Geometry of Isophote Curves

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Abstract. We consider the intensity surface of a 2D image, we study the evolution of the symmetry sets (and medial axes) of 1-parameter families of iso-intensity curves. This extends the investigation done on 1-parameter families of smooth plane curves (Bruce and Giblin, Giblin and Kimia, etc.) to the general case when the family of curves includes a singular member, as will happen if the curves are obtained by taking plane sections of a smooth surface, at the moment when the plane becomes tangent to the surface.

Keywords and Phrases: Isophote curve, symmetry set, medial axis, skeleton, vertex, inflexion, shape analysis.

1 Introduction

Image data is often thought of as a collection of pixel values \( I : \mathbb{Z}^2 \mapsto \mathbb{Z}_+ \). The physical information is better captured by embedding the pixel values in the real plane, as the pixilation and quantization are artifacts of the camera, hence \( I : \mathbb{R}^2 \mapsto \mathbb{R}_+ \). The geometrical information of an image is even better captured looking at the level sets \( I(x) = I_0 \), for all \( I_0 \in \mathbb{R}_+ \), that is, looking at the isophote curves of the image.

Shape analysis using point-based representations or medial representations (such as skeletons) has been widely applied on an object level demanding object segmentation from the image data. We propose to combine the object representation using a skeleton or symmetry set representation and the appearance modelling by representing image information as a collection of medial representations for the level-sets of an image. As the level \( I_0 \) changes, the curves change like sections of a smooth surface by parallel planes.

The qualitative changes in the medial representation of families of isophotes fall into two types: (1) those for which the isophotes remain nonsingular (see for example [3, 8]) and (2) those for which one isophote at least is singular. The symmetry set (SS) of a plane curve is the closure of the set of centres of circles which are tangent to the curve at two or more different places. The medial axis (MA) is the subset of the SS consisting of the closure of the locus of centres of circles which are maximal, (maximal means that the minimum distance from the centre to the curve equals the radius). Our aim is to extend the investigation to the case (2) when the family includes singular curves, as is the case when one of the plane sections is tangent to the surface so that this section is a singular
curve. The final goal is to represent image smooth surfaces by the collection of all medial repren
tations of isophotes, forming a singular surface in scale space.

In this article, which is theoretical in nature, we work with the full SS, and consider the transitions which occur in the SS of a family of plane sections of a generic smooth surface in 3-space, as the plane moves through a position where it is tangent to the surface. We investigate the local geometry of these families of curves and track the evolution of some crucial features of the SS and MA. In particular, we will trace and classify the patterns of some special points, on the sections of a surface as the section passes through a tangential point, such as vertices (maxima and minima of curvature), inflexions, triples of points where a circle is tritangent and the pattern of the centre of such a circle, paires of points where a circle is bitangent with a higher order contact at one of them, etc. The vertices are crucial to the understanding of the SS since it has branches which end at the centres of curvature at vertices. From the way in which vertices behave we can deduce a good deal about the evolution of the SS and its local number of branches. The inflexions correspond to where the evolute of the curve, recedes to infinity. We also classify all possible scenarii of how vertices and inflexions are distributed along the level curves.

Last, we produce examples of SS and MA illustrating the cases.

We are concerned with the local behaviour of symmetry sets (SS) and medial axis (MA) of plane sections of generic smooth surfaces so we may assume that our surface $M$ is given by an equation $z = f(x, y)$ for a smooth function $f$, which will often be assumed to be a polynomial of sufficiently high degree. We shall take $M$ in Monge form, that is $f, f_x$ and $f_y$ all vanish at $(0,0)$.

2 Intrinsic Geometry of Generic Isophote Curves

This section describes the geometry of isophote curves evolving on a fixed smooth surface $M$, under a 1-parameter family of parallel plane sections. Namely, we shall examine closely the different configurations of vertices and inflexions on the sections on our surface. We will in particular concentrate on the evolution through a plane section which is tangent to $M$ at a point $p$, so that this section is singular. For a generic surface, three situations arise, according to the contact between the tangent plane and $M$ at $p$, as measured by the singularity type of the height function in the normal direction at $p$. See for example [12] for the geometry of these situations, and [4, 11] for an extensive discussion of the singularity theory.

- The contact at $p$ is ordinary (‘$A_1$ contact’), in which case the point is (i) elliptic or (ii) hyperbolic. The intersection of $M$ with its tangent plane at $p$ is locally an isolated point or a pair of transverse arcs.
- The contact is of type $A_2$, which means that $p$ is parabolic. The intersection of $M$ with its tangent plane at $p$ is locally a cusped curve.

\footnote{The genericity conditions will vary from case to case. See [6].}