ON IDEAL TRIANGULATIONS OF SURFACES UP TO BRANCHED TRANSIT EQUIVALENCES

RICCARDO BENEDETTI

Abstract. We consider triangulations of closed surfaces $S$ with a given set of vertices $V$; every triangulation can be branched that is enhanced to a $\Delta$-complex. Branched triangulations are considered up to the $b$-transit equivalence generated by $b$-flips (i.e. branched diagonal exchanges) and isotopy keeping $V$ pointwise fixed. We extend a well known connectivity result for ‘naked’ triangulations; in particular in the generic case when $\chi(S) < 0$, we show that branched triangulations are equivalent to each other if $\chi(S)$ is even, while this holds also for odd $\chi(S)$ possibly after the complete inversion of one of the two branchings. Moreover we show that under a mild assumption, two branchings on a same triangulation are connected via a sequence a inversions of ambiguous edges (and possibly the total inversion of one of them). A natural organization of the $b$-flips in subfamilies gives rise to restricted transit equivalences with non trivial (even infinite) quotient sets. We analyze them in terms of certain preserved structures of differential topological nature carried by any branched triangulations; in particular a pair of transverse foliations with determined singular sets contained in $V$, including as particular cases the configuration of the vertical and horizontal foliations of the square of an Abelian differential on a Riemann surface.

1. Introduction

Different notions of “decorated ideal triangulations” of 3-manifolds (branched, pre-branched, weakly branchedp) considered up to various transit equivalences naturally arise in the developments of quantum hyperbolic geometry (see for instance [1], [2], [3]) and in several other instances of 3D quantum invariants based on state sums over triangulations (see [1]). To understand the intrinsic content of the corresponding quotient sets is an interesting and non trivial task. This note arises as a simpler but non obvious 2D counterpart of similar questions, which also emerged within [1], Section 5, in a so called “holographic” approach to 3D non ambiguous structure.

Let $(S, V)$ be a compact closed connected smooth surface $S$ with a set $V$ of $n$ marked points and Euler-Poincaré characteristic $\chi(S)$, such that $\chi(S) - n < 0$. It is well known that $(S, V)$ carries ideal triangulations, say $T$. This means that $T$ is a possibly loose triangulation (self and multiple edge adjacency being allowed) whose set of vertices coincides with $V$. Clearly such ideal triangulations of $(S, V)$ share the same numbers of edges and triangles, $3(n - \chi(S))$ and $2(n - \chi(S))$ respectively. It is sometimes useful to consider an ideal triangulation $T$ as a way to realize $(S, V)$ by assembling $2(n - \chi(S))$ “abstract” triangles by gluing their “abstract” edges in pairs in such a way that no edge remains unglued. Ideal triangulations of $(S, V)$ are considered up to the ideal transit equivalence which is generated by isotopy fixing $V$ pointwise and the elementary diagonal exchange move also called flip. Denote by $T^{id}(S, V)$ the corresponding quotient set. The following is an important well known connectivity result.

Theorem 1.1. For every $(S, V)$, $T^{id}(S, V)$ consists of one point.

Proofs are available in several papers such as [9], [11], [12], [14].

Every “naked” triangulation $T$ carries some branchings $(T, b)$ (see Lemma 2.14), where by definition $b$ is a system of edge orientations which lifted to every abstract triangle $(t, b)$ of $T$ is induced by a (local) ordering of the vertices, so that every edge goes towards the biggest endpoint; equivalently $b$ promotes $T$ to be a $\Delta$-complex accordingly with [8], Chapter 2. It is easy to see that for every branching $(T, b)$,
every naked flip $T \rightarrow T'$ can be enhanced to some $b$-flip $(T, b) \rightarrow (T, b')$ such that every “persistent” edge in both $(T, b)$ and $(T', b')$ keeps the same orientation. Isotopy relatively to $V$ and $b$-flips generate the so called ideal $b$-transit equivalence and we denote by $\mathcal{B}^{id}(S, V)$ the corresponding quotient set. We define the symmetrized relation by adding to the generators the complete inversion that is we stipulate that every $(T, b)$ is equivalent to $(T, -b)$ where $-b$ is obtained by inverting all edge orientations of $b$, and we denote by $\tilde{\mathcal{B}}^{id}(S, V)$ the corresponding quotient sets. It is not hard to see that by setting $\sigma([T, b]) = [(T, -b)]$ it is well defined an involution on $\mathcal{B}^{id}(S, V)$ and that $\tilde{\mathcal{B}}^{id}(S, V) \sim \mathcal{B}^{id}(S, V)/\sigma$.

By the topological homogeneity of every surface, the cardinality of $\tilde{\mathcal{B}}^{id}(S, V)$ only depends on the topological type of $S$ and the number $n = |V|$; sometimes we will write $(S, n)$ instead of $(S, V)$. The following branched version of the above connectivity result is a main result of the present note.

**Theorem 1.2.** (1) If $S$ is orientable or if it is non orientable and $\chi(S)$ is even and strictly negative, then for every $(S, V)$, $\mathcal{B}^{id}(S, V)$ consists of one point.

(2) If $S$ is not orientable and either $\chi(S) = 0$ or $\chi(S)$ is odd, then for every $(S, V)$, $\tilde{\mathcal{B}}^{id}(S, V)$ consists of one point.

As $\tilde{\mathcal{B}}^{id}(S, V)$ is a quotient of $\mathcal{B}^{id}(S, V)$ by an involution, it follows that in case (2), $|\mathcal{B}^{id}(S, V)| \leq 2$.

**Conjecture 1.3.** If $S$ is not orientable and either $\chi(S) = 0$ or $\chi(S)$ is odd, then for every $(S, V)$, $|\mathcal{B}^{id}(S, V)| = 2$.

This will be confirmed at least for $\mathcal{B}^{id}(\mathcal{P}^2(\mathbb{R}), 2)$ (Proposition 3.3).

Assuming Theorem 1.1, we will provide two constructive proofs of Theorem 1.2 each with its subtleties and constructions of distinguished $b$-transits. A key ingredient will be the move of inverting an ambiguous edge (see Section 2.2) and in particular Theorem 3.2.

By adding to the $b$-flips the branched positive $0 \rightarrow 2$ $b$-bubble moves and their inverse (or equivalently the stellar $1 \rightarrow 3$ branched moves and their inverse), we get the completed $b$-transit equivalence with quotient set denoted by $\mathcal{B}(S, V)$. A positive bubble move produces an ideal triangulation of $(S, V')$ where $V'$ contains one further marked point of $S$: if it is part of a $b$-transit which connects two ideal branched triangulations of $(S, V)$, then it must be compensated later by a negative inverse move. We will see a quick direct proof of the following weaker connectivity result (no matter if $S$ is orientable or not).

**Proposition 1.4.** For every $(S, V)$, $\mathcal{B}(S, V)$ consists of one point.

We will see in Section 2 that $b$-flips can be naturally organized in some sub-families so that more restrictive transit equivalences can be defined, with non-trivial (actually infinite, see Remark 5.5) quotient sets. Another main theme (Sections 5) is to point out the intrinsic content of these various transit equivalences, that is some relevant structures on $(S, V)$, carried by every $(T, b)$, which are invariant under a given instance of transit equivalence. Theorem 1.2 itself should be enlightened by the mutations of such structures along any ideal $b$-transit.

**The dual viewpoint.** Let $S_V$ be the surface with $n$ boundary components ($n = |V|$) obtained by removing from $S$ a small open ball around each $v \in V$. For every ideal triangulation $T$ of $(S, V)$, the 1-skeleton $\theta = \theta_T$ of the dual cell decomposition is a generic (internal) spine of $S_V$. In fact $\theta$ is a graph with 3-valent vertices and $S_V$ is a ribbon graph which tickens $\theta$. If $(T, b)$ is branched, this promotes $\theta$ to be a transversely oriented train track $(\theta, b)$ - for simplicity we keep the same notation “$b$”.

**Remark 1.5.** If $S$ is oriented, then $(\theta, b)$ can be equivalently considered as an oriented train track by means of the following dual orientation convention:

At every transversal intersection point of $T$ and $\theta$, an oriented edge of $(T, b)$ followed by the dual oriented branch of $(\theta, b)$ realize the orientation of $S_V$ (that is every intersection number is equal to 1).

By definition, $(\theta, b)$ is a (transversely oriented) branched spine of $S_V$. Viceversa, every ribbon graph, $\hat{S}$ say, carried by a (possibly branched) spine $\theta$ as above gives rise to a (possibly branched) ideal triangulation $T = T_\theta$ of $(S, V)$ obtained by filling each boundary component of $\hat{S}$ with a punctured
2-disk. Flips and bubbles, possibly branched, can be equivalently rephrased in terms of (branched) spine moves. We will freely adopt both equivalent dual viewpoints.

**Remark 1.6.** Although they are equivalent, there is some qualitative difference between spines and triangulations. A flip is a *discrete* transition with a cell decomposition as intermediate “state” which is no longer a triangulation (it includes one quadrilateral). The corresponding spine transition can be realized by a *continuous* deformation passing through a non generic spine (with one 4-valent vertex).

## 2. Generalities on $b$-transit

An “abstract” $b$-flip acts on a quadrilateral $Q$ endowed with a branched triangulation $(t_1 \cup t_2, b)$ made by two triangles with one common edge $e = t_1 \cap t_2$ (a diagonal of the quadrilateral). A $b$-flip produces another branched triangulation $(t_1' \cup t_2', b')$ of $Q$ made by two triangles having as common edge $e' = t_1' \cap t_2'$ the other diagonal of $Q$, while $b$ and $b'$ coincide on the persistent edges which form the boundary of $Q$. An abstract $b$-flip can be applied at every couple of abstract triangles of any branched ideal triangulation $(T, b)$ of any $(S, V)$, (partially) glued in $T$ along a common edge. When we say that a $b$-flip verifies a certain property we mean that this holds “universally” for every $(S, V)$ and every triangulation $(T, b)$ at which the flip operates.

### 2.1. A combinatorial classification of $b$-flips.

For every branched triangulation $(t_1 \cup t_2, b)$ of $Q$ as above there are either one or two ways to enhance the naked flip $t_1 \cup t_2 \rightarrow t_1' \cup t_2'$ to a $b$-flip $(t_1 \cup t_2, b) \rightarrow (t_1' \cup t_2', b')$. This last is sometimes denoted by $f_{e, b, b'}$ while the underlying naked flip is denoted by $f_e$. Then we can distinguish a few families of $b$-flips. The classification and even the terminology below could sound a bit arbitrary at this point. This will be clarified later.

![Figure 1. Branched flips.](image)

**Definition 2.1.**

1. A $b$-flip $f_{e, b, b'}$ is *forced* if it is the unique branched flip which enhances $f_e$, starting from $(t_1 \cup t_2, b)$.
2. A $b$-flip $f_{e, b, b'}$ is *non ambiguous* if both $f_{e, b, b'}$ and the inverse $b$-flip $f_{e', b', b}$ are forced.
3. A $b$-flip $f_{e, b, b'}$ is *forced ambiguous* if it is forced but the inverse $b$-flip is not.
4. A $b$-flip $f_{e, b, b'}$ is said a *sliding flip* ($s$-flip) if at least one among $f_{e, b, b'}$ or its inverse $f_{e', b', b}$ is forced.
5. A $b$-flip $f_{e, b, b'}$ is *totally ambiguous* (also called a *bump flip*) if noone among $f_{e, b, b'}$ and $f_{e', b', b}$ is forced.
In Figure 1 we show typical samples of $b$-flips in accordance with the above classification. We have labelled by 1 the corner of each branched triangle formed by the two edges that carry the prevalent orientation. Here 1 is just a highlighting label. For its meaning if $1 \in \mathbb{Z}/2\mathbb{Z}$, we refer to Section 5 of [1]. We note that the above classification of the $b$-flips is invariant under total inversion.

To stress it, in Figure 2 we show the dual pictures in terms of branched spines, provided that we have performed the total inversion on the $b$-flips of Figure 1; instead of the transverse orientations, we prefer (as it is easier) to indicate the local orientations on the dual train-tracks, by stipulating that all these pictures are planar, the plane $\mathbb{R}^2$ is oriented by the standard basis and we apply the orientation convention fixed in Remark 1.5. The $s$-flips as well as the $na$-flips (together with isotopy relative to $V$) generate restricted $s$- and $na$-ideal transit equivalence with respective quotient sets denoted $\mathcal{S}^{id}(S, V)$ and $\mathcal{N}A^{id}(S, V)$. Clearly there are surjective projections

$$\mathcal{N}A^{id}(S, V) \to \mathcal{S}^{id}(S, V) \to \mathcal{B}^{id}(S, V)$$

in particular the last quotient map is obtained by adding the bump $b$-flips to the sliding ones.

**Some characterizations.** For every branched triangulation $(T, b)$ of $(S, V)$, for every vertex $v$ of $T$, the number of corners at $v$ in its star labelled by 1 (as above) is even, say $2d_b(v)$. In fact given the star of $v$ an auxiliary orientation, the 1-labelled corners at $v$ belong to triangles $(t, b)$ whose $b$-orientations alternate with respect to the reference one. It is clear that

$$\chi(S) = |V| - \sum_v d_b(v) = \sum_v (1 - d_b(v)) .$$

We easily have that $b$-flips are characterized by the following property.

**Proposition 2.2.** A (abstract) $b$-flip $f_{e,b}$ is an $s$-flip if and only if for every $(S, V)$ and for every application of the flip on triangulations of $(S, V)$, $(T, b) \to (T', b')$, we have that for every $v \in V$, $d_b(v) = d_{b'}(v)$.

Note that when $|V| = 1$ the conclusion holds for every $b$-flip, not necessarily an $s$-flip, but this is not "universally" true.

Suppose now that $S$ is oriented. The orientation corresponds to a unique simplicial fundamental $\mathbb{Z}$-2-cycle

$$f(T, b) = \sum_t \ast_{(t, b)}(t, b)$$
where every $*_{(t,b)} \in \{\pm 1\}$. Set

$$
\epsilon_\pm = \epsilon_\pm(T, b) := |\{t; *_{(t,b)} = \pm 1\}|.
$$

We have

**Lemma 2.3.** For every $(T, b)$, $\epsilon_+ = \epsilon_-$.  
Proof. Around every $v \in V$, the $2d(v)$ 1-colored corners necessarily belong to triangles with alternating signs. As every triangle contains one 1-colored corner the result follows.

The above Lemma corroborates the validity of Theorem 1.2. For we easily check that every $b$-flip preserves the value of $\epsilon_+ - \epsilon_-$. On the other hand

$$(\epsilon_+ - \epsilon_-)(T, b) = -(\epsilon_+ - \epsilon_-)(T, -b) .$$

Denote by $S_\pm = S_\pm(T, b)$ the union of triangles such that $*_{(t,b)} = \pm 1$. Then $S$ decomposes as

$$S = S_+ \cup S_- .$$

Denote by $\partial S_\pm$ the boundary 1-cycle of the simplicial 2-cochain supported by $S_\pm$.

We have

**Proposition 2.4.** A $b$-flip $f_{e,b}$ is non ambiguous if and only if it is an $s$-flip and for every $(S, V)$ and for every application of the flip on triangulations of $(S, V)$, $(T, b) \to (T', b')$, we have that $S_\pm(T, b) = S_\pm(T', b')$, hence also $\partial S_\pm$ is preserved.

**Remark 2.5.** Also branched bubble and stellar $1 \leftrightarrow 3$ moves admit a sliding vs bump classification. This leads to the corresponding completed sliding equivalence with quotient set $S(M)$ which projects onto $B(M)$ (see [1], Section 5.3).

2.2. Inversion of an ambiguous edge.

**Definition 2.6.** (1) Let $T$ be a naked ideal triangulation of $(S, V)$. An edge $e$ of $T$ is said trapped if it results by the identification of two edges of one “abstract” triangle. Otherwise, $e$ is said untrapped. A trapped edge corresponds to a one vertex loop in the dual spine $\theta$.

(2) Given a branching $(T, b)$, a $b$-oriented edge $e$ is said ambiguous in $(T, b)$ if by inverting its orientation we keep a branched triangulation $(T', b')$.

**Lemma 2.7.** If $e$ is ambiguous and untrapped in $(T, b)$, then $(T, b)$ and $(T', b')$ (as in (2) of Definition 2.6) are connected by two $b$-flips.

**Proof.** Denote by $f_{e,b,b'}$ a $b$-flip that enhances the naked flip $f_e$ with inverse naked flip $f_{e'}$. Then we easily see that the untrapped edge $e$ is ambiguous if and only if either $f_{e,b,b'}$ is forced ambiguous or it is totally ambiguous. Hence $f_{e,b,b'}$ followed by $f_{e',b',b''}$ convert $(T, b)$ to $(T', b')$.

Then we can add the elementary move of inverting any untrapped ambiguous edge without changing the ideal $b$-transit equivalence.

**Remark 2.8.** Every trapped edge $e$ of $(T, b)$ is ambiguous but there is not any apparent local sequence of $b$-flips that inverts $e$.

We have

**Lemma 2.9.** (1) For every $T$ as above there is a sequence of flips $T \Rightarrow T'$ such that $T'$ does not contain trapped edges. (2) If $T$ and $T'$ do not contain trapped edges then they can be connected by a sequence of flips through triangulations without trapped edges.

**Proof.** The vertex of a loop in the spine $\theta$ which is dual to a trapped edge of $T$ is connected by an edge to the rest of the spine. By performing the dual flip at this edge we remove the loop without introducing new ones. If such a loop appear in a sequence of flips connecting $T$ and $T'$ as in (2), then we can follow it till it disappears so that we can eventually remove it from the sequence.
2.3. **Bad nutshells.** A positive naked $0 \rightarrow 2$ bubble produces a so called *nutshell* made by two triangles identified along two common edges. Not every branched nutshell $(N, b)$ supports a negative $2 \rightarrow 0$ $b$-bubble.

**Definition 2.10.** A branched nutshell $(N, b)$ is *bad* if the two boundary edges form an *oriented* circle. Otherwise $(N, b)$ is a *good* nutshell.

The following Lemma is immediate.

**Lemma 2.11.** (1) If $(N, b)$ is a bad nutshell, then the central vertex is necessarily either a pit or a source.
(2) $(N, b)$ is good if and only if it supports a negative $b$-bubble move.
(3) Two different good nutshells $(N, b)$ and $(N, b')$ sharing the same oriented boundary edges are connected by either one or two consecutive inversions of internal (hence untrapped) ambiguous edges.

A positive naked $1 \rightarrow 3$ move produces a so called *triangular star*. Similarly as before, not every branched triangular star, say $(\mathcal{S}, b)$, supports a negative $b$-$3 \rightarrow 1$ move.

**Definition 2.12.** A branched triangular star $(\mathcal{S}, b)$ is *bad* if the three boundary edges form an *oriented* circle. Otherwise $(\mathcal{S}, b)$ is *good*.

Easily we have

**Lemma 2.13.** (1) If $(\mathcal{S}, b)$ is bad, then the central vertex is necessarily either a pit or a source.
(2) $(\mathcal{S}, b)$ is good if and only if it supports a negative $b$-$3 \rightarrow 1$ move.
(3) Two good $b$-triangular stars $(\mathcal{S}, b)$ and $(\mathcal{S}, b')$ sharing the same oriented boundary edges are connected by a finite sequence of consecutive inversions of internal (hence untrapped) ambiguous edges.

2.4. **Existence of branched triangulations.** We have

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**Figure 3.** Existence of branched triangulations.

**Figure 4.** Existence of branched triangulations.
Lemma 2.14. Every ideal triangulation $T$ of $(S, V)$ can be branched.

Proof. Thanks to Theorem 1.1 and due to the fact that for every branched triangulation $(T, b)$, every naked flip $T \to T'$ can be enanch to a $b$-flip $(T, b) \to (T', b')$, it is enough to show that every $(S, V)$ admits a branched triangulation. By using the $b$-bubbles we see that if $(S, V)$ admits such a triangulation, then this holds for every $(S, V')$ such that $|V'| \geq |V|$. Then it is enough to show that for every $S$ there exists a branched triangulation $(T, b)$ of $(S, V)$ such that $n = |V|$ is the minimum for which $\chi(S) - n < 0$. There are of course several ways to do it. We indicate a way which will be suited for further use. If $S$ is a sphere then $n = 3$, and a branched triangulation is obtained by gluing two copies of a branched triangle along the common boundary (see the left side of the first row of Figure 3). If $S = \mathbb{P}^2(\mathbb{R})$ is a projective plane, then $n = 2$ and we can use the realization of $\mathbb{P}^2(\mathbb{R})$ by identifying the two edges of a bigon and triangulate it with one internal vertex - see the right side of the first row of Figure 3. For all other $S$, $n = 1$. If $S = \mathbb{P}^2(\mathbb{R}) \# \mathbb{P}^2$ is a Klein bottle, that is the connected sum of two projective planes, we get a branched triangulation of $(S, V)$ starting from a realization of $S$ by identifying the boundary of a quadrilateral obtained by gluing two “truncated bigons” - see the left side of the second row of Figure 3 at the middle of the row we show another branched triangulation starting from a realization of the bottle by identifying in pairs the opposite edges of a quadrilateral; similarly on the right side we suggest a triangulation of the torus $S^1 \times S^1$. In Figure 4 we show the elementary bricks in order to realize all other cases. These bricks are branched triangulations of certain surfaces with boundary. In the first row we see either a one or twice-pierced torus that is a torus from which one has removed respectively one or two open 2-disks; they are obtained by means of a one or twice-truncated quadrilateral with opposite edge identified in pairs. In the second row, right side we see a similar realization of a one-pierced Klein bottle; on the left a one-pierced projective plane given by means of a truncated bigon with the two possible branchings. The non oriented edges are ambiguous so that their orientation can be chosen arbitrarily. If $S$ is orientable of genus $g > 1$ we can realize it by means of a chain of $g - 2$ twice-pierced tori capped by two one-pierced ones. If $S$ is non orientable and $\chi(S) = 2 - r < 0$ is odd, set $g = \frac{r - 1}{2}$; then we can obtain $S$ by means of chain of $g - 2$ twice-pierced tori capped by a one-pierced torus and a one-pierced projective plane. If $\chi(S) - r < 0$ is even, set $g = \frac{r - 2}{2}$, then we obtain $S$ by means of chain of $g - 2$ twice-pierced tori capped by a one-pierced torus and a one-pierced Klein bottle.

□

Definition 2.15. The boundary of every brick as above is union of loops with one vertex. The corresponding edge of the triangulation is called a connection edge.

For every triangulation of $S$ obtained so far, the connection edges become separating loops that decompose $S$ by the bricks; in Figure 4 these edges correspond to the ambiguous edges on the boundary of the truncated quadrilaterals or to the non ambiguous edge in the branched truncated bigons.

2.5. Preliminary reductions to face the ideal $b$-transit equivalence. We assume Theorem 1.1 Isotopy relative to $V$ will be understood. At the end of this section we will obtain a quick proof of Proposition 1.4.

Lemma 2.16. The following facts are equivalent to each other:

1. $\mathcal{B}^{id}(S, V)$ consists of one point.
2. For every naked ideal triangulation $T$ of $(S, V)$, every two branchings $(T, b)$ and $(T', b')$ are connected by a chain of $b$-flips.
3. There exists a naked ideal triangulation $T$ of $(S, V)$ such that every two branchings $(T, b)$ and $(T', b')$ are connected by a chain of $b$-flips.

Proof. Obviously (1) $\Rightarrow$ (2) $\Rightarrow$ (3). In order to prove (3) $\Rightarrow$ (1), we argue similarly to the proof of Lemma 2.14. Let $(T_1, b_1)$ and $(T_2, b_2)$ be ideal triangulations of $(S, V)$. By Theorem 1.1, there are naked ideal $b$-transits $T_j \Rightarrow T$, $j = 1, 2$. There is no obstruction to enhance them to sequences of $b$-flips $(T_j, b_j) \Rightarrow (T, b'_j)$. The Lemma follows immediately.

□
Similar statements hold for both the symmetrized and completed $b$-transit equivalences.

**Definition 2.17.** Given $(T, b)$ and $(T, b')$, denote by $\delta(b, b')$ the set of edges of $T$ at which $b$ and $b'$ are opposite. An edge $e \in \delta(b, b')$ is said *disoriented*.

The previous considerations suggest two possible “strategies” in order to prove that $B^{id}(S, V)$ (or $\tilde{B}^{id}(S, V)$ or $B(S, V)$) consists of one point.

(A) For a given $(S, V)$ detect a distinguished naked triangulation $T$ for which one can check directly that (3) of Lemma 2.16 holds.

(B) To point out a few procedures such that for every couple $(T, b)$ and $(T, b')$ such that $\delta(b, b')$ is non empty, we can apply one of them producing $b$-transits $(T, b) \Rightarrow (T', b_1)$ and $(T, b') \Rightarrow (T', b_2)$ such that

$$|\delta(b_1, b_2)| < |\delta(b, b')|.$$

**Remark 2.18.** We understand the difference between the two strategies if we apply them in order to realize a $b$-transit between two *arbitrary* triangulations $(T_1, b_1)$ and $(T_2, b_2)$ of some $(S, V)$. Via A, if $T$ is the distinguished triangulation we must preliminarily connect both $(T_1, b_1)$ and $(T_2, b_2)$ with some $(T, b)$ and $(T, b')$ respectively, by applying twice Theorem 1.1. Via B, it is enough to connect $(T_1, b_1)$ to some $(T, b) := (T_2, b)$ and set $(T, b') = (T_2, b_2)$. This might be relevant in terms of computational cost.

Strategy B, suitably adapted, works quickly on $B(S, V)$.

**Proof of Proposition 3.3.** Given $(T, b)$ and $(T, b')$ as above perform on both triangulations a bump $b$-(1 $\rightarrow$ 3) move at every triangle of $T$ in such a way that all the new oriented edges point toward the new internal vertex. We get in this way $(T', b_1)$ and $(T', b_2)$ such that $\delta(b_1, b_2) = \delta(b, b')$. We realize now that every $e \in \delta(b, b')$ is untrapped and ambiguous in both $(T', b_1)$ and $(T', b_2)$. So we conclude by several applications of Lemma 2.17 (by the way, in the present situation every inversion of $e$ is obtained by a sequence of two bump $b$-flips).

We will implement both strategies and get the two promised proofs of Theorem 1.2

3. * Proof of the main Theorem

By implementing strategy A we will actually obtain stronger results. We have:

**Theorem 3.1.** (i) For every $(S, V)$ as in (1) of Theorem 1.2, there exists a distinguished triangulation $T$ such that every $(T, b)$ and $(T, b')$ can be explicitly connected by a sequence of inversions of untrapped ambiguous edges.

(ii) For every $(S, V)$ as in (2) of Theorem 1.2, there exists a distinguished triangulation $T$ such that for every $(T, b)$ and $(T, b')$ either $(T, b)$ and $(T, b')$ or $(T, -b)$ and $(T, b')$ can be explicitly connected by a sequence of inversions of untrapped ambiguous edges.

For every $(S, V)$ let us said *inversive* any triangulation $T$ which verifies the conclusions of Theorem 3.2 in accordance to the two cases. Finally we have:

**Theorem 3.2.** For every $(S, V)$, every triangulation $T$ without trapped edges is inversive.

**Proof of Theorem 3.2.** For every $(S, V)$ there is a minimum $n_S$ such that $\chi(S) - n_S < 0$. The proof is by induction on $n \geq n_S$.

**Initial step:** $(S, n_S)$.

- $(P^4(\mathbb{R}), 2)$. We use the naked triangulation $T$ of Figure 2. Let $(T, b)$ and $(T, b')$ be supported by $T$. By total inversion we can assume that $b$ and $b'$ agree on the internal edges so that the internal vertex is a pit. The boundary edges of the nutshell lift an ambiguous edge of both $(T, b)$ and $(T, b')$; if $b \neq b'$ we conclude by inverting it in $b$. Then $B^{id}(P^2(\mathbb{R}), V)$ consists of one point.

**Proposition 3.3.** $|B^{id}(P^2(\mathbb{R}), 2)| = 2$. 
Proof. Let \( (T, b) \) and \( (T, b') \) be as above such that \( \delta(b, b') \) consists of the two internal edges. If we flip an internal edge of \( (T, b) \) we produce a trapped edge; in order to get the same naked configuration we must flip the same edge in \( (T, b') \) and \( |\delta| \) is unchanged. If we flip the edge of \( (T, b) \) which lifts to the boundary edges of the nutshell, we get a triangulation \( (T_1, b_1) \) which is abstractly like \( (T, b') \) with respect to another nutshell, but the two vertices exchange their role, so \( (T_1, b_1) \) cannot be relatively isotopic to \( (T, b') \).

\( (S^2, 3) \). Take \( T \) made by two triangles glued along the common boundary as in the first row of Figure 3. Every branched \( (T, b) \) is determined by a labelling of the vertices by 0, 1, 2. Fix a \( (T, b_0) \), then all \( (T, b) \) are indexed by the elements \( \sigma \) of the symmetric group \( \Sigma_3 \), so that \( (T, b_0) \) corresponds to the identity. This group is generated by the transpositions \( \{(01), (12)\} \). If \( b = b_{\sigma} \), write \( \sigma \) as a product of minimal number of these generators. Every such a sequence of transpositions corresponds to a sequence of inversions of ambiguous edges going from \( (T, b_0) \) to \( (T, b) \).

In all other cases \( n_S = 1 \).

• Let \( S = S^1 \times S^1 \). Consider \( T \) as in the second row of Figure 3. Every branching of \( T \) is uniquely encoded by a total order of the three vertices of one of the abstract triangles of \( T \). Then we can manage similarly as for \( (S^2, 3) \) by checking that also in this case every transposition in a product of the generators corresponds to the inversion of an ambiguous edge.

• Let \( S \) be a Klein bottle. Refer to Figure 3. If we use the triangulation \( T \) made by two truncated bigons, it is immediate that it carries exactly two branchings say \( (T, b) \) and \( (T, -b) \). If we use as \( T \) the other triangulation, we see that it carries four branchings, distributed into two pairs \( \{(T, b), (T, b')\}, \{(T, -b), (T, -b')\} \) such that \( (T, b') \) is obtained from \( (T, b) \) via the inversions of an ambiguous edge.

Let us face now the remaining generic cases such that \( \chi(S) < 0 \).

Case (1) Let \( S \) be either orientable with \( \chi(S) < 0 \) or non orientable with \( \chi(S) < 0 \) and even. Take the naked triangulation \( T \) depicted in the proof of Lemma 2.13 let \( (T, b_0) \) be a branched triangulation contructed therein. Let \( (T, b) \) any branched triangulation supported by \( T \). This determine a system of orientations on the family of connection edges (Definition 2.16). Every connection edge is ambiguous in \( (T, b_0) \) so that up to some inversions, we can assume that \( (T, b_0) \) and \( (T, b) \) share such a partial system of orientations. Let us cut now \( S \) along the connection edges. By restriction we get a family of pairs of branched triangulated bricks \( (B, b_0) \) and \( (B, b) \) which coincide at every connection edge. It is enough to show that every \( (B, b) \) is connected to \( (B, b_0) \) by a sequence of inversions of ambiguous edges. This can be checked case by case. We have three types of \( B \), the one-epierced torus or Klein bottle and the twice-pierced torus. A priori, for every one-pierced brick we have two local configurations of \( (B, b_0) \) at the connection edges; for the twice-pierced torus there are four. For every brick and every pair of opposite local configurations, we see by means of the total inversion that the desired result holds for one if and only if it holds for the other. Then we are actually reduced to study one configuration in the one-pierced cases, two in the twice-pierced one. We organize the discussion as follows, referring to Figure 4.

• Denote by \( t \) the top (abstract) triangle of \( (B, b_0) \). Encode \( (t, b_0) \) by labelling its vertices by 0, 1, 2, say \( v_0, v_1, v_2 \); do the same for the bottom triangle \( (t, b_0) \), getting \( v_0', v_1', v_2' \). In the one-pierced brick \( v_0 = v_0' \). Every (abstract) edge of \( (B, b_0) \) has two vertices belonging to \( \{v_0, v_1, v_2, v_0', v_1', v_2'\} \). For every \( (B, b) \), every oriented edge \( e, b \) will be denoted by its vertices, \( e = ab \), written in the order so that the orientation emanates from the initial vertex \( a \) toward the final vertex \( b \). For every one-pierced brick, we stipulate that the connection edge in \( (B, b_0) \) has \( v_2 \) as initial vertex. For the twice-pierced torus we stipulate that in \( (B, b_0) \) the pairs of connection edges is either \( (v_2v_2', v_0v_0') \) or \( (v_2'v_2, v_0v_0) \). We note that having fixed orientation of the connection edges, \( b_0 \) is completely determined by \( (t, b_0) \), that is this propagates in a unique way to a global branching.

• For every permutation \( \sigma \in \Sigma_3 \) we consider the corresponding branched triangulated triangle \( (t, b_0) \) and we list all the extensions to a global branching say \( (B, b_{\sigma}) \), if any. Of course \( (B, b_0) \) corresponds to the identity.
• By varying $\sigma \in \Sigma_3$, the so obtained $(B_\sigma, b)$ cover all possible branchings $(B, b)$ and we have to manage in order to connect $(B, b_0)$ with every $(B, b_\sigma)$. It is convenient to start with the generating transpositions $\sigma = (0, 1), (1, 2)$, and express all other $\sigma$ as a product of three or two generators.

Let us pass now to the actual verifications.

**The one-pierced torus.**

$\sigma = (0, 1)$: there a unique extension $(B, b_{(0,1)})$ which differs from $(B, b_0)$ by the inversion of the ambiguous edge $v_1v_0$.

$\sigma = (1, 2)$: there are several extensions. There is only one containing the edge $v_0v_2$ and this differs from $(B, b_0)$ by the inversion of the ambiguous edge $v_2v_1$. There are two extensions containing the edge $v_2v_0$ which differ from each other by the inversion of the ambiguous edge $v_0v_2'$. In the one containing $v_0v_2' = (1)\sigma$, $v_2v_0$ is ambiguous, hence by inverting it we are in the first case.

$\sigma = (0, 1, 2) = ((1, 2)(0, 1))$: there are two extensions which differ from each other by the inversion of the ambiguous edge $v_0v'_1$. In the one containing $v_0v'_2$, the edge $v_0v_2$ is ambiguous, hence possibly by inverting it we reach the case $(B, b_{(1,2)})$.

$\sigma = (0, 2, 1) = (0, 1)(1, 2)$: the discussion is similar to the one for $(0, 1, 2)$; up to some inversion of ambiguous edges we reach the case $(B, b_{(0,1)})$.

$\sigma = (0, 2) = (0, 1)(1, 2)(1, 0)$: there only one extension in which $v_0v'_1$ is ambiguous. By inverting it we reach the case $(B, b_{(0,2,1)})$.

The first verification is complete.

**The one-pierced Klein bottle.**

$\sigma = (0, 1)$: there a unique extension $(B, b_{(0,1)})$ which differs from $(B, b_0)$ by the inversion of the ambiguous edge $v_0v_1$.

$\sigma = (1, 2)$: there are no extensions.

$\sigma = (0, 1, 2)$: there is only one extension $(B, b_{(0,1,2)})$. The following sequence of inversions of ambiguous edges realizes a transit from this extension to $(B, b_{(0,1)})$ (we indicate the initial orientation before the inversion): $v_0v_1, v'_2v_0, v_2v_1, v_2v_0$.

$\sigma = (0, 2, 1)$: there is only one extension. The following sequence of inversions of ambiguous edges realizes a transit to $(B, b_0)$: $v_2v_0, v_1v_0$.

$\sigma = (0, 2)$: there are two extensions which differ to each other by the ambiguous edge $v_0v'_2$. In the ones containing the oriented $v'_2v_0$, the edge $v_1v_0$ is ambiguous. By inverting it we reach $(B, b_{(0,1,2)})$.

The second verification is complete.

**The twice-pierced torus.** We have two cases depending on the orientation either $(v_2v'_2, v_0v'_0)$ or $(v_2v'_2, v'_0v_0)$ of the two connection edges.

**Subcase** $(v_2v'_2, v_0v'_0)$

$\sigma = (0, 1)$: There are two extensions which differ by the ambiguous edge $v'_0v'_2$. In the ones containing the oriented edge $v'_0v'_2$, $v_1v_0$ is ambiguous and possibly inverting it we reach $(B, b_0)$.

$\sigma = (1, 2)$: this is very similar to the case $(0, 1)$.

$\sigma = (0, 1, 2)$: there are several extensions. There is only one containing $v'_2v_0$ which is ambiguous. In the ones containing $v_0v'_2$ both $v_0v_2$ and $v'_0v'_2$ are ambiguous, then after at most two inversions we reach $(B, b_{(1,2)})$.

$\sigma = (0, 2, 1)$: this is very similar to the case $(0, 1, 2)$; via a sequence of inversions we reach now $(B, b_{(0,1,1)})$.

$\sigma = (0, 2)$: there are four extensions which differ by suitable inversions of the edges $v_1v_2$ and $v_0v'_2$ which are both ambiguous. Then up to such inversion we reach $(B, b_{(0,1)})$.

**Subcase** $(v_2v'_2, v'_0v_0)$

At this point the fourth verification is a routine, we leave it to the reader.

**Case (2)** Let $S$ be not orientable such that $\chi(S) < 0$ and odd. We manage as in Case (1). The only difference is that the capping pierced Klein bottle is replaced with a pierced projective plane. Again
we use the triangulations depicted in the proof of Lemma 2.13. Up to total inversion we can assume that \((T, b_0)\) and \((T, b)\) coincide on the capping truncated bigon. The rest of the proof is unchanged. The proof of Theorem 5.1 for \((S, n_S)\) is now complete.

The **inductive step.** Let us face first the generic case \(\chi(S) < 0\). So we have proved the result for \((S, 1)\), and we want to prove it for every \((S, n)\) by induction on \(n \geq 1\). We define the distinguished triangulation for \((S, n)\) by modifying the one used for \((S, 1)\) as follows:

- The pierced Klein bottle and the twice-pierced torus bricks are unchange.
- We modify only the the one-pierced torus brick, say \(B_1\), used when \(n = 1\) in order that \(B_n\) carries all further \(n - 1\) vertices. We do it inductively as follows: \(B_1\) is triangulated by say \(T_1\) as above; the naked triangulation \(T_n\) of \(B_n\) is obtained from \(T_{n-1}\) by performing a \(1 \rightarrow 3\) move on the triangle which contains the connection edge.
- In the treatment of \(n = 1\) we have also indicated a reference branched brick \((B, b_0)\); we define inductively the reference branching \((B_n, b_n)\) for every \(n \geq 1\) as follows: set \((B_1, b_1) := (B, b_0)\) and recall that it is completely determined by a suitable total order of the vertices of the top triangle (labelled by \(0, 1, 2\)); \((B_n, b_n)\) is uniquely determined by extending the ordered set of vertices which defines \(b_{n-1}\) by adding the new vertex produced by the \(1 \rightarrow 3\) move and stipulating that it is the smallest one (the vertices are labelled by \(0, 1, 2, \ldots, n-1, n, n+1, n+2\) and the labels of the vertices relative to \(b_{n-1}\) shift by one).

We can fix the orientation of the connection edge, say \(v_{n+2}v'_{n+2}\). Consider any \((B_n, b)\). There are two possibilities:

(a) The new vertex of \(B_n\) with respect to \(B_{n-1}\) has a good triangular star in \((B_n, b)\). Then \(b\) restricts to a branching \((B_{n-1}, b)\). By induction, this is connected to \((B_{n-1}, b_{n-1})\) by a sequence of inversions of ambiguous edges. Finally we conclude by applying Lemma 2.13 to the innermost triangular star.

(b) The innermost triangular star as above is bad in \((B_n, b)\). Consider first \((B_2, b)\); we readily see that \(v_2\) is necessarily either a pit or a source. In any case \(v_nv_{n+2}\) is ambiguous; by inverting it the triangular star becomes good and we reach the case (a). In general we can assume by induction that the triangular star of \(v_1\) with respect to the restriction of \(b\) to \(B_{n-1}\) is good, so that we can apply the above reasoning to the triangular star of \(v_0\) in \((B_n, b)\) and reach again the case (a).

The proof of Theorem 5.1 in the generic cases is now complete. \qed

For the remaining cases such that \(\chi(S) \geq 0\) we limit ourselves to some indications.

- \((S^2, n), \ n \geq 3\). Denote by \(T_3\) the triangulation used above for \(n = 3\). Select one triangle \(t\) and one edge \(e\). For every \(n > 3\), the distinguished triangulation \(T_n\) for \((S^2, n)\) is obtained by induction on \(n\) by performing a \(1 \rightarrow 3\) move on the triangle of \(T_{n-1}\) which is contained in \(t\) and contains \(e\). In particular \(T_4\) corresponds to the triangulation of the boundary of a tetrahedron. Every \((T_4, b)\) is determined by a labelling of the vertices by \(0, 1, 2, 3\). Fix a \((T, b)\); then the branchings are indexed by the elements of the symmetric group \(\Sigma_4\). This is generated by the transpositions \((0, 1), (1, 2), (2, 3)\). Write every \(\sigma\) as a product of these generators with minimal number of terms. This corresponds to a sequence of inversions of ambiguous edges connecting \((T, b)\) and \((T, b_{\sigma})\). For \(n > 4\) we argue by induction on \(n\).

- \((S^1 \times S^1, n)\) or \((\mathbb{P}^2(\mathbb{R})\# \mathbb{P}^2(\mathbb{R}), n), \ n \geq 1\). In both cases we start with the triangulation say \(T_1\) used for \(n = 1\). Then (referring to Figure 3) \(T_3\) is obtained from \(T_{n-1}\) by performing a \(1 \rightarrow 3\) move on the triangle contained in the top triangle and containing the diagonal edge of \(T_1\).

- \((\mathbb{P}^2(\mathbb{R}), n), \ n \geq 2\). We start with \(T_2\) used for \(n = 2\). Referring to Figure 3 \(T_3\) is obtained by performing a bubble move at the internal edge on the left side. Denote by \(t\) the new triangle contained in the top half of \(T_2\) and by \(e\) its edge contained in the interior of this top-half. Then, for \(n > 3, T_n\) is obtained from \(T_{n-1}\) by performing a \(1 \rightarrow 3\) move on the triangle of \(T_{n-1}\) which contains \(e\).

The proof of Theorem 5.1 is now complete. \qed

**Proof of Theorem 5.2**
Case (a): $\chi(S)$ is not strictly negative and odd. The distinguished inversive triangulations $T_n$ of $(S, n)$ constructed in the proof of Theorem 3.21 have no trapped edges. Let $T$ be any other triangulation without trapped edges. We know that there is a sequence of flips $T_n \Rightarrow T$ through triangulations without trapped edges. Denote by $l$ the number of flips. We work by induction on $l$. Let $T'$ be obtained by performing the first $l - 1$ flips. By induction the theorem holds for $T'$. Hence we are reduced to check the case $l = 1$. We fix the notations as follows: $e$ is the flipping edge, that is a diagonal of a quadrilateral $Q = t_1 \cup t_2$ in $T_n$, $t_1 \cap t_2 = e$; $e'$ denote the other diagonal of $Q$, that is the edge of $T$ which replaces $e$. Let $(T_n, b)$ and $(T_n, b')$ be connected by a sequence of $k$ inversions of ambiguous edges. Let $(T, b), (T, b')$ be obtained by $b$-enhancing in some way the flip $(T_n, b) \rightarrow T$ and $(T_n, b') \rightarrow T$. We want to modify the sequence in order to get one connecting this branchings of $T$. We note that if a $b$-flip is not forced then the two possibilities are related by inverting an ambiguous edge, so this is essentially immaterial for our discussion. If an inversion concerns an edge not contained in $(Q, b)$ then it makes sense also on $(T, b)$. By these remarks and working by induction on $k$, we are reduced to analyze the inversion of an edge $e^*$ contained in $(Q, b)$. There are a few possibilities.

- $e^* = e$: then the $b$-flip is either totally ambiguous (i.e. bump) or forced ambiguous, depending if the vertices of $(Q, b)$ opposite to $e$ are either both a pit (resp. source) or one is a pit and the other a source. So in the first case we possibly replace the inversion of $e$ with the inversion of $e'$.

Assume now that $e^* \neq e$.

- $e$ is ambiguous in $(Q, b)$ as above. If the flip is bump, there are two possibilities for $e^*$. Then the inversion of $e^*$ can be performed on $(T, b)$, possibly after having inverted the ambiguous edge $e'$. If the flip is forced ambiguous, again there are two possibilities for $e^*$ and in every case the inversion of $e^*$ can be performed on $(T, b)$.

- $e$ is ambiguous in one of the two triangles of $(Q, b)$ and non ambiguous in the other. The flip is non ambiguous. There are three possibilities for $e^*$. We readily check that in every case the inversion of $e^*$ can be performed on $(T, b)$.

- $e$ is non ambiguous in both triangles of $(Q, b)$. The flip is not forced. We check that the inversion of $e^*$ can be performed on $(T, b)$, possibly after having inverted the ambiguous edge $e'$.

This complete the proof in Case (a).

Case (b): $\chi(S)$ is strictly negative and odd. The distinguished triangulations $T_n$ of $(S, n)$ have one trapped edge carried by the one-pieced projective plane. Let $T_n^*$ be obtained by flipping its connection edge. $T_n^*$ does not contain any trapped edge and arguing similarly as above, we see that it is inversive. Then the proof is like in Case (a), by using $T_n^*$.

Theorem 3.21 is achieved.

\[ \square \]

4. B-proof of the main Theorem

We are going to implement strategy B. For simplicity we will deal only with the generic case $\chi(S) < 0$. Strictly speaking Remark 2.18 will apply to the case of orientable $S$, as in the non orientable case we will actually adopt a mixture of A and B.

4.1. B-proof when $S$ is orientable. Let $(T, b)$ and $(T, b')$ triangulations of $(S, V)$ such that $\delta(b, b')$ is non empty. By using Lemma 2.25 and Lemma 2.26 it is not restrictive to deal under the following:

Initial assumptions:  (1) $T$ does not contain trapped edges;
(2) Every disoriented edge $e \in \delta(b, b')$ is non ambiguous in both $(T, b)$ and $(T, b')$.

At first we analyze the effects of flipping $e \in \delta(b, b')$ in both $(T, b)$ and $(T, b')$ looking for a decreasing of $|\delta|$ if any. Let $t_1$ and $t_2$ be the two triangles of $T$ which share $e$. As $e$ is non ambiguous in both triangulations, then $e$ is non ambiguous in at least one of the branched triangles $(t_j, b)$ and similarly for the $(t_j, b')$'s. There are two possibilities:

- There is at least one triangle, say $t_1$, such that $e$ is non ambiguous in both $(t_1, b)$ and $(t_1, b')$, so that necessarily $(t_1, b') = (t_1, -b)$.
(2) $e$ is non ambiguous (resp. ambiguous) in $(t_1, b)$ (resp. $(t_2, b)$) while $e$ is non ambiguous (resp. ambiguous) in $(t_2, b')$ (resp. $(t_1, b')$).

Let us analyze the first case.

**Case (1).** Concerning $t_2$, there are three possibilities: it can contains either $k = 0, 1$ or 2 further edges belonging to $\delta(b, b')$. Note that $k = 2$ if and only if $b' = -b$ on the whole of $t_1 \cup t_2$. We say that the disoriented $e$ is (1)bad if $k = 2$ and $e$ is ambiguous in $(t_2, b)$ (hence in $(t_2, b')$). In all other cases we say that $e$ is (1)good. We have

**Lemma 4.1.** Let $e \in \delta(b, b')$ be (1)good. Then by flipping $e$ in both $(T, b)$ and $(T, b')$ we get $(T', b_1)$ and $(T', b_2)$ which still satisfy our assumtions and such that $|\delta|_{decreases by 1}. There are not compatibly branched nutshell containing $e$, hence the flip does not create any trapped edge (see the first row of Figure 5).

**Proof.** First we note that if $f_e$ creates a trapped edge, then necessarily $e$ is internal to some naked nutshell say $N$ in $T$. Now we analyze the situation case by case according to the value of $k = 0, 1, 2$.

![Figure 5. Flipping an untrapped edge, Case (1).](image)

(1) If $k = 0$ both $f_{e, b_1}$ and $f_{e, b_2}$ are non ambiguos and $|\delta|$ decreases by 1. There are not compatibly branched nutshell containing $e$, hence the flip does not create any trapped edge (see the first row of Figure 5).

(2) If $k = 1$, we can choose $f_{e, b_1}$ and $f_{e, b_2}$ in such a way that $|\delta|$ decreases by 1 and the new edge is ambiguous in one of the triangulations obtained so far (see Figure 6, second row). We claim that $f_e$ does not create any trapped edge. Otherwise one among the branched nutshell $(N, b)$ and $(N, b')$ would be good with internal vertex which is either a pit or a source. Then (recall Lemma 2.11 (3)) it would contain an ambiguous internal edge $\tilde{e}$ belonging to $\delta(b, b')$ against our assumption.

![Figure 6. Flipping an untrapped edge, Case (2).](image)
(3) If \( k = 2 \) and \( e \) is \((1)\)good, then necessarily \( e \) is non ambiguous also in \( (t_2, b) \) (hence in \( (t_2, b') \)). We can choose \( f_e, b_1 \) and \( f_e, b_2 \) in such a way that \(|\delta|\) decreases by 1 and the new edge is ambiguous in both triangulations obtained so far (see the left side of the third row of Figure 4). If \( e \) would belong to a naked nutshell in \( T \), then both \( (N, b) \) and \( (N, b') \) are good with internal vertex which is either a pit or a source, and we can argue as above. So \( f_e \) does not create any trapped edge.

The Lemma is proved. 

Concerning the \((1)\)bad situation, we readily realize that

**Lemma 4.2.** If \( e \) is \((1)\)bad, then by flipping \( e \) we keep the same value of \(|\delta|\). The new edge is non ambiguous in both triangulations obtained so far (see the right side of the third row of Figure 4). If the flip creates a trapped edge, then both nutshells \( (N, b) \) and \( (N, b') \) are bad and all edges of \( N \) belong to \( \delta(b, b') \)

Let us turn now to the second case.

**Case (2).** It is easy to see that necessarily the boundary of \( t_1 \cup t_2 \) contains exactly a couple \( e_1 \subset t_1, e_2 \subset t_2 \) of edges which do not belong to \( \delta(b, b') \). There are two possibilities, see Figure 4: 

(i) \( e_1 \) and \( e_2 \) are consecutive edges in the (abstract) quadrilateral \( t_1 \cup t_2 \). In this case we can flip \( e \) and \(|\delta|\) decreases by 1. If this creates a trapped edge, then as usual \( e \) is an internal edge of a nutshell \( N \), and both \( (N, b) \) and \( (N, b') \) are bad with the two boundary edges which do not belong to \( \delta(b, b') \).

(ii) \( e_1 \) and \( e_2 \) are opposite edges in the (‘abstract”) quadrilateral \( t_1 \cup t_2 \) and their orientations are necessarily compatible, that is they extend to an orientation of the whole boundary of the quadrilateral. Then by flipping \( e \) we keep the same value of \(|\delta|\). The flip does not create any trapped edge.

If we are in case (i) and the flip does not create a trapped edge, then we say that \( e \) is \((2)\)good. The other cases are \((2)\)bad. Let us call generically \textit{good} an edge \( e \in \delta(b, b') \) which is either \((1)\)good or \((2)\)good. Otherwise let us say that it is \textit{bad}. Summarizing the above discussion we can strenghten the “Initial assumptions” as follows:

**Lemma 4.3.** In order to prove Theorem 1.2 it is not restrictive to deal under the following

**All-bad assumptions:**

\((1)\) \( T \) does not contain trapped edges;

\((2)\) Every edge \( e \in \delta(b, b') \) is non ambiguous in both \( (T, b) \) and \( (T, b') \);

\((3)\) Every edge in \( \delta(b, b') \) is bad.

Now we show that the all-bad assumptions are quite constraining.

**Lemma 4.4.** Let \( (T, b) \) and \( (T, b') \) satisfy the all-bad assumptions. Then every \( e \in \delta(b, b') \) is actually \((2)\)bad.

**Proof.** Assume that there exists a \((1)\)bad edge \( e \in \delta(b, b') \). This propagates to all edges of \( T \): eventually \( b' = -b \) and all edges should be \((1)\)bad. We want to show that this is impossible. We note that there are not adjacent (necessarily bad) nutshells because a common boundary edge should be ambiguous, hence good. Let \( v \) be a vertex of \( T \) which is not the center of a nutshell and analyse the possible configurations of its (abstract) developed star \( St(v, b) \) \( (T, b) \) (the one in \( (T, b') \) is obtained by just reversing the orientations).

**Claim:** Every such a star \( St(v, b) \) has the following qualitative configuration. Every edge in the boundary of the star is ambiguous in the respective triangle. \( St(v, b) \) can contains an even number of bad nutshells sharing the vertex \( v \) (necessarily even because otherwise the star would contain a good edge) and the orientations of their boundaries alternate (with respect to the reference orientation of \( S \)). The edges of the boundary of the star between two consecutive nutshells have compatible orientations as well as the internal edges of their respective triangles are all either ingoing or outgoing with respect to the central vertex \( v \). Moving along the boundary of the star, boundary orientations and “in-out” types switch each time we pass a nutshell. In particular if there are no nutshells, then the boundary of \( St(v, b) \) is an oriented circles.
Proof of the Claim. Assume that there is an edge \( e \) in the boundary of \( St(v, b) \) which is non ambiguous in the relative triangles. Let us try to complete the star by moving along its boundary in the direction of the orientation of \( e \). Possibly after some boundary edges which are oriented like \( e \) and are ambiguous in the respective triangle (with internal edges pointing towards the central vertex \( v \)), we necessarily find either a boundary edge \( e' \) which is non ambiguous in the respective triangle and has opposite orientation with respect to \( e \), or a bad nutshell (whose boundary orientation is uniquely determined). We see that in both case there is an internal edge which is ambiguous in the star, hence good.

\[ \square \]

Now we can conclude by noticing that for every \( St(v, b) \) with the properties stated in the Claim there is a vertex \( v' \) in the boundary of the star such that the boundary of \( St(v', b) \) contains an edge which is non ambiguous in the relative triangle of \( St(v', b) \). Lemma 4.4 is proved.

\[ \square \]

Remark 4.5. The hypothesis that \( S \) is orientable has been already employed in order to limit the way a trapped edge can be produced; it will be important also in the rest of the discussion; the key point is that it prevents that the stars of the disoriented (2)bad edges (in an all-bad configuration) glue each other at edges not belonging to \( \delta(b, b') \) producing a Möbius strip. For example, the opposite edges not belonging to \( \delta(b, b') \) in a basic (2)bad configuration \( (t_1 \cup t_2, b), (t_1 \cup t_2, b') \) cannot be identified.

Definition 4.6. (1) A terminal (2)bad type is either:
- A couple of (2)bad nutshell \((N, b), (N, b')\) such that \(|\delta(b, b')| = 2\) and the boundary edges do not belong to \(\delta(b, b')\).
- A couple of triangulated annuli \((A, b), (A, b')\) obtained from a basic (2)bad configuration \((t_1 \cup t_2, b), (t_1 \cup t_2, b')\) by identifying the opposite boundary edges of the quadrilateral which belong to \(\delta(b, b')\). For the resulting triangulations we have \(|\delta(b, b')| = 2\) and the boundary of \((A, b)\) (similarly for \((A, b')\)) is formed by two circles each containing one vertex and endowed with opposite orientations.

(2) A couple \((T, b) (T, b')\) is said terminal all-bad if verify the all-bad assumptions and every disoriented edge \(e \in \delta(b, b')\) is contained in a terminal (2)bad type.

![Figure 7. Terminal move.](image)

We have

Lemma 4.7. In order to prove Theorem 1.2 it is not restrictive to deal with terminal all-bad couples \((T, b), (T, b')\).

Proof. Let \((T, b), (T, b')\) verify the all-bad assumptions. If \((T, b)\) presents a pattern as in the top of Figure 7 (we stipulate that the dashed edges do not belong to \(\delta(b, b')\)) we can perform the sequence of \(b\)-flips suggested by descending the rows of the picture (the corresponding flips on \((T, b')\) are understood).
We eventually get \((T',b_1)\) and \((T',b_2)\) such that \(|\delta(b_1,b_2)| = |\delta(b,b')| - 1\) and the number of \((2)\)bad \(t_1 \cup t_2\) decreases by 1. We stop when we reach a terminal configuration. \(\square\)

We can state now the conclusive lemma.

**Lemma 4.8.** Let \((T,b)\) and \((T,b')\) be a terminal all-bad couple. Then we can find sequence of inversions of ambiguous edges \((T,b) \Rightarrow (T,\tilde{b}), (T,b') \Rightarrow (T,\tilde{b}')\) such that \(|\delta(\tilde{b},\tilde{b}')| < |\delta(b,b')|\).

By iterating all the above procedure starting from \((T,\tilde{b}), (T,\tilde{b}')\) we eventually get \(|\delta| = 0\) and the main theorem follows.

**Proof of Lemma 4.8.** Let \(e\) be an edge not belonging to \(\delta(b,b')\) and contained in the star of a disoriented \(\tilde{e}\) in \(\delta(b',b')\). If \(e\) is ambiguous, let us invert it in both \((T,b)\) and \((T,b')\). If \(e\) was in the boundary of a bad nutshell, after the inversion the nutshell becomes good and we can apply Lemma 2.11. If \(e\) was in the boundary of a bad configuration \((t_1 \cup t_2, b), (t_1 \cup t_2, b')\), then after the inversion \(\tilde{e}\) becomes ambiguous (recall Remark 4.5) and we can invert it in \((T,b)\) to decrease \(|\delta|\) by 1. So, if \(e\) is ambiguous we have done. Assume that \(e\) is not ambiguous. Then \(e\) is the edge of a (abstract) triangle which is entirely formed by edges not belonging to \(\delta(b,b')\). Let \(v\) be the vertex of this triangle which does not belong to \(e\). We realize that there is an internal edge of \(St(v,b)\) which is ambiguous. Then by successive inversions of ambiguous edges we eventually make \(e\) ambiguous as we can conclude as above.

The B-proof in the orientable case is complete.

**4.2. AB-proof in the non orientable case.** In the non orientable case we do not use as initial situation an arbitrary couple of triangulations (without trapped edges) \((T,b), (T,b')\) of \((S,V)\). We partially specialize the choice by requiring that \(T\) respects a decomposition \(S = S_0 \# N\) where \(S_0\) is orientable and \(N\) is either a Klein bottle or a projective plane. More precisely we require that \(T\) is the union of a one-pierced non orientable brick with the distinguished triangulation already used in the proof of Theorem 3.1 and an arbitrary triangulation of a one-pierced \(S_0\), provided that they coincide at the connection loop of the non orientable brick. Then the orientable B-proof applies with minor changes to the one-pierced \(S_0\). \(\square\)

**5. On the ideal sliding equivalence**

The sliding transits have been already considered in Section 5 of [1]. There we have been mainly concerned with the completed sliding equivalence, more precisely with triangulations of a given oriented surface with arbitrary number of vertices, considered all together. Here we stress the ideal set up, we consider also non orientable surfaces and we introduce some new constructions (for instance the so-called horizontal foliation). The basic idea is that every branched triangulation of \((S,V)\) carries some remarkable structures of differential topological type which are preserved by the ideal sliding while they can be widely modified by the bump transits accordingly with Theorem 1.2.

**5.1. The transverse foliations carried by a branched triangulations.** Given \((S,V)\) we recall that \(S\) denotes the surface with boundary obtained by removing a family of disjoint open 2-disks centred at each \(v \in V\). We are going to consider possibly singular foliations on \(S\) or \(S\). Every such a foliation can be obtained by integration of some field of tangent direction (a tangent vector field if the leaves are oriented). We will say that two foliations are homotopic (isotopic) if they are obtained by integration of homotopic (isotopic) fields. Let \((T,b)\) be a branched triangulation of \((S,V)\). First we are going to show that \((T,b)\) carries in a canonical way a pair of regular transverse foliations \((V,H)\) on \(S\), called respectively the vertical and the horizontal foliations. \(V\) is always oriented. If \(S\) is oriented then also \(H\) can be oriented in such a way that every intersection point has intersection number equal to 1. \(V\) and \(H\) can be extended to singular foliations \(v\) and \(h\) defined on the whole of \(S\). They share the singular set \(Z\) which consists of the vertices \(v \in V\) where the index \(1 - d_0(v) \neq 0\). The two foliations are transverse on \(S' \setminus Z\). If \(S\) is oriented, then both \(v\) and \(h\) are oriented; if the singularity indices are all non positive, so that \(S\) is of genus \(g \geq 1\), then \((v,h)\) looks like the couple of vertical
and horizontal foliations of the square of an Abelian differential on a Riemann surface. Let us pass to the actual definition.

**Figure 8.** The vertical foliation on $S_V$.

**The vertical foliation** $\mathcal{V} = \mathcal{V}(T, b)$. Let $(T, b)$ be as above. Up to isotopy, the intersection of $S_V$ with every (abstract) triangle $t$ of $T$ is a “truncated triangle” $\bar{t}$, i.e. a hexagon with 3 internal “long” edges (each one contained in an edge of $T$) and 3 “short” edges contained in the boundary $\partial S_V$. The short edges are in bijection with the corners of triangles of $T$; some are labelled by 1 in accordance with the associated corners. The union of the hexagons form a cell decomposition of $S_V$, by the restriction to the long edges of the gluing in pairs of the abstract edges of $T$. The union of the short edges form a triangulation of $\partial S_V$. By using the branching $b$, we are going to endow every hexagon $\bar{t}$ of an oriented foliation $\mathcal{V}(\bar{t}, b)$. These constitute the tiles of a puzzle that once assembled realizes $\mathcal{V}$. This is illustrated on the top-left of Figure 8. In fact $\mathcal{V}(\bar{t}, b)$ is the restriction to $(\bar{t}, b)$ of a classical Whitney field which can be defined explicitly in terms of barycentric coordinates (see [7]). In the dual viewpoint, recall that the spine $(\theta, b)$ of $S_V$ is an embedded transversely oriented train track in $S_V$; the foliation $\mathcal{V}$ is positively transverse to it (on the top-right of Figure 8 we see the dual picture corresponding to the puzzle tile). The foliation $\mathcal{V}$ has remarkable properties.

**Definition 5.1.** A traversing foliation on $S_V$ is a foliation with oriented leaves such that:

1. Every leaf of $\mathcal{F}$ is a closed interval which intersects transversely $\partial S_V$ at its endpoints.
2. There is a non empty finite set of exceptional leaves of $\mathcal{F}$ which are simply tangent to $\partial S_V$ at finite number of points.

$\mathcal{F}$ is generic if every exceptional leaf is tangent to the boundary at one point.

Then it is not hard to see that:

**Proposition 5.2.** (1) The vertical foliation $\mathcal{V}$ associated to a branched spine $(T, b)$ of $(S, V)$ is a generic traversing foliation on $S_V$. The exceptional leaves of $\mathcal{V}$ are in bijection with the 1-labelled short edges of $\partial S_V$: every exceptional leaf is tangent to the interior of the associated edge. $\mathcal{V}$ is uniquely determined up to isotopy.

(2) The dual branched spine $(\theta, b)$ of $S_V$ intersects transversely all leaves of $\mathcal{V}$. Every exceptional leaf intersects transversely $\theta$ at two points. A leaf passing through a singular point of $\theta$ is generic and intersects $\theta$ at one point. A generic leaf intersects $\theta$ at one or two points. In the second case it is contained in a quadrilateral in $S_V$ vertically bounded by an exceptional leaf and a leaf passing through a singular point of $\theta$.

(3) Every generic traversing foliation on $S_V$ can be realized as the vertical foliation of some branched spine of $(S, V)$.

**Boundary bicoloring.** Given a traversing foliation $\mathcal{F}$ of $S_V$, denote by $X = X_\mathcal{F} \subset \partial S_V$ the set of tangency points of the exceptional leaves. $\mathcal{F}$ determines a bicoloring of the components of $\partial S_V \setminus X$, denoted by $\partial \mathcal{F}$; let us say that a component $c$ is white (resp. black) if the foliation is ingoing (outgoing).
along $c$. If $S$ is oriented the color can be encoded by an orientation, in the sense that a black component keeps the boundary orientation of $\partial S_V$ (according to the usual rule “first the outgoing normal”), while a white one has the opposite orientation. In the bottom of Figure 8 we see the oriented enhancement of the $\mathcal{V}$-tile (we stipulate that in the picture the $b$-orientation of the triangle agrees with the orientation of $S_V$; we obtain the picture for the negative branching $-b$ by just inverting all arrows).

![Figure 8](image1)

**Figure 8.** The horizontal foliation on $S_V$.

The horizontal foliation $\mathcal{H} = \mathcal{H}(T, b)$. Alike $\mathcal{V}$ we define $\mathcal{H}$ as the result of a puzzle. On the top of Figure 9 we show the foliated hexagon and the corresponding dual picture. In general $\mathcal{H}$ is not oriented. If $S$ is oriented then $\mathcal{H}$ is oriented as well; on the bottom of Figure 9 we show the oriented version of the tile. Now we realize that:

1. $\mathcal{V}$ and $\mathcal{H}$ are transverse foliations.
2. Let $Y$ be the union of the 1-labelled short edges. Then every component of $\partial S_V \setminus Y$ is contained in a leaf of $\mathcal{H}$, while $\mathcal{H}$ is transverse to the interior of every 1-labelled short edge.
3. If $S$, hence $\mathcal{H}$, is oriented then every boundary component in a leaf is oriented like the leaf and is contained in a component of $\partial S_V \setminus X$; these orientations propagate to the whole components of $\partial S_V \setminus X$ and reproduce the bicoloring orientation. $\mathcal{V}$ intersects $\mathcal{H}$ with intersection number equal to 1 everywhere.
4. The pair $(\mathcal{V}, \mathcal{H})$ is uniquely defined up to isotopy.

![Figure 9](image2)

**Figure 9.** The horizontal foliation on $S_V$.

If $S$ is oriented there is another way to realize the foliations $(\mathcal{V}, \mathcal{H})$ by using the 2D case of a result of [13]. This can be described as follows. Take a (non embedded) copy $\theta^*$ of the oriented train track $\theta$ and
Consider the oriented branched surface $F := \theta^* \times [-1, 1]$; this carries the vertical foliation $V^*$ with leaves of the form $\{x\} \times [-1, 1]$ and the horizontal one $H^*$ with branched leaves of the form $\theta^* \times \{y\}$. Then one can find an embedding of $S_V$ into $F$ which preserves the orientation and such that $V$ is just the restriction of $V^*$ to $S_V$. This is suggested in Figure 10. Also $H^*$ restricts to a regular foliation of $S_V$ which becomes our final horizontal foliation $H$ after a suitable homotopy.

**Extension to singular foliations.** Let $(S, V)$ be as usual. Let us define some notions. A function

$$i : V \to \{n \in \mathbb{Z} | n \leq 1\}$$

is *admissible* if

$$\chi(S) = \sum_v i(v).$$

For every admissible function $i$, a *vertical foliation of type $i$* on $(S, V)$ is an oriented singular foliation $v$ that verifies by definition the following properties:

1. The singular set $Z$ of $v$ consists of the $v \in V$ such that $i(v) \neq 0$.
2. If $i(v) \neq 0, 1$, then the local model of $v$ at $v$ is given by the vertical foliation at 0 of the quadratic differential $z^{-2i(v)}dz^2$. If $i(v) = 1$, the local model is given by the integral lines at 0 of the gradient of either the function $|z|^2$ or $-|z|^2$.

An *horizontal foliation of type $i$* on $(S, V)$ is a non oriented singular foliation $h$ that verifies by definition the following properties:

1. The singular set $Z$ of $h$ consists of the $v \in V$ such that $i(v) \neq 0$.
2. If $i(v) \neq 0, 1$, then the local model of $h$ at $v$ is given by the horizontal foliation at 0 of the quadratic differential $z^{-2i(v)}dz^2$. If $i(v) = 1$, the local model is given by the level curves at 0 of the function $|z|^2$.

A *transverse pair of foliations of type $i$* is a pair $(v, h)$ such that

1. $v$ and $h$ are vertical and horizontal foliations of type $i$ respectively.
2. The two foliations are transverse on $S \setminus Z$ and, case by case, the above local models at the singular points hold simultaneously for both $v$ and $h$.
3. If $S$ is oriented we require furthermore that also $h$ is oriented in such a way that $v$ and $h$ intersect everywhere with intersection number equal to 1.

**Lemma 5.3.** For every ammissible function $i$ there are transverse pairs of foliations of type $i$ on $(S, V)$.

**Proof.** It is enough to prove that there exists a vertical foliation of type $i$, for we can take as transverse horizontal foliation the orthogonal one with respect to a suitable auxiliary riemannian metric on $S$. Let $S_Z$ be the surface with boundary obtained by removing from $S$ a small 2-disk around every $v \in Z$. Consider the foliation on a neighbourhood of $\partial S_Z$ determined by the $i$-local model at singular points. By a simple variation of Hopf theorem we realizes that it extends to the whole of $S_Z$ without introducing new singularities.

Such transverse pairs of type $i$ are considered up homotopy through transverse pairs of type $i$ which is locally an isotopy at the singular points. We denote by

$$\mathcal{TP}(S, V, i)$$

the so obtained quotient set. Set

$$\mathcal{TP}(S, V) = \cup_i \mathcal{TP}(S, V, i).$$

Finally denote by

$$\mathfrak{S}(S_V)$$

the quotient set of the set of generic traversing foliations on $S_V$ considered up to homotopy through traversing (not necessarily generic) foliations.

The following theorem summarizes some main features of the $s$-transit equivalence.
Theorem 5.4. For every \((S, V)\):

1. The correspondence \((T, b) \rightarrow V(T, b)\) induces a well defined bijection
   \[ \tau : S^{id}(S, V) \rightarrow \mathfrak{X}(S_{V}). \]

2. For every \((T, b)\) consider the admissible function \(i_{b}(v) = 1 - d_{b}(v)\). Then the associated pair of transverse foliations \((V(T, b), H(T, b))\) on \(S_{V}\) extends to a transverse pair \((v(T, b), h(T, b))\) of type \(i_{b}\) on \((S, V)\) in such a way that this induces a well defined map
   \[ p : S^{id}(S, V) \rightarrow TP(S, V) . \]

3. Fix a base point \(v_{0} \in V\). Assume that the set of admissible functions such that \(i_{b}(v_{0}) = 0\) is non empty and denote by \(TP_{0}(S, V)\) the corresponding subset of \(TP(S, V)\). Then the set of triangulations \((T, b)\) of \((S, V)\) such that \(i_{b}(v_{0}) = 0\) is non empty and denoting by \(S^{id}_{0}(S, V)\) the corresponding subset of \(S^{id}(S, V)\) we have that the restricted map \(p : S^{id}_{0}(S, V) \rightarrow TP_{0}(S, V)\) is bijective.

Proof. The fact that the map \(\tau\) in (1) is well defined and onto follows just by looking at the sliding flips and from (3) of Proposition 5.2.

Once the extension \((V(T, b), H(T, b)) \rightarrow (v(T, b), h(T, b))\) will be established just below, the fact that the map \(p\) of (2) is well defined follows from the fact that \(\tau\) is well defined.

The fact that \(\tau\) in (1) is injective as well as item (3) are actually simpler 2D versions of results established in [5] and [6] for branched spines of 3-manifolds with non empty boundary. In [5] we essentially considered the case of closed manifolds, that is when the boundary consists of one 2-sphere. In [6] we faced the general case with minor changes. So we limit here to illustrate the main points, referring to the harder proofs in 3D.

The injectivity of \(\tau\) is the 2D analogous of Theorem 4.3.3 of [6]. By transversality we can assume that the homotopy is generic, that is, it contains only a finite number of non generic traversing foliations, each one containing one exceptional leaf which is tangent at two points of \(\partial S_{V}\). Then we analyze how two generic traversing foliations close to a non generic one are related to each other and we realize that the sliding \(b\)-flips cover all possible configurations.

As for the extension of \((V, H)\) we look at the configuration of this foliations at each component \(C\) of \(\partial S_{V}\) which is also the boundary of a small disk \(D\) in \(S\) centred at one vertex \(v \in V\) (see Figure 11). If \(i_{b}(v) = 0\), there are exactly two exceptional leaves of \(V\) tangent to \(C\); then we can extend \(V\) without singularities through \(D\) respecting the bicoloring of \(C\) and manage similarly on \(H\). In the other case we easily realize that up to isotopy the configuration of \((V, H)\) at \(C\) is the restriction of the configuration of the local models that we can assume to be carried by \(D\). The arbitrary choices in implementing the construction are immaterial if \((v, h)\) are considered up to kind of homotopy stated above.
Nevertheless, again when every vertical foliation \( v \) occurring in \( TP_0(S, V) \) is onto. The key point is to prove that every vertical foliation \( v \) occurring in \( TP_0(S, V) \) is realized by means of a triangulation \( (T, b) \) with the given distribution of \( d_b(v) \)'s. This is the 2D counterpart of Proposition 5.1.1 of \[3\]. The proof is based on Ishii's notion of flow spines \[10\]. Let \( A := V \setminus \{v_0\} \) and define \( S_A \) as usual, so that \( S_V \subset S_A \). In general \( v \) is not traversing \( S_A \). Qualitatively the idea is to detect an embedded 2-disk \( D \) in the interior of \( S_A \), centred at some point \( v_0' \), such that \( v \) becomes traversing \( S_V \) and generic, where \( V' = \{v_0'\} \cup A \), and such that only two exceptional leaves are tangent to \( \partial D \). This is carried by a branched triangulation of \((S, V')\). Finally, via the homogeneity of \( S \), we get the desired triangulation \((T, b)\) of \( S_V \). The injectivity of the restricted map \( p \) is the counterpart of Theorem 5.2.1 of \[3\]. The basic idea is to 'cover' any homotopy connecting \( v(T, b) \) with \( v(T', b') \) with a chain of flow-spines connecting \((T, b)\) with \((T', b')\) such that the traversing foliation associated to one is homotopic through traversing foliation to the traversing foliations of the subsequent.

\[\Box\]

**Remark 5.5.** Every \( TP(S, V, i) \) is an affine space over \( H_1(S_Z; \mathbb{Z}) \), \( Z \) being the singular set prescribed by \( i \). So in general \( p : S_0^d(S, V) \to TP_0(S, V) \) is a bijection between infinite sets.

### 5.2. On the non ambiguous transit.

In 3D the notion of non ambiguous structure, defined indeed in terms of transit of \emph{pre-branchings} rather than of branchings \[1, 4\], gives rise to non trivial examples of intrinsic interest. In 2D the intrinsic content of the \emph{na}-relation is not so evident.

**Example 5.6.** Let \( S \) be the torus and \( |V| = 1 \). Let \( T \) be the triangulation of \((S, V)\) as in the proof of Proposition \[2, 4\]. One checks by direct inspection that for every \( b \), \( T \) no edge of \( T \) supports a non ambiguous flip. This holds for \emph{every} triangulation \( T' \) of \((S, V)\) because all these triangulations are equivalent to each other up to diffeomorphism of \((S, V)\). Hence in this case the \emph{na}-transit equivalence is nothing else than the identity relation. On the other hand, we check that the branchings on \( T \) which share the same decomposition \( S = S_+ \cup S_- \) are not \emph{s}-equivalent to each other.

Referring to Proposition \[2, 4\] one would conjecture, better ask whether two branched triangulations \((T, b)\) and \((T', b')\) of \((S, V)\) are \emph{na}-equivalent if and only if they are \emph{s}-equivalent and share the decomposition \( S = S_+ \cup S_- \), provided that \( S \) is oriented. However, while by (3) of Theorem \[5\] the quotient set \( S_0^d(S, V) \) has a nice intrinsic content, the decomposition \( S = S_+ \cup S_- \) is not very transparent. Nevertheless, again when \( S \) is oriented, we will point out a further structure preserved by the \emph{na}-transits with a bit more intrinsic flavour. We set \( H^\Delta_{\text{na}} \) to denote the simplicial (or cellular) homology of a complex, \( H_\ast \), the singular homology of the underlying topological space. Similarly we set \( H^\Delta_{\text{na}} \), \( H^\ast_{\text{na}} \) for the cohomology. The inclusion of \( \theta \) in \( S_V \) induces an isomorphism

\[ H^\Delta_{\text{na}}(\theta, b; \mathbb{R}) \cong H_1(S_V; \mathbb{R}) \]

and via elementary Poincaré duality we have

\[ H^\Delta_{\text{na}}(\theta, b; \mathbb{R}) \cong H^1_{\text{na}}(T, b; \mathbb{R}) \cong H^1(S; \mathbb{R}) \]

Moreover, \( H^\Delta_{\text{na}}(\theta, b; \mathbb{R}) = Z^\Delta_{\text{na}}(\theta, b; \mathbb{R}) \) where this last denotes the space of 1-cycles on \((\theta, b)\). Every \( z \in Z^\Delta_{\text{na}}(\theta, b; \mathbb{R}) \) consists in giving each \( b \)-oriented edge \( e \) of \((\theta, b)\) a weight \( z(e) \in \mathbb{R} \) in such a way that at every switching vertex of \( \theta \) the three weights around the vertex verify the corresponding switching condition of the form \( z(e_0) = z(e_1) + z(e_2) \). These cycles transit along every \( b \)-flip, so that for every composite \( b \)-transit \((T, b) \Rightarrow (T', b') \) it is defined an isomorphism

\[ \alpha : Z^\Delta_{\text{na}}(\theta, b; \mathbb{R}) \to Z^\Delta_{\text{na}}(\theta', b'; \mathbb{R}) \]

Set

\[ \mathcal{M} = \mathcal{M}_{(T, b)} = \{ z \in Z^\Delta_{\text{na}}(\theta, b; \mathbb{R}) \mid \forall e \in \theta^{(1)}, \ z(e) \geq 0 \}; \quad \mathcal{M}^+ = \{ z \in Z^\Delta_{\text{na}}(\theta, b; \mathbb{R}) \mid \forall e \in \theta^{(1)}, \ z(e) > 0 \}. \]

Every \( z \in \mathcal{M} \) can be interpreted as a \emph{transverse measure} on the horizontal foliation \( \mathcal{H} \) or on the singular foliation \( h \). By taking into account the arbitrary choices in the realizations of \((V, \mathcal{H})\) and \((v, (\mathcal{N}, \mathcal{H})\) we radily have
Proposition 5.7. (1) For every \( z \in M \), the measured foliations \((H, z), (\mathfrak{h}, z)\) are uniquely determined up to measure equivalence (in particular this means that \((\mathfrak{h}, z)\) well define a measure spectrum on the set \( S \) of isotopy classes of simple closed curves on \( S \)).

(2) If we denote by \( M(\mathfrak{h}) \) the set of transverse measures on \( \mathfrak{h} \) up to measure equivalence, the above correspondence well defines a map

\[
m = m_{(T,b)} : M_{(T,b)} \to M(\mathfrak{h}).
\]

After a look at the na-transits we readily have

Proposition 5.8. If \( (T, b) \) and \( (T', b') \) are na-transit equivalent, then the maps \( m_{(T,b)} \) and \( m_{(T',b')} \) have the same image. More precisely, there is a bijection \( \alpha : M_{(T,b)} \to M_{(T',b')} \) such that \( \alpha(M^+_{(T,b)}) = M^+_{(T',b')} \) and \( m_{(T,b)} = m_{(T',b')} \circ \alpha \).

References

[1] S. Baseilhac, R. Benedetti, *Non ambiguous structures on 3-manifolds and quantum symmetry defects*, Quantum Topology, Volume 8, Issue 4, 2017, pp. 749–846.
[2] S. Baseilhac, R. Benedetti, *On the quantum Teichmüller invariants of fibred cusped 3-manifolds*, Geometriae Dedicata, 197(1), 1–32, 2018.
[3] S. Baseilhac, R. Benedetti, *Analytic families of quantum hyperbolic invariants*, Algebraic and Geometric Topology, 15 (2015) 1983–2063.
[4] R. Benedetti, *Ideal triangulations of 3-manifolds up to decorated transit equivalences*, preprint on arXiv, 2019.
[5] R. Benedetti, C. Petronio, *Branched Standard Spines of 3-manifolds*, Lect. Notes Math. 1653, Springer (1997).
[6] R. Benedetti, C. Petronio, *Reidemeister-Turaev torsion of 3-dimensional Euler structures with simple boundary tangency and pseudo-Legendrian knots*, Manuscripta Math. 106 (1) (2001) 13–61.
[7] S. Halperin, D. Toledo, *Stiefel-Whitney homology classes*, Ann. of Math. (2) 96 (1972), 511–525.
[8] A. Hatcher, *Algebraic Topology*, 2001.
[9] A. Hatcher, *On triangulations of surfaces*, Topology Appl. 40 (1991), no. 2, 189–194.
[10] I. Ishii, *Flows and spines*, Tokyo J. Math. 9 (1986) 505–525.
[11] M. Lackenby, *Taut ideal triangulations of 3-manifolds*, Geom. Topol. 4 (2000) 369–395.
[12] L. Mosher, *Tiling the projective foliation space of a punctured surface*, Trans. Amer. Math. Soc. 306 (1988), no. 1, 1–70.
[13] D. Gillman, D. Rolfsen, *The Zeeman conjecture for standard spines is equivalent to the Poincaré conjecture*, Topology Vol. 22, no. 3, (1983) 315–323.
[14] S. Tillmann, S. Wong, *An algorithm for the Euclidean cell decomposition of a cusped strictly convex projective surface*, J. Comput. Geom. 7 (2016), no. 1, 237–255.