Local times of subdiffusive biased walks on trees

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Summary. Consider a class of null-recurrent randomly biased walks on a super-critical Gaton-Watson tree. We obtain the scaling limits of the local times and the quenched local probability for the biased walk in the sub-diffusive case. These results are a consequence of a sharp estimate on the return time, whose analysis is driven by a family of concave recursive equations on trees.

Keywords. Biased random walk on the Galton–Watson tree, local time, concave recursive equations.

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1 Introduction

We are interested in a randomly biased walk \((X_n)_{n \geq 0}\) on a supercritical Galton–Watson tree \(T\), rooted at \(\emptyset\). For any vertex \(x \in T\setminus\{\emptyset\}\), denote by \(\leftarrow x\) its parent. Let \(\omega := (\omega(x, \cdot), x \in T)\) be a sequence of vectors such that for each vertex \(x \in T\), \(\omega(x, y) \geq 0\) for all \(y \in T\) and \(\sum_{y \in T} \omega(x, y) = 1\). We assume that \(\omega(x, y) > 0\) if and only if either \(\leftarrow x = y\) or \(\leftarrow y = x\).

For the sake of presentation, we add a specific vertex \(\leftarrow \emptyset\), considered as the parent of \(\emptyset\). Let us stress that \(\emptyset \notin T\). We define \(\omega(\leftarrow \emptyset, \emptyset) := 1\) and modify the vector \(\omega(\emptyset, \cdot)\) such that \(\omega(\emptyset, \leftarrow \emptyset) > 0\) and \(\omega(\emptyset, \leftarrow \emptyset) + \sum_{x: \leftarrow x = \emptyset} \omega(\emptyset, x) = 1\).

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For given $\omega$, $(X_n, n \geq 0)$ is a Markov chain on $\mathbb{T} \cup \{\emptyset\}$ with transition probabilities $\omega$, starting from $\emptyset$; i.e. $X_0 = \emptyset$ and

$$P_{\omega}(X_{n+1} = y \mid X_n = x) = \omega(x, y).$$

For any vertex $x \in \mathbb{T}$, let $(x^{(1)}, \ldots, x^{(\nu_x)})$ be its children, where $\nu_x \geq 0$ is the number of children of $x$. Define $A(x) := (A(x^{(i)}), 1 \leq i \leq \nu_x)$ by

$$A(x^{(i)}) := \frac{\omega(x, x^{(i)})}{\omega(x, \leftarrow)}, \quad 1 \leq i \leq \nu_x.$$

When all $A(x^{(i)}) = \lambda$ with some positive constant $\lambda$, the walk is called $\lambda$-biased walk on a Galton-Watson tree and was studied in detail by Lyons, Pemantle and Peres [19, 20]. We mention that several conjectures in [20] still remain open and we refer to Aidekon [3] for an explicit formula on the speed of the $\lambda$-biased walk and the references therein for recent developments.

When $A(x^{(i)})$ is also a random variable, the couple $(\mathbb{T}, \omega)$ is a marked tree in the sense of Neveu [22], and the biased walk $X$ can be reviewed as a random walk in random environment.

Let us assume that $A(x), x \in \mathbb{T}$ (including $x = \emptyset$) are i.i.d., and denote the vector $A(\emptyset)$ by $(A_1, \ldots, A_\nu)$ for notational convenience. As such, $\nu \equiv \nu_\emptyset$ is the number of children of $\emptyset$. Denote by $P$ the law of $\omega$ and define $P(\cdot) := \int P_\omega(\cdot) P(d\omega)$. In the literature of random walk in random environment, $P_\omega$ is referred to the quenched probability whereas $P$ is the annealed probability.

Define

$$\psi(t) := \log \mathbb{E}\left(\sum_{i=1}^\nu A_i^t\right) \in (-\infty, \infty], \quad \forall t \in \mathbb{R}.$$ 

In particular, $\psi(0) = \log \mathbb{E} (\nu) > 0$ since $\mathbb{T}$ is supercritical. Assume that

$$\sup\{t > 0 : \psi(t) < \infty\} > 1.$$

We shall consider the case when $(X_n)$ is null-recurrent and sub-diffusive. Lyons and Pemantle [18] gave a precise recurrence/transience criterion for randomly biased walks on an arbitrary infinite tree. Their results, together with Menshikov and Petritis [21] and Faraud [10], imply that $(X_n)$ is null recurrent if and only if $\inf_{0 \leq t \leq 1} \psi(t) = 0$ and $\psi'(1) \leq 0$. There are two different situations in the null-recurrent case: Either $\psi'(1) = 0$, then $(X_n)$ has a slow-movement behavior (see [11], and [16] for the localization of $X_n$ and
the study of the local times processes), or $\psi'(1) < 0$, then $(X_n)$ is sub-diffusive (see [15]). Therefore we assume throughout this paper

\begin{equation}
\inf_{0 \leq t \leq 1} \psi(t) = 0 \quad \text{and} \quad \psi'(1) < 0. \tag{1.1}
\end{equation}

Let us introduce a parameter

$$\kappa := \inf\{t > 1 : \psi(t) = 0\} \in (1, \infty],$$

with $\inf \emptyset := \infty$. We furthermore assume the following conditions:

\begin{equation}
\begin{cases}
E\left(\sum_{i=1}^{\nu} A_i\right)^\kappa + E\left(\sum_{i=1}^{\nu} A_i^\kappa \log_+ A_i\right) < \infty, & \text{if } 1 < \kappa \leq 2, \\
E\left(\sum_{i=1}^{\nu} A_i\right)^2 < \infty, & \text{if } \kappa \in (2, \infty],
\end{cases} \tag{1.2}
\end{equation}

with $\log_+ x := \max(0, \log x)$ for any $x > 0$, and

\begin{equation}
\text{the support of } \sum_{i=1}^{\nu} \delta_{\{\log A_i\}} \text{ is non-lattice when } 1 < \kappa \leq 2. \tag{1.3}
\end{equation}

Figure 1: Case $\inf_{0 \leq t \leq 1} \psi(t) = 0$ and $\psi'(1) < 0$: $\kappa \in (1, \infty)$ and $\kappa = \infty$

It was shown in [15] that if $\nu$ equals some constant (i.e. $T$ is a regular tree), then

$$\lim_{n \to \infty} \frac{1}{\log n} \log \max_{0 \leq i \leq n} |X_i| = 1 - \max\left(\frac{1}{2}, \frac{1}{\kappa}\right), \quad \mathbb{P}\text{-a.s.}$$

When $\kappa$ is sufficiently large (say $\kappa \in (5, \infty]$), Faraud [10] proved an invariance principle for the biased walk $X$, based on the techniques of Peres and Zeitouni [24]; some recent developments cover the whole region $\kappa \in (2, \infty]$ (see Aïdékon and de Raphélis [4] for the convergence to Brownian tree).
The biased walk on a Galton-Watson tree has also attracted many attentions from other directions: In the transient case, Aïdékon [1, 2] dealt with a leafless Galton-Watson tree, whereas Hammond [14] established stable limit laws for the walk on a supercritical Galton-Watson tree with leaves, which can be considered as a counterpart of Ben Arous, Gantert, Fribergh and Hammond [8]. When the tree is sub-critical, Ben Arous and Hammond [9] obtained power laws for the tails of \( E_\omega(T^+_\emptyset) \) and the convergence in law of \( T^+_\emptyset \) under a suitable conditional probability, where \( T^+_\emptyset \) denotes the return time to \( \emptyset \):

\[
T^+_\emptyset := \inf\{n \geq 1 : X_n = \emptyset\}.
\]

In the above-mentioned works [14, 8, 9], the authors explored the link between the biased walk \((X_n)\) and the trap models (cf. Ben Arous and Cerný [7]) to get various scaling limits, and an important step is the estimate on the return time to the trap entrance in their models.

We investigate here the return time \( T^+_\emptyset \) in the scope of limit theorems for the local time process of \( X \). It turns out the parameter \( \kappa \) plays a crucial role. Indeed, define \((M_n)\) by

\[
M_n := \sum_{|x| = n} \prod_{\emptyset < y \leq x} A(y), \quad n \geq 1,
\]

where here and in the sequel, \(|x|\) is the generation of \( x \) in \( T \) and we adopt the partial order: \( y < x \) means that \( y \) is ancestor of \( x \) [we write \( y \leq x \) iff either \( y < x \) or \( y = x \)]. Since \( \psi(1) = 0 \), it is easy to check that \((M_n)\) is a martingale, which in the language of branching random walk is called the additive martingale (cf. Shi [27] further properties on \((M_n))\). Define

\[
P^*(\bullet) := P\left( \bullet \mid T = \infty \right),
\]

where \( \{T = \infty\} \) denotes the event that the system survives forever. Let \( M_\infty := \lim_{n \to \infty} M_n \) be the almost sure limit of the nonnegative martingale \((M_n)\). Then under (1.1) and (1.2) [the condition (1.2) is more than necessary to ensure the non-triviality of \( M_\infty \)], \( P^*\)-a.s. \( M_\infty > 0 \); If furthermore (1.3) is satisfied (for \( 1 < \kappa \leq 2 \)), then

\[
P\left( M_\infty > x \right) \sim c_M x^{-\kappa},
\]

with some positive constant \( c_M \) (see Liu [17] Theorems 2.0 and 2.2).

The main estimate on the return time reads as follows. Denote by \( f(x) \sim g(x) \) as \( x \to x_0 \) if \( \lim_{x \to x_0} f(x)/g(x) = 1 \).
Theorem 1.1 Assume (1.1), (1.2) and (1.3). We have that $\mathbb{P}^*(d\omega)$-a.s.,

$$
\frac{1}{\omega(\emptyset, \emptyset) M_\infty} P_\omega(T^+_\emptyset > n) \sim \begin{cases} 
  c_1 n^{-1/\kappa}, & \text{if } 1 < \kappa < 2, \\
  c_2 (n \log n)^{-1/2}, & \text{if } \kappa = 2, \\
  c_3 n^{-1/2}, & \text{if } \kappa \in (2, \infty],
\end{cases}
$$

with

- $c_1 := \frac{1}{\Gamma(1 - 1/\kappa)} 2^{1/\kappa} (c_M \kappa \mathbb{B}(2 - \kappa, \kappa - 1))^{-1/\kappa}$,
- $c_2 := (\pi c_M)^{-1/2}$,
- $c_3 := \left( \frac{2}{\pi} \frac{1 - \mathbb{E}(\sum_{i=1}^\nu A_i^2)}{\mathbb{E}(\sum_{1 \leq i < j \leq \nu} A_i A_j)} \right)^{1/2}$,

where $\mathbb{B}$ denotes the Beta function and $c_M > 0$ is given in (1.6).

As a consequence, we get the asymptotic behaviors of the local times process:

$$
L^x_n := \sum_{i=1}^n 1_{(X_i = x)}, \quad n \geq 1, x \in \mathbb{T}.
$$

We shall restrict our attentions to the local times at the root. It was implicitly contained in [15, 6] that for any $\kappa \in (1, \infty]$, $\mathbb{P}$-almost surely on $\{T = \infty\}$,

$$
L^\emptyset_n = n^{\max(1/\kappa, 1/2) + o(1)}.
$$

Based on Theorem 1.1, we can get more precise information on $L^\emptyset_n$. For any $0 < \alpha < 1$, denote by $S_\alpha$ a positive stable random variable, independent of $(\omega, \mathbb{T})$, whose law is determined by the Laplace transform: $\mathbb{E} e^{-\lambda S_\alpha} = e^{-\lambda^\alpha}, \forall \lambda \geq 0$. It is easy to see, for instance by comparing their Laplace transforms, that $S_{1/2} \overset{(\text{law})}{=} \frac{1}{2\sqrt{\pi}} S_1$, where $S_1$ denotes a standard gaussian random variable, centered and with variance 1, independent of $(\omega, \mathbb{T})$.

Corollary 1.2 Under the same assumptions as in Theorem 1.1, $\mathbb{P}^*(d\omega)$-a.s., the following convergences in law hold under $P_\omega$:

(i) if $1 < \kappa < 2$, then

$$
\frac{L^\emptyset_n}{n^{1/\kappa}} \overset{(\text{law})}{\longrightarrow} \frac{1}{\omega(\emptyset, \emptyset) M_\infty} c_1 \Gamma(1 - 1/\kappa) (S_{1/\kappa})^{-1/\kappa},
$$
(ii) if \( \kappa = 2 \), then
\[
\frac{L_n^\varnothing}{\sqrt{n \log n}} \xrightarrow{(\text{law})} \frac{1}{\omega(\varnothing, \varnothing) M_\infty} \frac{2^{1/2}}{c_2 \pi^{1/2}} |\mathcal{N}|;
\]

(iii) if \( 2 < \kappa \leq \infty \), then
\[
\frac{L_n^\varnothing}{\sqrt{n}} \xrightarrow{(\text{law})} \frac{1}{\omega(\varnothing, \varnothing) M_\infty} \frac{2^{1/2}}{c_3 \pi^{1/2}} |\mathcal{N}|.
\]

By the classical fluctuation theory on the random walk in the domain of attraction, it is straightforward to deduce from Theorem 1.1 the almost sure limits on \( L_n^\varnothing \): for instance, we have the following law of iterated logarithm:

Corollary 1.3 Under the same assumptions as in Theorem 1.1, for any \( \kappa \in (1, \infty] \), there exists a random variable \( \Upsilon_\kappa \) only depending on \( (\omega, \mathbb{T}) \) such that \( \mathbb{P}^*(\Upsilon_\kappa > 0) = 1 \), and on the set \( \{ \mathbb{T} = \infty \} \),
\[
\limsup_{n \to \infty} \frac{L_n^\varnothing}{f_\kappa(n)} = \Upsilon_\kappa, \quad \mathbb{P}-\text{almost surely},
\]
where
\[
f_\kappa(n) := \begin{cases} n^{1/\kappa}(\log \log n)^{1-1/\kappa}, & \text{if } 1 < \kappa < 2, \\ n^{1/2}(\log n)^{1/2}(\log \log n)^{1/2}, & \text{if } \kappa = 2, \\ n^{1/2}(\log \log n)^{1/2}, & \text{if } \kappa \in (2, \infty]. \end{cases}
\]

Combining the estimates on the local times and the reversibility of the biased walk, we obtain the following estimates on the local probability.

Corollary 1.4 Under the same assumptions as in Theorem 1.1, \( \mathbb{P}^*(d\omega) \)-almost surely, for \( n \to \infty \) along the sequence of even numbers, we have

(i) if \( 1 < \kappa < 2 \), then
\[
P_\omega(X_n = \varnothing) \sim \frac{1}{\omega(\varnothing, \varnothing) M_\infty} \frac{2 \mathbb{E}(S_\kappa^{-1/\kappa})}{c_1 \kappa \Gamma(1 - 1/\kappa)} n^{1/\kappa - 1};
\]

(ii) if \( \kappa = 2 \), then
\[
P_\omega(X_n = \varnothing) \sim \frac{1}{\omega(\varnothing, \varnothing) M_\infty} \frac{2}{\pi c_2} n^{-1/2} (\log n)^{1/2};
\]

(iii) if \( 2 < \kappa \leq \infty \), then
\[
P_\omega(X_n = \varnothing) \sim \frac{1}{\omega(\varnothing, \varnothing) M_\infty} \frac{2}{\pi c_3} n^{-1/2}.
\]
2 Outline of the proof

For any \( x \in \mathbb{T} \), let \( P_{x,\omega} \) be the law of the biased walk \( X \) starting from \( X_0 := x \). Denote by \( E_{x,\omega} \) the expectation under the probability measure \( P_{x,\omega} \). In particular, we have \( P_{\emptyset,\omega} \equiv P_\omega \) and \( E_{\emptyset,\omega} \equiv E_\omega \). Let

\[
T_x := \inf\{ n \geq 0 : X_n = x \}, \quad x \in \mathbb{T},
\]

be the first hitting time of \( x \). Clearly for \( n > 2 \),

\[
P_\omega \left( T_\emptyset > n \right) = \sum_{|u|=1} \omega(\emptyset, u) P_{u,\omega} \left( T_\emptyset > n - 1 \right)
\]

(2.1)

By Tauberian theorems, the asymptotic behaviors of \( P_{u,\omega}(T_\emptyset > n - 1) \), are characterized by that of \( E_{u,\omega}(e^{-\lambda T_\emptyset}) \) as \( \lambda \to 0 \). More generally, we define for any \( \lambda > 0 \) and \( x \in \mathbb{T} \),

\[
\beta_\lambda(x) := 1 - E_{x,\omega} \left( e^{-\lambda(1+T_x)} \right), \quad x \in \mathbb{T},
\]

(2.2)

where as before, \( \overset{\to}{x} \) denotes the parent of \( x \). It is easy to see that \( \beta_\lambda(\cdot) \) satisfies the following recursive iteration equations:

**Fact 2.1** For any \( x \in \mathbb{T} \) and \( \lambda > 0 \), we have

\[
\beta_\lambda(x) = \frac{(1 - e^{-2\lambda}) + \sum_{i=1}^{\nu_x} A(x^{(i)}) \beta_\lambda(x^{(i)})}{1 + \sum_{i=1}^{\nu_x} A(x^{(i)}) \beta_\lambda(x^{(i)})},
\]

We mention that conditioned on \( ((A(x^{(i)}))_{1 \leq i \leq \nu_x}, \nu_x) \), \( (\beta_\lambda(x^{(i)}), 1 \leq i \leq \nu_x) \) are i.i.d. and are distributed as \( \beta_\lambda(\emptyset) \).

**Proof of Fact 2.1.** This fact is an easy application of Markov property. We give the proof for the sake of completeness. For use later, we define for any \( n \geq 1, \lambda > 0 \) and \( x \in \mathbb{T} \) and \( |x| \leq n \),

\[
\beta_{n,\lambda}(x) := 1 - E_{x,\omega} \left( e^{-\lambda(1+T_x)} 1_{(\tau_n > T_x)} \right),
\]

(2.3)

where

\[
\tau_n := \inf\{ k \geq 0 : |X_k| = n \},
\]

denotes the first time when \( X \) hits the \( n \)-th generation of the tree \( \mathbb{T} \).
Clearly $\beta_{n,\lambda}(x) = 1$ for all $|x| = n$ and for $|x| < n$, we have by the Markov property that

$$\beta_{n,\lambda}(x) = 1 - \left(\sum_{i=1}^{\nu_x} \omega(x, x^{(i)}) e^{-\lambda E_{\omega,x^{(i)}} e^{-\lambda(1+T_x)} 1(T_x < \tau_n)} + \omega(x, \bar{x}) e^{-2\lambda}\right)$$

$$= 1 - \left(\sum_{i=1}^{\nu_x} \omega(x, x^{(i)}) (1 - \beta_{n,\lambda}(x^{(i)}))(1 - \beta_{n,\lambda}(x)) + \omega(x, \bar{x}) e^{-2\lambda}\right).$$

After simplifications, we get that

$$\beta_{n,\lambda}(x) = \frac{(1 - e^{-2\lambda}) + \sum_{i=1}^{\nu_x} A(x^{(i)}) \beta_{n,\lambda}(x^{(i)})}{1 + \sum_{i=1}^{\nu_x} A(x^{(i)}) \beta_{n,\lambda}(x^{(i)})}, \quad |x| < n.$$

Letting $n \to \infty$, $\beta_\lambda(x) = \lim_{n \to \infty} \beta_{n,\lambda}(x)$ and we get Fact \ref{fact1}. \hfill \Box

For brevity, we make a change of variable $\varepsilon = 1 - e^{-2\lambda}$, by defining

$$B_\varepsilon(x) := \sum_{i=1}^{\nu_x} A(x^{(i)}) \beta_{\frac{1}{\varepsilon} \log 1/(1-\varepsilon)}(x^{(i)}), \quad x \in \mathbb{T}, \quad 0 < \varepsilon < 1,$$

then

$$B_\varepsilon(x) = \sum_{i=1}^{\nu_x} A(x^{(i)}) \frac{\varepsilon + B_\varepsilon(x^{(i)})}{1 + B_\varepsilon(x^{(i)})},$$

where as for $\beta_\lambda(x)$, conditioned on $(A(x^{(i)}))_{1 \leq i \leq \nu_x}$, $(B_\varepsilon(x^{(i)}), 1 \leq i \leq \nu_x)$ are i.i.d. and are distributed as $B_\varepsilon(\emptyset)$.

The main estimate in the proof of Theorem \ref{thm1} will be the following result:

**Proposition 2.2** Assume \ref{assumption1}, \ref{assumption2} and \ref{assumption3}. As $\varepsilon \to 0$, the following convergences hold $\mathbb{P}$-almost surely as well as in $L^p(\mathbb{P})$ for any $1 < p < \min(\kappa, 2)$:

(i) if $1 < \kappa < 2$, then

$$\varepsilon^{-1/\kappa} B_\varepsilon(\emptyset) \to c_4 M_\infty,$$

where $c_4 := (c_M \kappa \mathbb{B}(2 - \kappa, \kappa - 1))^{-1/\kappa}$, $\mathbb{B}$ denotes the Beta function and $c_M > 0$ is given in \ref{eqn4}.

(ii) if $\kappa = 2$, then

$$\left(\frac{\varepsilon}{\log 1/\varepsilon}\right)^{-1/2} B_\varepsilon(\emptyset) \to (2c_M)^{-1/2} M_\infty.$$

(iii) if $\kappa \in (2, \infty]$, then

$$\varepsilon^{-1/2} B_\varepsilon(\emptyset) \to c_5 M_\infty,$$

where $c_5 := \left(\frac{1 - \mathbb{E}(\sum_{i=1}^{\nu_x} A_i^2)}{\mathbb{E}(\sum_{i=1}^{\nu_x} A_i^2)}\right)^{1/2}.$
Recall that $\mathbb{P}$-a.s., $\{M_\infty > 0\} = \{T = \infty\}$. It is straightforward to see that on $\{T = \infty\}^c$, the biased walk $X$ is a Markov chain with finite states, hence $B_\varepsilon(\emptyset) = O(\varepsilon)$ as $\varepsilon \to 0$.

Let us give the proofs of Theorem 1.1 and Corollaries 1.2, 1.3 and 1.4, by admitting Proposition 2.2.

Proof of Theorem 1.1. By (2.2) and (2.5), we deduce from the usual Abel transform that if $\lambda > 0$ is such that $\varepsilon = 1 - e^{-2\lambda}$, then

$$B_\varepsilon(\emptyset) = (1 - e^{-\lambda}) \sum_{|u|=1} A(u) \sum_{n=0}^\infty e^{-\lambda n} P_{u,\omega}(T_{\emptyset} \geq n).$$

In view of (2.1),

$$\sum_{|u|=1} A(u) P_{u,\omega}(T_{\emptyset} \geq n) = P_{\omega}(T^+_{\emptyset} > n)/\omega(\emptyset, \emptyset).$$

It follows that

$$\sum_{n=0}^\infty e^{-\lambda n} P_{\omega}(T^+_{\emptyset} > n) = \omega(\emptyset, \emptyset) \frac{B_\varepsilon(\emptyset)}{1 - e^{-\lambda}},$$

with $\varepsilon = 1 - e^{-2\lambda}$. By the Tauberian theorem ([12], pp. 447, Theorem 5), we immediately obtain Theorem 1.1. □

Proof of Corollary 1.2. Define for $k \geq 1$,

$$T_{\emptyset}^{(k)} := \inf\{n > T_{\emptyset}^{(k-1)} : X_n = \emptyset\},$$

the $k$-th return to $\emptyset$ (with $T_{\emptyset}^{(0)} := 0$). Under $P_\omega$, $T_{\emptyset}^{(k)}$ is the sum of $k$ i.i.d. copies of $T^+_{\emptyset}$, which is in the domain of attraction of a stable law of index $\max(1/\kappa, 1/2)$. We claim that if for some $0 < \alpha < 1$ and some slowly varying function $\ell(n)$,

$$(2.7) \quad P_\omega(T^+_{\emptyset} > n) \sim \frac{1}{\Gamma(1 - \alpha)} n^{-\alpha} \ell(n),$$

then under $P_\omega$,

$$(2.8) \quad n^{-\alpha} \ell(n) L_n^{(\text{law})} \xrightarrow{n} (S_\alpha)^{-\alpha},$$

with $S_\alpha$ the stable law defined in Theorem 1.1. In fact, to get (2.8), we apply [12] (Theorem 2, pp.448) to see that under $P_\omega$,

$$\frac{T_{\emptyset}^{(k)}}{(k \ell(k^{1/\alpha}))^{1/\alpha}} \xrightarrow{(\text{law})} S_\alpha.$$
with $S_\alpha$ a positive stable variable of index $\alpha$: $E e^{-\lambda S_\alpha} = e^{-\lambda^\alpha}$ for any $\lambda > 0$. Using the fact that $P_\omega(L_n^\varnothing \geq k) = P_\omega(T_n^{(k)} \leq n)$ for $k \geq 1, n \geq 1$, we easily deduce that for any $z > 0$,

$$P_\omega\left(\frac{L_n^\varnothing}{n^\alpha/\ell(n)} \geq z\right) \to \mathbb{P}\left(S_\alpha \leq z^{-1/\alpha}\right),$$

showing (2.8). Clearly, (2.8) implies Corollary 1.2. □

Proof of Corollary 1.3. It suffices to apply Fristed and Pruitt ([13], Theorem 5) to $T_n^{(k)}$ (under $P_\omega$).

Proof of Corollary 1.4. Under the framework (2.7), we remark that $n^{-\alpha} \ell(n) L_n^\varnothing$ is bounded in $L^p(\mathbb{P}_\omega)$ for any $p > 0$. In fact,

$$E_\omega\left(L_n^\varnothing\right)^p \leq \sum_{k=0}^\infty p k^{p-1} P_\omega\left(L_n^\varnothing \geq k\right) = \sum_{k=0}^\infty p k^{p-1} P_\omega\left(T_n^{(k)} \leq n\right).$$

Observe that $P_\omega\left(T_n^{(k)} \leq n\right) \leq P_\omega\left(T_n^+ \leq n\right)^k \leq e^{-k P_\omega(T_n^+ > n)}$. Hence

$$E_\omega\left(L_n^\varnothing\right)^p \leq \sum_{k=0}^\infty p k^{p-1} e^{-k P_\omega(T_n^+ > n)}.$$

Since $\sum_{k=0}^\infty p k^{p-1} e^{-kx} \leq C_p x^{-p}$ for all $0 < x \leq 1$ and some constant $C_p$, we get that $P_\omega\left(T_n^+ > n\right) \times L_n^\varnothing$ is bounded in $L^p$ for any $p > 0$. This together with (2.8) imply that

$$E_\omega\left(L_n^\varnothing\right) \sim E\left((S_\alpha)^{-\alpha}\right) \frac{n^\alpha}{\ell(n)}, \quad n \to \infty.$$

Under $P_\omega$, the Markov chain $X$ is reversible and it is well-known (see e.g. Saloff-Coste ([26], Lemma 1.3.3 (1), page 323)) that $k \to P_\omega(X_{2k} = \varnothing)$ is non-increasing. Therefore the Tauberian theorem ([12], formula (5.26), pp.447) yields that

$$P_\omega(X_n = \varnothing) \sim 2 \alpha E\left((S_\alpha)^{-\alpha}\right) \frac{n^{\alpha-1}}{\ell(n)},$$

for $n \to \infty$ along the sequence of even numbers [the factor 2 comes from the periodicity]. Corollary 1.4 follows. □

The rest of this paper is devoted to the proof of Proposition 2.2, which will be mainly driven by the recursive equations (2.6). Aldous and Bandyopadhyay [5] pointed out the variety of contexts where the recursive equations have arisen in various models on tree, see also Peres and Pemantle [23] for the studies of a family of concave recursive iterations.
using the potential theory. We analyze here the equations (2.6) in the spirit of [15] by establishing some comparison inequalities on the concave iteration.

The key point in the proof of Proposition 2.2 will be the asymptotic behavior of $E(B_ε(∅))$. In Section 3, we obtain the lower bound for $E(B_ε(∅))$ for all $κ ∈ (1, ∞]$ and get the convergence in law for $ε^{-1/2}B_ε(∅)$ for $κ ∈ (2, ∞]$. The upper bound of $E(B_ε(∅))$ will be presented in Section 4, where we shall complete the proof of Proposition 2.2 by establishing the almost sure convergence of $B_ε(∅)$ to $M_∞$.

Throughout this paper, $C, C'$ and $C''$ (eventually with some subscripts) denote some unimportant constants whose values may vary from one paragraph to another.

### 3 Concave recursions on trees

Let $0 < ε < 1$. By (2.6), $B_ε(∅)$ is a nonnegative solution of the following equation in law:

$$B_ε \overset{\text{law}}{=} \sum_{i=1}^\nu A_i \frac{ε + B_ε(i)}{1 + B_ε(i)},$$

where as before, $(A_i, 1 ≤ i ≤ ν) ≡ (A(x), |x| = 1)$ and conditioned on $(A_i, 1 ≤ i ≤ ν), B_ε(i)$ are i.i.d., and are distributed as $B_ε$. We recall that $E(\sum_{i=1}^\nu A_i) = 1$ and $E(\sum_{i=1}^\nu A_i^κ) = 1$ if $κ < ∞$.

It is easy to get the uniqueness among the nonnegative solutions. Indeed, If $B_ε$ and $\tilde{B}_ε$ are two nonnegative solutions, then in some enlarged probability space, we can find a coupling of $(A_i, 1 ≤ i ≤ ν), (B_ε, B_ε(i), 1 ≤ i ≤ ν)$ and $(\tilde{B}_ε, \tilde{B}_ε(i), 1 ≤ i ≤ ν)$ such that the equation (3.1) hold a.s. for $B_ε$ and $\tilde{B}_ε$. Since $B_ε$ is stochastically dominated by $\sum_{i=1}^\nu A_i$ hence integrable, we get that $E|B_ε - \tilde{B}_ε| ≤ E\frac{ε + B_ε}{1 + B_ε} - \frac{ε + \tilde{B}_ε}{1 + \tilde{B}_ε} ≤ (1 - ε)E|B_ε - \tilde{B}_ε|$ which implies that $B_ε = \tilde{B}_ε$ and the claimed uniqueness in law. Therefore we write indistinguishably $B_ε ≡ B_ε(∅)$.

This section is devoted to the asymptotic behaviors of $E(B_ε)$ as $ε → 0$. Specifically, if $κ ∈ (2, ∞]$ which is the easier case, we shall obtain an exact asymptotic of $E(B_ε)$ as $ε → 0$, whereas for $κ ∈ (1, 2]$ we shall get a lower bound, the correspondent upper bound will be proved in Section 4.

First we check that $B_ε → 0$ in $L^1(P)$. Notice that $E(B_ε) = E\frac{ε + B_ε}{1 + B_ε}$ (since $E(\sum_{i=1}^\nu A_i) = 1$), which after simplification gives that

$$E\left(\frac{B_ε^2}{1 + B_ε}\right) = ε E\left(\frac{1}{1 + B_ε}\right).$$
Lemma 3.1

Let \( \xi \) be a nonnegative random variable with finite mean. We shall use several times the following inequality in [15], formula (3.3):

\[
\mathbf{E}(B_\varepsilon) = \mathbf{E} \left[ \frac{\varepsilon + B_\varepsilon}{1 + B_\varepsilon} \right] \leq \varepsilon + \mathbf{E} \left[ \frac{B_\varepsilon}{1 + B_\varepsilon} \right] \leq \varepsilon + \left( \mathbf{E} \left[ \frac{B_\varepsilon^2}{(1 + B_\varepsilon)^2} \right] \right)^{1/2},
\]

which in view of (3.2) yield that for any \( \kappa \in (1, \infty) \),

\[
(3.3) \quad \mathbf{E}(B_\varepsilon) \leq 2\varepsilon^{1/2}, \quad 0 < \varepsilon \leq 1.
\]

The above upper bound is sharp (up to a constant) only in the case \( \kappa \in (2, \infty] \). To obtain the lower bound on \( \mathbf{E}(B_\varepsilon) \), we shall need some inequalities on the concave iteration. Let us adopt the following notation in the rest of this paper:

\[
\langle \xi \rangle := \frac{\xi}{\mathbf{E}(\xi)},
\]

for any nonnegative random variable \( \xi \) with finite mean [as such, \( \mathbf{E}(\langle \xi \rangle)^p = \mathbf{E}(\xi)^p / (\mathbf{E}(\xi))^p \)].

Lemma 3.1

Let \( \phi : \mathbb{R}_+ \to \mathbb{R}_+ \) be a convex \( C^1 \)-function. For any nonnegative random variable \( \xi \) with finite mean and any \( 0 \leq \varepsilon < 1 \), we have

\[
\mathbf{E}\phi \left( \langle \varepsilon + \xi \rangle \right) \leq \mathbf{E}\phi \left( \langle \xi \rangle \right).
\]

Proof: We shall use several times the following inequality in [15], formula (3.3): Let \( x_0 \in \mathbb{R}_+ \) and let \( I \subset \mathbb{R}_+ \) be an open interval containing \( x_0 \). Assume that \( h : I \times \mathbb{R}_+ \to (0, \infty) \) is a Borel function such that \( \frac{\partial h}{\partial x} \) existe and

- \( \mathbf{E}[h(x_0, \xi)] < \infty \) and \( \mathbf{E} \left[ \phi \left( \langle h(x_0, \xi) \rangle \right) \right] < \infty \);
- \( \mathbf{E} \left[ \sup_{x \in I} \left( \frac{\partial h}{\partial x}(x, \xi) \right) + |\phi'| \left( \langle h(x, \xi) \rangle \right) \right] \left( \mathbf{E} \left[ \frac{\partial h(x, \xi)}{\mathbf{E}[h(x, \xi)]} \right] + \frac{h(x, \xi)}{\mathbf{E}[h(x, \xi)]^2} \right) \mathbf{E} \left[ \phi \left( \langle \frac{\partial h(x, \xi)}{\mathbf{E}[h(x, \xi)]} \rangle \right) \right] < \infty \);
- both \( y \to h(x_0, y) \) and \( y \to \frac{\partial}{\partial x} \log h(x, y)|_{x=x_0} \) are monotone on \( \mathbb{R}_+ \).

Then depending on whether \( h(x_0, \cdot) \) and \( \frac{\partial}{\partial x} \log h(x_0, \cdot) \) have the same monotonicity,

\[
(3.4) \quad \frac{d}{dx} \mathbf{E}\phi \left( \langle h(x, \xi) \rangle \right) \bigg|_{x=x_0} \geq 0, \quad \text{or} \quad \leq 0.
\]

Applying (3.4) to \( h(x, y) := \frac{x+y}{1+y} \), \( 0 < x < 1 \) and \( y \geq 0 \). For any fixed \( x_0 \in (0, 1) \), \( h(x_0, \cdot) \) is non-decreasing whereas \( \frac{\partial}{\partial x} \log h(x_0, \cdot) = \frac{1}{x_0 \cdot} \) is non-increasing. Therefore \( x_0 \in (0, 1) \mapsto \mathbf{E}\phi \left( \langle h(x_0, X) \rangle \right) \) is non-increasing. It follows that for any \( 0 < \varepsilon < 1 \),

\[
\mathbf{E}\phi \left( \langle \varepsilon + \xi \rangle \right) \leq \mathbf{E}\phi \left( \langle \xi \rangle \right).
\]

Now we take \( h(x, y) := \frac{y}{1+xy} \) for \( x \in (0, 1) \) and \( y \geq 0 \) in (3.4) and get that \( x \in (0, 1) \mapsto \mathbf{E}\phi \left( \langle \frac{\xi}{1+x\xi} \rangle \right) \) is non-increasing. Hence \( \mathbf{E}\phi \left( \langle \frac{\xi}{1+x\xi} \rangle \right) \leq \mathbf{E}\phi \left( \langle \xi \rangle \right) \) which gives the Lemma. \( \square \)
Lemma 3.2 Assume (1.1) and (1.2). For any \( p \in (1, 2] \cap (1, \kappa) \), there exists some positive constant \( C = C_{p, \kappa} \) such that for any \( 0 < \varepsilon < 1 \),

\[
(3.5) \quad E\left( \langle B_\varepsilon \rangle^p \right) \leq C
\]

\[
(3.6) \quad E\left( \frac{B_\varepsilon^2}{1 + B_\varepsilon} \right)^p \leq C.
\]

Proof of Lemma 3.2 The proof of (3.5) was already given in ([15], Proposition 5.1) in the case that \( \nu \) equals some integer larger than 2. The same proof can be adopted to the case of random \( \nu \) and we skip the details.

To prove (3.6), we apply (3.4) to \( h(x, y) = \frac{x^2}{x+y}, x > 0, y \geq 0 \). For any \( x_0 > 0 \), \( h(x_0, \cdot) \) is increasing whereas \( y \mapsto \frac{\partial}{\partial x} \log h(x_0, y) = -\frac{1}{x_0+y} \) is also increasing; Hence the function \( x \in \mathbb{R}^+ \mapsto E\left( \frac{B_\varepsilon^2}{x+B_\varepsilon} \right)^p \), is non-decreasing on \( x \). Since \( E(B_\varepsilon) \leq 1 \), we have

\[
E\left( \frac{B_\varepsilon^2}{1+B_\varepsilon} \right)^p \leq E\left( \frac{B_\varepsilon^2}{x+B_\varepsilon} \right)^p = E\left( \frac{B_\varepsilon}{1+B_\varepsilon} \right)^p.
\]

Observe that \( E\left( \frac{(B_\varepsilon)^2}{1+(B_\varepsilon)^2} \right)^p \leq E((B_\varepsilon)^p) \leq C \), whereas by Jensen’s inequality [the function \( x \mapsto \frac{x^2}{1+x} \) is convex], \( E\left( \frac{(B_\varepsilon)^2}{1+(B_\varepsilon)^2} \right) \geq \frac{1}{2} \) [recalling \( E(B_\varepsilon) = 1 \)], we get that

\[
E\left( \frac{(B_\varepsilon)^2}{1+(B_\varepsilon)^2} \right)^p = \frac{E\left( \frac{(B_\varepsilon)^2}{1+(B_\varepsilon)^2} \right)^p}{\left( E\left( \frac{(B_\varepsilon)^2}{1+(B_\varepsilon)^2} \right) \right)^p} \leq 2C,
\]

yielding (3.6) by eventually choosing a larger constant. \( \square \)

To get a lower bound of \( E(B_\varepsilon) \), we shall use the following comparison lemma:

Lemma 3.3 Assume (1.1) and (1.2). For any \( a > 0 \), let \( \phi_a(x) := \frac{x^2}{a+x} \) for any \( x \geq 0 \). We have

\[
E\phi_a\left( \langle B_\varepsilon \rangle \right) \leq E\phi_a(M_\infty).
\]

Proof of Lemma 3.3 Let \( a > 0 \). It is elementary to check that the function \( \phi_a \) is convex. Moreover, for any \( b \geq 0 \) and \( t > 0 \), the function \( x \mapsto \phi_a(b+tx) \) is still convex. By Lemma 3.1 we get that for any \( b \geq 0 \) and \( t > 0 \),

\[
(3.7) \quad E\phi_a\left( b + t \langle \frac{\varepsilon + \xi}{1+\xi} \rangle \right) \leq E\phi_a(b + t \langle \xi \rangle).
\]

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Recall \((2.3)\). Choose \(\lambda\) such that \(1 - e^{-2\lambda} = \varepsilon\). Define
\[
B_n(x) := \sum_{i=1}^{\nu_x} A(x^{(i)}) \beta_{n,\lambda}(x^{(i)}), \quad \forall |x| \leq n.
\]
Then \(B_\varepsilon = B_\varepsilon(\emptyset) = \lim_{n \to \infty} B_n(\emptyset)\), \(\mathbb{P}\)-almost surely. For any \(|x| < n\), we deduce from \((2.3)\) that
\[
B_n(x) = \sum_{i=1}^{\nu_x} A(x^{(i)}) \frac{\varepsilon + B_n(x^{(i)})}{1 + B_n(x^{(i)})}.
\]
Since \(\mathbb{E}(\sum_{i=1}^{\nu_x} A(x^{(i)})) = 1\), we get that
\[
\langle B_n(x) \rangle = \sum_{i=1}^{\nu_x} A(x^{(i)}) \left( \frac{\varepsilon + B_n(x^{(i)})}{1 + B_n(x^{(i)})} \right).
\]
Applying \((3.7)\) to \(\xi = B_n(x^{(1)}), t = A(x^{(1)})\) and \(b := 1_{(\nu_x \geq 2)} \sum_{i=2}^{\nu_x} A(x^{(i)}) \left( \frac{\varepsilon + B_n(x^{(i)})}{1 + B_n(x^{(i)})} \right)\) and conditioning on \((t, b)\), we have that
\[
\mathbb{E} \phi_\alpha(\langle B_n(x) \rangle) \leq \mathbb{E} \phi_\alpha \left( 1_{(\nu_x \geq 2)} \sum_{i=2}^{\nu_x} A(x^{(i)}) \left( \frac{\varepsilon + B_n(x^{(i)})}{1 + B_n(x^{(i)})} \right) + A(x^{(1)}) \langle B_n(x^{(1)}) \rangle \right).
\]
In the right-hand-side of the above inequality, applying \((3.7)\) successively to \(B_n(x^{(2)}), ..., B_n(x^{(\nu_x)})\) with obvious choices of \(t\) and \(b\), we get that for any \(|x| < n\),
\[
\mathbb{E} \phi_\alpha \left( \langle B_n(x) \rangle \right) \leq \mathbb{E} \phi_\alpha \left( \sum_{i=1}^{\nu_x} A(x^{(i)}) \langle B_n(x^{(i)}) \rangle \right).
\]
Notice by definition \(B_n(x) = \sum_{i=1}^{\nu_x} A(x^{(i)})\) for \(|x| = n - 1\). By iterating the above inequalities, we get that
\[
\mathbb{E} \phi_\alpha \left( \langle B_n(\emptyset) \rangle \right) \leq \mathbb{E} \phi_\alpha \left( \sum_{|x| = n} \prod_{\emptyset < y \leq x} A(y) \right) = \mathbb{E} \phi_\alpha(M_n).
\]
Lemma \((3.3)\) follows by letting \(n \to \infty\). \(\square\)

**Lemma 3.4** Assume \((1.1), (1.2)\) and \((1.3)\). We have
\[
\liminf_{\varepsilon \to 0} \varepsilon^{-1/\kappa} \mathbb{E}(B_\varepsilon) \geq c_4, \quad \text{if } 1 < \kappa < 2,
\]
\[
\liminf_{\varepsilon \to 0} \left( \frac{\log 1/\varepsilon}{\varepsilon} \right)^{1/2} \mathbb{E}(B_\varepsilon) \geq (2c_M)^{-1/2}, \quad \text{if } \kappa = 2,
\]
\[
\liminf_{\varepsilon \to 0} \varepsilon^{-1/2} \mathbb{E}(B_\varepsilon) \geq c_5, \quad \text{if } \kappa = 2,
\]
where \(c_4\) and \(c_5\) are as in Proposition \((2.2)\) and \(c_M > 0\) is given in \((1.6)\).
Proof of Lemma 3.4. If $\kappa \in (2, \infty]$, we remark that $E(\sum_{i=1}^{\nu} A_i^2) < 1$ and $E(M_\infty^2) = \frac{E(\sum_{i=1}^{\nu} A_i^2)}{1 - E(\sum_{i=1}^{\nu} A_i^2)}$. It is elementary to check that as $a \to \infty$,

$$E\left[\frac{M_\infty^2}{a + M_\infty}\right] \sim \begin{cases} c_M c_\kappa a^{1-\kappa}, & \text{if } 1 < \kappa < 2, \\ 2 c_M \log a, & \kappa = 2, \\ \frac{1}{a} E(M_\infty^2), & \kappa \in (2, \infty], \end{cases}$$

(3.11)

where for $1 < \kappa < 2$, $c_\kappa := \int_0^\infty dy \frac{(2+y)^{\kappa+1}}{(1+y)^2} = \kappa B(2 - \kappa, \kappa - 1)$.

Recall from (3.2) that $E(B_{\varepsilon}^2) = \varepsilon E\left(\frac{1}{1 + B_{\varepsilon}}\right)$ which can be re-written as

$$E\left[\frac{\langle B_{\varepsilon}\rangle^2}{a + \langle B_{\varepsilon}\rangle}\right] = a \varepsilon E\left(\frac{1}{1 + B_{\varepsilon}}\right) \sim a \varepsilon, \quad \varepsilon \to 0,$$

(3.12)

where $a \equiv a(\varepsilon) := 1/E(B_{\varepsilon}) \to \infty$ by (3.3). By Lemma 3.3,

$$E\left[\frac{\langle B_{\varepsilon}\rangle^2}{a + \langle B_{\varepsilon}\rangle}\right] \leq E\left[\frac{M_\infty^2}{a + M_\infty}\right].$$

Hence for $a = 1/E(B_{\varepsilon})$,

$$\liminf_{\varepsilon \to 0} \frac{1}{a} E\left[\frac{M_\infty^2}{a + M_\infty}\right] \geq 1,$$

which in view of (3.11) yield the Lemma.

We are ready to deal with the asymptotic behaviors of $B_{\varepsilon}$ when $\kappa \in (2, \infty]$:

**Proposition 3.5** Assume (1.1) and (1.2). If $\kappa \in (2, \infty]$, then under the probability $P$, as $\varepsilon \to 0$,

$$\varepsilon^{-1/2} B_{\varepsilon} \xrightarrow{\text{law}} c_5 M_\infty,$$

with $c_5 := \left(\frac{1 - E(\sum_{i=1}^{\nu} A_i^2)}{E(\sum_{i=1}^{\nu} A_i^2)}\right)^{1/2}$ as in Proposition 2.2. Moreover,

$$\lim_{\varepsilon \to 0} \varepsilon^{-1/2} E(B_{\varepsilon}) = c_5.$$

**Proof of Proposition 3.5** Based on the boundedness in $L^2$ of $\varepsilon^{-1/2} B_{\varepsilon}$ (cf. (3.5)), it suffices to prove the convergence in law. Let us first show the tightness of $\frac{B_{\varepsilon}}{\sqrt{\varepsilon}}$ as $\varepsilon \to 0$. By (3.3) and (3.10),

$$c_5 \leq \liminf_{\varepsilon \to 0} \frac{E(B_{\varepsilon})}{\varepsilon^{1/2}} \leq \limsup_{\varepsilon \to 0} \frac{E(B_{\varepsilon})}{\varepsilon^{1/2}} \leq 2.$$
In particular, under $P$, the family of the laws of $(\frac{B_{\epsilon}}{\sqrt{\epsilon}}, \epsilon \to 0)$ is tight. For any subsequence $\epsilon_n \to 0$ such that $\frac{B_{\epsilon_n}}{\sqrt{\epsilon_n}} \xrightarrow{(\text{law})} \xi$, for some nonnegative r.v. $\xi$. By (3.13), $\xi$ is not degenerate; moreover, we deduce from (3.1) that $\xi$ must satisfy the cascade equation:

$$\xi \sim \sum_{i=1}^{\nu} A_i \xi_i,$$

where conditioned on $(A_i)$, $\xi_i$ are i.i.d. copies of $\xi$. By the uniqueness of the solution (see Liu [17]), $\xi = c M_{\infty}$ for some positive constant $c$. We re-write (3.2) as

$$\mathbb{E}\left(\frac{(\frac{B_{\epsilon}}{\sqrt{\epsilon}})^2}{1 + B_{\epsilon}}\right) = \mathbb{E}\left(\frac{1}{1 + B_{\epsilon}}\right),$$

which by Fatou’s lemma along the subsequence $\epsilon_n \to 0$, gives that $c^2 \mathbb{E}(M_{\infty}^2) \leq 1$, i.e. $c \leq (\mathbb{E}(M_{\infty}^2))^{-1/2} = c_5$. This in view of the lower bound in (3.13) imply that $c = c_5$. Then we have proved that any subsequence of $\frac{B_{\epsilon}}{\sqrt{\epsilon}}$ converges to the same limit $c_5 M_{\infty}$, which gives the Proposition. □

4 Proof of Proposition 2.2

To prove Proposition 2.2, it suffices to show the following two statements: As $\epsilon \to 0$,

$$\mathbb{E}(B_{\epsilon}) \sim \begin{cases} 
  c_4 \epsilon^{1/\kappa}, & \text{if } 1 < \kappa < 2, \\
  (2cM)^{-1/2} \left(\frac{\epsilon}{\log \frac{1}{\epsilon}}\right)^{1/2}, & \text{if } \kappa = 2, \\
  c_5 \epsilon^{1/2}, & \text{if } \kappa \in (2, \infty]. 
\end{cases}$$

(4.1)

$$\langle B_{\epsilon}(\emptyset) \rangle \to M_{\infty}, \quad P\text{-a.s.}$$

(4.2)

The $L^p$-convergence will follow from (4.2) and the $L^p$-boundedness of $\langle B_{\epsilon}(\emptyset) \rangle$ given in (3.5).

Some preparations first. Using the elementary inequality: $\frac{\epsilon + x}{1 + x} = x + \frac{\epsilon x}{1 + x} \geq x - \frac{x^2}{1 + x}$, we deduce from (2.6) that

$$\langle B_{\epsilon}(x) \rangle = \sum_{y: y=x} A(y) \frac{1}{\mathbb{E}(B_{\epsilon})} \frac{\epsilon + B_{\epsilon}(y)}{1 + B_{\epsilon}(y)}$$

(4.3)

$$\geq \sum_{y: y=x} A(y) \langle B_{\epsilon}(y) \rangle - \sum_{y: y=x} A(y) \Delta(y),$$
with
\[
\Delta(y) := \frac{1}{\mathbb{E}(B_\varepsilon)} \frac{B_\varepsilon(y)^2}{1 + B_\varepsilon(y)},
\]
where as before \( B_\varepsilon \equiv B_\varepsilon(\emptyset) \), and conditioned on \((A(y), y \leftarrow x, \nu_x)\), \( \Delta(y) \) are i.i.d. copies of \( \Delta := \frac{1}{\mathbb{E}(B_\varepsilon)} \frac{B_\varepsilon(y)^2}{1 + B_\varepsilon(y)} \).

To get an upper bound, we dominate \( \varepsilon + B_\varepsilon(y) \) by \( \varepsilon + B_\varepsilon(y) \), and get that
\[
\langle B_\varepsilon(x) \rangle \leq \sum_{y, \tilde{y} = x} A(y) \langle B_\varepsilon(y) \rangle + \frac{\varepsilon}{\mathbb{E}(B_\varepsilon)} \sum_{y, \tilde{y} = x} A(y).
\]

By iterating (4.3), we get that for any \( m \geq 1 \),
\[
\langle B_\varepsilon \rangle \equiv \langle B_\varepsilon(\emptyset) \rangle \geq \sum_{|x| = m} \prod_{\emptyset < y \leq x} A(y) \langle B_\varepsilon(x) \rangle - \Theta_m,
\]
with
\[
\Theta_m := \sum_{k=1}^{m} \sum_{|x| = k} \prod_{\emptyset < y \leq x} A(y) \Delta(x),
\]
where conditioned on \((V(x), |x| \leq m), (B_\varepsilon(x), \Delta(x))\) are i.i.d. copies of \((B_\varepsilon, \Delta)\). Remark that
\[
\mathbb{E}(\Delta) = \frac{1}{\mathbb{E}(B_\varepsilon)} \mathbb{E} \left[ \frac{B_\varepsilon^2}{1 + B_\varepsilon} \right] = \frac{\varepsilon}{\mathbb{E}(B_\varepsilon)} \leq \frac{\varepsilon}{\mathbb{E}(B_\varepsilon)}.
\]
Consequently,
\[
\mathbb{E}(\Theta_m) = m \mathbb{E}(\Delta) \leq \frac{\varepsilon m}{\mathbb{E}(B_\varepsilon)}, \quad \forall m \geq 1.
\]

Similarly, by iterating (4.4), we get that for any \( m \geq 1 \),
\[
\langle B_\varepsilon \rangle \leq \sum_{|x| = m} \prod_{\emptyset < y \leq x} A(y) \langle B_\varepsilon(x) \rangle + \frac{\varepsilon}{\mathbb{E}(B_\varepsilon)} \sum_{k=1}^{m} M_k,
\]
where as before, \( M_k := \sum_{|x| = k} \prod_{\emptyset < y \leq x} A(y) \).

Observe that for any \( m \geq 1 \), \( M_\infty = \sum_{|x| = m} \prod_{\emptyset < y \leq x} A(y) M^{(x)}_\infty \), where conditioned on \((A(x), |x| \leq m), M^{(x)}_\infty\) are i.i.d. copies of \( M_\infty \). Let
\[
Y_m := \sum_{|x| = m} \prod_{\emptyset < y \leq x} A(y) (\langle B_\varepsilon(x) \rangle - M^{(x)}_\infty).
\]
The following fact is due to Petrov [25], pp. 82, (2.6.20): Let \( k \geq 1 \) and \( 1 \leq p \leq 2 \). Let \( \xi_1, \ldots, \xi_k \) be independent random variables such that \( \mathbb{E}(|\xi_i|^p) < \infty \) and \( \mathbb{E}(\xi_i) = 0 \) for all \( 1 \leq i \leq k \). Then
\[
\mathbb{E} \left( |\xi_1 + \cdots + \xi_k|^p \right) \leq 2 \sum_{i=1}^{k} \mathbb{E}(|\xi_i|^p).
\]
Applying (4.8) to $Y_m$ yields that for any $p \in (1, \kappa) \cap (1, 2]$,
\[
E\left(|Y_m|^p\right) \leq 2E\left(\sum_{|x|=m \ni y \leq x} (A(y))^p\right) E\left(|\langle B_\varepsilon \rangle - M_\infty|^p\right)
\]
by using the fact that $E(|\langle B_\varepsilon \rangle - M_\infty|^p)$ is bounded as $\varepsilon \to 0$.

To prove (4.11), we only need to get an upper bound of $E(B_\varepsilon)$ in the case $1 < \kappa \leq 2$.

We present a preliminary lemma:

**Lemma 4.1** Assume (1.1) and (1.2). Let $1 < \kappa \leq 2$. For any $0 < \delta < 1$ and $b > 1$, there exists some $\varepsilon_0 \equiv \varepsilon_0(b, \delta) > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and $\varepsilon^{1-1/\kappa}(\log 1/\varepsilon)^2 \leq r \leq \varepsilon^{-b}$, we have
\[
P\left(|\langle B_\varepsilon \rangle - M_\infty|^p\right) \leq C(E\nu \sum_{i=1}^m A_i^p)^m,
\]

**Proof of Lemma 4.1.** Fix a small $\delta > 0$. Let $m \geq 1$. Since $\langle B_\varepsilon \rangle \geq M_\infty + Y_m - \Theta_m$, we get that for any $r > 0$,
\[
P\left(\langle B_\varepsilon \rangle > r\right) \geq P\left(M_\infty > (1 + \delta)r\right) - P\left(|Y_m| > \frac{\delta}{2} r\right) - P\left(\Theta_m > \frac{\delta}{2} r\right).
\]
Choose $r$ such that
\[
\frac{\delta}{4} r \geq \frac{\varepsilon m}{E(B_\varepsilon)}.
\]
Let $p \in (1, \kappa)$. By (4.6), we have
\[
P\left(\Theta_m > \frac{\delta}{2} r\right) \leq P\left(\Theta_m - E(\Theta_m) > \frac{\delta}{4} r\right) \leq \left(\frac{\delta}{4} r\right)^{-p} E|\Theta_m - E(\Theta_m)|^p.
\]
By the convexity,
\[
E|\Theta_m - E(\Theta_m)|^p \leq E\left|\sum_{k=1}^m \sum_{|x|=k \ni y \leq x} \prod_{|x|=k \ni y \leq x} A(y)(\Delta(x) - E(\Delta(x)))\right|^p
\]
\[
\leq m^{p-1} \sum_{k=1}^m E\left|\sum_{|x|=k \ni y \leq x} \prod_{|x|=k \ni y \leq x} A(y)(\Delta(x) - E(\Delta(x)))\right|^p
\]
\[
\leq 2m^{p-1} \sum_{k=1}^m E\left(\sum_{|x|=k \ni y \leq x} A(y)^p\right) E|\Delta - E(\Delta)|^p,
\]

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where the last inequality is a consequence of the use of (4.8). Notice that $E(\langle \Delta \rangle^p) = E\left(\frac{B^2}{1+B}\right)^p \leq C$ by (3.6). Hence $E|\Delta - E(\Delta)|^p \leq C' (E(\Delta))^p \leq C' \left(\frac{\varepsilon}{E(B_\varepsilon)}\right)^p$. It follows that for any $p \in (1, \kappa)$,

\begin{equation}
E(\Theta_m - E(\Theta_m))^p \leq C'' m^{p-1} \left(\frac{\varepsilon}{E(B_\varepsilon)}\right)^p,
\end{equation}

by using the fact that $\sum_{k=1}^m E\left(\sum_{|x|=k} \prod_{0 < y \leq x} A(y)^p\right) = \sum_{k=1}^m (E \sum_{i=1}^\nu A_i)^k$ is bounded by some constant for all $m \geq 1$. Then for any $p \in (1, \kappa)$, if $\frac{\delta}{4} r \geq \frac{\varepsilon m}{E(B_\varepsilon)}$, then

$$P\left(\Theta_m > \frac{\delta}{2} r\right) \leq C (\delta r)^{-p} m^{p-1} \left(\frac{\varepsilon}{E(B_\varepsilon)}\right)^p.$$ 

For the term $Y_m$, we use (4.9) and get that

$$P\left(|Y_m| > \frac{\delta}{2} r\right) \leq C (\delta r)^{-p} (E \sum_{i=1}^\nu A_i)^m.$$ 

Notice that $E(\sum_{i=1}^\nu A_i^p) < 1$ for $p \in (1, \kappa)$ ($\kappa \leq 2$). By choosing $m := \lfloor C_p \log_{1/\varepsilon} \frac{1}{\varepsilon} \rfloor$ with a sufficiently large constant $C_p'$, we get that for any $p \in (1, \kappa)$ and for all $r$ such that $\frac{\delta}{4} r \geq C_p' \frac{\varepsilon \log\frac{1}{\varepsilon}}{E(B_\varepsilon)}$, 

$$P\left(\langle B_\varepsilon \rangle > r\right) \geq P\left(M_\infty > (1 + \delta) r\right) - (\delta r)^{-p} C_p (\log \frac{1}{\varepsilon})^{p-1} \left(\frac{\varepsilon}{E(B_\varepsilon)}\right)^p \geq P\left(M_\infty > (1 + \delta) r\right) - (\delta r)^{-p} \varepsilon^{p-p/\kappa + o(1)},$$

where the last inequality follows from the lower bound of $E(B_\varepsilon)$ in Lemma 3.4.

By the lower tail of $M_\infty$ (see Liu [17], Theorem 2.2), there exists some positive constant $C$ such that

$$P\left(M_\infty > (1 + \delta) r\right) \geq C r^{-\kappa}, \quad \forall r \geq 1, 0 < \delta < 1,$$

which easily yields Lemma 4.1 since $p$ can chosen as close to $\kappa$ as possible.

We now have all ingredients to give the proof of (4.1).

**Proof of (4.1):** By using Proposition 3.5, it remains to show the upper bound of $E(B_\varepsilon)$ when $1 < \kappa \leq 2$.

Recall from (3.12) that

\begin{equation}
E(\frac{(B_\varepsilon)^2}{E(B_\varepsilon)} + B_\varepsilon) \sim \frac{\varepsilon}{E(B_\varepsilon)}, \quad \varepsilon \to 0.
\end{equation}
Let $b = 1/(\kappa - 1)$ and $0 < \delta < 1$. Applying Lemma 4.1 to $\delta \leq r \leq \varepsilon^{-b}$, we have by the integration by parts that (with $a := \frac{1}{E(B_\varepsilon)}$)

$$
\mathbb{E}\frac{\langle B_\varepsilon \rangle^2}{a + \langle B_\varepsilon \rangle} \geq \int_\delta^{\varepsilon^{-a}} \frac{r(2a + r)}{(a + r)^2} \mathbb{P}(\langle B_\varepsilon \rangle > r)dr
$$

$$
\geq (1 - \delta) \int_\delta^{\varepsilon^{-a}} \frac{r(2a + r)}{(a + r)^2} \mathbb{P}(M_\infty > (1 + \delta)r)dr
$$

$$
= (1 - \delta) \mathbb{E}\left[ \frac{M_\infty^2}{a + M_\infty} 1_{\{\delta \leq M_\infty \leq \varepsilon^{-a}\}} \right].
$$

Observe that $\mathbb{E}\left[ \frac{M_\infty^2}{a + M_\infty} 1_{\{M_\infty \leq \delta\}} \right] \leq \frac{\delta^2}{a}$, and

$$
\mathbb{E}\left[ \frac{M_\infty^2}{a + M_\infty} 1_{\{M_\infty > \varepsilon^{-b}\}} \right] \leq C \int_{\varepsilon^{-b}}^{\infty} \frac{r^2}{a + r} r^{-\kappa - 1} dr \leq \frac{C}{\kappa - 1} \varepsilon^{b(\kappa - 1)}.
$$

Therefore

$$
\mathbb{E}\left[ \frac{M_\infty^2}{a + M_\infty} 1_{\{\delta \leq M_\infty \leq \varepsilon^{-a}\}} \right] \geq \mathbb{E}\left[ \frac{M_\infty^2}{a + M_\infty} \right] - \frac{\delta^2}{a} - \frac{C}{\kappa - 1} \varepsilon^{b(\kappa - 1)},
$$

with $a := \frac{1}{E(B_\varepsilon)}$. In view of (3.11), we get that for any $1 < \kappa \leq 2$, as $\varepsilon \to 0$,

$$
\mathbb{E}\left[ \frac{M_\infty^2}{a + M_\infty} 1_{\{\delta \leq M_\infty \leq \varepsilon^{-a}\}} \right] \sim \mathbb{E}\left[ \frac{M_\infty^2}{a + M_\infty} \right] \sim \begin{cases} 
C M c_* a^{-1 - \kappa}, & \text{if } 1 < \kappa < 2, \\
2 C M \log \frac{a}{\varepsilon}, & \text{if } \kappa = 2,
\end{cases}
$$

which together with (4.11) and (4.12) yield the desired upper bound for $\mathbb{E}(B_\varepsilon)$. Thus we get (4.1).

Now we are ready to prove (4.2):

**Proof of (4.2):** Assembling (4.5) and (4.7), we get that for any $m \geq 1$,

$$
|\langle B_\varepsilon(\varnothing) \rangle - M_\infty| \leq \frac{\varepsilon}{\mathbb{E}(B_\varepsilon)} \sum_{k=1}^m M_k + |Y_m| + \Theta_m.
$$

Denote by $\| \cdot \|_p$ the $L^p$-norm with respect to $\mathbb{P}$. It follows that for any $p \in (1, \kappa) \cap (1, 2]$ and $m \geq 1$,

$$
\|\langle B_\varepsilon(\varnothing) \rangle - M_\infty\|_p \leq \frac{\varepsilon}{\mathbb{E}(B_\varepsilon)} \sum_{k=1}^m \|M_k\|_p + \|Y_m\|_p + \|\Theta_m\|_p
$$

$$
\leq C \frac{\varepsilon}{\mathbb{E}(B_\varepsilon)} m + C \left( \mathbb{E} \sum_{i=1}^\nu A_i^p \right)^{m/p}
$$
by using (4.9), (4.6), (4.10) and the fact that sup \( k \geq 1 \|M_k\|_p < \infty \). Since \( E \sum_{i=1}^\nu A_i^p < 1 \), we choose \( m = \lceil C' \log 1/\varepsilon \rceil \) with a sufficiently large constant \( C' \) and use the lower bound of \( E(B_\varepsilon) \) in Lemma 3.3. This leads to that for any \( p \in (1, \kappa) \cap (1, 2] \)

\[ \left\| \langle B_\varepsilon(\emptyset) \rangle - M_\infty \right\|_p \leq \varepsilon^{1+o(1)}. \]

Let \( \varepsilon_n := n^{-2} \). The convergence part of the Borel-Cantelli lemma yields that as \( n \to \infty \),

\[ \langle B_{\varepsilon_n}(\emptyset) \rangle \to M_\infty, \quad P\text{-a.s.} \]

Observe that \( \varepsilon \to B_\varepsilon \) is non-increasing, hence for any \( \varepsilon_n \leq \varepsilon < \varepsilon_{n-1} \), \( \langle B_{\varepsilon_n}(\emptyset) \rangle \frac{E(B_{\varepsilon_n})}{E(B_{\varepsilon_{n-1}})} \leq \langle B_\varepsilon(\emptyset) \rangle \leq \langle B_{\varepsilon_n}(\emptyset) \rangle \frac{E(B_{\varepsilon_{n-1}})}{E(B_{\varepsilon_n})} \), which readily yield (4.2). This completes the proof of Proposition 2.2. \( \Box \)

References

[1] Aïdékon, E. (2008). Transient random walks in random environment on a Galton–Watson tree. *Probab. Theory Related Fields* **142**, 525–559.

[2] Aïdékon, E. (2010). Large deviations for transient random walks in random environment on a Galton–Watson tree. *Ann. Inst. H. Poincaré Probab. Statist.* **46**, 159–189.

[3] Aïdékon, E. (2011+). Speed of the biased random walk on a Galton–Watson tree. *Probab. Theory Related Fields* (to appear); arXiv:1111.4313

[4] Aïdékon, E. and de Raphélis, L. (2015+). (In preparation)

[5] Aldous, D. and Bandyopadhyay, A. (2005). A survey of max-type recursive distributional equations. *Ann. Appl. Probab.* **15**, 1047–1110.

[6] Andreoletti, P. and Debs, P. (2014). The number of generations entirely visited for recurrent random walks on random environment. *J. Theoret. Probab.* **27**, 518–538.

[7] Ben Arous, G. and Cerny, J. (2006). Dynamics of trap models. *École de Physique des Houches, Session LXXXIII*, Mathematical Statistical Physics, 331–391. Elsevier, Amsterdam.

[8] Ben Arous, G., Fribergh, A., Gantert, N. and Hammond, A. (2012). Biased random walks on a Galton-Watson tree with leaves. *Ann. Probab.* **40**, 280–338.

[9] Ben Arous, G. and Hammond, A. (2012). Randomly biased walks on subcritical trees. *Comm. Pure Appl. Math* **65**, 1481–1527.

[10] Faraud, G. (2011). A central limit theorem for random walk in a random environment on marked Galton-Watson trees. *Electron. J. Probab.* **16**, 174–215.
[11] Faraud, G., Hu, Y. and Shi, Z. (2012). Almost sure convergence for stochastically biased random walks on trees. *Probab. Theory Related Fields* **154**, 621–660.

[12] Feller, W. (1971). *An Introduction to Probability and its Applications.* Vol. II. Second edition. Wiley, New York.

[13] Fristedt, B.E. and Pruitt, W.E. (1971). Lower Functions for Increasing Random Walks and Subordinators. *Z. Wahrsche. verw. Geb.* **18**, 167–182.

[14] Hammond, A. (2011+). Stable limit laws for randomly biased walks on supercritical trees. *Ann. Probab.* (to appear)

[15] Hu, Y. and Shi, Z. (2007). A subdiffusive behaviour of recurrent random walk in random environment on a regular tree. *Probab. Theory Related Fields* **138**, 521–549.

[16] Hu, Y. and Shi, Z. (2014+). Localizing biased random walks on trees. (preprint).

[17] Liu, Q.S. (2000). On generalized multiplicative cascades. *Stoch. Proc. Appl.* **86**, 263–286.

[18] Lyons, R. and Pemantle, R. (1992). Random walk in a random environment and first-passage percolation on trees. *Ann. Probab.* **20**, 125–136.

[19] Lyons, R., Pemantle, R. and Peres, Y. (1995). Ergodic theory on Galton–Watson trees: speed of random walk and dimension of harmonic measure. *Ergodic Theory Dynam. Systems* **15**, 593–619.

[20] Lyons, R., Pemantle, R. and Peres, Y. (1996). Biased random walks on Galton–Watson trees. *Probab. Theory Related Fields* **106**, 249–264.

[21] Menshikov, M.V. and Petritis, D. (2002). On random walks in random environment on trees and their relationship with multiplicative chaos. In: *Mathematics and Computer Science II (Versailles, 2002)*, pp. 415–422. Birkhäuser, Basel.

[22] Neveu, J. (1986). Arbres et processus de Galton-Watson. *Annales Inst. Henri Poincaré* Série B, **22** pp. 199–207.

[23] Pemantle, R. and Peres, Y. (2010). The critical Ising model on trees, concave recursions and nonlinear capacity. *Ann. Probab.*

[24] Peres, Y. and Zeitouni, O. (2008). A central limit theorem for biased random walks on Galton-Watson trees. *Probab. Theory Related Fields* **140**, 595–629.

[25] Petrov, V.V. (1995). *Limit Theorems of Probability Theory.* Clarendon Press, Oxford.

[26] Saloff-Coste, L. (1997). *Lectures on Finite Markov Chains*, École d’Été de Saint-Flour XXVI (1996), *Lecture Notes in Math.* **1665**, pp. 301–413. Springer, Berlin.

[27] Shi, Z. (2014+). *Branching Random Walks*, École d’Été de Saint-Flour XLII (2012), (to appear).