Realization, Characterization, and Detection of Novel Superfluid Phases with Pairing between Unbalanced Fermion Species

Kun Yang

National High Magnetic Field Laboratory and Department of Physics,
Florida State University, Tallahassee, Florida 32306, USA

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Abstract

In this chapter we review recent experimental and theoretical work on various novel superfluid phases in fermion systems, that result from pairing fermions of different species with unequal densities. After briefly reviewing existing experimental work in superconductors subject to a strong magnetic field and trapped cold fermionic atom systems, we discuss how to characterize the possible pairing phases based on their symmetry properties, and the structure/topology of the Fermi surface(s) formed by the unpaired fermions due to the density imbalance. We also discuss possible experimental probes that can be used to directly detect the structure of the superfluid order parameter in superconductors and trapped cold atom systems, which may establish the presence of some of these phases unambiguously.

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I. INTRODUCTION AND BRIEF REVIEW OF EXPERIMENTAL WORK

It is well known that superfluidity and superconductivity in fermionic systems result from pairing of fermions, and the Bose condensations of these so-called Cooper pairs. In a specific fermion system, Cooper pairs are often made of fermions of different species; for example in superconductors they are electrons of opposite spins. Thus the most favorable situation for pairing is when the two species of fermions have the same density, so that there is no unpaired fermion in the ground state. The physics of pairing and resultant superfluidity under such condition is well described by the highly successful Bardeen-Cooper-Schrieffer (BCS) theory. It has been a long-standing fundamental question as to what kind of pairing states fermions can form when the two fermion species have different densities. A closely related issue is that in any paired or superfluid state formed under such situation, some of the majority fermions will necessarily be unpaired; thus a related question is how the system accommodates these unpaired fermions. An early suggestion was due to Fulde and Ferrell\textsuperscript{1}, and Larkin and Ovchinnikov\textsuperscript{2}, who argued that the Cooper pairs may condense into either a single finite momentum state, or a state that is a superposition of finite-momentum states. Such states are known collectively as the Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state; they break translation and rotation symmetries. More recently other suggestions have been put forward, including deformed Fermi surface pairing (DFSP)\textsuperscript{3,4} and breached pairing (BP)\textsuperscript{5,6,7} states, each with their distinct symmetry properties.

Experimentally, the issue of unbalanced pairing arise in several different contexts. Historically, it first arose in the context of a singlet superconductor subject to a large Zeeman splitting. The Zeeman splitting could be due to either a strong external magnetic field, or an internal exchange field (in the case of a ferromagnetic metal/superconductor). Under such a strong magnetic or exchange field, there is a splitting between the Fermi surfaces of spin-up and -down electrons. The original FFLO proposal was advanced in this context. However the FFLO state has not been observed in conventional low-$T_c$ superconductors. The reason for that, we believe, is because these superconductors are mostly three-dimensional, and the magnetic field that gives rise to Zeeman splitting also has a very strong orbital effect, which suppresses the superconductivity before the Zeeman effect becomes significant. The situation has changed recently, as experimental results suggestive of the FFLO state in heavy-fermion, organic, and high-$T_c$ superconductors have been found\textsuperscript{8,9,10,11,12,13,14,15}. 


These compounds are quasi-one or quasi-two-dimensional, thus the orbital effect is weak when the magnetic field is aligned in the conducting plane or along the chain; as a consequence the upper critical field of the superconductor is comparable to or exceeds the so-called Pauli paramagnetic limit, a field at which the Zeeman splitting becomes comparable to the superconducting gap. The the FFLO state becomes possible in such cases. Recent experimental results in CeCoIn$_5$, a quasi-2D d-wave superconductor, are particularly encouraging, as various experimental probes, including specific heat, thermal conductivity, ultrasound, penetration depth and NMR all identify a novel new phase in a region of the temperature-magnetic field phase diagram where the FFLO phase is expected theoretically. This system is by far the most promising candidate for the realization of the FFLO phase in superconductors at this point.

More recently, fermion pairing and superfluidity have become the focus of experimental work on trapped cold atom systems. Compared to electronic superconductors, one big advantage of such systems is that the strength of the pairing interaction can be very well controlled by manipulating the so-called Feshbach resonance, and one can explore a wide range of interaction strength from the weak coupling BCS regime to the strong coupling regime in which pairs of fermionic atoms form closely bound bosonic molecules (the so-called BEC regime). Another, perhaps more important advantage, is that experimentalists can induce and control the imbalance by simply mixing atoms of different species with different numbers. In contrast the imbalance in superconductors (Zeeman splitting) is due to an external magnetic field; the field, however, also brings the orbital effect that complicates the situation considerably. Indeed, in the very first such experiments unequal numbers of two hyperfine states of fermionic $^6$Li atoms were mixed and scanned across a Feshbach resonance. In these experiments it was found that paired and unpaired fermions phase separate when the imbalance is large. In one of the experiments it was found that the fermions do not phase separate when the imbalance is sufficiently small; further experiments are needed to clarify the nature of the state in such a situation.

Another place where pairing between unbalanced fermion species arises is quark and nucleon pairing in high density quark or nuclear matter, such as in the core of a neutron star. There the origin of density imbalance is due to the difference in the rest mass of quarks or nucleons that form the pairs; when the different pairing species are in chemical equilibrium (meaning they have the same chemical potential), their Fermi momenta and therefore den-
sities are different. The physics of quark and nucleon pairing have been previously reviewed in Ref. 27, and is also covered with great detail in other chapters of this volume.

II. CHARACTERIZATION OF PHASES BASED ON SYMMETRY AND TOPOLOGY OF FERMI SURFACE(S)

As discussed in Section I, a number of possible phases have been proposed theoretically in fermionic superfluids with unbalanced pairing species. The purpose of this section is to classify these phases based on their symmetry and other properties, and discuss the relations between these phases based on such classification. This also gives us insight into the nature of the phase transitions between various phases. We note that classification (or characterization) of classical phases and phase transitions are based on Landau theories, whose forms are completely determined by the symmetry properties of the phases involved; thus our classification based on symmetry considerations are complete at finite temperature \((T)\). It has been realized recently, however, that such classification may not be complete for quantum phases and phase transitions\(^{28}\); additional classification schemes may be necessary at \(T = 0\). Here we propose that in these pairing phases the structure and in particular, topology of the Fermi surfaces formed by unpaired fermions can be used as additional classification scheme to characterize phases and phase transitions. Most of the ideas behind such considerations were originally presented in Refs. 29,30,31, which we review below.

We begin with symmetry considerations, and start our discussion from the FFLO state, which has the longest history of studies. Following the superconductivity terminology, throughout the rest of this chapter we will use “spin” indices \(\sigma = \uparrow, \downarrow\) to label the two different species of fermions that form Cooper pairs, “Zeeman splitting” \(\Delta \mu = \mu_{\uparrow} - \mu_{\downarrow}\) to represent the chemical potential difference between the two fermion species that form Cooper pairs, and “magnetization” \(m\) to represent their density difference. When \(\Delta \mu \neq 0\), up- and down-spin electrons form Fermi seas with different Fermi momenta \(p_{F_{\uparrow}}\) and \(p_{F_{\downarrow}}\) in the normal state; it was thus suggested\(^{1,2}\) that when pairing interaction is turned on, the initial pairing instability is for fermions with opposite spins on their respective Fermi surfaces to pair up and form a Cooper pair with a net momentum \(p \approx p_{F_{\uparrow}} - p_{F_{\downarrow}}\). This results in a pairing order parameter \(\Delta(r)\) that is oscillatory in real space, with period \(2\pi\hbar/p\). In general the structure of \(\Delta(r)\) is characterized not just by a single momentum \(p\), but also by
its higher harmonic components. More detailed mean-field study\textsuperscript{32} suggested the following real space picture for FFLO state: it is a state with a finite density of uniformly spaced domain walls; across each domain wall the order parameter $\Delta$ (which is real in the mean-field theory) changes sign, and the excess magnetization due to spin imbalance are localized along the domain walls, where $\Delta$ (which is also the gap for unpaired fermions) vanishes; see Fig. 1a of Ref. \textsuperscript{29} for an illustration\textsuperscript{33}. Thus the total magnetization is proportional to the domain wall density. This picture was made more precise by an exact solution in one-dimension (1D) based on bosonized description of spin-gapped Luttinger liquids\textsuperscript{34} (also referred to as the Luther-Emery liquid in condensed matter literature), where the domain walls are solitons of the sine-Gorden model that describes the spin sector; each soliton carries one half-spin. While quantum and thermal fluctuations do not allow true long-range order in 1D, such order can be stabilized by weak interchain couplings\textsuperscript{34}. Coming back to isotropic high D cases, it is clear that the presence and ordering of these domain walls break rotation symmetry, and translation symmetry in the direction perpendicular to the walls, although translation symmetry along the wall remains intact. Thus the symmetry properties of the FFLO state is identical to that of the smectic phase of liquid crystals (smectic-A phase to be more precise)\textsuperscript{35}.

Once the mean-field FFLO state is identified with the smectic phase of liquid crystals based on symmetry considerations, one can borrow insights as well as known results on the thermodynamic phases of liquid crystals to the present problem\textsuperscript{29}. In classical liquid crystals it is known that as one increases thermal fluctuations, the broken symmetries of the smectic phase are restored in the following sequence\textsuperscript{35}: the translation symmetry is restored first when the smectic melts into a nematic that breaks the rotation symmetry only, and then the nematic melts into an isotropic liquid that has no broken spatial symmetry. We thus expect the same sequence of phases and phase transitions occur in superfluids with unbalanced fermion pairing, as we increase the strength of either thermal or quantum fluctuations. Very interestingly, the nematic and isotropic phases have precisely the same symmetry properties as the deformed Fermi sea pairing (DFSP)\textsuperscript{3,4} and breached pairing (BP)\textsuperscript{5,6,7} states, which were proposed as alternative phases for unbalanced pairing that compete with the FFLO phase. In the DFSP phase, the (originally mismatched) Fermi surfaces of the majority and minority fermion species deform spontaneously, so that they match in certain regions in momentum space to facilitate pairing; the rotation symmetry is broken by the Fermi
surface distortion, but the translation symmetry remains intact (see Fig. 2 of Ref. 3 for an illustration). In the BP phase, an isotropic shell in momentum space is used to accommodate the excess magnetization, while pairing occurs in the rest of the momentum space; both rotation and translation symmetries are intact in this phase. It should be noted that the variational states studied in Refs. 3, 4, 5, 6, 7 are quite simple and essentially of mean-field type; they look quite different from the real space picture developed in Ref. 29 (see its Fig. 1) based on considerations of fluctuation effects. We would like to emphasize, however, that it is the common symmetry properties that allowed us to identify the DFSP state as the nematic phase, and the BP state as the isotropic phase of the liquid crystal. The symmetry considerations also suggest a unified understanding of all three of these states as different phases of a liquid crystal. We note in passing that very similar considerations have also led to deeper understanding of different phases in cuprate superconductors\textsuperscript{36} and quantum Hall liquids\textsuperscript{37,38}.

Coming back to the smectic state, one should in principle also consider the possibility that further symmetry breaking occurs along directions parallel to the domain walls; this will result in breaking of translation symmetries in all directions, and result in a crystal version of the FFLO state\textsuperscript{33}.

As usual the strength of thermal fluctuation is controlled by temperature ($T$). The quantum fluctuations (QF), on the other hand, are controlled by the strength of pairing interactions; QF is weaker for weak pairing interactions (the BCS regime, where the superfluid state is well described by the mean-field theory), while stronger for strong pairing interaction (the BEC regime). In Ref. 29 a phase diagram has been proposed based on such considerations, in which an infinitesimal density imbalance is present (assuming there is no phase separation in this case, and the smectic phase is the least symmetric phase that is realized), while both temperature and pairing interaction strength are varied. As discussed earlier, the pairing interaction strength can be controlled by manipulating the Feshbach resonance in trapped cold atom systems; thus one may be able to explore the entire phase diagram in such systems.

At finite temperature, all phases and phase transitions are classical, and are fully characterized by symmetries. We thus conclude that our classification of the possible pairing phases based on symmetry is complete, and the crystal (FFLO), smectic (FFLO), nematic (DFSP) and isotropic (BP) phases discussed above exhaust all possible phases in this case.
Furthermore, symmetry also dictates the nature of the phase transitions. Again, based on known results from studies of classical liquid crystals, we expect the transition between nematic (DFSP) and isotropic (BP) phases to be generically first order, while the transition between nematic (DFSP) and smectic (FFLO) phases is most likely 2nd order. A direct transition between FFLO (either crystal or smectic) and isotropic (BP) phases is unlikely; should such a transition occur, it will be first order.

At zero temperature, the possible phases are characterized by the ground state of the system, and the low-lying excitations above it, which are intrinsically quantum mechanical. It has become increasingly clear in recent years that characterization of quantum phases based on symmetry alone is often insufficient, and additional characterization schemes are needed to classify such “quantum” or “topological” order. At present we do not yet have a complete classification scheme for quantum order. In the following we will argue that in the problem of unbalanced pairing discussed here, one may use the properties of the Fermi surface(s) formed by unpaired fermions, especially their topology, to characterize all the possible phases; combined with symmetry properties discussed above, they most likely provide a complete classification scheme. We note that Fermi surfaces are sharp and well-defined objects only at $T = 0$; finite $T$ smears Fermi surfaces, and as a consequence they are no longer well-defined.

It is intuitively clear that in the presence of imbalance, some of the majority fermions will be unpaired; these unpaired fermions will form a Fermi sea of its own with at least one, but possibly more Fermi surfaces. Recently this intuitive picture has been made quantitatively precise in the form of a mathematically rigorous theorem, which is a generalization of the Luttinger’s theorem for normal metals to the case with pairing interaction and superfluidity. The theorem makes distinction between two different cases:

(i) In the absence of superfluidity, there are two Fermi surfaces for spin-up and -down fermions, whose volumes are individually conserved:

$$
N_\uparrow = \frac{A}{(2\pi)^d} \Omega_\uparrow, \quad N_\downarrow = \frac{A}{(2\pi)^d} \Omega_\downarrow,
$$

where $d$ is the dimensionality, $A$ is the (real space) volume of the system, and $\Omega$ is the (momentum space) volume enclosed by the Fermi surface. We emphasize that this is an exact result that applies even when the pairing interaction is so strong that some of the fermions may form very closely bound pairs or “molecules”; in this case one might intuitively
expect that these fermions in closely bound states would not contribute to the Fermi surface volume. Our result indicate that as long as there is no superfluidity (or the pairs do not condense), it is the total numbers of fermions that dictate the volumes of Fermi surfaces.

(ii) In the presence of superfluidity, or when Cooper pairs Bose condense and the U(1) symmetry associated with charge conservation is spontaneously broken, the spin-up and -down Fermi surface volumes are no longer individually conserved. However their difference remains to be conserved, and is dictated by the imbalance:

$$\Delta N = N_\uparrow - N_\downarrow = \frac{A}{(2\pi)^d}(\Omega_\uparrow - \Omega_\downarrow).$$

In this case we can have either one or two Fermi surfaces; when there is only one Fermi surface we simply have $\Omega_\downarrow = 0$.

In our discussion so far we have assumed the system to be uniform or translationally invariant. These results, however, can be generalized to cases with spontaneously broken translational symmetry, which is the case for the FFLO state. In such cases, the Fermi surface volumes are well-defined modulo the Brillouin zone volume $\Omega_B$; as a consequence all of our statements on the constraints on Fermi surface volumes are modulo $\Omega_B$. The situation is identical to electrons moving in a periodic potential considered by Luttinger originally.\(^{39}\)

The theorem discussed above, in particular the constraint of Eq. (2), dictates that the ground states of the systems considered here can be characterized by their Fermi surfaces, and there must be gapless quasiparticle excitations near these Fermi surfaces. We can thus use the Fermi surfaces as an additional classification scheme for the unbalanced pairing phases at $T = 0$, and expect the following generic cases:

(i) One Fermi surface for spin-up fermions. In this case its volume is fixed to be

$$\Delta N = N_\uparrow - N_\downarrow = \frac{A}{(2\pi)^d}\Omega_\uparrow.$$  

(ii) Two Fermi surfaces, whose volumes are not fixed individually, but their difference are fixed by Eq. (2).

(iii) No Fermi surface. In this case Eq. (2) indicates $\Delta N = 0$, so there is no imbalance.

We believe combining the symmetry property (crystal, smectic, nematic or isotropic) with the number of Fermi surfaces, we have an essentially complete characterization of all the possible quantum pairing phases. As an example, we expect two possible isotropic phases. The breached pair (BP) phase (also known as Sarma phase\(^{40}\)), in its original form, has
two Fermi surfaces, which may be stable at weak coupling. On the other hand we expect an isotropic phase with a single Fermi surface at strong coupling\textsuperscript{29,41,42}. For the nematic case, we can in principle again have one or two Fermi surfaces; in this case the pairing order parameter is uniform and does not break any spatial symmetry, while the Fermi surface(s) should be anisotropic and break rotation symmetry spontaneously. As already mentioned, in the FFLO phase, due to the broken translation symmetry, Brillouin zones form and the Fermi surface(s) are folded into a Brillouin zone.

The transitions between different phases with different numbers of Fermi surfaces have been discussed in Refs. \textsuperscript{30,31}.

III. DETECTION OF PHASES BASED ON “PHASE SENSITIVE” EXPERIMENTAL PROBES

As discussed in the previous section, the possible phases for systems with pairing between unbalanced fermion species can be characterized by (i) their symmetry properties, especially those associated with the spatial structure of the pairing order parameter; and (ii) in the case $T = 0$, the structure and in particular, topology of the Fermi surfaces formed by unpaired fermions. Thus to experimentally identify a phase unambiguously, one needs to have experimental methods that probe (i) and/or (ii) directly. While there have been quite a few experiments that study possible FFLO phases in various systems, and the studies of CeCoIn$_5$ are getting more and more detailed, none of the existing experiments probes (i) or (ii) directly. In the following we will discuss a few possible experiments that probe either (i) or (ii), in either electronic superconductors or trapped cold atom systems.

A. Detecting Spatial Structure of Pairing Order Parameter in Superconductors Using Phase Sensitive Experimental Probes

In this subsection we will briefly discuss three possible experimental methods that directly probe the spatial structure of the pairing order parameter, in the FFLO state of electronic superconductors, which we have considered recently\textsuperscript{43,44,45}. We also discuss a proposed experiment\textsuperscript{46} that probes physics similar to that of Ref. \textsuperscript{43}, as well as the possibility of using neutron or muon scattering to detect the spin structure of the FFLO state.
Josephson Effect between a BCS and an FFLO superconductor — In Ref. 43 we demonstrated that one can use the Josephson effect between an FFLO superconductor and a BCS superconductor to measure the momenta of (in principle) all the Fourier components of the pairing order parameter of the FFLO superconductor. The idea behind this proposal is quite simple. Consider a two-dimensional BCS superconductor, described by a spatially dependent superconducting order parameter $\Psi_{BCS}(r)$, which is coupled to a two-dimensional FFLO superconductor, described by an order parameter $\Psi_{FFLO}(r)$. We consider the two Josephson junction geometries shown in Figure 1 of Ref. 43. Since the physics for the two geometries are similar we focus our discussion on geometry of Fig. 1a of Ref. 43, in which the two superconductors are stacked on top of each other. In the Ginsburg-Landau description, the Josephson coupling term in the free energy takes the form (in the absence of any magnetic flux going through the junction, or in between the two superconductors)

$$H_J = -t \int d^2r [\Psi_{FFLO}^*(r) \Psi_{BCS}(r) + c.c.]$$

(4)

where $t$ is the Josephson coupling strength. In the ground state of a BCS superconductor, $\Psi_{BCS}(r) = \psi_0$ is a constant. However, in an FFLO superconductor the order parameter is a superposition of components carrying finite momenta:

$$\Psi_{FFLO}(r) = \sum_m \psi_m e^{i k_m \cdot r}$$

(5)

and is oscillatory in space. In the absence of magnetic flux inside the junction, the total Josephson current is

$$I_J = \text{Im} \left[ t \int d^2r \Psi_{BCS}^*(r) \Psi_{FFLO}(r) \right] = \sum_m \text{Im} \left[ t \psi_0^* \psi_m \int d^2r e^{i k_m \cdot r} \right]$$

(6)

Clearly, due to the oscillatory nature of the integrand, the Josephson current is suppressed in such a junction.

Mathematically, the reason that the Josephson current is suppressed here is similar to the suppression of Josephson current by an applied magnetic field in an ordinary Josephson junction between two BCS superconductors. However, the physics is very different: here the suppression is due to the spatial oscillation of the order parameter in the FFLO state, while in the case of ordinary Josephson junction in a magnetic field, the phase of the Josephson tunneling matrix element is oscillatory (in a proper gauge choice). Nevertheless, the mathematical similarity allows these two effects to cancel each other and restore the Josephson
current, by applying an appropriate amount of magnetic flux through the junction, and the amount of flux that restores the Josephson effect is a direct measure of the momentum of one of the Fourier components of the pairing order parameter of the FFLO superconductor. This was demonstrated in Ref. 43, and we refer the reader to this paper for detailed analyzes using both the effective Ginsburg-Landau description and microscopic theory, as well as for an alternative geometry. This idea has some similarity to the so called “phase sensitive” experiments that unambiguously determined the d-wave nature of the pairing order parameter of high Tc cuprate superconductors. However unlike the cuprate experiments that attempt to determine the internal structure of the Cooper pairs (or their angular momentum), here we use the sensitivity of the Josephson coupling to the phase of the pairing order parameter to determine its spatial structure, or the momentum of the Cooper pairs.

In the following we discuss a few practical issues that may arise when trying to implement this proposal experimentally.

(i) We want to use the Josephson effect to probe the spatial structure of the pairing order parameter of the FFLO superconductor, using the BCS superconductor (whose pairing order parameter is uniform in space) as a reference point. In order for this idea to work however, the BCS and FFLO superconductors should have the same internal structure for their pairing order parameter, i.e., the two superconductors should be both s-wave or both d-wave etc, otherwise the Josephson current will vanish simply due to the mismatch in internal symmetry. As noted earlier, the most promising candidate for FFLO state thus far is CeCoIn$_5$, which is a d-wave superconductor. Thus to implement this idea on CeCoIn$_5$ one needs to use another d-wave superconductor for the reference BCS state. A natural choice is thus a cuprate superconductor, which has the additional advantage that it has a much bigger gap and higher Pauli limit than CeCoIn$_5$; thus when placed in a strong magnetic field (about 10T, necessary to drive CeCoIn$_5$ into the FFLO state), it is still in the BCS phase.

(ii) The key ingredient that makes the Josephson effect useful in the determination of the structure of the FFLO pairing order parameter is that one needs to adjust the magnetic flux in the junction to have the Josephson effect; the order parameter momentum can be determined from the magnetic flux. On the other hand we also need to put the superconductors in a strong magnetic field to drive one of them into the FFLO phase, unless it is a ferromagnetic superconductor that has a spontaneous magnetization. Thus the magnetic field that stabilizes the FFLO state may interfere with the flux through the junction. The
configuration that avoids this complication is the one depicted in Fig. 1b of Ref. 43, in which the BCS and FFLO superconductors, both assumed to be (quasi) two-dimensional, are placed side-by-side. The advantage of this configuration is that the magnetic field that stabilizes the FFLO state is an in-plane magnetic field, which does not contribute to the flux through the junction that controls the Josephson effect. As a result the in-plane field and (out of plane) flux through the junction can be tuned independently.

(iii) In an infinite system, which was analyzed in Ref. 43, the Josephson current is exactly zero unless the relative phase oscillation between the BCS and FFLO superconductors are canceled exactly by the phase oscillation in Josephson coupling induced by the flux through the junction. In real systems the junction has a finite size; we thus expect a Fraunhofer pattern in the flux-dependence of the Josephson current, which is peaked at a finite flux strength determined by the momentum of pairing order parameter of the FFLO superconductor.

**Exotic Vortex Structure of FFLO Superconductors**—The FFLO state is stabilized by the Zeeman effect of an external magnetic field. On the other hand the field can also generate an orbital effect; for example in a purely 2D system, the Zeeman effect is determined by the total magnetic field, while the orbital effect is generated by the out-of-plane component of the magnetic field, when it is nonzero. Thus the relative importance between the Zeeman and orbital effect can be controlled by the angle between the magnetic field and the 2D plane. The orbital effect generates vortex states, which can be used to detect FFLO physics. The idea here goes back to an early observation by Bulaevskii, who pointed out that depending on the interplay between the orbital and Zeeman effects of the magnetic field, the order parameter of a FFLO state near its upper critical field can correspond to a high Landau level (LL) index Cooper pair wave function. Recent work on the FFLO vortex lattice structure (VLS) in specific situations has demonstrated that these high LL index VLS’s can be very different from the triangular lattice Abrikosov VLS favored by lowest Landau level (LL) Cooper pairs. However it remained a challenging task to determine the vortex lattice structure for FFLO superconductors under general conditions. The difficulty has its origin in the complicated Ginsburg-Landau theory appropriate for FFLO superconductors:

\[
F \propto |(-\nabla^2 - q^2)\psi|^2 + a|\psi|^2 + b|\psi|^4 + c|\psi|^2|\nabla \psi|^2 + d[(\psi^*)^2(\nabla \psi)^2 + \psi^2(\nabla \psi^*)^2] + e|\psi|^6 + \cdots, \tag{7}
\]

where \(a, b, c, d, e\) and \(q\) are parameters that depend on both temperature and Zeeman split-
The fundamental difference between FFLO and BCS superconductors is expressed by the first term in $F$ which describes the kinetic energy of the order parameter; in an FFLO superconductor this term is minimized when the order parameter carries a finite wave vector (or momentum) $q$. Thus far we have only taken into account the Zeeman effect of the external magnetic field; for a 2D superconductor with the field $B$ tilted out of system plane, orbital coupling must be accounted for by performing a minimal substitution $\nabla \psi \rightarrow D \psi = (\nabla - 2ieA/c)\psi$ with $\nabla \times A = B_\perp \hat{z}$. This leads to Landau quantization of the kinetic energy term, namely the eigenvalues of $D^2 = (\nabla - 2ieA/c)^2$ are $-(2n + 1)/\ell^2$, where $\ell = \sqrt{\hbar c/2eB}$ is the Cooper pair magnetic length and $n = 0, 1, 2, \cdots$ is the LL index.

There are two specific sources of difficulty, compared to the vortex states of a BCS superconductor, that results in the Abrikosov lattice. (i) For a BCS superconductor the kinetic energy is minimized by $n = 0$, i.e., $\psi$ is a lowest LL wave function. For an FFLO superconductor, however, the kinetic energy is minimized by the index $n$ that minimizes $|(2n + 1)/\ell^2 - q^2|$. This can lead to high Landau level wave functions which are much more complicated. (ii) For the BCS case, one only needs to minimize the $|\psi|^4$ term in Eq. (7). For FFLO superconductors however, due to the fact that the order parameter carries a finite momentum, there are additional quartic terms (which involve spatial gradients) that make substantial contribution to the free energy, and higher order ($|\psi|^6$ and beyond) terms need to be kept because very often the quartic terms make negative contributions. Fortunately, the complicated high LL wave functions have been studied in great detail in the context of quantum Hall effect\textsuperscript{56}. In Ref. \textsuperscript{44} we have used techniques developed in the studies of quantum Hall effect to advance a very efficient method to evaluate the free energy (7) for the high LL wave function $\psi$, and minimize it to determine the optimal VLS. The method is somewhat technical and we refer the readers to Ref. \textsuperscript{44} for details. More importantly, from the details of the VLS one can extract the LL index $n$, from which we can get an estimate of the order parameter momentum: $q \approx \sqrt{2n + 1}/\ell$.

**Spectra of Andreev Surface Bound State of d-wave FFLO Superconductors Probed by Tunneling** — The idea here is specific to d-wave superconductors, which CeCoIn$_5$ is believed to be. In a d-wave superconductor, the sign of the pairing order parameter depends on the direction. As a consequence of this there exist low-energy quasiparticle states that are bound to the surface of a superconductor\textsuperscript{57}. These so-called Andreev surface bound states (ASBS) result from the change of sign of pairing order parameter when a quasi-
particle bounces off the surface; they give rise to a zero bias conductance peak (ZBCP) in the tunneling spectrum between a normal metal and the d-wave superconductor (with proper orientation), separated by a potential barrier. The ZBCP was recently observed in CeCoIn$_5$. In Ref. 45, we find the spectrum of ASBS changes when the d-wave superconductor is driven into the FFLO state, and depends on the momentum of the pairing order parameter. In particular, this leads to a shift and split of the ZBCP in the tunneling spectrum, with the split proportional to the order parameter momentum. This provides yet another way to measure the order parameter momentum using tunneling.

**Other Possible Experiments** — In Ref. 46, Bulaevskii and coworkers proposed using interlayer transport in quasi-2D superconductors in the presence of an in-plane magnetic field to detect the FFLO state. The idea bears some similarity to that of Ref. 43: in the superconducting phase, interlayer transport is dominated by Josephson tunneling; for the FFLO state the order parameter has spatial modulation, and the Josephson effect is enhanced when the order parameter modulation is commensurate with the phase modulation of the interlayer Josephson coupling due to the in-plane field. The authors of Ref. 46 have worked out the commensuration condition based on certain assumption on the spatial structure of the order parameter, under which the interlayer transport is enhanced (i.e., enhanced critical current or conductance). Experimentally one can tune the in-plane magnetic field to look for such enhancement associated with the commensuration, from which the wave vector of the order parameter may be extracted. This experiment is also “phase-sensitive”.

In addition to probing the spatial structure of the superconducting order parameter directly using the experiments discussed above, one can also try to probe the spatial distribution of the unpaired spin-up electrons in the FFLO state, which is closely related to the order parameter structure. For example, for the one-dimensional, Larkin-Ovchinnikov type order parameter structure (or the smectic phase), one expect the unpaired spin-up electrons to localize along the domain walls where the order parameter changes sign; thus the periodicity (or wave length) of the spin modulation should be one-half of that of the superconducting order parameter. The spatial structure of the spins can be detected from elastic neutron or muon scattering experiments. While such experiments do not directly probe the order parameter structure and are thus not “phase-sensitive”, being able to detecting the spin structure should also provide convincing evidence for the FFLO state, and allow us to extract the order parameter structure from it.
B. Detection of Novel Pairing Phases in Cold Atom Systems

As discussed in Section I, recent experiments\textsuperscript{24,25} have started to explore trapped cold atom systems with pairing between atoms of different species (or hyperfine quantum number) and unequal densities. Such systems have also generated strong theoretical interest recently\textsuperscript{41,42,60,61,62,63,64,65}. In particular, the possibility of realizing the FFLO state has been discussed, and in Ref. \textsuperscript{61} it was suggested that it can be detected by imaging the density profiles of each of the pairing species, which should be oscillatory in real space for the FFLO state. In another paper\textsuperscript{65}, it was suggested that radio-frequency spectroscopy can be used to detect both phase separation and the FFLO state. In Ref. \textsuperscript{62} we proposed two alternative methods to detect the FFLO state, which directly probes the momenta of the Cooper pairs, using the methods advanced in Refs. \textsuperscript{66,67,68}. In Ref. \textsuperscript{66} one projects the Cooper pairs of a BCS state onto molecules by sweeping the tuning field through the Feshbach resonance, and then removes the trap and uses time-of-flight (TOF) measurement to determine the molecular velocity distribution and the condensate fraction. One can do exactly the same experiment on the FFLO state; the fundamental difference here is that in this case because the Cooper pairs carry intrinsic (non-zero) momenta, the condensate will show up as peaks corresponding to a set of finite velocities in the distribution. Another method to detect the Cooper pairs is to study the correlation in the shot noise of the fermion absorption images in TOF\textsuperscript{68}, first proposed in Ref. \textsuperscript{67}. In Ref. \textsuperscript{68} the shot noise correlation clearly demonstrates correlation in the occupation of $\mathbf{k}$ and $-\mathbf{k}$ states in momentum space when weakly bound diatom molecules are dissociated and the trap is removed. In principle the same measurement can be performed on fermionic superfluid states, and for an FFLO state, it would reveal correlation in the occupation of $\mathbf{k}$ and $-\mathbf{k} + \mathbf{q}$ states, where $\mathbf{q}$ is one of the momenta of the pairing order parameter\textsuperscript{62}. Both methods allow one to directly measure $\mathbf{q}$, which defines the FFLO state. These methods are unique to the cold atom systems; very similar ideas have also been discussed in Ref. \textsuperscript{42}.

As discussed in section II, in addition to the spatial structure of the pairing order parameter, we also need to detect the structure and in particular the topology of the Fermi surface(s) of the unpaired fermions. As discussed in Ref. \textsuperscript{31}, this can be detected from the momentum distributions of the atoms, using TOF after removing the trap. Such a measurement was recently performed in a gas of $^{40}\text{K}$ across a Feshbach resonance\textsuperscript{69}, and hopefully
will be performed in systems with unbalanced pairing in the future. In the experiment of Ref. 69, the effect of the trap on the momentum distribution appears to be quite strong, such that the discontinuity in momentum distribution gets wiped out even for non-interacting fermions. We hope that by manipulating the form of the trap potential, its effect can be minimized so that discontinuities in momentum distribution associated with Fermi surfaces can be detected in future experiments; this would probably require a trap potential that is flat inside the trap and rises very fast near the boundary. It has also been pointed out that in the deformed Fermi surface pairing state, the TOF experiment will find anisotropy in the distribution of the fermion velocity. Again one needs to carefully analyze the effect of the trapping potential in this case.

We note in passing that the possibility of detecting some of the novel unbalanced phases in nuclear matter has been discussed recently, and the possibility that at large imbalance the system may switch from s-wave to p-wave pairing, including its detection.

IV. SUMMARY

In this chapter we have discussed how to characterize and detect various possible phases that may result from pairing fermions with different species and density imbalance. We argue in section II that all the possible phases may be completely characterized by (i) the spatial structure of the pairing order parameter; and (ii) the structure and in particular, topology of the Fermi surfaces formed by unpaired fermions.

These novel pairing phases may be realized in spin-singlet superconductors subject to a Zeeman splitting between electron spin states (either due to an external magnetic field or spontaneous magnetization), trapped cold atom systems, and high density quark/nuclear matter. For superconductors, the best case so far is a quasi-two-dimensional heavy fermion superconductor CeCoIn$_5$, where evidence for the realization of FFLO state has been found when it is subject to a large in-plane magnetic field. While the existing evidence from various experiments are quite strong, they are all circumstantial in the sense that they do not directly probe the spatial structure of the pairing order parameter. We hope some of the “phase sensitive” experiments we discussed in section IIIA will lead to definitive proof of the FFLO state in this or other superconductors. Experimental work on unbalanced pairing in trapped cold atom systems have just started. Thus far clear evidence of phase separation.
between paired and unpaired fermions have been found\cite{1,2} when the imbalance is large. Further work is needed to clarify whether some of the novel pairing phases discussed in this and other chapters in this book are realized at small imbalance, and the methods discussed in section IIIB will hopefully be useful in that task.

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