Characters of (relatively) integrable modules over affine Lie superalgebras

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Abstract. In the paper we consider the problem of computation of characters of relatively integrable irreducible highest weight modules \( L \) over finite-dimensional basic Lie superalgebras and over affine Lie superalgebras \( \mathfrak{g} \). The problem consists of two parts. First, it is the reduction of the problem to the \( \mathfrak{g} \)-module \( \mathcal{F}(L) \), where \( \mathfrak{g} \) is the associated to \( L \) integral Lie superalgebra and \( \mathcal{F}(L) \) is an integrable irreducible highest weight \( \mathfrak{g} \)-module. Second, it is the computation of characters of integrable highest weight modules. There is a general conjecture concerning the first part, which we check in many cases. As for the second part, we prove in many cases the KW-character formula, provided that the KW-condition holds, including almost all finite-dimensional \( \mathfrak{g} \)-modules when \( \mathfrak{g} \) is basic, and all maximally atypical non-critical integrable \( \mathfrak{g} \)-modules when \( \mathfrak{g} \) is affine with non-zero dual Coxeter number.

Keywords and phrases: basic Lie superalgebra, defect, dual Coxeter number, affine Lie superalgebra, odd reflection, integrable highest weight module over basic and affine Lie superalgebra, maximally atypical module, KW-condition, Enright functor, relatively integrable module, character formula

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Given a symmetrizable Kac–Moody algebra \( \mathfrak{g} \) with a Cartan subalgebra \( \mathfrak{h} \) and an irreducible non-critical highest weight \( \mathfrak{g} \)-module \( L = L(\lambda) \), one constructs the associated integral Kac–Moody algebra \( \mathfrak{g}^{\lambda} \) as follows. Let \( \Delta_{re} \subset \mathfrak{h}^* \) be the set of real roots of \( \mathfrak{g} \) and let

\[
\Delta_{re}(\lambda) = \{ \alpha \in \Delta_{re} \mid 2(\lambda, \alpha)/(\alpha, \alpha) \in \mathbb{Z} \}
\]

be the set of integral real roots. Then \( \Delta_{re}(\lambda) \) is the set of real roots of a Kac–Moody algebra \( \mathfrak{g}^{\lambda} \) with the same Cartan subalgebra \( \mathfrak{h} \). An important result of representation theory is the following relation between the characters of highest weight \( \mathfrak{g} \)-module \( L \) and the (non-critical) highest weight \( \mathfrak{g}^{\lambda} \)-module \( \overline{L} = \overline{L}(\lambda + \rho - \overline{\rho}) \):

\[
Re^\rho \ ch \ L(\lambda) = \overline{Re}^\overline{\rho} \ ch \ \overline{L}(\lambda + \rho - \overline{\rho}),
\]

where \( R \) and \( \overline{R} \) denote the Weyl denominators, and \( \rho \) and \( \overline{\rho} \) denote the Weyl vectors (see [F], [KT1], [KT2] and references there).

In the case when the \( \mathfrak{g}^{\lambda} \)-module \( \overline{L}(\lambda + \rho - \overline{\rho}) \) is integrable, its character is given by the Weyl–Kac character formula [K2], hence (1) gives an explicit formula for \( ch \ L(\lambda) \).

A \( \mathfrak{g} \)-module is called relatively integrable if the \( \mathfrak{g}^{\lambda} \)-module \( \overline{L}(\lambda + \rho - \overline{\rho}) \) is integrable; and it is called admissible if, in addition, the \( \mathbb{Q} \)-span of the set of roots of \( \mathfrak{g}^{\lambda} \) coincides with \( \mathbb{Q} \Delta \). In particular, if \( \lambda = \overline{\rho} - \rho \), we obtain from (1) that the character is given by a product:

\[
ch \ L(\overline{\rho} - \rho) = e^{\overline{\rho} - \rho} R^{-1} \overline{R}.
\]