New Families of Integrable Two-Dimensional Systems with Quartic Second Integrals

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The method introduced in [11] and [12] is extended to construct new families of several-parameter integrable systems, which admit a complementary integral quartic in the velocities. A list of 14 systems is obtained, of which 12 are new. Each of the new systems involves a number of parameters ranging from 7 up to 16 parameters entering into its structures. A detailed preliminary analysis of certain special cases of one of the new systems is performed, aimed at obtaining some global results. We point out twelve combinations of conditions on the parameters which characterize integrable dynamics on Riemannian manifolds as configuration spaces. Very special 7 versions of the 12 cases are interpreted as new integrable motions with a quartic integral in the Poincaré half-plane. A byproduct of the process of solution is the construction of 12 Riemannian metrics whose geodesic flow is integrable with a quartic second integral.

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1. Introduction

Over the last few decades, enormous efforts have been dedicated to answering the fundamental question whether a mechanical system is integrable and how one can find integrals of the motion if they exist? In fact, there is no systematic method for doing that, even for integrals of the simplest functional form, polynomial in the velocities and for the simplest configuration space, the 2D Euclidean plane. The only possible way is to compare one’s system with available tables of known integrable cases in different areas of interest. A fairly complete review of methods and the small list of known integrable potentials in the Euclidean plane with an integral polynomial in the velocities up to 1986 can be found in Hietarinta’s review article [1]. The list of cases added after that date is even smaller. Just a few more cases of systems in the plane were obtained in few works (see e.g. [2–5]).

The matter becomes much harder for integrable systems whose configuration space is more general, e.g., Riemannian 2D manifolds. For a long time the list of those cases consisted of separable (Liouville) systems and a few known cases of rigid body dynamics.

The method introduced by Yehia in [6] has been most successful in constructing new families of integrable two-dimensional mechanical systems with second integrals polynomial in velocities with degree ranging up to six: quadratic [7, 8], cubic [6, 9], and quartic [10, 11]. Most known cases with a quartic integral were recovered as special cases corresponding to certain choices of the parameters from the so-called master system involving 21 arbitrary parameters. Another system with 16 free parameters was obtained in [12]. The results of [11] and [12] have not only restored the famous Kowalevski integrable case of rigid body dynamics [14] and the case, due to Chaplygin, of motion of a body in a liquid [15], but also introduced several new integrable cases that generalized those two cases by adding certain terms to the potential in each case [10–12] and [16].

Yehia’s method consists in two steps. The first is constructing the basic system integrable on its zero-energy level and the second is the interpretation of the energy constant and the standard time variable. This usually gives the freedom to introduce several additional parameters to the structure of the system. More details on this can be found in [10, 12, 17].

The present paper is devoted to construction of integrable systems which admit an integral quartic in velocities. It is a continuation of [10] and [12]. Systematic application of an extension of the method of the last papers resulted in the construction of 14 systems with a quartic invariant, of which 12 systems are new. The new systems involve several parameters, ranging in number up to 16 parameters. Those systems are here classified. Due to the excessive abundance of parameters, the full analysis of the results, finding all simpler integrable systems that come out from them and possible interpretations would require a considerable amount of effort. In the last section of this paper we chose one of the new systems and isolated various sets of conditions on parameters which admit interpretation as integrable particle dynamics on certain Riemannian manifolds. Six cases are expressed explicitly in isometric coordinates and complementary integrals for them are given. Integrable geodesic flows on those manifolds come out as a byproduct of the dynamics. More special versions of the system are given interpretations as dynamics of a particle on simple manifolds, including 7 cases on the pseudo-sphere or, in other geometric terms, the Poincaré half-plane.

For all of the key formulae in this paper, the constancy of the integrals of motion in virtue of the corresponding Lagrangian equations of motion has been checked directly using computer algebra programs.
1.1. Formulation of the problem

Consider the natural conservative mechanical system described by the Lagrangian

\[ L = \frac{1}{2} \sum_{i,j=1}^{2} a_{ij} \dot{q}_i \dot{q}_j - V, \]  

(1.1)

where \( a_{ij}, V \) are certain functions of the generalized coordinates \( q_1, q_2 \) only. Clearly, the system (1.1) admits the energy integral

\[ H = \frac{1}{2} \sum_{i,j=1}^{2} a_{ij} \dot{q}_i \dot{q}_j + V = h, \]  

(1.2)

where \( h \) denotes the arbitrary energy parameter. The most general form of a quartic integral of (1.1) is

\[ I = \sum_{i=0}^{4} C_{4i} q_i \dot{q}_i^4 - i + \sum_{i=0}^{2} C_{2i} q_i \dot{q}_i^2 + C_0, \]  

(1.3)

where \( C_{ij}, C_0 \) are functions in \( q_1, q_2 \).

The problem is to determine the 13 unknown functions \( \{ a_{ij} \}, V, \{ C_{4i} \}, \{ C_{2i} \}, C_0 \) such that \( dI/dt = 0 \) in virtue of the equations of motion derived from the Lagrangian (1.1).

As was shown in [6] and recently in [12], whenever a natural 2D mechanical system admits an integral of motion quartic in velocities, this system can always be reduced in certain isometric coordinates \( \xi, \eta \) and a time parametrization \( \tau \) to a fictitious system described by the Lagrangian

\[ L = \frac{1}{2} (\dot{\xi}^2 + \dot{\eta}^2) + U, \quad U = \Lambda (h - V), \]  

(1.4)

restricted to its zero-energy level

\[ \dot{\xi}^2 + \dot{\eta}^2 + 2U = 0, \]  

(1.5)

where the prime denotes differentiation with respect to the parameter \( \tau \), \( h \) and \( V \) are the energy constant and the potential function of the original system, and \( \Lambda (\xi, \eta) \) is a conformal factor which depends on the metric of the configuration space. The quartic integral is simultaneously written in the following simple form involving only three unknown functions instead of nine in (1.3):

\[ I = \xi^4 + P \xi^2 + Q \xi \eta + R = \text{const}. \]  

(1.6)

All the functions involved are expressed in terms of an auxiliary function \( F(\xi, \eta) \), which is a solution of the nonlinear equation

\[ \frac{\partial^2 F}{\partial \xi \partial \eta} \left( \frac{\partial^4 F}{\partial \xi^4} - \frac{\partial^4 F}{\partial \eta^4} \right) + 3 \left( \frac{\partial^3 F}{\partial \xi^3} \frac{\partial^4 F}{\partial \xi^2 \partial \eta} - \frac{\partial^3 F}{\partial \eta^3} \frac{\partial^4 F}{\partial \eta^2 \partial \xi} \right) + 2 \left( \frac{\partial^2 F}{\partial \xi^2} \frac{\partial^4 F}{\partial \xi^3 \partial \eta} - \frac{\partial^2 F}{\partial \eta^2} \frac{\partial^4 F}{\partial \eta^3 \partial \xi} \right) = 0, \]  

(1.7)

which is called the resolving equation. In terms of \( F \), three of the unknown functions of the problem, namely, \( P, Q \) and \( U \), are expressed as

\[ P = \frac{\partial^2 F}{\partial \xi^2}, \quad Q = -\frac{\partial^2 F}{\partial \xi \partial \eta}, \quad U = -\frac{1}{4} \left( \frac{\partial^2 F}{\partial \xi^2} + \frac{\partial^2 F}{\partial \eta^2} \right), \]  

(1.8)

while the function \( R \) is given, up to an additive constant, by the quadrature

\[ R(\xi, \eta) = -\int Q \frac{\partial U}{\partial \xi} d\eta - \int \left[ 2F \frac{\partial U}{\partial \xi} + Q \frac{\partial U}{\partial \eta} + 2U \frac{\partial Q}{\partial \eta} \right] d\xi, \]  

(1.9)
where \([\eta_0]\) means that the expression in the bracket is computed for \(\eta\) taking an arbitrary constant value \(\eta_0\) (say).

The set of solutions of (1.7) generates all systems of the type (1.4) having an integral of the form (1.6) on the zero level of their energy integral. Affecting all possible conformal mappings of the complex \(\zeta = \xi + i\eta\) plane followed by a general point transformation to the generalized coordinates \(q_1, q_2\) with a suitable change of the time variable, we obtain all systems of the general form on two-dimensional Riemannian (or pseudo-Riemannian) manifolds, having a quartic integral on the zero level of their energy integral, i.e. conditional systems.

The original system can now be expressed in terms of the coordinates \(\xi, \eta\) and the natural time \(t\) by the Lagrangian
\[
L^* = \frac{1}{2} \Lambda \left( \dot{\xi}^2 + \dot{\eta}^2 \right) - V. \tag{1.10}
\]
The quartic integral now takes the form
\[
I^* = \Lambda^4 \dot{\xi}^4 + \Lambda^2 \left( P \dot{\xi}^2 + Q \dot{\xi} \dot{\eta} \right) + R = \text{const.} \tag{1.11}
\]

**Remark.** The resolving equation (1.7) was explicitly found in [11], where a solution was constructed leading to an integrable system termed “the master” system involving 21 arbitrary parameters and admitting an integral quartic in the velocities. It was used to build other integrable systems in [12] and [13]. In [29] the same equation was found to be satisfied by a function that determines the conformal factor of a metric which is periodic in two dimensions (metric on a torus) and admits a quartic integral. The authors of [29] do not notice [11–13] nor the concrete systems explicitly determined in them.

### 1.2. The choice of \(\Lambda\)

To construct systems that are integrable on all energy levels, the functions \(U\) obtained from (1.8) must have a structure in which the energy constant \(h\) of the original system enters linearly as a parameter. Any parameter that appears only as a linear multiplier in a certain term of the potential can be identified as the energy parameter \(h\) and its cofactor as the function \(\Lambda\), and we can proceed through an inverse time transformation to construct a set of general integrable systems valid on arbitrary energy levels, i.e., unconditional systems. In the general situation, however, \(U\) is assumed to have a set of linear multipliers \(h_i\). Then the Lagrangian can be written as (see, e.g., [12, 17])
\[
L = \frac{1}{2} \left( \xi'^2 + \eta'^2 \right) - \sum_{i=1}^{n} h_i U_i(\xi, \eta), \tag{1.12}
\]
which admits the quartic integral (1.6) on the zero level of the energy integral
\[
H = \frac{1}{2} \left( \xi'^2 + \eta'^2 \right) + \sum_{i=1}^{n} h_i U_i(\xi, \eta) = 0. \tag{1.13}
\]
Introducing new arbitrary parameters \(\alpha_i, \beta_i\) into (1.12) by the substitution \(h_i = \alpha_i - \beta_i h\), we get
\[
L = \frac{1}{2} \left( \xi'^2 + \eta'^2 \right) - \left( \sum_{i=1}^{n} \beta_i U_i \right) \left( \sum_{i=1}^{n} \alpha_i U_i \right) - h \tag{1.14}
\]
\[
\text{Making the change of the independent variable } \tau \text{ to the original time } t \text{ according to the relation } \tag{1.15}
\]
\[
t = \int \sum_{i=1}^{n} \beta_i U_i \, d\tau,
\]
we obtain the Lagrangian $L_1 = L^* + h$, where

$$L^* = \frac{1}{2} \left( \sum_{i=1}^{n} \beta_i U_i \right) \left( \dot{\xi}^2 + \dot{\eta}^2 \right) - \sum_{i=1}^{n} \alpha_i U_i \sum_{i=1}^{n} \beta_i U_i,$$  \hspace{1cm} (1.16)

while the energy integral (1.13) is transformed to

$$\frac{1}{2} \left( \sum_{i=1}^{n} \beta_i U_i \right) \left( \dot{\xi}^2 + \dot{\eta}^2 \right) + \sum_{i=1}^{n} \alpha_i U_i = h.$$  \hspace{1cm} (1.17)

The second integral of $L_1$ is obtained from (1.6).

Now, discarding the free additive parameter $h$ from $L_1$ reduces it to $L^*$. Since the zero level of energy integral of $L_1$ is the $h$-level for $L^*$, as determined by (1.17), the Lagrangian $L^*$ admits the second integral (1.11) on its $h$-level of energy. Finally, one can use the energy integral (1.17) to eliminate $h$ from (1.11) and then get a form of the second integral free of the energy parameter.

### 2. New solutions of the resolving equation

The discussion in [11] (see also [12]) showed that, in certain circumstances, the original isometric variables $\xi, \eta$ are not practically suitable for solving the equation, and that the symmetric separation solution discussed in [11] can be more conveniently expressed in the coordinates $p$ and $q$ defined by

$$\xi = \int_{p}^{z} \frac{dz}{\sqrt{a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0}},$$

$$\eta = \int_{q}^{z} \frac{dz}{\sqrt{a_4 z^4 + b_3 z^3 + b_2 z^2 + b_1 z + b_0}},$$  \hspace{1cm} (2.1)

where $a_4, a_3, a_2, a_1, a_0, b_3, b_2, b_1, b_0$ are arbitrary constants. The integrable system constructed in [11], the master system, represents the solution of (1.7) when

$$F(\xi, \eta) = \iint f(\xi) \, d\xi \, d\eta + \iint g(\eta) \, d\eta \, d\eta + \nu pq,$$  \hspace{1cm} (2.2)

where

$$f(\xi) = \frac{\frac{1}{4} b_3 p^3 + 4 A p^2 + 4 C_1 p + 4 C_0}{\sqrt{a_4 p^4 + a_3 p^3 + a_2 p^2 + a_1 p + a_0}},$$

$$g(\eta) = \frac{\frac{1}{4} a_3 q^3 + 4 A q^2 + 4 D_1 q + 4 D_0}{\sqrt{a_4 q^4 + b_3 q^3 + b_2 q^2 + b_1 q + b_0}},$$  \hspace{1cm} (2.3)

where $\nu, A, C_1, C_0, D_1, D_0$ are arbitrary parameters. Another solution of the resolving equation is obtained in [12] assuming $F$ in the form

$$F(\xi, \eta) = \iint f(\xi) \, d\xi \, d\eta + \iint g(\eta) \, d\eta \, d\eta + \nu_1 p^2 q^2,$$  \hspace{1cm} (2.4)

where $\nu$ and $\nu_1$ are arbitrary constants. The above two choices led to the construction of two systems integrable on all energy levels and involving a total number of parameters 21 and 16, respectively. Special cases of the two systems admit an interpretation in particle and rigid body dynamics (see [11, 12] for details).
The main object of the present work is to extend further the method of [11, 12] to construct and classify integrable systems corresponding to the generalized ansatz

\[ F(\xi, \eta) = \int \int f(\xi) \, d\xi d\xi + \int \int g(\eta) \, d\eta d\eta + \sum_{i=2}^{4} \sum_{j=2}^{4} \nu_{i-j} p^{i} q^{i-j}, \tag{2.5} \]

possibly with certain restrictions on the parameters involved in \( f(\xi), g(\eta) \) as given in (2.3).

Substituting (2.5) into equation (1.7) and making use of (2.1), we get a polynomial expression of the sixth degree in \( p, q \) that must vanish. This yields a system of 27 polynomial equations in the 26 parameters of the problem; \( \{A, C_0, C_1, D_0, D_1, a_i (i = 0, \ldots, 4), b_j (j = 0, \ldots, 3), \nu_{ij} (2 \leq i \leq j \leq 4)\} \). The system of polynomial equations is solved using the MAPLE computer algebra package and we obtained 59 distinct solutions, i.e., 59 working combinations of the parameters that may lead to the construction of integrable systems with a quartic second integral. It turned out that for 17 solutions the corresponding integrable systems are separable and then they admit integrals quadratic in velocities and will not be considered further. Moreover, due to the symmetric way in which groups of parameters are associated to the variables \( p, q \), there exist 16 symmetry relations between the remaining 42 solutions. This reduced the number of independent solutions to 26. Finally, it turned out that 13 cases can be obtained by assuming special values of parameters in the other 13. Thus, the final number of different systems is thirteen, the number we are going now to classify and put in a form as simple as possible.

3. The basic integrable systems

In this section we tabulate the 13 basic integrable systems with a quartic integral. Those are the conditional ones, valid on their zero-energy levels. We first note that

1. The first 5 systems could be expressed explicitly in terms of the Cartesian coordinates \( \xi, \eta \) in a Euclidean plane. This possibility was ignored, since the resulting expressions contained rational powers, which makes the potential and the complementary integral more complicated.

2. Each of the two square root terms figuring in systems from number 6 to 13 can be independently written with a negative sign. This is an obvious consequence of the invariance of equation (1.7) under each of the transformations \( \xi \rightarrow i\xi \) and \( \eta \rightarrow i\eta \). Note that the same change should be carried out simultaneously in the Lagrangians and in the corresponding integrals. We have also ignored this possibility. The main reason is that negative signs in the kinetic energy terms mean transition to pseudo-Riemannian spaces contrary to our interest in classical systems.

3. The system No. 3 (given by the Lagrangian (3.3)) already involves an additional arbitrary parameter \( d \), which can be interpreted as an energy parameter. This system is thus unconditionally integrable, but still we can add more parameters to its structure in the next section.

4. The two systems by numbers 11 and 13 were given earlier in [12] and [11], respectively. They are given here only for completeness of the results.

For each case in Table I we give the Lagrangian and the complementary integral valid on its zero-energy level. The systems are classified in Table I according to the number of arbitrary parameters entering into their structure.
Table 1. Basic (conditional) systems

1. \[ L = \frac{1}{2} \left( p^2 p'^2 + q^4 q'^2 \right) - (p^2 + q^2) \left\{ a \left[ (p^2 - q^2)^4 \left( 5 \left( \frac{q}{p} - \frac{p}{q} \right)^2 + \frac{\delta}{p^2 q^2} \right) + 12p^4 - 16p^2 q^2 + 12q^4 - \delta \right] \right\} + b \left( \frac{p}{q} - \frac{q}{p} \right)^2 + \frac{c}{p^2 q^2} \}, \]

\[ I = \left\{ \frac{1}{2} \left( p^4 + 119a p^10 + 27a(5q^4 - \delta)p^6 - \left[ a \left( 195q^8 - 105q^4 - \delta^2 \right) + b \right] p^2 + \frac{aq^2(5q^4 + \delta) + bq^4 + c}{p^2} \right)^2 - 4p^2 q^3 \left\{ a \left[ 15q^8 - 2(39q^4 - \delta)p^4 + 15q^8 + 2q^4 \delta \right] + b \right\} p'q' + 2964a^2 p^{10} - 1800q^2 a^2 p^8 - 4a^2 (12495q^4 - 94\delta)p^{16} + 48a^2 q^2 (39q^4 - \delta)p^4 + 4a \left[ a (17130q^8 + 1240\delta q^4 - 13\delta^2) + 122b \right] p^{12} - 16a q^2 \left[ a(3267q^8 - 126\delta q^4 + 23\delta) + 15b \right] p^{10} + 4a \left[ 17130aq^{12} - 5484a\delta q^6 + (141a\delta^2 - 282b)q^4 - a\delta^3 - 6b\delta - 78c \right] p^8 + 32aq^2 (15aq^8 + 2a\delta q^4 + b)(39q^4 - \delta)p^6 - \left\{ a(49980aq^{16} - 4960a\delta q^{12} - 564(a^2 - 2b)q^6 + 24(a\delta^3 + 106\delta + 10c)q^4 + 4\delta(b\delta + 4c)) - 4b^2 \right\} p^4 - 8q^2 (15aq^8 + 2a\delta q^4 + b^2) p^2 + 4q^4 \left\{ a \left[ 741aq^{16} + 94a\delta q^{12} - (13a\delta^2 - 122b)q^8 - (a\delta^3 + 6b\delta - 78c)q^4 - b\delta^2 - 4c\delta \right] + b^2 \right\} \right\} \]

(3.1)

2. \[ L = \frac{1}{2} \left( p p'^2 + q q'^2 \right) - \left( \frac{1}{q} + \frac{1}{p} \right) \left\{ (p + q)^4 \left[ a \left( 5p^2 + 6pq + 5q^2 \right) + b \right] + c(p + q)^2 + d \right\}, \]

\[ I = \left\{ \frac{1}{2} \left( p^2 p'^2 + 31ap^5 + 5(27aq^2 + b)p^3 + (85aq^4 + 10bq^2 + 3c)p + \frac{5aq^6 + bq^4 + c^2q^2 + d}{p} \right)^2 - 4pq \left\{ a \left( 5p^2 + 3q^2 \right) (3p^2 + 5q^2) \right\} + 2 \left( p^2 + q^2 \right) b + c \right\} p'q' - 1292a^2 p^{10} - 1800a^2 q p^8 - 12a(1085aq^2 + 34b)p^8 - 480aq(17aq^2 + b)p^7 - 8 \left\{ a(4355aq^4 + 380bq^2 + 33c) + 4b^2 \right\} p^6 - 16q \left\{ a(803aq^4 + 98bq^2 + 15c) + 2b^2 \right\} p^5 - 8 \left\{ a(355qa^2 q^2 + 674aqb^2 + 159ac + 20b^2)q^3 + 17ad + 5bc \right\} p^3 - 32 \left( 255a^2 q^6 + 49abq^4 + (17ac + 2b^2)q^2 + bc \right) p^3 - 3 \left\{ 3255a^2 q^6 + 760abq^6 + 2159ac + 20b^2)q^3 + 4(15ad + 7bc)q^2 + 4bd + 3c^2 \right\} p^2 - 8q \left( 225a^2 q^6 + 60abq^6 + 2(15ac + 2b^2)q^3 + 4bcq^2 + c^2 \right) p \]

\[ - 4q^2 \left\{ 323a^2 q^8 + 102abq^6 + 2(33ac + 4b^2)q^4 + 2(17ad + 5bc)q^2 + 4bd + 3c^2 \right\}. \]

(3.2)
3. \[ L = \frac{1}{2} \left( p^4 p^2 + q^4 q^2 \right) - a \left[ \frac{9p^6 + 2q^6}{2p^2} + 30\delta p^2 q^3 + 3\delta^2 \left( 9p^6 + 64q^6 \right) \right] - \frac{b}{p^2} - c (16\delta q^3 + 3p^2) - d. \]

\[ I = \left[ \frac{1}{2} p^4 p^2 + 9a\delta p^6 + 3(10a\delta q^3 + c)p^2 + 4a\delta^2 + \frac{b}{p^2} \right]^2 - 6ap^4 q^3 (3\delta p^4 + 4q^4)p'q' - 16a'\delta^2 p^{10} \]

\[ - \frac{27a}{4} (128a\delta^3 q^3 + 16c\delta^2 + 3a)p^8 - 108a^2 \delta q^3 p^6 - 9a(128a\delta^3 q^8 + 24c\delta q^3 + d)p^4 - 18a^2 q^6 p^2 - 12a q^2 (8a\delta q^6 + c\delta^2 + 4\delta). \]

(3.3)

4. \[ L = \frac{1}{2} (p^2 p^2 + q^2 q^2) - a(p + q)^3 \left[ \frac{(p + q)^4}{pq} + 4(p - q)^2 \right] - \frac{b(p + q)^5}{pq} - c \left( \frac{p^2 + q^2}{p} \right) - d \left( \frac{1}{q} + \frac{1}{p} \right) - e(p + q). \]

\[ I = \left( \frac{1}{2} pp^2 + 11ap^5 + (27aq^2 + 5b)p^3 + (25aq^4 + 10bq^2 + c)p + \frac{a\delta q^4 + bq^4 + c\delta^2 + d}{p} \right)^2 \]

\[ - 4pq \left[ 3ap^3 + 2(5aq^2 + b)p^2 + 3aq^4 + 2bq^2 + c \right] p'q' - 76a^2 p^{10} - 72a q^9 - 4a(231aq^2 + 26b)p^8 - 96aq(5aq^2 + b)p^7 \]

\[ - 4 \left[ a(518aq^4 + 200bq^2 + 15c + e) + 8\delta^2 \right] p^5 - 16 \left[ a(59aq^4 + 26bq^2 + 3c) + 2\delta^2 \right] b^5 \]

\[ - 4 \left[ 518a^2 q^6 + 316abq^4 + (33ac + 15ae + 40\delta^2)q^2 + 10ad + 7bc + be \right] p^4 - 32 \left[ 15a\delta q^6 + 13abq^4 + (5ac + 2b^2)q^2 + bc \right] p^3 \]

\[ - 4 \left[ 231a^2 q^8 + 20abq^6 + (33ac + 15ae + 40\delta^2)q^4 + (12ad + 10bc + 6be)q^2 + 4bd + ce \right] p^2 \]

\[ - 8q \left[ 9a^2 q^8 + 12abq^6 + 2(3ac + 2b^2)q^4 + 4bcq^2 + c^2 \right] p \]

\[ - 4q^2 \left[ 19a^2 q^8 + 26abq^6 + (15ac + ae + 8b^2)q^4 + (10ad + 7bc + be)q^2 + 4bd + ce \right]. \]

(3.4)

5. \[ L = \frac{1}{2} \left( p^4 p^2 + q^4 q^2 \right) - \left( \delta p^2 + q^2 \right) \left\{ a \left[ \frac{4(\delta p^2 + q^2)^2}{q} - \frac{\delta p^4}{p} \right] \right\} + \left( \frac{\delta q^2}{p^2} + \frac{p^2}{q^2} \right) + c \right\} - \frac{d}{p^2} - \frac{e}{q^2}, \]

\[ I = \left( \frac{1}{2} p^4 p^2 + a(4\delta^3 + 1)p^6 + (10a\delta q^4 + c\delta)p^2 + \frac{5q^4 (a\delta q^4 + b) + d}{p^2} \right)^2 - 4a\delta p^4 q^3 (2a\delta q^4 + 2aq^4 + b)p'q' \]

\[ - 32a^2 \delta p^{12} - 32a^2 \delta q^2 p^{10} - 4a \delta \left[ 8aq^4 (4\delta^3 + 1) + 8b\delta^2 + c \right] p^8 - 32a^2 \delta^2 q^2 (2a\delta q^4 + b)p^6 \]

\[ - 4\delta \left[ 8a^2 \delta q^8 (\delta^3 + 4) + 2a^4 (4\delta^3 + 3c\delta + 4b) + b^2 \delta^2 + 4ac\delta + bc \right] p^4 \]

\[ - 8\delta^2 q^2 (2a\delta q^4 + b)^2 p^2 - 4q^4 \left\{ a \left[ 8a\delta^2 q^8 + \delta (c\delta + 8b)q^4 + 4d \right] + \delta c\delta + b^2 \right\}. \]

(3.5)
6. \[ L = \frac{1}{2} \left( \frac{p'^2}{\sqrt{\alpha_2 p^2 + a_0}} + \frac{q'^2}{\sqrt{\alpha_2 q^2 + a_0}} \right) - \mu a_2 p^3 q^2 \left[ 5\mu (3a_2 p^2 + 2a_0) + \mu_1 a_2 p \right] + q \left[ 15\mu a_2 p^3 + 6\mu_1 a_2 p^2 + 2(2\mu_0 + A)p + 4\mu_1 a_0 \right] + \mu a_2 p^4 + \mu_1 a_2 p^3 + \mu A p + C_1 p + C_0 \]
\[ \sqrt{\alpha_2 p^2 + a_0} \]
\[ \mu a_2 p^3 q^2 + p^2 \left[ 5\mu (3a_2 q^2 + 2a_0) + \mu_1 a_2 q \right] + p \left[ 15\mu a_2 q^3 + 6\mu_1 a_2 q^2 + 2(2\mu_0 + A)q + 4\mu_1 a_0 \right] + \mu a_2 q^4 + \mu_1 a_2 q^3 + \mu A q + C_1 q + C_0, \]
\[ \sqrt{\alpha_2 q^2 + a_0} \]
\[ I = \frac{a_2^2}{a_2 p^2 + a_0} \left\{ p'^2/2 + \mu a_2 q^3 p + q^2 \left[ 5\mu (3a_2 p^2 + 2a_0) + \mu_1 a_2 p \right] + q \left[ 15\mu a_2 p^3 + 6\mu_1 a_2 p^2 + (2\mu_0 + A)p + 4\mu_1 a_0 \right] + \mu a_2 p^4 \\
+ \mu_1 a_2 p^3 + \mu A p + C_1 p + C_0 \right\} - \mu_2 a_2 q^3 \left[ 3\mu a_2 q^2 + 2a_2 q(5\mu_0 + \mu_1) + 3\mu a_2 p^2 + 2\mu_1 a_2 p - 10\mu_0 + A \right] p' q' \]
\[ + \mu_2 a_2 q^3 \left[ 5\mu_0 (50A - 210\mu_0) + 27\mu^2 a_2 \right] p^2 + 2a_2 (\mu(50A - 210\mu_0) + 11\mu_1 A) p \]
\[ - 4\mu \left( 5\mu_0^2 (2A^2) + 2\mu_0 (C_1^2 + 8\mu_1^2 a_0) + 20\mu_2 A C_0 + A^2 \right) + 2a_2 (\mu(11A - 20\mu_0) + 5\mu_1 a_2 p^3) + 4a_2 a_2 (15\mu(C_1 - 2\mu_0 a) + 11\mu_1 A) p^2 + 2 \left[ \mu_2 a_2 (3a_2 C_0 - 5\mu_0 a) + 3a_2 a_2 C_1 + A^2 \right] p + C_1 (A - 10\mu_0 a) + 4\mu_1 a_2 C_0 \right] q - \mu_2 a_2 q^3 \left[ 3\mu_0 (50A - 210\mu_0) + 11\mu_1 A \right] p^2 + 4a_2 (5\mu_2 a_0 - 90\mu_0 a + 2A) + 2a_2 (8\mu_1 A + C_1) + A^2 \right\} p^2 \]
\[ 2a_2 (C_1 (A - 10\mu_0 a) + 4\mu_2 a_2 C_0) p \]
(3.6)

7. \[ L = \frac{1}{2} \left( \frac{p'^2}{\sqrt{\alpha_2 p^2 + a_0}} + \frac{q'^2}{\sqrt{\alpha_2 q^2 + b_0}} \right) - \mu a_2 p^3 q^2 \left[ 5\mu (3a_2 p^2 + 2a_0) + \mu_1 a_2 p \right] + q \left[ 15\mu a_2 p^3 + 6\mu_1 a_2 p^2 + 2(2\mu_0 + A)p + 4\mu_1 a_0 \right] + \mu a_2 p^4 + \mu_1 a_2 p^3 + \mu A p + C_1 p + C_0 \]
\[ \sqrt{\alpha_2 p^2 + a_0} \]
\[ \mu a_2 p^3 q^2 + p \left[ 5\mu (3a_2 q^2 + 2b_0) + A q \right] + \mu a_2 q^4 + \mu A q^2 + C_1 q + D_0, \]
\[ \sqrt{\alpha_2 q^2 + b_0} \]
\[ I = \frac{a_2^2}{a_2 p^2 + a_0} \left\{ p'^2/2 + \mu a_2 q^3 p + q^2 \left[ 5\mu (3a_2 p^2 + 2a_0) + \mu_1 a_2 p \right] + q \left[ 15\mu a_2 p^3 + 6\mu_1 a_2 p^2 + (2\mu_0 + A)p + 4\mu_1 a_0 \right] + \mu a_2 p^4 \\
+ \mu_1 a_2 p^3 + \mu A p + C_1 p + C_0 \right\} - 2a_2 \left[ 2\mu_2 (p + q) + A \right] p' q' \]
\[ - 2a_2 (10\mu_2 a_0^2 + 11\mu_2 A p^2 + 2(A^2 + 3\mu_2 a_2 p + A C_0 + 4\mu_0 a) q - \mu_2 a_2 p^4 - 2\mu_2 A p^3 - a_2 \left[ 2\mu_2 (8\mu_0 + A) + C_1 \right] + A^2 \right\} p^2 \]
\[ 2a_2 (C_1 A + 4\mu_2 D_0) p. \]
(3.7)
8. \[ L = \frac{1}{2} \left[ \frac{\dot{p}^2}{\sqrt{2p^2 + a_1 p + a_0}} + \frac{\dot{q}^2}{\sqrt{b q + b_0}} \right] - \frac{\mu q (b_1^2 q^2 + 3b_0 b_1 q + 3b_0^2) (2a_2 p + a_1) + 81 \mu b_1^2 p^3 + 9b_1^2 A p^2 + C_1 p + C_0}{\sqrt{2p^2 + a_1 p + a_0}} \]

\[ - (b_1 q + b_0)^{3/2} (27 \mu b_1 p + A), \]

\[ I = \frac{\dot{p}^2 / 2 + \mu q (b_1^2 q^2 + 3b_0 b_1 q + 3b_0^2) (2a_2 p + a_1) + 81 \mu b_1^2 p^3 + 9b_1^2 A p^2 + C_1 p + C_0}{a_2 p^2 + a_1 p + a_0} \]

\[ - 12 \mu (b_1 q + b_0)^2 p' q' - 72 \mu^2 (b_1 q + b_0)^{3/2} \sqrt{a_2 p^2 + a_1 p + a_0} - 4 \mu \{ q (b_1^2 q^2 + 3b_0 b_1 q + 3b_0^2) [\mu a_2 q (b_1^2 q^2 + 3b_0 b_1 q + 3b_0^2) + C_1] + 9b_1 p (b_1 q + b_0)^3 (27 \mu b_1 p + 2 A) \}. \]

(3.8)

9. \[ L = \frac{1}{2} \left[ \frac{\dot{p}^2}{\sqrt{2p^2 + a_1 p + a_0}} + \frac{\dot{q}^2}{\sqrt{b q^2 + b_0}} \right] - \frac{\mu q (q + \mu_1) (2a_2 p + a_1) + 4 A p^2 + C_1 p + C_0}{\sqrt{a_2 p^2 + a_1 p + a_0}} - 8 \mu p - A p (q + \mu_1) - D_0, \]

\[ I = \frac{\dot{p}^2 / 2 + \mu q (q + \mu_1) (2a_2 p + a_1) + 4 A p^2 + C_1 p + C_0}{a_2 p^2 + a_1 p + a_0} \]

\[ - 4 (2 q + \mu_1) p' q' - 8 \mu^2 (2 q + \mu_1)^2 \sqrt{a_2 p^2 + a_1 p + a_0} \]

\[ - 4 \mu \{ q (q + \mu_1) [a_2 \mu q (q + \mu_1) + 8 A p + C_1] + 16 \mu p^2 + (\mu_1^2 A + 4 D_0) p \}. \]

(3.9)

10. \[ L = \frac{1}{2} \left[ \frac{\dot{p}^2}{\sqrt{2p^2 + a_0}} + \frac{\dot{q}^2}{\sqrt{b q^2 + b_0}} \right] - \frac{\mu q^2 (3a_2 p^2 + 2a_0) + \mu_1 a_2 p q + \mu b_2 p^4 + A p^2 + C_0}{\sqrt{a_2 p^2 + a_0}} \]

\[ - \frac{\mu q^2 (3b_2 q^2 + 2b_0) + \mu_1 b_2 q p + \mu a_2 q^4 + A q^2 + D_0}{\sqrt{b q^2 + b_0}}, \]

\[ I = \frac{\dot{p}^2 / 2 + \mu q^2 (3a_2 p^2 + 2a_0) + \mu_1 a_2 p q + \mu b_2 p^4 + A p^2 + C_0}{a_2 p^2 + a_0} \]

\[ - 2 (\mu_2 q p + \mu_1) p' q' - 2 (2 \mu_2 q p + \mu_1)^2 \sqrt{a_2 p^2 + a_0} \sqrt{b q^2 + b_0} \]

\[ - 4 \mu^2 (3a_2 p^2 + 2a_0) q^4 - 8 a_2 \mu_1 p q^3 - [4 \mu (3b_2 q^4 + 2 A p^2 + C_0) + \mu_1^2 a_2^2] q^4 - 4 \mu p (2 \mu b_2 p^2 + A) q - 4 \mu^2 b_0 p^4 \]

\[ - (\mu_1^2 b_2 + 4 \mu D_0) p^2. \]

(3.10)
New Families of Integrable Two-Dimensional Systems with Quartic Second Integrals

\[ L = \frac{1}{2} \left( p^2 \right) \]

11. \[ L = \frac{1}{2} \left( p^2 + q^2 \right) \]

12. \[ L = \frac{1}{2} \left( \sqrt{a^2 p^2 + a p + a_0} + \sqrt{b^2 q^2 + b q + b_0} \right) \]

13. \[ L = \frac{1}{2} \left( \sqrt{a^2 p^2 + a p + a_0} + \sqrt{b^2 q^2 + b q + b_0} \right) \]

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4. Classification of the unconditional integrable systems

In Table II below we list the most general deformations of the basic integrable systems of the preceding section into their unconditional counterparts, valid for arbitrary initial conditions. Those systems are constructed in the way described in Section 2. For each system we give the number of parameters in its structure, the final form of the Lagrangian $L^*$ (written simply as $L$) and the conformal factor $\Lambda$. The complementary integral $I^*$ will not be written down. It can be obtained for each case from the corresponding integral $I$ of the corresponding basic system by performing three steps:

1. Substituting $p'$ and $q'$ by $\Lambda\dot{p}$ and $\Lambda\dot{q}$, respectively.

2. Changing the energy-like parameters in $I$ according to:

$$\mu = \nu - \alpha h, \quad \mu_1 = \gamma - \beta h, \quad A = h_2 - \alpha_2 h,$$
$$C_1 = h_1 - \alpha_1 h, \quad C_0 = h_0 - \alpha_0 h, \quad D_1 = k_1 - \beta_1 h, \quad D_0 = k_0 - \beta_0 h. \quad (4.1)$$

3. The total energy parameter $h$, appearing in $I$ after the last substitutions, is replaced by the energy integral corresponding to the Lagrangian $L^*$.

Remark. The potential of the system number 9 in Table I involves several parameters in a linear way, but it is a bilinear function in the parameters $\mu_1, A$ and thus one can use at a time either $\mu_1$ or $A$ as an energy-like parameter. Thus, this system generates the two distinct unconditional systems occupying numbers 9 and 10 in Table II. We do not know at the moment if systems 9 and 10 come out as special cases of a more general system involving more parameters than both of them.

Systems occurring in Table II as numbers 12 and 14 were obtained and partially investigated earlier in [12] and [11], respectively. The remaining 12 systems are new.

Each of the 14 unconditional integrable systems pointed out in this section has a Lagrangian of the form

$$L = T - V$$
$$= \frac{1}{2} \Lambda \left( \frac{\dot{p}^2}{\sqrt{P_1}} + \frac{\dot{q}^2}{\sqrt{Q_1}} \right) - V, \quad (4.16)$$

where $\Lambda, V, P_1$ and $Q_1$ are the polynomials of degree 4 or less in $p, q$ given in Table II. For each system this Lagrangian depends on a number of parameters, ranging from 7 to 21. Only a part, 4 to 15, of those parameters enter in the part $T$ of the Lagrangian quadratic in the velocities which may be interpreted as the kinetic energy of the system and hence characterize the configuration space of that system or the metric

$$ds^2 = \Lambda \left( \frac{dp^2}{\sqrt{P_1}} + \frac{dq^2}{\sqrt{Q_1}} \right) \quad (4.17)$$
on that space. In view of the abundance of parameters in the structure of the systems, the full classification of those systems according to the geometry of their configuration spaces or to possible physical interpretations seems formidable at the moment and needs a huge effort for the whole set of systems. Examples of what can be performed on each of the systems:

1. To decide whether a given Lagrangian function describes a viable classical system, its kinetic-energy part $T$ (equivalently, the metric (4.17)) should be positive definite in the velocities. This can happen only in a part of the space of parameters, and mostly requires redefinition of the configuration space to suit various versions of one and the same system.
Table II. Unconditional generalization

1. Number of parameters: 7

\[
L = \frac{1}{2} \Lambda \left( p^4 \dot{p}^2 + q^4 \dot{q}^2 \right) - \frac{1}{\Lambda} \left( p^2 + q^2 \right) \left\{ h_1 \left[ \left( p^2 - q^2 \right)^4 \left[ 5 \left( \frac{p}{q} - \frac{q}{p} \right)^2 + \frac{\delta}{p^2 q^2} \right] + \left( 12p^4 - 16p^2 q^2 + 12q^4 - \delta \right) \right] \\
+ h_2 \left( \frac{p}{q} - \frac{q}{p} \right)^2 + \frac{h_3}{p^2 q^2} \right\}.
\]

\[
\Lambda = (p^2 + q^2) \left\{ \alpha_1 \left( p^2 - q^2 \right)^4 \left[ 5 \left( \frac{p}{q} - \frac{q}{p} \right)^2 + \frac{\delta}{p^2 q^2} \right] + \left( 12p^4 - 16p^2 q^2 + 12q^4 - \delta \right) \right] + \alpha_2 \left( \frac{p}{q} - \frac{q}{p} \right)^2 + \frac{\alpha_3}{p^2 q^2} \right\}.
\] (4.2)

2. Number of parameters: 8

\[
L = \frac{1}{2} \Lambda \left( p^2 \dot{p}^2 + q^2 \dot{q}^2 \right) - \frac{1}{\Lambda} \left\{ \left( \frac{1}{q} + \frac{1}{p} \right) \left\{ \left( p + q \right)^4 \left[ h_1 \left( 5p^2 + 6pq + 5q^2 \right) + h_2 \right] + h_3 (p + q)^2 + h_4 \right\} \right\},
\]

\[
\Lambda = \left( \frac{1}{q} + \frac{1}{p} \right) \left( p + q \right)^4 \left\{ \alpha_1 \left( 5p^2 + 6pq + 5q^2 \right) + \alpha_2 \right\} + a_3 (p + q)^2 + a_4.
\] (4.3)

3. Number of parameters: 9

\[
L = \frac{1}{2} \Lambda \left( p^4 \dot{p}^2 + q^4 \dot{q}^2 \right) - \frac{1}{\Lambda} \left\{ h_1 \left[ \frac{9p^6 + 2q^6}{2p^2} + 30\delta p^2 q^2 + \delta^2 \left( 9p^6 + 64q^6 \right) \right] + \frac{h_2}{p^2} + h_3 \left( 16\delta q^3 + 3p^2 \right) + h_4 \right\},
\]

\[
\Lambda = \alpha_1 \left[ \frac{9p^6 + 2q^6}{2p^2} + 30\delta p^2 q^2 + \delta^2 \left( 9p^6 + 64q^6 \right) \right] + \frac{\alpha_2}{p^2} + \alpha_3 \left( 16\delta q^3 + 3p^2 \right) + \alpha_4.
\] (4.4)

4. Number of parameters: 10

\[
L = \frac{1}{2} \Lambda \left( p^2 \dot{p}^2 + q^2 \dot{q}^2 \right) - \frac{1}{\Lambda} \left\{ h_1 (p + q)^3 \left[ \left( \frac{p + q}{pq} \right)^4 + 4(p - q)^2 \right] + \frac{h_2 (p + q)}{pq} + h_3 \left( \frac{p^2}{q} + \frac{q^2}{p} \right) + h_4 \left( \frac{1}{q} + \frac{1}{p} \right) + h_5 (p + q) \right\},
\]

\[
\Lambda = \alpha_1 (p + q)^3 \left[ \left( \frac{p + q}{pq} \right)^4 + 4(p - q)^2 \right] + \frac{\alpha_2 (p + q)}{pq} + \alpha_3 \left( \frac{p^2}{q} + \frac{q^2}{p} \right) + \alpha_4 \left( \frac{1}{q} + \frac{1}{p} \right) + \alpha_5 (p + q).
\] (4.5)
5. Number of parameters: 11

\[
L = \frac{1}{2} \Lambda \left( p^2 q^2 + q^4 q^2 \right) - \frac{1}{2} \Lambda \left\{ \bar{\delta} p^2 + q^2 \right\} \left\{ h_1 \left[ 4 \left( \frac{\bar{\delta} p^2 + q^2}{p} \right)^2 + \left( \frac{p^3}{q} - \frac{\bar{\delta} p^3}{p} \right)^2 \right] + h_2 \left( \frac{\delta q^2}{p} + \frac{\bar{\delta} p^2 + q^2}{q} \right) + h_3 \right\} + h_4 \left( \frac{\bar{\delta} q^2}{p^2} + \frac{\delta p^2 + q^2}{q^2} \right),
\]

\[
\Lambda = \left( \bar{\delta} p^2 + q^2 \right) \left\{ \alpha_1 \left[ 4 \left( \frac{\bar{\delta} p^2 + q^2}{p} \right)^2 + \left( \frac{p^3}{q} - \frac{\bar{\delta} p^3}{p} \right)^2 \right] + \alpha_2 \left( \frac{\delta q^2}{p} + \frac{p^2}{q^2} \right) + \alpha_3 \right\} - \frac{\alpha_4}{p^2} - \frac{\alpha_5}{q^2} \tag{4.6}
\]

6. Number of parameters: 12

\[
L = \frac{1}{2} \Lambda \left[ \sqrt{a_2 p^2 + a_0} + \frac{\bar{q}^2}{\sqrt{a_2 q^2 + a_0}} \right] - \frac{\nu a_2 q^3 p + q^2 \left[ 5 \nu (3 a_2 p^2 + 2 a_0) + \gamma a_2 p \right] + q \left[ 15 \nu a_2 q^3 p + 6 \gamma a_2 p^2 + (2 \nu a_0 + h_2) p + 4 \gamma a_0 \right] + \nu a_2 p^4 + \gamma a_2 p^3 + h_2 p^2 + h_1 p + h_0}{\sqrt{a_2 p^2 + a_0}}
\]

\[
\Lambda = \frac{\alpha a_2 q^3 p + q^2 \left[ 5 \alpha (3 a_2 p^2 + 2 a_0) + \beta a_2 p \right] + q \left[ 15 \alpha a_2 q^3 p + 6 \beta a_2 p^2 + (2 \alpha a_0 + \alpha_2) p + 4 \beta a_0 \right] + \alpha a_2 p^4 + \beta a_2 p^3 + \alpha_2 p^2 + \alpha_1 p + \alpha_0}{\sqrt{a_2 q^2 + a_0}}
\]

\[
\frac{\alpha a_2 p^3 q + p^2 \left[ 5 \alpha (3 a_2 q^2 + 2 a_0) + \beta a_2 q \right] + p \left[ 15 \alpha a_2 q^3 p + 6 \beta a_2 q^2 + (2 \alpha a_0 + \alpha_2) q + 4 \beta a_0 \right] + \alpha a_2 q^4 + \beta a_2 q^3 + \alpha_2 q^2 + \alpha_1 q + \alpha_0}{\sqrt{a_2 q^2 + a_0}}. \tag{4.7}
\]

7. Number of parameters: 13

\[
L = \frac{1}{2} \Lambda \left[ \sqrt{a_2 p^2 + a_0} + \frac{\bar{q}^2}{\sqrt{a_2 q^2 + b_0}} \right] - \frac{\nu a_2 q^3 p + q \left[ 2 \nu (3 a_2 p^2 + 2 a_0) + h_2 p \right] + \nu a_2 p^3 + h_2 p^2 + h_1 p + h_0}{\sqrt{a_2 p^2 + a_0}}
\]

\[
\Lambda = \frac{\alpha a_2 q^3 p + q \left[ 2 \alpha (3 a_2 p^2 + 2 a_0) + \alpha_2 p \right] + \alpha a_2 p^3 + \alpha_2 p^2 + \alpha_1 p + \alpha_0}{\sqrt{a_2 p^2 + a_0}}
\]

\[
+ \frac{\alpha a_2 p^3 q + q \left[ 2 \alpha (3 a_2 q^2 + 2 b_0) + \alpha_2 q \right] + \alpha a_2 q^3 + \alpha_2 q^2 + \alpha_1 q + \beta_0}{\sqrt{a_2 q^2 + b_0}}. \tag{4.8}
\]
8. Number of parameters: 13

\[
L = \frac{1}{2} \Lambda \left[ \frac{\dot{p}^2}{\sqrt{a_2 p^2 + a_1 p + a_0}} + \frac{\dot{q}^2}{\sqrt{b_1 q + b_0}} \right]
- \frac{1}{\Lambda} \left[ \nu q (b_1^2 q^2 + 3b_0 b_1 q + 3b_0^2) \frac{(2a_2 p + a_1)}{\sqrt{a_2 p^2 + a_1 p + a_0}} + 81 \nu b_1^2 p^3 + 9b_0^2 h_2 p^2 + h_1 p + h_0 - (b_1 q + b_0)^{3/2} (27 \nu b_1 p + h_2) \right],
\]
\[
\Lambda = \alpha q (b_1^2 q^2 + 3b_0 b_1 q + 3b_0^2) \frac{(2a_2 p + a_1) + 81 \nu b_1^2 p^3 + 9b_0^2 \alpha p^2 + \alpha_1 p + \alpha_0}{\sqrt{a_2 p^2 + a_1 p + a_0}} + (b_1 q + b_0)^{3/2} (27 \alpha b_1 p + \alpha_2).
\]

9. Number of parameters: 13

\[
L = \frac{1}{2} \Lambda \left[ \frac{\dot{p}^2}{\sqrt{a_2 p^2 + a_1 p + a_0}} + q^2 \right] - \frac{1}{\Lambda} \left[ \nu q (q + \gamma) (2a_2 p + a_1) + 4 A p^2 + h_3 p + h_0 \frac{\sqrt{a_2 p^2 + a_1 p + a_0}}{\sqrt{a_2 p^2 + a_1 p + a_0}} + 8 \mu p + A q (q + \gamma) + k_0 \right],
\]
\[
\Lambda = \beta \mu q (2a_2 p + a_1) + \alpha_1 p + \alpha_0 \frac{\sqrt{a_2 p^2 + a_1 p + a_0}}{\sqrt{a_2 p^2 + a_1 p + a_0}} + \beta A q + \beta_0.
\]

10. Number of parameters: 14

\[
L = \frac{1}{2} \Lambda \left[ \frac{\dot{p}^2}{\sqrt{a_2 p^2 + a_1 p + a_0}} + q^2 \right] - \frac{1}{\Lambda} \left[ \nu q (q + \mu_1) (2a_2 p + a_1) + 4 h_2 p^2 + h_3 p + h_0 \frac{\sqrt{a_2 p^2 + a_1 p + a_0}}{\sqrt{a_2 p^2 + a_1 p + a_0}} + 8 \nu p + h_2 q (q + \mu_1) + k_0 \right],
\]
\[
\Lambda = \alpha q (q + \mu_1) (2a_2 p + a_1) + 4 \alpha_2 p^2 + \alpha_1 p + \alpha_0 \frac{\sqrt{a_2 p^2 + a_1 p + a_0}}{\sqrt{a_2 p^2 + a_1 p + a_0}} + 8 \alpha p + \alpha_2 q (q + \mu_1) + \beta_0.
\]

11. Number of parameters: 14

\[
L = \frac{1}{2} \Lambda \left[ \frac{\dot{p}^2}{\sqrt{a_2 p^2 + a_0}} + \frac{\dot{q}^2}{\sqrt{b_2 q^2 + b_0}} \right]
- \frac{1}{\Lambda} \left[ \nu q^2 (3a_2 p^2 + 2a_0) + \gamma \alpha_0 q p + \nu b_2 p^4 + h_2 p^2 + h_0 + \nu p^2 (3b_2 q^2 + 2h_0) + \gamma b_2 p^2 + \nu b_2 q^4 + h_2 q^2 + k_0 \right],
\]
\[
\Lambda = \alpha q^2 (3a_2 p^2 + 2a_0) + \beta \alpha_0 p q + \alpha b_2 p^4 + \alpha_2 p^2 + \alpha_0 \frac{\sqrt{a_2 p^2 + a_0}}{\sqrt{b_2 q^2 + b_0}} + \alpha p^2 (3b_2 q^2 + 2h_0) + \beta b_2 p q + \alpha_2 q^2 + \alpha_2 q^2 + \beta_0 \frac{\sqrt{a_2 p^2 + a_0}}{\sqrt{b_2 q^2 + b_0}}.
\]
12. Number of parameters: 15

\[
L = \frac{1}{2} \lambda \left[ \frac{\dot{p}^2}{2 \sqrt{a_4 p^4 + a_2 p^2 + a_0}} + \frac{\dot{q}^2}{2 \sqrt{a_4 q^4 + b_2 q^2 + b_0}} \right] - \frac{1}{\lambda} \left[ \frac{\nu q^2 (4 a_4 p^4 + 3 a_2 p^2 + 2 a_0) + \gamma q p (2 a_4 p^2 + a_2) + \nu b_2 p^4 + h_2 p^2 + h_0}{\sqrt{a_4 p^4 + a_2 p^2 + a_0}} \right. \\
+ \left. \nu p^2 (4 a_4 q^4 + 3 b_2 q^2 + 2 b_0) + \gamma p q (2 a_4 q^2 + b_2) + \nu a_2 q^4 + h_2 q^2 + k_0} \right], \\
\Lambda = \frac{\alpha q^2 (4 a_4 p^4 + 3 a_2 p^2 + 2 a_0) + \beta q p (2 a_4 p^2 + a_2) + \alpha b_2 p^4 + \alpha_2 p^2 + \alpha_0}{\sqrt{a_4 p^4 + a_2 p^2 + a_0}} \\
+ \frac{\alpha p^2 (4 a_4 q^4 + 3 b_2 q^2 + 2 b_0) + \beta p q (2 a_4 q^2 + b_2) + \alpha a_2 q^4 + \alpha_2 q^2 + \beta_0}{\sqrt{a_4 q^4 + b_2 q^2 + b_0}} \tag{4.13}
\]

13. Number of parameters: 16

\[
L = \frac{1}{2} \lambda \left[ \frac{\dot{p}^2}{\sqrt{a_3 p^4 + a_2 p^2 + a_1 p + a_0}} + \frac{\dot{q}^2}{\sqrt{b_2 q^2 + b_0}} \right] \\
- \frac{1}{\lambda} \left[ \frac{4 \nu q^2 (3 a_3 p^2 + 2 a_2 p + a_1)}{\sqrt{a_3 p^4 + a_2 p^2 + a_1 p + a_0}} + 64 \nu b_2 p^3 + 4 h_2 p^2 + h_1 p + h_0 + 16 \nu p (3 b_2 q^2 + 2 b_0) + \nu a_3 q^4 + h_2 q^2 + k_0, \right] \\
\Lambda = \frac{4 a q^2 (3 a_3 p^2 + 2 a_2 p + a_1) + 64 \nu b_2 p^3 + 4 a_2 p^2 + \alpha_1 p + \alpha_0}{\sqrt{a_3 p^4 + a_2 p^2 + a_1 p + a_0}} + \frac{16 \alpha p (3 b_2 q^2 + 2 b_0) + \alpha a_3 q^4 + \alpha_2 q^2 + \beta_0}{\sqrt{b_2 q^2 + b_0}} \tag{4.14}
\]

14. Number of parameters: 21.

\[
L = \frac{1}{2} \lambda \left[ \frac{\dot{p}^2}{\sqrt{a_4 p^4 + a_3 p^3 + a_2 p^2 + a_1 p + a_0}} + \frac{\dot{q}^2}{\sqrt{a_4 q^4 + b_3 q^3 + b_2 q^2 + b_1 q + b_0}} \right] \\
- \frac{1}{\lambda} \left[ \frac{\nu q (4 a_4 p^3 + 3 a_3 p^2 + 2 a_2 p + a_1)}{\sqrt{a_4 p^4 + a_3 p^3 + a_2 p^2 + a_1 p + a_0}} + \frac{\nu p (4 a_4 q^3 + 3 b_3 q^2 + 2 b_2 q + b_1) + \nu a_3 q^3 + h_2 q^2 + k_1 q + k_0}{\sqrt{a_4 q^4 + b_3 q^3 + b_2 q^2 + b_1 q + b_0}} \right], \\
\Lambda = \frac{\alpha q (4 a_4 p^3 + 3 a_3 p^2 + 2 a_2 p + a_1) + \beta b_3 p^3 + \alpha_2 p^2 + \alpha_1 p + \alpha_0}{\sqrt{a_4 p^4 + a_3 p^3 + a_2 p^2 + a_1 p + a_0}} + \frac{\alpha p (4 a_4 q^3 + 3 b_3 q^2 + 2 b_2 q + b_1) + \alpha a_3 q^3 + \alpha_2 q^2 + \beta_1 q + \beta_0}{\sqrt{a_4 q^4 + b_3 q^3 + b_2 q^2 + b_1 q + b_0}} \tag{4.15}
\]
2. From every mechanical system which is integrable on a Riemannian manifold one can obtain an integrable geodesic flow on that manifold as a special case, as established in [17], simply by applying Mupertuis’ principle and setting all coefficients in the potential part to zero values.

3. In virtue of the invariance of Eq. (1.7) under transformations of the type $\eta \rightarrow i\eta$ and $\xi \rightarrow i\xi$ one of the square roots in (4.17) can be replaced by its negative and thus leading to a pseudo-Riemannian counterpart of every case of a Riemannian metric.

4. As an additional step one can calculate the Gaussian curvature of the metric (4.17) according to the formula

$$\kappa = -\frac{1}{2\Lambda} \left\{ \sqrt{P_1} \frac{d}{dp} \left[ \sqrt{P_1} \frac{d}{dp} (\ln \Lambda) \right] + \sqrt{Q_1} \frac{d}{dq} \left[ \sqrt{Q_1} \frac{d}{dq} (\ln \Lambda) \right] \right\}$$

(4.18)

and check for cases of zero or constant curvature. This will pose some conditions on the parameters of the system and the configuration space reduces to one of the simplest manifolds, like a plane, a sphere or a pseudo-sphere. Possible also are cases of indefinite metric. Those may be interpreted as cases on a Minkowski plane or some pseudo-Riemannian manifolds generalizing it.

Those steps were partially performed for the two systems (numbers 12, 14) in [11] and [12], where several new systems living in the Euclidean plane were recovered as special cases corresponding to particular values of the parameters. New generalizations of the well-known integrable cases of Kowalevski, Chaplygin and Goryachev in rigid body dynamics were also obtained as special cases in [10–12].

5. Example. The system 13: Special cases and applications

In the present section we perform a somewhat detailed analysis of one of the new systems, namely, system number 13 of Table II, as an illustration of what one can do for other systems.

The system number 13 (4.14) involves 16 arbitrary parameters in its structure. Of them 12 parameters enter in the metric (4.17) for this system and determine its geometry, which widely changes with the change of numerical values of the parameters.

We rewrite the system 13 in the form

$$L = \frac{1}{2} \Lambda \left[ \dot{\xi}^2 + \dot{\eta}^2 \right]$$

$$= \frac{1}{\Lambda} \left[ 4\nu q^2 (3a_3 p^2 + 2a_2 p^2 + a_1) + 64\nu b_2 p^3 + 4h_2 p^2 + h_1 p + h_0 \right. \right.$$  

$$+ \left. 16\nu p \left( 3b_2 q^2 + 2b_0 \right) + \nu a_3 q^4 + h_2 q^2 + k_0 \right] \right. \right.$$  

$$\sqrt{a_3 p^3 + a_2 p^2 + a_1 p + a_0}$$  

$$\sqrt{b_2 q^2 + b_0}$$  

$$\Lambda = \frac{4\alpha q^2 (3a_3 p^2 + 2a_2 p + a_1) + 64\alpha b_2 p^3 + 4\alpha_2 p^2 + \alpha_1 p + \alpha_0 \sqrt{a_3 p^3 + a_2 p^2 + a_1 p + a_0}}{\sqrt{b_2 q^2 + b_0}}$$  

$$+ \frac{16\alpha p (3b_2 q^2 + 2b_0) + \alpha a_3 q^4 + \alpha_2 q^2 + \beta_0}{\sqrt{b_2 q^2 + b_0}}.$$
in which \( p, q \) are related to \( \xi, \eta \) by

\[
\xi = \int \frac{dp}{\sqrt{a_3 p^3 + a_2 p^2 + a_1 p + a_0}}, \quad \eta = \int \frac{dq}{\sqrt{b_2 q^2 + b_0}} \tag{5.1}
\]

We first note that

a. The generic algebraic curve

\[
X^4 = a_3 p^3 + a_2 p^2 + a_1 p + a_0 \tag{5.2}
\]

has genus 3 and in general the inverse function \( p(\xi) \) cannot be expressed in terms of single valued functions. A change in the situation can happen only through the coalescence of roots of the cubic polynomial in the following manner:

b. Two cases of genus 1.

1. \( a_3 = 0 \). One root of \( P_1 \) coalesces with the infinity. In that case the integral for \( \xi \) in (5.1) can be expressed in terms of the elliptic integrals of the first and second types of modulus \( \frac{1}{\sqrt{2}} \) (see, e.g., [25], formulas (3.181)), so that the inverse function is not single valued, but has a single-valued real branch.

2. \( a_3 \neq 0 \) and \( P_1 \) has two equal roots among three finite roots. In that case the integral for \( \xi \) in (5.1) can be expressed in terms of the elliptic integrals of the first and second types of modulus \( \frac{1}{\sqrt{2}} \). The inverse function is not single valued, but has a single-valued real branch.

c. All other coalescence cases lead to curves of genus 0 or to reducible curves and hence to algebraic expressions of the inverse function. These 4 cases can be put as in the following table:

| Infinite roots | Finite roots | Equal | Simple |
|----------------|-------------|-------|--------|
| 0              | 3           | 0     |        |
| 1              | 2           |       | 0      |
| 2              | 0           |       | 1      |
| 3              | 0           |       | 0      |

5.1. Cases not explicitly expressible in terms of isometric coordinates

To deal with a concrete real and physically plausible case of the system 13, one must fix a configuration space. This involves choosing a pair of intervals \( \delta_1 = [p_1, p_2] \) and \( \delta_2 = [q_1, q_2] \) whose end points are roots of the polynomials \( P_1 \) and \( Q_1 \), respectively, or \( \pm \infty \), so that each polynomial is nonnegative on the relevant interval. The number of possible intervals for each polynomial depends on the relevant combination of coefficients. We shall not consider all possible cases exhaustively, but on a representative basis.
5.1.1. The function \( q(\eta) \)

The polynomial \( Q_1 = b_2q^2 + b_0 \) acquires positive values on three types of intervals:

1. The first type: \( q(\eta) \) on a semiinfinite interval, when \( b_0 > 0, b_2 < 0 \).

On a finite interval we have

\[
\eta - \eta_1 = \int_{-q_1}^{q} \frac{dq}{\sqrt{b_2(q_1^2 - q^2)}}. \tag{5.3}
\]

It is easily seen that \( q \) is periodic in \( \eta \) with period

\[
T^* = 2 \int_{-q_1}^{q_1} \frac{dq}{\sqrt{b_2(q_1^2 - q^2)}} = \frac{4\sqrt{2q_1}}{\sqrt{|b_2|}} K \left( \frac{1}{\sqrt{2}} \right), \quad q_1 = \sqrt{\frac{b_0}{|b_2|}}. \tag{5.4}
\]

2. The second type: \( q(\eta) \) on a semi infinite interval, when \( b_0 < 0, b_2 > 0 \).

In this case the \( \eta \)-axis is mapped by two-to-one mapping on the semiinfinite interval.

3. The third type: \( q(\eta) \) on an infinite interval when \( b_0 > 0, b_2 > 0 \).

In this case the \( \eta \)-axis is mapped one-to-one on the \( p \)-axis.

5.1.2. The function \( p(\xi) \)

On the other hand, the integral for \( \xi \) admits the last three cases when \( a_3 = 0 \). To suppress repetition we suppose that \( a_3 > 0 \). The case \( (a_3 < 0) \) can be rendered to this by changing \( p \) to \( -p \). Denote by \( \Delta \) the discriminant of the cubic polynomial \( P_1 = a_3p^3 + a_2p^2 + a_1p + a_0 \).

Without passing through any coalescence of roots we have two possible cases:

a) \( \Delta \geq 0 \). In that case only one real root \( p_1 \) (say) exists. The motion is possible on the interval \([p_1, \infty)\).

b) \( \Delta < 0 \). In that case we have three real roots \( p_1, p_2, p_3 \) leading to two possible intervals \([p_1, p_2], [p_3, \infty)\),

so that motion is possible on the \( p \)-axis either on a finite or on a semiinfinite interval. Note that, when \( a_3 = 0 \), the integral \( \xi \) becomes form-identical to the integral for \( \eta \) and can be treated as in the following subsection.

1. The first type: \( p(\xi) \) on a finite interval.

Let us examine the behavior of the function \( \xi \) corresponding to motion on a finite interval \([p_1, p_2]\). Let us also fix a value \( \xi_1 \) of \( \xi \) at \( p_1 \), so that \( p(\xi_1) = p_1 \). In the vicinity of the point \((\xi_1, p_1)\) the function \( p(\xi - \xi_1) \) is even. In fact,

\[
\xi - \xi_1 = \int_{p_1}^{p} \frac{dp}{\sqrt{a_3(p - p_1)(p_2 - p)(p_3 - p)}} \\
\approx \frac{1}{\sqrt{a_3(p_2 - p_1)(p_3 - p_1)}} \int_{p_1}^{p} \frac{dp}{\sqrt{p - p_1}} = \frac{4}{3\sqrt[4]{a_3(p_2 - p_1)(p_3 - p_1)}}(p - p_1)^{3/4},
\]

so that \( p - p_1 = A(\xi - \xi_1)^{4/3}, A = \left(\frac{3}{4}\right)^{4/3} \sqrt[4]{a_3(p_2 - p_1)(p_3 - p_1)}. \)
The integral $\xi$ increases monotonically with $p$ varying from $p_1$ to $p_2$, where it takes the value

$$\xi_2 = \xi_1 + \int_{p_1}^{p_2} \frac{dp}{\sqrt{a_3(p-p_1)(p_2-p)(p_3-p)}}$$

(5.5)

The same analysis applies to the vicinity of the point $(\xi_2, p_2)$, where the function $p(\xi - \xi_2)$ is even. $\xi - \xi_2$ changes its sign from negative to positive as $p$ reaches $p_2$ and begins to decrease. As $p$ decreases from $p_2$, the function $\sqrt{a_3(p-p_1)(p_2-p)(p_3-p)}$ takes its values on its negative real branch, so that when $p$ comes to $p_1$, the value of $\xi$ becomes

$$\xi_1 + \int_{p_1}^{p_2} \frac{dp}{\sqrt{a_3(p-p_1)(p_2-p)(p_3-p)}} + \int_{p_2}^{p_1} \frac{dp}{-\sqrt{a_3(p-p_1)(p_2-p)(p_3-p)}}$$

$$= \xi_1 + 2 \int_{p_1}^{p_2} \frac{dp}{\sqrt{a_3(p-p_1)(p_2-p)(p_3-p)}}$$

(5.6)

After every cycle of $p$ on $[p - p_2]$, $\xi$ increases by the same amount $T = 2 \int_{p_1}^{p_2} \frac{dp}{\sqrt{a_3(p-p_1)(p_2-p)(p_3-p)}}$. Thus, we have shown that $p(\xi)$ (and also the function $\sqrt{a_3(p-p_1)(p_2-p)(p_3-p)}$) is a real single-valued periodic function of period $T$. Both the metric of the configuration space and the potential in (5.1) are thus periodic with the same period $T$ in $\xi$.

2. The second type: $p(\xi)$ on a semiinfinite interval.

On such an interval let $p(\xi_3) = p_3$, so that

$$\xi - \xi_3 = \int_{p_3}^{p} \frac{dp}{\sqrt{a_3(p-p_1)(p_2-p)(p_3-p)}}$$

(5.7)

In the same manner as above we show that $p - p_3$ is even in $\xi - \xi_3$. But this time, since the integrand behaves for large $p$ as $\frac{1}{\sqrt[4]{a_3(p_3 - 1)}}$, the integral on the whole interval is divergent.

This means that as $\xi$ increases monotonically from $-\infty$ to $\xi_3$ and then to $\infty$, the real function $p$ decreases from $\infty$ to $p_3$ and then increases to $\infty$, i.e., the semiinfinite interval is covered twice as the image of the $\xi$-axis.

5.2. Some Riemannian subsystems

The next step is to examine the conformal factor and to determine conditions on the parameters which lead to a positive metric. We look at the conformal factor $\Lambda$ in (5.1), namely,

$$\Lambda = \frac{4a_3q^2(3a_3p^2 + 2a_2p + a_1)}{a_3p^4 + a_2p^2 + a_1p + a_0} + \frac{64a_3b_2p^3 + 4a_2p^2 + a_1p + a_0}{a_3p^4 + a_2p^2 + a_1p + a_0}$$

$$+ \frac{16a_3b_2q^2 + 2a_2q + a_3q^4 + a_2q^2 + \beta_0}{\sqrt{b_2q^2 + b_0}}$$

(5.8)

as a function in the isometric variables $\xi, \eta$ for different combinations of interval types.
Assume $a_3 > 0$, $b_0 > 0$, $b_2 < 0$, $p_1 < p_2 < p_3$, $q_1 = \sqrt{-b_0/b_2}$ and let $p$ and $q$ vary on the finite intervals $\delta_1 = [p_1,p_2]$ and $\delta_2 = [-q_1,q_1]$. On those intervals the square root $\sqrt{a_3 p^3 + a_2 p^2 + a_1 p + a_0}$ is nonnegative on $\delta_1$ and attains zero value only at $p_1 + 0$, $p_2 - 0$ $(-q_1 + 0, q_1 - 0)$. The first term in $\Lambda$ can be rewritten as $\frac{4\alpha q^2 dP_1(p)}{\sqrt{P_1(p)}}$, where $P_1(p) = a_3 p^3 + a_2 p^2 + a_1 p + a_0 = a_3(p - p_1)(p_2 - p)(p_3 - p)$. This term tends to $\infty$ as $p \to p_1 + 0$ tends to $-\infty$ as $p \to p_2 - 0$. A necessary condition for $\Lambda$ to be positive is thus $\alpha = 0$. This renders the metric of the system (5.1) to the form

$$ds^2 = \left[\frac{4\alpha q^2 p^2 + \alpha_1 p + \alpha_0}{\sqrt{a_3(p - p_1)(p_2 - p)(p_3 - p)}} + \frac{q_1(\alpha_2 q^2 + \beta_0)}{\sqrt{b_0(q_1^2 - q^2)}}\right] (d\xi^2 + d\eta^2),$$

and thus we can recognize four cases in which this metric becomes Riemannian:

**Case 1:**
If in (5.9) $\alpha_2 > 0$, $\beta_0 > 0$, $16\alpha_0 \alpha_2 - \alpha_1^2 > 0$, then the metric (5.9) is Riemannian on the open rectangular domain $(p_1, p_2) \times (-q_1, q_1)$ in the $pq$-plane. In the $\xi \eta$-plane this metric is defined on a domain $\left(0,\frac{1}{2}T\right) \times \left(0,\frac{1}{2}T^*\right)$ where $T$ and $T^*$ are the periods of $p$ and $q$, respectively, provided we have taken $\xi_1 = p^{-1}(p_1) = 0$ and $\eta_1 = q^{-1}(-q_1) = 0$.

**Case 2:**
If in (5.9) $\beta_0 > 0$, $\alpha_2 = -\frac{\beta_0}{q_1^2} < 0$, and $p_1, p_2$ are roots of the quadric $4\alpha_2 p^2 + \alpha_1 p + \alpha_0$, then (5.9) becomes

$$ds^2 = \frac{\beta_0}{q_1} \left[\frac{4(p - p_1)(p_2 - p)}{q_1 \sqrt{a_3(p - p_1)(p_2 - p)(p_3 - p)}} + \frac{(q_1^2 - q^2)}{\sqrt{b_0(q_1^2 - q^2)}}\right] (d\xi^2 + d\eta^2)$$

$$= \frac{\beta_0}{q_1} \left[\frac{4}{q_1} \sqrt{\frac{(p - p_1)(p_2 - p)}{a_3(p_3 - p)}} + \frac{q_1^2 - q^2}{b_0}\right] (d\xi^2 + d\eta^2).$$

The singularities in the conformal factor in (5.9) are now removed and the metric becomes Riemannian on the closed domain $[p_1, p_2] \times [-q_1, q_1]$ and, consequently, defined and periodic in both directions on the whole $\xi \eta$-plane.

**Cases 3, 4:**
Those are obtained by removing the singularity only from one of the two terms in the conformal factor of (5.9) and thus we form a metric which is defined on an infinite strip, open from both sides and periodic in one direction.

**Remark.** It should be noted that the metric (5.9) and its special case (5.10) are Liouville metrics. Their geodesic flows are integrable with a quadratic complementary integral. However, the dynamical problems corresponding to those metrics are integrable with a genuine quartic integral since in (5.1) the potential remains inseparable.

**Remark.** The metric (5.10) defined on the whole $\xi \eta$-plane is periodic in both directions and hence gives a nontrivial example of a Riemannian metric defined on a 2-torus, on which the dynamics admits a quartic integral (On the status of the problem of constructing integrable geodesic flow on a 2-torus with a polynomial integral, see, e.g., [26–29]).
Table IV: Special cases expressed in isometric coordinates

1. \( a_0 = 1, b_2 = 16, a_3 = a_2 = a_1 = b_0 = 0; p = \xi, q = \eta \)

\[
L = \frac{1}{2} \left[ \alpha_2 (16 \xi^2 + \eta^2) + \alpha \xi (16 \xi^2 + 3 \eta^2) + \alpha_1 \xi + \frac{\beta_0}{\eta^2} + \alpha_0 \right] (\dot{\xi}^2 + \dot{\eta}^2) - \frac{h_2 (16 \xi^2 + \eta^2) + \nu \xi (16 \xi^2 + 3 \eta^2) + h_1 \xi + \frac{k_0}{\eta^2} + h_0}{\alpha_2 (16 \xi^2 + \eta^2) + \alpha \xi (16 \xi^2 + 3 \theta^2) + \alpha_1 \xi + \frac{\beta_0}{\theta^2} + \alpha_0},
\]

\[
I = \left\{ \left[ \alpha_2 (16 \xi^2 + \theta^2) + \alpha \xi (16 \xi^2 + 3 \theta^2) + \alpha_1 \xi + \frac{\beta_0}{\theta^2} + \alpha_0 \right]^2 \dot{\xi}^2 + 32 \mu \xi^3 + 32 A \xi^2 + 2 C_1 \xi + 2 C_0 \right\}^2
- 4 \mu \left[ \alpha_2 (16 \xi^2 + \theta^2) + \alpha \xi (16 \xi^2 + 3 \theta^2) + \alpha_1 \xi + \frac{\beta_0}{\theta^2} + \alpha_0 \right]^2 \theta^3 \dot{\xi} \dot{\theta} - \mu \left\{ \theta^4 \left[ 2 \mu (24 \xi^2 + \theta^2) + 32 A \xi + C_1 \right] + 16 D \xi \right\}. \tag{5.11}
\]

2. \( a_2 = b_2 = 16, a_3 = a_1 = a_0 = b_0 = 0; p = \xi^2, q = \eta^2 \)

\[
L = \frac{1}{2} \left\{ (4 \xi^2 + \theta^2) \left[ \alpha (2 \xi^2 + \theta^2) + \alpha_2 \right] + \frac{\alpha_0}{\xi^2} + \frac{\beta_0}{\theta^2} + \alpha_1 \right\} (\dot{\xi}^2 + \dot{\theta}^2) - \frac{(4 \xi^2 + \theta^2) \left[ \nu (2 \xi^2 + \theta^2) + h_2 \right] + \frac{h_0}{\xi^2} + \frac{k_0}{\theta^2} + h_1}{(4 \xi^2 + \theta^2) \left[ \alpha (2 \xi^2 + \theta^2) + \alpha_2 \right] + \frac{\alpha_0}{\xi^2} + \frac{\beta_0}{\theta^2} + \alpha_1},
\]

\[
I = \left\{ \left[ (4 \xi^2 + \theta^2) \left[ \alpha (2 \xi^2 + \theta^2) + \alpha_2 \right] + \frac{\alpha_0}{\xi^2} + \frac{\beta_0}{\theta^2} + \alpha_1 \right]^2 \dot{\xi}^2 + 2 \mu (8 \xi^4 + \theta^4) + 8 A \xi^2 + 2 C_1 + \frac{2 C_0}{\theta^2} \right\}^2
- 16 \mu \left\{ (4 \xi^2 + \theta^2) \left[ \alpha (2 \xi^2 + \theta^2) + \alpha_2 \right] + \frac{\alpha_0}{\xi^2} + \frac{\beta_0}{\theta^2} + \alpha_1 \right\}^2 \theta^3 \dot{\xi} \dot{\theta} - 4 \mu \left\{ \theta^4 \left[ \mu (48 \xi^4 + 8 \xi^2 \theta^2 + \theta^4) + 16 A \xi^2 - 2 C_1 \right] + 8 D \xi \right\}. \tag{5.12}
\]

3. \( a_3 = 256, b_0 = 1, a_2 = a_1 = a_0 = b_2 = 0; p = \xi^4, q = \eta \)

\[
L = \frac{1}{2} \left\{ (\xi^2 + 4 \eta^2) \left[ \alpha (\xi^2 + 2 \eta^2) + \alpha_2 \right] + \frac{\alpha_1}{\xi^2} + \frac{\alpha_0}{\xi^6} + \beta_0 \right\} (\dot{\xi}^2 + \dot{\eta}^2) - \frac{(\xi^2 + 4 \eta^2) \left[ \nu (\xi^2 + 2 \eta^2) + h_2 \right] + \frac{h_0}{\xi^2} + \frac{k_0}{\theta^2} + h_1}{(\xi^2 + 4 \eta^2) \left[ \alpha (\xi^2 + 2 \eta^2) + \alpha_2 \right] + \frac{\alpha_1}{\xi^2} + \frac{\alpha_0}{\xi^6} + \beta_0},
\]

\[
I = \left\{ \left[ (\xi^2 + 4 \eta^2) \left[ \alpha (\xi^2 + 2 \eta^2) + \alpha_2 \right] + \frac{\alpha_1}{\xi^2} + \frac{\alpha_0}{\xi^6} + \beta_0 \right]^2 \dot{\xi}^2 + 2 (6 \mu \theta^2 + A) \xi^2 + \frac{2 C_1}{\xi^2} + \frac{2 C_0}{\xi^6} \right\}^2
- 16 \mu \left\{ (\xi^2 + 4 \eta^2) \left[ \alpha (\xi^2 + 2 \eta^2) + \alpha_2 \right] + \frac{\alpha_1}{\xi^2} + \frac{\alpha_0}{\xi^6} + \beta_0 \right\}^2 \theta^3 \dot{\xi} \dot{\theta} - 2 \mu \left\{ 2 \xi^4 \left[ \mu (\xi^4 + 8 \xi^2 \theta^2 + 4 \theta^4) + 16 A \theta^2 + 2 D_0 \right] + 16 C_1 \theta^2 \right\}. \tag{5.13}
\]
4. $a_3 = 256$, $b_2 = 16$, $a_2 = a_1 = a_0 = b_0 = 0$; $p = \xi^4$, $q = \eta^2$

$$L = \frac{1}{2} \left\{ (\xi^2 + \theta^2) \left[ \alpha (\xi^2 + \theta^2)^2 + \alpha_2 \right] + \frac{\alpha_1}{\xi^2} + \frac{\alpha_0}{\theta^2} + \frac{\beta_0}{\theta^2} \right\} \left( \dot{\xi}^2 + \dot{\theta}^2 \right) - \frac{(\xi^2 + \theta^2) \left[ \nu (\xi^2 + \theta^2)^2 + h_2 \right] + \frac{h_1}{\xi^2} + \frac{h_0}{\theta^2} + \frac{k_0}{\theta^2}}{(\xi^2 + \theta^2) \left[ \alpha (\xi^2 + \theta^2)^2 + \alpha_2 \right] + \frac{\alpha_1}{\xi^2} + \frac{\alpha_0}{\theta^2} + \frac{\beta_0}{\theta^2}},$$

$$I = \left\{ \left[ (\xi^2 + \theta^2) \left[ \alpha (\xi^2 + \theta^2)^2 + \alpha_2 \right] + \frac{\alpha_1}{\xi^2} + \frac{\alpha_0}{\theta^2} + \frac{\beta_0}{\theta^2} \right]^2 \xi^2 + 2\xi^2 \left[ \mu (\xi^4 + 3\theta^4) + A \right] + \frac{2C_1}{\xi^2} + \frac{2C_0}{\theta^2} \right\}^2 - 16\mu \left\{ (\xi^2 + \theta^2) \left[ \alpha (\xi^2 + \theta^2)^2 + \alpha_2 \right] + \frac{\alpha_1}{\xi^2} + \frac{\alpha_0}{\theta^2} + \frac{\beta_0}{\theta^2} \right\}^2 \xi^3 \theta^3 \dot{\xi} \dot{\theta} - 16\mu \left\{ \alpha (\xi^4 + 2\xi^2 \theta^2 + 3\theta^4) + 2A \right\} + C_1 \theta^4 + D_0 \xi^4 \right\}.$$

(5.14)

5. $a_1 = \frac{256}{81}$, $b_3 = 1$, $a_2 = a_0 = b_2 = 0$; $p = \xi^{4/3}$, $q = \eta$

$$L = \frac{1}{2} \left[ \alpha_2 (9\xi^2 + 4\eta^2) + \alpha_1 \xi^{4/3} + \frac{\alpha (9\xi^2 + 4\eta^2) + \alpha_0}{\xi^{2/3}} + \beta_0 \right] \left( \dot{\xi}^2 + \dot{\eta}^2 \right) - \frac{h_2 (9\xi^2 + 4\eta^2) + h_1 \xi^{4/3} + \nu (9\xi^2 + 2\eta^2) + h_0 + k_0}{a_2 (9\xi^2 + 4\eta^2) + \alpha \xi^{2/3} + \frac{\alpha (9\xi^2 + 2\eta^2) + \alpha_0}{\xi^{2/3}} + \beta_0},$$

$$I = \left\{ \left[ \alpha_2 (9\xi^2 + 4\eta^2) + \alpha_1 \xi^{4/3} + \frac{\alpha (9\xi^2 + 2\eta^2) + \alpha_0}{\xi^{2/3}} + \beta_0 \right]^2 \xi^2 + 18A\xi^2 + 2C_1 \xi^{2/3} + \frac{2(2\mu \theta^2 + C_0)}{\xi^{2/3}} \right\}^2 - 48\mu \left[ \alpha_2 (9\xi^2 + 4\eta^2) + \alpha_1 \xi^{4/3} + \frac{\alpha (9\xi^2 + 2\eta^2) + \alpha_0}{\xi^{2/3}} + \beta_0 \right]^2 \xi^{1/3} \theta \dot{\xi} \dot{\theta} - 4\mu \left\{ 9\xi^{2/3} \left[ 2\xi^2 \theta^2 (8A\theta^2 + D_0) + \mu (9\xi^2 + 8\theta^2) \right] + 8C_1 \theta^2 \right\}.$$

(5.15)
6. \( a_1 = \frac{256}{81}, b_2 = 16, a_3 = a_2 = a_0 = b_0 = 0; \ p = \xi^{4/3}, q = \eta^2 \)

\[
L = \frac{1}{2} \left[ \alpha_2(9\xi^2 + \theta^2) + \alpha_1\xi^{2/3} + \frac{\alpha(81\xi^4 + 27\xi^2\theta^2 + \theta^4) + \alpha_0}{\xi^{2/3}} + \frac{\beta_0}{\eta^2} \right] (\dot{\xi}^2 + \dot{\eta}^2)
\]

\[
I = \left\{ \alpha_2(9\xi^2 + \theta^2) + \alpha_1\xi^{2/3} + \frac{\alpha(81\xi^4 + 27\xi^2\theta^2 + \theta^4) + \alpha_0}{\xi^{2/3}} + \frac{\beta_0}{\eta^2} \right\}^2 \xi^2 + 18A\xi^2 + 2C_1\xi^{2/3} + \frac{2\mu(81\xi^4 + \theta^4) + C_0}{\xi^{2/3}}
\]

\[
-48\mu \left[ \alpha_2(9\xi^2 + \theta^2) + \alpha_1\xi^{2/3} + \frac{\alpha(81\xi^4 + 27\xi^2\theta^2 + \theta^4) + \alpha_0}{\xi^{2/3}} + \frac{\beta_0}{\eta^2} \right]^2 \xi^{1/3}\theta^3\dot{\theta}
\]

\[
-16\mu \left( 9\xi^{2/3} \left[ \xi^{2/3}(2A\theta^4 + D_0) + \mu\theta^4(27\xi^2 + 2\theta^2) \right] + C_1\theta^4 \right).
\]

\[(5.16)\]
5.3. Cases explicitly expressible in terms of isometric coordinates

Here we shall go along a much simpler, but a more particular, path. In order to isolate simpler cases when the metric becomes Riemannian, we proceed as follows.

Those four cases combined with two possible cases of the second integral in (5.1) give a total of eight combinations of parameters of the system 13, which allow explicit expression of the system in terms of the isometric coordinates $\xi, \eta$. It turned out that one of the resulting eight systems is a special case of another one, and one system is a Liouville system. The quartic integral for this last system degenerates to the square of a quadratic one. Thus, we give the remaining six cases in the following table. For each system we give the relevant conditions on the coefficients, the transformation to isometric coordinates and the final form of the Lagrangian of the system.

5.4. Cases of motion on Riemannian manifolds

In the following table we list the conditions on the parameters in each of the above six cases, which make the kinetic energy of each system a smooth positive definite function (so that the metric becomes Riemannian) on some region of the $\xi \eta$ plane and provide that region for each case (the configuration space). A total of twelve cases are obtained. Subcases of the systems integrable on Riemannian manifolds are omitted from this table whenever they degenerate into Liouville systems, so that all the resulting systems admit genuine quartic integrals.

| Case | Conditions | Region where the metric is Riemannian |
|------|------------|--------------------------------------|
| 2    | $a_2 > 0, a > 0, a_1 > 0, a_0 > 0, \beta_0 > 0$ | The positive quadrant of the $\xi \eta$ plane |
|      | $a_2 > 0, a > 0, a_1 > 0, a_0 = 0, \beta_0 > 0$ | The upper half of the $\xi \eta$ plane |
|      | $a_2 > 0, a > 0, a_1 > 0, a_0 = \beta_0 = 0$ | The whole $\xi \eta$ plane |
| 3    | $a_2 > 0, a > 0, a_1 > 0, a_0 > 0, \beta_0 > 0$ | The right half of the $\xi \eta$ plane |
|      | $a_2 > 0, a > 0, a_1 = a_0 = 0, \beta_0 > 0$ | The whole $\xi \eta$ plane |
| 4    | $a_2 > 0, a > 0, a_1 > 0, a_0 > 0, \beta_0 > 0$ | Positive quadrant of the $\xi \eta$ plane |
|      | $a_2 > 0, a > 0, a_1 = a_0 = 0, \beta_0 > 0$ | The upper half of the $\xi \eta$ plane |
|      | $a_2 > 0, a > 0, a_1 > 0, a_0 > 0, \beta_0 = 0$ | The right half of the $\xi \eta$ plane |
|      | $a_2 > 0, a > 0, a_1 = a_0 = \beta_0 = 0$ | The whole $\xi \eta$ plane |
| 5    | $a_2 > 0, a > 0, a_1 > 0, a_0 > 0, \beta_0 > 0$ | The right half of the $\xi \eta$ plane |
| 6    | $a_2 > 0, a > 0, a_1 > 0, a_0 > 0, \beta_0 > 0$ | The positive quadrant of the $\xi \eta$ plane |
|      | $a_2 > 0, a > 0, a_1 > 0, a_0 > 0, \beta_0 = 0$ | The right half of the $\xi \eta$ plane |

5.5. Integrable geodesic flows on Riemannian manifolds

There has been great interest in the last decades in constructing metrics whose integrable flow admits an integral polynomial in momenta on certain Riemannian manifolds. Most of the interest was dedicated to the cases of a plane, sphere and a torus (see, e.g., [26, 27] and references...
therein). As said above and explained in [17], the problem of constructing integrable geodesic flow on a manifold is a quite special case of the corresponding integrable dynamical problem on the same manifold. The latter reduces to the first when the potential vanishes.

The twelve systems of Section 5.4 generate twelve Riemannian metrics, on each of which the geodesic flow is integrable with a quartic integral.

In fact, each of the metrics is given by

\[ ds^2 = 2T dt^2 = \Lambda (d\xi^2 + d\eta^2), \]  

(5.17)

where \( \Lambda \) is the relevant conformal factor for each case. Geodesics on each of those metrics are trajectories of the relevant mechanical system with zero potential, i.e., with all five parameters \( \nu = h_0 = h_1 = h_2 = k_0 = 0 \).

For example, the metric corresponding to the case 2a of Table V can be written as

\[ ds^2 = \left( 4\xi^2 + \eta^2 \right) \left[ \nu (2\xi^2 + \eta^2) + \alpha_0 \xi^{-2} + \beta_0 \eta^{-2} + \alpha_1 \right] (d\xi^2 + d\eta^2). \]  

(5.19)

5.6. Integrable Cases of motion in the Poincaré half-plane or the pseudo-sphere

Poincaré’s half-plane is one of the models of the hyperbolic (or Lobachevsky) plane. It has the property that its Gaussian curvature is constant and negative, usually normalized to \(-1\). The Poincaré half-plane can be realized (embedded) in 3D Euclidean space by means of the Beltrami pseudo-sphere, but it is simpler for the purpose of this paper to remain in the 2D setting.

One can directly notice that five of the six cases of Table IV cover special cases of motion on the Poincaré half-plane (or on a pseudo-sphere). Those are characterized by a conformal factor \( \Lambda \) that takes one of the forms \( \xi^2, \eta^2 \) or \( \frac{1}{\xi^2} + \frac{1}{\eta^2} \). The resulting 9 cases integrable on the Poincaré half-plane are summed in the following table. A change of variables is performed in some cases in order to keep a unified form of the metric and to facilitate comparison of cases.

5.7. Integrable motions on the plane

It is easy also to recognize that four of the six cases of Table IV above admit special cases with constant conformal factor in the metric. Those cases are interpreted as cases of motion in the plane:

1. \( \alpha_2 = \alpha_1 = \alpha = \beta_0 = 0 \) and \( \alpha_0 = 1 \) in (5.11) gives:

\[ V = \nu x \left( 16x^2 + 3y^2 \right) + h_2 \left( 16x^2 + y^2 \right) + h_1 x + \frac{k_0}{y^2}. \]

A shift in the coordinate \( x \) can eliminate the term \( h_1 x \) and then the potential becomes a special version of the potential (3.4.9) of [1].

2. \( \alpha_2 = \alpha_0 = \alpha = \beta_0 = 0 \) and \( \alpha_1 = 1 \) in (5.12) gives:

\[ V = (4x^2 + y^2) \left[ \nu (2x^2 + y^2) + h_2 \right] + \frac{h}{x^2} + \frac{k_0}{y^2}. \]

This potential is one parameter less than the potential (83) in [11].
### Table VI Cases integrable on the Poincaré half-plane

| Case | Conditions and Lagrangian |
|-------|---------------------------|
| 1     | $\alpha_2 = \alpha_1 = \alpha_0 = \alpha = 0, \beta_0 = 1$ |
|       | $L = \frac{1}{2\eta^2} \left( \dot{\xi}^2 + \eta^2 \right) - \eta^2 \left[ h_2 (16 \xi^2 + \eta^2) + \nu \xi (16 \xi^2 + 3 \eta^2) + h_1 \xi + h_0 \right]$ |
|       | $I = \left[ \frac{\dot{\xi}^2}{\eta^2} + 4 \nu \xi (\theta^2 + 8 \xi^2) + 32 h_2 \xi^2 + 2 h_1 \xi + 2 h_0 \right]^2 + \frac{4 \nu}{\theta^2} (2 \dot{\xi} \dot{\theta} - \theta \dot{\xi}) - \nu \theta^4 \left[ 2 \nu (\theta^2 + 8 \xi^2) + 16 h_2 \xi + h_1 \right]$ |
| 2     | $\alpha_2 = \alpha_1 = \alpha = \beta_0 = 0, \alpha_0 = 1$ |
| a     | $L = \frac{1}{2\eta^2} \left( \dot{\xi}^2 + \eta^2 \right) - \eta^2 \left\{ (4 \eta^2 + \xi^2) \left[ \nu (2 \eta^2 + \xi^2) + h_2 \right] + \frac{h_0}{\xi^2} + h_1 \right\}$ |
|       | $I = \left[ \frac{\dot{\xi}^2}{2\eta^2} - \xi^2 (2 \nu \theta^2 - h_2 - h_0) \right]^2 + \frac{4 \nu \xi^2 \dot{\xi} \dot{\theta}}{\theta^2} \left( 2 \dot{\theta} \xi - \xi \dot{\theta} \right) - \nu \theta^4 \left[ \nu \xi^4 (4 \theta^2 + \xi^2)^2 + 2 h_1 \xi^4 - 8 h_0 \theta^2 \right]$ |
| b     | $\alpha_2 = \alpha_1 = \alpha = 0, \alpha_0 = \beta_0 = 1$ |
|       | $L = \frac{1}{2\eta^2} \left( \dot{\xi}^2 + \eta^2 \right) - \eta^2 \left\{ (4 \eta^2 + \xi^2) \left[ \nu (2 \eta^2 + \xi^2) + h_2 \right] + \frac{h_0}{\xi^2} + h_1 \right\}$ |
|       | $I = \left[ \frac{\dot{\xi}^2}{2\eta^2} + \nu \left( \theta^4 + 4 \theta^2 \xi^2 + 8 \xi^4 \right) + 4 h_2 \xi^2 + \frac{h_0}{\xi^2} + h_1 \right]^2 - \frac{4 \nu \theta^4}{\theta^2} \left( \theta \dot{\xi} - \xi \dot{\theta} \right) - \nu \theta^4 \left[ \nu \left( \theta^2 + 4 \xi^2 \right)^2 + 8 h_2 \xi^2 + 2 h_1 \right]$ |
| c     | $\alpha_2 = \alpha_0 = \alpha = \beta_0 = 0, \alpha_0 = \beta_0 = 1$ |
|       | $L = \frac{1}{2\eta^2} \left( \dot{\xi}^2 + \eta^2 \right) - \eta^2 \left\{ \left( 5 + \frac{3 \xi}{\sqrt{\eta^2 + \xi^2}} \right) \left[ \nu \left( 3 \sqrt{\eta^2 + \xi^2} + \xi \right) + h_2 \right] + \frac{h_1 \eta^2 + h_0 \xi}{\sqrt{\eta^2 + \xi^2}} \right\}$ |
|       | $I = \left\{ \frac{\dot{\xi}}{\theta} \left( \xi \dot{\theta} - \theta \dot{\xi} \right) + \nu \left( 11 \theta^2 + 32 \xi^2 \right) + 10 h_2 \xi + h_1 + \frac{\nu \xi (29 \theta^2 + 32 \xi^2) + 3 h_2 (\theta^2 + 2 \xi^2) + h_1 \xi - h_0}{\sqrt{\xi^4 + \theta^2}} \right\}^2$ |
|       | $+ \left\{ \frac{16 h_2 \xi}{\theta^2} \left( \theta \dot{\xi} - \xi \dot{\theta} \right) \left( \sqrt{\xi^4 + \theta^2} - \xi \dot{\theta} \right) - 16 \nu \left( \sqrt{\xi^4 + \theta^2} - \xi \right)^2 \left[ \nu \left( 30 \xi \sqrt{\xi^4 + \theta^2} + 34 \xi^2 + 25 \theta^2 \right) + 8 h_2 \left( \sqrt{\xi^4 + \theta^2} + \xi \right) + 2 h_1 \right] \right\}$ |
| 3     | $\alpha_2 = \alpha_0 = \alpha = \beta_0 = 0, \alpha_1 = 1$ |
|       | $L = \frac{1}{2\eta^2} \left( \dot{\xi}^2 + \eta^2 \right) - \eta^2 \left\{ (\eta^2 + 4 \xi^2) \left[ \nu (\eta^2 + 2 \xi^2) + h_2 \right] + \frac{h_0}{\eta^6} \right\}$ |
|       | $I = \left\{ \frac{\dot{\xi}^2}{2\theta^4} + 4 \xi^2 \left[ \nu (\theta^2 + 2 \xi^2) + h_2 \right] + 2 h_0 \right\}^2 + \nu \left( \theta \dot{\xi} - 2 \xi \dot{\theta} \right)^2 + \frac{8 h_0 \xi^2}{\theta^2}$ |
\[ \begin{array}{|c|c|}
\hline
\text{a} & \alpha_2 = \alpha_0 = \alpha = \beta_0 = 0, \quad \alpha_1 = 1 \\
\hline
L & = \frac{1}{2\eta^2} \left( \dot{\xi}^2 + \eta^2 \right) - \eta^2 \left\{ (\eta^2 + \xi^2) \left[ \nu (\eta^2 + \xi^2)^2 + h_2 \right] + \frac{k_0}{\xi^2} + \frac{h_0}{\eta^2} \right\} \\
I & = \left\{ \frac{\dot{\xi}^2}{2\theta^4} + \xi^2 \left[ \nu (\theta^2 + \xi^2)^2 + h_2 \right] + \frac{k_0}{\xi^2} \right\}^2 + \frac{2\nu \xi^2}{\theta^2} \left[ \left( \theta \dot{\xi} - \xi \dot{\theta} \right)^2 + \frac{2h_0 \xi^2}{\theta^2} \right] \\
\hline
\text{b} & \alpha_2 = \alpha_1 = \alpha_0 = \alpha = 0, \quad \beta_0 = 1 \\
L & = \frac{1}{2\eta^2} \left( \dot{\xi}^2 + \eta^2 \right) - \eta^2 \left\{ (\xi^2 + \eta^2) \left[ \nu (\xi^2 + \eta^2)^2 + h_2 \right] + \frac{h_1}{\xi^2} + \frac{h_0}{\xi^2} \right\} \\
I & = \left\{ \frac{\dot{\theta}^2}{2\xi^4} + \nu \theta^2 (3\xi^4 + 2\xi^2 \dot{\theta}^2 + \dot{\theta}^4) + h_2 \theta^2 + \frac{h_1}{\theta^2} + \frac{h_0}{\theta^2} \right\}^2 - \frac{2\nu^3 \xi^2}{\theta^2} (2\xi \dot{\theta} - \theta \dot{\xi}) - 4\nu^4 \left[ \nu (\theta^4 + 2\theta^2 \xi^2 + \xi^4) + h_2 \xi^4 + h_1 \right] \\
\hline
\text{c} & \alpha_2 = \alpha_0 = \alpha = 0, \quad \beta_0 = \alpha_1 = 1 \\
L & = \frac{1}{2\eta^2} \left( \dot{\xi}^2 + \eta^2 \right) - \eta^2 \left\{ \nu (\eta^2 + \xi^2)^2 + h_2 \right\} - \frac{h_1 \xi + h_0 \eta^4 \left( \sqrt{\eta^2 + \xi^2} - \xi \right)^3 + k_0 \xi}{\sqrt{\eta^2 + \xi^2}} \\
I & = \left\{ \frac{\dot{\xi}}{\theta^3} \left( \theta \dot{\theta} + \xi \dot{\xi} \right) + 2\xi \left[ \nu (\theta^2 + \xi^2) + h_2 \right] + \frac{h_0 \left( \sqrt{\xi^2 + \theta^2} - \xi \right)^2 - \theta^4 (h_1 + k_0)}{\theta^4 \sqrt{\xi^2 + \theta^2}} \right\}^2 \\
& + \frac{2\nu}{\theta^2} \left[ \left( \theta \dot{\xi} - \xi \dot{\theta} \right)^2 + \frac{h_0}{\theta^2} \left( \sqrt{\xi^2 + \theta^2} - \xi \right)^4 \right] \\
\hline
\text{d} & \alpha_2 = \alpha_1 = \alpha_0 = \alpha = 0, \quad \beta_0 = 1 \\
L & = \frac{1}{2\eta^2} \left( \dot{\xi}^2 + \eta^2 \right) - \eta^2 \left[ h_2 (9\xi^2 + \eta^2) + h_1 \xi^{2/3} + \frac{\nu (81\xi^4 + 27\xi^2 \eta^2 + \eta^4) + h_0}{\xi^{2/3}} \right] \\
I & = \left\{ \frac{\dot{\xi}^2}{2\theta^{4/3}} + \nu \left( \theta^4 + 81\xi^4 \right) + h_0 \right\}_{\xi^{2/3}} + 9h_2 \xi^2 + \xi^{2/3} \left( 18\nu \theta^2 \xi^{2/3} + h_1 \right) \right\}^2 + \frac{6\nu \xi^{1/3} \dot{\theta}}{\theta^2} \left( 3\xi \dot{\theta} - 2\theta \dot{\xi} \right) \\
& - 4\nu \theta^3 \left( 9\xi^{2/3} \left[ \nu (2\theta^2 + 9\xi^2) + h_2 \xi^{2/3} \right] + h_1 \right) \\
\hline
\end{array} \]
3. \( \alpha_2 = \alpha_1 = \alpha_0 = \alpha = 0 \) and \( \beta_0 = 1 \) in (5.13) gives:

\[
V = (\xi^2 + 4\eta^2) [\nu(\xi^2 + 2\eta^2) + h_2] + \frac{h_1}{\xi^2} + \frac{h_0}{\xi^6},
\]

which is one parameter less than the potential obtained in [20].

4. \( \alpha_2 = \alpha_1 = \alpha_0 = \alpha = 0, \beta_0 = 1, \nu = 1/2 \) and \( \xi = x, \eta = y \) in (5.15) gives:

\[
V = 9x^{4/3}/2 + (y^2 + h_0) x^{-2/3} + h_1 x^{2/3} + h_2 (9x^2 + 4y^2),
\]

which is one parameter less than the generalization obtained in [20] and [22] of the original Holt-type potential [18, 19]

\[
V = 9x^{4/3}/2 + y^2x^{-2/3}.
\]

Thus, the four cases, being extremely special cases of the system 13, do not give new integrable cases.

5.8. Using the constant curvature criterion

Although it is not hard to calculate the Gaussian curvature of the configuration space of the system according to the formula (4.18) using the expression for \( \Lambda \) given in (4.14), we do not write it down for space consideration. The Gaussian curvature has been equated to a constant value \( \nu \), resulting in 19 sets of conditions on the 12 parameters of the system and the constant \( \nu \).

It turns out that for 9 sets of conditions \( \nu \) assumes arbitrary nonzero constant values (cases of integrable motion on spaces with constant curvature). Here a common situation is noted; the kinetic energies are positive definite (indicating a Riemannian configuration space) only of integrable motion on spaces with constant curvature). Here a common situation is noted; the kinetic energies are positive definite (indicating a Riemannian configuration space) only of integrable motion on spaces with constant curvature).

Although it is not hard to calculate the Gaussian curvature of the configuration space of the system according to the formula (4.18) using the expression for \( \Lambda \) given in (4.14), we do not write it down for space consideration. The Gaussian curvature has been equated to a constant value \( \nu \), resulting in 19 sets of conditions on the 12 parameters of the system and the constant \( \nu \).

It turns out that for 9 sets of conditions \( \nu \) assumes arbitrary nonzero constant values (cases of integrable motion on spaces with constant curvature). Here a common situation is noted; the kinetic energies are positive definite (indicating a Riemannian configuration space) only of integrable motion on spaces with constant curvature).

Thus, the four cases, being extremely special cases of the system 13, do not give new integrable cases.

5.8.1. One more integrable case on the Poincaré half-plane

We now give for this case the conditions on the parameters of the system 13, the final form of the integrable Lagrangian \( L \) and the second integral \( I \) quartic in velocities:

\[
b_0 = \alpha_2 = \alpha_1 = \alpha_0 = \alpha = 0, \beta_0 = 4, b_2 = 16; q = \eta^2.
\]

\[
L = \frac{1}{2\eta^2}\left(\xi^2 + \eta^2\right)
\]

\[
-\eta^2\left\{\frac{16\nu^2(3a_3p^2 + 2a_2p + a_1) + 4096\nu^3 + 16h_2p^2 + 4h_1p + h_0}{\sqrt{a_3p^3 + a_2p^2 + a_1p + a_0}} + \frac{\nu(a_3\eta^4 + 768p) + h_2}{1}
\right\},
\]

\[
I = \left\{\eta^{-4}p^2/2 + 16\nu\left[\eta^4(3a_3p^2 + 2a_2p + a_1) + 256p^3\right] + 16h_2p^2 + 4h_1p + h_0 + 512\nu^2p\eta^2\right\}^2
\]

\[
+256\nu^{-2}\eta(2p\eta - \eta)p - 32768\nu^2\eta^6 \sqrt{a_3p^3 + a_2p^2 + a_1p + a_0}
\]

\[
-256\nu^4 \left\{4\nu\left[\eta^4(2a_3p + a_2) + 256p^2\right] + 4h_2p + h_1\right\},
\]

(5.20)
As seen from the kinetic energy part of the Lagrangian, the system just described characterizes a several-parameter family of integrable motions in the Poincaré half-plane. The potential is expressed in terms of \( \eta \) and \( p \), but one has to invert (5.21) to obtain \( p = p(\eta) \). This situation has been met in Section 5.3 above, and if we look at special cases of explicit inversion we find that (5.20) unifies and includes as special versions the four cases under numbers 1, 2b, 4b and 6 in Table VI.

5.8.2. Cases in the Euclidean plane

For the other 10 sets of conditions with zero curvature it turned out that all but two of the corresponding essentially different integrable plane systems are listed in Section 5.7. For each of these two systems, we give the conditions on the system parameters, the coordinate transformations from \( p, q \) to polar coordinates \( r, \theta \) in the Euclidean plane and the integrable potential \( V \):

1. \( a_2 = a_1 = a_0 = b_0 = \alpha_2 = \alpha_1 = \alpha_0 = \beta_0 = 0, \ a_3 = 256, \ b_2 = 16, \ \alpha = \frac{1}{4} \);
   \[ p = r \cos^4 \theta, \ \ q = \sqrt{r} \sin^2 \theta. \]
   \[ V = \frac{h_2}{r} + \frac{k_0}{r^2 (1 - \cos \frac{\theta}{2})} + \frac{h_1}{r^2 (1 + \cos \frac{\theta}{2})} + \frac{h_0}{r^3 (1 + \cos \frac{\theta}{2})^3}. \]  
   (5.22)

This potential contains one parameter less than the potential of Table IV in [5].

2. \( a_2 = a_1 = a_0 = b_0 = \alpha_1 = \alpha_0 = \alpha = \beta_0 = 0, \ b_2 = \alpha_2 = 1, \ a_3 = 16 \);
   \[ p = r^2 \cos^4 \theta, \ \ q = r \sin^2 \frac{\theta}{2}. \]
   \[ V = \nu r^2 + \frac{k_0}{r^2 (1 - \cos \theta)} + \frac{h_1}{r^2 (1 + \cos \theta)} + \frac{h_0}{r^4 (1 + \cos \theta)^3}. \]  
   (5.23)

This potential contains one parameter less than the potential of Table III in [5].

6. Conclusion

In this paper the following main results are obtained:

1. A list of 13 natural systems integrable on their zero-energy level is given in Table I, together with their complementary integrals.

2. A list of 14 natural systems integrable unconditionally (for all initial conditions) is given in Table II. Among them 12 systems are new. The other two were presented earlier in 2006 and 2012. Each of the new systems involves a number of free parameters ranging from 7 to 16.

3. As an example of what can be done in the analysis of the new systems we have investigated special cases of one of them, namely, number 13, which turned out to be extremely rich. We have found the following special cases of it:

   (a) Several cases of integrable motions on Riemannian manifolds are pointed out.
   (b) Six integrable systems completely expressed in terms of the isometric coordinate.
(c) Twelve several-parameter systems integrable with a quartic integral on various Riemannian manifolds.

(d) Each one of the last twelve systems contains as a special case the integrable geodesic flow on its Riemannian configuration space.

(e) Seven new systems integrable on the Poincaré half-plane are given. Six of them are given explicitly in terms of the Cartesian coordinates of the plane, tabulated in Table VI.

(f) Six special cases describing motions in the Euclidean plane are also found, but they turned out to be special cases of already known integrable cases.

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