New Formulation for Coloring Circle Graphs and its Application to Capacitated Stowage Stack Minimization

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Abstract

A circle graph is a graph in which the adjacency of vertices can be represented as the intersection of chords of a circle. The problem of calculating the chromatic number is known to be NP-complete, even on circle graphs. In this paper, we propose a new integer linear programming formulation for a coloring problem on circle graphs. We also show that the linear relaxation problem of our formulation finds the fractional chromatic number of a given circle graph. As a byproduct, our formulation gives a polynomial-sized linear programming formulation for calculating the fractional chromatic number of a circle graph.

We also extend our result to a formulation for a capacitated stowage stack minimization problem.

1 Introduction

This paper addresses problems of coloring circle graphs. A circle graph is a graph in which the adjacency of vertices can be represented as the intersection of chords of a circle. It is well known that circle graphs and overlap graphs are of the same class (e.g., see [19, 20]).

In [14], Even and Itai studied the problem of realizing a given permutation through networks of queues in parallel and through a network of stacks in parallel. The problem was translated into a coloring problem of a circle graph (see also [20]). There are practical applications involving stack sorting including assigning incoming trains [9, 10] or trams [5] to tracks of
a switching yard or depot; parking buses in parking lots [17]; and stowage planning for container ships [4, 33]. König and Lübbecke [27] considered an algorithmic view towards stack sorting. Stack and queue layouts of graphs related to colorations of circle graphs are discussed in [12].

Even in the case of circle graphs, the problems of finding the chromatic number [18] and clique covering number [25] are NP-complete. Approximation algorithms for coloring circle graphs are proposed in [7, 31]. In a survey [13], Durán, Grippo, and Safe summarized structural results related to circle graphs and presented some open problems. Both the maximum clique problem and maximum independent set problem have polynomial time algorithms when restricted to circle graphs [2, 8, 19, 30, 32].

In this paper, we propose an integer linear programming formulation for coloring problems on circle graphs. We also show that the linear relaxation problem of our formulation finds the fractional chromatic number of a given circle graph. For a general graph, the problem of finding the fractional chromatic number is NP-complete [21]. Our proposal gives a polynomial-sized formulation for fractional coloring problems on circle graphs.

The reminder of this paper is organized as follows. The next section presents some notations and definitions. In Section 3, we propose a new formulation for coloring circle graphs. We discuss a relation between the linear relaxation of our formulation and fractional chromatic number in Section 4. Section 5 reports our computational experiments. In Section 6, we briefly discuss an extension of our formulation to a capacitated stowage stack minimization problem with zero rehandle constraint. Finally, Section 7 makes some closing remarks.

2 Notations and Definitions

Let $G = (V, E)$ be an undirected graph with a set of vertices $V$ and set of arcs $E$. A coloring of a graph is an assignment of a color to each vertex such that all adjacent vertices are of a different color. The smallest number of colors needed to color a graph $G$ is called its chromatic number, denoted by $\chi(G)$. The coloring problem has long been studied and is known to be NP-complete for general graphs [24]. An independent set is a subset of vertices in a graph such that no two are adjacent. The fractional chromatic number $\chi_f(G)$ is the smallest positive number $k \in \mathbb{R}_+$ for which there exists a probability distribution over the independent sets of $G$ satisfying the following; given an independent set $S$ drawn from the distribution, $\Pr[v \in S] \geq 1/k$ ($\forall v \in V$). Although its computation is NP-complete [21], the fractional chromatic
number of a general graph can be obtained using linear programming (see Subsection 4.2).

A clique is a subset of vertices such that its induced subgraph is complete. The clique number $\omega(G)$ of a given graph $G$ is the number of vertices in a maximum clique in $G$. It is known that $\omega(G) \leq \vartheta(G) \leq \chi_f(G) \leq \chi(G)$, where $\vartheta(G)$ denotes the Lovász number [29] of a given graph $G$. A pentagon graph (5-cycle) $C_5$, which is an example of a circle graph, satisfies $(\omega(C_5), \vartheta(C_5), \chi_f(C_5), \chi(C_5)) = (2, \sqrt{5}, 2.5, 3)$.

A circle graph is a graph in which the adjacency of vertices can be represented as the intersection of chords of a circle. The circle and chords corresponding to a given circle graph $G$ are called a circle diagram of $G$. Hereinafter, we assume that terminal points of chords in a circle diagram are mutually distinct. Figure 1 (a) and (b) show an example of a circle graph and its corresponding circle diagram, respectively.

![Circle Graph and Circle Diagram](image)

**Fig. 1:** An example of circle graph and related diagram.

It is well known that circle graphs and overlap graphs are of the same class (e.g., see [19, 20]). A graph is an overlap graph if its vertices are intervals on a line such that two vertices are adjacent if and only if the corresponding intervals partially overlap (that is, they have non-empty intersection), but neither contains the other. It is easy to construct a set of
intervals representing a given circle graph (overlap graph) from a corresponding circle diagram by a simple transformation: cutting the circumference of the circle at some point $p$ that is not an endpoint of a chord and unfolding it at that point (e.g., see Section 11.3 of [20]). Hereinafter, we assume that an input of a given circle graph $G = (V, E)$ is a corresponding interval representation $I(G) = \{I(j) \subseteq \mathbb{R} \mid j \in V\}$, where $I: j \mapsto [l_j, r_j]$. We also assume that all the terminal points of intervals in $I(G)$ are mutually distinct. Here, we note that an interval representation of a given circle graph is not unique. Figure 1(c) shows an interval representation of the circle graph in Figure 1(a).

Given an interval representation $I(G)$ of a circle graph $G = (V, E)$, we introduce a partial order $\preceq$ defined on the vertex set $V$. For any pair of vertices $i, j \in V$, we define $i \preceq j$ if and only if either $i = j$ or $r_i \leq l_j$ holds, where $I(i) = [l_i, r_i]$ and $I(j) = [l_j, r_j]$. Obviously, $(V, \preceq)$ is a partially ordered set. Although every chain of $(V, \preceq)$ is an independent set of $G$, the converse implication does not hold. Figure 2 shows a partially ordered set corresponding to the interval representation in Figure 1(c).

Fig. 2: Partially ordered set corresponding to the interval representation in Figure 1(c). There is an arrow from $i$ to $j$ if and only if $i \preceq j$.

### 3 Integer Linear Programming Formulation

Given a circle graph $G = (V, E)$ and corresponding interval representation $I(G)$, we introduce a directed graph $\Gamma$ as follows. The vertex-set of $\Gamma$ is defined by $V \cup \{0\}$, where 0 is an artificial vertex called a root. The arc-set of $\Gamma$, denoted by $A$, is defined by

$$A = \{(0, i) \mid i \in V\} \cup \{(i, j) \mid I(i) \supseteq I(j)\}.$$
The above definition implies that \( \Gamma \) is acyclic. Figure 1(d) shows a directed acyclic graph \( \Gamma \) defined by the interval representation in (c).

An arc subset \( T \subseteq A \) is called an arborescence if and only if \( |T| = |V| \) and each vertex \( i \in V \) has a unique incoming-arc in \( T \). When a given arborescence \( T \) has an arc \( (i,j) \), we say that \( j \) is a child of \( i \) and \( i \) is a (unique) parent of \( j \) with respect to \( T \). For any arborescence \( T \) and a vertex \( i \in V \cup \{0\} \), \( \text{Ch}(T, i) \) denotes the set of children of \( i \) with respect to \( T \).

In the following, we associate each coloring with an arborescence on \( \Gamma \). Let \( \phi : V \to \{1, 2, \ldots, c\} \) be a \( c \)-coloring of \( G \). For each vertex \( j \in V \), we define a parent of \( j \) with respect to \( \phi \), denoted by \( \text{Prt}(\phi, j) \), as follows: if \( V' = \{ i \in V \mid \phi(i) = \phi(j), I(i) \supseteq I(j) \} \) is empty, then we define \( \text{Prt}(\phi, j) = 0 \) (root); else, \( \text{Prt}(\phi, j) \) denotes a vertex in \( V \) corresponding to a unique (inclusion-wise) minimum interval in \( V' \). Given a coloring \( \phi \) of \( G \), \( T(\phi) \) denotes an arborescence \( \{(\text{Prt}(\phi, j), j) \in A \mid j \in V\} \). Figure 3(a) shows a 3-coloring and corresponding arborescence in \( \Gamma \).

![3-coloring and arborescence](image)

Fig. 3: An example of a 3-coloring and its corresponding arborescence.

**Lemma 3.1.** Let \( T \) be an arborescence of \( \Gamma \). Then, there exists a \( c \)-coloring \( \phi \) of a given circle graph \( G \) satisfying \( T = T(\phi) \) if and only if

**C1:** for each \( i \in V \), \( \text{Ch}(T, i) \) is a chain of \((V, \leq)\) or the empty set and
C2: the size of every antichain of \((V, \preceq)\) contained in \(\text{Ch}(T, 0)\) is less than or equal to \(c\).

Proof. It is obvious that the size of a minimum chain cover (partition) of a poset is greater than or equal to the size of a maximum antichain. Thus, the definition of \(T(\phi)\) implies that if there exists a \(c\)-coloring \(\phi\) of \(G\) satisfying \(T = T(\phi)\), then \(T\) satisfies C1 and C2.

We show the converse implication. In the following, we construct a \(c\)-coloring from an arborescence \(T\) satisfying C1 and C2. Because \(T\) satisfies C2, Dilworth’s theorem \([11]\) implies that there exists a set of (at most) \(c\) chains of \((V, \preceq)\) partitioning \(\text{Ch}(T, 0)\). Because every chain of \((V, \preceq)\) is an independent set of a given circle graph, we obtain a \(c\)-coloring of a sub-graph of \(G\) induced by \(\text{Ch}(T, 0)\) by assigning a color to each chain. For each vertex \(i \in \text{Ch}(T, 0)\), we assign a color of \(i\) to all the descendants of \(i\) with respect to \(T\). Denote the map (coloring) obtained above by \(\phi\). We only need to show that if \(\phi(j) = \phi(j')\) and \(j \neq j'\), then vertices \(j\) and \(j'\) are non-adjacent on a given circle graph \(G\). When \(I(j) \subseteq I(j')\) or \(I(j') \subseteq I(j)\), the non-adjacency is obvious. Otherwise, let \(r'\) be a unique lowest common ancestor of \(j\) and \(j'\) with respect to \(T\). We denote a child of \(r'\) that is an ancestor of \(j\) (or \(j'\)) by \(i\) (or \(i'\)), respectively. If \(r' \neq 0\), then C1 directly implies that \(I(i) \cap I(i') = \emptyset\). When \(r' = 0\), \(\phi(i) = \phi(j) = \phi(j') = \phi(i')\) implies that \(i\) and \(i'\) are contained in a mutual chain in \(\text{Ch}(T, 0)\) and thus \(I(i) \cap I(i') = \emptyset\). From the above, \(I(j) \subseteq I(i)\), \(I(j') \subseteq I(i)\), and \(I(i) \cap I(i') = \emptyset\) hold. As a consequence, we obtain the non-adjacency of \(j\) and \(j'\), because \(I(j) \cap I(j') = \emptyset\).

Let \(P \subseteq \mathbb{R}\) be a set of (positions of) terminal points of intervals in \(\mathcal{I}(G)\). Recall that terminal points of intervals in \(\mathcal{I}(G)\) are mutually distinct and thus \(|P| = 2|V|\). Let \(M = (m_{pi})\) be a 0-1 matrix whose entries are indexed by \(P \times V\), satisfying

\[
m_{pi} = \begin{cases} 
1 & \text{if } p \in I(i), \\
0 & \text{otherwise}. 
\end{cases}
\]

(Here, we note that \(M\) is a *clique matrix* (Section 3.4 of \([20]\)) of an interval graph corresponding to the set of intervals \(\mathcal{I}(G)\).) Obviously, \(M\) is an antichains-versus-vertices incidence matrix of poset \((V, \preceq)\). In addition, it is easy to see that all the maximal antichains are included. The definition of \(M\) directly implies the following lemma, which characterizes the size of a maximum antichain.

**Lemma 3.2.** For any vertex subset \(\widetilde{V} \subseteq V\), the size of a maximum antichain of \((V, \preceq)\) contained in \(\widetilde{V}\) is equal to the maximum components of vector \(M\vec{x}\), where \(\vec{x}\) is the (fixed) characteristic vector of \(\widetilde{V}\).
Proof is omitted.

Now, we give our formulation for a circle graph coloring problem. For any vertex \( i \in V \cup \{0\} \), we define a vertex subset \( V[i] = \{ j \in V \mid (i, j) \in A \} \). We define \( V^* = \{ i \in V \mid V[i] \neq \emptyset \} \). Here, we note that \( V[0] = V \) and \( 0 \not\in V^* \) hold. For each arc \((i, j) \in A\), we introduce a 0-1 variable \( x^i_j \). The vector of all 0-1 variables is denoted by \( x \in \{0, 1\}^A \). For any vertex \( i \in V^* \cup \{0\} \), \( x[i] \) denotes a subvector of \( x \) indexed by arcs emanating from \( i \), and \( M[i] \) denotes a submatrix of \( M \) consisting of column vectors of \( M \) indexed by \( V[i] \). Then, we have the following.

**Lemma 3.3.** Given a vector \( x \in \{0, 1\}^A \) and positive integer \( c \), the constraints
\[
M x[0] \leq c 1, \\
M[i] x[i] \leq 1 \quad (\forall i \in V^*), \\
\sum_{i:(i,j) \in A} x^i_j = 1 \quad (\forall j \in V)
\]
are satisfied if and only if \( T = \{ (i, j) \in A \mid x^i_j = 1 \} \) is an arborescence satisfying conditions C1 and C2.

Proof. Let \( T \) be an arborescence satisfying conditions C1 and C2. We set a vector \( x \in \{0, 1\}^A \) as the characteristic vector of \( T \). Then, it is obvious that \( x \) satisfies the above constraints.

Now, assume that \( x \in \{0, 1\}^A \) and a positive integer \( c \) satisfy the above constraints. We define \( T = \{ (i, j) \in A \mid x^i_j = 1 \} \). Then, constraints \( \sum_{i:(i,j) \in A} x^i_j = 1 \quad (\forall j \in V) \) directly imply that \( T \) is an arborescence of \( \Gamma \). From Lemma 3.2, the inequality \( M x[0] \leq c 1 \) implies that \( T \) satisfies condition C2. Similarly, Lemma 3.2 and \( M[i] x[i] \leq 1 \quad (\forall i \in V^*) \) imply that for any \( i \in V^* \), the size of a maximum antichain in \( Ch(T, i) \) is less than or equal to 1. From Dilworth’s theorem, \( Ch(T, i) \) becomes a chain (or the empty set). For any \( i \in V \setminus V^* \), \( Ch(T, i) = \emptyset \). Thus, \( T \) satisfies condition C1.

The above lemma directly implies the following formulation for a circle graph coloring problem:
CG : min. $c$

s.t. $Mx^0 \leq c1$,

$M[i]x[i] \leq 1$ (\(\forall i \in V^{\bullet}\)),

$\sum_{i(i,j) \in A} x^i_j = 1$ (\(\forall j \in V\)),

$x^i_j \in \{0, 1\}$ (\(\forall (i, j) \in A\)),

$c \in \mathbb{Z}_+.$

Lemma \[3.3\] directly implies the following.

**Theorem 3.4.** A pair \((\tilde{x}, \tilde{c}) \in \{0, 1\}^A \times \mathbb{Z}_+\) is optimal to CG if and only if \(\tilde{c} = \chi(G)\) and there exists a \(\tilde{c}\)-coloring \(\phi\) satisfying \(T(\phi) = \{(i, j) \in A \mid \tilde{x}^i_j = 1\}\).

Proof. Lemma \[3.1\] implies that a pair \((x, c) \in \{0, 1\}^A \times \mathbb{Z}_+\) is feasible to CG, if and only if, \(T = \{(i, j) \in A \mid x^i_j = 1\}\) and \(c\) satisfies conditions C1 and C2. Thus, the optimal value of CG is equal to \(\chi(G)\). We can construct a \(\chi(G)\)-coloring \(\phi\) from an optimal solution of CG by applying a technique described in the proof of Lemma \[3.1\] The inverse implication is clear. \(\square\)

## 4 Linear Relaxation of ILP formulation

In this section, we show that the linear relaxation problem of our formulation (CG) finds the fractional chromatic number of a given circle graph.

### 4.1 Maximum Weight Independent Set Problem

In this subsection, we discuss a maximum weight independent set problem defined on a given circle graph \(G = (V,E)\) with a given vertex weight function \(w : V \to \mathbb{R}\) (incidentally, negative vertex weights are permitted). For an artificial vertex 0, we define \(w(0) = 0\). We propose a linear programming formulation of the problem based on a dynamic programming technique [8] [15] [19]. Our linear programming formulation plays an important role in the next subsection.

A maximum weight independent set problem finds an independent set \(S\) of \(G\) that maximizes the weight \(\sum_{i \in S} w(i)\). Throughout this section, we assign a linear ordering on the vertex set by setting \(V = \{1, 2, \ldots, n\}\) such that if \(I(i) \supseteq I(j)\), then \(i \leq j\).
For any vertex \( i \in V^* \cup \{0\} \), \( G^{[i]} \) denotes the subgraph of \( G \) induced by vertex subset \( V^{[i]} \) (note that \( i \notin V^{[i]} \)). Every maximum weight independent set \( S \subseteq V \) satisfies the following: \( \forall i \in S \cap V^*, S \cap V^{[i]} \) is a maximum weight independent set in \( G^{[i]} \). Here, we introduce vertex weights defined by

\[
\ell_i = \begin{cases} 
w(i) + \max \left\{ \sum_{j \in V'} w(j) \mid V' \text{ is an independent set of } G^{[i]} \right\} & (\forall i \in V^*), \\
w(i) & (\forall i \in V \setminus V^*).
\end{cases}
\]

For any vertex subset \( S \subseteq V \), \( \max S \) denotes a set of vertices corresponding to (inclusion-wise) maximal intervals in \( \{I(i) \mid i \in S\} \). It is clear that if \( S \) is an independent set of \( G \), then \( \max S \) is a chain of \((V, \preceq)\). This property implies that the weight of a maximum weight independent set with respect to \((w(i) \mid i \in V)\) is equal to the weight of a maximum weight chain (of \((V, \preceq)\)) with respect to \((\ell_i \mid i \in V)\). By applying the above idea recursively, it is easy to see that \((\ell_i \mid i \in V \cup \{0\})\) satisfies the following formula

\[
\ell_i = \begin{cases} 
\ell_i + \max \left\{ \sum_{j \in V'} \ell_j \mid V' \subseteq V^{[i]}, \ V' \text{ is a chain of } (V, \preceq) \right\} & (\forall i \in V^* \cup \{0\}), \\
\ell_i & (\forall i \in V \setminus V^*),
\end{cases}
\]

where we define \( \ell(0) = 0 \), and \( \ell_0 \) denotes the weight of a maximum weight independent set of \( G \) with respect to \((w(i) \mid i \in V)\). Because the vertex set \( V = \{1, 2, \ldots, n\} \) satisfies “if \( I(i) \supseteq I(j) \), then \( i \leq j \),” the above formula calculates \((\ell_n, \ell_{n-1}, \ldots, \ell_1, \ell_0)\) sequentially. Figure 4 shows an example solution of recursive formula (2).

The following lemma summarizes the above discussion.

**Lemma 4.1.** The solution of recursive formula (2) satisfies that \( \ell_0 \) is equal to the weight of a maximum weight independent set with respect to \((w(i) \mid i \in V)\).

Next, we describe a linear programming formulation of a subproblem appearing in (2).

**Lemma 4.2.** Let \( \ell = (\ell_j \mid j \in V) \) represent given vertex weights. For any \( i \in V^* \cup \{0\} \),

\[
\max \left\{ \sum_{j \in V'} \ell_j \mid V' \subseteq V^{[i]}, \ V' \text{ is a chain of } (V, \preceq) \right\} = \max \left\{ \sum_{j \in V^{[i]}} \ell_j x_j^{[i]} \mid M^{[i]} x^{[i]} \leq 1, \quad x^{[i]} \geq 0 \right\},
\]

where \( x^{[i]} \) is a vector of continuous variables indexed by \( V^{[i]} \).

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Fig. 4: An example solution of recursive formula (2).

Proof. In the following, \( LC_i(\ell) \) denotes the linear programming problem on the right-hand-side of (3). First, we show that \( LC_i(\ell) \) has an optimal 0-1 vector solution. We say that a matrix has the consecutive 1’s property (for columns) if and only if 1’s occur consecutively in each column. Clearly, the matrix \( M[i] \) has the consecutive 1’s property \([16]\) (for columns) and is totally unimodular. The total unimodularity implies that a feasible region \( \{ x[i] | M[i]x[i] \leq 1, x[i] \geq 0 \} \) is a 0-1 polytope \([23, 22]\). Therefore, \( LC_i(\ell) \) has an optimal 0-1 vector solution, denoted by \( \hat{x}^i \). Lemma \[3.2\] and \( M[i] \hat{x}^i \leq 1 \) imply that the size of a maximum antichain contained in \( \hat{V} = \{ j \in V[i] | \hat{x}_j^i = 1 \} \) is less than or equal to 1. From Dilworth’s theorem, \( \hat{V} \) becomes a chain (or the empty set). \( \square \)

For any \( i \in V^* \cup \{0\} \), the dual of \( LC_i(\ell) \), denoted by \( DLC_i(\ell) \), is

\[
\text{DLC}_i(\ell): \min \sum_{p \in P} y_p^i \\
\text{s.t.} \sum_{p \in I(j)} y_p^i \geq \ell_j \ (\forall j \in V[i]), \\
y_p^i \geq 0 \ (\forall p \in P).
\]

We substitute \( DLC_i(\ell) \) for the maximum weight chain problem in \([2]\) to
obtain the following recursive formula:

\[
\ell_i = \begin{cases} 
    w(i) + \min \left\{ \sum_{p \in P} y_p^i \mid \sum_{p \in I(j)} y_p^i \geq \ell_j \quad (\forall j \in V^{[i]}), \right. \\
    \left. \quad (\forall i \in V^* \cup \{0\}), \right. \\
    w(i) \quad (\forall i \in V \setminus V^*). 
\end{cases}
\]  \hspace{1cm} (4)

Lemmas 4.1 and 4.2 and the strong duality theorem directly imply the following.

**Theorem 4.3.** The solution of recursive formula (4) satisfies that \( \ell_0 \) is equal to the weight of a maximum weight independent set with respect to vertex weights \( w = (w(i) \mid i \in V) \).

Considering all constraints appearing in (4), we construct the following linear programming problem:

\[
\text{ISD}(w) : \min \ell_0 = \sum_{p \in P} y_p^0 \\
\text{subject to} \quad \ell_i = w(i) + \sum_{p \in P} y_p^i \quad (\forall i \in V^*), \\
\ell_i = w(i) \quad (\forall i \in V \setminus V^*), \\
\sum_{p \in I(j)} y_p^i \geq \ell_j \quad (\forall (i,j) \in A), \\
y_p^i \geq 0 \quad (\forall (i,p) \in (V^* \cup \{0\}) \times P),
\]  \hspace{1cm} (5a) \hspace{1cm} (5b) \hspace{1cm} (5c) \hspace{1cm} (5d) \hspace{1cm} (5e)

where \( (\ell_0, \ell_1, \ldots, \ell_n) \) and \( (y_p^i \mid (i,p) \in (V^* \cup \{0\}) \times P) \) are vectors of continuous variables.

**Theorem 4.4.** The optimal value of \( \text{ISD}(w) \) is equal to the weight of a maximum weight independent set with respect to vertex weights \( w = (w(i) \mid i \in V) \).

Proof. Let \((\bar{\ell}, \bar{y})\) be a solution of (4) and \((\ell^*, y^*)\) be an optimal solution of \( \text{ISD}(w) \). Because \((\bar{\ell}, \bar{y})\) is feasible to \( \text{ISD}(w) \), \( \bar{\ell}_0 \geq \ell^*_0 \) holds.

For the remainder of this proof, we show the inequality \( \bar{\ell}_0 \leq \ell^*_0 \). Here, we note that we assign a linear ordering on the vertex set by setting \( V = \{1, 2, \ldots, n\} \) such that if \( I(i) \supseteq I(j) \), then \( i \leq j \). We show that \( \bar{\ell}_j \leq \ell^*_j \) for each \( j \in \{n, n-1, \ldots, 0\} \) by induction on \( j \). Clearly, vertex \( n \in V \setminus V^* \), and thus \( \ell^*_n = w(n) = \ell_n \). Assume that \( \bar{\ell}_j \leq \ell^*_j \) for each \( j \in \{n, n-1, \ldots, i+1\} \).

When \( i \in V \setminus V^* \), we obviously have \( \ell^*_i = w(i) = \ell_i \). We consider the case
that \( i \in V^* \cup \{0\} \). It is obvious that for any \( i \in V^* \cup \{0\} \), the subvector \((\tilde{y}^i_p)p\in P\) of \(\tilde{y}\) is optimal to problem DLC\(_i\)(\(\bar{\ell}\)). The subvector \((y^i_p)p\in P\) of \(y^*\) is feasible to problem DLC\(_i\)(\(\ell^*\)). The induction hypothesis implies that the feasible region of DLC\(_i\)(\(\tilde{\ell}\)) includes that of DLC\(_i\)(\(\ell^*\)). The subvector \((y^i_p)p\in P\) of \(y^*\) is feasible to DLC\(_i\)(\(\tilde{\ell}\)), and the corresponding objective value satisfies
\[
\sum_{p\in P} y^i_p \geq \text{(optimal value of DLC\(_i\)(\(\tilde{\ell}\)))} = \sum_{p\in P} \tilde{y}^i_p.
\]
Thus, we obtain
\[
\ell^*_i = w(i) + \sum_{p\in P} y^i_p \geq w(i) + \sum_{p\in P} \tilde{y}^i_p = \bar{\ell}_i,
\]
where we let \(w(0) = 0\) for simplicity.

From the above discussion, we have shown that \(\ell^*_0 = \bar{\ell}_0\). □

4.2 Fractional Coloring Problem

In this subsection, we discuss the fractional coloring problem. Given an undirected graph \(G = (V, E)\), \(F\) denotes the incidence matrix of independent sets of \(G\). The rows of \(F\) are indexed by \(V\), the columns of \(F\) are indexed by all the independent sets of \(G\), and each column vector is the incidence vector (characteristic vector) of a corresponding independent set. The fractional coloring problem is defined by
\[
\min \{1^\top q \mid Fq = 1, q \geq 0\},
\]
where the variable vector \(q\) is indexed by all the independent sets in \(G\), and \(1\) denotes the all-ones vector. The optimal value of the above problem is called the fractional chromatic number and is denoted by \(\chi_f(G)\). Generally, the above linear programming problem has an exponential number of variables. The dual of the above problem is
\[
\max \{w^\top 1 \mid w^\top F \leq 1^\top\},
\]
which finds a vertex weight \(w : V \to \mathbb{R}\) maximizing the total sum \(w^\top 1\) subject to the constraint that the weight of every independent set (with respect to \(w\)) is less than or equal to 1.

Let us discuss the case that a given graph \(G\) is a circle graph. Given a vertex weight function \(w : V \to \mathbb{R}\), the weight of every independent set
(with respect to $w$) is less than or equal to 1 if and only if the minimization problem $\text{ISD}(w)$ has a feasible solution whose objective value is less than or equal to 1. Then, the linear programming problem

$$\begin{align*}
\text{max.} & \quad w^\top 1 \\
\text{s.t.} & \quad \sum_{p \in P} y_p^0 \leq 1,
\end{align*}$$

(6)

with the constraints of the ISD problem,
gives a formulation for the fractional coloring problem on a given circle graph, where $(w(i) \mid i \in V)$, $(\ell_i \mid i \in V)$, and $(y_p^i \mid (i,p) \in (V^* \cup \{0\}) \times P)$ are vectors of continuous variables. Here, we note that $(w(i) \mid i \in V)$ is a given vector of vertex weights in ISD($w$) and is a variable vector in the above problem.

We eliminate variables $(w(i) \mid i \in V)$ by applying equalities (5b) and (5c). Then, the objective function becomes

$$w^\top 1 = \sum_{i \in V^*} w(i) + \sum_{i \in V \setminus V^*} w(i)$$

$$= \sum_{i \in V^*} (\ell_i - \sum_{p \in P} y_p^i) + \sum_{i \in V \setminus V^*} \ell_i = \sum_{j \in V} \ell_j - \sum_{i \in V^*} \sum_{p \in P} y_p^i.$$

The obtained problem (6) transforms into

$$\text{FCP : max.} \quad \sum_{j \in V} \ell_j - \sum_{i \in V^*} \sum_{p \in P} y_p^i$$

$$\text{s.t.} \quad \sum_{p \in P} y_p^0 \leq 1,$$

$$\sum_{p \in I(j)} y_p^j \geq \ell_j \quad (\forall (i,j) \in A),$$

$$y_p^j \geq 0 \quad (\forall (i,p) \in (V^* \cup \{0\}) \times P),$$

where $(\ell_i \mid i \in V)$ and $(y_p^i \mid (i,p) \in (V^* \cup \{0\}) \times P)$ are vectors of continuous variables.

It is easy to check that FCP is the dual of the linear relaxation problem of CG obtained by substituting $x_j^i \geq 0$ and $c \geq 0$ for $x_j^i \in \{0,1\}$ and $c \in \mathbb{Z}_+$, respectively. Summarizing the above discussion, we obtain the following theorem.
Theorem 4.5. Let LR be a linear relaxation problem of CG obtained by substituting \( x^i_j \geq 0 \) and \( c \geq 0 \) for \( x^i_j \in \{0, 1\} \) and \( c \in \mathbb{Z}_+ \), respectively. Then, the optimal value of LR is equal to the fractional chromatic number of a given circle graph.

5 Computational Experiments

In our experiments, we compared the computational time required to find an optimal solution of our formulation CG, a classical coloring problem formulation

\[
\text{CL : min} \sum_{c=1}^{C} y_c,
\]

s.t. \( x_{ic} \leq y_c \) \( (\forall i \in V, \forall c \in \{1, 2, \ldots, C\}) \),

\( x_{ic} + x_{i'c} \leq 1 \) \( (\forall c \in \{1, 2, \ldots, C\}, \forall \{i, i'\} \in E) \),

\( \sum_{c=1}^{C} x_{ic} \geq 1 \) \( (\forall i \in V) \),

\( x_{ic} \in \{0, 1\} \) \( (\forall i \in V, \forall c \in \{1, 2, \ldots, C\}) \),

\( y_c \in \{0, 1\} \) \( (\forall c \in \{1, 2, \ldots, C\}) \),

and an asymmetric representative formulation [6]

\[
\text{AS : min} \sum_{i \in V} x_{ii}
\]

s.t. \( x_{ij} = x_{ji} = 0 \) \( (\forall \{i, j\} \in E) \),

\( x_{ji} = 0 \) \( (\forall i, \forall j \in V, \ i < j) \),

\( x_{ij} + x_{ik} \leq x_{ii} \) \( (\forall \{i, j, k\} \subseteq V, \{j, k\} \in E) \),

\( \sum_{i \in V} x_{ij} = 1 \) \( (\forall j \in V) \),

\( x_{ij} \leq x_{ii} \) \( (\forall i, \forall j \in V) \),

\( x_{ij} \in \{0, 1\} \) \( (\forall (i, j) \in V^2) \),

where the vertex set \( V = \{1, 2, \ldots, n\} \) satisfies \( \deg(1) \geq \deg(2) \geq \cdots \geq \deg(n) \) (\( \deg(v) \) denotes the degree of vertex \( v \in V \)). We set the constant \( C \) in the classical formulation to the number of colors required in a coloring obtained by the First Fit heuristic (e.g., see [3]). All the experiments were conducted on a PC running the Windows 10 Pro operating system with
an Intel(R) Core(TM) i7-7700 @3.60GHz processor and 32 GB RAM. All instances were solved using CPLEX 12.8.0.0 implemented in Python 3.6.5 and NumPy 1.17.2.

We generated instances of circle graphs as follows. First, we randomly shuffled the numbers \{1, 2, \ldots, 2|V|\} using the “random.shuffle()” command of the NumPy Python module. We repeatedly removed the first two numbers \(x, y\) from the shuffled sequence and added a non-empty interval \([x, y]\) or \([y, x]\) to a set of intervals \(\mathcal{I}\). We constructed an overlap graph (circle graph) from the set of intervals \(\mathcal{I}\). For example, from a sequence \(5, 3, 1, 4, 6, 2\), we construct a set of intervals \(\mathcal{I} = \{[3, 5], [1, 4], [2, 6]\}\). For each \(n \in \{5, 10, 30, \ldots, 700\}\), we generated 100 circle graphs with \(n\) vertices and solved the coloring problems using CPLEX.

The results are summarized in Table 1. The CL, AS, Ours columns represent the classical formulation, asymmetric representative formulation, and our formulation (CG), respectively. The “# \(\omega = \chi\)” and “# \(\chi_f = \chi\)” columns list the numbers of instances (out of 100 generated instances) satisfying \(\omega(G) = \chi(G)\) and \(\chi_f(G) = \chi(G)\), respectively. The “max \(\chi - \chi_f\)” column lists the maximum values of \(\chi(G) - \chi_f(G)\) over the 100 generated instances. We omit the computational results of some cases, denoted by “-,” because of time limitation. Table 1 shows that our formulation solves coloring problems efficiently compared to other formulations.

We also generated hard instances \(G(m)\) proposed by Kostochka (Section 6 in [28]) for each \(m \in \{2, 3, \ldots, 15\}\). These results are summarized in Table 2. When we employed the classical formulation and/or asymmetric representative formulation, the execution time exceeded 1800s even in the case of \(G(4)\).

When we employed our formulation (CG), CPLEX found optimal solutions for all instances reported in Tables 1 and 2 at the root node (without any branching process). Tables 1 and 2 show that all the generated instances satisfy \(\chi(G) - 1 < \chi_f(G) \leq \chi(G)\).

Ageev [11] constructed a triangle-free graph \(G_A = (V, E)\) with chromatic number equal to 5, where \(|V| = 220\) and \(|E| = 1395\). We calculated the fractional chromatic number of the graph and obtained that \((\omega(G_A), \chi_f(G_A), \chi(G_A)) = (2, 3.623 \cdots, 5)\). In this case, the computational time required to solve problem CG was 72.672s and number of branching nodes generated by CPLEX was 10,906.
Table 1: Computational results for 100 randomly generated instances.

| | \(V\) | \(E\) | computation time [s] | # \(\omega = \chi\) | # \(\chi_f = \chi\) | max. \(\chi - \chi_f\) |
|---|---|---|---|---|---|---|
| CL | AS | Ours |
| 5 | 3.35 | 0.013 | 0.004 | 0.006 | 100 | 100 | 0.0 |
| 10 | 14.17 | 0.018 | 0.006 | 0.012 | 95 | 95 | 0.5 |
| 30 | 145.39 | 0.038 | 0.016 | 0.021 | 94 | 94 | 0.5 |
| 50 | 413.40 | 0.364 | 0.059 | 0.029 | 89 | 100 | 0.7 |
| 70 | 802.79 | 2.096 | 0.158 | 0.045 | 92 | 94 | 0.7 |
| 100 | 1633.43 | 28.889 | 0.715 | 0.087 | 89 | 91 | 0.7 |
| 150 | 3784.46 | - | 4.092 | 0.202 | 82 | 83 | 0.8 |
| 200 | 6681.80 | - | 22.357 | 0.384 | 77 | 87 | 0.7 |
| 250 | 10412.11 | - | - | 0.710 | 77 | 85 | 0.7 |
| 300 | 14918.57 | - | - | 1.220 | 75 | 82 | 0.7 |
| 400 | 26404.89 | - | - | 2.988 | 71 | 77 | 0.8 |
| 500 | 41084.46 | - | - | 6.360 | 66 | 77 | 0.8 |
| 600 | 60191.42 | - | - | 12.430 | 70 | 83 | 0.8 |
| 700 | 81421.61 | - | - | 25.163 | 58 | 75 | 0.8 |

6 Capacitated Stowage Stack Minimization

In this section, we briefly discuss an extension of our formulation to a capacitated stowage stack minimization problem. Let \(G = (V, E)\) and \(I(G)\) be a given circle graph and the corresponding interval representation, respectively. Throughout this section, \(H\) denotes a positive integer representing “capacity.” For any independent set \(S\) of \(G\), the height of \(S\) is equal to the size of a maximum antichain of \((V, \preceq)\) contained in \(S\). In this section, \(\Xi_H\) denotes the set of independent sets (of \(G\)) whose heights are less than or equal to \(H\). We introduce a 0-1 matrix \(F_H\) indexed by \(V \times \Xi_H\) whose columns are the incidence vectors of corresponding independent sets in \(\Xi_H\). We consider the following 0-1 integer programming problem:

\[ P_H: \min \{1^T q \mid F_H q = 1, q \in \{0, 1\}^{\Xi_H}\}, \]

where the variable vector \(q\) is indexed by \(\Xi_H\). The above problem is essentially equivalent to a capacitated stowage stack minimization problem with zero rehandle constraint [33].

We say that a \(c\)-coloring of \(G\) is \(H\)-admissible if each color class is an independent set in \(\Xi_H\). It is obvious that each feasible solution of \(P_H\) corresponds to an \(H\)-admissible coloring. In a similar manner to that in Section 3, we consider the following 0-1 integer programming problem:
Table 2: Computational results for hard instances $G(m)$.

| $m$ | $|V|$ | computation time $[s]$ | $\omega$ | $\chi_f$ | $\chi$ |
|-----|------|------------------------|----------|----------|--------|
| 2   | 24   | 0.031                  | 5        | 6.500    | 7      |
| 3   | 62   | 0.031                  | 8        | 10.833   | 11     |
| 4   | 122  | 0.141                  | 10       | 15.750   | 16     |
| 5   | 205  | 0.359                  | 13       | 21.000   | 21     |
| 6   | 316  | 1.468                  | 15       | 26.833   | 27     |
| 7   | 453  | 4.282                  | 18       | 32.857   | 33     |
| 8   | 617  | 13.531                 | 20       | 39.062   | 40     |
| 9   | 812  | 59.656                 | 22       | 45.611   | 46     |
| 10  | 1039 | 155.328                | 25       | 52.450   | 53     |
| 11  | 1294 | 767.766                | 28       | 59.318   | 60     |
| 12  | 1584 | 3878.625               | 31       | 66.500   | 67     |
| 13  | 1904 | 9235.875               | 33       | 73.731   | 74     |
| 14  | 2258 | 33087.641              | 35       | 81.143   | 82     |
| 15  | 2647 | 119019.062             | 38       | 88.733   | 89     |

we introduce a directed graph and associate an $H$-admissible coloring of $G$ with a directed tree.

First, we define a directed graph $\Gamma_H = (V_H, A_H)$ as follows. For each vertex $i \in V$, we construct a set of $H$ copies of $i$ denoted by $V_H(i) = \{(i \cdot 1), (i \cdot 2), \ldots, (i \cdot H)\}$. We introduce an artificial vertex $(0 \cdot 0)$ and define a vertex set $V_H = \bigcup_{i \in V} V_H(i) \cup \{(0 \cdot 0)\}$. The set of arcs $A_H$ is defined by

$$A_H = \{(0 \cdot 0), (i \cdot 1) \mid i \in V\} \cup \{(i \cdot h), (j \cdot h + 1) \mid I(i) \supseteq I(j), h \in \{1, 2, \ldots, H-1\}\}.$$ 

From this definition, it is clear that $\Gamma_H = (V_H, A_H)$ is a directed acyclic graph. Figure 5 shows $\Gamma_3$ corresponding to the interval representation in Figure 3(c).

Given an $H$-admissible coloring $\phi'$, we define a subset of directed edges of $A_H$ as follows. Let $T(\phi')$ be an arborescence of $\Gamma$, defined in Section 3. For each vertex $i \in V$, $\text{Hgt}(\phi', i)$ denotes the length (number of edges) in a unique path in $T(\phi')$ from the root 0 to $i$. Obviously, we have $\text{Hgt}(\phi', i) \leq H$ ($\forall i \in V$). We define a set of arcs $T_H(\phi')$ of $\Gamma_H$ by

$$T_H(\phi') = \{(i \cdot h - 1), (j \cdot h) \mid (i, j) \in T(\phi')\}$$ 

where $h = \text{Hgt}(\phi', j)$. Figure 5 shows an arc set $T_3(\phi)$ in $\Gamma_3$ corresponding to the 3-admissible 3-coloring $\phi$ in Figure 3(b).
Fig. 5: Arc subset $T_3(\phi)$ in $\Gamma_3$ corresponding to a 3-admissible 3-coloring $\phi$ in Figure 3(b).

For any vertex $(i \cdot h) \in V_H$, $\delta^I(i \cdot h)$ and $\delta^O(i \cdot h)$ denote a set of arcs in-coming to $(i \cdot h)$ in $A_H$ and a set of arcs emanating from $(i \cdot h)$ in $A_H$, respectively. Given an arc subset $T' \subseteq A_H$ and vertex $(i \cdot h) \in V_H$, we define a set of vertices $\text{Ch}(T', (i \cdot h)) = \{ j \in V \mid ((i \cdot h), (j \cdot h + 1)) \in T' \}$. Then, we have the following property.

**Lemma 6.1.** Let $T'$ be an arc subset of $\Gamma_H$. Then, there exists an $H$-admissible $c$-coloring $\phi'$ of a given circle graph $G$ satisfying $T' = T_H(\phi')$ if and only if

**D0:** for any vertex $i \in V$, $T'$ includes a unique arc in $\bigcup_{h=1}^H \delta^I(i \cdot h)$,

**D1:** for each $(i \cdot h) \in V_H \setminus \{(0 \cdot 0)\}$, $\text{Ch}(T', (i \cdot h))$ is a chain of $(V, \preceq)$ or the empty set; if $\text{Ch}(T', (i \cdot h))$ is a (non-empty) chain, then $T'$ contains a unique in-coming arc to $(i \cdot h)$, and
**D2:** the size of every antichain of $(V, \preceq)$ contained in $\text{Ch}(T', (0 \cdot 0))$ is less than or equal to $c$.

Proof (outline). Given a set of arcs $T' \subseteq A_H$ satisfying conditions D0, D1, and D2, we define a set of arcs $T''$ of $\Gamma$ by

$$T'' = \{(i,j) \in A \mid \exists h \in \{0,1,2,\ldots,H-1\}, ((i \cdot h), (j \cdot h + 1)) \in T'\}.$$  

Condition D0 implies that $T''$ is an arborescence of $\Gamma$. From conditions D1 and D2, $T''$ satisfies conditions C1 and C2 in Lemma 6.1, and thus there exists a $c$-coloring, denoted by $\phi''$, of $G$. The definition of $\Gamma_H$ and condition D1 imply that $H_{\text{gt}}(\phi'', i) \leq H$ ($\forall i \in V$), which implies that $\phi''$ is $H$-admissible.

The converse implication is obvious. □

Now, we give an integer linear programming formulation. For each arc $((i \cdot h), (j \cdot h + 1)) \in A_H$, we introduce a 0-1 variable $x_{i \cdot h}^j$. The vector of all 0-1 variables is denoted by $x \in \{0,1\}^{A_H}$. For any vertex $(i \cdot h) \in V_H$, $x^{[i-h]}$ denotes a subvector of $x$ indexed by set of arcs $\delta^O(i \cdot h)$, or $\delta^O(i \cdot h) = \emptyset$. Lemma 6.1 implies a new formulation of $P_H$ as follows:

$$\text{CG}_H: \text{min. } c$$

s.t. $Mx^{[0-0]} \leq c1,$

$$M^i x^{[i-h]} \leq \left(\sum_{(j-h-1) \in \delta^I(i-h)} x_{j}^{i-h-1}\right) 1 \quad (\forall (i \cdot h) \in V^* \times \{1,\ldots,H-1\}),$$

$$\sum_{h=1}^{H} \sum_{(i-h-1) \in \delta^I(j,h)} x_{j}^{i-h-1} = 1 \quad (\forall j \in V),$$

$$x_{j}^{i-h} \in \{0,1\} \quad (\forall ((i \cdot h), (j \cdot h + 1)) \in A_H),$$

$c \in \mathbb{Z}_+.$

It is not difficult to show the following.

**Theorem 6.2.** The optimal value of the linear relaxation problem of $\text{CG}_H$ is equal to the optimal value of the linear relaxation problem of $P_H$.

Proof is omitted.
7 Conclusion

In this paper, we proposed a new formulation for coloring circle graphs. Our formulation is based on an interval representation of a given circle graph and uses a hierarchical structure of a set of intervals corresponding to each independent set. By employing Dilworth’s theorem, we obtain a simple system of inequality constraints represented by a clique matrix of an interval graph defined by a given interval representation.

An advantage of our formulation is that the corresponding linear relaxation problem finds the fractional chromatic number of a given circle graph. Thus, our formulation also gives a polynomial-sized formulation for a fractional coloring problem on a circle graph.

We confirmed by computational experiments that a commercial IP solver can find a coloration quickly under our formulation. When we employed our formulation, CPLEX found optimal solutions for all the instances randomly generated in our computational experiments at the root node (without any branching process). The results of our computational experiments indicate that the chromatic number $\chi(G)$ of a circle graph $G$ is very close to its fractional chromatic number $\chi_f(G)$. We conjecture that there exists a constant $C$ satisfying $\chi(G) - C \leq \chi_f(G)$ for any circle graph $G$.

We extended our result to a formulation for a capacitated stowage stack minimization problem. Future work is required to evaluate the computational performance of the proposed formulation.

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