = 0$ [26].
(see [27] for a review). However, we point out here that in order to observe a non-trivial interplay between the effects of a non-vanishing codimension \( D \neq 0 \) and those of non-Gaussian fluctuations for \( d < 4 \) one would need to account for higher-order terms in the expansion of \( R \) around a Markovian process, unless the dependence of the correlation function \( C_Q(t, t') \) on the dimensionality \( d \) is known non-perturbatively as in the case \( n \to \infty \) discussed in [13] and in section 5 below. Indeed, the general structure of the correlation function of the normalized process is \( \langle X(t)X(t') \rangle = \langle t/t' \rangle^{-\mu(d, D)} F(t'/t; d, D) \) with \( t > t' \), where the exponent \( \mu(d, D) \) and the function \( F \) depend on the specific process under study and \( \mu(d, D) \) is determined such that \( F(0; d, D) \) is finite and non-zero (note that, by definition, \( F(1; d, D) = 1 \)). If \( F(x; d, D) \) turns out to be independent of \( x \) (and therefore \( F(x; d, D) \equiv 1 \)), then the process is Markovian with persistence exponent \( \mu(d, D) \) which, in turn, is related to known exponents via hyperscaling relations such as equations (12) and (32). In the limit \( n \to \infty \) this is actually the case for \( D = 0 \) and generic \( d \) (compare section 5), i.e., \( F(x; d, D) = 1 \). Accordingly, the expansion for small \( d \) of \( F(x; d, D) \) takes the form \( F(x; d, D) = 1 + D \partial_D F(x; d, 0) + O(D^2) \) and one can use the formula presented in [7, 9] (see equation (24) above) in order to calculate the correction \( R(d, D) = 1 + D \partial_D + O(D^2) \) to \( \mu(d, D) \) which determines the persistence exponent \( \theta(d, D) \equiv \mu(d, D) \times R(d, D) \). Here the deviation from the Markovian evolution is controlled in terms of the (small) parameter \( D \) and \( R \) has a non-trivial dependence on the dimensionality \( d \) of the model. This approach was adopted in [13] for \( m_0 = 0 \) and, below in section 5 for \( m_0 \neq 0 \). In the case of finite \( n \), instead, the correlation function \( C_Q(t, t') \) for \( d < 4 \) is typically known in a dimensional expansion around \( d = 4 \), up to a certain order in \( \epsilon = 4 - d \). As a consequence, \( F(x; d, D) = F(x; 4, D) - \epsilon \partial_D F(x; 4, D) + O(\epsilon^2) \), where the lowest-order term \( F(x; 4, D) \) corresponds to the Gaussian approximation, discussed above for \( n = 1 \). (The line of argument presented below actually applies also to those cases in which the first correction term for \( C_Q(t, t') \) and therefore for \( F(x; d, D) = 0 \) is of \( O(\epsilon^2) \) [7, 9].) In the generic case, \( F(x; 4, D) \) as a function of \( x \) is not constant, resulting in a non-Markovian process even for \( \epsilon = 0 \). However, the process turns out to be Markovian for \( D = 0 \), i.e., \( F(x; d = 4, D = 0) = 1 \), and therefore one is naturally led to perform a codimension expansion of \( F(x; 4, D) = 1 + D \partial_D F(x; 4, 0) + O(D^2) \) and, for consistency, of \( \partial_D F(x; 4, D) = \partial_D F(x; 4, 0) + O(D) \). Accordingly, one has \( F(x; d, D) = 1 + D \partial_D F(x; 4, 0) - \epsilon \partial_D F(x; 4, 0) + O(D^2, \epsilon^2, \epsilon D) \), where the deviations from the non-Markovian evolution are jointly controlled by \( D \) and \( \epsilon \). Due to the fact that the perturbative expression for \( R \) (see, e.g., equation (24)) is valid up to the first order in the deviation from the Markovian evolution, all the terms of \( O(D^2, \epsilon^2, \epsilon D) \) in \( F(x; d, D) \) can be neglected when calculating

\[
R(d = 4 - \epsilon, D) \equiv 1 + DA + \epsilon B + O(\epsilon^2, \epsilon D, D^2),
\]

where the coefficients \( A \) and \( B \) are given by the integrals of \( \partial_D F(x; 4, 0) \) and \( -\partial_D F(x; 4, 0) \), respectively, according to the rhs of equation (24) in which, for consistency, the value \( \mu(d = 4, D = 0) \) of \( \mu \) at the lowest order in \( \epsilon \) and \( D \) enters. Due to the double expansion in equation (35), the lowest-order correction to \( \mu(d, D) \) which results from both a finite codimension \( D \) and non-Gaussian fluctuations for \( d < 4 \) is given by the superposition of the two corresponding corrections taken separately, which—for the \( O(n) \) universality class—can be inferred from [13] and the present study \( (D > 0, \epsilon = 0) \), and from [7, 9, 16] \( (D = 0, \epsilon > 0) \), respectively.
Table 2. Estimates of the universal ratios $R_{0,∞}^{(ng)}$ (up to $O(D^2)$; see equation (38)), of the critical exponents (see, e.g., [27,28]), and of the resulting Markovian approximations $\mu_{0,∞}$ (equations (32) and (13)) for the persistence exponents $\theta_{0,∞}$ within the Ising universality class in two and three spatial dimensions. The analytical estimates $\theta_{0,∞}^{(ng)} \equiv R_{0,∞}^{(ng)} \times \mu_{0,∞}$ of $\theta_{0,∞}$ agree with those of [9, 16] for $D = 0$, whereas for $D = 1$ they yield the values reported in table 3.

| $d$  | $d = 2$          | $d = 3$          |
|------|------------------|------------------|
| $R_0^{(ng)}$ | 1.03 + 0.414$D$  | 1.01 + 0.414$D$  |
| $R_∞^{(ng)}$ | 1.05 + 0.723$D$  | 1.03 + 0.723$D$  |
| $\eta$ | 1/4              | 0.03             |
| $z$   | 2.17             | 2.04             |
| $\theta_{ls}$ | 0.38           | 0.14             |
| $\mu_0$ | 0.216 + 0.23$D$  | 0.377 + 0.245$D$ |
| $\mu_∞$ | 1.46 + 0.23$D$   | 1.74 + 0.245$D$  |

Here we focus on the $O(n = 1)$ Ising universality class and we consider both the cases $m_0 = 0$ and $m_0 \neq 0$, for which the constants $A$ and $B$ in equation (35) actually take different values. Comparing the expression (35) of $R$ for $\epsilon = 0$ with equations (28) and (34) one readily finds that

$$A = \begin{cases} \sqrt{2} - 1 = 0.414 \ldots & \text{for } m_0 = 0, \\ 0.723 \ldots & \text{for } m_0 \neq 0. \end{cases}$$  \hspace{1cm} (36)$$

Analogously, the coefficient $B$ for the cases $m_0 = 0$ and $m_0 \neq 0$ can be determined by comparison with the results for $R(4 - \epsilon, D = 0)$ presented in equation (12) of [16] and in equation (19) of [9], respectively:

$$B = \begin{cases} 0.00754 \ldots & \text{for } m_0 = 0, \\ 0.0273 \ldots & \text{for } m_0 \neq 0. \end{cases}$$  \hspace{1cm} (37)$$

(Note, however, that for $m_0 = 0$ the first term of the expansion of $R(4 - \epsilon, D = 0) - 1$ is of order $\epsilon^2$ [9].) These values result in

$$R(4 - \epsilon, D) = \begin{cases} R_0 \equiv 1 + D \times 0.414 \ldots + \epsilon^2 \times 0.00754 \ldots + O(D^2, \epsilon^2D, \epsilon^3) & \text{for } m_0 = 0, \\ R_∞ \equiv 1 + D \times 0.723 \ldots + \epsilon \times 0.0273 \ldots + O(D^2, \epsilon D, \epsilon^2) & \text{for } m_0 \neq 0, \end{cases}$$  \hspace{1cm} (38)$$

which clearly shows that the corrections due to a finite codimension are quantitatively more relevant than those due to non-Gaussian fluctuations. In particular the dependence of the latter on the dimensionality $d = 4 - \epsilon$ is weak enough that a simple extrapolation of equation (38) to $\epsilon = 2$ and $\epsilon = 1$ should provide quantitatively reliable estimates of $R$ as a function of $D$ for $d = 2$ and $d = 3$, respectively. These estimates, denoted by the superscript (ng), are reported in table 2. The Markovian approximations $\mu(d, D)$ of the persistence exponents in the two cases $m_0 \neq 0$ and $m_0 = 0$ are respectively given by equations (13) and (32), where the critical exponents $\eta$, $z$, and $\theta_{ls}$ in spatial dimension $d = 2$ and 3 (see, e.g., [27,28]) take the values which are reported in table 2 together with
The resulting expressions for $\mu_{0,\infty}$ as a function of $D$. The persistence exponents $\theta_0$ and $\theta_{\infty}$ can be estimated as $\theta_{0}^{\text{ng}} \equiv R_{0}^{\text{ng}} \times \mu_0$ and $\theta_{\infty}^{\text{ng}} \equiv R_{\infty}^{\text{ng}} \times \mu_\infty$, which are reported as dash–dotted lines in the two panels of figure 2 for $d = 2$ (gray) and $d = 3$ (black) together with the Monte Carlo and Gaussian estimates anticipated in the corresponding figure 1. The comparison among all these estimates will be discussed further below in section 4.3.

In order to improve on the analytical estimates of the persistence exponents and to go beyond the linear dependence on $d$ and $D$ expressed by equation (38) (see also table 2), one would need first of all an expression of $R$ which accounts for non-Markovian corrections beyond the leading order, i.e., an extension to higher orders of the perturbation theory developed in [7,9]. Secondly, an analytic expression for the contribution of non-Gaussian fluctuations to $C_Q(t, t')$ would be required. In the present work, instead of pursuing this strategy, we shall investigate the general features of the $D$-dependence of the persistence exponent $\theta_{\infty}$ on the basis of the Monte Carlo results presented in section 4 and of the analytical study of transverse fluctuations in the $O(n \to \infty)$ case, presented in section 5.

4. Monte Carlo results

In order to test our prediction of a temporal crossover in the critical persistence of manifolds for $\zeta < 0$ (see table 1) as well as the theoretical estimate of $\theta_{\infty}$ as a function of $d$ for $D = 1$—see equation (25) and table 2—we studied via Monte Carlo simulations the
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Figure 3. Persistence probability $P_c(t)$ of the magnetization of (a) a line and (b) a plane within, respectively, the two- and three-dimensional critical Ising models with Glauber dynamics for the two extreme values $m_0 = 0$ (top data set) and $m_0 = 1$ (bottom data set) of the initial magnetization $m_0$. These data sets correspond, respectively, to the two different regimes $t \ll \tau_m$ and $t \gg \tau_m$, within which $P_c(t)$ decays algebraically at large times but with distinct exponents $\theta_{\text{MC}}^0$ (top) and $\theta_{\text{MC}}^\infty$ (bottom), the best fit values of which are indicated by the associated straight lines. These numerical data provide clear evidence for the occurrence of the dynamical crossover discussed in section 2.

4.1. Line magnetization within the two-dimensional Ising–Glauber model

We first focus on the two-dimensional case of an $L \times L$ square lattice with periodic boundary conditions, for which the size $L$ ranges from 64 to 256 and which is such that the data presented below are not appreciably affected by the expected finite-size corrections. We study the temporal evolution of the fluctuating magnetization of a complete line (row or column, i.e., $d' = 1$) selected within the lattice and we calculate the associated persistence probability. Taking into account the values of the exponents $z$ and $\eta$ reported in table 2, this choice corresponds to $\zeta = (d - d' - 2 + \eta)/z \simeq -0.345 < 0$ and therefore the persistence probability of the line magnetization is expected to decay algebraically at large times. The system is initially prepared in a random configuration with $N_+$ up and $N_-$ down spins, where $N_{\pm} = L^2(1 \pm m_0)/2$. Then, at each subsequent time step, a site is randomly chosen and the move $s_i \mapsto -s_i$ is accepted or rejected according to Metropolis rates corresponding to the critical temperature $T = T_c$. One time unit corresponds to $L^2$ attempted updates of spins. The determination of the persistence probability $P_c(t)$ of the fluctuations requires also the knowledge of the global magnetization $M(t) = L^{-2}\langle \sum_i s_i \rangle$, which we obtained by averaging over 2000 realizations of the dynamics. For each of these realizations, we also choose a new random initial condition with fixed magnetization $m_0$. The persistence probability is then computed as the probability that the fluctuating magnetization of the line $L^{-1}\sum_{i \in \text{line}} s_i - M(t)$ has not changed sign between $t = 0$ and time $t$. This probability is determined on the basis of more than $10^6$ samples. In figure 3(a), we
Table 3. Monte Carlo and analytical estimates for the persistence exponents $\theta_0$ and $\theta_\infty$ of the magnetization of a manifold of codimension $D = 1$—i.e., a line in spatial dimension $d = 2$ or a plane in $d = 3$—for an Ising model with Glauber dynamics, relaxing from initial states with $m_0 = 0$ and $m_0 \neq 0$, respectively. The corresponding Monte Carlo estimates $\theta_0^{(MC)}$ have been obtained as described in section 4, whereas the analytical estimates $\theta_0^{(ng)} = R_0^{(ng)} \times \mu_0^{(ng)}$ follow from the direct extrapolation to $D = 1$ of the results of section 3.2, summarized in table 2. The values $\theta_0^{(MC)}$ for $d = 2$ and $d = 3$ indicated here are reported, respectively, as gray and black vertical bars for $D = 1$ in figures 1 and 2.

| $D$ | $d = 2$ | $d = 3$ |
|-----|--------|--------|
| $\theta_0^{(MC)}$ | 0.78(5) | 1.10(5) |
| $\theta_0^{(ng)}$ | 0.64 | 0.89 |
| $\theta_\infty^{(MC)}$ | 4.2(1) | 3.7(1) |
| $\theta_\infty^{(ng)}$ | 3.0 | 3.5 |

show the results of our simulations corresponding to $L = 256$ and to two different values of the magnetization $m_0$. The top curve refers to $m_0 = 0$, i.e., $\tau_m \to \infty$, such that the system is always in the regime $t \ll \tau_m$ investigated in [13]. These data are fully compatible with a power-law decay with an exponent $\theta_0^{(MC)}(d = 2, d' = 1) = 0.78(5)$ (see also table 3), which turns out to be in a rather good agreement with the value $\simeq 0.72$ reported in [13]. The bottom curve in figure 3(a), instead, corresponds to $m_0 = 1$. In this case, $\tau_m \ll 1$ and after a short initial transient, the system enters the regime $t \gg \tau_m$, within which the persistence probability is expected, according to section 2, to decay algebraically at large times with an exponent $\theta_\infty$. The numerical data in the figure are compatible with such a power-law decay, characterized by an exponent $\theta_\infty^{(MC)}(d = 2, d' = 1) \simeq 4.2(1)$ which is significantly larger than the one measured in the case $m_0 = 0$.

4.2. Plane magnetization within the three-dimensional Ising–Glauber model

In order to investigate the dependence of the persistence exponent on the space dimensionality $d$ at fixed codimension $D$, we extended the investigation presented above ($d = 2, D = 1$) to the three-dimensional case, focusing on the magnetization of a plane ($d = 3, D = 1$) within the Ising–Glauber model on a $L \times L \times L$ cubic lattice with periodic boundary conditions. The size $L$ ranges from 16 to 128 and it is such that the data presented below are not appreciably affected by the expected finite-size corrections. Taking into account the values of the exponents $z$ and $\eta$ reported in table 2, this choice corresponds to $\zeta = (D - 2 + \eta)/z \simeq -0.48 < 0$ and therefore the persistence probability is expected to decay algebraically at large times. In figure 3(b), we report the results of our simulations on a lattice with $L = 128$, for the two extreme values $m_0 = 0$ (top data set) and $m_0 = 1$ (bottom data set) of the initial magnetization $m_0$, as we did in panel (a) of the same figure for the case of the magnetization of a line within the two-dimensional model. At large times one clearly observes that the power of the algebraic decay changes when passing from $m_0 = 0$ to 1. In the former case (top curve) $t \ll \tau_m$ and the data are compatible.
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Figure 4. Critical persistence probability \( P_C(t) \) of the fluctuating magnetization of a plane within the three-dimensional Ising model for different values of the initial magnetization \( m_0 \). In panel (a) \( P_C \) is plotted as a function of time \( t \) and its long-time algebraic decay clearly shows a crossover, which is controlled by the initial value \( m_0 \) of the magnetization. The uppermost and lowermost data sets are the same as those presented in figure 3(b), whereas the associated straight lines indicate algebraic decays with the exponents \( \theta^{(ng)}_0 \) and \( \theta^{(ng)}_\infty \), respectively, analytically estimated in section 3.2 (see figure 2 and, to compare, table 3). In panel (b) the data presented in (a) are scaled according to the heuristic scaling ansatz discussed in the main text: the data for \( P_C(t) \times m_0^{-\theta_0(d=3,d'=2)/\kappa} \) should collapse onto a single master curve when plotted as a function of \( t \times m_0^{1/\kappa} \) with \( \theta_0(d=3,d'=2) = \theta^{(MC)}_0(d=3,d'=2) = 1.1 \) and \( \kappa = 0.34 \) (see equation (2)). However, this is clearly not the case and a significant dependence on \( m_0 \) is still observed in the resulting scaled curves.

Intermediate values \( 0 < m_0 < 1 \) of \( m_0 \) correspond to finite and non-vanishing \( \tau_m \) and therefore one expects \( P_C(t) \) to display the two different power-law behaviors described separately above within the two consecutive time ranges \( t \ll \tau_m \) and \( t \gg \tau_m \). Indeed, this is clearly displayed in figure 4(a) where we report the time dependence of the persistence probability \( P_C(t) \) of the magnetization of a plane in three dimensions, for various values of \( m_0 \): at relatively small times (i.e., larger than some microscopic scale but smaller than \( \tau_m \) ) \( P_C(t) \) decreases algebraically with the power \( \theta_0 \) characteristic of the decay of the curve corresponding to \( m_0 = 0 \). As time \( t \) increases and exceeds \( \tau_m \), however, one observes a crossover towards an algebraic decay with the power \( \theta_\infty \) characteristic of the decay of the persistence probability for \( m_0 = 1 \). (For comparison, in figure 4(a) we also indicate the straight lines corresponding to algebraic decays with the powers \( \theta_{0,\infty} = \theta^{(ng)}_{0,\infty} \) predicted with \( P_C(t) \sim t^{-\theta_0(d=3,d'=2)} \) with an exponent \( \theta^{(MC)}_0(d=3,d'=2) = 1.10(5) \). This value is significantly larger than the estimate \( \theta^{(MC)}_0(d=3,d'=2) \approx 0.88 \) preliminarily reported in [13]. However, such an estimate was obtained for rather small system sizes \( L = 15 \) and 31 and therefore it might be biased by finite-size effects. The bottom curve in figure 3(b) corresponds to the case \( t \gg \tau_m \) and the numerical data for the persistence probability still decay algebraically at large times, but with an exponent \( \theta^{(MC)}_\infty(d=3,d'=2) \approx 3.7(1) \) which is significantly larger than in the case \( m_0 = 0 \).
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in section 3.2 and reported in figure 2 and in table 3.) As the effect of a finite $m_0$ is to introduce a time scale $\tau_m$ into the problem, it is natural to wonder whether these curves corresponding to different values of $m_0$ are characterized by dynamical scaling, i.e., if they collapse onto a single master curve after a proper rescaling of the time $t$ and of the probability $P_c(t)$ involving $\tau_m$. The natural heuristic candidate for such a scaling form is $P_c(t) \sim \tau_m^{-\theta_0/\kappa}P(t/\tau_m)$, where, for consistency with the known behaviors for $t \ll \tau_m$ and $t \gg \tau_m$, one has $P(x \ll 1) \sim x^{-\theta_0}$ and $P(x \gg 1) \sim x^{-\theta_\infty}$. This scaling ansatz can be tested by plotting $P_m^{\theta_0}P_c(t)$ or, equivalently, $m_0^{-\theta_0/\kappa}P_c(t)$, as a function of $t/\tau_m$ or, equivalently, $m_0^{1/\kappa}t$ where the exponents $\theta_0 = \theta_0(d,d')$ and $\kappa(d)$ are the ones appropriate to the dimensionality of the model and of the manifold. In [16] it was shown that the persistence probability of the global magnetization ($d' = d$) of the two-dimensional Ising–Glauber model does indeed obey such a scaling form with the expected value of $\kappa = \theta_0 + \beta/\nu = 0.249$. This numerical value follows form equation (2), where one uses the values of the exponents reported in table 2 together with the fact that in two spatial dimensions $\beta/\nu = \eta/2$. Similarly, we have checked (data not shown) that the same scaling behavior holds for the global magnetization of the three-dimensional Ising model, with the expected value of $\kappa = 0.34(1)$, which follows from equation (2) with $\beta_\ast = 0.104(3)$ [29], $\beta = 0.3267(10)$, $\nu = 0.6301(8)$ [30], $z = 2.04(2)$ [29]. However, the present numerical data indicate that this heuristic scaling ansatz captures the actual behavior of the persistence probability for the fluctuating magnetization of a manifold neither in the case of a line in two dimensions nor in that of a plane in three dimensions. The data for a plane in three dimensions are shown in figure 4(a), where we plot $m_0^{-\theta_0/\kappa}P_c(t)$ as a function of $m_0^{1/\kappa}t$ with $\theta_0(d = 3, d' = 2) = \theta_0^{MC}(d = 3, d' = 2) = 1.1$ (see figure 3(b)) and $\kappa = 0.34$ as derived above: clearly the curves corresponding to different values of the magnetization $m_0$ do not collapse onto a single master curve. We have carefully checked that the absence of data collapse is not due to finite-size effects. This lack of scaling can be qualitatively understood on the basis of the fact that the persistence probability that we have discussed so far is, in fact, a special case of a two-time quantity. Indeed, consider the persistence probability $P_c(t,t')$, defined as the probability that the process $X(r,t)$ does not change sign between the times $t'$ and $t > t'$. In terms of this quantity, the persistence probability $P_c(t)$ studied above is given by $P_c(t) = P_c(t,t_n)$, where $t_n$ is some non-universal microscopic time scale set, e.g., by the elementary moves of the dynamics. The scaling form of the correlation function $(X(r,t)X(r,t')) \equiv C(t/t_m,t'/t_m)$ (see equation (17)) implies straightforwardly that $P_c(t,t') \equiv P(t/t_m,t'/t_m)$, where $\tau_m \propto m_0^{-1/\kappa}$. In addition, the scaling behavior of the correlation function becomes independent of $\tau_m$ within the following two different regimes—which can be investigated analytically by taking, respectively, the limits $\tau_m \rightarrow \infty$ and $\tau_m \rightarrow 0$: (I) $t' < t \ll \tau_m$ within which $(X(r,t)X(r,t')) = C^I(t/t')$ and (II) $\tau_m \ll t' < t$ within which $(X(r,t)X(r,t')) = C^{I1}(t/t')$. Corresondingly, the persistence probability $P_c(t,t')$ takes two different scaling forms $P$:

$$
\begin{align*}
(\text{I}) \quad & P_c(t,t') \equiv \bar{P}^I(t/t') \sim (t/t')^{-\theta_0(d,d')} \quad \text{for } t' < t \ll \tau_m, \\
(\text{II}) \quad & P_c(t,t') \equiv \bar{P}^{I1}(t/t') \sim (t/t')^{-\theta_\infty(d,d')} \quad \text{for } \tau_m \ll t' < t.
\end{align*}
$$

In passing we mention that two analogous regimes emerge in the study of the persistence of fluctuating interfaces, in which the role of $\tau_m$ is played by the equilibration time

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Different values of the initial magnetization $m_0$ collapse onto a single master curve when $P_c(t) \times m_0^{-\theta_0(d=3,d'=2)/k}$ is plotted as a function of $tm_0^{1/k}$ with $\theta_0(d=3,d'=2) = 1.1$ and $k = 0.54 \neq \kappa$.

$\tau_{eq} \sim L^2$ for a system of finite size $L$ [31]. As anticipated above, the case $m_0 = 0$ corresponds to regime (I) (equation (39)), and $m_0 = 1$ to regime (II) (equation (40)). While these two instances are well understood, our simulations for intermediate values of $m_0$ are likely to correspond to a third regime $t' \ll \tau_m \ll t$ which we are currently unable to describe analytically for generic $D > 0$ because the correlation function of the process cannot be made stationary due to its residual dependence on $\tau_m$. (Note that the introduction of the logarithmic time yields stationary correlation functions as long as they depend only on $t'$ or, more generally, on ratios $f(t')/f(t)$; see further below.) In particular there is no obvious reason why the scaling $P_c(t) \sim t^{-\theta_0}P(t/\tau_m)$ or, equivalently, $P_c(t) \sim m_0^{\theta_0(d,d')/\kappa}P_c(m_0^{1/\kappa}t)$—which well describes the behavior of $P_c(t)$ for $D = 0$ [16]—should actually hold in general and therefore one should not be surprised by the fact that this scaling form turns out to be violated in the case of figure 4(b). Within the Gaussian approximation, however, one can heuristically understand why the case $D = 0$ investigated in [16] is indeed special in this respect. In fact, from equation (17) one can see that only for $D = 0$ does the correlation function $\langle X(r,t)X(r,t') \rangle$ have the form $\langle X(r,t)X(r,t') \rangle = L(t'/\tau_m)/L(t/\tau_m)$, with $L(\tilde{t}) = \tilde{t}^{1/2} \sqrt{I_0(\ldots,t)}$, such that $L(\tilde{t}) \sim \tilde{t}^{1/2}$ for $\tilde{t} \ll 1$, whereas $L(\tilde{t}) \sim \tilde{t}^{2}$ for $\tilde{t} \gg 1$—we do not specify here the first argument of $I_0$ as $I_D(x,u)$ is actually independent of $x$ for $D = 0$; see equation (18). The persistence probability for such a process can be calculated exactly [3], after a mapping onto a stationary process: $P_c(t,t') = (2/\pi)\arcsin[L(t'/\tau_m)/L(t/\tau_m)]$. Therefore in the third regime $t' \ll \tau_m \lesssim t$ one has $P_c(t,t') \sim (t'/\tau_m)^{1/2}/L(t/\tau_m)$ which actually yields $P_c(t) = \bar{P}_c(t,t' = t_0) \sim \tau_m^{-\theta_0}P(t/\tau_m)$, i.e., the heuristic scaling form anticipated above, with the correct exponents $\theta_0 = 1/2$ and $\theta_\infty = 2$ [16]. Finally, even though such a scaling form does not work for $D > 0$, our numerical simulations indicate that data corresponding to different values of $m_0$ actually fall—to a rather good extent; see figure 5—onto a single master curve obtained by plotting $m_0^{\theta_0(d=3,d'=2)/k}P_c(t)$ as a function of $m_0^{1/k}t$ with

![Figure 5. Effective scaling of the Monte Carlo data presented in figure 4(a) for the critical persistence probability $P_c(t)$ of the fluctuating magnetization of a plane within the three-dimensional Ising model. Data corresponding to different values of the initial magnetization $m_0$ collapse onto a single master curve when $P_c(t) \times m_0^{-\theta_0(d=3,d'=2)/k}$ is plotted as a function of $tm_0^{1/k}$ with $\theta_0(d=3,d'=2) = 1.1$ and $k = 0.54 \neq \kappa.$](image-url)
\( \theta_0(d = 3, d' = 2) = 1.1 \) and \( k = 0.54 \neq \kappa \). However, the origin of this effective scaling is currently unclear and it certainly deserves further investigations.

### 4.3. Comparison between analytical and numerical results

The comparison between the analytical predictions of section 3.2 and the available numerical estimates of \( \theta_{0,\infty} \) discussed in sections 4.1 and 4.2 or reported in the literature is summarized in figure 2 and, for \( D = 1 \), also in table 3. In particular, panels (a) and (b) of figure 2 compare the various estimates of \( \theta_\infty \) and \( \theta_0 \), respectively, as functions of the codimension \( D \), for spatial dimension \( d = 2 \) (gray lines and markers) and \( d = 3 \) (black lines and markers). As already pointed out in [9, 16], for vanishing codimension \( D = 0 \) the agreement between the Monte Carlo estimate \( \theta_{\infty,0}^{(MC)} \) (vertical bars in figure 2) and the analytical estimates \( \theta_{\infty,0}^{(ng)} \) (dash–dotted lines) of \( \theta_{\infty,0} \) is very good for both \( d = 2 \) and \( 3 \). For \( D = 1 \), instead, the comparison is less clear. On the one hand, our Monte Carlo estimate of \( \theta_{\infty}^{(MC)}(d = 3, d' = 2) = 3.7(1) \) for \( d = 3 \) (indicated in figure 2(a) as a vertical black bar—see section 4.2 for details and table 3), is in a rather good agreement with the corresponding analytical estimate \( \theta_{\infty}^{(ng)}(d = 3, d' = 2) = 3.5 \) reported in table 3 (dash–dotted black line in figure 2(a)) and also with the result of the Gaussian approximation (solid and dashed lines). On the other hand, \( \theta_{0}^{(MC)}(d = 2, d' = 1) = 4.2(1) \) for \( d = 2 \) (vertical gray bar in figure 2(a); see section 4.1 for details) is significantly larger than the corresponding analytical estimate \( \theta_{\infty}^{(ng)}(d = 2, d' = 1) = 3.0 \) reported in table 3 (dash–dotted gray line in figure 2(a)). In addition, the fact that \( \theta_{\infty}^{(MC)}(d = 2, d' = 1) > \theta_{\infty}^{(ng)}(d = 3, d' = 2) \) is at odds with what is observed for all the other numerical and analytical estimates presented in this work (compare figure 6 for the \( O(n \to \infty) \) universality class), i.e., that \( \theta_{\infty,0}(d = 2, d' = 1) < \theta_{\infty,0}(d = 3, d' = 2) \). As far as \( \theta_0 \) for \( D = 1 \) is concerned (see figure 2(b)), we note that the preliminary Monte Carlo estimates of [13] (black and gray circles, corresponding to \( \theta_0^{(MC)}(d = 3, d' = 2) \simeq 0.88 \) and \( \theta_0^{(MC)}(d = 2, d' = 1) \simeq 0.72 \), respectively) are essentially compatible with the values of \( \theta_{\infty}^{(ng)} \) (dash–dotted line in figure 2(b); see also table 3) for both \( d = 2 \) (gray) and \( d = 3 \) (black). The Monte Carlo estimates \( \theta_0^{(MC)} \) discussed in sections 4.2 and 4.1 and reported in table 3, instead, turn out to be significantly larger than and marginally compatible with the previous ones for \( d = 3 \) and \( d = 2 \), respectively, and therefore they no longer agree with the corresponding analytical predictions provided by \( \theta_0^{(ng)} \). The reason for such a discrepancy might be that for \( D = 1 \) the actual contribution of the term of \( O(D) \) in \( \mu_0 \times \mathcal{R}_{0}^{(ng)} \) (see equation (38)) is a rather large fraction of the \( O(D^0) \) term (about 100% and 150% for \( d = 3 \) and \( d = 2 \), respectively), i.e., the dependence on \( D \) is so pronounced that a quantitatively reliable estimate of these exponents might require accounting for higher-order terms in the codimension expansion.

We conclude this section by discussing briefly the case \( 0 \leq \zeta \leq 1 \) (see table 1), that includes the instance \( d' = 1 \) of a line within the three-dimensional Ising model (i.e., \( D = 2 \)) for which \( \zeta = \eta/z \simeq 0.016 \) (see table 2). For \( 0 \leq \zeta \leq 1 \), our analysis indicates that the long-time behavior of \( P_c(t) \) is independent of the actual value \( m_0 \) (e.g., \( m_0 = 0 \) or 1) and is characterized by a stretched exponential behavior (table 1). Unfortunately, we have not been able to observe incontrovertibly such a stretched exponential law in our
simulations. This difficulty mirrors the one recently encountered in the numerical analysis of the persistence probability $P_c(t)$ of stationary processes characterized by two-time correlations with a power-law decay \( C_{st}(t_1, t_2) \sim |t_1 - t_2|^{-\zeta} \). Although that context is rather different from the present one, there it was observed that the convergence to the stretched exponential behavior—which is expected on the basis of the theorem by Newell and Rosenblatt [2]—is actually extremely slow. In addition, the pre-asymptotic behavior was shown to be increasingly important at the quantitative level as \( \zeta \) decreases [33]. Given these results and the extremely small value of \( \zeta = 0.016 \) in the case of present interest it is not surprising that the stretched exponential is rather difficult to observe. On the other hand, our numerical data indicate that the pre-asymptotic behavior of \( P_c(t) \) is affected by the actual value of the initial magnetization \( m_0 \). However, we have not attempted a more detailed and quantitative characterization of this pre-asymptotic regime, beyond this qualitative feature, which would require more extensive and dedicated simulations.

5. The \( O(n) \) model in the limit \( n \to \infty \)

Here we extend the previous analysis of the persistence probability of a manifold to the case of an \( n \)-component vector order parameter \( \varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \) with \( n > 1 \) and model A dynamics (see equations (4) and (5)). Even though, strictly speaking, this model is not relevant for the description of actual physical systems, the fact that it can be solved exactly in the limit \( n \to \infty \) leads to insight into the effects of nonlinear terms in the Langevin equation beyond perturbation theory. If the initial magnetization \( m_0 \) does not vanish the original \( O(n) \) symmetry of the model is explicitly broken and one has to distinguish between fluctuations \( \psi_\sigma(x, t) \) which are parallel to the average order parameter \( m(t) = \langle \varphi(x, t) \rangle \) and those, \( \psi_\pi(x, t) \), which are transverse to it. The different fluctuations \( \psi_\sigma(x, t) \) and \( \psi_\pi(x, t) \) are expected to have distinct persistence probabilities \( P_\sigma^c(t) \) and \( P_\pi^c(t) \) which, on the basis of the results of [22], can be shown to exhibit the same temporal crossovers as in table 1 due to \( m_0 \neq 0 \).

5.1. Persistence of the transverse modes in the \( O(n \to \infty) \) model with \( m_0 \neq 0 \)

We focus here on the limit \( n \to \infty \) and we consider the transverse modes only, for which the response and correlation functions can be calculated exactly [22]. The wavevector-dependent response function is given by (see equation (68) in [22])

\[
R_\mathbf{Q}^\pi(t > t', t') = \left. \frac{\delta \langle \psi^\pi(\mathbf{Q}, t) \rangle}{\delta h^\pi(-\mathbf{Q}, t')} \right|_{h^\pi = 0} = e^{-Q^2(t-s)} \left( \frac{t}{s} \right)^{\epsilon/4} \left( \frac{1 + t/\tau_m}{1 + t'/\tau_m} \right)^{1/2}, \tag{41}
\]

from which one obtains the correlation function (see equation (56) in [22]) as

\[
C_\mathbf{Q}^\pi(t, t') \equiv \langle \psi(\mathbf{Q}, t) \psi(-\mathbf{Q}, t') \rangle = 2 \int_0^{t_c} dt_1 R_\mathbf{Q}^\pi(t, t_1) R_\mathbf{Q}^\pi(t', t_1)
\]

\[
= \frac{2(t t')^{\epsilon/4}}{[(1 + t/\tau_m)(1 + t'/\tau_m)]^{1/2}} e^{-Q^2(t + t')} \int_0^{t_c} dt_1 (t_1)^{-\epsilon/2} (1 + t_1/\tau_m) e^{2Q^2 t_1}, \tag{42}
\]

where \( \epsilon = 4 - d \). Due to the residual \( O(n - 1) \) symmetry in the internal space of the transverse components, only correlation functions between the same components of the vector \( \psi^\pi \) can be non-vanishing. Accordingly, here and in the following we always refer
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to such correlations even if this is not explicitly indicated. For $m_0 \to \infty$, $\tau_{m} \sim m_0^{-2} \to 0$ and therefore one has

$$R_\mathbf{Q}(t > t', t') = e^{-Q^2(t-t') \left( \frac{t}{t'} \right)^{\epsilon/4-1/2}}$$

and

$$C_\mathbf{Q}^\pi(t_1, t_2) = 2(t_1 t_2)^{\epsilon/4-1/2} e^{-Q^2(t_1+t_2)} \int_0^{t_2} dt' e^{2Q^2t' (t')^{1-\epsilon/2}}.$$  \hfill (43)

Accordingly, the connected correlation function of the magnetization $\tilde{M}^\pi(r, t)$ of a manifold defined as in equation (7) with the substitution $\psi \to \psi^\pi$ is

$$C_{\tilde{M}}^\pi(t, t') = \left< \tilde{M}^\pi(r, t) \tilde{M}^\pi(r, t') \right> = \int \frac{d\mathbf{q}}{2\pi} C_{\mathbf{Q}^\pi=(0, \mathbf{q})}^\pi(t, t')$$

$$= 2c_D2^{-D/2}(tt')^{\epsilon/4-1/2}t^{2-(d+D)/2} \int_0^1 dy u^{1-\epsilon/2}(y - u)^{-D/2},$$

where $t > t'$, $y \equiv (1 + x^{-1})/2$, and $x = t'/t < 1$ ($c_D$ was defined after equation (16)). The normalized process $X^\pi(r, t) = \tilde{M}^\pi(r, t)/\left< \tilde{M}^\pi(r, t) \right>^{1/2}$ is therefore characterized by the correlation function

$$\left< X^\pi(r, t) X^\pi(r, t') \right> = (t/t')^{-(d+D)/4} \mathcal{A}^\pi(t'/t; d),$$

which is of the form (11) with $F_\infty(t/t') = \mathcal{A}^\pi(t'/t; d)$, where

$$\mathcal{A}^\pi(x; d) = x^{-D/2} \int_0^1 dy u^{-1+d/2}(y - u)^{-D/2} \int_0^1 du u^{-1+d/2}(1 - u)^{-D/2}$$

(44)

(45)

(46)

(47)

(48)

(49)

(50)

(51)

(52)

(53)

(54)

(55)

(56)
is a monotonically decreasing function of $d$, with $I_1 = 1.149 \ldots$, $I_2 = \sqrt{2} - 1/2 = 0.9142 \ldots$, $I_3 = 0.867 \ldots$, and $I_4 = 1/\pi + \pi/4 - 1/4 = 0.853 \ldots$. For later convenience we indicate here also the values of $I_d$ for some $d \leq 2$. This provides the estimate

$$\theta_\infty^{(1)} = \frac{d}{4} [1 + D \times I_d + \mathcal{O}(D^2)],$$

which, for $D = 0$, yields the value of the global persistence exponent reported for this model in [16]. As discussed in section 3, a different estimate of $\theta_\infty$ can be obtained by expanding only the ratio $\mathcal{R}_d$ to first order in the codimension $D$, while keeping the full $D$-dependence of the Markovian exponent $\mu_\infty = (d + D)/4$, which corresponds to $\mathcal{A}^e \equiv 1$ in equation (45) and to $\mathcal{A}_1^e(x; d) \mapsto \mathcal{A}_1^e(x; d) \equiv \mathcal{A}_1^e(x; d) - (\ln x)/4$ in equation (5). This yields $\theta_\infty^{(2)} = \mu_\infty \times \mathcal{R}$, with $\mathcal{R} = \mathcal{R} - D/d + \mathcal{O}(D^2)$, i.e.,

$$\theta_\infty^{(2)} = \frac{d + D}{4} [1 + D \times (I_d - 1/d) + \mathcal{O}(D^2)].$$

As expected, $\theta_\infty^{(2)}$ has the same small-$D$ expansion as $\theta_\infty^{(1)}$ up to $\mathcal{O}(D^2)$.

For completeness and comparison we briefly present here also the expansion for $\theta_0$ within the same $O(n \to \infty)$ model, which was first discussed in [13]. As the initial value $m_0$ of the magnetization vanishes, the original $O(n)$ symmetry is restored and there is no longer a distinction between the transverse ($\pi$) and longitudinal ($\sigma$) fluctuation modes. The correlation function $\langle X(r, t)X(r, t') \rangle$ of the normalized process associated with the magnetization $\tilde{M}(r, t)$ of the manifold can be obtained from equation (44), with $C_{\mathcal{Q}=0,q}(t, t', \frac{\epsilon}{2})$ given by the limit for $\tau_m \to \infty$ of equation (42):

$$C_{\mathcal{Q}=0,q}(t, t') = 2(\epsilon^2 t')^{\epsilon/4} e^{-Q^2(t+t')} \int_0^{t+t'} \frac{dt_1}{\epsilon^2} e^{2Q^2 t_1}.$$  

(53)

Interestingly, this expression of $C_{\mathcal{Q}=0,q}(t, t')$ with $\tau_m = \infty$ (and, analogously, the expression for $R_{\mathcal{Q}}(t, t')$; see equation (41)) can be formally obtained from the one corresponding to $\tau_m = 0$ (see equation (44)) by substituting in the latter $\epsilon$ with $\epsilon + 2$, i.e., $d$ with $d - 2$. Accordingly, we can take advantage here of the results reported above for the case $\tau_m = 0$ in order to discuss the persistence properties of the manifold for $m_0 = 0$. In particular, for the normalized process $X(r, t) = \tilde{M}(r, t)/\langle [\tilde{M}(r, t)]^2 \rangle^{1/2}$, one finds

$$\langle X(r, t)X(r, t') \rangle = \langle X^e(r, t)X^e(r, t') \rangle|_{\epsilon \to 0} = \frac{1}{(\epsilon^2 t')^{\epsilon/4}} e^{-Q^2(t+t')} \int_0^{t+t'} \frac{dt_1}{\epsilon^2} e^{2Q^2 t_1}.$$ 

(54)

on the basis of equation (45). This result is of the form (11) with $\mu_0(d, d') = (d + D - 2)/4$ and $F_0(t/t') = \mathcal{A}^e(t/t'; d - 2)$. As expected, for $d = 4$ the equations above yield the corresponding ones for the Gaussian model with $m_0 = 0$, which were discussed at the end of section 3.1. In addition, this expression for $\mu_0$ agrees with the hyperscaling relation (analogous to equation (13)) briefly mentioned after equation (31) because $\eta = 0$ and $\theta = (4 - d)/4$ for the present $O(n \to \infty)$ model [15]. On the basis of the mapping highlighted above, one can take advantage of equations (51) and (52) in order to determine the expansion of $\theta_0(d, d - D)$ in the codimension $D$:

$$\theta_0^{(1)} = \frac{d - 2}{4} [1 + D \times I_{d-2} + \mathcal{O}(D^2)].$$

(55)
**Figure 6.** Persistence exponents $\theta_{\infty,0}(d, d-D)$ of the transverse components of the global order parameter of a manifold of codimension $D$ as a function of $D$, for $d = 4$ (uppermost solid and dashed curves) and $d = 3$ (lowermost solid and dashed curves), within the $O(n \to \infty)$ model universality class with relaxational dynamics (equation (4)). Panels (a) and (b) refer to the case of non-vanishing and vanishing initial value $m_0$ of the order parameter, respectively, whereas the two different estimates $\theta_{\infty,0}^{[1]}$ (equations (51) and (55)) and $\theta_{\infty,0}^{[2]}$ (equations (52) and (56)) are indicated by the solid and the dashed curves, respectively. For $D \geq 2$, i.e., $\zeta \geq 0$ (shaded areas) the asymptotic decay of the persistence probability of this model is no longer algebraic. In panel (a), the uppermost solid and dashed curves ($d = 4$) are the same as those reported in figures 1(b) and 2(b). The persistence exponent of the longitudinal component of the global order parameter for $m_0 = 0$ is the same as the one for the transverse component (panel (b)), whereas for $m_0 \neq 0$ and $d = 4$ (Gaussian approximation) it is given by the solid and dashed lines in figures 1(a) and 2(a). The case of the longitudinal component for $m_0 = 0$ was also discussed in [13].

(which reproduces the expansion provided in [13] for $d = 4$, whereas for $d = 3$ this gives a coefficient $I_{1,4} = 0.287 \ldots$ which corrects the value $0.183615 \ldots$ reported therein for the first-order correction in $D$) and

$$\theta_0^{[2]} = \frac{d - 2 + D}{4} \{1 + D \times [I_{d-2} - 1/(d-2)] + O(D^2)\}. \quad (56)$$

In figures 6(a) and (b) we report the estimates $\theta_{\infty}^{[1,2]}$ for $\theta_{\infty}$ (equations (51) and (52)) and $\theta_0^{[1,2]}$ for $\theta_0$ (equations (55) and (56); see also [13]), respectively, as functions of the codimension $D$. In the two panels $\theta_{\infty,0}^{[1]}$ and $\theta_{\infty,0}^{[2]}$ are indicated as solid and dashed curves, respectively, for $d = 4$ (uppermost solid and dashed curves) and $d = 3$ (lowermost solid and dashed curves). Comparing the curves for $d = 3$ to those corresponding to $d = 4$ (Gaussian approximation), it turns out that the effect of non-Gaussian terms in the effective Hamiltonian (5) for $n \to \infty$ is to reduce the values of both $\theta_{\infty}$ and $\theta_0$ for generic values of $D$. This trend agrees with the one observed at the end of section 3.2 both for the longitudinal and the transverse components of the order parameter and based
on the results of a dimensional expansion around $d = 4$ for the case $D = 0$ and generic $n$. Even though a definitive statement in this respect would require a careful analysis, one might heuristically expect on the basis of the evidence collected here that, for a fixed codimension $D \neq 0$, the value of the persistence exponents $\theta_{\infty,0}$ decreases as $d$ decreases below the upper critical dimensionality $d = 4$ of the $O(n \to \infty)$ model, as happens for $D = 0$.

5.2. The relation between the spherical model and the $O(n \to \infty)$ model

We conclude this section by discussing the relation between the $O(n \to \infty)$ model studied in section 5.1 and the spherical model (see, e.g., [35]), and its implications for the persistence probability. It is well known that these two models are equivalent in equilibrium as there is a mapping between the corresponding free energies. This equivalence also extends to their equilibrium dynamics. However, when considering non-equilibrium properties some aspects of the spherical model require a careful consideration. In particular, non-Gaussian fluctuations of the Lagrange multiplier which is introduced in order to impose the pure relaxational dynamics have to be accounted for when calculating some correlation functions of global quantities—which correspond to vanishing wavevector $Q = 0$ (see [23,24]). In doing so, it turns out that the behavior of these quantities cannot be obtained as the limit for $Q \to 0$ of the correlation functions for $Q \neq 0$ (local), given that the associated non-connected part (if non-vanishing) alters the scaling behavior compared to the case $Q \neq 0$. (Here and in what follows we assume that the system is spatially homogeneous.)

Consider, for example, the correlation function of the local order parameter (i.e., of the magnetization). As long as the average of the local magnetization vanishes—which is the case if $m_0 = 0$—there are no differences between the connected and the non-connected correlation functions. However, as soon as the average of the local magnetization is non-zero (e.g., when the system is prepared in a magnetized initial state $m_0 \neq 0$), the connected correlation function of the magnetization differs from the non-connected one only at $Q = 0$ and the subtraction of the non-connected part alters its original scaling behavior. In turn, this subtraction affects also the scaling behavior of the response function. For a different observable, e.g., the energy, this difference in the scaling behavior might emerge also in the case of $m_0 = 0$. Focusing here on the correlation and response functions of the order parameter of the spherical model ($S$), one finds that for $m_0 = 0$, the correlation and response functions of global quantities can be simply obtained as the limits $Q \to 0$ of the local quantities corresponding to a non-vanishing $Q$ and that they coincide with the same quantities in the $O(n \to \infty)$ model. For $m_0 \neq 0$ and local space integrals of the order parameter (i.e., $Q \neq 0$, for which the non-connected part vanishes), instead, the correlation and response functions are given by (see equations (2), (13) and (22) in [24])

$$R^S_Q(t > t', t') = \left( \frac{t}{t'} \right)^{\rho/2} \left( \frac{1 + t'/\tau_m}{1 + t/\tau_m} \right)^{1/2} e^{-\omega(t-t')},$$

(57)

where $\omega \equiv \omega_Q = 2 \sum_{i=1}^{d} (1 - \cos Q_i)$, $\rho = (4 - d)/2 = \epsilon/2$ for $2 < d < 4$, whereas $\rho = 0$ for $d > 4$ (see equation (14) in [24]), $\tau_m = c m_0^{-2}$ (see equation (15) in [24]). The correlation

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function turns out to be, for \( t > t' \),
\[
C^S_{Q}(t, t') = T_c \times \frac{2(t't')^{d/2}}{[(1 + t/\tau_m)(1 + t'/\tau_m)]^{1/2}} e^{-\omega(t+t')} \int_0^{t'} dt_1 (t_1)^{-\rho}(1 + t_1/\tau_m)e^{2\omega t_1}. \tag{58}
\]

In equations (57) and (58), the superscript \( S \) indicates that the expression refers to the spherical model. Comparing equations (57) and (58) with equations (41) and (42), respectively, one concludes that, in the limit of small momenta (such that \( \omega = \omega_Q \simeq Q^2 \)),
\[
R^S_{Q \neq 0}(t, t') = R^\pi_{Q}(t, t') \quad \text{and} \quad C^S_{Q \neq 0}(t, t') = C^\pi_{Q}(t, t'), \tag{59}
\]
up to non-universal factors, for all dimensions \( d > 2 \) (one can easily check that the equality holds also for the case \( d > 4 \), for which the Gaussian approximation is exact), and for all values of the initial magnetization \( m_0 \neq 0 \). We emphasize that the left-hand sides of equation (59) refer to the order parameter of the spherical model, whereas the corresponding right-hand sides refer to the transverse fluctuations (\( \pi \)) of the order parameter of the \( O(n \to \infty) \) model. These equalities explain the fact—already pointed out right after equations (73) and (126) in [22] and at the end of section 5 in [24]—that the asymptotic value of the fluctuation-dissipation ratio for local integrals of the order parameter in the spherical model is the same as the corresponding one for transverse fluctuations in the \( O(n \to \infty) \) model.

Consider now the global order parameter, i.e., the integral over the whole space of the local order parameter, corresponding to the Fourier mode with \( Q = 0 \): its statistical average does not vanish for \( m_0 \neq 0 \) and therefore its connected two-time correlation function differs from the non-connected one and cannot be simply obtained as the limit for \( Q \to 0 \) of the same quantity for \( Q \neq 0 \). The explicit expression for the correlation \( C_{Q=0} \) and response \( R_{Q=0} \) functions of the global order parameter for \( d > 4 \) and arbitrary value of the magnetization \( m_0 \) were reported in [24] (see equations (48) and (47) therein):
\[
R^S_{Q=0}(t, t') = \left( \frac{1 + t'/\tau_m}{1 + t/\tau_m} \right)^{3/2}, \tag{60}
\]
and (written in a slightly different form compared to [24])
\[
C^S_{Q=0}(t, t') = T_c \times \frac{2 \int_0^{t'} dt_1 (1 + t_1/\tau_m)^3}{[(1 + t/\tau_m)(1 + t'/\tau_m)]^{3/2}}. \tag{61}
\]

By comparing these expressions with equations (57) and (58), one realizes that—as anticipated above—the former are not given by the limit \( Q \to 0 \) (i.e., \( \omega \to 0 \)) of the latter. On the other hand, the expressions for \( R^S_{Q=0} \) and \( C^S_{Q=0} \) for the spherical model for \( d > 4 \) are the same (up to an irrelevant multiplicative factor for \( C \)) as the one for the longitudinal fluctuations (\( \sigma \)) with \( Q = 0 \) of the order parameter of the \( O(n \to \infty) \) model, given by equations (58) and (59) of [19]:
\[
R^S_{Q=0}(t, t') = R^\sigma_{Q=0}(t, t') \quad \text{and} \quad C^S_{Q=0}(t, t') = C^\sigma_{Q=0}(t, t') \quad (d > 4), \tag{62}
\]
where, as the superscripts indicate, the left-hand sides refer to the spherical model whereas the right-hand sides refer to the longitudinal fluctuations in the \( O(n \to \infty) \) model. (Note
that for the case \( d > 4 \) currently discussed, there is no difference between the longitudinal fluctuations of the \( O(n) \) and \( O(1) \) model, and therefore the expressions for \( R_{Q=0}^m \) and \( C_{Q=0}^m \) can be read from [19], in which the latter, i.e., the Ising model, is actually studied.) Even though we have proven the relation (62) only in the case \( d > 4 \), one heuristically expects it to be valid also for \( d < 4 \), as was the case for the analogous relation (59). The expressions for the correlation and response functions of the global magnetization of the spherical model for \( 2 < d < 4 \) and a generic value of the initial magnetization \( m_0 \) (i.e., \( \tau_m \)) can be found in [24]. While the expression for \( R_{Q=0}^S \) is explicitly given by equation (58) therein,

\[
R_{Q=0}^S(t > t', t') = \left( \frac{t}{t'} \right)^{1-d/4} \left( \frac{1 + t'/\tau_m}{1 + t/\tau_m} \right)^{1/2} \left[ 1 - \frac{t/\tau_m}{1 + t/\tau_m} \left( 1 - \frac{t'}{t} \right)^{d/2-1} \right],
\]

(63)

the corresponding expression for the correlation function \( C_{Q=0}^S(t, t') \) is significantly more involved and it has been worked out explicitly only in the asymptotic regime \( t' \ll t \). For the \( O(n) \) model, instead, the response and correlation functions \( R_{Q=0}^m(t, t') \) and \( C_{Q=0}^m(t, t') \), respectively, have been calculated for \( d < 4 \) only at first order in the \( \epsilon \)-expansion around \( d = 4 \) (with \( \epsilon = 4 - d \)) and in the limit of large magnetization, i.e., for \( \tau_m \to 0 \). These expressions, reported in [22] can be compared in the limit \( n \to \infty \) with the first term of the expansion around \( d = 4 - \epsilon \) of the corresponding result for the spherical model. In particular, focusing on the response function for \( n \to \infty \), one has (see equations (88) and (89) in [22])

\[
R_{Q=0}^m(t > t', t') = \left( \frac{t}{t'} \right)^{3/2} \left\{ 1 - \frac{\epsilon}{4} \ln \frac{t}{t'} + \frac{\epsilon}{2} \left( \frac{t}{t'} - 1 \right) \ln \left( 1 - \frac{t'}{t} \right) \right\} + O(\epsilon^2)
\]
\[
= \left( \frac{t}{t'} \right)^{(2-\epsilon)/4} \left\{ 1 - \left( 1 - \frac{t'}{t} \right)^{1-\epsilon/2} \right\} + O(\epsilon^2),
\]

(64)

which is indeed equal to the first-order expansion of equation (63) around \( d = 4 \) and for \( \tau_m = 0 \). In order to do an analogous comparison for the correlation functions \( C_{Q=0}^S(t, t') \) and \( C_{Q=0}^m(t, t') \) for \( d < 4 \) one would have to calculate the limit \( n \to \infty \) of the results for \( C_{Q=0}^S(t, t') \) presented in [22] for \( \tau_m = 0 \) (in particular, see equations (93), (B31), (B24), and (B29) therein) and compare it with the limit \( \tau_m \to 0 \) of the correlation function \( C_{Q=0}^S(t, t') \) of the spherical model explicitly presented in [24] for the case \( t' \ll t \). However, this somewhat lengthy calculation can be avoided by noting that this equality up to \( O(\epsilon^2) \) is implied by the one between the response functions of the spherical and of the \( O(n \to \infty) \) model up to \( O(\epsilon^2) \), together with the fact that the asymptotic values of the (fluctuation-dissipation) ratio \([\partial t C(t, t')]/R(t, t')\) for \( t \gg t' \) and \( \tau_m \to 0 \) are the same in the two models (see right after equation (105) in [24]).

Summing up, the results of [19], [22]–[24] suggest the equalities of equations (59) and (62) for the order parameter response and correlation functions of the spherical and of the \( O(n \to \infty) \) models relaxing from an initial state with \( m_0 \neq 0 \). Whereas equation (59) can be explicitly checked for \( d > 2 \) and all values of the initial magnetization \( m_0 \neq 0 \), equation (62) is proven for \( d > 4 \) with generic values of \( \tau_m \) and for \( 2 < d < 4 \) with \( \tau_m = 0 \) but only in the limit \( t' \ll t \). If, as is likely, this relation extends to the remaining cases, it would be interesting to understand its deeper motivation as it connects, together with equation (59), different degrees of freedom in different models.
According to the correspondence highlighted above, for a spatially constant non-vanishing initial value of the order parameter $m_0$, the persistence properties of the global order parameter of the spherical model are (up to non-universal factors) the same as the ones for the global longitudinal fluctuations ($\sigma$) of the order parameter in the O($n \to \infty$) model. However, the persistence properties of the global order parameter of a manifold of non-vanishing codimension $D \neq 0$ in the spherical model are (up to non-universal factors) the same as the ones for the global transverse fluctuations ($\pi$) of the order parameter of the same manifold in the O($n \to \infty$) model. A consequence of this correspondence is that the small codimension expansion which typically allows for a perturbative access to the persistence exponent of the manifold ($D \neq 0$) within the spherical model is not an expansion about the persistence exponent of the global order parameter of the entire system ($D = 0$), as is highlighted by the fact that the former and the latter actually involve different degrees of freedom ($\pi$ and $\sigma$, respectively) in the corresponding O($n \to \infty$) model.

In passing we mention that the result reported in [16] for the global persistence exponent $\theta_\infty$ of the spherical model (i.e., with $D = 0$ and $m_0 \neq 0$) is incorrectly based on the expression of the (connected) correlation function $C_m(t, t')$ of the global magnetization for $t \gg t'$ reported in equation (8.108) of [23]. Indeed, the knowledge of $C_m(t \gg t', t')$ gives access only to the associated exponent $\mu_\infty = d/4 + 1$ within the Markovian approximation (which is correctly reported in [16]), whereas $\theta_\infty \neq \mu_\infty$ is actually determined by the full functional form of $C_m(t, t')$ for generic values of $t$ and $t'$, which is not explicitly provided in [23]. The correspondence discussed above conveniently provides some information on the global (i.e., $D = 0$) $\theta_\infty$ for the spherical model with $m_0 \neq 0$ on the basis of the corresponding results for the O($n \to \infty$) model presented in [16]: Indeed, it implies that $\theta_\infty^\sigma = \theta_\infty^\pi$, in which the left-hand side refers to the spherical model whereas the right-hand side refers to the longitudinal fluctuations of the O($n \to \infty$) model. For the O($n$) model one finds $\theta_\infty^\sigma = \mu_\infty^\sigma \times \mathcal{R}^\sigma$, where $\mu_\infty^\sigma = 1 + d/(2z)$ and $\mathcal{R}^\sigma = 1 + \epsilon\left([0.115\ldots+n0.131\ldots]/(8+n)\right)+\mathcal{O}(\epsilon^2)$ with $\epsilon = 4 - d$ [16], so, in the limit $n \to \infty$, $\mu_\infty^\sigma \to 1 + d/4 \times [1 + \epsilon \times 0.131\ldots + \mathcal{O}(\epsilon^2)]$ which does indeed show a non-Markovian correction. Even though it would be nice to have a direct check of this prediction on the basis of the results of [23] for generic $d$, the direct calculation of $\theta_\infty^\sigma$ turns out to be rather involved and it is beyond the scope of the present study.

6. Conclusions and perspectives

In summary, we have investigated both analytically and numerically the persistence probability $P_\zeta(t)$ of a $d'$-dimensional manifold within a $d$-dimensional system which relaxes at the critical point from an initial state with non-vanishing value of the magnetization or, more generally, order parameter $m_0$. Such a persistence probability is defined as the probability that the fluctuating order parameter does not cross its average value up to time $t$. Depending on the value of the parameter $\zeta \equiv (D - 2 + \eta)/z$, with $D = d - d'$ we found that, as in the case $m_0 = 0$, the long-time decay of $P_\zeta(t)$ is (i) exponential for $\zeta > 1$, (ii) stretched exponential for $0 \leq \zeta \leq 1$ and (iii) algebraic for $\zeta < 0$. While in the first two cases (i) and (ii) the asymptotic behavior of $P_\zeta(t)$ is not affected by a finite value of $m_0$, in the third case $\zeta < 0$ we demonstrated that $P_\zeta(t)$ exhibits a temporal crossover.
between an early-time and a distinct late-time algebraic decay, which are characterized by two different exponents \( \theta_0(d, d') \) and \( \theta_{\infty}(d, d') \), respectively, with \( \theta_{\infty}(d, d') > \theta_0(d, d') \) (see table 1). Analogously to the case \( D = 0 \), the crossover is controlled by the time scale \( \tau_m \propto m_0^{-1/\kappa} \). The analytic determination of the associated exponents \( \theta_{0,\infty}(d, d') \) is rather non-trivial already within the Gaussian approximation (which becomes exact for \( d > 4 \)) because the stochastic process under study turns out to be non-Markovian for \( D \neq 0 \). In order to calculate \( \theta_{\infty}(d, d') \) (which was already studied in [13]) we performed a perturbative expansion up to first order in the codimension \( D = d - d' \) of the manifold (see equations (26), (28), (33) and (34)). Then we presented a perturbative approach which allows one to calculate the effects of non-Gaussian fluctuations in a dimensional expansion around the space dimensionality \( d = 4 \). Combining these two expansions we obtained analytic estimates of \( \theta_\infty(d, d') \) and \( \theta_0(d, d') \) up to order \( O(D, \epsilon) \) (see section 3.2 and figure 2). In order to assess the reliability of these analytic estimates and to go beyond the perturbation theory, we studied the critical relaxation of the Ising model with Glauber dynamics on a \( d \)-dimensional hypercubic lattice. In particular we computed the persistence probability of the magnetization of a line for \( d = 2 \) and of a plane for \( d = 3 \)—corresponding to codimension \( D = 1 \). In both these cases we observed a temporal crossover in \( P_c(t) \) between two algebraic decays, as predicted by our analytical investigation and we determined the numerical estimates of the associated exponents \( \theta_\infty^{\text{MC}}(d = 2, d' = 1) \) and \( \theta_{\infty}^{\text{MC}}(d = 3, d' = 2) \) as well as more accurate estimates of \( \theta_0^{\text{MC}}(d = 2, d' = 1) \) and \( \theta_0^{\text{MC}}(d = 3, d' = 2) \). In the case of vanishing codimension \( D = 0 \)—primarily investigated in [16]—the agreements between the corresponding numerical and analytical estimates of \( \theta_{0,\infty} \) are very good in both two and three dimensions, as summarized here in figure 2(b). For \( D = 1 \), instead, \( \theta_\infty^{\text{MC}}(d = 3, d' = 2) \) is in rather good agreement with the corresponding analytical estimate, whereas \( \theta_\infty^{\text{MC}}(d = 2, d' = 1) \) is significantly larger than the value predicted by our perturbative calculation—see figure 2(a) and section 4.3 for the comparison. In addition, it turns out that \( \theta_{\infty}^{\text{MC}}(d = 3, d' = 2) < \theta_\infty^{\text{MC}}(d = 2, d' = 1) \) while all the analytical approaches presented here (including the analysis of the \( O(n \to \infty) \) model) suggest that the converse should be true. Our numerical simulations also unveiled a non-trivial scaling of the persistence probability with the characteristic time \( \tau_m \), which remains to be understood. These latter two intriguing features of the persistence probability of the manifold certainly deserve further investigations beyond the preliminary one presented here.

Finally, we have complemented our analysis of the persistence properties with a thorough comparison between the non-equilibrium dynamics of the \( O(n \to \infty) \) model and of the spherical model. While they are strictly equivalent as far as their equilibrium properties are concerned, we have pointed out that such an equivalence has to be carefully qualified when discussing their non-equilibrium dynamics. In particular, in the case of the critical relaxation from an initial state with non-vanishing order parameter, an unexpected connection emerges between the local order parameter of the spherical model and the transverse components of the order parameter in the \( O(n \to \infty) \) model as well as between the global order parameter of the former and the longitudinal components of the latter—see section 5.2 for details. This connection is related to the fact—pointed out in [23,24]—that within the spherical model the correlation function of global quantities (corresponding to a vanishing wavevector \( Q = 0 \)) cannot be...
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obtained as the limit \( Q \to 0 \) of the associated local correlations (corresponding to \( Q \neq 0 \)).

In view of the results presented here, it would certainly be interesting to analyze the consequences of a finite initial magnetization \( m_0 \) for other relevant properties which characterize the temporal evolution of the thermal fluctuations \( \delta m(t) \) of the magnetization of a manifold. As an example, it was recently shown in [37, 38] that the longest excursion \( l_{\text{max}}(t) \) between two successive zeros of a stochastic process up to time \( t \) is an interesting quantity which characterizes the ‘history’ of the stochastic process, and whose asymptotic behavior depends qualitatively on the value of the persistence exponent of the process being larger or smaller than a certain critical value \( \theta_c \) [37]. Given that, for the systems studied here, \( \theta_0(d, d') < \theta_c < \theta_\infty(d, d') \), one expects an intriguing dynamical crossover in the growth of the average \( \langle l_{\text{max}}(t) \rangle \) [37], which certainly deserves further investigations.

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