Existence of solutions for a nonlocal variational problem in $\mathbb{R}^2$ with exponential critical growth

Claudianor O. Alves
Universidade Federal de Campina Grande
Unidade Acadêmica de Matemática
CEP: 58429-900, Campina Grande - Pb, Brazil

Minbo Yang
Department of Mathematics, Zhejiang Normal University
Jinhua, Zhejiang, 321004, P. R. China.

Abstract
We study the existence of solution for the following class of nonlocal problem,

$$-\Delta u + V(x)u = \left( I_{\mu} * F(x, u) \right) f(x, u) \quad \text{in} \quad \mathbb{R}^2,$$

where $V$ is a positive periodic potential, $I_{\mu} = \frac{1}{|x|^{\mu}}$, $0 < \mu < 2$ and $F(x, s)$ is the primitive function of $f(x, s)$ in the variable $s$. In this paper, by assuming that the nonlinearity $f(x, s)$ has an exponential critical growth at infinity, we prove the existence of solutions by using variational methods.

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1 Introduction and main results
At the last years, many attention have been given to the problem

$$\begin{cases}
-\Delta u + V(x)u = \left( I_{\mu} * F(x, u) \right) f(x, u) & \text{in} \quad \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N), \\
u(x) > 0 & \text{for all} \quad x \in \mathbb{R}^N,
\end{cases} \quad (P)$$

where $0 < \mu < N$, $I_{\mu} = \frac{1}{|x|^{\mu}}$, $F(x, s)$ is the primitive function of $f(x, s)$ in the variable $s$ and $V, f$ are continuous verifying some conditions. Here $I_{\mu} * F(x, u)$ denotes the convolution between $I_{\mu}$ and $F(\cdot, u(\cdot))$.

This problem comes from looking for standing waves of the nonlinear nonlocal Schrödinger equation which is known to influence the propagation of electromagnetic

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*Partially supported by CNPq/Brazil 304036/2013-7, coalves@dme.ufcg.edu.br
†Supported by NSFC (11101374, 11271331) and CNPq/Brazil 500001/2013-8, mbyang@zjnu.edu.cn
waves in plasmas \[8\] and also plays an important role in the theory of Bose-Einstein condensation \[13\]. It is used in the description of the quantum theory of a polaron at rest by S. Pekar in 1954 \[23\] and the modeling of an electron trapped in its own hole in 1976 in the work of P. Choquard, in a certain approximation to Hartree-Fock theory of one-component plasma \[17\].

If \( F(x,s) = |s|^q \), then we arrive at the Choquard-Pekar equation,

\[
-\Delta u + V(x)u = \left( \frac{1}{|x|^\mu} \ast |u|^q \right) |u|^{q-2} u \quad \text{in} \quad \mathbb{R}^N.
\]

(1.1)

In the case \( N \geq 3 \), if \( V(x) = 1 \), Lieb \[17\] proved the existence and uniqueness, up to translations, of the ground state to equation (1.1). Later, in \[19\], Lions showed the existence of a sequence of radially symmetric solutions to this equation. Involving the properties of the ground state solutions, Ma and Zhao \[20\] proved that every positive solution of it is radially symmetric and monotone decreasing about some point, under the assumption that a certain set of real numbers, defined in terms of \( N, \mu \) and \( q \), is nonempty. Under the same assumption, Cingolani, Clapp and Secchi \[12\] proved the existence and multiplicity results in the electromagnetic case, and established the regularity and decay behavior at infinity of the ground state solutions of (1.1). Moroz and Van Schaftingen \[21\] eliminated this restriction and showed the regularity, positivity and radial symmetry of the ground states for the optimal range of parameters, and derived decay property at infinity as well. When \( V \) is a continuous periodic function with \( \inf_{\mathbb{R}^N} V(x) > 0 \), noticing that the nonlocal term is invariant under translation, one can obtain the existence result easily by applying the Mountain Pass Theorem, see \[1\] for example. For periodic potential \( V \) that changes sign and \( 0 \) lies in the gap of the spectrum of the Schrödinger operator \(-\Delta + V\), the problem is strongly indefinite it have been considered in \[9\]. In that paper, the existence of nontrivial solution with \( \mu = 1 \) and \( F(u) = u^2 \) have been obtained by using the reduction methods. For a general class of response function \( Q \) and nonlinearity \( f \), Ackermann \[1\] proposed an approach to prove the existence of infinitely many geometrically distinct weak solutions.

In the study made in the above papers, it was crucial the following Hardy-Littlewood-Sobolev inequality.

**Proposition 1.1.** \[18\] [Hardy – Littlewood – Sobolev inequality]:

Let \( t, r > 1 \) and \( 0 < \mu < N \) with \( 1/t + \mu/N + 1/r = 2 \). If \( f \in L^t(\mathbb{R}^N) \) and \( h \in L^r(\mathbb{R}^N) \), then there exists a sharp constant \( C(t, \mu, r) \), independent of \( f, h \), such that

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu} \leq C(t, \mu, r)|f|_t|h|_r.
\]

The above inequality permits to use variational method to get a solution for problem \((P)\), for a large class of nonlinearity \( f \), which has in general a subcritical growth. However, we can observe that the Hardy-Littlewood-Sobolev inequality also holds for \( N = 2 \), motivated by this fact, at least from a mathematical point of view, it seems to be interesting to ask if the existence of solution still holds for nonlinearities \( f \) having an "exponential subcritical growth" or "exponential critical growth" in \( \mathbb{R}^2 \), since a lot
of estimates made for the case $N \geq 3$ cannot be repeated easily for the case $N = 2$, when the nonlinearity $f$ has an exponential growth, because in dimension 2, it is well known that the Trudinger-Moser inequality is a crucial tool to work with this type of nonlinearity. Here, we focus our attention for more difficulty case, that is, the "exponential critical growth" in $\mathbb{R}^2$. However, we would point out that we cannot say that problem $(P)$ in $\mathbb{R}^2$ is a nonlinear Choquard equation, because in dimension 2 the kernel associated with a Choquard equation, namely the term $I_\mu$, must involve a logarithmic convolution potential, which does not occur in our problem.

Since we intend to work with nonlinearity with "exponential critical growth" in $\mathbb{R}^2$, we mean that the function $f(x, s)$ has an exponential critical growth when it behaves like $e^{\alpha s^2}$ as $|s| \to +\infty$. More exactly, there exists $\alpha_0 > 0$ such that

$$\lim_{|s| \to +\infty} \frac{|f(x, s)|}{e^{\alpha s^2}} = 0, \quad \forall \alpha > \alpha_0, \quad \text{and} \quad \lim_{|s| \to +\infty} \frac{|f(x, s)|}{e^{\alpha s^2}} = +\infty, \quad \forall \alpha < \alpha_0. \quad (1.2)$$

The above notation of criticality was introduced by Admurth and Yadava [4], see also de Figueiredo, Miyagaki and Ruf [10].

To work with problems where the nonlinearity has an exponential critical growth, one of the most important tools is the Trudinger-Moser inequality, which says that if $\Omega$ is a bounded domain in $\mathbb{R}^2$, then for all $\alpha > 0$ and $u \in H^1_0(\Omega)$, $e^{\alpha u^2} \in L^1(\Omega)$. Moreover, there exists a positive constant $C$ such that

$$\sup_{u \in H^1_0(\Omega) : \|\nabla u\|_2 \leq 1} \int_{\Omega} e^{\beta u^2} \leq C|\Omega| \quad \text{if} \quad \alpha \leq 4\pi,$$

where $|\Omega|$ denotes the Lebesgue measure of $\Omega$. This inequality is optimal, in the sense that for any growth $e^{\alpha u^2}$ with $\alpha > 4\pi$ the correspondent supremum is infinite. In the present paper, we are working in whole $\mathbb{R}^2$, this way, it is more convenient for us to use the following Trudinger-Moser type inequality in $H^1(\mathbb{R}^2)$ due to Cao [11], which is crucial for our variational arguments

**Lemma 1.2.** If $\alpha > 0$ and $u \in H^1(\mathbb{R}^2)$, then

$$\int_{\mathbb{R}^2} \left[ e^{\alpha |u|^2} - 1 \right] < \infty. \quad (1.3)$$

Moreover, if $\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1$, $\|u\|_{L^2(\mathbb{R}^2)} \leq M < \infty$ and $\alpha < \alpha_0 = 4\pi$, then there exists a constant $C$, which depends only on $M$ and $\alpha$, such that

$$\int_{\mathbb{R}^2} \left[ e^{\alpha |u|^2} - 1 \right] \leq C(M, \alpha). \quad (1.4)$$

We assume that $V : \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies:

(V) There are $\alpha > 0$ and a continuous 1-periodic continuous $V_0 : \mathbb{R}^2 \to \mathbb{R}$ such that

(1) $0 < \alpha \leq V(x) \leq V_0(x) \quad \forall x \in \mathbb{R}^2$

and

(2) $|V(x) - V_0(x)| \to 0 \quad |x| \to +\infty.$

3
In the sequel, $E$ denotes the Sobolev space $H^1(\mathbb{R}^2)$ equipped with the norm
\[
\|u\| := \left( \int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)|u|^2) \right)^{1/2}
\]
and $L^s(\mathbb{R}^2)$, for $1 \leq s \leq \infty$, denotes the Lebesgue space endowed with the usual norm $|\cdot|_s$.

Since the imbedding $H^1(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$ is continuous for any $p \in (2, +\infty)$, from the Hardy-Littlewood-Sobolev inequality, there is a best constant $S_p$ verifying
\[
S_p = \inf_{u \in E, u \neq 0} \left( \frac{\left( \int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)|u|^2) \right)^{1/2}}{\left( \int_{\mathbb{R}^2} (\mu * |u|^p)|u|^p \right)^{1/p}} \right).
\]

Moreover, a standard minimizing argument shows that there exists a positive radial function $u_p \in E$ such that $S_p$ is achieved by $u_p$.

Related to function $f$, we assume that there is a 1-periodic continuous function $\tilde{f}(x, s)$ such that:
\[
0 \leq \tilde{f}(x, s) \leq f(x, s) \leq C e^{4\pi s^2} \quad \forall s \geq 0. \quad (f_1)
\]

There holds
\[
\lim_{s \to 0} \frac{f(x, s)}{s^{2-\mu}} = 0, \quad \lim_{s \to 0} \frac{\tilde{f}(x, s)}{s^{2-\mu}} = 0. \quad (f_2)
\]

There exist $\theta \geq \tilde{\theta} > 2$, such that
\[
0 < \theta F(x, s) \leq 2f(x, s)s, \quad 0 < \tilde{\theta} \tilde{F}(x, s) \leq 2\tilde{f}(x, s)s \quad \forall s > 0, \quad (f_3)
\]
where $F(x, t) = \int_0^t f(x, s)ds$ and $\tilde{F}(x, s) = \int_0^s \tilde{f}(x, s)ds$.

There exists $p > \frac{4-\mu}{2}$, such that
\[
F(x, s) \geq C_p s^p \quad \forall s \geq 0 \quad (f_4)
\]
where
\[
C_p > \left( \frac{4\theta(p-1)}{(2-\mu)(\theta-2)} \right)^{\frac{p-1}{p}} S_p.
\]

For any fixed $x \in \mathbb{R}^2$, the functions
\[
s \to f(x, s), \quad \tilde{f}(x, s) \quad \text{are increasing}. \quad (f_5)
\]

Moreover, $\tilde{F}(x, s) > F(x, s)$ for any $s \neq 0$ and there exists $A \in L^\infty(\mathbb{R}^2)$ verifying
\[
A(x) \to 0 \quad \text{as } |x| \to \infty
\]
and
\[
|\tilde{f}(x, s) - f(x, s)| \leq A(x) \left( s^{\frac{2-\mu}{2}} + e^{4\pi s^2} \right) \quad \forall x \in \mathbb{R}^2 \quad \text{and} \quad \forall s \in \mathbb{R} \quad (f_6)
\]
The first result of this paper is associated with the periodic case, and it has the following statement

Theorem 1.3. Assume \( N = 2, 0 < \mu < 2 \), \((V - 1)\) with \( V = V_0 \) and \((f_1) - (f_3)\) with \( \dot{f} = f \). Then, \((P)\) has a ground state solution in \( H^1(\mathbb{R}^2) \).

Concerning with the problem where the nonlinearity is asymptotically periodic, our main result is the following

Theorem 1.4. Assume \( N = 2, 0 < \mu < 2 \), \((V)\) and \((f_1) - (f_6)\). Then, \((P)\) has a ground state solution in \( H^1(\mathbb{R}^2) \).

In this paper, we will use \( C, C_i \) to denote positive constants and \( B_R \) will denote the open ball centered at the origin with radius \( R > 0 \). If \( E \) is a real Hilbert space and \( I : E \to \mathbb{R} \) is a functional of class \( C^1(E, \mathbb{R}) \), we say that \((u_n) \subset E\) is a \((PS)_c\) sequence for \( I \), when \((u_n)\) satisfies

\[
I(u_n) \to c \text{ and } I'(u_n) \to 0 \text{ as } n \to \infty.
\]

Moreover, we say that \( I \) satisfies the \((PS)_c\), if any \((PS)_c\) sequence possesses a convergent subsequence.

To conclude this introduction, we would like to cite some recent works involving exponential critical growth for the elliptic problem of the form

\[
-\Delta u + V(x)u = f(x, u) \quad \text{in } \mathbb{R}^2.
\]

See for example, Adimurthi and K. Sandeep [3], Adimurthi and Yang [4], Albuquerque, Alves and Medeiros [5], do Ó, Medeiros and Severo [14], do Ó and de Souza [15], Li and Ruf [16] and their references.

2 Mountain Pass Geometry

Since we are going to study the existence of positive solution via variational method, we will assume that

\[
f(x, s) = 0 \quad \forall s \leq 0 \text{ and } \forall x \in \mathbb{R}^2.
\]

From \((f_1) - (f_3)\), for any \( \varepsilon > 0, p \geq 1 \) and \( \beta > 1 \), there exists \( C_\varepsilon > 0 \) such that

\[
|f(x, s)| \leq \varepsilon|s|^{\frac{\mu}{2}} + C(\varepsilon, p, \beta)|s|^{p-1}[e^{\beta 4\pi s^2} - 1] \quad \forall s \in \mathbb{R},
\]

and

\[
|F(x, s)| \leq \varepsilon|s|^{\frac{2-\mu}{2}} + C(\varepsilon, p, \beta)|s|^p[e^{\beta 4\pi s^2} - 1] \quad \forall s \in \mathbb{R}.
\]

From Lemma 1.2 and Hölder inequality, we deduce that \( F(x, u) \in L^{\frac{4}{4-\mu}}(\mathbb{R}^2) \) for any \( u \in H^1(\mathbb{R}^2) \). Then, applying Hardy-Littlewood-Sobolev inequality, with \( t = r = \frac{4}{4-\mu} \), we see that

\[
\left(I_\mu \ast F(x, u)\right) F(x, u) \in L^1(\mathbb{R}^2),
\]

and so, the energy functional \( I : E \to \mathbb{R} \) associated with problem \((SNE)\) given by

\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)|u|^2) - \frac{1}{2} \int_{\mathbb{R}^2} \left(I_\mu \ast F(x, u)\right) F(x, u)
\]

...
is well defined on $E$. Furthermore, $I \in C^1(E, \mathbb{R})$ and
\[ I'(u) \varphi = \int_{\mathbb{R}^2} (\nabla u \nabla \varphi + V(x) u \varphi) - \int_{\mathbb{R}^2} \left( I_\mu \ast F(x, u) \right) f(x, u) \quad \forall u, \varphi \in E. \]

Next, we will show that $I$ verifies the Mountain Pass Geometry.

**Lemma 2.1.** Assume $0 < \mu < 2$, $(f_1) - (f_3)$ and $(V - 1)$. Then,

1. There exist $\rho, \delta_0 > 0$ such that $I|_{S_p} \geq \delta_0 > 0$, $\forall u \in S_p = \{ u \in E : \|u\| = \rho \}$.
2. There is $e \in E$ with $\|e\| > \rho$ such that $I(e) < 0$.

**Proof.** (1). For any $\varepsilon > 0$, $p > 1$ and $\beta > 1$, there exists $C_\varepsilon > 0$ such that
\[ |F(x, s)| \leq \varepsilon |s|^\frac{4-\mu}{4-p} + C(\varepsilon, p, \beta) |s|^p [e^{4\beta s^2} - 1] \quad \forall s \in \mathbb{R}, \]
from where it follows
\[ |F(x, u)|^\frac{4}{4-p} \leq \varepsilon C |u|^{\frac{4-\mu}{4-p}} + C(\varepsilon, p, \beta) |u|^p [e^{4\beta s^2} - 1] \left| u \right|^{\frac{4}{4-p}}. \tag{2.1} \]

Since the imbedding $E \hookrightarrow L^p(\mathbb{R}^2)$ is continuous, for each $p \in (2, +\infty)$, there exists a constant $C_1 > 0$ such that
\[ \int_{\mathbb{R}^2} \left| u \right|^{\frac{4\beta}{4-p}} [e^{\beta 4\pi u^2} - 1] \left| u \right|^{\frac{4}{4-p}} \leq \left( \int_{\mathbb{R}^2} \left| u \right|^{\frac{8\beta}{4-p}} \right)^\frac{1}{4} \left( \int_{\mathbb{R}^2} [e^{\beta 4\pi u^2} - 1] \left| u \right|^{\frac{4}{4-p}} \right)^\frac{1}{4} \]
\[ \leq C_1 \|u\|^{\frac{4\beta}{4-p}} \left( \int_{\mathbb{R}^2} \left| e^{\frac{4\beta}{4-p} 4\pi u^2} - 1 \right| \right)^\frac{1}{4}. \]

Observing that
\[ \int_{\mathbb{R}^2} \left| e^{\frac{4\beta}{4-p} 4\pi u^2} - 1 \right| = \int_{\mathbb{R}^2} \left| e^{\frac{4\beta}{4-p} \|u\|^2 4\pi u^2} - 1 \right|, \]
fixing $\xi \in (0, 1)$ and $\frac{4\beta}{4-p} \|u\|^2 = \xi < 1$, the Lemma 1.2 gives
\[ \int_{\mathbb{R}^2} \left| e^{\xi 4\pi u^2} \left| u \right|^2 - 1 \right| \leq C_2 \quad \text{for} \quad \|u\| = \left( \frac{\xi (4 - \mu)}{4\beta} \right)^\frac{1}{2}, \]
for some positive constant $C_2$. Gathering the last estimate and (2.1), there exists $C_3 > 0$ such that
\[ |F(x, u)|^\frac{4}{4-p} \leq \varepsilon \|u\|^{\frac{4-\mu}{4-p}} + C_3 \|u\|^p \quad \text{for} \quad \|u\| = \left( \frac{\xi (4 - \mu)}{4\beta} \right)^\frac{1}{2}. \]

Thereby, by Hardy-Littlewood-Sobolev inequality,
\[ \int_{\mathbb{R}^2} \left( I_\mu \ast F(x, u) \right) F(x, u) \leq \varepsilon^2 C \|u\|^{4-\mu} + 2C_3 \|u\|^{2p} \quad \text{for} \quad \|u\| = \left( \frac{\xi (4 - \mu)}{4\beta} \right)^\frac{1}{2}, \]
and so,
\[ I(u) \geq \frac{1}{2} \|u\|^2 - \varepsilon^2 C \|u\|^{4-\mu} - C_3 \|u\|^{2p}. \]
Since \(0 < \mu < 2\) and \(p > 1\), (1) follows choosing \(\rho = \left(\frac{\xi(4-\mu)}{4^\beta}\right)^{\frac{1}{\beta}}\) with \(\xi \approx 0^+\).

(2) Fixing \(u_0 \in E\) with \(u_0^+(x) = \max\{u_0(x), 0\} \neq 0\), we set
\[
 A(t) = \Psi\left(\frac{tu_0}{\|u_0\|}\right) > 0 \quad \text{for} \quad t > 0,
\]
where
\[
 \Psi(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left( I_\mu * F(x, u) \right) F(x, u).
\]
A straightforward computation yields
\[
 \frac{A'(t)}{A(t)} \geq \frac{\theta}{t} \quad \text{for all} \quad t > 0.
\]
Then, integrating this over \([1, s\|u_0\|]\) with \(s > \frac{1}{\|u_0\|}\), we find
\[
 \Psi(su_0) \geq \Psi\left(\frac{u_0}{\|u_0\|}\right)\|u_0\|^\theta s^\theta.
\]
Therefore
\[
 \Phi(su_0) \leq C_1 s^2 - C_2 s^\theta \quad \text{for} \quad s > \frac{1}{\|u_0\|},
\]
and (2) holds for \(e = su_0\) with \(s\) large enough. \qed

By the Mountain Pass Theorem without \((PS)\) condition found in [27], there is an \((PS)_{c_V}\) sequence \((u_n) \subset E\), that is,
\[
 I(u_n) \to c_V \quad \text{and} \quad I'(u_n) \to 0,
\]
where \(c_V\) is the mountain pass level characterized by
\[
 0 < c_V := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))
\]
with
\[
 \Gamma := \{ \gamma \in \mathcal{C}^1([0,1], E) : \gamma(0) = 0 \quad \text{and} \quad I(\gamma(1)) < 0\}.
\]

The next lemma is crucial in our arguments, because it establishes an important estimate involving the level \(c_V\).

**Lemma 2.2.** The mountain pass level \(c_V\) satisfies \(c_V \in [\rho, \frac{(2-\mu)(\theta-2)}{8\theta})\). Moreover, the \((PS)_{c_V}\) sequence \((u_n)\) is bounded and its weak limit \(u\) satisfies \(I'(u) = 0\).

**Proof.** From \((f_3)\),
\[
 c_V = \lim \left( I(u_n) - \frac{1}{\theta} I'(u_n) u_n \right) \geq \left( \frac{1}{2} - \frac{1}{\theta} \right) \limsup \|u_n\|^2
\]
which means
\[
 \limsup \|u_n\|^2 \leq \frac{2\theta}{\theta - 2} c_V.
\]

7
Let \( u_p \in E \) be a positive radial function verifying
\[
S_p = \inf_{u \in E, u \neq 0} \left( \frac{\left( \int_{\mathbb{R}^2} (|\nabla u|^2 + |V|_{\infty}u^2) \right)^{1/2}}{\left( \int_{\mathbb{R}^2} (I_{\mu} \ast |u|^p) |u|^p \right)^{1/p}} \right).
\]

By \((f_4)\), it is easy to see that
\[
v_c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))
\leq \inf_{u \in E \setminus \{0\}} \max_{t \geq 0} I(tu)
\leq \max_{t \geq 0} I(tu_p)
\leq \max_{t \geq 0} \left\{ \frac{t^2}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + |V|_{\infty}u^2) - \frac{t^{2p}C_p^2}{2} \int_{\mathbb{R}^2} (I_{\mu} \ast |u|^p) |u|^p \right\}
= \frac{(p - 1)S_p^{2p - 1}}{2p^{p - 1} C_p^{p - 1}}
< \frac{(2 - \mu)(\theta - 2)}{8\theta}.
\]

Consequently, from \((2.4)\),
\[
\limsup \|u_n\|^2 < \frac{(2 - \mu)}{4}.
\]

In the following, we may assume that there are \( n_0 \in \mathbb{N} \) and \( m \in (0, \frac{(2 - \mu)}{4}) \), such that
\[
\|u_n\|^2 \leq m \quad \forall n \geq n_0. \tag{2.5}
\]
Without lost of generality, in what follows we suppose that \( n_0 = 1 \).

**Claim 2.3.** There exists \( C > 0 \) such that
\[
|I_{\mu} \ast F(x, u_n)|_{\infty} < C \quad \forall n \in \mathbb{N}.
\]

**Proof.** For each \( \beta > 1 \), there exists \( C_0 > 0 \) such that
\[
F(x, s) \leq C_0 \left( |s|^{\frac{4 - \mu}{2}} + |s| [e^{34\pi s^2} - 1] \right) \quad \forall s \in \mathbb{R}.
\]
Hence,
\[
|I_{\mu} \ast F(x, u_n)(x)| \leq \int_{\mathbb{R}^2} \frac{F(x, u_n)}{|x - y|^\mu}
= \int_{|x - y| \leq 1} \frac{|u_n|^{\frac{4 - \mu}{2}}}{|x - y|^\mu} + \int_{|x - y| \geq 1} \frac{|u_n| [e^{34\pi|u_n|^2} - 1]}{|x - y|^\mu}
\leq C_0 \int_{|x - y| \leq 1} \frac{|u_n|^{\frac{4 - \mu}{2}}}{|x - y|^\mu} + C_0 \int_{|x - y| \geq 1} \left( |u_n|^{\frac{4 - \mu}{2}} + |u_n| [e^{34\pi|u_n|^2} - 1] \right).
\]
Since 
\[ \frac{1}{|y|^\mu} \in L^{\frac{2+\delta}{\mu}} (B_1^c(0)) \quad \forall \ \delta > 0, \]
we take \( \delta \approx 0^+ \) verifying 
\[ q_{1,\delta} = \frac{(4 - \mu)}{2} \left( \frac{(2 + \delta)}{(2 + \delta) - \mu} \right) > 2. \]

Using Hölder inequality, we derive that 
\[ \int_{|x-y|\geq 1} \frac{|u_n|^{\frac{4-\mu}{2}}}{|x-y|^{\mu}} \leq C_0 \left( \int_{|x-y|\geq 1} |u_n|^{q_{1,\delta}} \right)^{\frac{(2+\delta)-\mu}{4-\mu}} \leq C_1 \ \forall n \in \mathbb{N}. \quad (2.6) \]

By (2.6), we can fix \( \beta > 1 \) close to 1, such that \( 2\beta m \in (0, 1) \). Then, by Trudinger-Moser inequality, there exists \( C_2 > 0 \) such that 
\[ \int_{|x-y|\geq 1} |u_n| \left[ e^{4\pi u_n^2} - 1 \right] \leq |u_n|_2 \int_{\mathbb{R}^2} \left( e^{2\beta m 4\pi \frac{u_n^2}{|u_n|^2} t} - 1 \right) \frac{1}{t} \leq C_2 \ \forall n \in \mathbb{N}. \quad (2.7) \]

Choosing \( t \in \left( \frac{2}{2-\mu}, \infty \right) \), we see that \( \frac{(4 - \mu)t}{2} > 2 \) and \( 1 - \frac{t\mu}{t-1} > -1 \). Thus, by Hölder inequality, 
\[ \int_{|x-y|\leq 1} \frac{|u_n|^{\frac{4-\mu}{2}}}{|x-y|^{\mu}} \leq \left( \int_{|x-y|\leq 1} |u_n|^{\frac{(4-\mu)t}{2}} \right)^{\frac{1}{t}} \left( \int_{|x-y|\leq 1} \frac{1}{|x-y|^{\frac{t\mu}{t-1}}} \right)^{\frac{t-1}{t}} \leq C_2 \left( \int_{|r|\leq 1} |r|^{1-\frac{t\mu}{t-1}} dr \right)^{\frac{t-1}{t}} \leq C_3 \ \forall n \in \mathbb{N}, \quad (2.8) \]

for some \( C_3 > 0 \). Now, for \( t > \frac{2}{2-\mu} \) and close to \( \frac{2}{2-\mu} \), we can assume that \( 2\beta m \in (0, 1) \). Thus, the Trudinger-Moser inequality and the boundedness of \( (u_n) \) in \( E \) combine to give 
\[ \int_{|x-y|\leq 1} \frac{|u_n| \left[ e^{4\pi u_n^2} - 1 \right]}{|x-y|^\mu} \leq \left( \int_{|x-y|\leq 1} |u_n| \left[ e^{4\pi u_n^2} - 1 \right] \right)^{\frac{1}{2}} \left( \int_{|x-y|\leq 1} \frac{1}{|x-y|^{\frac{t\mu}{t-1}}} \right)^{\frac{t-1}{2t}} \leq \left( \int_{|x-y|\leq 1} |u_n|^{2t} \right)^{\frac{1}{2t}} \left( \int_{|x-y|\leq 1} \left[ e^{2\beta m t 4\pi \frac{u_n^2}{|u_n|^2}} - 1 \right] \right)^{\frac{1}{2t}} \left( \int_{|r|\leq 1} |r|^{1-\frac{t\mu}{t-1}} dr \right)^{\frac{t-1}{2t}} \]

implying that there is \( C_4 > 0 \) such that 
\[ \int_{|x-y|\leq 1} \frac{|u_n| \left[ e^{4\pi u_n^2} - 1 \right]}{|x-y|^\mu} \leq C_4 \ \forall n \in \mathbb{N}. \quad (2.9) \]

Now, the claim follows from \((2.6)-(2.9)\). \( \square \)

**Claim 2.4.** Let \( (u_n) \) be the \((PS)_{cv}\) sequence with weak limit \( u \). Then, \( u \) satisfies \( I'(u) = 0 \).
Proof. Since \((u_n)\) is bounded in \(E\), going to a subsequence still denoted by \((u_n)\), there is \(u \in E\) such that
\[
u_n \rightharpoonup u \text{ in } E, u_n \to u \text{ in } L^q_{\text{loc}}(\mathbb{R}^2) \text{ for all } q \in [1, +\infty)\)
and \(u_n(x) \to u(x)\) a.e. in \(\mathbb{R}^2\).

For each \(\varphi \in C^\infty_0(\mathbb{R}^2)\), the Hardy-Littlewood-Sobolev inequality together with Hölder inequality lead to
\[
|\int_{\mathbb{R}^2} \left( I_\mu * f(x, u_n) \right) f(x, u_n) \varphi | \leq |f(x, u_n)| \frac{4}{4 + \mu} |f(x, u_n)| \frac{4r'}{4 + \mu} |\varphi| \frac{4r'}{4 + \mu}
\]
where \(r = \frac{4 - \mu}{2 + \mu}\) and \(r' > 1\) satisfies \(\frac{1}{r} + \frac{1}{r'} = 1\). From \((f_1)-(f_3)\), for each \(\varepsilon > 0\) and \(\beta > 1\), there is \(C(\varepsilon, \beta) > 0\) such that
\[
|f(x, s)| \leq \varepsilon |s| \frac{2 - \mu}{4 + \mu} + C(\varepsilon, \beta) |s| e^{\beta 4\pi s^2} - 1 \quad \forall s \in \mathbb{R},
\]
and
\[
|F(x, s)| \leq \varepsilon |s| \frac{2 - \mu}{4 + \mu} + C(\varepsilon, \beta) |s| e^{\beta 4\pi s^2} - 1 \quad \forall s \in \mathbb{R}.
\]
Then
\[
|f(x, u_n)| \frac{4}{4 + \mu} \leq \varepsilon |u_n|^{\frac{2 - \mu}{2 + \mu}} + C(\varepsilon, \beta) e^{\beta 4\pi u_n^2} - 1 \frac{4r'}{4 + \mu}
\leq C_1 \|u_n\|^{\frac{2 - \mu}{2 + \mu}} + C_1 \left( \int_{\mathbb{R}^2} e^{\beta r u_n^2} \|u_n\||u_n\|^2 \|u_n\|^2 - 1 \right)^{\frac{4r'}{4 + \mu}}.
\]
Now, choosing \(\beta > 1\) sufficiently close to 1, the estimate (2.5) and Trudinger-Moser combined give that \((f(x, u_n))\) is bounded in \(L^{\frac{4}{2 - \mu}}(\mathbb{R}^2)\). Moreover, with a similar argument, the sequence \((F(x, u_n))\) is also bounded \(L^{\frac{4}{2 + \mu}}(\mathbb{R}^2)\).

In the sequel, we will prove that for any \(\varphi \in C^\infty_0(\mathbb{R}^2)\), there limit below holds
\[
\int_{\mathbb{R}^2} \left( I_\mu * f(x, u_n) \right) f(x, u_n) \varphi \to \int_{\mathbb{R}^2} \left( I_\mu * f(x, u) \right) f(x, u) \varphi.
\]
In fact, for any \(\varphi \in C^\infty_0(\mathbb{R}^2)\),
\[
\left| \int_{\mathbb{R}^2} \left( I_\mu * f(x, u_n) \right) f(x, u_n) - (I_\mu * f(x, u)) f(x, u) \right| \varphi
\leq \left| \int_{\mathbb{R}^2} \left( I_\mu * f(x, u_n) \right) \left( f(x, u_n) - f(x, u) \right) \varphi \right|
+ \left| \int_{\mathbb{R}^2} \left( I_\mu * (f(x, u_n) - F(x, u)) \right) f(x, u) \varphi \right|.
\]
For the above first term, we recall that \((I_\mu * F(x, u_n))\) is bounded in \(L^\infty(\mathbb{R}^2)\). Then,
\[
\left| \int_{\mathbb{R}^2} \left( I_\mu * F(x, u_n) \right) \left( f(x, u_n) - f(x, u) \right) \varphi \right|
\leq C \left| \int_{\mathbb{R}^2} \left( f(x, u_n) - f(x, u) \right) \varphi \right|.
\]
Since \(u_n(x) \to u(x)\) a.e. in \(\mathbb{R}^2\), the continuity of \(f\) implies \(f(x, u_n(x)) \to f(x, u(x))\) a.e. in \(\mathbb{R}^2\). This fact combined with boundedness of \((f(x, u_n))\) in \(L^{\frac{4}{2 - \mu}}(\mathbb{R}^2)\) leads to
\[
f(x, u_n) \to f(x, u) \text{ in } L^{\frac{4}{2 - \mu}}(\mathbb{R}^2),
\]
from where it follows that
\[ \int_{\mathbb{R}^2} (f(x, u_n) - f(x, u)) \varphi \to 0. \]
Consequently,
\[ \left| \int_{\mathbb{R}^2} \left( I_\mu \ast F(x, u_n) \right) \left( f(x, u_n) - f(x, u) \right) \varphi \right| \to 0, \tag{2.12} \]
for any \( \varphi \in C_0^\infty(\mathbb{R}^2) \).

For the second term, notice that
\[ \left| \int_{\mathbb{R}^2} \left( I_\mu \ast (F(x, u_n) - F(x, u)) \right) f(x, u) \varphi \right| \]
\[ = \left| \int_{\mathbb{R}^2} (F(x, u_n) - F(x, u)) I_\mu \ast (f(x, u) \varphi) \right| \]
Since \( u_n(x) \to u(x) \) a.e. in \( \mathbb{R}^2 \), the continuity of \( F \) implies \( F(x, u_n(x)) \to F(x, u(x)) \) a.e. in \( \mathbb{R}^2 \). Using the boundedness of \( (F(x, u_n)) \) in \( L^{\frac{4}{1-\mu}}(\mathbb{R}^2) \), we conclude that
\[ F(x, u_n) \to F(x, u) \text{ in } L^{\frac{4}{1-\mu}}(\mathbb{R}^2). \]
As \( I_\mu \ast (f(x, u) \varphi) \in L^{\frac{4}{1-\mu}}(\mathbb{R}^2) \), we must have,
\[ \left| \int_{\mathbb{R}^2} \left( I_\mu \ast (F(x, u_n) - F(x, u)) \right) f(x, u) \varphi \right| \to 0, \tag{2.13} \]
for any \( \varphi \in C_0^\infty(\mathbb{R}^2) \). Now, the result follows by using the density of \( C_0^\infty(\mathbb{R}^2) \) in \( H^1(\mathbb{R}^2) \).

### 3 Proof of the main results

In this section, we will prove the Theorems 1.3 and 1.4.

#### 3.1 Proof of Theorem 1.3

Let \((u_n)_n\) be the \((PS)_{c_V}\) sequence. Since \((u_n)_n\) is bounded and \( \lim \sup \|u_n\|^2 < \frac{(2-\mu)}{4} \), we have either \((u_n)_n\) is vanishing, i.e., there exists \( r > 0 \) such that
\[ \lim \sup_{y \in \mathbb{R}^2} \int_{B_r(y)} |u_n|^2 = 0 \]
or non-vanishing, i.e., there exist \( r, \delta > 0 \) and a sequence \((y_n)_n \subset \mathbb{Z}^2\) such that
\[ \lim_{n \to \infty} \int_{B_r(y_n)} |u_n|^2 \geq \delta. \]
If \((u_n)_n\) is vanishing, then by Lion’s result, we have that
\[ u_n \to 0 \text{ in } L^s(\mathbb{R}^2), \quad 2 < s < +\infty. \]
Using Hardy-Littlewood-Sobolev inequality and \((f_3)\), we derive
\[
\left| \int_{\mathbb{R}^2} \left( I_\mu * F(x, u_n) \right) f(x, u_n) u_n \right| \leq C |F(x, u_n)| \frac{4}{4-\mu} |f(x, u_n) u_n| \frac{4}{4-\mu} \leq C |f(x, u_n) u_n| \frac{4}{4-\mu}.
\]
For any \(\varepsilon > 0\) and \(p > 1\), we have that \(C(\varepsilon, p, \beta) > 0\) such that
\[
|f(x, s)| \leq \varepsilon |s|^{\frac{2-\mu}{2}} + C(\varepsilon, p, \beta) |s|^{p-1} \left[ e^{\beta 4\pi s^2} - 1 \right] \quad \forall s \in \mathbb{R}.
\]
Then,
\[
|f(x, u_n) u_n|^{\frac{1}{4-\mu}} \leq \varepsilon |u_n|^{\frac{4-\mu}{2}} + C(\varepsilon, p, \beta) |u_n|^{\frac{4-\mu}{4-\mu}} \left( \int_{\mathbb{R}^2} \left[ e^{\frac{2\beta t}{4-\mu} u_n^2} - 1 \right] \right)^{\frac{4-\mu}{4\pi}}
\]
where \(t, t' > 1\) satisfying \(\frac{1}{4} + \frac{1}{t'} = 1\). Now, gathering \([2.5]\) and Trudinger-Moser inequality, if \(\beta, t > 1\), we fixed close to 1, we deduce that
\[
\left( \int_{\mathbb{R}^2} \left[ e^{\frac{4\beta t}{4-\mu} u_n^2} - 1 \right] \right) \leq C_1 \quad \forall n \in \mathbb{N},
\]
for some \(C_1 > 0\). Then,
\[
\left| \int_{\mathbb{R}^2} \left( I_\mu * F(x, u_n) \right) f(x, u_n) u_n \right| \leq \varepsilon |u_n|^{\frac{4-\mu}{2}} + C_2 |u_n|^{\frac{4-\mu}{4\pi t'}}.
\]
Since \(t > 1\) is close to 1, we have that \(\frac{4\pi t'}{4-\mu} > 2\). Consequently
\[
\int_{\mathbb{R}^2} \left( I_\mu * F(x, u_n) \right) f(x, u_n) u_n \to 0,
\]
implying that
\[
u_n \to 0 \quad \text{in} \quad E, \quad n \to \infty.
\]
Recalling that \(I\) is a continuous functional, we must have
\[
I(u_n) \to 0,
\]
from where it follows that \(c_V = 0\), which is a contradiction. Thereby, vanishing case does not hold.

From now on, we set \(v_n = u_n(\cdot - y_n)\). Therefore, \(|v_n| = |u_n|\) and
\[
\int_{B_r(0)} |v_n|^2 \geq \delta.
\]
Using the definition of \(I\), we see that \(I\) and \(I'\) are both invariant by \(\mathbb{Z}^2\)-translation. Then,
\[
I(v_n) \to c_V \quad \text{and} \quad I'(v_n) \to 0.
\]
Since \((v_n)\) is also bounded, we may assume \(v_n \rightharpoonup v\) in \(E\) and \(v_n \rightharpoonup v\) in \(L^2_{loc}(\mathbb{R}^2)\). From the last inequality \(v \neq 0\), and by the same arguments in Lemma \([2.2]\) we can assume that \(I'(v) = 0\).
Let $\mathcal{N}$ be the Nehari manifold defined by

$$\mathcal{N} = \{ u \in E : u \neq 0, I'(u)u = 0 \}.$$  

From $(f_5)$, it is standard to check that the mountain pass level can be characterized by

$$c_V = \inf_{u \in E \setminus \{0\}} \max_{t \geq 0} I(tu) = \inf_{u \in \mathcal{N}} I(u).$$

The above characterization together with $(f_3)$ give $I(v) = c_V$, showing that $v$ is a ground state solution.

3.2 Proof of Theorem 1.4

In what follows, we will denote by $\tilde{I} : H^1(\mathbb{R}^2) \to \mathbb{R}$ the energy functional associated with problem

$$\begin{cases} 
- \Delta u + V_0(x)u = \left( I_\mu * \tilde{F}(x, u) \right) \tilde{f}(x, u) & \text{in } \mathbb{R}^2, \\
u \in H^1(\mathbb{R}^2), \\
u(x) > 0 & \text{for all } x \in \mathbb{R}^2,
\end{cases}$$

where $0 < \mu < 2$, $\tilde{F}(x, s)$ is the primitive function of $\tilde{f}(x, s)$ in the variable $s$ and $V_0, \tilde{f}$ are continuous and 1-periodic. Thus,

$$\tilde{I}(u) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + V_0(x)|u|^2) - \frac{1}{2} \int_{\mathbb{R}^2} \left( I_\mu * \tilde{F}(x, u) \right) \tilde{F}(x, u) \forall u \in H^1(\mathbb{R}^2).$$

By Theorem 1.4 we know that there is $u_0 \in H^1(\mathbb{R}^2)$ such that

$$\tilde{I}'(u_0) = 0 \text{ and } \tilde{I}(u_0) = \tilde{c}_V,$$

where $\tilde{c}_V$ is the mountain pass level associated with $\tilde{I}$.

Using the same arguments explored in proof of Lemma 2.1 we can show that $I$ verifies the Mountain Pass Geometry. Consequently, there is a (PS) sequence $(u_n) \subset E$ such that

$$I(u_n) \to c_V \text{ and } I'(u_n) \to 0,$$

where $c_V$ is the mountain pass level characterized by

$$0 < c_V := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I(\gamma(t)) = \inf_{u \in E \setminus \{0\}} \max_{t \geq 0} I(tu) = \inf_{u \in \mathcal{N}} I(u) \quad (3.2)$$

with

$$\Gamma := \{ \gamma \in C([0, 1], E) : \gamma(0) = 0 \text{ and } I(\gamma(1)) < 0 \}$$

and

$$\mathcal{N} = \{ u \in E : u \neq 0, I'(u)u = 0 \}.$$

Therefore, by the above notations,

$$c_V = \inf_{u \in E \setminus \{0\}} \max_{t \geq 0} I(tu) \leq \max_{t \geq 0} I(t_0 u_0) = I(t_0 u_0) < \tilde{I}(t_0 u_0) = \max_{t \geq 0} \tilde{I}(tu_0) = \tilde{I}(u_0) = \tilde{c}_V,$$
that is,
\[ c_V < \tilde{c}_V. \]  
(3.3)

From Lemma 2.2
\[ \tilde{c}_V \in (0, \frac{(2 - \mu)(\theta - 2)}{8\theta}) \]
then,
\[ c_V \in (0, \frac{(2 - \mu)(\theta - 2)}{8\theta}). \]

Recalling that \( \theta \geq \tilde{\theta} \), it follows from \((f_3)\),
\[ \tilde{\theta}F(x, s) \leq f(x, s)s \quad \forall x \in \mathbb{R}^2 \text{ and } s \in \mathbb{R}. \]

Arguing as in the proof Lemma 2.2, it follows that \((u_n)\) is bounded in \(E\) with
\[ \limsup \|u_n\|^2 < \frac{(2 - \mu)}{4}. \]

Moreover, there is \(u \in E\) such that for a subsequence,
\[ u_n \rightharpoonup u \text{ in } E \text{ and } \tilde{I}'(u) = 0. \]

We claim that \(u \neq 0\). To see why, we will argue by contradiction, supposing that \(u = 0\).
Notice that, for any \(\varphi \in E\),
\[ I(u_n) = \tilde{I}(u_n) + \frac{1}{2} \int_{\mathbb{R}^2} \left( I_\mu * \tilde{F}(x, u_n) \right) \tilde{F}(x, u_n) - \frac{1}{2} \int_{\mathbb{R}^2} \left( I_\mu * F(x, u_n) \right) F(x, u_n) \]
and
\[ I'(u_n)\varphi = \tilde{I}'(u_n)\varphi + \int_{\mathbb{R}^2} \left( I_\mu * \tilde{F}(x, u_n) \right) \tilde{f}(x, u_n)\varphi - \int_{\mathbb{R}^2} \left( I_\mu * F(x, u_n) \right) f(x, u_n)\varphi. \]

By \((f_6)\), it is possible to prove that
\[ \int_{\mathbb{R}^2} \left( I_\mu * \tilde{F}(x, u_n) \right) \tilde{F}(x, u_n) - \int_{\mathbb{R}^2} \left( I_\mu * F(x, u_n) \right) F(x, u_n) \to 0 \]  
(3.4)
and
\[ \int_{\mathbb{R}^2} \left( I_\mu * \tilde{F}(x, u_n) \right) \tilde{f}(x, u_n)\varphi - \int_{\mathbb{R}^2} \left( I_\mu * F(x, u_n) \right) f(x, u_n)\varphi \to 0 \]  
(3.5)
uniformly for \(\|\varphi\| \leq 1, \varphi \in E\). Next, we will show only \(3.5\), because \(3.4\) follows with
the same type of argument. For any \(\varphi \in E\), considering \(r = \frac{4 - \mu}{2 - \mu}\) and \(r' > 1\) satisfying
\[ \frac{1}{r} + \frac{1}{r'} = 1, \]
we have
\[ \left| \int_{\mathbb{R}^2} \left( I_\mu * \tilde{F}(x, u_n) \right) \tilde{f}(x, u_n)\varphi - \int_{\mathbb{R}^2} \left( I_\mu * F(x, u_n) \right) f(x, u_n)\varphi \right| \]
\[ \leq \left| \int_{\mathbb{R}^2} \left( I_\mu * (\tilde{F}(x, u_n) - F(x, u_n)) \right) \tilde{f}(x, u_n)\varphi \right| \]
\[ + \left| \int_{\mathbb{R}^2} \left( I_\mu * F(x, u_n) \right) (\tilde{f}(x, u_n) - f(x, u_n))\varphi \right| \]
\[ \leq |\tilde{F}(x, u_n) - F(x, u_n)| \frac{4 - \mu}{4 - \mu} |f(x, u_n)| \frac{4r}{4 - \mu} |\varphi|_2 \]
\[ + |F(x, u_n)| \frac{4 - \mu}{4 - \mu} |\tilde{f}(x, u_n) - f(x, u_n)| \frac{4r}{4 - \mu} |\varphi|_2 \]
\[ \leq C|\tilde{F}(x, u_n) - F(x, u_n)| \frac{4 - \mu}{4 - \mu} |f(x, u_n)| \frac{4r}{4 - \mu} \]
\[ + C|F(x, u_n)| \frac{4 - \mu}{4 - \mu} |\tilde{f}(x, u_n) - f(x, u_n)| \frac{4r}{4 - \mu}. \]
Since \( \limsup \|u_n\|^2 < \frac{2-\mu}{4} \), the ideas used in previous section work to show that 
\((\hat{F}(x, u_n)) \) and 
\((\tilde{f}(x, u_n)) \) are bounded in \( L^{\frac{4}{1-\mu}}(\mathbb{R}^2) \) and \( L^{\frac{4\mu}{4-\mu}}(\mathbb{R}^2) \) respectively.

On the other hand, from \((f_0)\), there exist \( C > 0 \) such that

\[
|\tilde{f}(x, s) - f(x, s)| \leq C A(x) \left( |s| \frac{2-\mu}{2} + s [e^{\beta_4 s^2} - 1] \right) \quad \forall s \in \mathbb{R} \text{ and } x \in \mathbb{R}^2,
\]

leading to

\[
|\tilde{f}(x, u_n) - f(x, u_n)| \frac{4}{4-\mu} \leq C \left( \int_{\mathbb{R}^2} |A(x)| \frac{4}{4-\mu} |u_n|^2 \right)^{\frac{2-\mu}{4}} + C \left( \int_{\mathbb{R}^2} |A(x)u_n| \frac{4}{4-\mu} \left[ e^{\frac{4\beta_3}{4-\mu} \beta_4 u_n^2} - 1 \right] \right)^{\frac{2-\mu}{4}}
\]

\[
\leq C \left( \int_{\mathbb{R}^2} |A(x)| \frac{4}{4-\mu} |u_n|^2 \right)^{\frac{2-\mu}{4}} + C \left( \int_{\mathbb{R}^2} |A(x)u_n| \frac{4\mu}{4-\mu} \left( \int_{\mathbb{R}^2} e^{\left( e^{\frac{4\beta_3}{4-\mu} \beta_4 u_n^2} - 1 \right) \lambda_n} \right)^{\frac{2-\mu}{4}}
\]

Fixing \( t > 1 \) sufficiently close to 1, again by Lemma \[1.2\]

\[
\left( \int_{\mathbb{R}^2} e^{\frac{4\beta_3}{4-\mu} \beta_4 u_n^2} \right)^{\frac{2-\mu}{4}} \leq C \forall n \in \mathbb{N},
\]

for some \( C > 0 \). Since \( A \in L^\infty(\mathbb{R}^2) \) and \( A(x) \to 0 \) as \( |x| \to \infty \), it easy to obtain

\[
\left( \int_{\mathbb{R}^2} |A(x)| \frac{4}{4-\mu} |u_n|^2 \right)^{\frac{2-\mu}{4}} \to 0
\]

and

\[
\left( \int_{\mathbb{R}^2} |A(x)u_n| \frac{4\mu}{4-\mu} \right)^{\frac{2-\mu}{4}} \to 0.
\]

Therefore

\[
|\tilde{f}(x, u_n) - f(x, u_n)| \frac{4}{4-\mu} \to 0,
\]

and by a similar argument,

\[
|\tilde{F}(x, u_n) - F(x, u_n)| \frac{4}{4-\mu} \to 0,
\]

showing \[3.5\].

Consequently, the sequence \((u_n) \subset E \) satisfies

\[
\tilde{F}'(u_n) \to 0 \quad \text{and} \quad \tilde{F}(u_n) \to c_V,
\]

Repeating the arguments explored in the proof of Theorem \[1.3\] we will find a nontrivial critical point \( \tilde{u} \) of \( \tilde{F} \) verifying the estimate \( \tilde{F}(\tilde{u}) \leq c_V \). However, this is a contradiction, because by definition of \( c_V \), we must have

\[
\tilde{c}_V \leq \tilde{F}(\tilde{u})
\]

implying that \( \tilde{c}_V \leq c_V \), which is a absurd with \[3.3\]. Thereby, the weak limit \( u \) of the \((PS)_{c_V} \) is nontrivial, finishing the proof of Theorem \[1.4\]
4 Final comments

The positive solution $u$ obtained in Theorem 1.3 belongs to $L^\infty(\mathbb{R}^2)$ and decays to zero as $|x| \to \infty$. First of all, we would like to point out that the arguments used in the proof of Claim 2.3 implies that

$$I_\mu * F(x, v) \in L^\infty(\mathbb{R}^2) \ \forall v \in H^1(\mathbb{R}^2).$$

Thus, if $u$ is a solution of

$$-\Delta u + V(x)u = \left(I_\mu * F(x, u)\right) f(x, u) \ \text{in} \ \mathbb{R}^2,$$

since $V, I_\mu * F(x, u) \in L^\infty(\mathbb{R}^2)$ and $f(x, u) \in L^q(\mathbb{R}^2)$ for $q$ large enough, we deduce by the bootstrap arguments that $u \in L^\infty(\mathbb{R}^2)$ and

$$|u(x)| \to 0 \ \text{as} \ |x| \to +\infty.$$

We would like to point out that with few modifications, it is possible to prove the existence of solution for (SNE) for other classes of potentials $V$, such as:

1- First Case:  $V(x) = V(|x|)$ and $f(x, s) = f(|x|, s)$ for all $x \in \mathbb{R}^2$ and $s \in \mathbb{R}$.

2- Second Case:  For all $M > 0$, we assume that

$$|\{ x \in \mathbb{R}^2 : V(x) < M \}| < +\infty,$$

where $| \ |$ stands for the Lebesgue measure in $\mathbb{R}^2$.

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