Mixed State Entanglement of Assistance and the Generalized Concurrence

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We consider the maximum bipartite entanglement that can be distilled from a single copy of a multipartite mixed entangled state, where we focus mostly on $d \times d \times n$-dimensional tripartite mixed states. We show that this assisted entanglement, when measured in terms of the generalized concurrence (named G-concurrence) is (tightly) bounded by an entanglement monotone, which we call the G-concurrence of assistance. The G-concurrence is one of the possible generalizations of the concurrence to higher dimensions, and for pure bipartite states it measures the geometric mean of the Schmidt numbers. For a large (non-trivial) class of $d \times d$-dimensional mixed states, we are able to generalize Wootters formula for the concurrence into lower and upper bounds on the G-concurrence. Moreover, we have found an explicit formula for the G-concurrence of assistance that generalizes the expression for the concurrence of assistance for a large class of $d \times d \times n$-dimensional tripartite pure states.

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I. INTRODUCTION

Preparation of entanglement between distant parties is an important task required for quantum communication and quantum information processing [1]. In the last years there has been an intensive research in the field of quantum communication which yields a variety of methods to distribute bipartite entanglement, such as entanglement swapping [2], quantum repeaters [3], entanglement of assistance [4, 5], localizable entanglement [6] and remote bipartite entangled state preparation [7, 8]. Nevertheless, due to the lack of a complete understanding of mixed state entanglement and multi-partite entanglement, it is not always clear what is the optimal way to distribute entanglement among distant parties.

For example, suppose a supplier of entanglement, named “Sapna”, wishes to create entanglement between two distant parties, say, Alice and Bob. Suppose also that Sapna creates the entanglement between Alice and Bob in two steps: first, she starts with three entangled qudits in a pure state (such as the GHZ state) and sends, via quantum channels, two of the three qudits to Alice and Bob. Due to the operations of the quantum channels, the pure tripartite entangled state degrades to a mixed tripartite entangled state. Second, she performs a measurement on her qudit and sends the classical result to Alice and Bob. In this way, she can assist Alice and Bob to increase their shared entanglement.

For the more simple case in which Alice, Bob and Sapna sharing a pure tripartite state, the maximum average of entanglement that Sapna can prepare between Alice and Bob is called “Entanglement of Assistance” (EoA) [4, 5]. Very recently [9, 10], an explicit formula for the EoA has been found in the asymptotic limit of many copies of tripartite pure states. However, for a single copy, in general, it is very hard to find Sapna’s optimal measurement. Moreover, the optimal measurement depends on the choice of the measure of entanglement, and therefore it is extremely helpful to work with computationally manageable measure of entanglement.

In this paper, we find a tight upper bound on the amount of bipartite entanglement that can be distilled from a single copy of a $d \times d \times n$ tripartite mixed state. For $d = 2$ our bound is given in terms of the convex roof extension of the concurrence of assistance [11], and for $d > 2$ in terms of an entanglement monotone which we call the G-concurrence [8] since for a pure bipartite state it is equal to the geometric mean of the Schmidt numbers. We show that the G-concurrence (GC) is a computationally manageable measure of entanglement, and in fact, for a large class of states, we find an explicit formula for the G-concurrence of assistance (GCoA), as well as lower and upper bounds for the GC. These formulas, naturally generalize the concurrence of assistance [11] and Wootters formula [12] to higher dimensions. Furthermore, the generalization of the GCoA to multipartite mixed states yields an entanglement monotone which provides an upper bound on the localizable entanglement of the parties.

The paper is organized as follows: In section II we define assisted entanglement as the distillable bipartite entanglement from a single copy of tripartite mixed state and discuss its difference from entanglement of assistance. We then define the GCoA as well as the assisted G-concurrence. In section III, we show that the GCoA is an entanglement monotone and in section IV we show that it provides an upper bound on the assisted entanglement. Then, in section V we find explicit formulas and bounds for the GC and the GCoA. Finally, in section VI we end with a short summary and conclusions.
II. ASSISTED ENTANGLEMENT VERSUS
ENTANGLEMENT OF ASSISTANCE

A. Entanglement of Assistance of pure states

For a pure tripartite state, \(|\psi\rangle_{ABS}\), the EoA is defined by [3, 4]:

\[
E_a(\langle \psi \rangle_{ABS}) \equiv \max \sum_k p_k E(|\phi_k\rangle_{AB}),
\]

which is maximized over all possible decompositions of \(\rho_{AB} = \sum_k p_k |\phi_k\rangle_{AB} \langle \phi_k|\), where \(\rho_{AB} \equiv \text{Tr}_C (|\psi\rangle_{ABS} \langle \psi|)\).

In general, a distribution of states that maximizes Eq. (1) for a given entanglement measure \(E\) will not necessarily be the optimal distribution for a different measure. Therefore, the choice of measure is important and depends on the planned quantum information task by Alice and Bob subsequent to Sapna’s assistance.

Eq. (1) provides the maximum average entanglement that Sapna can create between Alice and Bob because any decomposition of \(\rho_{AB}\) can be realized by a generalized measurement performed by Sapna [3]. To see that (for more details see [13]), let us first write \(|\psi\rangle_{ABS}\) in the following Schmidt decomposition: \(|\psi\rangle_{ABS} = \sum_{k=1}^{n} \sqrt{p_k} |\phi_k\rangle_{AB} |k\rangle_S\), where \(n \leq d\) is the number of positive (non-zero) Schmidt numbers, \(p_k\). Hence, Alice and Bob share the state \(\rho_{AB} \equiv \sum_{k=1}^{n} p_k |\phi_k\rangle_{AB} \langle \phi_k|\).

Now, for a given \(m \times m\) unitary matrix \(U\) (\(m \geq n\)), we define the following \(m\) (possibly non-normalized and non-orthogonal) states:

\[
|v_i\rangle_S \equiv \sum_{k=1}^{n} U_{ik} |k\rangle_S, \quad l = 1, 2, ..., m.
\]

Since \(U\) is unitary we have \(\sum_{i=1}^{m} E_i = 1\), with \(E_i \equiv |v_i\rangle \langle v_i|\); that is, the operators \(\{E_i\}\) form a POVM with \(m\) elements. We take this POVM to describe Sapna’s measurement. It can be shown (see [13]) that any decomposition of \(\rho_{AB}\) can be realized by this measurement of Sapna with an appropriate choice of \(U\). Note that this POVM can be realized on the original system plus ancilla.

B. Assisted Entanglement

Here we define assisted entanglement, \(A_E\), as the maximum average of entanglement (measured with \(E\)) that Alice and Bob can share after a general tripartite LOCC. That is, in general, it is not known if the EoA, as defined in Eq. (1), always provides the maximum average of entanglement under general three-party LOCC (i.e. not only by Sapna’s measurement). Thus, for a single copy of pure tripartite states,

\[
E_a \leq A_E.
\]

First, note that from its definition as the maximum distillable bipartite entanglement, \(A_E\), is an entanglement monotone on tripartite pure and mixed states. Second, if there exist a measure, \(E\), for which \(E_a < A_E\) for some pure state, then in this case the EoA when measured with \(E\) can not be considered as an entanglement monotone for tripartite pure states. Thus, it is very important to find measures of entanglement such that for pure states \(E_a = A_E\). After the next subsection we show that the GC satisfies this requirement.

C. Definitions and notations

For a pure bipartite state, \(|\psi\rangle\), the GC is defined as the geometric mean of the (non-negative) Schmidt numbers

\[
G(|\psi\rangle) \equiv d(\lambda_0 \lambda_1 \cdots \lambda_{d-1})^{\frac{1}{d}} = d \left[\text{Det}(A_{11})\right]^{\frac{1}{d}},
\]

where the matrix elements of \(A\) are \(a_{ij} = \langle \psi| \phi_j \rangle |j\rangle\). For a mixed \(d \times d\)-dimensional bipartite state, \(\rho\), the GC is defined in terms of the convex roof extension:

\[
G(\rho) = \min \sum_i p_i G(|\psi_i\rangle) \left( \rho = \sum_i p_i |\psi_i\rangle \langle \psi_i| \right),
\]

where the minimum is taken over all decompositions of \(\rho\). In [8] it has been shown that the GC as defined in Eqs. (5) is a bipartite entanglement monotone. We now show that it is possible to define a tripartite entanglement monotone in terms of the GC.

Definition 1. (a) For pure \(d \times d \times n\)-dimensional tripartite states the G-concurrence of assistance (GCoA), \(G_a\), is defined by

\[
G_a(|\psi\rangle_{ABS}) = \max \sum_i p_i G(|\phi_i\rangle_{AB}),
\]

where the maximum is taken over all the decompositions of \(\rho_{AB} \equiv \text{Tr}_C |\psi\rangle_{ABS} \langle \psi| = \sum_i p_i |\phi_i\rangle_{AB} \langle \phi_i|\).

(b) For tripartite mixed states \(\rho_{ABS} = \sum_j q_j |\psi_j\rangle_{ABS} \langle \psi_j|\), the GCoA is defined in terms of the convex roof extension:

\[
G_a(\rho_{ABS}) = \min \sum_j q_j G_a(|\psi_j\rangle_{ABS}),
\]

where the minimum is taken over all decompositions, \(\{q_j, |\psi_j\rangle_{ABS}\}\), of \(\rho_{ABS}\).

Note that with this definition, only for pure tripartite states the GCoA is equal to the maximum possible average of G-concurrence that Alice and Bob can share after Sapna’s measurement.

Definition 2. The assisted G-concurrence, \(A_G\), is defined as the maximum average of G-concurrence that Alice and Bob can share after a general tripartite LOCC.

That is,

\[
A_G(\rho_{ABS}) = \max \sum p_k G(\sigma_k^{AB})
\]
where the maximum is taken over all possible probability distributions of mixed states, \( \{p_k, \sigma_k^{AB}\} \), shared between Alice and Bob after a general tripartite LOCC.

Note that from its definition the assisted G-concurrence can not increase by a general tripartite LOCC. That is, it is an entanglement monotone.

### III. THE G-CONCURRENCE OF ASSISTANCE

In this section we prove that the GCoA is an entanglement monotone. In particular, it follows that for pure states, \( A_G = G_a \). That is, the optimal tripartite LOCC protocol for preparing the maximum possible G-concurrence between Alice and Bob consists of only a single measurement by Sapna.

**Theorem 1.** The G-concurrence of assistance is an entanglement monotone for tripartite mixed states.

For for \( 2 \times 2 \times n \)-dimensional pure states this theorem has been proven in [13].

**Proof.** In order to prove that the GCoA is an entanglement monotone we need to prove two conditions [14]:

(i) For any quantum operation \( \varepsilon_k \) performed (locally) by Alice, Bob or Sapna,

\[
G_a(\rho_{ABS}) \geq \sum_k p_k G_a(\rho_k) ,
\]

where \( p_k \equiv \text{Tr} \varepsilon_k(\rho_{ABS}) \) and \( p_k \equiv \varepsilon_k(\rho_{ABS})/p_k \).

(ii) \( G_a(\rho_{ABS}) \) is a convex function of \( \rho_{ABS} \).

Condition (ii) is trivially satisfied due to the definition of the GCoA in terms of the convex roof extension (see Eq. (7)). In order to prove condition (i) we first assume that \( \varepsilon_k \) is performed locally by Alice and that the initial state is pure; i.e. \( \rho_{ABS} = |\psi\rangle_{ABS} \langle \psi| \). The most general local operation \( \varepsilon_k \) is given in terms of the Kraus operators \( \hat{M}_{k,j} \):

\[
\varepsilon_k(\rho_{ABS}) = \sum_j |\hat{M}_{k,j}\rangle \langle \psi| \hat{M}_{k,j}^\dagger ,
\]

where \( \sum_k \hat{M}_{k,j}^\dagger \hat{M}_{k,j} \leq I_A \) and \( I_A \) is the identity operator in Alice system. As the GC of any bipartite state \( |\phi\rangle_{AB} \) satisfies \( G(A \otimes B|\phi\rangle) = |\text{Det} \hat{A}|^{2/d} |\text{Det} \hat{B}|^{2/d} G(|\phi\rangle) \), we obtain

\[
\sum_k p_k G_a(\rho_k) = \sum_{k,j} |\text{Det} \hat{M}_{k,j}|^{2/d} G_a(|\psi\rangle_{ABS}) .
\]

Thus, condition (i) follows from the geometric-arithmetic inequality:

\[
\sum_{k,j} |\text{Det} \hat{M}_{k,j}|^{2/d} \leq \frac{1}{d} \sum_{k,j} \sqrt[2d]{|\text{Det} \hat{M}_{k,j}^\dagger \hat{M}_{k,j}|} \leq \frac{1}{d} \sum_{k,j} \text{Tr} \hat{M}_{k,j}^\dagger \hat{M}_{k,j} \leq 1 .
\]

We now consider the case in which the local operation \( \varepsilon_k \) is performed by Sapna. We define

\[
|\phi_{jk}\rangle \equiv \frac{1}{\sqrt{q_{jk}}} \hat{M}_{k,j} |\psi\rangle_{ABS} ,
\]

where the normalization factor \( q_{jk} \) is taken such that \( \langle \phi_{jk}|\phi_{jk}\rangle = 1 \). Note that

\[
\rho_k = \sum_j q_{jk} |\phi_{jk}\rangle \langle \phi_{jk}| .
\]

Thus,

\[
\sum_k p_k G_a(\rho_k) \leq \sum_{k,j} p_k q_{jk} G_a(|\phi_{jk}\rangle) = \sum_{j,k} G_a(\hat{M}_{k,j} |\psi\rangle_{ABS} \langle \psi|) \leq G_a(|\psi\rangle_{ABS}) .
\]

Now, for any quantity \( E \) which is defined on mixed states in terms of the convex roof extension, and that satisfy \( E(\rho_{ABS}) \geq \sum_k p_k E(\rho_k) \) for any tripartite pure state, it follows that also \( E(\rho_{ABS}) \geq \sum_k p_k E(\rho_k) \) (see theorem 2 in [14]). Thus, the GCoA is an entanglement monotone.

The definition of the GCoA can be extended to multipartite mixed states. For example, suppose Alice, Bob and two “suppliers” \( S_1 \) and \( S_2 \) sharing four entangled qudits in the state \( |\psi\rangle_{ABS,S_1,S_2} \). We then define the GCoA of the four qudit state as follows:

\[
G_a(|\psi\rangle_{ABS,S_1,S_2}) = \max \sum_k p_k G_a(|\psi^k\rangle_{ABS}) ,
\]

where the maximum is taken over all decompositions of \( \rho_{ABS,S_1,S_2} = \sum_k p_k |\psi^k\rangle_{ABS,S_1,S_2} \langle \psi^k| \), with \( |\psi_{ABS,S_1,S_2}\rangle \equiv \text{Tr}_{S_2} |\psi\rangle_{ABS,S_1,S_2} \langle \psi| \). Using Eq. (16) we can define the GCoA for four-partite mixed states in terms of the convex roof extension (cf Eq. (7)). Using the same lines of the proof above, it is possible to show that with this definition, the GCoA of the four-partite mixed states is also an entanglement monotone. Therefore, in this way we can define the GCoA for any multipartite mixed states and show that, indeed, it is an entanglement monotone. The definition of GCoA for multipartite states is useful especially in the context of localizable entanglement [8].

### IV. UPPER BOUND ON THE ASSISTED G-CONCURRENCE

The GCoA provides an upper bound on the amount of the average entanglement that Alice and Bob can share:

**Theorem 2.** Let Alice, Bob and Sapna share a tripartite mixed state \( \rho_{ABS} \). Then,

\[
A_G(\rho_{ABS}) \leq G_a(\rho_{ABS}) .
\]
Remark. The theorem above can trivially be extended to include more than 3 parties. For example, the GCoA as define in Eq. (10) for four parties (even if sharing a mixed entangled state) provides an upper bound on the localizable entanglement that Alice and Bob can share after local operations by $S_1$ and $S_2$.

Proof. Let $\{q_j, \sigma^j_{AB}\}$ be a probability distribution of tripartite mixed states obtained from $\rho_{ABS}$ after a general tripartite LOCC. Let us also assume that the LOCC protocol is optimal in the sense that

$$A_G(\rho_{ABS}) = \sum_j q_j G(\sigma^j_{AB}),$$

(18)

where $\sigma^j_{AB} = \text{Tr}_c \sigma^j_{ABC}$. Now, since the GCoA is an entanglement monotone,

$$G_a(\rho_{ABS}) \geq \sum_j q_j G_a(\sigma^j_{ABS}) = \sum_{ij} q_{ij} p_{ij} G_a(|\phi^{ij}_{ABS}\rangle),$$

(19)

where for each $j$, $\{p_{ij}, |\phi^{ij}_{ABS}\rangle\}$ is an optimal decomposition of $\sigma^j_{ABS}$. Note also that $\sigma^j_{AB} = \sum_{ij} p_{ij} \sigma^{ij}_{AB}$, where $\sigma^{ij}_{AB} = \text{Tr}_c |\phi^{ij}_{ABC}\rangle \langle \phi^{ij}_{ABC}|$. Thus,

$$G_a(\rho_{ABS}) \geq \sum_{ij} q_{ij} G_a(\sigma^{ij}_{AB}) \geq \sum_j q_j G(\sigma^j_{AB}).$$

(20)

That is, $G_a(\rho_{ABS}) \geq A_G(\rho_{ABS})$.

The bound above on the assisted GC is tight and, in fact, in some cases $G_a = A_G$. As mentioned in the beginning of section III, $G_a = A_G$ for pure tripartite states. Also, in some cases in which Alice, Bob and Sapna sharing a “slightly” mixed $2 \times 2 \times 2$-dimensional state, $G_a = A_G$ up to a second order. Moreover, the bound in Eq. (17) is stronger than the upper bound given in Eq. (8) for mixed state entanglement swapping.

In mixed state entanglement swapping, Sapna shares with Alice the state $\rho_{AS}$, and with Bob the state $\rho_{BS}$ (Sapna holds systems $S_1$ and $S_2$). In Ref. [8] it has been shown that the maximum average GC, $A_G$, that Alice and Bob can share after any tripartite LOCC is bounded by $A_G \leq G(\rho_{AS})G(\rho_{BS})$, and equality is achievable if $\rho_{AS}$ and $\rho_{BS}$ are pure. We now prove a stronger version of this result.

Proposition 3.

$$G_a(\rho_{AS} \otimes \rho_{BS}) \leq G(\rho_{AS})G(\rho_{BS}).$$

(21)

Thus, in general, $G_a$ provides a tighter upper bound on $A_G$.

Proof. Let $\{p_i, |\psi_i\rangle\}$ and $\{q_j, |\phi_j\rangle\}$ be optimal decompositions of $\rho_{AS}$ and $\rho_{BS}$, respectively, such that $G(\rho_{AS}) = \sum_i p_i G(|\psi_i\rangle)$ and $G(\rho_{BS}) = \sum_j q_j G(|\phi_j\rangle)$. Thus,

$$G_a(\rho_{AS} \otimes \rho_{BS}) \leq \sum_{i,j} p_i q_j G_a(|\psi_i\rangle|\phi_j\rangle) = G(\rho_{AS})G(\rho_{BS}),$$

(22)

where we have used the equality $G_a(|\psi_i\rangle|\phi_j\rangle) = G(|\psi_i\rangle)G(|\phi_j\rangle)$ (see Ref. [8]).

V. EXPLICIT FORMULAS AND BOUNDS

In this section we find explicit formulas and bounds for the GC and the GCoA of a pure $d \times d \times n$-dimensional pure state $|\psi\rangle_{ABS}$. We denote by $\rho$ the $d \times d \times n$-dimensional mixed state shared between Alice and Bob after tracing over Sapna’s system. For the GCoA of $\rho$ we find an explicit formula for a non-trivial (large) class of states which generalize to higher dimensions the formula for the concurrence of assistance [11]. For the GC of $\rho$ we find lower and upper bounds that generalize to higher dimensions Wootters formula for the concurrence [12].

We begin with a complete set of orthogonal $d \times d$-dimensional eigenstates, $|\phi_k\rangle$, corresponding to the non-zero eigenvalues of $\rho$, and sub-normalized such that $\langle \phi_k | \phi_k \rangle$ is the $k$th eigenvalue of $\rho$. That is,

$$\rho = \sum_{k=1}^n |\phi_k\rangle \langle \phi_k|,$$

(23)

where $n \leq d^2$ is the rank of $\rho$. Any other decomposition of $\rho = \sum_{l=1}^m |\chi_l\rangle \langle \chi_l|$ is given by [12]

$$|\chi_l\rangle = \sum_{k=1}^n U_{lk} |\phi_k\rangle,$$

(24)

where $m \geq n$ and $U$ is an $m \times m$ unitary matrix. We would like to emphasize here that the action of the unitary group is not on density matrices but rather on different ensembles of the same density matrix $\rho$.

When $d = 2$, one can define a symmetric “tensor” of rank 2 (see Ref. [12]):

$$t^{\phi_{k'}\phi_k}_{kk'} \equiv \langle \phi_k | \tilde{\phi}_{k'} \rangle,$$

(25)

where $|\tilde{\phi}_{k'}\rangle \equiv \sigma_y |\phi_{k'}\rangle$. We call $\tau^{\phi}$ a tensor, because under a change of decomposition $\tau^{\phi}$ transforms like a tensor. That is, for a general decomposition (see Eq. (21a)) we have (see also Ref. [12])

$$\tau^{\chi\chi'}_{kk'} \equiv \langle \chi_l | \tilde{\chi}_{l'} \rangle_{kk'} = \sum_{k=1}^n U_{lk} U_{l'k'} \tau^{\phi_{k'}\phi_k}_{kk'}.$$

(26)

Moreover, since the tensor $\tau^{\phi}$ is symmetric it can be diagonalized. We will show now that for $d > 2$, similarly, one can define a completely symmetric tensor (i.e. invariant under index permutations) of rank $d$.

In order to generalize $\tau^{\phi}_{kk'}$ (see Eq. (25)) into a tensor of rank $d$, let us first rewrite $\tau^{\phi}_{kk'}$ in terms of the coefficients $a^{ij}_{k}$, where

$$|\phi_k\rangle = \sum_{i=1}^d \sum_{j=1}^d a^{ij}_{k} |i\rangle |j\rangle.$$

(27)
Then it is easy to check that Eq. (25) can be written as ($d = 2$):

$$\tau_{k\hat{k}} = \langle \phi_k | \phi_{\hat{k}} \rangle = \det \begin{pmatrix} a_{11}^{k} & a_{12}^{k} \\ a_{21}^{k} & a_{22}^{k} \end{pmatrix} + \det \begin{pmatrix} a_{11}^{\hat{k}} & a_{12}^{\hat{k}} \\ a_{21}^{\hat{k}} & a_{22}^{\hat{k}} \end{pmatrix} .$$

(28)

Thus, in analogy, for $d > 2$, we define a completely symmetric tensor of rank $d$:

$$\tau_{k_1k_2\ldots k_d}^\phi = \frac{d^{d/2}}{d!} \sum_{\sigma} \det \begin{pmatrix} a_{11}^{\sigma(1)} & a_{12}^{\sigma(1)} & \cdots & a_{1d}^{\sigma(1)} \\ a_{21}^{\sigma(2)} & a_{22}^{\sigma(2)} & \cdots & a_{2d}^{\sigma(2)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1}^{\sigma(d)} & a_{d2}^{\sigma(d)} & \cdots & a_{dd}^{\sigma(d)} \end{pmatrix} ,$$

(29)

where $\sigma$ varies over all permutations on $d$ symbols and the factor $d^{d/2}/d!$ has been chosen such that $|\tau_{k\hat{k}k\ldots k}^\phi|^{2/d}$ is the GC of the sub-normalized state $|\phi_k \rangle$. It follows from its definition that $\tau_{k_1k_2\ldots k_d}^\phi$ is symmetric under any permutation of its indexes. In Appendix A we show that under a change of a decomposition, $\tau^\phi$ (in Eq. (28)) transforms like a tensor. That is, for a general decomposition (see Eq. (24)) we prove

$$\tau_{l_1l_2\ldots l_d}^\chi = \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \cdots \sum_{k_d=1}^{n} U_{l_1k_1}U_{l_2k_2}\ldots U_{l_dk_d} \tau_{k_1k_2\ldots k_d}^\phi .$$

(30)

Moreover, note that the G-concurrence of the ensemble of Eq. (28) is conveniently expressed as:

$$\sum_{k=1}^{n} G(|\phi_k \rangle) = \sum_{k=1}^{n} |\tau_{k\hat{k}k\ldots k}^\phi|^{2/d} .$$

(31)

Unlike tensors of rank 2, higher-order tensors (even completely symmetric) cannot be reduced in general to a diagonal form by orthogonal or unitary transformations. We therefore define a class of $d \times d$-dimensional density matrices, $D$, for which the $\tau$-tensors are diagonalizable. That is, $\rho \in D$ if there exist a unitary matrix, $U$, such that $\tau^\chi$ as in Eq. (30) is a diagonal tensor. For $d = 2$ all density matrices belong to $D$. However, for $d > 2$ it is not clear what is the size of the class $D$ and, in fact, very little is known on the diagonalization of a $d^2 \times d^2 \times \cdots \times d^2$ matrix. Nevertheless, note that even though $\tau^\phi$ is a tensor of rank $d$ (represented by a $d^2 \times d^2 \times \cdots \times d^2$ matrix), it is constructed from a $d \times d$-dimensional density matrix $\rho$. That is, it depends on less than $d^2$ parameters, whereas a general tensor of rank $d$ depends on $d^2d$ parameters. Thus, among all the completely symmetric tensors of rank $d$, the $\tau$-tensors are a very small group.

We hope to report in the future on how much smaller is the class of diagonal $\tau$-tensors relative to the group of all $\tau$-tensors. In the following we assume that $\tau_{k_1k_2\ldots k_d}^\phi$ can be diagonalized and therefore, without the loss of generality, we take it to be diagonal; i.e.

$$\tau_{k_1k_2\ldots k_d}^\phi = \lambda_k \delta_{k_1k_2} \cdots \delta_{k_d} ,$$

(32)

where, in general, $\lambda_k$ are complex numbers. However, we can take a diagonal unitary matrix $U$ (see Eq. (24)) with appropriate phases in the diagonal, in order to obtain a diagonal $\tau$-tensor with $|\lambda_k \rangle$ on the diagonal. Thus, without the loss of generality we assume that $\lambda_k$ are real and non-negative.

1. The G-concurrence of assistance

**Theorem 4.** If $\rho \in D$ then

$$G_a(\rho) = \sum_{k=1}^{n} \lambda_k^{2/d} .$$

(33)

**Proof.** We need to show that there is no decomposition with higher average of GC. The average G-concurrence of a general decomposition (see Eqs. (21,31)) is given by:

$$\langle G \rangle_X = \sum_{l=1}^{m} |\tau_{l\hat{l}l\ldots l}^\chi|^{2/d} = \sum_{l=1}^{m} \sum_{k=1}^{n} (U_{lk})^d \lambda_k^{2/d} .$$

(34)

Since $|a + b| \leq |a| + |b|$, where $0 \leq r \leq 1$ and $a, b$ are complex numbers with absolute value less than 1,

$$\langle G \rangle_X \leq \sum_{l=1}^{m} \sum_{k=1}^{n} |U_{lk}|^2 \lambda_k^{2/d} = \sum_{k=1}^{n} \lambda_k^{2/d} .$$

(35)

**Example 1.** Suppose Alice, Bob and Sapna sharing the state

$$|\psi\rangle_{ABS} = \sqrt{p_0}|000\rangle + \sqrt{p_1}|111\rangle + \sqrt{p_2}|222\rangle + \sqrt{p_3}|333\rangle .$$

(36)

Thus, Alice and Bob sharing the mixed state

$$\rho = \sum_{k=0}^{3} |\phi_k \rangle \langle \phi_k | ,$$

(37)

where $|\phi_k \rangle = \sqrt{p_k}|kkk\rangle$. According to Eq. (38), the super symmetric $\tau$-tensor that corresponds to the decomposition in Eq. (37) is given by:

$$\tau_{1234}^\phi = \frac{2}{3} \sqrt{p_1p_2p_3p_4} ,$$

(38)

where for any other set of indexes $(k_1, k_2, k_3, k_4)$ which is not a permutation of $(1, 2, 3, 4)$, $\tau_{k_1k_2k_3k_4}^\phi = 0$. According to Eq. (39), for a different decomposition, $\chi$, we have

$$\tau_{1234}^\chi = \tau_{1234}^\phi \sum_{\sigma} U_{1,\sigma(1)}U_{1,\sigma(2)}U_{1,\sigma(3)}U_{1,\sigma(4)} ,$$

(39)
where $\sigma$ varies over all the 24 permutations of 4 symbols. For the orthogonal matrix

$$U = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

we get a diagonal tensor $\tau^\wedge$. The 4 diagonal elements of $\tau^\wedge$ are all equal to $\lambda_k = \sqrt{\rho_1 \rho_2 \rho_3 \rho_4}$ ($k = 1, 2, 3, 4$). Hence, from theorem 4

$$G_a(|\psi\rangle_{ABS}) = 4 (p_1 p_2 p_3 p_4)^{1/4}.$$ (41)

2. The G-concurrence

In order to find the decomposition with the least average of GC we first prove the following lemma:

**Lemma 5.** If $\rho \in \mathcal{D}$, for any $d^2$ number of phases $\theta_1, \theta_2, ..., \theta_{d^2}$, there exist a decomposition such that the G-concurrence of each (sub-normalized) state in the decomposition is equal to $|\lambda_1 \exp(i\theta_1) + \lambda_2 \exp(i\theta_2) + ... + \lambda_{d^2} \exp(i\theta_{d^2})|^2 / d^2$ (if $n < d^2$ then $\lambda_k \equiv 0$ for $n < k \leq d^2$).

**Proof.** First, we take the unitary matrix $U_{lk} = \delta_{lk} \exp(i\theta_{lk})/d$ and substitute in Eq. (43) to get a decomposition with diagonal $\tau$-tensor whose diagonal elements are $\lambda_k \exp(i\theta_{lk})$. Then, we take a $d^2 \times d^2$-dimensional unitary matrix $U$ with the property that $(U_{lk})^d = 1/d^2$ for all $k$ and $l$. It is left to show that such a unitary matrix exists. In order to show that, we first define a set of $d^2 \times d^2$-dimensional unitary matrices $V^{(a,b)}$ ($a, b = 1, ..., d$) with elements

$$V^{(a,b)}_{jj'} = \frac{1}{d^2} \exp \left( \frac{2\pi i}{d} (j + a)(j' + b) \right) : j, j' = 1, 2, ..., d.$$ (42)

Then we define a $d^2 \times d^2$ matrix $U$:

$$U = \begin{pmatrix} V^{(1,1)} & V^{(1,2)} & ... & V^{(1,d)} \\ V^{(2,1)} & V^{(2,2)} & ... & V^{(2,d)} \\ ... & ... & ... & ... \\ V^{(d,1)} & V^{(d,2)} & ... & V^{(d,d)} \end{pmatrix}.$$ (43)

Note that each $V^{(a,b)}$ above is a $d \times d$ matrix. It is easy to check that the matrix $U$ as defined above is a $d^2 \times d^2$ unitary matrix with the desired property.

Note that the corollary above provides a necessary condition for separability of bipartite mixed states.

**Theorem 7.** If $\rho \in \mathcal{D}$ and $\lambda_1 > \lambda_2 + ... + \lambda_n$

$$\lambda_1^{2/d} - \sum_{k=2}^{n} \lambda_k^{2/d} \leq G(\rho) \leq \left| \lambda_1 - \sum_{k=2}^{n} \lambda_k \right|^{2/d}.$$ (44)

**Proof.** The upper bound follows from the lemma above for $\theta_1 = 0$ and $\theta_l = \pi$ ($2 \leq l \leq n$). In order to prove the lower bound, first note that

$$||a||^r - ||b||^r \leq |a+b|^r - |a|^r + |b|^r$$ (45)

where $0 \leq r \leq 1$ and $a$, $b$ are complex numbers with absolute value less than 1. Hence,

$$\sum_{l=1}^{m} \left( \sum_{k=1}^{n} |U_{lk}|^d \lambda_k \right)^{2/r} \geq \sum_{l=1}^{m} \left( |U_{11}|^2 \lambda_1^2 - \sum_{k=2}^{n} |U_{lk}|^d \lambda_k \right)^{2/r} \geq \lambda_1^{2/d} - \sum_{k=2}^{n} \lambda_k^{2/d}.$$ (46)

Note that for $d = 2$ both the lower and upper bounds reduce to Wootters formula of the concurrence $[12]$. Also, if $\lambda_k = 0$ for $k \geq 2$, then $G(\rho) = G_a(\rho) = \lambda_1^{2/d}$.

**Example 2.** Consider the $3 \times 3$-dimensional mixed state

$$\rho = p|\chi\rangle\langle\chi| + (1-p)|01\rangle\langle01|,$$ (47)

where $|\chi\rangle = (|00\rangle + |11\rangle + |22\rangle)/\sqrt{3}$ is a maximally entangled state. It is easy to check that for this state $\lambda_2 = \lambda_3 = 0$ and $\lambda_1 = p^{3/2}$. Therefore, in this case $G = G_a = p$.

**VI. SUMMARY AND CONCLUSIONS**

In summary, we have considered the maximum bipartite entanglement that can be distilled from a single copy of a multipartite mixed entangled state, where we focused mostly on tripartite mixed states. We have shown that this assisted entanglement, when measured in terms of the G-concurrence (as defined in Eqs. (43)), is (tightly) bounded by the entanglement monotone given in Eqs. (46), which we call the G-concurrence of assistance. The G-concurrence is one of the possible generalizations of the concurrence to higher dimensions, and for pure bipartite states it measures the geometric mean of the Schmidt numbers. For a large (non-trivial) class, $\mathcal{D}$, of $d \times d$-dimensional mixed states, we were able to generalize Wootters formula for the concurrence, where the concurrence is replaced with the G-concurrence. Moreover, we have found an explicit formula for the G-concurrence of assistance for tripartite pure states, $|\psi\rangle_{ABS}$, with $\rho_{AB} \in \mathcal{D}$ ($\rho_{AB} \equiv \text{Tr}_C|\psi\rangle_{ABS}\langle\psi|$).

In addition to GC and GCoA being valuable for finding upper bounds for the assisted entanglement of
three (or more) parties, for $2 \times 2 \times 2$-dimensional pure states the concurrence as well as the concurrence of assistance exhibits monogamy constraints (entanglement tradeoffs) \cite{11,12}. Similarly, we are willing to conjecture that the generalizations to a $d \times d \times d$ dimensional pure state, $|\psi\rangle_{\text{ABS}}$, also holds:

**Conjecture 8.**

$$[G(\rho_{AB})]^d + [G(\rho_{AS})]^d \leq [G_{A(\text{BS})}]^d$$

$$[G(\rho_{AB})]^d + [G(\rho_{AS})]^d \geq [G_{A(\text{BS})}]^d.$$  \hspace{1cm} (48)

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**APPENDIX A**

In this appendix we prove that the function, $\tau^\phi_{k_1k_2...k_d}$, as define in Eq. (29) is indeed a tensor. That is, under a change of a decomposition, $\tau^\phi$ transforms as in Eq. (30).

In order to prove Eq. (30) we start with the definition of $\tau^\phi$:

$$\tau^\phi_{k_1k_2...k_d} = \frac{d^d/2}{d!} \sum_{\sigma} \sum_{\pi} \varepsilon_\pi a_{k_1(1)\pi a_{k_2(2)}...a_{k_d(2)}},$$  \hspace{1cm} (A1)

where $\pi$ and $\sigma$ varies over all permutations on $d$ symbols and $\varepsilon_\pi$ is the signature of the permutation $\pi$.

Now, from Eqs. (24)\textendash(27) it follows that:

$$|\chi\rangle = \sum_{i=1}^{d} \sum_{j=1}^{d} b_{ij}^I |i\rangle |j\rangle, \quad b_{ij}^I = \sum_{k=1}^{d} U_{ik} a_{ij}^k.$$  \hspace{1cm} (A2)

Thus, the tensor $\tau^\phi$ is given by:

$$\tau^\phi_{11...d} = \frac{d^d/2}{d!} \sum_{\sigma} \sum_{\pi} \varepsilon_\pi a_{1(1)\pi a_{2(2)}...a_{d(2)}} \sum_{k=1}^{d} U_{ik} a_{ij}^k.$$  \hspace{1cm} (A3)

Now, since the indexes $k_1$, $k_2$, ..., $k_d$ are dummy indexes, for each permutation $\sigma$ in the above expression we can replace the term

$$U_{\sigma(1)} a_{1(1)} U_{\sigma(2)} a_{2(2)} U_{\sigma(d)} a_{d(2)}$$

with

$$U_{\sigma(1)} a_{1(1)} U_{\sigma(2)} a_{2(2)} U_{\sigma(d)} a_{d(2)}.$$  \hspace{1cm} (A4)

Thus, substituting all this in Eq. (A3) gives

$$\tau^\phi_{11...d} = \frac{d^d/2}{d!} \sum_{k_1=1}^{d} \sum_{k_2=1}^{d} ... \sum_{k_d=1}^{d} \sum_{\pi} \sum_{\sigma} \varepsilon_\pi a_{1(1)\pi a_{2(2)}...a_{d(2)}} \sum_{k=1}^{d} U_{ik} a_{ij}^k.$$  \hspace{1cm} (A5)

That is, $\tau^\phi$ transforms like a tensor.

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