Chiral transition, eigenmode localisation and Anderson-like models

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Localisation in the Dirac Spectrum

QCD transition: change in both confining and chiral properties, \( \sim \) same \( T_c \)

Above \( T_c \) lowest-lying Dirac eigenmodes are localised for \( \lambda < \lambda_c(T) \)

[García-García, Osborn (2007), Kovács, Pittler (2012), Cossu, Hashimoto (2016)]

Onset of localisation at the chiral crossover

[Garca–Garca, Osborn (2007), Kovács, Pittler (2012)]

Observed also with unimproved staggered fermions, \( N_T = 4 \): localised modes appear at the 1st-order PT [MG, Kovács, Katz, Pittler (2014)]

Understanding localisation might help to understand how chiral symmetry restoration and deconfinement are related

\[
\text{IPR} = \sum_x |\psi(x)|^4
\]

\[
\text{PR} = \text{IPR}^{-1}/V_4
\]

(fraction of 4-vol occupied by a mode)

\[
T \approx 2.6 T_c
\]
Anderson Model vs. High-Temperature QCD

Tight-binding Hamiltonian for “dirty” conductors [Anderson (1958)]

\[
H_{\vec{x},\vec{y}}^{\text{AM}} = \varepsilon_{\vec{x}} \delta_{\vec{x},\vec{y}} + \sum_{\mu=1}^{3} (\delta_{\vec{x}+\hat{\mu},\vec{y}} + \delta_{\vec{x}-\hat{\mu},\vec{y}})
\]

Random potential \(|\varepsilon_{\vec{x}}| \leq \frac{W}{2}\)

Anderson transition: eigenstates localised for \(E > E_c(W)\) (mobility edge)
Second-order phase transition with divergent \(\xi \sim |E - E_c|^{-\nu}\)

Localisation/delocalisation transition in the staggered QCD spectrum at \(\lambda_c(T)\): same universality class of the 3D Unitary Anderson Model

- Correlation-length critical exponents match

\[
\nu_{\text{UAM}} = 1.43(4) \quad \nu_{\text{QCD}} = 1.43(6)
\]

[Slevin, Ohtsuki (1999)] [MG, Kovács, Pittler (2014)]

- Multifractal eigenfunctions near localisation/delocalisation transition, multifractal exponents match [Ujfalusi, MG, Pittler, Kovács, Varga (2015)]
Anderson Model vs. High-Temperature QCD

Tight-binding Hamiltonian for “dirty” conductors [Anderson (1958)]

\[ H_{\vec{x},\vec{y}}^{\text{UAM}} = \varepsilon_{\vec{x}} \delta_{\vec{x},\vec{y}} + \sum_{\mu=1}^{3} (\delta_{\vec{x}+\hat{\mu},\vec{y}} + \delta_{\vec{x}-\hat{\mu},\vec{y}}) e^{i\phi_{\vec{x},\vec{y}}} \]

Random potential \(|\varepsilon_{\vec{x}}| \leq \frac{W}{2}\), random phases \(\phi_{\vec{y},\vec{x}} = -\phi_{\vec{x},\vec{y}}\)

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Why same universality class? Same RMT symmetry class (unitary), but
- UAM: 3D, diagonal noise controlled by $W$ (mainly)
- QCD: 4D, off-diagonal noise controlled by $T$ (as this controls $\lambda_c$)

QCD above $T_c$ is effectively 3D: time slices strongly correlated, quark eigenfunctions look qualitatively the same at all $t$
- Temporal gauge: $U_4(t, \vec{x}) = 1, \ 0 \leq t < N_T - 1, \ U_4(N_T - 1, \vec{x}) = P(\vec{x})$
- Wave functions obey effective boundary conditions involving the Polyakov line
  $$\psi(N_T, \vec{x}) = -P(\vec{x})\psi(0, \vec{x})$$
- $P(\vec{x})$ fluctuates in space, providing effective 3D diagonal disorder

3D noise present also below $T_c$, but boundary conditions are ineffective and QCD is effectively 4D there: why so?
“Hamiltonian” \( H = -iD_{\text{stag}} \) split into “free” + “interaction”, \( H = H_0 + H_I \)

\[
(H_0)_{xx'} = \frac{\eta_4(\vec{x})}{2i} \left[ U_4(t, \vec{x}) \delta_{t+1,t'} - U_{-4}(t, \vec{x}) \delta_{t-1,t'} \right] \delta_{\vec{x},\vec{x}'}
\]

\( (H_I)_{xx'} = \sum_{j=1}^3 \frac{\eta_j(\vec{x})}{2i} \left[ U_j(t, \vec{x}) \delta_{\vec{x}+\hat{j},\vec{x}'} - U_{-j}(t, \vec{x}) \delta_{\vec{x}-\hat{j},\vec{x}'} \right] \delta_{t,t'} \)

Temporal diagonal (td) gauge: \( U_4(N_T-1, \vec{x}) = P(\vec{x}) = \text{diag}(e^{i\phi_a(\vec{x})}) \)

Work in the basis of the “unperturbed” eigenvectors of \( H_0 \)

Eigenvectors of \( H_0 \):

\[
H_0 \psi_0^{\vec{x}a} = \eta_4(\vec{x}) \sin \omega_{ak}(\vec{x}) \psi_0^{\vec{x}a}
\]

Effective Matsubara frequencies:

\[
\omega_{ak}(\vec{x}) = \frac{1}{N_T} (\pi + \phi_a(\vec{x}) + 2\pi k)
\]

Indices: spatial site \( \vec{x} \), colour \( a = 1, \ldots, N_c \), temporal momentum \( k = 0, \ldots, N_T - 1 \)
Dirac-Anderson Hamiltonian II

In the basis of the “unperturbed” eigenvectors of $H_0$

$$H_{\vec{x},\vec{y}} = \delta_{\vec{x},\vec{y}} D(\vec{x}) + \sum_{j=1}^{3} \frac{\eta_j(\vec{x})}{2i} \left[ \delta_{\vec{x}+\hat{j},\vec{y}} V_+(\vec{x}) - \delta_{\vec{x}-\hat{j},\vec{y}} V_-(\vec{x}) \right]$$

3D Anderson-type Hamiltonian with internal degrees of freedom: colour $a = 1, \ldots, N_c$, temporal momentum $k = 0, \ldots, N_T - 1$

Diagonal noise (random on-site potential)

$$[D(\vec{x})]_{ak,bl} = \eta_4(\vec{x}) \sin \omega_{ak}(\vec{x}) \delta_{ab} \delta_{kl},$$

Off-diagonal noise (random hoppings)

$$[V_{\pm j}(\vec{x})]_{ak,bl} = \frac{1}{N_T} \sum_{t=0}^{N_T-1} e^{i\frac{2\pi t}{N_T} (l-k)} e^{i\frac{t}{N_T} [\phi_b(\vec{x} \pm \hat{j}) - \phi_a(\vec{x})]} \left[ U^{(td)}_{\pm j}(t, \vec{x}) \right]_{ab}$$

Unlike the AM, disorder strength bounded ($|\sin \omega_{ak}| \leq 1, \ V_{\pm j} \text{ unitary} \), but type of disorder different in the confined and deconfined phase.
Above $T_c$, $P(\vec{x})$ gets ordered along $1$ with “islands” of “wrong” $P(\vec{x}) \neq 1$

- $\sin \omega_{ak=0} |_{\phi_a=0} = \sin \frac{\pi}{N_T}$ provides an effective gap in the spectrum
- “wrong” $P(\vec{x})$ allows for smaller $\lambda \Rightarrow$ localising “trap” for eigenmodes

[Bruckmann, Kovács, Schierenberg (2011), MG, Kovács, Pittler (2015)]

Ordering of PL induces correlation across time slices $\rightarrow$ reduced mixing of temporal momentum components: if $\phi_a(\vec{x}) = 0$ and $U_{\pm j}(t, \vec{x}) = \bar{U}_{\pm j}(\vec{x})$

$$[V_{\pm j}(\vec{x})]_{ak,bl} = \left[ \bar{U}_{\pm j}(\vec{x}) \right]_{ab} \frac{1}{N_T} \sum_{t=0}^{N_T-1} e^{i \frac{2\pi t}{N_T} (l-k)} = \left[ \bar{U}_{\pm j}(\vec{x}) \right]_{ab} \delta_{kl}$$

- Strong mixing of t-mom components $\rightarrow$ single effectively 4D system
- Decoupling of t-mom components $\rightarrow$ $N_T$ effectively 3D systems

Should be relevant also to $\chi_{SB}$: below/above $T_c$

- high/low density of small “unperturbed” eigenvalues
- strong/weak “push” towards the origin due to mixing
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Should be relevant also to $\chi_{SB}$: below/above $T_c$
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Relevant features for $\chi$SB and localisation should be ordering of the diagonal noise and correlation of spatial links across time slices: check in a toy model

Hamiltonian with same structure of the Dirac-Anderson Hamiltonian

$$
\mathcal{H}^{toy}_{\vec{x},\vec{y}} = d(\vec{x})\delta_{\vec{x},\vec{y}} + \sum_{j=1}^{3} \frac{\eta_j(\vec{x})}{2i} \left( v_+ j(\vec{x})\delta_{\vec{x}+\hat{j},\vec{y}} - v_- j(\vec{x})\delta_{\vec{x}-\hat{j},\vec{y}} \right)
$$

Replace Polyakov-line phases with complex-spin phases $\phi^a_{\vec{x}}$, $\sum_a \phi^a_{\vec{x}} = 0$

$$
[d(\vec{x})]_{ak,bl} = \eta_4(\vec{x}) \sin \frac{\pi + \phi^a_{\vec{x}} + 2\pi k}{N_T} \delta_{ab} \delta_{kl}
$$

Spin dynamics mimicking that of Polyakov lines

$$
\beta H_{noise} = - \frac{\beta}{N_c} \sum_{\vec{x},j,a} \cos(\phi^a_{\vec{x}+\hat{j}} - \phi^a_{\vec{x}}) - \frac{2h}{N_c(N_c-1)} \sum_{\vec{x},a<b} \cos(\phi^a_{\vec{x}} - \phi^b_{\vec{x}})
$$

Ordered phase $\rightarrow$ $N_c$ vacua $\phi^a_{\vec{x}} = \frac{2\pi}{N_c} \forall a, \vec{x}$ (analogues of the center sectors)
Hopping terms from “toy” gauge links $u_j(t, \vec{x})$

$$[v_{\pm j}(\vec{x})]_{ak,bl} = \frac{1}{N_T} \sum_{t=0}^{N_T-1} e^{\frac{i 2\pi t}{N_T} (l-k)} e^{i \frac{t}{N_T} [\phi_b^{\pm j} - \phi_a^{\pm j}]} [u_{\pm j}(t, \vec{x})]_{ab}$$

“Toy” gauge links obey a simpler dynamics: Wilson action without spatial plaquettes, no fermion determinant, Polyakov line as external field

$$S_u = -\hat{\beta} \text{Re} \text{Tr} \sum_{\vec{x}} \sum_{j=1}^{3} \left\{ \left[ \sum_{t=0}^{N_T-2} u_j(t, \vec{x})u_j^{\dagger}(t+1, \vec{x}) \right] + u_j(N_T-1, \vec{x})p(\vec{x} + \hat{j})u_j^{\dagger}(0, \vec{x})p^{\dagger}(\vec{x}) \right\}$$

$$p(\vec{x}) = \text{diag}(e^{i \phi_{\vec{x}}^a})$$

Expectation values:

$$\langle \mathcal{O} \rangle = \frac{\int D\phi \ e^{-\beta H_{\text{noise}}[\phi]} \left[ \int Du e^{-S_u[\phi, u]} \mathcal{O}[\phi, u] \right]}{\int D\phi \ e^{-\beta H_{\text{noise}}[\phi]}}$$
Minimal Toy Model: $N_c = N_T = 2$

Single phase $\phi_{\vec{x}} = \phi_{\vec{x}}^1 = -\phi_{\vec{x}}^2$

$$\beta H_{\text{noise}}^{N_c=2} = -\beta \sum_{\vec{x},j} \cos(\phi_{\vec{x}+\vec{j}} - \phi_{\vec{x}})$$

$$- h \sum_{\vec{x}} \cos(2\phi_{\vec{x}})$$

$U(1)$ symmetry broken to $\mathbb{Z}_2$ at $h \neq 0$

$\beta < \beta_c / \beta > \beta_c$ disordered/ordered phase ($\sim$ confined/deconfined phase)

In general

$$\sin \omega_{ak'}(\vec{x}) = -\sin \omega_{ak}(\vec{x}) \quad \text{if} \quad k' = \frac{N_T}{2} + k \mod N_T$$

$$\sin \frac{-\phi + \pi + 2\pi k}{N_T} = \sin \frac{\phi + \pi + 2\pi \left(\frac{N_T}{2} - 1 - k\right)}{N_T}$$

For $N_c = N_T = 2$ only one relevant Matsubara frequency $\omega(\vec{x}) = \frac{\phi_{\vec{x}} + \pi}{2}$

“Unperturbed” eigenvalues: $\pm \eta_4(\vec{x}) \cos \frac{\phi_{\vec{x}}}{2}$
Numerical Results: Chiral Symmetry

Fix “gauge coupling” $\hat{\beta} = 5.0$ and symmetry-breaking term $h = 1.0$, study dependence on $\beta$ (i.e., ordering of the spin system)

“Chiral symmetry breaking” = nonzero spectral density at the origin

below $\beta_c$: nonzero $\rho(0)$, weakly dependent on $\beta$

above $\beta_c$: zero $\rho(0)$, $\rho(\lambda)$ suppressed as $\beta$ is increased

Disordered phase ($\beta < \beta_c$): chiral symmetry broken ($\chi_{SB}$)

Ordered phase ($\beta > \beta_c$): chiral symmetry restored ($\chi_{SR}$)
Numerical Results: Localisation

Detect localisation from the statistical properties of the spectrum: Poisson for localised, Gaussian Symplectic Ensemble of RMT for delocalised modes

Integrated unfolded level spacing distribution

\[ I_{0.5} = \int_0^{0.5} ds P_\lambda(s) \]

\[ s_i = \frac{\lambda_{i+1} - \lambda_i}{\langle \lambda_{i+1} - \lambda_i \rangle} \]

Disordered phase \((\beta < \beta_c)\): all modes delocalised

Ordered phase \((\beta > \beta_c)\): localisation of lowest modes

Toy model reproduces qualitatively the properties of the QCD spectrum and eigenmodes
Variations on the Toy Model: Tweaking the Hopping Term

No correlations among gauge links across time slices (ordered phase)

No hopping between different temporal momenta (disordered phase)

Mixing of t-mom components crucial for $\chi_{SB}$, reduced mixing for $\chi_{SR}$
Variations on the Toy Model: Tweaking the Diagonal Term

Periodic boundary condition in the temporal direction (ordered phase)

Increased magnetisation (disordered phase) $z_{\chi} = \cos \frac{\phi_{\chi}}{2} \rightarrow \frac{2z_{\chi}^2}{1+z_{\chi}^2}, \tau = 0.2$

Large enough density of small unperturbed eigenvalues needed for $\chi_{SB}$
Staggered Dirac operator equivalent to an Anderson-type Hamiltonian with internal degrees of freedom

Chiral symmetry restoration and localisation of lowest modes depend on the ordering of the Polyakov lines:
- reduced density of small unperturbed eigenvalues
- reduced mixing of temporal-momentum components

Tested in a toy model: properties of QCD eigenmodes reproduced, ordering and correlation of spatial links only relevant features

Open issues:
- Does localisation appear exactly at the transition in the toy model?
- Study in this formalism: imaginary chemical potential, magnetic field, adjoint fermions...
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