ADAPTIVE SPECTRAL TRANSFORMATIONS OF POISSON-CHARLIER MEASURES AND OPTIMAL THRESHOLD FACTORS OF ONE-STEP METHODS

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Abstract. Threshold factors govern the maximally allowable step-size at which positivity or contractivity preservation of explicit integration methods for initial value problems is guaranteed. An optimal threshold factor, \( R_{m,n} \), is defined as the largest possible positive real number for which there exists a real polynomial \( P \) of degree \( m \), absolutely monotonic over the interval \([−R_{m,n}, 0]\) and such that \( P(x) = \exp(x) + O(x^{n+1}) \) when \( x \to 0 \). In this paper, we show that the task of computing optimal threshold factors is, to a certain extent, equivalent to the problem of characterizing positive quadratures with integer nodes with respect to Poisson-Charlier measures. Using this equivalence, we provide sharp upper and lower bounds for the optimal threshold factors in terms of the zeros of generalized Laguerre polynomials. Also based on this equivalence, we propose a highly efficient and stable algorithm for computing these factors based on adaptive spectral transformations of Poisson-Charlier measures. The algorithm possesses the remarkable property that its complexity depends only on the order of approximation and thus is independent of the degree of the underlying polynomials. Our results are achieved by adapting and extending an ingenious technique by Bernstein in his seminal work on absolutely monotonic functions [7]. Moreover, the techniques introduced in this work can be adapted to solve the integer quadrature problem for any positive discrete multi-parametric measure supported on \( \mathbb{N} \) under some mild conditions on the zeros of the associated orthogonal polynomials.

Key words. optimal threshold factors, strong stability preserving, Poisson-Charlier polynomials, spectral transformation, integer quadratures

AMS subject classifications. Primary, 65M12, 65M20; Secondary, 65L06, 65D32

1. Introduction. Many explicit numerical schemes for solving initial value problems, when applied to a linear system of \( s \geq 1 \) ordinary differential equations

\[
\frac{d}{dt}U(t) = AU(t), \quad t \geq 0, \quad U(0) = u_0,
\]

where \( A \) is a real \( s \times s \) matrix and \( u_0 \in \mathbb{R}^s \), reduce to a scheme of the type

\[
u_k = \phi(hA)u_{k-1}, \quad k = 1, 2, 3, \ldots ,
\]

where \( h > 0 \) is the step-size, \( u_k \) is an approximation to \( U(kh) \), and \( \phi \) is a polynomial with real coefficients which satisfies

\[
\phi(x) = \exp(x) + O(x^{n+1}) \quad \text{when} \quad x \to 0
\]

for an integer \( n \geq 1 \). The greatest integer \( n \) for which (3) holds is a measure for the local accuracy of the numerical scheme (2).

The matrix \( A \) in (1) and the polynomial \( \phi \) in (2) being given, it is natural to ask for the maximally allowable step-size \( h \) at which the numerical scheme (2) preserves a given property of the exact solution to (1). Prior to giving two prominent examples illustrating such situations, we recall a few definitions. A \( C^\infty \) function \( f \) is said to be absolutely monotonic over an interval \([a, b]\) if, for any \( x \in [a, b] \) and for any non-negative integer \( k \), \( f^{(k)}(x) \geq 0 \). Denote by \( \Pi_{m,n} \), with \( m \geq n \), the set of polynomials \( \phi \) of degree \( m \) (\( m \geq 1 \)) satisfying condition (3). The threshold factor, \( R(\phi) \), of a

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polynomial $\phi$ in $\Pi_{m,n}$ is defined as

$$R(\phi) = \sup\{r \mid r = 0 \text{ or } r > 0 \text{ and } \phi \text{ is absolutely monotonic over } [-r, 0]\}.$$ 

Now, let us assume that the matrix $A$ in (2) preserves positivity, i.e., for every initial value $u_0 \in \mathbb{R}^s$ such that $u_0 \geq 0$ we have $U(t) \geq 0$ for $t \geq 0$. Here and everywhere else the inequalities should be interpreted component-wise. It is well known that the matrix $A$ preserves positivity if and only if it is a Metzler matrix, i.e., the off diagonal elements of $A$ are non-negative [8]. Moreover, it is shown in [8] that, given a Metzler matrix $A$, if the step-size $h$ in (2) satisfies

$$h \leq \frac{R(\phi)}{\alpha} \quad \text{with} \quad \alpha = \max_{a_{ii} \leq 0} |a_{ii}|,$$

then the numerical scheme (2) preserves positivity in the sense that for any initial value $u_0 \geq 0$, we have $u_k \geq 0$ for any $k \geq 1$. Moreover, the quantity $R(\phi)/\alpha$ is the supremum of all the step-sizes that preserve positivity for any Metzler matrix with diagonal elements satisfying $a_{ii} \geq -\alpha$.

Another example where the threshold factor $R(\phi)$ appears naturally is when the matrix $A$ is dissipative with respect to a given norm $|\cdot|$ in $\mathbb{R}^s$ i.e., for any initial value $u_0 \in \mathbb{R}^s$, the solution to (1) satisfies $|U(t)| \leq |u_0|$ for any $t \geq 0$. It is well known that the set of dissipative matrices coincides with the set of matrices satisfying the so-called circle condition, i.e., there exists a positive real number $\beta$ such that $||A + \beta I|| \leq \beta$ where $||\cdot||$ stands for the matrix norm induced by $|$ and $I$ stands for the identity matrix [34]. It is shown in [34] that if the step-size $h$ in (2) satisfies

$$h \leq \frac{R(\phi)}{\beta},$$

then the numerical scheme (2) preserves contractivity in the sense that, for any initial value $u_0$, we have $|u_k| \leq |u_0|$ for any $k \geq 1$. Moreover, the quantity $R(\phi)/\beta$ is the supremum of all the step-sizes that preserve contractivity for any matrix satisfying the circle condition $||A + \beta I|| \leq \beta$.

In many practical situations, it is essential to have some flexibility in the choice of the step-size $h$ while ensuring the preservation of specific properties of the exact solution. In this respect, conditions (4) and (5) suggest to take in (2) the polynomial $\phi$ that maximizes the value of $R(\phi)$. This motivates the introduction of the optimal threshold factor $R_{m,n}$ defined as

$$R_{m,n} = \sup\{R(\phi) \mid \phi \in \Pi_{m,n}\}.$$ 

In [22] Kraaijevanger shows that $0 < R_{m,n} \leq m - n - 1$ and that there exists a unique polynomial $\Phi_{m,n}$ in $\Pi_{m,n}$, called the optimal threshold polynomial, such that

$$R(\Phi_{m,n}) = R_{m,n}.$$ 

Various studies have investigated the size of the optimal threshold factors for one-step, multi-stages methods [22, 38, 24] and one stage, multi-step methods [28, 29, 27]. Systems of the type (1) also appear in semi-discretization, discontinuous Galerkin semi-discretization or spectral semi-discretization of partial differential equations [9, 16, 30, 18, 17]. Optimally contractive schemes for solving these systems are important in so far as they prevent the growth of propagated errors. Methods with optimal
threshold factors can be viewed as the linear counterpart of the extensively studied subject of strong stability preserving (SSP) methods [14, 17, 19, 20, 25, 26].

In the present paper we study the size of the optimal threshold factors of multistage, one-step methods. To present our main results, we first recall that the generalized Laguerre polynomials \( L_n^{(\gamma)} \) are orthogonal on the interval \([0, \infty)\) with respect to the weight \( x^\gamma e^x \), that is,

\[
\int_0^\infty L_n^{(\gamma)}(x)L_m^{(\gamma)}(x)x^\gamma e^{-x}dx = 0, \quad \text{if } n \neq m. \tag{7}
\]

The integral in (7) converges only if \( \gamma > -1 \). The zeros of Laguerre polynomials are positive real numbers and throughout this work we will denote by \( \ell_n^{(\gamma)} \) the smallest zero of the polynomial \( L_n^{(\gamma)} \). Let us also recall that Poisson-Charlier polynomials \( C_n(., R) \) are orthogonal polynomials with respect to the discrete Poisson-Charlier measure \( \mu_R \) given by

\[
\mu_R = e^{-R} \sum_{j=0}^{\infty} \frac{R^j}{j!} \delta_j, \tag{8}
\]

where \( \delta_j \) is the Dirac measure. In Section 2 and Section 3 we re-visit the work of Kraaijevanger [22] with a new formalism that fits best our narrative. In Section 4 we establish a connection between the task of computing the optimal threshold factors and the notion of positive quadratures with integer nodes with respect to Poisson-Charlier measures. This connection leads to our first main result.

**Theorem 1.** For any positive integers \( m \) and \( p \) such that \( m \geq 2p-1 \)

\[
R_{m,2p-1} \leq \ell_p^{(m-p)}, \tag{9}
\]

with equality if and only if the zeros of the Poisson-Charlier polynomial \( C_p(., \ell_p^{(m-p)}) \) are integers.

Table 1 shows some of the exact values of the optimal threshold factors \( R_{m,n} \) and the upper bound obtained in Theorem 1. The quality of the upper bound (9) is rather remarkable and surprising. An attempt at finding an equally satisfying lower bound

| \( \ell_p^{(m-2p+1)} \) | \( R_{m,2p-1} \) | \( \ell_p^{(m-p)} \) |
|---|---|---|
| 11.0108 | \( R_{20,5} = 12.5512 \) | 12.6118 |
| 19.1884 | \( R_{30,5} = 20.8355 \) | 20.8659 |
| 9.7026 | \( R_{22,7} = 11.8435 \) | 11.9237 |
| 23.6589 | \( R_{40,7} = 26.0713 \) | 26.0927 |
| 44.4670 | \( R_{65,7} = 47.0065 \) | 47.0267 |
| 3.6304 | \( R_{16,9} = 5.9337 \) | 6.0762 |
| 41.7638 | \( R_{67,9} = 45.0148 \) | 45.0533 |
| 10.5254 | \( R_{30,11} = 13.8617 \) | 13.9257 |
| 17.4568 | \( R_{40,11} = 21.0411 \) | 21.0911 |

**Table 1**

Upper and lower bounds for the optimal threshold factor \( R_{m,2p-1} \).
Theorem 2. For any positive integers \(m\) and \(p\) such that \(m \geq 2p - 1\)

\[
R_{m,2p-1} \geq \ell_p^{(m-2p+1)}.
\] (10)

Although the lower bound given in (10) is sharp (we have equality in (10) when \(p = 1\)), it is not as impressive as the upper bound obtained in (9) (see Table 1). Nevertheless, we give strong evidences of possible improvements of this lower bound. More precisely, we show that

\[
R_{m,2p-1} \geq \ell_p^{(m-p-\tau_p)},
\]

where

\[
\tau_p := \sup_{R > 0} \sup_{P \in \mathcal{C}_p} \left( \frac{\int tP(t)d\mu_R(t)}{\int P(t)d\mu_R(t)} - \lambda_{p,p}(R) \right),
\]

where \(\lambda_{p,p}(R)\) is the largest zero of the Poisson-Charlier polynomial \(C_p(.,R)\) and \(\mathcal{C}_p\) is the set of non-zero real polynomials of degree at most \(2p - 2\) that are non-negative on \(\mathbb{Z}\). The lower bound (10) is obtained by showing that \(\tau_p \leq p - 1\).

In [22] it is shown that the \(R_{m,n}\)-table of the optimal threshold factors enjoys a remarkable property of stabilization along the diagonal, i.e., for any non-negative \(d\), there exists an integer \(p = p(d)\) such that

\[
R_{p+d+k,p+k} = R_{p+d,p}, \quad \text{for all} \quad k \geq 1.
\] (11)

In Section 6, we adapt and extend an ingenious technique by Bernstein in his seminal work [7] to identify a structural property of the optimal threshold polynomials. This leads to the following important property of the optimal threshold factors that, in some sense, complements the diagonal stability property (11).

Theorem 3. The optimal threshold factors \(R_{m,n}\) \((m \geq n \geq 1)\) are algebraic numbers such that

\[
R_{m+1,2p} = R_{m,2p-1},
\]

for any positive integers \(m\) and \(p\) such that \(m \geq 2p - 1\). Moreover, the associated optimal threshold polynomials satisfy the relation

\[
\Phi_{m+1,2p}(x) = 1 + \int_0^x \Phi_{m,2p-1}(\xi)d\xi.
\] (13)

Note that (12) and (13) assert that it is enough to compute \(R_{m,n}\) and \(\Phi_{m,n}\) for odd integers \(n\) to obtain the whole \(R_{m,n}\)-table of optimal threshold factors and their associated optimal threshold polynomials. This is an essential property that will prove extremely useful in this work, as we shall reveal that it is more natural to study the optimal threshold factors \(R_{m,n}\) with odd integers \(n\) than with even integers \(n\).

The structural property of the optimal threshold polynomial asserts the following fundamental result.

Theorem 4. For any positive integers \(m\) and \(p\) such that \(m \geq 2p - 1\), the optimal threshold polynomial \(\Phi_{m,2p-1}\) has the form

\[
\Phi_{m,2p-1}(x) = \sum_{k=1}^{2p-1} a_k \left( 1 + \frac{x}{R_{m,2p-1}} \right)^{m_k},
\] (14)
where $\alpha_k, k = 1, 2, \ldots, 2p - 1$ are non-negative real numbers with $\alpha_{2p-1} > 0$ and the integers $0 \leq m_1 < m_2 < \ldots < m_{2p-1} \leq m$ satisfy
\begin{equation}
(15) \quad m_{2k} = m_{2k-1} + 1, \quad k = 1, 2, \ldots, p - 1 \quad \text{and} \quad m_{2p-1} = m.
\end{equation}

The structure (14) is further analyzed in Section 7 to reveal a set of rigid rules on the allowable values of the integers $m_i, i = 1, \ldots, 2p - 1$, in (15). This is achieved through a comprehensive study of specific spectral transformations of Charlier-Poisson measures. Our analysis leads to a highly efficient and stable algorithm for computing the optimal threshold factors and their associated optimal polynomials via adaptive spectral transformations of Poisson-Charlier measures. **The algorithm has the particularity that its complexity depends only on the order of approximation and not of the degree of the polynomials** and will be described in Section 8. To put into perspective the importance of the complexity of our algorithm, we compared the execution time of our algorithm with a recent algorithm in [24] (within the same computational environment). For instance, the computation of $R_{2000,7}$ took 1 hour 30 minutes with the algorithm described in [24], while it took 1.2 seconds with ours. Increasing the degree of the underlying polynomials, we found that the algorithm in [24] took about 12 hours 30 minutes for the computation of $R_{4000,7}$, while it took 0.58 seconds with ours. Moreover, our algorithm gives the exact values of the optimal threshold factors in the sense that $R_{m,n}$ is given as the zero, in a prescribed interval, of a polynomial of degree $n$ with integer coefficients that are computed exactly. Thus, the precision on the computed optimal threshold factors depends only on the selected root-finding algorithm. We conclude with future work in Section 9.

2. **Touchard Polynomials and optimal threshold factors.** Denote by $(x)_h$ the Pochhammer symbol, i.e., $(x)_h = x(x-1)\ldots(x-h+1)$ for $h \geq 1$ and $(x)_0 = 1$. The Stirling numbers, $s(n,k)$, of the first kind and the Stirling numbers, $S(n,k)$, of the second kind are defined as the coefficients in the expansions
\begin{equation}
(16) \quad (x)_n = \sum_{k=0}^{n} s(n,k)x^k; \quad x^n = \sum_{k=0}^{n} S(n,k)(x)_k, \quad n \geq 0; \text{ for any } x \in \mathbb{R}.
\end{equation}

The univariate Touchard\textsuperscript{1} polynomials $B_n$ are defined by $B_n(x) = \sum_{k=0}^{n} S(n,k)x^k$ and satisfy the recurrence
\begin{equation}
(17) \quad B_0(x) = 1, \quad B_{n+1}(x) = x \left( B_n(x) + B_n^{(1)}(x) \right),
\end{equation}
where the notation $F^{(k)}$ refers to the $k^{th}$ derivative of the function $F$. The following useful relations hold:
\begin{equation}
(18) \quad \sum_{k=0}^{n} s(n,k)B_k(x) = x^n; \quad \sum_{k=0}^{n} S(n,k)x^k = B_n(x).
\end{equation}

For a fixed real number $R$, we define the polynomials $\mathcal{H}_n(\cdot; R)$ by
\begin{equation}
(19) \quad \mathcal{H}_n(x; R) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} B_{n-k}(R)x^k.
\end{equation}

The following result is implicit in [22], however for the sake of completeness and also due to the difference between our presentation and the one in [22], we provide a proof for the result.

\textsuperscript{1}These polynomials were called Stirling polynomials in [22].
Proposition 5. Let $m$ and $n$ be two positive integers such that $m \geq n$ and let $R$ be a positive real number. The polynomial $H_n(\cdot; R)$ admits a representation of the form

$$H_n(x; R) = \sum_{i=1}^{s} \alpha_i (x - m_i)^n, \quad s \geq 1,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_s$ are non-negative numbers and where the integers $m_1, m_2, \ldots, m_s$ are such that $0 \leq m_1 < m_2 < \ldots < m_s \leq m$ if and only if the polynomial

$$\Phi(x) = \sum_{i=1}^{s} \alpha_i \left(1 + \frac{x}{R}\right)^{m_i}$$

is of degree at most $m$, is absolutely monotonic over the interval $[-R, 0]$ and it satisfies $\Phi(x) - e^x = O(x^{n+1})$ as $x \to 0$.

Proof. Let us assume that the polynomial $H_n(\cdot; R)$ admits a representation of the form (20). We have

$$\Phi^{(\ell)}(x) = \sum_{i=1}^{s} \alpha_i (m_i)_{\ell} R^{-\ell} \left(1 + \frac{x}{R}\right)^{m_i - \ell}.$$ 

Thus, $\Phi^{(\ell)}(-R) = 0$ if $\ell \notin \{m_1, m_2, \ldots, m_s\}$ and $\Phi^{(m_i)}(-R) = \alpha_i (m_i)_{m_i}/R^{m_i} \geq 0$ for $i = 1, 2, \ldots, s$. Therefore, the polynomial $\Phi$ is absolutely monotonic at $-R$ and hence is absolutely monotonic over the interval $[-R, 0]$ [23, see Lemma 4.3 in]. Moreover, from (19) and (20) we have $\sum_{i=1}^{s} \alpha_i m_i^{j} = B_j(R)$ for $j = 0, 1, \ldots, n$. Thus, using (16) and (18), we obtain for $\ell = 0, 1, \ldots, n$,

$$\Phi^{(\ell)}(0) = \sum_{i=1}^{s} \alpha_i (m_i)_{\ell} R^{-\ell} = \sum_{j=1}^{\ell} s(\ell, j) \sum_{i=1}^{s} \alpha_i m_i^{j} = \sum_{j=1}^{\ell} s(\ell, j) B_j(R) = 1. $$ 

Therefore, we have $\Phi(x) - e^x = O(x^{n+1})$ as $x \to 0$. Conversely, given real numbers $\alpha_1, \ldots, \alpha_s$ and given integers $0 \leq m_1 < m_2 < \ldots < m_s \leq m$, assume that the corresponding polynomial $\Phi$ in (21) is absolutely monotonic over $[-R, 0]$ and that it satisfies $\Phi(x) - e^x = O(x^{n+1})$ as $x \to 0$. Then necessarily the coefficients $\alpha_i, i = 1, 2, \ldots, s$ are non-negative. Denote by $A_n(\cdot; R)$ the polynomial $A_n(x; R) = \sum_{i=1}^{s} \alpha_i (x - m_i)^n$. Using (16) the coefficient $a_j$ attached to the monomial $x^{n-j}$ of the polynomial $A_n(\cdot; R)$ is given by

$$a_j = (-1)^j \binom{n}{j} \sum_{i=1}^{s} \alpha_i m_i^{j} = (-1)^j \binom{n}{j} \sum_{\ell=0}^{j} S(j, \ell) \sum_{i=1}^{s} \alpha_i (m_i)_{\ell}. $$

According to (22), we have $\sum_{i=1}^{s} \alpha_i (m_i)_{\ell} = R^\ell$. Thus, form (18) we obtain

$$a_j = (-1)^j \binom{n}{j} \sum_{\ell=0}^{j} S(j, \ell) R^\ell = (-1)^j \binom{n}{j} B_j(R).$$

Therefore, the coefficient $a_j$ coincide with the coefficient of $x^{n-j}$ of the polynomial $H_n(\cdot; R)$ given in (19). Hence, the polynomials $A_n(\cdot; R)$ and $H_n(\cdot; R)$ coincide. 

From the previous proposition, the optimal threshold factor $R_{m,n}$ defined in (6) can also be characterized as follows.
Corollary 6. Let \( m \) and \( n \) be positive integers such that \( m \geq n \). The optimal threshold factor \( R_{m,n} \) is the maximum of positive real numbers \( R \) for which the polynomial \( \mathcal{H}_n(x; R) \) admits a representation of the form

\[
\mathcal{H}_n(x; R) = \sum_{i=1}^{s} \alpha_i (x - m_i)^n, \quad s \geq 1,
\]

with integers \( 0 \leq m_1 < m_2 < \ldots < m_s \leq m \) and non-negative real numbers \( \alpha_1, \alpha_2, \ldots, \alpha_s \).

In [22] Kraaijevanger showed that the optimal threshold polynomial \( \Phi_{m,n} \) satisfies the property that at least \((m - n + 1)\) numbers of the sequence \( \{\Phi_{m,n}(-R_{m,n})\}_{k=0}^{m} \) vanish. In terms of the polynomial \( \mathcal{H}_n(x; R_{m,n}) \) this claim can be re-stated as saying that for \( R = R_{m,n} \) the number of summands in the right-hand side of (23) is at most \( n \). More precisely, we have the following theorem.

Theorem 7. For any positive integers \( m \) and \( n \) such that \( m \geq n \), there exist integers \( 0 \leq m_1 < m_2 < \ldots < m_n \leq m \) and non-negative real numbers \( \alpha_1, \alpha_2, \ldots, \alpha_n \) such that

\[
\mathcal{H}_n(x; R_{m,n}) = \sum_{i=1}^{n} \alpha_i (x - m_i)^n.
\]

Proof. Let us assume that \( \mathcal{H}_n(x; R_{m,n}) = \sum_{i=1}^{s} \alpha_i (x - m_i)^n \) with \( s > n \) and \( \alpha_i > 0, i = 1, 2, \ldots, s \). Write

\[
\sum_{i=1}^{n+1} \alpha_i (x - m_i)^n = \mathcal{H}_n(x; R_{m,n}) - \sum_{i=n+2}^{s} \alpha_i (x - m_i)^n.
\]

Equation (24) can be viewed as a linear system in \( (\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) \), i.e;

\[
\sum_{i=1}^{n+1} \alpha_i m_i^j = B_j(R_{m,n}) - \sum_{i=n+2}^{s} \alpha_i m_i^j, \quad j = 0, 1, \ldots, n,
\]

that has a positive solution, i.e., \( \alpha_i > 0 \) for \( i = 1, 2, \ldots, n+1 \). Therefore, there exists an \( \epsilon > 0 \) such that the regular linear system in \( (\beta_1, \beta_2, \ldots, \beta_{n+1}) \)

\[
\sum_{i=1}^{n+1} \beta_i (x - m_i)^n = \mathcal{H}_n(x; R_{m,n} + \epsilon) - \sum_{i=n+2}^{s} \alpha_i (x - m_i)^n
\]

also has a positive solution. Thus, we obtain

\[
\mathcal{H}_n(x; R_{m,n} + \epsilon) = \sum_{i=1}^{n+1} \beta_i (x - m_i)^n + \sum_{i=n+2}^{s} \alpha_i (x - m_i)^n.
\]

This contradicts the definition of \( R_{m,n} \) as given in Corollary 6.

3. Polar forms and Kraaijevanger’s algorithm. Polar forms (or blossoms) for polynomials [32] are crucial tools in various mathematical areas [1, 2, 3, 4]. They will prove helpful, even essential, at several places in this work. In the present section,
after a brief reminder of their definition, we will use them to describe the algorithm proposed by Kraaijevanger [22] for computing the optimal threshold factors.

**Notation:** Throughout the article, for any real number \( x \) and any non-negative integer \( k \), \( x^{[k]} \) will stand for \( x \) repeated \( k \) times.

**Definition 8.** Given a real polynomial \( P \) of degree at most \( n \), there exists a unique symmetric multi-affine function \( p(u_1, u_2, \ldots, u_n) \) such that \( p(x^{[n]}) = P(x) \) for any \( x \in \mathbb{R} \). The function \( p \) is called the blossom or the polar form of the polynomial \( P \).

The polar form of a polynomial \( P \) expressed in the monomial basis as \( P(x) = \sum_{k=0}^{n} a_k x^k \) is given by

\[
p(u_1, u_2, \ldots, u_n) = \sum_{k=0}^{n} a_k \sigma_k(u_1, u_2, \ldots, u_n),
\]

where \( \sigma_k \) refers to the normalized \( k \)-th elementary symmetric polynomial, i.e.,

\[
\sigma_k(u_1, u_2, \ldots, u_n) = \binom{n}{k}^{-1} \sum_{1 \leq j_1 < \ldots < j_k \leq n} u_{j_1} u_{j_2} \ldots u_{j_k}.
\]

Of special interest within this work are polynomials of the form

\[
P(x) = \sum_{k=1}^{s} \alpha_k (x - a_k)^n.
\]

Their polar forms are simply given by

\[
(25) \quad p(u_1, u_2, \ldots, u_n) = \sum_{k=1}^{s} \alpha_k \prod_{i=1}^{n} (u_i - a_k).
\]

We shall need the following proposition.

**Proposition 9.** Let \( P \) be a real polynomial of degree at most \( n \) and \( p \) its polar form. Given any pairwise distinct real numbers \( \xi_1, \ldots, \xi_k \), we have

\[
(26) \quad p(\xi_1, \xi_2, \ldots, \xi_k, x^{[n-k]}) = 0 \quad \text{for any} \quad x \in \mathbb{R},
\]

if and only if the polynomial \( P \) can be written in the form \( P(x) = \sum_{j=1}^{k} \alpha_j (x - \xi_j)^n \).

**Proof.** The function \( \tilde{P}(x) := p(\xi_1, \ldots, \xi_k, x^{[n-k]}) \) is a polynomial of degree at most \( n - k \). Select any pairwise distinct \( \xi_{k+1}, \ldots, \xi_n \) in \( \mathbb{R} \setminus \{\xi_1, \ldots, \xi_k\} \). Let us expand \( P \) as \( P(x) = A + \sum_{i=1}^{n} \alpha_i (x - \xi_i)^n \). Then, from (26) we obtain

\[
\tilde{P}(x) = A + \sum_{i=k+1}^{n} \beta_i (x - \xi_i)^{n-k}, \quad x \in \mathbb{R},
\]

with \( \beta_i := \alpha_i \prod_{j=1}^{k} (\xi_j - \xi_i), i = k + 1, \ldots, n \). Accordingly, the polynomial \( \tilde{P} \) is identically zero if and only all coefficients \( A, \beta_{k+1}, \ldots, \beta_n \) are zero, that is, if and only if \( A \) and \( \alpha_{k+1}, \ldots, \alpha_n \) are zero. The claim is proved. \( \square \)

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Now, we are in a position to describe the algorithm of Kraaijenvanger for computing the optimal threshold factor \( R_{m,n} \) and the associated polynomial \( \Phi_{m,n} \). If we write the polynomial \( H_n(;R) \) defined in (19) in the form

\[
H_n(x; R) = \sum_{i=1}^{n} \alpha_i (x - m_i)^n,
\]

then, by denoting \( h_n(u_1, u_2, \ldots, u_n; R) \) the value at \((u_1, u_2, \ldots, u_n)\) of the polar form of the polynomial \( H_n(;R) \) and applying (25), we obtain

\[
h_n(m_1, m_2, \ldots, m_n; R) = 0.
\]

Moreover, evaluating the polar form of both sides of (27) at \((m_1, m_2, \ldots, m_k-1, m + 1, m_{k+1}, \ldots, m_n)\) yields

\[
\alpha_k = \frac{h_n(m_1, m_2, \ldots, m_{k-1}, m + 1, m_{k+1}, \ldots, m_n; R)}{(m + 1 - m_k) \prod_{i=1, i \neq k} (m_i - m_k)}, \quad k = 1, 2, \ldots, n.
\]

Based on (28) and (29), an algorithm for computing \( R_{m,n} \) goes as follows:

- **Step 1**: Generate all integer sequences \( M = (m_1, m_2, \ldots, m_n) \) such that \( 0 \leq m_1 < m_2 < \ldots < m_n \leq m \). For each such sequence, find the positive numbers \( R \) satisfying (28) (if any). Note that for each such integer sequence \( M \), Equation (28) is a polynomial equation of degree \( n \) in \( R \).

- **Step 2**: For each of the real numbers \( R \) found in Step 1, check the non-negativity of the coefficients \( \alpha_k \) using equations (29). Retain the numbers \( R \) and the associated sequences \( M \) for which all the coefficients \( \alpha_k \) are non-negative.

- **Step 3**: \( R_{m,n} \) is the maximum of all the values \( R \) that survived elimination from Step 2. The optimal threshold polynomial is then given by

\[
\Phi_{m,n}(x) = \sum_{k=1}^{n} \alpha_k \left(\frac{x}{R_{m,n}}\right)^{m_k},
\]

where \((m_1, m_2, \ldots, m_n)\) the integer sequence associated with \( R_{m,n} \) and \( \alpha_k \) are the coefficients that were already computed using (29).

Evidently, the computational cost of the above algorithm grows exponentially in \( m \) and \( n \) and could only be used to compute the optimal threshold factors for small values of \( m \) and \( n \). As will be clear later, the above algorithm can be substantially improved by our results of Section 6 where we identify a structural property of the optimal threshold polynomials that considerably reduces the number of integer sequences \( M = (m_1, m_2, \ldots, m_n) \) to be considered in Step 1 of the algorithm. We will have further comments on this aspect of the algorithm but we should stress that in Section 8, we propose a highly efficient algorithm for the computation of \( R_{m,n} \) whose computational cost is independent of the integer \( m \). We would like to mention that a method of computing \( R_{m,n} \) and the associated optimal threshold polynomial based on linear programming is presented in [24]. However, the algorithm in question suffers from stability problems for large value of the integer \( m \).
4. Poisson-Charlier orthogonal polynomials and sharp upper bounds for the optimal threshold factors. In this section, we give a connection between Poisson-Charlier orthogonal polynomials and the polynomials \( H_n(\cdot; R) \) defined in (19). This will enable us to give a sharp upper bound for the optimal threshold factors in terms of the smallest zero of generalized Laguerre polynomials.

The monic Poisson-Charlier polynomials \( C_n(\cdot; R) \) are orthogonal with respect to the discrete Poisson-Charlier measure (8). Thus, they satisfy the orthogonality relations \[ \int C_n(t, R)C_m(t, R)d\mu_R(t) = \sum_{j=0}^{\infty} C_n(j, R)C_m(j, R)e^{-Rj}R_j^j = n!R_n\delta_{nm}. \]

The Poisson-Charlier polynomials satisfy the three-term recurrence relation (30) \[ tC_n(t, R) = C_{n+1}(t, R) + (R + n)C_n(t, R) + nRC_{n-1}(t, R). \]

It is well known that the moments of Poisson-Charlier measures are Touchard polynomials. However, as we were not able to find a reference for a proof of this fact, we include a simple one for the readers convenience.

**Proposition 10.** For any non-negative integer \( n \) and positive number \( R \), the following relations hold

\[ t^n = \sum_{j=0}^{n} \frac{B_j(R)}{j!} C_j(t, R) \quad \text{and} \quad \int t^n d\mu_R(t) = B_n(R). \]

**Proof.** The proof of the left identity in (31) proceeds by induction on the integer \( n \). The identity being trivial for \( n = 0 \), assume that it holds for any \( k \leq n \). The three-term recurrence relation (30) yields

\[ t^{n+1} = \sum_{j=0}^{n} \frac{B_j(R)}{j!} tC_j(t, R) = \sum_{j=0}^{n+1} a_j C_j(t, R), \]

with

\[ a_j = \frac{B_{j-1}(R)}{(j-1)!} + (R + j)\frac{B_{j}(R)}{j!} + (j + 1)R\frac{B_{j+1}(R)}{(j+1)!} \quad \text{for} \quad j \geq 0, \quad \text{with} \quad B_{-1} = 0. \]

Form the recurrence equation (17) of Touchard polynomials, we have

\[ B_{n+1}^{(j)}(R) = jB_{n}^{(j-1)}(R) + (j + R)B_{n}^{(j)}(R) + RB_{n}^{(j+1)}(R). \]

Thus, for \( j = 0, 1, \ldots, n \), \( a_j = \frac{B_{j+1}(R)}{j!} \). The right identity in (31) is a direct consequence of the left identity and of the orthogonality of the Poisson-Charlier polynomials. This concludes the proof.

**Corollary 11.** For any positive integer \( p \), the polynomials \( \mathcal{H}_{2p-1}(\cdot; R) \) defined in (19) can be expressed as

\[ \mathcal{H}_{2p-1}(x; R) = \int (x - t)^{2p-1} d\mu_R(t) = \sum_{k=1}^{p} \omega_k(x - \lambda_{k,p})^{2p-1}, \]
where \( \lambda_{1,p} < \lambda_{2,p} < \ldots < \lambda_{p,p} \) are the zeros of the Poisson-Charlier polynomial \( C_p(\cdot; R) \) and \((\omega_1, \omega_2, \ldots, \omega_p)\) are the positive weights of the \( p \)-point Gaussian quadrature with respect to the measure \( \mu_R \).

**Proof.** The first identity in (32) is valid when we replace \( 2p - 1 \) by any integer \( n \). Indeed, according to Proposition 10, we have

\[
\int (x-t)^n d\mu_R(t) = \sum_{k=0}^{n} (-1)^{n-k} \left( \frac{n}{k} \right) x^k \int t^{n-k} d\mu_R(t) = \sum_{k=0}^{n} (-1)^{n-k} \left( \frac{n}{k} \right) x^k B_{n-k}(R) = \mathcal{H}_n(x; R).
\]

The second identity in (32) is nothing but the \( p \)-point Gaussian quadrature with respect to the measure \( \mu_R \) applied to the polynomial \( P(x) = \int (x-t)^{2p-1} d\mu_R(t) \).

**Remark 12.** Writing \( \mathcal{H}_n(x; R) = \sum_{i=1}^{s} \alpha_i (x-m_i)^n \) is equivalent to saying that for any polynomial \( P \) of degree at most \( n \) we have

\[
\int P(t) d\mu_R(t) = \sum_{i=1}^{s} \alpha_i P(m_i).
\]

Therefore, according to Corollary 6, the optimal threshold factor \( R_{m,n} \) is the maximum of the real numbers \( R \) for which the corresponding Poisson-Charlie measure \( \mu_R \) admits a positive quadrature with integer nodes that are smaller or equal to \( m \) and which is exact for polynomials of degree at most \( n \).

We shall need the following proposition.

**Proposition 13.** Let us assume that the polynomial \( \mathcal{H}_{2p-1}(\cdot; R) \) is written as

\[
\mathcal{H}_{2p-1}(x; R) = \sum_{k=1}^{s} \beta_k (x-\rho_k)^{2p-1},
\]

with \( s \geq p \), \( 0 \leq \rho_1 < \rho_2 < \ldots < \rho_s \) and \( \beta_j > 0 \), \( j = 1, 2, \ldots, s \). Let \( \lambda_{p,p} \) the largest zero of the Poisson-Charlier polynomial \( C_p(\cdot; R) \). Then \( \rho_s \geq \lambda_{p,p} \) with equality if and only if \( s = p \) and the representation (33) coincides with the one in (32). Moreover, \( \rho_1 \leq \lambda_{1,p} \) where \( \lambda_{1,p} \) is the smallest zero of \( C_p(\cdot; R) \).

**Proof.** Denote by \( h_{2p-1}(\cdot; R) \) the polar form of the polynomial \( \mathcal{H}_{2p-1}(\cdot; R) \) and by \( \Lambda \) the finite sequence \( \Lambda = (\lambda_{1,p}, \lambda_{2,p}, \ldots, \lambda_{p-1,p}, \lambda_{1,p}, \lambda_{2,p}, \ldots, \lambda_{p-1,p}, \lambda_{p,p}) \) where \( \lambda_{1,p} < \lambda_{2,p} < \ldots < \lambda_{p,p} \) are the zeros of the Poisson-Charlier polynomial \( C_p(\cdot; R) \). From (32), we have

\[
h_{2p-1}(\Lambda; R) = 0.
\]

Moreover, from (33) we have

\[
h_{2p-1}(\Lambda; R) = \sum_{k=1}^{s} \beta_k (\lambda_{1,p} - \rho_k)^2 \ldots (\lambda_{p-1,p} - \rho_k)^2 (\lambda_{p,p} - \rho_k) = 0.
\]

Therefore, there exists an integer \( k \in \{1, 2, \ldots, s\} \) such that \( \rho_k \geq \lambda_{p,p} \). In particular, we have \( \rho_s \geq \lambda_{p,p} \). If \( \rho_s = \lambda_{p,p} \) then from (34) we remark that for \( k = 1, 2, \ldots, s-1, \)
(\rho_k - \lambda_{1,p}) \ldots (\rho_k - \lambda_{p-1,p}) = 0. In other words, \rho_1, \rho_2, \ldots, \rho_{s-1} are zeros of the polynomial \psi(x) = (x - \lambda_{1,p})(x - \lambda_{2,p}) \ldots (x - \lambda_{p-1,p}). Thus, we necessarily have s-1 = p-1 and \rho_k = \lambda_{k,p} for k = 1, 2, \ldots, p-1. Similarly, to prove that \lambda_{1,p} \geq \rho_1 we proceed as follows: Define by \Lambda_1 the finite sequence \Lambda_1 = (\lambda_{2,p}, \ldots, \lambda_{p,p}, \lambda_{2,p}, \ldots, \lambda_{p,p}, \lambda_{1,p}). From (32), we have

\[ h_{2p-1}(\Lambda_1; R) = 0. \]

Moreover, from (33) we have

\[ h_{2p-1}(\Lambda_1; R) = \sum_{k=1}^{s} \beta_k (\lambda_{2,p} - \rho_k)^2 \ldots (\lambda_{p,p} - \rho_k)^2 (\lambda_{1,p} - \rho_k) = 0. \]

Therefore, there exists an integer k \in \{1, 2, \ldots, s\} such that \rho_k \leq \lambda_{1,p}. In particular, we have \rho_1 \leq \lambda_{1,p}. □

**Corollary 14.** Let R_{max} be the unique real number such that the largest zero of the Poisson-Charlier polynomial \( C_p(., R_{max}) \) is equal to m. Then \( R_{m,2p-1} \leq R_{max} \) with equality if and only if all the zeros of \( C_p(., R_{max}) \) are integers.

**Proof.** For a real number R, let us denote by \lambda_{p,p}(R) the largest zero of the Poisson-Charlier polynomial \( C_p(., R) \). From the definition of \( R_{m,2p-1} \), there exist non-negative integers \( 0 \leq m_1 < m_2 < \ldots < m_s \leq m \) such that

\[ \mathcal{H}_{2p-1}(x; R_{m,2p-1}) = \sum_{k=1}^{s} \beta_k (x - m_k)^{2p-1} \]

with \( \beta_k > 0 \) for \( k = 1, 2, \ldots, s \) (s \leq 2p-1). If \( R_{m,2p-1} > R_{max} \) and since the zeros of Poisson-Charlier polynomials are strictly increasing functions of the parameter R (see [6]) , we deduce that \( \lambda_{p,p}(R_{m,2p-1}) > \lambda_{p,p}(R_{max}) = m \). However, from Proposition 13, we have \( m_s \geq \lambda_{p,p}(R_{m,2p-1}) \). Thus, we obtain \( m_s > m \) contradicting our initial assumption on \( m_s \). Moreover, from Proposition 13, \( \lambda_{p,p}(R_{m,2p-1}) = \lambda_{p,p}(R_{max}) \) or equivalently \( R_{m,2p-1} = R_{max} \) if and only if the two representations (35) and (32) coincide, or equivalently the zeros of the polynomial \( C_p(., R_{max}) \) are integers. □

**Example of Application:** Corollary 14 shows that, if for a positive real number R the zeros \( \lambda_{1,p} < \lambda_{2,p} < \ldots < \lambda_{p,p} \) of the polynomial \( C_p(., R) \) are integers then

\[ R_{\lambda_{p,p},2p-1} = R \quad \text{and} \quad \Phi_{\lambda_{p,p},2p-1}(x) = \sum_{k=1}^{p} \omega_k \left( 1 + \frac{x}{R_{\lambda_{p,p},2p-1}} \right)^{\lambda_{k,p}}, \]

where \( \omega_k, k = 1, 2, \ldots, p \) are the weights of the p-point Gaussian quadrature with respect to the measure \( \mu_{R_{\lambda_{p,p},2p-1}} \). As an application, we now prove the following theorem which was derived in [22] using a completely different method.

**Theorem 15.** For any integer \( m \geq 1 \), we have \( R_{m,1} = m \) and

\[ \Phi_{m,1}(x) = \left( 1 + \frac{x}{m} \right)^{m}. \]

For any square integer \( m \geq 3 \), we have \( R_{m,3} = m - \sqrt{m} \) and

\[ \Phi_{m,3}(x) = \frac{\sqrt{m}}{2\sqrt{m} - 1} \left( 1 + \frac{x}{m - \sqrt{m}} \right)^{m - 2\sqrt{m} + 1} + \frac{\sqrt{m} - 1}{2\sqrt{m} - 1} \left( 1 + \frac{x}{m - \sqrt{m}} \right)^{m}. \]
Proof. For any non-negative integer \( m \), we have \( C_1(t, m) = t - m \). Thus, according to (36) with \( \lambda_{1,1} = m \), we have \( R_{m,1} = m \). The expression of \( \Phi_{m,1} \) in (37) is a direct consequence (36). The degree 2 Poisson-Charlier polynomial is given by \( C_2(t, R) = t^2 - (2R + 1)t + R^2 \). Thus, for \( R = m - \sqrt{m} \) with \( m \geq 3 \) is a square integer, we have

\[
C_2(t, R) = t^2 - (2(m - \sqrt{m}) + 1)t + (m - \sqrt{m})^2 = (t - (m - 2\sqrt{m} + 1))(t - m).
\]

Thus, for these specific values of the parameter \( R \), the zeros of \( C_2(., R) \) are integers with \( m \) as the largest one. Therefore, according to (36), we have \( R_{m,3} = m - \sqrt{m} \). The expression of \( \Phi_{m,3} \) is a direct consequence of (36) once the weights of the 2-point Gaussian quadrature with respect to \( \mu_{m - \sqrt{m}} \) are computed explicitly. \( \square \)

Remark 16. It is an interesting problem to find all the real numbers \( R \) and positive integers \( p \) for which all the zeros of the Poisson-Charlier polynomial \( C_p(., R) \) are integers. For these cases, the optimal threshold factors and their associated optimal polynomials are easily computed via Gaussian quadratures. It may be possible that the only cases for which all the zeros of \( C_p(., R) \) are integers are actually the cases already cited in Theorem 15.

We are now in a position to prove Theorem 1 (see Introduction).

**Proof of Theorem 1:** As is well known, the Poisson-Charlier polynomials are linked to the generalized Laguerre polynomials [37] via the relation

\[
C_p(x, R) = p! L_p^{(x-p)}(R).
\]

From Corollary 14, \( R_{m,2p-1} \leq R_{\text{max}} \) where \( R_{\text{max}} \) is the unique real number for which the largest zero of \( C_p(., R_{\text{max}}) \) is equal to \( m \). In other words, and taking into account that the zeros of \( C_p(., R) \) are increasing functions on the parameter \( R \), \( R_{\text{max}} \) is the smallest real number satisfying \( C_p(m, R_{\text{max}}) = 0 \). Thus, due to (38), \( R_{\text{max}} \) is the smallest real number such that \( L_p^{(m-p)}(R) = 0 \), i.e., \( R_{\text{max}} = L_p^{(m-p)} \). This shows inequality (9). The claim about equality in (9) stated in Theorem 1 is a direct consequence of Corollary 14.

5. Lower bounds for the optimal threshold factors. The good quality of the sharp upper bound (9) to the optimal threshold factor \( R_{m,2p-1} \) (see Table 1) suggests the possibility of finding an equally satisfying lower bound for \( R_{m,2p-1} \). This section is an attempt to finding such lower bounds. The results of this section are based on the following characterization of the optimal threshold factors.

**Theorem 17.** Let \( R \) be a positive real number and \( m \geq n \) be two positive integers. Then \( R_{m,n} \geq R \) if and only if, for any polynomial \( f \) of degree at most \( n \) such that \( f(j) \geq 0 \) for \( j = 0,1,\ldots,m \), we have

\[
\int f(t)d\mu_R(t) \geq 0.
\]

**Proof.** Let us assume that \( R_{m,n} \geq R \). According to Corollary 6, the polynomial \( H_n(., R) \) has a representation of the form

\[
H_n(x; R) = \sum_{k=1}^s \beta_k(x - m_k)^n,
\]
with $0 \leq m_1 < m_2 < \ldots < m_s \leq m$ and $\beta_k \geq 0$ for $k = 0, 1, \ldots, s$. In other words, for any polynomial $f$ of degree at most $n$, we have

$$ \int f(t) d\mu_R(t) = \sum_{k=1}^{s} \beta_k f(m_k). $$

In particular, if the polynomial $f$ is such that $f(j) \geq 0$ for $j = 0, 1, \ldots, m$, then by (39), $\int f(t) d\mu_R(t) \geq 0$. Conversely, let us assume that, for any polynomial $f$ of degree at most $n$ such that $f(j) \geq 0$ for $j = 0, 1, \ldots, m$, we have $\int f(t) d\mu_R(t) \geq 0$. Let us write $f$ as $f(t) = \sum_{k=0}^{n} \gamma_k t^k$. Set $\gamma := (\gamma_0, \gamma_1, \ldots, \gamma_n)^T$ and by $A$ the $(n+1, m+1)$ matrix $A = (a_{ij})$ with $a_{ij} = j^i$ for $i = 0, 1, \ldots, n$ and $j = 0, 1, \ldots, m$. Clearly, we have $\gamma^T A = (f(0), f(1), \ldots, f(m))$. Moreover, if we denote by $b = (B_0(R), B_1(R), \ldots, B_n(R))^T$, then using the fact that the moments of the Poisson-Charlier measures are Touchard-Hermite, this is equivalent to the existence of a vector $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m)^T \geq 0$ such that $A \alpha = b$, which in turn is equivalent to the representation of the polynomial $H_n(x; R)$ as

$$ H_n(x; R) = \sum_{k=0}^{m} \alpha_k (x - k)^n. $$

Therefore, $R \leq R_{m,n}$. This concludes the proof.

We shall need the following definition.

**Definition 18.** A non-zero real polynomial $P$ is said to be admissible if it is non-negative on the set of the integers $\mathbb{Z}$, i.e.;

$$ P(j) \geq 0 \quad \text{for any} \quad j \in \mathbb{Z}. $$

Let $p$ be a positive integer. Denote by $C_p$ the set of all admissible polynomials of degree at most $2p - 2$. Define the following quantity

$$ \tau_p := \sup_{R>0} \sup_{P \in C_p} \left( \frac{\int tP(t) d\mu_R(t)}{\int P(t) d\mu_R(t)} - \lambda_{p,p}(R) \right), $$

where $\lambda_{p,p}(R)$ is the largest zero of the Poisson-Charlier polynomial $C_p(., R)$. In the rest of this section, we shall prove that the quantity $\tau_p$ is bounded above, and it is even smaller than $p - 1$. The relevancy of the quantity $\tau_p$ in establishing a lower bound for the optimal threshold factors is stated below.

**Theorem 19.** For any integers $m \geq 2p - 1$, we have

$$ R_{m,2p-1} \geq t_p^{(m-p-\tau_p)} $$

provided that $m - p - \tau_p > -1$.

**Proof.** Let $\bar{R}$ be the unique real number such that $\lambda_{p,p}(\bar{R}) = m - \tau_p$. Similar arguments as in the proof of Theorem 1 show that $\bar{R} = t_p^{(m-p-\tau_p)}$. Let $f$ be a polynomial of degree $2p - 1$ such that $f(j) \geq 0$ for $j = 0, 1, \ldots, m$. Then $f$ can be written as $f = f_1 f_2$ where $f_1$ is an admissible polynomial of degree $2s$ ($0 \leq s \leq p$).
with zeros in the interval \([0, m]\) (in case \(s = 0\), take \(f_1 \equiv 1\)) and \(f_2\) is a polynomial of degree \(2(p - s) - 1\) with no zeros in the interval \([0, m]\). Thus necessarily

\[
f_2(x) > 0 \quad \text{for any} \quad x \in [0, m].
\]

Denote by \(\tilde{\mu}_R\) the positive measure

\[
\tilde{\mu}_R = e^{-R} \sum_{j=0}^{\infty} \frac{f_1(j)\tilde{R}^j}{j!} \delta_j,
\]

and by \((\tilde{\pi}_1, \tilde{\pi}_2, \ldots, \tilde{\pi}_n, \ldots)\) a sequence of orthogonal polynomials associated with the measure \(\tilde{\mu}_R\). By Gauss quadrature with respect to the measure \(\tilde{\mu}_R\), we have

\[
\int f(t) d\tilde{\mu}_R(t) = \int f_2(t) f_1(t) d\tilde{\mu}_R(t) = \int f_2(t) d\tilde{\mu}_R(t) = \sum_{j=1}^{p-s} \beta_j f_2(\tilde{\lambda}_j),
\]

where \(\beta_j > 0, j = 1, \ldots, p - s\) and where \(\tilde{\lambda}_1 < \tilde{\lambda}_2 < \ldots < \tilde{\lambda}_{p-s}\) are the zeros of the orthogonal polynomial \(\tilde{\pi}_{p-s}\). If we show that \(\tilde{\lambda}_{p-s} \leq m\) then, on account of (41), the integral in (42) will be non-negative and by Theorem 17, we will have \(R_{m,2p-1} \geq \tilde{R} = \bar{R}(m-p-\tau_p)\). Let us thus assume the opposite, i.e., \(\tilde{\lambda}_{p-s} > m\). Define the admissible polynomial \(Q\) of degree \(2p - 2\) by

\[
Q(t) = \prod_{j=1}^{p-s-1} (t - \tilde{\lambda}_j)^2 f_1(t).
\]

By orthogonality with respect to the measure \(\tilde{\mu}_R\), we can state that

\[
\int (t - \tilde{\lambda}_{p-s})Q(t) d\tilde{\mu}_R = \int \tilde{\pi}_{p-s}(t) \prod_{j=1}^{p-s-1} (t - \tilde{\lambda}_j) d\tilde{\mu}_R = 0.
\]

In other words,

\[
\frac{\int tQ(t)d\tilde{\mu}_R(t)}{\int Q(t)d\tilde{\mu}_R(t)} - \lambda_{p,p}(\tilde{R}) = (\tilde{\lambda}_{p-s} - m) + \tau_p > \tau_p.
\]

This contradicts the definition of \(\tau_p\). Thus we conclude that \(\tilde{\lambda}_{p-s} \leq m\) and the proof is complete.

To give an upper bound for the quantity \(\tau_p\) defined in (40), we need several preliminary results. Let \(P\) be an admissible polynomial and denote by \((\Pi^k)_{k \geq 0}\) a sequence of orthogonal polynomials associated with the measure

\[
d\tilde{\mu}_R = e^{-R} \sum_{j=0}^{\infty} \frac{P(j)R^j}{j!} \delta_j.
\]

Moreover, denote by \((\Pi^+_k)_{k \geq 0}\) a sequence of orthogonal polynomials associated with the measure

\[
d\tilde{\mu}_R^+ = e^{-R} \sum_{j=0}^{\infty} \frac{P^+(j)R^j}{j!} \delta_j, \quad \text{where} \quad P^+(t) = P(t - 1).
\]

We have the following comparison result.
PROPOSITION 20. Denote by $\tilde{\lambda}_k$ (resp. $\tilde{\lambda}_k^+$), $k = 1, 2, \ldots, p$, the zeros of the orthogonal polynomial $\Pi_p$ (resp. $\Pi_p^+$). Then

$$\tilde{\lambda}_p^+ \leq \tilde{\lambda}_p + 1.$$  

Proof. Applying Gauss quadrature, we obtain

$$(43) \quad G_{2p-1}(x) := \int (x-t)^{2p-1} d\mu_R(t) = \sum_{i=1}^{p} \beta_i (x-\tilde{\lambda}_i)^{2p-1},$$

with $\beta_i > 0, i = 1, \ldots, p$. Consider the polynomial $G_{2p}^+$ of degree 2p defined by

$$(44) \quad G_{2p}^+(x) := \int (x-t)^{2p} d\mu_R^+(t)$$

and let $g_{2p}$ be its polar form. We have

$$g_{2p}^+(0, x^{2p-1}) = - \int t(x-t)^{2p-1} d\mu_R^+(t)$$

$$= - \sum_{j=0}^{\infty} \frac{j(x-j)^{2p-1} P(j-1) R^j}{j!} = -RG_{2p-1}(x-1).$$

Thus, using (43), we obtain $g_{2p}^+(0, \tilde{\lambda}_1 + 1, \ldots, \tilde{\lambda}_p + 1, x^{p-1}) \equiv 0$. Therefore, by Proposition 9, there exist real numbers $\alpha_0, \alpha_1, \ldots, \alpha_p$ such that

$$(45) \quad G_{2p}^+(x) = \alpha_0 x^{2p} + \sum_{k=1}^{p} \alpha_k (x-(\tilde{\lambda}_k + 1))^{2p}.$$  

Using the fact that

$$g_{2p}^+(\tilde{\lambda}_1 + 1[2], \ldots, \tilde{\lambda}_p + 1[2]) = \alpha_0 \prod_{k=1}^{p} (\tilde{\lambda}_k + 1)^2 = \int \prod_{k=1}^{p} (\tilde{\lambda}_k + 1 - t)^2 d\mu_R^+(t) \geq 0,$$

we can thus conclude that $\alpha_0 \geq 0$. Similarly, with $\Lambda = (0[2], \ldots, \tilde{\lambda}_i - 1 + 1[2], \tilde{\lambda}_{i+1} + 1[2], \ldots, \tilde{\lambda}_p + 1[2])$, we have

$$g_{2p}^+(\Lambda) = \alpha_i (\tilde{\lambda}_i + 1)^2 \prod_{k=1, k \neq i}^{p} (\tilde{\lambda}_k - \tilde{\lambda}_i)^2 = \int t^2 \prod_{k=1, k \neq i}^{p} (\tilde{\lambda}_k + 1 - t)^2 d\mu_R^+(t) \geq 0.$$  

Thus we conclude that $\alpha_i \geq 0$ for $i = 1, 2, \ldots, p$. Differentiating (44) and (45) with respect to the variable $x$, and applying Gauss quadrature, we obtain

$$(46) \quad \alpha_0 x^{2p-1} + \sum_{k=1}^{p} \alpha_k (x-(\tilde{\lambda}_k + 1))^{2p-1} = \sum_{i=1}^{p} \gamma_i (x-\tilde{\lambda}_i^+)^{2p-1},$$

with $\gamma_i > 0$ for $i = 1, 2, \ldots, p$. Evaluating the polar form of both side of (46) at $(\tilde{\lambda}_1^+[2], \ldots, \tilde{\lambda}_p^+[2], \tilde{\lambda}_p^+)$ shows that $\tilde{\lambda}_p^+ \leq \tilde{\lambda}_p + 1$.  

The only instance of the previous proposition that we shall need is the $p = 1$ case. The unique zero $\tilde{\lambda}_1$ of the polynomial $\Pi_1$ is given by the condition

$$\int (t-\tilde{\lambda}_1) d\mu_R = \int (t-\tilde{\lambda}_1) P(t) d\mu_R = 0.$$
Thus
\[
\bar{\lambda}_1 = \frac{\int tP(t)d\mu_R(t)}{\int P(t)d\mu_R(t)}.
\]

In this specific situation, Proposition 20 states that, for any admissible polynomial \( P \), we have
\[
\frac{\int tP(t)d\mu_R(t)}{\int P(t)d\mu_R(t)} \leq \frac{\int tP(t+1)d\mu_R(t)}{\int P(t+1)d\mu_R(t)} + 1.
\]

Iterating this inequality leads to the following result.

**Corollary 21.** Let \( R \) be a positive real number. Then for any admissible polynomial \( P \), and for any non-negative integer \( j \), we have
\[
\frac{\int tP(t)d\mu_R(t)}{\int P(t)d\mu_R(t)} \leq \frac{\int tP(t+j)d\mu_R(t)}{\int P(t+j)d\mu_R(t)} + j.
\]

We shall need the following proposition whose proof was kindly provided to us by Fedja Nazarov [5].

**Proposition 22.** Let \( P \) be an admissible polynomial of degree at most \( 2n \). Then the polynomial
\[
Q(t) = \sum_{k=0}^{n} \binom{n}{k}^2 P(t+k)
\]
is non-negative on the whole real line, i.e., \( Q(t) \geq 0 \) for all \( t \in \mathbb{R} \).

**Proof.** Let \( t_0 \) be an arbitrary real number in \( \mathbb{R}/\mathbb{Z} \). Denote by \( S \) the polynomial \( S(t) = P(t + t_0) \). We thus need to show that
\[
\sum_{k=0}^{n} \binom{n}{k}^2 S(k) \geq 0 \text{ under the hypothesis that } S(t) \geq 0 \text{ for any } t \in \Lambda,
\]
where \( \Lambda := -t_0 + \mathbb{Z} \). The set \( \Lambda \) can be viewed as \( \Lambda = \{ t \in \mathbb{R} \mid \cos(\pi t + \lambda) = 0 \} \), where for instance \( \lambda := \pi t_0 + \pi/2 \). Set \( N(t) := (t-1)(t-2)\ldots(t-n) \) and consider the meromorphic function
\[
F(z) = \frac{\tan(\pi z + \lambda) - \tan(\lambda)}{N(z)^2} S(z).
\]
The poles of \( F \) are simple and \( F(z) \) decays like \( |z|^{-2} \) on any large circle centered at zero and does not pass through the poles of the function \( \tan(\pi z + \lambda) \). Therefore, the sum of residues of the function \( F \) converges to zero. The residue of \( F \) at the zero \( k \) of \( N \) is given by
\[
\text{Res}_{z=k} F(z) = \frac{\pi}{(n!)^2 \cos^2(\lambda)} \binom{n}{k}^2 S(k), \quad k = 0, \ldots, n,
\]
while the residue of \( F \) at a pole \( t \in \Lambda \) is given by
\[
\text{Res}_{z=t} F(z) = -\frac{S(t)}{\pi N(t)^2}.
\]

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Let $\alpha$ be the unique real number such that $\ell_p^{(\alpha)} = R_{5, 3} \simeq 2.6506$. The monotonicity of the zero of Laguerre polynomials with respect to the parameter $\alpha \in (-1, \infty)$ enables us to conclude from (47) that $\tau_p \geq p + 1 - \alpha$. Some of the upper bounds to $\tau_p$ using this inequality are

$$\tau_5 \geq 0.74, \quad \tau_8 \geq 1.99, \quad \tau_{12} \geq 4.10, \quad \tau_{15} \geq 5.88, \quad \tau_{22} \geq 10.42.$$
6. Structural properties of the optimal threshold polynomials. In this section, we adapt and extend an ingenious technique by Bernstein [7] to identify a structural property of the optimal threshold polynomial $\Phi_{m,n}$ that will be fundamental throughout the rest of the paper. To ease our exposition we adopt the following terminology. When we write a polynomial in the form

$$\Phi(x) = \sum_{k=1}^{s} \alpha_k \left(1 + \frac{x}{R}\right)^{m_k} \text{ with } m_i \neq m_j, \text{ if } i \neq j; \ i, j = 1, 2, \ldots, s,$$

then we will call the integers $m_1, m_2, \ldots, m_s$ the **exponents** of $\Phi$ and the real numbers $\alpha_1, \alpha_2, \ldots, \alpha_s$ the **coefficients** of $\Phi$. For a given index $k$, we shall call $\alpha_k$ the coefficient associated with $m_k$ or simply the coefficient of $m_k$. We will use the term **missing exponents** for exponents $m_k$ whose associated coefficients $\alpha_k$ are equal to zero. If an exponent $m_k$ is a missing exponent in the representation (48) then it is in fact a virtual exponent and can be placed anywhere at will. Thus when we say that the sequence of exponents $(m_1, m_2, \ldots, m_s)$ satisfies a certain property $(P)$ we mean that we can find positions for the missing exponents such that the resulting sequence of exponents satisfies the property $(P)$. We use the expression **explicitly missing exponent** to refer to the fact that we have deleted the missing exponent from the exponents sequence. For instance, when we say that a finite sequence $(m_1, m_2, m_3, m_4)$ is given by $(1, 4, 5)$ then necessarily there is one explicitly missing exponent. We have purposely avoided the use of the terminology of principal polynomials as in [7] for the following reason: In Bernstein work, the sequence of exponents is not bounded above, while in our case all the exponents of the optimal threshold polynomial $\Phi_{m,n}$ are at most equal to $m$. It will be also helpful to explicitly state the following simple theorem showing that there are four different ways to look at the problem at hand. The proof being implicitly contained in the previous sections, we leave it to the reader.

**Theorem 25.** Let $m, n$ be positive integers such that $m \geq n$. Let $(\alpha_1, \alpha_2, \ldots, \alpha_s)$ be non-negative real numbers and $(m_1, m_2, \ldots, m_s)$ be pair-wise distinct non-negative integers. The following statements are equivalent.

(i) The polynomial $\mathcal{H}_n(x, \cdot)$ can be written as

$$\mathcal{H}_n(x, R) = \int (x - t)^n d\mu_R(t) = \sum_{k=1}^{s} \alpha_k (x - m_k)^n.$$  

(ii) The real numbers $\alpha_1, \ldots, \alpha_s$ and the integers $m_1, \ldots, m_s$ satisfy the system

$$\sum_{k=1}^{s} \alpha_k m_k^\ell = B_\ell(R) \text{ for } \ell = 0, 1, \ldots, n.$$  

(iii) The measure $\mu_R$ possesses a positive quadrature with integer nodes, i.e., for any polynomial $P$ of degree at most $n$, we have

$$\int P(t) d\mu_R(t) = \sum_{k=1}^{s} \alpha_k P(m_k).$$  

(iv) The polynomial $\Phi$ defined by

$$\Phi(x) = \sum_{k=1}^{s} \alpha_k \left(1 + \frac{x}{R}\right)^{m_k}.$$
is absolutely monotonic over the interval \([-R,0]\) and it satisfies \(\Phi(x) - e^x = O(x^{n+1})\) as \(x \to 0\).

Sometimes we shall refer to (49) or to (51) as being a system in \(\alpha_i, m_i, i = 1, \ldots, s\) when we actually mean the system (50). The following fundamental theorem is based on ideas by Bernstein in [7].

**Theorem 26.** For any positive integers \(m\) and \(p\) such that \(m \geq 2p - 1\), the optimal threshold polynomial \(\Phi_{m,2p-1}\) has the form

\[
\Phi_{m,2p-1}(x) = \sum_{k=1}^{2p} \alpha_k \left(1 + \frac{x}{R_{m,2p-1}}\right)^{m_k},
\]

where \(\alpha_k, k = 1, 2, \ldots, 2p\) are non-negative real numbers and where the integers \(0 \leq m_1 < m_2 < \ldots < m_{2p-1} < m_{2p} \leq m\) can be grouped in the form

\[
(q_1, q_1 + 1), (q_2, q_2 + 1), \ldots, (q_{p-1}, q_{p-1} + 1), q_p
\]

with one explicitly missing exponent (and possibly many missing exponents) or of the form

\[
(q_1, q_1 + 1), (q_2, q_2 + 1), \ldots, (q_{p-1}, q_{p-1} + 1), (q_p, q_p + 1)
\]

with at least one missing coefficient. In (53) and (54), the integers \(q_1, q_2, \ldots, q_p\) satisfy the inequalities

\[q_k + 1 < q_{k+1}, \quad k = 1, 2, \ldots, p - 1.\]

**Proof.** The strategy of the proof consists in showing that, if in the representation (52) of the optimal threshold polynomial, the sequence of integers \((m_1, m_2, \ldots, m_{2p-1}, m_{2p})\) satisfy none of the conditions (53) and (54) then, starting from this representation, we can construct another representation of the optimal threshold polynomial whose exponents satisfy either (53) or (54). This will eventually contradict the uniqueness of the optimal threshold polynomial and thus conclude the proof of the theorem. According to Theorem 25, equation (52) is equivalent to the linear system

\[
\begin{align*}
\alpha_1 + \alpha_2 + \ldots + \alpha_{2p} &= B_0(R_{m,2p-1}) \\
\alpha_1 m_1 + \alpha_2 m_2 + \ldots + \alpha_{2p} m_{2p} &= B_1(R_{m,2p-1}) \\
\vdots & \quad \vdots \\
\alpha_1 m_1^k + \alpha_2 m_2^k + \ldots + \alpha_{2p} m_{2p}^k &= B_k(R_{m,2p-1}) \\
\vdots & \quad \vdots \\
\alpha_1 m_1^{2p-1} + \alpha_2 m_2^{2p-1} + \ldots + \alpha_{2p} m_{2p}^{2p-1} &= B_{2p-1}(R_{m,2p-1})
\end{align*}
\]

Without loss of generality, we assume that \(\alpha_{2p} > 0\). In the above linear system, let us fix all the integers \(m_k, k < 2p\) and change continuously the value of \(m_{2p}\) viewed as a real number. The variation of the coefficients \(\alpha_1, \alpha_2, \ldots, \alpha_{2p}\) satisfies the linear system

\[
\begin{align*}
\frac{\partial \alpha_1}{\partial m_{2p}} + \frac{\partial \alpha_2}{\partial m_{2p}} + \ldots + \frac{\partial \alpha_{2p}}{\partial m_{2p}} &= 0 \\
m_1 \frac{\partial \alpha_1}{\partial m_{2p}} + m_2 \frac{\partial \alpha_2}{\partial m_{2p}} + \ldots + m_{2p} \frac{\partial \alpha_{2p}}{\partial m_{2p}} &= -\alpha_{2p} \\
\vdots & \quad \vdots \\
m_1^{2p-1} \frac{\partial \alpha_1}{\partial m_{2p}} + m_2^{2p-1} \frac{\partial \alpha_2}{\partial m_{2p}} + \ldots + m_{2p}^{2p-1} \frac{\partial \alpha_{2p}}{\partial m_{2p}} &= -(2m - 1)\alpha_{2p} m_{2p}^{2p-2}.
\end{align*}
\]
The solution to the above linear system is given by

\[
\frac{\partial \alpha_k}{\partial m_{2p}} = -\frac{\alpha_{2p}}{\Delta} \frac{\partial \Delta_m}{\partial m}(m_{2p}), \quad k = 1, 2, \ldots, 2p,
\]

where \(\Delta = \prod_{1 \leq i < j \leq 2p} (m_j - m_i)\) and \(\Delta_m(m)\) is given by the function determinant

\[
\Delta_m(m) = \left| \begin{array}{cccccc}
1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & m & m_{k+1} & \cdots & m_{2p} \\
m_{k-1} & m_{k-1} & \cdots & \cdots & \cdots \\
m_{2p-1} & m_{2p-1} & \cdots & \cdots & m_{2p-1}
\end{array} \right|
\]

For \(k < 2p\), the largest zero of \(\Delta_m\) is \(m_{2p}\). Thus the sign of \(\frac{\partial \Delta_m}{\partial m}(m_{2p})\) is the same as the sign of \(\Delta_m(m)\) for \(m > m_{2p}\). Thus this sign is positive for even \(k\) and negative for odd \(k\). Therefore, we conclude from (56) that

\[
(-1)^k \frac{\partial \alpha_k}{\partial m_{2p}} < 0, \quad k = 1, 2, \ldots, 2p.
\]

From (57) we infer that if we increase the value of \(m_{2p}\) and solve the corresponding linear system (55) then all the coefficients with odd index \(\alpha_{2k-1}, k = 1, \ldots, p\), will increase, while the coefficients with even index \(\alpha_{2k}, k = 1, \ldots, p\), will decrease. The opposite happens if we proceed by decreasing the value of \(m_{2p}\). Now, assuming that the exponents of the decomposition (52) satisfy neither (53) nor (54), consider the associated system (55). We start a descending process by decreasing the value of \(m_{2p}\) while avoiding that any of the coefficients \(\alpha_k, k = 1, 2, \ldots, 2p - 1\), obtained by solving (55), becomes negative. Noting that the missing exponents \(m_i\) from (52) are virtual and can be placed anywhere at will, we can easily deduce that a decrease of \(m_{2p}\) is impossible only if the exponents \((m_1, m_2, \ldots, m_{2p-1})\) can be grouped into integers of the form

\[
(q_1, q_1 + 1), (q_2, q_2 + 1), \ldots, (q_{p-1}, q_{p-1} + 1), \quad q_k + 1 < q_{k+1}, \quad k = 1, 2, \ldots, p-2,
\]

with one explicitly missing exponent (and possibly many missing exponents). From our hypothesis such a decrease of \(m_{2p}\) is then possible. Thus, we decrease the value of \(m_{2p}\) until one of the odd coefficients \(\alpha_{2k-1}\) vanishes. This eventually happens before the value of \(\alpha_{2p}\) vanishes due to the fact that if \(\alpha_{2p} = 0\) before any of the odd coefficients vanishes then it should have been zero before the start of the descending process. Thus once one of the odd coefficients \(\alpha_{2k-1}\) vanishes, we replace the corresponding virtual exponent \(m_{2k-1}\) by the largest integer \(q < m_{2p}\) such that there are an odd number of integers \(m_i\) between \(q\) and \(m_{2p}\) (assuming that such move is possible). Note that a further decrease of \(m_{2p}\) will now increase the new value of \(\alpha_{2k-1}\) as the index of its corresponding exponent is now even. We continue this descending process until no further decrease of \(m_{2p}\) is possible. This is the case only when the exponents \((m_1, m_2, \ldots, m_{2p-2})\) can be grouped into integers of the form (58). If at the end of the descending process, the real number \(\rho := m_{2p}\) is an integer then we have found a solution to our linear system (55) where the exponents satisfy condition (53) and the associated coefficients are non-negative. This contradicts the uniqueness of the optimal threshold polynomial. Let us assume now that at the end of the descending process, the real number \(\rho := m_{2p}\) is not an integer. Since, there is a least one missing exponent at the end of the descending process, we place this missing exponent at the
position of the largest integer \( q < m_{2p} \) that is not occupied by another exponent with positive coefficient. We have then a configuration of exponents of the form

\[
(q_1, q_1 + 1), (q_2, q_2 + 1) \ldots (q_{p-1}, q_{p-1} + 1), (\bar{q}, \rho),
\]

where \( \bar{q} \) is an integer and \( \bar{q} < \rho < \bar{q} + 1 \). Moreover, the coefficients associated with even index exponents are non-zero. Now we start an ascending process by increasing the value of \( \rho \) while solving the corresponding linear system (55). In doing so, the coefficients with even index will decrease, while the one with odd index will increase.

If we increase \( \rho \) until \( \bar{q} + 1 \) without any of the coefficients \( \alpha_k \) becomes negative then we would arrive to a solution of the linear system (55) where the exponents satisfy condition (54) and thus again in contradiction with the uniqueness of the optimal threshold polynomial. If during the ascending process of \( \rho \) we reach \( \bar{q} + 1 \), then we replace the associated exponent \( q_k + 1 \) by \( q_k - 1 \) and continue the ascending process. However, if the site \( q_k - 1 \) is already occupied by another exponent with positive coefficient, then we replace it by \( q_{k-2} - 1 \) instead and so on, and then continue the ascending process. The only case where an increase of \( \rho \) is no longer possible is when the exponents are grouped into pairs of the form

\[
(0, 1), (2, 3), \ldots, (2i - 2, 2i - 1), (q_{i+1}, q_{i+1} + 1) \ldots (q_{p-1}, q_{p-1} + 1), (\bar{q}, \rho),
\]

with the coefficient of one exponent among \( (1, 3, 5, \ldots, 2i - 1) \), say \( k \), equal to zero.

Let us show that a configuration such as (59) cannot be reached before \( \rho \) reaches \( \bar{q} + 1 \). Let us consider the configuration (59) with \( \rho < \bar{q} + 1 \). The associated polynomial

\[
\psi(t) = (t - \bar{q})(t - \rho) \prod_{j=0, j \neq k}^{2i-1} (t - j) \prod_{j=i+1}^{p-1} (t - q_j)(t - q_j - 1)
\]

is non-negative on \( \mathbb{N} \) and thus we have

\[
(60) \quad \int \psi(t)d\mu_{R_{m,2p-1}}(t) > 0.
\]

For the configuration (59), we have

\[
(61) \quad \mathcal{H}_{2p-1}(x; R_{m,2p-1}) = \int (x - t)^{2p-1}d\mu_{R_{m,2p-1}}(t) = \sum_{j=1}^{2p-1} \alpha_j (x - x_j)^{2p-1},
\]

where the \( x_j \)'s are the zeros of the polynomial \( \psi \) and \( \alpha_1, \alpha_2, \ldots \alpha_{2p-1} \) are non-negative numbers. Evaluating the polar form of both sides of (61), we obtain

\[
\int \psi(t)d\mu_{R_{m,2p-1}}(t) = h_{2p-1}(0, 1, \ldots, k - 1, k + 1, \ldots, q_{p-1} + 1, \bar{q}, \rho; R_{m,2p-1}) = 0.
\]

This contradicts (60). Therefore a configuration of the form (59) cannot be reached before \( \rho \) reaches \( \bar{q} + 1 \). Hence, once \( \rho \) reaches \( \bar{q} + 1 \), we obtain a solution of our linear system (55) where the exponents satisfy condition (54) and where the associated coefficients are non-negative. This again contradicts the uniqueness of the optimal threshold polynomial. This concludes the proof of the theorem. \( \square \)
Remark 27. In Theorem 15, we have shown that the exponents of the optimal threshold polynomial $\Phi_{m,3}$, with $m$ is a square integer are $m - 2\sqrt{m} + 1, m$. These exponents can be grouped in the form (53) as

$$(m - 2\sqrt{m} + 1, m - 2\sqrt{m} + 2), m$$

where $m - 2\sqrt{m} + 2$ is a missing exponent.

The representations (53) and (54) of the exponents of the optimal threshold polynomial are quite similar. For example, if the coefficient of $q_p$ (or of $q_p + 1$) is equal to zero in (54) then the two representations coincide. However, for instance a representation where the coefficients are grouped in the form

(62) $(2, 3), (5, 6), (10, 11), (12, 13)$

where the coefficients of the exponents $10, 11, 12, 13$ are positive (and at least one of the exponents $2, 3, 5, 6$ is missing) cannot be represented in the form (53). However, if the coefficients of $11, 12, 13$ are positive while the coefficient of $10$ is equal to zero, i.e., $10$ is a missing exponent, then we can re-write (62) in the form (53) via a shift in the indices, i.e.,

$(2, 3), (5, 6), (11, 12), 13.$

with $10$ is an explicitly missing exponent. According to this simple observation, we shall provide a more refined structural property of the optimal threshold polynomial $\Phi_{m,2p-1}$ by showing that its exponents can always be represented in the form (53) with $q_p = m$. For this we shall need the following proposition.

Proposition 28. The coefficients of each pair, in the two possible representations (53) and (54) of the exponents of the optimal threshold polynomial $\Phi_{m,2p-1}$, cannot be simultaneously equal to zero.

Proof. We give a proof for representations of the exponents of the form (53). Representations of the from (54) can be handled in a similar fashion. Let us assume that the coefficients $\alpha_{2\ell-1}$ and $\alpha_{2\ell}$ ($\ell \leq p-1$) associated with one of the pair $(q_\ell, q_\ell + 1)$ are both zero. Then we have

$$H_{2p-1}(x; R_{m,2p-1}) = \int (x - t)^{2p-1} d\mu_{R_{m,2p-1}}$$

(63)

$$= \alpha_{2p-1}(x - q_p)^{2p-1} + \sum_{k=1,k\neq\ell}^{p-1} \alpha_{2k-1}(x - q_k)^{2p-1} + \alpha_{2k}(x - q_k - 1)^{2p-1}.$$ 

Differentiating (63) with respect to $x$, we obtain

$$H_{2p-2}(x; R_{m,2p-1}) = \int (x - t)^{2p-2} d\mu_{R_{m,2p-1}}$$

(64)

$$= \alpha_{2p-1}(x - q_p)^{2p-2} + \sum_{k=1,k\neq\ell}^{p-1} \alpha_{2k-1}(x - q_k)^{2p-2} + \alpha_{2k}(x - q_k - 1)^{2p-2}.$$ 

Denote by $h_{2p-2}(-; R_{m,2p-1})$ the polar form of the polynomial $H_{2p-2}(x; R_{m,2p-1})$ and $\Lambda = (q_1, q_1 + 1, \ldots, q_\ell - 1, q_\ell - 1 + 1, \ldots, q_\ell + 1, q_\ell + 1 + 1, \ldots, q_p, q_p + 1)$. From (64) we obtain
the following contradiction
\[ h_{2p-2}(A; R_{m,2p-1}) = 0 = \int \prod_{k=1, k \neq \ell}^{p} (q_k - t)(q_k + 1 - t) d\mu_{R_{m,2p-1}} > 0. \]

This completes the proof. □

We are now in a position to give a refined structural property of the optimal threshold polynomial.

**Theorem 29.** For any positive integers \( m \) and \( p \) such that \( m \geq 2p-1 \), the optimal threshold polynomial \( \Phi_{m,2p-1} \) has the form
\[
\Phi_{m,2p-1}(x) = \sum_{k=1}^{2p} \alpha_k \left( 1 + \frac{x}{R_{m,2p-1}} \right)^{m_k},
\]
where \( \alpha_k, k = 1, 2, \ldots, 2p \) are non-negative real numbers and the integers \( 0 \leq m_1 < m_2 < \ldots < m_{2p-1} \leq m \) can be grouped into the form
\[ (q_1, q_1 + 1), (q_2, q_2 + 1), \ldots, (q_p, q_p + 1), q_p, \ q_p + 1 < q_{p+1}, k = 1, 2, \ldots, p-1, \]
with one explicitly missing exponent (and possibly many missing exponents). Moreover, we have \( q_p = m \) and its coefficient is positive.

**Proof.** From Theorem 26, we know that the exponents \((m_1, m_2, \ldots, m_{2p})\) can be represented in the form \((53)\) or \((54)\). Assume that a configuration of exponents of \( \Phi_{m,2p-1} \) of the form
\[
(q_1, q_1 + 1), \ldots, (q_k, q_k + 1), (q_{k+1}, q_{k+1} + 1),
(q_{k+1} + 2, q_{k+1} + 3), \ldots, (q_{2s}, q_{2s} + 1),
\]
with \( k \leq p - 1, k + 1 + s = p \) and where the coefficients associated with \( q_{k+1}, q_{k+1} + 1, \ldots, q_{2s} + 1 \) are positive with \( q_{k+1} - (q_k + 1) > 1 \) is possible. Then we can always assume that the coefficients associated with the first element of each pair in \((66)\) is positive using the following procedure: If, for example, the coefficient associated with the first element of a pair \((q_r, q_r + 1) (\ell \leq k)\) is equal to zero, then according to Proposition 28, the coefficient associated with \( q_\ell + 1 \) is positive. In this case we change the pair \((q_\ell, q_\ell + 1)\) into the pair \((q_\ell + 1, q_\ell + 2)\). If the site \( q_\ell + 2 \) is already occupied by an exponent with positive coefficient then we place \( q_\ell \) at \( q_\ell + 4 \) instead and so on.

The fact that we have assumed the existence of a least one free site between \( q_k + 1 \) and \( q_{k+1}, i.e., q_{k+1} - (q_k + 1) > 1 \), insures the success of such procedure. Now that the coefficients associated with the first element of each pair in \((66)\) is positive, we start decreasing the value of \( q_{k+1} + 2s + 1 \) and solve the associated linear system \((55)\).

As we have shown before, a decrease of \( q_{k+1} + 2s + 1 \) will increase the coefficients with even index and decrease the ones with odd index. Therefore, a small decrease of \( q_{k+1} + 2s + 1 \) say to \( q_{k+1} + 2s + 1 - \delta \) will render all the coefficients \( \alpha_i \) positive. At this stage, we increase the value of \( R_{m,2p-1} \) to \( R_{m,2p-1} + \epsilon \) in such a way that all the coefficients \( \alpha_i \) solution to the new linear system \((55)\) remain positive and then we bring \( q_{k+1} + 2s + 1 - \delta \) to \( q_{k+1} + 2s + 1 \) by the same ascending process as in the proof of Theorem 26. At the end of this procedure, we obtain a solution to the linear system \((55)\) with non-negative coefficients \( \alpha_i \) and with \( R_{m,2p-1} \) replaced by \( R_{m,2p-1} + \epsilon \). This contradicts the very definition of \( R_{m,2p-1} \). Therefore, the only
possible configurations of the exponents of the optimal threshold polynomial are the ones that are of the form (53) or of the form (66) with $k \leq p - 1$, $k + 1 + s = p$, but now the coefficient associated with $q_{k+1}$ must be equal to zero while the coefficients associated with $q_{k+1} + 1, q_{k+1} + 2, \ldots, q_{k+1} + 2s + 1$ must be positive. The latter configurations can be written in the form (53) by a single shift of the indices as

\[(q_1, q_1 + 1), \ldots, (q_k, q_k + 1), (q_k + 1, q_k + 1 + 2),
\]
\[(q_{k+1} + 3, q_{k+1} + 4), \ldots, (q_k + 2s - 1, q_k + 2s + 1), q_{k+1} + 2s.
\]

This proves the first part of the theorem. Let us now prove that in the representation (65) we have $q_p = m$ and that its associated coefficient is positive. According to what we have just proved, the optimal threshold polynomial has the form

\[(67) \quad \Phi_{m, 2p-1}(x) = \sum_{k=1}^{2p-1} \alpha_k \left(1 + \frac{x}{R_{m, 2p-1}}\right)^{m_k},
\]

where $(m_1, m_2, \ldots, m_{2p-1}) = (q_1, q_1 + 1, q_2, q_2 + 1, \ldots, q_p + 1, q_p)$. We proceed by contradiction and assume that $m_{2p-1} < m$. According to Theorem 25, the identity (67) is equivalent to

\[\mathcal{H}_{2p-1}(x; R) = \int (x - t)^{2p-1} d\mu_{R_{m, 2p-1}}(t) = \sum_{k=1}^{2p-1} \alpha_k (x - m_k)^{2p-1}.
\]

Consider the polynomial $F(x) := \mathcal{H}_{2p}(x, R_{m, 2p-1}) = \int (x - t)^{2p} d\mu_{R_{m, 2p-1}}(t)$ and denote by $f$ its polar form. We have

\[f(0, x^{[2p-1]}) = -R_{m, 2p-1} \int (x - 1 - t)^{2p-1} d\mu_{R_{m, 2p-1}}(t)
\]
\[= -R_{m, 2p-1} \sum_{k=1}^{2p-1} \alpha_k (x - (m_k + 1))^{2p-1}.
\]

Hence $f(0, m_1 + 1, m_2 + 1, \ldots, m_{2p-1} + 1) = 0$ and thus according to Proposition 9, there exist coefficients $\beta_0, \beta_1, \ldots, \beta_{2p-1}$ such that

\[F(x) = \beta_0 x^{2p} + \sum_{k=1}^{2p-1} \beta_k (x - (m_k + 1))^{2p}.
\]

Computing $f(0, x^{[2p-1]})$ from (69) and comparing with (68) yields

\[R_{m, 2p-1} \sum_{k=1}^{2p-1} \alpha_k (x - (m_k + 1))^{2p-1} = \sum_{k=1}^{2p} \beta_k (x - (m_k + 1))^{2p-1}.
\]

Thus $\beta_k = \alpha_k \frac{R_{m, 2p-1}}{m_k + 1} \geq 0$ for $k = 1, 2, \ldots, 2p$. Now we prove that $\beta_0 \geq 0$ as follows:

\[f(m_1 + 1, m_2 + 1, \ldots, m_{2p-1} + 1, m_{2p-1} + 2) = \beta_0 (m_{2p-1} + 2) \prod_{k=1}^{2p-1} (m_k + 1)
\]
\[= \left(\prod_{k=1}^{m_{2p-1} + 2 - t} (m_k - t)\right) \prod_{k=1}^{2p-1} (m_k + 1) \geq 0.
\]
Let the optimal threshold polynomial \( \Phi_m \) be an increasing function of the parameter \( m \).

The optimal threshold polynomial \( \Phi_1 \) is the unique positive zero of the polynomial equation

\[
\Phi_m(x) = \beta_0 + \sum_{k=1}^{2p-1} \beta_k \left( 1 + \frac{x}{R_{m,2p-1}} \right)^{m_k+1}.
\]

Since \( \beta_j \geq 0, j = 0, \ldots, 2p \) and \( m_j + 1 \leq m \) for \( j = 1, \ldots, 2p \), the representation (70) contradicts the uniqueness of the optimal threshold polynomial. Thus, we conclude that \( m_{2p-1} = m \) and \( \alpha_{2m-1} > 0 \).

We will find it convenient to re-write Theorem 29 in the equivalent form stated in Theorem 4 (See Introduction).

**Example 17.** Using the algorithm of Section 8, it can be shown that \( R_{200,5} \approx 175.8348 \) is the unique positive zero of the polynomial equation

\[
R^5 - 852R^4 + 291352R^3 - 49988400R^2 + 4303437600R - 148719648000 = 0.
\]

The optimal threshold polynomial \( \Phi_{200,5} \) is given by (14) with

\[
(m_1, m_2, m_3, m_4, m_5) = (154, 155, 176, 177, 200)
\]

and \( (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (0.1846, 0.0007, 0.3320, 0.3336, 0.1491) \). The structural property of \( \Phi_{200,5} \) confirms the statement given in Theorem 4.

Theorem 4 shows in particular that for any positive integer \( n \), \( R_{m,n} \) is a strictly increasing function of the parameter \( m \geq n \). We should also point out that the findings in Theorem 4 can be used to significantly improve Step 1 in Kraaijevanger’s algorithm as it considerably reduces the number of integer sequences to be considered in Step 1 of the algorithm.

As a first application of Theorem 4, we prove Theorem 3 (See Introduction)

**Proof of Theorem 3:** To prove that \( R_{m+1,2p} \leq R_{m,2p-1} \) we proceed as follows:

Let the optimal threshold polynomial \( \Phi_{m+1,2p} \) be written as

\[
\Phi_{m+1,2p}(x) = \sum_{k=1}^{2p} \beta_k \left( 1 + \frac{x}{R_{m+1,2p}} \right)^{m_k},
\]

with \( \beta_k \geq 0 \) for \( k = 1, 2, \ldots, 2p \) and \( 0 \leq m_1 < m_2 < \ldots < m_{2p} \leq m + 1 \). Since \( \Phi_{m+1,2p} \) belongs to \( \Pi_{m+1,2p} \), its derivative with respect to \( x \) belongs to \( \Pi_{m,2p-1} \).

Moreover, taking the derivative of (71), we obtain

\[
\frac{\partial \Phi_{m+1,2p}}{\partial x}(x) = \sum_{k=1}^{2p} m_k \beta_k \left( 1 + \frac{x}{R_{m+1,2p}} \right)^{m_k-1}.
\]
Accordingly, Corollary 6 enables us to conclude that \( R_{m+1,2p} \leq R_{m,2p-1} \). To prove that \( R_{m+1,2p} \geq R_{m,2p-1} \) we proceed exactly as in the proof of the second part of Theorem 29. Namely, from the optimal threshold polynomial

\[
\Phi_{m,2p-1}(x) = \sum_{k=1}^{2p-1} \alpha_k \left( 1 + \frac{x}{R_{m,2p-1}} \right)^{m_k},
\]

we construct the polynomial \( F \) in (69) with the properties

\[
F(x) := H_{2p}(x, R_{m,2p-1}) = \int (x-t)^{2p} d\mu_{R_{m,2p-1}}(t) = \beta_0 x^{2p} + \sum_{k=1}^{2p-1} \beta_k (x-(m_k+1))^{2p},
\]

with \( \beta_k \geq 0, k = 0, 1, \ldots, 2p \). The last identity shows, according to Corollary 6, that \( R_{m+1,2p} \geq R_{m,2p-1} \). The relation (13) between the optimal threshold polynomials is a direct and simple consequence of the equality \( R_{m,2p-1} = R_{m+1,2p} \). That \( R_{m,n} \) are algebraic numbers is a consequence of the fact that \( R_{m,2p-1} \) is a zero of the polynomial equation in \( R \) with integer coefficients

\[
h_{2p-1}(m_1, m_2, \ldots, m_{2p-2}, m; R) = 0,
\]

where \( h_{2p-1}(\cdot; R) \) the polar form of the polynomial \( H_{2p-1}(\cdot; R) \).

7. Spectral transformations and optimal threshold factors. In this section we use the structural property stated in Theorem 4 to gain more insights on the optimal threshold polynomial. This will lead to a highly efficient algorithm for the computation of the optimal threshold factors and their associated polynomials. From now on, we adopt the following notation and terminology. For an admissible polynomial \( \Omega \), we define the Christoffel transform measure \( \mu_R^\Omega \) of \( \mu_R \) by

\[
\mu_R^\Omega = e^{-R} \sum_{j=0}^\infty \Omega(j) \frac{R^j}{j!} \delta_j.
\]

The polynomial \( \Omega \) is called the annihilator polynomial for the measure \( \mu_R^\Omega \). The orthogonal polynomials associated to the measure \( \mu_R^\Omega \) are denoted by \( \Pi_{j,R}^\Omega, n \geq 0 \) and their zeros by \( \lambda_1^\Omega, n < \lambda_2^\Omega, n < \ldots < \lambda_n^\Omega, n \) or simply \( \lambda_1^\Omega < \lambda_2^\Omega < \ldots < \lambda_n^\Omega \) if the real number \( R \) is understood within the context.

We shall need the following theorem by Sylvester [36, 31]

**Theorem 30.** (Sylvester.) Suppose \( 0 \neq \beta_k \) for all \( k \) and \( \gamma_1 < \ldots < \gamma_r, r \geq 2 \), are real numbers such that

\[
Q(t) = \sum_{k=1}^{r} \beta_k (x-\gamma_k)^d
\]

does not vanish identically. Suppose the sequence \( (\beta_1, \ldots, \beta_r, (-1)^d \beta_1) \) has \( C \) changes of sign and \( Q \) has \( Z \) real zeros, counting multiplicities. Then \( Z \leq C \).

Using Theorem 4 and Sylvester’s theorem, we show the following.

**Proposition 31.** Let \( \Phi_{m,2p-1} \) be the optimal threshold polynomial with optimal threshold factor \( R_{m,2p-1} \)

\[
\Phi_{m,2p-1}(x) = \sum_{k=1}^{2p-1} \alpha_k \left( 1 + \frac{x}{R_{m,2p-1}} \right)^{m_k},
\]
with $0 \leq m_1 \leq m_2 \leq \ldots \leq m_{2p-1} = m$. Let $\lambda_{1,p} < \lambda_{2,p} < \ldots < \lambda_{p,p}$ be the zeros of the Poisson-Charlier polynomial $C(., R_{m,2p-1})$. Then there exist an odd index $j_1 \leq 2p-1$ and an integer $k_1 \leq p$ such that

$$m_{j_1} = \lfloor \lambda_{k_1,p} \rfloor \quad \text{and} \quad m_{j_1+1} = \lfloor \lambda_{k_1,p} \rfloor + 1,$$

where $\lfloor x \rfloor$ refers to the greatest integer not exceeding $x$.

Proof. By Gauss quadrature, we have

$$\int (x-t)^{2p-1} d\mu_{R_{m,2p-1}} = \sum_{i=1}^{2p-1} \omega_i (x-\lambda_{i,p})^{2p-1}; \quad \omega_i > 0 \quad \text{for} \quad i = 1, 2, \ldots, p. \quad (73)$$

Moreover, from Theorem 29, we can deduce that

$$\int (x-t)^{2p-1} d\mu_{R_{m,2p-1}} = \sum_{i=1}^{2p-1} \alpha_i (x-m_i)^{2p-1}, \quad (74)$$

where the integer sequence $(m_1, m_2, \ldots, m_{2p-3}, m_{2p-1})$ satisfies (15). Let us assume that there exists an integer $k$ ($k \leq p-1$) such that

$$m_i \notin \lfloor \lambda_{k,p}, \lambda_{k+1,p} \rfloor \quad \text{for any} \quad i = 1, 2, \ldots, 2p-1. \quad (75)$$

Let $j \leq p$ be a positive integer different from $k$. From (73) and (32), we have

$$\sum_{i=1}^{2p-1} \alpha_i (x-m_i)^{2p-1} - \sum_{i=1, i \neq j}^{p} \omega_i (x-\lambda_{i,p})^{2p-1} = \omega_j (x-\lambda_{j,p})^{2p-1}. \quad (76)$$

Eliminating the zero coefficients $\alpha_i$ (if any) from the left-hand side of (76) and in account of (75), we can easily show that, no matter how we place the integers $m_i$ relative to the real numbers $\lambda_{j,p}$ and write the left-hand side of (76) in the form (72), the number of changes of sign of the obtained sequence $(\beta_1, \ldots, \beta_r, (-1)^{2p-1}\beta_1)$ is less than $2p-1$. This contradicts Sylvester’s theorem since the number of zeros of the right-hand side of (76) is equal to $2p-1$, counting multiplicities. Thus, between $\lambda_{k,p}$ and $\lambda_{k+1,p}$ for $k = 1, 2, \ldots, p-1$, there is at least one integer from the sequence $(m_1, m_2, \ldots, m_{2p-3}, m_{2p-2}, m)$. Furthermore, since $m_1 \leq \lambda_1$, $\lambda_p \leq m$ (See Proposition 13) and due to (15), we conclude that there exist an odd index $j_1 \leq 2p-1$ and an integer $k_1 \leq p$ such that $m_{j_1} \leq \lambda_{k_1,p} \leq m_{j_1+1} = m_{j_1} + 1$. If $\lambda_{k_1,p} \neq m_{j_1+1}$ then $m_{j_1} = \lfloor \lambda_{k_1,p} \rfloor$ and $m_{j_1+1} = \lfloor \lambda_{k_1,p} \rfloor + 1$ and the claim is proved. If $\lambda_{k_1,p} = m_{j_1+1}$, we take the Christoffel transform $\mu_{\Omega_{R_{m,2p-1}}}$ of $\mu_{R_{m,2p-1}}$ where $\Omega(t) = (t-m_{j_1})(t-m_{j_1+1})$. Evaluating the polar form to both sides of (73) and (74) at $(x^{2p-3}, m_{j_1}, m_{j_1+1})$, we obtain

$$\int (x-t)^{2p-3} d\mu_{\Omega_{R_{m,2p-1}}} = \sum_{i=1, i \neq j_1, j_1+1}^{p-1} \tilde{\omega}_i (x-\lambda_{i,p})^{2p-3}, \quad (77)$$

with $\tilde{\omega}_i = \omega_i (m_{j_1} - \lambda_{i,p})(m_{j_1+1} - \lambda_{i,p}) > 0$ and

$$\int (x-t)^{2p-3} d\mu_{\tilde{\Omega}_{R_{m,2p-1}}} = \sum_{i=1, i \neq j_1, j_1+1}^{2p-1} \tilde{\alpha}_i (x-m_i)^{2p-3}, \quad (78)$$

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with \( \bar{\alpha}_i = (m_{j_1} - m_i)(m_{j_1+1} - m_i) > 0 \). Applying the same arguments as above to (77) and (78), i.e. applying Sylvester’s theorem and taking into consideration condition (15) and Proposition 13 \(^2\), enable us to derive the existence of an odd index \( j_2 \leq 2p - 1 \) \((j_2 \neq j_1)\) and an integer \( k_2 \leq p \) \((k_2 \neq k_1)\) such that \( m_{j_2} \leq \lambda_{k_2,p} \leq m_{j_2+1} = m_{j_2} + 1 \). If \( \lambda_{k_2,p} < m_{j_2+1} \) then the proposition is proved. If \( \lambda_{k_2,p} = m_{j_2+1} \) then we iterate the above process by taking the Christoffel transform of \( \mu_{R_m,2p-1} \) with respect to the annihilator polynomial \((t - m_{j_1})(t - m_{j_1+1})(t - m_{j_2})(t - m_{j_2+1})\) and so on. In the course of this iterative process we either find an odd index \( j_s \leq 2p - 1 \) and an integer \( k_s \leq p \) such that \( m_{j_s} \leq \lambda_{k_s,p} < m_{j_s+1} = m_{j_s} + 1 \) and in this case the proposition is proved, or we find that all the zeros \( \lambda_{1,p} < \lambda_{2,p} < \ldots < \lambda_{p,p} \) of the Charlier-Poisson polynomial \( C(.,R_{m,2p-1}) \) are integers with \( m_{2k-1} = \lambda_{k,p} \) for \( k = 1,2,\ldots,p \) and the coefficients associated with \( m_{2k}, k = 1,2,\ldots,p-2 \) are equal to zero. In this case we write the integer sequence \((m_1,m_2,\ldots,m_{2p-1})\) in a form that answers the claim of the proposition, i.e.,

\[
(\lambda_{1,p},\lambda_{1,p} + 1,\ldots,\lambda_{p-1,p},\lambda_{p-1,p} + 1,\lambda_{p,p}).
\]

That the integers in the representation (79) are pairwise distinct is a consequence of the well-known fact that there is at least one integer between two consecutive zeros of discrete orthogonal polynomials \([21]\).

One can iterate Proposition 31 as follows. We know from Proposition 31 that there exist an odd index \( j_2 \leq 2p - 1 \) and an integer \( k_1 \leq p \) such that \( m_{j_1} = \lfloor \lambda_{k_1,p} \rfloor \) and \( m_{j_1+1} = \lfloor \lambda_{k_1,p} \rfloor + 1 \). Define the annihilator polynomial \( \Omega_1(t) = (m_{j_1} - t)(m_{j_1+1} - t) \) By Gauss quadrature with respect to the Christoffel transform measure \( \mu_{R_{m,2p-1}}^{\Omega_1} \), we have

\[
\int (x-t)^{2p-3}d\mu_{R_{m,2p-1}}^{\Omega_1} = \sum_{i=1}^{p-1} \omega_i^1(x-\lambda_{i,p-1}^1)^{2p-3}, \quad \omega_i^1 > 0 \quad \text{for} \quad i = 1,2,\ldots,p-1.
\]

and evaluating the polar form to both sides of (35) at \((m_{j_1},m_{j_1+1},x^{[2p-3]}))\) yields

\[
\int (x-t)^{2p-3}d\mu_{R_{m,2p-1}}^{\Omega_1} = \sum_{i=1}^{p-1} \omega_i^1(x-m_i)^{2p-3},
\]

with \( \alpha_i^1 = a_i(m_{j_1} - m_i)(m_{j_1+1} - m_i) > 0 \). Thus using the same arguments as in the proof of Proposition 31, we conclude that there exist an odd integer \( j_2 \leq 2p - 3 \) \((j_2 \neq j_1, j_1+1)\) and an integer \( k_2 \leq p - 1 \) such that

\[
m_{j_2} = \lfloor \lambda_{k_2,p-1}^1 \rfloor \quad \text{and} \quad m_{j_2+1} = \lfloor \lambda_{k_2,p-1}^1 \rfloor + 1.
\]

We can again iterate the same argument this time with the annihilator polynomial \( \Omega_2(t) = (m_{j_1} - t)(m_{j_1+1} - t)(m_{j_2} - t)(m_{j_2+1} - t) \). Obviously, the above process terminates after \((p-1)\) iterations and leads to the following theorem.

**Theorem 32.** Let \( \Phi_{m,2p-1} \) be the optimal threshold polynomial

\[
\Phi_{m,2p-1}(x) = \sum_{k=1}^{2p-1} \alpha_k \left( 1 + \frac{x}{R_{m,2p-1}} \right)^{m_k}.
\]

\(^2\)In fact, a straightforward generalization of Proposition 13 with respect to Christoffel transform measures is needed here and is left to the reader as a simple exercise.
We can arrange the integer sequence \((m_1, m_2, \ldots, m_{2p-3}, m_{2p-2}, m)\) as
\[
(m_{j_1}, m_{j_1+1}, m_{j_2}, m_{j_2+1} \ldots m_{j_p}, m_{j_p+1}, m)
\]
such that for any \(1 \leq \ell \leq p\), there exists an integer \(1 \leq k_\ell \leq p - \ell + 1\) such that
\[
m_{j_\ell} = \lfloor \lambda_{k_\ell, p-\ell+1}^{\Omega_{\ell-1}} \rfloor \quad \text{and} \quad m_{j_\ell+1} = \lfloor \lambda_{k_\ell, p-\ell+1}^{\Omega_{\ell-1}} \rfloor + 1,
\]
where \((\lambda_{1, p-\ell+1}^{\Omega_{\ell-1}}, \lambda_{2, p-\ell+1}^{\Omega_{\ell-1}}, \ldots, \lambda_{p-\ell+1, p-\ell+1}^{\Omega_{\ell-1}})\) are the zeros of the degree \((p - \ell + 1)\) orthogonal polynomial \(\Pi_{p-\ell+1}^{\Omega_{\ell-1}}\) where the annihilator polynomial \(\Omega_{\ell-1}\) is given by \(\Omega_{\ell-1}(t) = \prod_{i=1}^{\ell-1}(m_j - t)(m_{j+1} - t)\) and \(\Omega_0(t) \equiv 1\).

Let us illustrate Theorem 32 by an example.

**Example 19.** Using the algorithm of Section 8, it can be shown that \(R_{14,7} \simeq 6.0907\) is the unique positive real zero of the polynomial equation
\[
R^7 - 28R^6 + 380R^5 - 3260R^4 + 19080R^3 - 75960R^2 + 189360R - 226800 = 0.
\]
Moreover, the exponents of the optimal threshold polynomial \(\Phi_{14,7}\) are
\[
(m_1, m_2, m_3, m_4, m_5, m_6, m_7) = (2, 3, 5, 6, 9, 10, 14).
\]

Figure 1 ((a);(1)) shows the location of the exponents \(m_i, i = 1, 2, \ldots, 7\) (blue bars) relative to the location of the zeros of the Poisson-Charlier polynomial \(C_4(., R_{14,7})\) (red bars). According to Theorem 32, there is a least one zero of \(C_4(., R_{14,7})\) that is between exponents of the form \((q, q+1)\) of the optimal threshold polynomial. In this specific case, each zero (except for the largest one) of \(C_4(., R_{14,7})\) is between two exponents of the form \((q, q+1)\) of \(\Phi_{14,7}\) (see Figure 1 ((a);(1))). Figure 1 ((a);(2)) shows the location of the zeros of the degree 3 orthogonal polynomial \(\Pi_{3}^{\Omega_{1}}\) of the Christoffel transform measure \(\mu_{\Pi_{3}^{\Omega_{1}}}^{\Omega_{1}}\) associated with the annihilator polynomial \(\Omega_{1}(t) = (m_1 - t)(m_2 - t)\) (red bars) and the location of the remaining exponents \((m_3, m_4, m_5, m_6, m_7)\) of \(\Phi_{14,7}\). Here again and in accordance with Theorem 32, there exists a least one zero of \(\Pi_{3}^{\Omega_{1}}\) that is between exponents of the form \((q, q+1)\) of the remaining exponents. In this specific example, each zero (except for the largest one) of \(\Pi_{3}^{\Omega_{1}}\) is between two exponents of the form \((q, q+1)\) (see Figure 1, (a),(2)). Similarly, Figure 1 ((a);(3)) shows the
zeros of $\Pi_{1,2}^{R}$ with annihilator polynomial $\Omega_2(t) = (t - m_1)(t - m_2)(t - m_3)(t - m_4)$ relative to the remaining exponents $(m_5, m_6, m_7)$. Finally as

$$h_7(m_1, m_2, m_3, m_4, m_5, m_6, m_7; R_{14,7}) = 0,$$

the zero of $\Pi_{3}^{\Omega}$ with $\Omega_3(t) = \prod_{i=1}^{R} (t - m_i)$ is equal to $m = 14$. This is confirmed by Figure 1, ((a);(4)). This specific example offers various choice for the exponents where we can perform the Christoffel transformation at each stage of the iterative process. This is in contrast with the case of $R_{15,7}$ and its associated polynomial $\Phi_{15,7}$.

Using the algorithm described in Section 8, we find that $R_{15,7} \approx 6.8035$ is the unique positive real zero of the polynomial equation

$$R^7 - 33R^6 + 516R^5 - 4956R^4 + 31500R^3 - 132300R^2 + 340200R - 415800 = 0.$$

The exponents of the optimal threshold polynomial $\Phi_{15,7}$ are

$$(m_1, m_2, m_3, m_4, m_5, m_6, m_7) = (2, 3, 6, 7, 10, 11, 15).$$

As can be seen from Figure 1 (b), at each stage of the iterative process described in Theorem 32 there exists only one zero of the Christoffel transform orthogonal polynomials that is between exponents of the form $(q, q + 1)$ of the optimal threshold polynomial.

### 7.1. Integral spectrum and integral spectral radius.

Let $\Omega$ be an admissible polynomial and $\lambda^{\Omega}_{i,n}(R), \ldots, \lambda^{\Omega}_{i,n}(R)$ the zeros of the orthogonal polynomial $\Pi_{n,\Omega}^{R}$. Theorem 33. The functions $\lambda^{\Omega}_{i,n}(R), k = 1, 2, \ldots, n$ are strictly increasing functions with respect to the parameter $R$.

**Proof.** The proof of the theorem follows a classical argument by Markov. By Gauss quadrature, for any polynomial $P$ of degree $2n - 1$, we have

$$\int P(t)d\mu_{R}^{\Omega}(t) = \sum_{i=1}^{n} \alpha_i(R)P(\lambda^{\Omega}_{i,n}(R)),$$

or equivalently,

$$e^{-R} \sum_{j=0}^{\infty} P(j)\Omega(j) \frac{R^j}{j!} = \sum_{i=1}^{n} \alpha_i(R)P(\lambda^{\Omega}_{i,n}(R)).$$

Differentiating (80) with respect to the parameter $R$, we obtain

$$- \int P(t)d\mu_{R}^{\Omega}(t) + e^{-R} \sum_{j=0}^{\infty} P(j + 1)\Omega(j + 1) \frac{R^j}{j!} =$$

$$\sum_{i=1}^{n} \frac{\partial \alpha_i(R)}{\partial R} P(\lambda^{\Omega}_{i,n}(R)) + \sum_{i=1}^{n} \alpha_i(R) \frac{\partial \lambda^{\Omega}_{i,n}(R)}{\partial R} P'(\lambda^{\Omega}_{i,n}(R)).$$

Now we specialize our analysis to the polynomial

$$P_k(t) = \frac{(\Pi_{n,\Omega}^{R}(t))^{2}}{t - \lambda^{\Omega}_{k,n}(R)}, \quad 1 \leq k \leq n.$$
Moreover, we have
\begin{equation}
\int P_k(t+1)\Omega(t+1)d\mu_R(t) = \alpha_k(R) \frac{\partial^{\lambda_{k,n}^\alpha(R)}}{\partial R} P_k(\lambda_{k,n}^\alpha(R)).
\end{equation}

Moreover, we have
\begin{align*}
\int P_k(t+1)\Omega(t+1)d\mu_R(t) &= \int P_k(t+1)\Omega(t+1)d\mu_R(t) - \frac{\lambda_{k,n}^\alpha(R)}{R} \int P_k(t)\Omega(t)d\mu_R(t) \\
&= \frac{1}{R} \int (t - \lambda_{k,n}^\alpha(R)) P_k(t)\Omega(t)d\mu_R(t) \\
&= \frac{1}{R} \int (\Pi_n^{R,\alpha}(t))^2 d\mu_R(t) > 0.
\end{align*}

Thus, from (82), we deduce that $\frac{\partial^{\lambda_{k,n}^\alpha(R)}}{\partial R} > 0$. This concludes the proof.

**Definition 34.** A finite sequence of non-negative integers $n = \{n_1, n_2, \ldots, n_{p-1}\}$ is called a $p$-configuration if for $k = 1, \ldots, p-1$
\begin{equation}
1 \leq n_k \leq p - k + 1.
\end{equation}

Given a positive real number $R$ and a $p$-configuration $n = \{n_1, n_2, \ldots, n_{p-1}\}$, we construct a sequence of integers $(m_1, m_2, \ldots, m_{2p-3}, m_{2p-2})$ iteratively as follows:
\begin{equation}
m_1 = |\lambda_{n_1,p}^\alpha(R)|; \quad m_2 = m_1 + 1,
\end{equation}
where $\lambda_{n_1,p}^\alpha(R) < \lambda_{n_2,p}^\alpha(R) < \ldots < \lambda_{n_{p-1},p}^\alpha(R)$ are the zeros of the Poisson-Charlier orthogonal polynomial $C_{\alpha}(., R)$. The integers $m_{2k-1}, m_{2k}$ are defined by
\begin{equation}
m_{2k-1} = |\lambda_{n_{2k-1},p-k+1}^\alpha(R)|; \quad m_{2k} = m_{2k-1} + 1,
\end{equation}
where the annihilator polynomial $\Omega_{k-1}$ is given by $\Omega_{k-1}(t) = \prod_{i=1}^{k-1} (m_{2i-1} - t)(m_{2i} - t)$

**Definition 35.** The integer sequence $\mathcal{M}(n, R) = \{m_1, m_2, \ldots, m_{2p-3}, m_{2p-2}\}$ obtained from (83) and (84) is termed the integral spectrum associated with the $p$-configuration $n$ and the positive real number $R$. Moreover, the unique zero $\rho(n, R)$ of the degree 1 polynomial orthogonal with respect to the measure $\mu_R^{\alpha}$ is called the spectral radius with respect to the couple $(n, R)$. The real number $\rho(n, R)$ is given explicitly by
\begin{equation}
\rho(n, R) = \frac{\int tP(t)d\mu_R(t)}{\int P(t)d\mu_R(t)} \quad \text{where} \quad P(t) = \prod_{j \in \mathcal{M}(n, R)} (t-j).
\end{equation}

**Example 21.** Figure 2 shows the integral spectrum (bleu bars) associated with $R = 5$ and the configurations $n_1 = \{1, 1, 1\}$ (Figure 2; a) and $n_2 = \{3, 1, 1\}$ (Figure 2; b). The red bars show the zeros of the orthogonal polynomials $\Pi_{4-k+1}^{R,\Omega_{k-1}}$ for $k = 1, 2, 3, 4$ respectively. In this example, we find
\begin{align*}
\mathcal{M}(n_1, R) &= \{1, 2, 4, 5, 8, 9\} \quad \text{and} \quad \rho(n_1, R) \simeq 12.4539, \\
\mathcal{M}(n_2, R) &= \{7, 8, 1, 2, 4, 5\} \quad \text{and} \quad \rho(n_1, R) \simeq 12.4134.
\end{align*}
Continuous and strictly increasing function of the parameter $n$. Therefore, $M$ shows that

Let $\epsilon > 0$ be a small enough positive real number. Thus an iterative use of Theorem 33 shows that

$$m_{2k-1} \leq \lambda_{n_k, p-k+1}^{\Omega_{k-1}}(R + \epsilon) < m_{2k}, \quad k = 1, 2, \ldots, p - 1.$$  

Let $\epsilon > 0$ be a small enough positive real number. Thus an iterative use of Theorem 33 shows that

$$m_{2k-1} \leq \lambda_{n_k, p-k+1}^{\Omega_{k-1}}(R + \epsilon) < m_{2k}, \quad k = 1, 2, \ldots, p - 1.$$  

Therefore, $M(n, R) = M(n, R + \epsilon)$ and the right-continuity of $\rho(n, R)$ follows from the fact that for any $r$ such that $R \leq r \leq R + \epsilon$, $\rho(n, r)$ is given by the continuous function in $r$

$$\frac{\rho(n, r)}{\int P(t)d\mu(t)} = \prod_{j \in \mathcal{M}_n(R)} (t - j).$$

Proving the left-continuity of the function $\rho(n, R)$ is technically more difficult. The main reason is that some of the left inequalities $(85)$ can become equalities. Suppose there exists a certain $k \leq p - 1$ such that $m_{2k-1} = \lambda_{n_k, p-k+1}^{\Omega_{k-1}}(R)$. Then, for any positive real number $\epsilon$, no matter how small it is, we have $M(n, R) \neq M(n, R - \epsilon)$ and thus we cannot use the same arguments as in the proof of the right-continuity. Therefore, if all the inequalities in $(85)$ as strict one for a given $R$ then we can use exactly the same arguments as in the proof of the right-continuity to show the left-continuity of the function $\rho(n, R)$ at $\bar{R}$. Let us now take an $R$ for which there exist integers $k \leq p - 1$ with $m_{2k-1} = \lambda_{n_k, p-k+1}^{\Omega_{k-1}}(R)$ where $(m_1, m_2, \ldots, m_{2p-3}, m_{2p-2}) = M(n, R)$. Denote by $\mathcal{I}$ the set of indices $k$ such that $m_{2k-1} = \lambda_{n_k, p-k+1}^{\Omega_{k-1}}(R)$. Define the real numbers $\tilde{M} = (\tilde{m}_1, \tilde{m}_2, \ldots, \tilde{m}_{2p-2})$ by

$$\tilde{m}_{2i-1} = m_{2i-1} \quad \text{for} \quad i = 1, 2, \ldots, p - 1,$$

$$\tilde{m}_{2i} = m_{2i} \quad \text{if} \quad 2i - 1 \notin \mathcal{I},$$

$$\tilde{m}_{2i} = m_{2i-1} - \frac{1}{2} \quad \text{if} \quad 2i - 1 \in \mathcal{I}.$$
Obviously, we have

\[ \tilde{m}_{2k-1} < \lambda_{n_k,p-k+1}^{\Omega_k-1}(R) \leq \tilde{m}_{2k}, \quad k = 1, 2, \ldots, p - 1. \]

We shall show that

\[ \rho(n, R) = \int \frac{tQ(t) d\mu_{R}(t)}{Q(t) d\mu_{R}(t)} \quad \text{with} \quad Q(t) = \prod_{k=1}^{2p-2} (t - \tilde{m}_k). \tag{87} \]

Once (87) shown, the proof of the left-continuity follows the same arguments as the one we used for the right-continuity. Namely, we take a small enough positive real number \( \epsilon \) and by an iterative application of Theorem 33 while taking into account inequalities (86), leads to

\[ \tilde{m}_{2k-1} < \lambda_{n_k,p-k+1}^{\Omega_k-1}(R - \epsilon) \leq \tilde{m}_{2k}, \quad k = 1, 2, \ldots, p - 1. \]

Therefore, the left-continuity becomes a consequence of the fact that for any real number \( r \) such that \( R - \epsilon \leq r \leq R \), \( \rho(n, r) \) is given by the continuous function in \( r \)

\[ \rho(n, r) = \int \frac{tQ(t) d\mu_{r}(t)}{Q(t) d\mu_{r}(t)} \quad \text{with} \quad Q(t) = \prod_{k=1}^{2p-2} (t - \tilde{m}_k). \]

To show (87) we proceed as follows. Let \( k \) be the smallest index in \( I \). By Gauss quadrature with respect to the measure \( \mu_R^{\Omega_k-1} \), we have

\[ \int (x - t)^2(p-k)+1 d\mu_{R}^{\Omega_k-1} = \sum_{j=1}^{p-k+1} \alpha_j (x - \lambda_{j,p-k+1}^{\Omega_k-1})^{2(p-k)+1}, \tag{88} \]

with \( \lambda_{n_k,p-k+1} = m_{2k-1} \) and \( \alpha_j > 0 \) for \( j = 1, 2, \ldots, p-k+1 \). Let \( \eta \) be a real number such that \( m_{2k-1} - 1 \leq \eta \leq m_{2k-1} + 1 \). Evaluating the polar form of both sides of (88) at \( (x^{2(p-k)-1}, m_{2k-1}, \eta) \), we obtain

\[ \int (x - t)^2(p-k)-1 (m_{2k-1} - t)(\eta - t) d\mu_{R}^{\Omega_k-1} = \sum_{j=1,j \neq n_k}^{p-k+1} \tilde{\alpha}_j (x - \lambda_{j,p-k+1}^{\Omega_k-1})^{2(p-k)-1}, \tag{89} \]

with \( \tilde{\alpha}_j = \alpha_j (m_{2k-1} - \lambda_{j,p-k+1}^{\Omega_k-1})(\eta - \lambda_{j,p-k+1}^{\Omega_k-1}) > 0 \) for \( j = 1, \ldots, p-k+1; j \neq n_k \). This positivity is a direct consequence of the fact that the orthogonal polynomial \( P_{\Omega_{k-1}}^{\eta} \) has no zero other than \( m_{2k-1} = \lambda_{n_k,p-k+1}^{\Omega_k-1} \) in the interval \([m_{2k-1} - 1, m_{2k-1} + 1]\). Identity (89) is thus the Gauss quadrature with respect to the Christoffel transform measure with annihilator polynomial \( \Omega_{\eta}(t) = (m_{2k-1} - t)(\eta - t) \Omega_{k-1}(t) \). This shows in particular that \( \lambda_{j,p-k+1}^{\Omega_{k-1}}, j = 1, 2, \ldots, p-k+1; j \neq n_k \) are the zeros of the orthogonal polynomial \( P_{\eta}^{\Omega_{k-1}} \) and that these zeros does not depend on the real number \( \eta \). From the definition of the integral spectrum \( M(n, R) \), if we take \( \eta = m_{2k} \), then up to an adequate normalization, we have

\[ P_{\eta}^{\Omega_{k-1}} = P_{j}^{\Omega_{k-1}}, \quad j = 1, 2, \ldots \]

If instead of \( \eta = m_{2k} \), we take \( \eta = m_{2k-1} - 1/2 \), then we still have (90) to an adequate normalization. Accordingly, changing the value of \( m_{2k} \) to \( \tilde{m}_{2k} = m_{2k-1} - 1/2 \) in the
integral spectrum \( \mathcal{M}(n, R) \) does not change the value of the spectral radius \( \rho(n, R) \) in the sense that we have

\[
\rho(n, R) = \frac{\int tQ_1(t)d\mu_R(t)}{\int Q_1(t)d\mu_R(t)} = \frac{\int tQ_2(t)d\mu_R(t)}{\int Q_2(t)d\mu_R(t)},
\]

where \( Q_1(t) = \prod_{j=1}^{2^p-2}(t-m_j) \) and \( Q_2(t) = (t-m_{2k})\prod_{j=1, j\neq 2k}^{2^p-2}(t-m_k) \). Applying iteratively the same arguments to each index in \( R \) (from the smallest index to the largest one), proves (87). To prove that the function \( \rho(n, R) \) is a strictly increasing function of the parameter \( R \) it is sufficient to prove that for any \( \epsilon > 0 \) small enough, we have \( \rho(n, R + \epsilon) > \rho(n, R) \). As already shown, we can always choose an \( \epsilon > 0 \) small enough such that

\[
\mathcal{M}(n, R + \epsilon) = \mathcal{M}(n, R).
\]

According to Theorem 33, \( \rho(n, R + \epsilon) > \rho(n, R) \) since for any \( R \leq r \leq R + \epsilon, \rho(n, r) \) is the unique zero of the orthogonal polynomial \( \Pi_{\nu-1}^{R_{\nu}} \) where \( \Omega_{\nu-1}(t) = \prod_{j=1}^{2\nu-2}(t-m_j) \).

Given a positive integer \( m \) and a \( p \)-configuration \( n \), with \( m \geq 2p - 1 \), then according to Proposition 36, there exists a unique real number \( R_{m,p}(n) \) such that

\[
\rho(n, R_{m,p}(n)) = m.
\]

The real number \( R_{m,p}(n) \) will be called the **optimal spectral radius** with respect to the \( p \)-configuration \( n \) and the integer \( m \). The associated integral spectrum

\[
\mathcal{M}(n, R_{m,p}(n)) = \{m_1, m_2, \ldots, m_{2p-2}, m_{2p-2}\}
\]

will be termed the **optimal integral spectrum** and denoted by \( M_{m,p}(n) \).

We shall need the following simple yet, important result.

**Proposition 37.** Let \( m \) and \( p \) be positive integers \( m \geq 2p - 1 \) and \( n \) be a \( p \)-configuration. If \( R_1 \) and \( R_2 \) are two real numbers such that

\[
M(n, R_1) = M(n, R_2) \quad \text{and} \quad \rho(n, R_1) \leq m \leq \rho(n, R_2),
\]

then the optimal integral spectrum with respect to \( n \) and to the integer \( m \) is given by \( M_{m,p}(n) = M(n, R_1) \) and the optimal spectral radius \( R_{m,p}(n) \) is the unique positive real zero in \([R_1, R_2]\) of the polynomial equation in \( R \)

\[
h_{2p-1}(m_1, m_2, \ldots, m_{2p-2}, m; R) = 0,
\]

where \( (m_1, m_2, \ldots, m_{2p-2}) = M_{m,p}(n) \) and \( h_{2p-1}(-, R) \) the polar form of the polynomial \( \mathcal{H}_{2p-1}(\cdot; R) \).

**Proof.** As shown in Proposition 36, the function \( \rho(n, R) \) is continuous and strictly increasing with respect to \( R \). Thus, from the inequalities (91), we conclude that there exists a real number \( R \) such that \( R_1 \leq R \leq R_2 \) such \( \rho(n, R) = m \). Moreover, we have

\[
m_{2k-1} \leq \lambda_{n_{p-k+1}}^{\Omega_{\nu-1}}(R_j) < m_{2k}, \quad k = 1, 2, \ldots, p - 1 \quad \text{and} \quad j = 1, 2.
\]

Thus an iterative use of Theorem 33 shows that for any real number \( R \) such that \( R_1 \leq R \leq R_2 \) we have \( M(n, R) = M(n, R_1) \). That the optimal spectral radius \( R_{m,p}(n) \) is the unique zero in \([R_1, R_2]\) of the polynomial equation (92) is a direct consequence of (28). This concludes the proof. 

\[\square\]
With the definitions and results shown above, Theorem 32 can be restated as follows.

**Theorem 38.** For positive integers $m$ and $p$ such that $m \geq 2p - 1$, there exists a least one $p$-configuration $n$ such that

$$R_{m,2p-1} = R_{m,p}(n),$$

and such that the optimal threshold polynomial is given by

$$\Phi_{m,2p-1}(x) = \sum_{k=1}^{2p-1} \alpha_k \left(1 + \frac{x}{R_{m,2p-1}}\right)^{m_k},$$

where $(m_1, m_2, \ldots, m_{2p-3}, m_{2p-2}) = \mathcal{M}_{m,p}(n)$ and $m_{2p-1} = m$.

The following theorem is an essential part in the algorithmic aspect for computing the optimal threshold factors. It roughly states that we do not need to check all the $p$-configurations to compute $R_{m,2p-1}$ and its associated optimal threshold polynomial.

**Theorem 39.** Let $m$ and $p$ be positive integers such that $m \geq 2p - 1$. Let $n$ be a $p$-configuration such that the system

$$(93) \quad \int (x-t)^{2p-1} d\mu_{R_{m,p}(n)} = \sum_{i=1}^{2p-1} \alpha_i (x-m_i)^{2p-1},$$

where $(m_1, \ldots, m_{2p-3}, m_{2p-2}) = \mathcal{M}_{m,p}(n)$ and $m_{2p-1} = m$, admits a non-negative solution in $(\alpha_1, \alpha_2, \ldots, \alpha_{2p-1})$. Then

$$R_{m,2p-1} = R_{m,p}(n) \quad \text{and} \quad \Phi_{m,2p-1}(x) = \sum_{k=1}^{2p-1} \alpha_k \left(1 + \frac{x}{R_{m,2p-1}}\right)^{m_k}.$$ 

**Proof.** That $R_{m,p}(n) \leq R_{m,2p-1}$ is a direct consequence of Corollary 6. For the sake of simplicity, in the rest of the proof, we denote $R_{m,p}(n)$ simply by $R$. Let $\epsilon > 0$ be a small enough real number such that

$$\mathcal{M}(n, R + \epsilon) = \{m_1, m_2, \ldots, m_{2p-3}, m_{2p-2}\},$$

and let $\rho(n, R + \epsilon)$ be the associated spectral radius. Then according to Proposition 36, $\rho(n, R + \epsilon) > m$, in other word

$$(94) \quad \frac{\int t \prod_{i=1}^{2p-2} (t-m_i) d\mu_{R+\epsilon}}{\int \prod_{i=1}^{2p-2} (t-m_i) d\mu_{R+\epsilon}} > m.$$ 

The polynomial

$$f(t) = (m-t) \prod_{k=1}^{2p-2} (t-m_k)$$

satisfies $f(j) \geq 0$ for $j = 0, 1, \ldots, m$. Moreover, using (94) we obtain

$$\int f(t) d\mu_{R+\epsilon} = m \int \prod_{k=1}^{2p-2} (t-m_k) d\mu_{R+\epsilon} - \int \prod_{k=1}^{2p-2} t(t-m_k) d\mu_{R+\epsilon} < 0.$$
Thus according to Theorem 17, \( R_{m,2p-1} \leq R + \epsilon \) for any \( \epsilon \) small enough. This completes the proof.

**Example 25.** To illustrate the importance of Theorem 39, in this example we compute the value \( R_{100,5} \) using the 3-configuration \( n = \{1, 1\} \). From the bounds (9) and (10), we have

\[
\ell_3^{(95)} = 81.1972 \leq R_{100,5} \leq 83.023 = \ell_3^{(97)}.
\]

Computing the associated integral spectrum and spectral radius, we obtain

\[
\mathcal{M}(n, \ell_3^{(95)}) = (66, 67, 81, 82) \quad \text{and} \quad \rho(n, \ell_3^{(95)}) = 98.01.
\]

and

\[
\mathcal{M}(n, \ell_3^{(97)}) = (68, 69, 83, 84) \quad \text{and} \quad \rho(n, \ell_3^{(97)}) = 100.02.
\]

As the integral spectrum associated with \( \ell_3^{(95)} \) and \( \ell_3^{(97)} \) are different, we cannot yet compute the optimal integral spectrum associated with the configuration \( n \) and the integer \( m = 100 \). Now, one can show that the integral spectrum associated with \( R = 82.7 \) is given by

\[
\mathcal{M}(n, 82.7) = (68, 69, 83, 84) \quad \text{and} \quad \rho(n, 82.7) = 99.66.
\]

Since \( \mathcal{M}(n, 82.7) = \mathcal{M}(n, \ell_3^{(97)}) \) and \( \rho(n, 82.7) \leq 100 \leq \rho(n, \ell_3^{(97)}) \) then, according to Proposition 36, the optimal integral spectrum associated with the 3-configuration \( n \) is \( (m_1, m_2, m_3, m_4) = (68, 69, 83, 84) \) and the optimal spectrum radius \( R_{100,3}(n) \) is given by the unique positive zero in \([82.7, 83.032]\) of the equation

\[
h_5(68, 69, 83, 84, 100; R) = 0, \quad \text{i.e.:} \quad R_{100,3}(n) \simeq 83.002.
\]

Moreover, the coefficients \( \alpha_1, \alpha_2, \ldots, \alpha_5 \), solution to the associated linear system (93) are given by

\[
(\alpha_1, \alpha_2, \ldots, \alpha_5) = (0.1188, 0.0765, 0.2095, 0.4539, 0.1413).
\]

Since these coefficients are non-negative, we conclude according to Theorem 39 that

\[
R_{100,5} = R_{100,3}(n) \simeq 83.002,
\]

or more precisely, \( R_{100,5} \) is the unique zero in the interval \([82.7, 83.023]\) of the polynomial equation

\[
R^5 - 394R^4 + 62544R^3 - 5001012R^2 + 201456936R - 3271262400 = 0.
\]

Moreover, the optimal threshold polynomial is given by

\[
\Phi_{100,5}(x) = \sum_{k=1}^{5} \alpha_k \left( 1 + \frac{x}{R_{100,5}} \right)^{m_k},
\]

where \( (m_1, m_2, m_3, m_4, m_5) = (68, 69, 83, 84, 100) \) and \( \alpha_k, k = 1, \ldots, 5 \), are given by (95). Note that if we want to compute \( R_{101,5} \), it is better to start with the bounds

\[
83.002 \simeq R_{100,5} < R_{101,5} \leq \ell_3^{(98)} \simeq 83.936.
\]
8. Algorithm for the computation of the optimal threshold factors.

Example 25 exhibits all the ingredients for computing the optimal threshold factor \(R_{m,2p-1}\) and its associated polynomial \(\Phi_{m,2p-1}\). In this section, we go into the details of the algorithm and how to improve several of its key aspects.

**Computation of the integral spectrum:** Given a \(p\)-configuration \(n\) and a real number \(R\), the computation of \(\mathcal{M}(n,R)\) and \(\rho(n,R)\) amounts to computing the zeros of the orthogonal polynomials associated with the Christoffel transform measure \(\mu_R^\Omega\) with an annihilator polynomial \(\Omega\) of the form \(\Omega(t) = \prod_{k=1}^{2s} (t - m_k)\). The orthogonal polynomials \((\Pi_n^\Omega)_{n \geq 1}\) with respect to \(\mu_R^\Omega\) can be constructed by means of the Christoffel formulas \([11]\)

\[
\Pi_n^\Omega(t) = \frac{1}{\Omega(t)} \begin{vmatrix}
    C_n(t,R) & C_{n+1}(t,R) & \cdots & C_{n+2s}(t,R) \\
    C_n(m_1,R) & C_{n+1}(m_1,R) & \cdots & C_{n+2s}(m_1,R) \\
    \vdots & \vdots & \ddots & \vdots \\
    C_n(m_{2s},R) & C_{n+1}(m_{2s},R) & \cdots & C_{n+2s}(m_{2s},R)
\end{vmatrix}
\]

However, a more efficient method for computing \((\Pi_n^\Omega)_{n \geq 1}\) and their zeros consists in deriving the three-term recurrence relation for \((\Pi_n^\Omega)_{n \geq 1}\) from the one of Poisson-Charlier polynomials. Recall that, if a set of orthogonal polynomials satisfies the three-term recurrence relation

\[
p_i(t) = (t - b_i)p_{i-1}(t) - g_ip_{i-2}(t), \quad i = 1, 2, \ldots,
\]

with \(p_0 = 1\) and \(p_{-1} = 0\), then the zeros of \(p_n\) are the eigenvalues of the tridiagonal matrix \(J_n\)

\[
J_n = \begin{vmatrix}
    b_0 & \sqrt{g_1} & & \\
    \sqrt{g_1} & b_1 & \sqrt{g_2} & \\
    & \sqrt{g_2} & \ddots & \\
    & & \ddots & \sqrt{g_{n-1}} \\
    & & & b_{n-1}
\end{vmatrix}
\]

Poisson-Charlier polynomials satisfy (96) with \(b_i = R + i - 1\) and \(g_i = (i - 1)R\). The orthogonal polynomials \((\Pi_n^\Omega)_{n \geq 1}\) satisfy the three-term recurrence relation

\[
\Pi_i^\Omega,R(t) = (t - B_i^{(2s)})\Pi_{i-1}^\Omega,R(t) - G_i^{(2s)}\Pi_{i-2}^\Omega,R(t), \quad i = 1, 2, \ldots,
\]

with \(\Pi_0^\Omega,R \equiv 1\) and \(\Pi_{-1}^\Omega,R \equiv 0\) and the coefficients \(B_i^{(2s)}\) and \(G_i^{(2s)}\) are computed using Algorithm 1 [11]

**Computation of the optimal integral spectrum:** Let \(m\) be a positive integer \((m \geq 2p - 1)\) and \(\mathbf{n}\) be a \(p\)-configuration. A consequence of Proposition 36 is that if \(R_1\) and \(R_2\) are two real numbers such that

\[
\mathcal{M}(\mathbf{n}, R_1) = \mathcal{M}(\mathbf{n}, R_2) \quad \text{and} \quad \rho(\mathbf{n}, R_1) \leq m \leq \rho(\mathbf{n}, R_2),
\]

then the optimal integral spectrum is given by \(\mathcal{M}_{m,p}(\mathbf{n}) = \mathcal{M}(\mathbf{n}, R_1)\) and the optimal spectral radius \(\mathcal{R}_{m,p}(\mathbf{n})\) is the unique positive real zero in \([R_1, R_2]\) of the polynomial equation

\[
h_{2p-1}(m_1, m_2, \ldots, m_{2p-2}, m; R) = 0,
\]
Algorithm 1 Computing the coefficients of the three-term recurrence relation of the orthogonal polynomials \((\Pi_n^\Omega)_{n \geq 1}\)

Define \(B_j^{(0)} = b_j = R + j - 1, G_j^{(0)} = g_j = (j - 1)R; \ j = 1, 2, \ldots\)

Define \(B_j^{(k)}; G_j^{(k)}; \ j = 1, 2, \ldots; \ k = 1, 2, \ldots, 2s\) by

\[
E_0 = 0
\]

\[
Q_j = B_j^{(k-1)} - E_{j-1} - m_k
\]

\[
E_j = G_j^{(k-1)}/Q_j
\]

\[
B_j^{(k)} = m_k + Q_j + E_j
\]

\[
G_j^{(k)} = Q_jE_{j-1}
\]

return \(B_j^{(2s)}, G_j^{(2s)}\)

where \((m_1, m_2, \ldots, m_{2p-2}) = \mathcal{M}(n)\). In our algorithm, a search for the real numbers \(R_1, R_2\) is performed via a dichotomy starting from the values \(\ell_p^{(m-2p+1)}\) and \(\ell_p^{(m-p)}\) obtained in our bounds of the optimal threshold factors. This is achieved using Algorithm 2.

Algorithm 2 Computation of the optimal integral spectrum

Define \(R_{\text{max}} = \ell_p^{(m-p)}\) and \(R_{\text{min}} = \ell_p^{(m-2p+1)}\)

while \(\mathcal{M}(n, R_{\text{max}}) \neq \mathcal{M}(n, R_{\text{min}})\) do

\[
R = (R_{\text{min}} + R_{\text{max}})/2
\]

Compute \(\mathcal{M}(n, R)\) and \(\rho(n, R)\)

if \(\rho(n, R) > m\) then

Set \(R_{\text{max}} = R, \mathcal{M}(n, R_{\text{max}}) = \mathcal{M}(n, R)\)

else

Set \(R_{\text{min}} = R, \mathcal{M}(n, R_{\text{min}}) = \mathcal{M}(n, R)\)

end if

end while

Compute \(\mathcal{R}_{m,p}(n)\) as the unique positive zero in \([R_{\text{min}}, R_{\text{max}}]\) of

\[
\mathcal{H}_{2p-1}(x; \mathcal{R}_{m,p}(n)) = \sum_{k=1}^{p} \omega_k(x - \lambda_k)^{2p-1},
\]

where \(\omega_k, \lambda_k\) are the weights and the nodes of the Gaussian quadrature with respect
to the measure $\mu_{R_{m,p}(n)}$. Thus using Equations (29), we obtain

$$\alpha_k = \sum_{i=1}^{p} \omega_i \prod_{j=1,j\neq k}^{2p} \frac{m_j - \lambda_i}{m_j - m_i}, \quad k = 1, 2, \ldots, 2p - 1,$$

where $m_{2p-1} = m$ and $m_{2p} = m + 1$. An efficient algorithm for computing the weights and nodes of Gaussian quadrature from the three-term recurrence relation is the Golub-Welsch algorithm which can be found in [13].

**Computation of the optimal threshold factors and associated polynomials:**

The general algorithm for computing $R_{m,2p-1}$ and $\Phi_{m,2p-1}$ is given in Algorithm 3. The initial $p$-configuration $n$ is chosen to be $\{1, 1, \ldots, 1\}$. This choice is motivated by

\begin{algorithm}
\begin{algorithmic}
\State Set $\alpha_k = -1, k = 1, 2, \ldots, 2p - 1$
\While{one of the $\alpha_k, k = 1, 2, \ldots, 2p - 1$ is negative}
\State Select a $p$-configuration $n$
\State Compute $M_{m,p}(n)$ and $R_{m,p}(n)$
\State Compute $\alpha_k; k = 1, 2, \ldots, 2p - 1$
\EndWhile
\State return $M_{m,p}(n), R_{m,p}(n), \alpha_k, k = 1, 2, \ldots, 2p - 1$
\end{algorithmic}
\end{algorithm}

the observation that this configuration always leads to the optimal threshold factors $R_{m,5}$ for $m = 5, 6, \ldots, 2000$, i.e.,

$$R_{m,5} = R_{m,3}(\{1, 1\}) \quad \text{for} \quad k = 5, 6, \ldots, 2000.$$ 

The values of $R_{m,n}$ for $n = 5, 7, 9, 11$ and $m = 5k, k = 1, 2, \ldots, 40$ are shown in Table 2. An asterisk indicates a value for which the $p$-configuration $\{1, 1, \ldots, 1\}$ fails to provide for the optimal threshold factor, i.e., for which

$$R_{m,2p-1} \neq R_{m,p}(\{1, 1, \ldots, 1\}).$$

The values of $R_{m,3}$ are not given in Table 2 since an explicit expression for these values is known and given in [22]. Note that, for each $m \geq 2p - 1$, there may exist many $p$-configurations $n$ such that

$$R_{m,2p-1} = R_{m,p}(n).$$

Algorithm 3 terminates once it reaches any such configurations. This partially explain the high efficiency of the algorithm. After the initialization of $n$, we change the $p$-configurations randomly. Note that the complexity of the algorithm is independent of the degree $m$ of the polynomials. To make this more explicit, let us mention that the computational burdens for computing either $R_{10,5}$ or $R_{10^{m},5}$ is exactly the same. This feature is absent in all the existing algorithms in the literature. For instance, in our experiment using Matlab in an Intel(R) Core(TM) 3.20 GHz environment, the computation of all the values $R_{m,5}, m = 6, 7, \ldots, 2000$ took less than 84 seconds, while the computation of all the values $R_{m,7}, m = 8, 9, \ldots, 2000$ took less than 110 seconds.

9. **Conclusion.** In this work we provide sharp upper and lower bounds for the optimal threshold factors of one-step methods. An efficient algorithm based on adaptive Christoffel transformations of Poisson-Charlier measure is proposed. A deep
understanding of the set of $p$-configurations that lead to the optimal threshold factor or at least an estimate of the number of such configurations is missing and we believe it to be a rather challenging problem. Moreover, the techniques introduced in this work can be adapted to solve the following integer quadrature problem: let $\mu_R$ be a discrete positive measure supported on $\mathbb{N}$, with finite moments of all orders and which depends on a parameter $R \in [0, \infty]$.

$$\mu_R = \sum_{j=0}^{\infty} a_j(R) \delta_j.$$ 

Let us further assume that the zeros of the orthogonal polynomials associated with $\mu_R$ are strictly increasing functions of the parameter $R$. Then, one can adapt the techniques introduced in this work to compute the quantity $R_{m,n}$ defined as the supremum of positive real numbers $R$ such that $\mu_R$ admits a positive quadrature with integer nodes less or equal to $m$ and which is exact for polynomials of degree at most $n$. Details of this proposed solution will appear elsewhere.

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| $m$ | 5     | 7     | 9     | 11    |
|-----|-------|-------|-------|-------|
| 5   | 4.8308| 3.3733| 2     | –     |
| 10  | 8.5757| 6.8035| 5.3363| 4.1000*|
| 20  | 12.5512| 10.3955| 8.6207| 7.1968|
| 25  | 16.6426| 14.1458| 12.1181*| 10.4401|
| 30  | 20.8355| 18.0383| 15.7996| 13.8617|
| 35  | 25.0687| 21.9991| 19.5069| 17.3889|
| 40  | 29.3824| 26.0713| 23.3347| 21.0411|
| 45  | 33.6959| 30.1565| 27.2467| 24.7358|
| 50  | 38.0717| 34.3177| 31.1936| 28.5616*|
| 55  | 42.4783| 38.5138| 35.2269| 32.3888|
| 60  | 46.9045| 42.7402| 39.2661| 36.3101|
| 65  | 51.3742| 47.0065*| 43.3863| 40.2498|
| 70  | 55.8354| 51.2895| 47.4942| 44.2407|
| 75  | 60.3433| 55.6066| 51.6671| 48.2524|
| 80  | 64.8353| 59.9393| 55.8486| 52.3018|
| 85  | 69.3699| 64.2863| 60.0554| 56.4079|
| 90  | 73.8924| 68.6808| 64.3111| 60.5126*|
| 95  | 78.4477| 73.0643| 68.5542| 64.6361|
| 100 | 83.0020| 77.4830| 72.8405| 68.7990|
| 105 | 87.5746| 81.8947| 77.1210| 72.9860*|
| 110 | 92.1624| 86.3262| 81.4319| 77.1740|
| 115 | 96.7503| 90.7940| 85.7766| 81.3905|
| 120 | 101.3649| 95.2447| 90.0975| 85.6207|
| 125 | 105.9591| 99.7148| 94.4558| 89.8748|
| 130 | 110.5757| 104.2063| 98.8407| 94.1471|
| 135 | 115.2069| 108.6953| 103.2033| 98.4163|
| 140 | 119.8350| 113.1950| 107.5950| 102.7054|
| 145 | 124.4725| 117.7118| 111.9993| 107.0167|
| 150 | 129.1137| 122.2256| 116.4054| 111.3141|
| 155 | 133.7623| 126.7511| 120.8271| 115.6493|
| 160 | 138.4238| 131.2950| 125.2615| 119.9974|
| 165 | 143.0776| 135.8267| 129.7071| 124.3378|
| 170 | 147.7428| 140.3765| 134.1436| 128.6815|
| 175 | 152.4202| 144.9467| 138.6046| 133.0649|
| 180 | 157.0889| 149.4953| 143.0705| 137.4426|
| 185 | 161.7686| 154.0689| 147.5367| 141.8242|
| 190 | 166.4561| 158.6436| 152.0123| 146.2122|
| 195 | 171.1420| 163.2216| 156.5046| 150.6200|
| 200 | 175.8348| 167.7992| 160.9934| 155.0274|

*Table 2*

Value of the optimal threshold factors $R_{m,2p−1}$ computed using Algorithm 3. An asterisk indicates a value for which the $p$-configuration $\{1,1,\ldots,1\}$ fails to provide for the optimal threshold factor.

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