Multiple Vertex Coverings by Specified Induced Subgraphs

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Abstract

Given graphs $H_1, \ldots, H_k$, let $f(H_1, \ldots, H_k)$ be the minimum order of a graph $G$ such that for each $i$, the induced copies of $H_i$ in $G$ cover $V(G)$. We prove constructively that $f(H_1, H_2) \leq 2(n(H_1) + n(H_2) - 2)$; equality holds when $H_1 = \overline{K}_2 = K_n$. We prove that $f(H_1, K_n) = n + 2\sqrt{\delta(H_1)n} + O(1)$ as $n \to \infty$. We also determine $f(K_{1,m-1}, K_n)$ exactly.

1 Introduction

Entringer, Goddard, and Henning [2] determined the minimum order of a simple graph in which every vertex belongs to both a clique of size $m$ and an independent set of size $n$. They obtained a surprisingly simple formula for this value, which they called $f(m, n)$ (an alternative proof using matrix theory appears in §).

Theorem 1.1 [2] For $m, n \geq 2$, $f(m, n) = \left\lceil (\sqrt{m} - 1 + \sqrt{n} - 1)^2 \right\rceil$.

Theorem 1.1 was motivated by a concept introduced by Chartrand et al. [1] called the framing number. A graph $H$ is homogeneously embeddable in a graph $G$ if, for all vertices $x \in V(H)$ and $y \in V(G)$, there exists an embedding of $H$ into $G$ as an induced subgraph that maps $x$ to $y$. The framing number $fr(H)$ is the minimum order of a graph in which $H$ is homogeneously embeddable. The framing number of a pair of graphs $H_1$ and $H_2$, written $fr(H_1, H_2)$, is the minimum order of a graph $G$ in which both $H_1$ and $H_2$ are homogeneously embeddable.

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embeddable. Thus \( fr(K_m, K_n) = f(m, n) \). Various results about the framing number were developed in [1]. The framing number of a pair of cycles is studied in [1].

When the graphs to be homogeneously embedded are vertex-transitive, it matters not which vertex of \( H \) is mapped to \( y \in V(G) \) as long as \( y \) belongs to some induced copy of \( H \) in \( G \). Determining the framing number for a pair of graphs becomes an extremal graph covering problem. We generalize this variation to more than two graphs.

**Definition 1.2** A graph is \((H_1, \ldots, H_k)\)-full if each vertex belongs to induced subgraphs isomorphic to each of \( H_1, \ldots, H_k \). We use \( f(H_1, \ldots, H_k) \) to denote the minimum order of an \((H_1, \ldots, H_k)\)-full graph.

Equivalently, a graph is \((H_1, \ldots, H_k)\)-full if for each \( i \), the induced subgraphs isomorphic to \( H_i \) cover the vertex set, so we think in terms of multiple coverings of the vertex set.

Because every vertex in a cartesian product belongs to induced subgraphs isomorphic to each factor, we have \( f(H_1, \ldots, H_k) \leq \prod_i n(H_i) \), where \( n(G) \) denotes the order of \( G \). In fact, \( f(H_1, \ldots, H_k) \) is much smaller. Our constructions in Section 2 yield \( f(H_1, \ldots, H_k) \leq 2 \sum_i (n(H_i) - 1) \). Also, if \( k - 1 \) is a prime power and \( n(H_i) < k \) for each \( i \), then \( f(H_1, \ldots, H_k) \leq (k - 1)^2 \). By Theorem [1], the first construction is optimal when \( k = 2 \) for \( H_1 = K_n \) and \( H_2 = K_n \). We also provide a construction when \( H_1 \) is arbitrary and \( H_2 = K_n \) that is asymptotically sharp up to an additive constant.

In Section 3, we prove a general lower bound in terms of the order of \( H_2 \), the maximum degree of \( H_2 \), and the minimum degree of \( H_1 \). In Section 4, we determine \( f(K_{1,m-1}, K_n) \) exactly (the related parameter \( f(K_m, K_n) \) is studied in [1]). In Section 5, we present several open problems.

Since \( f(H_1, \ldots, H_k) = f(\overline{H_1}, \ldots, \overline{H_k}) \), all our results yield corresponding results for complementary conditions. We note also that there is an \((H_1, \ldots, H_k)\)-full graph for each order exceeding the minimum, since duplicating a vertex in such a graph yields another \((H_1, \ldots, H_k)\)-full graph.

We consider only simple graphs, denoting the vertex set and edge set of a graph \( G \) by \( V(G) \) and \( E(G) \), respectively. The order of \( G \) is \( n(G) = |V(G)| \). We use \( N_G(v) \) for the neighborhood of a vertex \( v \in V(G) \) (the set of vertices adjacent to \( v \)), and we let \( N_G[v] = N_G(v) \cup \{v\} \). The degree of \( v \) is \( d_G(v) = |N_G(v)| \); we may drop the subscript \( G \). For \( S \subseteq V(G) \), we write \( d_S(v) \) for \( |N_G(v) \cap S| \). The independence number of \( G \) is the maximum size of a subset of \( V(G) \) consisting of pairwise nonadjacent vertices; it is denoted by \( \alpha(G) \). When \( S \subseteq V(G) \), we let \( N(S) = \bigcup_{v \in S} N(v) \) and let \( G[S] \) denote the subgraph induced by \( S \).

## 2 General Upper Bounds

Our upper bounds are constructive.

**Theorem 2.1** If \( H_1, \ldots, H_k \) are graphs, then \( f(H_1, \ldots, H_k) \leq 2 \sum_{i=1}^k (n(H_i) - 1) \).

**Proof:** We construct an \((H_1, \ldots, H_k)\)-full graph \( G \) with \( 2 \sum_{i=1}^k (n(H_i) - 1) \) vertices. For \( 1 \leq r \leq k \), let \( H_{r+k} \) be a graph isomorphic to \( H_r \). For \( r \in \{1, \ldots, 2k\} \), distinguish a
vertex \( u_r \) in \( H_r \), and let \( N_r = N_{H_r}(u_r) \) and \( H'_r = H_r - u_r \). Construct \( G \) from the disjoint union \( H'_1 + \cdots + H'_{2k} \) by adding, for each \( r \), edges making all of \( V(H'_r) \) adjacent to all of \( N_{r+1} \cup \cdots \cup N_{r+k-1} \), where the indices are taken modulo \( 2k \).

By construction, \( G \) has the desired order. For \( v \in V(H'_r) \) and \( 1 \leq j \leq k-1 \), we have \( G[v \cup V(H'_{r+j})] \cong H_{r+j} \) (again taking indices modulo \( 2k \)). Finally, \( V(H'_{r-1}) \) together with any vertex of \( V(H'_{r-1}) \) induces a copy of \( H_r \) containing \( v \).

Fig. 1 illustrates the construction of Theorem 2.1 in the case \( k = 2 \); an edge to a circle indicates edges to all vertices in the corresponding set.

As mentioned earlier, Theorem 2.1 yields sharp upper bounds when \( k = 2 \) by letting \( H_1 = K_n \) and \( H_2 = \overline{K}_n \). In general, as pointed out by a referee, the bounds can be off from the optimal by at least a factor of two. To describe the construction that improves Theorem 2.1 in some cases, we use resolvable designs. We phrase the constructions in the language of hypergraphs. A hypergraph \( H = (V, E) \) has vertex set \( V \) and edge set \( E \) consisting of subsets of \( V \). \( H \) is \( k \)-uniform if every edge has size \( k \), and \( H \) is \( k \)-regular if every vertex lies in exactly \( k \) edges. A matching \( M \) in \( H \) is a set of pairwise disjoint edges; \( M \) is perfect if the union of its elements is \( V \).

A Steiner system \( S(n, k, 2) \) is an \( n \)-vertex \( k \)-uniform hypergraph in which every pair of vertices appears together in exactly one edge. It is resolvable if the edges can be partitioned into perfect matchings. Ray-Chaudhuri and Wilson [8] showed that the trivial necessary condition \( n \equiv k \pmod{k^2-k} \) for the existence of a resolvable \( S(n, k, 2) \) is also sufficient when \( n \) is sufficiently large compared to \( k \).

**Theorem 2.2** If a resolvable Steiner system \( S(n, k-1, 2) \) exists and \( H_1, \ldots , H_t \) are graphs of order less than \( k \), where \( t \leq (n-1)/(k-2) \), then \( f(H_1, \ldots , H_t) \leq n \).

**Proof:** Duplicating vertices cannot decrease \( f \), so we may assume that \( n(H_i) = k-1 \) for each \( i \). Let \( V \) and \( E \) be the vertex set and edge set of the resolvable Steiner system
$S(n, k - 1, 2)$; we construct a graph $G$ on vertex set $V$. For $1 \leq i \leq t$, consider the $i$th perfect matching $M_i$ consisting of edges $E_i^1, \ldots, E_i^{n/(k-1)}$. For $j = 1, \ldots, n/(k-1)$, add edges within each $E_i^j$ to make a copy of $H_i$.

Since every pair of vertices lies in only one edge of $S(n, k - 1, 2)$, this construction is well defined. To see that the construction is $H_i$-full, consider an arbitrary $v \in V$. Exactly one of the $t$ edges containing $v$ belongs to the $i$th matching. This edge forms a copy of $H_i$ containing $v$.

In the special case when $n = (k - 1)^2$, such a resolvable Steiner system is an affine plane, denoted $\mathcal{H}_{k-1}$. It is well known (see, e.g., page 672 or [2], for example) that an affine plane $\mathcal{H}_{k-1}$ exists when $k - 1$ is a power of a prime. This yields the following.

**Corollary 2.3** If $\mathcal{H}_{k-1}$ exists and $n(H_i) < k$ for each $i$, then $f(H_1, \ldots, H_k) \leq (k - 1)^2$.

When $n(H_i) = k - 1$ for each $i$, Corollary 2.3 improves the bound in Theorem 2.1 (asymptotically) by a factor of two. When $k = 2$ and $H_2 = \overline{K}_n$, a slightly different construction gives nearly optimal bounds for each $H_i$ as $n \to \infty$. In Theorem 3.2, we shall prove that this construction is asymptotically optimal.

**Theorem 2.4** If $H$ has order $m$ and positive minimum degree $\delta$, then $f(H, \overline{K}_n) < n + 2\sqrt{dn} + 2\delta$ when $n \geq 9\delta(m - \delta - 1)^2$.

**Proof:** Let $x$ be a vertex of minimum degree $\delta$ in $H$. We construct an $(H, \overline{K}_n)$-full graph $G$ in terms of a parameter $r$ that we optimize later. Let $V(G) = U \cup W$, where $U = U_1 \cup \cdots \cup U_r$ and $W = W_1 \cup \cdots \cup W_r$. Let $W$ be an independent set of size $n - 1 + s$, where $s = \lfloor n/(r-1) \rfloor$. Let each $W_i$ have size $s - 1$ or $s$ (set $|W_i| = s - 1$ and put the remaining $n$ vertices equitably into $r - 1$ sets). For each $i$, set $G[U_i] \cong H[N(x)]$, and make all of $U_i$ adjacent to all of $W_i$.

Each $U_i \cup w$ with $w \in W_i$ induces $N_H[x]$; we add edges to complete copies of $H$. Let $m' = m - \delta - 1$. For $j \in \{1, 2, 3\}$, let $T_j$ consist of $m'$ vertices, one chosen from each of $U_{(j-1)m'+1}, \ldots, U_{jm'}$. This requires $r \geq 3m'$. Add edges within each $T_j$ so that $G[T_j] \cong H - N[x]$. For each $U_i$ that contains a vertex of $T_j$, add edges from $U_i$ to $T_{j+1}$ (indices modulo 3 here) so that $G[U_i \cup T_{j+1}] \cong H - x$. For $3m' + 1 \leq i \leq r$, add edges from $U_i$ to $T_1$ so that $G[U_i \cup T_1] \cong H - x$. This completes the construction of $G$, as sketched in Fig. 2; dots represent the vertices of $\bigcup T_j$, and arrows suggest the edges from $U_i$ to $T_{j+1}$.

To show that $G$ is $(H, \overline{K}_n)$-full, it suffices to consider $u \in U_i$ and $w \in W_i$. By construction, we have $G[\{w\} \cup U_i \cup T_j] \cong H$ for some $j$. The vertices of $W - W_i$ together with $u$ or $w$ form an independent set of size at least $n + s - 1 - s + 1 = n$.

We now choose $r$ to minimize the order of $G$, which equals $n - 1 + r + \lfloor n/(r-1) \rfloor$ can be made. This satisfies the requirement that $r \geq 3m'$ when $n \geq 9\delta(m - \delta - 1)^2$. With this value of $r$, the order of $G$ is at most $n + \delta(2 + \sqrt{n/\delta}) + \sqrt{dn}$, which equals the bound claimed.

In the optimized construction, each $|W_i|$ is about $r|U_i|$. This reflects the use of $W$ to form the large independent set. When $n$ is smaller than $9\delta(m')^2$, we still obtain an
improvement on Theorem 2.1 by setting \( r = 3m' \), where \( m' = m - \delta - 1 \). The resulting \((H, K_n)\)-full graph has order \( n - 1 + \lceil n/(3m' - 1) \rceil + 3\delta m' \), which is less than \( 2(n + m) \) when \( n \) is bigger than about \( 3\delta m' \).

3 A Lower Bound

In this section we prove a lower bound that holds when the maximum degree of \( H_2 \) is less than half the minimum degree of \( H_1 \).

**Theorem 3.1** Let \( H_1 \) and \( H_2 \) be graphs such that \( H_1 \) has minimum degree \( \delta \), and \( H_2 \) has order \( n \) and maximum degree \( \Delta \). If \( 2\Delta < \delta \), then

\[
f(H_1, H_2) \geq n + \left[ 2\sqrt{(n + \Delta)(\delta - 2\Delta)} \right] - (\delta - \Delta).
\]

**Proof:** Let \( G \) be an \((H_1, H_2)\)-full graph, and choose \( A \subset V(G) \) such that \( G[A] \cong H_2 \). Let \( v \) be a vertex in \( V(G) - A \) with the most neighbors in \( A \). Since \( G \) is \((H_1, H_2)\)-full, \( v \) belongs to a set \( B \subset V(G) \) such that \( G[B] \cong H_2 \). Let \( C = V(G) - (A \cup B) \). Let \( k = |A - B| \); we obtain a lower bound on \( |C| \) in terms of \( k \).

Let \( e \) be the number of edges with endpoints in both \( C \) and \( A \cap B \), and let \( d = |N(v) \cap A| \). Our lower bound on \( C \) arises from the computation below. The first inequality counts \( e \) by the \( n - k \) endpoints in \( A \cap B \); each lies in a copy of \( H_1 \) but has at most \( 2\Delta \) neighbors outside \( C \). The second inequality counts \( e \) by the endpoints in \( C \), using the choice of \( v \).
For the third inequality, note that $v$ has at most $\Delta$ neighbors in $B$ and then at most $k$ more in $A - B$.

$$(n - k)(\delta - 2\Delta) \leq e \leq d|C| \leq (k + \Delta)|C|.$$  

Using the resulting lower bound on $|C|$, we have

$$|V(G)| = |A \cup B| + |C| \geq n + k + \frac{(n - k)(\delta - 2\Delta)}{k + \Delta} = n - (\delta - \Delta) + (k + \Delta) + \frac{(n + \Delta)(\delta - 2\Delta)}{k + \Delta}.$$  

This expression is minimized by $k + \Delta = \sqrt{(n + \Delta)(\delta - 2\Delta)}$, yielding the desired bound.  

Corollary 3.2 If $H_1$ has minimum degree $\delta$, then $f(H_1, K_n) = n + 2\sqrt{\delta n} + O(1)$ as $n \to \infty$.

Proof: For $\delta > 0$, the upper bound follows from Theorem 2.4, while the lower bound follows by setting $H_2 = \overline{K_n}$ in Theorem 3.1. Now suppose that $\delta = 0$ and let $m = n(H_1)$. Let $\alpha(G, v)$ denote the maximum size of an independent set containing vertex $v$ in a graph $G$. Let $s = \min_{v \in V(H_1)} \alpha(H_1, v)$.

We claim that $f(H_1, \overline{K_n}) = n - s + m$ for $n \geq s$. For the lower bound, let $u$ be a vertex of $H_1$ such that $s = \alpha(H_1, u)$. Completing an independent $n$-set for a vertex playing the role of $u$ in a copy of $H_1$ requires adding at least $n - s$ vertices to the $m$ vertices of $H_1$. Since $H_1$ has at least one isolated vertex, adding these as isolated vertices yields an $(H_1, \overline{K_n})$-full graph, thus proving the upper bound also.

By taking complements, one immediately obtains the following corollary.

Corollary 3.3 If $\overline{H_1}$ has minimum degree $\delta$, then $f(H_1, K_n) = n + 2\sqrt{\delta n} + O(1)$ as $n \to \infty$.

4 Stars versus Independent Sets

In this section we determine $f(H_1, H_2)$ when $H_1$ is a star of order $m$ and $H_2$ is an independent set of order $n$. Let $S_m = K_{1, m-1}$. The problem is rather easy when $n < m$.

Claim 4.1 For $n < m$, $f(S_m, \overline{K_n}) = n + m - 1$, achieved by $K_{n, m-1}$.

Proof: The center of an $m$-star must lie in an independent $n$-set avoiding its neighbors, so $f(S_m, \overline{K_n}) \geq n + m - 1$ for all $n$. When $n < m$, the graph $K_{n, m-1}$ is $(S_m, \overline{K_n})$-full.

The problem behaves much differently when $n \geq m$. First we provide a construction.

Lemma 4.2 For $n \geq m \geq 2$,

$$f(S_m, \overline{K_n}) \leq n + \min_k \max \left\{ k + \left\lceil \frac{n - 1}{k} \right\rceil, 2m - 3 - k \right\}.$$  

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Proof: We define a construction $G$ with parameters $r$ and $k$. Let $V(G)$ be the disjoint union of $X$ and $Y$, where $|X| = r$ and $|Y| = n - 1 + k$. Let $G[X] = K_{\lceil r/2 \rceil, \lfloor r/2 \rfloor}$, and let $Y$ be an independent set. Give $k$ neighbors in $Y$ to each vertex in $X$, arranged so that $G$ is bipartite and has no isolated vertices.

With $k \geq 1$, the size chosen for $Y$ ensures that each vertex lies in an independent $n$-set. Keeping $G$ bipartite requires $n - 1 \geq k$. This ensures that each vertex of $X$ lies at the center of an induced star of order $k + 1 + \lceil r/2 \rceil$. Thus we require

$$r/2 \geq m - 1 - k.$$

(A)

Ensuring that the stars cover $Y$ requires

$$(r - 1)k \geq n - 1.$$  

(B)

Given $n \geq m \geq 2$, we choose $r, k$ to minimize $n - 1 + k + r$, the order of $G$. Rewrite (A) as $r - 1 \geq 2m - 3 - 2k$. Both (A) and (B) impose lower bounds on $r - 1$ in terms of $k, m, n$; we set $r - 1 = \max\{\lceil (n - 1)/k \rceil, 2m - 3 - 2k\}$. This yields the one-variable minimization in the statement of the lemma.

In fact, the construction of Lemma 4.2 is optimal for all $n \geq m$. We begin the proof of optimality with a lower bound that differs from the upper bound by at most 1.

Lemma 4.3 For $n \geq m \geq 2$,

$$f(S_m, \overline{K}_n) \geq n + \min_d \max \left\{d - 1 + \left\lceil \frac{n}{d} \right\rceil, 2m - 2 - d \right\}.$$

Proof: We strengthen the general argument of Theorem 3.1. Let $G$ be an $(S_m, \overline{K}_n)$-full graph. Let $d$ be the maximum of $|N(v) \cap T|$ such that $v \in V(G)$ and $T$ is an independent $n$-set in $G$. Let $A$ be an independent $n$-set and $x$ a vertex such that $|N(x) \cap A| = d$.

As in the proof of Theorem 3.1, we choose $B$ to be an independent $n$-set containing $x$, let $C = V(G) - (A \cup B)$, and let $k$ be the size of $A - B$. With $\delta = 1$ and $\Delta = 0$, the argument applied there to the edges joining $C$ and $A \cap B$ yields

$$n - k \leq d|C| \leq k|C|.$$
Since \( d \leq k \), we obtain \( |V(G)| \geq n + d - 1 + \lfloor n/d \rfloor \).

To complete the proof, we must show that \( |V(G)| \geq n + 2m - 2 - d \). As observed in the proof of Claim 4.1, \( f(S_m, \overline{K_n}) \geq n + m - 1 \) always. Thus we may assume that \( d < m - 1 \). In proving a lower bound, we may also assume that \( G \) is a minimal \( (S_m, \overline{K_n}) \)-full graph. In particular, if we delete any edge of \( G \), then the resulting graph is not \( S_m \)-full. Let \( R_1, \ldots, R_t \) be a collection of induced stars of order at least \( m \) that cover \( V(G) \). By the minimality of \( G \), the vertices that are not centers of these stars form an independent set.

We consider two cases.

Case 1: The centers of \( R_1, \ldots, R_t \) form an independent set. In this case, \( G \) is a bipartite graph with bipartition \( X, Y \), where \( X \) is the set of centers of \( R_1, \ldots, R_t \) and \( Y \) is the set of leaves of \( R_1, \ldots, R_t \). By the definition of \( d \) and the restriction to \( d < m - 1 \), we have \( |Y| < n \).

Let \( x \) be the center of \( R_1 \), let \( I \) be an independent \( n \)-set containing \( x \), and let \( j = |I \cap X| \).

Each vertex of \( I \cap X \) has at least \( m - 1 \) neighbors in \( Y - I \). Since \( |Y - I| < n - (n-j) = j \) and there are at least \( j(m-1) \) edges from \( I \cap X \) to \( Y - I \), some \( y \in Y - I \) is incident to at least \( m - 1 \) of these edges. This gives \( y \) at least \( m - 1 > d \) neighbors in \( I \), contradicting the choice of \( d \). Thus this case cannot occur when \( d < m - 1 \).

Case 2. The centers of \( R_1, \ldots, R_t \) do not form an independent set. By the minimality of \( G \), each edge of \( G \) is needed to complete some induced star of order at least \( m \) centered at one of its endpoints. We may assume that the centers \( x \) of \( R_1 \) and \( y \) of \( R_2 \) are adjacent and that \( R_1 \) needs the edge \( xy \) to reach order \( m \). This implies that \( y \) is not adjacent to any leaf of \( R_1 \). In particular, the \( m - 2 \) or more additional vertices that complete \( R_2 \) are distinct from those in \( R_1 \), and \( |V(R_1) \cup V(R_2)| \geq 2m - 2 \).

Now let \( I \) be an independent \( n \)-set containing \( x \). The vertices of \( R_1 \cup R_2 \) in \( I \) are all neighbors of \( y \), and hence there are at most \( d \) of them. Thus \( |V(G)| \geq n - d + 2m - 2 \).

When \( d \) in the formula of Lemma 4.3 equals \( k \) in the formula of Lemma 4.2, the resulting values differ by at most one. A closer look at the one-variable optimization shows that the lower bound and the upper bound differ by at most one.

**Theorem 4.4** For \( n \geq m \geq 2 \), the construction of Lemma 4.2 is optimal.

**Proof:** We prove that the lower bound of Lemma 4.3 can be improved to match the upper bound of Lemma 4.2.

Choose \( A, B, C, d, k \) as in the proof of Lemma 4.3. If \( d \leq k - 1 \) or if there are at most \( (d-1)|C| \) edges between \( C \) and \( A \cap B \), then we obtain \( |C| \geq (n-k)/(k-1) \), which yields \( |V(G)| \geq n + k - 1 + (n-1)/(k-1) \). Also \( 2m - 2 - d \geq 2m - 2 - k \). Setting \( k' = k - 1 \) now yields \( |V(G)| \geq n + \max\{k' + \lfloor (n-1)/k' \rfloor, 2m-3-k' \} \). Hence the construction is optimal unless there is another construction satisfying \( d = k \) and having more than \( (d-1)|C| \) edges between \( C \) and \( A \cap B \) (thus there is a \( z \in C \) with \( d_{A \cap B}(z) \geq d \)).

More precisely, for every independent set \( A \) of size \( n \), every vertex \( x \notin A \) with \( d_A(x) = d \), and every independent set \( B \) of size \( n \) containing \( x \), the following holds:

\[
B \supseteq A - N(x) \quad (\ast)
\]

Choose \( z \in C \) with \( d_{A \cap B}(z) = d \), and let \( B' \) be an independent set of size \( n \) containing \( z \). Letting \( (z, A, B') \) play the role of \( (x, A, B) \) in \( (\ast) \) implies that \( B' \supseteq A - N(z) \supseteq \)
A − B. On the other hand, letting \((z, B, B')\) play the role of \((x, A, B)\) in \((*)\) implies that \(B' \supseteq B - N(z) \supseteq B - A\). This implies that \((A - B) \cup (B - A)\) is an independent set, a contradiction.

Fig. 4. Final proof of the lower bound.

It is worth noting what the result of the one-variable optimization is in terms of \(m\) and \(n\). In particular, the construction achieves a lower bound resulting from Theorem 1.1 when \(n > 1 + (4/9)(m - 2)^2\).

**Remark 4.5** If \(n > 1 + (4/9)(m - 2)^2\), then \(f(S_m, \overline{K}_n) = n + \lceil 2\sqrt{n-1} \rceil\).

If \(m \leq n \leq 1 + (4/9)(m - 2)^2\), then \(f(S_m, \overline{K}_n) = n + \lceil \frac{1}{4}(3\beta - \sqrt{\beta^2 - 8})\sqrt{n-1} \rceil\), where \(2m - 3 = \beta\sqrt{n-1}\) with \(\beta > 3\).

**Proof:** By Theorem 1.4, it suffices to minimize over \(k\) in Lemma 4.2. The term \(2m - 3 - k\) is linear. The term \(k + \lceil (n-1)/k \rceil\) is minimized when \(k = \lceil \sqrt{n-1} \rceil\), where it equals \(2\sqrt{n-1}\). (When \(k = \lceil \sqrt{n-1} \rceil\), we let \(n-1 = k^2 - r\) with \(r < 2k - 1\); both formulas yield \(2k - 1\) when \(r \geq k\) and \(2k\) when \(r < k\).)

When \(2m - 3 - \lceil \sqrt{n-1} \rceil \leq \lceil 2\sqrt{n-1} \rceil\), the construction yields \(f(S_m, \overline{K}_n) \leq n + \lceil 2\sqrt{n-1} \rceil\). Since every vertex of an induced star belongs to an induced edge, Theorem 1.1 yields \(f(S_m, \overline{K}_n) \geq f(K_2, \overline{K}_n) \geq n + \lceil 2\sqrt{n-1} \rceil\).

For smaller \(n\), the construction is optimized by choosing \(x\) so that \(x + (n - 1)/x = 2m - 3 - x\) and letting \(k = \lfloor x \rfloor\). The number of vertices is then \(2m - 3 - k\). For large \(m\) and \(n\), we can approximate the result by ignoring integer parts and defining \(\beta\) by \(2m - 3 = \beta\sqrt{n-1}\). The solution then occurs at \(x = \frac{1}{4}(\beta + \sqrt{\beta^2 - 8})\sqrt{n-1}\), and we invoke Theorem 1.4.

5 Open Problems

We list several open questions. The first is the most immediately appealing, suggested by comparing Theorem 1.1 and Theorem 2.1.
1. Among all choices of an $m$-vertex graph $H_1$ and an $n$-vertex graph $H_2$, is it true that $f(H_1, H_2)$ is maximized when $H_1$ is a clique and $H_2$ is an independent set?

2. Let $G$ be an $S_m$-full graph in which the deletion of any edge produces a graph that is not $S_m$-full. Is it true that $G$ must be triangle-free? \[1\]

3. Among random graphs, what order is needed so that almost every graph is $(H_1, \ldots, H_k)$-full?

4. Distinguish a root vertex in each of $H_1, \ldots, H_k$. An $(H_1, \ldots, H_k)$-root-full graph is an $(H_1, \ldots, H_k)$-full graph in which each vertex appears as the root in some induced copy of each $H_i$. Is it possible to bound the minimum order of such a graph (for arbitrary choice of roots) in terms of $f(H_1, \ldots, H_k)$? (suggested by Fred Galvin)

5. Similarly, one could require induced copies of each $H_i$ so that for each $v \in V(G)$ and $x \in H_i$, some copy of $H_i$ occurs with $v$ playing the role of $x$. The minimum order of such a graph is the framing number $fr(H_1, \ldots, H_k)$. How large can $fr(H_1, \ldots, H_k)$ be as a function of $f(H_1, \ldots, H_k)$? (suggested by Mike Jacobson)

6. **Acknowledgments**

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1 this has recently been proved positively in \[1\]