GLOBAL SOLUTIONS TO THE VLASOV-POISSON-LANDAU SYSTEM

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Abstract. Based on the recent study on the Vlasov-Poisson-Boltzmann system with general angular cutoff potentials [3, 4], we establish in this paper the global existence of classical solutions to the Cauchy problem of the Vlasov-Poisson-Landau system that includes the Coulomb potential. This then provides a different approach on this topic from the recent work [8].

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1. Introduction

In plasma physics, the binary grazing collision between particles (e.g., electrons and ions) can be modeled by the Landau operator, cf. [9]. There are different approaches in establishing the mathematical theories on the Landau equation, see [1, 2, 7, 10, 12, 14] and references therein. Recently, Guo [8] made progress in proving the global existence of classical solutions to the Vlasov-Poisson-Landau (called VPL in the sequel for simplicity) system in a periodic box. Precisely, he successfully constructed global unique solutions to the Cauchy problem for initial data which have small weighted \( H^2 \) norms, but can have large \( H^N \) \((N \geq 3)\) norms with high velocity moments, that includes the most important case of Coulomb potential. This result is highly non-trivial on this important topic. Note that the same approach is used in [13] for the problem in the whole space.

Based on our recent study on the Vlasov-Poisson-Boltzmann system with general angular cutoff potentials [3, 4], we establish in this short paper the global existence of classical solutions to the Cauchy problem on the VPL system in the whole space \( \mathbb{R}^3 \). Hence, it provides an alternative approach for the study on this topic compared to [8]. We emphasize that the main motivation of the paper is to clarify how the approach that we developed in [3, 4] can be applied to the VPL system so that we will not pursue here the optimal regularity and velocity moments on initial data.

Consider the following Cauchy problem for the VPL system which describes the dynamics of electrons with a constant ion background profile in the whole space,

\[
\begin{align*}
\partial_t f + \xi \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_\xi f &= Q(f, f), \\
\Delta_x \phi &= \int_{\mathbb{R}^3} f \, d\xi - 1, \quad \phi(x) \to 0 \text{ as } |x| \to \infty, \\
f(0, x, \xi) &= f_0(x, \xi).
\end{align*}
\]

Here, \( f = f(t, x, \xi) \geq 0 \) represents the density distribution function of the particles (e.g., electrons) located at \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \) with velocity \( \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \) at time \( t \geq 0 \). The potential function \( \phi = \phi(t, x) \) generating the self-consistent electric field \( \nabla_x \phi \) is coupled with \( f(t, x, \xi) \) through the Poisson

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where the constant ion background profile is normalized to be unit. $Q$ is the Landau collision operator defined by

$$Q(f, g) = \nabla_{\xi} \cdot \left\{ \int_{\mathbb{R}^3} B(\xi - \xi') [f(\xi') \nabla_{\xi} g(\xi) - \nabla_{\xi} f(\xi') g(\xi)] d\xi' \right\}$$

$$= \sum_{i, j=1}^{3} \partial_i \int_{\mathbb{R}^3} B^{ij}(\xi - \xi')[f(\xi') \partial_j g(\xi) - \partial_j f(\xi') g(\xi)] d\xi'$$

with

$$B^{ij}(\xi) = \left( \delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2} \right) |\xi|^{\gamma+2}, \quad -3 \leq \gamma < -2.$$  

Here and in the sequel, we use $\partial_i = \partial_{\xi_i}$ for brevity. Note that $\gamma = -3$ corresponds to the Coulomb potential.

Let $M = (2\pi)^{-3/2} e^{-|\xi|^2/2}$ be the normalized Maxwellian. By setting $f(t, x, \xi) - M = M^{1/2} u(t, x, \xi)$, the Cauchy problem \eqref{1.1} becomes

$$\begin{aligned}
\partial_t u + \xi \cdot \nabla_x u + \nabla_x \phi \cdot \nabla_\xi u - \frac{1}{2} \nabla_x \phi u - \nabla_x \phi \cdot \xi M^{1/2} + Lu = \Gamma(u, u),
\end{aligned}$$

$$\Delta_x \phi = \int_{\mathbb{R}^3} M^{1/2} u \, d\xi, \quad \phi(x) \to 0 \text{ as } |x| \to \infty,$$

$$u(0, x, \xi) = u_0(x, \xi) = M^{-1/2}(f_0 - M),$$

where

$$Lu = -M^{-\frac{1}{2}} \left\{ Q(M, M^{1/2} u) + Q(M^{1/2} u, M) \right\}, \quad \Gamma(u, u) = M^{-\frac{1}{2}} Q(M^{1/2} u, M^{1/2} u)$$

are the linearized and nonlinear Landau collision terms, respectively.

In order to state the global existence of solutions to \eqref{1.2}, we need the following notations. The Landau collision frequency is given by

$$\sigma^{ij}(\xi) = B^{ij} * M(\xi) = \int_{\mathbb{R}^3} B^{ij}(\xi - \xi') M(\xi') \, d\xi'.$$

Similar to \eqref{3.4}, we introduce the time-velocity weight corresponding to the Landau operator:

$$w_{\tau, \lambda}(t, \xi) = \langle \xi \rangle^{\gamma+2} e^{\frac{\Lambda(\xi)^2}{2(1+|\xi|^2)}}, \quad \langle \xi \rangle = \sqrt{1 + |\xi|^2}, \quad \tau \in \mathbb{R}, \quad 0 \leq \lambda \ll 1, \quad \vartheta > 0.$$  

(1.3)

Note that $w_{\tau, \lambda}$ depends also on the parameter $\vartheta$. As in \eqref{2} and \eqref{12}, we define the energy norm and the corresponding dissipation rate norm, respectively, by

$$|u(x)|_{\tau, \lambda}^2 = \int_{\mathbb{R}^3} w_{\tau, \lambda}^2(t, \xi) |u|^2 \, d\xi, \quad ||u||_{\tau, \lambda}^2 = \int_{\mathbb{R}^3} |u(x)|_{\tau, \lambda}^2 \, dx,$$

and

$$|u(x)|_{\sigma, \tau, \lambda}^2 = \sum_{i, j=1}^{3} \int_{\mathbb{R}^3} w_{\tau, \lambda}^2(t, \xi) \left\{ \sigma^{ij} \partial_i u \partial_j u + \sigma^{ij} \frac{\xi_i \xi_j}{2} |u|^2 \right\} \, d\xi, \quad ||u||_{\sigma, \tau, \lambda}^2 = \int_{\mathbb{R}^3} |u(x)|_{\sigma, \tau, \lambda}^2 \, dx.$$

Moreover, for an integer $N \geq 0$ and a constant $L \geq 1$, we define the energy norm of a given $u = u(t, x, \xi)$ involving the space-velocity derivatives and the time-weighted energy norm, respectively, by

$$\|u(t)\|_{N, \tau, \lambda}^2 = \sum_{|\alpha| + |\beta| \leq N} \left\| \partial_\alpha^\beta u(t) \right\|_{|\beta| - \ell, \tau, \lambda}^2 + \left\| \nabla_x \phi(t) \right\|_{H_N}^2,$$

(1.4)

$$X_{N, \ell, \lambda}(t) = \sup_{0 \leq s \leq t} \|u(s)\|_{N, \ell, \lambda}^2 + \sup_{0 \leq s \leq t} (1 + s)^\frac{\zeta}{2} \|u(s)\|_{N, \ell - 1, \lambda}^2 + \sup_{0 \leq s \leq t} (1 + s)^{2(1+\vartheta)} \|\nabla_x^2 \phi(s)\|_{H_{N-1}}^2.$$

(1.5)

Here, $\phi$ is determined by $u$ through

$$\phi(t, x) = -\frac{1}{4\pi |x|^2} \ast_x \int_{\mathbb{R}^3} M^{1/2} u(t, x, \xi) \, d\xi.$$  

(1.6)
If \( \lambda = 0 \) or \( \tau = 0 \), then we drop the corresponding parameter in the subscript, for example \( w_r = w_{r,0} \), \( \|u\|_{\sigma} = \|u\|_{\sigma,0,0} \).

The result of this paper can now be stated as follows. More details will be explained at the end of this section.

**Theorem 1.1.** Let \(-3 \leq \gamma < -2, N \geq 8, \ell_0 > \frac{3}{2}, \ell \geq 1 + \max \left\{ N, \frac{2}{\gamma} - \frac{1}{\gamma + 2} \right\}, 0 < \lambda \ll 1, \) and \( \vartheta = \frac{\gamma - 4}{\vartheta + 4} \in \left[ \frac{3}{14}, \frac{1}{4} \right] \). Assume that \( f_0 = M + M^{1/2} u_0 \geq 0 \) and

\[
\iint_{\mathbb{R}^3 \times \mathbb{R}^3} M^{1/2} u_0 \, dx d\xi = 0.
\]

There exist constants \( \epsilon_0 > 0, C_0 > 0 \) such that if

\[
Y_{N,\ell,\lambda}(0) = \sum_{|\alpha| + |\beta| \leq N} \| \partial_\alpha \vartheta \|_{|\beta| - \ell,\lambda} + \left\| \left( 1 + |x| + |\xi| \right) \frac{\gamma + 2 \vartheta}{\lambda} \right\|_{Z_1} \leq \epsilon_0,
\]

then the Cauchy problem \( \text{(1.2)} \) admits a unique global solution \( u(t, x, \xi) \) satisfying \( f(t, x, \xi) = M + M^{1/2} u(t, x, \xi) \geq 0 \) and

\[
\sup_{t \geq 0} X_{N,\ell,\lambda}(t) \leq C_0 Y_{N,\ell,\lambda}(0)^2.
\]

The proof of Theorem 1.1 is basically along the same line as [3] for the study on the Vlasov-Poisson-Boltzmann system with angular cutoff soft potentials. We point out here the main differences from [3]. First of all, corresponding to the dissipation property of the linearized Landau operator stated in Lemma 2.1 in the next section, the exponent in the algebraic part of the weight function \( 1/\ell \) is chosen to be \( (\gamma + 2)\tau \). In this way, to compensate one order of derivative in the velocity variable, the extra velocity moment \( \langle \xi \rangle^{-(\gamma + 2)} \) grows slower than \( \langle \xi \rangle^2 \) at large velocity when \(-3 \leq \gamma < -2\). This can be used to control the growth in the velocity variable when dealing with the weighted estimate on the nonlinear term \( \nabla_x \phi \cdot \nabla_\xi u \). The technique used here is different from the one in [8], where the velocity diffusion dissipation from the Landau operator was used.

Another difference concerns the time-decay estimate on the potential force \( \nabla_x \phi \). In the case of the Vlasov-Poisson-Boltzmann system, the nonlinear term contains at most one order derivative on the perturbation in the term \( \nabla_x \phi \cdot \nabla_\xi u \), and hence the time-decay estimate on \( \| \nabla^2 \phi \|_{H^{N-1}} \sim \|a\|^2_{H^{N-1}} \), particularly on \( \|\nabla_x^{-1}u\|^2 \), can be obtained in terms of the total energy functional \( \|u(t)\|_{N,\ell-1} \). However, the nonlinear Landau operator contains second order of differentiation in the velocity variable. To overcome this, we will use the high order energy functional \( E^h_{N,\ell,\lambda}(t) \). To obtain the time-decay of \( E^h_{N,\ell,\lambda}(t) \), the balance of \( \langle \xi \rangle^2/(1 + t)^{1+\vartheta} \) and \( \langle \xi \rangle^{\gamma+2} \) with \( \gamma < -2 \) is used to get a time-decay coefficient in the dissipation term.

That is,

\[
\min_{\xi \in \mathbb{R}^3} \left\{ \langle \xi \rangle^{\gamma+2}, \frac{\langle \xi \rangle^2}{(1 + t)^{1+\vartheta}} \right\} = (1 + t)^{(1+\vartheta)/(\gamma+2)}
\]

leads to

\[
D_{N,\ell,q}(t) \geq \kappa(1 + t)^{(1+\vartheta)/(\gamma+2)} E^h_{N,\ell,q}(t).
\]

Note that \( D_{N,\ell,q}(t) \) and \( E^h_{N,\ell,q}(t) \) will be defined in Sections 3 and 4 respectively. In this way, as in [12], the time-decay on \( E^h_{N,\ell,\lambda}(t) \) for a proper choice of \( 0 < \vartheta \leq 1/4 \) in terms of \( \gamma \) will be given in the Step 3 for the proof of Theorem 1.1 in Section 3.4.

The rest of this paper is arranged as follows. In the next two sections, we will state some lemmas related to the basic properties of \( L \) and \( \Gamma(\cdot, \cdot) \), and also the weighted estimates on the nonlinear terms. In Section 3, we will give the proof of Theorem 1.1.

**Notations.** Throughout this paper, \( C \) denotes some generic positive (generally large) constant and \( \kappa \) denotes some generic positive (generally small) constant, where both \( C \) and \( \kappa \) may take different values in different places. \( A \sim B \) means \( \kappa A \leq B \leq \frac{1}{\kappa} A \). We use \( L^2 \) to denote the usual Hilbert spaces \( L^2 = L^2_{x,\xi} \), \( L^2_x \) or \( L^2_\xi \) with the norm \( \| \cdot \| \), and use \( \langle \cdot, \cdot \rangle \) to denote the inner product over \( L^2_{x,\xi} \) or \( L^2_\xi \). For \( q \geq 1 \), the mixed velocity-space Lebesgue space \( Z_q = L^2_q(L^2_\xi) = L^2_q(\mathbb{R}^3 \times \mathbb{R}^3) \) is used. For multi-indices \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) and \( \beta = (\beta_1, \beta_2, \beta_3) \), \( \partial^{\alpha}_\beta = \partial^{\alpha_1}_x \partial^{\alpha_2}_x \partial^{\alpha_3}_\xi \partial^{\beta_1}_x \partial^{\beta_2}_x \partial^{\beta_3}_\xi \). The length of \( \alpha \) is \( |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \) and similar for \( |\beta| \).
2. Preliminary

In this section, we will state two lemmas about some basic properties of the Landau operator. Given a vector-valued function \( u = (u_1, u_2, u_3) \), define

\[
P_\xi u = \frac{\xi \otimes \xi}{|\xi|^2} u = \left\{ \frac{\xi}{|\xi|} \cdot u \right\} \frac{\xi}{|\xi|}, \quad i.e., \quad (P_\xi u)_i = \left\{ \sum_{j=1}^{3} \frac{\xi_j u_j}{|\xi|} \right\} \frac{\xi_i}{|\xi|}.
\]

Concerning the equivalent characterization of the dissipation rate and the dissipative property of the linearized Landau operator, the following lemma was proved in [7].

**Lemma 2.1** ([7]). By using the above notations, we have

(i) \[
|u_{\sigma,\tau,\lambda}|^2 \sim \left| (1 + |\xi|) \frac{\tau}{\tau} P_\xi \nabla_\xi u \right|^2_{\tau,\lambda} + \left| (1 + |\xi|) \frac{\tau}{\tau} (I - P_\xi) \nabla_\xi u \right|^2_{\tau,\lambda} + \left| (1 + |\xi|) \frac{\tau}{\tau} u \right|^2_{\tau,\lambda}. \]

(ii) \( \langle Lu, v \rangle = \langle u, Lv \rangle, \) \( \langle Lu, u \rangle \geq 0, \)

\[
\ker L = \text{span} \left\{ M^{1/2}, \xi \right\}_{i \leq 3}, |\xi|^2 M^{1/2} \right\},
\]

and, \( \langle Lu, u \rangle = 0 \) if and only if \( u = Pu, \) where

\[
P u = \left\{ a(t, x) + b(t, x) \cdot \xi + c(t, x) \left( |\xi|^2 - 3 \right) \right\} M^{1/2},
\]

\[
a = \int_{\mathbb{R}^3} M^{1/2} u d\xi, \quad b_i = \int_{\mathbb{R}^3} \xi^i M^{1/2} u d\xi, \quad 1 \leq i \leq 3, \quad c = \frac{1}{6} \int_{\mathbb{R}^3} \left( |\xi|^2 - 3 \right) M^{1/2} u d\xi.
\]

Moreover, there exists \( \kappa_0 > 0 \) such that

\[
\langle Lu, u \rangle \geq \kappa_0 \| (I - P) u \|_{2}^2.
\]

The following lemma states the weighted estimate on \( Lu \) and \( \Gamma(u, u) \). Notice that since \( \lambda \geq 0 \) is small enough, the coefficient \( \lambda/(1 + t)^{\alpha} \) in front of \( \langle \xi \rangle^2 \) in the exponential part of the weight function \( w_{\tau,\lambda} \) is small uniformly in \( t \geq 0 \). Then, the proof of the following lemma follows directly from the argument used in [12].

**Lemma 2.2** ([12]). (i) There exist \( \kappa > 0 \) and \( C > 0 \), such that

\[
\langle w^2_{\tau,\lambda}(t, \xi) Lu, u \rangle \geq \kappa |u|^2_{\sigma,\tau,\lambda} - C |\chi_{|\xi| \leq 2C} u|^2_{\tau}.
\]

Let \( |\beta| > 0, \tau = |\beta| - \ell \) with \( \ell \geq 0 \). For \( \eta > 0 \) small enough there exists \( C_\eta > 0 \) such that

\[
\langle w^2_{\tau,\lambda}(t, \xi) \partial_\beta Lu, \partial_\beta u \rangle \geq \kappa |\partial_\beta u|^2_{\sigma,\tau,\lambda} - \eta \sum_{|\beta'| = |\beta|} |\partial_\beta' u|^2_{\sigma,\tau,\lambda} - C_\eta \sum_{|\beta'| < |\beta|} \partial_\beta u |_{|\beta'| - \ell, \lambda}.
\]

(ii) Let \( N \geq 8, |\alpha| + |\beta| \leq N, \tau = |\beta| - \ell \) with \( \ell \geq 0 \). Then

\[
\langle w^2_{\tau,\lambda}(t, \xi) \partial_\beta \Gamma(u_1, u_2), \partial_\beta u_3 \rangle 
\]

\[
\leq C \sum_{|\alpha'| + |\beta'| \leq N} \left\{ \left| \partial_{\beta'} u_1 \right|_{\tau} \left| \partial_{\beta - \beta'} u_2 \right|_{\sigma,\tau,\lambda} + \left| \partial_{\beta'} u_1 \right|_{\sigma,\tau,\lambda} \right\} \left| \partial_{\beta'} u_3 \right|_{\sigma,\tau,\lambda}.
\]

3. Basic Lemmas

To prove Theorem 1.1 for brevity, we only focus on obtaining the uniform-in-time \textit{a priori} estimate on the solution \( u(t, x, \xi) \) to the Cauchy problem ([12]). Recall [14]. The main goal is to construct an energy functional \( \mathcal{E}_{N,\ell,\lambda}(t) \sim \| u(t) \|_{2}^{2} \) such that the following estimate holds. Similar to [15], define

\[
\bar{X}_{N,\ell,\lambda}(t) = \sup_{0 \leq s \leq t} \mathcal{E}_{N,\ell,\lambda}(s) + \sup_{0 \leq s \leq t} (1 + s)^{2} \mathcal{E}_{N,\ell-1,\lambda}(s) + \sup_{0 \leq s \leq t} (1 + s)^{2(1+\theta)} \left\| \nabla_{\ell} \phi(s) \right\|_{H_{N-1}^{-1}}^{2}.
\]

Then, by assuming that \( \bar{X}_{N,\ell,\lambda}(t) \) is small in \( 0 \leq t < T \) for given \( T > 0 \), for the proper choice of parameters \( N, \ell, \lambda, \theta \), we shall prove that

\[
\bar{X}_{N,\ell,\lambda}(t) \leq C \left\{ Y_{N,\ell,\lambda}(0)^{2} + \bar{X}_{N,\ell,\lambda}(t)^{2} \right\}
\]

(3.2)
holds for $0 \leq t < T$. Recall that $\mathcal{Y}_{N,\ell,\lambda}(0)$ given in (1.7) depends only on initial data $u_0$. For later use, corresponding to the energy functional $\mathcal{E}_{N,\ell,\lambda}(t)$, we also define the functional for the energy dissipation rate

$$\mathcal{D}_{N,\ell,\lambda}(t) = \sum_{|\alpha| + |\beta| \leq N} \| \partial_{\beta}^\alpha \{ I - P \} u(t) \|_{H^1_{\sigma,|\beta|-\ell,\lambda}}^2 + \| a \|^2 + \sum_{|\alpha| \leq N-1} \| \nabla_x \partial^{\alpha} (a, b, c) \|^2$$

$$+ \frac{1}{(1 + t)^{1+\theta}} \sum_{|\alpha| + |\beta| \leq N} \| \partial_{\beta}^\alpha \{ I - P \} u(t) \|_{H^1_{\sigma,|\beta|+\frac{1}{1+\theta},\ell,\lambda}}^2. \quad (3.3)$$

The proof of (3.2) will be given in the next section. Here, we prepare some more estimates on the nonlinear terms. The proofs follow from the arguments used in [1] for the Vlasov-Poisson-Boltzmann system and hence most of details will be omitted for brevity. However, when necessary, we will point out the key points in the proof and the main differences from [4].

**Lemma 3.1.** (i) Let $N \geq 4$, $\ell \geq 0$, $0 < \lambda \ll 1$. Then

$$|\Gamma(u, u, u)| \leq C \left\{ \mathcal{E}_{N,\ell,\lambda}(t) \right\}^{1/2} \left\{ \| \nabla_x (a, b, c) \|^2_{H^1_{\ell,\lambda}} + \| \{ I - P \} u \|^2_{H^1_{\ell,\lambda}} \right\}, \quad (3.4)$$

$$\left\langle \Gamma(u, u, w^2_{|\beta|-\ell,\lambda}(t, \xi) \{ I - P \} u) \right\rangle \leq C \left\{ \mathcal{E}_{N,\ell,\lambda}(t) \right\}^{1/2} \left\{ \| \nabla_x (a, b, c) \|^2_{H^1_{\ell,\lambda}} + \| \{ I - P \} u \|^2_{H^1_{\ell,\lambda}} \right\}. \quad (3.5)$$

(ii) Let $N \geq 8$, $1 \leq |\alpha| + |\beta| \leq N$, $\ell \geq |\beta|$ and $0 < \lambda \ll 1$. For $u = u(t, x, \xi)$, define $u_{\alpha\beta}$ as $u_{\alpha\beta} = \partial^{\alpha} u$ if $|\beta| = 0$ and $u_{\alpha\beta} = \partial^{\beta} \{ I - P \} u$ if $|\beta| \geq 1$. Then

$$\left\langle \partial_{\beta}^\alpha \Gamma(u, u, w^2_{|\beta|-\ell,\lambda}(t, \xi) u_{\alpha\beta}) \right\rangle \leq C \left\{ \mathcal{E}_{N,\ell,\lambda}(t) \right\}^{1/2} \left\{ \| \nabla_x (a, b, c) \|^2_{H^1_{N-1,\lambda}} + \mathcal{D}_{N,\ell,\lambda}(t) \right\}. \quad (3.6)$$

**Proof.** Apply the decomposition

$$\Gamma(u, u) = \Gamma(Pu, Pu) + \Gamma(Pu, \{ I - P \} u) + \Gamma(\{ I - P \} u, Pu) + \Gamma(\{ I - P \} u, \{ I - P \} u). \quad (3.7)$$

All the inner products on the left-hand side of (3.4), (3.5) and (3.6) corresponding to the fourth term of (3.7) can be estimated directly by using (2.4). And for those from the first three terms of (3.7), we recall the expression of $\Gamma(u_1, u_2)$ from [7, 12].

$$\Gamma(u_1, u_2) = \partial_t \left\{ B^{ij} \ast \left[ M^{1/2} u_1 \right] \right\} \partial_j u_2 - \left\{ B^{ij} \ast \left[ \frac{\xi}{2} M^{1/2} u_1 \right] \right\} \partial_j u_2$$

$$- \partial_t \left\{ B^{ij} \ast \left[ M^{1/2} \partial_j u_1 \right] \right\} u_2 + \left\{ B^{ij} \ast \left[ \frac{\xi}{2} M^{1/2} \partial_j u_1 \right] \right\} u_2,$$

and also recall [24] for the expression of $Pu$. Thus, similar to the proof of (2.4) in [12], one can follow [4] to obtain the estimates on these terms. This then completes the proof of Lemma 3.1. \qed

**Lemma 3.2.** (i) Let $u$ be the solution to (1.2). Then

$$\left\langle \frac{1}{2} \xi \cdot \nabla_x \phi u, u \right\rangle \leq \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |b|^2 (a + 2c) dx$$

$$+ C \left\{ \| (a, b, c) \|^2_{H^2} + \| \nabla_x \phi \|^2_{H^1} + \| \nabla_x \phi \cdot \| \nabla_x b \| \right\} \left\{ \| \nabla_x (a, b, c) \|^2 + \| \{ \xi \} \frac{d}{dt} \{ I - P \} u \|^2 \right\}$$

$$+ C \| \nabla_x \phi \|^2_{H^1} \left\{ \{ \xi \}^{1/2} \| \{ I - P \} u \|^2 \right\}. \quad (3.8)$$

(ii) Let $N \geq 4$, $1 \leq |\alpha| \leq N$, and $\ell \geq 0$. Then

$$\left\langle \partial^\alpha \left( \frac{1}{2} \xi \cdot \nabla_x \phi u \right), w^2_{-\ell,\lambda}(t, \xi) \partial^\alpha u \right\rangle \leq C \| \nabla_x \phi \|^2_{H^{N-1}} \sum_{1 \leq |\alpha| \leq N} \left\{ \left\| \{ \xi \}^{1/2} \{ I - P \} u \right\|^2_{-\ell,\lambda} + \| \partial^\alpha (a, b, c) \|^2 \right\}. \quad (3.9)
(iii) Let $N \geq 4$, $1 \leq |\alpha| + |\beta| \leq N$, $|\beta| \geq 1$, and $\ell \geq |\beta|$. Then
\[
\left\langle \partial^\alpha \left( \frac{1}{2} \xi \cdot \nabla_x \phi (I - P) u \right), w_{|\beta| - \ell, \lambda}^2 (t, \xi) \partial_\beta^\alpha (I - P) u \right\rangle
\leq C \left\| \nabla_x^2 \phi \right\|_{H^{N-1}} \sum_{|\alpha| + |\beta| \leq N} \left\| (\xi) \partial_\beta^\alpha (I - P) u \right\|_{|\beta| - \ell, \lambda}^2. \tag{3.10}
\]

Proof. To prove \((3.8)\), notice that since $u$ is the solution to \((1.2)\), $u$ satisfies the momentum equation
\[
\partial_t b + \nabla_x (a + 2c) + \nabla_x \cdot \int_{\mathbb{R}^3} \xi \otimes \xi (I - P) u \, d\xi - \nabla_x \phi a = \nabla_x \phi.
\]
Then \((3.8)\) follows from using the decomposition $u = Pu + (I - P)u$ as in \((3.4)\) and the Cauchy-Schwarz inequality, where for low order term $\left( \frac{1}{2} \xi \cdot \nabla_x \phi \right) Pu$, as in \([4]\), we need to take the velocity integration and then replace $\nabla_x \phi$ in terms of the equation above. It is then straightforward to obtain both \((3.9)\) and \((3.10)\) by using the Leibnitz rule, Hölder and Sobolev inequalities. The details proof of the lemma are omitted.

Lemma 3.3. (i) Let $-3 \leq \gamma < -2$, $N \geq 4$, $1 \leq |\alpha| \leq N$, and $\ell \geq 0$. Then
\[
\left\langle \partial^\alpha \left( \nabla_x \phi \cdot \nabla \xi \right), w_{\gamma \ell, \lambda}^2 (t, \xi) \partial^\alpha u \right\rangle
\leq C \left\| \nabla_x^2 \phi \right\|_{H^{N-1}} \left\{ \sum_{|\alpha| + |\beta| \leq N, |\beta| \leq 1} \left\| (\xi) \partial_\beta^\alpha (I - P) u \right\|_{|\beta| - \ell, \lambda}^2 + \left\| \nabla_x (a, b, c) \right\|_{H^{N-1}}^2 \right\}. \tag{3.11}
\]

(ii) Let $-3 \leq \gamma < -2$, $1 \leq |\alpha| + |\beta| \leq N$, $|\beta| \geq 1$, and $\ell \geq |\beta|$. Then
\[
\left\langle \partial^\alpha \left( \nabla_x \phi \cdot \nabla \xi \right), w_{|\beta| - \ell, \lambda}^2 (t, \xi) \partial_\beta^\alpha (I - P) u \right\rangle
\leq C \left\| \nabla_x^2 \phi \right\|_{H^{N-1}} \sum_{|\alpha| + |\beta| \leq N} \left\| (\xi) \partial_\beta^\alpha (I - P) u \right\|_{|\beta| - \ell, \lambda}^2. \tag{3.12}
\]

Proof. The proof is similar to the one for \((3.9)\) and \((3.10)\) through the Leibnitz rule, integrations by part in $\xi$, the Hölder and Sobolev inequalities. The difference lies in the fact that $\nabla_x \phi \cdot \nabla \xi u$ and $\nabla_x \phi \cdot \nabla \xi (I - P) u$ involve the first order derivative in the velocity variable so that the velocity weight has to be carefully distributed. In fact, to control those inner products on the left-hand side of \((3.9)\) and \((3.10)\), for $|\beta| \geq 0$ and $\ell \geq |\beta|$, it suffices to have
\[
w_{|\beta| - \ell, \lambda}^2 (t, \xi) \lesssim \left\{ (\xi) w_{|\beta| + 1, |\beta| - \ell, \lambda} (t, \xi) \right\} \cdot \left\{ (\xi) w_{|\beta| - \ell, \lambda} (t, \xi) \right\}, \quad |\beta| \leq |\beta|,
\]
which due to the definition of $w_{\gamma \ell, \lambda}$, is equivalent to require
\[
2 + (\gamma + 2) + (\gamma + 2)(|\beta|' - |\beta|) \geq 0, \quad |\beta|' \leq |\beta|.
\]
Since $-3 \leq \gamma < -2$, the above inequality is satisfied and this then it completes the proof of Lemma \((3.2)\) cf. \([4]\) for more details.

Lemma 3.4. Let $G = \frac{1}{2} \xi \cdot \nabla_x \phi u - \nabla_x \phi \cdot \nabla \xi u + \Gamma(u, u)$. Let $-3 \leq \gamma < -2$, $N \geq 4$, $\ell_0 > \frac{1}{2}$, and $\ell - 1 \geq \max \left\{ N, \frac{\ell_0}{2} - \frac{1}{\gamma + 2} \right\}$. Then
\[
\left\| (\xi) \cdot \nabla x \phi G(t) \right\|_{Z_1} + \sum_{|\alpha| \leq 1} \left\| (\xi) \cdot \nabla x \phi \partial^\alpha G(t) \right\| \leq C E_{N, \ell - 1}(t). \tag{3.13}
\]

Proof. Let $G_1 = \frac{1}{2} \xi \cdot \nabla_x \phi u - \nabla_x \phi \cdot \nabla \xi u$. Similar to \([4]\), as long as $\ell - 1 \geq \max \left\{ N, \frac{\ell_0}{2} - \frac{1}{\gamma + 2} \right\}$ and $-3 \leq \gamma < -2$, we have
\[
\left\| (\xi) \cdot \nabla_x \phi G_1(t) \right\|_{Z_1} \leq C \left\| \nabla_x \phi \right\| \left\{ \left\| w_{-\frac{1}{2} \ell_0}(t, \xi) \nabla \xi u \right\| + \left\| w_{-\frac{1}{2} \ell_0 + \frac{1}{\gamma + 2}}(t, \xi) u \right\| \right\} \leq C E_{N, \ell - 1}(t)
\]
and
\[
\sum_{|\alpha| \leq 1} \left\| (\xi) \cdot \nabla_x \phi \partial^\alpha G_1(t) \right\| \leq C \left\| \nabla_x \phi \right\|_{H^3} \sum_{|\alpha| + |\beta| \leq 2} \left\| w_{|\beta| - (\ell - 1)}(t, \xi) \partial_\beta^\alpha u \right\| \leq C E_{N, \ell - 1}(t).
Corresponding to $\Gamma(u, u)$, for $|\alpha| \leq 1$,
\[
\langle \xi \rangle \frac{-2\ell \ell}{\ell^2} \partial^\alpha \Gamma(u, u) = w_{\ell} \langle t, \xi \rangle \partial^\alpha \Gamma(u, u) = \sum_{|\alpha_1| \leq |\alpha|} C_{\alpha_1}^\alpha G_{\alpha_1}
\]
with
\[
G_{\alpha_1} = w_{\ell} \left\{ B^{ij} \ast \partial_i \left[ M^{1/2} \partial^\alpha_1 u \right] \right\} \partial_j \partial^\alpha_1 u + w_{-\ell} \left\{ B^{ij} \ast \left[ \frac{\xi}{2} M^{1/2} \partial^\alpha_1 u \right] \right\} \partial_j \partial^\alpha_1 u
\]
\[
-w_{\ell} \left\{ B^{ij} \ast \left[ \frac{\xi}{2} M^{1/2} \partial^\alpha_1 u \right] \right\} \partial_j \partial^\alpha_1 u
\]
\[
-w_{-\ell} \left\{ B^{ij} \ast \partial_i \left[ M^{1/2} \partial_j \partial^\alpha_1 u \right] \right\} \partial^\alpha_1 u - w_{-\ell} \left\{ B^{ij} \ast \left[ M^{1/2} \partial_j \partial^\alpha_1 u \right] \right\} \partial_i \partial^\alpha_1 u
\]
\[
+w_{-\ell} \left\{ B^{ij} \ast \left[ \frac{\xi}{2} M^{1/2} \partial_j \partial^\alpha_1 u \right] \right\} \partial^\alpha_1 u,
\]
where the Einstein summation convention for repeated indices has been used. We estimate the second term $G_{\alpha_1}^{(2)}$ of $G_{\alpha_1}$ first. By the Cauchy-Schwarz inequality,
\[
\left\{ B^{ij} \ast \left[ M^{1/2} \partial^\alpha_1 u \right] \right\} \leq \left[ \left\{ B^{ij} \right\}^{1/2} \ast M^{1/2} \right]^{1/2} \left\{ M^{1/4} \partial^\alpha_1 u \right\}_{L^2_{x_t}} \leq C \langle \xi \rangle^{\gamma + 2} \partial^\alpha_1 u \langle t \rangle^{-(\ell - 1)},
\]
so that
\[
\left| G_{\alpha_1}^{(2)} \right| \leq C w_{\ell} \langle \partial^\alpha_1 u \rangle_{\langle t \rangle^{-(\ell - 1)}} \left| \partial_i \partial_j \partial^\alpha_1 u \right|.
\]
This implies
\[
\left\| G_{\alpha_1}^{(2)} \right\|_{Z_1} + \left\| G_{\alpha_1}^{(2)} \right\| \leq C \left\| w_{\ell} \right\|_{Z_1} + \left\| \partial^\alpha_1 u \right\|_{\langle t \rangle^{-(\ell - 1)}} \left\| \partial_i \partial_j \partial^\alpha_1 u \right\|_{L^2_{x_t}} + C \sup_{x \in \mathbb{R}^3} \left| \partial^\alpha_1 u \right|_{\langle t \rangle^{-(\ell - 1)}} \times \left\| w_{\ell} \right\|_{Z_1} \leq C E_{N, \ell - 1}(t),
\]
because $-\frac{\ell}{\ell} + 1 \geq 2 - (\ell - 1)$ which is equivalent to $\ell - 1 \geq \frac{\ell - 1}{2} + 1$. It is direct to verify that for all other terms of $G_{\alpha_1}$, the same estimate still holds and hence it proves (5.13) for the part $\Gamma(u, u)$ in $G$. Thus, the proof of lemma is completed. 

For later use, we also need the time-decay property of the linearized Landau system
\[
\partial_t u + \xi \cdot \nabla_x u - \nabla_x \phi \cdot \xi M^{1/2} = Lu
\]
with initial data given by $u_0(x, \xi)$, where $\phi$ is defined by (1.6). Denote the solution operator by $A(t)$.

**Lemma 3.5.** Set $\mu = \mu(\xi) := \langle \xi \rangle^{-\frac{m}{\ell}}$. Let $-3 \leq \gamma < -2$, $\ell \geq 0$, $\ell_0 > 3/2$, $\alpha \geq 0$, $m = |\alpha|$, and $\sigma_m = \frac{3}{4} + \frac{m}{2}$. Assume
\[
\int_{\mathbb{R}^3} a_0 \, dx = 0, \quad \int_{\mathbb{R}^3} (1 + |x|)|a_0| \, dx < \infty,
\]
and
\[
\left\| \mu^{\ell + \ell_0} u_0 \right\|_{Z_1} + \left\| \mu^{\ell + \ell_0} \partial^\alpha u_0 \right\| < \infty.
\]
Then, the evolution operator $A(t)$ satisfies
\[
\left\| \mu^{\ell + \ell_0} A(t) u_0 \right\| + \left\| \partial^\alpha \nabla_x A^{-1} \mathbf{P}_0 A(t) u_0 \right\|
\leq C(1 + t)^{-\sigma_m} \left( \left\| \mu^{\ell + \ell_0} u_0 \right\|_{Z_1} + \left\| \mu^{\ell + \ell_0} \partial^\alpha u_0 \right\| + \left\| (1 + |x|)a_0 \right\|_{L^1_{x_t}} \right)
\]
for any $t \geq 0$, where $\mathbf{P}_0$ is the projection given by $\mathbf{P}_0 u = a(t, x) M^{1/2}$.

The proof of Lemma 3.5 is completely the same as the one in [H] and thus it is omitted for brevity. In fact, what we have changed in the Landau case is the definition of $\mu(\xi)$ due to the coercivity estimate
\[
\langle Lu, u \rangle \gtrsim \langle \{ I - \mathbf{P} \} u \rangle_{\alpha_0}^2
\]
and the property
\[
\langle \{ I - \mathbf{P} \} u \rangle_{\alpha_0}^2 \gtrsim \langle \langle \xi \rangle^{\frac{m}{\ell}} \{ I - \mathbf{P} \} u \rangle^2 = |\mu^{-1}(I - \mathbf{P}) u|^2.
\]
In addition, to prove Lemma 3.4 we need to use the weighted estimates on the linearized Landau operator $L$ in terms of (2.2); see also 12 and 11.

4. Global existence

In the last section, we will give the proof of the main theorem in this paper.

Proof of Theorem 1.1. The proof is divided into the following three steps.

Step 1. Suppose that $u(t, x, \xi)$ is a smooth solution to the Cauchy problem (1.2) satisfying that $\tilde{X}_{N, \ell, \lambda}(t)$ is small in $0 \leq t < T$ for a given $T > 0$, where parameters $N, \ell, \lambda, \vartheta$ are to be determined to ensure that this smallness holds global in time. Let $-3 \leq \gamma < -2, N \geq 8, \ell \geq N, 0 < \lambda \ll 1$, and $\vartheta > 0$. Then, we claim that there is $E_{N, \ell, \lambda}(t)$ such that for $0 \leq t < T$,

$$
\frac{d}{dt} E_{N, \ell, \lambda}(t) + \kappa D_{N, \ell, \lambda}(t) \leq 0,
$$

and also there is a high order energy functional $E_{N, \ell, \lambda}(t)$ satisfying

$$
E_{N, \ell, \lambda}(t) \sim \sum_{|\alpha| + |\beta| \leq N} \| \partial^\alpha_x \{ (I - P) u(t) \} \|_{H^{-1}}^2 + \| \nabla_x \phi \|_{H^{-1}}^2 + \sum_{|\alpha| \leq N-1} \| \nabla_x \partial^\alpha (a, b, c) \|_2^2,
$$

such that for $0 \leq t < T$,

$$
\frac{d}{dt} E_{N, \ell, \lambda}(t) + \kappa D_{N, \ell, \lambda}(t) \leq C \| \nabla_x (a, b, c) \|_2^2.
$$

Note that this implies that

$$
\sup_{0 \leq s \leq t} E_{N, \ell, \lambda}(s) \leq E_{N, \ell, \lambda}(0).
$$

Proof of (4.1). First notice that the a priori assumption implies

$$
\| \nabla_x \phi \|_{H^{-1}} \leq \frac{C \delta}{(1 + t)^\delta},
$$

where $\delta > 0$ is sufficiently small. Then, with the help of (2.2), (2.3), and Lemmas 3.1, 3.2 and 3.3 following the argument in [11], one can obtain (4.1) by standard energy method. Here, we only point out the key points in the proof. As in [7], regarding the estimates on the linear transport terms, we can use

$$
w_{2, -\ell, \lambda}^2(t, \xi) = w_{2, -\ell, \lambda}^2(t, \xi) \leq C \langle \xi \rangle \frac{\gamma + 2}{(1 + t)^\delta} w_{2, -\ell, \lambda}^2(t, \xi),
$$

where $\beta = \beta_1 + \beta_2, |\beta_1| = 1$. Moreover, due to the time-dependent exponential factor $\exp\{\lambda(\xi - 2)^2/(1 + t)\}$ in the velocity weight function, the weighted energy estimates indeed generate the last part of the energy dissipation rate given in (4.3). And this part can be used to absorb those nonlinear terms with possible velocity growth that appear on the right-hand of (3.8)-(3.10) and (3.11)-(3.12) by using (4.5). In addition, to obtain the dissipation of the macroscopic component $(a, b, c)$ in $D_{N, \ell, \lambda}(t)$, one can apply the following fluid-type moment system, cf. [7],

$$
\begin{aligned}
\partial_t a + \nabla_x \cdot b &= 0, \\
\partial_t b + \nabla_x (a + 2c) + \nabla_x \cdot (\Theta((I - P) u) - \nabla_x \phi) &= \nabla_x \phi \cdot a, \\
\partial_t c + \frac{1}{3} \nabla_x \cdot b + \frac{5}{3} \nabla_x \cdot (a + 2c) &= 3 \nabla_x \phi \cdot b, \\
\Delta x \phi &= a,
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_t \Theta_{ij}((I - P) u) &= \partial_x b_j + \partial_x b_i - \frac{10}{3} \delta_{ij} \nabla_x \cdot b - \frac{10}{3} \delta_{ij} \nabla_x \cdot (\Lambda((I - P) u) u) \\
&= \Theta_{ij}(r + G) - \frac{10}{3} \delta_{ij} \nabla_x \phi \cdot b, \\
\partial_t \Lambda_i((I - P) u) &= \partial_t c = \Lambda_i(r + G),
\end{aligned}
$$

with $r = -\xi \cdot \nabla_x \{ (I - P) u \} + Lu$ and $G$ given in Lemma 3.4 where as in [6], we use the notations

$$
\Theta_{ij}(u) = \int_{\mathbb{R}^3} (\xi_\xi \xi_j - 1) M^{1/2} u d\xi, \quad \Lambda_i(u) = \frac{1}{10} \int_{\mathbb{R}^3} (|\xi|^2 - 5) |\xi| M^{1/2} u d\xi.
$$
Finally, as in [1], \( \mathcal{E}_{N,t,\lambda}(t) \) can be defined by

\[
\mathcal{E}_{N,t,\lambda}(t) = \mathcal{E}_{N}^\text{int}(t) + \sum_{1 \leq |\alpha| \leq N} \int_{\mathbb{R}^3} \left| \nabla^\alpha \phi \right|^2 + \left| \nabla^\alpha \phi_{x} \right|^2 - \int_{\mathbb{R}^3} |b|^2(a + 2c)\, dx + \mathcal{E}_{N}^\text{ext}(t)
\]

together with

\[
\mathcal{E}_{N}^\text{int}(t) = \sum_{|\alpha| \leq N-1} \int_{\mathbb{R}^3} \nabla^\alpha c \cdot A(\nabla^\alpha \{I - P\} u)\, dx - \kappa \sum_{|\alpha| \leq N-1} \int_{\mathbb{R}^3} \nabla^\alpha a \nabla^\beta u \cdot dx
\]

where \( \kappa < 1 \) is a small constant, \( C_m > 0 \) are properly chosen constants, and \( M_i \) \( i = 1, 2, 3 \) are constants large enough.

**Proof of (4.3).** Compared with (4.1), the main difference in the proof of (4.3) comes from the estimation on \( \|\{I - P\} u\| \) and \( \|\{I - P\} u\|_{t,\lambda}. \) Notice that the time evolution of \( \{I - P\} u \) satisfies

\[
\partial_t \{I - P\} u + \xi \cdot \nabla_x \{I - P\} u + \nabla_x \phi \cdot \nabla_x \xi \{I - P\} u + L\{I - P\} u = \Gamma(u, u) + \frac{1}{2} \xi \cdot \nabla_x \phi \{I - P\} u + [P, T_\phi] u,
\]

where \([A, B] = AB - BA\) denotes the commutator of two operators \( A, B, \) and \( T_\phi \) is given by

\[
T_\phi = \xi \cdot \nabla_x + \nabla_x \phi \cdot \nabla_x - \frac{1}{2} \xi \cdot \nabla_x \phi.
\]

Therefore, (4.3) follows from the similar weighted energy estimates as in [5].

**Step 2.** By Lemma 3.2, it is straightforward to verify

\[
\|a, b, c(t)\| + \|\nabla_x \phi(t)\| \leq C(1 + t)^{-\frac{1}{2}} \left\{ Y_{N,t,\lambda}(t) + \tilde{X}_{N,t,\lambda}(t) \right\}
\]

and

\[
\|\nabla_x (a, b, c(t))\| \leq C(1 + t)^{-\frac{1}{2}} \left\{ Y_{N,t,\lambda}(t) + \tilde{X}_{N,t,\lambda}(t) \right\}
\]

for \( 0 \leq t < T \). Then, as in [1], by the time-weighted estimate, we have from (4.1) and (4.0) that

\[
\sup_{0 \leq s \leq t} (1 + s)^{\frac{3}{2}} \mathcal{E}_{N,t,\lambda}(t) \leq C \left\{ Y_{N,t,\lambda}(t)^2 + \tilde{X}_{N,t,\lambda}(t)^2 \right\}.
\]

Here, notice that we have used

\[
\langle \xi \rangle \overset{\text{def}}{=} \omega_{|\alpha|-1,\lambda}(t, \xi) \sim \omega_{|\beta|+\frac{1}{2}-1,\lambda}(t, \xi),
\]

so that

\[
\mathcal{E}_{N,t,\lambda}(t) + \|a, b, c(t)\|^2 + \|\nabla_x \phi(t)\|^2 \gtrsim \mathcal{E}_{N,t,\lambda}(t).
\]

**Step 3.** Observe that

\[
\min_{\xi \in \mathbb{R}^3} \left\{ \langle \xi \rangle^{\gamma+2}, \frac{\langle \xi \rangle^2}{(1 + t)^{\gamma+2}} \right\} = (1 + t)\frac{(1 + \gamma)(1 + \gamma + 2)}{2},
\]

where the equality is taken when \( \langle \xi \rangle^{\gamma+2} = \langle \xi \rangle^2/(1 + t)^{\gamma+2}, \) i.e., \( \langle \xi \rangle = (1 + t) \frac{1}{1 + \gamma}. \) Set \( p = 1 + \frac{(1 + \gamma)(1 + \gamma + 2)}{2} \). Then, by the choice of \( \theta \) given in Theorem 1.1 since \( -3 \leq \gamma < -2 \), one has \( 1/14 \leq \theta < 1/4 \) and \( p = \frac{1}{2} + 2\theta \) with \( 0 < p < 1 \). Therefore, recalling (4.2) for the equivalent property of \( \mathcal{E}_{N,t,\lambda}(t) \), it follows from (4.3) that

\[
\frac{d}{dt} \mathcal{E}_{N,t,\lambda}(t) + \kappa p(1 + t)^{p-1} \mathcal{E}_{N,t,\lambda}(t) \leq C \|\nabla_x (a, b, c)\|^2,
\]

Theorem 1.1 holds.
which after multiplying it by $e^{\kappa(1+t)p}$ and taking the time integration over $[0,t]$, implies

$$
\mathcal{E}^{th}_{N,\ell,\lambda}(t) \leq \mathcal{E}^{th}_{N,\ell,\lambda}(0)e^{-\kappa(1+t)p} + Ce^{-\kappa(1+t)p}\int_0^t \| \nabla_x (u, b, c)(s) \|^2 e^{\kappa(1+s)p} \, ds.
$$

Using (4.7) and

$$
\int_0^t (1+s)^{-\frac{3}{2}} e^{\kappa(1+s)p} \, ds \leq C(1+t)^{-(\frac{3}{2}+p)e^{\kappa(1+t)p}},
$$

one has

$$
\mathcal{E}^{th}_{N,\ell,\lambda}(t) \leq C(1+t)^{-2(1+p)} \left\{ \mathcal{E}^{th}_{N,\ell,\lambda}(0) + Y_{N,\ell,\lambda}(0)^2 + \bar{X}_{N,\ell}(t)^2 \right\}.
$$

Noticing $\| \nabla_x^2 \phi \|^2_{H_{N-1}} \leq C\mathcal{E}^{th}_{N,\ell,\lambda}(t)$, the above inequality implies

$$
\sup_{0 \leq s \leq t} (1+s)^{2(1+\vartheta)} \| \nabla_x^2 \phi(s) \|^2_{H_{N-1}} \leq C \sup_{0 \leq s \leq t} (1+s)^{2(1+\vartheta)} \mathcal{E}^{th}_{N,\ell,\lambda}(t) \leq C \left\{ Y_{N,\ell,\lambda}(0)^2 + \bar{X}_{N,\ell}(t)^2 \right\}. \quad (4.9)
$$

Recall (3.1) and notice $\bar{X}_{N,\ell}(t) \leq \bar{X}_{N,\ell,\lambda}(t)$. (4.9) together with (4.4) and (4.8) then prove (3.2) which is equivalent to (1.8). Hence, it completes the proof of Theorem 1.1.

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