ON THE DISTRIBUTION OF ADDITIVE TWISTS OF THE
DIVISOR FUNCTION AND HECKE EIGENVALUES

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1. Introduction

Let \( f \) be an \( SL_2(\mathbb{Z}) \) cusp form of weight \( k \), and suppose it has Fourier expansion

\[
f(z) = \sum_{n \geq 1} \lambda_f(n) n^{k-1} e(nz)
\]

for \( z \) in the upper half plane. In this paper, one of our main objects of interest will be the exponential sum

\[
S_f(\alpha; X) = \sum_{n \leq X} \lambda_f(n) e(n \alpha).
\]

Jutila \[6\] showed that this sum is \( O(\sqrt{X}) \) uniformly in \( \alpha \) and therefore exhibits considerable oscillation. By Plancherel and (14.56) in \[5\]

\[
\int_0^1 |S_f(\alpha)|^2 d\alpha = \sum_{n \leq X} |\lambda_f(n)|^2 = c_f X + O(X^{3/5})
\]

for some \( c_f > 0 \), so it is clear that Jutila’s bound is sharp. By Hölder’s inequality, it follows from these estimates that for all \( s > 0 \)

\[
\int_0^1 \left| \sum_{n \leq X} \lambda_f(n) e(n \alpha) \right|^s d\alpha \asymp X^{s-1/2}.
\]

It is desirable to know whether one can determine more information about the distribution of this exponential sum.

Another related exponential sum is

\[
S_d(\alpha; X) = \sum_{n \leq X} d(n) e(n \alpha)
\]

where

\[
d(n) = \sum_{d \mid n} 1
\]

is the divisor function.

The properties of this exponential sum are of interest in applications of the circle method. It behaves differently from the exponential sum \( S_f(\alpha; X) \) due to the positivity of its coefficients. For \( s > 2 \), due to this positivity, the contribution of \( \alpha \) near 0 (and more generally near rationals with small denominator) determine the size of the \( L^s \) norm. Using the circle method, finding asymptotics of the form

\[
\int_0^1 |S_d(\alpha; X)|^s d\alpha \sim C_s X^{s-1} (\log X)^s
\]
is then quite straightforward. See [10] for a proof of this for higher divisor functions, where the situation is similar. Asymptotics in the case \(s = 2\) quickly follow from Plancherel, as we have

\[
\int_0^1 |S_d(\alpha; X)|^2 d\alpha = \sum_{n \leq X} d(n)^2 \sim \frac{1}{\pi^2} X (\log X)^3.
\]

Lower moments are significantly more difficult, as for \(s < 2\) one expects a nontrivial contribution from the minor arcs as well. Until now, the only result for moments in this range was for the \(L^1\)-norm and due to Goldston and the author in [2], where it is shown that

\[
\sqrt{X} \ll \int_0^1 |S_d(\alpha)| d\alpha \ll \sqrt{X} \log X.
\]

In this paper, we are able to find asymptotics for the \(L^s\)-norm of \(S_d(\alpha; X)\) for all \(0 < s < 2\). This, combined with the aforementioned results for higher moments resolves the problem of finding asymptotics for all moments of \(S_d(\alpha; X)\). Using the same method, we are able to show the following similar result for all moments of \(S_f(\alpha; X)\) as well in Theorem 1.

**Theorem 1.** For \(0 < s < 2\), we have that for \(\star \in \{d, f\}\), with \(f\) a holomorphic cusp form for \(SL_2(\mathbb{Z})\)

\[
\int_0^1 |S_\star(\alpha; X)|^s d\alpha = C_s^\star X^{\frac{s}{2}} + O(X^{\frac{s}{2} - \eta_1 s(2 - s)})
\]

for some \(C_s^\star, \eta_1 > 0\). Furthermore, for \(s \geq 2\), we also have that

\[
\int_0^1 |S_f(\alpha; X)|^s d\alpha = C_s^f X^{\frac{s}{2}} + O_{s, f}(X^{\frac{s}{2} - \eta_2})
\]

for some \(\eta_2 > 0\).

This result has several interesting corollaries in the case \(\star = f\). Note that from the bound \(S_f(\alpha; X) \ll \sqrt{X}\), [11], and Hölder, we have that \(\exp(-C_1 s) \leq C_1^f \leq \exp(C_2 s)\) for some \(C_1, C_2 > 0\). Then, by the method of moments (one may apply Theorem 9.2 in [4], for example), we obtain a limiting distribution for the magnitude of \(\frac{1}{\sqrt{X}} S_f(\alpha; X)\) sum as \(X \to \infty\).

**Corollary 2.** Suppose \(f\) is a holomorphic cusp form for \(SL_2(\mathbb{Z})\). Let \(A_X\) be the random variable given by

\[
\frac{1}{\sqrt{X}} \left| \sum_{n \leq X} \lambda_f(n) e(n\alpha) \right|,
\]

where \(\alpha\) is chosen uniformly at random from \([0, 1]\). Then, there is a random variable \(A\) so that \(A_X\) converges to \(A\) in distribution as \(X \to \infty\).

In particular, it follows that there exists a compactly supported measure \(\mu\) on \([0, \infty)\) so that for any continuous \(f : [0, \infty) \to \mathbb{R}\), we have that

\[
\lim_{X \to \infty} \int_0^1 f \left( X^{-\frac{s}{2}} \left| \sum_{n \leq X} \lambda_f(n) e(n\alpha) \right| \right) d\alpha = \int f d\mu.
\]

If we restrict the second part of the main theorem to the case \(s = 2r\) for \(r\) a positive integer, we also obtain the following by orthogonality:
Corollary 3. For all positive integers \( r \), we have that
\[ \lambda(f(n_1)\ldots f(n_r)) \overline{\lambda(f(m_1)\ldots f(m_r))} = C_q^{2r} X^r + O(X^{r-\eta_2}). \]
for some constant \( C_q^{2r} > 0 \).

Our proof of the main theorem proceeds via an iterative method, which we sketch below. For \( Q \approx X^{\frac{1}{2}+\delta} \), the \( L^s \) integral may be approximated by
\[ \frac{1}{Q^2} \sum_{q \sim Q} \sum_{a(q)} S_\star \left( \frac{a}{q} X \right) |^s \]
up to some constant factor (the range of \( s \) where this works depends on the choice of \( \star \)). In our case, we achieve this via a version of Jutila’s variant of the circle method in [7]. Applying Voronoi summation, this roughly reduces to dealing with
\[ \frac{1}{Q^2} \sum_{q \sim Q} \sum_{a(q)} S_\star \left( \frac{a}{q^2} \right) |^s X. \]

Since \( q \) is now much larger than the length of the sum, the inner sum amounts to integration over \([0,1]\), and can be shown to be roughly
\[ \varphi(q) \int_0^1 \left| S_\star \left( \frac{a}{q^2} \right) \right|^s d\alpha. \]

Ignoring the factor of \( \varphi(q) \) for purpose of this discussion, note the inner sum varies very little as \( q \) varies by small amounts, and so one obtains that
\[ \frac{1}{Q^2} \sum_{q \sim Q} \sum_{a(q)} \left| S_\star \left( \frac{a}{q^2} \right) \right|^s \approx \frac{c_1}{Q} \int_{q \sim Q} \int_0^1 \left| S_\star \left( \frac{a}{q^2} \right) \right|^s d\alpha d\tilde{q} \]
for some \( c_1 > 0 \). The same method applied starting with sums of length \( X' \approx X \) with \( Q' = \sqrt{X'/X} Q \) yields the same quantity times \((X'/X)^{\frac{\delta}{2}}\). The following approximate functional equation is obtained:

Proposition 4. For \( X \approx X', 0 < s < 2, \star \in \{d,f\} \), we have that for some \( \eta_1 > 0 \)
\[ \int_0^1 \left| \frac{1}{\sqrt{X}} S_\star (\alpha) \right|^s d\alpha = \int_0^1 \left| \frac{1}{\sqrt{X'}} S_\star (\alpha) \right|^s d\alpha + O(X^{-\eta_1 s(2-s)}). \]

Furthermore, for \( s \geq 2 \), for some \( \eta_2 > 0 \), we have that
\[ \int_0^1 \left| \frac{1}{\sqrt{X}} S_f (\alpha) \right|^s d\alpha = \int_0^1 \left| \frac{1}{\sqrt{X'}} S_f (\alpha) \right|^s d\alpha + O(X^{-\eta_2 s}). \]

Iterating this yields the main theorem. The details of how this implies the main theorem are shown in Section 3 and the proof of Proposition 4 in Section 4. The next section is devoted to a few technical results that are used in the proof.

We remark that our result is related to a result of Jurkat and van Horne [8] in which a limiting distribution is found for magnitude of the exponential sum
\[ \sum_{n \leq X} e(n^2 \alpha). \]
Jurkat and van Horne’s methods appear to be quite different from ours, though both our result and their result involve applying a Farey dissection and using Poisson summation (Voronoi summation in our case) on what remains.

We expect that using our methods applied to the exponential sum with coefficients equal to $1 \ast \chi_4$, with $\chi_4$ the character of conductor 4, we can strengthen Theorem 4 of [8]. Specifically, one has that for $0 < s < 4$ and some $\delta > 0$:

$$
\int_0^1 \left| \sum_{n \leq X} e(n^2 \alpha) \right|^s d\alpha = c_s X^{s/2} + O(X^{s/2(1-(4-s)\delta)}),
$$

with $c_s$ as in Theorem 4 of [8]. We expect our methods also apply to the case of exponential sums with coefficients the Fourier coefficients of Maass forms and half integral weight forms, as we use nothing about $f$ besides its modularity via Voronoi summation. Such improvements and generalizations should be straightforward and we leave the details to the interested reader.

1.1. Notation and conventions. As usual, we use Vinogradov’s notation $A \ll B$ (equivalently $B \gg A$) to denote that $|A| \leq CB$ for some constant $C > 0$. When we use $\varepsilon$ in a statement, we mean that the statement holds for all $\varepsilon > 0$. For the purposes of this paper, this $C$ will depend only on $f, s, \varepsilon$, unless specified otherwise. Any further dependencies will be specified in subscript beneath the $\ll$. We write $A \sim B$ to denote $A \ll B, B \ll A$. In addition, we write $a \sim A$ to denote $A < a \leq 2A$. We write

$$
\sum_{a(q)}
$$

to denote a sum over $0 \leq a < q$ with $(a, q) = 1$. For convenience, for $\ast \in \{d, f\}$, we write

$$
\lambda_\ast(n) = \begin{cases} 
\lambda(n) & \ast = d \\
\lambda_f(n) & \ast = f 
\end{cases}
$$

2. Standard technical lemmas

We shall use Voronoi summation as stated below, along with some properties of the integral transforms involved. These are well-known, and the final bounds follow from repeated integration by parts and trivial bounds (see §2 of [1], for example).

**Proposition 5.** Let $w$ be smooth and supported on positive reals, $q \geq 1$ be prime, and $(a, q) = 1$. Then, we have

$$
\sum_{n \geq 1} d(n) e\left(\frac{an}{q} \right) w\left(\frac{n}{X}\right) = \frac{1}{q} \int (\log(x/q^2) + 2\gamma) w\left(\frac{n}{X}\right) dx
$$

$$
+ \frac{X}{q} \sum_{n \geq 1} d(n) e\left(-\frac{\overline{a} n}{q}\right) I_d w\left(\frac{n}{q^2/X}\right),
$$

$$
\sum_{n \geq 1} \lambda_f(n) e\left(\frac{an}{q} \right) w\left(\frac{n}{X}\right) = \frac{X}{q} \sum_{n \geq 1} \lambda_f(n) e\left(-\frac{\overline{a} n}{q}\right) I_f w\left(\frac{n}{q^2/X}\right)
$$

where for $\ast \in \{d, f\}$,

$$
I_d w(x) = \int w(x)(4K_0(4\pi \sqrt{x}) - 2\pi Y_0(4\pi \sqrt{x})) dx,
$$

$$
I_f w(x) = \int w(x)(4K_0(4\pi \sqrt{x}) - 2\pi Y_0(4\pi \sqrt{x})) dx,
$$

$$
Y_0(x) = \frac{\sin x}{x}
$$

and

$$
K_0(x) = \frac{1}{2\pi} \int_0^{\infty} e^{-tx} \cosh(tx) dt.
$$
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\[ \mathcal{I}_f w(x) = \int w(x) J_{k-1}(4\pi \sqrt{x}) dx. \]

These transforms also satisfy the property that if \( w \) is supported on values \( \approx 1 \) and \( w^{(j)} \ll H^j \) for \( j \geq 1 \), then \( \mathcal{I}_w \ll 1 \), and \( (\mathcal{I}_w)^{(j)} \ll H^j \). Also, for \( \delta > 0, |x| \geq H^{1+\delta} \), we have that \( w^{(j)}(x) \ll \delta H^{-A} \).

We shall also require and prove a modified version of Jutila’s circle method \([7]\).

It slightly improves the error term of Jutila’s result slightly in some cases, at the cost of requiring a smoothing.

**Proposition 6.** Let \( Q \) be a set of integers \( \approx Q \). Let \( \Delta = HQ^2 \) for some \( H \gg 1 \).

Also, suppose that \( \phi \) is some nonzero smooth compactly supported function on \( \mathbb{R} \).

Write

\[ L = \sum_{q \in Q} \phi(q), \]

\[ \tilde{\chi}(\alpha) = \frac{1}{\phi(0)} \Delta L \sum_{q \in Q} \sum_q^* \phi(\Delta^{-1}(\alpha - a/q)). \]

Then, we have that

\[ \int_0^1 |1 - \tilde{\chi}(\alpha)|^2 d\alpha \ll \frac{Q^4}{HL^2} + \frac{Q^{2+\varepsilon}}{L^2}. \]

**Proof.** By Poisson summation, for \( (a, q) = 1 \) we have

\[ \phi(\Delta^{-1}(\alpha - a/q)) = \Delta \hat{\phi}(0) + \Delta \sum_{|\ell| > 0} \hat{\phi}(\Delta \ell) e\left(\frac{a\ell}{q}\right) e(-\ell \alpha), \]

so

\[ \tilde{\chi}(\alpha) - 1 = \frac{1}{\phi(0) L} \sum_{q \in Q} \sum_q^* \sum_{|\ell| > 0} \hat{\phi}(\Delta \ell) e\left(\frac{a\ell}{q}\right) e(-\ell \alpha) \]

\[ = \frac{1}{\phi(0) L} \sum_{|\ell| > 0} \hat{\phi}(\Delta \ell) e(-\ell \alpha) \sum_{q \in Q} c_q(\ell) \]

where

\[ c_q(\ell) = \sum_{a(q)}^* e\left(\frac{a\ell}{q}\right) \]

is the usual Ramanujan sum. Using the fact that \( c_q(\ell) = \sum_{d|q,\ell} \mu(q/d) \), we obtain that

\[ \tilde{\chi}(\alpha) - 1 = \frac{1}{L} \sum_{|\ell| > 0} \hat{\phi}(\Delta \ell) e(\ell \alpha) \sum_{d|\ell} \sum_{d_1 \in Q} \mu(q_1). \]
By Plancherel, and the bound $\hat{\phi(t)} \ll_A (1 + |t|)^{-A}$ which holds for all $A > 0$, it follows that for some sufficiently large $C$

$$\int_0^1 |\tilde{\chi}(\alpha) - 1|^2 d\alpha = \frac{1}{\phi(0)^2 L^2} \sum_{|\ell| > 0} |\hat{\phi}(\Delta \ell)|^2 \left( \sum_{d|\ell} \sum_{d_1 \in \mathbb{Q}} \mu(q_1) \right)^2 \ll \frac{1}{L^2} \sum_{|\ell| > 0} |\hat{\phi}(\Delta \ell)|^2 \left( \sum_{d|\ell} \frac{Q}{d} \right)^2 \ll \frac{1}{L^2} \sum_{K} \frac{1}{K^{10}} \sum_{|\ell| \leq K \Delta^{-1}} \left( \sum_{d|\ell} \frac{Q}{d} \right)^2,$$

where $K$ runs over powers of two. Note that

$$\sum_{|\ell| \leq K \Delta^{-1}} \left( \sum_{d|\ell} \frac{Q}{d} \right)^2 = \sum_{d_1, d_2 \in \mathbb{Q}} \frac{Q^2}{d_1 d_2} \sum_{|\ell| \leq K \Delta^{-1}} \sum_{d|\ell} 1 \ll K \sum_{d_1, d_2 \in \mathbb{Q}} \frac{Q^2}{d_1 d_2} \left( \frac{\Delta^{-1}}{|d_1, d_2|} + 1 \right) \ll KQ^2 \Delta^{-1} \sum_{d_1, d_2 \in \mathbb{Q}} \frac{(d_1, d_2)}{d_1^2 d_2^2} + KQ^{2+\varepsilon} \ll KQ^2 \Delta^{-1} \sum_{d_1, d_2 \in \mathbb{Q}} \frac{1}{d_1^2 d_2^2} \sum_{a|d_1, d_2} \varphi(a) + KQ^{2+\varepsilon} \ll KQ^2 \Delta^{-1} \sum_a \frac{\phi(a)}{a^3} \sum_{d_1, d_2} \frac{1}{d_1^2 d_2^2} + KQ^{2+\varepsilon} \ll K(Q^2 \Delta^{-1} + Q^{2+\varepsilon}).$$

The desired result follows upon summing over $K$. 

$$\square$$

3. PROOF OF THE MAIN THEOREM

In this section, we prove the main theorem assuming Proposition 4 and in the following section, we prove Proposition 1. Iterating Proposition 4, we obtain that for all $0 < s < 2, Y \geq X$

$$\int_0^1 \left| \frac{1}{\sqrt{X}} S_*(\alpha; X) \right|^s d\alpha = \int_0^1 \left| \frac{1}{\sqrt{Y}} S_*(\alpha; Y) \right|^s d\alpha + O(X^{-\eta s(2-s)})$$

and that for $s \geq 2$

$$\int_0^1 \left| \frac{1}{\sqrt{X}} S_f(\alpha; X) \right|^s d\alpha = \int_0^1 \left| \frac{1}{\sqrt{Y}} S_f(\alpha; Y) \right|^s d\alpha + O(X^{-\eta s}).$$

In particular, the sequence

$$X \mapsto \int_0^1 \left| \frac{1}{\sqrt{X}} \sum_{n \leq X} \lambda_*(n) e(n\alpha) \right|^s d\alpha$$

is a Cauchy sequence for all $s \geq 2$ when $* = f$ and for $0 < s < 2$ for general $* \in \{d, f\}$. 
Taking the limit as $Y \to \infty$ in (3.1) and (3.2), we have that for $0 < s < 2, \, \ast \in \{d, f\}$
\[
\int_{0}^{1} \left| \frac{1}{\sqrt{X}} S_{\ast}(\alpha; X) \right|^{s} d\alpha = C_{\ast}^{s} + O(X^{-\eta_{1}(2-s)}),
\]
for some constants $C_{\ast}^{s}, \eta_{1} > 0$, and for $s \geq 2$
\[
\int_{0}^{1} \left| \frac{1}{\sqrt{X}} S_{f}(\alpha; X) \right|^{s} d\alpha = C_{f}^{s} + O(X^{-\eta_{2}})
\]
for some constants $C_{f}^{s}, \eta_{2} > 0$.

Thus, Theorem 1 follows if we can show that
\[
\int_{0}^{1} \left| \frac{1}{\sqrt{X}} S_{d}(\alpha; X) \right|^{s} d\alpha \gg 1 \quad (s > 0)
\]
(3.3) follows when $\ast = f$ from Hölder with the bounds $S_{f}(\alpha; X) \ll \sqrt{X}$ (Proposition 14) and (1.1). However, one does not have such bounds for $S_{d}$, so the rest of this section is dedicated to the case of $\ast = d$.

In [2], it was shown that
\[
\int_{0}^{1} |S_{d}(\alpha; X)| d\alpha \gg \sqrt{X}.
\]
It follows from Hölder that (3.3) holds for $s \geq 1$. Thus, it remains to show:

**Proposition 7.** We have
\[
\int_{0}^{1} |S_{d}(\alpha; X)|^{s} d\alpha \gg X^{\frac{s}{2}}.
\]
for $s < 1$.

**Proof.** The proof of this follows from Voronoi summation along with the large sieve to deal with the error terms introduced.

Let $w$ be some smooth function satisfying $\mathbb{I}_{[X^{-\frac{1}{10}}, 1 - X^{-\frac{1}{10}}]} \leq w \leq \mathbb{I}_{[1/X, 1]}$ with $w^{(j)}(x) \ll_j X^{-\frac{j}{10}}$ for all $j$. We may then smooth $S_{d}(\alpha; X)$ by replacing it with
\[
\tilde{S}_{d}(\alpha; X) = \sum_{n} d(n) e(n\alpha) w\left(\frac{n}{X}\right)
\]
since by Parseval and Cauchy-Schwarz
\[
\left| \int_{0}^{1} |S_{d}(\alpha; X)|^{s} - \left| \tilde{S}_{d}(\alpha; X) \right|^{s} d\alpha \right|
\leq \int_{0}^{1} \left| \sum_{n \in [1, X^{\frac{9}{10}}] \cup [X - X^{\frac{9}{10}}, X]} d(n) e(n\alpha) \left(1 - w\left(\frac{n}{X}\right)\right) \right|^{s} d\alpha
\leq \left( \sum_{n \in [1, X^{\frac{9}{10}}] \cup [X - X^{\frac{9}{10}}, X]} d(n)^{2} \right) \ll X^{\frac{9}{10} + \varepsilon},
\]
which is an acceptable error. It thus remains to show that
\[
\int_{0}^{1} |\tilde{S}_{d}(\alpha; X)|^{s} d\alpha \gg X^{\frac{s}{2}}.
\]
Take \( c \) to be some sufficiently small constant. Then, we have that
\[
\int_0^1 |S_d(\alpha; X)|^s d\alpha \geq \sum_{q \sim c \sqrt{X}} \sum_{a(q)} \int_{-c}^{c} \left| \tilde{S}_d \left( \frac{a}{q}; \beta; X \right) \right|^s d\beta.
\]
Now, note that by Proposition 5 (Voronoi summation), we have that
\[
S_d \left( \frac{a}{q}; \beta; X \right) = \frac{1}{q} \int (\log(x/q^2) + 2\gamma - 1)e(x\beta) w \left( \frac{x}{X} \right) dx
\]
\[
+ \frac{1}{q} \sum_{n} d(n)e \left( -\frac{an}{q} \right) I_d w_{\beta} \left( \frac{n}{q^2/X} \right).
\]
where
\[
w_{\beta}(x) = w(x)e(x\beta).
\]
If \( c \) is sufficiently small, then we have the bound \( \text{Re}(e(x\beta)) \geq 1 - c \) for \( 0 \leq x \leq X, |\beta| \leq \frac{c}{X} \). Therefore, for \( q \sim c\sqrt{X} \), we have that
\[
\left| \int (\log(x/q^2) + 2\gamma - 1)e(x\beta) w \left( \frac{x}{X} \right) dx \right|
\]
\[
\geq (1 - c) \int_{q^2}^{X-q^2} (\log(x/q^2) + 2\gamma - 1)dx - \int_1^{q^2} (\log(x^2/2) + 2\gamma - 1)dx
\]
\[
\geq (1 + O(c)) X.
\]
We remark that for \( |\beta| \leq \frac{c}{X} \) (which is so in our case), \( w_{\beta} \) satisfies the bounds
\[
w_{\beta}^{(j)}(x) \ll X^{\frac{j}{20}}
\]
and also for \( 1/X \leq x \leq 1 \)
\[
w_{\beta}'(x) \ll \begin{cases} X^{\frac{j}{20}} & \frac{1}{X} \leq x \leq X^{-\frac{j}{20}} \\ c & X^{-\frac{j}{20}} \leq x \leq 1 - X^{-\frac{j}{20}} \\ X^{\frac{j}{20}} & 1 - X^{-\frac{j}{20}} \leq x \leq 1. \end{cases}
\]
Note that the bounds on \( I_d w_{\beta} \) given by Proposition 5 imply that the contribution of terms \( n \geq X^{\frac{j}{20}} \) is \( \ll \lambda X^{-\lambda} \). It follows from (3.5) that
\[
\left| \tilde{S}_d \left( \frac{a}{q}; \beta; X \right) \right|^s \geq (1 + O(c)) \frac{X^s}{q^s} - \frac{1}{q^s} E(q, a; \beta)^s + O(X^{-2020})
\]
where
\[
E(q, a; \beta) = \left| \sum_{n \leq X^{\frac{j}{20}}} d(n)e \left( -\frac{an}{q} \right) I_d w_{\beta} \left( \frac{n}{q^2/X} \right) \right|
\]
By Hölder and the large sieve, we have that
\[
\sum_{a(q)}^* E(q, a; \beta)^s \leq \varphi(q)^{1 - \frac{1}{\hat{s}}} \left( \sum_{a(q)}^* E(q, a; \beta)^2 \right)^{\frac{\hat{s}}{2}}
\]
\[
\ll \varphi(q) \left( \sum_{n \leq X^{\frac{j}{20}}} d(n)^2 \left| I_d w_{\beta} \left( \frac{n}{q^2/X} \right) \right|^2 \right)^{\frac{\hat{s}}{2}}.
\]
Now, integrating by parts once yields that
\[ \int w' \left( \frac{x}{X} \right) B_1 \left( \frac{4\pi \sqrt{nx}}{q} \right) dx = \frac{q}{2\pi \sqrt{n}} \int \frac{1}{X} w' \left( \frac{x}{X} \right) B_1 \left( \frac{4\pi \sqrt{nx}}{q} \right) dx. \]

We have the standard bounds (see section 8.451 in [3], for example)
\[
B_1(x) \ll x^{-\frac{1}{2}} \quad (x \gg 1), \\
B_1(x) \ll x^{-1} \quad (x \ll 1),
\]
where we write \( B_\nu = 4(-1)^\nu K_\nu - 2\pi Y_\nu \), so it follows that
\[
\int w' \left( \frac{x}{X} \right) B_1 \left( \frac{4\pi \sqrt{nx}}{q} \right) dx \ll \frac{q}{n^{\frac{1}{2}}}.
\]

It follows that we have the bound
\[
\int w \left( \frac{x}{X} \right) B_0 \left( \frac{4\pi \sqrt{nx}}{q} \right) dx \ll \frac{q}{n^{\frac{1}{2}}}.
\]

Therefore, it follows that
\[
\sum_{a(q) \neq 0} E(q, a; \beta)^s \ll \varphi(q) \left( \frac{q}{\varphi(q)} \right)^{\frac{1}{2}} \left( \sum_{d(n) 2n^{-\frac{1}{2}}} q^2 \right)^s \ll \varphi(q) q^{2s} \cdot \frac{q}{\varphi(q)}.
\]

It follows that so long as \( c \) is sufficiently small, for \( q \sim c\sqrt{X} \)
\[
\sum_{a(q) \neq 0} \left| \mathcal{S}_d \left( \frac{a}{q} + \beta; X \right) \right|^s \geq (1 + O(c)) \varphi(q) X^s q^s + O \left( \varphi(q) q^s \frac{q}{\varphi(q)} \right)
\]
\[
\geq \varphi(q) X^s \cdot \left( \frac{1}{2} (2c)^{-s} + O \left( c^s \frac{q}{\varphi(q)} \right) \right).
\]

We thus obtain that
\[
\int_0^1 \left| \mathcal{S}_d \left( \frac{a}{q} + \beta; X \right) \right|^s \, d\alpha \geq X^s \int_{\frac{1}{2} \sim c \sqrt{X}} \varphi(q) \frac{1}{2} (2c)^{-s} + O(c^s q) d\beta + O(X^{-2020}).
\]

Since
\[
\sum_{q \sim c \sqrt{X}} \varphi(q) = (1 + o(1)) \frac{9}{\pi^2} c^2 X,
\]
we obtain
\[
\int_{\frac{1}{2} \sim c \sqrt{X}} \sum_{q \sim c \sqrt{X}} \varphi(q) \frac{1}{2} (2c)^{-s} + O(c^s q) d\beta \geq (1 + o(1)) 2c \frac{9}{\pi^2} c^2 \left( \frac{1}{2} (2c)^{-s} + O(c^s) \right).
\]

This is \( \gg c, s \) for \( c \) sufficiently small, so the desired result follows. \( \square \)
4. Proof of Proposition 8

Instead of showing Proposition 4, we show Proposition 8 from which Proposition 4 clearly follows.

**Proposition 8.** Let $\delta = \frac{1}{100}$. Suppose that $X' \asymp X$, and that $X_1 \asymp X^{2\delta}$. Also, suppose that $\kappa > 0$ is sufficiently small. Then, for $0 < s < 2$

\begin{equation}
\left\| \frac{1}{\sqrt{X'}} \sum_{n \leq X'} \lambda_\ast(n) e(n\alpha) \right\|^s \, d\alpha = \frac{2}{3} \int_1^2 \int_0^1 \left\| \frac{1}{t\sqrt{x_1}} \sum_n \lambda_\ast(n) e(n\alpha) I_{X_1} w \left( \frac{n}{X_1 t^2} \right) \right\|^s \, dt \, d\alpha + O(X^{-\kappa s(2-s)}), \tag{4.1}
\end{equation}

and for $s \geq 2$, we have that

\begin{equation}
\left\| \frac{1}{\sqrt{X'}} \sum_{n \leq X'} \lambda_f(n) e(n\alpha) \right\|^s \, d\alpha = \frac{2}{3} \int_1^2 \int_0^1 \left\| \frac{1}{t\sqrt{x_1}} \sum_n \lambda_f(n) e(n\alpha) I_{X_1} w \left( \frac{n}{X_1 t^2} \right) \right\|^s \, dt \, d\alpha + O(X^{-\kappa}). \tag{4.2}
\end{equation}

We now proceed to prove Proposition 8 in the remainder of this section. After some initial setup, we shall show (4.1), (4.2) separately. The proofs of the two are largely similar, with (4.2) being slightly simpler.

Let $\delta_1 = \delta^{100}$. Let $w$ be some smooth function satisfying $1_{[X^{-\delta_1},1]} \leq w \leq 1_{[C/X,1]}$ for some $C > 0$ with $w^{(j)}(x) \ll_j X^{\delta_1}$ for all $j$. Then, let

\begin{equation}
I_s^\ast = \int_0^1 \left\| \frac{1}{\sqrt{X'}} \sum_n \lambda_\ast(n) e(n\alpha) w \left( \frac{n}{X'} \right) \right\|^s \, d\alpha. \tag{4.4}
\end{equation}

We shall use the following lemma to show that working with this smoothed exponential sum results in an acceptable loss.

**Lemma 9.** We have that for $0 < s < 2$

\begin{equation}
I_s^\ast - \int_0^1 \left\| \frac{1}{\sqrt{X'}} \sum_n \lambda_\ast(n) e(n\alpha) \right\|^s \, d\alpha \ll X^{-\frac{\delta_1}{2} + s + \varepsilon}. \tag{4.6}
\end{equation}

Furthermore, for $s \geq 2$

\begin{equation}
I_s^\ast - \int_0^1 \left\| \frac{1}{\sqrt{X'}} \sum_n \lambda_f(n) e(n\alpha) \right\|^s \, d\alpha \ll X^{-\delta_1 + s + \varepsilon}. \tag{4.5}
\end{equation}

**Proof.** For $s \leq 1$, we have the inequality $||x|^s - |y|^s| \leq |x - y|^s$ for any $x, y \in \mathbb{C}$ so it follows by Cauchy-Schwarz and the bound $|\lambda_\ast(n)| \leq d(n)$ that

\begin{equation}
I_s^\ast - \int_0^1 \left\| \frac{1}{\sqrt{X'}} \sum_n \lambda_\ast(n) e(n\alpha) \right\|^s \, d\alpha \ll \int_0^1 \left\| \frac{1}{\sqrt{X'}} \sum_{n \in [0, X^{1-\delta_1}] \cup [X', \infty]} \lambda_\ast(n) e(n\alpha) \left( 1 - w \left( \frac{n}{X'} \right) \right) \right\|^s \, d\alpha \ll X^{-\frac{\delta_1}{2} + s + \varepsilon}. \tag{4.6}
\end{equation}
Now, for $1 < s \leq 2$, by Hölder and Plancherel, we have
\[
I_s^{s-1} \left| \frac{1}{\sqrt{X'}} \sum_{n \leq X'} \lambda_*(n)e(n\alpha) \right|^{s-1} d\alpha \ll X^\varepsilon.
\]
For $1 < s$, we also have that $|x|^s - |y|^s \ll (|x|^{s-1} + |y|^{s-1})|x - y|$, so for $1 < s < 2$, we have
\[
I_s - \int_0^1 \left| \frac{1}{\sqrt{X'}} \sum_{n \leq X'} \lambda_*(n)e(n\alpha) \right|^s d\alpha
\ll X^\varepsilon \int_0^1 \left| \frac{1}{\sqrt{X'}} \sum_{n \in [0,X'-X^{-\delta_1}] \cup [X'-X^{-\delta_1},X']} \lambda_*(n)e(n\alpha) \right|^s d\alpha
\ll X^{-\frac{2s}{2} + \varepsilon}.
\]
Combining (4.6), (4.7), we obtain (4.4).

To show (4.5), note that we have that by Proposition 14 and partial summation
\[
1 \sum_{n \leq X'} \lambda_*(n)e(n\alpha) w \left( \frac{n}{X'} \right) \ll 1.
\]
Then, for $s \geq 2$ we obtain from Cauchy-Schwarz and Plancherel that
\[
I_s^s - \int_0^1 \left| \frac{1}{\sqrt{X'}} \sum_{n \leq X'} \lambda_*(n)e(n\alpha) \right|^s d\alpha
\ll \frac{1}{\sqrt{X'}} \int_0^1 \left| \sum_{n \in [0,X'-X^{-\delta_1}] \cup [X'-X^{-\delta_1},X']} \lambda_*(n)e(n\alpha) \right|^s d\alpha
\ll X^{-\frac{2s}{2} + \varepsilon}
\]
so (4.5) follows from Cauchy-Schwarz and Plancherel.

4.1. Proof of (4.1). At this point we require the following estimate.

Lemma 10. Suppose that $Q \ll X^{\frac{1}{2} + \delta}, L = \sum_{q \leq Q} \varphi(q)$.

Then, we have that for $0 < s < 2$
\[
I_s^s = \frac{1}{L} \sum_{q \leq Q} \sum_{a(q)}^s \left| \sum_n \lambda_*(n)e \left( \frac{an}{q} \right) w \left( \frac{n}{X'} \right) \right|^s + O(X^{-\frac{s}{2}(2-s)} + \varepsilon).
\]

Proof. Let $\phi$ be some nonzero smooth function supported on $[1,2]$ with $\int \phi = 1$, and let
\[
H = X^{\delta}, \Delta = \frac{H}{Q^\varepsilon},
\]
\[
\tilde{\chi}(\alpha) = \frac{1}{\Delta L} \sum_{q \leq Q} \sum_{u(q)}^s \phi(\Delta^{-1}(\alpha - a/q)).
\]
Since the fractions $\left\{ \frac{a}{q} : q \in \mathbb{Q}, \langle a, q \rangle = 1 \right\}$ are at least $Q^{-2}$-spaced, we have the bound

$$\tilde{\chi}(\alpha) \ll \frac{1}{L \Delta} \cdot (1 + \Delta Q^2) \ll Q^\varepsilon$$

as $L \gg Q^2$.

Now, let

$$\tilde{I}_s = \int_0^1 \left| \frac{1}{\sqrt{X'}} \sum_n \lambda_s(n) e\left(\frac{n \alpha}{q}\right) \right|^s \tilde{\chi}(\alpha) \, d\alpha.$$ 

By Hölder, we have that

$$|I_s^* - \tilde{I}_s^*| \ll \left( \int_0^1 \left| \frac{1}{\sqrt{X'}} S_s(\alpha; X') \right|^2 \, d\alpha \right)^{\frac{s}{2}} \left( \int_0^1 |1 - \tilde{\chi}(\alpha)|^s \, d\alpha \right)^{1 - \frac{s}{2}}.$$ 

Note that for $0 < s < 2$, we have that $\frac{s}{2} > \frac{1}{2}$. By Plancherel and our pointwise bound on $\tilde{\chi}$, it follows that the above is

$$\ll X^\varepsilon \left( \int_0^1 |1 - \tilde{\chi}(\alpha)|^\frac{s}{2} \, d\alpha \right)^{1 - \frac{s}{2}}.$$ 

By Proposition 6 and Hölder, we may therefore conclude that

$$|I_s^* - \tilde{I}_s^*| \ll X^{-4s(2-s) + \varepsilon}.$$ 

By the definition of $\tilde{\chi}$, we have that

$$\tilde{I}_s = \int_1^2 \phi(t) \sum_{q \sim Q} \sum_{a(q)}^* \left| \frac{1}{\sqrt{X'}} \sum_n \lambda_s(n) e\left(\frac{an}{q}\right) \right|^s \, dt.$$ 

Now, note that for all $\beta \in [\Delta, 2\Delta]$, we have that

$$\frac{1}{\sqrt{X'}} \sum_n \lambda_s(n) e\left(\frac{an}{q}\right) e(n\beta) w\left(\frac{n}{X'}\right)$$

$$= \frac{1}{\sqrt{X'}} e(X'\beta) \sum_n \lambda_s(n) e\left(\frac{an}{q}\right) w\left(\frac{n}{X'}\right) + O(X\Delta E(q, a, \beta)),$$

where

$$E(q, a, \beta) = \sup_{1 \leq I \leq [1, X']} \left| \frac{1}{\sqrt{X'}} \sum_{n \in I} \lambda_s(n) e\left(\frac{an}{q}\right) w\left(\frac{n}{X'}\right) \right|.$$ 

In general, for positive $A, B$, we have the inequality $(A + O(B))^s = A^s + O(B^s + A^{s-\min(1,s)} B^{\min(1,s)})$. It follows that

$$\left| \frac{1}{\sqrt{X'}} \sum_n \lambda_s(n) e\left(\frac{an}{q}\right) e(n\beta) w\left(\frac{n}{X'}\right) \right|^s = \left| \frac{1}{\sqrt{X'}} \sum_n \lambda_s(n) e\left(\frac{an}{q}\right) w\left(\frac{n}{X'}\right) \right|^s$$

plus an error that is

$$\ll (X\Delta)^s E(q, a, \beta)^s.$$ 

if $s \leq 1$ and

$$\ll (X\Delta)^s E(q, a, \beta)^s + (X\Delta) E(q, a, \beta) \left| \sum_n \lambda_s(n) e\left(\frac{an}{q}\right) w\left(\frac{n}{X'}\right) \right|^{s-1}$$

$$\ll (X\Delta)^\frac{s}{2} E(q, a, \beta)^s.$$
if \(1 < s < 2\). Since \((X \Delta) \ll X^{-\delta}\), we have that both of these error terms are
\[
\ll (X \Delta)^{\frac{1}{2}} \left( E(q, a, \beta)^s + \left| \frac{1}{\sqrt{X}} \sum_{n} \lambda_*(n) e \left( \frac{an}{q} \right) w \left( \frac{n}{X^\gamma} \right) \right|^s \right).
\]

At this point, we shall use a maximal version of the large sieve of Montgomery (see Theorem 2 of [9]), which we state below:

**Proposition 11** (Maximal large sieve, [9]). For any sequence \(a : \mathbb{N} \to \mathbb{C}\) supported on an interval \(I\) of length \(N\), and \(\eta\)-separated frequencies \(\alpha_1, \ldots, \alpha_R\), we have that
\[
\sum_{x < r < X} \sup_{j < I \subset I} \left| \sum_{n \in J} a(n) e \left( n \alpha_j \right) \right|^2 \ll (\eta^{-1} + N) \sum_{n} |a(n)|^2
\]
where \(J\) ranges over subintervals of \(I\).

Then, by Hölder and the maximal large sieve, we have that
\[
\left(4.10\right) \quad \frac{1}{\varphi(q)} \sum_{\alpha(q)} E(q, a, \beta)^s + \left| \frac{1}{\sqrt{X}} \sum_{n} \lambda_*(n) e \left( \frac{an}{q} \right) w \left( \frac{n}{X^\gamma} \right) \right|^s \ll X^\varepsilon.
\]

Combining the above with (4.9), we obtain that
\[
\left(4.11\right) \quad I_* = \sum_{q \sim Q} \sum_{\alpha(q)} \frac{1}{\sqrt{X}^q} \sum_{n} \lambda_*(n) e \left( \frac{an}{q} \right) w \left( \frac{n}{X^\gamma} \right)^s + O(X^{-\frac{1}{3}(2-s)+\varepsilon}).
\]

Now, take \(Q = (X'X_1)^{\frac{1}{2}}\), and suppose that \(q \sim Q\). By Proposition [5] we have that (noting that the main term in the case of \(\ast = d\) can be easily absorbed into the error term)
\[
\frac{1}{\sqrt{X}} \sum_{n} \lambda_*(n) e \left( \frac{an}{q} \right) w \left( \frac{n}{X^\gamma} \right)^s = \frac{1}{\sqrt{X_1(q/Q)^2}} \sum_{n} \lambda_*(n) e \left( \frac{-\pi n}{q} \right) I_* w \left( \frac{n}{X_1(q/Q)^2} \right) + O(X^{-\frac{1}{3}+\varepsilon}).
\]

From the properties of \(I_* w\) noted in Proposition [5], we have that \(I_* w(x) \ll X^{-A}\) for \(x \geq X^{2\delta_1}\). Furthermore, we have the trivial bound \(I_* w(x) \ll 1\), and \((I_* w)^{(j)}(x) \ll X^{\delta_1}\) for \(j \geq 1\).

Thus, noting that \(a \mapsto -\pi\) is a permutation of \((\mathbb{Z}/q\mathbb{Z})^s\), combining the above with Lemma [10] (observing that \(Q^2/X' = X_1\))
\[
I_* = \frac{1}{I} \sum_{q \sim Q} \sum_{\alpha(q)} \left| \frac{1}{\sqrt{X_1(q/Q)^2}} \sum_{n \in X_1} \lambda_*(n) e \left( \frac{an}{q} \right) I_* w \left( \frac{n}{X_1(q/Q)^2} \right) \right|^s + O(X^{-\frac{1}{3}(2-s)+\varepsilon}).
\]

We now deal with the inner sum with the following lemma, which amounts to the observation that averaging the exponential sum over fractions \(\frac{a}{q}\) with \((a, q) = 1, 0 < a < q\) is essentially integration over \([0, 1]\). This is because these fractions are equidistributed on scales much smaller than the reciprocal of the length of the exponential sum since \(q\) is large.

This observation is stated more generally in the following lemma.
Lemma 12. Suppose that \( a : \mathbb{N} \rightarrow \mathbb{C} \) is a sequence of complex numbers satisfying \( |a(n)| \ll n^s \). Then, if \( q \geq Y^{10} \), \( 0 < s < 2 \), we have that

\[
\frac{1}{\varphi(q)} \sum_{a(q)} \left| \frac{1}{\sqrt{Y}} \sum_{n \leq Y} a(n) e\left(\frac{an}{q}\right) \right|^s = \int_0^1 \left| \frac{1}{\sqrt{Y}} \sum_{n \leq Y} a(n) e(n\alpha) \right|^s d\alpha + O(Y^{-s}).
\]

It should be possible to show this for \( q \geq Y^{1+\varepsilon} \) for any \( \varepsilon > 0 \) (and an error of \( Y^{-\varepsilon} \)), though our result suffices.

To prove this lemma, we first prove the following crude result on the equidistribution of reduced residue classes in short intervals.

Lemma 13. For \( q \geq 1, H \geq 1 \), we have that

\[
N_q([x, x+H]) = \frac{\varphi(q)}{q} H + O(d(q)).
\]

where

\[
N_q(I) = \sum_{n \in I} 1
\]

Proof. By Möbius inversion, we have that for any \( D \geq 1 \)

\[
\sum_{x \leq n \leq x+H} 1 = \sum_{d|n} \mu(d) \sum_{x \leq n \leq x+H} \frac{1}{d} = \sum_{d|n} \mu(d) \sum_{x \leq n \leq x+H} \frac{1}{d}
\]

\[
= \frac{H}{d} + O(1) = H \sum_{d|n} \frac{\mu(d)}{d} + O(d(q)).
\]

The desired result follows upon noting that \( \frac{\varphi(q)}{q} = \sum_{d|n} \frac{\mu(d)}{d} \). \( \square \)

Proof of Lemma 12. First, note that for all \( |\beta| \leq q^{-\frac{1}{2}} \), we have that

\[
\sum_{n \leq Y} a(n) e\left(\frac{an}{q}\right) e(n\beta) = \sum_{n \leq Y} a(n) e\left(\frac{an}{q}\right) + O(Y^2 |\beta|).
\]

From our bound on \( |\beta| \), the error term then must be \( \leq Y^{-3} \). It follows from Lemma 13 that

\[
\frac{1}{\varphi(q)} \sum_{a(q)} \left| \frac{1}{\sqrt{Y}} \sum_{n \leq Y} a(n) e\left(\frac{an}{q}\right) \right|^s
\]

\[
= \frac{1}{\varphi(q)} \sum_{a(q)} q^{\frac{1}{2}} \int_0^{q^{\frac{1}{2}}} \left| \frac{1}{\sqrt{Y}} \sum_{n \leq Y} a(n) e\left(\frac{an}{q}\right) e(n\beta) \right|^s d\alpha + O(Y^{-s})
\]

\[
= \frac{q^{\frac{1}{2}}}{\varphi(q)} \int_0^{1} \left| \frac{1}{\sqrt{Y}} \sum_{n \leq Y} a(n)e(n\alpha) \right|^s \left| N_q([qa, qa + q^{\frac{1}{2}}]) \right| d\alpha + O(Y^{-s})
\]

\[
= \int_0^{1} \left| \frac{1}{\sqrt{Y}} \sum_{n \leq Y} a(n)e(n\alpha) \right|^s d\alpha + O\left( q^{-1+\varepsilon} \int_0^{1} \left| \frac{1}{\sqrt{Y}} \sum_{n \leq Y} a(n)e(n\alpha) \right|^s d\alpha \right).
\]
By Hölder, the error is at most \( q^{-1+\varepsilon} \), and the desired result follows. \( \square \)

Applying Lemma 12, we obtain that for \( 0 < s < 2 \)

\[
I_s^* = \frac{1}{L} \int_{q<Q} \varphi(q) \int_0^1 \left| \frac{1}{\sqrt{X_1(q/Q)^2}} \sum_{n \leq X_1 X^{2s_1}} \lambda_s(n) e(n\alpha) \mathcal{I}_s w \left( \frac{n}{X_1(q/Q)^2} \right) \right|^s \, d\alpha
\]

\[
(4.12)
\]

We now show that the sum over \( q \sim Q \) may be turned into an integral over \([Q, 2Q]\) at a small loss. We first eliminate the \( \varphi(q) \) at the cost of averaging over \( q \) in a short interval. To see that this is so, consider some \( q \sim Q \) and a real \( \tilde{q} \in [Q, 2Q] \). Then, we have that

\[
\frac{1}{\sqrt{X_1(q/Q)^2}} = \frac{1}{\sqrt{X_1(\tilde{q}/Q)^2}} + O\left( |q - \tilde{q}| \frac{1}{Q \sqrt{X_1}} \right).
\]

Note that \( Q \sqrt{X_1} \gg X_1^{20} \). We also have that from the derivative bounds on \( \mathcal{I}_s w \) that for \( n \ll X_1 X^{2s_1} \)

\[
\mathcal{I}_s w \left( \frac{n}{X_1(q/Q)^2} \right) - \mathcal{I}_s w \left( \frac{n}{X_1(\tilde{q}/Q)^2} \right) \leq |q - \tilde{q}| \cdot \frac{X_1^{5s_1}}{Q}.
\]

Then, for all \( \alpha \in \mathbb{R} \)

\[
\frac{1}{\sqrt{X_1(q/Q)^2}} \sum_{n \leq X_1 X^{2s_1}} \lambda_s(n) e(n\alpha) \mathcal{I}_s w \left( \frac{n}{X_1(q/Q)^2} \right) = \frac{1}{\sqrt{X_1(\tilde{q}/Q)^2}} \sum_{n \leq X_1 X^{2s_1}} \lambda_s(n) e(n\alpha) \mathcal{I}_s w \left( \frac{n}{X_1(\tilde{q}/Q)^2} \right) + O\left( |q - \tilde{q}| \sqrt{X_1 X^{5s_1} Q^{-1}} \right)
\]

\[
= \frac{1}{\sqrt{X_1(\tilde{q}/Q)^2}} \sum_{n \leq X_1 X^{2s_1}} \lambda_s(n) e(n\alpha) \mathcal{I}_s w \left( \frac{n}{X_1(\tilde{q}/Q)^2} \right) + O\left( |q - \tilde{q}| \sqrt{X_1 X^{5s_1} Q^{-1}} \right).
\]

If \( |q - \tilde{q}| \ll Q^{1/2} \), note that we must have \( |q - \tilde{q}| \sqrt{X_1 X^{5s_1} Q^{-1}} \ll X^{-20s} \) so since \( \tilde{q}/q = 1 + O(Q^{-9/10}) \) (and since by Plancherel and Hölder, the below is \( \ll X^{\varepsilon} \)), we have

\[
\int_0^1 \left| \frac{1}{\sqrt{X_1(q/Q)^2}} \sum_{n \leq X_1 X^{2s_1}} \lambda_s(n) e(n\alpha) \mathcal{I}_s w \left( \frac{n}{X_1(q/Q)^2} \right) \right|^s \, d\alpha
\]

\[
= \int_0^1 \left| \frac{1}{\sqrt{X_1(\tilde{q}/Q)^2}} \sum_{n \leq X_1 X^{2s_1}} \lambda_s(n) e(n\alpha) \mathcal{I}_s w \left( \frac{n}{X_1(\tilde{q}/Q)^2} \right) \right|^s \, d\alpha + O(X^{-10s_8 + \varepsilon}).
\]

\[
(4.13)
\]

\[
= \frac{\tilde{q}}{q} \int_0^1 \left| \frac{1}{\sqrt{X_1(\tilde{q}/Q)^2}} \sum_{n \leq X_1 X^{2s_1}} \lambda_s(n) e(n\alpha) \mathcal{I}_s w \left( \frac{n}{X_1(\tilde{q}/Q)^2} \right) \right|^s \, d\alpha + O(X^{-10s_8 + \varepsilon}).
\]
Now, let \( \{I_j\}_{1 \leq j \leq K} \) be a partition of \( [Q, 2Q] \) into intervals of length \( \asymp Q^{\frac{1}{10}} \) for some \( K \asymp Q^{1-\frac{1}{10}} \). We obtain from (14.13), adding redundant averaging over \( \tilde{q} \sim Q \), that

\[
\frac{1}{L} \sum_{q \sim Q} \varphi(q) \int_0^1 \left| \frac{1}{\sqrt{X_1(q/Q)^2}} \sum_{n \leq X_1 X_2^{\delta_1}} \lambda_*(n) e(n\alpha) \mathcal{I}_s w \left( \frac{n}{X_1(q/Q)^2} \right) \right|^s \, d\alpha = \frac{1}{L} \sum_{j=1}^K \sum_{q \in I_j} \varphi(q) \int_0^1 \left| \frac{1}{\sqrt{X_1(q/Q)^2}} \sum_{n \leq X_1 X_2^{\delta_1}} \lambda_*(n) e(n\alpha) \mathcal{I}_s w \left( \frac{n}{X_1(q/Q)^2} \right) \right|^s \, d\alpha
\]

(4.14) \( \sim \frac{K}{L} \sum_{j=1}^K \varphi(q) \frac{1}{q} \left| \int_{I_j} \tilde{q} \int_0^1 [\ldots] d\alpha d\tilde{q} + O(X^{-10s+\varepsilon}) \right| \]

where \([\ldots]\) is

\[
\left| \frac{1}{\sqrt{X_1(q/Q)^2}} \sum_{n \leq X_1 X_2^{\delta_1}} \lambda_*(n) e(n\alpha) \mathcal{I}_s w \left( \frac{n}{X_1(q/Q)^2} \right) \right|^s.
\]

It follows from well-known elementary results that

\[
L = \frac{9}{\pi^2} Q^2 + O(Q \log Q), \sum_{n \in I_j} \varphi(q) = \frac{6}{\pi^2} |I_j| + O(\log Q).
\]

Thus (4.14) equals

\[
\frac{2}{3Q^2} \int_Q^{2Q} \tilde{q} \int_0^1 \left| \frac{1}{\sqrt{X_1(q/Q)^2}} \sum_{n \leq X_1 X_2^{\delta_1}} \lambda_*(n) e(n\alpha) \mathcal{I}_s w \left( \frac{n}{X_1(q/Q)^2} \right) \right|^s \, d\alpha d\tilde{q}
\]

plus an error of \( Q^{-1+\varepsilon} \ll X^{-\frac{1}{4}} \). By the change of variables \( t = \tilde{q}/Q \), this equals

\[
\frac{2}{3} \int_1^2 t \int_0^1 \left| \frac{1}{\sqrt{X_1 t^2}} \sum_{n \leq X_1 X_2^{\delta_1}} \lambda_*(n) e(n\alpha) \mathcal{I}_s w \left( \frac{n}{X_1 t^2} \right) \right|^s \, d\alpha dt.
\]

Completing the sum over \( n \) with the same method by which it was removed, and gathering up (4.14), (4.12), we obtain that for some \( \kappa > 0 \)

\[
\int_0^1 \left| \frac{1}{\sqrt{X_1}} \sum_{n \leq X_1} \lambda_*(n) e(n\alpha) \right|^s \, d\alpha = \frac{2}{3} \int_1^2 t \int_0^1 \left| \frac{1}{t\sqrt{X_1}} \sum_{n} \lambda_*(n) e(n\alpha) \mathcal{I}_s w \left( \frac{n}{X_1 t^2} \right) \right|^s \, d\alpha dt + O(X^{-\kappa s(2-s)}).
\]

for any \( X_1 \asymp X^{2\delta} \). The desired result follows.

4.2. **Proof of (4.2).** This section proceeds a similar fashion to the previous section, so we shall refer to it heavily and only indicate those places in which the two differ. What allows us to deal with higher moments is the following bound of Jutila (which improves by a factor of \( \log X \) on an older bound of Wilton, which we could have also used):

**Proposition 14.** For all \( L, R \), we have that

\[
\left| \sum_{L \leq n \leq R} \lambda_f(n) e(n\alpha) \right| \ll R^{\frac{1}{3}}.
\]
As a corollary of this and partial summation, it follows that for any $Y_1 \leq Y$
\[
\left| \sum_{Y_1 < n \leq Y} \lambda_f(n) e(n\alpha) w\left(\frac{n}{Y}\right) \right| \ll Y^{\frac{1}{2}}.
\]

For the rest of the section, we also suppose that $s \geq 2$. By (4.5), it suffices to show that for $X' \approx X$
\[
I_f^s = \frac{2}{3} \int_1^2 t \int_0^1 \left| \frac{1}{t \sqrt{X'}} \sum_n \lambda_f(n) e(n\alpha) \mathcal{L}_f \left( \frac{n}{X't^2} \right) \right|^s \, da \, dt + O(X^{-\eta})
\]
for some $\eta > 0$, and this is what the rest of the section is devoted to showing.

We now show the following analogue of (4.8)

**Lemma 15.** Suppose that
\[
Q \approx X^{\frac{1}{2} + \delta}, \quad L = \sum_{q \sim Q} \varphi(q).
\]

Then, we have that for $s \geq 2$
\[
(4.15) \quad I_f^s = \frac{1}{L} \sum_{q \sim Q} \sum_{a(q)}^* \left| \sum_n \lambda_f(n) e\left(\frac{an}{q}\right) w\left(\frac{n}{X'}\right) \right|^s + O(X^{-\frac{s}{2}}).
\]

**Proof.** Let $\phi$ be some nonzero smooth function supported on $[1, 2]$ with $\hat{\phi}(0) = 1$. Also, as in the previous subsection, let
\[
H = X^{\delta}, \Delta = \frac{H}{Q^2},
\]
\[
\tilde{\chi}(\alpha) = \frac{1}{\Delta L} \sum_{q \sim Q} \sum_{a(q)}^* \phi(\Delta^{-1}(\alpha - a/q)),
\]
\[
\tilde{I}_f^s = \int_0^1 \left| \frac{1}{\sqrt{X'}} \sum_n \lambda_f(n) e(n\alpha) w\left(\frac{n}{X'}\right) \right|^s \hat{\chi}(\alpha) \, d\alpha.
\]

Then, by Proposition 14 and partial summation, Proposition 6 and Cauchy Schwarz
\[
(4.16) \quad I_f^s - \tilde{I}_f^s \ll \int_0^1 \left| 1 - \hat{\chi}(\alpha) \right| d\alpha \ll \left( \int_0^1 \left| 1 - \hat{\chi}(\alpha) \right|^2 d\alpha \right)^{\frac{1}{2}} \ll X^{-\frac{3}{2}},
\]
so the error from replacing $I_f^s$ with $\tilde{I}_f^s$ is admissible. For $\beta \sim \Delta$, we have that by partial summation and Proposition 14
\[
\frac{1}{\sqrt{X'}} \sum_n \lambda_f(n) e\left(\frac{an}{q}\right) e(n\beta) w\left(\frac{n}{X}\right) = \frac{1}{\sqrt{X'}} \sum_n \lambda_f(n) e\left(\frac{an}{q}\right) w\left(\frac{n}{X}\right) + O(X^{-\delta})
\]
so
\[
\left| \frac{1}{\sqrt{X'}} \sum_n \lambda_f(n) e\left(\frac{an}{q}\right) e(n\beta) w\left(\frac{n}{X}\right) \right|^s = \left| \frac{1}{\sqrt{X'}} \sum_n \lambda_f(n) e\left(\frac{an}{q}\right) w\left(\frac{n}{X}\right) \right|^s + O(X^{-\delta})
\]
and the desired result follows from this and (4.16) immediately. \[\square\]
Take \( Q = (X'X_1)^{1/2} \). Applying Voronoi and bounds for tails of \( I_fw \) as in the previous section, we obtain that

\[
I_f^s = \frac{1}{L} \sum_{q=Q} \sum_{a(q)}^* \left| \frac{1}{\sqrt{X_1(q/Q)^2}} \sum_{n \leq X_1X_2} \lambda_f(n)e\left( \frac{an}{q} \right) I_fw\left( \frac{n}{X_1(q/Q)^2} \right) \right|^s.
\]

plus an error of \( O(X^{-\frac{s}{2}}) \)

We now require the following analogue of Lemma 12 for high moments.

**Lemma 16.** Suppose that \( q \geq (YY')^{20} \). Also, suppose that \( w \) is smooth and compactly supported away from 1 with \( \int |w'| \ll 1, |w^{(j)}| \ll Y' \). Then we have that for \( s \geq 1 \) and \( Y \) sufficiently large

\[
\frac{1}{\varphi(q)} \sum_{a(q)}^* \left| \frac{1}{\sqrt{Y}} \sum_{n \leq YY'2} \lambda_f(n)e\left( \frac{an}{q} \right) I_fw\left( \frac{n}{Y} \right) \right|^s
= \int_0^1 \left| \frac{1}{\sqrt{Y}} \sum_{n \leq YY'2} \lambda_f(n)e(n\alpha) I_fw\left( \frac{n}{Y} \right) \right|^s d\alpha + O((YY')^{-1}).
\]

**Proof.** Proceeding as in the proof of Lemma 12 we have that for \( |\beta| \ll q^{-\frac{1}{2}} \)

\[
\frac{1}{\sqrt{Y}} \sum_{n \leq YY'} \lambda_f(n)e\left( \frac{an}{q} \right) I_fw\left( \frac{n}{Y} \right)
= \frac{1}{\sqrt{Y}} \sum_{n \leq YY'} \lambda_f(n)e\left( \frac{an}{q} + \beta n \right) I_fw\left( \frac{n}{Y} \right) + O((YY')^{-3}).
\]

Now, using the bounds on \( I_fw \) in Proposition 4 we may complete the sum over \( n \) at the cost of an error of \( O((YY')^{-100}) \). Applying Voronoi summation to the complete sum yields that

\[
\frac{1}{\sqrt{Y}} \sum_{n \leq YY'} \lambda_f(n)e\left( \frac{an}{q} \right) I_fw\left( \frac{n}{Y} \right)
= \frac{1}{\sqrt{Y}} \sum_n \lambda_f(n)e\left( \frac{an}{q} \right) I_fw\left( \frac{n}{Y} \right) + O((YY')^{-100})
= \frac{1}{\sqrt{q^2/Y}} \sum_{n \geq q^2/Y} \lambda_f(n)e\left( \frac{an}{q} \right) w\left( \frac{n}{q^2/Y} \right) + O((YY')^{-100}).
\]

By Jutila’s bound (Proposition 14), we have that this is \( \ll 1 \), so it follows that since all \( \alpha \) are of the form \( \frac{a}{q} + \beta \) for some \( |\beta| \ll q^{-\frac{1}{2}} \) (by Lemma 13 for example), we have the pointwise bound

\[
(4.17) \frac{1}{\sqrt{Y}} \sum_{n \leq YY'} \lambda_f(n)e(n\alpha) I_fw\left( \frac{n}{Y} \right) \ll 1.
\]
Then, we obtain
\[ \left| \frac{1}{\sqrt{Y}} \sum_{n \leq YY'} \lambda_f(n)e \left( \frac{an}{q} \right) \mathcal{I}_f w \left( \frac{n}{Y} \right) \right|^s = \left| \frac{1}{\sqrt{Y}} \sum_{n \leq YY'} \lambda_f(n)e \left( \frac{an + \beta n}{q} \right) \mathcal{I}_f w \left( \frac{n}{Y} \right) \right|^s + O((YY')^{-3}). \]

Therefore, we have that
\[ \left| \frac{1}{\sqrt{Y}} \sum_{n \leq YY'} \lambda_f(n)e \left( \frac{an}{q} \right) \mathcal{I}_f w \left( \frac{n}{Y} \right) \right|^s = \left| \frac{1}{\sqrt{Y}} \sum_{n \leq YY'} \lambda_f(n)e \left( \frac{an}{q} \right) \mathcal{I}_f w \left( \frac{n}{Y} \right) \right|^s + O((YY')^{-3}). \]

Then, proceeding as in the proof of Lemma 12, we obtain that
\[ \frac{1}{\varphi(q)} \sum_{\alpha(q)} \left| \frac{1}{\sqrt{Y}} \sum_{n \leq YY'} \lambda_f(n)e \left( \frac{an}{q} \right) \mathcal{I}_f w \left( \frac{n}{Y} \right) \right|^s = \frac{q^{\frac{1}{2}}}{\varphi(q)} \int_0^1 \left| \frac{1}{\sqrt{Y}} \sum_{n \leq YY'} \lambda_f(n)e(n\alpha) \mathcal{I}_f w \left( \frac{n}{Y} \right) \right|^s |\mathcal{N}_q([q\alpha, q\alpha + q^{\frac{1}{2}}])| d\alpha + O((YY')^{-3}) \]
\[ = \int_0^1 \left| \frac{1}{\sqrt{Y}} \sum_{n \leq YY'} \lambda_f(n)e(n\alpha) \mathcal{I}_f w \left( \frac{n}{Y} \right) \right|^s d\alpha + O((YY')^{-3}). \]

The desired result follows.

Also, as in the previous subsection, by the pointwise bound (4.17), we have that for \( q \sim Q, \tilde{q} \in [Q, 2Q], \)
\[ \frac{1}{\sqrt{X_1(q/Q)^2}} \sum_{n \leq X_{1}X^{2s_1}} \lambda_f(n)e(n\alpha) \mathcal{I}_f w \left( \frac{n}{X_1(q/Q)^2} \right) = \frac{1}{\sqrt{X_1(\tilde{q}/Q)^2}} \sum_{n \leq X_{1}X^{2s_1}} \lambda_f(n)e(n\alpha) \mathcal{I}_f w \left( \frac{n}{X_1(\tilde{q}/Q)^2} \right) + O(\sqrt{X_1X^{5\delta}Q^{-1}}). \]

Then, we obtain
\[ \int_0^1 \left| \frac{1}{\sqrt{X_1(q/Q)^2}} \sum_{n \leq X_{1}X^{2s_1}} \lambda_f(n)e(n\alpha) \mathcal{I}_f w \left( \frac{n}{X_1(q/Q)^2} \right) \right|^s d\alpha = \int_0^1 \left| \frac{1}{\sqrt{X_1(\tilde{q}/Q)^2}} \sum_{n \leq X_{1}X^{2s_1}} \lambda_f(n)e(n\alpha) \mathcal{I}_f w \left( \frac{n}{X_1(\tilde{q}/Q)^2} \right) \right|^s d\alpha + O(X^{-10\delta}). \]

Combining with Lemma 16 exactly as in the previous section, we obtain that
\[ \int_0^1 \left| \frac{1}{\sqrt{X}} \sum_{n \leq X'} \lambda_f(n)e(n\alpha) \right|^s d\alpha = \frac{2}{3} \int_1^2 t \int_0^1 \left| \frac{1}{t\sqrt{X_1}} \sum_{n} \lambda_f(n)e(n\alpha) \mathcal{I}_f w \left( \frac{n}{X_1t^2} \right) \right|^s d\alpha dt + O(X^{-\kappa}) \]
for some \( \kappa > 0. \) The desired result follows.
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