Geometric Control for Load Transportation With Quadrotor UAVs by Elastic Cables

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Abstract—Groups of unmanned aerial vehicles (UAVs) are increasingly utilized in transportation tasks as the combined strength allows to increase the maximum payload. However, the resulting mechanical coupling of the UAVs imposes new challenges in terms of tracking control. Thus, we design a geometric trajectory tracking controller for the cooperative task of four quadrotor UAVs carrying and transporting a rigid body, which is attached to the quadrotors via inflexible elastic cables. The elasticity of the cables together with techniques of singular perturbation allows a reduction in the model to that of a similar model with inelastic cables. In this reduced model, we design a controller such that the position and attitude of the load exponentially converge to a given desired trajectory. We then show that this result leads to a uniformly converging tracking error for the original elastic model under some assumptions. Furthermore, under the presence of unstructured disturbances on the system, we show that the error is ultimately bounded with an arbitrarily small bound. Finally, a simulation illustrates the theoretical results.

Index Terms—Aerial systems, mechanics and control, motion and path planning, underactuated robots.

Nomenclature

Symbol Space Description
\( x_L \) \( \mathbb{R}^3 \) Position of the center of mass of the load in inertial frame.
\( v_L \) \( \mathbb{R}^3 \) Translational velocity of center of mass of the load in the inertial frame.
\( R_L \) \( \text{SO}(3) \) Attitude of the load in body frame.
\( \omega_L \) \( T\text{SO}(3) \) Angular velocity of the load in inertial frame.
\( m_L \) \( \mathbb{R} \) Mass of the load.

\( J_L \) \( \text{Sym}_{>0}(\mathbb{R}) \) Moment of inertia of the load.
\( x_Q \) \( \mathbb{R}^3 \) Position of quadrotor \( j \) in inertial frame.
\( R_j \) \( \text{SO}(3) \) Attitude of quadrotor \( j \).
\( \Omega_j \) \( \mathfrak{s}\text{o}(3) \) Angular velocity of quadrotor \( j \) in body frame.
\( m_Q \) \( \mathbb{R} \) Mass of quadrotors.
\( J_Q \) \( \text{Sym}_{>0}(\mathbb{R}) \) Moment of inertia of quadrotors.
\( u_j \) \( \mathbb{R}^3 \) Net thrust applied vertically in the body frame of quadrotor \( j \).
\( M_j \) \( \mathbb{R}^3 \) Moment vector in body frame of quadrotor \( j \).
\( L \) \( \mathbb{R} \) Rest length for elastic cables.
\( r_j \) \( \mathbb{R}^3 \) Unit vector from the center of mass of the load to quadrotor \( j \).
\( q_j \) \( S^2 \) Position vector of cable suspended from quadrotor \( j \).
\( l_j \) \( \mathbb{R} \) Length of elastic cable attached to quadrotor \( j \).
\( \omega_j \) \( T\mathbb{S}^2 \) Angular velocity of cable \( j \) in the inertial frame.
\( u^\perp \) \( \mathbb{R}^3 \) Component of \( u \) that is perpendicular to \( q_j \).
\( u^\parallel \) \( \mathbb{R}^3 \) Component of \( u \) that is parallel to \( q_j \).

I. INTRODUCTION

The use of aerial robots has become increasingly popular in the last years due to their superior mobility and versatility in individual and cooperative tasks (see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], and references therein). For instance, aerial robots can be utilized for monitoring [11], mapping [12], and agriculture tasks [13]. These vehicles are typically underactuated due to constructional reasons, which poses several challenges from the control perspective [14], [15]. Most of the control approaches for individual unmanned aerial vehicles (UAVs) are mainly based on feedback linearization [16] and backstepping methods [17], which are also analyzed in terms of stability, e.g., in [18] and [19].

Multiple aerial robots can be used to transport heavier payloads, thus expanding the capabilities of a single aerial robot [20], [21], [22], [23], [24], [25], [26]. In the aerial transportation task with multiple quadrotor UAVs considered in this work, a cable establishes a physical connection between the UAV and the cargo. Geometric control of single and multiple quadrotors with a suspended point-mass load with...
in [35], which studies the case of a single quadrotor transporting a point-mass load via an inelastic elastic cable. This is developed in Section III and employed in Section IV, where we utilize the geometric control scheme developed in [31] for the exponential tracking of the load position and attitude to some desired trajectories. The Lyapunov analysis is used to determine sufficient conditions for exponential tracking.

3) Here, Theorem 1 then proves the existence of stabilizing gains—for sufficiently small initial errors in the cables—which satisfy the conditions. We further outline a computationally feasible strategy that may be used to find stabilizing gains in practice. In Proposition 2, we then translate these results from the reduced model back to the elastic model.

4) We show that the control scheme is robust to disturbances acting on our system, provided that the initial errors are sufficiently small and gains are chosen appropriately. In particular, Theorem 2 shows that the same control scheme will yield uniform ultimate bounds in the case of unstructured bounded disturbances acting on the system. As such, this article serves to augment the robustness and applicability of the control scheme outlined in [31].

The rest of this article is structured as follows. In Section II, we model and derive the dynamical system describing the task of carrying and transport a rigid body load between the quadrotors by elastic cables. Section III reduces the dynamical model introduced in Section II by employing singular perturbation theory techniques. The main results of this work showing the convergence of the tracking error for the reduced and the actual system of quadrotors are given in Sections IV and IV-C. In Section IV-C, we introduce unstructured bounded disturbances to the reduced model and show that the same control scheme can be applied to achieve uniform ultimate boundedness. Finally, numerical simulations visualize the theoretical results.

II. Modelization and Control Equations

We begin by modeling and deriving the control system describing the transportation task between the quadrotors. This is done by constructing the total kinetic and potential energies of the system, in addition to the virtual work done by nonconservative forces and, subsequently, by using tools from variational calculus on manifolds [37], [41].

Consider four identical quadrotor UAVs transporting a rigid body of total mass \( m_L \in \mathbb{R}_{>0} \) and positive-definite inertia matrix \( J_L \in \mathbb{R}^{3 \times 3} \). The load is considered rigid and of uniform mass density. It is connected to the center of mass of each quadrotor via a massless inelastic elastic cable of rest length \( L \). A graphical description of the proposed system is visible in Fig. 1.

The configuration space of the mechanical system describing the cooperative task between quadrotors is given by

\[
Q = (\text{SO}(3) \times \mathbb{R}^3) \times (S^2 \times \mathbb{R})^4 \times (\text{SO}(3))^4
\]

where \( \text{SO}(3) \) denotes the special orthogonal group of \( 3 \times 3 \) rotation matrices and \( S^2 \) denotes the 2-sphere. The basic notation and methodology along this section are fairly

1) The modeling and subsequent derivation of the corresponding equations of motion for the cooperative task of four quadrotor UAVs transporting a rigid load via inelastic elastic cables. The modeling and dynamics are summarized in Proposition 1.

2) Reduction of these equations of motion to the case of inelastic cables via singular perturbation theory. This can be considered an extension of the results obtained
standard within the geometric control and classical mechanics literature, and we have attempted to use traditional symbols and definitions wherever feasible. The Nomenclature provides the symbols and geometric spaces that are used frequently throughout this article.

The system has 30 degrees of freedom—6 to each cable, and 3 for each quadrotor. Observe that the positions of the quadrotors do not appear in the configuration space, as they are uniquely defined in terms of the other state variables due to the constraints $x_{Q_j} = x_L + R_{L_j}l_j d_j$ for $j \in \{1, 2, 3, 4\}$. Meanwhile, we have 16 inputs to the system in the form of four thrust controls $f_j \in \mathbb{R}$, corresponding to the total lift force exerted on the quadrotors by the propellers, and four moment controls $M_j \in \mathbb{R}^3$, which are related to the torques induced on the quadrotors by the rotating propellers—for each $j \in \{1, 2, 3, 4\}$. Thus, the complete systems have 14 degrees of underactuation, with the quadrotor positions and attitudes being directly actuated. Note also that upon fixing a body frame to each quadrotor such that the vector $\vec{e}_3 = [0, 0, 1]^T$ points in the direction of the applied thrust, we may alternatively express the thrust controls $f_j$ via the vectors $u_j = f_j R_j e_3 \in \mathbb{R}^3$ in the inertial frame for $j \in \{1, 2, 3, 4\}$. We will utilize this representation frequently throughout this article. Alternatively, one may choose to control the total thrust of each propeller individually. However, we opt for the former approach both because it is more pervasive in the literature and because it leads nicely to the separation of the quadrotor’s attitude dynamics from the rest of the system’s dynamics. The thrust generated by the $i$th propeller along the $e_3$-axis can be determined by the total thrust and the moment controller [19].

The translational kinetic energy of each quadrotor can be described by $(1/2)m_Q ||\dot{x}_{Q_j}||^2$, where $m_Q$ denotes the mass of the quadrotor. As the quadrotors and load are rigid bodies, we further have rotational kinetic energy components in the total kinetic energy. Fixing a body frame to each quadrotor and denoting the angular velocity in this body frame by $\Omega_j \in \mathbb{R}^3$, the angular kinetic energy is given by $(1/2)\Omega_j^T J_Q \Omega_j$, where $J_Q$ is a symmetric positive-definite inertia tensor. The angular velocity $\dot{\Omega}_j$ is defined by the kinematic equation $\dot{R}_j = R_j \dot{\Omega}_j$, where $\mathbb{R}^3 \to SO(3)$ is the hat isomorphism, which maps vectors on $\mathbb{R}^3$ to the set $SO(3)$ of $(3 \times 3)$ skew-symmetric matrices, i.e.,

$$\Omega = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix} \mapsto \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix} := \hat{\Omega}.$$ 

We follow the same procedure for the load to obtain the angular kinetic energy as $(1/2)\Omega_L^T J_L \Omega_L$, where $J_L$ is the inertia tensor of the load (again a symmetric positive-definite matrix) and $\Omega_L$ is the angular velocity defined by $\dot{R}_L = R_L \dot{\Omega}_L$.

Elements in the tangent space $T_R SO(3)$ at $R \in SO(3)$ are identified with elements in $SO(3) \times \mathfrak{so}(3)$ by a left trivialization, that is, for $R \in SO(3)$, the map $(R, \dot{R}) \in T_R SO(3) \mapsto (R, R^{-1} \dot{R}) = : (R, \dot{\Omega}) \in SO(3) \times \mathfrak{so}(3)$ is a diffeomorphism (see [41] for details). Therefore, after a left trivialization of $T SO(3)$, the tangent bundle of $Q$, describing the state space of the system, can be identified as $TQ \cong (SO(3) \times \mathfrak{so}(3))^4 \times (S^2 \times TS^2 \times \mathbb{R} \times \mathbb{R})^4 \times (SO(3) \times \mathfrak{so}(3) \times \mathfrak{so}(3) \times \mathfrak{so}(3))$, where we have used that the tangent bundle of a finite-dimensional vector space $V$, i.e., $TV$, is isomorphic to $V \times V$.

Finally, the total kinetic energy of the system, $K : TQ \to \mathbb{R}$, is given by summing the respective translational and angular kinetic energies of the quadrotors and load, i.e.,

$$K = \frac{1}{2} m_L ||\dot{x}_L||^2 + \sum_{j=1}^4 \frac{1}{2} m_Q ||\dot{x}_{Q_j}||^2$$

Translational K.E.

$$+ \frac{1}{2} \Omega_L^T J_L \Omega_L + \sum_{j=1}^4 \frac{1}{2} \Omega_j^T J_Q \Omega_j.$$ Angular K.E.

Moreover, the total potential energy of the system, $U : Q \to \mathbb{R}$, is given by

$$U = \sum_{j=1}^4 \left( m_Q g e_3^T x_{Q_j} + m_L g e_3^T x_L + \frac{1}{2} k (L - l_j)^2 \right)$$

Gravitational P.E.

$$+ \frac{1}{2} k (L - l_j)^2$$

Elastic P.E.

which corresponds to the gravitational potential energies of the quadrotors and the load, as well as the elastic potential of the cables. Observe that here, we have fixed an inertial frame such that $e_3$ is oriented opposite to the direction of the gravitational acceleration. As usual, the Lagrangian of the system $L : TQ \to \mathbb{R}$ is defined by $L := K - U$.

Note that the control inputs take the form of nonconservative external forces so that we must use the Lagrange–d’Alembert variational principle (see [42] for instance)—with controls as virtual forces in our system—to obtain our system dynamics from the Lagrangian $L$. We further wish to add a nonconservative force corresponding to a damping in the elastic cables, that is, a velocity-dependent force that serves to reduce the amplitude of oscillations in the elastic cables. In particular, we will opt to make this force proportional to the velocity—as is standard for damped harmonic oscillators—with constant of proportionality $c > 0$, that is, we wish to minimize the action

$$A(c(t)) = \int_0^T L(c(t), \dot{c}(t)) \, dt$$

$$+ \sum_{j=1}^4 \int_0^T \left( ||f_j R_j e_3||^2_{\mathbb{R}^3} + ||M_j||_{\mathfrak{so}(3)}^2 c \right) dt.$$
over the space of smooth function from \([0, T]\) to \(Q\) with fixed endpoints, where \(\|\dot{M}_j\| = \sqrt{\text{Tr}(\dot{M}_j^T \dot{M}_j)} = (\dot{M}_j^T \dot{M}_j)^{1/2}\). Following similar arguments in [31] and [43, Proof of Proposition 1], we obtain the following system dynamics.

**Proposition 1:** Critical points of the action functional \(A\) for variations with fixed endpoints correspond with solutions of the controlled Euler–Lagrange equations

\[
\dot{x}_L = v_L, \quad \dot{R}_L = R_L \dot{\Omega}_L
\]

\[
m_{\text{eff}}(\dot{v}_L + g e_3) = \sum_{j=1}^{4} u_j + m_Q \ddot{\zeta}_j - m_Q R_L (\ddot{\Omega}_L^2 + \dot{\Omega}_L r_j)
\]

\[
J_{\text{eff}} \dot{\Omega}_L + \dot{\Omega}_L J_{\text{eff}} \Omega_L = \sum_{j=1}^{4} m_Q \dot{r}_j R_L^T \left( -g e_3 - \dot{v}_L + \ddot{\zeta}_j + \frac{1}{m_Q} u_j \right)
\]

\[
m_Q q_j \ddot{\zeta}_j = m_Q q_j \left( \dot{v}_L + R_L (\ddot{\Omega}_L^2 + \dot{\Omega}_L r_j) + g e_3 - \frac{1}{m_Q} u_j \right) - c l_j + k (L - l_j)
\]

\[
q_j \times \ddot{\zeta}_j = q_j \times \left( \dot{v}_L + R_L (\ddot{\Omega}_L^2 + \dot{\Omega}_L r_j) + g e_3 - \frac{1}{m_Q} u_j \right)
\]

\[
J_Q \dot{\Omega}_j = J_Q \dot{\Omega}_j \times \Omega + m_j, \quad \dot{R}_j = R_j \dot{\Omega}_j, \quad j = 1, \ldots, 4
\]

where \(m_{\text{eff}} := 4m_Q + m_L\), \(J_{\text{eff}} := J - \sum_{j=1}^{4} m_Q \dot{r}_j^2\), and \(\zeta_j := l_j q_j\).

**Proof:** See the Appendix.

**Remark 1:** Equation (1) describes the kinematics of the load’s position and attitude. Similarly, (2) and (3) describe the dynamics of the load’s position and attitude, respectively, and (6) corresponds to the dynamics and kinematics of the attitudes of each quadrotor.

Equation (4), indexed for \(j = 1, \ldots, 4\), describe the dynamics of the lengths of the elastic cables. This can be understood by observing that the projection of \(\ddot{\zeta}_j\) onto \(q_j\) preserves the acceleration of the length (which is inherently oriented along the cable) while removing the acceleration of the attitude from consideration with the identity \(q_j^T \ddot{q}_j = -||\dot{q}_j||^2\). Conversely, (5), indexed for \(j = 1, \ldots, 4\), describes the dynamics of the attitudes of the elastic cables, as the cross product with \(q_j\) preserves the acceleration of the cable attitude while annihilating the acceleration of the cable length.

**Remark 2:** Note that the proposed dynamics given by Proposition 1 can be easily extended to an arbitrary number \(n\) of quadrotors and elastic cables by making the range of sum in the kinetic and potential energies evolve from 1 to \(n\) instead of only from 1 to 4. We choose \(n = 4\) only for illustrative purposes on the transportation task, but all the results in this article follow exactly the same procedure for the case of \(n\) by appropriately change the range of the sums. Note that the generalization to \(n\) vehicles does not add coupling dynamics in the cable-load system since we translate the dynamics to the center of mass of the load independently of the point where the cables are attached. Note also that the analysis conducted in [31] does not incorporate the elasticity of the cables, which is the main difference with respect to the model proposed for our system. In addition, one can consider different kinds of objects. The difficulty here is the inertia mass of the load. In order to apply the results of our work with the proposed control strategy, the inertia mass of the load \(J_l\) must be a nonsingular matrix. For instance, in the case of transport of a rigid bar instead of a load, \(J_l\) is singular and the control design must be conducted in a different way as it was shown in our previous paper [43].

### III. Reduced Model

While the use of elastic cables provides the benefit of reducing impulsive forces on the load, large or rapid oscillations of the load can produce undesired aggressive movements, compromising the load. This can be combated by utilizing elastic cables with high stiffness and damping to guarantee the safety for the rigid load in the transportation task. Such a condition will commonly be fulfilled by the cables used in applications.

A benefit of this assumption is that we will be able to reduce the degrees of freedom in the original model via techniques from singular perturbation theory [44]. In fact, we will see that the reduced model is differentially flat. In other words, the states and inputs of the reduced model can be written as algebraic functions of 16 flat outputs and their derivatives—which dramatically reduces the difficulty involved in generating dynamically feasible trajectories for underactuated systems. On the other hand, the original elastic model is not differentially flat, which further justifies our desire to reduce the model.

In particular, we will consider the case that \(k = (\tilde{k}/\epsilon^2)\) and \(c = (\tilde{c}/\epsilon)\) with \(\tilde{k}, \tilde{c} > 0\) and \(\epsilon > 0\) sufficiently small, and we will show the dynamics approach that of the same model with inelastic cables (i.e., with \(l \equiv L\)) as \(\epsilon \to 0\). We further consider a change of variables of the form \(l_j = \epsilon^2 y_j + L_j\) and \(\dot{l}_j = \epsilon z_j\), which is motivated by observing that \(k (L - l_j) = \tilde{k} y_j\) and \(c l_j = \tilde{c} z_j\). From this, we can see that \(\dot{\zeta}_j = (\epsilon^2 y_j + L) q_j\). Therefore, \(\ddot{\zeta}_j = L \ddot{q}_j + \epsilon (\dot{z}_j q_j + z_j \dot{q}_j) + \epsilon^2 y_j \ddot{q}_j\). Making these substitutions into the dynamics described in Proposition 1, in addition to employing the equation \(\dot{q}_j = \omega_j \times q_j\), we obtain

\[
\dot{x}_L = v_L, \quad \dot{R}_L = R_L \dot{\Omega}_L, \quad \epsilon \dot{y}_j = z_j, \quad \dot{q}_j = \omega_j \times q_j
\]

\[
m_{\text{eff}}(\dot{v}_L + g e_3)
\]

\[
= \sum_{j=1}^{4} \left( u_j + m_Q R_L (\ddot{\Omega}_L^2 + \dot{\Omega}_L r_j) + m_Q (L \ddot{q}_j + \epsilon (\dot{z}_j q_j + z_j \dot{q}_j) + \epsilon^2 y_j \ddot{q}_j) \right)
\]

\[
J_{\text{eff}} \ddot{\Omega}_L + \dot{\Omega}_L J_{\text{eff}} \Omega_L = \sum_{j=1}^{4} m_Q \dot{r}_j R_L^T \left( -g e_3 - \dot{v}_L + L \ddot{q}_j + \epsilon (\dot{z}_j q_j + z_j \dot{q}_j) + \epsilon^2 y_j \ddot{q}_j \right)
\]
\[
m_Q L \ddot{z}_j = q_j^T \left( \ddot{z}_j q_j + \dddot{z}_j q_j + m_Q u_j + m_Q L \left( 1 + \varepsilon^2 y_j \right) \ddot{q}_j \right)
- m_Q \left( \ddot{v}_L - m_Q R_L (\ddot{\Omega}_L^2 + \ddot{\Omega}_L) r_j + g e_3 \right)
\]

\[
\dot{\omega}_j = \frac{1}{L} q_j \left( \ddot{v}_L - R_L (\ddot{\Omega}_L^2 + \ddot{\Omega}_L) r_j + g e_3 \right)
- \frac{1}{m_Q} u_j - \epsilon (\ddot{z}_j q_j + z_j w_j) - \epsilon \varepsilon^2 y_j \ddot{q}_j
\]

\[
J Q \ddot{\Omega}_j = J Q \dot{\Omega}_j \times \Omega_j + M_j, \quad \dot{\Omega}_j = R_j \dot{\Omega}_j, \quad j = 1, \ldots, 4.
\]

The previous system can be written as
\[
\dot{x} = f(t, x, z; \epsilon), \quad \dot{\epsilon} = g(t, x, z; \epsilon)
\] (7)

where \(f\) and \(g\) are smooth functions, \(x\) is the vector representing \((x_L, v_L, R_L, \Omega_L, q_j, \omega_j, R_j, \text{ and } \Omega_j)\), and \(z\) is the vector representing \((y_j, z_j)\), for \(j = 1, \ldots, 4\). Note that (7) is equivalent to the elastic model (1)–(6), just condensed and with a change of variables. In particular, (7) is known as the singular perturbation model [44], with the first equation describing the slow dynamics and the second describing the fast dynamics. Evaluating at \(\epsilon = 0\), the fast dynamics provide us with algebraic equations that can be solved to obtain \(z = h(t, x)\). In particular,

\[
z_j = 0
\]

\[
y_j = -\frac{m_Q}{k} q_j^T \left[ u_j + L \ddot{q}_j - \ddot{v}_L + R_L (\ddot{\Omega}_L^2 + \ddot{\Omega}_L) r_j - g e_3 \right].
\]

Substituting these equations back into the slow dynamics, we obtain the reduced (slow) model of the control system describing the cooperative task, given by

\[
\dot{x} = f(t, x, h(t, x), 0), \quad \text{i.e.,}
\]

\[
\dot{x} = v_L, \quad \dot{\omega}_L = R_L \dot{\Omega}_L, \quad \dot{\omega}_j = \omega_j \times q_j
\]

\[
m_{\text{eff}}(\ddot{v}_L + g e_3) = \sum_{j=1}^{4} \left( u_j - m_Q R_L (\ddot{\Omega}_L^2 + \ddot{\Omega}_L) r_j + m_Q L \ddot{q}_j \right)
\]

\[
\dot{\Omega}_L = J_{\text{eff}} \dot{\Omega}_L + \ddot{\Omega}_L J_{\text{eff}} \dot{\Omega}_L
\]

\[
= \sum_{j=1}^{4} m_Q R_j \ddot{q}_j \left( -g e_3 - \ddot{v}_L + L \ddot{q}_j + \frac{1}{m_Q} u_j \right)
\]

\[
\dot{\omega}_j = \frac{1}{L} q_j \left( \ddot{v}_L - R_L (\ddot{\Omega}_L^2 + \ddot{\Omega}_L) r_j + g e_3 - \frac{1}{m_Q} u_j \right)
\]

\[
J_Q \ddot{\Omega}_j = J_Q \dot{\Omega}_j \times \Omega_j + M_j, \quad \dot{\Omega}_j = R_j \dot{\Omega}_j, \quad j = 1, \ldots, 4.
\]

Remark 3: Observe that the reduced model (10)–(14) preserves the original 16 inputs of the system, but the configuration space has lost four degrees of freedom (namely the cable lengths). Hence, the reduced model has ten degrees of underactuation. In fact, it can be seen that the reduced model is equivalent to the original model with inelastic cables [31] (i.e., where \(l \equiv L\)). This system was shown to be differentially flat in [34]. From [31, Proposition 1], we know that achieving exponentially stable tracking of the reduced model on some set of initial conditions will guarantee exponentially stable tracking in some subset of those initial conditions—whose relative size depends on \(\epsilon\). Hence, we may work within this reduced model to design geometric controllers toward the end of tracking the position and attitude of the load.

Remark 4: Note that by setting \(l_j = L\) in the original model, it yields the reduced model. However, the advantage of using singular perturbation theory is that it allows us to obtain the results in the elastic model from results in the inelastic model. After designing the control scheme and proving exponential tracking in the reduced model, we will show in Proposition 2 that the integral curves of the elastic model (with the same control scheme) stay within a neighborhood of size \(O(\epsilon)\) of the integral curves of the unreduced model—where \(\epsilon\) is related to the elasticity of the cable. Hence, we can guarantee that if the elastic cables are stiff and have high damping (as is often the case in application), then the control scheme designed for the reduced model still guarantees near exponential tracking in the elastic model. This would not have been possible if we simply put \(l_j = L\).

IV. CONTROL DESIGN FOR POSITION AND ATTITUDE

TRAJECTORY TRACKING OF THE RIGID LOAD

In the following, we discuss the control design for the reduced model (10)–(14), that is, we provide a set of controllers \(u_j \in R^3\) such that the position and attitude of the load reach a desired position \(\tilde{x}_L \in R^3\) and attitude \(\tilde{R}_L \in SO(3)\) exponentially fast. The strategy is to decompose \(u_j\) into components that are parallel and perpendicular to the cable attitudes \(q_j\) via \(u_j = u_j^1 + u_j^2\), where \(u_j^1 = (q_j^T u_j) q_j\) \(u_j^2 = (I - q_j q_j^T) u_j\). This is motivated by the fact that only the \(u_j^1\) components influence the dynamics of the load, while only the \(u_j^2\) components influence the cables’ dynamics. Its graphical description is shown in Fig. 2.

In particular, we show that the reduced model is equivalent to the dynamical system discussed in [31] with \(n = 4\) quadrotors and \(l_j = L\) for \(j = 1, \ldots, 4\). As such, we may employ the control scheme designed there. After this, we will introduce configuration error functions for each state variable, from which we may derive our error dynamics.

Notice that (14) for quadrotor attitude is independent of the rest of the dynamics, and the moment controllers \(M_j\) appear exclusively within them. Moreover, these equations are just those that appear in [19] and [31], for which \(M_j\) was designed to attain almost-global exponential stability. We use the same controller for the attitude of the quadrotors and disregard the equations for the remainder of this article.

A. Error Dynamics and Control Design

We begin by further simplifying the dynamical system (10)–(13). In particular, we find an equation for \(\ddot{q}_j\) that we will
substitute into (11) and (12). By differentiating \( \dot{q}_j = \omega_j \times q_j \) and expanding it with the vector triple product identity, it can be shown that \( \dot{q}_j = \dot{\omega}_j \times q_j - [\omega_j \times q_j] \). Now, we may substitute (13) in for \( \dot{\omega}_j \) to find that

\[
L \ddot{q}_j = -q_j \times (L \dot{\omega}_j) - L ||\omega_j||^2 q_j

= (I - q_j q_j^T) \left[ \dot{v}_L - R_L (\bar{\Omega}_L + \dot{\Omega}_L) r_j + g \hat{e}_3 - \frac{1}{m_Q} u_j \right]

- L ||\omega_j||^2 q_j.
\]

Substituting this equation for \( m_Q L \ddot{q}_j \) into (11) and making use of the fact that \( m_{eff} = 4m_Q + m_L \), we obtain

\[
M_L (\dot{v}_L + g \hat{e}_3) = \sum_{j=1}^{4} \left[ m_Q q_j q_j^T R_L (\bar{\Omega}_j + \dot{\Omega}_L) r_j + u_j^T - m_Q L ||\omega_j||^2 q_j \right]
\]

where \( M_L = m_L I + \sum_{j=1}^{4} m_Q q_j q_j^T \). Repeating this procedure with (12) and making use of the fact that \( J_{eff} = J_L - \sum_{j=1}^{4} m_Q \dot{r}_j \) yields

\[
J_L \ddot{\Omega}_L + \dot{\Omega}_L J_L \Omega_L = \sum_{j=1}^{4} m_Q \dot{r}_j R_L \left[ q_j q_j^T R_L (\bar{\Omega}_j + \dot{\Omega}_L) r_j - q_j q_j^T (\dot{v}_L + g \hat{e}_3) \right]

- L ||\omega_j||^2 q_j + \frac{1}{m_Q} u_j.
\]

With (15) in place of (10) and (16) in place of (12), the reduced model is exactly the control system discussed in [31] with \( n = 4 \) and \( l_j = L \) for \( j = 1, \ldots, 4 \). As such, we may adopt the control scheme used in particular. In [31], we define our desired trajectories by \( \hat{x}_L, \hat{v}_L, \hat{R}_L, \hat{\Omega}_L, \hat{q}_j, \) and \( \hat{\omega}_j \) and define tracking errors as

\[
e_{x_L} = x_L - \hat{x}_L, \quad e_{v_L} = v_L - \hat{v}_L

e_{R_L} = \frac{1}{2} \left( \hat{R}_L^T R_L - R_L^T \hat{R}_L \right) \hat{v}, \quad e_{\Omega_L} = \Omega_L - R_L^T \hat{R}_L \hat{\Omega}_L

e_{q_j} = \tilde{q}_j \times q_j, \quad e_{\omega_j} = \omega_j + \hat{\omega}_j
\]

then, for some choice of gains \( k_{x_L}, k_{v_L}, k_{R_L}, k_{\Omega_L}, k_{q}, k_{\omega} \in \mathbb{R} \), the controllers take the form

\[
u_j^T = \mu_j + m_Q L ||\omega_j||^2 q_j + m_Q q_j q_j^T a_j

= m_Q \bar{L} \ddot{q}_j \left( -k_q e_{q_j} - k_\omega e_{\omega_j} - (q_j \times \omega_j) \right) + \frac{1}{2} \hat{\omega}_j^2
\]

where \( a_j = \dot{v}_L + g \hat{e}_3 + R_L \bar{\Omega}_L \dot{r}_j - R_L \hat{r}_j \hat{\Omega}_L \) and \( \mu_j \) is an additional controller, which satisfies

\[
u_j = q_j q_j^T \bar{\mu}_j
\]

We further define the desired cable attitudes

\[
\bar{q}_j = -\left( \frac{\mu_j}{||\mu_j||} \right) \quad \text{so that we have} \quad \bar{v}_L = \mu_j \quad \text{as} \quad \bar{q}_j = q_j.
\]

Notice that (20) and (21) can also be written in the form

\[
\mathcal{P} \text{diag} \left( R_L^T, \ldots, R_L^T \right) \begin{bmatrix} \bar{\mu}_1 \\ \vdots \\ \bar{\mu}_4 \end{bmatrix} = \begin{bmatrix} R_L^T F \\ M \end{bmatrix}
\]

where

\[
\mathcal{P} = \begin{bmatrix} I_{3 \times 3} & \cdots & I_{3 \times 3} \\ \hat{r}_1 & \cdots & \hat{r}_4 \end{bmatrix} \in \mathbb{R}^{6 \times 12}

\hat{F} = m_L \left( -k_{eL} e_{x_L} - k_{eL} e_{v_L} + \hat{v}_L + g \hat{e}_3 \right)

\tilde{M} = -k_{eL} e_{R_L} - k_{eL} e_{\Omega_L} + \left( R_L^T \hat{R}_L \hat{\Omega}_L \right)^T J_L R_L^T \hat{R}_L \hat{\Omega}_L

+ J_L R_L^T \hat{R}_L \hat{\Omega}_L.
\]

Then, if we assume that rank(\( \mathcal{P} \)) = 6 (which depends on the physical connection points of the cables to the load), there is guaranteed to exist a solution \( [\bar{\mu}_1 \cdots \bar{\mu}_4]^T \). However, in general, there will exist multiple solutions, so we choose the solution with minimal (Euclidean) norm. This is given by

\[
\begin{bmatrix} \bar{\mu}_1 \\ \vdots \\ \bar{\mu}_4 \end{bmatrix} = \text{diag}(R_L, \ldots, R_L) (\mathcal{P} \mathcal{P}^T)^{-1} \begin{bmatrix} R_L^T F \\ M \end{bmatrix}. \]

With these controllers, the error dynamics take the following form:

\[
e_{v_L} = - k_{eL} e_{v_L} - k_{eL} e_{x_L} - \frac{1}{m_Q} \sum_{j=1}^{4} (q_j^T \hat{\mu}_j) \hat{q}_j e_{q_j}
\]

\[
J_L e_{\Omega_L} = -k_{eL} e_{\Omega_L} - k_{eL} e_{R_L}

+ \left( J_L e_{\Omega_L} + (2J_L - \text{Tr}(J_L)I) R_L^T \hat{R}_L \hat{\Omega}_L \right) \hat{v}

- \sum_{j=1}^{4} \hat{r}_j R_L^T \left( \hat{q}_j \hat{\mu}_j \right) \hat{q}_j e_{q_j}
\]

\[
e_{\omega_j} = (q_j^T \hat{\omega}_j) q_j - k_{e\omega} e_{\omega_j} - k_{e\omega} e_{q_j},
\]

Remark 5: In the case that there are \( n \) quadrotors carrying the payload (\( n \) arbitrary), each with their respective connection points \( \hat{r}_i \) between the payload and their corresponding cable, we may simply replace the matrix \( \mathcal{P} \) via \( \mathcal{P} = \begin{bmatrix} I_{3 \times 3} & \cdots & I_{3 \times 3} \end{bmatrix} \in \mathbb{R}^{6 \times 3n} \). As before, we require that \( \mathcal{P} \) can be the maximal rank (which is again 6).

B. Theoretical Analysis

The goal now is to prove that the gains can be chosen in such a way that the error dynamics (23)–(25) have an exponentially stable equilibrium point at the origin. This will be accomplished via Lyapunov analysis. Before this, however, we will introduce two configuration errors—\( \Psi_{q_j} \) and \( \Psi_{R_L} \).

These are real-valued functions, which are positive-definite about the points \( q_j = \bar{q}_j \) and \( R_L = \bar{R}_L \) and are defined explicitly as \( \Psi_{q_j} = 1 - \bar{q}_j^T q_j \) and \( \Psi_{R_L} = (1/2) \text{Tr}(I - R_L^T R_L) \).
We now define a Lyapunov candidate $V : \mathcal{D} \to \mathbb{R}$ defined as

$$V = \frac{1}{2}||e_{vL}||^2 + \frac{1}{2}k_{xL}||e_{xL}||^2 + c_x e_T x e_{vl} + \sum_{j=1}^{4} \left[ \frac{1}{2}||e_{xj}||^2 + k_q \Psi_{qj} + c_q e_T x e_{oj} \right] + \frac{1}{2}L_k J_k e_{vl}$$

which is defined in the domain

$$\mathcal{D} = \{ (e_{xL}, e_{vL}, e_{xL}, e_{oj}, e_{oj}) \mid ||e_{xL}|| < e_{\max}, \Psi_{RL} < \Psi_{RL} < 1, \Psi_{qj} < \Psi_{qj} < 1 \}$$

where $c_x, c_q, c_R, \Psi_{qj}$, and $\Psi_{qj}$ are positive real numbers.

Before establishing sufficient conditions for the gains which ensure exponential stability of the zero equilibrium for the tracking error, we state the following lemma, which follows from [31].

**Lemma 1:** Consider the set of matrices (26)–(30), as shown at the bottom of the page, for $\alpha_j := (\psi_{qj}/(\psi_{xj} - \psi_{qj}))^{1/2}$, $\alpha_L := \psi_{RL}/(\psi_{RL} - \psi_{xL})^{1/2}$, $\gamma := (1/\gamma_{\text{min}}(P_{T}^{T} P_{T}))$, $\beta := m_{L} \gamma$, $\delta_j := m_{L} \gamma_{\text{min}}(P_{T}^{T} P_{T})^{1/2}$, $\sigma_j := (\delta_j/m_{L})$, $B$ is a positive constant, which can be obtained from $\bar{L}_x$ and $\bar{R}_x$, and $C_{qj}$ is a positive constant that can be obtained from $\hat{q}_j$. If $P_{xL}, P_{xR}, \tilde{P}_{xL}, \tilde{P}_{xR}$, and $P_{qj}$ are positive-definite matrices for $j = 1, \ldots, 4$, then the origin of the error dynamics (23)–(25) is an exponentially stable equilibrium point.

While Lemma 1 was obtained in [19] and [31], no rigorous study guaranteeing the existence of stabilizing gains, much less strategies to obtain said gains, was ever obtained. That is what we seek to do in the following theorem, which adopts a similar strategy to that found in [43], which studies the case of two quadrotor UAVs transporting a rigid rod.

**Theorem 1:** Consider the control system defined by (10)–(14) and control inputs determined by (19)–(21), with $u_j$ and $u_j^+$ given by (17) and (18). For sufficiently small $\alpha_j$, there exists gains $k_{xL}, k_{vL}, k_{RL}, k_{QL}, k_q$, and $k_{oq}, \gamma = 1, \ldots, 4$, such that the zero equilibrium of the tracking error $(e_{xL}, e_{vL}, e_{xL}, e_{oL}, e_{oL}, \psi_{qj})$ and $\psi_{qj}$ is exponentially stable.

**Proof:** It is easy to see that the matrices $P_{xL}, P_{xR}, P_{QL}, \tilde{P}_{QL}, \tilde{P}_{QL}$, and $P_{qj}$ are positive-definite for $c_x, c_R$, and $c_q$ sufficiently small. In particular, the matrices are simultaneously positive-definite for any choice of gains if and only if we take

$$c_x < \sqrt{k_x}, \quad c_q < \sqrt{k_q}, \quad c_R < \sqrt{k_R \lambda_{\text{min}}(J_L) / \lambda_{\text{max}}(J_L)}$$

Now, denote the symmetric part of $W_j$ by $\mathcal{W}_j = (1/2) (W_j + W_j^T)$ and similarly define the symmetric parts of the submatrices by $\mathcal{W}_{xL}, \mathcal{W}_{xR}, \mathcal{W}_{QL}, \mathcal{W}_{QR}$, and $\mathcal{W}_{QR}$. It is clear that $W_j$ can be expressed in the form $W_j = \left[ P_{ST} \mid S \right]$, where $P = \left[ W_{xj}^{1/2} \left( \begin{array}{cc} -W_{xj} & -W_{xj} \end{array} \right) \right]$, $S = \left( \begin{array}{cc} -W_{qj} & W_{qj} \end{array} \right)$, and $Q = W_{qj}$. Now, observe that $W_j$ can be decomposed as

$$W_j = \left[ P_{ST} \mid S \right] = \left[ \begin{array}{cc} I & S \end{array} \right] \left[ \begin{array}{cc} W_{ST} + S^T & S \end{array} \right] \left[ \begin{array}{c} W_{ST} \mid S \end{array} \right]$$

where $P = W_{ST} - S^T$ is often referred to as the Schur complement of $Q$. It then follows that $W_j > 0$ if and only if $P = W_{ST} - S^T > 0$ and $Q > 0$. Note that $P = W_{ST} - S^T$ can itself be expressed in form of a $4 \times 4$ block matrix given by

$$W_j = \left[ \begin{array}{cc} W_{xj} - \frac{1}{4} W_{xj} W_{qj} W_{xj} & -\frac{1}{4} W_{xj} W_{xj} W_{qj} \end{array} \right] \left[ \begin{array}{cc} W_{xj} & -\frac{1}{4} W_{xj} W_{qj} W_{xj} \end{array} \right]$$

where $P = W_{ST} - S^T$.
Repeating the previous analysis, but now on $P = SQ^{-1}S^T$, we find that $W_j = 0$ if and only if the following three conditions hold: 1) $W_{q_j} = 0$; 2) $WR_{j} - (1/4)W_qR_j W_{q_j} W_{q_j} R_j > 0$; and 3) $W_{x_j} - 1/4 W_{x_q} W_{q_j}^{-1} W_{x_j} - 1/4 W_{x_q} R_j W_{q_j}^{-1} W_{x_q} R_j < 0$.

This motivates the choice $k_0 = c_q k_q + c_0$, from which we obtain $\lambda_{\min}(W_{q_j}) = c_q (2c_k - k_q (1 + (1 - c_0))$. Now, let $C_j := \max_j = 1, \ldots, 4 |C_j|$, and choose $c_0 = (k_2 - C_j) (1 + k_q)$. For some fixed $\epsilon > 0$, we then require $k_q > C_j + \epsilon := k_q^*$, from which it follows that $c_q^* := (\epsilon/(1 + C_j + \epsilon)) < c_q < \sqrt{k_q}$, and $\lambda_{\min}(W_{q_j}) \geq c_q^* k_q > 0$. In summary, $P_q > 0$ and $W_{q_j} > 0$, provided that $k_q > k_q^*$. Consequently, $\lambda_{\max}(W^{-1}) = \lambda_{\min}(W_{q_j})^{-1}$ can be made arbitrarily small (but positive). Hence, for 2), we have $W_{x_q} > 0$ and $W_{q_j} > 0$, and since $W_{x_q} R_j$ and $W_{q_j}$ are independent, we can shrink the maximum eigenvalue of $W_{x_q} W^{-1} W_{q_j} W_{q_j}^{-1}$ by shrinking the maximum eigenvalue of $W_{q_j}^{-1}$. Moreover, observe that

$$2\lambda_{\min}(W_{q_j}) = (k_0 - c_q + c_q^* k_q)
- \sqrt{(k_0 - c_q - c_q k_q)^2 + c_q^2 (k_0 + c_0)^2}.$$
which again may be shrunk arbitrarily by shrinking the maximum eigenvalue of $\mathcal{V}_{q_j}^{-1}$. Finally, the third term $(1/4)\mathcal{V}_{q_j}(W_{R_j} - (1/4)\mathcal{V}_{q_j}W_{q_j}^{-1}\mathcal{V}_{q_j}W_{R_j})^{-1}\mathcal{V}_{q_j}$ may be written in the form $\alpha^2M^TAM$, where $M$ is independent of $\alpha$ and the terms of $A$ are at most of $\mathcal{O}(1/\alpha)$. It follows that if $\alpha$ is sufficiently small and $\lambda_{\text{max}}(W_{q_j})$ is sufficiently large, condition 3) holds and $\mathcal{V}_{q_j} > 0$.

Remark 6: We now wish to address the issue of control saturation, which is marked by the control magnitude becoming larger than permitted by the system’s physical constraints. From the control design conducted in Section IV-A, it is clear that saturation depends in part on the chosen desired trajectories. This is intuitively sensible, as trajectories with very high velocities, accelerations, and jerks will be impossible to track effectively. Beyond this, the control depends linearly on the gains. Assuming that sufficiently “nice” desired trajectories are chosen and initial configuration errors are sufficiently small to allow exponential convergence to said trajectories, it is reasonable to assume that control saturation may be avoided provided that the gains can be chosen sufficiently small. For simplicity, we assume that the initial configuration errors are zero, from which the result will immediately follow for sufficiently small initial errors.

For simplicity, let $x = \lambda_{\text{min}}(W_{q_j})$, $y = \lambda_{\text{min}}(W_{R_j})$, and $z = \lambda_{\text{min}}(W_{x_j})$, and choose the gains and Lyapunov constants so that they satisfy the conditions previously outlined in the proof of Theorem 1, that is, we have $k_q > k_q^*$, $k_{R_q} > k_{R_q}^*$, and $k_{L_q} > k_{L_q}^*$, which implies that $c_q > c_q^*$, $c_R > c_R^*$, and $c_L > c_L^*$, where $k_q^*, k_{R_q}^*, k_{L_q}^*, c_q^*, c_R^*, c_L^*$ are positive constants, which are bounded away from zero and depend only on the norms of the velocities, accelerations, and jerks of the chosen desired trajectories, the initial configuration errors (which we assume to be zero), and some fixed parameter $\epsilon > 0$. Moreover, as $B, C_q \rightarrow 0$ (which depends only on the chosen desired trajectories), $k_q \rightarrow \epsilon$, $k_{R_q} \rightarrow (1 + \epsilon)^2$, and $k_{L_q}$ tends to a quadratic polynomial in $\epsilon$. With these choices, we have

$$
k_q < \frac{x}{c_q^2}, \quad k_{R_q} < \frac{y}{c_R^2}, \quad k_{L_q} < \frac{z}{c_L^2}, \quad k_{\epsilon_q} < k_{\epsilon_q} + 1.
$$

Condition 2) from the proof of Theorem 1 will be satisfied if $y > (B^2(1 + 2c_q^2)/16x)$ (such a choice can always be made). Moreover, we see that as $B \rightarrow 0$ (which depends only on the desired trajectories chosen for $x_j$ and $R_j$), we need only take $x, y, z > 0$. Similarly, condition 3) from the proof of Theorem 1 will be satisfied if

$$z > \frac{B^2(1 + 2c_q^2)^2}{16x} - \frac{B^2(1 + 2c_q^2)(1 + 2c_q^2)^2}{16x(16x - B^2(1 + 2c_R^2)^2)}.
$$

As before, we see that if the desired trajectory data $B$ tend to zero, we need only take $x, y, z > 0$. Moreover, since the gains are bounded above by linear functions of the corresponding eigenvalues $x, y, z$, and below by polynomials in $\epsilon$ (which is a free parameter), it follows that the exponential convergence of the state variables to their desired trajectories can be obtained while also choosing the gains small enough to avoid control saturation, provided that the initial errors are sufficiently small and the desired trajectories are chosen sufficiently “nice,” as expected.

C. Control Design in the Presence of Unstructured Disturbances

It is well known that exponential stability is robust to small disturbances [45], and therefore, the control scheme developed in Section IV will hold, provided that the external disturbances and error measurements are sufficiently small. In order to understand the extent to which this robustness holds, we will introduce bounded unstructured disturbances to the reduced model (10)–(13) and test the previous control scheme in this new scenario.

In particular, for $j = 1, \ldots, 4$, we consider the following perturbed dynamical system:

$$\dot{x}_L = v_L, \quad \dot{R}_L = R_L\hat{\Omega}_L, \quad \dot{q}_j = \omega_j \times q_j
$$

$$m_{\text{eff}}(\dot{v}_L + \epsilon s)
$$

$$j = \sum_{j=1}^4 \left( u_j - m_{Q}R_L(\hat{\Omega}_{L}^2 + \Omega_{L}) \right) r_j + m_{Q}L_{\dot{q}_j} = \Delta x_L + \Delta R_L
$$

$$\omega_j = L^{-1}\dot{q}_j(\dot{v}_L - R_L(\hat{\Omega}_{L}^2 + \Omega_{L}) \dot{r}_j + g_3 - m_{Q}^{-1}u_j) + \Delta q_j
$$

where $\Delta x_L, \Delta R_L$, and $\Delta q_j$ are unstructured disturbances satisfying the bounds $||\Delta x_L|| \leq \tilde{x}_L, ||\Delta R_L|| \leq \tilde{R}_L$, and $||\Delta q_j|| \leq \tilde{q}_j$ for some real numbers $\tilde{x}_L, \tilde{R}_L$, and $\tilde{q}_j$.

Defining the controllers, configuration errors, and Lyapunov candidate $V$ as in Section IV, it is easy to see that the matrices $P_{x_L}, P_{q_L}, P_{R_L}, \dot{P}_{x_j}, \dot{P}_{q_j}$, and $\dot{P}_{q_j}$ remain unchanged. On the other hand, the upper bound on the time derivative of $V$ is modified as $\dot{V} \leq -\sum_{j=1}^4 z_j^T W_j z_j + E^T z$, where $E := ((c_q \delta_{x_L}/m_y), (\delta_{x_L}/m_y), (3c_R \delta_{R_L}/2m_L), (3c_L \delta_{L_R}/2m_L), (c_q \delta_{q_j}/m_Q L_c), (\delta_{q_j}/\tilde{q}_j L_c), z_j = \left[ |e_{x_L}|, |e_{q_j}|, |e_{R_L}|, |e_{\Omega_{L}}|, |e_{\dot{q}_j}|, |e_{\tilde{q}_j}| \right]$, and $W_j$ is as in Section IV. Fix $\epsilon > 0$. From Young’s inequality, we have $E^T z \leq (||E||^2/16\epsilon) + 4\epsilon ||z||^2$, and hence,

$$\dot{V} \leq -\sum_{j=1}^4 z_j^T (W_j - \epsilon I) z_j + \frac{||E||^2}{16\epsilon}.
$$

Analogous to before, we replace the matrix $W_j - \epsilon I$ with its symmetric part $W_j^\star := (1/2)(W_j + W_j^T) - \epsilon I$. Note that $\lambda_{\text{min}}(W_j^\star) = \min_{j=1,\ldots,4} \lambda_{\text{min}}(W_j^\star) - \epsilon$. From Theorem 1 and Remark 6, for sufficiently small $\alpha$, $\lambda_{\text{min}}(W_j^\star)$ can be made arbitrarily large by choosing the gains appropriately, and hence, we may choose them so that $\lambda_{\text{min}}(W_j^\star) > 0$. It follows that:

$$\lambda_{\text{min}}(P)||z||^2 \leq \dot{V} \leq \lambda_{\text{max}}(\tilde{P})||z||^2
$$

$$\dot{V} \leq -4\lambda_{\text{min}}(W_j^\star)||z||^2 + \frac{||E||^2}{16\epsilon}.$$
This implies that $\dot{V} \leq -(4\lambda_{\min}(W_P^*)/(\lambda_{\max}(P))V + (||E||^2/16\varepsilon))$ so that $\dot{V} < 0$ when $V > (\lambda_{\max}(P))/(\lambda_{\min}(W_P^*))(||E||^2/64\varepsilon) := \delta_1 > 0$. If we now define the set $S_1 := \{z \in D : V(z) < r\}$, where $r$ is some real number, then any trajectory starting in the open set $D \setminus S_1$ will converge exponentially to the region $S_1$, where $S_1$ denotes the topological closure of $S_1$. Since $V$ is continuous and positive, $S_1$ is some closed neighborhood of the origin. We formalize this result with the following theorem.

**Theorem 2:** Consider the system with disturbances defined by (31)–(34) with control inputs determined by (19)–(21), with $u_j^1$ and $u_j^2$ given by (17) and (18). For sufficiently small $\alpha$, there exists control gains $k_{\alpha t}, k_{\alpha z}, k_{\alpha R}, k_{\alpha L}, k_{\alpha q}$, and $k_{\alpha o}$, $j = 1, \ldots, 4$, such that the zero equilibrium of the tracking errors $e_{\alpha t}, e_{\alpha z}, e_{\alpha R}, e_{\alpha L}, e_{\alpha q},$ and $e_{\alpha o}$ is uniformly ultimately bounded. Moreover, the uniform bound can be made arbitrarily small, provided that $\alpha$ is sufficiently small.

### D. Control Design for the Elastic Model

We now wish to return to the elastic model. In particular, we will show that the control inputs (19)–(21), with $u_j^1$ and $u_j^2$ given by (17) and (18) used for the reduced (inelastic) model still yield favorable trajectory tracking in the elastic model. Before this, we seek to include the quadrotor attitude dynamics back into the problem so that we have a control scheme for the full reduced model when the origin of the boundary layer system and the error dynamics of the reduced model are exponentially stable—which follows immediately from Corollary 2 and Theorem 1. Formally stated, we have the following.

**Proposition 2:** Let the control inputs $u_j$ and $M_j$ as defined above has an exponentially stable equilibrium point at the origin. Khalil [46, Theorem 11.2] told us that the trajectories of the original model lie in a neighborhood of the trajectories of the reduced model when the origin of the boundary layer system and the error dynamics of the reduced model are exponentially stable—which follows immediately from Corollary 2 and Theorem 1. Formally stated, we have the following.

**Definition 7:** The boundary layer system for the singular perturbation problem given by (7) is defined as

$$\frac{\partial r}{\partial T} = g(t, x, r + h(t, x), 0)$$

where $r := z - h(t, x)$ with $h(t, x)$ as defined by (8) and $T := (t - t_0)/\varepsilon$ for $t_0$ the value of time from which we obtain our initial data.

The following corollary for the exponential stability of the boundary layer system (7) follows from the case of a single quadrotor transporting a point-load mass with an elastic cable (see [35, Lemma 2]).

**Corollary 2:** The boundary layer system for (7) with control inputs $u_j$ and $M_j$ as defined above has an exponentially stable equilibrium point at the origin.

### V. Numerical Validation

To validate the controller, we consider a numerical simulation with four quadrotors carrying a load with a mass of 1.5 kg and rigid body shape of rectangular box with length, width, and height are 1.0, 0.8, and 0.2 m, respectively. The mass of each quadrotor is $m_Q = 0.5$ kg. The rest length of the cables is set to $L = 1.0$ m and the load is attached in a symmetric manner to the cables. All further parameters of the configuration, initial values, and controller parameters are presented in Table I.

The numerical simulation was realized with 20 s of simulation, with time step of 0.1 ms implemented in Python on a computer with an Apple M1 Pro processor and 32 GB of RAM memory. The desired trajectory for position of the load used for this simulation is given by $x_{d, t}(t) = 1.2\sin(0.6\pi(t + 1.66)), x_{d, z}(t) = 4.2\cos(0.3\pi(t + 1.66)), x_{d, 3}(t) = 0.5\tan(h(t - 1.5) + 2.5,$ and the desired attitude by $R_d(t) = ([\tilde{e}_3]/|\tilde{e}_3|), (\tilde{e}_3\tilde{v}/|\tilde{e}_3\tilde{v}|), e_3)$, where $\tilde{v}$ is the desired attitude.
As written in Section IV, the quadrotor attitudes are independent of the rest of the dynamics and almost-global exponential stabilizing controllers for the attitude have been presented in [19] and [31]. Thus, we use the same controller for the attitude of the quadrotors and focus here on the discussion on the position results.

First, we start with a brief analysis of the reduced model and, afterward, evaluate the controller on the full model with different cable settings.

### A. Reduced Model With Inelastic Cables

Here, we perform a simulation of the reduced model (10)–(14) with the controller (17) and (18). Thus, the cables are considered fixed with length $L$. The values for the control gains as shown in Table I were selected to guarantee the global exponentially stability for the system by using conditions 1)–3) in the proof of Theorem 1. After 10 s, the system is perturbed by an external disturbance. As shown in Theorems 1 and 2, the tracking error is asymptotically stable (in the case of no disturbance) and ultimately bounded (in the case of a disturbance).

| Symbol | Value |
|--------|-------|
| $m_Q$  | 0.5 kg |
| $g$    | 9.81 m/s$^2$ |
| $r_1$  | $[0.5, 0.4, 0.1]$ |
| $r_3$  | $[-0.5, 0.4, 0.1]$ |

**Initials**

| $x_L(0)$ | $[0, 0, 0.3]^T$ m |
| $v_L(0)$ | $[-2, -3.5, 0]^T$ m/s |
| $R_L(0)$ | $[-0.49, 0.87, 0]$ |
| $q_L(0)$ | $[0.03, 0, 1]$ |

**Controller**

| $k_{x_L}$ | 8 |
| $k_{v_L}$ | 5.2 |
| $k_{q_L}$ | 2 |

**Cables**

| $L$ | 1.0 m |
| $k_{run}=1,2,3=\{10,8,20\}$ |

The resulting trajectory for the center of mass of the load is visualized in Fig. 4. Based on the trajectory, we can conclude the following two observations:

1) During the undisturbed time frame (0–10 s), the actual trajectory converges to the desired trajectory as proposed in Theorem 1.

2) If the system is disturbed (10–20 s), we can observe that the tracking error is increased in contrast to the undisturbed case but remains ultimately bounded as stated in Theorem 2.

### B. Original Model With Elastic Cables

Next, we evaluate the performance of the controller on the original model with elastic cables, as stated in Proposition 1. The setting of the load and quadcopters is equivalent to the simulation with the reduced model and all simulation parameters are provided in Table I. However, this model considers elastic cables and is evaluated on three different values for the stiffness and damping of the cables, in detail, $k_{run}=1,2,3=\{10,8,20\}$ and $c_{run}=1,2,3=\{5,30,3.3\}$, respectively.

The response of the controlled system with elastic cables is visualized in Fig. 4. Each column belongs to a different set of stiffness and damping as indicated by the corresponding titles. The top row shows the tracking error of the load. It can be observed that the tracking error is similar among the settings with oscillating behavior. Note that the frequencies and damping of the oscillations are impacted by: 1) the desired trajectory; 2) the properties of the cables; and 3) the selected feedback gains of the controller. By comparing the tracking error in Fig. 4 with the desired position in Fig. 3, it can be seen that the frequency of the oscillation in the $x$ and $y$ tracking error is synchronous to the desired trajectory. In contrast, the amplitude is impacted by the properties of the cables, see, e.g., the $x$-error and the length of the cables in the middle column of Fig. 4. As stated in Proposition 2, the tracking error remains bounded for all time. The second row visualizes the length of the four cables $l_1$–$l_4$ connected on the load and quadcopters. As expected, the dynamics of the cables mainly depends on their stiffness and damping properties. With decreased damping, more oscillations can be observed,
Fig. 4. Dynamics of the controlled model with elastic cables for different stiffness and damping values of the cables. Top row: tracking error of the load is similar among the different setting and remains bounded as proposed. Second row: oscillations of the cables mainly depend on their properties. Bottom row: trajectory of the load and the quadcopters. Video: https://youtu.be/0Pc_1ATqDlg.

as shown in the third column. Note that the controller (17) and (18) does not actively reduce the oscillation but guarantees boundedness of the tracking error. Finally, the bottom row displays the trajectory of the load (black) and the quadcopters (red, blue, green, and yellow) in the 3-D space. It can be observed that the four quadcopters anticipate the oscillation to reduce the tracking error of the load.

VI. CONCLUSION

We propose a geometric trajectory tracking controller for the cooperative task of four quadrotor UAVs transporting a rigid body load via inflexible elastic cables. This is handled in three stages.

1) Reduction of the model to that of a similar model with inelastic cables. We accomplish this by assuming sufficient stiffness and damping of the cables and utilizing the results of singular perturbation theory.

2) Analysis of a geometric tracking controller in the reduced model. The Lyapunov analysis is used to find sufficient conditions for stability, and Theorem 1 proves the existence of gains satisfying these conditions for sufficiently small initial errors in the cable attitudes.

3) Under the same control law—trajectories of the original (elastic) model converge uniformly to the trajectories of the reduced model as the stiffness and damping of the cables approach infinity.

4) Finally, we also extended the proposed approach to design a control law that guarantees bounded of the tracking error under unstructured bounded disturbances.

In our model, cables are attached to the center of each quadrotor. It would be interesting to explore in future work how to shift those attachment points and study how to deal with the resulting coupled systems—instead of a decoupled system. In addition, we are currently working to add uncertainties in order to further explore the robustness of the proposed controller. We also plan to study the construction of force variational integrators in optimal control problems, in a similar fashion to [47] and [48], dynamic interpolation problems [49], and obstacle avoidance problems [50] for the cooperative task between quadrotors UAVs presented in this article.

For future work, we are interested in testing the results experimentally for systems with elastic cables. Moreover,
we would like to better understand the dependency of the initial cable errors and gains on exponential stability so that safety guarantees are better understood. We would also like to consider the more physically realistic model with elastic initial cable errors and gains on exponential stability so that variations must be tangent vectors in the tangent spaces of the submanifolds of the configuration space in which the state variables live. In addition, they must vanish at the endpoints because tangent vectors on the tangent bundle of $C^\infty([0, T], Q, q_0, q_T)$ must satisfy such a condition (see, for instance, [37], [41], [51]).

In particular, we choose $\delta x_L \in \mathbb{R}^3$ and $\delta l_j \in \mathbb{R}$ arbitrary, $\delta q_j = (d/d\varepsilon)\varepsilon=0 \exp(\varepsilon \tilde{\xi})q_j = \tilde{\xi} \times q_j \in T_q S^2$, satisfying $\tilde{\xi} \cdot q_j = 0$ for arbitrary vectors $\tilde{\xi} \in \mathbb{R}^3$ and $j = 1, \ldots, 4$. By additionally defining the curves on the Lie algebra $\mathfrak{so}(3)$ given by $\tilde{\eta}_j = R_j^T \delta l_j \in \mathfrak{so}(3)$, it can be shown that (see, for instance, [51, Ch. 13]) $\delta \hat{\Omega}_j = \tilde{\eta}_j J\hat{\xi} + \hat{\eta}_j$ with $\tilde{\eta}_j$ satisfying $\tilde{\eta}_j(0) = \tilde{\eta}_j(T) = 0$ (since $\delta R_j(0) = \delta R_j(T) = 0$) for $j = 1, \ldots, 4$. Moreover, we have the following relations:

$$\begin{align*}
\delta x_{Q_j} &= \delta x_L + \delta R_L r_j - \delta l_j q_j - l_j \delta q_j \\
\delta x_{Q_j} &= \delta x_L + \delta R_L r_j - \delta l_i q_i - l_i \delta q_i - \delta l_i \dot{q}_i - l_i \delta \dot{q}_i.
\end{align*}$$

We wish to apply the Lagrange–d’Alembert variational principle for the Lagrangian $L$ and external forces. Therefore, our system dynamics must satisfy

$$\delta \int_0^T L(c(t), \dot{c}(t)) dt + \int_0^T \left( \delta x_{Q_j} \dot{u}_j + \left( R_j^T \delta R_j, \dot{M}_j \right) - c_l \delta l_j \right) dt = 0 \tag{37}$$

where the integral on the right represents the virtual work done by the thrust controls $u_j$, the moment controls $M_j \in \mathbb{R}^3$, and the spring damping.

Expanding the variations within (37), substituting the corresponding infinitesimal variations, and grouping like terms, we obtain

$$0 = \int_0^T \delta \dot{x}_L \left( m_{\text{eff}} \ddot{x}_L - m_Q \sum_{j=1}^4 (\ddot{\xi}_j - \ddot{R}_L r_j) \right)$$

$$+ \delta x_L \left( -m_{\text{eff}} \dot{g} \sum_{j=1}^4 \dot{u}_j \right) dt$$

$$+ \int_0^T \sum_{j=1}^4 \left[ m_Q \delta R_L r_j (\dot{x}_L + \dot{R}_L r_j - \ddot{\xi}_j) + \delta \hat{\Omega}_L^{T} J L. \right.$$}

$$- m_Q g e_T^T (\delta R_L r_j) + \sum_{j=1}^4 u_j^T \delta R_L r_j$$

$$- \sum_{j=1}^4 \int_0^T \left[ m_Q (\delta l_j) q_T^T \dot{x}_L \\
+ \delta l_j (m_Q q_T^T \dot{x}_L - m_Q g e_T^T q_j - k(l - l_j) \right.$$}

$$+ c_l \dot{q} + q_T^T u_j) dt$$

$$- \sum_{j=1}^4 \int_0^T \left[ \xi_j^T (q_j \times (m_Q l_j \dot{x}_L - m_Q g l_j e_3 + l_j u_j)$$

$$+ \dot{q}_j \times m_Q l_j \dot{x}_L) + \xi_j^T (q_j \times m_Q l_j \dot{x}_L) \right] dt$$

$$- \sum_{j=1}^4 \int_0^T \left[ \eta_j^T \left( J Q \dot{\Omega}_j + \eta_j^T (J Q \Omega_j \times \dot{\Omega}_j) + M_j \right) \right] dt$$

where $m_{\text{eff}} := 4m_Q + m_L$, $J_{\text{eff}} := J_L - \sum_{j=1}^4 m_Q \dot{r}_j^2$, and $\xi_j := lj q_j$.

Integrating by parts and applying the equality of mixed partial derivatives and the fact that variations vanish on the endpoints, we obtain

$$0 = \int_0^T \delta x_L \left[ m_{\text{eff}} (\ddot{x}_L + \dot{g} e_3) - m_Q (\ddot{\xi}_j + \ddot{R}_L r_j) - \dot{u}_j \right] dt$$

$$+ \int_0^T \sum_{j=1}^4 \eta_j^T \left[ J Q \dot{\Omega}_j - J Q \Omega_j \times \dot{\Omega}_j - M_j \right] dt$$

$$+ \int_0^T \eta_L^T \left[ J_{\text{eff}} (\Omega_L \times \dot{\Omega}_L) - \Omega_L \right.$$

$$\times \left( m_Q \dot{r}_j \left( \sum_{j=1}^4 \dot{\xi}_j + \dot{u}_j - \dot{g} e_3 - \dot{\xi}_L \right) \right) \right] dt$$

$$- \sum_{j=1}^4 \int_0^T \delta l_j \left[ m_Q q_T^T (\ddot{x}_L + R_L (\ddot{\Omega}_L^2 + \dot{\Omega}_L) r_j - \dot{\xi}_j + \dot{g} e_3) \right.$$}

$$- c_l \dot{q}_j + k(l - l_j) - q_T^T u_j \right] dt$$

$$- l_j \int_0^T \xi_j^T \left[ m_Q q_j \times (\ddot{x}_L + R_L (\ddot{\Omega}_L^2 + \dot{\Omega}_L) r_j$$

$$- c_l \dot{q}_j + k(l - l_j) - q_T^T u_j \right] dt.$$

Each of these integrals can be treated independently, as their respective variations are independent, that is, for the above equation to be satisfied, we necessarily have that each integral vanishes identically. Applying the fundamental lemma of the calculus of variations [52] to each integral yields the dynamical system

$$m_{\text{eff}} (\ddot{\xi}_L + \dot{g} e_3)$$

$$= \sum_{j=1}^4 \dot{u}_j + m_Q \ddot{\xi}_j - m_Q R_L (\ddot{\Omega}_L^2 + \dot{\Omega}_L) r_j$$

$$J_{\text{eff}} \dot{\Omega}_L + \dot{\Omega}_L J_{\text{eff}} \Omega_L$$

$$= \sum_{j=1}^4 m_Q \dot{r}_j R_j^T (-g e_3 - \dot{\xi}_L$$

$$+ \dot{\xi}_j + \frac{1}{m_Q} u_j).$$
\[m_Q q_j^T \ddot{q}_j = m_Q q_j^T \left( \ddot{v}_l + R_L (\ddot{\Omega}_L^2 + \dot{\Omega}_L) r_j + g e_3 - \frac{1}{m_Q} u_j \right) \]
\[-c l_j + k (L - l_j) \]
\[q_j \times \dot{q}_j = q_j \times \left( \ddot{v}_l + R_L (\ddot{\Omega}_L^2 + \dot{\Omega}_L) r_j + g e_3 - \frac{1}{m_Q} u_j \right) \]
\[J_Q \dot{\Omega}_j = J_Q \Omega_j \times \dot{\Omega}_j + M_j, \quad \hat{r}_j = R_j \Omega_j, \quad j = 1, \ldots, 4\]

where we have made the assumption that \(l_j \neq 0\). After implicitly defining the translational and angular velocities of the load with the kinematic equations \(\dot{v}_r = x_r\) and \(\dot{q}_r = \omega_r \times q_r\), and rearranging terms, we obtain the desired dynamical control system. \(\square\)

B. Proof Lemma 1

Proof: Observe that \(V_q\) defined as \(V_q := (1/2) ||e_{xz}||^2 + c_1 \dot{e}_{xz}^T e_{xz} + (1/2) k_{x_{\|}} ||e_{x_{\|}}||^2\) can be bounded from above and below as \((1/2) z_j^T \bar{P}_q z_j \leq V_q \leq (1/2) z_j^T \bar{P}_q z_j \leq V_q \leq (1/2) z_j^T \bar{P}_q z_j \leq V_q \)

\[z = \left[ ||e_{xz}|| \ ||e_{x_{\|}}|| \ ||e_{\Omega_L}|| \ ||e_{\Omega_L}|| \ ||e_{q_{\|}}|| \cdots ||e_{q_k}|| \right] \]
\[\bar{P} = \bar{P}_s \oplus \bar{P}_q \oplus \bar{P}_q \oplus \cdots \oplus \bar{P}_q, \quad \bar{P}_q = \bar{P}_s \oplus \bar{P}_q \oplus \bar{P}_q \oplus \cdots \oplus \bar{P}_q, \quad P = P_s \oplus P_q \oplus P_q \oplus \cdots \oplus P_q.\]

Next, note that by the invariance of scalar shifts of the scalar triple product and the fact that \(q_j^T e_{oj} = 0\), we have
\[\frac{d}{dt} \bar{q}_j = -\bar{q}_j^T \dot{q}_j - q_j^T \dot{q}_j = -\bar{q}_j^T (\omega_j \times q_j) - q_j^T (\ddot{\omega}_j \times q_j) = (\omega_j - \ddot{\omega}_j)^T e_{oj} = 0.\]

In addition, from the vector triple product, we see
\[\dot{e}_{q_j} = (\ddot{q}_j \times q_j) + (\ddot{q}_j \times q_j) \]
\[= (\ddot{q}_j \times q_j) \times q_j - (\ddot{q}_j \times q_j) \times q_j \]
\[= (\ddot{q}_j \times q_j) \times q_j - (\ddot{q}_j \times q_j) \times q_j \]
\[= (\ddot{q}_j \times q_j) \times q_j - (\ddot{q}_j \times q_j) \times q_j \]
\[= e_{oj} \times q_j + (\ddot{q}_j \times q_j) e_{oj} - q_j (\ddot{q}_j \times q_j) = e_{oj} \times q_j + (\ddot{q}_j \times q_j) e_{oj} + 2 \ddot{\omega}_j \times q_j \]
\[\leq ||e_{oj}||^2 + 2 (\ddot{\omega}_j \times q_j)^T e_{oj} \]

where \(C_q \leq 2 \sup ||\ddot{\omega}||\) is a nonnegative constant. Therefore, the time derivative of the proposed Lyapunov function is bounded as
\[\dot{V} \leq -(k_{v_z} - c_3)||e_{v_z}||^2 + c_x k_{v_z} ||e_{v_z}|| ||e_{v_z}|| + c_x k_{v_z} ||e_{v_z}||^2 \]
\[+ \sum_{j=1}^{4} (c_j B (c_j k_{x_{\|}} ||e_{x_{\|}}|| + c_k k_{x_{\|}} ||e_{x_{\|}}|| + k_k k_{x_{\|}} ||e_{x_{\|}}|| + k_v ||e_{v_z}||) \]
\[+ \sum_{j=1}^{4} (c_j B (c_j k_{x_{\|}} ||e_{x_{\|}}|| + c_k k_{x_{\|}} ||e_{x_{\|}}|| + k_k k_{x_{\|}} ||e_{x_{\|}}|| + k_v ||e_{v_z}||) \]

and
\[||e_{\Omega_L}|| + c_k ||e_{\Omega_L}|| \]

Applying these inequality directly to the above bound for \(V\), we find that \(\dot{V} \leq -\sum_{j=1}^{4} \bar{z}_j^T W \bar{z}_j\), where \(W = \left[ ||e_{x_{\|}}|| ||e_{x_{\|}}|| ||e_{x_{\|}}|| ||e_{x_{\|}}|| \right] \]

Hence, if each \(W\) is positive-definite, it follows that the origin is an exponentially stable equilibrium point. \(\square\)

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REFERENCES

[1] P. C. I. Pal unk and R. Fierro, “Agile load transportation: Safe and efficient load manipulation with aerial robots,” IEEE Robot. Autom. Mag., vol. 19, no. 3, pp. 69–79, Sep. 2012.
[2] K. K. Dhim an, M. Kothari, and A. Abhishek, “Autonomous load control and transportation using multiple quadrotors,” J. Aerosp. Inf. Syst., vol. 17, no. 8, pp. 417–435, Aug. 2020.
[3] G. Yu, D. Cabecinhas, R. Cunha, and C. Silvestre, “Aggressive maneuvers for a quadrotor-slung-load system through fast trajectory generation and tracking,” Auto. Robots, vol. 46, no. 4, pp. 499–513, Apr. 2022.
[4] F. Arab, F. A. Shirazi, and M. R. H. Yazdi, “Planning and distributed control for cooperative transportation of a non-uniform slung-load by multiple quadrotors,” Aeros. Sp. Technol., vol. 117, Oct. 2021, Art. no. 106917.
