On a delay Volterra-Stieltjes quadratic integral equation

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Abstract
In this work, we study the existence of at least one and exactly one continuous solution \( x \in C[0,T] \) for a delay quadratic integral equation of Volterra-Stieltjes type. The continuous dependent of the unique solution will be also proved. The delay Volterra quadratic integral equation of Chandrasekhar's type will be considered.

Keywords
Volterra-Stieltjes type, continuous solution, delay functional integral equation, continuous dependence.

AMS Subject Classification
74H10, 45G10, 47H30.

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In this paper, we study delay Volterra-Stieltjes quadratic integral equation, which includes many key integral and functional equations that arise in nonlinear analysis and its applications. We shall Schauder fixed point theorem instead of using the technique associated with measures of noncompactness.

Consider the delay quadratic integral equation

\[
x(t) = a(t) + \int_{0}^{\varphi_1(t)} f_1(t,s,x(s))d_sg_1(t,s) + \int_{0}^{\varphi_2(t)} f_2(t,s,x(s))d_sg_2(t,s), \quad t \in [0,T]
\] (1.1)

where \( g_i : [0,T] \times [0,T] \to R \) are nondecreasing in the second argument and the symbol \( d_s \) indicates the integration with respect to \( s \).

For the properties of Volterra-Stieltjes integral equations see [4]-[7].

The aim of this paper is to investigate the solvability of the delay Volterra-Stieltjes quadratic integral equation (1.1). The existence of at least one or exact one solution \( x \in C[0,T] \) of the delay quadratic integral equation (1.1) will be proved. The continuous dependence of the unique solution \( x \in C[0,T] \) of the delay functions \( \varphi_i(t) \) and the functions \( g_i(t,s) \) will be studied.

The delay Volterra quadratic integral equation of Chandrasekhar's
type [1]

\[ x(t) = a(t) + \int_0^t \frac{\varphi_1(t)}{t+s} k_1(t,s)x(s)ds \]

\[ \int_0^t \frac{\varphi_2(t)}{t+s} k_2(t,s)x(s)ds, \ t \in [0,T] \]  (1.2)

will be given as example.

### 2. Existence of at least one solution

Consider the quadratic integral equation (1.1) under the following assumptions:

(i) \( \varphi_i : [0,T] \to [0,T], \ i = 1,2, \varphi_i(t) \leq t \) are continuous and increasing

(ii) \( a : [0,T] \to [0,T] \) is continuous and \( \sup_i |a(t)| = a \)

(iii) \( f_i : [0,T] \times [0,T] \times \mathbb{R} \to \mathbb{R} \) are continuous and there exist the functions \( b_i \) and \( k_i \) such that

\[ |f_i(t,s,x)| \leq b_i(t,s) + k_i(t,s)|x| \]

where \( b_i, k_i : [0,T] \times [0,T] \to \mathbb{R} \) are continuous, \( b = \max \{b_1, b_2\} \), \( k = \max \{k_1, k_2\} \) and

\[ b_i = \sup_{t,s} b_i(t,s), \ k_i = \sup_{t,s} k_i(t,s) \]

(iv) The functions \( g_i : [0,T] \times [0,T] \times \mathbb{R} \to \mathbb{R}, \ i = 1,2 \) are continuous with

\[ \mu = \max \{\sup_i |g_i(t,\varphi_i(t))| + \sup_i |g_i(t,0)|, \ |a| [0,T]\} \]

(v) For all \( t_1, t_2 \in I \) such that \( t_1 < t_2 \) the functions \( s \to g_i(t_2,s) - g_i(t_1,s) \) are nondecreasing on \([0,T] \)

(vi) \( g_i(0,s) = 0 \) for any \( s \in [0,T] \)

(vii) \( 2b\mu k^2 < 1 \)

(viii) There exists a positive root \( r \) of the algebraic equation

\[ k^2 \mu^2 r^2 - (1 - 2b\mu k^2)r + (a + b^2 \mu^2) = 0. \]

Let \( C[0,T] \) be the Banach space of all continuous functions defined on \([0,T] \) with the norm

\[ ||x|| = \sup_{t \in [0,T]} |x(t)|. \]

For the existence of at least one solution of the quadratic integral equation (1.1), we have the following theorem.

**Theorem 2.3.** Let the assumptions (i)-(viii) be satisfied, then the functional integral equation (1.1) have at least one solution \( x \in C[0,T] \).

**Proof.** Define the operator

\[ Fx(t) = a(t) + \int_0^t \frac{\varphi_1(t)}{t+s} f_1(t,s,x(s)) \int_0^t \frac{\varphi_2(t)}{t+s} f_2(t,s,x(s)) \]

Define the set \( Q_r \) by

\[ Q_r = \{x \in C[0,T] : ||x|| \leq r\} \]

where \( r \) is a positive solution of the algebraic equation \( a + (b + kr)^2 \mu^2 = r \).

It is clear that the set \( Q \) is nonempty, bounded, closed, and convex set.

Now, let \( x \in Q_r \), then

\[ |Fx(t)| \leq |a(t)| + \int_0^t |f_1(t,s,x(s))| ds g_1(t,s) \int_0^t |f_2(t,s,x(s))| ds g_2(t,s) \]

\[ \leq a + \int_0^t |f_1(t,s,x(s))| ds g_1(t,s) \int_0^t |f_2(t,s,x(s))| ds g_2(t,s) \]

\[ \leq a + \int_0^t \left( b_1(t,s) + k_1(t,s)\right) g_1(t,s) ds \int_0^t \left( b_2(t,s) + k_2(t,s)\right) g_2(t,s) ds \]

\[ \leq a + \int_0^t \left( b + k\right) |x(t)| \int_0^t g_1(t,s) ds \]

\[ \int_0^t \left( b + k\right) |x(t)| \int_0^t g_2(t,s) ds \]

\[ \leq a + \int_0^t \left( b + k\right) |x(t)| \int_0^t g_1(t,s) ds \]

\[ \int_0^t \left( b + k\right) |x(t)| \int_0^t g_2(t,s) ds \]

\[ \leq a + (b + kr)(g_1(t,\varphi_1(t)) - g_1(t,0))(b + kr) \]

\[ (g_2(t,\varphi_2(t)) - g_2(t,0)) \leq a + (b + kr)^2 \mu^2 = r. \]

This proves that the operator \( F \) maps \( Q_r \) into itself and the class of functions \{Fx\} is uniformly bounded on \( Q_r \).

Let \( x \in Q_r \) and define

\[ \theta(\delta) = \sup_{t_1, t_2 \in [0,T]} \{ |f_1(t_2,s,x(s)) - f_1(t_1,s,x(s))| : t_1, t_2 \in [0,T], \]

\[ t_1 < t_2, |t_2 - t_1| < \delta \}, i = 1,2, \]

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then from the uniform continuity of the function $f_t : [0, T] \times [-r, r] \to R$, and assumption (iii), we deduce that $\theta(\delta) \to 0$, as $\delta \to 0$ independent of $x \in Q_T$.

Now, let $t_2, t_1 \in [0, T]$, such that $|t_2 - t_1| < \delta$, then we have

$$
|F(x(t_2) - F(x(t_1)) = |a(t_2) + \int_0^{\Phi_1(t_2)} f_1(t_2, s, x(s)) \, dg_1(t_2, s) \\
- \int_0^{\Phi_2(t_2)} f_2(t_2, s, x(s)) \, dg_2(t_2, s)) \\
- a(t_1) - \int_0^{\Phi_1(t_1)} f_1(t_1, s, x(s)) \, dg_1(t_1, s) \\
+ \int_0^{\Phi_2(t_1)} f_2(t_1, s, x(s)) \, dg_2(t_1, s) \\
- f_1(t_1, s, x(s)) \, dg_1(t_1, s) \\
+ f_2(t_1, s, x(s)) \, dg_2(t_1, s)| \\
|a(t_2) - a(t_1)| \\
+ |f_1(t_2, s, x(s)) \, dg_1(t_2, s) \\
- f_1(t_1, s, x(s)) \, dg_1(t_1, s) \\
+ f_2(t_2, s, x(s)) \, dg_2(t_2, s) \\
- f_2(t_1, s, x(s)) \, dg_2(t_1, s)| \\
\leq |a(t_2) - a(t_1)| \\
+ |f_1(t_2, s, x(s)) \, dg_1(t_2, s) \\
- f_1(t_1, s, x(s)) \, dg_1(t_1, s) \\
+ f_2(t_2, s, x(s)) \, dg_2(t_2, s) \\
- f_2(t_1, s, x(s)) \, dg_2(t_1, s)|
Applying Lebesgue dominated convergence theorem (see \[\text{(vii)-(vii)}\) be satisfied, if

\[|f_i(t,x(s))| - |f_i(t,x(s))| \leq |f_i(t,x(s)) - f_i(t,x(s))| \leq k|x|\]

From the assumption (iii) we have

\[|f_i(t,x(s))| - |f_i(t,x(s))| \leq |f_i(t,x(s)) - f_i(t,x(s))| \leq k|x|\]

then

\[|f_i(t,x(s))| \leq k|x| + b,\]

where \(b = \sup |f_i(t,x(s))|\). For the uniqueness of the solution of the functional integral equation (1.1) we have the following theorem.

**Theorem 3.1.** Let the assumptions (i)-(ii)-(iii)\(^*\)-(iv)-(v)-(vi)-(vii)-(viii) be satisfied, if \(2k\mu^2(b+kr) < 1\), then the solution \(x \in C[0,T]\) of the functional integral equation (1.1) is unique.

**Proof.** Let \(x_1, x_2\) be two solutions of the integral equation (1.1), then

\[|x_1(t) - x_2(t)| = |a(t) + \int_0^\Phi_1(t) f_1(t,x_n(s)) d_1g_1(t,s)\]

\[\int_0^\Phi_1(t) f_2(t,x_n(s)) d_1g_2(t,s)\]

\[= a(t) + \lim_{n \to \infty} \int_0^\Phi_1(t) f_1(t,x_n(s)) d_1g_1(t,s)\]

\[\int_0^\Phi_1(t) f_2(t,x_n(s)) d_1g_2(t,s)\]

Applying Lebesgue dominated convergence theorem (see [9]), then

\[= a(t) + \int_0^\Phi_1(t) f_1(t,x,\lim_{n \to \infty} x_n(s)) d_1g_1(t,s)\]

\[\int_0^\Phi_1(t) f_2(t,x,\lim_{n \to \infty} x_n(s)) d_1g_2(t,s)\]

\[= a(t) + \int_0^\Phi_1(t) f_1(t,x_0(s)) d_1g_1(t,s)\]

\[\int_0^\Phi_1(t) f_2(t,x_0(s)) d_1g_2(t,s)\] = \(F(x_0(t))\).

This means that the operator \(F\) is continuous on \(Q_r\).

Since all conditions of Schauder fixed point theorem (see [8]) are satisfied, then the operator \(F\) has at least one fixed point \(x \in Q_r\), and the quadratic integral equation (1.1) have at least one solution \(x \in C[0,T]\).

This completes the proof. \(\blacksquare\)
\[
+ \int_0^{\Phi_1(t)} f_1(t,s,x_1(s)) \, ds \leq |a(t) + \int_0^{\Phi_1(t)} f_1(t,s,x(s)) \, ds| \\
\int_0^{\Phi_2(t)} f_2(t,s,x_2(s)) \, ds \leq |a(t) - \int_0^{\Phi_1(t)} f_1(t,s,x^*(s)) \, ds| \\
- \int_0^{\Phi_1(t)} f_1(t,s,x_1(s)) \, ds \leq \left| \int_0^{\Phi_2(t)} f_2(t,s,x_2(s)) \, ds \right| \\
- \int_0^{\Phi_1(t)} f_1(t,s,x_1(s)) \, ds \leq \left| \int_0^{\Phi_2(t)} f_2(t,s,x_2(s)) \, ds \right| \\
+ \int_0^{\Phi_1(t)} f_1(t,s,x_1(s)) \, ds \leq \left| \int_0^{\Phi_2(t)} f_2(t,s,x_2(s)) \, ds \right| \\
+ \int_0^{\Phi_1(t)} f_1(t,s,x_1(s)) \, ds \leq \left| \int_0^{\Phi_2(t)} f_2(t,s,x_2(s)) \, ds \right|
\]

This means that \( x_1 = x_2 \) and the solution of the functional integral equation (1.1) is unique. \( \blacksquare \)

### 4. Continuous dependence of the solution

In this section we are going to study the continuous dependence of the unique solution \( x \in C[0,T] \) of the functional integral equation (1.1) on the delay functions and the functions \( g_i(t,s) \).

#### 4.1 Continuous dependence on the delay functions \( \Phi_i(t) \)

**Definition 4.1.** The solutions of the functional integral equation (1.1) depends continuously on the delay functions \( \Phi_i(t) \) if \( \forall \varepsilon > 0, \exists \delta > 0 \), such that if \( x, x^* \) are solutions of equation (1) related to functions \( \Phi_i \) and \( \Phi_i^* \), respectively, then

\[
|\Phi_i(t) - \Phi_i^*(t)| \leq \delta \Rightarrow \|x - x^*\| \leq \varepsilon.
\]

**Theorem 4.2.** Let the assumptions of Theorem 3 be satisfied, then the solution of the functional integral equation (1.1) dependence continuously on the delay functions \( \Phi_i(t) \).

**Proof.** Let \( \delta > 0 \) be given such that \( |\Phi_i(t) - \Phi_i^*(t)| \leq \delta \), \( \forall t \geq 0 \), then
This completes the proof.

Theorem 4.4. Let the assumptions of Theorem 3 be satisfied, then the solution of the delay quadratic functional integral equation (1.1) depends continuously on the functions \( g_i(t,s) \).

Proof.

\[
|x(t) - x^*(t)| = |a(t) + \int_0^t f_1(t,s,x(s)) \, ds \, g_1(t,s) + \int_0^t f_2(t,s,x(s)) \, ds \, g_2(t,s) - a(t) - \int_0^t f_1(t,s,x^*(s)) \, ds \, g_1^*(t,s) - \int_0^t f_2(t,s,x^*(s)) \, ds \, g_2^*(t,s)|
\]

\[
\leq \int_0^t \left| f_1(t,s,x(s)) \right| \, ds \, g_1(t,s) + \int_0^t \left| f_2(t,s,x(s)) \right| \, ds \, g_2(t,s)
\]

This implies that

\[
\|x - x^*\| \leq \left( \frac{(b+kr)\|g_1(t,\varphi_1(t)) - g_1(t,\varphi_1^*(t))\|}{1 - 2k(b+kr)\|\mu\|^2} \right) + \frac{(b+kr)\|g_2(t,\varphi_2(t)) - g_2(t,\varphi_2^*(t))\|}{1 - 2k(b+kr)\|\mu\|^2}
\]

But from the continuity of \( g_i \), we have

\[
|\varphi_i(t) - \varphi_i^*(t)| \leq \delta \Rightarrow |g_i(t,\varphi_i(t)) - g_i(t,\varphi_i^*(t))| < \varepsilon_i,
\]

then

\[
\leq \frac{(b+kr)\varepsilon_i + (b+kr)\varepsilon_i}{1 - 2k(b+kr)\|\mu\|^2} \leq \frac{2(b+kr)\varepsilon_i}{1 - 2k(b+kr)\|\mu\|^2} = \varepsilon.
\]

This completes the proof. \( \square \)

4.2 Continuous dependence on the functions \( g_i(t,s) \)

**Definition 4.3.** The solutions of the quadratic functional integral equation (1.1), is continuous dependence on the functions \( g_i(t,s) \), \( i = 1,2 \) if \( \forall \varepsilon > 0, \exists \delta > 0 \), such that if \( x, \ x^* \) are solutions of equation (1) related to functions \( g(t,s) \), and \( g(t,s)^* \), respectively,

\[
|g_i(t,s) - g_i^*(t,s)| \leq \delta \Rightarrow \|x - x^*\| \leq \varepsilon.
\]
then
\[ \| x - x^* \| \leq \frac{1}{2} \| x - x^* \| [1 - 2(b + kr)\mu^2 k] \leq 4(b + kr)^2 \mu \delta \]
and
\[ \| x - x^* \| \leq \frac{4(b + kr)^2 \mu \delta}{1 - 2k(b + kr)\mu^2} = \varepsilon. \]

This completes the proof. \( \square \)

**5. Example**

Let \( f_i(t, s, x(s)) = k_i(t, s)\| x(s) \| \) and let the functions \( g_i(t, s) \) be given by
\[
g_i(t, s) = \begin{cases} \frac{t \ln \frac{t+2}{t}}{t+2}, & \text{if } t \in (0, T] \\ 0, & \text{if } t = 0, \end{cases}
\]
then
\[
dg_i(t, s) = \frac{t}{t+s} ds
\]
and the assumptions (iii)-(vi) are satisfied (see [7]). Then our result can be applied to the delay Volterra quadratic integral equation of Chandrasekhar's type (1.2) and the unique solution of (1.2) depends continuously on the delay functions \( \varphi_i(t) \).

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