Boundary States of D-branes in $AdS_3$
Based on Discrete Series

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Abstract

We study D-branes in the Lorentzian $AdS_3$ background from the viewpoint of boundary states, emphasizing the role of open-closed duality in string theory. Employing the world sheet with Lorentzian signature, we construct the Cardy states with the discrete series. We show that they are compatible with (1) unitarity and normalizability, and (2) the spectral flow symmetry, in the open string spectrum. We also discuss their brane interpretation. We further show that in the case of superstrings on $AdS_3 \times S^3 \times T^4$, our Cardy states yield an infinite number of physical BPS states in the open string channel, on which the spectral flows act consistently.
1 Introduction

Closed string theory on the $AdS_3$ background has attracted a great deal of attention for several reasons. Among other things, it provides a non-trivial example of solvable string theories on a non-compact curved space-time, and it was studied for this reason in several pioneering works [1]. More recently, it has been studied from the viewpoint of the AdS/CFT correspondence [2] at the stringy level in Refs. [3, 4, 5] and also in many subsequent works. A detailed reference list is presented in, e.g., Ref. [6], and recent works on this subject are given in Ref. [7]. By contrast, there are comparably few works concerning the open string sectors of $AdS_3$ string theory, in other words, D-branes in the $AdS_3$ background [8, 9, 10, 11, 12, 13, 14].

The main purpose of this paper is to propose a boundary state description of D-branes in (super) string theory on the $AdS_3$ background, which is known to be described by the $SL(2; \mathbb{R})$ WZW model. The method for constructing the complete basis of boundary states in general WZW models, called Ishibashi states, is well known [15]. For this reason, it is easy to solve the general gluing conditions in any representation space of the current algebra. However, elucidating the brane interpretation of boundary states in the $SL(2; \mathbb{R})$ WZW model is still a difficult problem. The main difficulty originates from the fact that we have infinitely many representations in the physical Hilbert space and thus infinitely many Ishibashi states, which sharply contrasts with the situation in the $SU(2)$ WZW model. An important constraint to determine the spectrum of allowed boundary states is the Cardy condition [16], which embodies the open-closed string duality. In addition to the Cardy condition, we stipulate the following condition: Any states appearing in both open and closed string channels of the cylinder amplitudes must be consistent with the requirement of the no-ghost theorem and normalizability. This condition is the starting point of our investigation. We further assume consistency with the actions of the spectral flows, which is natural when comparing with the analysis of classical solutions presented in Refs. [11, 13].

This paper is organized as follows. In section 2, we briefly review some familiar results regarding D-branes in the $AdS_3$ background. In section 3, which is the main section of this paper, we study the boundary states constructed from the discrete series and discuss their brane interpretation. We further study the open string spectrum of the on-shell BPS states. Section 4 is devoted to discussion of several open problems.
2 D-Branes in $AdS_3$ space

In this section, we present a short review of the known results for D-branes in the $AdS_3$ space. The $AdS_3$ space can be defined as a hyper-surface in 4-dimensional flat space with signature $(2,2)$, such as

\[(X_0)^2 - (X_1)^2 - (X_2)^2 + (X_3)^2 = l_{AdS}^2, \tag{2.1}\]

where $l_{AdS}$ is the radius of the $AdS_3$ space. We set $l_{AdS} = 1$. This space is the $SL(2;\mathbb{R})$ group manifold, and a useful parametrization is given by

\[g = \begin{pmatrix} X_0 + X_1 & -X_2 + X_3 \\ -X_2 - X_3 & X_0 - X_1 \end{pmatrix}. \tag{2.2}\]

Another useful parametrization is given by

\[g = e^{i\sigma^2 \frac{z}{2}} e^{i\sigma^3 \rho} e^{i\sigma^2 \frac{\bar{z}}{2}}, \tag{2.3}\]

or equivalently,

\[X_0 = \cos t \cosh \rho, \quad X_1 = \cos \theta \sinh \rho, \quad X_2 = \sin \theta \sinh \rho, \quad X_3 = \sin t \cosh \rho, \tag{2.4}\]

which are called the “global coordinates”. The $AdS_3$ metric is written with this parametrization as

\[ds^2 = d\rho^2 - \cosh^2 \rho dt^2 + \sinh^2 \rho d\theta^2. \tag{2.5}\]

String theory in the group manifold can be described by the WZW model. We now consider the $SL(2;\mathbb{R})_{k+2}$ WZW model (we consider the level $k+2$ rather than the level $k$ only for convenience), whose action is given by

\[S = \frac{i(k+2)}{4\pi} \int d^2z \text{Tr}(g^{-1} \partial g \cdot g^{-1} \bar{\partial} g) + \frac{i(k+2)}{12\pi} \int_{\mathcal{B}} \text{Tr}(g^{-1} dg)^3, \tag{2.6}\]

where $\mathcal{B}$ is the manifold whose boundary is the world sheet. Because we will later consider the world sheet with Lorentzian signature, we use the light-cone coordinates $z = e^{i(t+\sigma)}$ and $\bar{z} = e^{i(t-\sigma)}$. This theory has the left-moving and right-moving conserved currents

\[j(z) = -\frac{k+2}{2} \partial g \cdot g^{-1} = j^a T^b \eta_{ab}, \quad \bar{j}(\bar{z}) = \frac{k+2}{2} g^{-1} \bar{\partial} g = \bar{j}^a T^b \eta_{ab}, \tag{2.7}\]

where we use the basis \[T^3 = \frac{1}{2} \sigma^2, \quad T^\pm = \frac{i}{2} (\sigma^3 \pm i\sigma^1), \tag{2.8}\]

\footnote{With the present convention, the space-time energy operator associated with the global coordinate $t$ is $\bar{j}_0 - j_0$.}
which satisfies the commutation relations

\[ [T^3, T^\pm] = \pm T^\pm, \quad [T^+, T^-] = -2T^3. \]  

(2.9)

The metric on this basis is defined by \( \eta^{ab} = -2\text{Tr}(T^a T^b) \) (and hence, the non-zero components of \( \eta^{ab} \) are \( \eta^{33} = -1, \, \eta^{+} = \eta^{-} = 2 \)), and we set \( \eta_{ab} = (\eta^{ab})^{-1} \).

D-branes in the WZW model are described by a world sheet with a boundary, as established in many works \[17\]. In the open string description, the gluing condition of the left and right moving currents is generally given by

\[ j = \omega(\tilde{j})|_{z=\tilde{z}}, \]  

(2.10)

where \( \omega \) is an automorphism of the \( SL(2; R) \) Lie algebra. Alternatively, by exchanging the roles of the world sheet coordinates \( \tau \) and \( \sigma \), we can rewrite this condition in the closed string picture as

\[ (j^a_n + \omega(\tilde{j})^a_{-n})|B\rangle = 0, \]  

(2.11)

where \( |B\rangle \) is a boundary state that corresponds to a D-brane. The world-volume of such a boundary state can be identified with the (twined) conjugacy class \[17\]

\[ \mathcal{C}^\omega(h) = \{ hg\omega(h)^{-1}, \forall h \in SL(2; R) \} . \]  

(2.12)

If \( \omega \) is an inner automorphism, we can set \( \omega = 1 \) with rotations of the currents, and the gluing condition (2.11) is thereby reduced to the simplest form,

\[ (j^a_n + \tilde{j}^a_{-n})|B\rangle = 0. \]  

(2.13)

We can parametrize the corresponding conjugacy classes as

\[ \text{Tr}g = 2X_0 = 2\tilde{C}, \]  

(2.14)

where \( \tilde{C} \) is a constant. In this parametrization, the hyper-surface (2.1) becomes

\[ (X_3)^2 - (X_1)^2 - (X_2)^2 = 1 - \tilde{C}^2, \]  

(2.15)

and this equation implies that the geometry of D-branes in the \( SL(2; R) \) group manifold is the \( dS_2 \) space (\( \tilde{C}^2 > 1 \)) or the hyperbolic plane (\( H_2 \)) (\( \tilde{C}^2 < 1 \)) \[8, 10\]. There are also cases of “degenerated D-branes”, whose shapes are the light-cone (\( \tilde{C}^2 = 1 \)) or the point (\( g = 1 \)).
was argued in Ref. [10] using the classical analysis of the DBI action that these D-branes are unphysical branes with supercritical electric fields.

If ω is an outer automorphism, we obtain a different geometry of the brane. Let us choose ω as

$$\omega(h) = \omega h \omega^{-1}, \quad \omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (2.16)$$

which defines an outer automorphism, since ω does not belong to SL(2; R). Then, the gluing condition becomes

$$\begin{cases} (j_n^3 - \tilde{j}_{-n}^3)|B\rangle = 0 \\ (j_n^\pm - \tilde{j}_{-n}^\pm)|B\rangle = 0, \end{cases} \quad (2.17)$$

and the “twined conjugacy classes” can be expressed as

$$C^\omega(h) = \{ h g (\omega h \omega^{-1})^{-1}, \forall h \in SL(2; R) \} = \{ h g \omega h^{-1} \cdot \omega^{-1}, \forall h \in SL(2; R) \}. \quad (2.18)$$

We can thus characterize these classes by a constant C as

$$\text{Tr}(g \omega) = -2X_2 = 2C. \quad (2.19)$$

The geometry of such D-branes is given by the hyper-surface (2.1)

$$(X_0)^2 + (X_3)^2 - (X_1)^2 = 1 + C^2, \quad (2.20)$$

which is an AdS2 space. As shown in Ref. [10], such AdS2-branes are physical D-branes with subcritical electric fields. We concentrate on this case in this paper.

It is known that the SL(2; R) WZW model possesses symmetry called spectral flow $U_w \otimes \tilde{U}_{\tilde{w}}$ (see, e.g., Ref. [18]), defined by the relations

$$\begin{cases} U_w j^3(z) U_w^{-1} = j^3(z) + \frac{k + 2w}{2z}, \\ U_w j^\pm(z) U_w^{-1} = z^{\mp w} j^\pm(z), \\ \tilde{U}_{\tilde{w}} \tilde{j}^3(\tilde{z}) \tilde{U}_{\tilde{w}}^{-1} = \tilde{j}^3(\tilde{z}) + \frac{k + 2\tilde{w}}{2\tilde{z}}, \\ \tilde{U}_{\tilde{w}} \tilde{j}^\pm(\tilde{z}) \tilde{U}_{\tilde{w}}^{-1} = \tilde{z}^{\mp \tilde{w}} \tilde{j}^\pm(\tilde{z}), \end{cases} \quad (2.21)$$

which are parametrized by integers w and \(\tilde{w}\). In Ref. [19] it is claimed that the closed string Hilbert space in the case of the universal cover of SL(2; R) should be extended by the spectral
flow with $w = -\tilde{w}$ (the “winding number”). A similar claim is made in Ref. [13] for the case of the open string Hilbert space.

In our convention for currents, the spectral flow $U_w \otimes \tilde{U}_{\tilde{w}}$ is represented by the transformation

\begin{equation}
  g(z, \tilde{z}) \mapsto \tilde{z}^{-wT^3} g(z, \tilde{z}) \tilde{z}^{-wT^3} \\
  \equiv e^{-\frac{i}{2}w\sigma^2} g(z, \tilde{z}) e^{-\frac{i}{2}\tilde{w}\sigma^2}.
\end{equation}

In this paper, we employ the single cover of $SL(2; \mathbb{R})$, which corresponds to the Wick rotation of thermal $AdS_3$ space. This is because we are interested in the open-closed string duality, in which the roles of the world sheet coordinates $\tau$ and $\sigma$ are exchanged, and hence we need to consider the winding sectors along not only the space-like circle but also the time-like circle in some situations. In this case we can a priori choose the left-moving and right-moving windings $w$ and $\tilde{w}$ independently. However, the requirement of consistency with the gluing condition yields some constraints on the windings $w$ and $\tilde{w}$. For the gluing condition (2.13), the allowed spectral flows should have the forms $U_w \otimes \tilde{U}_{-w}$, in other words,

\begin{equation}
  \begin{align*}
  t &\mapsto t + w\tau \\
  \theta &\mapsto \theta + w\sigma \\
  \rho &\mapsto \rho,
  \end{align*}
\end{equation}

in terms of the global coordinates (2.4) in the closed string channel. The spectral flows allowed for (2.17) are $U_w \otimes \hat{U}_w$, in other words,

\begin{equation}
  \begin{align*}
  t &\mapsto t + w\sigma \\
  \theta &\mapsto \theta + w\tau \\
  \rho &\mapsto \rho,
  \end{align*}
\end{equation}

which generates non-trivial winding sectors along the time-like circle.

3 Boundary states in string theory on $AdS_3$

In this section we attempt to construct consistent boundary states describing the D-branes in the $AdS_3$ space. As we stated in the Introduction, our main criterion is the open-closed 

\footnote{The time direction is uncompactified in the universal cover of $SL(2; \mathbb{R})$, and $w$ and $\tilde{w}$ must therefore be related [19].}
string duality with the requirement that the spectra in both open and closed string channels
be compatible with unitarity and normalizability and moreover the spectral flow symmetry.
Here, we treat only the states belonging to the principal discrete series (short string sector) and
leave the case of the principal continuous series to future works.

To fix a specific background, let us consider the superstring vacua $AdS_3 \times S^3 \times T^4$, which
is the most familiar example (see, e.g. Ref. [3]), although we shall mainly focus on the bosonic
sector. The $AdS_3$ sector is described by the $SL(2; \mathbb{R})_{k+2}$ super WZW model (where $k + 2$
is the level of bosonic current) and the $S^3$ sector is described by the $SU(2)_{k-2}$ super WZW
model. We here assume $k \in \mathbb{Z}$ and $k > 2$.

In string theory on Lorentzian $AdS_3$ it is known [19] that the physical Hilbert space
should be constructed using the representation spaces of discrete series with the constraints
of the no-ghost theorem [1, 20] and normalizability, and also using the continuous series. We
should incorporate the degrees of freedom of spectral flow for both of these representations,
as discussed in Ref. [19].

3.1 Open-closed duality on the Lorentzian world sheet

Before presenting our main analysis, we would like to make a few comments about the open-closed
duality on the world sheet with the Lorentzian signature, which we use in subsequent analysis.

In the case of a Euclidean world sheet, in the standard argument, the open-closed duality
for the cylinder amplitude is expressed as the relation

$$\int_0^\infty dT^{(c)}(\tau)^2 Z_{\text{closed}}(\tau) = \int_0^\infty \frac{dT^{(o)}(\tau)^2}{T^{(o)}} \eta(\tau)^2 Z_{\text{open}}(\tau),$$

where $\tau = iT^{(o)}$ denotes the open string modulus and $\tilde{\tau} \equiv -\frac{1}{\tau} = iT^{(c)}$ denotes the closed
string modulus. The factors of $\eta$-functions are the contributions from $bc$-ghosts.

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3In this paper we use the terminology “short string” and “long string”, following Ref. [19]. Here, a short
string is an excitation corresponding to the discrete series (with arbitrary windings $w$), and a long string
corresponds to the continuous series (with non-zero windings). We call the sectors with non-zero $w$ the
“winding strings” or “circular strings”.

4Throughout this paper we denote the moduli for the open string channel by $\tau$ and $z$ and those for the
closed string channel by $\tilde{\tau} \left( \equiv -\frac{1}{\tau} \right)$ and $\tilde{z} \left( \equiv \frac{z}{\tau} \right)$. We also write $\tilde{q} \equiv e^{2\pi i \tilde{\tau}}$, $\tilde{y} \equiv e^{2\pi i \tilde{z}}$, and so on.
From the simple identities
\[
\int_0^\infty dT^{(c)} = \int_0^\infty \frac{dT^{(o)}}{T^{(o)}}, \quad \eta(\tilde{\tau})^2 = -i\tau(\tau)^2 = T^{(o)}\eta(\tau)^2,
\]
we obtain
\[
Z_{\text{closed}}(\tilde{\tau}) = Z_{\text{open}}(\tau), \tag{3.3}
\]
which is the standard statement of the open-closed duality.

In the case of the Lorentzian world sheet, one must regard the moduli \(\tau\) and \(\tilde{\tau} (\equiv -\frac{1}{\tau})\) as real numbers. In this paper we adopt the convention \(\tau = -T^{(o)}(>0)\) and \(\tilde{\tau} = T^{(c)}(>0)\) (such that \(T^{(o)} = \frac{1}{\tau}\) holds as before). Then, (3.3) reduces to the relation
\[
Z_{\text{closed}}(\tilde{\tau}) = -iZ_{\text{open}}(\tau), \tag{3.4}
\]
instead of (3.3). Clearly, the same formula holds also for the superstring case.

### 3.2 Boundary states based on discrete series

We employ the (bosonic) \(SL(2; \mathbb{R})_{k+2}\) WZW model. Let \(\hat{D}^\pm_l\) be the representation space of discrete series (where + corresponds to the lowest weight representation, and − corresponds to the highest weight representation). We also express the representation transformed by the spectral flow as \(\hat{D}^{\pm(w)}_l\), as in Ref. [19]. The spectral flow parameter \(w (= \tilde{\omega} \text{ for the } AdS_2\text{-brane})\) is introduced as (2.21) in our convention. The unitarity and normalizability lead to the constraints \(-1 < l < k - 1\) for the \(l\)-quantum number [19, 20, 21]. Since we are here

5. With our convention, the conformal weight of zero-mode states is given by \(h = -\frac{l(l + 2)}{4k}\), and the \(j_0^3\) spectrum of the zero-mode states is \(j_0^3 = \pm \left(\frac{1}{2} + n + 1\right)\), \((n \in \mathbb{Z}_{\geq 0})\), for \(\hat{D}^\pm_l\) (\(l > -2\)). The double-sided representations (“degenerate representations”) have the zero-mode spectra \(j_0^3 = \frac{1}{2}, \frac{1}{2} - \frac{2}{2}, \ldots, -\frac{l}{2}\) \((l \in \mathbb{Z}_{\geq 0})\), which are natural analogs of the unitary representations of \(SU(2)\). On the other hand, the principal continuous series \(\hat{C}_{\lambda, \alpha}\) \((\lambda \in \mathbb{R}, 0 \leq \alpha < 1)\), which corresponds to the branch \(l = -1 + 2i\lambda\), i.e., \(h = \frac{1}{k} \left(\lambda^2 + \frac{1}{4}\right)\) and has the zero-mode spectrum \(j_0^3 = \alpha + n\), \((n \in \mathbb{Z})\).

6. The requirement of the no-ghost theorem is \(-2 < l < k\), and the requirement of normalizability is more severe. The wave function of discrete series behaves as \(\sim e^{-l(l+1)r}\), as we demonstrate below, and hence \(L^2\) normalizability requires \(l > -1\). Moreover, we assume that the spectral flow is the symmetry of the model, and thus the states characterized by \(l\) and \(k-l-2\) are related by this symmetry, and the requirement becomes \(-1 < l < k - 1\).
considering a single cover of the $AdS_3$ space, the allowed values of $l$ are $l = 0, 1, \ldots, k - 2$. Quite interestingly, this range is the same as that for the integral representations of $SU(2)_{k - 2}$.

The character with respect to the unflowed representation $\hat{D}_l^\pm$ is given by

$$\chi^\pm(l, \tau, z) \equiv \text{Tr}_{\hat{D}_l^\pm} q^{L_0 - \frac{c}{24}} y^3 = \frac{q^{\frac{(l+1)^2}{24} y^{l+1}}}{-i \theta_1(\tau, \pm z)} ,$$

where $q \equiv e^{2\pi i \tau}$, $y \equiv e^{2\pi i z}$. The character with respect to the flowed representation $\hat{D}_l^{\pm(w)}$ is given in, for example, Refs. [18, 19]. By using the relation between the unflowed and flowed currents $[21, 21]$, the character with respect to $\hat{D}_l^{\pm(w)}$ can be rewritten in the terms of the character (3.5), and we obtain

$$\chi^{\pm(w)}(l, \tau, z) \equiv \text{Tr}_{\hat{D}_l^{\pm(w)}} q^{L_0 - \frac{c}{24}} y^3 = (-1)^w q^{\frac{(l+1+kw)^2}{24} y^{l+1+kw}} - i \theta_1(\tau, \pm z) .$$

Let us start with the Ishibashi state $|l, \pm\rangle$ obtained with the representation $\hat{D}_l^\pm$ defined by the gluing condition (2.17) (with $a = 0$) describing the $AdS_3$-branes. Let $\mathcal{B}(\hat{D}_l^\pm)$ be an orthonormal basis of $\hat{D}_l^\pm$ composed of the eigenstates of $j_0^3$. We choose the phases of each bases $v \in \mathcal{B}(\hat{D}_l^\pm)$ so that the matrix elements of all the currents $j_n^3$ are real numbers. More precisely, we assume that $\langle v_1 | j_n^3 | v_2 \rangle = \langle v_2 | j_n^3 | v_1 \rangle$, $\langle v_1 | j_n^\pm | v_2 \rangle = \langle v_2 | j_n^\pm | v_1 \rangle$ for arbitrary $v_1, v_2 \in \mathcal{B}(\hat{D}_l^\pm)$. $\mathcal{B}(\hat{D}_l^\pm)$ is not uniquely defined even under this requirement. However, this ambiguity clearly causes no problem our analysis. Further, we denote the signature of norm $\langle v | v \rangle$ as $\epsilon_v$ for each $v \in \mathcal{B}(\hat{D}_l^\pm)$. (Recall that $\hat{D}_l^\pm$ is not a unitary representation of the affine algebra $\hat{SL}(2; \mathbb{R})_{k+2}$.)

With these preliminaries, we can explicitly write down the Ishibashi state that satisfies the gluing condition (2.17),

$$|l, \pm\rangle = \sum_{v \in \mathcal{B}(\hat{D}_l^\pm)} \epsilon_v |v\rangle \otimes \bar{v} .$$

The gluing condition (2.13) (for the conjugacy class with no twist) is also easily solved, and we find

$$|l, \pm\rangle = \sum_{v \in \mathcal{B}(\hat{D}_l^\pm)} \epsilon_v |v\rangle \otimes T(v) ,$$

where $T$ denotes the isomorphism $T : \hat{D}_l^\pm \xrightarrow{\sim} \hat{D}_l^\pm$, such that $T$ maps the lowest (highest) weight vector in $\hat{D}_l^+$ ($\hat{D}_l^-$) to the highest (lowest) weight vector in $\hat{D}_l^-$ ($\hat{D}_l^+$), and satisfies

$$T j_n^3 = -\tilde{j}_n^3 ,$$

$$T j_n^\pm = -\tilde{j}_n^\mp .$$

(3.9)
However, since the brane configuration described by (2.13) is known to be unphysical [10], we concentrate on the $AdS_2$-brane cases (2.17).

As pointed out above, the spectral flow compatible with the gluing condition (2.17) is of the type (2.24), and we can similarly obtain the Ishibashi states for the flowed representations:

$$|l, w, \pm \rangle_I = \sum_{v \in B(B_1^\pm(w))} \epsilon_v \langle v \rangle \otimes |v \rangle \ .$$

(3.10)

These states are characterized by the cylinder amplitudes

$$I_I(l, \pm, w|q^H(c) \tilde{y}^\beta \mid l', \pm, w')_I = \delta_{ll'} \delta_{ww'} \chi_l^{(w)}(\tilde{\tau}, \tilde{z}) \ ,$$

(3.11)

where $H(c) = \frac{1}{2}(L_0 + \tilde{L}_0 - \frac{c}{12})$ and $\chi_l^{(w)}(\tilde{\tau}, \tilde{z})$ is defined in (3.6).

Now, we consider the following boundary states, which will be used as the building blocks of our Cardy states:

$$|l, w\rangle_I = |l, w, +\rangle_I + |l, -w, -\rangle_I$$

$$\equiv |l, w, +\rangle_I + |k - 2 - l, -w + 1, +\rangle_I .$$

(3.12)

(These states are also used in Ref. [22].) For these states, we have

$$I_I(l, w|q^H(c) \tilde{y}^\beta |l', w')_I = \delta_{ll'} \delta_{ww'} (-1)^w \left[ \chi_l^{+(w)}(\tilde{\tau}, \tilde{z}) + \chi_l^{-(w)}(\tilde{\tau}, \tilde{z}) \right]$$

$$= \delta_{ll'} \delta_{ww'} (-1)^w \left[ q^{-\frac{(l+1-kw)^2}{4k}} y^{-\frac{l+1-kw}{2}} - i\theta_1(\tilde{\tau}, \tilde{z}) + q^{-\frac{(l+1-kw)^2}{4k}} y^{-\frac{l+1-kw}{2}} + i\theta_1(\tilde{\tau}, \tilde{z}) \right]$$

$$\equiv \delta_{ll'} \delta_{ww'} \chi_l^{(w)}(\tilde{\tau}, \tilde{z}) \ .$$

(3.13)

The right-hand side of the above identity (3.13) is well-defined in the limit $\tilde{z} \to 0$, where it becomes

$$I_I(l, w|q^H(c) |l', w')_I = \delta_{ll'} \delta_{ww'} (-1)^w \left[ -\frac{(l+1-kw)^2}{4k} \eta(\tilde{\tau})^3 \right]$$

$$\equiv \delta_{ll'} \delta_{ww'} \chi_l^{(w)}(\tilde{\tau}) .$$

(3.14)

In fact, the existence of $z$ makes the character (3.6) convergent, and if we take the naive limit $z \to 0$, this character would be divergent. However by using the combination (3.12), we

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7 It is interesting that the character $\chi_l^{(w=0)}(\tau, z)$ is formally equal to (the opposite sign of) the character of the degenerate representation treated in Ref. [12].
can remove this divergence. This divergence is due to the infinite dimensional representation of $SL(2, \mathbb{R})$ and this type of divergence can be removed using the usual regularization, like that of the $\zeta$ function. The combination $(3.12)$ cancels the divergence and also removes the regulator. (The detailed discussion about the subtlety of this character is given in Ref. [23].)

It is worthwhile to note that the summation of the characters $\sum_{w \in 2\mathbb{Z}} \chi^{(w)}_{SL(2)}(\tau, z)$ is formally quite similar to the well-known character formula of $\hat{SU}(2)_{k-2}$, as was pointed out in Ref. [18]. In fact, we have

$$\chi^{SL(2)}_{l}(\tau, z) \equiv \sum_{w \in 2\mathbb{Z}} \chi^{(w)}_{l}(\tau, z) = \frac{\Theta_{-(l+1),-k}(\tau, z) - \Theta_{l+1,-k}(\tau, z)}{i\theta_{1}(\tau, z)}, \quad (3.15)$$

and for $\hat{SU}(2)_{k-2}$, the character is written

$$\chi^{SU(2)}_{l}(\tau, z) = \frac{\Theta_{l+1,k}(\tau, z) - \Theta_{-(l+1),k}(\tau, z)}{i\theta_{1}(\tau, z)}. \quad (3.16)$$

This fact leads to the good modular property of the character $\chi^{SL(2)}_{l}$,

$$\chi^{SL(2)}_{L}(-\frac{1}{\tau}, z_{\tau}) = i^{k-2} \sum_{l=0}^{k-2} S^{(k-2)}_{Ll} \chi^{SL(2)}_{l}(\tau, z), \quad (3.17)$$

where

$$S^{(k-2)}_{Ll} \equiv \sqrt{\frac{2}{k}} \sin \left( \pi \frac{(L + 1)(l + 1)}{k} \right) \quad (3.18)$$

is the well-known matrix of the modular transformation for $\hat{SU}(2)_{k-2}$.

Of course, the power series of $(3.13)$ is divergent for the usual range of the modulus $\tau$ (i.e. $\text{Im} \, \tau > 0$), since they include the negative level theta functions. Therefore, we must here adopt the Lorentzian signature on the world sheet; that is, we must regard $\tau$ as a real number, as in the calculation of partition function in the appendix of Ref. [19]. The extra factor $i$ on the RHS of $(3.17)$ reflects this fact, and its existence matches the formula of the open-closed duality in the Lorentzian world sheet $(3.4)$. One may suppose that we still have a subtlety in $(3.17)$ (and also in the similar formulas given below), due to the fact that the power series does not absolutely converge. However, we can give this a strict meaning with the help of generalized functions. We here simply present such formal identities to avoid notational complexities. Their mathematically rigorous treatment is given in Appendix B.

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8The range of summation $w \in 2\mathbb{Z}$ in $\chi^{SL(2)}_{l}(\tau, z)$ indicates that we sum all the windings $w$, since we use the combination $(3.12)$ as the building blocks.
Now, we would like to construct the suitable Cardy states from the Ishibashi states (3.12). We start with the ansatz

$$|a\rangle_C = \sum_{l=0}^{k-2} \sum_{w \in \mathbb{Z}} \Psi_a(l, w)|l, w\rangle_I,$$

(3.19)

where the index $a$ runs over the set of allowed Cardy states, which should be fixed later. It should be emphasized that the summation over the winding $w$ is needed for the good modular property. From the unitarity bound for the closed string spectrum, we must take the range of the $l$-summation as $0 \leq l \leq k - 2$.

We first note the modular property of the character $\chi_l^{(w)}(\tau)$,

$$\sum_{w \in \mathbb{Z}} \sum_{l=0}^{k-2} \sqrt{2k} \sin \left( \frac{\pi (L + 1)(l + 1 - kw)}{k} \right) \chi_l^{(w)}(\tau) = -i \sum_{W \in \mathbb{Z}} \chi_L^{(W)}(\tilde{\tau}),$$

(3.20)

which can be derived by making use of the Poisson resummation formula. We can here assume $-1 < L < k - 1$ without loss of generality, because $L$ appears only in the combination $L' = L + 1 - kW$ in the character $\chi_L^{(W)}(\tau)$. Moreover, since the time direction is compactified, the character in the open string channel $\chi_L^{(W)}(\tau)$ must again be characterized by the discrete quantum number $L$. Therefore, we can assume $L = 0, 1, \ldots, k - 2$, and the Cardy condition for the ansatz (3.19) can be written

$$\Psi^*_a(l, w)\Psi_b(l, w) = \sum_{L=0}^{k-2} N_{ab}^{(o)}(L) \sqrt{2k} \sin \left( \frac{\pi (L + 1)(l + 1)}{k} \right),$$

(3.21)

where $N_{ab}^{(o)}(L) \in \mathbb{Z}_{\geq 0}$. This condition is formally the same as that of $\widehat{SU}(2)_{k-2}$, and the solution is well known [16]. Assuming the diagonal modular invariant, the solution is given by

$$|L\rangle_C = \sum_{l=0}^{k-2} \sum_{w \in \mathbb{Z}} \Psi_L(l, w)|l, w\rangle_I,$$

$$\Psi_L(l, w) = \frac{S_{Ll}}{S_{0l}} \frac{S_{Ll}^{(k-2)}}{S_{0l}^{(k-2)}} \left( \frac{2}{k} \right)^{1/4} \sin \left( \frac{\pi (L+1)(l+1)}{k} \right) \sqrt{\sin \left( \frac{\pi (l+1)}{k} \right)}.$$

(3.22)

Using the identity

$$\frac{S_{L1l}^{(k-2)} S_{L2l}^{(k-2)}}{S_{0l}^{(k-2)}} = \sum_{L=0}^{k-2} N_{L1, L2}^L S_{Ll}^{(k-2)},$$

(3.23)
where $N^L_{L_1, L_2}$ denotes the fusion matrix of $\tilde{SU}(2)_{k-2}$

$$N^L_{L_1, L_2} = \begin{cases} 
1 & \text{if } |L_1 - L_2| \leq L \leq \min(L_1 + L_2, 2(k-2) - L_1 - L_2) \\
& \text{and } L \equiv |L_1 - L_2| \mod 2 \\
0 & \text{otherwise}, 
\end{cases} \quad (3.24)$$

we obtain, as in the $SU(2)$ case,

$$C \langle L_1 | q^{H(c)} | L_2 \rangle_C = -i \sum_{L=0}^{k-2} N^L_{L_1, L_2} \chi^S_{L_1 L_2} (\tau), \quad (3.25)$$

where

$$\chi^S_{L_1 L_2} (\tau) = \sum_{W \in \mathbb{Z}} \chi^{(W)}_{L_1 L_2} (\tau) \equiv \sum_{W \in \mathbb{Z}} \left[ -(L + 1 - kW) \frac{q^{(L+1-kW)^2}}{4 \eta(\tau)^3} \right], \quad (3.26)$$

as defined above. The identity (3.25) corresponds to the relation of the open-closed duality (3.4) in which we are interested.

As we stated previously, the important criterion for the Cardy states is the requirement that the spectrum in both open and closed string channels be consistent with the unitarity bound. Moreover, it is quite natural to require that the density of states in the open string channel be invariant under the spectral flow, as claimed in Ref. [13] based on the analysis of the classical open string solutions. Equation (3.25) actually possesses these properties, and hence (3.22) is regarded as the desired solution of the Cardy condition.

### 3.3 Identification of $AdS_2$-branes with weak electric fields

It is discussed in Ref. [10] that the physical D-brane (D-string) in the $AdS_3$ background should be wrapped on the twined conjugacy class, which has the structure of the $AdS_2$ space. Such an $AdS_2$-brane can be expressed as the following simple equation with respect to the global coordinates of $AdS_3$ $(t, \theta, \rho)$ (2.4):

$$\sinh \rho \sin \theta = \text{const.} \equiv \sinh \psi_0. \quad (3.27)$$

Here $\psi_0$ parameterizes the location of the $AdS_2$-brane and corresponds physically to the strength of the electric field on that brane. In fact, we can readily find from (3.27) that $\rho \geq \rho_{\text{min}} \equiv \psi_0$, and thus $\psi_0$ parametrizes the point nearest to the center of the $AdS_3$ space, on the $AdS_2$-brane “bent” by the electric field.

We now attempt to identify the Cardy states (3.22) with the $AdS_2$-branes. Our arguments are summarized as follows:
1. From the construction and evaluation of the cylinder amplitudes (3.25), it is clear that our Cardy states (3.22) can interact only with the short string sectors in both open and closed string channels. Moreover, the fact that we are now taking the discrete values of \( l \) implies that only the string modes propagating within the range \( \rho \sim 1 \) (in the unit of the \( AdS_3 \) scale \( l_{AdS} \)) can interact with the Cardy states (3.22). In fact, the wave function of the string states corresponding to \( l \) is known to behave as \( \sim e^{-(l+1)^2} \rho \) with respect to the “radial coordinate” \( \rho \) (where \( l = -1 \) corresponds to the Breitenlohner-Freedman bound [25]). Therefore, since we are now working with \( l = 0, 1, \ldots, k-2 \), the corresponding wave functions exponentially damp at the length scale \( \rho \sim 1 \).

2. At least in the large \( k \) limit, our cylinder amplitudes must be interpreted as the summation of classical open string solutions. We have non-trivial winding sectors generated by spectral flows of the type \( t \rightarrow t + w\tau, \theta \rightarrow \theta + w\sigma \). Such classical solutions have been studied in several recent works [11, 13]. In particular, explicit forms of the classical open short string solutions connecting two \( AdS_2 \)-branes labeled \( \psi_1 \) and \( \psi_2 \) are given in Ref. [13] (up to the degrees of freedom of \( SL(2; \mathbb{R}) \) isometry). These are

\[
\begin{align*}
t &= (\alpha + w)\tau, \\
\theta &= (\alpha + w)\sigma + \theta_0, \\
\rho &= \rho_0 ,
\end{align*}
\]

with even winding \( w \in 2\mathbb{Z} \), and the parameters \( \alpha (0 \leq \alpha \leq 1), \theta_0 (0 \leq \theta_0 \leq 2\pi), \rho_0 (\geq 0) \) must satisfy

\[
\begin{align*}
\sinh \rho_0 \sin \theta_0 &= \sinh \psi_1 , \\
\sinh \rho_0 \sin(\alpha\pi + \theta_0) &= \sinh \psi_2 .
\end{align*}
\]

For odd winding \( w \in 2\mathbb{Z} + 1 \), we similarly have

\[
\begin{align*}
t &= (1 - \alpha + w)\tau, \\
\theta &= (1 - \alpha + w)\sigma + \pi - \theta_0 , \\
\rho &= \rho_0 ,
\end{align*}
\]

where \( \alpha, \theta_0, \rho_0 \) are the same as above.

\( ^9 \) The wave function of the discrete series behaves as \( \sim e^{-(l+2)^2}/g_s \), where \( g_s \) is the string coupling. Now it is given by \( g_s = e^{-\rho} \). (See, for example, Ref. [24].)
The building blocks of the open string amplitudes (3.25) are the characters \( \chi_{SL}^{(2)}(\tau) \) (3.26), which can be rewritten as

\[
\chi_{SL}^{(2)}(\tau) = \lim_{z \to 0} \left[ -\sum_{w \in 2\mathbb{Z}} q^{-(L+1-kw)^2} y^{L+1} \frac{-kw}{2i\theta_1(\tau, z)} + \sum_{w \in 2\mathbb{Z}+1} q^{-(k-L-1-kw)^2} y^{k-L-1} \frac{-kw}{2i\theta_1(\tau, z)} \right].
\] (3.31)

It is quite natural to interpret the zero-mode parts of these amplitudes as the summation over the classical solutions. In fact, one can easily find that the first term and the second term in (3.31) correspond nicely to the classical solutions (3.28) and (3.30), respectively, since the energy parameter \( \alpha \) is quantized as \( n/(k+2) \), because of the time-like compactification, and hence identified with \( (L+1)/k \) in the large \( k \) limit. (The classical solution (3.28) possesses the classical conformal weight \( -\frac{k+2}{4}\alpha^2 \approx -\frac{k}{4}\alpha^2 \), which should correspond to the quantum value \( -\frac{(L+1)^2}{4k} + \frac{1}{4k} \).

We can also check the consistency of spectral flows in the open and closed string channels. The spectral flows that generate the classical solutions (3.28) and (3.30) are equivalent to \( U_w \otimes \tilde{U}_w \) and (2.24) in the closed string channel, after exchanging the roles of \( \tau \) and \( \sigma \). These are in fact compatible with the gluing condition (2.17), as we noted above.

If we consider the \( dS_2 \)-branes instead of the \( AdS_2 \)-branes, classical open string solutions with non-trivial winding numbers are generated by the spectral flows: \( t \to t + w\sigma, \theta \to \theta + w\tau \). These are equivalent to (2.23) in the closed string channel and compatible with the gluing condition (2.13).

3. Recalling the previous results obtained for the \( SU(2) \) WZW model, it seems plausible to relate the labels of the Cardy states \( L_1 \) and \( L_2 \) with the parameters of the brane positions \( \psi_1 \) and \( \psi_2 \). However, we would immediately face an apparent contradiction. In (3.25), the \( L \)-value appearing in the open string channel has upper and lower bounds. However, the corresponding parameter \( \alpha \) in the classical solutions should not have such bounds depending on the brane positions \( \psi_1 \) and \( \psi_2 \), as discussed in Ref. [13]. How should we resolve this apparent contradiction? Recall the fact that our Cardy states only include the excitations of short strings propagating within the finite domain \( \rho \lesssim 1 \).
This implies that the classical solutions (3.28) and (3.30) we should compare with the open string spectrum have to satisfy the constraints \( \rho_0 \lesssim 1 \). We can hence expect the upper and lower bounds for \( \alpha \) in (3.28) and (3.30).

Interestingly, assuming the identification \( \psi_i \approx \pi \left( 1 - \frac{2L_i}{k} \right) \) for sufficiently small \( |\psi_i| \) (weak electric field) and requiring \( \sinh \rho_0 \lesssim \sinh 1 \sim 1 \), we can show by a simple geometrical argument that

\[
|L_1 - L_2| \lesssim k \alpha \lesssim \min(L_1 + L_2, 2k - L_1 - L_2).
\] (3.32)

This relation nicely reproduces the quantum truncation appearing in the cylinder amplitude (3.25). Based on these considerations, we assert that the quantum number \( L \) that labels the Cardy states should correspond to the parameter \( \psi_0 \) parametrizing the locations of the \( AdS_2 \)-brane. Our above observation supports this assertion, at least in cases in which \( |\psi_0| \ll 1 \).

Of course, since \( L \) can only take a value within the range \( L = 0, 1, \ldots, k-2 \), the number of allowed branes should be finite in our quantum analysis. (In particular, our Cardy states cannot describe the \( AdS_2 \)-brane with the strong electric field whose entire world-volume is located outside the \( AdS \) radius.) This situation is quite similar to that in the \( SU(2) \) case, and has its origin in the existence of the unitarity bound.

We also point out that the \( \mathbb{Z}_2 \) symmetry \( \psi_i \to -\psi_i \) corresponds to \( L_i \to k - 2 - L_i \), and the cylinder amplitudes (3.25) possess this symmetry, as expected. The existence of quantum truncation is again necessary for this to hold.

### 3.4 Space-time chiral primaries in the open string spectrum

Now, let us turn our attention to the superstring on the background \( AdS_3 \times S^3 \times T^4 \). Based on our analysis in the previous subsections and incorporating the \( SU(2) \) WZW model and free fermions, we can determine the open string spectrum in this superstring theory. We denote the free fermions along the \( SL(2; \mathbb{R}) \) directions as \( \psi^3 \) and \( \psi^\pm \) and those of \( SU(2) \) as \( \chi^3 \) and \( \chi^\pm \).

We focus on the special class of physical states - “space-time (anti) chiral primary states”, which are inevitably 1/2 BPS states. For the closed string sector, such BPS states (or the corresponding vertex operators) are important in the context of \( AdS_3/CFT_2 \) correspondence.
and are investigated in Refs. [26, 27, 28, 6]. We show that the BPS $AdS_2 \times S^2$-brane whose $SL(2, \mathbb{R})$ sector is described by our Cardy state contains an infinite number of such excitations compatible with the spectral flows.

The Cardy state of the total system should have the structure

$$| L, L', \ldots \rangle_C \equiv | L \rangle_C^{SU(2)} \otimes | L' \rangle_C^{SL(2)} \otimes \cdots ,$$

where $| L \rangle_C^{SU(2)}$ is the Cardy state of the $SU(2)$ sector, $| L' \rangle_C^{SL(2)}$ denotes the Cardy state defined in (3.22), and $\cdots$ represents the contributions from other sectors (free fermions and the $T^4$ sector). Considering a specific cylinder amplitude with respect to such a Cardy state, we typically obtain the amplitude as

$$\sum_{L,L',A} N^L_{L_1 L_2} N^{L'}_{L'_1 L'_2} \chi^S_{SU(2)}(\tau) \chi^S_{SL(2)}(\tau) \cdots ,$$

where $N^L_{L_1 L_2}$ and $N^{L'}_{L'_1 L'_2}$ denote again the fusion coefficients of $SU(2)_{k-2}$. The most important fact for our later discussion is that the character $\chi^S_{SL(2)}(\tau)$ (3.26) contains the contributions from the infinitely many sectors with non-trivial winding numbers. This leads us to a rich structure of the spectrum of BPS states.

Now, let us begin the analysis of on-shell BPS states. We only consider the NS sector here and work with the $(-1)$-picture. We also ignore the excitations along the $T^4$ direction for simplicity. First, we focus on the sector of primary states (in the usual sense of the world sheet) with $w = 0$, and consider the physical states including only one oscillator of free fermions ("level one states"). The on-shell condition stipulates that the spin of the $SL(2, \mathbb{R})$ sector must be equal to that of $SU(2)$. This means that we must look for the BRST invariant states within the Hilbert space, $\hat{L}_L \otimes \hat{D}_L^+ (\otimes$ Hilbert space of the free fermions) or $\hat{L}_L \otimes \hat{D}_L^-$, where $\hat{L}_L$ represents the integrable representation with the spin $L/2$ of $SU(2)_{k-2}$. For example, in the Hilbert space $\hat{L}_L \otimes \hat{D}_L^+$, generic physical states (level one) should have the form

$$\sum_{M,M',A} \alpha_{M,M',A} | L, M \rangle^{SU(2)} \otimes | L, M' \rangle^{SL(2)} \otimes \psi_{-1/2}^A | 0 \rangle_f \otimes c e^{-\phi} | 0 \rangle_{gh}$$

$$+ \sum_{M,M',a} \beta_{M,M',a} | L, M \rangle^{SU(2)} \otimes | L, M' \rangle^{SL(2)} \otimes \chi_{-1/2}^a | 0 \rangle_f \otimes c e^{-\phi} | 0 \rangle_{gh} \ , \quad (3.33)$$

where $| L, M \rangle^{SU(2)} (M = L, L - 2, \ldots, -L)$ and $| L, M' \rangle^{SL(2)} (M' = L + 2, L + 4, \ldots)$ are the primary states (zero-mode states) belonging to $\hat{L}_L$ and $\hat{D}_L^+$, respectively. Such physical states
are considered in Ref. [11], and it has been claimed that they correspond to fluctuations around the classical solution that make the quadratic variation of the DBI action vanish.

We now concentrate on the 1/2 BPS states, as previously mentioned. These are studied in Refs. [26, 27, 28, 6] as the space-time (anti) chiral primaries in the closed string sector. Among the level-one physical states, the chiral primaries are obtained by imposing the constraints $J_0^3 + K_0^3 = 0$ (where $J^A$ and $K^a$ are the total $SL(2; \mathbb{R})$ and $SU(2)$ currents including the fermionic contributions), and also the anti-chiral primaries are given by the constraints $J_0^3 - K_0^3 = 0$. Note that $-J_0^3$ corresponds to the space-time energy operator (or the space-time conformal weight) and $K_0^3$ evaluates the space-time R-charge.

The chiral primaries are as follows:

$|L, \psi^\pm \rangle^{(+)} \equiv |L, \pm L \rangle^{SU(2)} \otimes |L, \mp L \mp 2 \rangle^{SL(2)} \otimes \psi_{\pm 1/2}^0 \rangle_f \otimes c_1 e^{-\phi} \rangle_{gh}$,

$|L, \chi^\pm \rangle^{(+)} \equiv |L, \pm L \rangle^{SU(2)} \otimes |L, \mp L \mp 2 \rangle^{SL(2)} \otimes \chi_{\mp 1/2}^0 \rangle_f \otimes c_1 e^{-\phi} \rangle_{gh}$,  

(3.34)

Here, $|L, L \rangle^{SU(2)}$ ($|L, -L \rangle^{SU(2)}$) is the primary state of highest (lowest) weight in the spin $L/2$ integrable representation of $\hat{SU}(2)_{k-2}$, and similarly $|L, -L - 2 \rangle^{SL(2)}$ ($|L, L + 2 \rangle^{SL(2)}$) is the primary state of highest (lowest) weight in $\hat{D}_L^-(\hat{D}_L^+)$ of $\hat{SL}(2; \mathbb{R})_{k+2}$. The anti-chiral primaries are, similarly, given by

$|L, \psi^\pm \rangle^{(-)} \equiv |L, \mp L \rangle^{SU(2)} \otimes |L, \mp L \mp 2 \rangle^{SL(2)} \otimes \psi_{\mp 1/2}^0 \rangle_f \otimes c_1 e^{-\phi} \rangle_{gh}$,

$|L, \chi^\pm \rangle^{(-)} \equiv |L, \pm L \rangle^{SU(2)} \otimes |L, \pm L \mp 2 \rangle^{SL(2)} \otimes \chi_{\pm 1/2}^0 \rangle_f \otimes c_1 e^{-\phi} \rangle_{gh}$,  

(3.35)

Next, we consider the sectors with non-trivial winding number $w$. Generally, the spectral flows do not always map an on-shell state to another on-shell state, since they do not conserve the BRST charge. Fortunately, we find that the on-shell (anti) chiral states behave nicely under the spectral flow. To see this, we make use of an idea used in Ref. [28].

Recall the spectral flow (2.21) in the $SL(2; \mathbb{R})$ sector,

\[
\begin{align*}
U_w j_n^3 U_w^{-1} &= j_n^3 + \frac{k + 2}{2} w \delta_{n,0},
U_w j_n^\pm U_w^{-1} &= j_n^\pm.
\end{align*}
\]

(3.36)

One can explicitly check that these states (3.34) and (3.35) actually preserve a half of space-time SUSY when using the space-time SUSY generators constructed in Ref. [3]. Moreover, with the help of the Wakimoto free field representation, they can be shown to be the (anti) chiral primary states with respect to the full space-time superconformal generators, as discussed in Ref. [27].
We extend the action of the spectral flow operator $U_w$ to the other sectors as in Ref. \cite{28}:

\[
\begin{cases}
U_w^{(+)k_3}U_w^{(+)1} = k_3 - \frac{k - 2}{2}w\delta_{n,0}, \\
U_w^{(+)k_\pm}U_w^{(+)1} = k_\pm_{n\mp w},
\end{cases}
\tag{3.37}
\]

\[
\begin{cases}
U_w^{(+)\psi_3}U_w^{(+)1} = \psi_3, \\
U_w^{(+)\psi_\pm}U_w^{(+)1} = \psi_\pm_{n\mp w},
\end{cases}
\tag{3.38}
\]

\[
\begin{cases}
U_w^{(+)\chi_3}U_w^{(+)1} = \chi_3, \\
U_w^{(+)\chi_\pm}U_w^{(+)1} = \chi_\pm_{n\mp w},
\end{cases}
\tag{3.39}
\]

We also define

\[
\begin{cases}
U_w^{(-)k_3}U_w^{(-)1} = k_3 + \frac{k - 2}{2}w\delta_{n,0}, \\
U_w^{(-)k_\pm}U_w^{(-)1} = k_\pm_{n\pm w},
\end{cases}
\tag{3.40}
\]

\[
\begin{cases}
U_w^{(-)\chi_3}U_w^{(-)1} = \chi_3, \\
U_w^{(-)\chi_\pm}U_w^{(-)1} = \chi_\pm_{n\mp w},
\end{cases}
\tag{3.41}
\]

and the action of $U_w^{(-)}$ on the $\psi$ sector is defined to be the same as that of $U_w^{(+)1}$.

For the $SU(2)$ sector, \eqref{3.37} and \eqref{3.40} simply represent the actions of affine Weyl group, and we find

\[
U_w^{(+)1} \colon \hat{\mathcal{L}}_L \rightarrow \hat{\mathcal{L}}_L, \quad (w \in 2\mathbb{Z})
\]

\[
U_w^{(+)1} \colon \hat{\mathcal{L}}_L \rightarrow \hat{\mathcal{L}}_{k-2-L}, \quad (w \in 2\mathbb{Z} + 1)
\tag{3.42}
\]

Now, the important point here is that, as discussed in Ref. \cite{28}, the spectral flow operators $U_w^{(\pm)}$ have the properties

\[
U_w^{(\pm)}Q_{BRST}U_w^{(\pm)1} = Q_{BRST} - w\left[c(0)(j^3 \pm K^3)(0) + \sqrt{\frac{k}{2}}\eta(0)e^{\phi(0)}(\psi^3 \pm \chi^3)(0)\right],
\tag{3.43}
\]

\[
U_w^{(\pm)}(j_0^3 \pm K_0^3)U_w^{(\pm)1} = j_0^3 \pm K_0^3.
\tag{3.44}
\]

These properties imply that the action of $U_w^{(+)1}$ (or $U_w^{(-)1}$) is closed in the space of the on-shell space-time (anti) chiral primary states. We also note the identities

\[
U_w^{(+)1}|L, \psi^+(+) = |k - 2 - L, \chi^+(+)\rangle, \quad U_w^{(+)1}|L, \chi^+(+) = |k - 2 - L, \psi^-(+)\rangle,
\]

\[
U_w^{(-)1}|L, \psi^-(+) = |k - 2 - L, \chi^-(+)\rangle, \quad U_w^{(-)1}|L, \chi^-(+) = |k - 2 - L, \psi^-(+)\rangle.
\tag{3.45}
\]
Using these identities, the complete set of on-shell chiral primaries is given by

$$\{ |L, w, \psi^+\rangle^{(+)} \equiv U_w^{(+)}|L, \psi^+\rangle^{(+)}, |L, w, \chi^+\rangle^{(+)} \equiv U_w^{(+)}|L, \chi^+\rangle^{(+)} \} ,$$

$$(w \in \mathbb{Z}, L = 0, 1, \ldots, k - 2)$$  \hspace{1cm} (3.46)$$

and for the anti-chiral primaries, we similarly obtain

$$\{ |L, w, \psi^+\rangle^{(-)} \equiv U_w^{(-)}|L, \psi^+\rangle^{(-)}, |L, w, \chi^+\rangle^{(-)} \equiv U_w^{(-)}|L, \chi^+\rangle^{(-)} \} .$$

$$(w \in \mathbb{Z}, L = 0, 1, \ldots, k - 2)$$  \hspace{1cm} (3.47)$$

To summarize, we have obtained an infinite number of on-shell (anti) chiral primaries (3.46) (3.47). The $L$-value undergoes quantum truncation that depends on the choice of Cardy states of both $SU(2)$ and $SL(2; \mathbb{R})$ sectors. The spectral flows can act on this spectrum transitively for arbitrary even windings $w$, irrespective of the choice of the Cardy states. For odd windings, this is generally not the case, since the spectral flows act as the $\mathbb{Z}_2$-reflection $L \rightarrow k - 2 - L$, on both the $SU(2)_{k-2}$ and $SL(2; \mathbb{R})_{k+2}$ sectors. Related arguments based on the analysis of classical solutions are given in Refs. [11, 13].

We would like to point out that the spectrum of the closed string channel (Cardy states) also includes an infinite number of such chiral primary states with arbitrary winding numbers, which can be constructed in a manner similar to those for the open string spectrum given by (3.46) and (3.47). The essential point is that the spectral flows of the type (2.24) preserve the Cardy state of the $SL(2; \mathbb{R})$ sector (3.22) (up to the signature),

$$U_w \otimes \bar{U}_w |L\rangle^{SL(2)}_C = (-1)^{Lw} |L\rangle^{SL(2)}_C ,$$  \hspace{1cm} (3.48)$$

under our construction. Similar relations hold also for the other sectors (the $SU(2)$ sector and the sectors of free fermions).

### 3.5 Comments on boundary states based on continuous series

The above treatment may be incomplete, because long strings are not considered in either the open or closed string channel. The correct boundary states describing the $AdS_2$-branes are expected to have contributions also from the continuous series. The components of the boundary states obtained from the discrete series, which we constructed, describe the open short strings propagating in the domain bounded by the $AdS$ radius, and all the space-time chiral primary states (as both open and closed strings) appear in this sector.
It is not difficult to construct the Ishibashi states for the continuous series. However, there is a large ambiguity when we attempt to construct the Cardy states from them. We could expect open string excitations of the following types to result from such Cardy states constructed from the continuous series:

1. open long strings with arbitrary winding number \( w \),

2. open short strings with arbitrary winding number \( w \) that propagate in the domain \( \rho \approx 1 \) (the \( AdS \) length).

The classical solutions of open long strings with arbitrary winding numbers are explicitly constructed in Ref. \([13]\), and it is natural to expect the corresponding excitations (1) in the quantum open string spectrum. It can be easily shown from consideration of the modular weight that we must use boundary states constructed from the continuous series with no winding in order to obtain the summation over windings in the open string channel. Conversely, if we start with the boundary states with the summation over windings, we obtain the open string spectrum with no winding. Therefore, it seems difficult to naively construct the Cardy states so as to be consistent with the spectral flow symmetry in both open and closed string channels in this case. This is one of the main puzzles of this study and we need further detailed investigations to reach a definite conclusion.

To assume the second excitations (2) is also quite natural, since the Cardy states considered here are expected to interact with the strings that can propagate in the region far from the center, in contrast to those we discussed in the previous subsections. In fact, choosing some hyperbolic functions as the wave functions, we can obtain the open string spectrum of the discrete series, as presented in Ref. \([12]\). However, at least with a naive consideration, we would face the following difficulties if we start with the criterion used in the previous arguments:

1. It seems difficult to incorporate the non-trivial winding sectors in the open string channel.

2. It seems difficult to make the open string spectrum compatible with the unitarity bound.

For the reasons given above, it is unclear at this point whether we can reproduce a consistent open string spectrum from the boundary states constructed with the continuous series. This is an important problem, which should be resolved in future studies.\(^{11}\)

\(^{11}\)Recently, this problem has been discussed in Ref. \([24]\).
4 Discussion

In this paper we have studied the $AdS_2$-branes in the string theory on the $AdS_3$ background from the viewpoint of boundary states, emphasizing the role of open-closed duality in string theory. We have constructed the Cardy states from the discrete series. These states possess the following desirable properties:

1. They are compatible with the symmetry of the spectral flow.

2. They are consistent with the unitarity and normalizability condition in the open string spectrum.

We have found that the first property above yields a rich structure of physical BPS states both in the open and closed string channels; that is, the spectral flows consistently act on the spectrum of infinitely many space-time chiral primaries. Such physical BPS states are believed to play important roles in the context of $AdS_3/CFT_2$ correspondence. One interesting direction for future works is the attempt to describe the D-branes in $AdS_3$ string theory in the framework of the boundary CFT (in the sense of $AdS/CFT$ correspondence though this terminology is sometimes very confusing) rather than the world sheet CFT approach. The analysis of physical BPS states given in this paper should provide helpful insights for such studies.

The second property above originates from the quantum truncation, like the fusion rule in the $SU(2)$ WZW model. The existence of this truncation seems to lead to the “fuzziness” of brane dynamics, as in the $SU(2)$ case. Such fuzziness may seem peculiar, since $AdS_3$ and $AdS_2$ are non-compact spaces, in contrast to $SU(2)$ and $S^2$. In fact, the classical analysis given in Ref. [13] suggests that we should not have such a truncation in the open string spectrum. One possibility to resolve this apparent contradiction may be to claim that our Cardy states (3.22) define “fuzzy $AdS_2$-branes” that do not have counterparts in the classical brane geometry. More modestly speaking, (3.22) should correspond to the component of the Cardy states obtained from the discrete series, and we will have to further take account of the component obtained from the continuous series in order to construct the complete Cardy states describing the “classical” $AdS_2$-branes.

As discussed, the Cardy states (3.22) can interact only with the short strings propagating in the domain bounded by the $AdS$ radius. This fact is supposed to be the origin of the
fuzziness mentioned above. Also it is natural to hypothesize that the component obtained from
the continuous series effectively describes (1) open long strings that can reach the asymptotic
region (the boundary of $AdS_3$), and (2) open short strings propagating in region far from the
center. If this is indeed the case, we will be able to obtain boundary states reproducing the
classical geometry of $AdS_2$-branes. However, as we mentioned in the last part of section 3,
there are several puzzles concerning the Cardy states constructed from the continuous series,
and they have to be resolved in future studies.

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\section{A \hspace{1em} Notation}

The theta functions are defined as follows (where we define $q \equiv e^{2\pi i \tau}$ and $y \equiv e^{2\pi iz}$):

\begin{align}
\theta_1(\tau, z) &= i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n-1/2)^2/2} y^{-n+1/2} \\
&\equiv 2 \sin(\pi z) q^{1/8} \prod_{m=1}^{\infty} (1 - q^m)(1 - yq^m)(1 - y^{-1}q^m), \\
\theta_2(\tau, z) &= \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2} y^{-n+1/2} \\
&\equiv 2 \cos(\pi z) q^{1/8} \prod_{m=1}^{\infty} (1 - q^m)(1 + yq^m)(1 + y^{-1}q^m), \\
\theta_3(\tau, z) &= \sum_{n=-\infty}^{\infty} q^{n^2/2} y^n \equiv \prod_{m=1}^{\infty} (1 - q^m)(1 + yq^{m-1/2})(1 + y^{-1}q^{m-1/2}), \\
\theta_4(\tau, z) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2} y^n \equiv \prod_{m=1}^{\infty} (1 - q^m)(1 - yq^{m-1/2})(1 - y^{-1}q^{m-1/2}),
\end{align}

\begin{equation}
\Theta_{m,k}(\tau, z) = \sum_{n=-\infty}^{\infty} q^{k(n+m/2k)^2} y^{k(n+m/2k)}.
\end{equation}

We also set

\begin{equation}
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),
\end{equation}

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and often use the identity
\[ \partial_z \theta_1(\tau, z) |_{z=0} = 2\pi \eta(\tau)^3. \tag{A.4} \]

B Some Mathematical Comments

The aim of this appendix is to remove the mathematical subtlety in our argument in section 3. Here, it is useful to introduce the definition
\[ f_a(x) \overset{\text{def}}{=} \left( \frac{a}{\pi} \right)^{\frac{1}{4}} e^{-\frac{1}{2}ax^2}, \tag{B.1} \]
whose Fourier transform is given by \( f_{1/a} \). For positive real \( a \), the Poisson resummation formula leads to
\[ \sum_{n \in \mathbb{Z}} f_a(nL) = \frac{\sqrt{2\pi}}{L} \sum_{m \in \mathbb{Z}} f_{1/a}(2\pi m/L), \quad (L > 0) \tag{B.2} \]
which gives us the modular transformation formula of theta functions. However, since we are employing the world sheet with the Lorentzian signature in this paper, we face the case in which \( a \in i\mathbb{R} \). In this case, the formula (B.2) is not correct in a naive sense, because the power series on both sides do not absolutely converge. Nevertheless, we interpret it correctly meaning as an identity among the generalized functions (or distributions) \[30\]. Now, we briefly sketch how it works.

Let us consider the space of “rapidly decreasing functions” \( \mathcal{S}(\mathbb{R}) \). More precisely, \( f \in \mathcal{S}(\mathbb{R}) \) implies that \( f \) is a (\( C \)-valued) smooth function such that
\[ p_{m,n}(f) < +\infty, \quad (\forall m, n \in \mathbb{Z}_{\geq 0}) \]
\[ p_{m,n}(f) \overset{\text{def}}{=} \max_{x \in \mathbb{R}} \sup_{\alpha \leq m, \beta \leq n} |x^\alpha \partial^\beta f(x)|. \tag{B.3} \]
The function space \( \mathcal{S}(\mathbb{R}) \) is a countably normed space with respect to \( \{p_{m,n}\}_{m,n \in \mathbb{Z}_{\geq 0}} \), and the generalized functions should be defined as the elements of \( \mathcal{S}(\mathbb{R})^* \), i.e. continuous linear functionals on \( \mathcal{S}(\mathbb{R}) \). The Fourier transform \( \tilde{F} \) of \( F \in \mathcal{S}(\mathbb{R})^* \) is defined by the relation
\[ \langle \tilde{F}, \tilde{f} \rangle = \langle F, f \rangle, \tag{B.4} \]
for arbitrary \( f \in \mathcal{S}(\mathbb{R}) \), and \( \tilde{f} \) is its Fourier transform. In this sense, \( f_{ia} (a \in \mathbb{R}) \) can be regarded as a generalized function, and its Fourier transform is equal to \( f_{1/ia} \). The only non-trivial fact we use to prove this is that we have a dense subset of \( \mathcal{S}(\mathbb{R}) \) composed of
the functions of the form \( P(x)e^{-\frac{1}{2}x^2} \), where \( P(x) \) represents arbitrary polynomial (Hermite functions, essentially).

We can further obtain the identity

\[
\sum_{n \in \mathbb{Z}} f_{ia}(x + nL) = \frac{\sqrt{2\pi}}{L} \sum_{m \in \mathbb{Z}} f_{1/ia}(2\pi m/L)e^{2\pi im\nu} .
\]  

(B.5)

Both sides of this equation are well-defined as the generalized functions associated with the “periodic function version” of \( S(\mathbb{R}) \). To show this, we define we set \( S_L \) with period \( L \) as

\[
S_L \overset{\text{def}}{=} \left\{ f(x) = \sum_{n \in \mathbb{Z}} a_ne^{2\pi i\nu x}; \lim_{|n|\to\infty} |n^m a_n| = 0 , \ (\forall m \in \mathbb{Z}_{\geq 0}) \right\} ,
\]  

(B.6)

and both sides of (B.5) are then seen to be well-defined as elements of \( S_L^* \), owing to the property \( \left| \int_{-L/2}^{L/2} dx f(x)f(x) \right| < +\infty \) for arbitrary \( f \in S_L \). In particular, take an arbitrary series \( \{\rho_{\nu}\}_{\nu \in \mathbb{Z}_{>0}} \subset S_L \) such that \( \lim_{\nu \to +\infty} \rho_{\nu}(x) = \sum_{n \in \mathbb{Z}} \delta(x + nL) \) (for example, \( \rho_{\nu}(x) := \sum_{n \in \mathbb{Z}} \sqrt{\nu \pi} e^{-\nu(x+nL)^2} \)), then we obtain

\[
\sum_{n \in \mathbb{Z}} \int_{-L/2}^{L/2} dx \rho_{\nu}(x)f_{ia}(x + nL) = \frac{1}{L} \sum_{m \in \mathbb{Z}} \int_{-L/2}^{L/2} dx \rho_{\nu}(x)f_{1/ia}(m/L)e^{2\pi im\nu} ,
\]  

(B.7)

for arbitrary \( \nu \in \mathbb{Z}_{>0} \). This is a natural generalization of (B.2) and precisely the identity we want. The identities of the modular transformations presented in section 3 should be understood in this manner.

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