On the Design of Multi-Dimensional Irregular Repeat-Accumulate Lattice Codes

Min Qiu, Lei Yang, Yixuan Xie and Jinhong Yuan
School of Electrical Engineering and Telecommunications, The University of New South Wales, Sydney, Australia

Abstract—We propose and design the lattice codes with finite lattice constellations based on multi-dimensional (more than two dimensions) lattice partitions. The codes are constructed from non-binary irregular repeat-accumulate (IRA) codes. Most notably, we propose a novel encoding structure to ensure that the decoder’s messages exhibit permutation-invariance and symmetry properties. With these two properties, the densities of the messages in our iterative decoder can be well modeled by Gaussian distributions described by a single parameter. Under the Gaussian approximation, extrinsic information transfer charts for our multi-dimensional IRA lattice codes are developed and used for analysing the convergence behaviour and optimising the decoding threshold. Simulation results show that our proposed lattice codes outperform the previously designed lattice codes with two-dimensional lattice partitions.

I. INTRODUCTION

Lattice codes have been researched and studied because of their appealing algebraic structures and the capability of achieving the capacity on the additive white Gaussian noise (AWGN) channel [1]. There is tremendous work on constructing lattice codes which can be divided into two groups. The first group is to construct lattice codes directly in the Euclidean space. Some well-known examples are low-density lattice codes [2] and convolutional lattice codes [3]. Another group is to use capacity-achieving error correction codes to construct lattices including polar lattices [4], low-density parity-check (LDPC) lattices [5], [6], etc. Their construction methods involve Construction A [7] and Construction D [7]. These methods allow one to construct lattice codes with good error performances inherit from capacity-achieving linear codes. Most of the above designs have been shown to approach the Poltyrev limit [8] (i.e., the channel capacity without power restrictions) within 1 dB when the codeword length is long enough. In addition, all of these lattices can be decoded with efficient decoding algorithms.

Yet most of the available designs are based on infinite lattice constellations while their error performances are compared against Poltyrev limit. To put these lattice codes into practice, a power constraint must be satisfied. Moreover, most of the Construction A and Construction D lattice codes are designed based on one or two-dimensional (real dimension) lattice partitions. This can result in a shaping loss in error performance compared with using higher-dimensional lattice partitions [10]. Furthermore, most lattice codes have high complexity encoding structures due to the sparseness of their parity-check matrices which in general can lead to high-density generator matrices.

Recently, we have designed irregular repeat-accumulate (IRA) lattice network codes with finite constellations for two-way relay channels (TWRC) in [9]. Our schemes based on two-dimensional lattice partitions have achieved within 2 dB of the constellation constrained capacity of TWRC. That being said, our previous design cannot be directly extended to multi-dimensional lattice partitions because it requires the lattice partitions to form quotient rings whereas most multi-dimensional lattice partitions form additive quotient groups.

In this paper, we construct new multi-dimensional (more than two dimensions) IRA lattice codes based on finite constellations to further improve the error performances. We use Construction A as it has already been proved to be a simple and powerful tool for constructing capacity-achieving lattice codes [14]. Most notably, we propose a novel encoding structure to keep the design and analysis of multi-dimensional lattice codes simple. In particular, we prove that the proposed encoding structure introduces permutation-invariance and symmetry properties for the density distributions of the decoder’s messages. By exploring these two properties, we can use a Gaussian distribution characterised by a single parameter to model the soft information of the iterative decoder. This allows us to use two-dimensional extrinsic information transfer (EXIT) charts to design and analyse our lattice codes. In addition, we introduce a constraint in our encoding structure to maintain the linearity of the proposed codes. Numerical results show that our designed and optimised multi-dimensional lattice codes can achieve within 0.46 dB of the Shannon limit. At the same information rate, our proposed lattice codes significantly outperform previously designed lattice codes [9] with two-dimensional lattice partitions.

II. CONSTRUCTION OF MULTI-DIMENSIONAL IRA LATTICE CODES

A. Lattice Preliminaries

We first introduce some basic concepts of lattices and lattice codes [7] which will be used in the rest of the paper. An n-dimensional lattice $\Lambda$ is a discrete set of points $\lambda$ in $\mathbb{R}^n$ or $\mathbb{C}^n$. It can also be generated from an $k \times n$ ($k \leq n$) generator matrix $G_\Lambda$ via:

$$\Lambda = \{\lambda = bG_\Lambda, b \in \mathcal{P}^k\},$$

(1)
where \( P \) can be integers \( \mathbb{Z} \) or Gaussian integers \( \mathbb{Z}[i] = \{a + bi, a, b \in \mathbb{Z}\} \), etc. Here, \( \Lambda \) is a lattice point with dimension \( n \) or it can be deemed as a lattice codeword with length \( n \).

A nearest-neighbour lattice quantiser \( Q_\Lambda(x) \) maps a point \( x \) to its closest lattice point:

\[
Q_\Lambda(x) = \arg \min_{\lambda \in \Lambda} \|x - \lambda\|. \tag{2}
\]

The modulo-lattice operation with respect to \( \Lambda \) is defined as:

\[
x/\Lambda = x \mod \Lambda = x - Q_\Lambda(x). \tag{3}
\]

The modulo-lattice addition with respect to \( \Lambda \) is denoted by “\( \oplus \)’’ and defined as:

\[
\lambda_1 \oplus \lambda_2 = (\lambda_1 + \lambda_2) \mod \Lambda, \quad \lambda_1, \lambda_2 \in \Lambda. \tag{4}
\]

A lattice partition is formed by:

\[
\Lambda/\Lambda' = \{\lambda + \Lambda' : \lambda \in \Lambda\}, \tag{5}
\]

where \( \Lambda' \) is a sublattice of \( \Lambda : \Lambda' \subseteq \Lambda \). Note that for each \( \lambda \in \Lambda \), the set \( \Lambda + \Lambda' \) is a coset of \( \Lambda' \) in \( \Lambda \). The point \( \lambda \mod \Lambda' \) is called the coset leader of \( \lambda + \Lambda' \). We denote the set of coset leaders by: \( \Psi = \{\psi_0, \psi_1, \ldots, \psi_{M-1}\} \), where \( M \) is the cardinality of the lattice partition.

**B. Code Construction**

We construct lattices from non-binary IRA codes by using Construction A [7]. The error performances of Construction A lattices heavily depend on the underlying linear codes. Thus we choose IRA codes as they have been shown to be capacity approaching [11], [12] in AWGN channels and have lower encoding complexity than most LDPC encoders. In this work, we extend the concept of Construction A to a more generic case which is not just limited to real integer lattices. Denote an \((N, K)\) non-binary IRA codes over \(GF(p^m)\) by \(C\), where \( p \) is a prime number and \( m \) is a positive integer. The Construction A lattice \( \Lambda_C \) is generated via

\[
\Lambda_C = \{\lambda = \phi(C) + \xi \mathbb{R}^N\}, \tag{6}
\]

where \( \xi \in \mathbb{R} \) and \( \mathbb{R} \) is a lattice; \( \phi(.) \) is a homomorphism mapping function

\[
\phi : \mathbb{F}_p^m \rightarrow \mathbb{R}/\xi \mathbb{R}. \tag{7}
\]

Note that \( N \) in (6) should be a multiple of \( m \) in (7). Here the \( \mathbb{R} \)-lattice is partitioned into \( p^m \) numbers of cosets where each coset has a coset leader. For designing finite constellations, only coset leaders are used in transmission. In our design, we take into account transmitting complex signals. Thus, the information rate \( R \) for this Construction A lattice is

\[
R = \frac{K}{N} \cdot \frac{1}{n} \log_2(p^m), \tag{8}
\]

where \( n \) is the complex dimension of \( \mathbb{R} \).

We now design a specific example of using the \( D_4 \) lattice via Construction A. According to [7], the \( D_4 \) lattice is a four-dimensional lattice which has the highest sphere packing density in \( \mathbb{R}^4 \) and has a shaping gain about 0.37 dB. The generator matrix in integer lattice form:

\[
G_{D_4} = \begin{bmatrix}
-1 & -1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 1 & -1
\end{bmatrix}. \tag{9}
\]

The \( D_4 \) lattice can be identified as Hurwitz integers [15]:

\[
\mathbb{H} = \{a + bi + cj + dk | a, b, c, d \in \mathbb{Z} \text{ or } a, b, c, d \in \mathbb{Z} + \frac{1}{2}\}. \tag{10}
\]

Addition in \( \mathbb{H} \) is component wise whereas multiplication is non-commutative and defined based on the relations: \( i^2 = j^2 = k^2 = ijk = -1 \). Given \( A = a + bi + cj + dk \), the norm of \( A \) is:

\[
N(A) = a^2 + b^2 + c^2 + d^2 \in \mathbb{Z}.
\]

As an example, in (7), if we let \( \xi = 1 + 2i \), then the group homomorphism mapping function becomes:

\[
\phi : \mathbb{F}_2^2 \rightarrow \mathbb{H}/(1 + 2i)\mathbb{H}. \tag{11}
\]

To perform the partition, we follow the approach in [16] to develop the quantiser for \( D_4 \) and then apply (5). It should be noticed that the multiplication and division involved in this partition should follow quaternion arithmetic [15]. The cardinality of this partition is \( N(1 + 2i)^2 = 25 \). In this way, the \( D_4 \) lattice is partitioned into 25 cosets. It is also noteworthy that \( D_4 \) can be regarded as a complex lattice with two complex dimensions. Thus, for example, given a \( 1/2 \) IRA code, using (8) the information rate \( R = \frac{1}{2} \cdot \frac{1}{2} \log_2(5^2) = 1.161 \) bits/s/Hz.

**III. DESIGN OF MULTI-DIMENSIONAL IRA LATTICE CODES**

Non-binary IRA codes have been used in [9] to construct IRA lattice codes with two-dimensional lattice partitions. The previous design involves multiplication between the elements of a quotient ring formed by a two-dimensional lattice partition. However, most multi-dimensional lattice partitions form additive quotient groups which cannot be used in the previous case because only addition can be defined in additive groups. Therefore, we propose a novel design that only uses addition in the encoder. This allows the use of multi-dimensional lattice partitions.

**A. IRA Lattice Encoder**

The block diagram of the IRA lattice encoder is depicted in Fig. 1. The input to the encoder is a length \( K \) message \( u = [u_1, u_2, \ldots, u_K]^T \), where each element \( u_k \) for \( k = 1, 2, \ldots, K \) is taken from the set of coset leaders \( \Psi = \{\psi_0, \psi_1, \ldots, \psi_{p^m-1}\} \). This message \( u \) is fed into a repeater and repeated according to a discrete distribution of \( f_2, f_3, \ldots, f_I \), where \( f_i \geq 0 \) for \( i = 2, 3, \ldots, I \) and \( \sum_i f_i = 1 \). The number \( f_i \) represents the fraction of message symbols are repeated \( i \) times.

After repeating, the total number of symbols becomes \( L = K \sum_i f_i \). Next, the repeated symbols are interleaved and become \( z = [z_1, z_2, \ldots, z_L]^T \). A randomly generated sequence \( g = [g_1, g_2, \ldots, g_L]^T \), where \( g_i \in \Psi \) for \( l = 1, 2, \ldots, L \) is added to the interleaved sequence via \( z \oplus g \) in an element-wise manner. Note that \( \oplus \) is defined in (4).
The resultant symbols are combined according to a discrete distribution of $b_1, b_2, \ldots, b_J$, where $b_j \geq 0$ for $j = 1, 2, \ldots, J$ and $\sum_j b_j = 1$. The number $b_j$ represents the fraction of message symbols that are obtained from combining $j$ symbols from the interleaver and the corresponding $j$ elements. We randomly choose $\{c_1, c_2, \ldots, c_N\}$ from the set of coset leaders where elements are uniformly distributed over the set of coset leaders. The output sequence from the time-varying accumulator is $c = \{c_1, c_2, \ldots, c_N\}$. At time $n$ and $n \in \{1, 2, \ldots, N\}$, each symbol $c_n$ is generated by

$$c_n = (s_n \oplus (c_{n-1} \oplus g_n')) \oplus g_n'',$$

where the initial condition is given as $c_0 = 0$.

Finally, the lattice codeword $x$ is obtained by adding a length $N$ random-coset vector $c$ to a symbol-wise manner via $x = c \oplus r$. Elements of $r$ are uniformly distributed over the set of coset leaders $\Psi$. Before transmission, the average energy of codeword symbols is normalised to 1.

Although the four lattice sequences $g, g', g'', r$ and $c$ are random, they are assumed to be known at both transmitters and receivers prior to transmission. Ideally, they should be independent and identically distributed. However, due to the addition of $g, g'$ and $g''$ in our codes, we need to introduce a constraint on them to ensure the codes are linear. That is, for the $n$-th output element from the encoder, we must have

$$g_{t+j-1} \oplus \cdots \oplus g_t \oplus g_n' \oplus g_n'' = 0,$$

where $g_{t+j-1} \oplus \cdots \oplus g_t$ are the addition factors with respect to the first and last interleaved symbols at the $n$-th combiner. Note that this equation has $j + 2$ elements. We randomly choose any $j + 1$ elements out of these $j + 2$ elements to be random and uniformly distributed over the set of coset leaders $\Psi$. The last element is then determined by Equation (13).

It is noteworthy that our proposed lattice encoding structure is different from our previous design. Particularly, instead of using the modulo-lattice multiplication between encoder messages and random lattice sequences in [9], we introduce the $\oplus$ operation in the encoding process. We will prove and show in Section III-C that the proposed structure can guarantee the decoder's messages satisfying the permutation-invariance property. Without this property, their densities cannot be approximated by one-dimensional functions. This will give rise to an extremely high complexity of design and analysis.

**B. IRA Lattice Decoder**

Similar to conventional binary IRA codes, a belief propagation (BP) decoder can be employed to decode our multi-dimensional IRA lattice codes. The decoder is modified according to the new parity-check equations, i.e., at the $n$-th check node (combiner), the parity-check equation is

$$(c_{n-1} \oplus g_n') \oplus (c_n \oplus g_n'') = 0,$$

where $c_n' \oplus c_n = 0$ and $z_{t+j-1}$ and $z_t$ denote the first and the last interleaved symbol at the $n$-th combiner.

The decoder attempts to recover the source message $u$ from the noisy observation of the AWGN channel output $y$. Before decoding, we first need to compute the symbol-wise a posteriori probability (APP) of each coset leader and for each lattice codeword component $x_n$

$$P_{\psi_k}[n] = \frac{P(x_n = \psi_k | y_n)}{\sum_{\psi_k} P(x_n = \psi_k | y_n)},$$

where $k = 0, 1, \ldots, p^m - 1$ and $\psi_k$ is the $k$-th coset leader. To be more specific, the APP is calculated by

$$P_{\psi_k}[n] = \eta \exp \left(-\frac{\|y_n - \sqrt{2SNR}\psi_k\|^2}{2\sigma_n^2}\right),$$

where $\sigma_n^2$ is the variance of the channel noise and $\eta$ is the normalisation factor to ensure that $\sum_{k=0}^{p^m-1} P_{\psi_k}[n] = 1$. Note that $\psi_k$ and $y_n$ both are vectors with length equal to the dimension of the lattice. In our design example, $\psi_k$ is a $D_4$ lattice point with four dimensions. We perform the symbol-wise maximum-likelihood detection. However, practical systems can only transmit and receive two-dimensional signals. Thus, the detection is joint detection for two two-dimensional signals.

Before performing iterative decoding, the APP vectors are fed into a coset remover to obtain the APP vectors with respect to $c$ in (12). The resultant APP becomes $P'_{\psi_k}[n] = P_{\psi_k \oplus r}[n]$, where $\oplus$ denotes the modulo-lattice subtraction similar to (4). After all of these initialisation steps are completed, the final APP vectors are passed into the BP decoder. Due to the limited space, we omit the detailed algorithm here.

**C. EXIT Charts And Design Examples**

In this paper, the analysis of our multi-dimensional IRA lattice codes focus on the average behaviour of randomly selected codes from an ensemble of codes. First, we let $\alpha_i$
be the fraction of interleaver’s edges that connected to the information nodes (repeaters) with degree $i$ and let $\beta_i$ be the fraction of interleaver’s edges connected to the check nodes with degree $j + 2$. The additional “2” degrees is due to the deterministic connections from parity nodes (output of the accumulator) to check nodes. Given $(\alpha, \beta)$ and $\xi$ in (7), we define an $(\alpha, \beta, \xi, \mathbb{H})$ ensemble as the set of our multi-dimensional IRA lattice codes obtained via Construction A.

Now we briefly introduce the definition of symmetry and permutation-invariance properties of the decoder’s messages and explain how we can achieve these properties.

1) Symmetry: The probability of decoding error is equal for any transmitted codeword [13]. Recall in Section III-A, we add a random-coset vector $\mathbf{r}$ at the end of the encoder. The random-coset elements are randomly chosen and uniformly distributed over the set of coset leaders $\Psi$. Due to the isomorphism between finite fields and lattices in (7), our approach produces a similar effect to the output symmetry of non-binary coset LDPC codes in [13]. With this symmetry property, we can use all-zero lattice codewords in our EXIT chart analysis.

2) Permutation-invariance: Given a probability-vector random variable or a log-likelihood ratio (LLR)-vector random variable, each component of the random vector is identically distributed [13]. Recall in Section III-A, our codes have three randomly generated sequences added to the encoder messages. This leads to a symbol level permutation (the permutation from a coset leader to another coset leader) on the messages. Using our approach, we show that the densities of the messages in our BP decoder is permutation-invariant.

Define a probability-vector random variable $\mathbf{P} = [P_{\psi_0}, P_{\psi_1}, \ldots, P_{\psi_{p^m-1}}]$, where $P_{\psi_k}$ represents the probability of a lattice point being $\psi_k$ and satisfies $P_{\psi_k} \geq 0$, $\sum_{k=0}^{p^m-1} P_{\psi_k} = 1$. Given $\chi \in \Psi$, we define the $\oplus$ operation on $\mathbf{P}$ as:

$$\mathbf{P} \oplus \chi = [P_{\psi_0 \oplus \chi}, P_{\psi_1 \oplus \chi}, \ldots, P_{\psi_{p^m-1} \oplus \chi}].$$

(17)

Now we are ready to state the main theorem of the paper.

**Theorem 1.** Given a probability-vector random variable $\mathbf{P}$ and an $\chi \in \Psi$, the random vector $\mathbf{P} \oplus \chi$ is identically distributed with $\mathbf{P}$. Therefore $\mathbf{P}$ is permutation-invariant.

The proof of this theorem is similar to the proof in Appendix IV-D from [13]. Due to the limited space, we omit the detailed proof here. This theorem can be carried over straightforwardly to LLR representation. Thus we have the following lemma:

**Lemma 1.** An LLR-vector random variable $\mathbf{W}$ is permutation-invariant if the random vector $\mathbf{W} \oplus \chi$ is distributed identically with $\mathbf{W}$ for any fixed $\chi \in \Psi$.

Again the detail proof is omitted here. By adding the randomly generated lattice sequences, the LLR vectors in the iterative decoder can achieve permutation-invariance.

With the symmetry and permutation-invariance properties, the $p^m$-dimensional LLR can be modeled using a multivariate Gaussian distribution [13]

$$f_{\mathbf{W}}(\mathbf{w}) = \frac{1}{(2\pi)^{\frac{p^m}{2}} |\Sigma|^\frac{1}{2}} \exp \left( -\frac{1}{2} (\mathbf{w} - \mathbf{m})^T \Sigma^{-1} (\mathbf{w} - \mathbf{m}) \right),$$

(18)

with mean vector $\mathbf{m}$ where $m_i = \frac{c_i^2}{2}$ for $i = 1, 2, \ldots, p^m$ and covariance matrix $\Sigma$ where $\Sigma_{i,j} = \sigma^2$ if $i = j$ and $\frac{\sigma^2}{2}$ otherwise. As a result, the density of the $p^m$-dimensional LLR is completely described by a single parameter $\sigma$.

For the EXIT charts, we treat the variable nodes (VN) as a component decoder and the combiners and the time-varying accumulator as another component decoder. Based on [17], we developed the EXIT functions $I_A$ (a priori mutual information) and $I_E$ (extrinsic mutual information) for the input and output of variable-node decoder (VND) and the check-node decoder (CND), respectively. We employ the EXIT chart curve fitting technique [17] to find the optimal CN and VN degree distributions such that the area between the CND curve and the VND curve is minimised. The EXIT charts for our multi-dimensional IRA lattice codes over $\mathbb{H}/(1 + 2i)\mathbb{H}$ with code rates of $\frac{3}{2}$ and $\frac{1}{2}$ are shown in Fig. 2. In our design,

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{exit_chart.png}
\caption{EXIT Charts of optimised degree distributions for multi-dimensional IRA lattice codes with code rates $\frac{3}{2}$ and $\frac{1}{2}$.}
\end{figure}

the portion of degree 1 CN must larger than zero in order to ensure the decoder works in the first few iterations as the codes are nonsystematic. From Fig. 2, we can see that the VND curves for both code rates literally touches the CND curves for the range $[0, 1]$, which guarantees successful convergence and accurate decoding threshold.

We have adopted the proposed EXIT charts in designing the $(\alpha, \beta, \xi, \mathbb{H})$-lattice ensemble with code rate $\frac{3}{2}$ and $\frac{1}{2}$. The degree distributions and the decoding threshold are in Table I.

**IV. SIMULATION RESULTS**

In this section, we present our simulation results for our proposed multi-dimensional IRA lattice codes. The codes are designed over $\mathbb{H}/(1 + 2i)\mathbb{H}$. The codeword length was 100,000 symbols and the maximum number of iterations was set to
be 200. The performance for two designed code rates $\frac{1}{2}$ and $\frac{3}{2}$ is measured in terms of symbol error rate (SER) against SNR, which are depicted in Fig. 3. Based on the designed code rates, the corresponding information rates are calculated by (8) as $R_1 = 1.548 \text{ bits/s/Hz}$ and $R_2 = 1.161 \text{ bits/s/Hz}$, respectively. The unrestricted Shannon limit curves for these two information rates are plotted in the figure. We also show the SER performance of the previously proposed IRA lattice codes [9] based on the lattice partition $\mathcal{Z}[1]/(1+2i)\mathcal{Z}[1]$ in the figure for comparison because both partitions result in the same information rate. The unrestricted Shannon limit for $R_1$ is 2.84 dB. In this case, we observe that the gaps to the unrestricted Shannon limit and outperform the lattice coding schemes with two-dimensional lattice partitions.

**V. Conclusion**

In this paper, we designed new multi-dimensional IRA lattice codes with finite constellations. Most compellingly, we proposed a novel encoding structure to allow our codes to attain the permutation-invariance and symmetry properties in decoder’s messages. Under these properties, we used two-dimensional EXIT charts to analyse the convergence behaviour of our codes and to minimise the decoding threshold. Our design can employ any higher-dimensional lattice partitions. Numerical results show that our designed and optimised lattice codes can achieve within 0.46 dB of the Shannon limit and outperform the lattice coding schemes with two-dimensional lattice partitions.

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**TABLE I**

| Rates | Decoding Thresholds | Degree Distributions (i, $\alpha$) for variable nodes, (j, $\beta$) for check nodes |
|-------|---------------------|---------------------------------------------------------------|
| $\frac{1}{2}$ | 3.31 dB | $\alpha$: (2.0, 0.246085), (3.0, 0.237215), (5.0, 0.075854), (8.0, 0.199922), (19.0, 0.175951), (22.0, 0.053544) |
| $\frac{3}{2}$ | 1.26 dB | $\beta$: (1.0, 0.053861), (3.0, 0.946139) |

Fig. 3. SER of two designed multi-dimensional IRA lattice codes.

R $\frac{1}{2}$ is 2.84 dB. In this case, we observe that the gaps to the unrestricted Shannon limit when the SER at $10^{-5}$ is about 0.62 dB for our rate $\frac{1}{2}$ lattice code and 0.88 dB for the code in [9]. Thus, our newly designed lattice code is about 0.26 dB better than the lattice code with two-dimensional lattice partitions. The unrestricted Shannon limit for $R \frac{3}{2}$ is 0.92 dB. In this case, the gap between our lattice code and the unrestricted Shannon limit is about 0.46 dB. For the code in [9], the gap is 0.56 dB. Therefore, the proposed lattice code is about 0.1 dB better.

This figure also shows that the water-fall regions of the SER performance is about 0.15 dB to the predicted decoding thresholds as shown in Table I for various code rates. This indicates that the proposed EXIT chart analysis for our proposed multi-dimensional lattice codes is effective.