Representations of Classical Lie Algebras from their Quantum Deformations

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Abstract. We make use of a well-know deformation of the Poincaré Lie algebra in $p + q + 1$ dimensions ($p + q > 0$) to construct the Poincaré Lie algebra out of the Lie algebras of the de Sitter and anti de Sitter groups, the generators of the Poincaré Lie algebra appearing as certain irrational functions of the generators of the de Sitter groups. We have obtained generalizations of this “anti-deformation” for the $SO(p + 2, q)$ and $SO(p + 1, q + 1)$ cases with arbitrary $p$ and $q$ values. Combining known results on representations of $U_q(so(p, q))$ (for $q$ both generic and a root of unity) with our “anti-deformation” formulae, we get representations of classical Lie algebras which depend upon the deformation parameter $q$. Explicit results are given for the simplest example (of type $A_1$) i.e. that associated with $U_q(so(2, 1))$.‡

1. Introduction.

We start with a well-known deformation [1], [2] of the Poincaré Lie algebra in $p + q + 1$ dimensions ($p + q > 0$), which is defined in terms of the generators $L_{ij}$ of (pseudo) rotations and the translation generators $P_i$ by the following:

\[
L_{ij} \rightarrow L_{ij}, \\
P_i \rightarrow L_{p+q+1,i}^\pm = \frac{i}{2} Y [Q_2, P_i] + P_i
\]

where $Q_2 = \frac{1}{2} \sum_{i,j=0}^{p+q} L_{ij} L^{ij}$ is the second order Casimir operator of $SO_0(p + 1, q)$, and $Y$ satisfies $Y^2 = \pm \sum_{i,j=0}^{p+q} P_i P_i$. ($[\ , \ ]$ denotes commutator.) Choice of the plus sign in this equation for $Y^2$ leads to the Lie algebra of $SO_0(p + 2, q)$ and the minus sign gives the commutation relations of $SO_0(p + 1, q + 1)$. Now eqns. (1b±) may be considered as algebraic equations for the translation generators $P_i$ of the Poincaré group, and we may attempt to solve these equations for the $P_i$. The solution to this problem for $p = 0$, $q = 3$ and for the choice of eqn. (1b−) has been given by us in [1]. The general solution for the case of eqn. (1b+) ($p = 0$, $q = 3$) has been presented in [3]. We have also obtained a generalization of this “anti-deformation” to higher dimensions i.e. we have been able to solve eqns.(1b±) for the $P_i$ [3], but only by working in a particular class of irreducible representations, namely that which occurs in the decomposition of the left regular representation of $SO_0(p, q)$ groups on real hyperbolic spaces [4]. The proof of commutativity of the Poincaré translation generators for these higher dimensional cases makes use of an

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integral transform \[4\], which intertwines certain representations of \(SO_0(p, q)\) induced from the maximal parabolic subgroup with representations which are restrictions of the \(SO_0(p, q)\) left regular representation on eigenspaces of the Laplace-Beltrami operator on the hyperbolic space.

Here we report on some analogous findings for \(q\)-deformations of \(so(p+1, q+1)\) algebras in lowest dimensions i.e. for \(p + q + 1 = 2, 3\) and 4 \[5\] \[6\]. In particular, in the \(p = 1, q = 0\) case, we start with the Euclidean group in two dimensions \(E(2)\), with generators \(L_{12}\) (rotation generator) and \(P_i\) \((i = 1, 2)\) (translation generators), and define the following (c.f. \[5\]):

\[
\tilde{L}_{3i} = \left[\frac{[-iL_{21}]\sqrt{q}}{[2]\sqrt{q}}Y\right]^{2}, P_i + P_i, Y := \sqrt{\sum_{i=1}^{2} P_iP_i} \quad ([m]_q = \frac{q^{m/2} - q^{-m/2}}{q^{1/2} - q^{-1/2}}). \tag{1.2}
\]

We readily obtain the “anti-deformation” by solving eqns. (1.2) for the \(P_i\). Our results are given below in section 2.

2. An Embedding of \(E(2)\) into a skew field extension of \(U_q(so(2, 1))\).

The \(q\)-deformation \(U_q(so(3, \mathbb{Q}))\) is defined as the associative algebra over \(\mathbb{Q}\) with generators \(H, X^\pm\) and relations \[5\], \[6\]:

\[
[H, X^\pm] = \pm 2X^\pm, \tag{2.1a}
\]

\[
[X^+, X^-] = [H]_q. \tag{2.1b}
\]

Let \(I\) be the unit element in \(U_q(so(3, \mathbb{Q}))\), then the Casimir element of \(U_q(so(3, \mathbb{Q}))\) is

\[
\Delta_q = X^+X^- + ([\frac{1}{2}(H - I)]_q)^2 - \frac{1}{4} = X^-X^+ + ([\frac{1}{2}(H + I)]_q)^2 - \frac{1}{4}. \tag{2.2}
\]

The real form \(U_q(so(2, 1))\) of \(U_q(so(3, \mathbb{Q}))\) is defined as follows. The generators of \(U_q(so(2, 1))\) are given by the following expressions:

\[
L_{32} = -\frac{i}{2}(X^+ - X^-), \quad L_{13} = \frac{1}{2}(X^+ + X^-), \quad L_{21} = \frac{i}{2}H. \tag{2.3}
\]

Thus

\[
X^\pm = L_{13} \pm iL_{32}. \tag{2.4}
\]

The operators \(iL_{12}, iL_{13}, iL_{32}\) are preserved under the following antilinear anti-involution \(\omega\) of \(U_q(so(3, \mathbb{Q}))\)

\[
\omega(H) = H, \quad \omega(X^\pm) = -X^\mp. \tag{2.5}
\]

For the coproduct on \(U_q(so(3, \mathbb{Q}))\) we take: \[7\]

\[
\Delta(H) = H \otimes I + I \otimes H, \quad \Delta(X^\pm) = X^\pm \otimes q^{\frac{m}{2}} + q^{-\frac{m}{2}} \otimes X^\pm. \tag{2.6}
\]

The Lie algebra \(E(2)\) is the Lie algebra of the Euclidean group, \(E(2)\), which is the semidirect product of \(SO(2)\) with the group of translations of the plane, \(\mathbb{R}^2\). A basis for the Lie algebra \(E(2)\) consists of the generator of rotations \(L_{12}\) and two commuting translation generators \(P_i\) \((i = 1, 2)\). They satisfy the following commutation relations:

\[
[L_{12}, P_2] = P_1, \quad [L_{12}, P_1] = -P_2, \tag{2.7a}
\]

\[
[L_{12}, P_1] = P_2. \tag{2.7b}
\]
\[ [P_1, P_2] = 0. \quad (2.7b) \]

It is useful to work with the complexified translations generators, which are:
\[ P^\pm = -P_1 \pm i P_2. \quad (2.8) \]

We also define as above
\[ H = -2iL_{21}. \quad (2.9) \]

Then using (2.7) we verify that
\[ [H, P^\pm] = \pm 2P^\pm, \quad [P^+, P^-] = 0. \quad (2.10) \]

We now solve eqns. (1.2) for the \( P_i \), our solution expresses the translation generators of \( E(2) \) as irrational functions of \( U_q(so(2,1)) \). Thus it gives an embedding of \( E(2) \) into an algebraic extension \( K'(U_q(so(2,1))) \) of the skew field \( K(U_q(so(2,1))) \). Explicitly the solution is given by:
\[ P_1 = D^{-1} \left( \left\{ I - \frac{1}{2Y} \left[ \frac{H}{|H|} \right]_q \right\} L_{31} + \frac{i[2]_q}{2Y} \left[ \frac{H}{\sqrt{\tau}} \right]_q L_{32} \right), \quad (2.11a) \]
and
\[ P_2 = D^{-1} \left( \left\{ I - \frac{1}{2Y} \left[ \frac{H}{|H|} \right]_q \right\} L_{32} - \frac{i[2]_q}{2Y} \left[ \frac{H}{\sqrt{\tau}} \right]_q L_{31} \right), \quad (2.11b) \]
where
\[ D = -\frac{1}{4Y^2} \left\{ [H]_q - \left( \frac{[H]_q}{|H| \sqrt{\tau}} - 2Y^2 \right) \right\}. \quad (2.12) \]

Furthermore
\[ Y^2 = \Delta_q + \frac{1}{4} I. \quad (2.13) \]

One readily verifies that the \( P_i \) as defined by eqns. (2.11a) and (2.11b) satisfy the defining commutation relations for the translation generators of \( E(2) \), and verify that \( Y^2 = P^+ P^- \).

The embedding given by eqns. (1.2) extends to a homomorphism \( \tau \) from \( K'(U_q(so(2,1))) \) to \( K'(U(E(2))) \) (an algebraic extension of the skew field of \( U(E(2)) \)) \( (U(E(2))) \) is the enveloping algebra of \( E(2) \). In fact, since \( P_i \) in (2.11a) and (2.11b) commute, it is easy to see that \( \tau \) defined as \( \tau(X^\pm) = \tilde{X}^\pm \) and \( \tau(H) = H \) is an isomorphism. If we take the standard coproduct on \( U(E(2)) \) \( (9) \) and call it \( \tilde{\Delta} \), then one verifies that \( \tau(\Delta(X^\pm)) \neq \tilde{\Delta}(\tau(X^\pm)) \) even for \( q = 1 \). However, we can treat the tensor product of representations as in \( (6) \) where we gave a description of \( U_q(so(4, \mathcal{Q})) \) similar to the above description of \( U_q(so(3, \mathcal{Q})) \). (It is well-known that \( U_q(so(4, \mathcal{Q})) \) is constructed out of two mutually commuting pairs of \( U_q(so(3, \mathcal{Q})) \) \( (10) \).) There we introduced two commuting pairs of translation operators defined on the tensor product representation of two representations of \( U_q(so(3, \mathcal{Q})) \). They were defined implicitly by equations similar to eqns. (1.2), and, as above for \( U_q(so(3, \mathcal{Q})) \), we were able to solve the equations for these four translation operators.

A few comments about about the higher dimensional \( q \) deformed cases: the above remarks in the previous paragraph, outline the main ideas of our generalization to \( U_q(so(2,2)) \) and \( U_q(so(3,1)) \) \( (U_q(so(2,2)) \) and \( U_q(so(3,1)) \) are real forms of \( U_q(so(4, \mathcal{Q})) \).) We have also obtained a description of the Rac representation of \( U_q(so(3,2)) \) \( (11) \) along these lines. This uses the fact that the Rac representation remains irreducible under \( U_q(so(2,2)) \).
3. Representations

For $\sigma \in \mathcal{G}$ and for any $q \in \mathbb{Q}$ ($q \neq 0$ and not a root of unity) the following formulae define a representation $d\pi^{\sigma,q}$ of $U_q(\mathfrak{so}(3,\mathbb{Q}))$ \cite{12}:

$$d\pi^{\sigma,q}(H)|m> = 2m|m> , \quad d\pi^{\sigma,q}(X^\pm)|m> = [-\sigma \pm m]_q|m \pm 1> . \quad (3.1)$$

For $q^N \neq 1$ ($|q| = 1$): (1) $\sigma = i\rho - \frac{1}{2}$ ($\rho \in i\mathbb{R}$) and the representation space $\mathcal{D}^{i\rho - 1/2}$ is the linear span of the $|m> = (m = n + e, n = 0, \pm 1, \pm 2, \ldots)$, and $d\pi^{\sigma,q}$ is the (infinitesmally unitarizable) principal series of $U_q(\mathfrak{so}(2,1))$; (2) $\sigma = \epsilon \mod(2)$ and $\sigma = \ell$ with $\ell < -\frac{1}{2}$ and a) the representation space $X^{\ell,\epsilon}_+$ is the linear span of the above $|m> = m > \ell$, b) the representation space $X^{-\ell,\epsilon}_-$ is the linear span of the $|m> = m < \ell$. $d\pi^{\sigma,q}$ acts irreducibly on $X^{\ell,\epsilon}_\pm$. These give $q$ deformed discrete series of $U_q(\mathfrak{so}(2,1))$.

For $q^M = 1$ ($M \in \mathbb{Z}$, $M > 2$), let $q = e^{\frac{2\pi i}{M}}$ and set $M = m$ for odd, and set $M = \frac{m^2}{2}$ for $m$ even. Define $\sigma = \frac{1}{2}(d - 1) - \frac{1}{2}M$ ($d = 1, 2, \ldots M$) and let $V_d$ be linear span of the $|s_3> = (s_3 = \sigma, \sigma - 1, \ldots - (d - 1))$. The action $d\pi^{\sigma}$ of the basic generators $H$ and $X^\pm$ on $V_d$ is given by: \cite{13}

$$d\pi^{\sigma}(H)|s_3> = -2s_3|s_3> , \quad d\pi^{\sigma}(X^\pm)|s_3> = [-\sigma \pm s_3]_q|s_3 \pm 1> . \quad (3.2)$$

These finite dimensional highest weight modules are all infinitesmally unitary. For which of the above representations do eqns. (2.11) determine a representation of $\mathcal{E}(2)$ on the given representation space? The following theorem provides the answer to this question.

**Theorem:** For $q^N \neq 1$ we have representations of $\mathcal{E}(2)$ on $\mathcal{D}^{i\rho - 1/2}$ and on $X^{\ell,\epsilon}_\pm$ but the representation of $\mathcal{E}(2)$ is infinitesimally unitary only on $\mathcal{D}^{i\rho - 1/2}$. For $q^N = 1$ ($N \in \mathbb{Z}$, $N > 2$) none of the representations $d\pi^{\sigma,q}$ lead to representations of $\mathcal{E}(2)$ on $V_d$.

The main ingredient in the proof of the theorem involves determining the action of the operator $D$ of eqn. (2.12) on the given representation space, and, in particular, deciding whether zero lies in the resolvent set of the operator in its given action on the representation space.

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