Stress Assessment for 2D Corner based on Singularity Strength Method

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Abstract. Ship and box girder structural geometry contains sharp corners with singularity, which tends to produce fatigue cracks and cause structural damages. It’s not straightforward to assess FE analysis results of stiffened plating structures because of the stress singularities that are inherent in the overall geometry. A simple method to estimate the stress distribution using the singularity strength ‘as’ is proposed by the authors previously for a planar right angled cruciform which is limited to varies of forms. The aim of this paper is to expand the singularity strength theory to a 135-degree sheet corner through a series of FE analyses using ANSYS with no bending, thickness changes or overlapping plates. The estimation of the singularity strength ‘as’ is found based on the structure dimensions, consequently the stress distribution at the corner can be obtained easily from the singularity strength ‘as’. Finally the notch stress intensity factor (N-SIF) method is used to verify the singularity strength method for stress estimation.

1. Introduction
Idealized structure is designed to meet the cost and strength requirements in the marine engineering field. For example ships and marine structures usually consist of a large number of plates and stiffeners (plates) by welding together in order to achieve the requirements. These stiffened plate structures contain sharp internal corners as shown in Fig. 1. Stress concentration at these corners often makes the fatigue cracks initiate; these cracks propagate under fatigue loading and eventually cause the ship structure’s failure.

Fig.1 Key positions with stress singularity(Left: Transverse bulkhead structure of Tankers, Right: A typical cross-sectional view of hopper tanks of bulk carriers)
Based on elastic mechanics, stress at these structure corners is infinite which has a singularity. But FE analysis using ordinary plate or shell element does not give infinite stress. Stress results depend on the sizes of the elements [1] and can’t be directly used for fatigue analysis. Corner stress is usually estimated from the surface stresses by linear extrapolation from the distance to corner of 0.5z and 1.5z (z is the thickness), as shown in Fig.2. However, these extrapolation methods don’t have solid theoretical basis, the positions of the extrapolation points are usually related to thickness or being a fixed value without considering the structure sizes. There are other researches on the stress field around the corner [3,4,5], but the application is not very straightforward. This paper proposes a simple method using the singularity strength ‘as’ to estimate the stress distribution at the corner.

2. Corner stress singularity

For various forms of the corner structures, either butt welds or cross welds, the stress levels depend on the tip being sharp or frustrated. As we know, the case of the corner radius $\rho = 0$ is the most dangerous case where the stress concentration is very obvious and the structure is most prone to destruct, as shown in Fig.4. So this paper focuses on this most dangerous case where $\rho = 0$.

**Williams method.** Williams [2] considered that the stress singularity of the V-shaped sharp corner is similar to the crack tip stress singularity. The stress field around the corner is the summation of type-I (open type) and type-II (slip type) stress field. See Eq.1:

$$
\begin{align*}
\sigma_x &= a_1 f_{1,x} (\theta) + a_2 f_{2,x} (\theta) \\
\tau_{xy} &= \lambda f_{1,xy} (\theta) + \lambda f_{2,xy} (\theta)
\end{align*}
$$

Where:
- $x$ — The distance to the corner;
- $\lambda = \sin(\lambda q\pi) + \lambda \sin(q\pi) = 0$;

For details of $\lambda$, $\lambda$, $\lambda_1$, $\lambda_2$ and $f(\theta)$ please see [2].

Parameters are defined in Fig.4.
**Gross & Mendelson method.** They used SIF to describe the V-shaped sharp corner stress field and take the case of $\theta = 0$ to define the notch stress intensity factor N-SIF$^3$:

$$K_1 = \sqrt{2\pi} \lim_{r \to 0} (\sigma_\rho)_{\rho=\theta} r^{1+\lambda}$$

$$K_2 = \sqrt{2\pi} \lim_{r \to 0} (\tau_\rho)_{\rho=\theta} r^{1+\lambda}$$

By comparing the Eq.1-3, it’s easily to obtain the stress field represented by N-SIF, see Eq.4-5:

$$\begin{bmatrix} \sigma_y \\ \tau_{\rho\theta} \end{bmatrix}_{\rho=0} = \frac{1}{\sqrt{2\pi}} x^{1-\lambda} K_1 \begin{bmatrix} (1+\lambda_1\cos(1-\lambda_1)\theta) \\ (3-\lambda_1\cos(1-\lambda_1)\theta) \end{bmatrix} + \chi_1 \begin{bmatrix} \cos(1+\lambda_1)\theta \\ -\sin(1+\lambda_1)\theta \end{bmatrix}$$

$$\begin{bmatrix} \sigma_y \\ \tau_{\rho\theta} \end{bmatrix}_{\rho=0} = \frac{1}{\sqrt{2\pi}} (1-\lambda_2) x^{1-\lambda} K_2 \begin{bmatrix} -(1+\lambda_2)\sin(1-\lambda_2)\theta \\ -(3-\lambda_2)\sin(1-\lambda_2)\theta \end{bmatrix} + \chi_2 \begin{bmatrix} \sin(1+\lambda_2)\theta \\ \cos(1+\lambda_2)\theta \end{bmatrix}$$

**Lazzarin & Livieri method.** They linked N-SIF of stress distribution around the corner with the structure sizes and analyzed 135-degree sheet corner$^4$. See Fig.5. Under the tensile load for the 135-degree corner, its notch stress intensity factor (N-SIF) $K_1$ & $K_2$ are calculated as$^5$:

$$K_1 = k_1 \sigma_a r^{1-\lambda}$$

$$k_1 = 1.212 + 0.495 e^{-0.985 (2h/L)} - 1.259 e^{-1.120 (2h/L)} - 0.485 (2h/L)$$

$$K_2 = k_2 \sigma_a r^{1-\lambda}$$

$$k_2 = 0.508 - 0.797 e^{-1.959 (2h/L)} + 2.723 e^{-1.126 (2h/L)} - 0.769 (2h/L)$$

Where: $\sigma_a$ is the far-field stress. For a given size of 135-degree corner structure, $k_1$ & $k_2$ can be obtained from its dimensions ($t, h, L$) according to Eq.8 & Eq.9, by substituting them into the Eq.6 & Eq.7 obtaining $K_1$ & $K_2$ and then into the Eq.4 & Eq.5, the stress distribution of type-I and type-II in all directions around the corner can be obtained. Adding the two modes together gives the stress distribution of the 135-degree sheet corner at any direction.

**Singularity Strength method.** According to the studies of Lazzarin and Tovo$^4$, the stress distribution of type-I at a corner can be written as:

$$\begin{bmatrix} \sigma_y \\ \tau_{\rho\theta} \end{bmatrix} = \frac{1}{\sqrt{2\pi}} \frac{1}{x^n} \left[ C(\alpha, \theta) \cdot f(h, t, L) \right]^p \cdot \cdot t$$

Where:

$$C(\alpha, \theta) = \left[ \frac{(1+\lambda_1)\cos(1-\lambda_1)\theta + \chi_1 (1-\lambda_1)\cos(1+\lambda_1)\theta}{(1+\lambda_1) + \chi_1 (1-\lambda_1)} \right]^p$$

$$f(h, t, L) = k_1 r^{1-\lambda}, k_1 is a function of h, t, L;
\( p = 1 - \lambda \), for different corner angles, from Williams \cite{2}, is shown in Fig.3.

For a given corner \((2\alpha)\) at a given direction \((\theta)\), as shown in Fig.5, \( p \) & \( C (\alpha, \theta) \) are constants and \( \pi \) is a fixed value; \( k_1 \) is a function of \( h, t, L \). Taking all the constants into \( k_1 \), a new function can be defined as \( g (h,t,L) = C(\alpha,\theta) \cdot f (h,t,L)^{1/p} \cdot t \cdot \pi^{1/2p} \), then Eq. 10 becomes Eq.11:

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{x\theta}
\end{bmatrix} = \frac{1}{\sqrt{2} \cdot \pi^{1/2p}} \cdot [g(h,t,L)]^p
\]

(11)

To simplify the calculation of \( g(h,t,L) \), a new parameter ‘as’ is introduced and named as ‘singularity strength’ to describe the stress field at the corner, see Eq.12:

\[
\sigma_y = \sigma_{\infty} \cdot \text{as}^p
\]

(12)

By comparing Eq. 11 and Eq. 12, it can be seen that singularity strength \( as = g(h,t,L) \), which is a parameter with unit of length and determined by the structure sizes.

At distances larger than \( as/10 \) of a right angled cruciform corner or \( as/20 \) of a 135-degree sheet corner from the corner a more complicated formula is required to fit the stresses, which do not decay to 0 as predicted by Eq.12. The following is often a good fit:

\[
\sigma_y = \frac{\sigma_{\infty} \cdot \text{as}^p}{\sqrt{2} \cdot x^p}
\]

(13)

Where: \( q = 3p - 0.5 \), this is an empirical formula that is adapted from the formula by Paris & Sih \cite{6}. It blends the Eq.12 to a constant stress of \( \sigma_{\infty} \) instead of decaying to 0. For a 0-degree corner, which corresponds to a crack, \( p = 0.5 \) and the stress decreases at \( x (m) \) from the crack tip in proportion to \( 1/ x^{0.5} \). For a 90-degree corner \( p \) is reduced to 0.455 and for 135-degree corner \( p = 0.326 \). Stress plots for different corners are given in Fig.3.

**Estimation of Singularity Strength.** For a given corner, from the results above it can be seen that as long as the strength singularity ‘as’ is known, the stress distribution at the corner can then be determined from Eq.15. So it’s crucial to find a method to estimate the ‘as’ in order to calculate the stresses in a simple way. This paper studies the two typical corners in ship and marine structures, as shown in Fig. 6. A number of stress analyses are done using ANSYS for the two models, then the stresses for each case are plotted in MathCAD and the best fit ‘as’ value using least-square method is found for Eq. 13.

![Fig.6 The analysed edge detail](image-url)
The analysis of right angle corner singularity strength

The results of the analysis show that ‘as’ value can be determined by two characteristic quantities, its accuracy reflected by the extent of the value close to 1, see Fig.7. It’s concluded that the ‘as’ is the minimum of the two characteristic quantities, see Eq.14 for a right angle corner and Eq.15 for a 135-degree corner:

\[ as = \min \left( \frac{H}{2}, \frac{L}{25} \right) \]  

(14)

\[ as = \min \left[ \frac{t \left( \frac{H}{h} \right)^{0.1}}{6}, \frac{(2h + L)}{8} \right] \]  

(15)

For the 135-degree corner structure, ‘as’ can be estimated by Eq.17 with the the maximum error about 20%. But the calculation error of ‘as’is less than 8% in most cases. There are 160 models for calculating ‘as’, fig. 8 shows the ‘as’ differences between accurate fitting results from MathCAD and calculation results from Eq.17, difference more than 20% accounts 1.25% of the overall study, while difference less than 5% only had 66.25%. Therefore formulas that calculate ‘as’ are acceptable.

3. Stress Comparison

Comparison with ANSYS. By substituting the ‘as’ value estimated from Eq.16 or Eq.17 into Eq.15, the stress distribution along the stress line (as shown in Fig. 5) can be determined. Compared with the FE results obtained from ANSYS, the maximum error does not exceed 10%, as shown in Fig.9.
Comparison with N-SIF. As Lazzarin & Livieri method described, its scope need to meet $0 \leq L/t \leq 3.0$ and $0.25 \leq 2h/t \leq 2.5$ with the distance from the corner: $x \leq 0.1t$ ($t$, $h$, $L$ see Fig.6). Some cases meet the above conditions in 160 examples of this analysis, therefore compare these cases with N-SIF method, the results show in Fig.10:

1. While $1.5 \leq L/t \leq 3.0$, the difference near the corner is larger than remote with error about 12% between two methods. In terms of design with ‘as’ formula as standard, the stress distribution is somewhat safe, see Fig.10 -a.

2. While $0 \leq L/t \leq 1.5$, the difference remote the corner is larger than proximal end with error about 12%. As known the proximal region is the key motivation to crack initiation, here the stress distribution obtained by ‘as’ formula is somewhat dangerous, but the difference is within 5%, it can be used to calculate stress distribution, see Fig.10 -b.

Notice: when approaching the maximum boundary ($x=0.1t$) of N-SIF method, the error of stress results from N-SIF method is larger than proximal end ($x = 0$) itself [10], so comparison of the results presented here has a certain similarity.

4. Conclusions
The singularity strength ‘as’ is introduced to describe the stress field around the corners with stress concentration. Comparing with other methods, using the singularity strength to calculate the stresses at the corner is simpler. Singularity strength ‘as’ for 2D corners can be estimated from Eq.14 & Eq.15. Then the stress distribution can be obtained by Eq.15. Its accuracy has been preliminarily verified by
comparing with the ANSYS and N-SIF method. And the scope of application with proposed ‘as’
method to calculate the stress distribution is larger than the N-SIF method. In summary the results can
be the reference for fatigue and strength analysis.

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