Reach of Repulsion for Determinantal Point Processes in High Dimensions

François Baccelli    Eliza O’Reilly

Abstract

Goldman [8] proved that the distribution of a stationary determinantal point process (DPP) \( \Phi \) stochastically dominates that of its reduced Palm version \( \Phi^{0,1} \). Strassen’s theorem then implies the existence of a point process \( \eta \) such that \( \Phi = \Phi^{0,1} \cup \eta \) in distribution and \( \Phi^{0,1} \cap \eta = \emptyset \). The number of points in \( \eta \) and their location determine the repulsive nature of a typical point of \( \Phi \). In this paper, the repulsive behavior of DPPs in high dimensions is characterized using the measure of repulsiveness defined by the first moment measure of \( \eta \). Using this measure, it can be shown that many families of DPPs have the property that the total number of points in \( \eta \) converges in probability to zero as the space dimension \( n \) goes to infinity. This indicates that these DPPs behave similarly to Poisson point processes in high dimensions. Through a connection with deviation estimates for the norm of high dimensional vectors from their expectation, it is also proved that for some DPPs there exists an \( R^* \) such that the decay of the first moment measure of \( \eta \) is slowest in a small annulus around the sphere of radius \( \sqrt{nR^*} \). This \( R^* \) can be interpreted as the reach of repulsion of the DPP. The rates for these convergence results can also be computed in several cases. Examples of classes of DPP models exhibiting this behavior are presented and applications to high dimensional Boolean models and nearest neighbor distributions are given.

1 Introduction

Determinantal point processes (DPPs) are useful models for point patterns where the points exhibit some repulsion from each other, resulting in a more regularly spaced pattern than a Poisson point process. These models originally appeared in random matrix theory and the formalism was introduced by O. Macchi [18] who was motivated by modeling Fermionic particles in quantum mechanics. They have since been used in many applications, such as telecommunication networks, machine learning, ecology, etc. See [16], [13], [11], [14], and the references therein. This paper describes the repulsive behavior of stationary and isotropic DPPs as the space dimension goes to infinity. In the following, a ball with center at the origin and radius \( r \) in \( \mathbb{R}^n \) is denoted \( B_n(r) \). The \( \ell^2 \) vector norm will be denoted by \( | \cdot | \) and the \( L^2 \)-norm on the space \( L^2(\mathbb{R}^n) \) by \( || \cdot ||_2 \).

The geometry of high-dimensional spheres leads us to an asymptotic regime for point processes where the intensity grows exponentially with dimension and distances grow on the order of the square root of dimension. One can justify the interest of this scaling as follows. Let \( \{R_n\} \) be a sequence of positive real numbers. Stirling’s formula implies that

\[
\text{Vol}(B_n(R_n)) \sim \frac{1}{\sqrt{n\pi}} \left( \frac{2\pi e}{n} \right)^{\frac{n}{2}} R_n^n, \quad \text{as } n \to \infty.
\]

This volume decays sub-exponentially in \( n \) for any sequence of radii \( \{R_n\} \) such that \( R_n \ll \sqrt{n} \). For \( R_n = \sqrt{n}R \), the volumes grow to infinity or decrease to zero exponentially, depending on
whether $R$ is above or below a certain threshold. Now, consider a sequence of point processes $\Phi_n$ indexed by dimension, each with constant intensity $\rho_n$. A scaling is desired for $\rho_n$ and $R_n$ at which the ball of radius $R_n$ begins to contain points as $n$ goes to infinity, and such that the intensity $\rho_n$ plays a role. If $\rho_n = e^{\alpha n} \rho$ and $R_n = \sqrt{n} R$, there exists a threshold $\rho^* = -\frac{1}{2} \log(2\pi e) - \log R$ such that

$$E_n[\Phi_n(B_n(R_n))] \sim e^{n(\rho + \frac{1}{2} \log 2\pi e + \log R) + o(n)} \rightarrow \begin{cases} 0, & \rho < \rho^* \\ \infty, & \rho > \rho^* \end{cases} \text{ as } n \to \infty. \quad (1)$$

This justifies considering this regime where the intensities grow exponentially with dimension and distances grow with the square root of the dimension. This regime also naturally arises in information theory, and following [2], it will be called the Shannon regime. In this paper, the effect of repulsion in this regime is studied and the range and strength at which DPPs asymptotically exhibit repulsion between points is quantified.

Mention of these questions appear in [24], where the authors characterize a certain class of DPPs by an effective “hard-core” diameter $D$ that grows like $\sqrt{n}$, aligning with our observations. They observe that for $r \geq D$, the number of points in a ball of radius $r$ around a typical point cannot approach zero as dimension $n$ goes to infinity. This implies $D$ is an upper bound for the separation points of the DPP exhibit in high dimensions. Rather than resulting from the repulsive nature of the DPP, the separation is a result of the nature of high dimensional balls as described in the previous paragraph. However, the observation that $D$ is the maximal such separation follows from the fact that the expected number of points in a ball of radius $r \geq D$ surrounding a typical point is bounded away from zero, which is a feature of all DPPs, not just those studied in [24]. This fact does restrict the repulsive effect a point can have as dimension tends to infinity, and leads to similar high dimensional behavior as Poisson point processes. See the beginning of Section 3 and Appendix A for more details. In this paper, a more precise description of the repulsive behavior in high dimensions is given that is specific to the associated kernel of the DPP.

The measure of repulsiveness used in this paper is a refinement of the global measure of repulsiveness for stationary DPPs described in the on-line supplementary material to [14] (see [15]). In that work, the authors consider the measure

$$\gamma := \rho \int (1 - g(x)) \, dx, \quad (2)$$

where $g$ is the pair correlation function of the point process [5], and $\rho$ is the intensity. A point process is considered more repulsive the farther $g$ is away from 1; $g \equiv 1$ corresponds to a Poisson point process. Thus this measure is a natural way to quantify repulsiveness across the entire space. This measure is also the limit as $R$ goes to infinity of the difference between the expectation and the reduced Palm expectation of the number of points in a ball of radius $R$. The larger this difference is, the more effect the point at the origin under the Palm distribution has, and thus the more repulsive the DPP is.

Rather than studying the repulsive effect on the entire space, this measure can be refined in order to examine the repulsive effect of a point of the point process across some finite distance from the point. Goldman [8] proved that for a stationary DPP $\Phi$ satisfying certain conditions, $\Phi$ stochastically dominates $\Phi^{0,1}$, where $\Phi^{0,1}$ denotes a point process with the reduced Palm distribution of $\Phi$. Strassen’s theorem [17] then gives the existence of a coupling $(\hat{\Phi}, \hat{\Phi}^{0,1})$ of the two DPPs such that $\hat{\Phi}^{0,1} \subseteq \hat{\Phi}$ almost surely. This implies that there exists a point process $\eta$ such that

$$\Phi = \Phi^{0,1} \cup \eta \text{ in distribution, and } \Phi^{0,1} \cap \eta = \emptyset.$$
Thus, $\eta$ is the set of points of the point process that have to be removed from the stationary version due to repulsion when a point is “placed at” the origin. Although this is not discussed in [14], the measure of global repulsiveness considered in that paper corresponds to $\eta$ in the sense that for a stationary DPP $\Phi$ with intensity $\rho$,

$$\gamma = \mathbb{E}[\eta(R^n)].$$

In the following, the first moment measure of $\eta$ will be used as a measure of the repulsiveness of a DPP $\Phi$. One can examine the repulsive effect of a typical point over a finite distance $R$ using the quantity $\mathbb{E}[\eta(B_n(R))]$, the expected number of points in $\eta$ of distance less than $R$ from the origin.

Our main results describe the behavior of the first moment measure of $\eta$ in the Shannon regime. Consider a sequence of stationary DPPs $\{\Phi_n\}$, such that $\Phi_n \sim \text{DPP}(K_n)$ lies in $\mathbb{R}^n$. For each $n$, let $\eta_n$ be the point process such that $\Phi_n = \Phi_n^0 \cup \eta_n$ in distribution and $\Phi_n^0 \cap \eta_n = \emptyset$. The first observation, Lemma 3.1, is that the first moment measure of $\eta$ is a finite measure on $\mathbb{R}^n$. Thus, one can consider the quantity $\mathbb{E}_n[\eta_n(R^n)]$ and the probability measure $\frac{\mathbb{E}_n[\eta_n(R^n)]}{\mathbb{E}_n[\eta_n(R^n)]}$ on $\mathbb{R}^n$ that it defines to estimate the strength and reach of the repulsiveness of a DPP in any dimension.

First, the total mass of the first moment measure $\mathbb{E}_n[\eta_n(R^n)]$ gives a rough measure of the repulsiveness across all of space in dimension $n$. There exists a class of point processes such that $\mathbb{E}_n[\eta_n(R^n)] \to c$ for any $c \in (0, 1]$ as $n \to \infty$ that is discussed in the conclusion. It is often the case, however, that $\mathbb{E}_n[\eta_n(R^n)] \to 0$ as $n \to \infty$. In this case, Markov’s inequality and the coupling inequality imply that, in high dimensions, the total variation distance is small between $\Phi_n$ and $\Phi_n^0$. Indeed, if $(\hat{\Phi}_n, \hat{\Phi}_n^0)$ is the coupling given by Strassen’s theorem,

$$||\Phi_n - \Phi_n^0||_{TV} \leq \mathbb{P}_n(\hat{\Phi}_n \neq \hat{\Phi}_n^0) = \mathbb{P}_n(\eta_n(R^n) > 0) \leq \mathbb{E}_n(\eta_n(R^n)).$$

Since $\Phi$ and $\Phi_n^0$ have the same distribution if and only if $\Phi$ is Poisson by Slivnyak’s theorem [3], this says that such DPPs look increasingly like Poisson point processes as the space dimension increases.

However, the effect of the repulsion can still be observed. For this, we examine the probability measure $\frac{\mathbb{E}_n[\eta_n(R^n)]}{\mathbb{E}_n[\eta_n(R^n)]}$ on $\mathbb{R}^n$. Letting $X_n$ be a random vector in $\mathbb{R}^n$ with this probability distribution,

$$\mathbb{E}_n[\eta_n(B_n(\sqrt{n}R))] = \mathbb{E}_n[\eta_n(R^n)]\mathbb{P}\left(\frac{|X_n|}{\sqrt{n}} \leq R\right).$$

By normalizing the first moment measure of $\eta$ by its total mass, access to probabilistic methods is gained in order to describe the distribution of its mass through the $n$-dimensional random vector $X_n$. In many cases, the sequence $\frac{|X_n|^2}{\sqrt{n}}$ converges in probability to the limit of its mean as $n$ goes to infinity, and the rate of convergence can be estimated.

This behavior then provides insight into the nature of the repulsiveness of DPPs in high dimensions, see Propositions 3.2, 3.5, and 3.7. Indeed, if $\frac{|X_n|^2}{\sqrt{n}} \to R^* \in (0, \infty)$ in probability, then as $n \to \infty$,

$$\frac{\mathbb{E}_n[\eta_n(B_n(R\sqrt{n}))]}{\mathbb{E}_n[\eta_n(R^n)]} \to \begin{cases} 0, & R < R^* \\ 1, & R > R^* \end{cases}.$$ 

Here, $R^*$ is interpreted as the asymptotic reach of repulsion in the Shannon regime for these DPPs. This result implies that in high dimensions, a typical point has its strongest repulsive effect on points that are at a distance of $\sqrt{n}R^*$ away. If the total mass $\mathbb{E}_n[\eta_n(R^n)]$ decays to zero as $n$ increases, the repulsive effect is the strongest at distance $\sqrt{n}R^*$ in the sense that the
expected number of points of $\eta_n$ decreases the slowest in a thin annulus containing the sphere of radius $\sqrt{n}R^*$ as $n$ goes to infinity.

To determine if there exist sequences of DPPs exhibiting this behavior, a few families of DPPs are considered. The parametric families of DPP kernels presented in [3] and [14] provide large classes of DPPs that exhibit a wide range of repulsiveness and lead to many different examples of repulsive behavior, including counterexamples where no finite $R^*$ exists. Examining specific DPPs also gives computational results on the rates of convergence when a threshold does occur. For example, for Laguerre-Gaussian DPPs, the sequence $\frac{\|X_n\|^2}{n}$ satisfies a large deviations principle (see Lemma 4.1). As a consequence, the reach of repulsion $R^*$ becomes a phase transition for the exponential rate at which $E_n[\eta_n(B_n(R\sqrt{n}))] \to 0$ as $n \to \infty$ (see Proposition 4.2). For $R < R^*$ the rate is decreasing in $R$ and for $R > R^*$ the rate is at its slowest and no longer depends on $R$, a result justifying the characterization of $R^*$ as the asymptotic reach of repulsion for the sequence of DPPs.

Beyond Laguerre-Gaussian DPPs, different techniques can be applied depending on the kernels $\{K_n\}$ to determine if there is an asymptotically finite reach of repulsion. Three more classes of DPPs are studied: power exponential DPPs, Bessel-type DPPs, and normal-variance mixture DPPs. Power exponential DPPs are defined through the Fourier transform of their kernels, and it is shown they have a finite reach of repulsion in the Shannon regime, but only for certain parameters (see Proposition 4.4). Bessel-type DPPs are a more repulsive family that does not exhibit a finite reach of repulsion at this scaling (see Proposition 4.5). Finally, normal-variance mixture DPPs provide some additional examples of DPPs that exhibit a reach of repulsion in the Shannon regime, including the Cauchy models (see Proposition 4.7).

Finally, a couple of applications of these results are presented in Section 5. First, the nearest neighbor function in the Shannon regime is shown to have the same asymptotic behavior as the void function under certain conditions. Second, it can be shown that some threshold results in [2] for Poisson Boolean models can be extended to generalized Laguerre-Gaussian DPP Boolean models in the Shannon regime using the rates of convergence computed for these DPPs.

2 Preliminaries

Determinantal point processes are characterized by an integral operator $K$ with kernel $K$, and can be defined in terms of the densities of their factorial moment measures [3].

**Definition 2.1.** A simple, locally finite, spatial point process $\Phi$ on $\mathbb{R}^n$ is a determinantal point process with kernel $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ ($\Phi \sim DPP(K)$) if its product density functions exist for all order $k$ and satisfy

$$\rho^{(k)}(x_1, \ldots, x_k) = \det(K(x_i, x_j))_{1 \leq i, j \leq k}, \quad k = 1, 2, \ldots.$$ 

So, for all Borel measurable functions $h : (\mathbb{R}^n)^k \to [0, \infty)$,

$$E \left[ \sum_{X_1, \ldots, X_k \in \Phi} h(X_1, \ldots, X_k) \right] = \int_{(\mathbb{R}^n)^k} h(x_1, \ldots, x_k) \det(K(x_i, x_j))_{1 \leq i, j \leq k} dx_1 \ldots dx_k.$$ 

Note that the intensity function of $\Phi$ is given by $\rho(x) = K(x, x)$. The degenerate case where $K(x, y) = \delta_{(x=y)}$ coincides with a Poisson point process with unit intensity.

The following conditions on $K$ are imposed to ensure $\Phi \sim DPP(K)$ is well-defined. Let $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a continuous locally square integrable kernel and assume $K$ is symmetric, i.e. $K(x, y) = K(y, x)$. The kernel $K$ then defines a self-adjoint integral operator $K$ on $L^2(\mathbb{R}^n)$.
given by \( Kf(x) = \int K(x,y)f(y)dy \). For any compact set \( S \subset \mathbb{R}^n \), the restricted operator \( K_S \) given by
\[
K_S f(x) = \int_S K(x,y)f(y)dy, \quad x \in S,
\]
is also a compact operator. By the spectral theory for self adjoint compact operators, the spectrum of \( K_S \) consists solely of countably many eigenvalues \( \{ \lambda_k^S \}_{k \in \mathbb{N}} \) with an accumulation point only possible at zero. See [20] for more on compact operators. Assume also that \( K \) is locally trace class, meaning that for any compact \( S \subset \mathbb{R}^n \), \( \int_S K(x,x)dx < \infty \), i.e. the trace of \( K_S \) is finite. These conditions imply that for any compact \( S \subset \mathbb{R}^n \), the kernel \( K \) restricted to \( S \times S \) has a spectral representation
\[
K(x,y) = \sum_{k=1}^{\infty} \lambda_k^S \phi_k^S(x)\overline{\phi_k^S(y)}, \quad (x,y) \in S \times S,
\]
where \( \{ \phi_k^S \}_{k \in \mathbb{N}} \) are the eigenvectors of \( K_S \), and they form an orthonormal basis of \( L^2(S) \).

**Theorem 2.2.** (Theorem 4.5.5 in [11]) Under the conditions given above, a kernel \( K \) defines a determinantal process on \( \mathbb{R}^n \) if and only if the spectrum of \( K \) is contained in \( [0,1] \).

If \( K(x,y) = K_0(x-y) \), then \( \Phi \sim DPP(K) \) is stationary. In this case, the operator \( \mathcal{K} \) is the convolution operator \( \mathcal{K}(f) = K \ast f \) on \( L^2(\mathbb{R}^n) \). The intensity function \( \rho(x) \) is then constant and satisfies \( \rho = K_0(0) \). For these stationary DPPs, there is a simple spectral condition for existence.

**Theorem 2.3.** (Theorem 2.3 in [11]) Assume \( K_0 \) is a symmetric continuous real-valued function in \( L^2(\mathbb{R}^n) \). Let \( K(x,y) = K_0(x-y) \). Then DPP(\( K \)) exists if and only if \( 0 \leq \tilde{K}_0 \leq 1 \), where \( \tilde{K}_0 \) denotes the Fourier transform of \( K_0 \).

For the rest of this paper, when it is stated that \( \Phi \sim DPP(K) \) is stationary, it is assumed that \( K(x,y) = K_0(x-y) \) for \( K_0 \in L^2(\mathbb{R}^n) \), and \( K \) will be used to mean \( K_0 \). However, it is important to note that there exist stationary DPPs with kernels that are not of this form (see [11, p. 4.3.7]), but they are not considered here. In addition, when it is stated that \( \Phi \) is isotropic, it is meant that \( K_0(x) = R_0(|x|) \) and is thus invariant under rotations about the origin in \( \mathbb{R}^n \).

The Palm distribution of a stationary point process can be understood as the distribution of the process seen from a typical point. It can also be interpreted as the distribution of the point process conditioned on there being a point at the origin, see [1, Chapter 4]. The reduced Palm distribution is the Palm distribution with the point at the origin removed. The following lemma gives a useful result about the Palm distribution of DPPs.

**Theorem 2.4.** (Theorem 6.5 in [23]). Let \( \Phi \sim DPP(K) \) in \( \mathbb{R}^n \) be stationary, where \( K \) satisfies the conditions for existence of \( \Phi \). Let \( \rho = K(0) > 0 \). Then the reduced Palm distribution coincides with the distribution of a DPP with associated kernel
\[
K_0^r(x,y) = \frac{1}{K(0)} \det \begin{pmatrix} K(x-y) & K(x) \\ K(y) & K(0) \end{pmatrix} = K(x-y) - \frac{1}{\rho} K(x) K(y).
\]

The nearest neighbor function of a stationary point process \( \Phi \) in \( \mathbb{R}^n \) is defined as
\[
D(r) := \mathbb{P}(\Phi(B_n(r)) = 0),
\]
and gives the probability that a typical point’s nearest neighboring point is farther than distance \( r \) away. If \( \Phi \) is Poisson, Slivnyak’s theorem gives that \( D(r) = \mathbb{P}(\Phi(B_n(r)) = 0) = e^{-\mathbb{E}[\Phi(B_n(r))]} \).
For DPPs, Theorem 2.4 implies that \( D(r) = \P(\Phi^0(B_n(r)) = 0) \) for \( \Phi^0 \sim DPP(K_0^1) \), a DPP with the reduced Palm distribution of \( \Phi \). This probability then takes the form of a Fredholm determinant (see [23]). By the first moment inequality and Proposition 5.1 in [4], one also has the following bounds:

\[
1 - \E[\Phi^0(B_n(r))] \leq D(r) \leq \exp \left( -\E[\Phi^0(B_n(r))] \right). \tag{5}
\]

As mentioned in the introduction, Goldman [8] proved that under certain conditions, a stationary DPP \( \Phi \) stochastically dominates \( \Phi^0 \). Strassen’s Theorem [17] is then applied to obtain the following result.

**Theorem 2.5.** (Theorem 6 in [8]) Let \( \Phi \sim DPP(K) \), where \( K \) is continuous, and the spectrum of the integral operator \( \hat{K} \) with kernel \( K \) is contained in \([0, 1)\). Almost surely, there exists a point process \( \eta \) in \( \mathbb{R}^n \) such that \( \Phi = \Phi^0 \cup \eta \) in distribution, and \( \Phi^0 \cap \eta = \emptyset \).

This theorem says that there is a coupling of \( \Phi \) and \( \Phi^0 \) such that a point process with the distribution of \( \Phi^0 \) can be obtained from \( \Phi \) by simply removing a subset of points \( \eta \). This is a striking result, since the procedure does not include shifting any of the remaining points. The number and location of points in \( \eta \) thus characterize the repulsive nature of the DPP \( \Phi \), since these are the points that are “pushed out” by the point at zero under the Palm distribution. It also makes sense to compare the repulsiveness of DPPs using \( \eta \). For two stationary DPPs \( \Phi_1 \) and \( \Phi_2 \) with the same intensity, \( \Phi_1 \) is defined to be more repulsive than \( \Phi_2 \) if \( \E[\eta_1(\mathbb{R}^n)] > \E[\eta_2(\mathbb{R}^n)] \). This corresponds to the definition in [3] using the measure \( \gamma \). Note that for \( \eta \) to exist, one must exclude the interesting case where \( \hat{K}(x) \) attains a value of 1 for some \( x \).

### 3 Main Results

When considering the reach of repulsion of a DPP, it is natural to first consider the nearest neighbor function \( \rho^- \). For a stationary point process \( \Phi \), if \( D(r) \) is larger than \( \P(\Phi(B_n(r)) = 0) \) for all \( r \), then the point at zero under Palm is having a repulsive effect, and the difference measures the strength of repulsiveness. Determining the nature of the nearest neighbor function for DPPs in high dimensions was a topic of study in [24] and [21]. The following threshold behavior was observed for stationary DPPs. It is stated here for a sequence of DPPs in the Shannon regime.

For each \( n \), let \( \Phi_n \sim DPP(K_n) \) in \( \mathbb{R}^n \) be stationary with intensity \( K_n(0) = e^{n\rho} \) for some \( \rho \in \mathbb{R} \). Then, for \( \tilde{R} := \frac{1}{\sqrt{2\pi ne^{n^2}}} \),

\[
\lim_{n \to \infty} \P_n(\Phi^0_n(B_n(\sqrt{n}R)) = 0) = \begin{cases} 
1, & R < \tilde{R} \\
0, & R > \tilde{R}.
\end{cases} \tag{6}
\]

A quick proof of this fact is given in Appendix A.

This shows there is a natural separation of points for any stationary DPP regardless of the strength of the repulsiveness. Indeed, the same threshold behavior occurs if the elements of the sequence \( \{\Phi_n\} \) are stationary Poisson point processes, where it is a consequence of the asymptotic behavior of the volume of high dimensional balls. This observation shows that the repulsiveness of DPPs is not strong enough in high dimensions to see an effect on the threshold beyond what occurs naturally due to dimensionality.
The point process $\eta$ as defined in Theorem 2.3 gives an alternative characterization of the repulsiveness of a DPP. As described above, this point process consists of points that must be removed from a stationary DPP to obtain the reduced Palm distribution, i.e. the points a typical point of the DPP "pushes out". Although $\eta$ does not consist of enough points to change the threshold behavior of the nearest neighbor function from that of Poisson point processes, the first moment measure of $\eta$ can still measure some consequence of repulsiveness that follows from the determinantal structure.

Lemma 3.1. Let $\Phi \sim DPP(K)$ in $\mathbb{R}^n$ be stationary with intensity $K(0) = \rho$ and assume $0 \leq \hat{K} < 1$. Let $\eta$ be the point process such that $\Phi = \Phi^{0,1} \cup \eta$ in distribution and $\Phi^{0,1} \cap \eta = \emptyset$, where the existence of $\eta$ is ensured by Theorem 2.5. Then, $E[\eta(\mathbb{R}^n)] < 1$ and for all Borel sets $B \in \mathcal{B}(\mathbb{R}^n)$,

$$E[\eta(B)] = E[\eta(\mathbb{R}^n)]P(X \in B),$$

where $X$ is a random vector in $\mathbb{R}^n$ with density $K(x)^2 / ||K||_2^2$.

Proof. By Theorem 2.3 for any $B \in \mathcal{B}(\mathbb{R}^n)$,

$$E[\eta(B)] = E[\Phi(B)] - E[\Phi^{0,1}(B)]$$

$$= \int_B \rho dx - \int_B \left( \hat{K}(x-x) - \frac{1}{\rho} K(x)^2 \right) dx = \frac{1}{\rho} \int_B K(x)^2 dx,$$

i.e. the first moment measure of $\eta$ has a density with respect to Lebesgue measure equal to $\frac{1}{\rho} K(x)^2$. Then by the monotone convergence theorem,

$$E[\eta(\mathbb{R}^n)] = \lim_{R \to \infty} E[\eta(B_n(R))] = \frac{1}{\rho} \int_{\mathbb{R}^n} K(x)^2 dx = \frac{||K||_2^2}{\rho}.$$ 

By Parseval’s theorem and the assumption that $0 \leq \hat{K} < 1$,

$$E[\eta(\mathbb{R}^n)] = \frac{1}{\rho} \int_{\mathbb{R}^n} K(x)^2 dx = \frac{1}{\rho} \int_{\mathbb{R}^n} \hat{K}(\xi)^2 d\xi < \frac{1}{\rho} \int_{\mathbb{R}^n} \hat{K}(\xi) d\xi = \frac{K(0)}{\rho} = 1. \quad (8)$$

Finally, letting $X$ be a random vector in $\mathbb{R}^n$ with probability density $\frac{K(x)^2}{||K||_2^2}$, for all $B \in \mathcal{B}(\mathbb{R}^n)$,

$$E[\eta(B)] = \frac{||K||_2^2}{\rho} \int_B \frac{K(x)^2}{||K||_2^2} dx = E[\eta(\mathbb{R}^n)]P(X \in B).$$

In particular, one can consider $E[\eta(B_n(R))]$ as a measure of the repulsiveness across the distance $R$ from a typical point, i.e. the expected number of points to be "removed" within a distance $R$ of the typical point to obtain $\Phi^{0,1}$ from $\Phi$. Lemma 3.1 in combination with Markov’s inequality gives an upper bound on the probability a point is removed within a distance $R > 0$:

$$P(\eta(B_n(R)) > 0) < P(|X| \leq R).$$

Consider now the Shannon regime. For each $n$, let $\Phi_n \sim DPP(K_n)$ in $\mathbb{R}^n$ be stationary and isotropic with intensity $e^{\kappa p}$ for some $\rho \in \mathbb{R}$, and assume $0 \leq \hat{K}_n < 1$. Let $(\Omega_n, P_n, \hat{\Phi}_n, \hat{\Phi}_n^{0,1})$ be the coupling of $\Phi_n$ and $\Phi_n^{0,1}$ obtained from Theorem 2.3 where $\hat{\Phi}_n^{0,1} \subset \hat{\Phi}_n$, and define $\eta_n := \hat{\Phi}_n - \hat{\Phi}_n^{0,1}$.

Asymptotically, consider the repulsiveness at distances scaling on the order of $\sqrt{n}$ from a typical point. The following result shows that under certain limit conditions on the kernels of a sequence of DPPs, the repulsiveness measured by the first moment measure of $\eta_n$ is concentrated at a distance of $\sqrt{nR^*}$ for some $R^* \in (0, \infty)$ as $n$ goes to infinity.

The proofs of all of the following results are in the Appendix.
Proposition 3.2. For each \( n \), let \( \Phi_n \sim DPP(K_n) \) be a stationary and isotropic DPP in \( \mathbb{R}^n \), and assume \( 0 \leq K_n < 1 \). Let \( X_n \) be a random vector in \( \mathbb{R}^n \) with probability density \( \frac{K_n(x)^2}{\|K_n\|_2} \). Assume that

\[
\lim_{n \to \infty} \frac{\text{Var}(|X_n|^2)}{n^2} = 0,
\]

and that

\[
\lim_{n \to \infty} \left( \frac{E_n[|X_n|^2]}{n} \right)^{1/2} = R^* \in (0, \infty).
\]

Then,

\[
\lim_{n \to \infty} \frac{E_n[\eta_n(B(\sqrt{n}R))]}{E_n[\eta_n(R^n)]} = \begin{cases} 
0, & R < R^* \\
1, & R > R^*.
\end{cases}
\]

Remark 3.3. For general high-dimensional vectors \( X_n \), the concentration of \( \frac{|X_n|}{\sqrt{n}} \) near its mean has been well-studied (see [2], [12], [10]). The first condition of Proposition 3.2 implicitly requires that \( |X_n| \) has a finite fourth moment. If this is not the case, the conclusion may still hold if \( X_n \) is concentrated in a “thin shell”, i.e. there exists a sequence \( \{\varepsilon_n\} \) such that \( \varepsilon_n \to 0 \) as \( n \to \infty \) and for each \( n \),

\[
P\left( \left| \frac{|X_n|}{E|X_n|^2} - 1 \right| \geq \varepsilon_n \right) \leq \varepsilon_n.
\]

In [2, Proposition 3], it is proved that a sequence of vectors satisfies (12) if and only if \( |X_n| \) has a finite \( r \)th moment for \( r > 2 \), and for some \( 2 < p < r \),

\[
\left| \frac{(E|X_n|^p)^{1/p}}{(E|X_n|^2)^{1/2}} - 1 \right| \to 0 \text{ as } n \to \infty.
\]

For random vectors with log-concave distributions, the deviation estimate can be improved quite a bit from the estimate obtained through Chebychev’s inequality (see proof in Appendix B.1). A couple definitions are in order. A random vector \( X \) is isotropic if \( EX = 0 \) and \( E(X \otimes X) = I_n \). A random vector is said to be \( \psi_\alpha \) (\( \alpha \in [1, 2] \)) with constant \( b_\alpha \), if

\[
(E|X|^p)^{1/p} \leq b_\alpha \psi_\alpha^{1/\alpha}(E|X|^2)^{1/2} \text{ for all } p \geq 2, \text{ for all } y \in \mathbb{R}^n.
\]

The best known estimate is given by the following theorem in [10].

Theorem 3.4. (Guédon and Milman [10]) Let \( X \) denote an isotropic random vector in \( \mathbb{R}^n \) with log-concave density. In addition, assume \( X \) is \( \psi_\alpha \) (\( \alpha \in [1, 2] \)) with constant \( b_\alpha \). Then,

\[
P\left( \left| \frac{|X|}{\sqrt{n}} - 1 \right| \geq t \right) \leq C \exp\left( -c \left( \frac{n}{b_\alpha^2} \right)^{1/3} \min(t^3, t) \right).
\]

If a DPP \( \Phi_n \) in \( \mathbb{R}^n \) is isotropic, then \( X_n \) as defined in Proposition 3.2 has a radially symmetric density. Thus, \( X_n \) has the same distribution as the product \( R_nU_n \), where \( R_n \) is equal in distribution to \( |X_n| \), \( U_n \) is uniformly distributed on \( S^{n-1} \), and \( R_n \) and \( U_n \) are independent. To simplify the notation in the following, let \( \sigma_n^2 = E|X_n|^2 \) for each \( n \). If \( K_n^2 \) is log-concave, \( \frac{\sqrt{n}}{\sigma_n} X_n \) with \( X_n \) as defined above satisfies the conditions of Theorem 3.4 for each \( n \). This gives the following result.
Remark 3.6. The last conclusion of Proposition 3.5 about the rate also holds for \( \Phi \), and assume \( 0 = \hat{K}_n < 1 \). Let \( X_n \) be a random vector with density \( \frac{K_n(x)^2}{||K_n||^2} \) and suppose \( X_n \) is \( \psi_\alpha \) with constant \( b_\alpha \) for some \( \alpha \in [1,2] \). If \( K_n^2 \) is log-concave for all \( n \), then there exist absolute constants \( C,c \) such that for all \( \delta \in (0,1) \),

\[
\frac{E_n[\eta_n(B_n(\sigma_n(1 - \delta)))]}{E_n[\eta_n(R^n)]} \leq Ce^{-c\left(\frac{\delta}{\sqrt{\alpha}}\right)^\frac{\alpha}{2}},
\]

and for all \( \delta > 0 \),

\[
\frac{E_n[\eta_n(B_n(\sigma_n(1 + \delta)))]}{E_n[\eta_n(R^n)]} \leq Ce^{-c\left(\frac{\delta}{\sqrt{\alpha}}\right)^\frac{\alpha}{2}\min(\delta^3,\delta)}.
\]

If, in addition, \( \frac{\alpha}{\sqrt{n}} \to R^* \in (0,\infty) \) as \( n \to \infty \), then for all \( R < R^* \), there exists a constant \( C(R,b_\alpha,\alpha) > 0 \) such that

\[
\liminf_{n \to \infty} \frac{1}{n^{\alpha/2}} \ln \frac{E_n[\eta_n(B_n(\sqrt{n}R))]}{E_n[\eta_n(R^n)]} \geq C(R,b_\alpha,\alpha).
\]

Remark 3.8. The last conclusion of Proposition 3.5 about the rate also holds for \( \Phi \), if \( B_n(\sqrt{n}R) \) is replaced by \( B_n(\sqrt{n}R)^c = R^n \setminus B_n(\sqrt{n}R) \).

The stronger assumption of large deviation principle (LDP) concentration leads to an estimate of the exponential rate of convergence with speed \( n \) and an exact computation of the reach of repulsion \( R^* \).

Proposition 3.7. For each \( n \), let \( \Phi_n \sim DPP(K_n) \) be a stationary and isotropic DPP in \( \mathbb{R}^n \), and assume \( 0 \leq \hat{K}_n < 1 \). Let \( X_n \) be a random vector with density \( \frac{K_n(x)^2}{||K_n||^2} \) and suppose \( \frac{|X_n|}{\sqrt{n}} \) satisfies a LDP with strictly convex rate function \( I \). Then, for \( R^* \) such that \( I(R^*) = 0 \),

\[
\frac{E_n[\eta_n(B_n(\sqrt{n}R))]}{E_n[\eta_n(R^n)]} \to \begin{cases} 0 & , R < R^* \\ 1 & , R > R^* \end{cases}.
\]

Also, for \( R < R^* \),

\[
- \inf_{r < R} I(r) \leq \liminf_{n \to \infty} \frac{1}{n} \ln \frac{E_n[\eta_n(B_n(\sqrt{n}R))]}{E_n[\eta_n(R^n)]} \leq \limsup_{n \to \infty} \frac{1}{n} \ln \frac{E_n[\eta_n(B_n(\sqrt{n}R))]}{E_n[\eta_n(R^n)]} \leq - \inf_{r \leq R} I(r),
\]

and if the rate function \( I \) is continuous at \( R \),

\[
\lim_{n \to \infty} \frac{1}{n} \ln \frac{E_n[\eta_n(B_n(\sqrt{n}R))]}{E_n[\eta_n(R^n)]} = I(R).
\]

Remark 3.8. The second conclusion of Proposition 3.7 about the rate also holds for \( \Phi \), if \( B_n(\sqrt{n}R) \) is replaced by \( B_n(\sqrt{n}R)^c = R^n \setminus B_n(\sqrt{n}R) \).

As mentioned before, it may be the case that \( \eta_n \) contains points with probability decreasing to zero as \( n \) goes to infinity. However, when \( \eta_n \) does contain points, the distance of these points from the origin gives the reach of the repulsive effect of a typical point of \( \Phi_n \). If a sequence of DPPs in increasing dimensions exhibits a reach of repulsion \( R^* \), this says that the points of \( \eta_n \) are most likely to be near distance \( \sqrt{n}R^* \) away from the origin in high dimensions. If \( R^* \) is less than \( \hat{R} \), the nearest neighbor threshold, points are most likely to be removed at a distance where points of \( \Phi_n \) appear with probability decreasing to zero as \( n \) increases due to
dimensionality. If $R^*$ can reach past $\tilde{R}$, the points “pushed out” by repulsion are most likely to lie at a distance where points of $\Phi_n$ appear with high probability. Thus it is of interest to check whether there exist DPP models such that $R^*$ is greater than or equal to $\tilde{R}$, i.e. if $P(\Phi_n^{(1)}(B_n(\sqrt{n}R^*)) = 0) \to 0$ as $n \to \infty$. In Sections 4.1 and 4.2 examples of DPP models with this reach are provided.

The above results have strong assumptions, and open up a number of additional questions. The first question is whether the points of $\eta_n$ tend to lie at distances scaling with the square root of dimension, i.e. is the Shannon regime the right one to examine the repulsiveness between points of a family of DPPs in high dimensions? By the radial symmetry of the density of each $X_n$, the coordinates $\{X_{n,k}\}_{k=1}^n$ are identically distributed, and the sequence $|X_n|^2$ is the sequence of row sums of a triangular array of random variables with identically distributed rows. If the coordinate distributions depend on dimension in such a way that $E|X_n|^2 \neq O(n)$, then a different scaling is needed. The next question is how many and what kind of DPPs satisfy the conditions of Propositions 3.2, 3.5, and 3.7.

In the following, specific families are examined that illustrate both examples of DPP models satisfying the above results, as well as examples that do not. Where it is possible, computational results on the rates of convergence are obtained. These examples provide a window into the wide scope of repulsive behavior that can be described using this framework.

4 Examples

In this section, specific families of DPPs that were presented in [3] and [14] are defined and analyzed. The first task will be to compute the global repulsiveness of the DPPs as space dimension goes to infinity, i.e. the magnitude and behavior of $E_n[\eta_n(R^n)]$ as $n$ increases. We will see that for each of the examples provided in this section, $E_n[\eta_n(R^n)] \to 0$ as $n \to \infty$, but each class exhibits this convergence at different speeds. Then the goal is to determine if the DPP models satisfy the conditions of Propositions 3.2, 3.5, or 3.7. The first class, the Laguerre-Gaussian DPPs, provide a class of DPPs that satisfy the conditions of Proposition 3.7. Power-exponential DPPs, provide examples of models that satisfy the conditions of Proposition 3.2 for certain parameters. Thirdly, Bessel-type models are presented as examples of models that do not exhibit the concentration phenomenon and so do not have a reach of repulsion $R^*$. It is shown however that the repulsive effect of a typical point does occur within the Shannon regime and not at distances scaling faster than the square root of the space dimension. The last class presented, the normal-variance mixture models, include examples of DPPs satisfying Proposition 3.5, and the subclass of Cauchy models is shown to satisfy Proposition 3.2 for certain parameters.

4.1 Laguerre-Gaussian Models

The Laguerre-Gaussian DPP models as described in [3] are defined as follows. For all $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}^+$, let $P_m^\alpha(r) = \sum_{k=0}^m \binom{m+\alpha}{m-k} \frac{(-r)^k}{k!}$, for all $r \in \mathbb{R}$, denote the Laguerre polynomials. Then, $\Phi \sim DPP(K)$ is a Laguerre-Gaussian DPP in $\mathbb{R}^n$ if

$$K(x) = \frac{\rho}{(m-1+n/2)} m^{n/2} \left(\frac{1}{m} \frac{|x|^2}{\alpha}\right) e^{-\frac{|x|^2}{m}}, \quad x \in \mathbb{R}^n,$$

for some $m \in \mathbb{N}$, $\alpha \in \mathbb{R}^+$, and $\rho \in \mathbb{R}^+$. Note that $\rho = K(0)$ is the intensity of $\Phi$. In [3] it is mentioned that this kernel appears in approximation theory. The Fourier transform of the
The condition \(0 \leq \hat{K} < 1\) translates to a bound on \(\alpha\),
\[
\alpha^n < \frac{1}{\rho(m\pi)^{m/2}} \left( \frac{m-1+n/2}{m-1} \right).
\]

Now, let \(\eta\) be such that \(\Phi = \Phi^0 \cup \eta\) in distribution and \(\Phi^0 \cap \eta = \emptyset\). Direct calculations give that the global measure of repulsiveness is
\[
E[\eta(\mathbb{R}^n)] = \frac{1}{2}\|K\|_2^2 = \frac{\rho \alpha^n}{(m-1+n/2)^{m-1}} \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} \binom{m-1+n/2}{m-1-k} \binom{m-1+n/2}{m-1-j} \frac{(-1)^{k+j} \Gamma\left(\frac{3}{2}+k+j\right)}{k! j! 2^{k+j} \Gamma\left(\frac{3}{2}\right)}. \tag{13}
\]

By the upper bound on \(\alpha\),
\[
E[\eta(\mathbb{R}^n)] < 2^{-n/2} f(n, m), \tag{14}
\]

where
\[
f(n, m) = \frac{1}{(m-1+n/2)^{m-1}} \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} \binom{m-1+n/2}{m-1-k} \binom{m-1+n/2}{m-1-j} \frac{(-1)^{k+j} \Gamma\left(\frac{3}{2}+k+j\right)}{k! j! 2^{k+j} \Gamma\left(\frac{3}{2}\right)} = O(n^2),
\]
and as \(m \to \infty\), \(2^{-n/2} f(n, m) \to 1\). The larger \(m\) is, the closer \(E[\eta(\mathbb{R}^n)]\) is to its upper bound of 1, and as \(m \to 0\), the repulsiveness approaches 0. Thus, this class of DPPs covers a wide range of repulsiveness for fixed dimension. However for any fixed \(m\), the dominant behavior as \(n \to \infty\) is \(2^{-n/2}\).

For each \(n\), let \(\Phi_n \sim DPP(K_n)\) in \(\mathbb{R}^n\) be a Laguerre-Gaussian DPP with intensity \(K_n(0) = e^{np}\) and parameters \(m\) and \(\alpha\) such that \(0 < \alpha < \frac{1}{e^{\rho(m\pi)^{1/2}} \left( \frac{m-1+n/2}{m-1} \right)^{1/n}}\) for all \(n\). Since \(\left( \frac{m-1+n/2}{m-1} \right)^{1/n}\) decreases to 1 as \(n\) goes to infinity, the condition is simplified to \(0 < \alpha < \frac{1}{e^{\rho(m\pi)^{1/2}}}\).

Note that this scaling for the intensity is the right one for observing interactions between the parameters of the model. It provides a trade-off between how large the parameter \(\alpha\) can be and the magnitude of the logarithmic intensity \(\rho\). If the intensity did not grow as quickly with dimension, the upper bound on \(\alpha\) would depend less and less on changes in \(\rho\) as dimension increased, and if the intensity grew more quickly, the upper bound for \(\alpha\) would tend to zero as \(n\) goes to infinity.

Proposition 5.7 holds for this sequence of DPPs. Indeed, the next lemma shows that the sequence of \(\mathbb{R}^+\)-valued random variables \(\left\{\frac{|X_n|^2}{n}\right\}_n\) satisfies a LDP and then the contraction principle (see [3]) implies that the sequence \(\left\{\frac{|X_n|}{\sqrt{n}}\right\}_n\) satisfies a LDP as well.

**Lemma 4.1.** Fix \(m \in \mathbb{N}, \rho \in \mathbb{R}\), and let \(\alpha \in (0, e^{-\rho(m\pi)^{-1/2}})\). For each \(n\), let \(X_n\) be a random vector in \(\mathbb{R}^n\) with probability density \(K_n(x)^2 / ||K_n||^2\), where
\[
K_n(x) = e^{np} \left( \frac{m-1+n/2}{m-1} \right)^{m/2} \left( \frac{1}{m} \left| x^2 \alpha^2 \right| \right) e^{-|x^2/\alpha|^2}. \tag{15}
\]

Then, the sequence of \(\mathbb{R}^+\)-valued random variables \(\left\{\frac{|X_n|^2}{n}\right\}_n\) satisfies an LDP with rate function
\[
\Lambda^*(x) = \frac{2x}{\alpha^2 m} - \frac{1}{2} + \frac{1}{2} \log \left( \frac{\alpha^2 m}{4x} \right).
\]
Using this lemma, Proposition 3.7 implies that the asymptotic reach of repulsion \( R^* := m^{\frac{\alpha^2}{4}} \) exists, and the exponential rates can be determined. In addition, using (13), the exact rates of decay of the sequence \( \{E_n[\eta_n(B_n(\sqrt{n}R))]\} \) can be computed.

**Proposition 4.2.** Fix \( m \in \mathbb{N}, \rho \in \mathbb{R} \), and let \( \alpha \in (0, e^{-\rho(m\pi)^{-1/2}}) \). For each \( n \), let \( \Phi_n \sim DPP(K_n) \) where \( K_n \) is given by (15). Then,

\[
\lim_{n \to \infty} -\frac{1}{n} \log E_n[\eta_n(B_n(\sqrt{n}R))] = \begin{cases} 
-p - \frac{1}{2} \log 2\pi e + \frac{2R^2}{\alpha^2 m} - \log R, & 0 < R < \sqrt{m^{\frac{\alpha}{2}}} := R^* \\
-p - \log \alpha - \frac{1}{2} \log \frac{m^2}{\pi^2}, & R > \sqrt{m^{\frac{\alpha}{2}}} := R^*.
\end{cases}
\]

The rate decays as \( R \) increases to \( R^* := \sqrt{m^{\frac{\alpha}{2}}} \) and then for \( R > R^* \), the rate no longer depends on \( R \). This coincides with our interpretation of \( R^* \) as the asymptotic reach of repulsion of the sequence of DPPs.

For a fixed \( \alpha \), a larger \( m \) will give farther reach, and for a fixed \( m \), a larger \( \alpha \) will provide a farther reach. However, by the bound on \( \alpha \) needed for \( \Phi_n \) and \( \eta_n \) to exist for all \( n \), \( \alpha < \frac{1}{e^\rho(m\pi)^{1/2}} \), the following upper bound on the reach holds uniformly for all \( m \):

\[
R^* := \sqrt{m^{\frac{\alpha}{2}}} < \frac{1}{2e^\rho \pi^{1/2}}.
\]

Also note that the larger \( \rho \) is, the smaller the upper bound on the range of repulsion can be. This follows from the relationship between \( \alpha \) and \( \rho \): the higher the intensity, the smaller \( \alpha \) must be for the DPP to exist. Since a larger \( \alpha \) implies a larger values of \( E[\eta(R^n)] \), the parameter \( \alpha \) is associated with the strength of the repulsiveness. The relationship with \( \rho \) showcases the following tradeoff: the higher the intensity of the determinantal point process, the less repulsive it can be.

As mentioned in the previous section, it is of interest to know whether there is a range of parameters such that \( R^* \) is greater than \( \tilde{R} \), the threshold for the convergence of the nearest-neighbor function of \( \Phi \) (9). For Laguerre-Gaussian models, \( R^* := \sqrt{m^{\frac{\alpha}{2}}} \) is larger than \( \tilde{R} \) and the DPP exists if and only if \( \alpha \) satisfies

\[
\left( \frac{2}{e} \right)^{1/2} < e^\rho \sqrt{m\pi} \alpha < 1.
\]

Since the lower bound is strictly less than one, there is a non-empty range for \( \alpha \) such that the reach of repulsion reaches the described regime.

### 4.2 Power Exponential Spectral Models

The power exponential spectral models are defined through the Fourier transform of the kernel, which is an alternative way to define stationary and isotropic DPPs. For almost all of these models, there is no closed form for the kernel \( K \). Therefore one cannot check, for example, that the kernel is log-concave. However, using properties of the Fourier transform, a similar analysis of the repulsive behavior can still be performed and one can check if a sequence of these DPPs satisfies the conditions of Proposition 3.2.

A DPP associated with kernel \( K \) is a power exponential spectral model if the Fourier transform of \( K \) is

\[
\hat{K}(x) = \rho \frac{\Gamma\left(\frac{\nu}{2} + 1\right) \alpha^n}{\pi^{n/2} \Gamma\left(\frac{n}{2} + 1\right)} e^{-|\alpha x|^{\nu}}, \quad x \in \mathbb{R}^n,
\]

where \( \nu > 0 \) and \( \alpha > 0 \) are parameters, and \( \rho = K(0) \) is the intensity. When \( \nu = 2 \), a closed form expression for \( K \) does exist and is called the Gaussian kernel.
The condition $0 \leq \hat{K} < 1$ implies the following upper bound on $\alpha$:

$$\alpha^n < \frac{\Gamma\left(\frac{n}{2} + 1\right)\pi^{n/2}}{\rho \Gamma\left(\frac{n}{2} + 1\right)}.$$  

By Parseval’s theorem and a change of variables, the expected total number of points in $\eta$ is

$$E[\eta(R^n)] = \frac{1}{\rho} ||K||^2 = \frac{1}{\rho} ||\hat{K}||^2 = \frac{1}{\rho} \left(\frac{\Gamma\left(\frac{n}{2} + 1\right)\alpha^n}{\Gamma\left(\frac{n}{2} + 1\right)}\right)^2 \int_{R^n} e^{-2|\alpha x|^\nu} \, dx$$

$$= \rho \left(\frac{\Gamma\left(\frac{n}{2} + 1\right)\alpha^n}{\pi^{n/2} \Gamma\left(\frac{n}{\nu} + 1\right)}\right)^2 \frac{n \pi^{n/2}}{\Gamma\left(\frac{n}{\nu} + 1\right)} \int_0^\infty r^{\nu-1} e^{-2(\alpha r)^\nu} \, dr$$

$$= \frac{\rho}{\pi^{n/2} \Gamma\left(\frac{n}{\nu} + 1\right)^2} \frac{n}{\nu} \int_0^\infty t^{\nu-1} e^{-t} \, dt = 2^{-\nu/\nu} \frac{\rho \Gamma\left(\frac{n}{2} + 1\right)}{\pi^{n/2} \Gamma\left(\frac{n}{\nu} + 1\right)}.$$  \(\text{Equation (16)}\)

By the bound on $\alpha$,

$$E[\eta(R^n)] < 2^{-n/\nu}.$$  

For fixed $\nu$, this global measure of repulsion approaches 0 for large $n$, but for fixed dimension $n$, it approaches its upper bound of one for large $\nu$. Thus, this class covers a wide range of repulsiveness similar to the Laguerre-Gaussian DPPs. In addition, the decay is exponential as $n$ goes to infinity and for $\nu > 2$, the rate is smaller than for the Laguerre-Gaussian models, i.e. the decay is slower.

Now consider a sequence $\{\Phi_n\}$ such that $\Phi_n \sim DPP(K_n)$ in $R^n$ such that

$$\hat{K}_n(x) = e^{\rho x \frac{\Gamma\left(\frac{n}{2} + 1\right)\alpha^n}{\pi^{n/2} \Gamma\left(\frac{n}{\nu} + 1\right)} e^{-|\alpha_n x|^\nu}, \ x \in R^n,$$  \(\text{Equation (17)}\)

for some $\nu > 0$, $\rho \in R$, and $\alpha_n > 0$. The asymptotic expansion for the upper bound on $\alpha_n$ is

$$\alpha_n < \left(\frac{\Gamma\left(\frac{n}{2}+1\right)\pi^{n/2}}{e^{\rho \Gamma\left(\frac{n}{\nu} + 1\right)}}\right)^{1/n} \sim \left(\frac{\Gamma\left(\frac{n}{2}+1\right)\pi^{n/2}}{e^{\rho \Gamma\left(\frac{n}{\nu} + 1\right)}}\right)^{1/n} \sim e^{-\rho \frac{n}{\nu} \frac{1}{2} + \frac{1}{2} \nu} (\nu e)^{1/2} = O(n^{\frac{1}{2} - \frac{1}{2}}).$$  \(\text{Equation (18)}\)

This shows that for $\nu > 2$, $\alpha_n \to 0$ as $n \to \infty$. For fixed dimension $n$, as $\alpha_n$ approaches 0, the power exponential kernel approaches the kernel associated with a Poisson point process. In the following, only models with $\nu \leq 2$ are considered, to ensure the sequence of DPPs do not approach Poisson point processes in the limit as $n \to \infty$.

The following results show that if the parameters $\alpha_n$ grow appropriately with $n$, this sequence satisfies the assumptions of Proposition 3.2.

**Lemma 4.3.** Let $0 < \nu \leq 2$. For each $n$, let $X_n$ be a vector in $R^n$ with density $K^2/||K||^2$ such that $\hat{K}_n$ is given by \(17\). Assume $\alpha_n \sim \alpha n^{\frac{1}{2} - \frac{1}{2}}$ for $\alpha \in (0, \infty)$, and $\alpha_n < \left(\frac{\Gamma\left(\frac{n}{2}+1\right)\pi^{n/2}}{e^{\rho \Gamma\left(\frac{n}{\nu} + 1\right)}}\right)^{1/n}$. Then,

$$\lim_{n \to \infty} \frac{\text{Var}(|X_n|^2)}{n^2} = 0, \text{ and } \lim_{n \to \infty} \frac{E[|X_n|^2]}{n} = \alpha^2 \frac{(2\nu)^{2/\nu}}{16\pi^2}.$$  

Now, applying Proposition 3.2, the following holds for a sequence of power exponential DPPs in the Shannon regime.

**Proposition 4.4.** For each $n$, let $\Phi_n \sim DPP(K_n)$ where $\hat{K}_n$ satisfies the assumptions in Lemma 4.3. Then,

$$\lim_{n \to \infty} \frac{E_n[\eta_n(B_n(\sqrt{n}R))]}{E_n[\eta_n(R^n)]} = \begin{cases} 0 & , \ R < R^* \\ 1 & , \ R > R^* \end{cases},$$

for $R^* := \alpha \frac{(2\nu)^{1/\nu}}{4\pi}$.  

13
For $\nu \in (1,2]$, the reach of repulsion $R^*$ for the power exponential models can also reach past the nearest neighbor threshold $\hat{R}$. Indeed, for $\alpha_n \sim an^{\frac{1}{\nu} - \frac{1}{2}}$, $R^* := \alpha^{(2\nu)/4\pi}$ satisfies $P_n[\Phi_n(B_n(0, \sqrt{n}R^*))] = 0 \to 0$ as $n \to \infty$ if

$$
\alpha \frac{(2\nu)^{1/\nu}}{4\pi} > \frac{1}{\sqrt{2\pi e^\rho}}.
$$

By the asymptotic formula [18] for the upper bound of $\alpha_n$, $\alpha < e^{-\rho(\nu e)^{-1/\nu}}(2n)1/2 = \frac{\sqrt{2\pi e}}{e^\rho(\nu e)^{1/\nu}}$.

Thus, $R^*$ reaches the desired regime when $\alpha_n \sim an^{\frac{1}{\nu} - \frac{1}{2}}$ and

$$
\frac{4\pi}{(2\nu)^{1/\nu}e^\rho \sqrt{2\pi e}} < \alpha < \frac{\sqrt{2\pi e}}{e^\rho(\nu e)^{1/\nu}}.
$$

The interval is non-empty since the upper bound is strictly greater than the lower bound for $\nu > 1$.

### 4.3 Bessel-type Models

Another class of DPP models presented in [3] is the Bessel-type. This class is more repulsive than the previous two families of models. It will be shown that while the Shannon regime is the right scaling to examine the repulsiveness of this class in high dimensions, a sequence of these DPPs does not satisfy the conditions of Proposition 3.2.

A stationary DPP $\Phi$ in $\mathbb{R}^n$ has a Bessel-type kernel if for some $\sigma \geq 0$, $\alpha > 0$, and $\rho > 0$,

$$
K(x) = \rho^{2(\sigma+n)/2} \Gamma \left( \frac{\sigma + n + 2}{2} \right) J_{(\sigma+n)/2} \left( \frac{2|x| \sqrt{\sigma + n}}{2} \right).
$$

The Fourier transform of the kernel is

$$
\hat{K}(\xi) = \rho \left( \frac{2\pi n}{\sigma + n} \right)^{\sigma/2} \Gamma((\sigma + n)/2) \Gamma((\sigma + 2)/2) \left( 1 - \frac{2\pi^2 \alpha^2 |x|^2}{\sigma + n} \right)^{\sigma/2}.
$$

The bound $0 \leq \hat{K} < 1$ implies that

$$
\alpha^n < \frac{(\sigma + n)^{\sigma/2} \Gamma \left( \frac{\sigma + n + 2}{2} \right)}{\rho^{(\sigma + n)\sigma/2} \Gamma \left( \frac{\sigma + n + 2}{2} \right)}.
$$

Similarly to the previous examples, this family contains DPPs covering a wide range of repulsiveness measured by $\eta$, and as $n \to \infty$, they are more repulsive in the sense that $E[\eta(R^n)]$ decays slower. Indeed,

$$
E[\eta(R^n)] = \frac{1}{\rho} \int_{\mathbb{R}^n} K(x)^2 dx = \rho \left( \frac{\sigma + n}{\sigma + n} \right)^{\sigma/2} \Gamma((\sigma + n)/2) \Gamma((\sigma + 2)/2) \Gamma(\sigma + 1) \Gamma(\sigma + \frac{n}{2} + 1)
$$

and by the bound on $\alpha$,

$$
E[\eta(R^n)] < \frac{\Gamma(\sigma + 1) \Gamma(\frac{\sigma}{2} + \frac{n}{2} + 1)}{\Gamma(\frac{\sigma}{2} + 1) \Gamma(\sigma + \frac{n}{2} + 1)}.
$$

By Stirling’s formula, as $n \to \infty$, $\frac{\Gamma(\sigma + 1) \Gamma(\frac{\sigma}{2} + \frac{n}{2} + 1)}{\Gamma(\frac{\sigma}{2} + 1) \Gamma(\sigma + \frac{n}{2} + 1)} = O(n^{-\sigma/2}).$

These DPPs do not satisfy the conditions of Proposition 3.2 and so the concentration of the first moment measure does not occur, contrary to the first two families presented. However, the repulsive measure does not reach past the $\sqrt{n}$ scale in the sense of the following proposition.
Proposition 4.5. Let $\rho \in \mathbb{R}$, $\alpha > 0$, and $\sigma > 3$. For each $n$, let $\Phi_n \sim DPP(K_n)$ in $\mathbb{R}^n$ where

$$K_n(x) = e^{n\rho (\sigma + n + 2)} \left( \frac{\sigma + n + 2}{2} \right) \frac{J_{(\sigma + n)/2}(2|x|/\sqrt{\sigma + n}/2)}{(2|x|/\sqrt{\sigma + n}/2)^{(\sigma + n)/2}}, \ x \in \mathbb{R}^n.$$  

Then, for any $\beta > \frac{1}{2}$ and $R > 0$,

$$\frac{\mathbb{E}_n[\eta_n(B_n(R\gamma)^c)]}{\mathbb{E}_n[\eta_n(R^n)]} \rightarrow 0 \text{ as } n \rightarrow \infty.$$  

Remark 4.6. The proof of this result (see Appendix [4]) shows that this sequence of DPPs does not satisfy the concentration of measure based on Chebychev’s inequality in Proposition [7, 24]. For models where $1 < \sigma \leq 3$, one can use the more general result described in Remark [3, 3] to show that a thin-shell estimate does not occur for the sequence $|X_n|$ associated with these models.

4.4 Normal Variance Mixture Models

Another class of DPPs described in [14] are those with normal-variance mixture kernels. This is a large class of DPP models, and it is much more difficult to compute with them directly compared to the earlier classes. The few observations we can make about this class are mentioned here.

A DPP $\Phi$ with associated kernel $K$ is a normal-variance mixture DPP in $\mathbb{R}^n$ if

$$K(x) = \rho \frac{\mathbb{E}[W^{-n/2}e^{-|x|^2/(2W)}]}{\mathbb{E}[W^{-n/2}]}, \ x \in \mathbb{R}^n,$$

for some non-negative real-valued random variable $W$ such that $\mathbb{E}[W^{-n/2}] < \infty$. The Fourier transform of the kernel is

$$\hat{K}(\xi) = \rho \frac{(2\pi)^{n/2}}{\mathbb{E}[W^{-n/2}]} \mathbb{E}[\exp(-2\pi |\xi|^2 W)], \ \xi \in \mathbb{R}^n.$$  

Note that $\rho = K(0)$ is the intensity of $\Phi$. The bound $0 \leq \hat{K} < 1$ translates to the following bound on the intensity:

$$\rho < \mathbb{E}[W^{-n/2}]/(2\pi)^{n/2}.$$  

There are a few well-known examples of DPP models from this class. If $\sqrt{2W} = \alpha$, this is the Gaussian DPP model. If $W \sim \text{Gamma}(\nu + \frac{\alpha}{2}, 2\alpha^2)$, this is called the Whittle-Matérn model. The Cauchy model is given when $\frac{1}{W} \sim \text{Gamma}(\nu, 2\alpha^{-2})$. In both cases $\nu > 0$ and $\alpha > 0$ are parameters.

This is a large family of DPPs to consider, but they do not cover a wide range of repulsiveness as was the case for the previous families. Indeed, for any random variable $W$ in $\mathbb{R}^+$ such that $\mathbb{E}[W^{-n/2}] < \infty$, Parseval’s theorem, Jensen’s inequality, the upper bound on $\rho$, and Fubini’s theorem imply

$$\mathbb{E}[\eta(R^n)] = \frac{1}{\rho} \int_{\mathbb{R}^n} \hat{K}_n(x)^2 dx = \frac{1}{\rho} \int_{\mathbb{R}^n} \left( \rho \frac{(2\pi)^{n/2}}{\mathbb{E}[W^{-n/2}]} \mathbb{E}[e^{-2\pi^2 |x|^2 W}] \right)^2 dx 
\leq \frac{(2\pi)^{n/2}}{\mathbb{E}[W^{-n/2}]} \int_{\mathbb{R}^n} E[e^{-4\pi^2 |x|^2 W}] dx 
= \frac{(2\pi)^{n/2}}{\mathbb{E}[W^{-n/2}]} E \left( (4\pi W)^{-n/2} E \left[ (4\pi W)^{-n/2} \int_{\mathbb{R}^n} e^{-4\pi^2 |x|^2 W} dx \left| W \right) \right] \right) = \left( \frac{1}{2} \right)^{n/2}.$$  

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Is it difficult to make further general statements about this class with regards to the first moment measure of $\eta$ because the behavior of the sequence $\frac{|X_n|^2}{n}$ depends greatly on the distribution of the $R^+$-valued random variable $W$, leading to a wide variety of behaviors. The rest of the section will describe a few additional observations.

Consider a sequence of normal-variance mixture DPPs all associated with the same random variable $W$. If $W$ is a constant $\alpha$, the random variables $X_n$ become multivariate Gaussian vectors with mean zero and variance depending on $\alpha$. The scaled norms of these vectors are well-known to satisfy a LDP by the law of large numbers, since the coordinates are independent. This also corresponds to a Laguerre-Gaussian DPP with parameter $m = 2$.

It is also easy to identify a subclass of the normal-variance mixture models that satisfy Proposition 3.5. In [25], it is proved that if $\nu > 1$ is log-concave and $\nu^2 > 2$, the reach of repulsion $R^*$ has the upper bound $\tilde{R}$ for the nearest neighbor function, and so unlike in the case of Laguerre-Gaussian DPPs and power exponential DPPs, the reach of repulsion $R^*$ for a sequence of Cauchy models with fixed parameter $\nu$ cannot reach past $\tilde{R}$. The upper bound on $\alpha$ parameter grows appropriately with $n$. See [14] for more on these models.

**Proposition 4.7.** For each $n$, let $\Phi_n \sim DPP(K_n)$ be a Cauchy model in $\mathbb{R}^n$ with intensity $e^{\alpha n}$ and parameters $\nu > 0$ and $\alpha_n$, i.e. let

$$K_n(x) = \frac{e^{2\nu\rho}}{1 + \frac{|x|^2}{\alpha_n^{\nu+\frac{1}{2}}}}, \quad x \in \mathbb{R}^n.$$ 

Assume $\alpha_n \sim \alpha n^{1/2}$ as $n \to \infty$ for some $\alpha > 0$ such that $\alpha_n < \frac{\Gamma(\nu + \frac{1}{2})}{\sqrt{\pi e \Gamma(\nu)}}$ for each $n$. Then, for the random vector $X_n$ with density $\frac{K_n(x)}{\|K_n\|_2}$,

$$\lim_{n \to \infty} \frac{\text{Var}(|X_n|^2)}{n^2} = 0, \quad \text{and} \quad \lim_{n \to \infty} \frac{\mathbb{E}[|X_n|^2]}{n} = \alpha^2.$$ 

Thus, by Proposition 3.2 for $R^* := \alpha$,

$$\lim_{n \to \infty} \frac{\mathbb{E}_n[\eta_n(B_n(\sqrt{n}R))] \mathbb{E}_n[\eta_n(R^n)]}{\mathbb{E}_n[\eta_n(\mathbb{R}^n)]} = \begin{cases} 0, & R < R^* \\ 1, & R > R^* \end{cases}.$$ 

**Remark 4.8.** The upper bound on $\alpha_n$ has the following asymptotic expansion as $n \to \infty$:

$$\alpha_n < \frac{\Gamma(\nu + \frac{n}{2})}{\sqrt{\pi e \Gamma(\nu)}} \sim \frac{n^{1/2}}{\sqrt{2 \pi e \nu}}.$$ 

Thus, if $\alpha_n \sim \alpha n^{\frac{1}{2}}$, the reach of repulsion has the upper bound

$$R^* := \alpha < \frac{1}{\sqrt{2 \pi e \nu}}.$$
5 Applications

5.1 Rate of Convergence for the Nearest Neighbor Function in the Shannon Regime

In the following, the nearest neighbor function of DPPs is studied further. Under certain conditions, the nearest neighbor function does not only have the same threshold behavior as the void function, it also grows and decays at the same rate. This further signifies the Poisson-like behavior of DPPs in high dimensions. It is unknown if the rate changes for DPPs allowing eigenvalues of one.

**Proposition 5.1.** For each $n$, let $\Phi_n \sim DPP(K_n)$ be a stationary and isotropic DPP in $\mathbb{R}^n$ such that $0 \leq K_n \leq M < 1$. Let $R \in \mathbb{R}^+$. If $E_n[\eta_n(B_n(\sqrt{n}R))] \to 0$ as $n \to \infty$, then

$$P_n(\Phi_n^0(B_n(\sqrt{n}R)) = 0) \sim P_n(\Phi_n(B_n(\sqrt{n}R)) = 0) \quad \text{as} \quad n \to \infty.$$ 

**Proof.** Let $K_{n,B_n(\sqrt{n}R)}$ be the self-adjoint compact integral operator defined by

$$(K_{n,B_n(\sqrt{n}R)}f)(x) = \int_{B_n(\sqrt{n}R)} K_n(x - y)f(y)dx, \quad x \in B_n(\sqrt{n}R).$$

Suppose $\{\lambda_i^R\}$ and $\{\phi_i^R\}$ are the eigenvalues and eigenvectors of $K_{n,B_n(\sqrt{n}R)}$, i.e

$$(K_{n,B_n(\sqrt{n}R)}\phi_i^R)(x) = \lambda_i^R \phi_i^R(x) \quad \text{for all} \quad i.$$ 

By (3),

$$K_n(x) = \sum_{i=1}^{\infty} \lambda_i^R \phi_i^R(x) \phi_i^R(0), \quad x \in B_n(\sqrt{n}R).$$

Then, by Equation (19) in [8] and Lemma 7 in [8],

$$P_n(\Phi_n^0(B_n(\sqrt{n}R)) = 0) = P_n(\Phi_n(B_n(\sqrt{n}R)) = 0) \leq \frac{\sum_{i=1}^{\infty} \lambda_i^R |\phi_i^R(0)|^2}{\sum_{i=1}^{\infty} \lambda_i^R |\phi_i^R(0)|^2}.$$ 

Let $f_{B_n(\sqrt{n}R)}(x) = f(x)$ if $x \in B_n(\sqrt{n}R)$ and $f_{B_n(\sqrt{n}R)}(x) = 0$ otherwise. By the assumed bounds on $K_n$, the eigenvalues of $K_{n,B_n(\sqrt{n}R)}$ are all contained in $[0, M]$. Indeed, by the spectrum bound for a self-adjoint operator (see [20]) and Parseval’s theorem, for all $i$,

$$\lambda_i^R \leq \sup_{||f||_2 = 1} \int_{B_n(\sqrt{n}R)} (K_{n,B_n(\sqrt{n}R)}f)(y)f(y)dy = \sup_{||f||_2 = 1} \int_{B_n(\sqrt{n}R)} \int_{B_n(\sqrt{n}R)} K_n(x - y)f(x)f(y)dxdy = \sup_{||f||_2 = 1} \int_{\mathbb{R}^n} \tilde{K}_n(\xi)(f_{B_n(\sqrt{n}R)})^2(\xi)d\xi \leq M \sup_{||f||_2 = 1} ||f_{B_n(\sqrt{n}R)}||_2 \leq M,$$

and

$$\lambda_i^R \geq \inf_{||f||_2 = 1} \int_{B_n(\sqrt{n}R)} (K_{n,B_n(\sqrt{n}R)}f)(y)f(y)dy = \inf_{||f||_2 = 1} \int_{\mathbb{R}^n} \tilde{K}_n(\xi)(f_{B_n(\sqrt{n}R)})^2(\xi)d\xi \geq 0.$$
This implies that \( \frac{1}{1-\lambda_i^k} \leq \frac{1}{1-M} \) for all \( i \). Then,

\[
\left| \frac{P_n(\Phi_n^0(B_n(\sqrt{nR})) = 0)}{P_n(\Phi_n(B_n(\sqrt{nR})) = 0)} - 1 \right| = \left| \sum_{i=1}^{\infty} \frac{\lambda_i^R}{1-\lambda_i^R} |\phi_i^R(0)|^2 \right| - 1 = \left| \sum_{i=1}^{\infty} \frac{\lambda_i^R |\phi_i^R(0)|^2}{\sum_{i=1}^{\infty} \lambda_i^R |\phi_i^R(0)|^2} \right| \\
= \left| \sum_{i=1}^{\infty} \frac{(\lambda_i^R)^2 |\phi_i^R(0)|^2}{K_n(0)} \right| \leq \frac{1}{1-M} \left( \frac{1}{K_n(0)} \sum_{i=1}^{\infty} (\lambda_i^R)^2 |\phi_i^R(0)|^2 \right).
\]

Finally, note that

\[
\int_{B_n(\sqrt{nR})} K_n(x)^2 \, dx = \sum_{i=1}^{\infty} (\lambda_i^R)^2 |\phi_i^R(0)|^2.
\]

Since \( K_n(0) \) is the intensity of \( \Phi_n \), (7) gives

\[
\left| \frac{P_n(\Phi_n^0(B_n(\sqrt{nR})) = 0)}{P_n(\Phi_n(B_n(\sqrt{nR})) = 0)} - 1 \right| \leq \frac{1}{1-M} E_n[\eta_n(B_n(\sqrt{nR}))].
\]

Thus, if \( E_n[\eta_n(B_n(\sqrt{nR}))] \to 0 \) as \( n \to \infty \), then

\[
P_n(\Phi_n^0(B_n(\sqrt{nR})) = 0) \sim P(\Phi_n(B_n(\sqrt{nR})) = 0) \text{ as } n \to \infty.
\]

\[\square\]

### 5.2 Degree of Determinantal Boolean Models in the Shannon Regime

Poisson Boolean models in the Shannon regime were studied in [1], and the results on the degree threshold (see below) can be extended to Laguerre-Gaussian DPPs using Proposition 4.2.

The setting is the following: Consider a sequence of stationary DPPs \( \Phi_n \), indexed by dimension, where \( \Phi_n \sim DPP(K_n) \) in \( \mathbb{R}^n \). Assume that for each \( n \), \( K_n \) is continuous, symmetric, and \( 0 \leq K_n < 1 \). Let the intensity of \( \Phi_n \) be \( K_n(0) = e^{\rho_n} \). Let \( \Phi_n = \sum_k \delta_{T_n(k)} \) and \( R > 0 \). Then, consider the sequence of particle processes \( C_n = \bigcup_k B(T_n(k), \sqrt{nR}) \).

The degree of each process, called a determinantal Boolean model, is the expected number of balls that intersect the ball centered at zero under the reduced Palm distribution, i.e., \( E_n[\Phi_n^0(B_n(\sqrt{nR}))] \). In the case when \( \Phi_n \) is Poisson, \( E_n[\Phi_n(B(\sqrt{nR}))] = E[\Phi_n(B(\sqrt{nR}))] \) by Slivnyak’s theorem, and

\[
\lim_{n \to \infty} \frac{1}{n} \ln E_n[\Phi_n(B_n(\sqrt{nR}))] = \rho + \frac{1}{2} \log 2\pi e + \log R.
\]

To extend this result to DPPs, it is needed that as \( n \to \infty \),

\[
E_n[\Phi_n^0(B_n(\sqrt{nR}))] \sim E_n[\Phi_n(B_n(\sqrt{nR}))].
\]

Notice that

\[
\frac{E_n[\Phi_n^0(B_n(\sqrt{nR}))]}{E_n[\Phi_n(B_n(\sqrt{nR}))]} = 1 - \frac{E_n[\eta_n(B_n(\sqrt{nR}))]}{E_n[\Phi_n(B_n(\sqrt{nR}))]}.
\]

Thus, if \( \frac{E_n[\eta_n(B_n(\sqrt{nR}))]}{E_n[\Phi_n(B_n(\sqrt{nR}))]} \to 0 \) as \( n \to \infty \), then the degree of the determinantal Boolean model has the same asymptotic behavior as the Poisson Boolean model.

In the case of Laguerre-Gaussian kernels, this is the case, and the earlier results even provide the rate at which the quantity goes to zero, which exhibits a threshold at \( R^* \) as is expected.
Proposition 5.2. Let $m \in \mathbb{N}$ and $\rho \in \mathbb{R}$. For each $n$, let $\Phi_n \sim DPP(K_n)$ in $\mathbb{R}^n$ where

$$K_n(x) = \frac{e^{\rho x}}{(m-1)^{n/2}I_m^{-1/2} \left( \frac{x}{m} \right) e^{-\frac{|x|^2}{m}}}$$

and $\alpha$ is a parameter such that $0 < \alpha < \frac{1}{\sqrt{m_2}}$. Then,

$$\lim_{n \to \infty} -\frac{1}{n} \ln \frac{E_n[\eta_n(B_n(\sqrt{n}R))]}{E_n[\Phi_n(B_n(\sqrt{n}R))]} = \begin{cases} \frac{2R^2}{\alpha^2n}, & 0 < R < \sqrt{m_2} \\ \frac{1}{2} + log 2 - \log \alpha - \frac{1}{2} \log m + \log R, & R > \sqrt{m_2}. \end{cases}$$

Proof. By Proposition 4.2,

$$\lim_{n \to \infty} -\frac{1}{n} \ln E_n[\eta_n(B_n(\sqrt{n}R))] = \begin{cases} -\rho - \frac{1}{2} \log 2 \pi e + \frac{2R^2}{\alpha^2n} - \log R, & 0 < R < \sqrt{m_2} \\ -\log \alpha - \frac{1}{2} \log m + \log R, & R > \sqrt{m_2}. \end{cases}$$

Recall that $\lim_{n \to \infty} \frac{1}{n} \ln E_n[\Phi_n(B_n(\sqrt{n}R))] = \rho + \frac{1}{2} \log 2 \pi e + \log R$. Thus,

$$\lim_{n \to \infty} -\frac{1}{n} \ln \frac{E_n[\eta_n(B_n(\sqrt{n}R))]}{E_n[\Phi_n(B_n(\sqrt{n}R))]} = \lim_{n \to \infty} -\frac{1}{n} \ln E_n[\eta_n(B_n(\sqrt{n}R))] + \frac{1}{n} \ln E_n[\Phi_n(B_n(\sqrt{n}R))]

= \begin{cases} -\rho - \frac{1}{2} \log 2 \pi e + \frac{2R^2}{\alpha^2n} - \log R + \rho + \frac{1}{2} \log 2 \pi e + \log R, & 0 < R < \sqrt{m_2} \\ -\log \alpha - \frac{1}{2} \log m + \log R, & R > \sqrt{m_2}. \end{cases}$$

$$= \begin{cases} \frac{2R^2}{\alpha^2n}, & 0 < R < \sqrt{m_2} \\ \frac{1}{2} + \log 2 - \log \alpha - \frac{1}{2} \log m + \log R, & R > \sqrt{m_2}. \end{cases}$$

\[\square\]

6 Conclusion

By examining a measure of repulsiveness of DPPs, this paper provides insight into the high dimensional behavior of different families of DPP models. While it is clear that the families of DPPs presented in \cite{3} and \cite{14} become more and more similar to Poisson point processes as dimension increases, the reach of the small repulsive effect can still be quantified. By making a connection between the kernel of the DPP and the concentration in high dimensions of the norm of a random vector to its expectation, we have shown under certain conditions that there exists a distance on the $\sqrt{n}$ scale at which the repulsive effect of a point of the DPP model is strongest as the dimension $n$ increases. It has been illustrated that some families of DPPs exhibit this reach of repulsion and some do not.

Many questions remain concerning the range of possible repulsive behavior of DPPs in high dimensions. First, the results can be extended to scalings other than the Shannon regime in the following way. Assumptions \cite{9} and \cite{10} in Proposition 3.2 can be generalized to the assumptions that for some sequence $b_n$, $\frac{\text{Var}(X_n^2)}{b_n^2} \to 0$ as $n \to \infty$ and $\left( \frac{E[X_n^2]}{b_n} \right)^{1/2} \to R^*$ as $n \to \infty$. If $b_n \neq O(n)$, the result holds for a different scaling than the Shannon regime, and the repulsiveness is strongest near $R^*b_n^{1/2}$ in high dimensions. While this is precisely what is shown not to happen for the Bessel-type DPPs, examples of this generalization for $b_n = o(n)$ are provided by the power exponential DPPs when $\nu > 2$. However, as noted in the introduction, any distance scaling smaller than $\sqrt{n}$ will not reach the regime where the expected number
of points goes to infinity as dimension grows. Thus, this scaling appears less interesting. It would be very interesting to find a family of DPPs that exhibited the concentration at a scaling greater than $\sqrt{n}$, i.e. for $b_n \gg n$. It is unknown still if this is possible and if so, what kind of kernel would exhibit this behavior.

For all of the DPPs studied in this paper, the total mass of the first moment measure of $\eta_n$ decays to zero, i.e. $E_n[\eta_n(\mathbb{R}^n)] \to 0$ as $n \to \infty$. This is not always the case. For instance, there exists a sequence of DPPs in increasing dimensions such that for $c \in (0, 1)$, $E_n[\eta_n(\mathbb{R}^n)] = c$ for all $n$. Indeed, let $K_n \in L^2(\mathbb{R}^n)$ be such that its Fourier transform is

$$\hat{K}_n(\xi) = \sqrt{c}1_{B_n(r_n)}(\xi), \xi \in \mathbb{R}^n,$$

where $r_n \in \mathbb{R}^+$ is such that $\text{Vol}(B_n(r_n)) = K_n(0)$. Then,

$$E_n[\eta_n(\mathbb{R}^n)] = \frac{1}{K_n(0)} \int_{\mathbb{R}^n} K_n(x)^2 \, dx = \frac{1}{K_n(0)} \int_{\mathbb{R}^n} \hat{K}_n(\xi)^2 \, d\xi = \frac{c}{K_n(0)} \text{Vol}(B_n(r_n)) = c.$$ 

It would thus be useful to find a necessary and sufficient condition for $E_n[\eta_n(\mathbb{R}^n)]$ to converge to zero. However, if $E_n[\eta_n(\mathbb{R}^n)]$ does not converge to zero, this does not necessarily prevent $P_n(\eta_n(\mathbb{R}^n) = 0)$ from approaching 1 as $n$ goes to infinity. It would be interesting to find a sequence of DPPs where $P_n(\eta_n(\mathbb{R}^n) = 0)$ approaches some $c < 1$. This would require more knowledge of the joint distribution under the coupling given by Strassen’s, since we currently can only understand the behavior of the first moment measure and not higher moments of $\eta_n$.

There is an important class of stationary and isotropic DPPs that should be mentioned. Recall that in order for $\eta$ to be well-defined, it is required that the kernel $K$ associated with $\Phi$ satisfies $0 \leq \hat{K} < 1$. However, $\Phi$ still exists when $\hat{K}$ is allowed to attain the maximum value of 1. This corresponds to the case where the associated operator $\mathcal{K}$ has an eigenvalue of 1, and for the models studied in this paper, it is the case when the parameter achieves its upper bound. In this case, we can still define the measure of repulsiveness (2) even though it cannot be interpreted as the intensity measure of a point process $\eta$. Replacing $E[\eta(B)]$ with $\int_B (1-g(x)) \, dx$ for $B \in \mathcal{B}(\mathbb{R}^n)$, the main results (Propositions 3.2, 3.5 and 3.7) can be restated with the condition that $0 \leq \hat{K} \leq 1$. In this case, the reach of repulsion $\mathbb{R}^n$ is interpreted as the distance on the $\sqrt{n}$ scale at which the measure of repulsion is strongest.

A particularly interesting subclass of the DPPs described in the previous paragraph are the most repulsive stationary DPPs, introduced in the on-line supplementary material to [14], (see [13]). These DPPs maximize the measure of repulsiveness $\gamma$, and have a kernel $\hat{K}$ such that $\hat{K}$ is defined as in (19) but with $c = 1$. For the most repulsive DPPs, $\gamma = 1$ in any dimension. In addition, for a sequence of DPPs $\{\Phi_n\}_{n \in \mathbb{N}}$ where $\Phi_n$ is the most repulsive DPP in $\mathbb{R}^n$ with intensity $e^{\gamma \rho}$, $X_n$ as defined in Proposition 3.2 satisfies

$$E[|X_n|^2] = \int_{\mathbb{R}^n} |x|^2 \frac{K_n(x)^2}{\|K_n\|^2} \, dx = \frac{\Gamma \left( \frac{n}{2} + 1 \right)}{\pi^{n/2}} \int_{\mathbb{R}^n} |x|^2 \frac{J_{n/2}^2 \left( 2\sqrt{\pi \Gamma \left( \frac{n}{2} + 1 \right)^{1/n} e^\rho |x|} \right)}{|x|^n} \, dx$$

$$= \frac{\Gamma \left( \frac{n}{2} + 1 \right)}{\pi^{n/2}} \int_0^\infty r J_{n/2}^2 \left( 2\sqrt{\pi \Gamma \left( \frac{n}{2} + 1 \right)^{1/n} e^\rho r} \right) \, dr,$$

where $J_{\nu}$ is the Bessel function of the first kind of order $\nu$ (see [3]). By [19], Eq. 1.17.13, this integral does not converge, i.e. $|X_n|$ does not have a finite second moment.

**Acknowledgements**

The work of both authors was supported by a grant of the Simons Foundation (#197982 to UT Austin). The work of the second author was supported by the National Science Foundation Graduate Research Fellowship under Grant No. DGE-1110007.
Appendices

A Proof of (6)

For each \( n \), let \( \Phi_n \sim DPP(K_n) \) in \( \mathbb{R}^n \) be stationary with intensity \( K_n(0) = e^{n\rho} \). Recall from (1) that there exists a threshold \( \tilde{\rho} \) such that for \( \rho < \tilde{\rho} \), \( E_n[\Phi_n(B_n(\sqrt{n}R))] \to 0 \) and for \( \rho > \tilde{\rho} \), \( E_n[\Phi_n(B_n(\sqrt{n}R))] \to \infty \) as \( n \to \infty \). Fixing \( \rho \), one also has that for \( \tilde{R} := \frac{1}{\sqrt{2e}} e^{\frac{\rho}{2}} \),

\[
E_n[\Phi_n(B_n(\sqrt{n}R))] \to \begin{cases} 
0, & R < \tilde{R} \\
\infty, & R > \tilde{R}. 
\end{cases}
\]

By Theorem 2.4,

\[
E_n[\Phi_n(B_n(\sqrt{n}R))] - E_n[\Phi_0^{0,1}(B_n(\sqrt{n}R))] = \frac{1}{e^{n\rho}} \int_{B_n(\sqrt{n}R)} K_n(x)^2 \, dx
\]

Then, by Parseval’s theorem and Theorem 2.3,

\[
\frac{1}{e^{n\rho}} \int_{B_n(\sqrt{n}R)} K_n(x)^2 \, dx \leq \frac{1}{e^{n\rho}} \int_{\mathbb{R}^n} \hat{K}_n(\xi)^2 \, d\xi \leq \frac{1}{e^{n\rho}} \int_{\mathbb{R}^n} \hat{K}_n(\xi) \, d\xi = 1.
\]

Also, since \( \frac{1}{e^{n\rho}} \int_{B_n(\sqrt{n}R)} K_n(x)^2 \, dx \geq 0 \), the following bounds hold:

\[
E_n[\Phi_n(B_n(\sqrt{n}R))] - 1 \leq E_n[\Phi_0^{0,1}(B_n(\sqrt{n}R))] \leq E_n[\Phi_n(B_n(\sqrt{n}R))].
\]

Thus, the threshold remains the same for the reduced Palm expectation:

\[
E_n[\Phi_0^{0,1}(B_n(\sqrt{n}R))] \to \begin{cases} 
0, & R < \tilde{R} \\
\infty, & R > \tilde{R}. 
\end{cases}
\]

Finally, by (3),

\[
1 - E_n[\Phi_0^{0,1}(B(\sqrt{n}R))] \leq P_n(\Phi_0^{0,1}(B_n(\sqrt{n}R)) = 0) \leq \exp \left( -E_n[\Phi_0^{0,1}(B(\sqrt{n}R))] \right).
\]

Thus,

\[
P_n(\Phi_0^{0,1}(B_n(\sqrt{n}R)) = 0) \to \begin{cases} 
1, & R < \tilde{R} \\
0, & R > \tilde{R}, 
\end{cases}
\]

as \( n \to \infty \).

B Proof of Main Results

B.1 Proof of Proposition 3.2

The first assumption and Chebychev’s inequality implies that for any \( \delta > 0 \),

\[
P \left( \left| \frac{X_n}{n} - E[X_n/n] \right| > \delta \right) \leq \frac{\text{Var}(|X_n|^2)}{\delta^2 n^2} \to 0 \text{ as } n \to \infty.
\]

Thus, \( \frac{|X_n|^2}{n} - \frac{E|X_n|^2}{n} \to 0 \) in probability. The second assumption says that \( \frac{E|X_n|^2}{n} \to (R^*)^2 \) as \( n \to \infty \). Thus,
\[ \frac{|X_n|^2}{n} \rightarrow (R^*)^2 \] in probability.

Now, assume \( R < R^* \). Then, there exists \( \varepsilon > 0 \) such that \( R^2 = (R^*)^2 - \varepsilon \). Thus,

\[
P(|X_n| \leq \sqrt{nR}) = P\left( \frac{|X_n|^2}{n} \leq R^2 \right) = P\left( \frac{|X_n|^2}{n} \leq (R^*)^2 - \varepsilon \right)
\leq P\left( \left| \frac{|X_n|^2}{n} - (R^*)^2 \right| > \varepsilon \right) \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Second, assume \( R > R^* \). Then, there exists \( \varepsilon > 0 \) such that \( R^2 = (R^*)^2 + \varepsilon \), and

\[
P(|X_n| \leq \sqrt{nR}) = 1 - P\left( \frac{|X_n|^2}{n} > (R^*)^2 + \varepsilon \right) \geq 1 - P\left( \left| \frac{|X_n|^2}{n} - (R^*)^2 \right| > \varepsilon \right) \rightarrow 1.
\]

Then, by Lemma 3.1

\[
\frac{E_n[\eta_n(B_n(\sqrt{nR}))]}{E_n[\eta_n(R^n)]} = P\left( |X_n| \leq \sqrt{nR} \right) \rightarrow \begin{cases} 0, & R < R^* \\ 1, & R > R^* \end{cases}
\]

### B.2 Proof of Proposition 3.5

Recall \( \sigma_n^2 := E[|X_n|^2] \) and that the radial symmetry of the density of \( X_n \) implies \( \frac{\sqrt{n}}{\sigma_n}X_n \) is an isotropic vector. Then, applying Theorem 3.4 to \( \frac{\sqrt{n}}{\sigma_n}X_n \), we obtain that for any \( \delta > 0 \), there exist absolute constants \( C, c > 0 \) such that

\[
P\left( \left| \frac{|X_n|}{\sigma_n} - 1 \right| \geq \delta \right) \leq Ce^{-c\left( \frac{\sqrt{n}}{\sigma_n} \right)^{1\over 2} \min(\delta, \delta^3)}.
\]

Now, let \( \delta \in (0, 1) \). By Lemma 3.1

\[
\frac{E_n[\eta_n(B_n(\sigma_n(1 - \delta)))]}{E_n[\eta_n(R^n)]} = P\left( \frac{|X_n|}{\sigma_n} \leq 1 - \delta \right) \leq Ce^{-c\left( \frac{\sqrt{n}}{\sigma_n} \right)^{1\over 2} \delta^3},
\]

since \( \min(\delta^3, \delta) = \delta^3 \) for \( \delta \in (0, 1) \). Similarly, for any \( \delta > 0 \),

\[
\frac{E_n[\eta_n(B_n(\sigma_n(1 + \delta)^3))]}{E_n[\eta_n(R^n)]} = P\left( \frac{|X_n|}{\sigma_n} \geq 1 + \delta \right) \leq Ce^{-c\left( \frac{\sqrt{n}}{\sigma_n} \right)^{1\over 2} \min(\delta^3, \delta^9)}.
\]

Now, assume \( \frac{\sqrt{n}}{\sigma_n} \rightarrow R^* \in (0, \infty) \) as \( n \rightarrow \infty \). For \( R < R^* \), there exists \( \varepsilon \in (0, 1) \) such that \( R = R^*(1 - \varepsilon) \). Then, for \( n \) large enough, \( \frac{\sqrt{nR}}{\sigma_n} < 1 + \varepsilon \) and

\[
\frac{E_n[\eta_n(B_n(\sqrt{nR}))]}{E_n[\eta_n(R^n)]} = P\left( |X_n| \leq \sqrt{nR} \right) = P\left( \frac{|X_n|}{\sigma_n} \leq \sqrt{nR} \right)
\leq P\left( \left| \frac{|X_n|}{\sigma_n} - (1 + \varepsilon) \right| \leq \frac{1}{\sigma_n} \right) \leq C e^{-c\left( \frac{\sqrt{n}}{\sigma_n} \right)^{1\over 2} \varepsilon^2}.
\]

Thus for all \( R < R^* \), there exists a constant \( C(\varepsilon(R), \beta, \alpha) = \frac{C_3}{\varepsilon^2} \) such that

\[
\liminf_{n \rightarrow \infty} - \frac{1}{n^{1/2}} \ln \frac{E_n[\eta_n(B_n(\sqrt{nR}))]}{E_n[\eta_n(R^n)]} \geq C(\varepsilon(R), \beta, \alpha).
\]
B.3 Proof of Proposition 3.7

If $|X_n|/\sqrt{n}$ satisfies a large deviations principle with convex rate function $I$, then by definition,

$$-\inf_{r < R} I(r) \leq \liminf_{n \to \infty} \frac{1}{n} \ln P\left(\frac{|X_n|}{\sqrt{n}} \leq R\right) \leq \limsup_{n \to \infty} \frac{1}{n} \ln P\left(\frac{|X_n|}{\sqrt{n}} \leq R\right) \leq -\inf_{r \leq R} I(r).$$

Thus, by Lemma 3.1

$$-\inf_{r < R} I(r) \leq \liminf_{n \to \infty} \frac{1}{n} \ln \frac{E_n[\eta_n(B_n(\sqrt{n}R))]}{E_n[\eta_n(R^n)]} \leq \limsup_{n \to \infty} \frac{1}{n} \ln \frac{E_n[\eta_n(B_n(\sqrt{n}R))]}{E_n[\eta_n(R^n)]} \leq -\inf_{r \leq R} I(r).$$

By the assumption that the rate function $I$ is strictly convex, there exists a unique $R^*$ such that $I(R^*) = 0$. Note that the exponent is then zero for $R > R^*$. Thus,

$$\lim_{n \to \infty} \frac{E_n[\eta_n(B_n(\sqrt{n}R))]}{E_n[\eta_n(R^n)]} = \begin{cases} 0 & R < R^* \\ 1 & R > R^*. \end{cases}$$

Let $R < R^*$. If the rate function $I$ is continuous at $R$, then the above inequalities become equalities and

$$\lim_{n \to \infty} -\frac{1}{n} \ln \frac{E_n[\eta_n(B_n(\sqrt{n}R))]}{E_n[\eta_n(R^n)]} = I(R).$$

C. Proof of Proposition 4.1

The proof shows that the sequence of random variables satisfies the conditions of the Gärnter-Ellis theorem (see [6]). First,

$$E[e^{a|X_n|^2}] = \frac{e^{2np}}{\binom{m-1+n/2}{m-1}} \int \frac{I(s)}{R^n} e^{-(\frac{2}{m} - s)|x|^2} \left(L^{n/2}_{m-1} \left(\frac{1}{m} \frac{|x|^2}{\alpha^2}\right)\right)^2 dx.$$  

Writing out the polynomial, the integral $I$ above becomes

$$I(s) = \sum_{k=0}^{m-1} \sum_{j=0}^{m-1-k} \binom{m-1+k}{m-1-k} \binom{m-1}{m-1-j} \frac{(-1)^{k+j}}{k!j!} \frac{1}{(ma^2)^{k+j}} \int_{R^n} e^{-(\frac{2}{m} - s)|x|^2} |x|^{2k+2j} dx.$$  

A quick calculation shows that for $a > 0$,

$$\int_{R^n} e^{-a|x|^2} |x|^b dx = \frac{\pi^{n/2} \Gamma\left(n + b + \frac{1}{2}\right)}{a^{n+b} \Gamma\left(\frac{n+b}{2}\right)}.$$  

Then, if $s < \frac{2}{\alpha^2}$,

$$I(s) = \frac{\pi^{n/2}}{\frac{2}{\alpha^2} - s} \frac{1}{\Gamma\left(\frac{n}{2}\right)} \sum_{k=0}^{m-1} \sum_{j=0}^{m-1-k} \binom{m-1+k}{m-1-k} \binom{m-1}{m-1-j} \frac{(-1)^{k+j} \Gamma\left(n + k + j + \frac{1}{2}\right)}{k!j! \left(2 - s\alpha^2\right)^{k+j}},$$

and $I(s) = \infty$ otherwise. For each $k, j \in \mathbb{N}$,

$$\left(\binom{m-1+k}{m-1-j}\Gamma\left(\frac{n}{2} + k + j\right)\right) \sim \left(\frac{n}{2}\right)^{m-1-k} \left(\frac{n}{2}\right)^{m-1-j} \Gamma\left(\frac{n}{2}\right)^{k+j} = \left(\frac{n}{2}\right)^{2m-2} \Gamma\left(\frac{n}{2}\right),$$  

(21)
as \( n \to \infty \). So, \( I(s) \) has the following asymptotic expansion for \( s < \frac{2}{\alpha^2 m} \) as \( n \to \infty \):

\[
I(s) \sim \frac{\pi^{n/2}}{(\frac{2}{\alpha^2 m} - s)^{\frac{n}{2}}} \left( \frac{n}{2} \right)^{2m-\frac{n}{2}} \sum_{k=0}^{m-1} \frac{(-1)^{k+j}}{k! j!} \left( 2 - sm\alpha^2 \right)^{k+j}.
\]

By \((13)\) and \((21)\),

\[
\frac{1}{e^{2n\rho}}||K_n||^2 \sim \frac{\alpha^n}{(m-1+n/2)} \left( \frac{m\pi}{2} \right)^{2} \left( \frac{n}{2} \right)^{2m-2} \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} \frac{(-1)^{k+j}}{k! j!} \frac{1}{2^{k+j}},
\]

and hence,

\[
E[e^{s|X_n|^2}] \sim \left( 1 - \frac{s\alpha^2 m}{2} \right)^{m} \left( \frac{\sum_{k=0}^{m-1} \sum_{j=0}^{m-1} \frac{(-1)^{k+j}}{k! j!} \frac{1}{2^{k+j}}}{\sum_{k=0}^{m-1} \sum_{j=0}^{m-1} \frac{(-1)^{k+j}}{k! j!} \frac{1}{2^{k+j}}} \right),
\]

as \( n \to \infty \). Thus,

\[
\Lambda(s) = \lim_{n \to \infty} \frac{1}{n} \log E[e^{s|X_n|^2}] = -\frac{1}{2} \log \left( 1 - \frac{s\alpha^2 m}{2} \right) \text{ if } s < \frac{2}{\alpha^2 m},
\]

and is infinite otherwise. It is clear that \( 0 \in (D(\Lambda))^\circ \), where \( D(\Lambda) = \{ s \in \mathbb{R} : \Lambda(s) < \infty \} \). Thus, the Gärtner-Ellis conditions are satisfied. The rate function for the LDP is computed with the optimization

\[
\Lambda^*(x) = \sup_{\lambda \in \mathbb{R}} [x \lambda - \Lambda(\lambda)] = \sup_{\lambda \in \mathbb{R}} \left[ x \lambda + \frac{1}{2} \log \left( 1 - \frac{\lambda \alpha^2 m}{2} \right) \right].
\]

Then, since

\[
0 = \frac{d}{d\lambda} \left[ x \lambda + \frac{1}{2} \log \left( 1 - \frac{\lambda \alpha^2 m}{2} \right) \right] = x - \frac{\alpha^2 m}{4 - 2\alpha^2 m\lambda} \text{ if and only if } \lambda = \frac{\alpha^2 m}{4} - \frac{1}{2x},
\]

the rate function is

\[
\Lambda^*(x) = x \left( \frac{\alpha^2 m}{4} - \frac{1}{2x} \right) + \frac{1}{2} \log \left( 1 - \frac{\left( \frac{\alpha^2 m}{4} - \frac{1}{2x} \right) \alpha^2 m}{2} \right) = \frac{2x}{\alpha^2 m} - \frac{1}{2} + \frac{1}{2} \log \left( \frac{\alpha^2 m}{4x} \right).
\]

Note that \( \Lambda^*(x) = 0 \) if and only if \( x = m \frac{\alpha^2}{4} \). This implies that \( \lim_{n \to \infty} E\left( \frac{|X_n|^2}{n} \right) = m \frac{\alpha^2}{4} \).

D Proof of Proposition 4.2

Proof. For each \( n \), let \( X_n \) be a random vector in \( \mathbb{R}^n \) with density \( \frac{K_n^2}{||K_n||^2} \). By Lemma \ref{lemma:rate_function} for \( R < \sqrt{m \frac{\alpha^2}{4}} \),

\[
\lim_{n \to \infty} \frac{1}{n} \log P\left( \frac{|X_n|^2}{n} \leq R^2 \right) = \frac{2R^2}{\alpha^2 m} - \frac{1}{2} + \frac{1}{2} \log \left( \frac{\alpha^2 m}{4R^2} \right).
\]
Then by (13), \( E_n[\eta_n(R^n)] \sim \left( \frac{e^{\frac{\nu}{2}\pi^2 n^2 \nu}}{2} \right)^{n/2} \left( \sum \sum \frac{(-1)^{k+i}}{k!i!2^{k+i+j}} \right). \) Thus, by Lemma 3.1

\[
\lim_{n \to \infty} -\frac{1}{n} \log E_n[\eta_n(B_n(\sqrt{n}R))] = \lim_{n \to \infty} -\frac{1}{n} \log E_n[\eta_n(R^n)] + \lim_{n \to \infty} -\frac{1}{n} \log P \left( \frac{|X_n|^2}{n} \leq R^2 \right) = \begin{cases} -\rho - \log \alpha - \frac{1}{2} \log \left( \frac{m^2}{\alpha^2} \right) + \left( \frac{2\pi^2}{m^2} - \frac{1}{2} + \frac{1}{2} \log \left( \frac{2\pi}{R^2} \right) \right), & 0 < R < \sqrt{m^2} \\ -\rho - \log \alpha - \frac{1}{2} \log \left( \frac{m^2}{\alpha^2} \right), & R > \sqrt{m^2} \end{cases}
\]

Thus, by Lemma 3.1,

\[
\lim_{n \to \infty} -\frac{1}{n} \log E_n[\eta_n(B_n(\sqrt{n}R))] = \begin{cases} -\rho - \log \alpha - \frac{1}{2} \log \left( \frac{m^2}{\alpha^2} \right) - \log R, & 0 < R < \sqrt{m^2} \\ -\rho - \log \alpha - \frac{1}{2} \log \left( \frac{m^2}{\alpha^2} \right), & R > \sqrt{m^2} \end{cases}
\]

\[\square\]

### E Proof of Proposition 4.3

Since for all \( n, \tilde{K}_n \in C^2(R^n) \), Parseval’s theorem implies

\[
E[|X_n|^2] = \frac{1}{||\tilde{K}_n||^2} \int_{R^n} (|x|^2\tilde{K}_n(x))(\tilde{K}_n(x))dx = \frac{1}{||\tilde{K}_n||^2} \int_{R^n} \left( -\frac{\Delta \tilde{K}_n(\xi)}{(2\pi)^2} \right) \tilde{K}_n(\xi)d\xi. \tag{22}
\]

To compute the Laplacian of \( \tilde{K} \), we first see that for each \( i \),

\[
\frac{\partial^2}{\partial x_i^2} e^{-|x|^{\nu}} = \frac{\partial}{\partial x_i} \left( -\nu \alpha^\nu x_i |x|^{\nu-2} e^{-|x|^\nu} \right)
= -\nu \alpha^\nu |x|^{\nu-2} e^{-|x|^\nu} - \nu \alpha^\nu x_i \left( \frac{\partial}{\partial x_j} |x|^{\nu-2} \right) e^{-|x|^\nu} + (\nu \alpha^\nu x_i |x|^{\nu-2})^2 e^{-|x|^\nu}
= e^{-|x|^\nu} \left( -\nu \alpha^\nu |x|^{\nu-2} - \nu (\nu - 2) \alpha^\nu x_i^2 |x|^{\nu-4} + \nu^2 \alpha^{2\nu} x_i^2 |x|^{2\nu-4} \right)
= e^{-|x|^\nu} \left( \nu^2 \alpha^{2\nu} |x|^{2\nu-4} - \nu (\nu - 2) \alpha^\nu x_i^2 |x|^{\nu-4} - \nu \alpha^\nu |x|^{\nu-2} \right).
\]

Then,

\[
\Delta e^{-|x|^\nu} = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} e^{-|x|^{\nu}} = \sum_{i=1}^n e^{-|x|^\nu} \left( \nu^2 \alpha^{2\nu} |x|^{2\nu-4} - \nu (\nu - 2) \alpha^\nu x_i^2 |x|^{\nu-4} - \nu \alpha^\nu |x|^{\nu-2} \right)
= e^{-|x|^\nu} \left( |x|^2 (\nu^2 \alpha^{2\nu} |x|^{2\nu-4} - \nu (\nu - 2) \alpha^\nu |x|^{\nu-4} - n \nu \alpha^\nu |x|^{\nu-2} \right)
= e^{-|x|^\nu} \left( \nu^2 \alpha^{2\nu} |x|^{2\nu-4} - \nu (\nu - 2) \alpha^\nu + n \nu \alpha^\nu |x|^{\nu-2} \right).
\]

Thus by (22) and (16),

\[
E[|X_n|^2] = \frac{\Gamma(\frac{n}{\nu} + 1)\alpha_n^{\nu n/2}}{4\pi^{\frac{n}{\nu} / 2} \Gamma(\frac{n}{\nu} + 1)} \int_{R^n} e^{-2|\alpha_n x|^\nu} \left( -\nu \alpha^\nu |x|^{\nu-2} + (\nu (\nu - 2) \alpha^\nu + n \nu \alpha^\nu) |x|^{\nu-2} \right) dx
= \frac{\Gamma(\frac{n}{\nu} + 1)\alpha_n^{\nu n/2 + 2\nu}}{4\pi^{\frac{n}{\nu} / 2} \Gamma(\frac{n}{\nu} + 1)} \left( -\nu \alpha^\nu \int_{R^n} e^{-2|\alpha_n x|^\nu} |x|^{\nu-2} dx + (\nu - 2 + n) \int_{R^n} |x|^{\nu-2} e^{-2|\alpha_n x|^\nu} dx \right).
\]

Then, using (20),

\[
E[|X_n|^2] = n \frac{\alpha_n^{\nu n+2\nu}}{4\pi^{\frac{n}{\nu} / 2} \Gamma(\frac{n}{\nu} + 1)} \left[ -\frac{\nu \alpha^\nu \Gamma(\frac{n+2\nu-2}{\nu})}{\nu \Gamma(\frac{n+2\nu/2-2}{\nu})} + (\nu - 2 + n) \Gamma(\frac{n+2\nu}{\nu}) \right] + \frac{\nu \alpha^\nu \Gamma(\frac{n+2\nu}{\nu})}{\nu \Gamma(\frac{n+2\nu/2-2}{\nu})}
= n \frac{2^\nu |\alpha|^2}{4\pi^{\frac{n}{\nu} / 2} \Gamma(\frac{n}{\nu} + 1)} \left[ \frac{\nu - 2 + n}{\nu} \Gamma\left( \frac{n-2}{\nu} + 1 \right) - \frac{\nu}{4} \Gamma\left( \frac{n-2}{\nu} + 2 \right) \right].
\]
By the asymptotic formula for the Gamma function, as \( n \to \infty \),

\[
E[|X_n|^2] \sim n \alpha_n^2 \frac{2 \nu}{4 \pi^2} \left( \frac{\nu e}{n} \right)^{\frac{\nu}{2}} \left( \frac{2\pi (n-2)}{\nu} \right)^{\frac{(n-2)}{2}} \left[ \frac{n}{4} + \frac{\nu}{4} - \frac{1}{2} \right]
\]

\[
= n \alpha_n^2 \frac{2 \nu}{4 \pi^2} \left( \frac{\sqrt{n-2}}{\sqrt{n}} \left( 1 - \frac{2}{n} \right) \right)^{\frac{\nu}{2}} \left( \frac{n-2}{\nu} \right)^{-\frac{\nu}{2}} \left[ \frac{n}{4} + \frac{\nu}{4} - \frac{1}{2} \right] \sim n^{2-\nu} \alpha_n^2 (2\nu)^{2/\nu} \frac{2}{16\pi^2}.
\]

Since by assumption, \( \alpha_n \sim \alpha n^{\nu - \frac{\nu}{2}} \) for some constant \( \alpha \in (0, \infty) \), \( E[|X_n|^2] = O(n) \), and

\[
\lim_{n \to \infty} \frac{E[|X_n|^2]}{n} = \alpha^2 \frac{2\nu^{2/\nu}}{16\pi^2}.
\]

For the second moment of \( |X_n|^2 \), Parseval’s theorem is applied again and gives that

\[
E[(|X_n|^2)^2] = \int_{R^n} |x|^4 \frac{K_n(x)^2}{||K_n||_2^2} \, dx = \frac{1}{||K_n||_2^2} \int_{R^n} (|x|^2 K_n(x))^2 \, dx = \frac{1}{||K_n||_2^2} \int_{R^n} (\Delta K_n(x))^2 (2\pi)^4 \, d\xi.
\]

(23)

Then, by the above computation of the Laplacian of \( \hat{K} \), (20), and (16),

\[
E[(|X_n|^2)^2] = \frac{\Gamma(n+1)}{(2\pi)^{2n/\nu} \Gamma(n+1)} \int_{R^n} e^{-(2\alpha_n x)^{\nu}} (\nu \alpha_n^{\nu} |x|^{2\nu-2} - (\nu - 2 + n) |x|^{\nu-2})^2 \, dx
\]

\[
= \frac{\Gamma(n+1)}{(2\pi)^{2n/\nu} \Gamma(n+1)} \left[ \nu \alpha_n^{\nu} \Gamma(n+4) - 2(\nu - 2 + n) \Gamma(n+3) + (\nu - 2 + n)^2 \Gamma(n+2) \right]
\]

\[
= \frac{\Gamma(n+1)}{(2\pi)^{2n/\nu} \Gamma(n+1)} \left[ \nu \alpha_n^{\nu} \frac{n+4}{4} - 2(\nu - 2 + n) \Gamma(n+3) + (\nu - 2 + n)^2 \Gamma(n+2) \right]
\]

\[
= n \alpha_n^{2n/\nu} \frac{n+4}{4} \frac{\Gamma(n+4)}{(2\pi)^{2n/\nu} \Gamma(n+1)} \left[ \frac{n}{2} \left( \frac{n+4}{4} \right) \right] = \frac{n^2}{4} \alpha_n^{2n/\nu} \frac{\Gamma(n+4)}{(2\pi)^{2n/\nu} \Gamma(n+1)} \left[ \frac{n}{2} \left( \frac{n-4}{\nu} \right) \right]
\]

\[
\sim n^4 \alpha_n^{2n/\nu} \frac{\Gamma(n+4)}{(2\pi)^{2n/\nu} \Gamma(n+1)} \left[ \frac{n}{2} \left( \frac{n-4}{\nu} \right) \right] \sim n^4 \alpha_n^{2n/\nu} \frac{\Gamma(n+4)}{(2\pi)^{2n/\nu} \Gamma(n+1)} \left[ \frac{n}{2} \left( \frac{n-4}{\nu} \right) \right] \sim n^4 \alpha_n^{2n/\nu} \frac{\Gamma(n+4)}{(2\pi)^{2n/\nu} \Gamma(n+1)} \left[ \frac{n}{2} \left( \frac{n-4}{\nu} \right) \right] \sim n^4 \alpha_n^{2n/\nu} \frac{\Gamma(n+4)}{(2\pi)^{2n/\nu} \Gamma(n+1)} \left[ \frac{n}{2} \left( \frac{n-4}{\nu} \right) \right]
\]

(24)

Again, since \( \alpha_n \sim \alpha n^{\nu - \frac{\nu}{2}} \), \( E[(|X_n|^2)^2] = O(n^2) \), and

\[
\lim_{n \to \infty} \frac{E[(|X_n|^2)^2]}{n^2} = \alpha^4 \frac{(2\nu)^{4/\nu}}{16(2\pi)^4}.
\]

Note that this limit is exactly the square of the limit of the expectation of \( \frac{|X_n|^2}{n^2} \). Thus,

\[
\text{Var} \left( \frac{|X_n|^2}{n^2} \right) = \frac{E[(|X_n|^2)^2]}{n^2} - \left( \frac{E[|X_n|^2]}{n} \right)^2 \to 0 \text{ as } n \to \infty.
\]

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F  Proof of Proposition 4.5

First, for $k \geq 0$, we see that

\[
\int_{\mathbb{R}^n} |x|^k K(x)^2 \, dx = \int_{\mathbb{R}^n} |x|^k \left( e^{\pi \rho (\sigma + n)/2} \frac{(\frac{\sigma + n + 2}{2})}{(\frac{\sigma + n}{2})} \frac{J_{(\sigma + n)/2}(2|x/\alpha|\sqrt{\sigma + n}/2)}{(2|x/\alpha|\sqrt{\sigma + n}/2)^{\sigma + n/2}} \right)^2 \, dx \\
= e^{2\pi \rho (\sigma + n)} \frac{\Gamma\left(\frac{\sigma + n + 2}{2}\right)}{\Gamma\left(\frac{\sigma + n}{2}\right)} \frac{\alpha^{\sigma + n}}{2} \int_{\mathbb{R}^n} |x|^k \frac{J_{(\sigma + n)/2}(2|x/\alpha|\sqrt{\sigma + n}/2)^2}{(2|x/\alpha|\sqrt{\sigma + n}/2)^{(\sigma + n)/2}} \, dx \\
= e^{2\pi \rho (\sigma + n)} \frac{\Gamma\left(\frac{\sigma + n + 2}{2}\right)}{\Gamma\left(\frac{\sigma + n}{2}\right)} \frac{\alpha^{\sigma + n}}{2} \int_{\mathbb{R}^n} |x|^k \frac{J_{(\sigma + n)/2}(2|x/\alpha|\sqrt{\sigma + n}/2)^2}{(2|x/\alpha|\sqrt{\sigma + n}/2)^{(\sigma + n)/2}} \, dx.
\]

By the change of variables $y = \left(\frac{2}{\alpha} \sqrt{\frac{\sigma + n}{2}}\right) r$,

\[
e^{2\pi \rho (\sigma + n)} \frac{2\pi^{n/2} \Gamma\left(\frac{\sigma + n + 2}{2}\right)}{\Gamma\left(\frac{\sigma + n}{2}\right)} \frac{\alpha^{\sigma + n}}{2} \int_{0}^{\infty} \frac{2^{\sigma + k - 1}}{y^{\sigma + 1 - k}} \frac{J_{(\sigma + n)/2}(y)^2}{y^{\sigma + 1 - k}} \, dy.
\]

For $\sigma + 1 - k > 0$, from [19, p. 10.22.57],

\[
\int_{0}^{\infty} \frac{J_{(\sigma + n)/2}(y)^2}{y^{\sigma + 1 - k}} \, dy = \frac{\left(\frac{\sigma + 1 - k}{2}\right)^{-k-1}}{2^{\sigma - k + 1}} \frac{\Gamma\left(\frac{\sigma + n + 2}{2}\right)}{\Gamma\left(\frac{\sigma + n}{2}\right)} \frac{\Gamma\left(\begin{array}{c} \frac{n}{2} + \frac{k}{2} \\ \frac{n}{2} + \frac{k}{2} \end{array}\right)}{\Gamma\left(\frac{n}{2}\right)}
\]

and thus,

\[
\int_{\mathbb{R}^n} |x|^k K(x)^2 \, dx = e^{2\pi \rho (\sigma + n)} \frac{2\pi^{n/2} \Gamma\left(\frac{\sigma + n + 2}{2}\right)}{\Gamma\left(\frac{\sigma + n}{2}\right)} \frac{\alpha^{\sigma + n}}{2} \frac{\Gamma\left(\frac{n}{2} + \frac{k}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \frac{\Gamma\left(\sigma + 1 - k\right)}{\Gamma\left(\sigma + k + n\right)} \frac{\Gamma\left(\sigma + n\right)}{\Gamma\left(\sigma + n + 1\right)} \frac{\Gamma\left(\sigma + 1\right)}{\Gamma\left(\sigma + 1 + k\right)}
\]

Then, for each $k > 0$ and $\sigma > k - 1$,

\[
E[|X_n|^k] = \frac{1}{||X_n||^2} \int_{\mathbb{R}^n} |x|^k K_n(x)^2 \, dx
\]

\[
= \frac{(2\pi)^{\frac{k}{2}} \alpha^{\sigma + n + 2/2} \Gamma\left(\frac{\sigma + n + 2}{2}\right)}{(\sigma + n)^{\frac{k}{2}} \Gamma\left(\frac{n}{2}\right)} \frac{\Gamma\left(\sigma + 1 - k\right)}{\Gamma\left(\sigma + k + n\right)} \frac{\Gamma\left(\sigma + n\right)}{\Gamma\left(\sigma + n + 1\right)} \frac{\Gamma\left(\sigma + 1\right)}{\Gamma\left(\sigma + 1 + k\right)}
\]

\[
\approx \frac{\alpha^{k/2} \Gamma\left(\frac{n}{2}\right)^{\frac{k}{2}} \Gamma\left(\sigma + 1 - k\right) \Gamma\left(\frac{\sigma + 1}{2}\right)^2 \Gamma\left(\frac{\sigma + n}{2}\right)^2 \Gamma\left(\frac{\sigma + n + 2}{2}\right)}{(\sigma + n)^{k/2} \Gamma\left(\frac{n}{2}\right)^{\frac{k}{2}} \Gamma\left(\sigma + 1 - k\right) \Gamma\left(\frac{\sigma + 1}{2}\right)^2 \Gamma\left(\frac{\sigma + n}{2}\right)^2 \Gamma\left(\frac{\sigma + n + 2}{2}\right)}
\]

\[
= \frac{\alpha^{k/2} \Gamma\left(\frac{\sigma + 1}{2}\right)^2 \Gamma\left(\frac{\sigma + n}{2}\right)^2 \Gamma\left(\frac{\sigma + n + 2}{2}\right)}{(\sigma + n)^{k/2} \Gamma\left(\frac{n}{2}\right)^{\frac{k}{2}} \Gamma\left(\sigma + 1 - k\right) \Gamma\left(\frac{\sigma + 1}{2}\right)^2 \Gamma\left(\frac{\sigma + n}{2}\right)^2 \Gamma\left(\frac{\sigma + n + 2}{2}\right)}
\]

\[
\approx \frac{\alpha^{2k/2} \Gamma\left(\sigma + 1 - k\right) \Gamma\left(\frac{\sigma + 1}{2}\right)^2 \Gamma\left(\frac{\sigma + n}{2}\right)^2 \Gamma\left(\frac{\sigma + n + 2}{2}\right)}{(\sigma + n)^{k/2} \Gamma\left(\frac{n}{2}\right)^{\frac{k}{2}} \Gamma\left(\sigma + 1 - k\right) \Gamma\left(\frac{\sigma + 1}{2}\right)^2 \Gamma\left(\frac{\sigma + n}{2}\right)^2 \Gamma\left(\frac{\sigma + n + 2}{2}\right)}
\]

\[
\approx n^{k/2} \frac{\alpha^{2k/2} \Gamma\left(\sigma + 1 - k\right) \Gamma\left(\frac{\sigma + 1}{2}\right)^2 \Gamma\left(\frac{\sigma + n}{2}\right)^2 \Gamma\left(\frac{\sigma + n + 2}{2}\right)}{(\sigma + n)^{k/2} \Gamma\left(\frac{n}{2}\right)^{\frac{k}{2}} \Gamma\left(\sigma + 1 - k\right) \Gamma\left(\frac{\sigma + 1}{2}\right)^2 \Gamma\left(\frac{\sigma + n}{2}\right)^2 \Gamma\left(\frac{\sigma + n + 2}{2}\right)}
\]

\[
\approx n^{k/2} \frac{\alpha^{2k/2} \Gamma\left(\sigma + 1 - k\right) \Gamma\left(\frac{\sigma + 1}{2}\right)^2 \Gamma\left(\frac{\sigma + n}{2}\right)^2 \Gamma\left(\frac{\sigma + n + 2}{2}\right)}{(\sigma + n)^{k/2} \Gamma\left(\frac{n}{2}\right)^{\frac{k}{2}} \Gamma\left(\sigma + 1 - k\right) \Gamma\left(\frac{\sigma + 1}{2}\right)^2 \Gamma\left(\frac{\sigma + n}{2}\right)^2 \Gamma\left(\frac{\sigma + n + 2}{2}\right)}
\]
Then, for $\sigma > 1$,
\[
\lim_{n \to \infty} \frac{\mathbb{E}(|X_n|^2)}{n} = \frac{\alpha^2 \Gamma(\sigma - 1) \Gamma \left( \frac{\sigma}{2} + 1 \right)^2}{\Gamma \left( \frac{\sigma}{2} \right)^2 \Gamma(\sigma + 1)}.
\]
and for $\sigma > 3$,
\[
\lim_{n \to \infty} \frac{\mathbb{E}(|X_n|^4)}{n^2} = \frac{\alpha^4 \Gamma \left( \frac{\sigma}{2} + 1 \right)^2 \Gamma(\sigma - 3)}{2 \Gamma \left( \frac{\sigma}{2} - 1 \right)^2 \Gamma(\sigma + 1)}.
\]
Thus, if $\sigma > 3$,
\[
\lim_{n \to \infty} \frac{\text{Var}(|X_n|^2)}{n^2} = \frac{\alpha^4 \Gamma \left( \frac{\sigma}{2} + 1 \right)^2 \Gamma(\sigma - 3)}{4 \Gamma \left( \frac{\sigma}{2} - 1 \right)^2 \Gamma(\sigma + 1)} - \frac{\alpha^4 \Gamma(\sigma - 1)^2 \Gamma \left( \frac{\sigma}{2} + 1 \right)^4}{4 \Gamma \left( \frac{\sigma}{2} \right)^4 \Gamma(\sigma + 1)^2} > 0.
\]
Let $\beta > \frac{1}{2}$. By Chebychev’s inequality,
\[
\frac{\mathbb{E}_n[\eta_n(B_n(R^n)^\beta)]}{\mathbb{E}_n[\eta_n(R^n)]]} = P_n \left( |X_n|^2 \geq R^2 n^{2\beta} \right) = P_n \left( |X_n|^2 - \mathbb{E}|X_n|^2 \geq R^2 n^{2\beta} - \mathbb{E}|X_n|^2 \right) 
\leq \frac{\text{Var}(|X_n|^2)}{n^2 (R^2 n^{2\beta} - \mathbb{E}|X_n|^2 n^{2\beta - 1})} \to 0,
\]
since $2\beta - 1 > 0$.

### G Proof of Proposition 4.7

First, recall the beta function satisfies
\[
B(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1}dt = \int_0^\infty t^{x-1}(1+t)^{-x-y}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.
\]
Then, for any $k \geq 0$,
\[
\int_{\mathbb{R}^n} |x|^k K_n(x)^2 dx = \int_{\mathbb{R}^n} |x|^k \left( \frac{e^{2\pi \rho}}{(1 + \|x\|^2)^{2\nu + n}} \right) dx = e^{2\pi \rho} \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \int_0^\infty r^{n-1+k} \left( 1 + \frac{r^2}{\alpha_n^2} \right)^{-2\nu-n} dr
\]
\[
= e^{2\pi \rho} \frac{\pi^{n/2}}{\Gamma(\frac{n}{2})} \alpha_n^{n+k} \int_0^\infty t^{\frac{n-1}{2} + \frac{k}{2}} (1+t)^{-(2\nu+n)} dt
\]
\[
= e^{2\pi \rho} \frac{\pi^{n/2}}{\Gamma(\frac{n}{2})} \alpha_n^{n+k} B \left( \frac{n}{2} + \frac{k}{2}, 2\nu + \frac{n}{2} - \frac{k}{2} \right),
\]
Thus, the expectation of $|X_n|^2$ is
\[
\mathbb{E}(|X_n|^2) = \frac{1}{\|K_n\|^2} \int_{\mathbb{R}^n} |x|^2 K_n(x)^2 dx = \alpha_n^2 \frac{B(\frac{n}{2} + 1, 2\nu + \frac{n}{2} - 1)}{B(\frac{n}{2}, 2\nu + \frac{n}{2})} = \frac{\alpha_n^2 \Gamma(\frac{n}{2} + 1) \Gamma(2\nu + \frac{n}{2} - 1) \Gamma(n + 2\nu)}{\Gamma(n + 2\nu) \Gamma(\frac{n}{2}) \Gamma(2\nu + \frac{n}{2})} = \frac{n}{2} \frac{n}{2} \frac{2(\frac{n}{2} + 2\nu - 1)}{\frac{n}{2} + 2\nu - 1} = \frac{n}{2} \frac{n}{2} \frac{n}{2} \frac{n}{2} \frac{n}{2} - n - n - 2
\]
and
\[
\mathbb{E}(|X_n|^4) = \alpha_n^4 \frac{B(\frac{n}{2} + 2, 2\nu + \frac{n}{2} - 2)}{B(\frac{n}{2}, 2\nu + \frac{n}{2})} = \alpha_n^4 \frac{\Gamma(\frac{n}{2} + 2) \Gamma(2\nu + \frac{n}{2} - 2) \Gamma(n + 2\nu)}{\Gamma(n + 2\nu) \Gamma(\frac{n}{2}) \Gamma(2\nu + \frac{n}{2})} = \frac{(\frac{n}{2} + 1) \frac{n}{2}}{\frac{n}{2} + 2\nu - 2)(2\nu + \frac{n}{2} - 1)} = \frac{n}{2} \frac{n}{2} \frac{n}{2} \frac{n}{2} \frac{n}{2} \frac{n}{2} - n - n - 2
\]
Thus, by the assumption that $\alpha_n \sim \alpha n^{-\frac{1}{2}}$ as $n \to \infty$ for some $\alpha > 0$,
\[
\lim_{n \to \infty} \frac{\mathbb{E}(|X_n|^2)}{n} = \alpha^2 \text{ and } \lim_{n \to \infty} \frac{\text{Var}(|X_n|^2)}{n^2} = 0.
\]
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