Hesitant Fuzzy Graphs and Their Products

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ABSTRACT
In this article, the notion of hesitant fuzzy graph is introduced. It an extended structure of a fuzzy graph that gives more precision, flexibility and compatibility to a system when compared with the system that is designed using fuzzy graphs. Several operations on hesitant fuzzy graphs are defined, namely Cartesian product, composition, tensor product and normal product. Furthermore, the relationship between the degree of vertices of the hesitant fuzzy graphs and the hesitant fuzzy graph obtained by their different products is developed. These operations are highly used in computer science, geometry, algebra, operations research, etc.

1. Introduction
Concept of graph theory have applications in many areas of computer science, including data mining, image segmentation, clustering, image capturing, networking, etc. Classical graph theory is based on classical propositional logic. In many cases, human judgment and preference are ambiguous, vague and cannot be estimated with exact numeric value under many conditions, so the classical logic is not suitable to model real world situations. To solve the ambiguity and vagueness in information from human judgement and preference, classical logic is extended to fuzzy logic by Zadeh in 1965 [1]. It is obvious that much knowledge in real world situations is fuzzy rather than precise. Fuzzy logic was the base of several extensions of fuzzy set theory and it is successfully used in many discipline of theoretical and practical background [2–4]. For the basics of fuzzy logic several texts are available but here we recommend [5]. In 1975, Rosenfeld [6] discussed the concept of fuzzy graph whose basic idea was introduced by Kauffman [7] in 1973. The fuzzy relations between fuzzy sets were also considered by Rosenfeld and he developed the structure of fuzzy graphs, obtained analogs of several graphs theoretical concepts. The popularity and applicability of fuzzy logic attracted several researchers, resulting its various extensions and generalisations. Hesitant fuzzy set (HFS) is one such important extension and generalisation of fuzzy set theory [8]. It received lot of attention in clustering, optimisation, convexity, decision making, preference relations, data mining and aggregation operators. The fuzzy sets give the degree of membership, while HFSs give all the possible degrees of membership, which are independent from each other. A different perspective than the current one, of HFS in graph theory...
and decision making is presented in [9–13]. In this paper, the notion of hesitant fuzzy graphs (HFG) is discussed in general and broader prospective. We define some operations on HFG namely Cartesian product, tensor product, normal product and composition. Furthermore, studied about the degree of the vertex in HFG, which is obtained from two HFGs $G'$ and $G''$ using the operations Cartesian product, composition, tensor product and normal product. Real life application of these products are the great motivation of our work. After Rosenfeld [6] the fuzzy graph theory increases with its various types of branches, such as arcs in fuzzy graph [14], fuzzy tolerance graph [15], fuzzy threshold graphs [16], bipolar fuzzy graphs [17], highly irregular interval valued fuzzy graphs [18, 19], balanced interval valued fuzzy graphs [20], fuzzy $k$–competition graphs and $p$–competition fuzzy graphs [21], etc. A new concept of fuzzy colouring of fuzzy graph is given [23]. Ghorai and Pal introduced operations on $m$–polar fuzzy graphs [24], $m$–polar fuzzy planar graphs [25]. Nayeem and Pal introduced shortest path problem on a network with imprecise edge weight [26].

Rest of the article is organised as follows: In Section 2, we review the basic concept of graph theory and hesitant fuzzy sets. In Section 3, we proposed the concept of hesitant fuzzy graphs and degree of a vertex of this graph. In Section 4, is dedicated for the study of different types of products for hesitant fuzzy graphs and characterise the degree of vertex for these graphs. In Section 5, real life application is discussed for different products. Conclusion is given in the last section.

2. Preliminaries

In this section, we include some definition and initial concepts from graphs, and hesitant set theory.

2.1. Graphs

A graph $G = (V, E)$ consists of $V$, a nonempty set of vertices (or nodes) and $E$, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints. A graph is called simple if it has no loops and no multiple edges.

**Definition 2.1** ([27]): The collection of all the vertices adjacent to $v \in V$ is called its neighbourhood denoted by $N_G(v)$.

**Definition 2.2** ([27]): The degree of a vertex $v \in V$ is denoted by $\text{deg}(v)$ and is defined as the cardinality of the $N_G(v)$.

**Definition 2.3** ([28]): Let $G' = (V', E')$ and $G'' = (V'', E'')$ be any two graphs. Then their Cartesian product is the graph $G = G' \times G'' = (V, E)$, where $V = V' \times V''$ and $E = \{(u_1, v_1), (u_2, v_2)\} | u_1 \neq u_2, (v_1, v_2) \in E' \lor v_1 \neq v_2, (u_1, u_2) \in E'' \} \cup \{(u_1, v_1), (u_2, v_2)\} | u_1 = u_2, (v_1, v_2) \in E' \lor v_1 = v_2, (u_1, u_2) \in E'' \}.$

**Definition 2.4** ([28]): Let $G' = (V', E')$ and $G'' = (V'', E'')$ be any two graphs. Then the composition of $G'$ and $G''$ is the graph $G = G' \circ G'' = (V, E)$, where $V = V' \times V'' = V' \circ V''$ and $E = \{(u_1, v_1), (u_2, v_2)\} | u_1 = u_2, (v_1, v_2) \in E' \lor v_1 = v_2, (u_1, u_2) \in E'' \} \cup \{(u_1, v_1), (u_2, v_2)\} | u_1 \neq u_2, (v_1, v_2) \in E' \lor v_1 \neq v_2, (u_1, u_2) \in E'' \}.$
Definition 2.5 ([28]): Let \( G' = (V', E') \) and \( G'' = (V'', E'') \) be any two graphs. Then the tensor product of \( G' \) and \( G'' \) is the graph \( G = G' \otimes G'' = (V, E, \sigma' \otimes \sigma'') \), where \( V = V' \times V'' \) and \( E = \{ ((u_1, v_1), (u_2, v_2)) | (u_1, v_1) \in E', (u_2, v_2) \in E'' \} \).

Definition 2.6 ([28]): Let \( G' = (V', E') \) and \( G'' = (V'', E'') \) be any two graphs. Then the normal product of \( G' \) and \( G'' \) is the graph \( G = G' \ast G'' = (V, E) \), where \( V = V' \times V'' \) and \( E = \{ ((u_1, v_1), (u_2, v_2)) | (u_1, v_1) \in E', (v_1, v_2) \in E'' \) or \( v_1 = v_2, (u_1, u_2) \in E \} \cup \{ ((u_1, v_1), (u_2, v_2)) | (v_1, v_2) \in E'', (u_1, u_2) \in E' \} \).

All the graphs considered in this work are simple finite and undirected graphs. For further studies of undefined terms see [27, 29].

2.2. Hesitant Fuzzy Sets

Torra [8] proposed a hesitant fuzzy set (HFS), which is a more general fuzzy set and permits the membership to include a set of possible values.

Definition 2.7: Torra [8]. Let \( X \) be a fixed set, a hesitant fuzzy set \( A \) on \( X \) is defined in terms of a function \( h_A(x) \) that when applied to \( x \) returns a finite subset of \([0, 1]\).

To be easily understood, Xia and Xu [30] expressed the HFS by a mathematical symbol: \( E = \{ x, h_E(x) > | x \in X \} \), where \( h_E(x) \) is a set of some values in \([0, 1]\) and it is hesitant fuzzy element (HFE), denoting the possible membership degrees of the element \( x \in X \) to the set \( E \).

Definition 2.8 ([31]): Let \( A = \{ a_1, a_2, \ldots, a_n \} \) be a finite subset of the unit interval and \( r \in \mathbb{N}, A_r \in [0, 1]^m \) is the vector of \( n \) coordinates defined as

\[
A_r = (\hat{a}_1, \ldots, \hat{a}_1, \ldots, \hat{a}_2, \ldots, \hat{a}_2, \ldots, \hat{a}_n, \ldots, \hat{a}_n)
\]

Example 2.9: Let \( A = \{0.1, 0.4, 0.5\} \) and \( r = 3 \). Then

\[
A_3 = (0.1, 0.1, 0.1, 0.4, 0.4, 0.4, 0.5, 0.5, 0.5).
\]

Definition 2.10 ([31]): Let \( A = \{ a_1, a_2, \ldots, a_n \} \) and \( B = \{ b_1, b_2, \ldots, b_m \} \) be two finite subsets of the unit interval and \( lcm(n, m) \) be the least common multiple of \( n \) and \( m \). Rewriting \( A_{\frac{lcm(n,m)}{n}} = (c_1, c_2, \ldots, c_{lcm(n,m)}) \) and \( B_{\frac{lcm(n,m)}{m}} = (d_1, d_2, \ldots, d_{lcm(n,m)}) \).

\( A \leq B \) if and only if \( c_i \leq d_i \) for all \( i = 1, 2, \ldots, lcm(n, m) \).

Example 2.11: Let \( A = \{0.1, 0.4, 0.5\} \) and \( B = \{0.5, 0.6\} \). Then \( lcm(2, 3) = 6 \), and

\[
A_2 = (0.1, 0.1, 0.4, 0.4, 0.5, 0.5),
\]

\[
B_3 = (0.5, 0.5, 0.5, 0.6, 0.6, 0.6).
\]

Therefore, \( A \leq B \).

Definition 2.12 ([32, Definition 4]): Let \( A = \{ a_1, a_2, \ldots, a_n \} \) and \( B = \{ b_1, b_2, \ldots, b_m \} \) be two HFS then \( A \oplus B = \bigcup_{a_i \in A, b_j \in B} \{ a_i + b_j \} \).

For some well known operations of HFS we ref [32].
3. Hesitant Fuzzy Graph

In this section, we define the notion of hesitant fuzzy graph and furnish some examples in its support. We also define degree of hesitant fuzzy graph and present some examples.

**Definition 3.1:** A hesitant fuzzy graph is of the form \( G = (V, E, \sigma, \mu) \), with \( \sigma : V \rightarrow S_f[0, 1] \) and \( \mu : E \rightarrow S_f[0, 1] \), where \( S_f[0, 1] \) is the collection of all finite subsets of \([0, 1]\), \( \sigma \) and \( \mu \) are the membership functions of vertex set and edge set of the hesitant fuzzy graph, respectively.

An example of HFG is given in Figure 1, where a network of 7 companies is discussed and the mutually collaborated companies for business are connected with edges and companies represented as vertices. The membership value of the vertices represent the market worth of companies and the membership value of the edges represent the market worth of the companies joint ventures.

**Definition 3.2:** Let \( A = \{a_1, a_2, \ldots, a_n\} \) and \( B = \{b_1, b_2, \ldots, b_m\} \) be two HFS and \( \text{lcm}(n, m) \) be the least common multiple of \( n \) and \( m \). Rewriting \( A_{\frac{\text{lcm}(n,m)}{n}} = (c_1, c_2, \ldots, c_{\text{lcm}(n,m)}) \) and \( B_{\frac{\text{lcm}(n,m)}{m}} = (d_1, d_2, \ldots, d_{\text{lcm}(n,m)}) \). \( A_{\frac{\text{lcm}(n,m)}{n}} \land B_{\frac{\text{lcm}(n,m)}{m}} = (\min(c_1, d_1), \min(c_2, d_2), \ldots, \min(c_{\text{lcm}(n,m)}, d_{\text{lcm}(n,m)})) \).

Convert this vector to a set by eliminating the repetition of element if found then this set is \( A \land B \).

**Proposition 3.3:** Let \( A, B, C \in S_f[0, 1] \). Then following are true:

1. \( A \land B = B \land A \).
2. \( A \land (B \land C) = (A \land B) \land C \).
3. If \( A \preceq B \), then \( A \land B = A \).

![Figure 1. Hesitant fuzzy graph.](image-url)
Proof: Easy to prove.

Proposition 3.4: Let \( A, B, C \in S_f[0, 1] \). Then following are true:

1. \( A \oplus B = B \oplus A \)
2. \( A \oplus (B \oplus C) = (A \oplus B) \oplus C \)

Proof: Straightforward.

Example 3.5: Following the Example 2.11,

\[
A_2 \land B_3 = (0.1, 0.1, 0.4, 0.4, 0.5, 0.5).
\]

Therefore, \( A \land B = \{0.1, 0.4, 0.5\} \). Also,

\[
A \oplus B = \{0.1 + 0.5\} \cup \{0.1 + 0.6\} \cup \{0.4 + 0.5\} \cup \{0.4 + 0.6\} \cup \{0.5 + 0.5\} \cup \{0.5 + 0.6\}
\]

Implies

\[
A \oplus B = \{0.6, 0.7, 0.9, 1, 1.1\}
\]

Definition 3.6: Let \( G = (V, E, \sigma, \mu) \) be a hesitant fuzzy graph. Then the degree of a vertex \( u \) in \( G \) is denoted by \( d_G(u) \) and defined as

\[
d_G(u) = \bigoplus_{(u,v) \in E} \mu(u,v),
\]

where \( u \neq v \).

Example 3.7: Consider the HFG given in Figure 1. Then degree of some of its vertices are given below:

\[
d_G(v_6) = [0.3, 0.4] \oplus [0.5, 0.6, 0.7] = [0.8, 0.9, 1, 1.1]
\]

\[
d_G(v_2) = [0.1, 0.2] \oplus [0.6, 0.8] = [0.7, 0.8, 0.9, 1]
\]

\[
d_G(v_4) = [0.1, 0.9]
\]

4. Some Products of Hesitant Fuzzy Graphs

In this section, we define different type of products of hesitant fuzzy graphs and the degree of vertices in these products. We establish certain constraint relations between the degrees of vertices of hesitant fuzzy graphs and the degrees of vertices in the hesitant fuzzy graphs of their products.

4.1. Cartesian Product of Hesitant Fuzzy Graphs

Definition 4.1: The Cartesian product of two hesitant fuzzy graphs \( G' = (V', E', \sigma', \mu') \) and \( G'' = (V'', E'', \sigma'', \mu'') \) is defined as a hesitant fuzzy graph \( G = G' \times G'' = (V, E, \sigma \times \mu) \).
Figure 2. Cartesian Product of two hesitant fuzzy graphs.

\[ \sigma'', \mu' \times \mu'' \], where \( V = V' \times V'' \) and \( E = \{ ((u_1, v_1), (u_2, v_2)) | u_1 = u_2, (v_1, v_2) \in E'' \text{ or } v_1 = v_2, (u_1, u_2) \in E' \} \) with

\[ (\sigma' \times \sigma'')(u, v) = \sigma'(u) \land \sigma''(v) \]

and

\[(\mu' \times \mu'')(u_1, v_1, u_2, v_2) = \begin{cases} 
\sigma'(u_1) \land \mu''(v_1, v_2) & \text{if } u_1 = u_2, (v_1, v_2) \in E'' \\
\mu'(u_1, u_2) \land \sigma''(v_1) & \text{if } v_1 = v_2, (u_1, u_2) \in E'.
\end{cases} \]

An example of the Cartesian product of two HFGs is given in Figure 2, where we consider two networks \( G' \) and \( G'' \) with respective three and two vertices representing different companies. The membership value of the vertices represent the market worth of companies and the membership value of the edges represent the market worth of the companies joint ventures. It is required for the companies of the two networks to look for future merger with each other depending on their market worths. In order to achieve an option of merger of companies Cartesian product of \( G' \) and \( G'' \) shows that the relationship of \( v_2 \) with \( u_2 \) and \( u_3 \). Membership values of \((u_2, v_2)\) and \((u_3, v_2)\) are the highest with the relationship among these combinations.

**Definition 4.2:** Let \( G = G' \times G'' = (V, E, \sigma' \times \sigma'', \mu' \times \mu'') \) be Cartesian product of two hesitant fuzzy graphs \( G' = (V', E', \sigma', \mu') \) and \( G'' = (V'', E'', \sigma'', \mu'') \). Then, the degree of the vertex \((u_1, v_1) \in V\) is denoted by \( d_G(u_1, v_1) \) and defined as

\[
d_G(u_1, v_1) = \bigoplus_{((u_1, v_1), (u_2, v_2)) \in E} (\mu' \times \mu'')(u_1, v_1, u_2, v_2)
\]

\[
d_G(u_1, v_1) = \begin{cases} 
\bigoplus_{u_1 = u_2, (v_1, v_2) \in E''} \sigma'(u_1) \land \mu''(v_1, v_2) & \text{if } v_1 = v_2, (u_1, u_2) \in E' \\
\bigoplus_{v_1 = v_2, (u_1, u_2) \in E'} \mu'(u_1, u_2) \land \sigma''(v_1)
\end{cases}
\]
In the following theorem, the relation between degrees of two HFGs and the degree of the HFG obtained by their Cartesian product is developed.

**Theorem 4.3:** Let $G = G' \times G'' = (V, E, \sigma', \mu, \mu' \times \mu'')$ be Cartesian product of two hesitant fuzzy graphs $G' = (V', E', \sigma', \mu')$ and $G'' = (V'', E'', \sigma'', \mu'')$. If $\sigma' \geq \mu''$ and $\sigma'' \geq \mu'$, then $d_G(u_1, v_1) = d_{G'}(u_1) \oplus d_{G''}(v_1)$.

**Proof:** From Definition 4.2 of degree of a vertex in Cartesian product of $G'$ and $G''$, we have

$$d_G(u_1, v_1) = \bigoplus_{(u_1, v_1) \in E'} (\mu' \times \mu'')(u_1, v_2)$$

$$d_G(u_1, v_1) = \left\{ \bigoplus_{u_1 = u_2, (v_1, v_2) \in E''} \sigma'(u_1) \wedge \mu''(v_1, v_2) \right\} \oplus \left\{ \bigoplus_{v_1 = v_2, (u_1, u_2) \in E'} \mu'(u_1, u_2) \wedge \sigma''(v_1) \right\}.$$

Since $\sigma' \geq \mu''$ and $\sigma'' \geq \mu'$. Therefore,

$$d_G(u_1, v_1) = \left\{ \bigoplus_{u_1 = u_2, (v_1, v_2) \in E''} \mu''(v_1, v_2) \right\} \oplus \left\{ \bigoplus_{v_1 = v_2, (u_1, u_2) \in E'} \mu'(u_1, u_2) \right\}.$$

The following example is given in support of the above theorem. 

**Example 4.4:** In Figure 2, we have $\sigma' \geq \mu''$ and $\sigma'' \geq \mu'$. Therefore, from Definition 4.2: $d_{G' \times G''}(u_2, v_2) = \{0.1, 0.2\} \oplus \{0.1\} \oplus \{0.3\} = \{0.1, 0.2\} \oplus \{0.4\} = \{0.5, 0.6\}$. Also from Definition 3.6 $d_{G'}(u_2) \oplus d_{G''}(v_2) = \{(0.1, 0.2) \oplus \{0.3\}\} \oplus \{(0.1)\} = \{0.4, 0.5\} \oplus \{0.1\} = \{0.5, 0.6\}$.

Consequently, $d_{G' \times G''}(u_2, v_2) = d_{G'}(u_2) \oplus d_{G''}(v_2)$.

### 4.2. Composition of Hesitant Fuzzy Graphs

**Definition 4.5:** The composition of two hesitant fuzzy graphs $G' = (V', E', \sigma', \mu')$ and $G'' = (V'', E'', \sigma'', \mu'')$ is defined as a hesitant fuzzy graph $G = G' \circ G'' = (V, E, \sigma' \circ \sigma'', \mu' \circ \mu'')$, where $V = V' \times V'' = V' \circ V''$ and $E = \{(u_1, v_1), (u_2, v_2)\} | u_1 = u_2, (v_1, v_2) \in E''$ or $v_1 = v_2, (u_1, u_2) \in E'$ with

$$(\sigma' \circ \sigma'')(u, v) = \sigma'(u) \land \sigma''(v)$$

and

$$(\mu' \circ \mu'')(u_1, v_2) = \begin{cases} \sigma'(u_1) \land \mu''(v_1, v_2) & \text{if } u_1 = u_2, (v_1, v_2) \in E'' \\ \mu'(u_1, u_2) \land \sigma''(v_1) & \text{if } v_1 = v_2, (u_1, u_2) \in E' \\ \mu'(u_1, u_2) \land \sigma'(v_1) \land \sigma''(v_2) & \text{if } v_1 \neq v_2, (u_1, u_2) \in E'. \end{cases}$$
Figure 3. Composition of two hesitant fuzzy graphs.

An example of composition of two HFGs is given in Figure 3.

**Definition 4.6:** Let $G = G' \circ G'' = (V, E, \sigma' \circ \sigma'', \mu' \circ \mu'')$ be the composition of two hesitant fuzzy graphs $G' = (V', E', \sigma', \mu')$ and $G'' = (V'', E'', \sigma'', \mu'')$. Then the degree of the vertex $(u_1, v_1)$ in $V$ is denoted by $d_G(u_1, v_1)$ and defined as

$$d_G(u_1, v_1) = \bigoplus_{((u_1, v_1), (u_2, v_2)) \in E} (\mu' \circ \mu'')(u_1, v_1), (u_2, v_2))$$

$$d_G(u_1, v_1) = \bigoplus_{u_1 = u_2, (v_1, v_2) \in E''} \sigma'(u_1) \wedge \mu''(v_1, v_2)$$

$$\oplus \bigoplus_{v_1 = v_2, (u_1, u_2) \in E'} \mu'(u_1, u_2) \wedge \sigma''(v_1)$$

$$\oplus \bigoplus_{v_1 \neq v_2, (u_1, u_2) \in E'} \mu'(u_1, u_2) \wedge \sigma'(v_1) \wedge \sigma''(v_2).$$

In the following theorem, relationship between degree of vertices in two HFGs and the degree of vertices of the composition of their is established.

**Theorem 4.7:** Let $G = G' \circ G'' = (V, E, \sigma', \sigma'', \mu', \mu'')$ be composition of two hesitant fuzzy graphs $G' = (V', E', \sigma', \mu')$ and $G'' = (V'', E'', \sigma'', \mu'')$. If $\sigma' \geq \mu''$ and $\sigma'' \geq \mu'$. Then, $d_G(u_1, v_1) = |V''|d_{G'}(u_1) \oplus d_{G''}(v_1)$.

**Proof:** From the definition of degree of the vertex in composition of $G'$ and $G''$, we have

$$d_G(u_1, v_1) = \bigoplus_{((u_1, v_1), (u_2, v_2)) \in E} (\mu' \circ \mu'')(u_1, v_1), (u_2, v_2))$$

$$d_G(u_1, v_1) = \bigg\{ \bigoplus_{u_1 = u_2, (v_1, v_2) \in E''} \sigma'(u_1) \wedge \mu''(v_1, v_2) \bigg\}$$

$$\oplus \bigg\{ \bigoplus_{v_1 = v_2, (u_1, u_2) \in E'} \mu'(u_1, u_2) \wedge \sigma''(v_1) \bigg\}$$

$$\oplus \bigg\{ \bigoplus_{v_1 \neq v_2, (u_1, u_2) \in E'} \mu'(u_1, u_2) \wedge \sigma'(v_1) \wedge \sigma''(v_2) \bigg\}.$$
\[ \bigoplus \left\{ \bigoplus_{v_1 \neq v_2, (u_1, u_2) \in E'} \mu'(u_1, u_2) \land \sigma'(v_1) \land \sigma''(v_2) \right\}. \]

Using the given constraints \( \sigma' \geq \mu'' \) and \( \sigma'' \geq \mu' \), we have

\[
d_G(u_1, v_1) = \bigg\{ \bigg\{ \bigoplus_{v_1 = v_2, (u_1, u_2) \in E'} \mu'(u_1, u_2) \bigg\} \bigg\} \bigoplus \bigg\{ \bigg\{ \bigoplus_{v_1 \neq v_2, (u_1, u_2) \in E'} \mu'(u_1, u_2) \bigg\} \bigg\} \bigoplus \bigg\{ \bigg\{ \bigoplus_{v_1 \neq v_2, (u_1, u_2) \in E'} \mu''(v_1, v_2) \bigg\} \bigg\}. 
\]

\[
d_G(u_1, v_1) = |V''|d_G(u_1) \oplus d_{G''}(v_1). 
\]

**Example 4.8:** In Figure 3, we have \( \sigma' \geq \mu'' \) and \( \sigma'' \geq \mu' \). Now from Definition 4.6;

\[
d_{G \circ G'}(u_2, v_2) = \{0.1, 0.2\} \oplus \{0.1, 0.2\} \oplus \{0.1\} \oplus \{0.3\} \oplus \{0.3\} = \{0.1, 0.2\} \oplus \{0.1, 0.2\} \oplus \{0.7\} = \{0.2, 0.3, 0.4\} \oplus \{0.7\} = \{0.9, 1.0, 1.1\} \]

Also, \( |V''|d_G(u_2) \oplus d_{G''}(v_2) = 2(\{0.1, 0.2\} \oplus \{0.3\} \oplus \{0.1\}) = 2(\{0.4, 0.5\}) \oplus \{0.1\} = \{0.4, 0.5\} \oplus \{0.4, 0.5\} \oplus \{0.1\} = \{0.8, 0.9, 1\} \oplus \{0.1\} = \{0.9, 1.0, 1.1\}. \]

Consequently, \( d_{G \circ G'}(u_2, v_2) = |V''|d_G(u_2) \oplus d_{G''}(v_2). \)

**4.3. Tensor Product of Hesitant Fuzzy Graphs**

**Definition 4.9:** The tensor product of two hesitant fuzzy graphs \( G' = (V', E', \sigma', \mu') \) and \( G'' = (V'', E'', \sigma'', \mu'') \) is defined as a hesitant fuzzy graph \( G = G' \ast G'' = (V, E, \sigma', \sigma'', \mu', \mu'') \), where \( V = V' \times V'' = V' \ast V'' \) and \( E = \{(u_1, v_1), (u_2, v_2)\} \) for all \((u, v) \in V \)

\[
(\sigma' \ast \sigma'')(u, v) = \sigma'(u) \land \sigma''(v) \text{ for all } (u, v) \in V.
\]

and

\[
(\mu' \ast \mu'')(u_1, u_2) = \mu'(u_1, u_2) \land \mu''(v_1, v_2) \text{ for all } (u_1, u_2) \in E', (v_1, v_2) \in E''.
\]

Figure 4 explains the tensor product of two hesitant fuzzy graphs.
Definition 4.10: Let $G = G' * G'' = (V, E, \sigma' * \sigma'', \mu' * \mu'')$ be the tensor product of two hesitant fuzzy graphs $G' = (V', E', \sigma', \mu')$ and $G'' = (V'', E'', \sigma'', \mu'')$. Then the degree of the vertex $(u_1, v_1)$ in $V$ is denoted by $d_G(u_1, v_1)$ and defined as

$$d_G(u_1, v_1) = \bigoplus_{((u_1, v_1), (u_2, v_2)) \in E'\& (v_1, v_2) \in E''} (\mu' * \mu'')(u_1, v_1), (u_2, v_2)).$$

The degree of vertices of two HFGs are related to the degree of vertices of the HFG of their tensor product in following theorem.

Theorem 4.11: Let $G = G' * G'' = (V, E, \sigma' * \sigma'', \mu' * \mu'')$ be tensor product of two hesitant fuzzy graphs $G' = (V', E', \sigma', \mu')$ and $G'' = (V'', E'', \sigma'', \mu'')$. If $\mu'' \succeq \mu'$, then $d_G(u_1, v_1) = |N_{G'}(u_1)|d_{G''}(v_1)$ and if $\mu' \succeq \mu''$ then $d_G(u_1, v_1) = |N_{G''}(v_1)|d_{G'}(u_1)$.

Proof: From the definition of degree of the vertex in tensor product of $G'$ and $G''$, we have

$$d_G(u_1, v_1) = \bigoplus_{((u_1, v_1), (u_2, v_2)) \in E'\& (v_1, v_2) \in E''} \mu'(u_1, u_2) \land \mu''(v_1, v_2).$$

From the given constraints $\mu'' \succeq \mu'$ and Definition 3.6, we have,

$$d_G(u_1, v_1) = |N_{G'}(u_1)|\bigoplus_{(u_1, u_2) \in E'} \mu'(u_1, u_2).$$

Similarly, from $\mu' \succeq \mu''$, and Definition 3.6 we have

$$d_G(u_1, v_1) = |N_{G''}(v_1)|d_{G'}(u_1).$$

The following example elaborates the above theorem.
Example 4.12: In Figure 4, we have $\mu' \geq \mu''$. From Definition 4.1; $d_{G' \ast G''}(u_2, v_2) = \{0.1\} \oplus \{0.1\} = \{0.2\}$. Also, $|N_{G' \ast G''}(u_2, v_2)|d_{G''}(v_2) = 2(\{0.1\}) = \{0.2\}$. Consequently, $d_{G' \ast G''}(u_2, v_2) = |N_{G' \ast G''}(u_2, v_2)|d_{G''}(v_2)$. 

4.4. Normal Product of Hesitant Fuzzy Graphs

Definition 4.13: The normal product of two hesitant fuzzy graphs $G' = (V', E', \sigma', \mu')$ and $G'' = (V'', E'', \sigma'', \mu'')$ is defined as a hesitant fuzzy graph $G = G' \ast G'' = (V, E, \sigma' \ast \sigma'', \mu' \ast \mu'')$, where $V = V' \times V''$ and $E = \{(u_1, v_1), (u_2, v_2)\} | u_1 = u_2, (v_1, v_2) \in E''$ or $v_1 = v_2, (u_1, u_2) \in E'$ with

$$(\sigma' \ast \sigma'')(u, v) = \sigma'(u) \land \sigma''(v)$$

and

$$(\mu' \ast \mu'')(u_1, v_1), (u_2, v_2)) = \begin{cases} \sigma'(u_1) \land \mu''(v_1, v_2) & \text{if } u_1 = u_2, (v_1, v_2) \in E'' \\ \mu'(u_1, u_2) \land \sigma''(v_1) & \text{if } v_1 = v_2, (u_1, u_2) \in E' \\ \mu'(u_1, u_2) \land \mu''(v_1, v_2) & \text{if } (v_1, v_2) \in E'', (u_1, u_2) \in E'. \end{cases}$$

The normal product of two HFGs is given in Figure 5.

Definition 4.14: Let $G = G' \ast G'' = (V, E, \sigma' \ast \sigma'', \mu' \ast \mu'')$ be the normal product of two hesitant fuzzy graphs $G' = (V', E', \sigma', \mu')$ and $G'' = (V'', E'', \sigma'', \mu'')$. Then, the degree of the vertex $(u_1, v_1)$ in $V$ is denoted by $d_G(u_1, v_1)$ and defined as

$$d_G(u_1, v_1) = \bigoplus_{((u_1,v_1),(u_2,v_2)) \in E} (\mu' \ast \mu'')(u_1, v_1), (u_2, v_2))$$

$$d_G(u_1, v_1) = \bigoplus_{u_1 = u_2, (v_1, v_2) \in E''} \sigma'(u_1) \land \mu''(v_1, v_2)$$

$$\bigoplus_{v_1 = v_2, (u_1, u_2) \in E'} \mu'(u_1, u_2) \land \sigma''(v_1)$$

Figure 5. Normal product of two hesitant fuzzy graphs.
The degree of vertices in the normal product of two HFGs are related to the degree of vertices of the HFG in following theorem.

**Theorem 4.15:** Let \( G = G' \bullet G'' = (V, E, \sigma', \sigma'', \mu', \mu'') \) be normal product of two hesitant fuzzy graphs \( G' = (V', E', \sigma', \mu') \) and \( G'' = (V'', E'', \sigma'', \mu'') \). If \( \sigma' \geq \mu'' \) and \( \sigma'' \geq \mu', \mu'' \geq \mu' \). Then \( d_G(u_1, v_1) = |V''|d_{G'}(u_1) \oplus d_{G''}(v_1) \).

**Proof:** From the definition of degree of the vertex in normal product of two HFGs \( G' \) and \( G'' \), we have

\[
\begin{align*}
d_G(u_1, v_1) &= \bigoplus_{(u_1, v_1), (v_2, v_2) \in E} (\mu' \bullet \mu'')(u_1, v_1, (u_2, v_2)) \\
d_G(u_1, v_1) &= \left\{ \bigoplus_{u_1 = u_2, (v_1, v_2) \in E''} \sigma'(u_1) \land \mu''(v_1, v_2) \right\} \\
&\quad \bigoplus \left\{ \bigoplus_{v_1 = v_2, (u_1, u_2) \in E'} \mu'(u_1, u_2) \land \mu''(v_1, v_2) \right\} \\
&\quad \bigoplus \left\{ \bigoplus_{(u_1, u_2) \in E'} \mu'(u_1, u_2) \right\}
\end{align*}
\]

Since the given constraint are \( \sigma' \geq \mu'' \) and \( \sigma'' \geq \mu' \). Therefore,

\[
\begin{align*}
d_G(u_1, v_1) &= \left\{ \bigoplus_{(v_1, v_2) \in E''} \mu''(v_1, v_2) \right\} \\
&\quad \bigoplus \left\{ \bigoplus_{v_1 = v_2, (u_1, u_2) \in E'} \mu'(u_1, u_2) \right\} \\
&\quad \bigoplus \left\{ \bigoplus_{(u_1, u_2) \in E'} \mu'(u_1, u_2) \right\}
\end{align*}
\]

The relation discussed between degrees in above theorem is explained in the following example.
Example 4.16: In Figure 5, we have $\sigma' \geq \mu''$ and $\sigma'' \geq \mu'$, $\mu'' \geq \mu'$. From Definition 4.14; $d_{G' \cdot G''}(u_2, v_2) = \{0.1, 0.2\} \oplus \{0.1, 0.2\} \oplus \{0.5\} \oplus \{0.3\} \oplus \{0.3\} = \{0.1, 0.2\} \oplus \{0.1, 0.2\} \oplus \{1.1\}$ $\equiv \{0.2, 0.3, 0.4\} \oplus \{1.1\} = \{1.3, 1.4, 1.5\}$. Now $|V''|d_G(u_2) \oplus d_{G''}(v_2) = 2\{0.4, 0.5\} \oplus \{0.5\}$ $\equiv \{0.4, 0.5\} \oplus \{0.4, 0.5\} \oplus \{0.5\} = \{0.8, 0.9, 1\} \oplus \{0.5\} = \{1.3, 1.4, 1.5\}$.

Consequently, $d_{G' \cdot G''}(u_2, v_2) = |V''|d_G(u_2) \oplus d_{G''}(v_2)$.

5. Real Life Application

Here, we include a real life problem of optimising companies merger problem. As an illustrative case, consider two networks $G'$ and $G''$ with respective three and two vertices representing different companies as shown in Figure 2. The membership value of the vertices represent the market worth of companies and the membership value of the edges represent the market worth of the companies joint ventures. It is required for the companies of the two networks to look for future merger with each other depending on their market worths. In order to achieve an optimal merger of companies different network(graph) products are presented here. The resultant graph will equip the companies owners to optimally decide the merger out of all possibilities.

As in Figure 2, Cartesian product of $G'$ and $G''$ shows that the relationship of $v_2$ with $u_2$ and $u_3$ is the most suitable combination with respect to membership values and vertices and edges. Membership values of $(u_2, v_2)$ and $(u_3, v_2)$ are the highest with the relationship among these combinations.

As in Figure 3, composition of $G'$ and $G''$ shows that the relationships of $v_2$, $v_1$ with $u_2$ and $u_3$ are the most suitable combination with respect to membership values and vertices and edges. Membership values of $(u_2, v_2)$, $(u_3, v_2)$ and $(u_2, v_1)$, $(u_3, v_1)$ are the highest with the relationship among these combinations (membership values of edges).

As in Figure 4, Tensor product of $G'$ and $G''$ shows that the relationships of $v_2$, $v_1$ and $u_2$, $u_3$, respectively are the most suitable combination with respect to membership values and vertices and edges. Membership values of $(u_2, v_2)$, $(u_3, v_1)$ are the highest with the relationship among these combinations (membership value of edge).

As in Figure 5, Normal product of $G'$ and $G''$ shows that the relationships of $v_2$, $v_1$ with $u_2$ is the most suitable combination with respect to membership values and vertices and edges. Membership values of $(u_2, v_2)$, $(u_2, v_1)$ are the highest with the relationship among these combinations (membership value of edge).

So from the results discussed in these products we can see that the different products generate the different results of combination of companies. With the help of these structures someone can judge the which kind of linkage should be developed to enhance the stability and progress of several companies in a field.

6. Conclusion and Future Work

Fuzzy graph theory has numerous applications in modern sciences and technology, especially in the fields of operations research, neural networks, artificial intelligence and decision making. Hesitant fuzzy set is a generalisation of the notion of fuzzy set. The hesitant fuzzy models give more precision, flexibility and compatibility to the system as compared to the classical and fuzzy models. We have introduced the concepts of certain products in hesitant fuzzy graphs in this paper. We plan to extend our research of hesitant fuzzy graphs
to coloring problems, shortest path problems, labelling of hesitant fuzzy graphs, hesitant hypergraphs and roughness. In future, real life applications of hesitant fuzzy graphs can be explored, like; decision making, clustering, data mining and networking.

This work can be generalised for several different popular extensions of fuzzy sets and their real life applications can be developed according to the need of research and problems.

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