ON DIRECT AND INVERSE POLETSKY INEQUALITY WITH A TANGENTIAL DILATATION ON THE PLANE

EVGENY SEVOST’YANOV, VALERY TARGONSKII

July 22, 2022

Abstract

This article is devoted to the study of mappings defined in the region on the plane. Under certain conditions, the upper estimate of the distortion of the modulus of families of paths is obtained. Similarly, the upper estimate of the modulus of the families of paths in the pre-image under the mapping is also obtained.

2010 Mathematics Subject Classification: Primary 30C65; Secondary 31A15, 31B25

1 Introduction

Some times ago we obtained various inequalities for the distortion of the modulus of families of paths (see, e.g., [SalSev1, Sev1, Sev2] and [SST]). As a rule, we deal with the so-called outer or inner dilatations, the use of which is generally accepted (see, e.g., [Va Theorem 34.4], [MRSY, Theorems 8.5–8.6]). We note that there are also some other characteristics that describe the distortion of the module under mappings, such as, for example, tangential dilatations (see, e.g., [RSY1, Theorem 2.17], [RSY2, Theorem 4.2]). Their use may turn out to be more expedient, it may be connected with theorems on the existence of solutions of differential equations in partial derivatives. At the same time, inner and outer dilatations, which also play a role in a similar context, may not satisfy the appropriate conditions necessary for their application.

The main goal of this manuscript is to obtain new estimates of the distortion of the modulus of families of paths for using tangential dilatations. The article is divided into two parts. The first part concerns the inverse estimates of the modulus of families of paths, the second part deals with direct estimates. The results of the article mainly are proved for mappings with branching.
Here are the necessary definitions. Let \( X \) and \( Y \) be two spaces with measures \( \mu \) and \( \mu' \), respectively. We say that a mapping \( f : X \to Y \) has \( N \)-property of Luzin, if from the condition \( \mu(E) = 0 \) it follows that \( \mu'(f(E)) = 0 \). Similarly, we say that a mapping \( f : X \to Y \) has \( N' \)-Luzin property, if from the condition \( \mu'(E) = 0 \) it follows that \( \mu(f^{-1}(E)) = 0 \). Given a mapping \( f : D \to \mathbb{C} \), a set \( E \subset D \subset \mathbb{C} \) and a point \( y \in \mathbb{C} \), we define a multiplicity function \( N(y, f, E) \) as a number of pre-images of a point \( y \) in \( E \), i.e.,

\[
N(y, f, E) = \text{card} \{ z \in E : f(z) = y \},
\]

\[
N(f, E) = \sup_{y \in \mathbb{C}} N(y, f, E).
\]

A mapping \( f : D \to \mathbb{C} \) is called discrete if the image \( \{ f^{-1}(y) \} \) of each point \( y \in \mathbb{C} \) consists of isolated points, and open if the image of any open set \( U \subset D \) is an open set in \( \mathbb{C} \). A mapping \( f \) of \( D \) onto \( D' \) is called closed if \( f(E) \) is closed in \( D' \) for any closed set \( E \subset D \) (see, e.g., [Vil chap. 3]). Observe that, \( N(f, D) < \infty \) for any open discrete and closed mappings \( f \) of a domain \( D \) (see [MS Lemma 3.3]).

Let \( \Gamma \) be a family of paths \( \gamma : (a, b) \to \mathbb{C} \) (or dashed lines \( \gamma : \bigcup_{i=1}^{\infty} (a_i, b_i) \to \mathbb{C} \)). A Borel function \( \rho : \mathbb{C} \to \mathbb{R}^+ \) is called admissible for the family \( \Gamma \), write \( \rho \in \text{adm} \Gamma \), if

\[
\int_{\gamma} \rho(z) |dz| \geq 1
\]

for any locally rectifiable path (dashed line) \( \gamma \in \Gamma \). Given \( p \in (1, \infty) \), the quantity

\[
M_p(\Gamma) = \inf_{\rho \in \text{adm} \Gamma} \int_{\mathbb{C}} \rho^p(z) \, dm(z)
\]

is called \( p \)-modulus of \( \Gamma \). Let \( z_0 \in \mathbb{C}, 0 < r_1 < r_2 < \infty \) and

\[
B(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| < r \}, \quad S(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| = r \},
\]

\[
A = A(z_0, r_1, r_2) = \{ z \in \mathbb{C} : r_1 < |z - z_0| < r_2 \}.
\]

Given sets \( E, F \subset \overline{\mathbb{C}} \) and a domain \( D \subset \mathbb{C} \) we denote \( \Gamma(E, F; D) \) a family of all paths \( \gamma : [a, b] \to \overline{\mathbb{C}} \) such that \( \gamma(a) \in E, \gamma(b) \in F \) and \( \gamma(t) \in D \) for \( t \in [a, b] \). If \( f : D \to \mathbb{C}, z_0 \in f(D) \setminus \{ \infty \} \) and \( 0 < r_1 < r_2 < r_0 = \sup_{y \in f(D)} |y - y_0| \), then we define \( \Gamma_f(y_0, C_1, C_2) \) a family of all paths \( \gamma \) in \( D \) such that \( f(\gamma) \in \Gamma(C_1, C_2, A(y_0, r_1, r_2)) \). Let \( Q_* : \mathbb{C} \to [0, \infty] \) be a Lebesgue measurable function. We say that \( f \) satisfies the inverse Poletsky inequality at \( y_0 \in f(D) \setminus \{ \infty \} \) with respect to \( \alpha \)-modulus, \( \alpha > 1 \), if the relation

\[
M_{\alpha}(\Gamma_f(y_0, C_1, C_2)) \leq \int_{A(y_0, r_1, r_2) \cap f(D)} Q_*(y) \cdot \eta^\alpha(|y - y_0|) \, dm(y)
\]

(1.5)
for any $0 < r_1 < r_2 < d_0 := \text{dist}(z_0, \partial D)$, all continua $C_1 \subset B(z_0, r_1)$, $C_2 \subset D \setminus B(z_0, r_2)$ and any Lebesgue measurable function $\eta : (r_1, r_2) \rightarrow [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r) \, dr \geq 1. \quad (1.6)$$

Let $y \in \mathbb{C}$ does not belong to the set $f(A)$, where $A$ is the set of all points $z \in D$ where the mapping $f : D \rightarrow \mathbb{C}$ has no a total differential, or $J(z, f) = 0$. Given $p > 1$, we set

$$Q(y) := K_{CT, \rho, y_0}(y, f) := \sum_{z \in f^{-1}(y)} \left( \frac{|f'(z)|^{-1} |y - y_0|}{|J(z, f)|} \right)^p, \quad (1.7)$$

The will call the function $K_{CT, \rho, y_0}$ in (1.7) is co-tangential dilatation of $f$ of the order $p$ at $y_0$. The following theorem holds.

**Theorem 1.1.** Let $1 < \alpha \leq 2$, let $y_0 \in f(D) \setminus \{\infty\}$, $r_0 = \sup_{y \in f(D)} |y - y_0| > 0$, and let $f : D \rightarrow \mathbb{C}$ be an open discrete and closed mapping that is differentiable almost everywhere and has $N$-Luzin property with respect to the Lebesgue measure in $\mathbb{C}$. Assume that $\overline{D}$ is compact in $\mathbb{C}$, at the same time, $m(f(B_*)) = 0$, where $B_*$ is a set of points $z \in D$ where $f$ has a total differential and $J(z, f) = 0$. Suppose that, any path $\alpha$ with $f \circ \alpha \subset S(y_0, r)$, $S(y_0, r) = y_0 + re^{i\varphi/r}$, $\varphi \in [0, 2\pi r)$, is locally rectifiable for almost all $r \in (\epsilon, r_0)$ and, in addition, $f$ has $N^{-1}$-property on $S(y_0, r) \cap f(D)$ for almost all $r \in (\epsilon, r_0)$ with respect to $\mathcal{H}^1$ on $S(y_0, r)$. If

$$Q(y) := K_{CT, \alpha^{-1}, y_0}(y, f) := \sum_{z \in f^{-1}(y)} \left( \frac{|f'(z)|^{-1} |y - y_0|}{|J(z, f)|} \right)^{\frac{\alpha}{\alpha - 1}} \in L^{\alpha-1}(f(D)),$$

then $f$ satisfies inverse Poletsky’s inequality with respect to $\alpha$-modulus at the point $y_0$ for

$$Q_\ast(y) := N^\alpha(f, D) \cdot Q^{\alpha-1}(y).$$

Let us move on to the formulation of the results regarding the analogue of Poletsky’s inequality. Let $z \in D \subset \mathbb{C}$ be a point where $f : D \rightarrow \mathbb{C}$ has partial derivatives with respect to $x$ and $y$, where $z = x + iy$, $i^2 = -1$, and $J(z, f) \neq 0$, where $J(z, f)$ denotes the Jacobian of the mapping $f$ at the point $z$. Let $z_0 \in \overline{D}$ and let $p > 1$. Then we put

$$K_{T, p, z_0}(z, f) := \left( \frac{|f'(z)|^{-1} |z - z_0|}{|J(z, f)|} \right)^p \quad (1.8)$$

at the non-degenerate differentiability point $z$ of $f$, and $K_{T, p, z_0}(z, f) = 0$ otherwise. The quantity $K_{T, p, z_0}(z, f)$ in (1.8) is called the tangential dilation of order $p$ of the mapping $f$ at the point $z$ relative to the point $z_0$. 


Remark 1.1. We set
\[ D_f(z, z_0) = \frac{1 - \frac{z - z_0}{z - z_0} \mu(z)}{1 - |\mu(z)|^2}, \]
(1.9)
at the points \( z \) of the non-degenerate differentiability of \( f \), where \( \mu(z) = \frac{f_y}{f_x} \), \( f_x \neq 0 \), \( \mu(z) = 0 \) при \( f_z = 0 \), \( f_z = (f_x - if_y)/2 \), \( f_z = (f_x + if_y)/2 \), \( z = x + iy \), \( i^2 = -1 \) (see e.g. [RSY] Lemma 2.10 or [MRSY, Lemma 11.2]). Let us note the following remark concerning the connection of quantities in (1.7) and (1.9).

It may be shown that the relation
\[ K_{CT,2,z_0}(f(z), f^{-1}) = D_f(z, z_0), \quad y_0 = f(z_0), \]
(1.10)
holds whenever \( f \) is a homeomorphism and \( f \) is non-degenerate differentiable at \( z \in D \), see [MRSY] formulae (11.35), (11.43)]. Moreover, for \( p = 2 \), \( D_f(z, z_0) = K_{T,2,z_0}(z, f) \) at the non-degenerate differentiability points (see [MRSY relation (11.35)]).

In what follows, \( C^k_{\partial}(U) \) denotes the space of functions \( u : U \to \mathbb{R} \) with by a compact carrier in \( U \), which have \( k \) partial continua derivatives in \( U \). Recall the concept of Sobolev classes, see [Re] Section 2, Ch. I]. Let \( U \) be an open set, \( U \subset \mathbb{C} \), \( u : U \to \mathbb{R} \) is some function such that \( u \in L^1_{\text{loc}}(U) \). Suppose there is a function \( v \in L^1_{\text{loc}}(U) \) such that equality
\[ \int_U \frac{\partial \varphi}{\partial x_i}(z)u(z) \, dm(z) = -\int_U \varphi(z)v(z) \, dm(z) \]
is performed for any function \( \varphi \in C^1_{\partial}(U), i = 1, 2 \). In this case, we will say that the function \( v \) is a generalized derivative of the first order function \( u \) with respect to \( x_i \) and denote it by \( \frac{\partial u}{\partial x_i}(z) := v \). A function \( u \in W^{1,1}_{\text{loc}}(U) \), if \( u \) has generalized partial derivatives with respect to all variables in \( U \), are locally integrable in \( U \).

A mapping \( f : D \to \mathbb{C} \), \( f(z) = u(z) + iv(z) \), belongs to to *Sobolev class* \( W^{1,1}_{\text{loc}} \), write \( f \in W^{1,1}_{\text{loc}}(D) \), if \( u \) and \( v \) have generalized partial derivatives of the first order, locally integrable in \( D \). We write \( f \in W^{1,k}_{\text{loc}}(D) \), \( k \in \mathbb{N} \), if \( u \) and \( v \), are also locally integrable in degree \( k \).

Recall that a mapping \( f \) between domains \( D \) and \( D' \) has a finite distortion, if \( f \in W^{1,1}_{\text{loc}} \) and, in addition, there exists a function \( K(z) \), \( K(z) < \infty \) a.e., such that
\[ \|f'(z)\|^2 \leq K(z) \cdot J(z, f) \]
for almost all \( z \in D \), where \( \|f'(z)\| = |f_x| + |f_z| \).

Let \( Q : \mathbb{C} \to \mathbb{R} \) be a Lebesgue measurable function such that \( Q(z) \equiv 0 \) for \( z \in \mathbb{C} \setminus D \). Let \( z_0 \in \overline{D} \), \( z_0 \neq \infty \). Given \( \alpha \geq 1 \), a mapping \( f : D \to \mathbb{C} \) is called a ring \( Q \)-mapping at a point \( z_0 \in \overline{D} \setminus \{\infty\} \) with respect to \( \alpha \)-modulus, if the condition
\[ M_\alpha(f(\Gamma(C_1, C_2, D))) \leq \int_{\mathbb{A}\cap D} Q(z) \cdot \eta^\alpha(|z - z_0|) \, dm(z) \]
(1.11)
holds for any $0 < r_1 < r_2 < d_0 := \text{dist}((z_0, \partial D))$, all continua $C_1 \subset \overline{B(z_0, r_1)}$, $C_2 \subset D \setminus B(z_0, r_2)$ and any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r) \, dr \geq 1. \quad (1.12)$$

Recall that, a pair $E = (A, C)$ is said to be a condenser, if $A$ is an open set in $\mathbb{C}$, and $C$ is a non-empty compact subset of $A$. Denote by $dm(z)$ the element of the Lebesgue measure in $\mathbb{C}$, $W_0(E) = W_0(A, C)$ is a family of all absolutely continuous on lines (ACL) functions $u : A \rightarrow \mathbb{R}$ with a compact support in $A$ such that $u(z) \geq 1$ on $C$. The quantity

$$\text{cap}_p E = \text{cap}_p (A, C) = \inf_{u \in W_0(E)} \int_A |\nabla u|^p \, dm(z), \quad (1.13)$$

is called $p$-capacity of $E$. The following result holds.

**Theorem 1.2.** Let $f : D \rightarrow \mathbb{C}$ be a homeomorphism with a finite distortion and let $1 < \alpha \leq 2$. Set $p = \frac{\alpha}{\alpha - 1}$. Suppose that $K_{T,p,z_0}^\alpha(z, f) \in L^1(D)$, where $K_{T,p,z_0}(z, f)$ is defined in $(1.8)$. Then $f$ satisfies the relation $(1.11)$ at the point $z_0 \in D \setminus \{\infty\}$ at $Q(z) = K_{T,p,z_0}^\frac{1}{\alpha - 1}(z, f)$.

**Theorem 1.3.** Let $f : D \rightarrow \mathbb{C}$ be an open, discrete and closed bounded mapping with finite distortion such that $N(f, D) < \infty$. Let $1 < \alpha \leq 2$ and $z_0 \in D$. Set $p = \frac{\alpha}{\alpha - 1}$. Suppose that $K_{T,p,z_0}(z, f) \in L^1_{\text{loc}}(D)$, where $K_{T,p,z_0}(z, f)$ is defined in $(1.8)$. Then $f$ satisfies the relation

$$\text{cap}_\alpha f(\mathcal{E}) \leq \int_A N^{\alpha - 1}(f, D)K_{T,p,z_0}^\alpha(z, f) \cdot \eta^\alpha(|z - z_0|) \, dm(z)$$

for $\mathcal{E} = (B(z_0, r_2), B(z_0, r_1))$, $A = A(z_0, r_1, r_2)$, $0 < r_1 < r_2 < \varepsilon_0 := \text{dist}(z_0, \partial D)$, and any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ which satisfies the relation $(1.12)$.

**Theorem 1.4.** Let $f : D \rightarrow \mathbb{C}$ be an open, discrete and closed mapping with finite distortion, and $1 < \alpha \leq 2$. Suppose that $z_0 \in \partial D$, $p = \frac{\alpha}{\alpha - 1}$ and that $K_{T,p,z_0}(z, f) \in L^1(D)$, where $K_{T,p,z_0}(z, f)$ is defined in $(1.8)$. Then, for any $\varepsilon_0 < d_0 := \sup_{z \in D} |z - z_0|$ and a compact set $C_2 \subset D \setminus B(z_0, \varepsilon_0)$ there is $\varepsilon_1$, $0 < \varepsilon_1 < \varepsilon_0$, such that the relation

$$M_\alpha(f(\Gamma(C_1, C_2, D))) \leq \int_A N^{\alpha - 1}(f, D)K_{T,p,z_0}^\alpha(z, f)\eta^\alpha(|z - z_0|) \, dm(z) \quad (1.14)$$

holds for any $\varepsilon \in (0, \varepsilon_1)$ and any $C_1 \subset \overline{B(z_0, \varepsilon)} \cap D$, where $A(z_0, \varepsilon, \varepsilon_1)$ is defined in $(1.3)$, and $\eta : (\varepsilon, \varepsilon_1) \to [0, \infty]$ is arbitrary Lebesgue measurable function that satisfies the relation $(1.12)$. 

2 Preliminaries

The following important information concerning the capacity of a pair of sets with respect to a domain can be found in the paper of W. Ziemer [Zi]. Let $G$ be a bounded domain in $\mathbb{C}$ and $C_0, C_1$ be non-intersecting compact sets, which belong to the closure $G$. Let us set $R = G \setminus (C_0 \cup C_1)$ and $R^* = R \cup C_0 \cup C_1$. Given $p > 1$, we define the $p$-capacity of the pair $C_0, C_1$ relative to the closure $G$ by the equality

$$C_p[G, C_0, C_1] = \inf_{R} \int |\nabla u|^p dm(z),$$

where the exact lower bound is taken over all functions $u$, continuous in $R^*$, $u \in ACL(R)$, such that $u = 1$ on $C_1$ and $u = 0$ on $C_0$. The specified functions are called admissible for $C_p[G, C_0, C_1]$. We will say that the set $\sigma \subset \mathbb{C}$ separates $C_0$ from $C_1$ in $R^*$, if $\sigma \cap R$ is closed in $R$ and there are disjoint sets $A$ and $B$, open relative to $R^* \setminus \sigma$, such that $R^* \setminus \sigma = A \cup B$, $C_0 \subset A$ and $C_1 \subset B$. Let $\Sigma$ denote the class of all sets that separate $C_0$ and $C_1$ in $R^*$. Given a number $p' = p/(p-1)$, we define the quantity

$$\overline{M}_{p'}(\Sigma) = \inf_{\rho \in \text{adm} \Sigma} \int_{\mathbb{C}} \rho^{p'} dm(z)$$

where the inclusion $\rho \in \text{adm} \Sigma$ denotes that $\rho$ is an Borel measurable function in $\mathbb{C}$ such that

$$\int_{\sigma \cap R} \rho \, d\mathcal{H}^1 \geq 1 \quad \forall \sigma \in \Sigma. \quad (2.2)$$

Note that, by Ziemer’s result

$$\overline{M}_{p'}(\Sigma) = C_p[G, C_0, C_1]^{-1/(p-1)}, \quad (2.3)$$

see [Zi] theorem 3.13] for $p = 2$ and [Zi] p. 50] for $1 < p < \infty$. In addition, by the Hesse’s result

$$M_p(\Gamma(E, F, D)) = C_p[D, E, F], \quad (2.4)$$

where $(E \cup F) \cap \partial D = \emptyset$ (see [Hes], Theorem 5.5]). Shlyk showed that the requirement $(E \cup F) \cap \partial D = \emptyset$ can be removed, in other words, equality (2.4) holds for arbitrary disjoint nonempty compact sets $E, F \subset \overline{D}$ (see [Sh], Theorem 1)).

We say that some property $P$ is satisfied for $p$-almost all paths in the domain $D$, if this property is satisfied for all paths in $D$, except, perhaps, some of their subfamily, $p$-modulus of which equals to zero. We say that a Lebesgue measurable function $\rho : \mathbb{C} \to \mathbb{R}^+$ is $p$-extensively admissible for a family $\Gamma$ of paths $\gamma$ (or dashed lines) in $\mathbb{C}$, abbr. $\rho \in \text{ext}_p \text{adm} \Gamma$, if the relation (1.2) holds for $p$-a.e. paths (dashed lines) $\gamma$ of $\Gamma$.

Let $E$ be a set in $\mathbb{C}$ and let $\gamma : \Delta \to \mathbb{C}$ be some path. Denote by $\gamma \cap E = \gamma(\Delta) \cap E$. Let the path $\gamma$ be locally rectifiable and the length function $l_\gamma(t)$ is defined above. Let us put

$$l(\gamma \cap E) := \text{mes}_1(E_\gamma), \quad E_\gamma = l_\gamma(\gamma^{-1}(E)).$$
Here, as everywhere above, mes$_1(A)$ denotes the length (linear Lebesgue measure) of the set $A \subset \mathbb{R}$. Note that

$$E_\gamma = \gamma_0^{-1}(E),$$

where $\gamma_0 : \Delta_\gamma \rightarrow \mathbb{C}$ is natural parametrization of the path $\gamma$, and that

$$l(\gamma \cap E) = \int_\gamma \chi_{E}(z) |dz| = \int_{\Delta_\gamma} \chi_{E_\gamma}(s) dm_1(s).$$

The following statement can be found in [MRSY, Lemma 8.1].

**Proposition 2.1.** Let $E$ be a subset of a domain $D \subset \mathbb{C}$, $n \geq 2$, $p \geq 1$. Then the set $E$ is Lebesgue measurable if and only if the set $\gamma \cap E$ is measurable for $p$-almost all paths $\gamma$ in $D$. Moreover, $m(E) = 0$ if and only if

$$l(\gamma \cap E) = 0$$

(2.5)

for $p$-almost all curves $\gamma$ in $D$.

**Remark 2.1.** Recall that the family of paths $\Gamma_1$ is called *shorter* compared to the family of paths $\Gamma_2$, is written $\Gamma_1 < \Gamma_2$, if for each path $\gamma_2 \in \Gamma_2$, $\gamma_2 : (a_2, b_2) \rightarrow \mathbb{C}$, there is a path $\gamma_1 \in \Gamma_1$, $\gamma_1 : (a_1, b_1) \rightarrow \mathbb{C}$, such that $\gamma_1(t) = \gamma_2(t)$ for all $t \in (a_1, b_1) \subset (a_2, b_2)$. From the definition of the modulus of families of paths it follows that the condition $\Gamma_1 < \Gamma_2$ implies that $M_p(\Gamma_1) \geq M_p(\Gamma_2)$ (see, e.g., [V], Theorem 6.2).

From these considerations it also follows that the Proposition 2.1 is true also for families of dashed lines, not only families of paths. Really, let the set $E$ be Lebesgue measurable, and let $\Gamma_0$ be a family of all dashed lines $\gamma : \bigcup_{i=1}^\infty (a_i, b_i) \rightarrow \mathbb{C}$, for which the measurability of the set $\gamma \cap E$ is violated. Then, by the countable additivity of the Lebesgue measure there is a family $\Gamma_1$ of paths $\gamma_k : (a_k, b_k) \rightarrow \mathbb{C}$ such that $\gamma_k \cap E$ is not measurable. However, from the definition of the modulus of families of paths (see the considerations given above), we have: $M_p(\Gamma_0) \leq M_p(\Gamma_1) = 0$, i.e., $M_p(\Gamma_0)$. The inverse statement, i.e., that the dimensionality of the set $E$ follows from the dimensionality of $\gamma \cap E$ for almost all of dashed lines $\gamma$, is obvious and follows directly from the Proposition 2.1.

The validity of the equality (2.5) for families of dashed lines is proved in the same way as the equivalence of the dimensionality of the set $\gamma \cap E$ to the dimensionality of the set $E$ itself.

The following statement is proved in [IS, lemma 4.1].

**Proposition 2.2.** Let $D$ be a domain in $\mathbb{C}$, $p > 1$, and $z_0 \in D$. If some property $P$ holds for $p$-almost circles $D(z_0, r) := S(z_0, r) \cap D$, $r \in (\varepsilon, \varepsilon_0)$, where "$p$-almost all" should be understood in the sense of the modulus of families of dashed lines lines, in addition, the set

$$E = \{r \in \mathbb{R} : P \quad \text{holds for} \quad S(z_0, r) \cap D\}$$
is Lebesgue measurable. Then $P$ also holds for almost all $D(z_0, r)$ with respect to the parameter $r \in (\varepsilon, \varepsilon_0)$. On the contrary, if $P$ holds for almost all circles $D(z_0, r) := S(z_0, r) \cap D$ with respect to the Lebesgue measure by $r \in (\varepsilon, \varepsilon_0)$, then $P$ also holds for $p$-almost all of the dashed lines $D(z_0, r) := S(z_0, r) \cap D$ in the sense of $p$-modulus for any $p > 1$.

The following statement can be found in [Vu, lemma 3.7].

**Proposition 2.3.** Let $f : G \to \mathbb{C}$ be an open, discrete and closed mapping, let $\beta : [a, b) \to f(G)$ be a path and let $k = N(f, G)$. Then there are paths $\alpha_j : [a, b) \to G$, $1 \leq j \leq k$, such that: (1) $f \circ \alpha_j = \beta$, (2) Card $\{j : \alpha_j(t) = z\} = i(z, f)$ at $z \in f^{-1}(|\beta|)$ and $t \in [a, b)$, where $i(z, f)$ is the local topological index of the mapping $f$ at points $z$ and (3) $\sum_{j=1}^{k} |\alpha_j| = f^{-1}(|\beta|)$.

We write $\alpha \subset \beta$, if $\beta|_J = \alpha$, where $\beta : I \to \mathbb{C}$ and $J$ is some interval, segment, or the semiinterval $I$. Let $f : D \to \mathbb{C}$ be a discrete mapping, $\beta : I_0 \to \mathbb{C}$ be a closed rectifiable path, and $\alpha : I \to D$ is a path such that $f \circ \alpha \subset \beta$. If the length function $l_\beta : I_0 \to [0, l(\beta)]$ is constant on a certain interval $J \subset I$, then $\beta$ is constant on $J$ and due to the discreteness of the mapping $f$, the path $\alpha$ is also constant on $J$. Therefore, there exists a unique function $\alpha^* : l_\beta(I) \to D$ such that $\alpha = \alpha^* \circ (l_\beta|_I)$. We say that $\alpha^*$ is the $f$-representation of the path $\alpha$ with respect to $\beta$. Similarly, one can define $f$-representations for arbitrary rectifiable paths, not only closed ones.

The proof of the following lemma is based on the approach used when establishing lower estimates of the distortion of the modulus of families paths, (see, e.g., [RSY], Theorem 2.17 and [SY7], Theorem 4.1; see also [KRSS], Theorem 5 and [Sev1], Theorem 4]).

**Lemma 2.1.** Let $p > 1$, $D$ be a domain in $\mathbb{C}$, $f : D \to \mathbb{C}$ be an open, discrete and closed mapping which is differential almost everywhere and has $N$-property of Luzin with respect to the Lebesgue measure in $\mathbb{C}$. Let $y_0 \in \bar{f}(D) \setminus \{\infty\}$, $r_0 = \sup_{y \in \bar{f}(D)} |y - y_0| > 0$, $0 < \varepsilon < \varepsilon_0 < r_0$. Assume that, $m(f(B_1)) = 0$, where $B_1$ is the set of all points $z$ in which $f$ has a total differential and $J(z, f) = 0$. Denote by $\Sigma_\varepsilon$ the family of all sets of the form

$$\{ f^{-1}(S(y_0, r)) \}, \quad r \in (\varepsilon, \varepsilon_0).$$

Assume that, for almost all $r \in (\varepsilon, \varepsilon_0)$, $f$-has the $N^{-1}$-property on $S(y_0, r)$ with respect to the Hausdorff measure $\mathcal{H}^1$ on $S(y_0, r)$ and in addition, any path $\alpha$ with $f \circ \alpha \subset S(y_0, r)$ is locally rectifiable. Then

$$\widetilde{M}_p(\Sigma_\varepsilon) \geq \frac{1}{N^p(f, D)} \inf_{\rho \in \text{ext adm}_p, f(\Sigma_\varepsilon)} \int_{f(D) \cap A(y_0, \varepsilon, \varepsilon_0)} \frac{\rho^p(y)}{Q(y)} \, dm(y),$$

where $Q$ is defined by the relation

$$Q(y) := K_{CT,p,y_0}(y, f) := \sum_{z \in f^{-1}(y)} \left( \frac{|f'(z)|^{-1} (y - y_0)}{|y - y_0|} \right)^p.$$
Proof. We denote by \( B \) the (Borel) set of all points \( z \in D \), where the mapping \( f \) has the total differential \( f'(z) \) and \( J(z, f) \neq 0 \). By Kirszbraun’s theorem and by the uniqueness of the approximative differential (see, e.g., [Fe, 2.10.43 and Theorem 3.1.2]) it follows that the set \( B \) is a countable union Borel sets \( B_k, k = 1, 2, \ldots \), such that the mapping \( f_k = f|_{B_k} \) are bilipschitz homeomorphisms (see [Fe, Lemma 3.2.2 and Theorems 3.1.4 and 3.1.8]). Without loss of generality, we may assume that the sets \( B_k \) are pairwise disjoint. We also denote by \( B_* \) the set of all points \( z \in D \), where \( f \) has a total differential, however \( J(z, f) = 0 \).

Since the set \( B_0 := D \setminus (B \cup B_*) \) has a Lebesgue measure zero, and \( f \) has the N-Luzin property, then \( m(f(B_0)) = 0 \). Now, by Proposition 2.1 and Remark 2.1 \( l(S_r \cap f(B_0)) = 0 \) for \( p \)-almost all circles \( S_r := S(y_0, r) \cap f(D) \) centered at the point \( y_0 \), where ”almost all” should be understood in the sense of \( p \)-modulus of families of paths. Similarly, \( l(S_r \cap f(B_*)) = 0 \) for \( p \)-almost all such circles of \( S_r \).

Let us fix the circle

\[
\tilde{\gamma}(t) = S(y_0, r) = re^{it} + y_0, \quad t \in [0, 2\pi).
\]

Then \( S_r := S(y_0, r) \cap f(D) \) is a family of dashed lines

\[
\tilde{\gamma}_i : \bigcup_{i=1}^{\infty} (a_i, b_i) \to f(D), \quad (a_i, b_i) \subset \mathbb{R}, \quad i = 1, 2, \ldots.
\]

Let us denote this family of dashed lines by \( \Gamma \). Now we fix some point \( \omega_0^i \in f(D) \) which belongs to a dashed line \( \tilde{\gamma}_i : (a_i, b_i) \to f(D) \) such that \( \tilde{\gamma}_i(t_i) = \omega_0^i \) for some \( t_i \in (a_i, b_i) \). Note that \( N(f, D) < \infty \), where \( N(f, D) \) is defined by the ratio \( (1.1) \) (see [MS, Lemma 3.3]).

By Proposition 2.3 the path \( \tilde{\gamma}_i|_{[t_i, b_i]} \) has \( k := N(f, D) \) total liftings \( \gamma_{il} \to D, 1 \leq l \leq k \), such that card \( \{ j : \gamma_{il}(t) = z \} = i(z, f) \) at \( z \in f^{-1}(|\tilde{\gamma}_i|_{[t_i, b_i]}) \) and \( t \in [t_i, b_i] \), where \( \sum_{l=1}^{k} |\gamma_{il}| = f^{-1}(|\tilde{\gamma}_i|_{[t_i, b_i]}) \), which start at some points \( z_i, 1 \leq i \leq k \). Similarly, by Proposition 2.3 any path \( \tilde{\gamma}_i|_{[a_i, t_i]} \) has \( k := N(f, D) \) complete liftings \( \gamma_{il} \to D, 1 \leq l \leq k \), such that card \( \{ j : \gamma_{il}(t) = z \} = i(z, f) \) at \( z \in f^{-1}(|\tilde{\gamma}_i|_{[a_i, t_i]}) \) and \( t \in (a_i, t_i] \), where \( \sum_{l=1}^{k} |\gamma_{il}| = f^{-1}(|\tilde{\gamma}_i|_{[a_i, t_i]}) \), starting at some points \( z_i, 1 \leq i \leq k \). By uniting the paths \( \gamma_{il} \) and \( \gamma_{il}^2 \), we obtain the paths \( \gamma_{il} : (a_i, b_i) \to D, l = 1, 2, \ldots, k \), which have the following properties:

\[
(1) \quad f \circ \gamma_{il} = \tilde{\gamma}_i, \quad l = 1, 2, \ldots, k,
\]

\[
(2) \quad \text{card} \{ j : \gamma_{il}(t) = z \} = i(z, f) \quad z \in f^{-1}(|\tilde{\gamma}_i|), \quad t \in (a_i, t_i],
\]

\[
(3) \quad \bigcup_{l=1}^{k} |\gamma_{il}| = f^{-1}(|\tilde{\gamma}_i|).
\]
By the condition of the lemma, the path $\gamma_{il}^*(s)$ is locally rectifiable for almost all $r$. Then also $\gamma_{il}^*(s)$ is locally rectifiable for $p$-almost all $S_r \in \Gamma$ (see Proposition 2.2). For the dashed lines considered in the proof of Lemma 2.1 and their families, see the following Figure and diagram. Denote by $\Gamma_0$ the family of all $S_r$, for which $l(S_r \cap f(B_0)) > 0$, or $l(S_r \cap f(B_*)) > 0$.

Figure 1: To the proof of Lemma 2.1

Figure 2: To the proof of Lemma 2.1

or the corresponding path $\gamma_{il}^*(s)$ is not locally rectifiable. Now let $S_r \in \Gamma \setminus \Gamma_0$ and let $\tilde{\gamma}_i$ be arbitrary dashed lines in $S_r$. Let us first consider the case when this a path is rectifiable for the same $r$. Let $\rho \in \widetilde{\text{adm}} \Sigma_\varepsilon$. We put

$$
\tilde{\rho}(y) = \begin{cases} 
\sup_{z \in f^{-1}(y)} \rho(z) \left| (f'(z))^{-1} \frac{y-y_0}{|y-y_0|} \right|, & y \in f(D \setminus B_0) \setminus f(B_*), \\
0, & \text{in other cases.}
\end{cases}
$$

Note that $\tilde{\rho}(y) = \sup_{k \in \mathbb{N}} \tilde{\rho}_k(y)$, where

$$
\tilde{\rho}_k(y) = \begin{cases} 
\rho(f_k^{-1}(y)) \left| (f'(f_k^{-1}(y)))^{-1} \frac{y-y_0}{|y-y_0|} \right|, & y \in f(D \setminus B_0) \setminus f(B_*), \\
0, & \text{in other cases.}
\end{cases}
$$
therefore, the function \( \tilde{\rho}(y) \) is Borel, (see, e.g., \cite[Theorem I (8.5)]{Sa} or \cite[Section 2.3.2]{Fe}). Denote, as usual, \( \tilde{\gamma}_i^0 = \tilde{\gamma}_i^0(s) \) is normal representation of the path \( \tilde{\gamma}_i \), namely,

\[
\tilde{\gamma}_i^0(t) = \tilde{\gamma}_i^0 \circ s_i(t),
\]

were \( s_i(t) \) denotes the length of \( \tilde{\gamma} \) on \((a_i, b_i)\), and \( s \in (0, l(\tilde{\gamma}_i)) \) and \( l(\tilde{\gamma}_i) \) denotes the length of \( \tilde{\gamma}_i \). More precisely,

\[
\tilde{\gamma}_i^0(s) = y_0 + re^{i(s+s_i)/r}, \quad r = \text{const}, \quad s \in (0, l(\tilde{\gamma}_i)),
\]

(2.10)

where \( s_i \in \mathbb{R} \) is some parameter such that \( \tilde{\gamma}_i^0(s_i) = a_i \).

Since \( S_r \in \Gamma \setminus \Gamma_0 \), then \( \tilde{\gamma}_i^0(s) \notin f(B_0) \) for almost all \( s \in (0, l(\tilde{\gamma}_i)) \). Then

\[
\int_{\tilde{\gamma}_i} \tilde{\rho}(y) \, dy = \int_0^{l(\tilde{\gamma}_i)} \tilde{\rho}(\tilde{\gamma}_i^0(s)) \, ds = \frac{1}{k} \sum_{l=1}^k \int_0^{l(\tilde{\gamma}_i)} \tilde{\rho}(\tilde{\gamma}_i^0(s)) \, ds,
\]

(2.11)

where \( k = N(f, D) \). Let the paths \( \gamma_{il} \) satisfying properties (1), (2) and (3) mentioned above, and let \( \gamma_{il}^*(s) \) be their \( f \)-images, that is, \( f(\gamma_{il}^*(s)) = \tilde{\gamma}_i^0(s) \) at all \( s \in l(0, l(\tilde{\gamma}_i)) \). Since \( \tilde{\gamma}_i^0(s) \notin f(B_0) \) for almost all \( s \in (0, l(\tilde{\gamma}_i)) \), it follows that \( \gamma_{il}^*(s) \notin B_0 \) for almost all \( s \in (0, l(\tilde{\gamma}_i)) \). Therefore, points \( f^{-1}(\tilde{\gamma}_i^0(s)) = \{\gamma_{il}^0(s), \gamma_{il}^0(s), \ldots, \gamma_{il}^0(s)\} \) are different for almost all \( s \in (0, l(\tilde{\gamma}_i)) \). Due to the definition of \( \tilde{\rho} \) in (2.8), for each \( 1 \leq l \leq k \) we obtain that:

\[
\int_0^{l(\tilde{\gamma}_i)} \tilde{\rho}(\tilde{\gamma}_i^0(s)) \, ds = \int_0^{l(\tilde{\gamma}_i)} \sup_{\gamma_{il}^* \in f^{-1}(\tilde{\gamma}_i^0(s))} \rho(z) \left| f'(z) \right|^{-1} \frac{\gamma_{il}^0(s) - y_0}{|\tilde{\gamma}_i^0(s) - y_0|} \, ds \geq
\]

\[
\int_0^{l(\tilde{\gamma}_i)} \rho(\gamma_{il}^*(s)) \left| f'(\gamma_{il}^*(s)) \right|^{-1} \frac{\gamma_{il}^0(s) - y_0}{|\gamma_{il}^0(s) - y_0|} \, ds = \int_0^{l(\tilde{\gamma}_i)} \rho(\gamma_{il}^*(s)) \left| f'(\gamma_{il}^*(s)) \right|^{-1} \frac{e^{i(s+s_i)/r}}{s_i} \, ds.
\]

(2.12)

Since \( f(\gamma_{il}^0(s)) = y_0 + re^{i(s+s_i)/r} \), in addition, \( \gamma_{il}^*(s) \in D \setminus (B_0 \cup B_1) \) for \( p \)-almost all \( \tilde{\gamma}_i \in \Gamma \) and for almost all \( s \in l(0, l(\tilde{\gamma}_i)) \), by the differentiability theorem for superposition of mappings

\[
f'(\gamma_{il}^0(s))\gamma_{il}^*(s) = e^{i(s+s_i)/r}.
\]

(2.13)

By the relation (2.13), applying the matrix \( (f'(\gamma_{il}^*(s)))^{-1} \) to both parts, we will have that

\[
\gamma_{il}^*(s) = (f'(\gamma_{il}^0(s)))^{-1} e^{i(s+s_i)/r}.
\]

(2.14)

Therefore, by (2.12) and (2.13) we obtain that

\[
\int_0^{l(\tilde{\gamma}_i)} \tilde{\rho}(\tilde{\gamma}_i^0(s)) \, ds \geq \int_0^{l(\tilde{\gamma}_i)} \rho(\gamma_{il}^*(s)) |\gamma_{il}^*(s)| \, ds.
\]

(2.15)
In this case, by the relation (2.11) it follows that
\[
\int_{\tilde{\gamma}_i} \tilde{\rho}(y) |dy| \geq \frac{1}{k} \sum_{l=1}^{k} \int_{0}^{l(\tilde{\gamma}_i)} \rho(\gamma^*_u(s)) |\gamma^*_u(s)| \, ds.
\] (2.16)

Let \( s_* = s_*(s) \) denote the length of the path \( \gamma^*_u(s) \) on the segment \([0, s]\). Note that the function \( s_* = s_*(s) \) is absolutely continuous for almost all dashed lines \( \gamma_i \). Indeed, let \( E \subset [0, l(\tilde{\gamma}_i)] \) has a zero Lebesgue measure, i.e., \( m_1(E) = 0 \). Then by [Fe, theorem 3.2.5] the set \( \{ y \in \mathbb{C} : \exists s \in E : \tilde{\gamma}_i(s) = y \} \) has \( H^1 \)-measure is zero. Since, by the condition of the lemma, the mapping \( f \) has \( N^{-1} \)-property on almost all spheres \( S(y_0, r) \), the set \( f^{-1}(\tilde{\gamma}_i(E)) \) is also of \( H^1 \)-measure is zero. However, in this case, by [Fe, theorem 3.2.5] the set \( E_* := \{ s \in [0, l(\gamma^*_u)] : \gamma^*_u(s) \in f^{-1}(\tilde{\gamma}_i(E)) \} \) has the linear measure zero. Note that \( E_* = s_*(E) \), therefore, the function \( s_* = s_*(s) \) has Luzin \( N \)-property. Then, since by the assumption the path \( \gamma^*_u(s) \) is rectifiable for almost all \( r \), according to [Fe] Theorem 2.10.13 the function \( s_* = s_*(s) \) is absolutely continuous for almost all dashed lines \( \gamma_i \), which had to be proved.

In this case, the path \( \gamma^*_u(s) \) is also absolutely continuous, because \( \gamma^*_u(s) = \gamma^*_0(s_*(s)) \), and the path \( \gamma^*_0 \) is absolutely continuous (and even Lipschitz) with respect to its natural parameter \( s_* \). Then, by [Fe] theorem 3.2.5
\[
\int_{0}^{l(\tilde{\gamma}_i)} \rho(\gamma^*_u(s)) |\gamma^*_u(s)| \, ds = \int_{|\gamma^*_u|} \rho(z) N(\gamma^*_u, [0, l(\tilde{\gamma}_i)], z) \, dH^1(z) \geq \int_{|\gamma^*_u|} \rho(z) \, dH^1(z).
\] (2.17)

Then by (2.16) and (2.17) it follows that
\[
\int_{\tilde{\gamma}_i} \tilde{\rho}(y) |dy| \geq \frac{1}{k} \sum_{l=1}^{k} \int_{|\gamma^*_u|} \rho(z) \, dH^1(z).
\] (2.18)

We sum the ratio (2.18) over \( i = 1, 2, \ldots \) Due to the relation (3) on the page 10 and since \( \rho \in \text{adm} \Sigma_e \), we have that
\[
\int_{S_r} \tilde{\rho}(y) |dy| = \sum_{i=1}^{\infty} \int_{\tilde{\gamma}_i} \tilde{\rho}(y) |dy| \geq \frac{1}{k} \sum_{i=1}^{\infty} \sum_{l=1}^{k} \int_{|\gamma^*_u|} \rho(z) \, dH^1(z) \geq 1/k
\] (2.19)

for \( p \)-almost all \( S_r \subset \Gamma \), that is, \( k \cdot \tilde{\rho} \in \text{ext adm}_p \Gamma \).

Recall that the relation (2.19) has been proved under the condition that all paths \( \gamma^*_u \) are rectifiable. The general case, when they are only locally rectifiable, comes from (2.19) by taking any rectifiable subpaths \( \tilde{\gamma}_i \subset \gamma_i \) instead of \( \gamma_i \). As a result of considerations similar to those carried out above, we will obtain the relation
\[
\sum_{i=1}^{\infty} \int_{\tilde{\gamma}_i} \tilde{\rho}(y) |dy| \geq \frac{1}{k} \sum_{i=1}^{\infty} \sum_{l=1}^{k} \int_{|\gamma^*_u|} \rho(z) \, dH^1(z),
\] (2.20)
where \( f \circ \gamma_u^* = \tilde{\gamma}_i^0 \), \( \gamma_u^* \subset \gamma_u^i \). In this case, we are left just to go to sup over all \( \tilde{\gamma}_i \subset \tilde{\gamma}_i. \) to obtain (2.19) in the ratio (2.20).

For the matrix \( f'(z) \), let \( \lambda_1(z) \leq \lambda_2(z) \) be the so-called principal numbers, i.e., \( f'(z)e = \lambda_i(z)e_i \) for some orthonormal systems of vectors \( e_1, e_2 \) and \( \tilde{e}_1, \tilde{e}_2 \) (see, e.g., [Re, Lemma 4.2, §4, Ch. I]). Then \(|J(z, f)| = \lambda_1(z) \cdot \lambda_2(z) \) (see relation (4.5) again), so that \( \lambda_1(z) > 0 \) for each \( z \in D \setminus (B_0 \cup B_s) \). In this case, \( \frac{1}{\lambda_2(z)} \leq \frac{1}{\lambda_1(z)} \) are the main numbers for the inverse mapping \((f'(z))^{-1}\). Since \( m(f(B_0) \cup f(B_s)) = 0 \), for almost all \( y \in f(B_k) \) we have that

\[
\left( \frac{1}{\lambda_2(f_k^{-1}(y))} \right) \geq \frac{1}{\lambda_2(f_k^{-1}(y))} > 0. \tag{2.21}
\]

Since \( \bar{\rho}^p(y) = \sup_{k \in \mathbb{N}} \bar{\rho}^p_k(y) \leq \sum_{k=1}^{\infty} \bar{\rho}^p_k(y) \) and \( m(f(B_s)) = m(f(B_0)) = 0 \), by (2.21) we obtain that

\[
\int_{f(D)} \frac{\bar{\rho}^p(y)}{Q(y)} \ dm(y) \leq \sum_{k=1}^{\infty} \int_{f(B_k)} \frac{\bar{\rho}^p_k(y)}{Q(y)} \ dm(y) \leq \sum_{k=1}^{\infty} \int_{f(B_k)} \rho^p(f_k^{-1}(y)) \left| J(y, f_k^{-1}) \right| \ dm(y) = \sum_{k=1}^{\infty} \int_{f(B_k)} \rho^p(f_k^{-1}(y)) \left| J(y, f_k^{-1}) \right| \ dm(y). \tag{2.22}
\]

Using the change of variables on each \( B_k, k = 1, 2, \ldots \), see, e.g., [Fe, theorem 3.2.5], we obtain that

\[
\int_{f(B_k)} \rho^p(f_k^{-1}(y)) \left| J(y, f_k^{-1}) \right| \ dm(y) = \int_{B_k} \rho^p(z) \ dm(z). \tag{2.23}
\]

It follows by (2.22) and (2.23) that

\[
\int_{f(D)} \frac{\bar{\rho}^p(y)}{Q(y)} \ dm(y) \leq \sum_{k=1}^{\infty} \int_{B_k} \rho^p(z) \ dm(z). \tag{2.24}
\]

Summing (2.24) over \( k = 1, 2, \ldots \) and using the countable additivity of the Lebesgue integral (see, e.g., [Sa, theorem I.12.3]), we obtain that

\[
\int_{f(D)} \frac{1}{Q(y)} \bar{\rho}^p(y) \ dm(y) \leq \int_{D} \rho^p(z) \ dm(z). \tag{2.25}
\]

Taking inf in (2.25) by all functions \( \rho \in \widetilde{\text{adm}} \Sigma_e \), we obtain that

\[
\int_{f(D)} \frac{1}{Q(y)} \bar{\rho}^p(y) \ dm(y) \leq \widetilde{M}_{\rho}(\Sigma_e). \]
Multiplying the last relation on \( N^p(f, D) \), we obtain that
\[
\int_{f(D)} \frac{N^p(f, D)}{Q(y)} \tilde{\rho}^p(y) \cdot dm(y) \leq N^p(f, D) \cdot \tilde{M}_p(\Sigma_\epsilon) .
\]

Denote by \( \tilde{\rho}_1(y) := N(f, D) \cdot \tilde{\rho}(y) \), we obtain from the last relation that
\[
\int_{f(D)} \frac{\tilde{\rho}_1^p(y)}{Q(y)} dm(y) \leq N^p(f, D) \cdot \tilde{M}_p(\Sigma_\epsilon) .
\] (2.26)

Since by the proving above \( \tilde{\rho}_1(y) = N(f, D) \tilde{\rho} \in \text{ext adm}_p f(\Sigma_\epsilon) \), it follows from (2.26) that (2.7) holds, as required. The lemma is proved. \( \Box \)

3 Proof of Theorem 1.1

Let \( Q_* : D \to [0, \infty] \) be a Lebesgue measurable function. Denote by \( q_{x_0}(r) \) the integral average of \( Q_*(x) \) under the sphere \(|x - x_0| = r\),
\[
q_{x_0}(r) := \frac{1}{2\pi} \int_{0}^{2\pi} Q_*(x_0 + re^{i\theta}) d\theta .
\] (3.1)

Below we also assume that the following standard relations hold: \( a/\infty = 0 \) for \( a \neq \infty \), \( a/0 = \infty \) for \( a > 0 \) and \( 0 \cdot \infty = 0 \) (see, e.g., [Sa, §3, section I]). The following conclusion was obtained by V. Ryazanov together with the author in the case \( p = 2 \), see, e.g., [MRSY, Lemma 7.4] or [RS, Lemma 2.2]. In the case of an arbitrary \( p > 1 \), see, for example, [SalSev2, Lemma 2].

**Proposition 3.1.** Let \( p > 1, n \geq 2, x_0 \in \mathbb{C}, r_1, r_2 \in \mathbb{R}, r_1, r_2 > 0 \), and let \( Q_*(x) \) be a Lebesgue measurable function, \( Q_* : \mathbb{C} \to [0, \infty], Q_* \in L^1_{\text{loc}}(\mathbb{C}) \). We put
\[
I = I(x_0, r_1, r_2) = \int_{r_1}^{r_2} \frac{dr}{r^{n-1} q_{x_0}^{p-1}(r)},
\]
and let \( q_{x_0}(r) \) be defined by (3.1). Then
\[
\frac{2\pi}{r^{p-1}} \leq \int_{A} Q_*(x) \cdot \eta^p(|x - x_0|) dm(x)
\] (3.2)
for any Lebesgue measurable function \( \eta : (r_1, r_2) \to [0, \infty] \) such that
\[
\int_{r_1}^{r_2} \eta(r) \, dr = 1,
\] (3.3)
where \( A = A(x_0, r_1, r_2) \) is defined in (1.4).
Remark 3.1. Note that, if \( (3.2) \) holds for any function \( \eta \) with a condition \( (3.3) \), then the same relationship holds for any function \( \eta \) with the condition \( (1.6) \). Indeed, let \( \eta \) be a nonnegative Lebesgue function that satisfies the condition \( (1.6) \). If \( J := \int_{r_1}^{r_2} \eta(t) \, dt < \infty \), then we put \( \eta_0 := \eta / J \). Obviously, the function \( \eta_0 \) satisfies condition \( (3.3) \). Then the relation \( (3.2) \) gives that

\[
\frac{2\pi}{\text{I}^{p-1}} \leq \frac{1}{\text{I}^p} \int_A Q_\ast(x) \cdot \eta^p(|x - x_0|) \, dm(x) \leq \int_A Q_\ast(x) \cdot \eta^p(|x - x_0|) \, dm(x)
\]

because \( J \geq 1 \). Let now \( J = \infty \). Then, by [Sa, Theorem I.7.4], a function \( \eta \) is a limit of a nondecreasing nonnegative sequence of simple functions \( \eta_m, m = 1, 2, \ldots \). Set \( J_m := \int_{r_1}^{r_2} \eta_m(t) \, dt < \infty \) and \( w_m(t) := \eta_m(t)/J_m \). Then, it follows from \( (3.3) \) that

\[
\frac{2\pi}{\text{I}^{p-1}} \leq \frac{1}{\text{I}^p} \int_A Q_\ast(x) \cdot \eta^p_m(|x - x_0|) \, dm(x) \leq \int_A Q_\ast(x) \cdot \eta^p_m(|x - x_0|) \, dm(x) \tag{3.4}
\]

because \( J_m \to J = \infty \) as \( m \to \infty \) (see [Sa, Lemma I.11.6]). Thus, \( J_m \geq 1 \) for sufficiently large \( m \in \mathbb{N} \). Observe that, a functional sequence \( \psi_m(x) = Q_\ast(x) \cdot \eta^p_m(|x - x_0|), m = 1, 2, \ldots \), is nonnegative, monotone increasing and converges to a function \( \psi(x) := Q_\ast(x) \cdot \eta^p(|x - x_0|) \) almost everywhere. By the Lebesgue theorem on the monotone convergence (see [Sa, Theorem I.12.6]), it is possible to go to the limit on the right side of the inequality \( (3.4) \), which gives us the desired inequality \( (3.2) \).

Proof of Theorem 1.1 Fix \( y_0 \in \overline{f(D)} \setminus \{\infty\}, 0 < r_1 < r_2 < r_0 = \sup_{y \in f(D)} |y - y_0|, C_1 \subset B(y_0, r_1) \cap f(D) \) and \( C_2 \subset f(D) \setminus B(y_0, r_2) \). Set

\[
C_0 := f^{-1}(C_1), \quad C_0^* := f^{-1}(C_2)
\]

(see Figure 3). Observe that \( C_0 \) and \( C_1 \) are disjoint compact sets in \( D \), see [Vu, Theorem 3.3]. Besides that, \( C_1 \) and \( C_2 \) are non empty by the choice of \( r_0, r_1 \) and \( r_2 \).

Let us to show that a set \( \sigma_r := f^{-1}(S(y_0, r)) \) separates \( C_0 \) from \( C_0^* \) in \( D \) for any \( \sigma_r \in (r_1, r_2) \). Indeed, \( \sigma_r \) is closed in \( D \) as a preimage of a closed set \( S(y_0, r) \) under the continuous mapping \( f \) (see, e.g., [Ku, Theorem 1.IV.13, Ch. 1]). In particular, \( \sigma_r \) is also closed with respect to \( R := D \setminus (C_0 \cup C_0^*) \). We put

\[
A := f^{-1}(B(y_0, r))
\]

and

\[
B := D \setminus f^{-1}(B(y_0, r)).
\]

Observe that, \( A \) and \( B \) are not empty by the choice of \( r_0, r_1, r_2 \) and \( r \). Since \( f \) is continuous, \( f^{-1}(B(y_0, r)) \) and \( D \setminus f^{-1}(B(y_0, r)) \) are open in \( D \). In other words, \( A \) and \( B \) are open in

\[
R^* := R \cup C_0 \cup C_1 = D.
\]
Figure 3: To the proof of Theorem 1.1

Note that \( A \cap B = \emptyset \), and \( R^* \setminus \sigma_r = A \cup B \). Let \( \Sigma_{C_0, C_0^*} \) be the family of all sets separating \( C_0 \) and \( C_0^* \) in \( R^* \). In this case, by the equations of Ziemer and Hesse, see (2.3) and (2.4), respectively, we obtain that

\[
M_\alpha(\Gamma_f(y_0, C_1, C_2)) = \left( \tilde{M}_p(\Sigma_{r_1, r_2}) \right)^{1-\alpha}, \tag{3.5}
\]

where \( \alpha = \frac{p}{p-1} \). Then by Lemma 2.1 and by the relation (3.5), we obtain that

\[
M_\alpha(\Gamma_f(y_0, r_1, r_2)) \leq \inf_{\rho \in \text{ext adm} f(\Sigma_x)} \int_{f(D) \cap A(y_0, r_1, r_2)} \frac{\rho^p(y)}{N_p(f, D) \cdot Q(y)} \, dm(y) \left( \frac{1}{p-1} \right), \tag{3.6}
\]

where \( Q(y) := K_{CT,p,y_0}(y, f) := \sum_{z \in f^{-1}(y)} \left( \frac{|f'(z)|^{-1} \cdot \frac{y_0 - y}{|y_0 - y|}}{\alpha \left| f'(z) \right|} \right)^p \). Using the second remote formula in the proof of Theorem 9.2 in [MRSY], we obtain that

\[
\inf_{\rho \in \text{ext adm} f(\Sigma_x)} \int_{f(D) \cap A(y_0, r_1, r_2)} \frac{\rho^p(y)}{N_p(f, D) \cdot Q(y)} \, dm(y) = \int_{r_1}^{r_2} \left( \inf_{\alpha \in I(r)} \int_{S(y_0,r) \cap f(D)} \frac{\alpha^p(y)}{N_p(f, D) \cdot Q(y)} \, \mathcal{H}^1(y) \right) \, dr, \tag{3.7}
\]

where \( I(r) \) denotes the set of all measurable functions on \( S(y_0, r) \cap f(D) \) such that

\[
\int_{S(y_0,r) \cap f(D)} \alpha(x) \, \mathcal{H}^1 = 1.
\]
Then, choosing \( X = S(y_0, r) \cap f(D) \), \( \mu = \mathcal{H}^1 \) and \( \varphi = \frac{1}{\partial} \) in [MRSY, Lemma 9.2], we obtain that
\[
\int_{r_1}^{r_2} \left( \inf_{x \in f(r)} \int_{S(y_0, r) \cap f(D)} \frac{\alpha^p(y)}{Q(y)} \, d\mathcal{H}^1 \right) \, dr = \int_{r_1}^{r_2} \frac{dr}{\|Q\|_s(r)},
\]
where \( \|Q\|_s(r) = \left( \int_{S(y_0, r) \cap f(D)} Q^s(x) \, d\mathcal{H}^1 \right)^{1/s} \) and \( s := \frac{1}{p-1} = \alpha - 1 \). Thus, by (3.6), (3.7) and (3.8) we obtain that
\[
M_{\alpha}(\Gamma_f(y_0, r_1, r_2)) \leq \left( \int_{r_1}^{r_2} \frac{dr}{\|Q\|_s(r)} \right)^{-\frac{1}{p-1}}
= \frac{N^{\alpha}(f, D) \cdot 2\pi}{\left( \int_{r_1}^{r_2} \frac{dr}{\|Q\|_s(r)} \right)^{\frac{\alpha - 1}{\alpha - 1}}} = \frac{N^{\alpha}(f, D) \cdot 2\pi}{\left( \int_{r_1}^{r_2} \frac{dr}{\|Q\|_s(r)} \right)^{\frac{\alpha - 1}{\alpha - 1}}}, \tag{3.9}
\]
where \( q_{y_0}(r) = \frac{1}{2\pi r} \int_{S(y_0, r)} \tilde{Q} \, d\mathcal{H}^1 \) and \( \tilde{Q}(y) = \begin{cases} Q^{\alpha - 1}(y), & y \in f(D), \\ 0, & y \not\in f(D) \end{cases} \). Finally, it follows from (3.9) and Proposition 3.1 that the relation
\[
M_{\alpha}(\Gamma_f(y_0, r_1, r_2)) \leq \int_{A(y_0, r_1, r_2) \cap f(D)} N^{\alpha}(f, D) \cdot Q^{\alpha - 1}(y) \cdot \eta^{\alpha}(\|y - y_0\|) \, dm(y)
\]
holds for a function \( Q(y) := K_{CT, \frac{\alpha}{\alpha - 1}, y_0}(y, f) := \sum_{z \in f^{-1}(y)} \left( \frac{1}{|f'(z)|} \frac{y - y_0}{|y - y_0|} \right)^{\frac{\alpha}{\alpha - 1}} \), that is desired conclusion. \( \square \)

4 Proofs of Theorems 1.2–1.4

The next class of mappings is a generalization of quasiconformal mappings in the sense of Gehring’s ring definition (see [Ge]; it is the subject of a separate study, see, e.g., [MRSY], Chapter 9). Let \( D \) and \( D' \) be domains in \( \mathbb{C} \). Suppose that \( x_0 \in \overline{D} \setminus \{\infty\} \) and \( Q: D \to (0, \infty) \) is a Lebesgue measurable function. A function \( f: D \to D' \) is called a lower \( Q \)-mapping at a point \( x_0 \) relative to the \( p \)-modulus if
\[
M_p(f(\Sigma_\varepsilon)) \geq \inf_{\rho \in \text{extr}_{p, \text{adm}} \Sigma_\varepsilon} \int_{D \setminus A(x_0, \varepsilon, r_0)} \frac{\rho^p(x)}{Q(x)} \, dm(x) \tag{4.1}
\]
for every spherical ring \( A(x_0, \varepsilon, r_0) = \{ x \in \mathbb{C} : \varepsilon < |x - x_0| < r_0 \}, r_0 \in (0, d_0), d_0 = \sup_{x \in D} |x - x_0|, \) where \( \Sigma_\varepsilon \) is the family of all intersections of the spheres \( S(x_0, r) \) with the
domain \( D, r \in (\varepsilon, r_0) \). If \( p = 2 \), we say that \( f \) is a lower \( Q \)-mapping at \( x_0 \). We say that \( f \) is a lower \( Q \)-mapping relative to the \( p \)-modulus in \( A \subset \overline{D} \) if (4.1) is true for all \( x_0 \in A \).

The following statement can be proved much as Theorem 9.2 in [MRSY], so we omit the arguments.

**Lemma 4.1.** Let \( D, D' \subset \overline{\mathbb{C}} \), let \( x_0 \in \overline{D} \setminus \{\infty\} \), and let \( Q \) be a Lebesgue measurable function. A mapping \( f : D \to D' \) is a lower \( Q \)-mapping relative to the \( p \)-modulus at a point \( x_0 \), \( p > 1 \), if and only if \( M_p(f(\Sigma_\varepsilon)) \geq \int_0^r \frac{d\varepsilon}{\|Q\|_s(r)} \) for all \( \varepsilon \in (0, r_0), \ r_0 \in (0, d_0) \),

\[
d_0 = \sup_{x \in D} |x - x_0|, \quad s = \frac{1}{p - 1},
\]

where, as above, \( \Sigma_\varepsilon \) denotes the family of all intersections of the spheres \( S(x_0, r) \) with \( D, r \in (\varepsilon, r_0) \), \( \|Q\|_s(r) = \int_{D(x_0, r)} Q^s(x) dA \) is the \( L_s \)-norm of \( Q \) over the set \( D(x_0, r) = \{x \in D : |x - x_0| = r\} = D \cap S(x_0, r) \).

The following statement holds, cf. [KR, Theorem 2.1], [SSP, Lemma 2.3], [Sev2, Lemma 2] and [RSY3, Theorem 4.1].

**Theorem 4.1.** Let \( p > 1 \) and let \( f : D \to \mathbb{C} \) be an open discrete mapping of a finite distortion such that \( N(f, D) < \infty \). Then \( f \) satisfies the relation (4.1) at any \( z_0 \in \overline{D} \) for \( Q(z) = N(f, D) : K_{T,p,z_0}(z, f) \), where

\[
Q(z) := K_{T,p,z_0}(z, f) := \left( \left( \left| f'(z) \right| \frac{z - z_0}{|z - z_0|} \right)^p \right)\frac{1}{|J(z, f)|}
\]

and \( N(f, D) \) is defined in (4.1).

**Proof.** The proof of this Theorem uses the scheme outlined in [Sev1, Theorem 4], cf. [RSY2, Theorem 4.1]. Observe that, \( f = \varphi \circ g \), where \( g \) is some homeomorphism and \( \varphi \) is an analytic function, see ([St, 5.III.V]). Thus, \( f \) is differentiable almost everywhere (see, e.g., [LV, Theorem III.3.1]). Let \( B \) be a Borel set of all points \( z \in D \), where \( f \) has a total differential \( f'(z) \) and \( J(z, f) \neq 0 \). Observe that, \( B \) may be represented as a countable union of Borel sets \( B_l, l = 1, 2, \ldots \), such that \( f_l = f|_{B_l} \) are bi-lipschitzian homeomorphisms (see [Bel] items 3.2.2, 3.1.4 and 3.1.8)). Without loss of generality, we may assume that the sets \( B_l \) are pairwise disjoint. Denote by \( B_* \) the set of all points \( z \in D \) in which \( f \) has a total differential, however, \( f'(z) = 0 \).

Since \( f \) is of finite distortion, \( f'(z) = 0 \) for almost all \( z \), where \( J(z, f) = 0 \). Thus, by the construction the set \( B_* := D \setminus (B \cup B_*) \) has a zero Lebesgue measure. Therefore, by Proposition 2.1 and Remark 2.1 \( l(B_0 \cap S_r) = 0 \) for \( p \)-almost all circles \( S_r := S(z_0, r) \) centered at \( z_0 \in \overline{D} \). Observe that, a function \( \psi(r) := H^1(B_0 \cap S_r) = l(B_0 \cap S_r) \) is Lebesgue measurable by the Fubini theorem, thus, the set \( E = \{r \in \mathbb{R} : l(B_0 \cap S_r) = 0\} \) is Lebesgue measurable.

Now, by Proposition 2.2 we obtain that

\[
l(B_0 \cap S_r) = 0 \quad \text{for almost any} \ r \in \mathbb{R}.
\]

Let us fix the circle

\[
\gamma(t) = S(z_0, r) = re^{it} + y_0, \quad t \in [0, 2\pi). 
\]
Then \( S_r := S(y_0, r) \cap D \) is a family of dashed lines

\[
\gamma_i : \bigcup_{i=1}^{\infty} (a_i, b_i) \rightarrow D, \quad (a_i, b_i) \subset \mathbb{R}, \quad i = 1, 2, \ldots
\]

Consider the normal representation \( \gamma_i^0 \) for \( \gamma_i \), namely,

\[
\gamma_i^0(t) = z_0 + r e^{i(t + t_0)/r}, \quad t \in (0, l(\gamma_i^0)),
\]

where \( t_0 \in \mathbb{R} \) is a number such that \( \gamma_i^0(t_0) = \gamma_i(a_i) \), and the number \( l(\gamma_i^0) \) denotes the length of \( \gamma_i \) (see [Va, Definition 2.5]).

Let \( \Gamma \) be a family of all intersections of circles \( S_r, \ r \in (\varepsilon, \varepsilon_0) \), \( \varepsilon_0 < d_0 = \sup_{z \in D} |z - z_0| \), with a domain \( D \). Given an admissible function \( \rho_* \in \text{adm} f(\Gamma) \), \( \rho_* \equiv 0 \) outside of \( f(D) \), we set \( \rho \equiv 0 \) outside of \( D \) and on \( B_0 \cup B_* \), and

\[
\rho(z) := \rho_*(f(z)) \left| f'(z) \frac{z - z_0}{|z - z_0|} \right| \quad \text{for} \ z \in D \setminus B_0.
\]

Denote by

\[
\gamma_i(B_0 \cup B_*) = \{ t \in (0, l(\gamma_i)) : \gamma_i(t) \in B_0 \cup B_* \}.
\]

Using (4.3) and theorem on the derivative of superpositions of mappings, we obtain that

\[
\int_{\gamma_i} \rho(z) \, |dz| = \int_{[0, l(\gamma_i)] \setminus \gamma_i(B_0 \cup B_*)} \rho(\gamma_i^0(t)) \, dt + \int_{\gamma_i(B_0 \cup B_*)} \rho(\gamma_i^0(t)) \, dt = 0
\]

\[
= \int_{[0, l(\gamma_i)] \setminus \gamma_i(B_0 \cup B_*)} \rho_*(f(z_0 + re^{i(t + t_0)/r})) \left| f'(z_0 + re^{i(t + t_0)/r})re^{i(t + t_0)/r} \right| \, dt
\]

\[
= \int_{[0, l(\gamma_i)] \setminus \gamma_i(B_0 \cup B_*)} \rho_*(f(z_0 + re^{i(t + t_0)/r})) \left| f(z_0 + re^{i(t + t_0)/r})(t) \right| \, dt = 0
\]

\[
= \int_{\gamma_i} \rho_*(f(z_0 + re^{i(t + t_0)/r})) \left| f(z_0 + re^{i(t + t_0)/r})(t) \right| \, dt = \int_{f(\gamma_i)} \rho_*(w) \, |dw|
\]

for a.e. \( r \), because \( f \in W^{1,1}_{\text{loc}} \) and, consequently, the path \( f(\gamma_i(t)) \) is absolutely continuous over \( t \) for a.e. \( r \) (see [Ma, Theorem 1.1] and [Va, Theorem 4.1]). Here we have used the fact this \( z_0 + re^{i(t + t_0)/r} \not\in B_0 \) for a.e. \( r \) and for a.e. \( t \), that follows from (4.2). Similarly, since by the condition \( f \) is a mapping with a finite distortion, that implies that \( (f(z_0 + re^{i(t + t_0)/r})(t))' = 0 \) for any \( t \) such that \( z_0 + re^{i(t + t_0)/r} \in B_* \), that has been used in (4.4), as well. The relation (4.4) implies that

\[
\int_{\gamma_i} \rho(z) \, |dz| = \int_{f(\gamma_i)} \rho_*(w) \, |dw|
\]
for a.e. $r$. Summing the last relation over all $i = 1, 2, \ldots$, we obtain that
\[
\int_{S_r} \rho(z) |dz| = \int_{f(S_r)} \rho_s(w) |dw| \geq 1
\]
for a.e. $r$ and, consequently, for $p$-a.e. $S_r$ in the sense of $p$-modulus (see Proposition 2.2). Thus, $\rho \in \text{ext}_p \text{adm } \Gamma$.

Using the change of variables on $B_l$, $l = 1, 2, \ldots$ (see, e.g., [Fe, Theorem 3.2.5]), by the countable additivity of the Lebesgue integral we obtain that
\[
\int_D \frac{\rho^p(z)}{K_{T,p,\nu}(z,f)} \, dm(z) = \sum_{l=1}^{\infty} \int_{B_l} \rho^p(f(z)) |J(z,f)| \, dm(z) =
\]
\[
= \sum_{l=1}^{\infty} \int_{f(B_l)} \rho^p_s(y) \, dm(y) \leq \int_{f(D)} N(f,D) \rho^p_s(y) \, dm(y),
\]
as required. $\Box$

We also need the following statement given in [Ri, Proposition 10.2, Ch. III].

**Proposition 4.1.** Let $E = (A, C)$ be a condenser in $\mathbb{C}$ and let $\Gamma_E$ be the family of all paths of the form $\gamma : [a, b] \to A$ with $\gamma(a) \in C$ and $|\gamma| \cap (A \setminus F) \neq \emptyset$ for every compact $F \subset A$. Then $\text{cap}_q E = M_q(\Gamma_E)$.

An analogue of the following assertion has been proved several times earlier under slightly different conditions, see [SevSkv, Lemma 4.2] and [Sev1, Lemma 5]. In the formulation given below, this result is proved for the first time.

**Lemma 4.2.** Let $D$ be a domain in $\mathbb{C}$, let $p > 1$, let $x_0 \in D$ and let $f : D \to \mathbb{C}$ be an open and discrete mapping satisfying the relation (4.1) at a point $x_0$. Assume that $Q : D \to [0, \infty]$ is a Lebesgue measurable function which is locally integrable in the degree $s = \frac{1}{p-1}$ in $D$. Then the relation
\[
\text{cap}_c f(\mathcal{E}) \leq \int_{A} Q^*(x) \cdot \eta^\alpha(|x - x_0|) \, dm(x)
\]  
holds for $\alpha = \frac{p}{p-1}$ and $Q^*(x) = Q_{r_1,r_2}^{\frac{1}{p-1}}(x)$, where $\mathcal{E} = (B(x_0, r_2), B(x_0, r_1))$, $A = A(x_0, r_1, r_2)$, $0 < r_1 < r_2 < \varepsilon_0 := \text{dist}(x_0, \partial D)$, and $\eta : (r_1, r_2) \to [0, \infty]$ may be chosen as arbitrary nonnegative Lebesgue measurable function satisfying the relation (1.12).

**Proof.** Observe that $s = \alpha - 1$. By Lemma 2 in [SalSev2], it is sufficiently to prove that
\[
\text{cap}_c f(\mathcal{E}) \leq \frac{2\pi}{f_{\ast \alpha - 1}},
\]
where $\mathcal{E}$ is a condenser $\mathcal{E} = (B(x_0, r_2), B(x_0, r_1))$, and $q_{s_0}^*(r)$ denotes the integral average of $Q^{\alpha-1}(x)$ under $S(x_0, r)$,
\[
q_{s_0}(r) = \frac{1}{2\pi r} \int_{S(x_0, r)} Q(x) \, d\mathcal{H}^1,
\]
where
\[ I^* = I^*(x_0, r_1, r_2) = \int_{r_1}^{r_2} \frac{dr}{r \sqrt{\frac{1}{q r_0} - 1}}. \]

Let \( \varepsilon \in (r_1, r_2) \) and let \( B(x_0, \varepsilon) \). We put \( C_0 = \partial f(B(x_0, r_2)), C_1 = f(B(x_0, r_1)), \sigma = \partial f(B(x_0, \varepsilon)) \). Since \( f \) is continuous in \( D \), the set \( f(B(x_0, r_2)) \) is bounded.

Since \( f \) is continuous, \( f(B(x_0, r_1)) \) is a compact subset of \( f(B(x_0, \varepsilon)) \), and \( f(B(x_0, \varepsilon)) \) is a compact subset of \( f(B(x_0, r_2)) \). In particular,
\[ \overline{f(B(x_0, r_1)) \cap \partial f(B(x_0, \varepsilon))} = \emptyset. \]

Let, as above, \( R = G \setminus (C_0 \cup C_1), G := f(D), \) and \( R^* = R \cup C_0 \cup C_1 \). Then \( R^* \). Observe that, \( \sigma \) separates \( C_0 \) from \( C_1 \) in \( R^* = G \). Indeed, the set \( \sigma \cap R \) is closed in \( R \), besides that, if \( A := G \setminus f(B(x_0, \varepsilon)) \) and \( B = f(B(x_0, \varepsilon)) \), then \( A \) and \( B \) are open in \( G \setminus \sigma, C_0 \subset A, C_1 \subset B \) and \( G \setminus \sigma = A \cup B \).

Let \( \Sigma \) be a family of all sets, which separate \( C_0 \) from \( C_1 \) in \( G \). Below by \( \bigcup_{r_1 < r < r_2} \partial f(B(x_0, r)) \) or \( \bigcup_{r_1 < r < r_2} f(S(x_0, r)) \) we mean the union of all Borel sets into a family, but not in a theoretical-set sense (see [Z1 item 3, p. 464]). Let \( \rho \in \overline{\text{adm}} \bigcup_{r_1 < r < r_2} \partial f(B(x_0, r)) \) in the sense of the relation (2.2). Then \( \rho \in \text{adm} \bigcup_{r_1 < r < r_2} \partial f(B(x_0, r)) \) in the sense of (1.2). By the openness of the mapping \( f \) we obtain that \( \partial f(B(x_0, r)) \subset f(S(x_0, r)) \), therefore, \( \rho \in \text{adm} \bigcup_{r_1 < r < r_2} f(S(x_0, r)) \) and, consequently, by (2.1)

\[
\overline{M_p}(\Sigma) \geq \overline{M_p}\left( \bigcup_{r_1 < r < r_2} \partial f(B(x_0, r)) \right) \geq \overline{M_p}\left( \bigcup_{r_1 < r < r_2} f(S(x_0, r)) \right) \geq M_p\left( \bigcup_{r_1 < r < r_2} f(S(x_0, r)) \right) \quad (4.7)
\]

However, by (2.3) and (2.4) we obtain that
\[
\frac{1}{(M_\alpha(\Gamma(C_0, C_1, G)))^{1/(\alpha-1)}} = \overline{M_p}(\Sigma). \quad (4.8)
\]

Let \( \Gamma_{f(\mathcal{E})} \) be a family of all paths which correspond to the condenser \( f(\mathcal{E}) \) in the sense of Proposition 4.1, and let \( \Gamma_{f(\mathcal{E})}^* \) be a family of all rectifiable paths of \( \Gamma_{f(\mathcal{E})} \). Now, observe that, the families \( \Gamma_{f(\mathcal{E})} \) and \( \Gamma(C_0, C_1, G) \) have the same families of admissible functions \( \rho \). Thus,
\[
M_\alpha(\Gamma_{f(\mathcal{E})}) = M_\alpha(\Gamma(C_0, C_1, G)).
\]
By Proposition 4.1 we obtain that \( M_\alpha(\Gamma_f(\mathcal{E})) = \text{cap}_\alpha f(\mathcal{E}) \). By (4.8) we obtain that
\[
\left( \tilde{M}_p(\Sigma) \right)^{\alpha-1} = \frac{1}{\text{cap}_\alpha f(\mathcal{E})}.
\] (4.9)

Finally, by (4.7) and (4.9) we obtain that
\[
\text{cap}_\alpha f(\mathcal{E}) \leq \frac{1}{M_\alpha \left( \bigcup_{r_1 < r < r_2} f(S(x_0, r)) \right)^{\alpha-1}}.
\]

By Lemma 4.1 we obtain that
\[
\text{cap}_\alpha f(\mathcal{E}) \leq \frac{1}{\int_{r_1}^{r_2} d\rho} = \frac{1}{I^{* \alpha-1}},
\]
as required. □

The following result is proved in [Sev3, Theorem 5].

**Proposition 4.2.** Let \( x_0 \in \partial D \), let \( f : D \to \mathbb{C} \) be an open, discrete and closed bounded lower \( Q \)-mapping with a respect to \( p \)-modulus in \( D \subset \mathbb{C}, Q \in L_{\text{loc}}^1(\mathbb{C}) \), \( 1 < p \), and \( \alpha := \frac{p}{p-1} \). Then for any \( \varepsilon_0 < d_0 := \sup_{x \in D} |x - x_0| \) and any compactum \( C_2 \subset D \setminus B(x_0, \varepsilon_0) \) there is \( \varepsilon_1, 0 < \varepsilon_1 < \varepsilon_0 \), such that the inequality
\[
M_\alpha(f(\Gamma(C_1, C_2, D))) \leq \int_{A(x_0, \varepsilon, \varepsilon_1)} Q_1^{\frac{1}{p-1}}(x) \eta^\alpha(|x - x_0|) \, dm(x),
\] (4.10)
holds for any \( \varepsilon \in (0, \varepsilon_1) \) and any compactum \( C_1 \subset \overline{B(x_0, \varepsilon)} \cap D \), where \( A(x_0, \varepsilon, \varepsilon_1) = \{ x \in \mathbb{C} : \varepsilon < |x - x_0| < \varepsilon_1 \} \) and \( \eta : (\varepsilon, \varepsilon_1) \to [0, \infty] \) is any nonnegative Lebesgue measurable function such that
\[
\int_{\varepsilon}^{\varepsilon_1} \eta(r) \, dr = 1.
\] (4.11)

**Remark 4.1.** Note that, if (1.11) holds for any function \( \eta \) with a condition (1.11), then the same relationship holds for any function \( \eta \) with the condition (1.12). Indeed, let \( \eta \) be a nonnegative Lebesgue function that satisfies the condition (1.12). If \( J := \int_{r_1}^{r_2} \eta(t) \, dt < \infty \), then we put \( \eta_0 := \eta/J \). Obviously, the function \( \eta_0 \) satisfies condition (4.11). Then the relation (4.10) gives that
\[
M_\alpha(f(\Gamma(C_1, C_2, D))) \leq \int_{A} Q(x) \cdot \eta^\alpha(|x - x_0|) \, dm(x) \leq \int_{A} Q(x) \cdot \eta^\alpha(|x - x_0|) \, dm(x)
\] (4.12)
because $J \geq 1$. Let now $J = \infty$. Then, by [Sa, Theorem I.7.4], a function $\eta$ is a limit of a nondecreasing nonnegative sequence of simple functions $\eta_m$, $m = 1, 2, \ldots$. Set $J_m := \int_{r_1}^{r_2} \eta_m(t) \, dt < \infty$ and $w_m(t) := \eta_m(t)/J_m$. Then, similarly to (4.12) we obtain that

$$M_\alpha(f(\Gamma(C_1, C_2, D))) \leq \frac{1}{J_m} \int_{A} Q(x) \cdot \eta_m^\alpha(\|x - x_0\|) \, dm(x) \leq \int_{A} Q(x) \cdot \eta_m^\alpha(\|x - x_0\|) \, dm(x), \tag{4.13}$$

because $J_m \rightarrow J = \infty$ as $m \rightarrow \infty$ (see [Sa, Lemma I.11.6]). Thus, $J_m \geq 1$ for sufficiently large $m \in \mathbb{N}$. Observe that, if a functional sequence $\varphi_m(x) = Q_*(x) \cdot \eta_m^\alpha(\|x - x_0\|)$, $m = 1, 2, \ldots$, is nonnegative, monotone increasing and converges to a function $\varphi(x) := Q_*(x) \cdot \eta^\alpha(\|x - x_0\|)$ almost everywhere. By the Lebesgue theorem on the monotone convergence (see [Sa, Theorem I.12.6]), it is possible to go to the limit on the right side of the inequality (4.13), which gives us the desired inequality (1.11).

The following result is proved in [Sev, Theorem 6].

**Proposition 4.3.** Let $x_0 \in \partial D$, and let $f : D \rightarrow \mathbb{C}$ be a bounded lower $Q$-homeomorphism with respect to $p$-modulus in a domain $D \subset \mathbb{C}$, $Q \in L^1_{\text{loc}}(\mathbb{C})$, $p > 1$ and $\alpha := \frac{p-1}{p-1}$. Then $f$ is a ring $Q^{\frac{1}{p-1}}$-homeomorphism with respect to $\alpha$-modulus at this point, where $\alpha := \frac{p-1}{p-1}$.

**Proof of Theorem 1.2.** Fix $z_0 \in D$. Two situations are possible: when $z_0 \in D$, and when $z_0 \in \partial D$. Let $z_0 \in D$. Set $p := \frac{\alpha}{\alpha-1}$, then $\alpha = \frac{p}{p-1}$. Due to Theorem 4.1, $f$ satisfies the relation (4.11) with

$$Q(z) := N(f, D) \cdot K_{T, p, z_0}(z, f) = N(f, D) \cdot \frac{\left(\left|f'(z)\right| \frac{z-z_0}{|z-z_0|}\right)^p}{|J(z, f)|}. \tag{4.14}$$

Now, by Lemma 4.2 $f$ satisfies the relation (4.5) with

$$Q^*(z) := K_{T, p, z_0}^{\alpha-1}(z, f) := \left(\frac{\left|f'(z)\right| \frac{z-z_0}{|z-z_0|}}{|J(z, f)|}\right)^{\alpha-1}. \tag{4.15}$$

Observe that the relation

$$M_\alpha(f(\Gamma(C_1, C_2, D))) \leq \text{cap}_\alpha(f(B(z_0, r_2)), f(B(z_0, r_1))) \tag{4.16}$$

holds for all $0 < r_1 < r_2 < d_0 := \text{dist}(z_0, \partial D)$ and for any continua $C_1 \subset B(z_0, r_1)$, $C_2 \subset D \setminus B(z_0, r_2)$. Indeed, $f(\Gamma(C_1, C_2, D)) > \Gamma_{f(E)}$, where $E := (f(B(z_0, r_2)), f(B(z_0, r_1)))$, and $\Gamma_{f(E)}$ is a family from Proposition 4.1 for the condenser $f(E)$. The relation (4.16) finishes the proof for the case $z_0 \in D$. Let now $z_0 \in \partial D$. Again, by Theorem 4.1, $f$ satisfies the relation (4.11) with $Q(z)$ from (4.14). Now $f$ satisfies the relation (1.11) with $Q^*(z)$ from the relation (4.15) by Proposition 4.3. □
Proof of Theorem 1.3 directly follows by Theorem 4.1 and Lemma 4.2.

Proof of Theorem 1.4 directly follows by Theorem 4.1 and Proposition 4.3.

References

[Fe] Federer, H.: Geometric Measure Theory. – Springer, Berlin etc., 1969.

[Ge] Gehring, F.W.: Rings and quasiconformal mappings in space. - Trans. Amer. Math. Soc. 103, 1962, 353–393.

[Hes] Hesse, J.: A $p$-extremal length and $p$-capacity equality. - Ark. Mat. 13, 1975, 131–144.

[IS] Ilyutko, D., E. Sevost’yanov: Boundary behaviour of open discrete mappings on Riemannian manifolds. - Sb. Math. 209:5, 2018, 605–651.

[KR] Kovtonyuk, D., V. Ryazanov: New modulus estimates in Orlicz-Sobolev classes. - Annals of the University of Bucharest (mathematical series) 5 (LXIII), 2014, 131–135.

[KRSS] Kovtonyuk, D., V. Ryazanov, R. Salimov and E. Sevost’yanov: Toward the theory of Orlicz-Sobolev classes. - St. Petersburg Math. J. 25:6, 2014, 929–963.

[Ku] Kuratowski, K. Topology, vol. 1. - Academic Press, New York, 1968.

[LV] Lehto, O. and K. Virtanen: Quasiconformal Mappings in the Plane. - Springer, New York etc., 1973.

[MRSY] Martio, O., V. Ryazanov, U. Srebro, and E. Yakubov: Moduli in modern mapping theory. - Springer Science + Business Media, LLC, New York, 2009.

[MS] Martio, O., U. Srebro U.: Periodic quasimeromorphic mappings. - J. Analyse Math. 28, 1975, 20–40.

[Ma] Maz’ya, V.: Sobolev Spaces. - Springer-Verlag, Berlin, 1985.

[Re] Reshetnyak Yu. G. Space Mappings with Bounded Distortion. – Transl. of Math. Monographs, 73, AMS, 1989.

[Ri] Rickman, S.: Quasiregular mappings. – Springer-Verlag, Berlin, 1993.

[RS] Ryazanov, V.I. and E.A. Sevost’yanov: Equicontinuous classes of ring $Q$-homeomorphisms. - Siberian Math. J. 48:6, 2007, 1093–1105.

[RSY1] Ryazanov, V., U. Srebro and E. Yakubov: On ring solutions of Beltrami equations. - J. d’Anal. Math. 96, 2005, 117–150.

[RSY2] Ryazanov, V., R. Salimov R. and E. Yakubov: On Boundary Value Problems for the Beltrami Equations. - Contemporary Mathematics Volume 591, 2013, 211–242.
[Sa] Saks, S.: Theory of the Integral. – Dover Publ. Inc., New York, 1964.

[SalSev1] Sevost’yanov, E., R. Salimov: On a Väisälä-type inequality for the angular dilatation of mappings and some of its applications. - Journal of Mathematical Sciences 218:1, 2016, 69–88.

[SalSev2] Salimov, R.R., E.A. Sevost’yanov: On equicontinuity of one family of inverse mappings in terms of prime ends. - Ukr. Math. Zh. 70:9, 2019, 1456–1466.

[Sev1] Sevost’yanov, E: On the local behavior of Open Discrete Mappings from the Orlicz-Sobolev Classes. - Ukr. Math. J. 68:9, 2017, 1447–1465.

[Sev2] Sevost’yanov, E.A.: Boundary behavior and equicontinuity for families of mappings in terms of prime ends. - St. Petersburg Math. J. 30:6, 2019, 973–1005.

[Sev3] Sevost’yanov, E.: The inverse Poletsky inequality in one class of mappings. - Journal of Mathematical Sciences 264:4, 2022, 455–470.

[SSP] Sevost’yanov, E., R. Salimov, E. Petrov: On the removable of singularities of the Orlicz-Sobolev classes. - J. Math. Sci. 222:6, 2017, 723–740.

[SST] Sevost’yanov, E.A, R.R. Salimov and V. Targonskii: On modulus inequality of the order p for the inner dilatation. - https://arxiv.org/abs/2204.07870 .

[SevSkyv] Sevost’yanov, E.A., S.A. Skvortsov: On the local behavior of the Orlicz-Sobolev classes. - Journ. of Math. Sciences 224:4, 2017, 563–581.

[ST] Sevost’yanov, E. and V. Targonskii: On the inverse Poletsky inequality with cotangent dilatation. - https://arxiv.org/abs/2206.09869 .

[St] Stoïlov, S.: Leçons sur les principes topologiques de la théorie des fonctions analytiques. - Gauthier-Villars, Paris, 1956.

[Shl] Shlyk, V.A.: The equality between p-capacity and p-modulus. - Siberian Mathematical Journal 34:6, 1993, 1196–1200.

[Va] Väisälä J.: Lectures on n-dimensional quasiconformal mappings. - Lecture Notes in Math. 229, Springer-Verlag, Berlin etc., 1971.

[Vu] Vuorinen, M.: Exceptional sets and boundary behavior of quasiregular mappings in n-space. - Ann. Acad. Sci. Fenn. Ser. A I. Math. Dissertationes 11, 1976, 1–44.

[Zi1] Ziemer, W.P.: Extremal length and conformal capacity. - Trans. Amer. Math. Soc. 126:3, 1967, 460–473.

[Zi2] Ziemer, W.P.: Extremal length and p-capacity. - Michigan Math. J. 16, 1969, 43–51.
CONTACT INFORMATION

Evgeny Sevost’yanov
1. Zhytomyr Ivan Franko State University,
   40 Bol’shaya Berdichevskaya Str., 10 008 Zhytomyr, UKRAINE
2. Institute of Applied Mathematics and Mechanics
   of NAS of Ukraine,
   1 Dobrovol’skogo Str., 84 100 Slavyansk, UKRAINE
esevostyanov2009@gmail.com

Valery Targonskii
Zhytomyr Ivan Franko State University,
40 Bol’shaya Berdichevskaya Str., 10 008 Zhytomyr, UKRAINE
w.targonsk@gmail.com