LOWER BOUNDS OF GRADIENT’S BLOW-UP FOR THE LAMÉ SYSTEM WITH PARTIALLY INFINITE COEFFICIENTS

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ABSTRACT. In composite material, the stress may be arbitrarily large in the narrow region between two close-to-touching hard inclusions. The stress is represented by the gradient of a solution to the Lamé system of linear elasticity. The aim of this paper is to establish lower bounds of the gradients of solutions of the Lamé system with partially infinite coefficients as the distance between the surfaces of discontinuity of the coefficients of the system tends to zero. Combining it with the pointwise upper bounds obtained in our previous work, the optimality of the blow-up rate of gradients is proved for inclusions with arbitrary shape in dimensions two and three. The key to show this is that we find a blow-up factor, a linear functional of the boundary data, to determine whether the blow-up will occur or not.

1. INTRODUCTION AND MAIN RESULTS

In high-contrast composites, it is a common phenomenon that high concentration of extreme mechanical loads occurs in the narrow regions between two adjacent inclusions. Extreme loads are always amplified by such microstructure, which will cause failure or fracture initiation. Stimulated by the well-known work of Babuška, et al. [10], where the Lamé system of linear elasticity was assumed and the initiation of damage and fracture in composite materials was computationally analyzed, we consider the Lamé system with partially infinite coefficients to characterize high-contrast composites. The gradient of the solution exhibits singular behavior with respect to the distance between hard inclusions. This paper is a continuation of [14, 15], where a pointwise upper bound of the gradient of solution is established by an iteration technique with respect to the energy, as the distance (say, $\varepsilon$) between the surfaces of discontinuity of the coefficients of the system tends to zero.

The main purpose of this paper is to show the blow-up rates obtained in [14, 15] are actually optimal, by establishing the lower bounds on the gradients of solutions of the Lamé system with partially infinite coefficients in two physically relevant dimensions $d = 2, 3$. Namely, the optimal blow-up rates are, respectively, $\varepsilon^{-1/2}$ in dimension $d = 2$, $(|\varepsilon| \log \varepsilon)^{-1}$ in dimension $d = 3$. Usually, it is not easy to obtain a lower bound. The novelty of this paper is that we introduce a blow-up factor defined by a solution of the limit case when two inclusions touch each other, which is a linear functional of the boundary data to determine whether or not the blow-up to occur. Physically, this factor seems much natural. Here new difficulties need to be overcome, and a number of refined estimates are used in our proof. The introduced methodology allows us define an analogous blow-up factor for the

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perfect conductivity problem considered in [11] and give a new and simple proof for the lower bound estimates.

There have been many works on the analogous question for the following scalar equation with Dirichlet boundary condition, called the conductivity problem,

\[
\begin{cases}
\nabla \cdot (a_k(x) \nabla u_k) = 0 & \text{in } \Omega, \\
u_k = \varphi & \text{on } \partial \Omega,
\end{cases}
\]

(1.1)

where \( \Omega \) is a bounded open set of \( \mathbb{R}^d, d \geq 2 \), containing two \( \varepsilon \)-apart convex inclusions \( D_1 \) and \( D_2 \), \( \varphi \in C^2(\partial \Omega) \) is given, and

\[a_k(x) = \begin{cases} 
k \in (0, \infty) & \text{in } D_1 \cup D_2, \\
1 & \text{in } \Omega \setminus \overline{D_1 \cup D_2}.
\end{cases}\]

For touching disks \( D_1 \) and \( D_2 \) in dimension \( d = 2 \), Bonnetier and Vogelius [18] first proved that \( |\nabla u_k| \) remains bounded. The bound depends on the value of \( k \). Li and Vogelius [38] extended the result to general divergence form second order elliptic equations with piecewise smooth coefficients in all dimensions, and they proved that \( |\nabla u| \) remains bounded as \( \varepsilon \to 0 \). Li and Nirenberg [37] further extended the results in [38] to general divergence form second order elliptic systems including systems of elasticity.

The estimates in [37] and [38] depend on the ellipticity of the coefficients. If ellipticity constants are allowed to deteriorate, the situation is quite different. When \( k = \infty \), the \( L^\infty \)-norm of \( |\nabla u_\infty| \) for the solutions \( u_\infty \) of (1.1) generally becomes unbounded as \( \varepsilon \) tends to 0. The blow-up rate of \( |\nabla u_\infty| \) is respectively \( \varepsilon^{-1/2} \) in dimension \( d = 2 \), \( (\varepsilon|\log \varepsilon|)^{-1} \) in dimension \( d = 3 \), and \( \varepsilon^{-1} \) in dimension \( d \geq 4 \). See Bao, Li and Yin [11], as well as Budiansky and Carrier [20], Markenscoff [41], Ammari, Kang, and Lim [7], Ammari, Kang, Lee, Lee and Lim [8] and Yun [45, 46]. Further, more detailed, characterizations of the singular behavior of \( \nabla u_\infty \) have been obtained by Ammari, Ciraolo, Kang, Lee and Yun [9], Ammari, Kang, Lee, Lim and Zribi [9], Bonnetier and Triki [16, 17] and Kang, Lim and Yun [29, 30]. For related works, see [2, 4–6, 12, 13, 17, 19, 21–28, 32–36, 39, 40, 43, 47] and the references therein.

We follow the notations of [14, 15]. Let \( \Omega \subset \mathbb{R}^d \) be a bounded open set with \( C^2 \) boundary, and \( D_1 \) and \( D_2 \) are two disjoint convex open sets in \( \Omega \) with \( C^{2, \gamma} \) boundaries, \( 0 < \gamma < 1 \), which are \( \varepsilon \) apart and far away from \( \partial \Omega \), that is,

\[
\overline{D_1}, \overline{D_2} \subset \Omega, \quad \text{the principle curvatures of } \partial D_1, \partial D_2 \geq \kappa_0 > 0, \\
\varepsilon := \text{dist}(D_1, D_2) > 0, \quad \text{dist}(D_1 \cup D_2, \partial \Omega) > \kappa_1 > 0,
\]

(1.2)

where \( \kappa_0, \kappa_1 \) are constants independent of \( \varepsilon \). We also assume that the \( C^{2, \gamma} \) norms of \( \partial D_i \) are bounded by some constant independent of \( \varepsilon \). This implies that each \( D_i \) contains a ball of radius \( r_0^* \) for some constant \( r_0^* > 0 \) independent of \( \varepsilon \). Denote

\[
\Omega := \Omega \setminus \overline{D_1 \cup D_2}.
\]

Assume that \( \overline{\Omega} \) and \( D_1 \cup D_2 \) are occupied, respectively, by two different isotropic and homogeneous materials with different Lamé constants \((\lambda, \mu)\) and \((\lambda_1, \mu_1)\). Then the elasticity tensors for the inclusions and the background can be written, respectively, as \( C^1 \) and \( C^0 \), with

\[
C_{ij,kl}^1 = \lambda_1 \delta_{ij} \delta_{kl} + \mu_1 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),
\]
and
\[ C^{ij}_{kl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) , \]
where \( i, j, k, l = 1, 2, 3 \) and \( \delta_{ij} \) is the Kronecker symbol: \( \delta_{ij} = 0 \) for \( i \neq j \), \( \delta_{ij} = 1 \) for \( i = j \). Let \( u = (u_1, u_2, \cdots, u_d)^T : \Omega \to \mathbb{R}^d \) denote the displacement field. For a given vector valued function \( \varphi \), we consider the following Dirichlet problem for the Lamé system
\[
\begin{aligned}
\begin{cases}
\nabla \cdot \left( \chi_{\Omega} C^0 + \chi_{D_1 \cup D_2} C^1 \right) e(u) = 0, & \text{in } \Omega, \\
\varphi, & \text{on } \partial \Omega ,
\end{cases}
\end{aligned}
\]
where \( \chi_D \) is the characteristic function of \( D \subset \mathbb{R}^d \),
\[ e(u) = \frac{1}{2} \left( \nabla u + (\nabla u)^T \right) \]
is the strain tensor.

Assume that the standard ellipticity condition holds for (1.3), that is,
\[ \mu > 0, \quad d\lambda + 2\mu > 0; \quad \mu_1 > 0, \quad d\lambda_1 + 2\mu_1 > 0. \]
(1.4)
For \( \varphi \in H^1(\Omega; \mathbb{R}^d) \), it is well known that there exists a unique solution \( u \in H^1(\Omega; \mathbb{R}^d) \) of the Dirichlet problem (1.3). More details can be found in the Appendix in [14].

Let \( \Psi := \left\{ \psi \in C^1(\mathbb{R}^d; \mathbb{R}^d) \mid 2e(\psi) = \nabla \psi + (\nabla \psi)^T = 0 \right\} \) be the linear space of rigid displacement in \( \mathbb{R}^d \). With \( e_1, \cdots, e_d \) denoting the standard basis of \( \mathbb{R}^d \),
\[ \left\{ e_i, x_j e_k - x_k e_j \mid 1 \leq i \leq d, \ 1 \leq j < k \leq d \right\} \]
is a basis of \( \Psi \). Denote this basis of \( \Psi \) as \( \{ \psi^\alpha \} \), \( \alpha = 1, 2, \cdots, \frac{d(d+1)}{2} \). If \( \xi \in H^1(D; \mathbb{R}^d) \), \( e(\xi) = 0 \) in \( D \), and \( D \subset \mathbb{R}^d \) is a connected open set, then \( \xi \) is a linear combination of \( \{ \psi^\alpha \} \) in \( D \).

For fixed \( \lambda \) and \( \mu \) satisfying (1.4), denoting \( u_{\lambda_1, \mu_1} \) the solution of (1.3). Then, as proved in the Appendix in [14],
\[
\begin{aligned}
\begin{cases}
L_{\lambda, \mu} u := \nabla \cdot (C^0 e(u)) = 0, & \text{in } \Omega , \\
u|_+ = u|_-, & \text{on } \partial D_1 \cup \partial D_2 , \\
e(u) = 0, & \text{in } D_1 \cup D_2 , \\
\int_{\partial D_i} \frac{\partial u}{\partial n^+} \cdot \psi^\beta = 0, & i = 1, 2, \beta = 1, 2, \cdots, \frac{d(d+1)}{2} , \\
u = \varphi , & \text{on } \partial \Omega ,
\end{cases}
\end{aligned}
\]
(1.5)
where
\[ \frac{\partial u}{\partial n^+} := (C^0 e(u)) \vec{n} = \lambda (\nabla \cdot u) \vec{n} + \mu (\nabla u + (\nabla u)^T) \vec{n}, \]
and \( \vec{n} \) is the unit outer normal of \( D_i, i = 1, 2 \). Here and throughout this paper the subscript \( \pm \) indicates the limit from outside and inside the domain, respectively.

The existence, uniqueness and regularity of weak solutions of (1.5), as well as a variational formulation, can be found in the Appendix in [14]. In particular, the
\(H^1\) weak solution is in \(C^1(\Omega; \mathbb{R}^d) \cap C^1(D_1 \cup D_2; \mathbb{R}^d)\). The solution is the unique function which has the least energy in appropriate functional spaces, that is,

\[
I_{\infty}[u] = \min_{v \in \mathcal{A}} I_{\infty}[v], \quad I_{\infty}[v] := \frac{1}{2} \int_{\Omega} \left( C^0 e(v), e(v) \right) dx,
\]

where

\[
\mathcal{A} := \{ u \in H^1_{\varphi}(\Omega; \mathbb{R}^d) \mid e(u) = 0 \text{ in } D_1 \cup D_2 \}.
\]

It is well known, see [44], that for any open set \(O\) and \(u, v \in C^2(O)\),

\[
\int_{O} (C^0 e(u), e(v)) dx = - \int_{O} (\mathcal{L}_{\lambda, \mu} u) \cdot v + \int_{\partial O} \frac{\partial u}{\partial \nu} \cdot v.
\]

Throughout the paper, unless otherwise stated, \(C\) denotes a constant, whose values may vary from line to line, depending only on \(d, \kappa_0, \kappa_1, \gamma, \delta_0\), and an upper bound of the \(C^2\) norm of \(\partial \Omega\) and the \(C^{2, \gamma}\) norms of \(\partial D_1\) and \(\partial D_2\), but independent of \(\varepsilon\). Also, we call a constant having such dependence a universal constant. Let

\[
\rho_d(\varepsilon) = \begin{cases} 
\sqrt{\frac{\varepsilon}{2}}, & d = 2, \\
\frac{1}{\log \varepsilon}, & d = 3.
\end{cases}
\]

In order to show the optimality of the blow-up rate, we first recall the following upper bound estimates established in [14, 15].

**Theorem A.** (Upper Bounds, [14, 15]) For \(d = 2, 3\), assume that \(\Omega, D_1, D_2, \varepsilon\) are defined in (1.2), \(\lambda\) and \(\mu\) satisfy (1.7) for some \(\delta_0 > 0\), and \(\varphi \in C^2(\partial \Omega; \mathbb{R}^d)\). Let \(u \in H^1(\Omega; \mathbb{R}^d) \cap C^1(\Omega; \mathbb{R}^d)\) be the solution of (1.3). Then for \(0 < \varepsilon < 1/2\), we have

\[
\| \nabla u \|_{L^\infty(\Omega; \mathbb{R}^d)} \leq \frac{C \rho_d(\varepsilon)}{\varepsilon} \| \varphi \|_{C^2(\partial \Omega; \mathbb{R}^d)},
\]

where \(C\) is a universal constant.

**Remark 1.1.** Since \(D_1\) and \(D_2\) are two strictly convex subdomains of \(\Omega\), there exist two points \(P_1 \in \partial D_1\) and \(P_2 \in \partial D_2\) such that

\[
\text{dist}(P_1, P_2) = \text{dist}(\partial D_1, \partial D_2) = \varepsilon.
\]

Use \(P_1P_2\) to denote the line segment connecting \(P_1\) and \(P_2\). The proof of Theorem A actually gives us the following stronger estimates for \(x \in \Omega\):

\[
|\nabla u(x)| \leq \begin{cases} 
\frac{C}{\varepsilon + \text{dist}(x, P_1P_2)} ||\varphi||_{C^2(\partial \Omega; \mathbb{R}^d)}, & d = 2; \\
\left( \frac{C}{|\log \varepsilon| (\varepsilon + \text{dist}(x, P_1P_2))} + \frac{C\text{dist}(x, P_1P_2)}{\varepsilon + \text{dist}(x, P_1P_2)} \right) ||\varphi||_{C^2(\partial \Omega; \mathbb{R}^d)}, & d = 3.
\end{cases}
\]
The pointwise upper bound in (1.3) shows that the gradient $|\nabla u(x)|$ (at least the right hand side of (1.3)) would achieve its maximum on the segment $P_1P_2$ if the blow-up occurred. However, whether the blow-up occurs or not depends totally on the boundary data $\varphi$ for given domain $\Omega$, $D_1$ and $D_2$ with suitable smoothness. Therefore, in order to show that the blow-up rate of the gradients obtained in Theorem A is optimal, it is necessary to establish the lower bound of $|\nabla u(x)|$ on the segment $P_1P_2$ with the same blow-up rate.

To this aim, a key ingredient is to find a function $u^*_b$, one part of the limit function of $u$ as $\varepsilon$ tends to zero. Denote $D^*_1 := \{ x \in \mathbb{R}^d \mid x + P_1 \in D_1 \}$ and $D^*_2 := \{ x \in \mathbb{R}^d \mid x + P_2 \in D_2 \}$. Set $\Omega^* := D \setminus D^*_1 \cup D^*_2$. Let $u^*_b$ be the solution of the following boundary value problem:

$$
\begin{cases}
  L_{\lambda, \mu} u^*_b = 0, & \text{in } \Omega^*, \\
  u^*_b = \sum_{\alpha=1}^d C^\alpha \psi^\alpha, & \text{on } \partial D^*_1 \cup \partial D^*_2, \\
  u^*_b = \varphi(x), & \text{on } \partial D,
\end{cases}
\tag{1.10}
$$

where the constants $C^\alpha$, $\alpha = 1, 2, \cdots, d$, are determined later. We remark that $u^*_b$ is smooth near the origin by theorem 1.1 in [35]. In order to capture the lower bound of $|\nabla u|$, we now introduce a vector-valued linear functional of $\varphi$,

$$
b^*_b[\varphi] := \int_{\partial D^*_1} \frac{\partial u^*_b}{\partial \nu} \cdot \psi^\beta, \quad \beta = 1, 2, \cdots, d.
\tag{1.11}
$$

Notice that these quantities are independent of $\varepsilon$. It will be turn out that they will determine whether or not the blow-up to occur. We call them blow-up factors. Their important role will be shown in next section. The main result of this paper is the following lower bounds of $|\nabla u|$ on $P_1P_2$.

**Theorem 1.2.** (Lower Bounds for $d = 2, 3$). For $d = 2, 3$, under the assumptions as in Theorem A, let $u \in H^1(\Omega; \mathbb{R}^d) \cap C^1(\overline{\Omega}; \mathbb{R}^d)$ be a solution to (1.3). Then if there exists a $\varphi$ such that $b^*_{k_1}[\varphi] \neq 0$ for some integer $1 \leq k_0 \leq d$, then for sufficiently small $0 < \varepsilon < 1/2$,

$$
|\nabla u(x)| \geq \frac{\rho_d(\varepsilon)}{C \varepsilon} |b^*_{k_1}[\varphi]|, \quad \text{for } x \in P_1P_2,
$$

where $C$ is a universal constant.

**Remark 1.3.** Theorem 1.2, together with Theorem A, shows that the optimal blow-up rate of $|\nabla u|$ is $\frac{\rho_d(\varepsilon)}{\varepsilon}$, namely, $\varepsilon^{-1/2}$ in dimension $d = 2$, $(\varepsilon \log \varepsilon)^{-1}$ in dimension $d = 3$. These generic blow-up rates are actually the same as the scalar case [11], as well as expected in [14, 15]. Of course, we also can define $b^*_b[\varphi]$ on the boundary $\partial D^*_2$. We would like to point out that Kang and Yu [31] proved the blow-up rate $\varepsilon^{-1/2}$ is optimal under a stronger assumption that inclusions are of $C^{3,\alpha}$ in dimension two by using a singular function. The method is totally different with ours. Here we only assume that $\partial D_1$ and $\partial D_2$ are of $C^{2,\gamma}$ as before.
For the convenience of application, we give the corresponding results for the following perfect conductivity problem

\[
\begin{aligned}
\Delta u &= 0, & \text{in } \tilde{\Omega}, \\
u_u &= u|_+, & \text{on } \partial D_1 \cup \partial D_2, \\
\nabla u &= 0, & \text{in } D_1 \cup D_2, \\
\int_{\partial D_i} \frac{\partial u}{\partial n} |_+ &= 0, & i = 1, 2, \alpha = 1, 2, \ldots, \frac{d(d+1)}{2}, \\
u &= \varphi, & \text{on } \partial \Omega.
\end{aligned}
\] (1.12)

The proof is much simpler and shorter than that for the elasticity case. An analogous blow-up factor is defined by

\[
b^*_1[\varphi] := \int_{\partial D^*_1} \frac{\partial u^*}{\partial \nu}.
\] (1.13)

where \( u^* \) satisfies

\[
\begin{aligned}
\Delta u^* &= 0, & \text{in } \tilde{\Omega}^*, \\
u^* &= C^*, & \text{on } \partial D^*_1 \cup \partial D^*_2, \\
u^* &= \varphi, & \text{on } \partial \Omega.
\end{aligned}
\]

and the constant \( C^* \) is uniquely determined by minimizing the energy

\[
\int_{\tilde{\Omega}^*} |\nabla u|^2 dx,
\]
in an admissible function space

\[
A_0 := \{ u \in H^1(\Omega) \mid \nabla u = 0 \text{ in } D_1^* \cup D_2^*, \text{ and } u = \varphi \text{ on } \partial \Omega \}.
\]

**Theorem 1.4.** (Lower Bounds for perfect conductivity problem). For \( d = 2, 3 \), under the assumptions for the domain as in Theorem A, let \( u \in H^1(\Omega) \cap C^1(\tilde{\Omega}) \) be a solution to (1.12). Then if there exists a \( \varphi \) such that \( b^*_1[\varphi] \neq 0 \), then for sufficiently small \( 0 < \varepsilon < 1/2 \),

\[
|\nabla u(x)| \geq \frac{\rho_d(\varepsilon)}{C\varepsilon} \| b^*_1[\varphi] \|, \text{ for } x \in T_1 T_2,
\]

where \( C \) is a universal constant.

**Remark 1.5.** We remark that the quantity \( b^*_1[\varphi] \) is independent of \( \varepsilon \). This is an essential difference with \( Q_\varepsilon[\varphi] \) defined in [11]. On the other hand, one can see that \( u^* \) is smooth near the origin while \( v^*_i, i = 1, 2, \) in the definition of \( Q_\varepsilon[\varphi] \) are singular at the origin. So the definition of \( b^*_1[\varphi], [1.13] \), is more natural from physical viewpoint and it is easier to check whether it equals to zero or not, in these two physically related dimensions, although it can not be used to deal with higher dimensions cases so far.

The rest of this paper is organized as follows. In Section 2 we list several known results from [14, 15] about \( |\nabla v^*_i| \) and \( C^\alpha_i \), after we decompose the solution \( u = \sum_{\alpha=1}^{d(d+1)/2} (C^\alpha_1 v^*_1 + C^\alpha_2 v^*_2) + v_0 \) in \( \Omega \), see (2.1) below. By a careful observation of the structure of a system of linear equations for \( C^\alpha_i \), we find a quantity \( b^*_1[\varphi] \), which turns out to be convergent to the blow-up factor \( b^*_1[\varphi] \). This is the heart of this paper. The proof is technical and carried out in Section 3. Section 4 is devoted to proving Theorem 1.2 in dimension two and the proof of Theorem 1.2 in dimension three is given in Section 5. Finally, we prove Theorem 1.4 in Section 6.
to give a new and more simple proof of results in [11] for the perfect conductivity in dimension two and three, especially for the lower bound estimates of $|\nabla u|$.

2. Preliminaries and the blow-up factor

In this section we first introduce a decomposition of the solution of (1.5). In Subsection 2.2 we choose a new system of linear equations for $C_1^\alpha$ from the whole system to solve $C_1^\alpha - C_2^\alpha$, $\alpha = 1, 2, 3$. It is a different way from the selection made in [14, 15] that just allows us to obtain upper bound estimates. While this selection makes it possible to introduce the blow-up factor $b^\beta_1[\varphi]$ in Subsection 2.3 to get a lower bound of $|\nabla u|$. In the end, we list several preliminary results from our earlier papers [14, 15] to make our paper self-contained and our exposition clear.

2.1. Decomposition of $u$. By the third line of (1.5), $u$ is a linear combination of $\{\psi_\alpha\}$ in $D_1$ and $D_2$, respectively. By using continuity, we decompose the solution of (1.5), as in [14], as follows:

$$u = \sum_{\alpha=1}^{d(d+1)/2} C_1^\alpha v_1^\alpha + \sum_{\alpha=1}^{d(d+1)/2} C_2^\alpha v_2^\alpha + v_0, \quad \text{in } \tilde{\Omega},$$

where $v_\alpha^\alpha \in C^1(\tilde{\Omega}; \mathbb{R}^d)$, $i = 1, 2$, $\alpha = 1, 2, \cdots, \frac{d(d+1)}{2}$, and $v_0 \in C^1(\tilde{\Omega}; \mathbb{R}^d)$ are respectively the solution of

$$\begin{cases} L_{\lambda, \mu} v_\alpha^\alpha = 0, & \text{in } \tilde{\Omega}, \\ v_\alpha^\alpha = \psi_\alpha, & \text{on } \partial D_i, \\ v_\alpha^\alpha = 0, & \text{on } \partial D_j \cup \partial \Omega, j \neq i, \end{cases}$$

and

$$\begin{cases} L_{\lambda, \mu} v_0 = 0, & \text{in } \tilde{\Omega}, \\ v_0 = 0, & \text{on } \partial D_1 \cup \partial D_2, \\ v_0 = \varphi, & \text{on } \partial \Omega. \end{cases}$$

The constants $C_i^\alpha := C_i^\alpha(\varepsilon), i = 1, 2$, $\alpha = 1, 2, \cdots, \frac{d(d+1)}{2}$, are uniquely determined by $u$.

By the decomposition (2.1), we write

$$\nabla u = \sum_{\alpha=1}^d (C_1^\alpha - C_2^\alpha) \nabla v_1^\alpha + \sum_{\alpha=1}^{d(d+1)/2} \sum_{\alpha=1}^{d(d+1)} C_i^\alpha \nabla v_i^\alpha + \nabla u_b, \quad \text{in } \tilde{\Omega},$$

where

$$u_b := \sum_{\alpha=1}^d C_2^\alpha v^\alpha + v_0, \quad v^\alpha = v_1^\alpha + v_2^\alpha.$$

It is obvious that $v^\alpha$, $\alpha = 1, 2, \cdots, d$, verifies

$$\begin{cases} L_{\lambda, \mu} v^\alpha = 0, & \text{in } \tilde{\Omega}, \\ v^\alpha = \psi^\alpha, & \text{on } \partial D_1 \cup \partial D_2, \\ v^\alpha = 0, & \text{on } \partial \Omega. \end{cases}$$

Notice that from theorem 1.1 in [35],

$$\|\nabla v^\alpha\|_{L^\infty(\tilde{\Omega}^*)} \leq C, \quad \text{and } \|\nabla v_0\|_{L^\infty(\tilde{\Omega}^*)} \leq C,$$
since the displacement takes the same constant value on the the boundaries of both
inclusions. So that $|\nabla u|$ is also bounded.

2.2. A selected system of linear equations for $C_i^\alpha$. By the linearity of $e(u)$ and decomposition (2.4),

$$e(u) = \sum_{\alpha=1}^{d} \left( C_{1}^{\alpha} - C_{2}^{\alpha} \right) e(v_{1}^{\alpha}) + \sum_{i=1}^{2} \sum_{\alpha=d+1}^{d(d+1)} C_{i}^{\alpha} e(v_{i}^{\alpha}) + e(u_0), \quad \text{in } \tilde{\Omega}.$$  

It follows from the forth line of (1.5) that

$$\sum_{\alpha=1}^{d} \left( C_{1}^{\alpha} - C_{2}^{\alpha} \right) \int_{\partial D_j} \frac{\partial v_{1}^{\alpha}}{\partial \nu} \cdot \psi^\beta + \sum_{i=1}^{2} \sum_{\alpha=d+1}^{d(d+1)} C_{i}^{\alpha} \int_{\partial D_j} \frac{\partial v_{i}^{\alpha}}{\partial \nu} \cdot \psi^\beta + \int_{\partial D_j} \frac{\partial u_b}{\partial \nu} \cdot \psi^\beta = 0, \quad j = 1, 2, \quad \beta = 1, 2, \cdots, \frac{d(d+1)}{2}. \quad (2.7)$$

Denote, for $i, j = 1, 2, \alpha, \beta = 1, 2, \cdots, \frac{d(d+1)}{2}$,

$$a_{ij}^{\alpha \beta} := -\int_{\partial D_j} \frac{\partial v_{i}^{\alpha}}{\partial \nu} \cdot \psi^\beta, \quad b_j^\beta := b_j^\beta[\varphi] = \int_{\partial D_j} \frac{\partial u_b}{\partial \nu} \cdot \psi^\beta. \quad (2.8)$$

Then (2.7) can be written as

$$\begin{cases}
\sum_{\alpha=1}^{d} \left( C_{1}^{\alpha} - C_{2}^{\alpha} \right) a_{11}^{\alpha \beta} + \sum_{i=1}^{2} \sum_{\alpha=d+1}^{d(d+1)} C_{i}^{\alpha} a_{i1}^{\alpha \beta} - b_1^\beta = 0, \\
\sum_{\alpha=1}^{d} \left( C_{1}^{\alpha} - C_{2}^{\alpha} \right) a_{12}^{\alpha \beta} + \sum_{i=1}^{2} \sum_{\alpha=d+1}^{d(d+1)} C_{i}^{\alpha} a_{i2}^{\alpha \beta} - b_2^\beta = 0,
\end{cases} \quad \beta = 1, 2, \cdots, \frac{d(d+1)}{2}. \quad (2.9)$$

We select $\beta = 1, 2, \cdots, \frac{d(d+1)}{2}$ for $j = 1$ and $\beta = d + 1, \cdots, \frac{d(d+1)}{2}$ for $j = 2$ to solve $C_{1}^{\alpha} - C_{2}^{\alpha}, \alpha = 1, 2, \cdots, d$. One can see that this selection is different with that in [4,14]. For simplicity, we denote it in block matrix

$$AX := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

where

$$A_{11} := \left( a_{11}^{\alpha \beta} \right)_{\alpha, \beta=1,2,\cdots,d}, \quad A_{12} := \left( a_{11}^{\alpha \beta}, a_{12}^{\alpha \beta} \right)_{\alpha=1,2,\cdots,d; \beta=d+1,\cdots,\frac{d(d+1)}{2}},$$

$$A_{21} := \left( a_{11}^{\alpha \beta} \right)_{\alpha=d+1,\cdots,\frac{d(d+1)}{2}; \beta=1,2,\cdots,d}, \quad A_{22} := \left( a_{11}^{\alpha \beta}, a_{12}^{\alpha \beta} \right)_{\alpha, \beta=d+1,\cdots,\frac{d(d+1)}{2}},$$

$$X_1 = \begin{pmatrix} C_{1}^1 - C_{2}^1, C_{1}^2 - C_{2}^2, \cdots, C_{1}^d - C_{2}^d \end{pmatrix}^T,$$

$$X_2 = \begin{pmatrix} C_{1}^{d+1}, \cdots, C_{1}^{\frac{d(d+1)}{2}}, C_{2}^{d+1}, \cdots, C_{2}^{\frac{d(d+1)}{2}} \end{pmatrix}^T,$$

and

$$B_1 = \begin{pmatrix} b_1^1, b_1^2, \cdots, b_1^d \end{pmatrix}^T, \quad B_2 = \begin{pmatrix} b_2^{d+1}, \cdots, b_2^{\frac{d(d+1)}{2}} \end{pmatrix}^T.$$
Since $A$ is positive definite, we can solve $X_1$ by Cramer's rule. Then the quantities of $b_1^\beta$, $\beta = 1, 2, \cdots, d$, will play a key role to determine whether $C_1^\alpha - C_2^\alpha$, $\alpha = 1, 2, \cdots, d$, equal to zero or not. In next subsection, we will show that $b_1^\beta$ is convergent to the blow-up factor $b_{s_1}$, for $\beta = 1, 2, \cdots, d$.

2.3. $b_1^\beta[\varphi]$ convergent to the blow-up factor $b_{s_1}^\beta[\varphi]$. Recalling the definitions of $b_1^\beta[\varphi]$ and $b_{s_1}^\beta[\varphi]$, (2.8) and (1.11), respectively, we have $b_1^\beta \to b_{s_1}^\beta$, as $\varepsilon \to 0$, for $\beta = 1, 2, \cdots, d$.

**Proposition 2.1.** For $d = 2, 3$, and $\beta = 1, 2, \cdots, d$,

$$\left| b_1^\beta[\varphi] - b_{s_1}^\beta[\varphi] \right| \leq C \max\{\varepsilon^{1/3}, \rho_d(\varepsilon)\} \left( \|\varphi\|_{L^1(\partial D)} + |\partial D| \right).$$

Consequently,

$$b_1^\beta[\varphi] \to b_{s_1}^\beta[\varphi], \quad \text{as } \varepsilon \to 0, \quad \beta = 1, 2, \cdots, d.$$

To prove this convergence, similar to (2.5) and (2.3), we define their limit cases, respectively, for $\alpha = 1, 2, \cdots, d$,

$$\begin{cases}
L_{\lambda, \mu, \nu} v^{\alpha} = 0, & \text{in } \tilde{\Omega}^*, \\
v^{\alpha} = \psi^{\alpha}, & \text{on } \partial D_1 \cup \partial D_2, \\
v^{\alpha} = 0, & \text{on } \partial \Omega. 
\end{cases} \quad (2.10)$$

and

$$\begin{cases}
L_{\lambda, \mu, \nu} v_0^* = 0, & \text{in } \tilde{\Omega}^*, \\
v_0^* = 0, & \text{on } \partial D_1 \cup \partial D_2, \\
v_0^* = \varphi, & \text{on } \partial \Omega. 
\end{cases} \quad (2.11)$$

Then

$$u_b^* = \sum_{\alpha=1}^d C_\alpha v^{\alpha} + v_0^*.$$ 

It follows from theorem 1.1 in [35] that

$$\|\nabla v^{\alpha}\|_{L^\infty(\tilde{\Omega}^*)} \leq C, \quad \text{and} \quad \|\nabla v_0^*\|_{L^\infty(\tilde{\Omega}^*)} \leq C. \quad (2.12)$$

So that $|\nabla u_b^*|$ is also bounded. We shall prove that $u_b^*$ is actually the limit of $u_b$ later. The proof of Proposition 2.1 will be given in the next section.

To complete this section, we fix our notations and list several known results of [14, 15] for later use. We use $x = (x', x_d)$ to denote a point in $\mathbb{R}^d$, where $x' = (x_1, x_2, \cdots, x_{d-1})$. By a translation and rotation if necessary, we may assume without loss of generality that the points $P_1$ and $P_2$ in (1.8) satisfy

$$P_1 = \left(0', -\frac{\varepsilon}{2}\right) \in \partial D_1, \quad \text{and} \quad P_2 = \left(0', \frac{\varepsilon}{2}\right) \in \partial D_2.$$ 

Fix a small universal constant $R$, such that the portion of $\partial D_1$ and $\partial D_2$ near $P_1$ and $P_2$, respectively, can be represented by

$$x_d = \frac{\varepsilon}{2} + h_1(x'), \quad \text{and} \quad x_d = -\frac{\varepsilon}{2} + h_2(x'), \quad \text{for } |x'| < 2R. \quad (2.13)$$

Then by the smoothness assumptions on $\partial D_1$ and $\partial D_2$, the functions $h_1(x')$ and $h_2(x')$ are of class $C^{2,\gamma}(B_R(0'))$, satisfying

$$\frac{\varepsilon}{2} + h_1(x') > -\frac{\varepsilon}{2} + h_2(x'), \quad \text{for } |x'| < 2R,$$
\[ h_1(0') = h_2(0') = 0, \quad \nabla h_1(0') = \nabla h_2(0') = 0, \quad (2.14) \]
\[ \nabla^2 h_1(0') \geq \kappa_0 I, \quad \nabla^2 h_2(0') \leq -\kappa_0 I, \quad (2.15) \]
and
\[ \|h_1\|_{C^{2,\gamma}(B_2^*\mathbb{R}^n)} + \|h_2\|_{C^{2,\gamma}(B_2^*\mathbb{R}^n)} \leq C. \quad (2.16) \]

In particular, we only use a weaker relative strict convexity assumption of \( \partial D_1 \) and \( \partial D_2 \), that is
\[ h_1(x') - h_2(x') \geq \kappa_0 |x'|^2, \quad \text{if } |x'| < 2R. \quad (2.17) \]

For \( 0 \leq r \leq 2R \), denote
\[ \Omega_r := \left\{ (x', x_d) \in \mathbb{R}^d \mid -\frac{\varepsilon}{2} + h_2(x') < x_d < \frac{\varepsilon}{2} + h_1(x'), \ |x'| < r \right\}. \]

For \( |z'| \leq 2R \), we always use \( \delta \) to denote
\[ \delta := \delta(z') = \varepsilon - h_1(z') - h_2(z'). \]

By \((2.14)-(2.17)\),
\[ \frac{1}{C} (\varepsilon + |z'|^2) \leq \delta(z') \leq C (\varepsilon + |z'|^2). \quad (2.18) \]

We now list the following estimates of \(|\nabla v_\alpha^0|\) and \(C^\alpha_{i,d}\) from [14, 15].

**Lemma 2.2.** ([14, 15]) Under the hypotheses of Theorem A, and the normalization \( \|\varphi\|_{C^2(\partial\Omega)} = 1 \), let \( v_\alpha^0 \) and \( v_\beta^0 \) be the solution to \((2.2)\) and \((2.3)\), respectively. Then for \( 0 < \varepsilon < 1/2 \), we have
\[ \|\nabla v_0\|_{L^\infty(\tilde{\Omega})} \leq C; \quad (2.19) \]
\[ \|\nabla v_\alpha^0\|_{L^\infty(\tilde{\Omega})} \leq C, \quad \alpha = 1, 2, \cdots, d; \quad (2.20) \]
\[ \frac{1}{C(\varepsilon + |x'|^2)} \leq |\nabla v_i^0(x)| \leq \frac{C}{\varepsilon + |x'|^2}, \quad i = 1, 2, \alpha = 1, 2, \cdots, d, \ x \in \tilde{\Omega}; \quad (2.21) \]
\[ |\nabla v_i^0(x)| \leq \frac{C|x'|}{\varepsilon + |x'|^2} + C, \quad i = 1, 2, \alpha = d + 1, \cdots, d, x \in \tilde{\Omega}; \quad (2.22) \]
and
\[ |C_\alpha| \leq C, \quad i = 1, 2, \alpha = 1, 2, \cdots, d, \quad (2.23) \]

for \( d = 2, 3, \)
\[ |C_1^\alpha - C_2^\alpha| \leq C \rho_d(\varepsilon), \quad \alpha = 1, 2, \cdots, d. \quad (2.24) \]

**Remark 2.3.** Estimate \((2.24)\) can also be proved in Proposition \((3.4)\) and Proposition \((5.4)\) below in a different way. It tells us that as \( \varepsilon \to 0 \), in dimensions two and three the difference \( |C_1^\alpha - C_2^\alpha| \to 0, \alpha = 1, 2, \cdots, d \), which allows us to prove that \( b^\beta_{1,\varepsilon}(\varphi) \) can be convergent to the blow-up factor \( b^\beta_{1,\varepsilon}(\varphi) \). But in higher dimensions \( d \geq 4 \), so far we do not know whether \( |C_1^\alpha - C_2^\alpha| \) tends to 0 or not as \( \varepsilon \to 0 \). For more details, see the proof of Lemma \((3.2)\) in Section \((3)\) where we prove that \( C_1^\alpha \) and \( C_2^\alpha \) have the same limit \( C^\alpha \), for \( \alpha = 1, 2, \cdots, d \).
3. Proof of Proposition 2.1

We introduce a scalar auxiliary function \( \bar{u} \in C^2(\mathbb{R}^d) \) as before, such that \( \bar{u} = 1 \) on \( \partial D_1 \), \( \bar{u} = 0 \) on \( \partial D_2 \cup \partial \Omega \),

\[
\bar{u}(x) = \frac{x_d - h_2(x')}{\varepsilon + h_1(x') - h_2(x')}, \quad \text{in } \Omega_{2R},
\]

and

\[
\|\bar{u}\|_{C^2(\mathbb{R}^d \setminus \Omega_R)} \leq C.
\]

We use \( \bar{u} \) to define vector-value auxiliary functions

\[
\bar{u}_i^\alpha = \bar{u} \psi^\alpha, \quad \alpha = 1, 2, \ldots, d, \quad \text{in } \tilde{\Omega}.
\]

Thus, \( \bar{u}_i^\alpha = v_i^\alpha \) on \( \partial \tilde{\Omega} \). Similarly, we define

\[
\tilde{u}_i^\alpha = \bar{u} \psi^\alpha, \quad \alpha = 1, 2, \ldots, d, \quad \text{in } \tilde{\Omega},
\]

such that \( \tilde{u}_i^\alpha = v_i^\alpha \) on \( \partial \tilde{\Omega} \), where \( u \) is a scalar function in \( C^2(\mathbb{R}^d) \) satisfying \( u = 1 \) on \( \partial D_2 \), \( u = 0 \) on \( \partial D_1 \cup \partial \Omega \), \( u(x) = 1 - \bar{u} \), in \( \Omega_{2R} \), and \( \|u\|_{C^2(\mathbb{R}^d \setminus \Omega_R)} \leq C \).

A direct calculation gives, in view of (2.14)-(2.17), that

\[
|\partial x_i \bar{u}(x)| \leq \frac{C|x_k|}{\varepsilon + |x'|^2}, \quad k = 1, 2, \ldots, d - 1, \quad \partial x_d \bar{u}(x) = \frac{1}{\delta(x')}, \quad x \in \Omega_R.
\]

Thus, for \( i = 1, 2, \ldots, d, \)

\[
|\nabla \cdot \tilde{u}_i^\alpha(x)| \leq \frac{C}{\sqrt{\varepsilon + |x'|^2}} \quad \text{and} \quad \frac{1}{C(\varepsilon + |x'|^2)} \leq \partial x_j \bar{u}_i^\alpha(x) \leq \frac{C}{\varepsilon + |x'|^2}, \quad x \in \Omega_R.
\]

We need the following Lemma to prove Proposition 2.1.

Lemma 3.1. (\cite{14,13}) Assume the above, let \( v_i^\alpha \in H^1(\tilde{\Omega}; \mathbb{R}^d) \) be the weak solution of (2.2) with \( \alpha = 1, 2, \ldots, d \). Then for \( i = 1, 2, \alpha = 1, 2, \ldots, d \),

\[
|\nabla(v_i^\alpha - \bar{u}_i^\alpha)(x)| \leq \begin{cases} C, & |x'| \leq \sqrt{\varepsilon}, \\ \frac{C}{|x'|}, & \sqrt{\varepsilon} \leq |x'| \leq R, \end{cases} \quad \forall x \in \Omega_R,
\]

and

\[
\|\nabla(v_i^\alpha - \bar{u}_i^\alpha)\|_{L^\infty(\tilde{\Omega} \setminus \Omega_R)} \leq C.
\]

Let \( u^* \) be the solution of the following Dirichlet boundary value problem:

\[
\begin{aligned}
\mathcal{L}_{\lambda, \mu} u^* &= 0, & \text{in } \tilde{\Omega}^*, \\
u^* &= \sum_{\alpha=1}^{d(d+1)/2} C_{\alpha} \psi^\alpha, & \text{on } \partial D_1^* \cup \partial D_2^*, \\
\int_{\partial D_2^*} \partial \psi^\alpha \cdot \psi^\beta + \int_{\partial D_2^*} \partial \psi^\alpha \cdot \psi^\beta &= 0, & \beta = 1, 2, \ldots, \frac{d(d+1)}{2}, \\
u^* &= \varphi(x), & \text{on } \partial D,
\end{aligned}
\]

where \( C_{\alpha}, \alpha = 1, 2, \ldots, d(d+1)/2, \) are uniquely determined by minimizing the energy

\[
\int_{\tilde{\Omega}^*} \left( C^{(0)} e(v), e(v) \right) dx
\]

in an admission function space

\[
\mathcal{A} := \{ v \in H^1(\Omega; \mathbb{R}^d) \mid e(v) = 0 \text{ in } D_1^* \cup D_2^*, \text{ and } v = \varphi \text{ on } \partial \Omega \},
\]
and \( u^* \) is the limit of \( u \) in the sense of variation. If \( \varphi = 0 \), then by a variational argument, there is only trivial solution \( u^* = 0 \) for (3.6). Hence, \( C_i^\alpha = 0, \alpha = 1, 2, \cdots, d(d+1)/2 \). Generally, if \( \varphi \neq 0 \), we have

**Lemma 3.2.** For \( d = 2, 3 \), under the hypotheses of Theorem A, and the normalization \( ||\varphi||_{C^2(\partial \Omega)} = 1 \), let \( C_i^\alpha \) and \( C_\alpha^* \) be defined in (2.1) and (3.6), respectively. Then

\[
\frac{1}{2}(C_1^\alpha + C_2^\alpha) - C_\alpha^* \leq C \rho_d(\varepsilon), \quad \alpha = 1, 2, \cdots, d.
\]

Consequently, in view of (2.14),

\[
|C_\alpha^* - C_i^\alpha| = \frac{1}{2}|C_1^\alpha - C_2^\alpha| + C \rho_d(\varepsilon) \leq C \rho_d(\varepsilon), \quad i = 1, 2, \alpha = 1, 2, \cdots, d. \quad (3.7)
\]

The proof will be given later. We first use it to prove Proposition 2.1.

**Proof of Proposition 2.1.** We here prove the case \( \beta = 1 \) for instance. The other cases are the same. Set

\[
b_{1,0} = \int_{\partial D_1} \frac{\partial u}{\partial v} \cdot \psi = \int_{\partial D_1} \frac{\partial v_0}{\partial v} \cdot \psi + \sum_{\alpha=1}^{d} C_2^\alpha \int_{\partial D_1} \frac{\partial v_{1,\alpha}}{\partial v} \cdot \psi.
\]

\[
:= b_{1,0}^1 + \sum_{\alpha=1}^{d} C_2^\alpha b_{1,\alpha}^1. \quad (3.8)
\]

**STEP 1.** First, for \( b_{1,0}^1 \). It follows from the definitions of \( v_0 \) and \( v_1 \) and the integration by parts formula (1.6) that

\[
b_{1,0} = \int_{\partial D_1} \frac{\partial v_0}{\partial v} \cdot \psi = \int_{\partial \Omega} (C^0, e(v_1), e(v_0)) = \int_{\partial \Omega} \frac{\partial v_1}{\partial v} \cdot \varphi.
\]

Similarly,

\[
b_{1,1}^* := \int_{\partial D_1^*} \frac{\partial v_{1,1}^*}{\partial v} \cdot \psi = \int_{\partial \Omega} \frac{\partial v_{1,1}^*}{\partial v} \cdot \varphi,
\]

where \( v_{1,1}^* \) satisfies

\[
\begin{cases}
L_{\lambda, \mu} v_{1,1}^* = 0, & \text{in } \bar{\Omega}^*, \\
v_{1,1}^* = \psi^1, & \text{on } \partial D_1^* \setminus \{0\}, \\
v_{1,1}^* = 0, & \text{on } \partial D_2^* \cup \partial \Omega.
\end{cases} \quad (3.9)
\]

Thus,

\[
b_{1,0}^1 - b_{1,1}^* = \int_{\partial \Omega} \frac{\partial (v_1 - v_{1,1}^*)}{\partial v} \cdot \varphi. \quad (3.10)
\]

It suffices to estimate \( |\nabla (v_1 - v_{1,1}^*)| \) on the boundary \( \partial \Omega \).
STEP 1.1. In order to estimate the difference $v^1_1 - v^s_1$, we introduce two auxiliary functions

\[
\bar{u}_1 = \begin{pmatrix} \bar{u} \\ 0 \\ \vdots \end{pmatrix}, \quad \text{and} \quad \bar{u}^s_1 = \begin{pmatrix} \bar{u}^s \\ 0 \\ \vdots \end{pmatrix},
\]

where $\bar{u}$ is defined by (3.1), and $\bar{u}^s$ satisfies $\bar{u}^s = 1$ on $\partial D^*_1 \setminus \{0\}$, $\bar{u}^s = 0$ on $\partial D^*_2 \cup \partial \Omega$, and $\bar{u}^s = x_d - h_2(x')$ on $\Omega^*_R$, $\|\bar{u}^s\|_{C^2(\bar{\Omega} \setminus \Omega^*_R)} \leq C$.

By making use of (2.14), (2.15), and (2.18), we obtain, for $r < R$,

\[
\left| \nabla' \bar{u}^1_1(x) \right| \leq \frac{C}{|x'|}, \quad \text{and} \quad \frac{1}{C|x'|^2} \leq \partial_{x_d} \bar{u}^1_1(x) \leq \frac{C}{|x'|^2}, \quad x \in \Omega^*_R. \tag{3.11}
\]

Applying Lemma 3.1 to (3.9) leads to

\[
\left| \nabla' (\bar{u}^1_1)_1 - (\bar{u}^s_1)_1 \right| = \left| \nabla' (\bar{u} - \bar{u}^s) \right| \leq \frac{C}{|x'|}, \tag{3.12}
\]

and

\[
\left| \partial_{x_d} (\bar{u}^1_1)_1 - (\bar{u}^s_1)_1 \right| = \frac{1}{\varepsilon + h_1(x') - h_2(x')} - \frac{1}{h_1(x') - h_2(x')} \leq \frac{C\varepsilon}{|x'|^2(\varepsilon + |x'|^2)}. \tag{3.13}
\]

Applying Lemma 3.1 to (3.9) leads to

\[
|\nabla (v^1_1 - \bar{u}^1_1)(x)| \leq \frac{C}{|x'|}, \quad x \in \Omega^*_R; \tag{3.14}
\]

and in view of (3.11),

\[
|\nabla' v^1_1(x)| \leq \frac{C}{|x'|}, \quad |\partial_{x_d} v^1_1(x)| \leq \frac{C}{|x'|^2}, \quad x \in \Omega^*_R. \tag{3.15}
\]

STEP 1.2. Next, we estimate the difference $v^1_1 - v^s_1$. Notice that $v^1_1 - v^s_1$ satisfies

\[
\begin{cases}
\mathcal{L}_{\lambda, \mu}(v^1_1 - v^s_1) = 0, & \text{in } V := \Omega \setminus (D^*_1 \cup D^*_2), \\
v^1_1 - v^s_1 = \psi^1 - v^1_1, & \text{on } \partial D^*_i \setminus D^*_i, \ i = 1, 2, \\
v^1_1 - v^s_1 = v^1_1 - \psi^1, & \text{on } \partial D^*_i \setminus (D^*_i \cup \{0\}), \ i = 1, 2, \\
v^1_1 - v^s_1 = 0, & \text{on } \partial \Omega.
\end{cases}
\]

Define a cylinder

\[
\mathcal{C}_r := \left\{ x \in \mathbb{R}^d \mid |x'| < r - \frac{\varepsilon}{2} + 2 \min_{|x'| = r} h_2(x') \leq x_d \leq \frac{\varepsilon}{2} + 2 \max_{|x'| = r} h_1(x') \right\},
\]

for $r < R$. We first estimate $|v^1_1 - v^s_1|$ on $\partial (D^*_1 \cup D^*_2) \setminus \mathcal{C}_r$, where $0 < \gamma < 1/2$ to be determined later. For $\varepsilon$ sufficiently small, in view of the definition of $v^s_1$,

\[
|\partial_{x_d} v^s_1(x)| \leq C, \quad x \in \bar{\Omega} \setminus \Omega^*_R.
\]
By using mean value theorem, we have, for $x \in \partial D_1 \setminus D_1^*$,
\[
|(v_1^1 - v_1^{*1})(x', x_d)| = |(v_1^1 - v_1^{*1})(x', x_d)|
\leq |v_1^{*1}(x', x_d) - v_1^{*1}(x', x_d)| \leq C \varepsilon. \tag{3.15}
\]

For $x \in \partial D_1^* \setminus (D_1 \cup C_{\varepsilon^*})$, using mean value theorem again and (2.21),
\[
|(v_1^1 - v_1^{*1})(x', x_d)| = |v_1^1(x', x_d) - v_1^{*1}(x', x_d + \varepsilon)|
\leq \frac{C \varepsilon}{\varepsilon + |x'|^2} \leq C \varepsilon^{1-2\gamma}. \tag{3.16}
\]

Similarly, for $x \in \partial D_2 \setminus D_2^*$,
\[
|(v_1^1 - v_1^{*1})(x', x_d)| \leq C \varepsilon; \tag{3.17}
\]
for $x \in \partial D_2^* \setminus (D_2 \cup C_{\varepsilon^*})$, by (2.21),
\[
|(v_1^1 - v_1^{*1})(x', x_d)| \leq C \varepsilon^{1-2\gamma}. \tag{3.18}
\]

For $x \in \Omega_R$ with $|x'| = \varepsilon^*$, it follows from (3.4), (3.12), and (3.13) that
\[
|\partial_{x_d}(v_1^1 - v_1^{*1})(x', x_d)| = |\partial_{x_d}(v_1^1 - \bar{u}_1^1) + \partial_{x_d}(\bar{u}_1 - v_1^{*1}) + \partial_{x_d}(\bar{u}_1^{*1} - v_1^{*1})|(x', x_d)
\leq \frac{C \varepsilon}{|x'|^2(\varepsilon^* + |x'|^2)} + \frac{C}{|x'|}
\leq \frac{C}{\varepsilon^{2\gamma-1}} + \frac{C}{\varepsilon^\gamma}.
\]

Thus, for $x \in \Omega_R$ with $|x'| = \varepsilon^*$, recalling (3.18), we have
\[
|(v_1^1 - v_1^{*1})(x', x_d)| = |(v_1^1 - v_1^{*1})(x', x_d) - (v_1^1 - v_1^{*1})(x', h_2(x'))| + C \varepsilon^{1-2\gamma}
\leq |\partial_{x_d}(v_1^1 - v_1^{*1})|_{|x'|=\varepsilon^*} \cdot (h_1(x') - h_2(x')) + C \varepsilon^{1-2\gamma}
\leq \left(\frac{C}{\varepsilon^{2\gamma-1}} + \frac{C}{\varepsilon^\gamma}\right) \cdot \varepsilon^{2\gamma} + C \varepsilon^{1-2\gamma}
\leq C(\varepsilon^{1-2\gamma} + \varepsilon^\gamma). \tag{3.19}
\]

Letting $1 - 2\gamma = \gamma$, we take $\gamma = 1/3$. Combining (3.15), (3.16) and (3.19), and recalling $v_1^1 - v_1^{*1} = 0$ on $\partial \Omega$, we obtain
\[
|(v_1^1 - v_1^{*1})(x)| \leq C \varepsilon^{1/3}, \quad x \in \partial(V \setminus C_{\varepsilon^*}).
\]

Now applying the maximum principle for Lamé systems on $V \setminus C_{\varepsilon^*}$ (see, e.g. [12]) yields
\[
|(v_1^1 - v_1^{*1})(x)| \leq C \varepsilon^{1/3}, \quad \text{in } V \setminus C_{\varepsilon^*}.
\]

Then, using the standard boundary gradient estimates for Lamé system (see [1]),
\[
|\nabla (v_1^1 - v_1^{*1})(x)| \leq C \varepsilon^{1/3}, \quad \text{on } \partial \Omega.
\]

Therefore, recalling (3.10),
\[
|b_1^{1,0} - b_1^{*,1}| \leq \left| \int_{\partial \Omega} \frac{\partial (v_1^1 - v_1^{*1})}{\partial \nu} \cdot \varphi \right| \leq C \varepsilon^{1/3} \|\varphi\|_{L^1(\partial \Omega)}. \tag{3.20}
\]
STEP 2. Secondly, for \( b_1^{1,\alpha}, \alpha = 1, 2, \cdots, d \). The proof is essentially the same. It follows from the definitions of \( v^\alpha, \alpha = 1, 2, \cdots, d \), (2.5) that
\[
\begin{cases}
L_{\lambda, \mu}(v^\alpha - \psi^\alpha) = 0, & \text{in } \bar{\Omega}, \\
v^\alpha - \psi^\alpha = 0, & \text{on } \partial D_1 \cup \partial D_2, \\
v^\alpha - \psi^\alpha = -\psi^\alpha, & \text{on } \partial \Omega.
\end{cases}
\] (3.21)

Recalling the definitions of \( v_1^1 \), and using the integration by parts formula (1.3), we have, for \( \alpha = 1, 2, \cdots, d \),
\[
b_1^{1,\alpha} = \int_{\partial D_1} \left( \frac{\partial (v^\alpha - \psi^\alpha)}{\partial \nu} \right) + v_1^1 = \int_{\Omega} (C^0 e(v_1^1), e(v^\alpha - \psi^\alpha)) = \int_{\partial \Omega} \frac{\partial v_1^1}{\partial \nu} \cdot (-\psi^\alpha).
\]
Similarly, for \( \alpha = 1, 2, \cdots, d \),
\[
b_1^{1,\alpha} = \int_{\partial D_1} \left( \frac{\partial (v^\alpha - \psi^\alpha)}{\partial \nu} \right) + \psi_1^1 = \int_{\Omega} (C^0 e(v_1^1), e(v^\alpha - \psi^\alpha)) = \int_{\partial \Omega} \frac{\partial v_1^1}{\partial \nu} \cdot (-\psi^\alpha).
\]

In view of \( v_1^1 = 0 \) on \( \partial \Omega \), by using a standard boundary estimate for elliptic system (see. [11]), it is easy to see that
\[
|b_1^{1,\alpha}| \leq C\|\psi^\alpha\|_{L^1(\partial \Omega)} = C|\partial \Omega|, \quad \alpha = 1, 2, \cdots, d.
\] (3.22)

By applying the same argument above, we have
\[
|b_1^{1,\alpha} - b_1^{1,\alpha}| = \left| \int_{\partial \Omega} \frac{\partial (v^\alpha - \psi^\alpha)}{\partial \nu} \cdot (\psi_1^1) \right| \leq C^1/\beta|\partial \Omega|, \quad \alpha = 1, 2, \cdots, d.
\] (3.23)

STEP 3. Finally, recalling (3.8), using (3.7) and (2.23), and substituting (3.20), (3.22), (3.23) and (3.7), we have
\[
|b_1^1 - b_1^1| = \left| \int_{\partial D_1} \frac{\partial v^b}{\partial \nu} \cdot \psi_1^1 - \int_{\partial D_1} \frac{\partial \psi^b_1}{\partial \nu} \right| = \left| \left( b_{1,0}^1 + \sum_{\alpha=1}^d C_2^\alpha b_{1,1}^{1,\alpha} \right) \right|
\]
\[
\leq \left| b_{1,0}^1 \right| + \left| \sum_{\alpha=1}^d C_2^\alpha \right| \left| b_{1,1}^{1,\alpha} \right| + \sum_{\alpha=1}^d \left| C_2^\alpha - C_2^* \right| \left| b_{1,1}^{1,\alpha} \right|
\]
\[
\leq C \max\{\varepsilon^{1/3}, \rho_{\lambda}(\varepsilon)\} \left( \|\varphi\|_{L^1(\partial \Omega)} + |\partial \Omega| \right).
\]

The proof of Proposition 2.1 is completed.

**Proof of Lemma 3.2**

STEP 1. Systems of \((C_1^\alpha + C_2^\alpha)/2\) and \(C_2^\alpha\). Recalling the original decomposition (2.1) and the forth line of (1.3), we have
\[
\begin{cases}
\sum_{\alpha=1}^{d(d+1)/2} C_1^\alpha a_{11}^{1,\alpha} + \sum_{\alpha=1}^{d(d+1)/2} C_2^\alpha a_{21}^{1,\alpha} - \tilde{b}_1 = 0, \\
\sum_{\alpha=1}^{d+1/2} C_1^\alpha a_{12}^{1,\alpha} + \sum_{\alpha=1}^{d+1/2} C_2^\alpha a_{22}^{1,\alpha} - \tilde{b}_2 = 0,
\end{cases}
\]
(3.24)
where
\[
\tilde{b}_j^\beta = \int_{\partial D_j} \frac{\partial \psi_0}{\partial \nu} \cdot \psi_0^\beta.
\]
For the first equation of (3.24),
\[
\sum_{\alpha=1}^{d(d+1)/2} (C_1^\alpha + C_2^\alpha) a_{11}^{\alpha\beta} - \sum_{\alpha=1}^{d(d+1)/2} C_2^\alpha (a_{21}^{\alpha\beta} - a_{11}^{\alpha\beta}) - \bar{b}_1^{\beta} = 0,
\]
and
\[
\sum_{\alpha=1}^{d(d+1)/2} (C_1^\alpha + C_2^\alpha) a_{21}^{\alpha\beta} + \sum_{\alpha=1}^{d(d+1)/2} C_1^\alpha (a_{11}^{\alpha\beta} - a_{21}^{\alpha\beta}) - \bar{b}_1^{\beta} = 0.
\]
Adding these two equations together leads to
\[
\sum_{\alpha=1}^{d(d+1)/2} (C_1^\alpha + C_2^\alpha) (a_{11}^{\alpha\beta} + a_{21}^{\alpha\beta}) + \sum_{\alpha=1}^{d(d+1)/2} C_2^\alpha (a_{11}^{\alpha\beta} - a_{21}^{\alpha\beta}) = 0. \quad (3.25)
\]
Similarly, for the second equation of (3.24),
\[
\sum_{\alpha=1}^{d(d+1)/2} (C_1^\alpha + C_2^\alpha) (a_{12}^{\alpha\beta} + a_{22}^{\alpha\beta}) + \sum_{\alpha=1}^{d(d+1)/2} C_1^\alpha (a_{12}^{\alpha\beta} - a_{22}^{\alpha\beta}) = 0. \quad (3.26)
\]
A further combination of (3.25) and (3.26) together yields
\[
\sum_{\alpha=1}^{d(d+1)/2} \frac{C_1^\alpha + C_2^\alpha}{2} \left( \sum_{i,j=1}^{2} a_{ij}^{\alpha\beta} \right) + \sum_{\alpha=1}^{d(d+1)/2} \frac{C_1^\alpha - C_2^\alpha}{2} (a_{11}^{\alpha\beta} - a_{21}^{\alpha\beta}) + (\bar{b}_1^{\beta} + \bar{b}_2^{\beta}) = 0. \quad (3.27)
\]
Recalling that
\[
u^* = \sum_{\alpha=1}^{d(d+1)/2} C_2^\alpha \nu^\alpha + \nu_0^{\alpha},
\]
where \nu^\alpha and \nu_0^{\alpha} are, respectively, defined by (2.10) and (2.11). From the third line of (3.6), we have
\[
\sum_{\alpha=1}^{d(d+1)/2} C_2^\alpha \left( \int_{\partial D_1^\alpha} \frac{\partial \nu^\alpha}{\partial \nu} \cdot \psi^\beta + \int_{\partial D_2^\alpha} \frac{\partial \nu^\alpha}{\partial \nu} \cdot \psi^\beta \right) + \left( \int_{\partial D_1^\alpha} \frac{\partial \nu_0^{\alpha}}{\partial \nu} \cdot \psi^\beta + \int_{\partial D_2^\alpha} \frac{\partial \nu_0^{\alpha}}{\partial \nu} \cdot \psi^\beta \right) = 0, \quad \beta = 1, 2, \cdots, \frac{d(d+1)}{2}. \quad (3.28)
\]
**STEP 2. Closeness.** Next, comparing (3.27) with (3.28), we will prove
\[
\left| \sum_{i,j=1}^{2} a_{ij}^{\alpha\beta} - \sum_{i=1}^{2} \int_{\partial D_i^\alpha} \frac{\partial \nu_0^{\alpha}}{\partial \nu} \cdot \psi^\beta \right| \leq C \rho_d(\varepsilon), \quad \alpha, \beta = 1, 2, \cdots, \frac{d(d+1)}{2}; \quad (3.29)
\]
and
\[
\left| \sum_{i=1}^{2} b_i^{\beta} - \sum_{i=1}^{2} \int_{\partial D_i^\alpha} \frac{\partial \nu_0^{\alpha}}{\partial \nu} \cdot \psi^\beta \right| \leq C \rho_d(\varepsilon), \quad \beta = 1, 2, \cdots, \frac{d(d+1)}{2}. \quad (3.30)
\]
We only prove (3.29) for instance. The proof of (3.30) is the same. By the definition of $v^\alpha$, (3.21),

$$a^{\alpha\beta} := \sum_{i,j=1}^{2} a^{\alpha\beta}_{ij} = \int_{\partial D_1} \frac{\partial v^\alpha}{\partial v^\beta} |_{+} \cdot \psi^\beta + \int_{\partial D_2} \frac{\partial v^\alpha}{\partial v^\beta} |_{+} \cdot \psi^\beta = \int_{\partial D_1} \frac{\partial (v^\alpha - \psi^\alpha)}{\partial v^\beta} |_{+} \cdot \psi^\beta + \int_{\partial D_2} \frac{\partial (v^\alpha - \psi^\alpha)}{\partial v^\beta} |_{+} \cdot \psi^\beta = \int_{\partial \Omega} \frac{\partial v^\beta}{\partial v^\beta} |_{+} \cdot (-\psi^\alpha).$$

Similarly, by (2.10),

$$a^{*\alpha\beta} := \int_{\partial D_1} \frac{\partial v^{*\alpha}}{\partial v^\beta} |_{+} \cdot \psi^\beta + \int_{\partial D_2} \frac{\partial v^{*\alpha}}{\partial v^\beta} |_{+} \cdot \psi^\beta = \int_{\partial \Omega} \frac{\partial v^{*\beta}}{\partial v^\beta} |_{+} \cdot (-\psi^\alpha).$$

Thus,

$$|a^{\alpha\beta} - a^{*\alpha\beta}| = \left| \int_{\partial \Omega} \frac{\partial (v^\beta - v^{*\beta})}{\partial v^\beta} |_{+} \cdot (-\psi^\alpha) \right|. \quad (3.31)$$

Now we use the argument in STEP 1 and STEP 2 of the proof of Proposition 2.1 to prove that

$$|v^\beta - v^{*\beta}| \leq C\varepsilon, \quad \text{in } V, \quad \text{for } \beta = 1, 2, \ldots, d(d+1)/2. \quad (3.32)$$

Denote $\Gamma_{11} := \partial D_i \setminus \partial D_i$ and $\Gamma_{12} := \partial D_i \setminus \partial D_i^*$, $i = 1, 2$. Then $\partial V = \cup_{i,j=1,2} \Gamma_{ij} \cup \partial \Omega$. Setting

$$\phi^\beta(x) := v^\beta(x) - v^{*\beta}(x),$$

then $\mathcal{L}^{\alpha\beta} \phi^\beta = 0$ in $V$. It is easy to see that $\phi^\beta = 0$ on $\partial \Omega$. For $\beta = 1, 2, \ldots, d$, on $\Gamma_{11}$, by mean value theorem, and (2.10), we have

$$|\phi^\beta|_{\Gamma_{11}} = |v^\beta - v^{*\beta}|_{\Gamma_{11}} = |v^\beta - \psi^\beta|_{\Gamma_{11}} = |v^\beta(x',x_d) - v^\beta(x',x_d + \varepsilon/2)|_{\Gamma_{11}} = |\nabla v^\beta(\xi)|_{\Gamma_{11}} \leq C\varepsilon,$$

where $\xi \in D_1 \setminus D_i$. Similarly, by (2.12),

$$|\phi^\beta|_{\Gamma_{12}} = |v^\beta - v^{*\beta}|_{\Gamma_{12}} = |v^\beta - \psi^\beta|_{\Gamma_{12}} = |v^\beta(x',x_d) - v^\beta(x',x_d - \varepsilon/2)|_{\Gamma_{12}} = |\nabla v^\beta(\xi)|_{\Gamma_{12}} \leq C\varepsilon,$$

for some $\xi \in D_1 \setminus D_i$. For $\beta = d+1, \ldots, d(d+1)/2$, on $\Gamma_{11}$,

$$|\phi^\beta|_{\Gamma_{11}} = |v^\beta - v^{*\beta}|_{\Gamma_{11}} = |v^\beta - \psi^\beta|_{\Gamma_{11}} = |v^\beta - (\psi^\beta + \varepsilon)|_{\Gamma_{11}} = |v^\beta(x',x_d) - v^\beta(x',x_d + \varepsilon/2)|_{\Gamma_{11}} + \varepsilon = |\nabla v^\beta(\xi)|_{\Gamma_{11}} + \varepsilon \leq C\varepsilon,$$

where $\xi \in D_1 \setminus D_i$. On $\Gamma_{12}$ is the same. By the same way,

$$|\phi^\beta|_{\Gamma_{21} \cup \Gamma_{22}} \leq C\varepsilon, \quad \text{for } \beta = 1, 2, \ldots, d(d+1)/2.$$

Applying the maximum principle to $\phi^\beta$ on $V$ ( [42]), yields (5.32).

Then, using the standard boundary gradient estimates for Lamé system again,

$$|\nabla \phi^\beta| \leq C\varepsilon, \quad \text{on } \partial \Omega.$$
Therefore, recalling \[3.31\],
\[
|a^{\alpha \beta} - a^{\alpha \beta}| \leq C\varepsilon \|\psi\|_{L^1(\partial \Omega)}, \quad \alpha, \beta = 1, 2, \ldots, \frac{d(d + 1)}{2}.
\]

**STEP 3. Invertibility of the coefficients matrix** \(a^{\alpha \beta}\). On the other hand,
\[
a^{\alpha \beta} = \int_{\partial D_i^*} \frac{\partial v^{*\alpha}}{\partial \nu} + \cdot \psi^\beta + \int_{\partial D_2^*} \frac{\partial v^{*\alpha}}{\partial \nu} + \cdot \psi^\beta = \int_{\overline{\Omega}^*} (C^0 e(v^{*\alpha}), e(v^{*\beta})).
\]

We claim that \(a^{\alpha \beta}\) is positive definite, so invertible. Moveover, there exists a universal constant \(C\) such that
\[
\sum_{\alpha, \beta = 1}^{d(d+1)/2} a^{\alpha \beta} \xi^\alpha \xi^\beta \geq \frac{1}{C}, \quad \forall \mid \xi \mid = 1.
\]  
(3.33)

Indeed, if \(e \left( \sum_{\alpha = 1}^{d+1/2} \xi^\alpha v^{*\alpha} \right) = 0\) in \(\overline{\Omega}^*\), then \(\sum_{\alpha = 1}^{d+1/2} \xi^\alpha v^{*\alpha} = \sum_{\alpha = 1}^{d+1/2} a_\alpha \psi^\alpha\) in \(\overline{\Omega}^*\), for some constants \(a_\alpha\). Since \(\sum_{\alpha = 1}^{d+1/2} \xi^\alpha v^{*\alpha}\) is a polynomial, then \(a_1 = a_2 = \cdots = a_{d+1/2} = 0\). Hence, \(\mid \xi \mid = 0\) by using \(v^{*\alpha} = \psi^\alpha\) on \(\partial D_i^*\). This is a contradiction.

**STEP 4. Completion.** Finally, we notice from \(\bar{u} = 1 - u\) in \(\Omega_R\) that \(\nabla \bar{u}^\alpha = -\nabla u^\alpha\) in \(\Omega_R\), \(\alpha = 1, 2, \ldots, (d+1)/2\). Then making use of \[3.4\] and \[3.5\],
\[
|a_{11}^{\alpha \beta} - a_{22}^{\alpha \beta}| \leq \int_{\Omega_R} \left( C^0 e(v^\alpha_1), e(v^\beta_1) \right) - \int_{\Omega_R} \left( C^0 e(v^\alpha_2), e(v^\beta_2) \right)
\]
\[
+ \int_{\Omega \setminus \Omega_R} \left( C^0 e(v^\alpha_1), e(v^\beta_1) \right) - \int_{\Omega \setminus \Omega_R} \left( C^0 e(v^\alpha_2), e(v^\beta_2) \right)
\]
\[
\leq \int_{\Omega_R} \left( C^0 e(\bar{u}^\alpha_1), e(\bar{u}^\beta_1) \right) - \int_{\Omega_R} \left( C^0 e(\bar{u}^\alpha_2), e(\bar{u}^\beta_2) \right)
\]
\[
+ \int_{\Omega_R} \left( C^0 e(v^\alpha_1 - \bar{u}^\alpha_1), e(v^\beta_1) \right) - \int_{\Omega_R} \left( C^0 e(v^\alpha_2 - \bar{u}^\alpha_2), e(v^\beta_2) \right)
\]
\[
\leq C.
\]
So that, by \[3.4\],
\[
\left| \frac{C_{11}^{\alpha \beta} - C_{22}^{\alpha \beta}}{2} (a_{11}^{\alpha \beta} - a_{22}^{\alpha \beta}) \right| \leq C \rho_d(\varepsilon), \quad \text{for} \quad \alpha = 1, 2, \ldots, d.
\]  
(3.34)

It follows from \[3.33\] and \[3.29\] that for sufficiently small \(\varepsilon\), \((a^{\alpha \beta})\) is also invertible. So that for sufficiently small \(\varepsilon\), in view of \[3.30\] and \[3.34\] for \(\alpha = 1, 2, \ldots, d\), it follows from comparing \[3.27\] and \[3.28\] that the proof of Lemma \[3.2\] is finished.

4. Proof of Theorem 1.2 in Dimension Two

In this section, we first give an improvement of estimates for \(|C_{11}^\alpha - C_{22}^\alpha|, \alpha = 1, 2\), especially including a lower bound, which contains a non-zero factor \(b_1^\alpha\) (\(b_1^\alpha\) is its limit). This is due to a careful selection from the whole system of \(C_{11}^\alpha\), \[2.9\], although it seems a little different with that in \[14\]. From it we can see the role of the blow-up factor \(b_1^\alpha\) in such singularity analysis of \(|\nabla u|\).
Proposition 4.1. If $b_i^\alpha \neq 0$, then

$$\frac{\sqrt{\varepsilon}}{C} |b_i^\alpha[\varphi]| + o(\sqrt{\varepsilon}) \leq |C_i^\alpha - C_i^\alpha| \leq C\sqrt{\varepsilon}, \quad \alpha = 1, 2. \quad (4.1)$$

In order to solve $C_1^1 - C_2^1$ and $C_1^2 - C_2^2$ from (4.2), we choose $\beta = 1, 2, 3$, for $j = 1$, and $\beta = 3$ for $j = 2$. Then

$$AX = \begin{pmatrix} a_{11}^{11} & a_{12}^{12} & a_{13}^{13} & a_{14}^{14} \\ a_{21}^{11} & a_{22}^{12} & a_{23}^{13} & a_{24}^{14} \\ a_{31}^{11} & a_{32}^{12} & a_{33}^{13} & a_{34}^{14} \\ a_{31}^{21} & a_{32}^{22} & a_{33}^{23} & a_{34}^{24} \end{pmatrix} \begin{pmatrix} C_1^1 - C_2^1 \\ C_1^2 - C_2^2 \\ C_1^3 \\ C_2^3 \end{pmatrix} = \begin{pmatrix} b_1^1 \\ b_2^2 \\ b_3^3 \end{pmatrix}.$$

4.1. Refined estimates in dimension $d = 2$. We first give the following refined estimates for $a_{ij}^{\alpha\beta}$.

Lemma 4.2. $A$ is positive definite, and

$$\frac{1}{C\sqrt{\varepsilon}} \leq a_{ii}^{\alpha\alpha} \leq \frac{C}{\sqrt{\varepsilon}}, \quad \alpha = 1, 2; \quad (4.2)$$

$$|a_{ii}^{12}| = |a_{ii}^{21}| \leq C|\log \varepsilon|; \quad (4.3)$$

$$\frac{1}{C} \leq a_{ii}^{33} \leq C, \quad i = 1, 2; \quad (4.4)$$

$$|a_{12}^{33}| = |a_{21}^{33}|, |a_{13}^{33}| = |a_{23}^{33}|, |a_{33}^{33}| \leq C, \quad i = 1, 2, \alpha = 1, 2; \quad (4.5)$$

and

$$|b_j^\beta| \leq C. \quad (4.6)$$

Remark 4.3. Estimates (4.2), (4.4), (4.5) are the same as in [14], we omit their proof here. Estimate (4.3) is an improvement of (4.13) in [13], $|a_{ii}^{12}| = |a_{ii}^{21}| \leq C\varepsilon^{-1/4}$. While (4.6) needs to be shown since the definition of $b_j^\beta$ is different with that in [14]. So we only prove (4.3) and (4.6) below.

Proof of Lemma 4.2. For (4.3), we take $a_{11}^{11}$ for instance. By definition (2.8),

$$a_{11}^{11} = -\int_{\partial D_1} \frac{\partial v_1}{\partial \nu} \cdot \psi^2 = -\int_{\partial D_1} \frac{\partial \bar{u}_1}{\partial \nu} \cdot \psi^2 - \int_{\partial D_1} \frac{\partial (v_1 - \bar{u}_1)}{\partial \nu} \cdot \psi^2$$

$$:= -I - II,$$

where

$$I = \int_{\partial D_1} \frac{\partial \bar{u}_1}{\partial \nu} \cdot \psi^2 = \int_{\partial D_1 \cap C_R} \frac{\partial \bar{u}_1}{\partial \nu} \cdot \psi^2 + \int_{\partial D_1 \setminus C_R} \frac{\partial \bar{u}_1}{\partial \nu} \cdot \psi^2 := I_R + O(1),$$

and by using (3.4),

$$|II| \leq C.$$

On boundary $\partial D_1$,

$$n_1 = \frac{\partial x_1, h_1(x_1)}{\sqrt{1 + |\partial x_1, h_1(x_1)|^2}}, \quad n_2 = \frac{1}{\sqrt{1 + |\partial x_1, h_1(x_1)|^2}}.$$
Recalling
\[ \bar{u}^1_1 = \begin{pmatrix} \bar{u} \\ 0 \end{pmatrix}, \quad \nabla \bar{u}^1_1 = \begin{pmatrix} \partial_{x_1} \bar{u} & \partial_{x_2} \bar{u} \\ 0 & 0 \end{pmatrix}, \]
then
\[ (\nabla \bar{u}^1_1 + (\nabla \bar{u}^1_1)^T) \bar{u} = \begin{pmatrix} 2\partial_{x_1} \bar{u} n_1 + \partial_{x_2} \bar{u} n_2 \end{pmatrix}. \]
Thus,
\[ I_R = \int_{\partial D_1 \cap C_R} \frac{\partial \bar{u}^1_1}{\partial \nu} \cdot \psi^2 \]
\[ = \int_{\partial D_1 \cap C_R} \left( \lambda (\nabla \cdot \bar{u}^1_1) \bar{u} + \mu (\nabla \bar{u}^1_1 + (\nabla \bar{u}^1_1)^T) \bar{u} \right) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} dS \]
\[ = \int_{\partial D_1 \cap C_R} \lambda (\partial_{x_1} \bar{u}) n_2 + \mu \partial_{x_2} \bar{u} n_1 dS. \]
So that
\[ |I_R| \leq \int_{\partial D_1 \cap C_R} |\lambda (\partial_{x_1} \bar{u}) n_2 + \mu \partial_{x_2} \bar{u} n_1| dS \]
\[ \leq C \int_{|x_1| \leq \epsilon} \frac{|x_1|}{d} dx_1 \]
\[ \leq C |\log \epsilon|. \]
Therefore
\[ |a_{i1}^2| \leq |I| + |II| \leq C |\log \epsilon|. \]
By definition (2.28), and \( u_B u = \sum_{\alpha=1}^2 C_2^\alpha v^\alpha + v_0 \), we have
\[ b^\beta_i = \int_{\partial D_1} \frac{\partial u_1}{\partial \nu} \cdot \psi^\beta \]
\[ = C_2^1 \int_{\partial D_1} \frac{\partial v^1}{\partial \nu} \cdot \psi^\beta + C_2^2 \int_{\partial D_2} \frac{\partial v^2}{\partial \nu} \cdot \psi^\beta \]
\[ = C_2^1 I_1 + C_2^2 I_2 + I_0. \]
By using integration by parts, (1.10), and (2.19),
\[ |I_0| = \int_{\partial D_1} \frac{\partial v_0}{\partial \nu} \cdot \psi^\beta = \int \left( C e(v_0, e(v_1^\beta)) \right) \leq C. \]
Similarly, by using (2.20), we have
\[ |I_1|, |I_2| \leq C. \]
Combining these with (2.28), \( |C_2^\alpha| \leq C \), the proof of (4.4) is finished.
Finally, we prove that \( A \) is positive definite. For \( \xi = (\xi_1, \xi_2, \xi_3, \xi_4)^T \neq 0 \), by elliptic condition, (1.4), we have
\[ \xi^T A \xi = \int_{\Omega} \left( C \left( e(\xi_1 v_1 + \xi_2 v_1^2 + \xi_3 v_1^3 + \xi_4 v_2^3), e(\xi_1 v_1 + \xi_2 v_1^2 + \xi_3 v_1^3 + \xi_4 v_2^3) \right) \right) \]
\[ \geq \int_{\Omega} |e(\xi_1 v_1 + \xi_2 v_1^2 + \xi_3 v_1^3 + \xi_4 v_2^3)|^2 > 0. \]
In the last inequality, we used the fact that \( e(\xi_1 v_1 + \xi_2 v_1^2 + \xi_3 v_1^3 + \xi_4 v_2^3) \) is not identically zero. Indeed, if an element \( \xi \in \Psi \) vanishes at two distinct points \( \bar{x}_1, \bar{x}_2 \), then \( \xi \equiv 0 \), see lemma 6.1 in [15]. Namely, if \( e(\xi_1 v_1 + \xi_2 v_1^2 + \xi_3 v_1^3 + \xi_4 v_2^3) = \)
0, then $\xi_1 v_1^3 + \xi_2 v_2^3 + \xi_3 v_3^3 + \xi_4 v_4^3 = \sum_{\alpha=1}^{3} a_\alpha \psi^\alpha$ for some constants $a_\alpha$. Since $\xi_1 v_1^3 + \xi_2 v_2^3 + \xi_3 v_3^3 + \xi_4 v_4^3|_{\partial \Omega} = 0$, and $\{\psi^\alpha\}_{\alpha=1,2,3}$ is linear independent, it follows that $a_1 = a_2 = a_3 = 0$. Since $\xi_1 v_1^3 + \xi_2 v_2^3 + \xi_3 v_3^3 + \xi_4 v_4^3|_{\partial D_2} = \xi_4 \psi^3$, so that $\xi_4 = 0$. While, by $\xi_1 v_1^3 + \xi_2 v_2^3 + \xi_3 v_3^3 + \xi_4 v_4^3|_{\partial D_1} = \xi_1 \psi^1 + \xi_2 \psi^2 + \xi_3 \psi^3$ and the same reason, $\xi_1 = \xi_2 = \xi_3 = 0$. This is a contradiction. 

From the fact that $A$ is positive definite, we know that its principle minor $A_{22} = \begin{pmatrix} a_{11} & a_{13} \\ a_{13} & a_{22} \end{pmatrix}$ is also positive definite. Furthermore, we have

**Lemma 4.4.** There is a universal constant $C$, independent of $\varepsilon$, such that

$$\det A_{22} = a_{11}^3 a_{22} - a_{12}^3 a_{21} > \frac{1}{C}. $$

**Proof.** From elliptic condition, (4.4), it suffices to prove that

$$\int_\Omega |e(\xi_1 v_1^3 + \xi_2 v_2^3)|^2 > \frac{1}{C}, \quad \forall \xi = (\xi_1, \xi_2)^T, \ |\xi| = 1. $$

Indeed, if not, then there exist a sequence $\varepsilon_k \to 0^+$, $|\xi_k| = 1$, such that

$$\int_\Omega |e(\xi_1^k v_1^3 + \xi_2^k v_2^3)|^2 \to 0, \quad \text{as} \ i \to \infty. \quad (4.7)$$

Here we add superscript $\varepsilon_k$ to denote the solution of (4.4) when $\text{dist}(D_1, D_2) = \varepsilon_k$.

Since $v_i^{3,\varepsilon_k} \equiv 0$ on $\partial \Omega$, it follows from the second Korns inequality (see Theorem 2.5 in [14]) that there exists a constant $C$, independent of $\varepsilon_k$, such that

$$\|v_i^{3,\varepsilon_k}\|_{H^1(\bar{\Omega} \setminus B_{\bar{r}}; \mathbb{R}^d)} \leq C, \quad i = 1, 2, $$

for some $\bar{r} > 0$ (say, $\bar{r} = R/2$). Then there exists a subsequence, still denote $\{v_i^{3,\varepsilon_k}\}$, such that

$$v_i^{3,\varepsilon_k} \to v_i^{*3}, \quad \text{in} \ H^1(\bar{\Omega}^* \setminus B_{\bar{r}}; \mathbb{R}^d), \quad \text{as} \ k \to +\infty, \quad i = 1, 2. $$

It follows from (4.7) that there exists $\xi^*$ such that

$$\xi^k \to \xi^*, \quad \text{as} \ k \to +\infty, \quad \text{with} \ |\xi^*| = 1, $$

and

$$\int_{\bar{\Omega}^* \setminus B_{\bar{r}}} |e(\xi_1^* v_1^{3} + \xi_2^* v_2^{3})|^2 = 0, $$

where $v_i^{*3}$, $i = 1, 2$, are defined by

$$\begin{cases}
L^\omega v_i^{*3} = 0, & \text{in} \ \bar{\Omega}^*, \\
v_i^{*3} = \psi^\alpha, & \text{on} \ \partial D^* \setminus \{0\}, \\
v_i^{*3} = 0, & \text{on} \ \partial D^* \cup \partial \Omega, \ j \neq i,
\end{cases} \quad (4.8)$$

with $\alpha = 3$. This implies that

$$e(\xi_1^* v_1^{*3} + \xi_2^* v_2^{*3}) = 0, \quad \text{in} \ \bar{\Omega}^* \setminus B_{\bar{r}}. $$

Hence,

$$\xi_1^* v_1^{*3} + \xi_2^* v_2^{*3} = \sum_{\alpha=1}^{3} a_\alpha \psi^\alpha, \quad \text{in} \ \bar{\Omega}^* \setminus B_{\bar{r}}, $$

for some constants $a_\alpha$, $\alpha = 1, 2, 3$. 
Since \(\xi_1^* v_1^* + \xi_2^* v_2^*\big|_{\partial \Omega} = 0\), we have \(a_1 = a_2 = a_3 = 0\). Indeed, if an element \(v \in \Psi\) vanishes at two distinct points \(\bar{x}^1, \bar{x}^2\), then \(v \equiv 0\), see lemma 6.1 in [15]. Namely, restricting on one part of \(\partial D_2^*\), we have \(\xi_2^* = 0\). Restricting on one part of \(\partial D_1^*\), we have \(\xi_1^* = 0\). This is a contradiction with \(|\xi^*| = 1\). The proof is finished. □

**Proof of Proposition 4.1.** By the estimates in Lemma 4.2, it follows that \(A\) is positive definite, and

\[
\frac{1}{\sqrt{\varepsilon}} \leq \det A \leq \frac{C}{\varepsilon}.
\]

So that \(A\) is invertible, and

\[
C_1^1 - C_2^1 = \frac{b_1^1 c_{12} (a_1^{33} a_2^{33}) - a_1^{33} a_2^{33}}{\det A} + o(\sqrt{\varepsilon}),
\]

and

\[
C_1^2 - C_2^2 = \frac{b_1^2 c_{12} (a_1^{33} a_2^{33}) - a_1^{33} a_2^{33}}{\det A} + o(\sqrt{\varepsilon}).
\]

On one hand, it is easy to obtain from Lemma 4.2 again that the upper bound

\[
|C_1^\alpha - C_2^\alpha| \leq C \sqrt{\varepsilon}, \quad \alpha = 1, 2.
\]

On the other hand, since \(a_1^{33} a_2^{33} - a_1^{33} a_2^{33} \geq \frac{1}{C}\), then if \(b_1^\alpha \neq 0\), then

\[
|C_1^\alpha - C_2^\alpha| = \frac{\sqrt{\varepsilon}}{C} |b_1^\alpha| + o(\sqrt{\varepsilon}).
\]

Thus, Proposition 4.1 is proved. □

Finally, we give the proof of Theorem 1.2.

**Proof of Theorem 1.2 in dimension two.** By Proposition 2.1, we have

\[
b_1^\alpha - b_2^\alpha = O(\varepsilon^{1/3}), \quad \alpha = 1, 2.
\]

Then if there is an \(k_0 \in \{1, 2\}\) such that \(b_{\alpha_1}^{k_0} \neq 0\), then for sufficiently small \(\varepsilon\),

\[
|\nabla u(x)| \geq \left| \sum_{\alpha=1}^{2} (C_1^\alpha - C_2^\alpha) \nabla v_1^\alpha(x) \right| - C \geq \frac{|b_{\alpha_1}^{k_0}|}{C \sqrt{\varepsilon}} \geq \frac{|b_{\alpha_1}^{k_0} + O(\varepsilon^{1/3})|}{C \sqrt{\varepsilon}} \geq \frac{|b_{\alpha_1}^{k_0}|}{C \sqrt{\varepsilon}}.
\]

The proof is finished. □

**5. Proof of Theorem 1.2 in dimension three**

In this section, we are devoted to proving Theorem 1.2 in dimension three. Similarly, as in Section 4, we first give lower and upper bounds of estimates for \(|C_1^\alpha - C_2^\alpha|\), \(\alpha = 1, 2, 3\). Here the selection from the whole system for \(C_1^\alpha\), \((2.9)\), is different from that in [15].

**Proposition 5.1.** If \(b_1^\alpha[\varphi] \neq 0\), then

\[
\frac{1}{C |\log \varepsilon|} |b_1^\alpha[\varphi]| + O(|\log \varepsilon|^{-2}) \leq |C_1^\alpha - C_2^\alpha| \leq \frac{C}{|\log \varepsilon|}, \quad \alpha = 1, 2.
\]
5.1. Finer Estimates in Dimension \( d = 3 \). In order to solve \( C_1^1 - C_2^1, C_1^2 - C_2^2, \) and \( C_1^3 - C_2^3 \) from \((2.9)\), we take \( \beta = 1, 2, \cdots, 6 \) for \( j = 1 \) and \( \beta = 4, 5, 6, \) for \( j = 2 \). Then

\[
AX = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},
\]

where

\[
A_{11} = \begin{pmatrix} a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \\ a_{11}^{21} & a_{11}^{22} & a_{11}^{23} \\ a_{11}^{31} & a_{11}^{32} & a_{11}^{33} \end{pmatrix}, \quad A_{12} = \begin{pmatrix} a_{12}^{1 & \alpha} \\ a_{12}^{2 & \alpha} \\ a_{12}^{3 & \alpha} \end{pmatrix}, \quad \alpha = 1, 2, 3; \quad \beta = 4, 5, 6.
\]

\[
A_{21} = \begin{pmatrix} a_{21}^{1 & \alpha} \\ a_{21}^{2 & \alpha} \\ a_{21}^{3 & \alpha} \end{pmatrix}, \quad A_{22} = \begin{pmatrix} a_{22}^{1 & \alpha} & a_{22}^{1 & \beta} \\ a_{22}^{2 & \alpha} & a_{22}^{2 & \beta} \\ a_{22}^{3 & \alpha} & a_{22}^{3 & \beta} \end{pmatrix}, \quad \alpha, \beta = 4, 5, 6.
\]

\[
X_1 = (C_1^1 - C_2^1, C_1^2 - C_2^2, C_1^3 - C_2^3)^T, \quad X_2 = (C_1^4, C_1^5, C_1^6, C_2^4, C_2^5, C_2^6)^T,
\]

and

\[
B_1 = \begin{pmatrix} b_{1}^1, b_{1}^2, b_{1}^3 \end{pmatrix}^T, \quad B_2 = \begin{pmatrix} b_{1}^4, b_{1}^5, b_{1}^6, b_{2}^4, b_{2}^5, b_{2}^6 \end{pmatrix}^T.
\]

**Lemma 5.2.** \((13)\) \( A \) is positive definite, and

\[
\frac{|\log \varepsilon|}{C} \leq a_{11}^{\alpha & \alpha} \leq C |\log \varepsilon|, \quad \alpha = 1, 2, 3;
\]

\[
\frac{1}{C} \leq a_{ii}^{\alpha & \alpha} \leq C, \quad \alpha = 4, 5, 6, \quad i = 1, 2;
\]

and

\[
|a_{ij}^{\alpha & \beta}| = |a_{ji}^{\beta & \alpha}| \leq C, \quad \alpha, \beta = 1, 2, \cdots, \frac{d(d+1)}{2}, \quad \alpha \neq \beta.
\]

\[
|b_{\beta}| \leq C, \quad \beta = 1, 2, \cdots, \frac{d(d+1)}{2}.
\]

Therefore,

\[
\frac{|\log \varepsilon|^3}{C} \leq \det A_{11} \leq C |\log \varepsilon|^3.
\]

**Proof.** The estimate of \( b_{\beta}^i \) can be proved by a very similar way as in the proof of Lemma 4.2 We omit it here. \( \square \)

**Lemma 5.3.** There is a universal constant \( C \) such that, for any \( \xi = (\xi^1, \xi^2, \cdots, \xi^6)^T \neq 0, \)

\[
\xi^T A_{22} \xi > \frac{1}{C} |\xi|^2.
\]

**Proof.** The proof is similar to that of Lemma 4.3 We omit the limit process, since it is the same. After it, if there exists a vector \( \xi^* = (\xi^1_*, \xi^2_*, \cdots, \xi^6_*)^T \) with \( |\xi^*| = 1, \) such that

\[
eq 0, \quad \text{in } \Omega^* \setminus B_r,
\]

\[
eq 0, \quad \text{in } \Omega^* \setminus B_r.
\]

\[
eq 0, \quad \text{in } \Omega^* \setminus B_r.
\]
where $v_i^{*,\beta}$ are defined in (5.8) for $\beta = 4, 5, 6$. Indeed, by using Lemma 6.1 in [13] again, if an element $v \in \Psi$ vanishes at three distinct points $\bar{x}_1, \bar{x}_2$, and $\bar{x}_3$, which are not on a plane, then $v \equiv 0$. Namely, there exist $a_\alpha, \alpha = 1, 2, \cdots, 6$, such that

$$\sum_{\alpha=1}^{3} \xi_\alpha v_1^{*(\alpha+3)} + \sum_{\alpha=1}^{6} \xi_\alpha v_2^{*\alpha} = \sum_{\alpha=1}^{6} a_\alpha \psi^{\alpha}.$$ 

Recalling that $v_i^{*\alpha} = 0$ on $\partial \Omega$, we have $a_\alpha = 0, \alpha = 1, 2, \cdots, 6$. Restricting on one part of $\partial D_1^*$, in view of the linear independence of $\psi^4, \psi^5$ and $\psi^6$ on $\partial D_1^*$, we have $\xi_1^* = \xi_2^* = \xi_3^* = 0$. By the same reason on $\partial D_2, \xi_4^* = \xi_5^* = \xi_6^* = 0$. This is a contradiction with $|\xi^*| = 1$. \hfill \Box

Proof of Proposition 5.1. By the estimates in Lemma 5.2, it follows that

$$\frac{|\log \varepsilon|^3}{C} \leq \det A \leq C|\log \varepsilon|^3,$$

and $A$ is invertible, then by Cramer’s rule,

$$C_1 - C_2 = \frac{b_1 a_{11}^2 a_{12}^3}{\det A} \log \varepsilon + O(|\log \varepsilon|^{-2}),$$

$$C_1^2 - C_2^2 = \frac{b_1^2 a_{11}^2 a_{12}^2}{\det A} \log \varepsilon + O(|\log \varepsilon|^{-2}),$$

and

$$C_1^3 - C_2^3 = \frac{b_1^3 a_{11}^1 a_{12}^2}{\det A} \log \varepsilon + O(|\log \varepsilon|^{-2}).$$

On one hand, it is easy to obtain from Lemma 5.2 again that the upper bound

$$|C_1^{\alpha} - C_2^{\alpha}| \leq \frac{C}{|\log \varepsilon|}, \quad \alpha = 1, 2, 3.$$ 

On the other hand, since $\det A_{22} \geq \frac{1}{C}$, then if $b_1^\gamma \neq 0$, then

$$|C_1^{\alpha} - C_2^{\alpha}| \geq \frac{|b_1^\gamma|}{C|\log \varepsilon|} + O(|\log \varepsilon|^{-2}).$$

Thus, Proposition 5.1 is proved. \hfill \Box

Proof of Theorem 1.2 in dimension three. For $d = 3$, using the fact that $|\nabla v_i^\beta(0', x_d)| = 0$ for $|x_d| < \varepsilon/2$, $\beta = 4, 5, 6$, and estimates (2.19), (2.20), (2.22), and (5.1), we have for $x = (0', x_d) \in \bar{\Omega}$,

$$|\nabla u(x)| = \left| \sum_{\alpha=1}^{3} (C_1^{\alpha} - C_2^{\alpha}) \nabla v_1^{\alpha} + \sum_{\alpha=1}^{6} C_2^{\alpha} \nabla (v_1^{\alpha} + v_2^{\alpha}) + \sum_{i=1}^{2} \sum_{\alpha=4}^{6} C_i^{\alpha} \nabla v_i^{\alpha} + \nabla v_0(x) \right|$$

$$\geq \left| \sum_{\alpha=1}^{3} (C_1^{\alpha} - C_2^{\alpha}) \nabla v_1^{\alpha}(x) \right| - \left( \sum_{\alpha=1}^{3} |C_2^{\alpha}| \left| \nabla (v_1^{\alpha} + v_2^{\alpha})(x) \right| + |\nabla v_0(x)| + C \right)$$

$$\geq \left| \sum_{\alpha=1}^{3} (C_1^{\alpha} - C_2^{\alpha}) \nabla v_1^{\alpha}(x) \right| - C$$

$$\geq \left| \sum_{\alpha=1}^{3} (C_1^{\alpha} - C_2^{\alpha}) \nabla u_1^{\alpha}(x) \right| - C.$$ 

(5.2)
For $|x_d| < \varepsilon/2$,

$$\sum_{\alpha=1}^{2} (C_1^\alpha - C_2^\alpha) \nabla \bar{u}_1^\alpha(0', x_d) = \frac{1}{\varepsilon} \begin{pmatrix}
C_1^1 - C_2^1 \\
C_1^2 - C_2^2 \\
C_1^3 - C_2^3
\end{pmatrix}.$$  \tag{5.3}

Therefore, it suffices to obtain a positive lower bound of $|C_1^1 - C_2^1|$, $|C_1^2 - C_2^2|$ or $|C_1^3 - C_2^3|$.

If $b^{k_0}_{k_1} \neq 0$, for some integer $1 \leq k_0 \leq 3$, then it follows from Proposition 2.1 that there exists a universal constant $C_0 > 0$ and a sufficiently small number $\varepsilon_0 > 0$, such that, for $0 < \varepsilon < \varepsilon_0$,

$$|b^{k_0}_{k_1}| > \frac{1}{C_0}.$$  \nonumber

By \eqref{eq:6.1}, for sufficiently small $\varepsilon$,

$$|C_1^1 - C_2^1| \geq \frac{|b^{k_0}_{k_1}|}{C_0} \geq \frac{|b^{k_0}_{k_1}|}{C_0 |\log \varepsilon|}.$$  \nonumber

Combining it with \eqref{eq:6.1} and \eqref{eq:5.3} immediately yields that

$$|\nabla u(0', x_d)| \geq \frac{|b^{k_0}_{k_1}|}{C_0 |\log \varepsilon|}, \quad |x_d| < \varepsilon/2.$$  \nonumber

Theorem 1.2 is thus established.  \hfill \Box

6. PROOF OF THEOREM 1.4

6.1. Decomposition of $u$. We make use of the following decomposition as in \cite{11},

$$u = C_1v_1 + C_2v_2 + v_0, \quad \text{in } \tilde{\Omega},$$  \tag{6.1}

where $v_i \in C^1(\Omega)$, $i = 1, 2, 0$, are, respectively, the solutions of

$$\begin{cases}
\Delta v_i = 0, & \text{in } \tilde{\Omega}, \\
v_i = 1, & \text{on } \partial D_i, \\
v_i = 0, & \text{on } \partial \tilde{\Omega} \setminus \partial D_i,
\end{cases}$$  \tag{6.2}

and

$$\begin{cases}
\Delta v_0 = 0, & \text{in } \tilde{\Omega}, \\
v_0 = 0, & \text{on } \partial D_1 \cup \partial D_2, \\
v_0 = \varphi, & \text{on } \partial \Omega.
\end{cases}$$  \tag{6.3}

The constants $C_i := C_i(\varepsilon)$, $i = 1, 2$, in \eqref{eq:6.1}, are uniquely determined by $u$.

Denote

$$u^b := C_2(v_1 + v_2) + v_0,$$

then $u^b$ verifies

$$\begin{cases}
\Delta u^b = 0, & \text{in } \tilde{\Omega}, \\
u^b = C_2, & \text{on } \partial D_1 \cup \partial D_2, \\
u^b = \varphi, & \text{on } \partial \Omega.
\end{cases}$$  \tag{6.4}
By using theorem 1.1 in [35] again, we have
\[ |\nabla u^b| \leq C, \] (6.5)
where $C$ is a universal constant, independent of $\varepsilon$.

In view of the decomposition (6.1), we write
\[ \nabla u = (C_1 - C_2) \nabla v_1 + \nabla u^b, \quad \text{in } \tilde{\Omega}. \] (6.6)

It follows from the forth line of (1.12) that
\[ (C_1 - C_2) \int_{\partial D_j} \frac{\partial v_1}{\partial \nu} \bigg|_+ + \int_{\partial D_j} \frac{\partial u^b}{\partial \nu} \bigg|_+ = 0, \quad j = 1, 2. \] (6.7)

Denote $a_{ij} := -\int_{\partial D_j} \frac{\partial v_i}{\partial \nu} \bigg|_+ \nabla v_j \cdot \nabla v_j$, $b_j := b_j[\varphi] = \int_{\partial D_j} \frac{\partial u^b}{\partial \nu} \bigg|_+$, $i, j = 1, 2$.

Then (6.7) can be written as
\[ \begin{aligned}
& a_{11}(C_1 - C_2) = b_1, \\
& a_{12}(C_1 - C_2) = b_2.
\end{aligned} \] (6.8)

Recalling the definitions of $v_1, v_2$ and by using the integration by parts, we have
\[ a_{ij} = -\int_{\partial D_j} \frac{\partial v_i}{\partial \nu} \bigg|_+ \nabla v_j \cdot \nabla v_j, \quad b_j = \int_{\partial D_j} \frac{\partial u^b}{\partial \nu} \bigg|_+, \quad i, j = 1, 2. \]

By the same argument for the estimate of $|\nabla v_1|$ in Proposition [2.2] or the estimate of $|\nabla v_1|$ in Proposition 3.1 in [36], one can see
\[ \|\nabla (v_1 - \bar{u})\|_{L^\infty(\tilde{\Omega}_{R/2})} \leq C, \quad \|\nabla (v_2 - \bar{u})\|_{L^\infty(\tilde{\Omega}_{R})} \leq C. \] (6.9)

So that
\[ \frac{1}{C(x + |x'|^2)} \leq |\nabla v_i(x)| \leq \frac{C}{\varepsilon + |x'|^2}, \quad i = 1, 2, \quad x \in \Omega_R, \] (6.10)

and
\[ \|\nabla v_i\|_{L^\infty(\tilde{\Omega} \setminus \Omega_{R/2})} \leq C, \quad i = 1, 2. \] (6.11)

Therefore,
\[ \frac{1}{C \rho_d(\varepsilon)} \leq a_{11} \leq \frac{C}{\rho_d(\varepsilon)}, \quad \frac{1}{C \rho_d(\varepsilon)} \leq -a_{12} \leq \frac{C}{\rho_d(\varepsilon)}, \] (6.12)

and in view of (6.5),
\[ |b_j| \leq C, \quad j = 1, 2. \] (6.13)

6.2. Proof of Theorem 1.4

Proof of Theorem 1.4. If $b_1 = 0$, then by solving (6.8), we have $C_1 - C_2 = 0$, since $a_{11} > 0$. Hence, it follows from (6.6) and (6.5) that $|\nabla u| = |\nabla u^b| \leq C$. Therefore, blow-up does not occur.

If $b_1 \neq 0$, then solving (6.8), we have
\[ |C_1 - C_2| = \frac{|b_1|}{a_{11}}. \] (6.14)

In view of (6.12) and (6.13), we have
\[ |C_1 - C_2| \leq C \rho_d(\varepsilon), \] (6.15)
and
\[ |C_1 - C_2| \geq C|b_1|\rho_d(\varepsilon), \quad \text{if } b_1 \neq 0. \] (6.16)

It follows from (6.6), (6.5), (6.10) and (6.15) that
\[ |\nabla u| \leq |C_1 - C_2||\nabla v_1| + \left| \nabla u^b \right| \leq \frac{C\rho_d(\varepsilon)}{\varepsilon + |x'|^2}, \quad \text{in } \Omega_R, \] (6.17)

and on the other hand, by (6.16),
\[ |\nabla u(0', x_d)| \geq |(C_1 - C_2)\nabla v_1(0', x_d)| - C \geq \frac{|b_1|\rho_d(\varepsilon)}{C\varepsilon}, \quad |x_d| < \frac{\varepsilon}{2}. \] (6.18)

By using Proposition 6.1 below, \( b_j \rightarrow b^*_j \) as \( \varepsilon \rightarrow 0 \), it follows from (6.18) that for sufficiently small \( \varepsilon \),
\[ |\nabla u(x)| \geq \frac{|b_1|\rho_d(\varepsilon)}{C\varepsilon} \geq \frac{|b^*_1|\rho_d(\varepsilon)}{C\varepsilon}. \]

Thus, \( b^*_1 \) is a blow-up factor. If \( |b^*_1| \neq 0 \), then we obtain the lower bound of \( |\nabla u(x)| \) on \( P_1P_2 \) and complete the proof of Theorem 1.4. \( \square \)

6.3. The blow-up factors \( b^*_j[\varphi] \). In order to characterize the limit properties of \( b_1 \) (or \( b_2 \)), we consider the following limit problem. Let \( u^* \) be the solution of
\[
\begin{cases}
\Delta u^* = 0, & \text{in } \Omega^*, \\
u^* = C^*, & \text{on } \partial D^*_1 \cup \partial D^*_2, \\
\int_{\partial D^*_1} \frac{\partial u^*}{\partial \nu} + \int_{\partial D^*_2} \frac{\partial u^*}{\partial \nu} = 0, \\
u^* = \varphi, & \text{on } \partial \Omega.
\end{cases}
\] (6.19)

where, by the uniqueness of the solution and (6.15), we have \( C^* = \frac{1}{2}(C_1 + C_2) \).

Define
\[ b^*_j := b^*_j[\varphi] = \int_{\partial D^*_j} \frac{\partial u^*}{\partial \nu}. \]

Then

Proposition 6.1. For \( d = 2, 3 \),
\[ |b_j - b^*_j| = \left| \int_{\partial D^*_j} \frac{\partial u^b}{\partial \nu} ds - \int_{\partial D^*_j} \frac{\partial u^*}{\partial \nu} ds \right| \leq C\sqrt{\rho_d(\varepsilon)}, \quad j = 1, 2. \] (6.20)

To prove Proposition 6.1, we need the following lemma.

Lemma 6.2. Let \( C_1 \) and \( C_2 \) be defined in (6.1) and \( C^* \) be in (6.19). We have
\[ \left| \frac{C_1 + C_2}{2} - C^* \right| \leq C\rho_d(\varepsilon). \] (6.21)

As a consequence, combining it with (6.13), we have
\[ |C_i - C^*| \leq \left| C_i - \frac{C_1 + C_2}{2} \right| + \left| \frac{C_1 + C_2}{2} - C^* \right| \leq C\rho_d(\varepsilon), \quad i = 1, 2. \] (6.22)

The proof of Lemma 6.2 will be given later. We first use it to prove Proposition 6.1.
respectively, by using integration by parts, we have
\[ \phi(x) := u^b(x) - u^*(x), \]
and then \( \Delta \phi = 0 \) in \( V \). It is easy to see that \( \phi = 0 \) on \( \partial \Omega \). On \( \Gamma_{11} \), by mean value theorem, (6.5) and (6.22), we have
\[ |\phi|_{\Gamma_{11}} = |u^b - u^*|_{\Gamma_{11}} = |C_2 + |\nabla u^b(\xi)|\varepsilon - C^*| \leq C \rho_d(\varepsilon) + C \varepsilon \leq C \rho_d(\varepsilon), \]
where \( \xi \in D_1^* \setminus D_1 \). Similarly,
\[ |\phi|_{\Gamma_{12}} = |u^b - u^*|_{\Gamma_{12}} \leq |C_2 - C^* - |\nabla u^*(\xi)||\varepsilon| \leq C \rho_d(\varepsilon) + C \varepsilon \leq C \rho_d(\varepsilon), \]
for some \( \xi \in D_1 \setminus D_1^* \). By the same way,
\[ |\phi|_{\Gamma_{21} \cup \Gamma_{22}} \leq C \rho_d(\varepsilon). \]

We now apply the maximum principle to \( \phi \) on \( V \),
\[ |\phi| \leq C \rho_d(\varepsilon), \quad \text{on } V. \quad (6.23) \]

Denote
\[ \Omega^+ := V \cap \{ x \in \Omega \mid x_d > 0 \}, \quad \text{and } \partial \Omega^+ := \{ x \in \partial \Omega \mid x_d > 0 \}, \]
and \( \gamma = \{ x_d = 0 \} \cap \Omega \). Since \( u^b \) and \( u^* \) are harmonic in \( \Omega^+ \setminus D_1 \) and \( \Omega^+ \setminus D_1^* \), respectively, by using integration by parts, we have
\[ \int_{\partial D_1} \frac{\partial u^b}{\partial \nu} = \int_{\partial \Omega^+} \frac{\partial u^b}{\partial \nu} + \int_{\gamma} \frac{\partial u^b}{\partial \nu}, \]
and
\[ \int_{\partial D_1^*} \frac{\partial u^*}{\partial \nu} = \int_{\partial \Omega^+} \frac{\partial u^*}{\partial \nu} + \int_{\gamma} \frac{\partial u^*}{\partial \nu}. \]
Thus,
\[ \int_{\partial D_1} \frac{\partial u^b}{\partial \nu} - \int_{\partial D_1^*} \frac{\partial u^*}{\partial \nu} dS = \int_{\partial \Omega^+} \frac{\partial \phi}{\partial \nu} + \int_{\gamma} \frac{\partial \phi}{\partial \nu}. \]

Divide \( \gamma \) into three pieces: \( \gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \), where
\[ \gamma_1 := \{(x', 0) \mid |x'| \leq \sqrt{\rho_d(\varepsilon)}\}, \quad \gamma_2 := \{(x', 0) \mid \sqrt{\rho_d(\varepsilon)} < |x'| < R\}, \quad \gamma_3 := \gamma \setminus (\gamma_1 \cup \gamma_2). \]

Write
\[ \int_{\gamma} \frac{\partial \phi}{\partial \nu} = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} \frac{\partial \phi}{\partial \nu} := I + II + III. \]

First, for \( (y', 0) \in \gamma_1 \), since \( |\nabla u^b|, |\nabla u^*| \leq C \) in \( \Omega_R \), so \( |\nabla \phi| \leq C \) in \( \Omega_R \). Hence
\[ |I| \leq C(\sqrt{\rho_d(\varepsilon)})^{d-1} \leq C \rho_d(\varepsilon)^{(d-1)/2}. \]

For \( (y', 0) \in \gamma_2 \), there exists a \( r > \frac{1}{\sqrt{d}} |y'|^d \) for some \( C > 1 \) such that \( B_r(y', 0) \subset V \). It then follows from the standard gradient estimates for harmonic function and (6.22) that
\[ |\nabla \phi(y', 0)| \leq \frac{C \rho_d(\varepsilon)}{|y'|^d}, \]
and
\[ |II| \leq C \rho_d(\varepsilon) \int_{\sqrt{\rho_d(\varepsilon)} < |y'| < R} \frac{1}{|y'|^d} dS \leq C \sqrt{\rho_d(\varepsilon)}. \]

Proof of Proposition 6.1. Recall \( V = \Omega \setminus \bigcup D_1 \cup D_2 \cup D_1^* \cup D_2^* \), and \( \Gamma_{11} := \partial D_1^* \setminus \partial D_i \) and \( \Gamma_{12} := \partial D_i \setminus \partial D_i^* \), \( i = 1, 2 \). Then \( \partial V = \bigcup_{i,j=1,2} \Gamma_{ij} \cup \partial \Omega \). Setting
\[ \phi(x) := u^b(x) - u^*(x), \]
For $(y',0) \in \gamma_3$, there is a universal constant $r > 0$ such that $B_r(x) \subset V$ for all $x \in \gamma_3$. So we have from (6.23) that for any $x \in \gamma_3$,

$$|\nabla \varphi| \leq \frac{C\varepsilon}{r} \leq C\varepsilon,$$

and

$$|\text{III}| \leq C\varepsilon.$$

Finally, using the standard boundary gradient estimates for $\varphi$ and (6.23), we have

$$\left| \int_{\partial \Omega^+} \frac{\partial \phi}{\partial \nu} \right| \leq C\rho d(\varepsilon).$$

Thus, we have (6.20). The proof is completed. $\square$

Remark 6.3. From the proof of Proposition 6.1, one can see that the convergence of $b_j$ depends on the estimates (6.15), $|C_1 - C_2| \leq C\rho d(\varepsilon) \to 0$. However, in higher dimensions $d \geq 4$, we still do not have such closeness of $C_1$ and $C_2$. However, from (6.14), it is not difficult to find a boundary data $\varphi$ such that $|b_1[\varphi]| \geq \frac{1}{C}$ for some universal constant $C$, although $b_1^*[\varphi]$ is not necessarily its limit. Thus, we also can have a lower bound estimate,

$$|\nabla u(x)| \geq \frac{1}{C\varepsilon}.$$

Proof of Lemma 6.2. We decompose $u^*$ into

$$u^* = C^* v_1^* + v_0^*, \quad \text{in } \tilde{\Omega}^*,$$

where $v_1^*, v_0^* \in C^1(\tilde{\Omega})$ are, respectively, the solutions of

$$\begin{cases}
\Delta v_1^* = 0, & \text{in } \tilde{\Omega}^*, \\
v_1^* = 1, & \text{on } \partial D_1^* \cup \partial D_2^*, \\
v_1^* = 0, & \text{on } \partial \Omega,
\end{cases} \quad (6.24)$$

and

$$\begin{cases}
\Delta v_0^* = 0, & \text{in } \tilde{\Omega}^*, \\
v_0^* = 0, & \text{on } \partial D_1^* \cup \partial D_2^*, \\
v_0^* = \varphi, & \text{on } \partial \Omega.
\end{cases} \quad (6.25)$$

From the third line of (6.19), we have

$$C^* \left( \int_{\partial D_1^*} \frac{\partial v_1^*}{\partial \nu} + \int_{\partial D_2^*} \frac{\partial v_1^*}{\partial \nu} \right) + \left( \int_{\partial D_1^*} \frac{\partial v_0^*}{\partial \nu} + \int_{\partial D_2^*} \frac{\partial v_0^*}{\partial \nu} \right) = 0 \quad (6.26)$$

Let $v_1, v_2$ and $v_0$ be defined in (6.2) and (6.3). We claim that

$$\left| \int_{\partial D_i} \frac{\partial (v_1 + v_2)}{\partial \nu} \right| + \left| \int_{\partial D_i^*} \frac{\partial v_1^*}{\partial \nu} \right| \leq C\rho_d(\varepsilon), \quad i = 1, 2, \quad (6.27)$$

and

$$\left| \int_{\partial D_i} \frac{\partial v_0}{\partial \nu} \right| + \left| \int_{\partial D_i^*} \frac{\partial v_0^*}{\partial \nu} \right| \leq C\rho_d(\varepsilon), \quad i = 1, 2. \quad (6.28)$$

As in the proof of Proposition 6.1 letting

$$\phi_1 := (v_1 + v_2) - v_1^*,$$
then \( \Delta \phi_1 = 0 \) in \( V \), and \( \phi_1 = 0 \) on \( \partial \Omega \). Since \( v_1 + v_2 \) satisfies
\[
\begin{cases}
\Delta (v_1 + v_2) = 0, & \text{in } \tilde{\Omega}, \\
v_1 + v_2 = 1, & \text{on } \partial D_1 \cup \partial D_2, \\
v_1 + v_2 = 0, & \text{on } \partial \Omega,
\end{cases}
\tag{6.29}
\]
it follows from theorem 1.1 in [35] that
\[
|\nabla (v_1 + v_2)| \leq C, \quad \text{in } \tilde{\Omega},
\tag{6.30}
\]
since the potential takes the same constant value on boundaries of both partials. Because of the same reason,
\[
|\nabla v_1^*| \leq C, \quad \text{in } \tilde{\Omega}^*.
\tag{6.31}
\]
Thus, on \( \Gamma_{11} \), by using mean value theorem and (6.30), we have
\[
|\phi_1|_{\Gamma_{11}} = |(v_1 + v_2) - v_1^*|_{\Gamma_{11}} = |(v_1 + v_2) - 1|_{\Gamma_{11}} \\
= |(v_1 + v_2) - (v_1 + v_2)(x', x_d + \varepsilon)|_{\Gamma_{11}}
\leq |\nabla (v_1 + v_2)(\xi)|\varepsilon \leq C\varepsilon,
\]
for some \( \xi \in \tilde{\Omega} \); similarly, using (6.31),
\[
|\phi_1|_{\Gamma_{12}} = |(v_1 + v_2) - v_1^*|_{\Gamma_{12}} = |1 - v_1^*|_{\Gamma_{12}} \\
= |v_1^*(x', x_d - \varepsilon) - v_1^*|_{\Gamma_{12}} = |\nabla v_1^*(\xi)|\varepsilon \leq C\varepsilon,
\]
for some another \( \xi \in \tilde{\Omega}^* \). By the same way,
\[
|\phi_1|_{\Gamma_{21} \cup \Gamma_{22}} \leq C\varepsilon.
\]
We now apply the maximum principle to \( \phi_1 \) on \( V \), instead of (6.23), we have
\[
|\phi_1| \leq C\varepsilon, \quad \text{on } V.
\tag{6.32}
\]
Therefore, by the same process as in the rest of the proof of Proposition 6.1, for \( i = 1 \) is proved. The proofs of claim (6.27) for \( i = 2 \) and (6.28) are similar.

In view of the decomposition (6.1), the forth line of (1.12), we have
\[
C_1 \int_{\partial D_j} \frac{\partial v_1}{\partial \nu} |_+ + C_2 \int_{\partial D_j} \frac{\partial v_2}{\partial \nu} |_+ + \int_{\partial D_j} \frac{\partial u_0}{\partial \nu} |_+ = 0, \quad j = 1, 2.
\tag{6.33}
\]
That is,
\[
\begin{cases}
a_{11} C_1 + a_{12} C_2 = \tilde{b}_1, \\
a_{21} C_1 + a_{22} C_2 = \tilde{b}_2.
\end{cases}
\]
So that
\[
(a_{11} + a_{21}) C_1 + (a_{12} + a_{22}) C_2 + \tilde{b}_1 + \tilde{b}_2 = 0.
\]
Since \( a_{12} = a_{21} \), it follows that
\[
(a_{11} + a_{21})(C_1 + C_2) + (a_{22} - a_{11}) C_2 + \tilde{b}_1 + \tilde{b}_2 = 0.
\]
Similarly,
\[
(a_{12} + a_{22})(C_1 + C_2) - (a_{22} - a_{11}) C_1 + \tilde{b}_1 + \tilde{b}_2 = 0.
\]
Adding these two equations together and dividing it by two yields

\[(a_{11} + a_{21} + a_{12} + a_{22})\frac{(C_1 + C_2)}{2} + (a_{22} - a_{11})\frac{(C_2 - C_1)}{2} + \bar{b}_1 + \bar{b}_2 = 0.\]

That is,

\[
\begin{align*}
\left( \int_{\partial D_1} \frac{\partial (v_1 + v_2)}{\partial \nu} \bigg|_{+} + \int_{\partial D_2} \frac{\partial (v_1 + v_2)}{\partial \nu} \bigg|_{+} \right) \frac{(C_1 + C_2)}{2} \\
+ \left( \int_{\partial D_1} \frac{\partial v_0}{\partial \nu} \bigg|_{+} + \int_{\partial D_2} \frac{\partial v_0}{\partial \nu} \bigg|_{+} \right) + \left( \int_{\Omega} |\nabla v_2|^2 - \int_{\Omega} |\nabla v_1|^2 \right) \frac{(C_2 - C_1)}{2} &= 0.
\end{align*}
\]

(6.34)

Recalling \(u = 1 - \bar{u} \in \Omega_R\) and using estimates (6.10) and (6.11) leads to

\[
\left| \int_{\Omega} |\nabla v_2|^2 - \int_{\Omega} |\nabla v_1|^2 \right| \\
\leq \left| \int_{\Omega_R} |\nabla v_2|^2 - \int_{\Omega_R} |\nabla v_1|^2 \right| + \left| \int_{\Omega_R \setminus \Omega_R} |\nabla v_2|^2 - \int_{\Omega_R \setminus \Omega_R} |\nabla v_1|^2 \right| \\
\leq \left| \int_{\Omega_R} |\nabla u|^2 - \int_{\Omega_R} |\nabla \bar{u}|^2 \right| + \left| \int_{\Omega_R} |\nabla (v_2 - u)|^2 - \int_{\Omega_R} |\nabla (v_1 - \bar{u})|^2 \right| + C \\
\leq C.
\]

By using the integration by parts and the definition of \(v^*_1\), we have

\[
\int_{\partial D_1} \frac{\partial v^*_1}{\partial \nu} \bigg|_{+} + \int_{\partial D_2} \frac{\partial v^*_1}{\partial \nu} \bigg|_{+} = \int_{\Omega_R} |\nabla v^*_1|^2 > 0.
\]

Therefore, by the claim above, (6.34) can be written as

\[
\begin{align*}
\left( \int_{\partial D_1} \frac{\partial v^*_1}{\partial \nu} \bigg|_{+} + \int_{\partial D_2} \frac{\partial v^*_1}{\partial \nu} \bigg|_{+} + O(\rho_d(\varepsilon)) \right) \frac{(C_1 + C_2)}{2} \\
+ \left( \int_{\partial D_1} \frac{\partial v^*_0}{\partial \nu} \bigg|_{+} + \int_{\partial D_2} \frac{\partial v^*_0}{\partial \nu} \bigg|_{+} + O(\rho_d(\varepsilon)) \right) + O(\rho_d(\varepsilon)) &= 0.
\end{align*}
\]

Comparing it with (6.20), the proof of (6.21) is finished. \(\Box\)

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**References**

1. S. Agmon; A. Douglis; L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. Commun. Pure Appl. Math. 12 (1959), 623-727.
2. H. Ammari; E. Bonnetier; F. Triki; M. Vogelius, Elliptic estimates in composite media with smooth inclusions: an integral equation approach. Ann. Sci. éc. Norm. Supér. (4) 48 (2015), no. 2, 453-495.
3. H. Ammari; G. Ciraolo; H. Kang; H. Lee; K. Yun, Spectral analysis of the Neumann-Poincaré operator and characterization of the stress concentration in anti-plane elasticity. Arch. Ration. Mech. Anal. 208 (2013), 275-304.
4. H. Ammari; H. Dassios; H. Kang; M. Lim, Estimates for the electric field in the presence of adjacent perfectly conducting spheres. Quart. Appl. Math. 65 (2007), 339-355.

5. H. Ammari; P. Garapon; H. Kang; H. Lee, A method of biological tissues elasticity reconstruction using magnetic resonance elastography measurements. Quart. Appl. Math. 66 (2008), no. 1, 139-175.

6. H. Ammari; H. Kang; K. Kim; H. Lee, Strong convergence of the solutions of the linear elasticity and uniformity of asymptotic expansions in the presence of small inclusions. J. Differential Equations 254 (2013), 4446-4464.

7. H. Ammari; H. Kang; M. Lim, Gradient estimates to the conductivity problem. Math. Ann. 332 (2005), 277-286.

8. H. Ammari; H. Kang; H. Lee; J. Lee; M. Lim, Optimal estimates for the electrical field in two dimensions. J. Math. Pures Appl. 88 (2007), 307-324.

9. H. Ammari; H. Kang; H. Lee; M. Lim and H. Zribi, Decomposition theorems and fine estimates for electrical fields in the presence of closely located circular inclusions, J. Differential Equations 247 (2009), 2897-2912.

10. I. Babuška; B. Andersson; P. Smith; K. Levin, Damage analysis of fiber composites. I. Statistical analysis on fiber scale. Comput. Methods Appl. Mech. Engrg. 172 (1999), 27-77.

11. E.S. Bao; Y.Y. Li; B. Yin, Gradient estimates for the perfect conductivity problem. Arch. Ration. Mech. Anal. 193 (2009), 195-226.

12. E.S. Bao; Y.Y. Li; B. Yin, Gradient estimates for the perfect and insulated conductivity problems with multiple inclusions. Comm. Partial Differential Equations 35 (2010), 1982-2006.

13. J.G. Bao; H.J. Ju; H.G. Li, Optimal boundary gradient estimates for Lam systems with partially infinite coefficients. Adv. Math. 314 (2017), 583-629.

14. J.G. Bao; H.G. Li; Y.Y. Li, Gradient estimates for solutions of the Lamé system with partially infinite coefficients. Arch. Ration. Mech. Anal. 215 (2015), no. 1, 307-351.

15. J.G. Bao; H.G. Li; Y.Y. Li, Gradient estimates for solutions of the Lamé system with partially infinite coefficients in higher dimensions greater than two. Adv. Math. 305 (2017), 298-338.

16. E. Bonnetier and F. Triki, Pointwise bounds on the gradient and the spectrum of the Neumann-Poincaré operator: the case of 2 discs, Multi-scale and high-contrast PDE: from modeling, to mathematical analysis, to inversion, 81-91, Contemp. Math., 577, Amer. Math. Soc., Providence, RI, 2012.

17. E. Bonnetier; F. Triki, On the spectrum of the Poincaré variational problem for two close-to-touching inclusions in 2D. Arch. Ration. Mech. Anal. 209 (2013), no. 2, 541-567.

18. E. Bonnetier; M. Vogelius, An elliptic regularity result for a composite medium with “touching” fibers of circular cross-section, SIAM J. Math. Anal. 31 (2000) 651-677.

19. M. Briane; Y. Capdeboscq; L. Nguyen, Interior regularity estimates in high conductivity homogenization and application. Arch. Ration. Mech. Anal. 207 (2013), no. 1, 75-97.

20. B. Budiansky; G.F. Carrier, High shear stresses in stiff fiber composites. J. App. Mech. 51 (1984), 733-735.

21. H.J. Dong, Gradient estimates for parabolic and elliptic systems from linear laminates. Arch. Ration. Mech. Anal. 205 (2012), no. 1, 119-149.

22. H. Kang; H. Lee; K. Yun, Optimal estimates and asymptotics for the stress concentration between closely located stiff inclusions. Math. Ann. 363 (2015), no. 3-4, 1281-1306.
29. H. Kang; M. Lim; K. Yun, Asymptotics and computation of the solution to the conductivity equation in the presence of adjacent inclusions with extreme conductivities. J. Math. Pures Appl. (9) 99 (2013), 234-249.
30. H. Kang; M. Lim; K. Yun, Characterization of the electric field concentration between two adjacent spherical perfect conductors. SIAM J. Appl. Math. 74 (2014), no. 1, 125-146.
31. H. Kang, S. Yu: Quantitative characterization of stress concentration in the presence of closely spaced hard inclusions in two-dimensional linear elasticity. to appear in Arch. Ration. Mech. Anal., arXiv:1707.02207
32. J.B. Keller, Conductivity of a medium containing a dense array of perfectly conducting spheres or cylinders or nonconducting cylinders. J. Appl. Phys. 34 (1963), 991-993.
33. J.B. Keller, Stresses in narrow regions, Trans. ASME J. Appl. Mech. 60 (1993), 1054-1056.
34. H.G. Li; Y.Y. Li, Gradient estimates for parabolic systems from composite material. Sci. China Math. 60 (2017), no. 11, 2011-2052.
35. H.G. Li; Y.Y. Li; E.S. Bao; B. Yin, Derivative estimates of solutions of elliptic systems in narrow regions. Quart. Appl. Math. 72 (2014), no. 3, 589-596.
36. H.G. Li; L.J. Xu, Optimal estimates for the perfect conductivity problem with inclusions close to the boundary. SIAM J. Math. Anal. 49 (2017), no. 4, 3125-3142.
37. Y.Y. Li; L. Nirenberg, Estimates for elliptic systems from composite material. Comm. Pure Appl. Math. 56, (2003) 892-925.
38. Y.Y. Li; M. Vogelius, Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients. Arch. Rational Mech. Anal. 135 (2000), 91-151.
39. M. Lim; K. Yun, Blow-up of electric fields between closely spaced spherical perfect conductors. Comm. Partial Differential Equations 34 (2009), 1287-1315.
40. M. Lim; K. Yun, Strong influence of a small fiber on shear stress in fiber-reinforced composites. J. Diff. Equa. 250 (2011), 2402-2439.
41. X. Markenscoff, Stress amplification in vanishingly small geometries. Computational Mechanics 19 (1996), 77-83.
42. V.G. Maz'ya, A.B. Movchan and M.J. Nieves, Uniform asymptotic formular for Greens tensors in elastic singularly perturbed domains, Asym. Anal. 52 (2007), 173-206.
43. V. Maz'ya, S. Nazarov, and B. Plamenevskij, Asymptotic Theory of Elliptic Boundary Value Problems in Singularly Perturbed Domains, Vol. II, Oper. Theory Adv. Appl. 112, Birkhuser Verlag, Basel, 2000.
44. O.A. Oleinik; A.S. Shamaev; G.A. Yosifian, Mathematical problems in elasticity and homogenization. Studies in Mathematics and Its Applications, 26. North-Holland, Amsterdam, (1992).
45. K. Yun, Optimal bound on high stresses occurring between stiff fibers with arbitrary shaped cross-sections. J. Math. Anal. Appl. 350 (2009), 306-312.
46. K. Yun, Estimates for electric fields blown up between closely adjacent conductors with arbitrary shape. SIAM J. Appl. Math. 67 (2007), 714-730.
47. K. Yun, Two types of electric field enhancements by infinitely many circular conductors arranged closely in two parallel lines. Quart. Appl. Math. 75 (2017), no. 4, 649-676.