DYNAMICAL SYSTEMS AND POISSON STRUCTURES

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Abstract

We first consider the Hamiltonian formulation of $n = 3$ systems in general and show that all dynamical systems in $\mathbb{R}^3$ are bi-Hamiltonian. An algorithm is introduced to obtain Poisson structures of a given dynamical system. We find the Poisson structures of a dynamical system recently given by Bender et al. Secondly, we show that all dynamical systems in $\mathbb{R}^n$ are $(n - 1)$-Hamiltonian. We give also an algorithm, similar to the case in $\mathbb{R}^3$, to construct a rank two Poisson structure of dynamical systems in $\mathbb{R}^n$. We give a classification of the dynamical systems with respect to the invariant functions of the vector field $\vec{X}$ and show that all autonomous dynamical systems in $\mathbb{R}^n$ are super-integrable.
1. Introduction

Hamiltonian formulation of $n = 3$ systems has been intensively considered in the last two decades. Works [1], [2] on this subject give a very large class of solutions of the Jacobi equation for the Poisson matrix $J$. Recently generalizing the solutions given in [1] we gave the most general solution of the Jacobi equation in $\mathbb{R}^3$, [3]. Matrix $J = (J^{ij})$, $i, j = 1, 2, \ldots, n$ defines a Poisson structure in $\mathbb{R}^n$ if it is skew-symmetric, $J^{ij} = -J^{ji}$, and its entries satisfy the Jacobi equation

$$J^{il} \partial_l J^{jk} + J^{lj} \partial_l J^{ki} + J^{lk} \partial_l J^{ij} = 0, \quad (1)$$

where $i, j, k = 1, 2, \ldots, n$. Here we use the summation convention, meaning that repeated indices are summed up. We showed in [3] that the general solution of the above equation (1) in the case $n = 3$ has the form

$$J^{ij} = \mu \varepsilon^{ijk} \partial_k \Psi, \quad i, j = 1, 2, 3, \quad (2)$$

where $\mu$ and $\Psi$ are arbitrary differentiable functions of $x^i, t$, $i = 1, 2, 3$ and $\varepsilon^{ijk}$ is the Levi-Civita symbol. Here $t$ should be considered as a parameter. In the same work we have also considered a bi-Hamiltonian representation of Hamiltonian systems. It turned out that any Hamiltonian system in $\mathbb{R}^3$ has a bi-Hamiltonian representation.

In the present paper we prove that any $n$-dimensional dynamical system

$$\dot{x} = \vec{X}(x^1, x^2, \ldots, x^n, t), \quad (3)$$

where $\vec{x} = (x^1, x^2, \ldots, x^n)$, is Hamiltonian, that is, has the form

$$\dot{x}^i = J^{ij} \partial_j H, \quad i = 1, 2, \ldots, n, \quad (4)$$

where $J = (J^{ij})$ is a Poisson matrix and $H$, as well as $J^{ij}$, are differentiable functions of the variables $x_1, x_2, \ldots, x_n, t$. Moreover, we show that the system (3) is $(n - 1)$-Hamiltonian. This problem in the case $n = 3$ was considered in [4], [5] where authors start with an invariant of the dynamical system as a Hamiltonian and then proceed by writing the system in the form (4) and imposing conditions on $J$ so that it satisfies the Jacobi equation. But proofs given in these works are, as it seems to us, incomplete and not satisfactory.
Using (2) for matrix $J$ we can write equation (4) in $\mathbb{R}^3$ as

$$\dot{x} = \mu \vec{\nabla} \Psi \times \vec{\nabla} H.$$  \hspace{1cm} (5)

Let $\vec{X}$ be a vector field in $\mathbb{R}^3$. If $H_1$ and $H_2$ are two invariant functions of $\vec{X}$, i.e., $\vec{X}(H_\alpha) = X^i \partial_j H_\alpha = 0, \ \alpha = 1, 2$, then $\vec{X}$ is parallel to $\vec{\nabla} H_1 \times \vec{\nabla} H_2$. Therefore

$$\vec{X} = \mu \vec{\nabla} H_1 \times \vec{\nabla} H_2,$$  \hspace{1cm} (6)

where the function $\mu$ is a coefficient of proportionality. The right-hand side of equation (6) is in the same form as the right-hand side of equation (5), so $\vec{X}$ is a Hamiltonian vector field. We note that the equation which allows to find the invariants of a vector field $\vec{X}$ is a first order linear partial differential equation. We remark here that dynamical systems in $\mathbb{R}^3$ differ from the dynamical systems in $\mathbb{R}^n$ for $n > 3$. We know the general solution (2) of the Jacobi equation (1) in $\mathbb{R}^3$. In $\mathbb{R}^n$, as we shall see in the last section, we know only the rank 2 solutions of the Jacobi equations for all $n$.

An important difference of our work, contrary to other works in the subject, is that in the construction of the Poisson structures we take into account the invariant functions of the vector field $\vec{X}$ rather than the invariants (constants of motion) of the dynamical system. The total time derivative of a differentiable function $F$ in $\mathbb{R}^n$ along the phase trajectory is given by

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \vec{X} \cdot \vec{\nabla} F.$$  \hspace{1cm} (7)

An invariant function of the vector field $\vec{X}(x^1, x^2, \ldots, x^n, t)$, i.e., $\vec{X} \cdot \vec{\nabla} F = 0$, is not necessarily an invariant function (constant of motion) of the dynamical system. For autonomous systems where $\vec{X} = \vec{X}(x^1, x^2, \ldots, x^n)$ these invariant functions are the same. We give a representation of the vector field $\vec{X}$ in terms of its invariant functions. We show that all autonomous dynamical systems are super-integrable. A key role plays the existence of $n - 1$ functionally independent solutions $\zeta_\alpha(x^1, x^2, \ldots, x^n, t), (\alpha = 1, 2, \ldots, n - 1)$ of the linear partial differential equation

$$\vec{X} \cdot \vec{\nabla} \zeta \equiv X^1 \frac{\partial \zeta}{\partial x^1} + X^2 \frac{\partial \zeta}{\partial x^2} + \cdots + X^n \frac{\partial \zeta}{\partial x^n} = 0,$$  \hspace{1cm} (8)
where $X^i = X^i(x^1, x^2, \ldots, x^n, t), \ i = 1, 2, \ldots, n$, are given functions (see [6]-[8]). For all $\alpha = 1, 2, \ldots, n - 1$, $\vec{\nabla} \zeta_\alpha$ is perpendicular to the vector field $\vec{X}$. This leads to the construction of the rank 2 Poisson tensors for $n > 3$:

$$
J^{ij} = \mu \epsilon^{\alpha_1 \alpha_2 \cdots \alpha_{n-2}} \epsilon^{ijj_1 \cdots j_{n-2}} \partial_{j_1} \zeta_{\alpha_1} \partial_{j_2} \zeta_{\alpha_2} \cdots \partial_{j_{n-2}} \zeta_{\alpha_{n-2}},
$$

(9)

where $i, j = 1, 2, \ldots, n$, and $\alpha = 1, 2, \ldots, n - 1$. Here $\epsilon^{ijj_1 \cdots j_{n-2}}$ and $\epsilon^{\alpha_1 \alpha_2 \cdots \alpha_{n-2}}$ are Levi-Civita symbols in $n$ and $n - 1$ dimensions respectively. Any dynamical system with the vector field $\vec{X}$ possesses Poisson structures in the form given in (9). Hence we can give a classification of dynamical systems in $\mathbb{R}^n$ with respect to the invariant functions of the vector field $\vec{X}$. There are mainly three classes where the super-integrable dynamical systems constitute the first class. By the use of the invariant functions of the vector field $\vec{X}(x^1, x^2, \ldots, x^n, t)$ in general we give a Poisson structure in $\mathbb{R}^n$ which has rank 2. For autonomous systems, the form (9) of the above Poisson structure first was given in the works [11] and [12].

Our results in this work are mainly local. This means that our results are valid in an open domain of $\mathbb{R}^n$ where the Poisson structures are different from zero. In [3] we showed that the Poisson structure (2) in $\mathbb{R}^3$ preserves its form in the neighborhood of irregular points, lines and planes.

In the next section we give new proofs of the formula (2) and prove that any dynamical system in $\mathbb{R}^3$ is Hamiltonian. So, following [3] we show that any dynamical system in $\mathbb{R}^3$ is bi-Hamiltonian. Applications of these theorems to several dynamical systems are presented. Here we also show that the dynamical system given by Bender at al [10] is bi-Hamiltonian. In section 3 we discuss Poisson structures in $\mathbb{R}^n$. We give a representation of the Poisson structure in $\mathbb{R}^n$ in terms of the invariant functions of the vector field $\vec{X}$. Such a representation leads to a classification of dynamical systems with respect to these functions.

2. Dynamical Systems in $\mathbb{R}^3$

Although the proof of (2) was given in [3], here we shall give two simpler proofs. The first one is a shorter proof than the one given in [3]. In the sequel we use the notations $x^1 = x, x^2 = y, x^3 = z$.

**Theorem 1.** All Poisson structures in $\mathbb{R}^3$ have the form (2), i.e., $J^{ij} =$
\( \mu \epsilon^{ijk} \partial_k H_0 \). Here \( \mu \) and \( H_0 \) are some differentiable functions of \( x^i \) and \( t \), \((i = 1, 2, 3)\)

**Proof.** Any skew-symmetric second rank tensors in \( \mathbb{R}^3 \) can be given as

\[
J^i_j = \epsilon^{ijk} J_k, \quad i, j = 1, 2, 3,
\]

(10)

where \( J_1, J_2 \) and \( J_3 \) are differentiable functions in \( \mathbb{R}^3 \) and we assume that there exists a domain \( \Omega \) in \( \mathbb{R}^3 \) so that these functions do not vanish simultaneously. When (10) inserted into the Jacobi equation (1) we get

\[
\vec{J} \cdot (\vec{\nabla} \times \vec{J}) = 0,
\]

(11)

where \( \vec{J} = (J_1, J_2, J_3) \) is a differentiable vector field in \( \mathbb{R}^3 \) not vanishing in \( \Omega \). We call \( \vec{J} \) as the Poisson vector field. It is easy to show that (11) has a local scale invariance. Let \( \vec{J} = \psi \vec{E} \), where \( \psi \) is an arbitrary function. If \( \vec{E} \) satisfies (11) then \( \vec{J} \) satisfies the same equation. Hence it is enough to show that \( \vec{E} \) is proportional to the gradient of a function. Using freedom of local scale invariance we can take \( \vec{E} = (u, v, 1) \) where \( u \) and \( v \) are arbitrary functions in \( \mathbb{R}^3 \). Then (11) for vector \( \vec{E} \) reduces to

\[
\partial_y u - \partial_x v - v \partial_z u + u \partial_z v = 0,
\]

(12)

where \( x, y, z \) are local coordinates. Letting \( u = \frac{\partial f}{\rho} \) and \( v = \frac{\partial f}{\rho} \), where \( f \) and \( \rho \) are functions of \( x, y, z \) we get

\[
\partial_x f \partial_y (\rho - \partial_z f) - \partial_y f \partial_z (\rho - \partial_z f) = 0.
\]

(13)

General solution of this equation is given by

\[
\rho - \partial_z f = h(f, z),
\]

(14)

where \( h \) is an arbitrary function of \( f \) and \( z \). Then the vector field \( \vec{E} \) takes the form

\[
\vec{E} = \frac{1}{\partial_z f + h} (\partial_z f, \partial_y f, \partial_z f + h).
\]

(15)

Let \( g(f, z) \) be a function satisfying \( g_{,z} = h \partial_f g \). Here we note that \( \partial_z g(f, z) = \frac{\partial g}{\partial f} \partial_z f + g_{,z} \) where \( g_{,z} = \partial_s g(f(x, y, z), s)|_{s=z} \). Then (15) becomes

\[
\vec{E} = \frac{1}{(\partial_z f + h) \partial_f g} \vec{\nabla} g,
\]

(16)
which completes the proof. Here \( \partial f \, g = \frac{\partial g}{\partial f} \).

The second proof is an indirect one which is given in [8] (Theorem 5 in this reference).

**Definition 2.** Let \( \vec{F} \) be a vector field in \( \mathbb{R}^3 \). Then the equation \( \vec{F} \cdot d\vec{x} = 0 \) is called a Pfaffian differential equation. A Pfaffian differential equation is called integrable if the 1-form \( \vec{F} \cdot d\vec{x} = \mu dH \), where \( \mu \) and \( H \) are some differentiable functions in \( \mathbb{R}^3 \).

Let us now consider the Pfaffian differential equation with the Poisson vector field \( \vec{J} \) in (10)

\[
\vec{J} \cdot d\vec{x} = 0.
\]

For such Pfaffian differential equations we have the following result (see [8]).

**Theorem 3.** A necessary and sufficient condition that the Pfaffian differential equation \( \vec{J} \cdot d\vec{x} = 0 \) should be integrable is that \( \vec{J} \cdot (\nabla \times \vec{J}) = 0 \).

By (11), this theorem implies that \( \vec{J} = \mu \nabla \Psi \).

A well known example of a dynamical system with Hamiltonian structure of the form (4) is the Euler equations.

**Example 1.** The Euler equations [6] are

\[
\begin{align*}
\dot{x} &= \frac{I_2 - I_3}{I_2 I_3} y z, \\
\dot{y} &= \frac{I_3 - I_1}{I_3 I_1} x z, \\
\dot{z} &= \frac{I_1 - I_2}{I_1 I_2} x y,
\end{align*}
\]

where \( I_1, I_2, I_3 \in \mathbb{R} \) are some (non-vanishing) real constants. This system admits Hamiltonian representation of the form (4). The matrix \( J \) can be defined in terms of functions \( \Psi = H_0 = -\frac{1}{2}(x^2 + y^2 + z^2) \) and \( \mu = 1 \), and we take \( H = H_1 = \frac{x^2}{2I_1} + \frac{y^2}{2I_2} + \frac{z^2}{2I_3} \).

Writing the Poisson structure in the form (2) allows us to construct bi-Hamiltonian representations of a given Hamiltonian system.
Definition 4. Two Poisson structures $J_0$ and $J_1$ are compatible, if the sum $J_0 + J_1$ defines also a Poisson structure.

Lemma 5. Let $\mu, H_0$, and $H_1$ be arbitrary differentiable functions. Then the Poisson structures $J_0$ and $J_1$ given by $J_0^{ij} = \mu \epsilon^{ijk} \partial_k H_0$ and $J_1^{ij} = -\mu \epsilon^{ijk} \partial_k H_1$ are compatible.

This suggests that all Poisson structures in $\mathbb{R}^3$ have compatible companions. Such compatible Poisson structures can be used to construct bi-Hamiltonian systems (for Hamiltonian and bi-Hamiltonian systems see [6],[9] and the references therein).

Definition 6. A Hamiltonian equation is said to be bi-Hamiltonian if it admits compatible Poisson structures $J_0$ and $J_1$ with the corresponding Hamiltonian functions $H_1$ and $H_0$ respectively, such that

$$\frac{dx}{dt} = J_0 \nabla H_1 = J_1 \nabla H_0.$$ (19)

Lemma 7. Let $J_0$ be given by (2), i.e., $J_0^{ij} = \mu \epsilon^{ijk} \partial_k H_0$, and let $H_1$ be any differentiable function, then the Hamiltonian equation

$$\frac{dx}{dt} = J_0 \nabla H_1 = J_1 \nabla H_0 = \mu \vec{\nabla} H_1 \times \vec{\nabla} H_0,$$ (20)

is bi-Hamiltonian with the second Poisson structure given by $J_1$ with entries $J_1^{ij} = -\mu \epsilon^{ijk} \partial_k H_1$ and the second Hamiltonian $H_0$.

Let us prove that any dynamical system in $\mathbb{R}^3$ has Hamiltonian form.

Theorem 8. All dynamical systems in $\mathbb{R}^3$ are Hamiltonian. This means that any vector field $\vec{X}$ in $\mathbb{R}^3$ is Hamiltonian vector field. Furthermore all dynamical systems in $\mathbb{R}^3$ are bi-Hamiltonian.

Proof. Let $\zeta$ be an invariant function of the vector field $\vec{X}$, i.e., $X(\zeta) \equiv \vec{X} \cdot \vec{\nabla} \zeta = 0$. This gives a first order linear differential equation in $\mathbb{R}^3$ for $\zeta$. For a given vector field $\vec{X} = (f, g, h)$ this equation becomes

$$f(x, y, z, t) \frac{\partial \zeta}{\partial x} + g(x, y, z, t) \frac{\partial \zeta}{\partial y} + h(x, y, z, t) \frac{\partial \zeta}{\partial z} = 0,$$ (21)
where \( x, y, z \) are local coordinates. From the theory of first order linear partial differential equations \([6], [7], [8]\) the general solution of this partial differential equation can be determined from the following set of equations

\[
\frac{dx}{f(x, y, z, t)} = \frac{dy}{g(x, y, z, t)} = \frac{dz}{h(x, y, z, t)}. \tag{22}
\]

There exist two functionally independent solutions \( \zeta_1 \) and \( \zeta_2 \) of (22) in an open domain \( D \subset \mathbb{R}^3 \) and the general solution of (21) will be an arbitrary function of \( \zeta_1 \) and \( \zeta_2 \), i.e., \( \zeta = F(\zeta_1, \zeta_2) \). This implies that the vector field \( \vec{X} \) will be orthogonal to both \( \nabla \zeta_1 \) and \( \nabla \zeta_2 \). Then \( \vec{X} = \mu (\nabla \zeta_1) \times (\nabla \zeta_2) \). Hence the vector field \( \vec{X} \) is Hamiltonian by (5). \( \square \)

This theorem gives also an algorithm to find the Poisson structures or the functions \( H_0, H_1 \) and \( \mu \) of a given dynamical system. The functions \( H_0 \) and \( H_1 \) are the invariant functions of the vector field \( \vec{X} \) which can be determined by solving the system equations (22) and \( \mu \) is determined from

\[
\mu = \frac{\vec{X} \cdot \vec{X}}{\vec{X} \cdot (\nabla H_0 \times \nabla H_1)}. \tag{23}
\]

Note that \( \mu \) can also be determined from

\[
\mu = \frac{X^1}{\partial_2 H_0 \partial_3 H_1 - \partial_3 H_0 \partial_2 H_1}, \tag{24}
\]

\[
= \frac{X^2}{\partial_3 H_1 \partial_2 H_1 - \partial_2 H_0 \partial_3 H_1}, \tag{25}
\]

\[
= \frac{X^3}{\partial_1 H_0 \partial_2 H_1 - \partial_2 H_0 \partial_1 H_1}. \tag{26}
\]

**Example 2.** As an application of the method described above we consider Kermac-Mckendric system

\[
\dot{x} = -rxy, \\
\dot{y} = rxy - ay, \\
\dot{z} = ay, \tag{25}
\]

where \( r, a \in \mathbb{R} \) are constants. Let us put the system into Hamiltonian form. For the Kermac-Mckendric system, equations (22) become

\[
\frac{dx}{-rxy} = \frac{dy}{rxy - ay} = \frac{dz}{ay}. \tag{26}
\]
Here $a$ and $r$ may depend on $t$ in general. Adding the numerators and denominators of (26) we get
\[
\frac{dx}{-rx} = \frac{dx + dy + dz}{0}. 
\] (27)

Hence $H_1 = x + y + z$ is one of the invariant functions of the vector field. Using the first and last terms in (26) we get
\[
\frac{dx}{-rx} = \frac{dz}{a}, 
\] (28)
which gives $H_0 = rz + a \ln x$ as the second invariant function of the vector field $\vec{X}$. Using (23) we get $\mu = xy$. Since $\vec{X} = \mu \vec{\nabla} H_0 \times \vec{\nabla} H_1$, the system admits a Hamiltonian representation where the Poisson structure $J$ is given by (2) with $\mu = xy$, $\Psi = H_0 = rz + a \ln x$, and the Hamiltonian is $H_1 = x + y + z$.

**Example 3.** The dynamical system is given by
\[
\begin{align*}
\dot{x} &= yz(1 + 2x^2 \frac{N}{D}), \\
\dot{y} &= -2xz(1 - y^2 \frac{N}{D}), \\
\dot{z} &= xy(1 + 2z^2 \frac{N}{D}),
\end{align*} 
\] (29)
where $N = x^2 + y^2 + z^2 - 1$, $D = x^2y^2 + y^2z^2 + 4x^2z^2$. This example was obtained by Bender et all [10] by complexifying the Euler system in Example 1. They claim that this system is not Hamiltonian apparently bearing in mind the more classical definition of a Hamiltonian system. Using the Definition 6 we show that this system is not only Hamiltonian but also bi-Hamiltonian.

We obtain that
\[
H_0 = \frac{(N + 1)^2}{D} N, \quad H_1 = \frac{x^2 - z^2}{D} (2y^2z^2 + 4x^2z^2 + y^4 + 2x^2y^2 - y^2). \quad (30)
\]

Here
\[
\mu = \frac{D^2}{4[3D^2 + DP + Q]}, \quad (31)
\]
where
\[
\begin{align*}
P &= -2x^4 + 4y^4 - 4x^2y^2 + x^2 - 2y^2 - 4y^2z^2 + 14z^4 + z^2, \\
Q &= -2x^8 + 12x^6z^2 + 2x^6 - 20x^4z^4 - 6x^4z^2 - 52x^2z^6 - 6x^2z^4 \\
&\quad + y^8 - y^6 + 4y^4z^4 - 16y^2z^6 - 2z^8 + 2z^6. \quad (32)
\end{align*}
\]
Indeed these invariant functions were given in [10] as functions $A$ and $B$. The reason why Bender et al [10] concluded that the system in Example 3 is non-Hamiltonian is that the vector field $\vec{X}$ has nonzero divergence. It follows from $\vec{X} = \mu \nabla H_0 \times \nabla H_1$ that $\nabla \cdot \left( \frac{1}{\mu} \vec{X} \right) = 0$. When $\mu$ is not a constant the corresponding Hamiltonian vector field has a nonzero divergence.

Remark 1. With respect to the time dependency of invariant functions of the vector field $\vec{X}$ dynamical systems in $\mathbb{R}^3$ can be split into three classes.

Class A. Both invariant functions $H_0$ and $H_1$ of the vector field $\vec{X}$ do not depend on time explicitly. In this case both $H_0$ and $H_1$ are also invariant functions of the dynamical systems. Hence the system is super-integrable. All autonomous dynamical systems such as the Euler equation (Example 1) and the Kermac-Mckendric system (Example 2) belong to this class.

Class B. One of the invariant functions $H_0$ and $H_1$ of the vector field $\vec{X}$ depends on $t$ explicitly. Hence the other one is an invariant function also of the dynamical system. When $I_1, I_2$ and $I_3$ in Example 1 are time dependent the Euler system becomes the member of this class. In this case $H_0$ is the Hamiltonian function and $H_1$ is the function defining the Poisson structure. Similarly, in Example 2 we may consider the parameters $a$ and $r$ as time dependent. Then Kermac-Mckendric system becomes also a member of this class.

Class C. Both $H_0$ and $H_1$ are explicit functions of time variable $t$ but they are not the invariants of the system. There may be invariants of the dynamical system. Let $F$ be such an invariant. Then

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H_1\}_0 = \frac{\partial F}{\partial t} + \{F, H_0\}_1 = 0,$$  \hspace{1cm} (33)

where for any $F$ and $G$

$$\{F, G\}_\alpha = J^{ij}_\alpha \partial_i F \partial_j G, \quad \alpha = 0, 1.$$  \hspace{1cm} (34)

3. Poisson structures in $\mathbb{R}^n$

Let us consider the dynamical system
\[
\frac{dx^i}{dt} = X^i(x^1, x^2, \ldots, x^n, t), \quad i = 1, 2, \ldots, n. \tag{35}
\]

**Theorem 9.** All dynamical systems in \( \mathbb{R}^n \) are Hamiltonian. Furthermore all dynamical systems in \( \mathbb{R}^n \) are \((n - 1)\)-Hamiltonian.

**Proof.** Extending the proof of Theorem 8 to \( \mathbb{R}^n \) consider the linear partial differential equation (8). There exist \( n - 1 \) functionally independent solutions \( H_\alpha, (\alpha = 1, 2, \ldots, n - 1) \) of this equation (which are invariant functions of the vector field \( \vec{X} \)) [6]-[8]. Since \( \vec{X} \) is orthogonal to the vectors \( \vec{\nabla}H_\alpha, (\alpha = 1, 2, \ldots, n - 1) \), we have

\[
\vec{X} = \mu \begin{vmatrix}
\vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \\
\partial_1 H_1 & \partial_2 H_1 & \cdots & \partial_n H_1 \\
\vdots & \vdots & \ddots & \vdots \\
\partial_1 H_{n-1} & \partial_2 H_{n-1} & \cdots & \partial_n H_{n-1}
\end{vmatrix}, \tag{36}
\]

where the function \( \mu \) is a coefficient of proportionality and \( \vec{e}_i \) is \( n \)-dimensional unit vector with the \( i \)th coordinate 1 and remaining coordinates 0. Therefore

\[
X^i = \mu \epsilon^{i j_1 j_2 \cdots j_{n-1}} \partial_{j_1} H_1 \partial_{j_2} H_2 \cdots \partial_{j_{n-1}} H_{n-1}. \tag{37}
\]

Hence all dynamical systems (35) have the Hamiltonian representation

\[
\frac{dx^i}{dt} = J^{ij}_\alpha \partial_j H_\alpha, \quad i = 1, 2, \ldots, n, \quad (\text{no sum on } \alpha) \tag{38}
\]

with

\[
J^{ij}_\alpha = \mu \epsilon^{\alpha_1 \alpha_2 \cdots \alpha_{n-2}} \epsilon^{i j_1 j_2 \cdots j_{n-2}} \partial_{j_1} H_{\alpha_1} \partial_{j_2} H_{\alpha_2} \cdots \partial_{j_{n-2}} H_{\alpha_{n-2}}, \tag{39}
\]

where \( i, j = 1, 2, \ldots, n, \alpha = 1, 2, \ldots, n - 1 \). Here \( \epsilon^{i j_1 \cdots j_{n-2}} \) and \( \epsilon^{\alpha_1 \alpha_2 \cdots \alpha_{n-2}} \) are Levi-Civita symbols in \( n \) and \( n - 1 \) dimensions respectively. The function \( \mu \) can be determined, for example, from

\[
\mu = \frac{X^1}{\begin{vmatrix}
\partial_2 H_1 & \cdots & \partial_n H_1 \\
\vdots & \ddots & \vdots \\
\partial_2 H_{n-1} & \cdots & \partial_n H_{n-1}
\end{vmatrix}} \tag{40}
\]
It can be seen that the matrix $J_\alpha$ with the entries $J^{ij}_\alpha$ given by (39) defines a Poisson structure in $\mathbb{R}^n$ and since

$$J_\alpha \cdot \nabla H_\beta = 0, \quad \alpha, \beta = 1, 2, \ldots, n-1,$$

with $\beta \neq \alpha$, the rank of the matrix $J_\alpha$ equals 2 (for all $\alpha = 1, 2, \ldots, n-1$).

In (38) we can take any of $H_1, H_2, \cdots, H_{n-1}$ as the Hamilton function and use the remaining $H_k$’s in (39). We observe that all dynamical systems (35) in $\mathbb{R}^n$ have $n - 1$ number of different Poisson structures in the form given by (39). The same system may have a Poisson structure with a rank higher than two. The following example clarifies this point.

**Example 4.** Let

$$\dot{x}_1 = x_4, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = -x_2, \quad \dot{x}_4 = -x_1.$$  

(42)

Clearly this system admits a Poisson structure with rank four

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad H = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + x_4^2).$$

(43)

The invariant functions of the vector field $\vec{X} = (x_4, x_3, -x_2, -x_1)$ are

$$H_1 = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + x_4^2),$$

(44)

$$H_2 = \frac{1}{2}(x_2^2 + x_3^2),$$

(45)

$$H_3 = x_1 x_3 - x_2 x_4.$$  

(46)

Then the above system has three different ways of representation with the second rank Poisson structures

$$J^{ij}_1 = \mu \epsilon^{ijkl} \partial_k H_1 \partial_l H_2, \quad H = H_3,$$

(47)

$$J^{ij}_2 = -\mu \epsilon^{ijkl} \partial_k H_1 \partial_l H_3, \quad H = H_2,$$

(48)

$$J^{ij}_3 = \mu \epsilon^{ijkl} \partial_k H_2 \partial_l H_3, \quad H = H_1,$$

(49)

where $\mu(x_1 x_2 + x_3 x_4) = 1$. These Poisson structures are compatible not only pairwise but also triple-wise. This means that any linear combination of
these structures is also a Poisson structure. Let \( J = \alpha_1 J_1 + \alpha_2 J_2 + \alpha_3 J_3 \) then it is possible to show that

\[
J^{ij} = \mu e^{ijkl} \partial_k \tilde{H}_1 \partial_l \tilde{H}_2,
\]

where \( \tilde{H}_1 \) and \( \tilde{H}_2 \) are linear combinations of \( H_1, H_2 \) and \( H_3 \),

\[
\tilde{H}_1 = H_1 - \frac{\alpha_3}{\alpha_2} H_2, \quad \tilde{H}_2 = \alpha_1 H_2 - \alpha_2 H_3 \quad \text{if} \quad \alpha_2 \neq 0,
\]

\[
\tilde{H}_1 = \alpha_1 H_1 - \alpha_3 H_2, \quad \tilde{H}_2 = H_2 \quad \text{if} \quad \alpha_2 = 0.
\]

**Definition 10.** A dynamical system (35) in \( \mathbb{R}^n \) is called super-integrable if it has \( n - 1 \) functionally independent first integrals (constants of motion).

**Theorem 11.** All autonomous dynamical systems in \( \mathbb{R}^n \) are super-integrable.

**Proof.** If the system (35) is autonomous, then the vector field \( \vec{X} \) does not depend on \( t \) explicitly. Therefore each of the invariant functions \( H_\alpha, (\alpha = 1, 2, \cdots, n-1) \) of the vector field \( \vec{X} \) is a constant of motion of the system (35).

Some (or all) of the invariant functions \( H_\alpha, (\alpha = 1, 2, \cdots, n-1) \) of the vector field \( \vec{X} \) may depend on \( t \). Like in \( \mathbb{R}^3 \) we can classify the dynamical systems in \( \mathbb{R}^n \) with respect to the invariant functions of the vector field \( \vec{X} (x^1, x^2, \cdots, x^n, t) \).

**Class A.** All invariant functions \( H_\alpha, (\alpha = 1, 2, \cdots, n-1) \) of the vector field \( \vec{X} \) do not depend on \( t \) explicitly. In this case all functions \( H_\alpha, (\alpha = 1, 2, \cdots, n-1) \) are also invariant functions (constants of motion) of the dynamical system. Hence the system is super-integrable. In the context of the the multi- Hamiltonian structure, such systems were first studied by [12] and [11]. The form (39) of the Poisson structure was given in these works. Its properties were investigated in [13].

**Class B.** At least one of the invariant functions \( H_\alpha, (\alpha = 1, 2, \cdots, n-1) \) of the vector field \( \vec{X} \) does not depend on \( t \) explicitly. That function is an invariant function also of the dynamical system.
Class C. All $H_\alpha$, $(\alpha = 1, 2, \cdots, n - 1)$ are explicit function of time variable $t$ but they are not the invariants of the system. There may be invariants of the dynamical system. Let $F$ be such an invariant. Then

$$\frac{dF}{dt} \equiv \frac{\partial F}{\partial t} + \{F, H_\alpha\}_\alpha = 0, \quad \alpha = 1, 2, \cdots, n - 1 \quad (53)$$

where for any $F$ and $G$

$$\{F, G\}_\alpha \equiv J^i_j \alpha \partial_i F \partial_j G, \quad \alpha = 0, 1, \cdots, n - 1. \quad (54)$$

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