1. Introduction

In 1974, Levinson \cite{Le} proved that $1/3$ of the zeros of the Riemann zeta function $\zeta(s)$ lie on the critical line. Apparently his work has a reputation for being difficult, and many textbook authors (\cite{T}, \cite{I}, \cite{KV}, \cite{IK}) present Selberg’s method \cite{S} instead (which gives a very small positive percent of zeros). Here we show how innovations in the subject can greatly simplify the proof of Levinson’s theorem.

To set some terminology, let $N(T)$ denote the number of zeros $\rho = \beta + i\gamma$ with $0 < \gamma < T$, and let $N_0(T)$ denote the number of such critical zeros with $\beta = 1/2$. Define $\kappa$ by $\kappa = \liminf_{T \to \infty} \frac{N_0(T)}{N(T)}$. Levinson’s result is that $N_0(T) > \frac{1}{3}N(T)$ for $T$ sufficiently large.

The basic technology to prove that many zeros lie on the critical line is an asymptotic for a mollified second moment of the zeta function (and its derivative). This is well-known, and clear presentations can be found in various sources (\cite{Le}, \cite{C1}, etc.). We briefly summarize the setup. Let $L = \log T$, and suppose $Q(x)$ is a real polynomial satisfying $Q(0) = 1$. Set

$$V(s) = Q \left( -\frac{1}{L} \frac{d}{ds} \right) \zeta(s).$$

Levinson’s original approach naturally had $Q(x) = 1 - x$, but Conrey \cite{C2} showed how more general choices of $Q$ can be used to improve results. For historical comparison we shall eventually choose $Q(x) = 1 - x$. Let $\sigma_0 = \frac{1}{2} - R/L$ for $R$ a positive real number to be chosen later, $M = T^\theta$ for some $0 < \theta < \frac{1}{2}$, and $P(x) = \sum_j a_j x^j$ be a real polynomial satisfying $P(0) = 0, P(1) = 1$. Suppose that $\psi$ is a mollifier of the form

$$\psi(s) = \sum_{h \leq M} \frac{\mu(h)}{h^{s+\frac{1}{2}-\sigma_0}} P \left( \frac{\log M/h}{\log M} \right),$$

Again, for historical reasons we eventually take $P(x) = x$. The conclusion is that

$$\kappa \geq 1 - \frac{1}{R} \log \left( \frac{1}{T} \int_1^T |V\psi(\sigma_0 + it)|^2 dt \right) + o(1).$$

The evaluation of the mollified second moment of zeta appearing in (1.1) is considered to be the difficult part of Levinson’s proof (taking up over 30 pages in \cite{Le}). Conrey and Ghosh \cite{CG} gave a simpler proof. Here we show how to obtain this asymptotic in an easier way.

**Theorem 1.** We have

$$\frac{1}{T} \int_1^T |V\psi(\sigma_0 + it)|^2 dt = c(P, Q, R, \theta) + o(1),$$

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as $T \to \infty$, where

$$c(P, Q, R, \theta) = 1 + \frac{1}{\theta} \int_0^1 \int_0^1 e^{2Re\left(\frac{d}{dx} e^{R\theta x} P(x + u)Q(v + \theta x)\right)}_{x=0}^x 2^dudv.$$

With $P(x) = x$, $Q(x) = 1 - x$, $R = 1.3$, $\theta = 0.5$, and using any standard computer package,

$$c(P, Q, R, \theta) = 2.35 \ldots, \quad \text{and} \quad \kappa \geq 0.34 \ldots.$$

2. A smoothing argument

To simplify forthcoming arguments, it is preferable to smooth the integral in (1.2). Suppose that $w(t)$ is a smooth function satisfying the following properties:

(2.1) $0 \leq w(t) \leq 1$ for all $t \in \mathbb{R}$,

(2.2) $w$ has compact support in $[T/4, 2T]$,

(2.3) $w^{(j)}(t) \ll \Delta^{-j}$, for each $j = 0, 1, 2, \ldots$, where $\Delta = T/L$.

Theorem 2. For any $w$ satisfying (2.1)-(2.3), and $\sigma = 1/2 - R/L$,

$$\int_{-\infty}^{\infty} w(t)|V\psi(\psi + it)|^2 dt = c(P, Q, R, \theta)\hat{w}(0) + O(T/L),$$

uniformly for $R \ll 1$, where $c(P, Q, R, \theta)$ is given by (1.3).

We briefly explain how to deduce Theorem 1 from Theorem 2. By choosing $w$ to satisfy (2.1)-(2.3) and in addition to be an upper bound for the characteristic function of the interval $[T/2, T]$, and with support in $[T/2 - \Delta, T + \Delta]$, we get

$$\int_{T/2}^T |V\psi(\sigma_0 + it)|^2 dt \leq c(P, Q, R, \theta)\hat{w}(0) + O(T/L).$$

Note $\hat{w}(0) = T/2 + O(T/L)$. We similarly get a lower bound. Summing over dyadic segments gives the full integral.

3. The mean-value results

Rather than working directly with $V(s)$, instead consider the following general integral:

$$I(\alpha, \beta) = \int_{-\infty}^{\infty} w(t)\zeta(\frac{1}{2} + \alpha + it)\zeta(\frac{1}{2} + \beta - it)|\psi(\sigma_0 + it)|^2 dt,$$

where $\alpha, \beta \ll L^{-1}$ (with any fixed implied constant). The main result is

Lemma 3. We have

$$I(\alpha, \beta) = c(\alpha, \beta)\hat{w}(0) + O(T/L),$$

uniformly for $\alpha, \beta \ll L^{-1}$, where

$$c(\alpha, \beta) = 1 + \frac{1}{\theta} \frac{d^2}{dxdy}M^{-\beta x - \alpha y} \int_0^1 \int_0^1 T^{-\alpha + \beta + \psi(x + u)P(y + u)} du \bigg|_{x=y=0}.$$

Proof that Lemma 3 implies Theorem 2. Define $I_{\text{smooth}}$ to be the left hand side of (2.4). Then

$$I_{\text{smooth}} = Q\left(-\frac{1}{L\frac{d}{d\alpha}}\right)Q\left(-\frac{1}{L\frac{d}{d\beta}}\right)I(\alpha, \beta)\bigg|_{\alpha=\beta=-R/L}.$$

We first argue that we can obtain $c(P, Q, R, \theta)$ by applying the above differential operator to $c(\alpha, \beta)$. Since $I(\alpha, \beta)$ and $c(\alpha, \beta)$ are holomorphic with respect to $\alpha, \beta$ small, the derivatives
appearing in (3.4) can be obtained as integrals of radii $\asymp L^{-1}$ around the points $-R/L$, from Cauchy’s integral formula. Since the error terms hold uniformly on these contours, the same error terms that hold for $I(\alpha, \beta)$ also hold for $I_{\text{smooth}}$.

Next we check that applying the differential operator to $c(\alpha, \beta)$ does indeed give (1.3). Notice the simple formula

$$Q\left(-\frac{1}{\log T} \frac{d}{d\alpha}\right) X^{-\alpha} = Q\left(\frac{\log X}{\log T}\right) X^{-\alpha}. \tag{3.5}$$

Using (3.5) we have

$$Q\left(-\frac{1}{L} \frac{d}{d\alpha}\right) Q\left(-\frac{1}{L} \frac{d}{d\beta}\right) c(\alpha, \beta)$$

$$= 1 + \frac{1}{\theta} \frac{d^2}{dxdy} M^{-\beta x - \alpha y} \int_0^1 \int_0^1 T^{-v(\alpha + \beta)} P(x + u) P(y + u) Q(v + x\theta) Q(v + y\theta) du dv \bigg|_{x=y=0},$$

which after evaluating at $\alpha = \beta = -R/L$ and simplifying becomes

$$1 + \frac{1}{\theta} \frac{d^2}{dxdy} e^{R\theta(x+y)} \int_0^1 \int_0^1 e^{2Rv} P(x + u) P(y + u) Q(v + x\theta) Q(v + y\theta) du dv \bigg|_{x=y=0}.$$ 

This simplifies to give the right hand side of (1.3), as desired. $\square$

4. Two lemmas

A variation on the standard approximate functional equation ([IK] Theorem 5.3) gives

**Lemma 4.** Let $G(s) = e^s p(s)$ where $p(s) = \frac{(\alpha + \beta)^2 - (2s)^2}{(\alpha + \beta)^2}$, and define

$$V_{\alpha,\beta}(x, t) = \frac{1}{2\pi i} \int_{(1)} G(s) s g_{\alpha,\beta}(s, t) x^{-s} ds, \quad g_{\alpha,\beta}(s, t) = \pi^{-s} \frac{\Gamma\left(\frac{s + x + i t}{2}\right) \Gamma\left(\frac{s + x - i t}{2}\right)}{\Gamma\left(\frac{s + \beta + i t}{2}\right) \Gamma\left(\frac{s + \beta - i t}{2}\right)}.$$

Furthermore, set

$$X_{\alpha,\beta,t} = \pi^{\alpha + \beta} \frac{\Gamma\left(\frac{s - \alpha - it}{2}\right) \Gamma\left(\frac{s - \beta + it}{2}\right)}{\Gamma\left(\frac{s + \alpha + it}{2}\right) \Gamma\left(\frac{s + \beta - it}{2}\right)}.$$

Then if $\alpha, \beta$ have real part less than $1/2$, and for any $A \geq 0$, we have

$$\zeta\left(\frac{1}{2} + \alpha + it\right) \zeta\left(\frac{1}{2} + \beta - it\right) = \sum_{m,n} \frac{1}{m^{\frac{1}{2} + \alpha} n^{\frac{1}{2} + \beta}} \left(\frac{m}{n}\right)^{-it} V_{\alpha,\beta}(mn, t)$$

$$+ X_{\alpha,\beta,t} \sum_{m,n} \frac{1}{m^{\frac{1}{2} - \alpha} n^{\frac{1}{2} - \beta}} \left(\frac{m}{n}\right)^{-it} V_{-\alpha,\beta}(mn, t) + O_A((1 + |t|)^{-A}).$$

**Remark.** Stirling’s approximation gives for $t$ large and $s$ in any fixed vertical strip

$$X_{\alpha,\beta,t} = \left(\frac{t}{2\pi}\right)^{-\alpha - \beta} (1 + O(t^{-1})),$$

$$g_{\alpha,\beta}(s, t) = \left(\frac{t}{2\pi}\right)^s (1 + O(t^{-1}(1 + |s|^2))).$$

Furthermore, for any $A \geq 0$ and $j = 0, 1, 2, \ldots$, we have uniformly in $x,$

$$t^j \frac{\partial^j}{\partial t^j} V_{\alpha,\beta}(x, t) \ll_{A,j} (1 + |t/x|)^{-A}. \tag{4.3}$$
Lemma 5. Suppose \( w \) satisfies (2.1)–(2.3), and that \( h,k \) are positive integers with \( hk \leq T^{2\theta} \) with \( \theta < 1/2 \), and \( \alpha, \beta \ll L^{-1} \). Then

\[
(4.4) \quad \int_{-\infty}^{\infty} w(t) \left( \frac{h}{k} \right)^{-it} \zeta(\frac{1}{2} + \alpha + it) \zeta(\frac{1}{2} + \beta - it) dt = \sum_{h_m = k_n} \frac{1}{m^{\frac{1}{2} + \alpha} n^{\frac{1}{2} + \beta}} \int_{-\infty}^{\infty} V_{\alpha,\beta}(mn,t) w(t) dt \]
\[
+ \sum_{h_m = k_n} \frac{1}{m^{\frac{1}{2} - \beta} n^{\frac{1}{2} - \alpha}} \int_{-\infty}^{\infty} V_{\beta,\alpha}(mn,t) X_{\alpha,\beta,t} w(t) dt + O_{A,\theta}(T^{-A}).
\]

Proof. We apply Lemma 4 to the left hand side. It suffices by symmetry to consider the first part of the approximate functional equation, giving

\[
\sum_{m,n} \frac{1}{m^{\frac{1}{2} + \alpha} n^{\frac{1}{2} + \beta}} \int_{-\infty}^{\infty} w(t) \left( \frac{hm}{kn} \right)^{-it} V_{\alpha,\beta}(mn,t) dt.
\]

The terms with \( hm = kn \) visibly give the first term on the right hand side of (4.4). By combining (2.3) with (4.3), note that we have uniformly in \( x \) that

\[
\frac{\partial}{\partial t} w(t)V_{\alpha,\beta}(x,t) \ll_{j,A} (1 + |x/T|)^{-A} \Delta^{-j}.
\]

Hence for \( hm \neq kn \), we have by repeated integration by parts that

\[
\int_{-\infty}^{\infty} w(t) \left( \frac{hm}{kn} \right)^{-it} V_{\alpha,\beta}(mn,t) dt \ll_{j,A} \frac{(1 + mn)^{-A}}{\Delta |\log \frac{hm}{kn}|^j}.
\]

Say \( hm \geq kn + 1 \). Then

\[
|\log \frac{hm}{kn}| \geq \log \left( 1 + \frac{1}{kn} \right) \geq \frac{1}{2kn} \geq \frac{1}{2\sqrt{hkmn}}.
\]

The same inequality holds in case \( kn \geq hm + 1 \), by symmetry. The error terms from \( hm \neq kn \) are then easily bounded by \( O(T^{-A}) \) for arbitrarily large \( A \). \( \square \)

5. Proof of Lemma 3

Inserting the definition of the mollifier \( \psi \), we have

\[
I(\alpha, \beta) = \sum_{h,k \leq M} \frac{\mu(h)\mu(k)}{\sqrt{hk}} P\left( \log \frac{M}{h} \right) P\left( \log \frac{M}{k} \right) \int_{-\infty}^{\infty} w(t) \left( \frac{h}{k} \right)^{-it} \zeta(\frac{1}{2} + \alpha + it) \zeta(\frac{1}{2} + \beta - it) dt.
\]

According to Lemma 5 we write \( I(\alpha, \beta) = I_1(\alpha, \beta) + I_2(\alpha, \beta) + O(T^{-A}) \). Explicitly,

\[
(5.1) \quad I_1(\alpha, \beta) = \sum_{h,k \leq M} \frac{\mu(h)\mu(k)}{\sqrt{hk}} P\left( \log \frac{M}{h} \right) P\left( \log \frac{M}{k} \right) \sum_{h_m = k_n} \frac{1}{m^{\frac{1}{2} + \alpha} n^{\frac{1}{2} + \beta}} \int_{-\infty}^{\infty} V_{\alpha,\beta}(mn,t) w(t) dt.
\]

Notice that \( I_2(\alpha, \beta) \) is obtained by replacing \( \alpha \) with \( -\beta \), \( \beta \) with \( -\alpha \), and multiplying by \( X_{\alpha,\beta,t} = T^{-\alpha - \beta} (1 + O(L^{-1})) \). That is, \( I(\alpha, \beta) = I_1(\alpha, \beta) + T^{-\alpha - \beta} I_1(-\beta, -\alpha) + O(T/L) \).

Lemma 6. We have \( I_1(\alpha, \beta) = c_1(\alpha, \beta) \hat{w}(0) + O(T/L) \), uniformly on any fixed annuli such that \( \alpha, \beta \gg L^{-1} \), \( |\alpha + \beta| \gg L^{-1} \), where

\[
(5.2) \quad c_1(\alpha, \beta) = \left. \frac{1}{(\alpha + \beta) \log M} \frac{d^2}{dxdy} M^{\alpha x + \beta y} \int_0^1 P(x + u)P(y + u)du \right|_{x=y=0}.
\]
Remark. Note that \(c_1(\alpha, \beta)\) can be alternatively expressed as

\[
(5.3) \quad c_1(\alpha, \beta) = \frac{1}{(\alpha + \beta) \log M} \int_0^1 (P'(u) + \alpha \log M P(u))(P'(u) + \beta \log M P(u))du.
\]

We prove Lemma 6 in Section 6.

Proof that Lemma 6 impliesLemma 3. By adding and subtracting the same thing, we have

\[
I(\alpha, \beta) = [I_1(\alpha, \beta) + I_1(-\beta, -\alpha)] + I_1(-\beta, -\alpha)(T^{-\alpha-\beta} - 1) + O(T/L).
\]

We treat the two terms above differently.

We first compute the term in brackets using (5.3), getting

\[
c_1(\alpha, \beta) + c_1(-\beta, -\alpha) = \int_0^1 2P'(u)P(u)du = 1.
\]

As for the second term, we have from (5.2) that

\[
(T^{-\alpha-\beta} - 1)c_1(-\beta, -\alpha) = \frac{1 - T^{-\alpha-\beta}}{(\alpha + \beta) \log M} \frac{d^2}{dx^2} M^{-\beta x - \alpha y} \int_0^1 P(x + u)P(y + u)du \bigg|_{x=y=0}.
\]

Note that

\[
\frac{1 - T^{-\alpha-\beta}}{(\alpha + \beta) \log M} = \frac{1}{\theta} \int_0^1 T^{-\nu(\alpha + \beta)} dv.
\]

Gathering the formulas gives (3.3) although with the additional restriction that \(|\alpha + \beta| \gg L^{-1}\). However, the holomorphy of \(I(\alpha, \beta)\) and \(c(\alpha, \beta)\) with \(\alpha, \beta \ll L^{-1}\) implies that the error term is also holomorphic in this region. The maximum modulus principle extends the error term to this enlarged domain.

6. Proof of Lemma 6

A Mellin formula gives for \(1 \leq h \leq M\) and \(i = 1, 2, \ldots\)

\[
(6.1) \quad \left(\frac{\log M/h}{\log M}\right)^i = \frac{i!}{(\log M)^i} \frac{1}{2\pi i} \int_{(1)} \left(\frac{M}{h}\right)^v dv.
\]

Using (6.1) and (4.1) in (5.1), we have

\[
I_1(\alpha, \beta) = \int_{-\infty}^\infty w(t) \sum_{i,j} \frac{a_i \alpha_i j! \mu(h) \mu(k)}{(\log M)^{i+j}} \sum_{h_m = kn} \frac{\mu(h) \mu(k)}{h^{\frac{1}{2}k^2} m^{\frac{1}{2}+\alpha} n^{\frac{1}{2}+\beta}}
\]

\[
\left(\frac{1}{2\pi i}\right)^3 \int_{(1)} \int_{(1)} \int_{(1)} \frac{M^{u+v}}{h^v k^u} \frac{g_{\alpha, \beta}(s, t) G(s)}{(mn)^s} ds ds ds du dv du dv.
\]

We compute the sum over \(h, k, m, n\) as follows

\[
(6.2) \quad \sum_{h_m = kn} \frac{\mu(h) \mu(k)}{h^{\frac{1}{2}+v} k^{\frac{1}{2}+u} m^{\frac{1}{2}+\alpha+s} n^{\frac{1}{2}+\beta+s}} = \frac{\zeta(1+u+v)\zeta(1+\alpha+\beta+2s)}{\zeta(1+\alpha+\alpha+s)\zeta(1+\beta+\beta+s)} A_{\alpha, \beta}(u, v, s),
\]

where the arithmetical factor \(A_{\alpha, \beta}(u, v, s)\) is given by an absolutely convergent Euler product in some product of half planes containing the origin. Next we move the contours to \(\text{Re}(u) = \text{Re}(v) = \delta\), and then \(\text{Re}(s) = -\delta + \varepsilon\) (for \(\delta > 0\) sufficiently small so that the arithmetical factor is absolutely convergent), crossing a pole at \(s = 0\) only since \(G(s)\) vanishes at the pole.
of \( \zeta(1 + \alpha + \beta + 2s) \). Since \( M \leq T^\theta \) with \( \theta < \frac{1}{2} \), and \( t \geq T/2 \), the new contour of integration gives \( O(T^{1-\epsilon}) \) for sufficiently small \( \epsilon > 0 \), using (4.2). Thus

\[
I_1(\alpha, \beta) = \bar{w}(0)\zeta(1 + \alpha + \beta) \sum_{i,j} \frac{a_i a_j i^j!}{(\log M)^{i+j}} J_{\alpha,\beta}(M) + O(T^{1-\epsilon}),
\]

where

\[
J_{\alpha,\beta}(M) = \left(\frac{1}{2\pi i}\right)^2 \int_{(\epsilon)} \int_{(\epsilon)} M^{u+v} \frac{\zeta(1 + u + v) A_{\alpha,\beta}(u, v, 0)}{\zeta(1 + \alpha + u)\zeta(1 + \beta + v)} \frac{du}{u + 1} \frac{dv}{v + 1}.
\]

Lemma 7. We have, uniformly for \( \alpha, \beta \ll L^{-1} \),

\[
J_{\alpha,\beta}(M) = \frac{(\log M)^{i+j-1}}{i!j!} \frac{d^2}{dx dy} M^{\alpha+\beta} \int_0^1 (x + u)^i(y + u)^j du \Bigg|_{x=y=0} + O(L^{i+j-2}).
\]

Proof of Lemma 7. We begin by using the Dirichlet series for \( \zeta(1 + u + v) \) and reversing the order of summation and integration to get

\[
J_{\alpha,\beta}(M) = \sum_{n \leq M} \frac{1}{n} \left(\frac{1}{2\pi i}\right)^2 \int_{(\epsilon)} \int_{(\epsilon)} \frac{M^u A_{\alpha,\beta}(u, v, 0)}{\zeta(1 + \alpha + u)\zeta(1 + \beta + v)} \frac{du}{u + 1} \frac{dv}{v + 1}.
\]

Using the standard zero-free region of \( \zeta \) and upper bound on \( 1/\zeta \) (see [11], Theorem 3.8 and (3.11.8)), we obtain that \( J_{\alpha,\beta}(M) \) equals the residue at \( u = v = 0 \) plus an error of size

\[
\sum_{n \leq M} \frac{1}{n} (1 + \log \frac{M}{n})^{-2} \ll 1 \ll L^{i+j-2}.
\]

For computing the residue we take contour integrals of radius \( \asymp L^{-1} \) and use the Taylor approximation

\[
\frac{A_{\alpha,\beta}(u, v, 0)}{\zeta(1 + \alpha + u)\zeta(1 + \beta + v)} = (\alpha + u)(\beta + v)A_{0,0}(0, 0, 0) + O(L^{-3}).
\]

We show in Section 7 below that \( A_{0,0}(0, 0, 0) = 1 \), a result we now use freely. Thus

\[
J_{\alpha,\beta}(M) = \sum_{n \leq M} \frac{1}{n} \left(\frac{1}{2\pi i}\right)^2 \phi \left(\frac{M}{n}\right)^u (\alpha + u) \frac{du}{u + 1} \phi \left(\frac{M}{n}\right)^v (\beta + v) \frac{dv}{v + 1} + O(L^{i+j-2}),
\]

where the contours are circles of radius 1 around the origin.

We compute these two integrals exactly. Suppose \( a > 0 \). Then

\[
\frac{1}{2\pi i} \phi \left(\frac{M}{n}\right)^u \frac{du}{u + 1} = \frac{d}{dx} \left[ e^{ax} \frac{1}{2\pi i} \phi (ae^x)^u \frac{du}{u + 1} \right]_{x=0} = \frac{1}{i!} \frac{d}{dx} e^{ax}(x + \log a)^i \Bigg|_{x=0}.
\]

Thus

\[
J_{\alpha,\beta}(M) = \frac{1}{i!j!} \frac{d^2}{dx dy} e^{\alpha x + \beta y} \sum_{n \leq M} \frac{1}{n} (x + \log(M/n))^i (y + \log(M/n))^j \Bigg|_{x=y=0} + O(L^{i+j-2}).
\]

Note that

\[
\frac{d}{dx} e^{ax}(x + \log(M/n))^i \Bigg|_{x=0} = \frac{(\log M)^i}{\log M} \frac{d}{dx} M^{ax} \left(\frac{x + \log(M/n)}{\log M}\right)^i \Bigg|_{x=0},
\]
so that by summing over $i$ and $j$ we have

$$J_{\alpha,\beta}(M) = \frac{(\log M)^{i+j-2}}{i!j!} \frac{d^2}{dx dy} M^{\alpha x + \beta y} \sum_{n \leq M} \frac{1}{n} \left( x + \frac{\log(M/n)}{\log M} \right)^i \left( y + \frac{\log(M/n)}{\log M} \right)^j \bigg|_{x=y=0}$$

$$+ O(L^{i+j-2}).$$

By the Euler-Maclaurin formula, we can replace the sum over $n$ by a corresponding integral without introducing a new error term (this requires some thought). That is,

$$J_{\alpha,\beta}(M) = \frac{(\log M)^{i+j-2}}{i!j!} \frac{d^2}{dx dy} M^{\alpha x + \beta y} \int_1^M r^{-1} \left( x + \frac{\log(M/r)}{\log M} \right)^i \left( y + \frac{\log(M/r)}{\log M} \right)^j \bigg|_{x=y=0}$$

$$+ O(L^{i+j-2}).$$

Changing variables $r = M^{1-u}$ and simplifying finishes the proof. \hfill \Box

7. THE ARITHMETICAL FACTOR

Here we verify that $A_{0,0}(0,0,0) = 1$ as claimed in the proof of Lemma [7]. The proof is surprisingly easy. We show that $A_{0,0}(s,s,s) = 1$ for all $\operatorname{Re}(s) > 0$. From (6.2) we have

$$A_{0,0}(s,s,s) = \sum_{hm=kn} \frac{\mu(h)\mu(k)}{(hkmn)^{\frac{1}{2}+s}},$$

noting that the ratios of zeta’s on the right hand side of (6.2) cancel. The result now follows instantly from the Möbius formula.

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