Identities between Quantum Field Theories in Different Dimensions

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ABSTRACT

Review of some old and relatively new ideas surrounding the subject of AdS/CFT correspondence, generalized tau-functions and possible equivalences between a priori different quantum field theories.

This talk is motivated by the recent discussions around the subject of the “ADS/CFT correspondence”, discovered in [1] and reformulated in [2], [3]. In [3] the issue was actually reduced to the problem of various representations of generalized \( \tau \)-functions, which has been encountered in various other contexts during last years. Particular subject of AdS/CFT correspondence emphasizes the possibility to represent one and the same effective action in terms of quantum field theory models in different dimensions, and it is this aspect of \( \tau \)-function theory that will be briefly reviewed in the present notes.

1. The AdS/CFT correspondence itself was discovered in the context of brane theory. The simplest view on branes is in the framework of “tomography approach” (generalized Radon transform) [4]. One can study any given (quantum) theory, e.g. the entire “Theory of Everything” in its \( D \)-dimensional \( (D=10 \text{ or } 11) \) phase, by looking at the propagation of probe objects in the background fields. Then some basic properties of entire theory, i.e. the properties of background fields, can be recovered from the behavior peculiarities of the probe objects.

If probe objects are particles (0-branes), this idea is realized in conventional tomography devices, used in modern medicine. Inverse Radon transformation of this type is somewhat complicated: to recover the pattern of the body one needs to collect information by sending rays from all possible directions and examine them all together. One can instead make use of the quantum properties of particles, namely their wave properties leading to interference and diffraction, and obtain a holography picture in codimension one (on a “screen”) still carrying complete information about the full multidimensional structure in the bulk. Inverse holography is practically simple (it is enough to shed the light on holography plate) but formally it is still a sophisticated transformation. In any case
tomography with the help of particles does not translate description of original multidimensional theory into a pure particle \((d = 1 + 0)\) problem: additional structures like screens of codimension one are always needed. Moreover, if we switch from technological applications to the fundamental theories and fundamental “laws of nature”, they do not seem to turn into anything reasonably understandable under such particle-tomography transformation.

It is well known that if the probe particles are substituted by probe “relativistic” strings (1-dimensional objects with constant tension), quantum tomography becomes very efficient exactly in application to the fundamental theories. Namely, the fundamental equations of motion of original bulk theory (like Einstein equation \(R_{\mu \nu} = 0\) etc) turn into a symmetry principle for the quantum theory of probe strings: effective \(d = 2\) theory on the string world sheet becomes conformal invariant (under an infinite symmetry of local Weyl transformations of the world sheet metric \(g_{ab}(z) \rightarrow \lambda(z)g_{ab}(z)\)). This symmetry ensures decoupling of non-unitary excitations in effective theory on the world sheet, which would be associated with instability of the probe string in of-shell external fields of the bulk theory. In other words, the fundamental laws (equations of motion of the bulk theory) are the necessary conditions for stability of probe strings. This observation implies that strings (1-branes) are in fact among the true excitations (quasiparticles) of the bulk theory. Of course this does not exclude particles (the 0-branes) as other possible quasiparticles: it is just more sophisticated task to formulate the conditions of their stability, associated symmetry principle is space-time gauge invariance and it is not (yet?) reduced to any simple property of the world-line effective theory. In other words, no simple way is known to formulate gauge symmetry in terms of the first-quantized particles, but when particles are substituted by strings gauge invariance becomes related to the requirement of \(d = 2\) conformal invariance.

An old natural question is what happens when probe strings (1-branes) are further substituted by relativistic membranes (2-branes with constant tension) and higher-dimensional \(p\)-branes. For some time, because of concentration on the beautiful studies of strings, this subject was not in the center of investigation and only recently it attracted new attention, when it became widely recognized that stringy objects can not represent the full set of quasiparticles of the Theory of Everything in all its possible phases. Unfortunately, approaches developed in application to strings are not quite sufficient to attack the problems of higher-dimensional branes (as ordinary field theory technique developed for the study of particles is not quite sufficient for exhaustive description of strings).

The key element of the string theory is the possibility to switch from the Nambu-Goto quantum measure

\[
e^{-Area} = \exp \left( - \int \sqrt{\det [G_{\mu \nu}(x)]} \partial_a x^\mu \partial_b x^\nu |d^2 z\right) \tag{1}
\]

in summation over string \(d = 2\) world sheets embedded into D-dimensional
space-time to the Polyakov measure like

$$\exp \left( - \int [G_{\mu\nu}(x) \partial_\alpha x^\mu \partial_\beta x^\nu] g^{ab} \sqrt{g} d^2 z \right)$$  \tag{2}$$

in summation over embeddings and over world-sheet geometries $g_{ab}$ (including world-sheet topologies). This is not a formal transformation (like it is in the case of particles), but rather a physical principle, which allows one to introduce a relevant quantum object which can really play the role of a stable quasiparticle – at least in the adequate (on-shell) background (bulk) geometries $G_{\mu\nu}(x)$.

Such redefinition is even more important for membranes and generic branes. The stringy measure $e^{-\text{Area}}$ reflects the fact that the string energy is proportional to its length. Similarly, for membrane the naive measure $e^{-\text{Volume}}$ would follow from the fact that membrane energy is proportional to its area. However, such membrane can not exist as a stable quantum object: unlike string, membrane can be strongly deformed without changing its area (a very high but narrow pick can have tiny area) and such strong fluctuations can not be damped by the naive measure. Thus an analogue of the physical substitute (1) → (2) is even more important for generic branes than it was for strings.

Unfortunately, the brane analogue of the measure (2),

$$\mu_{p}\{A\} = \exp \left( - \sum_I A_I \int O_I(\Phi(z)) d^{p+1}z \right)$$  \tag{3}$$

is not yet discovered and there is no available way for deductive presentation of the theory of probe branes. One can instead try various guesses and use them in the search for the proper principles dictating the choice of the set of the fields $\Phi(z)$ on the brane world-volume, and the vertex operators $O_I(\Phi(z))$, associated with particular background fields $A_I$ of the main theory in the bulk.

The first immediate guess is that effective world-volume theory is much simpler in particularly adjusted backgrounds than in generic circumstances. For example, the string model is considerably simplified if $G_{\mu\nu}(x)$ in (2) is not just some solution to Einstein equations $R_{\mu\nu} = 0$, but if it is, say, the flat solution, $G_{\mu\nu}(x) = \delta_{\mu\nu}$. Then quantum theory with the weight (3) becomes essentially a theory of free $d = 2$ fields $x^\mu(z)$ and all the correlators in this theory can be obtained by application of Wick theorem. One can imagine that the same should be true for generic branes: in certain backgrounds the brane model is drastically simplified and it can even happen that the relevant quantum theory becomes that of the free fields and some sort of Wick theorem is applicable. The most straightforward way to find such distinguished backgrounds would be to look at the back reaction of probe brane on the bulk theory. The probe object is of course a source of the fields in the bulk, but normally “probe” means that these emitted fields are neglected. Still the shape of emitted fields can be looked at in order to determine distinguished backgrounds: background of its own fields...
is normally the one in which the object feels “most comfortable”. This is a kind of the necessary condition for the self-consistency of a quasiparticle definition: it (quasiparticle) can be destabilized by arbitrarily imposed external fields, but it should not be self-destroyable by its own fields. Of course, this is not a rigorously formulated statement, but this argument provides a possible direction for the search of distinguished backgrounds, presumably preferred by the brane theory. The \( \text{AdS}_{p+2} \times \text{S}^{D-p-2} \) backgrounds for supersymmetric branes are exactly of this type (one should take into account that supersymmetric Dirichlet branes are not only gravitating but also charged w.r.t various gauge fields, since charges are integer-valued, gravitation effects can not be negligible, therefore preferred backgrounds are not just flat, but rather maximally symmetric non-trivial geometries, composed of spheres and AdS spaces).

The second immediate guess is that the scalar free fields are not the most natural fields beyond two space-time dimensions, and effective world-volume theories in \( d = p + 1 \) dimensions should rather contain \( \left( \frac{p+1}{2} \right) \)-forms (i.e. vector fields for 3-branes, 2-forms for 5-branes etc). If \( k \)-forms are gauge fields, then for \( k \geq 2 \) they are naturally abelian and therefore free. However, for \( k = 1 \) at least the non-abelian gauge symmetry can also occur (what is the proper language to describe non-abelian 2-forms, if any, is still unclear). Remarkably, non-abelian Yang-Mills fields (gauge 1-forms) become essentially free (i.e. the Wick theorem holds) not only when the gauge group \( U(N) \) is small, \( N = 1 \), but also in the opposite limit \( N = \infty \).

2. This follows from the old t’Hooft’s calculus \([3]\). The Green functions of bosonic fields like \( A^\alpha_\mu(z) \) are basically given by the Coulomb law:

$$\langle A^\alpha_\mu(z_1)A^\beta_\nu(z_2) \rangle \sim \left( \delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2} \right) \frac{\delta^{ab}}{|z_1 - z_2|^{d-2}}$$

(4)

However, \( A^\alpha_\mu(z) \) is not the relevant operator in Yang-Mills theory. The simplest gauge invariant operator is rather \( \text{Tr} F^2_{\mu\nu}(z) \) with \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \). This operator contains three different contributions: quadratic, cubic and quartic in \( A \). Accordingly, there are three contributions to the pair correlators of such operators which differ not only in their dependencies on the world-volume coordinates \( z \), but also in those on the coupling constant \( g^2 \) and the size \( N \) of the gauge group. The leading term (with the minimal number of \( A \)-loops) in quadratic contribution is

$$\sim \frac{g^4 N^2}{|z_1 - z_2|^{2d}}$$

(5)

that in the cubic one is

$$\sim \frac{g^6 N^3}{|z_1 - z_2|^{3d-4}}$$

(6)
and that in the quartic one is
\[ \sim \frac{g^8 N^4}{|z_1 - z_2|^{4(d-2)}} \]  

(7)

The \( g^2 \) and \( N \) dependencies of various contributions to the multipoint correlator \( \prod_{i=1}^{2n} \text{Tr} F_{\mu \nu}^2 (z_i) \) are evaluated with the help of Euler theorem, \( V - E + L = 2l \):
\[ g^{2E} N^L = N^{2l-V} \prod_k (g^2 N)^{kV_k/2} \]  

(8)

Here \( V, E, L \) stand for the number of vertices, edges (links) and loops in the Feynman diagram, \( V = \sum_k V_k \) where \( V_k \) is the number of vertices with multiplicity \( k = 2, 3, 4 \) and \( l \) denotes the number of connected components of the diagram.

From (8) it follows that at large \( N \) the maximally disconnected diagrams (with maximal \( l \)) are dominant. Since minimal number of operators \( \text{Tr} F_{\mu \nu}^2 (z) \) in connected sub-diagram is two, we conclude that the 2\( n \)-fold correlator in the large \( N \) limit is reduced to the product of \( n \) elementary pairwise correlators. This is already almost the Wick theorem. One needs only to get rid of sophisticated combinatorial factors (and the mixture of different \( z_i \)-dependencies if \( d \neq 4 \)), which occur if all the three contributions (5-7) are simultaneously taken into account. The three contributions are all comparable in the standard t’Hooft’s limit \( N \to \infty \), with \( g^2 N \) finite. Further simplifications arise (and Wick theorem indeed holds) in two other “double-scaling” large-\( N \) limits, when \( g^2 N \to 0 \) (“abelian” large-\( N \) where multiplicities \( k \)’s should be minimized to \( k = 2 \)) or \( g^2 N \to \infty \) (strongly non-abelian “long wave” or Maldacena’s limit where \( k \)’s should be maximized to \( k = 4 \) and all the derivatives of \( A \)-fields are neglected). The elementary pairwise correlators in these two cases are given by (5) and (7) respectively.

3. Occurrence of the Wick theorem implies the possibility to introduce some new free fields \( \varphi (z) \) instead of original composite operators \( \text{Tr} F_{\mu \nu}^2 (z) \). However, their elementary pairwise correlators – both (5) and (7) – are somewhat unusual from the point of view of the \( z \)-dependencies: both degrees \( 2d \) and \( 4(d - 2) \) (which occasionally coincide for \( d = 4 \), i.e. exactly when vector fields should be most relevant) can seem somewhat un-natural for the free fields in \( d \) space-time dimensions. Remarkably, at least one of these degrees, \( 2d \) possesses a natural interpretation. Namely, the Green function of free scalars in \( d + 1 \)-dimensional space \( \text{AdS}_{d+1} \) (i.e. the space with the metric \( ds^2 = \frac{dt^2}{t^4} + |z|^2 \) and the Ricchi tensor \( g_{mn} = \Lambda g_{mn} \)) is equal to
\[ \langle \varphi (t, z_1) \varphi (0, z_2) \rangle \sim \frac{t^{d+1}}{(t^2 + |z_1 - z_2|^2)^{d/2}} \]  

(9)

which in the limit \( t \to 0 \), i.e. on the \( d \)-dimensional boundary of \( \text{AdS}_{d+1} = \text{AdS}_{p+2} \), turns – after appropriate rescaling of fields – into exactly the relevant formula \( |z_1 - z_2|^{-2d} \).
4. Certain evidence in support of “holography principle” in quantum gravity comes from considerations of propagation in classical gravitational fields. The basic idea is to note that geodesics are non-straight lines in the presence of gravitating bodies and combine this with the fact that all physical objects are in fact gravitating. When combined, these two observations imply that projection from the space on its boundary (a “screen” of codimension one) can have certain peculiarities when gravity is taken into account. The main phenomenon is illustrated in Fig.1. Ordinary projection by the rays orthogonal to the screen leads to information loss, since objects (point A) in the shadow of object B are not seen. However, if one takes into account the fact that the object B is gravitating, the point A can still be seen in orthogonal projection, since the rays (of light) are no longer straight. Certain calculations involving black holes demonstrate that the idea can be formulated in a self-consistent way and indeed reflects some essential property of general relativity.

Of course, shadows are not really a problem for collecting information about the bulk on the codimension-one screen. Shadows are problem for orthogonal projection, but not for anything else. For example, if the bodies are themselves emitting light (in all directions) or image in the scattered light is considered (when light can be reflected in all directions) no finite-size object can prevent another object from being seen somewhere on the infinite screen (Fig.2). However, such image exists only if one does not impose the requirement for rays to be orthogonal to the screen. This restriction can seem artificial, but without it one will have problems with separation of overlapping images of different objects – and information will be actually lost. To recover the information one can either rely upon quantum (interference) properties of the rays, as one does in laser holography, or stay in the classical framework but detect not only the image itself, but also the direction of the ray. It would be even easier to simply
fix the direction, imposing, for example, the restriction of orthogonality. However, then one returns to the problem of shadows and to above observation that gravitational effects allow one to overcome this problem.

As a next step, one can ask what is the mathematical language adequate for description of these ideas. If one detects the end-point of the ray on the screen and its direction, one actually deals with Cauchy problem for propagating particles/ rays. The relevant statement is that – unless in specifically bad circumstances – the bulk picture (solution to Laplace-like equation) is uniquely specified by the boundary conditions on the function and on its first derivatives. If, however, one imposes orthogonality restriction the story is somewhat different. The mathematical problem is now rather that of analytical continuation, and the solution is uniquely specified by the boundary condition itself.

The main problem with considerations of this section is that they are essentially classical – as are most of their modernized versions and generalizations involving various configurations of BPS branes. The simplest question to address is what is the way to describe fluctuations of vacuum configurations, like propagating gravitons, in the projected picture. There is hardly any satisfactory answer to this question at this moment. Clearly, the picture is very well suited for description of the topological phase of quantum gravity – which can hopefully be constructively developed in the close future, but what is the way to apply it to realistic gravity, which – at least naively – is not quite in a topological phase? Our naive vacuum in gravity spontaneously breaks gauge invariance and gauge-non-invariant excitations like individual gravitons seem to be relevant for the usual description of physics. As usual, the gauge invariant description is available, but looks a little bit artificial for description of most phenomena of ordinary physics – and transition from this gauge invariant (topological) description to the ordinary one remains somewhat obscure. Still it should exist, and information is not really lost in this transition – as it is not lost in transition from solutions to Laplace equation in the bulk to the boundary conditions in Cauchy or analytical-continuation problems.

5. Topological model is defined by a functional integral which is independent of the metric (normally, of the metric in the space-time where the quantum field theory is considered). This is the context, where one naturally assumes that quantum gravity is topological – since the result of integration (averaging) over all metrics is presumably independent of any individual metric. This is of course not so obvious if one tries to give any accurate definition of the integral – and there are various non-trivialities even in the simplest case of 2-dimensional gravity, which is partly understood. Even if one forgets about such problems, there are technical difficulties which still prevent one from dealing with most interesting questions about quantum gravity and its topological nature. For practical purposes one substitutes the study of gravities (averages over metrics) by that of a different class of topological theories, which can be more accurately called cohomological models. It is still an open question, whether the properties of cohomological models and gravities are similar and what – if any – are the
possible differences.

Cohomological theory is usually considered as reduction of some larger model with original Hilbert space $\mathcal{H}$ and a nilpotent operator $Q$ acting on this space,

$$Q : \mathcal{H} \to \mathcal{H}, \quad Q^2 = 0$$

(10)

Then the Hilbert space of cohomological theory per se is the one of cohomologies of $Q$:

$$h = \text{Ker } Q / \text{Coker } Q$$

(11)

Normally if original model is defined as a field theory on a compact manifold, associated cohomological one has small (finite-dimensional) Hilbert space $h$ and has not too many chances to resemble any field ordinary field theory with infinite-dimensional Hilbert space (every particle is an infinite collection of oscillators with different frequencies and every oscillator has infinitely many states). However, things change drastically when the space is non-compact: boundaries usually increase cohomologies and cohomological theory in non-compact space-time can be big enough to resemble (or even become equivalent) to an ordinary model of quantum field theory.

The simplest example of cohomology increase arises when one makes a puncture on the Riemann surface (complex curve). Then $\partial$-cohomologies, which were (finite) collections of holomorphic forms on original surface are enlarged by inclusion of infinitely many meromorphic forms with poles at the puncture. This example has been intensively exploited in the theory of open strings [10, 11] and it can serve to illustrate once again the equivalence between a model at the boundary and in the bulk and the way in which information about the bulk is contained in the boundary model. The relevant bulk/boundary relation in open string theory is provided by analytical continuation.

6. Let us consider a Riemann surface $\mathcal{C}$ with holes, $\partial \mathcal{C} = \Gamma = \sum_{i=1}^{n} \Gamma_i$. Then functional integral

$$\mathcal{Z}\{\phi_0\} = \int \mathcal{D}\phi \exp \left(-\int_{\mathcal{C}} \partial \bar{\partial} \phi\right)$$

(12)

which in the case of a closed Riemann surfaces defines the determinant of Laplace operator ($\mathcal{Z} \sim \sqrt{\det N_0/|\det \partial|^2}$), depends not only on the moduli of $\mathcal{C}$ but also on the boundary conditions $\phi|_{\Gamma} = \phi_0$. Substituting $\phi = \phi_{cl} + \varphi$ with $\phi_{cl}$ being solution to Laplace equation $\partial \bar{\partial} \phi_{cl} = 0$ on $\mathcal{C}$ with boundary condition $\phi_{cl}|_{\Gamma} = \phi_0$, so that $\varphi|_{\Gamma} = 0$, one gets:

$$\mathcal{Z}\{\phi_0\} = \sqrt{\frac{\det N_0}{\det -\Delta_0}} \exp \left(-S_{cl}\{\phi_0\}\right)$$

(13)
where $\text{det}_- \Delta_0$ stands for determinant of the scalar Laplace operator with Dirichlet boundary conditions and

$$S_{cl}\{\phi_0\} \equiv \int_C \partial \phi_{cl} \partial \bar{\phi}_{cl} = \oint_{\Gamma} \phi_0(\partial_x \phi)_0 =$$

$$= \oint_{\Gamma} \phi_0(x) \frac{\partial^2 G_D(x, y)}{\partial n_x \partial n_y} \phi_0(y)$$  \hspace{1cm} (14)$$

The classical solution $\phi_{cl}$ is constructed with the help of the Dirichlet Green function $G_D(x, y)$ of the scalar Laplace operator, which satisfies $G_D(x, y)|_{x \in \Gamma} = 0$, $G_D(x, y)|_{y \in \Gamma} = 0$:

$$\phi_{cl}(x) = \oint_{y \in \Gamma} \frac{\partial G_D(x, y)}{\partial n_y} \phi_0(y)$$ \hspace{1cm} (15)$$

where $\partial/\partial n_y$ is derivative in the normal direction to $\Gamma$. In the simplest case of an upper half-plane with a straight line as a boundary, the r.h.s. of (14) is

$$\oint_{\Gamma} \oint_{\Gamma} (\phi_0(x) - \phi_0(y))^2 (x - y)^2 dxdy$$ \hspace{1cm} (16)$$

For generic Riemann surfaces with holes explicit expression involves special functions made from the Prime bi-differential $E(x, y)$ \hspace{1cm} [12] - like the ordinary Green function \hspace{1cm} [13]  

\[ \frac{\partial^2 G_0(x, y)}{\partial x \partial y} = \langle \partial \varphi(x) \partial \varphi(y) \rangle = \partial_x \partial_y \log E(x, y) \] \hspace{1cm} (17)$$

The Dirichlet Green function $G_D$ can be constructed with the help of the double (image) technique.

Any Riemann surface with boundaries (in fact, also any non-oriented Riemann surface) can be represented as a factor $\mathcal{C} = D/Z_2$ of a closed Riemann surface $D$, a double, over an antiholomorphic $Z_2$ mapping $z \to \bar{z}$. For example, an annulus is a factor of a rectangular torus with $\tau = it$ over $Z_2$ mapping $z \to \bar{z}$. The stationary points (in this example they lie on the circles $\text{Im } z = 0$ and $\text{Im } z = t/2$) compose the boundary $\Gamma = \partial \mathcal{C}$. (If there are no stationary points, the factor is non-oriented surface, e.g. the same torus factored over $z \to \bar{z} + \frac{1}{2}$ is non-oriented but closed Klein bottle.) The genus of the double is $g_D = 2g_C + n - 1$, where $n$ is the number of components of the boundary $\Gamma = \partial \mathcal{C}$. Doubles are not generic closed Riemann surfaces, as in above example the dimension of the moduli space of doubles is twice smaller than the dimension of entire moduli space of the genus $g_D$ surfaces. Green function $G_D(x, y|\mathcal{C})$ on a surface $\mathcal{C}$ with boundaries is immediately obtained from the ordinary one $G_0(x, y|D)$ on its double:

$$G_D(x, y|\mathcal{C}) = G_0(x, y|D) - G_0(*x, y|D) - G_0(x, *y|D) + G_0(*x, *y|D)$$ \hspace{1cm} (18)$$

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and on the boundary (i.e. for $x = \ast x$, $y = \ast y$)

\[
\frac{\partial^2 G_D(x, y|C)}{\partial n_x \partial n_y} = 4 \frac{\partial^2 G_0(x, y|D)}{\partial x \partial y}
\]  

(19)

The r.h.s. is given in (17).

The theory of open strings has various applications. The one relevant for our purposes is application to the theory of closed strings, allowing one to express string measures on various Riemann surfaces in terms of the same kind of variables – points of the Universal Grassmannian which can be used in the role of the universal module space [14]. The basic idea is to make a small hole in the closed Riemann surface. Given Krichever data [14] – the complex curve $C$, a puncture $z_0$ on it and coordinates $z - z_0$ in the vicinity of the puncture – one can consider the set of meromorphic functions (or forms) with poles at $z_0$ puncture and look at their expansion near $z_0$. One can choose the standard basis in the space of such meromorphic functions (actually the best choice would be 1/2-differentials, but we neglect such details here):

\[
f_n(z) = \frac{1}{(z - z_0)^n} + \sum_{m \geq 0} A_{nm}(z - z_0)^m
\]  

(20)

Matrices $A_{nm}$ can be used to introduce coordinates on the universal moduli space, and all the relevant quantities are of course invariant under the changes of bases and coordinates (i.e. the moduli space is actually a universal Grassmannian). In particular determinant of Laplace operator is given by [10]:

\[
det D\bar{\partial} \sim \det(1 - AA^\dagger)
\]  

(21)

This formalism can be considered as representation of the “bulk theory” (of free fields on open Riemann surface, i.e. of $d = 2$ fields in non-trivial gravitational backgrounds) in terms of the “boundary theory” of the $d = 1$ “fields” $A_{mn}$ (with Universal Grassmannian as a target space). Expressions like (21) provide the description of “quantum effects in the bulk” (determinant knows about all the excitations in the bulk, not just about the “classical configurations” – the zero modes) in terms of the boundary theory. Thus we see that entire information about the quantum theory in the bulk is preserved at the boundary. In multidimensional case the similar statement would be that gravitons (and other particles in the bulk) can be reconstructed from the data at the boundary.

7. More examples of identities between field theories in different dimensions are provided by group theory (which is nowadays understood to be more or less equivalent to integrability theory). We shall briefly touch two examples of this kind: coming from geometrical quantization and from representation theory for non-compact groups.

The basic chain of relations from the theory of geometrical quantization
(Kirillov-Kostant construction) which will be of interest for us is:

\[ \text{Tr} R_1 = \int \mathcal{D}g(t) \exp \oint_{S^1} d^{-1} \Omega_R = \int_{\phi|_{\partial D} = X_R} \mathcal{D}\phi(t,r) \mathcal{D}A(t,r) \exp \left( \int_D \text{Tr} \phi\mathcal{F} + \oint_{\partial D} \text{Tr} \phi\mathcal{A} \right) \]  

(22)

At the l.h.s. here stands the dimension \( d_R = \text{Tr} R_1 \) of representation \( R \) of a Lie algebra \( \mathcal{G} \). It can be considered as a quantity in \( d = 0 \) theory (like matrix model). The first relation (22) expresses it in terms of a \( d = 1 \) functional integral over the group elements \( g \) on a circle with the action which is the \( d^{-1} \) of Kirillov-Kostant form \( \Omega_R \). For \( \mathcal{G} = SU(N), \Omega_R \) can be explicitly written as

\[ \Omega_R = \text{Tr} X_R g^{-1} dg \wedge g^{-1}dg, \]  

(24)

and

\[ d^{-1} \Omega_R = -\text{Tr} X_R g^{-1}dg \]  

(25)

with some constant matrix \( X_R \), which specifies the representation \( R \). The closed 2-form \( \Omega_R \) is degenerate, it becomes non-degenerate when restricted to an “orbit”, associated with representation \( R \). In other words, for given \( X_R \) the action \( d^{-1} \Omega_R \) is in fact independent of some of the variables \( g \), i.e. the \( d = 1 \) theory in (22) is a sort of a gauge theory. Note that in such formulation all the dependence on representation \( R \) is concentrated in the action, not in boundary conditions.

The second relation (23) expresses the same quantity in terms of \( d = 2 \) gauge model. The gauge field \( A \) is a 1-form on the disc \( D \) with values in adjoint representation of \( \mathcal{G} \), its curl is a 2-form \( \mathcal{F} = dA + A^2 \). The field \( \phi \) is a scalar in adjoint representation of \( \mathcal{G} \). The boundary term in the action, \( \oint_{\partial D} \text{Tr} \phi A \), is gauge invariant because \( \phi = X_R = \text{const} \) on the boundary \( \partial D = S^1 \). Integral over \( \phi \) provides a \( \delta \)-function of \( \mathcal{F} \), which implies that \( A \) is a pure gauge: \( A = g^{-1}dg \). Now particular representation \( R \) is specified by the boundary condition.

Relations (22)-(23) are easily generalized to character formulas (i.e. to represent \( \text{Tr} R g \) for the group element \( g \neq 1 \)) and to quantum groups. An interesting question is if this chain of relations can be continued further to higher dimensions, in the spirit of hierarchy of anomalies [16], making use of the operators \( K \) and \( k \), inverse to the nilpotent external derivative \( d \) and the boundary operator \( \partial \),

\[ Kd + dK = I, \]
\[ k\partial + \partial k = I \]  

(26)

8. Another way to change the dimension is to apply (23) per se to the case of affine (Kac-Moody) algebras. Then all dimensions are effectively increased by unity and (23) turns into relation between \( d = 2 \) Wess-Zumino-Novikov-Witten (WZNW) model and \( d = 3 \) Chern-Simons theory. Indeed, it
is well known \[17\] that Kirillov-Kostant 1-form \(d^{-1}\Omega\) in the case of affine algebra provides exactly the WZNW action. As to \(\text{Tr} \phi^a F^a\), in the case of the current-algebra realization of affine algebra the index \(a\) includes now a continuous loop parameter \(s\), so that \(\text{Tr} \phi^a F^a \rightarrow \oint ds \text{Tr} \phi^a F^a(s)\). As to \(F^a(s) = \partial_t A^a_t(s) - \partial_r A^a_r(s) + [A_t, A_r]^a(s)\), the commutator now contains a piece \(k f^{abc} (A^b_t(s) \partial_s A^c_r(s) - A^b_r(s) \partial_s A^c_t(s))\) proportional to the central charge \(k\). It remains to change the notation \(\phi(s) = A_s(s)\) in order to recognize that \(\text{Tr} \phi^a F^a\) in the case of affine algebras is just the Chern-Simons action on the \(d = 3\) manifold with coordinates \(t, r, s\) and the boundary at \(r = 1\). If the \(d = 2\) WZNW theory is formulated on a Riemann surface with coordinates \(t, s\), the Chern-Simons theory, associated to it by \(\tau\) is defined on the \(d = 3\) manifold filling the Riemann surface (i.e. the surface is its only boundary).

This identity between the WZNW and Chern-Simons is a simple example of the general “AdS/CFT correspondence” \[3\] which states that with any Lie algebra with generators \(J_\alpha\) one can associate a “topological” theory of gauge fields \(A\) on a non-compact manifold \(M\),

\[
\tau\{A\} \equiv \langle \exp \sum_\alpha A_\alpha J_\alpha \rangle = \int DA e^{S_{top}\{A\}} \bigg|_{A|\partial M = \{A_\alpha\}}
\]

so that the boundary conditions for \(A\) at the boundary \(\partial M\) of \(M\) are expressed through the generating parameters \(A_\alpha\). One of the names to the average at the l.h.s. of \(\tau\) is (generalized) \(\tau\)-function and (perhaps for somewhat sophisticated groups) it can be considered as providing an effective action of (actually, any) quantum field theory, while \(\{A_\alpha\}\) can be considered as some particular choice of coupling constants (comp. with \(\tau\)). Then relation \(\tau\) implies that the \(\tau\)-function at the l.h.s. can be represented as a functional integral over gauge fields in some gauge topological field theory in non-compact space time. In other words, there is supposedly a mapping from the space of \(d\)-dimensional QFT models into that of topological \(d + 1\)-dimensional models and it is established through the study of integrable structure of effective actions. Moreover, according to \(\tau\), integrable properties should be exhibited not only in effective action’s dependence on the coupling constants but also in that on the boundary conditions (the choice of vacuum). This suggestion can provide a new powerful tool for the study of (generalized) \(\tau\)-functions, which – if attacked straightforwardly – is a difficult task already for 2-loop algebras.

9. Let us consider relation \(\tau\) for the case of affine algebra \(G = U(1)_{k=1}\) with unit central charge \(k = 1\). Then the average at the l.h.s. can be represented as a correlator in the theory of free fermions on a Riemann surface \(C\),

\[
\langle \ldots \rangle = \int D\tilde{\psi} D\psi \ (\ldots) \ \exp \int_C \tilde{\psi} \partial \bar{\psi}
\]

and the generators \(J\) constitute holomorphic current \(J(z) = \tilde{\psi}(z)\). As explained above, the topological theory at the r.h.s. of \(\tau\) in this case is abelian
Chern-Simons model on the $d = 3$ manifold $M$, obtained by “filling” the Riemann surface $C = \partial M$, and (27) in this case is nothing but (23). Boundary conditions at $C$ can be imposed only on one of components of the gauge field $A$, the others remain unconstrained. The relevant choice is the antiholomorphic component $\bar{A}(z, \bar{z})$ ($z$ are coordinates on $C$).

Finally, (27) for affine algebra $\hat{U}(1)$ with the central charge $k = 1$ acquires the form:

$$\tau\{\bar{A}|C\} = \langle \exp \int_{C = \partial M} \bar{A}J \rangle = \int_{A|_{\partial M = C = \bar{A}}} \mathcal{D}A \exp \left( \int_M AdA + \oint_{\partial M} A\bar{A} \right)$$

(29)

The most straightforward proof of this relation is just independent calculation of both sides, which gives:

$$\tau\{\bar{A}|C\} = \theta\vec{e} \left( \int_C \bar{A}\vec{\omega} \right) \cdot \exp\int_C \int_C \bar{A}(x,y)G(x,y)\bar{A}(y)$$

(30)

with the Green function $\Psi\vec{a}(x,y)$, expressed through Szogo kernel

$$G(x,y) = \Psi\vec{a}(x,y)\Psi_{-\vec{a}}(x,y) = \partial_x\partial_y \log E(x,y) + \sum_{i,j} \omega_i\omega_j \partial^2_{ij} \log \theta_{\vec{a}}(\vec{x} - \vec{y})$$

(31)

expressed through Szogo kernel

$$\Psi\vec{a}(x,y) = \frac{\theta_\vec{x}(\vec{x} - \vec{y} + \vec{a})}{\theta_\vec{x}(\vec{a})E(x,y)} = \frac{\theta_\vec{a}(\vec{x} - \vec{y})}{\theta_\vec{a}(\vec{0})E(x,y)}$$

(32)

The characteristic

$$\vec{a} = \vec{e} - \vec{e}_* + \int_C \bar{A}\vec{\omega}$$

(33)

is expressed through the even half-integer theta-characteristic $\vec{e}$, used to define the free-fermion model on $C$, some odd half-integer theta-characteristic $\vec{e}_*$ used

$^1$ On the fermionic side one should use conservation of the current, $\bar{\delta}J(z) = 0$ to express surface integral through contour integrals, transform exponent to the normal form and make use of the expression

$$\langle : \exp \left( \delta_1 \int_{A_1} J + \epsilon_1 \int_{B_1} J \right) : \rangle = \theta_\vec{e} \left( \vec{e}\vec{T} + \vec{a} \right)$$

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to define the theta-function $\theta_\ast$, $\theta_\ast = \theta_\ast(\vec{e} - \vec{e}_\ast)$, and the integral over $\mathcal{C}$ of the product of $\overline{A}(z, \bar{z})$ times holomorphic $(1,0)$-differentials $\overline{\omega}(z)$.

The l.h.s. of (29) is nothing but a $\tau$-function of KP/Toda-family. Conventional Krichever’s $\tau$-function [15] appears when

$$\overline{A} = \partial \left( \sum_{k=1}^{\infty} \frac{t_k}{(z-z_0)^k} \right) = \sum_{k=1}^{\infty} \frac{(-)^k t_k}{(k-1)!} \delta^{k-1}(z-z_0)$$

so that

$$\langle \exp \int_\mathcal{C} \overline{A} J \rangle = \langle \exp \sum_{k=1}^{\infty} t_k J_k^{(z_0)} \rangle$$

As usual, in addition to the time-variables $t_k$, it depends on the set of Krichever data [15]: a Riemann surface $\mathcal{C}$, a point $z_0$ on it and coordinates $z$ in the vicinity of $z_0$. Given this data, one can define $J(z) = \sum_k J_k^{(z_0)}(z-z_0)^{k-1}dz$. If $\text{supp}(\overline{A})$ consists of $n$ points $z_{01}, \ldots, z_{0n}$,

$$\overline{A} = \partial \sum_{i=1}^{n} \left( \sum_{k=1}^{\infty} \frac{t_k^{(i)}}{(z-z_{0i})^k} \right)$$

we get the so-called $n$-component KP/Toda $\tau$-function ($n = 1$ is the KP and $n = 2$ the ordinary “Toda-lattice” case). In this sense we have at the l.h.s. of (29) a generic ($\infty$-component) KP/Toda $\tau$-function. As every (generalized) $\tau$-function, it satisfies bilinear Hirota equation [18]. The Miwa transform, which produces an insertion of $\psi(\lambda)$ or $\tilde{\psi}(\lambda)$ in the fermionic correlator, is just a shift

$$\overline{A} \rightarrow \tilde{\overline{A}} + \frac{1}{\bar{z} - \lambda}$$

Thus – as a simple example of “AdS/CFT correspondence” – we see that partition function of Chern-Simons theory on a manifold with a boundary is – as a function of the boundary conditions – a $\tau$-function of the KP/Toda family (i.e. associated with the theory of free fermions on Riemann surfaces). This example reveals – in a simple situation – integrable properties of effective actions as functionals of the boundary conditions (the moduli of vacua manifolds). This supplements the usual conjecture [4, 20] about integrable dependence of effective actions on the coupling constants.

10. Another relation between the “AdS/CFT correspondence” and group theory is provided by representation theory of non-compact groups. The most peculiar phenomenon here is the occurrence of “singletons”, which can be considered as fundamental representations (i.e. all other relevant representations belong to some power of the fundamental one) and at the same time can be interpreted as localized at the boundary of the non-compact homogeneous space.
where the group is acting. There are plenty of interesting speculations relating
the singleton-like phenomena to Kaluza-Klein theories [21] though not all the
sides of the story are fully clarified. The story is closely related to the theory
of Harish-Chandra functions and can again be attacked by the methods of
geometrical quantization [22].

The simplest case where the phenomenon is already present (at least if some-
what obscure notion of localization at the boundary is substituted by the leading
asymptotics at the boundary) is AdS$_2$: the homogeneous space of the $d = 2$ con-
formal group $SO(d - 1, 2) = SO(1, 2) \cong SL(2, R)$, i.e. the Lobachevsky plane.

Lobachevsky plane can be represented as an upper half-plane $y > 0$, of the
complex plane with coordinate $z = x + iy$, metric $ds^2 = \frac{dzd\bar{z}}{y^2}$ and Laplace
operator:

$$\Delta = y^2 \partial \partial$$

Eigenfunctions of Laplace operator are [23]

$$f_p(x, y) = \int da f(a) \left( \frac{(az + 1)(a\bar{z} + 1)}{y} \right)^p$$

with any function $f(a)$. The corresponding eigenvalue is $\lambda = p(p - 1)$. Representation is unitary if $p = -\frac{1}{2} + in$ with integer $n$. Since

$$f_p(x, y) = y^{-p} \int da f(a)(ax + 1)^{2p}(1 + O(y))$$

the leading asymptotics at the boundary $y = 0$ of the entire set of eigenfunctions
with the given eigenvalue is just the set $F_{2p}$ of homogeneous functions of weight
$2p$ (under rational transformations $z \rightarrow \frac{az+b}{cz+d}$). For example, for integer $p$ this
set is just the linear space of all the polynomials (in $x$) of degree $2p$. Remarkably
(and obviously) for integer $2p$

$$F_{2p} = F_2^{\otimes p}$$

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