A NOTE ON THE OFF-DIAGONAL MUCKENHOUPT-WHEEDEN CONJECTURE

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Abstract. We obtain the off-diagonal Muckenhoupt-Wheeden conjecture for Calderón-Zygmund operators. Namely, given $1 < p < q < \infty$ and a pair of weights $(u, v)$, if the Hardy-Littlewood maximal function satisfies the following two weight inequalities:

\[
M : L^p(v) \rightarrow L^q(u) \quad \text{and} \quad M : L^q\left(u^{1-q'}\right) \rightarrow L^{q'}\left(v^{1-q'}\right),
\]

then any Calderón-Zygmund operator $T$ and its associated truncated maximal operator $T_\ast$ are bounded from $L^p(v)$ to $L^q(u)$. Additionally, assuming only the second estimate for $M$ then $T$ and $T_\ast$ map continuously $L^p(v)$ into $L^{q,\infty}(u)$. We also consider the case of generalized Haar shift operators and show that their off-diagonal two weight estimates are governed by the corresponding estimates for the dyadic Hardy-Littlewood maximal function.

1. Introduction and Main results

In the 1970s, Muckenhoupt and Wheeden conjectured that given $p$, $1 < p < \infty$, a sufficient condition for the Hilbert transform to satisfy the two weight norm inequality

\[
H : L^p(v) \rightarrow L^p(u)
\]

is that the Hardy-Littlewood maximal operator satisfy the pair of norm inequalities

\[
M : L^p(v) \rightarrow L^p(u),
\]

\[
M : L^{p'}\left(u^{1-p'}\right) \rightarrow L^{p'}\left(v^{1-p'}\right).
\]

Moreover, they conjectured that the Hilbert transform satisfies the weak-type inequality

\[
H : L^p(v) \rightarrow L^{p,\infty}(u)
\]

provided that the maximal operator satisfies the second “dual” inequality. Both of these conjectures readily extend to all Calderón-Zygmund operators (see the definition below). Very recently, both conjectures were disproved:
the strong-type inequality by Reguera and Scurry [11] and the weak-type inequality by the first author, Reznikov and Volberg [5].

Remark 1.1. A special case of these conjectures, involving the $A_p$ bump conditions, has been considered by several authors: see [1, 2, 3, 4, 5, 9].

In this note we prove the somewhat surprising fact that the Muckenhoupt-Wheeden conjectures are true for off-diagonal inequalities. Our main result is Theorem 1.2 below. We also prove an analogous result for the Haar shift operators (the so-called dyadic Calderón-Zygmund operators) with the Hardy-Littlewood maximal operator replaced by the dyadic maximal operator: see Theorem 1.3 below.

To state our results we first give some preliminary definitions. By weights we will always mean non-negative, measurable functions. Given a pair of weights $(u, v)$, hereafter we will assume that $u > 0$ on a set of positive measure and $u < \infty$ a.e., and $v > 0$ a.e. and $v < \infty$ on a set of positive measure. We will also use the standard notation $0 \cdot \infty = 0$.

Calderón-Zygmund operators. A Calderón-Zygmund operator $T$ is a linear operator that is bounded on $L^2(\mathbb{R}^n)$ and

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy, \quad f \in L^\infty_c(\mathbb{R}^n), \quad x \notin \text{supp } f,$$

where the kernel $K$ satisfies the size and smoothness estimates

$$|K(x, y)| \leq \frac{C}{|x - y|^n}, \quad x \neq y,$$

and

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq C \frac{|x - x'|^6}{|x - y|^{n+3}},$$

for all $|x - y| > 2|x - x'|$.

Associated with $T$ is the truncated maximal operator

$$T_* f(x) = \sup_{0 < \epsilon < \epsilon' < \infty} \left| \int_{\epsilon < |x - y| < \epsilon'} K(x, y)f(y)dy \right|.$$

Let $M$ denote the Hardy-Littlewood maximal operator, that is,

$$Mf(x) = \sup_{Q} \frac{1}{|Q|} \int_{Q} |f(y)|dy = \sup_{Q} \frac{1}{|Q|} \int_{Q} |f(y)|dy.$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes.

**Theorem 1.2.** Given a Calderón-Zygmund operator $T$, let $1 < p < q < \infty$ and let $(u, v)$ be a pair of weights. If the maximal operator satisfies

$$M : L^p(v) \rightarrow L^q(u) \quad \text{and} \quad M : L^{q'}(u^{1-q'}) \rightarrow L^{p'}(v^{1-p'}),$$

then

$$\|Tf\|_{L^q(u)} \leq C\|f\|_{L^p(v)} \quad \text{and} \quad \|T_* f\|_{L^q(u)} \leq C\|f\|_{L^p(v)}.$$
Analogously, if the maximal operator satisfies

\begin{equation}
M : L^q (u^{1-q'}) 
\rightarrow L^{p'} (v^{1-p'}),
\end{equation}

then

\begin{equation}
\|Tf\|_{L^q,\infty (u)} \leq C \|f\|_{L^p (v)} \quad \text{and} \quad \|T_\ast f\|_{L^q,\infty (u)} \leq C \|f\|_{L^p (v)}.
\end{equation}

If the pairs of weights \((u,v)\) satisfy any of the conditions in (1.1), then the weights \(u\) and \(v^{1-p'}\) are locally integrable. This is a consequence of a characterization of the two weight norm inequalities for the maximal operator due to Sawyer [12]. He proved that the \(L^p - L^q\) inequality holds if and only if for every cube \(Q\),

\[
\left( \int_Q M (v^{1-p'} \chi_Q) (x)^q u(x) \, dx \right)^{1/q} \leq C \left( \int_Q v(x)^{1-p'} \, dx \right)^{1/p} < \infty,
\]

and the \(L^q - L^p'\) inequality holds if and only if

\[
\left( \int_Q M (u \chi_Q) (x)^{p'} v(x)^{1-p'} \, dx \right)^{1/p'} \leq C \left( \int_Q u(x) \, dx \right)^{1/q'} < \infty.
\]

It is straightforward to construct pairs of weights that satisfy these conditions. For instance, in \(\mathbb{R}\) both of these conditions follow easily for every \(1 < p < q < \infty\) and the pair of weights \((u,v)\) with \(u = \chi_{[0,1]}\) and \(v^{-1} = \chi_{[2,3]}\) (i.e., \(v = 1\) in \([2,3]\) and \(v = \infty\) elsewhere). Indeed, we only need to check Sawyer’s inequalities for cubes \(Q\) that intersect both \([0,1]\) and \([2,3]\), in which case we have \(M (\chi_{[2,3]\cap Q})(x) \leq \|2,3\cap Q\|\) for every \(x \in [0,1]\cap Q\), and \(M (\chi_{[0,1]\cap Q})(x) \leq \|[0,1]\cap Q\|\) for every \(x \in [2,3]\cap Q\). These readily imply the desired estimates.

**Dyadic Calderón-Zygmund operators.** A generalized dyadic grid \(\mathcal{D}\) in \(\mathbb{R}^n\) is a set of generalized dyadic cubes with the following properties: if \(Q \in \mathcal{D}\) then \(\ell(Q) = 2^k, k \in \mathbb{Z}\); if \(Q, R \in \mathcal{D}\) and \(Q \cap R \neq \emptyset\) then \(Q \subset R\) or \(R \subset Q\); the cubes in \(\mathcal{D}\) with \(\ell(Q) = 2^{-k}\) form a disjoint partition of \(\mathbb{R}^n\) (see [9] and [10] for more details).

We say that \(g_Q\) is a generalized a Haar function associated with \(Q \in \mathcal{D}\) if

(a) \(\text{supp}(g_Q) \subset Q\);

(b) if \(Q' \in \mathcal{D}\) and \(Q' \subset Q\), then \(g_Q\) is constant on \(Q'\);

(c) \(\|g_Q\|_{\infty} \leq 1\).

Given a dyadic grid \(\mathcal{D}\) and a pair \((m,k) \in \mathbb{Z}^2_+\), a linear operator \(S\) is a generalized Haar shift operator (that is, a dyadic Calderón-Zygmund operator) of complexity type \((m,k)\) if it is bounded on \(L^2(\mathbb{R}^n)\) and

\[
Sf(x) = \sum_{Q \in \mathcal{D}} S_Q f(x) = \sum_{Q \in \mathcal{D}} \sum_{Q' \in \mathcal{D}_m(Q)} \frac{\langle f, \eta_Q^{Q'} \rangle}{|Q|} g_Q^{Q'}(x),
\]

where \(\mathcal{D}_m(Q)\) is the set of generalized Haar shifts of \(Q\) with complexity \(m\) and \(\eta_Q^{Q'}\) is the characteristic function of the generalized Haar cube \(Q'\).
where $\mathcal{Q}_j(Q)$ stands for the dyadic subcubes of $Q$ with side length $2^{-j}\ell(Q)$, $g_{Q''}^\mathcal{Q}'$ is a generalized a Haar function associated with $Q'$ and $g_{Q''}^\mathcal{Q}'$ is a generalized a Haar function associated with $Q''$. We say that the complexity of $\mathcal{S}$ is $\kappa = \max(m, k)$. We also define the truncated Haar shift operator

$$S_\epsilon f(x) = \sup_{0 < \epsilon < \epsilon' < \infty} |S_{\epsilon, \epsilon'} f(x)| = \sup_{Q \in \mathcal{Q}, \epsilon \leq \ell(Q) \leq \epsilon'} \left| \sum_{Q \in \mathcal{Q}} S_Q f(x) \right|.$$ 

An important example of a Haar shift operator on the real line is the Haar shift (also known as the dyadic Hilbert transform) $H^\mathcal{Q}$, defined by

$$H^\mathcal{Q} f(x) = \sum_{I \in \Delta} \langle f, h_I \rangle (h_{I_+}(x) - h_{I_-}(x)),$$

where, given a dyadic interval $I$, $I_+$ and $I_-$ are its right and left halves, and

$$h_I(x) = |I|^{-1/2} (\chi_{I_-}(x) - \chi_{I_+}(x)).$$

After renormalizing, $h_I$ is a Haar function on $I$ and one can write $H^\mathcal{Q}$ as a generalized Haar shift operator of complexity 1. These operators have played a very important role in the proof of the $A_2$ conjecture: see [4, 6, 7] and the references they contain for more information.

Associated with the dyadic grid $\mathcal{Q}$ is the dyadic maximal function

$$M_\mathcal{Q} f(x) = \sup_{x \in Q \in \mathcal{Q}} \int_Q |f(y)| \, dy.$$ 

Note that $M_\mathcal{Q}$ is dominated pointwise by the Hardy-Littlewood maximal operator.

We can now state our result for dyadic Calderón-Zygmund operators.

**Theorem 1.3.** Let $\mathcal{S}$ be a generalized Haar shift operator of complexity $\kappa$. Given $1 < p < q < \infty$ and a pair of weights $(u, v)$, if the dyadic maximal operator satisfies

(1.5) $M_\mathcal{Q} : L^p(v) \to L^q(u)$ and $M_\mathcal{Q} : L^q(u^{1-q'}) \to L^{p'}(v^{1-p'})$,

then

(1.6) $\|S f\|_{L^q(u)} \leq C \kappa^2 \|f\|_{L^p(v)}$ and $\|S_\epsilon f\|_{L^q(u)} \leq C \kappa^2 \|f\|_{L^p(v)}$.

Analogously, if the dyadic maximal operator satisfies

(1.7) $M_\mathcal{Q} : L^q(u^{1-q'}) \to L^{p'}(v^{1-p'})$

then

(1.8) $\|S f\|_{L^q, \infty(u)} \leq C \kappa^2 \|f\|_{L^p(v)}$ and $\|S_\epsilon f\|_{L^q, \infty(u)} \leq C \kappa^2 \|f\|_{L^p(v)}$. 

2. Proofs of the Main results

Proof of Theorem 1.2. We will prove our estimates for $T_*$; the ones for $T$ are completely analogous.

Given a dyadic grid $\mathcal{D}$ we say that $\{Q_j^k\}_{j,k}$ is a sparse family of dyadic cubes if for any $k$ the cubes $\{Q_j^k\}_{j}$ are pairwise disjoint; if $\Omega_k := \bigcup_j Q_j^k$, then $\Omega_{k+1} \subset \Omega_k$; and $|\Omega_{k+1} \cap Q_{j,k}| \leq \frac{1}{2}|Q_j^k|$. Given $\mathcal{D}$ and a sparse family $\mathcal{S} = \{Q_j^k\}_{j,k} \subset \mathcal{D}$, define the positive dyadic operator $\mathcal{A}$ by

$$\mathcal{A} f(x) = \mathcal{A}_{\mathcal{D}, \mathcal{S}} f(x) = \sum_{j,k} f_{Q_j^k} \chi_{Q_j^k}(x)$$

where $f_Q = f_Q f(y)dy$.

For our proof we will use the main result in [9, 10]. Given a Banach function space $X$ and a non-negative function $f$,

$$(2.1) \quad \|T_* f\|_X \leq C(T, n) \sup_{\mathcal{D}, \mathcal{S}} \|\mathcal{A}_{\mathcal{D}, \mathcal{S}} f\|_X,$$

where the supremum is taken over all dyadic grids $\mathcal{D}$ and sparse families $\mathcal{S} \subset \mathcal{D}$. To prove Theorem 1.2 we apply this result with $X = L^p(u)$ or $X = L^{u, \infty}(u)$; it will then suffice to show that our assumptions on $M$ guarantee that $\mathcal{A}_{\mathcal{D}, \mathcal{S}}$ satisfies the corresponding two weight inequalities.

To prove this fact we will use a result by Lacey, Sawyer and Uriate-Tuero [8]. Given a sequence of non-negative constants $\alpha = \{\alpha_Q\}_{Q \in \mathcal{D}}$, define the positive operator

$$T_\alpha f(x) = \sum_{Q \in \mathcal{D}} \alpha_Q f_Q \chi_Q(x).$$

Further, given $R \in \mathcal{D}$ we define the “outer truncated” operator

$$T_\alpha^R f(x) = \sum_{Q \in \mathcal{D} \atop Q \supset R} \alpha_Q f_Q \chi_Q(x).$$

In [8] it was shown that for all $1 < p < q < \infty$, $T_\alpha : L^p(v) \to L^q(u)$ if and only if there exist constants $C_1$ and $C_2$ such that for every $R \in \mathcal{D}$

$$(2.2) \quad \left( \int_{\mathbb{R}^n} T_\alpha^R (v^{1-p'}(\chi_R))(x)^q u(x)dx \right)^{\frac{1}{q}} \leq C_1 \left( \int_R v(x)^{1-p'}dx \right)^{\frac{1}{p}},$$

and

$$(2.3) \quad \left( \int_{\mathbb{R}^n} T_\alpha^R (u \chi_R)(x)^p v(x)^{1-p'}dx \right)^{\frac{1}{p'}} \leq C_2 \left( \int_R u(x)dx \right)^{\frac{1}{q'}}.$$

Furthermore, for $1 < p < q < \infty$, $T_\alpha : L^p(v) \to L^{q, \infty}(u)$ holds if and only if there exists a constant $C_3$ such that for every $R \in \mathcal{D}$, (2.3) holds.

We can apply these results to the operator $\mathcal{A} = \mathcal{A}_{\mathcal{D}, \mathcal{S}}$ where $\mathcal{D}$ and $\mathcal{S}$ are fixed, since $\mathcal{A} = T_\alpha$ with $\alpha_Q = 1$ if $Q \in \mathcal{S}$ and $\alpha_Q = 0$ otherwise. Fix $R \in \mathcal{D}$; to estimate $\mathcal{A}^R$, take the increasing family of cubes $R = R_0 \subsetneq R_1 \subsetneq R_2 \subsetneq \ldots$ with $R_k \in \mathcal{D}$ and $\ell(R_k) = 2^k \ell(R)$. Define $R_{-1} = \emptyset$. Note that
supp $\mathcal{A} \subset \cup_{k \geq 0} R_k$. Then for every non-negative function $f$ and for every $x \in R_k \setminus R_{k-1}$ with $k \geq 0$ we have that

$$0 \leq \mathcal{A} R(f \chi_R)(x) \leq \sum_{j=0}^{\infty} (f \chi_R) R_j \chi_{R_j}(x) = f_R \sum_{j=k}^{\infty} 2^{-jn} \lesssim f_R 2^{-kn} = (f \chi_R) R_k \leq M_\mathcal{D}(f \chi_R)(x).$$

Consequently, for every $x \in \mathbb{R}^n$,

$$0 \leq \mathcal{A} R(f \chi_R)(x) \lesssim M_\mathcal{D}(f \chi_R)(x) \leq M(f \chi_R)(x). \tag{2.4}$$

Inequality (2.4) together with our hypothesis (1.1) implies (2.2) and (2.3). Therefore, we have that $\mathcal{A} : L^p(v) \to L^q(u)$ with constants depending on the dimension, $p, q$ and the implicit constants in (1.1). Therefore, by Lerner’s estimate (2.1) we get $T : L^p(v) \to L^q(u)$ as desired.

For the weak-type estimates we proceed in the same manner, using the fact that (1.3) yields (2.4) and therefore $\mathcal{A} : L^p(v) \to L^{q, \infty}(u)$. This in turn implies, by Lerner’s estimate (2.1) applied to $X = L^{q, \infty}(u)$, that $T : L^p(v) \to L^{q, \infty}(u)$.

**Proof of Theorem 1.3.** Fix $\mathcal{D}$ and a generalized Haar shift operator of complexity $\kappa$. As before we can work with $S_\mathcal{D}$. We can repeat the previous argument except that we want to keep the fixed dyadic structure $\mathcal{D}$. A careful examination of [9, Section 5] shows that, given a Banach function space $X$, we have

$$\|S_\mathcal{D} f\|_X \leq C_n \kappa^2 \sup_\mathcal{D} \|A_{\mathcal{D}, \mathcal{S}} f\|_X, \quad f \geq 0, \tag{2.5}$$

where the supremum is taken over all sparse families $\mathcal{J} \subset \mathcal{D}$. We emphasize that in [9, Section 5] there is an additional supremum over the dyadic grids $\mathcal{D}$. This is because at some places the dyadic maximal operator is majorized by the regular Hardy-Littlewood maximal operator and the latter is in turn controlled by a sum of $A_{\mathcal{D}, \mathcal{S}}$ for $2^n$ dyadic grids $\mathcal{D}_n$. However, keeping $M_\mathcal{D}$ one can easily show that (2.5) holds. Details are left to the interested reader.

Given (2.5), we fix a sparse family $\mathcal{J} \subset \mathcal{D}$ and write $\mathcal{A} = A_{\mathcal{J}, \mathcal{S}}$. Arguing exactly as before we obtain (2.4). Thus, (1.5) implies (2.2) and (2.3) and therefore the result from [8] yields $\mathcal{A} : L^p(v) \to L^q(u)$ with constants depending on the dimension, $p, q$ and the implicit constants in (1.5). Combining this with Lerner’s estimate (2.5) applied to $X = L^{q, \infty}(u)$ we conclude as desired that $S_\mathcal{D} : L^p(v) \to L^{q, \infty}(u)$. We get the weak-type estimate by adapting the above proof in exactly the same way.

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