Bifurcations of Invariant Torus and Knotted Periodic Orbits in Generalized Hopf-Langford Type Equations

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Bifurcations of Invariant Torus and Knotted Periodic Orbits in Generalized Hopf-Langford Type Equations

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Abstract In this paper, we study the bifurcations of invariant torus and knotted periodic orbits for generalized Hopf-Langford type equations. By using bifurcation theory of dynamical systems, we obtain the exact explicit form of the heteroclinic orbits and knot periodic orbits. Moreover, under small perturbation, we prove that the perturbed planar system has two symmetric stable limit cycles created by Poincare bifurcations. Therefore, the corresponding three-dimensional perturbed system has an attractive invariant rotation torus.

Keywords bifurcation · exact solution · knotted periodic orbit · invariant torus · generalized Hopf-Langford type equations

1 Introduction

In 2018, Yang et al. [1] studied the complex dynamics for a generalized Langford system
\begin{align}
\frac{dx_1}{dt} &= ax_1 + bx_2 + x_1x_3, \\
\frac{dx_2}{dt} &= cx_1 + dx_2 + x_2x_3, \\
\frac{dx_3}{dt} &= ex_3 - (x_1^2 + x_2^2 + x_3^2),
\end{align}
(1)
using averaging theory and bifurcation theory of three-dimensional phase space. For \(d = a, c = -b, e = -2a\) and \(ab \neq 0\), they proved that system (1) simultaneously has two heteroclinic cycles and a periodic orbit (Corollary 1, page 2262 in [1]).
Recently, Nikolov and Vassilev [2] continue to consider the following generalized Hopf-Langford system [3]

\[
\begin{align*}
\frac{dx_1}{dt} &= ax_1 - bx_2 + x_1x_3, \\
\frac{dx_2}{dt} &= bx_1 + ax_2 + x_2x_3, \\
\frac{dx_3}{dt} &= cx_3 - (x_1^2 + x_2^2 + x_3^2),
\end{align*}
\]

and showed that system (2) turns out to be equivalent to the nonlinear damped force-free Duffing oscillator. It follows from [4,5] that, for \(c = -2a\), the Duffing oscillator is integrable and equivalent to a first-order equation. The solutions of this first-order equation are explicitly expressed in [6,7], and thus they gave the periodic orbits expressed by means of elementary and Jacobi elliptic functions.

In this paper, we study the following more generalized Hopf-Langford type system

\[
\begin{align*}
\frac{dx}{dt} &= \alpha x - \omega y + xz, \\
\frac{dy}{dt} &= \omega x + \alpha y + yz, \\
\frac{dz}{dt} &= cz - a(x^2 + y^2) - bz^2.
\end{align*}
\]

By the cylinder coordinates transformation \(x = \rho \cos \theta, y = \rho \sin \theta\), system (3) is rewritten as

\[
\begin{align*}
\frac{d\rho}{dt} &= \rho(\alpha + z), \\
\frac{dz}{dt} &= cz - a\rho^2 - bz^2, \\
\frac{d\theta}{dt} &= \omega.
\end{align*}
\]

The first equation of system (4) implies that \(z = \frac{\dot{\rho}}{\rho} - \alpha, z' = \frac{1}{\rho^2}(\rho \rho'' - (\rho')^2)\). Thus, it follows from the second equation of (4) that

\[
\rho \rho'' + (b - 1)(\rho')^2 - (2b\alpha + c)\rho \rho' + \alpha(c + b\alpha)\rho^2 + a\rho^3 = 0.
\]

When \(b = 1\), equation (5) is just the Duffing oscillator as follows:

\[
\rho'' - (2\alpha + c)\rho' + \alpha(c + \alpha)\rho + a\rho^3 = 0.
\]

For \(b \neq 0\) and \(b \neq -1\), by the transformation \(\alpha + z \rightarrow -z\), system (4) becomes

\[
\begin{align*}
\frac{d\rho}{dt} &= -\rho z, \\
\frac{dz}{dt} &= \alpha(b\alpha + c) + (2b\alpha + c)z + a\rho^2 + bz^2, \\
\frac{d\theta}{dt} &= \omega.
\end{align*}
\]

When \(2b\alpha + c = 0\), i.e., \(\alpha = \frac{-c}{2b}\), system (7) becomes

\[
\begin{align*}
\frac{d\rho}{dt} &= -\rho z, \\
\frac{dz}{dt} &= -\frac{c^2}{4b} + a\rho^2 + bz^2, \\
\frac{d\theta}{dt} &= \omega
\end{align*}
\]

with the first integral:

\[
H(\rho, z) = \rho^2 \left(\frac{c^2}{4b} + \frac{a}{b + 1}\rho^2 - \frac{c^2}{4b^2}\right) = h.
\]
When $2\alpha + c = \epsilon \lambda, \epsilon \ll 1$, we discuss the planar perturbation system

$$\frac{d\rho}{dt} = -\rho z, \quad \frac{dz}{dt} = -\frac{c^2}{4b} + a\rho^2 + bz^2 - \epsilon z(\mu z^2 - \lambda). \quad (10)$$

This paper is organized as follows. In section 2, we discuss exact explicit solutions of system (7) and knot periodic orbits when $\alpha = -\frac{c}{2b}$. In section 3, we investigate the bifurcations of invariant torus of system (10) when $b = 1$. In section 4, we derive the exact solutions of system (3) when $b = 1, \alpha = -2c$. The main results are stated in Theorem 1-4 below.

**Remark 1** We obtain the explicit analytical form of two heteroclinic cycles in the following equation (20), which aren’t given in [1].

**Remark 2** Comparing with the method of direct using known solutions of Duffing oscillator in [2], bifurcation theory can provide more information of the solutions for understanding the dynamics of the Hopf-Langford type system, such as heteroclinic cycles, limit cycles and unbounded solutions. This is the reason why we study the more generalized Hopf-Langford type system (3).

## 2 Exact explicit solutions of system (7) and knot periodic orbits when $\alpha = -\frac{c}{2b}$

In the $(\rho, z)$–phase plane, we consider planar dynamical system

$$\frac{d\rho}{dt} = -\rho z, \quad \frac{dz}{dt} = -\frac{c^2}{4b} + a\rho^2 + bz^2. \quad (11)$$

Clearly, $\rho = 0$ is an invariant straight line solution of equation (11). Assume that $a > 0, b > 0, c > 0$. Then, system (11) has four equilibrium points at $C\left(-\frac{c}{2\sqrt{ab}}, 0\right), D\left(\frac{c}{2\sqrt{ab}}, 0\right)$ and $A\left(0, \frac{c}{b}\right), B\left(0, -\frac{c}{2b}\right)$. $C$ and $D$ are centers, while $A$ and $B$ are saddle points. For $H(\rho, z) = h$ defined by (9), we have $h_c = H\left(-\frac{c}{2\sqrt{ab}}, 0\right) = -\frac{c^2}{\sqrt{3}b(b+1)}, h_s = H\left(0, \frac{c}{2b}\right) = 0$. The phase portrait of system (11) is shown in Fig.1.

![Fig. 1: The phase portraits of system (11) for a fixed parameter group (a, b, c).](image)
It follows from Fig.1 that when \( h \in (h_c, 0) \), there exist two symmetric families \( \{I^h_1\} \) and \( \{I^h_2\} \) of closed orbits enclosing the equilibrium points \( C \) and \( D \), respectively. When \( h = 0 \), there exist two symmetric heteroclinic orbits \( I^0_{1,2} \) connecting two equilibrium points \( A \) and \( B \). To find exact solutions of system (11), from (9) and the first equation of (11), we know that

\[
\sqrt{\frac{a}{b+1}} t = \int_{\rho_0}^{\rho} \frac{\rho^{b-1} d\rho}{\sqrt{\frac{b+1}{a} h - \frac{(b+1)c^2}{4ab^2} \rho^2 - \rho^2 + 2}}. \tag{12}
\]

2.1 The case \( b = 1 \)

(i) Corresponding to the two families \( \{I^h_1\} \) of closed orbits defined by \( H(\rho, y) = h, h \in (h_c, 0) \), (12) can be written as

\[
\sqrt{\frac{a}{2}} t = \int_{\rho_0}^{\rho} \frac{d\rho}{\sqrt{\frac{1}{a} h - \frac{1}{4a} \rho^2 - \rho^2 + 2}},
\]

where \( \rho_0 = \frac{1}{4a} (c^2 + \Delta), \rho_0^2 = \frac{1}{4a} (c^2 + \Delta), \Delta = c^4 + 32ah \). It gives rise to the parametric representations of \( \{I^h_1\} \) as follows:

\[
\rho(t) = \pm \frac{\rho_0}{\text{dn}(\Omega_1 t, k)}, \quad z(t) = \frac{\Omega_1 k^2 \text{sn}(\Omega_1 t, k) \text{cn}(\Omega_1 t, k)}{\text{dn}(\Omega_1 t, k)}, \tag{13}
\]

where \( k^2 = 1 - \frac{c^2}{\rho_0^2} = \frac{2c^2}{c^2 + \sqrt{\Delta}}, \Omega_1 = \frac{c^2}{\sqrt{\Delta}}, \text{sn}(\cdot, k), \text{cn}(\cdot, k), \text{dn}(\cdot, k) \) are Jacobian elliptic functions (see [8]). It is easy to see that when \( h \rightarrow h_c, k \rightarrow 0 \); when \( h \rightarrow 0, k \rightarrow 1 \). Notice that \( h = \frac{c^4}{4a} \left( \frac{k^4}{(2-k^2)^2} - 1 \right) = h(k) \). For \( \rho(t), z(t) \) defined by (13), we can write them as \( \rho(t) = \rho_k(t), z(t) = z_k(t), k \in (0, 1) \).

(ii) Corresponding to the two heteroclinic orbits \( \{I^0_{1,2}\} \) defined by \( H(\rho, y) = 0 \), (12) becomes

\[
\sqrt{\frac{a}{2}} t = \int_{\rho_0}^{\rho_M} \frac{d\rho}{\sqrt{\frac{1}{a} \rho^2 - \rho^2}},
\]

where \( \rho_M^2 = \frac{c^2}{2a} \). It gives rise to the parametric representations of \( I^0_{1,2} \) as follows:

\[
\rho = \rho_0(t) = \frac{c}{\sqrt{2a}} \text{sech} \left( \frac{c t}{2} \right), \quad z = z_0(t) = \frac{c}{2} \tanh \left( \frac{c t}{2} \right). \tag{14}
\]

2.2 The case \( b = 2 \)

(i) Corresponding to the two families \( \{I^h_{1,2}\} \) of closed orbits defined by \( H(\rho, y) = h, h \in \left( -\frac{c^2}{30v_{2a}}, 0 \right) \), (12) can be written as

\[
\sqrt{\frac{4a}{3}} t = \int_{\rho_0}^{\rho} \frac{d\rho^2}{\sqrt{\frac{3}{a} h + \frac{3c^2}{16a} (\rho^2 - (\rho^2))}} = \int_{\rho_0^2}^{\rho^2} \frac{d\rho^2}{\sqrt{(\rho_0^2 - \rho^2)(\rho^2 - \rho_0^2)(\rho^2 + \rho_0^2)}}.
\]
Thus, we have the following parametric representations of \( \{ T^1_{\Gamma} \} \):

\[
\rho(t) = \pm \left( -\rho_c^2 + \frac{\rho_b^2 + \rho_c^2}{\text{dn}(\Omega_1 t, k)} \right)^{1/2}, \quad z(t) = \frac{\Omega_2 (\rho_c^2 - \rho_b^2) \text{sn}(\Omega_2 t, k) \text{cn}(\Omega_2 t, k)}{(\rho_b^2 + \rho_c^2) - \rho_c^2 \text{dn}^2(\Omega_2 t, k)},
\]

where \( \Omega_2 = \sqrt{\frac{n(\rho_b^2 + \rho_c^2)}{4}}, k^2 = \frac{\rho_b^2 - \rho_c^2}{\rho_b^2 + \rho_c^2} \).

(ii) Corresponding to the two heteroclinic orbits \( \{ T^0_{\Gamma} \} \) defined by \( H(p, y) = 0 \), (12) becomes

\[
\frac{d^2}{dt^2} t = \rho_M^2 \left( \rho_M \rho_0^2 \sqrt{\rho_0^2 - \rho_M^2} \right)^{1/2},
\]

where \( \rho_M^2 = \frac{3a^2}{16b} \). It gives rise to the parametric representations of \( T^0_{\Gamma} \) as follows:

\[
\rho(t) = \rho_M \text{sech} \left( \frac{c}{4} t \right), \quad z(t) = \frac{c}{4} \tanh \left( \frac{c}{4} t \right).
\]

2.3 The knot periodic orbits of system (8)

In the three-dimensional phase space, when \( b = 1 \), system (8) has the exact solutions:

\[
x(t) = \frac{p_0 \cos(\omega t)}{\text{dn}(\Omega_1 t, k)}, \quad y(t) = \frac{p_0 \sin(\omega t)}{\text{dn}(\Omega_1 t, k)}, \quad z(t) = \frac{\Omega_1 k^2 \text{sn}(\Omega_2 t, k) \text{cn}(\Omega_2 t, k)}{\text{dn}(\Omega_1 t, k)},
\]

with the initial condition \((x(0), y(0), z(0)) = (p_0, 0, 0)\). When \( b = 2 \), system (8) has the exact solutions:

\[
x(t) = \left( -\rho_c^2 + \frac{\rho_b^2 + \rho_c^2}{\text{dn}(\Omega_2 t, k)} \right)^{1/2} \cos(\omega t),
\]

\[
y(t) = \left( -\rho_c^2 + \frac{\rho_b^2 + \rho_c^2}{\text{dn}(\Omega_2 t, k)} \right)^{1/2} \sin(\omega t),
\]

\[
z(t) = \frac{\Omega_2 (\rho_c^2 - \rho_b^2) \text{sn}(\Omega_2 t, k) \text{cn}(\Omega_2 t, k)}{(\rho_b^2 + \rho_c^2) - \rho_c^2 \text{dn}^2(\Omega_2 t, k)},
\]

with the initial condition \((x(0), y(0), z(0)) = (p_0, 0, 0)\).

Note that the functions \( \cos(\omega t) \) and \( \sin(\omega t) \) have the period \( T_0 = \frac{2\pi}{\omega} \), and the function \( \text{dn}(\Omega_j t, k), j = 1, 2 \), has the period \( T_j(k) = \frac{2K(k)}{\omega_j} \), where \( K(k) \) is the complete elliptic integral of the first kind with the modulo \( k \). Generally, \( T_j(k) \) and \( T_0 \) are irreducible, therefore, for given \( k \in (0, 1) \) and \( \omega \), the functions defined by (17) and (18) are quasi-periodic solutions of system (8) which lie on the invariant torus consisting of the rotating surface created by \( T^0_{\Gamma} \).

We notice that a link \( L \) is a collection of pairwise disjoint oriented simple closed curves embedded in (oriented) \( S^3 \). A link of type \( L \) is its equivalence class under the relation \( L \sim L' \), i.e. there is an orientation preserving homeomorphism \( h : (S^3, L) \to (S^3, L) \).

A knot is a link consisting of a single component. The genus of \( L \) is the smallest integer \( g \) such that \( L \) is the boundary of an embedded orientable surface \( M \subset S^3 \), where \( M \) has genus \( g \). A braid on \( p \) strands is a presentation
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of a knot that its projection onto the plane passes in the same direction about the origin \( p \) times. If the genus of a knot \( K \) is one, i.e. \( K \) lies in a torus \( T_2 \), we say that \( K \) is a torus knot. A \((m, n)\)-torus knot is a special braid which winds on \( T_2 \) in the longitudinal direction \( m \) times, and in the latitudinal direction \( n \) times. Different pairs \((m, n)\) correspond to distinct type of knots. Obviously, \( m \) is also the number of strands of a torus knot (see [9] and the references therein).

Since \( \frac{2\pi}{T_j(k)} \leq T_j(k) < \infty \), for every fixed \( k \in (0, 1) \) and every relatively prime positive integer pair \((m, n)\), there is a unique \( \omega \) such that

\[
\frac{2K(k)}{\Omega_j} = \frac{2m\pi}{n\omega}, \quad \text{i.e.,} \quad \omega = \frac{2m\pi\Omega_j}{2nK(k)}. \tag{19}
\]

In this case, the functions defined by (17) give rise to a periodic solution of (8) with \((m, n)\)-torus knot type and the period \( nT_j(k) \). This knotted periodic solution lies on the invariant torus of (8) consisting of the rotating surface created by the periodic orbit \( \Gamma^b_h, h = h(k) \).

On the other hand, because \( k \to 1, K(k) \to \infty \), for a fixed \( \omega > 0 \), there are infinitely many \((m, n)\)-pairs such that the relationship (19) holds. It implies that for \( b = 1, 2 \), system (8) has infinitely many types of \((m, n)\)-torus knot of periodic orbits. In addition, for two fixed values of \( \omega \) and \( n \), as \( k \to 1, m \to \infty \), it follows that an infinite sequence of \((m, n)\)-torus knot periodic orbits approximates to the aperiodic heteroclinic manifold of (8) given by the rotating surface created by \( \Gamma_0^{b}, \) in which there exists the heteroclinic orbit (see Fig.2)

\[
x(t) = \frac{c}{\sqrt{2a}} \sech \left( \frac{c}{2} t \right) \cos(\omega t), \quad y(t) = \frac{c}{\sqrt{2a}} \sech \left( \frac{c}{2} t \right) \sin(\omega t), \quad z(t) = \frac{c}{2} \tanh \left( \frac{c}{2} t \right) \tag{20}
\]

or

\[
x(t) = \rho_M \sech \left( \frac{c}{4} t \right) \cos(\omega t), \quad y(t) = \rho_M \sech \left( \frac{c}{4} t \right) \sin(\omega t), \quad z(t) = \frac{c}{4} \tanh \left( \frac{c}{4} t \right). \tag{21}
\]

Fig. 2: The heteroclinic orbits of system (8) in three-dimensional phase.

From the above discussion, we have proved the following conclusion.
Theorem 1 Suppose that $a > 0, c > 0, b = 1$ or $2$ in system (8).

1. System (8) has infinitely many $(m, n)$-torus knots of periodic orbits, which lie on an invariant torus family of (8).

2. For a given pair $(m, n)$ and $k \in (0, 1)$, there exists a unique $\omega$, such that system (8) has $(m, n)$-torus knot of periodic solution.

3. For a given integer $n$, when $k \to 1$, there exists a sequence of $\omega$s satisfying (19), such that there is an infinite sequence of $(m, n)$-torus knot periodic orbits which approximate the aperiodic heteroclinic manifold of (8).

Fig. 3: The knotted periodic orbits of system (8) in three-dimensional phase space.

3 Bifurcation of invariant torus of system (10) when $b = 1$

In this section, in the $(\rho, z)$-plane, we consider the solutions of system (10). To study the existence of bifurcation of limit cycles, we investigate the Melnikov function

$$M_1(h) = \int_{\Gamma} (P_1 Q_0 - Q_1 P_0) \exp \left( -\int_0^t \left( \frac{\partial P_0}{\partial \rho} + \frac{\partial Q_0}{\partial z} \right) dt \right) dt,$$

where $P_0 = -\rho z, Q_0 = -\frac{c^2}{2} + a\rho^2 + b z^2, P_1 = 0, Q_1 = z(\mu z^2 - \lambda)$. By the bifurcation theory of dynamical systems (see [10,11]), if $M_1(h)$ has a simple zero $h_0$, then, near the closed orbit $I_2^0$ of unperturbed system, there exists a limit cycle of system (10). For our unperturbed system (10), we have $z(t) = -\frac{\rho(t)}{\rho(0)}$, hence,

$$\exp \left( -\int_0^t \left( \frac{\partial P_0}{\partial \rho} + \frac{\partial Q_0}{\partial z} \right) dt \right) = \exp \left( \int_0^t z(t) dt \right) = \exp \left( \int_0^t d \ln \rho(t) \right) = \frac{\rho(t)}{\rho_0}.$$
Along the closed orbit $I_3^c$, calculating the Melnikov function, we get from (9) that $z_k^2(t) = \frac{h(k)}{\mu k} - \frac{\mu}{2} \rho_k^2 + \frac{c^2}{4}$, thus,

$$M_1(h) = \frac{1}{\rho_k} \int_0^{T_1(k)} \rho_k^2(t) z_k^2(t)|-\mu z_k^2(t) + \lambda| dt$$

$$= \frac{1}{\rho_k} \left[ -\mu \int_0^{T_1(k)} \rho_k^2 \left( \frac{h(k)}{\rho_k^2} - \frac{\mu}{2} \rho_k^2 + \frac{c^2}{4} \right)^2 dt + \lambda \int_0^{T_1(k)} \rho_k^2 \left( \frac{h(k)}{\rho_k^2} - \frac{\mu}{2} \rho_k^2 + \frac{c^2}{4} \right) dt \right],$$

where $\rho_k(t), z_k(t)$ are given by (13). We define a detection function

$$\lambda(h(k)) = \frac{\mu \psi(h(k))}{\phi(h(k))} = \frac{\mu \int_0^{T_1(k)} \rho_k^2 \left( \frac{h(k)}{\rho_k^2} - \frac{\mu}{2} \rho_k^2 + \frac{c^2}{4} \right)^2 dt}{\int_0^{T_1(k)} \rho_k^2 \left( \frac{h(k)}{\rho_k^2} - \frac{\mu}{2} \rho_k^2 + \frac{c^2}{4} \right) dt}.\quad (23)$$

Let $u = \Omega_4 t$, then

$$\phi(h(k)) = \int_0^{2K(k)} \left[ h(k) + \frac{c^2 \rho_k^2}{4dn^2(u,k)} - \frac{a \rho_k^2}{2dn^2(u,k)} \right] du$$

$$= \frac{\rho_k^2}{1-k^2} \left[ 2c^2 - \frac{2(2-k^2)}{3(1-k^2)} a \rho_k^2 \right] E(k) + \left[ 2h(k) + \frac{a \rho_k^2}{3(1-k^2)} \right] K(k),$$

where $E(k)$ is the complete elliptic integral of the second kind with the modulo $k$;

$$\psi(h(k)) = \int_0^{2K(k)} \left[ \frac{h^2(k)}{\rho_k^2} dn^2(u,k) + \frac{c^2 h(k)}{4} + \left( \frac{c^2}{10} - ah(k) \right) \frac{a \rho_k^2}{4dn^2(u,k)} \right] du,$$

$$= \left[ \frac{2h^2(k)}{\rho_k^2} + \frac{c^2 h(k)}{4} - \frac{c^2}{10} a \rho_k^2 \right] E(k)$$

$$+ \left[ c^2 h(k) + \frac{a c^2 \rho_k^2}{4(1-k^2)} \right] K(k).\quad (25)$$

For a fixed parameter group $(a, b, c, \mu)$, for example, $a = 1.3, b = 1, c = 1.6, \mu = 1$, we have the graphs of the functions $\phi(h(k)), \psi(h(k))$ and $\lambda = \frac{\mu \psi(h(k))}{\phi(h(k))}$, which are shown in Fig. 4.

Fig. 4: The graphs of the detection curves of system (10).
Corresponding to the heteroclinic orbit $I^0_{2}$, we consider the Melnikov function

\[ M^\infty(\mu, \lambda) = \int_{-\infty}^{\infty} \rho_0^2(t) z_0^2(t)[-\mu z_0^2(t) + \lambda]dt \]

\[ = -\mu \int_{-\infty}^{\infty} \rho_0^2(t) \left(-\frac{a}{2} \rho_0^2(t) + \frac{c^2}{4}\right)^2 dt + \lambda \int_{-\infty}^{\infty} \rho_0^2(t) \left(-\frac{a}{2} \rho_0^2(t) + \frac{c^2}{4}\right) dt \]

\[ = -\frac{\mu c^2}{32} \int_{0}^{\infty} \left(\frac{1}{2} \text{sech}^6\left(\frac{1}{2} ct\right) - \text{sech}^4\left(\frac{1}{2} ct\right) + \text{sech}^2\left(\frac{1}{2} ct\right)\right)dt 
+ \frac{\lambda c^4}{16} \int_{0}^{\infty} \left(-\text{sech}^4\left(\frac{1}{2} ct\right) + 2\text{sech}^2\left(\frac{1}{2} ct\right)\right)dt = \frac{\lambda c^4}{16} \left[-\frac{7c^2}{30} + \frac{4\lambda}{3}\right]. \tag{27} \]

Let $M^\infty(\mu, \lambda) = 0$, it follows that $\lambda = \frac{7}{30} c^2 \mu$. This is the bifurcation parameter value of the existence of heteroclinic orbit of the perturbed system when $k \to 1$ in $\lambda = \frac{\psi(h(k))}{\psi(h(k))}$. By the above discussion, we have the bifurcations of phase portraits of system (10) shown in Fig.5.

![Fig. 5: The bifurcations of phase portraits of system (10).](image)

To sum up, by the theory of limit bifurcations [10], we have the following conclusion.

**Theorem 2** For $0 < \epsilon \ll 1, \lambda_0 < \lambda < \frac{7}{30}c^2 \mu$, where $\lambda_0 = \lambda(a, c)$ is a negative constant, system (10) has two symmetric stable limit cycles $L_{1,2}$ created by Poincare bifurcations from unperturbed periodic orbits $I^0_{1,2}$ of system (10)$_{\epsilon=0}$. As $\lambda$ increases, limit cycles expand to the limit curves of heteroclinic loops.

We next consider the three-dimensional system

\[ \frac{d\rho}{dt} = -\rho z, \quad \frac{dz}{dt} = -\frac{c^2}{4b} + a\rho^2 + bz^2 - \epsilon(\mu z^2 - \lambda), \quad \frac{d\theta}{dt} = \omega. \tag{28} \]

In the three-dimensional phase space, two symmetric stable limit cycles $L_{1,2}$ consist of an attractive invariant rotation torus. Hence, we have the following conclusion:
Theorem 3 For $0 < \epsilon \ll 1$ and a given $\lambda, \lambda_0 < \lambda < \frac{7}{40} c^2 \mu$, where $\lambda_0 = \lambda(a,c)$ is a negative constat, system (10) has an attractive invariant rotation torus. As $\lambda$ increases to the value $\lambda = \frac{7}{40} c^2 \mu$, this family of invariant rotation torus approximates to the aperiodic heteroclinic manifold.

4 The exact solutions of system (3) when $b = 1, \alpha = c$ or $\alpha = -2c$

In this section, we discuss the exact solutions of the cubic nonlinear oscillator with damping (6). Equation (6) is equivalent to the planar dynamical system

$$\frac{d\rho}{dt} = y, \quad \frac{dy}{dt} = (2\alpha + c)y - \alpha(c + \alpha)\rho - a\rho^3. \quad (29)$$

When $\alpha = c$ or $\alpha = -2c$, system (29) is integrable (see [2,12]). In two cases, system (29) becomes

$$\frac{d\rho}{dt} = y, \quad \frac{dy}{dt} = 3cy - 2c^2 \rho - a\rho^3. \quad (30)$$

or

$$\frac{d\rho}{dt} = y, \quad \frac{dy}{dt} = -3cy + 2c^2 \rho - a\rho^3. \quad (31)$$

We only consider the solutions of system (31). For system (30), the study is similar.

System (31) has the first integral depending on $t$ as follows:

$$H_{cub}(\rho,y,t) = \left( \frac{1}{2} (y + c\rho)^2 - \frac{1}{4} a\rho^4 \right) \exp(4t) = h, \quad (32)$$

where $h$ is a real constant. Clearly, for any $a > 0$, system (31) has three equilibrium points at $O(0,0)$ and $A_{\mp} \left( \mp \sqrt{\frac{2c^2}{a}}, 0 \right)$.

Let $M(\rho_j, 0)$ be the coefficient matrix of the linearized system of (29) at the equilibrium point $(\rho_j, 0)$. Then, we have

$$\det M(0,0) = 2c^2, \quad \det M \left( \mp \sqrt{\frac{2c^2}{a}}, 0 \right) = -4c^2, \quad \text{Trace} M(0,0))^2 - 4J(0,0) = c^2.$$ 

Hence, the equilibrium point $O(0,0)$ is an unstable node point. $A_{\mp}$ are two saddle points.

Based on the above qualitative analysis, the phase portrait of system (31) is presented in Fig. 6 (a).
When \( H_{\text{cub}}(\rho, y, t) = 0 \) implies that there are four invariant curve solutions of system (31) given by \( y = -c\rho \pm \sqrt{\frac{1}{2}a\rho^2} \), which consist of six orbits of system (31). Their graphs are shown in Fig. 6 (b).

Let us consider the explicit parametric representations of orbits defined by \( H_{\text{cub}}(\rho, y, t) = h \).

(i) With regard to the heteroclinic orbit connecting the equilibrium points \( O \) and \( A_+ \) defined by \( y = -c\rho + \sqrt{\frac{1}{2}a\rho^2} \), by using the first equation of system (31) we obtain

\[
\rho(t) = \frac{c}{\left(\sqrt{\frac{2}{a}} + \rho_0 \exp(-ct)\right)},
\]

where

\[
\rho_0 = \frac{1}{h_0} \left(-c + \sqrt{\frac{1}{2}a\rho_0^2}\right) > 0, \quad 0 < \rho_0 < \sqrt{\frac{2c^2}{a}}.
\]

We see from (33) that when \( t \to -\infty \), \( \rho(t) \to 0 \), and when \( t \to \infty \), \( \rho(t) \to \sqrt{\frac{2c^2}{a}} \). Similarly, we have the exact solution of the heteroclinic orbit connecting the equilibrium points \( O \) and \( A_- \) defined by \( y = -c\rho - \sqrt{\frac{1}{2}a\rho^2} \).

(ii) Corresponding to the curves defined by \( H_{\text{cub}}(\rho, y, t) = h, h \neq 0 \), by making the transformations

\[
w = \frac{1}{\sqrt{2}} \rho \exp(c\xi), \quad z = -\frac{1}{c} \exp(-c\xi),
\]

the first integral \( H_{\text{cub}}(\rho, y) = h \) defined by (32) can be converted into the form

\[
\left(\frac{dw}{dz}\right)^2 - aw^4 = h.
\]

From (34), for \( h > 0 \) we have

\[
z - z_0 = \int_w^\infty \frac{dw}{\sqrt{h\sqrt{1 + \left(\frac{1}{4}w^4\right)}}} = (ah)^{-\frac{1}{4}} \int_w^\infty \frac{d\hat{w}}{\sqrt{\hat{w}}},
\]
It further gives

\[ w(z) = \left( \frac{h}{a} \right)^{\frac{1}{4}} \hat{w} = \left( \frac{h}{a} \right)^{\frac{1}{4}} \left( \frac{1 + \text{cn}(\Omega_1(h)(z - z_0), k_1)}{1 - \text{cn}(\Omega_1(h)(z - z_0), k_1)} \right)^{\frac{1}{2}}, \]

where \( \Omega_1(h) = 2(a|h)^{\frac{1}{4}}, k_1 = \frac{1}{\sqrt{2}} \) and \( z_0 \) is an arbitrary constant. Thus, we obtain

\[ \phi(\xi) = \sqrt{2} \left( \frac{h}{a} \right)^{\frac{1}{4}} \left( \frac{1 + \text{cn} (\Omega_1(h) \left( -\frac{1}{c} \exp (-c\xi) - z_0 \right), k_1)}{1 - \text{cn} (\Omega_1(h) \left( -\frac{1}{c} \exp (-c\xi) - z_0 \right), k_1)} \right)^{\frac{1}{2}} \exp (-c\xi). \]  

(35)

From (34), for \( h < 0 \) we have

\[ z - z_0 = \int_{-w_0}^{w} \frac{dw}{\sqrt{|h|} \sqrt{\left( \frac{a}{|h|} w^4 - 1 \right)}} = (a|h|)^{-\frac{1}{4}} \int_{1}^{\tilde{w}} \frac{d\tilde{w}}{\sqrt{\tilde{w}^4 - 1}}, \]

which implies that

\[ \phi(\xi) = \sqrt{2} \left( \frac{|h|}{a} \right)^{\frac{1}{4}} \text{nc} \left\{ \Omega_2(h) \left[ -\frac{1}{c} \exp (-c\xi) - z_0 \right], k_1 \right\}, \]  

(36)

where \( \Omega_2(h) = 2(a|h)^{\frac{1}{4}} \) and \( z_0 \) is an arbitrary constant. Formulas (35) and (36) give rise to the parametric representations of all orbits of system (31), which approach to the invariant curve solution determined by \( y = -c\rho \pm \sqrt{\frac{1}{2}a\rho^2} \) and the node point \( O \) in Fig.6 (a). These parametric representations are not associated with any periodic orbit.

To sum up, we have the following results.

**Theorem 4** For system (31), under the parametric conditions \( c > 0, a > 0 \) we obtain that

1. system (31) has a heteroclinic orbit given by (33).
2. system (31) has unbounded solutions given by (35) and (36).

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**Compliance with ethical standards**

**Conflict of interest**

The authors declare that they have no conflict of interest.
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Figure 1

The phase portraits of system (11) for a fixed parameter group (a, b, c).
Figure 2

The heteroclinic orbits of system (8) in three-dimensional phase.

(a) $k = 0.3, \omega = 1.5549, (m, n) = (11, 5)$.  

(b) $k = 0.9, \omega = 0.4935, (m, n) = (17, 25)$.

(c) $k = 0.995, \omega = 3.523, (m, n) = (25, 3)$.  

(d) $k = 0.999, \omega = 7498, (m, n) = (5, 7)$.

Figure 3

The knotted periodic orbits of system (8) in three-dimensional phase space.

(a) $\phi(h(k))$.  

(b) $\psi(h(k))$.  

(c) $\lambda = \frac{\psi(h(k))}{\phi(h(k))}$.

Figure 4
The graphs of the detection curves of system (10).

Figure 5

The bifurcations of phase portraits of system (10).

(a) Phase portrait of system (31).  (b) The level curves defined by $H_{\text{cub}}(\rho, y) = 0$.

Figure 6
The phase portrait and level curves of system (31).