Codimension two branes and distributional curvature

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Abstract
In general relativity, there is a well-developed formalism for working with the approximation that a gravitational source is concentrated on a shell, or codimension one surface. In contrast, there are obstacles to concentrating sources on surfaces that have a higher codimension, for example, a string in a spacetime with a dimension greater than or equal to four. Here it is shown that, by giving up some of the generality of the codimension one case, curvature can be concentrated on submanifolds that have codimension two. A class of metrics is identified such that (1) the scalar curvature and Ricci densities exist as distributions with support on a codimension two submanifold, and (2) using the Einstein equation, the distributional curvature corresponds to a concentrated stress-energy with equation of state $p = -\rho$, where $p$ is the isotropic pressure tangent to the submanifold, and $\rho$ is the energy density. This is the appropriate stress-energy to describe a self-gravitating brane that is governed by an area action, or a braneworld deSitter cosmology. The possibility of having a different equation of state arise from a wider class of metrics is discussed.

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1. Introduction
Working with sources that are concentrated on lower-dimensional surfaces is an approximate description that is used to advantage in analyzing many problems in classical physics, whether it is modeling a section of plastic wrap as a two-dimensional membrane, an electric current as a line or a Newtonian mass as a point. In these examples, one locates the source by a delta-function, which is well defined as a distribution in a fixed geometry. The situation is more complicated in general relativity, in which the geometry is a dynamical field. And yet, there are problems of interest in which sources for the gravitational field are naturally viewed as being concentrated on a submanifold—cosmic strings, branes in string-motivated gravity and braneworld cosmologies. Codimension one branes in general relativity work fine. The Israel junction formalism [1] gives a prescription for constructing a solution to the Einstein equation with a shell source. Instead of being smooth, the metric is only continuous across the
shell, and has a jump in its normal derivative, which is interpreted via the Einstein equation as due to a shell of stress energy.

However if a brane is self-gravitating, and is concentrated on a lower-dimensional surface, then problems arise. There is no prescription for sources concentrated on submanifolds with codimension greater than one. Indeed, in [2] it was shown that for metrics that are well enough behaved so that the Riemann tensor exists as a distribution, in general, curvature can only be concentrated on codimension one surfaces. In the work presented here, by 'giving up a little', and assuming some added structure on the spacetime, we present a construction that is tailored to allowing concentrated curvature on a codimension two surface.

This work was motivated in part by known analytic solutions that do describe stress energy concentrated on a codimension two surface. The most famous example is \((3+1)\)-dimensional flat space minus a wedge, which is a model for the spacetime outside a straight cosmic string. In this picture, the stress-energy of the string is concentrated on a codimension two submanifold, the \((1+1)\)-dimensional axis of the string. This simple model matches the properties displayed by the solution for a finite width cosmic string composed of gauge and scalar fields [3]. Outside the core of the string, the metric approaches flat space minus a wedge exponentially fast. Inside the core, to leading order the equation of state of the matter is \(p = -\rho\), where \(p\) is the pressure tangent to the string, and \(\rho\) is the energy density. An example of a finite width brane cosmology is given in [4].

A second analytic example is provided by static Kaluza–Klein Killing bubbles [5, 6]. A Killing bubble is a minimal surface that arises as the fixed surface of a spacelike Killing field. In these solutions the bubble, at fixed time, is a \((D - 3)\)-dimensional sphere, where \(D\) is the spacetime dimension. The static Kaluza–Klein bubble metric has the form

\[
ds^2 = -dt^2 + f(R)k^2 d\phi^2 + \frac{1}{f(R)} dR^2 + R^2 d\Omega^2_{D-3},
\]

where \(f(R) = 1 - R_0/R\), \(0 \leq \phi \leq 2\pi\) and \(0 < k \leq 1\). The Killing vector \((\partial/\partial\phi)\) vanishes at \(R = R_0\), which is a minimal \((D - 3)\)-sphere with non-zero area. This is the Killing bubble. The two-dimensional space orthogonal to the bubble, here the \(R - \phi\) plane, is generally taken to be smooth, which requires \(k = 1\). To see this, one expands the metric near the bubble,

\[
ds^2 \rightarrow -dt^2 + k^2 y^2 d\phi^2 + dy^2 + R_0^2 d\Omega^2_{D-3}.
\]

For \(k = 1\), the spatial geometry has the form of a smooth \(R^2\) orthogonal to the bubble. With the Kaluza–Klein boundary conditions on the metric (1), this is usually described as a cigar cross a sphere, and the minimal sphere is at the tip of the cigar. However, the solutions make sense even if the \(R^2\) has a missing angle with \(k < 1\). Then the smooth cigar is replaced by a cone with the bubble at the tip. The missing angle geometry may be interpreted as a \(p = -\rho\) source that wraps the minimal sphere [6], analogous to the idealized cosmic string or a Euclidean black hole vortex [7]. Metrics having the same limiting form as (2) have been exploited in building braneworld cosmologies with two extra dimensions [8].

Therefore, on one hand, the picture that a metric locally of the form (2) describes matter concentrated on the \(D - 2\) submanifold at the tip of the cone \(y = 0\), has been used in many models. On the other hand, the result of [2] tells us that in general, the Riemann tensor of such a metric is not well defined as a distribution. The construction here aims at resolving these statements, and at generalizing the situation with the known solutions. We identify curvature invariants of the spacetime that are well defined as distributions in the codimension two case. This is sufficient to work out the effective distributional stress energy, without having to use a smooth fill-in, such as was done for a straight cosmic string using the Abelian Higgs model [3].
We will proceed as follows. Let \( B_0 \) be a codimension two submanifold on which curvature is to be concentrated, and assume that locally there is a codimension two foliation of surfaces \( B_\epsilon \) that includes \( B_0 \). Each surface has two normal forms, which are used to split the spacetime into surfaces ‘tangent to’ \( B_0 \), and ‘normal to’ it. The singular behavior is confined to the metric on the two-dimensional surfaces normal to \( B_0 \). We specify conditions such that (1) the spacetime scalar curvature density, and the Ricci density, exists as distributions concentrated on the codimension two submanifold, and (2) using the Einstein equation, the concentrated curvature corresponds to a concentrated stress-energy with equation of state \( p = -\rho \). The conditions needed are summarized in the preface to equation (13).

The assumptions made are simple ones, and yield a family of metrics that includes the analytic examples, but is more general as there are no symmetry requirements on the metric. Necessary and sufficient conditions are given in appendix A for the metric to have the assumed form, which is a particular block diagonal structure. Further, the calculation elucidates how this form of the metric implies that the equation of state on the brane is \( p = -\rho \). This is an issue of interest in the context of braneworld cosmologies—one would like the brane to contain other types of stress-energy as well. Various approaches to this have been studied, including adding higher derivative terms [9], and adding a thickness to the brane [10]. In our concluding section, we discuss generalizations that may yield other equations of state on the brane.

The construction presented here has similarities to stress focusing that occurs in crumpled membranes [11]. If external forces are applied to a thin material, the membrane bends in response. However, if one crumples the membrane, forcing it to occupy a smaller region, then one or more vertices appear at which stretching of the material occurs. Bending generates extrinsic curvature, which is cheaper energetically, whereas stretching corresponds to focused intrinsic curvature. This is more expensive energetically, but is the only option under certain external forces. In the gravitational construction here, the extrinsic curvatures are assumed to be well behaved, but focused curvature occurs in the two-dimensional space normal to the brane. An explicit example of crumpling in the gravitational context, with codimension two vertices connected by codimension one ridges, has been constructed in [12].

The plan of this paper is as follows. Section 2 sets up features of the geometry near \( B_0 \). These properties are used in section 3 to compute the scalar curvature and Ricci tensor densities, to show that they are concentrated on \( B_0 \), and to derive the equation of state for the concentrated stress-energy. Section 4 briefly discusses the next step of finding solutions to the vacuum Einstein equation with concentrated sources, and the role played by minimal surfaces. Section 5 mentions some open questions. Some technical points are deferred to the appendices. We use the convention that Latin letters run over all values \( a, b = 0, \ldots, D - 1 \). Greek indices run over \( D - 2 \) coordinates tangent to the surface, \( \alpha, \beta, = 0, 3, \ldots, D - 1 \), while \( i, j = 3, \ldots, D - 1 \) denote spatial coordinates tangent to the surface. Capital roman letters index the two normal directions \( I, J = 1, 2 \).

2. Geometry near the bubble

Let a spacetime \( \mathcal{M} \) contain a codimension two spacelike submanifold \( B_0 \) on which curvature is to be concentrated. We will refer to \( B_0 \) as ‘the bubble’, since the Killing bubbles referred to in the introduction provide analytic examples of the more general construction developed here. Assume that the bubble arises as the intersection of the level surfaces of two smooth functions, \( x^I = 0, I = 1, 2 \). Let \( x^\alpha, \alpha = 0, 3, \ldots, D - 1 \) be a choice of \( D - 2 \) other good coordinates on \( \mathcal{M} \) in a neighborhood of \( B_0 \). Since the analysis in this section applies to a neighborhood of \( B_0 \), we will not keep repeating this phrase, but that will be understood. The
forms \( n(I) = dx^I \) at \( x^I = 0 \) are two normals to \( B_0 \), though not necessarily unit. We also assume that the family of codimension two surfaces \( B_\epsilon \), defined by \( x^I = \epsilon^I \), is a foliation by smooth spacelike submanifolds. Let \( B_{ab}(x^a, x^I) \), evaluated at \( x^I = \epsilon^I \), be a smooth family of metrics on the \( B_\epsilon \). The coordinate vectors \( \left\{ \frac{\partial}{\partial x^a} \right\} \) are a basis for the tangent space \( T(B_\epsilon) \), and the forms \( dx^{(a)} \) are a basis for the dual space \( T^* B_\epsilon \) of the submanifolds.

We split the spacetime metric as

\[
g_{ab} = B_{ab} + \sigma_{ab}
\]

with \( \sigma_{ab} B^{bc} = 0 \). Since the curvature is nonlinear in the metric and its inverse, one has to start with a metric field that is better behaved to end up with distributional curvature. So the metric \( g_{ab} \) itself is not a distribution, that is, it is not already concentrated on a submanifold. Specifically, \( B_{ab} \) is assumed to be smooth, and \( \sigma_{ab} \) is a tensor field that is smooth in any region bounded away from \( B_0 \). The value of \( \sigma_{ab} \) at \( B_0 \) is given by its limiting value as \( x^I \to 0 \), which may be zero or infinity. This is different from the assumption made in [2] that the metric is regular. Confining the singular behavior to the normal plane is similar to the ‘normal dominated singularity’ approach of Israel, in analyzing line sources in four dimensions [13].

We require that the divergence is mild enough that local volumes are finite. Let \( V_l \) be a \((D-1)\)-dimensional spatial volume that contains all or part of \( B_0 \), so \( V_l : \{ 0 \leq x^I \leq l^I, 0 \leq x^j \leq x_{\text{max}}^j \} \). Let \( g_{ab}^{(D-1)} \) denote the spacetime metric restricted to \( V_l \). Then we shall assume that the volume of \( V_l \) is finite,

\[
\int_{V_l} \sqrt{g_{ab}^{(D-1)}} < \infty
\]

and that for any smooth function \( F \) integrated over \( V_l \), then in the limit \( l^I \to 0 \), this integral vanishes

\[
\lim_{l \to 0} \int_{V_l} \sqrt{g_{ab}^{(D-1)}} F = 0.
\]

This latter condition rules out the volume element itself acting as a delta-function.

The simplest construction is when the spacetime can be foliated by a second family of two-dimensional submanifolds \( N \) which are normal to the \( B_\epsilon \), with \( \sigma_{ab} \) the metric on \( N \). In this paper we will assume this holds, and that the spacetime metric is locally block diagonal. Hence

\[
dx^2 = \sigma_{IJ} dx^I dx^J + B_{ab} dx^a dx^b.
\]

The two-dimensional metric \( \sigma_{IJ} \) can always be written locally in conformally flat coordinates,

\[
d\sigma^2 = \sigma_{IJ} dx^I dx^J = \Omega^2 (\delta_{IJ} dy^I dy^J) = \Omega^2 (dr^2 + r^2 d\phi^2)
\]

with \( r^2 = \delta_{IJ} y^I y^J \) and \( 0 \leq \phi \leq 2\pi \). The simplification that we have gained is that the conditions for distributional curvature on the bubble can be stated in terms of conditions on the single function \( \Omega \). We will see that curvature can be concentrated on \( B_0 \) for appropriate choice of \( \Omega \). From the details of the arguments below, we expect that it is possible to generalize to a non-block diagonal metric, but still retaining the split (3).

When does one expect the block diagonal form to hold? Let \( g \) be a metric that is smooth in any neighborhood not including \( B_0 \). Choose positive \( \epsilon^I \). In appendix A, we show that a necessary and sufficient condition for the existence of coordinates so that \( g \) can be put in block form (6) in a neighborhood of \( B_\epsilon \) is dictated by Frobenius’ theorem. The condition is that the commutator of the normal vector fields \( \vec{n}^{(I)} = g^{ab} \frac{\partial}{\partial x^a} \) closes, see equation (A.4). Therefore we need that \( \vec{n}^{(I)} \) commute in a neighborhood \( 0 < \epsilon^I \leq \epsilon_0^I \), for some \( \epsilon_0^I \), for the metric to be able to be put in the form (6).
Lastly, we note that with the assumption that $\sigma_{ab}$ is the metric on a submanifold $N$, then the finite volume condition (4) can be stated in terms of finite areas of $N$. Let $D_l$ be the disc $0 \leq x^I \leq l^I$ located at some point $x^a$ on the minimal surface. We require that

$$A(D_l) = \int_{D_l} \sqrt{\sigma} < \infty$$

and that $A(D_l) \to 0$ as $l^I \to 0$. If $\Omega^2 \to r^{-2\mu}$, then finite area requires $\mu < 1$.

### 3. Curvature

The strategy for concentrating curvature on the codimension two minimal surface is to isolate the singular behavior in the two-dimensional metric $\sigma_{IJ}$, that is, in the conformal factor $\Omega$. $\Omega$ can be chosen in such a way that appropriate spacetime curvature tensor densities are well defined as distributions with support on $B_0$. To do this, we relax the assumptions of [2] on the spacetime metric. There $g_{ab}$ was defined to be a regular metric in a region if (i) it and its inverse exist everywhere, (ii) they are locally bounded and (iii) the weak first derivative of $g_{ab}$ exists and is square integrable. These conditions were chosen to ensure that the Riemann tensor is defined as a distribution; that is, for any smooth tensor density $s_{abcd}$, the integral $\int R_{abcd} s^{abcd}$ exists. Further, the outer product of the curvature tensor and the metric are distributions, and hence the usual contractions of Riemann are as well.

Geroch and Traschen [2] also note that under weaker conditions on the metric, particular curvature invariants can be defined as distributions. An example given is two-dimensional flat space minus a wedge, for which the conformal factor and scalar curvature densities of $\sigma_{ab}$ are

$$\Omega^2 = r^{-2\mu}, \quad \mu < 1$$

$$\sqrt{\sigma} R[\sigma] = 4\pi \mu \delta^{(2)}(y^I).$$

For $0 < \mu < 1$ this is the metric of a cone, while if $\mu < 0$ the curvature is negative—there is an extra angle. For brevity, we will refer to this metric as a cone in either case. $\Omega$ satisfies the finite area condition as long as $\mu < 1$. It turns out that the sign of $\mu$ determines the sign of the energy density on $B_0$.

Now, although $\sigma$ is not regular (unless $\mu = 0$), its scalar curvature density is the familiar, flat space, two-dimensional delta function, and hence makes sense as a distributional density. However, not all the curvature invariants are well defined. $R[\sigma]$ is zero as a distribution, as it gives zero integrated against a smooth tensor density, and the Riemann tensor is infinite. So we will proceed to make use of the well-behaved features of the two-dimensional curvature density, but as part of a higher-dimensional spacetime. The way that the singular two-dimensional metric contributes to the full spacetime curvature is controlled by splitting the spacetime metric into submanifolds parallel to $B_0$, and orthogonal to it. A different approach was taken in [14]. Here the Riemann tensor is defined as a distribution for a wider class of ‘semi-regular’ metrics, which includes flat space minus a wedge.

We start with the Gauss–Codazzi relations for smooth metrics. Splitting the metric as in (3), the various projections of the Riemann tensor for $g$ can be written in terms of the Riemann tensors for $B$ and $\sigma$, plus extrinsic curvature terms. The needed details are given in (B.3). The spacetime scalar curvature is

$$R[g] = R[\sigma] + R[B] + \Sigma_i \left( (K^{(I)})^2 - K^{(I)} K_{ab}^{(I)} \right)$$

$$+ \lambda_a \lambda^a = \lambda_{ab} \lambda_{ab} + 2(\nabla_a \pi^a - \nabla_a \lambda^a).$$

(10)
Here $K_{ab}^{(I)}$, $\pi^a$ and $\lambda_{abc}$ are extrinsic curvature tensors; the general definitions are given in Equations (B.1), (B.2). For a metric of the block diagonal form (6), (7), one finds

$$K_{ab}^{(I)} = \frac{1}{2\Omega} \partial_I B_{ab} \quad \text{and} \quad \lambda_{IJ} = \sigma_{IJ} \frac{\partial_{\Omega}}{\Omega}. \tag{11}$$

The decomposition of the scalar curvature in (10), and of the Ricci tensor in (16) are derived assuming the spacetime metric is smooth. Here, we proceed to use these equalities as true when integrated. That is, if the integral of the right-hand side is finite, we equate that to the integral of the left-hand side, as in equations (13) and (15) below.

Substituting the extrinsic curvatures into the terms of $R[g]$ gives

$$\Sigma \left[ (K^{(I)})^2 - K_{ab}^{(I)} K_{ab}^{(I)ab} - B_{ab} \Sigma \left[ \left( B^{\beta \mu} B^{\beta \nu} \partial_I B_{\alpha \beta} \right) \partial_I B_{\mu \nu} \right] \right] - B_{ab} \Sigma \left[ \left( B^{\beta \mu} B^{\beta \nu} \partial_I B_{\alpha \beta} \right) \partial_I B_{\mu \nu} \right] \tag{12}$$

Next, integrate $R[g]$ over $V_l$. In the limit that $l^I \to 0$, since volumes are locally finite, any bounded term on the right-hand side of (12) will give zero. Also, any term in the integrand that is of the form $\Omega^{-2} \times$ smooth will give zero upon integration, since $\sqrt{\Omega} = r^{\Omega}$. Since $B_{ab}$ is assumed to be smooth, $R[B]$ does not contribute. The terms that come from $\lambda_{abc}$ depend on $\Omega^{-2} B^{ab} \partial_i \Omega \partial_j \Omega$ and $\Omega^{-2} B^{ab} \Omega \partial_i \partial_j \Omega$, and so far we have not made any assumptions about this behavior. Different assumptions may give different results of interest. The choice that we make here is that these terms are bounded. It will be shown below that this choice will give a $p = -\rho$ equation of state for the concentrated curvature.

**Summarizing.** Let the metric near $B_0$ have the block diagonal form (6), (7). $B_{ab}$ is assumed to be smooth. $\sigma_{ab}$ is smooth in any region not including $B_0$. $\sigma_{ab}$ may approach zero or infinity on $B_0$, but volumes are finite, as stated in (4) or (8). Assume that $\Omega^{-2} B^{ab} \partial_i \Omega \partial_j \Omega$ and $\Omega^{-2} B^{ab} \Omega \partial_i \partial_j \Omega$ are bounded (including on $B_0$). Lastly, assume that $|r \partial_i \Omega| < r^{1+\epsilon}$ with $\epsilon > 0$. This last assumption is needed for deriving (17) below.

All of these assumptions are satisfied by the cone metric with a position-dependent amplitude, $\Omega = (x^a)r^{-\mu}$ with $\mu < 1$.

Let $V_l$ be a $(D - 1)$-dimensional spatial volume that contains all or part of $B_0$ as above, $V_l : 0 \leq x^i \leq l^I, 0 \leq x^i < x_{max}^i$. Integrate the scalar curvature (12) over $V_l$ and take the limit as $l^I \to 0$, which means that the spatial volume collapses to the codimension two surface $B_0$. Let $A(B_0)$ be the area of $B_0$ if it is compact, otherwise $A$ is the area of some subset. Then

$$\lim_{l \to 0} \int_{V_l} \sqrt{g}^{D-1} R[g] = A(B_0) \lim_{l \to 0} \int_{D^l} \sqrt{\sigma} R[\sigma]$$

$$= -2A(B_0) \lim_{l \to 0} \int_{D^l} dy^1 dy^2 \sqrt{ln} \Omega \tag{13}$$

where we have substituted the expression for the two-dimensional curvature $R[\sigma] = -\frac{1}{\sigma^{\nu}} V^2 \ln \Omega$. For the cone metric this becomes

$$\int_{V_l} R[g] \to 4\pi \mu A(B_0). \tag{14}$$

Hence the scalar curvature of the spacetime metric has support on the codimension two surface $B_0$. Equations (13) and (14) are one of the main results of this paper.

An effective stress-energy associated with this concentrated curvature is defined by integrating components of the Einstein tensor over $V_l$ and letting $l \to 0$. Let $u^a$ and $x^a$
be unit timelike and spacelike vectors that are tangent to $B_0$. So $B_a^a u^a = u^a$ and $B_a^a x^a = x^a$.

(There are $D - 3$ independent such spacelike vectors, but here we practice index suppression.)

Define an effective stress-energy concentrated on the surface by

$$8\pi\rho = \lim_{l \to 0} \int_{V_l} u^a u^b G_{ab} = \lim_{l \to 0} \int \left( u^a u^b R_{ab} + \frac{1}{2} R \right)$$

$$8\pi p_t = \lim_{l \to 0} \int_{V_l} x^a x^b G_{ab} = \lim_{l \to 0} \int \left( x^a x^b R_{ab} - \frac{1}{2} R \right)$$

and similarly for the other components. The Ricci and scalar curvatures are of the spacetime metric $g_{ab}$.

So the next step is to compute the components of the Ricci tensor tangent to the surfaces $B_\epsilon$, $B_\mu^m R_{\mu mn} = B_a^m B_b^n (B_{bd} + \sigma_{bd}) R_{mbnd}[g]$. Using the first two equations of (B.3) these terms become

$$B_a^m B_b^n B_{bd} R_{mbnd}[g] = R_{\alpha \gamma}[B] + \Sigma_I (K_a^{\beta(I)} K_{\alpha \gamma}^{(I)} - K_{\gamma(I)} K_{\alpha}^{(I)})$$

$$B_a^m B_b^n \sigma_{bd} R_{mbnd}[g] = \Sigma_I (K_a^{\beta(I)} K_{\alpha \gamma}^{(I)}) + B_a^\alpha \sigma_{\beta} (2\Omega^{-1} \partial_\Omega + 2\Gamma_{\beta \rho} \Omega^{-1} \partial_\rho \Omega - \Omega^{-2} \partial_\alpha \partial_{\beta} B_{\mu \rho})$$

where $K_a^{(I)}$ is given in (11).

Integrate these equations over $V_l$ and take the limit $l \to 0$. Then under the assumptions stated for (13), all of the terms on the right-hand side of (16) give zero. Hence in computing the stress-energy (15) on $B_0$:

$$8\pi\rho = -8\pi p_t = \lim_{l \to 0} \frac{1}{2} \int_{V_l} R[g] = A(B_0) \lim_{l \to 0} \frac{1}{2} \int_{D_l} R[\sigma].$$

When the normal plane has a cone metric with $\Omega = S(x^\alpha) r^{-\mu}$ this becomes

$$\rho = -p = \frac{1}{4} \mu A(B_0).$$

Equations (17) and (18) giving the equation of state of the concentrated curvature are the second main result of this paper.

What we have learned is that with the conditions stated for (13), the Ricci tensor projected tangent to $B_\mu$ does not contribute to the integrated Einstein tensor, as the volume collapses to the surface. Only the scalar curvature term contributes, giving an effective equation of state $p = -\rho$. This is reminiscent of the way that an effective cosmological constant (which has $p = -\rho$) arises from a scalar field when the Lagrangian is potential dominated. This result is very different from the codimension one case. Shells can have any effective stress-energy, by appropriately choosing the jump in the extrinsic curvature across the shell. So, the question arises whether it is possible to get other types of stress-energy on a codimension two brane [9, 10]. To get a different equation of state, the Ricci tensor must contribute to the concentrated stress energy in (15). We defer further discussion to the section 5 on open questions.

Actually, it remains to check the other projections of the Einstein tensor. It turns out that one additional type of term arises, $\Omega^{-1} (\partial_\rho \Omega / \Omega) (\partial_\gamma \Omega / \Omega)$. This integrates to zero as $l^I \to 0$, if $|r \partial_\Omega \Omega| < r^{-1+\epsilon}$. Hence with the same assumptions as for (13), these other components of the Einstein equation integrate to zero in the limit $l^I \to 0$.

A couple of details are worth mentioning, as they may be relevant to possible extensions to higher codimension. The most singular piece comes from the $\sigma_a^m \sigma_a^n G_{mn}[g]$ projection, which includes $R_{ab}[\sigma] - 1/2 \sigma_{ab} R[\sigma]$. However, for $\sigma_{ab}$ two dimensional, this term vanishes, which is special for two dimensions. The next most singular piece comes from the cross terms of the Ricci tensor $\sigma_a^m B_{a}^n R_{mn}$. To analyze this term [16] was used. These mixed components contain
the term discussed in the previous paragraph. There is one other potentially unbounded term, 
\[ \sigma_{p}^{m} \pi_{m} = - \Sigma_{l} \sigma_{p}^{m} \kappa^{l} \nabla_{p} \pi_{m}^{(l)} . \]
Simple power counting implies that the term can be mildly divergent, although it integrates to zero with the stated conditions. However, using the fact that the commutator of the \( \hat{\epsilon}^{(l)} \) closes, one can check that this term is actually finite, and is zero if the commutator vanishes\(^1\). So it may be that geometrical criteria also tame some of the other terms.

4. Minimal surfaces and solutions

The focus of this paper has been to identify a class of metrics that describes curvature concentrated on codimension two surfaces. It turns out that a set of natural assumptions imply that the concentrated source has an equation of state \( p = -\rho \). Though we have not addressed the questions of finding solutions to the vacuum Einstein equation, one expects that on solutions \( B_{0} \) will be a minimal surface. This is because a \( p = -\rho \) test brane is governed by an area action, and propagates as a minimal surface. So it is reasonable to guess that if this sort of brane is self-gravitating, a consistent configuration would be for the location of the brane to be a minimal surface. For example, in the two analytic solutions discussed in this paper, \( B_{0} \) is a minimal surface. In both examples, the effective stress energy is of the form \( \mu B_{ab} \) on \( B_{0} \), and the spacetime is vacuum elsewhere. To look for other solutions, one would need to pick a manifold and \( B_{0} \), then solve the vacuum Einstein equation, with the metric having the allowed singular behavior near \( B_{0} \). Our assumption that the metric is block diagonal limits the class of solutions, but hopefully that assumption will be relaxed in future work.

This raises the question of whether a surface \( B_{0} \) can be described as ‘minimal’, if the geometry is allowed to be singular as in (13). Hence we close by showing that even though \( \sigma_{ab} \) is allowed to be mildly singular, the description of \( B_{0} \) as a minimal surface does make sense.

A minimal surface has the property that if it is deformed, the area does not change to first order. Since the area is unchanged under tangential deformations, only variations off the original surface must be considered. The usual procedure is to compute \( \delta A \) under deformations in the directions \( n^{a(l)} \). In this paper, the normal forms have been defined so that they are well behaved, but the vectors \( n^{a(l)} = g^{ab} h_{b}^{(l)} \) may diverge or be zero on \( B_{0} \). However, the coordinate vector fields \( \left( \frac{\partial}{\partial x^{a}} \right) \) are smooth, and have an off the surface component, as \( \left( \frac{\partial}{\partial x^{a}} \right) h_{b}^{(l)} = \delta_{j}^{b} \) which is non-zero. Hence it is sufficient to require that \( \delta A = 0 \) under deformations along the vector field \( \xi = \lambda \Sigma_{l} h_{b}^{(l)} \frac{\partial}{\partial x^{b}}, \lambda \ll 1 \), where the \( h_{b}^{(l)} \) are arbitrary smooth functions.

The deformation under \( \xi \) defines a new surface \( B_{\lambda} \) parameterized as \( x^{l} = \lambda \hat{\xi}^{l}(u^{b}), x^{a} = u^{a} + \lambda \hat{\xi}^{a}(u^{b}) \), where the \( u^{b} \) are coordinates on the surface. To find the new area, one needs to compute \( \det g|_{l} = \det g|_{0}(1 + B^{ab} r \delta g_{ab}|_{l}) \), where evaluation on \( B_{\lambda} \) is indicated by the index \( \lambda \). There are two contributions to \( \delta g_{ab}|_{l} \), coming from the change in the metric components and from the change in the differentials. Substituting into the general form of the metric (A.1), one finds

\[ \frac{dx^{2}}{|_{l}} - \frac{dx^{2}}{|_{0}} = 2 \lambda (\hat{\xi}^{b} \partial_{b} B_{\mu \nu} + 2 \partial_{(\mu} \hat{\xi}^{\beta} g_{\alpha \beta)}) du^{\alpha} du^{\mu} . \]

Hence \( \sqrt{- \det g|_{l}} - \sqrt{- \det g|_{0}} = \sqrt{- \det g|_{0}(1 + B^{ab} r \delta g_{ab}|_{l})} \). Since this must be true for arbitrary fields \( \xi \), we find the generalization of the codimension one condition, that

\(^{1}\) Of course, this statement only makes sense if the unit basis forms \( \hat{\epsilon}^{(l)} \) are defined, which may not be true on the bubble. However, since the term only involves first derivatives of \( \Omega \), it cannot contain distributional curvature; that is, the value of its integral over \( V_{l} \) is independent of the value on \( B_{0} \).
\( B^{\alpha\beta} \mathcal{L}_{\xi} B_{\alpha\beta} = 0 \). In the case that unit normal vectors are well behaved, this becomes the condition that the two independent extrinsic curvature tensors are traceless on the minimal surface, \( B^{\alpha\beta} K^{(\alpha\beta)} = 0 \), see equation (B.2). For a block diagonal metric, this means that \( B_{\alpha\beta} \) must be quadratic in the transverse conformal coordinates \( x^I \) about the bubble at \( x^I = 0 \).

5. Open questions

A calculation for future work is to require that the metric (6) is a solution to the vacuum Einstein equation, with the correct boundary conditions on \( B_0 \) to give a distributional curvature. It would be interesting to see if there are other solutions than the known symmetrical cases. This might be facilitated by using the equations in the form (12). A second technical issue is to drop the restriction of a block diagonal metric. In dynamical situations, there will certainly be cross-terms in the metric. Does this destroy the model of concentrated curvature? Is topology important, so that compact \( B_0 \) are stable against dynamics, whereas for an infinite cosmic string one is forced to a finite width description?

An important issue is whether a codimension two submanifold can have concentrated stress-energy that is different from a cosmological constant on \( B_0 \). Recall that this form arises because with our assumptions (listed before equation (13)) only the scalar curvature contributes to the components of the distributional Einstein tensor that are tangent to \( B_0 \). To get a different equation of state, the Ricci tensor must also contribute in (15). Could this occur? A delta-function type contribution to the integral arises from a term in which two derivatives in the normal direction act on a function, and then choosing the function to be appropriately singular. This structure occurs in the scalar curvature of the transverse metric \( R[\sigma] \), and yields the results of this paper. Inspection of (16) shows that there are no other terms with this structure. If one wants the metric on \( B_0 \) to be smooth, this means one has to look at other components of the metric \( g_{1\alpha} \), which are not tangent to \( B_0 \). Hence one must study metrics that are not block diagonal. Turning to the more general Gauss–Codazzi relations in (B.3), one finds that there do occur terms with two normal derivatives on the functions \( g_{1\alpha} \). It would be interesting to see if these terms can indeed give a distributional contribution to the Ricci tensor on \( B_0 \).

Two types of questions that motivated this work were to model branes wrapping cycles of compact submanifolds and braneworld cosmologies. If the observed universe is confined to a brane in a spacetime with ten dimensions, then one is interested in concentrating curvature on a submanifold with codimension six. Likewise, there are many solutions to ten and eleven-dimensional supergravity that represent black branes wrapping cycles of submanifolds, but one might ask if there are solutions that represent branes which are not collapsed to black branes? These configurations typically involve wrapping gauge potentials on submanifolds with codimension greater than two. For either of these problems, one needs a formalism which applies to curvature concentrated on surfaces with higher codimension than one or two.

It is not obvious that there is a generalization to codimension greater than two. The fact that the plane normal to \( B_0 \) is two-dimensional was used heavily in the current construction. Most significantly, the famous flat space minus a wedge metric was exploited, as this metric has a curvature density that is a delta-function. Taking a solid angle out of flat three-dimensional space does not give a similar result—the curvature is not focused. In three or more spatial dimensions the expectation is that concentrated matter collapses to a black object. However, if our universe really is ten (or eleven) dimensional, it is worth pursuing these questions.

The relevance of these issues hinges on (i) how useful the thin-object model is in gravity shells, and (ii) how one thinks of branes in the context of classical gravity.
mechanically, particles are not points and a string is not a line. But what is the classical
description of a stack of branes? In a classical gravitational calculation, must branes leap from
being test objects, existing on lower-dimensional submanifolds, to being black branes? Or is
there a middle ground, and if so, is the stack of branes thin in a classical sense?

‘I leave it to whomsoever it may concern, whether the tendency of this work be altogether
to recommend quantum tyranny, or reward classical disobedience’ [17].

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Appendix A. Frobenius’ theorem and criteria to block diagonalize the metric

In general, the metric has the form near $x^I = 0$

$$ds^2 = g_{IJ} dx^I dx^J + 2g_{Ia} dx^I dx^α + B_{αβ} dx^α dx^β.$$  \hfill (A.1)

Suppose that the mixed terms $g_{Ia}$ are nonzero. We want to find a new set of coordinates $x'^α$
such that

$$0 = g^{αI} = \frac{∂x'^I}{∂x^α} g^{cd}, \quad I = 1, 2.$$  \hfill (A.2)

We look for a solution of the form $x'^I = x^I$ and $x'^α = F^α(x^b)$. After some algebra, equations (A.2) become

$$\vec{n}^{(I)}(F^α) \equiv g^{Ia} \frac{∂}{∂x^α}(F^α) = 0, \quad I = 1, 2,$$  \hfill (A.3)

where we have extended the definition of the normals off the surface as vector fields using the
spacetime metric, $n^{αI} = g^{Ia} \frac{∂}{∂x^α}$.

Frobenius’ theorem addresses the question of whether there exist solutions $F^α$. Assume
that a specification of two smooth vector fields $\{\vec{n}^{(1)}, \vec{n}^{(2)}\}$ is given in a neighborhood of the
bubble. Then Frobenius’ theorem states that the vector fields are integrable if and only if the
commutator of the vector fields closes,

$$[\vec{n}^{(1)}, \vec{n}^{(2)}] = f^1 \vec{n}^{(1)} + f^2 \vec{n}^{(2)}.$$  \hfill (A.4)

for some functions $f^I$.

In order that solutions exit to (A.3), it is necessary that the $\vec{n}^{(I)}$ satisfy the Frobenius
condition (FC), equation (A.4). For necessity, suppose that there is a solution $F^α$ to (A.3).
Then it must also be true that $0 = [\vec{n}^{(1)}, \vec{n}^{(2)}]F^α$. However, if the vectors $\vec{E}^{(I)}$ do not satisfy
the FC, then this statement is inconsistent.

For sufficiency, suppose that the $\vec{n}^{(I)}$ satisfy the FC. Then Frobenius’ theorem states that
the $\vec{n}^{(I)}$ are a basis for the tangent space of a smooth submanifold $N$. Choose good coordinates
$y^I$ on $N$. Then the coordinate basis vectors $\frac{∂}{∂y^I}$ form another basis for $T(N)$, and so each
$\vec{n}^{(I)}$ is a linear combination of the $\frac{∂}{∂y^I}$. The equations (A.3) become equivalent to the set

$$\frac{∂}{∂y^α} F^α = 0.$$  \hfill (A.3)

Choose $D - 2$ other coordinates $y^α$, so that $(y^I, y^α)$ are good coordinates on $M$,
in a neighborhood of $N$. Then the solutions to (A.3) are just functions which are independent
of the coordinates on $N$, for example, $F^α = y^α$.

Hence a necessary and sufficient condition that the metric can be put in block diagonal
form is that the normal vectors satisfy the FC (A.4).
Appendix B. Gauss–Codazzi details

This appendix assembles various pieces of the Gauss–Codazzi formalism that are used in the text. Here it is assumed that the metric is smooth. We follow the notation of [15].

The metric is split as in equation (3), \( \sigma_{ab} = B_{ab} + \sigma_{ab} \). There are two independent extrinsic curvature tensors,

\[
\pi_{ab}^c = B_{ab}^c B_{cd} \sigma_d \nabla_m B_d^m \quad \text{and} \quad \lambda_{ab}^c = \sigma_a^m \sigma_c^d B_d^m \nabla_a B_n^n \tag{B.1}
\]

\( \pi_{ab}^c \) is orthogonal to \( B \) in its third index, and tangent to \( B \) in its first two. The opposite is true for \( \lambda_{ab}^c \)—it is tangent to \( B \) in its third index, and orthogonal to \( B \) in its first two.

The assumption that the surfaces \( B \) are submanifolds implies that \( \pi \) is symmetric in its first two indices. \( \lambda \) is symmetric only if \( \sigma \) is the metric for a submanifold as well (as is assumed in the body of the paper).

The three index tensor \( \pi \) can be made to look more like the familiar extrinsic curvature for a codimension one submanifold as follows. Let \( \tilde{\epsilon}^{i(1)} \) be a set of orthonormal normal forms on \( B \), so that \( \sigma_{ab} \) can be expanded as \( \sigma_{ab} = \Sigma_i \tilde{\epsilon}_{ai} \tilde{\epsilon}_{bj}^{(1)} \). Substituting into the definition of \( \pi \), and using \( \nabla_m B_d^m = -\nabla_m \sigma_d^m \), one finds

\[
\pi_{ab}^c = -\Sigma_i K_{ab}^i \tilde{\epsilon}_c^{(1)}, \quad \text{where} \quad K_{ab}^i = B_d^i \nabla_a B_n^n. \tag{B.2}
\]

We will need the Gauss–Codazzi relations for the projections of the Riemann tensor [15]:

\[
B_{a}^{c} B_{d}^{f} B_{e}^{m} R_{cdef}^{\mu} \left[ g \right] = R_{abcd} \left[ B \right] - \pi_{ac} \sigma_{d} \tilde{\epsilon}^{(f)}_{b} + \pi_{bd} \sigma_{c} \tilde{\epsilon}^{(f)}_{a} - \sigma_{ac} \sigma_{bd} \tilde{\epsilon}^{(f)} \tag{B.3}
\]

These imply the following relation for the scalar curvatures:

\[
R \left[ g \right] = R \left[ B \right] + R \left[ \sigma \right] + \nabla_a \pi^a - \pi_{abc} \pi^{abc} + \lambda_{a} \sigma_{a} - \lambda_{abc} \lambda_{abc} + 2 \left( \nabla_a \pi^a - \nabla_a \lambda_a \right). \tag{B.4}
\]

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