JACOBIAN SYZYGIES AND PLANE CURVES WITH MAXIMAL GLOBAL TJURINA NUMBERS

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Abstract. First we give a sharp upper bound for the cardinal $m$ of a minimal set of generators for the module of Jacobian syzygies of a complex projective reduced plane curve $C$. Next we discuss the sharpness of an upper bound, given by A. du Plessis and C.T.C. Wall, for the global Tjurina number of such a curve $C$, in terms of its degree $d$ and of the minimal degree $r \leq d-1$ of a Jacobian syzygy. We give a homological characterization of the curves whose global Tjurina number equals the du Plessis-Wall upper bound, which implies in particular that for such curves the upper bound for $m$ is also attained. Finally we prove the existence of curves with maximal global Tjurina numbers for certain pairs $(d, r)$. Moreover, we conjecture that such curves exist for any pair $(d, r)$, and that, in addition, they may be chosen to be line arrangements when $r \leq d-2$.

1. Introduction

Let $S = \mathbb{C}[x, y, z]$ be the polynomial ring in three variables $x, y, z$ with complex coefficients, and let $C : f = 0$ be a reduced curve of degree $d$ in the complex projective plane $\mathbb{P}^2$. We denote by $J_f$ the Jacobian ideal of $f$, i.e. the homogeneous ideal in $S$ spanned by the partial derivatives $f_x, f_y, f_z$ of $f$, and by $M(f) = S/J_f$ the corresponding graded quotient ring, called the Jacobian (or Milnor) algebra of $f$. Consider the graded $S$-module of Jacobian syzygies of $f$, namely

$$ AR(f) = \{(a, b, c) \in S^3 : af_x + bf_y + cf_z = 0\}.$$ 

We say that $C : f = 0$ is an $m$-syzygy curve if the graded $S$-module $AR(f)$ is generated by $m$ homogeneous syzygies, say $\rho_1, \rho_2, ..., \rho_m$, with $m$ minimal possible, of degrees $d_j = \deg \rho_j$ ordered such that

$$ 1 \leq d_1 \leq d_2 \leq ... \leq d_m.$$ 

In fact, the case $d_1 = 0$ occurs if and only if $C$ is a union of lines through a point, a situation which is not considered in the sequel. We call these degrees the exponents of the curve $C$ and the syzygies $\rho_1, ..., \rho_m$ a minimal set of generators for the module $AR(f)$. Note that $d_1 = mdr(f)$ is the minimal degree of a non trivial Jacobian

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syzygy in $AR(f)$. The minimal possible value for $m$ is two, since the $S$-module $AR(f)$ has rank two for any $f$. The curve $C$ is called free when $m = 2$, since then $AR(f)$ is a free module, see for such curves [1, 7, 11, 13, 26, 27, 28, 29]. We prove in Proposition 2.1, which is our first result, that for a reduced, degree $d$ curve $C$, one has

$$m \leq d_1 + d_2 - d + 3 \leq d + 1.$$  

Note that for the case $C$ a line arrangement, the slightly stronger inequality $m \leq d - 1$ was known, see [14, Corollary 1.3]. Moreover, all these inequalities, involving the integers $m, d_1, d_2$ and $d$, are sharp, see Remark 2.2.

Recall that the global Tjurina number $\tau(C)$ of the plane curve $C : f = 0$ can be defined as either the degree of the Jacobian ideal $J_f$, or as the sum of the Tjurina numbers of all the singularities of the curve $C$. With this notation, it was shown by A. du Plessis and C.T.C. Wall that one has the following result, see [17, Theorem 3.2], and also [19, Theorem 20] for a new approach.

**Theorem 1.1.** Let $C : f = 0$ be a reduced plane curve of degree $d$ and let $r = d_1$ be the minimal degree of a non-zero syzygy in $AR(f)$. Then the following hold.

1. If $r < d/2$, then $\tau(C) \leq \tau(d, r)_{\text{max}} = (d - 1)(d - r - 1) + r^2$ and the equality holds if and only if the curve $C$ is free.
2. If $d/2 \leq r \leq d - 1$, then $\tau(C) \leq \tau(d, r)_{\text{max}}$, where, in this case, we set

$$\tau(d, r)_{\text{max}} = (d - 1)(d - r - 1) + r^2 - \binom{2r - d + 2}{2}.$$ 

In this note we investigate for which curves one has equality in the above result. To have a name, we call such curves maximal Tjurina curves of type $(d, r)$. Note that for any pair $(d, r)$, with $1 \leq r < d/2$, the existence of maximal Tjurina curves of type $(d, r)$, i.e. of free curves with these invariants, follows from [10]. The characterization and the existence of maximal Tjurina curves of type $(d, r)$, with $d/2 \leq r \leq d - 1$, is our main concern in this note, and hence we assume from now on that $d/2 \leq r$. In the third section we derive a homological characterization of these maximal Tjurina curves, see Theorem 3.1. If we set $m = 2r - d + 3$, this result says that a maximal Tjurina curve of type $(d, r)$ is exactly an $m$-syzygy curve, with exponents

$$d_1 = d_2 = \cdots = d_m = r.$$ 

In particular, a maximal Tjurina curve has the largest number $m$ of generators for the module $AR(f)$ allowed by the first inequality in (1.1). In the last section we describe some existence results for such curves. For $m = 3$, the minimal possible value, a maximal Tjurina curve of type $(d, r)$ is exactly a nearly free curve of degree $d = 2r$, with exponents $d_1 = d_2 = d_3 = r$, see subsection 3.1 and hence the existence of maximal Tjurina curves for any type $(d, r) = (2r, r)$ follows again from [10].

For $m = 4$, a maximal Tjurina curve of type $(d, r)$ is a 4-syzygy curve with $d = 2r - 1$ and exponents $d_1 = d_2 = d_3 = d_4 = r$. These curves are related to nearly cuspidal rational curves, i.e. to rational curves having only unibranch singularities,
except from one singularity which has 2 branches, see subsection 4.2. It is conjectured
that a nearly cuspidal rational curve $C$ satisfies the inequalities $m \leq 4$ and
\[
\tau(d, r)_{\text{max}} - 2 \leq \tau(C) \leq \tau(d, r)_{\text{max}},
\]
see [10], and moreover $\tau(C) = \tau(d, r)_{\text{max}}$ when $m = 4$. Proposition 4.3 below describe
a sequence of nearly cuspidal rational curves $C_d$ which are maximal Tjurina curves
of type $(2r - 1, r)$ for any $r \geq 3$.

At the other extreme, for $m = d + 1$, the maximal possible value for $m$, we notice
that in this case $r = d - 1$ and an example of maximal Tjurina curve of type $(d, d - 1)$
is given by any maximal nodal rational curve of degree $d$, see Proposition 4.10.

If we go one step back, for $m = d - 1$, we have $r = d - 2$ and an example of
maximal Tjurina curve of type $(d, d - 2)$ is given by any generic arrangement of $d$
lines in $\mathbb{P}^2$, see Proposition 4.6. If we go back one more step, namely for $m = d - 3$,
and hence $r = d - 3$, we describe a sequence of line arrangements $C_d$ which are very
likely maximal Tjurina curves of type $(d, d - 3)$ for any $d \geq 6$, and we check this
claim for $d \in [6, 15]$ using SINGULAR, see [4].

In view of all these examples, we offer the following.

**Conjecture 1.2.** For any integer $d \geq 3$ and for any integer $r$ such that $d/2 \leq r \leq d - 1$, there are maximal Tjurina curves of type $(d, r)$. Moreover, for $d/2 \leq r \leq d - 2$, there are maximal Tjurina line arrangements of type $(d, r)$.

In other words, the du Plessis-Wall inequality in Theorem 1.1 is sharp for any pair
$(d, r)$ as above. The fact that line arrangements seem to give examples of maximal
Tjurina curves of type $(d, r)$ for any $r < d - 1$ may encourage further study of the
deep relation between the combinatorics of a line arrangement $A : f = 0$ and the
integer $r = mdr(f)$, see the end of subsection 4.4 for a brief discussion of this point.

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2. ON THE NUMBER OF GENERATORS OF THE JACOBIAN SYZYGIES

Consider the general form of the minimal graded resolution for the graded $S$-
module $M(f)$, the Milnor algebra of a curve $C : f = 0$, that is assumed to be not free, namely
\[
(2.1) \quad 0 \to \oplus_{i=1}^{m-2} S(-e_i) \to \oplus_{j=1}^m S(1 - d - d_j) \to S^3(1 - d) \to S,
\]
with $m \geq 3$, $e_1 \leq \ldots \leq e_{m-2}$ and $d_1 \leq \ldots \leq d_m$. Since the kernel of the map
$S^3(1 - d) \to S$ is precisely $AR(f)(1 - d)$, we see that $d_1 \leq \ldots \leq d_m$ are the exponents
of $C$ as in the Introduction. It follows from [22, Lemma 1.1] that one has
\[
(2.2) \quad e_j = d + d_{j+2} - 1 + \epsilon_j,
\]
for $j = 1, \ldots, m - 2$ and some integers $\epsilon_j \geq 1$. Using [22, Formula (13)], it follows
that one has
\[
(2.3) \quad d_1 + d_2 = d - 1 + \sum_{i=1}^{m-2} \epsilon_j.
\]
It is known that, for a reduced degree $d$ curve $C$, one has
\[ d_m \leq 2d - 4, \]
see [3, Corollary 11], as well as [20, Corollary 12] for the case of a quasi-complete
intersection ideal replacing $J_f$ and also [2, Theorem 9.4] for an even more general case.
For the case $C$ a line arrangement, one has the much stronger inequality $d_m \leq d - 2$,
see [25, Corollary 3.5]. The inequality $d_m \leq d - 1$ holds more generally for curves $C$
having as irreducible components only rational curves, see [16, Corollary 5.2]. The
first main result of this note is the following.

**Proposition 2.1.** Let $C : f = 0$ be an $m$-syzygy curve of degree $d \geq 3$, with
exponents $1 \leq d_1 \leq d_2 \leq \cdots \leq d_m$. Then $m$, the cardinal of a minimal set of
generators for the module $AR(f)$, satisfies the inequalities
\[ m \leq d_1 + d_2 - 3 \leq d_1 + 2 \leq d + 1. \]

This result was obtained independently by Philippe Ellia, see Corollary 5 (i) and
Theorem 7 (ii) in [20], with a different approach and in a more general setting: the
Jacobian ideal $J_f$ is replaced by a quasi-complete intersection.

**Proof.** If $m = 2$, then the curve $C$ is free, and there is nothing to prove, since for
such curves $d_1 + d_2 = d - 1$. Assume that $m \geq 3$. The first claim follows from the
equality (2.3), which yields $m - 2 \leq d_1 + d_2 - d + 1$, since all the numbers $\epsilon_j$
are strictly positive integers. To get the other two inequalities, recall that [16, Theorem
2.4] implies that
\[ d_1 \leq d_2 \leq d - 1. \]

\[ \Box \]

**Remark 2.2.** (i) One has the equality $m = d + 1$ in Proposition 2.1 for some curves,
in particular for any maximal nodal rational curve, see Example 4.8, Example 4.9
and Proposition 4.10 below. Moreover, the equality $m = d_1 + d_2 - d + 3$ holds for any
maximal Tjurina curve of type $(d, r)$, with $r = mdr(f) \geq d/2$, as shown in Theorem
3.1 below. In this case $d_1 = d_2 = r$.

(ii) Recall that for a generic, i.e. nodal, line arrangement $C$ in $\mathbb{P}^2$, one has $m = d - 1$
and $d_m = d - 2$, see [25, Corollary 3.5]. The fact that for any line arrangement $C$,
one has $m \leq d - 1$, see [14, Corollary 1.3], can in fact be proven using the same idea
as in the proof of Proposition 2.1. Indeed, if the the arrangement is not generic, it
follows that it has a point of multiplicity $m \geq 3$. Then [8, Theorem 1.2] implies that
\[ d_1 \leq d - m \leq d - 3, \]
and hence $d_1 + d_2 \leq 2d - 4$, since $d_2 \leq d - 1$ by [16, Theorem 2.4]. This yields
$m \leq d - 1$ in this case.

(iii) Note that, for a uninodal plane curve of degree $d$, the module $AR(f)$ has 4
minimal generators, of degrees $d - 1, d - 1, d - 1, 2d - 4$. Such curves provide also
examples for the equality $d_m = 2d - 4$. 
3. A characterization of maximal Tjurina curves

We recall now the construction of the Bourbaki ideal $B(C, \rho_1)$ associated to a degree $d$ reduced curve $C : f = 0$ and to a minimal degree non-zero syzygy $\rho_1 \in AR(f)$, see [15]. For any choice of the syzygy $\rho_1 = (a_1, b_1, c_1)$ with minimal degree $r = d_1$, we have a morphism of graded $S$-modules

(3.1) \[ S(-r) \xrightarrow{u} AR(f), \quad u(h) = h \cdot \rho_1. \]

For any homogeneous syzygy $\rho = (a, b, c) \in AR(f)_m$, consider the determinant $\Delta(\rho) = \det M(\rho)$ of the $3 \times 3$ matrix $M(\rho)$ which has as first row $x, y, z$, as second row $a_1, b_1, c_1$ and as third row $a, b, c$. Then it turns out that $\Delta(\rho)$ is divisible by $f$, see [7], and we define thus a new morphism of graded $S$-modules

(3.2) \[ AR(f) \xrightarrow{v} S(r - d + 1), \quad v(\rho) = \Delta(\rho)/f, \]

and a homogeneous ideal $B(C, \rho_1) \subset S$ such that $im v = B(C, \rho_1)(r - d + 1)$. It is known that the ideal $B(C, \rho_1)$, when $C$ is not a free curve, defines a 0-dimensional subscheme $Z(C, \rho_1)$ in $\mathbb{P}^2$, which is locally a complete intersection, see [15, Theorem 5.1]. Using this construction, we can prove the following characterization of maximal Tjurina curves, which is our second main result in this paper.

Theorem 3.1. Let $C : f = 0$ be a reduced plane curve of degree $d$, let $r = mdr(f)$ be the minimal degree of a non-zero syzygy in $AR(f)$ and assume $d/2 \leq r \leq d - 1$. Then $\tau(C) \leq \tau(d, r)_{\text{max}}$, and, if equality holds, then the minimal resolution of the graded $S$-module $AR(f)$ has the form

\[ 0 \rightarrow S(-r - 1)^{m-2} \rightarrow S(-r)^m \rightarrow AR(f) \rightarrow 0, \]

where $m = 2r - d + 3$. In particular, the exponents of the curve $C$ are given by

\[ d_1 = d_2 = \cdots = d_m = r. \]

Conversely, if $C : f = 0$ is a reduced plane curve of degree $d$, which has exponents

\[ d_1 = d_2 = \cdots = d_m = r, \]

with $m = 2r - d + 3$, then the curve $C : f = 0$ is a maximal Tjurina curve of type $(d, r)$.

Proof. Since the quotient $S^3/AR(f)$ is torsion free, it follows that the ideal $I = B(C, \rho_1)$ is saturated, and hence $P = S/I$ is a Cohen-Macaulay module. This fact has two consequences. First the Hilbert function $H_P(k) = \dim P_k$ is increasing. By definition, all the generators of $I$ have degree at least $2r - d + 1$, and hence we get

(3.3) \[ \binom{2r - d + 2}{2} = \dim S_{2r-d} = \dim P_{2r-d} \leq H_P(k), \]

for large $k$. On the other hand, on has

(3.4) \[ H_P(k) = \deg Z(C, \rho_1) = (d - 1)^2 - r(d - r - 1) - \tau(C), \]
for large $k$, see [15, Theorem 5.1]. The last two relations imply the du Plessis-Wall inequality. Moreover, we see that we have equality for the curve $C$ if and only if

$$\left(\frac{2r - d + 2}{2}\right) = \dim P_{2r - d} = H_P(k),$$

for all $k \geq 2r - d$. Since $P$ is a Cohen-Macaulay module, it follow that

$$\operatorname{reg}(P) = 2r - d,$$

where $\operatorname{reg}(P)$ denotes the Castelnuovo-Mumford regularity of the $S$-module $P$, see [18, Theorem 4.2]. The minimal resolution of $P$ has the form

$$0 \to \oplus_j S(-a_{2,j}) \to \oplus_j S(-a_{1,j}) \to S \to P \to 0,$$

where $a_{1,j} \geq 2r - d + 1$ are the degrees of the generators for the ideal $I$. It follows that all these generators must have degree $a_{1,j} = 2r - d + 1$, since by definition

$$\operatorname{reg}(P) = \max_{i,j}(a_{i,j} - 1).$$

In order to have $H_P(2r - d) = H_P(2r - d + 1)$, we need exactly

$$m' = \dim S_{2r-d+1} - \dim S_{2r-d} = 2r - d + 2$$

generators for $I$. It follows that the above minimal resolution for $P$ yields the following minimal resolution

$$0 \to S(d - 2 - 2r)^{m'-1} \to S(d - 1 - 2r)^{m'} \to I \to 0,$$

for the ideal $I = B(C, \rho_1)$. Using the exact sequence

$$0 \to S(-r) \to AR(f) \to B(C, \rho_1)(r - d + 1) \to 0,$$

it follows that $AR(f)$ is minimally generated by $m = m' + 1$ generators, all of degree $r$, the first one being $\rho_1$, and then $\rho_j$ for $j = 2, ..., m$ being chosen such that their images under $v$ generate the ideal $I$. Moreover, each of the $m' - 1$ relations among the generators of $I$ will give rise to a relation, with linear coefficients, among the syzygies $\rho_i$. It follows that the minimal resolution of the $S$-module $AR(f)$ is given by

$$0 \to S(-r - 1)^{m-2} \to S(-r)^{m} \to AR(f) \to 0.$$  

To prove the converse, it is enough to show that our hypothesis implies that the minimal resolution of the $S$-module $AR(f)$ has the form above. Indeed, the minimal resolution of the $S$-module $AR(f)$ determines both $r = m\text{d}_r(f)$ and $\tau(C)$, e.g. using the exact sequence in Corollary [3.4] below. To show that the minimal resolution of the $S$-module $AR(f)$ has the form above, we use the formula [2.3]. This formula implies that $\epsilon_j = 1$ for any $j$, and hence all the second order syzygies of $AR(f)$ have the same degree

$$\epsilon'_1 = \epsilon'_2 = \cdots = \epsilon'_{m-2} = r + 1.$$  

This implies that $e_j = e'_j + d - 1 = r + d$, for all $1 \leq j \leq m - 2$, which ends the proof of Theorem [3.1]. \qed
Remark 3.2. If we denote by $E_C$ the rank two vector bundle on $\mathbb{P}^2$ associated to the graded $S$-module $AR(f)$, then Theorem 3.1 implies the existence of an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-r - 1)^{m-2} \to \mathcal{O}_{\mathbb{P}^2}(-r)^m \to E_C \to 0,$$

for any maximal Tjurina curve $C$.

Recall the following definition, see [7, 11].

**Definition 3.3.** (i) the coincidence threshold

$$ct(f) = \max\{q : \dim M(f)_k = \dim M(f_s)_k \text{ for all } k \leq q\},$$

with $f_s$ a homogeneous polynomial in $S$ of the same degree $d$ as $f$ and such that $C_s : f_s = 0$ is a smooth curve in $\mathbb{P}^2$.

(ii) the stability threshold $st(f) = \min\{q : \dim M(f)_k = \tau(C) \text{ for all } k \geq q\}$.

**Corollary 3.4.** Let $C : f = 0$ be a reduced plane curve of degree $d \geq 3$, let $r = mdr(f)$ be the minimal degree of a non-zero syzygy in $AR(f)$. If $C$ is a maximal Tjurina curve, then the minimal resolution of the corresponding graded Milnor algebra $M(f)$, regarded as an $S$-module, has the form

$$0 \to S(-d - r)^{m-2} \to S(1 - r - d)^m \to S(1 - d)^3 \to S \to M(f) \to 0.$$

In particular, one has

$$ct(f) = st(f).$$

**Proof.** The first claim follows from the obvious exact sequence

$$0 \to AR(f)(1 - d) \to S(1 - d)^3 \to S \to M(f) \to 0,$$

using Theorem 3.1. For the second claim, note that we have

$$\dim M(f)_k = \dim S_k - 3 \dim S_{k+1-d} + m \dim S_{k+1-d-r} - (m-2) \dim S_{k-d-r} =$$

$$= \binom{k+2}{2} - 3\binom{k+3-d}{2} + m\binom{k+3-d-r}{2} - (m-2)\binom{k+2-d-r}{2},$$

if and only if $k \geq d+r-2$, where the binomial coefficients are regarded as polynomials in $k$ given by the usual formulas. It follows that $st(f) = d + r - 2$. On the other hand, it is known that

$$ct(f) = d - 2 + mdr'(f),$$

where $mdr'(f)$ is the minimal degree of a syzygy in $AR(f)$ which is not in the submodule $KR((f) \subset AR(f)$ generated by the Koszul relations $(f_y,-f_x,0)$, $(f_x,0,-f_x)$ and $(0,f_x,-f_y)$, see [3, Formula (1.3)]. If $r < d - 1$, then clearly $mdr'(f) = r$ and the last claim is proved. If $r = d - 1$ and $C$ is a maximal Tjurina curve, then $AR(f)$ is generated by $2r - d + 3 = d + 1 > 3$ elements, so at least one of them is not in the 3-dimensional vector space $KR(f)_{d-1}$. This implies again $mdr'(f) = r$ and the last claim is proved in this case also. \qed
Remark 3.5. There are curves $C : f = 0$ which are not maximal Tjurina curves, but which satisfy the equality $ct(f) = st(f)$. To see this, consider a uninodal curve $C$ of degree $d$, as in Remark 2.2 (iii) above, for which

$$ct(f) = st(f) = 3(d - 2).$$

When $d > 3$, such a curve is not a maximal Tjurina curve.

4. Existence of maximal Tjurina curves when $2r \geq d$

Our discussion is a case-by-case analysis, according to the positive integer $m = 2r - d + 3$.

4.1. Maximal Tjurina curves in the case $m = 3$, minimal value for $m$. This corresponds to the case $d = 2r$ even, and the global Tjurina number is given by

$$\tau(C) = (d - 1)(d - r - 1) + r^2 - 1.$$  

Then it follows from [7] that this equality occurs exactly when $C$ is a nearly free curve, with exponents $d_1 = d_2 = d_3 = r$. Examples of such nearly free curves, both irreducible and line arrangements, are given in [10], for any pair $(d, r) = (2r, r)$.

4.2. Maximal Tjurina curves in the case $m = 4$. In this case the degree $d = 2r - 1$ is odd, and according to Theorem 3.1, the exponents are $d_1 = d_2 = d_3 = d_4 = r$. Such curves have occurred in [16, Theorem 3.11], and examples for the pairs $(d, r) \in \{(5, 3), (7, 4), (9, 5)\}$ are given in [16, Example 3.12].

The following example gives a sequence of maximal Tjurina curves which are in the same time rational nearly cuspidal curves.

Proposition 4.3. Let $d = 2r - 1 \geq 5$ be an odd integer and set

$$C_d : f_d = (y^3 - x^2 z)x^{r-3}y^{r-1} + x^d + y^d = 0.$$  

Then the plane curve $C_d$ is a maximal Tjurina curve of type $(d, r)$ for any odd degree $d \geq 5$. Moreover, any curve $C_d$ is rational, has a unique singular point, namely $p = (0 : 0 : 1)$, and the plane curve singularity $(C_d, p)$ has two branches.

Proof. The minimal degree syzygy for $f_d$ is given by

$$\rho_1 = (0, x^{r-1}y, (r + 2)x^{r-3}y^3 + (2r - 1)y^r - (r - 1)x^{r-1}z),$$

and hence indeed $mdr(f_d) = r$. The curve $C_d$ is clearly rational, since we can express $z$ as a rational function of $x$ and $y$. The Milnor number $\mu(C_d, p)$ can be easily computed, since the singularity $(C_d, p)$ is Newton nondegenerate and commode, see [23]. It follows that

$$\mu(C_d, p) = 4r^2 - 10r + 5.$$  

Since $C_d$ is rational, we have for the $\delta$-invariant the following equality

$$\delta(C_d, p) = \frac{(d - 1)(d - 2)}{2} = (r - 1)(2r - 3).$$

It follows that the number of branches of the singularity $(C_d, p)$ is

$$2\delta(C_d, p) - \mu(C_d, p) + 1 = 2.$$
and hence $C_d$ is a nearly cuspidal rational curve. Apply now [16, Theorem 5.5] with $d' = r - 1$, and conclude that $C$ is a 4-syzygy curve with exponents $d_1 = d_2 = d_3 = d_4 = r$. The last claim in Theorem 3.1 implies that $C$ is indeed a maximal Tjurina curve of type $(d, r)$.

\[ \square \]

4.4. Maximal Tjurina curves in the case $m = d - 3$. In this case $r = d - 3 \geq d/2$ and a direct computation shows that

$$\tau(d, d - 3)_{\text{max}} = \binom{d + 1}{2} - 3.$$  

We construct a sequence of line arrangements $C_d : f_d = 0$ such that $C_d$ consists of $d$ lines, has only double and triple points, $\tau(C_d) = \tau(d, d - 3)_{\text{max}}$ and $r = \text{mdr}(f_d) = d - 3$. We start with

$$C_6 : f_6 = xy(x - y)(x + \frac{y}{2} - z)(\frac{x}{2} + y - z)(\frac{x}{2} + \frac{y}{3} - z) = 0.$$  

Consider the points $A_k = (2^k : 0 : 1)$ and $B_k = (0 : 3^k : 1)$ for $k \geq k$. Note that the last factor $\ell_1 = (\frac{x}{2} + \frac{y}{3} - z)$ gives an equation for the line $L_1 = A_1B_1$ determined by the points $A_1$ and $B_1$. For $d = 7$, we add the line $L_2 = B_1A_2$, creating a new triple point at $B_1$, hence

$$C_7 : f_7 = f_6(\frac{x}{4} + \frac{y}{3} - z) = 0.$$  

Then we add the line $L_3 = A_2B_2$, creating a new triple point at $B_2$, hence

$$C_8 : f_8 = f_7(\frac{x}{4} + \frac{y}{9} - z) = 0.$$  

We continue in this way, adding the lines $L_4 = B_2A_3$, $L_5 = A_3B_3$, ..., $L_{2k+1} = A_{k+1}B_{k+1}$ for $k \geq 1$, in an obvious zig-zag manner. In other words, we set inductively

$$f_d = f_{d-1}\ell_{d-5},$$  

for any $d \geq 7$, where $\ell_{d-5} = 0$ is an equation for the line $L_{d-5}$. Since at each step of this inductive construction we create exactly one triple point and a number of nodes, it is easy to prove that $\tau(C_d) = \tau(d, d - 3)_{\text{max}}$ for any $d \geq 6$.

The claim $\text{mdr}(f_d) = d - 3$ is more difficult to check. Indeed, Ziegler’s celebrated example of two arrangements $\mathcal{A} : f = 0$ and $\mathcal{A}' : f' = 0$, both consisting of 9 lines and having only double and triple points, with isomorphic intersection lattices, and $\text{mdr}(f) = 6$ and $\text{mdr}(f') = 5$, shows that the invariant $\text{mdr}(f)$ is not combinatorial, see [30] and [6, Remark 8.5]. We have checked the claim $\text{mdr}(f_d) = d - 3$ for all degrees $d$ with $6 \leq d \leq 13$, using SINGULAR.

4.5. Maximal Tjurina curves in the case $m = d - 1$. In this case $r = d - 2$. Here are some examples.

$$\begin{align*}
 (d, r) &= (6, 4) \text{ and } f = (y^2z - x^3)^2 + x^6 + y^6 + xy^5. \\
 (d, r) &= (7, 5) \text{ and } f = (y^2z - x^3)^2y + x^7 + y^7. \\
 (d, r) &= (8, 6) \text{ and } f = (y^2z - x^3)^2xy + x^8 + y^8. 
\end{align*}$$
\[(d, r) = (9, 7) \text{ and } f = (y^3z + x^4)(x^3z + y^4)y + x^9 + y^9.\]

\[(d, r) = (10, 8) \text{ and } f = (y^2z - x^3 + x^2y)^3y + x^{10} + y^{10}.\]

The fact that these curves are maximal Tjurina curves can be checked using a computer algebra software, for instance SINGULAR. In this case \(r = d - 2 \geq d/2\), and a direct computation shows that

\[\tau(d, d - 2)_{\text{max}} = \binom{d}{2}.\]

One has the following result;

**Proposition 4.6.** Let \(C\) be a generic arrangement of \(d\) lines in \(\mathbb{P}^2\). Then \(C\) is a maximal Tjurina curve of type \((d, d - 2)\).

**Proof.** First note that \(\tau(d, d - 2)_{\text{max}} = \tau(C)\), since \(C\) has only nodes as singularities and their number is given by \(\binom{d}{2}\). It remains to recall that any reducible nodal curve \(C : f = 0\) has \(d_1 = \text{mdr}(f) = d - 2\), see [9, Theorem 4.1]. \(\square\)

4.7. Maximal Tjurina curves in the case \(m = d + 1\), maximal value for \(m\).

In this case \(r = d - 1\), and we treat first two cases, according to the parity of \(d\). The following examples have been checked using SINGULAR.

**Example 4.8.** Let \(d = 2p \geq 4\) be an even integer and set

\[f = (x^2 - yz)^{p - 1}yz + x^d + y^d.\]

Then the plane curve \(C : f = 0\) is a maximal Tjurina curve of type \((d, d - 1)\) for any even degree \(d = 2p\) with \(2 \leq p \leq 15\).

**Example 4.9.** Let \(d = 2p + 1 \geq 5\) be an odd integer and set

\[f = (x^2 - yz)^{p - 1}xyz + x^d + y^d.\]

Then the plane curve \(C : f = 0\) is a maximal Tjurina curve of type \((d, d - 1)\) for any odd degree \(d = 2p + 1\) with \(2 \leq p \leq 15\).

Recall that for any \(d \geq 2\) there are irreducible, rational, nodal curves of degree \(d\). They have exactly \((d - 1)(d - 2)/2\) nodes and no other singularities, see [21, 24]. For these curves, which are called maximal nodal curves in [24], we have the following result.

**Proposition 4.10.** Let \(C\) be a maximal nodal curve of degree \(d\). Then \(C\) is a maximal Tjurina curve of type \((d, d - 1)\).

**Proof.** First note that \(\tau(d, d - 1)_{\text{max}} = (d - 1)(d - 2)/2\), hence it remains to recall that an irreducible nodal curve \(C : f = 0\) has \(d_1 = \text{mdr}(f) = d - 1\), see [9, Theorem 4.1]. \(\square\)
References

[1] E. Artal Bartolo, L. Gorrochategui, I. Luengo, A. Melle-Hernández, On some conjectures about free and nearly free divisors, in: Singularities and Computer Algebra, Festschrift for Gert-Martin Greuel on the Occasion of his 70th Birthday, pp. 1–19, Springer (2017).

[2] M. Chardin, Some results and questions on Castelnuovo–Mumford regularity. In: Syzygies and Hilbert functions, 1–40, Lect. Notes Pure Appl. Math., 254, Chapman & Hall/CRC, Boca Raton, FL, 2007.

[3] A. D. R. Choudary, A. Dimca, Koszul complexes and hypersurface singularities, Proc. Amer. Math. Soc. 121(1994), 1009–1016.

[4] W. Decker, G.-M. Greuel, G. Pfister and H. Schönenmann. SINGULAR 4-0-1 — A computer algebra system for polynomial computations, available at http://www.singular.uni-kl.de (2014).

[5] A. Dimca, Syzygies of Jacobian ideals and defects of linear systems, Bull. Math. Soc. Sci. Math. Roumanie Tome 56(104) No. 2, 2013, 191–203.

[6] A. Dimca, Hyperplane Arrangements: An Introduction, Universitext, Springer, 2017.

[7] A. Dimca, Freeness versus maximal global Tjurina number for plane curves, Math. Proc. Cambridge Phil. Soc. 163 (2017), 161–172.

[8] A. Dimca, Curve arrangements, pencils, and Jacobian syzygies, Michigan Math. J. 66 (2017), 347–365.

[9] A. Dimca, G. Sticlaru, Koszul complexes and pole order filtrations, Proc. Edinburg. Math. Soc. 58(2015), 333–354.

[10] A. Dimca, G. Sticlaru, On the exponents of free and nearly free projective plane curves, Rev. Mat. Complut. 30(2017), 259–268.

[11] A. Dimca, G. Sticlaru, Free divisors and rational cuspidal plane curves, Math. Res. Lett. 24(2017), 1023–1042.

[12] A. Dimca, G. Sticlaru, Free and nearly free curves vs. rational cuspidal plane curves, Publ. RIMS Kyoto Univ. 54 (2018), 163–179.

[13] A. Dimca, G. Sticlaru, On the freeness of rational cuspidal plane curves, Moscow Math. J. 18(2018), 659–666.

[14] A. Dimca, G. Sticlaru, Saturation of Jacobian ideals: some applications to nearly free curves, line arrangements and rational cuspidal plane curves, arXiv: 1711.02595v2.

[15] A. Dimca, G. Sticlaru, On the jumping lines of bundles of logarithmic vector fields along plane curves, arXiv: 1804.06349.

[16] A. Dimca, G. Sticlaru, Plane curves with three syzygies, plus-one generated curves, and nearly cuspidal curves, arXiv: 1810.11766.

[17] A.A. du Plessis, C.T.C. Wall, Application of the theory of the discriminant to highly singular plane curves, Math. Proc. Cambridge Phil. Soc., 126(1999), 259-266.

[18] D. Eisenbud, The Geometry of Syzygies: A Second Course in Algebraic Geometry and Commutative Algebra, Graduate Texts in Mathematics, Vol. 229, Springer 2005.

[19] Ph. Ellia, Quasi complete intersections and global Tjurina number of plane curves, arXiv:1901.00809.

[20] Ph. Ellia, Quasi complete intersections in $\mathbb{P}^2$ and syzygies, arXiv:1902.05472.

[21] J. Harris, On the Severi problem, Invent. Math. 84 (1986), 445-461.

[22] S. H. Hassanzadeh, A. Simis, Plane Cremona maps: Saturation and regularity of the base ideal, J. Algebra 371 (2012), 620–652.

[23] A.G. Kouchnirenko, Polyèdres de Newton et nombres de Milnor, Invent. Math. 32 (1976), 1–31.

[24] M. Oka, On Fermat curves and maximal nodal curves, Michigan Math. J. 53 (2005), 459–477.

[25] H. Schenck, Elementary modifications and line configurations in $\mathbb{P}^2$, Comm. Math. Helv. 78 (2003), 447–462.
[26] A. Simis, Differential idealizers and algebraic free divisors, in: *Commutative Algebra: Geometric, Homological, Combinatorial and Computational Aspects*, Lecture Notes in Pure and Applied Mathematics (Eds. A. Corso, P. Gimenez, M. V. Pinto and S. Zarzuela), Chapman & Hall, Volume 244 (2005), 211–226.

[27] A. Simis, The depth of the Jacobian ring of a homogeneous polynomial in three variables, Proc. Amer. Math. Soc., 134 (2006), 1591–1598.

[28] A. Simis, S.O. Tohăneanu, Homology of homogeneous divisors, Israel J. Math. 200 (2014), 449-487.

[29] S.O. Tohăneanu, On freeness of divisors on $\mathbb{P}^2$. Comm. Algebra 41 (2013), no. 8, 2916–2932.

[30] G. Ziegler, Combinatorial construction of logarithmic differential forms, Adv. Math. 76 (1989), 116-154.

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