COVARIANT FUZZY OBSERVABLES AND COARSE-GRAINING

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Abstract. A fuzzy observable is regarded as a smearing of a sharp observable, and the structure of covariant fuzzy observables is studied. It is shown that the covariant coarse-grainings of sharp observables are exactly the covariant fuzzy observables. A necessary and sufficient condition for a covariant fuzzy observable to be informationally equivalent to the corresponding sharp observable is given.

1. Introduction

The discovery of the proper mathematical formulation of the notion of a physical quantity (observable) in quantum mechanics as a normalized positive operator measure instead of the more traditional spectral measure (normalized projection measure) has provided, among many other advantages, a quantitative frame to investigate various formulations of the idea of an imprecise measurement of a physical quantity. In this paper we study the structure of the so-called fuzzy observables, and, especially, covariant fuzzy observables.

The notion of a fuzzy observable will be formulated as a smearing of a sharp observable (projection measure). The definition of a fuzzy observable and some of its interpretations and motivations are given in Section 2. In Section 3 we recall some results concerning covariant observables. In Section 4 we show that another related notion, the so-called coarse-graining, agrees with the notion of a fuzzy observable in the covariant case. The structure of covariant fuzzy observables is described. The norm-1-property of an observable (the possibility that with a suitable preparation its measurement outcome probabilities can be made arbitrarily close to one) as well as the regularity (the Boolean structure of its range) are operationally important properties. In Section 5 we study consequences of these properties for fuzzy observables. Since fuzzy observables are coarse-grainings of sharp observables, the question arises under what conditions the state distinction power of a fuzzy observable is the same as that of the corresponding sharp observable. This will be answered in Section 5.3. In the final Section 6
we shall illustrate some of the results of the paper in terms of familiar examples.

2. Fuzzy observables: definition and motivations

Let \( H \) be a complex separable Hilbert space and \( \mathcal{L}( \mathcal{H} ) \) the set of bounded operators on \( \mathcal{H} \). Let \( \Omega \) be a nonempty set and \( \mathcal{F} \) a \( \sigma \)-algebra of subsets of \( \Omega \). We say that a set function \( E : \mathcal{F} \rightarrow \mathcal{L}( \mathcal{H} ) \) is an operator measure, if it is \( \sigma \)-additive with respect to the strong (or \( \sigma \) equivalently, weak) operator topology. The operator measure \( E \) is positive if \( E(X) \geq 0 \) for all \( X \in \mathcal{F} \), and normalized if \( E(\Omega) = I \). If the normalized operator measure \( E \) is projection valued, that is, \( E(X)^2 = E(X) \) for all \( X \in \mathcal{F} \), it is a spectral measure. We call a positive normalized operator measure \( E : \mathcal{F} \rightarrow \mathcal{L}( \mathcal{H} ) \) an observable and a spectral measure a sharp observable. For an observable \( E : \mathcal{F} \rightarrow \mathcal{L}( \mathcal{H} ) \) and a unit vector \( \psi \in \mathcal{H} \) we let \( p^E_\psi \) denote the probability measure on \( \Omega \) defined by

\[
p^E_\psi(X) = \langle \psi | E(X) \psi \rangle, \quad X \in \mathcal{F}.
\]

To distinguish sharp observables from general observables, we reserve the letter \( P \) for a spectral measure. If \( \Omega \) is a topological space, we denote by \( \mathcal{B}( \Omega ) \) the Borel \( \sigma \)-algebra of \( \Omega \). For a selfadjoint operator \( A \) in \( \mathcal{H} \) we let \( P^A : \mathcal{B}( \mathbb{R} ) \rightarrow \mathcal{L}( \mathcal{H} ) \) denote its spectral measure. In such a situation we may also refer to \( A \) as the sharp observable.

**Definition 1.** Let \( P : \mathcal{F} \rightarrow \mathcal{L}( \mathcal{H} ) \) be a sharp observable and consider a mapping \( \nu : \Omega \times \mathcal{F} \rightarrow [0, 1] \), denoted also by \((\omega, X) \mapsto \nu_\omega(X)\), with the following properties:

(i) for every \( \omega \in \Omega \), \( \nu_\omega \) is a probability measure;
(ii) for every \( X \in \mathcal{F} \), the mapping \( \omega \mapsto \nu_\omega(X) \) is measurable.

The integral

\[
E(X) = \int_{\Omega} \nu_\omega(X) \, dP(\omega), \quad X \in \mathcal{F}
\]

defines by the dominated convergence theorem a positive normalized operator measure. We call it a fuzzy observable with respect to \( P \), and we say that the mapping \( \nu \) is a confidence measure.

There are at least two interpretations for the operator measure \( E \) defined in Equation (1), depending on whether one wants to think of fuzziness as belonging to points or to events. We take a brief look at these two viewpoints.

Since realistic measurements always have some unsharpness or imprecision, one may think that the points of \( \Omega \) are to be replaced by
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We can write the sharp observable $P$ in the form

$$P(X) = \int_{\Omega} \delta_\omega(X) \, dP(\omega),$$

(2)

where $\delta_\omega$ is the Dirac measure at a point $\omega \in \Omega$. Comparing Equations (1) and (2), we notice that the fuzzy observable $E$ is formed by replacing at every point $\omega \in \Omega$ the point measure $\delta_\omega$ by the probability measure $\nu_\omega$. Thus unsharpness related to a point $\omega$ is quantified by a probability measure $\nu_\omega$. This is the viewpoint e.g. in [2], [3] and [19].

Another interpretation arises from Zadeh’s notion of a fuzzy set and a fuzzy event, [21], [22]. We recall that a fuzzy set in $\Omega$ is a mapping $\tilde{X}: \Omega \to [0, 1]$, and its values $\tilde{X}(\omega)$ are called the membership degrees of the points $\omega$. Ordinary sets are identified with their characteristic functions. Let $\mu$ be a probability measure on $\mathcal{F}$. Following Zadeh, a fuzzy set is called a fuzzy event if the corresponding function is measurable. The probability $\mu(\tilde{X})$ of a fuzzy event $\tilde{X}$ is defined as the integral $\int_{\Omega} \tilde{X}(\omega) \, d\mu(\omega)$. Following this way of thought, it is natural to define $P(\tilde{X})$ as the operator

$$P(\tilde{X}) = \int_{\Omega} \tilde{X}(\omega) \, dP(\omega).$$

Let now $\nu$ be a confidence measure. For a fixed set $X \in \mathcal{F}$ the mapping $\omega \mapsto \nu_\omega(X)$ is a fuzzy event. Thus $\nu$ can be seen as a function which maps measurable sets to fuzzy events. If $\tilde{X}$ is the fuzzy event $\omega \mapsto \nu_\omega(X)$, Equation (1) can be written as

$$E(X) = P(\tilde{X}).$$

We may thus conclude that in any vector state $\psi \in \mathcal{H}$, $\|\psi\| = 1$, the fuzzy observable $E$ gives for an event $X$ the same probability as the sharp observable $P$ gives for a fuzzy event $\tilde{X}$.

To expand on the physical motivation of the present study, we recall a typical situation of measurement theory, described by the so-called standard measurement model [4], [5]. We denote by $\mathcal{H}$ the Hilbert space of the system to be measured and by $\mathcal{K}$ the Hilbert space of the measurement apparatus. The aim is to measure a sharp observable $A$ of the object system (a selfadjoint operator on $\mathcal{H}$) using a measurement interaction of the form

$$U = e^{i\lambda A \otimes B},$$

where $B$ is a sharp observable of the apparatus (a selfadjoint operator on $\mathcal{K}$) and $\lambda \in \mathbb{R}$ a coupling constant. Let $\phi \in \mathcal{K}$ be an initial vector state of the apparatus and denote $\phi_\lambda = e^{i\lambda B} \phi$ for all $a \in \mathbb{R}$. Let $Z$ be
the sharp pointer observable (selfadjoint operator on $\mathcal{K}$) being applied. The probability reproducibility condition

$$\langle \varphi | E(X) | \varphi \rangle = \langle U(\varphi \otimes \phi) | I \otimes P^Z(X) U(\varphi \otimes \phi) \rangle,$$

required to hold for all vector states $\varphi \in \mathcal{H}$ and $X \in B(\mathbb{R})$, defines the actually measured observable $E : B(\mathbb{R}) \to L(\mathcal{H})$ of the object system, and a direct computation shows that

$$E(X) = \int_{\mathbb{R}} \langle \phi_{\lambda a} | P^Z(X) \phi_{\lambda a} \rangle \, dP^A(a).$$

For a fixed $a$, the mapping $X \mapsto \langle \phi_{\lambda a} | P^Z(X) \phi_{\lambda a} \rangle$ is a probability measure and for a fixed $X$, the mapping $a \mapsto \langle \phi_{\lambda a} | P^Z(X) \phi_{\lambda a} \rangle$ is continuous. Thus, the mapping $(a, X) \mapsto \langle \phi_{\lambda a} | P^Z(X) \phi_{\lambda a} \rangle$ is a confidence measure. One concludes that the standard measurement model $\langle K, \phi, U, Z \rangle$ for a sharp observable $A$ determines a fuzzy observable with respect to $A$.

### 3. Covariant fuzzy observables

The behavior of an observable under appropriate transformations, like space-time transformations, determines to a large extent the structure of an observable. In such a relativistic approach, developed systematically by George W. Mackey, one takes covariance as a defining property of an observable. In many relevant cases (e.g. position, momentum, spin), although not in some (e.g. phase, time), one has a unique covariant sharp observable, which is also called the corresponding canonical observable. Our investigation in the rest of this paper will concentrate on those covariant observables which are fuzzy observables with respect to a canonical observable. In this section we introduce further notation and recall some results, which, if not otherwise stated, can be found e.g. in [12] and [13].

Let $G$ be a locally compact topological group, which is Hausdorff and satisfies the second axiom of countability. Then $G$ is also metrizable and there exists a left invariant metric which induces its topology (see e.g. [18]). We fix a closed normal (proper) subgroup $H$ of $G$, and from now on $\Omega$ denotes the quotient group $G/H$. The groups $H$ and $\Omega$ are locally compact, Hausdorff and second countable topological spaces in a natural way. Moreover, the canonical projection

$$\pi : G \to \Omega, \ g \mapsto gH$$

is a continuous and open homomorphism. We fix (arbitrary) left invariant Haar measures of $G, H$ and $\Omega$ and denote them by $\mu_G, \mu_H$, and $\mu_\Omega$. Let $d_G, d_H$, and $d_\Omega$ denote left invariant metrics. We denote by
(g, ω) ↦ g · ω the action of G on Ω. If g ∈ G, ω ∈ Ω, X ⊂ Ω, we write

\[ g \cdot X = \{ g \cdot x | x \in X \} \]

\[ \omega X = \{ \omega x | x \in X \} \].

**Definition 2.** Let U be a strongly continuous unitary representation of G in a Hilbert space \( \mathcal{H} \). We say that the observable \( E : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H}) \) is \( G \)-covariant with respect to \( U \) if

\[ U(g)E(X)U(g)^* = E(g \cdot X) \quad (5) \]

for all \( g \in G, X \in \mathcal{B}(\Omega) \).

In the sequel we will need the following result concerning the structure of \( G \)-covariant observables. Let \( E \) be a \( G \)-covariant observable. Then for any \( X \in \mathcal{B}(\Omega) \),

\[ E(X) = O \text{ if and only if } \mu_{\Omega}(X) = 0. \quad (6) \]

For a proof see \cite{14}, Prop. 8.

We will assume that there exists a \( G \)-covariant sharp observable \( P : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H}) \) with respect to \( U \). By the Imprimitivity Theorem of Mackey \cite{17} this is the case if and only if \( U \) is equivalent to a representation induced from some representation of the subgroup \( H \). Moreover, the pair \((U, P)\) is equivalent to the canonical system of imprimitivity. This means that there is a separable Hilbert space \( \mathcal{K} \) such that we may (and always will) assume \( \mathcal{H} \) to be the space \( L^2(\Omega, \mu_{\Omega}, \mathcal{K}) \) of \( \mathcal{K} \)-valued weakly measurable square-integrable functions and \( P(X)\psi = \chi_X \psi \) for all \( X \in \mathcal{B}(\Omega) \) and \( \psi \in \mathcal{H} \) (for a convenient summary see e.g. \cite{11}). Throughout this article \((U, P)\) is a fixed canonical system of imprimitivity, if not otherwise stated. If \( E \) is a \( G \)-covariant observable and fuzzy with respect to \( P \), we call \( E \) a covariant fuzzy observable, for short.

**Lemma 1.** Let \( \rho : \mathcal{B}(\Omega) \to [0, 1] \) be a probability measure. The mapping \( \tilde{\rho} : \Omega \times \mathcal{B}(\Omega) \to [0, 1] \) defined by \( \tilde{\rho}(\omega, X) = \rho(\omega^{-1}X) \) is a confidence measure.

**Proof.** The mapping \( \tilde{\rho} \) clearly satisfies the condition (i) in Definition \( \| \) so we concentrate on the condition (ii). For any bounded Borel function \( f : \Omega \to \mathbb{C} \), let us define the function \( F_f : \Omega \to \mathbb{C} \) by

\[ F_f(\omega) = \int f(\omega \eta) \, d\rho(\eta). \]

In this notation \( \rho(\omega^{-1}X) = F_{xX}(\omega) \) so we have to show that the collection \( \mathcal{M} = \{ X \in \mathcal{B}(\Omega) | F_{xX} \text{ is measurable} \} \) coincides with \( \mathcal{B}(\Omega) \).

First, if \( f \) belongs to the set \( C_c(\Omega) \) of continuous functions with compact support, then \( f \) is left uniformly continuous and \( F_f \) is continuous. In particular, \( F_f \) is measurable.
Let us then consider the case where \( f = \chi_A \) for some open set \( A \subset \Omega \). As \( \Omega \) is locally compact and second countable, the set \( A \) is \( \sigma \)-compact. Therefore, we may choose an increasing sequence \( (K_n) \) of compact sets such that \( K_n \subset A \) and \( \bigcup_n K_n = A \). Using Urysohn’s Lemma we find for each \( n \) a function \( f_n \in C_c(\Omega) \) such that \( 0 \leq f_n(\omega) \leq 1 \) for all \( \omega \in \Omega \), \( f_n(\omega) = 1 \) for all \( \omega \in K_n \) and \( f_n(\omega) = 0 \) for all \( \omega \notin A \). The function \( \chi_A \) is the pointwise limit of the sequence \( (f_n) \). By the dominated convergence theorem

\[
F_{\chi_A}(\omega) = \int \chi_A(\omega \eta) \, d\rho(\eta) = \int \lim_{n \to \infty} f_n(\omega \eta) \, d\rho(\eta) = \lim_{n \to \infty} F_{f_n}(\omega),
\]

and so \( F_{\chi_A} \) is measurable. Thus any open set belongs to \( \mathcal{M} \).

Denote by \( \mathcal{A} \) the collection of those sets \( X \subset \Omega \) for which \( \chi_X \) is of the form \( \sum_{i=1}^n \epsilon_i \chi_{A_i} \) for some open sets \( A_i \) and numbers \( \epsilon_i = \pm 1 \). Clearly, \( \mathcal{A} \) is closed under complements and intersections and hence, it is an algebra. As \( F_{\sum_{i=1}^n \epsilon_i \chi_{A_i}} = \sum_{i=1}^n \epsilon_i F_{\chi_{A_i}} \) and each \( F_{\chi_{A_i}} \) is measurable, \( \mathcal{A} \subset \mathcal{M} \).

An argument analogous to (7) shows that \( \mathcal{M} \) is a monotone class. By the monotone class lemma [12, p. 66] the monotone class \( \mathcal{M}_0 \) generated by \( \mathcal{A} \) coincides with the \( \sigma \)-algebra \( \mathcal{B}_0 \) generated by \( \mathcal{A} \). Since \( \mathcal{B}_0 = \mathcal{B}(\Omega) \) and \( \mathcal{M}_0 \subset \mathcal{M} \subset \mathcal{B}(\Omega) \) we have \( \mathcal{M} = \mathcal{B}(\Omega) \).

**Proposition 1.** Let \( \rho : \mathcal{B}(\Omega) \to [0,1] \) be a probability measure. Then the observable \( E^\rho \) defined by

\[
E^\rho(X) = \int_\Omega \rho(\omega^{-1}X) \, dP(\omega), \quad X \in \mathcal{B}(\Omega),
\]

is a covariant fuzzy observable. Moreover, for any unit vector \( \psi \in \mathcal{H} \) the probability measure \( p^E_{\psi} \) takes the form

\[
p^E_{\psi} = p^P_{\psi} \ast \rho,
\]

where \( p^P_{\psi} \ast \rho \) is the convolution of the measures \( p^P_{\psi} \) and \( \rho \).

**Proof.** It follows directly from Lemma \[\square\] that \( E^\rho \) is a fuzzy observable with respect to \( P \). Let \( g \in G, X \in \mathcal{B}(\Omega) \), and \( \psi \in \mathcal{H} \). Since \( P \) is
G-covariant, we have

\[ \langle \psi | U(g) E_\rho(X) U(g)^* \psi \rangle = \int_\Omega \rho(\omega^{-1} X) \, dp_\psi^E(\omega) = \int_\Omega \rho((g^{-1} \cdot \omega)^{-1} X) \, dp_\psi^E(\omega) = \int_\Omega \rho(\omega^{-1} (g \cdot X)) \, dp_\psi^E(\omega) = \langle \psi | E_\rho(g \cdot X) \psi \rangle. \]

Thus \( E \) is \( G \)-covariant.

The last claim follows from the fact that for any \( X \in \mathcal{B}(\Omega) \),

\[ p_\psi^E(X) = \int_\Omega \rho(\omega^{-1} X) \, dp_\psi^E(\omega) = \int_\Omega \int_\Omega \chi_X(\omega \eta) \, d\rho(\eta) \, dp_\psi^E(\omega). \]

The Dirac measure \( \delta_\omega \) at a point \( \omega \in \Omega \) yields a fuzzy observable \( E_{\delta_\omega} \), which is the translated spectral measure

\[ X \mapsto P(X \omega^{-1}), \]

also denoted by \( P_\omega \). This differ from the canonical spectral measure \( P \) only by 'the choice of the origin'. Equation (10) can be written in the form

\[ E_\rho(X) = \int_\Omega P(X \omega^{-1}) \, d\rho(\omega) = \int_\Omega P_\omega(X) \, d\rho(\omega). \]

Thus the operators \( E_\rho(X) \) are weighted means of the translated projections \( P_\omega(X), \omega \in \Omega \).

**Example 1.** Let \( \rho \) be a probability measure \( \rho \) with a finite support \( \text{supp} \ (\rho) = \{ \omega_1, \omega_2, \ldots, \omega_n \} \) and \( \rho(\{\omega_i\}) = \lambda_i, i = 1, \ldots, n \). Equation (10) becomes

\[ E_\rho(X) = \sum_{i=1}^n \lambda_i P_{\omega_i}(X). \]

As \( \sum_{i=1}^n \lambda_i = 1 \), we see that the fuzzy observable \( E_\rho \) is a convex combination of the sharp observables \( P_{\omega_1}, \ldots, P_{\omega_n} \). Therefore, we may think of \( E_\rho \) as a mixture of translated sharp observables.

**Example 2.** Let \( \rho \) be a probability measure and \( \tilde{\rho} \) the corresponding confidence measure (defined in Lemma 1). Assume that \( \rho \) is absolutely continuous with respect to the Haar measure \( \mu_\Omega \) and let \( f \) be the corresponding Radon-Nikodym derivative. Define the function \( \tilde{f} \) by \( \tilde{f}(\omega) = f(\omega^{-1}) \). For any set \( X \in \mathcal{B}(\Omega) \) the confidence measure \( \tilde{\rho} \) defines a fuzzy event \( \tilde{X} \), as in Section 2. The fuzzy event \( \tilde{X} \) can be
written simply as a convolution of the characteristic function \( \chi_X \) and the function \( \tilde{f} \), since
\[
\tilde{X}(\omega) = \tilde{\rho}_\omega(X) = \rho(\omega^{-1}X) = \int \chi_{\omega^{-1}X}(\eta) f(\eta) \, d\mu(\eta)
\]
\[
= (\chi_X * \tilde{f})(\omega).
\]

4. Coarse-graining

Let \( \mathcal{T}(\mathcal{H}) \) be the set of all trace-class operators on \( \mathcal{H} \) and \( M(\Omega) \) the set of all complex Radon measures on \( \Omega \). An observable \( E : \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H}) \) defines a linear mapping
\[
V_E : \mathcal{T}(\mathcal{H}) \to M(\Omega), \quad V_E(T)(X) = \text{tr}[TE(X)]. \tag{12}
\]
In particular, if \( T \) is a state, i.e. a positive trace one operator, then \( V_E(T) \) is a probability measure. Moreover, if \( T \) is a vector state \( T = |\psi\rangle\langle\psi| \), then \( V_E(T) = p^E_\psi \). The mapping \( V_E \) characterizes the observable \( E \) completely in the sense that \( V_E = V_{E'} \) implies \( E = E' \).

**Definition 3.** Let \( E_1 \) and \( E_2 \) be observables from \( \mathcal{B}(\Omega) \) to \( \mathcal{L}(\mathcal{H}) \) and \( V_{E_1}, V_{E_2} \) the corresponding mappings. We say that the observable \( E_2 \) is **coarser** than \( E_1 \), if there exists a linear map \( W : M(\Omega) \to M(\Omega) \), such that

(i) for any probability measure \( m \in M(\Omega) \), \( W(m) \) is a probability measure;
(ii) \( V_{E_2} \) is the composite mapping of \( V_{E_1} \) and \( W \), that is, \( V_{E_2} = W \circ V_{E_1} \).

If \( E_2 \) is coarser than \( E_1 \) we also say that \( E_2 \) is a **coarse-graining** of \( E_1 \).

Using the Jordan decomposition of a measure one infers from (i) that \( W \) is a bounded linear map.

If \( E_2 \) is a coarse-graining of \( E_1 \), for any two states \( S, T \in \mathcal{T}(\mathcal{H})^+_1 \) the following implication holds:
\[
V_{E_1}(S) = V_{E_1}(T) \Rightarrow V_{E_2}(S) = V_{E_2}(T). \tag{13}
\]
This means that the state distinction power of \( E_2 \) does not exceed that of \( E_1 \).

**Example 3.** Fuzzy observables are coarser than the corresponding sharp observable. Indeed, any confidence measure \( \nu : \Omega \times \mathcal{B}(\Omega) \to [0,1] \) induces a linear mapping
\[
W_\nu : M(\Omega) \to M(\Omega), \quad W_\nu(m)(X) = \int_\Omega \nu_\omega(X) \, dm(\omega).
\]
The observable $E$ corresponding to the map $W_{\nu} \circ V_{P}$ is the fuzzy observable given by equation (1).

If $E$ is a $G$-covariant observable, equation (6) implies that for any $T \in \mathcal{T}(\mathcal{H})$, the measure $V_{E}(T)$ is absolutely continuous with respect to $\mu_{\Omega}$. Thus, we may identify $V_{E}(T)$ with its Radon-Nikodým derivative with respect to $\mu_{\Omega}$ and hence the mapping $V_{E}$ with the corresponding mapping from $\mathcal{T}(\mathcal{H})$ to $L^{1}(\Omega)$. For our subsequent needs we state and give a proof of the following known result.

**Lemma 2.** The mapping $V_{P} : \mathcal{T}(\mathcal{H}) \rightarrow L^{1}(\Omega)$ corresponding to the canonical observable $P$ is surjective.

**Proof.** Let $f$ be a function in $L^{1}(\Omega)$ and write it as $f = f_{1} - f_{2} + i(f_{3} - f_{4})$, where $f_{n}, n = 1, 2, 3, 4$, are nonnegative functions and belong to $L^{1}(\Omega)$. Fix a unit vector $\varphi \in \mathcal{K}$ and for each $n = 1, 2, 3, 4$ denote $\psi_{n}(\omega) = \sqrt{f_{n}(\omega)}\varphi$. Then $\psi_{n}$ belongs to $L^{2}(\Omega, \mu_{\Omega}, \mathcal{K})$ and $V_{P}(\langle \psi_{n} \rangle \langle \psi_{n} \rangle) = f_{n}$. The operator

$$T = \langle \psi_{1} \rangle \langle \psi_{1} \rangle - \langle \psi_{2} \rangle \langle \psi_{2} \rangle + i\langle \psi_{3} \rangle \langle \psi_{3} \rangle - i\langle \psi_{4} \rangle \langle \psi_{4} \rangle$$

is a trace-class operator and $V_{P}(T) = f$. □

**Proposition 2.** Let $E$ be a $G$-covariant observable. If $E$ is coarser than $P$, then $E = E_{\rho}$ for some probability measure $\rho : \mathcal{B}(\Omega) \rightarrow [0, 1]$.

**Proof.** Assume that $E$ is coarser than $P$. By definition there exists a linear map $W : L^{1}(\Omega) \rightarrow L^{1}(\Omega)$ such that

$$V_{E} = W \circ V_{P}. \quad (14)$$

For any $\xi \in \Omega$ and $f \in L^{1}(\Omega)$, we define $(\Lambda_{\xi}f)(\omega) = f(\xi \omega)$. Then $\Lambda_{\xi} : L^{1}(\Omega) \rightarrow L^{1}(\Omega)$ is a linear mapping. We will show that for all $\xi \in \Omega$,

$$\Lambda_{\xi} \circ W = W \circ \Lambda_{\xi}. \quad (15)$$

Together with a result of Wendel [23, Theorem 1] as formulated in [16, p. 376] this then implies that $W$ is of the form $W(f) = f \ast \rho$ for some probability measure $\rho$. By equation (14) this means that $E = E_{\rho}$.

To prove (15), fix $\xi \in \Omega$ and choose $g \in G$ such that $\pi(g) = \xi$. Since $E$ is covariant, we get

$$\Lambda_{\xi}V_{E}(T) = V_{E}(U(g)^{*}TU(g)) \quad (16)$$

for all $T \in \mathcal{T}(\mathcal{H})$. The same is true for $V_{P}$, that is,

$$\Lambda_{\xi}V_{P}(T) = V_{P}(U(g)^{*}TU(g)). \quad (17)$$

Comparing equations (14), (16) and (17), we get

$$\Lambda_{\xi} \circ W \circ V_{P} = W \circ \Lambda_{\xi} \circ V_{P}. \quad (18)$$


As the mapping $V_P$ is surjective, equation (15) follows from (18).

□

Corollary 1. Let $E$ be a $G$-covariant observable. The following are equivalent:

(i) $E$ is a fuzzy observable with respect to $P$;
(ii) $E$ is coarser than $P$;
(iii) There is a probability measure $\rho$ such that $E = E_\rho$.

Proof. In Example 3 it was shown that (i) implies (ii). By Proposition 2 (ii) implies (iii). That (iii) implies (i) was shown in Proposition 1. □

5. Properties of covariant fuzzy observables

In this section we study the norm-1-property as well as the regularity of a covariant fuzzy observable and we find a necessary and sufficient condition for a fuzzy observable to be informationally equivalent to its sharp counterpart.

5.1. The norm-1-property. Sharp observables $P$, i.e. spectral measures, have the property that the norm of any $P(X) \neq O$ is one. Spectral measures are not the only observables having this property, called the norm-1-property in [14]. Some examples are given in [14]. If an observable $E$ has the norm-1-property, then for each $E(X) \neq O$ there is a sequence of vector states $(\varphi_n)$ such that $\lim_{n \to \infty} P^E_{\varphi_n}(X) = 1$. The following proposition shows that among the covariant fuzzy observables only the translated spectral measures have the norm-1-property.

Proposition 3. Let $E$ be a $G$-covariant observable which is fuzzy with respect to $P$. Then $E$ has the norm-1-property if and only if $E = P_\omega$ for some $\omega \in \Omega$.

Proof. If $E$ is a spectral measure, then it has the norm-1-property. Assume now that $\rho$ is a probability measure on $\Omega$ and $E$ is defined by equation (8). Then $E = P_\omega$ if and only if $\rho = \delta_\omega$. Therefore it is enough to show that if $E$ has the norm-1-property, then supp ($\rho$) is a one point set.

For any $X \in B(\Omega)$,

$$\|E(X)\| = \|\int_{\Omega} \rho(\omega^{-1}X)dP(\omega)\| \leq \|P(X)\| \cdot \sup_{\omega \in \Omega} \rho(\omega^{-1}X) \leq \sup_{\omega \in \Omega} \rho(\omega X).$$

Thus $E$ cannot have the norm-1-property if $\sup_{\omega \in \Omega} \rho(\omega X) < 1$ for some $X \in B(\Omega)$, $\mu_\Omega(X) > 0$. (By equation (8), $E(X) \neq O$ if and only if $\mu_\Omega(X) \neq 0$.)

Assume now that there are at least two points $x$ and $y$ in supp ($\rho$) and denote $r = d_\Omega(x,y)$. Then the sets $B(x; \frac{r}{2})$ and $B(y; \frac{r}{2})$ are disjoint.
and have positive $\rho$-measure. Denote $m = \min\{\rho(B(x; \frac{y}{x})), \rho(B(y; \frac{z}{y}))\}$. Because the metric $d_\Omega$ is left invariant, we have $\omega B(x; \frac{y}{x}) = B(\omega x; \frac{y}{x})$ for any $\omega \in \Omega$. Moreover, for any $\omega \in \Omega$, we have $B(\omega x; \frac{y}{x}) \cap B(x; \frac{z}{y}) = \emptyset$ or $B(\omega x; \frac{y}{x}) \cap B(y; \frac{z}{y}) = \emptyset$. This implies that $\rho(\omega B(x; \frac{y}{x})) \leq 1 - m$ for all $\omega \in \Omega$ so that $\sup_{\omega \in \Omega} \rho(\omega B(x; \frac{y}{x})) < 1$. □

5.2. Regularity. An observable $E$ is called regular, if for each $O \neq E(X) \neq I$, the spectrum of $E(X)$ extends both below and above $\frac{1}{2}$. For such an observable, for each (nontrivial) $E(X)$ there are vector states $\varphi$ such that $p^E_\varphi(X) > 1/2$ and $p^E_\varphi(X') < 1/2$, a property which resembles the fact that for each (nontrivial) projection $P(X)$ there is a $\varphi$ such that $p^\varphi_P(X) = 1$ and $p^\varphi_P(X') = 0$. If an observable $E$ is regular, then its range $\{E(X)|X \in \mathcal{F}\}$ is a Boolean system with respect to the natural order and complement of the set of effects restricted to the range $[1]$.

Clearly, an observable having the norm-1-property is also regular. Next we give an example of covariant fuzzy observable which is regular and which is not a spectral measure.

Example 4. Let $\omega_1, \omega_2 \in \Omega$ be distinct points. Let $E$ be a proper convex combination of the translated spectral measures $P_{\omega_1}$ and $P_{\omega_2}$, that is,
\[
E(X) = tP_{\omega_1}(X) + (1 - t)P_{\omega_2}(X), \quad X \in \mathcal{B}(\Omega),
\]
for some $t \in (\frac{1}{2}, 1)$. For any $X \in \mathcal{B}(\Omega)$, the projections $P_{\omega_1}(X)$ and $P_{\omega_2}(X)$ commute, so that also their product $Q := P_{\omega_1}(X)P_{\omega_2}(X)$ is projection. The differences $P_{\omega_1}(X) - Q$ and $P_{\omega_1}(X) - Q$ are mutually orthogonal projections, and the spectral decomposition of the effect $E(X)$ is
\[
E(X) = t(P_{\omega_1}(X) - Q) + (1 - t)(P_{\omega_2}(X) - Q) + Q.
\]
Assume that the spectrum of $E(X)$ lies under $\frac{1}{2}$. This means that $P_{\omega_1}(X) - Q = Q = O$, implying also that $P_{\omega_1}(X) = O$ and by equation (6), $\mu_\Omega(X\omega_1^{-1}) = 0$. In the right translations the sets of $\mu_\Omega$-measure zero remain $\mu_\Omega$-measure zero. Hence, $\mu_\Omega(X\omega_2^{-1}) = 0$, and by equation (6), $P_{\omega_2}(X) = O$. Therefore, the spectrum of any $E(X) \neq O$ extends above $\frac{1}{2}$. If $O \neq E(X) \neq I$, then also the complement effect $E(X') = I - E(X)$ has $t$ or $1$ in spectrum. This implies that either $0$ or $1 - t$ belongs in the spectrum of $E(X)$, showing that the observable $E$ is regular.

Even though there are regular fuzzy observables, regularity is a surprisingly strong condition. Proposition 4 shows that many typical fuzzy observables (see Section 6.2) are not regular.
Proposition 4. Let $\Omega$ be non-discrete and $\rho : \mathcal{B}(\Omega) \to [0,1]$ a probability measure. If $\rho$ is absolutely continuous with respect to the Haar measure $\mu_\Omega$, then the fuzzy observable $E_\rho$ is not regular.

Proof. As in the proof of Proposition 3, we get the inequality
\[ ||E_\rho(X)|| \leq \sup_{\omega \in \Omega} \rho(\omega X) \] (19)
for any $X \in \mathcal{B}(\Omega)$. For every $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that $\rho(\omega X) < \epsilon$ whenever $\mu_\Omega(\omega X) < \delta_\epsilon$ (see [12, Theorem 3.5]). Hence, $\mu_\Omega(X) < \delta_\epsilon$ implies that $\rho(\omega X) < \epsilon$ for every $\omega \in \Omega$. Since $\Omega$ is not discrete, the Haar measure $\mu_\Omega$ has arbitrarily small positive values. Thus we can choose $X_1 \in \mathcal{B}(\Omega)$ such that $0 < \mu_\Omega(X_1) < \delta_1^2$. Now $\rho(\omega X) < \frac{1}{2}$ for every $\omega \in \Omega$ and this together with equation (19) gives $||E_\rho(X_1)|| < \frac{1}{2}$. Moreover, $E_\rho(X_1) \neq O$ by equation (6).

\[ \Box \]

5.3. Informational equivalence. We shall next investigate the state distinction power of fuzzy observables. In general, two observables $E_1, E_2$ are called informationally equivalent if for any two states $S, T \in \mathcal{T}(\mathcal{H})_1^+$,
\[ V_{E_1}(S) = V_{E_1}(T) \iff V_{E_2}(S) = V_{E_2}(T). \]
This means that informationally equivalent observables have the same ability to distinguish between different states. Clearly, $E_1$ and $E_2$ are informationally equivalent if and only if for any two unit vectors $\varphi, \psi \in \mathcal{H}$ we have
\[ p_{\varphi}^{E_1} = p_{\psi}^{E_1} \iff p_{\varphi}^{E_2} = p_{\psi}^{E_2}. \]

Since a fuzzy observable is a coarse-graining of a sharp observable, its ability to distinguish states does not exceed that of the sharp observable. However, a fuzzy observable can be informationally equivalent to the underlying sharp one. Proposition 5 deals with this matter.

Let the group $\Omega = G/H$ be Abelian. If $\rho$ is a probability measure on $\Omega$, its Fourier-Stieltjes transform $\hat{\rho}$ is the bounded continuous function on the dual group $\hat{\Omega}$, defined by
\[ \hat{\rho}(\xi) = \int_{\Omega} \langle \omega, \xi \rangle d\rho(\omega) = \int_{\Omega} \langle \omega, \xi^{-1} \rangle d\rho(\omega), \]
where $\langle \omega, \xi \rangle = \xi(\omega)$ is the value of the character $\xi$ at the point $\omega$.

Proposition 5. The observables $P$ and $E_\rho$ are informationally equivalent if and only if $\text{supp} (\hat{\rho}) = \hat{\Omega}$. 

Proof. Assume that the latter condition is fulfilled. Let $\varphi, \psi \in \mathcal{H}$ be two unit vectors such that $\hat{p}_\varphi = \hat{p}_\psi$. Using equation (20) and taking Fourier-Stieltjes transforms we get $\hat{p}_\varphi^p(\xi) = \hat{p}_\psi^p(\xi)$ for all $\xi \in \Omega$. This implies that $\hat{p}_\varphi^p(\xi) \in$ a dense set. Since the functions $\hat{p}_\varphi$ and $\hat{p}_\psi$ are continuous, it follows that $\hat{p}_\varphi^p = \hat{p}_\psi^p$, and so $p^p = p^p$. Thus the observables $P$ and $E^p$ are informationally equivalent.

Assume now that $\text{supp}(\hat{\rho}) \subseteq \Omega$. We have to show that there are unit vectors $\psi, \varphi \in L^2(\Omega, \mu, \mathcal{K})$ such that $p^p_\psi \neq p^p_\varphi$ but $p^E_\psi = p^E_\varphi$. Let us denote by $U$ the complement of $\text{supp}(\hat{\rho})$. Fix $\hat{\omega} \subseteq \Omega$ such that $\hat{\omega}^{-1}$ belongs to $U$. Then the set $\hat{\omega}U$ is an open neighborhood of the neutral element $e$ and there is a symmetric open set $V$ such that $e \in V$ and $VV \subseteq \hat{\omega}U$. According to Urysohn’s lemma there exists a continuous function $f : \hat{\Omega} \to [0, 1], f \neq 0$, having compact support such that $\text{supp}(f) \subseteq V$. The convolution $f * f^*$ is a positive definite continuous function with compact support contained in $\hat{\omega}U$. By the Fourier inversion theorem there is a positive function $F' \in L^1(\Omega)$ such that $\hat{F}' = f * f^*$. The product $F := \hat{\omega}^{-1}F'$ is in $L^1(\Omega)$ and

$$\hat{F}(\xi) = (f * f^*)(\hat{\omega} \xi).$$

It follows that the support of $\hat{F}$ is contained in $U$ and therefore $\hat{\rho} \hat{F} = 0$. This means that

$$\rho * F = 0. \quad (20)$$

We now write $F = F_1 - F_2 + iF_3 - iF_4$, where the $F_j, j = 1, 2, 3, 4$, are nonnegative functions and belong to $L^1(\Omega)$. Since $F \neq 0$, either $F_1 \neq F_2$ or $F_3 \neq F_4$. We consider the case $F_1 \neq F_2$, the other one being similar.

Equation (20) implies that

$$\rho * F_1 = \rho * F_2. \quad (21)$$

As $\|\rho * F_1\|_1 = \|F_1\|_1$ and similarly $\|\rho * F_2\|_1 = \|F_2\|_1$, we have $\|F_1\|_1 = \|F_2\|_1$. Therefore we may normalize $\|F_1\|_1 = \|F_2\|_1 = 1$ keeping equation (21) valid. Fix a unit vector $\vartheta \in \mathcal{K}$ and denote $\psi(\omega) = \sqrt{F_1}(\omega) \vartheta$ and $\varphi(\omega) = \sqrt{F_2}(\omega) \vartheta$ for all $\omega \in \Omega$. Then $\psi, \varphi \in L^2(\Omega, \mu, \mathcal{K})$ are states, $p^p_\psi \neq p^p_\varphi$ and $p^E_\psi = p^E_\varphi$.

**Remark 1.** In [2] the Euclidean covariant localization observables were studied. If $E$ is a Euclidean covariant localization observable, then it is in particular covariant under translations. For a translationally covariant fuzzy observable Proposition 3 is applicable. Our result therefore shows that in [2] Theorem 2 one needs the additional assumption that
the Fourier-Stieltjes transform of the probability measure in question has all of $\mathbb{R}^3$ as its support. The same observation applies e.g. to [6, p. 124] where [2] is quoted.

6. Examples

In this section we illustrate the concepts and results of the previous sections by the Stern-Gerlach experiment and by some localization observables.

6.1. The Stern-Gerlach experiment. In the most simplified, though prototypical textbook description of a Stern-Gerlach experiment a spin-$\frac{1}{2}$ atom prepared in a vector state $\phi_{\text{orbit}} \otimes \psi_{\text{spin}}$ is directed (say, horizontally) to a (say, vertical) Stern-Gerlach magnetic field. Writing $\psi_{\text{spin}} = c_\uparrow \psi_\uparrow + c_\downarrow \psi_\downarrow$, the action of the magnet on the atom is described by a unitary transformation coupling its spin and spatial degrees of freedom:

$$\phi_{\text{orbit}} \otimes \psi_{\text{spin}} \mapsto c_\uparrow \phi_\uparrow \otimes \psi_\uparrow + c_\downarrow \phi_\downarrow \otimes \psi_\downarrow.$$ 

The deflected atom hits a screen producing a spot on it. The counting of the spots (that is, the collecting of the measurement statistics of up and down deflected atoms) can be modelled by projection operators $P_\uparrow$ and $P_\downarrow$ representing the localization of the atom in the upper or lower half planes of the screen. The relevant measurement outcome probabilities are then

$$p_\uparrow = |c_\uparrow|^2 \langle \phi_\uparrow | P_\uparrow \phi_\uparrow \rangle + |c_\downarrow|^2 \langle \phi_\downarrow | P_\uparrow \phi_\downarrow \rangle,$$

$$p_\downarrow = |c_\uparrow|^2 \langle \phi_\uparrow | P_\downarrow \phi_\uparrow \rangle + |c_\downarrow|^2 \langle \phi_\downarrow | P_\downarrow \phi_\downarrow \rangle.$$ 

When these probabilities are expressed in terms of a two-valued spin observable $E$ and the incoming spin state $\psi_{\text{spin}}$, that is, $\langle \psi_{\text{spin}} | E_\uparrow \psi_{\text{spin}} \rangle := p_\uparrow$ and $\langle \psi_{\text{spin}} | E_\downarrow \psi_{\text{spin}} \rangle := p_\downarrow$, the measured spin observable $E$ has the form

$$E_\uparrow = \langle \phi_\uparrow | P_\uparrow \phi_\uparrow \rangle P[\psi_\uparrow] + \langle \phi_\downarrow | P_\uparrow \phi_\downarrow \rangle P[\psi_\downarrow],$$

$$E_\downarrow = \langle \phi_\uparrow | P_\downarrow \phi_\uparrow \rangle P[\psi_\uparrow] + \langle \phi_\downarrow | P_\downarrow \phi_\downarrow \rangle P[\psi_\downarrow].$$

If $\langle \phi_\uparrow | P_\uparrow \phi_\uparrow \rangle = \langle \phi_\downarrow | P_\downarrow \phi_\downarrow \rangle = 1$ and $\langle \phi_\downarrow | P_\uparrow \phi_\downarrow \rangle = \langle \phi_\uparrow | P_\uparrow \phi_\uparrow \rangle = 0$, we get the sharp spin observable $s_\uparrow$ with the spectral projections $E_\uparrow = P[\psi_\uparrow]$ and $E_\downarrow = P[\psi_\downarrow]$. In general, the measured spin observable $E$ is a fuzzy observable with respect to $s_\uparrow$. Apart from the trivial observable $E_\uparrow = E_\downarrow = \frac{1}{2} I$, the measured fuzzy spin observables are informationally equivalent to the sharp one, $s_\uparrow$. Being a two-valued observable $E$ is regular. The norms of $E_\uparrow$ and $E_\downarrow$ are their larger eigenvalues which are equal to one only when these effects are the spin projections. (A more realistic account of a Stern-Gerlach experiment as a spin measurement
experiment can be read, for instance, in [6], but the basic picture is essentially the same.)

6.2. Localization on \( \mathbb{R} \). The usual position observable on \( \mathbb{R} \) is the spectral measure \( P : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(L^2(\mathbb{R})) \), \( P(X)\psi = \chi_X\psi \). It is covariant under the regular representation \( U : \mathbb{R} \to \mathcal{U}(\mathcal{H}) \), \( [U(x)\psi](y) = \psi(y - x) \). The representation \( U \) is induced from the trivial representation of the trivial subgroup \( \{0\} \). All \( \mathbb{R} \)-covariant localization observables are characterized explicitly in [8]. We emphasize that there exist also other covariant position observables than the sharp and fuzzy localization observables. Even non-commutative covariant localization observables exist.

Davies [9] calls an observable \( E_f : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(L^2(\mathbb{R})) \), \( E_f(X) = (f * \chi_X)(P) \), where \( f \) is a probability density function, an approximate position observable. This is a covariant fuzzy observable, the probability measure being \( \hat{f}d\mu \), where \( \hat{f}(x) = f(-x) \) (see Example 2). According to Proposition 5 the approximate position observable \( E_f \) is informationally equivalent to the sharp position observable \( P \) if and only if \( \text{supp}(\hat{f}) = \mathbb{R} \). For example, if \( f \) is a Gaussian function, then the Fourier transform of \( f \) is also Gaussian and \( E_f \) is informationally equivalent to \( P \). Also the functions \( \frac{1}{2a}\chi_{[-a,a]} \), \( a > 0 \), yield approximate position observables that are informationally equivalent to \( P \). An example of an approximate position observable which is not informationally equivalent to \( P \), is obtained if one takes \( f(x) = \sin^2(2\pi x)/2\pi^2 x^2 \). In this case \( \hat{f}(\xi) = \max(1 - \frac{1}{4}|\xi|, 0) \) and \( \text{supp}(\hat{f}) = [-2, 2] \).

6.3. Localization on \( \mathbb{T} \). Let us denote by \( \mathbb{T} \) the set of all complex numbers modulus one. It is a compact Abelian group and the Haar measure \( \mu = \mu_\mathbb{T} \) of \( \mathbb{T} \) is just the normalized and translated Lebesgue measure. By a \( \mathbb{T} \)-covariant localization observable we mean an observable \( E : \mathcal{B}(\mathbb{T}) \to \mathcal{L}(L^2(\mathbb{T})) \) satisfying the covariance condition

\[
U(a)E(X)U(a)^* = E(aX),
\]

where \( [U(a)\psi](z) = \psi(a^{-1}z) \).

The family \( \{e_k\}_{k \in \mathbb{Z}} \) of functions

\[
e_k : \mathbb{T} \to \mathbb{C}, \quad e_k(a) = a^k,
\]

is an orthonormal basis for \( L^2(\mathbb{T}) \). Theorem 1 of [7] shows that any \( \mathbb{T} \)-covariant localization observable \( E \) has the form

\[
E(X) = \sum_{n,m \in \mathbb{Z}} c_{n,m} \int_X z^{m-n} d\mu(z) \langle e_n | e_m \rangle, \quad X \in \mathcal{B}(\mathbb{T}),
\]

where the numbers \( c_{n,m} \) satisfy the following two conditions:
(a) \( c_{n,n} = 1 \) for all \( n \in \mathbb{Z} \);
(b) \( \sum_{n,m=-k}^{k} c_{n,m} |e_n\rangle \langle e_m| \geq O \) for all \( k \in \mathbb{N} \).

Note that the canonical spectral measure \( P \) corresponds to the matrix \((c_{n,m})\), where \( c_{n,m} = 1 \) for all \( n, m \in \mathbb{Z} \).

This simple structure of \( \mathbb{T} \)-covariant localization observables gives us a possibility to investigate further the fuzzy observables on \( \mathbb{T} \). We say that the observables \( E_1 \) and \( E_2 \) commute if \( E_1(X)E_2(Y) = E_2(Y)E_1(X) \) for all \( X, Y \in \mathcal{B}(\mathbb{T}) \). An observable \( E \) is commutative if it commutes with itself.

Lemma 3. A \( \mathbb{T} \)-covariant localization observable \( E \) corresponding to the matrix \((c_{n,m})\) commutes with \( P \) if and only if

\[
\forall n, m, k \in \mathbb{Z} \quad c_{n+k,m+k} = c_{n,m} \tag{23}
\]

Proof. The proof is similar to the proof of Proposition 1 in [7], which characterizes commutative \( \mathbb{T} \)-covariant localization observables. Define

\[
\mu_{n,m,Y} := \langle e_n|P(X)E(Y) - E(Y)P(X)|e_m\rangle
\]

for all \( n, m \in \mathbb{Z} \) and \( X, Y \in \mathcal{B}(\mathbb{T}) \). Now

\[
\langle e_n|P(X)E(Y)|e_m\rangle = \sum_{s=-\infty}^{\infty} c_{s,m} \int_{X} z^{s-n} \, d\mu(z) \int_{Y} z^{m-s} \, d\mu(z)
\]

and

\[
\langle e_n|E(Y)P(X)|e_m\rangle = \sum_{s=-\infty}^{\infty} c_{n,s} \int_{X} z^{m-s} \, d\mu(z) \int_{Y} z^{s-n} \, d\mu(z)
\]

It follows that for all \( k \in \mathbb{Z} \),

\[
\int_{\mathbb{T}} z^{-k} \, d\mu_{n,m,Y}(z) = [c_{n+k,m} - c_{n,m-k}] \int_{Y} z^{m-n-k} \, d\mu(z).
\]

If \( P(X)E(Y) = E(Y)P(X) \) for all \( X, Y \in \mathcal{B}(\mathbb{T}) \), equation (23) is satisfied. Also, if (23) is valid, then

\[
\mu_{n,m,Y}(X) = \sum_{k=-\infty}^{\infty} [c_{n+k,m} - c_{n,m-k}] \int_{X} z^{k} \, d\mu(z) \int_{Y} z^{m-n-k} \, d\mu(z) = 0
\]

for all \( n, m \in \mathbb{Z} \). \( \square \)

Proposition 6. A \( \mathbb{T} \)-covariant localization observable \( E \) is fuzzy with respect to \( P \) if and only if \( E \) commutes with \( P \).
Proof. It is clear that if $E$ is fuzzy with respect to $P$, then it also commutes with $P$. Let now $E$ be a $\mathbb{T}$-covariant localization observable that commutes with $P$. Then according to Lemma 3 the matrix elements $c_{n,m}$ of $E$ satisfy $c_{n+k,m+k} = c_{n,m}$ for all $n, m, k \in \mathbb{Z}$. Let us define the function $\Phi: \mathbb{Z} \to \mathbb{C}$ by $\Phi(k) = c_{k,0}$. Since $c_{n,m} = c_{n-m,0}$ for all $n, m \in \mathbb{Z}$, the function $\Phi$ determines all the matrix elements $c_{n,m}$.

Moreover, $\Phi$ has following properties:

(i) $\Phi(0) = 1$;
(ii) $\Phi(-k) = \Phi(k)$ for all $k \in \mathbb{Z}$;
(iii) $|\Phi(k)| \leq 1$ for all $k \in \mathbb{Z}$;
(iv) $\sum_{n,m=-N}^{N} d_n \Phi(n-m)d_m \geq 0$ for all sequences $(d_n)_{n \in \mathbb{Z}} \subset \mathbb{C}$ and every $N \in \mathbb{N}$.

Condition (iv) means that $\Phi$ is positive definite. Herglotz’s theorem \cite{16} (see e.g. \cite{13} p. 95, \cite{16} p. 293) states that in this situation there is a unique positive measure $\rho \in M(\mathbb{T})$ such that $\Phi$ is given by

$$\Phi(k) = \int_{\mathbb{T}} \langle k, z \rangle d\rho(z) = \int_{\mathbb{T}} z^k d\rho(z). \tag{24}$$

Because

$$\rho(\mathbb{T}) = \int_{\mathbb{T}} d\rho(z) = \Phi(0) = 1,$$

$\rho$ is a probability measure. Now

$$\langle e_n|E(X)|e_m \rangle = \Phi(n-m) \int_{X} z^{m-n} d\mu(z)$$

$$= \int_{\mathbb{T}} y^{n-m} d\rho(y) \int_{\mathbb{T}} \chi_X(z) z^{m-n} d\mu(z)$$

$$= \int_{\mathbb{T}} \int_{\mathbb{T}} (y^{-1}z)^{m-n} \chi_X(z) d\mu(z) d\rho(y)$$

$$= \int_{\mathbb{T}} \int_{\mathbb{T}} z^{m-n} \chi_X(yz) d\mu(z) d\rho(y)$$

$$= \int_{\mathbb{T}} \int_{\mathbb{T}} z^{m-n} \left( \int_{\mathbb{T}} \chi_{z^{-1}X}(y) d\rho(y) \right) d\mu(z)$$

$$= \int_{\mathbb{T}} \int_{\mathbb{T}} z^{m-n} \rho(z^{-1}X) d\mu(z),$$

so $E$ can be written as

$$E(X) = \int_{\mathbb{T}} \rho(z^{-1}X) dP(z). \tag{25}$$

□
We finally note that not all commutative $\mathbb{T}$-covariant localization observables are fuzzy. This can be demonstrated by the following example, due to A. Toigo [20]. Set $c_{n,n} = 1$ for all $n \in \mathbb{Z}$ and for all $n, m \in \mathbb{Z}, n \neq m$ define

$$c_{n,m} = \begin{cases} 
1, & \text{if } n \text{ and } m \text{ are even;} \\
0, & \text{otherwise.}
\end{cases}$$

Then $(c_{n,m})$ satisfies conditions (a) and (b) stated after eq. [22], and using Proposition 1 in [7] it is easy to see that the localization observable $E$ defined by $(c_{n,m})$ is commutative. Since, for example, $c_{2,4} = 1$ and $c_{3,5} = 0$, Lemma 3 shows that $E$ does not commute with $P$ and therefore, is not fuzzy.

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