SHARP BOUNDS FOR FRACTIONAL OPERATOR WITH L^{r',r'}-HÖRMANDER CONDITIONS

GONZALO H. IBAÑEZ-FIRNKORN, MARÍA SILVINA RIVEROS, AND RAÚL E. VIDAL

Abstract. In this paper we prove the sharp boundedness for a fractional type operator given by a kernel that satisfy a L^{r',r'}-Hörmander conditions and a fractional size condition, where 0 < \alpha < n and 1 < r' \leq \infty. To prove this result we use a new appropriate sparse domination which we provide in this work. For the case r' = \infty we recover the sharp boundedness for the fractional integral, I_\alpha, proved in [Lacey, M. T., Moen, K., Pérez, C., Torres, R. H. (2010). Sharp weighted bounds for fractional integral operators. Journal of Functional Analysis, 259(5), 1073-1097.]

1. Introduction and main results

We recall that given 0 < \alpha < n, the fractional integral operator I_\alpha on \mathbb{R}^n is defined by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy.$$ 

This operator is bounded from L^p(dx) into L^q(dx) provided that 1 < p < \frac{n}{\alpha} and \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} (see [14] for this result).

In the study of weighted estimates for the fractional integral, the class of weights considered is the A_p,q introduced by Muckenhoupt and Wheeden in [12]. We recall that w is a weight if w is a locally integrable non-negative function. Given 1 < p < q < \infty, we say that the weight w is in the class A_p,q if

$$[w]_{A_{p,q}} := \sup_Q \left( \frac{1}{|Q|} \int_Q w^{q} \right) \left( \frac{1}{|Q|} \int_Q w^{-p'} \right)^{q/p'} < \infty.$$ 

If w \in A_{p,q} then w^q \in A_{1+q/p'} with [w]_{1+q/p'} = [w]_{A_{p,q}} and w^{p'} \in A_{1+p'/q} with [w^{p'}]_{1+p'/q} = [w]^{p'/q}_{A_{p,q}} where A_s denotes the classical Muckenhoupt class of weights. Observe that w \in A_{p,p} is equivalent to w^p \in A_p. We recall A_\infty = \bigcup_{p \geq 1} A_p, and the statement w \in A_{\infty,\infty} is equivalent to w^{-1} \in A_1.

During the last decade, there are several works devoted to the study of quantitative weighted estimates, in other words, in those papers the authors study how these estimates depend on the weight constant [w]_{A_p} or [w]_{A_{p,q}}. The A_2 Theorem, namely the linear dependence on the A_2 constant for Calderón-Zygmund operators, proved by Hytönen in [4], can be consider the most representative in this line. In the case of fractional integrals, the sharp dependence of the A_{p,q} constants was obtained by Lacey Moen, Pérez and Torres [7]. The precise statement is the following

\begin{itemize}
  \item 2010 Mathematics Subject Classification. 42B20, 42B25.
  \item Key words and phrases. Fractional operators, fractional L^{r',r'}-Hörmander’s condition, sharp weights inequalities, sparse operators.
  \item The authors are partially supported by CONICET and SECYT-UNC.
\end{itemize}
Theorem 1.1. \[7\] Let \(0 < \alpha < n\), \(1 < p < \frac{\alpha}{\alpha}\) and \(\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}\). If \(w \in A_{p,q}\) then

\[
\|I \alpha f\|_{L^p(w^q)} \leq c_{\alpha,w}[w]_{A_{p,q}}^{(1-\frac{\alpha}{p})\max\{1,\frac{1}{n}\}}\|f\|_{L^p(w^q)},
\]

and the estimate is sharp in the sense that the inequality does not hold if we replace the exponent of the \(A_{p,q}\) constant by a smaller one.

Recall that for \(0 \leq \alpha < n\), \(1 \leq r < \infty\) and \(f \in L^1_{\text{loc}}(dx)\), the \(M_{\alpha,r}\) maximal operator is defined by

\[
M_{\alpha,r}f(x) = \sup_{B \ni x} |B|^{\alpha/n}\|f\|_{L^r(B)} = \sup_{B \ni x} \left(\frac{1}{|B|^{1-\alpha/n}} \int_B |f|^r\right)^{1/r},
\]

where the supremum is taken over all the balls \(B\).

In this paper we consider the following fractional operator. Let \(0 < \alpha < n\) and \(1 \leq r < \infty\). For any \(f \in L^1_{\text{loc}}(dx)\) we define

\[
(1.1) \quad T_{\alpha}f(x) = \int K_{\alpha}(x,y)f(y)dy,
\]

where \(K_{\alpha} \in S_{\alpha,r'} \cap H_{\alpha,r'}\) and \(\frac{1}{r'} = 1 - \frac{1}{r}\).

We define these conditions. For this, first we introduce some notation: \(|x| \sim s\) means \(s < |x| \leq 2s\) and we write \(\|f\|_{r,|x|\sim s} = \|f\chi_{|x|\sim s}\|_{r,B(0,2s)}\).

The function \(K_{\alpha}\) is said to satisfy the fractional size condition, \(K_{\alpha} \in S_{\alpha,r'}\), if there exists a constant \(C > 0\) such that

\[
\|K_{\alpha}\|_{r,|x|\sim s} \leq Cs^{\alpha-n}.
\]

When \(r' = 1\) we write \(S_{\alpha,r'} = S_{\alpha}\). Observe that if \(K_{\alpha} \in S_{\alpha}\), then there exists a constant \(c > 0\) such that

\[
\int_{|x|\sim s} |K_{\alpha}(x)|dx \leq cs^\alpha.
\]

The function \(K_{\alpha}\) satisfies the \(L^{r,r'}\)-Hörmander condition \((K_{\alpha} \in H_{\alpha,r'})\), if there exist \(c_{r'} > 1\) and \(C_{r'} > 0\) such that for all \(x\) and \(R > C_{r'}|x|\),

\[
\sum_{m=1}^{\infty} (2^m R)^{n-\alpha}\|K_{\alpha}(\cdot,x) - K_{\alpha}(\cdot,|x|\sim 2^m R)\|_{r',|y|\sim 2^m R} \leq C_{r'}.
\]

We say that \(K_{\alpha} \in H_{\alpha,\infty}\) if \(K_{\alpha}\) satisfies the previous condition with \(\|\cdot\|_{L^{\infty,|x|\sim 2^m R}}\) in place of \(\|\cdot\|_{r',|x|\sim 2^m R}\).

Observe that if \(r = 1\) and \(k_{\alpha}(x) = |x|^{n-\alpha}\) then \(r' = \infty\) and \(T_{\alpha} = I_{\alpha}\) the fractional integral.

This type of operator has been studied by Kurtz in [8]. He prove the following result

Theorem 1.2. \[8\] Let \(0 < \alpha < n\) and \(1 \leq r < n/\alpha\). Let \(K_{\alpha} \in H_{\alpha,r'}\) and suppose \(T_{\alpha}\) is bounded from \(L^r(dx)\) into \(L^q(dx)\) for \(\frac{1}{r} = \frac{1}{q} - \frac{\alpha}{n}\). Then there exists a constant \(C\), such that for \(f \in L^1_{\text{loc}}\)

\[
M_r(T_{\alpha}f)(x) \leq CM_{\alpha,r}f(x).
\]

From this result and the good-\(\lambda\) technique, it follows that

Proposition 1.3. Let \(0 < \alpha < n\) and \(1 \leq r < n/\alpha\). If \(K_{\alpha} \in H_{\alpha,r'}\) and \(T_{\alpha}\) is bounded from \(L^r(dx)\) into \(L^q(dx)\) for \(\frac{1}{r} = \frac{1}{q} - \frac{\alpha}{n}\), then there exists a constant \(C_{\alpha}\), depending on \(\alpha\), such that for \(f \in L^1_{\text{loc}}(dx)\) and \(w^r \in A_{\frac{n}{\alpha} - r}\)

\[
\sup_{\lambda > 0} \lambda^{\frac{n}{\alpha} - r} \{ x \in \mathbb{R}^n : |(T_{\alpha}f)(x)| > \lambda \} \leq C_{\alpha} \int |f|^r w^r.
\]
The idea of the proof is the same as the one given in Theorem 3.6 in [13], so we do not give it.

Theorem 1.4. Let \( 0 < \alpha < n \) and let \( T_\alpha \) defined as in (1.1). Let \( 1 \leq r < p < n/\alpha \). Suppose \( K_\alpha \in S_{\alpha,r'} \cap H_{\alpha,r'} \). If \( w^r \in A_{\frac{p}{r},1} \), then there exists a constant \( C_w \), independent of \( f \) but depending on \( w \), such that

\[
\| T_\alpha f \|_{L^{q}(w^q)} \leq C_w \| f \|_{L^{p}(w^p)}.
\]

The main result in this paper is the dependence of the constant \( [w^r]_{A_{\frac{p}{r},1}} \), in this last inequality. The result is the following

Theorem 1.5. Let \( 0 < \alpha < n \) and let \( T_\alpha \) defined as in (1.1). Let \( 1 \leq r < p < n/\alpha \) and \( 1/q = 1/p - \alpha/n \). Suppose \( K_\alpha \in S_{\alpha,r} \cap H_{\alpha,r} \). If \( w^r \in A_{\frac{p}{r},1} \), then

\[
\| T_\alpha f \|_{L^{q}(w^q)} \leq c_n[w^r]_{A_{\frac{p}{r},1}} \max \{1, \frac{1}{\alpha} \left( 1 + \frac{(\alpha/r')}{p} \right) \} \| f \|_{L^{p}(w^p)}.
\]

This estimate is sharp in the following sense

Proposition 1.6. Let \( 0 < \alpha < n \) and let \( T_\alpha \) defined as in (1.1). Let \( 1 \leq r < p < n/\alpha \) and \( 1/q = 1/p - \alpha/n \). Suppose \( K_\alpha \in S_{\alpha,r} \cap H_{\alpha,r} \). If \( \Phi : [1, \infty) \to (0, \infty) \) is a monotone function such that

\[
\| T_\alpha \|_{L^{p}(w^p) \to L^{q}(w^q)} \lesssim \Phi([w^r]_{A_{\frac{p}{r},1}}),
\]

for all \( w^r \in A_{\frac{p}{r},1} \), then \( \Phi(t) \geq t^{\max \{1, \frac{1}{\alpha} \left( 1 + \frac{(\alpha/r')}{p} \right) \}} \).

The paper continues in the following way: in Section 2 we prove the sparse domination for \( T_\alpha \). In Section 3 we obtain the \( L^p(w^p) - L^q(w^q) \) boundedness of the sparse operator with the dependence of the \( [w^r]_{A_{\frac{p}{r},1}} \) constant. Finally in Section 4 we give some examples to prove that the dependency of the constant given in Section 3 is optimal.

2. Sparse domination for \( T_\alpha \)

In this section we present a sparse domination results for the operator \( T_\alpha \). First we recall some definitions.

Given a cube \( Q \in \mathbb{R}^n \), we denote by \( D(Q) \) the family of all dyadic cubes with respect \( Q \), that is, the cube obtained subdividing repeatedly \( Q \) and each of its descendant into \( 2^n \) subcubes of the same side lengths.

Given a dyadic family \( D \) we say that a family \( \mathcal{F} \subset D \) is a \( \eta \)-sparse family with \( 0 < \eta < 1 \), if for every \( Q \in \mathcal{F} \), there exists a measurable set \( E_Q \subset Q \) such that \( \eta |Q| \leq |E_Q| \) and the family \( \{E_Q\}_{Q \in \mathcal{F}} \) are pairwise disjoint.

Relying upon ideas in [1], [5], [10] and [11], it is posible to obtain a pointwise sparse domination that covers the fractional operators that we are considering. The precise statement is the following

Theorem 2.1. Let \( 0 < \alpha < n \), \( 1 \leq r < \infty \) and let \( T_\alpha \) defined as in (1.1). Suppose \( K_\alpha \in S_{\alpha,r'} \cap H_{\alpha,r'} \). For any \( f \in L^1_{r'}(\mathbb{R}^n) \), there exist \( 3^n \) sparse families such that for a.e. \( x \in \mathbb{R}^n \),

\[
|T_\alpha f(x)| \leq c \sum_{j=1}^{3^n} \sum_{Q \in \mathcal{F}_j} |Q|^{\alpha/n} \| f \|_{L^{r'}(\mathcal{F}_j)} \chi_Q(x) := c \sum_{j=1}^{3^n} A_{\alpha,\mathcal{F}_j} f(x).
\]
The grand maximal truncated operator $M_{T_{\alpha}}$ is defined by
\[ M_{T_{\alpha}}f(x) = \sup_{x \in Q_0} \text{ess sup}_{\xi \in Q} |T_{\alpha}(f\chi_{3Q}')(\xi)|, \]
where the supremum is taken over all the cubes $Q \subset \mathbb{R}^n$ containing $x$. For the proof of the preceding theorem we need to show that $M_{T_{\alpha}}$ maps $L^r(dx)$ to $L^{\frac{n}{1-r}}(dx)$. Also we need a local version of this maximal operator which is defined, for a cube $Q_0 \subset \mathbb{R}^n$, as
\[ M_{T_{\alpha},Q_0}f(x) = \sup_{x \in Q_0} \text{ess sup}_{\xi \in Q} |T_{\alpha}(f\chi_{3Q_0\backslash 3Q})|, \]

**Lemma 2.2.** Let $0 < \alpha < n$, $1 \leq r < \infty$, $Q_0 \subset \mathbb{R}^n$ be a cube and $T_{\alpha}$ defined as in (1.1). Suppose $K_{\alpha} \in S_{\alpha,r'} \cap H_{\alpha,r}$. The following estimates hold:

1. for a.e. $x \in Q_0$
\[ |T_{\alpha}(f\chi_{3Q_0})(x)| \leq M_{T_{\alpha},Q_0}f(x), \]
2. for all $x \in \mathbb{R}^n$
\[ M_{T_{\alpha}}(f)(x) \lesssim M_{\alpha,r}(x) + T_{\alpha}(|f|)(x). \]

From the last estimate and Proposition 1.3 it follows that $M_{T_{\alpha}}$ is bounded from $L^r(dx)$ to $L^{\frac{n}{1-r}}(dx)$.

**Proof.** (1) Let $Q(x,s)$ a cube centered at $x$ with length $s$ such that $Q(x,s) \subset Q$
\[ |T_{\alpha}(f\chi_{3Q_0})(x)| \leq |T_{\alpha}(f\chi_{3Q(x,s)})(x)| + |T_{\alpha}(f\chi_{3Q_0 \backslash 3Q(x,s)})(x)|. \]

For the first term, let us consider $B(x,R)$ with $R = 3\sqrt{n}s$ then $3Q(x,s) \subset B(x,R)$. As $K_{\alpha} \in S_{\alpha,r'} \cap H_{\alpha,r}$, we have
\[
|T_{\alpha}(f\chi_{3Q(x,s)})(x)| \leq |T_{\alpha}(f\chi_{B(x,R)})(x)| \leq \int_{B(x,R)} |K_{\alpha}(x-y)||f(y)|dy \\
= \sum_{m=0}^{\infty} \frac{|B(x,2^{-m}R)|}{|B(x,2^{-m}R)|} \int_{B(x,2^{-m}R)} \chi_{B(x,2^{-m}R)}(y)\chi_{B(x,2^{-m}R)}|K_{\alpha}(x-y)||f(y)|dy \\
\leq \sum_{m=0}^{\infty} |B(x,2^{-m}R)||K_{\alpha}||r',|x|_{\infty}^{2^{-m+1}R}||f||_{r,B(x,2^{-m}R)} \\
\leq cM_{r}(f)(x)\sum_{m=0}^{\infty} (2^{-m}R)^{n}(2^{-m}R)^{\alpha-n} \\
= cM_{r}(f)(x)R^\alpha \sum_{m=0}^{\infty} (2^{-m})^n = cM_{r}(f)(x)R^\alpha.
\]

Then,
\[ |T_{\alpha}(f\chi_{3Q_0})(x)| \leq c\alpha s^\alpha M_{r}f(x) + M_{T_{\alpha},Q_0}f(x). \]
Letting $s \to 0$, we have the desired estimate.

(2) Let $x \in \mathbb{R}^n$ and let $Q$ be a cube containing $x$. Let $B_x$ be a ball with radius $R$ such that $3Q \subset B_x$. For every $\xi \in Q$, we have
\[
|T_{\alpha}(f\chi_{\mathbb{R}^n \backslash 3Q})(\xi)| \leq |T_{\alpha}(f\chi_{\mathbb{R}^n \backslash B_x})(\xi)| - |T_{\alpha}(f\chi_{\mathbb{R}^n \backslash B_x})(\xi)| + |T_{\alpha}(f\chi_{\mathbb{R}^n \backslash 3Q})(\xi)| + |T_{\alpha}(f\chi_{\mathbb{R}^n \backslash B_x})(\xi)| \\
\leq |T_{\alpha}(f\chi_{\mathbb{R}^n \backslash B_x})(\xi)| - T_{\alpha}(f\chi_{\mathbb{R}^n \backslash B_x})(\xi)| + |T_{\alpha}(f\chi_{\mathbb{R}^n \backslash 3Q})(\xi)| + |T_{\alpha}(f\chi_{\mathbb{R}^n \backslash B_x})(\xi)|.
\]
For the first term, as $K_α ∈ H_{α,r}$, we get
\[
|T_α(fχ_{R^n \setminus B_x})(ξ) - T_α(fχ_{R^n \setminus B_x})(x)| ≤ \int_{R^n \setminus B_x} |K_α(ξ - y) - K_α(x - y)||f(y)|dy
\]
\[
= \sum_{m=1}^{∞} \frac{2^m B_x}{2^{m+1} B_x} \int_{2^{m+1} B_x \setminus 2^m B_x} |K_α(ξ - y) - K_α(x - y)||f(y)|dy
\]
\[
≤ \sum_{m=1}^{∞} (2^m R)^n |K_α(ξ - ·) - K_α(x - ·)||r,|y|<2^m R||f||r,2^{m+1} B_x
\]
\[
≤ \sum_{m=1}^{∞} (2^m R)^{n-α} |K_α(ξ - ·) - K_α(x - ·)||r,|y|<2^m R M_{α,r} f(x)
\]
\[
≤ c_r M_{α,r} f(x).
\]

For the second term, observe that there exists $l ∈ N$ such that $B(x, 2^{-l} R) ⊂ 3Q$, then, as $K_α ∈ S_α$, we obtain
\[
|T_α(fχ_{B_x \setminus 3Q})(ξ)| ≤ \int_{B_x \setminus 3Q} |K_α(x - y)||f(y)|dy
\]
\[
≤ \sum_{m=0}^{l-1} \int_{B(x, 2^{-m} R) \setminus B(x, 2^{-m-1} R)} |K_α(x - y)||f(y)|dy
\]
\[
≤ \sum_{m=0}^{l-1} |B(x, 2^{-m} R)||K_α||r,|x|<2^{-m-1} R||f||r,B(x, 2^{-m} R)
\]
\[
≤ c \sum_{m=0}^{l-1} (2^{-m} R)^n (2^{-m} R)^{α-n} ||f||r,B(x, 2^{-m} R)
\]
\[
≤ c M_{α,r} f(x).
\]

Finally we get
\[
|T_α(fχ_{R^n \setminus 3Q})(ξ)| ≤ M_{α,r} f(x) + T_α(|f|)(x).
\]

The following lemma is the so called $3^n$ dyadic lattices trick. This result was established in [9] and affirms:

**Lemma 2.3.** [9] Given a dyadic family $D$ there exist $3^n$ dyadic families $D_j$ such that
\[
\{3Q : Q ∈ D\} = \bigcup_{j=1}^{3^n} D_j,
\]
and for every cube $Q ∈ D$ we can find a cube $R_Q$ in each $D_j$ such that $Q ⊂ R_Q$ and $3l_Q = l_R_Q$.

**Proof of Theorem 2.1** We claim that for any cube $Q_0 ∈ R^n$, there exists a $1$-sparse family $F ⊂ D(Q_0)$ such that for a.e. $x ∈ Q_0$,
\[
|T_α(fχ_{3Q_0})(x)| \lesssim \sum_{Q ∈ F} |3Q|^{|α|/n} ||f||r,3Q χ_Q(x).
\]

Suppose that we have already proved the claim (2.1). Let us take a partition of $R^n$ by cubes $Q_j$ such that $\text{supp}(f) ⊂ 3Q_j$ for each $j$. We can do it as follows. We start with a cube $Q_0$
such that \( \text{supp}(f) \subset Q_0 \), and cover \( 3Q_0 \setminus Q_0 \) by \( 3^n - 1 \) congruent cubes \( Q_j \), each of them satisfies \( Q_0 \subset 3Q_j \). We do the same for \( 9Q_0 \setminus 3Q_0 \) and so on. The union of all those cubes will satisfy the desired properties.

We apply the claim (2.1) to each cube \( Q_j \). Then we have that since \( \text{supp}(f) \subset 3Q_j \) the following estimate holds a.e. \( x \in Q_j \)

\[
|T_\alpha f(x)| \chi_{Q_j}(x) = |T_\alpha(f \chi_{3Q_0})(x)| \lesssim \sum_{Q \in F_j} |3Q|^{n/n} \|f\|_{r,3Q} \chi_Q(x),
\]

where each \( \mathcal{F}_j \subset \mathcal{D}(Q_j) \) is a \( \frac{1}{2} \)-sparse family. Taking \( \mathcal{F} = \bigcup_j \mathcal{F}_j \), we have that \( \mathcal{F} \) is a \( \frac{1}{2} \)-sparse family and for a.e. \( x \in \mathbb{R}^n \)

\[
|T_\alpha f(x)| \lesssim \sum_{Q \in \mathcal{F}} |3Q|^{n/n} \|f\|_{r,3Q} \chi_Q(x).
\]

From Lemma 2.3 it follows that there exists \( 3^n \) dyadic families such that for every cube \( Q \) of \( \mathbb{R}^n \) there is a cube \( R_Q \in \mathcal{D}_j \) for some \( j \) for which \( 3Q \subset R_Q \) and \( |R_Q| \leq 3^n |3Q| \). Setting \( \mathcal{F}_j = \{ R_Q \in \mathcal{D}_j : Q \in \mathcal{F} \} \),

and since \( \mathcal{F} \) is a \( \frac{1}{2} \)-sparse, we obtain that for each family \( \mathcal{F}_j \) is \( \frac{1}{2} \)-sparse. Then we have that

\[
|T_\alpha f(x)| \lesssim \sum_{j=1}^{3^n} \sum_{Q \in \mathcal{F}_j} |Q|^{n/n} \|f\|_{r,Q} \chi_Q(x).
\]

Proof of the claim (2.1). To prove the claim it is suffices to show following recursive estimate: there exists a countable family \( \{ P_j \}_j \) of pairwise disjoint cube in \( \mathcal{D}(Q_0) \) such that \( \sum_j P_j \leq \frac{1}{2} |Q_0| \) and

\[
|T_\alpha(f \chi_{3Q_0})(x)| \chi_{Q_0}(x) \leq c|3Q_0|^{n/n} \|f\|_{r,3Q_0} \chi_{Q_0}(x) + \sum_j |T_\alpha(f \chi_{3P_j})(x)| \chi_{P_j}(x),
\]

for a.e. \( x \in Q_0 \). Iterating this estimate we obtain (2.1) with \( \mathcal{F} \) being the union of all the families \( \{ P_{jk} \} \) where \( \{ P_{j0} \} = \{ Q_0 \} \), \( \{ P_{j1} \} = \{ P_j \} \) and the \( \{ P_{jk} \} \) are the cubes obtained at the \( k \)-th stage of the iterative process. It is also clear that \( \mathcal{F} \) is a \( \frac{1}{2} \)-sparse family. Indeed, for each \( P_{jk} \) it is suffices to choose

\[
E_{P_{jk}} = P_k \setminus \bigcup_j P_{j+1}.
\]

Let us prove the recursive estimate (2.2). Observe that for any family \( \{ P_j \} \subset \mathcal{D}(Q_0) \) of disjoint cubes, we have

\[
|T_\alpha(f \chi_{3Q_0})(x)| \chi_{Q_0}(x) \leq |T_\alpha(f \chi_{3Q_0})(x)| \chi_{Q_0 \cup \bigcup_j P_j}(x) + \sum_j |T_\alpha(f \chi_{3Q_0})(x)| \chi_{P_j}(x)
\]

\[
\leq |T_\alpha(f \chi_{3Q_0})(x)| \chi_{Q_0 \cup \bigcup_j P_j}(x) + \sum_j |T_\alpha(f \chi_{3Q_0 \cup 3P_j})(x)| \chi_{P_j}(x) + \sum_j |T_\alpha(f \chi_{3P_j})(x)| \chi_{P_j}(x),
\]

for almost every \( x \in \mathbb{R}^n \). So it is suffices to show that we can choose a countable family \( \{ P_j \}_j \) of pairwise disjoint cube in \( \mathcal{D}(Q_0) \) such that \( \sum_j P_j \leq \frac{1}{2} |Q_0| \) and for, a.e.\( x \in Q_0 \) we have,

\[
|T_\alpha(f \chi_{3Q_0})(x)| \chi_{Q_0 \cup \bigcup_j P_j}(x) + \sum_j |T_\alpha(f \chi_{3Q_0 \cup 3P_j})(x)| \chi_{P_j}(x) \lesssim |3Q_0|^{n/n} \|f\|_{r,3Q_0} \chi_{Q_0}(x).
\]
Now we define the set $E$ as
\[ E = \{ x \in Q_0 : M_{T_0,Q_0} f(x) > \beta_n c |3Q_0|^{\alpha/n} \| f \|_{L^r(3Q_0)} \}, \]
by Lemma 2.2 we can choose $\beta_n$ such that $|E| \leq \frac{1}{2^{n+1}} |Q_0|$.

We apply Calderón-Zygmund decomposition to the function $\chi_E$ on $Q_0$ at height $\lambda = \frac{1}{2^{n+1}}$. Then, there exist a family $\{ P_j \} \subset \mathcal{D}(Q_0)$ of pairwise disjoint cubes such that
\[ \left\{ x \in Q_0 : \chi_E(x) > \frac{1}{2^{n+1}} \right\} = \bigcup_j P_j. \]

From this it follows that $|E \setminus \bigcup_j P_j| = 0$,
\[ \sum_j |P_j| \leq 2^{n+1} |E| \leq \frac{1}{2} |Q_0|, \]
and
\[ \frac{1}{2^{n+1}} \leq \frac{|P_j \cap E|}{|P_j|} \leq \frac{1}{2}, \]
from which it follows that $|P_j \cap E^c| > 0$.

Since $P_j \cap E^c \neq \emptyset$, we have $M_{T_0,Q_0}(f)(x) \leq \beta_n c |3Q_0|^{\alpha/n} \| f \|_{L^r(3Q_0)}$ for some $x \in P_j$ and this implies
\[ \text{ess sup}_{\xi \in P_j} |T_\alpha (f \chi_{3Q_0 \setminus P_j})(\xi)| \leq \beta_n c |3Q_0|^{\alpha/n} \| f \|_{L^r(3Q_0)}, \]
which allows us to control the second term in (2.3).

By (1) in Lemma 2.2 for a.e. $x \in Q_0$ we have
\[ |T_\alpha (f \chi_{3Q_0})(x)| \leq M_{T_0,Q_0} f(x) \chi_{Q_0 \setminus \bigcup_j P_j}(x). \]
Since $|E \setminus \bigcup_j P_j| = 0$ and by the definition of $E$, for a.e. $x \in Q_0 \setminus \bigcup_j P_j$ we obtain
\[ M_{T_0,Q_0}(f)(x) \leq \beta_n c |3Q_0|^{\alpha/n} \| f \|_{L^r(3Q_0)}. \]

Then, for a.e. $x \in Q_0 \setminus \bigcup_j P_j$ we get
\[ |T_\alpha (f \chi_{3Q_0})(x)| \leq \beta_n c |3Q_0|^{\alpha/n} \| f \|_{L^r(3Q_0)}. \]

Thus we obtain the estimate in (2.3).

\[ \square \]

3. Sharp bounds for norm inequality

Since the sparse domination is a pointwise estimate, it is suffices to prove Theorem 1.5 and Proposition 1.6 for the sparse operator $A^\alpha_{r,\mathcal{D}}$ for any sparse family $\mathcal{D}$.

**Theorem 3.1.** Let $0 \leq \alpha < n$, $1 \leq r < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. If $w^r \in A^p_{\infty,\mathcal{D}}$, then
\[ \| A^\alpha_{r,\mathcal{D}} f \|_{L^p(w^r)} \leq C_n \| w^r \|_{A^p_{\infty,\mathcal{D}}} \| f \|_{L^p(w^r)}, \]
this estimate is sharp in the sense of Proposition 1.6.

**Remark 3.2.** The first approximation of this type for the fractional integral, using the sparse technique, appears in [2]. In this paper the author does not prove the sharpness of the constant. In the case $r = 1$, the fractional integral $I_\alpha$, we obtain the same sharp bound as in [7]. If $\alpha = 0$, we get the same sharp bound as in [8].
We consider the following sparse operator defined in $\tilde{A}_{s,y}(\sigma)$, for $0 < s < \infty$ and $0 < \beta \leq 1$

$$\tilde{A}_{s,y}^\beta g(x) = \left( \sum_{Q \in y} \left( |Q|^{-\beta} \int_Q g \right)^s \chi_Q(x) \right)^{1/s}.$$ 

Theorem 3.3. Let $1 \leq r < p \leq q < \infty$, and $0 < \beta \leq 1$. Let $u, \sigma \in A_\infty$ be two weights. The sparse operator $A_{s,y}^\beta(\sigma)$ maps $L^p(\sigma) \to L^q(u)$ if and only if the two-weight $A_{p,q}^\beta$-characteristic

$$[u, \sigma]_{A_{p,q}^\beta(\sigma)} := \sup_{Q \in y} |Q|^{-\beta} u(Q)^{1/q} \sigma(Q)^{1/p'},$$

is finite, and in this case

$$1 \leq \frac{\|\tilde{A}_{s,y}^\beta(\sigma)\|_{L^p(\sigma) \to L^q(u)}}{[u, \sigma]_{A_{p,q}^\beta(\sigma)}} \lesssim [\sigma]_{A_\infty}^{1/q} + [u]_{A_\infty}^{\frac{1}{p}-\frac{1}{q}}.$$ 

Proof of Theorem 3.3. Let $\sigma = w^{-(p/r)}r$. Observe that

$$A_{r,y}^\alpha(f) = \left( \tilde{A}_{1/r,y}^{1-\alpha/n}(f_r) \right)^{1/r}.$$ 

Then,

$$\|A_{r,y}^\alpha(f)\|_{L^q(w^q)} = \|A_{1/r,y}^{1-\alpha/n}(f_r)\|_{L^q(w^q)} = \|A_{1/r,y}^{1-\alpha/n}(f_r \sigma^{-1})\|_{L^{q/r}(w^q)} \lesssim [w^q, \sigma]_{A_{p,q}^{1-\alpha/n}(\sigma)} \left( [\sigma]_{A_\infty}^{1/q} + [w^q]_{A_\infty}^{1/q} \right) \|f\|_{L^p(w^q)}.$$ 

Now observe that,

$$[w^q, \sigma]_{A_{p,q}^{1-\alpha/n}(\sigma)} = [w^r]_{A_{p,q}^{r/(p/r)}}^{r/q}.$$ 

and

$$\|f\|_{L^p(w^q)} = \|f\|_{L^p(w^q)}.$$ 

Since

$$[\sigma]_{A_{1+(p/r)'r/q}} = [w^q]_{A_{p,q}}^{(p/r)'r/q} = [w^r]_{A_{p,q}}^{r/(p/r)},$$

we have

$$\|A_{r,y}^\alpha(f)\|_{L^q(w^q)} \lesssim [w^r]_{A_{p,q}}^{1/q} \left( [\sigma]_{A_\infty}^{r/q} + [w^q]_{A_\infty}^{1/q} \right)^{1/r} \|f\|_{L^p(w^q)} \leq [w^r]_{A_{p,q}}^{1/q} \left( [w^r]_{A_{p,q}}^{(p/r)'r/q} + [w^r]_{A_{p,q}}^{1/q} \right)^{1/r} \|f\|_{L^p(w^q)} \leq [w^r]_{A_{p,q}}^{1/q+\max\{\alpha/(q)\}} \|f\|_{L^p(w^q)} \leq [w^r]_{A_{p,q}}^{\max\{1-\alpha/(q)\}} \|f\|_{L^p(w^q)}.$$ 

where the last inequality holds by $(1 + (p/r)'r/q) = (1 - \alpha/(q))(p/r)'$ and $1/q + 1/p' = 1 - \alpha/n$. \qed
4. Example

In this section, we prove the Proposition 1.6 for the sparse operator $A = A_{r,q}^\alpha$.

**Proof.** Let $0 < \epsilon < 1$. If

$$w_\epsilon(x) = |x|^{\frac{n-\alpha}{r}} \quad \text{and} \quad f(x) = |x|^{\frac{\alpha}{r}} \chi_{B(0,1)},$$

then

$$[w_\epsilon^q]_{A_{p,q}^\alpha} \approx \epsilon^{-\frac{q}{r(p/r)}} \quad \text{and} \quad \|fw_\epsilon\|_{L^p} \approx \epsilon^{-1/p}.$$  

Let $\{Q_k\}$ the cube of center 0 and length $2^{-k}$, observe that $B(0,1) \subset Q_0$. Observe that $\{Q_k\}$ is a $\frac{1}{2}$-sparse family, we can take $E_Q = Q_k \setminus Q_{k-1}$.

Now, if $x \in E_{2-k}, k \in \mathbb{N}$

$$Af(x) \geq |Q_k|^{\alpha/n-1/r} \left( \int_{Q_k} |y|^{q-1} \right)^{1/r} \geq |Q_k|^{\alpha/n-1/r} \left( \int_{E_{Q_k}} |y|^{q-1} \right)^{1/r} \geq (2^{-kn})^{\alpha/n-1/r} \left( \frac{2^{-k-1}\epsilon}{\epsilon} \right)^{1/r} \geq \epsilon^{-1/r} |x|^\alpha/n-1/r+\epsilon/r.$$  

Therefore,

$$\int Af^qw_\epsilon^q \geq \sum_{k=1}^\infty \int_{E_{Q_k}} Af^qw_\epsilon^q \geq \epsilon^{-q/r} \int_{B(0,\frac{1}{2})} |x|^{q(\alpha/n-1/r+\epsilon/r)+\frac{n-\alpha}{r(p/r)}} \, dx \approx \epsilon^{-q/r-1},$$

since $q(\alpha/n - 1/r + \epsilon/r) + q\frac{n-\alpha}{r(p/r)} \leq \epsilon q/p$. Then

$$\epsilon^{-\frac{q}{r(p/r)}} \leq \epsilon^{-1/r-1/q+1/p} \leq \|A\|_{L^p(w_\epsilon^q)} \leq \Phi(\epsilon^{-\frac{q}{r(p/r)}}).$$

Now, take $t = \epsilon^{-\frac{q}{r(p/r)}}$,

$$\Phi(t) \geq f(p/r)^{1/q(1-\alpha/n)}.$$  

Let $0 < \epsilon < 1$. If

$$w_\epsilon(x) = |x|^{\frac{n-\alpha}{q}} \quad \text{and} \quad f(x) = |x|^{\frac{\alpha}{r}} \chi_{B(0,1)},$$

then

$$[w_\epsilon^q]_{A_{p,q}^\alpha} \approx \epsilon^{-1} \quad \text{and} \quad \|fw_\epsilon\|_{L^p} \approx \epsilon^{-1/p}.$$  

Since $1/r + 1/q = 1/r - \alpha/n + 1/p \geq 1/p$,

$$\int f^pw_\epsilon^p = \int_{B(0,1)} |x|^{p\epsilon(1/r+1/q)-n(1/r+1/q)} \leq \int_{B(0,1)} |x|^{p\epsilon(1/r+1/q)-n/p} \approx \epsilon^{-1}.$$  

Now, if $x \in Q_0$,

$$Af(x) \geq |Q_0|^{\alpha/n-1/r} \left( \int_{Q_0} |y|^{q-1} \right)^{1/r} \geq \left( \frac{1}{\epsilon} \right)^{1/r} \approx \epsilon^{-1/r} \geq \epsilon^{-1}.$$
Then, since $B(0, 1) \subset Q_0$
\[
\int A f^q w^q \gtrsim \epsilon^{-q} \int_{B(0,1)} |x|^{-n} dx \gtrsim \epsilon^{-q-1},
\]
then,
\[
\epsilon^{-1-1/q} \lesssim \|Af\|_{L^q(w^q)} \lesssim \Phi(\epsilon^{-1}) \|f\|_{L^p(w^p)} \lesssim \Phi(\epsilon^{-1}) \epsilon^{-1/p}.
\]
Now, if we take $t = \epsilon^{-1}$ then
\[
t^{1-\alpha/n} \lesssim \Phi(t).
\]

\begin{thebibliography}{10}

[1] Accomazzo, N., Martínez-Perales, J. C., and Rivera-Ríos, I. P. On bloom type estimates for iterated commutators of fractional integrals. \textit{arXiv preprint arXiv:1712.06923} (2017).

[2] Cruz-Uribe, D. Elementary proofs of one weight norm inequalities for fractional integral operators and commutators. In \textit{Harmonic Analysis, Partial Differential Equations, Banach Spaces, and Operator Theory (Volume 2)}. Springer, 2017, pp. 183–198.

[3] Fackler, S., and Hyönen, T. P. Weighted estimates for integral operators on local BMO type spaces. \textit{New York J. Math} 24 (2018), 21–42.

[4] Hytönen, T. P. The sharp weighted bound for general Calderón–Zygmund operators. \textit{Annals of mathematics} (2012), 1473–1506.

[5] Ibañez-Firnkorn, G. H., and Rivera-Ríos, I. P. Sparse and weighted estimates for generalized Hörmander operators and commutators. \textit{arXiv preprint arXiv:1704.01018} (2017).

[6] Kurtz, D. S. Sharp function estimates for fractional integrals and related operators. \textit{Journal of the Australian Mathematical Society} 49, 1 (1990), 129–137.

[7] Lacey, M. T., Moen, K., Pérez, C., and Torres, R. H. Sharp weighted bounds for fractional integral operators. \textit{Journal of Functional Analysis} 259, 5 (2010), 1073–1097.

[8] Lerner, A. K. On pointwise estimates involving sparse operators. \textit{New York J. Math} 22 (2016), 341–349.

[9] Lerner, A. K., and Nazarov, F. Intuitive dyadic calculus: the basics. \textit{Expositiones Mathematicae} (2018).

[10] Lerner, A. K., Ombrosi, S., and Rivera-Ríos, I. P. On pointwise and weighted estimates for commutators of Calderón-Zygmund operators. \textit{arXiv preprint arXiv:1604.01334} (2016).

[11] Li, K. Sparse domination theorem for multilinear singular integral operators with $T$-Hörmander condition. \textit{arXiv preprint arXiv:1606.03925} (2016).

[12] Muckenhoupt, B., and Wheeden, R. Weighted norm inequalities for fractional integrals. \textit{Transactions of the American Mathematical Society} 192 (1974), 261–274.

[13] Riveros, M., and Urciuolo, M. Weighted inequalities for some integral operators with rough kernels. \textit{Open Mathematics} 12, 4 (2014), 636–647.

[14] Stein, E. M. \textit{Singular integrals and differentiability properties of functions}, vol. 30 of \textit{Princeton Mathematical Series}. Princeton university press, 1970.

G. H. Ibañez Firnkorn, FAMAF, Universidad Nacional de Córdoba, CIEM (CONICET), 5000 Córdoba, Argentina
\textit{E-mail address:} gibanez@famaf.unc.edu.ar

M. S. Riveros, FAMAF, Universidad Nacional de Córdoba, CIEM (CONICET), 5000 Córdoba, Argentina
\textit{E-mail address:} sriveros@famaf.unc.edu.ar

R. E. Vidal, FAMAF, Universidad Nacional de Córdoba, CIEM (CONICET), 5000 Córdoba, Argentina
\textit{E-mail address:} vidal@famaf.unc.edu.ar