THE LARGEST SIZE OF CONJUGACY CLASS AND THE $p$-PARTS OF FINITE GROUPS

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Abstract. Let $p$ be a prime and let $P$ be a Sylow $p$-subgroup of a finite nonabelian group $G$. Let $bcl(G)$ be the size of the largest conjugacy class of the group $G$. We show that $|P/O_p(G)| < bcl(G)$ if $G$ is not abelian.

1. Introduction

Throughout this paper, $G$ is a finite group. Let $P$ be a Sylow $p$-subgroup of a finite nonabelian group $G$, let $b(G)$ denote the largest irreducible character degree of $G$, and let $bcl(G)$ denote the size of the largest conjugacy classes of a finite group $G$.

It is known for finite groups that $b(G)$ is connected with the structure of $G$. In [2] Gluck proved that in all finite groups the index of the Fitting subgroup $F(G)$ in $G$ is bounded by a polynomial function of $b(G)$. For solvable groups, Gluck further shows that $|G:F(G)| \leq b(G)^{13/2}$ and conjectured that $|G:F(G)| \leq b(G)^2$. In [8], this bound was improved to $|G:F(G)| \leq (b(G)^p/p)^{1/p}$.

When we focus on a single prime, a stronger bound can be found. In [10], Qian and Shi showed that if $G$ is any finite group, then $|P/O_p(G)| < b(G)^2$ and $|P/O_p(G)| \leq b(G)$ if $P$ is abelian. Recently, the authors [11] improved the previous result, and showed that for a finite nonabelian group $G$, $|P/O_p(G)| \leq (b(G)^p/p)^{1/p}.$

Since there is some analogy between conjugacy class sizes and character degrees of a finite group, one may ask: do there exist some similar results for conjugacy class sizes?

Inspired by the results in [10], He and Shi [3 Theorem A] showed that for any finite group $|P/O_p(G)| < bcl(G)^2$ and $|P/O_p(G)| \leq bcl(G)$ if $P$ is abelian. In [7], Liu and Song improved the previous bound by showing that $|P/O_p(G)| \leq (bcl(G)^p/p)^{1/p}$ for a finite nonabelian group $G$. Yang [14] recently strengthened the bound to $|P/O_p(G)| \leq bcl(G)$ when $p$ is an odd prime but not a Mersenne prime. In this paper we remove the extra conditions and show that as long as $G$ is nonabelian, then we will always have $|P/O_p(G)| < bcl(G)$. This strengthened all the previously mentioned results in this paragraph. We also show that the bound is the best possible.

2. Main Theorem

While the proofs in [5,7,14] mainly use the consequences of some orbit theorems of linear group actions, it seems one has to get a weaker bound or to exclude some important cases due to the limit of those orbit theorems. However, by using the consequence of the $k(GV)$ problem, we are able to achieve the best possible bound.

The $p$-solvable case of the famous Brauer’s $k(B)$ conjecture was discovered to be equivalent to the $k(GV)$ problem (Fong [1], Nagao [9]). Namely, when a finite group acts coprimely on a
finite vector space $V$, the number of conjugacy classes of the group $G \times V$ is less than or equal to $|V|$, the number of elements in the vector space. Thompson, Robinson, Maggard, Gluck, Schmid [12, 3, 13] and many others have contributed to this well-known problem, and the key to the solution is to study the orbit structure of the linear group actions. The following could be viewed as a generalization of a special case of the $k(GV)$ problem by Guralnick and Robinson. [4]. The proof does not use the full strengthen of the $k(GV)$ problem, only the special case Knörr [6] proved a while back where the acting group is nilpotent.

**Lemma 2.1.** Let $G$ be a finite solvable group such that $G/F(G)$ is nilpotent, then we have $k(G) \leq |F(G)|$.

**Proof.** This is [4 Lemma 3].

**Lemma 2.2.** Let $G$ be a finite nilpotent group that acts faithfully and coprimely on an abelian group $V$, and we consider the semidirect product $G \times V$, then the largest conjugacy class size in $G \times V$ is of size greater than $|G|$.

**Proof.** By Lemma 2.1 we know that the number of conjugacy classes in $G \times V$ is less than or equal to $|V|$. Also the identity is a conjugacy class of size 1, and the result follows.

**Lemma 2.3.** Let $G$ be a Sylow $p$-subgroup of a permutation group of degree $n$. Then $|G| \leq 2^{n-1}$.

**Proof.** This result is well known (cf. [7 Lemma 5]).

**Proposition 2.4.** Let $G$ be one of the nonabelian simple groups and $P \in \text{Syl}_p(\text{Aut}(G))$ for some prime $p$. Then $\text{bcl}(G) > 2|P|$.

**Proof.** It was stated in [14 Proposition 2.6] that $\text{bcl}(G) \geq 2|P|$ but a close examination of the proof indeed shows that $\text{bcl}(G) > 2|P|$.

We now prove the main result.

**Theorem 2.5.** Let $p$ be a prime and let $P$ be a Sylow $p$-subgroup of a finite nonabelian group $G$. Let $\text{bcl}(G)$ be the size of the largest conjugacy class of the group $G$. Then $|P/O_p(G)| < \text{bcl}(G)$.

**Proof.** We will work by induction on $|G|$. Note that for any subgroup or quotient group $L$ of $G$, $\text{bcl}(G) \geq \text{bcl}(L)$.

Clearly we may assume that $O_p(G) = 1$. Assume that $\Phi(G) > 1$. Since $O_p(G) = 1$, we see that $\Phi(G)$ is a $p'$-group since $\Phi(G)$ is nilpotent. Let $T$ be the pre-image of $O_p(G/\Phi(G))$ in $G$. It is clear that $T = \Phi(G)Q$ where $Q$ is a Sylow $p$-subgroup of $T$. By the Frattini’s argument, we have that $G = N_G(Q)T = N_G(Q)\Phi(G)Q = N_G(Q)$, and thus $Q$ is a normal subgroup of $G$. Thus we know that $Q = 1$ and $O_p(G/\Phi(G)) = 1$. Hence we may assume that $\Phi(G) = 1$.

Assume that all minimal normal subgroups of $G$ are solvable. Let $F$ be the Fitting subgroup of $G$. Since $\Phi(G) = O_p(G) = 1$, $G = F \rtimes A$ is a semidirect product of an abelian $p'$-group $F$ and a group $A$.

Clearly, $C_G(F) = C_A(F) \times F$ and $C_A(F) \leq G$. Since $F$ contains all the minimal normal subgroups of $G$, we conclude that $C_A(F) = 1$, and hence, $C_G(F) = F$. Let us investigate the subgroup $K = PF$. Since $O_p(K)$ centralizes $F$ and hence $O_p(K) \leq C_G(F) = F$, it follows that $O_p(K) = 1$. 


By induction, we may assume that $G = K = PF$. Observe that $G = PF$ is solvable and $P$ acts faithfully on the abelian $p'$-group $F$. By Lemma 2.2, we know the result follows.

Now we assume that $G$ has a nonsolvable minimal normal subgroup $V$. Let $V = V_1 \times \cdots \times V_k$, where $V_1, \ldots, V_k$ are isomorphic nonabelian simple groups. Let us investigate the subgroup $K = P(V \times C_G(V))$.

Since $V$ is a direct product of nonabelian simple groups, $O_p(V) = 1$. This implies that $V \cap O_p(K) = 1$. Since $V$ and $O_p(K)$ are both normal in $K$, $O_p(K)$ centralizes $V$, so $O_p(K) \leq C_G(V)$, and hence $O_p(K) \leq O_p(C_G(V))$. Since $C_G(V)$ is normal in $G$, we see that $O_p(C_G(V)) \leq O_p(G) = 1$. Thus, we conclude that $O_p(K) = 1$. Therefore we may assume by induction that $G = P(V \times C_G(V))$.

Set $|C_G(V)|_p = p^n$, $|G/C_G(V)|_p = p^v$.

Clearly $O_p(C_G(V)) = 1$. If $C_G(V)$ is not abelian, then by induction there exists $t \in C_G(V)$ such that $|t^{C_G(V)}| \geq |C_G(V)|_p = p^n$. If $C_G(V)$ is abelian, then clearly and $p^n = 1$. Thus in all cases, we can find and $t \in C_G(V)$ such that $|t^{C_G(V)}| \geq |C_G(V)|_p = p^n$.

Let $x_i \in V_i$ such that $|x_i^{V_i}| = bcl(V_i)$ and set $x = x_1 \cdots x_k$. Clearly $x \in V$ and $|x^V| = bcl(V_1)^k = bcl(V)$. Note that $G/(V \times C_G(V)) \leq Out(V) \cong Out(V_1) \wr S_k$, $G/C_G(V) \leq Aut(V) \cong Aut(V_1)^k$. By Lemma 2.3 we have $p^v = |G/C_G(V)|_p \leq 2^{k-1}(|Out(V_1)|_p)^k$.

By Proposition 2.4, we have

$$bcl(V_1) > 2|Aut(V_1)|_p,$$

thus

$$bcl(V) > (2|Aut(V_1)|_p)^k \geq |S_k|_p |Aut(V_1)|_p^k \geq |G/C_G(V)|_p = p^v,$$

and then

$$bcl(G) \geq bcl(V \times C_G(V)) \geq bcl(V) \cdot bcl(C_G(V)) > |G|_p,$$

and we are done. \hfill \Box

Remark: We provide a family of examples to show that our result is the best possible. Let $G = K \times V$ where $|V|$ is a Fermat prime, $|K| = |V| - 1 = 2^n$ and $K$ acts fixed point freely on $V$. Note that $bcl(G) = |V|$ and $|G/O_2(G)|_2 = 2^n = |V| - 1$.

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