Continuous-time random walk with a superheavy-tailed distribution of waiting times

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We study the long-time behavior of the probability density associated with the decoupled continuous-time random walk which is characterized by a superheavy-tailed distribution of waiting times. It is shown that if the random walk is unbiased (biased) and the jump distribution has a finite second moment then the properly scaled probability density converges in the long-time limit to a symmetric two-sided (an asymmetric one-sided) exponential density. The convergence occurs in such a way that all the moments of the probability density grow slower than any power of time. As a consequence, the reference random walk can be viewed as a generic model of superslow diffusion. A few examples of superheavy-tailed distributions of waiting times that give rise to qualitatively different laws of superslow diffusion are considered.

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I. INTRODUCTION

Almost a half century ago, Montroll and Weiss [1] introduced a special class of cumulative processes, the so-called continuous-time random walks (CTRWs). Due to its simplicity and flexibility, the CTRW approach has become an important tool for the analysis of various stochastic systems. It is especially efficient for studying the systems whose average behavior differs from the classical one characterized by the normal, linear dependence of the mean-square displacement (variance) on time. In particular, the CTRW approach is extensively used to determine the laws of anomalous diffusion occurring in these systems (see, e.g., Refs. [2–4]).

One of the most important characteristics of the CTRW is the long-time behavior of the probability density \( P(x,t) \) associated with the position \( x(t) \) of a walking particle. It is this density that describes in full detail the diffusion properties of particles, including the asymptotic behavior of the variance (i.e., the diffusion law). Moreover, the probability density \( P(x,t) \) at \( t \to \infty \) plays a significant role in establishing the connection between the CTRW and the fractional time diffusion equation [5].

Within the CTRW framework, the probability density \( P(x,t) \) of the particle position is completely characterized by the joint probability density of the waiting times \( \tau_n \) \( (n = 1, 2, \ldots) \), i.e., random times between the particle jumps, and the jump magnitudes \( x_n \). For the decoupled CTRW, when the sets \( \{ \tau_n \} \) and \( \{ x_n \} \) are statistically independent of each other, \( P(x,t) \) depends only on the probability densities \( p(\tau) \) and \( w(x) \) of \( \tau_n \) and \( x_n \), respectively. In this case the long-time behavior of \( P(x,t) \) is determined by the main characteristics of \( p(\tau) \) and \( w(x) \) [6–8]. For example, if the second moment of \( w(x) \) exists and \( p(\tau) \) at \( \tau \to \infty \) is given by the asymptotic formula \( p(\tau) \sim h/\tau^{1+\alpha} \) with \( h \) being a positive parameter and \( 0 < \alpha < 1 \), then the properly scaled probability density \( P(x,t) \) at \( t \to \infty \) converges either to a symmetric two-sided (for an unbiased walk) or to an asymmetric one-sided (for a biased walk) stable density [6].

A more general class of waiting-time probability densities is described by the asymptotic relation \( p(\tau) \sim h(\tau)/\tau^{1+\alpha} \) \( (\tau \to \infty) \), where a positive function \( h(\tau) \) is assumed to be slowly varying at infinity. Due to this property the function \( h(\tau) \) does not strongly influence the long-time behavior of \( x(t) \) if \( \alpha > 0 \) [9]. However, at \( \alpha = 0 \) the situation changes drastically. From a mathematical point of view, this is a consequence of the fact that the waiting-time densities characterized by the condition \( \alpha = 0 \) belong to a class of superheavy-tailed densities [10, 11] whose all fractional moments are infinite. It is expected, therefore, that in this case the long-time behavior of the CTRW is mainly controlled by the asymptotic form of slowly varying function \( h(\tau) \) at \( \tau \to \infty \). The laws of both biased and unbiased superslow diffusion [12], i.e., diffusion of particles whose mean-square displacement grows slower than any power of time, corroborate this statement.

Some important features of the long-time behavior of the probability density \( P(x,t) \) have already been established by Havlin and Weiss [13] for a very special class of slowly varying functions with \( h(\tau) \sim a/\ln^{1+n} \tau \) \( (a > 0, \nu > 0) \) as \( \tau \to \infty \). They have shown, in particular, that if the CTRW is decoupled and unbiased and the second moment of the jump probability density \( w(x) \) is finite then \( P(x,t) \) at \( t \to \infty \) has symmetric exponential tails. In contrast, in the present paper we provide a detailed analysis of the long-time behavior of \( P(x,t) \) for a whole class of slowly varying functions \( h(\tau) \) that support the normalization condition of \( p(\tau) \) and consider both biased and unbiased versions of the decoupled CTRW.

The paper is organized as follows. In Sec. [11], we reproduce the basic equations of the decoupled CTRW and specify a class of superheavy-tailed densities of waiting times. The main results of the paper are obtained in Sec. [111]. Here, we determine the long-time behavior of
the probability density of the particle position and establish its limiting form for a properly scaled spatial variable. Moreover, we use the two-sided exponential jump density to verify our results and consider a few illustrative examples of superheavy-tailed densities of waiting times. In Sec. IV we calculate the moments of the particle position at long times and show that they grow slower than any power of time. In the same section, we derive the most general form of the laws of superslow diffusion and calculate the laws that correspond to the illustrative examples. Our findings are summarized in Sec. V.

II. SPECIFICATION OF THE MODEL

The CTRW is defined as the cumulative continuous-time jump process 2 3

\[ X(t) = \sum_{n=1}^{N(t)} x_n, \]  

(2.1)

where \( X(t) \) (\( X(0) = 0 \)) can be considered as the walker (particle) position, \( x_n \in (-\infty, \infty) \) is the random magnitude of the \( n \)th jump of a particle, and \( N(t) = 0, 1, 2, \ldots \) is the random number of jumps up to time \( t \) (if \( N(t) = 0 \) then \( X(t) = 0 \)). It is assumed that both the magnitudes \( x_n \) and the waiting times \( \tau_n \) (i.e., times between successive jumps) are independent random variables distributed with the probability densities \( p(x) \) and \( p(\tau) \), respectively. In the decoupled case, when the sets \( \{x_n\} \) and \( \{\tau_n\} \) of the variables \( x_n \) and \( \tau_n \) are independent of each other, the distribution of particles is completely determined by these densities. Specifically, in the Fourier-Laplace space the probability density \( P(x, t) \) of the particle position \( X(t) \) is given by the Montroll-Weiss equation \[ \] 1

\[ P_{ks} = \frac{1 - p_s}{s(1 - p_s w_k)}. \]  

(2.2)

Here, the indexes \( k \) and \( s \) denote the Fourier and Laplace transforms, \( u_k = \mathcal{F}\{u(x)\} = \int_{-\infty}^{\infty} dx e^{ikx} u(x) \) and \( v_s = \mathcal{L}\{v(t)\} = \int_{0}^{\infty} dt e^{-st} v(t) \) (\( \text{Res} > 0 \)), of the corresponding functions. Applying to Eq. (2.2) the inverse Fourier and Laplace transforms defined as \( u(x) = \mathcal{F}^{-1}\{u_k\} = (1/2\pi) \int_{-\infty}^{\infty} dk e^{-ikx} u_k \) and \( v(t) = \mathcal{L}^{-1}\{v_s\} = (1/2\pi i) \int_{c-i\infty}^{c+i\infty} dse^{st} v_s \) (the real parameter \( c \) is assumed to be larger than the real parts of all singularities of \( v_s \)), respectively, one obtains

\[ P(x, t) = \mathcal{L}^{-1}\left\{ \frac{1 - p_s}{s} \mathcal{F}^{-1}\left\{ \frac{1}{1 - p_s w_k} \right\} \right\}. \]  

(2.3)

It can also be shown \[ \] 14 16 that the probability density \( P(x, t) \) satisfies the integral equation

\[ P(x, t) = V(t)\delta(x) + \int_{0}^{t} d\tau \int_{-\infty}^{\infty} dx' P(x', \tau) \times p(t - \tau)w(x - x'), \]  

(2.4)

where \( \delta(x) \) is the Dirac \( \delta \) function and

\[ V(t) = 1 - \int_{0}^{t} d\tau p(\tau) = \int_{t}^{\infty} d\tau p(\tau) \]  

(2.5)

\( (V(0) = 1, V(\infty) = 0) \) is the complementary cumulative distribution function of waiting times, that is also known as the survival or exceedance probability. The first term in the right-hand side of this equation corresponds to the situation, which is realized with probability \( V(t) \), when there are no jumps up to time \( t \). While the exact solution of Eq. (2.4) was obtained for a special case of jump and waiting-time densities \[ \] 17, the long-time behavior of \( P(x, t) \) was analyzed for a much wider class of these densities. This has been done by using the representation \[ \] 2.3 4,7 and by applying the limit theorems of probability theory \[ \] 8 18. The first approach is based on the Tauberian theorem for the Laplace transforms (see, e.g., Ref. 19), which states that if the function \( v(t) \) is ultimately monotone and

\[ v_s \sim \frac{1}{s^\gamma} L\left( \frac{1}{s} \right) \]  

(0 < \gamma < \infty) as \( s \to 0 \) then

\[ v(t) \sim \frac{t^{\gamma - 1}}{\Gamma(\gamma)} L(t) \]  

(2.7)

as \( t \to \infty \). Here, \( \Gamma(\gamma) \) is the gamma function and \( L(t) \) is a slowly varying function at infinity, i.e., \( L(\mu t) \sim L(t) \) as \( t \to \infty \) for all \( \mu > 0 \). We note that, in contrast to the inverse Laplace transform where the parameter \( s \) is complex with \( \text{Res} > 0 \), in Eq. (2.6) this parameter is assumed to be real and positive.

In this paper, we use the Laplace transform

\[ P_s(x) = \frac{1 - p_s}{s} \delta(x) + \frac{1 - p_s}{s} p_x \mathcal{F}^{-1}\left\{ \frac{w_k}{1 - p_s w_k} \right\}, \]  

(2.8)

which follows from Eq. (2.2), and the Tauberian theorem to study the long-time behavior of the probability density \( P(x, t) \) in the case of superheavy-tailed distributions of waiting times. More precisely, we consider a class of probability distributions whose densities have the following asymptotic behavior:

\[ p(\tau) \sim \frac{h(\tau)}{\tau} \]  

(2.9)

(\( \tau \to \infty \)), where a positive function \( h(\tau) \) varies slowly at infinity. Since for each slowly varying function \( h(\tau) \) the condition \( \tau^\rho h(\tau) \to 0 \) (\( \rho > 0 \)) holds as \( \tau \to \infty \) \[ \] 20, all fractional moments of \( p(\tau) \) are infinite [i.e., \( T_0 = \int_{0}^{\infty} d\tau \tau^\rho p(\tau) = \infty \)]. It should be noted that the slowly varying function \( h(\tau) \) in Eq. (2.9) is not arbitrary: it must be compatible with the normalization condition \( T_0 = \int_{0}^{\infty} d\tau \tau^\rho p(\tau) = 1 \), which in turn implies that \( h(\tau) = o(1/\ln \tau) \) as \( \tau \to \infty \). Finally, we assume here that the first two moments of the jump density \( w(x) \), \( l_1 = \int_{-\infty}^{\infty} dx xw(x) \) and \( l_2 = \int_{-\infty}^{\infty} dx x^2w(x) \), exist. As will be shown below, in this case only these characteristics of \( w(x) \) influence the long-time behavior of \( P(x, t) \).
III. LONG-TIME BEHAVIOR OF P(x,t)

According to the Tauberian theorem, the long-time behavior of the probability density P(x,t) of the particle position is determined by the asymptotic behavior of the Laplace transform P_s(x), see Eq. (2.8), at s → 0. In order to find P_s(x) for small s, let us first determine the two terms of the asymptotic expansion of the Laplace transform p_s = \int_0^\infty d\tau e^{-\tau s}p(\tau). With this purpose, we use the exceedance probability (3.1) to represent p_s as

\[ p_s = 1 - \int_0^\infty dq e^{-qV}\left(\frac{q}{s}\right). \] (3.1)

An important feature of the exceedance probability, which follows from the asymptotic formula (2.9), is that it is a slowly varying function at infinity, i.e., V(\mu t) \sim V(t) as t \to \infty. Therefore, at s \to 0 the function V(q/s) in Eq. (3.1) can be replaced by V(1/s), yielding

\[ p_s \sim 1 - V\left(\frac{1}{s}\right). \] (3.2)

Since p_s and w_k tend to 1 as s and k tend to 0, at small s the main contribution to the inverse Fourier transform in Eq. (2.8) comes from the close vicinity of the point k = 0. Due to this fact, w_k can be approximated by a few terms of the expansion of w_k as k → 0. Taking into account that the first two moments of w(x), l_1 and l_2, are assumed to exist, we obtain

\[ w_k \sim 1 + il_1k - \frac{1}{2}l_2k^2 \] (k → 0).

Then, using Eqs. (3.2) and (3.3), we can find the asymptotic behavior of \( F^{-1}\left\{w_k/(1-p_s w_k)\right\}\) as s → 0 by replacing w_k/(1-p_s w_k) by \([l_2 k^2/2 + V(1/s)]^{-1}\) in the unbiased case (when \(l_1 = 0\)) and by \([-il_1k + V(1/s)]^{-1}\) in the biased case (when \(l_1 \neq 0\)).

For the former case one has [21]

\[ F^{-1}\left\{\frac{w_k}{1-p_s w_k}\right\} \sim F^{-1}\left\{\frac{1}{l_2 k^2/2 + V(1/s)}\right\} \sim e^{-|x|\sqrt{2V(1/s)/l_2}}. \] (3.4)

According to this result, the first term in the right-hand side of Eq. (2.8) can be neglected in the limit s → 0. Indeed, the integral of this term over some interval containing the point x = 0 is equal to \((1-p_s)/s \sim V(1/s)/s\), while Eq. (3.4) shows that the integral of the second term is proportional to 1/s. Therefore, since V(1/s) → 0 as s → 0, the main term of the asymptotic expansion of P_s(x) takes the form

\[ P_s(x) \sim \frac{1}{s} \sqrt{\frac{V(1/s)}{2l_2}} e^{-|x|\sqrt{2V(1/s)/l_2}}. \] (3.5)

(s → 0). Because the exceedance probability V(t) varies slowly at infinity, it is not difficult to show that also does the function \( \sqrt{V(t)/2l_2} e^{-|x|\sqrt{2V(t)/l_2}}\). Hence, the Tauberian theorem is applicable to this case and, in accordance with Eq. (3.4), leads to the following behavior of P(x,t) at long times:

\[ P(x,t) \sim \sqrt{\frac{V(t)}{2l_2}} e^{-|x|\sqrt{2V(t)/l_2}}, \] (3.6)

where, in accordance with Eq. (2.9),

\[ V(t) \sim \int_t^\infty d\tau \frac{1}{\tau} h(\tau). \] (3.7)

It should be stressed that in spite of the fact that the asymptotic formula (3.6) is derived considering a small vicinity of the point k = 0, it is valid for all x. The reason is that it is this vicinity which is responsible for the asymptotic behavior of P_s(x) as s → 0. We note in this context that the asymptotic formula for P(x,t) obtained in [13] for a special case of the exceedance probability is applicable for all x, not only for |x| → ∞ as it was assumed in Ref. [13]. Finally, introducing the variable y = x\sqrt{2V(t)/l_2}, we make sure that the limiting distribution of the scaled particle position Y(t) = X(t)/\sqrt{2V(t)/l_2} is described by the symmetric two-sided exponential density

\[ P(y) = \lim_{t \to \infty} \int_{-\infty}^{l_2/2V(t)} P\left(\sqrt{\frac{l_2}{2V(t)}} y, t\right) = e^{-|y|/2}. \] (3.8)

For the biased CTRW, when \(l_1 \neq 0\), we obtain [21]

\[ F^{-1}\left\{\frac{w_k}{1-p_s w_k}\right\} \sim F^{-1}\left\{-il_1k + V(1/s)\right\} \sim e^{-xV(1/s)/l_1} \theta(l_1x), \] (3.9)

where \(\theta(x) = 0\) if x < 0 and \(\theta(x) = 1\) if x > 0. It is important to emphasize that the condition \(F^{-1}\{w_k/(1-p_s w_k)\} \approx 0\), which occurs at \(l_1x < 0\), is a direct consequence of the used approximation [see the first line of Eq. (3.4)]. In fact, the term \([-il_1k + V(1/s)]^{-1}\) is the principal part of the Laurent expansion of \(w_k/(1-p_s w_k)\) in the vicinity of the point k = -V(1/s)/l_1 → 0. Taking the next term of this expansion, \([il_1k+c]^{-1}(c > 0)\), which is responsible for the behavior of \(F^{-1}\{w_k/(1-p_s w_k)\}\) at \(l_1x < 0\), one gets \(F^{-1}\{w_k/(1-p_s w_k)\} \sim e^{xc}/l_1\) \((x|l_1| < 0)\). However, since for small s the condition \(c/V(1/s) \gg 1\) holds, \(F^{-1}\{w_k/(1-p_s w_k)\}\) as a function of x at \(l_1x > 0\) varies much slower than it does for \(l_1x < 0\). Therefore, in the limit s → 0 the contribution coming from the region with \(l_1x < 0\) is negligible, and the inverse Fourier transform in Eq. (2.8) can be safely approximated by Eq. (3.9) [see also Sec. III A].

For the same reason as before, in the biased case the first term in the right-hand side of Eq. (2.8) can also be
neglected in the limit $s \to 0$. Therefore, using Eqs. (2.8), (3.2) and (3.9), one gets

$$P_s(x) \sim \frac{1}{s|l_1|}V\left(\frac{1}{s}\right)e^{-xV(1/s)/l_1}\theta(l_1x) \quad (3.10)$$

($s \to 0$), and the Tauberian theorem yields

$$P(x,t) \sim \frac{V(t)}{|l_1|}e^{-xV(t)/l_1}\theta(l_1x) \quad (3.11)$$

($t \to \infty$). Accordingly, the limiting distribution of the scaled particle position $Y(t) = X(t)V(t)/l_1$ is described by the asymmetric one-sided exponential density

$$P(y) = \lim_{t \to \infty} \frac{|l_1|}{V(t)}P\left(\frac{l_1}{V(t)}y,t\right) = e^{-y}\theta(y). \quad (3.12)$$

Equations (3.8) and (3.12) are the main results of this paper. They show that in the case of the decoupled CTRWs with superheavy-tailed distributions of waiting times the limiting distribution of the properly scaled particle position is described by either the two-sided or the one-sided exponential density. Remarkably, these two- and one-sided exponential densities correspond to all unbiased and all biased CTRWs, respectively.

A. Testing jump density

The limiting probability densities (3.8) and (3.12) were obtained from the approximation of the inverse Fourier transform in Eq. (2.8) by the asymptotic formulas (3.4) and (3.9). Although the applicability of this approximation is well-grounded, it is desirable to check the validity of Eqs. (3.8) and (3.12) by using an exact expression for that transform. To this end, we consider the two-sided exponential jump density

$$w(x) = \frac{\kappa_+\kappa_-}{\kappa_+ + \kappa_-} \begin{cases} e^{-x\tilde{k}_+}, & x \geq 0 \\ e^{x\tilde{k}_-}, & x < 0 \end{cases} \quad (3.13)$$

($\kappa_+ > 0$). Taking into account that the Fourier transform of this density is given by

$$w_k = \frac{\kappa_+\kappa_-}{(k+i\kappa_+)(k-i\kappa_-)}, \quad (3.14)$$

we obtain

$$w_k = \frac{\kappa_+\kappa_-}{1-p_s|\kappa_+|}(k+i\kappa_+)(k-i\kappa_-), \quad (3.15)$$

where

$$\tilde{k}_\pm = \mp \frac{1}{2}(\kappa_- - \kappa_+) + \frac{1}{2}\sqrt{(\kappa_- - \kappa_+)^2 + 4(1-p_s)\kappa_+\kappa_-}. \quad (3.16)$$

Since the right-hand sides of Eqs. (3.14) and (3.15) are similar, the inverse Fourier transform of $w_k/(1-p_sw_k)$ is of the form of $w(x)$; that is,

$$\mathcal{F}^{-1}\left\{\frac{w_k}{1-p_sw_k}\right\} = \frac{\kappa_+\kappa_-}{\kappa_+ + \kappa_-} \begin{cases} e^{-x\tilde{k}_+}, & x \geq 0 \\ e^{x\tilde{k}_-}, & x < 0 \end{cases}, \quad (3.17)$$

and so Eq. (2.8) in the reference case reads

$$P_s(x) = \frac{1}{s}\delta(x) + \frac{(1-p_s)p_s}{s}\frac{\kappa_+\kappa_-}{\kappa_+ + \kappa_-}\left\{e^{-x\tilde{k}_+}, x \geq 0 \right. \left. e^{x\tilde{k}_-}, x < 0 \right\}. \quad (3.18)$$

If $\kappa_+ = \kappa_- = \kappa$ then Eq. (3.13) yields $l_1 = 0$ (i.e., the corresponding CTRW is unbiased). In this case $\tilde{k}_+ = \tilde{k}_- = \kappa \sqrt{1-p_s} \sim \kappa V^{1/2}(1/s)$, $l_2 = 2/\kappa^2$, and Eq. (3.18) at $s \to 0$ reduces to Eq. (3.5). Thus, the limiting probability density which corresponds to Eq. (3.18) with $\kappa_+ = \kappa_- = \kappa$ is given by Eq. (3.8), as it should be for the unbiased case. If $\kappa_+ \neq \kappa_-$ then the biased CTRW with $l_1 = (\kappa_- - \kappa_+)/\kappa_+\kappa_-\kappa$ occurs. According to Eq. (3.19), in this case we have

$$\tilde{k}_\pm \sim \kappa_\pm\kappa_-|l_1|\theta(\mp l_1) + \frac{1}{|l_1|}V\left(\frac{1}{s}\right)\theta(\pm l_1) \quad (3.19)$$

as $s \to 0$. Using this result and Eq. (3.18), it is not difficult to find the asymptotic formula for $P_s(x)$ which, by applying the Tauberian theorem, gives

$$P(x,t) \sim \frac{\kappa_-\kappa_-|l_1|V(t)}{\kappa_+\kappa_-|l_1|^2 + V(t)} \left\{e^{-x^2V(t)/l_1}, l_1x > 0 \right. \left. e^{x^2\kappa_-l_1}, l_1x < 0 \right\} \quad (3.20)$$

as $t \to \infty$. While at $l_1x > 0$ the asymptotic formulas (3.21) and (3.11) are asymptotically equivalent, at $l_1x < 0$ they are different. However, as it was mentioned earlier, this difference does not affect the long-time distribution of the scaled particle position $Y(t) = X(t)V(t)/l_1$. Indeed, taking into account that the exponential term $e^{y\kappa_\pm l_1^2}$/t $V(t)$ at $y < 0$ tends to zero as $t \to \infty$, the limiting probability density $P(y)$ associated with probability density $P(x,t)$ from Eq. (3.20) is given by Eq. (3.12).

B. Illustrative examples of waiting-time densities

The long-time behavior of the exceedance probability $V(t)$ is the most important characteristic of the CTRWs with superheavy-tailed distributions of waiting times. As follows from Eqs. (3.6) and (3.11), it is this behavior that is responsible for the long-time behavior of the probability density $P(x,t)$. Below we consider three illustrative examples of the waiting-time probability density $p(\tau)$ which lead to different asymptotic formulas for $V(t)$. As a first example, we consider the probability density

$$p(\tau) = \frac{qb^{1/q}}{\Gamma(1/q, b\ln^2 \eta)} \exp\left[-b\ln^2(\eta + \tau)\right] \quad (3.21)$$

where $\Gamma(a, x) = \int_x^\infty \gamma^{-a-1} \right] \Gamma(\mu, a, 0) = \Gamma(\mu)$, and the conditions $\eta > 1$, $b > 0$ and $0 < q < 0$ are assumed to hold. The conditions $\eta > 1$, $b > 0$ and $q > 0$ are responsible for the
positivity and normalization of \( p(\tau) \), and the condition \( q < 1 \) guarantees that \( h(\tau) \) is a slowly varying function. The last inequality follows from that the limit
\[
\lim_{\tau \to \infty} \frac{h(\mu \tau)}{h(\tau)} = \lim_{\tau \to \infty} \exp(-b q \ln \mu \ln^{q-1} \tau) \tag{3.22}
\]
equals 1 for all \( \mu > 0 \), i.e., the function \( h(\tau) \) is slowly varying at infinity, only if \( q < 1 \). According to the definition (2.3), the exceedance probability that corresponds to the probability density (3.21) reads
\[
V(t) = \frac{\Gamma[1/q, b \ln^q(\eta + t)]}{\Gamma(1/q, b \ln^q \eta)}. \tag{3.23}
\]

Therefore, using the asymptotic formula \( \Gamma(a, x) \sim e^{-x} x^{a-1} \) as \( x \to \infty \), one gets in the long-time limit
\[
V(t) \sim \frac{b^{1/q-1}}{\Gamma(1/q, b \ln^q \eta)} \exp(-b \ln^q t) \ln^{1-q} t. \tag{3.24}
\]

Our second example is the probability density
\[
p(\tau) = \frac{(r-1) \ln \tau^{-\eta}}{(\eta + \tau) \ln (\eta + \tau)} \tag{3.25}
\]
where \( r > 1 \) and \( \eta > 1 \). The main feature of this density is that its right tail is heavier than in the previous case; that is, \( p(\tau) \) at \( \tau \to \infty \) tends to zero slower than that in Eq. (3.21). The exceedance probability associated with the probability density (3.25) has the form
\[
V(t) = \left( \frac{\ln \eta}{\ln (\eta + t)} \right)^{r-1}, \tag{3.26}
\]
and so
\[
V(t) \sim \left( \frac{\ln \eta}{\ln t} \right)^{r-1} \tag{3.27}
\]
as \( t \to \infty \). Comparing the asymptotic formulas (3.21) and (3.27), we conclude that the exceedance probability in Eq. (3.26) decreases slower than the exceedance probability in Eq. (3.23).

Finally, as a third example, we consider the waiting-time probability density
\[
p(\tau) = \frac{(r-1)(\ln \ln \eta)^{r-1}}{(\eta + \tau) \ln (\eta + \tau) \ln \ln (\eta + \tau)}, \tag{3.28}
\]
(\( \eta, r > 1 \)), whose right tail is heavier than for \( p(\tau) \) from Eq. (3.26). In this case the exceedance probability is also calculated exactly, yielding
\[
V(t) = \left( \frac{\ln \ln \eta}{\ln \ln (\eta + t)} \right)^{r-1} \tag{3.29}
\]
and
\[
V(t) \sim \left( \frac{\ln \ln \eta}{\ln \ln t} \right)^{r-1} \tag{3.30}
\]
as \( t \to \infty \). We note that the above examples are illustrative and they do not exhaust all possible behaviors of the exceedance probability \( V(t) \) at long times. Moreover, a class of slowly varying functions \( V(t) \) contains infinitely many functions that, at \( t \to \infty \), decrease both faster and slower than a given function from this class.

IV. MOMENTS OF THE PARTICLE POSITION

The moments \( M_n(t) \) \((n = 1, 2, \ldots)\) of the particle position are defined in the usual way:
\[
M_n(t) = \int_{-\infty}^{\infty} dx x^n P(x, t). \tag{4.1}
\]

Their behavior at long times can be easily determined by using the limiting probability densities (3.8) and (3.12). Specifically, in the case of unbiased CTRWs for the even moments we obtain
\[
M_{2n}(t) = \left( \frac{l_2}{2V(t)} \right)^{n+1/2} \int_{-\infty}^{\infty} dy y^n P \left( \frac{l_2}{2V(t)} y, t \right)
\sim \left( \frac{l_2}{2V(t)} \right)^{n} \int_0^{\infty} dy y^n e^{-y}. \tag{4.2}
\]

Taking into account that \( \int_0^{\infty} dy y^n e^{-y} = \Gamma(2n + 1) = (2n)! \), the above result becomes
\[
M_{2n}(t) \sim \left( \frac{l_2}{2V(t)} \right)^{n} (2n)!. \tag{4.3}
\]

Since in this case \( P(x, t) \) is a symmetric function of \( x \), all odd moments equal zero: \( M_{2n-1}(t) = 0 \).

For the biased CTRWs Eqs. (4.1) and (3.12) give
\[
M_n(t) = \left( \frac{l_1}{V(t)} \right)^n \int_{-l_1}^{l_1} dy y^n P \left( \frac{l_1}{V(t)} y, t \right)
\sim \left( \frac{l_1}{V(t)} \right)^n \frac{l_1}{|l_1|} \int_{-l_1}^{l_1} dy y^n e^{-y\theta(y)}. \tag{4.4}
\]

Therefore, since \( \int_{-l_1}^{l_1} dy y^n e^{-y\theta(y)} = l_1 n! / |l_1| \), the asymptotic formula (4.3) reduces to
\[
M_n(t) \sim \left( \frac{l_1}{V(t)} \right)^n n!. \tag{4.5}
\]

An important feature of the moments of the particle position is that they are slowly varying functions at infinity. This follows from the fact [20] that if some function [in our case \( V(t) \)] is slowly varying then so does any power of this function. Due to this feature, in the long-time limit the condition \( M_n(t) / t^\rho \to 0 \) holds for all \( \rho > 0 \) [20]. It shows that the moments \( M_n(t) \) grow to infinity (because \( V(t) \) tends to zero as \( t \to \infty \)) slower than any positive power of time. Hence, the CTRW with superheavy-tailed distributions of waiting times can be viewed as a generic model of superslow diffusion.
A. Laws of superslow diffusion

The character of diffusion is usually determined by the law of diffusion; that is, the long-time behavior of the mean-square displacement or variance

$$\sigma^2(t) = M_2(t) - M_2^2(t).$$

Using the asymptotic formulas (4.3) and (4.5), from the definition (4.6) we find the diffusion laws

$$\sigma^2(t) \sim \begin{cases} \frac{l_1^2}{V^2}(t), & l_1 \neq 0 \\ l_2/V(t), & l_1 = 0 \end{cases}$$

(4.7)

for both unbiased ($l_1 = 0$) and biased ($l_1 \neq 0$) CTRWs with superheavy-tailed distributions of waiting times. Since in this case the variance $\sigma^2(t)$ is a slowly varying function, it grows to infinity slower than any positive power of $t$, i.e., superslow diffusion occurs. As is clear from Eq. (4.7), the biased superslow diffusion is faster than the unbiased one. We note that the same laws of superslow diffusion have also been derived by using another approach [12], which does not involve explicitly the long-time behavior of the probability density $P(x,t)$.

Although superslow diffusion has been observed in very different systems, all previously known diffusion laws are given by a power function of the logarithm of time: $\sigma^2(t) \propto \ln^n t$ ($n > 0$). The best known example of this type of diffusion is the Sinai diffusion [23] for which $n = 4$. Other examples have been observed, e.g., in resistor networks [24], two-dimensional lattices [25, 26], charged polymers [27], aperiodic environments [28], iterated maps [29], Langevin dynamics [30], and fractional kinetics [31]. In contrast, the asymptotic formulas (4.7) define a large class of laws of superslow diffusion, which includes $\sigma^2(t) \propto \ln^n t$ as a very particular case. This class consists of the slowly varying functions that tend to infinity as time evolves.

According to Eq. (4.7), the diffusion laws are completely determined by the asymptotic behavior of the exceedance probability $V(t)$. In particular, for the examples considered in Sec. III B the laws of superslow diffusion can be obtained by the direct substitution of the asymptotic formulas (3.24), (3.27) and (3.30) into Eq. (4.7). For the first example we obtain

$$\sigma^2(t) \sim \begin{cases} \left( \frac{l_1 \Gamma(1/q,b \ln^q \eta)}{b^{1/q-1} \ln^{2(1-\alpha)} t} \right)^2 e^{2b \ln^q t}, & l_1 \neq 0 \\ \frac{l_2 \Gamma(1/q,b \ln^q \eta)}{b^{1/q-1} \ln^{1-q} t}, & l_1 = 0. \end{cases}$$

(4.8)

The second example leads to the following law of superslow diffusion:

$$\sigma^2(t) \sim \begin{cases} \left( \frac{l_1}{\ln^{1-\eta} \eta} \right)^2 \ln^{2(r-1)} t, & l_1 \neq 0 \\ \frac{l_2}{\ln^{r-1} \eta} \ln^{r-1} t, & l_1 = 0, \end{cases}$$

(4.9)

which for $l_1 = 0$ was first derived in Ref. [13]. In this example, that reproduces the diffusion laws with $\sigma^2(t) \propto \ln^2 t$, diffusion occurs slower than in the previous one. Finally, the third example yields

$$\sigma^2(t) \sim \begin{cases} \frac{l_1}{(\ln \ln \eta)^{r-1}} (\ln t)^{2(r-1)}, & l_1 \neq 0 \\ \frac{l_2}{(\ln \ln \eta)^{r-1}} (\ln t)^{r-1}, & l_1 = 0, \end{cases}$$

(4.10)

i.e., the variance grows slower than that in Eq. (4.9). We recall that these examples are illustrative because infinitely many different laws of superslow diffusion exist.

1. Specific application

Now, as an application of our results, we will show that particles moving under a constant force $f(> 0)$ through a randomly layered medium can exhibit the biased superslow diffusion. According to [22], the influence of this medium on the particle dynamics can be described by the piecewise constant random force $g(x) = g^{(n)}$, where $x \in (nl,(n+1)l)$, $n = 0,1,\ldots$, $f$ is the layer thickness, and $g^{(n)}$ are independent random variables distributed with the probability density $u(g)$ in the interval $[-g_0,g_0]$. In the overdamped regime the particle position $x_t(x_0 = 0)$ is governed by the equation of motion $\ddot{x}_t = f + g(x_t)$ with $\nu$ being the damping coefficient. At $f \geq g_0$ the long-time solution of this equation can be approximated by the walker position $X(t)$ if the jump density is given by $w(x) = \delta(x - l)$ and the waiting-time density $p(\tau)$ is associated with the probability density of the random time $\tau^{(n)}$ that a particle spends moving from the point $nl$ to the point $(n+1)l$ [32, 33]. Since $\tau^{(n)} = \nu l/(f + g^{(n)})$, the probability density $u(g)$ which characterizes the medium disorder is expressed through $p(\tau)$ as follows:

$$u(g) = \frac{\nu l}{(f + g)^2} p\left( \frac{\nu l}{f + g} \right).$$

(4.11)

It should be noted that the waiting-time density $p(\tau)$ is defined in the interval $[\tau_{\min}, \tau_{\max}]$, where $\tau_{\min} = \nu l/(f + g_0)$ and $\tau_{\max} = \nu l/f - g_0$.

It was shown in Ref. [33] that the biased diffusion of particles can be anomalous only if $f = g_0$. In this case the character of anomalous diffusion is determined by the behavior of $p(\tau)$ as $\tau \to \infty$ (we recall that $\tau_{\max} = \infty$ at $f = g_0$) or, as it follows from Eq. (4.11), by the behavior of $u(g)$ as $g \to -g_0 + 0$. Thus, taking into account the asymptotic formula (2.9), one can see that if

$$u(g) \sim \frac{1}{g_0 + g} h\left( \frac{1}{g_0 + g} \right)$$

(4.12)

($g \to -g_0 + 0$) then the biased diffusion of particles moving under a force $f = g_0$ in a randomly layered medium is superslow with $\sigma^2(t) \sim l^2/V^2(t)$ as $t \to \infty$ [since
$w(x) = \delta(x - l)$, we have $l_1 = l$. In particular, Eq. (4.12) for $p(\tau)$ from Eq. (3.25) yields

$$u(g) \sim \frac{(r - 1) \ln^{-1} \gamma}{\ln[(g_0 + g)\ln(g_0 + g)]}, \quad (4.13)$$

and so the law of superslow diffusion in such a medium is given by Eq. (4.9) with $l_1 = l \neq 0$.

V. CONCLUSIONS

We have studied the long-time behavior of the decoupled CTRW with a superheavy-tailed distribution of waiting times. We assume that the distribution function of waiting times belongs to a class of slowly varying functions and the jump distribution has a finite second moment. At these conditions, the limiting probability density of the properly scaled particle position has been derived for both unbiased and biased CTRWs. In the former case the limiting density is given by the symmetric two-sided exponential density, while in the latter case it takes the form of the asymmetric one-sided exponential density. We have used these limiting densities to calculate the moments of the particle position at long times. It has been shown that all the moments are expressed through the exceedance probability and grow slower than any positive power of time.

Due to the last property, the decoupled CTRW with a superheavy-tailed distribution of waiting times can be viewed as a generic model of superslow diffusion. We have derived the most general form of the laws of superslow diffusion and have shown that, in the cases of unbiased and biased diffusion, the variance of the particle position is inversely proportional to the first and second powers of the exceedance probability, respectively. The laws of superslow diffusion that correspond to the illustrative examples of superheavy-tailed distributions of waiting times have also been determined. Finally, we have applied the obtained results to show that the biased diffusion of particles moving under a constant force in a randomly layered medium can be superslow.

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