Abstract

In this paper we describe two new computational operators, called complex entropic form (CEF) and generalized complex entropic form (GEF), for pattern characterization of spatially extended systems. Besides of being a measure of regularity, both operators permit to quantify the degree of phase disorder associated with a given gradient field. An application of CEF and GEF to the analysis of the gradient pattern dynamics of a logistic Coupled Map Lattice is presented. Simulations using a Gaussian and random initial condition, provide interesting insights on the system gradual transition from order/symmetry to disorder/randomness.

Key words: complex entropic form; nonlinear coupled map lattices; gradient dynamics; phase disorder; pattern characterization

1 Introduction

Nonlinear spatially extended dynamical systems yield complex amplitude patterns which arise from the coupled dynamics of their different regions. Experiments in a variety of areas have shown spatio-temporal complexity, notably in fluid flows, diffusive-reactive chemical systems, optical electronics and laser physics. As reported by many authors (e.g. [1]), these systems cover a wide range of scales, from millimeters in optical electronics experiments to thousands of kilometers in natural systems. In all this cases it is important to quantify the degree of local complexity in order to characterize the spatial patterns and also study their time evolution. Consequently, one of the main

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tasks in computational physics today is to define suitable measures to characterize the dynamics of such complex systems. As stressed by Chatè [2], the current understanding of spatio-temporal disorder in extended systems is still very limited due to the lack of quantitative, meaningful, and sufficiently universal patterns and regimes characterizations.

Recently, the analysis of the gradient field of the amplitude envelope of dynamical extended systems has been shown to be a useful tool for understanding complex regimes as intermittency and localized turbulence [3]. However, the classical measures of complexity usually discharge the directional information contained in a vector field. In this paper we describe two new computational operators, called complex entropic form (CEF) and generalized complex entropic form (GEF), for pattern characterization of spatially extended systems. Besides of being a measure of regularity, both operators permit to quantify the degree of phase disorder associated with a given gradient field. Phase disorder plays an important role in the analysis of pattern changing frequency in the amplitude domain [4]. For illustration purposes, we apply CEF and GEF to the analysis of the gradient pattern dynamics of a logistic Coupled Map Lattice.

2 Complex Entropic Form (CEF)

If we consider spatially extended systems in two dimensions (x,y), their energy amplitude distribution is represented by the envelope A(x,y), which can be approximated by a matrix of amplitudes $A = \{a_{k,l}\}$, with $K \times L$ pixels. Note that a dynamical sequence of matrices can be related to a temporal evolution of an envelope $A(x,y,t)$. From the definition of $A$, it is possible to represent the gradient field of the amplitude envelope by $G \equiv \nabla(A) = \{z_{k,l}\}$, where $z_{k,l}$ is a complex number with $\text{Re}(z_{k,l}) = a_{k,l+1} - a_{k,l-1}$ and $\text{Im}(z_{k,l}) = a_{k+1,l} - a_{k-1,l}$.

Given the gradient field matrix $\nabla(A)$, we now define the Complex Entropic Form (CEF) operator as:

$$S_c(G) \equiv - \sum_{k,l} \frac{|z_{k,l}|}{|z|} \ln \left( \frac{|z_{k,l}|}{|z|} \right) = - \sum_{k,l} \frac{|z_{k,l}|}{|z|} \ln \left( \frac{|z_{k,l}|}{|z|} \right) - i \sum_{k,l} \frac{|z_{k,l}|}{|z|} \phi_{k,l} \quad (1)$$

where $|z| = \sum_{k,l} |z_{k,l}|$, $|z_{k,l}|$ and $\phi_{k,l}$ are respectively the modulus and the argument of $z_{k,l}$ and $i = \sqrt{-1}$.

From the above definition, we immediately verify that the $\text{Re}(S_c)$ corresponds to the classical Shannon’s entropy measure $S$ of the moduli $|z_{k,l}|$ and that $\text{Im}(S_c)$ represents the weighted average phase of the gradient field. Conse-
quently, a random generated pattern whose gradient field displays no dominant direction, will have \( Im(S_c) \approx 0 \) and \( Re(S_C) \) close to its maximum value, \( Re(S_c) = \log(M) \), where \( M = K \times L \) is the number of elements in the gradient matrix. On the other hand, intermittencies and symmetry breaking patterns lead to, respectively, a decrease on the value of \( Re(S_c) \) and to a non null \( Im(S_c) \).

### 3 Generalized Complex Entropic Form (GEF)

A more general entropic form that also incorporates the phase information of the gradient field can be obtained from a generalization of the concept of degeneracy \( W \), given by the multinomial coefficient formula, and normally used to deduce the expression of Shannon’s entropy of positive scalar fields, such as images [5]. Considering the gradient matrix \( G \) defined in the previous section, \( W(z_{1,1}, \ldots, z_{K,L}) \) may be generalized as follows:

\[
W(z_{1,1}, \ldots, z_{K,L}) = \frac{\Gamma(z)}{\Gamma(z_{1,1}) \ldots \Gamma(z_{K,L})}, \tag{2}
\]

where \( z = \sum_{k,l} z_{k,l} \). Using Stirling’s approximation, we immediately have

\[
z^{-1} \ln W \longrightarrow S_z = - \sum_{k,l} \frac{z_{k,l}}{z} \ln \left( \frac{z_{k,l}}{z} \right). \tag{3}
\]

This expression, that we hereafter call Generalized Complex Entropic Form (GEF), displays interesting properties [6], several of which it shares with Shannon’s entropy. In particular, we can easily verify that \( S_z \) is invariant under rotation and scaling of the vector field, and that \( Im(S_z) = 0 \) and \( Re(S_z) = S(|z_{k,l}|) \), when all \( z_{k,l} \) have the same direction (in other words, no information is conveyed by the phases when they are all the same).

### 4 Application

In order to illustrate the performance of the computational operators described in the previous sections, we applied CEF and GEF to the characterization of amplitude pattern formation (fragmentation, symmetry breaking and phase disorder) of a logistic extended Coupled Map Lattice (CML). Additional results of the performance of these operators in different contexts (osmosedimentation process [7], wavelet spectra of temporal series [8], Swift-Hohenberg dynamics [9], porous silicon surface patterns [10]) are reported elsewhere.
CML is one of the most important systems that have been introduced to study the dynamics of spatio-temporal complexity in the amplitude domain [11]. The explicit form of a global CML is given by

\[
x_{n+1} = (1 - \epsilon)f(x_n) + \frac{\epsilon}{N} \sum_{l}^{M} f(x_n(l)),  \tag{4}
\]

where \(n\) refers to a discrete time step and \(M = K = L\) is the lattice size. Usually, the function \(f(x)\) is chosen to be the well known dissipative chaotic logistic map, \(f(x) = 1 - \rho x^2\).

From Eq. (4), considering the matrix of amplitudes \(A = \{a_{k,l}\}\), we have

\[
a_{n+1,k,l} = (1 - \epsilon)a_{n,k,l} + \frac{\epsilon}{4}(f(a_{n,k+1,l}) + f(a_{n,k,l+1}) + f(a_{n,k-1,l}) + f(a_{n,k,l-1})), \tag{5}
\]

where \(\epsilon\) is the coupling constant and \(f(a) = \rho a(1 - a)\), with \(\rho\) being a real constant (the chaotic control parameter).

We solved Eq. (5), for \(\epsilon = 0.5\), \(\rho = 4.0\) (chaotic regime), and an aspect ratio (ratio between largest and smallest system characteristic scales) of \(\Gamma = 400\). We started the simulations from two different initial conditions: (i) a Gaussian, symmetric initial distribution, and (ii) a random distribution. Figure 1 (a,b,c,d) shows four frames \((n = 5, 20, 35, 50)\) of the evolution of the gradient field and the corresponding intensity contours plots for the Gaussian initial condition. Figures 2 and 3 depicts respectively the variation of \(Re(S_z)\) and \(Re(S_c)\), for test cases (i) and (ii).

While the results of the random test case, as expected, do not display any particular characteristic, from the evolution of \(Re(S_z)\) for the Gaussian initial distribution (Figure 2) we identify three well defined periods, corresponding to different phase disorder pattern regimes. For \(n \leq 5\), the value of \(Re(S_z)\) is very high (but fast decreasing), since the gradient field still presents a high degree of symmetry. For \(5 \leq n \leq 20\), as the Gaussian perturbation reaches the boundaries of the system, the symmetries of the initial condition are almost completely lost. Moreover, during this period, the system have not yet acquired the symmetries, in a statistical sense, that characterizes a random field and, thus, \(Re(S_z)_G < Re(S_z)_R\). Finally, for \(n > 20\), the gradient field becomes increasingly random. However, since the system still carries a memory of the original symmetry, the value of \(Re(S_z)_G\) is higher than \(Re(S_z)_R\) (see [12] for a similar analysis, concerning the loss and recovery of symmetries of Navier-Stokes equation for increasing values of Reynolds number). The evolution of \(Re(S_c)\) (Figure 3), although with less detail, also shows the system gradual transition from order/symmetry (lower \(Re(S_c)\) value) to disorder/randomness (higher \(Re(S_c)\) value).
5 Concluding Remarks

In this paper we introduced two new computational operators, called complex entropic form (CEF) and generalized complex entropic form (GEF), for pattern characterization of spatially extended systems. The real part of CEF corresponds to the classical Shannon's entropy measure, while its imaginary part represents the weighted average phase of the gradient field. The definition of GEF is obtained from a generalization of the concept of degeneracy. Besides of being a measure of regularity or smoothness of the embedding amplitude envelope, both operators provide information on the degree of phase disorder of the corresponding gradient field.

For illustration purposes, we applied CEF and GEF to the analysis of the gradient pattern dynamics of a logistic Coupled Map Lattice. Simulations using a Gaussian and random initial condition, provided interesting insights on the the system gradual transition from order/symmetry to disorder/randomness.

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**Figure Captions**

**Fig. 1** Four frames ($n = 5, 20, 35, 50$) showing the evolution of the gradient field and intensity contours plots for the solution of logistic Coupled Map with Gaussian initial condition, $\Gamma = 400$, $\epsilon = 0.5$ and $\rho = 4.0$.

**Fig. 2** Evolution of $Re(S_z)$ as a function of the time index $n$, for Gaussian and random initial conditions.

**Fig. 3** Evolution of $Re(S_c)$ as a function of the time index $n$, for Gaussian and random initial conditions.
$Re(S_z)$

$Re(S_c)$

Time index, $n$