A Simple Solution to the Cylindrical Indentation of an Elastic Compressible Thin Layer Resting on a Rigid Substrate

ZQ Wang, ZL Dan and J Wu*

College of Aerospace Engineering, Chongqing University, Chongqing, China, 400044

*Corresponding author email: jwu@cqu.edu.cn

Abstract. In this paper, an analytical model is presented to study the contact that recedes between an elastic thin film that could be compressed and a substrate of rigidity. The surface of rigidity was formed due to cylindrical indentation. The substrate was assumed to be a rough surface without any friction. Further, the contact width of the substrate was derived, and the relationship between the compression force, compression depth, and the compression width was determined using the energy method. Finally, the obtained results were validated using finite element analysis.

Keywords: Receding contact; Contact mechanics thin layer; Cylindrical indentation.

1. Introduction

The cylindrical contact method is predominantly used in most engineering applications. The cylindrical indentation is an essential methodology used for the evaluation of the mechanical properties of thin films [1, 2]. In indentation analysis, it is generally considered that the film is either attached to the substrate or has sliding contact with the substrate [3, 4]. However, from a practical perspective, it is observed that while the elastic film rests on the substrate and is indented on its upper surface, a part of the layer would detach from the substrate (Figure 1), which is termed as receding contact [5, 6].

The receding contact problem attracts considerable attention and is solved numerically in many literatures [5-11], which often involves complex mathematical derivations and provides implicit numerical relationships between the compression force, width, and depth. From the literature, it was observed that contact loss is not much considered in simple analytical methods of cylindrical indentation of elastic thin films.

In this paper, a simple method to the receding contact problem in the cylindrical indentation analysis is proposed, and several explicit relationships between the compression width, depth, and force are obtained. Further, the plane strain Kerr-type relation [2, 12], which is widely used for thin films, is employed. Furthermore, explicit relationships are obtained based on the energy method. Moreover, the analytical results are demonstrated and verified using finite element analysis (FEA).
Figure 1. Schematic representation of a cylindrical indenter pressed into an elastic thin film built on a substrate.

2. Mathematical Formulation

The geometrical structure of the indenter-film system is shown in Figure 1. In Figure 1, the thickness of the compressible elastic film is $t$; the radius of the frictionless rigid cylindrical indenter is $R$; the half-width of the indentation zone between the indenter and the film is $a$; the compression depth is $\delta$, and the half-width of the compression area between the layer and the rigid substrate is $b$. The layer loses contact with the substrate when $|x| > b$ due to indentation. An analytical asymptotic result in the following assuming $t \ll a \ll R$, $a \approx b$ is derived below.

2.1. The Kerr-type Model of the Lower Compression Area ($x \leq b$)

Under the plane-strain assumption, the differential relationship between the indenter pressure $p$ in the compression area and the normal displacement of the film $w$ is given by the Kerr-type model [12] as shown in equation (1).

$$w - A_1 t^2 \left( \frac{d^2}{dx^2} \right) \approx B_1 \frac{t}{E} p - B_2 \frac{t}{E} \left( \frac{d^2}{dx^2} \right) p$$ (1)

In equation (1), $E$ refers to Young’s modulus, $A_1$, $B_1$, and $B_2$ refer to the real constants which are obtained using Poisson’s ratio $\nu$. When there is no friction between the film and the substrate, equation (2) is obtained, whereas when the substrate is very rough, equation (3) is obtained [12].

$$A_1 = \frac{1}{3}, \quad B_1 = 1 - \nu^2, \quad B_2 = \frac{(1 - \nu^2)}{3}$$ (2)

$$A_1 = \frac{1}{1 - \nu}, \quad B_1 = \frac{(1 - 2\nu)(1 + \nu)}{1 - \nu}, \quad B_2 = \frac{(3 - 4\nu)(1 + \nu)}{3(1 - \nu)}$$ (3)

During the indentation process, the force starts from zero and gradually increases to $F$, and the contact width between the layer and the substrate increases from 0 to $b$. In this work, the displacements outside the contact area were considered as rigid-body displacements and were neglected [13].

2.2. Compression Pressure $p(x)$ of the Compression Area ($x \leq a$)

In the compression area ($x \leq a$), the normal deflection $w(x)$ of the top surface of the layer in the downward direction is as shown in equation (4).

$$w = \delta + \left( R^2 - x^2 \right)^{1/2} - R \approx \delta - \frac{x^2}{2R}, \quad x \leq a$$ (4)

From equation (1) and equation (4), equation (5) is obtained.
\[
B_1 \frac{t}{E} p - B_2 \frac{t^2}{E} \left( \frac{d^2}{dx^2} \right) p = \delta - \frac{x^2}{2R} + \frac{A t^2}{R}; \quad p = 0, \quad \text{at} \quad x = a
\]

From Figure 1, it is observed that \( p(x) = p(-x) \), thus equation (6) is obtained.

\[
p / E = c_1 \cosh \left[ \frac{1}{t} \left( \frac{B_1}{B_2} \right)^{1/2} x \right] + c_2 + c_3 x^2
\]

In equation (6), \( c_1, c_2, \) and \( c_3 \) are given in terms of \( \delta \) and \( a \) by using equation (7).

\[
c_1 = -\left( c_2 a^2 + c_3 \right) \cosh \left[ \frac{1}{t} \left( \frac{B_1}{B_2} \right)^{1/2} a \right]
\]

In equation (7), \( c_1 \) refers to the general solution of equation (6), whereas \( c_2 \) and \( c_3 \) refer to the particular solutions.

**2.3. Indenter-layer System’s Total Energy**

The total potential energy of the indenter-layer system \( U \) is as shown in equation (8)

\[
U = U_e - F \delta
\]

In equation (8), \( U \) refers to the energy per unit thickness, \( F \) refers to the contact force per unit thickness, and the strain energy is as shown in equation (9).

\[
U_e = \frac{d}{2} \int_0^a p(x) w(x) dx
\]

Since \( t \ll a \), from equations (6)-(9), equation (10) is obtained.

\[
U = E \delta \left\{ c_1 \left( \frac{B_1}{B_2} \right)^{1/2} \sinh \left[ \frac{a}{t} \left( \frac{B_1}{B_2} \right)^{1/2} \right] + \frac{c_2 a + 1}{3} c_3 a^3 \right\}
\]

By substituting equation (7) in equation (10), equation (11) is obtained.

\[
\frac{U}{E} = \frac{\delta}{B_1 t} \left( a \delta - \frac{a^3}{6R} \right) - \frac{a^3}{2B_1 t R} \left( \frac{\delta - a^2}{3} \right) \frac{F \delta}{E}
\]

**2.4. Relationships between the Compression Force, Width and Depth**

According to the principle of stationary total potential energy, the higher-order term is ignored, and the following results are obtained as shown in equation (12).

\[
\frac{\partial U}{\partial a} = \left( \frac{\delta}{B_1 t} - \frac{a^2}{2B_1 t R} \right) \left( \delta - \frac{a^2}{2R} \right) = 0
\]

From equation (12), the \( \delta-a \) relation as shown in equation (13) is obtained.

\[
\delta = \frac{a^2}{2R}
\]

When \( \partial U / \partial \delta = 0 \), equation (11) and equation (13), the \( F-a \) relationship and the \( F-\delta \) relationship are obtained as shown in equation (13) and equation (14) respectively.
\[ F = \frac{2a^3}{3BrR} \]  

(14)

\[ F = \frac{32^{\frac{1}{3}}\delta^2R^2}{3Br} \]  

(15)

2.5. The Width of the Compression Zone

It is derived in the electronic supplementary material, that the half-contact width between the thin film and the substrate \( b = a \), for both smooth and infinitely rough substrates.

3. Results and Discussions

3.1. Finite Element Model

In order to validate the obtained results, FEA was performed using ABAQUS software. The radius of the cylindrical indenter was taken as 8 mm, and the thickness of the film was 5 \( \mu m \). Further, 40,000 elements were used while meshing. The maximum depth of the indenter was 0.18 \( \mu m \). The Coulomb-type friction factor \( \mu \) was set relatively large to model the “infinitely rough condition”.

3.2. The Contact Width

Figure 2 shows the contact width obtained from the analytical model and the FEA. From Figure 2, it is observed that the results obtained by the analytical model and the FEA are similar. It is also observed that as \( a \) increases, the relative error decreases gradually. Further, under frictionless conditions, the relative error is found to be lesser than 2% and under stick conditions, it is lesser than 5%.
3.3. F-a and F-δ relations
Figures 3-6 show the relationships between the compression force $F$, the half compression width $a$, and the compression depth $\delta$. From the figures, it is observed that the relative error decreases as $\delta$ (or $a$) increases. Further, when the layer’s Poisson’s ratio $\nu = 0.4$, the relative error is found to be larger than $\nu = 0.2$. Furthermore, from Figure 3 and Figure 5, it is observed that, under smooth conditions, the relative error is lesser than 5% when $a > 5t$, and lesser than 3% when $a > 8t$. From Figure 4 and Figure 6, it is observed that, under stick conditions, the relative error is lesser than 5% when $\delta > 0.03t$.

Figure 3. F-a relation under frictionless contact condition.

Figure 4. F-a relation under stick contact condition.
Figure 5. F-δ relation under frictionless contact condition.

Figure 6. F-δ relation under stick contact condition.

4. Conclusions
In this paper, a simple method was presented to analyze the contact that recedes between an elastic compressible thin film and a rigid substrate that was formed by cylindrical indentation. The following conclusions were obtained.
1. Using the energy method, the relationships between the indentation force, depth, and width were obtained.
2. The compression width between the layer and the substrate was found to be equal to the indentation width between the film and the indenter.
3. When the substrate contact surface was smooth and the Poisson’s ratio of the layer was relatively less, it was found that the relative error was less.
4. The results obtained using the analytical method and the FEA are found to be similar.

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