Abstract. In recent work, M. Schneider and the first author studied a curious class of integer partitions called “sequentially congruent” partitions: the $m$th part is congruent to the $(m+1)$th part modulo $m$, with the smallest part congruent to zero modulo the number of parts. Let $p_S(n)$ be the number of sequentially congruent partitions of $n$, and let $p□(n)$ be the number of partitions of $n$ wherein all parts are squares. In this note we prove bijectively, for all $n \geq 1$, that $p_S(n) = p□(n)$. Our proof naturally extends to show other exotic classes of partitions of $n$ are in bijection with certain partitions of $n$ into $k$th powers.

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1. Euler, Cauchy, and sequentially congruent partitions

Let $P$ denote the set of integer partitions. Euler connected partitions to analysis with his generating function for the partition function $p(n)$, which gives the number of partitions of $n \geq 0$:

$$\prod_{k=1}^{\infty} \frac{1}{1-q^k} = \prod_{k=1}^{\infty} \left( \sum_{j=0}^{\infty} q^{jk} \right) = \sum_{n=0}^{\infty} p(n) q^n,$$

where $q \in \mathbb{C}$, $|q| < 1$, and we define $p(0) := 1$. Another important classical result is the Cauchy product formula giving the product of two convergent power series:

$$\left( \sum_{i=0}^{\infty} a_i q^i \right) \left( \sum_{j=0}^{\infty} b_j q^j \right) = \sum_{n=0}^{\infty} q^n \sum_{k=0}^{n} a_k b_{n-k}.$$

This result extends to the product $\prod_k \left( \sum_{j=0}^{\infty} a_{k,j} q^j \right)$ of any number of power series (even infinitely many), with more complicated multiple sums comprising the resulting coefficients.

Now we consider what happens if we interpret the middle product of (1.1) in terms of (1.2), that is, if we expand the product of geometric series via the Cauchy product formula, instead of collecting coefficients to count partitions of $n$ as in Euler’s treatment. Since each geometric series $\sum_j q^{jk} = \sum_i a_{i,k} q^i$ has coefficient $a_{i,k} = 1$ if $k$ divides $i$ and $a_{i,k} = 0$ otherwise, this calculation yields

$$\prod_{k=1}^{\infty} \left( \sum_{j=0}^{\infty} q^{jk} \right) = \sum_{n=0}^{\infty} q^n \sum_{k_2=1}^{n} \sum_{k_3=1}^{k_2} \sum_{k_4=1}^{k_3} \ldots \sum_{k_n=1}^{k_{n-1}} \{ \text{either 0 or 1} \},$$

where the coefficient of $q^n$ on the right-hand side is an $(n-1)$-tuple sum whose summands enumerate some class of partitions induced by particular combinations of the indices of the multiple sum. But what are these seemingly-complicated partitions?

1See [1] for further reading about integer partitions.

2See [7], Cor. 2.9, for explicit treatment of a more general case of (1.3).
In fact, the right-hand side of (1.3) enumerates certain sequentially congruent partitions, a subset of $\mathcal{P}$ studied by M. Schneider and the first author in [6].

**Definition 1.1.** A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$, is sequentially congruent if:

1. $\lambda_i \equiv \lambda_{i+1} \pmod{i}$ for $1 \leq i \leq r-1$; and
2. $\lambda_r \equiv 0 \pmod{r}$.

We let $\mathcal{S} \subset \mathcal{P}$ denote the set of sequentially congruent partitions.

**Example 1.1.** The partition $(20, 17, 15, 9, 5)$ is sequentially congruent, because $20 \equiv 17 \pmod{1}$ trivially, $17 \equiv 15 \pmod{2}$, $15 \equiv 9 \pmod{3}$, $9 \equiv 5 \pmod{4}$, and finally $5 \equiv 0 \pmod{5}$.

With such strict congruence conditions imposed on the parts, sequentially congruent partitions seem to comprise a somewhat artificial subset of $\mathcal{P}$. However, in [6] they are found to fit nicely into partition theory, as the following theorem exemplifies.

**Theorem** (Schneider–Schneider). The set $\mathcal{S}$ enjoys a natural bijection with the set $\mathcal{P}$. Moreover, the number of sequentially congruent partitions with largest part $n$ is equal to $p(n)$.

This bijection can be inferred by comparing coefficients on the right-hand sides of (1.1) and (1.3). In [6], two different bijective maps between $\mathcal{S}$ and $\mathcal{P}$ are given, producing two combinatorial proofs of the theorem. Surprisingly, composing either map with the inverse of the other map produces partition concatenation, i.e., the interchange of rows and columns in the Young diagram of $\lambda$. Of course, conjugation preserves partition size $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_r$, thus the aforementioned composite map takes the set of partitions of size $n$ to itself.

Presented with these size-$n$ correspondences, an obvious question to ask now is: what is the number $p_S(n)$ of sequentially congruent partitions of size $n$? Interestingly, a different bijection involving $\mathcal{S}$ gives a very satisfying answer.

**Theorem 1.2.** The set $\mathcal{S}$ enjoys a natural bijection with the set $\mathcal{P}_\square$ of partitions whose parts are all perfect squares. Moreover, let $p_\square(n)$ be the number of partitions of $n$ into squares. Then for all $n \geq 1$, we have

$$p_S(n) = p_\square(n).$$

Historically, sums of squares are of deep interest in arithmetic. The clay tablet known as Plimpton 322 contains lists of Pythagorean triples dating back to ancient Babylon [5], and there are many examples in the partitions literature of connections to sums of squares (see e.g. [2,4,8]). It is interesting to see the seemingly-artificial set $\mathcal{S}$ fitting nicely, once again, into the theory of numbers.

Combining Theorem 1.2 with the aforementioned theorem from [6] allows one to define a set of partitions into squares which is equinumerous with the number of partitions of $n$. Let

$$\lambda = ((1^2)^{e_1} (2^2)^{e_2} (3^2)^{e_3} \cdots (r^2)^{e_r}) \in \mathcal{P}_\square$$

be a partition into squares given in frequency notation (as defined in (2.1) below), and let us define the following “weighted frequency” statistic $\mathcal{L}$ on partitions into squares:

$$L(\lambda) = \sum_{i \geq 1} i \cdot e_i.$$  

**Corollary 1.3.** The set $\mathcal{P}_\square$ enjoys a natural bijection with the set $\mathcal{P}$. Moreover, the number of partitions $\lambda$ whose parts are all perfect squares with $L(\lambda) = n$ is equal to $p(n)$.

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3We note then that the case $k = 2, M = 1$ of Wright [8], Thm. 2, gives the asymptotic estimate $p_S(n) \sim \frac{Be^{C\sqrt{n}/3}}{n^{1/6}}$ with constants $B := \left(\frac{\zeta(3/2)^4}{442\zeta(2)^2}\right)^{1/6}, C := \frac{3}{2} \left(\frac{\zeta(3/2)^2}{2}\right)^{1/3}$, where $\zeta$ is the Riemann zeta function.
2. Proofs, example, and extension of our result

We present a bijective proof of Theorem 1.2. Rather than proving the theorem directly from Young diagrams or other combinatorial devices (which seems possible), we proceed by considering the set of conjugates of the partitions in $S$, that is, partitions formed by conjugation of the Young diagrams of sequentially congruent partitions. These conjugate partitions are also studied in [6], where they are called frequency congruent partitions.

Let $m_i = m_i(\lambda) \geq 0$ denote the frequency (or “multiplicity”) of $i \in \mathbb{N}$, the number of repetitions of $i$ as a part of partition $\lambda$.

**Definition 2.1.** A partition $\lambda$ is frequency congruent if for all $i \geq 1$,

$$m_i(\lambda) \equiv 0 \pmod{i}.$$

We let $F \subset P$ denote the set of frequency congruent partitions.

We also recall the alternative “frequency notation” representation of a partition:

$$\lambda = (1^{m_1} 2^{m_2} 3^{m_3} \ldots i^{m_i} \ldots),$$

where only finitely many of the frequencies $m_i$ are nonzero.

**Proof of Theorem 1.2 and Corollary 1.3.** We begin by observing that the sets $S$ and $F$ are in bijection through conjugation, which preserves size and interchanges the largest part and number of parts. Therefore, if $p_F(n)$ denotes the number of frequency congruent partitions of $n$, we have

$$p_S(n) = p_F(n),$$

and if $\lambda \in S$ has largest part equal to $k$, then the conjugate $\lambda^* \in F$ has $k$ total parts.

Let $\lambda$ be a partition of size $n$ which is sequentially congruent. Then the conjugate $\lambda^*$ of $\lambda$ is frequency congruent, thus each part divides its own frequency:

$$\lambda^* = (1^{e_1} 2^{e_2} \ldots i^{e_i} \ldots)$$

for some $e_i \geq 0$. Now, let us define a map $\pi : F \rightarrow P$ taking $\lambda^* \in F$ to the partition

$$\pi(\lambda^*) = ((1^{e_1}) (2^{e_2}) \ldots (i^{e_i}) \ldots) \in P,$$

noting that $|\lambda^*| = |\pi(\lambda^*)| = n$ since for each $i \geq 1$, the sum of corresponding parts is preserved:

$$i + i + \cdots + i = i \cdot (i + i + \cdots + i) = i^2 + i^2 + \cdots + i^2.$$

The map is clearly reversible, so $\pi$ is a bijection. Therefore, we have

$$p_F(n) = p_P(n).$$

Comparing (2.2) and (2.6) completes the proof of Theorem 1.2.

The proof of Corollary 1.3 follows by noting that $L(\pi(\lambda^*))$ is the number of parts (or “length”) of $\lambda^*$. The statistic $L(\pi(\lambda^*))$ can be interpreted as a weighted sum of the number of squares of each size that appear in the Young diagram of the partition $\lambda^*$, as illustrated in Example 2.2 below.

**Remark.** One could prove Theorem 1.2 analytically using the generating function proof of Corollary 4.3 in [6], noting the left-hand product can be interpreted as generating either frequency congruent partitions or partitions into squares.

Here we give a concrete example of the bijection proved above.
Example 2.2. Begin with the frequency congruent partition
\[ \phi = (1^7 2^6 4^8 5^5) \in \mathcal{F}. \]
By conjugation, \( \phi \) is mapped to the sequentially congruent partition
\[ (2.8) \quad (26, 19, 13, 13, 5) \in \mathcal{S}. \]
Moreover, using the “squaring” map \( \pi \) defined in (2.4), \( \phi \) is mapped to the partition
\[ (2.9) \quad (1^7 4^3 16^2 25^1) \in \mathcal{P}. \]
When we compose these two maps (and ignore the intermediary frequency congruent partition \( \phi \) with which we began), we see the bijection
\[ (2.10) \quad (26, 19, 13, 13, 5) \leftrightarrow (1^7 4^3 16^2 25^1). \]
This bijection is visually evident in the Young diagram of the sequentially congruent partition, which we shade to highlight that it breaks down into a concatenation of perfect squares:

\[ (2.11) \]

Remark 2.1. Conversely, any collection of perfect squares may be concatenated in similar fashion (top edges aligned, weakly decreasing areas from left to right) to produce the Young diagram of a sequentially congruent partition.

We now define a certain refinement of sequentially congruent partitions amenable to a combinatorial proof like the proof of Theorem 1.2 above, to show how similar methods yield other bijections of a similar flavor.

Definition 2.3. Let \( \mathcal{S}(j, k) \subset \mathcal{P} \) be the subset of partitions \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) satisfying:
1. \( \lambda_i - \lambda_{i+1} = j \cdot i^k \) for \( 1 \leq i \leq r - 1 \); and
2. \( \lambda_r = j \cdot r^k \).

Theorem 2.4. The partitions \( \lambda \in \mathcal{S}(j, k) \) of size \( n \) are in bijection with partitions of \( n \) into \( (k+1) \)th powers where each part occurs exactly \( j \) times.

Proof of Theorem 2.4. A very similar map to \( \pi \), which we will denote by \( \pi_{j,k} \), can be employed to give this result. Notice that the conjugate \( \lambda^* \) of a partition \( \lambda \in \mathcal{S}(j, k) \) is a partition where \( m_i(\lambda^*) = j \cdot i^k \epsilon_i \), with \( \epsilon_i = \epsilon_i(\lambda^*) = 1 \) if \( i \) is a part of \( \lambda^* \) and \( \epsilon_i = 0 \) otherwise. Then we define
\[ (2.12) \quad \pi_{j,k}(\lambda^*) = \pi_{j,k}((1^j 1^j \epsilon_1 2^j 2^j \epsilon_2 3^j 3^j \epsilon_3 \ldots i^j i^j \epsilon_i \ldots)) = ((1^{k+1})^j \epsilon_1 (2^{k+1})^j \epsilon_2 (3^{k+1})^j \epsilon_3 \ldots (i^{k+1})^j \epsilon_i \ldots). \]
Noting that the map \( \pi_{j,k} \) is both size-preserving and reversible, much like the map \( \pi \), completes the proof of the bijection.

It seems worthy of further investigation, to apply partition-theoretic maps like those presented here to study sums of squares and higher powers.

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References

[1] G. E. Andrews. *The Theory of Partitions*, Encyclopedia of Mathematics and its Applications, vol. 2, Addison–Wesley, Reading, MA, 1976. Reissued, Cambridge University Press, 1998.

[2] E. T. Bell. “Square-partition congruences.” *Bulletin of the American Mathematical Society* **29.8** (1923): 349-355.

[3] M. D. Hirschhorn and J. A. Sellers. “Some relations for partitions into four squares.” *Proceedings of the International Workshop on Special Functions, Asymptotics, Harmonic Analysis, and Mathematical Physics*, World Scientific, 2000.

[4] M. D. Hirschhorn and J. A. Sellers. “On a problem of Lehmer on partitions into squares.” *The Ramanujan Journal* **8.3** (2004): 279-287.

[5] E. Robson. “Words and pictures: New light on Plimpton 322.” *The American Mathematical Monthly* **109.2** (2002): 105-120.

[6] M. Schneider and R. Schneider. “Sequentially congruent partitions and related bijections.” *Annals of Combinatorics* **23.3-4** (2019): 1027-1037.

[7] R. Schneider. “Partition zeta functions.” *Research in Number Theory* **2.1** (2016): 9.

[8] E. M. Wright. “Asymptotic partition formulae. III. Partitions into k-th powers.” *Acta Mathematica* **63** (1934): 143-191.

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