ON THE CAUCHY PROBLEM FOR THE
ZAKHAROV-RUBENCHIK/ BENNEY-ROSKES SYSTEM

Dedicated to Professor Vladimir Georgiev on the occasion of his sixtieth birthday

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Abstract. We address various issues concerning the Cauchy problem for the Zakharov-Rubenchik system (known as the Benney-Roskes system in water waves theory), which models the interaction of short and long waves in many physical situations. Motivated by the transverse stability/instability of the one-dimensional solitary wave (line solitary), we study the Cauchy problem in the background of a line solitary wave.

1. Introduction. This paper is concerned with various issues concerning the Cauchy problem for the two or three-dimensional Zakharov-Rubenchik (or Benney-Roskes) system and its perturbation by a line soliton. The Zakharov-Rubenchik system is fundamental, being a "generic" asymptotic system in the so-called modulation regime (slowly varying envelope of a fast oscillating train) and it was actually derived in various physical contexts. Moreover it contains in various limits the classical (scalar) Zakharov system (coupling a nonlinear Schrödinger equation and a wave equation, see (3) below) and the Davey-Stewartson systems (coupling a nonlinear Schrödinger equation and an elliptic equation). We refer to [52] for more details on the formal derivation of those systems and on the physical background.

The Davey-Stewartson system was first derived formally in the context of water waves in [14, 1, 15] (see also [11, 12] for a derivation of Davey-Stewartson systems in a different context). However, as noticed in [25] it is less general than the Benney-Roskes system (6) below in the sense that the initial conditions for the acoustic type components have to be prepared to obtain an approximation of the full water waves system.

We refer to [13] for a rigorous justification of the Zakharov limit of the Zakharov-Rubenchik system and to [36] for the Schrödinger limit of the Zakharov-Rubenchik system in the one-dimensional case and for well-prepared initial data.

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The Zakharov-Rubenchik/Benney-Roskes system is thus richer than those simpler models and should capture more of the original dynamics. It was introduced in [53] (see also the survey article [52]) to describe the interaction of spectrally narrow high-frequency wave packets of small amplitude with low-frequency acoustic type oscillations. The analysis is general and carried out in the Hamiltonian formalism and yields the following universal system

$$
\begin{aligned}
\psi_t + v_y \psi_x + i \frac{\omega''}{2} \psi_{xx} + i \frac{v_y}{2k} \Delta_{\perp} \psi - i(q|\psi|^2 + \beta \rho + \alpha \phi_x) \psi &= 0, \\
\rho_t + \rho_0 \Delta \phi + \alpha(|\psi|^2)_x &= 0, \\
\phi_t + \frac{c^2}{\rho_0} \rho + \beta |\psi|^2 &= 0,
\end{aligned}
$$

where $v_y$, $\omega''$, $k$, $q$, $\alpha$, $\rho_0$, $c$ are parameters. The two last equations describe the acoustic type waves and $\Delta_{\perp} = \partial_y^2 + \partial_z^2$ or $\partial_y^2$, $\Delta = \Delta_{\perp} + \partial_z^2$.

In two space dimensions a more specific (formal) derivation in the context of surface water waves is displayed in [6] and rigorously justified in [25], see below for a more precise description.

In the notations of [41] (see also [40] where it is used in the context of Alfvén waves in dispersive MHD), the Zakharov-Rubenchik system has the form

$$
\begin{aligned}
\psi_t - \sigma_3 \psi_x - i \delta \psi_x - i \sigma_1 \Delta_{\perp} \psi + i \{ \sigma_2 |\psi|^2 + W(\rho + D\phi_x) \} \psi &= 0, \\
\rho_t + \Delta \phi + D(|\psi|^2)_x &= 0, \\
\phi_t + \frac{1}{M^2} \rho + |\psi|^2 &= 0,
\end{aligned}
$$

where $\psi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, $\rho, \phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $d = 2, 3$ describe the fast oscillating and, resp., acoustic type waves.

Here $\sigma_1, \sigma_2, \sigma_3 = \pm 1$, $W > 0$ measures the strength of the coupling with acoustic type waves, $M > 0$ is a Mach number, $D \in \mathbb{R}$ is associated to the Doppler shift due to the medium velocity and $\delta \in \mathbb{R}$ is a nondimensional dispersion coefficient.

When $\alpha = 0$ (resp. $D = 0$) in (1) (resp. (2)) the Zakharov-Rubenchik system reduces to the classical (scalar) Zakharov system (see eg Chapter V in [46]). More precisely, in the framework of (1), one gets

$$
\begin{aligned}
\psi_t + v_y \psi_x + i \frac{\omega''}{2} \psi_{xx} + i \frac{v_y}{2k} \Delta_{\perp} \psi = i(q|\psi|^2 + \beta \rho) \psi, \\
\rho_{tt} - c^2 \Delta \rho = \beta \rho_0 \Delta |\psi|^2,
\end{aligned}
$$

which is a form of the 2 or 3D Zakharov system. Note however that the second order operator in the first equation is not necessarily elliptic.

The local well-posedness in $H^s(\mathbb{R}^d) \times H^{s-1/2}(\mathbb{R}^d) \times H^{s+1/2}(\mathbb{R}^d)$ with $s > \frac{d}{2}$, $d = 2, 3$ for (2), (1) was obtained in [41] by using the local smoothing property of the free Schrödinger operator after reducing the system to a quasilinear (non local) Schrödinger equation. Since it uses dispersive properties of the free Schrödinger group that are valid only in the whole space the proof does not extend to the Cauchy problem posed on the torus $\mathbb{T}^d$ or "semi-periodically" in $\mathbb{R}^{d-1} \times \mathbb{T}$, the latter situation being relevant for transverse stability issues. On the other hand, when applied to the Benney-Roskes system (5) below, it provides $^1$ an existence

\footnote{Roughly speaking, the idea in [41] is to reduce the system to a (nonlocal) quasilinear Schrödinger equation. When $\epsilon$ is taken into account, the crucial dispersive smoothing estimate on the Schrödinger group has a $1/\epsilon$ factor while the nonlinear term has a $\epsilon$ factor.}
time of order $O(1)$, while an existence time of order $O(1/\epsilon)$ is needed to fully justify the Benney-Roskes as a water wave model on the correct time scales (see [24]).

Local well-posedness of the Zakharov-Rubenchik/Benney-Roskes system was also obtained in [31], for $s > 2$ with the additional condition $\delta \sigma_1 > 0$ (that is the second order operator in the first equation of (2), (1) is elliptic) by using an energy method inspired by the work of Schochet-Weinstein in [45] on the nonlinear Schrödinger limit of the Zakharov system. The method used in [31] and [45] consists in rewriting the Zakharov system (or the Zakharov-Rubenchik system) as a dispersive (skew-adjoint) perturbation of a symmetric nonlinear hyperbolic system and it uses only the algebraic structure of the system. A shortcoming of the method is that one has to prepare the initial data.

We will see that, when the small parameter $\epsilon$ is included, this method provides also the existence on the time scale $O(1)$ in the context of water waves (see the Benney-Roskes system (6) below) and moreover that it can be applied to the system obtained from (2) which is satisfied by a (localized) perturbation of a line soliton. Also, since it does not use any dispersive property of the Schrödinger group, it applies to the Cauchy problem in $T^d$ or $\mathbb{R}^d - 1 \times T$, a situation that has not been addressed before (see on the other hand [7, 8] for the periodic Zakharov system).

Thus, none of the two aforementioned methods seems to give the expected existence time scale for the Benney-Roskes system. Nevertheless they provide different results for Zakharov-Rubenchik type systems. The "dispersive method" used in [41] works only in $\mathbb{R}^d$ but does not need the Schrödinger part of the system to be "elliptic" (that is it does not need the condition $\delta \sigma_1 > 0$). Also it lowers the regularity on the initial data (an effect of the dispersive smoothing effect) and could be applied as well to (possibly non physical) nonlinear perturbations of the system.

On the other hand, the Schochet-Weinstein type "hyperbolic like" methods allow to deal with the periodic or semi-periodic cases, but are relatively rigid (they rely on the algebraic structure of the system) and require initial data in the "hyperbolic space" $H^s(\mathbb{R}^d), s > \frac{d}{2} + 1$.

The situation is better understood in spatial dimension one. Oliveira [35] proved the local (thus global using the conservation laws below) well-posedness in $H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})$. This result was improved in [26] where in particular global well-posedness was established in the energy space $H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})$.

It is worth noticing that (2) possesses two conserved quantities, the $L^2$ norm
\[
\int |\psi(x,y_\perp,t)|^2 = \int |\psi(x,y_\perp,0)|^2,
\]
where $y_\perp = y$ or $(y,z)$, and, after the change of variable $(x,t) \rightarrow (x + \sigma_3 t, t)$, the Hamiltonian
\[
E(t) = \int_{\mathbb{R}^d} \left( \frac{\delta}{2} |\psi_x|^2 + \frac{\sigma_1}{2} |\nabla_\perp \psi|^2 + \frac{\sigma_2}{4} |\psi|^4 + \frac{W}{4M^2} \rho^2 + \frac{W}{4} |\nabla \phi|^2 + \frac{W}{2} (\rho + D\phi_x)|\psi|^2 \right).
\]

\begin{align}
E(t) &= E(0).
\end{align}

The conservation laws are used in [41] to obtain global weak solutions under suitable assumptions on the coefficients. We will use them in Section 5 to prove the global existence of weak solutions of the systems obtained by perturbing a line (dark) soliton. Note also that the conservation laws can be used to get the global well-posedness of the Zakharov-Rubenchik, Benney-Roskes system in space dimension one (see [35]).
As aforementioned, in the context of water waves, the Zakharov-Rubenchik system is known as the Benney-Roskes system and it was formally derived in [6]. We follow here the notations in [25], where a rigorous derivation is performed.

\[ k = |k|e_x, \quad \omega(k) = \omega(|k|), \]
\[ \omega = \omega(|k|), \quad \omega' = \omega'(|k|), \quad \omega'' = \omega''(|k|), \]
where \( \omega(|\xi|) = \left( g + \frac{\sigma}{\rho} |\xi|^2 \right) |\xi| \tanh(\sqrt{\mu}|\xi|) \right)^{1/2} \]
is the dispersion relation of water waves and where \(|k|\) is a fixed wave number, \(g\) is the gravity, \(\sigma \geq 0\) is a surface tension coefficient, \(\rho\) is the density of the water and \(\mu\) is the shallowness parameter (square of the typical fluid depth over a typical horizontal scale) which is large or infinite in the deep water models and \(\alpha = -\frac{g}{8\pi} (1 - \sigma^2)^2\).

The small parameter \(\epsilon\) is the wave steepness that is the ratio of a typical amplitude of the wave over a typical horizontal scale. Recall ([25]) that the typical time scale for the solutions of (6) below is \(1/\epsilon\) and so it is crucial to establish the well-posedness on those time scales.

The Benney-Roskes equations can then be written in 2 dimensions as follows

\[
\begin{align*}
\partial_t \psi_{01} + \omega \partial_x \psi_{01} - i\epsilon \left( \frac{1}{2} (\omega'' \partial_x^2 + \frac{\omega'}{\omega} \partial_y^2) \psi_{01} ight. \\
+ i\epsilon \left( |k| \partial_x \psi_{00} + \frac{|k|^2}{2\omega} (1 - \sigma^2) \zeta_{10} + 2\frac{|k|^4}{\omega} (1-\alpha) |\psi_{01}|^2 \right) \psi_{01} = 0, \\
\partial_t \zeta_{10} + \sqrt{\mu} \Delta \psi_{00} = -2\omega |k| \partial_x (|\psi_{01}|^2), \\
\partial_t \psi_{00} + \zeta_{10} = -|k|^2 (1-\sigma^2) |\psi_{01}|^2.
\end{align*}
\]

It is known (see eg [25] Chapter 8) that \(\omega' > 0\), while for purely gravity waves (\(\sigma = 0\)) \(\omega < 0\) and the Schrödinger equation in the Benney-Roskes system is "non elliptic".

On the other hand, in the presence of a surface tension, the condition \(\omega'' > 0\) is possible as shown in the following computation.

For simplicity of notations, we will consider \(\omega\) of the following form instead of (5)

\[ \omega(r) = \left((1 + \gamma r^2) r \tanh(\sqrt{\mu}r)\right)^{1/2} \]
where \(\gamma > 0\) which depends on \((g, \rho)\) and is proportional to \(\sigma\) and \(r = |k|\). We have

\[ \omega'(r) = \left((1 + 3\gamma r^2) \tanh(\sqrt{\mu}r) + \sqrt{\mu}(r + \gamma r^3) \sech^2(\sqrt{\mu}r)\right) \times \frac{1}{2} \left((r + \gamma r^3) \tanh(\sqrt{\mu}r)\right)^{-1/2} \]
and

\[ \omega''(r) = -\frac{1}{4} \left((1 + 3\gamma r^2) \tanh(\sqrt{\mu}r) + \sqrt{\mu}(r + \gamma r^3) \sech^2(\sqrt{\mu}r)\right)^2 \left((r + \gamma r^3) \tanh(\sqrt{\mu}r)\right)^{-3/2} - \frac{1}{2} \left((r + \gamma r^3) \tanh(\sqrt{\mu}r)\right)^{-1/2} \left(6\gamma r \tanh(\sqrt{\mu}r) + 2\sqrt{\mu}(1 + 3\gamma r^2) \sech^2(\sqrt{\mu}r) - 2\mu(r + \gamma r^3) \tanh(\sqrt{\mu}r) \sech^2(\sqrt{\mu}r)\right). \]
We see that $\omega'(r) > 0$ with $\gamma, r > 0$, we thus will look for $r$ such that with fixed $\gamma$, $\omega''(r) > 0$. We assume that $\sqrt{\mu r} \gg 1$ implying that $\text{sech}(\sqrt{\mu r}) \approx 0$ and $\tanh(\sqrt{\mu r}) \approx 1$. Therefore we only need to choose $r$ large enough so that

$$12\gamma r > \frac{(1 + 3\gamma r^2)^2}{r + \gamma r^3}$$

or

$$3\gamma^2 r^4 + 6\gamma r^2 > 1.$$ 

In order to apply the Schochet-Weinstein method we will need the condition $\delta \sigma_1 > 0$ and we will only consider the Zakharov-Rubenchik (or Benney-Roskes) system of the form of (2) satisfying this condition.

The one-dimensional Zakharov system possesses solitary wave solutions and in [35], Oliveira proved their orbital stability. One motivation of the present paper was the study of their transverse stability. The transverse instability of the line solitary wave for some two dimensional models such as the nonlinear Schrödinger equation (NLS), the Kadomtsev-Petviashvili equation (KP) and some general "abstract" Hamiltonian systems have been carried out extensively in [42, 43, 44, 29, 30].

It is thus of interest to study the transverse stability of the line soliton for the two dimensional model (2) and the first step is to study the Cauchy problem of a localized perturbation of (2) by a line soliton. Another possibility is to consider $y$ or $(y, z)$-periodic perturbations of the line solitary wave, a first step being to establish the well-posedness of the Cauchy problem for the Zakharov-Rubenchik, Benney-Roskes system in $\mathbb{R}^d \times \mathbb{T}$, $d = 1, 2$, which could not result from the methods used in [41], but in the way we achieve here and also in the pure periodic case $\mathbb{T}^{d+1}$.

In order to unify the notation, we will rewrite the Benney-Roskes system (6) in the form of (2). We replace $(\psi_{01}, |k|^2(1 - \sigma^2)^{1/2}, |k|^2(1 - \sigma^2)^{1/2})$ by $(\psi, \rho, \phi)$ and after calculating the corresponding coefficients, we have:

$$\begin{cases}
\sigma_3 = -\omega', \\
\delta = \frac{\epsilon \omega''}{2}, \\
\sigma_1 = \frac{\epsilon \omega'}{\omega}, \\
\sigma_2 = \frac{2\epsilon |k|^4(1 - \alpha)}{\omega}, \\
W = \frac{\epsilon |k|^4(1 - \sigma^2)^2}{2\omega} \sqrt{\mu}, \\
D = \frac{2\omega}{|k|(1 - \sigma^2)\sqrt{\mu}}, \\
M = \mu^{-1/4}.
\end{cases} \tag{7}$$

The paper is organized as follows. In the next Section we reformulate the existence of one-dimensional solitary waves (bright and dark) in our framework. In Section 3 we use the Schochet-Weinstein method to prove a local existence for the Benney-Roskes/ Zakharov-Rubenchik system, keeping the small parameter $\epsilon$ which is relevant for deep water waves. In Section 4 we consider the case of a localized perturbation of a line solitary wave. Finally we prove in Section 5 the global existence of weak solutions perturbing a dark solitary wave.

We conclude the paper by a list of open questions.

**Notation.**
- $\partial_x$ or $(\cdot)_x$ will be used to denote the derivative with respect to variable $x$.
- $H^s(D), s \in \mathbb{R}$ denotes the classical Sobolev space in the domain $D$.
- $\|\cdot\|_X$: The norm in a functional space $X$.
- $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier and inverse Fourier transform respectively.
- $(\xi) = \sqrt{1 + |\xi|^2}$ for $\xi \in \mathbb{R}^n$ and $\sigma(D)$ denotes the Fourier multiplier with the symbol $\sigma(\xi)$.
- $\Re$ and $\Im$ denote the real part and imaginary part of a complex number respectively.
2. **Existence of one dimensional solitary waves.** In this section, we reframe the proof of the existence of 1-d solitary waves in \([35]\) in our setting. The 1-d Zakharov-Rubenchik system has the form

\[
\begin{align*}
\psi_t - \sigma_3 \psi_x - i \delta \psi_{xx} + i \left\{ \sigma_2 |\psi|^2 + W(\rho + D\phi_x) \right\} \psi &= 0, \\
\rho_t + \phi_{xx} + D(|\psi|^2)_x &= 0, \\
\phi_t + \frac{1}{M^2} \rho + |\psi|^2 &= 0.
\end{align*}
\]  
(8)

Setting \( \tilde{\phi} = \phi_x \), (8) becomes

\[
\begin{align*}
\psi_t - \sigma_3 \psi_x - i \delta \psi_{xx} + i \left\{ \sigma_2 |\psi|^2 + W(\rho + D\tilde{\phi}) \right\} \psi &= 0, \\
\rho_t + \tilde{\phi}_x + D(|\psi|^2)_x &= 0, \\
\tilde{\phi}_t + \frac{1}{M^2} \rho_x + (|\psi|^2)_x &= 0.
\end{align*}
\]  
(9)

Let \( c \geq 0 \), we look for solutions of the system (9) of the form

\[
\left( e^{i\lambda t} K(x - ct), a|K(x - ct)|^2, b|K(x - ct)|^2 \right).
\]

From the last two equations of (9) we deduce that

\[
a = -\frac{(1 + cD)}{1/M^2 - c^2} \quad \text{and} \quad b = -\frac{(c + D/M^2)}{1/M^2 - c^2}.
\]  
(10)

Then the first equation of (9) is equivalent to

\[
\delta \dddot{K} - i(c + \sigma_3) \dddot{K} - \lambda K = (\sigma_2 + W(a + bD)) |K|^2 K.
\]

Set

\[
R(x) = e^{-i(c+\sigma_3)x/2\delta} K(x),
\]
then

\[
\delta \ddot{R} + \left( \frac{(c + \sigma_3)^2}{4\delta} - \lambda \right) R = (\sigma_2 + W(a + bD)) |R|^2 R.
\]  
(11)

The equation (11) has a positive solution, which is unique up to a translation, if:

\[
\begin{align*}
\frac{1}{\delta} \left( \frac{(c + \sigma_3)^2}{4\delta} - \lambda \right) &< 0, \\
\frac{1}{\delta} (\sigma_2 + W(a + bD)) &< 0
\end{align*}
\]

or equivalently

\[
\begin{align*}
\frac{1}{\delta} \left( \frac{(c + \sigma_3)^2}{4\delta} - \lambda \right) &< 0, \\
\frac{1}{\delta} \left( \sigma_2 - \frac{W(1 + D^2/M^2 + 2cD)}{1/M^2 - c^2} \right) &< 0.
\end{align*}
\]  
(12)

We see that if \( c \to (1/M)^- \) and \( \lambda \) is large enough then (12) holds assuming that \( W > 0 \) and \( \delta > 0 \) which holds true in both models (2) and (6).

In this case,

\[
R(x) = \sqrt{\frac{2}{\sigma_2 + W(a + bD)}} \left( \frac{(c + \sigma_3)^2}{4\delta} - \lambda \right) \sech \left( \frac{1}{\delta} \left( \frac{(c + \sigma_3)^2}{4\delta} - \lambda \right) x \right).
\]  
(13)
Otherwise, if
\[
\begin{aligned}
\frac{1}{\delta} \left( \frac{(c + \sigma_3)^2}{4\delta} - \lambda \right) > 0, \\
\frac{1}{\delta} (\sigma_2 + W(a + bD)) > 0,
\end{aligned}
\]
or equivalently
\[
\begin{aligned}
\frac{1}{\delta} \left( \frac{(c + \sigma_3)^2}{4\delta} - \lambda \right) > 0, \\
\frac{1}{\delta} \left( \sigma_2 - \frac{W(1 + D^2/M^2 + 2cD)}{1/M^2 - c^2} \right) > 0.
\end{aligned}
\]

(14)

In the context of water waves (Benney-Roskes system) there is a regime where the condition (14) holds. In particular, if we choose \( c > \frac{1}{M} \) and \( \lambda < \frac{(c + \sigma_3)^2}{4\delta} \) then (14) holds, since from (7) we know that \( \delta, \sigma_2, W \) and \( D \) are positive. If \( c = 0 \), then (14) is equivalent to
\[
\begin{aligned}
\frac{\sigma_3^2}{4\delta} > \lambda, \\
\frac{2|k|^4(1 - \alpha)}{\omega} - \frac{|k|^4(1 - \sigma_2^2)^2}{2\omega} - \frac{2\omega|k|^2}{\sqrt{\mu}} > 0.
\end{aligned}
\]

Since \( \alpha < 0 \), and in the context of water wave, the surface tension is small, then one has \( \frac{2|k|^4(1 - \alpha)}{\omega} > \frac{|k|^4(1 - \sigma_2^2)^2}{2\omega} \). Therefore, if \( \mu \) is large enough (which occurs in the context of deep water waves) then the above conditions hold.

In this case,
\[
R(x) = \sqrt{\frac{1}{\sigma_2 + W(a + bD)} \left( \frac{(c + \sigma_3)^2}{4\delta} - \lambda \right)} \tanh\left( \sqrt{-\frac{1}{2\delta} \left( \frac{(c + \sigma_3)^2}{4\delta} - \lambda \right) x} \right)
\]

(15)

Then the system (9) has two kind of solitary waves corresponding to the two conditions (12) and (14):
\[
(e^{i\lambda t} e^{i(c + \sigma_3)x/2\delta} R(x - ct), a R^2(x - ct), b R^2(x - ct)).
\]

Recalling that \( \tilde{\phi} = \phi_x \), the solutions of system (8) should have thus the form
\[
Q = (e^{i\lambda t} e^{i(c + \sigma_3)x/2\delta} R(x - ct), a R^2(x - ct), b P(x - ct)),
\]

(16)

where
\[
P(x) = \frac{\alpha^2}{\beta} \tanh(\beta x)
\]

with
\[
\alpha = \sqrt{\frac{2}{\sigma_2 + W(a + bD)} \left( \frac{(c + \sigma_3)^2}{4\delta} - \lambda \right)}, \quad \beta = \sqrt{-\frac{1}{\delta} \left( \frac{(c + \sigma_3)^2}{4\delta} - \lambda \right)}
\]

in the case \( R(x) \) is given by (13),
\[
P(x) = \frac{\alpha^2}{\beta} (\beta x - \tanh(\beta x))
\]

with
\[
\alpha = \sqrt{\frac{1}{\sigma_2 + W(a + bD)} \left( \frac{(c + \sigma_3)^2}{4\delta} - \lambda \right)}, \quad \beta = \sqrt{\frac{1}{2\delta} \left( \frac{(c + \sigma_3)^2}{4\delta} - \lambda \right)}
\]

in the case \( R(x) \) is given by (15).
Remark 1. Similarly to the case of the cubic nonlinear Schrödinger equation, we will call the 1-d solitary wave corresponding to the condition (12) and (14) the "bright", "dark" soliton respectively.

3. The Z-R/B-R system. As aforementioned the asymptotic model (6) is a good approximation of the full water wave system on a time scale $O(1/\epsilon)$ (see [25] page 233). It is thus crucial to prove the well-posedness of the Cauchy problem on time scales of order $1/\epsilon$.

However, the existence time obtained by using the method in [41] does not reach the $O(1/\epsilon)$ time scale (as we already mentioned it is of order $O(1)$).

In this section, we give the proof of the local well-posedness for (2) in two dimensional case by using Schochet-Weinstein method in [31] but keeping the parameter $\epsilon$ in (6) to estimate the existence time obtained by this method. It turns out, however, that one does not improve upon the previously known $O(1)$ result (see however the comments in Introduction).

We consider the following system

\[
\begin{aligned}
\psi_t - \sigma_3 \psi_x - i\epsilon \delta \psi_{xx} - i\epsilon \sigma_1 \psi_{yy} + i\epsilon \left( \sigma_2 |\psi|^2 + W(\rho + D\phi_x) \right) \psi &= 0, \\
\rho_t + \Delta \phi + D(|\psi|^2)_x &= 0, \\
\phi_t + \frac{1}{M^2} \rho + |\psi|^2 &= 0,
\end{aligned}
\]

with initial data $(\psi, \rho, \phi)(t=0) = (\psi_0, \rho_0, \phi_0)$ for which we obtain a local existence result:

**Theorem 3.1.** Let $\delta \sigma_1 > 0$, $s > 2$. For any initial data $(\psi_0, \rho_0, \phi_0) \in H^{s+1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2) \times H^{s+1}(\mathbb{R}^2)$, there exist $T > 0$ independent of $\epsilon \in (0, 1]$ such that (17) admits a unique solution $(\psi, \rho, \phi) \in C(0, T; H^{s+1}(\mathbb{R}^2)) \times C(0, T; H^s(\mathbb{R}^2)) \times C(0, T; H^{s+1}(\mathbb{R}^2))$.

**Remark 2.** With some minor changes, one obtains the same result in the three-dimensional case, that is $\psi_{yy}$ replaced by $\Delta \psi$.

**Remark 3.** The above theorem and its proof are valid *mutatis mutandi* in a periodic $(\mathbb{T}^d)$, $d = 2, 3,$ or semi-periodic $(\mathbb{R}^{d-1} \times \mathbb{T})$ setting.

**Proof.** We follow closely the proof in [31], Section 3.3, but we keep track of the parameter $\epsilon$.

We first rewrite (17) as a dispersive perturbation of a symmetric hyperbolic system. We take the time derivative of the second and the third equation of the system (2). This allows to decouple the linear parts of those equations

\[
\begin{aligned}
\psi_t - \sigma_3 \psi_x - i\epsilon \delta \psi_{xx} - i\epsilon \sigma_1 \psi_{yy} + i\epsilon \left( \sigma_2 |\psi|^2 + W(\rho + D\phi_x) \right) \psi &= 0, \\
\rho_{tt} - \Delta \left( \frac{1}{M^2} \rho + |\psi|^2 \right) + D(|\psi|^2)_x &= 0, \\
\phi_{tt} - \frac{1}{M^2} \left( \Delta \phi + D(|\psi|^2)_x \right) + (|\psi|^2)_t &= 0.
\end{aligned}
\]

We set

\[
U = W\rho + WD\phi_x.
\]
We then get a coupled system for \( \psi \) and \( \mathcal{U} \)
\[
\begin{align*}
\psi_t - \sigma_3 \psi_x - i e \delta \psi_{xx} - i e \sigma_1 \psi_{yy} + i e (\sigma_2 |\psi|^2 + \mathcal{U}) \psi &= 0, \\
\mathcal{U}_t - \frac{1}{M^2} \Delta \mathcal{U} - W \Delta (|\psi|^2) + 2 DW(|\psi|^2)_x - \frac{W D^2}{M^2} (|\psi|^2)_{xx} &= 0. 
\end{align*}
\tag{20}
\]

Following the idea in [45] for the Zakharov system, we define the following auxiliary (vector valued) function \( V \) as the unique solution of the following equation
\[
V_t = \frac{1}{M} \nabla \mathcal{U} + WM \nabla (|\psi|^2) + \left( \frac{W D^2}{M} (|\psi|^2)_x, 0 \right)^T,
\tag{21}
\]
with the initial data given by
\[
V(t = 0) = V_0 = -MW \nabla \phi_0 - \frac{W D}{M} (\rho_0, 0)^T.
\tag{22}
\]

Therefore, we obtain the equivalent first order system
\[
\begin{align*}
\psi_t - \sigma_3 \psi_x - i e \delta \psi_{xx} - i e \sigma_1 \psi_{yy} + i e (\sigma_2 |\psi|^2 + \mathcal{U}) \psi &= 0, \\
\mathcal{U}_t - \frac{1}{M} \nabla \cdot V + 2 WD(|\psi|^2)_x &= 0, \\
V_t - \frac{1}{M} \nabla V - WM \nabla (|\psi|^2) - \left( \frac{W D^2}{M} (|\psi|^2)_x, 0 \right)^T &= 0.
\end{align*}
\tag{23}
\]

We now set
\[
\mathcal{D} = \mathcal{U} + \frac{\sigma_1}{2} |\psi|^2,
\]
then the system (23) becomes
\[
\begin{align*}
\psi_t - \sigma_3 \psi_x - i e \delta \psi_{xx} - i e \sigma_1 \psi_{yy} + i e \left( (\sigma_2 - \frac{\sigma_1}{2}) |\psi|^2 + \mathcal{D} \right) \psi &= 0, \\
\mathcal{D}_t - \frac{1}{M} \nabla \cdot V - \frac{\sigma_1}{2} \left( (|\psi|^2)_t - \sigma_3 (|\psi|^2)_x \right) + \left( 2 WD - \frac{\sigma_1 \sigma_3}{2} \right) (|\psi|^2)_x &= 0, \\
V_t - \frac{1}{M} \nabla \mathcal{D} - \left( \frac{2 WD^2 + M^2}{2M} - \frac{\sigma_1}{2} (|\psi|^2)_x, \frac{2 WM^2 - \sigma_1}{2M} (|\psi|^2)_y \right)^T &= 0.
\end{align*}
\tag{24}
\]

For simplicity of notation we set
\[
c_1 = 2 WD - \frac{\sigma_1 \sigma_3}{2}, \; c_2 = \frac{2 WD^2 + M^2 - \sigma_1}{2M}, \; c_3 = \frac{2 WM^2 - \sigma_1}{2M}.
\]

Furthermore, we split the \( \psi \) in real and imaginary part
\[
\psi = F + i G
\]
and
\[
\nabla \psi = H + i L = (H_1, H_2)^T + i (L_1, L_2)^T.
\]

Multiplying the first equation of (24) by \( \bar{\psi} \) and taking the real part, we deduce that
\[
(|\psi|^2)_t - \sigma_3 (|\psi|)_x = i e \delta \psi_{xx} \bar{\psi} - i e \delta \bar{\psi}_{xx} \psi + i e \sigma_1 \psi_{yy} \bar{\psi} - i e \sigma_1 \bar{\psi}_{yy} \psi
\]
\[
= 2 e \delta (G \partial_x H_1 - F \partial_x L_1) + 2 e \sigma_1 (G \partial_y H_2 - F \partial_y L_2).
\tag{25}
\]

We insert (25) into (24) and then separate the real and imaginary parts of the first equation of the system. We furthermore apply the spatial gradient to the
We then set $A \equiv 0$ and we then set the following system

\begin{align*}
H_1 - \sigma_3 H_x + \epsilon \delta L_{xx} + \epsilon \sigma_1 L_{yy} - \epsilon G \nabla D &= 0, \\
L_1 - \sigma_3 L_x - \epsilon \delta H_{xx} - \epsilon \sigma_1 H_{yy} + \epsilon F \nabla D &= 0,
\end{align*}

\begin{align*}
F_t - \sigma_3 F_x + \epsilon \delta G_{xx} + \epsilon \sigma_1 G_{yy} - \epsilon \left( \left( \sigma_2 - \frac{\sigma_1}{2} \right) \left( F^2 + G^2 \right) + D \right) G &= 0, \\
G_t - \sigma_3 G_x - \epsilon \delta F_{xx} - \epsilon \sigma_1 F_{yy} + \epsilon \left( \left( \sigma_2 - \frac{\sigma_1}{2} \right) \left( F^2 + G^2 \right) + D \right) F &= 0,
\end{align*}

\begin{align*}
\mathcal{D}_t - \frac{1}{M} \nabla \cdot V - \epsilon \sigma_1 \delta \left( G \partial_x H_1 - F \partial_x L_1 \right) - \epsilon \sigma_1^2 \left( G \partial_y H_2 - F \partial_y L_2 \right) &+ 2c_1 \left( H_1 F + L_1 G \right) = 0, \\
V_t - \frac{1}{M} \nabla D - 2 \left( c_2 \left( H_1 F + L_1 G \right), c_3 \left( H_2 F + L_2 G \right) \right)^T &= 0.
\end{align*}

Since $\sigma_1 \delta > 0$, we can perform the following change of variables

\begin{align*}
H^* &= \left( \sqrt{\delta \sigma_1} H_1, \sigma_1 H_2 \right)^T, \\
L^* &= \left( \sqrt{\delta \sigma_1} L_1, \sigma_1 L_2 \right)^T,
\end{align*}

and we then set $U = (H^*, L^*, F, G, D, V)^T$.

Therefore, (17) is rewritten as a dispersive (skew adjoint) perturbation of a symmetric hyperbolic system given by

\begin{align*}
U_t + (\epsilon A_1(U) + B_1) U_x + (\epsilon A_2(U) + B_2) U_y + C(U) U &= -K_1 U_{xx} - K_2 U_{yy},
\end{align*}

where $A_1$, $A_2$, $B_1$ and $B_2$ are symmetric matrices, $K_1$ and $K_2$ are skew symmetric matrices.

\begin{align*}
A_1(U) &= \begin{pmatrix}
0_{3 \times 3} & 0_{3 \times 3} & M_1(U)^T & 0_{3 \times 2} \\
0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} & 0_{3 \times 2} \\
M_1(U) & 0_{1 \times 3} & 0_{1 \times 2} & 0_{1 \times 2} \\
0_{2 \times 3} & 0_{2 \times 3} & 0_{2 \times 1} & 0_{2 \times 2}
\end{pmatrix},
\end{align*}

with

\begin{align*}
M_1(U) &= (-\sqrt{\delta \sigma_1} G, 0, \sqrt{\delta \sigma_1} F),
\end{align*}

\begin{align*}
A_2(U) &= \begin{pmatrix}
0 & 0_{1 \times 3} & 0_{1 \times 2} & 0 & 0_{1 \times 2} \\
0_{4 \times 1} & 0_{3 \times 3} & 0_{3 \times 2} & M_2(U)^T & 0_{3 \times 2} \\
0_{2 \times 1} & 0_{2 \times 3} & 0_{2 \times 2} & 0_{2 \times 1} & 0_{2 \times 2} \\
0 & M_2(U) & 0_{1 \times 2} & 0 & 0_{1 \times 2} \\
0_{2 \times 1} & 0_{2 \times 3} & 0_{2 \times 2} & 0_{2 \times 1} & 0_{2 \times 2}
\end{pmatrix},
\end{align*}

with

\begin{align*}
M_2(U) &= (-\sigma_1 G, 0, \sigma_1 F).
\end{align*}
Now note that \( C(U) \) contains the term that is independent of \( \epsilon \) which is

\[
C_1(U) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-2c_1}{\sqrt{\delta_1}} F & 0 & \frac{-2c_1}{\sqrt{\delta_1}} G & 0 & 0 \\
\frac{-2c_2}{\sqrt{\delta_2}} F & 0 & \frac{-2c_2}{\sqrt{\delta_2}} G & 0 & 0 \\
\frac{1}{\sigma_1} \frac{-2c_3}{F} & 0 & \frac{1}{\sigma_1} \frac{-2c_3}{G} & 0 & 0 \\
\end{pmatrix}_{1 \times 5}.
\]

Next, we prove that if the initial data \( U(0, x, y) = U_0 \in (H^s(\mathbb{R}^2))^9, s > 2 \), then there exists \( T = T(\|U_0\|_{H^s(\mathbb{R}^2)})^9 \) such that equation (32) has a unique solution in \( L^\infty([0, T], (H^s(\mathbb{R}^2))^9) \).

The proof of the existence of a solution is standard and proceeds via a classical iteration scheme for symmetric hyperbolic system (see [28, 23]). The presence of \( C_1(U) \) unfortunately, leads to the existence time of order \( O(1) \).

The uniqueness of solution of (32) is classically obtained by estimating the difference of two solutions since the dispersive part does not contribute to the \( L^2 \) energy estimate.

The last step will be to recover the solution of (17) from the solution of (32).

To this end, the initial data \( U_0 = (H_0, L_0, F_0, G_0, D_0, V_0) \) should be chosen as \( F_0 + iG_0 = \psi_0, H_0 + iL_0 = \nabla \psi_0, D_0 = W(\rho_0 + D\phi_0) \) and \( V_0 = -MW\nabla \phi_0 - \frac{W}{\sigma_1} (\rho_0, 0)^T \).

First, set \( \psi = F + iG \), then (28) and (29) imply that

\[
\psi_t - \sigma_3 \psi - i\epsilon \delta \psi_{xx} - i\epsilon \sigma_1 \psi_{yy} + i\epsilon \left( (\sigma_2 - \frac{\sigma_1}{2}) |\psi|^2 + D \right) \psi = 0.
\]

From (26)-(29) we can derive an \( L^2 \) estimate of \( \mathcal{W} = (\nabla F - H, \nabla G - L) \), that implies

\[
\|\mathcal{W}(t)\|_{L^2}^2 \leq e^{Ct} \|\mathcal{W}(0)\|_{L^2}^2,
\]

thus, \( \nabla \psi = H + iL \).

Then (30)-(31) implies the last two equations of (24).

Next, we are going to recover \((\rho, \phi)\). Let \( \phi \) be the unique solution of the following linear wave equation

\[
\phi_{tt} - \frac{1}{M^2} \left( \Delta \phi + D(|\psi|^2)_{x} \right) + (|\psi|^2)_t = 0,
\]

with initial data given by \( \phi(t = 0) = \phi_0 \) and \( \phi_t(t = 0) = -\frac{1}{M^2} \rho_0 - |\psi_0|^2 \).

Define

\[
\rho = \frac{1}{W}(\mathcal{U} - WD\phi_x),
\]

with

\[
\mathcal{U} = D - \frac{\sigma_1}{2} |\psi|^2.
\]

Then we get that \((\psi, \rho, \phi)\) solves (18) uniquely with respect to the given initial data.

Next, let \( \tilde{\phi}(t, x, y) \) be the unique solution of the differential equation

\[
\tilde{\phi}_t + \frac{1}{M^2} \rho + |\psi|^2 = 0,
\]
with initial data \( \hat{\phi}(t = 0) = \phi_0 \).

From the second equation in (18) we get

\[
\rho_{tt} + \Delta \hat{\phi} + D|\psi|^2_t = 0,
\]

and integrating this in time and using the initial data, we get the following system

\[
\begin{cases}
\rho_t + \Delta \hat{\phi} + D(|\psi|^2)_x = 0, \\
\hat{\phi}_t + \frac{1}{M^2} \rho + |\psi|^2 = 0.
\end{cases}
\]

Taking the time derivative of the second equation in (33) we get

\[
\hat{\phi}_{tt} - \frac{1}{M^2} (\Delta \hat{\phi} + D(|\psi|^2)_x) + (|\psi|^2)_t = 0.
\]

Note that the initial data is also given by \( \hat{\phi}(t = 0) = \phi_0 \) and \( \hat{\phi}_t(t = 0) = -\frac{1}{M^2} \rho_0 - |\psi_0|^2 \). Therefore we have \( \hat{\phi} = \phi \), achieving to prove that \((\psi, \rho, \hat{\phi})\) solves the original Zakharov-Rubenchik system.

\[
\square
\]

4. The perturbed Z-R/B-R system. In this section, we consider the Cauchy problem for (2) when it is perturbed by the line solitary wave \( Q \) given by (16). That means, we will find solutions of (2) of the form \((\psi + \phi_1, \rho + \phi_2, \phi + \phi_3)\), where we denote \( Q = (\phi_1, \phi_2, \phi_3) \). The new system reads

\[
\begin{aligned}
\psi_t - \sigma_3 \psi_x - i \epsilon \hat{\psi}_{xx} &- i \epsilon \sigma_1 \psi_{yy} + i \epsilon \{ \sigma_2 |\psi|^2 + 2 \sigma_2 \Re(\phi_1 \hat{\psi}) + W(\rho + D \phi_3) \} \phi_1 \\
&+ i \epsilon \{ \sigma_2 |\psi + \phi_1|^2 + W(\rho + \phi_2 + D(\phi + \phi_3)_x) \} \psi = 0, \\
\rho_t + \Delta \phi + D(|\psi|^2 + 2 \Re(\phi_1 \hat{\psi}))_x = 0, \\
\phi_t + \frac{1}{M^2} \rho + |\psi|^2 + 2 \Re(\phi_1 \hat{\psi}) = 0.
\end{aligned}
\]

(34)

**Remark 4.** A natural way to solve the Cauchy problem for (34) would be to use the “dispersive method” in [41]. However the fact that the line soliton does not decay to 0 in the transverse direction leads to a difficulty when dealing with a new nonlocal linear term and seems to preclude to extend this method in a straightforward way. Therefore we will apply the method of the previous section to this case.

In the first step, we need to rewrite (34) in the form of a skew-adjoint perturbation of a symmetric hyperbolic system.

Using the fact that \( Q = (\phi_1, \phi_2, \phi_3) \) is also a solution of (17) with \( \epsilon = 1 \) then by the same calculations as in the previous section, we obtain that \((H_r, L_r, F_r, G_r, D_r, V_r)^T\) is a solution of (26)-(31) with \( \epsilon = 1 \), where

\[
\begin{aligned}
H_r &= \nabla(\Re \phi_1), \\
L_r &= \nabla(\Im \phi_1), \\
F_r &= \Re \phi_1, \\
G_r &= \Im \phi_1, \\
D_r &= U_r + \frac{\sigma_1}{2} |\phi_1|^2 \text{ with } U_r = W \phi_2 + WD(\phi_3)_x, \\
V_r &= -WM (|(\phi_3)_x, 0)^T - \frac{WD}{M} ((\phi_2)_x, 0)^T.
\end{aligned}
\]

(35)

Similarly, if \((\psi, \rho, \phi)\) is a solution of (34) or if \((\psi + \phi_1, \rho + \phi_2, \phi + \phi_3)\) is a solution of (17) (with \( \epsilon = 1 \)), then \((H, L, F, G, D, V)^T\) is a solution of (26)-(31) with \( \epsilon = 1 \),
where

\[
\begin{align*}
\tilde{H} &= \nabla(\Re(\psi + \phi_1)), \quad \tilde{L} = \nabla(\Im(\psi + \phi_1)), \\
\tilde{F} &= \Re(\psi + \phi_1), \quad \tilde{G} = \Im(\psi + \phi_1), \\
\tilde{D} &= \tilde{U} + \frac{\sigma_1}{2} |\psi + \phi_1|^2 \text{ with } \tilde{U} = W(\rho + \phi_2) + WD(\phi + \phi_3)_x, \\
\tilde{V}_t - \frac{1}{M} \nabla \tilde{U} - WM\nabla(|\psi + \phi_1|^2) - \left( \frac{WD^2}{M} (|\psi + \phi_1|^2)_x, 0 \right)^T &= 0 \\
\text{with } \tilde{V}(t = 0) &= \left( -WM\nabla(\phi + \phi_3) - \frac{WD}{M} (\rho + \phi_2, 0)^T \right)(t = 0).
\end{align*}
\]

We now set \((H, L, F, G, D, V)^T = (\tilde{H}, \tilde{L}, \tilde{F}, \tilde{G}, \tilde{D}, \tilde{V})^T - (H_r, L_r, F_r, G_r, D_r, V_r)^T,\) more precisely, we have

\[
\begin{align*}
H &= \nabla(\Re(\psi)), \quad L = \nabla(\Im(\psi)), \\
F &= \Re(\psi), \quad G = \Im(\psi), \\
D &= \mathcal{U} + \frac{\sigma_1}{2} (|\psi|^2 + 2\Re(\phi_1 \tilde{\psi})) \quad \text{with } \mathcal{U} = W\rho + WD\phi_x, \\
V_t - \frac{1}{M} \nabla \mathcal{U} - WM\nabla(|\psi|^2 + 2\Re(\phi_1 \tilde{\psi})) - \left( \frac{WD^2}{M} (|\psi|^2 + 2\Re(\phi_1 \tilde{\psi}))_x, 0 \right)^T &= 0 \\
\text{with } V(t = 0) &= V_0 = \left( -WM\nabla\phi - \frac{WD}{M} (\rho, 0)^T \right)(t = 0).
\end{align*}
\]

Combining (26)-(31), (35) and (36), it transpires that \((H, L, F, G, D, V)^T\) is a solution of

\[
\begin{align*}
H_t - \sigma_3 H_x + \delta L_{xx} + \sigma_1 L_{yy} - (G + G_r)\nabla D + R_1 &= 0, \\
L_t - \sigma_3 L_x - \delta H_{xx} - \sigma_1 H_{yy} + (F + F_r)\nabla D + R_2 &= 0, \\
F_t - \sigma_3 F_x + \delta G_{xx} + \sigma_1 G_{yy} + R_3 &= 0, \\
G_t - \sigma_3 G_x - \delta F_{xx} - \sigma_1 F_{yy} + R_4 &= 0, \\
D_t - \frac{1}{M} \nabla \cdot V - \sigma_1 \delta ((G + G_r)\partial_x H_1 - (F + F_r)\partial_x L_1) \\
&\quad - \sigma_1^2 ((G + G_r)\partial_y H_2 - (F + F_r)\partial_y L_2) + R_5 &= 0, \\
V_t - \frac{1}{M} \nabla D + R_6 &= 0,
\end{align*}
\]

where

\[
R_1 = -G\nabla D_r - \bar{D} L - D L_r \\
- \left( \sigma_2 - \frac{\sigma_1}{2} \right) \left( (\tilde{F}^2 + \tilde{G}^2) L + (F^2 + G^2 + 2FF_r + 2GG_r)_r \right)_r \\
+ 2G \left( H\tilde{F} + \tilde{G}\tilde{L} \right) + 2G_r \left( \tilde{H}F + HF_r + \tilde{L}G + LG_r \right).
\]
\[ \mathcal{R}_2 = \nabla \mathcal{D} + \hat{D} + \hat{D}H, \]
\[ + \left( \sigma_2 - \frac{\sigma_1}{2} \right) \left( \hat{F}^2 + \hat{G}^2 + (F^2 + G^2 + 2FF_r + 2GG_r) H_r \right) \]
\[ + 2F \left( \hat{H}F + \hat{G}L \right) + 2F_r \left( \hat{H}F + HF_r + \hat{G}L + GL_r \right) \],
\[ \mathcal{R}_3 = - \left( \sigma_2 - \frac{\sigma_1}{2} \right) \left( \hat{F}^2 + \hat{G}^2 + \hat{D} \right) G, \]
\[ - \left( \sigma_2 - \frac{\sigma_1}{2} \right) (F^2 + G^2 + 2FF_r + 2GG_r + \hat{D}) G_r, \]
\[ \mathcal{R}_4 = \left( \sigma_2 - \frac{\sigma_1}{2} \right) \left( \hat{F}^2 + \hat{G}^2 + \hat{D} \right) F, \]
\[ + \left( \sigma_2 - \frac{\sigma_1}{2} \right) (F^2 + G^2 + 2FF_r + 2GG_r + \hat{D}) F_r, \]
\[ \mathcal{R}_5 = \left( 2c_1 H_1 + \sigma_1 \delta_3 (\phi_1)_{xx} \right) F + \left( 2c_1 L - \sigma_1 \delta \Re(\phi_1)_{xx} \right) G \]
\[ + 2c_1 (H_1 F_r + L_1 G_r), \]
\[ \mathcal{R}_6 = -2 \left( c_2 (\hat{H}_1 F + F_r H_1 + \hat{L}_1 G + G_r L_1), c_3 (\hat{H}_2 F + \hat{L}_2 G + F_r H_2 + G_r L_2) \right)^T. \]

Similarly to the last section, if \( \rho_1 \delta > 0 \), we can change variables as follows

\[ H^* = (\sqrt{\rho_1} H_1, \sigma_1 H_2)^T \]
\[ \text{and then we set } U = (H^*, L^*, F, G, D, \mathcal{D})^T. \]

Therefore, the perturbation of (2) by the line solitary wave \( Q \) is rewritten as a dispersive perturbation of a symmetric hyperbolic system given by

\[ U_t + (A_1(U) + B_1(\phi_1) + C_1) U_x + (A_2(U) + B_2(\phi_1) + C_2) U_y \]
\[ + C(U, Q) U = -K_1 U_{xx} - K_2 U_{yy}, \]

where, with \( j \in \{1, 2\} \), \( A_j, B_j, C_j \) are symmetric matrices, \( K_j \) are skew symmetric and \( C_j \) are constant matrices. \( A_j \) have the same form as in the proof of Theorem 3.1 and \( B_j \) have the form

\[ B_1(\phi_1) = \begin{pmatrix} 0_{3 \times 3} & 0_{3 \times 3} & N_1(\phi_1)^T & 0_{3 \times 2} \\ 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} & 0_{3 \times 2} \\ N_1(\phi_1) & 0_{1 \times 3} & 0_{1 \times 2} & 0_{1 \times 2} \\ 0_{2 \times 3} & 0_{2 \times 3} & 0_{2 \times 1} & 0_{2 \times 2} \end{pmatrix}, \]

with

\[ N_1(\phi_1) = (-\sqrt{\rho_1} G_r, 0, \sqrt{\delta \sigma_1} F_r), \]

\[ B_2(\phi_1) = \begin{pmatrix} 0 & 0_{1 \times 3} & 0_{1 \times 2} & 0_{1 \times 2} \\ 0_{3 \times 1} & 0_{3 \times 3} & 0_{3 \times 2} & N_2(\phi_1)^T & 0_{3 \times 2} \\ 0_{2 \times 1} & 0_{2 \times 3} & 0_{2 \times 2} & 0_{2 \times 1} & 0_{2 \times 2} \\ 0 & N_2(\phi_1) & 0_{1 \times 2} & 0_{1 \times 2} \\ 0_{2 \times 1} & 0_{2 \times 3} & 0_{2 \times 2} & 0_{2 \times 1} & 0_{2 \times 2} \end{pmatrix}, \]

with

\[ N_2(\phi_1) = (-\sigma_1 G_r, 0, \sigma_1 F_r). \]
Furthermore, note that the matrix $C(U, Q)$ contains the term which depends only on $(\phi_1, \phi_2, \partial_x \phi_3)$ making the following analysis hold for both cases when $Q$ is the bright or the dark soliton.

We have written the perturbation of (2) by the line solitary wave $Q$ in the form of a symmetric hyperbolic system. Applying the same method as in the proof of Theorem 3.1 we obtain the following result.

**Theorem 4.1.** Let $\delta \sigma_1 > 0$ and $s > 2$. For any initial data $(\psi_0, \rho_0, \phi_0) \in H^{s+1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2) \times H^{s+1}(\mathbb{R}^2)$, there exists $T > 0$ such that (2), when it is perturbed by the line soliton $Q = (\phi_1, \phi_2, \phi_3)$, admits a unique solution $(\psi, \rho, \phi) \in C([0, T]; H^{s+1}(\mathbb{R}^2)) \times C([0, T]; H^s(\mathbb{R}^2)) \times C([0, T]; H^{s+1}(\mathbb{R}^2))$.

**Proof.** The proof of Theorem 4.1 and Theorem 3.1 are essentially the same except the estimates for the terms $B_j(\phi_1)$ and $C(Q)$.

In the proof of Theorem 3.1, in order to estimate the derivative of order $s$, we use the commutator estimate and the Bessel potential $J^s(1 - \Delta)^{s/2}$. Although, in this case, the 1-D soliton solution $Q$ does not decay in the “$y$” direction which makes that argument not true. Therefore, we use the fact that $J^s \sim J^s_x + J^s_y$, where $J^s_x = F^{-1} (\xi_1) F$, $J^s_y = F^{-1} (\xi_2) F$. Hence, we only need to estimate $J^s_x U$ and $J^s_y U$ instead of $J^s U$. Since $Q$ is independent of $y$, $J^s_y$ is harmless and since $\| J^s_x U \|_{L^2_x} = \| J^s_y U \|_{L^2_y}$, we can apply the commutator estimate in one dimensional case.

The rest of the proof proceeds exactly as in the proof of Theorem 3.1. Again we emphasize that the same result holds true *mutatis mutandi* in a $\mathbb{R} \times T$ setting, a framework that would be needed to study the stability of the line soliton with respect to periodic transverse perturbations.

5. **Global solution.** In this section we will establish the conservation of energy for the perturbation of (2) by the line soliton $Q$ given in Section 2 and as a consequence, the existence of a global weak solution when $Q$ is the dark soliton.

In order to make the calculation easier, we will consider the solution of the form

$$(e^{iM} e^{i\frac{\sigma_2}{2\delta^2}} \psi(x, y, t), \rho(x, y, t), \phi(x, y, t)),$$

then the 1-d solitary wave has the following form

$$Q = (\phi_1, \phi_2, \phi_3) = (R(x), aR^2(x), bP(x)),$$

with $R(x), P(x)$ are given in Section 2 (note that this trick will not affect the analysis in section 4). Then the system (34) becomes

$$
\begin{align*}
\psi_t + i(\lambda - \frac{\sigma_2^2}{4\delta}) \psi - i\delta \psi_{xx} - i\sigma_1 \psi_{yy} &+ i \left\{\sigma_2 |\psi|^2 + W(\rho + D\phi_x) + 2\sigma_2 \phi_1 \Re(\psi) + \sigma_2 |\phi_1|^2 + W(\phi_2 + D\phi_x)\right\} \psi \\
\rho_t + \Delta \phi + D(|\psi|^2 + 2\phi_1 \Re(\psi)) & = 0, \\
\phi_t + \frac{1}{M^2} \rho + |\psi|^2 + 2\phi_1 \Re(\psi) & = 0.
\end{align*}
$$

(40)

In this section we will establish the energy conservation for (40) when it is perturbed by a line soliton $Q$ and the existence of a global weak solution when $Q$ is the dark soliton.
Theorem 5.1. Let \((\psi, \rho, \phi)\) be a solution of the system (40) obtained in Theorem 4.1, defined in the time interval \([0, T]\). Then the quantity

\[
E = (\lambda - \frac{\sigma_2^2}{4\delta}) \|\psi\|_{L^2}^2 + \delta \|\psi_x\|_{L^2}^2 + \sigma_1 \|\psi_y\|_{L^2}^2 + \frac{\sigma_2}{2} \|\psi^2 + 2\phi_1 \Re(\psi)\|_{L^2}^2 + \sigma_2 \|\phi_1\|_{L^2}^2 + \frac{W}{2M^2} \|\rho\|_{L^2}^2 + \frac{W}{2} \|\nabla \phi\|_{L^2}^2 \tag{41}
\]

is conserved for \(t \in [0, T]\).

Proof. We multiply the first equation in (40) by \(\partial_t \bar{\psi}\), integrate the result and take its imaginary part to get successively

\[
(\lambda - \frac{\sigma_2^2}{4\delta}) \Re \int \psi \bar{\psi}_t = \frac{1}{2} (\lambda - \frac{\sigma_2^2}{4\delta}) \int (|\psi|^2)_t, \tag{42}
\]

\[
- \delta \Re \int \psi_{xx} \bar{\psi}_t = \sigma_1 \Re \int \psi_{yy} \bar{\psi}_t = \frac{1}{2} \sigma_1 \int (|\psi|^2)_t, \tag{43}
\]

\[
\Re \int (\sigma_2 |\psi|^2 + W(\rho + D\phi_x) + 2\sigma_2 \phi_1 \Re(\psi) + \sigma_2 |\phi_1|^2 + W(\phi_2 + D\phi_x \phi_3)) \psi \bar{\psi}_t
= \frac{1}{2} \int (\sigma_2 |\psi|^2 + W(\rho + D\phi_x) + 2\sigma_2 \phi_1 \Re(\psi) + \sigma_2 |\phi_1|^2 + W(\phi_2 + D\phi_x \phi_3)) (|\psi|^2)_t 
= \frac{1}{2} \int \sigma_2 (|\psi|^4)_t + W(\rho + D\phi_x) (|\psi|^2)_t + 2\sigma_2 \phi_1 \Re(\psi) (|\psi|^2)_t 
+ (\sigma_2 |\phi_1|^2 + W(\phi_2 + D\phi_x \phi_3)) (|\psi|^2)_t, \tag{45}
\]

\[
\Re \int (\sigma_2 |\psi|^2 + W(\rho + D\phi_x) + 2\sigma_2 \phi_1 \Re(\psi)) \phi_1 \bar{\psi}_t
= \int \sigma_2 |\psi|^2 (\phi_1 \Re(\psi))_t + W(\rho + D\phi_x) (\phi_1 \Re(\psi))_t + \sigma_2 (|\phi_1 \Re(\psi)|^2)_t. \tag{46}
\]

Combining (42), (43), (44), (45) and (46) we obtain

\[
0 = \frac{d}{dt} \int \left( \frac{1}{2} (\lambda - \frac{\sigma_2^2}{4\delta}) |\psi|^2 + \frac{\delta}{2} |\psi_x|^2 + \frac{\sigma_1}{2} |\psi_y|^2 + \frac{\sigma_2}{4} |\psi|^4 
+ \sigma_2 |\psi|^2 \phi_1 \Re(\psi) + \frac{1}{2} (\sigma_2 |\phi_1|^2 + W(\phi_2 + D\phi_x \phi_3)) |\psi|^2 + \sigma_2 |\phi_1 \Re(\psi)|^2 
+ \frac{1}{2} W(\rho + D\phi_x) (|\psi|^2 + 2\phi_1 \Re(\psi)) \right) 
- \frac{1}{2} \int W(\rho_t + D\phi_{xt}) (|\psi|^2 + 2\phi_1 \Re(\psi)).
\]

From the second and the third equation in (40), we get

\[
\int \rho_t (|\psi|^2 + 2\phi_1 \Re(\psi)) = - \int \rho_t (\phi_t + \frac{1}{M^2} \rho) = - \int \rho_t \phi_t - \frac{1}{2M^2} \int \rho_t^2,
\]
Theorem 5.2. Assume that

Finally, we get the conserved energy

and

That implies

Finally, we get the conserved energy

Note that we got rid of the terms involving \( \phi_2 \) and \( \phi_3 \) since by (10) one has \( a + Db = -(M^2 + D^2) \) implying that \( \phi_2 + D\partial_x \phi_3 = -(M^2 + D^2)\phi_1^2 \).

Theorem 5.2. Assume that \( Q \) is the dark soliton given by (15) with wave speed \( c = 0 \).

i) Let \((\psi, \rho, \phi)\) be the solution of (40) obtained by Theorem 4.1 with existence time interval \([0, T]\). Then for all \( t \in [0, T] \) we have

\[
\|\psi(t)\|_{H^1} + \|\rho(t)\|_{L^2} + \|\phi(t)\|_{H^1} \leq C(t).
\]

(47)

ii) For any \((\psi_0, \rho_0, \phi_0) \in H^1 \times L^2 \times H^1\), there exists a global weak solution \((\psi, \rho, \phi)\) of (40) such that for any \( T > 0 \)

\[
\psi, \phi \in L^\infty([0, T]; H^1), \quad \rho \in L^\infty([0, T]; L^2)
\]

\[
\psi_t, \rho_t \in L^\infty([0, T]; H^{-1}), \quad \phi_t \in L^\infty([0, T]; L^2).
\]

(48)
Proof. i) For any \( \varepsilon \in (0, 1) \), using Cauchy’s inequality we have

\[
\left| \int W \rho (|\psi|^2 + 2\phi_1 \Re (\psi)) \right| \leq \frac{W}{2M^2} (1 - \varepsilon) \int \rho^2 + \frac{M^2 W}{2(1 - \varepsilon)} \int (|\psi|^2 + 2\phi_1 \Re (\psi))^2
\]

and

\[
\left| \int W D\phi_x (|\psi|^2 + 2\phi_1 \Re (\psi)) \right| \leq \frac{W}{2} (1 - \varepsilon) \int |\phi_x|^2 + \frac{W D^2}{2(1 - \varepsilon)} \int (|\psi|^2 + 2\phi_1 \Re (\psi))^2,
\]

then

\[
\left| \int W (\rho + D\phi_x) (|\psi|^2 + 2\phi_1 \Re (\psi)) \right|
\leq \frac{W(M^2 + D^2)}{2(1 - \varepsilon)} \left\| |\psi|^2 + 2\phi_1 \Re (\psi) \right\|_{L^2}^2 + \frac{W}{2M^2} (1 - \varepsilon) \|\rho\|_{L^2}^2
+ \frac{W}{2} (1 - \varepsilon) \|\nabla \phi\|_{L^2}^2.
\]

Note that we are considering the stationary dark soliton, so that the condition (14) becomes

\[
\begin{align*}
\left\{ \frac{1}{\delta} \left( \frac{\sigma_2^2}{4\delta} - \lambda \right) & > 0, \\
\frac{1}{\delta} (\sigma_2 - W(M^2 + D^2)) & > 0,
\end{align*}
\]

so there exists \( \varepsilon > 0 \) small enough such that

\[
\sigma_2 > \frac{W(M^2 + D^2)}{1 - \varepsilon}.
\]

Therefore, the conservation law (41) implies

\[
\frac{1}{2} (\sigma_2 - \frac{W(M^2 + D^2)}{1 - \varepsilon}) \left\| |\psi|^2 + 2\phi_1 \Re (\psi) \right\|_{L^2}^2 + \frac{\varepsilon W}{2M^2} \|\rho\|_{L^2}^2 + \frac{\varepsilon W}{2} \|\nabla \phi\|_{L^2}^2
\leq E + (\frac{\sigma_2^2}{4\delta} - \lambda) \left\| |\psi|^2 \right\|_{L^2}^2.
\]

From now we will fix such an \( \varepsilon \) and define

\[
c_1 = \frac{1}{2} (\sigma_2 - \frac{W(M + D^2)}{1 - \varepsilon}), \quad c_2 = \frac{\varepsilon W}{2M}, \quad c_3 = \frac{\varepsilon W}{2}.
\]

The first equation of (40) implies

\[
\frac{1}{2} \frac{d}{dt} \left\| |\psi|^2 \right\|_{L^2}^2 = 3 \int (\sigma_2 (|\psi|^2 + 2\phi_1 \Re (\psi)) + W(\rho + D\phi_x))\phi_1 \overline{\psi}
\leq c_1 \left\| |\psi|^2 + 2\phi_1 \Re (\psi) \right\|_{L^2}^2 + c_2 \|\rho\|_{L^2}^2 + c_3 \|\phi_x\|_{L^2}^2
+ \left( \frac{\sigma_2^2}{4c_1} + \frac{W^2}{4c_2} + \frac{W^2 D^2}{4c_3} \right) \|\phi_1 \Im (\psi)\|_{L^2}^2.
\]

Combining with (49) we get

\[
\frac{1}{2} \frac{d}{dt} \left\| |\psi|^2 \right\|_{L^2} \leq E + (\frac{\sigma_2^2}{4\delta} - \lambda) \left\| |\psi|^2 \right\|_{L^2}
+ \left( \frac{\sigma_2^2}{4c_1} + \frac{W^2}{4c_2} + \frac{W^2 D^2}{4c_3} \right) \|\phi_1 \Im (\psi)\|_{L^2}^2.
\]
Since \( \phi_1 \in L^\infty \), \( \| \phi_1 \Im(\psi) \|_{L^2} \) is under control and we recall that \( \frac{\sigma_1^2}{4b} - \lambda > 0 \). Hence, by Gronwall’s inequality we can bound \( \| \psi(t) \|_{L^2} \) by a constant depending on \( t \). The bounds on \( \| \nabla \psi \|_{L^2} \), \( \| \rho \|_{L^2} \), \( \| \nabla \phi \|_{L^2} \) then follow from the energy conservation.

ii) We shall use a classical compactness method (see for instance [27]). The estimates in (i) prove that regular solutions of (40) are uniformly bounded in \( T > 0 \). Therefore, using (40) and Sobolev’s theorem we infer that (40) is uniformly bounded, independently of \( \eta \), in the space
\[
L^\infty((0, T); H^1) \times L^\infty((0, T); L^2) \times L^\infty((0, T); H^1).
\]
(50)

Therefore, using (40) and Sobolev’s theorem we infer that (\( \partial_t \psi_m, \partial_t \rho_m, \partial_t \phi_m \)) is bounded in
\[
L^\infty((0, T); H^{-1}) \times L^\infty((0, T); H^{-1}) \times L^\infty((0, T); L^2).
\]

Hence, up to a subsequence, one can assume that
\[
\begin{align*}
\psi_m &\to \psi \text{ in } L^\infty((0, T); H^1) \text{ weak*}, \\
\rho_m &\to \rho \text{ in } L^\infty((0, T); L^2) \text{ weak*}, \\
\phi_m &\to \phi \text{ in } L^\infty((0, T); H^1) \text{ weak*}.
\end{align*}
\]
(51)

By Aubin-Lions lemma one can furthermore assume that up to a subsequence
\[
\psi_m \to \psi \text{ in } L^p_{\text{loc}}([0, T]; L^q_{\text{loc}}(\mathbb{R}^2)),
\]
(52)

for any \( 2 \leq p, q < \infty \). Similar convergence results hold true for \( \phi_m, \rho_m \).

These convergences allow to pass to the limit in the distribution sense in (40) for (\( \psi_m, \rho_m, \phi_m \)), proving that \((\psi, \rho, \phi)\) satisfies (40) in \( L^\infty((0, T); H^{-1}) \times L^\infty((0, T); H^{-1}) \times L^\infty((0, T); L^2) \). The initial condition makes sense since
\[
(\psi, \rho, \phi) \in C_w([0, T]; H^1) \times C_w([0, T]; L^2) \times C_w([0, T]; H^1).
\]

\( \square \)

6. Solitary wave solutions in higher dimension. Let consider now solitary waves solutions of (2), i.e. solutions of the form \( (e^{\omega t}\psi(x+\sigma_3 t, y), \phi(x+\sigma_3 t, y), \eta(x+\sigma_3 t, y)) \), \( \omega \in \mathbb{R}, \psi \in H^1(\mathbb{R}^2) \), yielding the system
\[
\begin{align*}
-\omega \psi + \delta \psi_{xx} + \sigma_1 \psi_{yy} - (\sigma_2 - WM^2) |\psi|^2 \psi - c \phi_x \psi &= 0, \\
(1 - \sigma_2^2 M^2) \phi_{xx} + \phi_{yy} + (\sigma_2 - WM^2) |\psi|^2 \phi_x &= 0,
\end{align*}
\]
(53)

which is similar to the equation for the solitary wave solutions of the elliptic/hyperbolic-elliptic Davey-Stewartson systems in the terminology of [16]. By Pohozaev type arguments one obtains (see [17] for similar arguments) that non trivial solutions to (53) cannot exist when \( \delta \sigma_2 < 0 \).

On the other hand the existence of non trivial solutions to (53) has been established ([9]) in the focusing case
\[
\delta \sigma_1 > 0, \quad c < 0, \quad cW(\sigma_2 - WM^2) < 0.
\]

Various stability and instability results of solutions to (53) have been obtained in [10, 32, 33, 34] in the context of the Davey-Stewartson systems, but no similar
results seemed to be known when they are viewed as solutions to the Zakharov-Rubenchik systems. In particular one does not know if the solutions of (53) are constrained minimizers of the Zakharov-Rubenchik system.

According to the Davey-Stewartson case, one could conjecture that those localized solitary waves are unstable.

7. Conclusion and open questions. We have addressed in this paper some issues on the Zakharov-Rubenchik, Benney-Roskes systems. Many questions remain unsolved for those important systems and we indicate a few below.

1. Justify rigorously the limit of ZR (BR) systems to the Davey-Stewartson systems. This is non trivial (because of a boundary layer at $t = 0$ in analogy (but more delicate) to the Schrödinger limit of the Zakharov system (see [38, 39]).

2. The present work can be viewed as a preliminary step towards the study of the transverse stability/instability of the ZR or BR one dimensional dark or bright solitary wave. Perturbations could be localized in $(x, y)$ or periodic in $y$. The Cauchy problem was addressed in the present paper in both cases.

In both the functional settings, we plan to come back to those transverse stability issues a subsequent work, in the spirit of [42, 43, 44, 29, 30].

3. It is known ([19, 20]) that (radially symmetric) solutions of the Zakharov system may blow up in finite time. Such a result is unknown for the Zakharov-Rubenchik, Benney-Roskes system and it would be interesting to see if the results in [19, 20] extend to (1).

4. The existence result Theorem 3.1 is established when $\delta \sigma_1 > 0$, that is when the Schrödinger equation in (1) is not an "non elliptic" one in the terminology of [16]. This condition is never satisfied in the context of purely gravity waves water waves (see [25] and the discussion above) and it would interesting to relax it.

5. We recall that an existence result on time scales of order $1/\epsilon$ is needed to fully justify the Benney-Roskes system. Obtaining such a result is still a challenging open problem.

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