A Note on the Size-Sensitive Packing Lemma

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Abstract

We show that the size-sensitive packing lemma follows from a simple modification of the standard proof, due to Haussler and simplified by Chazelle, of the packing lemma.

1 Introduction

In 1995 Haussler [4] proved the following interesting theorem.

Theorem 1 (Packing lemma [5, Lemma 5.14, p. 156]). Let \((X, \mathcal{P})\) be a set-system on \(n\) elements, and with VC-dimension at most \(d\). Let \(\delta\) be an integer, \(1 \leq \delta \leq n\), such that \(|\Delta(R, S)| \geq \delta\) for every \(R, S \in \mathcal{P}\), where \(\Delta(R, S) = (R \setminus S) \cup (S \setminus R)\). Then \(|\mathcal{P}| = O((n/\delta)^d)\).

Haussler’s proof is a beautiful application of the probabilistic method, and was simplified by Chazelle [1].

We refer to the discussion in [5] for motivations and applications. Recently much effort has been devoted to finding size-sensitive generalizations of this result. After a series of partial bounds [3, 6], the following statement has been recently established in [2], via two different proofs (one building on Haussler’s original proof while the other extends Chazelle’s proof):

Theorem 2 (Size sensitive packing lemma). Let \((X, \mathcal{P})\) be a set-system on \(n\) elements, and let \(d, d_1, k, \delta > 0\) be integers. Assume the system has VC-dimension at most \(d\). Further, assume that for any set \(Y \subseteq X\) the number of sets in \(\mathcal{P}|_Y\) of size at most \(r\) is at most \(f(|Y|, r) = O(|Y|^{d_1} r^{d-d_1})\). If \(|\Delta(R, S)| \geq \delta\) for every \(R, S \in \mathcal{P}\) and \(|S| \leq k\) for all \(S \in \mathcal{P}\), then \(|\mathcal{P}| = O(n^{d_1} k^{d-d_1} / \delta^d)\).

The objective of this present note is to point out that Theorem 2 with a simple trick, is an immediate consequence of the textbook proof [5] of Theorem 1.

Haussler and Chazelle’s Proof. We rewrite the main step in their proof in a slightly more general form:

Lemma 1.1 (Proof of Packing Lemma 5.14, pp. 157–159). Let \((X, \mathcal{P})\) be a set-system on \(n\) elements. Let \(d, \delta\) be two integers such that the VC-dimension of \(\mathcal{P}\) is at most \(d\), and \(|\Delta(S, T)| \geq \delta\) for all \(S, T \in \mathcal{P}\). Then \(|\mathcal{P}| \leq 2 \cdot E[|\mathcal{P}|_{A'}]|\), where \(A'\) is a random sample of size \(\frac{Adn}{\delta} - 1\).

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\[\text{\footnotesize \textsuperscript{1}}\text{In what follows below, we refer the reader to Matousek’s textbook [5 Section 5.3] for notations and proofs.}\]
Proof. Let $W, W_1, A, s = \frac{4dn}{\delta}$ be precisely as defined in the textbook [5] Proof of Packing Lemma 5.14, pp. 157–159], where the following relations are proven: \( i \) \( 2d|P| \geq E[W] = s \cdot E[W_1] \), where the expectation is over the choice of $A$, and \( ii \) \( E[W_1 | A' = Y] \geq \frac{\delta}{n} (|P| - |P_{|Y|})) \), where $A'$ is set to a fixed $Y$, and the expectation is over the choice of a random element in $X \setminus Y$. Together they imply the lemma:

$$2d|P| \geq E[W] = s \cdot E[W_1] \geq s \cdot \left( \sum_{Y \subseteq X \setminus Y} \frac{\delta}{n} (|P| - |P_{|Y|}) \cdot Pr[A' = Y] \right) = 4d|P| - 4dE[|P_{A'}|].$$

The proof in [5] Proof of Packing Lemma 5.14, pp. 157–159 uses the primal shatter lemma to conclude

Lemma 5.14, pp. 157–159, where the following relations are proven:

Proof.

The somewhat subtle key idea that was missed by earlier work [2, 3] is that it is fine if $c|P|_{s-1}$ is set to a fixed $s$. Together they imply the lemma:

\(2d|P| \geq E[W] = s \cdot E[W_1] \geq s \cdot \left( \sum_{Y \subseteq X \setminus Y} \frac{\delta}{n} (|P| - |P_{|Y|}) \cdot Pr[A' = Y] \right) = 4d|P| - 4dE[|P_{A'}|].$$

The proof in [5] Proof of Packing Lemma 5.14, pp. 157–159 uses the primal shatter lemma to conclude the proof of Theorem 1. $|P_{A'}| = O((4dn/\delta)^d)$ for any $A'$ of size $s - 1$. Now we show that the proof of Theorem 2 is also a similar step away, by using instead the size-sensitive bound given in the assumption.

Proof of Theorem 2. Let $A' \subseteq X$ be a random sample of size $\frac{4dn}{\delta}$ $- 1$. Let $P_1 = \{ S \in P \text{ s.t. } |S \cap A'| \geq 3 \cdot 4dk/\delta \}$. Note that $E[|S \cap A'|] \leq 4dk/\delta$ as $|S| \leq k$ for all $S \in P$. By Markov’s inequality, for any $S \in P$, $Pr[S \in P_1] = Pr[|S \cap A'| > 3 \cdot 4dk/\delta] \leq 1/3$. Thus

$E[|P_{A'}|] \leq E[|P_1|] + E[(P \setminus P_1)_{A'}|] \leq \sum_{S \in P} Pr[S \in P_1] + f(|A'|, 12dk/\delta) \leq \frac{|P|}{3} + O((\frac{4dn}{\delta})^{d_1}(\frac{12dk}{\delta})^{d-d_1})$

where the projection size of $P \setminus P_1$ to $A'$ is bounded by $f(\cdot, \cdot)$. Applying Lemma 1 finishes the proof.

Discussion. Besides a dramatically shorter proof, our proof also improves the constants in the bounds. Furthermore, we have shown that it follows without any modification or addition to the Chazelle-Haussler proof. The somewhat subtle key idea that was missed by earlier work [2, 3] is that it is fine if $E[|P_{A'}|]$ is bounded in terms of $c|P|$ for a small-enough constant $c$, as in any case it would be absorbed by the LHS of the equation in Lemma 11. This allows us to replace the complicated technical machinery developed in earlier work (iterative processes, Chernoff bounds for hypergeometric series, complicated probabilistic computations) for over 20 pages-long proofs by a mere Markov’s inequality.

References

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