CHARACTERIZATIONS OF FUNCTIONS IN WANDERING SUBSPACES OF THE BERGMAN SHIFT VIA THE HARDY SPACE OF THE BIDISC

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Abtract. Let $W$ be the corresponding wandering subspace of an invariant subspace of the Bergman shift. By identifying the Bergman space with $H^2(D^2) \ominus [z-w]$, a sufficient and necessary conditions of a closed subspace of $H^2(D^2) \ominus [z-w]$ to be a wandering subspace of an invariant subspace is given also, and a functional characterization and a coefficient characterization for a function in a wandering subspace are given. As a byproduct, we proved that for two invariant subspaces $M, N$ with $M \supseteq N$ and $\dim(N \ominus BN) < \infty$ $\dim(M \ominus BM) = \infty$, then there is an invariant subspace $L$ such that $M \supseteq L \supseteq N$. Finally, we define an operator from one wandering subspace to another, and get a decomposition theorem for such an operator which is related to the universal property of the Bergman shift.

1. Introduction

Let $D$ be the open unit disk in the complex plane $C$, $T$ is the unit circle which is the boundary of $D$. $H^2(T)$ denotes the Hardy space on $T$. Let $dA$ denote Lebesgue area measure on $D$, normalized so that the measure of $D$ equals 1. The Bergman space $L^2_a(D)$ is the Hilbert space consisting of the analytic functions on $D$ that are also in the space $L^2(D, dA)$ of square integrable functions on $D$. The operator of multiplication by the coordinate function $z$ on the Bergman space is called the Bergman shift. The torus $T^2$ is the Cartesian product $T \times T$. The Hardy space $H^2(T^2)$ over the bidisk is $H^2(T) \otimes H^2(T)$.

For an operator $T$ on a Hilbert space $H$, the structure of an operator is an important research subject in operator theory. A subspace $M$ of $H$ is called an invariant subspace of $T$ if $TM \subseteq M$. The famous invariant subspace problem is one of the central problems in operator theory. If $M$ is an invariant subspace of $T$, $M \ominus TM$ is called the wandering subspace of $T$ on $M$. The famous Beurling theorem says that all invariant subspaces of the unilateral shift which can be identified as multiplication by $z$ on the Hardy space are generated by their wandering subspaces with dimension 1. The study of the Bergman shift attract great attentions since its universal property. The remarkable Beurling type theorem of Aleman, Richter and Sundberg implies every invariant subspace of the Bergman shift is generated by its wandering subspace with possible dimension from 1 to $\infty$.

For each integer $n \geq 0$, let

$$p_n(z, w) = \sum_{i=0}^{n} z^i w^{n-i}.$$
Rudin [10] defined $\mathcal{H}$ to be the subspace of $H^2(T^2)$ spanned by functions $\{p_n\}_{n=0}^\infty$. Thus every function in $\mathcal{H}$ is symmetric with respect to $z$ and $w$. Let $[z - w]$ denote the closure of $(z - w)H^2(T^2)$ in $H^2(T^2)$. As every function in $[z - w]$ is orthogonal to each $p_n$, it is well-known and easy to see that

$$H^2(T^2) = \mathcal{H} \oplus [z - w].$$

For a subspace $\mathcal{M}$ of $L^2(T^2)$ and let $P_M$ denote the orthogonal projection from $L^2(T^2)$ onto $\mathcal{M}$. The Toeplitz operator on $H^2(T^2)$ with symbol $f$ in $L^\infty(T^2)$ is defined by

$$T_f(h) = P_{H^2(T^2)}(fh),$$

for $h$ in $H^2(T^2)$. It is not difficult to see that $T_z$ and $T_w$ are a pair of doubly commuting pure isometries on $H^2(T^2)$. It is easy to check that

$$P_{H^2(T^2)}T_z|_{\mathcal{H}} = P_{H^2(T^2)}T_w|_{\mathcal{H}}.$$

Let $B$ denote the operator above. It was shown explicitly in [11] and implicitly in [4] that $B$ is unitarily equivalent to the Bergman shift, the multiplication operator by the coordinate function $z$ on the Bergman space $L^2_a(D)$ via the following unitary operator $U : L^2_a(D) \to \mathcal{H}$,

$$Uz^n = \frac{p_n(z, w)}{n + 1}.$$

So the Bergman shift is lifted up as the compression of an isometry on a nice subspace $\mathcal{H}$ of $H^2$. In the rest of the paper we identify the Bergman shift with the operator $B$, and so an invariant subspace of the Bergman shift can be identified as a subspace of $\mathcal{H}$ which is invariant under $B$.

In last twenty years, it become an important tool to use the theory of multivariable operators and functions to study a single operator and the functions of one variable in the study of the Bergman shift. The idea is to lift the Bergman shift up as the compression of a commuting pair of isometries on a subspace of the Hardy space of the bidisk which was given in Rudin’s book [10], used in studying the Hilbert modules by R. Douglas and V Paulsen [4], operator theory in the Hardy space over the bidisk by R. Douglas and R. Yang [5, 16, 15], the lattice of the invariant subspaces of the Bergman shift by S. Richter, the reducing subspaces of the Bergman shift by Zheng, Guo, Zhong, and the authors [7, 14, 13], the Beurling type theorem of the Bergman shift by Zheng and the fisrt author [12] etc. In this paper, the idea is used to study the characterization of a function in the wandering subspace of any invariant subspace of the Bergman shift.

2. CHARACTERIZATIONS OF A WANDERING SUBSPACE

For a function in $\mathcal{H}$, we have the following seires representation.

**Proposition 2.1.** For any $q(z, w) \in \mathcal{H}$, then

$$q(z, w) = \sum_{j=0}^\infty z^j T_w^{*j} q(0, w)$$

**Proof.** The identity follows from

$$q(z, w) = \sum_{n=0}^\infty a_n \frac{p_n(z, w)}{\sqrt{n + 1}} = \sum_{n=0}^\infty a_n \frac{1}{\sqrt{n + 1}} \sum_{j=0}^n z^j w^{n-j}$$

$$= \sum_{j=0}^\infty z^j \sum_{n=j}^\infty a_n \frac{1}{\sqrt{n + 1}} w^{n-j}.$$
since \( q(z, w) \in \mathcal{H} \), and

\[
T^* w q(0, w) = T^* w \sum_{n=0}^{\infty} \frac{d_n}{\sqrt{n+1}} w^n = T^* w \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} w^{n-j}.
\]

By Proposition 2.1, we can get following characterization of a function in the wandering subspace for an invariant subspace.

**Theorem 2.2.** If \( M \) is an invariant subspace of the Bergman shift \( B \), and \( q(z, w) \) is a function in the wandering subspace \( M \ominus BM \) of \( M \), then

\[
\sum_{j=0}^{\infty} |T^* w q(0, w)|^2 = \text{constant}, \quad \forall w \in \partial \mathbb{D}.
\]

**Proof.** Since \( q(z, w) \in M \ominus BM \) and \( M \perp [z - w] \), for any \( k \neq 0 \), we have

\[
0 = \langle z^k q(z, w), q(z, w) \rangle_{H^2(\mathbb{T}^2)}
= \langle T^k q(z, w), q(z, w) \rangle_{H^2(\mathbb{T}^2)}
= \sum_{j=0}^{\infty} \langle w^j T^* w q(0, w), T^* w q(0, w) \rangle_{H^2(\mathbb{T})}
= \frac{1}{2\pi} \int_0^{2\pi} e^{ik\theta} \sum_{j=0}^{\infty} T^* w q(0, w) T^* w q(0, w) d\theta
= \frac{1}{2\pi} \int_0^{2\pi} e^{ik\theta} \sum_{j=0}^{\infty} |T^* w q(0, w)|^2 d\theta
\]

by the proposition. It means that \( \sum_{j=0}^{\infty} |T^* w q(0, w)|^2 \) is orthogonal to \( w^k \) for any \( k \neq 0 \), so it must be a constant. \( \square \)

One the other side, the converse of the above theorem is true also.

**Theorem 2.3.** If \( W \) is a closed subspace of \( \mathcal{H} \), and every function \( q(z, w) \) in \( W \) satisfies

\[
\sum_{j=0}^{\infty} |T^* w q(0, w)|^2 = \text{constant}, \quad \forall w \in \partial \mathbb{D}.
\]

Then there is a minimal invariant subspace \( M \) of \( B \) such that

\[
M \ominus BM = W.
\]

**Proof.** In fact, let \( M = \text{span} B^* W : j = 0, 1, 2, \ldots \), then \( M \) is an invariant subspace and it is clear that

\[
W \subseteq M \ominus BM.
\]
Now if \(q_0 \in (M \ominus BM) \ominus W\), by definition of \(M\), there exist polynomials \(p_n\) such that 
\[
\sum_{j=1}^{N} p_j(B)q_n \to q_0, \quad \text{where } q_n \in W.
\]
So 
\[
\|q_0\|_{H^2(T^2)}^2 = \lim_{N \to \infty} \left( \sum_{j=0}^{N} p_n(B)q_n, q_0 \right) 
\]
\[
= \lim_{N \to \infty} \sum_{j=0}^{N} (q_n, (p_n(B))^* q_0) 
\]
\[
= 0
\]
since \((p_n(B))^* q_0 \perp M\) and \(q_n \perp q_0\). \(\square\)

It is easy to get a necessary and sufficient conditions of a closed subspace in \(H\) to be the wandering subspace of an invariant subspace \(M\).

**Corollary 2.4.** An closed subspace \(W\) of \(H\) is the wandering subspace of an invariant subspace \(M\) if and only if every function \(q(z, w)\) in \(W\) satisfies 
\[
\sum_{j=0}^{\infty} |T_{w}^{*j} q(0, w)|^2 = \text{constant}, \quad \forall w \in \partial D.
\]

**Remark 2.5.** It is easy to check that if \(q_1, q_2\) satisfy (1), then every function in \(S \text{pan}\{q_1, q_2\}\) satisfies (1) if and only if 
\[
\sum_{j=0}^{\infty} T_{w}^{*j} q_1(0, w)T_{w}^{*j} q_2(0, w) = \text{constant}, \quad \forall w \in \partial D.
\]

The famous invariant subspace problem is equivalent to the universal property of the Bergman shift, i.e., for two invariant subspaces \(M, N\) with \(M \supseteq N\) and \(dim(N \ominus BN) = dim(M \ominus BM) = \infty\), is there an invariant subspace \(L\) such that \(M \supseteq L \supseteq N\)? And so it is natural to ask for two invariant subspaces \(M, N\) with \(M \supseteq N\), is there an invariant subspace \(L\) such that \(M \supseteq L \supseteq N\)? We get the following interesting theorem in a special case.

**Theorem 2.6.** For two invariant subspaces \(M\) and \(N\) of \(B\), if \(M \supseteq N\) and \(N \ominus BN\) is finite dimensional, and \(dim(M \ominus BM) = \infty\). Then there is an invariant subspace \(L\) of \(B\) such that 
\[
N \subseteq L \subseteq M.
\]

**Proof.** Assume that \(dim(N \ominus BN) = K\) and let \(\{q_j\}_{j=1}^{K}\) be an orthonormal basis of \(N \ominus BN\).

We can choose an function \(\tilde{q}\) with norm 1 in \(M \ominus BM\) which is orthogonal to \(q_j\) for \(1 \leq j \leq K\) since \(dim(M \ominus BM) = \infty\). Let \(W\) be the closed linear span of \(\{q_j\}_{j=1}^{K}\) and \(\tilde{q}\).

Next we show that every function \(q\) in \(W\) satisfying 
\[
\sum_{j=0}^{\infty} |T_{w}^{*j} q(0, w)|^2 = \text{constant}, \quad \forall w \in \partial D.
\]
For simplicity and without loss of generality, we prove the statement for the case $K = 1$. In this case $q = \lambda q_1 + \mu \overline{q}$ where $\lambda, \mu$ are two constants, and so

\[
\sum_{j=0}^{\infty} |T_w^j q(0, w)|^2 = \sum_{j=0}^{\infty} |\lambda T_w^j q_1(0, w) + \mu T_w^j \overline{q}(0, w)|^2
\]

\[
= \sum_{j=0}^{\infty} |T_w^j q_1(0, w)|^2 + \sum_{j=0}^{\infty} |T_w^j \overline{q}(0, w)|^2
\]

\[
= 2 \text{Re} \left( \sum_{j=0}^{\infty} T_w^j q_1(0, w) T_w^j \overline{q}(0, w) \right)
\]

\[
= \text{constant},
\]

by Theorem 2.2 and Remark 2.5 since $q_1$ and $\overline{q}$ are in wandering subspaces $N \subseteq BN$ and $M \subseteq BM$ respectively.

Then there is a minimal invariant subspace $L$ of $B$ such that

\[
L \oplus BL = W,
\]

by Theorem 2.3. It is clear that $N \subseteq L \subseteq M$, and so the theorem follows. \qed

Remark 2.7. If on the Dirichlet space $D(\mathbb{D})$, we define the following inner product

\[
\langle f, g \rangle_{D(\mathbb{D})} = \frac{1}{\pi} \int_{\mathbb{D}} (w f(w))^* \overline{w g(w)} dA(w)
\]

for $f, g \in D(\mathbb{D})$. Then $q_f = \sum_{j=0}^{\infty} z^j T_w^j f$ and $q_g = \sum_{j=0}^{\infty} z^j T_w^j g$ are in $H$. Moreover, if $f(w) = \sum_{k=0}^{\infty} f_k w^k$ and $g(w) = \sum_{k=0}^{\infty} g_k w^k$, then we have

\[
\langle q_f, q_g \rangle_{H^2(\mathbb{T})} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \langle T_w^j f, T_w^j g \rangle_{H^2(\mathbb{T})} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} f_k \overline{g_k} = \sum_{k=0}^{\infty} (k+1) f_k \overline{g_k}
\]

\[
= \langle f, g \rangle_{D(\mathbb{D})}.
\]

Theorem 2.8. If a closed subspace $W$ of $H$ is the wandering subspace of an invariant subspace $M$ if and only if every function $q(z, w) = \sum_{j=0}^{\infty} z^j T_w^j q(0, w)$ in $W$ satisfies

\[
\begin{cases}
\sum_{j=0}^{\infty} j q_{j+1} \overline{q_j} = 0 \\
\sum_{j=0}^{\infty} j^2 q_{j+1} \overline{q_j} = 0
\end{cases} \quad \forall k > 0
\] (2)

where $q(0, w) = \sum_{k=0}^{\infty} q_k w^k$.

Proof. By Theorem 2.2, $W$ is the wandering subspace of an invariant subspace $M$ if and only if every function $q(z, w)$ in $W$ satisfies

\[
\sum_{j=0}^{\infty} |T_w^j q(0, w)|^2 = \text{constant}, \quad \forall w \in \partial \mathbb{D},
\]
It is equivalent to that for any \( k > 0 \),

\[
\frac{1}{2\pi} \int_0^{2\pi} w^k \sum_{j=0}^{\infty} |T_w^j q(0, w)|^2 d\theta = 0
\]

\[
\Leftrightarrow 0 = \frac{1}{2\pi} \int_0^{2\pi} w^k \sum_{j=0}^{\infty} T_w^j q(0, w) T_w^j q(0, w) d\theta
\]

\[
= \sum_{j=0}^{\infty} (T_w^j q(0, w), \overline{T_w^j q(0, w)})
\]

\[
= \sum_{j=0}^{\infty} (T_w^j q(0, w), T_w^{j+k} q(0, w))
\]

\[
= \langle q(0, w), T_w^k q(0, w) \rangle
\]

\[
= \sum_{j=0}^{\infty} q_j \overline{q}_{j+k}.
\]

For \( k < 0 \), we get the second equation. \( \square \)

**Remark 2.9.** The theorem is a coefficient description of a function in wandering subspace.

**Remark 2.10.** Define a map from the Dirichlet space \( \mathcal{D} \) to \( \mathcal{H} \) by

\[
f \mapsto \sum_{j=0}^{\infty} z^j T_w^j f(w).
\]

The map is an isometry one-to-one by remark 2.5, the inverse map is \( q(z, w) \mapsto q(0, w) \).

Moreover, \( T_w \) on \( \mathcal{H} \) and \( T_w^* \) on \( \mathcal{D} \) satisfy the following identity

\[
\langle q_j, T_w^* q_k \rangle_{H^1(\mathcal{D})} = \langle f, T_w^k g \rangle, \quad \forall f, g \in \mathcal{D}
\]

It shows that \( B^*|_{H^1} \cong T_{w|_{\mathcal{D}}}^* \). It does not mean that \( B|_{H^1} \) is \( P_\mathcal{D} T_{w|_{\mathcal{D}}} \). Indeed, \( B|_{H^1} \) is \( (T_{w|_{\mathcal{D}}}^*)^* \).

\( T_{w|_{\mathcal{D}}}^* \) is the adjoint of \( T_w \) in the Dirichlet space, but \( B^*|_{H} \) is the adjoint of \( P_{H} T_{\mathcal{D}|_{H}} \) in the Hardy space \( H^2(\mathbb{D}) \).

Wandering subspace of the Bergman shift has possible dimension from 1 to \( \infty \), which is different to Hardy space on the unit disc. The following corollary give a possible construction of wandering subspace with dimension greater than 1.

**Corollary 2.11.** A closed subspace \( \mathcal{W} \) of \( \mathcal{H} \) is the wandering subspace of an invariant subspace of \( B \) if and only if for \( q_1(z, w) = \sum_{j=0}^{\infty} z^j T_w^j q_1(0, w) \) and \( q_2(z, w) = \sum_{j=0}^{\infty} z^j T_w^j q_2(0, w) \) in \( \mathcal{W} \), the following identity holds

\[
\sum_{j=0}^{\infty} T_w^j q_1(0, w) T_w^j q_2(0, w) \in H^1(\mathbb{D})
\]
Proof. It is sufficient to show that (3) is holomorphic, which is equivalent to show that the Laurent coefficients with negative index are zero. As a fact, $\forall k > 0$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} w^k \sum_{j=0}^{\infty} T_w^j q_1(0, w) \overline{T_w^{j'} q_2(0, w)} d\theta = \frac{1}{2\pi} \sum_{j=0}^{\infty} \int_0^{2\pi} T_w^j q_1(0, w) \overline{w^{j'} T_w^{j'} q_2(0, w)} d\theta$$

$$= \sum_{j=0}^{\infty} \langle T_w^j q_1(0, w), w^{j'} T_w^{j'} q_2(0, w) \rangle_{L^2(\mathbb{T})}$$

$$= \sum_{j=0}^{\infty} \langle T_w^j q_1(0, w), T_w^{j+k} q_2(0, w) \rangle_{L^2(\mathbb{T})}$$

$$= \langle q_1(z, w), T_w^{k} q_2(z, w) \rangle_{H^p(\mathbb{T}^2)} = 0,$$

since $q_1(z, w)$ is orthogonal to $T_w^{k} q_2(z, w)$. \hfill $\square$

3. Wandering subspaces with infinite dimension

Now we turn to study the internal structure of a wandering subspace $W$ for the case of $\dim W = \infty$.

**Proposition 3.1.** If $M$ is an invariant subspace of $B$, $\dim(M \oplus BM) = \infty$, and $\{q_j\}_{j=1}^{\infty}$ is an orthonormal basis of $M \oplus BM$. Then

$$\sum_{j=0}^{\infty} T_w^j q_k(0, w) \overline{T_w^{j'} q_{k'}(0, w)} = \begin{cases} 1 & \text{if } k = k' \\ 0 & \text{if } k \neq k' \end{cases} \quad \forall w \in \mathbb{T}.$$  

**Proof.** For case $k = k'$, it is just Equation (1). For case $k \neq k'$, note that $q_j \in W$, we have for any $m > 0$

$$0 = \langle z^m q_k, q_{k'} \rangle_{H^p(\mathbb{T}^2)} = \langle q_k, T_z^{m} q_{k'} \rangle_{H^p(\mathbb{T}^2)}$$

$$= \langle q_k, T_w^{m} q_{k'} \rangle_{H^p(\mathbb{T}^2)}$$

$$= \sum_{j=0}^{\infty} \langle T_w^j q_k(0, w), T_w^{m+j} q_{k'}(0, w) \rangle_{H^p(\mathbb{T}^2)}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} w^m \sum_{j=0}^{\infty} T_w^j q_k(0, w) \overline{T_w^{j'} q_{k'}(0, w)} d\theta.$$  

For $m < 0$, it is similar to show that $\frac{1}{2\pi} \int_0^{2\pi} w^m \sum_{j=0}^{\infty} T_w^j q_k(0, w) \overline{T_w^{j'} q_{k'}(0, w)} d\theta = 0$. Finally, $m = 0$, it is zero since $q_k \perp q_{k'}$. So we have proved that for any $m \in \mathbb{Z}$,

$$\frac{1}{2\pi} \int_0^{2\pi} w^m \sum_{j=0}^{\infty} T_w^j q_k(0, w) \overline{T_w^{j'} q_{k'}(0, w)} d\theta = 0,$$

and so

$$\sum_{j=0}^{\infty} T_w^j q_k(0, w) \overline{T_w^{j'} q_{k'}(0, w)} = 0, \quad \forall w \in \partial \mathbb{D}, k \neq k'.$$  

$\square$
Definition 3.2. For any \( q(z, w) = \sum_{j=0}^{\infty} z^j T_w^j q(0, w) \), define \( L_w : W \mapsto \ell^2(\mathbb{N}, H^2(\mathbb{T})) \), \( \forall w \in \partial \mathbb{D} \) as follows:

\[
L_w q(z, w) = (q(0, w), T_w q(0, w), \cdots, T_w^j q(0, w), \cdots)
\]

\( L_w \) is defined on \( \partial \mathbb{D} \) pointwise. And if \( q_1, q_2 \) in a wandering subspace \( W \) and \( q_1 \) is orthogonal to \( q_2 \) in \( H^2(\mathbb{T}^2) \), then \( (L_w q_1, L_w q_2)_{\ell^2(\mathbb{D})} = 0 \), \( \forall w \in \partial \mathbb{D} \). \( W \) can be viewed as the subspace of \( \ell^2(\mathbb{N}, H^2(\mathbb{T})) \) under the map \( L_w \).

**Definition 3.3.** If \( M \) and \( N \) are two invariant subspace of \( B \), \( M \supseteq N \), \( \dim(M \ominus BM) = \dim(N \ominus BN) \), and \( \{q_k\}_{k=1}^{\infty} \) and \( \{q_k\}_{k=1}^{\infty} \) are orthonormal basis of \( N \ominus BN \) and \( M \ominus BM \) respectively. Define \( T_w \) as

\[
T_w(L_w q_k) = L_w \tilde{q}_k, \quad \forall k \geq 1, \forall w \in \partial \mathbb{D}.
\]

**Proposition 3.4.** The operator \( T_w \) is an isometry from \( W_N \) to \( W_M \).

**Proof.** Firstly, \( T_w \) is defined for basis of \( W_N \), it can be extended to \( W_N \). \( T_w \) is an isometry by Proposition 2.10. \( \square \)

**Proposition 3.5.** If \( M \) and \( N \) are two invariant subspaces of \( B \), \( M \supseteq N \) and \( \dim \mathcal{W}_N = \dim \mathcal{W}_M = \infty \). Then for any unitary operator \( U : W_N \rightarrow W_M \), there is a unitary operator \( V : \ell^2(W_N) \rightarrow \ell^2(W_M) \) such that \( T_w U = VT_w \), i.e. the following diagram is commutative.

\[
\begin{array}{ccc}
\mathcal{W}_N & \xrightarrow{U} & \mathcal{W}_M \\
T_w \downarrow & & \downarrow T_w \\
\ell^2(W_N) & \xrightarrow{V} & \ell^2(W_M)
\end{array}
\]

**Proof.** It is clear. \( \square \)

If \( M \) and \( N \) are two invariant subspaces of \( B \), for convenience, \( T_w^{(MN)} \) is used to denote the isometry \( T_w \) from \( W_M \) to \( W_N \).

**Theorem 3.6.** If \( M, L \) and \( N \) are three invariant subspaces of \( B \), \( M \supseteq L \supseteq N \), and \( \dim \mathcal{W}_M = \dim \mathcal{W}_L = \dim \mathcal{W}_N \), then we have the following decomposition

\[
T_w^{(MN)} = T_w^{(NL)} T_w^{(ML)}.
\]

**Proof.** Assume that \( \{q_k^{(M)}\}_{k=1}^{\infty} \), \( \{q_k^{(L)}\}_{k=1}^{\infty} \), \( \{q_k^{(N)}\}_{k=1}^{\infty} \) are an orthonormal basis of \( \mathcal{W}_M \), \( \mathcal{W}_L \) and \( \mathcal{W}_N \) respectively, by definition, we have

\[
\begin{align*}
T_w^{(MN)}(L_w q_k^{(M)}) &= L_w q_k^{(N)} \\
T_w^{(ML)}(L_w q_k^{(M)}) &= L_w q_k^{(L)} \\
T_w^{(LN)}(L_w q_k^{(N)}) &= L_w q_k^{(N)}
\end{align*}
\]

The decomposition follows from a direct computation. \( \square \)

**Remark 3.7.** The theorem shows that for two invariant subspaces \( M \) and \( N \) with \( M \supseteq N \), if there is an invariant subspace \( L \) such that \( M \supseteq L \supseteq N \), then the holomorphic isometric valued function \( T_w^{(MN)} \) has a decomposition. So by the universal property of the Bergman shift, the famous invariant subspace problem is equivalent to the decomposition problem of the holomorphic isometric valued function \( T_w^{(MN)} \).

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