Exact solutions for a family of discretely spiked harmonic oscillators

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1 Introduction

The name *spiked oscillator* generally refers to Hamiltonians with potentials containing a parabolic term $x^2$ describing ordinary quantum oscillator, and a singular term $1/x^\alpha$ forming a sharp spike at $x = 0$. A typical Hamiltonian of a spiked oscillator is of the form given by Harrell[6]:

$$H = -\frac{d}{dx^2} + x^2 + \frac{\lambda}{x^\alpha} \quad 0 \leq x < \infty.$$  

It has found variety of uses in atomic, molecular, nuclear and particle physics since it provides the simplest realistic model of interaction potentials due to its repulsive core $x^{-\alpha}$.

Some forms of spiked oscillators have exact solutions, such as the one described by one dimensional Hamiltonian, examined four decades ago by Goldman and Krivchenkov[4]

$$H = -\frac{d}{dx^2} + V_0\left(\frac{a}{x} - \frac{x}{a}\right)^2$$

whose exact solutions $\phi_n(x) \in [0, \infty)$ satisfy Dirichlet’s boundary condition $\phi_n(0) = 0$.

Models with exact solutions can serve as starting points for analyses of related cases where exact solutions do not exist and where a perturbative analysis is required[3]. A good example can be found in the papers of Hall et al[1][2], who slightly modified the above Hamiltonian, computed a basis given by a set of exact orthonormal eigenfunctions $\phi_n(x)$ for $L^2(0, \infty)$, and then used this basis in variational analysis of the Hamiltonian

$$H = -\frac{d}{dx^2} + Bx^2 + \frac{A}{x^2} + \frac{\lambda}{x^\alpha}.$$  

They argued that their singular basis forms much better starting point for perturbation techniques than the basis of the ordinary harmonic oscillator, as used in earlier work of Aguilera-Navarro et al[5], who had applied a perturbative variational analysis to the lowest eigenvalue of the Harrel’s Hamiltonian.

In this context, we present an analysis of family of following spiked Hamiltonians
\( H_l = -\frac{d^2}{dx^2} + x^2 + \frac{l(l + 1)}{x^2} \),

parametrized by discrete values of \( l = 0, 1, 2... \) Depending on interpretation of variable \( x \), this form can either describe a family of one dimensional linear oscillators, or a family of \( N \) dimensional radial oscillators for odd \( N = 1, 3, 5... \). What is traditionally known as the three dimensional isotropic radial oscillator, with known exact solutions, can be reduced to a special case of \( N = 1 \). We shall show that solutions for radial oscillators in even dimensions \( N = 2, 4, 6... \) do not form well defined orthogonal bases and, as such, they are of no practical interest.

The solutions we provide are exact and much simpler than any of those described in this introduction, and therefore they are good candidates for orthonormal bases that could be used in perturbative analyses of oscillators with spiked potentials other than \( x^{-2} \).

The factorization formalism developed here is completely algebraic and somehow related to methods of Superstring Quantum Mechanics (SUSY QM). Modern algebraic factorization methods are overviewed in the paper of Rosas-Ortiz[7] and closely related SUSY methods are described for example in introductory parts of the papers by Junker and Roy [11][12], and Lévai and Roy[13].

The common premise of all those methods is the use of algebraic recursive manipulations on two Hamiltonians: one with known spectrum and known eigenvectors and another which is isospectral to the former but whose eigenvectors are to be found. The methodology developed here uses the same premise, but goes a bit further by manipulating not just two Hamiltonians but the uncountable set of related Hamiltonians \( \{H_l\} \). On the other hand, the SUSY and similar methods rely on general solution of underlying Ricatti equation, which is completely ignored in the approach presented in this paper.

Quantum models with exact solutions play an important role and there is a renewed interest in enrichment of the traditional narrow set of exactly solvable models, such as harmonic oscillator, hydrogen-like potential, Morse potential and square well potential. It is only fair to mention other related methods, which admit certain type of non-hermitian Hamiltonians with complex-valued potentials giving rise to real energy spectrum. For example, Cannata et al[10] use Darboux method to derive a general class of complex potentials with energy spectra identical to that of regular harmonic oscillator. Bender et al[8][9] replaces the requirement of Hamiltonian hermiticity by weaker requirement of \( PT\)-symmetry, which is satisfactory condition for real energy spectra. They examine complex harmonic and anharmonic os-
cillators, complex square wells, etc. This avenue is being explored by many other researchers.

To complete this introduction, we just mention in passing that exactly solvable problems can be categorized in one of three classes: Exactly solvable (ES), Quasi exactly solvable (QES) and Conditionally exactly solvable (CES).

In the next section we present an overview of our findings, while the remaining sections provide specific details.

2 Outline

The first Hamiltonian, \( H_0 \), from the family of discretely spiked oscillators \([\text{III}]\) describes the well known harmonic oscillator, without the spike, of known discrete energy spectrum and known ladder operators \( a_0 \) and \( a_0^\dagger \). Aside from this simple case with parabolic potential, the potentials of remaining members of this family have strong discontinuities at \( x = 0 \) and do not resemble the first case at all. Yet they all are isospectral; that is, they all have the discrete energy spectra identical to that of the simple harmonic oscillator. The only difference is in the definition of their ground state energies, \( E_{l,0} = 2l + 1 \), in units of \( \frac{1}{2} \omega \hbar \).

We have chosen to denote the corresponding ground states as \( |0,0\rangle \), \( |1,0\rangle \), \( |2,0\rangle \), etc. for \( l = 0, 1, 2... \), respectively. Table \([\text{III}]\) visualizes the relationship between energy states and energies for the entire family of discretely spiked oscillators.

For each Hamiltonian \( H_l \) a pair of intertwining operators can be defined

\[
\begin{align*}
 b_{l+1}^\dagger |l, k\rangle &\propto |l+1, k\rangle \\
 b_{l+1} |l+1, k\rangle &\propto |l, k\rangle
\end{align*}
\]

which connect eigenstates of two ‘neighbours’, \( H_l \) and \( H_{l+1} \). Both represent actions corresponding to diagonal traversal of the Table \([\text{III}]\) in NE and SW directions, respectively.

In addition to intertwining operators, we can also define the ladder operators \( a_l \) and \( a_l^\dagger \) that allow for traversing of the Table \([\text{III}]\) in vertical direction. The annihilation operator changes the state \( |l, k\rangle \) with energy \( E_{l,k} \) to the state \( |l, k-1\rangle \) of lower energy \( E_{l,k-1} \)

\[
\begin{align*}
 a_l |l, 0\rangle &= 0 \\
 a_l |l, k\rangle &= \alpha_l |l, k-1\rangle.
\end{align*}
\]

Similarly, the creation operator changes the state \( |l, k\rangle \) with energy \( E_{l,k} \) to the state \( |l, k+1\rangle \) of higher energy \( E_{l,k+1} \)

\[
\begin{align*}
 a_l^\dagger |l, k\rangle &= \alpha_l^\dagger |l, k+1\rangle
\end{align*}
\]
Table 1: States and energies of discretely spiked harmonic oscillators. Some states are unphysical and will be later rejected.

| $l = 0$ | $l = 1$ | $l = 2$ | $l = \ldots$ | Energy |
|---------|---------|---------|-------------|--------|
| $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $\ldots$ | $\ldots$ | $|2, k\rangle$ | $|2, k - 1\rangle$ | $2k + 5$ |
| $|0, k\rangle$ | $|1, k - 1\rangle$ | $|2, k - 1\rangle$ | $2k + 3$ |
| $|0, k - 1\rangle$ | $|1, k - 2\rangle$ | $|2, k - 2\rangle$ | $2k + 1$ |
| $|0, k - 2\rangle$ | $\ldots$ | $\ldots$ | $2k - 3$ |
| $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ | $\ldots$ |
| $\ldots$ | $\ldots$ | $|2, 2\rangle$ | $|2, 1\rangle$ | $9$ |
| $\ldots$ | $|1, 2\rangle$ | $|2, 1\rangle$ | $7$ |
| $|0, 2\rangle$ | $|1, 1\rangle$ | $|2, 0\rangle$ | $5$ |
| $|0, 1\rangle$ | $|1, 0\rangle$ | $\ldots$ | $3$ |
| $|0, 0\rangle$ | $\ldots$ | $\ldots$ | $1$ |

where $\alpha_l$ and $\alpha'_l$ are the normalization constants.

Known ladder operators $a_0$ and $a_0^\dagger$ for the classical harmonic oscillator are used to recursively prove the existence of similar operators for the remaining Hamiltonians $H_l$. The are given by formulas

$$a_{l+1}^\dagger = b_{l+1}^\dagger a_l^\dagger b_{l+1}$$

$$a_{l+1} = b_{l+1} a_l b_{l+1}$$

The condition $a_l |l, 0\rangle = 0$ allows to generate explicit form of the state $|l, 0\rangle$ in position representation; that is the waveform function $\phi_{l,0}(x)$ of the ground state. For $l = 0$, this is the bell-shaped function, $\phi_{0,0} \propto \exp\left(-\frac{x^2}{2}\right)$, subject to normalization. Having computed the ground waveform $\phi_{l,0}$, one can use the creation operator $a_l^\dagger$ to recursively generate all other eigenstates $|l, k\rangle$, or corresponding waveforms $\phi_{l,k}(x)$.

This seemingly completes the scheme since, thanks to the pairs of ladder and intertwining operators, the entire Table 1 can be created – starting from the left lower corner and zigzagging one step a time, with or without the help of ladder operators $a_1^\dagger$, $a_2^\dagger$, etc.

However, we must also require that each set of state vectors is a basis in its own space; that is, they all are mutually orthogonal and normalized

$$\langle l, j | l, k \rangle = \delta_{jk}.$$
But some of the wavefunctions are singular at \( x = 0 \), since they contain factors \( \frac{1}{x}, \frac{1}{x^2}, \) etc. Depending on whether the Hamiltonians describe linear or radial oscillators and, in the latter case, depending on the space dimension, some such functions are not square-integrable and therefore they must be rejected as unphysical. Those wave functions that are physical must additionally pass the test of orthogonality in order to qualify as members of useful bases. To take it all into account we therefore specialize the generic scheme and draw additional conclusions specific to dimensionality of the problem. All of this is supported by a Haskell program, which tests the theory and provides useful and flexible computational tool.

In the following sections we shall present a detailed account of this outline.

3 Operators

We shall define two pairs of state generation operators: intertwining operators, affecting the quantum number \( l \), and ladder operators affecting the quantum number \( k \). The former correspond to diagonal traversal of Table 1 and the latter – for the vertical traversal.

3.1 Intertwining operators

Let’s define two operators \( b_l \) and its adjoint \( b_l^\dagger \)

\[
\begin{align*}
  b_l &= \frac{d}{dx} + \beta_l(x), \\
  b_l^\dagger &= -\frac{d}{dx} + \beta_l(x) \\
  \beta_l(x) &= x + \frac{l}{x}.
\end{align*}
\]

Function \( \beta(x) \) and operator \( \frac{d}{dx} \) do not commute

\[
[\beta_l, \frac{d}{dx}] = -\beta'_l(x) = \frac{l}{x^2} - 1,
\]

and, as a consequence, the commutator \([b_l, b_l^\dagger]\) is not zero either, since

\[
\begin{align*}
  b_l^\dagger b_l &= -\frac{d^2}{dx^2} + \beta_l^2(x) - \beta'_l(x) = H_l + (2l - 1), \\
  b_l b_l^\dagger &= -\frac{d^2}{dx^2} + \beta_l^2(x) + \beta'_l(x) = H_{l-1} + (2l + 1),
\end{align*}
\]

6
where

\[ H_{-1} = H_0 = -\frac{d^2}{dx^2} + x^2, \]

and where the remaining Hamiltonians are defined by (1).

With the help of the above equations we can establish two very useful formulas, which we will use later:

\[
\begin{align*}
H_l b_l^\dagger - b_l^\dagger H_{l-1} &= 2b_l^\dagger, \\
H_{l-1} b_l - b_l H_l &= -2b_l.
\end{align*}
\]  

(7)  

(8)

These are quasi commutative relations, connecting two “neighbouring” Hamiltonians. Let us now consider specific cases of \( \beta_l(x) \).

### 3.1.1 Ordinary oscillator

The case \( l = 0 \) is very special since

\[
\begin{align*}
b_0^\dagger b_0 &= H_0 - 1, \\
b_0 b_0^\dagger &= H_{-1} + 1 \\
&= H_0 + 1.
\end{align*}
\]

Both equations refer to the same Hamiltonian of ordinary harmonic oscillator, and the operators \( b_0 \) and \( b_0^\dagger \) degenerate to the familiar ladder operators

\[
\begin{align*}
a_0^\dagger &= b_0^\dagger = -\frac{d}{dx} + x, \\
a_0 &= b_0 = \frac{d}{dx} + x,
\end{align*}
\]  

(9)  

(10)

which have these important properties:

\[
\begin{align*}
[a_0, a_0^\dagger] &= 2, \\
[H_0, a_0] &= -2a_0, \\
[H_0, a_0^\dagger] &= 2a_0^\dagger.
\end{align*}
\]  

(11)  

(12)  

(13)

Consequently, the energies are given by \( E_{0,k} = 2k + 1 \) for \( k = 0, 1, 2... \), in units of \( \frac{1}{2}\omega\hbar \). We will skip other details, since this is the very known case.
3.1.2 Discretely spiked oscillators

The interesting part begins here. First, let us prove that $b_{l+1}^\dagger |l, k\rangle$ is an eigenvector of Hamiltonian $H_{l+1}$ by assuming that this eigen equation holds

$$H_l |l, k\rangle = E_{l,k} |l, k\rangle$$

and making use of formula (7):

$$H_{l+1} b_{l+1}^\dagger (l, k) = H_{l+1} b_{l+1}^\dagger |l, k\rangle = b_{l+1}^\dagger H_l |l, k\rangle + 2b_{l+1}^\dagger |l, k\rangle = (E_{l,k} + 2)b_{l+1}^\dagger (l, k).$$

The operator $b_{l+1}^\dagger$ transforms the state $|l, k\rangle$ with energy $E_{l,k}$ into the state $|l+1, k\rangle$ with energy $E_{l+1,k} = E_{l,k} + 2$

$$|l+1, k\rangle = \frac{1}{\sqrt{2k + 4l + 4}} b_{l+1}^\dagger |l, k\rangle. \quad (14)$$

This corresponds to the diagonal move in NE direction in Table I. One can verify the normalization factor in equation (14) by comparing the norm of its both sides, making use of (6) and noticing that the energy of $l$-th oscillator in the state $|l, k\rangle$ is given by

$$E_{l,k} = 2(l + k) + 1. \quad (15)$$

The latter can be easily established with the help of Table I.

Similarly, we can prove that state $b_{l+1} |l+1, k\rangle$ is the eigenvector of Hamiltonian $H_l$: $H_l(b_{l+1} |l+1, k\rangle) = (b_{l+1} H_{l+1} - 2b_{l+1}) |l+1, k\rangle$$

$$= (E_{l+1,k} - 2)(b_{l+1} |l+1, k\rangle).$$

The operator $b_{l+1}$ transforms the state $|l+1, k\rangle$ with energy $E_{l+1,k}$ into the state $|l, k\rangle$ with energy $E_{l,k} = E_{l+1,k} - 2$

$$|l, k\rangle = \frac{1}{\sqrt{2k + 4l + 4}} b_{l+1} |l+1, k\rangle. \quad (16)$$

This corresponds to the diagonal move in SW direction in Table I.
3.2 Ladder operators

We shall now define two operators

\[ a_{l+1}^\dagger = b_{l+1}^\dagger a_l^\dagger b_{l+1} \quad (17) \]

and

\[ a_{l+1} = b_{l+1}^\dagger a_l b_{l+1} \quad (18) \]

and prove that they indeed have the expected properties of ladder operators. We shall first prove by induction, that

\[ [H_l, a_l^\dagger] = 2a_l^\dagger \quad (19) \]

This holds true for the case \( l = 0 \), as seen in (13). Assuming now, that the above is true for \( l - 1 \), the proof is as follows:

\[
H_l a_l^\dagger = H_l b_l^\dagger a_{l-1}^\dagger b_l \\
\uparrow \text{from (17)} \\
= (b_l^\dagger H_{l-1} + 2b_l^\dagger) a_{l-1}^\dagger b_l \\
\uparrow \text{from (7)} \\
= b_l^\dagger (a_{l-1}^\dagger H_{l-1} + 2a_{l-1}^\dagger) b_l + 2a_l^\dagger \\
\uparrow \text{from (13) by induction} \\
= b_l^\dagger a_{l-1}^\dagger H_{l-1} b_l + 4a_l^\dagger \\
= b_l^\dagger a_{l-1}^\dagger (b_l H_l - 2b_l) + 4a_l^\dagger \\
\uparrow \text{from (8)} \\
= b_l^\dagger a_{l-1}^\dagger b_l H_l + 2a_l^\dagger \\
= a_l^\dagger H_l + 2a_l^\dagger \\
\uparrow \text{from (17)}
\]

Similarly, we can prove by induction that the following holds

\[ [H_l, a_l] = -2a_l. \quad (20) \]

It immediately follows from (19) and (20) that operators \( a_l^\dagger \) and \( a_l \) create and annihilate eigenstates of Hamiltonian \( H_l \), respectively

\[
H_l a_l^\dagger |l, k\rangle = a_l^\dagger H_l |l, k\rangle + 2a_l^\dagger |l, k\rangle \\
= (E_{l,k} + 2)a_l^\dagger |l, k\rangle,
\]

\[
H_l a_l |l, k\rangle = a_l H_l |l, k\rangle - 2a_l |l, k\rangle \\
= (E_{l,k} - 2)a_l |l, k\rangle.
\]
Finally, we can prove by induction that if \( a_0|0,0\rangle = 0 \) then similar relation holds for any other oscillator \((l > 0)\):

\[ a_l|l,0\rangle = 0. \]

4 Wavefunctions

4.1 Haskell module Spike

The remaining part of this paper refers to results obtained with the help of a little program, \texttt{Spike.hs}\footnote{The program is available at \url{http://www.numeric-quest.com/haskell/Spike.hs}.} written in functional lazy language Haskell\footnote{See home page of Haskell at \url{http://www.haskell.org}.}. The program is used for generation of wavefunctions, for normalization of solutions in n-dimensional spaces and, generally, for testing the theory presented in the previous sections. The program performs some algebraic manipulation of Laurent series, such as computation of derivatives, integrals, sums and products, and directly implements raising and intertwining operators previously discussed. It is best used in interpreting environment\footnote{Haskell interpreter Hugs is available for free at \url{http://www.haskell.org/hugs/}. It runs on any major platform.}, since one can evaluate any formula from module \textit{Spike} in any order of one’s choice. Module \textit{Spike} relies on other standard Haskell modules and on our module \textit{Fraction.hs}\footnote{Module Fraction is available at \url{http://www.numeric-quest.com/haskell/Fraction.hs}.}. For interesting introduction to Haskell as a coding tool for scientific applications see\cite{14}.

4.2 Laurent series

Any analytic function \( f(z) \) in an open annulus \( r < |z - z_0| < R \) can be expressed as the sum of two series

\[
f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} a_{-j} (z - z_0)^{-j}.
\]

Such an expansion, containing negative as well as positive powers is called \textit{Laurent series} for \( f(z) \) in this annulus. In Haskell we will represent it as a pair of two lists

\[
f = L \langle a_0, a_1, a_2, ... \rangle, \langle a_{-1}, a_{-2}, ... \rangle,
\]

where coefficients of expansions can be any numbers belonging to class \textit{Num}, such as \textit{Integer}, \textit{Double}, \textit{Complex Double}, etc.
The following defines the type of data structure representing Laurent series:

```haskell
data Laurent a = L [a] [a],
```

where `a` is a type variable of some numeric kind. It is convenient to choose it as `Integer` (of unlimited size) since this leads to highly accurate results. For computations requiring normalization the data `Fraction` will be used instead of traditional `Double` since this guarantees very good accuracy, with relatively small performance loss.

As it will soon become clear, wavefunctions of spiked oscillators can be represented as products of two functions

$$
\phi_{l,k}(x) = f_{l,k}(x)e^{-x^2/2}
$$

(21)

where `f_{l,k}(x)` is a real function of `x` and can be expanded as Laurent series of type `Laurent Integer`.

### 4.3 One step operators

Application of the intertwining operator to function (21)

$$
b^\dagger_l \phi_{l,k}(x) = b^\dagger_l (f_{k,l}(x)e^{-x^2/2})
$$

$$
= (-f'_{k,l}(x) + 2xf_{k,l}(x) + \frac{l}{x}f_{k,l}(x))e^{-x^2/2}
$$

$$
= (-f'_{k,l}(x) + p_l(x)f_{k,l}(x))e^{-x^2/2},
$$

results in manipulation of two Laurent series `p_l` and `f_{k,l}`, where `p_l(x)` is a short Laurent series, represented in Haskell as `L [0,2] [l]`. The expression `(-f'_{k,l}(x) + p_l(x)f_{k,l}(x))` is implemented in Haskell as function `b'`. It takes as an input an integer `l` and a Laurent series `f_{l,k}` of type `Laurent Integer`, and then produces another Laurent series by invoking few primitives, such as multiplication, addition and differentiation defined for such series. Notice the `gcd normalization` of the results, which keeps the size of integers in check and also provides a canonical standard for comparison of different methods of computations.

```haskell
b' :: Integer -> Laurent Integer -> Laurent Integer
b' l f = gcdNormalize (p * f - diff f)
  where
    p = L [0,2] [l]
```

11
Implementation $b' \ell$ of operator $b_\ell^\dagger$ is the only Haskell function needed to generate all eigenfunctions for all spiked harmonic oscillators. But to complement the theory developed in the previous sections, below are the remaining operators. Firstly, here is function $b \ell$ corresponding to intertwining operator $b_\ell$

$$b \colon \text{Integer} \to \text{Laurent Integer} \to \text{Laurent Integer}$$

$$b \ell f = \text{gcdNormalize} \left( p \ast f + \text{diff} f \right)$$

where

$$p = L \left[ \begin{array}{c} 1 \\ \end{array} \right].$$

Next is the implementation of the raising operator $a^\dagger$:

$$a' \colon \text{Integer} \to \text{Laurent Integer} \to \text{Laurent Integer}$$

$$a' 0 = b' 0$$

$$a' \ell = b' \ell . a' \left( \ell - 1 \right) . b \ell$$

Notice that function $a' 0 = b' 0$ is a special case. It corresponds to the raising operator $a_0^\dagger = b_0^\dagger$ – exactly as specified in the theoretical sections. Finally, here is the implementation of the lowering ladder operator $a_\ell$:

$$a \colon \text{Integer} \to \text{Laurent Integer} \to \text{Laurent Integer}$$

$$a 0 = b 0$$

$$a \ell = b' \ell . a \left( \ell - 1 \right) . b \ell.$$

### 4.4 Cumulative operators

Since the one-step operators from the previous section are to be applied recursively to some base functions it is convenient to define cumulative versions of these operators.

The cumulative intertwining operator, $\text{twine}' \ell$, is a composition of (NE) operators $b_{\ell}^\dagger b_{\ell-1}^\dagger \ldots b_1^\dagger$

$$\text{twine}' \colon \text{Integer} \to \text{Laurent Integer} \to \text{Laurent Integer}$$

$$\text{twine}' 0 = \text{id}$$

$$\text{twine}' \ell = b' \ell . \text{twine}' \left( \ell - 1 \right)$$

The cumulative raising ladder operator, $\text{ladder}' \ell k$, is a composition of $k$ raising ladder operators $a_\ell^\dagger a_\ell^\dagger a_\ell^\dagger \ldots$

$$\text{ladder}' \colon \text{Integer} \to \text{Integer} \to \text{Laurent Integer} \to \text{Laurent Integer}$$

$$\text{ladder}' \ell 0 = \text{id}$$

$$\text{ladder}' \ell 1 = a' \ell$$

$$\text{ladder}' \ell k = a' \ell . \text{ladder}' \ell \left( k - 1 \right).$$
4.5 Generating wavefunctions

Using these tools we can easily generate sets of eigenfunctions for spiked oscillators. When \( l = 0 \) the first wavefunction is given by \( e^{-x/2} \) and that implies that \( f_{0,0} \) is simply 1, and its corresponding Laurent series is \( L[1][] \). Here [] represents an empty list – meaning no negative powers at all. By executing

\[
\text{iterate (a' 0) (L [1] [])}
\]

we can generate infinite list of gcd-normalized eigenfunctions for ordinary harmonic oscillator. If we want to retain six, say, wave functions we need to cut this infinite list down to six elements

\[
\text{take 6 (iterate (a' 0) (L [1] [])).}
\]

A slightly prettified output of the above looks like this:

\[
\begin{align*}
[ &\quad \text{-- Meaning:} \\
L &\ [1] \quad [], \ -- \ 1 \\
L &\ [0,1] \quad [], \ -- \ x \\
L &\ [-1,0,2] \quad [], \ -- \ -1 + 2x^2 \\
L &\ [0,-3,0,2] \quad [], \ -- \ -3x + 2x^3 \\
L &\ [3,0,-12,0,4] \quad [], \ -- \ 3 - 12x^2 + 4x^4 \\
L &\ [0,15,0,-20,0,4] \quad [], \ -- \ 15x - 20x^3 + 4x^5.
\end{align*}
\]

It is evident that these solutions do not contain negative powers of \( x \). In fact they represent (gcd normalized) Hermite polynomials in disguise, and this is exactly what one would expect from ordinary harmonic oscillator. To test implementation of the lowering ladder operator \( a_0 \), execute

\[
\text{take 6 (iterate (a 0) (L [0,15,0,-20,0,4])).}
\]

This should recreate the above list in reverse. In addition, the expression

\[
\text{a 0 (L [1] [])}
\]

generates a special kind of Laurent series

\[
L [] [],
\]

which is the Laurent series for \( \text{function zero} \).

To find eigenfunctions of the first spiked oscillator \( (l = 1) \) we have two choices. Firstly, we can apply operator \( b_1^\dagger \), or function \( b' 1 \) to every eigenfunction of ordinary oscillator, by executing
map (b' 1) (take 6 $ iterate (a' 0) (L [1] [])).

Alternatively, we can iteratively apply function $ a' 1 $ to the first eigenfunction $ f_{1,0} = b' 1 (L [1][])$ of the oscillator $l = 1$

take 6 $ iterate (a' 1) f$ where $ f = b' 1 (L [1] [])$.

Either way, the output is as follows

\[
\begin{align*}
[ & \quad \text{-- Meaning:} \\
L [0,2] & \quad [1], \quad \text{-- } 2x + 1/x \\
L [0,0,1] & \quad [], \quad \text{-- } x^2 \\
L [0,-4,0,4] & \quad [-1], \quad \text{-- } -4x + 4x^3 - 1/x \\
L [0,0,-5,0,2] & \quad [], \quad \text{-- } -5x^2 + 2x^4 \\
L [0,18,0,-36,0,8] & \quad [3], \quad \text{-- } 18x - 36x^3 + 8x^5 + 3/x \\
L [0,0,35,0,-28,0,4] & \quad [], \quad \text{-- } 35x^2 - 28x^4 + 4x^6 \\
\end{align*}
\]

This time, every other solution contains term with $1/x$, and as such it might or might not be acceptable as physical solution, depending on the space dimension under consideration. More about it in later sections.

To generate similar list for the second spiked oscillator ($l = 2$) we need to apply two functions $ b' 1 $ and $ b' 2 $ in succession to a list of eigenfunctions of ordinary harmonic oscillator

\[
\begin{align*}
\text{map (b' 2 \cdot b' 1) (take 6 $ iterate (a' 0) (L [1] []))}, \\
\text{or, alternatively, apply the raising ladder function } a' 2 \text{ to the lowest eigen-} \\
\text{function of the oscillator } l = 2 \\
take 6 $ iterate (a' 2) f$ where $ f = (b' 2 \cdot b' 1) (L [1] [])$.
\end{align*}
\]

Both methods lead to the same output:

\[
\begin{align*}
[ & \\
L [4,0,4] & \quad [0,3], \\
L [0,0,0,1] & \quad [], \\
L [-6,0,-12,0,8] & \quad [0,-3], \\
L [0,0,0,-7,0,2] & \quad [], \\
L [24,0,72,0,-96,0,16] & \quad [0,9], \\
L [0,0,0,63,0,-36,0,4] & \quad [] \\
\] \\
\]

Here is a general formula for generation of infinite lists of wavefunctions for spiked harmonic oscillators $ l = 0, 1, 2.. $
wavefunctions :: Integer -> Laurent Integer
wavefunctions l = map (twine' l) $ iterate (a' 0) (L [1] [])

To generate just one eigenfunction $f_{l,k}$ the following Haskell function can be used

wavefunction :: Integer -> Integer -> Laurent Integer
wavefunction l k = twine' l (ladder' 0 k (L [1] []))

5 Normalization and orthogonalization

We have shown that the solutions for spiked oscillators are of the form $\phi_{l,k} = f_{l,k}(x)e^{-x^2/2}$, where $f_{l,k}(x)$ are real functions of $x$ and, specifically, they are some Laurent series – admitting both positive and negative powers of $x$. For a specific case of ordinary oscillator, these are the Hermite polynomials, with positive powers only.

The scheme presented so far accepts all solutions, but does not check whether they are square-integrable and, in addition, whether they obey the orthogonality relations

$$\langle l, i | l, j \rangle = \delta_{i,j}.$$ 

Due to presence of the term $\frac{1}{x}$ in the definition of $b_l^\dagger$ some of the wavefunctions $\phi_{l,k}$ are tainted by powers of $\frac{1}{x}$ and as such might not be admissible as physical solutions for a dimension under consideration. Terms such as

$$\frac{1}{x}e^{-x^2/2},$$

when considered in one dimension, imply normalization integrals such as

$$\int_{-\infty}^{+\infty} \frac{1}{x^2}e^{-x^2} dx$$

which are not integrable. On the other hand, if variable $x$ is treated as three dimensional radius then the normalization integral like this

$$\int_{0}^{+\infty} \frac{1}{x^2}e^{-x^2} 4\pi x^2 dx,$$

is integrable.
5.1 Admissible physical solutions

A general pattern of physically acceptable solutions is as follows:

- All odd eigenfunctions $\phi_{l,k}$ for $k = 1, 3, 5...$ of any oscillator $l = 0, 1, 2...$ are physically acceptable, regardless the space dimension.

- All even eigenfunctions $\phi_{l,k}$ for $k = 0, 2, 4...$ of any oscillator $l = 0, 1, 2...$ are acceptable as physical solutions only when $N \geq 2l + 1$, where $N$ is the space dimension.

The above statements can be verified by Haskell function $\text{physicalPattern}$, which produces infinite list of booleans – each stating whether a solution $k$ for oscillator $l$ and space dimension $N$ is admissible as physical solution

$$\text{physicalPattern} :: \text{Integer} \to \text{Integer} \to \text{[Bool]}$$

$$\text{physicalPattern} = \text{map (isPhysical n) (wavefunctions l)}$$

$$\text{isPhysical} :: \text{Integer} \to \text{Laurent Integer} \to \text{Bool}$$

$$\text{isPhysical n f} =$$

$$\quad | \text{length us'} == 0 = \text{True}$$

$$\quad | \text{otherwise} = \text{False}$$

$$\quad \text{where}$$

$$\quad \text{L us us'} = f * f * \text{dV}$$

$$\quad \text{dV} = \text{diff$\text{L}$ ((take (fromIntegral n)$\text{L}$ (\text{take (fromIntegral l)$\text{L}$ (\text{repeat 0)$\text{L}$ \text{[]})))}$ \text{[]}$$

For example, even solutions for oscillator $l = 2$ are unphysical in three dimensional space, $N = 3$, as seen below (Haskell lists are traditionally indexed from zero)

$$\text{take 6$\text{physicalPattern}$ 3 2}$$

$$=> \text{[False,True,False,True,False,True]}.$$

5.2 Linear normalization

In order to normalize the eigenfunctions of one dimensional linear spiked oscillators and to test their orthogonality conditions we need algebraic formulae for the values of improper integrals

$$P_n = \int_{-\infty}^{+\infty} x^n e^{-x^2} dx.$$ 

They are as follows
\[ P_0 = \sqrt{\pi} \]
\[ P_1 = 0 \]
\[ P_n = \frac{n-1}{2} P_{n-2}. \]

Notice that integrals with odd powers of \( x^n \) are all zero.

All solutions for the ordinary harmonic oscillator (\( l = 0 \)) are physically acceptable since their Laurent series do not contain coefficients of negative powers of \( x \). What’s more, they form the orthogonal system of eigenfunctions

\[ \langle 0, k | 0, k' \rangle = \delta_{kk'} \]

Physically acceptable solutions for the remaining, spiked, oscillators (\( l > 0 \)) are those with odd values of \( k \), \( k = 1, 3, 5... \) Each oscillator \( l \) has its own orthogonal basis. This can be easily checked with the help of the Haskell function \textit{bracket}, the scalar product of two (not normalized) wavefunctions – each annotated by a pair of integers \((l, k)\).

\begin{verbatim}
bracket :: Integer -> (Integer,Integer)
    -> (Integer, Integer) -> Fraction
bracket n (l, k) (l', k')
    = scalarProduct n f g
    where
        f = wavefunction l k
        g = wavefunction l' k'
\end{verbatim}

The integer \( n \) specifies the space dimension \( n = 1, 2, 3... \) when computing volume integrals for radial cases. But we can use the same function for computing linear integrals as well – by setting, somewhat artificially, this argument to zero, \( n = 0 \). For example

\[
decimal 8 \$ bracket 0 (7,1) (7,1) => 14034.40729347
\]
\[
decimal 8 \$ bracket 0 (7,1) (7,5) => 0
\]

Notice the conversion from fractional to decimal representation, specified here with accuracy to eight decimal places. All internal computations are performed on integers of unlimited size and on fractions made of such integers. The accuracy can be extremely good, depending on the value of \( eps \), set at the top of module \textit{Spike}. It is only for presentation purposes that we convert fractions to decimal representation.

The following table summarizes the case of linear spiked oscillators
5.3 Radial normalization

Scalar product of two basis vectors for radial oscillators can be defined by a generic form of improper volume integral

\[
\langle l, i | l, j \rangle = \int_0^{+\infty} f_{l,i}(x) f_{l,j}(x) e^{-x^2} dV, \tag{22}
\]

where \(dV\) represents volume of an elementary sphere in \(N\)-dimensional space. Using general formula for a volume of a sphere with radius \(x\) in \(N\) dimensions

\[
V(N) = \frac{\pi^{N/2}}{\Gamma(1 + N/2)} x^N,
\]

where

\[
\begin{align*}
\Gamma(0) &= 1 \\
\Gamma(1/2) &= \sqrt{\pi} \\
\Gamma(1) &= 1 \\
\Gamma(N) &= (N - 1)\Gamma(N - 1),
\end{align*}
\]

the elementary volume \(dV\), and thus volume integrals (22) can be easily computed. Specifically, the ‘volume’ of sphere with radius \(x\) in dimensions \(N = 1, 2, 3\) are given by \(2x, \pi x^2, 4/3\pi x^3\), etc. To simplify Haskell computations we will ignore constant factors in definition of the \(N\)-dimensional volume and define \(dV\) simply as \(dV = dx, xdx, x^2dx\), etc., for \(N = 1, 2, 3\), respectively.

5.3.1 One dimensional

The ‘volume’ integral in one dimension is twice the value of the one dimensional integral over \([0, +\infty)\). This is, generally, not the same as the improper integral over \((-\infty, +\infty)\) – unless the function being integrated is even: \(g(-x) = g(x)\).

The recurrence formulae for integrals \(P_n = 2 \int_0^{+\infty} x^n e^{-x^2} dx\), are given below.

\[
\begin{align*}
P_0 &= \sqrt{\pi} \\
P_1 &= 1 \\
P_n &= \frac{n-1}{2} P_{n-2}.
\end{align*}
\]
As opposed to the case of one dimensional linear normalization the integrals $P_n$ for odd powers of $n$ do not vanish. The case $l = 0$ admits all solutions $k = 0, 1, 2, \ldots$, but further investigation reveals that they do not form the basis since scalar products of one odd and one even eigenfunctions is not equal to zero, as in this example, 

$$\langle 0, 0 | 0, 1 \rangle = 0.797884.$$ 

However, the set of solutions for $l = 0$ can be split into two sets: one even ($k = 0, 2, 4, 6, \ldots$) and one odd ($k = 1, 3, 5, \ldots$) – each forming the basis 

$$\langle 0, k | 0, k' \rangle = \delta_{kk'}.$$ 

Solutions for the remaining oscillators ($l > 0$) are restricted to odd eigenfunctions ($k = 1, 3, 5, \ldots$), because even eigenfunctions are unphysical. As a result we can summarize this case as follows:

**Even $k$ basis**

| $l$ | $k$ | $E_{0,k}$ |
|-----|-----|---------|
| 0   | 0, 2, 4... | 1, 5, 9... |

**Odd $k$ bases**

| $l$ | $k$ | $E_{0,k}$ | $E_{1,k}$ | $E_{2,k}$ |
|-----|-----|---------|---------|---------|
| 0   | 1, 3, 5... | 3, 7, 11... |
| 1   | 1, 3, 5... | 5, 9, 13... |
| 2   | 1, 3, 5... | 7, 11, 15... |

This table is particularly interesting because it summarizes what is traditionally known as *three dimensional isotropic harmonic oscillator* [15]. When expressed in spherical coordinates and after separation of variables the radial part of its Hamiltonian becomes

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} r + \frac{1}{2} m \omega^2 r^2 + \frac{l(l+1)\hbar^2}{2mr^2},$$

which can be further reduced to the form [1] after representing the radial solutions as 

$$R_{l,k} = \frac{1}{r} \phi_{l,k}(r).$$

Functions $\phi_{l,k}$ correspond to $f_{l,k} e^{-r^2/2}$ investigated in this paper, and the three dimensional normalization of solutions $R_{l,k}$ becomes one dimensional normalization of $\phi_{l,k}$.
$$\int_0^\infty R_{l,k}R_{l,k'} 4\pi r^2 \, dr = 4\pi \int_0^\infty \phi_{l,k}\phi_{l,k'} \, dr.$$ 

According to (15) energies are given by $E_{l,k} = 2(l + k + 1)$, in units of $\hbar\omega$. It is customary to represent this as $\varepsilon_n = (n + 3/2)\hbar\omega$, where $n = l + k - 1$—taking one of the values $n = 0, 1, 2, 3...$

With every $n$, we can associate either one or many pairs of $(l, k)$, which generate the states $|l, k\rangle$ corresponding to the same energy $\varepsilon_n$

$$n = 0 \implies |0, 1\rangle$$
$$n = 1 \implies |1, 1\rangle$$
$$n = 2 \implies |0, 3\rangle, |2, 1\rangle$$
$$n = 3 \implies |1, 3\rangle, |3, 1\rangle$$
$$n = 4 \implies |0, 5\rangle, |2, 3\rangle, |4, 1\rangle$$
$$... \implies ...$$

The energy levels are degenerate; when $n$ is even then there are $n + 1$ states

$$|0, n + 1\rangle, |2, n - 1\rangle, |4, n - 3\rangle...$$

and when $n$ is odd then there are $\frac{n+1}{2}$ states

$$|1, n\rangle, |3, n - 2\rangle, |5, n - 4\rangle...$$

associated with each energy level. When $2l + 1$ possible eigenvalues of $L_3$ are also considered for each $l$ then the total degeneracy of energy levels becomes $\frac{1}{2}(n+1)(n+2)$. The solutions obtained by other means (see for example [15]) are exactly the same as the ones presented in this paper.

### 5.3.2 Two dimensional

According to the condition $N \geq 2l + 1$, we can accept both even and odd solutions for oscillator $l = 0$, while the even solutions for the remaining oscillators ($l > 0$) must be rejected as unphysical. However, it seems that the two dimensional normalization does not lead to any well defined pattern of orthogonal bases and therefore we reject this case as ill-specified. But similar normalization in the next dimension ($N = 3$) provides us with several usable bases, as shown below.
5.3.3 Three dimensional

In accordance with the general pattern of physical admissibility of even wavefunctions, \( N \geq 2l + 1 \), the first two oscillators, \( l = 0 \) and \( l = 1 \), admit both even and odd solutions \( k = 0, 1, 2, \ldots \). For the remaining oscillators, \( l \geq 2 \), only the odd solutions \( k = 1, 3, 4, \ldots \) are acceptable.

For each \( l \), the following pattern of orthogonality relations emerges

\[
\langle l, k|l, k' \rangle = \delta_{kk'} \quad \text{for } |k' - k| = 0, 4, 8, 12, \ldots
\]

Consequently, the eigenvectors of the first two oscillators can be partitioned into four bases - two even and two odd, while each of the remaining oscillators, \( l \geq 2 \), has two orthogonal bases. The following four tables summarize the three dimensional normalization.

**Even \( k \) bases**

| \( l = 0 \) | \( k = 0, 4, 8, \ldots \) | \( E_{0,k} = 1, 9, 17, \ldots \) |
| \( l = 1 \) | \( k = 0, 4, 8, \ldots \) | \( E_{1,k} = 3, 11, 19, \ldots \) |

| \( l = 0 \) | \( k = 2, 6, 10, \ldots \) | \( E_{0,k} = 5, 13, 21, \ldots \) |
| \( l = 1 \) | \( k = 2, 6, 10, \ldots \) | \( E_{1,k} = 7, 15, 23, \ldots \) |

**The first odd \( k \) basis**

| \( l = 0 \) | \( k = 1, 5, 9, \ldots \) | \( E_{0,k} = 3, 11, 19, \ldots \) |
| \( l = 1 \) | \( k = 1, 5, 9, \ldots \) | \( E_{1,k} = 5, 13, 21, \ldots \) |
| \( l = 2 \) | \( k = 1, 5, 9, \ldots \) | \( E_{2,k} = 7, 15, 23, \ldots \) |
| ... | ... | ... |

Its degenerated energy levels are given by

\[
\varepsilon_n = (n + \frac{3}{2})\hbar \omega,
\]

where \( n = l + k - 1 = 0, 1, 2, \ldots \)

**The second odd \( k \) basis**

| \( l = 0 \) | \( k = 3, 7, 11, \ldots \) | \( E_{0,k} = 7, 15, 23, \ldots \) |
| \( l = 1 \) | \( k = 3, 7, 11, \ldots \) | \( E_{1,k} = 9, 17, 25, \ldots \) |
| \( l = 2 \) | \( k = 3, 7, 11, \ldots \) | \( E_{2,k} = 11, 19, 27, \ldots \) |
| ... | ... | ... |
Its degenerated energy levels are given by

$$\varepsilon_n = (n + \frac{7}{2})\hbar \omega$$

where \(n = l + k - 3 = 0, 1, 2, 3\ldots\)

### 5.3.4 N dimensional

Specific cases, examined in the last few sections, exhibit certain pattern, which can be generalized on \(N\) dimensions and directly verified by Haskell function \textit{bracket}.

In each \(N\) dimensional space there exists a threshold integer number \(l' = (N - 1)/2\), such that for all \(l \leq l'\) all wavefunctions \(k = 0, 1, 2\ldots\) can be accepted as physical. Above this threshold the even solutions \(k = 0, 2, 4\) must be rejected as unphysical. However, it appears that radial normalization in any of the even dimensions \(N = 2, 4, 6\ldots\) does not lead to properly defined basis or bases. In contrary, for each of the odd dimensions \(N = 1, 3, 5\ldots\) there is a clear pattern of several staggered orthogonal bases. Half of them are even in \(k\) and they are subjected to the limitation of physicality. The other half are orthogonal bases with odd vectors \(k\) and they are all physical and well defined.

For \(N = 1\) the vectors belonging to the same basis are enumerated by \(\Delta k = 2\). For \(N = 3\) the staggering of bases is given by \(\Delta k = 4\), for \(N = 5\) it is \(\Delta k = 6\), etc.

### 6 Conclusions

We have shown that spiked oscillators described by Hamiltonians \((\text{[1]}\)) have simple and exact eigenfunctions, subject to further normalization restrictions, which are specific to a problem dimension. One such restriction is related to integrability of the solutions and the other to selection of orthogonal bases. The form \((\text{[1]}\)) can be interpreted either as a family of one dimensional linear spiked oscillators, or families of radial spiked oscillators in odd \(N\) dimensional spaces, where \(N = 1, 3, 5\ldots\) Even dimensions \(N = 2, 4, 6\ldots\) must be rejected because they do not give rise to orthogonal bases.

Each Hamiltonian, \(H_l\), has an uncountable number of eigenvectors \(|l, k\rangle\), where \(k = 0, 1, 2\ldots\). Generally, all solutions indexed by odd \(k = 1, 3, 5\ldots\) are integrable and constitute one basis or several interleaved bases. In contrary, only a limited number of oscillators \(l\) admit even eigenfunctions \(k = 0, 2, 4\ldots\) for a given odd dimension \(N = 1, 3, 5\ldots - each forming a basis or a set of interleaved bases.
Each Hamiltonian \([\Pi]\) leads to an equidistant energy spectrum, isomorphic to a spectrum of ordinary quantum oscillator, but subjected to restrictions described above. Our factorization method which is akin to \textit{SUSY} method, and which relies on two kinds of operators: \textit{intertwing operators} \(b_1^\dagger\) and \(b_1\), and \textit{ladder operators} \(a_l^\dagger\) and \(a_l\), is purely algebraic. But while the \textit{SUSY} method correlates solutions of two Hamiltonians with isomorphic spectra, our method extends this approach on infinite set of Hamiltonians. The theory is augmented by very simple Haskell program, which directly implements these operators, generates the eigenfunctions, tests integrability of the solutions and verifies orthogonality conditions for wavefunctions enumerated by quantum numbers \(l\) and \(k\).

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