THE GLOBAL WELL-POSEDNESS OF THE KINETIC CUCKER-SMALE FLOCKING MODEL WITH CHEMOTACTIC MOVEMENTS

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Abstract. We present a coupled kinetic-macroscopic equation describing the dynamic behaviors of Cucker-Smale (in short C-S) ensemble undergoing velocity jumps and chemotactic movements. The proposed coupled model consists of a kinetic C-S equation supplemented with a turning operator for the kinetic density of C-S particles, and a reaction-diffusion equation for the chemotactic density. We study a global existence of strong solutions for the proposed model, when initial data is sufficiently regular, compactly supported in velocity and has finite mass and energy. The turning operator can screw up the velocity alignment, and result in a dispersed state. However, under suitable structural assumptions on the turning kernel and ansatz for the reaction term, the effects of the turning operator can vanish asymptotically due to the diffusion of chemical substances. In this situation, velocity alignment can emerge algebraically slow. We also present parabolic and hyperbolic Keller-Segel models with alignment dissipation in two scaling limits.

1. Introduction. The emergence of collective behaviors such as velocity alignment is ubiquitous in an ensemble of self-propelled particles. Here the jargon “velocity alignment” represents some phenomenon in which the velocities of self-propelled particles tend to a common value asymptotically using only limited environmental information and simple rules [44]. In literature, the jargons such as swarming and herding are also used to represent similar phenomena. In recent years, several mathematical models have been proposed for the modeling of velocity alignment and coordinated control in literature [2, 3, 4, 5, 11, 13, 15, 17, 20, 21, 30, 33, 34, 35, 38, 40, 41, 45], and they have been extensively investigated due to their potential

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engineering applications in sensor networks and the formation control of robots and unmanned aerial vehicles [33, 35]. In this paper, our main interest lies in the particle model, namely Cucker-Smale (C-S) model which was proposed by Cucker and Smale a decade ago in [11]. The C-S model is a continuous-time dynamical system given by the second-order system of ordinary differential equation for position. The jargon “C-S particle” denote a point particle whose mechanical state is governed by the C-S model.

When the number of C-S particles is sufficiently large, the dynamics of the C-S ensemble can be effectively approximated by its corresponding mean-field model (see [20, 21]). Let \( f = f(x, v, t) \) be a one-particle probability density function of the C-S ensemble. Then, the phase-space evolution of the kinetic density function \( f \) is governed by the following kinetic C-S equation:

\[
\partial_t f + v \cdot \nabla_x f + \kappa_0 \nabla_v \cdot \left( F_a[f] f \right) = 0, \quad x, v \in \mathbb{R}^d, \ t > 0,
\]

where \( \kappa_0 \) is a nonnegative constant representing the coupling strength between particles, and the term \( F_a[f] \) is a velocity alignment force that induces an asymptotic velocity alignment among C-S particles. Here, we assume that the nonnegative communication weight function \( \psi \) is bounded from above and away from zero: there exist positive constants \( \psi_m \) and \( \psi_M \) such that

\[
0 < \psi_m \leq \psi(r) \leq \psi_M < \infty, \quad r \geq 0, \quad \psi(\cdot) \in \mathcal{C}^1(\mathbb{R}_+).
\]

Thus, our communication setting in (2) can be viewed as a perturbation of the all-to-all communication weight function, i.e., \( \psi = \text{constant} \), and the condition (2) has been employed in literature [2, 3, 4, 16, 17, 18] on the C-S flocking. In fact, the positive lower bound for \( \psi \) is crucial for the emergence of velocity alignment, when the model lacks a uniform compact support in position.

Note that the equation (1) can naturally arise from the kinetic description of the ensemble of mechanical C-S particles (see [20]), and (1), and its variants have been extensively studied in literature from diverse perspectives, e.g., global existence theories of classical solutions [5, 21], measure-valued solutions [20], coupling with fluid equations [2, 3, 4], and the macroscopic C-S model and its asymptotic justification [20].

In this paper, we address two issues in relation with the kinetic model (1). First, we propose a new kinetic model for C-S particles with chemotactic movements with velocity jumps and attraction toward chemotactic substances. In chemotaxis literature [9, 10, 42], it is commonly reported that some bacteria exhibit an aggregation dynamics and chemotactic movements. For this, we employ the idea from the kinetic Keller-Segel model [36], i.e., we add a velocity jump process and chemotactic movements to R.H.S. of the kinetic C-S model (1) and a suitable field equation for chemotactic density via the turning operator. Second, we study a global existence of strong solution to the proposed model.

The main results of this paper is two-fold. First, we provide a coupled kinetic-parabolic PDE model describing the dynamics of C-S particles with chemotactic movements. For the dynamics of the chemotactic density, we use a reaction-diffusion equation. More precisely, to register the abrupt changes in velocities due to the attraction by the chemotactic substance, we introduce a nonlocal turning operator \( T[S](f) \) whose kernel depends on the density of chemotactic substances (in short
chemotactic density). Following the spirit of the Boltzmann equation, we supplement the turning operator $T[S](f)$ on the right-hand side of (1) to obtain
\[
\partial_t f + v \cdot \nabla_x f + \kappa_0 \nabla_v \cdot (F_a[f]f) = \kappa_1 T[S](f).
\]
(3)

To close the dynamics of $f$ in (3), we need to add the spatio-temporal evolution of the chemotactic density $S = S(x,t)$. To do this, we use the reaction-diffusion equation: for $\tau = 0, 1$,
\[
\tau \partial_t S - \Delta S = \kappa_2 \varphi(S, \rho), \quad \rho = \int_{\mathbb{R}^d} f dv.
\]
(4)

Finally, we combine (3) and (4) to obtain a coupled PDE system with chemotactic movement:
\[
\begin{align*}
\partial_t f + v \cdot \nabla_x f + \kappa_0 \nabla_v \cdot (F_a[f]f) &= \kappa_1 T[S](f), & x,v \in \mathbb{R}^d, \ t > 0, \\
\partial_t S - \Delta S &= \kappa_2 \varphi(S, \rho), & \rho = \int_{\mathbb{R}^d} f dv,
\end{align*}
\]
(5)

where $\varphi = \varphi(S, \rho)$ is a reaction term representing chemical interactions between the C-S particle and chemical substances. In the case that the velocity alignment mechanism in (5) is turned off, i.e., $\kappa_0 = 0$, the coupled system (5) is reduced to the kinetic Keller-Segel model [6, 8, 13, 24, 23, 25, 26, 32]. Second, we provide a global existence of strong solution to the coupled system (5) as follows. We first recall the concept of strong solution as follows.

**Definition 1.1.** For a positive constant $T \in (0, \infty)$, the pair $(f, S)$ is a strong solution of the system (5) if and only if the following conditions hold:

1. The solution $(f, S)$ has the following regularity:
   \[
   f \in W^{1, \infty}(0, T); L^\infty(\mathbb{R}^{2d}) \cap L^\infty([0, T); (L^1_+ \cap W^{1, \infty})(\mathbb{R}^{2d})),
   
   S \in W^{1, \infty}(0, T); L^\infty(\mathbb{R}^d) \cap L^\infty([0, T); (L^1_+ \cap W^{2, \infty})(\mathbb{R}^d)).
   \]

Here $L^1_+$ means $f$ and $S$ are non-negative integrable functions.

2. For any test function $\phi(x, v, t) \in C^1_c(\mathbb{R}^d \times \mathbb{R}^d \times [0, T))$, $f$ satisfies the weak formula below
   \[
   \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_t f \left( \phi + v \cdot \nabla_x \phi + \kappa_0 \nabla_v \phi \cdot (F_a[f]f) + \kappa_1 T[S](f) \right) dv dx = 0,
   \]
   for $x,v \in \mathbb{R}^d, \ t > 0$.

3. For any test function $\eta(x, t) \in C^1_c(\mathbb{R}^d \times [0, T))$, $S$ satisfies
   \[
   -\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \tau \partial_t \eta S + \nabla_x \eta \cdot \nabla_x S + \kappa_2 \eta \varphi(S, \rho) \right) dv dx = \int_{\mathbb{R}^d} \tau S_0 \eta(x, v, 0) dv dx.
   \]

When the initial data $(f_0, S_0)$ is sufficiently regular and has finite velocity moments, and the reaction term $\varphi(S, \rho)$ has a specific ansatz, we show that there exists a global existence of strong solutions (see Theorem 4.1).

The rest of the paper is organized as follows. In Section 2, we briefly review a kinetic C-S model and discuss the our proposed model (5). We also list a sufficient assumption for the global existence of strong solutions to (5) with asymptotic velocity alignment property. In Section 3, we provide a local existence of a strong solution. To do this, we use a standard successive approximation and several a priori estimates. In Section 4, we discuss the global existence of strong solutions.
to the coupled model (5) by excluding the possibility of the finite-time blow up of \( f \) and \( S \) and their derivatives (see Theorem 4.1). Finally, Section 5 is devoted to summary of main results of this paper. In appendix, we present two macroscopic models using the scaling limits. In Appendix A, we present the drift-diffusion system using a formal parabolic limit. In Appendix B, we derive a hyperbolic model involving velocity alignment force and chemotactic movements using a hyperbolic scaling.

Notation: For any measurable functions \( f = f(x,v) \) and \( u = u(x) \) defined on \( \mathbb{R}^{2d} \) and \( \mathbb{R}^d \), we set
\[
\|f\|_{L^p} := \|f\|_{L^p(\mathbb{R}^{2d})}, \quad \|u\|_{L^p} := \|u\|_{L^p(\mathbb{R}^d)}, \quad 1 \leq p \leq \infty.
\]

2. Preliminaries. In this section, we briefly discuss a kinetic C-S model and a generalized kinetic C-S model incorporating velocity jumps and chemotactic movement, and present several a priori estimates for the proposed model (5).

2.1. A kinetic C-S model. Let \( x_i \) and \( v_i \) be the position and velocity of the \( i \)-th particle unit mass, respectively. Then, the dynamics of mechanical state \((x_i,v_i)\) is governed by the following \( N \)-body system:
\[
\begin{align*}
\frac{dx_i}{dt} &= v_i, \quad t > 0, \quad i = 1, \cdots, N, \\
\frac{dv_i}{dt} &= \frac{\kappa_0}{N} \sum_{j=1}^{N} \psi(|x_j - x_i|)(v_j - v_i),
\end{align*}
\]
(6)

When the number of particles is sufficiently large, i.e., \( N \gg 1 \), the numerical integration of (6) is almost impossible to implement. Thus, we are forced to approximate the system (6) using a corresponding mean-field kinetic model (see [21, 20, 31] for formal and rigorous derivations). Let \( f = f(x,v,t) \) be a one-particle probability density function of C-S particles (or cells) at position \( x \in \mathbb{R}^d \) and time \( t \in \mathbb{R}_+ \) with velocity \( v \in \mathbb{R}^d \). Then, the spatio-temporal evolution of \( f \) is described by the Vlasov-McKean type equation:
\[
\begin{align*}
\partial_t f + v \cdot \nabla_x f + \kappa_0 \nabla_v \cdot (F_a[f]f) &= 0, \quad x \in \mathbb{R}^d, \quad t > 0, \\
F_a[f](x,v,t) &= -\int_{\mathbb{R}^{2d}} \psi(|x - y|)(v - v_*)f(y,v_*,t)dv_*dy.
\end{align*}
\]
(7)

For notational simplicity, we suppress \( t \)-dependence in \( f \) and use a handy notation for velocity moments of \( f \): for \( x, v \in \mathbb{R}^d \), \( t \geq 0 \),
\[
f(x,v) := f(x,v,t) \quad \text{and} \quad \int_{\mathbb{R}^{2d}} v^mf dvdx := \int_{\mathbb{R}^{2d}} v^m f(x,v) dvdx.
\]

Next, we discuss the basic properties of the velocity alignment term \( F_a[f]f \) in (7) in the following lemma.

Lemma 2.1. Let \( f = f(x,v,t) \) be a strong solution to (7) which decays to zero sufficiently fast at infinity:
\[
\int_{\mathbb{R}^{2d}} (1 + |v|^2) f dvdx < \infty, \quad t \geq 0.
\]
Then, we have
\[ \int_{\mathbb{R}^{2d}} v \nabla_v \cdot (F_a[f]f) dv dx = -d \int_{\mathbb{R}^{2d}} \psi(|x-y|)(v - v_*) f(y, v_*) f(x, v) dv_* dv dy dx. \]
\[ \int_{\mathbb{R}^{2d}} |v|^2 \nabla_v \cdot (F_a[f]f) dv dx = \int_{\mathbb{R}^{2d}} \psi(|x-y|)|v - v_*|^2 f(y, v_*) f(x, v) dv_* dv dy dx. \]

Proof. (i) We used the relation
\[ v \nabla_v \cdot (F_a[f]f) = \nabla_v (v \otimes F_a[f]f) - dF_a[f]f. \]
to obtain
\[ \int_{\mathbb{R}^{2d}} v \nabla_v \cdot (F_a[f]f) dv dx = -d \int_{\mathbb{R}^{2d}} \psi(|x-y|)(v - v_*) f(y, v_*) f(x, v) dv_* dv dy dx. \]
(ii) Again, we use the relation
\[ |v|^2 \nabla_v \cdot (F_a[f]f) = \nabla_v (|v|^2 F_a[f]f) - 2v \cdot F_a[f]f \]
to find
\[ \int_{\mathbb{R}^{2d}} |v|^2 \nabla_v \cdot (F_a[f]f) dv dx = 2 \int_{\mathbb{R}^{2d}} \psi(|x-y|)|v - v_*| f(y, v_*) f(x, v) dv_* dv dy dx \]
\[ = -2 \int_{\mathbb{R}^{2d}} \psi(|x-y|)v_\ast \cdot (v - v_*) f(y, v_*) f(x, v) dv_* dv dy dx \]
\[ = - \int_{\mathbb{R}^{2d}} \psi(|x-y|)|v - v_*|^2 f(y, v_*) f(x, v) dv_* dv dy dx. \]

Next, we set the first three velocity moments \( M_0, M_1 \) and \( M_2 \) which represent the total mass, momentum, and twice the value of energy, respectively:
\[ M_0[f] := \int_{\mathbb{R}^{2d}} f dv dx, \quad M_1[f] := \int_{\mathbb{R}^{2d}} v f dv dx \quad \text{and} \quad M_2[f] := \int_{\mathbb{R}^{2d}} |v|^2 f dv dx. \]

Based on the estimates in Lemma 2.1, we obtain the conservation of momentum and dissipation of energy for (7) in the following proposition.

**Proposition 1.** Let \( f = f(x, v) \) be a strong solution to (7) which decays to zero sufficiently fast at infinity:
\[ \int_{\mathbb{R}^{2d}} (1 + |v|^2) f dv dx < \infty, \quad t \geq 0. \]

Then, we obtain the conservation of momentum and dissipation of energy: for a.e. \( t > 0 \),
\[ (i) \quad \frac{d}{dt} M_0[f] = 0, \quad \frac{d}{dt} M_1[f] = 0. \]
\[ (ii) \quad \frac{d}{dt} M_2[f] = - \int_{\mathbb{R}^{2d}} \psi(|x-y|)|v - v_*|^2 f(y, v_*) f(x, v) dv_* dv dy dx. \]

Proof. (i) The conservation of mass follows from the far-field decay condition of \( f \) and divergence form of (7). For the conservation of momentum, we multiply (7) by \( v \) and we integrate the resulting equation over \( \mathbb{R}^{2d} \) and use Lemma 2.1(i) to find
\[ \frac{d}{dt} \int_{\mathbb{R}^{2d}} v f(x, v) dv dx = \int_{\mathbb{R}^{2d}} v \nabla_v \cdot (F_a[f]f) dv dx \]
\[ = -d \int_{\mathbb{R}^{2d}} \psi(|x-y|)(v - v_*) f(y, v_*) f(x, v) dv_* dv dy dx. \]
Note that \( \psi(|x-y|)(v-v_*)f(y, v_*, t)f(x, v, t) \) is skew-symmetric in the transformation:
\[
(x, v) \iff (y, v_*).
\]
Therefore, the right-hand side of (9) becomes zero. This implies the conservation of momentum.

(ii) We multiply (7) by \(|v|^2\) and integrate the resulting equation over \(\mathbb{R}^{2d}\) to obtain the dissipation of the energy.

Next, we discuss a priori velocity alignment estimate following the idea in [21]. For this, we introduce a Lyapunov functional \(L_0[f]\) measuring the velocity dispersion of the kinetic density \(f\):
\[
L_0[f(t)] := \int_{\mathbb{R}^{2d}} |v-v_c(t)|^2 f(x, v) dv dx, \quad t \geq 0,
\]
where \(v_c\) is the average velocity defined by
\[
v_c(t) := \frac{\int_{\mathbb{R}^{2d}} v f dv dx}{\int_{\mathbb{R}^{2d}} f dv dx}.
\]

Then, it follows from the conservations of mass and momentum in Proposition 1 that we have
\[
v_c(t) = v_c(0) \quad \text{and} \quad L_0[f(t)] := \int_{\mathbb{R}^{2d}} |v-v_c(0)|^2 f(x, v) dv dx, \quad t \geq 0.
\]

As discussed in [16], the functional \(L_0\) measures the variance of the velocity around the average velocity \(v_c\). More precisely, zero convergence of \(L_0[f(t)]\) as \(t \to \infty\) implies the formation of velocity alignment in probability. This can be seen as follows. From Chebyshev inequality, for any \(\varepsilon > 0\), we have
\[
L_0[f(t)] = \int_{\mathbb{R}^{2d}} |v-v^c(t)|^2 f dv dx \geq \int_{|v-v^c(t)| > \varepsilon} |v-v^c(0)|^2 f dv dx \\
\geq \varepsilon^2 \int_{|v-v^c(t)| > \varepsilon} f dv dx = \varepsilon^2 \mathbb{P}[|v-v^c(0)| > \varepsilon] = \varepsilon^2 |v-v^c(0)| > \varepsilon] \leq \frac{1}{\varepsilon^2} \lim_{t \to \infty} L_0[f(t)] = 0.
\]

**Corollary 1** ([21]). Suppose that the communication weight \(\psi\) has a positive lower bound
\[
\inf_{s \geq 0} \psi(s) \geq \psi_m > 0.
\]
Then, for any strong solution \(f = f(x, v)\) to (7) decaying to zero at infinity sufficiently fast:
\[
\int_{\mathbb{R}^{2d}} (1 + |v|^2) f(x, v) dv dx < \infty, \quad t \geq 0,
\]
we have an exponential velocity alignment:
\[
L_0[f(t)] := \int_{\mathbb{R}^{2d}} |v|^2 f dv dx.
\]

**Proof.** Because (7) conserves the total momentum, we may assume that \(v^c(0) = 0\), and the functional \(L_0(t)\) becomes twice that of the total energy:
\[
L_0[f(t)] := \int_{\mathbb{R}^{2d}} |v|^2 f dv dx.
\]
From the estimate (ii) in Lemma 2.1,
\[
\frac{dL_0}{dt} = - \int_{\mathbb{R}^d} |v|^2 \nabla v \cdot (F_a[f]f) dx
= - \int_{\mathbb{R}^d} \psi(|x-y|)|v - v_\ast|^2 f(y, v_\ast) f(x, v) dv dy dx
\leq -2\psi_m L_0,
\]
where we have used a lower bound of \(\psi\) and zero momentum. Then, Gronwall’s inequality yields the desired estimate. \(\square\)

2.2. Modeling of chemotactic movements. In this subsection, we introduce a generalized kinetic C-S model with chemotactic movements. As reported in [22, 29, 46], collective behaviors in animal groups often emerge via individual interactions (communications). In 2007, Cucker-Smale [11] introduced a simple analytical model for flocking behavior of animal groups, which generalizes Vicsek’s model [45]. After the Cucker-Smale’s seminal work, most works on the flocking adopted their simple velocity alignment mechanism, which results in the smooth change of individual velocities. However, as we often see in the school of fish in aquarium, fish change their velocity abruptly, i.e., they have several components such as free swimming, tumbling and swarming etc in their motion. To model abrupt change of velocity, we adopt turning operator which is commonly used in kinetic Keller-Segel models [6, 7, 9, 25, 26, 36, 37]. As far as the authors know, there is only one work [19] to incorporating the velocity alignment and turning operator to model the collective behaviors in phototactic Cyanobacteria. The authors presented a coupled model without any theoretical studies on the coupled system.

Next, we describe the velocity jump process via the turning operator: for a given \((x,t) \in \mathbb{R}^d \times \mathbb{R}_+\), we set \(T[\mathcal{S}]f\) to denote the rates of jumps from \(v'\) to \(v\) and vice versa, respectively:
\[
v' \overset{T[\mathcal{S}]}{\Rightarrow} v, \quad \text{and} \quad v' \overset{T^+[\mathcal{S}]}{\Rightarrow} v.
\]

Let \(S = S(x,t)\) be the concentration of the chemotactic substance. Then, the contribution of the rate of change in \(f\) along the particle trajectory due to chemotactic movements will be registered by the turning operator \(T[\mathcal{S}]f\) (see [1, 12, 13, 27, 28]):
\[
T[\mathcal{S}](f) = T^+[\mathcal{S}](f) - T^-[\mathcal{S}](f),
\]
\[
T^+[\mathcal{S}](f)(x,v,t) := \int_{\mathbb{R}^d} T[\mathcal{S}](x,t,v,v') f(x,v') dv',
\]
\[
T^-[\mathcal{S}](f)(x,v,t) := \int_{\mathbb{R}^d} T^+[\mathcal{S}](x,t,v,v') f(x,v') dv' = \lambda[\mathcal{S}](x,v,t) f(x,v),
\]
where the quantity \(\lambda[\mathcal{S}]\) denotes the turning frequency. Now, we combine (7) and (11) to obtain a kinetic chemotaxis-Cucker-Smale (CCS) model (5).

2.2.1. Propagation of velocity moments. Due to the effects of chemotactic attraction, the propagation of the moments of (5) will be different from the kinetic Cucker-Smale model (7), which has been discussed in Proposition 1. For the simplicity of notation, we set
\[
M_i(t) := M_i[f(t)], \quad i = 0, 1, 2.
\]
Lemma 2.2. Let \( f = f(x, v, t) \) be a strong solution to (5) which decays to zero at infinity sufficiently fast:
\[
\int_{\mathbb{R}^d} (1 + |v|^2) f dv dx < \infty.
\]
Then, for \( t > 0 \), we have
\[
(i) \quad \frac{dM_0(t)}{dt} = 0, \quad \frac{dM_1(t)}{dt} = \kappa_1 \int_{\mathbb{R}^d} v T[S](f) dv dx.
\]
\[
(ii) \quad \frac{dM_2(t)}{dt} = -\kappa_0 \int_{\mathbb{R}^d} \psi(|x - y|)|v - v_s|^2 f(y, v_*) f(x, v) dv dy + \kappa_1 \int_{\mathbb{R}^d} |v|^2 T[S](f) dv dx.
\]

Proof. (i) For the estimates of \( M_0 \), we use similar calculation in Lemma 2.1 to obtain
\[
\frac{dM_0(t)}{dt} = \int_{\mathbb{R}^d} T[S](f) dv dx = \int_{\mathbb{R}^d} T^+ [S](f) dv dx - \int_{\mathbb{R}^d} T^- [S](f) dv dx. \tag{12}
\]
On the other hand, we use the identity \( T^*[S](x, t, v, v') := T[S](x, t, v', v) \) and definition of turning kernel in (11) to see
\[
\int_{\mathbb{R}^d} T^+ [S](f) dv dx = \int_{\mathbb{R}^d} T^- [S](f) dv dx. \tag{13}
\]
We combine (12) and (13) to derive a conservation of mass \( \frac{dM_0}{dt} = 0 \). For the estimate on momentum \( M_1 \), we again use Proposition 1 and the fact that the convection and the alignment forcing terms do not affect the total momentum. Therefore, we have
\[
\frac{dM_1}{dt} = \frac{d}{dt} \int_{\mathbb{R}^d} vf(x, v) dv dx = \kappa_1 \int_{\mathbb{R}^d} v T[S](f) dv dx.
\]
(ii) We multiply (5) by \( |v|^2 \) and integrate the resulting relation with respect to \((x, v)\) to obtain
\[
\frac{dM_2(t)}{dt} = -\kappa_0 \int_{\mathbb{R}^d} \psi(|x - y|)|v - v_s|^2 f(y, v_*) f(x, v) dv dy + \kappa_1 \int_{\mathbb{R}^d} |v|^2 T[S](f) dv dx.
\]
\( \square \)

Remark 1. In general, the coupled system (5) does not conserve total momentum, and the energy itself may not be monotone in time unlike the kinetic C-S model.

3. Local existence of strong solutions. In this section, we first briefly describe an assumption, based on which we present the local existence of strong solutions.

3.1. Assumptions. In this section, we first give structural ansatz for the turning kernel \( T[S] \) and reaction term \( \varphi(S, \rho) \) for the local existence theory to (5).

• (A1): The dimension \( d \) of the spatial domain is any positive integer and (5) is a reaction-diffusion equation.
strong solutions to \((5)\) by performing the following four steps.

3.2. Evolution of solutions.

The velocity is a variable and the support of this variable may change during the evolution of solutions. In \([36,37]\), the authors studied a Boltzmann type equation where the velocity alignment force is absent. Therefore, micro-velocity becomes a variable and has compact supports in \(v\) and \(v'\). Under the assumption \((A2)\) - \((A3)\), the turning kernel has compact support in the velocity domain. In \([36,37]\), the authors studied a Boltzmann type equation where the velocity alignment force is absent. Therefore, micro-velocity becomes a parameter and the support of velocity will not change. However, in our model, the velocity is a variable and the support of this variable may change during the evolution of solutions.

Under the assumption \((A1)\) - \((A3)\), system \((5)\) becomes

\[
\begin{align*}
\partial_t f + v \cdot \nabla_x f + \kappa_0 \nabla_v \cdot (F_v[f]f) &= \kappa_1 T[S](f), \quad x,v \in \mathbb{R}^d, \quad t > 0, \\
\partial_t S - \Delta S &= -\kappa_2 S \rho.
\end{align*}
\]

Our first main result is concerned with the local existence of strong solutions.

**Theorem 3.1** (Local existence of strong solution). Suppose that the assumption \((A1)\) - \((A3)\) holds, and initial data \(f_0, S_0\) satisfy

\[
\begin{align*}
f_0 &\in (L_+^1 \cap W^{1,\infty})(\mathbb{R}^{2d}), \quad S_0 \in (L_+^1 \cap W^{2,\infty})(\mathbb{R}^d), \\
|\mathcal{M}_1(0)| + M_2(0) &< \infty, \quad \text{supp}_x f_0(x,\cdot) \subset K \subset \subset \mathbb{R}^d, \quad \text{for } x \in \mathbb{R}^d.
\end{align*}
\]

Then, there exists a local-in-time strong solution \((f, S)\) to \((5)\) and a positive constant \(T^*\) such that

\[
\begin{align*}
f &\in W^{1,\infty}(0,T^*; L^\infty(\mathbb{R}^{2d})) \cap L^\infty(0,T^*; L_+^1 \cap W^{1,\infty})(\mathbb{R}^{2d})), \\
S &\in W^{1,\infty}(0,T^*; L^\infty(\mathbb{R}^d)) \cap L^\infty(0,T^*; L_+^1 \cap W^{2,\infty})(\mathbb{R}^d)), \\
\text{supp}_x f(x,\cdot, t) &\subset K' \subset \subset \mathbb{R}^d, \quad \text{for } x \in \mathbb{R}^d, \quad t \in [0,T),
\end{align*}
\]

where \(K\) is a compact set containing \(B_{R_v}(0)\) and \(K'\) is another compact set containing \(K\).

**Proof.** Since the proof is rather lengthy, we will present its proof in the following subsection.

\[\square\]

3.2. Proof of Theorem 3.1. In this subsection, we present the local existence of strong solutions to \((5)\) by performing the following four steps.

- **Step A:** Construction of approximate solutions using the standard successive approximations.
- **Step B:** Derivation of a priori estimates for approximate solutions.
- **Step C:** Establishment of the convergence of approximate solutions.
- **Step D:** Verification that the limit function is in fact our desired strong solution.
3.2.1. Construction of approximate solutions. In this part, we present a sequence of approximate solutions \( \{(f^n, S^n)\}_{n=0}^{\infty} \) for system \((16)\) as follows.

\( \diamond \) Initial step \((n = 0)\): We set
\[
f^0 := f_0, \quad S^0 := S_0,
\]
where \((f_0, S_0)\) represents the initial data.

\( \diamond \) Inductive step \((n \geq 1)\): Suppose that the \((n - 1)\)-th iterate \((f^{n-1}, S^{n-1})\) has been constructed. Then, with this \((n - 1)\)-th iterate \((f^{n-1}, S^{n-1})\), we define the \(n\)-th iterate \((f^n, S^n)\) as a solution of the Cauchy problem to the following linear system:
\[
\begin{align*}
\partial_t f^n + v \cdot \nabla_x f^n + \kappa_0 \partial_v \psi_0 \cdot (F_n[f^{n-1}]f^n) &= \kappa_1 \left( T^+[S^{n-1}](f^{n-1}) - \lambda[S^{n-1}]f^n \right), \\
\partial_t S^n - \Delta S^n &= -\kappa_2 S^n \rho^{n-1}, \\
f^n(x, v, 0) &= f_0(x, v), \quad S^n(x, 0) = S_0(x), \quad x, v \in \mathbb{R}^d, \quad t = 0,
\end{align*}
\]
Note that \((17)\) can be rewritten in the following quasi-linear form:
\[
\partial_t f^n + v \cdot \nabla_x f^n + \kappa_0 F_n[f^{n-1}] \cdot \nabla_v f^n \\
= \left( \kappa_0 \partial_v \psi_0 \cdot (F_n[f^{n-1}]) - \kappa_1 \lambda[S^{n-1}] \right) f^n + \kappa_1 T^+[S^{n-1}](f^{n-1}).
\]
From the method of characteristics for \((18)\), it follows that for a given \(x, v \in \mathbb{R}^d\) at time \(t\), consider the the forward characteristics:
\[
(x^n(t), v^n(t), f^n(t)) := (x^n(t; 0, x, v), v^n(t; 0, x, v), f^n(x^n(t; 0, x, v), v^n(t; 0, x, v), t)),
\]
which are given by the solution of the system:
\[
\begin{align*}
\frac{d}{dt} x^n(t) &= v^n(t), \quad t > 0, \\
\frac{d}{dt} v^n(t) &= -v^n(t) \int_{\mathbb{R}^d} \psi(|x^n(t) - y|) f^{n-1}(y, v^*) dv^* dy \\
&\quad + \int_{\mathbb{R}^d} \psi(|x^n(t) - y|) v^* f^{n-1}(y, v^*) dv^* dy, \\
\frac{d}{dt} f^n(t) &= \left[ \kappa_0 d \int_{\mathbb{R}^d} \psi(|x^n(t) - y|) f^{n-1}(y, v^*) dv^* dy \\
&\quad - \kappa_1 \lambda[S^{n-1}] |x^n(t), v^n(t), t| f^n(t) + \kappa_1 T^+[S^{n-1}](f^{n-1})(x^n(t), v^n(t), t) \\
&\quad = h(x^n(t), v^n(t), t) f^n(t) + \kappa_1 T^+[S^{n-1}](f^{n-1})(x^n(t), v^n(t), t). \quad (19)
\end{align*}
\]
Then, it is easy to see that equation \((19)\) can be rewritten as
\[
f^n(t) = f_0(x, v)e_t^s \int_0^t h(x(s), v(s), s) ds \\
+ \kappa_1 \int_0^t e^{-\int_s^t h(x(\tau), v(\tau), \tau) d\tau} T^+[S^{n-1}](f^{n-1})(x^n(s), v^n(s), s) ds. \quad (20)
\]
Note that the approximation scheme \((17)\) guarantees the positivity of \(f^n\). Hence, if the limit function for the sequence \(\{f^n\}\) exists, then it will be nonnegative as well. Moreover, the non-negativity of \(S^n\) follows from the maximum principle of elliptic equation.
3.2.2. A priori estimates for approximate solutions. Suppose that the initial data satisfy

\[
   f_0 \in (L^1_+ \cap W^{1,\infty})(\mathbb{R}^d), \quad S_0 \in (L^1_+ \cap W^{2,\infty})(\mathbb{R}^d),
\]

\[
   \text{supp}_x f(x, \cdot, t) \subset K \subset \subset \mathbb{R}^d, \quad \text{for } x \in \mathbb{R}^d, t \in [0, T).
\]

Then, we have

\[
   f^n \in W^{1,\infty}([0, T); L^\infty(\mathbb{R}^{2d})) \cap L^\infty([0, T); (L^1_+ \cap W^{1,\infty})(\mathbb{R}^{2d}))
\]

\[
   S^n \in W^{1,\infty}([0, T); L^\infty(\mathbb{R}^d)) \cap L^\infty([0, T); (L^1_+ \cap W^{2,\infty})(\mathbb{R}^d)).
\]

Before we begin technical estimates, we briefly discuss how to obtain a local solution: Suppose that \( f^{n-1} \) is in the desired function space in (22), and that it has finite moments

\[
   |M^{n-1}_1| := |M_1[f^{n-1}]| < \infty \quad \text{and} \quad M^{n-1}_2 := M_2[f^{n-1}] < \infty,
\]

and compact support with respect to \( v \). We solve the linear system (17) using the method of characteristics, and we further show the regularity of the solution. For this, we show the following statements:

- For small \( T > 0 \), we show that both \( M^n_1 \) and \( M^n_2 \) are bounded in the time-interval \([0, T)\).
- We show that \( f^n \) and \( S^n \) are locally bounded in the function space (22), and thus we can extract a weak* limit in their space.
- We show that \( f^n \) and \( S^n \) form Cauchy sequences in \( L^\infty \) space, and we construct a weak solution of the nonlinear system. As the weak* limit is unique, this constructed solution and the weak* limit coincide and thus have regularity.

**Lemma 3.2.** For \( T \in (0, \infty) \), suppose that the initial data \((f_0, S_0)\) satisfy (21), and the \((n - 1)\)-th iterate \((f^{n-1}, S^{n-1})\) satisfies

\[
   \begin{align*}
   (i) & \quad f^{n-1} \in W^{1,\infty}([0, T); L^\infty(\mathbb{R}^{2d})) \cap L^\infty([0, T); (L^1_+ \cap W^{1,\infty})(\mathbb{R}^{2d})), \\
   (ii) & \quad S^{n-1} \in W^{1,\infty}([0, T); L^\infty(\mathbb{R}^d)) \cap L^\infty([0, T); (L^1_+ \cap W^{2,\infty})(\mathbb{R}^d)), \\
   (iii) & \quad \sup_{t \in [0, T]} |M^{n-1}_i(t)| < \infty, \quad i = 1, 2; \\
   (iv) & \quad \text{supp}_x f^{n-1}(x, \cdot, t) \subset K' \subset \subset \mathbb{R}^d, \quad \text{for } x \in \mathbb{R}^d, t \in [0, T).
   \end{align*}
\]

Then, we have

\[
   \begin{align*}
   (i) & \quad f^n \in W^{1,\infty}([0, T); L^\infty(\mathbb{R}^{2d})) \cap L^\infty([0, T); (L^1_+ \cap W^{1,\infty})(\mathbb{R}^{2d})), \\
   (ii) & \quad \text{supp}_x f^n(x, \cdot, t) \subset K \subset \subset \mathbb{R}^d, \quad \text{for } x \in \mathbb{R}^d, t \in [0, T); \\
   (iii) & \quad \sup_{t \in [0, T]} |M^n_i(t)| < \infty, \quad i = 1, 2 \\
   (iv) & \quad S^n \in W^{1,\infty}([0, T); L^\infty(\mathbb{R}^d)) \cap L^\infty([0, T); (L^1_+ \cap W^{2,\infty})(\mathbb{R}^d)).
   \end{align*}
\]

**Proof.** We split the proof into several pieces.

(i) (Regularity of \( f^n \)): We integrate the first equation in (17) over \( \mathbb{R}^{2d} \) to obtain:

\[
   \frac{d}{dt} \int_{\mathbb{R}^{2d}} f^n v dx \leq CM^{n-1}_0.
\]
Thus, we have \( f^n \in L^1(\mathbb{R}^d) \). Consider a backward equation of the characteristic curve:

\[
\frac{d}{d\tau} (x^n(\tau; t, x, v), v^n(\tau; t, x, v)) = \mathbf{F} (x^n(\tau; t, x, v), v^n(\tau; t, x, v), \tau), \quad 0 \leq \tau \leq t, \quad x^n(t; t, x, v) = x \quad \text{and} \quad v^n(t; t, x, v) = v.
\]

(24)

Where \( \mathbf{F} := (F_1, F_2) \) is the vector field

\[
F_1(x^n, v^n) = v^n,
\]

\[
F_2(x^n, v^n) = -v^n \int_{\mathbb{R}^d} \psi(|x^n - y|) f^{n-1}(y, v^*) dv^* dy + \int_{\mathbb{R}^d} \psi(|x^n - y|) v^n f^{n-1}(y, v^*) dv^* dy.
\]

Then, it is easy to verify that \( \mathbf{F} \) is \( C^1 \) with respect to \( x^n \) and \( v^n \). Hence, we can conclude that \( (x^n(\tau; t, x, v), v^n(\tau; t, x, v)) \) are \( C^1 \) with respect to \( (x, v) \). Then we denote for short that \( x^n(\tau) = x^n(\tau; t, x, v), v^n(\tau) = v^n(\tau; t, x, v) \) and consider the equation

\[
\frac{d}{d\tau} f^n(x^n(\tau), v^n(\tau), \tau) = \left[ \kappa_0 d \int_{\mathbb{R}^d} \psi(|x^n(\tau) - y|) f^{n-1}(y, v^*, \tau) dv^* dy \\
- \kappa_1 \lambda[S^{n-1}](x^n(\tau), v^n(\tau), t) f^n(x^n(\tau), v^n(\tau), \tau) \\
+ \kappa_1 T^+[S^{n-1}](f^{n-1})(x^n(\tau), v^n(\tau), \tau)
\right] = h^n(\tau) f^n(x^n(\tau), v^n(\tau), \tau) + \kappa_1 T^+[S^{n-1}](f^{n-1})(x^n(\tau), v^n(\tau), \tau).
\]

Here we use \( h^n(\tau) \) to represent

\[
\left[ \kappa_0 d \int_{\mathbb{R}^d} \psi(|x^n - y|) f^{n-1}(y, v^*, \tau) dv^* dy - \kappa_1 \lambda[S^{n-1}](x^n, v^n, t) \right].
\]

Therefore, we can write \( f^n(x, v, t) \) in terms of an integral form as following

\[
f^n(x, v, t) = f_0(x^n(0; t, x, v), v^n(0; t, x, v)) e^{\int_0^t h^n(x, v, \tau) d\tau} \\
+ \kappa_1 \int_0^t e^{\int_s^t h^n(x(\tau), v^n(\tau), \tau) d\tau} T^+[S^{n-1}](f^{n-1})(x^n(s), v^n(s), s) ds
\]

(25)

Thus \( f(x, v, t) \) is Lipschitz continuous with respect to \( t \). On the other hand, it follows from the regularity of \( S^{n-1}, T^+[S^n](f^{n-1}), x^n(\tau; x, v, t), \psi \) and \( v^n(\tau; x, v, t) \) that we have

\( f^n \in W^{1,\infty}([0, T]; L^\infty(\mathbb{R}^d)) \cap L^\infty([0, T]; (L^1_+ \cap W^{1, \infty})(\mathbb{R}^d)) \).

(ii) (Boundedness of \( M^n \)): It follows from (19)\( _2 \) that the \( j \)-th component of \( v^n \) satisfies

\[
\frac{d}{dt} v^n_j(t) = -v^n_j(t) \int_{\mathbb{R}^d} \psi(|x^n(t) - y|) f^{n-1}(y, v^*) dv^* dy \\
+ \int_{\mathbb{R}^d} \psi(|x^n(t) - y|) v^n_j f^{n-1}(y, v^*) dv^* dy \\
\leq -\psi_m||f_0||_{L^1} v^n_j(t) + C\psi_M||f_0||_{L^1},
\]

(26)

where we have used the facts:

\[
\psi_m \leq \psi(|x^n(t) - y|) \leq \psi_M, \quad \text{supp} f^{n-1}(x, \cdot, t) \subset \subset \mathbb{R}^d.
\]
Now, we apply Gronwall’s lemma to (26) to determine the boundedness of the velocity support:

$$|v^n_j(t)| \leq C(T).$$

Next, we consider the second moments $M^n_2$. From the definition of $M^n_2$, we have

$$\frac{d}{dt}M^n_2(t) \leq 2\kappa_0 \psi_M M_0 (M^n_2 - M^n_0) + \kappa_1 \int_{\mathbb{R}^d} |v|^2 \tau^{+} [S^{n-1}](f^{n-1})|dvdx.$$

Then, we use the assumptions (23) for $S^{n-1}$, $f^{n-1}$, $M^n_2$, and the compact support of turning operator $\tau$ with respect to $v$ and $v'$ to obtain

$$\frac{d}{dt}M^n_2(t) \leq 2\kappa_0 \psi_M M_0 M^n_0 + C.$$

Therefore, we have

$$M^n_2(t) \leq e^{2\kappa_0 \psi_M M_0 t} + \frac{C}{2\kappa_0 \psi_M M_0} \left(e^{2\kappa_0 \psi_M M_0 T} - 1\right) < \infty, \quad t \in [0, T).$$

Again it follows from the relation $M^n_1 \leq \sqrt{M_0 M^n_2}$ that we also have

$$\sup_{t \in [0, T]} |M^n_1(t)| < \infty.$$

(iii) (Regularity of $S^n$): We use the defining equation for $v^n(t)$ in (19) to show that $f^n$ has compact support with respect to $v$. Then, there exists a positive constant $C$ such that

$$|\nabla_x \rho^n(x, t)| \leq \int_{\mathbb{R}^d} |\nabla_x f^n(x, v, t)|dv \leq C ||\nabla_x f^n||_{L^\infty},$$

$$|\partial_v \rho^n| \leq \int_{\mathbb{R}^d} |\partial_v f^n(x, v, t)|dv \leq C ||\partial_v f^n||_{L^\infty}.$$

Thus, we have

$$\rho^n(x, t) \in W^{1, \infty}([0, T); L^\infty(\mathbb{R}^d)) \cap L^\infty([0, T); (L^1_+ \cap W^{1, \infty})(\mathbb{R}^d)).$$

Finally, we prove the regularity of $S^n$ by showing the following a priori estimates. First, since $S^n$ satisfies a linear parabolic equation (17), we can use the Green’s function of heat equation and the non-negative property of $\rho^{n-1}$ and $S^n$ to obtain

$$S^n(x, t) \leq \int_{\mathbb{R}^d} \frac{1}{(4\pi t)^{d/2}} e^{-\frac{(x-y)^2}{(4t)^2}} S_0(y)dy \leq ||S_0||_{L^\infty}.$$

Thus we know $S^n \in L^\infty([0, T); (L^1_+ \cap L^\infty)(\mathbb{R}^d))$. For the first order derivative, we can use Green’s function of heat equation to obtain

$$||\partial_x S^n(x, t)||_{L^\infty} \leq \int_{\mathbb{R}^d} \frac{1}{(4\pi t)^{d/2}} e^{-\frac{(x-y)^2}{(4t)^2}} ||\partial_x S_0||_{L^\infty} dy$$

$$+ \int_0^t \int_{\mathbb{R}^d} |\partial_y \left( \frac{1}{(4\pi(t-s))^{d/2}} e^{-\frac{(y-y)^2}{(4(t-s))^{d/2}}} \right) | \rho^{n-1}(x-y, s) S^n(x-y, s) dyds$$

$$\leq ||\partial_x S_0||_{L^\infty} + \int_0^t \int_{\mathbb{R}^d} |\partial_y \left( \frac{1}{(4\pi(t-s))^{d/2}} e^{-\frac{(y-y)^2}{(4(t-s))^{d/2}}} \right) | dsdy ||\rho^{n-1}||_{L^\infty} ||S^n||_{L^\infty}$$

$$\leq ||\partial_x S_0||_{L^\infty} + C(T) ||\rho^{n-1}||_{W^{1, \infty}} ||S^n||_{W^{1, \infty}}.$$

We can do second order derivative estimate similarly to obtain

$$||\partial^2_{x,x} S^n(x, t)||_{L^\infty} \leq ||\partial^2_{x,x} S_0||_{L^\infty} + C(T) ||\rho^{n-1}||_{W^{1, \infty}} ||S^n||_{W^{1, \infty}}.$$
Therefore, we finally obtain the desired regularity of $S^n$ as following

$$S^n \in W^{1,\infty}([0,T); L^\infty(\mathbb{R}^d)) \cap L^\infty([0,T); (L^1_+ \cap W^{2,\infty})(\mathbb{R}^d)).$$

\[ \square \]

3.2.3. Convergence of approximate solutions. In this part, we show that the limit of $f^n$ and $S^n$ exists, when $n$ tends to infinity. First, we show that $f^n$ and $S^n$ are uniformly bounded in a short-time interval. Let $L$ be a positive constant such that

$$\|f_0\|_{W^{1,\infty}} + \|S_0\|_{W^{2,\infty}} + M_0(0) + |M_1(0)| + M_2(0) + 1 \ll L < \infty. \quad (27)$$

**Lemma 3.3.** There exists a positive constant $T^*$ such that if

$$\max\{|f^n|_{W^{1,\infty}}, |S^n|_{W^{2,\infty}}, M_0^{n-1}, |M_1^{n-1}|, M_2^{n-1}\} < L, \quad 0 \leq t \leq T^*,$$

we have

$$\max\{|f^n|_{W^{1,\infty}}, |S^n|_{W^{2,\infty}}, M_0^n, |M_1^n|, M_2^n\} < L, \quad 0 \leq t \leq T^*. $$

**Proof.** We will show that the lower-order estimate and the higher-order estimate can be obtained with the same method.

- **Step A** (zeroth order estimate of $f^n$): It follows from (20) that

  $$|f^n(x, v, t)| \leq \|f_0\|_{L^\infty}e^{(\psi M_0 + C\kappa_1 L^2)t^* + C\kappa_1 T^* e^{(\psi M_0 + C\kappa_1 L^2)t^*}} L^3, \quad 0 \leq t \leq T^*. $$

  Note that for $T^* = 0$, it follows from (27) that

  $$|f^n(x, v, t)| \leq \|f_0\|_{L^\infty} < L. $$

  From the continuity of $f^n$ with respect to time $t$, we can find $T^*$ independent of $n$ such that

  $$\|f^n(t)\|_{L^\infty} < L, \quad 0 \leq t \leq T^*. $$

- **Step B** (uniform bound of $M^n$): Directly calculation shows that

  $$\frac{d}{dt} M_0^n(t) \leq \kappa_1 \int_{\mathbb{R}^d} |T^+[S^{n-1}](f^n-1)| dv dx,$$

  $$\frac{d}{dt} M_2^n(t) \leq 2\kappa_0 \psi M_0 M_1^n + M_0^{n-1} M_2^n + \kappa_1 \int_{\mathbb{R}^d} |v|^2 |T^+[S^{n-1}](f^n-1)| dv dx$$

  $$\leq 2\kappa_0 \psi M_0 (LM_2^n + LM_0^n) + \kappa_1 \int_{\mathbb{R}^d} |v|^2 |T^+[S^{n-1}](f^n-1)| dv dx. \quad (28)$$

  Since the turning effect exists only for low-velocity particles due to (14), and the turning kernel has the estimate (14), we have

  $$\int_{\mathbb{R}^d} |v|^2 |T^+[S^{n-1}](f^n-1)| dv dx$$

  $$\leq \int_{\mathbb{R}^d} |v|^2 S^{n-1}(x - v', t) S^{n-1}(x + v, t) f^{n-1}(x, v', t) dv' dv dx$$

  $$\leq \int_{\mathbb{R}^d} R^2 L^2 f^{n-1}(x, v', t) dv' dv dx$$

  $$\leq R^2 L^2 |B_{R^2}(0)| M_0^{n-1} \leq |B_{R^2}(0)| R^2 L^3.$$

  Similarly, we have

  $$\int_{\mathbb{R}^d} |T^+[S^{n-1}](f^n-1)| dv dx \leq |B_{R^2}(0)| L^3. \quad (30)$$
Therefore, we combine (28), (29) and (30) to obtain a differential inequality:

$$\frac{d}{dt} \left( M_0^n(t) + M_2^n(t) \right) \leq 2\kappa_0 \psi_M L \left( M_0^n + M_2^n \right) + \kappa_1 (1 + R_0^n) |B_{R_0}(0)| L^3.$$  

Then, we immediately obtain the estimate of $M_0^n$ and $M_2^n$ as below:

$$\left( M_0^n(t) + M_2^n(t) \right) \leq \left( M_0^n(0) + M_2^n(0) \right) e^{2\kappa_0 \psi_M L t} + \int_0^t e^{2\kappa_0 \psi_M L (t-s)} \kappa_1 (1 + R_0^n) |B_{R_0}(0)| L^3 ds.$$

Hence, we can apply the similar method as that used for $f^n$ to imply that for proper $T^*$ independent of $n$, we have

$$M_0^n(t) + M_2^n(t) < L, \quad 0 \leq t \leq T^*.$$  

Then, we apply the relation $M_i^n \leq \sqrt{M_0^n M_2^n}$ to conclude that

$$|M_i^n(t)| < L, \quad 0 \leq t \leq T^*, \quad i = 0, 1, 2.$$

- Step C (uniform compact velocity support): Now, we can investigate the equation of $v^n(t)$ in (19):

$$\frac{d}{dt} v^n(t) = - v^n(t) \int_{\mathbb{R}^d} \psi(|x^n(t) - y|) f^{n-1}(y, v^*) dv^* dy$$

$$+ \int_{\mathbb{R}^d} \psi(|x^n(t) - y|) v^* f^{n-1}(y, v^*) dv^* dy.$$  

We multiply $v^n$ to (31) and denote the $\ell^2$ norm of $v^n$ by $|v^n|$. Then we obtain

$$\frac{d}{dt} |v^n(t)| \leq - |v^n(t)| \int_{\mathbb{R}^d} \psi(|x^n(t) - y|) f^{n-1}(y, v^*) dv^* dy$$

$$+ \left| \int_{\mathbb{R}^d} \psi(|x^n(t) - y|) v^* f^{n-1}(y, v^*) dv^* dy \right|$$

$$\leq \psi_M \int_{\mathbb{R}^d} |v^* f^{n-1}(y, v^*)| dv^* dy \leq \psi_M \sqrt{M_0^{n-1} M_2^{n-1}} \leq \psi_M L^2.$$

Therefore, we obtain the estimate of $|v^n|$ as below:

$$|v^n(t)| \leq |v^n(0)| + \psi_M L^2 T^*, \quad 0 \leq t \leq T^*.$$  

This shows that for sufficiently small $T^*$

$$|v^n(t)| - \psi_M L^2 T^* \leq |v^n(\tau)|, \quad 0 \leq \tau \leq t \leq T^*.$$  

Therefore, we can find a large enough compact set $K'$ such that if $v^n(t) \notin K'$, then $v^n(\tau) \notin K$, $0 \leq \tau \leq t \leq T^*$.

Since the initial datum for $f^n$ is $f_0$ which has a compact support. Thus, $|v^n(0)|$ has a common upper bound for any $n$. Moreover, due to the choice of $K$, the support of turning operator is contained in $K$. Therefore, for any $v^n(t) \notin K'$, we have $v^n(\tau) \notin K$ along the characteristic curve from 0 to $t$. Thus according to (25), we have

$$\frac{d}{dt} f^n(\tau) = 0, \quad 0 \leq \tau \leq t \leq T^*.$$  

Hence, we have

$$f^n(t) = f^n(0) = 0.$$
This shows that the support of $f^n$ with respect to $v$ is contained in $K'$ and thus uniformly bounded for $0 \leq t \leq T^*$. Due to this uniformly compact support, we have for $0 \leq t \leq T^*$,

$$||\rho^n||_{L^\infty} = ||\int_{\mathbb{R}^d} f^n dv||_{L^\infty} \leq (|v^n(0)| + \psi_M L^2 T^*)^d ||f^n||_{L^\infty} \leq (|v^n(0)| + \psi_M L^2 T^*)^d L.$$  

- Step D (lower-order estimate of $S^n$): First, we check the zeroth-order estimate of $S^n$, and we find that

$$S^n(x, t) \leq \int_{\mathbb{R}^d} \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{(x-y)^2}{4t}} S_0(y) dy.$$  

The right hand side is the convolution of heat kernel and $S_0$. Therefore, we apply the maximum principle to show that

$$||S^n(t)||_{L^\infty} \leq ||S_0||_{L^\infty} < L, \quad 0 \leq t \leq T^*.$$  

Next, we check the first-order estimate of $S^n$. For each $i$, we have for any $0 \leq t \leq T^*$,

$$||\partial_i S^n(x, t)||_{L^\infty} \leq \int_{\mathbb{R}^d} \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{(x-y)^2}{4t}} ||\partial_i S_0||_{L^\infty} dy  
+ \int_0^t \int_{\mathbb{R}^d} |\partial_i \left( \frac{1}{(4\pi (t-s))^{\frac{d}{2}}} e^{-\frac{(y-s)^2}{4(t-s)}} \right) \rho^{n-1}(x-y, s) S^n(x-y, s) dy ds  
\leq ||\partial_i S_0||_{L^\infty} + \int_0^t \int_{\mathbb{R}^d} |\partial_i \left( \frac{1}{(4\pi (t-s))^{\frac{d}{2}}} e^{-\frac{(y-s)^2}{4(t-s)}} \right) | ds d\rho^{n-1} ||S^n||_{L^\infty}  
< ||\partial_i S_0||_{L^\infty} + L^2 \int_0^t \int_{\mathbb{R}^d} |\partial_i \left( \frac{1}{(4\pi (t-s))^{\frac{d}{2}}} e^{-\frac{(y-s)^2}{4(t-s)}} \right) | ds dy  
\leq ||\partial_i S_0||_{L^\infty} + CL^2 \int_0^t \frac{1}{\sqrt{t-s}} ds \leq ||\partial_i S_0||_{L^\infty} + \sqrt{C} L^2.$$

Therefore, we can choose $T^*$ sufficiently small to guarantee

$$||\nabla_x S^n(x, t)||_{L^\infty} < L, \quad 0 \leq t \leq T^*.$$  

- Step F (higher-order estimate): We can take the derivative of the equation of $f^n$ and $S^n$ to obtain the partial differential equation of $\partial_x f^n$, $\partial_v f^n$ and $\nabla_{x, v}^2 S^n(x, t)$. Then we apply the same method from step A to step E and obtain the estimates of $||\partial_x f^n||_{L^\infty}$, $||\partial_v f^n||_{L^\infty}$ and $||\nabla_{x, v}^2 S^n(x, t)||$.

Next, we will show that the sequences $\{f^n\}$ and $\{S^n\}$ are Cauchy in the space $L^\infty(\mathbb{R}^{2d})$. We use (17) to derive the equation for $f^{n+1} - f^n$:

$$\partial_t (f^{n+1} - f^n) + v \cdot \nabla_x (f^{n+1} - f^n) + \kappa_0 \nabla_v \cdot (F_v[f^n][f^{n+1} - f^n] - F_v[f^{n-1}][f^n])  
+ \kappa_1 (\lambda[S^n][f^{n+1} - \lambda[S^{n-1}][f^n]) = \kappa_1 (T^+[S^n](f^n) - T^+[S^{n-1}](f^{n-1})).$$  

(32)
Then, we can rewrite the relation (32) as
\[
\partial_t (f^{n+1} - f^n) + v \cdot \nabla_x (f^{n+1} - f^n) + \kappa_0 F_a [f^{n-1}] \cdot \nabla_v (f^{n+1} - f^n) = -\kappa_0 \nabla_v \cdot \left( (F_a[f^n] - F_a[f^{n-1}]) f^{n+1} \right) - \kappa_0 (\nabla_v \cdot F_a[f^{n-1}]) (f^{n+1} - f^n) \\
- \kappa_1 (\lambda [S^n] - \lambda [S^{n-1}]) f^{n+1} - \kappa_1 \lambda [S^{n-1}] (f^{n+1} - f^n) \\
+ \kappa_1 (T^+[S^n] - T^+[S^{n-1}]) (f^n) + \kappa_1 T^+[S^{n-1}] (f^n - f^{n-1}) \\
=: \sum_{i=1}^6 I_{1i}.
\] (33)

Because we have proved the uniform compact support of \( f^n \) with respect to \( v \), we have \( |x^n| \leq (|v^n(0)| + \psi_M L^2 T^*) T^* \) by finite speed of propagation. Thus, we can assume \( T^* \) sufficiently small so that \( 0 \leq t \leq T^* < L \) and
\[
\max_{v \in \text{supp} f(x,t)} |v| \leq |v^n(0)| + \psi_M L^2 T^*, \\
\max_{x \in \text{supp} f(\cdot,v,t)} |x| \leq (|v^n(0)| + \psi_M L^2 T^*) T^* < L.
\] (34)

Lemma 3.4. Let \( (f^n, S^n) \) be a sequence of approximate solutions defined by (17). Then, the sequence \( (f^n, S^n) \) is Cauchy for sufficiently small \( T^* \) in the space:
\[
(f^n, S^n) \in L^\infty([0, T^*); L^\infty(\mathbb{R}^d)) \times L^\infty([0, T^*); L^\infty(\mathbb{R}^d)).
\]

Proof. We split the estimate into two parts.
- Case A (Estimate of \( ||f^{n+1} - f^n||_{L^\infty} \)): In this part, we estimate \( I_{1i} \) in (33) separately.
  - (Estimate of \( I_{11} \)): We apply the compact support (34) of the velocity, position, and the higher order estimate in Lemma 3.3 to obtain
    \[
    |I_{11}| = \kappa_0 \left| \nabla_v \cdot \left( (F_a[f^n] - F_a[f^{n-1}]) f^{n+1} \right) \right| \\
    \leq d \kappa_0 \psi_M \left( \int_{\mathbb{R}^d} |f^n - f^{n-1}| dv dx \right) f^{n+1} \\
    + \sum_{i=1}^d 2 L \kappa_0 \psi_M \left( \int_{\mathbb{R}^d} |f^n - f^{n-1}| dv dx \right) |\partial_{v_i} f^{n+1}| \\
    \leq C L^3 ||f^n - f^{n-1}||_{L^\infty},
    \] (35)

where we used Lemma 3.3 and the relation (34) to find
\[
|v - v^*| < 2L, \quad \text{for} \quad v, v^* \in \text{supp} f(x, \cdot, t), \\
|f^{n+1}| < L, \quad |\partial_{v_i} f^{n+1}| < L, \quad \text{and} \quad \int_{\mathbb{R}^d} |f^n - f^{n-1}| dv dx \leq C L^2 ||f^n - f^{n-1}||_{L^\infty}.
\]
  - (Estimate of \( I_{12} \)): By directly calculation, we have
    \[
    |\nabla_v \cdot F_a[f^{n-1}]| \leq \psi_M M_0 f^{n-1},
    \]
    and this yields
    \[
    |I_{12}| = \left| - \kappa_0 (\nabla_v \cdot F_a[f^{n-1}]) (f^{n+1} - f^n) \right| \leq C L ||f^{n+1} - f^n||_{L^\infty}. \] (36)
Therefore, we have
\[ |\lambda[S^n] - \lambda[S^{n-1}]| = \left| \int_{\mathbb{R}^d} (T[S^n] - T[S^{n-1}])dv' \right| \]
\[ \leq \int_{\mathbb{R}^d} |S^n(x - v)S^n(x + v') - S^{n-1}(x - v)S^{n-1}(x + v')|dv' \]
\[ \leq \int_{\mathbb{R}^d} |S^n(x - v)(S^n(x + v') - S^{n-1}(x + v'))|dv' \]
\[ + \int_{\mathbb{R}^d} |S^{n-1}(x + v')(S^n(x - v) - S^{n-1}(x - v))|dv' \]
\[ \leq C||S^n - S^{n-1}||_{L^\infty}. \] (37)

Therefore, we have
\[ ||I_{13}|| \leq \kappa_1||\lambda[S^n] - \lambda[S^{n-1}]||f^{n+1} \leq CL||S^n - S^{n-1}||_{L^\infty}. \] (38)

(Estimate of $I_{13}$): It follows from (11) that we have the estimate of $\lambda[S^n]$ as below
\[ ||\lambda[S^n]|| = \int_{\mathbb{R}^d} S(x - v)S(x + v')dv' \leq C||S(x - v)||_{L^\infty} \leq CL. \]

Therefore, we have
\[ ||I_{14}|| \leq \kappa_1||\lambda[S^{n-1}]||f^{n+1} - f^n \leq CL^2||f^{n+1} - f^n||_{L^\infty}. \] (39)

(Estimate of $I_{14}$): It follows from (11) that we have the estimate of $\lambda[S^n]$ as below
\[ ||\lambda[S^n]|| = \int_{\mathbb{R}^d} S(x - v)S(x + v')dv' \leq C||S(x - v)||_{L^\infty} \leq CL. \]

Finally, we integrate (33) along the characteristic curve and combine estimates (35), (36), (38), (39), (40), and (41) to obtain
\[ ||(f^{n+1} - f^n)(t)||_{L^\infty} \leq \int_0^t C(L + L^2)||f^{n+1} - f^n||_{L^\infty}ds \]
\[ + \int_0^t \left[ C(L + L^2 + L^3)||f^n - f^{n-1}||_{L^\infty} + CL||S^n - S^{n-1}||_{L^\infty} \right]ds. \] (42)

Then, Gronwall’s lemma and $f^{n+1}(x, v, 0) = f^n(x, v, 0)$ imply
\[ ||(f^{n+1} - f^n)(t)||_{L^\infty} \leq e^{C(L+L^2)t} \int_0^t C(L + L^2 + L^3)||f^n - f^{n-1}||_{L^\infty}d\tau \]
\[ + e^{C(L+L^2)t} \int_0^t CL||S^n - S^{n-1}||_{L^\infty}d\tau. \] (43)

We next return to the estimate of $||S^{n+1} - S^n||_{L^\infty}$. 
• Case B (Estimate of $S^{n+1} - S^n$): We consider the representation formula for $S^n$ to obtain

$$|(S^{n+1} - S^n)(x, t)|$$

$$\leq \int_0^t \int_{\mathbb{R}^d} \frac{1}{(4\pi(t-s))^\frac{d}{2}} e^{-\frac{(x-y)^2}{4(t-s)}} |\rho^n(y,s)S^{n+1}(y,s) - \rho^{n-1}(y,s)S^n(y,s)| dyds$$

$$\leq \int_0^t \int_{\mathbb{R}^d} \frac{1}{(4\pi(t-s))^\frac{d}{2}} e^{-\frac{(x-y)^2}{4(t-s)}} \left( \int_{\mathbb{R}^d} f^n dv S^{n+1}(y,s) - \int_{\mathbb{R}^d} f^{n-1} dv S^n(y,s) \right) dyds$$

$$\leq \int_0^t \int_{\mathbb{R}^d} \frac{1}{(4\pi(t-s))^\frac{d}{2}} e^{-\frac{(x-y)^2}{4(t-s)}} \left( \int_{\mathbb{R}^d} (f^n(y,y,s) - f^{n-1}(y,y,s)) dv S^{n+1}(y,s) \right) dyds$$

$$+ \int_0^t \int_{\mathbb{R}^d} \frac{1}{(4\pi(t-s))^\frac{d}{2}} e^{-\frac{(x-y)^2}{4(t-s)}} \left( \int_{\mathbb{R}^d} f^{n-1}(y,y,s) dv (S^{n+1}(y,s) - S^n(y,s)) \right) dyds$$

$$\leq C_1 \int_0^t \left( ||f^n - f^{n-1}(s)||_{L^\infty} + ||(S^{n+1} - S^n)(s)||_{L^\infty} \right) ds.$$ (44)

This yields

$$||S^{n+1} - S^n(t)||_{L^\infty} \leq C_2 e^{C_1 t} \int_0^t \left( ||f^n - f^{n-1}(s)||_{L^\infty} \right) ds.$$ (44)

Finally, we combine (43) and (44), and apply Lemma 3.3 to obtain

$$||f^{n+1} - f^n||_{L^\infty} + ||S^{n+1} - S^n||_{L^\infty}$$

$$\leq \int_0^t \left[ (C(L + L^2 + L^3)e^{(CL+CL^2)T^*} + CLe^{CLT^*})||f^n - f^{n-1}||_{L^\infty} \right. $$

$$\left. + (CLe^{CL+CL^2)T^*})||S^n - S^{n-1}||_{L^\infty} \right] dt$$

$$\leq (C(L + L^2 + L^3)e^{(CL+CL^2)T^*} + CLe^{CLT^*})$$

$$\times \int_0^t \left[ ||f^n - f^{n-1}||_{L^\infty} + ||S^n - S^{n-1}||_{L^\infty} \right] dt$$

(45)

$$\leq (C(L + L^2 + L^3)e^{(CL+CL^2)T^*} + CLe^{CLT^*})^{n(T^*)^n}$$

$$\times \sup_{0 \leq t \leq T^*} \left[ ||f^1 - f^0||_{L^\infty} + ||S^1 - S^0||_{L^\infty} \right]$$

$$\leq \frac{4L(C(L + L^2 + L^3)e^{(CL+CL^2)T^*} + CLe^{CLT^*})^{n(T^*)^n}}{n!}.$$ (45)

The term on the right hand side makes up a convergent series, from which we conclude that the sequences $f^n$ and $S^n$ are Cauchy sequences. \qed

We next return to the proof of Theorem 3.1.

Proof of Theorem 3.1: Now, we can construct a limit function

$$(f, S) \in L^\infty([0, T^*); L^\infty(\mathbb{R}^d)) \times L^\infty([0, T^*); L^\infty(\mathbb{R}^d)).$$

This gives a solution to (5) in the distribution sense. Moreover, we have uniform bounds for $f^n$ and $S^n$ in the spaces:

$$f^n \in W^{1,\infty}([0, T^*); L^\infty(\mathbb{R}^d)) \cap L^\infty([0, T^*); (L^1_+ \cap W^{1,\infty}(\mathbb{R}^d)),$$

$$S^n \in W^{1,\infty}([0, T^*); L^\infty(\mathbb{R}^d)) \cap L^\infty([0, T^*); (L^1_+ \cap W^{2,\infty}(\mathbb{R}^d)).$$
Thus, there exists a weak* limit \((f^\infty, S^\infty)\) such that
\[
\begin{align*}
f^\infty \in W^{1,\infty}([0, T^*); L^\infty(\mathbb{R}^d)) \cap L^\infty([0, T^*); (L^1_+ \cap W^{1,\infty})(\mathbb{R}^d)), \\
S^\infty \in W^{1,\infty}([0, T^*); L^\infty(\mathbb{R}^d)) \cap L^\infty([0, T^*); (L^1_+ \cap W^{2,\infty})(\mathbb{R}^d)).
\end{align*}
\]
Because \(f^n\) and \(S^n\) are Cauchy, the limit functions \((f, S)\) coincide with the weak* limit \((f^\infty, S^\infty)\). Thus, we have
\[
\begin{align*}
f \in W^{1,\infty}([0, T^*); L^\infty(\mathbb{R}^d)) \cap L^\infty([0, T^*); (L^1_+ \cap W^{1,\infty})(\mathbb{R}^d)), \\
S \in W^{1,\infty}([0, T^*); L^\infty(\mathbb{R}^d)) \cap L^\infty([0, T^*); (L^1_+ \cap W^{2,\infty})(\mathbb{R}^d)).
\end{align*}
\]
This finish the proof of Theorem 3.1.

4. Global existence of strong solutions. In this section, we show that the local strong solution will never blow up in any finite-time interval; thus the local strong solution can be extended to the global strong solution by the continuous induction argument. More precisely, the main result of this paper is given as follows.

**Theorem 4.1.** Under the same assumptions as those in Theorem 3.1, there exists a global strong solution \((f, S)\) to \((5)\) satisfying for any \(T > 0\) that
\[
\begin{align*}
f &\in W^{1,\infty}([0, T); L^\infty(\mathbb{R}^d)) \cap L^\infty([0, T); (L^1_+ \cap W^{1,\infty})(\mathbb{R}^d)), \\
S &\in W^{1,\infty}([0, T); L^\infty(\mathbb{R}^d)) \cap L^\infty([0, T); (L^1_+ \cap W^{2,\infty})(\mathbb{R}^d)).
\end{align*}
\]
Proof. Since the proof is rather lengthy, we postpone the proof to the end of this section.

In the following subsections, we present a series of a priori estimates.

4.1. A priori estimates. In this subsection, we provide several a priori estimates.

**Lemma 4.2.** Let \(S = S(x, t)\) be a strong solution to \((5)\) in the time interval \([0, T)\). Then, we have for \(t \in (0, T)\):
\[
||S(t)||_{L^1} \leq ||S_0||_{L^1}, \quad \sup_{0 \leq t \leq T} ||S(t)||_{L^\infty} \leq ||S_0||_{L^\infty}, \quad ||S(t)||_{L^\infty} \leq (4\pi t)^{-\frac{d}{2}} ||S_0||_{L^1}.
\]
Proof. (i) Integrating the equation of \(S\) with respect to \(x\), we obtain
\[
\frac{d}{dt} ||S(t)||_{L^1} \leq 0.
\]
Thus, we have \(||S(t)||_{L^1} \leq ||S_0||_{L^1}||.

(ii) Note that \(S\) satisfies
\[
\partial_t S - \Delta S = -\kappa_2 S\rho, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}_+,
\]
\[
S(x, 0) = S_0, \quad x \in \mathbb{R}^d.
\]
By Duhamel’s principle for the inhomogeneous heat equation, we have
\[
S(x, t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{(x-y)^2}{4t}} S_0(y) dy
\]
\[
- \kappa_2 \int_0^t \int_{\mathbb{R}^d} \frac{1}{(4\pi (t-s))^{\frac{d}{2}}} e^{-\frac{(x-y)^2}{4(t-s)}} S(y, s) \rho(y, s) dy ds.
\]

We next use \(\frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{(x-y)^2}{4t}} dy = 1\) and the positivity of \(\rho\) and \(S\) in \((46)\) to obtain
\[
S(x, t) \leq \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{(x-y)^2}{4t}} S_0(y) dy \leq ||S_0||_{L^\infty}, \quad x \in \mathbb{R}^d, \quad t > 0.
\]
Lemma 4.3. For a positive constant $T \in (0, \infty)$, let $(f, S)$ be a strong solution to (16). Then, we have the following estimates:

$$\sup_{0 \leq t < T} |M_1(t)| \leq C(|M_1(0)| + 1) \quad \text{and} \quad \sup_{0 \leq t < T} M_2(t) \leq C(M_2(0) + 1),$$

where $C$ is a positive constant that is independent of $t$.

Proof. (i) (The estimate of $M_1$): We multiply $v$ by the first equation of (16) and integrate the resulting equation with respect to $x$ and $v$ to obtain:

$$|M_1(t)| = \left| M_1(0) + \kappa_1 \int_0^t \int_{\mathbb{R}^d} v T[S](f) dv dx ds \right|$$

$$\leq |M_1(0)| + 2\kappa_1 M_0 R_v |B_{R_v}| \int_0^t \|S(s)\|_{L^\infty}^2 ds$$

$$\leq |M_1(0)| + 2\kappa_1 R_v |B_{R_v}| M_0 \left( \|S_0\|_{L^\infty}^2 + \frac{1}{(d - 1)(4\pi)^{d/2}} \|S_0\|_{L^1}^2 \right),$$

where we used the estimate in Lemma 4.2, and $V = B_{R_v}$ is defined in (42) in Section 3.1. This yields the desired result.

(ii) (The estimate of $M_2$): We multiply $|v|^2$ by the first equation of (16) and integrate the resulting relation to obtain:

$$\frac{dM_2}{dt} = -\kappa_0 \int_{\mathbb{R}^d} \psi(|x - y|)|v - v_*|^2 f(y, v_*) f(x, v) dv dy dx$$

$$+ \kappa_1 \int_{\mathbb{R}^d} |v|^2 T[S](f) dv dx$$

$$\leq -2\kappa_0 \psi_m M_0 M_2(t) + 2\kappa_0 \psi_m |M_1(t)|^2$$

$$+ \kappa_1 \int_{\mathbb{R}^d} \int_{V \times V} |v|^2 S(x + v) S(x - v') f(x, v') dv' dv dx. \tag{47}$$

The second term in (47) can be estimated as follows.

$$\int_{\mathbb{R}^d} \int_{V \times V} \|v|^2 S(x + v) S(x - v') f(x, v') dv' dv dx \leq R_v^2 |B_{R_v}| \|S_0\|_{L^\infty}^2 M_0(0). \tag{48}$$

We combine (47) and (48) to obtain

$$\frac{dM_2}{dt} \leq -\kappa_0 \psi_m 2M_0 M_2(t) + 2\kappa_0 \psi_m |M_1(t)|^2 + \kappa_1 R_v^2 |B_{R_v}| \|S_0\|_{L^\infty}^2 M_0(0).$$

This yields

$$M_2(t) \leq M_2(0)e^{-2\kappa_0 \psi_m t} + O(1) \int_0^t e^{-(2\kappa_0 \psi_m)(t-s)} \left( 1 + M_2^2(s) \right) ds \leq M_2(0) + O(1).$$

Thus, we have the desired estimate:

$$M_2(t) \leq C(M_2(0) + 1), \quad t \in [0, T).$$
where the constant $C$ depends on $|B_{R_v}|, R_v, \psi_m, \kappa_0, \kappa_1, d, ||S_0||_{\infty}, ||S_0||_{L^1}$, and $M_2(0)$. \hfill \square

Next, we estimate the size of the velocity support of $f$. For this, we consider the following backward characteristic equation: for $x,v \in \mathbb{R}^d$ and $0 \leq \tau \leq t < T$,
\[
\frac{dx(\tau; t, x, v)}{d\tau} = v(\tau), \quad \frac{dv(\tau)}{d\tau} = F_a[f](x(\tau; t, x, v), v(\tau; t, x, v), \tau),
\]
(49)
\[x(t) = x, \quad v(t) = v.\]

**Lemma 4.4.** For $T \in (0, \infty)$, let $(f, S)$ be a strong solution to (5) in the time interval $[0, T)$, and the support of $f_0$ in the velocity domain is contained in a compact set independent of $x$. Then, there exists a compact domain $D \subset \mathbb{R}^d$ such that for $(x, t) \in \mathbb{R}^d \times [0, T)$,
\[
supp_v(f(x, \cdot, t)) \subset D.
\]

**Proof.** We set a sequence of increasing balls:
\[D_n := B(0; r_n), \quad r_n := n \max \left\{ \frac{1}{\psi_m M_0}, C(|M_1(0)| + 1), R_v \right\}, \quad n \geq 1.
\]
Suppose that the initial configuration has compact support in the velocity domain
\[
supp_v(f_0) \subset D_{n_1}, \quad \text{for some } n_1 \geq 2.
\]
We claim:
\[f(x, v, t) \equiv 0, \quad \text{for } v \notin D_{n_2} \quad \text{and} \quad n_2 \geq n_1 + 1.\] (50)
This immediately show that $\text{supp}_v(f(x, \cdot, t)) \subset D$. Next, we prove the claim by contradiction. Suppose we have
\[
\text{supp}_v(f_0) \subset D_{n_1} \quad \text{and} \quad f(x, v, t) \neq 0 \quad \text{at} \quad (x, v, t), \quad v \notin D_{n_2}.\]
(51)
Then we notice from the definition of $D_n$ that, for any $v \notin D_{n_1}$, $v$ is out of $V$ which is the support of the turning kernel. Thus, only alignment force affect the dynamic and we have
\[
\frac{d}{d\tau} |v(\tau; t, x, v)| \leq -\psi_m M_0 |v(\tau; t, x, v)| \leq 0, \quad \text{whenever} \quad |v(\tau; t, x, v)| \in D_{n_1}^c.
\]
(52)
On the other hand, due to the continuity of $v(\tau; t, x, v)$ and the fact $\text{supp}_v(f_0) \subset D_{n_1} \subset D_{n_2}$, there must be $\tau_0$ such that
\[
v(\tau; t, x, v) \in D_{n_1}^c \cap D_{n_2}, \quad \tau_0 \leq \tau \leq t.
\]
From (52), we immediately obtain
\[
|v| = |v(t; t, x, v)| \leq |v(\tau_0; t, x, v)|.
\]
(53)
However, the facts $v \in D_{n_2}^c$ and $v(\tau_0; t, x, v) \in D_{n_1}^c \cap D_{n_2}$ imply that
\[
|v| \geq r_{n_2} > |v(\tau_0; t, x, v)|,
\]
which is contradictory to (53). Therefore, assumption (51) is not true and we finish the proof. \hfill \square

**Remark 2.** For $D$ obtained in the above lemma, we actually have $V \subset D$, where $V$ is the turning kernel effect region in assumption $(A2)$, which was discussed in Section 3.1.
4.2. Velocity alignment estimate. In this subsection, we present the a priori velocity alignment estimate. For the velocity alignment estimate of model (5), we cannot use the Lyapunov functional (10) as we do not know a priori the asymptotic velocity of the particles. This is due to the lack of the conservation of momentum. Hence, we instead use a new functional $L_1$:

$$L_1(t) := \int_{\mathbb{R}^d} |v - v_*|^2 f(y,v) f(x,v) dv dvdydx, \quad t \geq 0.$$ 

Then, it is easy to see that the functional $L_1$ can be rewritten as follows.

$$L_1(t) := 2(M_0 M_2 - |M_1|^2).$$

**Lemma 4.5.** Let $(f,S)$ be a strong solution to (5). Then, the Lyapunov functional $L_1(t)$ decays to zero in that

$$L_1(t) \leq O(1)(1 + t)^{-d}, \quad t \geq 0.$$

**Proof.** We use Lemma 2.2 and Lemma 4.2 to see the time-variation of $L_1(t)$:

$$\frac{dL_1}{dt} = 2M_0 \frac{dM_2}{dt} - 4 \left( \frac{dM_1}{dt} \right) M_1$$

$$= -2M_0 \kappa_0 \int_{\mathbb{R}^d} \psi(|x - y|)|v - v_*|^2 f(y,v) f(x,v) dv dvdydx$$

$$+ 2M_0 \kappa_1 \int_{\mathbb{R}^d} |v|^2 T[S](f) dv dx - 4 \kappa_1 \left( \int_{\mathbb{R}^d} v T[S](f) dv dx \right) M_1$$

$$\leq -2M_0 \kappa_0 \psi_m L_1 + 2M_0 \kappa_1 \int_{\mathbb{R}^d} |v|^2 T[S](f) dv dx - 4 \kappa_1 \left( \int_{\mathbb{R}^d} v T[S](f) dv dx \right) M_1$$

$$\leq -2M_0 \kappa_0 \psi_m L_1 + 2M_0 R_v^2 \kappa_1 \int_{\mathbb{R}^d} |T[S](f)| dv dx + 4 \kappa_1 R_v \int_{\mathbb{R}^d} |T[S](f)| dv dx M_1$$

$$\leq -2M_0 \kappa_0 \psi_m L_1 + \left( 2M_0 R_v^2 \kappa_1 + 4 \kappa_1 R_v \sqrt{M_0 M_2} \right) \int_{\mathbb{R}^d} |T[S](f)| dv dx$$

$$\leq -2M_0 \kappa_0 \psi_m L_1 + 2M_0 R_v^2 \kappa_1 |B_{R_v}(0)| \int_{\mathbb{R}^d} f dv |S|_{L^\infty}^2$$

$$+ 4 \kappa_1 R_v \sqrt{CM_0(M_2(0) + 1)} |B_{R_v}(0)| \int_{\mathbb{R}^d} f dv |S|_{L^\infty}^2$$

$$\leq -2M_0 \kappa_0 \psi_m L_1 + O(1)(1 + t)^{-d}.$$

The last inequality follows the decay rate of $||S||_{L^\infty}$ in Lemma 4.2. From Duhammel’s principle, the above relation yields

$$L_1(t) \leq O(1)(1 + t)^{-d}.$$ 

4.3. Proof of Theorem 4.1. In this subsection, we provide the proof of Theorem 4.1 with respect to the global existence of strong solutions. In Theorem 3.1, we have already seen that the coupled system (5) yields the local existence of strong solutions. Thus, for the global existence, we will extend these local solutions to global ones by showing the finiteness of the following quantity in any finite-time interval: for any $T \in (0, \infty),$$$

$$\sum_{0 \leq |\alpha| \leq 2} ||\nabla_x^\alpha S(t)||_{L^\infty} + \sum_{0 \leq |\alpha| + |\beta| \leq 1} ||\nabla_x^\alpha \nabla_v^\beta f(t)||_{L^\infty} < \infty.$$
Lemma 4.6. Assume that the assumptions (A1) - (A3) hold, and the initial data 
\( (f_0, S_0) \) satisfy 
\[
    f_0 \in (L^1_+ \cap L^\infty)(\mathbb{R}^{2d}) \quad \text{and} \quad S_0 \in (L^1_+ \cap L^\infty)(\mathbb{R}^{d}).
\]
Then, for any \( T \in (0, \infty) \), there exists a positive constant \( C \) and time dependent constant \( C(T) \) such that 
\[
    \|S(t)\|_{L^p} \leq C \quad \text{and} \quad \|f(t)\|_{L^\infty} \leq C(T)\|f_0\|_{L^\infty}, \quad t \in (0, T).
\]

Proof. (i) (Estimate of \( \|S\|_{L^p} \)): For \( p = 1 \) and \( \infty \), the desired estimates follow from Lemma 4.2. On the other hand, for \( p \in (1, \infty) \), we use the interpolation lemma to find 
\[
    \|S(t)\|_{L^p} \leq C.
\]
(ii) (Estimate of \( \|f\|_{L^\infty} \)): We first multiply \( p|p|^{p-1} \) by the kinetic equation (5), and integrate with respect to \( x \) and \( v \) to obtain
\[
    \frac{d}{dt} \|f\|_{L^p}^p \leq -(p-1) \int_{\mathbb{R}^{2d}} f^p \nabla_v \cdot (F_a[f]) dv dx + \kappa_1 p \int_{\mathbb{R}^{d}} T[S] f^p dv dx
\]
\[
    =: I_{21} + p \kappa_1 I_{22}. \tag{54}
\]

- (Estimate of \( I_{21} \)): We use 
\[
    - \nabla_v \cdot (F_a[f]) = \kappa_0 \nabla_v \cdot \left( \int_{\mathbb{R}^{2d}} \psi(|x-y|)(v-v_\ast) f(y, v_\ast) dv_x dv_y \right)
\]
\[
    = \kappa_0 \int_{\mathbb{R}^{2d}} \psi(|x-y|) f(y, v_\ast) dv_x dv_y \leq \kappa_0 \psi_M M_0
\]
to find 
\[
    |I_{21}(t)| = \left| \int_{\mathbb{R}^{2d}} (p-1) f^p \nabla_v \cdot (F_a[f]) dv dx \right| \leq \kappa_0 \psi_M M_0 (p-1) ||f||_{L^p}^p. \tag{55}
\]

- (Estimate of \( I_{22} \)): We use the assumption (A2) in Section 3.1 to obtain 
\[
    |I_{22}(t)| \leq \int_{\mathbb{R}^{d}} \left( \int_{\mathbb{R}^{d}} S(x-v', t) f' dv' \int_{\mathbb{R}^{d}} S(x+v, t) f^{p-1} dv \right) dx
\]
\[
    \leq \left[ \int_{\mathbb{R}^{d}} \left( \int_{\mathbb{R}^{d}} S(x-v', t) f' dv' \right)^p dx \right]^{\frac{1}{p}} \left[ \int_{\mathbb{R}^{d}} \left( \int_{\mathbb{R}^{d}} S(x+v, t) f^{p-1} dv \right)^{\frac{p}{p-1}} dx \right]^{\frac{p-1}{p}}
\]
\[
    \leq \left[ \int_{\mathbb{R}^{d}} \left( |S(t)| \right)^p_{L^{\frac{p}{p-1}}} \left( |f(x, t)| \right)^p_{L^{\frac{p}{p-1}}} dx \right]^{\frac{1}{p}} \left[ \int_{\mathbb{R}^{d}} \left( |S(t)| \right)^p_{L^\frac{p}{p-1}} \left( |f(t)| \right)^p_{L^\frac{p}{p-1}} dx \right]^{\frac{p-1}{p}}
\]
\[
    = ||S(t)||_{L^{\frac{p}{p-1}}} ||f(t)||_{L^p} ||f(t)||_{L^{\frac{p}{p-1}}}^{p-1}
\]
\[
    \leq ||S(t)||_{L^{\frac{p}{p-1}}} ||f(t)||_{L^p} ||f(t)||_{L^{\frac{p}{p-1}}}^{p-1}
\]
\[
    =: C ||f(t)||_{L^p}^p. \tag{56}
\]

Thus, we have 
\[
    |I_{22}(t)| \leq C ||f(t)||_{L^p}^p. \tag{56}
\]

Finally, in (54), we combine (55) and (56) to obtain 
\[
    \frac{d}{dt} \|f\|_{L^p}^p \leq \kappa_0 \psi_M M_0 (p-1) ||f||_{L^p}^p + C p \kappa_1 ||f(t)||_{L^p}^p.
\]

This yields 
\[
    \frac{d}{dt} \|f\|_{L^p} \leq \left( \kappa_0 \psi_M M_0 \frac{p-1}{p} + C \kappa_1 \right) \|f\|_{L^p}.
\]
Then, it follows from Gronwall’s lemma that we have
\[ \|f(t)\|_{L^p} \leq \|f_0\|_{L^p} \exp \left[ \int_0^t \left( \kappa_0 \psi_M M_0 \frac{p-1}{p} + C_{\kappa_1} \right) \, d\tau \right]. \] (57)
We let \( p \to \infty \) to obtain
\[ \|f(t)\|_{L^\infty} \leq \exp \left[ \int_0^t (\kappa_0 \psi_M M_0 + \kappa_1 C) \, d\tau \right] \|f_0\|_{L^\infty} \leq C(T) \|f_0\|_{L^\infty}. \]

**Lemma 4.7.** For \( i = 1, \cdots, d \), we have
\[ \|\partial_{x_i} S(t)\|_{L^\infty} \leq \|\partial_{x_i} S_0\|_{L^\infty} + C(t) \|S_0\|_{L^\infty} \|f_0\|_{L^\infty}, \quad t \geq 0. \]
Proof. Denote the diameter of the velocity support by \( \mathcal{V}(t) \). Then, it follows from Lemma 4.4 that
\[ \mathcal{V}(t) \leq C \mathcal{V}(0). \]
Then, we have
\[
\begin{align*}
|\partial_{x_i} S(x, t)| &\leq \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{(x-y)^2}{4t}} \partial_{x_i} S_0(y) \, dy \\
&\quad + \int_0^t \left( \int_{\mathbb{R}^d} \frac{1}{(4\pi(t-s))^{\frac{d}{2}}} \left( e^{-\frac{(x-y)^2}{4(t-s)}} \right)_{x_i} S(y, s) \rho(y, s) \, dy ds \right) \\
&\quad \leq \|\partial_{x_i} S_0\|_{L^\infty} + \int_0^t \left( \int_{\mathbb{R}^d} \frac{1}{(4\pi(t-s))^{\frac{d}{2}}} \left( e^{-\frac{(x-y)^2}{4(t-s)}} \right)_{x_i} S(y, s) \rho(y, s) \, dy ds \right) \\
&\quad \leq \|\partial_{x_i} S_0\|_{L^\infty} + \mathcal{O}(1) \int_0^t \|S(s)\|_{L^\infty} \|\rho(s)\|_{L^\infty} \frac{1}{\sqrt{t-s}} ds \\
&\quad \leq \|\partial_{x_i} S_0\|_{L^\infty} + C(t) \|S_0\|_{L^\infty} \|f_0\|_{L^\infty}. \quad (58)
\end{align*}
\]
Thus, \( \|\partial_{x_i} S\|_{L^\infty} \) will not blow up in finite time.

**Lemma 4.8.** For \( T \in (0, \infty) \), we have
\[ \sup_{0 \leq t < T} \sum_{|\alpha| + |\beta| = 1} \|\nabla_x^\alpha \nabla_v^\beta f(t)\|_{L^\infty} < \infty. \]
Proof. We estimate the terms one-by-one:
\[ \|\nabla_x f(t)\|_{L^\infty} \quad \text{and} \quad \|\nabla_v f(t)\|_{L^\infty}. \]
• Case A (Estimate of \( \|\nabla_x f\|_{L^\infty} \)): Note that \( \partial_{x_i} f \) satisfies
\[
\begin{align*}
\partial_t (\partial_{x_i} f) + v \cdot \nabla_x (\partial_{x_i} f) + F_a[f] \cdot \nabla_v (\partial_{x_i} f) &= -\nabla_v \cdot (\partial_{x_i} (L[f]) f) - (\nabla_v \cdot L[f]) \partial_{x_i} f \\
&\quad + \int_{\mathbb{R}^d} ((\partial_{x_i} T) f' + T \partial_{x_i} f' - (\partial_{x_i} T^*) f - T^* \partial_{x_i} f) \, dv'. \quad (59)
\end{align*}
\]
Then, we integrate the equation (59) along the particle trajectory
\[ [x(t), v(t)] := [x(t; 0, x, v), v(t; 0, x, v)] \]
to obtain
$$\left| \partial_{x_i} f(x(t), v(t), t) \right|$$
$$\leq \left| \partial_{x_i} f_0(x, v) \right| + \int_0^t \left( |\nabla_v \cdot (\partial_{x_i} (F_a[f]) f) | + | (\nabla_v \cdot F_a[f]) \partial_{x_i} f | \right) ds$$
$$+ \int_0^t \int_{\mathbb{R}^d} | (\partial_{x_i} T) f' | + | T \partial_{x_i} f' | + | (\partial_{x_i} T^*) f | + | (T^* \partial_{x_i} f) | (x(s), v(s), s) dv' ds.$$  \hfill (60)

\begin{itemize}
  \item Case A.1: For the term $|\nabla_v \cdot (\partial_{x_i} (F_a[f]) f)$, we have
    $$|\nabla_v \cdot (\partial_{x_i} (F_a[f]) f) | \leq |(\partial_{x_i} \nabla_v \cdot F_a[f]) f| + |(\partial_{x_i} F_a[f]) \cdot \nabla_v f|$$
    $$\leq C(\psi) M_0 ||f||_{L^\infty} + C(\psi)(\sqrt{M_2(t) M_0} + M_0 |v(t)|)||\nabla_v f||_{L^\infty}$$
    $$\leq C(\psi) M_0 ||f||_{L^\infty} + C(\psi)(\sqrt{M_2(t) M_0} + C M_0 V_0)||\nabla_v f||_{L^\infty}.\quad (61)$$

  \item Case A.2: For $|(\nabla_v \cdot F_a[f]) \partial_{x_i} f|$, we have
    $$|(\nabla_v \cdot F_a[f]) \partial_{x_i} f| \leq C(\psi) M_0 ||\partial_{x_i} f||_{L^\infty}.\quad (62)$$

  \item Case A.3: For $\partial_{x_i} T^* f'$, from Assumption (14), Lemma 4.6, and Lemma 4.7, we have
    $$|\partial_{x_i} T[S]| \leq |\nabla_x (S(x + v, t) S(x - v', t))|$$
    $$\leq 2 ||S_0||_{L^\infty} (||\partial_{x_i} S_0||_{L^\infty} + C(t) ||S_0||_{L^\infty} ||f_0||_{L^\infty}).\quad (63)$$
\end{itemize}

Thus, in (60), we combine (61), (62), and (63) to find
$$\left| \partial_{x_i} f(x(t), v(t), t) \right|$$
$$\leq \left| \partial_{x_i} f_0(x, v) \right| + \int_0^t \left[ C(\psi) M_0 ||f(s)||_{L^\infty} 
+ C(\psi)(\sqrt{M_2(s) M_0} + C M_0 V_0)||\nabla_v f(s)||_{L^\infty} \right] ds + \int_0^t C(\psi) M_0 ||\partial_{x_i} f(s)||_{L^\infty} ds$$
$$+ C \int_0^t \left[ ||S_0||_{L^\infty} (||\partial_{x_i} S_0||_{L^\infty} + C(s) ||S_0||_{L^\infty} ||f_0||_{L^\infty}) ||f(s)||_{L^\infty} \right] ds$$
$$+ C \int_0^t ||S_0||_{L^2} ||\partial_{x_i} f(s)||_{L^\infty} ds.$$  \hfill (64)

This yields
$$||\nabla_x f(t)||_{L^\infty}$$
$$\leq ||\nabla_x f_0||_{L^\infty} + \int_0^t C_1(s)||\nabla_v f(s)||_{L^\infty} ds + \int_0^t C_2(s)||\nabla_x f(s)||_{L^\infty} ds + C_3(t).$$

- Case B (Estimate of $||\nabla_v f||_{L^\infty}$): Similar to Case A, we have
  $$||\nabla_v f(t)||_{L^\infty} \leq ||\nabla_v f_0||_{L^\infty} + \int_0^t C_4(s)||\nabla_v f(s)||_{L^\infty} ds + \int_0^t C_5(s)||\nabla_x f(s)||_{L^\infty} + C_6(t).$$

  From the above estimates, we apply Gronwall’s inequality to conclude that $||\nabla_x \nabla_x^i f(t)||_{L^\infty}$ will not blow up in finite-time, where $i + j = 1$. \hfill \Box

Finally, we need to consider the second-order regularity of the chemical concentration $S(x, t)$.

**Lemma 4.9.** The quantity $||\Delta_x S(t)||_{L^\infty}$ remains finite in any finite-time interval.
Proof. We use the relation (58) and Lemma 4.8 to rewrite $\Delta_x S$ as follows.

$$|\partial^2_{x_i} S(x,t)| \leq \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{(x-y)^2}{4t}} \partial_{y_i} S_0(y) dy$$

$$+ \left| \int_0^t \int_{\mathbb{R}^d} \frac{1}{(4\pi (t-s))^{\frac{d}{2}}} e^{-\frac{(x-y)^2}{4(t-s)}} (S(y,s) \rho(y,s)) y_i dy ds \right|$$

$$ \leq O(1) \frac{1}{\sqrt{t}} ||\partial_{x_i} S_0||_{L^\infty}$$

$$+ C \int_0^t \frac{1}{\sqrt{t-s}} \left( ||S_{x_i}(s)||_{L^\infty} ||f(s)||_{L^\infty} + ||S(s)||_{L^\infty} ||f_{x_i}(s)||_{L^\infty} \right) ds. \quad (65)$$

From the previous estimates for $||\nabla_x S||_{L^\infty}, ||\nabla_x f||_{L^\infty}, \text{and } ||f||_{L^\infty}$, we can conclude that $||\Delta_x S||_{L^\infty}$ will not blow up in finite time. \hfill $\square$

Now, we are ready to present the proof of Theorem 4.1.

Proof of Theorem 4.1: We combine Lemmata 4.6, 4.7, 4.8, and 4.9 to obtain a strong solution in any time period $[0, T)$:

$$f \in W^{1,\infty}([0, T); \mathbb{L}^\infty(\mathbb{R}^{2d})) \cap L^\infty([0, T); (L^1_+ \cap W^{1,\infty})(\mathbb{R}^{2d}))$$

$$S \in W^{1,\infty}([0, T); \mathbb{L}^\infty(\mathbb{R}^{d})) \cap L^\infty([0, T); (L^1_+ \cap W^{2,\infty})(\mathbb{R}^{d}))$$

if the assumptions of the initial data and the turning kernel in Theorem 4.1 hold. Moreover, by applying Lemma 4.5, we can obtain asymptotic behavior of the strong solution. More precisely, flocking emerges asymptotically:

$$\int_{\mathbb{R}^{4d}} |v - v_*|^2 f(y, v_*) f(x, v) dv_* dv dx \leq O(1) (1 + t)^{-d}. \quad (65)$$

Remark 3. The chemical diffusion may attract particles to positions far away from the initial position, although the possibility of particles moving towards infinity is very small. Thus, the support of the kinetic density $f = f(x, v, t)$ with respect to $x$ is unbounded.

5. Conclusion. In this paper, we presented a new mathematical modeling of the spatio-temporal dynamics of C-S flocking particles with chemotactic movements. Our combined model consists of two coupled equation. For the evolution of the kinetic density, we employed the kinetic C-S model which is a Vlasov-McKean type equation, whereas for the chemical density, we used the standard reaction-diffusion equation. These are coupled through the turning operator which represent the abrupt change of velocities of flocking particles. For this coupled system, we provided a global well-posedness of strong solution and presented a velocity alignment estimate for the special choice of reaction term. In particular, for velocity alignment estimate, we employed a robust Lyapunov functional approach measuring the velocity variation, and we showed that under suitable setting on the communication function and reaction term, we show that this functional tends to zero algebraically fast. This zero convergence implies the velocity alignment in probability. Finally, we also provide two macroscopic models using two scaling limits, parabolic limit and hyperbolic limit from the proposed coupled model, respectively and provide analytical forms for transport coefficients. There are still lots of interesting problems to be explored in future. For example, we assume that the positive lower bound for the communication weight function to show that the constructed strong solution satisfies velocity alignment. Moreover, to figure out the dynamic effects of singular...
communication weight functions will be also an interesting problem. We will leave
this issues for a future work.

Appendix A. A formal parabolic limit. In this appendix, we present a formal
parabolic limit, which is called the drift-diffusion limit for the generalized kinetic
C-S model following the presentations given in [7, 14, 26].

Recall a generalized kinetic C-S model with chemotaxis:
\[
\partial_t f + v \cdot \nabla_x f + \kappa_0 \nabla_v \cdot (F_u[f]f) = \kappa_1 T[S](f), \quad (x, v, t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+,
\]
\[
\partial_t S - \Delta S = \kappa_2 \varphi(S, \rho), \quad \rho = \int_{\mathbb{R}^d} f dv.
\]
(66)

A.1. Drift-diffusion limit. We rescale time and space variables as follows.
\[(x, v, t) \rightarrow (\varepsilon x, \varepsilon v, \varepsilon^2 t), \quad \text{for } \varepsilon > 0.\]

Then, we obtain a rescaled system with small parameter \(\varepsilon\):
\[
\begin{cases}
\varepsilon \partial_t f_\varepsilon + \kappa_0 \left( v \cdot \nabla_x f_\varepsilon + \nabla_v \cdot (F_u[f_\varepsilon]f_\varepsilon) \right) = \frac{\kappa_1}{\varepsilon} T_\varepsilon[S_\varepsilon](f_\varepsilon), \\
\partial_t S_\varepsilon - \Delta S_\varepsilon = \kappa_2 \varphi(S_\varepsilon, \rho_\varepsilon), \quad \rho_\varepsilon = \int_{\mathbb{R}^d} f_\varepsilon dv,
\end{cases}
\]
(67)

After performing relatively lengthy formal calculations, we can show that the macroscopic system arising from the drift-diffusion limit (from (66)) is given by the
generalized Keller-Segel system with flocking mechanism:
\[
\begin{align*}
\partial_t \rho + \nabla_x \cdot \left( \Gamma[S] \rho + A[S] \rho L[\rho] \right) &= \nabla_x \cdot \left( D[S] \nabla_x \rho \right), \\
\partial_t S - \Delta S &= \kappa_2 \varphi(S, \rho),
\end{align*}
\]
where transport coefficients \(\Gamma[S], A[S],\) and \(D[S]\) are given by explicit expressions
depending on \(S\) (see (77)).

We now begin the procedure of determining the drift-diffusion limit. We first
introduce hydrodynamic observables (local mass and current densities): for \((x, t) \in \mathbb{R}^d \times \mathbb{R}_+\),
\[
\rho_\varepsilon(x, t) := \int_{\mathbb{R}^d} f_\varepsilon dv, \quad j_\varepsilon(x, t) := \int_{\mathbb{R}^d} v f_\varepsilon dv.
\]

We integrate the first equation in (67) with respect to the variable \(v\) over \(\mathbb{R}^d\) to
obtain the continuity equation:
\[
\begin{align*}
\varepsilon \partial_t \rho_\varepsilon + \nabla_x \cdot j_\varepsilon &= 0, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}_+, \\
\partial_t S_\varepsilon - \Delta S_\varepsilon &= \kappa_2 \varphi(S_\varepsilon, \rho_\varepsilon), \quad \rho_\varepsilon = \int_{\mathbb{R}^d} f_\varepsilon dv.
\end{align*}
\]
(68)

We next proceed to perform a formal asymptotic analysis for (68) using the following
ansatzes:
\[
\begin{align*}
f_\varepsilon &= f_0 + \varepsilon f_1 + \mathcal{O}(\varepsilon^2), \quad \rho_\varepsilon = \rho_0 + \varepsilon \rho_1 + \mathcal{O}(\varepsilon^2), \\
S_\varepsilon &= S_0 + \varepsilon S_1 + \mathcal{O}(\varepsilon^2), \quad T_\varepsilon[S] = T_0[S] + \varepsilon T_1[S] + \mathcal{O}(\varepsilon^2).
\end{align*}
\]
(69)

For \(S_0\) in (69), the leading order dynamics for the second equation is clearly given
by the equation:
\[
\partial_t S_0 - \Delta S_0 = \kappa_2 \varphi(S_0, \rho_0), \quad \rho_0 = \int_{\mathbb{R}^d} f_0 dv.
\]
(70)
On the other hand, we substitute the ansatz (69) into (68) to obtain
\[ \varepsilon (\partial_t \rho_0 + \varepsilon \partial_t \rho_1 + O(\varepsilon^2)) + \nabla_x \cdot \left( \int_{\mathbb{R}^d} v f_0 dv + \varepsilon \int_{\mathbb{R}^d} v f_1 dv + O(\varepsilon^2) \right) = 0. \]
By comparing the terms in \( O(1) \) and \( O(\varepsilon) \), we can see that
\[ \int_{\mathbb{R}^d} v f_0 dv = 0, \quad \partial_t \rho_0 + \nabla_x \cdot \left( \int_{\mathbb{R}^d} v f_1 dv \right) = 0. \quad (71) \]
Thus, once we have an expression for \( f_1 \) in terms of \( \rho_0 \) and \( S_0 \), we will obtain the desired leading-order dynamics for \( \rho_0 \) and \( S_0 \). To derive such a representation for \( f_1 \), we return to (67), and substitute the ansatzes (69) and (70) into (67) to obtain
\[ \varepsilon^2 \partial_t \left( \partial_t f_0 + \varepsilon \partial_t f_1 + O(\varepsilon^2) \right) + \varepsilon \left( v \cdot \nabla_x f_0 + \varepsilon v \cdot \nabla_x f_1 + O(\varepsilon^2) \right) + \varepsilon \kappa_0 \left( \nabla_v \cdot \left( (F_a[f_0] + \varepsilon F_a[f_1] + O(\varepsilon^2))(f_0 + \varepsilon f_1 + O(\varepsilon^2)) \right) \right) = \kappa_1 \left( T_0[S_0](f_0) + \varepsilon T_1[S_0](f_0) + \varepsilon T_0[S_0](f_1) + \varepsilon T_0[S_0,S_1](f_0) + O(\varepsilon^2) \right), \]
where \( T_0[S_0,S_1] \) is a turning operator whose kernel is the Fréchet derivative of \( T_0 \) with respect to \( S \), and it is evaluated at \( S_0 \) in the direction of \( S_1 \).

We now compare terms in \( O(1), O(\varepsilon), O(\varepsilon^2), \cdots \).
\( O(1) \): \( T_0[S_0](f_0) = 0 \),
\( O(\varepsilon) \): \( v \cdot \nabla_x f_0 + \varepsilon \kappa_1 \left( T_1[S_0](f_0) + T_0[S_0](f_1) + T_0[S_0,S_1](f_0) \right) \).

At this stage, we assume that there exists a bounded equilibrium velocity distribution \( F = F(v) > 0 \) that is independent of \( x, t \), and \( S \) satisfying the detailed balance principle and suitable normalization conditions:
\[ T_0[S_0]F' = T_0^* S_0 F, \quad \int_{\mathbb{R}^d} F(v) dv = 1, \quad \int_{\mathbb{R}^d} v F(v) dv = 0. \quad (72) \]
Then, it follows from the entropy inequality [7] that the kernel of \( T_0[S_0] \) is spanned by \( F \). Thus, we can set
\[ f_0(x,v,t) = \rho_0(x,t) F(v). \quad (73) \]
It is easy to see that the ansatz (73) satisfies the equation (72), and
\[ F_a[f_0] = -\kappa_0 \int_{\mathbb{R}^{2d}} \psi(|x-y|)(v-v_*) \rho_0(y,t) F(v_*) dv_* dy \]
\[ = -\kappa_0 \int_{\mathbb{R}^d} \psi(|x-y|) \rho_0(y,t) \left( \int_{\mathbb{R}^d} (v-v_*) F(v_*) dv_* \right) dy \]
\[ = -\kappa_0 v \int_{\mathbb{R}^d} \psi(|x-y|) \rho_0(y,t) dy \]
\[ = -\kappa_0 v L[\rho_0], \]
where \( L[\rho_0] \) is given by the relation:
\[ L[\rho_0](x,t) := \int_{\mathbb{R}^d} \psi(|x-y|) \rho_0(y,t) dy. \]
On the other hand, it follows from (72) and (74) that
\[ T_0[S_0](f_1) = v \cdot \nabla_x f_0 + \nabla_v \cdot (F_a[f_0] f_0) = -\kappa_1 T_1[S_0](f_0) \]
\[ = (vF) \cdot \nabla_x \rho_0 - \kappa_0 (\nabla_v \cdot (vF)) \rho_0 L[\rho_0] - \kappa_1 (T_1[S_0](F)) \rho_0. \]
Now, we assume that the following equations are solvable:

\[ T_0[S_0](\kappa) = vF, \quad T_0[S_0](\mu) = \nabla_v \cdot (vF), \quad T_0[S_0](\Theta) = T_1[S_0](F). \]  

(75)

Then, with \( \kappa = \kappa[S_0] \in \mathbb{R}^d, \mu = \mu[S_0], \Theta = \Theta[S_0] \), we have

\[
f_1(x, v, t) = \kappa(x, v, t) \cdot \nabla_x \rho_0(x, t) - \kappa_0 \mu(x, v, t) \rho_0 L[\rho_0] - \kappa_1 \Theta(x, v, t) \rho_0 + \rho_1(x, t) F(v).
\]

(76)

We again substitute (76) into (71) to obtain

\[
\partial_t \rho_0 + \nabla_x \cdot \left( \Gamma[S_0] \rho_0 + A[S_0] \rho_0 L[\rho_0] \right) = \nabla_x \cdot \left( D[S_0] \nabla_x \rho_0 \right),
\]

where the transport coefficients \( \Gamma[S_0], A[S_0], \) and \( D[S_0] \) are defined by the following relations: for \((x, t) \in \mathbb{R}^d \times \mathbb{R}_+^+\),

\[
\Gamma[S_0](x, t) := -\kappa_1 \int_{\mathbb{R}^d} v \Theta[S_0](x, v, t) dv,
\]

\[
A[S_0](x, t) := -\kappa_0 \int_{\mathbb{R}^d} v \mu(x, v, t) dv,
\]

(77)

\[
D[S_0](x, t) := -\int_{\mathbb{R}^d} v \otimes \kappa[S_0](x, v, t) dv.
\]

A.2. Transport coefficients. In this part, we explicitly calculate the transport coefficients for some specific turning kernel [7]. We assume that the turning process is a Poisson process with the rate:

\[
\lambda[S] := \int_{\mathbb{R}^d} T^*[S] dv'.
\]

(78)

Then, we can interpret \( T^*[S]_{\lambda[S]} \) as the probability of obtaining a change in its velocity from \( v \) to \( v' \) at position \( x \) for a particle (or cell) at position \( x \), velocity \( v \) at time \( t \). In this setting, the equilibrium turning kernel \( T_0[S] \) is given by

\[
T_0[S](x, v, v', t) = \lambda[S](x, t) F(v), \quad \lambda[S] > 0.
\]

(79)

Under the setting (78) and (79), the transport coefficients \( \Gamma \) and \( D \) are already given in [7]. Thus, we mainly focus on the structural ansatz for \( A[S] \). Then, the leading order turning operator \( T_0[S] \) becomes

\[
T_0[S](f) = \int_{\mathbb{R}^d} (T_0[S] f' - T_0[S] f) dv' = \int_{\mathbb{R}^d} (\lambda[S](x, t) F(v) f(v') dv' - f(v) \int_{\mathbb{R}^d} T_0[S] dv') = \lambda[S] \rho F - \lambda[S] f = \lambda[S](\rho F - f).
\]

For \( g \) with \( \int_{\mathbb{R}^d} g dv = 0 \), we consider the problem:

\[
T_0[S](f) = g, \quad \int_{\mathbb{R}^d} f dv = 0.
\]

This yields

\[
\rho = 0, \quad f = -\frac{g}{\lambda[S]}.
\]

We now apply the above result to (75) to see that

\[
\kappa[S_0] = -\frac{v F}{\lambda[S_0]}, \quad \mu[S_0] = -\frac{\nabla_v \cdot (v F)}{\lambda[S_0]}, \quad \Theta = -\frac{T_1[S_0](F)}{\lambda[S_0]}.
\]

(80)
Finally, it follows from (77) and (80) that we have
\[
\Gamma[S_0](x,t) := \frac{\kappa_1}{\lambda[S_0]} \int_{\mathbb{R}^d} vT_1[S_0](F)dv,
\]
\[
A[S_0](x,t) := \frac{\kappa_0}{\lambda[S_0]} \int_{\mathbb{R}^d} v\nabla_x \cdot (vF)dv,
\]
\[
D[S_0](x,t) := \frac{1}{\lambda[S_0]} \int_{\mathbb{R}^d} v \otimes vFdv.
\]

**Appendix B. A formal hydrodynamic limit.** In this appendix, we study the hyperbolic limit for (66). To do this, we rescale the phase-space variables and time:
\[
(x,v,t) \rightarrow (\varepsilon x, \varepsilon v, \varepsilon t), \quad \text{for } \varepsilon > 0.
\]
Under the above hyperbolic scaling (81), system (66) becomes
\[
\begin{cases}
\partial_t f_{\varepsilon} + \nabla_x \cdot (vf_{\varepsilon}) + \kappa_0 \nabla_v \cdot (\mu[u][f_{\varepsilon}]f_{\varepsilon}) = \frac{\kappa_1}{\varepsilon} T_{\varepsilon}[S_{\varepsilon}](f_{\varepsilon}), \\
\partial_t S_{\varepsilon} - \Delta S_{\varepsilon} = \kappa_2 \varphi(S_{\varepsilon}, \rho_{\varepsilon}), \quad \rho_{\varepsilon} = \int_{\mathbb{R}^d} f_{\varepsilon} dv.
\end{cases}
\]
We assume that the turning operator $T_{\varepsilon}[S](f)$ takes the following form:
\[
T_{\varepsilon}[S](f) = T_0[S](f) + \varepsilon T_1[S](f),
\]
where $T_0[S]$ measures the dominant part of the turning mechanism that is mainly due to the tumble process in the absence of a chemical substance. In contrast, $T_1[S]$ is the perturbation due to chemical signals, and the parameter $\varepsilon$ is a time scale that is proportional to the turning frequency. Below, we impose some conditions on $T_i[S]$, $i = 0, 1$, which we obtained from [14].
  
- (B1): The turning operators $T_i[S]$ preserve the local mass:
  \[
  \int_{\mathbb{R}^d} T_0[S](f) dv = 0, \quad \int_{\mathbb{R}^d} T_1[S](f) dv = 0.
  \]
  
- (B2): The dominant turning operator $T_0[S]$ preserves the local flux density:
  \[
  \int_{\mathbb{R}^d} vT_0[S](f) dv = 0.
  \]
  
- (B3): For all $\rho \in [0, \infty)$ and $u \in \mathbb{R}^d$, there exists a unique function $F_{\rho,u} \in L^1((1 + |v|)dv)$ such that
  \[
  T_0[S](F_{\rho,u}) = 0, \quad \int_{\mathbb{R}^d} F_{\rho,u}(v) dv = \rho, \quad \int_{\mathbb{R}^d} vF_{\rho,u}(v) dv = \rho u.
  \]
We now introduce the first three velocity moments for $f$:
\[
\rho := \int_{\mathbb{R}^d} dv, \quad \rho u := \int_{\mathbb{R}^d} vf dv, \quad \rho E := \frac{1}{2} \int_{\mathbb{R}^d} |v|^2 dv.
\]
We set
\[
f(x,v,t) = F_{\rho,u}(v) + \varepsilon f_1(x,v,t).
\]
Then, the distribution $f_1$ satisfies
\[
\int_{\mathbb{R}^d} f_1 dv = \int_{\mathbb{R}^d} vf_1 dv = 0.
\]
• (Derivation of the continuity equation): We integrate the first equation in (82) to obtain
\[ \partial_t \rho + \nabla \cdot (\rho u) = 0. \]

• (Derivation of the momentum equation): We multiply \( v \) by the first equation in (82) to get
\[
\begin{align*}
\partial_t (\rho u) + \nabla \cdot \left( \int_{\mathbb{R}^d} v \otimes v f dv \right) &= -\kappa_0 \int_{\mathbb{R}^d} \psi(|x - y|) (u(x,t) - u(y,t)) \rho(x,t) \rho(y,t) dy + \int_{\mathbb{R}^d} v T_1[S] f dv.
\end{align*}
\]

We now use the ansatz for \( f \) as in [14]:
\[ f(x, v, t) = F_{\rho(x,t),u(x,t)}(v) + \varepsilon f_1(x, v, t). \]

We substitute this ansatz (84) into the following terms in (83) to obtain
\[
\begin{align*}
\int_{\mathbb{R}^d} v \otimes v f dv &= \int_{\mathbb{R}^d} (v - u) \otimes (v - u) F_{\rho,u} dv + nu \otimes u + O(\varepsilon) \\
&= P + nu \otimes u + O(\varepsilon), \quad (85)
\end{align*}
\]

where \( P \) is the stress tensor defined by
\[ P(x, t) := \int_{\mathbb{R}^d} (v - u(x,t)) \otimes (v - u(x,t)) F_{\rho,u} dv. \]

Then, we combine (84) and (85) to obtain the momentum balance equations after \( \varepsilon \to 0 \):
\[
\begin{align*}
\partial_t (\rho u) + \nabla \cdot \left( nu \otimes u + P \right) &= \int_{\mathbb{R}^d} (v - u) T_1[S] f dv - \kappa_0 \int_{\mathbb{R}^d} \psi(|x - y|) (u(x,t) - u(y,t)) \rho(x,t) \rho(y,t) dy.
\end{align*}
\]

Finally, we combine (82), (86), and the equation for \( S \) to obtain the hydrodynamic model:
\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho u) &= 0, \\
\partial_t (\rho u) + \nabla \cdot \left( nu \otimes u + P \right) &= \int_{\mathbb{R}^d} (v - u) T_1[S] F_{\rho,u} dv - \kappa_0 \int_{\mathbb{R}^d} \psi(|x - y|) (u(x,t) - u(y,t)) \rho(x,t) \rho(y,t) dy, \\
\partial_t S - \Delta S &= \kappa_2 \varphi(S, \rho).
\end{align*}
\]

As an explicit example of \( F_{\rho,u} \), we adopt the example in [12, 14], leading to the Cattaneo system with flocking dissipation:
\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho u) &= 0, \\
\partial_t (\rho u) + C \nabla \rho &= -\mu_1 \rho u + \mu_2 \rho \nabla S - \kappa_0 \int_{\mathbb{R}^d} \psi(|x - y|) (u(x,t) - u(y,t)) \rho(x,t) \rho(y,t) dy, \\
\partial_t S - \Delta S &= \kappa_2 \varphi(S, \rho).
\end{align*}
\]
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