One dimensional Dirac-Moshinsky oscillator-like system and isospectral partners

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Abstract. Two different exactly solvable systems are constructed using the supersymmetric quantum mechanics formalism and a pseudoscalar one-dimensional version of the Dirac-Moshinsky oscillator as a departing system. One system is built using a first-order SUSY transformation. The second is obtained through the confluent supersymmetry algorithm. The two of them are explicitly designed to have the same spectrum as the departing system and pseudoscalar potentials.

Keywords: Dirac equation, Dirac-Moshinsky oscillator, pseudoscalar potential, confluent SUSY algorithm

1. Introduction

Supersymmetric quantum mechanics (SUSY) is a tool that allows to find new exactly solvable stationary Schrödinger equations. Moreover, the SUSY formalism provides a method to obtain, in principle, a quantum system with a desired spectrum. To apply it we need a solvable initial system and at least one of its solutions, as a result a new system, often referred as SUSY partner, and its solutions are generated [9,11,16]. This tool has been applied to a large variety of quantum interactions such as the harmonic oscillator [15], the hydrogen atom [12,21], the Pöschl-Teller potentials [6], among many others. Moreover, it has been possible to extend the SUSY formalism so it can be used to find solutions of different kind of equations like the Fokker-Planck equation [22,24], the time dependent Schrödinger equation [3], mass position dependent Schrödinger equation [13], Painlevé equations [4] and Dirac equations [5,7,8,10,14,18,20].

The purpose of this work is to apply the SUSY technique to a one dimensional Dirac Moshinsky oscillator-like system [17,23] to obtain new isospectral families using the supersymmetric quantum mechanic formalism. In particular we will focus on two versions of this technique: the first-order SUSY quantum mechanics will be used to generate a one-parameter family of isospectral systems, this part is closely related to Bogdan Mielnik’s seminal article on isospectral Hamiltonians to the harmonic oscillator, and second, a three parametric family will be generated through the confluent supersymmetry algorithm.

This paper is organized as follows. Section 2 gives a brief review of the first-order and the confluent case of supersymmetric quantum mechanics. In section 3 the confluent supersymmetry algorithm for Dirac equations is described. Section 4 is devoted to present and apply the SUSY
formalism to a one dimensional version of the Dirac-Moshinsky oscillator in order to find new isospectral systems. Our conclusions will be presented in the last section.

2. Supersymmetric quantum mechanics
In this section we explain how to make a SUSY transformations focusing on two types of them. The simplest case of this technique is the first-order SUSY transformation and it permits to add a new level below the ground state energy, to delete the ground state energy or to create a new isospectral system. The second type is the confluent SUSY transformation and it allows to add an energy level in any position, to delete any existing level and also new isospectral systems can be generated.

Consider the following one dimensional stationary Schrödinger equation

\[ \Psi'' + (\epsilon - U_0)\Psi = 0, \]  

(1)

where the energy \( \epsilon \) is a real-valued constant, \( U_0 = U_0(x) \) is the potential energy and the prime denotes derivative with respect to the position. Through a SUSY transformation a new Schrödinger equation and its solutions can be obtained if the solutions of the original equation (1) are known.

2.1. First-order SUSY transformation
To apply a first-order SUSY transformation suppose that we know a solution \( u \) for the equation

\[ u'' + (\lambda - U_0)u = 0, \]  

(2)

where \( \lambda \) is a real constant known as factorization energy, then the function

\[ \Phi = \frac{u'}{u} \Psi - \Psi' \]  

(3)

is solution of the Schrödinger equation

\[ \Phi'' + (\epsilon - U_1)\Phi = 0, \]  

(4)

where

\[ U_1 = U_0 - 2 (\ln u)'' \]  

(5)

is the transformed potential, also called SUSY partner of \( U_0 \). In the case \( \epsilon = \lambda, \ \Psi = u \), the function (3) gives the trivial solution, it can be shown by direct substitution that the function

\[ \Phi = \frac{1}{u} \]  

(6)

is solution of (4) under these circumstances. When we consider boundary conditions for quantum mechanical problems with discrete spectra, the eigenvalues of the two Schrödinger equations (1) and (4) are exactly the same or at most they differ only in the lowest eigenvalue or ground state energy. If the goal is to obtain regular transformed potentials, this technique imposes the restriction \( \lambda \leq \epsilon_0 \) where \( \epsilon_0 \) is the ground state energy of the original system.
2.2. Confluent SUSY transformation

In order to apply the second-order SUSY transformation we need two transformations functions $u_1$ and $u_2$ satisfying the following set of equations

\begin{align}
  u_1'' + (\lambda - U_0)u_1 &= 0, \\
  u_2'' + (\lambda - U_0)u_2 &= u_1,
\end{align}

this set is referred as a Jordan chain. Once the two transformation functions are known we can construct the function

$$
\Phi = \frac{W_{u_1, u_2}}{W_{u_1, u_2}} \Psi
$$

where $W$ represent the Wronskian of the functions in its index. The function $\Phi$ is a solution of the Schrödinger equation

$$
\Phi'' + (\epsilon - U_2)\Phi = 0
$$

where the potential $U_2$ is given by

$$
U_2 = U_0 - 2(\ln W_{u_1, u_2})''.
$$

As in the first order transformation, a special case is presented when $\lambda = \epsilon$ and $\Psi = u_1$; direct substitution shows that

$$
\Phi = \frac{u_1}{W_{u_1, u_2}}
$$

is a non trivial solution of (10) in this situation. For quantum mechanical problems with discrete spectra, equations (1) and (10) with same boundary conditions could have the same spectrum or, in the more general case, the spectra will differ in only one eigenvalue. The main difference between the first-order and the confluent transformations is that in the last case the restriction for the values of $\lambda$ disappears.

3. Dirac equation with pseudoscalar potentials and the confluent supersymmetry algorithm

The stationary Dirac equation in one spatial dimension is given by [18,19]

$$
i \sigma_2 \Psi' + (V_0 - E) \Psi = 0,
$$

where $\sigma_2$ stands for the second Pauli matrix, $E$ denotes a real-valued number, and the spinor $\Psi = (\Psi_1, \Psi_2)^T$ represents the solution. Furthermore, we assume the potential $V_0$ to be of pseudoscalar form, that is,

$$
V_0 = m \sigma_3 + q_0 \sigma_1,
$$

for a constant, positive mass $m$, Pauli matrices $\sigma_1$, $\sigma_3$, and a function $q_0 = q_0(x)$. In components, the Dirac equation (13) for the potential (14) can be written as

\begin{align}
\Psi_1' - q_0 \Psi_1 + (E + m) \Psi_2 &= 0, \\
\Psi_2' + q_0 \Psi_2 - (E - m) \Psi_1 &= 0.
\end{align}

Solving the first of these equations for $\Psi_2$ gives the relation

$$
\Psi_2 = \frac{1}{E + m} (q_0 \Psi_1 - \Psi_1')
$$

Substitution of this expression into (16) leads to the equation

$$
\Psi_1'' + (E^2 - m^2 - q_0^2 - q_0') \Psi_1 = 0.
$$
Hence, any spinor that solves the initial Dirac equation (13), is also a solution to equations (17), (18) and vice versa. Let us now rewrite (18) as follows
\[ \Psi_1'' + (\epsilon - U_0) \Psi_1 = 0, \] (19)
where \( \epsilon = E^2 - m^2 \) and \( U_0 = q_0^2 + q_0' \). Comparing equations (17) with (3) we can see that the component \( \Psi_2 \) can be obtained as a first-order SUSY transformation where the function \( q_0 \) can be written as
\[ q_0 = u'. \] (20)

The equation fulfilled by \( u \) can be found substituting the relation (20) in the definition of the potential \( U_0 = q_0^2 + q_0' \), the resulting equation is
\[ u'' - U_0 u = 0, \] (21)
i.e. the transformation function \( u \) is a solution of the Schrödinger equation (19) when \( \epsilon = \lambda = 0 \). Thus, in any stationary Dirac equation in one dimension with pseudoscalar potential the component \( \Psi_1 \) satisfies a Schrödinger equation and the \( \Psi_2 \) can be obtained through a first-order SUSY transformation where the factorization constant \( \lambda \) is equal to zero.

Now, to obtain a new exactly solvable Dirac equation with a pseudoscalar potential in the form
\[ i \sigma_2 \Phi' + (V_1 - E) \Phi = 0, \] (22)
where \( \Phi = (\Phi_1, \Phi_2)^T \), \( E \) a real-valued constant and the pseudoscalar potential as
\[ V_1 = m \sigma_3 + q_1 \sigma_1, \] (23)
we can use the solutions of (13) and the second-order SUSY formalism to obtain the component \( \Phi_1 \), then through a first-order SUSY transformation the function \( q_1 \) and the component \( \Phi_2 \) can be constructed.

Let us assume that we have found three transformation functions \( u_1, u_2, u_3 \), the first two are solution of the Jordan chain (7), (8) while \( u_3 \) is solution of (2) for \( \lambda = 0 \). Then the function \( \Phi_1 \), given by (9) is a solution of the Schrödinger equation
\[ \Phi_1'' + (\epsilon - U_2) \Phi_1 = 0. \] (24)
The potential \( U_2 \) is given by the expression (11), that is,
\[ U_2 = U_0 - 2 \left[ \ln (W_{u_1,u_2}) \right]''. \] (25)

Analogous to (20), we can find \( q_1 \) with a solution of (24) transforming \( u_3 \) as given in (9), for instance
\[ q_1 = \frac{W_{u_1,u_2,u_3}'}{W_{u_1,u_2,u_3}} - \frac{W_{u_1,u_2}'}{W_{u_1,u_2}}. \] (26)

The component \( \Phi_2 \) can be obtained applying a first-order SUSY transformation to the functions \( \Psi_1 \) in the following way
\[ \Phi_2 = \frac{1}{E + m} (q_1 \Phi_1 - \Phi_1'). \] (27)

In the last step it remains to convert (26) into a transformed pseudoscalar Dirac potential \( V_1 \), given by
\[ V_1 = m \sigma_3 + q_1 \sigma_1. \] (28)
The explicit form of this potential is obtained after insertion of (26). We get

$$V_1 = m \sigma_3 + \left[ \frac{W_{u_1,u_2,u_3}}{W_{u_1,u_2,u_3}} - \frac{W'_{u_1,u_2}}{W_{u_1,u_2}} \right] \sigma_1. \quad (29)$$

The family of pseudoscalar potentials (29) enters in the transformed Dirac equation as follows

$$i \sigma_2 \Phi' + (V_1 - E) \Phi = 0. \quad (30)$$

The first component $\Phi_1$ of the solution spinor $\Phi = (\Phi_1, \Phi_2)^T$ of our equation (30) is given in (9), while the second component $\Phi_2$ is in (27).

4. One dimensional Dirac-Moshinsky oscillator and isospectral SUSY partners

Consider the following one-dimensional Dirac-Moshinsky oscillator-like equation:

$$i \sigma_2 \Psi' + (x \sigma_1 + m \sigma_3 - E) \Psi = 0, \quad x \in (-\infty, \infty) \quad (31)$$

where $\Psi = (\Psi_1, \Psi_2)^T$ is a two component spinor and the boundary conditions to be fulfilled are

$$\lim_{|x| \to \infty} \Psi_1 = \lim_{|x| \to \infty} \Psi_2 = 0. \quad (32)$$

Comparing (13) and (14) with (31) it can be seen that for this system $q_0 = x$. Furthermore, from (18), the equation satisfied by the component $\Psi_1$ of the spinor $\Psi$ is

$$\Psi_1'' + (E^2 - m^2 - x^2 - 1) \Psi_1 = 0, \quad (33)$$

and with the substitution $\epsilon = E^2 - m^2$ and $U_0 = x^2 + 1$ it has the standard form of a Schrödinger equation (see (1)). Considering the boundary conditions (32), the eigenvalues of (33) are

$$\epsilon_n = 2n + 2, \quad n = 0, 1, 2, \ldots, \quad (34)$$

and the corresponding eigenfunctions are

$$\Psi_{1,n} = \frac{1}{\sqrt{\pi^{1/2} 2^n n!}} \exp \left( -\frac{x^2}{2} \right) H_n(x) \quad (35)$$

where $H_n$ are the Hermite polynomials [2]. The second component can be found using the expression (17) and simplified using properties of the Hermite polynomials as

$$\Psi_{2,n} = \frac{1}{E_n + m} (x \Psi_{1,n} - \Psi_{1,n}') = \frac{\sqrt{2(n + 1)}}{E_n + m} \Psi_{1,n+1}. \quad (36)$$

The eigenvalues then are given by solving for $E$ the relation $\epsilon = E^2 - m^2$:

$$|E_n| = \sqrt{2n + 2 + m^2}. \quad (37)$$

For this one dimensional Dirac-Moshinsky like system both components (35) and (36) of the spinor $\Psi$ are solution of the harmonic oscillator Schrödinger equation. Figure 1 shows the function $q_0 = x$ and three normalized probability densities $P = \Psi^T \Psi$ for $n = 0$ (blue curve), $n = 1$ (purple curve) and $n = 2$ (green curve).
4.1. Isospectral Dirac problems using the first-order SUSY.

To obtain a new exactly solvable Dirac equation

\[ i\sigma_2 \tilde{\Psi}' + (\tilde{V} - E)\tilde{\Psi} = 0, \quad x \in (-\infty, \infty) \]

\[ \lim_{|x| \to \infty} \tilde{\Psi}_1 = \lim_{|x| \to \infty} \tilde{\Psi}_2 = 0. \]  

(38)

with the spectrum (37) using the first order transformation we can keep one component of the spinor \(\tilde{\Psi}\) as given by (35) or (36) and then build the second with a general 1-SUSY transformation. In this work we start by taking \(\tilde{\Psi}_1 = \Psi_1\). Now, in order to construct the component \(\tilde{\Psi}_2\) for the new system we need a transformation function \(u\) satisfying

\[ u'' - (x^2 + 1)u = 0. \]  

(39)

The general solution of this equation is

\[ u = \exp\left(\frac{x^2}{2}\right) \left[ C_1 + C_2 \frac{\sqrt{\pi}}{2} \text{Erf}(x) \right], \]  

(40)

where Erf stands for the Error function [2]. The function \(\tilde{q}_0\) involved in the new pseudoscalar potential will be given by (20) as

\[ \tilde{q}_0 = \frac{u'}{u} = x + \frac{2C_2 \exp\left(-\frac{x^2}{2}\right)}{2C_1 + C_2 \sqrt{\pi} \text{Erf}(x)}. \]  

(41)

In order to avoid singularities for the new function \(\tilde{q}_0\) we should set \(C_1 = 1\) and \(|C_2| < 2/\sqrt{\pi}\). Then the new Dirac equation is

\[ i\sigma_2 \tilde{\Psi}' + \left\{ m\sigma_3 + \left[ x + \frac{2C_2 \exp\left(-\frac{x^2}{2}\right)}{2 + C_2 \sqrt{\pi} \text{Erf}(x)} \right] \sigma_1 - E \right\} \tilde{\Psi} = 0, \quad x \in (-\infty, \infty) \]  

(42)
and its boundary conditions as in (38). The second component is obtained with the relation (17) and can be expressed as

\[
\tilde{\Psi}_{2,n} = \frac{1}{E_n + m} \left[ \sqrt{2(n + 1)} \Psi_{1,n+1} + \frac{2C_2 \exp \left( -\frac{x^2}{2} \right)}{2 + C_2 \sqrt{\pi} \operatorname{Erf}(x)} \Psi_{1,n} \right].
\] (43)

The eigenvalues of the new Dirac equation (42) are given in (37), i.e. the system (31) and the family of equations (42) characterized by the continuous parameter \( C_2 \) are isospectral.

Figure 2 shows on the left the plot of the function \( q_0 = x \) (blue curve) and the function \( \tilde{q}_0 \) given by (41) when \( C_1 = 1 \) and \( C_2 = -1.1 \) (purple curve). On the right the normalized probability densities \( \tilde{P} = \tilde{\Psi}^2 \tilde{\Psi} \) for \( n = 0 \) (blue curve), \( n = 1 \) (purple curve) and \( n = 2 \) (green curve) when \( C_2 = -1.1 \).

4.2. Isospectral partners of the Dirac Moshinsky Oscillator using confluent SUSY transformations

To obtain new isospectral and exactly solvable Dirac problems we can use the confluent supersymmetry algorithm for Dirac equations. The new Dirac equation will have the form

\[
 i\sigma_2 \Phi' + (V_1 - E)\Phi = 0, \quad x \in (\pm \infty, \infty) \\
 \lim_{|x| \to \infty} \Phi_1 = \lim_{|x| \to \infty} \Phi_2 = 0,
\] (44)

where \( \Phi = (\Phi_1, \Phi_2)^T \) and \( V_1 \) is a pseudoscalar potential with a parametrizing function \( q_1 \) to be found. First we can apply the second-order confluent algorithm to the Schrödinger equation (33) to obtain the equation to be fulfilled by \( \Phi_1 \), then applying a first order SUSY transformation the parametrizing function \( q_1 \) involved in the new potential \( V_1 \) and the component \( \Phi_2 \) of the spinorial solution can be constructed.
To proceed we need the third transformation function, where the new potential \( U \) variation-of-constants formula. Now, through the confluent SUSY transformation we can obtain of the homogeneous equation (8) while the last term is a particular solution obtained using the integrable solution of the Schrödinger equation (7). The first two terms of \( u_2 \) are solution the system (7)-(8) with \( U \) we obtain

\[
u_1 = \Psi_{1,k}, \quad k = 0, 1, 2, \ldots,
\]

\[
u_2 = \Psi_{1,k} + C_3 \Psi_{1,k} \int_{-\infty}^{x} \frac{1}{\Psi_{1,k}^2} ds + \Psi_{1,k} \int_{-\infty}^{x} \frac{\int_{-\infty}^{t} \Psi_{1,k}^2 ds}{\Psi_{1,k}^2} dt
\]

where \( C_3 \) is an arbitrary constant. We are using as transformation function \( u_1 \) a square integrable solution of the Schrödinger equation (7). The first two terms of \( u_2 \) are solution the variation-of-constants formula. Now, through the confluent SUSY transformation we can obtain

\[
\Phi''_1 + (\epsilon - U_1)\Phi_1 = 0,
\]

where the new potential \( U_1 \) is given by (11). Using the transformation functions in (45) and (46) we obtain

\[
U_1 = x^2 + 1 - 2 \left[ \ln \left( C_3 + \int_{-\infty}^{x} \Psi_{1,k}^2 dx \right) \right]''.
\]

With the indicated integration limits and since \( \Psi_{1,k} \) is a normalized eigenfunction of (33) we can ensure that the potential \( U_1 \) is free of singularities when \( C_3 \in (-\infty, -1) \cup (0, \infty) \). The solution of (47) can be written in terms of (35) and the transformation functions \( u_1 \) and \( u_2 \) as

\[
\Phi_1 = \frac{W_{u_1, u_2, \Psi_1}}{W_{u_1, u_2}} = \left( \epsilon_k - \epsilon + \frac{\Psi_{1,k} \Psi_{1,k}'}{C_3 + \int_{-\infty}^{x} \Psi_{1,k}^2 dx} \right) \Psi_1 - \left( \frac{\Psi_{1,k}^2}{C_3 + \int_{-\infty}^{x} \Psi_{1,k}^2 dx} \right) \Psi_1'.
\]

Note that the Wronskian in the numerator was simplified by Laplace expansion and (19) was used for further simplification. The previous formula is valid when \( \epsilon \neq \epsilon_k \), in case the the factorization energy \( \epsilon_k \) coincides with the energy \( \epsilon \) we have to consider (12), then

\[
\Phi_1 = \frac{\Psi_{1,k}}{C_3 + \int_{-\infty}^{x} \Psi_{1,k}^2 dx}.
\]

To proceed we need the third transformation function, \( u_3 \) satisfying (2) when \( \lambda = 0 \), its solution is

\[
u_3 = \exp \left( \frac{x^2}{2} \right) \left[ 1 + C_4 \frac{\sqrt{\pi}}{2} \text{Erf}(x) \right],
\]

where \( C_4 \) is an arbitrary constant. Now, using the Jordan chain (7)-(8) and (2) the Wronskian \( W_{u_1, u_2, u_3} \) and its first derivative can be expressed as

\[
W_{u_1, u_2, u_3} = -\Psi_{1,k}^2 u_3' - \epsilon_k \left( C_3 + \int_{-\infty}^{x} \Psi_{1,k}^2 dx \right) + \Psi_{1,k} \Psi_{1,k} \right) u_3,
\]

\[
W_{u_1, u_2, u_3}' = \epsilon_k \left( C_3 + \int_{-\infty}^{x} \Psi_{1,k}^2 dx \right) - \Psi_{1,k} \Psi_{1,k} \right) u_3' + \Psi_{1,k}^2 u_3,
\]

these two expressions have to be substituted in (26) to find

\[
q_1 = \frac{W_{u_1, u_2, u_3}}{W_{u_1, u_2, u_3}} - \left( \frac{\Psi_{1,k}^2}{C_3 + \int_{-\infty}^{x} \Psi_{1,k}^2 dx} \right).
\]
The component \( \Phi_2 \) of the spinor \( \Phi \) is given by (27) as
\[
\Phi_2 = \frac{1}{E + m} \left( q_1 \Phi_1 - \Phi'_1 \right),
\] (54)

unfortunately, the explicit form of this function is very long, such that we do not show it here but the explicit expressions for the functions involved are in (49), (50) and (53).

The function \( q_1 \) is plotted in Fig 3 (purple curve). The figure shows how the parametrizing function \( q_0 \) (blue curve) is deformed after the confluent SUSY transformation. Also, on the right, you can find the first three normalized probability densities \( \hat{P} = \Phi^T \Phi \). The parameters characterizing this particular transformation are \( \epsilon_k = \epsilon_1 \), i.e. we are using the first excited level in the Schrödinger scheme as transformation function, the constant \( C_3 = 0.04 \) (see (46)) and finally the third parameter appearing in (51) is \( C_4 = 1 \). Note that with the selected parameter the spinor component \( \Phi_1 \) is given by (49) for \( n = 0, 2, 3, 4, \ldots \) and by (50) when \( n = 1 \). The function \( q_1 \) was constructed using a non-standard SUSY transformation in the sense that a factorization energy \( \epsilon_k \) above the ground state energy was used.

![Figure 3](image.jpg)

**Figure 3.** On the left, function \( q_1 \) (purple curve) and as reference \( q_0 \) (blue curve). On the right, probability densities \( \hat{P} \) for \( n = 0 \) (blue curve), \( n = 1 \) (purple curve) and \( n = 2 \) (green curve). The parameters used are \( \epsilon_k = \epsilon_1 = 4 \), \( C_3 = 0.04 \) and \( C_4 = 1 \).

5. Concluding remarks
Two different exactly solvable families of systems were constructed departing from a Dirac-Moshinsky oscillator like system in one dimension using supersymmetric quantum mechanics. They were explicitly built to be isospectral. The first family obtained is characterized by a continuous parameter and, as the Dirac-Moshinsky Oscillator is related to the harmonic oscillator in non-relativistic quantum mechanics, the new systems are related to the Abraham-Moses-Mielnik oscillator. The second generated family is characterized by three parameters, two of them are continuous and the other one is discrete. This last family was obtained using the confluent supersymmetry algorithm and expressions for the solutions were given.

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