Geometric quantization of relativistic Hamiltonian mechanics

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Abstract. A relativistic Hamiltonian mechanical system is seen as a conservative Dirac constraint system on the cotangent bundle of a pseudo-Riemannian manifold. We provide geometric quantization of this cotangent bundle where the quantum constraint serves as a relativistic quantum equation.

We are based on the fact that both relativistic and non-relativistic mechanical systems on a configuration space $Q$ can be seen as conservative Dirac constraint systems on the cotangent bundle $T^*Q$ of $Q$, but occupy its different subbundles. Therefore, one can follow suit of geometric quantization of non-relativistic time-dependent mechanics in order to quantize relativistic mechanics.

Recall that, given a symplectic manifold $(Z, \Omega)$ and a Hamiltonian $H$ on $Z$, a Dirac constraint system on a closed imbedded submanifold $i_N : N \to Z$ of $Z$ is defined as a Hamiltonian system on $N$ provided with the pull-back presymplectic form $\Omega_N = i_N^*\Omega$ and the pull-back Hamiltonian $i_N^*H$. Its solution is a vector field $\gamma$ on $N$ which fulfils the equation

$$\gamma]|\Omega_N + i_N^*dH = 0.$$ 

Let $N$ be coisotropic. Then a solution exists if the Poisson bracket $\{H, f\}$ vanishes on $N$ whenever $f$ is a function vanishing on $N$. It is the Hamiltonian vector field of $H$ on $Z$ restricted to $N$.

A configuration space of non-relativistic time-dependent mechanics (henceforth NRM) of $m$ degrees of freedom is an $(m + 1)$-dimensional smooth fibre bundle $Q \to \mathbb{R}$ over the time axis $\mathbb{R}$.

It is coordinated by $(q^\lambda) = (q^0, q^i)$, where $q^0$ is the standard Cartesian coordinate on $\mathbb{R}$. Let $T^*Q$ be the cotangent bundle of $Q$ equipped with the induced coordinates $(q^\lambda, p_\lambda = \dot{q}_\lambda)$ with respect to the holonomic coframes $\{dq^\lambda\}$. Provided with the canonical symplectic form

$$\Omega = dp_\lambda \wedge dq^\lambda,$$

(1)

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the cotangent bundle \( T^*Q \) plays the role of a homogeneous momentum phase space of NRM. Its momentum phase space is the vertical cotangent bundle \( V^*Q \) of \( Q \to \mathbb{R} \) coordinated by \((q^\lambda, q^i)\). A Hamiltonian \( \mathcal{H} \) of NRM is defined as a section \( p_0 = -\mathcal{H} \) of the fibre bundle \( T^*Q \to V^*Q \). Then the momentum phase space of NRM can be identified with the image \( N \) of \( \mathcal{H} \) in \( T^*Q \) which is the one-codimensional (consequently, coisotropic) imbedded submanifold given by the constraint

\[
\mathcal{H}_T = p_0 + \mathcal{H}(q^\lambda, p_k) = 0.
\]

Furthermore, a solution of a non-relativistic Hamiltonian system with a Hamiltonian \( \mathcal{H} \) is the restriction \( \gamma \to N \cong V^*Q \) of the Hamiltonian vector field of \( \mathcal{H}_T \) on \( T^*Q \). It obeys the equation \( \gamma_\mathcal{H} \Omega_N = 0 \) \([4, 11]\). Moreover, one can show that geometric quantization of \( V^*Q \) is equivalent to geometric quantization of the cotangent bundle \( T^*Q \) where the quantum constraint \( \hat{\mathcal{H}}_T \psi = 0 \) on sections \( \psi \) of the quantum bundle serves as the Schrödinger equation \([3, 4]\). This quantization is a variant of quantization of presymplectic manifolds via coisotropic imbeddings \([3]\).

A configuration space of relativistic mechanics (henceforth RM) is an oriented pseudo-Riemannian manifold \((Q,g)\), coordinated by \((q^\lambda)\). Its momentum phase space is the cotangent bundle \( T^*Q \) provided with the symplectic form \( \Omega \) \((1)\). Note that one also considers another symplectic form \( \Omega + F \) where \( F \) is the strength of an electromagnetic field \([12]\). A relativistic Hamiltonian is defined as a smooth real function \( H \) on \( T^*Q \) \([7, 10, 11]\). Then a relativistic Hamiltonian system is described as a Dirac constraint system on the subspace \( N \) of \( T^*Q \) given by the equation

\[
H_T = g_{\mu\nu} \partial^\mu H \partial^\nu H - 1 = 0.
\]

Similarly to geometric quantization of NRM, we provide geometric quantization of the cotangent bundle \( T^*Q \) and characterize a quantum relativistic Hamiltonian system by the quantum constraint

\[
\hat{H}_T \psi = 0.
\]

We choose the vertical polarization on \( T^*Q \) spanned by the tangent vectors \( \partial^\lambda \). The corresponding quantum algebra \( \mathcal{A} \subset C^\infty(T^*Q) \) consists of affine functions of momenta

\[
f = a^\lambda(q^\mu)p_\lambda + b(q^\mu)
\]

on \( T^*Q \). They are represented by the Schrödinger operators

\[
\hat{f} = -ia^\lambda \partial_\lambda - i2 \partial_\lambda a^\lambda - i4 a^\lambda \partial_\lambda \ln(-g) + b, \quad g = \det(g_{\alpha\beta})
\]
in the space $\mathbb{C}^\infty(Q)$ of smooth complex functions on $Q$.

Note that the function $H_T$ need not belong to the quantum algebra $\mathcal{A}$. Nevertheless, one can show that, if $H_T$ is a polynomial of momenta of degree $k$, it can be represented as a finite composition

$$H_T = \sum_i f_{1i} \cdots f_{ki}$$

of products of affine functions (4), i.e., as an element of the enveloping algebra $\overline{\mathcal{A}}$ of the Lie algebra $\mathcal{A}$ [3]. Then it is quantized

$$H_T \mapsto \hat{H}_T = \sum_i \hat{f}_{1i} \cdots \hat{f}_{ki}$$

as an element of $\overline{\mathcal{A}}$. However, the representation (6) and, consequently, the quantization (7) fail to be unique.

Let us provide the above mentioned formulation of classical RM as a constraint autonomous mechanics on a pseudo-Riemannian manifold $(Q, g)$ [2, 7, 8]. Note that it need not be a space-time manifold.

The space of relativistic velocities of RM on $Q$ is the the tangent bundle $TQ$ of $Q$ equipped with the induced coordinates $(q^\lambda, \dot{q}^\lambda)$ with respect to the holonomic frames $\{\partial_\lambda\}$. Relativistic motion is located in the subbundle $W_g$ of hyperboloids

$$g_{\mu\nu}(q)\dot{q}^\mu \dot{q}^\nu - 1 = 0$$

of $TQ$. It is described by a second order dynamic equation

$$\ddot{q}^\lambda = \Xi^\lambda(q^\mu, \dot{q}^\mu)$$

on $Q$ which preserves the subbundle (8), i.e.,

$$(\dot{q}^\lambda \partial_\lambda + \Xi^\lambda \partial_\lambda)(g_{\mu\nu}\dot{q}^\mu \dot{q}^\nu - 1) = 0, \quad \partial_\lambda = \partial/\partial \dot{q}^\lambda.$$  

This condition holds if the right-hand side of the equation (9) takes the form

$$\Xi^\lambda = \{_{\mu}^\lambda_{\nu}\} \dot{q}^\mu \dot{q}^\nu + F^\lambda,$$

where $\{_{\mu}^\lambda_{\nu}\}$ are Cristoffel symbols of a metric $g$, while $F^\lambda$ obey the relation $g_{\mu\nu}F^\mu \dot{q}^\nu = 0$. In particular, if the dynamic equation (4) is a geodesic equation

$$\ddot{q}^\lambda = K^\lambda_\mu \dot{q}^\mu$$

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with respect to a (non-linear) connection

\[ K = dq^\lambda \otimes (\partial_\lambda + K_\lambda^\mu \hat{\partial}_\mu) \]

on the tangent bundle \( TQ \to Q \), this connections splits into the sum

\[ K_\mu^\lambda = \{ \mu^\lambda , \nu \} \dot{q}^\nu + F_\mu^\lambda \] (10)

of the Levi–Civita connection of \( g \) and a soldering form

\[ F = g^{\lambda \nu} F_{\mu \nu} dq^\mu \otimes \dot{\partial}_\lambda, \quad F_{\mu \nu} = -F_{\nu \mu}. \]

As was mentioned above, the momentum phase space of RM on \( Q \) is the cotangent bundle \( T^*Q \) provided with the symplectic form \( \Omega \) (11). Let \( H \) be a smooth real function on \( T^*Q \) such that the morphism

\[ \tilde{H} : T^*Q \to TQ, \quad \dot{q}^\mu = \partial^\mu H \] (11)

is a bundle isomorphism. Then the inverse image \( N = \tilde{H}^{-1}(W_g) \) of the subbundle of hyperboloids \( W_g \) (8) is a one-codimensional (consequently, coisotropic) closed imbedded subbundle of \( T^*Q \) given by the constraint \( H_T = 0 \) (2). We say that \( H \) is a relativistic Hamiltonian if the Poisson bracket \( \{ H, H_T \} \) vanishes on \( N \). This means that the Hamiltonian vector field

\[ \gamma = \partial^\lambda H \partial_\lambda - \partial_\lambda H \partial^\lambda \] (12)

of \( H \) preserves the constraint \( N \) and, restricted to \( N \), it obeys the Hamilton equation

\[ \gamma \rvert \Omega_N + i^*_N dH = 0 \] (13)

of a Dirac constraint system on \( N \) with a Hamiltonian \( H \).

The morphism (11) sends the vector field \( \gamma \) (12) onto the vector field

\[ \gamma_T = \dot{q}^\lambda \partial_\lambda + (\partial^\mu H \partial^\lambda \partial_\mu H - \partial_\mu H \partial^\lambda \partial^\mu H) \hat{\partial}_\lambda \]

on \( TQ \). This vector field defines the second order dynamic equation

\[ \ddot{q}^\lambda = \partial^\mu H \partial^\lambda \partial_\mu H - \partial_\mu H \partial^\lambda \partial^\mu H \] (14)

on \( Q \) which preserves the subbundle of hyperboloids (8).
Example 1. The following is a basic example of relativistic Hamiltonian systems. Put

\[ H = \frac{1}{2m} g^{\mu\nu} (p_\mu - b_\mu)(p_\nu - b_\nu), \]

where \( m \) is a constant and \( b_\mu dq^\mu \) is a covector field on \( Q \). Then \( H_T = 2m^{-1}H - 1 \) and \( \{H, H_T\} = 0 \). The constraint \( H_T = 0 \) defines a closed imbedded one-codimensional subbundle \( N \) of \( T^*Q \). The Hamilton equation \((13)\) takes the form \( \gamma | \Omega_N = 0 \). Its solution \((12)\) reads

\[ \dot{q}^\alpha = \frac{1}{m} g^{\alpha\nu} (p_\nu - b_\nu), \]

\[ \dot{p}_\alpha = -\frac{1}{2m} \partial_\alpha g^{\mu\nu} (p_\mu - b_\mu)(p_\nu - b_\nu) + \frac{1}{m} g^{\mu\nu} (p_\mu - b_\mu) \partial_\alpha b_\nu. \]

The corresponding second order dynamic equation \((14)\) on \( Q \) is

\[ \ddot{q}^\lambda = \{\mu, \nu\} \dot{q}^\mu \dot{q}^\nu - \frac{1}{m} g^{\lambda\nu} F_{\mu\nu} \dot{q}^\mu, \]

\[ \{\mu, \nu\} = -\frac{1}{2} g^{\lambda\beta} (\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}), \quad F_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu. \]

It is a geodesic equation with respect to the affine connection

\[ K^\lambda_\mu = \{\mu, \nu\} \dot{q}^\nu - \frac{1}{m} g^{\lambda\nu} F_{\mu\nu} \]

of type \((11)\). For instance, let \( g \) be a metric gravitational field and let \( b_\mu = eA_\mu \), where \( A_\mu \) is an electromagnetic potential whose gauge holds fixed. Then the equation \((15)\) is the well-known equation of motion of a relativistic massive charge in the presence of these fields.

Turn now to quantization of RM. We follow the standard geometric quantization of the cotangent bundle \([1, 12, 13]\). Because the canonical symplectic form \( \Omega \) \((1)\) on \( T^*Q \) is exact, the prequantum bundle is defined as a trivial complex line bundle \( C \) over \( T^*Q \). Note that this bundle need no metaplectic correction since \( T^*X \) is endowed with canonical coordinates for the symplectic form \( \Omega \). Thus, \( C \) is a quantum bundle. Let its trivialization

\[ C \cong T^*Q \times \mathbb{C} \]

hold fixed, and let \((q^\lambda, p_\lambda, c), c \in \mathbb{C}\), be the associated bundle coordinates. Then one can treat sections of \( C \) \((14)\) as smooth complex functions on \( T^*Q \). Note that another trivialization of \( C \) leads to an equivalent quantization of \( T^*Q \).
The Kostant–Souriau prequantization formula associates to each smooth real function $f \in C^\infty(T^*Q)$ on $T^*Q$ the first order differential operator
\[
\hat{f} = -i\nabla_{\vartheta_f} + f
\]  
(17)
on sections of $C$, where $\vartheta_f = \partial^\lambda f \partial_\lambda - \partial_\lambda f \partial^\lambda$ is the Hamiltonian vector field of $f$ and $\nabla$ is the covariant differential with respect to a suitable $U(1)$-principal connection $A$ on $C$. This connection preserves the Hermitian metric $g(c, c') = cc'$ on $C$, and its curvature form obeys the prequantization condition $R = i\Omega$. For the sake of simplicity, let us assume that $Q$ and, consequently, $T^*Q$ is simply connected. Then the connection $A$ up to gauge transformations is
\[
A = dp_\lambda \otimes \partial^\lambda + dq_\lambda \otimes (\partial_\lambda + icp_\lambda \partial_c),
\]  
(18)and the prequantization operators (17) read
\[
\hat{f} = -i\vartheta_f + (f - p_\lambda \partial^\lambda f).
\]  
(19)

Let us choose the vertical polarization on $T^*Q$. It is the vertical tangent bundle $VT^*Q$ of the fibration $\pi : T^*Q \to Q$. As was mentioned above, the corresponding quantum algebra $\mathcal{A} \subset C^\infty(T^*Q)$ consists of affine functions $f$ of momenta $p_\lambda$. Its representation by operators (19) is defined in the space $E$ of sections $\rho$ of the quantum bundle $C$ of compact support which obey the condition $\nabla_{\vartheta} \rho = 0$ for any vertical Hamiltonian vector field $\vartheta$ on $T^*Q$. This condition takes the form
\[
\partial_\lambda f \partial^\lambda \rho = 0, \quad \forall f \in C^\infty(Q).
\]
It follows that elements of $E$ are independent of momenta and, consequently, fail to be compactly supported, unless $\rho = 0$. This is the well-known problem of Schrödinger quantization which is solved as follows [1, 3].

Let $i_Q : Q \to T^*Q$ be the canonical zero section of the cotangent bundle $T^*Q$. Let $C_Q = i_Q C$ be the pull-back of the bundle $C$ (16) over $Q$. It is a trivial complex line bundle $C_Q = Q \times \mathbb{C}$ provided with the pull-back Hermitian metric $g(c, c') = cc'$ and the pull-back
\[
A_Q = i_Q A = dq^\lambda \otimes (\partial_\lambda + icp_\lambda \partial_c)
\]
of the connection $A$ (18) on $C$. Sections of $C_Q$ are smooth complex functions on $Q$, but this bundle need metaplectic correction.
Let the cohomology group $H^2(Q;\mathbb{Z}_2)$ of $Q$ be trivial. Then a metalinear bundle $\mathcal{D}$ of complex half-forms on $Q$ is defined. It admits the canonical lift of any vector field $\tau$ on $Q$ such that the corresponding Lie derivative of its sections reads

$$L_\tau = \tau^\lambda \partial_\lambda + \frac{1}{2} \partial_\lambda \tau^\lambda.$$  

Let us consider the tensor product $Y = C_Q \otimes \mathcal{D}$ over $Q$. Since the Hamiltonian vector fields

$$\vartheta_f = a^\lambda \partial_\lambda - (p_\mu \partial_\lambda a^\mu + \partial_\lambda b) \partial^\lambda$$

of functions $f$ are projected onto $Q$, one can assign to each element $f$ of the quantum algebra $\mathcal{A}$ the first order differential operator

$$\hat{f} = (-i \nabla_{\pi \vartheta_f} + f) \otimes \text{Id} + \text{Id} \otimes \mathcal{L}_{\pi \vartheta_f} = -ia^\lambda \partial_\lambda - \frac{i}{2} \partial_\lambda a^\lambda + b$$

on sections $\rho_Q$ of $Y$. For the sake of simplicity, let us choose a trivial metalinear bundle $\mathcal{D} \rightarrow Q$ associated to the orientation of $Q$. Its sections can be written in the form $\rho_Q = (-g)^{1/4} \psi$, where $\psi$ are smooth complex functions on $Q$. Then the quantum algebra $\mathcal{A}$ can be represented by the operators $\hat{f}$ in the space $C^\infty(Q)$ of these functions. It is easily justified that these operators obey the Dirac condition

$$[\hat{f}, \hat{f}'] = -i \{\hat{f}, \hat{f}'\}. $$

Remark 2. One usually considers the subspace $E_Q \subset C^\infty(Q)$ of functions of compact support. It is a pre-Hilbert space with respect to the non-degenerate Hermitian form

$$\langle \psi | \psi' \rangle = \int_Q \overline{\psi} \psi' (-g)^{1/2} d^{m+1}q$$

It is readily observed that $\hat{f}$ are symmetric operators $\hat{f} = \hat{f}^*$ in $E_Q$, i.e., $\langle \hat{f} \psi | \psi' \rangle = \langle \psi | \hat{f} \psi' \rangle$. In RM, the space $E_Q$ however gets no physical meaning.

As was mentioned above, the function $H_T$ need not belong to the quantum algebra $\mathcal{A}$, but a polynomial function $H_T$ can be quantized as an element of the enveloping algebra $\overline{\mathcal{A}}$ by operators $\overline{H_T}$. Then the quantum constraint (3) serves as a relativistic quantum equation.
Example 3. Let us consider a massive relativistic charge in Example 1 whose relativistic Hamiltonian is

\[ H = \frac{1}{2m} g^{\mu\nu} (p_\mu - eA_\mu)(p_\nu - eA_\nu). \]

It defines the constraint

\[ H_T = \frac{1}{m^2} g^{\mu\nu} (p_\mu - eA_\mu)(p_\nu - eA_\nu) - 1 = 0. \tag{20} \]

Let us represent the function \( H_T \) as the symmetric product

\[ H_T = \frac{(-g)^{-1/4}}{m} \cdot (p_\mu - eA_\mu) \cdot (-g)^{1/4} \cdot g^{\mu\nu} \cdot (-g)^{1/4} \cdot (p_\nu - eA_\nu) \cdot \frac{(-g)^{-1/4}}{m} - 1 \]

of affine functions of momenta. It is quantized by the rule \( \partial_\alpha \), where

\[ (-g)^{1/4} \circ \partial_\alpha \circ (-g)^{-1/4} = -i\partial_\alpha. \]

Then the well-known relativistic quantum equation

\[ (-g)^{-1/2}[(\partial_\mu - ieA_\mu)g^{\mu\nu}(-g)^{1/2}(\partial_\nu - ieA_\nu) + m^2]\psi = 0. \tag{21} \]

is reproduced up to the factor \((-g)^{-1/2}\).

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