Bayesian Estimation and Prediction
Based on Progressively First Failure Censored Scheme from a Mixture of Weibull and Lomax Distributions

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Abstract

This paper develops Bayesian estimation and prediction, for a mixture of Weibull and Lomax distributions, in the context of the new life test plan called progressive first failure censored samples. Maximum likelihood estimation and Bayes estimation, under informative and non-informative priors, are obtained using Markov Chain Monte Carlo methods, based on the symmetric square error Loss function and the asymmetric linear exponential (LINEX) and general entropy loss functions. The maximum likelihood estimates and the different Bayes estimates are compared via a Monte Carlo simulation study. Finally, Bayesian prediction intervals for future observations are obtained using a numerical example.

Key Words: Mixture model; Progressive First Failure Censored Scheme; Loss Function; Maximum Likelihood Estimation; Bayesian Estimation and Prediction; Markov Chain Monte Carlo.

1. Introduction

Mixtures models have received great attention from analysts in the recent years due to their important role in life testing and reliability. In many applications, mixture models are used to analyze random duration in possibly heterogeneous populations, statistical analysis and machine learning such as modeling, classification, and survival analysis. Attention has been paid by some authors to the finite mixtures to discuss lifetime distributions, [see Everitt and Hand (1981), Titterington et al. (1985), McLachlan and Basford (1988), Lindsay (1995), McLachlan and Peel (2000)]. Also, mixture distributions have been considered extensively by several researchers using both classical and Bayesian techniques, [for example Shawky and Bakoban (2009), Abu-Zinadah (2010), Erisoglu et al. (2011), Feroze and Aslam (2014), Daniyal and Rajab (2015), Elshahat and Mahmoud (2016), Mahmoud et al. (2017)].

The Weibull distribution has been widely used in modeling of lifetime event data; this is due to the variety of shapes of the probability density function (pdf) based on its parameters. The Weibull distribution has been shown to be useful for modeling and analyzing lifetime data in the applied engineering sciences [see Murthy et al. (2003)]. The Lomax distribution, sometimes called Pareto of the second kind, has a considerable importance in the field of life testing because of its uses to fit business failure data [see Lomax (1954)].

A random variable X is said to have a mixture of two component Weibull and Lomax distribution if its probability density function (pdf) is given by
\[ f(x) = \sum_{j=1}^{2} p_j f_j(x), \quad j = 1,2. \] (1)

\[ f_1(x) = \alpha_1 \theta_1 x^{\theta_1 - 1} e^{-\alpha_1 x^{\theta_1}} \]
\[ f_2(x) = \alpha_2 \theta_2 (1 + \theta_2 x)^{-(\alpha_2+1)} \]

where \( x > 0, (\alpha_j > 0, \theta_j > 0), j = 1, 2. \) The mixing proportions \( p_j \) are such that \( 0 \leq p_j \leq 1, \sum_{j=1}^{2} p_j = 1. \)

The corresponding cumulative distribution function (cdf) and reliability function, respectively are given by

\[ F(x) = \sum_{j=1}^{2} p_j F_j(x), \quad j = 1,2. \] (2)

\[ F_1(x) = 1 - e^{-\alpha_1 x^{\theta_1}} \]
\[ F_2(x) = 1 - (1 + \theta_2 x)^{-\alpha_2} \]

and

\[ R(x) = \sum_{j=1}^{2} p_j R_j(x), \quad j = 1,2. \] (3)

\[ R_1(x) = e^{-\alpha_1 x^{\theta_1}} \]
\[ R_2(x) = (1 + \theta_2 x)^{-\alpha_2} \]

Common censoring schemes of type I and type II censoring do not allow units to be removed from the test at any other point than the final termination point. Therefore, the focus in the last few years has been on progressive censoring due to its flexibility that allows the experimenter to remove active units during the experiment. Progressive censoring has been studied by many authors. Some of the early work on progressive censoring are Cohen (1963), Mann (1971), Viveros and Balakrishnan (1994), Balasooriya et al. (2000), Balakrishnan and Aggarwala (2000) and Balakrishnan (2007) have presented an elaborate overview of various developments in progressive censoring data.

The experimental time is usually an important concern for the life test designers. Although conventional censoring schemes can shorten the duration of a life test, the experimental time is still too long that cannot be waited for when the units are highly reliable. Johnson (1964) introduced the first failure censoring by grouping the test units into several sets. The experimenter runs all test units simultaneously until the first failure in one of the sets. Some references that discussed this type of first failure are Balasooriya (1995), Wu et al. (2003) and Wu and Yu (2005).

Wu and Kus (2009) described new life test scheme, progressively first failure censoring scheme by combining the concept of first failure censoring with the progressive censoring. Many authors have discussed inference for different distributions based on this scheme introduced by Huang and Wu (2011), Soliman et al. (2012), Abou-Elheggag (2013), Javadkhani et al. (2014), Kim and Han (2015), Dube et al. (2016) and Rashad et al. (2017).

The prediction problem of the future samples based on censored data is an important topic in statistics. The prediction techniques are used in medicine, engineering, business and other areas as well. Several authors studied Bayesian prediction for future observation; [see Al-Hussaini et al. (2001), Jaheen (2003)].

The objective of this work is to apply the Bayesian procedure to estimate the parameters and obtain two sample prediction bounds for future observations from the proposed model, based on progressive first failure censoring scheme. The rest of this paper is organized as follows: In Section 2, the progressive first-failure censoring scheme is described. In Section 3, we obtain maximum likelihood estimators of the parameters. Different loss functions are presented in Section 4. The Bayesian estimation is discussed in Section 5, using Markov Chain Monte Carlo technique. In Section 6, Monte Carlo simulation study is conducted to compare the performance of different estimation methods. Bayesian prediction with numerical data are presented in Section 7. Finally, we conclude the paper in Section 8.

2. A Progressive First-Failure Censoring Scheme

In a life-testing experiment, suppose sample of \( n \) independent groups with \( k \) items in each group are put in a life test. After units have failed, each item can be attributed to the appropriate subpopulation. Thus if the \( m \) are failed
during the interval \((0, x_{(m)})\): \(r_1\) from the first subpopulation and \(r_2\) from the second subpopulation. When the first failure is observed, \(R_1\) groups and the group with observed failure are randomly removed. At the second observed failure, \(R_2\) groups and the group with observed failure are randomly removed. This experiment terminates at the time when the \(m^{th}\) failure is observed and the remaining \(R_m\) groups and the group with observed failure are all removed. Here \(x_{1,m,n,k}^R < x_{2,m,n,k}^R < \ldots < x_{m,m,n,k}^R\) are known as progressive first failure censored orders statistics with the progressive censoring scheme \(R = (R_1, R_2, ..., R_m)\); \((m \leq n)\). Let \(x_{ij}\) denote the failure of the \(j^{th}\) unit that belongs to the \(i^{th}\) subpopulation and \(x_{ij} \leq x_{(m)}\); \(j = 1, 2, ..., m_i; m = r_1 + r_2\). where \(x_{(m)}\) denotes the failure time of the \(m^{th}\) unit. For a two component mixture model, the likelihood function, is defined as

\[
L(x_1, x_2, ..., x_m) = C k^m \prod_{j=1}^{r_1} p_1 f_1(x_{1j}) \prod_{j=1}^{r_2} p_2 f_2(x_{2j}) \prod_{j=1}^{m} \left[1 - F(x_j)\right]^{k(R_j+1)-1}
\]

\[0 < x_{1,m,n,k}^R < x_{2,m,n,k}^R < \ldots < x_{m,m,n,k}^R < \infty\] (4)

where \(C = n(n - R_1 - 1)(n - R_1 - R_2 - 2) ... (n - \sum_{j=1}^{m-1} R_j - m + 1)\), \(p_1 = p\), \(p_2 = 1 - p\). Note that if \(k = 1\) then, sampling scheme reduces to the progressively type II censoring, a first failure censored scheme when \(R = (0, ..., 0)\), a usual type II censored scheme when \(k = 1\) and \(R = (0, ..., 0)\), and complete sample case if \(k = 1\) and \(R = (0, ..., 0)\), with \(n = m\). It should be noted that \(x_{1,m,n,k}^R, x_{2,m,n,k}^R, ..., x_{m,m,n,k}^R\) can be viewed as a progressive type II censored sample from a population with distribution function \(1 - (1 - F(x))^k\). For this reason, results extended to progressive first- failure censored scheme easily. The progressive first-failure censored plan has advantages in terms of reducing the test time, in which more items are used but only \(m\) of \(n \times k\) items are failures.

3. Maximum Likelihood Estimation (MLE)

Let \(x_{i,m,n,k}^R, i = 1, 2, ..., m\), with \(x_{1,m,n,k}^R \leq x_{2,m,n,k}^R, ..., \leq x_{m,m,n,k}^R\) denote the progressively first failure censored from the mixture of Weibull and Lomax distributions, with censored scheme \(R = (R_1, R_2, ..., R_m)\). Substituting (1) and (2) into (4), the likelihood function based on a progressive first failure censored sample is given by

\[
L(p, \alpha_1, \alpha_2, \theta_1, \theta_2 | x) = C k^m \prod_{j=1}^{r_1} p_1 \alpha_1 \theta_1 x_{1j}^{\theta_1-1} e^{-\alpha_1 x_{1j}^\theta_1} \prod_{j=1}^{r_2} p_2 \alpha_2 \theta_2 (1 + \theta_2 x_{2j})^{-(\alpha_2+1)}
\]

\[
\times \prod_{j=1}^{m} \left[p_1 R_1(x_j) + p_2 R_2(x_j)\right]^{(k(R_j+1)-1)}
\]

Assuming that the parameters \(\theta_1\) and \(\theta_2\) are known, the likelihood function (5) reduces to

\[
L(p, \alpha_1, \alpha_2 | x) \propto C k^m \prod_{i=1}^{2} (p_i \alpha_i)^r_i \prod_{j=1}^{r_1} x_{1j}^{\theta_1-1} \times e^{-\alpha_1 \sum_{j=1}^{r_1} x_{1j}^{\theta_1}} \prod_{j=1}^{r_2} (1 + \theta_2 x_{2j})^{-(\alpha_2+1)}
\]

\[
\times \prod_{j=1}^{m} \left[p_1 R_1(x_j) + p_2 R_2(x_j)\right]^{(k(R_j+1)-1)}
\]

(6)

Thus, the log-likelihood function of the parameters \(\alpha_1, \alpha_2\) and \(p\) are given by
\[ \ln L(p, \alpha_1, \alpha_2 | x) \propto \ln C + m \ln k + \sum_{i=1}^{2} \{r_i \ln p_i + r_i \ln \alpha_1\} - \alpha_1 \sum_{j=1}^{r_1} x_{ij}^{\theta_1} - (\alpha_2 + 1) \sum_{j=1}^{r_2} \ln (1 + \theta_2 x_{2j}) \]
\[ + \sum_{j=1}^{m} \{k(R_j + 1) - 1\} \ln \left[ p_1 e^{-\alpha_1 x_{1j}^{\theta_1}} + p_2 (1 + \theta_2 x_j)^{-\alpha_2} \right] \]

(7)

Partially differentiating equation (7) with respect to \( \alpha_1, \alpha_2 \) and \( p \) maximum likelihood estimates (MLEs) can obtained by solving the equations after equating it to zero

\[ \frac{\partial \ln L(p, \alpha_1, \alpha_2)}{\partial \alpha_1} = \frac{r_1}{\alpha_1} - \sum_{j=1}^{r_1} x_{1j}^{\theta_1} - \sum_{j=1}^{m} \frac{\{k(R_j + 1) - 1\} p_1 x_{1j}^{\theta_1} e^{-\alpha_1 x_{1j}^{\theta_1}}}{p_1 e^{-\alpha_1 x_{1j}^{\theta_1}} + p_2 (1 + \theta_2 x_j)^{-\alpha_2}} \]

\[ \frac{\partial \ln L(p, \alpha_1, \alpha_2)}{\partial \alpha_2} = \frac{r_2}{\alpha_2} - \sum_{j=1}^{r_2} \ln (1 + \theta_2 x_{2j}) - \sum_{j=1}^{m} \frac{\{k(R_j + 1) - 1\} p_2 (1 + \theta_2 x_j)^{-\alpha_2} \ln (1 + \theta_2 x_j)}{p_1 e^{-\alpha_1 x_{1j}^{\theta_1}} + p_2 (1 + \theta_2 x_j)^{-\alpha_2}} \]

\[ \frac{\partial \ln L(p, \alpha_1, \alpha_2)}{\partial p} = \frac{r_1}{p_1} - \frac{r_2}{p_2} + \sum_{j=1}^{m} \frac{\{k(R_j + 1) - 1\} \{e^{-\alpha_1 x_{1j}^{\theta_1}} - (1 + \theta_2 x_j)^{-\alpha_2}\}}{p_1 e^{-\alpha_1 x_{1j}^{\theta_1}} + p_2 (1 + \theta_2 x_j)^{-\alpha_2}} \]

4. Loss Function

In decision theory, the loss criterion is specified in order to obtain the best estimator. Three loss functions are proposed, symmetric (square error) loss function and asymmetric (LINEX and general entropy) loss functions, as follows:

- Squared error loss function: A simple, and very common loss function is defined by
  \[ L_1(\hat{\theta}, \theta) = c(\hat{\theta} - \theta)^2; \ c \text{ is a constant} \]

  which is symmetrical in nature and gives equal weight to overestimation as well as under estimation. However, in real applications, estimation of reliability and failure rate function, an overestimate is more serious than the underestimates. The use of asymmetric loss function might be inappropriate as has been recognized by Basu and Ebrahimi (1991).

- Linear exponential loss function (LINEX): One of the most commonly used asymmetric loss functions, introduced by Varian (1975) under the assumption that the minimal loss occurs at \( \hat{\theta} = \theta \), it can be expressed as
  \[ L_2(\Delta) \propto e^{-q\Delta} - q\Delta - 1, \ \Delta = \hat{\theta} - \theta, \ q \neq 0 \]

where \( q \) determines the shape of the loss function. If \( q > 0 \) means overestimation and underestimation if \( q < 0 \), but in a situation where \( q \approx 0 \), the LINEX loss is almost symmetric and approaches square error loss function.

Under the above loss function, the Bayes estimator \( \hat{\theta}_{\text{LINEX}} \) of \( \theta \) can be obtained as

\[ \hat{\theta}_{\text{LINEX}} = -\frac{1}{q} \ln [E(e^{-q\theta}|x)] \]

provided that the expected value with respect to the posterior function of \( \theta, E(e^{-q\theta}|x) \) exists and is finite.
5. Bayesian Estimation

In this section, we derive Bayes estimators of the parameters $\alpha_1$, $\alpha_2$ and $p$ of the considered model based first failure progressively censored sample. Assuming the following independent prior distributions for the parameters $\alpha_1 \sim \text{Gamma}(\alpha_1, b_1)$, $\alpha_2 \sim \text{Gamma}(\alpha_2, b_2)$, and $p \sim \text{Beta}(c, d)$ for the mixing parameter $p$. The joint prior distribution of $\alpha_1$, $\alpha_2$ and $p$ is

$$
\pi(\alpha_1, \alpha_2, p) = \pi_1(\alpha_1)\pi_2(\alpha_2)\pi_3(p)
$$

where

$$
\pi(\alpha_i) \propto \alpha_i^{a_i-1}e^{-b_i\alpha_i}; \quad \alpha_i > 0, b_i > 0; \quad i = 1, 2
$$

and

$$
\pi_3(p) \propto p_{1}^{c-1}p_{2}^{d-1}
$$

The joint density function of $\alpha_1$, $\alpha_2$, $p$ and the sample $x$ can be written as follows

$$
P(p, \alpha_1, \alpha_2, x) \propto \left( \prod_{i=1}^{2} p_i^{r_i+1} \alpha_i^{r_i} \times e^{-\sum_{j=1}^{r_i} b_i^{1+1} \alpha_i^{1+1}} \prod_{j=1}^{r_2} (1 + \theta_2 x_j)^{-1-\alpha_2} \prod_{j=1}^{m} \left[p_1 r_1(x_j) + p_2 r_2(x_j)\right]^{k_{\theta_1}-1} \right)
$$

$$
\times \left( p_1^{c-1}p_2^{d-1} \prod_{i=1}^{2} \alpha_i^{a_i-1} e^{-b_i \alpha_i} \right)
$$

$$
P(p, \alpha_1, \alpha_2, x) \propto p_1^{r_1+1}p_2^{r_2+1} \prod_{i=1}^{2} \alpha_i^{r_i+a_i-1} \prod_{j=1}^{m} \left[p_1 r_1(x_j) + p_2 r_2(x_j)\right]^{k_{\theta_2}-1}
$$

(9)

where

$$
\varphi_1 = b_1 + \sum_{j=1}^{r_1} \vartheta_i \quad \text{and} \quad \varphi_2 = b_2 + \sum_{j=1}^{r_2} \ln(1 + \theta_2 x_j)
$$

Based on Equation (9), the joint posterior density function of $\alpha_1$, $\alpha_2$ and $p$, is given by

$$
P(p, \alpha_1, \alpha_2|x) = \frac{P(p, \alpha_1, \alpha_2, x)}{\int_0^1 \int_0^\infty \int_0^\infty P(p, \alpha_1, \alpha_2, x) d\alpha_1 d\alpha_2 dp}
$$

(10)
Thus, under the squared error, LINEX and general entropy loss functions, the Bayes estimators of any function of 
\( (p, \alpha_1, \alpha_2) \), say \( \phi(p, \alpha_1, \alpha_2) \), is

\[
\hat{\phi}_{SE}(\alpha_1, \alpha_2) = \frac{\int_0^1 \int_0^{\infty} \phi(p, \alpha_1, \alpha_2) P(p, \alpha_1, \alpha_2, x) \, dx \, dp}{\int_0^1 \int_0^{\infty} P(p, \alpha_1, \alpha_2, x) \, dx \, dp} 
\]

(11)

\[
\hat{\phi}_{LINEX}(\alpha_1, \alpha_2) = -\frac{1}{q} \ln \left[ \frac{\int_0^1 \int_0^{\infty} e^{-q \phi(p, \alpha_1, \alpha_2)} P(p, \alpha_1, \alpha_2, x) \, dx \, dp}{\int_0^1 \int_0^{\infty} P(p, \alpha_1, \alpha_2, x) \, dx \, dp} \right] 
\]

(12)

\[
\hat{\phi}_{GE}(\alpha_1, \alpha_2) = \left[ \frac{\int_0^1 \int_0^{\infty} \phi(p, \alpha_1, \alpha_2)^{-q} P(p, \alpha_1, \alpha_2, x) \, dx \, dp}{\int_0^1 \int_0^{\infty} P(p, \alpha_1, \alpha_2, x) \, dx \, dp} \right]^{-\frac{1}{q}} 
\]

(13)

The ratio of the integrals in Equations (11), (12) and (13) cannot be obtained in a closed form. Therefore, the Markov Chain Monte Carlo technique will be used to approximate the integrals.

5.1. Markov Chain Monte Carlo method (MCMC)

In this subsection, we apply the importance sampling technique to obtain the approximate Bayes estimates. This technique needs no calculation of the normalizing constant. The joint posterior density function of \( \alpha_1, \alpha_2 \) and \( p \) given data \( \mathbf{x} \) can be written in the form

\[
P(p, \alpha_1, \alpha_2 | \mathbf{x}) \propto \beta(p; r_1 + c, r_2 + d) \gamma_G(\alpha_1; r_1 + \alpha_1, \varphi_1) \gamma_G(\alpha_2; r_2 + \alpha_2, \varphi_2) h(\alpha_1, \alpha_2, p) 
\]

(14)

where \( \varphi_1, \varphi_2 \) are defined as above, and

\[
h(\alpha_1, \alpha_2, p) = \prod_{j=1}^{m} \left[ p_1 R_1(x_j) + p_2 R_2(x_j) \right]^{(k R_j + 1) - 1} 
\]

(15)

According to the importance sampling technique, the approximate Bayes estimators, based on the three loss functions, can be computed by the following algorithm

**Step1:** \( p \sim Beta(r_1 + c, r_2 + d) \)

\[ \alpha_1 \sim Gamma(r_1 + \alpha_1, \varphi_1) \]

\[ \alpha_2 \sim Gamma(r_2 + \alpha_2, \varphi_2) \]

**Step2:** Repeat this procedure to obtain important sample \( \{ (p_i, \alpha_{1i}, \alpha_{2i}), i = 1, ..., N \} \)

**Step 3:** Calculate Bayes estimator of \( \phi(p, \alpha_1, \alpha_2) \), under squared error, LINEX and general entropy loss functions, respectively by

\[
\hat{\phi}_{SE}(\alpha_1, \alpha_2) = \frac{\sum_{i=1}^{N} \phi(p_i, \alpha_{1i}, \alpha_{2i}) h(\alpha_{1i}, \alpha_{2i}, p_i)}{\sum_{i=1}^{N} h(\alpha_{1i}, \alpha_{2i}, p_i)} 
\]

\[
\hat{\phi}_{LINEX}(\alpha_1, \alpha_2) = -\frac{1}{q} \ln \left[ \frac{\sum_{i=1}^{N} e^{-q \phi(p_i, \alpha_{1i}, \alpha_{2i})} h(\alpha_{1i}, \alpha_{2i}, p_i)}{\sum_{i=1}^{N} h(\alpha_{1i}, \alpha_{2i}, p_i)} \right] 
\]
\[ \phi_{GE}(p, \alpha_1, \alpha_2) = \left[ \frac{\sum_{i=1}^{N} \phi(p, \alpha_i, \alpha_2)^{-h} h(\alpha_1, \alpha_2, p_i)}{\sum_{i=1}^{N} h(\alpha_1, \alpha_2, p_i)} \right]^{\frac{1}{h}} \]

6. Comparison Study

In this section, we present the numerical results of a simulation study to compare the performance of the various estimates for different combinations of \((n, m, k)\). We consider different sampling schemes \(R = (R_1, \ldots, R_m)\) as follow

| \((k,n,m)\) | Scheme | \(R_i\) |
|------------|--------|--------|
| (2,50,30)  | I      | (20,29,0) |
|           | II     | (29,0,20)  |
|           | III    | (10,28,0,10) |
| (2,50,40)  | I      | (10,39,0)  |
|           | II     | (39,0,10)  |
|           | III    | (5,38,0,5)  |
| (4,75,45)  | I      | (30,44,0)  |
|           | II     | (44,0,30)  |
|           | III    | (15,43,0,15) |
| (4,75,55)  | I      | (20,54,0)  |
|           | II     | (54,0,20)  |
|           | III    | (10,53,0,10) |

To carry out this comparative study, we follow the following steps:

- In this study the following parameters values were used \((\alpha_1, \alpha_2, p) = (1,2,0.40)\) along with \((\theta_1, \theta_2) = (0.5,1.5)\). The values chosen for the constants \(q\) are \((0.5, -0.5)\) for the LINEX and for \(h\) are \((1, -1)\) for general entropy loss functions. The following values are used for the hyper parameters \((\alpha_1 = 3, \alpha_2 = 2, b_1 = 2, b_2 = 1, c = 2 and d = 4)\) for informative prior. In case of non-informative prior, we take \(((\alpha_1 = \alpha_2 = b_1 = b_2 = 0), (c = d = 1))\).

- Applying the algorithms of Balakrishnan and Sandhu (1995) to generate a progressive first-failure censored sample from mixture Weibull and Lomax distributions for different values of \((n,m,k,R)\).

- Based on the progressive first-failure censored data, maximum likelihood estimates and Bayes estimates of the parameters are calculated according to Section 3 and Section 5, respectively.

- The above steps are repeated 1000 times, and computation off the average of the estimates and mean square error for different values \((n,m,k,R)\) are presented in Tables (2-8). The computations are done using Mathematica 10.0.

Table (2-8) indicates that the Bayes estimates perform better under informative prior than non-informative prior for all different loss functions. In most cases, notice that the performance of the Bayes estimates under informative prior are smaller than the maximum likelihood estimates in terms of MSE. Also, the Bayesian estimates under general entropy loss function in case of the value \((h = -1)\) are almost the same as the estimates under squared error loss function. The mean square error of all estimates decreases when the sample size \(n\) and effective sample size \(m\) increase. Also, when the value of the group size \(k\) increases, the mean square error decreases.
| SC     | Maximum Likelihood Estimation | \(k, n, m\) | \(k, n, m\) | \(k, n, m\) |
|--------|--------------------------------|-------------|-------------|-------------|
|        | \(\alpha_1\)                 | \(\alpha_2\) | \(p\)       | \(\alpha_1\) | \(\alpha_2\) | \(p\)       | \(\alpha_1\) | \(\alpha_2\) | \(p\)       |
| I      | 1.03676 (0.02118)             | 2.20610 (0.19323) | 0.40994 (0.00065) | 1.04959 (0.01516) | 2.25190 (0.16074) | 0.40806 (0.00037) | 0.65021 (0.15266) | 1.43521 (0.41863) | 0.41540 (0.00031) |
| II     | 0.74133 (0.11755)             | 1.70425 (0.26606) | 0.48890 (0.01020) | 0.76587 (0.11302) | 1.73385 (0.27612) | 0.46558 (0.00654) | 0.81885 (0.07292) | 1.76575 (0.16383) | 0.41292 (0.00043) |
| III    | 0.75646 (0.11778)             | 1.73760 (0.27541) | 0.47448 (0.00804) | 0.86449 (0.07537) | 1.90287 (0.22749) | 0.44234 (0.00369) | 0.78610 (0.08594) | 1.70020 (0.20451) | 0.41297 (0.00036) |
|        |                               |             |             |              |                 |              |             |                 |             |
| I      | 0.64870 (0.14971)             | 1.42957 (0.41584) | 0.41550 (0.00029) | 0.77324 (0.08521) | 1.67761 (0.19722) | 0.41426 (0.00037) | 0.73617 (0.10280) | 1.60063 (0.25517) | 0.41424 (0.00034) |
Table 3: Average values and corresponding MSE of parameter $\alpha_1$ based informative prior

| $(k, n, m)$ | SC | Square Error | LINEX | General Entropy |
|-------------|----|--------------|-------|-----------------|
|             |    | q = 0.5      | q = −0.5 | h = 1 | h = −1 |
| (2,50,30)   | I  | 1.11252      | 1.09040 (0.09090) | 1.13601 | 1.03598 (0.06846) | 1.11252 | 1.11252 |
|             | II | 1.15429      | 1.13055 (0.11326) | 1.17962 | 1.07533 (0.08309) | 1.15429 | 1.1326 |
|             | III| 1.12124      | 1.09866 (0.10132) | 1.14529 | 1.04441 (0.07510) | 1.12124 | 0.10132 |
| (2,50,40)   | I  | 1.10453      | 1.08770 (0.07751) | 1.12217 | 1.04578 (0.06132) | 1.10453 | 0.07751 |
|             | II | 1.12562      | 1.10799 (0.08511) | 1.14411 | 1.06522 (0.06573) | 1.12562 | 0.08511 |
|             | III| 1.10145      | 1.08427 (0.08140) | 1.11947 | 1.04173 (0.06437) | 1.10145 | 0.08140 |
| (4,75,45)   | I  | 1.07658      | 1.06246 (0.06134) | 1.09126 | 1.02571 (0.05110) | 1.07658 | 0.06134 |
|             | II | 1.09767      | 1.08266 (0.06848) | 1.1133 | 1.04457 (0.05519) | 1.09767 | 0.06848 |
|             | III| 1.07333      | 1.05884 (0.06380) | 1.0898 | 1.02105 (0.05318) | 1.07333 | 0.06380 |
| (4,75,55)   | I  | 1.06722      | 1.0553 (0.05298)  | 1.06695 | 1.02382 (0.04493) | 1.06722 | 0.05298 |
|             | II | 1.08769      | 1.0754 (0.06191)  | 1.1008 | 1.04378 (0.05185) | 1.08769 | 0.06191 |
|             | III| 1.0921       | 1.07964 (0.05829) | 1.10498 | 1.04769 (0.04796) | 1.0921 | 0.05829 |

Table 4: Average values and corresponding MSE of parameter $\alpha_2$ based informative prior

| $(k, n, m)$ | SC | Square Error | LINEX | General Entropy |
|-------------|----|--------------|-------|-----------------|
|             |    | q = 0.5      | q = −0.5 | h = 1 | h = −1 |
| (2,50,30)   | I  | 2.03588      | 1.98295 (0.19947) | 2.09315 | 1.93307 (0.18304) | 2.03588 | 0.19947 |
|             | II | 2.11918      | 2.06157 (0.23750) | 2.18172 | 2.01164 (0.20143) | 2.11918 | 0.23750 |
|             | III| 2.05643      | 2.00280 (0.19529) | 2.11441 | 1.95296 (0.17461) | 2.05643 | 0.19529 |
| (2,50,40)   | I  | 2.04134      | 2.00055 (0.16174) | 2.08452 | 1.96233 (0.14898) | 2.04134 | 0.16174 |
|             | II | 2.07451      | 2.03293 (0.15337) | 2.11866 | 1.99493 (0.13651) | 2.07451 | 0.15337 |
|             | III| 2.06282      | 2.02129 (0.15368) | 2.10691 | 1.98290 (0.13874) | 2.06282 | 0.15368 |
| (4,75,45)   | I  | 1.98372      | 1.94923 (0.13120) | 2.0201 | 1.91489 (0.12977) | 1.98372 | 0.13120 |
|             | II | 2.05776      | 2.02084 (0.14037) | 2.09658 | 1.98651 (0.12785) | 2.05776 | 0.14041 |
|             | III| 2.03646      | 2.00075 (0.13702) | 2.07408 | 1.96694 (0.12822) | 2.03646 | 0.13702 |
| (4,75,55)   | I  | 2.01132      | 1.98235 (0.10798) | 2.04153 | 1.95408 (0.10426) | 2.01132 | 0.10798 |
|             | II | 2.07333      | 2.04255 (0.13141) | 2.10543 | 2.01449 (0.11908) | 2.07333 | 0.13141 |
|             | III| 2.0087       | 1.97964 (0.11706) | 2.03899 | 1.95149 (0.11251) | 2.0087 | 0.11706 |
Table 5: Average values and corresponding MSE of parameter $p$ based informative prior

| $(k, n, m)$ | SC  | Square Error | LINEX   | General Entropy |
|------------|-----|--------------|---------|-----------------|
|            |     | $q = 0.5$    | $q = -0.5$ | $h = 1$ | $h = -1$ |
| (2,50,30)  | I   | 0.3863       | 0.38708  | 0.39018 | 0.37137 | 0.38863 |
|            |     | (0.0553)     | (0.00555) | (0.00551) | (0.00655) | (0.00553) |
|            | II  | 0.39062      | 0.38908  | 0.39218 | 0.37344 | 0.39062 |
|            |     | (0.0610)     | (0.00611) | (0.00609) | (0.00707) | (0.00610) |
|            | III | 0.38925      | 0.38771  | 0.39080 | 0.37208 | 0.38925 |
|            |     | (0.0606)     | (0.00607) | (0.00604) | (0.00605) | (0.00606) |
| (2,50,40)  | I   | 0.39290      | 0.39167  | 0.39413 | 0.37961 | 0.39290 |
|            |     | (0.0438)     | (0.00439) | (0.00437) | (0.00494) | (0.00438) |
|            | II  | 0.39150      | 0.39027  | 0.39279 | 0.37804 | 0.39150 |
|            |     | (0.0473)     | (0.00474) | (0.00472) | (0.00536) | (0.00473) |
|            | III | 0.38871      | 0.38749  | 0.38993 | 0.37535 | 0.38871 |
|            |     | (0.0481)     | (0.00483) | (0.00479) | (0.00550) | (0.00481) |
| (4,75,45)  | I   | 0.39352      | 0.39240  | 0.39464 | 0.38148 | 0.39352 |
|            |     | (0.0393)     | (0.00394) | (0.00392) | (0.00439) | (0.00393) |
|            | II  | 0.39256      | 0.39145  | 0.39368 | 0.38061 | 0.39256 |
|            |     | (0.0412)     | (0.00413) | (0.00412) | (0.00460) | (0.00412) |
|            | III | 0.39140      | 0.39029  | 0.39252 | 0.37936 | 0.39140 |
|            |     | (0.0405)     | (0.00406) | (0.00404) | (0.00456) | (0.00405) |
| (4,75,55)  | I   | 0.38898      | 0.38805  | 0.38991 | 0.37894 | 0.38898 |
|            |     | (0.0358)     | (0.00359) | (0.00356) | (0.00402) | (0.00358) |
|            | II  | 0.39275      | 0.39181  | 0.39369 | 0.38272 | 0.39175 |
|            |     | (0.0367)     | (0.00368) | (0.00367) | (0.00405) | (0.00367) |
|            | III | 0.39222      | 0.39128  | 0.39316 | 0.38217 | 0.39222 |
|            |     | (0.0363)     | (0.00364) | (0.00362) | (0.00401) | (0.00363) |

Table 6: Average values and corresponding MSE of parameter $\alpha_i$ based non-informative prior

| $(k, n, m)$ | SC  | Square Error | LINEX   | General Entropy |
|------------|-----|--------------|---------|-----------------|
|            |     | $q = 0.5$    | $q = -0.5$ | $h = 1$ | $h = -1$ |
| (2,50,30)  | I   | 1.06434      | 1.03748  | 1.09373 | 0.96989 | 1.06434 |
|            |     | (0.11318)    | (0.09795) | (0.13197) | (0.09849) | (0.11318) |
|            | II  | 1.11986      | 1.09008  | 1.15247 | 1.02246 | 1.11968 |
|            |     | (0.15991)    | (0.13418) | (0.19673) | (0.11777) | (0.15986) |
|            | III | 1.08625      | 1.05873  | 1.11633 | 0.99166 | 1.08625 |
|            |     | (0.13244)    | (0.11403) | (0.15731) | (0.10380) | (0.13244) |
| (2,50,40)  | I   | 1.06124      | 1.04194  | 1.08166 | 0.99239 | 1.06124 |
|            |     | (0.08708)    | (0.0777) | (0.09873) | (0.07124) | (0.08708) |
|            | II  | 1.07316      | 1.05355  | 1.09388 | 1.00400 | 1.07316 |
|            |     | (0.09799)    | (0.08794) | (0.11019) | (0.08007) | (0.09799) |
|            | III | 1.07992      | 1.05973  | 1.10142 | 1.00927 | 1.07992 |
|            |     | (0.09728)    | (0.08659) | (0.11057) | (0.07843) | (0.09728) |
| (4,75,45)  | I   | 1.03387      | 1.01782  | 1.05070 | 0.97434 | 1.03387 |
|            |     | (0.06658)    | (0.06128) | (0.07308) | (0.05856) | (0.06658) |
|            | II  | 1.06191      | 1.04502  | 1.07963 | 1.00096 | 1.06191 |
|            |     | (0.07635)    | (0.06956) | (0.08453) | (0.06410) | (0.07635) |
|            | III | 1.0442       | 1.02785  | 1.06137 | 0.98452 | 1.0442 |
|            |     | (0.07317)    | (0.06716) | (0.08047) | (0.06305) | (0.07317) |
| (4,75,55)  | I   | 1.02123      | 1.00857  | 1.03437 | 0.97383 | 1.02123 |
|            |     | (0.05676)    | (0.05311) | (0.06113) | (0.05103) | (0.05676) |
|            | II  | 1.07138      | 1.05751  | 1.08581 | 1.02165 | 1.07138 |
|            |     | (0.06791)    | (0.06232) | (0.07443) | (0.05711) | (0.06791) |
|            | III | 1.04105      | 1.02818  | 1.05439 | 0.99337 | 1.04105 |
|            |     | (0.05995)    | (0.05590) | (0.06471) | (0.05307) | (0.05995) |
Table 7: Average values and corresponding MSE of parameter $\alpha_2$ based non-informative prior

| $(k,n,m)$ | SC | Square Error | LINEX | General Entropy |
|-----------|----|--------------|-------|-----------------|
|           |    | $q = 0.5$    | $q = -0.5$ | $h = 1$ | $h = -1$ |
| $(2,50,30)$ | I  | 2.05088 (0.25008) | 1.99093 (0.21861) | 2.11627 (0.29671) | 1.93594 (0.22475) | 2.05088 (0.25008) |
|           | II | 2.10759 (0.31001) | 2.04345 (0.26424) | 2.17814 (0.37649) | 1.98835 (0.26617) | 2.10759 (0.31001) |
|           | III| 2.10062 (0.27298) | 2.03715 (0.22998) | 2.17026 (0.33621) | 1.98188 (0.23176) | 2.10062 (0.27298) |
| $(2,50,40)$ | I  | 2.06909 (0.20625) | 2.02334 (0.18368) | 2.11791 (0.23678) | 1.98215 (0.18473) | 2.06909 (0.20625) |
|           | II | 2.10569 (0.23132) | 2.05883 (0.20762) | 2.15573 (0.27302) | 2.01818 (0.20652) | 2.10569 (0.23132) |
|           | III| 2.08595 (0.21484) | 2.0393 (0.18857)  | 2.13576 (0.26746) | 1.99824 (0.18837) | 2.08595 (0.21484) |
| $(4,75,45)$ | I  | 2.01609 (0.17404) | 1.9777 (0.16088)  | 2.05673 (0.19249) | 1.94115 (0.16490) | 2.01609 (0.17404) |
|           | II | 2.06383 (0.18269) | 2.02361 (0.16494) | 2.10634 (0.20625) | 1.98697 (0.16564) | 2.06383 (0.18269) |
|           | III| 2.0391 (0.16029)  | 2.0006 (0.14681)  | 2.07976 (0.17880) | 1.96445 (0.14903) | 2.0391 (0.16029)  |
| $(4,75,55)$ | I  | 2.02569 (0.12196) | 1.99425 (0.11356) | 2.05854 (0.13338) | 1.96426 (0.11492) | 2.02569 (0.12196) |
|           | II | 2.06601 (0.13954) | 2.03336 (0.12752) | 2.1001 (0.15496)  | 2.0033 (0.12710)  | 2.06601 (0.13954) |
|           | III| 2.04398 (0.13454) | 2.01167 (0.12416) | 2.07784 (0.14837) | 1.98154 (0.12468) | 2.04398 (0.13454) |

Table 8: Average values and corresponding MSE of parameter $p$ based non-informative prior

| $(k,n,m)$ | SC | Square Error | LINEX | General Entropy |
|-----------|----|--------------|-------|-----------------|
|           |    | $q = 0.5$    | $q = -0.5$ | $h = 1$ | $h = -1$ |
| $(2,50,30)$ | I  | 0.40647 (0.00763) | 0.40472 (0.00759) | 0.40823 (0.00678) | 0.38753 (0.00823) | 0.40647 (0.00763) |
|           | II | 0.41368 (0.00714) | 0.41191 (0.00708) | 0.41545 (0.00721) | 0.39487 (0.00743) | 0.41368 (0.00714) |
|           | III| 0.40721 (0.00687) | 0.40546 (0.00683) | 0.40897 (0.00692) | 0.38829 (0.00742) | 0.40721 (0.00687) |
| $(2,50,40)$ | I  | 0.40503 (0.00533) | 0.40367 (0.00531) | 0.40639 (0.00536) | 0.39058 (0.00567) | 0.40503 (0.00533) |
|           | II | 0.40356 (0.00527) | 0.40220 (0.00526) | 0.40492 (0.00530) | 0.38911 (0.00565) | 0.40356 (0.00527) |
|           | III| 0.40329 (0.00541) | 0.40194 (0.00539) | 0.40464 (0.00543) | 0.38897 (0.00577) | 0.40329 (0.00541) |
| $(4,75,45)$ | I  | 0.40534 (0.00491) | 0.40411 (0.00489) | 0.40658 (0.00494) | 0.39241 (0.00515) | 0.40534 (0.00491) |
|           | II | 0.40310 (0.00476) | 0.40278 (0.00473) | 0.40521 (0.00477) | 0.39121 (0.00502) | 0.40310 (0.00476) |
|           | III| 0.402959 (0.00450) | 0.40174 (0.00448) | 0.40418 (0.00452) | 0.39016 (0.00477) | 0.402959 (0.00450) |
| $(4,75,55)$ | I  | 0.40293 (0.00429) | 0.40193 (0.00427) | 0.403941 (0.00430) | 0.39243 (0.00448) | 0.40293 (0.00429) |
|           | II | 0.40366 (0.00409) | 0.40265 (0.00408) | 0.40466 (0.00411) | 0.39316 (0.00427) | 0.40366 (0.00409) |
|           | III| 0.40375 (0.00413) | 0.40274 (0.00411) | 0.40476 (0.00414) | 0.39322 (0.00431) | 0.40375 (0.00413) |
7. Bayesian Prediction For Future Observations

In this section, the Bayesian two sample prediction of a future order statistics is considered based on the observed progressive first failure censored data \( \bar{X} \). Based on a random sample of size \( n \) drawn from a population with pdf (1), a future unobservable independent random sample of size \( m \) from the same population is under consideration. Let \( Y_s \) represents the \( s^{th} \) ordered statistic in the future sample, \( 1 \leq s \leq m \). The \( s^{th} \) order statistic in a sample of size \( m \) represents the life length of a \((m - s + 1)\) out of \( m \) system. The distribution function of \( Y_s \) the ordered future sample is given, [See Arnold et al.(1992) and Jaheen (2003)], by

\[
F_{Y_s}(p, \alpha_1, \alpha_2) = \sum_{l=0}^{m} \binom{m}{l} \left[ F_X(Y_s|p, \alpha_1, \alpha_2) \right]^l \left[ 1 - F_X(Y_s|p, \alpha_1, \alpha_2) \right]^{m-l}
\]

\[
= \sum_{l=0}^{m} \sum_{j_1=0}^{l} \binom{m}{l} \binom{l}{j_1} (-1)^{j_1} [R(Y_s)]^{m-l+j_1}
\]

(16)

where \( F_X(p, \alpha_1, \alpha_2) \) is the distribution function of the mixture model and \( R(Y_s) \) is the reliability function of the mixture model after replacing \( x \) by \( y_s \).

Using the binomial expansion for \([R(Y_s)]^{m-l+j_1}\), we get

\[
[R(Y_s)]^{m-l+j_1} = \left[ p_1 e^{-\alpha_1 y_s^{\alpha_1}} + p_2 (1 + \theta_2 y_s)^{-\alpha_2} \right]^{m-l+j_1}
\]

\[
= \sum_{j_2=0}^{m-l+j_1} \binom{m-l+j_1}{j_2} p_1^{\alpha_1} p_2^{\alpha_2} (e^{-\alpha_1 y_s^{\alpha_1}}) \delta_1 (1 + \theta_2 y_s)^{-\alpha_2 j_2}
\]

where \( \delta_1 = m - l + j_1 - j_2 \)

Therefore,

\[
F_{Y_s}(p, \alpha_1, \alpha_2) = \sum_{l=0}^{m} \sum_{j_1=0}^{l} \sum_{j_2=0}^{m-l+j_1} \binom{m}{l} \binom{l}{j_1} \binom{m-l+j_1}{j_2} (-1)^{j_1} p_1^{\alpha_1} p_2^{\alpha_2} (e^{-\alpha_1 y_s^{\alpha_1}}) \delta_1 + \theta_2 y_s)^{-\alpha_2 j_2}
\]

(17)

The Bayes predictive pdf of \( Y_s \) given \( x \) is defined by:

\[
f^\star(y_s|x) = \int \int \int f(y_s|p, \alpha_1, \alpha_2) P(p, \alpha_1, \alpha_2|x) d\alpha_1 d\alpha_2 dp
\]

(18)

where \( P(p, \alpha_1, \alpha_2|x) \) is the joint posterior density function for parameters \( \alpha_1, \alpha_2 \) and \( p \) and \( f(y_s|p, \alpha_1, \alpha_2) \) is the pdf of \( s^{th} \) component in a future sample.

Therefore, using the Bayesian predictive density of \( Y_s \), for a given value \( \nu \), we obtain
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\[ Pr[y_s \geq \nu | x] = \int_{\nu}^{\infty} f^*(y_s | x) dy_s \]

\[ = 1 - \int_{0}^{1} \int_{0}^{\infty} \int_{0}^{\infty} F_{\gamma_s}(\nu | p, \alpha_1, \alpha_2) P(p, \alpha_1, \alpha_2 | x) d\alpha_1 d\alpha_2 dp \]

(19)

where \( F_{\gamma_s}(\nu | p, \alpha_1, \alpha_2) \) is the cumulative distribution of the \( s \)th component in a future sample as given by (17). This cannot be evaluated analytically. Thus, the MCMC sampling procedure described in Subsection 5.1 is applied.

Based on importance samples \( \{(p_i, \alpha_{1i}, \alpha_{2i}), i = 1, \ldots, N\} \). A simulator of the predictive distribution of \( y_s \), as

\[ Pr[y_s \geq \nu | x] = 1 - \sum_{i=1}^{N} F_{\gamma_s}(\nu | p_i, \alpha_{1i}, \alpha_{2i}) w_i \]

(20)

where

\[ w_i = \frac{h(\alpha_{1i}, \alpha_{2i}, p_i)}{\sum_{i=1}^{N} h(\alpha_{1i}, \alpha_{2i}, p_i)}, i = 1, \ldots, N \]

A 100 \( \gamma \)% prediction interval for \( y_s \) is given by

\[ P[L(x) < y_s < U(x)] = \gamma \]

where \( L(x) \) and \( U(x) \) are obtained respectively by solving the following two nonlinear equations:

\[ Pr[y_s > L(x) | x] = \frac{1 + \gamma}{2} \quad \text{and} \quad Pr[y_s > U(x) | x] = \frac{1 - \gamma}{2} \]

(21)

These equations cannot be solved analytically, using Mathematica software program.

7.1. Numerical Example

This section presents a numerical example to illustrate the methodology for the proposed estimates based on real data. The data set is an uncensored data set consisting of 66 observations on breaking stress of carbon fibers (in Gba). This data was presented by Al-Babtain et al. (2015) and is as follows: 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 3.56, 4.42, 2.43, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 1.57, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.32, 2.85, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.89, 2.88, 2.82, 2.05, 3.65, 3.75, 2.73, 2.95, 2.97, 3.39, 2.96, 2.35, 2.55, 2.59, 2.03, 1.61, 2.12, 3.15, 1.08, 2.56, 1.80, 2.53. Now, we assume that the observations are randomly grouped into 22 groups with \( k = 3 \) observations within each group, with \( m = 16, n = 22, k = 3 \), the grouped data set are as follows:

\[ \{3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 3.56, 4.42, 2.43, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 1.57, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.32, 2.85, 1.25, 4.38, 1.84\} \]

\[ \{0.39, 1.08, 1.47, 1.61, 1.69, 1.80, 1.87, 1.89, 2.05, 2.35, 2.41, 2.43, 2.50, 2.55, 2.81, 2.95\} \]

Taking \( \theta_1 = 1.5, \theta_2 = 1.2, p = 0.60 \), the following progressively first failure censored mixture real life data is
0.39, 1.08, 1.61, 1.80, 1.87, 1.89, 2.05, 2.35, 2.43, 2.55, 2.95 \quad (r_1 = 11)
1.47, 1.69, 2.41, 2.50, 2.81 \quad (r_2 = 5)

Based on these data, we compute predictive interval of future sample. According the algorithm in Subsection 5.1, we generate 1000 MCMC samples. The 90% and 95% predictive interval for the future observation $Y_s$ are given by solving Equation (21) numerically. The results are presented in Table (9) and Table (10).

### Table 9: Two sample prediction intervals for the future observation $Y_s$ in case informative prior

| $s$ | (Lower, Upper) | Length |
|-----|----------------|--------|
| 1   | (0.01984,0.92570) | 0.90586 |
| 2   | (0.06398,1.20602) | 1.14204 |
| 3   | (0.28126,2.16251) | 1.88125 |
| 4   | (0.46879,3.13421) | 2.66542 |
| 5   | (0.76259,4.29607) | 3.53348 |
| 6   | (1.0686,6.3282) | 5.26134 |

| $s$ | (Lower, Upper) | Length |
|-----|----------------|--------|
| 1   | (0.01413,1.06036) | 1.04623 |
| 2   | (0.05280,1.50413) | 1.45133 |
| 3   | (0.26863,2.46712) | 1.89849 |
| 4   | (0.43263,3.13654) | 2.70391 |
| 5   | (0.68720,5.66177) | 4.97457 |
| 6   | (0.95036,7.56534) | 6.61498 |

### Table 10: Two sample prediction intervals for the future observation $Y_s$ in case non-informative prior

| $s$ | (Lower, Upper) | Length |
|-----|----------------|--------|
| 1   | (0.02533,0.97698) | 0.95165 |
| 2   | (0.08472,1.71576) | 1.63104 |
| 3   | (0.29504,2.34837) | 2.05333 |
| 4   | (0.62261,3.7236) | 3.10099 |
| 5   | (1.03315,5.41041) | 4.37726 |
| 6   | (1.15488,7.61791) | 6.46303 |

| $s$ | (Lower, Upper) | Length |
|-----|----------------|--------|
| 1   | (0.02160,1.10856) | 1.08696 |
| 2   | (0.07490,1.98037) | 1.90547 |
| 3   | (0.28267,2.82536) | 2.54269 |
| 4   | (0.56385,4.30136) | 3.73751 |
| 5   | (0.89780,6.29315) | 5.39535 |
| 6   | (1.03551,8.91873) | 7.88322 |

### 8. Conclusion

Based on progressively first-failure censored scheme, in this paper, we have addressed the estimation and prediction problems of the mixture of Weibull and Lomax distributions. The Bayes estimates cannot be obtained in explicit form so importance samples procedure is used to draw MCMC samples. Also, the same MCMC method is used for computing two sample predictive intervals. An example using real data set was used for illustration.

### References

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