WEAK-$L^p$ BOUNDS FOR THE CARLESON AND WALSH-CARLESON OPERATORS

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ABSTRACT. We prove a weak-$L^p$ bound for the Walsh-Carleson operator for $p$ near 1, improving on a theorem of Sjölin [16]. We relate our result to the conjectures that the Walsh-Fourier and Fourier series of a function $f \in L \log L(T)$ converge for almost every $x \in T$.

1. MOTIVATION AND MAIN RESULT

The $L^p(T)$, $1 < p < \infty$ boundedness of the Carleson maximal operator

$$Cf(x) = \sup_{n \in \mathbb{N}} \left| \text{p.v.} \int_T f(x - t) e^{2\pi int} \frac{dt}{t} \right|, \quad x \in T,$$

first proved in [3, 9], entails as a consequence the almost everywhere convergence of the sequence $S_n f$ of partial Fourier sums for each $f \in L^p(T)$. A natural question (posed for instance by Konyagin in [11]) is whether, given an Orlicz function $\Phi(t)$ such that $L^1(T) \subsetneq L^\Phi(T) \subsetneq L^p(T)$ for all $p > 1$, it is true that

$$\|Cf\|_{1, \infty} \leq c \|f\|_{L^\Phi(T)},$$

so that (equivalently) $S_n f$ converges almost everywhere to $f$ whenever $f \in L^\Phi(T)$. It is a result of Antonov [1] that (1) holds true for $\Phi(t) = t \log(e + t) \log\log(e^{e^t} + t)$. Antonov’s proof makes use of an approximation technique relying on the smoothness of the Dirichlet kernels to upgrade the restricted weak-type estimate of Hunt [9]

$$\|C1_E\|_{p, \infty} \leq c \frac{p^2}{p - 1} |E|^\frac{1}{p} \quad \forall E \subset T, \quad \forall 1 < p < \infty,$$

(2)

to the mixed bound

$$\|Cf\|_{1, \infty} \leq c \|f\|_1 \log \left( e + \frac{\|f\|_{1, \infty}}{\|f\|_1} \right),$$

(3)

which, in turn, yields that $C : L \log L \log\log L(T) \to L^{1, \infty}(T)$, in view of the log-convexity of the latter space. We remark that a larger quasi-Banach rearrangement invariant space QA such that $C : QA \to L^{1, \infty}(T)$ holds was found in [2]; in [4] it is shown that, however, Antonov’s space is the largest (in a suitable sense) Orlicz space $L^\Phi(T)$ such that the embedding $L^\Phi(T) \hookrightarrow QA$ holds. We further note that the results of [1, 2] have been reproved by Lie [12], where (3) is obtained directly, without the use of approximation techniques.

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The above mentioned results strongly suggest that (1) holds for the space $L \log L(\mathbb{T})$ as well. If the "$L \log L$ conjecture" were true, a consequence would be the unrestricted version of Hunt's estimate (2):

$$\|Cf\|_{p,\infty} \leq \frac{c}{p-1} \|f\|_p, \quad \forall \, 1 < p \leq 2. \tag{4}$$

On the other hand, a suitable choice of $p \in (1, 2)$ in (4) yields (3) directly, and in turn, recovers (1) for Antonov's $\Phi$; thus, the weak-$L^p$ estimate (4) arises naturally as an intermediate result between the conjectured $L \log L(\mathbb{T})$ bound in (1) and the presently known best Orlicz space bound. That the $L \log L$ conjecture implies (4) is a particular case of the following observation, due to Andrei Lerner (personal communication). Assuming (1) holds for a given $\Phi$, one has the pointwise inequality $M^p((Cf)^\sharp) \leq (M \Phi f)^\sharp$, the latter being the local Orlicz maximal function associated to $\Phi$ [8, Proposition 5.2]. It follows that

$$\|Cf\|_{p,\infty} \leq c\left(\left(\sup_{t \geq 1} \frac{\Phi(t)}{t^p}\right)^{\frac{1}{p}}\right) \|f\|_p, \quad \forall \, 1 < p \leq 2. \tag{5}$$

Using Antonov's $\Phi(t) = t \log(e + t) \log \log(e^e + t)$ in (5) leads to

$$\|Cf\|_{p,\infty} \leq \frac{c}{p-1} \log \log \left(e^e + \frac{1}{p-1}\right) \|f\|_p, \quad \forall \, 1 < p \leq 2; \tag{6}$$

to the best of the author's knowledge, there seems to be no better weak-$L^p$ bound than (6) in the current literature, and in particular the validity of (4), which can be thought of as a weakening of the $L \log L$ conjecture, is open.

The main new result of this article is that the analogue of (4) actually holds for the Walsh-Fourier analogue of the Carleson operator, which is often thought of as a discrete model of the Fourier case: see [19, Chapter 8] for the relevant definitions.

**Theorem 1.1.** Denote by $W_n f(x)$ the $n$-th partial Walsh-Fourier sum of $f \in L^1(\mathbb{T})$. There exists an absolute constant $c > 0$ such that the Walsh-Carleson maximal operator

$$Wf(x) := \sup_{n \in \mathbb{N}} |W_n f(x)|, \quad x \in \mathbb{T}$$

satisfies the operator norm bound

$$\|W\|_{L^p(\mathbb{T}) \rightarrow L^{p,\infty}(\mathbb{T})} \leq \frac{c}{p-1}, \quad \forall \, 1 < p \leq 2. \tag{7}$$

**Remark (Previous results and sharpness).** Theorem 1.1 is a strengthening of the Walsh analogue of (2), obtained by Sjölin in [16], and recovers the correspondent version of (3), first established in [17], without the need for approximation techniques developed therein. The bound $W : L \log L \log \log \log L(\mathbb{T}) \rightarrow L^{1,\infty}(\mathbb{T})$, which is the Walsh case of Antonov's result, follows as a further consequence. Furthermore, if we assume that the Walsh case of the $L \log L$ conjecture is sharp, in the sense that there exists no Young function $\Phi$ with $W : L^\Phi(\mathbb{T}) \rightarrow L^{1,\infty}(\mathbb{T})$ and such that $\limsup_{t \rightarrow \infty} (t \log(e + t)^{-1} \Phi(t)) = 0$, then the bound (7) is sharp, up to a doubly logarithmic term in $(p-1)^{-1}$; see [6, Section 2] for details.

**Remark (The $L \log L$ conjecture and weak-$L^p$ bounds for the lacunary Carleson operator).** It is conjectured in [11, Conjecture 3.2] that the subsequence $S_{nj} f$ of the partial Fourier sums of $f \in L \log \log L(\mathbb{T})$ converges almost everywhere whenever $n_j$ is a lacunary sequence of
integers, in the sense that \( n_{j+1} \geq \theta n_j \) for all \( j \) and for some \( \theta > 1 \); if true, this result would be sharp. This is equivalent to the conjecture that the \textit{lacunary} Carleson maximal operator

\[
C_{\{n_j\}}f(x) = \sup_{j \in \mathbb{N}} \left| \text{p.v.} \int_{\mathbb{T}} f(x-t) e^{2\pi i n_j t} \frac{dt}{t} \right|, \quad x \in \mathbb{T},
\]

satisfies

\[
\|C_{\{n_j\}}f\|_{L^p}\mathcal{L}(\mathbb{T})} \leq c \|f\|_{L^p(\mathbb{T})}, \quad \forall \|f\|_{L^p(\mathbb{T})},
\]

for \( \Phi(t) = t \log\log(e^t + t) \), with constant \( c > 0 \) depending only on the lacunarity constant \( \theta \) of the sequence \( \{n_j\} \). By (5), if the above conjectured bound held true, the weak-\( L^p \) estimate

\[
\|C_{\{n_j\}}f\|_{L^p(\mathbb{T})} \leq c \log(e + (p - 1)^{-1}) \|f\|_{L^p(\mathbb{T})}, \quad \forall 1 < p \leq 2
\]

would follow. The current best result [6, 13] is that (8) holds with

\[
\Phi(t) = t \log\log(e^t + t) \log\log\log(e^t + t).
\]

However, we remark that the argument for the main theorem in [13] can be suitably reformulated to prove the stronger (9) in place of the main result therein (which is an estimate of the same type as (3), with a log log in place of the log). Therefore, the weaker form of Konyagin’s \( L \log\log L \) conjecture given by (9) holds true. Finally, we mention that the Walsh analogue of (9) is explicitly proved in [6].

We give the proof of Theorem 1.1 in the upcoming Section 2. For the convenience of the reader, we provide an appendix, claiming no originality, containing a step-by-step account of the changes needed in the argument for the main theorem of [13] to obtain the weak-\( L^p \) bound (9) for the lacunary Carleson operator.

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### 2. Proof of Theorem 1.1

As usual, we will prove (7) by relying on the (Walsh) phase plane model sums (see for instance [18, 19]). We remark that the main technical tool which is not present in the classical works we mentioned is a discrete variant of the multi-frequency Calderón-Zygmund decomposition of [14] (Lemma 2.2 below). Similar arguments have already found ample use in the treatment of discrete modulation-invariant singular integrals [15, 7, 5, 6].

Let \( \mathcal{D} \) be the standard dyadic grid on \( \mathbb{R}_+ \); below, we indicate with \( S \) an (arbitrary) finite collection of \textit{bitiles}, that is rectangles \( s = I_s \times \omega_s \subset \mathcal{D} \times \mathcal{D} \) with \( |\omega_s| = 2|I_s|^{-1} \). Denoting by \( \omega_{s_1}, \omega_{s_2} \), respectively, the left and right dyadic child of \( \omega_s \), each bitile \( s \) is thought of as the union of the two \textit{tiles} (dyadic rectangles of area 1) \( s_1 = I_s \times \omega_{s_1}, s_2 = I_s \times \omega_{s_2} \). Writing \( W_n \) for the \( n \)-th Walsh character on \( \mathbb{T} \), the Walsh wave packet time-frequency adapted to a tile \( t = I_t \times \omega_t \) is then defined as

\[
w_t(x) = \text{DiL}_{|I_t|}^2 \text{Tr}_{L} W_n(x) = |I_t|^{-1/2} W_n\left(\frac{x - \inf I_t}{|I_t|}\right), \quad n_t := |I_t|\inf \omega_t.
\]

The model sums for the Walsh-Carleson maximal operator \( W \) are then given by

\[
W_{S}f(x) = \sum_{s \in S} \varepsilon_s \langle f, w_{s_1} \rangle w_{s_1}(x) 1_{\omega_{s_2}}(N(x)),
\]
where \( N : \mathbb{R}^+ \to \mathbb{R}^+ \) is an (arbitrary) measurable choice function, and \( \{ \varepsilon_j \} \in \{-1,0,1\}^S \). By the reduction given in e.g. [18, 19], Theorem 1.1 is a consequence of the bound (\( p' \) is the Hölder dual of \( p \))

\[
\| W_s f \|_{p,\infty} \lesssim p' \| f \|_p, \quad \forall 1 < p \leq 2;
\]

in (10) and in what follows, the constants implied by the almost inequality signs are meant to be absolute (in particular, independent on \( S, N \) and \( \{ \varepsilon_j \} \)) and may vary at each occurrence. Observe that (10) is recovered by taking \( G = \{ |W_s f| > \lambda \}, g(x) = 1_{G'}(x) \exp(-i \arg(W_s f(x))) \) in the bound

\[
|\langle W_s f, g \rangle| \lesssim p' \| f \|_p |G|^{\frac{1}{p'}}, \quad \forall |g| \leq 1_{G'},
\]

where \( G' \subset G \) is a suitably chosen (possibly depending on \( f \)) major subset of \( G \): that is, \( |G| \leq 4|G'| \). By (dyadic) scale-invariance of the family of operators \( \{ W_S \} \) over all choices of \( S \subset \mathcal{D} \times \mathcal{D} \) and measurable functions \( N \), and by linearity in \( f \), it suffices to prove (11) in the case \( \| f \|_p = 1, 1 \leq |G| < 4 \), to which we turn in Subsection 2.2. In the upcoming Subsection 2.1, we recall some tools of discrete time-frequency analysis.

### 2.1. Trees, size and density.

We will use the well-known Fefferman order relation on either tiles or bitiles: \( s \ll s' \) if \( I_s \subset I_{s'} \) and \( \omega_s \supset \omega_{s'} \). We say that \( S \) is a convex collection of bitiles if \( s, s' \in S, s \ll s'' \ll s' \) implies \( s'' \in S \). It is no restriction to prove (10) under the further assumption that \( S \) is convex, and we do so. A convex collection of bitiles \( T \subset S \) is called tree with top bitile \( s_T \) if \( s \ll s_T \) for all \( s \in T \). We simplify notation and write \( I_T := I_{s_T}, \omega_T = \omega_{s_T} \). We will call forest a collection of (convex) trees \( T \in \mathcal{F} \), and will make use of the quantity

\[
tops(\mathcal{F}) := \sum_{T \in \mathcal{F}} |I_T|.
\]

Given a measurable function \( N : \mathbb{R} \to \mathbb{R} \) and \( G \subset \mathbb{R} \), define

\[
dense_G(S) = \sup_{s \in S} \sup_{s' \in S, s \ll s'} \frac{|G \cap I_{s'} \cap N^{-1}(\omega_{s'})|}{|I_{s'}|}.
\]

Furthermore, for \( f \in L^2(\mathbb{T}) \), we set

\[
\size_f(S) = \sup \max_{s \in S, j=1,2} \frac{|\langle f, w_s \rangle|}{|I_s|^\frac{1}{2}}.
\]

We observe that \( \size, \dense \) are monotone increasing with respect to set inclusion. One has \( \dense_G(S) \leq 1 \) for each \( G \subset \mathbb{R} \), and it is immediate to see that

\[
\size_f(S) \leq \inf_{s \in S} \sup_{x \in I_s} M_p f(x).
\]

where \( M_p, 1 \leq p < \infty \), denotes the (dyadic) \( p \)-th Hardy-Littlewood maximal function. Finally, we recall verbatim a result from [7] (Lemma 2.13 therein).

**Lemma 2.1.** Let \( h \in L^2(\mathbb{R}) \) and \( \mathcal{F} \) be a forest with \( \dense_G(\mathcal{F}) \leq \delta \), \( \tops(\mathcal{F}_G) \leq \delta^{-1}|G| \). Then for all \( g : \mathbb{R} \to \mathbb{C}, |g| \leq 1 \_G \),

\[
|\langle W_s h, g \rangle| \lesssim \min \left\{ \size_f(\mathcal{F}) |G|, \delta^\frac{1}{2} \sqrt{|G|||h||_2} \right\}.
\]
2.2. **Proof of** (11). Recall that we are assuming \( \|f\|_p = 1, 1 \leq |G| < 4. \) For an appropriate (absolute) choice of \( c > 0, \)
\[
|E := \{M_pf \geq c\}| \lesssim c^{-p}\|M_pf\|_p^p \lesssim \frac{1}{4}.
\]
Set \( G' := G \setminus E; \) by the above, \( |G'| \geq \frac{1}{2}, \) so that \( G' \) is a major subset of \( G. \) Since \( w_{s_1}(x)1_{\omega_{s_1}}(N(x)) \) is supported inside \( I_s, \) we have that \( \langle w_{s_1}, g \rangle = 0 \) when \( |g| \leq 1_{G'} \) and \( I_s \cap G' = \emptyset. \) This means that
\[
\langle W_s f, g \rangle = \langle W_{S_{good}} f, g \rangle, \quad S_{good} := \{s \in S : I_s \cap E^c \neq \emptyset\}.
\]
Therefore, from now on, we will just replace \( S \) by \( S_{good} \) in (11). Note that, as a consequence of (12) and of the definition of \( S_{good}, \) we have size\(_f(S_{good}) \lesssim 1. \)

The next step is to apply the density decomposition lemma (for instance, [7, Lemma 2.6]) to \( S_{good}, \) writing
\[
S_{good} = \bigcup_{\delta \in 2^{-N}} \mathcal{F}_\delta, \quad \text{size}_f(\mathcal{F}_\delta) \lesssim 1, \quad \text{dense}_G(\mathcal{F}_\delta) \leq \delta, \quad \text{tops}(\mathcal{F}_\delta) \lesssim \delta^{-1}|G|.
\]
We claim the single forest estimate
\[
|\langle W_{\mathcal{F}_\delta} f, g \rangle| \lesssim \delta^{-\frac{1}{2}}.
\]
Assuming (16) holds true,
\[
|\langle W_{S_{good}} f, g \rangle| \lesssim \sum_{\delta \in 2^{-N}} |\langle W_{\mathcal{F}_\delta} f, g \rangle| \lesssim \sum_{\delta \in 2^{-N}} \delta^{-\frac{1}{p'}} \lesssim p',
\]
that is, we have proved (11). The remainder of the section is then devoted to the proof of the single forest estimate (16). The key tool is provided by the Lemma below.

**Lemma 2.2.** For each \( \delta \in 2^{-N}, \) there is a function \( h_\delta \) such that
\[
\|h_\delta\|_2 \lesssim \delta^{-\frac{1}{2} + \frac{1}{p'}}, \quad \langle f, w_{s_1} \rangle = \langle h_\delta, w_{s_1} \rangle \quad \forall s \in \mathcal{F}_\delta.
\]

In particular, we see from Lemma 2.2 that \( \langle W_{\mathcal{F}_\delta} f, g \rangle = \langle W_{\mathcal{F}_\delta} h_\delta, g \rangle \) and that \( \text{size}_f(\mathcal{F}_\delta) = \text{size}_f(\mathcal{F}_\delta) \lesssim 1; \) therefore, we may use Lemma 2.1 to bound
\[
|\langle W_{\mathcal{F}_\delta} f, g \rangle| = |\langle W_{\mathcal{F}_\delta} h_\delta, g \rangle| \lesssim \delta^{\frac{1}{2}} |G|^\frac{1}{2} \|h_\delta\|_2 \lesssim \delta^{\frac{1}{2}},
\]
which is (16). We have thus completed the proof of Theorem 1.1, up to showing Lemma 2.2 holds true.

2.3. **Proof of Lemma 2.2.** This argument is analogous to [5, Lemma 5.1]. We argue under the additional assumption that \( f \) is supported on \( E = \{M_pf \geq c\}; \) the general case requires only trivial modifications. Let \( I \in I_1 \) be the maximal dyadic intervals of \( E; \) for each \( I \in I_1, \) let \( t \in T_I \) be the collection of all tiles having \( I_t = I \) and which are comparable under \( \ll \) to some tile \( s_1 \in \mathcal{F}_\delta. \) The tiles of \( T_I \) are obviously pairwise disjoint.

The definition of \( S_{good} \) ensures that, whenever \( I_t \cap I \neq \emptyset \) for some \( s \in S_{good} \) and \( I \in I, \) the inclusion \( I \subset I_t \) must hold. It follows that if \( t \in T_I, s_1 \in \{s_1 : s \in T \in \mathcal{F}_\delta\} \) are related under \( \ll, \) then \( t \ll s_1. \) By standard properties of the Walsh wave packets, \( w_{s_1} \) is a scalar multiple of \( w_t \) on \( I; \) in particular, \( w_{s_1} 1_I \) belongs to \( H_I, \) the subspace of \( L^2(I) \) spanned by \( \{w_t : t \in T_I\}. \) A
further consequence is that, if \( N_I \) is the number of trees \( T \in \mathcal{F}_\delta \) with \( I \subset I_T \), we have \( \#T_I \leq N_I \). For \( v \in H_I \), we have the inequality
\[
\|v\|_{L^p(I)} \lesssim N_I^{\frac{1}{p} - \frac{1}{p'}} \|v\|_{L^q(I)}.
\]
Since \( \|f\|_{L^p(I)} \lesssim 1 \) by maximality of \( I \) in \( E \), it then follows that
\[
(f, v)_{L^2(I)} \leq \|f\|_{L^p(I)} \|v\|_{L^{p'}(I)} \lesssim N_I^{\frac{1}{p} - \frac{1}{p'}} \|v\|_{L^2(I)} \quad \forall v \in H_I,
\]
and consequently \( h_I \), the projection of \( fI \) on \( H_I \), satisfies \( \|h_I\|_{L^2(I)} \lesssim N_I^{\frac{1}{p} - \frac{1}{p'}} \). Setting \( h_\delta := \sum_{I \in \mathcal{F}_\delta} h_I \), we see that
\[
\|h_\delta\|_2^2 = \sum_{I \in \mathcal{F}_\delta} \|h_I\|_{L^2(I)}^2 \lesssim \sum_{I \in \mathcal{F}_\delta} |I| N_I^{\frac{1}{p} - \frac{1}{p'}} \lesssim \left( \sum_{I \in \mathcal{F}_\delta} 1_{|I|} \right)^{\frac{1}{p'}} \left( \sum_{I \in \mathcal{F}_\delta} |I| \right)^{\frac{1}{p}} \lesssim \delta^{-1} \delta^\theta;
\]
in the last step, we made use of the bound on tops from (15), and of (13) to estimate the sum over \( I \). Finally, in view of the above discussion, if \( s \in T \in \mathcal{F}_\delta \)
\[
\langle f, w_{s_1} \rangle = \sum_{I \in \mathcal{F}_\delta} \langle fI, w_{s_1} \rangle = \sum_{I \in \mathcal{F}_\delta} \langle fI, c w_{t(s_1;I)} \rangle = \sum_{I \in \mathcal{F}_\delta} \langle h_I, w_{s_1} \rangle = \langle h_\delta, w_{s_1} \rangle
\]
where \( t(s_1;I) \) is the unique (if any) element \( t \) of \( T_I \) with \( t \ll s_1 \). This finishes the proof of the lemma.

**Appendix A. Proof of the weak-\( L^p \) bound (9)**

This appendix is a re-elaboration of the proof of the main theorem of [13], whose content is that the maximal operator \( C_{\{n_I\}} \) associated to the \( \theta \)-lacunary sequence \( \{n_I\} \) satisfies
\[
\|C_{\{n_I\}}\|_{L^1 \to L^\infty} \lesssim \|f\|_1 \log \log \left( e + \frac{\|f\|_\infty}{\|f\|_1} \right).
\]
Our aim is to prove the stronger bound (9), that is
\[
\|C_{\{n_I\}}f\|_{p, \infty} \leq c \log(e + (p - 1)^{-1})\|f\|_p, \quad \forall 1 < p \leq 2;
\]
the \( p = 2 \) case can be obtained by standard Littlewood-Paley theory techniques, so that, by interpolation, it suffices to argue for \( 1 < p < 4/3 \) (say).

We claim no originality, essentially following step by step the proof in [13], the only difference being that our \( (f, \lambda) \) decomposition (in the terminology of [13, Subsection 3.2]) is based on \( M_p \) rather than on \( M_1 \), reflecting the assumption \( f \in L^p(\mathbb{T}) \). This allows for the use of (the dual of) Zygmund’s inequality in the form

\[
(A.1) \quad \left( \sum_{k \geq 1} \left| \int_{\mathbb{T}} f(x)e^{-i\xi_k x} \, dx \right|^2 \right)^{\frac{1}{2}} \lesssim p' \|f\|_{L^p(\mathbb{T})}
\]
for each \( \theta \)-lacunary sequence \( \xi_k \) (not necessarily of integers) with \( \xi_1 \geq 4\theta^{-1} \), and \( 1 < p \leq 2 \), with implicit constant independent on all but \( \theta \). The statement given in (A.1) above can be found e.g. in [10]; we note that, although the proof in [10] can be modified so that \( \sqrt{p'} \) appears in place of \( p' \), this is not allowing for any essential improvement in the result we are aiming for.

\footnote{Here \( p' = p/(p - 1) \in (2, \infty) \) is the Hölder dual of \( p \).}
A.1. Discretization. Let $\mathcal{D}$ be the standard dyadic grid on $\mathbb{R}$ and $\mathcal{D}_T$ be its restriction to $T$; we indicate by $S$ the collection of tiles $s = I_s \times \omega_s \subset \mathcal{D}_T \times \mathcal{D}$ with $|I_s| = |\omega_s|^{-1}$. For a given measurable $N$ function on $T$ with range contained in $\{n_j\}$, and $s \in S$, set $E(s) := \{x \in I_s : N(x) \in \omega_s\}$. Further, let $\psi$ be a smooth function supported on $[2,8]$ and such that
\[
\sum_{k \geq 0} 2^k \psi(2^k x) = \frac{1}{x}, \quad \forall x \in T \setminus \{0\},
\]
and define
\[
T_s f(x) = \left( \int_T e^{-iN(x)p} |I_s|^{-1} \psi(|I_s|^{-1}(x - t)) f(t) \, dt \right) 1_{E(s)}(x).
\]
Then, for a suitable choice of $N$ as above,
\[
|C_{n_j} f(x)| \leq 2 \left| \sum_{s \in S} T_s f(x) \right|, \quad x \in T;
\]
to prove (9), it will thus suffice to show that for all measurable $N$ with lacunary range $\{n_j\}$, each $1 < p \leq \frac{4}{3}$, $f \in L^p(T)$ of unit norm and all $G \subset T$ there exists a subset $G' \subset G$ with $|G| \leq 4 |G'|$ such that
\[
(A.2) \quad \left| \left\langle \sum_{s \in S} T_s f, g \right\rangle \right| \lesssim \log(e + p') |G|^{\frac{1}{p'}} \quad \forall |g| \leq 1_{G'}
\]
with implicit constant independent of all but (possibly) the lacunarity constant $\theta$.

Observe that $T_s$ is supported on $E(s) \subset I_s$, and $T_s^*$ is supported on $17I_s \setminus 3I_s$; by further cutting (smoothly) $\psi$ into 32 pieces, we can assume that $T_s^*$ is supported on the interval $I_s^* = I_s + j|I_s|/4$ for some fixed integer $j \in (-40, -8) \cup [8,40)$. Furthermore, there is no loss in generality by working with $j = 8$, and we do so, so that $I_s^* = I_s + 2|I_s|$ from now on.

A.2. Proof of (A.2): main reductions. Let now $f \in L^p(T)$ of unit norm and $G \subset T$ be given. The first step in the proof of (A.2) is the definition of the major set $G'$ as
\[
(A.3) \quad G' := G \setminus 1000 \{ M_p f(x) \geq c |G|^{-\frac{1}{p'}} \};
\]
one obtains that $|G| \leq 4 |G'|$ by using the weak-$L^p$ boundedness of $M_p$ and suitably choosing an absolute constant $c > 0$.

The next task is to decompose the collection of tiles $S$ (roughly) according to the local $L^p$-norm of $f$ on $I_s$. Define for each $k \geq 0$,
\[
\mathcal{F}_k := \{ M_p f(x) \geq c 2^{-k} |G|^{-\frac{1}{p'}} \} = \bigcup_{I \in \mathcal{F}_k} I
\]
$I \in \mathcal{F}_k$ being the maximal dyadic intervals of $\mathcal{F}_k$; we note that, by the maximal theorem and by maximality of $I$
\[
(A.4) \quad |\mathcal{F}_k| \lesssim 2^{kp} |G|, \quad \|f\|_{L^p(I)} \lesssim 2^{-k} |G|^{-\frac{1}{p'}};
\]
the notation $L^p(I)$ stands for $L^p(I; dx/|I|)$. We partition the tiles of $S$ by making use of the subcollections
\[
S_{k, \omega} := \{ s_0 = I_{s_0} \times \omega_{s_0} : I_{s_0} \in \mathcal{F}_k, 0 \in 2\omega_{s_0} \}, \quad k \geq 0
\]
as follows:

$$
S = S_{\text{clust}} \cup \left( \bigcup_{k \geq 0} S(k) \right) \cup \left( \bigcup_{k \geq 0} \bigcup_{s \in S(k)} S_{k,1}(s_o) \right) \cup \left( \bigcup_{k \geq 0} \bigcup_{s \in S_{k,2}(s_o)} S_{k,2}(s_o) \right),
$$

where

$$
S_{\text{clust}} := \{ s : 10 \theta \omega_s \not\in \emptyset \};
$$

$$
S(0) := \{ s \not\in S_{\text{clust}} : I_s^* \subset \bar{F}_0 \},
$$

$$
S(k) := \{ s \not\in S_{\text{clust}} : I_s^* \subset \bar{F}_k, I_s^* \cap \bar{F}_{k-1} = \emptyset \},
$$

$$
S_{k,1}(s_o) := \{ s \not\in S_{\text{clust}} : I_s^* \supset I_{s_o}, 2 \omega_s \cap 2 \omega_{s_o} \not\in \emptyset, \text{ either } I_s^* \cap \bar{F}_{k+1} = \emptyset \text{ or } I_s^* \subset \bar{F}_{k+1} \},
$$

$$
S_{k,2}(s_o) := \{ s \not\in S_{\text{clust}} : I_s^* \supset I_{s_o}, 2 \omega_s \cap 2 \omega_{s_o} = \emptyset, \text{ either } I_s^* \cap \bar{F}_{k+1} = \emptyset \text{ or } I_s^* \subset \bar{F}_{k+1} \}.
$$

With the above decomposition in hand, (A.2) will follow by combining the bounds of the following proposition. Note that the choice $p \leq \frac{4}{3}$ guarantees that the summation index exponent $(\frac{p}{2} - 1)$ appearing in (A.7) below (as well as in the sequel) is uniformly bounded away from zero.

**Proposition A.1.** Let $g$ be a subindicator function supported on $G'$ defined above. We have the estimates

\begin{align}
(A.5) & \quad \left| \left( \sum_{s \in S_{\text{clust}}} T_s f, g \right) \right| \lesssim |G|^{\frac{1}{p}}, \\
(A.6) & \quad \left( \sum_{s \in S(0)} T_s f, g \right) = 0, \\
(A.7) & \quad \left| \left( \sum_{s \in S(k)} T_s f, g \right) \right| \lesssim 2^{(\frac{p}{2} - 1)k} |G|^{|s|}, \quad k \geq 1, \\
(A.8) & \quad \left| \left( \sum_{k \geq 0} \sum_{s \in S_{k,1}(s_o)} \sum_{s \in S_{k,2}(s_o)} T_s f, g \right) \right| \lesssim |G|^{\frac{1}{p}}, \\
(A.9) & \quad \left| \left( \sum_{k \geq 0} \sum_{s \in S_{k,1}(s_o)} \sum_{s \in S_{k,2}(s_o)} T_s f, g \right) \right| \lesssim \log(e + p') |G|^{\frac{1}{p}}.
\end{align}

**Proof of (A.5).** This estimate follows from the well-known fact that the operator $\sum_{s \in S_{\text{clust}}} T_s$, akin to a maximally truncated Hilbert transform, is weak-$L^p$ bounded with operator norm independent of $1 < p \leq 2$. \hfill \Box 

**Proof of (A.6).** Note that if $s \in S(0)$, $I_s$ (the support of $T_s f$) is contained in $30 \bar{F}_0$, while $g$ is supported away from $1000 \bar{F}_0$. \hfill \Box

**Proof of (A.7).** Here we use that the support of $\sum_{s \in S(k)} T_s^*$ is contained in $\bar{F}_k$, and that $\|f\|_{L^\infty(I)} \lesssim 2^{-k} |G|^{-\frac{1}{p}}$ whenever $I \cap \bar{F}_{k-1} = \emptyset$, so that

\[ \left| \left( \sum_{s \in S(k)} T_s f, g \right) \right| \lesssim 2^{-k} |G|^{-\frac{1}{p}} \left\| \sum_{s \in S(k)} T_s^* g \right\|_2 \lesssim 2^{(\frac{p}{2} - 1)k} |G|^{\frac{1}{p}}, \]

using (A.4) and the $L^2$-boundedness of $\sum_{s \in S(k)} T_s^*$, which is essentially an adjoint (discretized) Carleson operator. \hfill \Box

In the next subsection, we give the proof of (A.9), which is the only term for which (A.1) is needed. The proof of (A.8) requires only minimal modifications from the argument used for [13, Proposition 3]; we omit the details.
A.3. **Proof of (A.9).** We begin with some notation: we write $A_t(h)$ for the average of $h \in L^1(I)$ on $I$; for a tree $T$ (see [12] for the definition in this context), we write $(T)_0$ for the shift of $T$ to the zero frequency.

We perform a further decomposition of the sum in (A.9). To begin with, observe that for fixed $k, s_o \in S_k, o$ there exists a $\theta$-lacunary sequence $\{\xi_\ell = \xi_{i}(s_o): \ell \geq 1\}$ such that $S_{k,2}(s_o)$ can be organized into the union of maximal trees $T_\ell(s_o)$ each with top frequency $\xi_\ell$, and such that $\xi_1 \geq 4\theta^{-1}|I_{s_o}|^{-1}$ (this point is granted by the requirement $2\omega_s \cap 2\omega_s = \emptyset$ for each $s \in S_{k,2}(s_o)$).

This said, we have that

$$\sum_{k \geq 0, s_o \in S_k, o} \sum_{s \in S_{k,2}(s_o)} \sum_{\ell} T_\ell^s = \sum_{k \geq 0, s_o \in S_k, o} \sum_{\ell} T_{T_\ell(s_o)}^s, \quad T_{T_\ell(s_o)}^s := \sum_{s \in T_\ell(s_o)} T_\ell^s,$$

and we define

$$T_{T_\ell(s_o)}^s g(x) := e^{-i\xi_\ell x} A_t(T_{T_\ell(s_o)}^s)g(x), \quad R_{T_\ell(s_o)}^s g := T_{T_\ell(s_o)}^s g - T_{T_\ell(s_o)}^s g;$$

(our $T$ stands for the notation $T_\ell$ in [13]). With the above splitting in hand, the bound (A.9) is obtained by combining the two bounds of the proof below.

**Proposition A.2.** *We have the estimates*

(A.10) \[ \left| \left\langle f, \sum_{k \geq 0, s_o \in S_k, o} \sum_{\ell} R_{T_\ell(s_o)}^s g \right\rangle \right| \lesssim |G|^{p'}, \]

(A.11) \[ \left| \left\langle f, \sum_{k \geq 0, s_o \in S_k, o} \sum_{\ell} T_{T_\ell(s_o)}^s g \right\rangle \right| \lesssim \log(e + p') |G|^{p'}. \]

**Proof of (A.10).** Perusal (with minimal changes) of the proofs of [13, Lemmata 1 and 2].

**Proof of (A.11).** We perform here one last decomposition of our set of tiles $S$. Referring to the mass decomposition recalled in [13, Subsection 3.1] (see also [12, Section 5]), we write $S = \bigcup_{n \in \mathbb{N}} S^n$, with (in particular) $|E(s)| \leq 2^{-n}|I_s|$ for all $s \in S^n$. It is important to observe that the mass decomposition above is independent of $f$.

For each $k, s_o \in S_k, o$, we set $S_{k,2}^n(s_o) := S_{k,2}(s_o) \cap S^n$; as above, $S_{k,2}(s_o)$ can be partitioned into a union of maximal trees $T_\ell^s(s_o)$ each with top frequency $\xi_\ell$. The sequence $\{\xi_\ell\}$ we obtain is actually a subsequence of the lacunary sequence defined in the previous section: we avoid the subsequence notation for simplicity. Estimate (A.11) will be then obtained by summation over $n$ of the inequality

(A.12) \[ \left| \left\langle f, \sum_{k \geq 0, s_o \in S_k, o} \sum_{\ell} T_{T_\ell^s(s_o)}^s g \right\rangle \right| \lesssim \min\{1, 2^{-\frac{n}{2}} p'\} |G|^{\frac{1}{p'}}. \]

The left estimate in (A.12) follows by repeating the proof of part (b) of the main theorem in [12] (with minimal modifications). From now on, we devote ourselves to the proof of the right estimate in (A.12). Define the square function

$$S_{k,2}^n(s_o)(g)(x) := \left( \sum_{\ell} |T_{T_\ell^s(s_o)}^s(g)(x)|^2 \right)^{1/2}.$$

It is a consequence of the analysis carried out in [12] that

(A.13) \[ \left\| \sum_{s_o \in S_{k,2}^n} 1_{I_{s_o}} S_{k,2}^n(s_o)(g) \right\|_2 \lesssim 2^{-\frac{n}{2}} |G|^{\frac{1}{2}}. \]
In the same spirit of [13, Lemma 3], we have that, for \( s_0 \in S_{k,0} \), we have the estimate
\[
(A.14) \quad \left| \left< f, \sum_{\ell} T_{T_0}^s (s_0) g \right> \right| \lesssim p' 2^{-k} |G|^{-\frac{1}{p}} \|I_{s_0} \|_2 \|1_{I_{s_0}} S_{k_2}^u (s_0) (g)\|_2.
\]

To get (A.14), the only modification to the proof of [13, Lemma 3] which we need is bounding
\[
\frac{1}{|I_{s_0}|} \int_{I_{s_0}} f(y) e^{i \xi \cdot y} \, dy \right|^2 \lesssim p' \|f\|_{L^p(I_{s_0})} \lesssim p' 2^{-k} |G|^{-\frac{1}{p}}
\]
which is the scaled version of inequality (A.1). At this point, taking advantage of (A.14), subsequently making use of Cauchy-Schwarz together with disjointness in \( \{s_0 \in S_{k,0}\} \) of the supports of \( 1_{I_{s_0}} S_{k_2}^u (s_0) (g) \), following up with (A.13) and finally relying on (A.4) in the last step, we obtain
\[
\left| \left< f, \sum_{s_0 \in S_{k,0}} \sum_{\ell} T_{T_0}^s (s_0) g \right> \right| \lesssim p' 2^{-k} |G|^{-\frac{1}{p}} \sum_{s_0 \in S_{k,0}} \|I_{s_0} \|_2 \|1_{I_{s_0}} S_{k_2}^u (s_0) (g)\|_2
\]
\[
\leq p' 2^{-k} |G|^{-\frac{1}{p}} \left( \sum_{s_0 \in S_{k,0}} \|I_{s_0}\| \right)^{\frac{1}{2}} \|1_{I_{s_0}} S_{k_2}^u (s_0) (g)\|_2
\]
\[
\lesssim p' 2^{-k} 2^{-\frac{k}{2}} |G|^{\frac{1}{p}} \|f\|_{L^p} \lesssim p' 2^{-\frac{k}{2}} |G|^{\frac{1}{p}} 2^\frac{k}{2} (\frac{1}{p} - 1)^k.
\]
The right estimate in (A.12) finally follows by summing up over \( k \) the above display. \( \square \)

This concludes the proof of Proposition A.2, and in turn, of (A.9).

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