The Effects of Superhigh Magnetic Fields on Equations of States of Neutron Stars

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By introducing Dirac δ-function in superhigh magnetic field, we deduce a general formula for pressure of degenerate and relativistic electrons, $P_e$, which is suitable for superhigh magnetic fields, discuss the quantization of Landau levels of electrons, and consider the quantum electrodynamic(QED) effects on the equations of states (EOSs) for different matter systems. The main conclusions are as follows: the stronger the magnetic field strength, the higher the electron pressure becomes; compared with a common radio pulsar, a magnetar could be a more compact oblate spheroid-like deformed neutron star due to the anisotropic total pressure; and an increase in the maximum mass of a magnetar is expected because of the positive contribution of the magnetic field energy to the EOS of the star. Since this is an original work in which some uncertainties could exist, to further modify and perfect our theory model should be considered in our future studies.

1 Introduction

Pulsars are among the most mysterious objects in the universe that provide natural laboratory for investigating the nature of matter under extreme conditions, and are universally recognized as normal neutron stars (NSs), but sometimes have been argued to be quark stars (Du et al. 2009, Lai et al. 2013, Xu et al. 2013). The equation of state (EoS) of matter under exotic conditions is an important tool for understanding the nuclear force and for astrophysical applications. The Fermi energy of relativistic electrons $E_F(e)$ is one of the most important and indispensable physical parameters in EoS, and affects direct weak-interaction processes including modified URCA reactions, electron capture (e.g., Gao et al. 2011a, 2011b, 2011c, 2011d, 2012a, 2012b; Liu 2012, 2013, 2014, 2015; Du et al. 2014). These influences will change intrinsic EoS, interior structure and heat evolution, and even affect the whole properties of the star.

As we know, for degenerate and relativistic electrons in $\beta-$equilibrium, the distribution function $f(E_e)$ obeys Fermi-Dirac statistics: $f(E_e) = 1/(\exp[(E_e - \mu_e)/kT] + 1)$, $k$ represents Boltzmann’s constant, and $\mu_e$ is the electron chemical potential. If $T \to 0$, $\mu_e$ is also called the electron Fermi energy, $E_F(e)$, which presents the energy of highest occupied states for electrons. The electron Fermi energy $E_F(e)$ has the simple form

$$E_F(e) = (p_F^2(e)c^2 + m_e^2c^4)^{1/2},$$

with $p_F(e)$ being the electron Fermi momentum.

In the context of general relativity principle, the matter density is defined as: $\rho = \varepsilon/c^2$, $\varepsilon$ is the total energy density, including the rest-mass energies of particles. Using the basic thermodynamics, we obtain the relation of the total matter pressure $P$ and matter density $\rho$ in a common NS,

$$P(n_B) = n_B^2 \frac{d(\varepsilon/n_B)}{dn_B},$$

$$\rho(n_B) = \varepsilon(n_B)/c^2, \quad \Rightarrow P = P(\rho).$$

From the above equation, it is obvious that $P$ solely depends on $\rho$. Theoretically, we can obtain the value of $E_F(e)$ by solving EOS in a specific matter model. The pressure of degenerate and relativistic electrons, $P_e$, is another important and indispensable physical parameter in EoSs of a NS. $P_e$ is one of important dynamical pressures against a NS’s gravitational collapse, and affects the structures and properties of the star, substantially.

Thompson and Duncan (1996) predicted that superhigh magnetic fields (MFs) could exist in the interiors of magnetars with a typical surface dipolar MF, $B \sim 10^{14}$ to $10^{15}$ G (Thompson & Duncan 1996). Superhigh MFs have effects on EoSs of a NS, as well as on its spin-down evolution (e.g., Gao et al. 2014, 2015). Recently, Franzon et al. (2015) studied the effects of strong MFs on hybrid stars by using a full

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general-relativity approach, and pointed that the MF could cause the stellar central density to be reduced, inducing major changes in the populated degrees of freedom and, potentially, converting a hybrid star into a hadronic star. In accordance with the popular point of view, the stronger the MF strength, the lower the electron pressure becomes. With respect to this view, we cannot directly verify it by experiment in actual existence, owing to lack of such high MFs on the earth. After a careful check, we found that popular methods of calculating $E_F(x)$ the electron Fermi energy are contradictory to the quantization of electron Landau levels. In an extremely strong MF, the Landau column becomes a very long narrow cylinder along MF. If we consider Dirac $\delta$-function in superhigh MFs, all the results should be reconsidered.

In Sec. 2, we deduce an equation of $P_e$ in superhigh MFs; in Sec. 3, we consider QED effects on EOSs of NS matter, and discuss an anisotropy of the total pressure; In Sec. 4 we discuss our future work on improving our model, and present conclusions in Sec. 5.

## 2 Deduction of the pressure of electrons in superhigh MFs

The relativistic Dirac-Equation for the electrons in a uniform external magnetic field along the $z$–axis gives the electron energy level

$$E_e = [m_e^2c^4 + (1 + \nu \frac{2B}{B_{cr}}) + p_z^2c^2]^{1/2},$$

(3)

where $\nu = n + \frac{1}{2} + \sigma$ is the quantum number, $n$ the Landau level number, $\sigma = \pm \frac{1}{2}$ the spin quantum number (Canuto & Ventura 1977), and $p_z$ is the $z$-component of electron momentum, and may be treated as a continuous function. Combining $B_{cr} = m_e^2c^3/\mu_e$ with $\mu_e = e\hbar/2mc$ gives

$$E_e^2 = m_e^2c^4 + p_z^2c^2 + 2mc^2\mu_eB = m_e^2c^4 + p_z^2c^2 + p_{\perp}^2c^2,$$

where $\mu_e$ is the magnetic moment of an electron, and $p_{\perp} = m_e(2\nu B^*)^{1/2}$. The maximum electron Landau level number $n_{max}$ is uniquely determined by the condition $|p_{\perp}(z)c|^2 \geq 0$ (Lai & Shapiro 1991, Gao et al. 2013), where $p_{\perp}$ is the Fermi momentum along the $z$–axis. The expression for $\nu_{max}$ can be expressed as

$$\nu_{max}(\sigma = -\frac{1}{2}) = \text{Int} \frac{1}{2B} \left( \frac{E_F(e)}{m_e c^2} \right)^2 - 1 - \left( \frac{p_{\perp}}{m_e c} \right)^2\frac{1}{2} \left( \frac{1}{2} - \frac{1}{2} \right)$$

$$\nu_{max}(\sigma = \frac{1}{2}) = \text{Int} \frac{1}{2B} \left( \frac{E_F(e)}{m_e c^2} \right)^2 - 1 - \left( \frac{p_{\perp}}{m_e c} \right)^2\frac{1}{2}$$

Thus, Eq.(16) can be rewritten as

$$\nu_{max}(\sigma = -\frac{1}{2}) = \text{Int} \frac{1}{2B} \left( \frac{E_F(e)}{m_e c^2} \right)^2 - 1 - \left( \frac{p_{\perp}}{m_e c} \right)^2\frac{1}{2}$$

$$\nu_{max}(\sigma = \frac{1}{2}) = \text{Int} \frac{1}{2B} \left( \frac{E_F(e)}{m_e c^2} \right)^2 - 1 - \left( \frac{p_{\perp}}{m_e c} \right)^2\frac{1}{2}$$

According to the definition of $E_F(e)$ in Eq.(1), we obtain $E_F(e) \equiv p_F(e)c$ if electrons are super-relativistic ($E_F(e) \gg m_e c^2$). In the presence of a superhigh MF, $B \gg B_{cr}, E_F(e) \gg m_e c^2$, we have

$$\nu_{max}(\sigma = -\frac{1}{2}) = \text{Int} \frac{1}{2B} \left( \frac{E_F(e)}{m_e c^2} \right)^2 - 1 - \left( \frac{p_{\perp}}{m_e c} \right)^2\frac{1}{2}$$

$$\nu_{max}(\sigma = \frac{1}{2}) = \text{Int} \frac{1}{2B} \left( \frac{E_F(e)}{m_e c^2} \right)^2 - 1 - \left( \frac{p_{\perp}}{m_e c} \right)^2\frac{1}{2}.$$

The maximum of $p_{\perp}$ for electrons in a superhigh MF is

$$p_{\perp}^2(m_{\text{max}}) = 2\nu_{max}^2m_\text{e}c^2\mu_eB,$$

where the relation of $2\mu_eB_{cr}/m_e c^2 = 1$ is used. Inserting Eq.(6) into Eq.(7) gives

$$p_{\perp}^2(m_{\text{max}}) = 2 \times \frac{1}{2B} \left( \frac{E_F(e)}{m_e c^2} \right)^2 - 2m_e c^2\mu_eB$$

$$\zeta \equiv B_{cr} \times \left( \frac{E_F(e)}{m_e c^2} \right)^2 \times 2m_e c^2\mu_eB \equiv E_F(e) \zeta,$$

(8)

In superhigh MFs, $E_F(e)$ is determined by

$$E_F(e) \equiv 43.44 (\frac{B}{B_{cr}})^{1/4} \left( \frac{\rho Y_e}{\rho_0 0.0535} \right)^{1/4} \text{MeV},$$

(9)

where $\rho_0 = 2.8 \times 10^{-3} \text{ g cm}^{-3}$ is the standard nuclear density (Gao et al. 2012b). Thus, we obtain

$$p_{\perp}(m_{\text{max}}) = p_F(e) \equiv \frac{E_F(e)}{e c}$$

$$= 43.44 \times (\frac{Y_e}{\rho_0 0.0535} \rho \frac{B}{B_{cr}})^{1/4} \text{MeV} / c (B^* \geq 1),$$

(10)

As pointed out above, when $n = 0$, the electron Landau level is non-degenerate, and $p_{\perp}$ has its maximum $p_{\perp}(m_{\text{max}})$,

$$p_{\perp}(m_{\text{max}}) = p_F(e) \equiv \frac{E_F(e)}{e c}$$

$$= 43.44 \times (\frac{Y_e}{\rho_0 0.0535} \rho \frac{B}{B_{cr}})^{1/4} \text{MeV} / c (B^* \geq 1),$$

(11)

From Eq.(9) and Eq.(10), it’s obvious that $p_{\perp}(m_{\text{max}}) = p_{\perp}(m_{\text{max}}) = p_F(e)$. The reason for this is that in the interior of a magnetar, electrons are degenerate and super-relativistic, and can be approximately treated as an ideal Fermi gas with equivalent pressures in all directions, though the existence of Landau levels. The equation of $P_e$ in a superhigh magnetic field is consequently given by

$$P_e = \frac{2}{3} \hbar c \int_0^{p_F(e)} \frac{p_e^2c^2}{(p_e^2c^2 + m_e^2c^4)^{1/2}} 4\pi p_e^2 dp_e$$

$$= 1.142 \times 10^{29} \phi(x_e) \text{ dynes cm}^{-2}$$

(12)

where $\phi(x_e)$ is the electron Compton wavelength, $x_e = \frac{h}{m_e c} \rho \frac{B}{B_{cr}}$, and $\phi(x_e)$ is the polynomial $\phi(x_e) = 86.77 \times (x_e + 1)^2 \left( \frac{2}{\pi} - 1 \right) + \ln [x_e + (1 + x_e^2)^{1/2}]$.

When $\rho \geq 10^7 \text{ g cm}^{-3}, x_e \gg 1$, and $\phi(x_e) \rightarrow \frac{x_e^4}{12\pi^2}$. Thus, Eq.(11) can be rewritten as

$$P_e \approx 6.266 \times 10^{10} \rho \frac{B}{B_{cr}} \frac{Y_e}{0.0535} \text{ dyne cm}^{-2}.$$ 

(13)

It is obvious that $P_e$ increases sharply with increasing $B$ when the values of $Y_e$ and $\rho$ are given.
3 Magnetic effects on EoSs

3.1 Magnetic effects on the EoS of BPS model

By introducing the lattice energy, Baym, Pethick & Sutherland (1971) (hereafter “BPS model”) improved on Salpeter’s treatment (Salpeter 1961), and described the nuclear composition and EoS for catalyzed matter in complete thermodynamic equilibrium below $\rho_d$. BPS model is one of most successful models describing matter of the outer crust. According to BPS model, the matter energy density is given by

$$\varepsilon = n_N(W_N(A, Z) + \varepsilon_L(Z, n_e) + \varepsilon_e(n_e)), \quad (14)$$

where $n_N$ is the number density of nuclei, $W_N(A, Z)$ is the mass-energy per nucleus (including the rest mass of $Z$ electrons and $A$ nucleons); $\varepsilon_e$ is the free electron energy including the rest mass of electrons in a unit volume; $\varepsilon_L$ is the bcc Coulomb lattice energy per nucleus,

$$\varepsilon_L = -1.4442^{2/3} e^2 c^2 n_e^{4/3}, \quad (15)$$

where the relations of $n_N = n_B/A$ and $n_e = Zn_N$ are used. The matter pressure $P$ of the system is given by

$$P = P_e + P_L = P_e + \frac{1}{3} \varepsilon_L. \quad (16)$$

For a magnetic field $B^* \gg 1$, $P_e$ in Eq.(15) is given by Eq.(12). Based on the above equations, we plot one schematic diagrams of QED effects on the EOS of BPS model, as shown in Fig.1.

![Fig.1](image)

The relation of dynamic pressure $P$ and density $\rho$ in BPS model.

$$P = P_e + P_L = P_e + \frac{1}{3} \varepsilon_L. \quad (16)$$

3.2 The QED effects on the EOS of ideal npe gas

We consider a homogenous ideal npe gas under $\beta$-equilibrium, and adopt ST83 approximation (Shapiro & Teukolsksee 1983) corresponding to the weak-field limit as the main method to treat EoS of this system in the density range of $0.5 \sim 2.5 \rho_0$ where electrons are relativistic, neutrons and protons are non-relativistic. When neutron pressure dominates, $\rho \approx m_n n_n$, then $n_n = 1.7 \times 10^{38} (\rho_0) \text{ cm}^{-3}$ (Shapiro & Teukolsksee 1983); employing $\beta$-equilibrium and charge neutrality gives $n_p = n_e = 9.6 \times 10^{35} (\rho_0)^2 \text{ cm}^{-3}$; $\beta$-equilibrium implies energy conservation and momentum conservation $(p_V(p) = p_V(e))$, we get $E_V(e) = \mu_n = E_V(n) = p_V(n)/2m_n = 60(\rho_0)^{2/3}$ MeV, and $\mu_p = E_V(p) = p_V(p)/2m_p = 1.9(\rho_0)^{1/3}$ MeV; the isotropic matter pressure $P$ is given by

$$P = P_e + P_p + P_n = \frac{m_e c^2}{\lambda^3} \phi(x_e) + \frac{m_p c^2}{\lambda^3} \phi(x_p) + \frac{m_n c^2}{\lambda^3} \phi(x_n), \quad (17)$$

where $x_p = p_V(p) = \sqrt{2m_p \mu_p} = \sqrt{\frac{2m_p}{m_p c^2}}$, the expression of $\phi(x_p)$ is completely similar to that of $\phi(x_n)$.

Based on the above results, we gain the following useful formulae:

$$P_p = 1.169 \times 10^{30} \left(\frac{\rho}{\rho_0}\right)^{7/4} \text{ dynes cm}^{-2},$$

$$P_e = 1.825 \times 10^{31} \left(\frac{\rho}{\rho_0}\right)^{7/4} \text{ dynes cm}^{-2},$$

$$P_n = 6.807 \times 10^{33} \left(\frac{\rho}{\rho_0}\right)^{7/4} \text{ dynes cm}^{-2},$$

$$Y_e = \frac{n_e}{n_p + n_e} \approx \frac{n_n}{n_n} = 0.005647 \left(\frac{\rho}{\rho_0}\right). \quad (18)$$

Our methods to treat EOS of an ideal npe gas (system) under $\beta$-equilibrium in superhigh MFs are introduced as follows: Combining Eq.(9) with momentum conservation gives the chemical potential $\mu_p = E_V(p) = 1.005(\frac{B}{B_{cr}}\rho Y_e)\text{ MeV}$, and the non-dimensional variable $x_p = \sqrt{\frac{2m_p}{m_p c^2}} \approx 4.626 \times 10^{-2}(\frac{B}{B_{cr}}\rho Y_e(0.0535)^{1/2})^{1/2}$; Then we get

$$x_n = \sqrt{\left(1 - \frac{1}{m_n c^2}\right)(2 \times (43.44(\frac{B}{B_{cr}}\rho Y_e(0.0535))^{1/4}}$$

$$-1.29 + 1.005(\frac{B}{B_{cr}}\rho Y_e(0.0535))^{1/2})^{1/2}. \quad (19)$$

The $\beta$-equilibrium condition gives the expression for the isotropic matter pressure $P$,

$$P = \frac{m_e c^2}{\lambda^3} \phi(x_e) + \frac{m_p c^2}{\lambda^3} \phi(x_p) + \frac{m_n c^2}{\lambda^3} \phi(x_n)$$

$$= 6.266 \times 10^{30} \left(\frac{\rho}{\rho_0}\right)(\frac{B}{B_{cr}})\frac{Y_e}{0.0535} + 2.324 \times 10^{26}$$

$$\times \left(\frac{B}{B_{cr}}\rho Y_e(0.0535)^{1/2}\right)^{1/2} + 1.624 \times 10^{38}$$

$$\times \frac{1}{15\pi^2}(x_n^5 - \frac{5}{14} x_n^7 + \frac{5}{24} x_n^9) \text{ dynes cm}^{-2}, \quad (20)$$

where $x_n$ is determined by Eq.(19). The above equation always approximately hold in an ideal npe gas when $B^* \gg 1$ and $\rho \sim 0.5\rho_0 - 2\rho_0$. Based on the above equations, we plot two schematic diagrams of QED effects on EOS of this npe gas, as shown in Figs.2-3. Both $P_p$ and $P_n$ increase obviously with $\rho$ and $B$ for an ideal npe gas.
3.3 The QED effects on the total matter pressure and total energy density

As discussed above, the pressures of fermions increase with \( B \), the total matter pressure increases with \( B \). Due to a positive co-relation between the total energy density \( \varepsilon \) and the total matter pressure, \( \varepsilon \) also increases with \( B \).

The stable configurations of a NS can be obtained from the well-known hydrostatic equilibrium equations of Tolman, Oppenheimer and Volkov (TOV) for the pressure \( P(r) \) and the enclosed mass \( m(r) \),

\[
\frac{dP(r)}{dr} = - \frac{G(m(r) + 4\pi r^3 P(r)/c^2)(\rho + P(r)/c^2)}{r(r - 2Gm/r/c^2)} \\
\frac{dm(r)}{dr} = 4\pi \rho r^2 \tag{21}
\]

where \( G \) is the gravitational constant. For a chosen central value of \( \rho \), the numerical integration of Eq.(21) provides the mass-radius relation. In Eq.(21), the pressure \( P(r) \) is the gravitational collapse pressure, and always balanced by the total dynamics pressure, \( P \); the central density \( \rho \) is proportional to the matter energy density \( \varepsilon \); the enclosed mass, \( m(r) \), increases with the central density \( \rho \) when \( r \) is given.

As we know, the magnetic effects can give rise to an anisotropy of the total pressure of the system to become anisotropic (Bocquet et al. 1995, Paulucci et al. 2011). The total energy momentum tensor due to both matter and magnetic field is to be given by

\[
T^{\mu \nu} = T_m^{\mu \nu} + T_B^{\mu \nu},
\]

where,

\[
T_m^{\mu \nu} = \epsilon_m u^\mu u^\nu - P_m (g^{\mu \nu} - u^\mu u^\nu),
\]

and

\[
T_B^{\mu \nu} = \frac{B^2}{4\pi} \left( u^\mu u^\nu - \frac{1}{2} g^{\mu \nu} \right) - \frac{B^\mu B^\nu}{4\pi} .
\]

The first term in Eq.(24) is equivalent to magnetic pressure, while the second term causes the magnetic tension. Due to an excess negative pressure or tension along the direction to the magnetic field, the component of \( T_B^{\mu \nu} \) along the field, \( T_B^{zz} \), is negative. Thus, the total pressure in the parallel direction to MF can be written as

\[
P_\parallel = P_m - \frac{B^2}{8\pi} ,
\]

and that perpendicular to MF, \( P_\perp \), is written as

\[
P_\perp = P_m + \frac{B^2}{8\pi} - MB ,
\]

where \( M \) is the magnetization of the system, and \( MB \) is the magnetization pressure (Perez et al. 2008, Ferrer et al. 2010). In this work magnetars universally have typical dipole MFs \( \sim (10^{14} - 10^{15}) \) G and inner field strengths not more than \( 10^{17} \) G, under which the system magnetic moment satisfies \( M < B \), a condition that can be justified for any medium that is not ferromagnetic, the effect of AMMs of nucleons on the EOS are insignificant and thus ignored (Ferrer et. al. 2015). It’s obvious that the total pressure of the system becomes anisotropic, that is \( P_\perp > P_\parallel \), which could lead to the Earth-like oblatening effect.

According to our calculations, when \( B^\theta = 100 \), \( P_m \sim 10^{33} - 10^{34} \) dynes cm\(^{-2} \) and \( B^\phi \sim 10^{29} - 10^{30} \). Hence, in this presentation, we consider that the component of the total energy momentum tensor along the symmetry axis becomes positive, \( T^{zz} > 0 \), since the total matter pressure increases more rapidly than the magnetic pressure. We propose that the component of the total energy momentum tensor along the symmetry axis becomes positive, since \( P_m \) always grows more rapidly than the magnetic pressure. The magnetic tension along the direction to the magnetic field will be responsible for deforming a magnetar along MF, and turns the star into a kind of oblate spheroid. Be note that such a deformation in shape might even render a more compact magnetar endowed with canonical strong surface fields \( B \sim 10^{14-15} \) G. Also, such a deformed magnetar could have a more massive mass because of the positive contribution of the magnetic field energy to EOSs of a magnetar.

4 To modify \( P_e \) in superhigh MFs

According to atomic physics physics, the higher the orbit quantum number \( l \), the larger the probability of an electron’s transition (this transition is referred to the transition
from a higher energy level into a lower energy level) is. Analogous to atomic energy level, in a superhigh MF, the easier an electron’s transition from a higher Landau level into a lower Landau level, Thus, the higher the electron Landau level number \( n \), the lower the stability of the Landau level. Owing to the uncertainties of microscopic states, we introduce a new quantity, \( g_n \), the stability coefficient of electron Landau level in a superhigh MF, and assume that \( g_n \) decreases with \( n \) as an exponential form,

\[
g_n = g_0 n^{-\alpha},
\]

where \( g_0 \) is the stability coefficient of the ground-state Landau level of electrons, \( \alpha \) is the Landau level stability index, and is restricted to be \( \alpha < 0 \). From Eq.(27), it is obvious that \( g(n) \) is a function of \( n \) and \( \alpha \), and the higher \( n \) is, the smaller \( g_n \) is (except for \( g_1 = g_0 \)).

According to the Pauli exclusion principle, electron energy state number in a unit volume, \( N_{\text{pha}} \), should be equal to electron number in a unit volume, \( n_e \). Considering the electron Landau level stability coefficient \( g_n \), and summing over electron energy states in a 6-dimension phase space, we can express \( N_{\text{pha}} \) as follows:

\[
N_{\text{pha}} = n_e = N_A \rho Y_e \\
= \frac{2\pi}{h^3} \int dp_z \sum_{n=0}^{\infty} \sum_{\sigma} g_n \\
\times \int \delta \left( \frac{p_z}{m_e c} - [(2n + \sigma)B^*]^{\frac{1}{2}} \right) dp_{\perp},
\]

where \( N_A \) is the Avogadro constant. When \( n_m \geq 6 \), the summation formula can be approximately replaced by the following integral equation

\[
\sum_{n=0}^{n_m} n^{\alpha+\frac{3}{2}} \approx \int_{0}^{n_m} n^{\alpha+\frac{3}{2}} dn = \frac{2}{2\alpha+3} n_m^{\alpha+\frac{3}{2}}.
\]

Thus, Eq.(28) can be rewritten as

\[
N_{\text{pha}} = N_A \rho Y_e = \frac{2\pi}{h^3} \frac{1}{2\alpha+3} \sqrt{B^*} ( \frac{m_e c}{h} )^3 g_0 \\
\int_{0}^{p_{\text{F}}} \left( \frac{E_F(e)}{m_e c^2} \right)^2 - 1 - \left( \frac{p_z}{m_e c} \right)^2 \frac{1}{2} \int dp_{\perp}.
\]

After a complicated deduction process, we get an non-dimensional momentum

\[
x_e = \frac{p_{\text{F}(e)}}{m_e c} = C \left[ \frac{Y_e}{0.05} \frac{\rho}{0.05} \right]^{\frac{1}{2(1-\alpha)\pi}} \frac{1}{g_0 (1-\alpha)} (B^*)^{\frac{\alpha+\frac{1}{2}}{2(1-\alpha)\pi}},
\]

where \( C \) is a constant, which is determined by

\[
C = \left( \frac{0.05\rho N_A (2\alpha+3)}{2^{2(1-\alpha)\pi}} \right)^{\frac{1}{2(1-\alpha)\pi}} \left( \frac{h}{m_e c} \right)^{\frac{1}{2(1-\alpha)\pi}},
\]

\[
\approx (337.12) \left( \frac{1-\alpha}{2\alpha+3} \right)^{\frac{1}{2(1-\alpha)\pi}} \left( \frac{h}{m_e c} \right)^{\frac{1}{2(1-\alpha)\pi}},
\]

with \( p_{\text{F}(e)}^2 = (1-t)^{1/2} + \alpha \), and \( t = p_z^2/(2E_F(e)) \). If \( \alpha \) and \( g_0 \) are determined, the expressions of \( E_F(e) \) and \( p_{\text{F}(e)} \) in superhigh MFs will be modified accordingly. To exactly determine the values of \( \alpha \) and \( g_0 \) is an interesting and important task, but is beyond of this paper. Since this is an original work in which some uncertainties could exist, to further modify and perfect our model should be considered in our future studies, especially to further investigate QED effects on the EoSs using an improved expression of \( p_{\text{F}(e)} \) in a superhigh MF.

5 Conclusions

In this presentation, we derived a general expression for electron pressure, which holds in a superhigh MF, considered QED effects on EoSs of neutron star matter, and discussed an anisotropy of the total pressure in superhigh MFs. Compared with a common pulsar, a magnetar could be a more compact oblate spheroid-like deformed NS, due to the anisotropic total pressure; an increase in the maximum mass of a magnetar is expected because of the positive contribution of the magnetic field energy to EoS.

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