Chapter 4

Operator Models for Nevanlinna Functions

The classes of Weyl functions and more generally of Nevanlinna functions will be studied from the point of view of reproducing kernel Hilbert spaces. It is clear from Chapter 2 that every Weyl function is a uniformly strict Nevanlinna function and it is one of the main objectives here to show that also the converse is true: every uniformly strict Nevanlinna function is a Weyl function. The model space is built as a reproducing kernel Hilbert space of holomorphic functions. A brief introduction to reproducing kernel Hilbert spaces is given in Section 4.1. Using the Nevanlinna kernel in Section 4.2 multiplication operators by the independent variable are studied and a boundary triplet whose Weyl function is the original Nevanlinna function is constructed. For scalar Nevanlinna functions an alternative model in an $L^2$-space is given in Section 4.3. The uniqueness of these constructions will also be discussed in detail. An extension of the operator model in Section 4.2 to Nevanlinna functions which are not necessarily uniformly strict, and to Nevanlinna families is provided in Section 4.4. This also includes a discussion of generalized resolvents, and as a byproduct one obtains the Sz.-Nagy dilation theorem. The connection with extension theory is given via the compressed resolvents of self-adjoint relations in the Krein–Naimark formula in Section 4.5. It will be shown that for every Nevanlinna family there is a self-adjoint exit space extension whose compressed resolvent is parametrized by the Nevanlinna family. Closely connected is the discussion about the orthogonal coupling of two boundary triplets in Section 4.6, which also complements the considerations in Section 2.7.

4.1 Reproducing kernel Hilbert spaces

The following discussion of reproducing kernel Hilbert spaces is focused on what is needed in this text. Within these bounds there is a complete treatment for the reader’s convenience. In the first definition one restricts attention to open
sets, as the emphasis will be on reproducing kernel Hilbert spaces of holomorphic functions.

**Definition 4.1.1.** Let $\Omega \subset \mathbb{C}$ be an open set and let $\mathcal{G}$ be a Hilbert space. A mapping

$$K(\cdot, \cdot) : \Omega \times \Omega \to \mathcal{B}(\mathcal{G}) \quad (4.1.1)$$

is called a $\mathcal{B}(\mathcal{G})$-valued kernel on $\Omega$. The kernel $K(\cdot, \cdot)$, the kernel $K$ for short, is said to be

(i) **nonnegative**, if for any finite set of points $\lambda_1, \ldots, \lambda_n \in \Omega$ and any choice of vectors $\varphi_1, \ldots, \varphi_n \in \mathcal{G}$ the $n \times n$ matrix

$$\begin{pmatrix} (K(\lambda_i, \lambda_j)\varphi_j, \varphi_i)_{\mathcal{G}} \end{pmatrix}_{i,j=1}^n$$

is nonnegative;

(ii) **symmetric**, if $K(\lambda, \mu)^* = K(\mu, \lambda)$ for all $\lambda, \mu \in \Omega$;

(iii) **holomorphic** on $\Omega$, if the mapping $\lambda \mapsto K(\lambda, \mu)$ is holomorphic on $\Omega$ for each $\mu \in \Omega$;

(iv) **uniformly bounded on compact subsets of $\Omega$**, if for any compact set $K \subset \Omega$ one has $\sup_{\lambda \in K} \|K(\lambda, \lambda)\| < \infty$.

Note that the first two items in this definition are not independent. Nonnegativity is the stronger condition.

**Lemma 4.1.2.** Let $K(\cdot, \cdot)$ be a $\mathcal{B}(\mathcal{G})$-valued kernel on $\Omega$ as in (4.1.1). If $K(\cdot, \cdot)$ is nonnegative, then $K(\cdot, \cdot)$ is symmetric.

**Proof.** Let $\varphi, \psi \in \mathcal{G}$ and $\lambda, \mu \in \Omega$. According to (i), the $2 \times 2$ matrix

$$\begin{pmatrix} (K(\lambda, \lambda)\varphi, \varphi)_{\mathcal{G}} & (K(\lambda, \mu)\psi, \varphi)_{\mathcal{G}} \\ (K(\mu, \lambda)\varphi, \psi)_{\mathcal{G}} & (K(\mu, \mu)\psi, \psi)_{\mathcal{G}} \end{pmatrix}$$

is nonnegative, and, hence hermitian. In particular, this implies that

$$(K(\lambda, \mu)\psi, \varphi)_{\mathcal{G}} = (K(\mu, \lambda)\varphi, \psi)_{\mathcal{G}} = (\psi, K(\mu, \lambda)\varphi)_{\mathcal{G}}$$

for all $\varphi, \psi \in \mathcal{G}$. This gives $K(\lambda, \mu)^* = K(\mu, \lambda)$, so that the kernel $K(\cdot, \cdot)$ is symmetric. \qed

The kernels described in Definition 4.1.1 form the basis of the theory of reproducing kernel Hilbert spaces. They arise naturally in the following context. Let $\Omega \subset \mathbb{C}$ be an open set and let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space of functions defined on $\Omega$ with values in a Hilbert space $\mathcal{G}$. The Hilbert space $\mathcal{H}$ is called a **reproducing kernel Hilbert space** if for all $\mu \in \Omega$ the operation of point evaluation

$$f \in \mathcal{H} \mapsto f(\mu) \in \mathcal{G}$$
is bounded. In other words, for each \( \mu \in \Omega \) the linear operator \( E(\mu) : \mathcal{H} \to \mathcal{G} \), defined by \( E(\mu)f = f(\mu) \), belongs to \( \mathbf{B}(\mathcal{H}, \mathcal{G}) \).

In the next theorem a kernel is related to a Hilbert space of functions in which point evaluation is bounded.

**Theorem 4.1.3.** Let \( \mathcal{G} \) be a Hilbert space and assume that \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) is a Hilbert space of \( \mathcal{G} \)-valued functions on an open set \( \Omega \subset \mathbb{C} \) such that point evaluation is bounded for all \( \mu \in \Omega \). Define the corresponding kernel \( K(\cdot, \cdot) \) by

\[
K(\lambda, \mu) = E(\lambda)E(\mu)^* \in \mathbf{B}(\mathcal{G}), \quad \lambda, \mu \in \Omega.
\]

Then the following statements hold:

(i) For \( f \in \mathcal{H} \) one has the reproducing kernel property

\[
\langle f, K(\cdot, \mu)\varphi \rangle = (f(\mu), \varphi)_{\mathcal{G}}, \quad \varphi \in \mathcal{G}, \quad \mu \in \Omega. \tag{4.1.2}
\]

(ii) The identity

\[
\langle K(\cdot, \nu)\eta, K(\cdot, \mu)\varphi \rangle = (K(\mu, \nu)\eta, \varphi)_{\mathcal{G}}
\]

is valid for all \( \nu, \mu \in \Omega \) and \( \eta, \varphi \in \mathcal{G} \).

(iii) \( K(\cdot, \cdot) \) is nonnegative and symmetric.

(iv) \( \mathcal{H} = \text{span}\{K(\cdot, \mu)\varphi : \mu \in \Omega, \ \varphi \in \mathcal{G} \} \).

(v) If the \( \mathcal{G} \)-valued functions in \( \mathcal{H} \) are holomorphic on \( \Omega \), then \( K(\cdot, \cdot) \) is holomorphic and uniformly bounded on compact subsets of \( \Omega \).

**Proof.** (i) & (ii) Note that for \( f \in \mathcal{H}, \varphi \in \mathcal{G}, \) and \( \mu \in \Omega \) one has

\[
(f(\mu), \varphi)_{\mathcal{G}} = (E(\mu)f, \varphi)_{\mathcal{G}} = \langle f, E(\mu)^*\varphi \rangle
\]

and observe that \( E(\mu)^*\varphi \) is a function in \( \mathcal{H} \) whose value at \( \lambda \in \Omega \) is given by

\[
(E(\mu)^*\varphi)(\lambda) = E(\lambda)E(\mu)^*\varphi = K(\lambda, \mu)\varphi.
\]

This implies that \( K(\cdot, \cdot) \) has the reproducing kernel property (4.1.2). The identity in (ii) follows with the special choice \( f(\cdot) = K(\cdot, \nu)\eta \).

(iii) To see that \( K(\cdot, \cdot) \) is a nonnegative kernel it suffices to observe that the matrix

\[
((K(\lambda_i, \lambda_j)\varphi_j, \varphi_i)_{\mathcal{G}})_{i,j=1}^n = ((E(\lambda_i)E(\lambda_j)^*\varphi_j, \varphi_i)_{\mathcal{G}})_{i,j=1}^n
\]

is nonnegative. Lemma 4.1.2 implies that \( K(\cdot, \cdot) \) is symmetric.

(iv) In order to see that the subspace \( \text{span}\{K(\cdot, \mu)\varphi : \mu \in \Omega, \ \varphi \in \mathcal{G} \} \) is dense in \( \mathcal{H} \), assume that there is an element \( f \in \mathcal{H} \) such that \( \langle f, K(\cdot, \mu)\varphi \rangle = 0 \) for all \( \varphi \in \mathcal{G} \)
and \(\mu \in \Omega\). But then \((f(\mu), \varphi)_\mathcal{G} = 0\) for all \(\varphi \in \mathcal{G}\) and \(\mu \in \Omega\). Hence, \(f(\mu) = 0\) for all \(\mu \in \mathcal{G}\), i.e., \(f\) is the null function, which completes the argument.

(v) To see that \(K(\cdot, \cdot)\) is uniformly bounded on compact subsets of \(\Omega\), note first that

\[
\|K(\lambda, \lambda)\| = \|E(\lambda)E(\lambda)^*\| = \|E(\lambda)\|^2. \tag{4.1.3}
\]

Now observe that for all \(f \in \mathcal{H}\) and \(\varphi \in \mathcal{G}\),

\[
(E(\lambda)f, \varphi)_\mathcal{G} = (f(\lambda), \varphi)_\mathcal{G}.
\]

Since by assumption the function \(\lambda \mapsto f(\lambda)\) from \(\Omega\) to \(\mathcal{G}\) is holomorphic, it follows that the mapping \(\lambda \mapsto E(\lambda)\) from \(\Omega\) to \(\mathcal{B}(\mathcal{G}; \mathcal{G})\) is holomorphic, which implies that \(\lambda \mapsto \|E(\lambda)\|\) is continuous. Hence, for any compact set \(K \subset \Omega\) there is some \(M' \geq 0\) such that

\[
\sup_{\lambda \in K} \|E(\lambda)\| \leq M'.
\]

Therefore, (4.1.3) shows that the kernel \(K(\cdot, \cdot)\) is uniformly bounded on compact subsets of \(\Omega\).

It has been shown in Theorem 4.1.3 that a Hilbert space of \(\mathcal{G}\)-valued functions in which point evaluation is bounded gives rise to a nonnegative kernel that possesses the reproducing kernel property in (4.1.2). Now it will be shown, conversely, that any nonnegative kernel \(K(\cdot, \cdot)\) gives rise to such a reproducing kernel Hilbert space. Assume that \(K(\cdot, \cdot)\) is some nonnegative kernel on \(\Omega\) with values in \(\mathcal{B}(\mathcal{G})\). Then \(K(\cdot, \cdot)\) is automatically symmetric by Lemma 4.1.2. Consider the linear space of functions from \(\Omega\) into \(\mathcal{G}\) generated by \(K(\cdot, \cdot)\) via

\[
\mathcal{H}(K) := \text{span} \{ \lambda \mapsto K(\lambda, \mu)\varphi : \mu \in \Omega, \varphi \in \mathcal{G} \}. \tag{4.1.4}
\]

Define the form \(\langle \cdot, \cdot \rangle\) on generating elements by

\[
\langle K(\cdot, \nu)\eta, K(\cdot, \mu)\varphi \rangle := (K(\mu, \nu)\eta, \varphi)_\mathcal{G}, \quad \nu, \mu \in \Omega, \quad \eta, \varphi \in \mathcal{G}, \tag{4.1.5}
\]

and extend it to a form on \(\mathcal{H}(K)\) by

\[
\langle \sum_{j=1}^m \alpha_j K(\cdot, \nu_j)\varphi_j, \sum_{i=1}^m \beta_i K(\cdot, \mu_i)\psi_i \rangle = \sum_{i,j=1}^{n,m} (K(\mu_i, \nu_j)\alpha_j \beta_i, \varphi_j, \psi_i)_{\mathcal{G}}, \tag{4.1.6}
\]

where \(\alpha_j, \beta_i \in \mathbb{C}, \nu_j, \mu_i \in \Omega\), and \(\varphi_j, \psi_i \in \mathcal{G}\) for \(j = 1, \ldots, n, i = 1, \ldots, m\).

In particular, one has by (4.1.5) for all \(f \in \mathcal{H}(K)\)

\[
\langle f, K(\cdot, \mu)\varphi \rangle = (f(\mu), \varphi)_\mathcal{G}, \quad \mu \in \Omega, \quad \varphi \in \mathcal{G}. \tag{4.1.7}
\]

Thus, the definition of the form \(\langle \cdot, \cdot \rangle\) in (4.1.6) implies the reproducing kernel property in (4.1.7); the kernel \(K(\cdot, \cdot)\) is called a reproducing kernel, relative to the linear space \(\mathcal{H}(K)\) in (4.1.4). It will now be shown that the form defined by (4.1.5) or (4.1.6) is actually a scalar product.
Lemma 4.1.4. Let $\Omega \subset \mathbb{C}$ be an open set, let $\mathcal{S}$ be a Hilbert space, and let the kernel $K(\cdot, \cdot)$ in (4.1.1) be nonnegative. Define the space $\tilde{H}(K)$ by (4.1.4) and define the form $\langle \cdot, \cdot \rangle$ on $\tilde{H}(K)$ as in (4.1.5) and (4.1.6). Then $\tilde{H}(K)$ is a pre-Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$. 

Proof. A straightforward calculation shows that $\langle \cdot, \cdot \rangle$ is a well-defined sesquilinear form on $\tilde{H}(K)$. By Lemma 4.1.2, the kernel $K(\cdot, \cdot)$ is symmetric and this yields that $\langle \cdot, \cdot \rangle$ is symmetric. In order to show that $\langle \cdot, \cdot \rangle$ is nonnegative on $\tilde{H}(K)$, observe that

$$\langle \sum_{j=1}^{n} \alpha_j K(\nu_j, \nu_j) \varphi_j, \sum_{i=1}^{n} \alpha_i K(\nu_i, \nu_i) \varphi_i \rangle = \sum_{i,j=1}^{n} (K(\nu_i, \nu_j) \alpha_j \alpha_i) g$$

for all $\nu_1, \ldots, \nu_n \in \Omega$, $\varphi_1, \ldots, \varphi_n \in \mathcal{S}$, and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. Clearly, the last term is equal to

$$\begin{pmatrix} (K(\nu_1, \nu_1) \varphi_1, \varphi_1) g & \cdots & (K(\nu_1, \nu_n) \varphi_n, \varphi_1) g \\ \vdots & \ddots & \vdots \\ (K(\nu_n, \nu_1) \varphi_1, \varphi_n) g & \cdots & (K(\nu_n, \nu_n) \varphi_n, \varphi_n) g \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \alpha.$$ 

The assumption that the kernel $K(\cdot, \cdot)$ is nonnegative means that the $n \times n$ matrix

$$[(K(\nu_i, \nu_j) \varphi_j, \varphi_i) g]_{i,j=1}^{n}$$

is nonnegative. Thus for a typical element

$$f = \sum_{j=1}^{n} \alpha_j K(\nu_j, \nu_j) \varphi_j \in \tilde{H}(K)$$

one sees that $\langle f, f \rangle \geq 0$ and hence $\langle \cdot, \cdot \rangle$ is a nonnegative symmetric sesquilinear form on $\tilde{H}(K)$. In particular, $\langle \cdot, \cdot \rangle$ satisfies the Cauchy–Schwarz inequality. This implies that $\langle \cdot, \cdot \rangle$ is positive definite. In fact, if $\langle f, f \rangle = 0$ for some $f \in \tilde{H}(K)$, then

$$|\langle f, g \rangle|^2 \leq \langle f, f \rangle \langle g, g \rangle = 0 \quad \text{for all} \quad g \in \tilde{H}(K).$$

Hence, with $g = K(\cdot, \mu) \psi$, $\mu \in \Omega$, $\psi \in \mathcal{S}$, the reproducing kernel property (4.1.7) shows that

$$0 = \langle f, K(\cdot, \mu) \psi \rangle = (f(\mu), \psi) g.$$ 

Thus, $f(\mu) = 0$ for all $\mu \in \Omega$ and so $f = 0 \in \tilde{H}(K)$. Summing up, it has been shown that $\langle \cdot, \cdot \rangle$ is a positive definite symmetric sesquilinear form on $\tilde{H}(K)$, that is, $\langle \cdot, \cdot \rangle$ is a scalar product and $(\tilde{H}(K), \langle \cdot, \cdot \rangle)$ is a pre-Hilbert space. \qed
In the following theorem it is shown that a nonnegative kernel $\mathcal{H}(\cdot, \cdot)$ on $\Omega$ produces a Hilbert space $\mathcal{H}(\mathcal{K})$, as a completion of $\mathcal{H}(\mathcal{K})$, of functions on $\Omega$ for which point evaluation is a continuous map. Moreover, if the kernel is holomorphic and uniformly bounded on compact subsets of $\Omega$, then the functions in the resulting Hilbert space are holomorphic.

**Theorem 4.1.5.** Let $\mathcal{G}$ be a Hilbert space, let $\mathcal{H}(\cdot, \cdot)$ be a nonnegative kernel on the open set $\Omega \subset \mathbb{C}$, and let the form $\langle \cdot, \cdot \rangle$ on $\mathcal{H}(\mathcal{K})$ be defined as in (4.1.5) and (4.1.6). Then the following statements hold:

(i) The completion $\mathcal{H}(\mathcal{K})$ of the pre-Hilbert space $(\mathcal{H}(\mathcal{K}), \langle \cdot, \cdot \rangle)$ can be identified with a Hilbert space of $\mathcal{G}$-valued functions defined on $\Omega$.

(ii) For $f \in \mathcal{H}(\mathcal{K})$ one has the reproducing kernel property

$$\langle f, \mathcal{K}(\cdot, \mu) \varphi \rangle = (f(\mu), \varphi)_{\mathcal{G}}, \quad \mu \in \Omega, \quad \varphi \in \mathcal{G}.$$ 

(iii) For $\lambda \in \Omega$ the point evaluation $E(\lambda) : \mathcal{H}(\mathcal{K}) \to \mathcal{G}$, $f \mapsto E(\lambda)f = f(\lambda)$ is a continuous linear mapping and

$$\mathcal{K}(\lambda, \mu) = E(\lambda)E(\mu)^*, \quad \lambda, \mu \in \Omega.$$ 

(iv) If the kernel $\mathcal{H}(\cdot, \cdot)$ is holomorphic and uniformly bounded on every compact subset of $\Omega$, then the functions in $\mathcal{H}(\mathcal{K})$ are holomorphic on $\Omega$.

**Proof.** (i) Let $(\mathcal{H}(\mathcal{K}), \langle \cdot, \cdot \rangle)$ be the Hilbert space that is obtained when one completes the pre-Hilbert space $(\mathcal{H}(\mathcal{K}), \langle \cdot, \cdot \rangle)$. It will be shown that the elements in $\mathcal{H}(\mathcal{K})$ can be identified with $\mathcal{G}$-valued functions on $\Omega$. For this let $f \in \mathcal{H}(\mathcal{K})$ and fix some $\lambda \in \Omega$. Consider the functional

$$\Psi_{f,\lambda} : \mathcal{G} \to \mathbb{C}, \quad \varphi \mapsto \langle \mathcal{K}(\cdot, \lambda) \varphi, f \rangle.$$ 

Then an application of the Cauchy–Schwarz inequality shows that

$$|\Psi_{f,\lambda}(\varphi)|^2 = |\langle \mathcal{K}(\cdot, \lambda) \varphi, f \rangle|^2$$

$$\leq \|\mathcal{K}(\cdot, \lambda)\varphi\|^2 \|f\|^2$$

$$= \langle \mathcal{K}(\cdot, \lambda) \varphi, \mathcal{K}(\cdot, \lambda) \varphi \rangle \|f\|^2$$

$$= (\mathcal{K}(\lambda, \lambda) \varphi, \varphi)_{\mathcal{G}} \|f\|^2$$

$$\leq \|\mathcal{K}(\lambda, \lambda)\| \|f\|^2 \|\varphi\|^2_{\mathcal{G}},$$

and hence $\Psi_{f,\lambda}$ is continuous. By the Riesz representation theorem, there is a unique vector $\psi_{f,\lambda} \in \mathcal{G}$ such that

$$(\varphi, \psi_{f,\lambda})_{\mathcal{G}} = \Psi_{f,\lambda}(\varphi) = \langle \mathcal{K}(\cdot, \lambda) \varphi, f \rangle, \quad \varphi \in \mathcal{G}.$$
Let $\mathfrak{F}(\Omega, \mathcal{G})$ be the space of all $\mathcal{G}$-valued functions defined on $\Omega$, and consider the mapping $\iota : \mathfrak{H}(K) \to \mathfrak{F}(\Omega, \mathcal{G})$, $f \mapsto \iota(f)$, where $\iota(f)(\lambda) := \psi_{f,\lambda}$.  

(4.1.9)  

It follows from the definition of $\iota$ and $\psi_{f,\lambda}$ that  

\[
\begin{align*}
(\iota(f)(\lambda), \varphi)_{\mathcal{G}} &= (\psi_{f,\lambda}, \varphi)_{\mathcal{G}} = \langle f, K(\cdot, \lambda)\varphi \rangle, \\
\end{align*}
\]

(4.1.10)  

and this equality also shows that $\iota$ is a linear mapping.

The mapping $\iota$ in (4.1.9) is injective. To see this, assume that $\iota(f) = 0$ for some $f \in \mathfrak{H}(K)$. This means $\iota(f)(\lambda) = 0$ for all $\lambda \in \Omega$, and (4.1.10) implies $\langle f, K(\cdot, \lambda)\varphi \rangle = 0$ for all $\lambda \in \Omega$ and $\varphi \in \mathcal{G}$. Since the linear span of the functions $K(\cdot, \lambda)\varphi$ forms the dense subspace $\mathfrak{H}(K)$ of $\mathfrak{H}(K)$, it follows that $f = 0$, that is, $\iota$ is injective.

Observe that for $f \in \mathfrak{H}(K)$ it follows from (4.1.10) and the reproducing kernel property (4.1.7) that  

\[
(\iota(f)(\lambda), \varphi)_{\mathcal{G}} = \langle f, K(\cdot, \lambda)\varphi \rangle = (f(\lambda), \varphi)_{\mathcal{G}}, \quad \varphi \in \mathcal{G},
\]

for all $\lambda \in \Omega$, and hence $\iota(f) = f$ for $f \in \mathfrak{H}(K)$. In other words, $\iota$ restricted to the dense subspace $\mathfrak{H}(K)$ is the identity, so that $\iota(\mathfrak{H}(K)) = \mathfrak{H}(K)$.

Finally, item (i) follows when the subspace $\text{ran } \iota$ of $\mathfrak{F}(\Omega, \mathcal{G})$ is equipped with the scalar product induced by $\mathfrak{H}(K)$, that is, for $\tilde{f}, \tilde{g} \in \text{ran } \iota$ define  

\[
\langle \tilde{f}, \tilde{g} \rangle := \langle \iota^{-1} \tilde{f}, \iota^{-1} \tilde{g} \rangle.
\]

Then $\iota$ is a unitary mapping from the Hilbert space $(\mathfrak{H}(K), \langle \cdot, \cdot \rangle)$ onto the Hilbert space $(\text{ran } \iota, \langle \cdot, \cdot \rangle)$.  

(ii) After identifying $\iota(f)$ and $f \in \mathfrak{H}(K)$ as in (i), the reproducing kernel property is immediate from (4.1.10).

(iii) With the identification from (ii) observe that for all $\lambda \in \Omega$ and $\varphi \in \mathcal{G}$ the mapping  

\[
f \mapsto (f(\lambda), \varphi)_{\mathcal{G}}
\]

(4.1.11)  

is continuous on $\mathfrak{H}(K)$. In fact, this follows from (ii) and the computation  

\[
|\langle f(\lambda), \varphi \rangle_{\mathcal{G}}|^2 = |\langle f, K(\cdot, \lambda)\varphi \rangle|^2 \\
\leq \langle f, f \rangle \langle K(\cdot, \lambda)\varphi, K(\cdot, \lambda)\varphi \rangle \\
= (K(\lambda, \lambda)\varphi, \varphi)_{\mathcal{G}} \| f \|^2.
\]

For a fixed $\lambda \in \Omega$ the mapping  

\[
E(\lambda) : \mathfrak{H}(K) \to \mathcal{G}, \quad f \mapsto E(\lambda)f = f(\lambda),
\]
is closed. To see this, suppose that \( f_n \to f \) in \( \mathcal{H}(K) \) and \( E(\lambda)f_n \to \psi \) in \( \mathcal{G} \). As \( \text{dom} \, E(\lambda) = \mathcal{H}(K) \), it follows that \( f \in \text{dom} \, E(\lambda) \) and the continuity of \((4.1.11)\) then yields

\[
(\psi, \varphi)_\mathcal{G} = \lim_{n \to \infty} (E(\lambda)f_n, \varphi)_\mathcal{G} = \lim_{n \to \infty} (f_n(\lambda), \varphi)_\mathcal{G} \\
= (f(\lambda), \varphi)_\mathcal{G} = (E(\lambda)f, \varphi)_\mathcal{G}
\]

for all \( \varphi \in \mathcal{G} \). This shows \( E(\lambda)f = \psi \) and hence \( E(\lambda) \) is a closed operator. Since \( \text{dom} \, E(\lambda) = \mathcal{H}(K) \), the closed graph theorem implies that \( E(\lambda) \) is continuous.

It remains to check the identity \( K(\lambda, \mu) = E(\lambda)E(\mu)^* \) for \( \lambda, \mu \in \Omega \). For this let \( \varphi, \psi \in \mathcal{G}, \lambda, \mu \in \Omega \), and note that \( E(\mu)^* \varphi \in \mathcal{H}(K) \) is a function in the variable \( \lambda \). Hence, \( E(\lambda)E(\mu)^* \varphi = (E(\mu)^* \varphi)(\lambda) \), the reproducing kernel property, and the symmetry of the kernel \( K(\cdot, \cdot) \) imply

\[
(E(\lambda)E(\mu)^* \varphi, \psi)_\mathcal{G} = ((E(\mu)^* \varphi)(\lambda), \psi)_\mathcal{G} \\
= \langle E(\mu)^* \varphi, K(\cdot, \lambda)\psi \rangle \\
= \langle \varphi, E(\mu)K(\cdot, \lambda)\psi \rangle_\mathcal{G} \\
= \langle \varphi, K(\lambda, \mu)\psi \rangle_\mathcal{G} \\
= \langle K(\lambda, \mu)\varphi, \psi \rangle_\mathcal{G},
\]

which shows that \( K(\lambda, \mu) = E(\lambda)E(\mu)^* \).

(iv) Let \( f \in \mathcal{H}(K) \) and choose a sequence \( f_n \in \mathcal{H}(K) \) such that \( f_n \to f \) in \( \mathcal{H}(K) \). By assumption, the functions \( K(\cdot, \mu)\varphi \) are holomorphic on \( \Omega \), and hence so are the functions \( f_n \). Now let \( K \subset \Omega \) be a compact set and let \( \sup_{\lambda \in K} \| K(\lambda, \lambda) \| = M_K \).

Then for \( \lambda \in K \) one gets

\[
\|(f(\lambda) - f_n(\lambda), \varphi)_\mathcal{G} = |\langle f - f_n, K(\cdot, \lambda)\varphi \rangle| \\
\leq \|f - f_n\| \|K(\cdot, \lambda)\varphi\| \\
\leq \|f - f_n\| \|K(\lambda, \lambda)\varphi\|^{1/2}_\mathcal{G} \\
\leq M_K^{1/2} \|f - f_n\| \|\varphi\|_\mathcal{G},
\]

and hence \( (f_n(\cdot), \varphi)_\mathcal{G} \to (f(\cdot), \varphi)_\mathcal{G} \) uniformly on \( K \) for all \( \varphi \in \mathcal{G} \). As \( K \) is an arbitrary compact subset of \( \Omega \), it follows that the function \( \lambda \mapsto (f(\lambda), \varphi)_\mathcal{G} \) is holomorphic on \( \Omega \) for all \( \varphi \in \mathcal{G} \). This implies that \( f \) is holomorphic. \( \square \)

If the kernel \( K(\cdot, \cdot) \) is holomorphic and uniformly bounded on every compact subset of \( \Omega \), then the elements in the reproducing kernel Hilbert space \( \mathcal{H}(K) \) can be described as holomorphic functions from \( \Omega \) to \( \mathcal{G} \) which satisfy an additional boundedness condition involving the kernel \( K(\cdot, \cdot) \).

**Theorem 4.1.6.** Let \( \mathcal{G} \) be a Hilbert space, assume that \( K(\cdot, \cdot) \) is a nonnegative holomorphic kernel on the open set \( \Omega \subset \mathbb{C} \) which is uniformly bounded on every compact subset of \( \Omega \), and let \((\mathcal{H}(K), \langle \cdot, \cdot \rangle)\) be the associated reproducing kernel
Hilbert space. Then \( f \in \mathcal{H}(K) \) with \( \| f \| \leq \gamma \) if and only if \( f : \Omega \to \mathcal{G} \) is holomorphic and the \( n \times n \) matrix

\[
\gamma^2 [(K(\nu_i, \nu_j) \varphi_j, \varphi_i)_\mathcal{G}]_{i,j=1}^n - [(f(\nu_i), \varphi_i)_\mathcal{G}, (f(\nu_j), \varphi_j)_\mathcal{G}]_{i,j=1}^n \tag{4.1.12}
\]

is nonnegative for all \( n \in \mathbb{N}, \nu_1, \ldots, \nu_n \in \Omega, \) and \( \varphi_1, \ldots, \varphi_n \in \mathcal{G} \).

**Proof.** In order to prove the necessary and sufficient conditions it is helpful to note that the formulation of the condition (4.1.12) is based on the following identities. For the reproducing kernel \( K(\cdot, \cdot) \) one has, as in (4.1.8),

\[
\sum_{i,j=1}^n \left( (K(\nu_i, \nu_j) \varphi_j, \varphi_i)_\mathcal{G} \right) \alpha_j \alpha_i = \left\| \sum_{j=1}^n \alpha_j K(\cdot, \nu_j) \varphi_j \right\|^2 \tag{4.1.13}
\]

for all \( \nu_1, \ldots, \nu_n \in \Omega, \varphi_1, \ldots, \varphi_n \in \mathcal{G}, \) and \( \alpha_1, \ldots, \alpha_n \in \mathbb{C} \). Furthermore, for a function \( f : \Omega \to \mathcal{G} \) which is holomorphic one has

\[
\sum_{i,j=1}^n \left( (f(\nu_i), \varphi_i)_\mathcal{G}, (f(\nu_j), \varphi_j)_\mathcal{G} \right) \alpha_j \alpha_i = \sum_{j=1}^n \left( f(\nu_j), \alpha_j \varphi_j \right)_\mathcal{G}^2 \tag{4.1.14}
\]

for all \( \nu_1, \ldots, \nu_n \in \Omega, \varphi_1, \ldots, \varphi_n \in \mathcal{G}, \) and \( \alpha_1, \ldots, \alpha_n \in \mathbb{C} \).

Assume that the function \( f : \Omega \to \mathcal{G} \) is holomorphic and there exists \( \gamma > 0 \) such that the \( n \times n \) matrix (4.1.12) is nonnegative for all \( n \in \mathbb{N}, \nu_1, \ldots, \nu_n \in \Omega, \) and \( \varphi_1, \ldots, \varphi_n \in \mathcal{G} \). Together with (4.1.13) and (4.1.14), this implies that the relation from \( \mathcal{H}(K) \) to \( \mathbb{C} \), spanned by the elements

\[
\left\{ \sum_{j=1}^n \alpha_j K(\cdot, \nu_j) \varphi_j, \sum_{j=1}^n (\alpha_j \varphi_j, f(\nu_j))_\mathcal{G} \right\},
\]

where \( \nu_1, \ldots, \nu_n \in \Omega, \varphi_1, \ldots, \varphi_n \in \mathcal{G}, \) and \( \alpha_1, \ldots, \alpha_n \in \mathbb{C} \), is a bounded functional with bound \( \gamma \). Furthermore, it is densely defined on \( \mathcal{H}(K) \), so that it admits
a uniquely defined bounded linear extension defined on all of \( \mathcal{S}(K) \). This functional is represented by a unique element \( F \in \mathcal{S}(K) \) with \( \|F\| \leq \gamma \) via the Riesz representation theorem. In particular this means that
\[
(\varphi, f(\nu))_G = \langle K(\cdot, \nu)\varphi, F \rangle, \quad \nu \in \Omega, \ \varphi \in \mathcal{S},
\]
whereas by the reproducing kernel property one has
\[
\langle K(\cdot, \nu)\varphi, F \rangle = \langle F, K(\cdot, \nu)\varphi \rangle = (\varphi, F(\nu))_G, \quad \nu \in \Omega, \ \varphi \in \mathcal{S}.
\]
Combining the last two identities one concludes that \( f = F \), which gives that \( f \in \mathcal{S}(K) \) and \( \|f\| \leq \gamma \).

For the converse statement, assume that \( f \in \mathcal{S}(K) \) and \( \|f\| \leq \gamma \). Then the function \( f : \Omega \to \mathcal{S} \) is holomorphic and for all \( \nu_1, \ldots, \nu_n \in \Omega, \ \varphi_1, \ldots, \varphi_n \in \mathcal{S}, \) and \( \alpha_1, \ldots, \alpha_n \in \mathbb{C} \) one has by means of (4.1.14) and the fact that \( f \in \mathcal{S}(K) \)
\[
\sum_{i,j=1}^{n} \left( (f(\nu_i), \varphi_i)_G (\overline{f(\nu_j), \varphi_j})_G \right) \alpha_j \alpha_i \leq \sum_{j=1}^{n} (f(\nu_j), \varphi_j)_G \left| \sum_{j=1}^{n} \alpha_j K(\cdot, \nu_j)\varphi_j \right|^2 \leq \||f||^2 \left\| \sum_{j=1}^{n} \alpha_j K(\cdot, \nu_j)\varphi_j \right\|^2.
\]
Together with (4.1.13) and \( \|f\| \leq \gamma \) this gives (4.1.12).

Due to the holomorphy it is sometimes convenient to consider a set of functions \( \lambda \mapsto K(\lambda, \mu)\varphi, \ \varphi \in \mathcal{S}, \) on a determining set of points \( \mu \in \Omega \).

**Corollary 4.1.7.** Let \( K(\cdot, \cdot) \) be a nonnegative holomorphic kernel on an open set \( \Omega \subset \mathbb{C} \) which is uniformly bounded on every compact subset of \( \Omega \), and let \( \mathcal{S}(K) \) be the associated reproducing kernel Hilbert space. Let \( D \subset \Omega \) be a set of points which has an accumulation point in each connected component of \( \Omega \). Then
\[
\mathcal{S}(K) = \text{span} \{ \lambda \mapsto K(\lambda, \mu)\varphi : \mu \in D, \ \varphi \in \mathcal{S} \}.
\]

**Proof.** The inclusion (\( \supseteq \)) is obvious from (4.1.4). To show the inclusion (\( \subseteq \)), it suffices to verify that the linear space
\[
\text{span} \{ \lambda \mapsto K(\lambda, \mu)\varphi : \mu \in D, \ \varphi \in \mathcal{S} \}
\]
is dense in $\mathcal{H}(K)$. Therefore, let $f \in \mathcal{H}(K)$ be orthogonal to this set. Then

$$0 = \langle f, K(\cdot, \mu) \varphi \rangle = (f(\mu), \varphi)_{\mathcal{G}}$$

for all $\mu \in \mathcal{D}$ and $\varphi \in \mathcal{G}$, and hence $f(\mu) = 0$ for all $\mu \in \mathcal{D}$. Since $f \in \mathcal{H}(K)$ is holomorphic on $\Omega$, the assumption on $\mathcal{D}$ now implies that $f(\lambda) = 0$ for all $\lambda \in \Omega$. Hence, $f = 0$ and the proof is complete. 

Let $K(\cdot, \cdot)$ be a nonnegative holomorphic kernel on an open set $\Omega$. If $\Omega' \subset \mathbb{C}$ is an open set such that $\Omega \subset \Omega'$ and that $K'(\cdot, \cdot)$ is a nonnegative holomorphic kernel on $\Omega'$ extending $K(\cdot, \cdot)$, then the functions in the reproducing kernel Hilbert space $\mathcal{H}(K)$ may be seen as restrictions to $\Omega$ of the functions in the reproducing kernel Hilbert space $\mathcal{H}(K')$.

**Proposition 4.1.8.** Let $K(\cdot, \cdot)$ be a nonnegative holomorphic kernel on an open set $\Omega \subset \mathbb{C}$ which is uniformly bounded on every compact subset of $\Omega$. Assume that $\Omega' \subset \mathbb{C}$ is an open set such that $\Omega \subset \Omega'$ and that $K'(\cdot, \cdot)$ is a nonnegative holomorphic kernel on $\Omega'$ which is uniformly bounded on every compact subset of $\Omega'$ and which is equal to $K(\cdot, \cdot)$ on $\Omega$. Then

$$\mathcal{H}(K) = \{ f|_\Omega : f \in \mathcal{H}(K') \}.$$

**Proof.** Consider the linear space of functions from $\Omega'$ into $\mathcal{G}$ generated by $K'(\cdot, \cdot)$ via

$$\hat{\mathcal{H}}(K') := \text{span} \{ \lambda \mapsto K'(\lambda, \mu) \varphi : \mu \in \Omega', \varphi \in \mathcal{G} \}.$$

It is clear that the analogous linear space

$$\hat{\mathcal{H}}(K) := \text{span} \{ \lambda \mapsto K(\lambda, \mu) \varphi : \mu \in \Omega, \varphi \in \mathcal{G} \}$$

is contained in $\hat{\mathcal{H}}(K')$ in the sense that each function $\lambda \mapsto K(\lambda, \mu) \varphi$ with $\mu \in \Omega$ is the restriction to $\Omega$ of the function $\lambda \mapsto K'(\lambda, \mu) \varphi$. Hence, a continuity argument shows the inclusion

$$\mathcal{H}(K) \subset \{ f|_\Omega : f \in \mathcal{H}(K') \}.$$

For the opposite inclusion consider $f|_\Omega : \Omega \to \mathbb{C}$ for some $f \in \mathcal{H}(K')$, and set $\gamma = \| f \|$. Then, by Theorem 4.1.6, the matrix

$$\gamma^2 \left[ (K'(\nu_i, \nu_j) \varphi_i, \varphi_j)_{\mathcal{G}} \right]_{i,j=1}^n - \left[ (f(\nu_i), \varphi_i)_{\mathcal{G}} (f(\nu_j), \varphi_j)_{\mathcal{G}} \right]_{i,j=1}^n$$

is nonnegative for all $n \in \mathbb{N}$, $\nu_1, \ldots, \nu_n \in \Omega'$, and $\varphi_1, \ldots, \varphi_n \in \mathcal{G}$. In particular,

$$\gamma^2 \left[ (K(\nu_i, \nu_j) \varphi_i, \varphi_j)_{\mathcal{G}} \right]_{i,j=1}^n - \left[ (f(\nu_i), \varphi_i)_{\mathcal{G}} (f(\nu_j), \varphi_j)_{\mathcal{G}} \right]_{i,j=1}^n$$

is nonnegative for all $n \in \mathbb{N}$, $\nu_1, \ldots, \nu_n \in \Omega$, and $\varphi_1, \ldots, \varphi_n \in \mathcal{G}$. Another application of Theorem 4.1.6 implies $f|_\Omega \in \mathcal{H}(K)$. 

Under suitable circumstances multiplication of a given reproducing kernel by an operator function gives rise to a new reproducing kernel. In the following proposition this fact and the relation between the corresponding reproducing kernel Hilbert spaces are explained.
**Proposition 4.1.9.** Let $\mathcal{G}$ be a Hilbert space, assume that $K(\cdot, \cdot)$ is a nonnegative kernel on $\Omega$, and let $(\mathcal{H}(K), \langle \cdot, \cdot \rangle)$ be the associated reproducing Hilbert space. Let $\Phi : \Omega \to \mathcal{B}(\mathcal{G})$ be such that $0 \in \rho(\Phi(\lambda))$ for all $\lambda \in \Omega$. Then

$$K_\Phi(\lambda, \mu) = \Phi(\lambda)K(\lambda, \mu)\Phi(\mu)^* \quad (4.1.15)$$

is a nonnegative kernel on $\Omega$ and the corresponding reproducing Hilbert space $(\mathcal{H}(K_\Phi), \langle \cdot, \cdot \rangle_\Phi)$ is unitarily equivalent to $\mathcal{H}(K)$ via the mapping

$$M_\Phi : \mathcal{H}(K) \to \mathcal{H}(K_\Phi), \quad f \mapsto \Phi f.$$

Moreover, if $K(\cdot, \cdot)$ is holomorphic and uniformly bounded on every compact subset of $\Omega$, and $\Phi$ is holomorphic, then also $K_\Phi(\cdot, \cdot)$ is holomorphic and uniformly bounded on every compact subset of $\Omega$.

**Proof.** The definition (4.1.15) leads to the identity

$$\left( (K_\Phi(\lambda_i, \lambda_j) \varphi_j, \varphi_i)_{\mathcal{G}} \right)_{i,j=1}^n = \left( (K(\lambda_i, \lambda_j) \Phi(\lambda_j)^* \varphi_j, \Phi(\lambda_i)^* \varphi_i)_{\mathcal{G}} \right)_{i,j=1}^n.$$

Hence, the nonnegativity of $K(\cdot, \cdot)$ implies that $K_\Phi(\cdot, \cdot)$ is a nonnegative kernel. Moreover, (4.1.15) shows that for all $\mu \in \Omega$ and $\varphi, \psi \in \mathcal{G}$

$$\Phi(\cdot)K(\cdot, \mu)\varphi = \Phi(\cdot)K(\cdot, \mu)\Phi(\mu)^*\Phi(\mu)^{-*}\varphi = K_\Phi(\cdot, \mu)\Phi(\mu)^{-*}\varphi.$$

Hence, $M_\Phi$ maps $\mathcal{H}(K)$ onto $\mathcal{H}(K_\Phi)$. The identity

$$\langle \Phi(\cdot)K(\cdot, \mu)\varphi, \Phi(\cdot)K(\cdot, \nu)\psi \rangle_\Phi = \langle K_\Phi(\cdot, \mu)\Phi(\mu)^{-*}\varphi, K_\Phi(\cdot, \nu)\Phi(\nu)^{-*}\psi \rangle_\Phi = \langle \Phi(\nu)K(\nu, \mu)\varphi, \Phi(\nu)^{-*}\psi \rangle_{\mathcal{G}} = \langle K(\nu, \mu)\varphi, \psi \rangle_{\mathcal{G}} = \langle K(\cdot, \mu)\varphi, K(\cdot, \nu)\psi \rangle,$$

which is valid for all $\mu, \nu \in \Omega$ and all $\varphi, \psi \in \mathcal{G}$, shows that the mapping $M_\Phi$ from $\mathcal{H}(K)$ onto $\mathcal{H}(K_\Phi)$ is an isometry. Its unique bounded linear extension gives a unitary mapping from $\mathcal{H}(K)$ onto $\mathcal{H}(K_\Phi)$. In order to see that this extension $M_\Phi$ acts as multiplication by $\Phi$ on all functions in $\mathcal{H}(K)$, let $f \in \mathcal{H}(K)$ and choose a sequence $f_n \in \mathcal{H}(K)$ such that $f_n \to f$ in $\mathcal{H}(K)$. By isometry, the sequence $M_\Phi f_n$ converges to $M_\Phi f$ in $\mathcal{H}(K_\Phi)$. Observe that the approximating sequence $(f_n)$ satisfies for all $\varphi \in \mathcal{G}$

$$\langle (M_\Phi f_n)(\cdot), K_\Phi(\cdot, \mu)\varphi \rangle_\Phi = \langle \Phi(\cdot)f_n(\cdot), K_\Phi(\cdot, \mu)\varphi \rangle_\Phi = \langle \Phi(\mu)f_n(\mu), \varphi \rangle_{\mathcal{G}} = \langle f_n(\mu), \Phi(\mu)^*\varphi \rangle_{\mathcal{G}} = \langle f_n(\cdot), K(\cdot, \mu)\Phi(\mu)^*\varphi \rangle.$$
Hence, taking limits one sees that
\[
\langle (M\Phi f)(\cdot), K\Phi (\cdot, \mu)\varphi \rangle_{\Phi} = \langle f(\cdot), K(\cdot, \mu)\Phi(\mu)^*\varphi \rangle,
\]
and therefore
\[
\left( (M\Phi f)(\mu), \varphi \right)_G = \left( (M\Phi f)(\cdot), K\Phi (\cdot, \mu)\varphi \right)_{\Phi} = \left( f(\cdot), K(\cdot, \mu)\Phi(\mu)^*\varphi \right)_G = \left( \Phi(\mu)f(\mu), \varphi \right)_G
\]
for all \( \varphi \in \mathcal{G} \). This shows \((M\Phi f)(\mu) = \Phi(\mu)f(\mu)\) for all \( \mu \in \Omega \).

The last assertion on the holomorphy and uniform boundedness of \( K\Phi(\cdot, \cdot) \) on compact subsets of \( \Omega \) is clear. \( \square \)

### 4.2 Realization of uniformly strict Nevanlinna functions

The aim of this section is to show that every operator-valued uniformly strict Nevanlinna function can be realized as the Weyl function corresponding to a boundary triplet. The reproducing kernel Hilbert space associated with a given uniformly strict Nevanlinna function will serve as a model space. The uniqueness of the model is discussed as well.

Let \( \mathcal{G} \) be a Hilbert space and let \( M \) be a \( \mathcal{B}(\mathcal{G}) \)-valued Nevanlinna function. The associated **Nevanlinna kernel**
\[
N_M(\cdot, \cdot) : \Omega \times \Omega \to \mathcal{B}(\mathcal{G})
\]
with \( \Omega = \mathbb{C} \setminus \mathbb{R} \) is defined by
\[
N_M(\lambda, \mu) := \frac{M(\lambda) - M(\mu)^*}{\lambda - \overline{\mu}}, \quad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}, \quad \lambda \neq \overline{\mu}, \quad (4.2.1)
\]
and \( N_M(\lambda, \overline{\lambda}) = M'(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R} \). Then clearly the kernel \( N_M \) is symmetric. The kernel \( N_M \) is holomorphic, since
\[
\lambda \mapsto N_M(\lambda, \mu)
\]
is holomorphic for each \( \mu \in \mathbb{C} \setminus \mathbb{R} \). Moreover, from the definition of \( N_M \) one sees immediately that
\[
N_M(\lambda, \lambda) = \frac{\text{Im} M(\lambda)}{\text{Im} \lambda}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]
and hence \( N_M(\lambda, \lambda) \geq 0, \lambda \in \mathbb{C} \setminus \mathbb{R} \). In the next theorem it turns out that the kernel \( N_M \) is, in fact, nonnegative on \( \mathbb{C} \setminus \mathbb{R} \). Note also that the kernel \( N_M \) is uniformly bounded on compact subsets of \( \mathbb{C} \setminus \mathbb{R} \) since
\[
\|N_M(\lambda, \lambda)\| \leq \frac{\|M(\lambda)\|}{|\text{Im} \lambda|}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]
Theorem 4.2.1. Let $M$ be a $\mathcal{B}(\mathcal{G})$-valued Nevanlinna function. Then the kernel $N_M$ in (4.2.1) is nonnegative.

Proof. The function $M$ has the integral representation

$$M(\lambda) = \alpha + \lambda \beta + \int_{\mathbb{R}} \left( \frac{1}{t - \lambda} - \frac{t}{1 + t^2} \right) d\Sigma(t), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

with self-adjoint operators $\alpha, \beta \in \mathcal{B}(\mathcal{G})$, $\beta \geq 0$, and a nondecreasing self-adjoint operator function $\Sigma : \mathbb{R} \rightarrow \mathcal{B}(\mathcal{G})$ such that

$$\int_{\mathbb{R}} \frac{d\Sigma(t)}{1 + t^2} \in \mathcal{B}(\mathcal{G}),$$

where the integral in (4.2.2) converges in the strong topology; cf. Theorem A.4.2.

For any $n \in \mathbb{N}$, points $\lambda_1, \ldots, \lambda_n \in \mathbb{C} \setminus \mathbb{R}$, and elements $\varphi_1, \ldots, \varphi_n \in \mathcal{G}$ it follows from (4.2.2) that

$$(N_M(\lambda_i, \lambda_j) \varphi_j, \varphi_i)_\mathcal{G} = \left( \left( \int_{\mathbb{R}} \frac{1}{t - \lambda_i} \frac{1}{t - \lambda_j} d\Sigma(t) \right) \varphi_j, \varphi_i \right)_\mathcal{G}.$$

The first matrix on the right-hand side in (4.2.3) is nonnegative as for any vector $(x_1, \ldots, x_n)^T \in \mathbb{C}^n$ and $\varphi = \sum x_i \varphi_i$ the nonnegativity of the operator $\beta$ implies

$$\begin{pmatrix} (\beta \varphi_1, \varphi_1)_\mathcal{G} & \cdots & (\beta \varphi_n, \varphi_1)_\mathcal{G} \\ \vdots & \ddots & \vdots \\ (\beta \varphi_1, \varphi_n)_\mathcal{G} & \cdots & (\beta \varphi_n, \varphi_n)_\mathcal{G} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \left( \beta \varphi, \varphi \right)_\mathcal{G} \geq 0.$$
and when $\max |t_i - t_{i-1}|$ tends to zero these finite Riemann–Stieltjes sums converge to

$$\sum_{i,j=1}^{n} \int_{a}^{b} \frac{1}{t - \lambda_i} \frac{1}{t - \lambda_j} \, d(\Sigma(t)x_j \varphi_j, x_i \varphi_i) \geq 0.$$ 

Hence, also (4.2.4) is nonnegative; thus, both matrices on the right-hand side in (4.2.3) are nonnegative, and so is their sum, i.e., the kernel $N_M$ is nonnegative. □

According to Theorem 4.1.5, the nonnegative kernel $N_M$ gives rise to a Hilbert space of holomorphic $\mathcal{G}$-valued functions, which will be denoted by $\mathcal{H}(N_M)$, with inner product $\langle \cdot, \cdot \rangle$; cf. Section 4.1. Recall that the reproducing kernel property

$$\langle f, N_M(\cdot, \mu) \varphi \rangle = (f(\mu), \varphi)_{\mathcal{G}}, \quad \varphi, \mu \in \mathbb{C} \setminus \mathbb{R}, \quad (4.2.5)$$

holds for all functions $f \in \mathcal{H}(N_M)$. The main results in this section concern a Nevanlinna function $M$ and the construction of a self-adjoint relation which represents $M$ in a sense to be explained. The construction will involve the associated reproducing kernel space $\mathcal{H}(N_M)$.

Let $M$ be a (not necessarily uniformly strict) $\mathcal{B}(\mathcal{G})$-valued Nevanlinna function. The first main objective in this section is the construction of a minimal model in which the function

$$\lambda \mapsto -(M(\lambda) + \lambda)^{-1}$$

is realized as the compressed resolvent of a self-adjoint relation. The uniqueness of the construction will be discussed after the theorem. Note that the definition of the self-adjoint relation involves multiplication by the independent variable; however, the resulting functions do not necessarily belong to $\mathcal{H}(N_M)$.

**Theorem 4.2.2.** Let $M$ be a $\mathcal{B}(\mathcal{G})$-valued Nevanlinna function and let $\mathcal{H}(N_M)$ be the associated reproducing kernel Hilbert space. Denote by $P_{\mathcal{G}}$ the orthogonal projection from $\mathcal{H}(N_M) \oplus \mathcal{G}$ onto $\mathcal{G}$ and let $\iota_{\mathcal{G}}$ be the canonical embedding of $\mathcal{G}$ into $\mathcal{H}(N_M) \oplus \mathcal{G}$. Then

$$\tilde{A} = \{ \left( f, \varphi' \right) \in \mathcal{H}(N_M), \varphi, \varphi' \in \mathcal{G} : f(\xi) \varphi - \xi f(\xi) = M(\xi) \varphi - \varphi' \} \quad (4.2.6)$$

is a self-adjoint relation in the Hilbert space $\mathcal{H}(N_M) \oplus \mathcal{G}$ and the compressed resolvent of $\tilde{A}$ onto $\mathcal{G}$ is given by

$$P_{\mathcal{G}}(\tilde{A} - \lambda)^{-1} \iota_{\mathcal{G}} = -(M(\lambda) + \lambda)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (4.2.7)$$

Furthermore, the self-adjoint relation $\tilde{A}$ satisfies the following minimality condition:

$$\mathcal{H}(N_M) \oplus \mathcal{G} = \text{span} \{ \mathcal{G}, \text{ran} (\tilde{A} - \lambda)^{-1} \iota_{\mathcal{G}} : \lambda \in \mathbb{C} \setminus \mathbb{R} \}. \quad (4.2.8)$$
Proof. Step 1. The relation $\tilde{A}$ in (4.2.6) contains an essentially self-adjoint relation. Indeed, define the relation $B$ in $\mathcal{H}(N_M) \oplus G$ by

$$B = \text{span} \left\{ \left( \frac{N_M(\cdot, \bar{\mu})\varphi}{M(\mu)\varphi}, \frac{\mu N_M(\cdot, \bar{\mu})\varphi}{M(\mu)\varphi} \right) : \varphi \in \mathcal{G}, \mu \in \mathbb{C} \setminus \mathbb{R} \right\}.$$ 

It follows from the definition of $N_M$ in (4.2.1) that

$$\mu N_M(\xi, \bar{\mu})\varphi - \xi N_M(\xi, \bar{\mu})\varphi = M(\mu)\varphi - M(\xi)\varphi, \quad \varphi \in \mathcal{G}.$$ 

Therefore, one sees from (4.2.6) that $B \subset \tilde{A}$. It remains to show that $B$ is essentially self-adjoint.

The symmetry of $B$ is easily verified: it follows from the definition in (4.2.1) and the reproducing kernel property (4.2.5) that

$$\left( \frac{\mu N_M(\cdot, \bar{\mu})\varphi}{M(\mu)\varphi}, \frac{\varphi}{-\varphi} \right) - \left( \frac{N_M(\cdot, \bar{\nu})\psi}{M(\nu)\psi}, \frac{\varphi}{-\varphi} \right) = \langle \mu N_M(\cdot, \bar{\mu})\varphi, N_M(\cdot, \bar{\nu})\psi \rangle - (M(\mu)\varphi, \psi)_G - \langle N_M(\cdot, \bar{\nu})\varphi, \nu N_M(\cdot, \bar{\nu})\psi \rangle + (\varphi, M(\nu)\psi)_G = 0$$

for all $\varphi, \psi \in \mathcal{G}$ and all $\mu, \nu \in \mathbb{C} \setminus \mathbb{R}$. This identity implies that $B$ is symmetric in $\mathcal{H}(N_M) \oplus G$.

To see that $B$ is essentially self-adjoint, it now suffices to establish that $\text{ran}(B - \lambda_0)$ is dense in $\mathcal{H}(N_M) \oplus G$ for arbitrary $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$. Observe that it follows from the definition that

$$\text{ran}(B - \lambda_0) = \text{span} \left\{ \left( \frac{(\mu - \lambda_0) N_M(\cdot, \bar{\mu})\varphi}{M(\mu) + \lambda_0}\varphi \right) : \varphi \in \mathcal{G}, \mu \in \mathbb{C} \setminus \mathbb{R} \right\}.$$ 

The choice $\mu = \lambda_0$ together with the fact $-\lambda_0 \in \rho(M(\lambda_0))$ (see Definition A.4.1) imply that $\text{ran}(M(\lambda_0) + \lambda_0) = \mathcal{G}$ and hence

$$\{0\} \oplus G \subset \text{ran}(B - \lambda_0).$$

Therefore, also the elements of the form

$$\left( \frac{N_M(\cdot, \bar{\mu})\varphi}{0}, \varphi \in \mathcal{G}, \mu \in \mathbb{C} \setminus \mathbb{R}, \mu \neq \lambda_0, \right.$$

belong to $\text{ran}(B - \lambda_0)$. Moreover, since the set

$$\text{span} \left\{ N_M(\cdot, \mu)\varphi : \varphi \in \mathcal{G}, \mu \in \mathbb{C} \setminus \mathbb{R}, \mu \neq \lambda_0 \right\}$$

is dense in $\mathcal{H}(N_M)$, see Corollary 4.1.7, it follows from (4.2.9) and (4.2.10) that $\text{ran}(B - \lambda_0)$ is dense in $\mathcal{H}(N_M) \oplus G$ for all $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$. 


Now let \( \overline{B} \) be the closure of the symmetric relation \( B \). It is clear that \( \overline{B} \) is symmetric and that \( \operatorname{ran}(\overline{B} - \lambda_0) \) is closed (see Proposition 1.4.4 and Lemma 1.2.2). Hence, it follows from the above considerations that \( \operatorname{ran}(\overline{B} - \lambda_0) = \mathcal{H}(N_M) \oplus \mathcal{G} \), and Theorem 1.5.5 yields that \( \overline{B} \) is self-adjoint in \( \mathcal{H}(N_M) \oplus \mathcal{G} \).

**Step 2.** The relation \( \tilde{A} \) is self-adjoint. To prove this, it suffices to establish that the closure of \( B \) coincides with the relation \( \tilde{A} \).

First one shows that the relation \( \tilde{A} \) is closed. To see this, let

\[
\begin{pmatrix} f_n \varphi_n \\ \varphi_n' \end{pmatrix} \to \begin{pmatrix} f \varphi \\ f' \end{pmatrix}, \quad \begin{pmatrix} f_n \varphi_n \\ -\varphi_n' \end{pmatrix} \to \begin{pmatrix} f \varphi \\ -\varphi' \end{pmatrix}
\]

in

\[
\begin{pmatrix} \mathcal{H}(N_M) \oplus \mathcal{G} \\ \mathcal{G} \end{pmatrix} \times \begin{pmatrix} \mathcal{H}(N_M) \oplus \mathcal{G} \\ \mathcal{G} \end{pmatrix},
\]

where the sequence on the left-hand side belongs to \( \tilde{A} \), so that

\[
f_n'(\xi) - \xi f_n(\xi) = M(\xi)\varphi_n - \varphi_n', \quad \xi \in \mathbb{C} \setminus \mathbb{R}.
\]

Then taking limits in the last identity and using the continuity of point evaluation in \( \mathcal{H}(N_M) \), see Theorem 4.1.5, leads to the identity

\[
f'(\xi) - \xi f(\xi) = M(\xi)\varphi - \varphi', \quad \xi \in \mathbb{C} \setminus \mathbb{R}.
\]

Therefore, the relation \( \tilde{A} \) is closed.

Since \( B \subset \tilde{A} \) and \( \tilde{A} \) is closed, one has \( \overline{B} \subset \tilde{A} \). Hence, to see that \( \tilde{A} = \overline{B} \) it suffices to prove the inclusion \( \tilde{A} \subset \overline{B} \). As \( \overline{B} \) is self-adjoint, it suffices to show \( \tilde{A} \subset B^* \). For this let

\[
\begin{pmatrix} f \\ \varphi \end{pmatrix} \in \tilde{A},
\]

Then \( f, f' \in \mathcal{H}(N_M), \varphi, \varphi' \in \mathcal{G}, \) and

\[
f'(\xi) - \xi f(\xi) = M(\xi)\varphi - \varphi', \quad \xi \in \mathbb{C} \setminus \mathbb{R}.
\]

For an element in \( B \) of the form

\[
\begin{pmatrix} N_M(\cdot, \mu)\psi \\ -\psi \end{pmatrix}, \begin{pmatrix} \mu N_M(\cdot, \mu)\psi \\ M(\mu)\psi \end{pmatrix}
\]

with \( \psi \in \mathcal{G} \) and some \( \mu \in \mathbb{C} \setminus \mathbb{R} \) it follows that

\[
\begin{pmatrix} f' \varphi \\ -\varphi' \end{pmatrix}, \begin{pmatrix} N_M(\cdot, \mu)\psi \\ -\psi \end{pmatrix} - \begin{pmatrix} f \varphi \\ M(\mu)\psi \end{pmatrix}
\]

\[
= \begin{pmatrix} f'(\mu) - \mu f(\mu) + \varphi' - M(\mu)\varphi, \psi \end{pmatrix} \in \mathcal{G}
\]

\[
= \begin{pmatrix} M(\mu)\varphi - \varphi' + \varphi' - M(\mu)\varphi, \psi \end{pmatrix} = 0,
\]

where (4.2.11) with \( \xi = \mu \) was used. This implies \( \tilde{A} \subset B^* \) and thus \( \tilde{A} = \overline{B} \). Hence, \( \tilde{A} \) is self-adjoint in \( \mathcal{H}(N_M) \oplus \mathcal{G} \).
Step 3. It remains to establish the identities (4.2.7) and (4.2.8). Both are direct consequences of (4.2.6). In fact, let \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) and note that
\[
(\tilde{A} - \lambda)^{-1} = \left\{ \left( \begin{array}{c} f' - \lambda f \\ -\varphi' + \lambda \varphi \end{array} \right) : f, f' \in \mathcal{H}(N_M), \varphi, \varphi' \in \mathcal{G}, f'(\xi) - \xi f(\xi) = M(\xi) \varphi - \varphi' \right\}
\]
and hence
\[
(\tilde{A} - \lambda)^{-1} \iota_{\mathcal{G}} = \left\{ -\varphi' + \lambda \varphi : f \in \mathcal{H}(N_M), \varphi' \in \mathcal{G}, f'(\xi) - \xi f(\xi) = M(\xi) \varphi - \varphi' \right\}.
\]
The condition \( f' = \lambda f \) yields \( (\lambda - \xi)f(\xi) = M(\xi) \varphi - \varphi' \), \( \xi \in \mathbb{C} \setminus \mathbb{R} \), and setting \( \xi = \lambda \) one obtains \( \varphi' = M(\lambda) \varphi \). Conversely, if \( (\lambda - \xi)f(\xi) = (M(\xi) - M(\lambda)) \varphi \) and \( \varphi' = M(\lambda) \varphi \) and \( f' = \lambda f \), then \( f'(\xi) - \xi f(\xi) = M(\xi) \varphi - \varphi' \) for \( \xi \in \mathbb{C} \setminus \mathbb{R} \). Therefore,
\[
(\tilde{A} - \lambda)^{-1} \iota_{\mathcal{G}} = \left\{ -(M(\lambda) + \lambda) \varphi : f \in \mathcal{H}(N_M), \varphi \in \mathcal{G}, (\lambda - \xi)f(\xi) = (M(\xi) - M(\lambda)) \varphi \right\}
\]
and this yields \( P_{\mathcal{G}}(\tilde{A} - \lambda)^{-1} \iota_{\mathcal{G}} = -(M(\lambda) + \lambda)^{-1} \); recall that \( -\lambda \in \rho(M(\lambda)) \) for all \( \lambda \in \mathbb{C} \setminus \mathbb{R} \). Hence, (4.2.7) is shown. Moreover, from (4.2.1) it follows that
\[
\text{ran} \ P_{\mathcal{G}(N_M)}(\tilde{A} - \lambda)^{-1} \iota_{\mathcal{G}} = \{ f \in \mathcal{H}(N_M) : (\lambda - \xi)f(\xi) = (M(\xi) - M(\lambda)) \varphi, \varphi \in \mathcal{G} \}
= \{ -N_M(\xi, \lambda) \varphi : \varphi \in \mathcal{G}, \xi \in \mathbb{C} \setminus \mathbb{R}, \xi \neq \lambda \}
\]
and hence
\[
\text{span} \ \{ \text{ran} \ P_{\mathcal{G}(N_M)}(\tilde{A} - \lambda)^{-1} \iota_{\mathcal{G}} : \lambda \in \mathbb{C} \setminus \mathbb{R} \} = \mathcal{H}(N_M)
\]
by Theorem 4.1.5 and Corollary 4.1.7. This implies (4.2.8). \( \square \)

The model and the self-adjoint relation in Theorem 4.2.2 are unique up to unitary equivalence. This is a consequence of the following general equivalence result.

**Theorem 4.2.3.** Let \( \mathcal{G}, \mathcal{H} \), and \( \mathcal{H}' \) be Hilbert spaces and let \( \tilde{A} \) and \( \tilde{A}' \) be self-adjoint relations in the product spaces \( \mathcal{H} \oplus \mathcal{G} \) and \( \mathcal{H}' \oplus \mathcal{G} \), respectively. Denote by \( P_{\mathcal{G}} \) and \( P'_{\mathcal{G}} \) the orthogonal projections from \( \mathcal{H} \oplus \mathcal{G} \) and \( \mathcal{H}' \oplus \mathcal{G} \) onto \( \mathcal{G} \), respectively, and let \( \iota_{\mathcal{G}} \) and \( \iota'_{\mathcal{G}} \) be the corresponding canonical embeddings. Assume that \( \tilde{A} \) satisfies the minimality condition
\[
\mathcal{H} \oplus \mathcal{G} = \text{span} \ \{ \mathcal{G}, \text{ran} \ (\tilde{A} - \lambda)^{-1} \iota_{\mathcal{G}} : \lambda \in \mathbb{C} \setminus \mathbb{R} \} \quad (4.2.12)
\]
and that \( \tilde{A}' \) satisfies the minimality condition
\[
\mathcal{H}' \oplus \mathcal{G} = \text{span} \ \{ \mathcal{G}, \text{ran} \ (\tilde{A}' - \lambda)^{-1} \iota'_{\mathcal{G}} : \lambda \in \mathbb{C} \setminus \mathbb{R} \} \quad (4.2.13)
\]
4.2. Realization of uniformly strict Nevanlinna functions

Furthermore, assume that

\[ P_S(\tilde{A} - \lambda)^{-1}t_g = P_S'(\tilde{A}' - \lambda)^{-1}t'_g, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \]  

(4.2.14)

Then \( \tilde{A} \) and \( \tilde{A}' \) are unitarily equivalent, that is, there exists a unitary operator \( U \in \mathbf{B}(\mathfrak{H} \oplus \mathfrak{S}, \mathfrak{H}' \oplus \mathfrak{S}) \) such that \( \tilde{A}' = U \tilde{A} U^* \).

**Proof.** Note that the elements of the form

\[ \sum_{j=1}^{n} (\alpha_j \varphi_j + \beta_j (\tilde{A} - \lambda_j)^{-1} \psi_j), \]  

(4.2.15)

where \( \varphi_j, \psi_j \in \mathfrak{S}, \alpha_j, \beta_j \in \mathbb{C}, \lambda_j \in \mathbb{C} \setminus \mathbb{R} \) for \( j = 1, \ldots, n \), and \( n \in \mathbb{N} \) are arbitrarily chosen, form a dense subspace of the Hilbert space \( \mathfrak{H} \oplus \mathfrak{S} \) by the assumption (4.2.12). Likewise, the elements of the form

\[ \sum_{j=1}^{n'} (\alpha'_j \varphi'_j + \beta'_j (\tilde{A}' - \lambda'_j)^{-1} \psi'_j), \]  

(4.2.16)

where \( \varphi'_j, \psi'_j \in \mathfrak{S}, \alpha'_j, \beta'_j \in \mathbb{C}, \lambda'_j \in \mathbb{C} \setminus \mathbb{R} \) for \( j = 1, \ldots, n' \), and \( n' \in \mathbb{N} \) are arbitrarily chosen, form a dense subspace of the Hilbert space \( \mathfrak{H}' \oplus \mathfrak{S} \) by the assumption (4.2.13).

Define the linear relation \( U \) from \( \mathfrak{H} \oplus \mathfrak{S} \) to \( \mathfrak{H}' \oplus \mathfrak{S} \) as the linear span of all pairs of the form

\[ \left\{ \sum_{j=1}^{n} (\alpha_j \varphi_j + \beta_j (\tilde{A} - \lambda_j)^{-1} \psi_j), \sum_{j=1}^{n} (\alpha_j \varphi_j + \beta_j (\tilde{A}' - \lambda_j)^{-1} \psi_j) \right\}, \]

where \( \varphi_j, \psi_j \in \mathfrak{S}, \alpha_j, \beta_j \in \mathbb{C}, \lambda_j \in \mathbb{C} \setminus \mathbb{R} \) for \( j = 1, \ldots, n \), and \( n \in \mathbb{N} \) are arbitrarily chosen. Then according to (4.2.15) and (4.2.16) the relation \( U \) has a dense domain and a dense range. To show that the relation \( U \) is isometric, i.e., \( \|h'\| = \|h\| \) for all \( \{h, h'\} \in H \), one has to verify that

\[
\left( \sum_{j=1}^{n} (\alpha_j \varphi_j + \beta_j (\tilde{A} - \lambda_j)^{-1} \psi_j), \sum_{i=1}^{n} (\alpha_i \varphi_i + \beta_i (\tilde{A}' - \lambda_i)^{-1} \psi_i) \right) = \left( \sum_{j=1}^{n} (\alpha_j \varphi_j + \beta_j (\tilde{A} - \lambda_j)^{-1} \psi_j), \sum_{i=1}^{n} (\alpha_i \varphi_i + \beta_i (\tilde{A} - \lambda_i)^{-1} \psi_i) \right) \]

To see this, it suffices to observe that (4.2.14) implies

\[
((\tilde{A}' - \lambda_j)^{-1} \psi_j, \varphi_i)_\mathfrak{S} = (P_S'(\tilde{A}' - \lambda_j)^{-1} \psi_j, \varphi_i)_\mathfrak{S} = (P_S(\tilde{A} - \lambda_j)^{-1} \psi_j, \varphi_i)_\mathfrak{S} = (\tilde{A} - \lambda_j)^{-1} \psi_j, \varphi_i)_\mathfrak{S},
\]
and, likewise, by symmetry,
\[(\varphi_j, (\tilde{A}' - \lambda_i)^{-1}\psi_i)_g = (\varphi_j, (\tilde{A} - \lambda_i)^{-1}\psi_i)_g.\]

Moreover, using the resolvent identity, one sees that for \(\lambda_j \neq \overline{\lambda}_i\) (4.2.14) implies
\[
((\tilde{A}' - \lambda_j)^{-1}\psi_j, (\tilde{A}' - \lambda_j)^{-1}\psi_i)_g = ((\tilde{A}' - \lambda_j)^{-1}(\tilde{A}' - \overline{\lambda}_i)^{-1}\psi_j, \psi_i)_g
\]
\[
= (\lambda_j - \overline{\lambda}_i)^{-1}[( (\tilde{A}' - \lambda_j)^{-1}\psi_j, \psi_i)_g - ((\tilde{A}' - \overline{\lambda}_i)^{-1}\psi_j, \psi_i)_g]
\]
\[
= (\lambda_j - \overline{\lambda}_i)^{-1}[( (\tilde{A} - \lambda_j)^{-1}\psi_j, \psi_i)_g - ((\tilde{A} - \overline{\lambda}_i)^{-1}\psi_j, \psi_i)_g]
\]
\[
= ((\tilde{A} - \lambda_j)^{-1}(\tilde{A} - \lambda_j)^{-1}\psi_j, \psi_i)_g
\]
\[
= ((\tilde{A} - \lambda_j)^{-1}\psi_j, (\tilde{A} - \lambda_j)^{-1}\psi_i)_g,
\]

and a limit argument together with the continuity of the resolvent shows that the same is true in the case \(\lambda_j = \overline{\lambda}_i\). Thus, the relation \(U\) is isometric; hence it is a well-defined isometric operator and the closure of \(U\), denoted again by \(U\), is a unitary operator from \(\tilde{\mathcal{H}} \oplus \mathcal{G}\) onto \(\tilde{\mathcal{H}}' \oplus \mathcal{G}\).

Next it will be shown that \(\tilde{A}\) and \(\tilde{A}'\) are unitarily equivalent under \(U\). To see this, one needs to show
\[
U(\tilde{A} - \lambda)^{-1} = (\tilde{A}' - \lambda)^{-1}U
\]
(4.2.17)
for some \(\lambda \in \mathbb{C} \setminus \mathbb{R}\); cf. Lemma 1.3.8. Since all operators involved are bounded, it suffices to check this identity on a dense set of \(\tilde{\mathcal{H}} \oplus \mathcal{G}\); thus, in fact, it suffices to check this only for the elements in (4.2.15). Observe that for elements in (4.2.15) with \(\lambda \neq \lambda_j\) and \(\gamma := \frac{\beta_j}{\lambda_j - \lambda}\) one has, again by the resolvent identity,
\[
U(\tilde{A} - \lambda)^{-1}(\alpha_j \varphi_j + \beta_j (\tilde{A} - \lambda_j)^{-1}\psi_j)
\]
\[
= U((\tilde{A} - \lambda)^{-1}(\alpha_j \varphi_j - \gamma_j \psi_j) + \gamma_j (\tilde{A} - \lambda_j)^{-1}\psi_j)
\]
\[
= (\tilde{A}' - \lambda)^{-1}(\alpha_j \varphi_j - \gamma_j \psi_j) + \gamma_j (\tilde{A}' - \lambda_j)^{-1}\psi_j
\]
\[
= (\tilde{A}' - \lambda)^{-1}(\alpha_j \varphi_j + \beta_j (\tilde{A}' - \lambda_j)^{-1}\psi_j)
\]
\[
= (\tilde{A}' - \lambda)^{-1}U(\alpha_j \varphi_j + \beta_j (\tilde{A} - \lambda_j)^{-1}\psi_j),
\]

and by the continuity of the resolvent the same relation holds also for \(\lambda = \lambda_j\). Thus, (4.2.17) has been established.

The second main result in this section concerns a minimal model for uniformly strict Nevanlinna functions. By means of Theorem 4.2.2 it will be shown that every uniformly strict Nevanlinna function \(\mathcal{M}\) is the Weyl function corresponding to a boundary triplet of a simple symmetric operator in the Hilbert space \(\tilde{\mathcal{H}}(\mathbb{N}_M)\). After this result it will be shown that every boundary triplet producing the same Weyl
function is unitarily equivalent to the boundary triplet in this construction. Note that the description of \((S_M)^*\) involves functions which do not necessarily belong to \(H(N_M)\); however the definition of \(S_M\) concerns functions which remain in \(H(N_M)\) after multiplication by the independent variable.

**Theorem 4.2.4.** Let \(M\) be a uniformly strict \(B(G)\)-valued Nevanlinna function and let \(H(N_M)\) be the associated reproducing kernel Hilbert space. Then

\[
S_M = \{ \{f, f'\} \in H(N_M)^2 : f'(\xi) = \xi f(\xi) \} \tag{4.2.18}
\]

is a closed simple symmetric operator in \(H(N_M)\) and its adjoint is given by

\[
(S_M)^* = \{ \{f, f'\} \in H(N_M)^2 : f'(\xi) - \xi f(\xi) = M(\xi)\varphi - \varphi', \varphi, \varphi' \in \mathcal{S} \}. \tag{4.2.19}
\]

Moreover, the mappings

\[
\Gamma_0 \widehat{f} = \varphi \quad \text{and} \quad \Gamma_1 \widehat{f} = \varphi', \quad \widehat{f} \in (S_M)^*,
\]

are well defined and \(\{\mathcal{S}, \Gamma_0, \Gamma_1\}\) is a boundary triplet for \((S_M)^*\). The corresponding \(\gamma\)-field is given by

\[
\gamma(\lambda)\varphi = -N_M(\cdot, \lambda)\varphi \tag{4.2.21}
\]

and the corresponding Weyl function is given by \(M\).

**Proof.** By Theorem 4.2.2, the relation

\[
\widetilde{A} = \left\{ \left( \begin{array}{c} f \\ \varphi \end{array} \right), \left( \begin{array}{c} f' \\ -\varphi' \end{array} \right) : f, f' \in H(N_M), \varphi, \varphi' \in \mathcal{S}, \right\}
\]

is self-adjoint in \(H(N_M) \oplus \mathcal{S}\). The relations in (4.2.18) and (4.2.19) are defined in the component space \(H(N_M)\); the condition that \(M\) is uniformly strict makes it possible to connect them with \(\widetilde{A}\).

**Step 1.** The relation \(S_M\) in (4.2.18) is a closed symmetric operator with adjoint \((S_M)^*\) given by (4.2.19). First observe that the relation \(S_M\) in (4.2.18) satisfies

\[
S_M = \widetilde{A} \cap \left( \left( H(N_M) \setminus \{0\} \right) \times \left( H(N_M) \setminus \{0\} \right) \right), \tag{4.2.22}
\]

when the space \(H(N_M) \times \{0\}\) is identified with \(H(N_M)\). Since \(\widetilde{A}\) is self-adjoint, (4.2.22) implies that \(S_M\) is closed and symmetric, and it is clear from (4.2.18) that \(S_M\) is an operator. In order to find the adjoint of \(S_M\), let the relation \(T_M\) be defined by

\[
T_M = \{ \{f, f'\} \in H(N_M)^2 : f'(\xi) - \xi f(\xi) = M(\xi)\varphi - \varphi', \varphi, \varphi' \in \mathcal{S} \}.
\]

Observe that

\[
\{g, g'\} \in (T_M)^* \iff \left\{ \begin{array}{c} g \\ 0 \end{array} \right, \left\{ \begin{array}{c} g' \\ 0 \end{array} \right\} \in \widetilde{A}^* = \widetilde{A} \iff \{g, g'\} \in S_M,
\]
which leads to 

\[(T_M)^* = (S_M)^*.\]

Hence, to conclude (4.2.19), it suffices to show that \(T_M\) is closed. To see this, assume that \(\{f_n, f_n'\} \in T_M\) converges in \(S_2(N_M)\)\(^2\) to \(\{f, f'\}\). Then there exists a sequence \(\{\varphi_n, \varphi'_n\} \in \mathbb{G}\) such that

\[f'_n(\xi) - \xi f_n(\xi) = M(\xi)\varphi_n - \varphi'_n, \quad \xi \in \mathbb{C} \setminus \mathbb{R}.\]

By Theorem 4.1.5 (iii), the point evaluation is continuous, so that

\[f_n(\xi) \to f(\xi) \quad \text{and} \quad f'_n(\xi) \to f'(\xi).\]

Hence,

\[M(\xi)\varphi_n - \varphi'_n \to f'(\xi) - \xi f(\xi)\]

for all \(\xi \in \mathbb{C} \setminus \mathbb{R}\). Taking \(\xi = \lambda_0\) and \(\xi = \tilde{\lambda}_0\) with \(\lambda_0 \in \mathbb{C} \setminus \mathbb{R}\) one sees that

\[(M(\lambda_0) - M(\tilde{\lambda}_0))\varphi_n \to f'(\lambda_0) - \lambda_0 f(\lambda_0) - (f'((\tilde{\lambda}_0) - \tilde{\lambda}_0 f(\tilde{\lambda}_0))\]

and using that \(\text{Im } M(\lambda_0)\) is boundedly invertible (since \(M\) is assumed to be uniformly strict), it follows that

\[\varphi_n \to \varphi \quad \text{and hence also} \quad \varphi'_n \to \varphi'\]

for some \(\varphi, \varphi' \in \mathbb{G}\). Therefore,

\[f'(\xi) - \xi f(\xi) = M(\xi)\varphi - \varphi', \quad \xi \in \mathbb{C} \setminus \mathbb{R}.\]

In other words, \(\{f, f'\} \in T_M\), and hence \(T_M\) is closed.

**Step 2.** The mappings in (4.2.20) form a boundary triplet for \((S_M)^*\). First note that they are single-valued. Indeed, assume that \(\{f, f'\} \in (S_M)^*\) is the trivial element, then \(M(\xi)\varphi - \varphi' = 0\) for the corresponding elements \(\varphi, \varphi' \in \mathbb{G}\) and all \(\xi \in \mathbb{C} \setminus \mathbb{R}\). Taking \(\xi = \lambda_0\) and \(\xi = \tilde{\lambda}_0\) with \(\lambda_0 \in \mathbb{C} \setminus \mathbb{R}\) and using the fact that \(\ker (\text{Im } M(\lambda_0)) = \{0\}\), one concludes that \(\varphi = 0\) and \(\varphi' = 0\).

To verify the abstract Green identity, let \(\hat{f} = \{f, f'\}, \hat{g} = \{g, g'\} \in (S_M)^*\). Then

\[f'(\xi) - \xi f(\xi) = M(\xi)\varphi - \varphi' \quad \text{and} \quad g'(\xi) - \xi g(\xi) = M(\xi)\psi - \psi',\]

for some \(\varphi, \varphi', \psi, \psi' \in \mathbb{G}\) and some \(\xi \in \mathbb{C} \setminus \mathbb{R}\); moreover it is clear that

\[\left\{ \left( \begin{array}{c} f \\ f' \end{array} \right), \left( \begin{array}{c} -\varphi' \\
\end{array} \right) \right\} = \left\{ \left( \begin{array}{c} g \\ g' \end{array} \right), \left( \begin{array}{c} -\psi' \\
\end{array} \right) \right\} \in \hat{A}.
\]

Since \(\hat{A}\) is self-adjoint and thus symmetric, one sees that

\[\langle f', g \rangle - \langle f, g' \rangle = (\varphi', \psi)_G - (\varphi, \psi')_G = (\Gamma_1 \hat{f}, \Gamma_0 \hat{g})_G - (\Gamma_0 \hat{f}, \Gamma_1 \hat{g})_G.\]
Thus, the abstract Green identity is satisfied. It remains to show that $\Gamma$ maps onto $\mathcal{G}^2$, which then implies that $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for $(S_M)^*$.

First, it will be shown that ran$\Gamma$ is dense in $\mathcal{G}^2$. Suppose that $\{(\alpha', \alpha)\} \in \mathcal{G}^2$ is orthogonal to ran$\Gamma$, that is, $(\alpha', \varphi) + (\alpha, \varphi') = 0$ for all $\{\varphi, \varphi'\} \in \text{ran} \Gamma$.

It follows that $\{(0, \alpha), (0, \alpha')\} \in \tilde{A}^* = \tilde{A}$ and hence $M(\xi)\alpha + \alpha' = 0$ for all $\xi \in \mathbb{C} \setminus \mathbb{R}$. Now as above one concludes that $\alpha = 0$ and $\alpha' = 0$. Therefore, ran$\Gamma$ is dense in $\mathcal{G}^2$.

Next, it will be shown that ran$\Gamma$ is closed. For this consider again the self-adjoint relation $\tilde{A}$ as a subspace of $(\mathcal{H}(N_M) \oplus \mathcal{G})^2$ and define the orthogonal projections $P$ and $I - P$ by

$$P : \left( \begin{pmatrix} \mathcal{H}(N_M) \mathcal{G} \\ \mathcal{G} \end{pmatrix} \right) \times \left( \begin{pmatrix} \mathcal{H}(N_M) \mathcal{G} \\ \mathcal{G} \end{pmatrix} \right) \to \left( \begin{pmatrix} \mathcal{H}(N_M) \{0\} \\ \{0\} \mathcal{G} \end{pmatrix} \right) \times \left( \begin{pmatrix} \mathcal{H}(N_M) \{0\} \\ \{0\} \mathcal{G} \end{pmatrix} \right),$$

and

$$I - P : \left( \begin{pmatrix} \mathcal{H}(N_M) \mathcal{G} \\ \mathcal{G} \end{pmatrix} \right) \times \left( \begin{pmatrix} \mathcal{H}(N_M) \mathcal{G} \\ \mathcal{G} \end{pmatrix} \right) \to \left( \begin{pmatrix} \{0\} \mathcal{G} \\ \mathcal{G} \end{pmatrix} \right) \times \left( \begin{pmatrix} \{0\} \mathcal{G} \\ \mathcal{G} \end{pmatrix} \right),$$

respectively. Then $P\tilde{A} = T_M = (S_M)^*$ is closed and hence, by Lemma C.4,

$$\tilde{A} \hat{\oplus} \ker P = \tilde{A} \hat{\oplus} \left( \begin{pmatrix} \{0\} \mathcal{G} \\ \mathcal{G} \end{pmatrix} \right)$$

is closed. Since $\tilde{A}$ is self-adjoint, it follows from this and (1.3.5) that

$$\tilde{A} \hat{\oplus} \left( \begin{pmatrix} \mathcal{H}(N_M) \mathcal{G} \\ \mathcal{G} \end{pmatrix} \right) \times \left( \begin{pmatrix} \mathcal{H}(N_M) \mathcal{G} \\ \mathcal{G} \end{pmatrix} \right) = \tilde{A} \hat{\oplus} \ker (I - P)$$

is closed. By Lemma C.4, the relation $(I - P)\tilde{A}$ is closed. Observe that $(I - P)\tilde{A}$ is given by

$$\{(\varphi, -\varphi') \in \mathcal{G}^2 : f'(\xi) - \xi f(\xi) = M(\xi)\varphi - \varphi', f, f' \in \mathcal{H}(N_M)\}$$

and hence (4.2.23) is closed. In other words, ran$(\Gamma_0, -\Gamma_1)^\top$ is closed or, equivalently, ran$(\Gamma_0, \Gamma_1)^\top$ is closed.

**Step 3.** The $\gamma$-field and Weyl function corresponding to the boundary triplet (4.2.20) are given by $\gamma$ in (4.2.21) and $M$, respectively, and the symmetric operator $S_M$ in (4.2.18) is simple.

To establish the assertion about $M$ being the Weyl function, fix $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and let $f_\lambda \in \mathcal{H}(S_M)^*$. Then $f'_\lambda(\xi) = \lambda f_\lambda(\xi)$ for all $\xi \in \mathbb{C} \setminus \mathbb{R}$, and by (4.2.19)

$$(\lambda - \xi) f_\lambda(\xi) = M(\xi) \varphi_\lambda - \varphi'_\lambda, \quad \xi \in \mathbb{C} \setminus \mathbb{R},$$

(4.2.24)
where \(\varphi_\lambda = \Gamma_0 \hat{f}_\lambda\) and \(\varphi'_\lambda = \Gamma_1 \hat{f}_\lambda\). The choice \(\xi = \lambda\) in (4.2.24) shows \(M(\lambda)\varphi_\lambda = \varphi'_\lambda\) and hence

\[M(\lambda)\Gamma_0 \hat{f}_\lambda = \Gamma_1 \hat{f}_\lambda.\]

As this is true for all \(\hat{f}_\lambda \in \hat{\mathfrak{H}}(\mathcal{S}_M)^*\) and all \(\lambda \in \mathbb{C} \setminus \mathbb{R}\), one concludes that \(M\) is the Weyl function corresponding to the boundary triplet (4.2.20).

To compute the \(\gamma\)-field and to show that \(\mathcal{S}_M\) is simple, assume again that \(\hat{f}_\lambda \in \hat{\mathfrak{H}}(\mathcal{S}_M)^*\). Then (4.2.24) with \(\xi \neq \lambda\) implies that

\[f_\lambda(\xi) = \frac{M(\xi)\varphi_\lambda - \varphi'_\lambda}{\lambda - \xi} = -\frac{M(\xi) - M(\lambda)}{\xi - \lambda}\varphi_\lambda = -\mathcal{N}_M(\xi, \lambda)\varphi_\lambda.\]

Hence, the \(\gamma\)-field corresponding to the boundary triplet (4.2.20) is given by (4.2.21), and since the elements \(f_\lambda(\cdot) = \mathcal{N}_M(\cdot, \lambda)\varphi_\lambda \in \ker((\mathcal{S}_M)^* - \lambda)\) form a dense set in the Hilbert space \(\mathfrak{H}(\mathcal{N}_M)\) (see Theorem 4.1.5 and Corollary 4.1.7), it follows from Corollary 3.4.5 that \(\mathcal{S}_M\) is simple. \(\square\)

**Corollary 4.2.5.** Let \(\{\mathfrak{S}, \mathfrak{G}_0, \Gamma_1\}\) be the boundary triplet from Theorem 4.2.4 for \((\mathcal{S}_M)^*\). Then

\[A_0 = \ker \Gamma_0 = \{\{f, f'\} \in \mathfrak{H}(\mathcal{N}_M)^2 : f'(\xi) - \xi f(\xi) = \varphi', \varphi' \in \mathfrak{G}\}\]

and

\[A_1 = \ker \Gamma_1 = \{\{f, f'\} \in \mathfrak{H}(\mathcal{N}_M)^2 : f'(\xi) - \xi f(\xi) = M(\xi)\varphi, \varphi \in \mathfrak{G}\}\]

are self-adjoint relations in \(\mathfrak{H}(\mathcal{N}_M)\).

It follows from Theorem 4.2.4 that \(\text{mul} ((\mathcal{S}_M)^*)\) is the linear space spanned by all linear combinations \(M(\cdot)\varphi - \varphi', \varphi, \varphi' \in \mathfrak{G}\), which belong to the Hilbert space \(\mathfrak{H}(\mathcal{N}_M)\). Likewise, it follows from Corollary 4.2.5 that \(\text{mul} A_0\) consists of all constant functions in \(\mathfrak{H}(\mathcal{N}_M)\), while \(\text{mul} A_1\) consists of all linear combinations \(M(\cdot)\varphi, \varphi \in \mathfrak{G}\), which belong to the Hilbert space \(\mathfrak{H}(\mathcal{N}_M)\).

The construction of the boundary triplet in Theorem 4.2.4 is unique up to unitary equivalence. More precisely, if \(\mathcal{S}\) is a simple symmetric operator in a Hilbert space \(\mathfrak{H}\) and there is a boundary triplet for \(\mathcal{S}^*\) with the same Weyl function \(M\) as in Theorem 4.2.4, then the boundary triplets are unitarily equivalent in the sense of Definition 2.5.14 (where \(\mathfrak{G} = \mathfrak{G}'\)). This is a consequence of the following general equivalence result, which is a further specification of Theorem 4.2.3.

**Theorem 4.2.6.** Let \(\mathcal{S}\) and \(\mathcal{S}'\) be closed simple symmetric operators in Hilbert spaces \(\mathfrak{H}\) and \(\mathfrak{H}'\), respectively. Let \(\{\mathfrak{G}, \Gamma_0, \Gamma_1\}\) and \(\{\mathfrak{G}', \Gamma'_0, \Gamma'_1\}\) be boundary triplets for \(\mathcal{S}^*\) and \((\mathcal{S}')^*\) with \(\gamma\)-fields \(\gamma\) and \(\gamma'\), respectively. Assume that the corresponding Weyl functions \(M\) and \(M'\) coincide. Then the boundary triplets \(\{\mathfrak{G}, \Gamma_0, \Gamma_1\}\) and \(\{\mathfrak{G}', \Gamma'_0, \Gamma'_1\}\) are unitarily equivalent by means of a unitary operator \(U : \mathfrak{H} \rightarrow \mathfrak{H}'\) which is determined by the property

\[U \gamma(\lambda) = \gamma'(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.\]  

(4.2.25)
Proof. The basic idea of the proof follows the proof of Theorem 4.2.3. By assumption, the Weyl functions $M$ and $M'$ of the boundary triplets $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ and $\{\mathcal{G}, \Gamma_0', \Gamma_1'\}$ coincide. It follows from Proposition 2.3.6 (iii) that the corresponding $\gamma$-fields $\gamma$ and $\gamma'$ satisfy the identity

$$
\gamma(\mu)^* \gamma(\lambda) = \frac{M(\lambda) - M(\mu)^*}{\lambda - \mu} = \frac{M'(\lambda) - M'(\mu)^*}{\lambda - \mu} = \gamma'(\mu)^* \gamma'(\lambda)
$$

(4.2.26)

for all $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, $\lambda \neq \mu$, and $\gamma(\lambda)^* \gamma(\lambda) = \gamma'(\lambda)^* \gamma'(\lambda)$ follows by continuity. Define the linear relation $U$ from $\mathcal{H}$ to $\mathcal{H}'$ as the linear set of all pairs of the form

$$\left\{ \sum_{j=1}^{n} \alpha_j \gamma(\lambda_j) \varphi_j, \sum_{j=1}^{n} \alpha_j \gamma'(\lambda_j) \varphi_j \right\},$$

where $\varphi_j \in \mathcal{G}$, $\alpha_j \in \mathbb{C}$, $\lambda_j \in \mathbb{C} \setminus \mathbb{R}$ for $j = 1, \ldots, n$, and $n \in \mathbb{N}$ are arbitrarily chosen. It is clear from the definition of $U$ that its domain is given by

$$\text{dom } U = \text{span} \left\{ \text{ran } \gamma(\lambda) : \lambda \in \mathbb{C} \setminus \mathbb{R} \right\}
= \text{span} \left\{ \ker ((S)^* - \lambda) : \lambda \in \mathbb{C} \setminus \mathbb{R} \right\},$$

and its range is given by

$$\text{ran } U = \text{span} \left\{ \text{ran } \gamma'(\lambda) : \lambda \in \mathbb{C} \setminus \mathbb{R} \right\}
= \text{span} \left\{ \ker ((S')^* - \lambda) : \lambda \in \mathbb{C} \setminus \mathbb{R} \right\},$$

which are dense in $\mathcal{H}$ and $\mathcal{H}'$, respectively, since $S$ and $S'$ are both simple by assumption; cf. Definition 3.4.3 and Corollary 3.4.5. From (4.2.26) it follows that the relation $U$ is isometric; hence, it is a well-defined isometric operator. Therefore, $U$ extends by continuity to a unitary operator from $\mathcal{H}$ to $\mathcal{H}'$, denoted again by $U$. From

$$U \gamma(\lambda) \varphi = \gamma'(\lambda) \varphi, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad \varphi \in \mathcal{G},
$$

(4.2.27)

it follows that for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the restriction $U : \ker (S^* - \lambda) \rightarrow \ker ((S')^* - \lambda)$ is unitary. Thus, (4.2.27) implies by Proposition 2.3.6 (ii) that

$$\Gamma \{ \gamma(\lambda) \varphi, \lambda \gamma(\lambda) \varphi \} = \{ \varphi, M(\lambda) \varphi \}
= \{ \varphi, M'(\lambda) \varphi \}
= \Gamma' \{ \gamma'(\lambda) \varphi, \lambda \gamma'(\lambda) \varphi \}
= \Gamma' \{ U \gamma(\lambda) \varphi, \lambda U \gamma(\lambda) \varphi \},$$

and, in particular,

$$\Gamma_0 \{ \gamma(\lambda) \varphi, \lambda \gamma(\lambda) \varphi \} = \Gamma_0' \{ U \gamma(\lambda) \varphi, \lambda U \gamma(\lambda) \varphi \},
\Gamma_1 \{ \gamma(\lambda) \varphi, \lambda \gamma(\lambda) \varphi \} = \Gamma_1' \{ U \gamma(\lambda) \varphi, \lambda U \gamma(\lambda) \varphi \},
$$

(4.2.28)

for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $\varphi \in \mathcal{G}$. 

Now let $A_0 = \ker \Gamma_0$ and $A'_0 = \ker \Gamma'_0$. Then the property (4.2.27) and Proposition 2.3.2 (ii) imply

$$U \left( I + (\lambda - \mu)(A_0 - \lambda)^{-1} \right) \gamma(\mu) = U \gamma(\lambda)$$

$$= \gamma'(\lambda)$$

$$= (I + (\lambda - \mu)(A'_0 - \lambda)^{-1}) \gamma'(\mu)$$

$$= (I + (\lambda - \mu)(A'_0 - \lambda)^{-1}) U \gamma(\mu)$$

for all $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$, and hence

$$U(A_0 - \lambda)^{-1} \gamma(\mu) = (A'_0 - \lambda)^{-1} U \gamma(\mu), \quad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}.$$

Since $S$ is simple, one sees that span \{ran $\gamma(\mu) : \mu \in \mathbb{C} \setminus \mathbb{R}$\} is a dense subspace of $\mathcal{H}$ and hence

$$U(A_0 - \lambda)^{-1} = (A'_0 - \lambda)^{-1} U, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

and

$$U(A_0 - \lambda)^{-1} U^* = (A'_0 - \lambda)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad (4.2.29)$$

follow. Therefore, by Lemma 1.3.8, the self-adjoint relations $A'_0$ and $A_0$ are unitarily equivalent, that is,

$$A'_0 = \{ \{U f, U f'\} : \{f, f'\} \in A_0 \}. \quad (4.2.30)$$

This immediately yields

$$\Gamma'_0 \{U f, U f'\} = 0 \quad \text{and} \quad \Gamma_0 \{f, f'\} = 0, \quad \{f, f'\} \in A_0. \quad (4.2.31)$$

Furthermore, each $\{f, f'\} \in A_0$ can be written as

$$\{f, f'\} = \{(A_0 - \lambda)^{-1} U^* g, (I + \lambda(A_0 - \lambda)^{-1}) U^* g\}$$

for some $g \in \mathcal{H}$, so that by means of (4.2.29), Proposition 2.3.2 (iv), and (4.2.27) one obtains that

$$\Gamma'_1 \{U f, U f'\} = \Gamma'_1 \left\{ (A_0 - \lambda)^{-1} U^* g, U(I + \lambda(A_0 - \lambda)^{-1}) U^* g \right\}$$

$$= \Gamma'_1 \{ (A'_0 - \lambda)^{-1} g, (I + \lambda(A'_0 - \lambda)^{-1}) g \}$$

$$= \gamma'(\lambda)^* g$$

$$= \gamma(\lambda)^* U^* g$$

$$= \Gamma_1 \left\{ (A_0 - \lambda)^{-1} U^* g, (I + \lambda(A_0 - \lambda)^{-1}) U^* g \right\}.$$

Therefore,

$$\Gamma'_1 \{U f, U f'\} = \Gamma_1 \{f, f'\}, \quad \{f, f'\} \in A_0. \quad (4.2.32)$$

To see that the boundary triplets are unitarily equivalent, first recall that $A_0$ and $A'_0$ are unitarily equivalent, see (4.2.30), and that

$$\tilde{\mathcal{R}}_\lambda((S')^*) = \{ \{U f_\lambda, \lambda U f_\lambda\} : \{f_\lambda, \lambda f_\lambda\} \in S^* \};$$
4.2. Realization of uniformly strict Nevanlinna functions

The direct sum decompositions

\[(S')^* = A_0' \oplus \mathfrak{N}_\lambda((S')^*) \quad \text{and} \quad S^* = A_0 \oplus \mathfrak{N}_\lambda(S^*) \quad (4.2.33)\]

for \(\lambda \in \rho(A_0) = \rho(A_0')\) from Theorem 1.7.1 now show that

\[(S')^* = \{\{Uf, Uf'\} : \{f, f'\} \in S^*\}, \quad (4.2.34)\]

so that \(S^*\) and \((S')^*\) are unitarily equivalent. It follows from (4.2.33) and the equalities (4.2.28), (4.2.31), and (4.2.32) that

\[\Gamma_0'\{Uf, Uf'\} = \Gamma_0\{f, f'\} \quad \text{and} \quad \Gamma_1'\{Uf, Uf'\} = \Gamma_1\{f, f'\}, \quad \{f, f'\} \in S^*. \quad (4.2.35)\]

According to Theorem 2.5.1 and Proposition 2.5.3 every operator matrix

\[W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \in B(G \times G, G \times G) \quad (4.2.36)\]

with the properties (2.5.1) gives rise to a boundary triplet \(\{\mathcal{G}, (\Gamma_M)_0, (\Gamma_M)_1\}\) for \((S_M)^*\) via \((\Gamma_M)' = W\Gamma_M\), that is,

\[\begin{pmatrix} (\Gamma_M)_0' \\ (\Gamma_M)_1' \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} (\Gamma_M)_0 \\ (\Gamma_M)_1 \end{pmatrix}, \quad (4.2.37)\]

and the corresponding \(\gamma\)-field and Weyl function are then given by

\[\gamma'(\lambda) = \gamma(\lambda)(W_{11} + W_{12}M(\lambda))^{-1} \quad (4.2.38)\]

and

\[M'(\lambda) = (W_{21} + W_{22}M(\lambda))(W_{11} + W_{12}M(\lambda))^{-1}. \quad (4.2.39)\]

The function \(M' = W[M]\) in (4.2.39), being a Weyl function, is a uniformly strict \(B(G)\)-valued Nevanlinna function. Let \(\mathcal{F}(N_{M'})\) be the associated reproducing kernel Hilbert space. Then according to Theorem 4.2.4

\[S_{M'} = \{\{F, F'\} \in \mathcal{F}(N_{M'})^2 : F'(\xi) = \xi F(\xi)\}\]

is a closed simple symmetric operator in \(\mathcal{F}(N_{M'})\) and its adjoint is given by

\[(S_{M'})^* = \{\{F, F'\} \in \mathcal{F}(N_{M'})^2 : F'(\xi) - \xi F(\xi) = M'(\xi)\psi - \psi', \psi, \psi' \in \mathcal{G}\}. \quad (4.2.40)\]
The corresponding boundary triplet \( \{ \mathcal{S}, (\Gamma_{M'})_0, (\Gamma_{M'})_1 \} \) is given by
\[
(\Gamma_{M'})_0 \hat{F} = \psi, \quad (\Gamma_{M'})_1 \hat{F} = \psi', \quad \hat{F} = \{ F, F' \} \in (S_{M'})^*,
\]
and according to Theorem 4.2.4 the corresponding Weyl function is \( M' \). In the next proposition it will explained how this model for \( M' \) is connected with the space \( \mathcal{S}(\mathbb{N}_M) \) and the transformed boundary triplet \( \{ \mathcal{S}, (\Gamma_{M'})_0, (\Gamma_{M'})_1 \} \) for \( (S_{M'})^* \) in (4.2.37). The unitary map \( \Phi \) in (4.2.43) below provides the unitary equivalence between the boundary triplets in the sense of Theorem 4.2.6.

**Proposition 4.2.7.** Let \( M \) be the Weyl function in Theorem 4.2.4 with boundary triplet \( \{ \mathcal{S}, (\Gamma_M)_0, (\Gamma_M)_1 \} \) and let \( W \) be of the form (4.2.36) with the properties in (2.5.1) so that \( \{ \mathcal{S}, (\Gamma_M)_0, (\Gamma_M)_1 \} \) in (4.2.37) is a boundary triplet for \( (S_M)^* \) with corresponding Weyl function \( M' \). Then the kernels \( N_M \) and \( N_{M'} \) are connected via
\[
N_{M'}(\lambda, \mu) = \Phi(\lambda)N_M(\lambda, \mu)\Phi(\mu)^*,
\]
where \( \Phi: \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{S}) \) is a holomorphic function given by
\[
\Phi(\lambda) = (W_{11} + M(\lambda)W_{12})^{-1}.
\]
Furthermore, \( \hat{f} = \{ f, f' \} \in (S_M)^* \) if and only if \( \hat{F} = \{ \Phi f, \Phi f' \} \in (S_{M'})^* \), and the boundary triplets in (4.2.37) and (4.2.41) are connected via
\[
(\Gamma_{M'})_0 \hat{f} = (\Gamma_{M'})_0 \hat{F} \quad \text{and} \quad (\Gamma_{M'})_1 \hat{f} = (\Gamma_{M'})_1 \hat{F}
\]
for \( \hat{f} \in (S_M)^* \) and \( \hat{F} = \{ \Phi f, \Phi f' \} \in (S_{M'})^* \).

**Proof.** To establish (4.2.42), note that
\[
N_{M'}(\lambda, \mu) = \frac{M'(\lambda) - M'(\mu)^*}{\lambda - \bar{\mu}} = \frac{M'(\bar{\mu}) - M'(\bar{\lambda})^*}{\bar{\mu} - \lambda} = \gamma'(\bar{\lambda})^*\gamma'(\bar{\mu}),
\]
and hence, in view of (4.2.38) and (4.2.43),
\[
N_{M'}(\lambda, \mu) = (W_{11} + M(\lambda)W_{12})^{-1} \gamma(\bar{\lambda})^*\gamma(\bar{\mu})(W_{11} + W_{12}M(\bar{\mu}))^{-1}
= \Phi(\lambda)\gamma(\bar{\lambda})^*\gamma(\bar{\mu})\Phi(\mu)^*
= \Phi(\lambda)N_M(\lambda, \mu)\Phi(\mu)^*
\]
for all \( \lambda, \mu \in \mathbb{C} \setminus \mathbb{R} \).

Therefore, according to Proposition 4.1.9, each \( \{ F, F' \} \in \mathcal{H}(N_{M'})^2 \) is of the form
\[
\{ F, F' \} = \{ \Phi f, \Phi f' \}, \quad \{ f, f' \} \in \mathcal{H}(N_M)^2,
\]
and conversely. Let \( \{ F, F' \} \in \mathcal{H}(N_{M'})^2 \) and \( \{ f, f' \} \in \mathcal{H}(N_M)^2 \) be connected by (4.2.45), then
\[
F'(\xi) - \xi F(\xi) = M'(\xi)\psi - \psi' \iff f'(\xi) - \xi f(\xi) = M(\xi)\varphi - \varphi',
\]
where $\varphi, \varphi' \in \mathcal{S}$ and $\psi, \psi' \in \mathcal{S}$ are related by
\[
\begin{pmatrix} \varphi' \\ \varphi \end{pmatrix} = W^{-1} \begin{pmatrix} \psi \\ \psi' \end{pmatrix} = \begin{pmatrix} W^*_{22} & -W^*_{12} \\ -W^*_{21} & W^*_{11} \end{pmatrix} \begin{pmatrix} \psi \\ \psi' \end{pmatrix}.
\tag{4.2.47}
\]

In fact, if $F'(\xi) - \xi F(\xi) = M'(\xi)\psi - \psi'$, then it follows from (4.2.45), (4.2.39), and (4.2.43) that
\[
f'(\xi) - \xi f(\xi) = \Phi(\xi)^{-1}(F'(\xi) - \xi F(\xi))
= \Phi(\xi)^{-1}(M'(\xi)\psi - \psi')
= \Phi(\xi)^{-1}(M'\xi)^*\psi - \psi'
= \Phi(\xi)^{-1}((W^*_{11} + M(\xi)W^*_{12})^{-1}(W^*_{11} + M(\xi)W^*_{12})\psi - \psi')
= (W^*_{21} + M(\xi)W^*_{22})\psi - (W^*_{11} + M(\xi)W^*_{12})\psi'
= M(\xi)(W^*_{21}\psi - W^*_{12}\psi') - (-W^*_{21}\psi + W^*_{11}\psi')
= M(\xi)\varphi - \varphi',
\]

where (4.2.47) was used in the last equality. Conversely, if $\{f, f'\} \in \mathcal{S}(N_M)^2$ and $f'(\xi) - \xi f(\xi) = M(\xi)\varphi - \varphi'$, then a similar computation shows that $\{F, F'\}$ in (4.2.45) satisfy $F'(\xi) - \xi F(\xi) = M'(\xi)\psi - \psi'$ with $\psi, \psi'$ from (4.2.47).

Comparing (4.2.19) and (4.2.40), it follows from the equivalence (4.2.46) that
\[
(S_M)^* = \{ \{\Phi f, \Phi f'\} : \{f, f'\} \in (S_M)^* \}.
\]
Moreover, from (4.2.35) and the model in Theorem 4.2.4 one then concludes
\[
(\Gamma_M)_{0}\hat{f} = \varphi = W^*_{22}\psi - W^*_{12}\psi'
\text{ and } (\Gamma_M)_{1}\hat{f} = \varphi' = -W^*_{21}\psi + W^*_{11}\psi'
\]
for $\hat{f} \in (S_M)^*$, that is,
\[
\begin{pmatrix} (\Gamma_M)_{0}\hat{f} \\ (\Gamma_M)_{1}\hat{f} \end{pmatrix} = \begin{pmatrix} W^*_{22} & -W^*_{12} \\ -W^*_{21} & W^*_{11} \end{pmatrix} \begin{pmatrix} \psi \\ \psi' \end{pmatrix}, \quad \hat{f} \in (S_M)^*.
\]
or, equivalently,
\[
\begin{pmatrix} W^*_{11} & W^*_{12} \\ W^*_{21} & W^*_{22} \end{pmatrix} \begin{pmatrix} (\Gamma_M)_{0}\hat{f} \\ (\Gamma_M)_{1}\hat{f} \end{pmatrix} = \begin{pmatrix} \psi \\ \psi' \end{pmatrix}, \quad \hat{f} \in (S_M)^*;
\]
\cf{2.5.1}. The result (4.2.44) now follows from (4.2.37) and (4.2.41). \qed

Finally, note that in Theorem 4.2.2 a model was constructed for a $B(\mathcal{S})$-valued Nevanlinna function $M$. Under the extra condition that the Nevanlinna function $M$ is uniformly strict, Theorem 4.2.4 provides by means of this model a boundary triplet for $(S_M)^*$ for which $M$ is the Weyl function. Observe that with this boundary triplet the self-adjoint relation $\tilde{A}$ in Theorem 4.2.2 in the Hilbert
space $\mathcal{H}(N_M) \oplus \mathcal{S}$ can be written by means of (4.2.19) and (4.2.20) in the alternative way
\[
\tilde{A} = \left\{ \left( \begin{array}{c} f \\ \Gamma_0 \hat{f} \end{array} \right), \left( \begin{array}{c} f' \\ -\Gamma_1 \hat{f} \end{array} \right) \right\} : \hat{f} = \{f, f'\} \in (S_M)^* \right\}.
\]
This representation is in fact the counterpart of the observations concerning Weyl functions in Proposition 2.7.8.

### 4.3 Realization of scalar Nevanlinna functions via $L^2$-space models

In the case of a scalar Nevanlinna function $M$ one may also construct a minimal model via the corresponding integral representation
\[
M(\lambda) = \alpha + \beta \lambda + \int_{\mathbb{R}} \left( \frac{1}{t^2 - \lambda^2} - \frac{t}{1 + t^2} \right) d\sigma(t), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.3.1}
\]
Here the constants $\alpha$, $\beta$, and the nondecreasing function $\sigma$ satisfy
\[
\alpha \in \mathbb{R}, \quad \beta \geq 0, \quad \int_{\mathbb{R}} \frac{1}{1 + t^2} d\sigma(t) < \infty. \tag{4.3.2}
\]
It is a consequence of this representation that
\[
\text{Im} \ M(\lambda) = \beta + \int_{\mathbb{R}} \frac{1}{|t - \lambda|^2} d\sigma(t), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]
Therefore, a scalar Nevanlinna function $M$ is equal to the real constant $\alpha$ if and only if $M$ is not uniformly strict, i.e., if and only if $\text{Im} \ M(\lambda) = 0$ for some, and hence for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Under the assumption that the Nevanlinna function is not constant, a model involving the integral representation is constructed in this section. Moreover, a concrete natural isomorphism between the new model space and the reproducing kernel Hilbert space $\mathcal{H}(N_M)$ in Theorem 4.2.4 will be given.

The new model is build in the Hilbert space $L^2_{d\sigma}(\mathbb{R})$ consisting of all (equivalence classes of) complex $d\sigma$-measurable functions $f$ such that $\int_{\mathbb{R}} |f|^2 d\sigma < \infty$, equipped with the scalar product
\[
(f, g)_{L^2_{d\sigma}(\mathbb{R})} := \int_{\mathbb{R}} f(t)\overline{g(t)} d\sigma(t), \quad f, g \in L^2_{d\sigma}(\mathbb{R}).
\]
The following observations will be used in the construction of the model. Under the integrability condition on $\sigma$ in (4.3.2) one has that
\[
\frac{t}{1 + t^2}, \quad \frac{1}{1 + t^2} \in L^2_{d\sigma}(\mathbb{R}), \tag{4.3.3}
\]
4.3. Realization of scalar Nevanlinna functions via $L^2$-space models

and, in addition,

$$f(t), tf(t) \in L^2_{d\sigma}(\mathbb{R}) \quad \Rightarrow \quad f(t) \in L^1_{d\sigma}(\mathbb{R}). \quad (4.3.4)$$

It is first assumed for convenience that the linear term in the integral representation (4.3.1) is absent, that is, $\beta = 0$. The general case $\beta \neq 0$ will be discussed afterwards in Theorem 4.3.4. The usual notation for general elements $\{f, f'\}$ will also be used here; the reader should be aware that $f'$ is not the derivative here.

**Theorem 4.3.1.** Let $M$ be a scalar Nevanlinna function of the form (4.3.1) with $\beta = 0$ and assume that $M$ is uniformly strict, that is, $M$ is not identically equal to a constant. Then

$$S = \left\{ \{f(t), tf(t)\} : f(t), tf(t) \in L^2_{d\sigma}(\mathbb{R}), \int_{\mathbb{R}} f(t) \, d\sigma(t) = 0 \right\} \quad (4.3.5)$$

is a closed simple symmetric operator in $L^2_{d\sigma}(\mathbb{R})$ and its adjoint is given by

$$S^* = \left\{ \{f(t), f'(t)\} : f(t), f'(t) \in L^2_{d\sigma}(\mathbb{R}), tf(t) - f'(t) = c \in \mathbb{C} \right\}. \quad (4.3.6)$$

Moreover, the mappings

$$\Gamma_0 \hat{f} = c \quad \text{and} \quad \Gamma_1 \hat{f} = \alpha c + \int_{\mathbb{R}} \frac{tf'(t) + f(t)}{1 + t^2} \, d\sigma(t), \quad \hat{f} = \{f, f'\} \in S^*, \quad (4.3.7)$$

are well defined and $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for $S^*$. The corresponding $\gamma$-field is given by the mapping

$$c \mapsto f_\lambda(t) = \frac{c}{t - \lambda} \in \ker(S^* - \lambda), \quad (4.3.8)$$

and the corresponding Weyl function is $M$.

**Proof.** The proof consists of two steps. In Step 1 it will be shown that $S$ in (4.3.5) is a closed symmetric operator and that $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for its adjoint $S^*$ in (4.3.6). Moreover, it will be shown that the $\gamma$-field is given by (4.3.8) and that $M$ is the corresponding Weyl function. In Step 2 the simplicity of $S$ is concluded from Corollary A.1.5.

**Step 1.** The right-hand side of (4.3.6) is a relation which satisfies the conditions in Theorem 2.1.9. To see this, denote the relation on the right-hand side of (4.3.6) by $T$ and think of $\Gamma_0$ and $\Gamma_1$ as being defined on $T$. Observe that the mapping $\Gamma_1$ is well defined thanks to (4.3.3). Likewise, the operator $S$ is well defined due to (4.3.4).

First, it is clear that $A_0 = \ker \Gamma_0 \subset T$ is the maximal multiplication operator by the independent variable in $L^2_{d\sigma}(\mathbb{R})$:

$$(A_0 f)(t) = tf(t), \quad \text{dom } A_0 = \{ f(t) \in L^2_{d\sigma}(\mathbb{R}) : tf(t) \in L^2_{d\sigma}(\mathbb{R}) \}.$$
Since $A_0$ is a self-adjoint operator in $L^2_{d\sigma}(\mathbb{R})$, one sees that condition (i) in Theorem 2.1.9 is satisfied.

Next, one has that $\Gamma$ is surjective. For this, note that, by (4.3.6), the defect subspace $\hat{N}_\lambda(T)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, of $T$ consists of elements of the form
\[
\hat{f}_\lambda = \left\{ \frac{c_\lambda}{t - \lambda}, \frac{\lambda c_\lambda}{t - \lambda} \right\} : c_\lambda \in \mathbb{C},
\]
which after a simple computation gives
\[
\Gamma_0 \hat{f}_\lambda = c_\lambda \quad \text{and} \quad \Gamma_1 \hat{f}_\lambda = M(\lambda)c_\lambda,
\]
or, in other words,
\[
\Gamma_1 \hat{f}_\lambda = M(\lambda)\Gamma_0 \hat{f}_\lambda. \tag{4.3.10}
\]

Using (4.3.10) one now observes that
\[
\begin{pmatrix}
\Gamma_0 (\hat{f}_\lambda c_\lambda + \hat{f}_\lambda c_\lambda) \\
\Gamma_1 (\hat{f}_\lambda c_\lambda + \hat{f}_\lambda c_\lambda)
\end{pmatrix} =
\begin{pmatrix}
c_\lambda + c_\lambda \\
M(\lambda)c_\lambda + M(\lambda)c_\lambda
\end{pmatrix}
\begin{pmatrix}
1 \\
M(\lambda)
\end{pmatrix}
\begin{pmatrix}
c_\lambda \\
c_\lambda
\end{pmatrix}.
\]
This shows that $\Gamma$ is surjective; just note that $\lambda \in \mathbb{C} \setminus \mathbb{R}$ implies that $\text{Im } M(\lambda) \neq 0$. Hence, condition (ii) in Theorem 2.1.9 is satisfied.

Finally, the abstract Green identity for $T$ and $\Gamma$ in Theorem 2.1.9 (iii) will be exhibited. For this purpose, let $\hat{f} = \{f, f'\}, \hat{g} = \{g, g'\} \in T$, and assume that $tf(t) - f'(t) = c$ and $tg(t) - g'(t) = d$ for some $c, d \in \mathbb{C}$. Then a calculation shows that
\[
\Gamma_1 \hat{f} \widehat{\Gamma_0 g} - \Gamma_0 \hat{f} \widehat{\Gamma_1 g}
\Rightarrow
\begin{align*}
&= \left( \alpha c + \int_\mathbb{R} \frac{tf'(t) + f(t)}{1 + t^2} \, d\sigma(t) \right) \widehat{d} - c \left( \alpha d + \int_\mathbb{R} \frac{tg'(t) + g(t)}{1 + t^2} \, d\sigma(t) \right) \\
&= \int_\mathbb{R} \frac{tf'(t) + f(t)}{1 + t^2} \, d\sigma(t) - c \left( \int_\mathbb{R} \frac{tg'(t) + g(t)}{1 + t^2} \, d\sigma(t) \right).
\end{align*}
\]
The last line gives, after substitution of $c$ and $d$,
\[
\begin{align*}
\int_\mathbb{R} & \frac{tf'(t) + f(t)}{1 + t^2} \left( tg(t) - g'(t) \right) \, d\sigma(t) - \int_\mathbb{R} \left( tf(t) - f'(t) \right) \frac{tg'(t) + g(t)}{1 + t^2} \, d\sigma(t) \\
&= \int_\mathbb{R} f'(t)g(t) \, d\sigma(t) - \int_\mathbb{R} f(t)g'(t) \, d\sigma(t) \\
&= (f, g)_{L^2_{d\sigma}(\mathbb{R})} - (f, g')_{L^2_{d\sigma}(\mathbb{R})}.
\end{align*}
\]
Hence, also condition (iii) in Theorem 2.1.9 is satisfied.
4.3. Realization of scalar Nevanlinna functions via $L^2$-space models

Therefore, all conditions of Theorem 2.1.9 have been verified. As a consequence the relation $\ker \Gamma_0 \cap \ker \Gamma_1$ is closed and symmetric. It coincides with $S$ in (4.3.5), since for $\hat{f} \in T$ one has

$$\Gamma_0 \hat{f} = \Gamma_1 \hat{f} = 0 \quad \text{if and only if} \quad f'(t) = tf(t) \quad \text{and} \quad \int_{\mathbb{R}} f(t) \, d\sigma(t) = 0.$$ 

Thus, it follows from Theorem 2.1.9 that the adjoint of the closed symmetric operator $S$ in (4.3.5) is given by $T$ and hence has the form (4.3.6), and, moreover, \{\mathbb{C}, \Gamma_0, \Gamma_1\} is a boundary triplet for $S^*$. As a byproduct of (4.3.9) and (4.3.10) one sees that the corresponding $\gamma$-field is given by (4.3.8) and that the corresponding Weyl function coincides with $M$.

**Step 2.** It remains to show that the operator $S$ in (4.3.5) is simple. To see this, assume that there is an element $g \in L^2_{d\sigma}(\mathbb{R})$ which is orthogonal to all elements $f_\lambda \in \ker (S^* - \lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, that is,

$$\int_{\mathbb{R}} \frac{1}{t - \lambda} g(t) \, d\sigma(t) = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$ 

Then $g = 0$ in $L^2_{d\sigma}(\mathbb{R})$ by Corollary A.1.5. Thus, the linear span of the defect spaces $\ker (S^* - \lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, is dense in $L^2_{d\sigma}(\mathbb{R})$ and now Corollary 3.4.5 implies that the symmetric operator $S$ is simple. This completes the proof of Theorem 4.3.1. $\square$

Note that in the model in Theorem 4.3.1 the self-adjoint extension $A_0$ is equal to the operator of multiplication by the independent variable. The closed minimal operator $S$ is not densely defined if and only if the constant functions belong to $L^2_{d\sigma}(\mathbb{R})$ or, equivalently, $\sigma$ is a finite measure.

According to Theorem 4.2.6 the $L^2_{d\sigma}(\mathbb{R})$-space model for the function $M$ and the model in Theorem 4.2.4 are unitarily equivalent thanks to the simplicity of the underlying symmetric operators. A concrete unitary map will be provided in the following proposition.

**Proposition 4.3.2.** Let $M$ be the Nevanlinna function in (4.3.1) with $\beta = 0$, and let $\mathcal{H}(N_M)$ be the associated reproducing kernel Hilbert space. Then the operator $V : L^2_{d\sigma}(\mathbb{R}) \rightarrow \mathcal{H}(N_M)$ defined by the rule

$$f \mapsto -\int_{\mathbb{R}} \frac{1}{t - \xi} f(t) \, d\sigma(t), \quad \xi \in \mathbb{C} \setminus \mathbb{R},$$

is unitary. Moreover, under this mapping the boundary triplets in Theorem 4.3.1 and Theorem 4.2.4 are unitarily equivalent.

**Proof.** It suffices to show that the operator in (4.3.11) satisfies (4.2.25); cf. Theorem 4.2.6. In fact, recall that the $\gamma$-field corresponding to the boundary triplet in Theorem 4.2.4 at a point $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is given by the mapping

$$c \mapsto -c N_M(\cdot, \bar{\lambda}) \in \ker ((S_M)^* - \lambda).$$

(4.3.12)
The $\gamma$-field corresponding to the boundary triplet in Theorem 4.3.1 at a point $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is given by the mapping
\[
c \mapsto f_\lambda(t) = \frac{c}{t - \lambda} \in \ker(S^* - \lambda).
\] (4.3.13)

It follows from the integral representation (4.3.1) with $\beta = 0$ that
\[
N_M(\xi, \lambda) = M(\xi) - M(\lambda) = \int_{\mathbb{R}} \frac{1}{t - \xi} \frac{1}{t - \lambda} d\sigma(t).
\] In view of (4.3.11) and (4.3.13), it follows that
\[
(Vf_\lambda)(\xi) = -\int_{\mathbb{R}} \frac{1}{t - \xi} \frac{c}{t - \lambda} d\sigma(t) = -c N_M(\xi, \lambda),
\]
and taking into account (4.3.12) one concludes that (4.2.25) is satisfied. Hence, Theorem 4.2.6 ensures that the operator $V$ in (4.3.11) is well defined and unitary, and the boundary triplets in Theorem 4.3.1 and Theorem 4.2.4 are unitarily equivalent. \hfill \Box

The special case of a rational Nevanlinna function serves as an illustration of Theorem 4.3.1. In this situation the measure $d\sigma$ in (4.3.1) has only finitely many point masses and the space $L^2_{d\sigma}(\mathbb{R})$ can be identified with $\mathbb{C}^n$.

**Example 4.3.3.** Let $\alpha_1 \in \mathbb{R}$, $n \in \mathbb{N}$, $\gamma_1, \ldots, \gamma_n > 0$, and
\[-\infty < \delta_1 < \delta_2 < \cdots < \delta_n < \infty,
\]
and consider the rational complex Nevanlinna function
\[
N(\lambda) = \alpha_1 + \sum_{i=1}^{n} \frac{\gamma_i}{\delta_i - \lambda}, \quad \lambda \neq \delta_i, \ i = 1, \ldots, n.
\] (4.3.14)

Define a nondecreasing step function $\sigma : \mathbb{R} \to [0, \infty)$ by
\[
\sigma(t) = \begin{cases} 0, & t \in (-\infty, \delta_1], \\
\gamma_1, & t \in (\delta_1, \delta_2], \\
\gamma_1 + \gamma_2, & t \in (\delta_2, \delta_3], \\
& \ldots \\
\gamma_1 + \gamma_2 + \cdots + \gamma_n, & t \in (\delta_n, \infty), \end{cases}
\]
and consider the corresponding $L^2$-space $L^2_{d\sigma}(\mathbb{R})$ with the scalar product
\[
(f, g)_{L^2_{d\sigma}(\mathbb{R})} = \int_{\mathbb{R}} f(t)\overline{g(t)} d\sigma(t) = \sum_{i=1}^{n} \gamma_i f(\delta_i)\overline{g(\delta_i)}.
\]
The rational Nevanlinna function $N$ in (4.3.14) admits the integral representation

$$N(\lambda) = \alpha + \int_{\mathbb{R}} \left( \frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) d\sigma(t)$$

as in (4.3.1), where

$$\alpha := \alpha_1 + \int_{\mathbb{R}} \frac{t}{1+t^2} d\sigma(t) = \alpha_1 + \sum_{i=1}^{n} \gamma_i \frac{\delta_i}{1+\delta_i^2}.$$

The Hilbert space $L_{d\sigma}^2(\mathbb{R})$ can be identified with $(\mathbb{C}^n, (\cdot, \cdot)_{\gamma})$, where

$$(x, y)_{\gamma} := \sum_{i=1}^{n} \gamma_i x_i \bar{y}_i, \quad x = (x_1, \ldots, x_n)^{\top}, \quad y = (y_1, \ldots, y_n)^{\top} \in \mathbb{C}^n,$$

via the unitary mapping

$$L_{d\sigma}^2(\mathbb{R}) \ni f \mapsto \begin{pmatrix} f(\delta_1) \\ \vdots \\ f(\delta_n) \end{pmatrix},$$

and the maximal operator of multiplication by the independent variable in $L_{d\sigma}^2(\mathbb{R})$ is unitarily equivalent to the diagonal matrix

$$\begin{pmatrix} \delta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_n \end{pmatrix}.$$  

(4.3.15)

As

$$\int_{\mathbb{R}} f(t) d\sigma(t) = \sum_{i=1}^{n} \gamma_i f(\delta_i) = \begin{pmatrix} f(\delta_1) \\ \vdots \\ f(\delta_n) \end{pmatrix},$$

the simple symmetric operator $S$ in Theorem 4.3.1 is unitarily equivalent to the restriction of the diagonal matrix in (4.3.15) to the orthogonal complement of the subspace span$(1, \ldots, 1)^{\top}$. Furthermore, $S^*$ corresponds to the relation

$$\{ \{ f, \tilde{f} \} \in \mathbb{C}^n \times \mathbb{C}^n : \delta_i f(\delta_i) - f'(\delta_i) = c \in \mathbb{C}, \ i = 1, \ldots, n \}$$

and the boundary triplet $\{ \mathbb{C}, \Gamma_0, \Gamma_1 \}$ in Theorem 4.3.1 is of the form

$$\Gamma_0 \tilde{f} = c, \quad \Gamma_1 \tilde{f} = \alpha c + \sum_{i=1}^{n} \frac{\delta_i f'(\delta_i) + f(\delta_i)}{1+\delta_i^2}, \quad \tilde{f} = \{ f, f' \} \in S^*.$$

According to Theorem 4.3.1, the corresponding Weyl function is the rational Nevanlinna function $N$ in (4.3.14).
Now the general case of a scalar Nevanlinna function of the form (4.3.1) with $\beta > 0$ will be addressed.

**Theorem 4.3.4.** Let $M$ be a scalar Nevanlinna function of the form (4.3.1) with $\beta > 0$. Then

$$S = \left\{ \left( \begin{array}{c} f(t) \\ tf(t) \\ h' \end{array} \right), f(t), tf(t) \in L^2_{d\sigma}(\mathbb{R}), \int_{\mathbb{R}} f(t) d\sigma(t) = -\beta^{1/2}h' \right\}$$

is a closed simple symmetric operator in $L^2_{d\sigma}(\mathbb{R}) \oplus \mathbb{C}$ and its adjoint is given by

$$S^* = \left\{ \left( \begin{array}{c} f(t) \\ f'(t) \\ h' \end{array} \right), f(t), f'(t) \in L^2_{d\sigma}(\mathbb{R}), h, h' \in \mathbb{C}, \right\}$$

Moreover, for $\hat{f} \in S^*$ the mappings

$$\Gamma_0 \hat{f} = \beta^{-1/2}h \quad \text{and} \quad \Gamma_1 \hat{f} = \alpha \beta^{-1/2}h + \int_{\mathbb{R}} tf'(t) + f(t) + 1 + t^2 d\sigma(t) + \beta^{1/2}h'$$

are well defined and $\{ \mathbb{C}, \Gamma_0, \Gamma_1 \}$ is a boundary triplet for $S^*$. The corresponding $\gamma$-field is given by the mapping

$$c \mapsto f_\lambda(t) = \left( \begin{array}{c} c \\ t - \lambda \\ \beta^{1/2}c \end{array} \right) \in \ker (S^* - \lambda) \quad (4.3.16)$$

and the corresponding Weyl function is given by $M$.

**Proof.** The proof is similar to the one of Theorem 4.3.1, thus only a brief sketch will be given. Denote the right-hand side of the formula for $S^*$ by $T$ and think of $\Gamma_0$ and $\Gamma_1$ as being defined on $T$. It is clear that $A_0 = \ker \Gamma_0 \subset T$ is the orthogonal componentwise sum of the maximal multiplication operator by the independent variable in $L^2_{d\sigma}(\mathbb{R})$ and the purely multivalued part $\{0\} \times \mathbb{C}$. Hence, $A_0$ is a self-adjoint relation. To show that $\Gamma$ is surjective, note that the defect subspace $\hat{\mathcal{N}}_\lambda(T)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, of $T$ consists of elements of the form

$$\hat{f}_\lambda = \left\{ \left( \begin{array}{c} c_\lambda \\ t - \lambda \\ \beta^{1/2}c_\lambda \end{array} \right), \left( \begin{array}{c} \lambda c_\lambda \\ t - \lambda \\ \lambda \beta^{1/2}c_\lambda \end{array} \right) \right\} \in \hat{\mathcal{N}}_\lambda(S^*),$$

which gives

$$\Gamma_0 \hat{f}_\lambda = c_\lambda \quad \text{and} \quad \Gamma_1 \hat{f}_\lambda = M(\lambda)c_\lambda.$$

Again, as in the proof of Theorem 4.3.1 it follows that $\Gamma$ is surjective. It can be checked by straightforward calculation as in the proof of Theorem 2.1.9 that the abstract Green identity is satisfied. Thus, by Theorem 2.1.9, one concludes that $T$ is the adjoint of the closed symmetric relation $S$ and that $\Gamma_0$ and $\Gamma_1$ define a
boundary triplet for $S^*$. Hence, the statements about the $\gamma$-field and the Weyl function follow.

To show the simplicity of $S$, assume that there is an element orthogonal to $\mathcal{H}_\lambda(S^*)$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, i.e., there exists an element $g \in L^2_{d\sigma}(\mathbb{R})$ and a constant $\gamma \in \mathbb{C}$ such that
$$\int_\mathbb{R} \frac{1}{t-\lambda} g(t) d\sigma(t) = \gamma, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$ 
For $\lambda = iy$ and $y \to \infty$ it follows that $\gamma = 0$ and hence Corollary A.1.5 implies $g = 0$. Therefore, the closed linear span of all $\mathcal{H}_\lambda(S^*)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, is equal to $L^2_{d\sigma}(\mathbb{R}) \oplus \mathbb{C}$, and, as a consequence, the closed symmetric operator $S$ is simple. \qed

In the situation of Theorem 4.3.4 one sees that $S$ is a nondensely defined operator and that $\text{mul} S^*$ is spanned by the vector
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in L^2_{d\sigma}(\mathbb{R}) \oplus \mathbb{C}. \quad (4.3.17)$$
Now $A_0$ is the only self-adjoint extension of $S$ which is multivalued: it is the orthogonal sum of multiplication by the independent variable in $L^2_{d\sigma}$ and the space spanned by (4.3.17).

**Proposition 4.3.5.** Let $M$ be the Nevanlinna function in (4.3.1) with $\beta > 0$ and let $\mathcal{H}(N_M)$ be the associated reproducing kernel Hilbert space. Then the operator $V : L^2_{d\sigma}(\mathbb{R}) \oplus \mathbb{C} \to \mathcal{H}(N_M)$ given by the rule
$$\begin{pmatrix} f \\ h \end{pmatrix} \mapsto -\beta^{1/2} h - \int_\mathbb{R} \frac{1}{t-\xi} f(t) d\sigma(t)$$
is unitary. Moreover, under this mapping the boundary triplets in Theorem 4.3.4 and Theorem 4.2.4 are unitarily equivalent.

**Proof.** The proof is similar to the one of Proposition 4.3.2 and will be sketched briefly. Recall that the $\gamma$-field corresponding to the boundary triplet in Theorem 4.2.4 at a point $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is given by the mapping
$$c \mapsto -cN_M(\cdot, \lambda) \in \ker ((S_M)^* - \lambda), \quad (4.3.18)$$
while the $\gamma$-field corresponding to the boundary triplet in Theorem 4.3.4 at a point $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is given by the mapping
$$c \mapsto f_\lambda(t) = \begin{pmatrix} c \\ \frac{t - \lambda}{\beta^{1/2} c} \end{pmatrix} \in \ker (S^* - \lambda). \quad (4.3.19)$$
It follows from the integral representation (4.3.1) that
$$N_M(\xi, \lambda) = \frac{M(\xi) - M(\lambda)}{\xi - \lambda} = \beta + \int_\mathbb{R} \frac{1}{t - \xi} \frac{1}{t - \lambda} d\sigma(t).$$
Hence, (4.3.18)–(4.3.19) and the fact that

\[(Vf_\lambda)(\xi) = -c\beta - \int_\mathbb{R} \frac{1}{t-\xi} \frac{c}{t-\lambda} \, d\sigma(t) = -c N_M(\xi, \lambda),\]

imply that the property (4.2.25) holds. This implies that the boundary triplets in Theorem 4.3.4 and Theorem 4.2.4 are unitarily equivalent.

□

In the following it is briefly explained how the self-adjoint multiplication operator in \(L^2_{d\sigma}(\mathbb{R})\) and the model discussed in this section (in the case \(\beta = 0\)) are connected with the spectral theory and the limit properties of the Weyl function in Chapter 3. For this assume that \(\sigma : \mathbb{R} \to \mathbb{R}\) is a nondecreasing function such that

\[\int_\mathbb{R} \frac{1}{1+t^2} \, d\sigma(t) < \infty\]

and consider the self-adjoint multiplication operator

\[(A_0f)(t) = tf(t), \quad \text{dom } A_0 = \{f \in L^2_{d\sigma}(\mathbb{R}) : t \mapsto tf(t) \in L^2_{d\sigma}(\mathbb{R})\}\]

in \(L^2_{d\sigma}(\mathbb{R})\). Then it is known from Example 3.3.7 that the spectrum \(\sigma(A_0)\) coincides with the set of growth points of the function \(\sigma\), see (3.2.1), and the same is true for the absolutely continuous part \(\sigma_{ac}\), singular continuous part \(\sigma_{sc}\), and singular part \(\sigma_s\) of \(\sigma\). On the other hand, the one-dimensional restriction

\[S = \left\{\{f(t), tf(t)\} : f(t), tf(t) \in L^2_{d\sigma}(\mathbb{R}), \int_\mathbb{R} f(t) \, d\sigma(t) = 0\right\}\]

of \(A_0\) in Theorem 4.3.1 is a closed simple symmetric operator in \(L^2_{d\sigma}(\mathbb{R})\) and \(\{C, \Gamma_0, \Gamma_1\}\) in (4.3.7) is a boundary triplet for \(S^*\) in (4.3.6) with \(A_0 = \ker \Gamma_0\) and corresponding Weyl function

\[M(\lambda) = \alpha + \int_\mathbb{R} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2}\right) \, d\sigma(t), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad (4.3.20)\]

where \(\alpha\) is an arbitrary real number in the definition of the boundary map \(\Gamma_1\) in (4.3.7). Hence, the results on the description of the spectrum of \(A_0\) via the limit properties of the Weyl function from Section 3.5 and Section 3.6 apply in the present situation. For example, Theorem 3.6.5 shows that

\[\sigma_{ac}(A_0) = \text{clos}_{ac} \left(\{x \in \mathbb{R} : 0 < \text{Im } M(x+i0) < \infty\}\right),\]

which is also clear from Theorem 3.2.6 (i), taking into account (3.1.25) and Corollary 3.1.8 (ii). Similar observations can be made for the other spectral subsets. In other words, in the special situation where \(A_0\) is the self-adjoint multiplication operator in \(L^2_{d\sigma}(\mathbb{R})\) the general description of the spectrum of \(A_0\) and its subsets in Chapter 3 in terms of the limit properties of the associated Weyl function in (4.3.20) agrees with Example 3.3.7.
4.4 Realization of Nevanlinna pairs and generalized resolvents

In this section the model from Section 4.2 for Nevanlinna functions will be extended to the general setting of Nevanlinna pairs and of generalized resolvents. As a byproduct the extended model leads to the Sz.-Nagy dilation theorem.

Let $\mathcal{G}$ be a Hilbert space and let $\{A, B\}$ be a Nevanlinna pair of $\mathcal{B}(\mathcal{G})$-valued functions; cf. Section 1.12. The associated Nevanlinna kernel $N_{A,B}$

$$N_{A,B}(\cdot, \cdot) : \Omega \times \Omega \to \mathcal{B}(\mathcal{G})$$

is defined on $\Omega = \mathbb{C} \setminus \mathbb{R}$ by

$$N_{A,B}(\lambda, \mu) = \frac{B(\overline{\lambda})^*A(\overline{\mu}) - A(\overline{\lambda})^*B(\overline{\mu})}{\lambda - \overline{\mu}}, \quad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}, \quad \lambda \neq \overline{\mu}, \quad (4.4.1)$$

and $N_{A,B}(\lambda, \overline{\lambda}) = B'(\overline{\lambda})^*A(\lambda) - A'(\overline{\lambda})^*B(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}$. Then clearly the kernel $N_{A,B}$ is symmetric. Recall that $\lambda \mapsto A(\lambda)$ and $\lambda \mapsto B(\lambda)$ are holomorphic mappings on $\mathbb{C} \setminus \mathbb{R}$. Hence,

$$\lambda \mapsto N_{A,B}(\lambda, \mu)$$

is holomorphic for each $\mu \in \mathbb{C} \setminus \mathbb{R}$, that is, the kernel $N_{A,B}$ is holomorphic. Moreover, it follows from (4.4.1) and Definition 1.12.3 that

$$N_{A,B}(\lambda, \lambda) = \frac{\text{Im}(A(\overline{\lambda})^*B(\lambda))}{\text{Im} \lambda} \geq 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$ 

In the next theorem it is shown that the kernel $N_{A,B}$ is, in fact, nonnegative on $\mathbb{C} \setminus \mathbb{R}$. Note also that the kernel $N_{A,B}$ is uniformly bounded on compact subsets of $\mathbb{C} \setminus \mathbb{R}$ since

$$\|N_{A,B}(\lambda, \lambda)\| \leq \frac{\|A(\overline{\lambda})^*\|\|B(\lambda)^*\|}{\|\text{Im} \lambda\|}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$ 

**Theorem 4.4.1.** Let $\{A, B\}$ be a Nevanlinna pair in $\mathcal{G}$. Then the kernel $N_{A,B}$ is nonnegative.

**Proof.** To see this, let $N$ be a uniformly strict $\mathcal{B}(\mathcal{G})$-valued Nevanlinna function and let $\varepsilon > 0$. Then $\varepsilon N$ is again a uniformly strict Nevanlinna function. Define the function $S_\varepsilon$ by

$$S_\varepsilon(\lambda) = -A(\lambda)((\varepsilon N(\lambda)A(\lambda) + B(\lambda))^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$ 

By Proposition 1.12.6, $S_\varepsilon$ is a Nevanlinna function. A calculation shows that the Nevanlinna kernel associated with the function $S_\varepsilon$ is of the form

$$N_{S_\varepsilon}(\lambda, \mu) = (\varepsilon N(\overline{\lambda})A(\overline{\lambda}) + B(\overline{\lambda}))^{-*} \cdot [N_{A,B}(\lambda, \mu) + \varepsilon A(\overline{\lambda})^*N(\lambda, \mu)A(\overline{\mu})] (\varepsilon N(\overline{\mu})A(\overline{\mu}) + B(\overline{\mu}))^{-1}. \quad (4.4.2)$$
Observe that for any $\varepsilon > 0$ the kernel $\mathbf{N}_S$ is nonnegative since $S_\varepsilon$ is a Nevanlinna function. The identity (4.4.2) shows that the kernel
\[
\mathbf{N}_{A,B}(\lambda, \mu) + \varepsilon A(\overline{\lambda})^* \mathbf{N}_{N}(\lambda, \mu)A(\overline{\mu})
\] (4.4.3)
is nonnegative for any $\varepsilon > 0$.

To show that the kernel $\mathbf{N}_{A,B}$ is nonnegative, assume the contrary, i.e., assume that $\mathbf{N}_{A,B}$ is not nonnegative. Then it follows from the definition of nonnegativity that there exist $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C} \setminus \mathbb{R}$, elements $\varphi_1, \ldots, \varphi_n \in \mathcal{S}$, and a vector $c \in \mathbb{C}^n$, such that
\[
((\mathbf{N}_{A,B}(\lambda_i, \lambda_j) \varphi_j, \varphi_i))_{i,j=1}^n c, c = -x < 0.
\]
Since $-x > 0$ and the kernel $\mathbf{N}_N$ is nonnegative, one can choose $\varepsilon > 0$ so small that
\[
0 \leq \varepsilon ((A(\overline{x})^* \mathbf{N}_N(\lambda_i, \lambda_j)A(\overline{\lambda}_j) \varphi_j, \varphi_i))_{i,j=1}^n c, c < -x.
\]
Combining these results one arrives at the inequality
\[
((\mathbf{N}_{A,B}(\lambda_i, \lambda_j) + \varepsilon A(\overline{x})^* \mathbf{N}_N(\lambda_i, \lambda_j)A(\overline{\lambda}_j)) \varphi_j, \varphi_i)_{i,j=1}^n c, c < 0,
\]
which contradicts the nonnegativity of the kernel in (4.4.3). Thus, the kernel $\mathbf{N}_{A,B}$ is nonnegative. \hfill \Box

Let $\{A, B\}$ be a Nevanlinna pair in $\mathcal{S}$. According to Theorem 4.1.5, with the nonnegative kernel $\mathbf{N}_{A,B}$ there is associated a Hilbert space of holomorphic $\mathcal{S}$-valued functions, which will be denoted by $\mathcal{S}(\mathbf{N}_{A,B})$, with inner product $\langle \cdot, \cdot \rangle$; cf. Section 4.1. Recall that the reproducing kernel property
\[
\langle f, \mathbf{N}_{A,B}(\cdot, \mu) \varphi \rangle = (f(\mu), \varphi)_{\mathcal{S}}, \quad \varphi \in \mathcal{S}, \quad \mu \in \mathbb{C} \setminus \mathbb{R},
\]
holds for all functions $f \in \mathcal{S}(\mathbf{N}_{A,B})$. The following realization result extends Theorem 4.2.2 to the case of Nevanlinna pairs. It follows from Theorem 4.2.3 that this construction is unique up to unitary equivalence. Note in this context that a Nevanlinna function $M$ always gives rise to a Nevanlinna pair $\{I, M\}$.

**Theorem 4.4.2.** Let $\{A, B\}$ be a Nevanlinna pair in $\mathcal{S}$ and let $\tau = \{A, B\}$ be the corresponding Nevanlinna family. Let $\mathcal{S}(\mathbf{N}_{A,B})$ be the associated reproducing kernel Hilbert space generated by $\{A, B\}$. Denote by $P_\mathcal{S}$ the orthogonal projection from $\mathcal{S}(\mathbf{N}_M) \oplus \mathcal{S}$ onto $\mathcal{S}$ and let $\iota_\mathcal{S}$ be the canonical embedding of $\mathcal{S}$ into $\mathcal{S}(\mathbf{N}_M) \oplus \mathcal{S}$. Then
\[
\mathcal{A}_{A,B} = \left\{ \left\{ \begin{pmatrix} f \\ \varphi \end{pmatrix}, \begin{pmatrix} f' \\ -\varphi' \end{pmatrix} \right\} : f, f' \in \mathcal{S}(\mathbf{N}_{A,B}), \varphi, \varphi' \in \mathcal{S}, \quad f'(\xi) - \xi f(\xi) = B(\overline{\xi})^* \varphi - A(\overline{\xi})^* \varphi' \right\}
\]
is a self-adjoint relation in the Hilbert space $\mathcal{S}(\mathbf{N}_{A,B}) \oplus \mathcal{S}$ and the compressed resolvent of $\mathcal{A}_{A,B}$ onto $\mathcal{S}$ is given by
\[
P_\mathcal{S}(\mathcal{A}_{A,B} - \lambda)^{-1} \iota_\mathcal{S} = - (\tau(\lambda) + \lambda)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (4.4.4)
\]
Furthermore, the self-adjoint relation $\tilde{A}_{A,B}$ satisfies the following minimality condition:

$$\mathcal{H}(N_{A,B}) \oplus \mathcal{G} = \text{span} \{ \mathcal{G}, \text{ran} (\tilde{A}_{A,B} - \lambda)^{-1} \mathcal{I}_G : \lambda \in \mathbb{C} \setminus \mathbb{R} \}. \quad (4.4.5)$$

**Proof.** The proof is almost the same as the proof of Theorem 4.2.2; therefore, only the main elements are recalled and the details are left to the reader.

**Step 1.** Use the Nevanlinna pair $\{A,B\}$ to define the auxiliary relation $B$ in $\mathcal{H}(N_{A,B}) \oplus \mathcal{G}$ by

$$B = \text{span} \left\{ \left( \begin{array}{c} N_{A,B}(\cdot, \overline{\mu})\varphi \\ -A(\mu)\varphi \\ B(\mu)\varphi \end{array} \right) : \varphi \in \mathcal{G}, \mu \in \mathbb{C} \setminus \mathbb{R} \right\}.$$

It is a direct computation to show that $B \subset \tilde{A}_{A,B}$. Likewise, by a similar computation one verifies that $B$ is symmetric in $\mathcal{H}(N_{A,B}) \oplus \mathcal{G}$. Observe that for $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$ one has

$$\text{ran} (B - \lambda_0) = \text{span} \left\{ \left( \begin{array}{c} (\mu - \lambda_0)N_{A,B}(\cdot, \overline{\mu})\varphi \\ (B(\mu) + \lambda_0A(\mu))\varphi \end{array} \right) : \varphi \in \mathcal{G}, \mu \in \mathbb{C} \setminus \mathbb{R} \right\}.$$

Therefore, choosing $\mu = \lambda_0$ and taking into account that $\text{ran} (B(\lambda_0) + \lambda_0A(\lambda_0)) = \mathcal{G}$ by Definition 1.12.3 and Lemma 1.12.5 it follows that $\{0\} \oplus \mathcal{G} \subset \text{ran} (B - \lambda_0)$; hence also the elements of the form

$$\left( \begin{array}{c} N_{A,B}(\cdot, \overline{\mu})\varphi \\ 0 \end{array} \right), \quad \varphi \in \mathcal{G}, \mu \in \mathbb{C} \setminus \mathbb{R}, \mu \neq \lambda_0,$$

belong to $\text{ran} (B - \lambda_0)$. It follows from Corollary 4.1.7 that $\text{ran} (B - \lambda_0)$ is dense in $\mathcal{H}(N_{A,B}) \oplus \mathcal{G}$, and thus $B$ is essentially self-adjoint.

**Step 2.** One verifies in the same way as in the proof of Theorem 4.2.2 that $\tilde{A}_{A,B}$ is closed and that $\tilde{A}_{A,B} \subset B^*$. Since $B \subset \tilde{A}_{A,B}$ and $B$ is self-adjoint, it follows that $\tilde{A}_{A,B}$ is self-adjoint in $\mathcal{H}(N_{A,B}) \oplus \mathcal{G}$.

**Step 3.** The statement (4.4.4) follows in a similar way as in Theorem 4.2.2. In fact, observe that $(\tilde{A}_{A,B} - \lambda)^{-1}$ consists of all the elements

$$\left\{ \left( \begin{array}{c} f' - \lambda f \\ -\varphi' - \lambda \varphi \end{array} \right), \left( \begin{array}{c} f \\ \varphi \end{array} \right) \right\}, \quad f, f' \in \mathcal{H}(N_{A,B}), \varphi, \varphi' \in \mathcal{G},$$

for which

$$f'(\xi) - \xi f(\xi) = B(\xi)^*\varphi - A(\xi)^*\varphi', \quad \xi \in \mathbb{C} \setminus \mathbb{R}. \quad (4.4.6)$$

Hence, the compression $P_{\mathcal{G}}(\tilde{A}_{A,B} - \lambda)^{-1} \mathcal{I}_G$ is formed by the pairs

$$\{-\varphi' - \lambda \varphi, \varphi, \varphi' \in \mathcal{G}, \varphi, \varphi' \in \mathcal{G}, \} \quad (4.4.7)$$
which satisfy (4.4.6) for some \( f, f' \in \mathcal{H}(N_{A,B}) \) and, in addition, \( f'(\xi) = \lambda f(\xi) \) for \( \xi \in \mathbb{C} \setminus \mathbb{R} \). This implies that (4.4.6) becomes

\[
(\lambda - \xi)f(\xi) = B(\xi)^* \varphi - A(\xi)^* \varphi', \quad \xi \in \mathbb{C} \setminus \mathbb{R},
\]

and the choice \( \xi = \lambda \) gives

\[
B(\lambda)^* \varphi = A(\lambda)^* \varphi'. \tag{4.4.8}
\]

On the other hand, as \( \tau(\lambda) = \{ \{ A(\lambda)\psi, B(\lambda)\psi : \psi \in \mathcal{G} \} \} \), one has by the symmetry property of the Nevanlinna family \( \tau \) and (1.10.3) that

\[
\tau(\lambda) = \tau(\lambda)^* = \{ \{ \psi, \psi' \} : B(\lambda)^* \psi = A(\lambda)^* \psi' \},
\]

and since the pair \( \{ \varphi, \varphi' \} \) in (4.4.7) satisfies (4.4.8), it follows that \( \{ \varphi, \varphi' \} \in \tau(\lambda) \). Hence, \( -\varphi' - \lambda \varphi, \varphi \in -(\tau(\lambda) + \lambda)^{-1} \), which yields the inclusion

\[
P_{\mathcal{G}}(\tilde{A}_{A,B} - \lambda)^{-1} \iota_{\mathcal{G}} \subset -(\tau(\lambda) + \lambda)^{-1}.
\]

Since the compressed resolvent of \( \tilde{A}_{A,B} \) and \( -(\tau(\lambda) + \lambda)^{-1} \) are both everywhere defined and bounded operators (4.4.4) follows.

Finally, the minimality condition (4.4.5) is shown in the same way as in the proof of Theorem 4.2.2. \( \square \)

Theorem 4.4.2 provides a representation of the resolvent of the Nevanlinna family \( \tau \) in terms of the model for the Nevanlinna pair \( \{ A, B \} \). Now let \( \{ C, D \} \) be a Nevanlinna pair which is equivalent to \( \{ A, B \} \):

\[
C(\lambda) = A(\lambda)X(\lambda) \quad \text{and} \quad D(\lambda) = B(\lambda)X(\lambda), \tag{4.4.9}
\]

where \( X(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R} \), is a bounded and boundedly invertible holomorphic operator function in \( \mathbf{B}(\mathcal{H}) \); cf. Section 1.12. Then the kernels \( N_{A,B} \) and \( N_{C,D} \) of the Nevanlinna pairs in (4.4.9) are connected by

\[
N_{C,D}(\lambda, \mu) = X(\lambda)^* N_{A,B}(\lambda, \mu) X(\mu).
\]

The following special case is of interest.

**Lemma 4.4.3.** Let \( \{ A, B \} \) be a Nevanlinna pair in \( \mathcal{H} \) and consider the bounded and boundedly invertible holomorphic operator function \( X(\lambda) = (B(\lambda) + \lambda A(\lambda))^{-1}, \lambda \in \mathbb{C} \setminus \mathbb{R} \). Then the Nevanlinna pair \( \{ C, D \} \) in (4.4.9) satisfies

\[
C(\lambda)^* = C(\lambda), \quad D(\lambda)^* = D(\lambda), \quad \text{and} \quad D(\lambda) + \lambda C(\lambda) = I,
\]

and for \( \lambda, \mu \in \mathbb{C} \setminus \mathbb{R} \) the corresponding Nevanlinna kernel can be written as

\[
N_{C,D}(\lambda, \mu) = \frac{D(\lambda)C(\mu)^* - C(\lambda)D(\mu)^*}{\lambda - \tilde{\mu}} = \frac{C(\mu)^* - C(\lambda)}{\lambda - \tilde{\mu}} - C(\lambda)C(\mu)^*.
\]
Let again \{A, B\} be a Nevanlinna pair, let \(\tau\) be the corresponding Nevanlinna family, and consider a Nevanlinna pair \{C, D\} which is equivalent to \{A, B\} via (4.4.9), so that it generates the same Nevanlinna family \(\tau\). Then according to Theorem 4.4.2,

\[
\tilde{A}_{C, D} = \left\{ \left( \begin{array}{cc} F & F' \\ \psi & -\psi' \end{array} \right) \right\} : F, F' \in \mathcal{H}(N_{C, D}), \psi, \psi' \in \mathcal{G}, \quad F' - \xi F = D(\xi)^*\psi - C(\xi)^*\psi' \quad \quad (4.4.11)
\]

is a self-adjoint relation in the Hilbert space \(\mathcal{H}(N_{C, D}) \oplus \mathcal{G}\) and the compressed resolvent of \(\tilde{A}_{C, D}\) onto \(\mathcal{G}\) is given by

\[
P_\mathcal{G}(\tilde{A}_{C, D} - \lambda)^{-1} = -(\tau(\lambda) + \lambda)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]

Furthermore, the self-adjoint relation \(\tilde{A}_{C, D}\) satisfies the following minimality condition:

\[
\mathcal{H}(N_{C, D}) \oplus \mathcal{G} = \text{span} \left\{ \mathcal{G}, \text{ran}(\tilde{A}_{C, D} - \lambda)^{-1} : \lambda \in \mathbb{C} \setminus \mathbb{R} \right\}.
\]

The explicit connection between the various models involving these kernels in Theorem 4.4.2 now depends on Proposition 4.1.9. The corresponding self-adjoint relations are then unitarily equivalent in the sense of Definition 1.3.7 and Lemma 1.3.8.

**Lemma 4.4.4.** Let the Nevanlinna pairs \{A, B\} and \{C, D\} be equivalent in the sense of (4.4.9). Then the mapping \(U\) defined by

\[
U : \left( \begin{array}{c} \mathcal{H}(N_{A, B}) \\mathcal{G} \\ \mathcal{G} \end{array} \right) \rightarrow \left( \begin{array}{c} \mathcal{H}(N_{C, D}) \\mathcal{G} \\ \mathcal{G} \end{array} \right), \quad \left( \begin{array}{c} f(\xi) \\ \varphi \end{array} \right) \mapsto \left( \begin{array}{c} X(\xi)^* f(\xi) \\ \varphi \end{array} \right), \quad (4.4.12)
\]

is unitary. Moreover, the self-adjoint relation \(\tilde{A}_{A, B}\) in \(\mathcal{H}(N_{A, B}) \oplus \mathcal{G}\) in Theorem 4.4.2 and the self-adjoint relation \(\tilde{A}_{C, D}\) in \(\mathcal{H}(N_{C, D}) \oplus \mathcal{G}\) in (4.4.11) are unitarily equivalent under the mapping \(U\), that is, \(\tilde{A}_{C, D} = U \tilde{A}_{A, B} U^*\).

**Proof.** In the identity (4.4.10) set \(\Phi(\lambda) = X(\xi)^*\) with \(X(\lambda)\) as in (4.4.9). Since \(X(\lambda)\) is boundedly invertible one may apply Proposition 4.1.9 and hence \(U\) in (4.4.12) is unitary. Now consider an element

\[
\left\{ \left( \begin{array}{c} f \\ \varphi \end{array} \right), \left( \begin{array}{c} f' \\ -\varphi' \end{array} \right) \right\} \in \tilde{A}_{A, B},
\]

so that

\[
f'(\xi) - \xi f(\xi) = B(\xi)^*\varphi - A(\xi)^*\varphi', \quad \xi \in \mathbb{C} \setminus \mathbb{R}.
\]

Then with \(F(\xi) = X(\xi)^* f(\xi)\) and \(F'(\xi) = X(\xi)^* f'(\xi)\) it follows that

\[
F'(\xi) - \xi F(\xi) = X(\xi)^* B(\xi)^* \varphi - X(\xi)^* A(\xi)^* \varphi' = D(\xi)^* \varphi - C(\xi)^* \varphi'.
\]

This implies

\[
\left\{ U \left( \begin{array}{c} f \\ \varphi \end{array} \right), U \left( \begin{array}{c} f' \\ -\varphi' \end{array} \right) \right\} = \left\{ \left( F \\ \varphi \right), \left( F' \right) \right\} \in \tilde{A}_{C, D}.
\]
One verifies in the same way that every element
\[ \left\{ \left( F\varphi, F'\varphi' \right) \right\} \in \widetilde{A}_{C,D} \]
can be written in the form
\[ \left\{ U\left( f\varphi, f'\varphi' \right) \right\} \text{ for some } \left\{ \left( f\varphi, f'\varphi' \right) \right\} \in \widetilde{A}_{A,B}. \]

This shows that the self-adjoint relations \( \widetilde{A}_{A,B} \) and \( \widetilde{A}_{C,D} \) are unitarily equivalent under the mapping \( U \); cf. Definition 1.3.7. \( \square \)

The discussions in this section so far centered mainly on Nevanlinna pairs and will now be put in a slightly different context.

**Definition 4.4.5.** Let \( \mathcal{H} \) be a Hilbert space and let \( R \) be a \( \mathcal{B}(\mathcal{H}) \)-valued function defined on \( \mathbb{C} \setminus \mathbb{R} \). Then \( R \) is called a *generalized resolvent* if it has the following properties:

(i) \( \lambda \mapsto R(\lambda) \) is holomorphic on \( \mathbb{C} \setminus \mathbb{R} \);
(ii) \( R(\lambda)^* = R(\bar{\lambda}), \lambda \in \mathbb{C} \setminus \mathbb{R} \);
(iii) \[ \frac{\text{Im} R(\lambda)}{\text{Im} \lambda} - R(\lambda)R(\lambda)^* \geq 0, \lambda \in \mathbb{C} \setminus \mathbb{R}. \]

With the function \( R \) one associates the kernel \( R_R \)
\[ R_R(\cdot, \cdot) : \Omega \times \Omega \to \mathcal{B}(\mathcal{H}), \]
defined on \( \Omega = \mathbb{C} \setminus \mathbb{R} \) by
\[ R_R(\lambda, \mu) = \frac{R(\lambda) - R(\mu)^*}{\lambda - \bar{\mu}} - R(\lambda)R(\mu)^*, \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}, \lambda \neq \mu, \quad (4.4.13) \]
and \( R(\lambda, \bar{\lambda}) = R'(\lambda) - R(\lambda)^2, \lambda \in \mathbb{C} \setminus \mathbb{R} \). Then clearly the kernel \( R_R \) is symmetric. Since \( \lambda \mapsto R(\lambda) \) is holomorphic, the mapping \( \lambda \mapsto R_R(\lambda, \mu) \) is holomorphic for each \( \mu \in \mathbb{C} \setminus \mathbb{R} \), that is, the kernel \( R_R \) is holomorphic. Note also that the kernel \( R_R \) is uniformly bounded on compact subsets of \( \mathbb{C} \setminus \mathbb{R} \) since
\[ \|R_R(\lambda, \lambda)\| \leq \frac{\|R(\lambda)\|}{|\text{Im} \lambda|} + \|R(\lambda)\|^2, \lambda \in \mathbb{C} \setminus \mathbb{R}. \]

For \( R_R \) to be a reproducing kernel in the sense of Theorem 4.1.5 one needs non-negativity.

**Lemma 4.4.6.** Let \( R : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{H}) \) be a generalized resolvent. Then the kernel \( R_R(\cdot, \cdot) \) is nonnegative.
Proof. Introduce the pair of $\mathcal{B}(\mathcal{H})$-valued functions $C$ and $D$ by
\[
C(\lambda) = -R(\lambda) \quad \text{and} \quad D(\lambda) = I + \lambda R(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]
Since $R$ is a generalized resolvent, a straightforward computation shows that \{\$C, D\$\} is a Nevanlinna pair and that the kernels satisfy
\[
N_{C,D}(\lambda, \mu) = R(\lambda, \mu), \quad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}; \quad (4.4.14)
\]
cf. (4.4.1) and (4.4.13). Now it follows from Theorem 4.4.1 (with $\mathcal{G} = \mathcal{H}$) that the kernel $R(\cdot, \cdot)$ is nonnegative. \qed

Let $R : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{H})$ be a generalized resolvent. By Theorem 4.1.5, the corresponding nonnegative kernel $R$ induces a Hilbert space of holomorphic $\mathcal{H}$-valued functions, which will be denoted by $\mathcal{H}(R)$, with inner product $\langle \cdot, \cdot \rangle$; cf. Section 4.1. Recall that the reproducing kernel property
\[
\langle f, R(\cdot, \mu)\phi \rangle = (f(\mu), \phi)_{\mathcal{H}}, \quad \phi \in \mathcal{H}, \mu \in \mathbb{C} \setminus \mathbb{R},
\]
holds for all functions $f \in \mathcal{H}(R)$. The following result gives a representation of the function $R$.

Corollary 4.4.7. Let $R : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{H})$ be a generalized resolvent and let $\mathcal{H}(R)$ be the associated reproducing kernel Hilbert space. Denote by $P_{\mathcal{H}}$ the orthogonal projection from $\mathcal{H}(R) \oplus \mathcal{H}$ onto $\mathcal{H}$ and let $\iota_{\mathcal{H}}$ be the canonical embedding of $\mathcal{H}$ into $\mathcal{H}(R) \oplus \mathcal{H}$. Then
\[
\tilde{A}_R = \left\{ \left( \begin{array}{c} f \\ h \\ (-h') \end{array} \right) : f, f' \in \mathcal{H}(R), h, h' \in \mathcal{H}, \right. \left. f'(\xi) - \xi f(\xi) = (I + \xi R(\xi))h + R(\xi)h' \right\}
\]
is a self-adjoint relation in the Hilbert space $\mathcal{H}(R) \oplus \mathcal{H}$ and the compressed resolvent of $\tilde{A}_R$ onto $\mathcal{H}$ is given by
\[
P_{\mathcal{H}}(\mathcal{H}(R) - \lambda)^{-1} \iota_{\mathcal{H}} = R(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]
Furthermore, the self-adjoint relation $\tilde{A}_R$ satisfies the following minimality condition:
\[
\mathcal{H}(R) \oplus \mathcal{H} = \text{span} \left\{ \mathcal{H}, \text{ran} \left( \begin{array}{c} \mathcal{H} \\ \tilde{A}_R - \lambda \end{array} \right)^{-1} \iota_{\mathcal{H}} : \lambda \in \mathbb{C} \setminus \mathbb{R} \right\}. \quad (4.4.15)
\]
Proof. Let $R : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{H})$ be a generalized resolvent and consider the Nevanlinna pair $\{C, D\}$ defined by
\[
\{C(\lambda), D(\lambda)\} = \{-R(\lambda), I + \lambda R(\lambda)\};
\]
cf. the proof of Lemma 4.4.6. Then the kernels $N_{C,D}$ and $R$ coincide by (4.4.14) and hence one has $\mathcal{H}(N_{C,D}) = \mathcal{H}(R)$. Now Theorem 4.4.2 (with $\mathcal{G} = \mathcal{H}$) can be applied to the Nevanlinna family $\tau(\lambda) = \{C(\lambda), D(\lambda)\}$. It follows that $\tilde{A}_R := \tilde{A}_{C,D}$
is a self-adjoint relation in the Hilbert space $\mathcal{H}(R_R) \oplus \tilde{\mathcal{H}}$ and that its compressed resolvent is given by

$$P_\mathcal{H}(\tilde{A}_R - \lambda)^{-1}t_\mathcal{H} = -(\tau(\lambda) + \lambda)^{-1} = R(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

where the fact that $\tau(\lambda) + \lambda = \{-R(\lambda), I\}$ was used in the last equality. Moreover, the minimality condition (4.4.15) holds. □

By Corollary 4.4.7, every generalized resolvent can be interpreted as a compressed resolvent of a self-adjoint relation. Such compressed resolvents have been discussed briefly in the context of the Kreĭn formula in Section 2.7 and will be further studied in Section 4.5. The next theorem complements Corollary 4.4.7 by providing equivalent conditions. In particular, generalized resolvents or, equivalently, compressed resolvents, are characterized as Stieltjes transforms of nondecreasing families of nonnegative contractions. As a simple consequence one obtains the Sz.-Nagy dilation theorem in Corollary 4.4.9.

**Theorem 4.4.8.** Let $\mathcal{H}$ be a Hilbert space and let $R : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{H})$ be an operator function. Then the following statements are equivalent:

(i) The function $R$ is a generalized resolvent.

(ii) There exist a Hilbert space $\mathcal{K}$ and a self-adjoint relation $\tilde{A}$ in the space $\mathcal{H} \oplus \mathcal{K}$ such that

$$R(\lambda) = P_\mathcal{H}(\tilde{A} - \lambda)^{-1}t_\mathcal{H}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. $$

Furthermore, the self-adjoint relation $\tilde{A}$ satisfies the following minimality condition:

$$\mathcal{H} \oplus \mathcal{K} = \text{span} \{\mathcal{H}, \text{ran}(\tilde{A} - \lambda)^{-1}t_\mathcal{H} : \lambda \in \mathbb{C} \setminus \mathbb{R}\}. $$

(iii) There exists a nondecreasing function $\Sigma : \mathbb{R} \to \mathcal{B}(\mathcal{H})$, whose values are nonnegative contractions, such that

$$R(\lambda) = \int_\mathbb{R} \frac{1}{t - \lambda} d\Sigma(t), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. $$

**Proof.** (i) $\Rightarrow$ (ii) This follows directly from Corollary 4.4.7.

(ii) $\Rightarrow$ (iii) Since $\tilde{A}$ is self-adjoint, one can write

$$(\tilde{A} - \lambda)^{-1} = \int_\mathbb{R} \frac{1}{t - \lambda} dE(t), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

with the spectral measure $E(\cdot)$ of $\tilde{A}$; cf. (1.5.6). The function $t \mapsto E((\infty, t))$ is a nondecreasing family of orthogonal projections from $\mathbb{R}$ to $\mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ and one has that

$$R(\lambda) = P_\mathcal{H}(\tilde{A} - \lambda)^{-1}t_\mathcal{H} = \int_\mathbb{R} \frac{1}{t - \lambda} dP_\mathcal{H}E(t)t_\mathcal{H}. $$
4.4. Realization of Nevanlinna pairs and generalized resolvents

Now define $\Sigma(t) = P_\mathcal{H}E((-(\infty,t)])$, which is a nondecreasing family of nonnegative contractions from $\mathbb{R}$ to $\mathcal{B}(\mathcal{H})$ that satisfies $\int_\mathbb{R} d\Sigma(t) \in \mathcal{B}(\mathcal{H})$ and the estimate $\| \int_\mathbb{R} d\Sigma(t) \| \leq 1$.

(iii) $\Rightarrow$ (i) It is clear that the function $R : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathcal{H})$ is holomorphic and satisfies $R(\lambda) = R(\lambda)^*$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Moreover, it follows from Proposition A.5.4 that

$$\frac{\text{Im} R(\lambda)}{\text{Im} \lambda} - R(\lambda)R(\lambda)^* = \frac{\text{Im} R(\bar{\lambda})}{\text{Im} \bar{\lambda}} - R(\bar{\lambda})^* R(\bar{\lambda}) \geq 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

which implies that $R$ is a generalized resolvent. $\square$

The next corollary is a variant of the dilation theorem, which goes back to M.A. Naimark and B. Sz.-Nagy; here it is obtained from Theorem 4.4.8 and the Stieltjes inversion formula.

**Corollary 4.4.9.** Let $\Sigma : \mathbb{R} \to \mathcal{B}(\mathcal{H})$ be a left-continuous nondecreasing function, whose values are nonnegative contractions, such that

$$\int_\mathbb{R} d\Sigma(t) \in \mathcal{B}(\mathcal{H}), \quad \left\| \int_\mathbb{R} d\Sigma(t) \right\| \leq 1, \quad \text{and} \quad \Sigma(\infty) = 0.$$

Then there exist a Hilbert space $\mathfrak{K}$ and a left-continuous nondecreasing function $E : \mathbb{R} \to \mathcal{B}(\mathcal{H} \oplus \mathfrak{K})$, whose values are orthogonal projections, such that

$$\Sigma(t) = P_\mathcal{H}E(t)_{\mathcal{H}}, \quad t \in \mathbb{R}.$$

**Proof.** Associate with $\Sigma$ the function

$$R(\lambda) = \int_\mathbb{R} \frac{1}{t - \lambda} d\Sigma(t), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (4.4.16)$$

By Theorem 4.4.8, there exists a Hilbert space $\mathfrak{K}$ and a self-adjoint relation $\tilde{A}$ in $\mathcal{H} \oplus \mathfrak{K}$ such that the compression of the resolvent of $\tilde{A}$ onto $\mathcal{H}$ is given by

$$P_\mathcal{H}(\tilde{A} - \lambda)^{-1}_{\mathcal{H}} = R(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (4.4.17)$$

Let $E(\cdot)$ be the spectral measure of $\tilde{A}$ and let $t \mapsto E((-(\infty,t)))$ be the corresponding spectral function, which is left-continuous and satisfies $\lim_{t \to -\infty} E((-(\infty,t))) = 0$.

As in the proof of Theorem 4.4.8 one has

$$P_\mathcal{H}(\tilde{A} - \lambda)^{-1}_{\mathcal{H}} = P_\mathcal{H} \left( \int_\mathbb{R} \frac{1}{t - \lambda} dE(t) \right)_{\mathcal{H}} = \int_\mathbb{R} \frac{1}{t - \lambda} dP_\mathcal{H}E(t)_{\mathcal{H}}.$$

Taking into account (4.4.16) and (4.4.17), it follows that

$$\int_\mathbb{R} \frac{1}{t - \lambda} dP_\mathcal{H}E(t)_{\mathcal{H}} = \int_\mathbb{R} \frac{1}{t - \lambda} d\Sigma(t), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$
and hence for all \( h \in H \)
\[
\int_{\mathbb{R}} \frac{1}{t - \lambda} d(E(t)\iota_{\mathcal{H}}h, \iota_{\mathcal{H}}h) = \int_{\mathbb{R}} \frac{1}{t - \lambda} d(\Sigma(t)h, h), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]
Since the functions \( t \mapsto (E(t)\iota_{\mathcal{H}}h, \iota_{\mathcal{H}}h) \) and \( t \mapsto (\Sigma(t)h, h) \) are left-continuous, and
\[
\lim_{t \to -\infty} \left( E((\infty, t))\iota_{\mathcal{H}}h, \iota_{\mathcal{H}}h \right) = 0 = (\Sigma(\infty)h, h),
\]
the Stieltjes inversion formula in Corollary \ref{cor:stieltjes} yields \((E(t)\iota_{\mathcal{H}}h, \iota_{\mathcal{H}}h) = (\Sigma(t)h, h)\) for all \( t \in \mathbb{R} \) and \( h \in \mathcal{H} \). This leads to the assertion. \( \square \)

### 4.5 Kreǐn’s formula for exit space extensions

Let \( S \) be a closed symmetric relation in the Hilbert space \( \mathcal{H} \), let \( \{ G, \Gamma_0, \Gamma_1 \} \) be a boundary triplet for \( S^* \), \( A_0 = \ker \Gamma_0 \), and let \( \gamma \) and \( M \) be the corresponding \( \gamma \)-field and Weyl function, respectively. Suppose that \( \tilde{A} \) is a self-adjoint extension of \( S \) in \( H \oplus H' \), where \( H' \) is the exit space. It was shown in Theorem \ref{thm:krein-naimark} that there exists a Nevanlinna family \( \tau(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R} \), in \( \mathcal{G} \) such that
\[
P_{\mathcal{H}}(\tilde{A} - \lambda)^{-1}\iota_{\mathcal{H}} = (A_0 - \lambda)^{-1} - \gamma(\lambda)(M(\lambda) + \tau(\lambda))^{-1}\gamma(\lambda^*)^*, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]
holds. This is Kreǐn’s formula for the compressed resolvents of self-adjoint exit space extensions (as studied by M.A. Naǐmark); it is also referred to as Kreǐn–Naǐmark formula in this text; cf. Section \ref{sec:krein-naimark}.

The goal of this section is to show the converse statement. More precisely, it will be proved that for every Nevanlinna family \( \tau(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R} \), in the Hilbert space \( \mathcal{G} \) there exists a self-adjoint exit space extension \( \tilde{A} \) of \( S \) such that the compressed resolvent of \( \tilde{A} \) onto \( \mathcal{H} \) is given by the Kreǐn–Naǐmark formula. The following result is a first step.

**Lemma 4.5.1.** Let \( S \) be a closed symmetric relation in \( \mathcal{H} \), let \( \{ G, \Gamma_0, \Gamma_1 \} \) be a boundary triplet for \( S^* \) with \( A_0 = \ker \Gamma_0 \), and let \( \gamma \) and \( M \) be the corresponding \( \gamma \)-field and Weyl function, respectively. Let \( \tau = \{ A, B \} \) be a Nevanlinna family in \( \mathcal{G} \) and define \( R(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R} \), by
\[
R(\lambda) = (A_0 - \lambda)^{-1} - \gamma(\lambda)(M(\lambda) + \tau(\lambda))^{-1}\gamma(\lambda^*)^*, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]
Then the kernel
\[
R_R(\lambda, \mu) = \frac{R(\lambda) - R(\mu)^*}{\lambda - \mu} - R(\lambda)R(\mu)^*, \quad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}, \quad \lambda \neq \mu,
\]
satisfies
\[
R_R(\lambda, \mu) = W(\lambda)N_{A,B}(\lambda, \mu)W(\mu)^*,
\]
where
\[ W(\lambda) = \gamma(\lambda)(M(\overline{\lambda})A(\overline{\lambda}) + B(\overline{\lambda}))^{-1}. \] (4.5.4)

In particular, the kernel \( R \) is nonnegative, symmetric, holomorphic, and uniformly bounded on compact subsets of \( \mathbb{C} \setminus \mathbb{R} \).

**Proof.** Step 1. For \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) introduce the following notations
\[ R_0(\lambda) = (A_0 - \lambda)^{-1} \quad \text{and} \quad Q(\lambda) = (M(\lambda) + \tau(\lambda))^{-1}, \]
so that \( R \) in (4.5.1) is given by
\[ R(\lambda) = R_0(\lambda) - \gamma(\lambda)Q(\lambda)\gamma(\overline{\lambda})^*. \]
Rewrite the kernel \( R_R(\cdot, \cdot) \) in (4.5.2) in terms of this notation:
\[
R_R(\lambda, \mu) = \frac{1}{\lambda - \overline{\mu}} \left( R_0(\lambda) - R_0(\mu)^* - \gamma(\lambda)Q(\lambda)\gamma(\overline{\lambda})^* + \gamma(\mu)Q(\mu)^*\gamma(\mu)^* \right) \\
- \left( R_0(\lambda) - \gamma(\lambda)Q(\lambda)\gamma(\overline{\lambda})^* \right) \left( R_0(\mu)^* - \gamma(\mu)Q(\mu)^*\gamma(\mu)^* \right) \\
= \frac{1}{\lambda - \overline{\mu}} \left( -\gamma(\lambda)Q(\lambda)\gamma(\overline{\lambda})^* + \gamma(\mu)Q(\mu)^*\gamma(\mu)^* \right) \\
+ R_0(\lambda)\gamma(\overline{\mu})Q(\mu)^*\gamma(\mu)^* + \gamma(\lambda)Q(\lambda)\gamma(\overline{\lambda})^*R_0(\mu)^* \\
- \gamma(\lambda)Q(\lambda)\gamma(\overline{\lambda})^*\gamma(\overline{\mu})Q(\mu)^*\gamma(\mu)^*.
\]
Recall that, by Proposition 2.3.2 (ii) and Proposition 2.3.6 (iii),
\[ R_0(\lambda)\gamma(\overline{\mu}) = \frac{\gamma(\lambda) - \gamma(\overline{\mu})}{\lambda - \overline{\mu}}, \quad \gamma(\overline{\lambda})^*R_0(\mu)^* = \frac{\gamma(\overline{\lambda})^* - \gamma(\mu)^*}{\lambda - \overline{\mu}}, \]
and
\[
\gamma(\overline{\lambda})^*\gamma(\overline{\mu}) = (\gamma(\overline{\mu})^*\gamma(\overline{\lambda}))^* = \left( \frac{M(\overline{\lambda}) - M(\overline{\mu})^*}{\overline{\lambda} - \mu} \right)^* = \frac{M(\lambda) - M(\mu)^*}{\lambda - \overline{\mu}}.
\]
Therefore, the kernel \( R_R(\cdot, \cdot) \) has the form
\[
R_R(\lambda, \mu) = \frac{1}{\lambda - \overline{\mu}}\gamma(\lambda) \left[ Q(\mu)^* - Q(\lambda) \right] \\
+ Q(\lambda)M(\mu)^*Q(\mu)^* - Q(\lambda)M(\lambda)Q(\mu)^* \gamma(\mu)^*. \] (4.5.5)

**Step 2.** Express the identity (4.5.5) in terms of the Nevanliina pair \( \{A, B\} \), representing the Nevanliina family \( \tau \). For this, consider the equivalent Nevanliina pair \( \{C, D\} \) as in Lemma 4.4.3, that is,
\[ C(\lambda) = A(\lambda)X(\lambda) \quad \text{and} \quad D(\lambda) = B(\lambda)X(\lambda), \]
where $X(\lambda) = (B(\lambda) + \lambda A(\lambda))^{-1}$, so that
\[
N_{C,D}(\lambda, \mu) = \frac{D(\lambda)C(\mu)^* - C(\lambda)D(\mu)^*}{\lambda - \overline{\mu}}.
\] (4.5.6)

Observe that $Q(\lambda)$ can be written in terms of $\tau = \{C, D\}$ as
\[
Q(\lambda) = C(\lambda)(M(\lambda)C(\lambda) + D(\lambda))^{-1};
\] cf. (1.12.10). It follows that
\[
Q(\lambda) = Q(\lambda)^* = \left(\frac{M(\lambda)}{\lambda}ight) - C(\lambda)(M(\lambda)C(\lambda) + D(\lambda))^{-1};
\] (4.5.7)

and
\[
Q(\mu)^* = Q(\mu) = C(\mu)^*(\frac{M(\mu)}{\mu}) - C(\mu)^*(\frac{M(\mu)}{\mu})^{-*}.
\] (4.5.8)

Inserting the expressions (4.5.7) and (4.5.8) in (4.5.5) one arrives after a straightforward computation at the identity
\[
R_R(\lambda, \mu) = Z(\lambda)N_{C,D}(\lambda, \mu)Z(\mu)^*,
\]
where the factor $Z(\lambda)$ is given by
\[
Z(\lambda) = \frac{\gamma(\lambda)(C(\lambda)M(\lambda) + D(\lambda))^{-1}}{\lambda - \overline{\mu}}.
\] (4.5.9)

Recall that $N_{A,B}$ and $N_{C,D}$ are related via (4.4.10). Therefore, with (4.5.6) and (4.5.9) one obtains the identity (4.5.3), where
\[
W(\lambda) = \gamma(\lambda)(C(\lambda)M(\lambda) + D(\lambda))^{-1}(B(\lambda) + \lambda A(\lambda))^{-*}.
\] (4.5.10)

The proof is finished by writing the normalized pair $\{C, D\}$ in (4.5.10) in terms of the pair $\{A, B\}$. Observe that since $X(\lambda) = (B(\lambda) + \lambda A(\lambda))^{-1}$, the symmetry property $B(\lambda)^*A(\lambda) = A(\lambda)^*B(\lambda)$ of the Nevanlinna pair yields
\[
(B(\lambda)^* + \lambda A(\lambda)^*)(C(\lambda)M(\lambda) + D(\lambda))
= (B(\lambda)^* + \lambda A(\lambda)^*)(A(\lambda)X(\lambda)M(\lambda) + B(\lambda)X(\lambda))
= A(\lambda)^*(B(\lambda) + \lambda A(\lambda))X(\lambda)M(\lambda) + B(\lambda)^*(B(\lambda) + \lambda A(\lambda))X(\lambda)
= A(\lambda)^*M(\lambda) + B(\lambda)^*,
\]

which gives (4.5.4).

It follows from (4.5.3) and (4.5.4) that the kernel $R_R$ in (4.5.2) is nonnegative, symmetric, holomorphic, and uniformly bounded on compact subsets of $\mathbb{C} \setminus \mathbb{R}$. □

Lemma 4.5.1 shows that $R_R(\cdot, \cdot)$ is a reproducing kernel. Therefore, one may apply Theorem 4.4.8.
**Theorem 4.5.2.** Let $S$ be a closed symmetric relation, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $S^*$ with $A_0 = \ker \Gamma_0$, and let $\gamma$ and $M$ be the corresponding $\gamma$-field and Weyl function, respectively. Let $\tau$ be a Nevanlinna family in $\mathcal{G}$. Then there exist an exit Hilbert space $\mathfrak{H}$ and a self-adjoint relation $\tilde{A}$ in $\mathfrak{H} \oplus \mathfrak{H}'$ such that $\tilde{A}$ is an extension of $S$ and the compressed resolvent of $\tilde{A}$ is given by the Krein–Naïmark formula:

$$P_\mathcal{H}(\tilde{A} - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \frac{1}{\gamma(\lambda)(M(\lambda) + \tau(\lambda))^{-1} \gamma(\tilde{\lambda})^*}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (4.5.11)$$

Furthermore, the self-adjoint relation $\tilde{A}$ satisfies the following minimality condition:

$$\mathfrak{H} \oplus \mathfrak{H}' = \overline{\text{span}} \{\mathfrak{H}, \text{ran}(\tilde{A} - \mu)^{-1} \iota_\mathcal{H} : \mu \in \mathbb{C} \setminus \mathbb{R}\}. \quad (4.5.12)$$

**Proof.** Define the function $R$ as in Lemma 4.5.1. Then $R$ is a $\mathcal{B}(\mathfrak{H})$-valued holomorphic function on $\mathbb{C} \setminus \mathbb{R}$ which satisfies $R(\lambda) = R(\lambda)^*$ and so, by Lemma 4.5.1, $R$ is a generalized resolvent. Hence, by Theorem 4.4.8, the function $R$ is a compressed resolvent, that is, there exist a Hilbert space $\mathfrak{H}$ and a self-adjoint relation $A$ in $\mathfrak{H} \oplus \mathfrak{H}'$ such that

$$R(\lambda) = P_\mathfrak{H}(A - \lambda)^{-1} \iota_\mathcal{H}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R};$$

this implies (4.5.11) Moreover, it follows from Theorem 4.4.8 that $\tilde{A}$ satisfies the minimality condition (4.5.12).

It remains to prove that $S \subset \tilde{A}$. Observe first that by (4.5.11) the Štraus family corresponding to $\tilde{A}$ satisfies

$$T(\lambda) = \{\{R(\lambda)h, (I + \lambda R(\lambda))h\} : h \in \mathfrak{H}\}$$

$$\subset \{R_0(\lambda)h, (I + \lambda R_0(\lambda))h : h \in \mathfrak{H}\} \oplus \{\{\gamma(\lambda)\varphi, \lambda \gamma(\lambda)\varphi\} : \varphi \in \mathcal{G}\}.$$  

Since each relation on the right-hand side is contained in $S^*$, so is the relation $T(\lambda)$. As $T(\lambda)^* = T(\tilde{\lambda})$, it follows that $S \subset T(\lambda)$. Now let $\{f, f'\} \in S$ so that for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there exists $h \in \mathfrak{H}$ such that

$$\left\{\left(\begin{array}{c} f \\ h \end{array}\right), \left(\begin{array}{c} f' \\ \lambda h \end{array}\right)\right\} \in \tilde{A}.$$  

The relation $\tilde{A}$ is self-adjoint and, in particular, symmetric. Therefore, one sees that $(f', f) + \lambda(h, h) \in \mathbb{R}$, while by definition $(f', f) \in \mathbb{R}$. Since $\lambda \in \mathbb{C} \setminus \mathbb{R}$, it follows that $h = 0$, and thus

$$\left\{\left(\begin{array}{c} f \\ 0 \end{array}\right), \left(\begin{array}{c} f' \\ 0 \end{array}\right)\right\} \in \tilde{A}.$$  

This shows that $S \subset \tilde{A}$. \qed
4.6 Orthogonal coupling of boundary triplets

In this section a different look is taken at the Kreĭn–Naĭmark formula. By means of an abstract coupling method for direct orthogonal sums of symmetric relations and corresponding boundary triplets, a particular self-adjoint extension \( \tilde{A} \) of the direct sum is identified, and it is shown that the compressed resolvent of \( \tilde{A} \) is of the same form as in the Kreĭn–Naĭmark formula. When combined with Theorem 4.2.4, this coupling procedure provides a constructive approach to the exit space extension in Theorem 4.5.2 in the special case where the Nevanlinna family \( \tau \) is a uniformly strict Nevanlinna function.

First a slightly more general, abstract point of view is adopted. In the following let \( S \) and \( T \) be closed symmetric relations in the Hilbert spaces \( \mathcal{H} \) and \( \mathcal{H}' \), respectively, and assume that the defect numbers of \( S \) and \( T \) coincide:

\[
n_+(S) = n_-(S) = n_+(T) = n_-(T) \leq \infty.
\]

Let \( \{S, \Gamma_0, \Gamma_1\} \) be a boundary triplet for \( S^* \) with \( A_0 = \ker \Gamma_0 \) and let \( \{S, \Gamma'_0, \Gamma'_1\} \) be a boundary triplet for \( T^* \) with \( B_0 = \ker \Gamma'_0 \). Then it is easy to see that the direct orthogonal sum \( S \oplus T \) is a closed symmetric relation in \( \mathcal{H} \oplus \mathcal{H}' \) and \( \{S \oplus S, \Gamma_0, \Gamma_1\} \), where

\[
\begin{align*}
\bar{\Gamma}_0 \begin{pmatrix} f \\ g \end{pmatrix} &= \begin{pmatrix} \Gamma_0 f \\ \Gamma_0' g \end{pmatrix} \quad \text{and} \quad \bar{\Gamma}_1 \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \Gamma_1 f \\ \Gamma_1' g \end{pmatrix}, \quad f \in S^*, \; g \in T^*, \quad (4.6.1)
\end{align*}
\]

is a boundary triplet for \( (S \oplus T)^* = S^* \oplus T^* \), and that

\[
\tilde{A}_0 := A_0 \oplus B_0 = \ker \bar{\Gamma}_0 \quad (4.6.2)
\]

is a self-adjoint extension of \( S \oplus T \) in \( \mathcal{H} \oplus \mathcal{H}' \). Furthermore, if \( \gamma \) and \( \gamma' \) denote the \( \gamma \)-fields corresponding to the boundary triplets \( \{S, \Gamma_0, \Gamma_1\} \) and \( \{S, \Gamma'_0, \Gamma'_1\} \), and \( \mathcal{M} \) and \( \tau \) are the Weyl functions corresponding to \( \{S, \Gamma_0, \Gamma_1\} \) and \( \{S, \Gamma'_0, \Gamma'_1\} \), respectively, then it is clear that for \( \lambda \in \rho(\tilde{A}_0) = \rho(A_0) \cap \rho(B_0) \) the \( \gamma \)-field \( \tilde{\gamma} \) and the Weyl function \( \tilde{M} \) corresponding to the boundary triplet \( \{S \oplus S, \tilde{\Gamma}_0, \tilde{\Gamma}_1\} \) have the forms

\[
\begin{align*}
\tilde{\gamma}(\lambda) &= \begin{pmatrix} \gamma(\lambda) & 0 \\ 0 & \gamma'(\lambda) \end{pmatrix} \quad \text{and} \quad \tilde{M}(\lambda) = \begin{pmatrix} M(\lambda) & 0 \\ 0 & \tau(\lambda) \end{pmatrix}. \quad (4.6.3)
\end{align*}
\]

Let \( \tilde{A} \) be a self-adjoint extension of \( S \oplus T \) in \( \mathcal{H} \oplus \mathcal{H}' \). Then Kreĭn’s formula in Theorem 2.6.1 has the form

\[
(\tilde{A} - \lambda)^{-1} = (\tilde{A}_0 - \lambda)^{-1} + \tilde{\gamma}(\lambda)(\tilde{\Theta} - \tilde{M}(\lambda))^{-1}\tilde{\gamma}(\lambda)^* \quad (4.6.4)
\]

for all \( \lambda \in \rho(\tilde{A}) \cap \rho(\tilde{A}_0) \), where \( \tilde{\gamma} \) and \( \tilde{M} \) denote the \( \gamma \)-field and the Weyl function corresponding to the boundary triplet \( \{S \oplus S, \tilde{\Gamma}_0, \tilde{\Gamma}_1\} \) in (4.6.3). If \( \tilde{\Theta} = \{A, B\} \) with \( A, B \in \mathcal{B}(S \oplus S) \), then

\[
(\tilde{A} - \lambda)^{-1} = (\tilde{A}_0 - \lambda)^{-1} - \tilde{\gamma}(\lambda)A(\tilde{M}(\lambda)A - B)^{-1}\tilde{\gamma}(\lambda)^*, \quad (4.6.5)
\]
see Corollary 2.6.3. Writing $\mathcal{A}$ and $\mathcal{B}$ as block operators

$$\mathcal{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where $A_{ij}, B_{ij} \in \mathcal{B}(H)$, it follows that

$$\tilde{M}(\lambda)\mathcal{A} - \mathcal{B} = \begin{pmatrix} M(\lambda)A_{11} - B_{11} & M(\lambda)A_{12} - B_{12} \\ \tau(\lambda)A_{21} - B_{21} & \tau(\lambda)A_{22} - B_{22} \end{pmatrix},$$

so that

$$\mathcal{A}(\tilde{M}(\lambda)\mathcal{A} - \mathcal{B})^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} M(\lambda)A_{11} - B_{11} & M(\lambda)A_{12} - B_{12} \\ \tau(\lambda)A_{21} - B_{21} & \tau(\lambda)A_{22} - B_{22} \end{pmatrix}^{-1}.$$

The following proposition exhibits a particular self-adjoint extension of $\mathcal{S} \oplus T$ in $\mathcal{H} \oplus \mathcal{H}'$.

**Proposition 4.6.1.** Let $S$ and $T$ be closed symmetric relations in the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ with boundary triplets $\{\mathcal{S}, \Gamma_0, \Gamma_1\}$ and $\{\mathcal{S}', \Gamma_0', \Gamma_1'\}$ as above, respectively. Then

$$\tilde{A} = \left\{ \left( \frac{f}{g} \right) \in S^* \oplus T^* : \Gamma_0 f = \Gamma_0' g, \Gamma_1 f = -\Gamma_1' g \right\}$$

is a self-adjoint relation in $\mathcal{H} \oplus \mathcal{H}'$ and for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the resolvent of $\tilde{A}$ has the form

$$(\tilde{A} - \lambda)^{-1} = (\tilde{A}_0 - \lambda)^{-1} - \tilde{\gamma}(\lambda) \left( \begin{pmatrix} M(\lambda) + \tau(\lambda) \end{pmatrix}^{-1} M(\lambda) + \tau(\lambda) \right)^{-1} \tilde{\gamma}(\lambda)^*,$$

where $\tilde{A}_0$ and $\tilde{\gamma}$ are as in (4.6.2), and $M$ and $\tau$ denote the Weyl functions corresponding to $\{\mathcal{S}, \Gamma_0, \Gamma_1\}$ and $\{\mathcal{S}', \Gamma_0', \Gamma_1'\}$, respectively.

**Proof.** Consider the boundary triplet $\{\mathcal{S} \oplus \mathcal{S}', \tilde{\Gamma}_0, \tilde{\Gamma}_1\}$ in (4.6.1) and observe that the relation

$$\tilde{\Theta} := \left\{ \left( \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \begin{pmatrix} \psi' \\ -\varphi' \end{pmatrix} \right) : \varphi, \psi \in \mathcal{S} \right\}$$

is self-adjoint in $\mathcal{S} \oplus \mathcal{S}$. Hence, by Corollary 2.1.4 and (4.6.1),

$$\left\{ \left( \frac{f}{g} \right) \in S^* \oplus T^* : \tilde{\Gamma} \left( \frac{f}{g} \right) = \left( \begin{pmatrix} \Gamma_0 f \\ \Gamma_0' g \end{pmatrix}, \begin{pmatrix} \Gamma_1 f \\ \Gamma_1' g \end{pmatrix} \right) \in \tilde{\Theta} \right\} \subset S^* \oplus T^*$$

is a self-adjoint relation $\mathcal{H} \oplus \mathcal{H}'$. Now it follows from the particular form of $\tilde{\Theta}$ in (4.6.5) that the self-adjoint relation in (4.6.6) coincides with $\tilde{A}$ in (4.6.4).

Next the resolvent of $\tilde{A}$ will be computed. Recall first that Krein’s formula in Theorem 2.6.1 implies

$$(\tilde{A} - \lambda)^{-1} = (\tilde{A}_0 - \lambda)^{-1} - \tilde{\gamma}(\lambda)(\tilde{\Theta} - \tilde{M}(\lambda))^{-1}\tilde{\gamma}(\lambda)^*$$

(4.6.7)
for all \( \lambda \in \rho(\tilde{A}) \cap \rho(\tilde{A}_0) \), where \( \tilde{\gamma} \) and \( \tilde{M} \) denote the \( \gamma \)-field and the Weyl function corresponding to \( \{ \mathcal{S} \oplus \mathcal{S}, \tilde{\Gamma}_0, \tilde{\Gamma}_1 \} \) in (4.6.3). From (4.6.5) and (4.6.3) one obtains

\[
(\tilde{\Theta} - \tilde{M}(\lambda))^{-1} = \left\{ \left( \begin{array}{c} \psi - M(\lambda) \varphi \\ -\psi - \tau(\lambda) \varphi \end{array} \right), \left( \begin{array}{c} \varphi \\ \tau(\lambda) \varphi \end{array} \right) \right\} : \varphi, \psi \in \mathcal{S}.
\]

Setting \( \phi := \psi - M(\lambda) \varphi \) and \( \chi := -\psi - \tau(\lambda) \varphi \) one has \( \phi + \chi = -(M(\lambda) + \tau(\lambda)) \varphi \). For \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) it follows from Lemma 1.11.5 (see also Proposition 1.12.6) that \( (M(\lambda) + \tau(\lambda))^{-1} \in \mathcal{B}(\mathcal{S}) \), and hence

\[
\varphi = -(M(\lambda) + \tau(\lambda))^{-1} \phi - (M(\lambda) + \tau(\lambda))^{-1} \chi, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]

This yields

\[
(\tilde{\Theta} - \tilde{M}(\lambda))^{-1} = -\left( \begin{array}{cc} (M(\lambda) + \tau(\lambda))^{-1} & (M(\lambda) + \tau(\lambda))^{-1} \\ (M(\lambda) + \tau(\lambda))^{-1} & (M(\lambda) + \tau(\lambda))^{-1} \end{array} \right),
\]

and the statement about the resolvent of \( \tilde{A} \) follows from (4.6.7). \( \square \)

The compressions of the resolvent of the self-adjoint relation \( \tilde{A} \) in (4.6.4) to \( \tilde{\mathcal{S}} \) and \( \tilde{\mathcal{S}}' \) are of interest. Note that the resolvent of \( \tilde{A}_0 \) in (4.6.2) is given by the direct orthogonal sum of the resolvents of \( A_0 \) and \( B_0 \), and hence for \( \lambda \in \rho(\tilde{A}_0) \) the compressions to \( \tilde{\mathcal{S}} \) and \( \tilde{\mathcal{S}}' \) are

\[
P_{\tilde{\mathcal{S}}}(\tilde{A}_0 - \lambda)^{-1} \iota_{\tilde{\mathcal{S}}} = (A_0 - \lambda)^{-1} \quad \text{and} \quad P_{\tilde{\mathcal{S}}'}(\tilde{A}_0 - \lambda)^{-1} \iota_{\tilde{\mathcal{S}}'} = (B_0 - \lambda)^{-1},
\]

respectively. The next statement follows directly from Proposition 4.6.1 and (4.6.3).

**Corollary 4.6.2.** Let \( S \) and \( T \) be closed symmetric relations in the Hilbert spaces \( \tilde{\mathcal{S}} \) and \( \tilde{\mathcal{S}}' \) with boundary triplets \( \{ \mathcal{S}, \Gamma_0, \Gamma_1 \} \) and \( \{ \mathcal{S}, \Gamma_0', \Gamma_1' \} \), and corresponding \( \gamma \)-fields and Weyl functions \( \gamma, \gamma' \) and \( M, \tau \), respectively. Then for all \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) the following statements hold:

(i) The compression of the resolvent of the self-adjoint relation \( \tilde{A} \) in (4.6.4) to \( \tilde{\mathcal{S}} \) is given by

\[
P_{\tilde{\mathcal{S}}}((\tilde{A} - \lambda)^{-1} \iota_{\tilde{\mathcal{S}}} = (A_0 - \lambda)^{-1} - \gamma(\lambda)(M(\lambda) + \tau(\lambda))^{-1}(A - \lambda)^*.
\]

(ii) The compression of the resolvent of the self-adjoint relation \( \tilde{A} \) in (4.6.4) to \( \tilde{\mathcal{S}}' \) is given by

\[
P_{\tilde{\mathcal{S}}'}((\tilde{A} - \lambda)^{-1} \iota_{\tilde{\mathcal{S}}'} = (B_0 - \lambda)^{-1} - \gamma'(\lambda)(M(\lambda) + \tau(\lambda))^{-1}\gamma'(A - \lambda)^*.
\]

Corollary 4.6.2 and Proposition 4.6.1 can also be viewed as an alternative approach to the Krein–Naimark formula in the special case where the Nevanlinna family \( \tau \) in Theorem 4.5.2 is a uniformly strict Nevanlinna function. In fact, according to Theorem 4.2.4 every uniformly strict \( \mathcal{B}(\mathcal{S}) \)-valued Nevanlinna function
can be realized as a Weyl function, that is, there exist a (reproducing kernel) Hilbert space $\mathcal{H}' (= \mathcal{H}(N_\tau))$, a closed simple symmetric operator $T (= S_\tau)$ in $\mathcal{H}'$, and a boundary triplet $\{\mathcal{G}, \Gamma_0', \Gamma_1\}$ for the adjoint $T^*$ such that $\tau$ is the corresponding Weyl function. In this situation the relation $\widetilde{A}$ in (4.6.4) is self-adjoint in $\mathcal{H} \oplus \mathcal{H}' = \mathcal{H}(N_\tau)$ and its compressed resolvent in Corollary 4.6.2 coincides with the one in the Krein–Naïmark formula in Theorem 4.5.2. Summing up, the following special case of Theorem 4.5.2 is a consequence of the coupling method in Proposition 4.6.1 and Corollary 4.6.2.

**Corollary 4.6.3.** Let $\mathcal{S}$ be a closed symmetric relation, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $\mathcal{S}^*$ with $A_0 = \ker \Gamma_0$ and let $\gamma$ and $M$ be the corresponding $\gamma$-field and Weyl function, respectively. Let $\tau$ be a uniformly strict $\mathcal{B}(\mathcal{G})$-valued Nevanlinna function. Then there exist an exit Hilbert space $\mathcal{H}'$ and a self-adjoint relation $\widetilde{A}$ in $\mathcal{H} \oplus \mathcal{H}'$ such that $\widetilde{A}$ is an extension of $\mathcal{S}$ and the compressed resolvent of $\widetilde{A}$ is given by the Krein–Naïmark formula:

$$P_{\mathcal{H}'}(\widetilde{A} - \lambda)^{-1}I_{\mathcal{H}'} = (A_0 - \lambda)^{-1} - \gamma(\lambda)(M(\lambda) + \tau(\lambda))^{-1}\gamma(\lambda)^*, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. $$

Furthermore, the self-adjoint relation $\widetilde{A}$ satisfies the minimality condition

$$\mathcal{H}' = \overline{\operatorname{span}} \left\{ \ker (\widetilde{T}^* - \mu) : \mu \in \mathbb{C} \setminus \mathbb{R} \right\}. $$

**(Proof.** All statements except the minimality condition (4.6.8) follow from Proposition 4.6.1, Corollary 4.6.2, and Theorem 4.2.4 as explained above. For (4.6.8) recall first that the closed symmetric operator $S_\tau (= T)$ in Theorem 4.2.4 is simple, and hence

$$\mathcal{H}' = \overline{\operatorname{span}} \left\{ \ker (T^* - \mu) : \mu \in \mathbb{C} \setminus \mathbb{R} \right\} = \overline{\operatorname{span}} \left\{ \operatorname{ran} \gamma'(\mu) : \mu \in \mathbb{C} \setminus \mathbb{R} \right\}. $$

It follows from Proposition 4.6.1 that

$$P_{\mathcal{H}'}(\widetilde{A} - \mu)^{-1}I_{\mathcal{H}'} = -\gamma'(\mu)(M(\mu) + \tau(\mu))^{-1}\gamma(\mu)^*, $$

and since $\operatorname{ran} \gamma(\mu)^* = \mathcal{G}$ and $\operatorname{dom}(M(\mu) + \tau(\mu)) = \mathcal{G}$, one sees that

$$\operatorname{ran} (P_{\mathcal{H}'}(\widetilde{A} - \mu)^{-1}I_{\mathcal{H}'}) = \operatorname{ran} \gamma'(\mu), \quad \mu \in \mathbb{C} \setminus \mathbb{R}. $$

With (4.6.9) one then concludes that

$$\mathcal{H}' = \overline{\operatorname{span}} \left\{ \operatorname{ran} (P_{\mathcal{H}'}(\widetilde{A} - \mu)^{-1}I_{\mathcal{H}'}) : \mu \in \mathbb{C} \setminus \mathbb{R} \right\}, $$

which in turn yields (4.6.8). $\square$

In the next proposition a particular boundary triplet $\{\mathcal{G} \oplus \mathcal{G}, \hat{\Gamma}_0, \hat{\Gamma}_1\}$ is specified such that the self-adjoint relation $\widetilde{A}$ in (4.6.4) coincides with the kernel of the boundary mapping $\hat{\Gamma}_0$. The corresponding Weyl function $\hat{M}$ is useful for the spectral analysis of $\widetilde{A}$; cf. Chapter 6.
Proposition 4.6.4. Let $S$ and $T$ be closed symmetric relations in the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ with boundary triplets $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ and $\{\mathcal{G}', \Gamma'_0, \Gamma'_1\}$ and corresponding Weyl functions $M$ and $\tau$, respectively, as in the beginning of this section. Then $\{\mathcal{G} \oplus \mathcal{G}', \hat{\Gamma}_0, \hat{\Gamma}_1\}$, where

$$
\hat{\Gamma}_0 \left( \begin{array}{c} f \\ \bar{g} \end{array} \right) = \left( \begin{array}{c} -\Gamma_1 \hat{f} - \Gamma'_1 \hat{g} \\ \Gamma_0 \hat{f} - \Gamma'_0 \hat{g} \end{array} \right) \quad \text{and} \quad \hat{\Gamma}_1 \left( \begin{array}{c} f \\ \bar{g} \end{array} \right) = \left( \begin{array}{c} \Gamma_0 \hat{f} \\ -\Gamma'_1 \hat{g} \end{array} \right), \quad \hat{f} \in S^*, \hat{g} \in T^*,
$$

is a boundary triplet for $S^* \oplus T^*$ such that the self-adjoint relation $\tilde{A}$ in (4.6.4) corresponds to the boundary mapping $\hat{\Gamma}_0$, that is,

$$
\tilde{A} = \ker \hat{\Gamma}_0.
$$

The Weyl function of $\{\mathcal{G} \oplus \mathcal{G}', \hat{\Gamma}_0, \hat{\Gamma}_1\}$ is given by

$$
\tilde{M}(\lambda) = - \begin{pmatrix} M(\lambda) & -I \\ -I & -\tau(\lambda)^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} -(M(\lambda) + \tau(\lambda))^{-1} & (M(\lambda) + \tau(\lambda))^{-1} \tau(\lambda) \\ \tau(\lambda)(M(\lambda) + \tau(\lambda))^{-1} & \tau(\lambda)(M(\lambda) + \tau(\lambda))^{-1}M(\lambda) \end{pmatrix}
$$

for $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Proof. Instead of a direct proof the assertions will be obtained as consequences of the results in Section 2.5. For this consider the boundary triplet $\{\mathcal{G} \oplus \mathcal{G}, \hat{\Gamma}_0, \hat{\Gamma}_1\}$ in (4.6.1) with the Weyl function $\tilde{M}$ given in (4.6.3), let

$$
\mathcal{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} I & 0 \\ I & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & I \\ 0 & -I \end{pmatrix},
$$

and observe that $\tilde{\Theta} = \{\mathcal{A}, \mathcal{B}\}$, with $\tilde{\Theta}$ in (4.6.5). It is easy to see that $\mathcal{A}$ and $\mathcal{B}$ satisfy the conditions in Corollary 2.5.11. Therefore, $\{\mathcal{G}', \hat{\Gamma}_0, \hat{\Gamma}_1\}$, where

$$
\hat{\Gamma}_0 = \mathcal{B}^* \hat{\Gamma}_0 - \mathcal{A}^* \hat{\Gamma}_1 \quad \text{and} \quad \hat{\Gamma}_1 = \mathcal{A}^* \hat{\Gamma}_0 + \mathcal{B}^* \hat{\Gamma}_1,
$$

is a boundary triplet with corresponding Weyl function

$$
\tilde{M}(\lambda) = (\mathcal{A}^* + \mathcal{B}^* \tilde{M}(\lambda)) (\mathcal{B}^* - \mathcal{A}^* \tilde{M}(\lambda))^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
$$

It follows that

$$
\hat{\Gamma}_0 \left( \begin{array}{c} \hat{f} \\ \bar{\hat{g}} \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} -\Gamma_1 \hat{f} - \Gamma'_1 \hat{g} \\ \Gamma_0 \hat{f} - \Gamma'_0 \hat{g} \end{array} \right), \quad \hat{f} \in S^*, \hat{g} \in T^*,
$$

and

$$
\hat{\Gamma}_1 \left( \begin{array}{c} \hat{f} \\ \bar{\hat{g}} \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \Gamma_0 \hat{f} + \Gamma'_0 \hat{g} \\ \Gamma_1 \hat{f} - \Gamma'_1 \hat{g} \end{array} \right), \quad \hat{f} \in S^*, \hat{g} \in T^*.
$$
Furthermore, it is easily seen from the above that
\[ \tilde{M}(\lambda) = \begin{pmatrix} 1 & 1 \\ M(\lambda) & -\tau(\lambda) \end{pmatrix} \begin{pmatrix} -M(\lambda) & -\tau(\lambda) \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ M(\lambda) & -\tau(\lambda) \end{pmatrix} \begin{pmatrix} -(M(\lambda) + \tau(\lambda))^{-1} \tau(\lambda) \\ -(M(\lambda) + \tau(\lambda))^{-1} -(M(\lambda) + \tau(\lambda))^{-1} M(\lambda) \end{pmatrix}, \]
where the last step used the identity
\[ M(\lambda)(M(\lambda) + \tau(\lambda))^{-1} \tau(\lambda) = \tau(\lambda)(M(\lambda) + \tau(\lambda))^{-1} M(\lambda). \]

Thus, it is clear that
\[ \tilde{M}(\lambda) = \begin{pmatrix} -2(M(\lambda) + \tau(\lambda))^{-1} \\ (\tau(\lambda) - M(\lambda))(M(\lambda) + \tau(\lambda))^{-1} \\ 2M(\lambda)(M(\lambda) + \tau(\lambda))^{-1} \tau(\lambda) \end{pmatrix} \]
holds for all \( \lambda \in \mathbb{C} \setminus \mathbb{R} \). Now let
\[ D = \frac{1}{\sqrt{2}} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = D^* \quad \text{and} \quad P = \frac{1}{2} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \]
and apply Corollary 2.5.5 to conclude that
\[ \hat{\Gamma}_0 = D^{-1} \tilde{\Gamma}_0 \quad \text{and} \quad \hat{\Gamma}_1 = D^* \tilde{\Gamma}_1 + PD^{-1} \tilde{\Gamma}_0 \]
give a boundary triplet for \( S^* \oplus T^* \). According to Corollary 2.5.5, the corresponding Weyl function is given by
\[ \hat{M}(\lambda) = D^* \tilde{M}(\lambda) D + P, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \]
and one verifies that the first identity in \((4.6.10)\) holds. It is straightforward to check the second identity in \((4.6.10)\). \(\square\)