Projective Spaces and Splitting of Madsen-Tillmann Spectra

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Abstract

We refine the classical splitting of $BSO(2n+1)_+$ off $BO(n)_+$ to show that $BSO(2n+1)_+$ splits off $MTO(2n)$. In complex case, we show that $BSU(n+1)_+$ splits off $MTU(n)$ whenever $p$ does not divide $(n-1)(n+1)$. As an immediate corollary, we show that $S^0$ splits off $MTO(2n)$, and $MTU(n)$ at these primes. When $n = 1$, this is a special case of the splittings obtained by Steinberg idempotents which will be discussed in a subsequent work.

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1 Introduction

Let $G$ denote $O$, $SO$ or $U$. The Madsen-Tillmann spectra $MTG(n)$’s ([4]) are the Thom spectra of the virtual bundle $-\gamma^n$ where $\gamma^n$ is the universal bundle over $BG(n)$. They classify fiber bundles whose fibers are homeomorphic to an $n$-dimensional manifold with $G$-structure (see e.g. [3] for a readable account). Their homology is easy to compute thanks to the Thom isomorphism, but there are still lots of things that remain unknown about their homotopy types. For example, we know very little about their homotopy groups. Of course, as the homology groups of these spectra are rather simple, their homotopy groups are expected to be difficult to determine. Moreover, the cohomology of the associated infinite loop spaces contain characteristic classes for such fibrations. Among these cohomology rings, so far, only $H^*_\Lambda(\Omega\infty MTG(n); \mathbb{Z}/p)$ with $i=1,2$ for an arbitrary prime $p$ [10] and $H^*_\Lambda(\Omega\infty MTU(1); \mathbb{Z}/p)$ for all primes [5] have been known.

In this note, we show that the well-known splitting of $BSO(2n+1)_+$ off $BO(2n)_+$ etc. using the Becker-Gottlieb transfer can be refined using the Mann-Miller-Miller transfer to produce a splitting of $BSO(2n+1)_+$ off $MTG(2n)$ etc., and discuss some consequences. Among other things, we give a (very simple) alternative proof of Theorem C of [10].

The case of $MTG(2)$ is a special case of the splitting obtained using the Steinberg idempotent, which we will discuss in a subsequent work [6]. Throughout the note, we identify a (pointed) space with its suspension spectrum, and for a space $X$, we note $X_+$ the space $X$ with the disjoint basepoint added.

2 Statement of the main results

Let $G$ be a compact Lie group, and $K$ its closed subgroup. Consider the fibration

$$G/K \longrightarrow BK \longrightarrow BG.$$ 

The composition

$$BG_+ \longrightarrow BK_+ \longrightarrow BG_+$$

where the first map is the associated Becker-Gottlieb transfer induces multiplication by the Euler characteristic of the homogeneous space $G/K$, $\chi(G/K)$ in $\tilde{H}^*(-; \Lambda)$ where $\Lambda$ is any Abelian group [11 Theorem 5.5]. So if $\chi(G/K)$ is not a multiple of $p$, we get a stable splitting of $BK_+$ off $BG_+$ when localised at $p$. This is well known, and in the case where $G$ is finite, has been used extensively to study the stable homotopy type of the classifying space $BG$ (e.g. [9]). The case when $G$ is not finite, is also well-known, and for example, it has been shown [12 Lemma 1] that $BSO(2n+1)_+$ splits off $BO(2n)_+$ (this splitting occurs without localization) and $BSU(n+1)$ splits off $BU(n)_+$ unless $p$ divides $n$. What is considered less often is the fact that the Becker-Gottlieb transfer factors through so-called Mann-Miller-Miller transfer, giving rise to a splitting of appropriate Thom spectra, which happen to be the Madsen-Tillmann spectra in favorable cases. More precisely, we will prove

**Theorem 2.1.** Let $(K, G)$ be a pair where

i). $K = O(2n), G = SO(2n+1)$

ii). $K = U(n), G = SU(n+1)$ with $p \nmid (n-1)(n+1)$

iii). $K = SO(2n), G = SO(2n+1)$ with $p$ odd
iv). \( K = O(2n), G = O(2n + 1) \) with \( p \) odd.

Then \( BG_+ \) splits off stably off \( MTK \), after \( p \)-localization in the cases ii)-iv).

As \( BSO(2n + 1) \) is homotopy equivalent to \( BO(2n + 1) \) at an odd prime, iv) follows from iii). However we mentioned it for the sake of completeness.

We also note that for any space \( X \), we have \( \Sigma^\infty(X_+) \cong \Sigma^\infty(X) \vee S^0 \), which implies the following.

**Corollary 2.2.** \( S^0 \) splits off \( MTO(2n) \) (without localization involved). Furthermore, \( S^0 \) splits off \( MT SO(2n) \) when \( p \) is an odd prime \( p \), and splits off \( MTU(n) \) when \( p \mid (n - 1)(n + 1) \), both after \( p \)-localization.

Although this sounds rather innocent, this is strong enough to lead to a new proof of \[10\] Theorem C] as we shall see later.

We will also provide a second proof of this Corollary using the Madsen-Tillmann-Weiss map, with slightly stronger result, namely

**Theorem 2.3.** \( S^0 \) splits off \( MT G(n) \) where \( G, n \) and \( p \) are as in Corollary 2.2. \( S^0 \) also splits off \( MTU(n) \) and \( MT Sp(n) \) if \( p \nmid n + 1 \), after \( p \)-localization.

At the relevant primes, the above results show the potential complication in computing \( \beta_\ast\pi_\ast MTG(n) \) whereas from homological point of view it shows that \( \mathbb{Z}/p \)-homology of \( \Omega^\infty MTG(n) \) contains a copy of \( H_\ast(Q_0S^0; \mathbb{Z}/p) \), which imply results on some families of characteristic classes as we will discuss in Sections 4 and 5.

### 3 Proof of the main results

In this section, we prove Theorem 2.1. From now on, we fix a prime \( p \), and spectra are understood to be localized at \( p \), unless otherwise specified.

We start from reviewing the Mann-Miller-Miller transfer \[8\]. Suppose we have an embedding of compact Lie groups \( K \subset G \) and a (virtual) bundle \( \alpha \rightarrow BG \). Then we get a transfer map

\[
BG^{\text{ad}_G + \alpha} \rightarrow BK^{\text{ad}_K + \alpha|_K}
\]

where, for a bundle \( \xi \) over a base space \( X \), \( X^\xi \) denotes the Thom spectrum of \( \xi \), \( \alpha|_K \) is the pull-back of \( \alpha \) by the obvious map \( BK \rightarrow BG \), and \( \text{ad}_H = EH \times_H \mathfrak{h} \rightarrow BH \) is the adjoint bundle over \( BH \). We note that if \( \alpha \) corresponds to a representation \( \phi \) of \( G \), then \( \alpha|_K \) corresponds to the restriction \( \phi|_K \), and that the adjoint bundle corresponds to the adjoint representation.

Now, if we take \( \alpha \) to be the \(-\text{ad}_G \), we obtain a map \( BG_+ \rightarrow BK^{\text{ad}_K - \text{ad}_G|_K} \). As the adjoint representation of \( K \) is a sub-representation of the restriction to \( K \) of the adjoint representation of \( G \), the bundle \( \text{ad}_G|_K - \text{ad}_K \) is a genuine bundle, thus we have a map of Thom spectra \( BK^{\text{ad}_K - \text{ad}_G|_K} \rightarrow BK^{(\text{ad}_K - \text{ad}_G|_K) + (\text{ad}_G|_K - \text{ad}_K)} \cong BK_+ \). The composition of those two maps \( BG_+ \rightarrow BK_+ \) can be easily seen to agree with the Becker-Gottlieb transfer. Therefore, we see that

**Proposition 3.1.** Let \( G \) be a compact Lie group, \( K \) its closed subgroup, such that \( \chi(G/K) \) is not divisible by \( p \). Then \( BG_+ \) splits off \( BK^{(\text{ad}_K - \text{ad}_G|_K)} \).

Now, through the usual embeddings we have diffeomorphisms \( O(2n + 1)/O(2n) \cong SO(2n + 1)/SO(2n) \cong S^{2n} \). Moreover, the embeddings \( O(n) \rightarrow SO(n + 1) \) and \( U(n) \rightarrow SU(n + 1) \) respectively determined by \( A \mapsto (\det A)(A \oplus 1) \) and \( A \mapsto A \oplus (\det A)^{-1} \) provide diffeomorphisms
SO(2n + 1)/O(2n) ≅ \mathbb{R}P^{2n} and U(n + 1)/SU(n) ≅ \mathbb{C}P^n. With the equalities \( \chi(S^{2n}) = 2, \chi(\mathbb{R}P^{2n}) = 1 \) (with any coefficient) and \( \chi(\mathbb{C}P^n) = n + 1 \), all it remains to prove Theorem 2.1 is to identify the representation \( \text{ad}_{G|K} - \text{ad}_K \). However, we have

\[
\begin{pmatrix} X & W \\ W & D \end{pmatrix} \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \begin{pmatrix} X^{-1} & W^{-1} \\ D \end{pmatrix} = \begin{pmatrix} XAX^{-1} & W^{-1}XB \\ (W^{-1}XB)^* & D \end{pmatrix},
\]

where \( X, A \) are \( m \times m \) matrices and \( W, D \) are \( 1 \times 1 \) matrices. Therefore, in the cases iii) and iv), we see that \( \text{ad}_{G|K} - \text{ad}_K \) is isomorphic to the canonical representation of \( K \). In the case i), the embedding that we have chosen for \( O(n) \subset SO(n + 1) \) instead of the traditional \( A \rightarrow A \oplus \text{det}(A) \) permits to have \( \text{ad}_{G|K} - \text{ad}_K \) isomorphic to the canonical representation of \( K \). As to the case ii), we see that the representation \( \text{ad}_{G|K} - \text{ad}_K \) is isomorphic to the tensor product (over \( \mathbb{C} \)) of the canonical representation with the inverse of the determinant representation. Thus we will need the following

**Lemma 3.2.** Let \( p \nmid n - 1 \). Then the homomorphism \( \varphi : A \mapsto \text{det}(A)^{-1}A \) induces a self homotopy equivalence of \( BU(n) \).

**Proof.** It suffices to show that it induces an automorphism on \( H^*(BU(n); \mathbb{Z}/p) \). Consider the following commutative diagram

\[
\begin{array}{ccc}
U(1)^n & \xrightarrow{\varphi} & U(1)^n \\
\downarrow & & \downarrow \\
U(n) & \xrightarrow{\varphi} & U(n)
\end{array}
\]

where the vertical arrows are the inclusions of the diagonal matrices with entries in \( U(1) \), \( \varphi \) is given by

\[
\varphi(e^{i\theta_1}, \ldots, e^{i\theta_n}) = (e^{i(\theta_1 - \theta)}, \ldots, e^{i(\theta_n - \theta)}) \text{ where } \theta = \theta_1 + \cdots + \theta_n.
\]

Now we see that \( H^*(B\varphi) \) on \( H^*(BU(1)^n; \mathbb{Z}/p) \cong \mathbb{Z}/p[x_1, \ldots, x_n] \) is given by \( H^*(B\varphi)(x_i) = x_i - c_1 \) with \( c_1 = x_1 + \cdots + x_n \). Thus by restricting to \( H^*(BU(n); \mathbb{Z}/p) \cong \mathbb{Z}/p[c_1, \ldots, c_n] \), we see that

\[
H^*(B\varphi)(c_1) = (1 - n)c_1, \quad H^*(B\varphi)(c_i) \equiv c_i \text{ mod } (c_1) \text{ for } i > 1.
\]

Thus \( H^*(B\varphi) \) is an automorphism if and only if \( p \) doesn’t divide \( n - 1 \).

Now, we note that the pull-back by \( \varphi \) of the canonical representation \( \gamma_n \) is just \( \text{det}^{-1} \otimes \gamma_n \), so using the same notation for the bundle and representation, we get a bundle map \( \text{det}^{-1} \otimes \gamma_n \rightarrow \gamma_n \) over the map \( \varphi \), and thus \( \text{det}^{-1} \otimes \gamma_n \rightarrow -\gamma_n \) as well. Since \( \varphi \) is a homotopy equivalence, we see that the map between the Thom spectra \( BU(n)^{-\text{det}^{-1} \otimes \gamma_n} \rightarrow BU(n)^{-\gamma_n} = MTU(n) \) is also a homotopy equivalence. Thus we get the desired splitting in the case ii) as well.

The above Lemma completes the proof of Theorem 2.1. Next, let’s prove Theorem 2.3. First suppose that \( M \) is a manifold with reduction of the structure bundle to \( G \). Denote by \( TM \) its tangent bundle, \( f : M \rightarrow BG \) its classifying map. Thus \( f^*(\gamma) = TM \) where \( \gamma \) is the universal vector bundle over \( BG \). So, the stable normal bundle \( -TM \) is a pull-back of \( -\gamma \) by \( f \). Thus one can consider the Thomified map \( M^{-TM} \rightarrow MTG \). Now, choose an embedding of the \( m \)-dimensional manifold \( M \) in an Euclidean space \( \mathbb{R}^{m+k} \), say \( i : M \rightarrow \mathbb{R}^{m+k} \). Then denoting \( \nu_i \)
the normal bundle of the embedding \( i \) which we identify with the tubular neighbourhood of \( M \) in \( \mathbb{R}^m+k \), the Thom-Pontrjagin construction provides a map

\[
S^{m+k} \rightarrow M^{\nu_i} = M^{\mathbb{R}^m+k-TM}.
\]

The Pontrjagin-Thom construction for the composition of the embedding \( M \rightarrow \mathbb{R}^{m+k} \) with the standard inclusion \( \mathbb{R}^{m+k} \rightarrow \mathbb{R}^{m+k+1} \) yields a map \( S^{m+k+1} \rightarrow M^{\mathbb{R}^{m+k+1}-TM} \) which is the suspension of the previous map. We then, by stabilising, obtain a stable map

\[
S^0 \rightarrow M^{-TM}.
\]

The composition \( S^0 \rightarrow M^{-TM} \rightarrow MTG \) is the Madsen-Tillmann-Weiss map associated to the fiber bundle \( M \times pt \rightarrow pt \) with \( TM \) having a \( G \)-structure. It is then a consequence of Hopf’s vector field theorem [1, Theorem 4] that the composition

\[
S^0 \rightarrow M^{-TM} \rightarrow MTG \rightarrow BG_+ \rightarrow S^0
\]

where \( MTG \rightarrow BG_+ \) is the zero section and \( BG_+ \rightarrow S^0 \) the collapse map, is a map of degree \( \chi(M) \). Therefore, if \( \chi(M) \) is not a multiple of \( p \), we obtain a splitting of \( S^0 \) off \( MTG \) using the Madsen-Tillmann-Weiss map. Noting that \( \mathbb{R}P^n, \mathbb{C}P^n \) and \( \mathbb{H}P^n \) have the tangent bundle with structure group \( O(n), U(n) \) and \( Sp(n) \) respectively, and \( \chi(\mathbb{R}P^{2n}) = 1, \chi(\mathbb{C}P^n) = \chi(\mathbb{H}P^n) = n+1 \), we get the desired result.

**Remark 3.3.** To conclude the section, we note that the twisted transfer map \( BG_+ \rightarrow MTK \) that we have used to split \( MTK \) in the proof of Theorem [2, Theorem 4] in the real cases, is essentially the same as the Madsen-Tillmann-Weiss map associated to the fibration \( G/K \rightarrow BK \rightarrow BG \), as both of these maps are essentially obtained from Pontrjagin-Thom collapse. This is not the case in general, however. More precisely, in the case of \( G = SU(n+1), K = U(n) \), consider the following diagram

\[
\begin{array}{cccccc}
BSU(n+1)_+ & \longrightarrow & BU(n)^{\text{det} \otimes \gamma_n} & \longrightarrow & BU(n)^{-\gamma_n} \\
\downarrow & & & & \downarrow \\
BU(n)_+ & \longrightarrow & BU(n)_+
\end{array}
\]

where the top left map is the Miller transfer, the composition of the top horizontal maps is \( MTW \) map, the composition of the top left map and left vertical map is the \( BG \)-transfer and the bottom horizontal map happens to be self equivalence if \( p \) doesn’t divide \( n-1 \), but it is not identity even in these case. So \( MTW \) map \( BG \rightarrow MTK \) composed with \( MTK \rightarrow BK \) is not the \( BG \) transfer; the case when \( p \) divides \( n-1 \), the difference becomes very important.

The example of \( G = SU(n+1), K = U(n) \) then shows that the composition \( BG_+ \rightarrow MTK_+ \rightarrow BK_+ \rightarrow BG_+ \) is not very easy to understand (the composition of the first two arrows is homotopic to the composition of Becker-Gottlieb transfer followed by a self map of \( BK_+ \), which happens to be self-homotopy equivalence in the cases we consider), thus the use of Mann-Miller-Miller transfer is more convenient.
4 Randall-Williams’ polynomial family

In [10], Randall-Williams defines a polynomial family in $H^*(\Omega^\infty MTO(2); \mathbb{Z}/2)$ as a pull-back of Stiefl-Whitney classes by the map

$$\Omega^\infty MTO(2) \rightarrow QBO(2)_+ \rightarrow QS^0 \rightarrow \mathbb{Z} \times BO \rightarrow \mathbb{Z} \times BO.$$

However, Corollary [2,2] implies that $QS^0$ splits off $\Omega^\infty MTO(2)$, so any family that is algebraically independent in $H^*(QS^0)$ is algebraically independent in $H^*(\Omega^\infty MTO(2); \mathbb{Z}/2)$. Thus we recover [10] Theorem C. As a matter of fact, we could recover [10] Theorem C even more simply. The surface bundle $\mathbb{R}P^2 \times S^0 \rightarrow S^0$ corresponds to a map $S^0 \rightarrow MTO(2)$ which is a section to the composition $MTO(2) \rightarrow BO(2)_+ \rightarrow S^0$ thus $S^0$ splits off as we have seen, this is enough to show that the family survives.

We restate the results for MTG spectra.

**Theorem 4.1.** Let $G$ be $O$, $SO$, $U$ or $Sp$. The composition

$$i \circ \text{collapse} \circ \omega : MTG(2n) \rightarrow BG(2n)_+ \rightarrow S^0 \rightarrow KO$$

induces an injection in mod 2 cohomology of infinite loop spaces

$$H^*(\mathbb{Z} \times BO; \mathbb{Z}/2) \hookrightarrow H^*(\Omega^\infty MTG(2n); \mathbb{Z}/2).$$

Thus if we define the characteristic class $\xi_i$ for manifold bundle $F \rightarrow E \xrightarrow{\pi} B$ by $\xi_i(E) = w_i(tr f^*_\pi(1))$ where $tr f$ is the transfer in KO-cohomology, then we have

$$\mathbb{Z}/2[\xi_1, \ldots, \xi_k, \ldots] \subset H^*(\Omega^\infty MTG(2n); \mathbb{Z}/2).$$

Note that in the case of $MTO(2)$, our $\xi_i$ is conjugate to $\chi_i$ defined in [10]. We also note that $tr f^*_\pi(1)$ is the virtual bundle defined by $\Sigma(-1)^i[H^*(F_i, \mathbb{R})]$ by [2] Theorem 6.1.

To prove the first part of Theorem, it suffices to note that $H_*(BO(1); \mathbb{Z}/2)$ generates $H_*(BO; \mathbb{Z}/2)$ as an algebra, and that the elements of $H_*(BO(1); \mathbb{Z}/2) \subset H_*(BO; \mathbb{Z}/2)$ is obtained by applying the Dyer-Lashof operations on the bottom class, thus as an algebra over $H_*(BO; \mathbb{Z}/2)$ is monogenic, which implies that the map $H_*(QS^0; \mathbb{Z}/2) \rightarrow H_*(BO \times Z; \mathbb{Z}/2)$ is surjective. Since the composition $QS^0 \rightarrow \Omega^\infty MTG(2n) \rightarrow QS^0$ is just the identity, we conclude that $H_*(\Omega^\infty MTG(2n); \mathbb{Z}/2)$ surjects to $H_*(QS^0; \mathbb{Z}/2)$. By dualizing we get the desired result. The second part follows from the first easily.

At prime 2, noting that $H^*(BU; \mathbb{Z}/2)$ injects to $H^*(BO; \mathbb{Z}/2)$, we can define a similar polynomial family in $H^*(\Omega^\infty MTG(2n); \mathbb{Z}/2)$ by using Chern classes instead of Stiefl-Whitney classes. Of course, this is of limited interest as the Chern classes are simply the square of Stiefl-Whitney classes. At odd primes, however, the use of Chern classes seems to be appropriate. That is, we have

**Theorem 4.2.** Let $G = O$, $SO$, $U$ or $Sp$. If $G = O$ or $SO$, suppose that $n$ is even. If $G = U$ or $Sp$, suppose that $p \nmid n + 1$. Then after localization at $p$, the composition

$$MTG(n) \rightarrow BG(n)_+ \rightarrow S^0 \rightarrow KU \rightarrow E(1)$$

induces an injection in mod $p$ cohomology of infinite loop spaces

$$H^*(\Omega^\infty E(1); \mathbb{Z}/p) \hookrightarrow H^*(\Omega^\infty MTG(n); \mathbb{Z}/p).$$

Thus if we define the characteristic class $\xi_i$ for manifold bundle $F \rightarrow E \xrightarrow{\pi} B$ by $\xi_i(E) = c_i(tr f^*_\pi(1))$ where $tr f$ is the transfer in KU-cohomology, then we have $\mathbb{Z}/p[\xi_1, \ldots, \xi_k, \ldots, (p-1) \epsilon k] \subset H^*(\Omega^\infty MTG(n); \mathbb{Z}/p)$. 

Before proceeding to the proof, note that the image of the map $H_*(QS^0; \mathbb{Z}/p) \to H_*(BU; \mathbb{Z}/p)$ is concentrated in degrees multiple of $2(p - 1)$, so this can’t be surjective, unless $p = 2$. Now, recall that $BU$ is a retract of $QCP^\infty$ [11]. As is shown in [7], the stable decomposition at $p$ of $QCP^\infty$ is compatible with that of $BU$ in Adams’ summands, and we have a spectrum $X$ such that $\tilde{H}^*(X) \cong \tilde{H}^*(\Sigma_p^\infty; \mathbb{Z}/p)$ if $*$ is a multiple of $2(p - 1)$ and $\tilde{H}^*(X; \mathbb{Z}/p) \cong 0$ otherwise, and that $\Omega^\infty E(1)$ is a retract of $\Omega^\infty X$. Now, it is easy to see that the homology of $B\Sigma_p$ maps surjectively to that of $H_*(X; \mathbb{Z}/p)$, which implies that the homology of $\Omega^\infty X$ is monogenic as an algebra over the Dyer-Lashof algebra. As $\Omega^\infty E(1)$ is a retract of $\Omega^\infty X$ we get the surjectivity of the induced map in homology. By dualizing, we get the first statement of Theorem. The second part follows as $H^*(\Omega^\infty_0 E(1); \mathbb{Z}/p) \cong \mathbb{Z}/p[c_1, \ldots, c_k|p - 1 \not| k].$

5 The universal characteristic classes

In [10], there is another family of characteristic classes considered. That is

Definition 5.1. A universal characteristic class is an element in the image of the map

$$H^*(BG(m); \mathbb{Z}/p) \to H^*(Q_0(BG(m)_+); \mathbb{Z}/p) \to H^*(\Omega^\infty_0 MTG(m); \mathbb{Z}/p).$$

Now consider the case where $G = O$, $m = 2n$, $p = 2$. The map $H^*(BSO(2n + 1); \mathbb{Z}/2) \to H^*(\Omega^\infty_0 MTO(2n); \mathbb{Z}/2)$ is injective, and although it is not a ring map, its natural right inverse is. Thus the universal characteristic classes that are images of the standard polynomial generators of $H^*(BSO(2n + 1); \mathbb{Z}/2)$ are algebraically independent. But of course the standard polynomial generators of $H^*(BSO(2n + 1); \mathbb{Z}/2)$ do not map to polynomial generators of $H^*(BO(2n); \mathbb{Z}/2)$, which makes things a little bit complicated.

For example, let’s take the case of $\Omega^\infty_0 MTO(2)$, $p = 2$. Then we have $H^*(BSO(3); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3]$, $H^*(BO(2); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, w_2]$, and the map $BO(2) \to BSO(3)$ induces a map $w_2 \mapsto w_2 + w_3^2$, $w_3 \mapsto w_1 w_2$. Although this doesn’t lead to a strong result concerning the universal characteristic classes, we still can derive from this the classes $\mu_{1,1}$ and $\mu_{0,1} + \mu_{1,0}^2$ as defined in [11] are algebraically independent.

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