Extremal Kerr/CFT correspondence of five-dimensional rotating (charged) black holes with squashed horizons

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Abstract

A new holographic duality named Kerr/CFT correspondence was recently proposed to derive the statistical entropy of four-dimensional extremal Kerr black holes via identifying the quantum states in the near-horizon region with those of two-dimensional conformal field theory living on the boundary. In this paper, we apply this method to investigate five-dimensional extremal Kerr and Cvetič-Youm black holes with squashed horizons in different coordinates and find that the near-horizon geometries are not affected by the squashing transformation. Our investigation shows that the microscopic entropies are in agreement with those given by Bekenstein-Hawking formula. In addition, we have also investigated thermodynamics of the general non-extremal Cvetič-Youm black holes with squashed horizons.

1 Introduction

Recently, a new duality called as Kerr/CFT correspondence has been put forward in [1] to evaluate the microscopic entropy of a four-dimensional extremal Kerr black hole. This

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method is very similar to the early work of Brown and Henneaux [2] except that the \( AdS_3 \) background is replaced by the near-horizon geometry of an extremal Kerr black hole previously obtained in Ref. [3]. It indicates that the generators of diffeomorphism, which preserves an appropriate boundary condition for the near-horizon geometry of an extremal Kerr black hole, are found to form a copy of Virasoro algebra in the two-dimensional conformal field theory (CFT). By virtue of the Cardy formula, the microscopic entropy that matches the one macroscopically calculated by the Bekenstein-Hawking area law can be derived in terms of a generalized temperature related to the Frolov-Thorne vacuum. Compared with the previous derivation [4] of the microscopic entropy of a five-dimensional extremal rotating black hole [5], this method does not use string theory or supersymmetry, like the work of [6]. It is worth noticing that this method is essentially different from the Carlip’s work [7] in which the boundary is also restricted to the horizon, and the statistical entropy of a general non-extremal black hole is derived via studying the relationship between black hole thermodynamics and the two-dimensional near-horizon CFT [7,8,9].

Inspired with the extremal Kerr/CFT correspondence proposed in [1], the method has been generalized to study the entropies of extremal black holes in a lot of theories such as the Einstein theory, string theory, and supergravity theory, as well as those solutions in diverse dimensions [10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27]. The Kerr/CFT correspondence has also been applied to derive the CFT entropy of extremal black rings [28] and that of NS5-branes [29]. In all these extensions, the central charges from the vector and scalar fields were usually neglected, but it was proven in [30] that they indeed make no contribution to the central charges. Subsequently, this method was applied [31] to the case of extremal black holes in Einstein-Gauss-Bonnet gravity theory, where it was demonstrated that the CFT entropy need not always be in agreement with that via the Wald formula. However, if one takes into consideration the higher-derivative corrections to the Einstein-Hilbert Lagrangian [32], they may coincide with each other. Furthermore, making use of the holographic renormalization group flow, the Kerr/CFT correspondence was extended in [33] to a more general gauge/gravity correspondence in the full bulk spacetime of extremal black holes. This correspondence has been checked [15,20,21,24] within the context of five-dimensional minimal supergravity. Recently we [20] have examined its validity in a five-dimensional, extremal, rotating (charged) black hole immersed in Gödel universe.

Although the Kerr/CFT correspondence has been checked in a wide range, a lot of
questions, such as how this duality takes place and properties of the conserved charges, are still open. Along these lines, progress has been made in dynamics of the near horizon geometry [34, 35, 36, 37, 38], boundary conditions [39, 40], applicability of the Cardy formula [41] and covariance of the conserved charges [42]. Other approaches to realize this dual can be found in [43, 44, 45].

On the other hand, squashed Kaluza-Klein black holes [46, 47, 48, 49] in five dimensions attracted a lot of recent interest due to their many different features from their un-squashed counterparts. All of them have a squashed three-sphere as their horizon topology, while at spatial infinity they possess the same asymptotic geometry, namely, a nontrivial $S^1$ bundle with constant fiber over the two-sphere in a four-dimensional Minkowski spacetime. With the presence of such an asymptotic structure, one should take into consideration the contribution of the gravitational tension to black hole thermodynamics [50, 51, 52]. What is more, when both the rotation and charge parameters [53], and/or an additional Gödel parameter [54, 55, 56, 57] appear, one has to modify the first law of the squashed black hole thermodynamics [57] by incorporating the effect of the dipole charge [58]. Some other interesting properties of these squashed Kaluza-Klein black holes can be found in [59, 60].

Therefore, it is important to check whether the Kerr/CFT correspondence still holds true or not under the squashing transformation [46]. In this paper, we shall apply the above Kerr/CFT correspondence to explore the entropies of an extremal squashed Kerr black hole [49] and its charged extension [53] — an extremal squashed Cvetič-Youm (CY) black hole in five dimensions. These black hole solutions are generated by performing a squashing transformation to the five-dimensional Myers-Perry black hole [61] with two equal rotation parameters and the CY black hole [62] by setting three charges equal. They are included as special cases of the general squashed version [51, 53, 56, 57] of the five-dimensional Einstein-Maxwell-Chern-Simons-Gödel black hole solution [63] when the Gödel parameter is set to zero. Remarkably, we find that after taking the near-horizon limit to the extremal squashed Kerr and CY black holes, the squashing transformation takes no effect on their near-horizon geometries, and the same still holds true for the extremal squashed black holes embedded in the Gödel universe.

Our purpose of this paper is to examine the Kerr/CFT correspondence for the squashed black holes in different coordinates and to see the central charges how to vary under the change of the angular coordinates. The rest part of this paper is organized as follows. In Sec. 2, we shall utilize the Kerr/CFT correspondence to calculate the entropy of a
five-dimensional extremal squashed Kerr black hole in the coordinates where the initial radial coordinate remains unchanged. In Sec. 3 to compare with the uncharged case, we investigate the extremal squashed CY black hole in a different coordinate system where the apparent singularity at spatial infinity is removed and the angular coordinates are the standard $2\pi$-period azimuthal ones. In Sec. 4 we will present the full thermodynamical properties of a general non-extremal squashed CY black hole and re-derive the generalized CFT temperature associated with the Frolov-Thorne vacuum by making use of the first law of black hole thermodynamics. A brief summary is given in the last section.

2 The extremal squashed Kerr black hole and its CFT duals

In Ref. [1], the microscopic entropy of a four-dimensional extremal Kerr black hole was obtained by identifying its near-horizon quantum states with those of a two-dimensional conformal field theory living on the infinite boundary of the black hole’s near-horizon geometry. Following this method, we shall firstly derive the entropy of a five-dimensional extremal Kerr black hole with a squashed $S^3$ horizon in this section.

2.1 Near-horizon geometry of a squashed Kerr black hole

Making use of the squashing transformation [46] to the five-dimensional Myers-Perry black hole with two equal angular momenta, one can obtain a new black hole solution [49] that possesses a squashed horizon. This squashed Kerr black hole is characterized by three parameters $(m,a,r_\infty)$ that correspond to the mass, angular momentum and the size of a $S^1$ fiber at spatial infinity, respectively. Its metric is given by

$$
\begin{align}
\text{ds}^2 &= -\frac{\chi^2 r^2 \Delta(\hat{r})}{B(\hat{r})} d\hat{t}^2 + \frac{K^2(\hat{r})}{\Delta(\hat{r})} d\hat{r}^2 + \frac{1}{4} \hat{r}^2 K(\hat{r})(d\theta^2 + \sin^2 \theta d\psi^2) \\
&\quad + \frac{1}{4} B(\hat{r}) \left\{ d\phi + \cos \theta d\psi - \chi [\omega(\hat{r}) - \omega(r_\infty)] d\hat{t} \right\}^2,
\end{align}
$$

where

$$
\begin{align}
\Delta(\hat{r}) &= \frac{(\hat{r}^2 - m)^2 - m(m - 2a^2)}{\hat{r}^4}, \\
B(\hat{r}) &= \frac{\hat{r}^4 + 2ma^2}{\hat{r}^2}, \quad \omega(\hat{r}) = \frac{4ma}{\hat{r}^4 + 2ma^2}, \\
K(\hat{r}) &= \frac{(r_\infty^2 - m)^2 - m(m - 2a^2)}{(r_\infty^2 - \hat{r}^2)^2}, \quad \chi = \frac{\sqrt{r_\infty^4 + 2ma^2}}{\sqrt{(r_\infty^2 - m)^2 - m(m - 2a^2)}}.
\end{align}
$$
In Eq. (2.1), the radial coordinate $\hat{r}$ and the Euler angles $(\theta, \psi, \hat{\phi})$ take $0 < \hat{r} < r_\infty$ and $(0 < \theta < \pi, 0 < \psi < 2\pi, 0 < \hat{\phi} < 4\pi)$, respectively. The line element (2.1) depicts a five-dimensional black hole with squashed $S^3$ horizons. It has the asymptotical structure that describes a twisted $S^1$ bundle over a four-dimensional Minkowski space-time when $\hat{r} \to r_\infty$. This can be seen more clearly by sending $\rho \to \infty$ after performing a coordinate transformation $\rho = \rho_0 \hat{r}^2/(r_\infty^2 - \hat{r}^2)^2$, where $\rho_0 = r_\infty \sqrt{\Delta(r_\infty)}/2$. Taking the $r_\infty \to \infty$ limit, one recovers the five-dimensional Kerr metric with two equal angular momenta. The dual CFT entropy of a five-dimensional extremal Kerr black hole with two different rotation parameters was investigated in [11].

The Hawking temperature, the entropy and the angular velocities of the event horizon can be computed as

$$T = \frac{\chi \hat{r}_+ \Delta'(\hat{r}_+)}{4\pi K(\hat{r}_+) \sqrt{B(\hat{r}_+)}} , \quad S = \frac{1}{2} \frac{\pi^2 r_\infty^2 K(\hat{r}_+) \sqrt{B(\hat{r}_+)}}{\sqrt{\Delta(r_\infty)}}, \quad (2.3)$$

$$\Omega_\psi(\hat{r}_+) = 0 , \quad \Omega_\phi(\hat{r}_+) = \chi \left[ \omega(\hat{r}_+) - \omega(r_\infty) \right], \quad (2.4)$$

where $t$ denotes the differentiation with respect to the coordinate $\hat{r}$, and $\hat{r}_+$ is the outer horizon given by $\hat{r}_+^2 = m + \sqrt{m(m - 2a^2)}$. To obtain a regular black hole solution, here and in what follows, we shall assume that $m \geq 2a^2$, $a > 0$, and $\hat{r}_+ \ll r_\infty$. In particular, when $m = 2a^2$, the black hole (2.1) becomes extremal.

Since our aim is to derive the entropy of an extremal squashed Kerr black hole via the CFT duality, we now try to explore its near-horizon geometry. To do so, we need to perform the following coordinate transformations:

$$\hat{r} = \sqrt{2}a(1 + \lambda r) , \quad \hat{t} = \frac{2}{\sqrt{r_\infty^2 + 4a^4}} \cdot \frac{t}{\lambda} , \quad \hat{\phi} = \phi + \frac{r_\infty^4 - 4a^4}{2(r_\infty^4 + 4a^4)} \cdot \frac{t}{\lambda} \quad (2.5)$$

After taking the $\lambda \to 0$ limit, the near-horizon geometry of an extremal squashed Kerr black hole reads

$$ds^2 = \frac{a^2}{2} \left( -r^2 dt^2 + \frac{dr^2}{r^2} + d\theta^2 + \sin^2 \theta d\psi^2 \right) + a^2 (d\phi + \cos \theta d\psi + r dt)^2 \quad (2.6)$$

which describes a three-sphere bundle over the $AdS_2$ space. It is worth noting that the near-horizon metric (2.6) of an extremal squashed Kerr black hole takes the same form as that of an extremal Kerr black hole without performing the squashing transformation since the squashing function $K \to 1$ at the near-horizon limit under the extremity condition $m = 2a^2$. In other words, we find that the squashing transformation does not change the near-horizon geometry of an extremal Kerr black hole.
2.2 Central charges and microscopic CFT entropy

Now, we pay our attention to calculate the central charges of the near-horizon metric (2.6). Before doing this, it is important to impose the appropriate boundary conditions at spatial infinity and find the asymptotical symmetry group that preserves these boundary conditions. Since the five dimensional near-horizon metric (2.6) possesses two \( U(1) \) isometries, we can choose two boundary conditions. Using \( h_{\mu \nu} \) to denote the metric deviation from the near horizon geometry (2.6), one of the boundary conditions is given by

\[
\begin{pmatrix}
  h_{tt} = O(r^2) & h_{tr} = O(1/r) & h_{t\phi} = O(1/r) & h_{t\psi} = O(r) \\
  h_{rr} = O(1/r) & h_{r\theta} = O(1/r) & h_{r\phi} = O(1/r) & h_{r\psi} = O(1/r) \\
  h_{\theta\theta} = O(1/r) & h_{\theta\phi} = O(1/r) & h_{\theta\psi} = O(1/r) \\
  h_{\phi\phi} = O(1) & h_{\phi\psi} = O(1) & h_{\psi\psi} = O(1/r)
\end{pmatrix}.
\]

(2.7)

The most general diffeomorphism that preserves such a boundary condition takes the form

\[
\zeta = \left[ C + O(r^{-3}) \right] \partial_t + \left[ -r \partial_\phi \epsilon(\phi) + O(1) \right] \partial_r + O(r^{-1}) \partial_\theta + \left[ \epsilon(\phi) + O(r^{-2}) \right] \partial_\phi + O(r^{-2}) \partial_\psi,
\]

(2.8)

where \( C \) is a constant and \( \epsilon(\phi) \) is an arbitrary function. If we exchange \( \phi \) and \( \psi \) in Eqs. (2.7) and (2.8), we obtain the other boundary condition and the corresponding general diffeomorphism.

Defining \( \epsilon(\phi) = -e^{-in\phi} \) and \( \epsilon(\psi) = -e^{-in\psi} \), where \( n \) are integers, the asymptotic symmetry group can be generated by a class of diffeomorphisms

\[
\zeta_n^{(1)} = -e^{-in\phi} \partial_\phi - inre^{-in\phi} \partial_r, \quad \zeta_n^{(2)} = -e^{-in\psi} \partial_\psi - inre^{-in\psi} \partial_r,
\]

(2.9)

which satisfy Virasoro algebras

\[
i \left[ \zeta_n^{(i)}, \zeta_n^{(j)} \right] = (l - n) \zeta_{l+n}^{(i)} \quad (i = 1, 2).
\]

(2.10)

It is worth noting that there exists another diffeomorphism \( \partial_t \) that preserves the boundary condition (2.7) and commutes with \( \zeta_n^{(i)} \). As shown in \([1]\), we impose a supplemental boundary condition that the conserved charge generating the diffeomorphism \( \partial_t \) vanishes to eliminate excitations above the extremity. Each diffeomorphism \( \zeta_n^{(i)} \) is associated to a conserved charge defined by \([30, 64]\)

\[
Q_{\zeta_n^{(i)}} = \int_{\partial\Sigma} k_{\zeta_n^{(i)}} [h, g],
\]

(2.11)
where \( \partial \Sigma \) is a spatial slice that extends to the infinity, \( h_{\mu \nu} = \mathcal{L}_z g_{\mu \nu} \) denotes the deviation from the background metric (2.6), and the 3-form \( k^{gr}_\zeta[h,g] \) is given by

\[
k^{gr}_\zeta[h,g] = -\frac{1}{96\pi} \sqrt{-g} \epsilon_{\alpha \beta \gamma \rho \sigma} \left[ \zeta^\rho \nabla^\sigma h - \zeta^\rho \nabla^\sigma h + \zeta^\rho \nabla^\rho h_{\sigma \nu} + \zeta^\rho \nabla^\rho h_{\sigma \nu} + \frac{1}{2} h \nabla^\rho \zeta^\sigma \\
- h^{\rho \nu} \nabla_\nu \zeta^\sigma + \frac{1}{2} h^{\rho \nu} (\nabla^\sigma \zeta_\nu + \nabla^\nu \zeta^\sigma) \right] dx^\alpha \wedge dx^\beta \wedge dx^\gamma , \tag{2.12}
\]

in which, \( \zeta = \zeta^{(i)}_n \). The Dirac brackets of the conserved charges corresponding to the diffeomorphisms \( \zeta^{(i)}_l \) and \( \zeta^{(i)}_n \) yield a common form of the Virasoro algebras with central terms

\[
\int_{\partial \Sigma} k^{gr}_\zeta[h,g] = -\frac{i}{12} c_i (l^3 + \beta l) \delta_{l+n,0} , \tag{2.13}
\]

where \( c_i \) denote the central charges corresponding to the diffeomorphisms \( \zeta^{(i)}_l \), \( \beta \) is a trivial constant since it can be absorbed by a shift in \( Q^{(z)}_{\zeta^{(i)}_n} \), and

\[
\mathcal{L}_{\zeta^{(i)}_l} g_{\rho \sigma} = \zeta^{(i)\nu}_n \partial_\nu g_{\rho \sigma} + g_{\nu \sigma} \partial_\rho \zeta^{(i)\nu}_n + g_{\nu \rho} \partial_\sigma \zeta^{(i)\nu}_n \tag{2.14}
\]

is the Lie derivative of the background metric (2.6) with respect to the vector field \( \zeta^{(i)}_n \). For the background metric (2.6), the conserved charges associated with the diffeomorphisms \( \zeta^{(1)}_l \) and \( \zeta^{(2)}_l \) can be computed as

\[
Q_{\zeta^{(1)}_l} = -i \pi a^3 (l^3 + 2l) \delta_{l+n,0} , \quad Q_{\zeta^{(2)}_l} = 0 . \tag{2.15}
\]

By comparison of Eq. (2.13) with (2.15), the central charges can be read off as

\[
c_1 = 12 \pi a^3 , \quad c_2 = 0 . \tag{2.16}
\]

Eq. (2.16) shows that the central charge associated with the coordinate \( \psi \) vanishes. This is attributed to the choice of the angle coordinates \( \theta, \psi \) and \( \hat{\phi} \). In fact, the metric (2.1) describes a black hole with two equal but opposite angular momenta. Our choice of the coordinates just makes the angular momentum related to the coordinate \( \psi \) disappear but the one corresponding to \( \hat{\phi} \) become double. If one adopts the coordinates given in [20], one can find that \( c_1 \) and \( c_2 \) have the same but nonzero values. To see this, we will adopt this kind of coordinates to calculate the CFT entropy of an extremal CY black holes with squashed horizons in the next section. What is more, Eq. (2.16) supports that the central charge got from an extremal black hole strongly relies on the rotational \( U(1) \) isometry.

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After obtaining the central charges of an extremal squashed Kerr black hole (2.6), in order to get its CFT entropy, we have to calculate the generalized temperature with respect to the Frolov-Thorne vacuum. The so-called CFT temperature can be evaluated by

$$T_F = -\lim_{\hat{r}_e \to r_e} \frac{T}{\Omega_{\hat{\phi}}(\hat{r}_e) - \Omega_{\hat{\phi}}(r_e)} = \frac{1}{2\pi},$$

(2.17)

where $r_e = \sqrt{m} = \sqrt{2}a$ is the degenerate horizon of an extremal squashed Kerr black hole. In favor of the Cardy formula for the CFT entropy at the temperature $T_F$, we can obtain the microscopic entropy of the extremal squashed Kerr black hole

$$S_{CFT} = \frac{\pi^2}{3} c_1 T_F = 2\pi^2 a^3,$$

(2.18)

which precisely agrees with the Bekenstein-Hawking entropy derived from Eq. (2.4).

3 The extremal squashed Cvetič-Youm black hole and its dual CFTs

In this section, we shall utilize the analysis parallel to the previous section to investigate the CFT entropy of an extremal squashed CY black hole [53] within a coordinate system that is regular at spatial infinity and possesses the angular coordinates different from those of an extremal squashed Kerr black hole. Thanks to the existence of an additional $U(1)$ gauge field, one has to consider its effect on the central charges. However, an explicit calculation shows that the contribution from the gauge field is zero. Besides, our calculations also prove that the squashing transformation still does not affect the near-horizon geometry of an extremal squashed CY black hole.

3.1 Squashed Cvetič-Youm black hole and its near-horizon geometry

The squashed CY black hole solution can be constructed by applying the squashing procedure [46, 49] to a five-dimensional rotating charged black hole solution with three equal $U(1)$ charges found by Cvetič and Youm [62] in string theory. The solution fulfils the complete Einstein-Maxwell-Chern-Simons equations in $D = 5$ minimal supergravity theory, whose bosonic part is described by the Lagrangian

$$I = \frac{1}{16\pi} \int d^5 x \left[ \sqrt{-g} (R - F_{\mu\nu} F^{\mu\nu}) - \frac{2}{3\sqrt{3}} e^{\mu\nu\alpha\beta\gamma} F_{\mu\nu} F_{\alpha\beta} A_{\gamma} \right],$$

(3.1)
where $\epsilon^{\mu\nu\alpha\beta\gamma}$ is the Levi-Civita tensor density. The metric of the squashed CY black hole and the gauge potential that solve the motion equations derived from the action (3.1) takes the following forms

\[
d s^2 = \frac{\Delta(r)}{B(r)} d\tilde{t}^2 + \frac{K^2(r)}{\Delta(r)} dr^2 + \frac{1}{4} r^2 K(r) (d\theta^2 + \sin^2 \theta d\hat{\psi}^2) + \frac{1}{4} B(r) [d\tilde{\phi} + \cos \theta d\hat{\psi} - F(r) d\tilde{t}]^2,
\]

(3.2)

\[
A = \sqrt{3} q^2 r^2 \left[ d\tilde{t} - a^2 \left( d\tilde{\phi} + \cos \theta d\hat{\psi} \right) \right],
\]

(3.3)

where

\[
\Delta(r) = \frac{(r^2 - m)^2 - (m - q)(m + q - 2a^2)}{r^4}, \quad K(r) = \frac{r_\infty^4 \Delta(r_\infty)}{(r_\infty^2 - r^2)^2},
\]

(3.4a)

\[
B(r) = r^2 + \frac{2(m - q)a^2}{r^2} - \frac{q^2 a^2}{r^4}, \quad F(r) = \frac{2a}{B(r)} \left( \frac{2m - q}{r^2} - \frac{q^2}{r^4} \right),
\]

(3.4b)

and the coordinates $(r, \theta, \tilde{\phi}, \hat{\psi})$ take the same ranges as those of a squashed Kerr black hole (2.1). The parameters $(m, a, q, r_\infty)$, which correspond to the mass, the angular momenta, the charge, and the spatial infinity, respectively, are required to satisfy $m > 0, m \geq q, m \geq 2a^2 - q$, and $0 < r_- \leq r_+ < r_\infty$ so that the metric can describe a regular black hole. The outer/inner horizons $r_\pm$ of the squashed CY black hole are determined by

\[
r_\pm^2 = m \pm \sqrt{(m - q)(m + q - 2a^2)}. \quad (3.5)
\]

When $r_\infty \to \infty$, the metric (3.2) becomes that of the original CY black hole, whose CFT entropies have been derived via the Kerr/CFT correspondence method in [14].

There exists an apparent singularity at $r = r_\infty$ in the coordinate system $(\tilde{t}, r, \theta, \tilde{\phi}, \hat{\psi})$. To remove this singularity and obtain the asymptotical structure, we now perform the coordinate transformation

\[
\hat{\rho} = \rho_0 \frac{r^2}{r_\infty^2 - r^2}, \quad \hat{\phi} = \tilde{\phi} - \Omega_\infty \tilde{t}, \quad \hat{\tilde{t}} = \tilde{t}/\chi,
\]

(3.6)

where

\[
\rho_0 = \frac{1}{2} r_\infty \sqrt{\Delta(r_\infty)}, \quad \chi = \frac{\sqrt{B(r_\infty)}}{2\rho_0}, \quad \Omega_\infty = F(r_\infty) = \frac{2a[(2m - q)r_\infty^2 - q^2]}{r_\infty^6 + 2(m - q)a^2 r_\infty^2 - q^2 a^2},
\]

(3.7)
and then recast the metric \((3.2)\) and the gauge potential \((3.3)\) into the forms

\[
ds^2 = -\frac{\chi^2 r_0^2 V(\hat{\rho})}{\hat{\rho}(\hat{\rho} + \rho_0)} d\hat{t}^2 + \frac{\dot{\hat{\rho}}(\hat{\rho} + \rho_0)}{V(\hat{\rho})} d\hat{\rho}^2 + \dot{\hat{\rho}}(\hat{\rho} + \rho_0)(d\theta^2 + \sin^2 \theta d\hat{\psi}^2) \\
+ h(\hat{\rho}) \left\{ d\hat{\phi} + \cos \theta d\hat{\psi} + \chi \left[ \Omega_\infty - \frac{g(\hat{\rho})}{h(\hat{\rho})} \right] dt \right\}^2,
\]

\[
A = \frac{q(\hat{\rho} + \rho_0)\sqrt{3}}{4\hat{\rho} r^2_\infty} \left[ (2 - a\Omega_\infty) dt - a(d\hat{\phi} + \cos \theta d\hat{\psi}) \right],
\]

where the functions \(V(\hat{\rho}), h(\hat{\rho}),\) and \(g(\hat{\rho})\) are given by

\[
V(\hat{\rho}) = \frac{[2(m - q)a^2 + q^2](\hat{\rho} + \rho_0)^2 - 2m r^2_\infty \hat{\rho} (\hat{\rho} + \rho_0) + r^4_\infty \rho^2}{4r^2_\infty \rho^2},
\]

\[
h(\hat{\rho}) = \frac{r^6_\infty \hat{\rho}^3 + a^2 (\hat{\rho} + \rho_0)^2 [2(m - q)r^2_\infty \hat{\rho} - q^2 (\hat{\rho} + \rho_0)]}{4r^4_\infty \hat{\rho}^2 (\hat{\rho} + \rho_0)},
\]

\[
g(\hat{\rho}) = \frac{a(\hat{\rho} + \rho_0) [(2m - q)r^2_\infty \hat{\rho} - q^2 (\hat{\rho} + \rho_0)]}{2r^4_\infty \rho^2}.\]

Clearly, the apparent singularity at \(r = r_\infty\) of the metric \((3.2)\) are removed in the new coordinate system \((\hat{t}, \hat{\rho}, \theta, \hat{\phi}, \hat{\psi})\). At spatial infinity \(\hat{\rho} \to \infty\), the asymptotic metric becomes

\[
ds^2 = -d\hat{t}^2 + d\hat{\rho}^2 + \hat{\rho}^2 (d\theta^2 + \sin^2 \theta d\hat{\psi}^2) + \frac{1}{4} B(r_\infty)(d\hat{\phi} + \cos \theta d\hat{\psi})^2.
\]

Therefore it is easy to see that the metric \((3.8)\) has the asymptotic geometry of a twisted \(S^1\) bundle over a four-dimensional Minkowski space-time when \(\hat{\rho} \to \infty\).

We further make the angle coordinate transformations \(\theta \to 2\theta, \hat{\phi} \to \hat{\phi} + \hat{\psi},\) and \(\hat{\psi} \to \hat{\phi} - \hat{\psi},\) and rewrite the metric \((3.8)\) and the gauge potential \((3.9)\) as

\[
ds^2 = -\frac{\chi^2 r_0^2 V(\hat{\rho})}{\hat{\rho}(\hat{\rho} + \rho_0)} d\hat{t}^2 + \frac{\dot{\hat{\rho}}(\hat{\rho} + \rho_0)}{V(\hat{\rho})} d\hat{\rho}^2 + \dot{\hat{\rho}}(\hat{\rho} + \rho_0)(d\theta^2 + \sin^2 \theta d\hat{\psi}^2) \\
+ 4\dot{\hat{\rho}}(\hat{\rho} + \rho_0)[d\theta^2 + \sin^2 \theta \cos^2 \theta (d\hat{\phi} - d\hat{\psi})^2] \\
+ 4h(\hat{\rho}) \left\{ \cos^2 \theta d\hat{\phi} + \sin^2 \theta d\hat{\psi} + \chi \left[ \Omega_\infty - \frac{g(\hat{\rho})}{h(\hat{\rho})} \right] dt \right\}^2,
\]

\[
A = \frac{q(\hat{\rho} + \rho_0)\sqrt{3}}{4\hat{\rho} r^2_\infty} \left[ (2 - a\Omega_\infty) dt - 2a(\cos^2 \theta d\hat{\phi} + \sin^2 \theta d\hat{\psi}) \right].
\]

After doing these, the coordinates \(\hat{\phi}\) and \(\hat{\psi}\) in Eqs. \((3.12)\) and \((3.13)\) become the standard \(2\pi\)-period azimuthal ones. At the outer horizon, the angular velocities with respect to them are equal and read

\[
\Omega_\phi(\hat{\rho}_+) = \Omega_\psi(\hat{\rho}_+) = \chi(\Omega_H - \Omega_\infty)/2,
\]

\[
(3.14)\]
where \( \hat{\rho}_+ = \rho_0 r_+^2 / (r_0^2 - r_+^2) \) is the outer horizon and \( \Omega_H = g(\hat{\rho}_+)/h(\hat{\rho}_+) \).

The Hawking temperature and the Bekenstein-Hawking entropy are

\[
T = \frac{\chi \rho_0 V'(\hat{\rho}_+)}{4\pi \hat{\rho}_+ (\hat{\rho}_+ + \hat{\rho}_0) \sqrt{h(\hat{\rho}_+)}} = \frac{\chi (r_+^2 - r_-^2)}{2\pi r_+^2 K(r_+) \sqrt{B(r_+)}},
\]

\[
S = 4\pi^2 \hat{\rho}_+ (\hat{\rho}_+ + \rho_0) \sqrt{h(\hat{\rho}_+)} = \frac{1}{2} \pi^2 r_+^2 K(r_+) \sqrt{B(r_+)},
\]

in which we have denoted \( \hat{t} = \partial_t \).

As before, our aim is to study the dual CFT entropies of an extremal squashed CY black hole. The extremity conditions are \( q = m \) or \( q = 2a^2 - m \). When \( q = m \), the extremal squashed CY black hole becomes an extremal BMPV black hole with squashed horizon. In the following, we shall derive the microscopic entropy of the extremal squashed CY black hole in these two extremal cases. We will first discuss the extremal case \( q = 2a^2 - m \), and leave the extremal case \( q = m \) for a separate subsection.

For the extremal case \( q = 2a^2 - m \), we impose the supplement condition \( a > 0 \) and \( m > a^2 \). In contrast with the case of an extremal squashed Kerr black hole, we shall implement our task on the basis of the metric (3.12) and the gauge potential (3.13). Under the extremity condition \( q = 2a^2 - m \), we obtain their near-horizon forms

\[
ds^2 = \frac{m}{4} \left( -\rho^2 dt^2 + \frac{d\rho^2}{\rho^2} \right) + m \left[ d\theta^2 + \sin^2 \theta \cos^2 \theta (d\phi - d\psi)^2 \right] + \frac{(m - a^2)(m + 2a^2)}{m^2} \left( \cos^2 \theta d\phi + \sin^2 \theta d\psi + k \rho dt \right)^2,
\]

\[
A = -w \cos^2 \theta \left( \hat{d} \phi + \frac{\sqrt{m-a^2}}{2a} \rho dt \right) - w \sin^2 \theta \left( \hat{d} \psi + \frac{\sqrt{m-a^2}}{2a} \rho dt \right),
\]

after we make use of the coordinate transformations

\[
\hat{\rho} = \rho_e (1 + \lambda \rho), \quad \hat{t} = \frac{m \sqrt{h(\rho_e)}}{4 \chi \rho_0 \rho_e} \frac{t}{\lambda},
\]

\[
\hat{\phi} = \phi + \chi (\Omega_H - \Omega_\infty) \hat{t}/2, \quad \hat{\psi} = \psi + \chi (\Omega_H - \Omega_\infty) \hat{t}/2,
\]

and send \( \lambda \to 0 \). In the above, \( \rho_e = \rho_0 m / (r_0^2 - m) \), or \( r_e = \sqrt{m} \), is the location of the event horizon of an extremal squashed CY black hole, while two constants \( k \) and \( w \) are given by

\[
k = \frac{a(3m - 2a^2)}{2(m + 2a^2) \sqrt{m - a^2}}, \quad w = \frac{\sqrt{3}a(2a^2 - m)}{2m}.
\]

From Eqs. (3.16) and (3.17), one can note that the squashing transformation still does not affect the near-horizon geometry of an extremal squashed CY black hole, since \( K(r_e) \to 1 \).
when $r \to r_e$. This fact can be easily found from the expression of the squashing function: $K(r) = (r_\infty^2 - r_e^2)(r_\infty^2 - r_\pm^2)/(r^2 - r_\infty^2)^2$, which guarantees $K \to 1$ once the near-horizon limit has been taken. Particularly, if we set $m = 2a^2$, then the near-horizon geometry of an extremal squashed CY black hole (3.16) reduces to that of an extremal squashed Kerr black hole presented in the last section.

### 3.2 Central charges and dual CFT entropies in the $q = 2a^2 - m$ case

In this subsection, we will calculate the dual CFT entropies on the basis of the metric (3.16) and the gauge potential (3.17). As before, we first impose the boundary conditions, which include the perturbations of the metric and the gauge field. For the metric fluctuations around the background metric (3.16), we choose the same boundary conditions as those of the five dimensional squashed extremal Kerr black hole in the previous section. For the gauge field, letting $a_\mu$ denote its perturbation, we impose the boundary condition

\[(a_t, a_r, a_\theta, a_\phi, a_\psi) \sim \mathcal{O}(r, r^{-2}, 1, r^{-1}, r^{-1}). \tag{3.20}\]

The diffeomorphisms and the $U(1)$ gauge transformations that preserve these boundary conditions are given by

\[
\begin{align*}
\zeta_n^{(1)} &= -e^{-i\phi}\partial_\phi - in\rho e^{-i\phi}\partial_\rho, & \Lambda_n^{(1)} &= -w \cos^2 \theta e^{-i\phi}, \\
\zeta_n^{(2)} &= -e^{-iv}\partial_\phi - in\rho e^{-iv}\partial_\rho, & \Lambda_n^{(2)} &= -w \sin^2 \theta e^{-iv},
\end{align*}\]

where $(n = 0, \pm 1, \pm 2, \cdots)$. These generators constitute the centerless Virasoro algebras

\[i\left[(\zeta_{l}^{(i)}, \Lambda_{l}^{(i)}), (\zeta_{n}^{(i)}, \Lambda_{n}^{(i)})\right] = (l - n)(\zeta_{l+n}^{(i)}, \Lambda_{l+n}^{(i)}), \quad (i = 1, 2). \tag{3.22}\]

The combined generator $(\zeta_{l}^{(i)}, \Lambda_{l}^{(i)}) \equiv (\zeta, \Lambda)$ possesses an associated conserved charge $Q_{\zeta,\Lambda}$ defined by

\[Q_{\zeta,\Lambda} = \int_{\partial\Sigma} \left(k_{\zeta}^{gr}[h, g] + k_{\zeta,\Lambda}^{en}[h, a; g, A] + k_{\zeta,\Lambda}^{es}[h, a; g, A]\right). \tag{3.23}\]

From the Lagrangian (3.1), we note that $k_{\zeta}^{gr}$ is still given by Eq. (2.12), while $k_{\zeta,\Lambda}^{en}$ and $k_{\zeta,\Lambda}^{es}$ read

\[
\begin{align*}
k_{\zeta,\Lambda}^{en} &= \frac{1}{48\pi} \sqrt{-g} \epsilon_{\alpha\beta\gamma\mu} \left[(2h^{\mu\lambda}F_{\lambda}^{\nu} - f^{\mu\nu} - \frac{1}{2}hF^{\mu\nu})\right] (A_\rho \zeta^\rho + \Lambda) - F^{\mu\nu}a_\rho \zeta^\rho \\
&\quad - 2\zeta^\mu F^{\nu\lambda}a_\lambda - a^\mu g^{\sigma\rho} (\mathcal{L}_{\zeta}A_\sigma + \partial_\sigma \Lambda) \right) dx^\alpha \wedge dx^\beta \wedge dx^\gamma, \\
k_{\zeta,\Lambda}^{es} &= \frac{1}{2\sqrt{3}\pi} a_\alpha F_{\beta\gamma} dx^\alpha \wedge dx^\beta \wedge dx^\gamma, \tag{3.24}\end{align*}\]
in which \( a_\mu = \mathcal{L}_\xi A_\mu + \partial_\mu \Lambda \), and \( f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \). Taking into account the contribution from the \( U(1) \) gauge transformations, the central term \([2.13]\) in the Virasoro algebras is now modified to

\[
\int_{\partial \Sigma} \left( k^g_{\xi} \left[ \mathcal{L}_\xi g, g \right] + k^m_{(\xi, \Lambda)} \left[ \mathcal{L}_\xi g, \mathcal{L}_\xi A + d\Lambda; g, A \right] + k^{cs}_{(\xi, A)} \left[ \mathcal{L}_\xi g, \mathcal{L}_\xi A + d\Lambda; g, A \right] \right),
\]

(3.26)

where \( (\tilde{\zeta}, \tilde{\Lambda}) \equiv (\zeta^{(i)}_n, \Lambda^{(i)}_n) \).

Based upon the metric \([3.16]\) and the gauge potential \([3.17]\), an explicit calculation shows that \( k^m_{(\xi, \Lambda)} \) and \( k^{cs}_{(\xi, A)} \) make no contribution to the central term \([3.26]\). Therefore, we only need to consider the contribution from the gravitational part \( k^g_{\xi} \) as before and get

\[
\int_{\partial \Sigma} k^g_{\xi(\xi)} \left[ \mathcal{L}_\xi(\xi) g, g \right] = -\frac{i}{4} \pi k (m + 2a^2) \sqrt{m - a^2} \left[ \frac{2}{m^3} + \frac{2(m - a^2)(m + 2a^2)^2}{m^3} \right] \delta_{l+n,0},
\]

(3.27)

from which the central charges \( c_1 \) and \( c_2 \), corresponding to the angle coordinates \( \phi \) and \( \psi \) respectively, can be read off as

\[
c_1 = c_2 = 3\pi k (m + 2a^2) \sqrt{m - a^2}.
\]

(3.28)

Note here that the central charges \( c_1 \) and \( c_2 \) are equal and nonzero. In contrast, if one adopts another different angular coordinates from those of Eq. \([3.16]\), one will find that one of the central charges of an extremal squashed CY black hole vanishes but the other does not. As a matter of fact, when we calculate the central charges of an extremal squashed CY black hole on the basis of the metric \([3.8]\) and the gauge potential \([3.9]\), which have the same angle coordinates as those of an extremal squashed Kerr metric \([2.6]\), we also obtain \( c'_1 = 0 \) and \( c'_2 = 2c_2 \), where \( c'_1 \) and \( c'_2 \) denote the central charges with respect to the angular coordinates \( \hat{\phi} \) and \( \hat{\psi} \) in Eq. \([3.8]\), respectively.

Bearing the above central charges in mind, we now have to calculate the dual CFT entropies of the extremal squashed CY black hole. In terms of the generalized CFT temperature

\[
T_1 = T_2 = -\lim_{\bar{\rho}_+-\rho_+} \frac{T}{\Omega_{\hat{\psi}}(\bar{\rho}_+) - \Omega_{\hat{\psi}}(\rho_+)} = \frac{1}{2\pi k},
\]

(3.29)

we can directly derive the microscopic entropy via each of two copies of Virasoro algebras by virtue of the Cardy formula and get

\[
S_{CFT} = \frac{\pi^2}{3} c_1 T_1 = \frac{\pi^2}{2} c_2 T_2 = \frac{\pi^2}{2} (m + 2a^2) \sqrt{m - a^2},
\]

(3.30)
which matches the Bekenstein-Hawking entropy got from Eq. (3.15b). In particular, if $m = 2a^2$, namely $q = 0$, Eq. (3.30) reproduces the dual CFT entropy (2.18) of an extremal squashed Kerr black hole.

### 3.3 Dual CFT entropies under the extremity condition $q = m$

In this subsection, we shall derive the dual CFT entropies of an extremal squashed CY black hole under the extremity condition $q = m$. In other words, we will take into consideration the case of an extremal BMPV black hole with a squashed horizon. After taking the near-horizon limit, the metric (3.12) and the potential (3.13) now become

$$
\begin{align*}
    ds^2 &= \frac{m}{4} \left( -\rho^2 dt^2 + \frac{d\rho^2}{\rho^2} \right) + m \left[ d\theta^2 + \sin^2 \theta \cos^2 \theta (d\phi - d\psi)^2 \right] \\
    &\quad + (m - a^2) \left[ \cos^2 \theta d\phi + \sin^2 \theta d\psi - \frac{a\sqrt{m - a^2}}{2(m - a^2)} \rho dt \right]^2, \\
    A &= -\frac{\sqrt{3}}{2} a \left[ \cos^2 \theta \left( d\phi + \frac{\sqrt{m - a^2}}{2a} \rho dt \right) + \sin^2 \theta \left( d\psi + \frac{\sqrt{m - a^2}}{2a} \rho dt \right) \right].
\end{align*}
$$

(3.31)

As before, the Chern-Simons term and the gauge field still make no contribution to the central charges. Hence we just need to calculate the gravitational part and get

$$
\int_{\partial \Sigma} k_{ij}^{\alpha \beta} \{ \mathcal{L}^{\alpha \beta} g, g \} = \frac{i}{8} \pi ma \left[ l^3 + \frac{2(m - a^2)}{m} l \right] \delta_{l+n,0},
$$

(3.33)

from which the central charges $c_1$ and $c_2$, associated with the angle coordinates $\phi$ and $\psi$ respectively, can be obtained as

$$
c_1 = c_2 = -\frac{3}{2} \pi ma.
$$

(3.34)

The generalized CFT temperature now turns to be

$$
T_1 = T_2 = -\lim_{\hat{\rho}_+ \to \hat{\rho}_c} \frac{T}{\Omega_{\phi}(\hat{\rho}_+ - \Omega_{\psi}(\hat{\rho}_c)} = -\frac{\sqrt{m - a^2}}{\pi a}.
$$

(3.35)

With the help of the Cardy formula, the dual CFT entropies of an extremal squashed BMPV black hole are presented as

$$
S_{CFT} = \frac{\pi^2}{3} c_1 T_1 = \frac{\pi^2}{3} c_2 T_2 = \frac{\pi^2}{2} m \sqrt{m - a^2}.
$$

(3.36)

This result agrees with the previous ones [15] [21] obtained for an extremal, supersymmetric BMPV black hole without making the squashing transformation.
4 Thermodynamics of general non-extremal squashed Cvetič-Youm black holes and the generalized CFT temperature

In this section, we shall discuss the thermodynamical properties of general non-extremal squashed Cvetič-Youm black holes and calculate the conserved charges based upon the metric (3.8) and the gauge potential (3.9). Making use of the first law of black hole thermodynamics, we will derive the generalized CFT temperature with respect to the Frolov-Thorne vacuum.

For the general non-extremal Cvetič-Youm with a squashed horizon, it is natural to choose the asymptotic metric (3.11) as the appropriate reference background solution; therefore, following the work of [50], we can get the generalized Abbott-Deser mass and angular momentum as follows:

\[
M = \frac{1}{4} \pi \chi \left\{ (r^4 - 3q^2)r^4 + 2m(r^4 - 2qa^2)r^2 - 4(m - q)^2(r^2 + a^2)a^2 
+ 2(m + q)q^2a^2 \right\} \times \left[ r^6 + 2(m - q)a^2r^2 - q^2a^2 \right]^{-1}, \tag{4.1}
\]

\[
J = J_\phi = \frac{\pi a[2(2m - q)r^6 - 3q^2r^4 + q^3a^2]}{8r^6}. \tag{4.2}
\]

The angular momentum along the \( \hat{\psi} \)-direction is zero. However, if we perform our calculations based upon the metric (3.12), we will find that the angular momenta \( J_\hat{\psi} = J_\phi = J \).

The generalized Abbott-Deser mass and angular momentum given above coincide with those obtained by the counterterm method [65], which corresponds to the results presented in [57] when the Gödel parameter \( j = 0 \).

Since the general non-extremal squashed CY black hole has the asymptotical geometry similar to that of a squashed Kaluza-Klein black hole in five dimensions, one should take into account the gravitational tension (per unit time), which can be computed via the counterterm method and given by

\[
T = \chi \left\{ 2(r^2 - m)r^8 + [(2m - q)r^2 - 2q^2]a^2 - q^4a^2 + 6(m - q)q^2a^4 
- 8(m - q)^2r^2a^4 \right\} \times \left\{ 4r^6 + 2(m - q)a^2r^2 - q^2a^2 \right\}^{-\frac{3}{2}}. \tag{4.3}
\]

The size of the extra dimension with a \( S^1 \) circle at infinity is

\[
\mathcal{L} = \frac{2\pi \sqrt{r^6 + 2(m - q)a^2r^2 - q^2a^2}}{r^2}. \tag{4.4}
\]

The Komar mass is related to the Abbott-Deser or counterterm mass by \( M_K = M - T\mathcal{L}/2 \).
In addition to the global charge — the electric charge, one more additional local charge — the dipole charge \[58\] enters into the first law of black hole thermodynamics also. The electric charge \(Q\) and the dipole charge \(D\) can be computed as

\[
Q = \frac{\sqrt{3}}{2} \pi q, \quad D = -\frac{\sqrt{3} q a}{4 r_\infty^2}.
\]

The electro-static potential \(\Phi\) can be obtained by

\[
\Phi = \xi^\mu A_\mu \bigg|_{r_+} - \xi^\mu A_\mu \bigg|_{r_\infty} = \frac{\sqrt{3} q \chi (r_+^4 - qa^2) (r_\infty^2 - r_+^2)}{2 r_\infty [r_+^6 + 2 (m - q) a^2 r_+^2 - q^2 a^2]},
\]

where \(\xi = \partial_t + \Omega \partial_\phi\) is the Killing vector normal to the horizon, in which \(\Omega = \chi (\Omega_H - \Omega_\infty)\) is the angular velocity measured in a non-rotating frame relative to infinity.

Finally, the dipole potential \(\Phi_D\) can be computed as

\[
\Phi_D = -\frac{3 \pi q a \chi}{r_+^4} \left\{ \frac{(r_\infty^4 - qa^2) [(2m - q) r_+^2 - q^2]}{r_+^6 + 2(m - q) a^2 r_+^2 - q^2 a^2} - \frac{3}{2} \right\}.
\]

With all the thermodynamical quantities in hand, it can be verified that they fulfill the differential and integral first laws of black hole thermodynamics

\[
dM = T dS + \Omega dJ + \Phi dQ + \Phi_D dD + T dL, \quad (4.8)
\]

\[
2M = 3TS + 3 \Omega J + 2 \Phi Q + \Phi_D D + T L. \quad (4.9)
\]

On the other hand, at the extremal limit, the differential and integral first laws become

\[
dM = \Omega^{(e)} dJ + \Phi^{(e)} dQ + \Phi_D^{(e)} dD + T dL, \quad (4.10)
\]

\[
2M = 3 \Omega^{(e)} J + 2 \Phi^{(e)} Q + \Phi_D^{(e)} D + T L, \quad (4.11)
\]

where \(\Omega^{(e)}, \Phi^{(e)},\) and \(\Phi_D^{(e)}\) can be obtained respectively from \(\Omega, \Phi,\) and \(\Phi_D\) via the replacement \(r_+ \rightarrow r_e = \sqrt{m}.$

Subtracting Eq. (4.8) from (4.10), and Eq. (4.9) from (4.11), we arrive at two thermodynamical relations

\[
dS = \frac{\Omega^{(e)} - \Omega}{T} dJ + \frac{\Phi^{(e)} - \Phi}{T} dQ + \frac{\Phi_D^{(e)} - \Phi_D}{T} dD = 2 \frac{dJ}{T_L} + \frac{dQ}{T_Q} + \frac{dD}{T_D}, \quad (4.12)
\]

\[
S = \frac{\Omega^{(e)} - \Omega}{T} J + 2 \frac{\Phi^{(e)} - \Phi}{3T} Q + \frac{\Phi_D^{(e)} - \Phi_D}{3T} D = 2 \frac{J}{T_L} + \frac{2Q}{3T_Q} + \frac{D}{3T_D}, \quad (4.13)
\]
which hold true in the sense of L’Hospital rule even at the non-extremal case. Taking the extremal limit \((r_+ \rightarrow r_e)\), we obtain the generalized CFT temperature

\[ T_L = T_1 = T_2 = \lim_{r_+ \rightarrow r_e} \frac{2T}{\Omega^{(e)} - \Omega}, \]  

(4.14)

where the factor ‘2’ is attributed to the fact we have adopted the metric \((3.12)\) and \(J_\psi = J_\phi = J\). For an extremal squashed CY black hole at the extremal case \((q = 2a^2 - m)\), we get

\[ T_L = \frac{1}{2\pi k}, \]  

(4.15)

while for an extremal squashed BMPV black hole where \(q = m\), we have

\[ T_L = -\frac{\sqrt{m - a^2}}{\pi a}. \]  

(4.16)

To end up with our discussions, we shall derive the generalized CFT temperature in another way by applying a similar thermodynamical relation that holds true for an extremal CY black hole without making the squashing transformation. This is possible because the squashing transformation does not affect the near-horizon geometry of an extremal squashed CY black hole. In other words, the near-horizon geometry of an extremal squashed CY black hole is identical to that of an extremal CY black hole.

For a general non-extremal CY black hole, its line element is given by Eq. \((3.2)\) with \(K(r) = 1\) or equivalently by setting \(r_\infty \rightarrow \infty\). At the extremal case, the relevant equation to determine the CFT temperature \(T_L\) is

\[ dS = 2\frac{dJ}{T_L} + \frac{dQ}{T_Q}, \]  

(4.17)

where \(S\), \(J\) and \(Q\) are the entropy, the angular momentum, and the electric charge of an extremal CY black hole. Under the extremity condition \((q = 2a^2 - m)\), they are

\[ S = \frac{\pi^2}{2}(m + 2a^2)\sqrt{m - a^2}, \quad J = \frac{\pi}{4}a(3m - 2a^2), \quad Q = \frac{\sqrt{3}\pi}{2}(2a^2 - m), \]  

(4.18)

where at the extremal BMPV case \((q = m)\), they become

\[ S = \frac{\pi^2}{2}m\sqrt{m - a^2}, \quad J = \frac{\pi am}{4}, \quad Q = \frac{\sqrt{3}\pi m}{2}. \]  

(4.19)

Substituting these quantities into Eq. \((4.17)\), we can re-derive the generalized CFT temperature \(T_L\) given by Eq. \((4.15)\) and \((4.16)\), respectively.
5 Conclusions

In this paper, we have applied the Kerr/CFT correspondence [1] to derive the microscopic entropies of the five-dimensional extremal squashed Kerr and CY black holes. After performing the near-horizon limit, we find that their near-horizon geometries are described by a squashed three-spheres over $AdS_2$. Since under the extremity conditions, the squashing function $K \rightarrow 1$ at the near-horizon limit, their near-horizon geometries take the same ones as those of the extremal black holes without the squashing transformation. When some suitable boundary conditions were imposed to the near-horizon geometries of the extremal black holes, there exist a class of diffeomorphisms (and the $U(1)$ gauge transformations for the charged case) that preserve these boundary conditions and generate two copies of centerless Virasoro algebras. By calculating the Dirac brackets of the central charges corresponding to the diffeomorphisms, one gets two copies of Virasoro algebras with nonzero central terms, from which the central charges can be read off. By virtue of the Cardy formula, together with the help of the generalized temperatures associated with the Frolov-Thorne vacuum, we are able to derive the dual CFT entropies that precisely agree with the Bekenstein-Hawking ones. It is worth noting that we have calculated the central charges of the squashed Kerr and CY black holes in different angle coordinates. Our results further support that the rotational $U(1)$ isometry plays a key role in determining the central charge of the extremal black holes.

In addition, the thermodynamic properties of the general non-extremal squashed CY black hole have been discussed. We have, for the first time, presented the explicit expression for the dipole potential, in addition to the dipole charge and the gravitational tension, that enters into the differential first law and the generalized Smarr relation. Based upon the first law of black hole thermodynamics, we have re-derived the generalized CFT temperature.

In our forthcoming paper [66], the complete thermodynamical properties of general non-extremal Kerr-Gödel and (charged) Einstein-Maxwell-Chern-Simons black holes with squashed horizons in Gödel universe have been studied. Besides, we will show that the method of the extremal Kerr/CFT correspondence can be also applicable to reproduce the Bekenstein-Hawking entropies in the extremal limit, although the computations become much more involved. A paper about these aspects is in preparation.

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