For each $t > 0$, up to the number $n = N(t)$, the exact estimations of all initial Taylor coefficients in the class $B_t$ were found, where $B_t$ is a set of holomorphic in unit disk functions $f$, $0 < |f| < 1$, $f(0) = e^{-t}$.

1. Let $\Delta := \{z \in \mathbb{C} : |z| < 1\}$, where $\mathbb{C}$ — a field of all complex numbers.

Class $B$ consists of holomorphic in $\Delta$ functions $w = f(z)$, not vanishing and such that $|f(z)| \leq 1$, $z \in \Delta$.

The Krzyz conjecture [1] consists in that for every $f \in B$ and all natural $n$ the Taylor coefficients $\{f\}_n$ of function $f$ satisfy to inequality $|\{f\}_n| \leq 2/e$. The extremals can be only rotations of the function $F^*(z^n, 1)$ in the planes $z$ and $w$, where

$$F^*(z, t) := e^{-tH(z)}, \quad H(z) := \frac{1 + z}{1 - z}, \quad t > 0. \quad (1)$$

The existence of extremals of this problem is obvious, since after addition of function $f(z) \equiv 0$ to the class $B$ it becomes a family of functions, compact in the topology of the local uniform convergence. However, extremal problems on the class $B$ are very complicated and, at present, the hypothesis remains unconfirmed.

Since the class $B$ is invariant under rotations in the planes of the variables $z$ and $w$, than we can restrict ourselves to studying only of functions, for which $0 < \{f\}_0 < 1$. Further, we can fix the parameter $t \in (0, +\infty)$ and set $\{f\}_0 = e^{-t}$; these subclasses we denote by $B_t$.

From geometrical considerations it is clear that every function of class $B_t$ can be represented in the form

$$f(z) = F^*(\omega(z), t), \quad \omega \in \Omega, \quad (2)$$

where class $\Omega$ consists of holomorphic in unit disk $\Delta$ functions $\omega(z)$ such, that $\omega(0) = 0$ and $|\omega(z)| < 1$, when $z \in \Delta$.

Note that for every $t > 0$ the formula (2) establishes a one-to-one correspondence between the classes $B_t$ and $\Omega$ (see [2]).

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If $t \in (0, 2]$ then the Krzyz conjecture can be refined: if $f \in B_t$ then, probably, the accurate estimates $|\{f\}_n| \leq |\{F^*\}_1(t)| = 2t/e^t$ are correct; the equality is delivered only by functions $F^*(e^{i\varphi}z^n, t)$, $n \in \mathbb{N}$, $\varphi \in \mathbb{R}$.

2. Let us dwell on representations of the form (2). And let the functions $g(z)$ and $G(z)$ be holomorphic in $\Delta$. Function $g(z)$ is called subordinate in the disk $\Delta$ for the function $G(z)$, if it can be represented in $\Delta$ in the form

$$g(z) = G(\omega(z)) = \sum_{n=0}^{\infty} \{G\}_n \omega(z)^n.$$

Then

$$g(z) = G(\omega(z)) = \sum_{n=0}^{\infty} \{G\}_n \omega(z)^n = \{G\}_0 + \sum_{n=1}^{\infty} \left( \sum_{j=1}^{n} \{G\}_j \{\omega^j\}_n \right) z^n.$$

Whence it appears $\{g\}_0 = \{G\}_0$ and

$$\{g\}_n(\omega) = \sum_{j=1}^{n} \{G\}_j \{\omega^j\}_n, \quad n \in \mathbb{N}, \quad \omega \in \Omega. \quad (3)$$

3. The class of all functions $f(z)$, regular and univalent in $\Delta$, with the normalization $f(0) = 0$, $f'(0) = 1$, mapping the disk $\Delta$ on convex domain, is denoted by $S^0$.

The set of all functions $h(z)$, with a positive in $\Delta$ real part and with the normalization $h(0) = 1$, mapping the unit disk $\Delta$ into right half-plane, is called the Caratheodory class and is denoted by $C$.

Between $C$ and $S^0$ there is the bijection $[2]$: 

$$h(z) = 1 + z \frac{f''(z)}{f'(z)}, \quad h \in C, \quad f \in S^0. \quad (4)$$

The following simple but important statement is hold $[2]$:

**Lemma 1** If the function $s(z) = \sum_{n=1}^{\infty} \{s\}_n z^n$, regular in $\Delta$, is subordinated to the function $S(z) \in S^0$, then the sharp estimates

$$|\{s\}_n| \leq |\{S\}_1| = 1, \quad n \in \mathbb{N},$$

are valid. Equality is achieved only on the functions $S(e^{i\varphi}z^n)$, $\varphi \in \mathbb{R}$, $n \in \mathbb{N}$.

Carathéodory and Töplitz $[3, 4]$ have completely solved the problem about the possibility of extension of a polynomial up to a function of the class $C$. Later Shur gave a constructive proof of this theorem $[5]$. 

2
Criterion 1 (Carathéodory, Töplitz) Let $\{h\}_1, \ldots, \{h\}_n, n \in \mathbb{N}$, are fixed complex numbers. Polynomial

$$R(z, n) := 1 + \sum_{k=1}^{n} \{h\}_k z^k$$

can be extended up to the function $h(z) = R(z, n) + O(z^{n+1}) \in C$ if and only if determinants

$$M_k := \det \{a_{ij}\}_{i,j=0}^k, \quad 1 \leq k \leq n,$$

$$a_{ii} = 2, \quad a_{ij} = \{h\}_{j-i}, \quad j > i, \quad a_{ij} = \overline{a_{ji}}, \quad j < i,$$

either all positive or positive to a certain number, from which are equal to zero. In the latter case, the extension is unique.

4. For further progress we need to study the Taylor coefficients of our majorant function $F^*(z, t)$ of class $B_t$. Everywhere below, we will not be interested in the zero coefficient of this function, since it is not included in the formula (3). The first coefficient $\{F^*\}_1(t)$ of function $F^*(z, t)$ is equal to $-2t/e^t$. We normalize the function $F^*(z, t)$ so, that the first coefficient in its Taylor expansion becomes equal to 1. Let us introduce the notation

$$F(z, t) := \frac{F^*(z, t)}{\{F^*\}_1(t)}. \quad (5)$$

The question arises: whether convex univalent functions $f \in S^0$ exist there, with some initial Taylor coefficients that match with all first coefficients of the function $F(z, t)$, except for the coefficient $\{F\}_0(t)$?

It is not hard to check. By substituting $f(z) = F(z, t)$ to the formula (4), we obtain

$$h(z) = 1 + 2z \left( \frac{1}{1 - z} - \frac{1}{(1 - z)^2} t \right).$$

From which we elementary derive remarkably simple formula

$$\{h\}_j = 2(1 -jt), \quad j \in \mathbb{N}. \quad (6)$$

We use the Carathéodory-Töeplitz extension criterion of polynomials up to a function of class $C$. Let us compute the principal minors $M_{j-1}$, for all $j \in \mathbb{N}$. Here the index $j - 1$ means, that the dimension of the corresponding to the minor $M_{j-1}$ matrix is equal to $j$. According to lemma 2, the formulation
and the proof of which can be found at the end of this work (see section 6), we have:

\[ M_{j-1} = 2^{2(j-1)}t^{j-1}(2 - (j - 1)t), \quad j \in \mathbb{N}. \] (7)

Further, it is obvious that the minors \( M_1, \ldots, M_{n-1} \) are not negative if and only if \( t \leq 2/(n - 1) \), for \( n > 1 \), and \( t > 2 \), for \( n = 1 \), or \( n \leq 2/t + 1 \). Note also, that if \( t = 2/(n - 1) \), \( n \in \mathbb{N} \setminus \{1\} \), the extension is unique.

Thus, we have

**Theorem 1** For every \( t > 0 \) and \( n \leq 2/t + 1 \), \( n \in \mathbb{N} \), the segment of Taylor expansion of the function \( F(z, t) \), first introduced in the formula (5),

\[- \text{polynom } P(z, t, n) := z + \sum_{k=2}^{n} \{F\}_k(t) z^k \quad \text{can be extended to the function } \]

\[ f(z) = P(z, t, n) + O(z^{n+1}) \in S^0. \] For \( t = 2/(n - 1) \), \( n \in \mathbb{N} \setminus \{1\} \), the extension is unique.

5. From lemma 1, theorem 1, formula (3) and normalization (5) follows a central for this work

**Theorem 2** For every \( t > 0 \), arbitrary \( n \leq 2/t + 1 \), \( n \in \mathbb{N} \), and each \( f^* \in B_t \), sharp estimates

\[ |\{f^*\}_n| \leq |\{F^*\}_1(t)| = \frac{2t}{e^t}, \quad n \in \{1, \ldots, N\}, \] (8)

are correct. Extremals in the estimates (8) are only the functions

\[ F^*(e^{i\varphi}z^n, t), \quad \varphi \in \mathbb{R}, \]

where the function \( F^* \) defined by (1).

**Proof.** We fix \( \omega \in \Omega \), \( t > 0 \) and \( n \leq 2/t + 1 \), \( N \in \mathbb{N} \).

Let us take a natural number \( n \), not exceeding the number \( N \). Using formula (3), we write \( n \)-th coefficient of function

\[ f(z) := F(\omega(z), t), \]

where \( F \) is defined in formula (5), in the form

\[ \{f\}_n = \sum_{j=1}^{n} \{F\}_j \{\omega^j\}_n. \]

Now we apply theorem 1 to \( n \)-th segment of Taylor expansion of function \( F(z, t) \), which we have denoted by \( P(z, t, n) \). Let \( S(z) \) — be an extension of
polynom $P(z, t, n)$ to function of class $S^0$. Then, using the formula (3), $n$-th coefficient of function

$$s(z) := S(\omega(z), t)$$

can be written as

$$\{s\}_n = \sum_{j=1}^{n} \{S\}_j \{\omega^j\}_n.$$  

From which, by lemma 1, we find that

$$|\{s\}_n| \leq 1.$$  

But $\{S\}_j := \{F\}_j(t)$, where $j \in \{1, \ldots, n\}$, therefore $\{f\}_n = \{s\}_n$, on basis of which we conclude, that

$$|\{f\}_n| \leq 1.$$  

Remembering about the normalization (5), we obtain the estimates (8). Accuracy of the estimates (8) and the form of extremal functions follow from lemma 1.

The theorem has been completely proved.

From theorem 2 it implies, that the smaller the number $t > 0$ we fix, the more Taylor coefficients we can estimate on the class $B_t$. In this case, our estimates are sharp in the sense, that the equality, in the inequality (8), is attained on functions $F^\ast(e^{i\varphi}z^n, t)$.

For example, if $t > 2$ we can estimate only one coefficient, for $t = 2$ — two coefficients, for $t = 1$ — three coefficients, and at $t = 1/2$ — five. And so on. Similar results were obtained in [6].

6. We eliminate the blank in the arguments given above. To do this we must only prove the validity of the formula (7).

**Lemma 2** If the coefficients $\{h\}_j$, $j \in \mathbb{N}$, are defined by formula (6), then for all integers $n \geq 0$

$$M_n = 2^{2n}t^n(2 - nt).$$

**Proof.** The minor $M_n/2^{n+1}$ is equal to the determinant

$$\begin{vmatrix}
1 & 1 - t & 1 - 2t & \ldots & 1 - (n - 1)t & 1 - nt \\
1 - t & 1 & 1 - t & \ldots & 1 - (n - 2)t & 1 - (n - 1)t \\
1 - 2t & 1 - t & 1 & \ldots & 1 - (n - 3)t & 1 - (n - 2)t \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 - (n - 1)t & 1 - (n - 2)t & 1 - (n - 3)t & \ldots & 1 - 2t & 1 - t \\
1 - nt & 1 - (n - 1)t & 1 - (n - 2)t & \ldots & 1 - t & 1 \\
\end{vmatrix}.$$
Subtracting from each row, except the first one, the previous one we obtain

$$\begin{bmatrix}
1 & 1-t & 1-2t & \ldots & 1-(n-1)t & 1-nt \\
-t & t & t & \ldots & t & t \\
-t & -t & t & \ldots & t & t \\
: & : & : & \ddots & : & : \\
-t & -t & -t & \ldots & t & t \\
-t & -t & -t & \ldots & -t & t \\
\end{bmatrix}$$

For each column, except for the last one, we add the last column

$$\begin{bmatrix}
2-nt & 1-(n+1)t & 1-(n+2)t & \ldots & 1-(2n-1)t & 1-nt \\
0 & 2t & 2t & \ldots & 2t & t \\
0 & 0 & 2t & \ldots & 2t & t \\
: & : & : & \ddots & : & : \\
0 & 0 & 0 & \ldots & 2t & t \\
0 & 0 & 0 & \ldots & 0 & t \\
\end{bmatrix} = 2^{n-1}t^n(2-nt).$$

7. We present an example of an extension. Let $t = 1/2$. By theorem 1, the desired extension is unique. Setting in the formula (4)

$$f(z) = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 - \frac{1}{24}z^4 - \frac{19}{120}z^5 + \ldots \in S^0$$

or using the formula (6), we obtain that

$$h(z) = 1 + z - z^3 - 2z^4 + \ldots \in C.$$ 

We know that the function

$$\omega(z) = \frac{1-h(z)}{1+h(z)} = -\frac{1}{2}z + \frac{1}{4}z^2 + \frac{3}{8}z^3 + \frac{9}{16}z^4 + \ldots$$

belongs to the class $\Omega$. It is also known (see [2, 5]) that

$$\omega(z) = \frac{\lambda}{\alpha_{n-1} + \alpha_{n-2}z + \ldots + \alpha_0z^{n-1}}.$$ 

In this case $\lambda = 1$ (see [2, 5]). Having found the parameters $\alpha_0, \ldots, \alpha_{n-1}$, we find

$$\omega(z) = \frac{1-z^2 - 2z^3}{-2 - z + z^3}.$$
Whence
\[ h(z) = \frac{1 + z - z^3 - z^4}{1 + z^4}. \]

Now we use the formula (4), by substituting the obtained expression for \( h(z) \) there. Well
\[ f(z) = \int_0^z \frac{\left( \frac{v^2 + \sqrt{2}v + 1}{v^2 - \sqrt{2}v + 1} \right)^{\sqrt{2}/4}}{\sqrt{1 + v^4}} \, dv. \]

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