Symmetry-Protected Nodal Phases in Non-Hermitian Systems

Jan Carl Budich¹, Johan Carlström², Flore K. Kunst², and Emil J. Bergholtz²

¹Institute of Theoretical Physics, Technische Universität Dresden, 01069 Dresden, Germany
²Department of Physics, Stockholm University, AlbaNova University Center, 106 91 Stockholm, Sweden

(Dated: October 3, 2018)

Non-Hermitian (NH) Hamiltonians have become an important asset for the effective description of various physical systems that are subject to dissipation. Motivated by recent experimental progress on realizing the NH counterparts of gapless phases such as Weyl semimetals, here we investigate how NH symmetries affect the occurrence of exceptional points (EPs), that generalize the notion of nodal points in the spectrum beyond the Hermitian realm. Remarkably, we find that the dimension of the manifold of EPs is generically increased by one as compared to the case without symmetry. This leads to nodal surfaces formed by EPs that are stable as long as a protecting symmetry is preserved, and that are connected by open Fermi volumes. We illustrate our findings with analytically solvable two-band lattice models in one and two spatial dimensions, and show how they are readily generalized to generic NH crystalline systems.

Introduction.— Nodal materials such as Dirac semimetals and Weyl semimetals are in the spotlight of current research due to their fascinating transport properties, including ramifications of quantum anomalies [1, 2]. The defining property of such intriguing phases of quantum matter is the topological stability of the nodal points in their band structure which may, as in the case of a spin-degenerate Dirac semimetal, rely on the presence of certain physical symmetries—a ubiquitous scenario in the context of topological phases known as symmetry protection [3–6].

Very recently, intense theoretical [7–35] and experimental [36–44] efforts have been made to extend the concept of topological band structures to non-Hermitian (NH) systems which play an important role in the effective description of dissipation effects in various physical situations, ranging from quasi-particles with a finite lifetime in strongly correlated solids to arrays of optical resonators subject to gain and loss [7, 45, 46]. There, the notion of nodal points is generalized to exceptional points (EPs) [47, 48] which, in sharp contrast to band touching points in Hermitian systems, reflect a defective, i.e. non-diagonalizable, non-Hermitian Hamiltonian [49–51]. Interestingly, in NH systems the occurrence of nodal band structures qualitatively changes as compared to their Hermitian counterparts [10–12, 36]. For example, the NH analog of a Weyl semimetal with stable EPs at isolated momenta is found in two spatial dimensions rather than three [10, 11, 36, 52]. Motivated by these insights and their recent experimental verification [36, 37], the purpose of this work is to shed light on the role of symmetry protection of EPs in NH systems, resulting in the discovery of new symmetry protected NH phases that have no immediate Hermitian analog.

Below, investigating the role of genuinely NH symmetries in nodal phases, we demonstrate that symmetry protected EPs in NH band structures generically form a surface (the analog of a nodal surface in the Hermitian context) the dimension of which is increased by one as compared to the case without symmetries. By this mechanism, nodal NH semimetals are promoted to symmetry protected NH metals, where the surfaces of EPs form the boundaries of open Fermi volumes characterized by a vanishing real part of the energy gap (see Eq. (1)). This figure was obtained from a model on the form $H = [2 - \cos k_x - \cos k_y] \sigma_x + \sigma_z/4$ which satisfies the $Q_4$-symmetry defined in Eq. (1).

Symmetries in non-Hermitian band structures.— The general problem of identifying the generic symmetries for NH systems, thus generalizing the celebrated ten-fold Altland-Zirnbauer classification from the Hermitian context [53, 54], has been addressed by Bernard and LeClair (BL) [55, 56]. Instead of ten symmetry classes, BL found a system of 43 classes, an extension based on additional...
symmetries that would not be generic in the Hermitian realm. These classes have been discussed in the context of edge states by Esaki et. al. [33] and steps toward classifying NH phases in terms of the BL classes were very recently taken by Lieu [34]. Here, we start by focusing on a concrete example of such a genuinely NH symmetry coined \( Q_+ \) to reveal novel symmetry protected NH nodal phases. The action of \( Q_+ \) on the NH Hamiltonian of a system is defined as

\[
H = qH^\dagger q^{-1}, \quad q^\dagger q^{-1} = qq^\dagger = I. \tag{1}
\]

Clearly, in the limit of a Hermitian Hamiltonian, \( Q_+ \) reduces to an ordinary unitary symmetry that commutes with the Hamiltonian, and hence is not considered in the Altland-Zirnbauer classification.

**Symmetry protected EPs in two-band models.**— We now illustrate the idea of symmetry protected EPs on the basis of lattice periodic NH two-band models, noting that, as we demonstrate below, our results can readily be generalized beyond this minimal framework. In reciprocal space, the Bloch Hamiltonian of a NH two-band model at lattice momentum \( k \) reads as

\[
H(k) = d(k) \cdot \sigma + d_0(k)\sigma_0, \tag{2}
\]

with the standard Pauli matrices \( \sigma \), the \((2 \times 2)\) identity matrix \( \sigma_0 \), and \( d \equiv d_R + id_I \) with \( d_R, d_I \in \mathbb{R}^3 \) parameterizing the Hermitian and anti-Hermitian part of the Hamiltonian, respectively. Here and in the following, we drop the momentum dependence of \( d \) for notational brevity. The eigenvalues of the Bloch Hamiltonian are given by \( E_{\pm} = d_0 \pm \sqrt{d_R^2 - d_I^2 + 2id_R \cdot d_I} \). Hence, EPs appear whenever

\[
d_R^2 - d_I^2 = 0, \quad d_R \cdot d_I = 0 \tag{3}
\]

are simultaneously satisfied. To find (second-order) EPs, we thus need to tune two parameters instead of the usual three parameters in the case of Hermitian systems. This is the basic reason why nodal NH phases, even in the absence of symmetries, occur in one dimension lower than their Hermitian counterparts.

Taking into consideration the symmetry relation \( Q_+ \) in Eq. (1), we find two inequivalent possibilities for \( q \), either \( q = \sigma_0 \) or \( q \) equals one of the Pauli matrices. In the first case, Eq. (1) reduces to \( H = H^\dagger \), constraining our Hamiltonian to the Hermitian realm, which is not the subject of our present interest. In the second case, we take \( q = \sigma_x \), such that Eq. (1) takes the explicit form \( H = \sigma_x H^\dagger \sigma_x \). This leads to the following constraints for the Hamiltonian in Eq. (2)

\[
d_x, d_0 \in \mathbb{R}, \quad d_y, d_z \in i\mathbb{R}. \tag{4}
\]

This means that the relation \( d_R \cdot d_I = 0 \) in Eq. (3) is trivially satisfied. Therefore, to find EPs in this case, we only need to solve \( d_R^2 - d_I^2 = 0 \), and we thus need to tune only one parameter instead of two. This observation has two important consequences. First, EPs, exceptional lines and exceptional surfaces generically appear in one-, two- and three-dimensional systems, respectively. This amounts to a further reduction by one of the spatial dimension in which nodal phases occur as compared to the NH case without symmetries. Second, the eigenvalue equation in the presence of \( Q_+ \) reduces to

\[
E_{\pm} = \pm \sqrt{d_R^2 - d_I^2}, \tag{5}
\]

where we have neglected \( d_0 \) as this term does not affect the gap \( \Delta E \) between the bands.

The implication of Eq. (5) is that the parameter space is divided into regions where the energy is either real, or purely imaginary, and the exceptional points thus form boundaries between Fermi volumes of vanishing energy, and regions of infinite quasi particle lifetimes where \( \text{Im}(E) = 0 \) (see Fig. 1 for an illustration). The inclusion of \( d_0 \) would just move these Fermi volumes away from \( \text{Re}(E) = 0 \), but the energy gap \( \Delta E \) would still have a vanishing real part.

We now elaborate on these insights with the help of concrete microscopic lattice models in one spatial dimension (1D) and in 2D on a square lattice with unit lattice constant. To this end, we consider

\[
d = (m + 1 - \cos(k_x) - \cos(k_y), i \sin(k_x), i \sin(k_y)) \tag{6}
\]

where \( m \in \mathbb{R} \). Clearly the Hamiltonian \( H = d \cdot \sigma \) resulting from Eq. (6) satisfies the symmetry relation \( Q_+ \) with \( q = \sigma_x \) (see Eq. (1) and Eq. (4)). To obtain our 1D model, we simply set \( k_y = 0 \) for now. If \( m = 0 \) as well, EPs occur whenever \( \cos(k_x) = \pm \sin(k_x) \), i.e. at \( k_x = \pi/4 (\text{mod} \pi/2) \). For \( m \neq 0 \), EPs satisfying the condition \( \cos(k_x)^2 - \sin(k_x)^2 = 0 \) (see Eq. 3) are still found for \( |m| \leq \sqrt{2} \), and the spectrum becomes fully gapped only for \( |m| > 2 \). Furthermore, it is clear that EPs at which the spectrum of \( H \) switches from being purely real to purely imaginary are always connected via the aforementioned open Fermi volumes, which in the present 1D case form open arcs. Thus, as long as the symmetry \( Q_+ \) is preserved, the EPs are stable in the sense that they can only be removed pairwise by bringing them together in momentum space and contracting the arcs. This behavior exemplifies and corroborates our claim of the existence of a stable nodal NH phase protected by the symmetry \( Q_+ \). To explicitly demonstrate the importance of this symmetry protection, we add a symmetry breaking perturbation \( H_y = i \delta \sigma_x \) with \( \delta > 0 \). Then, the overlap \( d_R \cdot d_I \) becomes finite and EPs can be continuously removed already at arbitrarily small \( \delta \) (cf. Eq. (3)).

Proceeding to the 2D case, we now also allow \( k_y \) to take arbitrary values in the first Brillouin zone, i.e. \( k_y \in \)}
[-π, π] in Eq. (6). A good quantity to understand the spectrum of our model is the squared energy $E^2_\pm$, which is real due to $d_R \cdot d_I = 0$, but not always positive, and has zeros at the EPs. In Fig. 2, we show the momentum dependence of $E^2_\pm$, for various values of $m$ demonstrating the rich physical phenomenology of our model (6): For $|m+1| < 1 + \sqrt{2}$, the spectrum exhibits a pair of closed lines of EPs connected by open 2D Fermi surfaces (see Fig. 2a), interrupted by the point $|m+1| = 2$, where one line of EPs collapses to a single Hermitian nodal point (see Fig. 2b). When further increasing $|m+1|$ beyond $1 + \sqrt{2}$, these lines of EPs split into four NH Fermi pockets (see Fig. 2c) and eventually disappear at $|m+1| = \sqrt{6}$, where the spectrum becomes fully gapped for all $|m+1| > 1 + \sqrt{6}$. Again, if we add the perturbation $H_b$, all EPs are removed at arbitrarily small $\delta$, highlighting the fact that the stability of the EP’s hinges on symmetry.

It is worth emphasizing that all the ingredients considered here, in particular all terms in our model (6), are available with state of the art experimental techniques: An imaginary $d_z$ is relatively easily achieved by staggered gain and loss terms acting on the two sublattices of the system [40, 41], and asymmetric, thus anti-Hermitian, hopping terms corresponding to imaginary $d_y$ (and $d_x$) have recently been realized in experiments [43, 44].

Generalization to many bands and other symmetries.— We now discuss the generalization of our results beyond the minimal setting of two band models as well as to other symmetries. Building upon the BL classification [55, 56], we consider the following system of generic NH symmetries:

- $P : H = -p H_p^{-1}, \quad p^2 = I$, \hspace{1cm} (7)
- $C : H = \epsilon c H^T c^{-1}, \quad c^T c^{-1} = \pm I$, \hspace{1cm} (8)
- $K : H = \epsilon k H^* k^{-1}, \quad k k^* = \pm I$, \hspace{1cm} (9)
- $Q : H = \epsilon q H^q q^{-1}, \quad q^q q^{-1} = I$, \hspace{1cm} (10)

where $p$, $c$, $k$, $q$ are understood to be unitary transformations, and $\epsilon_i = \pm 1$, $i = c, k, q$. To derive the implications of Eqs. (7-10) for the EPs of a NH system, it is helpful to consider the corresponding spectral symmetries. Specifically, we have $|H - E| = 0 \implies |\epsilon_i \Gamma H^T \Gamma^{-1} - E| = 0 \implies |\epsilon_i H^T - E| = 0 \implies |H - \epsilon_i E^T| = 0$, where we have used the fact that $\Gamma = p, c, k$ or $q$ is unitary, and where $H^T$, depending on which symmetry is considered, denotes $H$, $H^T$, $H^*$, or $H^\dagger$, respectively. This in turn gives rise to three principal spectral symmetries:

- $\{ E_i \} = \{-E_i \}: \quad P$ and $C$, $\epsilon_c = -1$ \hspace{1cm} (11)
- $\{ E_i \} = \{ E^*_i \}: \quad K$, $\epsilon_k = 1$ and $Q$, $\epsilon_q = 1$ \hspace{1cm} (12)
- $\{ E_i \} = \{-E^*_i \}: \quad K$, $\epsilon_k = -1$ and $Q$, $\epsilon_q = -1$ \hspace{1cm} (13)

where $\{ E_i \}$ denotes the set of (complex) energy eigenvalues. The constraint in Eq. (11) merely implies that the eigenvalues are symmetric around zero. This is in general not sufficient for the occurrence of symmetry protected EPs, as is immediately clear from the two-band case, where it simply forbids terms that break particle-hole symmetry, i.e. the presence of $d_0(k)$ in Eq. (2). The case of Eq. (12), where the spectrum is invariant under complex conjugation, is also relevant for NH systems with $\mathbb{PT}$ symmetry [48], and gives rise to generic exceptional sets of dimensionality $D - 1$ as we show in the following:

We can decompose the NH Hamiltonian into a Hermitian and an anti-Hermitian part according to

$$H = \alpha H_A + i \beta H_B, \quad H_A = H_A^\dagger, \quad H_B = H_B^\dagger \hspace{1cm} (14)$$

where $\beta = 0$ corresponds to the Hermitian limit where all eigenvalues must be real. Next we note, that if the spectrum of $H_B$ is non-degenerate, i.e. $|E_i - E_j| \geq \Delta_R > 0$, $i \neq j$, then the eigenvalues of $H$ must remain real in the proximity of $\beta = 0$. This is because the spectrum is a continuous function of model parameters, and complex conjugate pairs of eigenvalues cannot continuously

FIG. 2: Momentum dependence of the squared energy $E^2_\pm$ spectrum of the model defined in Eq. (6), which is real due to Eq. (5). The white contours hallmark the lines of EPs, separating regions with real energies $E_\pm$ (blue) from the open Fermi volumes characterized by $\text{Re}(E_\pm) = 0$ (orange). The plot parameters from left to right are $m = 0.25$, $m = 1.0$, $m = 1.42$. At $m = 1.0$ (panel b), the inner line of EPs collapses to a single Hermitian nodal point. At $m = 1.42$ (panel c), the EPs form a pattern reminiscent of Fermi pockets with a fourfold rotational symmetry.
emerge from a set of non-degenerate real energies with gaps bounded by $\Delta_R$. Hence, there exists a finite $\beta_0 > 0$ such that the eigenvalues are real for $|\beta| \leq \beta_0$, see also Fig. 3 for a concrete example of a four-band system.

If we on the other hand consider the anti-Hermitian limit $\alpha \to 0$, while $\beta$ remains finite, then the spectrum becomes purely imaginary. By analogy to the Hermitian limit, if the the spectrum is non-degenerate, i.e. $|\lambda_i - \lambda_j| \geq \Delta_I > 0$, then these must remain open for a finite range $|\alpha| \leq \alpha_c$.

It thus follows that there exists finite neighborhoods in parameter space where pairs of eigenvalues are either real, or complex and related by conjugation respectively. At the boundaries between these regions, both the real and imaginary parts of a gap between two of the energy levels must necessarily vanish, meaning that second order exceptional points form $D - 1$ dimensional surfaces that are topologically stable in the presence of the spectral symmetry (12). This generalizes the above discussion of two-band models to generic NH band structures with a symmetry of the form (12).

Finally, we note that the spectral symmetry (13), where the set of eigenvalues is odd under complex conjugation, has similar consequences on the occurrence of symmetry protected EPs as that of (12). This is because for a Hamiltonian $H$ satisfying Eq. (12), it is clear that $iH$ satisfies Eq. (13). Hence, the EPs once again form $D - 1$ dimensional surfaces that separate domains in parameter space. However, in the case of Eq. (13), the EPs are either characterized by purely imaginary eigenvalues, or complex eigenvalues that appear in pairs such that $\text{Re}(\lambda_i) = -\text{Re}(\lambda_j)$.

\[ \text{Re}(\lambda) \quad \text{Im}(\lambda) \quad \text{Exceptional points} \]

\[ \alpha \]

\[ -1 \quad 1 \]

\[ -2 \quad 2 \]

FIG. 3: Real and imaginary parts of the spectrum of a four band model with the spectral symmetry (12). In accordance with Eq. (14), the model becomes anti-Hermitian when $\alpha \to 0$, whilst $\alpha \to \pm \infty$ corresponds to the Hermitian limit. The bifurcations are exceptional points that separate regions with complex eigenvalues related by conjugation at small $\alpha$, from those with a real spectrum when $\alpha$ is large. The model was obtained by randomly generating $H_A$, $H_B$ in Eq. (14), subject to the symmetry constraints given by Eq. (10), where $q = \tau_3 \sigma_0$ and $\epsilon = +1$.

Concluding discussion.— In this work, we have investigated the impact of symmetries on nodal non-Hermitian band structures. Remarkably, even symmetries that are redundant in the Hermitian limit are shown to have a profound impact on NH band structures; they reduce the number of parameters that need to be tuned in order to find generic EPs from two to one, hence separating them even further from the generic nodal points in Hermitian systems that required three tuning parameters. The $Q_+$ symmetry in Eq. (1) provides an emblematic example of this where the Hermitian limit is trivial while the non-Hermitian theory is very rich with generic symmetry protected EPs as we discussed in one and two spatial dimensions. In particular, the EPs are generally accompanied by what we dub Fermi volumes: open regions of vanishing real part of the energy gap which have the same dimension as the system itself. These Fermi volumes may be readily observed with scattering experiments similar to those that recently reported on the observation of two-dimensional bulk Fermi arcs in photonic crystals [36]. Our systematic approach does not only reveal new phenomena, but also puts well established knowledge into a coherent theoretical framework, for instance the ubiquitous occurrence of EPs in $\mathcal{PT}$ symmetric systems where tuning of only a single parameter is enough to generically probe an exceptional point.

The enriching effects of non-Hermiticity found here for nodal systems stand in strong contrast to the outcome of a very recent study in the context of gapped NH band-structures [8], where it has been found that the standard Altland-Zirnbauer classification [53] is reduced from ten to seven distinct classes, and an accompanying sharp drop in the number of gapped topological phases, including their complete absence in two dimensions has been reported. That we instead find non-Hermiticity to make the problem richer stems from the fact that we consider inherently non-Hermitian symmetries, namely the Bernard-LeClair classes, and that we study nodal rather than gapped phases.

Finally, we note that the ingredients needed to create and manipulate the nodal phases discussed here are already available in a number of experimental settings ranging from photonic crystals, to micro mechanical resonators and arrays of classical waveguides. Hence, our work provides a basis for the experimental search of such phases as well as for further in-depth theoretical explorations of the rich variety of nodal NH phases.

Acknowledgments.— F.K.K., J.C., and E.J.B. are supported by the Swedish research council (VR) and the Wallenberg Academy Fellows program of the Knut and Alice Wallenberg Foundation. J.C.B. acknowledges financial support from the German Research Foundation (DFG) through the Collaborative Research Centre SFB 1143.
[1] N.P. Armitage, E.J. Mele, and A. Vishwanath, Weyl and Dirac semimetals in three-dimensional solids, Rev. Mod. Phys. 90, 015001 (2018).
[2] T. O. Wehling, A. M. Black-Schaffer, and A. V. Balatsky, Dirac materials, Adv. Phys. 67, 1 (2014).
[3] M.Z. Hasan and C.L. Kane, Colloquium: Topological insulators, Rev. Mod. Phys. 82, 3045 (2010).
[4] X.-L. Qi and S.C. Zhang, Topological insulators and superconductors, Rev. Mod. Phys. 83, 1057 (2011).
[5] A. Bansil, H. Lin, and T. Das, Colloquium: Topological band theory, Rev. Mod. Phys. 88, 021004 (2016).
[6] C.-K. Chiu, J. C. Y. Teo, A. P. Schnyder, and S. Ryu, Classification of topological quantum matter with symmetries, Rev. Mod. Phys. 88, 035005 (2016).
[7] V. M. Martinez Alvarez, J. E. Barrios Vargas, M. Berdakin, L. E. F. Fox Torres, Topological states of non-Hermitian systems, arXiv:1805.08200.
[8] Z. Gong, Y. Ashida, K. Kawabata, T. Takasan, S. Higashikawa, and M. Ueda, Topological Phases of Non-Hermitian Systems, Phys. Rev. X 8, 031079 (2018).
[9] F.K. Kunst, E. Edvardsson, J.C. Budich, and E.J. Bergholtz, Biaxorothogonal Bulk-Boundary Correspondence in Non-Hermitian Systems, Phys. Rev. Lett. 121, 026808 (2018).
[10] V. Kozii and L. Fu, Non-Hermitian Topological Theory of Finite-Lifetime Quasiparticles: Prediction of bulk Fermi Arc due to Exceptional Point, arXiv:1708.05841 (2017).
[11] J. Carlström and E.J. Bergholtz, Exceptional Links and Twisted Fermi Ribbons in Non-Hermitian Systems, arXiv:1807.03330.
[12] Y. Xu, S.-T. Wang, and L.-M. Duan, Weyl Exceptional Rings in a Three-Dimensional Dissipative Cold Atomic Gas, Phys. Rev. Lett. 118, 045701 (2018).
[13] Z. Yang and J. Hu, Nodal Line Semimetals under non-Hermitian Perturbations-Emerging Hopf-Link Exceptional Line Semimetals, arXiv:1807.05661 (2018).
[14] T. Yoshida, R. Peters, and N. Kawakami, Non-Hermitian perspective of the band structure in heavy-fermion systems, Phys. Rev. B 98, 035141 (2018).
[15] Y. Xiong, Why does bulk boundary correspondence fail in some non-hermitian topological models, Journal of Physics Communications 2, 035043 (2018).
[16] S. Yao, F. Song, and Z. Wang, Non-Hermitian Chern bands and Chern numbers, Phys. Rev. Lett. 121, 136802 (2018).
[17] S. Yao and Z. Wang, Edge states and topological invariants of non-Hermitian systems, Phys. Rev. Lett. 121, 086803 (2018).
[18] K. Kawabata, K. Shizaki, and M. Ueda, Non-Hermitian Chern insulator, arXiv:1805.09632 (2018).
[19] C. H. Lee and R. Thomale, Anatomy of skin modes and topology in non-Hermitian systems, arXiv:1809.02125 (2018).
[20] T.E. Lee, Anomalous Edge State in a Non-Hermitian Lattice, Phys. Rev. Lett. 116, 133903 (2016).
[21] D. Leykam, K.Y. Bliokh, C. Huang, Y.D. Chong, and F. Nori, Edge Modes, Degeneracies, and Topological Numbers in Non-Hermitian Systems, Phys. Rev. Lett. 118, 040401 (2017).
[22] C. Yin, H. Jiang, L. Li, R. Lu, and S. Chen, Geometrical meaning of winding number and its characterization of topological phases in one-dimensional chiral non-Hermitian systems, arXiv:1802.04160 (2018).
[23] B. Zhu, R. Liu, and S. Chen, PT symmetry in the non-Hermitian Su-Schrieffer-Heeger model with complex boundary potentials, Phys. Rev. A 89, 062102 (2014).
[24] S. Lieu, Topological phases in the non-Hermitian Su-Schrieffer-Heeger model, Phys. Rev. B 97, 045106 (2018).
[25] S. Malzard, C. Poli, and H. Schomerus, Topologically Protected Defect States in Open Photonic Systems with Non-Hermitian Charge-Conjugation and Parity-Time Symmetry, Phys. Rev. Lett. 115, 200402 (2015).
[26] V.M. Martinez Alvarez, J.E. Barrios Vargas, and L.E.F. Foa Torres, Non-Hermitian robust edge states in one dimension: Anomalous localization and eigenspace condensation at exceptional points, Phys. Rev. B 97, 121401(R) (2018).
[27] H. Shen, B. Zhen, and L. Fu, Topological Band Theory for Non-Hermitian Hamiltonians, Phys. Rev. Lett. 120, 146402 (2018).
[28] R. Wang, X.Z. Zhang, and Z. Song, Dynamical topological invariant for non-Hermitian Rice-Mele model, arXiv:1804.09975 (2018).
[29] M.S. Rudner and L.S. Levitov, Topological Transition in a Non-Hermitian Quantum Walk, Phys. Rev. Lett. 102, 065703 (2009).
[30] C. Yuce, Topological phase in a non-Hermitian PPT symmetric system, Phys. Lett. A 379, 1213 (2015).
[31] S. Malzard and H. Schomerus, Bulk and edge-state arcs in non-hermitian coupled-resonator arrays, Phys. Rev. A 98, 033807 (2018).
[32] A.K. Harter, T.E. Lee, and Y.N. Joglekar, PPT-breaking threshold in spatially asymmetric Aubry-And Harper models: Hidden symmetry and topological states, Phys. Rev. A 93, 062101 (2016).
[33] K. Esaki, M. Sato, K. Hasebe, and M. Kohmoto, Edge states and topological phases in non-Hermitian systems, Phys. Rev. B 84, 205128 (2011).
[34] S. Lieu, Topological symmetry classes for non-Hermitian models and connections to the bosonic Bogoliubov-de Gennes equation, arXiv:1807.03320.
[35] K. Kawabata, S. Higashikawa, Z. Gong, Y. Ashida, and M. Ueda, Topological Unification of Time-Reversal and Particle-Hole Symmetries in Non-Hermitian Physics, arXiv:1804.04676.
[36] L. Lu, Z. Wang, D. Ye, L. Ran, L. Fu, J. D. Joannopoulos, and M. Soljačić, Observation of bulk Fermi arc and polarization half charge from paired exceptional points, Science 359, 1009 (2019).
[37] A. Cerjan, S. Huang, K. P. Chen, Y. Chong, and M. C. Rechtsman, Experimental realization of a Weyl exceptional ring, arXiv:1808.09541 (2018).
[38] S. Weimann, M. Kremer, Y. Plotnik, Y. Lumer, S. Nolte, K.G. Makris, M. Segev, M. C. Rechtsman, and A. Szameit, Topologically protected bound states in photonic parity-time-symmetric crystals, Nature Materials 16, 433 (2017).
[39] J. M. Zeuner, M. C. Rechtsman, Y. Plotnik, Y. Lumer, S. Nolte, K.G. Makris, M. Segev, M. C. Rechtsman, and A. Szameit, Topologically protected bound states in photonic parity-time-symmetric crystals, Nature Mater. 16, 433 (2017).
[41] C. Poli, M. Bellec, U. Kuhl, F. Mortessagne, and H. Schomerus, Selective enhancement of topologically induced interface states in a dielectric resonator chain, Nature Commun. 6, 6710 (2015).
[42] M. A. Bandres, S. Wittek, G. Harari, M. Parto, J. Ren, M. Segev, D. N. Christodoulides, and M. Khajavikhan, Topological insulator laser: Experiments, Science 359, eaar4005 (2018).
[43] B Peng, S.K. Özdemir, M. Liertzer, W. Chen, J. Kramer, H. Yilmaz, J. Wiersig, S. Rotter, and L. Yang, Chiral modes and directional lasing at exceptional points, PNAS 113, 684 (2016).
[44] W. Chen, S.K. Özdemir, G. Zhao, J. Wiersig, and L. Yang, Exceptional points enhance sensing in an optical microcavity, Nature 548, 192 (2017).
[45] L. Lu, J.D. Joannopoulos, and M. Soljačić, Topological photonics, Nature Photonics 8, 821 (2014).
[46] R. El-Ganainy, K.G. Makris, M. Khajavikhan, Z.H. Musslimani, S. Rotter, and D.N. Christodoulides, Non-Hermitian physics and PT symmetry, Nature Phys. 14, 11 (2018).
[47] C.M. Bender and S. Boettcher, Real Spectra in Non-Hermitian Hamiltonians Having PT Symmetry, Phys. Rev. Lett. 80, 5243 (1998).
[48] W. D. Heiss, The physics of exceptional points, Journal of Physics A. 45, 444016 (2012).
[49] C.M. Bender, Making sense of non-Hermitian Hamiltonians, Rep. Prog. Phys. 70, 947 (2007).
[50] D.C. Brody, Biorthogonal quantum mechanics, J Phys. A: Math. Theor. 47, 035305 (2013).
[51] I. Rotter, A non-Hermitian Hamilton operator and the physics of open quantum systems, J Phys. A: Math. Theor. 42, 153001 (2009).
[52] M. Berry, Physics of Nonhermitian Degeneracies, Czechoslovak Journal of Physics 54, 1039 (2004).
[53] A. Altland and M.R. Zirnbauer, Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures, Phys. Rev. B 55, 1142 (1997).
[54] A.P. Schnyder, S. Ryu, A. Furusaki, and A.W.W. Ludwig, Classification of topological insulators and superconductors in three spatial dimensions, Phys. Rev. B 78, 195125 (2008).
[55] D. Bernard and A. LeClair, “A classification of non-hermitian random matrices,” in Statistical Field Theories, edited by A. Cappelli and G. Mussardo (Springer Netherlands, Dordrecht, 2002) pp. 207-214.
[56] D. Bernard and A. LeClair, A Classification of Non-Hermitian Random Matrices, arXiv:cond-mat/0110649 (2001).