Complete nonmeasurability in regular families

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Abstract. We show that for a σ-ideal $\mathcal{I}$ with a Borel base of subsets of an uncountable Polish space, if $\mathcal{A}$ is (in several senses) a "regular" family of subsets from $\mathcal{I}$ then there is a subfamily of $\mathcal{A}$ whose union is completely nonmeasurable i.e. its intersection with every Borel set not in $\mathcal{I}$ does not belong to the smallest σ-algebra containing all Borel sets and $\mathcal{I}$. Our results generalize results from [3] and [4].

1. Notation and Terminology

Throughout this paper, $X, Y$ will denote uncountable Polish spaces and $\mathcal{B}(X)$ the Borel σ-algebra of $X$. We say that the ideal $\mathcal{I}$ on $X$ has Borel base if every element $A \in \mathcal{I}$ is contained in a Borel set in $\mathcal{I}$. (It is assumed that an ideal is always proper.) The ideal consisting of all countable subsets of $X$ will be denoted by $[X]^{\leq \omega}$ and the ideal of all meager subsets of $X$ will be denoted by $\mathbb{K}$. Let $\mu$ be a continous probability measure on $X$. The ideal consisting of all $\mu$-null sets will be denoted by $L_\mu$. By the following well known result, $L_\mu$ can be identified with the σ-ideal of Lebesgue null sets.

Theorem 1.1 ([6], Theorem 3.4.23). If $\mu$ is a continous probability on $\mathcal{B}(X)$, then there is a Borel isomorphism $h : X \to [0,1]$ such that for every Borel subset $B$ of $[0,1]$, $\lambda(B) = \mu(h^{-1}(B))$, where $\lambda$ is a Lebesgue measure.

Definition 1.1. We say that $(Z, \mathcal{I})$ is Polish ideal space if $Z$ is Polish uncountable space and $\mathcal{I}$ is a σ-ideal on $Z$ having Borel base and containing all singletons. In this case, we set

$$\mathcal{B}_+(Z) = \mathcal{B}(Z) \setminus \mathcal{I}.$$ 

A subset of $Z$ not in $\mathcal{I}$ will be called a $\mathcal{I}$-positive set; sets in $\mathcal{I}$ will also be called $\mathcal{I}$-null. Also, the σ-algebra generated by $\mathcal{B}(Z) \cup \mathcal{I}$ will be denoted by $\overline{\mathcal{B}}(Z)$, called the $\mathcal{I}$-completion of $\mathcal{B}(Z)$.

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It is easy to check that \( A \in \overline{B}(Z) \) if and only if there is an \( I \in \mathcal{I} \) such that \( A \oplus I \) (the symmetric difference) is Borel.

**Example 1.1.** Let \( \mu \) be a continuous probability measure on \( X \). Then \((X, [X]^\omega), (X, \mathcal{K}), (X, \mathcal{L}_\mu)\) are Polish ideal spaces.

**Definition 1.2.** A Polish ideal group is 3-tuple \((G, \mathcal{I}, +)\) where \((G, \mathcal{I})\) is Polish ideal space and \((G, +)\) is an abelian topological group with respect to the Polish topology of \( G \).

**Definition 1.3.** Let \((X, \mathcal{I})\) be a Polish ideal space and \( A \subseteq X \). We say that \( A \) is \( \mathcal{I} \)-nonmeasurable, if \( A \notin \overline{B}(X) \). Further, we say that \( A \) is completely \( \mathcal{I} \)-nonmeasurable if \( \forall B \in \mathcal{B}_+(X) \ A \cap B \neq \emptyset \wedge A^c \cap B \neq \emptyset \).

Clearly every completely \( \mathcal{I} \)-nonmeasurable set is \( \mathcal{I} \)-nonmeasurable. In the literature, completely \([X]^\omega \)-nonmeasurable sets are called Bernstein sets. Also, note that \( A \) is completely \( \mathcal{L}_\mu \)-nonmeasurable if and only if the inner measure of \( A \) is zero and the outer measure one.

For any set \( E \), \(|E|\) will denote the cardinality of \( E \).

Let \((X, \mathcal{I})\) be a Polish ideal space and \( F \subseteq \mathcal{I} \). We set

\[
\text{add}(\mathcal{I}) = \min\{|A| : A \subseteq \mathcal{I} \wedge \bigcup A \notin \mathcal{I}\}
\]
\[
\text{cov}(\mathcal{I}) = \min\{|A| : A \subseteq \mathcal{I} \wedge \bigcup A = X\}
\]
\[
\text{cov}(\mathcal{F}) = \min\{|A| : A \subseteq \mathcal{F} \wedge \bigcup A = X\}
\]
\[
\text{cov}_{\mathcal{I}}(\mathcal{I}) = \min\{|A| : A \subseteq \mathcal{I} \wedge \exists B \in \mathcal{B}_+(X) B \subseteq \bigcup A\}
\]
\[
\text{cov}_{\mathcal{F}}(\mathcal{F}) = \min\{|A| : A \subseteq \mathcal{F} \wedge \exists B \in \mathcal{B}_+(X) B \subseteq \bigcup A\}
\]

An ideal \( \mathcal{I} \) is c.c.c. if every family of pairwise disjoint non-empty \( \mathcal{I} \)-positive Borel sets is countable. Now let \((X, \mathcal{I})\) be a Polish ideal space with \( \mathcal{I} \) c.c.c. and \( A \subseteq X \).

Let \( \mathcal{A} \) be a maximal family of pairwise disjoint \( \mathcal{I} \)-positive Borel sets contained in \( A^c \). Set \( B = (\bigcup \mathcal{A})^c \). Then \( B \) is Borel, \( A \subseteq B \) and for every Borel set \( C \supseteq A \), \( B \setminus C \in \mathcal{I} \).

Any such set \( B \) is called a Borel envelope of \( A \) and will be denoted by \([A]_{\mathcal{I}}\). Note that a Borel envelope of \( A \) is unique modulo \( \mathcal{I} \) and it is minimal (modulo \([A]_{\mathcal{I}}\)) Borel set containing \( A \).

It follows that \( \overline{B}(X) \) is Marczewski complete (see [6], p.114). Therefore, it is closed under Souslin operation (see [6], Theorem 3.5.22). It follows that if \( \mathcal{I} \) is also c.c.c., \( \overline{B}(X) \) contains all analytic sets.

For any set \( F \subseteq X \times Y \) and \( x \in X, y \in Y \) let

\[
F_x = \{y \in Y : (x, y) \in F\}
\]

and

\[
F^y = \{x \in X : (x, y) \in F\}.
\]

Further, for any \( T \subseteq Y \), we set

\[
F^{-1}(T) = \{x \in X : F_x \cap T \neq \emptyset\}.
\]
A multifunction $F : X \to Y$ is called $\mathcal{A}$-measurable if for every open set $U$ in $Y$, $F^{-1}(U) \in \mathcal{A}$, where $\mathcal{A}$ is a $\sigma$-algebra on $X$.

Let $\pi$ be a partition of $X$ and $A \subseteq X$. The smallest $\pi$-invariant subset of $X$ containing $A$ is called the saturation of $A$ and is denoted by $A^\ast$. Thus,

$$A^\ast = \bigcup \{ E \in \pi : E \cap A \neq \emptyset \}.$$

We call $\pi$ Borel measurable if the saturation of every open set is Borel; it is strongly Borel measurable if the saturation of every closed set is Borel measurable. Since $X$ is second countable, every strongly Borel measurable partition is Borel measurable. The rest of our notations and terminology are standard. For other notation and terminology in Descriptive Set Theory we follow [6].

2. Main results

The following results are the main results of the paper.

**Theorem 2.1.** Let $(X, \mathcal{I})$ be a Polish ideal space such that every set in $\mathcal{B}_+(X)$ contains a $\mathcal{I}$-positive closed set. Suppose $\mathcal{A}$ is a strongly Borel measurable partition of $X$ into $\mathcal{I}$-null closed sets. Then there is a subfamily $A_0 \subseteq A$ such that $\bigcup A_0$ is completely $\mathcal{I}$-nonmeasurable.

**Theorem 2.2.** Let $(X, \mathcal{I})$ be a Polish ideal space. Suppose $f : X \to Y$ is a $\mathcal{B}(X)$-measurable map such that for every $y \in Y$, $f^{-1}(y) \in \mathcal{I}$. Then there is a $T \subseteq Y$ such that $f^{-1}(T)$ is completely $\mathcal{I}$-nonmeasurable.

**Theorem 2.3.** Let $(X, \mathcal{I})$ be a Polish ideal space with $\mathcal{I}$ c.c.c. Let $F : X \to Y$ be a $\mathcal{B}(X)$-measurable multifunction such that for every $x \in X$, $F(x)$ is finite. Then there exists a $T \subseteq Y$ such that $F^{-1}(T)$ is completely $\mathcal{I}$-nonmeasurable.

**Theorem 2.4.** Let $(X, \mathcal{I})$ be a Polish ideal space with $\mathcal{I}$ c.c.c. Suppose $F$ is an analytic subset of $X \times Y$ satisfying the following conditions:

1. $(\forall y \in Y)(F^y \in \mathcal{I})$;
2. $X \setminus \pi_X(F) \in \mathcal{I}$, where $\pi_X : X \times Y \to X$ is the projection map;
3. $(\forall x \in X)(|F_x| < \omega)$.

Then there exists a $T \subseteq Y$ such that $F^{-1}(T)$ is completely $\mathcal{I}$-nonmeasurable.

These results generalize results from [3] and [4]. In the next section, we present the proofs of our theorems.

3. Proofs of the main results

One of the key ideas of this paper is the following theorem (see [4]). For reader’s convenience we will give the proof of it.

**Theorem 3.1.** Let $(X, \mathcal{I})$ be a Polish ideal space. Assume that a family $\mathcal{A} \subseteq \mathcal{I}$ satisfies the following conditions:
Then there exists a subfamily $\mathcal{A}_0 \subseteq \mathcal{A}$ such that $\bigcup \mathcal{A}_0$ is completely $\mathcal{I}$–nonmeasurable.

**Proof.** First of all, we can assume that $Z = \emptyset$ in the second assumption. Now, let us enumerate the family of all positive Borel sets with respect to the ideal $\mathcal{I}$ i.e. $\mathcal{B}_+(X) = \{B_\alpha : \alpha < 2^\omega\}$. By transfinite induction we will construct a sequence $\langle (d_\xi, A_\xi) : \xi < 2^\omega \rangle$ satisfying the following conditions

1. $A_\xi \cap B_\xi \neq \emptyset$,
2. $d_\xi \notin \bigcup_{\alpha < 2^\omega} A_\alpha$.

Assume that we have constructed a sequence $\langle (d_\xi, A_\xi) : \xi < \alpha \rangle$. Since $\bigcup_{\xi < \alpha} \{A \in \mathcal{A} : d_\xi \in A\}$ does not cover any positive Borel set, we are able to find $a_\alpha \in B_\alpha \setminus \bigcup_{\xi < \alpha} \{A \in \mathcal{A} : d_\xi \in A\}$. Let $A_\alpha$ be any element of $\mathcal{A}$ such that $a_\alpha \in A_\alpha$ and find $d_\alpha \in B_\alpha \setminus \bigcup_{\xi \leq \alpha} A_\xi$. It finishes $\alpha$ step of our construction.

Now, let us define $\mathcal{A}_0 = \{A_\xi : \xi \in 2^\omega\}$. For every positive Borel set we have that $\bigcup \mathcal{A}_0 \cap B \neq \emptyset$ and $\{d_\xi : \xi \in 2^\omega\} \cap B \neq \emptyset$. Moreover, $\{d_\xi : \xi \in 2^\omega\} \cap \bigcup \mathcal{A}_0 = \emptyset$. It shows that $\bigcup \mathcal{A}_0$ is completely $\mathcal{I}$–nonmeasurable. $\square$

**Remark 3.1.** We can replace the last assumption in Theorem 3.1 by the set theoretic assumption $\text{cov}_h(\mathcal{I}) = 2^\omega$.

As a corollary we have:

**Corollary 3.1 (ZFC+CH).** Let $(X, \mathcal{I})$ be a Polish ideal space. Let $\mathcal{A} \subseteq \mathcal{I}$ be a point-countable family i.e. $\forall x \in X \ |\{A \in \mathcal{A} : x \in A\}| \leq \omega$ and $\bigcup \mathcal{A} = X$. Then there exists a subfamily $\mathcal{A}_0 \subseteq \mathcal{A}$ such that $\bigcup \mathcal{A}_0$ is completely $\mathcal{I}$–nonmeasurable.

It is also known that above corollary is independent from ZFC theory (see [5]).

**Proof of Theorem 2.1.** By Theorem 3.1, it is sufficient to prove that $\text{cov}_h(\mathcal{A}) = 2^\omega$. Towards proving this, take any $B \in \mathcal{B}_+(X)$. Let $F \subseteq B$ be a $\mathcal{I}$–positive closed set. Let

$$\pi = \{E \cap F : E \in \mathcal{F}\}.$$ 

Note that $\pi$ is uncountable and strongly Borel measurable partition of $F$ into closed sets. Since every strongly Borel measurable partition is Borel measurable, it is Borel measurable. Hence, it admits a Borel cross-selection $S$ (see [6], Theorem 5.4.3, see [1]). Clearly $S$ is uncountable and, therefore of cardinality $2^\omega$. This implies that $|\pi| = 2^\omega$. $\square$

As a corollary we get the following result for Polish groups:
Corollary 3.2. Let \((G, \mathcal{I}, +)\) be a compact Polish ideal group. Suppose \(\mathcal{I}\) is closed under translations. Assume that each set from \(\mathcal{B}_+(G)\) contains a \(\mathcal{I}\)-positive closed set. Let \(H \subset G\) be a perfect subgroup and \(H \in \mathcal{I}\). Then there exists a \(T \subset G\) such that \(T + H\) is completely \(\mathcal{I}\)-nonmeasurable in \(G\).

**Proof.** This follows from Theorem 2.1 by taking \(\mathcal{A}\) to be the set of all left cosets of \(H\). \(\square\)

To prove Theorem 2.2, we need the following result from [3].

**Theorem 3.2** (Brzuchowski, Cichoń, Grzegorek, Ryll-Nardzewski). Let \((X, \mathcal{I})\) be a Polish ideal space and \(A \subset X\) a point-finite cover of \(X\). Then there is a subfamily \(A_0 \subset A\) whose union is not in \(\mathcal{B}(X)\).

**Proof of Theorem 2.2.** Fix a countable base \(\{U_n\}\) for the topology of \(Y\). For each \(n\), let \(I_n \in \mathcal{I}\) such that \(f^{-1}(U_n) \triangle I_n\) is Borel. Let \(X' = X \setminus \bigcup_n I_n\). Then \(f : X' \to Y\) is Borel. Thus, without any loss of generality, we assume that \(f\) is Borel measurable.

Now, let \(B \in \mathcal{B}_+(X)\). Set
\[
A = \pi_Y((B \times Y) \cap \text{graph}(f)).
\]
Then \(A\), being analytic, is either countable or of cardinality \(2^\omega\). If \(A\) were countable, \(B\) is covered by countable subfamily of \(\mathcal{I}\), a contradiction, Thus, \(\text{cov}\{f^{-1}(y) : y \in Y\} = 2^\omega\). Our result now follows from Theorem 3.1. \(\square\)

**Theorem 3.3.** Let \((X, \mathcal{I})\) be a Polish ideal space. Let \(I \in \mathcal{I}\) and \(f : X \setminus I \to Y\) a Borel map such that for every \(y \in Y\), \(f^{-1}(y)\) is \(\mathcal{I}\)-null. Then there is a \(T \subset Y\) such that \(f^{-1}(T)\) is completely \(\mathcal{I}\)-nonmeasurable set.

**Proof.** Let \(B \supseteq I\) be a Borel \(\mathcal{I}\)-null set. Now apply Theorem 2.2 to \(f\lfloor (X \setminus B)\). \(\square\)

The next theorem is a technical result which helps us to prove stronger theorems in case \(\mathcal{I}\) is c.c.c.

**Theorem 3.4.** Let \((X, \mathcal{I})\) be a Polish ideal space with \(\mathcal{I}\) c.c.c. Assume that we have a family \(\mathcal{F} \subset \mathcal{I}\) satisfying the following conditions:

1. \(\mathcal{F}\) is point-finite;
2. \((\forall B \in \mathcal{B}_+(X))(B \subset [\bigcup \mathcal{F}]_\mathcal{I} \rightarrow |\{F \in \mathcal{F} : F \cap B \neq \emptyset\}| = 2^\omega)\).

Then there exists a subfamily \(\mathcal{F}' \subset \mathcal{F}\) such that \([\bigcup \mathcal{F}']_\mathcal{I}\) is completely \(\mathcal{I}\)-nonmeasurable in \([\bigcup \mathcal{F}]_\mathcal{I}\).

**Proof.**

**Step 1.** There exists a subfamily \(\mathcal{F}_0 \subset \mathcal{F}\) having the following properties

1. \([\bigcup \mathcal{F}_0]_\mathcal{I} = [\bigcup \mathcal{F}]_\mathcal{I}\),
2. \((\forall B \in \mathcal{B}_+(X))(B \subset \bigcup \mathcal{F}_0 \rightarrow \min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{F}_0 \land B \subset \bigcup \mathcal{A}\} = 2^\omega)\).
We finish the construction if

\[ \bigcup_{i<n} A_i \]

is maximal element in the family \{ \bigcup A | A < 2^\omega \} \subseteq \mathcal{F} \setminus \bigcup_{i<n} A_i \}.

Notice that the existence of the maximal element in the family \{ \bigcup A | A < 2^\omega \} \subseteq \mathcal{F} \setminus \bigcup_{i<n} A_i \} is implied by the c.c.c property of the ideal \mathcal{I}.

We finish the construction if \{ \bigcup A | A < 2^\omega \} \subseteq \mathcal{F} \setminus \bigcup_{i<n} A_i \} = \emptyset. Our construction has to end up after finitely many steps. Notice that \bigcup_{i<n} A_i \neq \emptyset. So, assuming that there is infinitely many \mathcal{A}_n’s we find a point \( x \in X \) which belongs to infinitely many \bigcup A_i’s. Then \( x \) belongs to infinitely many members of \mathcal{F}, what gives a contradiction with point-finiteness of the family \mathcal{F}. So, our construction ends up after \( k \) steps (\( k < \omega \)).

Now, put \( \mathcal{F}_0 = \mathcal{F} \setminus \bigcup \{ A_n : n \leq k \} \). It is a desired family.

\[ \square \]

**Step 2.** There exists a subfamily \( \mathcal{F}' \subseteq \mathcal{F}_0 \) such that \( \mathcal{F}' \) is completely \( \mathcal{I} \)-nonmeasurable in \( \bigcup \mathcal{F}_0 \).

**Proof.** Let us enumerate two families of positive Borel sets. Namely,

\[ \mathcal{B}_0 = \{ B^0_\alpha : \alpha < 2^\omega \} = \{ B \in \mathcal{B}_+ (X) : B \subseteq [\bigcup \mathcal{F}_0] \setminus [\bigcup \mathcal{F}_0] \} \]

\[ \mathcal{B}_1 = \{ B^1_\alpha : \alpha < 2^\omega \} = \{ B \in \mathcal{B}_+ (X) : B \subseteq [\bigcup \mathcal{F}_0] \} \]

By transfinite induction we construct a sequence

\[ ((F^0, F^1, d_\xi) \in \mathcal{F}_0 \times \mathcal{F}_0 \times B^1_\xi : \xi < 2^\omega) \]

satisfying the following conditions

1. \( F^0 \cap B^0 \neq \emptyset, \quad F^1 \cap B^1 \neq \emptyset \),
2. \( d_\xi \notin \bigcup_{\xi < 2^\omega} (F^0 \cup F^1) \).

Assume that we have constructed a sequence \((F^0, F^1, d_\xi) \in \mathcal{F}_0 \times \mathcal{F}_0 \times B^1_\xi : \xi < \alpha \). Since \(|\{ F \in \mathcal{F}_0 : d_\xi \in F \text{ for some } \xi < \alpha \}| < 2^\omega \), we are able to find \( F^0_\alpha, F^1_\alpha \) such that \( F^0_\alpha \cap F^1_\alpha \neq 0 \), \( F^0_\alpha \cap B^0 \neq \emptyset, F^1_\alpha \cap B^1 \neq \emptyset \). What is more \( \bigcup \{ F^0_\xi, F^1_\xi : \xi \leq \alpha \} \) does not cover \( B^1_\alpha \). So, we can pick \( d_\alpha \in B^1_\alpha \setminus \bigcup \{ F^0_\xi, F^1_\xi : \xi \leq \alpha \} \). It finishes \( \alpha \) step of our construction.

Now, let us define \( \mathcal{F}' = \{ F^0_\xi, F^1_\xi : \xi < 2^\omega \} \). We have that \( \bigcup \mathcal{F}' \) has not empty intersection with any positive Borel set contained in \( \bigcup \mathcal{F}_0 \) and \( \{ d_\xi : \xi < 2^\omega \} \) has not empty intersection with every positive Borel set contained in \( \mathcal{F}_0 \). Moreover, \( \{ d_\xi : \xi < 2^\omega \} \cap \bigcup \mathcal{F}' = \emptyset \) It implies that \( \bigcup \mathcal{F}' \) does not contain any positive Borel set. It shows that \( \bigcup \mathcal{F}' \) is completely \( \mathcal{I} \)-nonmeasurable in \( \mathcal{F}_0 \).

\[ \square \]
Since $\bigcup F = \bigcup F_0$, it finishes the proof.

**Remark 3.2.** Assuming that $\text{cov}(\mathcal{I}) > \omega_1$ we can prove the same theorem for wider class of families. Namely, it is enough to assume that a family $\mathcal{F} \subseteq \mathcal{I}$ is point-countable, i.e. $(\forall x \in X)(|\{F \in \mathcal{F} : x \in f\}| \leq \omega$. Since $\text{cov}(\mathcal{I}) > \omega_1$, there is a point which belongs to $\omega_1$ many Borel sets with the same envelope.

**Proof of Theorem 2.3.** By an argument contained in the proof of Theorem 2.2 without loss of generality, we can assume that $F^{-1}(U)$ is Borel for every open set $U$ in $Y$. Fix any $B \in \mathcal{B}_+(X)$. By Kuratowski–Ryll-Nardzewski selection theorem (see [6], Theorem 5.2.1, see [2]), $F[B]$ admits a Borel selection $s$. The range of $s$, being uncountable, is of cardinality $2^\omega$. This implies that the condition (2) of Theorem 3.4 is satisfied by $\mathcal{F} = \{F^{-1}(y) : y \in Y\}$. Since each $F(x)$ is finite, $\mathcal{F}$ is point-finite. The result now follows from Theorem 3.4.

**Proof of Theorem 2.4.** Without loss of generality, we can assume that $\pi_X(F) = X$. Since $\mathcal{I}$ is c.c.c., every analytic set in $X$ is in $\mathcal{B}(X)$ (see Section 1). It follows that $F$ is the graph of $\mathcal{B}(X)$-measurable, finite set valued multifunction. The result follows from Theorem 2.3.

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