RATIONAL CURVES 
ON FIBERED CALABI–YAU MANIFOLDS 
(WITH AN APPENDIX BY ROBERTO SVALDI)

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Abstract. We show that a smooth projective complex manifold of dimension greater than two endowed with an elliptic fiber space structure and with finite fundamental group always contains a rational curve, provided its canonical bundle is relatively trivial. As an application of this result, we prove that any Calabi–Yau manifold that admits a fibration onto a curve whose general fiber is an abelian variety always contains a rational curve.

1. Introduction

The first goal of this paper is to prove the following result about the existence of rational curves on an elliptically fibered projective manifold \( X \) with some restriction on its fundamental group. We always assume that the dimension of \( X \) is greater than two.

**Theorem 1.1.** Let \( X \) be a smooth projective manifold with finite fundamental group. Suppose there exists a projective variety \( B \) and a morphism \( f: X \to B \) such that the general fiber has dimension one. Suppose, moreover, that there exists a line bundle \( L \) on \( B \) such that \( K_X \cong f^*L \). Then \( X \) does contain a rational curve.

An elliptic fiber space is a projective variety endowed with a fibration of relative dimension one such that its general fiber is an elliptic curve. The assumption on the canonical bundle in the statement readily implies, by adjunction, that the manifold \( X \) in the statement is indeed an elliptic fiber space.

By a Calabi–Yau manifold we mean a smooth projective manifold \( X \) with trivial canonical bundle \( K_X \cong \mathcal{O}_X \) and finite fundamental group. Calabi–Yau manifolds are of interest in both algebraic geometry and theoretical physics. In particular, the problem of determining whether they do contain rational curves is important in string theory (see for instance [Wit86, DSWW86], where the physical relevance of rational curves on Calabi–Yau manifolds is discussed). Moreover, a folklore conjecture in algebraic geometry predicts the existence of rational curves on every Calabi–Yau manifold, see for instance [Miy95, Question 1.6], and [MP97, Problem 10.2]. Already in dimension three, the conjecture is open. There are results for high Picard rank (see [Wil89] and [HBW92]), in the case of existence of a non-zero, effective, non-ample divisor (see [Pet91] and [Ogu93]), and in the case of existence of a non-zero, nef, non-ample divisor (see [DF14]). For a more detailed discussion of...
the three dimensional case we refer to the Appendix by R. Svaldi. Almost nothing is known in higher dimension.

Here, we consider the case of an elliptic Calabi–Yau manifold, i.e. a Calabi–Yau manifold which is also an elliptic fiber space. Quoting Kollár [Kol15], “F-theory posits that the hidden dimensions constitute a Calabi–Yau 4-fold X that has an elliptic structure with a section”.

As an immediate corollary of Theorem 1.1, we obtain that on elliptic Calabi–Yau’s, there is always at least one rational curve. It is a generalization of [Ogu93, Theorem 3.1, Case ν(X, L) = 2], who treated therein the three dimensional case.

**Corollary 1.2.** Let X be an elliptic Calabi–Yau manifold. Then X always contains a rational curve.

**Remark 1.3.** Calabi–Yau in dimension two are just K3 surfaces, that are known to contain rational curves thanks to the Bogomolov–Mumford Theorem, [MM83].

Conjecturally, a Calabi–Yau manifold X is elliptic if and only if there exists a (1,1)-class α ∈ H^2(X, Q) such that α is nef of numerical dimension ν(α) = dim X − 1 (this enters in the circle of ideas around the Generalized Abundance Conjecture for Calabi–Yau manifolds). Recall that the numerical dimension of a nef class α is the biggest integer m such that self-intersection α^m is non zero in H^{2m}(X, Q). This conjecture is known to hold true, under the further assumption that α^{dim X − 2} · c_2(X) ̸= 0, for threefolds by the work of [Wil94, Ogu93] and in all dimensions by [Kol15, Corollary 11] (see also Section 4). We can thus state the following numerical sufficient criterion for the existence of rational curves on Calabi–Yau manifolds.

**Corollary 1.4.** Let X be a Calabi–Yau manifold. Suppose that X possesses a (1,1)-class α ∈ H^2(X, Q) such that

- the class α is nef,
- the numerical dimension ν(α) of α is dim X − 1,
- the intersection product α^{dim X − 2} · c_2(X) is non zero.

Then X is elliptic and therefore it contains a rational curve.

In order to discuss another corollary, let us recall the following. Suppose that X is a projective manifold with semi-ample canonical bundle, i.e. some tensor power of K_X is globally generated, then there exists on X an algebraic fiber space structure (see Subsection 1.1 for the definition) φ: X → B, called the semi-ample Iitaka fibration. This algebraic fiber space has the property that dim B = κ(X) and that there exists an ample line bundle A over B such that K_X^{⊗ℓ} ∼ φ^∗A (see [Las94, Theorem 2.1.27]). Here, ℓ is the exponent of the sub-semigroup of natural numbers m such that K_X^{⊗m} is globally generated. In particular, if every sufficiently large power of K_X is free, then m = 1.

**Corollary 1.5.** Let X be a smooth projective manifold with finite fundamental group and Kodaira dimension κ(X) = dim X − 1. Suppose that K_X is semi-ample of exponent ℓ = 1. Then X contains a rational curve.

Observe that the hypothesis of semi-ampleness of K_X can actually be relaxed in nefness. This is because the numerical dimension of a nef line bundle is always greater than or equal to its Kodaira–Iitaka dimension, so that ν(K_X) ≥ dim X − 1. Now, if ν(K_X) = dim X then K_X would be big and thus X of general type, contradicting κ(X) = dim X − 1. Thus, we necessarily have ν(X) = κ(X) = dim X − 1, and so K_X is semi-ample by [Kaw85, Theorem 1.1]. On the other hand, the hypothesis on the exponent ℓ = 1 of K_X cannot be dropped, since in general it is not automatic.
The second goal of the paper is to prove an application of Theorem 1.1, where we deal with Calabi–Yau manifolds endowed with a fibration onto a curve whose fibers are abelian varieties.

**Theorem 1.6.** Let $X$ be a Calabi–Yau manifold that admits a fibration onto a curve whose general fibers are abelian varieties. Then $X$ does contain a rational curve.

For explicit examples of fibrations as in Theorem 1.6 we refer to [Ogu93, Theorem 4.9] and to [GP01].

**Acknowledgements.** This project started during the visit of the third–named author at the Institut de Mathématiques de Jussieu–Paris Rive Gauche and continued during her visit at the Department of Mathematics at the University of Trento. She wishes to thank both institutes for the warm hospitality.

The first–named author warmly thanks A. Höring, A.-S. Kaloghiros and A. Rapagnetta for useful conversations along the years around the subject presented here.

The second–named and the third–named authors would like to thank M. Andreatta, E. Ballico, C. Ciliberto, E. Floris, V. Lazić and A. Petracci for important comments and discussions.

All authors warmly thank R. Svaldi for nice conversations on the subject of the present paper.

1.1. **Notation and conventions.** We work over the field of complex numbers $\mathbb{C}$. An algebraic fiber space is a surjective proper mapping $f: X \to Y$ of projective varieties such that $f_*\mathcal{O}_X \cong \mathcal{O}_Y$; in particular it has connected fibers and if $X$ is normal, so is $Y$. Given a holomorphic (proper) surjective map $f: X \to Y$ of smooth complex manifolds, we say that $y \in Y$ is a regular value for $f$ if for all $x \in f^{-1}(y)$ the differential $df(x): T_{X,x} \to T_{Y,y}$ is surjective; the set of singular values for $f$, i.e. the complement of the set of regular values for $f$, is a proper closed analytic subset of $Y$.

2. **Proof of Theorem 1.1**

In this section we shall prove the following result, which will readily imply Theorem 1.1.

**Theorem 2.1.** Let $X$ be a smooth projective $n$ dimensional manifold. Suppose that $X$ is simply connected and that it is endowed with an elliptic fiber space structure $\phi: X \to B$. Suppose moreover that there exist a line bundle $L$ on $B$ such that $K_X \cong \phi^*L$. Then $X$ does contain a rational curve.

Let us first show how Theorem 2.1 implies Theorem 1.1.

**Proof of Theorem 1.1.** Let $f: X \to B$ as in Theorem 1.1 and consider the universal cover $\tilde{X} \to X$ of $X$. It is of course smooth and projective. Next, consider the Stein factorization of $f \circ \pi$

$$
\tilde{X} \xrightarrow{\phi} B' \xrightarrow{\nu} B,
$$

so that $\phi$ has connected fibers, and $B'$ is a normal projective variety. Since $K_{\tilde{X}} \cong \pi^*K_X$, we obtain that $K_{\tilde{X}} \cong (f \circ \pi)^*L = (\nu \circ \phi)^*L = \phi^*L'$, where $L'$ is the line bundle on $B'$ given by $\nu^*L$. Moreover, $\phi: \tilde{X} \to B'$ is an elliptic fiber space since by construction it is a fiber space whose general fiber has dimension one and, moreover, has trivial canonical bundle. Indeed, let $B^0 \subset B'$ the non-empty Zariski open set of regular point of $B'$ which are also regular values for $\phi$, so that $\phi^0 = \phi|_{\phi^{-1}(B^0)}$ is
a proper holomorphic submersion. Then over \( \tilde{X}' = \phi^{-1}(B') \), the relative tangent bundle sequence
\[
0 \to T_{\tilde{X}'/B'} \to T_{\tilde{X}'} \to (\phi^\ast T_{B'}) \to 0
\]
is a short exact sequence of vector bundles. Restricting to one fiber \( E \) and taking the determinant of the dual sequence gives a (non canonical) isomorphism
\[
K_E \simeq K_{\tilde{X}'}|_E = K_{\tilde{X}}|_E \simeq \phi^\ast L|_E \simeq O_E,
\]
and thus \( E \) is an elliptic curve. Therefore, Theorem 2.1 applies to \( \phi: \tilde{X} \to B' \) and we deduce that \( \tilde{X} \) contains a rational curve \( \tilde{R} \subset \tilde{X} \). But then \( R = \pi(\tilde{R}) \subset X \) is a rational curve in \( X \).

We now start the proof of Theorem 2.1. We first observe that thanks to the following result of Kawamata, which we state in a slightly simplified version, one can suppose that the fibers of \( \phi \) are all one dimensional.

**Theorem 2.2 (Kawamata [Kaw91]).** Let \( f: X \to Y \) be a surjective projective morphism, where \( X \) is smooth and \(-K_X \) is \( f \)-nef (that is, it intersects non negatively the curves which are contracted by \( f \)). Then any irreducible component of \( \text{Exc}(f) = \{ x \in X \mid \dim f^{-1}(f(x)) > \dim X - \dim Y \} \) is uniruled.

Notice that, if the exceptional set \( \text{Exc}(\phi) \) is not empty, then we obtain at once infinitely many rational curves.

Next, we look at the proper subvariety \( Z \subset B \) consisting of all the singular points of \( B \) and all the singular values of \( \phi \). We also call \( B^0 \) the Zariski open set complement of \( Z \) in \( B \), and \( X^0 = \phi^{-1}(B^0) \). Thus, the restriction \( \phi^0 = \phi|_{X^0}: X^0 \to B^0 \) is a proper surjective submersion.

**Lemma 2.3.** The subvariety \( Z \) has at least one irreducible component of codimension one in \( B \).

**Proof.** Suppose the contrary. Then by equidimensionality of the fibers, the complement of \( X^0 \) in \( X \) has codimension at least two. In particular, \( \pi_1(X^0) \simeq \pi_1(X) = \{1\} \). Since \( \phi^0 \) is a proper holomorphic surjective submersion with connected fibers, by Ehresmann’s theorem it is a differentiable fiber bundle and thus a Serre fibration. In particular, the long exact sequence for the homotopy groups of the fibration tells us that \( B^0 \) is simply connected. Now, by [Del68], in this situation the Leray spectral sequence
\[
E_2^{p,q} = H^p(B^0, R^q\phi^0_\ast \mathbb{Q}) \Rightarrow H^{p+q}(X^0, \mathbb{Q})
\]
degenerates at \( E_2 \). Since \( B^0 \) is simply connected, the locally constant sheaves \( R^q\phi^0_\ast \mathbb{Q} \) are indeed constant, isomorphic to \( H^q(E, \mathbb{Q}) \) where \( E \) is the (diffeomorphism class of) a fiber, i.e. a one dimensional complex torus. Now, take \( p + q = 1 \) to get that the graded module associated to some filtration of \( H^1(X^0, \mathbb{Q}) \simeq \{0\} \) has the non zero factor \( H^0(B^0, \mathbb{Q}^2) \simeq \mathbb{Q}^2 \). This is a contradiction. \( \square \)

**Remark 2.4.** Using a result of [KL09] it is possible to conclude as well in a stronger form, as follows. Consider the holomorphic function \( J: B^0 \to \mathbb{C} \) given by the \( j \)-invariant of the (elliptic) fibers. Since we suppose that \( B^0 \) has at least codimension two complement in \( B \) and \( B \) is normal, \( J \) extends to a holomorphic function \( B \to \mathbb{C} \), which must be constant. Thus, all fibers over \( B^0 \) are isomorphic and by the Grauert–Fischer theorem the family \( \phi^0: X^0 \to B^0 \) is locally holomorphically trivial. We are thus in position to apply [KL09, Lemma 17] which gives, since \( B^0 \) is moreover simply connected, that \( \phi^0: X^0 \to B^0 \) is globally holomorphically trivial. In particular \( X^0 \simeq E \times B^0 \) and thus \( X^0 \) cannot be simply connected.
Following [Ogu93], we shall reduce our situation to the surface case by picking a general curve in $B$ and use Kodaira’s canonical bundle formula to study singular fibers. So, let $H$ be a very ample line bundle over $B$, positive enough in order to ensure that $H^\otimes(n-2) \otimes \mathcal{O}_B(L)$ is generated by global sections. Observe that, since $\phi : X \to B$ is an algebraic fiber space, then $H^0(X, \phi^* H) \simeq H^0(B, \phi_* (\phi^* H)) \simeq H^0(B, H \otimes \phi_* \mathcal{O}_X) \simeq H^0(B, H)$. In particular, general elements in the linear system $|H|$ are also general members of $|\phi^* H|$. Now, take a curve $C \subset B$ which is a general complete intersection of divisors in $|H|$, and the surface $S \subset X$ cut out by the pull-back of such divisors to $X$. By Bertini’s theorem, $C$ is normal, hence smooth, and $S$ is smooth, too. Let us still call, by abuse of notation, $\phi : S \to C$ the restriction of $\phi$ to $S$. The surface $S$ is an elliptic surface, which we can suppose to be relatively minimal (otherwise we would have found a rational curve on $S$ and hence on $X$).

Since $C$ is general, and the singular locus $B_{\text{sing}}$ of $B$ is of codimension two, we can suppose that $C \cap B_{\text{sing}} = \emptyset$. Next, pick a divisorial irreducible component $Z_0$ of $Z$, which always exists thanks to the above lemma. Then $C$ must necessarily intersect $Z_0$, since $C \cdot Z_0 = H^{n-2} \cdot Z_0 > 0$. This means that $\phi : S \to C$ must always have at least one singular fiber. Our goal is now to show that such a fiber can never be a multiple fiber and that a singular (non multiple) fiber must necessarily contain an irreducible component which is rational.

Let us start with the following.

**Lemma 2.5.** The canonical bundle $K_S$ of $S$ is globally generated.

**Proof.** Let $H_1, \ldots, H_{n-2} \in |\phi^* H|$ be the smooth divisors in general position which cut out $S$. Then by iterating the adjunction formula we find

\begin{equation}
K_S \simeq (K_X \otimes \phi^* H^{\otimes(n-2)})|_S \simeq \phi^* (\mathcal{O}_B(L) \otimes H^{\otimes(n-2)})|_S.
\end{equation}

Thus, $K_S$, as a restriction of a pull-back of a globally generated line bundle, is globally generated itself. \qed

Now, recall the (weaker form of the) canonical bundle formula for relatively minimal elliptic fibrations such that its multiple fibers are $S_{c_1} = m_1 F_1, \ldots, S_{c_k} = m_k F_k$, which reads (see [BHPVdV04, Corollary V.12.3]):

\begin{equation}
K_S \simeq \phi^* G \otimes \mathcal{O}_S \left( \sum_{i=1}^k (m_i - 1) F_i \right),
\end{equation}

where $G$ is some line bundle living over $C$.

**Proposition 2.6.** The elliptic surface $S$ does not have any multiple fiber.

**Proof.** Indeed, on the one hand the restriction to $K_S$ to any subscheme of $S$ is globally generated, since $K_S$ is globally generated itself. On the other hand, the canonical bundle formula (3) together with [BHPVdV04, Lemma III.8.3] tell us that the restriction $\mathcal{O}_{F_i} ((m_i - 1) F_i)$ of $K_S$ to $F_i$ would have no sections, since $\mathcal{O}_{F_i} (F_i)$ is torsion of order $m_i$. \qed

To conclude the proof, we now have to examine singular but not multiple fibers, following Kodaira’s table [BHPVdV04, Section V.7]. So, let $F$ be such a fiber.

(i) If $F$ is irreducible, then it is necessarily rational with a node, or rational with a cusp. In both cases we find a (singular) rational curve on $S$, and hence on $X$.

(ii) If $F$ is reducible, then it is of the form $F = \sum m_i F_i$, with $F_i$ irreducible and reduced, and $\gcd\{m_i\} = 1$. Since $F^2 = \sum m_i m_j F_i \cdot F_j = 0$ and we have at least two components, it follows that there exists at least one irreducible
component, say $F_1$, with negative self-intersection. But since $K_S$ is a pull-back by $(2)$ and $F_1$ is contained in a fiber, then $K_S \cdot F_1 = 0$. It follows by adjunction that $F_1$ is a smooth rational $(-2)$-curve.

3. Proof of Theorem 1.6

In this section, after collecting all the essential ingredients, we prove Theorem 1.6. Two standard tools to produce rational curves on a smooth projective variety are the uniruledness of the exceptional loci (see Kawamata’s Theorem 2.2) and the logarithmic version of the Cone Theorem (see for instance [Mat02, Theorem 7-2-2]). We combine them in the following lemma (vaguely inspired by the key lemma in [Wil89]).

Lemma 3.1. Let $X$ be a Calabi–Yau manifold such that there exists $D$ a non-ample divisor on $X$ satisfying $D^n > 0$. Assume that $h^i(X, mD) = 0$ for $i > 1$ and for $m$ large enough. Then $X$ does contain a rational curve.

Proof. We first observe that we may assume $D$ non-nef: otherwise $D$ nef and $D^n > 0$ implies that $D$ is big (see for instance [Laz04, Theorem 2.2.16]). Then $D$ is semiample by the Basepoint-free Theorem (see for instance [KM98, Theorem 3.3]) and there exists a multiple of $D$ that defines a surjective generically 1-1 morphism $g: X \to Y$. Since $D$ is non-ample, the exceptional locus of $g$ is non-empty, so we can conclude thanks to Kawamata’s Theorem that $\text{Exc}(g)$ is uniruled.

Now we prove that it is possible to choose $m > 0$ such that $mD$ is effective. Indeed, by the Hirzebruch–Riemann–Roch Theorem, we have
\[
\chi(O_X(mD)) = \text{deg } (\text{ch}(mD) \cdot \text{td}(T_X))_n.
\]
Since $\text{ch}(mD) = \sum_{k=0}^{\infty} \frac{m^k D^k}{k!}$ because $mD$ is a line bundle, we obtain for $m$ sufficiently large
\[
\chi(O_X(mD)) \sim \frac{m^n D^n}{n!} > 0.
\]
By assumption,
\[
\chi(O_X(mD)) = h^0(O_X(mD)) - h^1(O_X(mD)) \leq h^0(O_X(mD)).
\]
Then $h^0(O_X(mD)) > 0$ and $mD$ is effective. Since $mD$ is not nef, then we can conclude thanks to the logarithmic version of the Cone Theorem. □

We are going to apply Lemma 3.1 to $D_{a,b} := aH - bF$, where $H$ is an ample divisor on $X$ and $F$ is the generic fiber of a fibration of $X$ onto a curve. The point is to obtain an asymptotic vanishing of the higher cohomology of $D$ for some $a$, $b$ fixed. Indeed, we are able to prove a slightly stronger statement, where we obtain a uniform vanishing for every $b$.

Lemma 3.2. Let $X$ be an $n$-dimensional projective manifold that admits a fibration onto a curve, let $H$ be an ample divisor on $X$ and let $F$ be the generic fiber of the fibration. Let $m_0 \in \mathbb{N}$ such that
\begin{itemize}
  \item $h^i(X, mH) = 0$, $i > 0$,
  \item $h^i(F, mH) = 0$, $i > 0$,
\end{itemize}
for all $m \geq m_0$. Then $h^i(X, mH - kF) = 0$, $i > 1$, for all $m \geq m_0$ and for all $k \in \mathbb{N}$.

In particular, for any positive integer $a$, $b$, the divisor $D_{a,b} = aH - bF$ satisfies the assumption of Lemma 3.1.

It is of course always possible to choose $m_0$ satisfying the hypothesis of Lemma 3.2 thanks to Serre vanishing.
Proof. Given the standard exact sequence
\[ 0 \to \mathcal{O}_X(mH - kF) \to \mathcal{O}_X(mH) \to \mathcal{O}_{kF}(mH) \to 0, \]
it is enough to show that for every \( i > 1 \) we have
\[ h^{i-1}(kF, \mathcal{O}_{kF}(mH)) = 0. \]
Indeed, let \( D \) be an effective Cartier divisor on \( X \) and let \( \mathcal{I} \) be the ideal sheaf of \( D \) in \( X \), i.e. \( \mathcal{I} = \mathcal{O}_X(-D) \). Then for any \( k \in \mathbb{N} \), we have \( \mathcal{O}_{(k+1)D} = \mathcal{O}_X/\mathcal{I}^{k+1} \) and \( \mathcal{O}_{kD} = \mathcal{O}_X/\mathcal{I}^k \). Let \( K \) be defined by the short exact sequence
\[ 0 \to K \to \mathcal{O}_{(k+1)D} \to \mathcal{O}_{kD} \to 0. \]
There is a natural sheaf isomorphism
\[ K = \mathcal{I}^k/\mathcal{I}^{k+1} = \mathcal{I}^k \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I} = \mathcal{O}_X(-kD) \otimes_{\mathcal{O}_X} \mathcal{O}_D = \mathcal{O}_D(-kD). \]
Hence we obtain the exact sequence
\[ 0 \to \mathcal{O}_D(-kD) \to \mathcal{O}_{(k+1)D} \to \mathcal{O}_{kD} \to 0 \]
and tensoring by \( \mathcal{O}_X(mH) \) we get
\[ 0 \to \mathcal{O}_D(mH - kD) \to \mathcal{O}_{(k+1)D}(mH) \to \mathcal{O}_{kD}(mH) \to 0. \]
Now, if we set \( D = F \), since \( F \) is a fiber we have
\[ \mathcal{O}_F(mH - kF) = \mathcal{O}_F(mH), \]
so we can easily conclude by induction on \( k \).

Remark 3.3. The required weaker vanishing condition of Lemma 3.1 for the divisor \( D_{a,b} \) also follows from the next theorem, that can be seen as a consequence of the classical Andreotti–Grauert finiteness theorem.

Theorem 3.4 (see [Dem12, Chapter VII, Theorem 5.1]). Let \( X \) be an \( n \)-dimensional compact complex manifold, \( s \) a positive integer and \( F \) a hermitian line bundle such that its Chern curvature \( i\Theta(F) \) has at least \( n - s + 1 \) positive eigenvalues at every point of \( X \). Then there exists a positive integer \( l_0 \) such that
\[ H^q(X, F^\otimes l) = 0, \text{ for all } l \geq l_0 \text{ and } q \geq s. \]
Indeed, let \( \pi : X \to C \) be a fibration onto a curve and let \( F \) be its generic fiber. Then as a divisor \( F \) is the pull-back of a point \( p \in C \). Call \( A = \mathcal{O}_C(p) \) the corresponding ample line bundle, so that \( \mathcal{O}_X(F) \simeq \pi^*A \). Put a metric \( h_A \) on \( A \) whose Chern curvature \( i\Theta(A) \) is positive. Now take any ample divisor \( H \) on \( X \) and take a positively curved metric \( h_H \) on \( \mathcal{O}_X(H) \). We claim that for any integers \( a, b > 0 \), the line bundle (associated to the divisor) \( D_{a,b} = aH - bF \) has a metric whose Chern curvature has at least \( n - 1 \) positive eigenvalues at every point of \( X \).

Indeed, the metric \( h_{a,b} = h_H^a \otimes \pi^*h_A^{-b} \) on \( \mathcal{O}_X(D_{a,b}) \) is such that
\[ \Theta(\mathcal{O}_X(D_{a,b})) = m\Theta(\mathcal{O}_X(H)) - k\pi^*\Theta(A). \]
When one evaluates \( i\Theta(\mathcal{O}_X(D_{a,b})) \) (seen as a hermitian form) on a non zero tangent vector \( v \) on \( X \) one gets then
\[ i\Theta(\mathcal{O}_X(D_{a,b}))(v) = m i\Theta(\mathcal{O}_X(H))(v) - k i\Theta(A)(d\pi(v)). \]
So, if \( d\pi(v) = 0 \), then \( i\Theta(\mathcal{O}_X(D_{a,b}))(v) > 0 \). It is then enough to observe that at every point \( x \in X \), the kernel of \( d\pi \) is at least of dimension \( n - 1 \).

Thus, we can apply Theorem 3.4 to deduce that
\[ H^q(X, \mathcal{O}_X(\ell D_{a,b})) = 0, \text{ for all } \ell \geq \ell_0 \text{ and } q \geq 2. \]
We also need the following remark.
Remark 3.5. We first recall that $c_2(X)$ is non-zero, otherwise by the Beauville–Bogomolov decomposition theorem $X$ would be a finite unramified quotient of a torus and its fundamental group would contain a free abelian group of rank $2n$, contradicting our assumption of simply connectedness. Given the fibration $f : X \to C$, thanks to the relative tangent sequence (1) we get

$$0 \to T_F \to T_{X|\nu} \to \mathcal{O}_F \to 0$$

where $F$ is the generic fiber of the fibration. Hence we deduce

$$c_2(X) \cdot F = c_2(T_{X|\nu}) = c_2(T_F) = c_2(F).$$

Since $F$ is an abelian variety, we have $c_2(X) \cdot F = c_2(F) = 0$ as a cycle.

Finally we prove Theorem 1.6.

Let $F$ be the generic fiber of the fibration and let $H$ be an ample divisor on $X$. We consider the affine line of divisors (with rational slope) $N_t = H - tF$ for $t \in \mathbb{Q}$. If we let

$$t_0 = \frac{H^n}{nH^{n-1} \cdot F} \in \mathbb{Q},$$

then we have $(N_t)^n > 0$ for each $t < t_0$ and $(N_{t_0})^n = 0$. Now there are two possible cases:

(I) $N_t$ is nef for each $t < t_0$.

(II) There exists a $t < t_0$ such that $N_t$ is non-nef.

Let us focus on case (I) first. Since being nef is a closed condition, $N_{t_0}$ is also nef. Next, it is easy to verify that $(N_{t_0})^{n-1} \cdot H = (1 - \frac{a}{n})H^n > 0$. Finally, according to Remark 3.5, $c_2(X) \cdot F = 0$, hence we can conclude that $c_2(X) \cdot (N_{t_0})^{n-2} = c_2(X) \cdot H^{n-2} > 0$ thanks to [Miy87, Theorem 1.1], because $H$ is ample and $c_2(X) \neq 0$, see Remark 3.5. Hence from [Kol15, Corollary 11], it follows that $N_{t_0}$ is semiample and since we have $\nu(N_{t_0}) = n - 1$ we can conclude thanks to Corollary 1.2.

Let us consider now case (II). Let $a$ and $b$ positive natural numbers such that $\bar{t} = \frac{a}{n}$. Then we have that $aN_{\bar{t}} = aH - bF = D_{a,b}$ is as in Lemma 3.2 and since $(aN_{\bar{t}})^n = a^n(N_{\bar{t}})^n > 0$ we can conclude thanks to Lemma 3.1.

4. A sufficient numerical criterion by Kollár

To finish with, since we used it in a prominent way all along the paper, for the sake of completeness we give an overview of Kollár’s proof of the abundance-type result for nef line bundles of numerical dimension $n - 1$ [Kol15, Corollary 11], in the slightly less general setting we need here (cf. Corollary 1.4). So, let $X$ be a smooth $n$-dimensional projective manifold with trivial canonical bundle $K_X \simeq \mathcal{O}_X$, and suppose that $L \to X$ is a nef holomorphic line bundle of numerical dimension $\nu(L) = n - 1$. One wants to show that $L$ is semi-ample. For this purpose, it is sufficient to show that the Kodaira–Iitaka dimension $\kappa(L)$ of $L$, which is in general less than or equal to its numerical dimension, satisfies indeed $\kappa(L) = \nu(L)$. For, in this case one can then apply the following theorem, which we state in an oversimplified version.

**Theorem 4.1** (see [Kaw85,KMM87] and also [Fuj11, Theorem 1.1]). Let $X$ be a smooth projective manifold with trivial canonical bundle, and $L \to X$ a nef line bundle. Then $L$ is semi-ample if and only if its Kodaira–Iitaka dimension equals its numerical dimension.

Observe that, in the theorem above, the “only if” part is straightforward, and the importance of this statement really relies upon the “if” part. Now, the asymptotic
Riemann–Roch formula for $L$ reads, under our assumptions:

$$
\chi(X, L^\otimes m) = \frac{L^{n-2} \cdot c_2(X)}{12(n-2)!} m^{n-2} + O(m^{n-3}).
$$

On the other hand, the Kawamata–Viehweg vanishing theorem [Dem12, Special case 6.13] gives, since $\nu(L) = n-1$ and $K_X \simeq O_X$, that $H^q(X, L^\otimes m) = \{0\}$ for all $m \geq 1$ and all $q > 1$. Therefore, we obtain

$$
h^0(X, L^\otimes m) \geq h^0(X, L^\otimes m) - h^1(X, L^\otimes m)
$$

$$
\quad = \chi(X, L^\otimes m) = \frac{L^{n-2} \cdot c_2(X)}{12(n-2)!} m^{n-2} + O(m^{n-3}).
$$

Thus, if $L^{n-2} \cdot c_2(X) > 0$, we get that $\kappa(L) \geq n-2$. By [Miy87], it always holds true that $L^{n-2} \cdot c_2(X) \geq 0$, and therefore if $L^{n-2} \cdot c_2(X) \neq 0$, then indeed we have that $\kappa(L) \geq n-2$.

We now proceed by contradiction and suppose that $\kappa(L) = n-2 < n-1 = \nu(L)$. Let’s consider the Iitaka fibration (cf. [La04, Theorem 2.1.33]) associated to $L$. Namely, there exists for any large $m$ divisible enough a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\phi_m} & X_m \\
\downarrow{\phi} & & \downarrow{\phi} \\
Y & \xrightarrow{\nu_m} & Y_m
\end{array}
$$

of rational maps and morphisms, where the horizontal maps are birational, $u_\infty$ is a morphism and $\phi_\infty$ determines an algebraic fiber space structure on $X_\infty$. Here, $\phi_m$ is the rational map associated to the linear system $|L^\otimes m|$. One has that $\dim Y_\infty = \kappa(X, L) = n-2$ and moreover, if we set $L_\infty = u_\infty^* L$ and take $S \subset X_\infty$ to be a very general fiber of $\phi_\infty$ (so that $S$ is in particular is a smooth surface), then $\kappa(S, L_\infty|_S) = 0$. Observe that one can suppose $X_\infty$ to be smooth and hence $Y_\infty$ normal.

Next, remark that, since $(u_\infty)_* O_{X_\infty} \simeq O_X$, one has that $\kappa(L_\infty) = \kappa(L) = n-2$. Moreover, since $u_\infty$ is a proper surjective map and $X_\infty$ is projective, $L_\infty$ is nef and $\nu(L_\infty) = \nu(L) = n-1$. Another remark is that $K_{X_\infty}$ is linearly equivalent to the effective divisor given by $E := \{\det du_\infty = 0\}$.

**Lemma 4.2.** The surface $S$ has Kodaira dimension zero.

**Proof.** The short exact sequence induced by the differential of $\phi_\infty$ gives $K_S \simeq K_{X_\infty}|_S$. Since $K_{X_\infty} \sim E \geq 0$ is effective and $S$ is not contained in $E$, it follows that $\kappa(S) \geq 0$. We now claim that $\kappa(S, K_S \otimes L_\infty|_S) = 0$. This is enough to conclude. Indeed, since (some power of) $L_\infty|_S$ is effective, then

$$
\kappa(S) = \kappa(S, K_S) \leq \kappa(S, K_S \otimes L_\infty|_S) = 0.
$$

It follows that $\kappa(S) = 0$.

We now come back to the claim. Observe that, by the projection formula, $(u_\infty)_* O_{X_\infty}(K_{X_\infty}^\otimes \otimes L_\infty^\otimes) \simeq O_X(L^\otimes m)$, since we have that $O_X(K_X) \simeq O_X$ and thus $(u_\infty)_* O_{X_\infty}(K_{X_\infty}^\otimes \otimes L_\infty^\otimes) \simeq O_X$. This implies that the map $\psi_m$ induced by the linear system $|K_{X_\infty}^\otimes \otimes L_\infty^\otimes|$ is nothing but the composite $\phi_m \circ u_\infty$. To conclude, it suffices to follow [verbatim [La04, Proof of Theorem 2.1.33, Step 3]], replacing $L$ by $K_{X_\infty} \otimes L_\infty$ (observe that, by [La04, Remark 2.1.34], a posteriori we get that $\phi_\infty$ is then also the Iitaka fibration associated to $K_{X_\infty} \otimes L_\infty$). 

Now, take $D$ to be an effective divisor in the linear system $|L^\otimes m|$, $m \gg 1$. Then either $D$ is $\phi_\infty$-horizontal, i.e. $\phi_\infty(D) = Y_\infty$, or it is $\phi_\infty$-vertical, i.e. $\phi_\infty(D) \subset Y_\infty$. In the first case, we shall obtain a contradiction by showing that this would imply
\( \kappa(S, L_\infty|s) \geq 1; \) in the second case, we shall get that \( \nu(L_\infty) < n - 1, \) which is again a contradiction.

The horizontal case. In this case, we are in the situation where the restriction \( D_S \) of \( D \) to \( S \) is a nef, effective, non zero divisor. In particular \( D_S \) is not numerically trivial. Let \( S_{\text{min}} \) be the minimal model of \( S \), obtained by contracting all \((-1)\)-curves on \( S \). Call \( \mu: S \to S_{\text{min}} \) the corresponding morphism. Observe that \( D_S \) cannot be entirely contracted by \( \mu \), since in this case one would have \( D_S^2 < 0 \), contradicting the nefness of \( D_S \). Indeed, take any ample Cartier divisor \( H \) on \( S_{\text{min}} \). Its pull-back \( \mu^* H \) to \( S \) is big and nef so that \( \mu^* H^2 > 0 \); on the other hand, if \( D_S \) were entirely contracted by \( \mu \), we would have \( \mu^* H \cdot D_S = 0 \) since \( \nu(H) \) would be trivial. Since \( D_S \) is not numerically trivial then the Hodge Index Theorem would give, as claimed, \( D_S^2 < 0 \). Thus, the proper push-forward \( \mu_* D_S =; D_{S_{\text{min}}} \) is again a nef effective non zero divisor on \( S_{\text{min}} \). Observe that \( \kappa(D_S) \geq \kappa(D_{S_{\text{min}}}) \).

Now, since \( \kappa(S_{\text{min}}) = \kappa(S) \), \( \kappa(S) = 0 \) by Lemma 4.2 and \( K_{S_{\text{min}}} \) is nef, then, by abundance for surfaces we see that \( K_{S_{\text{min}}} \) is semi-ample, and thus a torsion line bundle. In particular, \( \kappa(D_{S_{\text{min}}}) = \kappa(K_{S_{\text{min}}} + D_{S_{\text{min}}}) \). Now, there exists a (small) positive rational number \( \varepsilon \) such that \( (S_{\text{min}}, \varepsilon D_{S_{\text{min}}}) \) is a klt pair, and moreover \( K_{S_{\text{min}}} + \varepsilon D_{S_{\text{min}}} \) is nef. Thus, by log-abundance for surfaces \( K_{S_{\text{min}}} + \varepsilon D_{S_{\text{min}}} \) is semi-ample and it is not numerically trivial by construction. This implies that \( \kappa(K_{S_{\text{min}}} + \varepsilon D_{S_{\text{min}}}) \geq 1 \). But

\[
\kappa(D_S) \geq \kappa(D_{S_{\text{min}}}) = \kappa(\varepsilon D_{S_{\text{min}}}) = \kappa(K_{S_{\text{min}}} + \varepsilon D_{S_{\text{min}}}) \geq 1,
\]

contradiction.

The vertical case. We finally treat the case where \( \phi_\infty(D) \subset Y_\infty \). The hypothesis implies that the coherent sheaf \( (\phi_\infty)_* O_Y(-D) \) on \( Y_\infty \) is non zero. Thus, by the Cartan–Serre–Grothendieck Theorem, there exists an ample line bundle \( A \to Y_\infty \) such that \( H^0(Y_\infty, A \otimes (\phi_\infty)_* O_Y(-D)) \neq \{0\} \). By the projection formula we thus get \( H^0(X_\infty, \phi_\infty^* A \otimes O_X(-D)) \neq \{0\} \). In particular, \( \phi_\infty^* A \) is linearly equivalent to \( D + F \), where \( F \) is an effective divisor on \( X_\infty \).

Therefore, for all integer \( r \geq 1 \), we have

\[
\phi_\infty^* A^r - D^r = (\phi_\infty^* A - D) \cdot \left( \sum_{i=0}^{r-1} \phi_\infty^* A^i \cdot D^{r-i} \right)
\]

\[
= F \cdot \left( \sum_{i=0}^{r-1} \phi_\infty^* A^i \cdot D^{r-i} \right) \geq 0,
\]

since both \( \phi_\infty^* A \) and \( D \) are nef and \( F \) is effective. In particular,

\[
0 \leq D^{n-1} = \phi_\infty^* A^{n-1} = 0,
\]

since \( \phi_\infty^* A \) comes from an \((n-2)\)-dimensional variety. Thus, we have that \( \nu(L_\infty) = \nu(D) < n - 1 \), which gives the desired contradiction.

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APPENDIX
RATIONAL CURVES ON CALABI–YAU THREEFOLDS

ROBERTO SVALDI

Even though it is widely believed that every Calabi–Yau threefold should contain rational curves, when $h^{i,0} = 0$, $i = 1, 2$ and the fundamental group is trivial, nonetheless this appears to be a very difficult problem.

Existence of rational curves is fully established only in dimension 2 thanks to Bogomolov and Mumford, while only partial results are known in dimension 3, as we discuss below.

The aim of this appendix is to explain what is known so far in dimension 3 and to illustrate how the problem of the existence of rational curves on Calabi–Yau threefolds is strictly intertwined to several other problems and conjectures regarding different aspects of the geometry of these varieties.

In his study of the birational structure of Calabi–Yau threefolds, Oguiso proposed the following weakened version of the classical question on the existence of rational curves.

**Conjecture 0.1.** [Ogu93, Conjecture, p. 456] Let $X$ be a Calabi–Yau threefold. Assume that there exists on $X$ a non-trivial Cartier divisor $D$ whose first Chern class is contained in the boundary of the nef cone of $X$.
Then $X$ contains a rational curve.

The existence of a Cartier divisor $D$ as in the statement of the conjecture is a rather strong assumption. First of all, it requires that $h^2(X) > 1$. It also implies that the boundary of the nef cone contains rational points other than the origin.

Let us also point out that the lack of rational points in the boundary of the nef cone is an even more typical situation if we do not assume that $\pi_1(X) = \{1\}$ or $h^i(X, \mathcal{O}_X) = 0$, $0 < i < \dim X$, e.g. in the case of abelian varieties or as in the examples of hyperkähler manifolds constructed by Oguiso in [Ogu14]. Nonetheless, abelian varieties do not contain rational curves, while Verbitski proved in [Ver15] that no hyperkähler manifold is hyperbolic.

From the point of view of birational geometry, Conjecture 0.1 is strictly intertwined with the following folklore conjecture, which generalizes the well-known Abundance Conjecture in the context of Calabi–Yau varieties.

**Conjecture 0.2** (Generalized Abundance Conjecture). [Kol15, Conjecture 51]
Let $X$ be a Calabi–Yau manifold and let $L$ be a nef Cartier divisor on $X$.
Then $L$ is semi-ample, i.e. $|nL|$ is base point free for $n \gg 0$.

Let us remind the reader that when $X$ is three dimensional and there exists $m \in \mathbb{N}$ such that $|mD|$ is non-empty then the Generalized Abundance Conjecture is known to hold true.

**Theorem 0.3.** [KMM94, Corollary, p. 100] Let $X$ be a Calabi–Yau threefold and let $D$ be a nef divisor which is effective, i.e. $H^0(X, \mathcal{O}_X(D)) \neq 0$.
Then $D$ is semi-ample.
In higher dimension, when $D$ is not a big divisor, the only known (partial) results towards a solution of Conjecture \ref{conj:main} are in dimension 4, \cite{fuk}. On a Calabi–Yau threefold $X$, if we assume that Conjecture \ref{conj:main} holds then also Conjecture \ref{conj:main2} holds. In fact, let $\phi : X \to Y$ be the morphism induced by some sufficiently high multiple of $D$. That is either a birational morphism, or a $K3$ abelian surface fibration or an elliptic fibration.

In the first case, the exceptional locus of the birational modification is known to be uniruled. For a $K3$ fibration, it is well known that the general fiber will contain rational curves. In the other cases, the results in \cite[Appendix]{ou93} imply the existence of rational curves on $X$ at once.

In a fundamental series of works, e.g. \cite{wil92}, \cite{wil89}, \cite{wil94}, Wilson studied the structure of the nef cone of a Calabi–Yau threefold. In particular, Wilson showed that whenever the dimension of the second cohomology group of $X$ is sufficiently big then there exists a birational morphism $f : X \to Y$. As explained above, the exceptional locus of $f$ is uniruled.

If there are no rational curves on $X$, then any effective Cartier divisor $E$ on $X$ is automatically nef and hence semi-ample. In fact, for $\varepsilon$ sufficiently small the pair $(X, \varepsilon E)$ is klt. If $E$ is not nef then by running the $K_X + \varepsilon E$-MMP, again, we find a birational morphism whose exceptional locus is uniruled. Hence, the nef cone, $\text{Nef}(X)$, and the pseudoeffective cone, $\overline{\text{Eff}}(X)$, coincide.

Therefore, it suffices to focus on the case where $D^3 = 0$, i.e. when the cohomology class associated to $D$ lies on the cubic hypersurface $$W := \{ T \in H^2(X, \mathbb{R}) \mid T^3 = 0 \} \subset H^2(X, \mathbb{R}).$$

This is the starting point of the proof of the following result, due to Diverio and Ferretti, which partially solves the conjecture of Oguiso.

**Theorem 0.4.** \cite[Theorem 1.2]{df14} Let $X$ be a Calabi–Yau threefold. Assume that there exists on $X$ a non-trivial Cartier divisor $D$ contained in the boundary of the nef cone of $X$.

Then $X$ contains a rational curve, provided $h^2(X) \geq 5$.

A proof of the above theorem can be carried out via a careful analysis of the possible configurations of the pair given by hypersurface $\widetilde{W} \subset \mathbb{P}(H^2(X, \mathbb{R}))$ – the projectivization $W$ – together with the point represented by the first Chern class of $D$ and the hyperplane $c_2(X)^\perp := \{ H \in \mathbb{P}(H^2(X, \mathbb{R}) \mid c_2(X) \cdot H = 0 \}$. In fact, the second Chern class of $X$, $c_2(X)$ has non-negative intersection with nef divisors, by the Miyaoka–Yau inequality. Using Kawamata–Viehweg vanishing and the Riemann–Roch formula, it is easy to see that when $c_2(X) \cdot D > 0$, then some multiple of $D$ is effective and by Theorem \ref{thm:main} it is also semi-ample. Hence it is natural to assume that $D \in c_2(X)^\perp$.

If one further assumes that $D^2 = 0$, then $D$ represents a singular point of $\widetilde{W}$. Taking a general line through $D$ in $\mathbb{P}(H^2(X, \mathbb{R}))$, it is not hard to see that, when $h^2(X) > 5$, the boundary of the nef cone must contain another integral divisor which is semi-ample. Thus, there exists a rational curve on $X$.

The final case to consider then is when $D^3 = 0 = c_2(X) \cdot D$, $D^2 \neq 0$. Under these assumptions, the Riemann–Roch formula implies that $\chi(X, \mathcal{O}_X(mD)) = 0$, $\forall m \in \mathbb{Z}$ and it is not clear in general how to produce sections in any of the linear systems $|mD|$. In \cite{df14}, the authors carefully analyze this framework using techniques partly mutated by the work of Wilson to conclude the proof of 0.4.

In his analysis of the birational geometry of Calabi–Yau threefolds, Wilson established the following result, which describes what would happen in case a nef divisor $D$ had numerical dimension $2$ and was not semi-ample. Of course, should
the Generalized Abundance Conjecture hold true, such a situation would never be attained.

**Theorem 0.5.** ([Wil94], [Wil98]) Let $X$ be a Calabi–Yau threefold and $D$ a nef Cartier divisor on $X$ with $D^3 = 0 = D \cdot c_2(X)$ and $D^2 \neq 0$. Let $E_1, \ldots, E_r$ denote the (necessarily finitely many) surfaces $E$ on $X$ with $D|_E \equiv 0$. Then $D$ is semi-ample except possibly for the case when the topological Euler characteristic $e(X) = 2r$ and each $E_i$ is rational.

Using Theorem 0.5, it is immediate to see that, at least when $e(X) \neq 0$, then $X$ will contain rational curves. In fact, when $e(X) < 0$ Theorem 0.5 automatically implies the semi-ampleness of $D$, while if $D$ is not semi-ample and $e(X) > 0$, then the surfaces $E_i, 1 \leq i \leq e(X)/2$, are rational and in particular contain infinitely many rational curves. Thus, we can extend Theorem 0.4 to enlarge the class of varieties for which Conjecture 0.1 holds.

**Theorem 0.6.** Let $X$ be a Calabi–Yau threefold. If there exists on $X$ a non-zero nef non-ample Cartier divisor, then $X$ contains a rational curve, provided the topological Euler characteristic $e(X) \neq 0$. When $e(X) = 0$, then rational curves exist provided again that $h^2(X) \geq 5$.

**Remark 0.7.** By the Gauss-Bonnet formula, the hypothesis on the vanishing of the topological Euler characteristic, $e(X) = 0$, is equivalent to the vanishing of the top Chern class of $X$, $c_3(X) = 0$.

Finally, let us explain how Conjecture 0.1 is also strictly related to another important conjecture regarding Calabi–Yau threefolds, the so-called Cone Conjecture.

**Conjecture 0.8** (Kawamata-Morrison Cone Conjecture). Let $X$ be a Calabi–Yau threefold and let $\text{Aut}(X)$ be the group of automorphisms $X$.

Then there exists a rational polyhedral cone $\Pi$ contained in the cone spanned by Chern classes of nef effective Cartier divisors, $\text{Nef}^e := \text{Nef}(X) \cap \text{Eff}(X)$, which is a fundamental domain for the action of $\text{Aut}(X)$ on $\text{Nef}^e$.

In particular, the above conjecture predicts that if $\text{Aut}(X)$ is finite then the cone on which that group acts should be rational polyhedral. It follows that the divisors contained in the facets of the nef cone of $X$ not lying on $c_2(X)^\perp$, i.e. all but at most one facet of $\text{Nef}(X)$, will contain semi-ample divisors on $X$. Thus, $X$ will contain rational curves.

If Conjecture 0.8 holds then also also Conjecture 0.1 holds in some of the cases that are not covered by Theorems 0.4 or 0.6.

**Proposition 0.9.** Let $X$ be a Calabi–Yau threefold. Assume that Conjecture 0.8 holds true. Then, Conjecture 0.1 holds true in the following cases:

- $\rho(X) = 2$;
- $\rho(X) = 3$ and $W$ is not the union of a line and a conic.
- $\rho(X) = 4$ and $W$ is not the union of a plane and a quadric or $W$ is not a singular cubic.

**Proof.** When $h^2(X) = 2$, $W$ is decomposed into three real lines, $l_1, l_2, l_3$ and at most two of those, say $l_1, l_2$ are defined over the reals and conjugated under the action of $\text{Gal}(\mathbb{R}/\mathbb{Q})$. As $\text{Nef}(X) = \text{Eff}(X)$ and by the assumption of Conjecture 0.1 the extremal ray $\mathbb{R}_{>0}E$ of $\text{Nef}(X)$ other than $\mathbb{R}_{>0}D$ must lie on either one of $l_1$ or $l_2$. This is a simple consequence of Campana–Peternell extension of Kleiman’s ampleness criterion. The matrix corresponding to the action of an element of $\text{Aut}(X)$ on $H^2(X, \mathbb{R})$ is actually defined over the integral underlying structure. The extremal ray $\mathbb{R}_{>0}D$ spans an eigenspace of eigenvalue 1 since it is the kernel of
the cap product with $c_2(X)$. This gives a contradiction as it implies that the class $E$ should be defined over $\mathbb{Q}$. A similar strategy can be used to prove more generally that if Conjecture 0.8 holds for $X$ a strict Calabi–Yau 3-fold of Picard number 2 then a rational curve always exists on $X$ — without assuming the existence of the nef divisor $D$, see [Ogu14, Corollary 1.6].

When $h^2(X) = 3$ and $W$ is the union of three distinct hyperplanes $H_1, H_2, H_3$, then either all of the hyperplanes are defined over $\mathbb{Q}$ and the boundary of $\text{Nef}(X)$ contains a nef divisor $E$ with $c_2(X) \cdot E > 0$ or, up to relabeling, $H_1$ is defined over $\mathbb{Q}$ and $H_2, H_3$ are defined over $\mathbb{R}$ and conjugated under the action of $\text{Gal}(\mathbb{R}/\mathbb{Q})$. In the latter case, it suffices to notice that the line $H_2 \cap H_3$ is defined over the rationals and it will be spanned by a nef divisor $F$ such that $c_2(X) \cdot F > 0$. If instead $\tilde{W}$ is geometrically irreducible, then it follows immediately that the action of $\text{Aut}(X)$ on the second cohomology embeds in the automorphisms of the pair $(\tilde{W}, c_2(X)_{\tilde{W}})$, that are finite. As we noted above this implies that the nef cone is polyhedral and in particular there are some nef non-ample divisors on $X$ that are semi-ample.

When $h^2(X) = 4$, we only need to check the case where $W$ is defined over $\mathbb{Q}$ and $\mathbb{P}(W)$ is a smooth cubic surface, since 3 hyperplanes can not bound a strongly rational cone in a 4 dimensional vector space. Under these assumptions, it suffices to notice that the automorphisms of a cubic surface are finite, by [MM63], and then conclude as above.

Even though we are not able to show that if Conjecture 0.8 holds then also Conjecture 0.1 holds, it is actually still possible to show that $X$ is actually not Kobayashi hyperbolic at least when its Picard number is greater than one.

**Proposition 0.10.** [Div12] *Let us assume that Conjecture 0.8 holds. Then the Kobayashi Conjecture is true in dimension three, except possibly if there exists a Calabi–Yau threefold of Picard number one which is hyperbolic.*

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