Quadratic-in-spin effects in the orbital dynamics and gravitational-wave energy flux of compact binaries at the 3PN order

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Abstract

We investigate the dynamics of spinning binaries of compact objects at the next-to-leading order in the quadratic-in-spin effects, which corresponds to the third post-Newtonian order (3PN). Using a Dixon-type multipolar formalism for spinning point particles endowed with spin-induced quadrupoles and computing iteratively in harmonic coordinates the relevant pieces of the PN metric within the near zone, we derive the post-Newtonian equations of motion as well as the equations of spin precession. We find full equivalence with available results. We then focus on the far-zone field produced by those systems and obtain the previously unknown 3PN spin contributions to the gravitational-wave energy flux by means of the multipolar post-Minkowskian wave generation formalism. Our results are presented in the center-of-mass frame for generic orbits, before being further specialized to the case of spin-aligned, circular orbits. Based on the energy balance equation, we derive the orbital phase of the binary at the order 3PN including all conservative spin effects and briefly discuss the relevance of the new terms.

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1. Introduction

Coalescing binary systems composed of stellar mass black holes and/or neutron stars are among the most promising sources for a first direct detection of gravitational waves (GW) by the network of ground-based interferometers formed by GEO-HF [1] and the advanced version of the detectors LIGO [2] and Virgo [3], which should resume their science runs from 2015, approaching gradually their design sensitivity, expected to be better by an order of magnitude than that of the first generation. The cryogenic detector KAGRA [4] will join them in a near future. Further ahead, the space-based observatory eLISA [5, 6]—a serious proposal for the mission recently announced by the European Space Agency—will allow us to scan a different frequency band where we expect to detect, notably, GW emitted by supermassive black-hole binaries before merger.

Extraction of the signal from the noisy data by means of matched filtering techniques and source parameter estimation both require an accurate modeling of the waveform. For binary systems of compact objects, the inspiralling phase of the coalescence can be modeled extremely well by resorting to the perturbative post-Newtonian (PN) scheme (see [7] for a review), in which all quantities of interest are expanded as formal series in powers of $1/c$. For non-spinning (NS) systems, the phase of the waveform is currently known up to the order 3.5PN (i.e. including corrections up to $1/c^7$), whereas the full polarizations have been obtained up to the order 3PN [8] (with the dominant quadrupole and octupole modes in the decomposition of the waveform in spin-weighted spherical harmonics known up to the order 3.5PN [9, 10]).

In recent years, motivated by astrophysical observations suggesting that black holes in our Universe can have significant spins, considerable effort has been devoted to investigating higher order corrections to the spin effects in the binary dynamics, mostly restricted to the conservative piece of the body evolution in the near zone. While for the neutron stars observed so far, the largest dimensionless spin magnitude ever measured [11] is only $\chi \sim 0.4$ (and may reasonably be assumed to be much smaller for typical expected observations), the spin of a black hole might be commonly close to its maximal value [12–15]. Then, its effect on the waveform can be fairly strong and, in particular, for spins misaligned with the orbital angular momentum of the system, the dynamics becomes much more involved as the orbital plane undergoes precession, resulting in large modulations of the waveforms [16, 17]. Even in the simpler case where the spins are aligned with the orbital angular momentum, they significantly affect the inspiral rate of the binary, i.e. the frequency evolution of the signal, starting at the 1.5PN order (see for instance [18] for a detailed study of the effect of the spin on the waveform quantified in terms of figures of merit relevant to data analysis). To make all factors $1/c$ appear explicitly in this paper, we rescale the physical spin variable $S_{\text{physical}}$ as

$$S = c S_{\text{physical}} = G m^2 \chi,$$

where $\chi$ is the dimensionless spin, with value 1 for an extremal Kerr black hole.

The calculation of the spin PN corrections to the conservative part of the dynamics and, to some extent, to the radiation field of the binary beyond the leading order contributions has been tackled using essentially three different approaches: (i) a Hamiltonian approach that strongly relies on the use of the (second) Arnowitt–Deser–Missner (ADM) gauge [19], and in which the dissipative part of the dynamics, demanding a special treatment, is generally
discarded (see however [20]), (ii) an effective field theory (EFT) Lagrangian formalism [21, 22], whose application to binary systems in general relativity has been actively developed since the mid-2000’s, and (iii) a post-Newtonian iteration scheme in harmonic coordinates (PNISH), reviewed in [7], which we follow in the present paper. The existence of those three independent methods permits important checks of calculations that are often tedious, whenever quantities are available at the same order in more than one formalism.

The binary dynamics at the spin–orbit level (i.e. linear-in-spin effects, which will be referred to as SO from now on) are known up to the order 3.5PN in both the PNISH and ADM approaches [23–28], and to the order 2.5PN in the EFT framework [29, 30]. On the other hand, quadratic-in-spin corrections (labeled as SS throughout the paper) have been obtained to the order 2PN in the PNISH formalism [31–33], while in both the ADM and EFT formalisms they are known up to the order 3PN [34–37], and even 4PN for the simpler $S_1S_2$ interactions [22, 34, 38, 39]. Higher-order-in-spin corrections have also been recently derived [37, 40–42]. As for the spin contributions to the radiation field, they have mostly been computed by using the same usual combination of the multipolar post-Minkowskian (MPM) and PNISH approaches as in the present paper, although partial results required for the calculation of the 3PN flux [43] and the 2.5PN waveform [44] have been obtained within the EFT approach. The energy flux of GW radiation is known up to the order 4PN at the SO level [45–47], whereas at the SS level only the leading order (2PN) terms were known until now [33]. Moreover, the leading order cubic-in-spin terms, which arise at 3.5PN, have been calculated very recently [42].

Our goal here will be to determine, within the PNISH approach, the 3PN (i.e. next-to-leading order) spin-spin corrections entering both the source dynamics (thereby providing an additional confirmation of the ADM and EFT results already available at this order) and most importantly the energy flux, thus completing the knowledge of all the spinning corrections to the phasing formula up to the 3PN order. At the next order 3.5PN, the only remaining unknown terms all come from a SS tail contribution. By contrast, the spin corrections to the full GW polarizations are only known to the poorer 2PN accuracy [33, 48] and we postpone to future work the task of obtaining all the corrections up to the order 3PN.

Our source modeling, as well as the one used in the EFT and ADM approaches, consists in representing each compact object as a (spinning) point particle whose internal structure is entirely parametrized by a set of effective multipole moments. The validity of this description, which makes the calculations tractable analytically, relies on (i) the compact character of the bodies, and (ii) the weak influence of their internal dynamics to their ‘global’ motion in general relativity, often referred to as the effacement principle [49]. The foundations of this formalism were laid down in the seminal works of Mathisson [50–52]. Later Papapetrou [53] found a particularly simple form for the evolution equations (which comprise both the equations of motion and of spin precession) for dipolar particles, i.e. at linear order in spins. His derivation was improved and rephrased in the language of distribution theory by Tulczyjew [54], whose method—systematically extensible beyond the dipolar model—has been recently applied at the quadrupolar level [55]. The dynamics of point particles with finite-size effects described by higher multipoles was thoroughly investigated by Dixon [56–59], who constructed an appropriate stress-energy ‘skeleton’ to encode information about the internal structure of the body while, on their side, Bailey & Israel proposed an elegant effective Lagrangian formulation [60]. Recently, Harte [61] showed how the formalism of Dixon could be extended to self-gravitating systems, by constructing appropriate effective momenta and effective multipole moments evolving in some effective metric.

In the present article, we are interested in the quadratic-in-spin contributions arising from the quadrupolar moment of the compact object in the case where it is adiabatically induced by
the spin \([32, 33, 35, 62]\), as well as the simpler contributions coming from products of SO corrections. This means that we shall implicitly assume that the relaxation time needed for the body, subject to an external perturbation, to recover equilibrium in its co-rotating frame is much shorter than the orbital period of the binary. The equilibrium states are then entirely determined by the mass and the spin of the object. On the other hand, because, in our source model, we replace extended bodies by point particles within a self-gravitating system, our approach must be regarded as an effective one and supplemented with some UV regularization procedure. A good choice is known to be dimensional regularization, with possible need of renormalization. We find however that, at this order, the so-called pure Hadamard–Schwartz prescription \([63]\) is sufficient, i.e. that dimensional regularization is not necessary.

The paper is organized as follows. In section 2, we explain how the dynamics of a test point particle endowed with a spin-induced quadrupolar structure moving in a curved background spacetime is described in the Dixon–Mathisson–Papapetrou formalism. We also write the equations of evolution for the particle worldline, as well as for the spin, under a convenient explicit form, and we define a spin vector of conserved Euclidean norm in terms of which our PN results shall be written. The validity of the model to describe the body dynamics in self-gravitating binaries is discussed. In section 3, dedicated to the computation of the next-to-leading order SS contributions to the PN equations of motion, we present expressions for the conserved energy in the center-of-mass frame, both for generic orbits and for the restricted case of circular orbits in the absence of precession. Finally, section 4 sketches the derivation of the next-to-leading order SS contributions to the GW flux and presents the full phase expression for non-precessing binaries at the order 3PN (apart from possible black-hole absorption effects). It also includes a discussion about the impact of our newly computed terms on the GW phasing in the frequency band of LIGO and Virgo. Because of the length of the equations, some results are relegated to appendices. Appendix A gives the explicit expressions for the relative acceleration, the precession vector, and the conserved angular momentum in the center-of-mass frame, and appendix B shows the relevant SS contributions to the source moments. We also give the explicit transformation between spin vector and spin tensor in appendix C, as well as the correspondence between our results and the ADM ones in appendix D.

We use the following conventions henceforth: \(\mathcal{O}(n)\) means \(\mathcal{O}(1/c^n)\), i.e. represents a contribution of the order \((n/2)\)PN at least. Greek indices denote spacetime coordinates, i.e. \(\mu = 0, 1, 2, 3\), while Latin indices are used for spatial coordinates, i.e. \(i = 1, 2, 3\). Symmetrization and anti-symmetrization are represented by, respectively, parenthesis and brackets around indices. We adopt the signature \((-\,+,\,+,\,+\,)\) and keep explicit both Newton’s constant \(G\) and the speed of light \(c\). Finally the covariant derivative along the worldline is written as \(\nabla_\mu = u^\rho \partial_\rho\), where \(u^\mu\) is the 4-velocity of the particle, defined such that \(u^\mu u_\mu = -1\).

2. Dynamics of quadrupolar particles

We shall now introduce the model we have adopted to represent the two spinning compact objects composing the binary as point particles. In section 2.1, we display the Dixon–Mathisson–Papapetrou evolution equations for test bodies at quadrupolar order, set the covariant spin supplementary condition (SSC), and discuss its consequences. In section 2.2, we rewrite the equations of motion in terms of the 4-velocity and introduce a conserved mass. Section 2.3 presents the construction of a spin vector with a conserved Euclidean norm and shows the precession equation it satisfies. Finally, section 2.4 explains to what extent the
Dixon–Mathisson–Papapetrou dynamics can be used for the companions of a self-gravitating binary.

2.1. The Dixon–Mathisson–Papapetrou framework

When describing the dynamics of a binary system of compact objects with masses \( m_A, A = 1, 2 \), in the context of the post-Newtonian approximation, it is physically sound to model the two companions as point particles. Indeed, the ratio of the radii \( R \sim G m_A / c^2 \) to the body separation \( r_{12} \) is of the order \( G m_A / (r_{12} c^2) \), and thus much smaller than 1. The dynamics of test point-like objects including finite size effects has been investigated extensively by Dixon [56–59], who generalized the Mathisson–Papapetrou equations for spinning particles [50, 51, 53, 64] by attaching arbitrary high-order moments to the individual bodies, beyond the monopole and the current dipole also referred to as the particle spin. This dynamics can also be derived from an effective Lagrangian-type approach for spinning particles, pioneered by Bailey and Israel [60] (see also an extensive study for special relativity in [65]) and later implemented in EFT [21, 22], where higher-order moments appear as parametrizing couplings in the action to the value of the Riemann tensor and its derivatives on the worldline.

The Dixon–Mathisson–Papapetrou equations of evolution for a spinning particle with quadrupolar structure read:

\[
\frac{D}{c} \frac{p^\alpha}{dr} = - \frac{1}{2c} R^a_{\lambda\mu\nu} u^b S^{\mu
u} \frac{c^3}{3} V_\rho R^a_{\lambda\mu\nu} J^{\rho\mu\nu},
\]

\[
\frac{DS^{\alpha\beta}}{c^2 dr} = 2 p^{[\alpha \mu} J^{\beta \nu]} + \frac{4c^4}{3} R_{\lambda\rho\sigma}^{[\alpha} J^{\beta \mu\nu]} J^{\lambda\rho\sigma],}
\]

where \( p^\alpha \) is the 4-momentum of the particle and \( u^i = dx^i/(c \, dt) \) the 4-velocity along the worldline. The anti-symmetric spin tensor \( S^{\alpha\beta} \) represents the effective 4-angular momentum of the object, while the (effective) mass and current type quadrupoles are encoded into the Dixon quadrupolar tensor \( J^{\rho\mu\nu} \), which is only constrained at this stage to have the same symmetry properties as \( R^{\rho\mu\nu} \).

The stress-energy tensor \( T^{\alpha\beta} \) of the model can be constructed after the Tulczyjew procedure, by making the only assumption that its support is point like with at most two derivatives acting on the Dirac distributions, in the three following steps [55]: (i) write the most general symmetric tensor that involves up to two (covariant) derivatives of the particle scalar density

\[
n = \int_{-\infty}^{+\infty} c \, dr \, \delta^4(x - y(\tau)) \sqrt{\gamma},
\]

where \( \delta^4(x - y(\tau)) \) is a 4-dimensional Dirac delta, with \( y(\tau) \) the particle worldline and \( x \) the field point; (ii) derive the hierarchy of equations verified by the coefficients of \( n \) in \( T^{\alpha\beta} \) due to the conservation equation \( \nabla_\nu T^{\nu\mu} = 0 \); (iii) constrain those coefficients by solving all algebraic equations, which leaves two sets of ordinary differential equations. Identifying these two equations to equations (2.1) yields the expression of \( T^{\alpha\beta} \) in terms of \( p^\alpha, S^{\alpha\beta} \) and \( J^{\rho\mu\nu} \):

\[
T^{\alpha\beta} = n \left[ p^{(\alpha \mu} J^{\beta)\rho\sigma} - \frac{1}{3} R_{\lambda\rho\sigma}^{(\alpha} J^{\beta)\lambda\rho\sigma} \frac{c^2}{3} \right]
\]

\[
- V_\rho \left[ n S^{(\alpha \mu \nu)} \right] - \frac{2}{3} V_\rho V_\sigma \left[ n c^2 J^{(\alpha \rho \sigma)} \right].
\]
It can be recovered with a smaller amount of calculation, further assuming that the system dynamics is governed by the effective Lagrangian of Bailey & Israel [60], by differentiating the resulting action with respect to the metric [42]. For a structureless particle, the spin and quadrupole tensors both vanish, so that, by virtue of equation (2.1b), \( p^\alpha \) is proportional to \( u^\alpha \), i.e. \( p^\alpha = m c u^\alpha \), and the stress-energy tensor (2.3) reduces to its standard expression for a test mass.

As the spin tensor \( S^{\alpha \beta} \) is anti-symmetric, it actually contains six degrees of freedom. Moreover, for an isolated body, the space-time components \( J^0_i \) of the total angular momentum \( J_i = S^{0i}/c \) in an appropriate asymptotically Minkowskian gauge represent the mass-type dipole of the object, and can thus always be taken to be zero. Similarly, for a test particle moving in a gravitational background, three degrees of freedom among those contained in the effective spin tensor are expected to be non-dynamical. They may be eliminated by fixing the ‘center-of-body’ reference point with the help of three independent space-time equations, globally referred to as the spin supplementary condition (SSC). The three remaining degrees of freedom correspond to the spatial components of the spin vector \( S^\alpha \).

Various choices of SSC are possible (see for instance [66]). Here we shall adopt, in keeping with previous works, the covariant (or Tulczyjew [54]) condition

\[
S^\mu p_\mu = 0. \tag{2.4}
\]

Assuming that the rotating bodies are always at equilibrium, we can reasonably expect their moments to depend on their masses, spins, as well as possible dimensionless parameters that characterize the internal structures. Notably, the spins may induce mass quadrupoles as they do for Kerr black holes. This effect produces spin-square type contributions that must be crucially taken into account at quadratic order in the spin variables. Tidal fields inside the bodies may also generate \( \ell \geq 2 \) multipoles, but their leading order contribution to the acceleration would be \( \sim \langle R_{1/2}/\ell^2 \rangle = \mathcal{O}(10) \) for a compact binary, so that they can safely be neglected in the present work.

As not all degrees of freedom in the Dixon quadrupole are physical, its value as a function of time cannot be uniquely determined by the internal dynamics of the body. In the adiabatic approximation, there exists a relation, valid along the particle worldline, between \( J^{i\rho \alpha} \), the 4-velocity \( u^\alpha \) and the spin tensor \( S^{\alpha \beta} \). It can be derived from an effective Lagrangian \( L_{SS} \) built to be the most general Lagrangian—modulo perturbative redefinitions of the gravitational field, terms in the form of a total time derivative, terms that vanish under some given SSC, and \( \mathcal{O}(S^3) \) remainders—with the properties of: (i) being quadratic in \( S^{\alpha \beta} \), (ii) depending on \( u^\alpha \), the metric \( g_{\alpha \beta} \), (derivatives of) the Riemann tensor, as well as some parameters characterizing the object [21, 33, 35]. As explained with more details in [33], we find, after redefining \( p^\alpha \), \( S^{\alpha \beta} \), that the stress-energy tensor associated with \( L_{SS} \) coincides with that of equation (2.3) provided \( J^{i\rho \alpha} \) is given by

\[
J^{i\rho \alpha} = \frac{3\kappa}{m^2} S^{\alpha \nu \beta \lambda} \xi_{\nu} [u^\rho] S^\beta_{\lambda}, \tag{2.5}
\]

at any instant. The above expression properly describes the presence of a non-vanishing spin-induced quadrupole, with the source dependent constant \( \kappa \) representing the quadrupolar polarizability, to be identified with the parameter \( a \) of [32]. This constant is exactly 1 for an isolated Kerr black hole whereas it is expected to remain close to unity for black holes in binary systems. For a neutron star, \( \kappa \) roughly ranges from 4 to 8. The mass parameter \( \tilde{m} \) is defined by \( p^2 \equiv (\rho^\mu p_\mu = -\tilde{m}^2 c^2 \). Notice that \( \tilde{m} \) is not a priori conserved. In fact, as shown below, its time derivative is quadratic in spin and cannot be ignored at our accuracy level.
even though in the above expression of $J_{\lambda\rho\alpha\beta}$, it may be replaced consistently with the conserved post-Newtonian mass $m$ defined by equation (2.11).

The (contravariant) 4-momentum is proportional to the 4-velocity of the particle when terms beyond linear order in the spins are neglected: $p^\alpha = \tilde{m} \ c \ u^\alpha + \mathcal{O}(S^2)$. Our first step will consist in expressing $p^\alpha$ as a function of $u^\alpha$ to quadratic order in the spins. We impose that the derivative along the worldline of the SSC (2.4) is zero, insert the equations of motion (2.1a) and (2.1b) into the resulting identity, and use the fact that $J_{\lambda\rho\alpha\beta} \sim \mathcal{O}(S^2)$ whereas $S_{\mu\nu} \sim \mathcal{O}(S^3)$. This yields, at quadratic order in spin,

$$p^\alpha = \frac{\tilde{m} \ c \ u^\alpha}{2\tilde{m} \ c^3} - \frac{S_{\alpha\beta} S^\mu\nu}{2\tilde{m} \ c^3} u^\beta R_{\rho\mu\alpha} + \frac{4c}{3} u_\beta R^{[\alpha}_{\lambda\mu\nu} J^{\beta]}_{\mu\nu} + \mathcal{O}(S^3).$$  (2.6)

We are now in a position to write the spin evolution equation in a more explicit way. In the Lagrangian formalism, the effective linear and angular momenta are defined in a way that guarantees the conservation of the spin magnitude [42, 62]. This conservation law is a remarkable feature of the spinning-particle dynamics. In our context, it will follow from equations (2.1) for some class of supplementary conditions. In fact, it can indeed be derived explicitly from those equations, for the form (2.5) of the quadrupole moment and the covariant SSC (2.4). By substituting the 4-momentum (2.6) into equation (2.1b) we get

$$\frac{DS_{\alpha\beta}}{c^2 \csc} = \frac{4c}{3} \left[ R^{[\alpha}_{\lambda\mu\nu} J^{\beta]}_{\mu\nu} + u_\alpha u^\beta R_{\rho\mu\alpha} J^{\rho}_{\mu\nu} - u_\alpha u^\beta J^{[\alpha}_{\lambda\mu\nu} R^{\beta]}_{\mu\nu} \right]$$

$$- u^4 R_{\rho\mu\alpha} u^\beta S_{\alpha\beta} S_{\mu\nu} \tilde{m} \ c^3 + \mathcal{O}(S^3).$$  (2.7)

If we contract this expression with $S_{\alpha\beta}$, we obtain $S_{\alpha\beta} DS_{\alpha\beta}/(c \csc) \sim \mathcal{O}(S^4)$ and, therefore, defining the spin magnitude as $s^2 = S_{\alpha\beta} S_{\alpha\beta}/2$,

$$\frac{ds}{d\csc} \sim \mathcal{O}(S^3).$$  (2.8)

This demonstrates that the spin magnitude is actually conserved at order $\mathcal{O}(S^2)$.

### 2.2. Conserved mass and evolution equations

Our next task is to investigate the issue of mass conservation at quadratic order in spins. For this purpose, let us compute the time derivative of the mass parameter $\tilde{m}$. Using the equation of motion (2.1a) and the Bianchi identities, we can write

$$-\tilde{m} \ c^2 \frac{d\tilde{m}}{d\csc} = \rho^\alpha \frac{Dp^\alpha}{c d\csc} = -\frac{\tilde{m} \ c^2}{6} u^\beta V_{\beta} R_{\rho\mu\alpha} J^{\rho\mu\nu} + \mathcal{O}(S^3).$$  (2.9)

As the time dependence of $J^{\rho\mu\nu}$ is through the 4-velocity and the spin tensor, i.e. $J^{\rho\mu\nu}(\tau) = J^{\rho\mu\nu}(u^\beta(\tau), S_{\alpha\beta}(\tau))$, the fact that $Du^\beta/(c \csc) \sim \mathcal{O}(S)$ and $DS_{\alpha\beta}/(c \csc) \sim \mathcal{O}(S^2)$ implies the approximate conservation of the Dixon quadrupole: $D J^{\rho\mu\nu}/(c \csc) \sim \mathcal{O}(S^3)$. Now, substituting $u^\beta V_{\beta}$ with $D/(c \csc)$, we can write down the equation

$$\frac{d}{d\csc} \left[ \tilde{m} - \frac{1}{6} R_{\rho\mu\alpha} J^{\rho\mu\nu} \right] = \mathcal{O}(S^3),$$  (2.10)
which finally allows us to define a conserved quantity $m$ as

$$m \equiv m - \frac{1}{6} R_{\rho \mu \nu} J^{\rho \mu \nu}.$$  

(2.11)

Hereafter, the constant parameter $m$ will be regarded as the effective mass of the particle. This mass is the one that appears in all our post-Newtonian results. By construction, it is conserved, like the spin magnitude. Substituting the expression (2.11) into equation (2.6) gives us the link between the 4-momentum and the 4-velocity:

$$p^\alpha = mc u^\alpha + \frac{c}{6} u^\alpha R_{\rho \mu \nu} J^{\rho \mu \nu} - \frac{S^\alpha S^\mu}{2 m c^2} u^\alpha R_{\beta \mu} u^\beta + \frac{4 c}{3} \mu R_{\beta \alpha \mu} J^{\beta \rho \mu} + \mathcal{O}(S^3),$$

(2.12)

where $m$ was just shown to be a constant parameter at order $\mathcal{O}(S^2)$. We are then in a position to rewrite the evolution equations for spinning particles to quadratic order in the spins, using the 4-velocity instead of the 4-momentum. Those are:

$$\frac{Du^\alpha}{d\tau} = \frac{u^\alpha}{2} \frac{DR_{\rho \mu \nu} S^\rho S^\mu}{m^2 c^4} - \frac{R_\alpha^{\mu \nu}}{2 m c} u^\nu S^\mu - \frac{c}{3} \frac{\nabla_\rho R_\mu R_\lambda J^{\rho \lambda \mu}}{m} + \mathcal{O}(S^3),$$

(2.13a)

$$\frac{DS^\alpha}{d\tau} = \frac{4 c^3}{3} \left[ \frac{R_\alpha^R R_\mu^R J^{\rho \mu}}{m} - \frac{u^\beta R_\beta}{} \frac{J^{\rho \mu}}{m} + \frac{4 u^\beta R_{\rho \mu \nu} J^{\rho \mu \nu}}{m + \mathcal{O}(S^3).} \right]$$

(2.13b)

2.3. Definition of a spin vector and equation of precession

From the anti-symmetric spin tensor $S^{\alpha \beta}$, we define the spin 4-covector $\tilde{S}_\alpha$ as

$$\tilde{S}_\alpha = -\frac{1}{2} \epsilon_{\alpha \beta \mu \nu} \frac{p^\beta}{m c} S^\mu,$$

(2.14)

where $\epsilon_{\alpha \beta \mu \nu}$ denotes the covariant Levi–Civita tensor, with $\eta_{\alpha \beta \mu \nu}$ being the completely anti-symmetric symbol that verifies $\eta_{0123} = 1$, and where $g = \det g_{\mu \nu}$ is the determinant of the metric tensor in generic coordinates. The tilde on this covariant spin vector will allow us to distinguish it from the Euclidean conserved-norm spin vector we shall introduce below. Notice that $\tilde{S}_\alpha$ automatically satisfies $\tilde{S}_\alpha p^\alpha = 0$ and thus carries 3 degrees of freedom as required. If we contract the above equation with $\epsilon^{\alpha \beta \mu \nu} p_\mu$ and use the SSC $p_\lambda S^{\lambda \mu} = 0$, we can invert equation (2.14) and obtain the spin tensor in terms of $\tilde{S}_\alpha$:

$$S^{\alpha \beta} = \epsilon^{\alpha \beta \mu \nu} \frac{p^\nu}{m c} \tilde{S}_\mu + \mathcal{O}(S^3).$$

(2.15)

Remembering that $p^\beta = mc u^\beta + \mathcal{O}(S^2)$ at the linear-in-spin level, it is straightforward to check that $\tilde{S}_\alpha S^\alpha = s^2$, by virtue of the relation $\epsilon_{\alpha \beta \mu \nu} \epsilon^{\lambda \mu \rho \sigma} = -4! \delta_\alpha^\lambda \delta_\beta^\mu \delta_\rho^\nu \delta_\sigma^\sigma$. To derive the evolution equation for the spin 4-covector, we differentiate equation (2.14) with respect to the proper time, which yields
In what follows, we shall explicitly resort to our particular form (2.5) for $J_{\lambda \rho \alpha \beta}$, relevant in the case of a spin-induced quadrupole. It will be convenient to investigate each term on the right-hand side of equation (2.16) individually. With our definition of $\tilde{S}_\alpha$, the first term there reads

$$\frac{1}{2} \epsilon_{\alpha \beta \mu \nu} \mathcal{R}_{\lambda \rho \alpha \beta} \tilde{S}^\lambda \tilde{S}^\rho \frac{u^\mu u^\nu}{m c} = -\frac{1}{2} \mu \epsilon_{\alpha \beta \gamma \delta} \tilde{S}_\gamma \frac{\mathcal{R}^\beta_{\lambda \rho \alpha \beta} \tilde{S}_\rho u^\mu u^\nu}{m c} + \mathcal{O}\left(S^3\right).$$

At this stage, it is useful to introduce the gravitomagnetic part of the Bel decomposition of the Riemann tensor

$$H_{ab} = 2 \ast R_{ab \mu \nu} u^\mu u^\nu,$$

where $\ast R$ is the dual of the Riemann tensor defined by

$$\ast R_{ab \mu \nu} = \frac{1}{2} \epsilon_{ab \mu \nu} R_{\alpha \beta \rho \sigma}.$$

Physically, the tensor $H_{ab}$ represents the tidal current-type quadrupole in the relativistic theory of tides. We can now put equation (2.16) in the form

$$\frac{1}{2} \epsilon_{\alpha \beta \mu \nu} \mathcal{R}_{\lambda \rho \alpha \beta} \tilde{S}^\lambda \tilde{S}^\rho \frac{u^\mu u^\nu}{m c} = \frac{1}{2} H_{ab} \tilde{S}_a \tilde{S}_b + \mathcal{O}\left(S^3\right).$$

Let us focus next on the second term on the right-hand side of equation (2.16). After substituting the value for $J_{\nu \lambda \sigma \rho}$ therein, we rewrite the resulting expression in terms of the gravitoelectric part of the Bel decomposition of the Riemann tensor

$$G_{ab} = -R_{ab \mu \nu} u^\mu u^\nu,$$

which is nothing but the tidal mass-type quadrupole generalizing that of Newtonian gravity (up to a factor $1/c^2$). Next, we directly replace the spin tensors with their corresponding spin covectors in equation (2.16), hence:

$$\frac{D \tilde{S}_a}{dr} = \frac{1}{2} H_{ab} \tilde{S}_b \frac{u^\mu u^\nu}{m c} - \kappa \epsilon_{\alpha \beta \mu \nu} u^\beta G^\mu \tilde{S}_a \tilde{S}^\nu \frac{m c}{m c} + \mathcal{O}\left(S^3\right).$$

Finally, after setting

$$\tilde{\Omega}_{ab} = \frac{\tilde{S}_a}{m c} \left[ u_{(a} H_{b)} \right] - \kappa \epsilon_{\alpha \beta \mu \nu} u^\beta G_{ab \mu \nu},$$

the spin precession equation for the covariant spin vector takes the form

$$\frac{D \tilde{S}_a}{dr} = \tilde{\Omega}_{ab} \tilde{S}^b + \mathcal{O}\left(S^3\right).$$

The anti-symmetric tensor $\tilde{\Omega}_{ab}$ may be interpreted as a spin-precession frequency tensor.

It remains to construct a spin 3-vector $S^i$ with conserved Euclidean norm. A ‘canonical’ construction is already explained in section 2.1 of [25], to which the reader may refer for further details. The precession vector governing the evolution of $S^i$ differs from that of [25], derived in the SO approximation, by additional terms that are quadratic in the spins.

The passage to spin 3-vectors is achieved by introducing a direct orthonormal tetrad $e_\alpha^\mu$.

The underlined index represents the vector label, which may be viewed as the tetrad index, spacetime indices being represented by Greek letters and spatial indices by Latin letters as
usual. Posing $e_\mu^d = u^\mu$, we see that
\[ \hat{S}_2 = \tilde{S}_2 e_\mu^d = \mathcal{O}(S^3), \]
which means that $\hat{S}_2$ may be neglected. The squared Euclidean norm of $\hat{S}_2$ is then given by
\[ \delta_{ab} \tilde{S}_a^a \tilde{S}_b^b = \gamma_{\mu
u} \tilde{S}_\mu^\mu \tilde{S}_\nu^\nu = \tilde{S}_\mu^\mu = s^2, \]
with $\gamma_{ab} = g_{ab} + u_a u_b$. In words, the spin vector $\hat{S}_2$ has a conserved Euclidean norm. To define the spin variable uniquely in some coordinate grid, we still need to specify the choice to be made for the spatial part of the tetrad. Considering that $\delta_{ab} e_\mu^a e_\nu^b = \gamma_{ab}$, a natural choice is to take for $e_\mu^a$ the unique symmetric positive-definite square root (in the matrix sense) of $\gamma_{ab}$. The complete expression for the tetrad is
\[ e_\mu^a = \left( \gamma_{\mu
u} - \gamma_{\mu0} \gamma_{0\nu} \right) e_\nu^a. \]
with $v^a \equiv c u^\mu / u^0$ denoting the coordinate velocity. After projection on the basis vectors (2.27), the precession equation for the spin vector becomes
\[ \frac{d\tilde{S}_2}{dt} = \left( \tilde{\omega}_{ab} + \tilde{\Omega}_{ab} \right) \tilde{S}_2, \]
where we have introduced the rotation coefficients for the tetrad
\[ \tilde{\omega}_{ab} = -e_\mu^d \frac{De_\mu^a}{dt}, \]
and where $\tilde{\Omega}_{ab} = \tilde{\Omega}_{\mu
u} e_\mu^a e_\nu^b$. Now, considering that $d/dt = u^0 \partial / \partial t$, it is convenient to define an anti-symmetric precession frequency tensor associated with the coordinate time as
\[ \Omega_{ab} = \frac{1}{u^0} \left( \tilde{\omega}_{ab} + \tilde{\Omega}_{ab} \right). \]
Since $\tilde{S}_2$ is negligible, the precession equation reduces to
\[ \frac{d\tilde{S}_2}{dt} = \Omega_{ab} \tilde{S}_a \mathcal{O}(S^3). \]

Moreover, from the equality $e_0^\mu = u^\mu$, it follows that the first term on the right-hand side of equation (2.23) vanishes when projected on spatial tetrad indices, so that
\[ \tilde{\Omega}_{ij} = -\kappa e_{ij}^{\ell k} \frac{\tilde{S}_k}{m c}, \]
where $e_{ij}^{\ell k}$ or $e_{ij}^{\ell k}$ (indifferently) denote the Euclidean Levi–Civita symbol, with normalization $e^{123} = e_{123} = 1$, which is linked to the four dimensional Levi–Civita tensor by the relation $e_{ij}^{\ell k} = e_{ij}^{\ell k}$. In the rest of the paper, we shall use a conserved Euclidean spin vector $S$ with spatial components $S^i$ in harmonic coordinates such that
\[ S^i \equiv \tilde{S}^i. \]
Because of the anti-symmetric character of $\Omega_{ij}$, we can finally rewrite the precession equation in terms of a precession vector $\Omega^i = -e_{ij}^{\ell k} \Omega_{j\ell k}/2$ as
\[
\frac{dS_i}{dr} = \epsilon_{ijk} \Omega^j S^k + \mathcal{O}(S^3).
\] (2.34)

It is the above precession vector \(\Omega^j\) that will be computed, along with the equation of motion, in section 3. Our results will be displayed either in terms of the vector \(S^i\) or of the spatial components of the spin tensor \(S^{ij}\).

2.4. Application to self-gravitating binary systems

Although the evolution equations (2.1) originally obtained by Dixon are only suitable to describe the dynamics of test particles, their rederivation based on the method of Tulczyjew or the Lagrangian approach of Bailey and Israel, regarded as effective field schemes, holds for self-gravitating \(N\) point-like body systems. Nonetheless, the validity of the point particle model breaks down at UV scales where the post-Newtonian expansion cannot be applied, i.e. for \(r_A \sim R_A\), with \(r\) being the distance between the particle representing the body \(A\) and the field point \(x\). In particular, some infinities arise when computing the gravitational field iteratively due to divergences at the particle positions \(y_A\). The situation is even worse as we make \(x\) tend towards \(y_A\).

As usual, those infinities are cured thanks to dimensional regularization, which preserves the invariance under diffeomorphism of general relativity, combined with some renormalization procedure. For an appropriate choice of the space dimension \(d\), the field remains weak near \(r_A = 0\) and can be computed perturbatively in the post-Newtonian approximation. We are confident that this leads to the correct PN dynamics because: (i) the result for the acceleration is unambiguous up to the order 3.5PN for binaries of spinning compact objects, (ii) it is equivalent to that obtained from other methods (see the review paper [7] for references), and notably from the approach à la Einstein–Infeld–Hoffmann used by Itoh [67] in the case of spinless bodies where no regularization is needed. Those cautions being taken, a self-gravitating system of \(N\) spinning bodies endowed with a quadrupolar structure may be modeled by means of the following effective stress-energy tensor, which generalizes that of equation (2.3):

\[
T^{\alpha\beta} = \sum_{A=1,2} \left[ n_A \left( p_A^{(\alpha} u_A^{\beta)} c + \frac{1}{3} R_{(\alpha}^{(\beta} \rho^\beta_\rho \sigma \rho \sigma c^2 \right) 
- V_{\rho} \left( n_A S_A^{(\alpha} u_A^{\beta)} \right) - \frac{2}{3} V_{\rho} V_{\sigma} \left( n_A c^2 J_A^{(\alpha} \sigma) \right) \right].
\] (2.35)

where the subscripts \(A\) indicate the particle labels.

The presence of poles \(\propto \epsilon^{-k}\) in the metric at a given post-Newtonian order, with \(\epsilon \equiv d - 3\) and \(k\) being a positive integer, may generate contributions in the source for the next order that could not be recovered by resorting to a purely three dimensional regularization. However, in the absence of such subtleties, the so-called pure Hadamard–Schwartz regularization [68] is sufficient to get the correct result. This prescription essentially relies on a specific use of the Hadamard partie finie regularization, which we shall briefly discuss now (the reader will find more details in [69]).

Let us consider a function \(F(x)\) with the same regularity properties as those arising in our problem, i.e smooth everywhere except at some singular points \(y_A\) \((A = 1, 2, \ldots, N)\) in the neighborhood of which its admits an expansion of the form
\[ F(x) = \sum_{p_0 \leq p \leq P} r_p^p f_p^p(n_A) + o(r_p^p) \]  

(2.36)

for any integer \( P \), with \( n_A = (x - y_A)/r_A \). Such a function is said to be of class \( \mathcal{F} \). Its Hadamard partie finie \((F)_A\) is then defined as the angular average of the finite part \( \delta^0(n_A) \):

\[ (F)_A = \int \frac{d\Omega_A}{4\pi} \delta^0(n_A), \]  

(2.37)

where \( d\Omega_A \) denotes the elementary solid angle with direction \( n_A \) centered on \( y_A \). The operation of taking the Hadamard partie finie is not distributive with respect to multiplication in the sense that, for another function \( G(x) \) of class \( \mathcal{F} \), \( (F)_A(G)_A \neq (FG)_A \) in general. Moreover, it does not respect the Lorentz invariance. Because of the first of those two unpleasant features, the so-defined regularization is fundamentally ambiguous as such. Howbeit, it can still be used in practical computations provided it is supplemented by some additional prescription. In the PNISH approach, the post-Newtonian metric is constructed iteratively with the help of PN potentials. Those are elementary bricks satisfying a wave-type equation (more details are provided in section 3). A convenient prescription is to define the value of a product \( FG \) of two potentials (or potential derivatives) evaluated at point \( y_A \) as \( (F)_A(G)_A \). Similarly, the regularized product of a potential \( F \) and an arbitrary smooth function \( \alpha(x) \) will be given by \( \alpha(y_A)F_A \).

Divergent integrals are cured by applying another kind of Hadamard partie finie regularization. The regularized value of an integral with class-\( \mathcal{F} \) integrand is calculated in three main steps: (i) balls of radius \( \eta \) centered on the singular points are extracted from the integration domain; (ii) terms that diverge near \( \eta = 0 \) are removed; (iii) one goes to the limit \( \eta \to 0 \). The singularities that generate poles in dimensional regularization produce logarithmic divergences in the Hadamard one. Those are associated with cutoff parameters \( s_A \) entering terms such as \( \ln(\eta/s_A) \). For consistency between the two kinds of Hadamard regularizations, all derivatives must be evaluated in the sense of distributions [69]. The action of the three dimensional Dirac delta \( \delta_A \equiv \delta^3(x - y_A) \) on test functions must also be generalized to \( \mathcal{F} \)-class functions by posing \( F\delta_A = (F)_A \delta_A \).

In this context, the pure Hadamard–Schwartz regularization is an ensemble of prescriptions designed to yield results that are ‘as close as possible’ to those obtained through dimensional regularization. These prescriptions demand: (i) to evaluate monomials of the form \( \alpha(x)F_1\ldots F_n \), where \( \alpha(x) \) is a smooth function and the \( F_i \)’s are (derivatives of) \( \mathcal{F} \)-class potentials, as \( \alpha(y_A)(F_1)_{A} \ldots (F_n)_{A} \); (ii) to evaluate divergent integrals by means of the Hadamard partie regularization for integrals; (iii) to extend the definition of \( \delta_A \) as explained above; (iv) to compute all derivatives in the sense of Schwartzian distributions.

The absence of logarithmic cut-offs in the SS piece of the metric up to the order 3PN suggests that dimensional regularization may safely be swapped for the pure Hadamard–Schwartz one at this accuracy level. The insensitivity of the calculations to the choice of regularization procedure has been checked explicitly by evaluating source terms of the type \( FG \delta_A \) in the stress-energy tensor as \( \langle FG \rangle_A \delta_A \), thus violating the pure Hadamard–Schwartz prescription. The results have always turned out to be unaffected by such modifications.

3. Next-to-leading order contributions to the post-Newtonian evolution

We now turn to the computation of the dynamics of a binary system in the post-Newtonian approximation, at the next-to-leading order for the quadratic-in-spin effects, i.e. at the order
1/c^6 (or 3PN) in the equations of motion and at the order 1/c^5 in the equations of precession. We will recover the results for the dynamics obtained in the ADM [62, 70–73] and EFT [21, 22, 35, 38, 39, 74] approaches, and extend them towards the completion of the calculation of the GW energy flux.

We start with some general definitions in section 3.1 and introduce a set of potentials parametrizing the PN metric. Next, in section 3.2, we express the quantities of interest in terms of these potentials. In section 3.3, we present their computation, and finally in section 3.4 the results obtained for the dynamics as well as various tests of their correctness. The lengthier calculations are all performed by means of the algebraic computing software Mathematica® supplemented by the tensor calculus package xAct [75].

3.1. General definitions

The two objects are represented as quadrupolar point particles as explained above. An important ingredient of the formalism is the treatment of the infinite self-field of the point particles, essentially represented by means of Dirac deltas, through the pure Hadamard–Schwartz regularization procedure discussed in section 2.4. The distributional contributions yielded by derivatives are handled by using the Gel’fand-Shilov formula [76]. We found that at this order in spin, we have to keep track of distributional contributions in the metric itself to obtain the correct result for the wave generation formalism, as will be detailed in section 4.2.

The equations of motion and the equations of precession for particle 1 (say) are determined by the expressions of its acceleration $A_1$ and its precession vector $\Omega_1$. The general structure of those quantities is as follows:

$$A_1 = A_{1\text{NS}} + \frac{1}{c^2} A_{1\text{IPN}} + \frac{1}{c^4} A_{1\text{2PN}} + \frac{1}{c^5} A_{1\text{2.5PN}} + \frac{1}{c^6} A_{1\text{3PN}}$$

$$\Omega_1 = \frac{1}{c^2} \Omega_{1\text{NS}} + \frac{1}{c^4} \Omega_{1\text{IPN}} + \frac{1}{c^5} \Omega_{1\text{2PN}} + \frac{1}{c^6} \Omega_{1\text{2.5PN}} + \mathcal{O}(7),$$

where the spin order in equation (3.1b) indicates the contribution in $\Omega_1$ itself, rather than in $S_1 = \Omega_1 \times S_1$ (notably the SO terms feature the constants $\kappa_{1,2}$ and actually correspond to SS terms in $S_1$). The 2.5PN NS terms in the acceleration are the first manifestation of radiation reaction.

We use the same notations as in previous works. Three-dimensional indices are raised or lowered with the Euclidean metric $\delta_{ij}$; we do not distinguish between upper and lower indices. We sometimes use boldface for Euclidean vectors. The positions and velocities of the two bodies are denoted by $y_1, y_2$ and $v_1, v_2$. Apart from the separation distance $n_2 = |y_2| = |y_1 - y_2|$ which we have already defined, we shall need the separation direction $n_2^{\leftrightarrow} = (y_1^i - y_2^i)/n_2$. The symbol $1 \leftrightarrow 2$ indicates the same expression as the one before it, with the label of the two particles exchanged. The results are expressed in terms of the spatial components $S_1^i, S_2^i$ of the spin tensors $S_1^{\mu}, S_2^{\mu}$, as well as the spin vectors $S_1^i, S_2^i$ of conserved Euclidean norm as defined above, in section 2.3. The mixed components $S_1^{0i}$ of the spin tensors can always be eliminated with the help of the SSC (2.4).

In harmonic (or DeDonder) gauge, the gravitational field equations can be rewritten as

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4} |\xi| T^{\mu\nu} + A^{\mu\nu}[h] \equiv \frac{16\pi G}{c^4} r^{\mu\nu},$$

(3.2)
where the stress-energy pseudo-tensor $\tau^{\mu\nu}$ includes both matter and field contributions through $T^{\mu\nu}$ and $A^\mu$, the latter source term being at least quadratic in $h^{\mu\nu}$. The field equations (3.2), when iterated order by order, yield a solution expressed formally in terms of a hierarchy of potentials of increasing complexity and post-Newtonian order (see [77, 78] for the precise definition of this iteration in the near-zone).

Since on the one hand we are working at the order $1/c^6$ in the equations of motion, and on the other hand the spin contributions always come at relative $1/c$ order at least, only the so-called 2PN metric and potentials (i.e. necessary for the 2PN NS case) are required. In fact, we will see below that, among the potentials arising at the order 2PN, only $\hat{X}$ turns out to be needed. For completeness, we quote here the result of the iteration for the 2PN metric, which reads

$$g_{00} = -1 + \frac{2}{c^2} V - \frac{2}{c^4} V^2 + \frac{8}{c^6} \left( \hat{X} + V_i V_i + \frac{V^3}{6} \right) + \mathcal{O}(8), \quad (3.3a)$$

$$g_{0i} = -\frac{4}{c^3} V_i - \frac{8}{c^5} \hat{R}_i + \mathcal{O}(7), \quad (3.3b)$$

$$g_{ij} = \delta_{ij} \left[ 1 + \frac{2}{c^2} V + \frac{2}{c^4} V^2 \right] + \frac{4}{c^4} \hat{W}_{ij} + \mathcal{O}(6). \quad (3.3c)$$

The potentials therein are defined as

$$V = \square_{\mathcal{R}}^{-1} \left[ -4\pi G \sigma \right], \quad (3.4a)$$

$$V_i = \square_{\mathcal{R}}^{-1} \left[ -4\pi G \sigma_i \right], \quad (3.4b)$$

$$\hat{X} = \square_{\mathcal{R}}^{-1} \left[ -4\pi G V \sigma_{ij} + \hat{W}_{ij} \partial_{ij} V + 2V_i \partial_j \partial_i V + V \partial_i^2 V \right] + \frac{3}{2} (\partial_i V)^2 - 2\partial_i V \partial_j \partial_j V, \quad (3.4c)$$

$$\hat{R}_i = \square_{\mathcal{R}}^{-1} \left[ -4\pi G \left( V \sigma_i - V_i \sigma \right) - 2\partial_k V \partial_i \partial_k V - \frac{3}{2} \partial_i V \partial_j V \right], \quad (3.4d)$$

$$\hat{W}_{ij} = \square_{\mathcal{R}}^{-1} \left[ -4\pi G \left( \sigma_{ij} - \delta_{ij} \sigma_k \right) - \partial_i V \partial_j V \right], \quad (3.4e)$$

where the $\sigma$, $\sigma_i$, $\sigma_{ij}$ quantities are convenient matter source densities, given by

$$\sigma = \frac{1}{c^2} \left( T^{00} + T^i_i \right), \quad \sigma_i = \frac{1}{c} T^{0i}, \quad \sigma_{ij} = T^{ij}, \quad (3.5)$$

while $\square_{\mathcal{R}}^{-1}$ stands for the PN-expanded retarded inverse d’Alembertian operator, acting on a function $f(x, t)$ as [77, 78]

$$\left( \square_{\mathcal{R}}^{-1} f \right)(x, t) = -\frac{1}{4\pi} \sum_{n \geq 0} \frac{(-1)^n}{n! c^n} \left( \frac{d}{dt} \right)^n \int_{B=0}^\infty \int d^3 \mathbf{x} \left( \frac{|x|}{r_0} \right)^B |x - x'|^{B-n} f(x', t). \quad (3.6)$$

---

7 Possible contributions to the metric of nonlinear tail terms, which are not made of (products of) elementary potentials defined by means of the operator $\square_{\mathcal{R}}^{-1}$, do not arise below the order 4PN [77, 78].
Here $F_{B=0}$ denotes the so-called finite part regularization, and $r_0$ is an associated arbitrary length scale. This regularization is used and described in section 4.1 for the wave generation formalism, but in equation (3.6) it cures the divergences of the near-zone post-Newtonian metric at infinity rather than the divergences of the multipolar far-zone expansion at the origin. At the order we are considering here, it does not matter for the equations of motion. In particular the final results are all independent of the scale $r_0$.

### 3.2. Matter source and equations of motion in terms of the potentials

In this section, we introduce convenient additional definitions for the matter source in the PN context. In the covariant expression (2.35), the worldline integration contained in the particle densities $n_A$ (see equation (2.2)) can be performed explicitly in a definite coordinate grid $(t, x)$. This results in

$$T^{\alpha\beta} = \sum_{A=1,2} \left[ U^{\alpha\beta}_A \delta_A + \nabla_\mu \left( U^{\alpha\beta}_A \delta_A \right) + \nabla_\nu \delta_A \left( U^{\alpha\beta}_A \delta_A \right) \right], \quad (3.7)$$

where we have defined

$$U^{\alpha\beta}_A = \frac{1}{u_A^{\alpha} \sqrt{-g}} \left( c \rho^{(a)} u^{(b)} + \frac{1}{3} R^{(a)}_{\lambda \mu \nu} J^{\beta (\lambda \mu \nu)} A \right), \quad (3.8a)$$

$$U^{\alpha\beta\mu}_A = \frac{1}{u_A^{\alpha} \sqrt{-g}} u_A^{\beta (\mu)}, \quad (3.8b)$$

$$U^{\alpha\beta\mu\nu}_A = \frac{1}{u_A^{\alpha} \sqrt{-g}} \left( \frac{2}{3} c^{2} J^{\mu (\alpha\beta) \nu A} \right). \quad (3.8c)$$

Here $u^0 = 1 / \sqrt{-g^{\alpha \mu} \rho^{(a) \nu} \rho^{(b) \mu}} \rho^{(c)}, \gamma^\mu = (c, \gamma^\nu)$ (so that $u^\mu = u^0 \gamma^\mu / c$), and the label index $A$ on metric-dependent quantities means that they are to be regularized according to Hadamard regularization at the location of the particle $A$. In terms of partial derivatives, we have

$$T^{\alpha\beta} = \sum_{A=1,2} \left[ T^{\alpha\beta}_A \delta_A + \frac{1}{\sqrt{-g}} \partial_\mu \left( T^{\alpha\beta}_A \delta_A \right) + \frac{1}{\sqrt{-g}} \partial_\nu \delta_A \left( T^{\alpha\beta}_A \delta_A \right) \right], \quad (3.9)$$

with

$$T^{\alpha\beta}_A = U^{\alpha\beta}_A + 2 \Gamma^{(a)}_{\lambda \mu} U^{\beta\mu}_A + \left( \partial_\nu \Gamma^{(a)}_{\lambda \mu} \right) U^{\beta (\lambda \mu \nu)} A$$

$$+ \Gamma^{(a)}_{\lambda \mu} U^{\beta \nu}_{\lambda \mu} + U^{\beta \nu}_{A \lambda \mu} \Gamma^{(a)}_{\lambda \mu \nu} - \Gamma^{(a)}_{\lambda \mu \nu} U^{\beta (\lambda \mu \nu)} A \right), \quad (3.10a)$$

$$T^{\alpha\beta\mu}_A = \sqrt{-g} \left( U^{\alpha\beta\mu}_A + \Gamma^{\mu}_{\lambda \kappa} U^{\alpha\beta\kappa}_A - 2 \Gamma^{(a)}_{\lambda \kappa} U^{(a) \beta \mu\kappa}_A \right), \quad (3.10b)$$

$$T^{\alpha\beta\mu\nu}_A = \sqrt{-g} U^{\alpha\beta\mu\nu}_A. \quad (3.10c)$$

By using equations (3.8), and the definitions of $\sigma, \sigma_i, \sigma_{ij}$ given in equations (3.5), we arrive at the following expressions in terms of the metric potentials

---

8 At the 3PN non-spinning order, the scale $r_0$ does appear in the final results for the dynamics, but it disappears when considering gauge-invariant expressions such as $E(\omega)$, the conserved energy as a function of the orbital frequency.
\[
\sigma = m_1 \delta_k \left[ 1 + \frac{1}{c^2} \left( \frac{3}{2} v_i^2 - V \right) + \frac{1}{c^4} \left( \frac{7}{8} v_i^4 - 4 (v_i^a V_a) - 2 \dot{W} + \frac{1}{2} v_i^2 V + \frac{1}{2} V^2 \right) \\
+ \frac{1}{m_1 c^3} \left( 2 \left( S^{ab}_{\mu} v^a_i \partial_{\mu} V \right) - 4 \left( S^{ab}_{\mu} \partial_{\mu} V \right) \right) + \frac{1}{c^6} \left( -8 (\vec{R}_a v_i^a) + \frac{11}{16} v_i^6 \right) \\
- 10 v_i^2 (v_i^a V_a) - 4 (V_i V_i') + 2 \left( v_i^c v_i^d \vec{W}_{\mu \nu} \right) - 3 v_i^2 \dot{W} - 8 \ddot{Z} + \frac{33}{8} v_i^4 V \\
- 4 (v_i^a V_a) V + 2 \dot{W} + \frac{11}{4} v_i^2 V^2 - \frac{1}{6} V^3 - 4 \dddot{X} - \frac{k_1}{2m_1^2} \left( S^{ba}_{\mu} S^{bn}_{\mu} \partial_{\nu} V \right) \right] \\
+ \frac{1}{\sqrt{-g}} \partial_k \left[ \delta_i \left[ -2 S^{ia}_{\mu} v^a_i \right. \\
\left. + \frac{S^{ia}_{\mu}}{c^3} \left( -4 V v_i^a + 4 V \right) \right] \right] \\
+ \frac{k_1}{m_1 c^6} \left( S^{ab}_{\mu} S^{ib}_{\nu} \partial_{\mu} V - \left( S^{ab}_{\mu} S^{ib}_{\nu} \right) \partial_{\nu} \dot{V} \right) \right] \\
+ \frac{k_1}{2m_1 c^6 \sqrt{-g}} \partial^3 \left[ \delta_i \left( -S^{ia}_{\mu} S^{ib}_{\nu} v^a_i \right) + \left( S^{ia}_{\mu} S^{ib}_{\nu} \right) v^b_i \right] \\
+ \frac{k_1}{m_1 c^6 \sqrt{-g}} \partial^3 \left[ \delta_1 \left( \frac{S^{ia}_{\mu} S^{ib}_{\nu}}{2c^3} + \frac{S^{ia}_{\mu}}{c^6} \left( S^{ib}_{\mu} S^{ia}_{\nu} \right) \left( \frac{3}{4} v_i^2 + \frac{3}{2} V \right) - \frac{1}{2} S^{ib}_{\mu} S^{ia}_{\nu} v^a_i v^b_i \right. \\
\left. + \frac{1}{2} \left( S^{ia}_{\mu} S^{ib}_{\nu} \right) v^a_i v^b_i + S^{ia}_{\mu} \left( -S^{ia}_{\mu} v_i^a v_i^b - S^{ia}_{\mu} v_i^a v_i^b \right) \right) \right] \\
+ 1 \leftrightarrow 2 + \mathcal{O}(7).
\]

\[
\sigma_i = m_1 \delta_i \left[ v_i^l + \frac{1}{c^2} \left( \frac{3}{2} v_i^2 v_i^l - V v_i^l \right) \right] \\
+ \frac{1}{c^4} \left( \frac{3}{8} v_i^4 v_i^l - 4 (v_i^a V_a) v_i^l - 2 \dot{W} v_i^l + \frac{3}{2} v_i^2 V v_i^l + \frac{1}{2} V^2 v_i^l \right) \\
+ \frac{1}{2c^3 \sqrt{-g}} \partial_i \left( \delta_1 S^{ia}_{\mu} v^a_i \right) + \frac{1}{\sqrt{-g}} \partial_i \left[ \delta_1 \left( \frac{S^{ia}_{\mu}}{2c} - \frac{S^{ia}_{\mu}}{2c^3} \right) \right] \\
- \frac{k_1}{2m_1 c^6 \sqrt{-g}} \partial_i \left( \delta_1 S^{ib}_{\mu} S^{ia}_{\nu} \right) \\
+ \frac{k_1}{m_1 c^4 \sqrt{-g}} \partial_i \left( \delta_1 \left( \frac{S^{ia}_{\mu} S^{ib}_{\nu}}{2c^3} + \frac{S^{ia}_{\mu}}{c^6} \left( S^{ib}_{\mu} S^{ia}_{\nu} \right) \left( \frac{3}{4} v_i^2 + \frac{3}{2} V \right) - \frac{1}{2} S^{ib}_{\mu} S^{ia}_{\nu} v^a_i v^b_i \right) \right) \\
+ 1 \leftrightarrow 2 + \mathcal{O}(5).
\]

\[
\sigma_g = m_1 \delta_i \left[ v_i^l v_i^l + \frac{1}{c^2} \left( \frac{3}{2} v_i^2 v_i^l - V v_i^l v_i^l \right) \right] \\
+ \frac{1}{c^4 \sqrt{-g}} \partial_i \left( \delta_1 \left( \frac{1}{2} S^{ia}_{\mu} v_i^l + \frac{1}{2} S^{ia}_{\mu} v_i^l \right) \right) + 1 \leftrightarrow 2 + \mathcal{O}(3).
\]
quantity in factor of a Dirac delta as regularized. Notice however that the $1/\sqrt{-g}$ prefactors must not be evaluated at point $y_i$, i.e. they are still functions of the field point $x$. These matter sources display explicit factors with spins and others without spin, but we should always keep in mind that there are secondary spin contributions coming from the potentials themselves.

Let us now turn to the expression of the equations of evolution in terms of the metric potentials (3.4). Using $\text{d}/\text{d}r = u^0 \text{d}/\text{d}r$ and the second Bianchi identity, the covariant equations of motion (2.1a) may be put in the form

$$\frac{dP_\alpha}{dt} = F_\alpha \equiv F^\mu_\alpha \tau^\nu P_\mu - \frac{1}{2mc} R_{\alpha\beta\gamma\delta} \tau^\nu S^\alpha_\beta - \frac{c^2}{6mu^0} f^\beta_\delta \tau^\nu V_\alpha R_{\beta\delta\gamma\nu} + \mathcal{O}(S^3),$$

(3.12)

where $P_\alpha = p_\mu/m$ can be read from equation (2.12). For a lower spatial index $\alpha = i$, we decompose $P_\alpha$ as (coming back to notations for the body 1)

$$R_{ii} = R_{ii}^{NS} + R_{ii}^{SS},$$
$$F_{ii} = F_{ii}^{NS} + F_{ii}^{SO} + F_{ii}^{SS}.$$ (3.13)

Here, the order in spin refers to the order in the spin tensor as it reads in the formulas (2.12) and (2.1a), but we should recall that there are also spin contributions coming from the potentials themselves, as well as from the replacement of accelerations using the equations of motion. We see from equation (2.12) that $P_i$ has no SO part in this sense, and its NS part comes from $P_i = u^0 g_{ii} v^i/c$. The NS part of $F_i$ comes from the usual connexion term in the geodesic equation, the first term in equation (3.12). As the NS and SO parts can already be found e.g. in equation (2.12) of [79] and in equation (3.7) of [24], we only display here the SS pieces:

$$P_i^{(SS)} = \frac{1}{m_i^2 c^6} \left\{ -S_i^{ab} S_i^{bd} \partial_a V + 2S_i^{bd} S_i^{da} \partial_d V + S_i^{bd} \left( S_i^{cd} v^c_i \partial_d V - 2S_i^{d} v^i_i \partial_d V \right) \right\}$$
$$= \frac{\kappa_i}{m_i^2 c^6} \left\{ \frac{3}{2} S_i^{ab} S_i^{bd} v^c_i \partial_d V - S_i^{ab} S_i^{bd} \partial_d v^c_i \right\} + \kappa_i \left[ \left( S_i^{ab} S_i^{bd} \partial_d V \right) v^c_i - S_i^{ab} S_i^{bd} \partial_d v^c_i \right]$$
$$+ \left( S_i^{ab} S_i^{bd} \right) \partial_d v^c_i V + \left( S_i^{ab} S_i^{bd} \right) v^c_i \partial_d V + S_i^{bd} S_i^{ab} v^c_i \partial_d V$$
$$+ 2S_i^{ab} S_i^{bd} v^c_i \partial_d V + S_i^{bd} S_i^{ab} \left( -2\partial_d V + 2\partial_d V \right) \}.$$ (3.14a)

$$F_{ii}^{(SS)} = \frac{\kappa_i}{2m_i^2 c^4} S_i^{ab} S_i^{ab} \partial_d V + \frac{\kappa_i}{m_i^2 c^6} \left[ \left( S_i^{ab} S_i^{bd} \right) v^c_i \partial_d V - \frac{3}{2} \left( S_i^{ab} S_i^{bd} \right) \partial_d V + \frac{3}{2} \left( S_i^{ab} S_i^{bd} \right) \partial_d v^c_i \right]$$
$$+ \frac{1}{2} \left( S_i^{ab} S_i^{bd} \right) v^c_i v^d_i \partial_d V - S_i^{bd} S_i^{ab} V + S_i^{bd} S_i^{ab} \left( v^c_i \partial_d V + V \right)$$
$$+ 4a_i V \partial_d V + S_i^{bd} S_i^{ab} \left( 2\partial_d V + \frac{3}{2} v^c_i \partial_d V + \frac{1}{2} V \right) \}.$$ (3.14b)
In these formulas, the potentials and their derivatives are to be understood as regularized at
the location of body 1.

For the equation of precession of the conserved-norm spin, we decompose similarly the
precession vector into a NS and a SO part, before replacement of the potentials. We obtain
\[
\Omega_i^* = \left( \Omega_i^* \right)_\text{NS} + \left( \Omega_i^* \right)_\text{SO},
\]
where \( \left( \Omega_i^* \right)_\text{NS} \) is given by equation (2.19) in \[25\] and where
\[
\left( \Omega_i^* \right)_\text{SO} = \frac{k_i}{m_i c^3} S_i^a \partial_a V + \frac{1}{m_i c^3} \left\{ \nu_i^k \left( S_i^a v_i^b \partial_{ab} V \right) - \frac{1}{2} \left( S_i^a v_i^b \right) \partial_i \partial_{ab} V + S_i^a v_i^b \partial_{ab} V \right\}
- \nu_i^k S_i^a \partial_a V + \frac{1}{2} \left( S_i^a v_i^b \right) \partial_{ab} V - \left( S_i^a v_i^b \right) \partial_{ab} V + S_i^a v_i^b \partial_{ab} V
+ \kappa \left\{ -\left( S_i^a \partial_a V \right) \nu_i^k - \frac{3}{2} \left( S_i^a v_i^b \partial_{ab} V \right)\nu_i^k - \left( S_i^a v_i^b \right) \partial_i \partial_{ab} V
+ S_i^a \left( 2 \left( v_i^a \partial_a \nu_i^k \right) + \left( \partial_a \nu_i^k \partial_{ab} \nu_i^k \right) + \partial_i \partial_{ab} \nu_i^k \right)
- 3 \left( S_i^a \partial_a V \right) \partial_i V - \frac{3}{2} \left( S_i^a v_i^b \right) \partial_{ab} V + S_i^a \left( 2 \partial_a \partial_a V + 2 \partial_i \partial_i V \right.
+ 2 \nu_i^k \partial_{ab} V + \frac{3}{2} \nu_i^k \partial_{ab} V - 3 \nu_i^k \partial_{ab} V - 4 \nu_i^k \partial_{ab} V + 2 \nu_i^k \partial_{ab} V \right) \right\}. \tag{3.16}
\]

The contributions featuring \( k_i \) come directly from the second term in equation (2.30), while
the other contributions come from the first term there. The time derivatives of the velocities that
enter the definition of the tetrad are replaced by the expression of the acceleration in terms of the
potentials, which include SO terms (as given for instance in section 3 of \[24\]).

### 3.3. Spin contributions in the metric potentials

We now investigate the spin contributions to the metric potentials introduced in section 3.1.
As we are effectively working at the next-to-leading order, calculating these contributions
from the results already presented above will be rather straightforward and we will only need
to resort to well-known techniques.

By inspection of the matter sources (3.11), one can see that the SO and SS contributions
to the metric potentials start at the following PN orders:\(^9\)
\[
V_i^\text{SO} = \mathcal{O}(3), \quad V_i^\text{SS} = \mathcal{O}(4),
\]
\[
V_i^\text{SO} = \mathcal{O}(1), \quad V_i^\text{SS} = \mathcal{O}(4),
\]
\[
\hat{X}_i^\text{SO} = \mathcal{O}(1), \quad \hat{X}_i^\text{SS} = \mathcal{O}(2),
\]
\[
\hat{R}_i^\text{SO} = \mathcal{O}(1), \quad \hat{R}_i^\text{SS} = \mathcal{O}(4),
\]
\[
W_{ij}^\text{SO} = \mathcal{O}(3), \quad W_{ij}^\text{SS} = \mathcal{O}(4). \tag{3.17}
\]

From equations (3.14) and (3.16), we see that it is sufficient to compute the new SS
contributions of the potentials \( V \) at the order 3PN (the leading order contribution at the order
2PN being already known, see e.g. \[33\]), \( V_i \) at the order 2PN, and \( \hat{X} \) at the order 1PN.

We turn now to the calculation of \( V_i^\text{SS} \) at the 3PN order which actually corresponds to the
relative 1PN order. Truncating equation (3.6) appropriately, we have (dropping the FP
\[B=0\])

\(^9\) It is implicitly understood that the orders \( n \) showed in the terms \( \mathcal{O}(n) \) below take their highest possible values.
Because we are working at the next-to-leading order, various indirect contributions appear. Aside from the SS terms generated directly by the SS terms of $\sigma$ given in equation (3.11a), there are contributions from the SO 0.5PN part of $V$ in the SO 2.5PN part of $\sigma$, from the SS 2PN part of $V$ in the NS 1PN part of $\sigma$, and from the acceleration replacement featuring the SS 2PN part of $a^i$ in the second time derivative of the integral of the NS Newtonian part of $\sigma$ (in the third term above).

In the following, to present the results in a more compact form, we adopt shortcut notations: for any vectors $a$, $b$ and spin tensors $S_{AB}^{ij}$, we define the scalars $\sigma_{ab}^{ij}$, $(S_a S_b) \equiv S_{AB}^{ik} S_{ij}^{jk}$, $(a S_a b) b^i$, and $(a S_a S_b) b^i$ (beware of the convention for the order of the indices on the spin tensors).

We replace in equation (3.18) the full expression (3.11a) for $\sigma$, perform integration by parts when derivatives of Dirac deltas appear, and compute the resulting integrals using Hadamard regularization, e.g. $\int d^3x \delta_F = (F)$. The metric potentials can be considered as regularized when appearing in factor of a Dirac delta in the integrals, according to the pure Hadamard–Schwartz rule [63].

An important point is that the derivatives have to be treated in a distributional sense. For the first time in our formalism, we have to take into account an essential distributional term in the potential $V$ itself. The leading order result is indeed

$$V^{SS} = \frac{Gk_1}{2e^4m_1} S_i^{ik} S_i^{jk} \delta_3 \left( \frac{1}{\eta_1} \right) + 1 \leftrightarrow 2 + \mathcal{O}(6),$$

which, along with a non-distributional contribution, yields a distributional term

$$V_{SS}^{\text{dist}} = \frac{Gk_1}{2e^4m_1} S_i^{ij} S_i^{jk} \left( \frac{4\pi}{3} \delta_1 \right) + 1 \leftrightarrow 2 + \mathcal{O}(6).$$

This distributional term will play no role in the derivation of the equations of motion themselves, but it will produce a net contribution when computing the mass source quadrupole moment, as explained below in section 4.2. Because, in this computation of the quadrupole moment, the $V$ potential is only needed at the 2PN order, we will not need to consider possible distributional terms at the higher 3PN order.

Gathering the different non-distributional contributions we obtain

$$V_{\text{non dist}}^{SS} = \frac{Gk_1}{2e^4m_1 r_1^3} \left( 3 \left( n_1 S_i S_i n_1 \right) - \left( S_i S_i \right) \right)$$

$$+ \frac{Gk_1}{e^4m_1} \left( S_i S_i \right) \left( 3 \left( n_1 \bar{v}_1 \right)^2 \frac{4r_1^3}{n_1^3} - \left( n_1 \bar{v}_1 \right)^2 \frac{4r_1^3}{n_1^3} + \frac{3Gm_2(n_{12} n_1)}{4r_1^3} + \frac{3Gm_2(n_{12} n_2)}{4r_1^3} \right)$$

$$+ \frac{3Gm_2}{2\eta r_1^3} - \frac{3Gm_2(n_{12} n_1)}{4r_1^3} - \frac{Gm_2}{2r_1^3 \eta_2} + \frac{Gm_2}{2r_1^3 \eta_2} - \frac{3 \left( n_1 S_i \bar{v}_1 \right)^2}{2r_1^3}$$

$$+ \frac{3Gm_2}{2\eta r_1^3} - \frac{3Gm_2(n_{12} n_1)}{4r_1^3} - \frac{Gm_2}{2r_1^3 \eta_2} + \frac{Gm_2}{2r_1^3 \eta_2} - \frac{3 \left( n_1 S_i \bar{v}_1 \right)^2}{2r_1^3}.$$
+ (n_{12} S_i n_{12}) \left( \frac{15 G m_2 (n_{12} n_1)}{4 r_{12}^4} - \frac{15 G m_2 (n_{12} n_2)}{4 r_{12}^4} - \frac{9 G m_2}{2 n_{12} r_{12}^3} - \frac{3 G m_2}{2 r_{12}^3} \right) \\
+ (n_i S_i n_i) \left( \frac{15 (n_i v_i)^2}{4 r_i^4} + \frac{3 (v_i v_i)}{r_i^3} + \frac{3 G m_2 (n_i n_1)}{4 r_i^2 r_{12}^2} + \frac{3 G m_2}{2 r_i^2 r_{12}^2} \right) \\
+ (n_{12} S_i S_{12}) \left( \frac{-3 G m_2}{2 r_{12}^4} + \frac{3 G m_2}{2 r_{12}^4} \right) + \left( \frac{v_i S_i S_i v_i}{r_i^3} \right) + \left( \frac{3 G m_2 (n_{12} S_i S_i n_{12})}{2 r_{12}^4} \right) \\
+ \frac{G}{r_i^5} \left( n_i S_i n_{12} \right) \left( \frac{-3 G (n_{12} n_1)}{2 r_{12}^4} + \frac{2 G}{r_{12}^3} \right) + \left( n_{12} S_i S_{12} \right) \left( \frac{15 G (n_{12} n_1)}{2 r_{12}^4} - \frac{3 G}{n_{12} r_{12}^3} \right) \\
+ \frac{G}{r_{12}^4} \left( S_i S_i \right) \left( \frac{-3 G (n_{12} n_1)}{2 r_{12}^4} + \frac{G}{r_{12}^3} \right) - \frac{3 G (n_{12} S_i S_i n_{12})}{2 r_{12}^4} \\
+ 1 \leftrightarrow 2 + O(8). \quad (3.21)

For the calculation of \( V_i^{SS} \) at its leading order 2PN, we proceed similarly as for \( V \), but keeping only the first term in the expansion (3.18). The calculation is simpler, with no indirect SS contributions. We get

\[
V_i^{SS} = \frac{G k_i}{2m c^4 r_i^4} (3(n_i S_i S_i n_1) - (S_i S_i) v_i^i - \frac{G k_i}{2m c^4} \frac{4\pi}{3} (S_i S_i) \delta_{ij} v_i^j + 1 \leftrightarrow 2 + O(6), \quad (3.22)
\]

where we have included for completeness a distributional term completely analogous to the one discussed above for the potential \( V^{SS} \), but that will not contribute in the rest of our calculations.

The computation of \( \hat{X}^{SS} \) is different, as it involves non-compact support terms. From equations (3.4), we see that the only 1PN SS contribution in \( \hat{X} \) is

\[
\hat{X}^{SS} = \square^{\alpha \beta} \left[ -2\partial_i V_i^{SO} \partial_j V_i^{SO} \right] + O(4), \quad (3.23)
\]

with the leading order SO part of \( V_i \) given by

\[
V_i^{SO} = \frac{G}{2c} S_i^0 \partial_i \left( \frac{1}{\eta} \right) + 1 \leftrightarrow 2 + O(3). \quad (3.24)
\]

We are working at the leading order here, so that we keep only the first term in the expanded inverse d’Alembertian operator, which is just an inverse Laplacian. For the cross term, with derivatives of both \( 1/\eta \) and \( 1/r_i \), we use the function \( g = \ln \left[ 1/\eta + r_i + \eta_i \right] \) which satisfies \( \Delta g = 1/(\eta_i r_i) \) (including the distributional part of the derivatives). With the notations \( \partial_i^1 = \partial/\partial \eta_i \), \( \partial_i^2 = \partial/\partial r_i \), we can write\(^{10}\)

\[
\Delta^{-1} \left[ \partial_{i} \left( \frac{1}{\eta} \right) \partial_{k} \left( \frac{1}{r_i} \right) \right] = \partial_i^1 \partial_i^2 \left[ \Delta^{-1} \left( \frac{1}{\eta_i r_i} \right) \right] = \partial_i^1 \partial_i^2 g. \quad (3.25)
\]

For the ‘self’ terms, we can ‘factorize’ the derivatives as explained in [81]. Since we may ignore contributions of the form \( \Delta^{-1} \left( \eta_i^p / r_i^q \delta_1 \right) \) for \( \ell + p \) even, we disregard possible distributional terms generated by space or time differentiation. After factorizing the derivatives,\(^{10}\) including properly the regularization \( F_{p=0} \) would yield an additional constant contribution [80], which would vanish after applying the derivatives.

\(^{10}\) Including properly the regularization \( F_{p=0} \) would yield an additional constant contribution [80], which would vanish after applying the derivatives.
we transform them into derivatives with respect to $y_{1,2}^i$ and apply $\Delta^{-1}$ straightforwardly on the argument. The corresponding formula is

$$
\Delta^{-1} \left[ \partial_j \left( \frac{1}{n} \partial_{i1} \left( \frac{1}{n} \right) \right) \right] = \frac{1}{128} \left[ 3 \partial_{i1} \left( \ln n \right) - 5 \left( \delta_{ij} \partial_{i1} + \delta_{ij} \partial_{i1} \right) \left( \frac{1}{r_1^2} \right) + 3 \left( \delta_{ik} \partial_{i1} + \delta_{ik} \partial_{i1} + \delta_{ik} \partial_{i1} \right) \left( \frac{1}{r_1^2} \right) + \left( \delta_{ij} \partial_{i1} + \delta_{ij} \partial_{i1} + \delta_{ij} \partial_{i1} \right) \Delta \left( \frac{1}{r_1^2} \right) \right].
$$

(3.26)

Gathering these contributions, we find the following simple expression for the leading-order SS part of the potential $\hat{X}$:

$$
\hat{X}^{ss} = -\frac{G^2}{2c^2 r_1^2} \left[ \frac{1}{4} \left( e_{ij} e_{ij} - (n_1 S_1 e_{ij} - n_1) \right) \right] - \frac{G^2}{2c^2 r_1^2} s_{ij}^{ss} s_{ij}^{ss} - 1 \leftrightarrow 2 + O(4),
$$

(3.27)

where we keep the derivatives in the second term unexpanded.

### 3.4. Results for the evolution equations

Using the results of the previous section and the NS and SO parts of the metric potentials that are already known, we are in a position to complete the calculation of the equations of motion and precession (3.14) and (3.16). The results for the accelerations $a_{1,2}$ and the precession vectors $\Omega_{1,2}$ must pass several tests checking their validity.

The first one is to make sure of the existence of a set of conserved quantities associated with the Poincaré invariance of the problem when radiation reaction is turned off, up to our approximation level: a conserved energy $E$, an angular momentum $J$, a linear momentum $P$, and a center-of-mass integral $G$. We were actually able to construct all those quantities explicitly by guess work. Their next-to-leading order quadratic-in-spin pieces were searched for in the form of sums of independent terms that: (i) are proportional to the appropriate power of $1/c$ and to non-negative powers of $G$, $1/n_2$, $m_1$ and $m_2$, (ii) have the required physical dimension, (iii) have the desired number of free indices, (iv) are proportional to the monomials obtained by contracting $n_1 S_1$, $v_1$, $S_1^{ij}$, $m_1$, $S_1^{ij}$, $\delta_{ij}$ and $\varepsilon_{ijk}$, (v) are quadratic in spins, (vi) satisfy the parity symmetry. The unknown coefficients of those terms were taken to be dimensionless parameters depending implicitly on the polarizabilities $\kappa_1$ and $\kappa_2$. These were uniquely determined by imposing that the first time derivatives of the conserved quantities vanish at the order $3PN$ when the radiation reaction effects are turned off. The higher-order terms in the precession equations intervene only in the conservation of the angular momentum, whereas the higher-order terms in the equations of motion intervene in all other conservation relations. We shall exhibit below the expression of the conserved energy, which will be later used to control the phase evolution of the binary in the case of circular orbits through the balance equation as explained in section 4.3. The expression of the conserved total angular momentum, which plays a role in controlling the precession effects of the system, is relegated to appendix A.

Another test consists in checking the Lorentz invariance of the dynamics, which must be manifest since the harmonic gauge choice is Lorentz-preserving. We use the same method as
in [23, 24], to which we refer the reader for more details, and find that our results pass this second test.

As the 3PN SS dynamics has been already investigated in both the EFT [21, 22, 35, 38, 39, 74] and the ADM [62, 70–73] approaches, we must be able to recover their results in our scheme. The equivalence between the ADM and EFT description has been shown to hold in [36, 37, 73], so that we will only compare our results to the ADM ones, in keeping with our previous works. We present this comparison, and the resulting transformation from harmonic to ADM variables, in appendix D. The agreement with the ADM results also validates the test-mass limit of ours.

Because the expressions we produced are rather lengthy, we shall give directly their reduced version in the center-of-mass (CM) frame. As in our previous works, this frame is defined as the one where the center-of-mass integral \( G_t \) (which is such that \( dG/dt = P \) and hence \( d^2G/dt^2 = 0 \)) vanishes. We define \( \mathbf{x} = \mathbf{r} = \mathbf{y}_1 - \mathbf{y}_2 \) the separation vector of the binary, \( \mathbf{v} = d\mathbf{x}/dt \) the relative velocity, \( m = m_1 + m_2 \) the total mass, \( \nu = m_1m_2/m^2 \) the symmetric mass ratio and \( \delta = (m_1 - m_2)/m \) the fractional mass difference. We also use, for convenience, the same spin variables as in the previous works [23, 45], namely

\[
S = S_1 + S_2, \quad \Sigma = m \left( \frac{S_2}{m_2} - \frac{S_1}{m_1} \right).
\]

The vectors \( S_1 \) and \( S_2 \) are the conserved-norm vectors constructed in section 2.3. Additionally, we shall use the notation \( \kappa_+ = \kappa_1 + \kappa_2 \) and \( \kappa_- = \kappa_1 - \kappa_2 \). Note that \( \kappa_+ = 2 \) and \( \kappa_- = 0 \) for two black holes.

The positions of the two point bodies in the new frame read

\[
y_1 = \frac{m_2}{m} \mathbf{x} + \frac{1}{c^2} \mathbf{z}, \quad y_2 = -\frac{m_1}{m} \mathbf{x} + \frac{1}{c^2} \mathbf{z},
\]

with \( z \) being a vector related to the center-of-mass integral \( G \). In general, when working at the \( n \)PN order, only the \( (n-1) \)PN expression of \( G \) (or \( z \)) is required. This can be checked explicitly from the Newtonian expressions in the general frame of the quantities of interest, as explained for instance in [25]. Thus, we would only need the SS 2PN expression of \( G \) in principle, but it turns out that there is no such contribution. We can therefore translate our results to the CM frame using simply the same rules as in previous works: namely, we need the NS 1PN and the SO 1.5PN terms in \( z \), as given in the section 3 of [25].

For the SS contributions to the conserved energy, we find

\[
E_{SS} = \frac{G
u}{c^4} \left[ \frac{1}{4} e^0_4 + \frac{1}{2} e_6^0 + \frac{1}{8} \frac{G m}{r} e_6 \right],
\]

with

\[
e^0_4 = S^2 \left( -2\kappa_+ - 4 \right) + (5\Sigma) \left( -2\delta\kappa_+ - 4\delta + 2\kappa_- \right) + \Sigma^2 \left( \delta\kappa_- - \kappa_+ \right) + \nu \left( 2\kappa_+ + 4 \right)
\]

\[
+ (nS)^2 \left( 6\kappa_+ + 12 \right) + (nS)^2 \left( 3\kappa_+ - 3\delta\kappa_- \right) + \nu \left( -6\kappa_+ - 12 \right)
\]

\[
+ (nS)(n\Sigma) \left( 6\delta\kappa_- + 12\delta - 6\kappa_- \right),
\]

\[
e^0_6 = S^2 \left[ (n\nu)^2 \left( 6\delta\kappa_- - 6\kappa_+ + 24 \right) + \nu \left( 6\kappa_+ + 12 \right) \right]
\]

\[
+ v^2 \left( -2\delta\kappa_- + 8\kappa_+ - 28 \right) + \nu \left( 2\kappa_+ + 4 \right) \right]
\]
The corresponding expressions for the relative acceleration $a = a_1 - a_2$, the precession vectors $\Omega_{1,2}$ and the total conserved angular momentum $J$ are provided in appendix A.

Finally, we further specialize our results to the case of circular, non-precessing orbits. As discussed in [33], we have in fact three classes of orbits for the conservative dynamics. The CM expression are valid for general orbits, for which we make no assumption on the presence of precession and/or eccentricity. Quasi-circular precessing orbits correspond to the case where we allow for a generic orientation of the spins, but assume that the separation is constant at the SO level; as soon as SS and higher-order-in-spin terms are included the radius and orbital frequency become also variable on an orbital timescale. In [33] the definition of such orbits was investigated by perturbing orbital averaged quantities. The third and simplest class of orbits is that of circular orbits where spins are aligned with the orbital angular momentum and where precession is absent. As working at the next-to-leading order makes their determination more complicated, we leave the investigation of quasi-circular orbits for future work and focus on the circular, spin-aligned, non-precessing case.
To present results for circular orbits, we use the same definitions as in previous works. We introduce a moving basis \((\mathbf{n}, \lambda, \ell)\), with \(\mathbf{n}\) denoting the unit vector along the separation vector \(\mathbf{x} = r \mathbf{n}\), \(\ell = \mathbf{n} \times \mathbf{v} / |\mathbf{n} \times \mathbf{v}|\) the normal to the orbital plane, and \(\lambda\) completing the triad. When neglecting radiation reaction and assuming the spins aligned with \(\ell\), the expressions for the relative velocity and acceleration become \(\mathbf{v} = ro\mathbf{n}\) and \(\mathbf{a} = -ro^2\mathbf{n}\), with \(o\) the orbital frequency defined by \(\mathbf{n} = o\lambda\). For the projected value of the (aligned or anti-aligned) spins along \(\ell\), we use the notation \(\mathbf{S}_\ell = \mathbf{S} \cdot \ell \equiv (\mathbf{S}_\ell)\). We also introduce the usual PN parameters \(Gm/rc^2\) and \(x = (Gm/rc^3)^{2/3}\), both of order 1PN. In the following, we only display the SS terms, and refer the reader to sections 9.3 and 11.3 of [7] for the NS and SO contributions, and to [42] for the newly computed cubic-in-spin contributions.

First, we relate \(r\) to \(o\) by means of the equations of motion. We obtain the following SS terms for the PN generalization of Kepler’s law:

\[
\gamma_{SS} = \frac{x}{G^2m^4} \left\{ x^2 \left[ S_\ell^2 \left( -\frac{\kappa_+}{2} - 1 \right) + S_\ell \Sigma_\ell \left( -\frac{\delta \kappa_+}{2} - \delta + \frac{\kappa_-}{2} \right) \right] + \Sigma_\ell^2 \left[ \left( \frac{\delta \kappa_-}{4} - \frac{\kappa_+}{4} \right) + \nu \left( \frac{\kappa_+}{2} + 1 \right) \right] \right\} + \cdots \tag{3.32}
\]

The result for the energy for circular, spin-aligned orbits is then

\[
E_{SS} = -\frac{1}{2} mc^2 \frac{x}{G^2m^4} \left\{ x^2 \left[ S_\ell^2 \left( -\kappa_+ - 2 \right) + S_\ell \Sigma_\ell \left( -\delta \kappa_+ - 2\delta + \kappa_- \right) \right] + \Sigma_\ell^2 \left[ \left( \frac{\delta \kappa_-}{2} - \frac{\kappa_+}{2} \right) + \nu \left( \kappa_+ + 2 \right) \right] \right\} + \cdots \tag{3.33}
\]

This expression can be shown to be in agreement, in the test-mass limit, with the energy of a test particle in circular equatorial orbits around a Kerr black hole [82]. It is a crucial ingredient to control the phase evolution through the balance equation (see section 4.3). The other
contributions to the binding energy at the 3PN order can be found in sections 9.3 and 11.3 of [7] (see also [83, 84] for the non-spin part and [45, 46] for the spin contributions).

4. Next-to-leading order contributions to the post-Newtonian GW energy flux

We now move to the computation of the 3PN spin-spin contribution to the energy flux radiated by the system. We start by briefly reviewing in section 4.1 the basic elements of the wave generation formalism that we need here, before providing in section 4.2 some intermediate results useful in the calculation of the source multipole moments that are required to this order. The explicit expressions for the moments in the CM frame are relegated to appendix B because of their lengths. Our explicit result for the GW flux is presented in section 4.3 for general orbits in the center-of-mass frame of the system and then reduced to the case of circular orbits in the configuration where the spins are aligned with the orbital angular momentum.

4.1. Formalism

We perform our calculation in the framework of the MPM approach to gravitational radiation. This formalism has been developed over many years, see e.g. [77, 85–89]. Since we will only use a simplified version, as we are working at next-to-leading order, we will refer the reader to [7] for a review, and give only a brief overview.

The asymptotic waveform is defined from the transverse-tracefree (TT) projection of the metric perturbation, in a suitable radiative coordinate system\(\mathcal{X}\), as its leading-order term in the \(1/R\) expansion when the distance \(R = |\mathcal{X}|\) to the source goes to infinity (keeping the retarded time \(\mathcal{R}_\text{R} \equiv T - R/c\) fixed). It can be parametrized using two sets of symmetric and trace-free (STF) radiative multipole moments, \(U_L\), of mass type, and \(V_L\), of current type, as

\[
h_{ij}^{\text{TT}} = \frac{4G}{c^2R} P_{ij}^{\text{TT}}(\mathcal{X}) \sum_{\ell=2}^{\infty} \frac{N_{\ell-2}}{\ell!} \left[ U_{\ell-2}(T_\mathcal{R}) - \frac{2\ell}{c(\ell + 1)} N_{\mathcal{R}} \varepsilon_{\alpha\beta\gamma\delta} V_{(\ell-2)\alpha\beta\gamma\delta}(T_\mathcal{R}) \right] + \mathcal{O} \left( \frac{1}{R^2} \right),
\]

where we denote by \(L = i_1 \ldots i_\ell\) a multi-index composed of \(\ell\) multipolar spatial indices \(i_1, \ldots, i_\ell\) ranging from 1 to 3. Similarly \(L - 1 = i_1 \ldots i_{\ell-1}\) and \(kL - 2 = k i_1 \ldots i_{\ell-2}\); \(N_L = N_{i_1} \ldots N_{i_\ell}\) is the product of \(\ell\) spatial vectors \(N_i\). The TT projection operator is denoted \(P_{ij}^{\text{TT}} = P_{ik} P_{jk} - \frac{1}{2} P_{ij}\), where \(P_{ij} = \delta_{ij} - N_i N_j\) is the projector orthogonal to the unit direction \(N = \mathcal{X}/R\) of the radiative coordinate system. Like in the rest of this paper, \(\varepsilon_{ijk}\) is the Levi–Civita antisymmetric symbol such that \(\varepsilon_{123} = 1\). The STF projection is indicated using brackets or a hat; thus \(U_L = \hat{U}_L = U_{(\ell)}\) and \(V_L = \hat{V}_L = V_{(\ell)}\) for STF moments. We denote time derivatives with a superscript \(n\).

In terms of the radiative moments, the energy flux into GW then reads

\[
\mathcal{F} = \sum_{\ell=2}^{\infty} G \frac{( \ell + 1)(\ell + 2)}{(\ell - 1)!\ell!(2\ell + 1)!!} \left[ U^{(1)}_L U^{(1)}_L + \frac{4\ell(\ell + 2)}{c^2(\ell - 1)(\ell + 1)!!(2\ell + 1)!!} V^{(1)}_L V^{(1)}_L \right].
\]

The \(U_L\) and \(V_L\) can be expressed as (nonlinear) functions of two sets of intermediate source rooted, so-called canonical moments \(M_L\) and \(S_L\), which are themselves related, through a gauge transformation, to a set of two so-called source multipole moments \(I_L, J_L\) parametrizing the most general solution to the Einstein equations outside the source (together with four gauge STF moments). The differences between \(M_L\) and \(I_L\) (or similarly between \(J_L\) and \(S_L\)
arise at the 2.5PN order (see for instance [8]). Since we are interested in SS effects, which always add at least a factor $1/c^2$, we can safely ignore their differences here. Using the same argument, we only need to consider the terms in the relation between the radiative moments and the canonical ones up to the order 2PN. Furthermore, we may also neglect the tail terms, which would only generate SS contributions at the order 3.5PN. Finally, we get the simple relation

\[
\begin{align*}
(U_{ij})_{SS} &= (T_{ij}^{(2)})_{SS} + \mathcal{O}(7), \\
(V_{ij})_{SS} &= (J_{ij}^{(2)})_{SS} + \mathcal{O}(7), \\
(U_{ijk})_{SS} &= (I_{ijk}^{(3)})_{SS} + \mathcal{O}(7).
\end{align*}
\]

(4.3a) (4.3b) (4.3c)

Noticing additionally that the leading order spin-spin contribution to any of the $I_L$ or $J_L$ (and their time derivatives) is of the order 2PN (as will be clear from the expressions in the next section), we write the spin-spin flux in terms of the relevant source moments as

\[
F_{SS} = \frac{G}{c^3} \left\{ \frac{1}{5} I_{ij}^{(3)} I_{ij}^{(3)} + \frac{1}{c^2} \left[ \frac{1}{189} I_{ij}^{(4)} I_{ij}^{(4)} + \frac{16}{45} J_{ij}^{(3)} J_{ij}^{(3)} \right] + \frac{1}{c^4} \left[ \frac{1}{84} J_{ij}^{(4)} J_{ij}^{(4)} \right] \right\}_{SS} + \mathcal{O}(7).
\]

(4.4)

Its computation requires the SS parts of $I_{ij}$ to the order 3PN, as well as those of $J_{ij}$ and $I_{ijk}$ to the order 2PN. We also need the NS parts of $I_{ij}$ up to the order 1PN, the NS pieces of $J_{ij}$ and $I_{ijk}$ at the Newtonian order, the SO contributions in $I_{ij}$ and $J_{ij}$ up to the order 1.5PN and the leading 0.5PN SO contribution to $J_{ijk}$. All those quantities are known from previous works. Remember that the spin–orbit contributions to mass (resp. current) type moments start at 1.5PN (resp. 0.5PN) order, and that time derivatives of NS (resp. spin–orbit) expressions generate spin-spin contributions with an additional order 2PN (resp. 1.5PN) at least.

The matching procedure at the core of the formalism then allows us to express the source moments as closed-form integrals over space [89]. Instead of reproducing the general expressions which can be found in equation (123) of [7], we directly display below the terms that contribute to the spin-spin corrections at the required orders. They read

\[
\begin{align*}
(I_{ij})_{SS} &= FP \int \frac{d^4x}{r_0^B} \left\{ \frac{r}{r_0} \left[ \Sigma + \frac{r^2}{14c^2} \Sigma^{(2)} \right] - \frac{20}{21c^2} \hat{x}_{ij} \Sigma^{(1)} \right\}_{SS} + \mathcal{O}(7),
\end{align*}
\]

(4.5a)

\[
\begin{align*}
(J_{ij})_{SS} &= FP \int \frac{d^4x}{r_0^B} \left\{ \frac{r}{r_0} \epsilon_{ab,<j} \epsilon_{i,a} \Sigma_{SS} \right\} + \mathcal{O}(5),
\end{align*}
\]

(4.5b)

\[
\begin{align*}
(I_{ijk})_{SS} &= FP \int \frac{d^4x}{r_0^B} \hat{x}_{ijk} \Sigma_{SS} + \mathcal{O}(5),
\end{align*}
\]

(4.5c)

where $FP_{B=0}$ denotes a finite part operation defined by analytic continuation in the complex plane for the parameter $B$ which deals here with the infrared divergences at infinity. An arbitrary scale $r_0$ is introduced, which will play no role in the present calculation and has to disappear from gauge-invariant results. The basic ‘building blocks’ $\Sigma$, $\Sigma_i$ and $\Sigma_{ij}$ entering the integrands are defined as
where \( \tau^{\mu \nu} \) has been defined itself in equation (3.2), and the overline indicates a post-Newtonian (near-zone) expansion. In identifying the relevant terms in equation (4.5a), we slightly anticipated on the results of the next subsection (see equation (4.7a)), using the fact that the SS contributions to \( \Sigma, \Sigma_i \), and \( \Sigma_{ij} \) all start at the 2PN order at least.

### 4.2. Computation of the source moments

To obtain the relevant SS contributions to the source moments, we first express the sources \( \Sigma, \Sigma_i \), and \( \Sigma_{ij} \) in terms of the potentials parametrizing the metric, as well as the matter sources \( \sigma, \sigma_i \), and \( \sigma_{ij} \) defined in (3.5) (the complete relations can be found, generalized to \( d \) dimensions, in [90]). Taking into account the order of the spin corrections in these quantities, the only terms that yield spin-spin contributions to the orders we are interested in are

\[
\Sigma^{SS} = \left\{ \left[ 1 + \frac{4V}{c^2} \right] \sigma - \frac{1}{\pi G c^2} \partial_i V \partial_j V + \frac{2}{\pi G c^4} \partial_i V \partial_j V \right\}^{SS} + O(7),
\]

\[
\Sigma_i^{SS} = \left\{ \left[ 1 + \frac{4V}{c^2} \right] \sigma_i - \frac{1}{\pi G c^2} \partial_k V \partial_j V \right\}^{SS} + O(5),
\]

\[
\Sigma_{ij}^{SS} = O(3).
\]

The integrals in equation (4.5a) can now be evaluated using the standard techniques described in [68, 69], handling the UV divergences of the integral through the Hadamard regularization and the IR divergences through the finite part operation \( \mathcal{F} \).

We highlight here that the distributional parts of the sources have to be treated with care. In particular, for the first time, we encountered the situation where these contributions in the metric itself (more precisely in the potential \( V \)), and not just those coming from derivatives applied to the metric, have to be crucially taken into account.

More specifically, the spin-spin leading order contribution in the potential \( V \), computed in equations (3.19) and (3.20), contains a term proportional to \( \delta_i \) which has to be accounted for when integrating the \( \partial_i V \partial_i V \) term of (4.7a) in (4.5a). In order to illustrate this further, let us focus on the second and third terms in \( \Sigma_{SS} \),

\[
\Sigma_{SS} = \frac{4V}{c^2} \sigma - \frac{1}{\pi G c^2} \partial_i V \partial_j V,
\]

which we can rewrite using the identity \( 2\partial_i \Lambda \partial_i B = \Delta (AB) - A \Delta B - B \Delta A \), and the fact that \( \Delta V = -4\pi G \sigma \) at leading order, as

\[
\Sigma_{SS} = -\frac{1}{2} \frac{1}{\pi G c^2} \Delta [V^2].
\]

By reinjecting this second form into (4.5a), integrating by parts, noticing that \( \Delta \delta_i = 0 \) and treating the surface terms as explained in the section IV D of [68], we readily see that \( \Sigma_{SS} \) actually gives a vanishing contribution to \( I_{ij} \). If on the other hand one uses (4.8) without including the distributional part of \( V \), one obtains an incorrect non-zero result.

Our explicit SS contributions to the source moments reduced to the center of mass are presented in appendix B.
4.3. GW energy flux

Using equation (4.4), our expressions for the source moments, as well as the equations of motion and precession obtained in section 3.4 to compute time derivatives, we can finally calculate the GW flux. We will give the result already reduced in the center-of-mass frame, and resort to the same notations as already introduced in section 3.4. We obtain

\[ F_{SS} = \frac{G^2 m^2 v^2}{5c^4 r^6} \left[ 1 + \frac{1}{3} f_{4}^0 + \frac{1}{21 \nu^2} \left( f_{6}^0 + \frac{G m}{r^2} f_{7}^1 + \frac{G^2 m^2}{r^2} f_{8}^2 \right) \right] \]  

(4.10)

with

\[
\begin{align*}
f_{4}^0 &= S^2 \left[ (n\nu)^2 \left( -312 \kappa_+ - 624 \right) + \nu^2 \left( 288 \kappa_+ + 576 \right) \right] \\
&\quad + (n \Sigma) (n \nu) \left( (n \nu)^2 \left( -312 \delta \kappa_+ - 624 \delta - 312 \kappa_- \right) + \nu^2 \left( 288 \delta \kappa_+ + 576 \delta - 288 \kappa_- \right) \right) \\
&\quad + \Sigma^2 \left[ (n \nu)^2 \left( 156 \delta \kappa_- - 156 \kappa_- + 18 \right) + \nu \left( 312 \kappa_+ + 624 \right) \right] \\
&\quad + \nu^2 \left( -144 \delta \kappa_- - 144 \kappa_- + 6 \right) + \nu \left( -288 \kappa_+ - 576 \right) \right] \\
&\quad + (n \Sigma)^2 \left[ (n \nu)^2 \left( 1632 \kappa_+ + 3264 \right) + \nu^2 \left( -1008 \kappa_+ - 2016 \right) \right] \\
&\quad + (n \Sigma) (n \nu) \left( (n \nu)^2 \left( 348 \delta \kappa_- - 348 \kappa_- - 12 \right) + \nu \left( 696 \kappa_+ + 1392 \right) \right) \\
&\quad + (n \nu) \left( (n \nu)^2 \left( -72 \delta \kappa_- + 72 \kappa_- + 2 \right) + \nu \left( -144 \kappa_- - 288 \right) \right) \\
&\quad + (n \Sigma)^2 \left( (n \nu)^2 \left( 1632 \delta \kappa_- + 3264 \delta - 1632 \kappa_- \right) + \nu^2 \left( -1008 \delta \kappa_- - 2016 \delta - 1008 \kappa_- \right) \right] \\
&\quad + (n \Sigma) (n \nu) \left( (n \nu)^2 \left( -348 \delta \kappa_- - 696 \delta + 348 \kappa_- \right) + (n \nu) \left( (n \nu)^2 \left( -348 \kappa_- - 696 \delta + 348 \kappa_- \right) + (n \nu) \left( (n \nu)^2 \left( 144 \delta \kappa_- + 288 \delta - 144 \kappa_- \right) \right) \right) \right) \\
&\quad + (n \Sigma)^2 \left( (n \nu)^2 \left( 2274 \delta \kappa_- + 12918 \kappa_- + 35436 \right) + \nu \left( -14112 \kappa_- - 28224 \right) \right] \\
&\quad + (n \nu)^2 \nu^2 \left( -2592 \delta \kappa_- - 17544 \kappa_- + 51984 \right) + \nu^2 \left( 17928 \kappa_- + 35856 \right) \\
&\quad + \nu^2 \left( 366 \delta \kappa_- + 5034 \kappa_- + 18276 \right) + \nu \left( -4584 \kappa_- - 9168 \right) \right] \\
&\quad + (n \Sigma) \left( (n \nu)^2 \left( -10644 \delta \kappa_- + 50652 \delta - 10644 \kappa_- \right) + \nu \left( -14112 \delta \kappa_- - 28224 \delta + 5016 \kappa_- \right) \right) \\
&\quad + (n \nu)^2 \nu^2 \left( -14952 \delta \kappa_- - 69672 \delta + 14952 \kappa_- \right) + \nu \left( 17928 \kappa_- + 35856 \delta - 7560 \kappa_- \right) \right]
\end{align*}
\]
\[ + v^4 \left( (4668\delta\kappa_+ + 20812\delta - 4668\kappa_-) + \nu \left( -4584\delta\kappa_+ - 9168\delta + 3120\kappa_- \right) \right) \]
\[ + \Sigma^2 \left[ (\nu v)^2 \left( (-5322\delta\kappa_+ + 5322\kappa_- + 9714) + \nu \left( 4782\delta\kappa_+ - 15426\kappa_- - 64788 \right) + \nu^2 (14112\kappa_+ + 28224) \right) \right. \]
\[ + \Sigma^2 \left( 7476\delta\kappa_+ - 7476\kappa_- - 14286 \right) + \nu \left( -6372\delta\kappa_+ + 21324\kappa_- + 86316 \right) + \nu^2 (17928\kappa_+ - 35856) \]
\[ + \nu^4 \left( (2334\delta\kappa_+ + 2334\kappa_- + 3796) + \nu \left( 1926\delta\kappa_+ - 6594\kappa_- - 23336 \right) + \nu^2 (4584\kappa_+ + 9168) \right) \]
\[ + (n\Sigma)^2 \left( (12930\delta\kappa_+ - 90570\kappa_- - 81570) + \nu \left( 71520\kappa_+ + 143040 \right) \right) \]
\[ + (n\Sigma)^2 \left( (8124\delta\kappa_+ - 81636\kappa_- + 65220) + \nu \left( -62976\kappa_- - 125952 \right) \right) \]
\[ + \nu^2 \left( (5700\delta\kappa_+ - 14778\kappa_- - 6546) + \nu \left( 13632\kappa_- + 27264 \right) \right) \]
\[ + (n\Sigma)(\nu\Sigma) \left( (19752\delta\kappa_+ - 51816\kappa_- + 16464) + \nu \left( -29184\kappa_- - 58368 \right) \right) \]
\[ + (n\Sigma)^2 \left( (9522\delta\kappa_+ - 19890\kappa_- - 180) + \nu \left( 4092\kappa_- + 8184 \right) \right) \]
\[ + (\nu\Sigma)^2 \left( (6378\delta\kappa_+ - 9114\kappa_- + 7794) + \nu \left( 5100\kappa_- + 10200 \right) \right) \]
\[ + \nu^2 \left( (1668\delta\kappa_+ - 324\kappa_- - 6478) + \nu \left( 120\kappa_- + 240 \right) \right) \]
\[ + (n\Sigma)^2 \left( (51750\delta\kappa_+ - 51750\kappa_- + 18420) \right) \]
\[ + \nu \left( -48690\delta\kappa_+ + 152190\kappa_- + 60960 \right) + \nu^2 \left( -71520\kappa_- - 143040 \right) \]
\[ + (n\Sigma)^2 \left( (44880\delta\kappa_+ + 44880\kappa_- - 8112) \right) \]
\[ + \nu \left( 39612\delta\kappa_+ - 129372\kappa_- - 79608 \right) + \nu^2 \left( 62976\kappa_- + 125952 \right) \]
\[ + \nu^4 \left( 7674\delta\kappa_+ - 7674\kappa_- + 3090) + \nu \left( -7386\delta\kappa_+ + 22734\kappa_- + 10884 \right) + \nu^2 \left( -13632\kappa_- - 27264 \right) \right) \]
\[ + (n\Sigma)(\nu\Sigma) \left( (35784\delta\kappa_+ + 35784\kappa_- - 33534) \right) \]
\[ + \nu \left( 34344\delta\kappa_+ - 105912\kappa_- + 48888 \right) + \nu^2 \left( 29184\kappa_- + 58368 \right) \]
\[ + (n\Sigma)^2 \left( (14706\delta\kappa_+ - 14706\kappa_- + 7782) + \nu \left( -11568\delta\kappa_+ + 40980\kappa_- - 7794 \right) \right) \]
\[ + \nu^2 \left( -4092\kappa_- + 8184 \right) \]
\[ + (\nu\Sigma)^2 \left( (7746\delta\kappa_+ - 7746\kappa_- + 14124) + \nu \left( -8928\delta\kappa_+ + 24420\kappa_- - 36432 \right) \right) \]
\[ + \nu^2 \left( -5100\kappa_- - 10200 \right) \]
\[ + \nu^2 \left( -672\delta\kappa_+ + 672\kappa_- - 2242) + \nu \left( 1608\delta\kappa_+ - 2952\kappa_- + 9788 \right) + \nu^2 \left( -120\kappa_- - 240 \right) \right) \]
\begin{align*}
&+ (n\Sigma)(n\Sigma) \left[ (nv)^4 \left( -103500\delta_\kappa - 70920\delta + 103500\kappa_\nu \right) \\
&+ \nu \left( 71520\delta_\kappa + 143040\delta + 123240\kappa_\tau \right) \\
&+ (nv)^2v^2 \left( 89760\delta_\kappa + 71808\delta - 89760\kappa_\nu \right) \\
&+ \nu \left( 62976\delta_\kappa - 125952\delta + 95472\kappa_\tau \right) \\
&+ v^4 \left( -15348\delta_\kappa + 8664\delta + 15348\kappa_\nu \right) \\
&+ \nu \left( 13632\delta_\kappa + 27264\delta - 15912\kappa_\tau \right) \right] \\
&+ (n\Sigma)(vS) \left[ (nv)^3 \left( 35784\delta_\kappa - 15402\delta - 35784\kappa_\nu \right) \\
&+ \nu \left( 14592\delta_\kappa - 29184\delta + 54096\kappa_\tau \right) \\
&+ (nv)v^2 \left( -14706\delta_\kappa + 8190\delta + 14706\kappa_\nu \right) \\
&+ \nu \left( 2046\delta_\kappa + 4092\delta - 21090\kappa_\tau \right) \right] \\
&+ (n\Sigma)(vS) \left[ (nv)^3 \left( 35784\delta_\kappa - 240\delta - 35784\kappa_\nu \right) \\
&+ \nu \left( 14592\delta_\kappa - 29184\delta + 54096\kappa_\tau \right) \\
&+ (nv)v^2 \left( -14706\delta_\kappa + 5124\delta + 14706\kappa_\nu \right) \\
&+ \nu \left( 2046\delta_\kappa + 4092\delta - 21090\kappa_\tau \right) \right] \\
&+ (vS)(vS) \left[ (nv)^3 \left( -15492\delta_\kappa + 23052\delta + 15492\kappa_\nu \right) \\
&+ \nu \left( 5100\delta_\kappa + 10200\delta - 30612\kappa_\tau \right) \\
&+ v^2 \left( 1344\delta_\kappa + 8188\delta - 1344\kappa_\nu \right) + \nu \left( 120\delta_\kappa + 240\delta + 6552\kappa_\tau \right) \right].
\end{align*}
After reduction to the case of spin-aligned, circular orbits, using the notations already introduced in section 3.4 for the energy, we obtain

\[
\mathcal{F}_{SS} = \frac{32\nu^2 \ell^5 S^5}{5 \frac{G}{G'} m^4} \left\{ x^2 \left[ S^2_\ell \left( 2\kappa_+ + 4 \right) + S_\ell \Sigma_\ell \left( 2\delta\kappa_+ + 4\delta - 2\kappa_- \right) \right] \right. \\
+ \left. \Sigma^2_\ell \left( -\delta\kappa_- + \kappa_+ + \frac{1}{16} \right) + \nu \left( -2\kappa_+ - 4 \right) \right) \right] \\
+ x^3 \left[ S^2_\ell \left( \frac{41\delta\kappa_-}{16} - \frac{271\kappa_+}{112} - \frac{5229}{504} \right) + \nu \left( -\frac{43\kappa_-}{4} - \frac{43}{2} \right) \right] \\
+ S_\ell \Sigma_\ell \left( \frac{-279\delta\kappa_+}{56} + \frac{817\delta}{56} + \frac{279\kappa_-}{56} \right) \\
+ \nu \left( -\frac{43\delta\kappa_+}{4} + \frac{43\delta}{2} + \frac{\kappa_-}{2} \right) \right].
\]

(4.11)
The full 3PN GW flux can be rebuilt by adding the above quadratic-in-spin piece to the non-spin [83, 84] and spin–orbit [45, 46, 91] parts, both explicitly displayed in [7]. Using this result as well as the expression of the orbital energy (3.33), we can write the balance equation \( \mathcal{P} = -dE/dt \) for circular orbits to find the phase evolution of the binary. Different ways of mixing analytical and numerical integration give rise to different approximants (see for instance [92] for a comparison). For simplicity, we will show here only the phasing formula for the TaylorT2 approximant: we re-expand \( \phi \) and integrate term by term to compute the orbital phase \( \phi \) (here \( \phi \) is the angle between the ascending node and the body separation, equal to half the GW phase of the leading 22 mode) as a function of \( \omega \) or equivalently of \( x \). We get for the SS contributions

\[
(\phi)_{SS} = -\frac{x^{-5/2}}{32\nu G^2m^3} \left\{ x^2 \left[ S_\ell^2 \left( -25\kappa_+ - 50 \right) + S_\ell^3 \left( -25\delta \kappa_+ - 50\delta + 25\kappa_- \right) \right] + \nu^2 \right. \\
+ S_\ell^2 \left( \frac{2215\delta \kappa_-}{48} + \frac{15635\kappa_+}{84} - \frac{31075\delta}{126} \right) + \nu \left( 30\kappa_- + 60 \right) \right. \\
+ \left. x^3 \left[ S_\ell^2 \left( \frac{2215\delta \kappa_-}{48} + \frac{15635\kappa_+}{84} - \frac{31075\delta}{126} \right) + \nu \left( 30\kappa_- + 60 \right) \right] + \nu^2 \left( -30\kappa_- - 60 \right) \right\},
\]

The known NS and SO contributions are summarized in sections 9.3 and 11.3 of [7], and additional cubic-in-spin 3.5PN contributions can be found in [42]. We give in table 1 the number of cycles of the signal resulting from each term in the phasing formula, for the frequency band of advanced LIGO/Virgo detectors. Notice however that these results are illustrative, as they are specific to the TaylorT2 approximant and as these number of cycles give only a rough idea of the relevance of these terms in actual data analysis applications.

We have checked that our result (4.12) is in agreement, in the limit of a test particle orbiting a Kerr black hole, with the result of [93] obtained in the framework of black-hole perturbation theory. We leave for future work the comparison with the so far incomplete
results (given only at the level of the multipole moments) of [43, 44]. Other natural extensions include the investigation of quasi-circular precessing orbits, the computation of the spherical harmonic decomposition of the waveform (or, equivalently, of the full polarizations \( h, h^\times \)), and the integration to the factorized waveforms of the Effective-One-Body formalism with spins (see e.g. [94–97]).

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Appendix A. Explicit results for the equations of evolution

We gather in this appendix explicit results for the equations of motion, the equations of precession, and conserved angular momentum obtained in section 3.4, which are too long to be shown in the main text. We present them already reduced to the center-of-mass frame.

For the precession vectors \( \Omega^i_{1,2} \), we have found simpler to keep the variables \( S_{1,2} \) and \( \kappa_{1,2} \) instead of \( S, \Sigma \) and \( \kappa, \kappa_\pm \). We get

\[
\left( \Omega^i_{1} \right)_{SO} = \frac{G}{c^3 r^3} \left[ w^0_{i,0} + \frac{1}{c} \left( \frac{w_{x,0}}{GM} + \frac{w_{x,i}}{r^2} \right) \right] + \mathcal{O}(7),
\]

\[
\text{(A.1)}
\]
where the contributions $w_{i,m}^j$ are given by

$$w_{i,0}^j = 3(\eta S_2)n^i - S_1^j + 3\kappa_1(nS_i)n^i \left( \frac{1}{X_1} - 1 \right)$$

$$w_{5,0}^i = S_1^i \left[ (nS_2)n^i \left( \frac{3\delta}{4} - \frac{3\nu}{2} + \frac{15}{4} \right) + \varphi^2 \left( -\frac{3\delta}{4} - \frac{\nu}{2} + \frac{9}{4} \right) \right] + n^i \left[ -\frac{15}{2} \kappa_1(nS_i)(nS_2) + \left( \frac{15\delta}{4} + \frac{15\nu}{2} - \frac{15}{4} \right) \right]

+ (nS_1)^2 \left( \frac{3\delta}{2} - \frac{3}{X_1} + \frac{9}{2} \right) + \kappa_1 \left( -\frac{3\delta}{2} - \frac{3\nu}{2} + \frac{9}{2X_1} - 3 \right) \right] + (nS_2)^2 \left( \frac{9\delta}{4} + \frac{2\nu}{2} + \frac{15}{4} \right) + (nS_2)(nS_2) \left( \frac{3\delta}{4} - \frac{3\nu}{2} - \frac{9}{4} \right)

+ (nS_2)(nS_2) \left( \frac{3\delta}{4} + \frac{3\nu}{2} - \frac{9}{4} \right) + \kappa_1 \left( \frac{3\nu}{2} - \frac{9}{2X_1} + \frac{9}{2} \right) \right],

$$w_{i,1}^j = n^i \left[ (nS_1) \left( \frac{5\delta}{4} + \frac{\nu}{2} - \frac{5}{4} \right) + \kappa_1 \left( \frac{9\delta}{4} - \frac{12}{X_1} + \frac{57}{4} \right) \right]

+ (nS_2) \left( -\frac{3\delta}{4} + \frac{\nu}{2} - \frac{39}{4} \right) + S_1^j \left[ \frac{13}{4} + \frac{\delta}{4} - \frac{\nu}{2} \right]. \quad (A.2)

with $X_1 = m_1/m = (1 + \delta)/2$. For the relative acceleration $a_i = a_i^1 - a_i^2$, we obtain (coming back to the $S_i, \Sigma$ and $\kappa_\pm$ variables)

$$\left( a_i^j \right)^{SS} = \frac{G}{c^4r^4m} \left[ \frac{1}{4} a_{i,0} + \frac{1}{c^2} \left( \frac{1}{8} a_{k,0} + \frac{1}{4} \right) \right] + O(8). \quad (A.3)$$

with

$$a_{i,0} = S^j \left[ (nS)(-12\kappa_- - 24) + (nS)(-6\delta\kappa_+ - 12\delta + 6\kappa_-) \right]

+ \Sigma^j \left[ (nS)(-6\delta\kappa_+ - 12\delta + 6\kappa_-) + (nS)(6\delta\kappa_- - 6\kappa_- + \nu (12\kappa_+ + 24)) \right]

+ n^j \left[ S^2 (-6\kappa_+ - 12) + \Sigma^2 \left( (3\delta\kappa_- - 3\kappa_+) + \nu (6\kappa_+ + 12) \right) + (nS)^2 (30\kappa_+ + 60) \right.

+ \Sigma (nS) (-6\delta\kappa_+ - 12\kappa_+ + 6\kappa_-) + (nS)^2 \left( (15\kappa_+ - 15\delta\kappa_- + \nu (30\kappa_- - 60)) \right)

+ (nS)(nS)(30\delta\kappa_+ + 60\delta - 30\kappa_-) \right].
\[ a_{q_0} = S \left[ (nS)(nv)^2 \left( (60\kappa_s - 60\delta \kappa_s) + \nu \left( 60\kappa_s + 120 \right) \right) + \nu^2 \left( (12\delta \kappa_s - 36\kappa_s + 48) + \nu (-72\kappa_s - 144) \right) \right] \]

\[ + (n\Sigma)(nv)^2 \left( (60\delta \kappa_s - 120\delta - 60\kappa_s) + \nu \left( 30\delta \kappa_s + 60\delta + 90\kappa_s \right) \right) + \nu^2 \left( (24\delta \kappa_s - 24\delta \kappa_s + \nu (-36\delta \kappa_s - 72\delta + 12\kappa_s) \right) \]

\[ + (n\Sigma)(nv)^2 \left( (30\delta \kappa_s - 30\kappa_s + 84) + \nu (-12\kappa_s - 24) \right) + (v\Sigma)(nv)^2 \left( -30\delta \kappa_s + 132\delta + 30\kappa_s \right) + \nu \left( -6\delta \kappa_s - 12\delta - 54\kappa_s \right) \]
\begin{equation}
\alpha_{\delta,1} = S^4 \left[ (nS) \left( ( -24\delta \kappa_+ + 72\kappa_+ + 164 ) + \nu \left( 36\kappa_+ + 72 \right) \right) \\
+ (nS) \left( ( 48\delta \kappa_+ + 72\delta - 48\kappa_+ ) + \nu \left( 18\delta \kappa_+ + 36\delta + 30\kappa_+ \right) \right) \\
+ \Sigma^2 \left[ (nS) \left( ( 48\delta \kappa_+ + 84\delta - 48\kappa_+ ) + \nu \left( 18\delta \kappa_+ + 36\delta + 30\kappa_+ \right) \right) \\
+ (n\Sigma) \left( ( 48\delta \kappa_+ - 48\delta \kappa_+ ) + \nu \left( 6\delta \kappa_+ - 102\kappa_+ - 148 \right) + \nu^2 \left( -36\delta \kappa_+ - 72 \right) \right) \right] \\
+ n^4 \left[ (nS)^2 \left( ( 48\delta \kappa_- + 192\kappa_+ + 420 ) + \nu \left( -96\kappa_+ - 192 \right) \right) \\
+ S^2 \left( ( -8\delta \kappa_+ + 40\kappa_+ + 72 ) + \nu \left( 20\kappa_+ + 40 \right) \right) \\
+ (nS) \left( ( -240\delta \kappa_+ - 396\delta + 240\kappa_- ) + \nu \left( -96\delta \kappa_+ - 192\delta - 96\kappa_- \right) \right) \\
+ (n\Sigma)^2 \left( ( 120\delta \kappa_- - 120\kappa_+ ) + \nu \left( 240\kappa_+ + 372 \right) + \nu^2 \left( 96\kappa_- + 192 \right) \right) \\
+ \Sigma^2 \left( ( 48\delta \kappa_+ + 72\delta - 48\kappa_+ ) + \nu \left( 20\delta \kappa_+ + 40\delta + 12\kappa_- \right) \right) \\
+ \Sigma^2 \left( ( 24\delta \kappa_- - 46\kappa_+ - 72 ) + \nu^2 \left( -20\kappa_- - 40 \right) \right) \right]. \tag{A.4}
\end{equation}

The conserved total angular momentum of the system contains SS contributions, but they arise only at the order $O(6)$ instead of the order $O(4)$ for the conserved energy. These terms are given in the center-of-mass frame by

\begin{equation}
J_{SS} = \frac{G\nu m}{4c^6 c^2} + O(8), \tag{A.5}
\end{equation}
with, using the triple product notation \((a, b, c) = a \cdot (b \times c)\),
\[ j = n \times v \left[ (nS)^2 \left( (12 - 18 \kappa_+ - 6 \delta \kappa_+ + 12 \kappa_-) + \nu \right) + (nS)(n\Sigma) \left( -9 \delta \kappa_+ + 3 \kappa_+ - 6 \delta \kappa_- + 12 \kappa_- \right) + (nS)(n\delta \kappa) \right] \]

\[ + n \times S \left[ (nS)(n\nu) \left( 6 \delta \kappa_+ + 12 \delta + 12 \kappa_- \right) + (n\Sigma)(n\nu) \left( -9 \delta \kappa_+ + 3 \kappa_+ - 6 \delta \kappa_- + 12 \kappa_- \right) + (v\Sigma)(n\delta \kappa) \right] \]

\[ + n \times \Sigma \left[ (nS)(n\nu) \left( 6 \delta \kappa_+ + 12 \delta + 12 \kappa_- \right) + (n\Sigma)(n\nu) \left( -9 \delta \kappa_+ + 3 \kappa_+ - 6 \delta \kappa_- + 12 \kappa_- \right) + (v\Sigma)(n\delta \kappa) \right] \]

\[ + v \times S \left[ (nS)(n\nu)(n\Sigma) \right] + \nu \times \Sigma \left[ (nS)(2 \delta + 6 \kappa_+ - 3 \kappa_-) ) \right] + \nu \times \Sigma \left[ (n\Sigma)(2 \delta + 6 \kappa_+ - 3 \kappa_-) \right] + \Sigma \left[-6(n, v, \Sigma) - 2 \delta(n, v, \Sigma) \right] \]

\[+ n \times S \left[ (nS)(n\nu)(n\Sigma) \right] \]

\[ + \frac{1}{mc^2} \left[ \sum_{i=0}^{\nu} \left( \frac{\nu}{2} \right) \left( \frac{\nu}{2} \right) \left( \frac{\nu}{2} \right) \right] \]

\[ + \frac{\nu}{84c^2} \left( \frac{\nu}{2} \right) \left( \frac{\nu}{2} \right) \left( \frac{\nu}{2} \right) \]

\[ + \frac{Gm}{r} \left( \frac{\nu}{2} \right) \left( \frac{\nu}{2} \right) \left( \frac{\nu}{2} \right) \]
with
\[i_{\Omega 0}^i = -S^{<i}S^{i>} (\delta \kappa_- + \kappa_+) + 4 S^{<i}S^{i>} \nu \kappa_- + \Sigma^{<i} \Sigma^{i>} \nu (\delta \kappa_- - \kappa_+),\]
\[i_{\Omega 0}^j = 29 S^{<i}S^{i>} \nu^2 (\delta \kappa_- - \kappa_+) + 58 S^{<i} \Sigma^{i>} \nu^2 ((\kappa_- - \delta \kappa_+) - 2\nu \kappa_-)\]
\[+ \Sigma^{<i} \Sigma^{i>} \nu^2 \left(29 (\delta \kappa_- - \kappa_+) + \nu (-29 \delta \kappa_- + 87 \kappa_+ + 140)\right)\]
\[+ 66 (\Sigma \nu)S^{<i} \Sigma^{i>} (\kappa_+ - \delta \kappa_-) + 66 (\Sigma \nu)S^{<i} \Sigma^{i>} ((\delta \kappa_+ - \kappa_-) + 2\nu \kappa_-)\]
\[+ 66 (\Sigma \nu)S^{<i} \Sigma^{i>} ((\delta \kappa_+ - \kappa_-) + 2\nu \kappa_-)\]
\[+ 22 S^2 \nu^{<i} \nu^{i>} ((\delta \kappa_- - \kappa_+) + 44 (\Sigma \Sigma) \nu^{<i} \nu^{i>}) ((\kappa_- - \delta \kappa_+) - 2\nu \kappa_-)\]
\[+ 2 \Sigma^2 \nu^{<i} \nu^{i>} ((\delta \kappa_- - \kappa_+) + \nu (-11\delta \kappa_- + 33 \kappa_+ + 14)).\]
\[i_{\Omega 1}^{ij} = 6 \left(nS^2\right) n^{<i} \nu^{i>} ((7\delta \kappa_- + 18 \kappa_+ + 36) - 40 \nu (\kappa_- + 2))\]
\[+ 2 S^2 n^{<i} \nu^{i>} ((-11\delta \kappa_- + 8 \kappa_+ - 48) + 30 \nu (\kappa_+ + 2))\]
\[+ 6 \left(nS\right) n^{<i} \nu^{i>} ((11\delta \kappa_+ + 36 \delta - 11 \kappa_-) + 4 \nu (-10\delta \kappa_- + 2\delta - 3 \kappa_-))\]
\[+ 3 \left(nS\right) n^{<i} \nu^{i>} ((11 \kappa_+ - \delta \kappa_-) + 2 \nu (13 \delta \kappa_- - 24 \kappa_+ - 36) + 80 \nu^2 (\kappa_- + 2))\]
\[+ 2 \left(nS\right) n^{<i} \nu^{i>} ((19 \delta \kappa_- - 48 \delta - 19 \kappa_-) + 2 \nu (15 \delta \kappa_- + 30 \delta + 7 \kappa_-))\]
\[+ (\Sigma \nu) n^{<i} \nu^{i>} ((19 \kappa_+ - \delta \kappa_-) - 2 \nu (4 \delta \kappa_- + 15 \kappa_+ + 8) - 60 \nu^2 (\kappa_+ + 2))\]
\[+ 12 \left(nS\right) n^{<i} \nu^{i>} ((2 \delta \kappa_- - 13 \kappa_+ - 22) + 5 \nu (\kappa_+ + 2))\]
\[+ 2 \left(nS\right) n^{<i} \nu^{i>} ((-9 \delta \kappa_- - 2 \delta + 9 \kappa_-) + 3 \nu (5 \delta \kappa_- + 10 \delta - 13 \kappa_-))\]
\[+ 2 S^{<i} \nu^{i>} (17 \delta \kappa_- + 109 \kappa_+ + 2 \left(nS\right) n^{<i} \nu^{i>} ((-45 \delta \kappa_- - 122 \delta + 45 \kappa_-) + 3 \nu (5 \delta \kappa_- + 10 \delta - 13 \kappa_-))\]
\[+ 6 \left(nS\right) n^{<i} \nu^{i>} ((15 \delta \kappa_- - \kappa_+) + \nu (-9 \delta \kappa_- + 39 \kappa_- + 44) - 10 \nu^2 (\kappa_- + 2))\]
\[+ 8 S^{<i} \nu^{i>} (23 (\delta \kappa_- - \kappa_-) - 17 \nu \kappa_-)\]
\[+ 2 \Sigma^{<i} \nu^{i>} (46 (\kappa_+ - \delta \kappa_-) + \nu (-17 \delta \kappa_- - 75 \kappa_- + 56)).\]

Finally, for the mass octupole moment, we find [42]
\[
\left(P^{ij}\right)_{SS} = \frac{3\nu r}{2c^4} n^{<i} (-2\kappa_- S^{<i} S^{i>} + 2 (\kappa_- - \delta \kappa_-) S^{<i} S^{i>} + (\delta \kappa_- + \kappa_+ + \Delta \kappa) (\delta \kappa_- - \kappa_+ - 2\nu \kappa_-) S^{<i} S^{i>}).
\]
Appendix C. Correspondence between the spin vector and spin tensor variables

This appendix provides the link between the spin tensor and the conserved-norm spin vector variables which we use to present our PN results. We recall that the spin tensor variable $\tilde{S}_{ij}$ is the spatial part, in harmonic coordinates, of the spin tensor introduced in section 2.1, and that the spin vector variable has been defined in section 2.3 as $\tilde{S}_\mu = \tilde{S}_\mu^i$, with $\tilde{S}_\mu$ given by equation (2.14) and $i$ being a spatial index referring to the tetrad $e_\mu^i$ constructed in the same section.

We display below the SS contributions to the expression of the spin vector in terms of the spin tensor, in the general frame. These contributions complete those computed at the SO order in [25], equations (B.1) (notice that the spin tensor components there were denoted as $\tilde{S}_{ij}$ instead of $S_{ij}$). We have

$$
\left( S^1_i \right)_{SS} = \frac{G}{c^2 s_{12}} \left[ 2n_{12}^a S^a_{ij} S^b_{kl} e^{ijkl} + \frac{1}{2} v^i_1 \left( n_{12}^a S^a_{ij} S^b_{kl} e^{ijkl} \right) - \frac{1}{2} v^k_2 \left( n_{12}^a S^a_{ij} S^b_{kl} e^{ijkl} \right) \right] - \frac{1}{2} n_{12}^a S^a_{ij} \left( \tilde{\omega}^b_{ik} e^{ijkl} \right) - S^{iab}_{kl} e^{ijkl} \left( n_{12}^a S^a_{ij} \right) + \frac{1}{2} n_{12}^a S^a_{ij} \left( \tilde{\omega}^b_{ik} e^{ijkl} \right) + S^{iab}_{kl} e^{ijkl} \left( n_{12}^a S^a_{ij} \right) + O(7),
$$

where we have authorized the repetition of indices appearing in scalar quantities enclosed with parenthesis. At this order, there appear $S_1 S_2$ terms only, and thus no $S_1^2, S_2^2$ terms.

Appendix D. Equivalence with ADM results for the dynamics

In this appendix, we compare our results for the dynamics with those previously obtained in the ADM [62, 70–73] and EFT [21, 22, 35, 38, 39, 74] approaches. As the equivalence of ADM and EFT results has been already demonstrated in [36, 37, 73], we actually restrict ourselves to the comparison of our findings with the ADM ones, in line with our previous works.

The two results have been obtained in different gauges, and the spin variables differ in their definition. It is thus important to take properly into account the transformation of the particle positions and spins from one formalism to the other. In the following, we will denote the ADM variables with an overbar and resort to the convenient notation $p^{\mu A} A = \pi^{\mu A} A$. Let us now introduce the contact transformation $Y_A(\vec{x}, \vec{p}, \vec{S})$ and the rotation vector $\theta_A(\vec{x}, \vec{p}, \vec{S})$ such that the harmonic variables are related to the ADM ones by

$$
y_A = Y_A(\vec{x}, \vec{p}, \vec{S}) + O(7),
$$

$$
S_A = S_A + \theta_A(\vec{x}, \vec{p}, \vec{S}) \times S_A + O(6).
$$

The ADM spin variables and ours have the same Euclidean norm $S_A \cdot S_A = S_A^2 = S_A^2$, which is precisely the conserved norm introduced in section 2.1. Since the first corrections enter as $\theta_A = O(4)$, we see that the transformation for the spins necessarily takes this form.

Now, if we denote by $A_A(\vec{x}, \vec{p}, \vec{S})$ and $\Omega_A(\vec{x}, \vec{p}, \vec{S})$ the function that converts to ADM variables the harmonic-coordinate acceleration and precession vector, and by $\Omega_A$ the precession vector of the ADM spins, such that $dS_A/dt = \Omega_A \times S_A$, the two relations to impose
for the dynamics to be equivalent are
\[ \{ A_A, H_{ADM} \} + \mathcal{O}(7), \quad (D.2a) \]
\[ \{ \theta_A, H_{ADM} \} + \theta_A \times \mathbf{\Omega} = \Omega_A(x, p, S) - \mathbf{\Omega}_A + \mathcal{O}(6), \quad (D.2b) \]
where \( H_{ADM} \) is the ADM Hamiltonian (which can be found for instance in section 6.2 of [62]) and \{ \cdot \} is the usual Poisson brackets extended to spin variables. Here the term \( \theta_A \times \mathbf{\Omega} \) is actually negligible, for \( \theta_A = \mathcal{O}(2) \) and \( \mathbf{\Omega}_A = \mathcal{O}(4) \).

We find that there is no contribution at leading order in the transformations (D1), i.e. \( (Y_A)_{SS} = \mathcal{O}(6) \) and \( (\theta_A)_{SO} = \mathcal{O}(5) \). Using the method of undetermined coefficients then leads to a unique solution for the higher-order terms in the transformations. For the rotation vector we obtain
\[ \theta_{SO} = \frac{G}{c^5 r_{12}^2} \left[ \frac{m_2}{m_1} \theta_{5,1}^{5,1} + \theta_{5,2}^{5,2} \right] + \mathcal{O}(7), \quad (D.3) \]
with (adopting the same notations as in the rest of the paper for scalar products)
\[ \theta_{5,1}^{5,1} = -\frac{3k_1}{2} \pi_{12} \left( \pi_{5,1} - (\pi_{12} \pi_2)(\pi_{12} \pi_1) \right) - \frac{3k_1}{2} \pi_{2} \left( \pi_{12} \pi_1 \right) - \frac{1}{2} \pi_{1} \left( \pi_{12} \pi_1 \right), \]
\[ \theta_{5,2}^{5,2} = -\frac{1}{2} \pi_{2} \left( \pi_{12} \pi_2 \right) + \frac{1}{2} \pi_{12} \left( \pi_{5,2} - 3(\pi_{12} \pi_2)(\pi_{12} \pi_1) \right) + \frac{1}{2} \pi_{1} \left( \pi_{12} \pi_1 \right). \quad (D.4) \]

We recall that SO terms in \( \theta \) actually correspond to SS effects in the dynamics. For the contact transformation, we arrive at the simple expression
\[ (Y_i)_{SS} = -\frac{Gm_2}{2m_1^2 c^5 r_{12}^2} \left[ \pi_{12} \left( \pi_{5,1} \right) - \pi_{12} \left( \pi_{5,1} \right) \right] + \mathcal{O}(8). \quad (D.5) \]
The relevant NS and SO contributions to these transformations are given for instance in [24] and [24, 25].

The existence of a solution relating our variables to the ADM ones validates our results, the problem of finding such transformations being largely over-constrained.

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