LIE ALGEBRA AUTOMORPHISMS IN CONFORMAL FIELD THEORY

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Abstract
The role of automorphisms of infinite-dimensional Lie algebras in conformal field theory is examined. Two main types of applications are discussed; they are related to the enhancement and reduction of symmetry, respectively. The structures one encounters also appear in other areas of physics and mathematics. In particular, they lead to two conjectures on the sub-bundle structure of chiral blocks, and they are instrumental in the study of conformally invariant boundary conditions.

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1 Observables and automorphisms in quantum field theory

In quantum theory, a physical system is described in terms of its space $\mathcal{H}$ of physical states – some vector space over $\mathbb{C}$, in many situations a Hilbert space – and of fields acting as operators on $\mathcal{H}$. Distinguished among the fields are the observables. In the quantum mechanical description of a point particle, for example, $\mathcal{H}$ is a Hilbert space and the observables are densely defined operators on $\mathcal{H}$. One may wish to endow the collection of all observables with the structure of a unital associative algebra. But since two operators can only be multiplied if the image of the first operator is contained in the range of the second, this idea is somewhat too naive already in quantum mechanics.

Once one studies quantum field theory rather than quantum mechanics, the algebraic structure must be further refined. For instance, there is a framework in which the local observables of relativistic quantum field theory give rise to a net of von Neumann algebras. Tentatively, the space $\mathcal{H}$ of physical states may then be regarded as a module over this (net of) algebra(s), which may decompose into a direct sum $\bigoplus_\mu \mathcal{H}_\mu$ of irreducible modules $\mathcal{H}_\mu$. But a refinement of this point of view proves to be necessary, involving the concept of superselection rules. This is briefly expressed by saying that vectors in different $\mathcal{H}_\mu$ must not be linearly combined; a precise formulation is that one must not regard the space of states as the direct sum of modules $\mathcal{H}_\mu$ over the observable algebra, but rather just as a collection of those modules. The spaces $\mathcal{H}_\mu$ and the associated representations $R_\mu$ of the observables are then called superselection sectors [SW]. Thus observables act as operators within the individual modules $\mathcal{H}_\mu$, whereas non-observable (‘charged’) fields operate between different modules. (An example of such a field operator is provided by the fermion field $\psi$, which changes the fermion number and thereby the superselection sector.)

The spaces $\mathcal{H}_\mu$ are not just modules over the observable algebra, but enjoy additional properties. For instance, to incorporate compatibility with the probabilistic features of quantum physics, they should be unitarizable. Also, one would like to have a sufficiently large observable algebra, and hence as a kind of naturality property requires the modules $\mathcal{H}_\mu$ to be simple. On the other hand, it is economic if the observable algebra is not too large either; this is implemented by a state-field correspondence according to which any vector in each of the spaces $\mathcal{H}_\mu$ is obtainable by acting with a unique field operator on a unique (up to a scalar) vacuum vector $v_\Omega$. The module for which $v_\Omega$ is a cyclic vector is called the vacuum superselection sector and is denoted by $\mathcal{H}_\Omega$. Now for any endomorphism $\varpi$ of the observable algebra and any representation $R$, the composition $R \circ \varpi$ is again a representation on the same vector space; it inherits crucial features from $R$, such as simplicity and, for suitable $\varpi$, unitarizability. Whether the representation $R \circ \varpi$ is physically interesting depends on the endomorphism $\varpi$ (and on $R$, of course). In quantum field theory, the issues of having an interesting representation and of having an interesting endomorphism are intimately related. Indeed, there should exist a collection of special endomorphisms $\varpi_\mu$ that satisfy $R_\mu \cong R_\Omega \circ \varpi_\mu$, with $R_\Omega$ the vacuum representation that corresponds to $\mathcal{H}_\Omega$. All these structures can be made explicit in the formalization of local observables by nets of von Neumann algebras alluded to above.

In that framework, via composition of the associated endomorphisms one can also endow the collection of representations $R_\mu$ with a tensor product, which is fully reducible. On the set
of isomorphism classes of simple modules $\mathcal{H}_\mu$ – to be referred as sectors, for short, and to be denoted by $\lambda, \mu, \ldots$ – this tensor product provides a multiplication called the fusion product and denoted by ‘$\ast$’. By full reducibility one can write

$$\lambda \ast \mu = \sum_{\nu \in I} N_{\lambda,\mu}^{\nu} \nu,$$

(1.1)

where $I$ stands for the set of sectors. The vacuum sector $\Omega$ is a unit element for the fusion product. Also, the evaluation on the unit element, $C_{\lambda,\mu} := N_{\lambda,\mu}^{\Omega}$, provides an involutive automorphism of the fusion rules (otherwise the QFT model wouldn’t possess sensible two-point correlation functions). We write $C: \mu \mapsto \mu^+, \text{i.e. } C_{\lambda,\mu} = \delta_{\lambda,\mu^+}$. In short, one has the structure of a fusion ring – a commutative associative unital ring $\mathcal{R}$ over $\mathbb{Z}$, with a distinguished basis in which the structure constants are non-negative integers $N_{\lambda,\mu}^{\nu}$, and with evaluation at the unit element being a conjugation. (Instead of $\mathcal{R}$, one may also consider $\mathcal{R} \otimes \mathbb{Z} \mathbb{C}$, called the fusion algebra. The relations (1.1) are also referred to as the fusion rules.)

Of special interest are rational theories, which possess only a finite number of sectors, $|I| < \infty$. A feature special to the rational case is the existence of a matrix $S$ that diagonalizes the fusion rules in the sense that

$$N_{\lambda,\mu}^{\nu} = \sum_{\kappa \in I} S_{\kappa,\lambda} S_{\kappa,\mu} S_{\kappa,\nu}^* / S_{\kappa,\Omega},$$

(1.2)

The two representations $R$ and $R \circ \omega$ are particularly close relatives when $\omega = \omega$ is an automorphism. Sectors associated via $R_\mu \cong R_\Omega \circ \omega_\mu$ to automorphisms $\omega_\mu$ are therefore distinguished; they are called simple currents. Trivially, there is always at least one simple current, the vacuum sector $\Omega$. The simple currents $J$ are the elements of the distinguished basis of the fusion ring $\mathcal{R}$ that are units of $\mathcal{R}$. They are characterized by $\sum_\mu N_{\lambda,\mu}^{\mu} = 1$ for all $\lambda \in I$; under the fusion product they form a subring of $\mathcal{R}$ that is the group ring of an abelian group. In rational theories, simple currents can also be characterized by $S_{J,\Omega} = S_{\Omega,\Omega}$.

Let us mention that the observables can be regarded as encoding part of the symmetry of a physical system. (They play in particular the role of a ‘spectrum generating symmetry’.) Now ‘symmetry’ is one of the most helpful concepts in physics, but it shares the fate of other physical notions in that by no means does it refer to a single well-identified concept, but rather encompasses a whole zoo of different ideas. E.g. a common incarnation of symmetry that is not covered by the above discussion of observables arises in the form of transformations that act non-trivially on the field variables used to describe a physical system, but leave the system itself invariant. Concrete examples include various groups of transformations of the canonical variables in Lagrangian mechanics and field theory.\footnote{The set of all such transformations then has the structure of a group. In quantum theory a Lagrangian description of a system need not exist, though. Accordingly, more general structures than groups can occur as symmetry transformations, see e.g. [FrRS, R1, MaS, FGV2, BNS]. Collectively these structures are often referred to as ‘quantum symmetries’ or (in a non-technical sense) quantum groups. While these issues do not concern us here, in any case we use the term ‘quantum field theory’ in a broad sense, not implying a Lagrangian setting, even not the existence of a classical limit (and hence in particular not restricted to the realm of perturbation theory).}
2 Lie algebras and their automorphisms in conformal field theory

We now focus our attention on a particular class of models – rational models of two-dimensional conformal field theory (CFT). The conformal symmetry (and extensions thereof) present in such models is encoded in a suitable algebra of observables. The structure of CFT is severely constrained by the geometry of the underlying world sheet, i.e. the two-dimensional manifold $X$ on which the field theory is defined. In particular, the observables split into quantities depending either analytically or anti-analytically on a local complex coordinate on the world sheet; these are referred to as chiral and anti-chiral observables, respectively. Studies of special classes of models (see e.g. [Wi2]) indicate that a similar splitting happens for other quantities such as correlation functions; this phenomenon is known as ‘holomorphic factorization’. In free field models, and consequently also in the string theories constructed from them, this amounts to a decomposition into ‘left-’ and ‘right-movers’.

Because of their interest in string theory, we consider manifolds $X$ that are compact surfaces of Euclidean signature. In order to constitute a valid world sheet for CFT, such a surface $X$ must be endowed with a conformal structure, i.e. an equivalence class, with respect to the local rescalings, of metrics. $X$ may have a boundary or be non-orientable. In the orientable case, together with a choice of orientation every conformal structure provides a complex structure. The best way to account for the splitting into chiral and anti-chiral observables on arbitrary world sheets is to replace the physical world sheet $X$ by its Schottky double cover $\hat{X}$, which is a complex curve. The theory to be studied on $\hat{X}$ is called chiral CFT, or CCFT for short; until the end of section 5, we will mainly deal with CCFT.

There exist several different formalizations of the notion of chiral CFT, which are not entirely equivalent. Here we adopt an approach that is well suited for various applications to string theory as well as to statistical mechanics and condensed matter physics. Its basic features are the following:

- The observables of a CCFT model have the structure of a rational vertex operator algebra (VOA), or conformal vertex algebra, $\mathfrak{A}$. Such algebras, in turn, can for instance be constructed from so-called Lie algebras $\mathfrak{L}$ of formal distributions.
- Along with the VOA $\mathfrak{A}$ comes its representation category $\text{Rep}(\mathfrak{A})$, which is a semi-simple tensor category.
- The isomorphism classes of simple $\mathfrak{A}$-modules are the sectors of the model.
- There is a lot more structure on the representation category $\text{Rep}(\mathfrak{A})$. It is conjectured that this can be encoded in so-called Moore–Seiberg data [MoS], [FK], [BK1], which equip $\text{Rep}(\mathfrak{A})$ with the structure of a modular tensor category.
- The presence of a modular tensor category is, in turn, the basis for the relation of CFT to three-dimensional topological field theory (TFT), see e.g. [FK, MoS, Wil, FSS], and thereby to invariants of knots and three-manifolds [Hu].

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2 Other approaches can e.g. be found in [FrRS, EK, R2] (local observables forming nets of von Neumann algebras over the circle) and in [GG]. For a functional-analytic interpretation of VOAs see [Hu2].

3 It is so far not known whether the representation category of every rational VOA indeed possesses all properties of a modular tensor category, though this property has been established for several classes of VOAs, compare e.g. [Hu, HL]. In the approach based on nets of von Neumann algebras, the presence of a (unitary) modular tensor category has been established in [KLM].
Note that this formalization does not entirely fit into the general field theoretic framework of section 1; for example we are not given a relation between the sectors and endomorphisms of the observables. Nevertheless the important features mentioned there are again present. For instance, the vertex operator map supplies a state-field correspondence in the vacuum sector. Also, a modular tensor category is in particular spherical, so it has a Grothendieck ring, and this ring precisely realizes the fusion rules. The tensor identity object is the vacuum sector \( \Omega \), for which the underlying vector space \( H_\Omega \) is the vector space of the VOA (\( H_\Omega \) naturally carries the structure of a module over the VOA itself). For more details we refer to \([Fr, BK2, SFW]\) and the literature cited there.

As we are concerned with rational theories, \( |I| < \infty \) there is a matrix \( S \) that diagonalizes the fusion rules, see (1.2). The requirement that the CCFT gives rise to a modular tensor category includes the statement that this matrix \( S \) and the diagonal matrix \( T \) – that has entries \( T_{\mu,\nu} = \delta_{\mu,\nu} T_{\mu} \), \( T_{\mu} = \exp(2\pi i (\Delta_{\mu} - c/24)) \), with \( \Delta_{\mu} \) the conformal weights of the sectors and \( c \) the eigenvalue of the Virasoro central charge – satisfy the defining relations \( S^4 = 1 \) and \((ST)^3 = S^2\) for the generators of \( SL(2,\mathbb{Z}) \).

Next we comment on the infinite-dimensional complex Lie algebras \( \mathfrak{L} \) of formal distributions from which one can construct vertex operator algebras. The idea in that construction is to define a VOA \( \mathfrak{A} \) such that \( \mathfrak{L} \) can be regarded as the Lie algebra of Laurent modes for the vertex operators associated to specific vectors in \( \mathfrak{A} \). The most prominent examples for such Lie algebras of formal distributions are the Virasoro algebra – the relevant vector in \( \mathfrak{A} \) is the Virasoro element \( v_{\text{vir}} \) –, the Heisenberg algebra, and untwisted affine Lie algebras. Various other classes of such Lie algebras \( \mathfrak{L} \) are known. Interesting results about the corresponding CFT models can often be derived by relating them to simpler models that are based on the algebras just mentioned, by special constructions like the ‘coset construction’ or ‘Hamiltonian reduction’. But the chiral algebras for these models are less understood and their representation theory is not fully developed.

One should think of \( \mathfrak{A} \) respectively \( \mathfrak{L} \) as expressing the chiral symmetries of a CFT model locally in a neighborhood of any point on the curve \( \hat{X} \). One also wants to express the symmetries globally on \( \hat{X} \); this is achieved by imposing appropriate invariance conditions, known as Ward identities, on the chiral blocks. When \( \mathfrak{A} \) is obtained from a Lie algebra \( \mathfrak{L} \) of formal distributions, then the Ward identities frequently reduce to a (co-)invariance condition with respect to another type of infinite-dimensional algebras \( \mathfrak{A}_{\hat{X},\vec{p}} \) which we like to call block algebras \( \mathfrak{B} \) (for more details, see section 4). The block algebras can be described in terms of globally defined quantities like functions and differential forms on \( \hat{X} \), or sections in corresponding bundles. The algebras \( \mathfrak{A} \) and \( \mathfrak{L} \) are related to them via Laurent expansion of such global quantities in a local holomorphic coordinate.

An important class of CFT models are the WZW models. They provide the input for the coset construction and for Hamiltonian reduction. WZW models possess a Lagrangian field theory realization, as sigma models with target space a (connected simply connected compact) real Lie group \( G \). The vertex operator algebra of a WZW model is generated by a Lie algebra

\[ \mathfrak{L} \]

However, many of the concepts and results for rational CFT in fact carry over to large classes of non-rational ones.

5 In \([BD]\), the term ‘chiral algebra’ is used for the objects \( \mathfrak{A}_{\hat{X},\vec{p}} \). This use of the term is unfortunately different from terminology in most of the physics literature, where it is the VOA \( \mathfrak{A} \) itself or the Lie algebra \( \mathfrak{L} \) of Laurent modes that is called chiral algebra.
of formal distributions, which is just an untwisted affine Lie algebra. We denote this affine Lie
algebra by $g$; the horizontal subalgebra $\bar{g}$ of $g$ is the complexification of the Lie algebra $\text{Lie}(G)$
of the group manifold $G$. The block algebras of WZW models are again infinite-dimensional
Lie algebras, obtained as tensor products of $\bar{g}$ with certain algebras of meromorphic functions
on $\hat{X}$, see formula (3.3).

Below we will examine two typical situations in which automorphisms enter in quantum
field theory:

1. The enhancement of symmetries.

2. The restriction of observables.

In CFT, the first issue arises in the form of extended chiral algebras, notably so-called simple
current extensions, and the second in the form of orbifold constructions. The operations of
forming the orbifold of a CFT with respect to a finite abelian group and of simple current
extension are actually each other’s inverse. (Non-abelian orbifolds are inverse to more general
extensions.) Thus in principle there is no need to study the two constructions separately.
Nevertheless in practice it is most sensible to do so. The reason is that some CFT models –
notably WZW models – are much better understood than others and hence serve as natural
starting points for each of the two constructions.

These matters will be studied in sections 3 and 5, respectively. Let us include into the list
as separate items also two more specific contexts – to be dealt with in sections 4 and 7 – in
which these structures are encountered:

3. The sub-bundle structure of chiral blocks.

4. Symmetry reduction caused by the presence of boundaries.

The role of automorphisms in these situations is the following.

1. Symmetry enhancement: Simple currents naturally correspond to (classes of) outer au-
tomorphisms of the Lie algebra $\mathcal{L}$ of formal distributions. (They do not preserve the
Virasoro algebra and hence do not give rise to automorphisms of the VOA $\mathfrak{A}$.)
For instance, in the WZW case, where $\mathcal{L} = g$ is an untwisted affine Lie algebra, it can be
proven [Fu, FGV] that (barring a single exceptional case present for $E_8$ level 2) there is
a natural isomorphism

$$\mathcal{G}(g) \cong \mathbb{Z}(G)$$

between the simple current group $\mathcal{G}$ and the center of the relevant simply connected Lie
group $G$.

2. Restriction of observables: The orbifold group is a group of automorphisms of $\mathcal{L}$. The
orbifold model should still possess conformal symmetry; accordingly, the fixed point set
of the orbifold group must contain the Virasoro subalgebra of $\mathcal{L}$. As a consequence, in
this case the automorphisms of $\mathcal{L}$ induce automorphisms of $\mathfrak{A}$.
3. Chiral blocks: Suitable outer automorphisms of \( \mathfrak{L} \) give rise to finite order automorphisms of the bundles of chiral blocks. These bundles then split into sub-bundles invariant under those automorphisms.

4. Symmetry breaking boundary conditions: In CFT they are partially characterized by the subalgebra of \( \mathfrak{A} \) respectively \( \mathfrak{L} \) that they preserve. (Again for preserving conformal invariance this must contain the Virasoro algebra.) Their analysis is simplified technically when that subalgebra can be characterized as the fixed point subalgebra under some group of automorphisms.

To summarize these introductory remarks, let us repeat that in CFT (and in other areas of physics as well) various structures related to automorphisms of Lie algebras appear naturally. Which automorphisms are of interest depends largely on the concrete system under study. In this paper we restrict our attention to the applications listed above. (The exposition mainly reviews the subject, but a few results, such as formula (3.10), have not appeared previously.) Before proceeding to those applications, we also should like to mention that in many circumstances, in particular when it comes to model-dependent issues, in the study of CFT the arguments are not, or not yet, rigorous, but involve some amount of heuristics – often concealed as physical intuition. Nevertheless, there do exist situations in which one can make precise statements and establish rigorous proofs. A well-known illustration is the WZW Verlinde formula, which will be mentioned in section 4. Below we will indicate the relevant literature where such proofs are available, but not go into the details of any of the proofs.

When studying the automorphisms of our interest we will also encounter the following general structures. First, given an \( \mathfrak{L} \)-representation \( R \) on the vector space \( V \) and an automorphism \( \omega \) of \( \mathfrak{L} \), there is a ‘twisted intertwiner’, i.e. a linear map \( \Theta_\omega \equiv \Theta_\omega^{(R)} \) from the \( \mathfrak{L} \)-module \( (R,V) \) to \( (R\circ \omega, V) \) satisfying

\[
(R\circ \omega)(x) \circ \Theta_\omega = \Theta_\omega \circ R(x)
\]

for all \( x \in \mathfrak{L} \). Second, when \( (R\circ \omega, V) \) happens to be isomorphic to \( (R, V) \) and when in addition \( (R, V) \) is a weight module with finite-dimensional weight spaces – for which the notion of a (formal) character \( \chi_V = \text{tr}_V q^H \) is available – then one also has a ‘twisted’ or ‘twining’ analogue

\[
\chi_\omega := \text{tr}_V \Theta_\omega q^H
\]

of \( \chi_V \). The twining characters (2.3) are generalized character-valued indices. Their coefficients in an expansion in powers of the formal variables \( q \) are elements in an extension of \( \mathbb{Z} \) by the eigenvalues of \( \omega \), i.e. in a cyclotomic field when \( \omega \) has finite order.

3 Enhancement of symmetries

Symmetry enhancement in quantum field theory refers to situations where one starts from a known model with symmetry \( \mathfrak{A} \) and attempts to obtain a new model with extended symmetry \( \mathfrak{A}^{\text{ext}} \supset \mathfrak{A} \). A general mechanism for doing so consists in promoting relatively local (charged) fields to observables. In CCFT, this amounts to include suitable intertwining operators into the vertex operator algebra \([DLM]\). As already announced, the extensions that we consider
here are extensions by simple current fields. That is, there is some simple current group $G$ of the original CFT model such that

$$\mathcal{H}_{G} = \bigoplus_{J \in G} \mathcal{H}_J$$

is the underlying vector space of the chiral algebra of the extended model. The chiral CFT model with enhanced symmetry is called the *simple current extension* of the original one.

A complete understanding of the extended chiral algebra $\mathfrak{A}^{\text{ext}}$ and its representation theory is in general not yet available. Among the ingredients of the extended model that one would like to establish are in particular:

- The label set $I^{\text{ext}} = \{ \mu^{\text{ext}} \}$ of $\mathfrak{A}^{\text{ext}}$-sectors.
- The decomposition $\mathcal{H}_{\mu^{\text{ext}}} = \bigoplus_{i} \mathcal{H}_{\mu_i}$ of $\mathfrak{A}^{\text{ext}}$-sectors viewed as $\mathfrak{A}$-sectors.
- The fusion rules $N_{\mu^{\text{ext}}}$ of the $\mathfrak{A}^{\text{ext}}$-model.
- The characters $\chi_{\mu^{\text{ext}}}$ of the $\mathfrak{A}^{\text{ext}}$-model and their modular $S$-transformation matrix.
- The chiral blocks and correlation functions of the $\mathfrak{A}^{\text{ext}}$-model.

To become more concrete, let us first collect a few basic facts [SY1, SY2] that are needed to understand simple current extensions. We start with a fusion ring containing a non-trivial group of simple currents. But the perspective is somewhat different from the exposition in section 2, in that the Verlinde conjecture (see below) is built in from the beginning; thus the matrix $S$ is provided by the modular transformation of characters, and then the fusion rules are *defined* through $S$ by formula (1.2).

The input information needed for the extension then consists of the following:

- A set $\{ \chi_{\mu} \}$ ($\mu \in I, |I| < \infty$) of functions of a complex variable $\tau$, convergent in the upper half-plane and forming a basis of a unitary module $V$ over $\text{SL}(2, \mathbb{Z})$ for which the generator $S$ (implementing $\tau \mapsto -1/\tau$) is symmetric and the generator $T$ (implementing $\tau \mapsto \tau+1$) is diagonal.
- A vacuum label $\Omega \in I$, satisfying $S_{\Omega, \mu} \in \mathbb{R}_{>0}$ for all $\mu \in I$.
- An involution $C: \mu \mapsto \mu^{\text{op}}$ on $I$ leaving $\Omega$ fixed and satisfying $CS = S^*$ and $CT = TC$.
- A subset $G \subseteq I$ such that $S_{J, \Omega} = S_{\Omega, J}$ and $T_{J} = T_{\Omega}$ for all $J \in G$.
- Numbers $N_{\lambda, \mu^{\text{op}}}$, defined by (1.2) through the modular transformation matrix $S$.
- A product `$\star$' on $\mathbb{C}[I]$ defined by formula (1.1).

Having these data at one’s disposal, many of the goals in the list given above can be reached. Concretely, one first proves [SY2]:

- $\mathbb{C}[I]$ endowed with the product `$\star$' is a fusion algebra $\mathcal{A}$.
- $G$ is a finite abelian group w.r.t. `$\star$' – the group of units in the basis $I$ of $\mathcal{A}$.
- $G$ organizes $I$ into orbits $[\mu] := \{ J\mu \mid J \in G \}$ with $J\mu := J \star \mu \in I$. The length of the orbits is $\ell_{\mu} = |G|/|S_{\mu}|$, with $S_{\lambda} := \{ J \in G \mid J\lambda = \lambda \}$ the stabilizer of $\lambda$ in $G$.
- The combination

$$Z^{(G)}(\tau) := \sum_{[\mu]} \left( |S_{\mu}| \cdot \left| \sum_{J \in G/S_{\mu}} \chi_{J, \mu}(\tau) \right|^2 \right)$$

is invariant under the $\text{PSL}(2, \mathbb{Z})$-action $\tau \mapsto \frac{a\tau + b}{c\tau + d}$.\footnote{Those notions appearing here that were not yet introduced will be explained later on.}
Furthermore, the following interpretation has been shown to be self-consistent in [FSS2] and, when combined with uniqueness results [Br, M¨ u] on the modularisation of premodular categories, can be established rigorously:

- $Z^{(G)}$ as given by (3.2) is the ‘diagonal modular invariant’

$$Z^{(G)}(\tau) = Z^{xt}(\tau) := \sum_{\mu \in \Gamma} |\chi^{xt}_\mu(\tau)|^2$$

(3.3)

for the simple current extension.

- The label set $I^{xt}$ consists of all equivalence classes of pairs $[\mu, \psi]$ with $\mu \in I$ obeying $T_{1\mu} = T_\mu$ and $\psi$ a character of the central stabilizer

$$U_\mu := \{ J \in \mathcal{S}_\mu : F_\mu(J, J') = 1 \forall J' \in \mathcal{S}_\mu \} \subseteq \mathcal{S}_\mu .$$

(3.4)

Here $F_\mu$ is a certain alternating bihomomorphism on $\mathcal{S}_\mu$ (see formula (3.10) below), and the classes $[\mu, \psi]$ are with respect to the action $(\mu, \psi) \mapsto (J\mu, F_\mu(J, \cdot)\psi)$ for $J \in \mathcal{G}$.

- The functions $\chi^{xt}$ are given by

$$\chi^{xt}_{[\mu, \psi]} = d_\mu \sum_{J \in \mathcal{G}/\mathcal{S}_\mu} \chi^{xt}_\mu$$

(3.5)

with $d_\mu$ the square root of the embedding index of $U_\mu \subseteq \mathcal{S}_\mu$. These functions can be interpreted as the characters of simple modules $\mathcal{H}^{xt}_{[\mu, \psi]}$ over the extended VOA.

- Thus the summands in the expression (3.2) are to be read as $|U_\mu| \cdot |\chi^{xt}_{[\mu, \psi]}|^2$, i.e. to each orbit $[\mu]$ there are associated $|U_\mu|$ many extended irreducible characters $\chi^{xt}_{[\mu, \psi]}$. Correspondingly, considered as an $\mathcal{A}$-module, the $\mathcal{A}^{xt}$-module $\mathcal{H}^{xt}_{[\mu, \psi]}$ decomposes as

$$\mathcal{H}^{xt}_{[\mu, \psi]} = \bigoplus_{J \in \mathcal{G}/\mathcal{S}_\mu} C^{d_\mu} \otimes \mathcal{H}_{J\mu} .$$

(3.6)

- The functions $\chi^{xt}_{[\mu, \psi]}$ (3.5) span a unitary $\text{SL}(2,\mathbb{Z})$-module. Their $S$-transformation matrix $S^{xt}$ is obtained by sandwiching certain matrices $S^1$ between characters of central stabilizers:

$$S^{xt}_{[\lambda, \psi_x],[\mu, \psi_\mu]} = \frac{|\mathcal{G}|}{|\mathcal{S}_x| |\mathcal{U}_x| |\mathcal{S}_\mu| |\mathcal{U}_\mu|}^{1/2} \sum_{J \in \mathcal{U}_x \cap \mathcal{U}_\mu} \psi_\mu(J) S^1_{\lambda, \mu} \psi_x(J)^* .$$

(3.7)

The matrices $S^1$ occurring here are those that describe the modular $S$-transformation of the one-point chiral blocks $B^{(i)}_J$ with insertion $J$ on an elliptic curve. (The rows and columns of $S^1$ are labelled by $\mu \in I \setminus \mathcal{S}_\mu \ni J$, and $S^1$ obeys the $\text{SL}(2,\mathbb{Z})$ relations as well as $(S^1)^t = S^{−1}$. $S^\Omega = S$.)

- $S^{xt}$ as given by (3.1) is proven to be unitary and symmetric and to satisfy the $\text{SL}(2,\mathbb{Z})$ relations. With the help of the computer program kac $\text{Sch}$ it was also checked in a huge...
number of examples that (3.7) produces non-negative integers when inserted into formula (1.2) [FSS2].

Several further comments are in order:

- There is evidence that for WZW models (and similarly for coset models), up to a calculable phase \( S^J \) is the Kac-Peterson matrix of the orbit Lie algebra that is associated [FSS1, FRS] to \( g \) and J via a folding of the Dynkin diagram. Hence for these models all \( S^J \) are known explicitly.

- By general results on the cohomology of finite abelian groups, the alternating two-cocycle \( F_\mu \) is the commutator cocycle for some \( F_\mu \in H^2(S_\mu, U(1)) \), i.e. satisfies \( F_\mu(J, J') = F_\mu(J, J') / F_\mu(J', J) \) [Ban, Mu, FS2]. Thus the group algebra \( C_U_\mu \) is isomorphic to the center of the twisted group algebra \( C_F_\mu S_\mu \), implying e.g. that the inclusion \( U_\mu \subseteq S_\mu \) is of square index \( d^2_\mu \), with \( d_\mu \) the dimension of the irreducible \( C_F_\mu S_\mu \)-representations. (The occurrence of non-trivial cohomology classes \( F_\mu \) can be regarded as a manifestation of the fact that in quantum theory symmetries are in general only realized projectively.)

- It follows directly from the definition of \( S^J \) as the modular S-transformation matrix for the chiral blocks \( B_1^{(1)} \) that

\[
\frac{S^K_{\lambda, \mu}}{S^{\Omega, \lambda}_\mu} = \frac{S_{\mu, \lambda}^{(1)}}{S_{\mu, \lambda}^{(1)}} \cdot F_{J, \lambda, \mu}^{(K)} \cdot \frac{S_{\mu, \lambda}^{(1)}}{S^{\Omega, \lambda}_\mu}, \tag{3.8}
\]

where \( F_{J, \lambda, \mu}^{(K)} \) is an entry of a so-called fusing matrix \( F \), i.e. a 6j-symbol of the modular tensor category of the original CFT. One can also show [FSS2] that

\[
S^J_{J, \lambda, \mu} = (T_\mu / T_{J, \mu}) \cdot F_\mu(J, J') \cdot S^J_{J, \lambda, \mu}, \tag{3.9}
\]

which by comparison with (3.8) tells us that

\[
F_\lambda(J, K) = F_{J, \lambda, \mu}^{(K)} \cdot \frac{S_{\mu, \lambda}^{(1)}}{S^{\Omega, \lambda}_\mu}. \tag{3.10}
\]

- In connection with boundary conditions also simple currents of conformal weight \( \Delta \in \mathbb{Z} + \frac{1}{2} \) turn out to be relevant. In that case \( F_\mu \) is no longer a commutator cocycle. But after combining it with so-called discrete torsion [KS], one arrives again at a cohomological interpretation [FHS^2W].

- Recall the relation (2.1) between the simple currents of the WZW model based on \( g \) and the center of the Lie group \( G \) (for which \( \text{Lie}(G) \) is the compact real form of the horizontal subalgebra \( \bar{g} \) of \( g \)). It says that (except for \( g = E_8 \) and level 2)

\[
\mathcal{G}(g) \cong \mathcal{Z}(G) \cong L^w / L^\vee \tag{3.11}
\]

with \( L^w \) and \( L^\vee \subseteq L^w \) the coweight and coroot lattices of \( \bar{g} \), respectively. (\( \mathcal{Z}(G) \), and hence \( \mathcal{G}(g) \), is also naturally isomorphic to the maximal normal abelian subgroup of the group of symmetries of the Dynkin diagram of the affine Lie algebra \( g \).) It is therefore not surprising that the matrix \( S^{xt} \) appears in the Verlinde formula for non-simply connected quotients \( \hat{G} \) of \( G \). For some of the matrix elements \( S^{xt} \) the prediction (3.7) was checked in [Be2] for \( G = \text{PGL}(n, \mathbb{C}) \) with

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8 This is based on the identity between twining characters for \( g \) and ordinary characters for the orbit Lie algebra, which was established in [FSS3, FRS] for Verma modules and simple highest weight modules. Results on other classes of modules can be found in [N, KKW, KN], and related work in [Schw, BFM, We].
n prime.

Among the labels $\mu^x = [\mu, \psi]$ associated to a given orbit $[\mu]$ there is no distinguished one. Accordingly, the degeneracy labels $\psi$ should better not be regarded as elements of the character group $U^*_\mu$, but rather as elements of the torsor over that group. Insisting on the description in terms of $U^*_\mu$, certain non-canonical basis choices are implied. A formulation free of such non-canonical choices can be achieved [Bai] by expressing $\mu^x$ through suitable idempotents in morphism spaces, similarly as in [Br, Mu].

4 Simple current automorphisms of block algebras and chiral blocks

The modular tensor category summarizes the basic quantities of a chiral CFT model, such as the set $I$ of sector labels, the fractional part of conformal weights $\Delta$, the fusion rules with their diagonalizing matrix $S$, and also the fusing matrices $F$. It contains more sophisticated information as well, in particular about the chiral blocks of the model on arbitrary curves $\hat{X}$. The chiral (or conformal) blocks $B_{\vec{\mu}}(\hat{X}_g)$ for a genus-$g$ curve $\hat{X}_g$ with distinct marked points $p_1, p_2, \ldots, p_m$ and associated sector labels $\mu_1, \mu_2, \ldots, \mu_m \in I$ are specific linear forms $B_{\vec{\mu}}: \vec{H}_{\vec{\mu}} \rightarrow \mathbb{C}$.

Namely, they form the space of co-invariants with respect to the action of the block algebra $\mathfrak{A}_{\hat{X}_g, \vec{\mu}}$ that expresses the chiral symmetries globally on $\hat{X}_g$:

$$\vec{H}_{\vec{\mu}} := \mathcal{H}_{\mu_1} \otimes \mathcal{H}_{\mu_2} \otimes \cdots \otimes \mathcal{H}_{\mu_m}.$$ (4.1)

Alternatively, by duality one may think of the chiral blocks as the invariants $((\vec{H}_{\vec{\mu}})^*)^{\mathfrak{A}_{\hat{X}_g, \vec{\mu}}}$ in the algebraic dual of $\vec{H}_{\vec{\mu}}$. In physical terms, the invariance condition amounts to imposing the Ward identities for the symmetries.

For WZW models, the block algebra is a Lie algebra, which is the tensor product [TUY, U, Bel]

$$\mathfrak{A}_{\hat{X}, \vec{\mu}} = \mathfrak{g} \otimes \mathcal{O}(\hat{X} \setminus \vec{p})$$ (4.3)

of the horizontal subalgebra $\mathfrak{g}$ of the affine Lie algebra $\mathfrak{g}$ with the associative algebra $\mathcal{O}(\hat{X} \setminus \vec{p})$ of meromorphic functions on $\hat{X}$ whose singularities are at most finite order poles at the marked points $p_s$. The action of the algebra (4.3) on $\vec{H}_{\vec{\mu}}$ follows via Laurent expansion of the functions in $\mathcal{O}(\hat{X} \setminus \vec{p})$ in local coordinates around the $p_s$.

The corresponding prescription in the general case is considerably more involved [Fr]. One introduces a bundle $\mathcal{A}$ over $\hat{X}$ whose fibers are isomorphic to the vector space $\mathcal{H}_\Omega$ of the VOA, and requires that for every $m$-tuple of vectors $v_i \in \mathcal{H}_{\mu_i}$ and every $w \in \mathcal{H}_\Omega$ the section $B_{\vec{\mu}}(v_1 \otimes \cdots \otimes v_m) \mathcal{Y}(w, \cdot) v_1 \otimes \cdots \otimes v_m$ (with $\mathcal{Y}$ a section of $\mathcal{A}^*$ that takes over the role of the state-field correspondence) in the restriction of $\mathcal{A}^*$ over local disks around the points $p_s$ can be extended to a global holomorphic section on the punctured curve $\hat{X}_g \setminus \vec{p}$. In view of this general
prescription, it is actually a non-trivial statement that in WZW models it is sufficient to impose only the Ward identities coming from the affine Lie algebra $\mathfrak{g}$.

The following properties of the block spaces $B_{\vec{\mu}}(\hat{X}_g)$ have been proven in the WZW case \cite{TUY} and are expected to hold for arbitrary (rational, and partly even for non-rational) CFT models:

- For fixed $\vec{\bar{\mu}} = (p_1, \ldots, p_m)$ and fixed moduli of $\hat{X}_g$, $B_{\vec{\mu}}(\hat{X}_g)$ is a finite-dimensional vector space.
- For fixed $\mu = (\mu_1, \ldots, \mu_m)$ and fixed genus $g$ these spaces fit together to the total space of a vector bundle $B_{\vec{\mu}}(g)$ of finite rank over the moduli space $\mathcal{M}_{g,m}$ of complex curves of genus $g$ with $m$ ordered marked points. In particular, the dimension of $B_{\vec{\mu}}(\hat{X}_g)$ depends only on $g$ and on $\vec{\mu}$, but neither on the moduli of the curve $\hat{X}_g$ nor on the positions of the marked points.
- The dimensions of the genus zero 3-point chiral blocks $B_{\lambda,\mu,\nu}^{(0)}$ are given by the fusion rules $N_{\lambda,\mu,\nu}^{(0)}$ \cite{TUY}, and those for higher genera and/or more insertions can be expressed through the fusion rules by simple factorization prescriptions. Together with (1.2), this results in the formula

$$\text{rank } B_{\vec{\mu}}^{(0)} = \sum_{\nu \in I} |S_{\Omega,\nu}|^{2-2g} \prod_{s=1}^{m} \frac{S_{\mu_s,\nu}}{S_{\Omega,\nu}},$$

where $S$ is the matrix that according to (1.2) diagonalizes the fusion rules.

- Each bundle $B_{\vec{\mu}}^{(0)}$ comes equipped with a projectively flat connection, the Knizhnik–Zamolodchikov connection. This implies a projective action of the mapping class group on $B_{\vec{\mu}}(\hat{X}_g)$.
- In the case of an elliptic curve and with a single ‘insertion’ $\mu_1 = \Omega$, this furnishes a unitary projective representation of the modular group $\text{PSL}(2,\mathbb{Z})$ on the 1-point chiral blocks $B_{\vec{\mu}}^{(1)} \cong \mathbb{C}[I]$.
- In a natural basis, the generator $S: \tau \mapsto -1/\tau$ of $\text{PSL}(2,\mathbb{Z})$ is represented by a symmetric matrix $S$. This matrix coincides with the diagonalizing matrix that appears in (1.4), justifying that we already used the same symbol for both of them.
- The Verlinde conjecture states that this matrix $S$ also coincides with the matrix describing the modular $S$-transformation $\tau \mapsto -1/\tau$ on the characters $\chi_{\mu}$. This conjecture is proven (see e.g. \cite{Be1, Fa, Te, Fi, So}) for WZW models for which the modular $S$-matrix is given by the Kac–Peterson formula.

There is no reason to expect that the vector bundles $B_{\vec{\mu}}^{(g)}$ are irreducible. Indeed any automorphism of $B_{\vec{\mu}}^{(g)}$ constitutes trivially a source for reducibility, namely into a direct sum of its eigenspaces. Such automorphisms can e.g. be inherited from suitable automorphisms of the relevant block algebra $\mathfrak{A}_{\vec{X},\vec{\bar{\mu}}}$. What is non-trivial is the observation that such automorphisms of $\mathfrak{A}_{\vec{X},\vec{\bar{\mu}}}$ arise naturally, namely as a consequence of the presence of simple currents.

This can be made explicit for WZW models, for which $\mathfrak{A}_{\vec{X},\vec{\bar{\mu}}}$ takes the form (1.3). Then there are automorphisms of $\mathfrak{A}_{\vec{X},\vec{\bar{\mu}}}$ coming from simple current automorphisms of the underlying affine Lie algebra $\mathfrak{g}$. They are constructed as follows \cite{FS1}. For $g = 0$ and any $m \geq 2$ one has a group of automorphisms of $\mathfrak{g}$ labelled by the set

$$\Gamma_w := \{ (\vec{\bar{\nu}}_1, \vec{\bar{\nu}}_2, \ldots, \vec{\bar{\nu}}_m) \in (L_w)^m \mid \sum_{s=1}^{m} \vec{\bar{\nu}}_s = 0 \}$$

($L_w$ the coweight lattice of $\mathfrak{g}$). These automorphisms $\sigma_{\vec{\bar{\nu}}}$ depend in addition on a sequence of pairwise distinct complex parameters $z_s$ ($s = 1, 2, \ldots, m$), one of which, say $z_1$, is singled out. $\sigma_{\vec{\bar{\nu}}}$ acts on the canonical central element $K$ and on the elements $H^i \circ f$ and $E^a \circ f$ of $\mathfrak{g} = \mathbb{C}(t) \oplus \mathbb{C}K$.

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(f ∈ ℂ((t))) and \( \{ H^i \mid i = 1,2,\ldots, \text{rank}(\mathfrak{g}) \} \cup \{ E^\alpha \mid \bar{\alpha} \text{ a g-root} \} \) a Cartan–Weyl basis of \( \mathfrak{g} \) as
\[
\sigma_\bar{\nu}(H^i \circ f) := H^i \circ f + K \sum_{s=1}^m \bar{\nu}_s \text{Res}(\varphi_{1,s} f),
\sigma_\bar{\nu}(E^{\beta} \circ f) := E^{\beta} \circ f \cdot \prod_{s=1}^m (\varphi_{s,\bar{\nu}})^{-1(\phi_s,\bar{\beta})}, \quad \sigma_\bar{\nu}(K) := K
\]
with
\[
\varphi_{1,s}(t) := (t + z_1 - z_s)^{-1}.
\]

Here \( \text{Res} \) denotes the residue of a formal Laurent series in \( t \), and we employ the notation \( f^\ell \) for the function with values \( f^\ell(t) = (f(t))^\ell \). The *multi-shift automorphisms* \((4.6)\) of \( \mathfrak{g} \) are close relatives of the ordinary shift automorphisms – studied e.g. in \([FH, GRM, LVW]\) – that can be recovered from formula \((4.6)\) by setting \( \bar{\nu}_s = 0 \) for \( s \neq 1 \) (which does not belong to \( \Gamma_w \), though).

The set \( \Gamma_w \) is an abelian group with respect to component-wise addition, isomorphic to \((L_w^\vee)^{m-1}\). The automorphisms \((4.6)\) form a group isomorphic to \( \Gamma_w \). Further, the function \( \varphi_{1,s} \) can be recognized as the local expansion at \( p_1 \) of the function \( \varphi_{(s)} \in \mathcal{F}_{\bar{\nu},\mathbb{C}P^1} \) defined by
\[
\varphi_{(s)}(z) := (z - z_s)^{-1},
\]
where \( z \) is a quasi-global coordinate on \( \mathbb{C}P^1 \) and \( z_s = z(p_s) \). It follows \([FS1]\) that to every \( \bar{\nu} \in \Gamma_w \) there is also associated an automorphism \( \tilde{\sigma}_{\bar{\nu}} \) of the block algebra \( \mathfrak{g} \otimes \mathcal{O}(\mathbb{C}P^1 \setminus \bar{\nu}) \), acting as
\[
\tilde{\sigma}_{\bar{\nu}}(H^i \circ f) = H^i \circ f, \quad \tilde{\sigma}_{\bar{\nu}}(E^{\beta} \circ f) = E^{\beta} \circ f \cdot \prod_{s=1}^m (\varphi_{(s)})^{-1(\phi_s,\bar{\beta})}.
\]

Up to central terms the local expansions of the automorphism \((4.9)\) at the marked points reproduce the automorphisms \((4.6)\) of \( \mathfrak{g} \).

Every automorphism \( \sigma_{\bar{\nu}} \) of \( \mathfrak{g} \) induces a permutation \( \sigma_{\bar{\nu}}^* \) of the label set \( I \). Further, for every \( \bar{\nu} \in \Gamma_w \) and every \( \bar{\mu} \in I^m \) there is an induced map
\[
\Theta_{\bar{\nu}} : \bar{\mathcal{H}}_{\bar{\mu}} \rightarrow \bar{\mathcal{H}}_{\bar{\sigma}_{\bar{\nu}}^* \bar{\mu}},
\]
unique up to a scalar, which is a twisted intertwiner in the sense of \((2.2)\). This descends to a map \( \Theta_{\bar{\nu}}^* \) from the chiral blocks \( B_{\bar{\mu}} \) to \( (\bar{\mathcal{H}}_{\bar{\sigma}_{\bar{\nu}}^* \bar{\mu}})^* \), which is in fact an isomorphism to \( B_{\bar{\sigma}_{\bar{\nu}}^* \bar{\mu}} \). The map on the co-invariants obtained this way from any *inner* automorphism of the block algebra is just the identity. Thus for the action on chiral blocks we are interested in the outer automorphism class of \( \bar{\sigma}_{\bar{\nu}} \) \((4.9)\). Now \( \bar{\sigma}_{\bar{\nu}} \) is outer iff at least one of the vectors \( \bar{\nu}_s \) in the tuple \( \bar{\nu} \in \Gamma_w \) is not a coroot, so the group of outer automorphism classes is the factor group
\[
\Gamma_{\text{out}} := \Gamma_w / \Gamma \quad \text{with} \quad \Gamma := \{(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m) \in (L^\vee)^m \mid \sum_{s=1}^m \bar{\beta}_s = 0\}.
\]
\( \Gamma_{\text{out}} \) is a finite abelian group, obeying (compare formula \((3.11)\))
\[
\Gamma_{\text{out}} \cong (L_w^\vee/L^\vee)^{m-1} \cong (\mathcal{G}((\mathfrak{g})))^{m-1},
\]

\(^9\) The central terms are needed in order to have an automorphism of the affine Lie algebra rather than only of the corresponding loop algebra. Accordingly, the centers of the \( m \) copies of \( \mathfrak{g} \) associated to the marked points \( p_s \) must be identified; then upon summing over all insertion points these terms cancel by the residue theorem.
with $\mathcal{G}$ the group of simple currents of the WZW model. Thus we can view the elements of $\Gamma_{\text{out}}$ as $m$-tuples of simple currents $J_s$ subject to $J_1*J_2*\cdots*J_m = \Omega$, and accordingly write the map $\Theta_{\tilde{\mu}}^*$ as

$$\Theta_{J_1,\ldots,J_m}^*: B_{\tilde{\mu}} \to B_{\tilde{\sigma} \tilde{\mu}}.$$  

(4.13)

When $J_s \in \mathcal{S}_{\mu_s}$ for all $s = 1, 2, \ldots, m$, then the map $\Theta_{J_1,\ldots,J_m}$ is an automorphism of $B_{\tilde{\mu}}$, and in particular one can study its trace. So for every $\tilde{\mu} \in \mathfrak{f}^m$ the group $\mathcal{S}_{\tilde{\mu}} := \Gamma_{\text{out}} \cap (\bigotimes_{s=1}^m \mathcal{S}_{\mu_s})$ acts on $B_{\tilde{\mu}}$ by the maps $\Theta_{J_1,\ldots,J_m}$. Since the twisted intertwining property determines the maps $\Theta_{\tilde{\mu}}$ (4.10), and hence also the maps $\Theta_{J_1,\ldots,J_m}$ (4.13), only up to a phase, our construction furnishes in general only a projective representation of $\mathcal{S}_{\tilde{\mu}}$ on $B_{\tilde{\mu}}$. We denote by $\mathcal{F}_{\tilde{\mu}}$ the class in $H^2(\mathcal{S}_{\tilde{\mu}}, U(1))$ that specifies the projectivity.

The cohomology class $\mathcal{F}_{\tilde{\mu}}$ depends on $\tilde{\mu}$, as well as a priori on the insertion points $\tilde{\mu}$. But one can choose [FS1] these phases in such a manner that $\mathcal{F}_{\tilde{\mu}}$ does in fact not depend on $\tau$ and $\tilde{\mu}$. Also, with this choice the map $\Theta_{J_1,\ldots,J_m}$ has finite order, and the choice is compatible with the Knizhnik–Zamolodchikov connection on $B_{\tilde{\mu}}$. Thus for every element $\tilde{J} \in \mathcal{S}_{\tilde{\mu}}$ we have a finite order automorphism of the chiral block bundle $B_{\tilde{\mu}}$ over the moduli space $\mathcal{M}_{0,m}$. We will use the same symbol $\Theta_{J_1,\ldots,J_m}$ for this bundle automorphism as for the vector space automorphisms of the individual fibers $B_{\tilde{\mu}}$ of the bundle.

We are interested in the ranks of the sub-bundles that are invariant under the action of $\Theta_{J_1,\ldots,J_m}$; they depend on the tuple $\tilde{J}$ of simple currents. A hint on how this dependence looks like can be obtained by inspecting the Verlinde formula for simple current extensions of WZW models, in which the matrix (4.7) appears. This leads to a concrete conjecture for the ranks of the sub-bundles in the case of genus 0, which can be extended to higher genera by analogy with the Verlinde formula.

We first formulate a conjecture for the cohomology class of $\mathcal{F}_{\tilde{\mu}}$, in terms of its commutator cocycle $F_{\tilde{\mu}}$: We propose that $F_{\tilde{\mu}}$ is the product of the commutator cocycles of the factors:

$$F_{\tilde{\mu}} = \prod_{s=1}^m F_{\mu_s},$$  

(4.14)

with $F_{\mu_s}$ given by (3.10). It is then natural to introduce the ‘central stabilizer’ (compare formula (3.4)) $\mathcal{U}_{\tilde{\mu}} := \{ \tilde{J} \in \mathcal{S}_{\tilde{\mu}} | F_{\tilde{\mu}}(\tilde{J}, \tilde{J}') = 1 \forall \tilde{J}' \in \mathcal{S}_{\tilde{\mu}} \}$, for which our conjecture amounts to

$$\mathcal{U}_{\tilde{\mu}} = \Gamma_{\text{out}} \cap (\bigotimes_{s=1}^m \mathcal{U}_{\mu_s}).$$  

(4.15)

For each character $\psi$ of $\mathcal{U}_{\tilde{\mu}}$, there is now an invariant sub-bundle $B_{\psi}$ of the bundle of chiral blocks. (A word of warning is in order: The correspondence of characters $\psi$ and sub-bundles is not canonical.\footnote{The requirement that $F_{\tilde{\mu}}$ be constant on $\mathcal{M}_{0,m}$ does not yet determine the phase choices for the maps $\Theta_{J_1,\ldots,J_m}$ uniquely. Different allowed choices result in different values for the traces. But one can show that this difference simply amounts to a relabelling of the eigenspaces. So again one deals with the torsor over the character group $\mathcal{U}_{\tilde{\mu}}^*$ rather than with $\mathcal{U}_{\tilde{\mu}}$ itself.}) Our (conjectural) formula for the rank of these sub-bundles, however, depends on precisely the same non-canonical choices.

Rather than describing the ranks of the sub-bundles directly, it proves to be more convenient to perform a Fourier transform over the central stabilizer $\mathcal{U}_{\tilde{\mu}}$ and to give instead the traces of the maps $\Theta_{J_1,\ldots,J_m}$. Our conjecture is then formulated as follows.
Conjecture 1: The trace on $B^{(g)}_{\mu}$ of the twisted intertwiner $\Theta^*_{J_1,\ldots,J_m}$ associated with the $m$-tuple $(J_1, J_2, \ldots, J_m) \in U_{\mu}$ is

$$\text{tr}_{B^{(g)}_{\mu}} \Theta^*_{J_1,\ldots,J_m} = \sum_{\nu \in I} |S_{\Omega,\nu}|^{2-2g} \prod_{s=1}^{m} \frac{S_{\mu_s,\nu}}{S_{\Omega,\nu}}.$$  (4.16)

Here $S^J$ is the matrix introduced in (3.7), i.e. the modular S-transformation matrix for the one-point chiral blocks $B_J^{(1)}$. Also recall that in WZW models $S^J$ is conjectured to coincide (up to a calculable phase) with the Kac-Peterson matrix for the orbit Lie algebra that is associated to $g$ and $J$; inserting this relation into (4.16), one arrives at a numerical prediction for $\text{tr}_{B^{(g)}_{\mu}} \Theta^*_{J_1,\ldots,J_m}$ that is as explicit as the Verlinde formula.

Note that the basic ingredients in the formula (4.16) are closely related to simple current extensions. Since that construction works for arbitrary CFT models, not only in the WZW case, the conjecture (4.16) can be extended to every (rational) CFT model, too. But already in the WZW case one is still far from being able to prove the formula. In particular, since $S^\Omega$ is the ordinary matrix $S$, for $J_1 = \cdots = J_m = \Omega$, (4.16) reduces to the Verlinde formula (4.4) for the trace of $\Theta^* = \text{id}$. So any proof of (4.16) is a fortiori also a proof of the Verlinde formula itself. On the other hand, as is immediately checked, (4.16) is compatible with factorization. Moreover, in the WZW case there is enormous numerical evidence. Namely, for very many cases it has been checked on the computer that the numbers obtained by Fourier transforming the traces (4.16) are non-negative integers, as required for the interpretation as dimensions of eigenspaces, even though they are obtained as complicated combinations of arbitrary complex numbers.

Surprisingly, these numerical studies reveal that already the traces themselves are integral (they may be negative, though), even when the order of the automorphism exceeds two. So far no explanation of this empirical observation is available. But there is some reminiscence with the interpretation of the dimensions of block spaces as the Euler number of a suitable (BGG-like) complex, with non-negativity implied by an additional acyclicity property \[ \text{[Te]} \]. One may hope that a similar description can be found for the traces as well.

5 Restriction of observables

We now switch to our second main topic. The basic idea is just the reverse of what we have done in section 3. One starts from a QFT model with observables $\mathfrak{A}$ and a group $G$ of automorphisms of $\mathfrak{A}$, and desires to construct the $G$-orbifold of the theory, by which one means some new QFT model whose observables are given by the fixed point subalgebra $\mathfrak{A}^G$ of $\mathfrak{A}$ with respect to $G$.

In CFT, where the observables are realized as a VOA, it is natural to demand that the orbifold is again a CFT model. This requires in particular that the fixed point algebra $\mathfrak{A}^G$ contains a Virasoro element, and in fact this must coincide with the Virasoro element $v_{\text{vir}}$ of $\mathfrak{A}$. Note that this condition implies that none of the elements of $G$ can be a (non-trivial) simple current automorphism.

\[ ^{11} \text{As a consequence of } |I| < \infty \text{ they are in fact integers in a cyclotomic extension of } \mathbb{Q}. \]
In the orbifold model, the $\mathfrak{A}$-representations $R$ and $R^\sigma \equiv R \circ \sigma$ with $\sigma \in G$ become indistinguishable, and hence describe the same sector (or collection of sectors) of the orbifold. On the other hand, when $R^\sigma$ is isomorphic to $R$ already as an $\mathfrak{A}$-representation, then $R$, even when it is $\mathfrak{A}$-irreducible, becomes reducible in the orbifold. Several distinct orbifold sectors are then obtained by splitting such an $\mathfrak{A}$-sector into eigenspaces of the twisted intertwiner $\Theta_\sigma$ that implements $\sigma$ on the sector. But in addition there are also further orbifold sectors that are not obtainable by decomposing $\mathfrak{A}$-sectors. They are called twisted sectors.

Again, the goal is to express all quantities of interest of the new theory, i.e. the orbifold, ‘in terms of’ the original model. Due to the presence of twisted sectors this can be quite hard. Fortunately, it can be expected that the characters of twisted sectors are obtainable via the modular $S$-transformation of (differences of) the characters of untwisted sectors. This relationship provides a powerful tool for concrete calculations, and we will assume that it is indeed satisfied. Once this assumption is made, a rigorous derivation of the results collected below is possible \cite{BFS}.

Every $\sigma \in G$ induces a permutation $\sigma^*$ of the label set $I$ such that $\mathcal{H}_\sigma^\mu \cong \mathcal{H}_{\sigma^*\mu}$ as $\mathfrak{A}$-modules; $\sigma^*$ is an automorphism of fusion rules. From now on we restrict our attention to orbifolds by a $\mathbb{Z}_2$-group. Then one must only distinguish between length-two $\sigma^*$-orbits of $\mathfrak{A}$-sectors and ‘fixed points’ $\mu = \sigma^*\mu$. One finds:

- Every length-two orbit $\{\mu, \sigma^*\mu\}$ gives rise to a single untwisted orbifold sector, which we denote by $(\mu, 0, 0)$ and whose character is $\chi_{\mu,0,0}^\sigma(\tau) = \chi_\mu(\tau)$.
- Every fixed point $\mu$ yields two distinct untwisted orbifold sectors $(\mu, \psi, 0)$, $\psi \in \{\pm 1\}$. Their characters read

$$\chi_{(\mu,\psi,0)}^O(\tau) = \frac{1}{2} \text{tr}_{\mathcal{H}_\mu} ( (1 + \psi \Theta_\sigma) q^{L_0-c/24}) = \frac{1}{2} (\chi_\mu(\tau) + \psi \eta_\mu^{-1} \chi^\sigma_\mu(2\tau)). \quad (5.1)$$

Here we write $\eta_\mu^{-1} \chi^\sigma_\mu(2\tau)$ for the twining character $\chi^\sigma_{\mu}(\tau)$ because for suitable choice of the phases $\eta_\mu$ the functions $\chi_\mu(\tau)$ introduced this way possess an expansion with integral powers of $q = e^{2\pi i \tau}$. Also, $T_{(\lambda,\psi,0)}^O = T_{\lambda}^*$.

- In addition, each fixed point also gives rise to two twisted orbifold sectors, which we label as $(\hat{\mu}, \psi, 1)$ with $\psi \in \{\pm 1\}$, with characters

$$\chi_{(\hat{\mu},\psi,1)}^O(\tau) = \frac{1}{2} \left( \chi_{\hat{\mu}}^{(1)}(\tau) \pm \psi (T_{\hat{\mu}}^{(1)})^{-1/2} \chi_{\hat{\mu}}^{(1)}(\frac{\tau+1}{2}) \right). \quad (5.2)$$

The labels $\hat{\mu}$ appearing here are in one-to-one correspondence with the fixed points $\mu$. But generically this correspondence is not canonical, so that in particular we cannot dispense of using two different kinds of labels for the functions $\chi^0$ and $\chi^{(1)}$.

- The functions $\chi_{\hat{\mu}}^{(1)}(\tau)$ possess an expansion with integral powers of $q$, too. The T-transformation of the twisted orbifold characters therefore reads

$$\chi_{(\hat{\mu},\psi,1)}^O(\tau+1) = \psi (T_{\hat{\mu}}^{(1)})^{1/2} \chi_{(\hat{\mu},\psi,1)}^O(\tau), \quad (5.3)$$

so that we have $T_{(\lambda,\psi,1)}^O = \psi (T_{\lambda}^{(1)})^{1/2}$.

- The twisted and untwisted sectors transform into each other under an $S$-transformation. For the functions $\chi^{(0)}$ and $\chi^{(1)}$, this amounts to the relations

$$\chi_\lambda^{(0)}(\frac{-1}{\tau}) = \sum_{\hat{\mu}} S_{\lambda,\hat{\mu}}^{(0)} \chi_{\hat{\mu}}^{(1)}(\tau) \quad \text{and} \quad \chi_\lambda^{(1)}(\frac{-1}{\tau}) = \sum_{\mu} S_{\lambda,\mu}^{(1)} \chi_\mu^{(0)}(\tau). \quad (5.4)$$

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with a unitary matrix $S^{(0)}$ and $S^{(1)}_{\lambda,\mu} = \eta^{-2\pi} S^{(0)}_{\mu,\lambda}$.

It follows that the modular $S$-transformation matrix of the orbifold is given by

$$S^{(p)}_{\lambda,\mu,0,0} = S_{\lambda,\mu} + S_{\lambda,\sigma^p \mu}, \quad S^{(p)}_{\lambda,\mu,0,0} = \frac{1}{2} S_{\lambda,\mu},$$

$$S^{(p)}_{\lambda,\mu,0,0} = S_{\lambda,\mu}, \quad S^{(p)}_{\lambda,\mu,0,0} = \frac{1}{2} \psi \eta^{-1} S^{(0)}_{\lambda,\mu},$$

$$S^{(p)}_{\lambda,\mu,0,0} = 0, \quad S^{(p)}_{\lambda,\mu,0,0} = \frac{1}{2} \psi \eta' P_{\lambda,\mu}$$

with

$$P := (T^{(1)})^{1/2} (S^{(0)})^t (\eta^{-1} T^{(0)})^2 S^{(0)} (T^{(1)})^{1/2}.$$  

$S^{(1)} S^{(0)}$ is an order-2 permutation which up to sign factors coincides with $P^2$, while $\eta^{-1} S^{(0)} S^{(1)} \eta$ is an order-2 permutation up to sign factors.

To compute the functions $\chi^{(0)}_\mu$ and their modular transformations, sufficiently detailed representation theoretic information is needed. Therefore we now specialize again to the WZW case. Then we are dealing with automorphisms of an untwisted affine Lie algebra $\mathfrak{g}$, which must preserve the (Sugawara) Virasoro algebra. Every such map comes from an automorphism, for brevity to be again denoted by $\sigma$, of the horizontal subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. When $\sigma$ has finite order, then in a suitable Cartan–Weyl basis of $\mathfrak{h}$ it can be written as

$$\sigma = \sigma_0 \circ \sigma_s,$$

with $\sigma_0$ a diagram automorphism and $\sigma_s = \exp(2\pi i \text{ad}_{H_s})$ an inner automorphism. $H_s \equiv (s, H)$ is an element of the Cartan subalgebra satisfying $\sigma(H_s) = H_s$. (The automorphisms of finite-dimensional simple Lie algebras have been classified [OV]. For a list of all order-2 automorphisms see e.g. table 1 of [BFSS].)

Further, we know that $\sigma_s^* = \text{id}$, hence $\sigma^* = \sigma_s^*$, and the phases $\eta_\mu$ are given by $\eta_\mu = \exp(2\pi i (s, \mu))$. Finally, the twining characters $\chi^{(0)}_\mu(\tau) = \eta^{-1}_\mu \chi^{(0)}_\mu(2\tau)$ coincide with ordinary characters of the orbit Lie algebra $\mathfrak{g}[\sigma]$ that is associated to $\mathfrak{g}$ and $\sigma$ and hence are known very explicitly. The orbit Lie algebras arising here turn out to be twisted affine Lie algebras. More precisely, $\mathfrak{g}[\sigma]$ isn’t isomorphic to any untwisted algebra (i.e. ‘genuinely twisted’) iff $\sigma$ is outer. For inner $\sigma$, where $\mathfrak{g}[\sigma]$ is isomorphic to an untwisted algebra, there is a canonical correspondence between fixed points $\mu$ and twisted sector labels $\hat{\mu}$.

Next, we exploit the behavior of the functions $\chi^{(0)}_\mu$ with respect to the Weyl group $W$ of $\mathfrak{g}$, respectively with respect to the subgroup

$$W_{\sigma_s} := \{ w \in W \mid w \circ \sigma_0 = \sigma_0 \circ w \},$$

which is isomorphic, as a Coxeter group, to the Weyl group of the orbit Lie algebra $\mathfrak{g}[\sigma]$. The functions $\chi^{(0)}$ and $\chi^{(1)}$ can then be shown to be quotients of alternating $W_{\sigma_s}$-sums of Theta functions. Except for $A_{2n}$ with outer automorphism, the result reads

$$\chi^{(0)}_\mu(2\tau) = \chi^{(0)}_\mu[0, s](\tau) \quad \text{and} \quad \chi^{(1)}_\mu(\tau) = \chi^{(1)}_\mu[0, 1](\tau)$$

with $\mu \in \hat{L}_{\sigma_s} / (\hat{W}_{\sigma_s} \ltimes h \hat{L}_{\sigma_s})$ and $\hat{\mu} \in \hat{L}_{\sigma_s} / (\hat{W}_{\sigma_s} \ltimes h \hat{L}_{\sigma_s})$. Here $\chi[s_1, s_2]$ are shifted characters defined by means of shifted Theta functions, and the relevant lattices are $\hat{L}_{\sigma_s} = \{ \sum n_i \alpha^{(i)\vee} \mid n_{\sigma_i} = n_i \in \mathbb{Z} \}$.
and $\hat{L}_{\sigma} = \{ \sum_i n_i \alpha^{(i)} | \ell_i n_i \in \mathbb{Z}, n_{\beta_i} = n_i \}$, with $\ell_i$ the orbit lengths with respect to the symmetry of the Dynkin diagram of $\hat{g}$ that corresponds to $\sigma_o$.

In the exceptional case $\hat{g} = A_{2n}$ with $\sigma = \sigma_c$ (‘charge conjugation’), owing to the minus sign that (only) in this case appears in the transformation $\sigma_o(E^\theta) = -E^\theta$ of the generator associated to the highest $\hat{g}$-root, the formulas (5.9) get replaced by

$$\chi^{(0)}_\mu(2\tau) = T^{(1/2)}_\mu \chi^{\sigma_o}_\mu [0,0](\tau + \frac{1}{2}) = \chi^{\sigma_o}_\mu [0, s_0](\tau), \quad \chi^{(1)}_\mu(\frac{1}{2}) = \chi^{\sigma_o}_{\mu} [s_0,0](\tau),$$

(5.10)

where $s_0$ is the $A_{2n}$-weight $s_0 = \frac{1}{2}(\Lambda(n) + \Lambda(n+1))$.

It is then not too difficult to read off the S- and T-transformations. One finds

$$S^{(0)}_{\lambda, \mu} = S^{\sigma_o}_{\lambda, \mu} \quad \text{and} \quad S^{(1)}_{\mu, \lambda} = \eta^{-2}_{\lambda} S^{\sigma_o}_{\lambda, \mu}$$

(5.11)

with $S^{\sigma_o}$ the S-matrix of the orbit Lie algebra $\hat{g}^{[\sigma_o]}$.\[2] and

$$T^{(1)}_\mu = \begin{cases} e^{2\pi i k(s_o, s_o)} e^{2\pi i (\mu, s_o)} (T^{(1)}_\mu)^{1/2} & \text{for } \hat{g} = A_{2n}, \quad \sigma_o = \sigma_c , \\ e^{2\pi i k(s, s)} e^{2\pi i (\mu, 2s)} (T^{(1)}_\mu)^2 & \text{else.} \end{cases}$$

(5.12)

To conclude this section, we remark that an action of the orbifold automorphisms $\sigma$ by twisted intertwiners $\Theta_\sigma$ can be defined on the $\mathfrak{A}$-modules $\mathcal{H}_\mu$, and correspondingly there is an action on the spaces of chiral blocks of the $\mathfrak{A}$-theory. Inspection of the fusion rules of the orbifold model [BFS], as obtained from the modular S-matrix $S^\sigma$ (5.3) by the Verlinde formula (1.2), then leads to a conjecture on the traces of the action of $\sigma^{\otimes m}$ on chiral blocks, similar to the formula (1.14):

Conjecture 2: For every automorphism $\sigma$ of $\mathfrak{L}$ that preserves the Virasoro element $v_{\nu_\mu}$, the trace of the induced map on $B_{\mu, \mathbb{C}P^1}$ is

$$\text{tr}_{B_{\mu, \mathbb{C}P^1}}(\Theta^{\sigma_{\sigma_1}, \ldots, \sigma_{\sigma_m}}_\sigma) = \sum_\kappa |S^{(0)}_{\kappa, \Omega}|^2 \prod_{i=1}^m S^{(0)}_{\kappa, \mu_i} S^{(0)}_{\kappa, \ell_i}.$$  

(5.13)

The matrix $S^{(0)}$ appearing here is given as in formula (5.11), i.e. coincides with the Kac-Peterson matrix of $\mathfrak{L} = \hat{g}$ for $\sigma$ inner, and with the Kac-Peterson matrix of the orbit Lie algebra $\hat{g}^{[\sigma]}$ if $\sigma$ is outer. (It appears in the modular S-matrix of the orbifold, see (5.5).)

The conjecture is formulated for WZW models, but just like in the case of formula (4.16) it originates from structures present in every rational CFT model and hence can again be extended to arbitrary such models. Note that in the present case the Fourier transformation between the dimensions of eigenspaces and the traces (5.13) is to be performed with respect to the full cyclic group generated by $\sigma$. No projectivity needs to be accounted for, since one and the same automorphism $\sigma$ is used at each insertion point and the second cohomology of cyclic groups is trivial.

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\[2\] This matrix expresses the characters of integrable modules over the twisted affine Lie algebra $\hat{g}^{[\sigma_o]}$ at $-1/\tau$ in terms of the characters at $\tau$ of another twisted affine Lie algebra $\hat{g}^{[\sigma_c]}$. In some cases that algebra coincides with $\hat{g}^{[\sigma]}$, but then on the S-transformed side one deals with a different set of representations.
6 Full conformal field theory

We now turn our attention to what in contradistinction to chiral CFT one calls full CFT. It is full CFT that is relevant to many applications, e.g. in statistical mechanics, condensed matter physics and string theory. (CCFT, on the other hand, appears in the description of the (fractional) quantum Hall effect.) The world sheet for full CFT is a real two-dimensional manifold $X$ with a conformal structure. $X$ may have a boundary, and it need not be orientable. Even when it is orientable, it does not come with a canonical choice of orientation.

In order to exploit the power of complex geometry, one likes to relate full CFT to CCFT. This is indeed possible. As far as the geometric aspects are concerned, the relationship is established by associating to $X$ its Schottky cover $\hat{X}$, a double cover branched over the boundary $\partial X$ from which $X$ is recovered as the quotient by an anti-conformal involution. For $\partial X = \emptyset$ the double is just the total space of the orientation bundle. (For orientable boundaryless $X$ this bundle is trivial, i.e. the disconnected sum of two copies of $X$, with opposite orientations.) At the field theory level the relationship turns out to be quite a bit more involved. But fortunately it can still be formulated in a model independent manner. In particular, the quantities of basic physical interest are the correlation functions, and these can be obtained as specific elements in spaces (respectively as sections in the bundles) of chiral blocks on $\hat{X}$. Which specific element in a block space constitutes a correlation function is determined by the following requirements:

- **Locality**: Correlation functions must be functions of the insertion points $p_s$ and (modulo the ‘Weyl anomaly’) of the moduli of $X$.
- **Factorization**: Correlation functions on world sheets of different topology must be compatible with the desingularization of singular curves $\hat{X}$.

The factorization (or gluing, or sewing) conditions arise because one allows for singular curves $\hat{X}$, possessing ordinary double points. When $\hat{X}'$ is a partial desingularization of $\hat{X}$ that resolves a double point $p \in \hat{X}$ in two points $p', p'' \in \hat{C}$, chiral factorization provides a canonical isomorphism $B_{\mu, \hat{X}} \cong \bigoplus_{\nu \in I} B_{\mu, \nu, \nu', \hat{X}'}$. This way one relates the chiral blocks at genus $g$ to the genus-zero blocks for

$$\mathcal{H}_{\mu_1} \otimes \mathcal{H}_{\mu_2} \otimes \cdots \otimes \mathcal{H}_{\mu_m} \otimes \left[ \bigoplus_{\lambda \in I} \mathcal{H}_\lambda \otimes \mathcal{H}_\lambda \right]^{\otimes g}.$$ (6.1)

The non-chiral factorization requires that the image of a correlation function under such an isomorphism is again a correlation function (for concrete formulas, see [F3S2]). In physical terms, this amounts to a restriction on the allowed intermediate states that contribute to singular limits of correlators, and is closely related to the causality of dynamics and to the existence of operator product expansions.

The system of locality and factorization conditions is largely overdetermined. Thus a priori the existence of solutions is questionable; but indeed, when the torus partition function is of charge conjugation type and as long as there are no symmetry breaking boundary conditions (see below), existence can be proven by explicit construction via the relation with three-dimensional TFT [F3S2]. There is also no guarantee of uniqueness. For closed orientable $X$ the solution is expected to be unique (and has been shown to be so for genus zero in large classes of models), once the modular invariant torus partition function

$$Z(\tau) = \sum_{\mu, \tilde{\mu} \in I} Z_{\mu, \tilde{\mu}} \chi_\mu(\tau) \overline{\chi_{\tilde{\mu}}(\tau)}$$ (6.2)
has been specified. In contrast, when one allows also for world sheets with boundary, uniqueness requires the specification of an allowed \textit{boundary condition} on each boundary component.

## 7 Conformal boundary conditions

The basic requirements in full CFT are the locality and factorization constraints. They allow e.g. to express all correlation functions on any closed orientable world sheet $X$ through the 3-point correlation functions on the sphere (in string theory these determine the couplings of closed string vertex operators) and the 1-point correlation functions on the torus (including in particular the torus partition function (6.2)). When $X$ has a boundary $\partial X$ and/or is non-orientable, there are further factorization conditions $[Ld, FPS, PSS1, PSS2]$, which also account for the possible insertion of so-called \textit{boundary fields} $\phi$. (Boundary fields correspond to open string vertex operators and can only be inserted on the boundary. The fields inserted in the interior of $X$ are called \textit{bulk fields}.) Use of these constraints allows one to express all correlation functions through the 3-point functions on the sphere, the torus partition function, as well as:

- The 1-point correlation functions for bulk fields on $\mathbb{R}P^2$.
- The 3-point correlation functions for boundary fields on the disk.
- The correlation functions for one bulk and one boundary field on the disk.

In physical terms, correlation functions are regarded as vacuum expectation values of time-ordered products of field operators. Since the insertion point of a bulk field on $X$ has two pre-images in $\hat{X}$, it accounts for two marked points on $\hat{X}$, to be labelled by two elements $\mu, \tilde{\mu} \in I$. In contrast, the insertion points for boundary fields lie on $\partial X$ and hence have a single pre-image in $\hat{X}$, so boundary fields carry a single chiral label $\mu \in I$. It is worth stressing that the boundaries of interest here are genuine, physical boundaries. They must not be confused with the boundaries of small disks around field insertions that one sometimes cuts out from $X$ in order to specify local coordinates around the insertion points.

A major new issue are the \textit{boundary conditions} that are to be specified at such physical boundaries.\footnote{Such world sheets play e.g. a role in the analysis of dissipative quantum mechanics [LS], of defects in systems of condensed matter physics, of percolation probabilities, and of string perturbation theory in the background of certain solitonic excitations, the so-called D-branes.} For investigating most aspects of boundary conditions one can concentrate on the 1-point functions $\langle \phi_{\mu,\tilde{\mu}} \rangle$ for bulk fields on the disk $D$ (this is the basis of the so-called boundary state formalism). As bulk fields carry two chiral labels $\mu, \tilde{\mu} \in I$, these correspond to 2-point blocks $B_{\mu,\tilde{\mu}}$ on $\hat{D} = \mathbb{C}\mathbb{P}^1$ and, due to $\dim(B_{\mu,\tilde{\mu}}) \in \{0,1\}$, are determined up to a scalar:

$$\langle \phi_{\mu,\tilde{\mu}}(v \otimes \tilde{v}) \rangle_A = R^a_{\mu,\tilde{\mu}} B_{\mu,\tilde{\mu}}(v \otimes \tilde{v}). \quad (7.1)$$

By factorization, the coefficients $R^a_{\mu,\tilde{\mu}}$ appearing here are interpreted as so-called \textit{reflection coefficients}. These arise in the operator product that heuristically describes with which strength the boundary vacuum field is excited when the bulk field $\phi_{\mu,\tilde{\mu}}$ approaches the boundary. One has $R^a_{\mu,\tilde{\mu}} = 0$ unless $\tilde{\mu} = \mu^+$; thus we concentrate on the latter situation, and abbreviate $R^a_{\mu,\mu^+} := R^a_{\mu}$.

\footnote{These, in turn, should not be mixed up with the periodic or twisted-periodic boundary conditions that are often studied in field theory. They specify the topological type of a bundle over the manifold and are not related to physical boundaries.}
The new label $a$ in formula (7.1) indicates that the 1-point function $\langle \phi_{\mu, \bar{\mu}} \rangle$ is not unique. Rather, it depends on the boundary condition to be attached to $\partial D$. In other words, the label $a$ distinguishes between distinct boundary conditions. A basic task is then to determine all conformally invariant boundary conditions that lead to correlation functions satisfying all consistency constraints. This task naturally splits into a chiral and a non-chiral part. At the chiral level, one wants to compute the ‘boundary blocks’ $B_{\mu, \bar{\mu}}$ as solutions to the appropriate Ward identities. The solution is known explicitly in many cases of interest, e.g. for Neumann or Dirichlet conditions of free boson CFTs and for maximally symmetric boundary conditions of WZW models. The resulting expressions are not particularly useful, though. Rather, what is important are their normalization and their factorization properties – issues that can be studied model independently.

At the level of full CFT, the goal is to determine the reflection coefficients $R_{\mu}^a$. This can be achieved by solving a specific factorization identity, obtained by comparison of two singular limits of the correlation function $\langle \phi_{\mu, \mu} + \phi_{\nu, \nu} \rangle_a$ for two bulk fields on $D$. An important observation is that this step is logically independent from the former. Thus one can classify boundary conditions even when the boundary blocks are not known concretely. In fact this can be done in a model independent manner, only making use of the fact that the underlying CCFT obeys all chiral consistency conditions.

To remain within the framework of CFT, one wants the boundary conditions to preserve the conformal symmetry, which means that the chiral blocks in the presence of the boundary still satisfy the Ward identities associated to the Virasoro algebra. But this requirement of conformal invariance turns out to be rather weak; typically it is obeyed by infinitely many boundary conditions, so that a classification is difficult. In contrast, a finite number of boundary conditions arises when one requires that they preserve even a subalgebra $\bar{\mathfrak{A}}$ of $\mathfrak{A}$ that is itself the VOA for some rational CFT. In the sequel we assume to be in this situation and regard $\bar{\mathfrak{A}}$ to be given. The simplest case are the bulk symmetry preserving boundary conditions, for which $\bar{\mathfrak{A}}$ is all of $\mathfrak{A}$.

Let us be more specific. The appropriate labels of the bulk fields on the disk depend on the sub-VOA $\bar{\mathfrak{A}}$ that is prescribed to be preserved. But as will be seen below, they are not just given by the sector labels $\bar{\mu}$ of the CCFT model that corresponds to $\bar{\mathfrak{A}}$; we denote the label set by $\tilde{I}$ and its elements by $\tilde{\mu}, \tilde{\nu}, ....$. Independently of the choice of $\bar{\mathfrak{A}}$, factorization can be seen to imply

$$R_{\lambda}^a R_{\mu}^a = \sum_{\tilde{\nu} \in \tilde{I}} \tilde{N}_{\lambda, \tilde{\mu}}^{\tilde{\nu}} R_{\tilde{\nu}}^a,$$

i.e. for every boundary condition $a$ the reflection coefficients $R_{\mu}^a$ with $\mu \in \tilde{I}$ furnish a one-dimensional irreducible representation of some algebra, called the classifying algebra and denoted by $\mathcal{C}(\mathfrak{A}, \bar{\mathfrak{A}})$. The structure constants $\tilde{N}_{\lambda, \tilde{\mu}}^{\tilde{\nu}}$ can be entirely expressed through CFT data that, manifestly, do not involve the boundary.

For $\bar{\mathfrak{A}} = \mathfrak{A}$ (bulk symmetry preserving boundary conditions), and with torus partition function given by charge conjugation, i.e. $Z_{\lambda, \mu} = \delta_{\lambda, \mu^c}$, one finds that $\tilde{I} = I$ and $\tilde{N}_{\lambda, \nu} = N_{\lambda, \nu}$, i.e. the classifying algebra $\mathcal{C}(\mathfrak{A}, \bar{\mathfrak{A}})$ coincides with fusion algebra of the CCFT. Thus in particular the basis elements of $\mathcal{C}(\mathfrak{A}, \bar{\mathfrak{A}})$ correspond to the sectors of the $\mathfrak{A}$-theory, the boundary labels $a$ are in the same set $I$, and $\mathcal{C}(\mathfrak{A}, \bar{\mathfrak{A}})$ is a semi-simple commutative associative algebra whose structure constants are expressible through a diagonalizing matrix $S$ as in (1.3).

For $\bar{\mathfrak{A}}$ strictly contained in $\mathfrak{A}$ – symmetry breaking boundary conditions – the factorization
arguments go through as well, so that one still deals with one-dimensional irreducible representations of a classifying algebra. But the details are more involved. A systematic solution has been achieved \cite{FS2} for all cases where the preserved sub-VOA $\mathfrak{A}$ is the fixed point algebra

$$\mathfrak{A} = \mathfrak{A}^{G}$$  \hspace{1cm} (7.3)

with respect to any finite abelian group $G$ of automorphisms of $\mathfrak{A}$, or in other words, when $\mathfrak{A}$ is the VOA of an abelian orbifold of the $\mathfrak{A}$-theory, as studied in section 5. To determine the label set $\bar{I}$ one considers the decomposition

$$\mathcal{H}_{\lambda} = \bigoplus_{\bar{\mu} \in \bar{I}} V_{\bar{\mu}} \otimes \bar{\mathcal{H}}_{\bar{\mu}}$$  \hspace{1cm} (7.4)

of simple $\mathfrak{A}$-modules as $\bar{\mathfrak{A}}$-modules. Preservation of $\mathfrak{A}$ by the boundary condition means that the relevant chiral blocks respect the Ward identities for $\mathfrak{A}$, but not in general those for $\bar{\mathfrak{A}}$, and hence the sector labels $\bar{\mu}$ of the $\bar{\mathfrak{A}}$-theory that appear in (7.4) – corresponding to untwisted orbifold sectors – are to be used. But in addition one and the same $\bar{\mathfrak{A}}$-module $\bar{\mathcal{H}}_{\bar{\mu}}$ will typically give rise to different chiral blocks when it appears in the decomposition of distinct $\mathfrak{A}$-modules $H_{\lambda}$. One concludes that the labels $\bar{\mu}$ are pairs $(\bar{\mu}, \psi_{\bar{\mu}})$ with a suitable degeneracy label $\psi_{\bar{\mu}}$.

One must also determine the appropriate set of boundary labels $a$. At the present state of affairs this still involves some heuristics.\footnote{This statement even applies to the symmetry preserving situation $\mathfrak{A} = \mathfrak{A}$ as studied in \cite{C} (except for some special classes of models where the structure constants $\hat{N}_{\lambda,\mu,\nu}$ have been computed \cite{PSS2} explicitly). In that case, however, many of the results can be made rigorous by employing the relation to TFT $\hat{F}S^{2}$. We expect that the methods of $\hat{F}S^{2}$ can be extended so as to establish a proof also for more general boundary conditions. Work in this direction is in progress.} The conclusions $\cite{FS2}$ are that the boundary labels are $G$-orbits $[\bar{\rho}, \psi_{\bar{\rho}}]$ of pairs that consist of an (untwisted or twisted) $\bar{\mathfrak{A}}$-sector label $\bar{\rho}$ and a character $\psi_{\bar{\rho}}$ of the central stabilizer $U_{\bar{\rho}}$, and that the degeneracy label $\psi_{\bar{\mu}}$ of the labels in $\bar{I}$ is a character of the (full) stabilizer $S_{\bar{\mu}}$. Here $\mathcal{G}$ is a simple current group that is necessarily present in the orbifold theory and which is naturally isomorphic to the character group $G^{*}$ of the orbifold group $G$. One thus deals with two different deviations from the labels appearing in simple current extensions, cf. the text before (7.4). Namely, for $\bar{\mu}$ there is a character of the full rather than the central stabilizer and no $\mathcal{G}$-orbit is to be formed, while $a$ has the same form as extended labels $\mu^{ex}$, but the restriction to $T_{\bar{\mu}} = T_{\mu}$ (that is, to untwisted orbifold sectors) is dropped. There are indications $\cite{FS2, BPPZ, FS3, SF}$ that boundary conditions violating this equality correspond to twisted or ‘solitonic’ representations of the bulk symmetry $\mathfrak{A}$.

Further, the reflection coefficients are seen to be quotients $R_{\mu}^{a} = \hat{S}_{\bar{\mu},a} / \hat{S}_{\bar{\Omega},a}$ with

$$\hat{S}_{(\bar{\mu}, \psi_{\bar{\mu}}),[\bar{\rho}, \psi_{\bar{\rho}}]} = \frac{|G|}{\left| S_{\bar{\mu}} / U_{\bar{\mu}} \right| \left| S_{\bar{\rho}} / U_{\bar{\rho}} \right|} \sum_{J \in S_{\bar{\mu}} \cap U_{\bar{\rho}}} \psi_{\bar{\mu}}(J) \psi_{\bar{\rho}}(J)^{*} S_{\bar{\mu},\bar{\rho}}^{J}. \hspace{1cm} (7.5)$$

Note that all input data appearing in this formula are purely chiral data.

Once these prescriptions for the labels and for the reflection coefficients are taken for granted, the classifying algebra can be analysed rigorously. Among the results $\cite{FS3}$ are the following:

- $\mathcal{C}(\mathfrak{A}, \bar{\mathfrak{A}})$ is a unital semi-simple commutative associative algebra.
The unit element is $\hat{\Omega} = \bar{\Omega}$, the vacuum sector of the $G$-orbifold.

The structure constants of $C(A, \bar{A})$ obey the Verlinde-like formula

$$\hat{N}_{\lambda, \mu, \nu} = \sum_a \hat{S}_{\lambda, a} \hat{S}_{\mu, a} \hat{S}_{\nu, a}. \tag{7.6}$$

(To raise the third index, one contracts with the inverse of $\hat{C}_{\lambda, \mu} = \hat{N}_{\lambda, \mu, \bar{\Omega}}$.)

- $C(A, \bar{A})$ contains the fusion rule algebra $A(A)$ as a subalgebra.
- More specifically, $C(A, \bar{A})$ is a direct sum of ideals $C^\sigma(A)$ with $\sigma \in G$. Each of them corresponds to boundary conditions of a definite automorphism type $\sigma$ (and depends only on the automorphism $\sigma$, but not on the specific orbifold group $G \ni \sigma$ considered). The latter is characterized by the fact that in the presence of such a boundary condition the 1-point functions of bulk fields can be expressed entirely through twisted boundary blocks $B_{\mu, \mu}^\sigma$ which satisfy $\sigma$-twisted Ward identities. Schematically,

$$B_{\mu, \mu}^\sigma \circ (Y_n \otimes 1 - (-1)^{\Delta_Y} 1 \otimes \sigma(Y_n)) = 0 \tag{7.7}$$

for every field $Y$, of conformal weight $\Delta_Y$, in the VOA $\mathfrak{A}$. Twisted boundary blocks are related to ordinary boundary blocks $B_{\mu, \mu}^\sigma$ as

$$B_{\mu, \mu}^\sigma = B_{\mu, \mu}^\theta \circ (\Theta_\sigma \otimes 1) \tag{7.8}$$

with $\Theta_\sigma$ the twisted intertwiner associated to $\sigma$.

Let us add a few further comments:

- Symmetry breaking boundary conditions play e.g. an important role for space-time supersymmetry breaking in superstring theory. In that context the preserved subalgebra $\bar{A}$ must contain the $N = 1$ world sheet superconformal symmetry, which is a gauge symmetry of (perturbative) superstring theory. On the other hand, in condensed matter applications one may wish to study boundary conditions that even break the conformal symmetry; these are not covered by the CFT constructions reported above.

- The boundary conditions preserving $A_G \subset A$ include of course all those which even preserve some intermediate algebra $A' \subset A \subset A$. By the Galois theory of vertex operator algebras [DM], there is a bijection between these intermediate algebras and the set of subgroups of $G$.

- That every boundary condition possesses an automorphism type is no longer true in general when $\bar{A} \neq A_G$. This property is only to be expected when $\bar{A}$ is an orbifold subalgebra, and it has only been proven for abelian orbifold groups $G$. In string theory, boundary conditions without automorphism type correspond to so-called non-BPS D-branes (for a review of the latter see e.g. [LR]).

- The one-point functions on $\mathbb{RP}^2$ have also been studied in the literature, see e.g. [FPS, FL, HSS]. Recently, closed formulas for the one-point functions on the disk and on $\mathbb{RP}^2$ have been found [FHSW] for the case of arbitrary torus partition functions that are related to simple currents.

- There is evidence [SF] that for arbitrary (rational) $\mathfrak{A}$ and arbitrary torus partition function the set of boundary conditions preserving $\bar{A}$ comes equipped with the structure of a spherical tensor category, called the ‘boundary category’, and that this category can be entirely
constructed from the modular tensor category associated to $\mathfrak{A}$. The structures arising in this context are familiar from subfactor theory [BE] and suggest an interpretation of the symmetry breaking boundary conditions in terms of suitable ‘solitonic’ representations of the bulk chiral algebra $\mathfrak{A}$. (When $\mathfrak{A}$ is an orbifold subalgebra $\mathfrak{A}^G$ of $\mathfrak{A}$, then these solitonic representations are nothing but the usual twisted sector representations \cite{DLM} of the orbifold model.)

The same arguments as for symmetry preserving boundary conditions show that the operator product of two boundary fields corresponds to the $6j$-symbols of the boundary category. The fusion rules of the boundary category coincide with the annulus coefficients, i.e. the coefficients in an expansion of the partition function on the annulus with respect to the characters of the solitonic representations.

Finally let us illustrate our results by exposing the boundary conditions of WZW models that preserve at least the fixed point algebra of $\mathfrak{A}$ under a $\mathbb{Z}_2$ orbifold group $G = \{\text{id}, \sigma\}$. (Other WZW-orbifolds are rather more complicated, but for the analysis of boundary conditions fortunately only partial results about the orbifolds are needed.) Using the information about the corresponding WZW-orbifolds collected in section 5, one finds that the elements of $\hat{I}$ can be suggestively written by using sector labels of the full bulk symmetry, according to

$$\lambda \text{ and } \sigma^* \lambda \text{ for } \sigma^* \lambda \neq \lambda, \quad \text{and} \quad (\lambda, \psi) \ (\psi \in \{\pm 1\}) \text{ for } \sigma^* \lambda = \lambda. \quad (7.9)$$

Similarly, the labels for the boundary conditions are

- $\mu$ for length-2 orbits $\{(\mu, 1, 0), (\mu, -1, 0)\}$ of untwisted orbifold sectors,
- $\mu$ and $\sigma^* \mu$ for $G$-fixed points $(\mu, 0, 0)$,
- $\dot{\mu}$ for length-2 orbits $\{((\mu, 1, 1), (\mu, -1, 1)\}$ of twisted orbifold sectors.

In this notation, the entries of the diagonalizing matrix $\hat{S}$ read

$$\hat{S}_{(\lambda, \psi), \mu} = S_{\lambda, \mu} \quad \text{for } \sigma^* \lambda = \lambda, \quad \hat{S}_{\lambda, \mu} = S_{\lambda, \mu} \quad \text{for } \sigma^* \lambda \neq \lambda. \quad (7.10)$$

For the structure constants of the classifying algebra one finds e.g.

$$\tilde{N}_{\lambda_1, \lambda_2}^{\lambda_3} = N_{\lambda_1, \lambda_2}^{\lambda_3}, \quad \tilde{N}_{(\lambda_1, \psi_1), (\lambda_2, \psi_2), (\lambda_3, \psi_3)} = N_{\lambda_1, \lambda_2}^{\lambda_3}, \quad \tilde{N}_{(\lambda_1, \lambda_2), (\lambda_2, \lambda_3)} = N_{\lambda_1, \lambda_2}^{\lambda_3}, \quad (7.11)$$

as well as

$$\tilde{N}_{(\lambda, \psi_1), (\lambda_2, \psi_2), (\lambda_3, \psi_3)} = N_{\lambda_1, \lambda_2, \lambda_3} + \frac{\psi_1 \psi_2 \psi_3}{\eta_{\lambda_1} \eta_{\lambda_2} \eta_{\lambda_3}} N_{(\lambda_1, \lambda_2, \lambda_3)} \quad (7.12)$$

with $N_{(\lambda_1, \lambda_2, \lambda_3)} := \sum_{\mu} S_{(\lambda_1, \mu), (\lambda_2, \mu), (\lambda_3, \mu)}^{(0)} / S_{(\lambda_1, \mu), (\lambda_2, \mu), (\lambda_3, \mu)}^{(0)}$. 

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