THE EXISTENCE OF CONJUGATE DEGRADABLE CHANNELS THAT ARE NOT DEGRADABLE

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Abstract. Conjugate degradable channels are channels whose quantum capacity is calculable. They were defined and studied in [1] where, however, only channels that are both degradable and conjugate degradable were found. In this paper we bring the very first example of conjugate degradable channels that are not degradable. We also identify the physical origin of these channels and show that they belong to the class of optimal universal asymmetric cloners. We thus not only positively answer the question whether conjugate degradable channels form a new class of channels with a single-letter capacity formula but as a side result we also calculate the quantum capacity of the optimal asymmetric $1 \rightarrow 1 + 1$ cloning channels.

INTRODUCTION AND TERMINOLOGY

Degradable quantum channels were first introduced in [2]. Loosely speaking, they are completely positive maps that are capable of simulating its own environment degrees of freedom by composing with another completely positive map called a degrading map. Their importance comes from the fact that if they describe the behavior of a physical system its quantum communication capabilities are fully understood. This statement essentially means that one is able to calculate the maximal rate at which quantum information can be reliably transmitted. The quantity that characterizes the channel’s ability to coherently transfer a quantum message is called the quantum channel capacity [3–5] and a great deal of effort has been invested in understanding its properties [6]. The problem with this truly fundamental quantity is that except for degradable channels it is virtually incalculable and often only a lower estimate called the coherent information [7] is known. On a more positive note, it has been shown that among natural physical processes there is plenty of them represented by a degradable quantum channel. The examples are the erasure channel [10], the amplitude-damping channel [11], the dephasing channel [2] and the qudit Unruh and optimal qudit cloning channels [12, 13]. The latter will eventually play an important role leading to the presented result. Degradable channels in low dimensions have also been studied systematically [14] and a summary of their many favorable properties can be found in [6, 15].

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Figure 1. The action of four maps $A, D, \overline{D}$ and $\overline{A}$ relating the output Hilbert spaces $A$ quantum channel (completely positive map) $N$ and its complementary channel $\overline{N}$ is investigated. The channel $N$ maps the input density matrices from the Hilbert space denoted by $A$ to the output Hilbert space $B$, whereas its complement's output is another Hilbert space $E$. The involutive map $C$ acts by transposing (or complex conjugating) the density matrix in the $B$ or $E$ space. Complex conjugation is implemented by an antiunitary operator and therefore it is a physically forbidden map.

It is highly desirable to extend the applicability of the single-letter quantum capacity formula to other than degradable quantum channels. Conjugate degradable channels identified here are the first such example. Let’s introduce some crucial terms already mentioned in the previous paragraph and also recall the definition of a degradable channel together with conjugate degradability [1] and two other related concepts. Let $U: \rho_A \mapsto \sigma_{BE}$ be a unitary capturing the evolution of a quantum state $\rho_A$ where the subscript denotes the corresponding Hilbert space. Then $N \overset{df}= \text{Tr}_E U$ is a quantum channel and $\overline{N} \overset{df}= \text{Tr}_B U$ is its complementary channel. By further denoting $C$ to be complex conjugation (density matrix transposition in a fixed basis) we have the following constructions:

(i) A quantum channel $N$ is degradable if there exists another channel $D$ such that $D \circ N = \overline{N}$. The map $D$ is called a degrading channel.

(ii) A quantum channel $N$ is conjugate degradable if there exists another channel $\overline{D}$ such that $\overline{D} \circ N = C \circ \overline{N}$. The map $\overline{D}$ is called a conjugate degrading channel.

(iii) A quantum channel $N$ is antidegradable if there exists another channel $A$ such that $N = A \circ \overline{N}$. The map $A$ is called an antidegrading channel.

(iv) A quantum channel $N$ is conjugate antidegradable if there exists another channel $\overline{A}$ such that $C \circ N = \overline{A} \circ \overline{N}$. The map $\overline{A}$ is called a conjugate antidegrading channel.

For a graphical depiction see Fig. 1. Note that we can trivially relabel the channels such that $M = \overline{N}$ and so $\overline{M} = N$. Then if $N$ is, for example, degradable it is equivalent to $M$ being antidegradable.

The quantum capacity of a noisy quantum channel $N$ defined as the maximal rate at which quantum information can be sent and perfectly recovered (in the units of bits
per channel) is calculated by
\[
Q(\mathcal{N}) = \lim_{n \to \infty} \frac{1}{n} \max_{\varrho} Q^{(1)}(\mathcal{N}^\otimes n(\varrho)) = \lim_{n \to \infty} \frac{1}{n} \max_{\varrho} [H(\mathcal{N}^\otimes n(\varrho)) - H(\widehat{\mathcal{N}}^\otimes n(\varrho))],
\]
where \(\varrho\) is an input state to \(n\) copies of the quantum channel \(\mathcal{N}\) and its complement \(\widehat{\mathcal{N}}\), the quantity \(Q^{(1)}(\mathcal{N}(\varrho))\) is called the one-shot quantum capacity also known as the coherent information and \(H(\varrho) = -\text{Tr}[\varrho \log \varrho]\) is the von Neumann entropy. The magic of degradable channels lies in the observation \([2]\) that
\[
Q(\mathcal{N}) = \max_{\varrho} Q^{(1)}(\mathcal{N}(\varrho)) = \max_{\varrho} [H(B)_\sigma - H(E)_\sigma],
\]
where the succinct notation on the right side stresses the fact that the coherent information is maximized over the input ensemble \(\varrho_A\) but it is evaluated on \(\sigma_{B(E)}\) living in the output Hilbert subspace \(B\) and \(E\) corresponding to \(\mathcal{N}\) and \(\widehat{\mathcal{N}}\), respectively. This is the content behind the statement that the channel capacity is single-letterized. The same magic happens for conjugate degradable channels whose quantum capacity is given by Eq. \([2]\) as well \([1]\). The similarity does not end here. If a channel is antidegradable, its quantum capacity is zero. Conjugate antidegradable channels satisfy the same property \([1]\).

TRANSPOSE DEPOLARIZING CHANNEL AND THE PHYSICAL INTERPRETATION OF ITS COMPLEMENTARY CHANNEL

The class of channels we will study here has been nicknamed the qudit transpose depolarizing (TD) channels \([16, 17]\). They are defined as
\[
\mathcal{T}(\varrho) = t \varrho^T + (1 - t) \frac{1}{d} \text{id},
\]
where \(t\) is a real parameter whose values lie in the interval \(\frac{1}{d} - \frac{1}{d} \leq t \leq \frac{1}{d} - \frac{1}{d}\) dictated by complete positivity of \(\mathcal{T}\), the superscript \(T\) denotes density matrix transposition and \(\text{id}\) stands for a \(d\)-dimensional identity operator. The instance of \(\mathcal{T}\) for \(t = -1/(d - 1)\) and \(d > 2\) is particularly significant. It is called the Werner-Holevo channel \([18]\) and it played a role as a counterexample to the \(p\)-norm additivity conjecture. It had been introduced even before the family of TD channels was formally defined \([16]\). We will present another interesting property of this channel family and show that in a certain — non-negligible — interval of \(t\) the \(d = 2,3\) complementary channels are conjugate degradable but not degradable. Equivalently, on the same interval the TD channels are not antidegradable but they are conjugate antidegradable.

Before we proceed let us mention an important feature of the qudit TD channels: they are covariant with respect to the unitary group \(SU(d)\). Denote \(g : SU(d) \to GL(\mathbb{C}^d)\) the fundamental representation of \(SU(d)\) and \(g^* : SU^*(d) \to GL(\mathbb{C}^d)\) the inequivalent (for \(d > 2\)) fundamental representation. Further denote \(h : SU(d) \otimes SU(d) \to GL(\mathbb{C}^{2d})\) to be the tensor product of two fundamental representations. It has been known \([17]\) that the qudit TD channel and its complement transform covariantly: \(\mathcal{T} \circ g = g^* \circ \mathcal{T}\) and \(\widehat{\mathcal{T}} \circ h = h \circ \widehat{\mathcal{T}}\). Hence the TD channel output transforms irreducibly but not its complement. Recall that \(SU(d) \otimes SU(d)\) splits into a direct sum of a completely symmetric and antisymmetric representation. It is well-known that a general covariant property usually simplifies the capacity calculations \([3]\) and our case won’t be different.

Based on the discussed symmetry properties we will show that a subset of the complementary channels to the TD channels has a direct physical interpretation: for
0 \leq t \leq 1/3 \) it corresponds to the optimal asymmetric cloner \([19, 20]\). It is this subset that in the qubit case \((d = 2)\) contains a parameter range of \(t\) where the complement is conjugate degradable but not degradable. Let \(\Pi_{\pm}\) be a projector onto a completely symmetric (+) and completely antisymmetric subspace (−). Then the action of the complementary channel on the input density matrix must be

\[
g \mapsto \bar{g} = ((\alpha + \beta)\Pi_{+} + (\alpha - \beta)\Pi_{-})(\text{id} \otimes g)((\alpha + \beta)\Pi_{+} + (\alpha - \beta)\Pi_{-})^\dagger. \tag{4}
\]

This is a linear positive map whose coefficients \(\alpha, \beta \in \mathbb{R}\) are chosen such that it corresponds to the notation of Ref. [21], where such a map has already been studied.

Focusing on the qubit case we find that for the map in Eq. (4) to be trace preserving, the coefficients are required to satisfy \(2(\alpha^2 + \alpha\beta + \beta^2) = 1\). The fact that this transformation also induces a completely positive map identical to the complement to the TD channel can be seen from a direct comparison of the complementary output density matrix

\[
\hat{T}(g) = \begin{bmatrix}
\frac{1}{2}(1+t)\theta_{11} & \frac{1+4t}{\sqrt{2}}\theta_{01} & 0 & \frac{\sqrt{1-3t} \sqrt{1+t}}{\sqrt{2}} \theta_{10} \\
\frac{\sqrt{1-3t}}{\sqrt{2}} \theta_{01} & \frac{1}{2}(1-t) & \frac{4t}{\sqrt{2}} \theta_{10} & \frac{-\sqrt{1-3t} \sqrt{1+t}}{\sqrt{2}} \theta_{00} \\
0 & \frac{1}{2}(1-t) & \frac{4t}{\sqrt{2}} \theta_{10} & \frac{-\sqrt{1-3t} \sqrt{1+t}}{\sqrt{2}} \theta_{00} \\
\frac{\sqrt{1-3t}}{\sqrt{2}} \theta_{01} & \frac{-\sqrt{1-3t} \sqrt{1+t}}{\sqrt{2}} \theta_{10} & \frac{1}{2}(1-t) & \frac{1}{2}\theta_{11}
\end{bmatrix}, \tag{5}
\]

with

\[
\rho = V\hat{a}V^\dagger = \begin{bmatrix}
(\alpha + \beta)^2 \theta_{11} & \frac{(\alpha + \beta)^2}{\sqrt{2}} \theta_{01} & 0 & \frac{\alpha^2 \beta^2}{\sqrt{2}} \theta_{10} \\
\frac{(\alpha + \beta)^2}{\sqrt{2}} \theta_{01} & \frac{1}{2}(\alpha + \beta)^2 \theta_{10} & \frac{(\alpha + \beta)^2}{\sqrt{2}} \theta_{00} & \frac{\alpha^2 \beta^2}{\sqrt{2}} \theta_{11} \\
0 & \frac{(\alpha + \beta)^2}{\sqrt{2}} \theta_{01} & \frac{(\alpha + \beta)^2}{\sqrt{2}} \theta_{10} & -\frac{(\alpha + \beta)^2}{\sqrt{2}} \theta_{00} \\
\frac{\alpha^2 \beta^2}{\sqrt{2}} \theta_{01} & \frac{-\alpha^2 \beta^2}{\sqrt{2}} \theta_{10} & \frac{1}{2}(\alpha - \beta)^2 \theta_{11} & \frac{1}{2}(\alpha - \beta)^2 \theta_{10}
\end{bmatrix}, \tag{6}
\]

where \(V\) is a unitary operator. By identifying \(\alpha + \beta = \sqrt{1+t}\) and \(\alpha - \beta = \sqrt{1-3t}\) we find \(t = 2\alpha\beta\). The complete positivity condition of the qubit TD channel (and its complement) dictates \(-1 \leq t < 1 \) and only for these values of \(t\) the hyperbolae given by \(t = 2\alpha\beta\) intersect the normalization ellipse \(2(\alpha^2 + \alpha\beta + \beta^2) = 1\). Hence the normalization condition found above automatically ensures the complete positivity of the trace-preserving map given by Eq. (1).

To provide the ultimate interpretation of a subset of the complementary qubit TD channels we notice that the coefficients \(\alpha\) and \(\beta\) turn out to be the parameters appearing in the optimal universal asymmetric \(1 \to 1 + 1\) cloner for qubits \([22, 23]\). There exists a fundamental trade-off for the quality of the clones and the role of \(\alpha, \beta\) is to “tune” how close in terms of fidelity one of the clones will be to the input state \([20]\). This correspondingly determines the best achievable quality of the other clone. For this purpose it is advantageous to introduce an asymmetry parameter \(0 \leq p \leq 1\) \([19]\) related to \(\alpha\) and \(\beta\) in the following way:

\[
\alpha^2 = \frac{p^2}{2(1 - p + p^2)}, \tag{7a}
\]

\[
\beta^2 = \frac{(1 - p)^2}{2(1 - p + p^2)}. \tag{7b}
\]

Hence \(t = p(1 - p)/(1 - p + p^2)\) and we get the optimal universal symmetric cloner for \(p = 1/2\) \([21]\) corresponding to \(t = 1/3\). The optimal maximally asymmetric universal qubit cloner is obtained for \(p = 0, 1\) where \(t = 0\). The reason for the excursion to the
A world of cloners is two-fold. First, we want to point out the physical interpretation of a subset of the complementary channels to the qubit TD channel that may not be immediately obvious. Second, in the next section we will see that in the parameter interval corresponding to the optimal asymmetric cloner there is a smaller subset of channels that are exclusively conjugate degradable. It is interesting to note that such channels has already been prepared in a laboratory [25].

**Exclusively conjugate degradable channels**

The whole task reduces to investigating whether the maps $\mathcal{D}, \mathcal{A}, \overline{\mathcal{D}}$ and $\overline{\mathcal{A}}$ are completely positive. To illustrate it on the degrading case, we are asking whether the degrading map $\mathcal{D} = \hat{N} \circ \mathcal{N}^{-1}$ is completely positive. The Choi-Jamiołkowski formalism [26, 27] is a suitable tool to investigate the complete positivity of $\mathcal{D}$ since it reduces to the positivity of the Choi matrix $\Phi_\mathcal{D} \overset{df}{=} (\text{id} \otimes \mathcal{D})(\Phi)$ corresponding to $\mathcal{D}$ where $\Phi$ is an unnormalized maximally entangled bipartite state. However, it is cumbersome to work in the Choi-Jamiołkowski picture when composing two channels. For that purpose it is advantageous to switch to yet another, closely related, representation of completely positive maps called the superoperator formalism, namely its matrix representation. This is in fact a more natural formalism (as stressed in [28]) for investigating the properties of physical maps transforming positive operators (density matrices) into positive operators by means of “vectorization” of a density matrix. The Choi-Jamiołkowski matrix is essentially the same object in disguise. The details can be found for example in [29] and it has been used for a similar task in [30]. We will write the matrix superoperators in the sans serif font. They are defined as $\mathcal{M}(|m\rangle\langle\mu|) \overset{df}{=} M_{\mu\nu}|n\rangle\langle\nu|$ where the first (second) pair of indices denotes the row (column) and the summation convention is adopted. The composition of two maps $\mathcal{K} \circ \mathcal{M}$ becomes a mere matrix multiplication written as $M K$ and the Choi matrix of $\mathcal{M}$ is obtained from its superoperator matrix $\mathcal{M}$ by reshuffling the two inner indices: $M_{\mu\nu} \rightarrow \Phi_\mathcal{M} \equiv M_{\mu\nu\mu\nu}$. For more details see [29].

The candidate for the degrading channel $\mathcal{D}$ obtained from $\mathcal{D} = T^{-1} \hat{T}$ and represented as the Choi matrix reads

$$
\Phi_\mathcal{D} = \begin{bmatrix}
\frac{t^2 - 1}{4t} & 0 & 0 & 0 & 0 & \frac{t + 1}{2\sqrt{2t}} & 0 & \frac{\sqrt{t(t+1)}}{2\sqrt{2t}} \\
0 & \frac{t + 1}{2\sqrt{2t}} & 0 & \frac{\sqrt{t(t+1)}}{2t} & 0 & 0 & \frac{t + 1}{2\sqrt{2t}} & 0 \\
0 & 0 & \frac{t^2 - 1}{4t} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{t^2}{4t} & \frac{1}{4}(1 - 3t) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{t + 1}{4t} & 0 & 0 & 0 \\
\frac{t + 1}{2\sqrt{2t}} & 0 & 0 & 0 & 0 & \frac{t + 1}{2\sqrt{2t}} & 0 & 0 \\
\frac{\sqrt{1 - t^2}}{2\sqrt{2t}} & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{t^2}}{4t} & \frac{1}{4}(1 - 3t)
\end{bmatrix}.
$$

(8)

Its distinct eigenvalues are

$$
\lambda_1^\mathcal{D} = \frac{(t + 1)^2}{4t},
$$

(9a)

$$
\lambda_{2,3}^\mathcal{D} = \frac{-t^2 - 1 \pm 2\sqrt{t^4 - t^3 - 3t^2 - t + 1}}{4t}.
$$

(9b)
The only point from the admissible interval \( t \in [-1, 1/3] \) where none of them are negative is \( t = -1 \). So except for this point the degrading map is unphysical, that is, non-complete positive and \( T \) is not degradable. On the other hand, the existence of a degrading channel for \( t = -1 \) is not surprising – it is due to the fact that the channel \( T \) is a mere unitary, its complement \( \hat{T} \) is a trivial trace map and so is the degrading channel. For \( t = 0 \) the degrading map is not well-defined and the TD channel \( 3 \) coincides with a completely depolarizing channel.

The superoperator for the candidate on the conjugate degrading Choi map \( \Phi_{\overline{D}} \) is given by \( \overline{D} = T^{-1} \hat{T}_C \). \( \hat{T}_C \) is the superoperator matrix corresponding to the complementary channel \( \hat{T} \) where the output density matrix is complex conjugated (see the magenta curve in Fig. 1 depicting the action of \( \overline{D} \) whose output differs from the output the complementary channel by complex conjugation \( C \)). The eigenvalues turn out to be

\[
\lambda_1^\overline{D} = \frac{t^2 - 1}{4t}, \quad (10a)
\]
\[
\lambda_{2,3}^\overline{D} = -\frac{t^2 + 2t + 1 \pm 2\sqrt{t^4 + t^2 - t^2 - t + 1}}{4t} \quad (10b)
\]

and again the only non-negative point is \( t = -1 \).

Interesting things start to happen when we compare antidegradability and conjugate antidegradability of \( T \). The Choi matrix corresponding to the superoperator \( A = \overline{T}^{-1} \overline{T} \) read:\(^2\)

\[
\Phi_A =
\begin{bmatrix}
\frac{3t^3 + t^2 + 1}{2(t-1)(3t^2 + 1)} & 0 & 0 & \sqrt{2(1-t)} & 0 & 0 & 0 & t\sqrt{-3t^2 - 2t + 1} \\
0 & \frac{3t^3 + t^2 + 1}{6t^3 + 6t^2 - 2t + 2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{6t^2 + 2} & 0 & 0 & \sqrt{2(1-t)} & \frac{t\sqrt{-3t^2 - 2t + 1}}{2 - 2t^2} & 0 \\
\sqrt{2(1-t)} & 0 & 0 & \frac{3t^3 - t^2 + 1}{-6t^3 + 6t^2 - 2t + 2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2(1-t)}} & 0 & 0 & \frac{t\sqrt{-3t^2 - 2t + 1}}{2t - 2t^2} & 0 \\
0 & 0 & 0 & 0 & \frac{3t^3 - t^2 + 1}{2t - 2t^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{2(1-t)} & \frac{t\sqrt{-3t^2 - 2t + 1}}{2t - 2t^2} & 0 \\
\frac{t\sqrt{-3t^2 - 2t + 1}}{\sqrt{2(1-t)}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2(1-t)}} & 0 \\
\end{bmatrix}
\]

(11)

whose distinct eigenvalues are

\[
\lambda_1^A = \frac{-3t^4 - 2t^3 - 2t - 1}{2(t - 1)(t + 1)(3t^2 + 1)}, \quad (12a)
\]
\[
\lambda_{2,3}^A = \frac{3t^4 + 2t^3 + 2t^2 + 2t - 1 \pm 2t\sqrt{-18t^6 - 6t^5 + t^4 - 8t^3 - 2t + 1}}{2(t - 1)(t + 1)(3t^2 + 1)}. \quad (12b)
\]

\(^2\)The inverse \( \overline{T}^{-1} \) is a generalized inverse whose uniqueness is guaranteed by certain criteria \(^3\) that are satisfied in our case.
The existence of a completely positive map (quantum channel) can be revealed by studying the positivity of the corresponding Choi matrix. The distinct eigenvalues \( \lambda_{1,2,3}^A \) from Eqs. (12) of the Choi matrix \( \Phi_A \) for an antidegrading map \( A \) of the qubit transpose depolarizing channel \( T \) are depicted on the left and the eigenvalues \( \lambda_{1,2,3}^{\overline{A}} \) from Eqs. (13) of \( \Phi^{\overline{A}} \) corresponding to the conjugate antidegrading map \( \overline{A} \) are on the right as a function of the (transpose) depolarizing parameter \( t \in [-1, 1/3] \) from Eq. (3).

The eigenvalues for the conjugate antidegrading candidate \( \Phi^{\overline{A}} \) obtained from the superoperator \( \overline{A} = \overline{T}^{-1}T_C \) are the following expressions

\[
\lambda_1^{\overline{A}} = \frac{3t^3 + t^2 + t - 1}{2(t - 1)(3t^2 + 1)},
\]

\[
\lambda_{2,3}^{\overline{A}} = \frac{3t^4 + 4t^3 + 1 \pm 2t\sqrt{-18t^6 - 12t^5 - 5t^4 - 4t^3 + 4t^2 + 3}}{-6t^4 + 4t^2 + 2}.
\]

To better see what is happening we first plot the eigenvalues from Eqs. (12) in Fig. 2(a) and we notice that in the interval \( t \in [-0.47, 0.26] \) the antidegrading map exists. We get our main result by comparing it with the eigenvalues from Eqs. (13) plotted in Fig. 2(b). In the parameter range \( t \in (0.26, 1/3) \), where \( T \) is not antidegradable, the conjugate antidegrading map \( \overline{A} \) exists. Henceforth, in this interval the complementary channel \( \overline{T} \) is not degradable but it is conjugate degradable. But this is a subset of the optimal universal asymmetric 1 \( \rightarrow \) 1 + 1 cloners as revealed in the previous section. So they are the ones responsible for exclusive conjugate degradability.

Additionally, in the interval \( t \in [-0.26, 0.26] \) the complementary channel \( \overline{T} \) is both degradable and conjugate degradable adding to the only known example of such channels so far that lies at \( t = 1/3 \) – this is precisely the optimal universal qubit 1 \( \rightarrow \) 2 cloning channel introduced in [12] and denoted by \( C_{1,2} \). Its output is given by density matrix Eq. (5) for \( t = 1/3 \) or, eventually, Eq. (6) where we set \( \alpha = \beta = 1/\sqrt{6} \). In this form it coincides with the density matrix from [12]. Furthermore, from Fig. 2 we confirm this result by observing that for \( t = 1/3 \) the channel \( \overline{T} \) is both degradable and conjugate degradable which again is in agreement with [12].

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\[ \text{The actual interval is } \left\{ \frac{\sqrt{9\sqrt{97} - 67} + \sqrt{9\sqrt{57} - 67}}{9 \sqrt{9\sqrt{97} - 67}}, \frac{\sqrt{9\sqrt{57} - 67} + \sqrt{9\sqrt{97} - 67} + 2\sqrt{\sqrt{9\sqrt{57} - 67} + \sqrt{9\sqrt{97} - 67}}}{2\sqrt{\sqrt{9\sqrt{57} - 67} + \sqrt{9\sqrt{97} - 67}}} \right\} \]

but we will refrain from these analytical but otherwise unwieldy expressions.
It is important to emphasize that our conjugate degradable but non-degradable channels are not trivial. As follows from Eqs. (9) and (10) the qubit TD channel $\mathcal{T}$ is nowhere degradable or conjugate degradable except for $t = -1$. Therefore the quantum capacity of $\mathcal{T}$ is expressible in a single-letter form and nonzero at the same time. As a sanity check we can also verify that the channel $\mathcal{T}$ is not PPT (i.e. generating a state with a positive partial transpose). These channels are either entanglement breaking [8] or entanglement binding [9] and, as such, their quantum capacity is zero. This is indeed not the case again except for the trivial trace map when $t = -1$.

Using the insight from [1] we can readily calculate the quantum capacity of $\mathcal{T}$. As revealed in Eq. (2) we have to perform the maximization of the coherent information only over a single copy of the channel. But the qubit TD channel $\mathcal{T}$ and its complement $\mathcal{T}$ are also $SU(2)$ covariant and this implies that the maximizing ensemble is a maximally mixed input state of one qubit $\rho = \text{id}/2$. Note that we have argued in the paragraph below Eq. (3) that the output of $\mathcal{T}$ does not transform irreducibly. This, however, does not limit the proof presented in [1]. Hence we may write

$$Q(\mathcal{T}) = H(\mathcal{T}(\text{id}/2)) - H(\mathcal{T}(\text{id}/2)) = -3 \frac{1 + t}{4} \log_2 \frac{1 + t}{4} - \frac{1 - 3t}{4} \log_2 \frac{1 - 3t}{4} - 1,$$

which is valid for $t \in [0.26, 1/3]$.

Finally, notice that for $t = 1/3$ the quantum capacity formula reduces to $Q(\mathcal{T}) = \log_2 3 - 1$ that was already found in [1] for the qubit cloning channel $\mathcal{C}_{1,2}$.

The results we have found for a subset of the complementary channels of the qubit TD channel seem to be rather generic. If we perform the same analysis for the qutrit TD channel and its complement by finding the Choi matrices for the (conjugate) antidegrading maps plus analytical expressions for their eigenvalues, we arrive at similar conclusions. Without going into detail we simply plot the eigenvalues of these maps in Fig. 3. The complementary channel $\mathcal{F}$ is conjugate degradable but not degradable in the parameter interval $t \in (0.19, 1/4)$. In the interval $t \in [-0.11, 0.19]$ the channel $\mathcal{F}$ is both degradable and conjugate degradable. This also happens for the endpoint

It is also valid on the interval $t \in [-0.47, 0.26]$ since the complementary channel $\mathcal{F}$ is degradable as can be seen in Fig. 2(a).
Degradable channels
Conjugate degradable channels

Channels with calculable (and positive) quantum capacity

**Figure 4.** The schematic relation between the two known classes of quantum channels whose quantum capacity possesses a tractable formula and is nonzero.

$t = 1/4$ as envisaged in [13] where $\hat{T}$ was identified as the optimal universal qutrit $1 \rightarrow 2$ cloner and its quantum capacity was calculated. Finally, the complementary channel $\hat{T}$ is not PPT at any point of the admissible interval for the (transpose) depolarizing parameter $t$.

**Discussion and open questions**

We have found the very first example of channels that are conjugate degradable but not degradable. Conjugate degradable channels are channels defined to be degradable up to complex conjugation of the channel output density matrix. Their importance comes from the fact that apart from degradable channels, conjugate degradable channels are the only example of channels whose quantum capacity is calculable and non-trivial (that is, nonzero). The channel class exhibiting these properties is the complementary channel to the qubit and qutrit transpose depolarizing channel. In the qubit case we identified these channels with a subset of the optimal universal asymmetric $1 \rightarrow 1 + 1$ cloners. It is likely that in the general qudit case the subset of the channels responsible for exclusive conjugate degradability will be recruited from the optimal qudit asymmetric cloners as well.

The obtained results have an impact on our current understanding of the quantum capacity landscape of noisy quantum channels. The new relations between channels whose quantum capacity is calculable is schematically depicted in Fig. 4. We not only considerably enlarged the set of channels whose quantum communication capabilities are fully understood but we also opened a whole new area of research with a number of unanswered questions. The most obvious one is how big is the set of exclusively conjugate degradable channels including their detailed mathematical properties. Furthermore, since we have also found a set of channels that are both degradable and conjugate degradable (adding to the only examples known prior to this work called the optimal $1 \rightarrow 2$ cloning channel $\mathcal{C}_{l_{1,2}}$ [1, 12], and the Unruh channel [13]), the question is how big the overlap between the two classes is. The intersection seems to be non-trivial as well since the cloning channel is disconnected from the rest of both degradable and conjugate degradable channels. Note that the diagram in Fig. 4 gets further complicated when the channels whose quantum capacity equals zero are incorporated. Their capacity is clearly calculable as well and the most studied examples are antidegradable.
channels. A class of antidegradable channels that happen to be degradable is known to exist, for example. Similarly, we do not see any reason why both conjugate degradable and conjugate antidegradable cannot exist as well but examples are yet to be found and the detailed mathematical properties are unknown. This is closely related to another question of how generic our example based on the transpose depolarizing channels is. We conjecture that the similar result holds for the complements of all qudit transpose depolarizing channels. But are there entirely different classes of channels that are exclusively conjugate degradable?

Next intriguing open problem seems to be the possibility of other forbidden operations in place of complex conjugation in Fig. 1. First task would be to define degradability “up to a forbidden operation” and investigate whether the quantum capacity of the channels with such a property is tractable. The positive answer could be followed by an effort to find some examples. For the success of this program, the existence of exclusively conjugate degradable channels revealed in this paper is encouraging. Forbidden operations we have in mind are positive maps that are not completely positive [29]. Complex conjugation alias density matrix transposition used for the definition of conjugate degradability is the simplest example of such a map. Finally, it may be rewarding to focus on the continuous-variable extension of the concept of exclusive conjugate degradability namely in the framework of Gaussian states and Gaussian quantum channels.

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