Sparse Graph Codes for Non-adaptive Quantitative Group Testing

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Abstract—This paper considers the problem of quantitative group testing with a non-adaptive testing strategy. In a quantitative group testing scheme, a set of tests are designed to identify defective items among a large population of items, where the outcome of a test shows the number of defective items in the tested group. There are two models for the defective items: (i) deterministic, and (ii) randomized. In the deterministic model, each defective item is defective with probability $\frac{k}{N}$, independent of the other items, where $N$ is the total number of items. In this work, we propose a non-adaptive quantitative group testing algorithm using sparse graph codes over bi-regular bipartite graphs with left-degree $\ell$ and right degree $r$ and binary $t$-error-correcting BCH codes. We show that for both the deterministic and randomized models, our algorithm requires at most $m = c(t)K(t \log(N) + 1) + 1$ tests to recover all the defective items with probability approaching one (as $N$ and $K$ grow unbounded), where $c(t)$ is a constant that depends only on $t$. The results of our theoretical analysis reveal that using a $t$-error-correcting binary BCH code for $t \in \{1, 2, 3\}$, when compared to $t \geq 4$, leads to a fewer number of tests. Simulation results show that the proposed strategy significantly reduces the required number of tests for identifying all the defective items with probability approaching one compared to a recently proposed scheme.

I. INTRODUCTION

We consider the Quantitative Group Testing (QGT) problem which is concerned with identifying defective items in a given population of items, where the result of a test reveals the number of defective items in the tested group. The objective of QGT is to design a test plan that identifies all the defective items using minimum number of tests. There are two models for the defective items: deterministic and randomized. In the deterministic model (a.k.a. the combinatorial model), the exact number of defective items, $K$, is known, whereas in the randomized model (a.k.a. the probabilistic model), each item is defective with probability $\frac{k}{N}$, independent of the other items, where $N$ is the total number of items [1].

There are two general categories of test strategies: non-adaptive and adaptive. In an adaptive scheme, each test depends on the outcomes of the previous tests. On the other hand, in a non-adaptive scheme, all tests are planned in advance. In other words, the result of one test does not affect the design of another test. Although, in general, adaptive algorithms require fewer tests, in most practical applications non-adaptive algorithms are preferred since they allow one to perform all tests at once in parallel.

A. Related Work and Applications

The QGT problem was first introduced by Shapiro in 1960 [2]. Shapiro studied the QGT problem for unknown $K$ (i.e., when the number of defective items, although fixed, is not known a priori), but later the QGT problem was studied for known $K$, e.g., in [3], [4]. Several non-adaptive and adaptive weighing strategies have been proposed, see, e.g., [5]–[7], and references therein. In this work, we focus on non-adaptive algorithms. As was shown in [8] and [9], any non-adaptive algorithm must perform at least $\frac{2K \log N}{\epsilon^2}$ tests. Various order-optimal or near-optimal non-adaptive strategies were previously proposed for unknown $K$, see, e.g., [9], and for known $K$, see, e.g., [5], [7]. The best known polynomial-time non-adaptive algorithms require $K \log N$ tests [9], [11]. Recently, a semi-quantitative group testing scheme based on sparse graph codes was proposed in [12], where the result of each test is an integer in the set $\{0, 1, 2, \cdots, L\}$. This strategy identifies $\left(1 - \epsilon\right)$ fraction of the defective items using $c(\epsilon, L)K \log(N)$ tests, where $c(\epsilon, L)$ is a constant that only depends on $\epsilon$ and $L$.

The QGT problem has been extensively studied for a wide range of applications, e.g., multi-access communication, spectrum sensing, and network tomography, to name a few (see, e.g., [6], and references therein).

B. Main Contributions

In this work, we propose a non-adaptive quantitative group testing strategy for both the deterministic and randomized models. We utilize sparse graph codes over bi-regular bipartite graphs with left-degree $\ell$ and right-degree $r$ and binary $t$-error-correcting BCH codes for the design of the proposed strategy, and leverage powerful density evolution techniques for the analysis. For both models of the defective items, our algorithm requires at most $m = c(t)K(t \log \left(\frac{N}{c(t)K} + 1\right) + 1) + 1$ tests to recover all the defective items with probability approaching one (as $N$ and $K$ grow unbounded), where $c(t)$ is a constant that depends only on $t$. The results of our theoretical analysis reveal that using a $t$-error-correcting binary BCH code for $t \in \{1, 2, 3\}$, when compared to $t \geq 4$, leads to a fewer number of tests.

Throughout the paper the base of log is 2, unless explicitly noted otherwise.
II. PROBLEM SETUP AND NOTATIONS

Throughout the paper, we use bold-face small and capital letters to denote vectors and matrices, respectively.

We define the following two models for the defective items.

Definition 1. (Deterministic Model) Among a set of $N$ items, there are exactly $K$ defective items.

Definition 2. (Randomized Model) Among a set of $N$ items, each item is defective with probability $\frac{K}{N}$, independently from the other items, for some given integer $1 \leq K \leq N$.

The objective of the quantitative group testing problem is to identify the defective items given the results of $m$ tests, where the result of each test shows the number of defective items in the testing pool.

Let vector $x \in \{0, 1\}^N$ denote the set of $N$ items in which the coordinates with value 1 correspond to the defective items. A non-adaptive group testing problem consisting of $m$ tests can be represented by a measurement matrix $A \in \{0, 1\}^{m \times N}$, where the $i$-th row of the matrix corresponds to the $i$-th test. That is, the coordinates with value 1 in the $i$-th row correspond to the items in the $i$-th test. The results of the $m$ tests can be expressed by vector $y \in \{Z_{\geq 0}\}^m$ where $Z_{\geq 0}$ denotes the set of non-negative integers, i.e.,

$$y = [y_1, \cdots, y_m]^T = Ax.$$

The goal is to design a testing matrix $A$ that has a small number of rows (tests), $m$, and can identify the defective items given the test results $y$.

III. PROPOSED ALGORITHM

A. Preliminaries

Consider an arbitrary $m \times n$ binary matrix $D$ such that $n > t$.

Definition 3. ($t$-separable matrix) A binary matrix $D \in \{0, 1\}^{m \times n}$ (for $n > t$) is $t$-separable over field $\mathbb{F}$ if the sum (over field $\mathbb{F}$) of any set of $t$ columns is distinct.

Example 1. Consider the following matrix.

$$D = \begin{bmatrix}
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix}$$

The matrix $D$ is 2-separable over real field $\mathbb{R}$, but it is not 2-separable over $\mathbb{F}_2$ since, for instance, the sum of the first and second columns over $\mathbb{F}_2$ is the same as the sum of the third and fourth columns over $\mathbb{F}_2$.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \oplus \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

By the definition, it can be easily seen that if a matrix $D$ (with $n$ columns) is $t$-separable over a field $\mathbb{F}$, then $D$ is also $i$-separable over $\mathbb{F}$ for any $1 \leq i < t < n$.

The test results vector, $y$, is sum of the columns in the testing matrix corresponding to the coordinates of the defective items. If we use a $t$-separable matrix over $\mathbb{R}$ as the testing matrix, the vector $y$ will be distinct for any set of $t$ defective items. Thus, a $t$-separable matrix over $\mathbb{R}$ can be used as the testing matrix for identifying $t$ defective items. The construction of a $t$-separable matrix for an arbitrary $t$ is an open problem. However, we can leverage the idea that the parity-check matrix of any binary $t$-error-correcting code is a $t$-separable matrix over $\mathbb{F}_2$. It should be noted that $t$-separability over $\mathbb{F}_2$ results in $t$-separability over $\mathbb{R}$. Hence, a possible choice for designing a $t$-separable matrix over $\mathbb{R}$ is utilizing the parity-check matrix of a binary $t$-error-correcting code.

In this work, we use binary BCH codes for this purpose. The key feature of the BCH codes which make them suitable for designing $t$-separable matrices is that it is possible to design binary BCH codes, capable of correcting any combination of $t$ or fewer errors.

Definition 4. [13] (Binary BCH codes) For any positive integers $m \geq 3$ and $t < 2^m - 1$, there exists a binary $t$-error-correcting BCH code with the following parameters:

$$\begin{cases}
 n = 2^m - 1 & \text{block length} \\
 n - k \leq mt & \text{number of parity-check digits} \\
 d_{\text{min}} \geq 2t + 1 & \text{minimum distance}
\end{cases}$$

The parity-check matrix of such a code is given by

$$H_t = \begin{bmatrix}
1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \\
1 & \alpha^3 & (\alpha^3)^2 & \cdots & (\alpha^3)^{n-1} \\
1 & \alpha^5 & (\alpha^5)^2 & \cdots & (\alpha^5)^{n-1} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 & (\alpha^{2t-1}) & (\alpha^{2t-1})^2 & \cdots & (\alpha^{2t-1})^{n-1}
\end{bmatrix},$$

where $\alpha$ is a primitive element in $GF(2^m)$.

Since each entry of $H_t$ is an element in $GF(2^m)$, it can be represented by an $m$-tuple over $GF(2)$. Thus, the number of rows in $H_t$ can be given by

$$R = tm = t \log(n + 1).$$

B. Encoding

The measurement matrix design for our scheme is based on an architectural philosophy that is proposed in [14] and [15]. The key idea is to design the measurement matrix based on a sparse left-and-right-regular bipartite graph and to apply a peeling-based iterative algorithm for recovering the defective items given the test results.

Let $G_{t,r}(N; M)$ be a bipartite graph where each of the $N$ left nodes is connected to $\ell$ right nodes uniformly at random, and each of the $M$ right nodes is connected to $r$ left nodes uniformly at random. Note that there are $N\ell$ edge connections from the left side and $Mr$ edge connections from the right side.

$$N\ell = Mr$$
Let \( T_G \in \{0, 1\}^{M \times N} \) be the adjacency matrix corresponding to a left-and-right-regular bipartite graph \( G_{\ell,r}(N, M) \), where each column in \( T_G \) corresponds to a left node and has exactly \( \ell \) ones, and each row corresponds to a right node and has exactly \( r \) ones. Let \( t_i \in \{0, 1\}^N \) denote the \( i \)-th row of \( T_G \) = \([t_1^T, t_2^T, \ldots, t_M^T]^T \). We assign \( s \) tests to each right node based on a signature matrix \( U \in \{0, 1\}^{s \times r} \times N \). This signature matrix is chosen as \( U = [1_{1 \times r}^T, H_i^T]^T \), where \( 1_{1 \times r} \) is an all-ones row of length \( r \), and \( H_i \in \{0, 1\}^{t \log(r+1) + r} \) is the parity-check matrix of a binary \( t\)-error-correcting BCH code. From (2), it can be easily seen that \( s = R + 1 = t \log(r + 1) + 1 \).

The measurement matrix is then given by \( A = [A_1^T, \ldots, A_M^T]^T \) where \( A_i \in \{0, 1\}^{s \times N} \) is a block matrix that defines the \( s \) tests at \( i \)-th right node. There are exactly \( r \) ones in each row \( t_i \) of \( T_G \); and the signature matrix \( U = [u_1, u_2, \ldots, u_r] \) has \( r \) columns. Note that \( u_i = [1, h_i]^T \) denotes the \( i \)-th column of \( U \), where \( h_i \) is the \( i \)-th column of \( H_i \). \( A_i \) is constructed by placing the \( r \) columns of \( U \) at the coordinates of the \( r \) ones of the row vector \( t_i \), and replacing zeros by all-zero columns, i.e.,

\[
A_i = [0, \ldots, 0, u_1, 0, \ldots, u_2, 0, \ldots, u_r]
\]  

where \( t_i = [0, \ldots, 0, 1, 0, \ldots, 1, 0, \ldots, 1] \).

The number of rows in the measurement matrix \( A \) that defines the total number of tests for the proposed scheme is \( m = M \times s \) where \( s = t \log(r + 1) + 1 \).

Example 2. Let the number of items be \( N = 14 \) and the number of right nodes be \( M = 4 \). And let \( G \) be a left-and-right-regular graph with the left degree \( \ell = 2 \) and the right degree \( r = 7 \). The signature matrix is constructed using the parity-check matrix of a 1-error-correcting BCH code, \( H_1 \in \{0, 1\}^{3 \times 7} \).

\[
H_1 = \begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 \ 0 & 1 & 0 & 1 & 1 & 0 & 1 \ 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix},
\]

where \( \alpha \in GF(2^5) \) is a root of the primitive polynomial \( \alpha^3 + \alpha + 1 = 0 \). The adjacency matrix \( T_G \) of a graph \( G_{2,7}(14, 4) \) and the signature matrix \( U = [1_{1 \times 7}, H_1^T]^T \) can be given by

\[
T_G = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix},
\]

\[
U = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.
\]

According to the construction process explained before, the testing matrix \( A \) is given by

\[
A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}.
\]

C. Decoding

Let the observation vector corresponding to the \( i \)-th right node be given by

\[
z_i = [z_{i,1}, z_{i,2}, \cdots, z_{i,s}]^T = A_i x, \quad i \in \{1, \cdots, M\}.
\]

Note that \( z_i = [y_{(i-1)\cdot 1}^T, \cdots, y_{i\cdot 1}^T]^T \).

Definition 5. (t-resolvable right node) A right node is called \( t \)-resolvable if \( t \) or fewer defective items are connected to it.

Lemma 1. Our algorithm can successfully detect and resolve all \( t \)-resolvable right nodes. In addition, for \( t \leq 4 \), the time complexity of resolving each \( t \)-resolvable right node is logarithmic in \( r \) (i.e., \( O(\log r) \)).

Proof: Divide \( z_i \) into two blocks, \( z_i = [z_i^{(1)}^T, z_i^{(2)}^T]^T \), where \( z_i^{(1)} = [z_{i,1}, \ldots, z_{i,s}]^T \) and \( z_i^{(2)} = [z_{i,2}, \cdots, z_{i,s}]^T \). We can rewrite (5) by placing \([1, h_i]^T \) at the coordinates of \( u_i \)'s in (4).

\[
\begin{bmatrix} z_i^{(1)} \\ z_i^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 1 \\ 0 & \cdots & 0 & h_1 & 0 & \cdots & h_2 & 0 & \cdots & h_r \end{bmatrix} x.
\]

The first block, \( z_i^{(1)} \), which is the first element of \( z_i \), shows the number of defective items connected to the \( i \)-th right node. Recall that the first row of the signature matrix is an all-ones vector. It means that there are \( r \) ones in the first row of every \( A_i \), \( i \in \{1, 2, \cdots, M\} \). Thus, all the \( r \) items connected to the \( i \)-th right node are included in the test corresponding to the first row of \( A_i \). The second block, \( z_i^{(2)} \), is equal to the sum of \( h_i \)'s corresponding to the defective items connected to the \( i \)-th right node. Since \( H_i \) is a \( t \)-separable matrix over \( \mathbb{R} \), given the number of defective items is \( j \leq t \), sum of \( j \) number of its columns over \( \mathbb{R} \) is distinct. The key idea behind the decoding technique is the existence of a one-to-one mapping between \( z_i^{(2)} \) and any set of \( j \) defective items.

For a general \( t \)-separable matrix \( H_i \), a pre-calculated table may be needed to locate the defective items connected to
the $i$-th right node. Given that the number of defective items connected to a right node is $j \leq t$, the required look up table is of size $\binom{t}{j} \approx r^j$, where $r$ is the number of columns in $H_i$. Thus, the time complexity of searching a look up table when resolving a $t$-resolvable right node is $O(r^t)$.

However, when $H_i$ is the parity check matrix of a $t$-error-correcting BCH code, decoding complexity is different. Although decoding can be accomplished in quasilinear time in general, for $t \leq 4$ the decoding complexity is only logarithmic in $r$ (i.e., $O(\log r)$).

The block vector $z_i^{(2)}$ under modulo 2 can be interpreted as the syndrome corresponding to an error pattern of Hamming weight $j \leq t$. The location of the $j$ errors ($j$ defective items) can be determined from $z_i^{(2)}$ under modulo 2 by first using a Berlekamp-Massey algorithm for finding the error locator polynomial. This step involves a time complexity of $O(t^2 \log r)$ (all computations are performed in a finite field of size $2^m = r + 1$). Once the error locator polynomial is determined, the roots of the error locator polynomial have to be found. A standard Chien search can be used to solve this step with complexity $O(t \log r)$; however, when $t \leq 4$, the Chien search can be avoided and the roots can be solved directly using the algorithm in [16] with a complexity that is only $O(t \log r)$. Therefore, for $t \leq 4$, the decoding complexity is only logarithmic in $r$ (i.e., $O(\log r)$). 

During each iteration, first, the decoding algorithm iterates through all the right node observation vectors $\{z_i\}^{N_i}_{i=1}$ and resolves all the $t$-resolvable right nodes. Then, given the identities of the recovered left nodes, the edges connected to these defective items are peeled off the graph, i.e., contributions of the recovered defective items will be removed from the unresolved right nodes so that new right nodes can be resolved. The decoding algorithm stops when there is no more $t$-resolvable right nodes.

**Example 3.** Consider the group testing problem in the Example 2. Let the number of defective items be $K = 3$ and let $x = [1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0]^T$, i.e., item 1, item 4, and item 10 are defective items. We show how the proposed scheme can identify the defective items. The result of the tests can be expressed as follows.

$$y = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = Ax = \begin{bmatrix} u_1 \\ u_5 \\ u_2 + u_5 \\ u_1 + u_2 \end{bmatrix}$$

Thus, the right-node observation vectors are as following.

- $z_1 = u_1 = [1, 0, 0, 1]^T$
- $z_2 = u_5 = [1, 1, 1, 0]^T$
- $z_3 = u_2 + u_5 = [2, 1, 2, 0]^T$
- $z_4 = u_1 + u_2 = [2, 0, 1, 1]^T$

Because the signature matrix is built using a 1-separable matrix, each right node can be resolved if it is connected to at most one defective item. Iteration 1: we first find the 1-resolvable right nodes. The first and second right nodes are 1-resolvable because $z_{1,1} = z_{2,1} = 1$. Using the BCH decoder algorithm, it can be found that the defective items connected to the first and second right nodes are item 1 and item 10, respectively. Now, we remove the contributions of the items 1 and 10 from the unresolved right nodes. The new observation vectors are as follows.

$$z_3 = u_2 = [1, 0, 1, 0]^T$$
$$z_4 = u_2 = [1, 0, 1, 0]^T$$

Iteration 2: it can be easily observed that the third and forth right nodes are 1-resolvable since $z_{3,1} = z_{4,1} = 1$. Using the BCH decoder algorithm, it will be revealed that the items 4 is the defective item connected to both right nodes 3 and 4. Since all the defective items are identified, the decoding algorithm terminates.

**IV. MAIN RESULTS**

In this section, we present our main results. Theorem 1 characterizes the required number of tests (for both the deterministic and randomized models) that guarantees the identification of all the defective items with probability approaching one as $N$ and $K$ grow unbounded. The proof of Theorem 1 is given in Sections IV-A and IV-B.

**Theorem 1.** For both the deterministic and randomized models of the defective items, the proposed scheme recovers all the defective items with probability approaching one (as $N$ and $K$ grow unbounded) with at most $m = c(t)K \left(t \log \left(\frac{KN}{c(t)}\right) + 1\right) + 1$ tests, where $c(t)$ is a constant that depends only on $t$. Table I shows the constant $c(t)$ for $t \in \{1, 2, \cdots, 8\}$.

| $t$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|-----|----|----|----|----|----|----|----|----|
| $c(t)$ | 1.222 | 0.597 | 0.388 | 0.294 | 0.239 | 0.202 | 0.176 | 0.156 |

Table I: The constant $c(t)$ and the optimal left degree $\ell^*$ for $t \in \{1, 2, \cdots, 8\}$.

**A. Analysis of the deterministic Model**

Let $N$ be the total number of items, out of which $K$ items are defective. As mentioned in Section III-C our scheme performs an iterative decoding procedure. In each iteration, the algorithm finds and resolves all the $t$-resolvable right nodes. At the end of each iteration, the decoder subtracts the contribution of the identified defective items from the unresolved right nodes. This process is repeated until there is no $t$-resolvable right nodes left in the graph. The fraction of the defective items remained unidentified at the end of the decoding algorithm can be analyzed using density evolution.

In the deterministic model, the exact number of the defective items is known and the values assigned to the defective and non-defective items are one and zero, respectively. Thus, the left-and-right-regular bipartite graph can be pruned. All the zero left nodes and their respective edges are removed from the graph. The new number of left nodes in the pruned
graph is $K$, but the left degree remains unchanged. On the other hand, the number of right nodes remains unchanged, but the resulting graph is not right-regular any longer. Let $\lambda \in \mathbb{R}_{>0}$ be the average right degree, i.e.,

$$\lambda = \frac{K\ell}{M}.$$  

Let $\rho_i(x) \triangleq \sum_{k=1}^{\min(K,r)} \rho_i x^{i-1}$ denote the right edge degree distribution, where $\rho_i$ is the probability that a randomly picked edge in the pruned graph is connected to a right node of degree $i$, and $\min(K,r)$ is the maximum degree of the right nodes. As shown in [15], in the limit $K,N \to \infty$, we have $\rho_i = e^{-\lambda} \frac{\lambda^{i-1}}{(i-1)!}$.

**Lemma 2.** Let $p_{ij}$ be the probability that a randomly chosen defective item is not recovered at iteration $j$ of the decoding algorithm; and let $q_j$ be the probability that a randomly picked right node is resolved at iteration $j$ of the decoding algorithm. The relation between $p_{ij}$ and $p_{j+1}$ is determined by the following density evolution equations:

$$q_j = \sum_{i=1}^{t} \rho_i + \sum_{i=t+1}^{\min(K,r)} \rho_i \left( \sum_{k=0}^{t-1} \binom{i-1}{k} p_j^k (1-p_j)^{i-k-1} \right),$$  

$$p_{j+1} = (1-q_j)^{\ell-1},$$  

where $t$ is the level of separability, and $\rho_i$ is the probability that a randomly picked edge in the pruned graph is connected to a right node of degree $i$.

**Proof:** As mentioned earlier, the pruned graph is left-regular and the degree of the left nodes is $\ell$ but the pruned graph is not right-regular any longer and the degree of the right nodes can be any integer in $\{0,1,\ldots,\min(K,r)\}$. A tree-like representation of the neighborhood of an edge between left node $v$ of degree $\ell$ and right node $c$ of degree $i$ is shown in Fig. 1. Left node $v$ sends a “not identified” message to right node $c$ at iteration $j+1$ if none of its other neighboring right nodes $\{c_i\}_{i=1}^{t-1}$ have been resolved at iteration $j$, i.e., $\bar{c}$. Right node $c$ of degree $i$ passes a “resolved” message to left node $v$ at iteration $j$ if 1) $i \leq t$, i.e., the number of defective items connected to this right node is $t$ or fewer; or 2) the number of the defective items connected to this right node is $t+1 \leq \min(K,r)$ but only $k \in \{0,1,\ldots,t-1\}$ defective items among them are unidentified, i.e., (7).

We can solve this problem numerically and attain the optimal $\ell^*$. Let $c(t) \triangleq \frac{\ell}{c(t)K}\beta$. The number of right nodes can be chosen as $M = c(t)K\beta$ for any $\beta > 1$ to guarantee that $M > c(t)K\beta$.

Substituting $M = c(t)K\beta$ in (3) results in $r = \frac{\ell N}{c(t)K\beta}$. Therefore, the total number of tests will become $m = M \times s = c(t)K\beta \left( t \log \left( \frac{\ell N}{c(t)K\beta} + 1 \right) + 1 \right)$.

**Lemma 3.** There exist some $\beta > 1$ such that

$$c(t)K \left( t \log \left( \frac{\ell N}{c(t)K} + 1 \right) + 1 \right) + 1 \geq c(t)K\beta \left( t \log \left( \frac{\ell N}{c(t)K\beta} + 1 \right) + 1 \right).$$  

We can solve this problem numerically and attain the optimal $\ell^*$. Let $c(t) \triangleq \frac{\ell}{c(t)K}\beta$. The number of right nodes can be chosen as $M = c(t)K\beta$ for any $\beta > 1$ to guarantee that $M > c(t)K\beta$.

Substituting $M = c(t)K\beta$ in (3) results in $r = \frac{\ell N}{c(t)K\beta}$. Therefore, the total number of tests will become $m = M \times s = c(t)K\beta \left( t \log \left( \frac{\ell N}{c(t)K\beta} + 1 \right) + 1 \right)$.
Proof: Let \( f(\beta) \equiv c(t)K \left( t \log \left( \frac{\ell N}{t(1-\beta)} + 1 \right) + 1 \right) \). We need to show that there exists some \( \beta > 1 \) such that \( f(1) + 1 \geq \beta f(\beta) \), or equivalently, \( \beta f(\beta) - f(1) \leq 1 \). Since \( f(\beta) \) is a monotone decreasing function of \( \beta \), \( f(\beta) < f(1) \) for \( \beta > 1 \). This inequality leads to \( \beta f(\beta) - f(1) < (\beta - 1)f(1) \). Hence, to guarantee that there exist some \( \beta > 1 \) such that \( \beta f(\beta) - f(1) \leq 1 \), it suffices to show that \( (\beta - 1)f(1) \leq 1 \) for some \( \beta > 1 \). It is easy to see that \( 1 < \beta \leq \frac{1}{f(1)} + 1 \) is the satisfactory range.

Here, we will give the complete analysis for the case of \( t = 1 \), and show how one can compute the constant \( c(t) \). The same procedure can be followed for any \( t > 1 \).

Consider the case \( t = 1 \). For this case, the density evolution equations (7) and (8) can be combined as \( p_{j+1} = (1 - \sum_{i=1}^{\min(K,r)} \rho_i (1 - p_j)^{i-1})^{\ell-1} \). Substituting \( \rho_i = e^{-\lambda \frac{i-1}{\ell}} \), we can rewrite this equation as \( p_{j+1} = \left( 1 - e^{-\lambda} \sum_{i=1}^{\min(K,r)} \frac{i-1}{\ell} (1 - p_j)^{i-1} \right)^{\ell-1} \). Letting \( \min(K,r) \to \infty \), we get
\[
p_{j+1} = \left( 1 - e^{-\lambda p_j} \right)^{\ell-1}.
\]

**Lemma 4.** Let \( p_{j+1} = (1 - e^{-\lambda p_j})^{\ell-1} \) and \( p_1 = 1 \). For any \( \ell \geq 2 \) and any real number \( \lambda > 0 \), the infinite sequence \( \{p_1, p_2, \ldots\} \) converges.

**Proof:** Note that every bounded and monotonic sequence converges. From the definition, it is obvious that \( 0 \leq p_j \leq 1 \) for any integer \( \ell \geq 2 \) and any real number \( \lambda > 0 \). Thus, it suffices to show the monotonicity of the sequence \( \{p_1, p_2, \ldots\} \). The proof is based on induction. It is easy to see that \( p_2 < p_1 \), i.e., \( (1 - e^{-\lambda})^{\ell-1} < 1 \). The induction hypothesis is that \( p_i < p_{i-1} \), i.e., \( (1 - e^{-\lambda p_{i-1}})^{\ell-1} < p_{i-1} \). We need to show that \( p_{j+1} < p_j \), i.e.,
\[
(1 - e^{-\lambda p_j})^{\ell-1} < p_j.
\]

We start from the induction hypothesis, \( (1 - e^{-\lambda p_{j-1}})^{\ell-1} < p_{j-1} \). It is easy to see that \( 1 - e^{-\lambda (1 - e^{-\lambda p_{j-1}})^{\ell-1}} < 1 - e^{-\lambda p_{j-1}} \), or equivalently,
\[
(1 - e^{-\lambda (1 - e^{-\lambda p_{j-1}})^{\ell-1}})^{\ell-1} < (1 - e^{-\lambda p_{j-1}})^{\ell-1}.
\]
Replacing \( (1 - e^{-\lambda p_{j-1}})^{\ell-1} \) by \( p_j \), we can rewrite (12) as \( (1 - e^{-\lambda p_j})^{\ell-1} < p_j \), and hence (11) holds.

**Lemma 5.** Let \( p_{j+1} = (1 - e^{-\lambda p_j})^{\ell-1} \) and \( p_1 = 1 \). Let \( p_\infty \) be the limit of the sequence \( \{p_1, p_2, \ldots\} \). Also, let
\[
\lambda_T \equiv \inf_{0 < x < 1} \left( \ln(1 - x^{\frac{1}{\ell}}) \right).
\]
Then, for any \( \ell \geq 2 \), we have
\[
\begin{cases}
p_\infty = 0, & 0 < \lambda < \lambda_T, \\
p_\infty > 0, & \lambda \geq \lambda_T.
\end{cases}
\]

**Proof:** By Lemma 4, \( p_\infty \) exists, and \( p_\infty \) must be a solution to the following equation.
\[
p_\infty = (1 - e^{-\lambda p_\infty})^{\ell-1} \quad (13)
\]
We first show that if \( 0 < \lambda < \lambda_T \), then \( p_\infty = 0 \). It suffices to show that for \( 0 < \lambda < \lambda_T \) and any integer \( \ell \geq 2 \), the only solution of (13) is \( p_\infty = 0 \). Obviously, \( p_\infty = 0 \) is a solution of (13) for any \( 0 < \lambda < \lambda_T \) and any integer \( \ell \geq 2 \). Thus, we need to show that for \( 0 < \lambda < \lambda_T \) and any integer \( \ell \geq 2 \), and any \( 0 < \epsilon < 1 \), we have \( \epsilon = (1 - e^{-\lambda_T})^{\ell-1} \).

The proof is by the way of contradiction. Suppose that \( \epsilon = (1 - e^{-\lambda_T})^{\ell-1} \) for some \( 0 < \epsilon < 1 \). This equation can be rewritten as \( \lambda = \frac{\ln(1 - \epsilon^{\ell-1})}{\ln(1 - \epsilon)} \). On the other hand, we know that \( \lambda < \lambda_T = \inf_{0 < x < 1} \left( \frac{\ln(1 - x^{\frac{1}{\ell}})}{-x} \right) \). Thus, we have \( \frac{\ln(1 - \epsilon^{\ell-1})}{\ln(1 - \epsilon)} < \inf_{0 < x < 1} \left( \frac{\ln(1 - x^{\frac{1}{\ell}})}{-x} \right) \) for some \( 0 < \epsilon < 1 \). Obviously, this inequality does not hold and we reach to a contradiction.

Now, we want to show that for \( \lambda > \lambda_T \), we have \( p_\infty > 0 \). It follows from (13) that \( \lambda = \frac{\ln(1 - \epsilon^{\ell-1})}{\ln(1 - \epsilon)} \). Hence, \( \lambda > \lambda_T \) implies that \( \frac{\ln(1 - \epsilon^{\ell-1})}{\ln(1 - \epsilon)} < \inf_{0 < x < 1} \left( \frac{\ln(1 - x^{\frac{1}{\ell}})}{-x} \right) \). Again, the proof is by the way of contradiction. Suppose that \( p_\infty = 0 \), i.e., the sequence \( \{p_1, p_2, \ldots\} \) converges to 0. Therefore, for any \( \delta > 0 \), there exist an \( \epsilon \in \mathbb{N} \) such that for any \( j \geq i \), \( |p_\infty - p_j| = p_j < \delta \). For an arbitrary \( 0 < \delta < 1 \), let \( \epsilon \) be such that \( p_\infty - p_{i-1} \geq \delta \) and \( p_j < \delta \) for \( j \geq i \). From (10), it is clear that \( p_i < \delta \) implies that \( (1 - e^{-\lambda p_{i-1}})^{\ell-1} < \delta \).

This inequality can be rewritten as \( \lambda < \frac{\ln(1 - \epsilon^{\ell-1})}{\ln(1 - \epsilon)} \). Using the facts that \( \lambda > \lambda_T \) and \( p_\infty - p_{i-1} \geq \delta \), we can have the following inequalities, respectively:
\[
\inf_{0 < x < 1} \left( \frac{\ln(1 - x^{\frac{1}{\ell}})}{-x} \right) < \frac{\ln(1 - \epsilon^{\ell-1})}{\ln(1 - \epsilon)} < \frac{\ln(1 - \epsilon^{\ell-1})}{-p_{i-1}}.
\]
\[
\frac{\ln(1 - \delta^{\frac{1}{\ell}})}{-p_{i-1}} \leq \frac{\ln(1 - \delta^{\frac{1}{\ell}})}{-\delta}.
\]
These inequalities lead to
\[
\inf_{0 < x < 1} \left( \frac{\ln(1 - x^{\frac{1}{\ell}})}{-x} \right) < \frac{\ln(1 - \epsilon^{\ell-1})}{-p_{i-1}} < \frac{\ln(1 - \epsilon^{\ell-1})}{-\delta}.
\]
Let \( f(x) \equiv \frac{\ln(1 - x^{\frac{1}{\ell}})}{-x} \) and let \( x_{\inf} \) be such that \( \inf_{0 < x < 1} f(x) = \frac{\ln(1 - x_{\inf}^{\ell})}{-x_{\inf}} \). Because \( \lim_{x \to 0^+} f(x) = \lim_{x \to 1^-} f(x) = +\infty \), it is obvious that \( 0 < x_{\inf} < 1 \). Take \( \delta = x_{\inf} \) for an arbitrary \( x_{\inf} \). Then, we will have
\[
\inf_{0 < x < 1} \left( \frac{\ln(1 - x^{\frac{1}{\ell}})}{-x} \right) = \ln(1 - \delta^{\frac{1}{\ell}}) \quad (17)
\]
From (16) and (17), we arrive at a contradiction since there exist a \( 0 < \delta < 1 \) such that (16) does not hold.

To solve the optimization problem in (9), for any integer \( \ell \geq 2 \) the corresponding \( \lambda_T \) must be computed. Then, the
optimal $\ell^*$ and $c(t) = \frac{\ell^*}{\lambda_T(\ell)}$ will be attained easily. The function $\frac{\ell}{\lambda_T(\ell)}$ for different values of $\ell$ when $t = 1$ is depicted in Fig. 3. As it can be seen from Fig. 3 for the case $t = 1$, the optimal left degree is $\ell^* = 3$.

B. Analysis of the randomized model

Let the number of items be $N$. Suppose that each item is defective with probability $\gamma = \frac{\ell}{N}$. For this model, unlike the deterministic model, the exact number of defective items is not known. Thus, it is not possible to prune the bipartite graph.

Lemma 6. Let $p_j$ be the probability that a randomly chosen defective item is not recovered at iteration $j$ of the decoding algorithm; and let $q_j$ be the probability that a randomly picked right node is not resolved at iteration $j$ of the decoding algorithm. The relation between $p_j$ and $p_{j+1}$ is then given by the following density evolution equations:

$$q_j = \sum_{i=0}^{\ell-1} \binom{r-1}{i} p_j^i (1-p_j)^{r-i-1}, \quad (18)$$

$$p_{j+1} = \gamma (1-q_j)^{\ell-1}, \quad (19)$$

where $t$, $\ell$, and $r$ are the level of separability for the signature matrix, the degree of left nodes, and the degree of right nodes, respectively.

Proof: Since the graph is not pruned, the graph remains left-and-right-regular with the left degree $\ell$ and right $r$. A tree-like representation of the neighborhood of an edge between left node $v$ and right node $c$ is depicted in Fig. 4. At the beginning of the decoding algorithm, each item can be defective with the probability $\gamma$. Left node $v$ sends a "not identified" message to right node $c$ at iteration $j+1$ if none of its other neighboring right nodes $\{c_i\}_{i=1}^{\ell-1} \setminus c$ have been resolved at iteration $i$, i.e. $(19)$. Right node $c$ passes a "resolved" message to left node $v$ at iteration $j$ if among the $r-1$ items connected to it only $i \in \{0, 1, \ldots, t-1\}$ items are unidentified, i.e., $(18)$. \hfill \square

Combining $(18)$ and $(19)$, we will have

$$p_{j+1} = \gamma \left(1 - \sum_{i=0}^{\ell-1} \binom{r-1}{i} p_j^i (1-p_j)^{r-i-1}\right)^{\ell-1}.$$  

Letting $r \to \infty$, and using the Poisson approximation, one can see that

$$p_{j+1} = \left(1 - \frac{1}{\ell} \sum_{i=0}^{\ell-1} \binom{r}{i} e^{-r} \frac{p_j^i}{i!}\right)^{\ell-1}. \quad (20)$$

Let $\phi_j \triangleq \frac{p_j}{\gamma}$ and $\psi \triangleq r\gamma$. We can rewrite $(20)$ as follows:

$$\phi_{j+1} = \left(1 - \frac{1}{\ell} \sum_{i=0}^{\ell-1} \phi_j^i e^{-\psi} \frac{1}{i!}\right)^{\ell-1}.$$  

We refer to this equation as normalized density evolution. Again, recall that the goal is to minimize the total number of tests $m = M \times s$, where $M$ is the number of right nodes and $s$ is the number of rows in signature matrix. Substituting $\gamma = \frac{\ell}{N}$ in $3$, the number of right nodes is given by $M = \frac{1}{\psi} K$. Then, using $\psi = r\gamma$, we can rewrite the number of right nodes as $M = \frac{\ell}{\psi} K$. For a given $t$, we can minimize the number of right nodes, $M = \frac{\ell}{\psi} K$, considering the constraint $\lim_{j \to \infty} \phi_j(\ell, \psi) = 0$, in order to minimize the total number of the tests. For any integer $\ell \geq 2$, let us define $\psi_T(\ell) = \sup\{\psi : \lim_{j \to \infty} \phi_j(\ell, \psi) = 0\}$. Then, for any integer $\ell \geq 2$ and $\psi < \psi_T(\ell)$, we will have $\lim_{j \to \infty} \phi_j(\ell, \psi) = 0$. Accordingly, for any integer $\ell \geq 2$ and any $M = \frac{\ell}{\psi} K > \frac{\ell}{\psi_T(\ell)} K$, it follows that $\lim_{j \to \infty} \phi_j = 0$. Our goal is then to compute

$$\min_{\ell \in \{2, 3, \ldots\}} \frac{\ell}{\psi_T(\ell)} K. \quad (21)$$

We can solve this problem numerically and attain the optimal $\ell^*$. Let $c(t) = \frac{\ell^*}{\psi_T(\ell^*)}$. The rest of the analysis follows from the exact same steps as in the analysis of the deterministic model.

Consider the case of $t = 1$. For this case, the normalized density evolution equation can be written as $\phi_{j+1} = (1 - e^{-\psi \phi_j})^{\ell-1}$. One can readily see the similarity between this equation and the equation $(10)$ (for the case
such a resemblance can be seen for all values of $t$. This implies that for any given $t$, the constant $c(t)$ will have the same value for both the deterministic and randomized models.

V. COMPARISON RESULTS

In this section we will evaluate the performance of the proposed algorithm based on our theoretical analysis and the Monte Carlo simulations.

Based on the results of the analysis provided in Theorem 1 and Table 1, Fig. 5 depicts the total number of tests ($m$) required to identify all the defective items for different values of $t$. The total number of items is considered to be $N = 2^{16}$. As it can be seen, when $t \in \{1, 2, 3\}$ the required number of tests for identifying all the defective items is less than that for larger values of $t$.

Using the Monte Carlo simulation, we also compare the performance of the proposed scheme for $t \in \{1, 2, 3\}$ with the performance of the Multi-Level Group Testing algorithm from [12], referred to as MLGT. The MLGT scheme, in general, is a semi-quantitative scheme where the result of each test is an integer in the set $\{0, 1, 2, \cdots, L\}$. Letting $L = \infty$, the MLGT scheme becomes a fully quantitative group testing scheme. Based on the optimization that we have performed, the optimal left degree for the MLGT scheme is $\ell^* = 3$ when $L \to \infty$. The average fraction of unidentified defective items for $K = 100$ defective items among a population of $N = 2^{16}$ items is shown in Fig. 6. As it can be observed, the proposed scheme for all three tested values of $t$ outperforms the MLGT scheme significantly. For instance, when the fraction of unidentified defective items is $2 \times 10^{-4}$, the required number of tests for the MLGT scheme ($\ell = 3$) is 3 times, 5 times, and 7 times more than that of the proposed scheme for $t = 1$, $t = 2$, and $t = 3$, respectively.

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