A NOTE ON THE SINGULARITY CATEGORY OF AN ENDOMORPHISM RING

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Abstract. We propose the notion of partial resolution of a ring, which is by definition the endomorphism ring of a certain generator of the given ring. We prove that the singularity category of the partial resolution is a quotient of the singularity category of the given ring. Consequences and examples are given.

1. Introduction

Let $A$ be a left coherent ring with a unit. Denote by $A$-mod the category of finitely presented left $A$-modules and by $D^b(A$-mod) the bounded derived category. Following [3, 13], the singularity category $D_{sg}(A)$ of $A$ is the Verdier quotient category of $D^b(A$-mod) with respect to the subcategory formed by perfect complexes. We denote by $q: D^b(A$-mod) $\rightarrow D_{sg}(A)$ the quotient functor. The singularity category measures the homological singularity of $A$.

Let $A M$ be a finitely presented $A$-module. Denote by $\text{add } M$ the full subcategory consisting of direct summands of finite direct sums of $M$. A finite $M$-resolution of an $A$-module $X$ means an exact sequence $0 \rightarrow N^n \rightarrow \cdots \rightarrow N^1 \rightarrow X$ with each $N^i \in \text{add } M$ which remains exact after applying the functor $\text{Hom}_A(M, -)$.

Recall that an $A$-module $M$ is a generator if $A$ lies in $\text{add } M$. We consider the opposite ring of its endomorphism ring $\Gamma = \text{End}_A(M)^{op}$, and $M$ becomes an $A$-$\Gamma$-bimodule. In particular, if $\Gamma$ is left coherent, we have the functor $M \otimes \Gamma - : \Gamma$-mod $\rightarrow A$-mod.

Let $M$ be a generator with $\Gamma = \text{End}_A(M)^{op}$. Following the idea of [19], we call $\Gamma$ a partial resolution of $A$ if $\Gamma$ is left coherent and any $A$-module $X$ has a finite $M$-resolution, provided that it fits into an exact sequence $0 \rightarrow X \rightarrow N^1 \rightarrow N^2 \rightarrow 0$ with $N^i \in \text{add } M$. We mention that similar idea might trace back to [1].

The following result justifies the terminology: the partial resolution $\Gamma$ has “better” singularity than $A$. The result is inspired by [19 Theorem 1.2], where the case that $A$ is an artin algebra with finitely many indecomposable Gorenstein projective modules is studied.

Proposition 1.1. Let $A$ be a left coherent ring, and let $\Gamma$ be a partial resolution of $A$ as above. Then there is a triangle equivalence

$$D_{sg}(\Gamma) \sim D_{sg}(A)/\langle q(M) \rangle$$

induced by the functor $M \otimes \Gamma -$.
Here, we identify the \( A \)-module \( M \) as the stalk complex concentrated at degree zero, and then \( q(M) \) denotes the image in \( D_{\text{sg}}(A) \). We denote by \( \langle q(M) \rangle \) the smallest triangulated subcategory of \( D_{\text{sg}}(A) \) that contains \( q(M) \) and is closed under taking direct summands. Then \( D_{\text{sg}}(A)/\langle q(M) \rangle \) is the corresponding Verdier quotient category.

The aim of this note is to prove Proposition \[10\] and discuss related results and examples for artin algebras.

2. The Proof of Proposition 1.1

To give the proof, we collect in the following lemma some well-known facts. Throughout, \( A \) is a left coherent ring and \( A M \) is a generator with \( \Gamma = \text{End}_A(M)^{\text{op}} \) left coherent.

Recall the functor \( M \otimes_\Gamma - : \Gamma\text{-mod} \to A\text{-mod} \). Denote by \( \mathcal{N} \) the essential kernel of \( M \otimes_\Gamma - \), that is, the full subcategory of \( \Gamma\text{-mod} \) consisting of \( rY \) such that \( M \otimes_\Gamma Y \simeq 0 \).

We will also consider the category \( A\text{-Mod} \) of arbitrary left \( A \)-modules. Note that the functor \( M \otimes_\Gamma - : \Gamma\text{-Mod} \to A\text{-Mod} \) is left adjoint to \( \text{Hom}_A(M, -) : A\text{-Mod} \to \Gamma\text{-Mod} \). Denote by \( \mathcal{N}' \) the essential kernel of \( M \otimes_\Gamma - : \Gamma\text{-Mod} \to A\text{-Mod} \). The functor \( \text{Hom}_A(M, -) \) induces an equivalence \( A \simeq \Gamma\text{-proj} \), where \( \Gamma\text{-proj} \) denotes the category of finitely generated projective \( \Gamma \)-modules.

For a class \( \mathcal{S} \) of objects in a triangulated category \( \mathcal{T} \), we denote by \( \langle \mathcal{S} \rangle \) the smallest triangulated subcategory that contains \( \mathcal{S} \) and is closed under taking direct summands. For example, the subcategory of \( D^b(A\text{-mod}) \) formed by perfect complexes equals \( \langle A \rangle \); here, we view a module as a stalk complex concentrated on degree zero.

**Lemma 2.1.** Let the \( A \)-module \( M \) be a generator and \( \Gamma = \text{End}_A(M)^{\text{op}} \). Then the following results hold.

1. the functor \( \text{Hom}_A(M, -) : A\text{-Mod} \to \Gamma\text{-Mod} \) is fully faithful;
2. the right \( \Gamma \)-module \( M_\Gamma \) is projective, and then the subcategory \( \mathcal{N} \) of \( \Gamma \)-mod is a Serre subcategory, that is, it is closed under submodules, factor modules and extensions;
3. the functor \( M \otimes_\Gamma - : \Gamma\text{-mod} \to A\text{-mod} \) induces an equivalence \( \Gamma\text{-mod}/\mathcal{N} \sim \to A\text{-mod} \), where \( \Gamma\text{-mod}/\mathcal{N} \) denotes the quotient category of \( \Gamma\text{-mod} \) with respect to the Serre subcategory \( \mathcal{N} \);
4. the functor \( M \otimes_\Gamma - : \Gamma\text{-mod} \to A\text{-mod} \) induces a triangle equivalence \( D^b(\Gamma\text{-mod})/\langle \mathcal{N} \rangle \sim \to D^b(A\text{-mod}) \).

**Proof.** (1) is contained in \[13\] Theorem X.4.1(i)], and (2) is contained in \[13\] Proposition IV. 6.7(i)]. In particular, the functor \( M \otimes_\Gamma - \) is exact. Then we recall that the essential kernel of any exact functor between abelian categories is a Serre subcategory. We infer from (1) an equivalence \( \Gamma\text{-mod}/\mathcal{N} \sim \to A\text{-mod} \); consult \[6\] Proposition I.1.3 and Subsection 2.5 d)]. Then (3) follows from \[11\] Proposition A.5. Thanks to (3), the last statement follows from a general result \[12\] Theorem 3.2. \( \Box \)

The argument in the proof of the following result is essentially contained in the proof of \[11\] Chapter III, Section 3, Theorem].

**Lemma 2.2.** Keep the notation as above. Then any \( \Gamma \)-module in \( \mathcal{N} \) has finite projective dimension if and only if \( \Gamma \) is a partial resolution of \( A \).

**Proof.** For the “if” part, assume that \( \Gamma \) is a partial resolution of \( A \), and let \( rY \) be a module in \( \mathcal{N} \), that is, \( M \otimes_\Gamma Y \simeq 0 \). Take an exact sequence \( 0 \to Y' \to P^1 \to Y \to 0 \).
\( P^2 \to Y \to 0 \) in \( \Gamma\text{-mod} \) such that each \( P^i \) is projective. Recall the equivalence \( \text{Hom}_A(M, -) \): add \( M \to \Gamma\text{-proj} \). Then there is a map \( g \colon N^1 \to N^2 \) with \( N^i \in \text{add} \, M \) such that \( \text{Hom}_A(M, g) \) is identified with \( f \). Then \( M \otimes \Gamma Y \simeq 0 \) implies that \( g \) is epic. Moreover, if \( X = \text{Ker} \, g \), then \( Y' \simeq \text{Hom}_A(M, X) \). Then by assumption \( X \) admits a finite \( M \)-resolution \( 0 \to N^{-n} \to N^{1-n} \to \cdots \to N^{-1} \to N^0 \to X \to 0 \) with each \( N^{-i} \in \text{add} \, M \). Applying \( \text{Hom}_A(M, -) \) to it, we obtain a finite projective resolution of \( Y' \). In particular, \( Y' \) has finite projective dimension. The “only if” part follows by reversing the argument. 

Recall that \( (A) \) in \( D^b(A\text{-mod}) \) equals the subcategory formed by perfect complexes. The singularity category is given by \( D^b(A\text{-proj}) = D^b(A\text{-mod})/\langle q(M) \rangle \). We denote by \( q \colon D^b(A\text{-mod}) \to D^b(A\text{-proj}) \) the quotient functor.

The following observation is due to \[9, \text{Proposition 3.3}\] in a slightly different setting. We include the proof for completeness.

**Lemma 2.3.** Keep the notation as above. Then the functor \( M \otimes \Gamma - \colon \Gamma\text{-mod} \to A\text{-mod} \) induces a triangle \( D^b(\Gamma\text{-mod})/\langle N \rangle \to D^b(A\text{-mod}) \) induced by \( M \otimes \Gamma - \). In particular, it sends \( \Gamma \) to \( M \). Hence, it induces a triangle equivalence

\[
D^b(\Gamma\text{-mod})/\langle N \rangle \sim D^b(A\text{-mod})/\langle M \rangle.
\]

Since \( A\text{-mod} \) is a generator, we have \( \langle A \rangle \subseteq \langle M \rangle \subseteq D^b(A\text{-mod}) \). It follows from \[20, \text{\S 2, 4-3 Corollaire}] that \( D^b(A\text{-mod})/\langle M \rangle = D^b(\Gamma)/\langle q(N) \rangle \). For the same reason, \( (D^b(\Gamma\text{-mod})/\langle N \rangle)/\langle \Gamma \rangle \) is identified with \( D^b(\Gamma)/\langle q(N) \rangle \). Then we are done. 

**Proof of Proposition 1.1** Recall that for an \( A \)-module \( X \), \( q(X) \simeq 0 \) in \( D^b(A\text{-mod}) \) if and only if \( X \) has finite projective dimension. Hence by Lemma 2.2 all objects in \( q(N) \) are isomorphic to zero. Then the result follows from Lemma 2.3. 

We observe the following immediate consequence.

**Corollary 2.4.** Let \( A \) be a left coherent ring and \( A\text{-mod} \) a generator. Assume that \( \Gamma = \text{End}_A(M)^{\text{op}} \) is left coherent such that each finitely presented \( \Gamma \)-module has finite projective dimension. Then \( \Gamma \) is a partial resolution of \( A \) and \( D^b(\Gamma) = (q(M)) \).

In the situation of this corollary, we might even call \( \Gamma \) a resolution of \( A \). Such a resolution always exists for any left artinian ring; see \[1, \text{Chapter III, Section 3, Theorem}].

**Proof.** The first statement follows from Lemma 2.2. By assumption, \( D^b(\Gamma) = 0 \) and thus the second statement follows from Proposition 1.1.

3. **Consequences and Examples**

We draw some consequences of Proposition 1.1 for artin algebras. In this section, \( A \) will be an artin algebra over a commutative artinian ring \( R \).

3.1. Recall that an \( A \) module \( M \) is Gorenstein projective provided that \( M \) is reflexive, \( \text{Ext}^i_A(M, A) = 0 = \text{Ext}^i_A(A, M^*) \) for all \( i \geq 1 \). Here, \( M^* = \text{Hom}_A(M, A) \). Any projective module is Gorenstein projective.

Denote by \( A\text{-proj} \) the full subcategory of \( A\text{-mod} \) consisting of Gorenstein projective modules. It is closed under extensions and kernels of surjective maps. In particular, \( A\text{-proj} \) is a Frobenius exact category whose projective-injective objects are precisely projective \( A \)-modules. Denote by \( A\text{-Gproj} \) the stable category;
it is naturally triangulated by [7, Theorem 1.2.8]. The Hom spaces in the stable
category are denoted by $\text{Hom}$.

Recall from [8, Theorem 1.4.4] or [9, Theorem 4.6] that there is a full
triangle embedding $F_A : A\text{-Gproj} \to D_{sg}(A)$ sending $M$ to $q(M)$. Moreover, $F_A$ is dense,
thus a triangle equivalence if and only if $A$ is Gorenstein, that is, the injective
dimension of the regular $A$-module has finite injective dimension on both sides.

A full subcategory $C \subseteq A\text{-Gproj}$ is thick if it contains all projective modules
and for any short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ with terms in $A\text{-Gproj}$, all
$M_i$ lie in $C$ provided that two of them lie in $C$. In this case, the stable category $C$ is a
triangulated subcategory of $A\text{-Gproj}$, and thus via $F_A$, a triangulated subcategory
of $D_{sg}(A)$.

**Proposition 3.1.** Let $AM$ be a generator which is Gorenstein projective such that
add $M$ is a thick subcategory of $A\text{-Gproj}$. Then $\Gamma = \text{End}_A(M)^{op}$ is a partial
resolution of $A$ and thus we have a triangle equivalence

$$D_{sg}(\Gamma) \to D_{sg}(A)/\text{add} M,$$

which is further triangle equivalent to $q(M)^\perp$.

Here, $q(M)^\perp$ denotes the full subcategory of $D_{sg}(A)$ consisting of objects $X$
with $\text{Hom}_{D_{sg}(A)}(q(M), X) = 0$. Since add $M$ is a thick subcategory of $A\text{-Gproj}$, it
follows that $q(M)^\perp$ is a triangulated subcategory of $D_{sg}(A)$.

**Proof.** For any exact sequence $0 \to X \to N^1 \to N^2 \to 0$ with $N^1 \in \text{add} M$, we have
that $X$ is Gorenstein projective, since $A\text{-Gproj}$ is closed under kernels of surjective maps.
Then by assumption, $X$ lies in $\text{add} M$, in particular, $X$ admits a finite $M$-resolution. Then $\Gamma$ is a partial resolution of $A$. We note that $(q(M)) = \text{add} M$. Then the first equivalence follows from Proposition [10, Proposition 1.14] immediately.

For the second one, we note by [13, Proposition 1.21] that for each object
$X \in D_{sg}(A)$, $\text{Hom}_{D_{sg}(A)}(q(M), X)$ is a finite length $R$-module and then the coho-
monological functor $\text{Hom}_{D_{sg}(A)}(q(M), -) : \text{add} M \to R\text{-mod}$ is representable. Hence,
the triangulated subcategory $\text{add} M$ of $D_{sg}(A)$ is right admissible in the sense of
[2]. Then the result follows from [2, Proposition 1.6].

A special case is of independent interest: an algebra $A$ is **CM-finite** provided
that there exists a module $M$ such that $A\text{-Gproj} = \text{add} M$. In this case, we obtain
a triangle equivalence

$$D_{sg}(\Gamma) \to D_{sg}(A)/A\text{-Gproj}.$$

In particular, $\Gamma$ has finite global dimension if and only if $A$ is Gorenstein. This
triangle equivalence is due to [11, Theorem 1.2].

For the following example, we recall that any semisimple abelian category $A$ has
a unique (trivial) triangulated structure with the translation functor given by any
auto-equivalence $\Sigma$. This triangulated category is denoted by $(A, \Sigma)$.

**Example 3.2.** Let $k$ be a field. Consider the algebra $A$ given by the following
quiver with relations $\{\beta \alpha \gamma \beta \alpha, \alpha \gamma \beta \alpha \gamma \beta\}$

![Quiver with relations](image)

Here, we write the concatenation of arrows from right to left. The algebra $A$ is
Nakayama with admissible sequence $(5, 6, 6)$.

For each vertex $i$, we associate the simple $A$-module $S_i$. Consider the unique
indecomposable module $S_2^{(3)}$ with top $S_2$ and length $3$; it is the unique indecomposable
non-projective Gorenstein projective $A$-module; see [5, Proposition 3.14(3)] or [14].
Set $M = A \oplus S_2^{[3]}$. Then add $M = A$-Gproj. The algebra $\Gamma = \operatorname{End}_A(M)\text{op}$ is given by the following quiver with relations $\{ab, b\alpha a, ba - \alpha \beta\}$

\[
\begin{array}{c}
1 \\
\downarrow a \\
\downarrow 2 \\
\downarrow \beta \\
\gamma \\
\end{array}
\]

Then by Proposition 3.7 we have triangle equivalences

\[
\mathcal{D}_{\operatorname{sg}}(\Gamma) \sim \mathcal{D}_{\operatorname{sg}}(A)/\text{A-Gproj} \sim q(S_2^{[3]})^\perp.
\]

Here, we recall that $q(S_2^{[3]})^\perp = q(\mathcal{M})^\perp$ is the triangulated subcategory of $\mathcal{D}_{\operatorname{sg}}(A)$ consisting of objects $X$ with $\operatorname{Hom}_{\mathcal{D}_{\operatorname{sg}}(A)}(q(S_2^{[3]}), X) = 0$. The singularity category $\mathcal{D}_{\operatorname{sg}}(A)$ is triangle equivalent to the stable category $\mathcal{A}'$-mod for an elementary connected Nakayama algebra $\mathcal{A}'$ with admissible sequence $(4, 4)$ such that $q(S_2^{[3]})$ corresponds to an $\mathcal{A}'$-module of length 2; see [5] Corollary 3.11.

Then explicit calculation in $\mathcal{A}'$-mod yields that $q(S_2^{[3]})^\perp$ is equivalent to $k \times k$-mod; moreover, the translation functor $\Sigma$ is induced by the algebra automorphism of $k \times k$ that switches the coordinates. In summary, we obtain a triangle equivalence

\[
\mathcal{D}_{\operatorname{sg}}(\Gamma) \sim (k \times k$-mod, $\Sigma).
\]

We mention that by [10] Theorem 1.1 any Gorenstein projective $\Gamma$-module is projective. In particular, the algebra $\Gamma$ is not Gorenstein.

3.2. Let $k$ be a field and $Q$ a finite quiver without oriented cycles. Consider the path algebra $kQ$ and the algebra $A = kQ[\epsilon] = kQ \otimes_k k[\epsilon]$ of dual numbers with coefficients in $kQ$, where $k[\epsilon]$ is the algebra of dual numbers. Then $A$ is Gorenstein. The stable category $\Gamma$-Gproj, which is equivalent to $\mathcal{D}_{\operatorname{sg}}(A)$, is studied in [10]; see [4] Section 5.

For an $A$-module $Y$, $\epsilon$ induces a $kQ$-module map $\epsilon_Y : Y \to Y$ satisfying $\epsilon_Y^2 = 0$. The cohomology of $Y$ is defined as $H(Y) = \operatorname{Ker} \epsilon_Y/\operatorname{Im} \epsilon_Y$. This gives rise to a functor

\[
H : \text{A-Gproj} \to kQ\text{-mod},
\]

which induces a cohomological functor $\text{A-Gproj} \to kQ$-mod. It follows that for each subcategory $\mathcal{C}$ of $kQ$-mod which is closed under extensions, kernels of surjective maps and cokernels of injective maps, the corresponding subcategory $H^{-1}(\mathcal{C})$ of $A$-Gproj is thick.

Recall that the path algebra $kQ$ is hereditary. For each $kQ$-module $X$, consider its minimal projective $kQ$-resolution $0 \to P^{-1} \to P^0 \to X \to 0$. Set $\eta(X) = P^{-1} \oplus P^0$; it is an $A$-module such that $\epsilon$ acts on $P^{-1}$ by $d$ and on $P^0$ as zero. If $X$ is indecomposable, the $A$-module $\eta(X)$ is Gorenstein projective which is indecomposable and non-projective. Observe that $H(\eta(X)) \simeq X$. Indeed, if $Y$ is an indecomposable non-projective Gorenstein projective $A$-module satisfying $H(Y) \simeq X$, then $Y$ is isomorphic to $\eta(X)$. For two $kQ$-modules $X$ and $X'$, we have a natural isomorphism

\[
(3.1) \quad \operatorname{Hom}_A(\eta(X), \eta(X')) \simeq \operatorname{Hom}_{\mathcal{A}Q}(X, X') \oplus \operatorname{Ext}_A^1(X, X').
\]

We refer the details to [10] Theorems 1 and 2.

Let $E$ be an exceptional $kQ$-module, that is, an indecomposable $kQ$-module such that $\operatorname{Ext}_A^1(E, E) = 0$. It is well known that the full subcategory $E^{\perp_{kQ}}$ of $kQ$-mod consisting of modules $X$ with $\operatorname{Hom}_{\mathcal{A}Q}(E, X) = 0 = \operatorname{Ext}_A^1(E, X)$ is equivalent to
Proposition 3.3. Keep the notation as above. Set $M = A \oplus \eta(E)$ and $\Gamma = \text{End}_A(M)^{\text{op}}$. The following statements hold.

1. $\text{add}M$ is a thick subcategory of $A\text{-Gproj}$, and the corresponding stable category $\text{add}M$ is triangle equivalent to $(\text{k-mod}, \text{Id}_k\text{-mod})$;
2. $q(M)^{\perp}$ is triangle equivalent to $kQ'[\epsilon]\text{-Gproj}$;
3. there is a triangle equivalence $D_{\text{sg}}(\Gamma) \sim kQ'[\epsilon]\text{-Gproj}$.

Proof. For (1), recall that $\text{End} kQ(E) \simeq k$. It follows that the subcategory $\text{add}E$ of $kQ\text{-mod}$ is closed under extensions, kernels of surjective maps and cokernels of injective maps. Observe that $\text{add}M = H^{-1}(\text{add}E)$. Then the thickness of $\text{add}M$ follows. The stable category $\text{add}M$ is given by $\text{add} \eta(E)$; moreover, by [5, 3.4] we have $\text{End}_A(\eta(E)) \simeq k$. We observe that the translation functor acts on $\text{add} \eta(E)$ as the identity; see [16, Proposition 4.10]. Then we infer (1).

We identify $D_{\text{sg}}(A)$ with $A\text{-Gproj}$. By [5, 3.1], $q(M)^{\perp} = q(\eta(E))^{\perp}$ equals the stable category of $H^{-1}(E^{\perp[0,1]} \text{-mod})$. Since $E^{\perp[0,1]}$ is equivalent to $kQ'[\epsilon]\text{-mod}$, it follows from [10, Theorem 1] that the stable category of $H^{-1}(E^{\perp[0,1]})$ is triangle equivalent to $kQ'[\epsilon]\text{-Gproj}$. This equivalence might also be deduced from the results in [4, Sections 3 and 5].

The last statement follows from (2) and Proposition 3.3. □

We conclude with an example of Proposition 3.3.

Example 3.4. Let $Q$ be the quiver $1 \overset{\alpha}{\rightarrow} 2$ and let $A = kQ[\epsilon]$. Denote by $S_1$ the simple $kQ$-module corresponding to the vertex 1; it is an exceptional $kQ$-module. Then $\eta(S_1) = kQ$ such that $\epsilon$ acts as the multiplication of $\alpha$ from the right. The algebra $\Gamma = \text{End}_A(A \oplus \eta(S_1))^{\text{op}}$ is given by the following quiver with relations $\{ \beta\alpha, \alpha\delta = \delta\gamma \}$

$1 \overset{\alpha}{\rightarrow} 2 \overset{\beta}{\rightarrow} 3$.

The corresponding quiver $Q'$ in Proposition 3.3 is a single vertex, and then the stable category $kQ'[\epsilon]\text{-Gproj}$ is triangle equivalent to $(\text{k-mod}, \text{Id}_k\text{-mod})$. Hence, by Proposition 3.3(3), we obtain a triangle equivalence $D_{\text{sg}}(\Gamma) \sim (\text{k-mod}, \text{Id}_k\text{-mod})$.

We mention that the simple $\Gamma$-module corresponding to the vertex 3 is localizable, whose corresponding left retraction $L(\Gamma)$ is an elementary connected Nakayama algebra with admissible sequence $(3, 4)$; see [5]. Then by [5] Lemma 3.12(2) and Proposition 2.6 any Gorenstein projective $\Gamma$-module is projective; in particular, $\Gamma$ is not Gorenstein. The above triangle equivalence might also be deduced from [5] Proposition 2.13 and Corollary 3.11.

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