ON A CERTAIN FAMILY OF INVERSE TERNARY CYCLOMATIC POLYNOMIALS

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ABSTRACT. We study a family of inverse ternary cyclotomic polynomials \( \Psi_{pqr} \) in which \( r \leq \varphi(pq) \) is a positive linear combination of \( p \) and \( q \). We derive a formula for the height of such polynomial and characterize all flat polynomials in this family.

1. Introduction

Let

\[
\Phi_n(x) = \prod_{1 \leq k \leq n, (k,n) = 1} (x - e^{2k\pi i/n}) = \sum_m a_n(m)x^m
\]

be the \( n \)th cyclotomic polynomial. The \( n \)th inverse cyclotomic polynomial is defined by the formula

\[
\Psi_n(x) = \frac{x^n - 1}{\Phi_n(x)} = \sum_m c_n(m)x^m.
\]

Like for cyclotomic polynomials, for odd primes \( p < q < r \), we say that \( \Psi_{pq} \) is binary, \( \Psi_{pqr} \) is ternary, etc.

Recall that the height of a given polynomial \( F \) is the maximal absolute value of its coefficients. We say that polynomial is flat, if its height equals 1. Traditionally we denote the height of \( \Phi_n \) by \( A(n) \) and the height of \( \Psi_n \) by \( C(n) \).

Ternary inverse cyclotomic polynomials were studied by P. Moree [5]. He proved that \( C(pqr) \leq p - 1 \) and for every prime \( p \geq 3 \) there are infinitely many pairs \((q,r)\) of primes for which \( C(pqr) = p - 1 \). Additionally he came up with the following bound ([5], Theorem 7):

\[
C(pqr) \leq \max\{\min\{p', q'\}, \min\{q - p', p - q'\}\} \text{ for } \deg \Psi_{pqr} < 2qr.
\]

He also found some flat inverse ternary cyclotomic polynomials.

Let us remark that the case \( r > \varphi(pq) = \deg \Phi_{pq} \) is trivial, because by the identity \( \Psi_{pqr}(x) = \Psi_{pq}(x^r)\Phi_{pq}(x) \) we have \( c_{pqr}(ar+b) = a_{pq}(b)c_{pq}(a) \) for \( a \geq 0 \) and \( 0 \leq b < r \). The coefficients of polynomials \( \Phi_{pq} \) and \( \Psi_{pq} \) are well known, so we can evaluate \( c_{pqr}(ar+b) \) easily.

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Although there is a substantial research on flat ternary cyclotomic polynomials [1, 2, 3], we do not know much about flat ternary inverse cyclotomic polynomials. Particularly, no infinite family of such polynomials in which \( r \leq \varphi(pq) \) was known so far.

In this paper we investigate polynomials \( \Psi_{pqr} \) in which \( r \leq \varphi(pq) \) is a positive linear combination of \( p \) and \( q \). For this specific type of polynomials we improve some of the results of P. Moree mentioned above. Our main result is the following theorem.

**Theorem 1.** Let \( r = \alpha p + \beta q \leq \varphi(pq) \), where \( \alpha, \beta > 0 \). Let also \( p' \in \{1, 2, \ldots, q-1\} \) be the inverse of \( p \) modulo \( q \) and \( q' \in \{1, 2, \ldots, p-1\} \) be the inverse of \( q \) modulo \( p \). Then

\[
C(pqr) = \max \left\{ \min \left\{ \left\lceil \frac{p'}{\alpha} \right\rceil, \left\lceil \frac{q'}{\beta} \right\rceil \right\}, \min \left\{ \left\lceil \frac{q-p'}{\alpha} \right\rceil, \left\lceil \frac{p-q'}{\beta} \right\rceil \right\} \right\}.
\]

The above formula is similar to the already mentioned one obtained by P. Moree. However, our theorem does not require the assumption \( \deg \Psi_{pqr} < 2qr \). We use Theorem 1 to characterize all flat inverse ternary cyclotomic polynomials \( \Psi_{pqr} \) in which \( r \) is a positive linear combination of \( p \) and \( q \).

**Theorem 2.** Let \( r = \alpha p + \beta q \leq \varphi(pq) \), where \( \alpha, \beta > 0 \). Then \( \Psi_{pqr} \) is flat if and only if at least one of the following conditions holds:

(a) \( \alpha \geq \max \{p', q-p', \} \),
(b) \( \beta \geq \max \{q', p-q', \} \),
(c) \( \alpha \geq p' \) and \( \beta \geq p-q' \),
(d) \( \alpha \geq q - p' \) and \( \beta \geq q' \),

where \( p' \) and \( q' \) are like in Theorem 1.

At first one may expect that the set of primes \( r \leq \varphi(pq) \) satisfying at least one of the conditions (a) – (d) of Theorem 2 is rather small and consists of primes which are relatively close to \( \varphi(pq) \) (if (a) or (b) holds, then \( r > pq/2 \)). Fortunately, the following theorem says that this is not the truth in general.

**Theorem 3.** Let \( S(p, q) \) denote the set of primes \( r \leq \varphi(pq) \) of form \( \alpha p + \beta q \), \( \alpha, \beta > 0 \), for which \( \Psi_{pqr} \) is flat. Then

(i) For every \( M > 0 \) and every \( p \) there exists such \( q \) that \( \#S(p, q) > M \).
(ii) For every \( \varepsilon > 0 \) there exists a triple \( (p, q, r) \) of primes such that \( r \in S(p, q) \) and \( r < \varepsilon \varphi(pq) \).

So we reveal a new, vast family of nontrivially flat ternary inverse cyclotomic polynomials.

Our paper is organized in the following way. In section 3 we derive a formula for \( c_{pqr}(m) \), the \( m \)th coefficient of \( \Psi_{pqr} \). In section 4 we prove Theorem 1. Finally, in section 5 we prove Theorems 2 and 3.

2. Preliminaries

In this section we recall some basic properties of standard and inverse cyclotomic polynomials. All of them can be found in [4] or [5]. Let us start
with the properties of their degrees:

\[
\deg \Phi_n = \varphi(n), \quad \deg \Psi_n = n - \varphi(n).
\]

Particularly \(\deg \Psi_{pqr} = qr + rp + pq - p - q - r + 1\).

We say that a polynomial \(F\) is reciprocal if \(F(x) = x^{\deg F} F(1/x)\) and anti-reciprocal if \(F(x) = -x^{\deg F} F(1/x)\). It is known that all cyclotomic polynomials except of \(\Phi_1\) are reciprocal and all inverse cyclotomic polynomials except of \(\Psi_1\) are anti-reciprocal.

In our investigations we need to know the coefficients of \(\Phi_{pq}\). The following lemma, proved in [4], derives a formula on \(a_{pq}(m)\).

**Lemma 4.** Let \(m \in \{0, 1, \ldots, pq - 1\}\) and let \(u, v\) be the unique numbers such that \(m \equiv up + vq \pmod{pq}\) and \(0 \leq u < q, 0 \leq v < p\). Then we have

\[
a_{pq}(m) = \begin{cases} 
1, & \text{if } u < p' \text{ and } v < q', \\
-1, & \text{if } u \geq p' \text{ and } v \geq q', \\
0, & \text{otherwise}.
\end{cases}
\]

Following P. Moree [5], we define the polynomial

\[
f_{pqr}(x) = (1 + x^r + \ldots + x^{(p-1)r})\Phi_{pq}(x) = \sum_m e_{pqr}(m)x^m.
\]

Let also

\[
\tau(pqr) = \deg f_{pqr} = (p - 1)r + \varphi(pq) = (p - 1)(q + r - 1).
\]

The next lemma was proved in [5].

**Lemma 5.** For primes \(p < q < r\) we have

\[
\Psi_{pqr}(x) = (x^{qr} - 1)f_{pqr}(x).
\]

By this lemma \(e_{pqr}(m) = e_{pqr}(m - qr) - e_{pqr}(m)\), so if we want to determine the coefficients of \(\Psi_{pqr}\), we need to know the coefficients of \(f_{pqr}\).

Let us remark that in the formula

\[
\Psi_{pqr}(x) = \frac{(1 - x)(1 - x^{qr})(1 - x^{rp})(1 - x^{pq})}{(1 - x^p)(1 - x^q)(1 - x^r)}
\]

one can replace the assumption that \(p, q, r\) are primes by the assumption that they are pairwise coprime. This way we receive the definition of inverse inclusion-exclusion polynomial \(\Psi_{p',q',r'}\). Theorems 1 and 2 hold also for inverse inclusion-exclusion polynomials and they can be proved by analogous methods.

### 3. Coefficients of \(f_{pqr}\)

The following lemma, partially proved in [5], derives a formula on coefficients of \(f_{pqr}\) in terms of coefficients of \(\Phi_{pq}\). We remark that this is true for all primes \(r\).

**Lemma 6.** The following equalities hold:

(i) if \(m < pr\), then \(e_{pqr}(m) = \sum_{j=0}^{\lfloor m/r \rfloor} a_{pq}(m - jr)\),
(ii) for \( pq < m < pr \) we have \( e_{pqr}(m) = e_{pqr}(m - r) \).

(iii) if \( pr \leq m \leq \tau(pqr) \), then \( e_{pqr}(m) = e_{pqr}(m') \), where \( m' = \tau(pqr) - m < pr \).

**Proof.** Case (i) follows directly from the definition of \( f_{pqr} \). The polynomial \( f_{pqr} \) is reciprocal as a product of reciprocal polynomials, so \( e_{pqr}(m) = e_{pqr}(\tau(pqr) - m) \). Because \( \tau(pqr) < 2pr \), for \( m \geq pr \) we have \( m' = \tau(pqr) - m < pr \), so (iii) holds. To prove (ii) we observe that for \( pq < m < pr \) we have \( a_{pq}(m) = 0 \) and then by (i)

\[
e_{pqr}(m) = a_{pq}(m) + \sum_{j=1}^{\lfloor m/r \rfloor} a_{pq}(m-jr) = \sum_{j=0}^{\lfloor (m-r)/r \rfloor} a_{pq}(m-r-jr) = e_{pqr}(m-r),
\]

which completes the proof. \( \Box \)

Now we use Lemmas 4 and 6 to determine coefficients of \( f_{pqr} \). We do it for the exponents not greater than \( pq \), since for greater ones we can use (ii) and (iii) of Lemma 6. In order to simplify the notation, for a finite set \( A \) we define

\[
\min_{\geq 0} A = \max\{0, \min A\}.
\]

**Theorem 7.** Let \( r = \alpha p + \beta q \leq \varphi(pq) \), \( \alpha, \beta > 0 \) and \( m < pq \). Put \( m = (a - 1)r + b \), where \( 0 \leq b < r \).

(i) If \( \frac{b}{b} \) for some integers \( 0 \leq u < q \) and \( 0 \leq v < p \), then

\[
e_{pqr}(m) = \min_{\geq 0} \left\{ a, \left\lfloor \frac{p'-u}{\alpha} \right\rfloor, \left\lfloor \frac{q'-v}{\beta} \right\rfloor \right\}.
\]

(ii) If \( b + pq = up + vq \) for some integers \( 0 \leq u < q \) and \( 0 \leq v < p \), then we define

\[
j_0 = \min \left\{ \left\lfloor \frac{q-u}{\alpha} \right\rfloor, \left\lfloor \frac{p-v}{\beta} \right\rfloor \right\}, \quad a^* = a - j_0,
\]

\[
(u^*, v^*) = \begin{cases} (u + j_0 \alpha - q, v + j_0 \beta) & \text{if } j_0 = \left\lfloor \frac{(q - u)}{\alpha} \right\rfloor, \\
(u + j_0 \alpha, v + j_0 \beta - p) & \text{if } j_0 = \left\lfloor \frac{(p - v)}{\beta} \right\rfloor.
\end{cases}
\]

We have \( e_{pqr}(m) = e_{pqr}^+(m) - e_{pqr}^-(m) \), where

\[
e_{pqr}^+(m) = \min_{\geq 0} \left\{ a^*, \left\lfloor \frac{p'-u^*}{\alpha} \right\rfloor, \left\lfloor \frac{q'-v^*}{\beta} \right\rfloor \right\},
\]

\[
e_{pqr}^-(m) = \min_{\geq 0} \left\{ \min \left\{ a, \left\lfloor \frac{q-u}{\alpha} \right\rfloor, \left\lfloor \frac{p-v}{\beta} \right\rfloor \right\}, - \max \left\{ 0, \left\lfloor \frac{p'-u}{\alpha} \right\rfloor, \left\lfloor \frac{q'-v}{\beta} \right\rfloor \right\} \right\}.
\]

One can easily prove that every \( 0 \leq b < pq \) can be written in exactly one of forms: \( up + vq \) or \( up + vq - pq \), where \( 0 \leq u < q \) and \( 0 \leq v < p \). So cases (i) and (ii) cover all possible values of \( b \).
Proof. First consider \( \mathbf{b} = \mathbf{u}p + \mathbf{v}q \). Then for \( 0 \leq j < a \) we have
\[
\mathbf{b} + j\mathbf{r} = (u + j\alpha)p + (v + j\beta)q.
\]
Notice that \( \mathbf{b} + j\mathbf{r} \leq \mathbf{b} + (a - 1)r = \mathbf{m} < \mathbf{pq} \), so \( u + j\alpha < \mathbf{q} \) and \( v + j\beta < \mathbf{p} \). By Lemma 4
\[
a_{\mathbf{pq}}(\mathbf{b} + j\mathbf{r}) = \begin{cases} 1, & \text{if } u + j\alpha < p' \text{ and } v + j\beta < q', \\ 0, & \text{otherwise}, \end{cases}
\]
where the case \( a_{\mathbf{pq}}(\mathbf{b} + j\mathbf{r}) = -1 \) was omitted, as the inequalities \( u + j\alpha \geq p' \) and \( v + j\beta \geq q' \) cannot hold at the same time (if they both held then we would have \( \mathbf{pq} > \mathbf{b} + j\mathbf{r} \geq p'\mathbf{p} + q'\mathbf{q} = \mathbf{pq} + 1 \), a contradiction). So for \( 0 \leq j < a \) we have
\[
a_{\mathbf{pq}}(\mathbf{b} + j\mathbf{r}) = \begin{cases} 1, & \text{if and only if both equalities } \\
0, & \text{if at least one equality does not hold} \end{cases}
\]
hold. Thus by Lemma 6
\[
e_{\mathbf{pqr}}(\mathbf{m}) = \sum_{j=0}^{a-1} a_{\mathbf{pq}}(\mathbf{b} + j\mathbf{r}) = \min_{j \geq 0} \left\{ a, \left\lceil \frac{p' - u}{\alpha} \right\rceil, \left\lceil \frac{q' - v}{\beta} \right\rceil \right\}
\]
as desired. We have proved (i).

Now consider \( \mathbf{b} + \mathbf{pq} = \mathbf{u}p + \mathbf{v}q \). For \( 0 \leq j < a \) we have
\[
\mathbf{b} + j\mathbf{r} = (u + j\alpha)p + (v + j\beta)q - \mathbf{pq}.
\]
As long as \( u + j\alpha < \mathbf{q} \) and \( v + j\beta < \mathbf{p} \) (or equivalently \( j < j_0 \)), \( \mathbf{b} + j\mathbf{r} \) is clearly not a nonnegative linear combination of \( \mathbf{p} \) and \( \mathbf{q} \), while for \( j \geq j_0 \) it is. Therefore we put
\[
-h^- = \sum_{j=0}^{\min\{j_0, a\}-1} a_{\mathbf{pq}}(\mathbf{b} + j\mathbf{r}), \quad h^+ = \sum_{j=\min\{j_0, a\}}^{a-1} a_{\mathbf{pq}}(\mathbf{b} + j\mathbf{r}).
\]
(if a sum is empty then it equals 0). By Lemma 6 we have \( e_{\mathbf{pqr}}(\mathbf{m}) = h^+ - h^- \), so we need to prove that \( h^- = e^{-}_{\mathbf{pqr}}(\mathbf{m}) \) and \( h^+ = e^{+}_{\mathbf{pqr}}(\mathbf{m}) \).

The value of \( h^- \) is easier to determine. For \( j < j_0 \) the inequalities \( u + j\alpha < p' \) and \( v + j\beta < q' \) cannot hold at the same time, because if \( kp + lq \geq \mathbf{pq} \), then \( k \geq p' \) or \( l \geq q' \). So for \( j < j_0 \) we receive
\[
a_{\mathbf{pq}}(\mathbf{b} + j\mathbf{r}) = \begin{cases} -1, & \text{if } u + j\alpha \geq p' \text{ and } v + j\beta \geq q', \\ 0, & \text{otherwise}. \end{cases}
\]
Therefore for \( 0 \leq j < \min\{j_0, a\} \) we have \( a_{\mathbf{pq}}(\mathbf{b} + j\mathbf{r}) = -1 \) if and only if both inequalities
\[
\frac{p' - u}{\alpha} \leq j < \frac{q - u}{\alpha}, \quad \frac{q' - v}{\beta} \leq j < \frac{p - v}{\beta}
\]
hold. Thus

\[ h^- = \min_{\geq 0} \left\{ \min \left\{ j_0, a, \left\lceil \frac{q - u}{\alpha} \right\rceil, \left\lceil \frac{p - v}{\beta} \right\rceil \right\} \right. \]
\[ \left. - \max \left\{ 0, \left\lceil \frac{p'}{\alpha} \right\rceil, \left\lceil \frac{q'}{\beta} \right\rceil \right\} \right\} \]
\[ = e_{pqr}(m), \]

by the definition of \( j_0 \).

Now we determine \( h^+ \). If \( j_0 \geq a \), then \( h^+ = 0 \), so further we assume that \( j_0 < a \). We have two analogous cases here: \( j_0 = \lceil (q - u)/\alpha \rceil \) and \( j_0 = \lceil (p - v)/\beta \rceil \). Let us consider the first one. For \( j_0 \leq j < a \) we have

\[ b + jr = (u + j\alpha - q)p + (v + j\beta)q \]

with \( 0 \leq u + j\alpha - q < q \) and \( v + j\beta < p \). We have \( u^* = u + j_0\alpha - q \) and \( v^* = v + j_0\beta \). Additionally put

\[ b^* = b + j_0r = (u + j_0\alpha - q)p + (v + j_0\beta)q = u^*p + v^*q. \]

Clearly \( b^* \leq b + (a - 1)r < pq \), so \( u^* < q \) and \( v^* < p \). Note that \( b^* \) does not have to be smaller than \( r \), however, we can still use the arguments from the proof of (i) to obtain that

\[ h^+ = \sum_{j=0}^{a^*-1} a_{pq}(b^* + jr) = e_{pqr}^+(m). \]

This formula remains correct even if \( j_0 \geq a \). Applying the analogous argument to the case \( j_0 = \lceil (p - v)/\beta \rceil \) we complete the proof. \( \square \)

From the results of this section, we obtained an algorithm which instantly computes the value of \( c_{pqr}(k) \). By Lemma 5 we have \( c_{pqr}(k) = e_{pqr}(k - qr) - e_{pqr}(k) \). Then using Lemma 6 we reduce computing \( e_{pqr}(k) \) to computing \( e_{pqr}(m_1) \) and \( e_{pqr}(m_2) \) for some \( m_1, m_2 < pq \). Finally we apply Theorem 7 to evaluate \( e_{pqr}(m_1) \) and \( e_{pqr}(m_2) \).

4. The Height of \( f_{pqr} \) and \( \Psi_{pqr} \)

In this section we evaluate the height of \( f_{pqr} \) and compare it with the height of \( \Psi_{pqr} \).

**Lemma 8.** Let \( H(pqr) \) denotes the height of \( f_{pqr} \). Then for \( r = \alpha p + \beta q \leq \varphi(pq), \alpha, \beta > 0 \), we have

\[ H(pqr) = \max \left\{ \min \left\{ \left\lceil \frac{p'}{\alpha} \right\rceil, \left\lceil \frac{q'}{\beta} \right\rceil \right\}, \min \left\{ \left\lceil \frac{q - p'}{\alpha} \right\rceil, \left\lceil \frac{p - q'}{\beta} \right\rceil \right\} \right\}. \]
Theorem 7. We receive the inequalities

Proof. By Lemma 6 we can restrict our considerations to \( m < pq \) and use Theorem 7. We have to prove that

\[
\begin{align*}
\tau &= \frac{p'}{\alpha} \leq \frac{q'}{\beta}, \\
-\epsilon_{\tau\eta}(m) &\leq \max_{\eta,\tau} \left\{ \min \left\{ \frac{q - u}{\alpha}, \frac{p - v}{\beta} \right\} - \max \left\{ \frac{p' - u}{\alpha}, \frac{q' - v}{\beta} \right\} \right\} \\
\end{align*}
\]

To complete the proof we will show that we have equalities: in the first inequality for \( m = m_1 \) and in the second one for \( m = m_2 \), where

\[
\begin{align*}
m_1 &= \left( \min \left\{ \frac{p'}{\alpha}, \frac{q'}{\beta} \right\} - 1 \right) r, \\
m_2 &= \left( \min \left\{ \frac{p' - p}{\alpha}, \frac{p - q'}{\beta} \right\} - 1 \right) r + 1.
\end{align*}
\]

Now we are ready to prove our main result.

Proof of Theorem 7. Let 

\[
\begin{align*}
\epsilon_{\tau\eta}(m) &= H(pqr) - H(qqr), \\
\end{align*}
\]

First let us verify that \( C(pqr) \geq H(pqr) \). Because \( f_{pqr} \) is reciprocal, \( |\epsilon_{pqr}(m)| = H(pqr) \) for some \( m \leq \tau(pqr)/2 \). Since \( \tau(pqr) < 2qr \), we have \( m < qr \) and hence \( \epsilon_{pqr}(m - qr) = 0 \). Thus by Lemma 5

\[
C(pqr) \geq |\epsilon_{pqr}(m)| = |\epsilon_{pqr}(m - qr) - \epsilon_{pqr}(m)| = | - \epsilon_{pqr}(m)| = H(pqr).
\]

The opposite inequality is much harder to prove. By Lemma 5 we have \( |\epsilon_{pqr}(m)| = | - \epsilon_{pqr}(m)| \leq H(pqr) \) for \( m < qr \). By the anti-reciprocity of \( \Psi_{pqr} \) we also have \( \epsilon_{pqr}(m) \leq H(pqr) \) for \( m > \tau(pqr) \). Therefore we can restrict our considerations to \( qr \leq m \leq \tau(pqr) \). In this case

\[
\begin{align*}
c_{pqr}(m) &= \epsilon_{pqr}(m - qr) - \epsilon_{pqr}(m) = \epsilon_{pqr}(m - qr) - \epsilon_{pqr}(\tau(pqr) - m) \\
&= \epsilon_{pqr}(m_1) - \epsilon_{pqr}(m_2),
\end{align*}
\]

Thus the proof is done.
where $m_1 = m - qr$ and $m_2 = \tau(pqr) - m$. Additionally,

$$m_1 + m_2 = \tau(pqr) - qr = \varphi(pq) - (q - 1 - p)r < pq.$$ 

Hence $0 \leq m_1 < pq$, $0 \leq m_2 < pq$ and so we can use Theorem 7. We will show that $e_{pqr}(m_1)$ and $e_{pqr}(m_2)$ cannot have opposite signs, which actually completes the proof. Without loss of generality we can assume that $e_{pqr}(m_1) > 0$.

For $i \in \{1, 2\}$ put

$$m_i = (a_i - 1)r + b_i, \quad 0 \leq b_i < r,$$

$$b_i \equiv u_ip + v_iq \pmod{pq}, \quad 0 \leq u_i < q, \quad 0 \leq v_i < p.$$

We have

$$m_2 = \tau(pqr) - qr - m_1 = (p - q - a_1)r + \varphi(pq) - b_1 = (t + p - q - a_1)r + b_2,$$

where $\varphi(pq) - b_1 = tr + b_2$ with $0 \leq b_2 < r$. Then $a_2 = t + p - q - a_1 + 1$.

Now we consider some cases, in which we determine different values of $u_2$, $v_2$.

**Case (1):** $b_1 = u_1p + v_1q$. Here

$$b_2 = \varphi(pq) - b_1 - tr = (p' - 1)p + (q' - 1)q - (u_1p + v_1q) - t(\alpha p + \beta q) = (p' - 1 - u_1 - t\alpha)p + (q' - 1 - v_1 - t\beta)q,$$

so $u_2 \equiv p' - 1 - u_1 - t\alpha \pmod{q}$ and $v_2 \equiv q' - 1 - v_1 - t\beta \pmod{p}$.

Both numbers $p' - 1 - u_1 - t\alpha$ and $q' - 1 - v_1 - t\beta$ cannot be negative at the same time, since $b_2 \geq 0$. If both are positive, then they equal $u_2$ and $v_2$ and $e_{pqr}(m_2) \geq 0$ by Theorem 7. Therefore we have to consider the situation in which one of these numbers is negative and one is positive. Without loss of generality, we assume that $p' - 1 - u_1 - t\alpha < 0$. Then

$$u_2 = q + p' - 1 - u_1 - t\alpha, \quad v_2 = q' - 1 - v_1 - t\beta, \quad b_2 = u_2p + v_2q - pq.$$

**Case (2):** $b_1 = u_1p + v_1q - pq$. Here $u_1 \geq p'$ or $v_1 \geq q'$. Without loss of generality we assume that $u_1 \geq p'$. Then

$$b_2 = \varphi(pq) - b_1 - tr = (p' - 1)p + (q' - 1)q - (u_1p + v_1q) + pq - t(\alpha p + \beta q) = (q + p' - 1 - u_1 - t\alpha)p + (q' - 1 - v_1 - t\beta)q.$$

By the similar argument to one used in case (1), we consider two subcases in which the signs of $q + p' - 1 - u_1 - t\alpha$ and $q' - 1 - v_1 - t\beta$ are opposite:

**Case (2a):** $u_2 = 2q + p' - 1 - u_1 - t\alpha$, $v_2 = q' - 1 - v_1 - t\beta$, $b_2 = u_2p + v_2q - pq$,

**Case (2b):** $u_2 = q + p' - 1 - u_1 - t\alpha$, $v_2 = p + q' - 1 - v_1 - t\beta$, $b_2 = u_2p + v_2q - pq$.

Now we show that in cases (1) and (2a) we have $e_{pqr}(m_2) \geq 0$. Note that $b_2 = u_2p + v_2q - pq$ and $v_2 = q' - 1 - v_1 - t\beta$ in both these cases, so we
estimate
\[
\max \left\{ 0, \left\lceil \frac{p' - u_2}{\alpha} \right\rceil, \left\lfloor \frac{q' - v_2}{\beta} \right\rfloor \right\} \geq \left\lceil \frac{q' - v_2}{\beta} \right\rceil = \left\lfloor \frac{v_1 + 1}{\beta} \right\rfloor + t \geq t,
\]
\[
\min \left\{ a_2, \left\lceil \frac{q - u_2}{\alpha} \right\rceil, \left\lfloor \frac{p - v_2}{\beta} \right\rfloor \right\} \leq a_2 = t + p - a_1 + 1 \leq t.
\]

Hence \(e_{pqr}(m_2) \geq e_{pqr}^-(m_2) = 0\) by Theorem 7 which completes the proof in cases (1) and (2a).

It remains to prove that \(e_2(m) \geq 0\) in case (2b). We will use variables \(u_1^*, v_1^*, a_1^*, j_{0,1}\), which we define like in part (ii) of Theorem 7. As we assumed \(e_{pqr}(m_1) > 0\), by Theorem 7 we have \(e_{pqr}^+(m_1) > 0\). Thus \(u_1^* < p'\) and \(v_1^* < q'\). So \(u_1^* \neq u_1 + j_{0,1}\alpha\) because \(u_1 \geq p'\). Therefore \(u_1^* = u_1 + j_{0,1}\alpha - q\) and \(v_1^* = v_1 + j_{0,1}\beta < q'\), which implies \(v_1 < q'\) and \(j_{0,1} = \lceil (q - u_1)/\alpha \rceil\).

Let us assume that \(e_{pqr}(m_2) < 0\). We will show that it leads to a contradiction. We have then \(e_{pqr}^+(m_1), e_{pqr}^-(m_2) > 0\) and so
\[
e_{pqr}^+(m_1) = \min \left\{ a_1^*, \left\lceil \frac{p' - u_1^*}{\alpha} \right\rceil, \left\lfloor \frac{q' - v_1^*}{\beta} \right\rfloor \right\} \leq \left\lfloor \frac{q' - v_1^*}{\beta} \right\rfloor - \left\lfloor \frac{q - u_1}{\alpha} \right\rfloor.
\]
Additionally,
\[
e_{pqr}^-(m_2) \leq \max \left\{ 0, \left\lceil \frac{p' - u_2}{\alpha} \right\rceil, \left\lfloor \frac{q' - v_2}{\beta} \right\rfloor \right\} - \left\lfloor \frac{p - v_2}{\beta} \right\rfloor \leq \left\lfloor \frac{p - v_2}{\beta} \right\rfloor - \left\lceil \frac{p' - u_2}{\alpha} \right\rceil = \left\lfloor \frac{v_1 + 1 - q'}{\beta} \right\rfloor - \left\lfloor \frac{u_1 + 1 - q}{\alpha} \right\rfloor.
\]

Let \(l_1 = q' - v_1\) and \(l_2 = q - u_1\). Combining the above bounds we conclude that
\[
e_{pqr}^+(m_1) + e_{pqr}^-(m_2) \leq \left( \left\lceil \frac{l_1}{\beta} \right\rceil + \left\lfloor \frac{1 - l_1}{\beta} \right\rfloor \right) - \left( \left\lceil \frac{l_2}{\alpha} \right\rceil + \left\lfloor \frac{1 - l_2}{\alpha} \right\rfloor \right).
\]

Since for all positive integers \(l\) and \(\gamma\) we have
\[
\left( \left\lceil \frac{l}{\gamma} \right\rceil + \left\lfloor \frac{1 - l}{\gamma} \right\rfloor \right) \in \{1, 2\},
\]
we receive \(e_{pqr}^+(m_1) + e_{pqr}^-(m_2) \leq 1\), contradicting \(e_{pqr}^+(m_1), e_{pqr}^-(m_2) > 0\). The proof is finally completed. \(\square\)

5. Flat Polynomials

Proof of Theorem 2 By Theorem 1 the polynomial \(\Psi_{pqr}\) is flat if and only if
\[
\max \left\{ \min \left\{ \left\lceil \frac{p'}{\alpha} \right\rceil, \left\lfloor \frac{q'}{\beta} \right\rfloor \right\}, \min \left\{ \left\lceil \frac{q - p'}{\alpha} \right\rceil, \left\lfloor \frac{p - q'}{\beta} \right\rfloor \right\} \right\} = 1.
\]
This is equivalent to
\[(\alpha \geq p' \text{ or } \beta \geq q') \text{ and } (\alpha \geq q - p' \text{ or } \beta \geq p - q'),\]
which by the logical distributive laws is equivalent to
\[(a) \text{ or } (b) \text{ or } (c) \text{ or } (d)\]
as desired. □

Proof of Theorem 3. Put \(q = tp + 1\). Then \(p' = q - t\) and \(q' = 1\), so (d) transforms into the condition
\[\alpha \geq t \text{ and } \beta \geq 1.\]
Therefore all primes \(r\) from the arithmetic progression
\[(q + tp, q + (t + 1)p, \ldots) = (2tp + 1, (2t + 1)p + 1, \ldots)\]
satisfy the condition (d). Recall that \(\pi(x; a, n)\) denotes the number of primes \(r \leq x\) satisfying \(r \equiv a \pmod{n}\). By the Dirichlet’s theorem on primes in arithmetic progressions and by the above observations, we have
\[
\#S(p, q) \geq \pi(\varphi(pq); 1, p) - \pi(2tp; 1, p)
\]
\[= \pi((p - 1)tp; 1, p) - \pi(2tp; 1, p)\]
\[\sim \frac{1}{p - 1} \left( \frac{(p - 1)tp}{\log t + \log(p^2 - p)} - \frac{2tp}{\log t + \log(2p)} \right)\]
\[\sim \frac{p(p - 3)}{p - 1} \frac{t}{\log t} \to \infty\]
with \(t \to \infty\), as we assumed \(p > 3\). Hence (1) is proved.
To prove (ii) we again put \(q = tp + 1\) and we assume that \(p > 5\), so
\[4tp + 1 = 4(q - 1) + 1 < \varphi(pq).\]
Then once more we use the Dirichlet’s theorem to show that the arithmetic progression
\[(2tp + 1, (2t + 1)p + 1, \ldots, 4tp + 1)\]
contains asymptotically \(\frac{2p}{p - 1} \frac{t}{\log t}\) primes as \(t \to \infty\), and hence for \(t\) large enough it contains at least one prime. If \(r\) is a prime contained in this progression, then \(r \in S(p, q)\) and
\[\frac{r}{\varphi(pq)} \leq \frac{4tp + 1}{(p - 1)tp} < \frac{5}{p}\]
Therefore for every prime \(p > 5\) there exist a prime \(q\) and a prime \(r \in S(p, q)\) such that \(r/\varphi(pq) < 5/p\). We can chose \(p\) arbitrarily, so the proof is completed. □

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References

[1] G. Bachman, Flat cyclotomic polynomials of order three, Bull. London Math. Soc. 38 (2006), 53–60.
[2] S. Elder, Flat cyclotomic polynomials: A new approach, arXiv:1207.5811 [math.NT].
[3] N. Kaplan, Flat cyclotomic polynomials of order three, J. Number Theory 127 (2007), 118–126.
[4] T.Y. Lam, K.H. Leung, On the cyclotomic polynomial $\Phi_{pq}(X)$, Amer.Math. Monthly 103 (1996), 562-564.
[5] P. Moree, Inverse cyclotomic polynomials, J. Number Theory 129 (2009), 667–680.

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