A Note on Sparse Minimum Variance Portfolios and Coordinate-Wise Descent Algorithms

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Abstract

In this short report, we discuss how coordinate-wise descent algorithms can be used to solve minimum variance portfolio (MVP) problems in which the portfolio weights are constrained by \( l_q \) norms, where \( 1 \leq q \leq 2 \). A portfolio which weights are regularised by such norms is called a sparse portfolio (Brodie et al., 2009), since these constraints facilitate sparsity (zero components) of the weight vector. We first consider a case when the portfolio weights are regularised by a weighted \( l_1 \) and squared \( l_2 \) norm. Then two benchmark data sets (Fama and French 48 industries and 100 size and BM ratio portfolios) are used to examine performances of the sparse portfolios. When the sample size is not relatively large to the number of assets, sparse portfolios tend to have lower out-of-sample portfolio variances, turnover rates, active assets, short-sale positions, but higher Sharpe ratios than the unregularised MVP. We then show some possible extensions; particularly we derive an efficient algorithm for solving an MVP problem in which assets are allowed to be chosen grouply.

1 Introduction

The short report discusses how coordinate-wise descent algorithms can be used to solve minimum variance portfolio (MVP) problems in which the portfolio weights are penalised by \( l_q \) norms, where \( 1 \leq q \leq 2 \). A portfolio which weights are regularised by such norms is called a sparse portfolio, since these constraints facilitate sparsity (zero components) of the weight vector. We first consider a special case when the portfolio weights are regularised by a weighted \( l_1 \) and squared \( l_2 \) norm:

\[
\min_w w^T \Sigma w + \lambda \| w \|_{l_1} + \lambda (1 - \alpha) \| w \|_{l_2}^2 \quad \text{subject to } w^T 1_p = 1,
\]

where \( \| w \|_{l_1} = \sum_{i=1}^p |w_i|, \| w \|_{l_2}^2 = \sum_{i=1}^p w_i^2, \lambda \in \mathbb{R}^+ \) and \( \alpha \in [0,1] \). We call \( \lambda \) the penalty parameter, and \( \alpha \) is the parameter for adjusting the relative weight of \( l_1 \) and squared \( l_2 \) norms. Suppose we have \( p \) assets, then \( \text{dim } (w) = 1 \times p \). We also have

\[
\Sigma = \begin{pmatrix}
\sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1p} \\
\sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{p1} & \sigma_{p2} & \cdots & \sigma_p^2
\end{pmatrix}
\]
as the covariance matrix of \( p \) asset returns, and it is a \( p \times p \) symmetric positive semidefinite matrix.

The problem (1) can be seen as an MVP problem with constraint \( \alpha \sum_{i=1}^{p} |w_i| + (1 - \alpha) \sum_{i=1}^{p} w_i^2 \leq c \), where \( c > 0 \) is some constant. Such a weighted \( l_1 \) and squared \( l_2 \) norm is called the elastic net constraint \cite{Zou2005} in regression-based variable selection problems. When \( \alpha = 0 \), the weights are only regularised by squared \( l_2 \) norm, and the optimal solution of (1) is the same as that of the unregularised MVP problem in which \( \Sigma \) is replaced by \( \Sigma + \lambda I_{p \times p} \). When \( \alpha = 1 \), the weights are only penalised by \( l_1 \) norm. Performances and properties of such \( l_1 \) norm constrained portfolio are documented in \cite{Brodie2009, DeMiguel2009, Fan2009} and \cite{Welsch2007}. When \( 0 < \alpha < 1 \), the solution of (1) is equivalent to that of the MVP problem with covariance matrix \( \Sigma + \lambda(1 - \alpha) I_{p \times p} \) and \( l_1 \) penalty \( \lambda \|w\|_1 \).

2 The Algorithm

Let \( \gamma \) be the Lagrange multiplier of the constraint \( w^T 1_p = 1 \). The Lagrangian function of problem (1) is given by

\[
L(w, \gamma; \Sigma, \lambda, \alpha) = w^T \Sigma w + \lambda \alpha \|w\|_1 + \lambda (1 - \alpha) \|w\|_2^2 - \gamma (w^T 1_p - 1) = w^T \Sigma w + \sum_{i=1}^{p} (\lambda \alpha |w_i| + \lambda (1 - \alpha) w_i^2 - \gamma w_i) + \gamma. \tag{2}
\]

At the stationary point, the following condition should hold,

\[
2w_i \sigma_i^2 + 2 \sum_{j \neq i}^p w_j \sigma_{ij} + 2\lambda (1 - \alpha) w_i - \gamma = -\lambda \alpha \text{ if } w_i > 0, \\
2w_i \sigma_i^2 + 2 \sum_{j \neq i}^p w_j \sigma_{ij} + 2\lambda (1 - \alpha) w_i - \gamma = \lambda \alpha \text{ if } w_i < 0, \\
\left| 2 \sum_{j \neq i}^p w_j \sigma_{ij} - \gamma \right| \leq \lambda \alpha \text{ if } w_i = 0, \tag{3}
\]

\( w^T 1_p = 1 \).

\cite{Friedman2007} show how coordinate-wise descent algorithms are powerful to solve regression-based variable selection problems with convex constraints. Suppose the objective function \( f(w) = f(w_1, \ldots, w_p) \) is differentiable and convex. The method starts by fixing \( w_i \), for \( i = 2, \ldots, p \), and then find a value of \( w_1 \) to minimise (or reduce) \( f(w) \). The next step is to fix \( w_i \), for \( i = 1, 3, \ldots, p \) and then find a value of \( w_2 \) to minimise (or reduce) the objective function and so on. When the iteration is done over all \( w_i \), we then go back to start the iteration again for \( w_1 \). The procedure repeats until we find the global minimum of \( f(w) \).

Theoretical validity of using coordinate-wise descent algorithms to solve regularised penalty problems can be found in \cite{Tseng2001}. Suppose

\[
f(w) = f_0(w) + \sum_{i=1}^{p} f_i(w_i).
\]
It can be shown that if \( f(\mathbf{w}) \) is bounded, \( f_0(\mathbf{w}) \) is differentiable and convex, and \( f_i(w_i) \) is also convex but not a function of \( w_j \) for any \( j \neq i, j = 1, \cdots, p \) (\( \sum_{i=1}^{p} f_i(w_i) \) is additively separable), then coordinate-wise descent algorithms are valid for solving \( \min_{\mathbf{w}} f(\mathbf{w}) \). The property of additive separability can be applied to when \( f_i(.) \) is a multivariate function, i.e. \( w_i \) is a vector. Then the property holds if \( w_i \) and \( w_j \) do not have any overlapped element.

It is easily seen that given \( \gamma, \Sigma, \lambda \) and \( \alpha \), \( L(\mathbf{w}, \gamma; \Sigma, \lambda, \alpha) - \gamma \) satisfies the above conditions if we set \( f_0(\mathbf{w}) = \mathbf{w}^T \Sigma \mathbf{w} \) and \( f_i(w_i) = \lambda \alpha |w_i| + \lambda (1 - \alpha) w_i^2 - \gamma w_i \). Therefore with fixed \( \gamma, \Sigma, \lambda \) and \( \alpha \), minimising \( (2) \) via a suitable coordinate-wise descent algorithm can attain its minimum. However, solved portfolio weights for such minimum may not satisfy the full investment constraint \( \alpha \). Note that in an unregularised MVP optimisation, through the adjustment of \( \gamma \), we can make the constraint \( \mathbf{w}^T \mathbf{1}_p = 1 \) hold. We can use the same trick here and finally achieve the optimal solution of \( (1) \).

Following the strategy, we propose the following form to update each \( w_i \),

\[
\begin{align*}
  w_i &\leftarrow ST \left( \frac{\gamma - 2 \sum_{j \neq i} w_j \sigma_{ij} + \lambda \alpha}{2(\sigma_i^2 + \lambda (1 - \alpha))} \right), \\
\end{align*}
\]

(4)

where

\[
ST(x, y) = \text{sign}(x) (|x| - y)_+
\]

is the soft thresholding function. It can be shown that with fixed \( \gamma, \Sigma, \lambda, \alpha \) and \( w_j, j = 1, \cdots, p, j \neq i \), \( (4) \) is at the stationary point of \( (2) \).

As mentioned, \( \gamma \) can be used to make the constraint \( \mathbf{w}^T \mathbf{1}_p = 1 \) hold. Our strategy for updating \( \gamma \) is to take advantage of this property. Let \( z_i = 2 \sum_{j \neq i} w_j \sigma_{ij} \). From \( (3) \), it can be shown that when \( w_i \neq 0 \),

\[
\begin{align*}
  w_i &= \frac{\gamma - z_i - \lambda \alpha}{2(\sigma_i^2 + \lambda (1 - \alpha))}, \quad \text{if } w_i > 0, \\
  w_i &= \frac{\gamma - z_i + \lambda \alpha}{2(\sigma_i^2 + \lambda (1 - \alpha))}, \quad \text{if } w_i < 0.
\end{align*}
\]

Let \( S^+ = \{i : w_i > 0\} \) and \( S^- = \{i : w_i < 0\} \). Then

\[
\mathbf{w}^T \mathbf{1}_p = \sum_{i=1}^{p} w_i
\]

\[
= \sum_{i \in S^+} w_i + \sum_{i \in S^-} w_i
\]

\[
= \sum_{i \in S^+} \frac{\gamma - z_i - \lambda \alpha}{2(\sigma_i^2 + \lambda (1 - \alpha))} + \sum_{i \in S^-} \frac{\gamma - z_i + \lambda \alpha}{2(\sigma_i^2 + \lambda (1 - \alpha))}
\]

\[
= \gamma \left( \sum_{i \in S^+ \cup S^-} \frac{1}{2(\sigma_i^2 + \lambda (1 - \alpha))} \right) - \sum_{i \in S^+ \cup S^-} \frac{z_i}{2(\sigma_i^2 + \lambda (1 - \alpha))} + \\
\lambda \alpha \left( \sum_{i \in S^-} \frac{1}{2(\sigma_i^2 + \lambda (1 - \alpha))} - \sum_{i \in S^+} \frac{1}{2(\sigma_i^2 + \lambda (1 - \alpha))} \right).
\]
By \( w^T 1_p = 1 \), we propose the following form to update \( \gamma \)

\[
\gamma \leftarrow \frac{1 + \sum_{i \in S^+ \cup S^-} \frac{z_i}{2(\sigma_i^2 + \lambda(1 - \alpha))} - \lambda \alpha \left( \sum_{i \in S^-} \frac{1}{2(\sigma_i^2 + \lambda(1 - \alpha))} - \sum_{i \in S^+} \frac{1}{2(\sigma_i^2 + \lambda(1 - \alpha))} \right)}{\left( \sum_{i \in S^+ \cup S^-} \frac{1}{2(\sigma_i^2 + \lambda(1 - \alpha))} \right)}.
\]

For stability of the algorithm, we set the initial values: \( w_1 = w_2 = \cdots = w_p = 1/p \), and \( \gamma > \lambda \). The updating process starts from \( w_1 \), then \( w_2, \ldots, \) and \( w_p \). The updated vector \( w \) is then used to update \( \gamma \). The process terminates until \( w \) and \( \gamma \) converge. The algorithm can be summarised as follows.

**Algorithm 1 Naive Coordinate-Wise Descent Updating for MVP Penalised by a Weighted \( l_1 \) and squared \( l_2 \) Norm**

1. Fix \( \lambda \) and \( \alpha \in [0, 1] \) at some constant levels.
2. Initialise \( w = \frac{1}{p} 1_p \) and \( \gamma = \lambda \times 1.1 \).
3. For \( i = 1, \ldots, p \),
   \[
   w_i \leftarrow \frac{S^T \left( \gamma - 2 \sum_{j \neq i} w_j \sigma_{ij}, \lambda \alpha \right)}{2(\sigma_i^2 + \lambda(1 - \alpha))}.
   \]
4. Let \( z_i = 2 \sum_{j \neq i} w_j \sigma_{ij} \). Update \( \gamma \) as
   \[
   \gamma \leftarrow \frac{1 + \sum_{i \in S^+ \cup S^-} \frac{z_i}{2(\sigma_i^2 + \lambda(1 - \alpha))} - \lambda \alpha \left( \sum_{i \in S^-} \frac{1}{2(\sigma_i^2 + \lambda(1 - \alpha))} - \sum_{i \in S^+} \frac{1}{2(\sigma_i^2 + \lambda(1 - \alpha))} \right)}{\left( \sum_{i \in S^+ \cup S^-} \frac{1}{2(\sigma_i^2 + \lambda(1 - \alpha))} \right)}.
   \]
5. Repeat 3 and 4 until \( w \) and \( \gamma \) converge.

Before the algorithm is used, one caution should be made here. The coordinate-wise descent algorithm is easily implemented. The convergence is very fast when \( \lambda \) is not small, since the resulted \( w \) is sparse. It is still true even when \( p \) is very large. A fast and stable convergence is very important for the following empirical studies, since we rebalance the portfolio quite often over a long period. However, when \( \lambda \) is too small, the resulted \( w \) will have only a few (or none) zero components; the convergence would become extremely slow. This perhaps is the main reason why the coordinate-wise type algorithms are often ignored in solving optimisation problem in which solution vectors are often dense.

### 3 Empirical Results

#### 3.1 Profiles of the Optimal Weights

We then use the algorithm to obtain the optimal sparse portfolios from real data. Figure 1 shows under different \( \alpha \) and \( \lambda \), profiles of portfolio weights, proportion of active portfolios: \( |S^+ \cup S^-| / p \), and proportion of shortsale portfolios: \( |S^+| / p \). We set \( \alpha = 0, 0.5, 1 \), and \( \lambda = 0, 0.5, \ldots, 15 \). The data used is monthly return data of Fama and French 25 portfolios formed on size and book-to-market. The period

\(^1\)Sample codes for \( R \) programme are available from the author.
is randomly selected 120 months (here is from November 1986 to October 1996). The Σ we calibrate into (1) is the sample covariance matrix of monthly returns during the selected period.

Corresponding to each \( \lambda \), sum of the optimal weights is 1. When \( \alpha = 1 \), we only have \( l_1 \) penalty active, and the profiles behave very similar as those of regression coefficients with the LASSO (Tibshirani, 1996): some of the weights are exactly zero when \( \lambda \gg 0 \). However, unlike the LASSO profiles, the profiles of the sparse portfolio weights do not all vanish to zero when \( \lambda \) goes large, since the constraint \( w^T 1_p = 1 \) needs to hold. When \( \alpha = 0.5 \), the profiles behave like those of regression coefficients with the elastic net. When \( \alpha = 0 \), it is equivalent to regularising the weights with squared \( l_2 \) norm, and the profiles behave like those of regression coefficients with the ridge regression.

For \( \alpha \neq 0 \), the proportion of active portfolios declines as the penalty parameter \( \lambda \) goes large. Active \( l_1 \) penalty facilitates the sparsity, so \( |S^+ \cup S^-|/p = 1 \) when \( \alpha = 0 \). If \( \lambda \) is large enough and \( \alpha = 1 \), it can be shown that the solution of (1) is the solution of the MVP problem with no-shortsale constraints. We have checked this property and find that when \( \alpha = 1 \) and \( \lambda > 5 \), the solution produced by the algorithm is almost the same as the optimal no-shortsale solution. Consequently, when \( \alpha = 1 \) and \( \lambda \) is large, all of the active portfolio weights tend to be positive. When \( \alpha = 0.5 \) and \( \lambda \) is large, no portfolio has any negative weight, but the number of active portfolios is different from the case of \( \alpha = 1 \). The result is not surprising, since \( \alpha = 0.5 \) is equivalent to replacing \( \Sigma \) with \( \Sigma + 0.5 \lambda I_{p \times p} \) in (1) but only having the \( l_1 \) penalty \( 0.5 \lambda \|w\|_{l_1} \) active. Therefore as \( \lambda \) goes large, the new problem will also have a new optimal no-shortsale solution.

3.2 A Comparison with Other Optimisation Solver

We then compare the solutions produced by Algorithms 1 with other optimisation package for solving problem (1). Figure 2 presents cumulative differences between the solutions from Algorithm 1 and cvx (Grant and Boyd, 2010). Let \( \hat{w}_{cd,t} \) and \( \hat{w}_{cvx,t} \) be the \( t \) period solution vectors produced by Algorithm 1 and cvx respectively. The cumulative difference is defined as,

\[
\sum_{t=\tau+1}^{T} \|\hat{w}_{cd,t} - \hat{w}_{cvx,t}\|_{l_1}.
\]

The data used is the same as in section 3.3 (\( \tau = 120 \), and \( T = 483 \) and 555 for 48 industries and 100 size and BM portfolios respectively, see below). We fix \( \alpha = 1 \) and vary \( \lambda \) at six different levels. As can be seen in Figure 2, the cumulative differences are small and decline with \( \lambda \) (but not monotonically). We also use cvx to obtain the optimal no-shortsale weights. Dash line (dot line) shows the cumulative difference between the no-shortsale solutions and \( \hat{w}_{cd,t} \) (\( \hat{w}_{cvx,t} \)) when \( \lambda = 30 \). It is a little bit surprising that \( \hat{w}_{cd,t} \) is even closer to the no-shortsale solutions than \( \hat{w}_{cvx,t} \) is.

3.3 Performances of the Sparse Portfolios

Next we look at how the sparse portfolios perform in real world. The data sets used are monthly return data of Fama and French 48 industry portfolios and 100 portfolios formed on size and book-to-market. The period we select for the 48 industry portfolios is from July of 1969 to September of 2009; for the 100 size and book-to-market portfolios, it is from July of 1963 to September of 2009. For the case of 48 industry portfolios, we do not find any missing data. However we find 89 missing data in the case of
100 size and book-to-market portfolios. If one month has missing data, we use equally weighted returns of other available portfolios at that month to replace the missing data.

We consider the cases of \( \alpha = 1 \) and \( \alpha = 0.5 \) with different \( \lambda \). As mentioned, when \( \alpha = 1 \), similar results already documented in previous research. However, those results are from different algorithms and different strategies for updating the portfolio. For example, Brodie et al. (2009) and Fan, Zhang, and Yu (2009) use modified Least Angle Regression algorithm (LARS) (Efron et al., 2004). Thus one of our focus is on whether the algorithm we use produces different numerical results. We also compare performances of the sparse portfolios with two benchmark portfolios: a naively diversified portfolio with equal weights \( 1/p \) and the no-shortsale portfolio. We rebalance the portfolios every month (monthly updating the weights). Figure 3 to Figure 6 show under different \( \lambda \), the (out of) sample variance of portfolio returns, Sharpe ratio, turnover rate, proportion of active portfolios, absolute position of shortsale portfolios, and the optimal \( \gamma \) from solving (1). Corresponding to each \( \alpha \), we vary \( \lambda \) at 15 different levels, ranging from 0 to 30.

We now introduce the ”rolling window” strategy used to update the portfolio and how we obtain these quantities. Let monthly return of asset \( i \) at period \( t \) be \( r_{i,t} \); \( \Sigma \) used to calibrate to solve (1) is the sample covariance matrix of the monthly returns from previous \( \tau = 120 \) months (from \( t - 120 \) to \( t - 1 \)).

We solve (1) at the end of period \( t - 1 \) to obtain the optimal weights for period \( t \). Let the optimal weight of asset \( i \) for period \( t \) be \( \hat{w}_{i,t} \). Then the (monthly) portfolio return at period \( t \) is

\[
\hat{r}_{\text{por},t} = \sum_{i=1}^{p} \hat{w}_{i,t} r_{i,t}.
\]

We use the 48 industry portfolios as an example. In the data set, we have totally \( T = 483 \) months, and \( t = 121, \ldots, 483 \). We use \( \text{Var}(\hat{r}_{\text{por},t}) \) to denote the sample variance of \( \hat{r}_{\text{por},t} \) over the \( T - t + 1 = 363 \) months. \( \text{Var}(\hat{r}_{\text{por},t}) \) is also called out of sample variance of the portfolio returns. Sharpe ratio \( \left( \text{SR}(\hat{r}_{\text{por},t}) \right) \) is the sample mean of \( \hat{r}_{\text{por},t} \) divided by its sample standard deviation over the \( T - t + 1 = 363 \) months. We calculate the other four quantities monthly and show their boxplots over the \( T - t + 1 = 363 \) months.

We then introduce the turnover rate we use here. Suppose at the end of period \( t - 1 \), we have wealth \( \theta_{t-1} \) to be invested on these assets. Suppose the weight needed to invest on asset \( i \) is \( \hat{w}_{i,t} \) for period \( t \). The value of holding asset \( i \) is then \( \theta_{t-1} \hat{w}_{i,t} (1 + r_{i,t}) \) at the end of period \( t \). The total wealth at period \( t \) is given by

\[
\theta_{t} = \sum_{i=1}^{p} \theta_{t-1} \hat{w}_{i,t} (1 + r_{i,t}) = \theta_{t-1} \left( \sum_{i=1}^{p} \hat{w}_{i,t} + \sum_{i=1}^{p} \hat{w}_{i,t} r_{i,t} \right) = \theta_{t-1} (1 + \hat{r}_{\text{por},t}).
\]

If the weight of asset \( i \) for \( t + 1 \) is \( \hat{w}_{i,t+1} \), then the amount of wealth to invest on asset \( i \) becomes
\[ \theta_t \hat{w}_{i,t+1}. \] We define the turnover rate of asset \( i \) needed for period \( t + 1 \) as

\[
to_{i,t+1} = \left| \frac{\theta_t \hat{w}_{i,t+1} - \theta_{t-1} \hat{w}_{i,t} (1 + r_{i,t})}{\theta_t} \right|.
\]

That is, the proportion of wealth at the end of period \( t \) needed to be invested on asset \( i \) in order to satisfy the amount \( \theta_t \hat{w}_{i,t+1} \). The portfolio turnover rate for period \( t + 1 \) is then defined as

\[
to_{por,t+1} = \sum_{i=1}^{p} to_{i,t+1}.
\]

Knowing \( to_{por,t+1} \) is useful for an investor to evaluate whether a strategy for updating the portfolio is worth to implement or not if transaction costs are taken into account. For example, if buying or selling a stock needs to pay the fees about 0.15% of total value of the stock, and all of the assets considered are stocks, then the expected fees are \( to_{por,t+1} \times 0.0015 \theta_t \).

The last two quantities are defined as the following. Let \( \hat{S}_t^+ = \{ i : \hat{w}_{i,t} > 0 \} \) and \( \hat{S}_t^- = \{ i : \hat{w}_{i,t} < 0 \} \).

Proportion of active portfolios at period \( t \) is defined as:

\[
PAC_t = \frac{\vert \hat{S}_t^+ \cup \hat{S}_t^- \vert}{p}.
\]

Absolute position of shortsale portfolios at period \( t \) is defined as sum of absolute values of the negative weights:

\[
APS_t = \sum_{i \in \hat{S}_t^-} \vert \hat{w}_{i,t} \vert.
\]

The empirical results can be summarised as follows.

1. When \( \alpha = 1 \), the results of \( \hat{Var}(\hat{\tilde{r}}_{por,t}) \) and \( \hat{SR}(\hat{\tilde{r}}_{por,t}) \) corresponding to different \( \lambda \) are pretty similar as those shown in Figure 2 of Indics \( \text{et al.} \) (2009) and Figure 7 of Fan, Zhang, and Yu (2009). When \( \lambda \) increases, the out of sample variance of the sparse portfolio declines to its minimum and then converges to the level when the no-shortsale portfolio is held. In the case of 48 industry portfolios, the \( 1/p \) portfolio has the largest \( \hat{Var}(\hat{\tilde{r}}_{por,t}) \), while in the case of 100 size and BM portfolios, the unregularised MVP (corresponding to \( \lambda = 0 \)) has the largest \( \hat{Var}(\tilde{r}_{por,t}) \). This seems to suggest that the naive diversification is a better way to reduce portfolio variance than the unregularised MVP when \( p \) is relatively large to sample size. But the portfolio variance of the \( 1/p \) portfolio are still larger than those which weights are regularised in the FF 100 size and BM case. For the case of \( \alpha = 0.5 \), all the relevant results are very similar to the case of \( \alpha = 1 \).

2. When \( \alpha = 1 \), Sharpe ratio increases monotonically with \( \lambda \) in the case of 48 industry portfolios, and achieves its maximum level when the no-shortsale portfolio is held. But in the case of 100 size and BM portfolios, it reaches its maximum value when \( \lambda \) is around 1.7, and then starts to decline to the level when the no-shortsale portfolio is held. It is unknown why the same method produces two different patterns of Sharpe ratio paths for the two different data sets. It also worth to note that when \( \alpha = 0.5 \), as for the 48 industry portfolios, if we set \( \lambda \geq 5 \), we can obtain slightly
better performances than those in the case of $\alpha = 1$; but this benefit does not happen in the case of 100 size and BM portfolios.

3. As for the issue of transaction costs, it can be seen that $1/p$ portfolio has the lowest turnover rate. The result is widely documented in previous research. However, for the sparse portfolio, we find that the turnover rate also monotonically decreases as $\lambda$ increases. This suggests regularisation on portfolio weights also facilitates their stabilities over time, and is helpful for an investor to make decision when transaction costs are taken into account.

4. When $\alpha = 1$, $PAC_t$ and $APS_t$ both decline monotonically with $\lambda$. For $PAC_t$, the reason is that $l_1$ penalty facilitates sparsity. For $APS_t$, the reason is that as $\lambda$ increases, the optimal solution of \[ \] will converges to the solution of no-shortsale constrained problem, and consequently the absolute position of shortsale declines to zero. For the case of $\alpha = 0.5$, $APS_t$ shows very similar behaviour as in the case of $\alpha = 1$. However, $PAC_t$ no longer declines monotonically with $\lambda$. It reaches its minimum value and then slightly increases as $\lambda$ goes large. In general, we have more active assets in the case of $\alpha = 0.5$ than $\alpha = 1$.

5. As a Lagrange multiplier, $\gamma$ seems not to be so interesting as the optimal portfolio weights. However, we conjecture that $\gamma$ may play an important role on controlling how many assets should be included in the portfolio. To know how many assets should be included in a portfolio is helpful on allocating sources to monitor the individual asset performances. The coordinate-wise descent algorithm used here cannot tell us exactly how many assets are already included during the processing. One way we can do is to adjust $\lambda$ to roughly know this when the algorithm is processing. Some algorithms in regression-based variable selection have such advantage; for example, Least Angle Regression (LARS). For a regression function, LARS selects only one covariate each step, and after $k$ steps, there will be exact $k$ covariates in the regression. Therefore we can exactly control the number of included variables by controlling the iterative steps of LARS. LARS algorithm does not need the penalty parameter $\lambda$. The tricky part of LARS lies in its search direction and length of each iterated step. [Efron et al. (2004)] show the length of each iterated step of LARS can be optimally determined by its search direction in order to satisfy the "one step, one variable in" property. As can be seen in the figures, $\gamma$ monotonically increases with $\lambda$. Note that $\lambda$ determines how many assets should be included in the portfolio. Therefore we wonder if we try to solve a sparse MVP problem in the way similar as LARS in regression-based variable selection, without $\lambda$, $\gamma$ can provide the information of how many assets already have been included in the portfolio during the iterations. Future research is needed to confirm this conjecture.

4 Some Extensions

The sparse MVP problem is very similar to the regression-based variable selection problem. One of the differences between the two problems is that in the sparse MVP problem, sum of the portfolio weights is required to be 1. If we can guarantee that the constraint is satisfied, then many techniques used in variable selections can be applied to the problem. In this section we show some possible extensions.
4.1 Mean-Variance Portfolio

Until now we only consider the problem of purely minimising portfolio variance rather than the following mean-variance portfolio optimisation,

\[
\min_w -\tau w^T \mu + w^T \Sigma w + \lambda \alpha \|w\|_1 + \lambda (1 - \alpha) \|w\|_2^2 \quad \text{subject to } w^T 1_p = 1,
\]

(5)

where \(\tau\) is the risk preference parameter, and \(\mu = (\mu_1, \ldots, \mu_p)^T\) is the vector of expected asset returns.

The problem (5) can be solved via a modified version of Algorithm 1 if we use the following update forms:

\[
w_i \leftarrow \frac{ST \left(\gamma - (\tau \mu_i + z_i), \lambda \alpha\right)}{2 \left(\sigma_i^2 + \lambda (1 - \alpha)\right)},
\]

\[
\gamma \leftarrow \frac{1 + \sum_{i \in S^+ \cup S^-} \frac{\tau \mu_i + z_i}{2 \left(\sigma_i^2 + \lambda (1 - \alpha)\right)}}{\left(\sum_{i \in S^+ \cup S^-} \frac{1}{2 \left(\sigma_i^2 + \lambda (1 - \alpha)\right)}\right)} - \lambda \alpha \left(\sum_{i \in S^+} \frac{1}{2 \left(\sigma_i^2 + \lambda (1 - \alpha)\right)} - \sum_{i \in S^-} \frac{1}{2 \left(\sigma_i^2 + \lambda (1 - \alpha)\right)}\right).
\]

Practically, to solve (5), we need to estimate \(\mu\), and it is not an easy task. One naive way to avoid such estimation is to specify \(\mu\) according to some prior beliefs; for example, if we set \(\mu = 0\), (5) becomes (1). DeMiguel et al. (2009) and Jagannathan and Ma (2003) demonstrate unsatisfactory empirical evidences of portfolio selections via the traditional mean-variance portfolio optimisation. This is why we mainly consider (1) instead of (5) in this report.

4.2 Weighted \(l_1\) Penalty

We can specify the penalty parameter according to our prior information about which assets are more important than the others. Considering a modified version of (1):

\[
\min_w w^T \Sigma w + \lambda \sum_{i=1}^p \eta_i |w_i| \quad \text{subject to } w^T 1_p = 1,
\]

(6)

where \(\eta_i\) is a nonnegative constant. The weighted \(l_1\) penalty is also used by adaptive LASSO in Zou (2006). The modified update forms are

\[
w_i \leftarrow \frac{ST \left(\gamma - z_i, \lambda \eta_i\right)}{2 \left(\sigma_i^2 + \lambda \eta_i\right)},
\]

\[
\gamma \leftarrow \frac{1 + \sum_{i \in S^+ \cup S^-} \frac{z_i}{2 \left(\sigma_i^2 + \lambda \eta_i\right)}}{\left(\sum_{i \in S^+ \cup S^-} \frac{1}{2 \left(\sigma_i^2 + \lambda \eta_i\right)}\right)} - \lambda \left(\sum_{i \in S^+} \frac{\eta_i}{2 \left(\sigma_i^2 + \lambda \eta_i\right)} - \sum_{i \in S^-} \frac{\eta_i}{2 \left(\sigma_i^2 + \lambda \eta_i\right)}\right).
\]

In Brodie et al. (2009), \(\eta_i\) is viewed as the transaction cost for the \(i\) th asset. One also can view \(\eta_i\) as an increasing function of risk of asset \(i\), such as its \(\sigma_i^2\) or beta.
4.3 Berhu Penalty

The coordinate-wise descent algorithm can be applied to other penalty forms. For example, the berhu penalty proposed by Owen (2006):

\[ \lambda \sum_{i=1}^{p} \left( |w_i| \mathbf{1}\{ |w_i| < \delta \} + \frac{w_i^2 + \delta^2}{2\delta} \mathbf{1}\{ |w_i| \geq \delta \} \right), \tag{7} \]

where \( \mathbf{1}\{ \cdot \} \) is the indicator function. The name "berhu" comes from that \( (7) \) is the reverse of Huber’s loss. The berhu penalty is convex and satisfies separability. It is also a comprise between \( l_1 \) and squared \( l_2 \) regularisation. If \( |w_i| \) is less than some criterion \( \delta > 0 \), it is regularised by \( l_1 \) norm. If \( |w_i| \) is no less than \( \delta \), then it is regularised by squared \( l_2 \) norm.

With the berhu penalty and fixed \( \delta \), the first order derivative of the Lagrangian function is given by

\[ 2w_i \sigma_i^2 + 2 \sum_{j \neq i}^p w_j \sigma_{ij} - \gamma + \lambda \text{sign}(w_i) \mathbf{1}\{ |w_i| < \delta \} + \lambda \frac{w_i}{\delta} \mathbf{1}\{ |w_i| \geq \delta \} = 0, \]

for \( i = 1, \ldots, p \). It can be shown that at the stationary point,

\[ 2w_i \sigma_i^2 + 2 \sum_{j \neq i}^p w_j \sigma_{ij} - \gamma = -\lambda \text{ if } 0 < w_i < \delta, \]

\[ 2w_i \sigma_i^2 + 2 \sum_{j \neq i}^p w_j \sigma_{ij} - \gamma = \lambda \text{ if } -\delta < w_i < 0, \]

\[ 2w_i \sigma_i^2 + 2 \sum_{j \neq i}^p w_j \sigma_{ij} - \gamma \leq \lambda \text{ if } w_i = 0, \]

\[ 2w_i \sigma_i^2 + 2 \sum_{j \neq i}^p w_j \sigma_{ij} - \gamma + \frac{\lambda w_i}{\delta} = 0 \text{ if } \delta \leq |w_i|, \]

\[ w^T \mathbf{1}_p = 1. \]

Fixing \( w_j \) for \( j = 1, \ldots, p \) and \( i \neq j \), we can solve the above equation for \( w_i \). When \( \delta \leq |w_i|, w_i = \frac{\gamma - z_i}{2\sigma_i^2 + \frac{\delta}{\lambda}}. \)

By \( 2\sigma_i^2 + \frac{\delta}{\lambda} > 0, \delta \leq |w_i| \) implies that \( |\gamma - z_i| \geq 2\sigma_i^2 \delta + \lambda \). When \( 0 < w_i < \delta, w_i = \frac{\gamma - z_i - \lambda}{2\sigma_i^2} \), and this implies that \( \gamma - z_i < 2\sigma_i^2 \delta + \lambda \). When \( -\delta < w_i < 0, w_i = \frac{\gamma - z_i + \lambda}{2\sigma_i^2} \), and it implies that \( \gamma - z_i > -2\sigma_i^2 \delta - \lambda \).

We therefore propose the following form to update each \( w_i \),

\[ w_i \leftarrow \frac{ST(\gamma - z_i, \lambda)}{2\sigma_i^2}, \text{ if } |\gamma - z_i| < 2\sigma_i^2 \delta + \lambda, \]

\[ w_i \leftarrow \frac{\gamma - z_i}{2\sigma_i^2 + \frac{\delta}{\lambda}}, \text{ if } |\gamma - z_i| \geq 2\sigma_i^2 \delta + \lambda. \]

Since the updating form of \( w_i \) is also linear in \( \gamma \), we can derive updating form of \( \gamma \) easily via \( w^T \mathbf{1}_p = 1. \)
Let $\Delta^- = \{i : |\gamma - z_i| < 2\sigma_i^2 \delta + \lambda\}$ and $\Delta^+ = \{i : |\gamma - z_i| \geq 2\sigma_i^2 \delta + \lambda\}$. Then

$$\mathbf{w}^T \mathbf{1}_p = \sum_{i=1}^{p} w_i$$

$$= \sum_{i \in S^+ \cap \Delta^-} w_i + \sum_{i \in S^- \cap \Delta^-} w_i + \sum_{i \in \Delta^+} w_i$$

$$= \sum_{i \in S^+ \cap \Delta^-} \frac{|\gamma - z_i - \lambda|}{2\sigma_i^2} + \sum_{i \in S^- \cap \Delta^-} \frac{|\gamma - z_i + \lambda|}{2\sigma_i^2} + \sum_{i \in \Delta^+} \frac{|\gamma - z_i|}{2\sigma_i^2 + \lambda}$$

$$= \gamma \left( \sum_{i \in (S^+ \cap \Delta^-) \cup (S^- \cap \Delta^-)} \frac{1}{2\sigma_i^2} + \sum_{i \in \Delta^+} \frac{1}{2\sigma_i^2 + \lambda} \right) - \sum_{i \in (S^+ \cap \Delta^-) \cup (S^- \cap \Delta^-)} \frac{z_i}{2\sigma_i^2} + \lambda \left( \sum_{i \in S^- \cap \Delta^-} \frac{1}{2\sigma_i^2} - \sum_{i \in S^+ \cap \Delta^-} \frac{1}{2\sigma_i^2} \right).$$

With the constraint $\mathbf{w}^T \mathbf{1}_p = 1$, we propose following form to update $\gamma$,

$$\gamma \leftarrow 1 + \frac{\left( \sum_{i \in (S^+ \cap \Delta^-) \cup (S^- \cap \Delta^-)} \frac{z_i}{2\sigma_i^2} + \sum_{i \in \Delta^+} \frac{z_i}{2\sigma_i^2 + \lambda} \right) - \lambda \left( \sum_{i \in S^- \cap \Delta^-} \frac{1}{2\sigma_i^2} - \sum_{i \in S^+ \cap \Delta^-} \frac{1}{2\sigma_i^2} \right)}{\left( \sum_{i \in (S^+ \cap \Delta^-) \cup (S^- \cap \Delta^-)} \frac{1}{2\sigma_i^2} + \sum_{i \in \Delta^+} \frac{1}{2\sigma_i^2 + \lambda} \right)}.$$

The algorithm for the berhu penalty is similar as Algorithm 1, with a modification on updating $w_i$ and $\gamma$. The algorithm can be summarised as follows.

**Algorithm 2 Naive Coordinate-Wise Descent Updating for MVP Constrained by the Berhu Penalty**

1. Fix $\lambda$ and $\delta$ at some constant levels.
2. Initialise $\mathbf{w} = \frac{1}{p} \mathbf{1}_p$ and $\gamma = \lambda \times 1.1$.
3. For $i = 1, \ldots, p$,
   
   if $|\gamma - z_i| < 2\sigma_i^2 \delta + \lambda$, $w_i \leftarrow \frac{ST (\gamma - z_i, \lambda)}{2\sigma_i^2}$,

   otherwise
   
   $$w_i \leftarrow \frac{\gamma - z_i}{2\sigma_i^2 + \lambda}.$$

4. Let $z_i = 2 \sum_{j \neq i} w_j \sigma_{ij}$. Update $\gamma$ as

   $$\gamma \leftarrow 1 + \frac{\left( \sum_{i \in (S^+ \cap \Delta^-) \cup (S^- \cap \Delta^-)} \frac{z_i}{2\sigma_i^2} + \sum_{i \in \Delta^+} \frac{z_i}{2\sigma_i^2 + \lambda} \right) - \lambda \left( \sum_{i \in S^- \cap \Delta^-} \frac{1}{2\sigma_i^2} - \sum_{i \in S^+ \cap \Delta^-} \frac{1}{2\sigma_i^2} \right)}{\left( \sum_{i \in (S^+ \cap \Delta^-) \cup (S^- \cap \Delta^-)} \frac{1}{2\sigma_i^2} + \sum_{i \in \Delta^+} \frac{1}{2\sigma_i^2 + \lambda} \right)}.$$

5. Repeat 3 and 4 until $\mathbf{w}$ and $\gamma$ converge.

Figure [I] shows when the berhu penalty is imposed, profiles of portfolio weights, proportion of active portfolios, and proportion of shortsale portfolios under different $\delta$ and $\lambda$. The data used is the same as in Figure [II]. As $\delta = 1$, the profiles are almost the same as those in the case when only $l_1$ constraint is active. However, if $\delta$ deviates from 1, the profiles look very different from the previous cases.
4.4 Adaptive Group Portfolio

Yuan and Lin (2006) propose the group LASSO which can select covariates "grouply" in a regression problem: either entire covariates in a certain group are all selected or all of them are dropped out. The penalty for variable selection in the group LASSO is the Euclidean norm $\| \cdot \|_2$. However, the group LASSO can only result sparsity between groups; it cannot facilitate sparsity within a group. Friedman et al. (2010) propose the group sparse LASSO in which the penalty is a hybrid form which combines with the group penalty and $l_1$ norm. They show that such setting can facilitate sparsity between groups as well as within a group.

Categorising assets into different groups based on certain characteristics of assets and then forming a portfolio according to these characteristics, is almost a standard process in portfolio management. This kind of strategic selecting assets of certain groups may be due to some practical reasons, e.g. financial regulations, investor’s risk preferences or a fund's own objective. However, such strategy often concentrates on certain groups of assets and ignore the benefits from more diversities. Thus in term of minimising the overall portfolio variance, it is not optimal. In the following, we try to take the idea of group selection, but at the same time, also consider to minimise the overall variance. To fairly compromise the two goals, at first, we do not to pre-specify which groups of assets should be more important, nor do we ditch boundaries of groups and purely minimise the portfolio variance. Instead we group these assets according to their common features, and then minimising the portfolio variance with the group penalty. Thus the typical outcome is: either some assets (not "all" if we take sparsity within a group into account) in a certain group are selected or all of them are dropped out. Indeed this may produce zero weights for one’s preferred groups of assets. However, if we can specify different penalties to different groups of assets, such individual preference can be satisfied easily. Our method allows us to do such setting.

Let $w_l = (w_{l1}, \ldots, w_{lK})$ be the portfolio weight vector for $K$ assets in group $l$, $l = 1, \ldots, L$. Without loss of generality, for each group, we assume the number of assets all equal to $K$. The case of different number of assets in different groups can be easily modified in our algorithm. However, we do not allow different groups can have the same assets. If different groups can have the same assets, the block coordinate descent method may not guarantee a stable convergence. Let $w = (w_1, \ldots, w_L)$, then the modified MVP problem is

$$\min_w w^T \Sigma w + \lambda_1 \sum_{l=1}^L \| w_l \|_{l_2, \Omega_l} + \lambda_2 \| w \|_{l_1}, \quad \text{subject to } w^T 1_p = 1, \quad (8)$$

where $\| w_l \|_{l_2, \Omega_l} = \sqrt{w_l^T \Omega_l w_l}$, and $\Omega_l$ is a kernel matrix, which is required to be symmetric and positive definite. By the definition, the Euclidean norm of $w_l$ can then be expressed as

$$\| w_l \|_{l_2} = \| w_l \|_{l_2, I_K \times K}$$

Suppose we have $p$ assets, then $p = L \times K$ and $\dim (w) = 1 \times p$. The penalty parameters, $\lambda_1$ and $\lambda_2$
are still required to be nonnegative. For convenience, we re-express $\Sigma$ as

$$\Sigma = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1L} \\
A_{21} & A_{22} & \cdots & A_{2L} \\
\vdots & \vdots & \ddots & \vdots \\
A_{L1} & A_{L2} & \cdots & A_{LL}
\end{pmatrix}.$$  

For $l, l' = 1, \ldots, L$, if $l = l'$, $A_{ll'}$ is the $K \times K$ covariance matrix for asset returns in group $l$. If $l \neq l'$, $A_{ll'}$ is a $K \times K$ matrix for covariances of asset returns between group $l$ and $l'$. For example, if $K = 5$ and $l \neq 1$,

$$A_{11} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{15} \\
\sigma_{21} & \sigma_2^2 & \cdots & \sigma_{25} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{51} & \sigma_{52} & \cdots & \sigma_5^2 \end{pmatrix} \quad \text{and} \quad A_{1l} = \begin{pmatrix} \sigma_{1,s} & \sigma_{1,s+1} & \cdots & \sigma_{1,s+4} \\
\sigma_{2,s} & \sigma_{2,s+1} & \cdots & \sigma_{2,s+4} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{5,s} & \sigma_{5,s+1} & \cdots & \sigma_{5,s+4} \end{pmatrix},$$

where $s = 5l - 4$.

Now we show that a block coordinate-wise descent algorithm can be used to solve the MVP problem when portfolio weights are penalised by the group penalty. Here we consider a special case of (8) in which $\lambda_2 = 0$ and $\Omega_l = A_{ll}$. We call the solved optimal solution "adaptive group portfolio". Let $\gamma$ be the Lagrange multiplier of constraint $w^T p$. With such setting, the update form for $w_l$ and $\gamma$ can be solved explicitly. Under this restriction, (8) becomes

$$\min_w w^T \Sigma w + \lambda_1 \sum_{l=1}^L \|w_l\|_{ll,A_{ll}} \quad \text{subject to} \quad w^T p = 1. \quad (9)$$

We can reparameterise $w_l$ as

$$w_l = A_{ll}^{-\frac{1}{2}} x_l,$$

where $x_l$ is also a $K \times 1$ vector. Therefore $\|w_l\|_{ll,A_{ll}} = \|x_l\|_{l_2}$. Let $x = (x_1, \ldots, x_L)$, then problem (9) becomes

$$\min_x x^T \Sigma' x + \lambda_1 \sum_{l=1}^L \|x_l\|_{l_2} \quad \text{subject to} \quad \sum_{l=1}^L x_l^T A_{ll}^{-\frac{1}{2}} 1_K = 1, \quad (10)$$

where

$$\Sigma' = \begin{pmatrix}
1 & A_{11}^{-\frac{1}{2}} A_{12} A_{22}^{-\frac{1}{2}} & \cdots & A_{11}^{-\frac{1}{2}} A_{1L} A_{LL}^{-\frac{1}{2}} \\
A_{22}^{-\frac{1}{2}} A_{21} A_{11}^{-\frac{1}{2}} & 1 & \cdots & A_{22}^{-\frac{1}{2}} A_{2L} A_{LL}^{-\frac{1}{2}} \\
\vdots & \vdots & \ddots & \vdots \\
A_{LL}^{-\frac{1}{2}} A_{LL}^{-\frac{1}{2}} A_{L1} A_{11}^{-\frac{1}{2}} & A_{LL}^{-\frac{1}{2}} A_{L2} A_{22}^{-\frac{1}{2}} & \cdots & 1
\end{pmatrix}.$$
At the stationary point, for $l = 1, \ldots, L$,

\[ 2x_l + 2A_{l1}^{-1} \left( \sum_{j \neq l}^L A_{lj} A_{jj}^{-\frac{1}{2}} x_j \right) + \lambda_1 \frac{x_l}{\|x_l\|_{l_2}} - \gamma A_{l1}^{-\frac{1}{2}} 1_K = 0, \text{ if } x_l \neq 0, \]

\[ 2A_{l1}^{-\frac{1}{2}} \left( \sum_{j \neq l}^L A_{lj} A_{jj}^{-\frac{1}{2}} x_j \right) + \lambda_1 s_l - \gamma A_{l1}^{-\frac{1}{2}} 1_K = 0, \text{ if } x_l = 0, \]

\[ \sum_{l=1}^L x_l^T A_{l1}^{-\frac{1}{2}} 1_K = 1. \]

where $s_l$ is a $K \times 1$ vector, and $\|s_l\|_{l_2} \leq 1$. Let $B_l = 2 \sum_{j \neq l}^L A_{lj} A_{jj}^{-\frac{1}{2}} x_j$ and $\Lambda_l(\gamma) = \left\|A_{l1}^{-\frac{1}{2}} \gamma 1_K - A_{l1}^{-\frac{1}{2}} B_l\right\|_{l_2}$.

The necessary and sufficient condition for $x_l = 0$ is

\[ \Lambda_l(\gamma) \leq \lambda_1. \]  

From (11), if $x_l \neq 0$,

\[ x_l = \frac{A_{l1}^{-\frac{1}{2}} (\gamma 1_K - B_l)}{2 + \frac{\lambda_1}{\|x_l\|_{l_2}}}. \]

As $x_l \neq 0$ implies $w_l \neq 0$, so

\[ w_l = A_{l1}^{-\frac{1}{2}} x_l = \frac{A_{l1}^{-1} (\gamma 1_K - B_l)}{2 + \frac{\lambda_1}{\|w_l\|_{l_2, A_{l1}}}}. \]

We now show $\|w_l\|_{l_2, A_{l1}}$ can be solved as a function of $\gamma$. To see this, we know that

\[ \left( 2 + \frac{\lambda_1}{\|w_l\|_{l_2, A_{l1}}} \right) A_{l1}^\frac{1}{2} w_l = A_{l1}^{-\frac{1}{2}} (\gamma 1_K - B_l), \]

and then

\[ w_l^T A_{l1}^\frac{1}{2} \left( 4 + \frac{4\lambda_1}{\|w_l\|_{l_2, A_{l1}}} + \frac{\lambda_1^2}{\|w_l\|_{l_2, A_{l1}}} \right) A_{l1}^\frac{1}{2} w_l = (\gamma 1_K - B_l)^T A_{l1}^{-1} (\gamma 1_K - B_l). \]

Thus we can obtain

\[ \left( 2 \|w_l\|_{l_2, A_{l1}} + \lambda_1 \right)^2 = \Lambda_l^2(\gamma). \]

By $\|w_l\|_{l_2, A_{l1}} > 0$ and $\lambda_1 \geq 0$, if $\Lambda_l - \lambda_1 > 0$, the solution for $\|w_l\|_{l_2, A_{l1}}$ is given by

\[ \|w_l\|_{l_2, A_{l1}} = \frac{\Lambda_l(\gamma) - \lambda_1}{2}. \]

Therefore if $w_l \neq 0$,

\[ w_l = \frac{1}{2} \left( 1 - \frac{\lambda_1}{\Lambda_l(\gamma)} \right) A_{l1}^{-1} (\gamma 1_K - B_l). \]

Combining with the group-level test condition (12), we propose the following form to update the group
weight vector $\mathbf{w}_l$,
\[
\mathbf{w}_l \leftarrow \frac{1}{2} \left( 1 - \frac{\lambda_1}{\Omega_l(\gamma)} \right) + A_{ll}^{-1} (\gamma \mathbf{1}_K - B_l).
\]

For updating $\gamma$, we can use the constraint $\mathbf{w}^T \mathbf{1}_p = 1$. However, since the weights are non-linear in $\gamma$, we cannot update $\gamma$ as in the previous cases. Let $S_l = \{ l : \mathbf{w}_l \neq 0 \}$. Practically, updated $\gamma$ can be solved by minimising
\[
\left( 1 - \sum_{l \in S_l} \left[ \frac{1}{2} \left( 1 - \frac{\lambda_1}{\Omega_l(\gamma)} \right) + A_{ll}^{-1} (\gamma \mathbf{1}_K - B_l) \right] \right)^2
\]
with respect to $\gamma$.

For stability of our algorithm, we set the initial values: $w_1 = w_2 = \cdots = w_p = 1/p$, and $\gamma > \lambda_1 \sqrt{\max_{i=1,\ldots,p} \sigma_i^2}$. The updating process starts from $\mathbf{w}_1$, then $\mathbf{w}_2$, ..., then $\mathbf{w}_L$. The updated vector $\mathbf{w}$ is then used to update $\gamma$. The process terminates until $\mathbf{w}$ and $\gamma$ converge. The algorithm can be summarised as follows.

**Algorithm 3 Block Coordinate-Wise Descent Updating for MVP Penalised by Weighted Euclidean Norm**

1. Fix $\lambda_1$ at some non-negative constant level.

2. Initialise $\mathbf{w} = \frac{1}{p} \mathbf{1}_{p \times 1}$ and $\gamma = \lambda_1 \sqrt{\max_{i=1,\ldots,p} \sigma_i^2} \times 1.1$.

3. Let $2 \sum_{j \neq l} A_{lj} \mathbf{w}_j = B_l$. For $l = 1, \ldots, L$,
\[
\mathbf{w}_l \leftarrow \frac{1}{2} \left( 1 - \frac{\lambda_1}{\Omega_l(\gamma)} \right) + A_{ll}^{-1} (\gamma \mathbf{1}_K - B_l).
\]

4. Update $\gamma$ by
\[
\gamma \leftarrow \arg \min_{\gamma} \left( 1 - \sum_{l \in S_l} \left[ \frac{1}{2} \left( 1 - \frac{\lambda_1}{\Omega_l(\gamma)} \right) + A_{ll}^{-1} (\gamma \mathbf{1}_K - B_l) \right] \right)^2.
\]

5. Repeat 3 and 4 until $\mathbf{w}$ and $\gamma$ converge.

We use Fama and French 100 size and BM ratio portfolios as an example. The data set contains value weighted returns for the intersections of 10 market cap portfolios and 10 BM ratio portfolios. We categorise these 100 portfolios via two different ways. The first method is to group them according to 10 market cap levels; and in each group, we have 10 different BM ratio portfolios. The second method is opposite: we group them according to 10 BM ratio levels; and in each group, we have 10 different market cap portfolios. Thus the two settings both have $L = K = 10$. The results are shown in Figure 8, 9, 10 and 11. We let "Size-BM" and "BM-Size" denote the first and second method respectively. All quantities are calculated via the same ways as in section 3.2. Sample size for estimating the sample covariance matrices is also 120.

Different methods for grouping result in similar reductions on out-of-sample portfolio variances, but significant differences for Sharp ratios. As can be seen in Figure 11, the two grouping methods also have different dynamics of $PAC_t$. Comparing with Figure 5 and Figure 6 when the penalty parameter becomes large, the Sharp ratios for Size-BM are slightly better than the cases of no grouping.
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Figure 1: The figure shows under different $\alpha$ and $\lambda$, profiles of portfolio weights, proportion of active portfolios, and proportion of shortsale portfolios. We vary $\lambda = 0, 0.5, \ldots, 15$. The data used is from November 1986 to October 1996 of Fama and French 25 portfolios formed on size and book-to-market ratio.
Figure 2: The figure presents cumulative differences between the solutions from Algorithm 1 and cvx (Grant and Boyd, 2010) for solving problem (1). We vary $\lambda$ at 6 different levels. The data used is monthly return data of Fama and French 48 industries and 100 size and book-to-market portfolios. The period we select for the 48 industry portfolios is from July of 1969 to September of 2009; for the 100 size and book-to-market portfolios, it is from July of 1963 to September of 2009.
Figure 3: The figure shows under different $\lambda$ and trading strategies, the (out of) sample variance of portfolio returns, Sharpe ratio, turnover rate, proportion of active portfolios, absolute position of shortsale portfolios, and the optimal $\gamma$ from solving (1). We vary $\lambda$ at 15 different levels, ranging from 0 to 30. The data used is monthly return data of Fama and French 48 industry portfolios. The period we select is from July of 1969 to September of 2009. In the box plots, ns and $1/p$ denote the no-shortsale and equally weighted portfolios respectively.
Figure 4: The figure shows under different λ and trading strategies, the (out of) sample variance of portfolio returns, Sharpe ratio, turnover rate, proportion of active portfolios, absolute position of shortsale portfolios, and the optimal γ from solving (1). We vary λ at 15 different levels, ranging from 0 to 30. The data used is monthly return data of Fama and French 48 industry portfolios. The period we select is from July of 1969 to September of 2009. In the box plots, ns and 1/p denote the no-shortsale and equally weighted portfolios respectively.
Figure 5: The figure shows under different $\lambda$ and trading strategies, the (out of) sample variance of portfolio returns, Sharpe ratio, turnover rate, proportion of active portfolios, absolute position of shortsale portfolios, and the optimal $\gamma$ from solving (1). We vary $\lambda$ at 15 different levels, ranging from 0 to 30. The data used is monthly return data of Fama and French 100 size and book-to-market portfolios. The period we select is from July of 1963 to September of 2009. In the box plots, ns and $1/p$ denote the no-shortsale and equally weighted portfolios respectively. In this case, if one month has missing data, we use equally weighted returns of other available portfolios at that month to replace the missing data.
Figure 6: The figure shows under different $\lambda$ and trading strategies, the (out of) sample variance of portfolio returns, Sharpe ratio, turnover rate, proportion of active portfolios, absolute position of shortsale portfolios, and the optimal $\gamma$ from solving (1). We vary $\lambda$ at 15 different levels, ranging from 0 to 30. The data used is monthly return data of Fama and French 100 size and book-to-market portfolios. The period we select is from July of 1963 to September of 2009. In the box plots, ns and 1/p denote the no-shortsale and equally weighted portfolios respectively. In this case, if one month has missing data, we use equally weighted returns of other available portfolios at that month to replace the missing data.
Figure 7: The figure shows profiles of portfolio weights, proportion of active portfolios, and proportion of shortsale portfolios under different $\delta$ and $\lambda$ when the berhu penalty is imposed. We vary $\lambda = 0, 0.5, \ldots, 15$. The data used here is the same as in Figure 1.
Figure 8: The figure shows under different $\lambda$, the (out of) sample variance of portfolio returns and Sharpe ratio for adaptive group portfolios. We vary $\lambda$ at 15 different levels, ranging from 0 to 30. The data used is monthly return data of Fama and French 100 size and book-to-market portfolios. The period we select is from July of 1963 to September of 2009. We categorise these 100 portfolios via two different ways as described in section 4.4. In this case, if one month has missing data, we use equally weighted returns of other available portfolios at that month to replace the missing data.
Figure 9: The figure shows under different $\lambda$ and trading strategies, turnover rates and proportion of active portfolios of adaptive group, no-shortsale and 1/p portfolios. We vary $\lambda$ at 15 different levels, ranging from 0 to 30. The data used is monthly return data of Fama and French 100 size and book-to-market portfolios. The period we select is from July of 1963 to September of 2009. We categorise these 100 portfolios via two different ways as described in section 4.4. In the box plots, ns and 1/p denote the no-shortsale and equally weighted portfolios respectively. In this case, if one month has missing data, we use equally weighted returns of other available portfolios at that month to replace the missing data.
Figure 10: The figure shows under different $\lambda$ and trading strategies, absolute positions of shortsale of adaptive group, no-shortsale and $1/p$ portfolios; and the optimal $\gamma$ from solving (9). We vary $\lambda$ at 15 different levels, ranging from 0 to 30. The data used is monthly return data of Fama and French 100 size and book-to-market portfolios. The period we select is from July of 1963 to September of 2009. We categorise these 100 portfolios via two different ways as described in section 4.4. In the box plots, ns and $1/p$ denote the no-shortsale and equally weighted portfolios respectively. In this case, if one month has missing data, we use equally weighted returns of other available portfolios at that month to replace the missing data.
Figure 11: The figure shows monthly dynamics of $PAC_t$ of two different adaptive portfolios Size-BM and BM-Size with different $\lambda_1$, from July of 1973 to September of 2009. The dot-dash line (blue), solid line (black), dash line (red) and dot line (green) in each graph correspond to $\lambda_1 = 0.2, 1.2, 5$ and 30.