Numerical analysis of forced vibrations of the Golden Gate suspension bridge in the case of the 1:1 internal resonance

A L Katembo, V V Kandu and M V Shitikova

Research Center on Dynamics of Solids and Structures, Voronezh State Technical University, Voronezh 394006, Russia

katembo2020@gmail.com

Abstract: Nonlinear force driven coupled vertical and torsional vibrations of suspension bridges, when the frequency of an external force is approaching one of the natural frequencies of the suspension system, which, in its turn, undergoes the conditions of the one-to-one internal resonance, are investigated. The generalized method of multiple time-scales is used as the method of solution. The damping characteristics are described by the rheological model involving a fractional derivative, which is interpreted as a fractional power of the differentiation operator. The main subject of this research is the numerical analysis of the influence of the fractional damping on the dynamic behavior of a suspension combined system.

1. Introduction
The suspension bridges are unique building structures, as they allow one not only to cover large spans, but also are economically viable. Compared to other types of bridges, suspension bridges have a number of technical and aesthetic advantages, that is why they are so widely used in the modern world. The history of suspension bridges met with the largest catastrophe in bridge construction - the collapse of the bridge over the Tacoma River (USA) in 1940 (Tacoma Bridge). In flexible suspension bridges under the action of various dynamic loads, such as moving load or wind, strong bending-torsional vibrations could occur, sometimes resulting in extremely large amplitudes complicating the normal operation of the bridge, and sometimes causing its destruction. Due to the low damping ability of the suspension bridges, the oscillations could be accompanied by the transfer of energy between different modes of vibrations for a long time even after unloading, which was the cause of their occurrence. This is explained by the phenomenon of internal resonance, when one of the frequencies of free bending vibrations is close in value to one of the natural frequencies of torsional vibrations, which in practice can occur quite often due to the density of the spectrum of the natural frequencies of suspension bridges, which largely depend on the geometric parameters of the bridge.

To analyze the phenomena of the internal resonance during dynamic response of suspension bridges, different mathematical models have been utilized. Thus, the continuous model proposed in [1] has been used in [2-6] to solve the system of nonlinear differential equations describing the dynamics of suspension bridges under one-to-one [2-6] and two-to-one [3-5] internal resonances by means of the multiple time scales perturbation technique [7]. The state-of-the-art survey of the internal phenomena
in suspension bridges was made by Shitikova and Rossikhin [8] in their plenary lecture at the 5th European Conference of Civil Engineering held in Florence, Italy in 2014. During this report, the authors passed aloud their opinion that the reason of failure of the Tacoma Narrows Bridge was connected with the internal resonance between vertical and torsional vibrations.

This idea was repeated a year later, in 2015, by Arioli and Gazzola [9], who trying to explain why did torsional oscillations suddenly appears before the Tacoma Narrows collapse found out that vertical oscillations had become large enough and switched to torsional ones. The four-degree-of-freedom model accounting for both the flexural-torsional motion of the bridge deck and for the transversal motion of a pair of hangers has been considered in [10], and the internal resonance between the modes of deck and hangers vibrations has been studied. Stability of dynamic response of suspension bridges with due account for the phenomenon of the internal resonance has been considered in [11].

Thus, the potential occurrence of internal resonance phenomena has been identified as the potential cause of critical dynamic states in long-span suspension bridges. Therefore, the task of studying the internal resonance in suspension bridges is very relevant and important.

The first field observations of the vibrations of the Golden Gate suspension bridge were made in the period from 1933 to 1942, when seismological instruments were installed on the piers, towers and cables to measure any vibration that might occur [12]. After the failure of the Tacoma Narrows Bridge in 1940, it was decided to install ten instruments for measuring the vertical movement of the bridge, which worked continuously until 1954. Vincent [13-15] analyzed these recordings of observations of the Golden Gate bridge vibrations, and the field observations of this bridge were further continued to [16-19]. Thus, the experimental data obtained in [19] showed that different vibrational modes feature different amplitude damping coefficients, and the order of smallness of these coefficients tells about low damping capacity of suspension combined systems, resulting in prolonged energy transfer from one partial subsystem to another. However, the analytical model described in [2] with its further extension in [3,4] allows one to analyze only free undamped vibrations of suspension bridges.

Nonlinear free damped vibrations of suspension bridges in the cases of the one-to-one internal resonance, when the natural frequency of a certain mode of vertical vibrations is close to the natural frequency of a certain mode of torsional vibrations, and the two-to-one internal resonance, when one natural frequency is nearly twice as large as another natural frequency, have been examined in [5] when damping features of the system are prescribed by the first derivative of the displacement with respect to time. It has been shown that for the both types of the internal resonance the damping coefficient does not depend on the natural frequency of vibrations, but this result is in conflict with the experimental data presented in [19] and [20].

To lead the theoretical investigations in line with the experiment, fractional derivatives were introduced in [21] for describing the processes of internal friction proceeding in suspension combined systems at nonlinear free vibrations. The nonlinear suspension bridge model put forward allows one to obtain the damping coefficient dependent on the natural frequency of vibrations. This model was further generalized in [22] by using two different fractional parameters for analyzing vertical and torsional vibrations of suspension bridges.

The model proposed in [21] for the analysis of free damped vibrations was generalized by the authors in [23] to the case of nonlinear forced vibrations of suspension bridges, when the frequency of the external force is close to one of the natural frequencies of the vertical vibrations of the suspension combined system, which undergoes conditions of internal resonance one-to-one. In this paper, numerical studies of the equations obtained in [23] are performed.

2. Problem formulation
Let us consider forced vibrations of a suspension bridge (Fig. 1) following [23], assuming that only two modes predominate in the vibrational process, namely: the vertical $n$-th mode with linear natural
frequency $\omega_{0n}$ and the torsional $m$-th mode with natural frequency $\Omega_{0m}$, then functions $\eta(z,t)$ and $\varphi(z,t)$ could be approximately defined as

$$\eta(z,t) - \nu_n(z)x_{1n}(t), \quad \varphi(z,t) - \Theta_m(z)x_{2m}(t),$$

(1)

where $x_{1n}$ and $x_{2m}$ are the generalized displacements, and $\nu_n(z)$ and $\Theta_m(z)$ are the natural modes of the two interacting modes of vibrations.

The resolving set of equations describing forced vibrations is written in a dimensionless form as [23]

$$\ddot{x}_{1n} + \alpha_{0n}^2 x_{1n} + \beta D_x^\gamma x_1 + a_{11} x_{1n} + a_{22} x_{2m} + (b_{11}^n x_{1n} + b_{22}^m x_{2m}) x_{1n} = \tilde{F} \cos(\omega_r t),$$

$$\ddot{x}_{2m} + \Omega_{0m}^2 x_{2m} + \beta D_x^\gamma x_2 + a_{12} x_{1n} x_{2m} + (c_{11}^m x_{1m} + c_{22}^m x_{2m}) x_{2m} = 0,$$

(2)

where the coefficients $a_{ij}, b_{ij}$ and $c_{ij}$ ($i=1,2; j=2$) depending on the coupled natural modes are defined in [2,5] (subsequently the indices $n$ and $m$ will be omitted for ease of presentation), $\tilde{F} = \text{const}$ and $\omega_r$ are, respectively, the amplitude and frequency of the external harmonic force, the terms $\beta D_x^\gamma x_1$ and $\beta D_x^\gamma x_2$ describe inelastic reaction of the system, $\beta$ is the viscosity coefficient, $D_x^\gamma x$ ($\gamma_1 \leq \gamma \leq \gamma_2$) is the fractional derivative which is interpreted as the fractional power of the differentiation operator [21] with the fractional parameter $\gamma$ (order of the fractional derivative).

$$D_x^\gamma x = \left( \frac{d}{dt} \right)^\gamma (0 < \gamma \leq 1).$$

(3)

It should be noted that the set of equations (2) in the case of free damped vibrations, i.e. when $F = 0$, for the first time was proposed in Rossikhin and Shitikova [21] utilizing the method of different time scales [7] by introducing the expansion of the fractional derivative in terms of multiple time scales. The history of the development of the generalized method of time scales was recently presented in [24].

Thus, utilizing this method, the system of equations for amplitudes $a_i$ and phases $\varphi_i$ ($i=1, 2$) of nonlinear vibrations in the case of the one-to-one internal resonance, i.e. when $\omega_b = \Omega_0 + \varepsilon^2 \sigma$, which is accompanied by the external resonance, i.e. when $\omega_r = \omega_b + \varepsilon^2 \sigma$, has the following form [23]:

\[\text{Figure 1. Scheme of a suspension bridge}\]
leads to the interaction of two vibration modes. In the present paper, the influence of the parameters of the fractional derivative viscoelastic model on forced vibrations of suspension bridges has been analyzed, when the motion of the suspension bridge is described by two nonlinear differential equations in the presence of internal resonance, which leads to the interaction of two vibration modes. The analysis has shown that dimensionless amplitudes of vertical vibrations are very sensitive to the action of the force.

The set of equations (4) describes the amplitude and phase modulation of forced nonlinear vibrations and could be investigated numerically.

3. Numerical analysis

For numerical studies of the influence of the parameters of the fractional derivative viscoelastic model on forced vibrations of suspension bridges, the fourth-order Runge-Kutta method was used in the «GNU Octave» system for numerical mathematics, for different values of the fractional parameter in the case of the internal resonance and for force.

Envelopes of the amplitudes of nonlinear vibrations of the Golden Gate Bridge in the case of internal resonance $\omega_0 = \Omega_0 = 2.16$ are depicted in Figure 2(a) for free vibrations and in Figure 2(b) for forced vibrations at $F=1$ at different magnitudes of the fractional parameter $\gamma_1 = \gamma_2 = \gamma = 0, 0.15, 0.5$. Reference to Figure 2 shows that the increase in the fractional parameter results in a significant decrease in dimensionless amplitudes of nonlinear oscillations. The energy exchange between the interacting modes takes place both in the case of undamped ($\gamma = 0$) and damped ($0 < \gamma \leq 1$) vibrations, and the action of the external force does not affect this phenomenon.

Dimensionless displacements of the considered bridge for forced vibrations are shown in Figure 3 for different levels of the external force magnitudes. From Figure 3 it is evident that the displacement $x_1$ is more susceptible to a higher vertical force $F \cos (\omega_j t)$ than $x_2$. This is due to the fact that $x_1$ and $x_2$ are responsible for vertical and torsional vibrations, respectively, whence it follows that the $x_2$-displacement is weakly sensitive to the increase in force $F$.

Figure 4 allows one to trace the influence of the level of the external force magnitude on the dimensionless amplitudes of vertical $a_1$ and torsional $a_2$ vibrations. From Figure 4 it could be seen that the magnitudes of the amplitudes of vertical vibrations are very sensitive to the action of the force.

4. Conclusion

\[ \dot{a}_1 + \frac{1}{2} \mu \omega_0^{\gamma_1} \sin \left( \frac{1}{2} \pi \gamma_1 \right) a_1 - \frac{1}{4} \Gamma_1 a_1^2 \sin \delta + \frac{1}{4} F \omega_0^{\gamma_1} \sin \phi_1 = 0, \]  
\[ \dot{a}_2 + \frac{1}{2} \mu \Omega_0^{\gamma_2} \sin \left( \frac{1}{2} \pi \gamma_2 \right) a_2 + \frac{1}{4} \Gamma_2 a_2^2 \sin \delta = 0, \]  
\[ \dot{\phi}_1 - \frac{1}{2} \mu \omega_0^{\gamma_1} \cos \left( \frac{1}{2} \pi \gamma_1 \right) - \sigma_1 - \lambda_1 a_1^2 - \lambda_2 a_2^2 + \frac{1}{4} \Gamma_1 a_1^2 \cos \delta + \frac{1}{4} F \omega_0^{\gamma_1} a_1^2 \cos \phi_1 = 0, \]  
\[ \dot{\phi}_2 - \frac{1}{2} \mu \Omega_0^{\gamma_2} \cos \left( \frac{1}{2} \pi \gamma_2 \right) - (\sigma_1 - \sigma) - \lambda_1 a_1^2 - \lambda_2 a_2^2 + \frac{1}{4} \Gamma_2 a_2^2 \cos \delta = 0, \]  
where a dot denotes differentiation with respect to $T_2$, $\delta = 2(\phi_2 - \phi_1)$ is the phase difference, $\sigma$ and $\sigma_1$ are detuning parameters, and the coefficients $\lambda_i$, $\Gamma_j$ and $c_{ij}$ ($i=1,2,3,4$; $j=1,2$) depending on the coupled natural modes are defined in [2,5].

In the present paper, the influence of the parameters of the fractional derivative viscoelastic model on the forced vibrations of suspension bridges has been analyzed, when the motion of the suspension bridge is described by two nonlinear differential equations in the presence of internal resonance, which leads to the interaction of two vibration modes. The analysis has shown that dimensionless amplitudes...
decrease with the increase in the fractional parameter \( \gamma \), and the vertical amplitude and hence vertical displacement are much more susceptible to the higher vertical external force than torsional amplitude.

Figure 2. Dimensionless amplitude vs. dimensionless time as the numerical solution of equations (4a, 4b). Free vibrations (a), forced vibrations (b) at \( F = 1 \), with the initial amplitude \( a_{10} = 0.3 \), blue line – \( a_1 \), orange line – \( a_2 \).
Figure 3. The time-dependence of the generalized displacements at different levels of external force magnitude for $\gamma_1 = \gamma_2 = 0$ and $\omega_0 = \Omega_n = 2.16$
Figure 4. Dependence of the dimensionless amplitudes $a_1$ (blue) and $a_2$ (orange) on the dimensionless time $T_2$ at different levels of the external force amplitude.

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