Class, degree and integral free forms for the family of Bour’s minimal surfaces

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September 29, 2018

Abstract
We consider the family of the Bour’s minimal surfaces in Euclidean 3-space, and compute their classes, degrees and integral free representations.

1 Introduction

A minimal surface in $\mathbb{E}^3$ is a regular surface for which the mean curvature vanishes identically.

Minimal surfaces applicable onto a rotational surface were first determined by E. Bour [1] in 1862. These surfaces have been called $\mathbb{B}_m$ (following Haag) to emphasize the value of $m$. Mathematicians have dealt with the $\mathbb{B}_m$ in the literature: E. Bour (1862), H.A. Schwarz (1875), A. Ribaucour (1882), A. Thybaut (1887), A. Demoulin (1897), L. Bianchi (1899), J. Haag (1906), G. Darboux (1914), E. Stübler (1914), J. K. Whittemore (1917), B. Gambier (1921), G. Calugareano (1938).

It was proven by Schwarz [10] that all real minimal surfaces applicable to rotational surfaces are given by Whittemore setting

$$\mathfrak{f}(s) = Cs^{m-2}$$

in the Weierstrass representation equations, where $s, C \in \mathbb{C}, m \in \mathbb{R}$, and $\mathfrak{f}(s)$ is an analytic function. For $C = 1, m = 0$ we obtain the catenoid, $C = i, m = 0$, the right helicoid, $C = 1, m = 2$, Enneper’s surface (see also [3, 7, 14]). A. Gray [4] gave the complex forms of the Bour’s curve and surface of value $m$.

2 Preliminaries

Let $\mathbb{E}^3$ be three dimensional Euclidean space with natural metric $\langle \cdot, \cdot \rangle = dx^2 + dy^2 + dz^2$. We will often identify $\mathbb{R}^3$ and $\mathbb{R}^4$ without further comment.
Let $\mathcal{U}$ be an open subset of $\mathbb{C}$. A minimal (or isotropic) curve is an analytic function $\Psi : \mathcal{U} \to \mathbb{C}^n$ such that $\Psi'(\zeta) \cdot \overline{\Psi'(\zeta)} = 0$, where $\zeta \in \mathcal{U}$, and $\Psi' := \frac{d\Psi}{d\zeta}$.

In addition, if $\Psi' \cdot \overline{\Psi'} = |\Psi'|^2 \neq 0$, then $\Psi$ is a regular minimal curve. We then have minimal surfaces in the associated family of a minimal curve, like as given by the following Weierstrass representation theorem for minimal surfaces.

**Theorem 1 (K. Weierstrass [12])**. Let $\mathcal{F}$ and $\mathcal{G}$ be two holomorphic functions defined on a simply connected open subset $\mathcal{U}$ of $\mathbb{C}$ such that $\mathcal{F}$ does not vanish on $\mathcal{U}$. Then the map

$$x(\zeta) = \text{Re} \int_{\zeta} \left( \frac{\mathcal{F}(1 - \mathcal{G}^2)}{\mathcal{G} \mathcal{F}} \right) d\zeta$$

is a minimal, conformal immersion of $\mathcal{U}$ into $\mathbb{C}^3$, and $x$ is called the Weierstrass patch.

**Lemma 2** Let $\Psi : \mathcal{U} \to \mathbb{C}^3$ minimal curve and write $\Psi' = (\varphi_1, \varphi_2, \varphi_3)$. Then

$$\mathcal{F} = \frac{\varphi_3 - i \varphi_2}{2} \quad \text{and} \quad \mathcal{G} = \frac{\varphi_1 - i \varphi_2}{\varphi_3}$$

give rise to the Weierstrass representation of $\Psi$. That is

$$\Psi' = \left( \frac{\mathcal{F}(1 - \mathcal{G}^2)}{\mathcal{F} \mathcal{G}} \right) , i \left( \frac{\mathcal{F}(1 + \mathcal{G}^2)}{\mathcal{F} \mathcal{G}} \right) .$$

In section 3, we give the family of Bour’s minimal surfaces $\mathcal{B}_m$. We obtain the class and degree of surface $\mathcal{B}_3$ (resp., $\mathcal{B}_4$) in section 4 (resp., in section 5). Finally, using the integral free form of Weierstrass, we find some algebraic functions for $\mathcal{B}_m$ ($m \geq 3, m \in \mathbb{Z}$) in the last section.

### 3 The family of Bour’s minimal surfaces $\mathcal{B}_m$

We consider the Bour’s curve of value $m$.

**Lemma 3** The Bour’s curve of value $m$

$$B_m(\zeta) = \left( \frac{\zeta^{m-1}}{m-1} - \frac{\zeta^{m+1}}{m+1}, i \left( \frac{\zeta^{m-1}}{m-1} + \frac{\zeta^{m+1}}{m+1} \right) , 2 \frac{\zeta^m}{m} \right)$$

is a minimal curve in $\mathbb{C}^3$, where $m \in \mathbb{R} - \{-1, 0, 1\}$, $\zeta \in \mathbb{C}$, $i = \sqrt{-1}$.

We have

$$B'_m \cdot B'_m = 0.$$  \hspace{1cm} (2)

Bour’s surface of value $m$ in $\mathbb{R}^3$ is

$$\mathcal{B}_m(\zeta) = \text{Re} \int B'_m(\zeta) d\zeta.$$  \hspace{1cm} (3)
Lemma 4  The Weierstrass patch determined by the functions

\[ \mathcal{F}(\zeta) = \zeta^{m-2} \quad \text{and} \quad \mathcal{G}(\zeta) = \zeta \]

is a representation of \( \mathcal{B}_m \).

Therefore, the associated family of minimal surfaces is described by

\[ \mathcal{B}(r, \theta; \alpha) = \text{Re} \int e^{-i\alpha} B_m' + \text{Im} \int B_m' \]

When \( \alpha = 0 \) (resp. \( \alpha = \pi/2 \)), we have the Bour’s surface of value \( m \) (resp. the conjugate surface \( \mathcal{B}_m^* \)).

The parametric equations of \( \mathcal{B}_m \), in polar coordinates \( \zeta = re^{i\theta} \), are

\[ \mathcal{B}_m(r, \theta) = \left( \frac{r^{m-1} \cos((m-1)\theta)}{m-1} - \frac{r^{m+1} \cos((m+1)\theta)}{m+1} \right), \quad \text{with Gauss map} \]

\[ n = \left( \frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right). \]

Remark 5  \( \mathcal{B}_m \), \( m \geq 3 \), \( m \in \mathbb{Z} \), has a branch point at \( \zeta = 0 \). Also, the total curvature of \( \mathcal{B}_3 \) is \(-4\pi\). Note that the catenoid and Enneper’s surface are the only complete regular minimal surfaces in \( \mathbb{E}^3 \) with finite total curvature \(-4\pi \) [8].

Remark 6  Ribaucour showed that each curve \( \mathcal{B}_m \mid_{r=r_0} \) lies on the quadric of revolution

\[ x^2 + y^2 + \frac{m^2}{m^2 - 1} z^2 = \left( \frac{r_0^{m-1}}{m-1} + \frac{r_0^{m+1}}{m+1} \right)^2. \]

Next, we will focus on the degree and class of surface \( \mathcal{B}_m \).

With \( \mathbb{R}^3 = \{ (x, y, z) \mid x, y, z \in \mathbb{R} \} \), the set of roots of a polynomial \( f(x, y, z) = 0 \) gives an algebraic surface. An algebraic surface is said to be of degree (or order) \( n \) when \( n = \deg(f) \).

The tangent plane on a surface \( x(u, v) = (x(u, v), y(u, v), z(u, v)) \) at a point \((u, v)\) is given by

\[ Xx + Yy + Zz + P = 0, \]

where the Gauss map is \( n = (X(u, v), Y(u, v), Z(u, v)) \), \( P = P(u, v) \). We have inhomogeneous tangential coordinates \( \overrightarrow{\mathbf{r}} = X/P, \overrightarrow{\mathbf{v}} = Y/P, \) and \( \overrightarrow{\mathbf{w}} = Z/P \). By eliminating \( u \) and \( v \), we obtain an implicit equation of \( x(u, v) \) in tangential
coordinates. The maximum degree of the equation gives class of \(x(u,v)\). See \cite{7}, for details.

General cases of degree and class of \(B_m\) were studied by Demoulin \cite{2}, Haag \cite{5}, Ribaucour \cite{9} and St" ubler \cite{11}. Using the binomial formula we obtain the following parametric equations of \(B_m(u,v)\):

\[
x = \text{Re} \left\{ \frac{1}{m-1} \left[ \sum_{k=0}^{m-1} \binom{m-1}{k} u^{m-1-k} (iv)^k \right] - \frac{1}{m+1} \left[ \sum_{k=0}^{m+1} \binom{m+1}{k} u^{m+1-k} (iv)^k \right] \right\},
\]

\[
y = \text{Re} \left\{ \frac{i}{m-1} \left[ \sum_{k=0}^{m-1} \binom{m-1}{k} u^{m-1-k} (iv)^k \right] + \frac{i}{m+1} \left[ \sum_{k=0}^{m+1} \binom{m+1}{k} u^{m+1-k} (iv)^k \right] \right\},
\]

\[
z = \text{Re} \left\{ \frac{2}{m} \sum_{k=0}^{m} \binom{m}{k} u^{m-k} (iv)^k \right\}.
\]

(7)

It is clear that \(\deg(x) = m+1, \deg(y) = m+1, \deg(z) = m\) (see also Table 1).

Ribaucour showed that if \(m = \frac{p}{q}\) then \(cl(B_m) = 2q(p+q), \ m \in \mathbb{Z}\) then \(\deg(B_m) = (m+1)^2\),

\(m < 1\) then \(cl(B_m) = \deg(B_m)\), \(m > 1\) then \(cl(B_m) < \deg(B_m)\).

Using eliminate methods we calculate the implicit equations, degrees and classes of the surfaces \(B_2, B_3, B_4\). Our findings agree with Ribaucour’s, and we give them in Table 1. For the surface \(B_2\) (i.e., Enneper’s surface, see Fig. 1, left two pictures), it is known that the surface has class 6, degree 9. So, it is also an algebraic minimal surface. For expanded results on \(B_2\), see \cite{7}.

4 Degree and class of \(B_3\)

The simplest Weierstrass representation \((\mathfrak{F}, \mathfrak{G}) = (\zeta, \zeta)\) gives the Bour’s minimal surface of value 3. In polar coordinates, the parametric equations of \(B_3\) (see Fig. 1, right two pictures) are

\[
B_3(r, \theta) = \begin{pmatrix}
\frac{r^2}{2} \cos(2\theta) - \frac{r^4}{4} \cos(4\theta) \\
-\frac{r^2}{2} \sin(2\theta) - \frac{r^4}{4} \sin(4\theta) \\
\frac{3}{4} r^3 \cos(3\theta)
\end{pmatrix},
\]

(8)

where \(r \in [-1,1], \ \theta \in [0, \pi]\). When \(r = 1\) on plane \(xy\), we have deltoid curve, which is a 3-cusped hypocycloid (Steiner’s hypocycloid (1856)), also called tricuspid, is discovered by Euler in 1745. The parametric form of the surface \(B_3\), in \((u,v)\) coordinates, is

\[
B_3(u,v) = \begin{pmatrix}
-\frac{u^4}{4} - \frac{u^6}{8} + \frac{3}{4} u^2 v^2 + \frac{v^2}{4} - \frac{v^4}{2} \\
-u^3 v + uv^3 - uv \\
\frac{2}{3} u^3 - 2uv^2
\end{pmatrix} = \begin{pmatrix}
x(u,v) \\
y(u,v) \\
z(u,v)
\end{pmatrix},
\]

(9)
where \( u, v \in \mathbb{R} \). Using the Maple eliminate codes we find the irreducible implicit equation of surface \( B_3 \) as follows:

\[
43046721 z^{16} - 859963392 x^4 y^2 z^4 \\
-1719926784 x^2 y^2 z^6 + 509607936 x^2 y^4 z^4 \\
+ 69 \text{ other lower order terms} = 0,
\]

and its degree is \( \text{deg}(B_3) = 16 \). Therefore, \( B_3 \) is an algebraic minimal surface.

To find the class of surface \( B_3 \), we obtain

\[
P(u, v) = \frac{(u^2 + v^2 + 2)(3uv^2 - u^3)}{6(u^2 + v^2 + 1)},
\]

and the inhomogeneous tangential coordinates

\[
\begin{align*}
\overline{u} &= \frac{12u}{(u^2 + v^2 + 2)(3uv^2 - u^3)}, \\
\overline{v} &= \frac{12v}{(u^2 + v^2 + 2)(3uv^2 - u^3)}, \\
\overline{w} &= \frac{6(u^2 + v^2 - 1)}{(u^2 + v^2 + 2)(3uv^2 - u^3)}.
\end{align*}
\]

In tangential coordinates \( \overline{u}, \overline{v}, \overline{w} \), the irreducible implicit equation of \( B_3 \) is

\[
9\overline{u}^6 + 72\overline{u}^5 + 144\overline{u}^4 + 288\overline{u}^3\overline{w}^2 + 192\overline{u}^2\overline{v}^2\overline{w}^2 + 8\overline{u}\overline{v}^2\overline{w}^2 + 81\overline{v}^4\overline{w}^2 + 864\overline{u}^2\overline{v}^2\overline{w}^2 - 864\overline{v}^6\overline{w}^2 = 0.
\]

Therefore, the class of the algebraic minimal surface \( B_3 \) is \( \text{cl}(B_3) = 8 \).

**Remark 7** Henneberg showed that a plane intersects an algebraic minimal surface in an algebraic curve \([7]\). Using the Gröbner eliminate method we find that the implicit equation of the curve \( B_3(r, 0) = \gamma(r) = \left( \frac{r^2}{2} - \frac{r^2}{4}, 0, \frac{2}{3} r^3 \right) \) (see Fig. 2, right two pictures) on the \( xz \)-plane is

\[
1024x^2 + 864xz^2 - 288z^2 + 81z^4 = 0,
\]

and its degree is \( \text{deg}(\gamma) = 4 \). So, we see that the \( xz \)-plane intersects the algebraic minimal surface \( B_3 \) in an algebraic curve \( \gamma(r) \) (see Fig. 2, left two pictures).

**Remark 8** The Bour’s minimal curve of value 3 is intersects itself three times along three straight rays, which meet an angle \( 2\pi/3 \) at the origin in \( \mathbb{E}^3 \). The surface \( B_3 \) has self-intersections along three linear rays \( u = 0, u \pm v\sqrt{3} = 0 \) at distinct distances from the branch point \( O(0, 0, 0) \), where \( \zeta = u + iv = re^{i\theta} \).
5 Degree and class of $\mathcal{B}_4$

The parametric form of $\mathcal{B}_4$ (see Fig. 3, left two pictures) is

$$\mathcal{B}_4 (r, \theta) = \left( \frac{r^3}{3} \cos (3\theta) - \frac{r^5}{5} \cos (5\theta), \frac{r^3}{3} \sin (3\theta) - \frac{r^5}{5} \sin (5\theta) \right), \quad (10)$$

where $r \in [-1, 1]$, $\theta \in [0, \pi]$. In $(u,v)$ coordinates, $\mathcal{B}_4$ has the form as follows

$$\mathcal{B}_4 (u,v) = \left( -u^2 v + \frac{1}{4} v^3 - u^4 v + 2u^2 v^3 - \frac{1}{3} v^5, \frac{1}{2} u^4 - 3u^2 v^2 + \frac{1}{2} v^4 \right), \quad (11)$$

where $u,v \in \mathbb{R}$. The implicit equation of $\mathcal{B}_4 (u,v)$, in cartesian coordinates $x,y,z$, is as follows

$$48466299163780426235904 \cdot 25 - 14790740711602913280000x^4z^{20} + 88744442696174796800000x^2y^2z^{20} - 14790740711602913280000y^4z^{20} - 2640558873378816000000000x^8z^{15} + 233 \text{ other lower order terms} = 0.$$  

Its degree is $\deg(\mathcal{B}_4) = 25$. Hence, $\mathcal{B}_4$ is an algebraic minimal surface. To find the class of surface $\mathcal{B}_4$ we obtain

$$P(u,v) = \frac{(3u^2 + 3v^2 + 5) (u^4 + 6u^2v^2 - u^4)}{30(u^2 + v^2 + 1)},$$

and the inhomogeneous tangential coordinates

$$\pi = \frac{60u}{(3u^2 + 3v^2 + 5) (v^4 + 6u^2v^2 - u^4)},$$

$$\nu = \frac{60v}{(3u^2 + 3v^2 + 5) (v^4 + 6u^2v^2 - u^4)},$$

$$\omega = \frac{30(u^2 + v^2 - 1)}{(3u^2 + 3v^2 + 5) (v^4 + 6u^2v^2 - u^4)}.$$  

So, the irreducible implicit equation of $\mathcal{B}_4$, in tangential coordinates $\pi, \nu, \omega$, is

$$900\pi^8 w + 15\pi^6 w^2 + 15\pi^4 w^2 - 180\pi^2 w^2 - 180\pi^2 v^2 w^2 + 3600\pi^2 v^6 w + 416\pi^6 v^6 - 3600\pi^2 v^6 - 3600\pi^2 v^6 w^3 - 176\pi^4 v^8 - 5400\pi^4 w^4 + 416\pi^6 w^4 - 900\pi^4 w^4 - 900\pi^4 + 16\pi^4 + 16 \pi^{20}$$

$$-900\pi^8 - 1440\pi^6 w^5 - 1440\pi^4 w^5 - 2400\pi^2 w^5 + 12000\pi^2 v^2 w^3 + 3600\pi^2 v^2 w^3 - 180\pi^2 v^2 w^3 - 176\pi^8 v^2 - 240\pi^6 w^3 - 900\pi^4 w^3 w + 1200\pi^2 w^3 + 570\pi^2 w^3 = 0.$$  

Hence, the class of the algebraic minimal surface $\mathcal{B}_4$ is $cl(\mathcal{B}_4) = 10$. We see that the family of $\mathcal{B}_m (u,v) = (x(u,v), y(u,v), z(u,v))$ are algebraic minimal surfaces, where $m \in \mathbb{Z}$, $m \geq 2$ (see Table 1).
6 Integral free form

Integral free form of the Weierstrass representation is

\[
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} = \text{Re} \left( \begin{pmatrix}
  (1 - w^2) \phi''(w) + 2w\phi'(w) - 2\phi(w) \\
  i \left[ (1 + w^2) \phi''(w) - 2w\phi'(w) + 2\phi(w) \right] \\
  2 \left[ w\phi''(w) - \phi'(w) \right]
\end{pmatrix} \right) \equiv \text{Re} \left( \begin{pmatrix}
  f_1(w) \\
  f_2(w) \\
  f_3(w)
\end{pmatrix} \right),
\]

(12)

where algebraic function \( \phi(w) \) and the functions \( f_i(w) \) are connected by the relation

\[
\phi(w) = \frac{1}{4} (w^2 - 1) f_1(w) - \frac{i}{4} (w^2 + 1) f_2(w) - \frac{1}{2} wf_3(w)
\]

(13)

for \( w \in \mathbb{C} \). Integral free form is suitable for algebraic minimal surfaces. For instance, \( \phi(w) = \frac{1}{4} w^3 \) give rise to Enneper’s minimal surface \( \mathcal{B}_2 \) (see [7]).

We obtain the function

\[
\phi(w) = \frac{1}{24} w^4
\]

(14)

leads to Bour’s minimal surface \( \mathcal{B}_3 \). We also obtain \( \phi_{\mathcal{B}_4}(w) = \frac{1}{60} w^5 \) for \( \mathcal{B}_4 \), \( \phi_{\mathcal{B}_5}(w) = \frac{1}{120} w^6 \) for \( \mathcal{B}_5 \), ...

\[
\phi_{\mathcal{B}_m}(w) = \frac{1}{(m - 1)m(m + 1)} w^{m+1}
\]

(15)

for \( \mathcal{B}_m \), where \( m \geq 2, m \in \mathbb{Z} \).

Remark 9 We find relations between degree of algebraic function \( \phi^2(w) \) in the integral free form and class of surfaces \( \mathcal{B}_m \), for integers \( m \geq 2 \). We know \( \phi_{\mathcal{B}_m}(w) = \frac{1}{(m - 1)m(m + 1)} w^{m+1} \) for \( \mathcal{B}_m \), \( m \geq 2, m \in \mathbb{Z} \). Therefore, we obtain \( \deg(\phi_{\mathcal{B}_2}^2) = 6 = \text{cl}(\mathcal{B}_2) \), \( \deg(\phi_{\mathcal{B}_3}^2) = 8 = \text{cl}(\mathcal{B}_3) \), \( \deg(\phi_{\mathcal{B}_4}^2) = 10 = \text{cl}(\mathcal{B}_4) \), ..., \( \deg(\phi_{\mathcal{B}_m}^2) = 2m + 2 = \text{cl}(\mathcal{B}_m) \).

We can see any other parametric eq. and also figure of surface \( \mathcal{B}_m \) for arbitrary \( m \in \mathbb{R} \) using Maple codes. For the figure of \( \mathcal{B}_5 \) (resp. \( \mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8, \mathcal{B}_9, \mathcal{B}_{10} \)), see Fig. 3, right two pictures (resp. Fig. 4, Fig. 5, Fig. 6).

Remark 10 We can calculate class of \( \mathcal{B}_m \) for integers \( m \geq 5 \), but not calculate degree using Maple codes. Calculation of degree is a time problem for software programmes.

Acknowledgements. The author (visiting as a post doctoral researcher of Katholieke Leuven University, Belgium in 2011-2012 academic year, and also of Kobe University, Japan at the end of 2013-2014 academic year) would like to thank the members of the geometry sections, Professor Franki Dillen (1963-2013), Professor Wayne Rossman, Dr. Ana Irina Nistor and Masashi Yasumoto for their valuable comments and hospitality.
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Table 1. Class and degree of $\mathfrak{B}_m(u,v)$, $m \geq 2$, $m \in \mathbb{Z}$

Figure 1. Left two: Enneper surface $\mathfrak{B}_2(r,\theta)$, right two: Bour surface $\mathfrak{B}_3(r,\theta)$

Figure 2. Left: Surface $\mathfrak{B}_3(r,\theta)$, right: its algebraic curve on the xz-plane

Figure 3. Left two: Surface $\mathfrak{B}_4(r,\theta)$, right two: Surface $\mathfrak{B}_5(r,\theta)$
Figure 4. Left two: Surface $\mathcal{B}_6 (r, \theta)$, right two: Surface $\mathcal{B}_7 (r, \theta)$

Figure 5. Left two: Surface $\mathcal{B}_8 (r, \theta)$, right two: Surface $\mathcal{B}_9 (r, \theta)$

Figure 6. Left: Surface $\mathcal{B}_{10} (r, \theta)$, right: its top view