Deriving a Simple Gradual Security Language

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Abstract
Abstracting Gradual Typing (AGT) is an approach to systematically deriving gradual counterparts to static type disciplines (Garcia et al. 2016). The approach consists of defining the semantics of gradual types by interpreting them as sets of static types, and then defining an optimal abstraction back to gradual types. These operations are used to lift the static discipline to the gradual setting. The runtime semantics of the gradual language then arises as reductions on gradual typing derivations.

To demonstrate the flexibility of AGT, we gradualize \( \lambda_{sec} \) (Zdancewic 2002), the prototypical security-typed language, with respect to only security labels rather than entire types, yielding a type system that ranges gradually from simply-typed to securely-typed. We establish noninterference for the gradual language, called \( \lambda_{\sec} \), using Zdancewic's logical relation proof method. Whereas prior work presents gradual security cast languages, which require explicit security casts, this work yields the first gradual security source language, which requires no explicit casts.

1. Introduction
Gradual typing has often been viewed as a means to combine the agility benefits of dynamic languages, like Python and Ruby with the reliability benefits of static languages like OCaml and Scala. This paper, in a line of work on the foundations of gradual typing, explores the idea that static and dynamic are merely relative notions.

This relativistic view of gradual typing is not new. Work on gradual information flow security by Disney and Flanagan (2011) and Fennell and Thiennam (2013) develop languages where only information-flow security properties are subject to a mix of dynamic and static checking. Baños Schweter et al. (2014) develop a language where only computational effect capabilities are gradually quantized. In each of these cases, the "fully-dynamic" corner of the gradual language is not dynamic at all by typical standards, but rather simply typed. However, those languages support seamless migration toward a more precise typing discipline that subsumes simple typing.

To explore this notion, we revisit the idea of gradual information-flow security. Our tool of inquiry is a new approach to the foundations of gradual typing called Abstracting Gradual Typing (AGT) (Garcia et al. 2016). AGT is a technique for systematically deriving gradual type systems by interpreting gradual types as sets of static types. That work developed a traditional gradual type system with subtyping, introducing an unknown type \(?\). But AGT was directly inspired by Baños Schweter et al. (2014), who used an early version of these techniques to gradualize only effects. However, they develop the dynamic semantics of gradual effects in the traditional ad hoc fashion. Here, we bring the approach full circle, deriving a complete static and dynamic semantics for a gradual counterpart to the \( \lambda_{sec} \) language of Zdancewic (2002).

In their simplest form, security-typed languages require values and types to be annotated with security labels, indicating their confidentiality level. The security type system guarantees noninterference, i.e., that more-confidential information does not alter the less-confidential results of any expression.

We prove that the resulting gradual language, called \( \lambda_{\sec} \), is not only safe in that it never unexpectedly crashes, but that it is sound in that it honours the information-flow invariants of the precisely typed terms. The former property is unsurprising, since even the most imprecisely typed program still maintains the simple typing discipline, which is enough to establish the safety of the operational semantics. The soundness of the language with respect to the security type discipline, i.e., that basic information flow properties are respected, is the key property.

The prior work in gradual security typing developed gradual cast languages, which require explicit type casts to connect imprecisely typed terms with precisely typed terms. This is akin to the intermediate languages of traditional gradually-typed languages. This work presents the first gradual source language, where no explicit casts are needed: they are introduced by the language semantics. Furthermore, following the AGT approach, the runtime semantics are induced by the proof of type safety for \( \lambda_{sec} \), yielding a crisp connection to that precise static type discipline.

As with the original work on AGT, we can straightforwardly establish proper adaptations of the refined criteria for gradually typed languages. We will do so in this ongoing work.

Ultimately this work views gradual typing as a theory of imprecise typing rather than dynamic checking. Indeed dynamic checking is an inevitable consequence of this approach, but the focus here is on the types and their meaning. We believe that this broader view of gradual typing can widen the reach of gradual typing beyond its current niche of interest among dynamic language enthusiasts. Furthermore, we believe that AGT generalizes the foundations of gradual typing enough to support a wide variety of gradual type disciplines.

2. The Static Language: \( \lambda_{sec} \)
We first present the \( \lambda_{sec} \) language, with some differences from the original presentation (Zdancewic 2002). The most notable changes are that the type system is syntax directed, and the runtime semantics are small-step structural operational semantics rather than big-step natural semantics.

Figure 1 presents the syntax and type system for \( \lambda_{sec} \). The language extends a simple typing discipline with a lattice of security labels \( \ell \). All program values are ascribed security labels, which are partial ordered \( \preceq \) from low security to high-security and include top and bottom security labels \( \top \) and \( \bot \). The \( \lambda_{sec} \) types \( S \) extend...
\[ \ell \in \text{LABEL, } S \in \text{TYPE, } x \in \text{VAR, } b \in \text{BOOL, } \oplus \in \text{BOOLOP} \]
\[ t \in \text{TERM, } r \in \text{RAWVALUE } v \in \text{VALUE } \Gamma \in \text{VAR} \xrightarrow{\text{in}} \text{TYPE} \]

\[ S ::= \text{Bool}_\ell \mid S \rightarrow_\ell S \quad (\text{types}) \]
\[ b ::= \text{true} \mid \text{false} \quad (\text{Booleans}) \]
\[ r ::= b \mid b : \text{Ax : T} \quad (\text{raw values}) \]
\[ v ::= \tau \quad (\text{values}) \]
\[ t ::= v \mid t \rightarrow_\ell t \mid \text{if then } t \text{ else } t \mid t :: S \quad (\text{terms}) \]
\[ \oplus ::= \Lambda \mid \vee \quad \implies (\text{operations}) \]

\[ \Gamma \vdash t : S \]

(\text{Sx}) \quad x : S \in \Gamma \quad (\text{Sb}) \quad \Gamma \vdash b : \text{Bool}_\ell \\
(\text{Sa}) \quad \Gamma, x : S_1 \vdash t : S_2 \quad (\text{Sa}) \quad \Gamma \vdash (\lambda x : S_1 \cdot t) : S_1 \rightarrow_\ell S_2 \\
(\text{Sp}) \quad \Gamma \vdash t_1 : \text{Bool}_\ell_1 \quad \Gamma \vdash t_2 : \text{Bool}_\ell_2 \quad (\text{Sp}) \quad \Gamma \vdash t_1 \oplus t_2 : \text{Bool}_\ell_1 \oplus \text{Bool}_\ell_2 \\
(\text{Sapp}) \quad \Gamma \vdash t_1 : S_{11} \rightarrow_\ell S_{12} \quad \Gamma \vdash t_2 : S_{21} \quad S_{2} : (S_{11} \rightarrow_\ell S_{12}) \times (S_{21} \rightarrow_\ell S_{22}) \quad S \vdash S_{11} \rightarrow_\ell S_{12} \times (S_{21} \rightarrow_\ell S_{22}) \quad S \vdash S_{

| V |
|---|
|\text{S} < : S |

\[ t \vdash t :

(\text{Sy}) \quad \Gamma \vdash t : S \quad (\text{frames})

Notions of Reduction
\[ b_1 \ell_1 \oplus b_2 \ell_2 \rightarrow (b_1 [[ b_2 ]] (\ell_1 \gamma_{\ell_2}) \]
\[ (\lambda x : S_{1} \cdot t) v \rightarrow v ((\lambda x : S_{1} \cdot t) v) \gamma_{\ell} \]
\[ \text{if true} \quad \text{then } t_1 \text{ else } t_2 \rightarrow t_1 \gamma_{\ell} \]
\[ \text{if false} \quad \text{then } t_1 \text{ else } t_2 \rightarrow t_2 \gamma_{\ell} \]

| v |
|---|

\[ r_{\ell_1} \gamma_{\ell_2} \rightarrow r_{(\ell_1 \gamma_{\ell_2})} \]

\[ t \rightarrow t \]

Reduction
\[ t_1 \rightarrow t_2 \]
\[ f[t_1] \rightarrow f[t_2] \]

Figure 2. \( \lambda_{\text{SEC}} \): Small-Step Dynamic Semantics

versa. Rule (Sif) specifies that the type of a conditional is the subtyping join \( \gamma \) of the types of the branches, further stamped to incorporate the confidentiality of the predicate expression’s label \( \ell \). The latter is necessary to forbid indirect flow of information through the conditional. As usual, the join of two function types is defined in terms of the meet \( \wedge \) of the argument types, which in turn relies on the label meet operator \( \wedge \). The (S:) rule introduces ascription, which can move the type of an expression to any supertype.

These syntax-directed typing rules define a type system that is sound and complete with respect to Zdancewic’s. The following propositions use \( \vdash \) (for the type system of Zdancewic 2002), and consider only terms without ascription (i.e., the common subset of the two systems).

Proposition 1. If \( \Gamma \vdash t : S \) then \( \Gamma \vdash^Z t : S \).

\[ \text{Proof. By induction on } \Gamma \vdash t : S. \]

Proposition 2. If \( \Gamma \vdash^Z t : S \) then \( \Gamma \vdash t : S' \) for some \( S' \vdash S \).

\[ \text{Proof. By induction on } \Gamma \vdash^Z t : S. \]

Dynamic semantics. The dynamic semantics of \( \lambda_{\text{SEC}} \) were originally presented as big step semantics (Zdancewic 2002). Figure 2 presents the equivalent small-step semantics. Of particular interest is the new label stamping form on terms, which we call term stamping \( t \gamma_{\ell} \). Term stamping allows small-step reduction to retain security information that is merged with the resulting value of the nested term.

This small-step semantics coincides with the big-step semantics of \( \lambda_{\text{SEC}} \) (Zdancewic 2002). Note that we establish the equivalence to the source \( \lambda_{\text{SEC}} \) language (Figure 1), i.e., without term stamping, since it is only needed internally to support small-step reduction. As usual, \( \rightarrow^* \) denotes the reflexive, transitive closure of \( \rightarrow \).

Proposition 3. \( t \vdash v \) if and only if \( t \rightarrow^* v \).

\[ \text{Proof.} \]
\[ \text{Case (only if). By induction on } t \vdash v, \text{using the admissibility of the } \downarrow \text{ rules in } \rightarrow^*. \]

1 We overload the join notation \( \gamma \) throughout, and rely on the context to disambiguate.
Case (if). By induction on the length of the reduction \( t \rightarrow^* v \). Straightforward case analysis on \( t \) using the admissibility of the inversion lemmas for \( \Downarrow \) in \( \rightarrow^* \).

\[ \square \]

3. Gradualizing \( \lambda_{SEC} \)

In gradualizing \( \lambda_{SEC} \), we could decide to support unknown information in both types and security labels. Here, to show the flexibility of the AGT approach, we gradualize \( \lambda_{SEC} \) only in terms of security labels, thereby supporting a gradual evolution between simply-typed programs and securely-typed programs.

3.1 Meaning of Unknown Security Type Information

To gradualize our security types, we introduce a notion of gradual labels, thereby supporting a gradual evolution between simply-typed programs and securely-typed programs.

**Definition 1** (Gradual labels). A gradual label \( \hat{\ell} \) is either a label \( \ell \) or the unknown label \( ?. \)

\[ \hat{\ell} \in \text{GLABEL} \]
\[ \hat{\ell} \triangleq \ell \mid ? \] (gradual labels)

As with static security typing, we develop gradual security types by assigning a gradual label to every type constructor.

**Definition 2** (Gradual security type). A gradual security type is a gradual type labeled with a gradual label:

\[ \tilde{S} \in \text{GTYPE} \]
\[ \tilde{S} \triangleq \text{Bool}_\hat{\ell} \mid \tilde{S} \rightarrow^\hat{\ell} \tilde{S} \] (gradual types)

To give meaning to gradual security types, we use the AGT approach of defining a concretization function that maps gradual security types to sets of static security types. This concretization is the natural lifting of a concretization function for gradual labels.

**Definition 3** (Label Concretization). Let \( \gamma_\ell : \text{GLABEL} \rightarrow \mathcal{P} (\text{LABEL}) \) be defined as follows:

\[ \gamma_\ell (\ell) = \{ \ell \} \]
\[ \gamma_\ell (?) = \text{LABEL} \]

We give meaning to the unknown label by saying that it represents any label. On the other hand, any static label represents only itself.

Since we are operating on complete lattices, the sound and optimal abstraction function from sets of labels to gradual labels is fully determined by the concretization. We characterize it below.

**Definition 4** (Label Abstraction). Let \( \alpha_\ell : \mathcal{P} (\text{LABEL}) \rightarrow \text{GLABEL} \) be defined as follows:

\[ \alpha_\ell (\{ \ell \}) = \ell \]
\[ \alpha_\ell (\emptyset) \text{ is undefined} \]
\[ \alpha_\ell (?) = ? \text{ otherwise} \]

**Proposition 4** (\( \alpha_\ell \) is Sound). If \( \hat{\ell} \) is not empty, then \( \hat{\ell} \subseteq \gamma_\ell (\alpha_\ell (\hat{\ell})) \).

**Proof.** By case analysis on the structure of \( \hat{\ell} \). If \( \hat{\ell} = \{ \ell \} \) then \( \gamma_\ell (\alpha_\ell (\{ \ell \})) = \gamma_\ell (\ell) = \{ \ell \} = \hat{\ell} \); otherwise \( \gamma_\ell (\alpha_\ell (\ell)) = \gamma_\ell (?) = \text{LABEL} \supseteq \hat{\ell} \).

\[ \square \]

**Proposition 5** (\( \alpha_\ell \) is Optimal). If \( \hat{\ell} \subseteq \gamma_\ell (\hat{\ell}) \) then \( \alpha_\ell (\hat{\ell}) \subseteq \hat{\ell} \).

**Definition 5** (Type Concretization). Let \( \gamma_S : \text{GTYPE} \rightarrow \mathcal{P} (\text{TYPE}) \) be defined as follows:

\[ \gamma_S (\text{Bool}_\ell) = \{ \text{Bool}_\ell \mid \ell \in \gamma_\ell (\hat{\ell}) \} \]
\[ \gamma_S (\tilde{S} \rightarrow^\ell \tilde{S}) = \gamma_S (\tilde{S}_1) \rightarrow^\ell \gamma_S (\tilde{S}_2) \]

where \( \tilde{S}_1 \rightarrow^\ell \tilde{S}_2 = \{ S_1 \rightarrow^\ell S_2 \mid S_1 \in \gamma_S (\tilde{S}_1), \ell \in \hat{\ell} \} \).

Having defined the meaning of gradual labels, we define the meaning of gradual security types via concretization.

**Definition 6** (Label and Type Precision). \( \gamma_S : \text{GTYPE} \rightarrow \mathcal{P} (\text{TYPE}) \) is less imprecise than \( \gamma_T : \text{GTYPE} \rightarrow \mathcal{P} (\text{TYPE}) \) if and only if \( \gamma_S (\tilde{S}) \subseteq \gamma_T (\tilde{T}) \).

\[ \ell \in \hat{\ell} \]
\[ \ell \subseteq \hat{\ell} \]

**Definition 7** (Type Abstraction). Let the abstraction function \( \alpha_S : \mathcal{P} (\text{TYPE}) \rightarrow \text{GTYPE} \) be defined as:

\[ \alpha_S ((\text{Bool}_\ell)) = \text{Bool}_{\alpha_\ell (\ell)} \]
\[ \alpha_S ((\tilde{S}_1 \rightarrow^{\ell_1} \tilde{S}_2)) = \alpha_S ((\tilde{S}_1)) \rightarrow^{\alpha_\ell (\ell_1)} \alpha_S ((\tilde{S}_2)) \]

\( \alpha_S (\tilde{T}) \) is undefined otherwise.

We can only abstract valid sets of security types, i.e. in which elements only defer by security labels.

**Definition 8** (Valid Type Sets).

\[ \text{valid} ((\text{Bool}_{\ell_1})) \]
\[ \text{valid} ((\tilde{S}_1)) \]
\[ \text{valid} ((\tilde{S}_2)) \]

**Proposition 6** (\( \alpha_S \) is Sound). If \( \text{valid} (\tilde{S}) \) then \( \tilde{S} \subseteq \gamma_S (\alpha_S (\tilde{S})) \).

**Proof.** By well-founded induction on \( \tilde{S} \) according to the ordering relation \( \tilde{S} \sqsubseteq \tilde{T} \) defined as follows:

\[ \text{dom} (\tilde{S}) \subseteq \tilde{S} \]
\[ \text{cod} (\tilde{S}) \subseteq \tilde{S} \]

Where \( \text{dom}, \text{cod} : \mathcal{P} (\text{GTYPE}) \rightarrow \mathcal{P} (\text{GTYPE}) \) are the collecting liftings of the domain and codomain functions \( \text{dom}, \text{cod} \) respectively, e.g.

\[ \text{dom} (\tilde{S}) = \{ \text{dom} (S) \mid S \in \tilde{S} \} \].
We then consider cases on $S$ according to the definition of $\alpha_S$.

**Case** ($\{\text{Bool}_{\ell_i}\}$).

$\gamma_S(\alpha_S(\{\text{Bool}_{\ell_i}\})) = \gamma_S(\text{Bool}_{\alpha(\{\text{Bool}_{\ell_i}\})})$

$= \{\text{Bool}_{\ell} \mid \ell \in \gamma_S(\alpha(\{\text{Bool}_{\ell_i}\}))\}$

$\geq \{\text{Bool}_{\ell_i}\}$ by soundness of $\alpha_{\ell_i}$.

**Case** ($\{\text{S}_1 \rightarrow_{\ell_i} \text{S}_2\}$).

$\gamma_S(\alpha_S(\{\text{S}_1 \rightarrow_{\ell_i} \text{S}_2\}))$

$= \gamma_S(\alpha_S(\{\text{S}_1\}) \rightarrow_{\alpha(\{\text{S}_1\})} \alpha_S(\{\text{S}_2\}))$

$= \gamma_S(\alpha_S(\{\text{S}_1\}) \rightarrow_{\alpha(\{\text{S}_1\})} \gamma_S(\alpha_S(\{\text{S}_2\}))$

$\geq \{\text{S}_1 \rightarrow_{\ell_i} \text{S}_2\}$

by induction hypotheses on $\{\text{S}_1\}$ and $\{\text{S}_2\}$, and soundness of $\alpha_{\ell_i}$.

**Proposition 7** ($\alpha_S$ is Optimal). If valid $(\hat{S})$ and $\hat{S} \subseteq \gamma_S(\hat{S})$ then $\alpha_S(\hat{S}) \subseteq \hat{S}$.

**Proof.** By induction on the structure of $\hat{S}$.

**Case** ($\{\text{Bool}_{\ell}\}$). $\gamma_S(\text{Bool}_{\ell}) = \{\text{Bool}_{\ell} \mid \ell \in \gamma_{\ell}(\hat{\ell})\}$

So $\hat{S} = \{\text{Bool}_{\ell} \mid \ell \in \gamma_{\ell}(\hat{\ell})\}$ for some $\ell \subseteq \gamma_{\ell}(\hat{\ell})$. By optimality of $\alpha_{\ell}$, $\alpha_{\ell}(\hat{\ell}) \subseteq \ell$, so $\alpha_{\ell}(\{\text{Bool}_{\ell} \mid \ell \in \hat{\ell}\}) = \text{Bool}_{\alpha_{\ell}(\hat{\ell})} \subseteq \text{Bool}_{\ell}$.

**Case** ($\{\text{S}_1 \rightarrow_{\ell_i} \text{S}_2\}$).

$\gamma_S(\alpha_S(\{\text{S}_1 \rightarrow_{\ell_i} \text{S}_2\}))$

$= \gamma_S(\alpha_S(\{\text{S}_1\}) \rightarrow_{\alpha(\{\text{S}_1\})} \alpha_S(\{\text{S}_2\}))$

$= \gamma_S(\alpha_S(\{\text{S}_1\}) \rightarrow_{\alpha(\{\text{S}_1\})} \gamma_S(\alpha_S(\{\text{S}_2\}))$

$\geq \{\text{S}_1 \rightarrow_{\ell_i} \text{S}_2\}$

by induction hypotheses on $\{\text{S}_1\}$ and $\{\text{S}_2\}$, and soundness of $\alpha_{\ell_i}$.

**3.2 Consistent Predicates and Operators**

Following the AGT approach, we lift predicates on labels and types to consistent predicates on the corresponding gradual labels and gradual types. Consistent predicates hold if some member of the collecting semantics satisfies the corresponding static predicate. We lift partial functions to gradual partial functions, as per the standard approach in abstract interpretation.

**Definition 9** (Consistent label ordering). $\ell_1 \preceq \ell_2$ if and only if $\ell_1 \subseteq \ell_2$ for some $(\ell_1, \ell_2) \in \gamma_T(\hat{\ell}_1) \times \gamma_T(\hat{\ell}_2)$.

Algorithmsically:

$$\begin{array}{ccc}
\preceq & \equiv & \preceq \\
\ell_1 & \neg \leftrightarrow & \ell_2 \\
\ell_1 & \preceq & \ell_2 \\
\end{array}$$

**Definition 10** (Gradual label join). $\ell_1 \vee \ell_2 = \alpha_\ell(\{\ell_1 \vee \ell_2 \mid (\ell_1, \ell_2) \in \gamma_T(\hat{\ell}_1) \times \gamma_T(\hat{\ell}_2)\})$.

Algorithmsically:

$$\begin{array}{c}
\top \vee ? = ? \vee \top = \top \\
\ell \vee ? = ? \vee \ell = ? \text{ if } \ell \neq \top \\
\ell_1 \vee \ell_2 = \ell_1 \vee \ell_2 \\
\end{array}$$

Both gradual label stamping and gradual join of security types are obtained by lifting their corresponding static operations.

**Definition 11** (Gradual label meet). $\ell_1 \wedge \ell_2 = \alpha_\ell(\{\ell_1 \wedge \ell_2 \mid (\ell_1, \ell_2) \in \gamma_T(\hat{\ell}_1) \times \gamma_T(\hat{\ell}_2)\})$.

Algorithmsically:

$$\begin{array}{c}
\bot \wedge ? = ? \wedge \bot = \bot \\
\ell \wedge ? = ? \wedge \ell = \ell \text{ if } \ell \neq \bot \\
\ell_1 \wedge \ell_2 = \ell_1 \wedge \ell_2 \\
\end{array}$$

We now lift subtyping to gradual security types.

**Definition 12** (Consistent subtyping). $S_1 \preceq S_2$ if and only if $S_1 \subseteq S_2$ for some $(S_1, S_2)$ in $\gamma_S(\hat{S}_1) \times \gamma_S(\hat{S}_2)$.

**3.3 Gradual Security Type System**

The gradual security type system is adapted from Figure 1 by lifting static types and labels to gradual types and labels, lifting partial functions on static types to partial functions on gradual types, and lifting predicates on types and labels to consistent predicates on gradual types and labels.

The AGT approach yields a gradual counterpart to an underlying static type system that satisfies a number of desirable properties. To state these properties, the following propositions use $\vdash_S$ to denote the $\lambda_S$ typing relation of Figure 1.

**Proposition 8** (Conservative Extension). For $t \in \text{TTERM}$, $\vdash_S t : S$ if and only if $\vdash t : S$.

**Proof.** By induction over the typing derivations. The proof is trivial because static types are given singleton meanings via concretization.

In the following proposition, precision on terms $\ell_1 \subseteq \ell_2$ is the natural lifting of type precision to terms.

**Proposition 9** (Static gradual guarantee). If $\vdash \ell_1 : S_1$ and $\ell_1 \subseteq \ell_2$, then $\vdash \ell_2 : S_2$ and $S_1 \subseteq S_2$.

**Proof.** Corollary of the corresponding proposition for open terms.

By induction on typing derivation of $\Gamma \vdash \ell_1 : S_1$ using the definition of $\ell_1 \subseteq \ell_2$.

**3.4 Dynamic Semantics of Gradual Security Typing**

**Interiors of consistent subtyping and label ordering.** The interior of a consistent judgment expresses the most precise deductible information about a consistent judgment. We define the interior of a judgment in terms of our abstraction.

**Definition 13** (Interior). Let $P$ be a binary predicate on static types. Then the interior of the judgment $P(\hat{T}_1, \hat{T}_2)$, notation $I_P(\hat{T}_1, \hat{T}_2)$, is the smallest tuple $(\hat{T}_1', \hat{T}_2') \subseteq (\hat{T}_1, \hat{T}_2)$ such that for $(\hat{T}_1, \hat{T}_2) \in \gamma_S(\hat{T}_1, \hat{T}_2)$ and $P(\hat{T}_1, \hat{T}_2)$, then $(\hat{T}_1, \hat{T}_2) \in \gamma_S(\hat{T}_1', \hat{T}_2')$.

It is formalized as follows:

$I_P(\hat{T}_1, \hat{T}_2) = \alpha_S(\{(\hat{T}_1, \hat{T}_2) \in \gamma_S(\hat{T}_1, \hat{T}_2) \mid P(\hat{T}_1, \hat{T}_2)\})$.

We use case-based analysis to calculate the algorithmic rules for the interior of consistent subtyping on gradual security types:

$$\begin{array}{c}
\vdash_S(\hat{S}_1, \hat{S}_2) = (\hat{S}_1', \hat{S}_2') \\
\vdash_S(\text{Bool}_{\hat{\ell}_1}, \text{Bool}_{\hat{\ell}_2}) = (\text{Bool}_{\hat{\ell}_1'}, \text{Bool}_{\hat{\ell}_2'}) \\
\vdash_S(\hat{S}_1, \hat{S}_2) = (\hat{S}_1', \hat{S}_2') \\
\vdash_S(\hat{S}_1, \hat{S}_2) = (\hat{S}_1', \hat{S}_2') \\
\vdash_S(\hat{S}_1, \hat{S}_2) = (\hat{S}_1', \hat{S}_2') \\
\vdash_S(\hat{S}_1, \hat{S}_2) = (\hat{S}_1', \hat{S}_2') \\
\vdash_S(\hat{S}_1, \hat{S}_2) = (\hat{S}_1', \hat{S}_2') \\
\vdash_S(\hat{S}_1, \hat{S}_2) = (\hat{S}_1', \hat{S}_2') \\
\vdash_S(\hat{S}_1, \hat{S}_2) = (\hat{S}_1', \hat{S}_2') \\
\end{array}$$
\[ \Gamma \vdash \ell : S \]

\[ (\mathcal{S}x) \quad x : \bar{S} \in \Gamma \quad \Gamma \vdash x : S \]

\[ (\mathcal{S}a) \quad \Gamma, x : \bar{S}_1 \vdash \ell \quad \bar{S}_2 \quad \Gamma \vdash (\lambda x : \bar{S}_1 . \ell) : \bar{S}_2 \]

\[ (\mathcal{S}b) \quad \Gamma \vdash \ell_1 : \text{Bool}_\Gamma \quad \Gamma \vdash \ell_2 : \text{Bool}_\Gamma \]

\[ (\mathcal{S}+\) \quad \Gamma \vdash \ell_1 \oplus \ell_2 : \text{Bool}_\Gamma \]

\[ (\mathcal{S}app) \quad \Gamma \vdash \ell : \text{Bool}_\Gamma \quad \Gamma \vdash \ell_1 : \bar{S}_1 \rightarrow \bar{S}_2 \quad \Gamma \vdash \ell_2 : \bar{S}_2 \rightarrow \ell \]

\[ (\mathcal{S}if) \quad \Gamma \vdash \ell_1 : \text{Bool}_\Gamma \quad \Gamma \vdash \ell_2 : \bar{S}_1 \rightarrow \bar{S}_2 \quad \Gamma \vdash \ell_3 : \bar{S}_2 \rightarrow \ell \]

\[ (\mathcal{S}if\) \quad \bar{S}_1 \rightarrow \ell \quad \bar{S}_2 \rightarrow \ell_1 \quad \bar{S}_3 \rightarrow \ell_2 \]

\[ \bar{S} \leq \bar{S} \]

\[ \text{Bool}_\Gamma \leq \text{Bool}_{\Gamma'} \]

Intrinsic terms. Fig. 4 presents the intrinsic terms for \( \lambda_{\sim} \). Note that we do not need to introduce term stamping in this language. Since terms are intrinsically typed and we have ascriptions, labels can be stamped at the type level.

Reduction. Evaluation uses the consistent transitivity operator \( \circ_{\leq} \) to combine evidences:

\[ (\bar{S}_1, \bar{S}_{21}) \circ_{\leq} (\bar{S}_{22}, \bar{S}_3) = \bigtriangleup_{\leq} (\bar{S}_1, \bar{S}_{21} \cap \bar{S}_{22}, \bar{S}_3) \]

First we calculate a recursive meet operator for gradual types:

\[ \text{Bool}_\Gamma \cap \text{Bool}_{\Gamma'} = \text{Bool}_{\text{\Gamma} \cap \text{\Gamma'}} \]

\[ (\bar{S}_1 \rightarrow \ell_1 \bar{S}_{21} \cap (\bar{S}_{22} \rightarrow \ell_2 \bar{S}_{22})) = (\bar{S}_{11} \cap \bar{S}_{21} \rightarrow (\bar{S}_{22} \cap \bar{S}_{22})) \]

\[ \bar{S} \cap \bar{S}' \text{ undefined otherwise} \]

Figure 3. \( \lambda_{\sim} \): Syntax and Static Semantics

The rules appeal to the algorithmic rules for the interior of consistent label ordering, calculated similarly:

\[ \{ \ell \neq \top \} \]

\[ \mathcal{I}_\prec (\ell, \top) = \langle \ell, \top \rangle \quad \mathcal{I}_\prec (\top, \top) = \langle \top, \top \rangle \]

\[ \ell \neq \bot \]

\[ \mathcal{I}_\prec (\top, \ell) = \langle \top, \ell \rangle \quad \mathcal{I}_\prec (\top, \bot) = \langle \bot, \bot \rangle \]

\[ \mathcal{I}_\prec (\ell, \ell) = \langle \ell, \ell \rangle \]
We calculate a recursive definition for $\Delta^\prec$ by case analysis on the structure of the second argument,

\[
\Delta^\prec(\bar{t}_1, \bar{t}_2, \bar{t}_3) = \langle \bar{t}_1, \bar{t}_3 \rangle
\]

with the following definition of $\Delta^\leq$, again calculated by case analysis on the middle granular label:

\[
\Delta^\leq(\bar{t}_1, \bar{t}, \bar{t}_3) = \langle \bar{t}_1, \bar{t} \rangle
\]

The reduction rules are given in Fig. 5. The evidence inversion functions reflect the contravariance on arguments and the need to stamp security labels on return types:

\[
\begin{align*}
idom((\bar{S}_1 \rightarrow_{\ell_p} \bar{S}_2, \bar{S}_1 \rightarrow_{\ell_p} \bar{S}_3')) &= (\bar{S}_1', \bar{S}_1) \\
\mathrm{icod}((\bar{S}_1 \rightarrow_{\ell_p} \bar{S}_2, \bar{S}_1' \rightarrow_{\ell_p} \bar{S}_3')) &= (\bar{S}_2' \gamma \ell', \bar{S}_2' \gamma \ell')
\end{align*}
\]

### 3.5 Example

Consider a simple lattice with two confidentiality levels, $L = \bot$ and $H = \top$, and the following extrinsic program definitions:

\[
\begin{align*}
f &\triangleq (\lambda x : \text{Bool}. x)_L & \text{a public channel} \\
g &\triangleq (\lambda x : \text{Bool}. x)_L & \text{an unknown channel that can be publicly used} \\
v &\triangleq \text{false}_H & \text{a private value}
\end{align*}
\]

$f, v$ does not type check, but $f \ (g \ v)$ does typecheck, even though it fails at runtime. Type checking yields the corresponding intrinsic definitions:

\[
\begin{align*}
(\bar{S}_f, \bar{S}_i) f \cdot \bar{S}_g (\langle \bar{S}_g, \bar{S}_g \rangle g \cdot \bar{S}_g (\langle H, H \rangle v))
\end{align*}
\]

where the intrinsic subterms are essentially identical to their extrinsic counterparts:

\[
\begin{align*}
f &\triangleq (\lambda x : \text{Boolean}. x)_L & \text{a public channel} \\
g &\triangleq (\lambda x : \text{Boolean}. x)_L & \text{an unknown channel that can be publicly used} \\
v &\triangleq \text{false}_H & \text{a private value}
\end{align*}
\]

For conciseness, we abbreviate $\text{Boolean}$ with $\bar{t}$, use $\bar{S}_f$ and $\bar{S}_g$ to refer to the types of $f$ and $g$, and elide the application operators $\cdot$. At each step, we use grey boxes to highlight the focus of reduction/rewriting, and underline the result.
4. Noninterference for $\lambda_{SEC}$

We establish noninterference for the gradual security language using logical relations, adapting the technique from Zdancewic (2002).

First, in the intrinsic setting, the type environment related to an open term $t$ is simply the set of (intrinsically-typed) free variables of the term, $\text{FV}(t)$. We use the metavariable $\Gamma \in \mathcal{P}(\text{VAR}_*)$ to denote such “type-environments-as-sets”.

Informally, the noninterference theorem states that a program with low (visible) output and a high (private) input can be run with different high-security values and, if terminating, will always yield the same observable value.\footnote{The label function returns the top-level security label of the given type: $\text{label}(\text{Bool}_\ell) = \ell$ and $\text{label}(\overline{S}_1 \rightarrow_{\ell} \overline{S}_2) = \ell$.}

**Theorem 10 (Noninterference).** If $t \in \text{TERM}_{\text{Bool}}$ with $\text{FV}(t) = \{ x \}$, $x \in \text{TERM}_{S}$, and $v_1, v_2 \in \text{TERM}_{S}$ with $\text{label}(\overline{S}) \neq \ell$, then

\[
t[v_1/x] \rightarrow^* v'_1 \land t[v_2/x] \rightarrow^* v'_2 \Rightarrow \text{bval}(v'_1) = \text{bval}(v'_2)
\]

**Proof.** The result follows by using the method of logical relations (following Zdancewic (2002)), as a special case of Lemma 15 below.

Note that we compare equality of base values at base types, stripping the checking-related information (labels, evidences and ascriptions): i.e. $\text{bval}(b_\ell) = b$ and $\text{bval}(\text{bool}_\ell : \overline{S}) = b$. Also, gradual programs can fail. We establish termination-insensitive noninterference, meaning in particular that any program may run into an error without violating noninterference.

**Definition 14 (Gradual security logical relations).** For an arbitrary element $\xi$ of the security lattice, the $\xi$-level gradual security relations are type-indexed binary relations on closed terms defined inductively as presented in Figure 6. The notation $v_1 \approx_{\xi} v_2 : \overline{S}$ indicates that $v_1$ is related to $v_2$ at type $\overline{S}$ when observed at the security level $\xi$. Similarly, the notation $t_1 \approx_{\xi} t_2 : C(\overline{S})$ indicates that $t_1$ and $t_2$ are related computations that produce related values at type $\overline{S}$ when observed at the security level $\xi$.

The logical relations are very similar to those of Zdancewic (2002), except for the points discussed above and the fact that we account for subtyping in the relation between values at a function type (recall that our type system is syntax-directed).

**Definition 15 (Secure program).** A well-typed program $t$ that produces a $\xi$-observable output of type $\overline{S}$ (i.e. $\text{label}(\overline{S}) \leq \xi$) is secure iff $t \approx_{\xi} : C(\overline{S})$.

**Definition 16 (Related substitutions).** Two substitutions $\sigma_1$ and $\sigma_2$ are related, notation $\Gamma \vdash \sigma_1 \approx_{\xi} \sigma_2$, if $\sigma_1 \models \Gamma$ and $\forall x \in \Gamma, \sigma_1(x) \approx_{\xi} \sigma_2(x) : \overline{S}$.

**Lemma 11 (Substitution preserves typing).** If $t \overline{S} \in \text{TERM}_{\overline{S}}$ and $\sigma \models \text{FV}(t) \overline{S}$ and $\sigma(t) \in \text{TERM}_{\overline{S}}$.

**Proof.** By induction on the derivation of $t \overline{S} \in \text{TERM}_{\overline{S}}$.

**Lemma 12 (Reduction preserves relations).** Consider $t_1, t_2 \in \text{TERM}_{\overline{S}}$. Posing $t_i \rightarrow^* t'_i$, we have $t_1 \approx_{\xi} t_2 : C(\overline{S})$ if and only if $t'_1 \approx_{\xi} t'_2 : C(\overline{S})$.

**Proof.** Direct by definition of $t_1 \approx_{\xi} t_2 : C(\overline{S})$ and transitivity of $\rightarrow^*$.

**Lemma 13 (Canonical forms).** Consider a value $v \in \text{TERM}_{\overline{S}}$. Then either $v = u$, or $v = \varepsilon u : \overline{S}$ with $u \in \text{TERM}_{\overline{S}}$, and $\varepsilon \vdash \overline{S}_1 \leq \overline{S}$. Furthermore:

1. If $\overline{S} = \text{Bool}_\ell$ then either $v = b_\ell$ or $v = \varepsilon b_\ell : \text{Bool}_\ell$.
2. If $\varepsilon \vdash \overline{S}_1 \rightarrow_{\ell} \overline{S}_2$ then either $v = (\lambda x \overline{S}_1. t \overline{S}_2) \varepsilon$ with $t \overline{S}_2 \in \text{TERM}_{\overline{S}_2}$ or $v = \varepsilon (\lambda x \overline{S}_1. t \overline{S}_2) \varepsilon : \overline{S}_1 \rightarrow_{\ell} \overline{S}_2$ with $t \overline{S}_2 \in \text{TERM}_{\overline{S}_2}$ and $\varepsilon \vdash \overline{S}_1 \rightarrow_{\ell} : \overline{S}_2 \leq \overline{S}_1 \rightarrow_{\ell} \overline{S}_2$.

**Proof.** By direct inspection of the formation rules of gradual intrinsic terms (Figure 4).

**Lemma 14 (Ascription preserves relation).** Suppose $\varepsilon \vdash \overline{S}_1 \leq \overline{S}_2$. Then either $v_1 \approx_{\xi} \overline{S}_1 \rightarrow_{\ell} v_2 : \overline{S}_2$.

**Proof.** Following Zdancewic (2002), the proof proceeds by induction on the judgment $\varepsilon \vdash \overline{S}_1 \leq \overline{S}_2$. The difference here is that consistent subtyping is justified by evidence, and that the terms have to be ascribed to exploit subtyping. In particular, case 1 above establishes a computation-level relation because each ascribed term $(\varepsilon u : \overline{S})$ may not be a value; each value $u_1$ is either a bare value $u_1$ or a casted value $\varepsilon u_1 : \overline{S}_1$, with $\varepsilon \vdash \overline{S}_1 \leq \overline{S}_2$. In the latter case, $(\varepsilon \varepsilon u : \overline{S}_1) : \overline{S}_1$ either steps to error (in which case the relation is vacuously established), or steps to $\varepsilon u : \overline{S}$, which is a value.

Noninterference follows directly from the following lemma, which establishes that substitution preserves the logical relations:

**Lemma 15 (Substitution preserves relations).** If $t \overline{S} \in \text{TERM}_{\overline{S}}$, $\Gamma = \text{FV}(t) \overline{S}$, and $\Gamma \vdash \sigma_1 \approx_{\xi} \sigma_2$, then $\sigma_1(t) \overline{S} \approx_{\xi} \sigma_2(t) \overline{S} : C(\overline{S})$.

**Proof.** By induction on the derivation that $t \in \text{TERM}_{\overline{S}}$. Considering the last step used in the derivation:

- Case (a): $t \overline{S} = b_\ell$. By definition of substitution, $\sigma_1(b_\ell) = \sigma_2(b_\ell) = b_\ell$. By definition, $b_\ell \approx_{\xi} b_\ell : \text{Bool}_\ell$ as required.

- Case (b): $t \overline{S} = \varepsilon u_1 : \overline{S}_1 \rightarrow_{\ell} \overline{S}_2$. By definition of substitution, assuming $u_1 \overline{S}_1 \notin \text{dom}(\sigma_1)$, and Lemma 11:

$$\sigma_1(t) \overline{S} = (\lambda x \overline{S}_1. \sigma_1(t) \overline{S}_2) \varepsilon \in \text{TERM}_{\overline{S}_2}$$

If $\varepsilon \not\approx_{\xi} \xi$ the result holds trivially because all function values are related in such a cases. Assume $\varepsilon \approx_{\xi} \xi$, and assume two values $v_1$ and $v_2$ such that $v_1 \approx_{\xi} v_2 : \overline{S}_1$, with $\varepsilon \vdash \overline{S}_1 \leq \overline{S}_1$. (We omit the $\approx_{\xi}$ operator in applications below since we simply pick
By backward preservation of the relations (Lemma 12), this implies

\[ v_1 \approx_{\zeta} v_2 : \text{Bool} \quad \iff \quad v_1 \in \text{TERM}_{\text{Bool}} \land \vec{\ell} \approx \zeta \quad \implies \quad \text{beval}(v_1) = \text{beval}(v_2) \]

\[ v_1 \approx_{\zeta} v_2 : \tilde{S}_1 \rightarrow \tilde{S}_2 \quad \iff \quad v_1 \in \text{TERM}_{\tilde{S}_1 \rightarrow \tilde{S}_2} \land \vec{\ell} \approx \zeta \quad \implies \quad \forall v_1, v_2, \tilde{S}_1', \tilde{S}_2' \text{ such that} \]

\[ \varepsilon_1 \vdash \tilde{S}_1 \rightarrow \tilde{S}_2 \subseteq \tilde{S}_1' \rightarrow \tilde{S}_2' \text{ and } v_2 \vdash \tilde{S}_2'' \subseteq \tilde{S}_1', \text{ we have:} \]

\[ \forall v_1' \approx_{\zeta} v_2' : \tilde{S}_1'. (v_1' @ \tilde{S}_1' \rightarrow \tilde{S}_2' \varepsilon v_2') \approx_{\zeta} (v_1 v_2 @ \tilde{S}_1' \rightarrow \tilde{S}_2' \varepsilon v_2') : \text{C} \tilde{S}_2 \vec{\gamma} \vec{\ell} \]

\[ t_1 \approx_{\zeta} t_2 : \text{C}(\tilde{S}) \quad \iff \quad t_1 \in \text{TERM}_{\tilde{S}} \land \varepsilon_{t_1} \vdash^{*} v_1 \land t_2 \vdash^{*} v_2 \implies v_1 \approx_{\zeta} v_2 : \tilde{S} \]

By definition of related computations:

\[ \sigma_1(t_{\tilde{S}_1}) \rightarrow^{*} v_{11} \land \sigma_2(t_{\tilde{S}_2}) \rightarrow^{*} v_{21} \implies v_{11} \approx_{\zeta} v_{21} : \tilde{S}_1 \]

\[ \sigma_1(t_{\tilde{S}_2}) \rightarrow^{*} v_{12} \land \sigma_2(t_{\tilde{S}_2}) \rightarrow^{*} v_{22} \implies v_{12} \approx_{\zeta} v_{22} : \tilde{S}_2 \]

By Lemma 13, each \( v_{ij} \) is either a boolean \( (b_{ij})_{i,j} \) or a casted boolean \( (b_{ij})_{i,j} : \tilde{S}_j \). In case a value \( v_{ij} \) is a casted value, then the whole term \( \sigma_j(t_{\tilde{S}_j}) \) can take a step by (R_{\text{g}}), combining \( \varepsilon \) with \( \varepsilon_{ij} \). Such a step either fails, or succeeds with a new combined evidence. Therefore, either:

\[ \sigma_j(t_{\tilde{S}_j}) \rightarrow^{*} \text{error} \]

In which case we do not care since we only consider termination-insensitive noninterference, or:

\[ \sigma_j(t_{\tilde{S}_j}) \rightarrow^{*} \varepsilon_1'(b_{11})_{i,j} \vdash^{\ell} \varepsilon_2'(b_{12})_{i,j} \]

\[ \rightarrow^{*} \varepsilon_1'(b_{11})_{i,j}^{\ell} \vdash \text{Bool}_{\ell} \]

with \( b_1 = b_{11} [\tilde{S}] b_{12} \) and \( \vec{\ell}_1 = \vec{\ell}_1 \vec{\gamma} \vec{\ell}_1 \). It remains to show that:

\[ (b_1)_{i,j}^{\ell} \approx_{\zeta} (b_2)_{i,j} : \text{Bool}_{\ell} \]

If \( \vec{\ell} \not\approx \zeta \), then the result holds trivially because all boolean values are related. If \( \vec{\ell} \approx \zeta \), then also \( \vec{\ell}_1 \approx \zeta \), which means by definition of \( \approx_{\zeta} \) on boolean values, that \( b_{11} \approx b_{21} \) and \( b_{12} \approx b_{22} \), so \( b_1 = b_2 \).

---

Case (app). \( t_{\tilde{S}} = \varepsilon t_{\tilde{S}_1} @ \tilde{S}_1 \rightarrow \tilde{S}_2 \varepsilon t_{\tilde{S}_2} \)

with \( \varepsilon_1 \vdash \tilde{S}_1 \subseteq \tilde{S}_1 \rightarrow \tilde{S}_2 \), \( \varepsilon_2 \vdash \tilde{S}_2 \subseteq \tilde{S}_1 \), and \( \tilde{S} = \tilde{S}_2 \vec{\gamma} \vec{\ell} \).

We omit the \( @ \tilde{S}_1 \rightarrow \tilde{S}_2 \) operator in applications below.

By definition of substitution and Lemma 11:

\[ \sigma_j(t_{\tilde{S}}) = \varepsilon_1 \sigma_j(t_{\tilde{S}_1}) \varepsilon_2 \sigma_j(t_{\tilde{S}_2}) \in \text{TERM}_{\tilde{S}} \]

By induction hypothesis:

\[ \sigma_1(t_{\tilde{S}_1}) \approx_{\zeta} \sigma_2(t_{\tilde{S}_1}) : \text{C}(\tilde{S}_1) \text{ and } \sigma_1(t_{\tilde{S}_2}) \approx_{\zeta} \sigma_2(t_{\tilde{S}_2}) : \text{C}(\tilde{S}_2) \]

By definition of related computations:

\[ \sigma_1(t_{\tilde{S}_1}) \rightarrow^{*} v_{11} \land \sigma_2(t_{\tilde{S}_2}) \rightarrow^{*} v_{21} \implies v_{11} \approx_{\zeta} v_{21} : \tilde{S}_1 \]

\[ \sigma_1(t_{\tilde{S}_2}) \rightarrow^{*} v_{12} \land \sigma_2(t_{\tilde{S}_2}) \rightarrow^{*} v_{22} \implies v_{12} \approx_{\zeta} v_{22} : \tilde{S}_2 \]

By definition of \( \approx_{\zeta} \) at values of function type, using \( \varepsilon_1 \) and \( \varepsilon_2 \) to justify the subtyping relations, we have:

\[ (\varepsilon_1 v_{11} \varepsilon_2 v_{12}) \approx_{\zeta} (\varepsilon_1 v_{21} \varepsilon_2 v_{22}) : \text{C}(\tilde{S}_2 \vec{\gamma} \vec{\ell}) \]
By definition of substitution:
\[ \sigma_i(t\tilde{S}) = \epsilon_1 \sigma_i(t\tilde{S}_1) \text{ if } \epsilon_1 \sigma_i(t\tilde{S}_2) \text{ else } \epsilon_3 \sigma_i(t\tilde{S}_3) \]

By the induction hypothesis we have that:
If \( \tilde{S} \not\approx \zeta \), then \( \sigma_1(t\tilde{S}) \approx \zeta = \sigma_2(t\tilde{S}) = C(S\tilde{1}) \) holds trivially because the \( \approx \zeta \) relations relate all such well-typed terms. Let us assume \( \ell \not\approx \zeta \).

By the induction hypothesis we have that:
\[ \sigma_1(t\tilde{S}_1) \approx \zeta \sigma_2(t\tilde{S}_2) = C(S\tilde{1}) \]
Assuming \( \sigma_1(t\tilde{S}_1) \mapsto^* \epsilon_{v11} \), by the definition of \( \approx \zeta \) we have:
\[ v_{11} \approx \zeta v_{21} = S\tilde{1} \]

By Lemma 13, each \( v_{11} \) is either a boolean \((b_{11})_{p_{11}}\) or a casted boolean \( \epsilon_{11}(b_{11})_{p_{11}} \). In either case, \( S\tilde{1} \approx \zeta \) \( \text{Bool}_{p_{11}} \), so by definition of \( \approx \zeta \) on boolean values, \( b_{11} = b_{21} \).

In case a value \( v_{11} \) is a casted value, then the whole term \( \sigma_i(t\tilde{S}) \) can take a step by \((Rg)\), combining \( \epsilon_i \) with \( \epsilon_{x1} \). Such a step either fails, or succeeds with a new combined evidence. Therefore, either:
\[ \sigma_i(t\tilde{S}) \mapsto^* \text{error} \]
in which case we do not care since we only consider termination-insensitive noninterference, or:
\[ \sigma_i(t\tilde{S}) \mapsto^* \text{ if } \epsilon_1(b_{11})_{p_{11}} \text{ then } \epsilon_2 \sigma_i(t\tilde{S}_2) \text{ else } \epsilon_3 \sigma_i(t\tilde{S}_3) \]

Because \( b_{11} = b_{21} \), both \( \sigma_1(t\tilde{S}) \) and \( \sigma_2(t\tilde{S}) \) step into the same branch of the conditional. Let us assume the condition is true (the other case is similar). Then:
\[ \sigma_i(t\tilde{S}) \mapsto \epsilon_2 \sigma_i(t\tilde{S}_2) = S\tilde{2} \]

By induction hypothesis:
\[ \sigma_1(t\tilde{S}_2) \approx \zeta \sigma_2(t\tilde{S}_2) = C(S\tilde{2}) \]
Assume \( \sigma_i(t\tilde{S}_2) \mapsto^* \epsilon_{v12} \), then \( v_{12} \approx \zeta v_{22} = S\tilde{2} \). Since \( \epsilon_2 \vdash S\tilde{2} \approx \zeta \)

Case (\( \vdash \)): \( t\tilde{S} = \epsilon_S S\tilde{1} \approx S \), with \( t\tilde{S}_1 \in \text{TERM}_{\tilde{S}_1} \) and \( \epsilon_S \vdash S\tilde{1} \approx \zeta \).

By definition of substitution:
\[ \sigma_i(t\tilde{S}) = \epsilon_S \sigma_i(t\tilde{S}_1) \approx S \]

By induction hypothesis:
\[ \sigma_i(t\tilde{S}_1) \approx \zeta \sigma_2(t\tilde{S}_1) = C(S\tilde{1}) \]

The result follows directly by Lemma 14.

\[ \square \]

5. Related Work and Conclusion

The design of a gradual security-typed language is a novel contribution. Despite the fact that both Disney and Flanagan (2011) and Fennell and Thiemann (2013) have proposed languages for security typing dubbed gradual, they do not propose gradual source languages. Rather, the language designs require explicit security casts—which can also be encoded with a label test expression in Jif (Zheng and Myers 2007). Furthermore, both designs treat an unlabeled type as having the top label, then allowing explicit casts downward in the security lattice. This design is analogous to the internal language of the quasi-static typing approach. In that approach, explicit casts work well, but the external language there accepts too many programs. That difficulty was the original motivation for consistency in gradual typing (Siek and Taha 2006).

Thiemann and Fennell (2014) develop a generic approach to gradualize annotated type systems. This is similar to security typing (labels are one kind of annotation), except that they only consider annotation on base types, and the language only includes explicit casts, like the gradual security work discussed above. They track blame and provide a translation that removes unnecessary casts.

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A. Auxiliary Proofs

Proposition 1. If \( \Gamma \vdash t : S \) then \( \Gamma \vdash^{Z} t : S \).

Proof. By induction on \( \Gamma \vdash t : S \).

As most of the type rules are identical, most of the cases are straightforward. The exceptions to this are the (Sif) and (Sapp) rules.

Case (Sif). Then
\[ \frac{\Gamma \vdash t_0 : \text{Bool}_2 \quad \Gamma \vdash t_1 : S_1 \quad \Gamma \vdash t_2 : S_2 \quad D_0 \quad D_1 \quad D_2}{\Gamma \vdash^{Z} t_1 : (S_1 \triangleright S_2) \quad \gamma \quad \ell} \]

By Lemma 16, \( S_1 < : (S_1 \triangleright S_2) \) and \( S_2 < : (S_1 \triangleright S_2) \), and by Lemma 17 \((S_1 \triangleright S_2) < : (S_1 \triangleright S_2) \quad \gamma \quad \ell\), therefore \( S_1 < : (S_1 \triangleright S_2) \quad \gamma \quad \ell \) and \( S_2 < : (S_1 \triangleright S_2) \quad \gamma \quad \ell \).

Combining these with the induction hypotheses, we get
\[ \begin{align*}
\frac{\Gamma \vdash^{Z} t_0 : \text{Bool}_2 \quad \Gamma \vdash^{Z} t_1 : (S_1 \triangleright S_2) \quad \gamma \quad \ell}{\Gamma \vdash^{Z} t_1 : S_2 \quad \gamma \quad \ell}
\end{align*} \]

\[ \begin{align*}
\frac{\epsilon_0 \quad \Gamma \vdash^{Z} t_1 : S_1 \quad S_1 < : (S_1 \triangleright S_2) \quad \gamma \quad \ell}{\epsilon_1 \quad \Gamma \vdash^{Z} t_1 : (S_1 \triangleright S_2) \quad \gamma \quad \ell}
\end{align*} \]

\[ \frac{\epsilon \quad \Gamma \vdash^{Z} t_0 : \text{Bool}_2}{\epsilon \quad \Gamma \vdash^{Z} t_2 : (S_1 \triangleright S_2) \quad \gamma \quad \ell} \]
Case (Sapp). Then \( t = t_1 t_2 \) and \( \Gamma \vdash t_1 \ t_2 : S_{12} \ \forall \ell \) for some \( S_{12} \) and \( \ell \) such that \( \Gamma \vdash t_1 : S_{11} \rightarrow \ell \ S_{12}, \Gamma \vdash t_2 : S_2 \) and \( S_2 < : S_{11} \). Using induction hypothesis on \( t_2 \) we know that \( \Gamma \vdash t_2 : S_2 \). As \( S_2 < : S_{11} \). Then by \( (\lambda_{\text{SEC-SUB}}) \) \( \Gamma \vdash t_2 : S_{11} \). Using induction hypothesis on \( t_1, \Gamma \vdash t_1 : S_{11} \rightarrow \ell \ S_{12}, \) then by \( (\lambda_{\text{SEC-APP}}) \) we conclude that \( \Gamma \vdash t_1 t_2 : S_{12} \ \forall \ell \).

Lemma 16. Let \( S_1, S_2 \in \text{TYPE} \). Then
1. If \( (S_1 \triangledown S_2) \) is defined then \( S_1 < : (S_1 \triangledown S_2) \).
2. If \( (S_1 \triangledown S_2) \) is defined then \( (S_1 \triangledown S_2) < : S_1 \).

Proof. We start by proving (1) assuming that \( (S_1 \triangledown S_2) \) is defined. We proceed by case analysis on \( S_1 \).

Case (\( \text{Bool} \)). If \( S_1 = \text{Bool}_{t_1} \) then as \( (S_1 \triangledown S_2) \) is defined then \( S_2 \) must have the form \( \text{Bool}_{t_2} \) for some \( t_2 \). Therefore \( (S_1 \triangledown S_2) = \text{Bool}_{(t_1 \triangledown t_2)} \). But by definition of \( \triangleleft \), \( t_1 \triangleleft (t_1 \triangledown t_2) \) and therefore we use \( (\triangleleft \text{Bool}) \) to conclude that \( \text{Bool}_{t_1} < : (S_1 \triangledown S_2) \), i.e. \( S_1 < : (S_1 \triangledown S_2) \).

Case (\( S \rightarrow S \)). If \( S_1 = S_{11} \rightarrow t_1 S_{12} \) then as \( (S_1 \triangledown S_2) \) is defined then \( S_2 \) must have the form \( S_{21} \rightarrow t_2 S_{22} \) for some \( S_{21}, S_{22} \) and \( t_2 \).

We also know that \( (S_1 \triangledown S_2) = (S_{11} \triangledown S_{21}) \rightarrow (t_1 \triangledown t_2) \rightarrow (S_{12} \triangledown S_{22}) \). By definition of \( \triangleleft \), \( t_1 \triangleleft (t_1 \triangledown t_2) \). Also, as \( (S_1 \triangledown S_2) \) is defined then \( (S_{11} \triangledown S_{21}) \) is defined. Using the induction hypothesis of (2) on \( S_{11}, (S_{11} \triangledown S_{21}) < : S_1 \). Also, using the induction hypothesis of (1) on \( S_{12} \) we also know that \( S_{12} < : (S_{12} \triangledown S_{22}) \). Then by \( (\triangleleft \rightarrow \rightarrow) \) we can conclude that \( S_{11} \rightarrow t_1 S_{12} < : (S_{11} \triangledown S_{21}) \rightarrow (t_1 \triangledown t_2) \rightarrow (S_{12} \triangledown S_{22}) \), i.e. \( S_1 < : (S_1 \triangledown S_2) \).

The proof of (2) is similar to (1) but using the argument that \( (t_1 \triangleleft t_2) \triangleleft t_1 \).

Lemma 17. Let \( S \in \text{TYPE} \) and \( \ell \in \text{LABEL} \). Then \( S \triangledown \ell \).

Proof. Straightforward case analysis on type \( S \) using the fact that \( \ell \triangleleft (\ell' \triangledown \ell) \) for any \( \ell' \).

Lemma 18. Let \( S_1, S_2 \in \text{TYPE} \) such that \( S_1 < : S_2 \), and let \( t_1, t_2 \in \text{LABEL} \) such that \( t_1 \triangleleft t_2 \). Then \( S_1 \triangledown t_1 \triangledown t_2 < : S_2 \triangledown t_1 \triangledown t_2 \).

Proof. Straightforward case analysis on type \( S \) using the definition of label stamping on types.

Proposition 2. If \( \Gamma \vdash Z \ t : S \) Then \( \Gamma \vdash t : S' \) for some \( S' \triangledown S \).

Proof. By induction on derivations of \( \Gamma \vdash Z \ t : S \).

We proceed by case analysis on \( t \) (modulo \( (\lambda_{\text{SEC-SUB}}) \)). As most of the type rules are identical, most of the cases are straightforward. The exception to this is case \( (\lambda_{\text{SEC-COND}}) \) and \( (\lambda_{\text{SEC-APP}}) \):

Case (\( \lambda_{\text{SEC-COND}} \)). Then \( t = \text{if } t' \text{ then } t_1 \text{ else } t_2 \) and \( \Gamma \vdash t' : \text{Bool}_{t'} \), \( \Gamma \vdash t_1 : S_1 \ \forall \ell \), \( \Gamma \vdash t_2 : S_2 \ \forall \ell \), \( S_1 \triangledown S_2 \).

Using induction hypothesis on the premises we also know that \( \Gamma \vdash t' : \text{Bool}_{t'} \) for some \( t' \triangledown \ell \), \( \Gamma \vdash t_1 : S_1' \triangledown S_1 \) for some \( S_1' < : S_1 \ \forall \ell \), \( \Gamma \vdash t_2 : S_2' \triangledown S_2 \) for some \( S_2' < : S_2 \ \forall \ell \).

By (Si), \( \Gamma \vdash \text{if } t' \text{ then } t_1 \text{ else } t_2 : (S_1' \triangledown S_2') \triangledown S_1 \). Then by Lemma 16 we know that \( (S_1' \triangledown S_2') \triangledown S_1 \triangledown S_1 \), and by Lemma 18 if we choose \( S' = (S_1' \triangledown S_2') \triangledown S_1 \), we conclude that \( (S_1' \triangledown S_2') \triangledown S_1 \triangledown S_2 \), i.e. \( S' < : S \).