The Weyl double copy in maximally symmetric spacetimes

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Abstract: Using the method we proposed previously, in which the spin-1/2 massless spinors (Dirac-Weyl fields) are regarded as basic units, we study the Weyl double copy for vacuum solutions with a cosmological constant. The result explicitly demonstrates that the single and zeroth copy satisfy conformally invariant field equations on conformally flat spacetime, based on the exact non-twisting vacuum type $N$ and vacuum type $D$ cases. Furthermore, irrespective of the presence of a cosmological constant, we show that the zeroth copy not only connects gravity fields with the single copy but also connects degenerate electromagnetic fields with Dirac-Weyl fields in the curved spacetime. Moreover, the study shows that the zeroth copy plays a vital role in time-dependent radiation spacetimes. In particular, for Robinson-Trautman ($\Lambda$) gravitational waves, as opposed to the single copy, we find that the zeroth copy carries extra information to indicate whether the sources of associated gravitational waves are time-like, null, or space-like.
1 Introduction

The double copy originates from the study of perturbative scattering amplitudes,[1–3], which brings forth a fascinating connection between gauge amplitudes and gravity amplitudes. Moreover, this idea has been extended to classical context. In Kerr-Schild coordinate system, a map between gravity solutions and gauge theory solutions was proposed, called Kerr-Schild double copy [4]. Then a wide array of such classes of spacetimes have been studied [5–16]. Inspired by this, a new type of double copy called Weyl double copy is drawing more attention [17–24]. It is represented by

$$\Psi_{ABCD} = \frac{\Phi_{(AB}\Phi_{CD)}}{S},$$

where $\Psi_{ABCD}$ is a Weyl spinor describing vacuum gravity fields, $\Phi_{AB}$ is an electromagnetic spinor—the simplest solution of gauge theory, and $S$ is an auxiliary scalar field. The latter two fields are called single copy and zeroth copy, respectively; both fields satisfy their field equations on Minkowski spacetime. Starting from the gravity fields, the Weyl double copy leads to a gauge field that is totally independent of the gravity scheme. Therefore, the Weyl double copy is regarded as a potential bridge connecting the gravity theory and gauge theory.

The Weyl double copy was proposed for the first time for vacuum type D solutions by Luna et al. [17]. Then, in spinor language, the relation was extended to non-twisting vacuum type N solutions by Godazgar et al. [21]. Making use of the pealing property [25, 26] of the Weyl tensor, they further showed that the Weyl double copy also holds asymptotically for algebraically general solutions [23]. In addition, at linearised lever,
the Weyl double copy was shown to hold for arbitrary Petrov type solutions using the twistor formalism [18, 19]. An extended Weyl double copy was also proposed recently for non-vacuum solutions whose Weyl spinor are decomposed into a sum of source terms [24]. Regarding the Dirac-Weyl (DW) spinors (spin-1/2 massless free-field spinors) as the basic units of other higher spin massless spinors, we systematically reconstructed the Weyl double copy for non-twisting vacuum type N and vacuum type D solutions in a preceding work [27]. We further found a map similar to the double copy between gravity field and gauge field for non-twisting vacuum type III case. However, the Weyl double copy for the exact vacuum solutions with a cosmological constant has not been studied before. This is the main aim of the present work. In fact, since 1998, by the observations of supernovae of Ia type [28, 29], studies have shown that the expansion of our universe is accelerating, which strongly supports the condition that the cosmological constant $\Lambda$ is nonzero and positive. On the other hand, although Anti-de Sitter (AdS) spacetime does not appear to have direct cosmological applications, it plays an important role in AdS/CFT correspondence. Therefore, it would be interesting to investigate the Weyl double copy in maximally symmetric spacetimes\(^1\), in which the Ricci scalar is a constant. Besides, there is another motivation for this work. For Kerr-Schild($\Lambda$) double copy [13], the single and zeroth copies satisfy different equations for time-dependent and time-independent solutions. This encourages us to investigate whether the Weyl double copy shares this property. In this paper, we shall give an explicit demonstration to show, different from the Kerr-Schild($\Lambda$) double copy, the single and zeroth copies of the Weyl double copy all satisfy the conformal invariant field equations on conformally flat spacetime, both for time-independent solutions and for time-dependent solutions. This result coincides with the statement of Ref. [18] in twistorial version. Some inspiring relations between the zeroth copy and transverse gravitational waves will also be discussed.

The structure of this paper is as follows. In Sec. 2, we will briefly review how to construct electromagnetic spinors in vacuum type N and type D spacetimes by regarding DW spinors as the basic units. Then, we will study the Weyl double copy on exact vacuum solutions with a cosmological constant in Sec. 3. The interpretations of the single copy and the zeroth copy will also be included. Discussion and conclusions are given in Sec. 4. The notation of this paper follows the conventions of Ref. [27].

2 Massless free-fields in spinor formalism

In this section, based on the method of the preceeding work [27], we will take a brief review how to construct electromagnetic spinors in order to verify the Weyl double copy.

In spinor formalism, spin-$n/2$ massless free-field equations have a simple form [30]

$$\nabla^{A_1A_1'}S_{A_1A_2...A_n} = 0,$$

\(2.1\)

where spinor $S_{A_1A_2...A_n}$ is totally symmetric.

\(^1\)Depending on the sign of the cosmological constant, there are three different maximally symmetric spaces: spherical space ($\Lambda > 0$), flat space ($\Lambda = 0$) and hyperbolic space ($\Lambda < 0$).
For spin-2 massless free-field, the typical object is the Weyl tensor $C_{abcd}$ whose spinor form reads

$$C_{abcd} = C_{AA'B'B''C'C'D'D''} = Ψ_{ABCD} \varepsilon_{A'B'B''C'C'D'D''} + \tilde{Ψ}_{A'B'B''C'C'D'D''} \varepsilon_{ABCD}. \quad (2.2)$$

It is easy to find that the Weyl spinor $Ψ_{ABCD}$ plays the same role as the Weyl tensor $C_{abcd}$. For a vacuum spacetime (with or without a cosmological constant $Λ$), the Einstein field equation is absorbed into the Bianchi identity with the form

$$\nabla^{AA'} Ψ_{ABCD} = 0. \quad (2.3)$$

This is nothing but a spin-2 massless free-field equation. Notably, the fact that this field equation keeps the same form whether we have a cosmological constant or not, inspires us to generalise the original Weyl double copy to the case with a cosmological constant. As is well known, with the aid of a null tetrad, ten independent real components of the Weyl tensor reduce to 5 independent complex scalars as defined in Ref.[31]. We can define them by making use of totally symmetric property of the Weyl spinor as follows,

$$ψ_0 = Ψ_{ABCD} ο_{A'B'C'D'} = C_{abcd} ℓ^a m^b n^c \bar{m}^d,$$
$$ψ_1 = Ψ_{ABCD} ο_{A'B'C'D'} = C_{abcd} ℓ^a m^b n^c \bar{m}^d,$$
$$ψ_2 = Ψ_{ABCD} ο_{A'B'C'D'} = C_{abcd} ℓ^a m^b \bar{m}^c n^d,$$
$$ψ_3 = Ψ_{ABCD} ο_{A'B'C'D'} = C_{abcd} ℓ^a n^b \bar{m}^c n^d,$$
$$ψ_4 = Ψ_{ABCD} ο_{A'B'C'D'} = C_{abcd} \bar{m}^a n^b \bar{m}^c n^d,$$ \quad (2.4)

where the second equations hold based on the definition of the null tetrad in the spinor bases

$$ℓ^a = ο^{A'} A', \quad n^a = ο^{A'} A', \quad m^a = ο^{A'} A', \quad \bar{m}^a = ο^{A'} A', \quad \bar{n}^a = ο^{A'} A'. \quad (2.5)$$

It is easy to find that the Weyl spinor $Ψ_{ABCD}$, whose spinor bases are

$$ℓ^a = ο_{A'} A', \quad n^a = ο_{A'} A', \quad m^a = ο_{A'} A', \quad \bar{m}^a = ο_{A'} A', \quad \bar{n}^a = ο_{A'} A'. \quad (2.5)$$

The spin coefficients are defined the same as in the preceding work [27]. (See also Ref. [32, 33] for more details.) To distinguish from other symbols in this paper, we use $*$ to mark these spin coefficients in the following, such as $κ^*$, $α^*$, $β^*$, etc. Expanding out the Weyl spinor, the general form reads

$$Ψ_{ABCD} = ψ_0 ο_{A'B'C'D'} - 4ψ_1 ο_{A'B'C'D'} + 6ψ_2 ο_{A'B'C'D'} - 4ψ_3 ο_{A'B'C'D'} + ψ_4 ο_{A'B'C'D'} \quad (2.7)$$

For vacuum type N and type D solutions, the Weyl spinors reduce to

$$\text{type } N: \quad Ψ_{ABCD} = ψ_4 ο_{A'B'C'D'}, \quad (2.8)$$
$$\text{type } D: \quad Ψ_{ABCD} = 6ψ_2 ο_{A'B'C'D'}, \quad (2.9)$$
For spin-$1/2$ massless free-field, the typical one is DW field. Its field equation is given by
\[ \nabla^{AA'}\xi_A = 0. \tag{2.10} \]

With the map proposed in the preceding work \[27\]
\[ \Psi_{ABCD} = \frac{\xi(A\xi B\xi C\xi D)}{S_{14}}, \tag{2.11} \]
where the four DW spinors on the right side can be chosen to be the same (depending on which type of spacetimes we are focusing on), we are now ready to derive the DW spinors in a particular vacuum spacetime with a cosmological constant. Correspondingly, the electromagnetic spinors in curved spacetime will be formulated.

Specifically, for vacuum type N solutions, the map Eq.(2.11) reduces to²
\[ \Psi_{ABCD} = \frac{\xi_A\xi_B\xi_C\xi_D}{S_{14}} = \frac{\xi_A\xi_B\xi_C\xi_D}{(S_{12})^3}, \tag{2.12} \]
from Eq.(2.8) one can see that \( \xi_A \) should have the form \( \xi_{oA} \). The independent dyad components of the Weyl field equation Eq.(2.3) then reads
\[ o_A \nabla^{AA'} \log \psi_4 + 4o_A \epsilon^B\nabla^{AA'} o_B - \iota_A \epsilon^B \nabla^{AA'} o_B = 0 \tag{2.13} \]
where we define the Weyl scalar \( \psi_4 \). Combining with the dyad component of the DW field equation
\[ o_A \nabla^{AA'} \log \xi + o_A \epsilon^B \nabla^{AA'} o_A - \iota_A \epsilon^B \nabla^{AA'} o_B = 0, \tag{2.14} \]
the auxiliary scalar \( S_{12} \) and the DW scalar will be identified by solving
\[ \ell \cdot \nabla \log S_{12} - \rho^* = 0, \quad m \cdot \nabla \log S_{12} - \tau^* = 0. \tag{2.15} \]
Since there is only one type of DW spinor \( \xi_A = \xi_{oA} \), correspondingly there is only one type of electromagnetic spinor, namely the degenerate electromagnetic spinor
\[ \Phi_{AB} = \frac{\xi_A\xi_B}{S_{12}} = \frac{\xi}{S_{12}} o_{AB} = \phi_{oA}o_B. \tag{2.16} \]
Since the electromagnetic tensor \( F_{ab} = F_{AA'BB'} = \Phi_{AB}\epsilon_{A'B'} + \Phi_{A'B}\epsilon_{AB} \), where \( \epsilon_{AB} = 2\epsilon_{[AB]} \), in the null tetrad we have
\[ F_{ab} = 2\phi_2\ell_{[a}m_{b]} + 2\tilde{\phi}_2\ell_{[a}\tilde{m}_{b]}. \tag{2.17} \]

For vacuum type D solutions, most of spacetimes we are familiar with belong to this class, such as Kerr (A)dS black holes, charged (A)dS black holes, NUT solutions, C-metric, etc. In this case, the map Eq.(2.11) reduces to
\[ \Psi_{ABCD} = \frac{\xi(A\xi B\eta C\eta D)}{S_{14}}, \tag{2.18} \]
²According to Ref. [27], we know that \( S_{12} = S_{24} = S_{14}^{1/3} \), where \( S_{mn} \) is an auxiliary scalar connecting a spin-$m/4$ massless free-field spinor with a spin-$n/4$ massless free-field spinor.
where we choose two DW spinors with the same coefficient, namely,

$$\xi_A = \xi o_A, \quad \eta_A = \xi i_A.$$  \hfill (2.19)

The dyad components of gravity field equation Eq.(2.3) are then given by

$$o_A \nabla^{AA'} \log(\Psi_2) - 3 \iota_A o^B \nabla^{AA'} o_B = 0,$$ \hfill (2.20)

$$\iota_A \nabla^{AA'} \log(\Psi_2) + 3 o_A i^B \nabla^{AA'} i_B = 0,$$ \hfill (2.21)

where we let the Weyl scalar \(\Psi_2 = 6 \psi_2\). Two dyad components of the DW field equations read

$$o_A \nabla^{AA'} \log(\xi) - 2 \iota_A o^B \nabla^{AA'} o_B = 0,$$ \hfill (2.22)

$$\iota_A \nabla^{AA'} \log(\xi) + 2 o_A i^B \nabla^{AA'} i_B = 0.$$ \hfill (2.23)

By making use of the map Eq.(2.18), the auxiliary scalar \(S_{14}\) and the DW scalar will be identified by solving

$$\ell \cdot \nabla \log S_{14} + 4 \epsilon^* - \rho^* = 0, \quad m \cdot \nabla \log S_{14} + 4 \beta^* - \tau^* = 0,$$

$$\bar{m} \cdot \nabla \log S_{14} - 2 \alpha^* + \pi^* = 0, \quad n \cdot \nabla \log S_{14} - 2 \gamma^* + \mu^* = 0.$$ \hfill (2.24)

Different from the type N case, since there are two different types of DW spinors, we thereby have two different types of electromagnetic spinors. Apart from the degenerate electromagnetic spinor we discussed above, the other type is a non-degenerate electromagnetic spinor,

$$\Phi_{1AB}^{(1)} = \Phi_{1oAB}^{(1)} = \frac{\xi^2 o_A i_B}{S_{12}^{(1)}}.$$ \hfill (2.25)

In order to distinguish two different types of electromagnetic spinors, we use upper index (1) to refer to non-degenerate spinors here and (0), (2) to refer to degenerate spinors \(\Phi_{0AB}^{(0)} = \phi_0 o_AB\) and \(\Phi_{2AB}^{(2)} = \phi_2 o_AB\) respectively. The dyad components of the non-degenerate electromagnetic field equation are given by

$$o_A \nabla^{AA'} \log \phi_1 - 2 \iota_A o^B \nabla^{AA'} o_B = 0,$$ \hfill (2.26)

$$\iota_A \nabla^{AA'} \log \phi_1 + 2 o_A i^B \nabla^{AA'} i_B = 0.$$ \hfill (2.27)

Substitution of the map Eq.(2.25) into the above equations and multiplying \(o_A\) and \(i_A\) respectively yields

$$\ell \cdot \nabla \log S_{12}^{(1)} + 2 \epsilon^* = 0, \quad m \cdot \nabla \log S_{12}^{(1)} + 2 \beta^* = 0,$$

$$\bar{m} \cdot \nabla \log S_{12}^{(1)} - 2 \alpha^* = 0, \quad n \cdot \nabla \log S_{12}^{(1)} - 2 \gamma^* = 0.$$ \hfill (2.28)

Solving the above equations, we are able to obtain \(S_{12}^{(1)}\) and the electromagnetic scalar \(\phi_1\) then will be determined from Eq.(2.25). In analogy to Eq.(2.17), the non-degenerate electromagnetic tensor in the null tetrad reads

$$F_{ab}^{(1)} = 2 \phi_1 \left( \ell_{[a} n_{b]} + \bar{m}_{[a} m_{b]} \right) + 2 \bar{\phi}_1 \left( \ell_{[a} m_{b]} + m_{[a} \bar{m}_{b]} \right).$$ \hfill (2.29)

\(^3\)The degenerate electromagnetic spinors such as Eq.(2.16) correspond to \(\Phi_{2AB}^{(2)}\), since there is only one type of the field in the type N case, it is not necessary to mark it there.
In fact, independent of considering type N or type D solutions, once DW spinors are identified, all electromagnetic fields (or other higher spin massless free-fields) that can live in the curved spacetime can principally be formulated with the aid of an auxiliary scalar field. To verify the Weyl double copy relation, we just need to find the curvature-independent electromagnetic fields\(^4\). If such electromagnetic fields do exist, they are nothing but the single copy, and the associated auxiliary scalar fields are the zeroth copy.

3 The Weyl double copy in maximally symmetric spacetimes

Regarding DW spinors as basic units, electromagnetic spinors living in a certain curved spacetime are constructed according to maps Eq.(2.16) and Eq.(2.25). Then, they are translated into tensor form and are expanded in terms of the products of the null tetrad bases, such as Eq.(2.17) and Eq.(2.29). As we will see later, except for the electromagnetic scalars, the products of the null tetrad bases of the degenerate electromagnetic tensors are independent of the curvature for non-twisting vacuum type N spacetimes, so are the non-degenerate electromagnetic tensors for vacuum type D cases. Therefore, once we prove that the electromagnetic scalars are independent of the curvature, we can conclude that such electromagnetic fields are curvature-independent, thus they should also satisfy the field equations on conformally flat spacetime. Surprisingly, one will discover that the associated scalar fields automatically satisfy their conformally invariant field equations on conformally flat spacetime. An explicit demonstration of the Weyl double copy for non-twisting vacuum type N and vacuum type D solutions is given in the following. The signature of the spacetime metric is chosen as \((+,-,-,-)\) in this work.

3.1 The case for non-twisting vacuum type N solutions

As the solutions of transverse gravitational waves, non-twisting vacuum type N solutions \((\Lambda)\) are composed of two classes\([34, 35]\), one is non-expanding Kundt\((\Lambda)\) class, the other is expanding Robinson-Trautman\((\Lambda)\) class.

3.1.1 The Kundt\((\Lambda)\) class

The metric in this case reads

\[
\text{d}s^2 = -F\text{d}u^2 + 2\frac{q^2}{p^2}\text{d}u\text{d}v - 2\frac{1}{p^2}\text{d}z\text{d}\bar{z},
\]

with

\[
p = 1 + \frac{\Lambda}{6}z\bar{z}, \quad q = \left(1 - \frac{\Lambda}{6}z\bar{z}\right)\alpha + \bar{\beta}z + \beta\bar{z},
\]

\[
F = \kappa\frac{q^2}{p^2}v^2 - \left(\frac{q^2}{p^2}\right)\frac{u}{v} - \frac{\text{d}}{\text{p}}H, \quad \kappa = \frac{\Lambda}{3}\alpha^2 + 2\beta\bar{\beta},
\]

\[
H = H(u, z, \bar{z}) = (f_{,z} + \bar{f}_{,\bar{z}}) - \frac{\Lambda}{3p}(\bar{\xi} + \xi\bar{f}).
\]

\(4\)Note we still call it curvature-independent if a field depends on the cosmological constant \(\Lambda\) but it does not depend on the other curvature parameters, such as some sources.
where \( f \) is an arbitrary complex function of \( \xi \) and \( u \), analytic in \( \xi \). And \( \alpha \) and \( \beta \) are two arbitrary real and complex functions of \( u \), respectively. In fact, according to Ref.[34], one can see the parameter \( \kappa \) is sign invariant. For the case \( \Lambda = 0 \), there are two classes of solutions—generalised pp-waves \( (\kappa = 0) \) and generalised Kundt waves \( (\kappa > 0) \). If our universe admits a positive cosmological constant, namely \( \Lambda > 0 \), there is no limit on \( \alpha \) and \( \beta \), and there is only one kind of solution—generalised Kundt waves. For the case \( \Lambda < 0 \), the values of parameters \( \alpha \) and \( \beta \) classify the metric into three types of solutions—generalised Kundt waves \( (\kappa > 0) \), generalised Siklos waves \( (\kappa = 0) \), and generalised pp-waves \( (\kappa < 0) \). We will soon see that the zeroth copy inherits this property of classifying the gravity solutions.

Choosing the null tetrad

\[
\ell = du, \quad n = -\frac{F}{2}du + \frac{q^2}{p^2}dv, \quad m = \frac{1}{p}d\bar{z},
\]

we have

\[
\Psi_4 = \frac{1}{72} \left( (\Lambda \xi \bar{z} + 6) \left( (\Lambda \xi \bar{z} - 6)\alpha - 6(\bar{z}\beta + z\bar{\beta}) \right) \right) \partial_3^3 \bar{f}.
\]

Recalling Eq.(2.17), it is easy to check

\[
2\ell[a]m_b = \begin{pmatrix} 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad 2\ell[a]\bar{m}_b = \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 \end{pmatrix},
\]

where \( I = \frac{6}{\Lambda + \Lambda \bar{z} \bar{z}} \). Both matrices do not depend on the curvature, so the formula of electromagnetic scalar will decide whether this kind of electromagnetic fields are dependent on the curvature or not. From Eq.(2.15), the auxiliary scalar \( S_{12} \) is solved by

\[
S_{12} = C(u, \bar{z}) \frac{\Lambda \bar{z} \bar{z} + 6}{(\Lambda \bar{z} \bar{z} - 6)\alpha - 6(\bar{z}\beta + z\bar{\beta})},
\]

where \( C(u, \bar{z}) \) is an arbitrary function of \( u \) and \( \bar{z} \). Clearly \( S_{12} \) itself is independent of the curvature. According to Eq.(2.12) and Eq.(2.16), the DW scalar \( \xi \) and the electromagnetic scalar \( \phi_2 \) are solved by

\[
\xi^4 = \frac{(6 + \Lambda \bar{z} \bar{z})^4}{72 \left( (\Lambda \bar{z} \bar{z} - 6)\alpha - 6(\bar{z}\beta + z\bar{\beta}) \right)^2} C(u, \bar{z})^3 \partial_3^3 \bar{f},
\]

\[
\phi_2 = \frac{(6 + \Lambda \xi \bar{z})}{6\sqrt{2}} \sqrt{C(u, \bar{z})} \partial_3^3 \bar{f}.
\]

The structure function \( \partial_3^3 \bar{f} \), which measures the value of \( \Psi_4 \), is absorbed by an arbitrary function \( C(u, \bar{z}) \). Electromagnetic scalar thus does not depend on \( \partial_3^3 \bar{f}(u, \bar{z}) \). So we obtain a curvature-independent degenerate electromagnetic field. It is easy to check that this type of electromagnetic field satisfies its field equation even for the case \( \partial_3^3 \bar{f} = 0 \). Namely,

\[
\bar{\nabla}_a F^{ab} = 0,
\]
where the symbol tilde denotes that the background is (A)dS spacetimes—conformally flat spacetimes—where we just need to let \( f = 1 \) in the original metric. In fact, there is a freedom to choose a polynomial function \( f = c_0(u) + c_1(u)\bar{z} + c_2(u)\bar{z}^2 \) as long as \( \partial_3^2 \bar{f}(u, \bar{z}) = 0 \), where \( c_i(u) \) are expanding parameters of \( \bar{z} \). Furthermore, with the fact that the Ricci scalar \( R = -4\Lambda \), it is easy to verify that the auxiliary scalar field \( S_{24} = S_{12} \) satisfies the conformally invariant scalar field equation not only on the curved spacetime but also on conformally flat spacetime. So we have

\[
\tilde{\nabla}^a \tilde{\nabla}_a S_{24} - \frac{1}{6} \bar{R} S_{24} = 0. \tag{3.10}
\]

When \( \Lambda \to 0 \), the result reduces to the Kundt (\( \Lambda = 0 \)) class, the single copy and the zeroth copy satisfy their field equations on Minkowski spacetime.

More interestingly, one can find that the single copy only confines the structure function. For example, \( f \) can just be a function of coordinates \( u \) and \( \bar{z} \), and it does not depend on the parameters \( \alpha, \beta, \) and \( \Lambda \). In contrast, the zeroth copy has a close relationship with \( \alpha, \beta, \) and \( \Lambda \). With a negative cosmological constant, combined with the introduction of the first paragraph of this section one can see that it is the zeroth copy that indicates which type of curved spacetimes they are mapping.

### 3.1.2 The Robinson-Trautman(\( \Lambda \)) class

One of the familiar metric of Robinson-Trautman (\( \Lambda \)) solutions is given by García Díaz and Plebański [34, 36]

\[
ds^2 = -2(A\bar{A} + \psi B)du^2 - 2\psi du dv - 2v\bar{A}dudz - 2vAdu d\bar{z} - 2v^2 dz d\bar{z},
\]

\[
A = \epsilon z - vf, \quad B = -\epsilon + \frac{v}{2}(f_{,z} + \bar{f}_{,\bar{z}}) + \frac{\Lambda}{6} v^2 \psi, \quad \psi = 1 + \epsilon z \bar{z}, \tag{3.11}
\]

where \( \epsilon = +1, 0, -1 \) corresponds to the fact that the source of the transverse gravitational waves is time-like, null, or space-like, respectively. This is consistent with the case \( \Lambda = 0 \) [37]. Notably, this metric only depends linearly on an arbitrary structure function \( f(u, z) \), this will help facilitate the following discussions.

Choosing the null tetrad

\[
\ell = du, \quad n = -(A\bar{A} + \psi B)du - \psi dv - A\bar{A} dz - vA\bar{A} d\bar{z}, \quad m = vd\bar{z}, \tag{3.12}
\]

we have

\[
\Psi_4 = \frac{(1 + \epsilon z \bar{z})\partial_3^2 \bar{f}}{2v}. \tag{3.13}
\]

In this case, the Weyl scalar does not even depend on the cosmological constant. Recalling Eq.(2.17), one observes

\[
2\ell_{[a} m_{b]} = \begin{pmatrix} 0 & 0 & 0 & v \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -v & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad 2\ell_{[a} \bar{m}_{b]} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.14}
\]
both terms are independent of the structure function $f(u, \xi)$. Solving Eq.(2.15) the auxiliary scalar field $S_{12}$ is given by

$$S_{12} = \frac{C(u, \bar{z})}{v(1 + \bar{\epsilon}z\bar{z})}. \quad (3.15)$$

Following Eq.(2.12) and Eq.(2.16), the DW scalar and the electromagnetic scalar are solved by

$$\xi^4 = \frac{C(u, \bar{z})^3 \partial_\bar{z}^2 \bar{f}}{2v^4(1 + \epsilon\xi\bar{z})}, \quad (3.16)$$

$$\phi_2 = \sqrt{\frac{C(u, \bar{z})\partial_\bar{z}^3 \bar{f}}{2v}}. \quad (3.17)$$

Clearly, the function $C(u, \bar{z})$ lets $\phi_2$ be independent of the structure function. Thus, the electromagnetic field also satisfies the field equation on conformally flat spacetime. We can further check that the auxiliary scalar filed $S_{24} (= S_{12})$ satisfies Eq.(3.10) both on the curved spacetime and on the conformally flat spacetime.

In addition, it is worth noting that the single copy does not depend on the parameter $\epsilon$. It is the zero copy that decides what kind of sources of gravitational waves they are mapping. For example, given the same electromagnetic field in conformally flat spacetime, the scalar field $S_{12}$ with $\epsilon = 1$ following the map Eq.(1.1) will lead to a class of transverse gravitational waves whose source is time-like. While another scalar field $S_{12}$ with $\epsilon = 0$ will lead to another class of transverse gravitational waves whose source is null.

So far, we only consider the time-dependent vacuum solutions. Next, we will investigate time-independent vacuum solutions by focusing on type D spacetimes. More interpretations about the single copy and the zeroth copy will be discussed later.

### 3.2 The case for vacuum type D solutions

#### 3.2.1 Kerr-(A)dS black holes

As we know, rotating black holes are believed as the most typical astrophysical black holes in the universe. It is necessary to take the case of Kerr-(A)dS black holes as a specific example to study the double copy relation before going to the most general vacuum type D solutions.

The metric of Kerr-(A)dS black holes reads [38]

$$\begin{align*}
\text{d}s^2 &= \frac{\mathcal{R}}{\rho^2}(\text{d}t - \frac{a}{\Sigma}\sin^2\theta \text{d}\phi)^2 - \frac{\rho^2}{\mathcal{R}}\text{d}r^2 - \frac{\rho^2}{\Theta}\text{d}\theta^2 \\
&\quad - \frac{\Theta}{\rho^2}\sin^2\theta(\text{d}t - \frac{r^2 + a^2}{\Sigma}\text{d}\phi)^2, \quad (3.18)
\end{align*}$$

where

$$\begin{align*}
\mathcal{R} &= (r^2 + a^2)(1 + \frac{r^2}{l^2}) - 2Mr, \quad \Theta = 1 - \frac{a^2}{l^2}\cos^2\theta, \quad (3.20) \\
\Sigma &= 1 - \frac{a^2}{l^2}, \quad \rho^2 = r^2 + a^2\cos^2\theta, \quad l = \sqrt{-3/\Lambda}, \quad (3.21)
\end{align*}$$
with mass $M/\Sigma^2$ and angular momentum $J = aM/\Sigma^2$. Clearly, $M$ and $a$ can be regarded as mass parameter and angular momentum parameter, respectively.

Since the metric has already been written in the orthogonal tetrad \{e^{i}\} ($i = 1, 2, 3, 4$) such that $ds^2 = (e^1)^2 - (e^2)^2 - (e^3)^2 - (e^4)^2$, the null tetrad \{e^i\} then is easily given under the transformation
\[
e^{i1} = \frac{1}{\sqrt{2}} (e^1 + e^2), \quad e^{i2} = \frac{1}{\sqrt{2}} (e^1 - e^2),
\]
\[
e^{i3} = \frac{1}{\sqrt{2}} (e^3 + ie^4), \quad e^{i4} = \frac{1}{\sqrt{2}} (e^3 - ie^4).
\]

We have
\[
e^{i1} = \ell = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{\Theta}{\rho^2}} dt + \sqrt{\frac{\rho^2}{\Theta}} dr - \sqrt{\frac{a}{\rho^2 \Sigma}} d\phi \right),
\]
\[
e^{i2} = n = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{\Theta}{\rho^2}} dt + \sqrt{\frac{\rho^2}{\Theta}} dr - \sqrt{\frac{a}{\rho^2 \Sigma}} d\phi \right),
\]
\[
e^{i3} = m = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{\Theta}{\rho^2}} a \sin \theta dt + i \sqrt{\frac{\rho^2}{\Theta}} d\theta - \sqrt{\frac{a}{\rho^2 \Sigma}} \sin \theta d\phi \right),
\]
\[
e^{i4} = \bar{m} = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{\Theta}{\rho^2}} a \sin \theta dt - i \sqrt{\frac{\rho^2}{\Theta}} d\theta - \sqrt{\frac{a}{\rho^2 \Sigma}} \sin \theta d\phi \right).
\]

So we obtain the Weyl scalar
\[
\Psi_2 = 6\psi_2 = \frac{6M}{(r + ia \cos \theta)^3}.
\]

According to Eq.(2.18), to identify the DW scalar we need to solve the auxiliary scalar field $S_{14}$. Using Eq.(2.24), it is not hard to obtain that
\[
S_{14} = \frac{C_1 \csc^2 \theta (r + ia \cos \theta)}{(a^2 \cos \theta^2 - l^2)(r^4 + l^2 r (r - 2M) + a^2 (r^2 + l^2))},
\]
where all of the constant coefficients have been absorbed by a constant of integration $C_1$.

The DW scalar then can be solved by
\[
\xi^2 = \sqrt{\Psi_2} S_{14} = \frac{\sqrt{6}C_1M \csc \theta}{(r + ia \cos \theta) \sqrt{(a^2 \cos \theta^2 - l^2)(r^4 + l^2 r (r - 2M) + a^2 (r^2 + l^2))}}.
\]

Note this is also the coefficient of DW tensor in the null tetrad [27], while we are not going to talk about DW tensor in more details in this paper, the only reason we show this is to construct electromagnetic scalar. With the help of an auxiliary scalar $S_{12}^{(1)}$, which is solved from Eq.(2.28)
\[
S_{12}^{(1)} = \frac{C_2 \csc \theta (r + ia \cos \theta)}{\sqrt{(l^2 - a^2 \cos \theta^2)(r^4 + l^2 r (r - 2M) + a^2 (r^2 + l^2))}},
\]
we obtain
\[
\phi_1 = \frac{\xi^2}{S_{12}^{(1)}} = \frac{i}{C_2} \frac{\sqrt{6}C_1M}{(r + ia \cos \theta)^2} \sim \frac{1}{(r + ia \cos \theta)^2}.
\]
One can observe that the mass parameter $M$ is absorbed by a constant of integration. So it actually does not depend on the source. In addition, it is easy to verify that $(l_{[a}m_{b]} + \tilde{m}_{[a}m_{b]})$ and $(l_{[a}n_{b]} + m_{[a}\tilde{m}_{b]})$ are all independent of the mass parameter $M$. Hence, from Eq. (2.29), we conclude that the non-degenerate electromagnetic field we construct is independent of the source. It even satisfies the conformally invariant field equation on conformally flat spacetime, where we just need to let $M = 0$ in the metric. What about the auxiliary scalar field $S_{24}^{(1,1)}$ associated to this electromagnetic field? Using the formula Eq. (1.1), one observes that

$$S_{24}^{(1,1)} = \frac{\phi^2}{\Psi_2} = -\frac{C_1}{(C_2)^2} \frac{1}{(r + ia \cos \theta)} \sim \frac{1}{r + ia \cos \theta},$$  \hspace{1cm} (3.32)

As we expect, this satisfies the conformally invariant scalar field equation

$$\tilde{\nabla}^a \tilde{\nabla}_a S_{24}^{(1,1)} - \frac{1}{6} \tilde{R} S_{24}^{(1,1)} = 0.$$  \hspace{1cm} (3.33)

When $\Lambda \to 0$, it satisfies the wave equation on Minkowski background. Thus, we have shown that the single copy and the zeroth copy of Kerr-AdS spacetimes satisfy their conformally invariant field equations on the conformally flat spacetime.

Moreover, in analogy to Eq. (2.15) of the type N case, one can show that

$$S_{12}^{(2)} = S_{12}^{(0)} \sim \frac{1}{r + ia \cos \theta}.$$  \hspace{1cm} (3.34)

Combining with Eq. (3.32), it is clear that the zeroth copy not only connects gravity fields with the single copy but also connects those degenerate electromagnetic fields with DW fields living in the curved spacetime. This property is consistent with the discovery of the preceding work [27] in the absence of the cosmological constant $\Lambda$.

### 3.2.2 The most general vacuum type D solutions

Now, we shall investigate the double copy relation for the most general vacuum type D solutions with a cosmological constant. The metric has been given by Plebanski and Demianski [39],

$$ds^2 = \frac{1}{(p + q)^2} \left( -\frac{1 + (pq)^2}{P} \frac{dp^2}{\sigma^2 + q^2 \tau^2} - \frac{P}{1 + (pq)^2} (d\sigma + q^2 d\tau)^2 - \frac{1 + (pq)^2}{L} dq^2 + \frac{L}{1 + (pq)^2} (-p^2 d\sigma + d\tau)^2 \right),$$  \hspace{1cm} (3.35)

where the structure functions read

$$P = (-\frac{\Lambda}{6} + \gamma) + 2np - \epsilon p^2 + 2mp^3 + (-\frac{\Lambda}{6} - \gamma) p^4,$$  \hspace{1cm} (3.37)

$$L = (-\frac{\Lambda}{6} - \gamma) + 2nq + \epsilon q^2 + 2mq^3 + (-\frac{\Lambda}{6} + \gamma) q^4,$$  \hspace{1cm} (3.38)

$m$ and $n$ are dynamical parameters measuring the curvature, $\epsilon$ and $\gamma$ are called kinematical parameters which will affect the properties of the solutions.
By choosing null tetrad
\[
\ell = \frac{1}{2(p + q)} \left( \sqrt{1 + (pq)^2} dq - p^2 \sqrt{\frac{\mathcal{L}}{1 + (pq)^2}} d\sigma + \sqrt{\frac{\mathcal{L}}{1 + (pq)^2}} d\tau \right),
\]
(3.39)
\[
n = \frac{1}{2(p + q)} \left( -\sqrt{1 + (pq)^2} dq - p^2 \sqrt{\frac{\mathcal{L}}{1 + (pq)^2}} d\sigma + \sqrt{\frac{\mathcal{L}}{1 + (pq)^2}} d\tau \right),
\]
(3.40)
\[
m = \frac{1}{2(p + q)} \left( \sqrt{1 + (pq)^2} dp + i \sqrt{\frac{\mathcal{P}}{1 + (pq)^2}} d\sigma + iq^2 \sqrt{\frac{\mathcal{P}}{1 + (pq)^2}} d\tau \right),
\]
(3.41)
\[
\bar{m} = \frac{1}{2(p + q)} \left( \sqrt{1 + (pq)^2} dp - i \sqrt{\frac{\mathcal{P}}{1 + (pq)^2}} d\sigma - iq^2 \sqrt{\frac{\mathcal{P}}{1 + (pq)^2}} d\tau \right),
\]
(3.42)
we obtain
\[
\Psi_2 = 6\psi_2 = -6(m + in) \left( \frac{p + q}{1 - ipq} \right)^3.
\]
(3.43)

Obviously, one can observe that the cosmological constant does not affect the Weyl scalar according to the above result. Solving Eq.(2.24), the auxiliary scalar field \( S_{14} \) is given by
\[
S_{14} = C_1 \frac{(p + q)^3(1 - ipq)}{\mathcal{P}(p)\mathcal{L}(q)},
\]
(3.44)
where \( C_1 \) is a constant of integration. Then from Eq.(2.18) we have
\[
\xi^2 = \frac{-6C_1(m + in)(p + q)^3}{\sqrt{\mathcal{P}(p)\mathcal{L}(q)} \left(1 - ipq\right)}.
\]
(3.45)
Recalling Eq.(2.29), one observes that
\[
2 \left( \ell_{[a}n_{b]} + \bar{m}_{[a}m_{b]} \right) = \begin{pmatrix} 0 & 0 & iA & ip^2A \\ 0 & 0 & \bar{q}A & -A \\ -iA & \bar{q}A & 0 & 0 \\ -ip^2A & A & 0 & 0 \end{pmatrix},
\]
(3.46)
and
\[
2 \left( \ell_{[a}n_{b]} + \bar{m}_{[a}m_{b]} \right) = \begin{pmatrix} 0 & 0 & -iA & -ip^2A \\ 0 & 0 & \bar{q}A & -A \\ iA & \bar{q}A & 0 & 0 \\ ip^2A & A & 0 & 0 \end{pmatrix},
\]
(3.47)
where \( A = \frac{(p + q)^2}{1 + pq^2} \). Clearly, they are independent of the dynamical parameters, that means, if the electromagnetic scalar is independent of the dynamical parameters as well, the electromagnetic field we construct will be independent of the dynamical parameters and the field equation will hold even on conformally flat spacetime. From Eq.(2.28), one obtains
\[
S_{12}^{(1)} = C_2 \frac{(p + q)(1 - ipq)}{\sqrt{\mathcal{P}(p)\mathcal{L}(q)}},
\]
(3.48)
where $C_2$ is a constant of integration. Following Eq.(2.25), the non-degenerate electromagnetic scalar $\phi_1$ is given by

\[
\phi_1 = \sqrt{-6C_1(m + in)} \frac{(p + q)^2}{C_2 (1 - ipq)^2} \sim \frac{(p + q)^2}{(1 - ipq)^2}.
\] (3.49)

One can see that the dynamical parameters measuring the curvature are absorbed by the constants of integration, so $\phi_1$ is independent of the dynamical parameters. Thus we have found a curvature-independent non-degenerate electromagnetic field. Correspondingly, the auxiliary scalar field $S_{24}^{(1)}$ is given by

\[
S_{24}^{(1)} = \left( \frac{\phi_1}{\Psi_2} \right)^2 = \frac{C_1}{(C_2)^2} \frac{p + q}{1 - ipq} \sim S_{24}^{(1,1)}.
\] (3.50)

It is easy to check that $S_{24}^{(1)}$ satisfies the conformal invariant equation Eq.(3.10) on conformally flat spacetime, where we just need to set $m = n = 0$.

Therefore, we conclude, for vacuum type D solutions with a cosmological constant, that the single copy and the zeroth copy satisfy their conformal invariant field equations on conformally flat spacetime. When $\Lambda \to 0$, the background goes back to Minkowski spacetime, which is consistent with the previous result [17].

In addition, similar to the case of Kerr-AdS spacetime, for the general vacuum type D solutions with or without a cosmological constant, we find that

\[
S_{12}^{(0)} = S_{12}^{(2)} \sim \frac{p + q}{1 - ipq} \sim S_{24}^{(1,1)}.
\] (3.51)

Namely, the zeroth copy not only connects gravity fields with the single copy but also connects those degenerate electromagnetic fields with DW fields living in the curved spacetime. Recalling the previous section, clearly, this property also holds for non-twisting type N solutions. While, as time-independent solutions, there is a different point from the type N cases, the zeroth copy now does not possess any extra information about the source compared with the single copy. This is directly reflected by the double copy scalar relation

$\Psi_2 = (\phi_2)^{3/2} = \left( S_{24}^{(1,1)} \right)^3$. Therefore, we find that only for the time-dependent solutions, the zeroth copy carries extra information about the source. This provides support for constructing other exact time-dependent radiation solutions for future work.

4 Discussion and Conclusions

In this paper, regarding DW spinors (massless spin-1/2 spinors) as basic units, we constructed curvature-independent electromagnetic fields in non-twisting vacuum type N and vacuum type D spacetimes with the presence of a cosmological constant. Regarding those fields as the single copies and the associated auxiliary scalar fields connecting the single copies to gravity fields as the zeroth copy, we verified the Weyl double copy relation in the maximally symmetric spacetimes. Different from the Kerr-Schild ($\Lambda$) double copy, for which the single and zeroth copies satisfy different equations on conformally flat spacetime for time-independent solutions and for time-dependent solutions [13], we found that both
for the time-dependent solutions (type N cases) and for the time-independent solutions (type D cases), the single and zeroth copies all satisfy corresponding conformally invariant field equations on conformally flat spacetime. When $\Lambda \to 0$, the result reduces to the original case; namely, they satisfy their field equations on Minkowski spacetime. From this point of view, the Weyl double copy appears as a more fundamental map between gravity theory and gauge theory. On the other hand, we found that the previous discovery[27] also holds in the presence of $\Lambda$. Specifically speaking, the zeroth copy not only connects gravity fields with the single copy but also connects degenerate electromagnetic fields with DW fields in the curved spacetime, both for non-twisting vacuum type N solutions and vacuum type D solutions with or without a cosmological constant. More interestingly, we found that the zeroth copy plays an important role for time-dependent radiation solutions (type N cases). Unlike the single copy, which only confines the form of the structure function, the zeroth copy carries the specific information indicating what kind of curved spacetimes they are mapping.

Especially for the Robinson-Trautman (\(\Lambda\)) class, we found it is the zeroth copy that decides whether the sources of associated gravitational waves are time-like, null, or spacetime. For Kundt(\(\Lambda\)) solutions, however, we are not very clear about the specific correspondence between the parameters $\alpha$ and $\beta$ showing in the zeroth copy and the source of Kundt(\(\Lambda\)) waves. Even though, there still exist some interesting relationships: when $\Lambda < 0$, depending on the values of the parameters $\alpha$ and $\beta$, the solutions are classified into three cases, generalised Kundt waves, generalised Siklos waves, and generalised pp-waves; when $\Lambda > 0$, the zeroth copy corresponds to only one type of solution—generalised Kundt waves. It seems that the zeroth copy does not play any role in this case; while different from the single copy, we need to emphasize that the zeroth copy is also related to $\Lambda$. So it is indeed affecting the classification of the spacetime. With further study on the parameters $\alpha$ and $\beta$, we believe that the physical interpretation of the zeroth copy will be more clear. In any case, we found that for time-dependent vacuum radiation solutions, the information describing the source of the transverse gravitational waves is encoded in the zeroth copy. This provides support for constructing other exact time-dependent radiation solutions for future work.

All in all, we have specifically shown that the single copy and the zeroth copy satisfy conformally invariant equations on conformally flat spacetime by focusing on non-twisting vacuum type N and vacuum type D solutions. Some new interpretations about double copy are given, especially for the zeroth copy. Next, it would be interesting to check whether the generalised Weyl double copy holds asymptotically for the algebraically general case with a cosmological constant. On the other hand, it is significant to investigate the applications of the Weyl double copy on astrophysical observations, such as the specific correspondence between the source of gravitational waves and the Weyl double copy. Since all of the discussion in this paper is limited to 4-dimensional spacetimes, it is also a meaningful work to generalise the study to high dimensional spacetimes.
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