SUBCONVEXITY BOUND FOR $GL(3) \times GL(2)$ L-FUNCTIONS IN
$GL(2)$ SPECTRAL ASPECT

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Abstract

Let $\pi$ be a Hecke-Maass cusp form for $SL(3, \mathbb{Z})$ and $f$ be a holomorphic cusp form for $SL(2, \mathbb{Z})$ of weight $k$ or a Hecke-Maass cusp form corresponding to the Laplacian eigenvalue $1/4 + \frac{k^2}{4}$, $k \geq 1$, for $SL(2, \mathbb{Z})$. In this paper, we prove the following subconvexity bound

$$L(1/2, \pi \times f) \ll_{\pi, \epsilon} k^{3/2} \frac{1}{\pi^\epsilon}.$$ 

1. Introduction

A degree $d$ automorphic $L$-function $L(s, F)$ associated to an automorphic form $F$ is a Dirichlet series with an Euler product of degree $d$ and satisfying some “nice” analytic properties. In fact, it has a meromorphic continuation to the whole complex plane $\mathbb{C}$ and its completed $L$-function satisfies a functional equation relating its value at $s$ to the value of the corresponding dual $L$-function at $1 - s$. One may apply the Phragmén-Lindelöf principle together with the functional equation to get an upper bound $L(1/2 + it, F) \ll_{d, \epsilon} (C(F, t))^{1/4 + \epsilon}$, for any $\epsilon > 0$, on the critical line $\Re s = 1/2$. Here $C(F, t)$ is a quantity, so-called the analytic conductor, which measures the complexity of the $L$-function and encapsulates the main parameters (level, spectral parameters, etc.) attached to the form $F$. The resulting bound is usually referred to as the convexity bound (or the trivial bound). It is conjectured, known as the Lindelöf Hypothesis, that the exponent $1/4$ can be reduced to 0. While the Lindelöf Hypothesis is still out of reach, breaking the convexity bound, i.e., reducing the exponent $1/4$ by any small quantity, known as the subconvexity problem, is a challenging yet an interesting problem.

For degree one $L$-functions ($\zeta(s)$ and $L(s, \chi)$), such estimates are known due to Weyl [33] and Hardy-Littlewood in the $t$-aspect and due to Burgess [4] in the level aspect. For degree two $L$-functions, the first subconvexity bound was achieved by Good [10] in the $t$-aspect, by Duke-Friedlander-Iwaniec [6], [7], [8] in the level aspect and by Iwaniec [14] in the spectral aspect. For degree three $L$-functions attached to self-dual forms, such estimates were first obtained by Li [24] in the $t$-aspect in a groundbreaking work. Li’s work was generalised to all $GL(3)$ forms by Munshi [26], by introducing a novel delta method which he also applied in resolving the subconvexity problem for $GL(3)$ $L$-functions in the twist aspect [27]. In $GL(3)$ spectral aspect, when spectral parameters of a $GL(3)$ form, $\pi$ say, are in generic position, subconvexity estimates for $L(1/2, \pi)$ were obtained by Blomer-Buttcane [3].

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For higher degree $L$-functions, the subconvexity problem becomes even more challenging, and hence it is still open except for a few particular cases of Rankin-Selberg convolution $L$-functions. For the Rankin-Selberg $L$-functions on $GL(2) \times GL(2)$, subconvexity bounds are known due to Michel-Venkatesh [20] in the $t$-aspect, Sarnak [30], and Lau-Liu-Ye [22] in the spectral aspect, and Kowalski-Michel-Vanderkam [19], Michel [19] and Harcos-Michel [11] in the level aspect. Some impressive subconvexity estimates were obtained by Bernstein-Reznikov [1] and Venkatesh [31] for the Rankin-Selberg triple $L$-functions on $GL(2)$.

We will now discuss a few known results for degree six Rankin-Selberg $L$-functions on $GL(3) \times GL(2)$. To start with, let $\pi$ be a normalized Hecke-Maass cusp form of type $(\nu_1, \nu_2)$ for $SL(3, \mathbb{Z})$. Let $f$ be a holomorphic cusp form with weight $k$ or Maass Hecke cusp form with the Laplace eigenvalue $1/4 + k^2$ for $SL(2, \mathbb{Z})$. The associated Rankin-Selberg $L$-series is given by

$$L(s, \pi \times f) = \sum_{n,r\geq1} \frac{\lambda_\pi(n,r) \lambda_f(n)}{(nr^2)^s}, \Re(s) > 1.$$ (1)

In a pioneering work, Li [24] studied the above series and obtained subconvexity for $L(1/2, \pi \times f)$ in the $GL(2)$ spectral aspect as well as subconvexity for $L(1/2 + it, \pi)$ for a self-dual form $\pi$ (also mentioned above). Her main theorem was the following:

**Theorem (X. Li.).** Let $\pi$ be a fixed self-dual Hecke-Maass cusp form for $SL(3, \mathbb{Z})$ and $u_j$ be an orthonormal basis of even Hecke-Maass cusp form for $SL(3, \mathbb{Z})$ corresponding to the Laplacian eigenvalue $1/4 + t_j^2$ with $t_j \geq 0$; then for $\epsilon > 0$, $T$ large and $T^{3/4 + \epsilon} \leq M \leq T^{1/2}$, we have

$$\sum_j e^{-(u_j-t_j)^2/M^2} L\left(\frac{1}{2}, \pi \times u_j\right) + \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{-(u_j-t_j)^2/M^2} \left| L\left(\frac{1}{2} - it, \pi\right) \right|^2 dt \ll_{\epsilon, \pi} T^{1+\epsilon} M,$$

where $t$ means summing over the orthonormal basis of even Hecke-Maass cusp forms.

As a corollary, she obtained $L(1/2, \pi \times u_j) \ll_{\epsilon, \pi} (1 + |t_j|)^{3/2 - 1/4 + \epsilon}$. She adapted Conrey-Iwaniec’s moment method (see [5]) to prove the above theorem. The fact $L(1/2, \pi \times u_j) \geq 0$ plays a crucial role in her approach. Unfortunately, the above fact does not hold for non-self-dual GL(3) forms, that is why she only dealt with the self-dual forms. Recently, Munshi [28], using his delta method, obtained subconvexity for $L(s, \pi \times f)$ in the $t$-aspect proving the following result:

$$L(1/2 + it, \pi \times f) \ll_{\epsilon, f, \pi} (1 + |t|)^{3/2 - 1/4 + \epsilon}.$$  

His method is insensitive to self-duality of the GL(3) forms. Thus he could obtain the above result for any GL(3) form. Using a similar approach, Sharma [32] and the author, Mallesham and Singh [17] proved subconvexity bounds in the twist and the GL(3) spectral aspect (in some cases), respectively. It is natural to ask a similar question in other aspects as well. As suggested by Munshi, we vary the GL(2) form and establish subconvexity for $L(1/2, \pi \times f)$ in the GL(2) spectral aspect. Our main theorem is the following:

**Theorem 1.** Let $\pi$ be a fixed Hecke-Maass cusp form for $SL(3, \mathbb{Z})$ and $f$ be a holomorphic cusp form with weight $k$ or a Hecke-Maass cusp form corresponding to the
Laplacian eigenvalue $1/4 + k^2$, $k \geq 1$, for $SL(2, \mathbb{Z})$. Then, for any $\epsilon > 0$, we have

$$L(1/2, \pi \times f) \ll_{\pi, \epsilon} k^{\frac{1}{4} + \frac{1}{4} + \epsilon}.$$  

**Remark 1.** (1) We follow Munshi [28] to prove the above theorem. As mentioned before, the method is insensitive to self-duality of the $GL(3)$ forms, we also obtain our result for any $GL(3)$ form. Thus we generalise Li’s main result [24]. Although our bound is weaker than her’s, it yields a subconvexity.

(2) Our arguments in the proof work for both Maass and holomorphic forms. For the exposition of the method, we will give details for holomorphic forms only.

(3) Our method also works for any fixed central value $1/2 + it$. In this case, the implied constant will depend polynomially on $t$. For simplicity, we take $t = 0$ in the proof.

If we take $\pi$ to be the minimal Eisenstein series with Langlands parameters $(\alpha_1, \alpha_2, \alpha_3)$ for $SL(3, \mathbb{Z})$ in [1], we observe that (see [9, p. 314])

$$L(s, \pi) = \prod_p \prod_{i=1}^3 (1 - p^{\alpha_i-s})^{-1} = \zeta(s - \alpha_1)\zeta(s - \alpha_2)\zeta(s - \alpha_3).$$

It is also well-known that

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \prod_{j=1}^2 (1 - \beta_{p,j}p^{-s})^{-1},$$

where $\beta_{p,1}\beta_{p,2} = 1$, $\beta_{p,1} + \beta_{p,2} = \lambda_f(p)$, and $\lambda_f(n)$ denote the normalised Fourier coefficients of $f$. Using the Rankin-Selberg theory (see [9, p. 379]), we get

$$L(s, \pi \times f) = \prod_p \prod_{i=1}^3 (1 - \beta_{p,1}p^{\alpha_i-s})^{-1} (1 - \beta_{p,2}p^{\alpha_i-s})^{-1} = L(s - \alpha_1, f)L(s - \alpha_2, f)L(s - \alpha_3, f).$$

Our method also fits for the above $L$-functions as well. Hence we also obtain the following result:

**Theorem 2.** Let $f$ be a holomorphic cusp form with weight $k$ or Hecke-Maass cusp form corresponding to the Laplacian eigenvalue $1/4 + k^2$, $k \geq 1$, for $SL(2, \mathbb{Z})$. Then, for $\epsilon > 0$, we have

$$L(1/2, f) \ll_{\epsilon} k^{\frac{1}{4} + \frac{1}{4} + \epsilon}.$$  

We end the introduction by commenting on the method. As a first step, we use the ‘conductor lowering’ trick as a device to separate the oscillations using the delta method due to Duke-Friedlander-Iwaniec (DFI) (see [6]). We now apply the $GL(3)$ and $GL(2)$ Voronoi formulae. A crucial observation, which was also present in [17], [28] and [32], that the $GL(2)$ and $GL(3)$ Voronoi formulae together tranform the Ramanujan sum

$$\sum_{a \mod q}^* e\left(\frac{(n - m)a}{q}\right),$$

arising from the DFI delta method, into

$$\sum_{a \mod q}^* S(\overline{a}, n; q)e\left(\frac{\overline{am}}{q}\right),$$
which boils down to an additive character $q e(\overline{m}n/q)$ with respect to $n$, plays a vital role to prove our main theorem. In fact, we save $\sqrt{q}$ extra after applying the Poisson summation formula in the $n$-variable due to the additive character (see subsection 3.2). The analysis of the integral transforms (see Section 6) is one of technical inputs of the article.

**Notation.** Throughout the paper, $e(x)$ means $e^{2\pi i x}$. By negligibly small we mean $O(k^{-A})$ for any large postive constant $A > 0$. In particular, we take $A = 2020$. The letter $\epsilon$ denotes arbitrarily small constant, not necessarily the same at different occurrences. The notation $\alpha \ll A$ will mean that for any $\epsilon > 0$, there is a constant $c$ such that $|\alpha| \leq c A^\epsilon$. We also ignore the dependence of the constant on $\pi$ and $\epsilon$, whenever it occurs. By $\alpha \asymp A$, we mean that $k^{-\epsilon} A \leq \alpha \leq k^\epsilon A$, also $\alpha \sim A$ means $A \leq \alpha < 2A$. For absolute explicit constant, we will write $c$ or $c_i$ in the whole paper.

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2. **Preliminaries**

In this section, we will recall some known results which we need in the proof.

2.1. **Holomorphic cusp forms on GL(2).** Let $f$ be a holomorphic Hecke eigenform of weight $k$ for the full modular group $\text{SL}(2, \mathbb{Z})$. The Fourier expansion of $f$ at the cusp $\infty$ is given by

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e(nz), \quad z \in \mathbb{H}.$$ 

We assume that $f$ is normalised so that $\lambda_f(1) = 1$. We have the well-known Deligne bound $|\lambda_f(n)| \leq d(n), n \geq 1$, where $d(n)$ is the divisor function. However, in our proof, we only need the Ramanujan bound on average:

$$\sum_{n \leq X} |\lambda_f(n)|^2 \ll_{\epsilon} X^{1+\epsilon},$$

for any $\epsilon > 0$. We now recall the Voronoi summation formula for the form $f$ which will be crucially used in our proof.

**Lemma 1.** Let $\lambda_f(n)$ be as above and $g$ be a smooth, compactly supported function on $(0, \infty)$. Let $a, q \in \mathbb{Z}$ with $(a, q) = 1$. Then we have

$$\sum_{n=1}^{\infty} \lambda_f(n) e\left(\frac{an}{q}\right) g(n) = \frac{1}{q} \sum_{n=1}^{\infty} \lambda_f(n) e\left(-\frac{dn}{q}\right) h\left(\frac{n}{q^2}\right),$$

where $ad \equiv 1 (\text{mod } q)$ and

$$h(y) = 2\pi i^k \int_{0}^{\infty} g(x) J_{k-1} (4\pi \sqrt{xy}) \, dx,$$
where $J_{k-1}$ is the usual $J$-Bessel function of order $k-1$.

Proof. See [16, Theorem A.4]. □

2.2. Maass cusp forms for GL(2). Let $f$ be a Hecke-Maass eigenform for $\text{SL}(2,\mathbb{Z})$ with Laplace eigenvalue $1/4 + \nu^2$, $\nu > 0$. The Fourier series expansion of $f$ at the cup $\infty$ is given by

$$f(z) = \sqrt{y} \sum_{n \neq 0} \lambda_f(n) K_{i\nu}(2\pi |n| y)e(nx),$$

where $K_{i\nu}(y)$ is the Bessel function of the third kind and $f$ is normalized so that $\lambda_f(1) = 1$. The Ramanujan-Petersson conjecture, which asserts that $|\lambda_f(n)| \leq d(n)$ is not known yet. However, we do not need of such individual bound for our proof. Rather, the following Ramanujan bound on average (see [14, Lemma 1])

$$\sum_{1 \leq n \leq X} |\lambda_f(n)|^2 \ll \nu^\epsilon X^{1+\epsilon},$$

for any $\epsilon > 0$, is sufficient for our purpose. We also have the following Voronoi summation formula for the Maass cusp forms, which is similar to the case of holomorphic cusps forms.

**Lemma 2.** Let $\lambda_f(n)$ be as above and $g$ be a smooth, compactly supported function on $(0, \infty)$. Let $a, q \in \mathbb{Z}$ with $(a, q) = 1$. Then we have

$$\sum_{n=1}^{\infty} \lambda_f(n) e\left(\frac{an}{q}\right) g(n) = \frac{1}{q} \sum \sum_{n=1}^{\infty} \lambda_f(n) e\left(\frac{+dn}{q}\right) H^\pm\left(\frac{n}{q^2}\right),$$

where $ad \equiv 1 \text{ mod } q$ and

$$H^-(y) = \frac{-\pi}{\sin(\pi \nu)} \int_0^\infty g(x) \{J_{2\nu} - J_{-2\nu}\} (4\pi \sqrt{xy}) \, dx,$$

$$H^+(y) = 4\epsilon_f \cosh(\pi \nu) \int_0^\infty g(x) K_{2\nu} (4\pi \sqrt{xy}) \, dx.$$

Here $\epsilon_f$ is the eigenvalue of $f$ under the reflection operator.

Proof. See [15, Theorem A.4]. □

2.3. Automorphic forms on GL(3). This section, except for the notations, is taken from [24]. Let $\pi$ be a Hecke-Maass cusp form of type $(\nu_1, \nu_2)$ for $\text{SL}(3,\mathbb{Z})$. Let $\lambda_\pi(n, r)$ denote the normalised Fourier coefficients of $\pi$. Let

$$\alpha_1 = -\nu_1 - 2\nu_2 + 1, \quad \alpha_2 = -\nu_1 + \nu_2 \quad \text{and} \quad \alpha_3 = 2\nu_1 + \nu_2 - 1$$

be the spectral parameters for $\pi$ (see [9]). Let $g$ be a compactly supported smooth function on $(0, \infty)$ and

$$\bar{g}(s) = \int_0^\infty g(x)x^{s-1} \, dx$$

be its Mellin transform. For $\ell = 0$ and $1$, we define

$$\gamma_\ell(s) := \frac{\pi^{-3s-\frac{3}{2}}}{2} \prod_{i=1}^{3} \frac{\Gamma\left(\frac{1+s+\alpha_i+\ell}{2}\right)}{\Gamma\left(-\frac{s+\alpha_i+\ell}{2}\right)}.$$
Set $\gamma_\pm(s) = \gamma_0(s) \mp \gamma_1(s)$ and let

\begin{equation}
\label{eqn:Gpm}
G_\pm(y) = \frac{1}{2\pi i} \int_{(0)} y^{-s} \gamma_\pm(s) \tilde{g}(-s) \, ds,
\end{equation}

where $\sigma > -1 + \max\{-\Re(\alpha_1), -\Re(\alpha_2), -\Re(\alpha_3)\}$. With the aid of the above terminology, we now state the GL(3) Voronoi summation formula.

**Lemma 3.** Let $g(x)$ and $\lambda_\pi(n, r)$ be as above. Let $a, q \in \mathbb{Z}$ with $q \geq 1, (a, q) = 1$, and $a \bar{a} \equiv 1(\mod q)$. Then we have

\begin{equation}
\sum_{n=1}^{\infty} \lambda_\pi(n, r)e\left(\frac{an}{q}\right) g(n) = q \sum_{\pm} \sum_{n_1|qr}^{\infty} \sum_{n_2=1}^{\infty} \lambda_\pi(n_1, n_2) S(r\bar{a}, \pm n_2; qr/n_1) G_\pm\left(\frac{n_1^2 n_2}{q^3 r}\right),
\end{equation}

where $S(a, b; q)$ is the Kloosterman sum which is defined as follows:

\begin{equation}
S(a, b; q) = \sum_{x \mod q}^* e\left(\frac{ax + b\bar{x}}{q}\right).
\end{equation}

**Proof.** See [21].

We need to extract the oscillations of the integral transform $G_\pm$. To this end, we have the following lemma:

**Lemma 4.** Let $G_\pm(x)$ be as above, and $g(x) \in C^\infty_c(X, 2X)$. Then for any fixed integer $K \geq 1$ and $xX \gg 1$, we have

\begin{equation}
G_\pm(x) = x \int_0^\infty g(y) \sum_{j=1}^{K} c_j(\pm) e\left(3(xy)^{1/3}\right) + d_j(\pm) e\left(-3(xy)^{1/3}\right) \, dy + O \left((xX)^{-K+1}\right),
\end{equation}

where $c_j(\pm)$ and $d_j(\pm)$ are some absolute constants depending on $\alpha_i$’s, for $i = 1, 2, 3$.

**Proof.** See [23, Lemma 6.1].

The following lemma is the well-known Ramanujan bound on average,

**Lemma 5.** We have

\begin{equation}
\sum_{n_1^2 n_2 \leq x} \sum_{n_1, n_2} |\lambda_\pi(n_1, n_2)|^2 \ll_{\pi} x,
\end{equation}

where the implied constant depends on the form $\pi$.

**Proof.** For the proof, we refer to Goldfeld’s book [9].

2.4. The delta method. Let $\delta : \mathbb{Z} \to \{0, 1\}$ be defined by

\begin{equation}
\delta(n) = \begin{cases} 
1 & \text{if } n = 0; \\
0 & \text{otherwise.}
\end{cases}
\end{equation}

The above function can be used to separate the oscillations involved in a sum, $\sum_{n \leq X} a(n) b(n)$, say. Furthermore, we seek a ‘nice’ Fourier expansion of $\delta(n)$. We mention here an expansion for $\delta(n)$ due to Duke, Friedlander and Iwaniec (see [15 Chapter 20]). Let $L \geq 1$ be a large real number. For $n \in [-2L, 2L]$, we have

\begin{equation}
\delta(n) = \frac{1}{Q} \sum_{1 \leq q \leq Q} \frac{1}{q} \sum_{a \mod q} e\left(\frac{na}{q}\right) \int_{\mathbb{R}} g(q, x) e\left(\frac{nx}{qQ}\right) \, dx,
\end{equation}

where $Q = \sigma_q(n)$ is the circle average.
Lemma 6. Let \( \delta \) be as above. Let \( L \geq 1 \) be a large parameter. Then, for \( n \in [-2L, 2L] \), we have

\[
\delta(n) = \frac{1}{Q} \sum_{1 \leq q \leq Q} \frac{1}{q} \sum_{a \mod q} e \left( \frac{na}{q} \right) \int_{\mathbb{R}} W(x/Q) g(q, x) e \left( \frac{nx}{qQ} \right) \, dx + O(L^{-2020}),
\]

where \( Q = 2L^{1/2} \), \( g \) is a function satisfying (31) and \( W(x) \) is a non-negative smooth bump function supported in \([-2, 2]\), with \( W(x) = 1 \) for \( x \in [-1, 1] \) and \( W^{(j)}(x) \ll_j 1 \), for \( j \geq 0 \).

\[\text{Proof. See [15, Chapter 20] and [12, Lemma 15].}\]

2.5. Bessel function. In this subsection, we will recall some well-known expansions of the Bessel functions of the first kind. For \( k \geq 2 \) an integer, let \( J_{k-1}(x) \) be the Bessel function of the first kind and of order \( k - 1 \), which is defined as

\[
J_{k-1}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e \left( \frac{(k-1)x \sin \tau}{2\pi} \right) \, d\tau,
\]

for any \( x \in \mathbb{R} \). In the analysis of integral transforms, we require a uniform asymptotic expansion of \( J_{k-1}(x) \) for large values of \( k \) and \( x \). The following lemma provides one such asymptotic expansion.

Lemma 7. Let \( x \geq (k-1)^{1+r/2} \) be a positive real number. Then, as \( k \to \infty \), we have

\[
J_{k-1}(x) = \left( \frac{2}{\pi(k-1)w} \right)^{1/2} \left[ \cos \left( (k-1)(w - \tan^{-1} w) - \frac{\pi}{4} \right) \sum_{j=0}^{\infty} \frac{P_j \left( \frac{1}{w - \tan^{-1} w} \right)}{(k-1)^j} \right] + \left( \frac{2}{\pi(k-1)w} \right)^{1/2} \left[ \sin \left( (k-1)(w - \tan^{-1} w) - \frac{\pi}{4} \right) \sum_{j=1}^{\infty} \frac{P_j \left( \frac{1}{w - \tan^{-1} w} \right)}{(k-1)^j} \right],
\]

where

\[ w = \left( \frac{x^2}{(k-1)^2} - 1 \right)^{1/2}, \]

and \( P_j \) is a polynomial of the degree \( j \) with coefficients which are bounded functions of \( k - 1 \) and \( \log(x/(k-1)) \) with \( P_0 \equiv 1 \).
Proof. Let \( x = (k - 1) \sec \beta \), with \( 0 < \beta < \pi/2 \). Thus, as \( x \geq (k - 1)^{1+\epsilon/2} \), we have \( \sec \beta \geq (k - 1)^{\epsilon/2} \) and

\[
\xi := (k - 1)(\tan \beta - \beta) \geq (k - 1)(\sqrt{(k - 1)^2 - 1} - \pi/2).
\]

Thus, on using formula (63) on page 58 of [18], we get

\[
J_{k-1}((k - 1) \sec \beta) = \left( \frac{2}{\pi(k - 1) \tan \beta} \right)^{1/2} \left[ \cos f_1(\beta) \sum_{j=0}^{\infty} \frac{P_j \left( \frac{1}{\tan \beta - \beta} \right)}{(k - 1)^j} \right] + \left( \frac{2}{\pi(k - 1) \tan \beta} \right)^{1/2} \left[ \sin f_1(\beta) \sum_{j=1}^{\infty} \frac{P_j \left( \frac{1}{\tan \beta - \beta} \right)}{(k - 1)^j} \right],
\]

where \( f_1(\beta) = (k - 1)(\tan \beta - \beta) - \pi/4 \), and \( P_j \) represents a polynomial of the degree \( j \) with coefficients which are bounded functions of \( k - 1 \) and \( \log \sec \beta \) with \( P_0 \equiv 1 \). Now substituting \( (k - 1) \sec \beta = x \) and \( \tan \beta = w \), we get the lemma. \( \square \)

The expansion (11) can be truncated at any stage to get

**Corollary 1.** Under the assumptions of Lemma 7 we have

\[
J_{k-1}(x) = \sum_{\pm} \sum_{j=0}^{2019} e^{\pm \frac{(k-1)(w-\tan^{-1} w)}{2\pi w^{1/2}(k - 1)^{j+1/2}}} \frac{1}{w^{j+1/2}} + O \left( \frac{1}{k^{2020}} \right).
\]

**Proof.** The statement follows directly from Lemma 7. \( \square \)

For \( 0 < x \leq (k - 1)^{1-\epsilon/2} \), we have the following lemma.

**Lemma 8.** Let \( x = (k - 1)z \) with \( 0 < z \ll (k - 1)^{-\epsilon/2} \). Then, as \( k \to \infty \), we have

\[
J_{k-1}(x) \ll \exp\{-1/(k - 1)/6\}.
\]

**Proof.** By Lemma 4.2 of [29], we have

\[
|J_{k-1}((k - 1)z)| \leq A_1(k - 1)^{-1/2}(1 - z^2)^{-1/4} \exp \left\{ -\frac{1}{3}(k - 1)(1 - z^2)^{3/2} \right\},
\]

for \( 0 < z \leq \sqrt{1 - \frac{1}{(k - 1)^{2/3}}} \), \( k \geq 16 \) and some absolute constant \( A_1 \). Note that, by assumption, we have \( z \leq (k - 1)^{-\epsilon} \). Thus, \( 1 - z^2 \geq 1/2^{2/3} \) as \( k \to \infty \), and we get

\[
|J_{k-1}((k - 1)z)| \leq A_2 2^{1/6} \exp \left\{ -\frac{1}{6}(k - 1) \right\}.
\]

Hence the lemma follows. \( \square \)

### 2.6. Stationary phase analysis

In this subsection, we will recall some facts about the exponential integrals of the form

\[
I = \int_a^b g(x)e(f(x))dx,
\]

where \( f \) and \( g \) are smooth real valued functions on \([a, b] \).
Lemma 9. Let $I$, $f$ and $g$ be as above. Let $V(g)$ denote the total variation of $g(x)$ on $[a,b]$ plus the maximum modulus of $g(x)$ on $[a,b]$. Then, if $f'$ is monotone and $|f'(x)| \geq \mu_1 > 0$ for $x \in [a,b]$, we have $I \ll V(g)/\mu_1$. For $r > 1$, let $|f^{(r)}(x)| \geq \mu_r > 0$. Then we have $I \ll _r V(g)/\mu_r^{1/r}$. Moreover, let $f'(x) \geq B$ and $f^{(j)}(x) \ll B^{1+\epsilon}$ for $j \geq 2$ together with $\text{Supp}(g) \subset (a,b)$ and $g^{(j)}(x) \ll_{a,b,j} 1$. Then we have

$$I \ll_{a,b,j,\epsilon} B^{-j+\epsilon}.$$

Proof. See [26] Subsection 2.2 and [13] Lemma 5.1.4.

We apply the above lemma for $r = 1$ whenever the phase function $f$ does not have any stationary point. We will also apply it for $r = 2$ and 3. In case there is a unique stationary point, we use the following stationary phase expansion.

Lemma 10. Let $I$, $f$ and $g$ be as above. Let $0 < \delta < 1/10$, $X$, $Y$, $U$, $Q > 0$, $Z := Q + X + Y + b - a + 1$, and assume that

$$Y \geq Z^{3\delta}, \ b - a \geq U \geq \frac{QZ^2}{\sqrt{Y}}.$$

Further, assume that $g$ satisfies

$$g^{(j)}(x) \ll_j \frac{X}{U^2}, \quad \text{for} \ j = 0, 1, 2, \ldots$$

Suppose that there exists a unique $x_0 \in [a,b]$ such that $f'(x_0) = 0$, and the function $f$ satisfies

$$f''(x) \gg \frac{Y}{Q^2}, \quad f^{(j)}(x) \ll_j \frac{Y}{Q^2}, \quad \text{for} \ j = 1, 2, 3, \ldots.$$

Then we have

$$I = \frac{e(f(x_0))}{\sqrt{f''(x_0)}} \sum_{n=0}^{3\delta^{-1}A} p_n(x_0) + O_{A,\delta}(Z^{-A}),$$

$$p_n(x_0) = \frac{e^{\pi i/4}}{n!} \left( \frac{i}{2f''(x_0)} \right)^n G^{(2n)}(x_0),$$

where $A > 0$ is arbitrary, and

$$G(x) = g(x)e(F(x)), \quad F(x) = f(x) - f(x_0) - \frac{1}{2}f''(x_0)(x-x_0)^2.$$

Furthermore, each $p_n$ is a rational function in $f'$, $f''$, $\ldots$, satisfying

$$\frac{d^j}{dx_0^j} p_n(x_0) \ll_{j,n} X \left( \frac{1}{U^2} + \frac{1}{Q^2} \right) \left( \left( \frac{U^2 Y}{Q^2} \right)^{-n} + Y^{-\frac{n}{3}} \right).$$

Proof. See [2] Lemma 8.1.

3. The set-up and outline of proof

Let $\pi$ and $f$ be defined as in Theorem 11. Let $\lambda_n(n,r)$ denote the normalised Fourier coefficients of the form $\pi$ (see [9] Chapter 6) and let $\lambda_f(n)$ denote the normalised Fourier coefficients of the form $f$ (see [13] Chapter 14). We are interested in analyzing the Rankin-Selberg $L$-series $L(s, \pi \times f)$ (defined in [11]) attached to $\pi$ and $f$ at the central point $1/2$. To study $L(1/2, \pi \times f)$, we first express it as a weighted Dirichlet series.
Lemma 11. Let $\theta$ be a positive real number such that $0 < \theta < 3/2$. Then, as $k \to \infty$, we have
\begin{equation}
L(1/2, \pi \times f) \ll k^\epsilon \sup_{r \leq k^\theta} \frac{1}{\epsilon} \sup_{r \leq N \leq k^{3+\epsilon}} \frac{|S_r(N)|}{N^{1/2}} + k^{(3-\theta)/2+\epsilon},
\end{equation}
where $S_r(N)$ is a sum of the form
\begin{equation}
S_r(N) := \sum_{n=1}^{\infty} \lambda_\pi(n, r)\lambda_f(n) V\left(\frac{n}{N}\right),
\end{equation}
for some smooth function $V$ supported in $[1, 2]$, satisfying $V^{(j)}(x) \ll j$ for $j \geq 0$ and normalised so that $\int V(y) dy = 1$.

Proof. Proof follows using the template given in [15, Theorem 5.3]. See also [28, p. 1546-1547].

Remark 2. Upon estimating $S_r(N)$ using Cauchy’s inequality and the Ramanujan bound on average (see [2], [3], [6]), we see that $L(1/2, \pi \times f) \ll \pi \varepsilon k^{3/2+\varepsilon}$. Hence, to establish subconvexity, we need to get some cancellations in the sum $S_r(N)$ for $N$, roughly, of size $k^5$. To this end, we will analyze $S_r(N)$ in the rest of the paper.

3.1. An application of the delta method. As a first step, following Munshi [28], we separate the oscillatory terms $\lambda_\pi(n, r)$ and $\lambda_f(n)$ involved in the sum $S_r(N)$. We use the delta method of Duke, Friedlander and Iwaniec as a device to separate these terms. We also apply the conductor lowering trick introduced by Munshi in [26]. For this purpose, we introduce an extra $t$-integral. In fact, we express $S_r(N)$ as
\begin{equation}
\frac{1}{T} \int_\mathbb{R} V\left(\frac{t}{T}\right) \sum_{n, m=1}^{\infty} \lambda_\pi(n, r)\lambda_f(m) \left(\frac{n}{m}\right)^{it} V\left(\frac{n}{N}\right) U\left(\frac{m}{N}\right) dt
\end{equation}
\begin{equation}
= \frac{1}{T} \int_\mathbb{R} V\left(\frac{t}{T}\right) \sum_{n, m=1}^{\infty} \delta(n - m)\lambda_\pi(n, r)\lambda_f(m) \left(\frac{n}{m}\right)^{it} V\left(\frac{n}{N}\right) U\left(\frac{m}{N}\right) dt,
\end{equation}
where $k^\epsilon < T < k^{1-\epsilon}$ is a parameter of the form $k^{1-\eta}$, for $\eta > 0$, which will be chosen later optimally, and $U$ is a smooth function supported in $[1/2, 5/2]$ with $U(x) = 1$ for $x \in [1, 2]$, and $U^{(j)}(x) \ll j$ for any integer $j \geq 0$. Consider the $t$-integral
\begin{equation}
\frac{1}{T} \int_\mathbb{R} V\left(\frac{t}{T}\right) \left(\frac{m}{n}\right)^{it} dt.
\end{equation}

On applying integration by parts repeatedly, we observe that the above integral is negligibly small unless $|n - m| \ll k^\varepsilon N/T$. Thus the $t$-integral reduces the size of the equation $n = m$. Thus, on applying Lemma 6 to (13) with $L = k^\varepsilon N/T$, and $Q = k^\varepsilon \sqrt{N/T}$, we see that $S_r(N)$ is transformed into
\begin{equation}
S_r(N) = \frac{1}{QT} \int_\mathbb{R} W(x/Q^2) \int_\mathbb{R} V\left(\frac{t}{T}\right) \sum_{1 \leq q \leq Q} \frac{g(q, x)}{q} \sum_{a \bmod q}^*
\times \sum_{n=1}^{\infty} \lambda_\pi(n, r)e\left(\frac{an}{q}\right) e\left(\frac{nx}{qQ}\right) n^{it} V\left(\frac{n}{N}\right)
\times \sum_{m=1}^{\infty} \lambda_f(m)m^{-it} e\left(\frac{-am}{q}\right) e\left(\frac{-mx}{qQ}\right) U\left(\frac{m}{N}\right) dt dx + O(k^{-2020}).
\end{equation}
3.2. Sketch of proof. In this subsection, we will discuss rough ideas to get non-trivial cancellations in \( S_r(N) \) given in \([11]\). For simplicity, we consider the generic case, i.e., \( N = k^3, \ r = 1 \) and \( q \sim Q = \sqrt{N/T} = k^{3/2}/T^{1/2} \). Thus \( S_r(N) \) is roughly given by

\[
\frac{1}{QT} \int_{T}^{2T} \sum_{q \sim Q} \sum_{n \sim N} \lambda_\pi(n, 1)n^\iota e \left( \frac{an}{q} \right) \sum_{m \sim N} \lambda_f(m) m^{-\iota} e \left( -\frac{am}{q} \right) dt.
\]

Note that we have ignored the \( x \)-integral, as it does not contribute in the generic case, and we have also suppressed all the weight functions. On estimating the above sum using Cauchy’s inequality and the Rankin-Selberg bound, we get

\[
S_r(N) \ll N^{2+\epsilon},
\]

for some \( \delta > 0 \). In the next step, we dualize the sum over \( n \) and \( m \) (See Section 4 for full details).

Consider the sum over \( n \)

\[
S_3 = \sum_{n \sim N} \lambda_\pi(n, 1)n^\iota e \left( \frac{an}{q} \right).
\]

On applying the GL(3) Voronoi summation formula to the above sum, we arrive at (see Lemma 12)

\[
S_3 \approx \frac{N^{2/3}}{q} \sum_{n_2 \sim Q^{3/3}/N} \lambda_\pi(1, n_2) n_2^{1/3} S(\bar{a}, \pm n_2; q) I_3(...),
\]

where \( I_3(...) \) is an integral transform in which we need to get square root cancellations, i.e., need to show \( I_3(...) \ll 1/\sqrt{T} \). Next we apply the GL(2) Voronoi formula to the sum over \( m \) and we get (see Lemma 13 for details)

\[
\sum_{m \sim N} \lambda_f(m) m^{-\iota} e \left( -\frac{am}{q} \right) \approx \frac{N}{q} \sum_{m \sim Q^{2k^2}/N} \lambda_f(m) e \left( \frac{\bar{a}m}{q} \right) I_2(...),
\]

where \( I_2(...) \) is an integral transform in which we need to get full cancellations, i.e., need to show \( I_2(...) \ll 1/k \). Next we analyse the sum over \( a \) which is given by

\[
e = \sum_{a \mod q}^* S(\bar{a}, n_2; q) e \left( \frac{\bar{a}m}{q} \right) \approx q e \left( -\frac{\bar{mn}_2}{q} \right).
\]

We observe that the above sum becomes an additive character with respect to \( n_2 \) (which saves us extra \( q \) when we apply the Poisson after Cauchy). Thus, we arrive at the following expression

\[
\frac{1}{QT} \frac{N}{Q^{2/T}Q} \sum_{q \sim Q} \sum_{n_2 \sim Q^{3/2}/N^{1/2}} \lambda_\pi(1, n_2) \sum_{m \sim k^2/T} \lambda_f(m) e \left( -\frac{\bar{mn}_2}{q} \right) \mathcal{J},
\]

where \( \mathcal{J} \) is an integral transform involving the \( t \)-integral, \( I_2(...) \) and \( I_3(...) \). We analyze it in Section 6. We observe that

\[
\mathcal{J} \ll T \frac{1}{\sqrt{T}} \frac{1}{\sqrt{T}k}.
\]

Note that a saving of \( \sqrt{T} \) comes from the \( t \)-integral, another saving of \( \sqrt{T} \) comes from the GL(3)-integral and the saving of \( k \) comes from the GL(2) integral. The
factor $T$ reflects the length of the $t$-integral. Thus, on plugging it in place of $\mathfrak{z}$, we see that
\[ S_r(N) \ll \sum_{q|Q} \sum_{n_2 \sim T^{3/2}N^{1/2}} |\lambda_z(1, n_2)| \left| \sum_{m \sim k^2/T} \lambda_f(m) e \left( \frac{-\bar{m}n_2}{q} \right) \mathfrak{z} \right| \]
\[ \ll QT^{3/2}N^{1/2} \frac{k^2}{T} \ll Nk. \]
Thus we now need to save $k^{1+\delta}$. Next we apply Cauchy’s inequality to the sum over $n_2$ to get rid of the GL(3) coefficients. Thus we arrive at (see Subsection 5.1)
\[ (T^{3/2}N^{1/2})^{1/2} \left( \sum_{n_2 \sim T^{3/2}N^{1/2}} \left| \sum_{q|Q} \sum_{m \sim k^2/T} \lambda_f(m) e \left( \frac{-\bar{m}n}{q} \right) \mathfrak{z} \right|^2 \right)^{1/2}. \]

The end game strategy is to apply the Poisson to the sum over $n_2$ (see Subsection 5.2). Opening the absolute value square followed by the Poisson, we observe that we save the whole length, i.e., $k^2Q/T$ in the zero-frequency ($n_2 = 0$ case) which suffices if $k^2Q/T > k^2$ which implies that $T < k$. On the other hand, in the non-zero frequencies ($n_2 \neq 0$ case), we save
\[ \frac{T^{3/2}N^{1/2}}{(Q^2T)^{1/2}}. \]
Here the factor $Q^2T$ in the denominator reflects the size of the conductor, which is given by
\[ \text{arithmetic conductor} \times \text{analytic conductor}.. \]
Note that the arithmetic conductor is of size $Q^2$ and the analytic conductor is of size $T$ (because $\mathfrak{z}$ oscillates like $n_2^T$ with respect to $n_2$). We also save $Q$ due to the presence of the additive character $e(-\bar{m}n/q)$. Thus the total saving in the non-zero frequencies turns out to be
\[ \frac{T^{3/2}N^{1/2}}{(Q^2T)^{1/2}} \times Q = T N^{1/2}, \]
which suffices if $TN^{1/2} > k^2$ which boils down to $T > k^{1/2}$. Hence we have the restriction $k^{1/2} < T < k$. In fact, the optimal choice for $T$ turns out to be $k^{41/51}$. Thus we get Theorem 1.

4. Applications of Voronoi formulae

In this section, we will analyse the sum over $n$ and $m$ in (14) using Voronoi summation formulae.

4.1. GL(3) Voronoi. Let’s consider the sum over $n$

\[ S_3 := \sum_{n=1}^{\infty} \lambda_z(n, r) e \left( \frac{an}{q} \right) e \left( \frac{nx}{qQ} \right) n^{it} V \left( \frac{n}{N} \right). \]
Recall that $N$ is of the form $N = 2^\alpha$, $\alpha \in [-1, \infty) \cap \mathbb{Z}$, such that $N \leq k^{3+\epsilon}/r^2$. We analyze $S_3$ using the GL(3) Voronoi summation formula (see Lemma 3). In the
present set-up, we have $g(n) = e(nz/qQ)n^iV(n/N)$ and $X = N$. Thus, on applying Lemma 3 to the above sum, we get
\begin{equation}
S_3 = q \sum_{\pm} \sum_{n_1|qr} \sum_{n_2=1}^{\infty} \frac{\lambda_\pi(n_1, n_2)}{n_1n_2} S(r \bar{a}, \pm n_2; qr/n_1) G_\pm(n_2^*) ,
\end{equation}
where $n_2^* := n_2^2/(q^3r)$ and $G_\pm(n_2^*)$ is the integral transform defined in \((5)\). Next we extract the oscillations of the integral transform $G_\pm(n_2^*)$ using Lemma 4 which gives us the following expression for $G_\pm(n_2^*)$ in the range $n_2^*N \gg k^e$:
\[ G_\pm(n_2^*) = n_2^* \int_0^\infty g(z) \frac{K_0}{q^2/3} \sum_{n_1|qr} \sum_{n_2=1}^{\infty} \frac{\lambda_\pi(n_1, n_2)}{n_1 n_2^{1/3}} S(r \bar{a}, \pm n_2; qr/n_1) I_3(n_1^2 n_2, q, x) , \]
where $K_0 = \left\lfloor \frac{6060}{r} + 5 \right\rfloor + 1$ with $\lfloor . \rfloor$ denoting the greatest integer function. From now on, we will continue our analysis with the terms corresponding to $j = 1$, as the other terms can be treated in a similar way and in fact, give us better estimates. Thus, on plugging the above expression corresponding to the term $j = 1$ into \((16)\), we arrive at
\begin{align*}
&\frac{N^{2/3+it}}{q^2/3} \sum_{\pm} \sum_{n_1|qr} n_1^{1/3} \sum_{n_2=1}^{\infty} \frac{\lambda_\pi(n_1, n_2)}{n_2^{1/3}} S(r \bar{a}, \pm n_2; qr/n_1) I_3(n_1^2 n_2, q, x) , \\
&= N^{2/3+it} \sum_{\pm} \sum_{n_1|qr} n_1^{1/3} \sum_{n_2=1}^{\infty} \frac{\lambda_\pi(n_1, n_2)}{n_2^{1/3}} S(r \bar{a}, \pm n_2; qr/n_1) I_3(n_1^2 n_2, q, x) + O(k^{-2020}),
\end{align*}
where
\begin{align*}
I_3(n_1^2 n_2, q, x) := \int_0^\infty V(z)z^{3}e \left( \frac{Nxz}{qQ} \pm \frac{3(Nn_1^2 n_2 z)^{1/3}}{q^{1/3}} \right) dz.
\end{align*}
On applying the change of variable $z \mapsto z^3$ followed by integration by parts (differentiating $3z^2V(z^3)z^{3}e(Nxz^3/qQ)$ and integrating $e(\pm 3(Nn_1^2 n_2 z)^{1/3}/q^{1/3})$) $j$-times to the above integral, we observe that
\[ \left| I_3(n_1^2 n_2, q, x) \right| \ll j \left( 1 + T + N|x| \right)^j \left( \frac{q^{1/3}}{(Nn_1^2 n_2)^{1/3}} \right)^j , \]
for any integer $j \geq 0$, and it is negligibly small if
\begin{equation}
n_1^2 n_2 \gg k^e \max \left\{ \frac{q^3T^3r}{N}, T^{3/2}N^{1/2}r \right\} =: N_0 .
\end{equation}
Now it remains to analyse $G_\pm(n_2^*)$ for $n_2^*N \ll k^e$, which is given as
\begin{align*}
G_\pm(n_2^*) &= \frac{1}{2\pi i} \int_{(\sigma)} (n_2^*)^{-s} \gamma_\pm(s) \tilde{g}(-s) ds \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} (n_2^*)^{-1} \gamma_\pm(\sigma + i\tau) \tilde{g}(-\sigma - i\tau) d\tau .
\end{align*}
We will analyse this case in Subsection 8.3. We conclude this subsection by summarising the above discussion in the following lemma.

**Lemma 12.** Let $S_3$ be as in \((15)\). Then, for $n_2^*N = n_1^2 n_2N/(q^3r) \gg k^e$, we have
\begin{align*}
S_3 &= \frac{N^{2/3+it}}{q^2/3} \sum_{\pm} \sum_{n_1|qr} n_1^{1/3} \sum_{n_2 < N_0/n_1^2} \frac{\lambda_\pi(n_1, n_2)}{n_2^{1/3}} S(r \bar{a}, \pm n_2; qr/n_1) I_3(n_1^2 n_2, q, x) \\
&\quad + \text{other lower order terms} + O(k^{-2020}) ,
\end{align*}
where $I_3(n_1^2n_2, q, x)$ is an integral transform defined in (17) and $N_0$ is as defined in (18). For the non-generic case $n_2^*N \ll k^\epsilon$, we have

$$S_3 = q \sum_{\pm} \sum_{n_1|qr \ n_2} \sum_{n_2} \frac{\lambda_n(n_1, n_2)}{n_1n_2} S(r\tilde{a}, \pm n_2; qr/n_1) G_{\pm}(n_2^*),$$

where $G_{\pm}(n_2^*)$ is as defined in (19).

From now on, we will proceed with the main term of (20).

4.2. GL(2) Voronoi. We now consider the sum over $m$ in (14), which is given as

$$S_2 := \sum_{m=1}^{\infty} \lambda_f(m)m^{-\epsilon}e\left(\frac{-am}{q}\right)e\left(\frac{-mx}{qQ}\right) U\left(m \frac{N}{q}\right).$$

On applying the GL(2) Voronoi summation formula (see Lemma 1) to the above sum with $g(m) = m^{-\epsilon}e(-mx/(qQ))U(m/N)$, we get

$$S_2 = \frac{2\pi^k N^{1-\epsilon}}{q} \sum_{m=1}^{\infty} \lambda_f(m)e\left(\frac{\tilde{am}}{q}\right) I_2(m, q, x),$$

where

$$I_2(m, q, x) := \int_0^\infty U(y)y^{-\epsilon}e\left(\frac{-Nxy}{qQ}\right) J_{k-1}\left(\frac{4\pi \sqrt{mNy}}{q}\right) dy.$$

We now analyse the above integral to determine the range of $m$. We claim that $I_2(m, q, x)$ is negligibly small unless

$$M := \frac{q^2(k-1)^2k^{-\epsilon}}{N} \leq m \leq k^\epsilon \max\left(\frac{(k-1)^2q^2}{N}, T\right) =: M_0.$$

In fact, in the range $m < M$, we have

$$4\pi \sqrt{mNy}/q < 4\pi \sqrt{5/2(k-1)^{1-\epsilon/2}} \ll (k-1)^{1-\epsilon/2}.$$

Thus, by Lemma 8 $I_2(m, q, x)$ is negligibly small.

Next we consider the range $m > M_0$ and we claim that $I_2(m, q, x)$ is also negligibly small. We note that $4\pi \sqrt{mNy}/q > (k-1)^{1+\epsilon/2}$. Thus we use Langer’s expansion (see Lemma 7) for $J_{k-1}$. On applying Corollary 1 with $x = 4\pi \sqrt{mNy}/q$, we see that $I_2(m, q, x)$, up to a negligible error term, is given by

$$\sum_{j=0}^{2019} \frac{1}{(k-1)^{j+1/2}} \int_0^\infty U_j(y)y^{-\epsilon}e\left(\frac{-Nxy}{qQ}\right)e\left(\pm \frac{(k-1)(w - \tan^{-1}w)}{2\pi}\right) dy,$$

where $U_j(y) = U(y)P_j((w - \tan^{-1}w)^{-1})w^{-1/2}$ with

$$w = \left(\frac{x^2}{(k-1)^2} - 1\right)^{1/2} = \left(\frac{16\pi^2 mNy}{q^2(k-1)^2} - 1\right)^{1/2},$$

and $P_j$ is a polynomial of the degree $j$ with coefficients which are bounded functions of $k$. Note that $w > ((k-1)^{\epsilon} - 1)^{1/2}$. Thus

$$w - \tan^{-1}w = w - \frac{\pi}{2} + \tan^{-1}\frac{1}{w} \approx w,$$
and \( U_j^{(i)}(y) \ll_i k^i \) for any integer \( i \geq 0 \). Next we apply integration by parts \( i \)-times to the \( y \)-integral and we get

\[
|I_2(m, q, x)| \ll_i \left( k^i + T + \frac{N|x|}{qQ} \right)^i \left( \frac{1}{(k - 1)\sqrt{mN/(q(k-1))}} \right)^i \leq \left( \frac{Tq}{\sqrt{M_0N}} + \frac{N}{Q\sqrt{M_0N}} \right)^i \ll \left( \frac{k'T}{k} + \frac{1}{k^i} \right)^i \ll \frac{1}{k^{ei}}.
\]

Upon taking \( i \) sufficiently large, we get the claim. We end this subsection by summarizing the above arguments in the following lemma.

**Lemma 13.** Let \( S_2 \) be the sum over \( m \) as given in (22). Then we have

\[
S_2 = \frac{2\pi i^{k^2}N^{1-it}}{q} \sum_{M \leq m \leq M_0} \lambda_f(m) e \left( \frac{am}{q} \right) I_2(m, q, x) + O(k^{-2020}),
\]

where

\[
I_2(m, q, x) = \int_0^\infty U(y) y^{-it} e \left( \frac{-Nxy}{qQ} \right) J_{k-1} \left( \frac{4\pi\sqrt{mNy}}{q} \right) dy,
\]

and \( M \) and \( M_0 \) are the ranges of \( m \) defined in (21).

5. Cauchy and Poisson

After the applications of the Voronoi formulae and applying Lemma 12 and Lemma 13 to (14), we find that the expression in (14), up to an error term to be treated in Section 5.3 has been essentially reduced to

\[
\frac{N^{5/3}}{QT^{2/3}} \sum_{1 \leq q \leq Q} \frac{1}{q^3} \sum_{a \mod q} \sum_{\pm n_1^{1/3}} \lambda_\pi(n_1, n_2) S(r\tilde{a}, \pm n_2; qr/n_1) \times \sum_{n_2 \ll N_0/n_1^2} \lambda_f(m) e \left( \frac{am}{q} \right) J_{\pm}(m, n_1^2 n_2, q),
\]

where

\[
J_{\pm}(m, n_1^2 n_2, q) = \int_{\mathbb{R}} \int_{\mathbb{R}} W(x/Q^r) g(q, x) I_2(m, q, x) I_3(n_1^2 n_2, q, x) V \left( \frac{t}{T} \right) dt \, dx.
\]

In this section, we will analyse (23) using Cauchy’s inequality and the Poisson summation formula.

5.1. Cauchy’s inequality. Splitting the sum over \( q \) into dyadic blocks \( q \sim C \), i.e., \( C \leq q < 2C \), \( C \ll Q \) and writing \( q = q_1 q_2 \) with \( q_1 \mid (n_1 r)^\infty \), \( (n_1, q_2) = 1 \), we see that the expression in (23) is dominated by

\[
\sup_{C \ll Q} \frac{N^{5/3} \log Q}{QT^{2/3} C^3} \sum_{\pm} \sum_{(n_1, r) \ll C} n_1^{1/3} \sum_{(n_1, r) \ll N_0/n_1^2} \sum_{q_2 \sim C/q_1} \left| \lambda_\pi(n_1, n_2) \right| \times \lambda_f(m) c_{\pm}(q, n_2, m) J_{\pm}(m, n_1^2 n_2, q),
\]
where the character sum \( C_\pm(q, n_2, m) = C_\pm(\ldots) \) is defined as
\[
C_\pm(\ldots) := \sum_{a \mod q} S(r\bar{a}, \pm n_2; qr/n_1) e \left( \frac{\pm an_2}{q} \right) = \sum_{d | q} \mu(d) \sum_{n_1 \equiv -m \mod d} e \left( \frac{\pm an_2}{qr/n_1} \right).
\]

Next we analyse the expression inside \(| | \). We first split the sum over \( m \) into dyadic blocks \( m \sim M_1, M \ll M_1 \ll M_0 \) and then apply Cauchy’s inequality to the sum over \( n_2 \) in (28) to arrive at
\[
(29) \quad S_r(N) \ll \sup_{M_1 \ll M_0 \ll M} \frac{N^{5/3}(QM_0)^{2}}{QT r^{2/3} C^3} \sum_{n_1} \Theta^{1/2} \sum_{n_2} \sqrt{\Omega_\pm},
\]
where
\[
(30) \quad \Theta = \sum_{n_2 \ll N_0/n_1^2} \frac{|\lambda_\pi(n_1, n_2)|^2}{n_2^{2/3}}.
\]
and
\[
(31) \quad \Omega_\pm = \sum_{n_2 \ll N_0/n_1^2} \left| \sum_{q_2-C/q_1} \sum_{m \sim M_1} \lambda_f(m) C_\pm(q, n_2, m) J_\pm(m, n_1^2 n_2, q) \right|^2,
\]
with
\[
(32) \quad \frac{(k-1)^2 C^2}{N} k^{-\epsilon} = M \ll M_1 \ll M_0 = k^{\epsilon} \max \left\{ \frac{(k-1)^2 C^2}{N}, T \right\},
\]
\[
N_0 = k^{\epsilon} \max \left\{ \frac{(CT)^3 r}{N}, T^{3/2} N^{1/2} r \right\}.
\]

5.2. **Poisson summation.** Next we apply the Poisson summation formula to the sum over \( n_2 \) with the modulus \( q := q_1 q_2 q_3 r/n_1 \) in (31). To this end, we first split the sum over \( n_2 \) into dyadic blocks \( n_2 \sim N/n_1^2, N \ll N_0 \). Then opening the absolute value square in (31), we arrive at
\[
\Omega_\pm = \sum_{q_2, q_3 \sim C/q_1} \sum_{m \sim M_1} \sum_{n_1} \lambda_f(m) \lambda_f(m') \Delta_\pm,
\]
where
\[
\Delta_\pm = \sum_{n_2 \in \mathbb{Z}} \phi \left( \frac{n_2^2 n_2}{N} \right) C_\pm(q, n_2, m) C_\pm(q', n_2, m') J_\pm(m, n_1^2 n_2, q) J_\pm(m', n_1^2 n_2, q'),
\]
\( q' = q_1 q_2 \) and \( \phi(w) \) is a non-negative smooth function supported on \([2/3, 3]\) with \( \phi(w) = 1 \) for \( w \in [1, 2] \) and \( \phi^{(j)}(w) \ll 1 \). Now applying the change of variable
\[
n_2 \rightarrow n_2 q + \beta, \quad \beta \mod q,
\]
we get the following expression for \( \Delta_\pm \):
\[
\Delta_\pm = \sum_{n_2 \in \mathbb{Z}} \phi \left( \frac{n_2^2 q + \beta}{N/n_1^2} \right) J_\pm(m, n_1^2 (n_2 q + \beta), q) J_\pm(m', n_1^2 (n_2 q + \beta), q').
\]
On applying the Poisson summation formula to the sum over $n_2$, we see that
\begin{equation}
\Omega_\pm = \sum_{N} \frac{\widetilde{N}}{n_1^2} \sum_{q_2, q_5' \sim C/q_1} \sum_{m, m' \sim M_1} \lambda_f(m) \lambda_f(m') \sum_{n_2 \in \mathbb{Z}} \mathcal{C}_\pm \mathcal{J}_\pm,
\end{equation}
where
\begin{equation}
\mathcal{C}_\pm = \frac{1}{q} \sum_{\beta \mod q} \mathcal{C}(q, \beta, m) \mathcal{C}(q', \beta, m') e \left( \frac{n_2 \beta}{q} \right)
= \sum_{d \mid q} d' \mu \left( \frac{q}{d} \right) \mu \left( \frac{q'}{d'} \right) \sum_{\alpha \mod qr/n_1} \sum_{\alpha' \mod q'r/n_1} 1,
\end{equation}
and
\begin{equation}
\mathcal{J}_\pm = \int_{\mathbb{R}} \phi(w) J_\pm(m, \tilde{N} w, q) \overline{J}_\pm(m', \tilde{N} w, q') e \left( -\frac{n_2 \tilde{N} w}{q_1 q_2' q_3} \right) \, dw.
\end{equation}

On estimating the sum over $\tilde{N}$, we get
\begin{equation}
\Omega_\pm \ll k^\epsilon \sup_{N \ll N_0} \frac{\widetilde{N}}{n_1^2} \sum_{q_2, q_5' \sim C/q_1} \sum_{m, m' \sim M_1} |\lambda_f(m)| |\lambda_f(m')| \sum_{n_2 \in \mathbb{Z}} |\mathcal{C}_\pm||\mathcal{J}_\pm|.
\end{equation}

6. Estimates for the integral transform

In this section, we will analyse the integral transform
\begin{equation}
\mathcal{J}_\pm = \int_{\mathbb{R}} \phi(w) J_\pm(m, \tilde{N} w, q) \overline{J}_\pm(m', \tilde{N} w, q') e \left( -\frac{n_2 \tilde{N} w}{q_1 q_2' q_3} \right) \, dw,
\end{equation}
where (see (27))
\begin{equation}
J_\pm(m, \tilde{N} w, q) = \int_{\mathbb{R}} \int_{\mathbb{R}} W(x/Q^\epsilon) g(q, x) I_3(m, q, x) I_3(\tilde{N} w, q, x) V \left( \frac{t}{T} \right) \, dt \, dx
= \int_{\mathbb{R}} W(x/Q^\epsilon) g(q, x) \int_{\mathbb{R}} V \left( \frac{t}{T} \right) \int_{0}^{\infty} U(y) y^{-it} \int_{0}^{\infty} V(z) z^{it} \\
\times e \left( \frac{N x (z - y)}{q Q} + \frac{3(N \tilde{N} w z)^{1/3}}{q r^{1/3}} \right) J_{k-1} \left( \frac{4 \pi \sqrt{n \tilde{N} w}}{q} \right) \, dz \, dy \, dt \, dx.
\end{equation}

and $J_\pm(m', \tilde{N} w, q')$ is similarly defined. We first analyse $J_\pm(m, \tilde{N} w, q)$.

Lemma 14. Let $J_\pm(m, \tilde{N} w, q)$ be as above. Then we have
\begin{equation}
J_\pm(m, \tilde{N} w, q) = \int_{\mathbb{R}} V \left( \frac{t}{T} \right) \int_{u < \frac{4 \pi \sqrt{m \tilde{N} w}}{q}} I_u I_\pm(m, \tilde{N} w, q) \, du \, dt + O(k^{-2020}),
\end{equation}
where $I_u$ and $I_\pm(m, \tilde{N} w, q)$ are the integrals defined in (42) and (43) respectively, with the weight function $U_{u, \epsilon}$ satisfying $U_{u, \epsilon}(y) \ll k^\epsilon$ for $j \geq 0$.

Proof. We consider two cases.
Case 1. $q \sim C \ll Q^{1-\varepsilon}$.

Consider the integral over $x$ in (38) which is given by

$$I_{z-y} = \int_{\mathbb{R}} W(x/Q^*) g(q,x) e\left(\frac{Nx(z-y)}{qQ}\right) dx$$

$$= Q^c \int_{\mathbb{R}} W(x) g(q,xQ^c) e\left(\frac{NxQ^c(z-y)}{qQ}\right) dx.$$ 

We now split the above integral as

$$\int_{\mathbb{R}} \ldots dx = \int_{-Q^{-2\varepsilon}}^{Q^{-2\varepsilon}} \ldots dx + \int_{D} \ldots dx,$$

where $D = [-2, 2]\setminus[-Q^{-2\varepsilon}, Q^{-2\varepsilon}]$. Note that, for $x \in [-Q^{-2\varepsilon}, Q^{-2\varepsilon}]$, we have

$$g(q,xQ^c) = 1 + h(q,xQ^c) = 1 + O\left(\frac{Q}{q} + \frac{1}{|x|}Q^c\right) = 1 + O(Q^{-2020}).$$

Thus, in this range, we can replace $g(q,xQ^c)$ by 1 at the cost of a negligible error term. Then by repeated integration by parts we see that the integral is negligibly small unless

$$|z-y| \ll k^{\varepsilon} C/(QT).$$

Now we consider the complementary range, i.e., $x \in D$. Note that, using the second property (see (8)) of $g(q,x)$, we have

$$x^j \frac{\partial^j}{\partial x^j} g(q,x) \ll \log Q \min\left\{\frac{Q}{q}, \frac{1}{|x|}\right\} \ll Q^{2\varepsilon}.$$ 

Thus, on using integration by parts repeatedly, we see that the integral is negligibly small unless (10) holds true.

Case 2. $q \sim C \gg Q^{1-\varepsilon}$.

In this case, we consider the $t$-integral in (38) which is given by

$$\int_{\mathbb{R}} V\left(\frac{t}{T}\right) \left(\frac{z}{y}\right)^{it} dt.$$ 

Now applying the change of variable $t \to tT$ followed by integration by parts repeatedly, we conclude that the $t$-integral is negligibly small unless

$$|z-y| \ll k'/T \ll k^{\varepsilon} C/(QT).$$ 

Next writing $z-y = u$ with $u \ll k^{\varepsilon} C/(QT)$ in (38), we see that

$$J_{\pm}(m, \tilde{N}w, q) = \int_{\mathbb{R}} V\left(\frac{t}{T}\right) \int_{u < k'/T} I_u I_{\pm}(m, \tilde{N}w, q) du dt + O(k^{-2020}),$$

where

$$I_u = \int_{\mathbb{R}} W(x/Q^c) g(q,x) e\left(\frac{N xu}{qQ}\right) dx,$$

and

$$I_{\pm}(m, \tilde{N}w, q) = \int_{0}^{\infty} U_{u,t}(y) e\left(\pm \frac{3(N \tilde{N}w(y+u))^{1/3}}{qr^{1/3}}\right) J_{k-1} \left(4\pi \sqrt{mnq} y\right) dy,$$
with $U_{u,t}(y) = U(y)V(y+u)(1+u/y)^t$. Note that
\[
\frac{\partial^j}{\partial y^j} \left( 1 + \frac{u}{y} \right)^t = \frac{\partial^j}{\partial y^j} \exp \left( it \log \left( 1 + \frac{u}{y} \right) \right) \ll_j k^j, \quad j \geq 0.
\]
Thus $U_{u,t}^{(j)}(y) \ll_j k^j$ for $j \geq 0$. Hence the lemma follows.

The analysis for $J_\pm(m', \tilde{N}w, q')$ is exactly same. Thus on plugging the expression of $J_\pm(m, \tilde{N}w, q)$ from (39) and a corresponding expression of $J_\pm(m', \tilde{N}w, q')$ into (37), we see that

\[
J_\pm = \int_{R} \mathcal{V} \left( \frac{t}{T} \right) \mathcal{V} \left( \frac{t'}{T} \right) \int_{u \ll \frac{k^C}{\sqrt{q'}}} \int_{w \ll \frac{k^C}{\sqrt{q'}}} I_u \tilde{I}_w \mathcal{J}_\pm du' dt' dt + O(k^{-2020}),
\]
where
\[
\mathcal{J}_\pm := \int_{R} \phi(w) I_\pm(m, \tilde{N}w, q) I_\pm(m', \tilde{N}w, q') e \left( -\frac{n_2 \tilde{N}w}{q_2 q'_1 \tau_1 n_1} \right) dw,
\]
which we will analyse now. We have the following proposition.

**Proposition 1.** Let $\mathcal{J}_\pm$ be as above. Then $\mathcal{J}_\pm$ is negligibly small unless
\[
n_2 \ll k^e \frac{CN^{1/3} \tau^{2/3} n_1}{q_1 N^{2/3}} := N_2,
\]
in which case we have
\[
\mathcal{J}_\pm \ll \frac{k^e C^2}{M_1 N}.
\]
Furthermore, if $q \sim C \gg k^{1+\varepsilon}$ and $n_2 \neq 0$, then we have
\[
\mathcal{J}_\pm \ll \frac{C r^{1/3} k^{2/3}}{k^2 (NN)^{1/3}}.
\]

Before proving the proposition, we will analyze $I_\pm(m, \tilde{N}w, q)$ and $I_\pm(m', \tilde{N}w, q')$. We have the following lemma.

**Lemma 15.** Let $I_\pm(m, \tilde{N}w, q)$ be the integral transform defined in (13). Let $b = 4\pi \sqrt{mN}/q$ and $a = a(q, r) := 3(NN)^{1/3}/(qr^{1/3}) \gg k^e$. Then $I_\pm(m, \tilde{N}w, q)$ is negligibly small unless $a \leq k^e b$. In the case when $a \leq k^{-e} b$, we have
\[
I_\pm(m, \tilde{N}w, q) \ll k^e/b.
\]
Furthermore, if $q \sim C \gg k^{1+\varepsilon}$, then $b \asymp k$ and we have
\[
I_\pm(m, \tilde{N}w, q) = \frac{e(f(\tau_0))}{\sqrt{f''(\tau_0)}} \frac{c_3 a_0^{9/2} w^{3/2}}{b^5 \tau_0^5 \sqrt{1 - \tau_0^2}} U_{u,t} \left( \left( \frac{4\pi a w^{1/3}}{3b \tau_0} \right)^6 \right) + \text{lower order terms} + O(k^{-2020}),
\]
where $\tau_0$ is the stationary point of the phase function
\[
f(\tau) = \frac{(k-1) \sin^{-1} \tau}{2\pi} + \frac{16\pi^2 a^3 w}{27b^2 \tau^2},
\]
which is given by (61) and $c_3 = c_2 e(1/8) = 3\sqrt{2}(4\pi/3)^5 e(1/4)$. In the remaining case, i.e., $k^{-} b \leq a \leq k^{+} b$, $I_{\pm}(m, \tilde{N} w, q)$ essentially looks like

$$\frac{c_2 a^{9/2} w^{3/2}}{b^5} \int_{b_1/2}^1 \frac{1}{\tau^5 \sqrt{1 - \tau^2}} U_{u,t} \left( \left( \frac{4\pi a w^{1/3}}{3b} \right)^6 \right) e(f(\tau)) \, d\tau,$$

where $b_1 := 4\pi (2/3)^{1/3} a/(3(2.5)^{1/6} b)$.

**Proof.** Let's recall from (43) that

$$I_{\pm}(m, \tilde{N} w, q) = \int_{1/2}^{5/2} U_{u,t}(y) e(\pm a w^{1/3}(y + u)^{1/3}) J_{k-1}(b \sqrt{y}) \, dy.$$ 

Consider the term $e(\pm a w^{1/3}(y + u)^{1/3})$. It can be written as

$$e(\pm a w^{1/3}(y + u)^{1/3}) = e(\pm a w^{1/3} y^{1/3}) e(\pm a w^{1/3} y^{1/3} ((1 + u/y)^{1/3} - 1)).$$

Note that

$$\frac{\partial^j}{\partial y^j} e(\pm a w^{1/3} y^{1/3} ((1 + u/y)^{1/3} - 1)) \ll k^{\delta j}, \quad j \geq 0.$$

This is obvious for $j = 0$. We will verify it for $j = 1$ (for other $j$, a similar calculation will follow). Let $h(y, w) := \pm a w^{1/3} y^{1/3} ((1 + u/y)^{1/3} - 1)$. Thus for $j = 1$ we have

$$\frac{\partial}{\partial y} e(h(y, w)) = e(h(y, w))(\pm a) w^{1/3} \left( \frac{(1 + u/y)^{1/3} - 1}{3y^{2/3}} - \frac{u}{3y^{5/3}(1 + u/y)^{2/3}} \right).$$

Thus, using $y, w \asymp 1$ and $(1 + u/y)^{1/3} - 1 \ll |u|$, we see that

$$\frac{\partial}{\partial y} e(h(y, w)) \ll a |u| \ll \frac{(N \tilde{N})^{1/3} Ck^\epsilon}{C r^{1/3} QT} \ll \frac{(N N_0)^{1/3} Qk^\epsilon}{Q r^{1/3} QT} \ll k^\epsilon,$$

where we used (32) to estimate $N_0$. Hence we can insert $e(h(y, w))$ into the weight function $U_{u,t}(y)$. Thus we arrive at the following expression:

$$I_{\pm} := I_{\pm}(m, \tilde{N} w, q) = \int_{1/2}^{5/2} U_{u,t}(y) e(\pm a w^{1/3} y^{1/3}) J_{k-1}(b \sqrt{y}) \, dy.$$ 

Notice the slight abuse of notation: the weight function $U_{u,t}$ in the above expression is different from the one in (50). To analyze (51) further, we use an integral representation of the Bessel function $J_{k-1}$. Thus, on applying (9) to the Bessel function $J_{k-1}$, we see that

$$I_{\pm} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-1)\tau} \int_{1/2}^{5/2} U_{u,t}(y) e(\pm a w^{1/3} y^{1/3} - b \sqrt{y} \sin \tau/2\pi) \, dy \, d\tau.$$

We now split the $\tau$-integral as follows:

$$\int_{-\pi}^{\pi} \, d\tau = \int_{0}^{\pi/2} \, d\tau + \int_{\pi/2}^{\pi} \, d\tau + \int_{0}^{\pi/2} \, d\tau + \int_{-\pi/2}^{-\pi} \, d\tau.$$

Let $I_{\pm}^{(i)}$ denote the $i$-th integral in the right hand side of the above expression for $i = 1, 2, 3$ and $4$. Let’s first consider $I_{\pm}^{(1)}$ which is defined as follows:

$$I_{\pm}^{(1)} = \frac{1}{2\pi} \int_{0}^{\pi/2} e^{i(k-1)\tau} \int_{1/2}^{5/2} U_{u,t}(y) e(\pm a w^{1/3} y^{1/3} - b \sqrt{y} \sin \tau/2\pi) \, dy \, d\tau.$$
Next we apply stationary phase analysis to the $y$-integral. By the change of variable $y \to y^3$, we arrive at the following expression of the $y$-integral:

$$
\int_{\sqrt{1/2}}^{\sqrt{5/2}} 3y^2 U_{a,t}(y^3)e \left( \pm aw^{1/3}y - by^{3/2} \sin \tau/2\pi \right) dy.
$$

Note that if we have negative sign with $a$, then the above integral is negligibly small by Lemma 9. Thus we proceed with the $y$-integral of $I_{a,t}^{(1)}$, which is given by

$$
\int_{\sqrt{1/2}}^{\sqrt{5/2}} 3y^2 U_{a,t}(y^3)e \left( aw^{1/3}y - by^{3/2} \sin \tau/2\pi \right) dy.
$$

Here the phase function is given by $f_1(y) = aw^{1/3}y - by^{3/2} \sin \tau/2\pi$. On computing the first order derivative, we see that the stationary point occurs at $y_0 = \left( \frac{4\pi aw^{1/3}}{3b \sin \tau} \right)^2$.

Note that

$$
\sqrt{1/2} \leq y_0 \leq \sqrt{5/2} \iff \frac{4\pi}{3} \frac{aw^{1/3}}{b(2.5)^{1/6}} \leq \sin \tau \leq \frac{4\pi}{3} \frac{aw^{1/3}}{b(0.5)^{1/6}}.
$$

Let $b_1 := \frac{4\pi}{3} \frac{2(3/1)^{1/3}}{b(2.5)^{1/6}}$ and $b_2 := \frac{4\pi}{3} \frac{3^{1/3}a}{b(0.5)^{1/6}}$. We consider three cases.

Case 1. $a \geq k^{-\epsilon} b$. In this case we have $b_1 \geq 2$. Thus there is no stationary point in the range $[(1/2)^{1/3}, (5/2)^{1/3}]$. Moreover,

$$
f_1^{(1)}(y) = aw^{1/3} - 3b \sqrt{y} \sin \tau/(4\pi) \gg b, \quad f_1^{(j)}(y) \ll b, \quad j \geq 2.
$$

Hence, by Lemma 9 the integral is negligibly small. This proves the first part of the lemma.

Case 2. $a \leq k^{-\epsilon} b$. In this case we have $0 < b_1/2 < b_2 \ll k^{-\epsilon} < 1$. We now split the $\tau$-integral in (52) as follows:

$$
\int_0^{\pi/2} \ldots d\tau = \int_0^{\sin^{-1}(b_1/2)} \ldots d\tau + \int_{\sin^{-1}(b_1/2)}^{\sin^{-1}(b_2)} \ldots d\tau + \int_{\sin^{-1}(b_2)}^{\pi/2} \ldots d\tau.
$$

Note that the first and the third integrals of the right side of the above expression are negligibly small due to absence of the stationary point. Hence it boils down to analyse the second integral which is given by

$$
\int_{\sin^{-1}(b_1/2)}^{\sin^{-1}(b_2)} e^{i(k-1)\tau} \int_{\sqrt{1/2}}^{\sqrt{5/2}} 3y^2 U_{a,t}(y^3)e \left( aw^{1/3}y - by^{3/2} \sin \tau/2\pi \right) dy d\tau.
$$

On applying the stationary phase analysis (see Lemma 10) to the $y$-integral, we see that, it is given by

$$
\frac{c_1 y_0^2 U_{a,t}(y_0^3)e(f_1(y_0))}{\sqrt{|f_1''(y_0)|}} + \text{lower order terms} + O(k^{-2020}),
$$

where $c_1 = 3e(1/8)$, $y_0 = \left( \frac{4\pi aw^{1/3}}{3b \sin \tau} \right)^2$ and $f_1(y) = aw^{1/3}y - by^{3/2} \sin \tau/2\pi$. We will proceed with the main term, as the other terms can be analysed similarly and in fact, give better bounds. Hence, on plugging the values of $y_0$, $f_1(y_0)$ and $f_1''(y_0)$, we essentially get the following expression for the $y$-integral:

$$
c_2 \frac{a^{9/2} w^{3/2}}{b^5 \sin^5 \tau} U_{a,t} \left( \left( \frac{4\pi aw^{1/3}}{3b \sin \tau} \right)^6 e \left( \frac{16\pi^2 a^3 w}{27b^2 \sin^2 \tau} \right),
$$

where $c_2$ is a constant.
where $c_2 = c_1 \sqrt{2}(4\pi/3)^5$. On plugging the above expression in place of the $y$-integral into (53), we arrive at

$$
\frac{c_2a^{9/2}w^{3/2}}{b^{5}} \int_{\sin^{-1}(b_2)}^{\sin^{-1}(b_2)} \frac{1}{\sin^3\tau} U_{u,t} \left( \frac{4\pi aw^{1/3}}{3b \sin \tau} \right)^6 e \left( \frac{(k-1)\tau}{2\pi} + \frac{16\pi^2a^3w}{27b^2\sin^2\tau} \right) d\tau.
$$

On applying the change of variable $\sin \tau \to \tau$, we arrive at

$$
\frac{c_2a^{9/2}w^{3/2}}{b^{5}} \int_{b_1/2}^{b_2} \frac{1}{\tau^3\sqrt{1-\tau^2}} U_{u,t} \left( \frac{4\pi aw^{1/3}}{3b \tau} \right)^6 e \left( \frac{(k-1)\sin^{-1}\tau}{2\pi} + \frac{16\pi^2a^3w}{27b^2\tau^2} \right) d\tau.
$$

Next we apply the second derivative bound to the above integral. Here the phase function is given by

$$
f(\tau) = \frac{(k-1)\sin^{-1}\tau}{2\pi} + \frac{16\pi^2a^3w}{27b^2\tau^2}.
$$

On computing the first and the second order derivatives, we see that

$$
f'(\tau) = \frac{(k-1)}{2\pi\sqrt{1-\tau^2}} - \frac{32\pi^2a^3w}{27b^2\tau^3},
$$

$$
f''(\tau) = \frac{(k-1)\tau}{2\pi(1-\tau^2)^{3/2}} + \frac{32\pi^2a^3w}{96\tau^4} \gg \frac{a^3}{b^2\tau^4} > \frac{b^2}{a}.
$$

Thus on applying Lemma 9 to (55), it is bounded above by

$$
\frac{\text{Var} g + \max |g|}{\min \sqrt{f''(\tau)}} \ll \frac{k^c a^{9/2}}{b^5(a/b)^5 \sqrt{b^2/a}} = \frac{k^c}{b},
$$

where $\text{Var} g$ denotes the total variation of the weight function

$$
g(\tau) = \frac{c_2a^{9/2}w^{3/2}U_{u,t}}{b^{5}\tau^3\sqrt{1-\tau^2}} \left( \frac{4\pi aw^{1/3}}{3b \tau} \right)^6.
$$

Hence, $I_1^{(1)} \ll k^c/b$. On analyzing other $I_1^{(i)}$’s in a similar fashion, we get

$$
I_\pm = I_\pm(m, N_0 w, q) \ll k^c/b.
$$

Now we proceed to prove (49). We will give details for $I_1^{(1)}$ only, as the analysis for other $I_1^{(i)}$ is similar. Let $q \sim C \gg k^{1+\epsilon}$. Note that this condition assures that $b \asymp k$, as, by (32), we have

$$
k^{-\epsilon}(k-1)^2C^2/N \ll M_1 \ll k^c \max ((k-1)^2C^2/N, T) \ll k^c(k-1)^2C^2/N,
$$

and hence

$$
a = \frac{3(N\tilde{N})^{1/3}}{qr^{1/3}} \ll \frac{(NN_0)^{1/3}}{qr^{1/3}} \ll (kT)^{1/2} = k^{1-\eta/2} < k \asymp b,
$$

as $T = k^{1-\eta} < k$. We now apply the stationary phase analysis to (55). The stationary point of the phase function $f(\tau)$ occurs at $\tau_0$, where $\tau_0$ satisfies

$$
\frac{(k-1)}{2\pi\sqrt{1-\tau_0^2}} = \frac{32\pi^2a^3w}{27b^2\tau_0^3} \iff \frac{\tau_0^3}{\sqrt{1-\tau_0^2}} = \left( \frac{4\pi}{3} \right)^3 \frac{a^3w}{b^2(k-1)}.
$$

Simplifying it further, we see that $\tau_0$ satisfies

$$
\tau^6 - c^2(1 - \tau^2) = 0,
$$
where \( c = c(w) := (\frac{4\pi}{3})^3 \frac{a b^3}{\omega (k - 1)}. \) Upon letting \( \tau^2 = \tau_1, \) the above equation reduces to the cubic polynomial equation \( \tau_1^3 - c^2 (1 - \tau_1) = 0, \) which can be solved using Cardano’s method. In fact, as the discriminant of the cubic is negative, it has only one real root which can be found as follows: Let \( \theta_1 + \theta_2 \) be the real root. Upon substituting it into the cubic, we get

\[
theta_1^2 + \theta_2^2 + (3\theta_1\theta_2 + c^2)(\theta_1 + \theta_2) - c^2 = 0,
\]

which leads to the following system of equations:

\[
3\theta_1\theta_2 + c^2 = 0, \quad \theta_1^2 + \theta_2^2 - c^2 = 0.
\]

Now using the formula

\[
(\theta_1^2 - \theta_2^2)^2 = (\theta_1^2 + \theta_2^2)^2 - 4\theta_1^2\theta_2^2,
\]

we see that the real root \( \theta_1 + \theta_2 \) is given by

\[
\sqrt[3]{\frac{c^2}{2} + \sqrt{\frac{c^4}{4} + \frac{c^6}{27}}} + \sqrt[3]{\frac{c^2}{2} - \sqrt{\frac{c^4}{4} + \frac{c^6}{27}}}
\]

Hence we get

\[
(59) \quad \tau_0 = \tau_0(w) = \left(3 \sqrt{\frac{c^2}{2} + \sqrt{\frac{c^4}{4} + \frac{c^6}{27}}} + \sqrt[3]{\frac{c^2}{2} - \sqrt{\frac{c^4}{4} + \frac{c^6}{27}}} \right)^{1/2}
\]

\[
(60) \quad = \sqrt[3]{\frac{c^2}{2} + \sqrt{\frac{c^4}{4} + \frac{c^6}{27}} \left( 1 - 3 \frac{c^2}{2} \left( \sqrt{\frac{c^4}{4} + \frac{c^6}{27}} - \frac{c^2}{2} \right)^{2/3} \right)^{1/2}}.
\]

Now expanding the above expression using the binomial theorem, we see that

\[
(61) \quad \tau_0 = \tau_0(w) = c_1 h(w) + c_3 h(w)^3 + c_5 h(w)^5 \ldots + c_{2n-1} h(w)^{2n-1} + \ldots,
\]

where \( c_i \)’s, \( i = 1, 3, 5, \ldots, \) are some non-zero explicit absolute constants and

\[
h(w) = \frac{aw^{1/3}}{b^{2/3}(k - 1)^{1/3}}.
\]

Note that the above series in (61) is convergent and each binomial expansion in (69) is justified as \( c \ll a^3/(b^2(k - 1)) \ll k^{-3n/2}. \) Next we analyse the higher order derivatives of the phase function \( f(\tau). \) On using (60) and computing other higher order derivatives of \( f(\tau), \) we get

\[
f''(\tau) \asymp b^2/a = a(a/b)^{-2}, \quad f'(\tau) \ll a(a/b)^{-1},
\]

\[
f^{(j)}(\tau) = \frac{(k - 1)}{2\pi} \frac{d^j - 2}{d\tau^j - 2} \frac{\tau}{(1 - \tau^2)^{3/2}} + \frac{32\pi^2 a^3 w}{9b^2} \frac{d^{j-2}(\tau^{-4})}{d\tau^{j-2}} \ll a(a/b)^{-j}, \quad j = 3, 4, \ldots,
\]

where we used the fact \( a \ll b \asymp k \) and

\[
\frac{d^{j-2}}{d\tau^{j-2}} \frac{\tau}{(1 - \tau^2)^{3/2}} \ll_j 1.
\]

On computing derivatives of the weight function

\[
g(\tau) = \frac{c_2 a^{9/2} w^{3/2} U_{\omega,1} \left( (4\pi a w^{1/3}/3b \tau)^{1/3} \right)}{b^5 \tau^5 \sqrt{1 - \tau^2}}.
\]
since \( \tau \asymp a/b \), we see that
\[ g^{(i)}(\tau) \ll a^{-1/2} (a/b)^{-i}, \quad i = 0, 1, 2, ... \]

Thus, on applying Lemma 110 with \( X = a^{-1/2}, Q = U = a/b \) and \( Y = a \) to the \( \tau \)-integral in (62), we get (63).

Case 3. \( k^{-\epsilon}b \leq a \leq k^\epsilon b \). In this case we can assume that \( b_1/2 < 1 \), otherwise, we get back to the starting point of the discussion in Case 1. Consider
\[ I^{(1)}_\pm = \frac{1}{2\pi} \int_0^{\pi/2} e^{i(k-1)\tau} \int_1^{5/2} U_{u,t}(y) e \left( \pm aw^{1/3}y^{1/3} - b\sqrt{y} \sin \tau/2\pi \right) dy d\tau. \]

We split the \( \tau \)-integral as follows:
\[ \int_0^{\pi/2} ... d\tau = \int_0^{\sin^{-1}(b_1/2)} ... d\tau + \int_{\sin^{-1}(b_1/2)}^{\pi/2} ... d\tau. \]

The first integral on the right side is negligibly small due to absence of the stationary point. Consider the second integral which is given by
\[ \int_{\sin^{-1}(b_1/2)}^{\pi/2} e^{i(k-1)\tau} \int_{\sqrt{1/\sqrt{2}}}^{\sqrt{5/2}} 3y^2 U_{u,t}(y) e \left( \pm aw^{1/3}y^{1/3} - b\sqrt{y} \sin \tau/2\pi \right) dy d\tau. \]

On analyzing the \( y \)-integral like Case 2, we get the lemma.

\[ \square \]

**Proof of Proposition 11** Recall from (51) that
\[ I_\pm (m, \tilde{N}w, q) = \int_1^{5/2} U_{u,t}(y) e \left( \pm aw^{1/3}y^{1/3} \right) J_{k-1} (b\sqrt{y}) dy. \]

Note that
\[ \frac{\partial^j}{\partial w^j} I_\pm (m, \tilde{N}w, q) \ll a^j, \quad j \geq 0. \]

Similarly it follows that
\[ \frac{\partial^j}{\partial w^j} I_\pm (m', \tilde{N}w, q') \ll a^j, \quad j \geq 0. \]

Hence, on applying integration by parts \( j \)-times to the \( w \)-integral in (45), we see that
\[ \Im_\pm \ll (k^\epsilon + a + a')^j \left( \frac{q_2 q_1 r n_1}{n_2 N} \right)^j \ll \left( \frac{(N\tilde{N})^{1/3}}{C r^{1/3}} \right)^j \left( \frac{C^2 r n_1}{q_1 n_2 N} \right)^j = \left( \frac{N^{1/3} C r^{2/3} n_1}{q_1 n_2 N^{2/3}} \right)^j. \]

Thus \( \Im_\pm \) is negligibly small if
\[ \frac{N^{1/3} C r^{2/3} n_1}{q_1 n_2 N^{2/3}} \ll 1 \quad \Leftrightarrow \quad n_2 \gg k^\epsilon C N^{1/3} r^{2/3} n_1 / q_1 N^{2/3}. \]

Next we prove \( \Im_\pm \ll k^\epsilon C^2 / (M_1 N) \).

Case 1. \( a \neq b \), i.e., \( a' \asymp a \ll k^{-\epsilon}b \asymp k^\epsilon b' \) or \( a' \asymp a \gg k^\epsilon b \asymp k^\epsilon b' \). In the case when \( a \gg k^\epsilon b \), on applying Lemma 115 to \( I_\pm (m, \tilde{N}w, q) \), we see that \( \Im_\pm \) is negligibly small. In the other case, i.e., \( a' \asymp a \ll k^{-\epsilon}b \asymp k^\epsilon b' \), on applying Lemma 115 to (45), we get
\[ \Im_\pm \ll \int_\mathbb{R} \phi(w) |I_\pm (m, \tilde{N}w, q)| |I_\pm (m', \tilde{N}w, q')| dw \ll \frac{k^\epsilon \langle b b' \rangle}{M_1 N}. \]
Case 2. \( a \asymp b \), i.e., \( k^{-r}b \leq a \leq k^r b \). On applying the last part of Lemma \([15]\) to \((45)\), we see that

\[
\mathfrak{J}_\pm \ll \frac{(aa')^{9/2}}{(bb')^5} \int_{b_{1/2}}^1 \int_{b_{1/2}}^1 \frac{1}{\tau^2 \sqrt{1 - \tau^2}} \frac{1}{\tau'^2 \sqrt{1 - \tau'^2}} \left| \int_{2/3}^3 g_3(\tau, \tau', w)e(wf_3(\tau, \tau')) \, dw \right| \, d\tau \, d\tau',
\]

where

\[
f_3(\tau, \tau') = \frac{16\pi^2a^3}{27b^2\tau^2} - \frac{16\pi^2a'^3}{27b^2\tau'^2} - \frac{n_2\tilde{N}}{q_2q'_2q_1r_{11}}
\]

and

\[
g_3(\tau, \tau', w) = \phi(w)w^3U_{u,t} \left( \frac{4\pi aw^{1/3}}{3b\tau} \right)^6 \tilde{U}_{u',r'} \left( \frac{4\pi a'w^{1/3}}{3b'\tau'} \right)^6.
\]

On applying the change of variable \( \tau \to 1/\sqrt{\tau}, \tau' \to 1/\sqrt{\tau'} \), we arrive at

\[
\mathfrak{J}_\pm \ll \frac{(aa')^{9/2}}{(bb')^5} \int_{1}^{1} \int_{1}^{1} \frac{\tau^{3/2}}{2\sqrt{\tau - 1}} \frac{\tau'^{3/2}}{2\sqrt{\tau' - 1}} \left| \int_{2/3}^3 g_3(1/\sqrt{\tau}, 1/\sqrt{\tau'}, w)e\left( \frac{16\pi^2a^3w}{27b^2} \right) f_4(\tau, \tau') \, dw \right| \, d\tau \, d\tau',
\]

where

\[
f_4(\tau, \tau') = \tau - \frac{a'^3b^2}{a^3b'^2} \tau' - \frac{27n_2\tilde{N}b^2}{16\pi^2q_2q'_2q_1r_{11}a^3}.
\]

Now using the change of variable

\[
\frac{a'^3b^2}{a^3b'^2} \tau' + \frac{27n_2\tilde{N}b^2}{16\pi^2q_2q'_2q_1r_{11}a^3} \to \tau',
\]

we arrive at the following \( w \)-integral

\[
\int_{2/3}^3 g_3(\ldots, w)e\left( \frac{16\pi^2a^3}{27b^2}(\tau - \tau') \right) \, dw,
\]

where \( g_3(\ldots, w) \) is given by

\[
\phi(w)w^3U_{u,t} \left( \frac{(4\pi a)^6\tau^3w^2}{(3b)^6} \right) \tilde{U}_{u',r'} \left( \frac{(4\pi a')^6w^2}{(3b')^6} \right) \left( \frac{a'^3b^2}{a^3b'^2} \tau' - \frac{27n_2\tilde{N}b^2}{16\pi^2q_2q'_2q_1r_{11}a^3} \right)^3.
\]

Note that

\[
\frac{\partial^j}{\partial w^j}g_3(\ldots, w) \ll_j k^j, \quad j \geq 0,
\]

as \( a \asymp b \) and

\[
\frac{a'^3b^2}{a^3b'^2} \tau' - \frac{27n_2\tilde{N}b^2}{16\pi^2q_2q'_2q_1r_{11}a^3} \ll k^r + \frac{(a + a')b^2}{a^3} \ll k^r,
\]

where, in the first inequality, we used \( \frac{n_2\tilde{N}}{q_2q'_2q_1r_{11}} \ll a + a' \), which follows by applying integration by parts to the \( w \)-integral in \((45)\). On applying integration by parts repeatedly, we see that the above integral is negligibly small unless

\[
|\tau - \tau'| \ll k^r b^2 / a^3.
\]
Now writing $\tau - \tau' = \tau_2$, with $\tau_2 \ll k' b^2 / a^3$, and estimating all the integrals in (66) trivially, we get
\[ \tilde{J}_\pm \ll \frac{(aa')^{9/2} k' b^2}{(bb')^5 a^3} \ll \frac{1}{(bb')^{1/2} b} \ll \frac{k' C^2}{M_1 N}, \]
where we used the fact $a' \asymp a \asymp b \asymp b'$. Hence we get (67).

Now we proceed to prove the last part. Let $q \sim C \gg k^{1+\epsilon}$. We also have $q' \sim C \gg k^{1+\epsilon}$. Note that in this situation we have $a \ll k^{-\epsilon} b$, $a' \ll k^{-\epsilon} b'$ and $b \asymp b' \asymp k$ (see (57) and (58)). On substituting the main term of $I_\pm(m, \tilde{N} w, q')$ from (61) and a similar expression for $I_\pm(m', \tilde{N} w, q')$ into (45), we arrive at the following expression:
\[ \frac{c_3^2 (aa')^{9/2}}{(bb')^5} \int_R \phi_1(w) e(f_5(w)) \, dw, \]
where
\[ \phi_1(w) = \frac{1}{\sqrt{f''(\tau_0)}} \frac{1}{\sqrt{1 - \tau_0^2}} \sqrt{f''(\tau_0)} \frac{1}{\sqrt{1 - \tau_0^2}} \times U_{u,t} \left( \frac{4\pi a w^{1/3}}{3b \tau_0} \right) \tilde{U}_{u',t} \left( \frac{4\pi a' w^{1/3}}{3b' \tau_0} \right)^6, \]
and
\[ f_5(w) = \frac{(k - 1)(\sin^{-1} \tau_0 - \sin^{-1} \tau_0')}{2\pi} + \frac{16\pi^2}{27} \left( \frac{a^3 w}{b^2 \tau_0^2} - \frac{a^3 w}{b^2 \tau_0^2} \right) \frac{\tilde{N} n_2 w}{q_2 q' q_1 r n_1}, \]
to which we apply the third derivative bound. Recall from (61) that
\[ \tau_0 = \tau_0(w) = c_1 b(w) + c_3(b(w))^3 + c_3(b(w))^5 + \cdots + c_{2n-1}(b(w))^{2n-1} + \cdots, \]
with
\[ b(w) = \frac{a w^{1/3}}{b^{2/3}(k - 1)^{1/3}}, \quad b = \frac{4\pi \sqrt{mN}}{q} \quad \text{and} \quad a = \frac{3(N\tilde{N})^{1/3}}{q r^{1/3}}. \]
and $\tau_0'$ is similarly defined. On applying the change of variable $w \to w^3$ in (68), we see that the phase function is given by
\[ \frac{(k - 1)(\sin^{-1} \tau_0(w^3) - \sin^{-1} \tau_0')}{2\pi} + \frac{16\pi^2}{27} \left( \frac{a^3 w^3}{b^2 \tau_0^2(w^3)} - \frac{a^3 w^3}{b^2 \tau_0^2(w^3)} \right) - \frac{\tilde{N} n_2 w^3}{q_2 q' q_1 r n_1}. \]
On applying the Taylor series expansion of $\sin^{-1} \tau_0(w^3)$, we see that
\[ \sin^{-1} \tau_0(w^3) = \tau_0(w^3) + (\tau_0(w^3))^3 / 6 + \cdots \]
\[ = d_1 b(w^3) + d_3(b(w^3))^3 + \cdots \]
\[ = d_1 b^{2/3}(k - 1)^{1/3} + d_3 b^3(k - 1) + \cdots, \]
where $d_1, d_3 \cdots$ are some absolute constants. Thus
\[ \frac{\partial^3}{\partial w^3} \sin^{-1} \tau_0(w^3) \ll \frac{a^3}{b^2(k - 1)}. \]
Similarly,
\[ \frac{\partial^3}{\partial w^3} \sin^{-1} \tau_0'(w^3) \ll \frac{a^3}{b^2(k - 1)}. \]
Next we consider $a^3 w^3/(b^2 \tau_0^2(w^3))$. On applying the Taylor series expansion, we get
\[
\frac{a^3 w^3}{b^2 \tau_0^2(w^3)} = \frac{(k - 1)(b(w^3))^3}{\tau_0^2(w^3)} = \frac{(k - 1)b(w^3)}{c_1^2} \left(1 + \frac{c_3(b(w^3))^3}{c_4 b(w^3)} + \ldots \right)
\]
\[
= \frac{(k - 1)b(w^3)}{c_1^2} \left(1 - \frac{2c_3(b(w^3))^3}{c_4 b(w^3)} - \ldots \right)
\]
\[
= \frac{(k - 1)}{c_4^2} \left(b(w^3) - \frac{2c_3(b(w^3))^3}{c_4 b(w^3)} - \ldots \right)
\]
Thus
\[
\frac{\partial^3 a^3 w^3}{\partial w^3 b^2 \tau_0^2(w^3)} \ll \frac{a^3}{b^2}.
\]
A similar analysis also gives us
\[
\frac{\partial^3 a^3 w^3}{\partial w^3 b^2 \tau_0^2(w^3)} \ll \frac{a^3}{b^2}.
\]
Hence, upon combining the above estimates, we conclude that
\[
\frac{\partial^3 f_5(w^3)}{\partial w^3} = O \left(\frac{a^3}{b^2} + \frac{a^3}{b^2} \right) - \frac{6\tilde{N}n_2}{q_2q_1 r n_1}.
\]
Since $n_2 \neq 0$, we note that
\[
\frac{a^3}{b^2} + \frac{a^3}{b^2} \ll \frac{N\tilde{N}}{C^r k^2} \ll \frac{(k^3/r^2)\tilde{N}}{C^2 r k^{3+\varepsilon}} \ll \frac{\tilde{N}}{k^c (C^2/q_1) r n_1} \ll \frac{\tilde{N} k^{-1} 6N|n_2|}{q_2q_1 r n_1}.
\]
In the first inequality, we used the fact $a \asymp a'$, $b \asymp b' \asymp k$. For the second inequality, we used $Nr^2 \ll k^{3+\varepsilon}$ and $C \gg k^{1+\varepsilon}$, while for the second last inequality, $(n_1, r) \geq n_1/q_1$, is being used. Hence we see that
\[
\left|\frac{\partial^3 f_5(w^3)}{\partial w^3}\right| = \left|O \left(\frac{a^3}{b^2} + \frac{a^3}{b^2}\right) - \frac{6\tilde{N}n_2}{q_2q_1 r n_1}\right| \gg \frac{a^3}{b^2} + \frac{a^3}{b^2} \gg \frac{N\tilde{N}}{C^3 k^2}.
\]
On computing the variation of $\phi_1(w)$ (see (69)), we note that
\[
\text{Var} \phi_1 \ll \frac{1}{\sqrt{b^2/a}} \left(\frac{a}{k}\right)^{3/2} \ll \frac{1}{\sqrt{b^2/a}} \left(\frac{a}{k}\right)^{3/2} \ll \frac{1}{\sqrt{b^2/a}} \left(\frac{a}{k}\right)^{10/3},
\]
where we used $f''(\tau_0) \asymp b^2/a$, $f''(\tau_0) \asymp b^2/a'$, $\tau_0 \asymp a/(b^{2/3}(k - 1)^{1/3}) \asymp a/k$ and $\tau_0' \asymp a'/k$. Hence, on applying the third derivative bound (see Lemma 9) to (68), we see that (68) is bounded by
\[
\frac{c_3^2 (a^4)^{9/2}}{(b^2 a')^5} \text{Var} \phi_1 + \max |\phi_1| \ll \frac{a^9}{b^{10} b^2/a} \left(\frac{a}{k}\right)^{10} \frac{1}{(NN)^{1/3}} \ll C r^{-1/3} k^{2/3}.
\]
Hence we get Proposition 11.

We conclude this section by giving the final estimation of the main integral $\mathcal{J}_\pm$ defined in (37) in the following corollary:

**Corollary 2.** Let $\mathcal{J}_\pm$ be the integral transform as defined in (37). Then we have
\[
\mathcal{J}_\pm \ll \frac{k^r C^4}{Q^2 M_1 N}.
\]
Furthermore, if $C \gg k^{1+\epsilon}$ and $n_2 \neq 0$, we have
\begin{equation}
\mathcal{J}_\pm \ll \frac{k^8 C^2}{Q^2} \frac{C r^{1/3} k^{2/3}}{k^2(NN)^{1/3}}.
\end{equation}

**Proof.** Let’s recall from (43) that
\begin{equation}
\mathcal{J}_\pm = \int_R \int_R V \left( \frac{t}{T} \right) V \left( \frac{t'}{T} \right) \int_{|u| \ll \frac{k^8 C}{Q^2}} \int_{|u'| \ll \frac{k^8 C}{Q^2}} I_u I_{u'} \mathcal{J}_\pm \d u \d u' \d t' \d t + O(k^{-2020}),
\end{equation}
where
\begin{equation}
I_u = \int_R W(x/Q') g(q, x) e\left( \frac{N x u}{q Q} \right) \d x,
\end{equation}
and $I_{u'}$ is similarly defined. On applying the bound $\mathcal{J}_\pm \ll k^8 C^2 / (M_1 N)$ from Proposition 1, we see that
\begin{equation}
|\mathcal{J}_\pm| \ll \frac{k^8 C^2}{M_1 N} \int_R \int_R V \left( \frac{t}{T} \right) V \left( \frac{t'}{T} \right) \int_{|u| \ll \frac{k^8 C}{Q^2}} \int_{|u'| \ll \frac{k^8 C}{Q^2}} |I_u||I_{u'}| \d u \d u' \d t' \d t.
\end{equation}
Note that
\begin{equation}
\int_{|u| \ll \frac{k^8 C}{Q^2}} |I_u| \d u \ll \int_{|u| \ll \frac{k^8 C}{Q^2}} \int_R W(x/Q') |g(q, x)| \d x \d u \ll \frac{k^8 C}{Q T} Q',
\end{equation}
where we used Property 4 (see (8)) of $g(q, x)$. The same bound holds for the $u'$-integral as well. Thus, on plugging these bounds into (75) and estimating the $t$ and $t'$-integral trivially, we get (73). On analysing the $u$, $u'$, $t$ and $t'$-integrals as above and applying the bound (48) from Proposition 1, we get the second part of the corollary. \square

7. Analysis of the zero frequency: $n_2 = 0$

With all the ingredients in hand, we now give final estimates for $S_r(N)$, given in (29), in the present and coming sections. The zero frequency case, i.e., $n_2 = 0$, needs to be analysed differently. Let $\Omega^0_\pm$ denote the contribution of the zero frequency to $\Omega_\pm$, given in (33), and let $S^0_r(N)$ denote the contribution of $\Omega^0_\pm$ to $S_r(N)$. We have the following lemma:

**Lemma 16.** Let $\Omega^0_\pm$ and $S^0_r(N)$ be defined as above. Then we have
\begin{equation}
\Omega^0_\pm \ll \frac{k^8 N_0 C^6 r}{q_1 n_1^2 Q^2 N(C + M_1)},
\end{equation}
and
\begin{equation}
S^0_r(N) \ll k^{r+1/2} N^{1/2} k^{3/2 - \eta/2},
\end{equation}
where $T = k^{1-\eta}$.

**Proof.** Let’s recall from (33) that
\begin{equation}
\Omega^0_\pm \ll k^8 \sup_{N \ll N_0} \sum_{q_2, q_2' \sim C/q_1} \sum_{m, m' \sim M_1} |\lambda_f(m)||\lambda_f(m')||\varepsilon_\pm||\mathcal{J}_\pm|.
\end{equation}
Consider the congruence condition
\begin{equation}
\pm \bar{\alpha} q_2' \pm \bar{\alpha}' q_2 \equiv n_2 \mod q_1 q_2 q_2' / n_1.
\end{equation}
given in the expression (34) of $C_\pm$. For $n_2 = 0$, it follows that $q_2 = q_2'$ and $\alpha = \alpha'$. Hence we get

$$C_\pm = \sum_{d,d'|q} \sum_{\alpha \mod qr/n_1} \mu \left( \frac{q}{d} \right) \mu \left( \frac{q'}{d'} \right) \sum^{\ast} 1$$

$$\ll \sum_{d,d'|q} \sum_{n_1|d/n_1, d'|n_1, d'} qr \ll \sum_{d,d'|q} qr([d,n_1, d']) \ll \sum_{d,d'|q} qr [d,d'].$$

On plugging the above expression and the bound $J_\pm \ll k^4 C^4/(Q^2 M_1 N)$ from Corollary 2 into (76), we get

$$\Omega^0_\pm \ll \frac{k^4 C^4}{Q^2 M_1 N} \sup_{N \ll N_0} \frac{N}{N_1^2} \sum_{q_2\sim C/q_1} qr \sum_{d,d'|q} (d,d') \sum_{m,m'\sim M_1} |\lambda_f(m)| |\lambda_f(m')|.$$

We use the inequality

$$|\lambda_f(m)| |\lambda_f(m')| \leq \frac{1}{2}(|\lambda_f(m)|^2 + |\lambda_f(m')|^2)$$

(77)

to count the number of $m$ and $m'$ as follows:

$$\sum_{m,m'\sim M_1} |\lambda_f(m)| |\lambda_f(m')| \ll \sum_{m\sim M_1} |\lambda_f(m)|^2 + \sum_{m,m'\sim M_1} (|\lambda_f(m)|^2 + |\lambda_f(m')|^2)$$

$$\ll k^4 M_1 + \sum_{m\sim M_1} |\lambda_f(m)|^2 \sum_{m'\sim M_1} 1$$

$$\ll k^4 M_1 (1 + M_1/(d,d')).$$

where we used the Ramanujan bound on average (see (2) and (3)). Thus we see that

$$\Omega^0_\pm \ll \frac{k^4 N_0 C^4}{n_1^2 Q^2 M_1 N} \sum_{q_2\sim C/q_1} qr \sum_{d,d'|q} (M_1(d,d') + M_1')$$

$$\ll \frac{k^4 N_0 C^4}{n_1^2 Q^2 M_1 N} \sum_{q_2\sim C/q_1} qr (M_1 q + M_1') \ll \frac{k^4 N_0 C^6}{q_1 n_1^2 Q^2 N} (C + M_1).$$

Hence we have the first part of the lemma. On substituting the above bound in place of $\Omega_{\pm}$ in (29), we get

$$S_r^0(N) \ll \sup_{M_1 \ll M_0} \frac{N^{5/3 + \epsilon}}{QT r^{2/3} C^3} \sum_{n_1 \ll M_1} \frac{1}{n_1^{1/3}} \Theta^{1/2} \sum_{n_1^{1/3}} \frac{C^3 (N_0 r)^{1/2}}{n_1 q_1^{1/2} Q \sqrt{N}} \left( \sqrt{M_1} + \sqrt{C} \right).$$

Executing the $q_1$-sum trivially and replacing the range for $n_1$ by the longer range $n_1 \ll Cr$, we get

$$S_r^0(N) \ll k^4 \sup_{M_1 \ll M_0} \frac{N^{2/3} (N_0 r)^{1/2}}{r^{2/3} \sqrt{N}} \sum_{n_1 \ll Cr} \frac{(n_1, r)^{1/2}}{n_1^{1/6}} \Theta^{1/2} \left( \sqrt{M_1} + \sqrt{C} \right).$$
Next we evaluate the $n_1$-sum, using Cauchy's inequality and the Ramanujan bound on average (see Lemma 5), as follows:

\[
\sum_{n_1 \leq C r} \frac{(n_1, r)^{1/2}}{n_1^{7/6}} \Theta^{1/2} \ll \left[ \sum_{n_1 \leq C r} \frac{(n_1, r)}{n_1} \right]^{1/2} \left[ \sum_{n_1 r n_2 \leq N_0} \left| \lambda_{\scriptscriptstyle n_1}(n_1, n_2) \right|^2 \right]^{1/2} \ll_{\varepsilon} N_0^{1/6+\varepsilon}.
\]

Thus we arrive at

\[
S^0 r(N) \ll k^r N^{2/3} N_0^{2/3} \left( \sqrt{M_0} + \sqrt{Q} \right).
\]

Note that

\[
Q = k^r \sqrt{N/T} \ll k^{3/2+\varepsilon}/\sqrt{T} \ll k^{2+\varepsilon}/T \ll k^{2+\varepsilon}Q^2/N.
\]

We also have $M_0 = k^r \max ((k-1)^2 C^2/N, T) \ll k^{2+\varepsilon}Q^2/N$ and

\[
N_0 = k^r \max \{ (CT)^3 r/N, T^{3/2} N^{1/2} \} \ll k^r (QT)^3 r/N \ll k^r T^{3/2} \sqrt{N_0}.
\]

Finally, upon using the above bounds in (79), we get

\[
S^0 r(N) \ll \frac{k^r N^{2/3} T N}{r^{1/6} \sqrt{N}} \frac{k Q}{\sqrt{N}} \ll k^{r+1/2} N^{1/2} k^{3/2-n/2}.
\]

Hence the lemma follows. \hfill \Box

8. Analysis of the non-zero frequencies: $n_2 \neq 0$

It now remains to estimate $S r(N)$ corresponding to the non-zero frequencies, i.e., $n_2 \neq 0$. We will consider two cases, small $q$'s and large $q$'s. To start with, we analyse the character sum $C_{\pm}$ given in (34). We have the following lemma which is taken from [28].

**Lemma 17.** Let $C_{\pm}$ be as in (34). Then, for $n_2 \neq 0$, we have

\[
C_{\pm} \ll \frac{q^2 r(m, n_1)}{n_1} \sum_{d_2 | (q_2, n_1 q_2 \mp mn_2)} \sum_{d_2' | (q_2', n_1 q_2 \mp m' n_2)} d_2 d_2'.
\]

**Proof.** Let's recall from (34) that

\[
C_{\pm} = \sum_{d \mid q} \sum_{d' \mid q'} d d' \mu \left( \frac{q}{d} \right) \mu \left( \frac{q'}{d'} \right) \sum_{\alpha \equiv m \mod d} \sum_{\alpha' \equiv m' \mod d'} \sum_{\beta \equiv q r / n_1} \sum_{\beta' \equiv q' r / n_1} 1,
\]

Using the Chinese Remainder theorem, we observe that $C_{\pm}$ can be dominated by a product of two sums $C_{\pm} \ll C_{\pm}^{(1)} C_{\pm}^{(2)}$, where

\[
C_{\pm}^{(1)} = \sum_{d_1 d_1' \mid q_1} \sum_{\beta \equiv \beta' \equiv 0 \mod d_1} \sum_{\beta \equiv \beta' \equiv 0 \mod d_1'} 1,
\]

\[
C_{\pm}^{(2)} = \sum_{d_2 d_2' \mid q_2} \sum_{\beta \equiv \beta' \equiv 0 \mod d_2} \sum_{\beta \equiv \beta' \equiv 0 \mod d_2'} 1.
\]
and
\[ C^{(2)}_{\pm} = \sum_{d_2 | q_2} \sum_{d_2' | q_2'} \sum_{\beta \mod q_2} \sum_{\beta' \mod q_2'} d_2 d_2' 1. \]

On applying (77) to (36), we see that \( \Omega \) and hence let
\[ 8.1. \quad S_r(N) \text{ for small } q. \] In this subsection, we will estimate \( S_r(N) \) for small values of \( q \). Let \( \Omega^{\neq 0}_ \) denote the part of \( \Omega \) (defined in (33)) which is complement to \( \Omega^0_ \) (contribution of \( n_2 \neq 0 \) ) and let \( S^{\neq 0}_r(N) \) denote the part of \( S_r(N) \) corresponding to \( \Omega^{\neq 0} \). We have the following lemma.

**Lemma 18.** Let \( \Omega^{\neq 0}_ \) and \( S^{\neq 0}_r(N) \) be as above. Then, for \( C \ll k^{1+\epsilon} \), we have
\[ (80) \quad \Omega^{\neq 0}_ \ll \frac{k^r_2 C^r(TN)^{1/2}}{n_1^2 q_1 q_2 M_1 N} \left( \frac{C M_1 n_1}{q_1} + M_1^2 \right). \]

Furthermore, let \( S^{\neq 0}_{r, \text{small}}(N) \) denote the contribution of \( C \ll k^{1+\epsilon} \) to \( S^{\neq 0}_r(N) \). Then we have
\[ (81) \quad S^{\neq 0}_{r, \text{small}}(N) \ll k^{3-\eta/2}, \]
where \( T = k^{1-\eta} \).

**Proof.** On applying (77) to (36), we see that \( \Omega^{\neq 0}_ \) is dominated by
\[ k^r \sup_{N < N_0} \frac{\tilde{N}}{n_1} \sum_{q_2, q_2' \sim C/q_1} \sum_{m, m' \sim M_1} (|\lambda_f(m)|^2 + |\lambda_f(m')|^2) \sum_{n_2 \in \mathbb{Z} - \{0\}} |C_{\pm}||J_{\pm}|. \]

We analyse the expression corresponding to \( |\lambda_f(m')|^2 \) only, since the calculations for the other expression is absolutely similar. Thus, on applying Lemma 17 and Corollary 2 we arrive at
\[ \frac{k^r q_1^2 q_2^4 C^r}{n_1^2 q_2^2 M_1 N} \sup_{N < N_0} \frac{\tilde{N}}{n_1} \sum_{q_2, q_2' \sim C/q_1} \sum_{m, m' \sim M_1} \sum_{d_2 | q_2} \sum_{d_2' | q_2'} d_2 d_2' \sum_{n_2 \in \mathbb{Z} - \{0\}} \sum_{n_1 q_2 + m_2 \equiv 0 \mod d_2} \sum_{n_1 q_2 + m_2' \equiv 0 \mod d_2'} |\lambda_f(m')|^2(m, n_1). \]
Writing \( q_2d_2 \) and \( q_2'd_2' \) in place of \( q_2' \) and \( q_2' \) respectively, we arrive at

\[
\sum_{d_2, d_2' \ll C/q_1} \frac{k\ell q_2 r C^4}{n_1^2 Q^2 M_1 N} \sup_{\tilde{N} \ll N_0} \tilde{N} \sum_{d_2, d_2' \ll C/q_1} d_2 d_2' \sum_{q_2 \sim \frac{C}{\sqrt{q_1}}} \sum_{q_2' \sim \frac{C}{\sqrt{q_1}}} \sum_{m, m' \sim M_1} \sum_{1 \leq |m| \leq N_2} \sum_{n_1 q_2 d_2' \equiv m n_2 \equiv 0 \mod d_2} | \lambda_f(m')^2 | (m, n_1).
\]

Fixing the parameters \((n_2, q_2, q_2', d_2, d_2', m')\), we count the number of \( m \) as follows:

\[
\sum_{n_1 q_2 d_2' \equiv m n_2 \equiv 0 \mod d_2} (m, n_1) = \sum_{\ell | n_1} \ell \sum_{m \sim M_1 / \ell} \frac{M_1}{\ell d_2 / (d_2, d_2' q_2', n_2)} \ll (d_2, d_2' q_2', n_2) \left( n_1 + \frac{M_1}{d_2} \right),
\]

where \( \ell \) is the inverse of \( \ell \) modulo \( d_2 \) which follows from the fact \((d_2, n_1) = 1\). On applying (83) with the bound \((d_2, n_2)(n_1 + M_1/d_2)\) and then executing the sum over \( q_2' \) in (82), we arrive at

\[
\sum_{1 \leq |n_2| \ll N_2} \sum_{m' \sim M_1} \sum_{d_2 \ll C/q_1} \frac{k\ell q_2 r C^4}{n_1^2 Q^2 M_1 N} \sup_{\tilde{N} \ll N_0} \tilde{N} \sum_{d_2 \ll C/q_1} d_2 \sum_{q_2 \sim \frac{C}{\sqrt{q_1}}} | \lambda_f(m')^2 | (d_2, n_2) \left( n_1 + \frac{M_1}{d_2} \right).
\]

We now count the number of \((d_2', m')\) following the arguments in [25, Section 6.1].

Case 1. \( n_1 q_2 d_2 \equiv m' n_2 \equiv 0 \mod d_2' \) but \( n_1 q_2 d_2' \equiv m' n_2 \neq 0 \). On switching the order of summations over \( d_2' \) and \( m' \), we see that the \( d_2' \)-sum is bounded above by \( d(|n_1 q_2 d_2 \pm m' n_2|) \ll k^\epsilon \), with \( d(n) \) being the divisor function. Thus (84) is bounded above by

\[
\sum_{1 \leq |n_2| \ll N_2} \sum_{m' \sim M_1} \sum_{d_2 \ll C/q_1} \frac{k\ell q_2 r C^4}{n_1^2 Q^2 M_1 N} \sup_{\tilde{N} \ll N_0} \tilde{N} \sum_{d_2 \ll C/q_1} d_2 \sum_{q_2 \sim \frac{C}{\sqrt{q_1}}} | \lambda_f(m')^2 | (d_2, n_2) \left( n_1 + \frac{M_1}{d_2} \right).
\]

On applying the Ramanujan bound on average to the \( m' \)-sum (see (2), (3)) and executing the \( n_2 \)-sum, we arrive at

\[
\sum_{d_2 \ll C/q_1} \frac{k\ell q_2 r C^4}{n_1^2 Q^2 M_1 N} \sup_{\tilde{N} \ll N_0} \tilde{N} N_2 M_1 \sum_{d_2 \ll C/q_1} \frac{C d_2}{q_1} \sum_{q_2 \sim \frac{C}{\sqrt{q_1}}} \left( n_1 + \frac{M_1}{d_2} \right).
\]

Now executing the remaining sums, we get the following expression

\[
\frac{k\ell r C^6}{n_1^2 Q^2 M_1 N} \sup_{\tilde{N} \ll N_0} \tilde{N} N_2 \left( \frac{C n_1 M_1}{q_1} + M_1^2 \right).
\]
On applying the bounds \( N_2 = k^e CN^{1/3} r^{2/3} n_1/(q_1 N^{2/3}) \) (see \( \Box \)) and \( N_0 \ll k^e T^{3/2} \sqrt{N} r \) (see \( \Box \)), we note that

\[
(86) \quad \sup_{\hat{N} < N_0} \hat{N} N_2 \ll k^e C^{r^{2/3} n_1 / q_1} (N \hat{N})^{1/3} \ll k^e C^{r^{2/3} n_1 / q_1} (NN_0)^{1/3} \ll k^e r n_1 (TN)^{1/2} C.
\]

Thus, in Case 1, we get the following bound for \( \Omega_{\pm} \)

\[
(87) \quad \frac{k^e r^2 C^7 (TN)^{1/2}}{n_1^2 q_1 Q^2 M_1 N} \left( \frac{Cn_1 M_1}{q_1} + M_1^2 \right).
\]

Case 2. \( n_1 q_2 d_2 \pm m'n_2 = 0 \). On applying \( \Box \) with the bound \( (d_2 q_2', n_2)(n_1 + M_1 / d_2) \) and switching some summations in \( \Box \), we arrive at

\[
(88) \quad \frac{k^e q_1^2 r^4 C^4}{n_1^3 Q^2 M_1 N} \sup_{\hat{N} < N_0} \hat{N} \sum_{d_2, d_2' \ll C/q_1} \sum_{q_2' \sim C \sim q_1} \sum_{m' \sim M_1} |\lambda_f(m')|^2 \\
\times \sum_{1 \leq |n_2| < N_0} \sum_{q_2, q_2' \sim C/d_2 q_1} (d_2 q_2', n_2) \left( n_1 + \frac{M_1}{d_2} \right).
\]

Fixing the tuple \((m', n_2, d_2)\), the number of \( q_2 \) turns out to be \( O(k^e) \) (as \( q_2 | m'n_2 \)).

Thus we arrive at

\[
\frac{k^e q_1^2 r^4 C^4}{n_1^3 Q^2 M_1 N} \sup_{\hat{N} < N_0} \hat{N} \sum_{d_2, d_2' \ll C/q_1} \sum_{q_2' \sim C \sim q_1} \sum_{m' \sim M_1} |\lambda_f(m')|^2 \\
\times \sum_{1 \leq |n_2| < N_0} \sum_{d_2 \ll C/q_1, d_2 | m'n_2} (d_2 q_2', n_2) \sum_{n_1 d_2 + M_1}.
\]

Now executing the sum over \( d_2 \), followed by the sum over \( n_2, m', q_2' \) and \( d_2' \), we see that the above expression is bounded above by

\[
\frac{k^e r C^6}{n_1^3 Q^2 M_1 N} \sup_{\hat{N} < N_0} \hat{N} N_2 \left( \frac{Cn_1 M_1}{q_1} + M_1^2 \right).
\]

Now estimating \( \hat{N} N_2 \) like Case 1, we get the first part of the lemma.

We will now prove \( \Box \). Consider the second term of the right hand side in \( \Box \).

On substituting it in place of \( \Omega_{\pm} \) in \( \Box \), we arrive at

\[
\sup_{M \ll M_1 \ll M_0} \frac{N^{5/3 + \epsilon}}{Q \sqrt{T r^{2/3} C^3}} \sum_{\pm \chi \ll C} \sum_{n_1^{1/3} \Theta^{1/2}} \sum_{n_1^{1/3} \Theta^{1/2}} \left( \frac{r^2 C^7 (TN)^{1/2} M_1}{n_1^2 q_1 Q^2 N} \right)^{1/2}
\ll \left( \sup_{M \ll M_1 \ll M_0} \frac{N^{5/3 + \epsilon}}{Q \sqrt{T r^{2/3} C^3}} \right)^{1/2} \frac{r (TN)^{1/4} C^{1/2} M_1^{1/2}}{Q \sqrt{N}} \sum_{n_1^{1/3} \ll C r} \sum_{n_1^{1/3} \ll C r} \frac{1}{Q_1^{1/2}}
\ll \left( \sup_{M \ll M_1 \ll M_0} \frac{N^{5/3 + \epsilon}}{Q \sqrt{T r^{2/3} C^3}} \right)^{1/2} \frac{r (TN)^{1/4} C^{1/2} M_1^{1/2}}{Q \sqrt{N}} \sum_{n_1^{1/3} \ll C r} \sum_{n_1^{1/3} \ll C r} \frac{1}{n_1^{7/6}} \Theta^{1/2}
\ll k^e r^{1/2} k^{3-\eta/2},
\]

where the two terms in the above expression are bounded by \( \ll k^e r^{1/2} k^{3-\eta/2} \).
where in the second last inequality, we used

\[ \sum_{n_1 \leq C_\varepsilon} \frac{\sqrt{(n_1, r)}}{n_1^{7/6}} \Theta^{1/2} \ll_{\varepsilon, \kappa} N_0^{1/6+\varepsilon} \]

from (18), \( C \ll k^{1+\varepsilon}, N_0 \ll k^r \sqrt{NT^{3/2}} \) and \( M_0 \ll k^{4+\varepsilon}/N \) as \( C \ll k^{1+\varepsilon} \).

Let’s now consider the first term in the right hand side of \( 8.1 \). We see that its contribution to \( S_r(N) \) in (29) is given by

\[ \sup_{M \leq M_1 \leq M_0} \sup_{C \leq k^{1+\varepsilon}} \frac{N^{5/3+\varepsilon}}{Q T r^{2/3} C^3} \sum_{n_1 \leq C} \sum_{\frac{n_1}{r}} n_1^{1/3} \Theta^{1/2} \sum_{\frac{n_1}{r} q_1 \neq 0} \frac{1}{\frac{n_1}{r} q_1 (n_1 r)^{\varepsilon}} \left( \frac{r^2 C^7 (TN)^{1/2} C}{n_1 q_1^2 Q^2 N} \right)^{1/2} \]

\[ \ll \sup_{M \leq M_1 \leq M_0} \sup_{C \leq k^{1+\varepsilon}} \frac{N^{5/3+\varepsilon}}{Q T r^{2/3} C^3} \frac{r(TN)^{1/4} C^{7/2} C^{1/2}}{Q \sqrt{N}} \sum_{n_1 \leq C_\varepsilon} n_1^{-1+6} \Theta^{1/2} \sum_{\frac{n_1}{r} q_1 \neq 0} \frac{1}{\frac{n_1}{r} q_1 (n_1 r)^{\varepsilon}} \]

\[ \ll k^{3-\eta/2} . \]

In the second last inequality, we used the bound

\[ \sum_{n_1 \leq C_\varepsilon} \frac{(n_1, r)}{n_1^{7/6}} \Theta^{1/2} \ll \left[ \sum_{n_1 \leq C_\varepsilon} \frac{(n_1, r)^2}{n_1} \right]^{1/2} \left[ \sum_{n_1 \leq C_\varepsilon} \sum_{n_2 \leq N_0} |\lambda(n_1, n_2)|^2 \right]^{1/2} \ll_{\varepsilon, \kappa} r^{1/2} N_0^{1/6+\varepsilon} . \]

Thus we have the lemma.

\[ \square \]

8.2. Estimates for generic \( q \). Now we tackle the case when \( C \gg k^{1+\varepsilon} \) and \( n_2 \neq 0 \). Let \( S_{r, \text{generic}}(N) \) denote the part of \( S_r^0(N) \) which is complement to \( S_{r, \text{small}}^0(N) \) (i.e., the contribution of \( C \gg k^{1+\varepsilon} \) and \( n_2 \neq 0 \) to \( S_r(N) \)). We have the following lemma.

**Lemma 19.** Let \( S_{r, \text{generic}}^0(N) \) be as above. Then we have

\[ S_{r, \text{generic}}^0(N) \ll N^{1/2} k^{3/2-1/6+3n/4} . \]

**Proof.** Let’s recall from the analysis of \( \Omega_{\pm}^0 \) in the proof of Lemma \( 18 \) that (see (53))

\[ \Omega_{\pm}^0 \ll \frac{k^r C^6}{n_1^3 Q^2 M_1 N} \sup_{S \leq N_0} \tilde{N} N_2 \left( \frac{C n_1 M_1}{q_1} + M_1^2 \right) . \]

To get this, we used the bound \( J_{\pm} \ll k^r C^4/(Q^2 M_1 N) \). For \( C \gg k^{1+\varepsilon} \), we have a better bound for \( J_{\pm} \) (see Corollary \( 2 \)). In fact, we have

\[ J_{\pm} \ll \frac{k^r C^2}{Q^2} \frac{C r^{1/3} k^{2/3} N}{k^2 (N N)^{1/3}} \times \frac{k^r C^4}{Q^2 M_1 N} \frac{C r^{1/3} k^{2/3} N}{(N N)^{1/3}} , \]

where we used \( \sqrt{M_1 N}/C \ll k \) for \( C \gg k^{1+\varepsilon} \). Thus, on applying the above bound, we see that

\[ \Omega_{\pm}^0 \ll \frac{k^r C^6}{n_1^3 Q^2 M_1 N} \times \frac{C r^{1/3} k^{2/3}}{k^2 (N N)^{1/3}} \times \sup_{N_2 \leq N_0} \frac{\tilde{N} N_2}{(N N)^{1/3}} \left( \frac{C n_1 M_1}{q_1} + M_1^2 \right) . \]
Recall from (86) that

\[ \sup_{N \ll N_0} \frac{\tilde{N} N_2}{(N N)^{1/3}} \ll k^{e} C r^{2/3} n_1 \]

and

\[ \sup_{N \ll N_0} \frac{\tilde{N} N_2}{(N N)^{1/3}} = \frac{N_0 N_2}{(N N_0)^{1/3}}. \]

Thus we see that

\[ \Omega_{\pm}^{\neq 0} \ll \frac{k^{e} r C^6}{n_1^3 Q^2 M_1 N} \times C r^{1/3} k^{2/3} \times \frac{N_0 N_2}{(N N_0)^{1/3}} \left( \frac{C n_1 M_1}{q_1} + M_1^2 \right). \]

On comparing it with (87), we observe that we have an extra factor

\[ \frac{C r^{1/3} k^{2/3}}{r^{1/3} (NT)^{1/2}} \ll \frac{Q k^{2/3}}{(NT)^{1/2}} = k^{e} r^{-1/3} \]

in this case. Hence, taking it into account, we get

\[ \Omega_{\pm}^{\neq 0} \ll \frac{k^{e} r C^7 (TN)^{1/2}}{n_1^2 q_1 Q^2 M_1 N} \times k^{-1/3} \left( \frac{C n_1 M_1}{q_1} + M_1^2 \right). \]

Note that

\[ \frac{C n_1}{q_1} + M_1 \ll \frac{Q n_1}{q_1} + M_0 \ll \frac{n_1 k^{e}}{q_1} \sqrt{\frac{N}{T}} + \frac{Q^2 k^{2+e}}{N} \ll (n_1, r) k^{e} \sqrt{\frac{N}{T}} + \frac{k^{2+e}}{T} \ll k^{2+e} T, \]

where we used \( M_0 \ll Q^2 k^{2+e} / N, N r^2 \ll k^{3+e}, Q = k^{e} \sqrt{N/T}, T \ll k \) and \( n_1 / q_1 \leq (n_1, r) \). Thus, on plugging the above bound into (95), we get

\[ \Omega_{\pm}^{\neq 0} \ll \frac{k^{e} r C^7 (TN)^{1/2}}{n_1^2 q_1 Q^2 N} \times k^{-1/3} \times \frac{k^{2+e}}{T}. \]

On substituting the above bound in place of \( \Omega_{\pm}^{\neq 0} \) in (29), we see that \( S_{r, \text{generic}}^{\neq 0} (N) \) is dominated by

\[ \sup_{C \ll Q} \frac{N^{5/3 + e}}{Q T r^{2/3} C^3} \sum_{\pm \frac{n_1 \Theta^{1/2}}{(n_1, r)}} \frac{n_1^{1/3} \Theta^{1/2}}{(n_1, r)} \sum_{\frac{n_1 \Theta^{1/2}}{(n_1, r)}} \left( \frac{r^2 C^7 (TN)^{1/2}}{n_1^2 q_1 Q^2 N} \right)^{1/2} \times \frac{k^{5/6 + e/2}}{\sqrt{T}} \]

\[ \ll \sup_{C \ll Q} \frac{N^{5/3 + e}}{Q T r^{2/3} C^3} \frac{r(TN)^{1/4} C^7}{Q \sqrt{N}} \sum_{n_1 \ll C r} n_1^{-2/3} \Theta^{1/2} \sum_{\frac{n_1 \Theta^{1/2}}{(n_1, r)}} \frac{1}{q_1^{1/2}} \times \frac{k^{5/6 + e/2}}{\sqrt{T}} \]

\[ \ll \sup_{C \ll Q} \frac{N^{5/3 + e}}{Q T r^{2/3} C^3} \frac{r(TN)^{1/4} C}{Q \sqrt{N}} \sum_{n_1 \ll C r} \frac{\sqrt{n_1, r}}{n_1^{7/6}} \Theta^{1/2} \times \frac{k^{5/6 + e/2}}{\sqrt{T}} \]

\[ \ll N^{1/2} k^{3/2 - 1/6 + 3e/4}. \]

Hence the lemma follows. \( \square \)
8.3. Estimates for the error term. In this subsection, we give estimates for $S_r(N)$ corresponding to the non-generic case $n_2^*N \ll k^r$ (see Lemma 12). Recall from (21) that, if $n_2^*N = n_1^2n_2N/(q^3r) \ll k^r$, then we have

$$S_3 = q \sum_{n_1 \mid qr} \sum_{n_2 \geq 1} \sum_{\lambda} \lambda(n_1, n_2) \frac{n_1 n_2}{n_2} S(r \hat{a}, \pm n_2; qr/n_1) G_\pm(n_2^*),$$

where $G_\pm(n_2^*)$ is as defined in (19). On plugging (96) and (25) in place of $S_3$ and $S_2$ respectively into (14) we arrive at

$$\frac{2\pi i^k N^{1-it}}{Q T} \sum_{1 \leq q \leq Q} \frac{1}{q} \sum_{n_1 \mid qr} \sum_{n_2 \leq N/n_1^2} \lambda(n_1, n_2) \frac{n_1 n_2}{n_2}$$

$$\times \sum_{M \leq m \leq M_0} \lambda_f(m) C_\pm(...) I_4(q, m, n_1^2n_2) + O(k^{-2020}),$$

where

$$C_\pm(...) := \sum_{\alpha \mod q}^* S(r \hat{a}, \pm n_2; qr/n_1) e\left(\frac{\alpha n_2}{q}\right)$$

$$= \sum_{d \mid q} d \mu\left(\frac{q}{d}\right) \sum_{\alpha \mod q}^* e\left(\pm \frac{\alpha n_2}{qr/n_1}\right)$$

$$\ll (n_1, m, q) \left(q + \frac{qr}{n_1}\right) \ll (n_1, m) \sqrt{(n_1, q)} \left(q + \frac{qr}{n_1}\right),$$

and

$$I_4(q, m, n_1^2n_2) = \int_\mathbb{R} W(x/Q') \int_\mathbb{R} V\left(\frac{t}{T}\right) g(q, x) I_2(m, q, x) G_\pm(n_1^2n_2) \, dt \, dx,$$

with

$$I_2(m, q, x) = \int_0^\infty U(y) y^{-it} e\left(-\frac{Nxy}{qQ}\right) J_{k-1}\left(\frac{4\pi \sqrt{mNy}}{q}\right) \, dy,$$

and

$$G_\pm(n_1^2n_2) = \frac{1}{2\pi i} \int_\sigma (n_1^2n_2)^{-s} \gamma_\pm(s) \tilde{g}(-s) \, ds$$

$$= \frac{N^{it}}{2\pi} \int_\infty^\infty \gamma_\pm(\sigma + it) \Gamma_{\sigma+it} \int_0^\infty e\left(\frac{z_1 Nx}{qQ}\right) V(z_1) z_1^{-\sigma-it+1} \frac{dz_1}{z_1} \, d\tau,$$

where $\sigma > -1 + \max\{-\Re(\alpha_1), -\Re(\alpha_2), -\Re(\alpha_3)\}$. On analysing the $x$-integral and the $t$-integral following Lemma 13, we get the following restriction

$$|z_1 - y| \ll k^r q/Q T.$$

Thus, on replacing $z_1$ by $y + u$ with $u \ll k^r q/Q T$, we essentially arrive at

$$I_4(q, m, n_1^2n_2) = \frac{1}{2\pi} \int_\infty^\infty \gamma_\pm(\sigma + it) \int_\mathbb{R} V\left(\frac{t}{T}\right) N^{it} \int_{u \ll k^r q} I_u I_5(m, q, u, \tau) \, du \, d\tau \, d\tau,$$

where

$$I_u = \int_\mathbb{R} W(x/Q') g(q, x) e\left(\frac{Nxu}{qQ}\right) \, dx,$$
and
\[ I_5(m, q, u, \tau) = \int_0^\infty U_{t,u,\tau}(y)y^{-is}J_{k-1}\left(\frac{4\pi\sqrt{mNy}}{q}\right) \, dy, \]
with \( U_{t,u,\tau}(y) = U(y)y^{-\sigma(1+u/y)^{-\sigma-\sigma+it}}. \) On analysing \( I_5(m, q, u, \tau) \) like \( I_\pm(m, \tilde{N}w, q) \) (see Lemma [15]), we get
\[ I_5(m, q, u, \tau) \ll \frac{k^\epsilon q^{3/2}}{(mN)^{1/4}}. \]

We now move the contour \( \sigma \) in \((99)\) to the left up to \( \sigma = -5/2 \) passing through the poles given by
\[ 1 + \sigma + \Re\alpha_i + \ell = 0 \iff \sigma = -1 - \Re\alpha_i - \ell. \]
Thus, on treating the \( u \) and \( t \)-integral trivially, we get
\[ I_4(q, m, n_1^2n_2) \ll (n_2^*N)^{5/2} \frac{k^\epsilon q^{3/2}}{Q(mN)^{1/4}} \int_{-\infty}^{\infty} |\gamma_\pm(-5/2 + i\tau)| \, d\tau \]
\[ + \frac{k^\epsilon q^{3/2}}{Q(mN)^{1/4}} + \sum_{\ell=0,1} \sum_{i=1}^3 (n_2^*N)^{1+\ell+\Re\alpha_i}. \]
Now using the Stirling bound
\[ |\gamma_\pm(-5/2 + i\tau)| \ll (1 + |\tau|)^3(-5/2+1/2) = (1 + |\tau|)^{-6}, \]
we arrive at
\[ I_4(q, m, n_1^2n_2) \ll \frac{k^\epsilon q^{3/2}}{Q(mN)^{1/4}} \left((n_2^*N)^{5/2} + \sum_{\ell=0,1} \sum_{i=1}^3 (n_2^*N)^{1+\ell+\Re\alpha_i}\right). \]

Note that \((n_2^*N)^{5/2} = (n_2^*N)^{1/4+9/4} \ll k^\epsilon (n_2^*N)^{1/4}, \) and
\[ \sum_{i=1}^3 (n_2^*N)^{1+\ell+\Re\alpha_i} = \sum_{i=1}^3 (n_2^*N)^{1/2+\beta_i} \ll k^\epsilon (n_2^*N)^{1/2} \ll k^\epsilon (n_2^*N)^{1/4} \]
as \( 1 + \ell + \Re\alpha_i = 1/2 + \beta_i \) for some \( \beta_i > 0. \) Thus we get
\[ I_4(q, m, n_1^2n_2) \ll \frac{k^\epsilon q^{3/2}}{Q(mN)^{1/4}} (n_2^*N)^{1/4} = \frac{k^\epsilon q^{3/4}(n_2^*N)^{1/4}}{Qm^{1/4}r^{1/4}}. \]
Thus, on plugging the above bound and the bound \((98)\) for \( C_\pm(...) \) into \((97)\) and then estimating the sum over \( m \) using the Ramanujan bound on average, we see that \((97)\)
is dominated by
\[ \sum_{1 \leq q \leq Q} \frac{N_0^{3/4}}{q^2Tr^{1/4}} \sum_{n_1|qr} \sum_{n_2 \ll \frac{\sqrt{q^*N}}{n_1^2n_2}} \frac{|\lambda_\pi(n_1, n_2)|}{n_1n_2} (n_1^2n_2)^{1/4} \sqrt{(n_1, q)} \left(1 + \frac{r}{n_1}\right). \]
We estimate the sum over $n_1$ and $n_2$ as follows:

$$\sum_{n_1|q'r_2 \ll n_1^{3/4} N^{3/4}} \sum_{n_2 < N^{3/4}} \frac{\lambda(n_1, n_2)}{|n_1 n_2|^{1/4}} \left( n_1^{2/3} q_2^{1/3} \right)^{1/4} \frac{1}{n_1} \left( 1 + \frac{r}{n_1} \right)$$

$$\ll \sum_{n_1|q'r_2 \ll n_1^{3/4} N^{3/4}} \sum_{n_2 < N^{3/4}} \frac{\lambda(n_1, n_2)}{|n_1 n_2|^{1/4}} \frac{r}{n_2}$$

$$\ll \left( \sum_{n_1 \leq N^{3/4}} \left( \sum_{r_2 < N^{3/4}} \frac{1}{r_2^{1/4}} \right)^{1/2} \left( \sum_{n_2 < N^{3/4}} \frac{1}{n_2} \right)^{1/2} \right)^{1/2} \frac{q^{3/2} r^{3/2}}{\sqrt{N}}.$$

Hence the contribution of the terms $n_1^2 n_2 N/(q^3 r) \ll k^e$ to $S_r(N)$ is dominated by

$$k^e \sum_{1 \leq q' \leq Q} \frac{NM_0^{3/4} q^{3/2} r^{3/2}}{Q^2 T r^{1/4} \sqrt{N}} \ll \sqrt{N} k^{1+2\epsilon+3\eta/8+\epsilon},$$

where we used $M_0 \ll k^{2+\epsilon}/T$ and $N r^2 \ll k^{3+\epsilon}$.

9. Conclusion: Proof of Theorem 1

We now pull together the bounds from Lemma 16, Lemma 18, Lemma 19 and (102) to get that

$$\frac{S_r(N)}{N^{1/2} k^{3/2+\epsilon}} \ll k^{-1/2+2\eta+3\eta/8} + r^{1/2} k^{-\eta/2} + r^{1/2} k^{3/2-\eta/2} N^{1/2} + k^{-1/6+3\eta/4}.$$

Using $k^{3-\theta} \ll N r^2 \ll k^{3+\epsilon}$ and $r \ll k^\theta$, we further get

$$\frac{S_r(N)}{N^{1/2} k^{3/2+\epsilon}} \ll k^{-1/2+2\eta+3\eta/8} + k^{\theta/2-\eta/2} + k^{2\eta-\eta/2} + k^{1/6+3\eta/4}.$$

Hence to get subconvexity, we need all of the above exponents to be negative. So the first and the third term gives $4/19 > \eta > 4\theta$, and consequently the third and the fourth terms dominate the rest. Thus the above bound reduces to

$$\frac{S_r(N)}{N^{1/2} k^{3/2+\epsilon}} \ll k^{2\theta-\eta/2} + k^{-1/6+3\eta/4}.$$

The optimal choice for $\eta$ is given by $\eta = 8\theta/5 + 2/15$. On plugging this in Lemma 11 we get

$$L(1/2, \pi \times f) \ll k^{3/2+6\theta/5-1/15+\epsilon} + k^{3/2-\theta/2+\epsilon},$$

and with the optimal choice $\theta = 2/51$, we obtain the bound given in Theorem 1.

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