STRUCTURE OF HIGHER GENUS GROMOV–WITTEN INVARIANTS OF QUINTIC 3-FOLDS

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Abstract. There is a set of remarkable physical predictions for the structure of BCOV's higher genus B-model of mirror quintic 3-folds which can be viewed as conjectures for the Gromov–Witten theory of quintic 3-folds. They are (i) Yamaguchi–Yau’s finite generation, (ii) the holomorphic anomaly equation, (iii) the orbifold regularity and (iv) the conifold gap condition. Moreover, these properties are expected to be universal properties for all the Calabi–Yau 3-folds. This article is devoted to proving first three conjectures.

The main geometric input to our proof is a log GLSM moduli space and the comparison formula between its reduced virtual cycle (reproducing Gromov–Witten invariants of quintic 3-folds) and its nonreduced virtual cycle [7]. Our starting point is a Combinatorial Structural Theorem expressing the Gromov–Witten cohomological field theory as an action of a generalized $R$-matrix in the sense of Givental. An $R$-matrix computation implies a graded finite generation property. Our graded finite generation implies Yamaguchi–Yau’s (nongraded) finite generation, as well as the orbifold regularity. By differentiating the Combinatorial Structural Theorem carefully, we derive the holomorphic anomaly equations. Our technique is purely A-model theoretic and does not assume any knowledge of B-model. Finally, above structural theorems hold for a family of theories (the extended quintic family) including the theory of quintic as a special case.

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1. Introduction

The computation of the Gromov–Witten (GW) theory of compact Calabi–Yau 3-folds is a central and yet difficult problem in geometry and physics. During last twenty years, it has attracted a lot of attention from both physicists and mathematicians. In the early 90s, the physicist Candelas and his collaborators [2] surprised the mathematical community by using mirror symmetry to derive a conjectural formula for a certain generating function (the $J$-function) of genus zero Gromov–Witten invariants of a quintic 3-fold in terms of the period integral (or the $I$-function) of its B-model mirror. The effort to prove the formula has directly lead to the birth of mirror symmetry as a mathematical subject. In a seminal work [1] in 1993, Bershadsky, Cecotti, Ooguri and Vafa (BCOV) introduced the higher genus B-model in physics in an effort to push mirror symmetry to higher genus. During the subsequent years, a series of conjectural formulae was proposed by physicists based on the BCOV B-model. Let $F_g(Q)$ be the generating function of genus $g$ GW-invariants of a quintic 3-fold. BCOV had already proposed a conjectural formula for $F_1$ and $F_2$ in their original paper (see also [34]). A conjectural formula for $F_3$ was proposed by Katz–Klemm–Vafa in 1999 [24]. Afterwards, it became clear that we need a better understanding of the structure of $F_g$. A fundamental physical prediction of Yamaguchi–Yau is that $F_g$ can be written as a polynomial of five generators constructed explicitly from the period integral of its mirror. Among the five generators, four of them are the holomorphic limit of certain non-holomorphic objects in the B-model. The famous holomorphic anomaly equation of BCOV can be recasted into equations determining the dependence on four of the generators. Abusing terminology, we still refer to them as holomorphic anomaly equations even though the four generators are holomorphic. The initial condition of the holomorphic anomaly equations is
expected to be a degree $3g - 3$ polynomial $f_g$ of a single variable $Z$. Two other physical predictions regarding the structure of $f_g$ are the conifold gap condition and the orbifold regularity which determine the lower and upper parts of $f_g$. Furthermore, the above structures of Gromov–Witten invariants are expected to be present in some fashion for all Calabi–Yau 3-folds, and hence can be considered as universal properties. Based on the above B-model structural predictions and an additional A-model conjecture called Castelnuovo bound, Klemm and his collaborators [23] derived a formula for $F_g$ for all $g \leq 51$!

The progress for mathematicians to prove these conjectures has been slow. The genus zero conjecture was proved by Givental [14] and Lian–Liu–Yau [28], and it was considered to be a major event in mathematics during the 90s. It took another ten years for Zinger to prove BCOV’s conjecture in genus one [36]. It took yet another ten years for the authors’ recent proof of the genus two BCOV conjecture [19]. The geometric input to our work on genus two is a construction of a certain reduced virtual cycle on an appropriate log compactification of the GLSM moduli space [7] (see also [9]). Its localization formula expresses the Gromov–Witten invariants of quintic 3-folds in terms of a graph sum of (rather mysterious) effective invariants and (rather well-understood) twisted GW-invariants of $\mathbb{P}^4$. This formula works in arbitrary genus as long as one can compute the effective invariants. In particular, there is only one effective invariant for $g = 2, 3$, and it can be computed from the $g = 2, 3$, degree zero GW-invariant. So in principle, we can push our technique to $g = 3$ to prove the conjectural formula of Katz–Klemm–Vafa. The main difficulty is directly related to the physical prediction of Yamaguchi–Yau that $F_g$ should be a polynomial of five generators. A direct generalization of our argument in [19] only implies that $F_g$ is a polynomial of nine generators. The appearance of extra generators is similar to the simpler case of the Gromov–Witten theory of an elliptic curve, where $F_g$ is a quasi-modular form of $SL_2(\mathbb{Z})$. On the other hand, the corresponding twisted theory of $O(3)$ on $\mathbb{P}^2$ is a quasi-modular form for $\Gamma_0(3)$. The ring of quasi-modular forms of $\Gamma_0(3)$ has more generators than that of $SL_2(\mathbb{Z})$! In the case of the quintic, the presence of the extra four generators increases the computational complexity significantly. We could try to prove the genus three formula by brute force but the proof would not be illuminating and is unlikely to generalize to higher genus. Our eventual goal is to reach to $g = 51$ and beyond. It is clear to us that we should first attack the set of physical predictions for the structures of $F_g$.

To describe the main idea, recall that the general twisted theory naturally depends on six equivariant parameters, five for the base $\mathbb{P}^4$ and one for the twist. It is complicated to study the general twisted theory, and therefore Zagier–Zinger [35] specialize the equivariant parameter to $(\lambda, \zeta \lambda, \zeta^2 \lambda, \zeta^3 \lambda, \zeta^4 \lambda, 0)$, where $\zeta$ is a primitive fifth root of unity. This theory is referred as formal quintic (see [27]). They show that the twisted theory is generated by the five generators predicted by physicists. Unfortunately, in our work, the natural specialization of equivariant parameters is $(0, 0, 0, 0, 0, t)$. The bulk of [19] is to show that the corresponding twisted theory for equivariant parameter $(0, 0, 0, 0, 0, t)$ has four extra generators. The main input of the current article is a comparison formula expressing the reduced virtual cycle of the Log GLSM moduli space in terms of its canonical (non-reduced) virtual cycle. The advantage of non-reduced theory is that
it admits a \((\mathbb{C}^\ast)^6\) action with six equivariant parameters which we can specialize to 
\((\lambda, \zeta \lambda, \zeta^2 \lambda, \zeta^3 \lambda, \zeta^4 \lambda, 0)\) as for the formal quintic.

To state our precise results, we need to setup some notation. Let \(X_5\) be a quintic 3-fold. Fix \((g, n)\) such that \(2g - 2 + n > 0\), and fix ambient classes \(\gamma_1, \ldots, \gamma_n \in H^\ast(X_5)\).

Let
\[
\Omega_{g,n}(\gamma_1, \ldots, \gamma_n) := \sum_{\beta = 0}^{\infty} Q^\beta \rho\left(\prod_{i=1}^{n} \text{ev}_i^\ast(\gamma_i) \cap [\overline{\mathcal{M}}_{g,n}(X_5, \beta)]^{\text{vir}}\right),
\]
where \(\rho: \overline{\mathcal{M}}_{g,n}(X_5, \beta) \to \overline{\mathcal{M}}_{g,n}\) is the forgetful map, be the generating series of Gromov–Witten classes defined by \(X_5\).

For \(g \geq 2\), let
\[
F_g(Q) := \int_{\overline{\mathcal{M}}_{g}} \Omega_{g,0}
\]
be the corresponding numerical generating series. In the cases \(g = 0\), and \(g = 1\) avoid unstable terms, we need markings (see below).

Let \(\mathcal{M}_{g,n}(\mathbb{P}^4, \mathcal{O}(5), d)\) be the GLSM moduli space for a quintic 3-fold, that is the moduli space of stable maps to \(\mathbb{P}^4\) with a \(p\)-field. In [7], we construct a certain logarithmic compactification \(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^4, \mathcal{O}(5), d, \nu)\) (see Section 2 for more details) where \(\nu\) is a partition representing the contact order (or relative condition). Traditionally, we call the case \(\nu = \emptyset\) the holomorphic theory, and the case \(\nu \neq \emptyset\) the meromorphic theory. The moduli space \(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^4, \mathcal{O}(5), d, \nu)\) is a proper Deligne–Mumford stack with a two-term perfect obstruction theory. Hence, it admits a virtual fundamental cycle \([\overline{\mathcal{M}}_{g,n}(\mathbb{P}^4, \mathcal{O}(5), d, \nu)]^{\text{vir}}\). The cycle \([\overline{\mathcal{M}}_{g,n}(\mathbb{P}^4, \mathcal{O}(5), d, \nu)]^{\text{vir}}\) is equivariant for both the \((\mathbb{C}^\ast)^5\) action of \(\mathbb{P}^4\) and the \(\mathbb{C}^\ast\) action on the \(p\)-field. The holomorphic theory is very special in the sense, that in [7] we construct a reduced virtual cycle \([\overline{\mathcal{M}}_{g,n}(\mathbb{P}^4, \mathcal{O}(5), d, \nu)]^{\text{red}}\).

The main result of [7] is that \([\overline{\mathcal{M}}_{g,n}(\mathbb{P}^4, \mathcal{O}(5), d, \nu)]^{\text{red}}\) computes the GW-invariants of quintic 3-folds. The boundary of \(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^4, \mathcal{O}(5), d, \nu)\) contains rubber moduli spaces \(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^4, \mathcal{O}(5), d, \mu, \nu)\) with their own virtual fundamental cycles. The key new higher genus information for quintic 3-folds are the effective invariants
\[
c_{g,d}^\text{eff} = \deg[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^4, \mathcal{O}(5), d, (2^{2g-2-5d}, \emptyset))]^{\text{red}} \in \mathbb{Q}
\]
for integral \(g, d \geq 0\) such that \(5d \leq 2g - 2\). In particular, the invariants \(c_{g,d}^\text{eff}\) are only defined when \(g \geq 1\), and when \(d \leq \frac{2g-2}{5}\). Furthermore, the \(c_{g,d}^\text{eff}\) are determined by the corresponding low degree Gromov–Witten invariants.

The comparison formula of [7] between the reduced and non-reduced cycle yields a formula of the form (the precise result is in Theorem 2.4)
\[
\Omega_{g,n} = \sum_{\Gamma} c_{g_1,d_1}^\text{eff} \cdots c_{g_k,d_k}^\text{eff} \Omega_{\Gamma},
\]
where \(\Gamma\) is a decorated bipartite graph, the \(c_{g_i,d_i}^\text{eff}\) are effective invariants associated to \(\infty\) vertices and \(\Omega_{\Gamma}\) is the remaining contributions which can be expressed in terms of non-reduced virtual cycles. We can replace \(c_{g,d}^\text{eff}\) in the above formula by formal parameters \(c_{g,d}\) and denote the resulting generating series by \(\Omega_{g,n}^\text{eff}\). We call \(\Omega_{g,n}^\text{eff}\) the extended quintic invariants.

\(^1\)By a dimension consideration, insertions of primitive classes are not very interesting.
family. We can then obtain the theory of $X_5$ by setting $c_{g,d} = c_{g,d}^{\text{eff}}$. By setting $c_{g,d} = 0$, we obtain the theory of holomorphic GLSM.

1.1. Graded finite generation and orbifold regularity. Let us consider the $I$-function (or period integral of its mirror) of the quintic 3-fold

$$I(q, z) = z \sum_{d \geq 0} q^d \prod_{k=1}^{5d} (5H + kz) \prod_{k=1}^{d} (H + kz)^5$$

where $H$ is a formal variable satisfying $H^4 = 0$. We separate $I(q, z)$ into components:

$$I(q, z) = z I_0(q) + I_1(q) H + z^{-1} I_2(q) H^2 + z^{-2} I_3(q) H^3$$

The genus zero mirror symmetry conjecture of quintic 3-folds can be phrased as a relationship

$$J(Q) = \frac{I(q)}{I_0(q)}$$

between the $J$- and $I$-function, involving the mirror map $Q = q e^{\tau_Q(q)}$, where

$$\tau_Q(q) = \frac{I_1(q)}{I_0(q)}.$$ 

Now we introduce the following degree $k$ “basic” generators

$$X_k := \frac{d^k}{dq^k} \left( \log \frac{I_0}{L} \right), \quad Y_k := \frac{d^k}{dq^k} \left( \log \frac{I_0 I_1}{L^2} \right), \quad Z_k := \frac{d^k}{dq^k} \left( \log (q^{\frac{k}{2}} L) \right),$$

where $I_{1,1} := 1 + q \frac{d}{dq} \tau_Q$, $L := (1 - 5q)^{-\frac{1}{4}}$ and $du := \frac{dI}{I_1} = \frac{L}{I_{1,1}} \frac{dQ}{Q}$. We often abbreviate $X = X_1$, $Y = Y_1$ and $Z = Z_1$.

To simplify later formulae, we introduce following basis:

$$\phi_k = I_{1,1} \cdots I_{k,k} L^{-k} H^k$$

and the normalized Gromov–Witten classes

$$\bar{\Omega}_{g,n}^c := 5^{g-1} (L/I_0)^{2g-2} \Omega_{g,n}^c,$$

where $I_{2,2} = L^5 I_{0}^{-2} I_{1,1}^{-2}$, $I_{3,3} = I_{1,1}$, $I_{4,4} = I_0$ [35].

One can apply localization formula to the non-reduced virtual cycles in the comparison formula to compute $\bar{\Omega}_{g,n}^c$, and specialize the equivariant parameter to the formal quintic parameter $(\lambda, \zeta \lambda, \zeta^2 \lambda, \zeta^3 \lambda, \zeta^4 \lambda, 0)$. The first main result of the article is a combinatorial structural theorem that packages the localization contributions into a Givental-style $R$-matrix action.

**Theorem 1.1.** (Combinatorial Structural Theorem)

$$\bar{\Omega}_{g,n}^c = \lim_{\lambda \to 0} \sum_{\Gamma \in G^\infty_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} \text{Cont}_\Gamma.$$ 

Here, $G^\infty_{g,n}$ is the set of genus $g$, $n$-marked stable graphs with the decorations:

- for each vertex $v$, we assign a label 0 or $\infty$;
- for each flag $f = (v, e)$ or $f = (v, l)$ where $v$ labeled by $\infty$, we assign a degree $\delta_f \in \mathbb{Z}_{\geq 1}$. 

For each $\Gamma \in G_{g,n}^\infty$, the contribution $\text{Cont}_{\Gamma}$ is defined from the $R$- and $S$-matrices of the formal quintic theory in an explicit formula similar to that of Givental’s $R$-matrix action (we refer to Section 4.2 for the precise formula).

We cannot directly apply the structural results of formal quintic proven in [26]. Nevertheless, after much computation, we can prove Yamaguchi–Yau’s prediction.

**Definition 1.2.** We introduce the ring of 6-generators
\[ \hat{R} := \mathbb{Q}[L^{-1}, \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}, Z] \]
the degree defined as follows
\[ \deg L^{-1} = 1, \quad \deg \mathcal{X}_k = k, \quad \deg \mathcal{Y} = 1, \quad \deg Z = 0. \]
We define Yamaguchi–Yau’s finite generation ring\(^2\) by
\[ R := \mathbb{Q}[\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}] \otimes \text{span}_{\mathbb{Q}} \{ L^{-a}Z^b : b \leq a \leq 5b \} \quad (2) \]
We denote by $R_k \subset R$ the linear subspace of degree $k$ elements.

This ring admits an additional structure: there exists a derivation $\partial_u$ acting on this ring, such that the ring is closed under $\partial_u$ and that $\partial_u$ increases the degree by 1. The explicit definition of $\partial_u$ is given in Lemma 5.4.

**Remark 1.3.** By setting $Z = L^5$, we can regard $R \subset \mathbb{Q}[L^{-1}, \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}, Z]$ as a subring of $\mathbb{Q}[\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}, L]$. Then we will lose the degree information.

**Theorem 1.4.** The following “graded finite generation properties” hold for the extended quintic family
\[ (1): \bar{\Omega}^{\text{e}}_{g,n}(\phi_{a_1}, \ldots, \phi_{a_n}) \in H^{2(3g-3+\sum a_i)}(\overline{\mathcal{M}}_{g,n}; \mathbb{Q}) \otimes R_{3g-3+\sum a_i} \]
\[ (2): \bar{F}_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} \bar{\Omega}^{\text{e}}_{g,n}(\phi_{1}^{\otimes n}) \in R_{3g-3+n} \]
Our graded finite generation theorem is much stronger than Yamaguchi–Yau’s original non-graded finite generation. For example, a direct consequence is another key structural prediction:

**Corollary 1.5.** (Orbifold regularity) Suppose that $f_g = L^{3g-3}\bar{F}_g|_{X=Y=0}$. Then we can write
\[ f_g = \sum_{i=0}^{3g-3} a_{i,g}Z^i \quad \text{with} \quad Z = L^5, \]
where $a_{i,g} = 0$ for $i \leq \lceil \frac{3g-3}{5} \rceil$.

**Remark 1.6.** $f_g$ represents the initial condition of the holomorphic anomaly equations. By using the boundary behavior of the large complex structure limit point, the conifold singularity and the orbifold regularity, it is expected to be a polynomial of $Z$ of degree $3g-3$. At this moment, it is beyond our ability to calculate $f_g$ directly even at $g=3$. There are two additional B-model structural predictions for $f_g(Z)$. The orbifold regularity claims the vanishing of the lower part of $f_g$. The other is the conifold gap condition, which determines the upper $2g-2$ coefficients of of $f_g$. As we see, for each $g$ there

\(^2\)From the definitions one can check that it is a subring of $\hat{R}$. 
are $3g - 2$ of initial conditions. By using the orbifold regularity and the degree zero Gromov-Witten invariants, the number of initial conditions is reduced by $\lceil \frac{3g - 3}{5} \rceil + 1$. By using the conifold gap condition, it is further reduced to $\lfloor \frac{2g - 2}{5} \rfloor$ many, the same as the number of effective invariants. It is natural to speculate that our approach also implies the conifold gap condition. We leave this for future research.

Remark 1.7. There is a different approach to the higher genus theory of quintic 3-folds by Chang–Guo–Li–Li–Liu. Recently, they posted a series of articles [5, 4, 3]. Among other things, they proved the original Yamaguchi–Yau (non-graded) prediction independently.

Remark 1.8. It is nontrivial to show [29] that the five generators are algebraic independent and hence the expression of $\bar{F}_g$ is unique. This is important for the statement of the holomorphic anomaly equation for which we need to consider derivatives with respect to generators. On the other hand, our proof of Theorem 1.4 gives a canonical expression of $\bar{\Omega}_c^n$ and $\bar{F}_g$ in terms of the generators. Hence, we do not need the algebraic independence of five generators.

1.2. Holomorphic anomaly equations. We now consider the holomorphic anomaly equations (HAE). It is clear that the choice (2) of the generators of $R$ is not canonical. Let us make a choice.

Definition 1.9. Suppose $S$ be a finite set that generates $R$. We pick a subset $S'$ of $S$ such that

$$S' \cap \mathbb{Q}[L] = \emptyset.$$ 

Let $R' = \mathbb{Q}[S'] \subset R$ be the subring generated by $S'$. We call $R'$ a choice of non-holomorphic subring, and we call the elements in the ring of the derivations of $R'$

$$\mathfrak{D}_{R'} \subset \mathfrak{D}_R$$

the vector fields in the non-holomorphic direction.

Remark 1.10. A natural choice of the non-holomorphic subring is

$$R' := \mathbb{Q}[\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}].$$

(Motivated by [34], we can pick another set of generators

$$S' := \{U := \mathcal{X}, \quad V := \mathcal{X} + \mathcal{Y}, \quad V_1 := \partial_u U + UV^2 - UV, \quad V_2 := (\partial_u + V) V_2 \}$$

of $R'$ defined in (3), such that $\mathbb{Q}[S'] = R'$. Here $\partial_u := \frac{d}{du}$.

Theorem 1.11. We introduce the derivations $\partial_1, \partial_0 \in \mathfrak{D}_{R'}$ to be

$$\partial_1 = \partial_U, \quad \partial_0 = \partial_{V_1} - U \partial_{V_2} - U^2 \partial_{V_3}.$$ 

Then we have the following holomorphic anomaly equations conjectured in [34]

$$-\partial_0 \bar{F}_g = \frac{1}{2} \bar{F}_{g-1,2} + \frac{1}{2} \sum_{g_1 + g_2 = g} \bar{F}_{g_1,1} \bar{F}_{g_2,1},$$

$$-\partial_1 \bar{F}_g = 0.$$ 

Note that we could consider $F_{g,n}$ as polynomials of either the generators $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}, L\}$ or the generators $\{V_1, V_2, V_3, U, L\}$.
Remark 1.12. The above holomorphic anomaly equation should be viewed as a higher genus generalization of the Picard–Fuchs equation. For example, for \( g \geq 2 \), it determines (see Remark 6.11) \( F_g \) inductively up to a holomorphic function in \( L \), which is referred to as the holomorphic anomaly in the physics literature. Our Graded Finite Generation Theorem implies that the holomorphic anomaly is a polynomial of degree \( 3g - 3 \), whose lower part vanishes (orbifold regularity).

1.3. Plan of the paper. The paper is organized as follows. In Section 2, we introduce the logarithmic compactification of the GLSM moduli space and the comparison formula between its reduced and nonreduced virtual cycle. In Section 3, we prove a geometric comparison between the formal quintic and the true quintic theory. The combinatorial structural theorem is proved in Section 4. Graded finite generation and orbifold regularity are proven in Section 5. The holomorphic anomaly equation is proven in Section 6. In the final section 7, we prove a technical result of the formal quintic theory which we used in the proof of graded finite generation. It implies a conjecture of Zagier–Zinger [35].

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2. Logarithmic GLSM Moduli Space and its Virtual Cycles

In this Section, we collect some results that will appear in [7, 8].

2.1. Moduli space. Let \( X \) be a smooth projective variety with a line bundle \( L \) admitting a section cutting out a smooth hypersurface \( Y \). We are mainly interested in the case that \( X = \mathbb{P}^4 \) and \( L = \mathcal{O}(5) \).

Given a map \( f: C \to X \) from a log-smooth curve \( C \), let \( \mathbb{P} \) be the projective bundle \( \mathbb{P}(\omega_{\log} \otimes f^*L' \oplus \mathcal{O}) \) equipped with the logarithmic structure pulled back from \( C \) and the divisorial log structure from the infinity section. Let \( \overline{\mathcal{M}}_{g,n}(X, L, \beta, \nu) \) be the moduli space of maps \( f: C \to X \) of degree \( \beta \) together with an aligned log-section \( \eta: C \to \mathbb{P} \) with contact order \( \nu \) along the infinity section, and with all markings mapping to the zero section, and such that \( \omega_{\log}^C \otimes f^*\mathcal{O}(1) \otimes \eta^*\mathcal{O}(0_\mathbb{P}) \), where \( 0_\mathbb{P} \) is the zero section of \( \mathbb{P} \), is ample for all sufficiently large \( k \). For the computation of the Gromov–Witten theory of \( Y \) it suffices to consider the case when \( \nu \) is the empty partition but in the proof of the Combinatorial Structural Theorem, we will also need to consider more general \( \nu \).

\footnote{This imposes that the partial order on the degeneracies of the irreducible components is actually a total order. This is a variant of the logarithmic moduli space that behaves well for the localization.}
**Figure 1.** The bipartite genus two graphs (for $\nu = \emptyset$). Each vertex is decorated by its genus when it is different from zero. Each edge is decorated by its degree when it is different from one. The curve class $\beta$ is distributed among the stable vertices $v \in V_0(\Gamma)$ (of which there is at most one).

**Theorem 2.1** ([7]). The space $\overline{M}_{g,n}(X, L, \beta, \nu)$ is a proper Deligne–Mumford stack.

There are evaluation maps $ev_i: \overline{M}_{g,n}(X, L, \beta, \nu) \to X$, and there is a forgetful map $p: \overline{M}_{g,n}(X, L, \beta, \nu) \to \overline{M}_{g,n+|\nu|}(X, \beta)$.

**2.2. Virtual cycles.** The moduli space $\overline{M}_{g,n}(X, L, \beta, \nu)$ has a canonical perfect obstruction theory defining a virtual cycle $[\overline{M}_{g,n}(X, L, \beta, \nu)]^{\text{vir}}$.

The maximal degeneracy defines a line bundle $L_{\text{max}}$ on $\overline{M}_{g,n}(X, L, \beta, \nu)$. In the case that $\nu = \emptyset$, in [7], a surjective homomorphism of sheaves $\sigma: \text{Ob}_{\overline{M}_{g,n}(X,L,\beta,\emptyset)} \to L_{\text{max}}^\vee$ is constructed using the section cutting out the hypersurface $Y$. Using this cosection-like homomorphism, an alternative reduced virtual cycle $[\overline{M}_{g,n}(X, L, \beta, \emptyset)]^{\text{red}}$ is constructed, which agrees with $[\overline{M}_{g,n}(X, L, \beta, \emptyset)]^{\text{vir}}$ on the locus where the log-section $\eta$ does not map any components of the source to the infinity section. In particular, the reduced virtual dimension is the same as the ordinary virtual dimension. The crucial property of the reduced virtual cycle is the following:

**Theorem 2.2** ([7]). We have

$$p_*[\overline{M}_{g,n}(Y, \beta)]^{\text{vir}} = (-1)^{1-g+j}c_1(L)p_*[\overline{M}_{g,n}(X, L, \beta, \emptyset)]^{\text{red}},$$

where on both sides $p$ denotes the corresponding forgetful map to $\overline{M}_{g,n}(X, \beta)$.

**2.3. Localization formula.** The moduli space $\overline{M}_{g,n}(X, L, \beta, \nu)$ admits a $\mathbb{C}^*$-action defined via scaling the log-section $\eta$. Both the ordinary and reduced perfect obstruction theories are equivariant with respect to this torus action. Hence, we can apply the virtual localization theorem [17] to compute their virtual classes.

We first consider the fixed loci of $\overline{M}_{g,n}(X, L, \beta, \nu)$. They are classified by a set of decorated bipartite graphs $\Gamma$, which we describe now. Let $V(\Gamma)$ and $E(\Gamma)$ be the sets of vertices and edges of $\Gamma$, respectively. There are various decorations:
Figure 2. The bipartite genus one graphs for the partition $\nu = (1)$

- the bipartite structure $V(\Gamma) = V_0(\Gamma) \sqcup V_\infty(\Gamma)$
- a genus mapping $g : V(\Gamma) \to \mathbb{Z}_{\geq 0}$
- a curve class mapping $\beta : V(\Gamma) \to \text{Pic}(X)^\vee$
- a fiber class mapping $\delta : E(\Gamma) \to \mathbb{Z}_{\geq 1}$
- a distribution of markings $m : \{1, \ldots, n\} \sqcup \nu \to V(\Gamma)$

The valence $n(v)$ of a vertex $v \in V(\Gamma)$ is the sum of the number of edges at $v$ and the number of preimages under $m$. We set for all $v \in V_\infty(\Gamma)$ that $\nu(v) = |m^{-1}(v)|$ and that $\mu(v)$ is the partition formed by $\delta(e)$ for all edges $e$ at $v$.

We require the decorated graphs to satisfy the following conditions:

- $\sum_v g(v) + h_1(\Gamma) = g$
- $\sum_v \beta(v) = \beta$
- $\forall v \in V_\infty(\Gamma) : |\mu(v)| - |\nu(v)| = 2g(v) - 2 + n(v) - \int_{\beta(v)} c_1(X)$
- $m(\{1, \ldots, n\}) \subseteq V_0(\Gamma), m(\nu) \subseteq V_\infty(\Gamma)$
- If for $v \in V(\Gamma)$, we have $(g(v), n(v), \beta(v)) = (0, 1, 0)$, then the unique incident edge $e$ has $\delta(e) > 1$.

In the case that $X = \mathbb{P}^4$, $L = \mathcal{O}(5)$, Figure 1 and Figure 2 show all localization graphs for $\overline{\mathcal{M}}_2(X, L, \beta, \emptyset)$ and $\overline{\mathcal{M}}_1(X, L, \beta, (1))$ which do not have any vertices $v \in V_0(\Gamma)$ with $(g(v), n(v)) = (0, 1)$. These are all localization graphs for the analogous moduli space defined using stable quotients. The full set of stable maps localization graphs is obtained by attaching additional genus zero vertices via degree-one edges to any $v \in V_\infty(\Gamma)$ and distributing $\beta$ among them and the original stable vertices in $V_0(\Gamma)$.

Given a decorated graph $\Gamma$, consider the fiber product

\[
\begin{array}{ccc}
\widetilde{M}_\Gamma & \longrightarrow & M_\Gamma = \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v), n(v)}(X, \beta(v)) \\
X^{[E(\Gamma)]} & \xrightarrow{\Delta} & X^{[E(\Gamma)]} \times X^{[E(\Gamma)]}
\end{array}
\]

where for unstable $v \in V_0(\Gamma)$ in the sense that $2g(v) - 2 + n(v) + 3 \int_{\beta(v)} c_1(X) \leq 0$, we define the corresponding factor in $M_\Gamma$ to be $X$. There is a gluing map

\[
t_\Gamma : \widetilde{M}_\Gamma \to \overline{\mathcal{M}}_{g,n+|\nu|}(X, \beta).
\]
We then have

$$p_*([\overline{\mathcal{M}}_{g,n}(X, L, \beta, \nu)]^\text{vir}) = \sum_{\Gamma} \frac{1}{|\text{Aut}(\Gamma)|} t_{\Gamma*} \Delta^1(C^\text{vir}_\Gamma)$$

(6)

where

$$C^\text{vir}_\Gamma = \prod_{v \in V(\Gamma)} \frac{c(-R\pi_*(\omega_{\pi} \otimes f^*L^v) \otimes [1])}{\prod_{e \in v} (t-\frac{e^{-\psi_e}((c_1(L))}{\delta(e)})} \cap [\overline{\mathcal{M}}_{g(v), n(v)}(X, \beta(v))]^\text{vir}$$

$$\prod_{v \in V_{\infty}(\Gamma)} p_* \left( \frac{1}{-t - c_1(L_{\min})} \overline{\mathcal{M}}_{g(v)}(X, L, \beta(v), \mu(v), \nu(v)) \right)^\text{vir} \prod_{e \in E(\Gamma)} \frac{1}{\delta(e)} \prod_{i=1}^{\delta(e)} \frac{1}{\delta(e)}$$

and

$$p_*([\overline{\mathcal{M}}_{g,n}(X, L, \beta, \emptyset)]^\text{red}) = \sum_{\Gamma} \frac{1}{|\text{Aut}(\Gamma)|} t_{\Gamma*} \Delta^1(C^\text{red}_\Gamma)$$

(7)

where

$$C^\text{red}_\Gamma = \prod_{v \in V(\Gamma)} \frac{c(-R\pi_*(\omega_{\pi} \otimes f^*L^v) \otimes [1])}{\prod_{e \in v} (t-\frac{e^{-\psi_e}((c_1(L))}{\delta(e)})} \cap [\overline{\mathcal{M}}_{g(v), n(v)}(X, \beta(v))]^\text{vir}$$

$$\prod_{v \in V_{\infty}(\Gamma)} p_* \left( \frac{t}{-t - c_1(L_{\min})} \overline{\mathcal{M}}_{g(v)}(X, L, \beta(v), \mu(v), \emptyset) \right)^\text{red} \prod_{e \in E(\Gamma)} \frac{1}{\delta(e)} \prod_{i=1}^{\delta(e)} \frac{1}{\delta(e)}.$$
The only case of unstable vertices $v$ at $\infty$ happens when $(g(v), n(v), \beta(v)) = (0, 2, 0)$ and $\nu(v) \neq \emptyset$. Then we define
\[
p_* \left( \frac{1}{-t - c_1(L_{\text{min}})} \llbracket \overline{\mathcal{M}}_{g(v)}(X, L, \beta(v), \mu(v), \nu(v)) \rrbracket^\text{vir} \right) = 1.
\]

2.4. Comparison of virtual cycles. The ordinary and reduced virtual cycle of the moduli space $\overline{\mathcal{M}}_{g,n}(X, L, \beta, \emptyset)$ only differ over the boundary $\partial \overline{\mathcal{M}}_{g,n}(X, L, \beta, \emptyset)$. As in a similar situation in [32], the boundary contribution can be made explicit.

**Theorem 2.3.** There is a reduced virtual cycle $[\partial \overline{\mathcal{M}}_{g,n}(X, L, \beta, \emptyset)]^\text{red}$ such that
\[
[\overline{\mathcal{M}}_{g,n}(X, L, \beta, \emptyset)]^\text{vir} = [\overline{\mathcal{M}}_{g,n}(X, L, \beta, \emptyset)]^\text{red} + [\partial \overline{\mathcal{M}}_{g,n}(X, L, \beta, \emptyset)]^\text{red}.
\]

We make the comparison more explicit by decomposing the boundary $\partial \overline{\mathcal{M}}_{g,n}(X, L, \beta, \emptyset)$ according to “tropical data”, which is in correspondence to exactly the same decorated graphs as those discussed in Section 2.3.

The graphs symbolize a decomposition of the source curve of the log-section into a part mapping directly into the infinity section, and one part not mapping directly into the infinity section. The vertices $v \in V_\infty(\Gamma)$ (respectively, $v \in V_0(\Gamma)$) make up the first (respectively, second) part. There is exactly one graph, the graph consisting of a single vertex $v \in V_0(\Gamma)$, that does not stand for any component of $\partial \overline{\mathcal{M}}_{g,n}(X, L, \beta, \emptyset)$. It instead corresponds to $\overline{\mathcal{M}}_{g,n}(X, L, \beta, \emptyset)$ itself.

Applying an additional comparison of connected and disconnected rubber virtual cycles, [8] arrives at:

**Theorem 2.4.**
\[
p_*([\overline{\mathcal{M}}_{g,n}(X, L, \beta, \emptyset)]^\text{red}) = \sum_{\Gamma} \frac{1}{|\text{Aut}(\Gamma)|} t_{\Gamma} \Delta'(C_{\Gamma}^\text{comp}),
\]
where
\[
C_{\Gamma}^\text{comp} = \prod_{v \in V_0(\Gamma)} p_*([\overline{\mathcal{M}}_{g(v), n(v)}(X, L, \beta(v), \nu(v))]^\text{vir}) \prod_{v \in V_\infty(\Gamma)} p_*(-[\overline{\mathcal{M}}_{g(v), n(v)}(X, L, \beta(v), \mu(v), \emptyset)]^\text{red}) \prod_{e \in E(\Gamma)} \delta(e).
\]

**Remark 2.5.** Note that $C_{\Gamma}^\text{comp} = p_*[\overline{\mathcal{M}}_{g,n}(X, L, \beta, \emptyset)]^\text{red}$ in the case that $\Gamma$ is the graph consisting of a single vertex $v \in V_0(\Gamma)$. The sum of the contributions of the other graphs is $p_*[\partial \overline{\mathcal{M}}_{g,n}(X, L, \beta, \emptyset)]^\text{red}$.

**Remark 2.6.** In the quintic case ($X = \mathbb{P}^4$, $L = \mathcal{O}(5))$, the classes $[\overline{\mathcal{M}}_{g,n}(X, L, \beta, \mu, \emptyset)]^\text{red}$ are determined from only $1 + \lfloor \frac{2g - 2}{5} \rfloor$ numbers in any genus: To see this, note first that $\mu$ is a partition of $2g - 2 + n - 5\beta$, and that the dimension of $[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^4, \mathcal{O}(5), \beta, \mu, \emptyset)]^\text{red}$ is $n - (2g - 2 - 5\beta)$.

Therefore, in the case that all parts of $\mu$ are equal to 2, the virtual dimension is zero, and we may define a constant $c_{g,\beta}^\text{eff}$ for $5\beta \leq 2g - 2$ by
\[
-[\overline{\mathcal{M}}_{g,2g - 2 - 5\beta}(\mathbb{P}^4, \mathcal{O}(5), \beta, (2g - 2 - 5\beta), \emptyset)]^\text{red} = c_{g,\beta}^\text{eff}[pt].
\]
Furthermore, in the case that some parts of $\mu$ are equal to 1, the reduced virtual class is pulled back under the map forgetting those markings. Therefore, the virtual class vanishes unless all parts of $\mu$ are equal to 1 or 2, and when $\mu = (2^{g-2-5\beta}, 1^k)$, we have

$$-\langle \overline{\mathcal{M}}_{g,2g-2-5\beta}(\mathbb{P}^4, \mathcal{O}(5), \beta, (2^{g-2-5\beta}, 1^k), \emptyset) \rangle_{\text{red}} = e\overline{g,\beta}^{\ast}[\text{pt}],$$

where $\pi_k$ is the map forgetting the last $k$ markings.

Therefore, $\langle \overline{\mathcal{M}}_{g,n}(X, L, \beta, \mu, \emptyset) \rangle_{\text{red}}$ are determined from the numbers $e\overline{g,\beta}$ for $5\beta \leq 2g-2$. We call $e\overline{g,\beta}$ the effective constants.

2.5. The extended quintic family. By Remark 2.6, the comparison formula in Theorem 2.4 involves effective constants $e\overline{g,\beta}$. Viewing the effective constants as parameters, we define the extended quintic family of theories. Given a choice $c$ of constants $c_{g,\beta}$ for $5\beta \leq 2g-2$, we formally define the extended quintic virtual class

$$[Q_{g,n,\beta}]_{\text{vir}} := \sum_{\Gamma} \frac{1}{|\text{Aut}(\Gamma)|} \prod_{e \in E(\Gamma)} \delta(e)$$

$$\times t_{\Gamma} \Delta^\dagger \left( \prod_{v \in V_0(\Gamma)} p_\ast([\mathcal{M}_{g(v),n(v)}(X, L, \beta(v), \nu(v))]_{\text{vir}} \prod_{v \in V_\infty(\Gamma)} c_{g(v),\beta(v),\mu(v)}) \right),$$

where we set

$$c_{g,\beta,\mu} := \begin{cases} 
0 & \text{if } \mu \text{ has a part of size } \geq 3, \\
 c_{g,\beta,\pi_k^{-1}(\mu) - (2g-2-5\beta)}[\text{pt}] & \text{otherwise}.
\end{cases}$$

This is a cycle on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^4, \beta)$.

3. From formal quintic to quintic

The goal of this section is to prove a graph sum formula expressing the extended quintic family in terms of the formal quintic. We will prove a cycle-valued formula that will be the an essential ingredient in the Combinatorial Structure Theorem for the quintic family that we will prove in the following Section 4.

3.1. Twisted virtual cycles. Compared to the previous section, we now specialize to the case that $X = \mathbb{P}^4$ and $L = \mathcal{O}(5)$.\footnote{The analysis of this and the following section can be performed analogously for the case where $X$ is any projective space and $L = \mathcal{O}(k)$ for some $k > 0.$} In this case, the localization formula of Section 2.3 involves the twisted virtual cycle

$$e(-R\pi_\ast(\omega_\pi \otimes f^\ast\mathcal{O}(-5)) \otimes [1]) \cap [\mathcal{M}_{g,n}(\mathbb{P}^4, \beta)]_{\text{vir}},$$

which we can rewrite via Serre duality as

$$e(-R\pi_\ast(\omega_\pi \otimes f^\ast\mathcal{O}(-5)) \otimes [1]) \cap [\mathcal{M}_{g,n}(\mathbb{P}^4, \beta)]_{\text{vir}}$$

$$= (-1)^{1-g+5\beta} e(R\pi_\ast(f^\ast\mathcal{O}(5)) \otimes [1]) \cap [\mathcal{M}_{g,n}(\mathbb{P}^4, \beta)]_{\text{vir}}. \quad (8)$$

In the following, we call the cohomological field theory defined using the virtual cycle

$$e(R\pi_\ast(f^\ast\mathcal{O}(5)) \otimes [1]) \cap [\mathcal{M}_{g,n}(\mathbb{P}^4, \beta)]_{\text{vir}}$$
the $t$-twisted theory.

This theory is closely related to the formal quintic theory (we also refer to it as the $\lambda$-twisted theory) of [20, 21, 27, 26, 35] in genus zero. To explain how they differ, it is useful to consider the fully equivariant twisted theory, which is defined in the same way as the $t$-twisted theory except that we use the $(\mathbb{C}^*)^5$-equivariant virtual cycle $[\overline{M}_{g,n}(\mathbb{P}^4, \beta)]^{\text{vir}}_{(\mathbb{C}^*)^5}$ of $\overline{M}_{g,n}(\mathbb{P}^4, \beta)$ with respect to the diagonal $(\mathbb{C}^*)^5$-action on $\mathbb{P}^4$. Let $\lambda_i$ denote the corresponding additional equivariant parameters.

Then the $t$-twisted theory can be obtained by specializing $\lambda_i = 0$. On the other hand, the $\lambda$-twisted theory is obtained by setting $t = 0$, which is possible since $c_1(O(5))$ is invertible in equivariant cohomology, as well as specializing $\lambda_i = \zeta^i \lambda$ where $\lambda$ is one parameter and $\zeta$ is a fifth root of unity. We accordingly define the formal quintic virtual cycle:

$$[\overline{M}_{g,n}(\mathbb{P}^4, \beta)]^{\text{vir}, \lambda} = \left( e(R\pi_* f^* O(5) \otimes [1]) \cap [\overline{M}_{g,n}(\mathbb{P}^4, \beta)]^{\text{vir}}_{(\mathbb{C}^*)^5} \right)_{t = 0, \lambda_i = \zeta^i \lambda}.$$

**Remark 3.1.** In genus zero, the two twisted theories admit non-equivariant limits $t = 0$, $\lambda = 0$, which agree. However, in higher genus, it is not possible to take these limits, and taking the $t^0$-coefficient in the Laurent expansion of the $t$-twisted theory will give a different result to the $\lambda^0$-coefficient in the $\lambda$-twisted theory.

### 3.2. Full torus localization.

In addition to the $\mathbb{C}^*$-action on $\overline{M}_{g,n}(\mathbb{P}^4, O(5), \beta, \nu)$ discussed in Section 2.3 we may also use the diagonal $(\mathbb{C}^*)^5$-action on $\mathbb{P}^4$ to define a $(\mathbb{C}^*)^6$-action on $\overline{M}_{g,n}(\mathbb{P}^4, O(5), \beta, \nu)$. The ordinary perfect obstruction theory of $\overline{M}_{g,n}(\mathbb{P}^4, O(5), \beta, \nu)$ is equivariant for the full $(\mathbb{C}^*)^6$-action, while (when defined), the reduced perfect obstruction theory is only $\mathbb{C}^*$-equivariant.

The localization formula of Section 2.3 can clearly be lifted to $(\mathbb{C}^*)^6$-equivariant cohomology. We note that this is not the same as the localization formula for the $(\mathbb{C}^*)^6$-action on $\overline{M}_{g,n}(\mathbb{P}^4, O(5), \beta, \nu)$ because we still use the fixed loci for the $\mathbb{C}^*$-action and not the (smaller) fixed loci of the $(\mathbb{C}^*)^6$-action.

We now consider the specialization of equivariant parameters

$$\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, t = (\lambda, \zeta \lambda, \zeta^2 \lambda, \zeta^3 \lambda, \zeta^4 \lambda, 0).$$

Under this specialization, the localization formula from Section 2.3 can be written as

$$p_*([\overline{M}_{g,n}(\mathbb{P}^4, O(5), \beta, \nu)]^{\text{vir}}) = \lim_{\lambda \to 0} \sum_{\Gamma} \frac{1}{|\text{Aut}(\Gamma)|} \Delta^{t_{\Gamma} \Delta^1}(C^{\text{vir}}_{\Gamma})$$

where

$$C^{\text{vir}}_{\Gamma} = \prod_{v \in V_5(\Gamma)} \left( \frac{(-1)^{1-g(v)+5\beta(v)}}{\prod_{e \in \text{at } v} (\frac{-\psi_e(5H)}{\delta(e)} - \psi_e)} \cap [\overline{M}_{g(v),n(v)}(\mathbb{P}^4, \beta(v))]^{\text{vir}, \lambda}_{(\mathbb{C}^*)^5} \right) \prod_{e \in E(\Gamma)} \frac{1}{\prod_{i=1}^{\delta(e) - ic\psi_e(5H)/\delta(e)}} \prod_{v \in V_{\infty}(\Gamma)} p_* \left( \frac{1}{\prod_{e \in \text{at } v} (\frac{-\psi_e(5H)}{\delta(e)} - \psi_e)} \right)$$

where and all other notation and exceptional cases are similar as in Section 2.3.
Figure 3. The tripartite graphs in genus two, part 1. The letters under each graph indicate which genus two bipartite graph is obtained when merging the upper two or the lower two level, respectively.

Figure 4. The tripartite graphs in genus two, part 2.

Remark 3.2. It is not obvious that we could perform the formal quintic specialization. The next lemma addresses invertibility of \( c_1(L_{\text{min}}) \). To see that \( -\alpha_e^*(5H) - \psi_e \) is invertible note that on each fixed locus for localization on \( \overline{\mathcal{M}}_{g,n}(\mathbb{P}^4, \beta) \), \( H \) specializes to an element of the form \( \zeta^i \lambda \), whereas \( \psi_e \) is either nilpotent or an element of the form \( (\zeta^j - \zeta^k) \lambda \) for \( j \neq k \).

Lemma 3.3 (\cite{8}). \( c_1(L_{\text{min}}) \) is invertible in the \( (\mathbb{C}^*)^5 \)-equivariant cohomology of \( \overline{\mathcal{M}}_{g,v}(\mathbb{P}^4, \mathcal{O}(5), \beta(v), \mu(v), \nu(v)) \).

3.3. Tripartite graphs. We now combine the comparison of Theorem 2.4 and more generally, the definition of the extended quintic family in Section 2.5 with the localization formula of Section 3.2 to the computation of \( \text{vir} \).

The comparison formula is already a sum over bipartite graphs. Applying localization to \( p_*(\overline{\mathcal{M}}_{g,v,n}(\mathbb{P}^4, \mathcal{O}(5), \beta(v), \mu(v), \nu(v)))^{\text{vir}} \) for each \( v \in V_0(\Gamma) \), we arrive at a sum over graphs with three different types of vertices.

The decorations of such a graph \( \Gamma \) consist of

- the tripartite structure \( V(\Gamma) = V_l(\Gamma) \sqcup V_m(\Gamma) \sqcup V_u(\Gamma) \)
• a genus mapping \( g : V(\Gamma) \rightarrow \mathbb{Z}_{\geq 0} \)
• a curve class mapping \( \beta : V(\Gamma) \rightarrow \text{Pic}(X)^t \)
• a fiber class mapping \( \delta : E(\Gamma) \rightarrow \mathbb{Z}_{\geq 1} \)
• a distribution of markings \( m : \{1, \ldots, n\} \rightarrow V(\Gamma) \)

When \( v \in V_l(\Gamma) \) (respectively, \( v \in V_m(\Gamma), v \in V_u(\Gamma) \)), we say that \( v \) is in the lower (respectively, middle, upper) level. Edges must always connect vertices of different levels. We can therefore decompose \( E(\Gamma) = E_{lm}(\Gamma) \sqcup E_{lu}(\Gamma) \sqcup E_{mu}(\Gamma) \) according to which levels are connected by an edge. For \( v \in V_m(\Gamma) \sqcup V_u(\Gamma) \), we set \( \mu(v) \) (respectively, \( \nu(v) \)) to be the partitions formed by the \( \delta(e) \) for all \( e \) at \( v \) connecting to a lower (respectively, upper) level.

Note that if \( v \in V_u(\Gamma) \), then \( \nu(v) = \emptyset \). They need to satisfy the constraints:

- \( E_{lu}(\Gamma) = \emptyset \)
- \( \sum_v g(v) + h^1(\Gamma) = g \)
- \( \sum_v \beta(v) = \beta \)
- \( \forall v \in V_m(\Gamma) \sqcup V_u(\Gamma) : |\mu(v)| - |\nu(v)| = 2g(v) - 2 + n(v) - \int_{\beta(v)} c_1(X) \)
- \( \text{If for } v \in V(\Gamma), \text{ we have } (g(v), n(v), \beta(v)) = (0, 1, 0), \text{ then the unique incident edge } e \text{ has } \delta(e) > 1. \)

We have listed all tripartite graphs in the case that \( (g, n) = (2, 0) \) (and we use a stable quotient theory) in Figure 3 and 4.

We define \( M_\Gamma, \tilde{M}_\Gamma \) and

\[ \iota_\Gamma : \tilde{M}_\Gamma \rightarrow \overline{M}_{g,n}(\mathbb{P}^4, \beta) \]

analogously as in Section 2.3 There is an additional case of an unstable vertex \( v \), that is, when \( v \in V_m(\Gamma) \) and \( (g(v), n(v), \beta(v)) = (0, 2, 0) \).

We can now write \( p_*([\overline{M}_{g,n}(\mathbb{P}^4, \mathcal{O}(5), \beta, \emptyset)]^{\text{red}}) \) as a sum over decorated tripartite graphs:

\[ [Q_{g,n,\beta}^c]^{\text{vir}} = \lim_{\lambda \rightarrow 0} \sum_{\Gamma} \frac{1}{|\text{Aut}(\Gamma)|} \iota_{\Gamma*} \Delta^I(C_\Gamma^3), \]

where

\[ C_\Gamma^3 = \prod_{v \in V_l(\Gamma)} (-1)^{1-g(v)+5\delta(v)} \prod_{e \text{ at } v} \left( \frac{-c_1(L_{\min})}{\delta(e)} - \psi_e \right) \cap [\overline{M}_{g,v,n}(\mathbb{P}^4, \beta(v))]^{\text{vir}, \lambda} \]

\[ \prod_{v \in V_m(\Gamma)} p_* \left( \frac{1}{-c_1(L_{\min})} [\overline{M}_{g,v}(\mathbb{P}^4, \mathcal{O}(5), \beta(v), \mu(v), \nu(v))]^{\text{vir}} \right) \]

\[ \prod_{v \in V_u(\Gamma)} \mathcal{C}_{g,v}(\beta(v), \mu(v)) \prod_{e \in E_{lm}(\Gamma)} \frac{1}{\delta(e)} \prod_{i=1}^{\delta(e)} \frac{1}{\delta(e)} \prod_{e \in E_{mu}(\Gamma)} \delta(e). \]

Here the notations are similar to those in Section 2.3, as are the contributions of unstable vertices \( v \in V_l(\Gamma) \) and \( v \in V_m(\Gamma) \).

We now rewrite the formula as a sum over bipartite graphs by merging the middle and upper levels. The result is a sum over decorated graphs as in Section 2.3

\[ [Q_{g,n,\beta}^c]^{\text{vir}} = \lim_{\lambda \rightarrow 0} \sum_{\Gamma} \frac{1}{|\text{Aut}(\Gamma)|} \iota_{\Gamma*} \Delta^I(C_\Gamma^2), \quad (10) \]
where
\[
C^2_{\Gamma} = \prod_{v \in V_0(\Gamma)} \prod_{e \text{ at } v} \left(\frac{(-1)^{1-g(v)+5\beta(v)}}{\delta(e)} - \psi_e\right) \cap \left[\mathcal{M}_{g(v),n(v)(\mathbb{P}^4,\beta(v))}\right]^{\text{vir},\lambda}
\]
\[
\prod_{v \in V_\infty(\Gamma)} C_{g(v),\beta(v),\mu(v)} \prod_{e \in E(\Gamma)} \frac{1}{\delta(e)} \prod_{i=1}^{\beta(e)} \frac{1}{\delta(e)}
\]

for classes \(C_{g,\beta,\mu} \in H_*(\mathcal{M}_{g,\mu,\beta}(\mathbb{P}^4,\beta))\) that themselves can be written as a sum over a class of bipartite decorated graphs \(\Gamma\) that we will describe now.

The decorations are given by:
- the bipartite structure \(V(\Gamma) = V_0(\Gamma) \sqcup V_\infty(\Gamma)\)
- a genus mapping \(g: V(\Gamma) \to \mathbb{Z}_{\geq 0}\)
- a curve class mapping \(\beta: V(\Gamma) \to \text{Pic}(X)^\vee\)
- a fiber class mapping \(\delta: E(\Gamma) \to \mathbb{Z}_{\geq 1}\)
- a distribution of markings \(m: \mu \to V_0(\Gamma)\)

We set for all \(v \in V_0(\Gamma)\) that \(\mu(v) = |m^{-1}(\mu)|\) and that \(\nu(v)\) is the partition formed by \(\delta(e)\) for all edges \(e\) at \(v\). For all \(v \in V_\infty(\Gamma)\), the partition formed by \(\delta(e)\) for all edges \(e\) at \(v\) is denoted by \(\mu(v)\), while we set that \(\nu(v) = \emptyset\).

We require the decorated graphs to satisfy the following conditions:
- \(\sum_v g(v) + h^1(\Gamma) = g\)
- \(\sum_v \beta(v) = \beta\)
- \(\forall v \in V_0(\Gamma): |\mu(v)| - |\nu(v)| = 2g(v) - 2 + n(v) - 5\beta(v)\)
- There are no vertices \(v \in V(\Gamma)\) such that \((g(v),n(v),\beta(v)) = (0,1,0)\).

We note that the conditions imply that \(\mu\) is a partition of \(2g - 2 - 5\beta\).

Then, we can define \(C_{g,\beta,\mu}\) via
\[
C_{g,\beta,\mu} = \sum_{\Gamma} \frac{1}{|\text{Aut}(\Gamma)|} t_{\Gamma^*} \Delta^!(C^m_{\Gamma^*}), \tag{11}
\]
where
\[ C^\mu = \prod_{v \in V_0(\Gamma)} p_*( \frac{1}{-c_1(L_{\min})} [\mathcal{M}_{g(v)}(\mathbb{P}^4, \mathcal{O}(5), \beta(v), \mu(v), \nu(v))]^{\text{vir}} ) \prod_{v \in V_\infty(\Gamma)} c_{g(v), \beta(v), \mu(v)} \prod_{e \in E(\Gamma)} \delta(e). \]

All in all, we have found a formula (10) expressing the extended quintic virtual cycle in terms of the formal quintic and cycles (11) that depend on the chosen element of the extended quintic family.

**Remark 3.4.** We could further extend the quintic family by allowing the \( C_{g, \beta, \mu} \) to be arbitrary cycles subject to the constraint that \( C_{g, \beta, \mu} \) is only nonzero if \( \mu \) is a partition of \( 2g - 2 - 5\beta \).

### 4. The combinatorial structure theorem

In Section 2.5, we introduced the extended quintic family. It contains the quintic and holomorphic GLSM theories. In this section, we prove a structural theorem for the extended quintic family, which sets the stage for the proof of the main theorems.

#### 4.1. Formal quintic CohFT

In Section 3.1, we introduced the \( \lambda \)-twisted virtual cycle \( [\mathcal{M}_{g,n}(\mathbb{P}^4, \beta)]^{\text{vir}, \lambda} \). We now define the corresponding cohomological field theory (CohFT). The state space is \( H = H^*(\mathbb{P}^4) \), the pairing is given by
\[ (\gamma_1, \gamma_2)^\lambda = \int_{\mathbb{P}^4(\mathbb{C}^*)^5} e(\mathcal{O}(5)) \gamma_1 \gamma_2 \bigg|_{\lambda_i = \lambda'}, \]
and the unit is \( 1 \), the unit in cohomology. We now define the \( \lambda \)-twisted cohomological field theory \( \Omega^\lambda \) by
\[ \Omega^\lambda_{g,n}(\gamma_1, \ldots, \gamma_n) = \sum_{\beta=0}^{\infty} q^\beta \rho_*( \prod_{i=1}^{n} \text{ev}^*_i(\gamma_i) \cap [\mathcal{M}_{g,n}(\mathbb{P}^4, \beta)]^{\text{vir}, \lambda} ), \]
where \( \rho: \mathcal{M}_{g,n}(\mathbb{P}^4, \beta) \rightarrow \mathcal{M}_{g,n} \) is the forgetful map. More generally, for any (formal) \( \tau \in H \), we may define the \( \tau \)-shifted \( \lambda \)-twisted cohomological field theory \( \Omega^\lambda_{\tau} \) by
\[ \Omega^\lambda_{g,n}(\gamma_1, \ldots, \gamma_n) = \sum_{\beta,k=0}^{\infty} \frac{q^\beta}{k!} \rho_* \pi^k_*( \prod_{i=1}^{n} \text{ev}^*_i(\gamma_i) \prod_{i=n+1}^{n+k} \text{ev}^*_i(\tau) \cap [\mathcal{M}_{g,n+k}(\mathbb{P}^4, \beta)]^{\text{vir}, \lambda} ), \]
where \( \pi^k: \mathcal{M}_{g,n+k}(\mathbb{P}^4, \beta) \rightarrow \mathcal{M}_{g,n}(\mathbb{P}^4, \beta) \) is the forgetful map.

The \( (\tau \text{-shifted}) \lambda \)-twisted cohomological field theory is semi-simple. Hence there exists a basis \( \{ e_\alpha \} \) of idempotents, and the Givental–Teleman reconstruction theorem [33, 31] applies. This theorem determines \( \Omega^\lambda \) from its degree-zero part (topological field theory) \( \omega \), and an \( R \)-matrix
\[ R_\tau(z) \in \text{End}(H) \otimes \mathbb{Q}(\lambda)[[q, z]] \]
satisfying the symplectic condition
\[ R_\tau(z) R^*_\tau(-z) = \text{Id}. \]
To motivate the extended $R$-matrix action, we recall the statement of the Givental–Teleman theorem. Let $G_{g,n}$ be the set of stable graphs of genus $g$ with vertices and $n$ legs. For any $\Gamma$ in $G_{g,n}^\infty$ or $G_{g,n}$, there is a corresponding gluing map

$$t_\Gamma: \overline{\mathcal{M}}_\Gamma := \prod_v \overline{\mathcal{M}}_{g(v), n(v)} \to \overline{\mathcal{M}}_{g,n}.$$

We then, for each $\Gamma \in G_{g,n}$, define the contribution

$$\text{Cont}_\Gamma: \mathbb{H}^\otimes n \to \overline{\mathcal{M}}_\Gamma$$

by the following contraction of tensors

1. at each flag $f = (v, l)$ with leg insertion $\gamma$, we place $R_\gamma^{-1}(z_f)\gamma$;
2. at each edge $e$ connecting two vertices $v_1, v_2$, we place
   $$\sum_{\alpha} e_\alpha \otimes e^\alpha - R_\gamma^{-1}(z_{f_1})e_\alpha \otimes R_\gamma^{-1}(z_{f_2})e^\alpha$$
   $$z_{f_1} + z_{f_2}$$
   where $f_1 := (v_1, e), f_2 := (v_2, e)$ and $\{e^\alpha\}$ is the dual basis to $\{e_\alpha\}$, and
3. at each vertex $v$ with $m$ flags $f_1, \ldots, f_m$, we place the map
   $$\gamma_1(z_{f_1}) \otimes \cdots \otimes \gamma_m(z_{f_m}) \mapsto T_\gamma(\overline{\omega}_{g(v), m}(\gamma_1(\overline{\psi}_1) \otimes \cdots \otimes \gamma_m(\overline{\psi}_m)))$$
   $$:= \sum_{k=0}^{\infty} \frac{1}{k!} \pi_{k+\omega_{g(v), m+k}} \left( \gamma_1(\overline{\psi}_1) \otimes \cdots \otimes \gamma_m(\overline{\psi}_m) \otimes \bigotimes_{i=1}^{k} T(\overline{\psi}_{m+i}) \right),$$
   where
   $$T(z) = z(1 - R_\gamma^{-1}(z)1),$$
   and the $\overline{\psi}$ are the (ancestor) psi classes.

**Theorem 4.1** (Givental–Teleman).

$$\Omega^\lambda_{g,n} = \sum_{\Gamma \in G_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} t_{\Gamma *} \text{Cont}_{\Gamma}$$

For later reference, we recall the definition of the twisted $J$-function and the (inverse) $S$-matrix. We allow for a more general shift $t(z) \in \mathbb{H}[[t]]$:

$$J_t(z) := z\mathbf{1} + t(-z) + \sum_{\beta, k=0}^{\infty} \frac{q^\beta}{k!} \text{ev}_{1*} \left( \frac{1}{z - \psi_1} \prod_{i=2}^{k+1} \text{ev}_{i*}(t(\psi_i)) \right) \cap [\overline{\mathcal{M}}_{0, k+1}(\mathbb{P}^4, \beta)]^{\text{vir,}\lambda}$$

$$S_t^{-1}(z)\gamma := \gamma + \sum_{\beta, k=0}^{\infty} \frac{q^\beta}{k!} \text{ev}_{1*} \left( \frac{1}{-z - \psi_1} \prod_{i=3}^{k+2} \text{ev}_{i*}(t(\psi_i)) \right) \cap [\overline{\mathcal{M}}_{0, k+2}(\mathbb{P}^4, \beta)]^{\text{vir,}\lambda}$$
4.2. The extended quintic family as a generalized $R$-matrix action. We use the extended quintic virtual cycle $[Q_{g,n,\beta}]^\text{vir}$ to define $(t$-shifted) extended quintic classes for any $\gamma_1, \ldots, \gamma_n \in H$. We use a slightly artificial definition in order to avoid making assumptions on the effective constants. We define

$$
\Omega_{g,n}^c(\gamma_1, \ldots, \gamma_n) := (I_0)^{2g-2+n} \sum_{\beta,k=0}^{\infty} \frac{q^{\beta}}{k!} (-1)^{1-g+5\beta} \rho_\pi \pi_k \left( \prod_{i=1}^{n} ev_i^*(\gamma_i) \prod_{i=n+1}^{n+k} ev_i^*(t(\psi_i)) \cap [Q_{g,n+k,\beta}]^\text{vir} \right),
$$

where we use the shift

$$
t = z(1 - I_0(q)) + I_1(q) H,
$$

and where $I_0(q)$ and $I_1(q)$ are as in the introduction. The choice is motivated by the Wall-Crossing Theorem [10] (see also [11]). The extended quintic classes are related to the quintic CohFT via

$$
\Omega_{g,n}(\gamma_1, \ldots, \gamma_n) = \Omega_{g,n}^{c,t}(\gamma_1, \ldots, \gamma_n)
$$

under the mirror transformation between $q$ and $Q$.

Although the extended quintic classes do not form a CohFT, in this section, we will express them in terms of a generalized $R$-matrix action. In the following sections, we will mostly work with the related shift

$$
\tau = \frac{I_1(q)}{I_0(q)} H.
$$

Before stating the generalized $R$-matrix action, we need to setup some notation. First, let $G_{g,n}^\infty$ be the set of stable graphs in $G_{g,n}$ with the decorations:

- for each vertex $v$, we assign a label $0$ or $\infty$;
- for each flag $f = (v,e)$ or $f = (v,l)$ where $v$ labeled by $\infty$, we assign a degree $\delta_f \in \mathbb{Z}_{\geq 1}$.

Then, if $v$ is a vertex of a graph $\Gamma \in G_{g,n}^\infty$, and $\gamma \in H$, we write

$$
\mathcal{R}_v^{-1}(z_f)\gamma := \begin{cases} 
R_v^{-1}(z_f)\gamma & \text{if } v \text{ is labeled by } 0, \\
S_v^{-1}(z_f)\gamma & \text{if } v \text{ is labeled by } \infty.
\end{cases}
$$

To define the generalized $R$-matrix action, define for each $\Gamma \in G_{g,n}^\infty$ a contribution

$$
\text{Cont}_\Gamma : H^{\otimes n} \to \overline{M}_\Gamma
$$

by the following construction:

1. at each flag $f = (v,l)$ with leg insertion $\gamma$, we place $\mathcal{R}_v^{-1}(z_f)\gamma$;
2. at each edge $e$ connecting two vertices $v_1, v_2$, we place

$$
\mathcal{Y}_{f_1,f_2}(z_{f_1}, z_{f_2}) := \sum_{\alpha} \delta_{v_1 v_2} e_\alpha \otimes e^\alpha - \mathcal{R}^{-1}_{v_1}(z_{f_1}) e_\alpha \otimes \mathcal{R}^{-1}_{v_2}(z_{f_2}) e^\alpha
$$

where $f_1 := (v_1, e), f_2 := (v_2, e)$ and

$$
\delta_{v_1 v_2} = \begin{cases} 
1 & \text{if } v_1, v_2 \text{ have both label } 0, \\
0 & \text{otherwise};
\end{cases}
$$

with

$$
\mathcal{R}_v^{-1}(z_f)\gamma := R_v^{-1}(z_f)\gamma \quad \text{if } v \text{ is labeled by } 0,
$$

and

$$
\mathcal{S}_v^{-1}(z_f)\gamma := S_v^{-1}(z_f)\gamma \quad \text{if } v \text{ is labeled by } \infty.
$$
(3) at each vertex $v$ with $m$ flags $f_1, \ldots, f_m$, we place the map
\[
\gamma_1(z_{f_1}) \otimes \cdots \otimes \gamma_m(z_{f_m}) \mapsto \begin{cases} 
T \omega_{g(v),m} \gamma_1(\tilde{\psi}_1) \otimes \cdots \otimes \gamma_m(\tilde{\psi}_m) & v \text{ is labeled by } 0, \\
(I_0)^{2g(v)-2+\ell(\mu)} J \Omega^\infty_{g(\mu),\mu}(\gamma_1(5H/\mu_1) \otimes \cdots \otimes \gamma_m(5H/\mu_m)) & v \text{ is labeled by } \infty. 
\end{cases}
\]
where $\mu$ is the partition of length $m$ formed by $\delta_f$ for all flags $f$, where
\[
\Omega^\infty_{g,\mu} : H^\otimes(\mu) \to H^*(\bar{M}_{g,\ell(\mu)})
\]
is a multilinear form determined from the effective invariants such that
\[
\Omega^\infty_{g,\mu} = 0 \quad \text{for} \quad 2g - 2 + \ell(\mu) - 5|\mu| < 0,
\]
and where
\[
J \Omega^\infty_{g,\mu}(\gamma_1 \otimes \cdots \otimes \gamma_m) = \sum'_{\mu'} \frac{1}{|\text{Aut}(\mu')|} \Omega^\infty_{g,\mu+\mu'} \left( \gamma_1 \otimes \cdots \otimes \gamma_m \otimes \bigotimes_{i=1}^{\ell(\mu')} J_\mu(-5H/\mu'_i) \right),
\]
in which $\sum'$ indicates that the sum is only over all partitions $\mu'$ with no part equal to one.

Remark 4.2. By Remark 3.2, the substitution $z_i = \frac{5H}{\mu_i}$ is well-defined. Also, note that the sum in the definition of $J \Omega^\infty_{g,\mu}$ has only finitely many nonzero terms.

Remark 4.3. The formal definition of $\Omega^\infty_{g,\mu}$ is in Section 4.3.1.

Theorem 4.4.
\[
\Omega_{g,n}^c = \lim_{\lambda \to 0} \sum_{\Gamma \in G_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} \Omega^c_{g,\mu}^{\lambda}(\Gamma)(\text{Cont}_\Gamma)
\]
Definition 4.5. We define a $\lambda$-equivariant version of $\Omega_{g,n}^c$ by
\[
\Omega_{g,n}^{c,\lambda} = \sum_{\Gamma \in G_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} \Omega^c_{g,\mu}^{\lambda}(\Gamma)(\text{Cont}_\Gamma)
\]
so that $\Omega_{g,n}^c = \lim_{\lambda \to 0} \Omega_{g,n}^{c,\lambda}$.

The proof of this theorem will occupy Section 4.3.

4.3. Proof of Theorem 4.4.

4.3.1. Step 1. For elements $\gamma_1, \ldots, \gamma_n \in H$, we apply (10) to compute $\Omega_{g,n}^{c,\lambda}(\gamma_1, \ldots, \gamma_n)$. We note that the bivalent graphs $\Gamma$ that (10) sums over may not be stable graphs in $G_{g,n}$, but may include unstable vertices of genus zero, with at most two markings. The conditions from Section 2.3 imply that any such unstable vertex $v$ satisfies $v \in V_0(\Gamma)$.

Contracting the unstable vertices produces a stable graph $\Gamma'$ from $\Gamma$. We label each vertex in $\Gamma'$ according to whether the corresponding vertex in $\Gamma$ lies in $V_0(\Gamma)$ or $V_\infty(\Gamma)$. Note that the flags $f$ of $\Gamma'$ at vertices labeled by $\infty$ are in bijective correspondence to the edges of $\Gamma$ that are not connected to a genus zero vertex of valence one. Using the fiber class mapping $\delta : E(\Gamma) \to \mathbb{Z}_{\geq 1}$, we define the degree $\delta_f$. In this way, we
assign to each decorated bivalent graph, a decorated dual graph in $G_{g,n}^\infty$ with the special property that there is no edge connecting two vertices of label 0. We note that $S^{-1}$ is the generating series of unstable vertices of genus zero with one edge and one marking, $J$ is the generating series of unstable vertices with a single edge, and

$$V_t(z_1, z_2) := \sum_{\alpha} e^\alpha \otimes e^\alpha - z_1 - z_2$$

$$+ \sum_{\beta=0}^{\infty} \sum_{\alpha_1, \alpha_2} \frac{q^\beta}{\beta!} \int_{\Lambda_{g,2}(\mathbb{P}^4, \beta)} \left( \frac{\text{ev}_1^*(c_{\alpha_1}) - \text{ev}_1^*(c_{\alpha_2})}{-z_1 - \psi_1 - z_2 - \psi_2} \prod_{i=3}^{k+2} \text{ev}_i^*(t(\psi_i)) \right) e^{\alpha_1} \otimes e^{\alpha_2}$$

is the generating series of unstable vertices with two edges.

Reorganizing (10) according to the decorated dual graph, we see that

$$\gamma_i, \Omega^t_z \in \mathcal{G}_{g,n}^\infty$$

where the contribution $\text{Cont}_1^1$ is defined by the following:

1. at each flag $f = (v, l)$ with insertion $\gamma$, we place $\mathcal{R}^{-1}_v(z_f)\gamma$ where

$$\mathcal{R}^{-1}_v(z)\gamma := \begin{cases} \gamma & \text{if } v \text{ is labeled by } 0, \\ S^{-1}_t(z)\gamma & \text{if } v \text{ is labeled by } \infty \end{cases}$$

2. at each edge $e$ connecting two vertices $v_1$ and $v_2$, we place

$$\begin{cases} 0, & \text{if } v_1 \text{ and } v_2 \text{ are labeled by } 0, \\ \sum_{\alpha} \frac{e^\alpha}{z_1 + z_2}, & \text{if exactly one of } v_1 \text{ and } v_2 \text{ is labeled by } 0, \\ V_t(z_{f_1}, z_{f_2}), & \text{if } v_1 \text{ and } v_2 \text{ are labeled by } \infty \end{cases}$$

where $f_1 := (v_1, e), f_2 := (v_2, e)$, and

3. at each vertex $v$ with $m$ flags $f_1, \ldots, f_m$, we place the map

$$\gamma_1(z_{f_1}) \otimes \cdots \otimes \gamma_m(z_{f_m}) \mapsto \begin{cases} \Omega_{g,v}^t(\gamma_1(\psi_1), \ldots, \gamma_m(\psi_m)) & \text{if } v \text{ is labeled by } 0, \\ J\Omega_{g,v}^t(\gamma_1(\psi_1), \ldots, \gamma_m(\psi_m)) |_{z_i=5H/\mu_i} & \text{if } v \text{ is labeled by } \infty, \end{cases}$$

where we set

$$\Omega_{g,n}(\gamma_1, \ldots, \gamma_n)$$

$$= (I_0)^{2g-2+n} \sum_{\beta,k=0}^{\infty} \frac{q^\beta}{k!} \pi^{z_k} \left( \prod_{i=1}^{n} \text{ev}_i^*(\gamma_i(\psi_i)) \prod_{i=n+1}^{n+k} \text{ev}_i^*(t(\psi_i)) \right) \cap [\mathcal{M}_{g,n+k}(\mathbb{P}^4, \beta)]^{\text{vir}, \lambda},$$

where

$$\Omega_{g,\mu}^\infty(\gamma_1, \ldots, \gamma_m) = \sum_{\beta \geq 0} (-1)^{1-g+5\beta} q^\beta \left( C_{g,\beta,\mu} \prod_{e=1}^{\ell(\mu)} \frac{1}{\prod_{i=2}^{\ell(\mu)} \mu_i} \gamma_1 \cdots \gamma_m \right)^\lambda.$$


and where

\[ J_{g,\mu}^{\infty,c}(\gamma_1 \otimes \cdots \otimes \gamma_m) \]

\[ = \sum_{\mu'} \frac{1}{|\text{Aut}(\mu')|} \Omega_{g,\mu+\mu'}^{\infty,c}(\gamma_1 \otimes \cdots \otimes \gamma_m \otimes \prod_{i=1}^{\ell(\mu')} (\delta_{1\mu_{m+i}} z_{m+i} + J_t(-z_{m+i}))|_{z_{m+i}=5H/\mu'}). \]

Here, we set \( C_{g,\beta,\mu} = 0 \), when \( 2g - 2 + \ell(\mu) - 5\beta \neq |\mu| \). Note that \( \Omega_{g,\mu}^{\infty} \) has absorbed a factor of \( t - 5H \) from the unstable vertex contributions and the factor relating the ordinary and twisted pairing of \( \mathbb{P}^4 \).

4.3.2. Step 2. We simplify the expression from Step 1 slightly, using the following three facts from Gromov–Witten theory.

First, the dilaton equation implies (see [10, Corollary 1.5])

\[ \Omega_{g,n}^{\lambda,t}(\gamma_1, \ldots, \gamma_n) = I_0^{-(2g-2+n)} \Omega_{g,n}^{\lambda}(\gamma_1, \ldots, \gamma_n), \]

\[ S_t^{-1}(z) = S_\tau^{-1}(z). \]

Second, there is the identity

\[ V_t(z_1, z_2) = -\sum_\alpha S_t^{-1}(z_1) e_\alpha \otimes S_t^{-1}(z_2) e_\alpha = -\sum_\alpha S_\tau^{-1}(z_1) e_\alpha \otimes S_\tau^{-1}(z_2) e_\alpha. \]

Third, using the ancestor-descendent comparison, we may replace the insertion

\[ \frac{\gamma}{-z - \psi} \]

involving a descendent psi-class \( \psi \) by the insertion

\[ \frac{S_t^{-1}(z)\gamma}{-z - \psi} = \frac{S_\tau^{-1}(z)\gamma}{-z - \psi} \]

involving the corresponding ancestor psi-class \( \bar{\psi} \).

We therefore may redefine \( \text{Cont}_1^T \) as follows:

1. at each flag \( f = (v, l) \) with insertion \( \gamma \), we place \( M_v^{-1}(z_f)\gamma \).
2. at each edge \( e \) connecting two vertices \( v_1 \) and \( v_2 \), we place

\[ \frac{-M_{v_1}^{-1}(z_{f_1}) e_\alpha \otimes M_{v_2}^{-1}(z_{f_2}) e_\alpha}{z_{f_1} + z_{f_2}} \]

3. at each vertex \( v \) with \( m \) flags \( f_1, \ldots, f_m \), we place the map

\[ \gamma_1(z_{f_1}) \otimes \cdots \otimes \gamma_m(z_{f_m}) \mapsto \begin{cases} \Omega_{g(v),m}^{\lambda}(\gamma_1(\bar{\psi}_1), \ldots, \gamma_m(\bar{\psi}_m)) & \text{if } v \text{ is labeled by } 0 \\ I_0^{2g(v)-2+|\ell(\mu)|} J_{g(v),\mu}^{\infty}(\gamma_1(z_1), \ldots, \gamma_m(z_m)) & \text{if } v \text{ is labeled by } \infty. \end{cases} \]
4.3.3. Step 3. We finally apply the Givental–Teleman classification to the computation of \( \Omega^\lambda_{g(m)} \) at every vertex \( v \) of \( \Gamma \) which has label 0. For every such vertex \( v \), we will have a sum over dual graphs \( \Gamma_v \). Replacing \( v \) in \( \Gamma \) by \( \Gamma_v \) for all \( v \) results in a new dual graph \( \Gamma' \). We can extend the decoration of \( \Gamma \) to the one of \( \Gamma' \) by requiring that all vertices in \( \Gamma_v \) should be labeled 0. Therefore, \( \Gamma' \) is an element of \( G^\infty_{g,n} \) with no further condition on the labeling.

We can therefore write

\[
\Omega_{g,m}^c(\gamma_1, \ldots, \gamma_m) = \lim_{\lambda \to 0} \sum_{\Gamma' \in G^\infty_{g,n}} \frac{1}{|\text{Aut}(\Gamma')|} t_\Gamma(\text{Cont}_{\Gamma'}^2),
\]

where \( \text{Cont}_{\Gamma'}^2 \) is defined in the following way:

1. at each flag \( f = (v,l) \) with insertion \( \gamma \), we place \( R^{-1}_{v}(z_f)\gamma \).
2. at each edge \( e \) connecting two vertices \( v_1 \) and \( v_2 \), we place

\[
\sum_{\alpha} \delta_{v_1v_2} e_\alpha \otimes e^\alpha - R^{-1}_{v_1}(z_{f_1})e_\alpha \otimes R^{-1}_{v_2}(z_{f_2})e^\alpha
\]

which is \( z_{f_1} + z_{f_2} \).
3. at each vertex \( v \) with \( m \) flags \( f_1, \ldots, f_m \), we place the map

\[
\gamma_1(z_{f_1}) \otimes \cdots \otimes \gamma_m(z_{f_m}) \mapsto
\begin{cases} 
T^\lambda_{g(v),m}(\gamma_1(\bar{\psi}_1), \ldots, \gamma_m(\bar{\psi}_m)) & \text{if } v \text{ is labeled by } 0 \\
I^2_{g(v)-2+\ell(\mu)} J^\infty_{g(v),m}(\gamma_1(z_1), \ldots, \gamma_m(z_m)) & \text{if } v \text{ is labeled by } \infty.
\end{cases}
\]

This is almost the same as the definition of the generalized \( R \)-matrix action. The only difference lies in the definition of \( J^\infty_{\gamma,\mu} \).

To complete the proof of Theorem 4.4, it suffices to prove the following Lemma.

\textbf{Lemma 4.6.} Setting \( H_\mu = -5H/\mu \), we have

\[
J_t(H_\mu) = H_\mu
\]

for \( \mu \leq 5 \). More generally, for any positive integer \( \mu \), the power series \( J_t(H_\mu) \) is a polynomial of degree \( \lfloor (\mu - 1)/5 \rfloor \).

\textbf{Proof.} By the genus zero mirror theorem

\[
J_t(z) = z \sum_{\beta=0}^{\infty} q^{\beta} \frac{\prod_{i=1}^{5\beta}(5H + iz)}{\prod_{i=1}^{\beta}((H + iz)^5 - \lambda^5)}.
\]

Therefore, we have

\[
J_t(H_\delta) = H_\delta \sum_{\beta=0}^{\lfloor (\mu - 1)/5 \rfloor} q^{\beta} \frac{\prod_{i=1}^{5\beta}(5\mu - 5i)}{\prod_{i=1}^{\beta}((\mu - 5i)^5 - \mu^5)}.
\]

\( \square \)

\textbf{Corollary 4.7.} For any partition \( \mu \) of size \( m \) and any \( \gamma_1, \ldots, \gamma_m \), the class

\[
J^\infty_{g,\mu}(\gamma_1, \ldots, \gamma_m)
\]
is a polynomial of $q$ of degree $\lfloor (2g - 2 + \ell(\mu) - |\mu|)/5 \rfloor$, and of codimension
$$3g - 3 + \sum_i \deg(\gamma_i).$$

Proof. Under the mirror map, we can simplify
$$J^{\infty,e} g,\mu \gamma_1, \ldots, \gamma_m = (C_{g,\mu}(q), \gamma_1 \cdots \gamma_m)^{\lambda},$$
where
$$C_{g,\mu}(q) = \sum_{\beta \geq 0} q^\beta (-1)^{1-g+5\beta} \prod_{e=1}^{\ell(\mu)} \frac{1}{i H_{i_1 \cdots i}}$$
$$\sum_{\mu'} \frac{1}{|\text{Aut}(\mu')|} \prod_{e=1}^{\ell(\mu')} \frac{1}{i H_{i_1 \cdots i}} C_{g,\beta,\mu+\mu'} \prod_{i=1}^{\ell(\mu')} J_t(H_{i}).$$

Recall that $C_{g,\beta,\mu+\mu'}$ vanishes unless
$$5\beta = 2g - 2 + \ell(\mu) + \ell(\mu') - |\mu| - |\mu'|.$$
Therefore, the summands vanish unless
$$\beta \leq (2g - 2 + \ell(\mu) + \ell(\mu') - |\mu| - |\mu'|)/5 \leq (2g - 2 + \ell(\mu) - |\mu|)/5.$$

\qed

5. Graded finite generation and orbifold regularity

As we mentioned in the introduction, a fundamental structural prediction is the finite generation property of $F_g$ of Yamaguchi–Yau [34], which says that $F_g$ is a polynomial of five generators. This can be thought as a higher dimensional generalization of the statement that $F_g$ is a quasi-modular form for an elliptic curve. Because of this, the finite generation property is often loosely referred to as the “modularity” of the Gromov–Witten theory of Calabi–Yau manifolds. Since we only have to compute finitely many coefficients of a polynomial, this key property immediately reduces the infinite number of degreewise computations to a finite computation! By introducing a grading, we prove a stronger version of finite generation which implies orbifold regularity. The main theorem of this section is the following:

**Theorem 5.1.** The following “finite generation properties” hold for the extended quintic family

1. $\bar{\Omega}^{e,\lambda}_{g,n}(\phi_{a_1}, \ldots, \phi_{a_n}) \in \bigoplus_{k \geq 0} \lambda^k H^{3g-3+\sum_i a_i} Q \otimes R_{3g-3+\sum_i a_i}$
2. $\bar{F}_{g,n} = 5^{g-1}(L/I_0)^{2g-2} I_{1,1}^{n} (Q \frac{d Q}{d t})^{n} F_g \in R_{3g-3+n}$

In the non-equivariant limit $\lambda \to 0$, we recover Theorem 1.4.
5.1. **CohFT of the \(\lambda\)-twisted invariants.** Recall the \(\lambda\)-twisted invariants defined in Section 4.1 for the mirror shift

\[
\tau = \frac{I_1(q)}{I_0(q)} H.
\]

Note that the \(\lambda\)-twisted theory is semi-simple. We will work with five bases of the state space \(H^*(\mathbb{P}^4):\)

1. The flat basis \(\{H^i\}_{i=0,...,4}\)
2. The normalized flat basis \(\{\phi_i\}_{i=0,...,4}\) (see the introduction)
3. An alternative normalized flat basis \(\{\bar{\phi}_i\}_{i=0,...,4}\), where
   \[
   \bar{\phi}_k = \frac{I_0}{L} \phi_k = \frac{I_0 I_{1,1} \cdots I_{k,k}}{L^{k+1}} H^k
   \]
4. The canonical basis \(\{e_\alpha\}\) (of idempotents)
5. The normalized canonical basis \(\{\bar{e}_\alpha\}\), where
   \[
   \bar{e}_\alpha := \Delta_\alpha^{1/2} e_\alpha,
   \]
   and
   \[
   \Delta_\alpha := (e_\alpha, e_\alpha)^{-1} = (\zeta^\alpha \lambda)^3 \frac{I_0^2}{L^2}.
   \]

We now collect basic formulae for the \(\lambda\)-twisted invariants (see [21, Section 6], and also [20, Section 7]):

- **Canonical basis and normalized canonical basis**
  \[
  e_\alpha = \frac{1}{5} \sum_i (\zeta^\alpha \lambda)^{-i} \phi_i, \quad \bar{e}_\alpha = \frac{1}{5} \sum_i (\zeta^\alpha \lambda)^{-i+\frac{3}{2}} \bar{\phi}_i.
  \]

- **Canonical coordinates**
  \[
  u_\alpha = \zeta^\alpha \lambda \int_0^q (L(q) - 1) \frac{dq}{q}.
  \]

- **The pairing \((,)^\lambda\) in the bases \(\{\phi_i\}\) and \(\{\bar{\phi}_i\}\) is given by the matrices**
  \[
  \begin{pmatrix}
  I_1^{2,1} & 1 & 1 \\
  1 & 1 & \lambda^5 \\
  \end{pmatrix}, \quad \begin{pmatrix}
  5 & \\
  5 & 5 & \lambda^5 \\
  \end{pmatrix},
  \]
  respectively.

- **The matrix of quantum multiplication by \(\hat{\tau} = H + q \frac{d}{dq} (\frac{I_1}{I_0})\) in the flat basis \(\{H^i\}_{i=0}^4\) is**
  \[
  \hat{\tau}^* = A := \begin{pmatrix}
  I_{1,1} & I_{1,2} & I_{1,3} & I_{1,4} & I_{1,5} \\
  I_{2,1} & I_{2,2} & I_{2,3} & I_{2,4} & I_{2,5} \\
  I_{3,1} & I_{3,2} & I_{3,3} & I_{3,4} & I_{3,5} \\
  I_{4,1} & I_{4,2} & I_{4,3} & I_{4,4} & I_{4,5} \\
  I_{5,1} & I_{5,2} & I_{5,3} & I_{5,4} & I_{5,5} \lambda^5
  \end{pmatrix}^T.
  \]
• Topological field theory

\[ \omega_{g,n}(\epsilon_{\alpha_1}, \ldots, \epsilon_{\alpha_n}) = \sum_\alpha \Delta^g_{\alpha} \prod_i \delta_{\alpha_i} \]

\[ \omega_{g,n}(\phi_{\alpha_1}, \ldots, \phi_{\alpha_n}) = \sum_\alpha \zeta^\alpha \sum_i a_i \lambda^3 \delta^{3 \alpha} + \sum_i (I_0/L)^{2g-2}. \]

We will also consider the following dual bases:

1. The dual flat basis \( \{ \hat{\phi}^i \}_{i=0, \ldots, 4} \), where we note that

\[ H^{-1} = H^4 H^{-5} = H^4 \lambda^{-5} \]

2. The alternative dual normalized flat basis \( \{ \phi^i \}_{i=0, \ldots, 4} \), where

\[ \bar{\phi}^i = \frac{1}{5} \bar{\phi}_{3-i}, \]

and \( \bar{\phi}_{-1} = \lambda^{-5} \bar{\phi}_4 \).

Note that if \( k = a + 5b \in \mathbb{Z} \) for \( a \in \{0, 1, 2, 3, 4\} \), we have \( H^k = \lambda^b H^a \). We analogously define \( \phi_k = \lambda^b \phi_a \) and \( \bar{\phi}_k = \lambda^b \bar{\phi}_a \). This is consistent with the above definition of \( \bar{\phi}_{-1} \).

5.2. Differential equations for S-matrix and R-matrix. Recall the S-matrix is the solution of the quantum differential equation (QDE) \[13, 25\]

\[ z \, dS_t(z) = dt \ast S_t(z). \]

By using the divisor equation, the QDE at \( t = \tau := I_1/I_0 \) is equivalent to the following

\[ zDS_{\tau}(z) + S_{\tau}(z) \cdot H = A \cdot S_{\tau}(z), \]

where \( A = \tau \ast \) is the quantum product matrix \[13\]. We will omit the subscript \( \tau \) in \( S_{\tau} \) in the rest of the paper.

**Lemma 5.2.** We consider the entries of the S-matrix and R-matrix

\[ S_{i\alpha}(z) := (\bar{\epsilon}_\alpha, S^*(z) H^i)^{\lambda}, \quad R_{i\alpha}(z) := (\bar{\epsilon}_\alpha, R^*(z) H^i)^{\lambda}. \]

The entries satisfy the following quantum differential equation

\[ \left( z \frac{d}{dq} + h_\alpha \right) S_{i\alpha}(z) = \sum_j A^i_j \cdot S_{j\alpha}(z) \]

\[ \left( z \frac{d}{dq} + L_\alpha \right) R_{i\alpha}(z) = \sum_j A^i_j \cdot R_{j\alpha}(z) \]

where \( h_\alpha = \zeta^\alpha \lambda, \quad L_\alpha = \zeta^\alpha \lambda L \) and \( A^i_j = \frac{1}{5} (A H^i, H^{3-j})^{\lambda} \) are the entries of the matrix \( A \) \[13\].

**Proof.** The equation for the \( S_{i\alpha} \) is just the component form of the equation \[14\].

By the following Birkhoff factorization

\[ S_{i\alpha}(z) C_{\alpha}(z)^{-1} = e^{a_\alpha / z} R_{i\alpha}(z) \]

and \[12\], the equation for the R-matrix follows. Here by the results of \[16, 12\]

\[ C_{\alpha}(z) = e^{\sum_{k>0, \beta \neq \alpha} \frac{B_k}{2k(2k-1)} \frac{z^{2k-1}}{(\zeta^\alpha \lambda - \zeta^\beta)^{2k-1}}} \cdot e^{\sum_{k>0} \frac{B_k}{2k(2k-1)} \frac{z^{2k-1}}{(-5\zeta^\alpha)^{2k-1}}} \]

\[ \Box \]
We denote by $\tilde{R}^*$ the matrix representation of $R^*$ under the normalized flat basis $\{\tilde{\phi}_i\}$, in other words

$$\tilde{R}_i^j(z) := (\tilde{\phi}_j, R^*(z)\tilde{\phi}_i)^\lambda,$$

and we also define a $\tilde{S}^*(z)$ matrix by

$$\tilde{S}_i^j(z) := \frac{1}{5}(H^{3-j}, S^*(z)\tilde{\phi}_i)^\lambda, \quad (\tilde{S}_i^j) := L^\delta \cdot \frac{1}{5}(H^{3-j}, S^*(H_\delta)\tilde{\phi}_i)^\lambda,$$

where $H_\delta = -5H/\delta$.

**Lemma 5.3.** We have the following properties

(a) \( (zD_C + \Lambda_{-1}) \tilde{R}^*(z) = \tilde{R}^*(z)\Lambda_{-1} \) \( (8) \)

(b) \( [z^k] \tilde{R}_i^j(z) = 0 \) if \( k - i + j \notin 5\mathbb{Z} \) \( (20) \)

(c) \( (\tilde{S}_i^j) = 0 \) if \( i \neq j \) \( (21) \)

where $\Lambda_{-1}$ is defined by $\Lambda_{-1} \cdot \tilde{\phi}_i := \tilde{\phi}_{i+1}$, $[z^k]f(z)$ is the coefficient of $z^k$ in a formal series $f(z)$ of $z$, and for any $M$

$$D_CM := \partial_u M - M \cdot C, \quad C = \text{diag}(\mathcal{X}, \mathcal{Y}, -\mathcal{Y}, -\mathcal{X}, 0)$$

**Proof.** Note that we have

$$\frac{1}{5}H^{3-j} = \frac{1}{5} \sum_\alpha (\zeta^\alpha \lambda)^{\frac{1}{5}-j} \hat{e}_\alpha |_{q=0}, \quad \tilde{\phi}_j = \frac{1}{5} \sum_\alpha (\zeta^\alpha \lambda)^{\frac{1}{5}-j} \hat{e}_\alpha,$$

Letting $S^j_i(z) := \frac{1}{5}(H^{3-j}, S^*(z)H^i)^\lambda$ and $R^j_i(z) := (\tilde{\phi}_j, R^*(z)H^i)^\lambda$, we have

\[
z \frac{1}{L} \frac{d}{dq} S^j_i(z) + \frac{1}{L} S^{j-1}_i(z) = \frac{I_{i+1,i+1}}{L} S^j_{i+1}(z),
\]

\[
z \frac{1}{L} \frac{d}{dq} R^j_i(z) + \frac{1}{L} R^{j-1}_i(z) = \frac{I_{i+1,i+1}}{L} R^j_{i+1}(z).
\]

By changing the basis on the right hand side to the alternative normalized flat basis as well, we arrive at

\[
z(\partial_u - \partial_u \log \frac{I_{0,i+1}}{L_{i+1}}) \tilde{S}^j_i(z) + L^{-1} \tilde{S}^{j-1}_i(z) = \tilde{S}^j_{i+1}(z)
\]

\[
z(\partial_u - \partial_u \log \frac{I_{0,i+1}}{L_{i+1}}) \tilde{R}^j_i(z) + \tilde{R}^{j-1}_i(z) = \tilde{R}^j_{i+1}(z)
\]

The second equality gives (18). The first equality gives

$$zD_C \tilde{S}^*(z) + H \cdot L^{-1} \tilde{S}^*(z) = \tilde{S}^*(z)\Lambda_{-1}.$$  \( (19) \)

By setting $z = H_\delta = -5H/\delta$ and by using $\partial_u \log L + (5L)^{-1} = Z$ we obtain (19).

For (b), the following $i = 0$ case will be proved in Theorem 7.4

\[
[z^k] \tilde{R}^j_i(z) = 0 \quad \text{if} \quad k + j \notin 5\mathbb{Z}.
\]

Then by using Part (a) we can deduce (b) for any $i > 0$ inductively.

We apply Lemma 4.6 to compute the entries $(S_\delta)_i^j$. We note that by the genus zero mirror theorem

$$J_i(H_\delta) = H_\delta \frac{S^*(H_\delta)}{I_0} = H_\delta \frac{S^*(H_\delta)\tilde{\phi}_0}{L}.$$  \( (22) \)
So by Lemma 4.6, \((\bar{S}_j)_{0} = 0\), unless \(j = 0\). As for \(\bar{R}\), we can now deduce Part (b) from Part (a) inductively.

### 5.3. Finite generation for \(S\)-matrix and \(R\)-matrix

As we will see soon, Lemma 5.3 gives us a canonical way to write each entry of the \(R\)-matrix as an element in the ring of six generators. We first note the following:

**Lemma 5.4.** There is a derivation \(\partial_a\) on the ring \(R\) of six generators compatible with the derivation \(\frac{d}{dq}\). Explicitly:

\[
\begin{align*}
\partial_a x_1 &= x_2, \\
\partial_a x_2 &= x_3, \\
\partial_a L^{-a}Z^b &= \frac{1}{5}L^{-a}Z^b(5b-a)L^{-1}(Z-1), \\
\partial_a y_1 &= -3x_2 - 3x_2 - \frac{3}{5}L^{-2}Z(Z-1) \\
\partial_a x_3 &= -4x_3 - 6x_2 x_2 - x_1^3 - \frac{3}{5}L^{-2}Z(Z-1)(x_1^2 + x_2) \\
&\quad - \frac{3}{25}L^{-3}Z(Z-1)(8Z-3)x_1 + \frac{1}{625}L^{-4} \cdot Z(-396Z^3 + 714Z^2 - 341Z + 23)
\end{align*}
\]

**Lemma 5.5.** Let \(\bar{R}_k := [z^k] \bar{R}(z)\). We have \(R_0 = \text{Id}\). For each \(k > 0\), all entries of the \(\bar{R}_k\)-matrix can be canonically expressed as an element of the degree \(k\) subspace \(R_k\) of the ring \(R\):

\[
[z^k] \bar{R}_i(z) \in \lambda_i^{k-j} R_k
\]

**Proof.** The statement for \([z^k] \bar{R}_0(z)\) follows from Lemma 7.4 (1) and Lemma 38 where we regard \(Z = L^5\) such that

\[
[z^k] \bar{R}_0(z) \in \lambda^{k-j} L^{-k} Q[Z]_{k/5 \leq d \leq k} \subset R_k
\]

The general case of the lemma follows then from Lemma 5.3 inductively, where we use Lemma 5.4 to canonically define \(\partial_a\). □

We move on to proving finite generation results for the specialized \(S\)-matrices.

**Lemma 5.6.** Following [35], let

\[
\tilde{I}(q, z) = z \sum_{d \geq 0} q^d \frac{\prod_{k=0}^{5d-1} (5H + k z)}{\prod_{k=1}^{d} ((H + k z)^5 - \lambda^5)}.
\]

Then, we have

\[
H^4 \cdot \tilde{I}(z) = I_0 \cdot S^*(z) \tilde{\phi}_4, \quad I(z) = \left(H + z q \frac{d}{dq}\right) \tilde{I}(z).
\]

**Proof.** This follows from [35 Section 2]. □

**Definition 5.7.** For any \(A \subset Q[L^{-1}, L, x_1, x_2, x_3, y]\), we introduce

\[A_{\text{reg}} := A \cap Q[L, x_1, x_2, x_3, y].\]

We also introduce

\[
\bar{R} := R[L^{-1}], \quad \bar{R}_k := \{f \in R[L^{-1}] \subset \bar{R} : \deg f = k\}
\]
Remark 5.8. Clearly, \( R = \overline{R} \cap \tilde{R}_{\text{reg}} \).

Lemma 5.9. Recall \( H_\delta := -\frac{5H}{2} \) and let (see (21))
\[
S_{\delta;j} := (S_\delta)^j = L^\delta \cdot \frac{I_0 \cdots I_{3-j}}{5L^{j+1}} (H^{3-j}, S^*(H_\delta)H^j)^\lambda,
\]
then by regarding \( Z = L^5 \) we have for \( i = 0, 1, 2, 3, 4, \)
\[
S_{\delta;i} \in (L^{\delta-i} \overline{R})_{\text{reg}} \otimes Q[Z]_{\leq \lfloor \frac{d-1}{5} \rfloor}
\]
where \( \overline{i} = i - 5\lfloor i/5 \rfloor \) denotes the remainder of \( i \) under division by 5, and \( Q[Z]_{\leq d} \) is the set of degree \( \leq d \) polynomials in \( Z \).

Proof. Indeed we have the following stronger result:
\[
\begin{align*}
S_{\delta;4} & \in L^\delta \cdot Q[Z]_{\leq \lfloor \frac{d}{5} \rfloor}, & S_{\delta;0} & \in \overline{L}^{\delta-i} \cdot Q[Z]_{\leq \lfloor \frac{d-1}{5} \rfloor}, \\
S_{\delta;1} & \in (L^{\delta-i} Q[L^{-1}, X, Z_1])_{\text{reg}} \otimes Q[Z]_{\leq \lfloor \frac{d-1}{5} \rfloor}, \\
S_{\delta;2} & \in (L^{\delta-i} Q[L^{-1}, X_2, Y, Z_1, Z_2])_{\text{reg}} \otimes Q[Z]_{\leq \lfloor \frac{d-1}{5} \rfloor}, \\
S_{\delta;3} & \in (L^{\delta-i} Q[L^{-1}, X_2, X_3, Z_1, Z_2, Z_3])_{\text{reg}} \otimes Q[Z]_{\leq \lfloor \frac{d-1}{5} \rfloor}.
\end{align*}
\]
The statement for \( S_{\delta;4} \) and \( S_{\delta;0} \) follows from the oscillatory integral analysis in Section 7 and from (21) and (22), respectively. The other properties follow from a direct computation via (19).

The fact that the function \( S_{\delta;3} \) does not involve \( Y \) is not important here. It will follow from the first equation in Example 6.6 (Note although stated for \( R \) there, the PDE holds also for \( S_\delta \)).

\[ \square \]

5.4. Proof of Theorem 5.1. We carefully look at the generalized \( R \)-matrix action from Section 4.2. We first consider the contribution \( \mathcal{F}_{f_1, f_2} \) of an edge \( e \) connecting \( v_1 \) and \( v_2 \) with corresponding flags \( f_1, f_2 \). We distinguish three cases:

- If both \( v_1 \) and \( v_2 \) are labeled by 0, the contribution is of the form
\[
\frac{1}{5} \sum_{k,l \geq 0} \sum_{a+b=k+l+1, b > l, i \in \mathbb{Z}^5} (-1)^{a+l} \tilde{R}_{a;3-i} \tilde{R}_{b;i} \tilde{\phi}_{3-i-a}^k \otimes \tilde{\phi}_{l-b}^j \phi_{f_1}^k \otimes \phi_{f_2}^j,
\]
\[
eq \frac{L_2^2}{L_2} \bigoplus_{k \geq 0} \left( R_{k+1} \otimes (H^{\otimes 2}[\lambda^{-5}])_{3-k-1} \otimes H^k(M_{g_{v_1}, n_{v_1}} \times M_{g_{v_2}, n_{v_2}}) \right),
\]
where \( \tilde{R}_{a;i} = [z^a] \tilde{R}_{i-a}^a \), and where \( (H^{\otimes 2}[\lambda^{-5}])_k \) denotes the subspace of elements of cohomological degree \( k \). The fact that only powers of \( \lambda \) divisible by 5 appear is due to the fact that the classes under consideration are symmetric in the equivariant parameters.
The contribution of a half-edge \( e \) is compatible with Theorem 5.1. We may therefore split the contribution of each edge into powers of \( \bar{\psi} \). Note that it suffices to only consider edge contributions not involving \( \bar{\psi} \) classes because each power of \( \bar{\psi} \) increases both cohomological degree and degree in \( R \), by one, which is compatible with Theorem 5.3. We may therefore split the contribution of each edge \( e = (v_1, v_2) \) into two half-edge contributions according to a choice

\[
\text{deg } \gamma_{(e,v_1)} + \text{deg } \gamma_{(e,v_2)} = 2, \quad k_{(e,v_1)} + k_{(e,v_2)} = 3.
\]

The contribution of a half-edge \( h = (e,v) \) with \( v \) of label 0 lies in

\[
\frac{I_0}{L} R_{k_h - \text{deg } \gamma_h} \otimes (H[\lambda^{-5}])_{\text{deg } \gamma_h},
\]

while the contribution of a half-edge \( h = (e,v) \) with \( v \) of label \( \infty \) lies in

\[
L^{-\delta_h} H^{\text{deg } \gamma_h} (L^{\delta_h - 5} R_{k_h})^\text{reg} \otimes Q[Z]_{\leq \lfloor (\delta_h - 1)/5 \rfloor}.
\]

We next consider the contribution of a leg \( l \) with insertion \( \phi_i \) at a vertex \( v \):• If \( v \) has label 0, the contribution is

\[
\sum_{k \geq 0} R_{k_i} \phi_{i - k} \bar{\psi}^k_l \in \sum_{k \geq 0} R_{k} \phi_{i - k} \bar{\psi}^k_l.
\]

• If \( v \) has label \( \infty \) and \( l \) has degree \( \delta \), the contribution is

\[
\frac{L}{I_0} L^{-\delta} (S_{\delta})^i_l H^i \in \sum_{k \geq 0} \lambda^5 R_{3g(v) - 3 + \sum_i a_i - 5k} \otimes H^{3g(v) - 3 + \sum_i a_i - 5k}(\bar{M}_{g(v),m}).
\]

We now consider the vertex contributions:

• The contribution at each vertex of label 0 with \( m \) half-edges is

\[
T_{\omega_g(v), m} \left( \bigotimes_{i=1}^m \phi_{a_i} \right) \in \left( \frac{I_0}{L} \right)^{2g(v) - 2} \bigoplus_{k \in Z} \lambda^5 R_{3g(v) - 3 + \sum_i a_i - 5k} \otimes H^{3g(v) - 3 + \sum_i a_i - 5k}(\bar{M}_{g(v),m}).
\]
This is because the left hand side is a sum of terms of the form
\[
(p_k)_{t=1}^{m} \phi_{a_t} \otimes (R_{l_j})_{t=1}^{k} \phi_{b_j} \phi_{m+j}
\]
\[
\otimes (R_{l_j})_{t=1}^{k} \phi_{b_j} \phi_{m+j+1}
\]
where \(3g(v) - 3 + \sum a_i - \sum b_j \) is divisible by 5 because both \(b_j + l_j\) and \(a_i + \sum b_j\) must be divisible by 5 in order for the term to be nonzero. After multiplying by \(1 = L^{-5k}Z^k \in \mathbb{R}_{5k}\), we may regard the degree in \(\mathbb{R}\) to be \(3g - 3 + \sum a_i\).

**•** The contribution of a vertex \(v\) of label \(\infty\) with \(m\) half-edges is
\[
J\Omega_{g,v}^{\infty,e} (H_{a_1}^{1}, \ldots, H_{a_m}^{m})
\]
\[
\in \left( \frac{I_0}{L} \right)^{2g(v) - 2 + m} L^{2g(v) - 2 + m} \bigoplus_{k \in \mathbb{Z}} \lambda^{5k} \mathbb{Q}[q]_{\leq d_v} \otimes H^{3g(v) - 3 + \sum a_i - 5k} \mathbb{Q}[M_{g,v}, m]
\]
where \(\nu_i = \mu_i - 1\) and \(d_v := \left\lfloor \frac{q \nu_i}{v} \right\rfloor\) with \(a_v := 2g(v) - 2 - |\nu|\).

We now consider the total contribution of a graph \(\Gamma\) to the computation of
\[
\mathcal{G}_{g,v}^{\infty,e}(\phi_{a_1}, \ldots, \phi_{a_n})
\]
We first consider the power of \(I_0/L\). We have a factor \((I_0/L)^{2g(v) - 2 + |E_v|}\) where \(E_v\) is the set of edges at \(v\) for every vertex \(v\). If \(v\) has label 0, this factor comes from both the vertex and the edges. If \(v\) has label \(\infty\), this factor comes from both the vertex and the legs. Combining these factors for all vertices \(v\) gives the desired global factor \((I_0/L)^{2g(v) - 2}\).

The overall cohomological degree is
\[
|E(\Gamma)| + \sum_v (3g(v) - 3) + \sum a_i + 2|E(\Gamma)| = 3g - 3 + \sum a_i,
\]
as expected.

We next observe that the total contribution is regular in \(L\). This is because negative powers of \(L\) only appear in the half-edge factors, and the \(q\)-polynomial in the contribution of a vertex \(v\) with label \(\infty\). These negative powers are absorbed by the power of \(L\) in the vertex term:
\[
L^{-|\nu|} L^{2g(v) - 2 + m} \mathbb{Q}[q]_{\leq d_v} = L^{a_v} \mathbb{Q}[q]_{\leq d_v} = L^{a_v} \mathbb{Q}[Z]_{\leq d_v}
\]
Combining at every vertex \(v\) with label \(\infty\), the factors involving \(L\), \(q\) and \(Z\) gives an element of
\[
L^{-|\nu| + \sum \nu_i + 2g(v) - 2 + |E(\nu)|} \mathbb{Q}[Z]_{\leq |\nu_i/5|} \mathbb{Q}[q]_{\leq d_v}
\]
\[
= L^{-3g(v) - 3} Z^{-\sum \nu_i/5} Z^{g(v) - 1} \mathbb{Q}[Z]_{\leq |\nu_i/5|} \mathbb{Q}[q]_{\leq d_v}
\]
\[
\subset L^{-3g(v) - 3} \mathbb{Q}[Z, Z^{-1}]_{\leq g(v) - 1} \subset \mathbb{R}_{3g(v) - 3}
\]
We therefore obtain an element of \(\mathbb{R}\) of degree
\[
\sum_v (3g(v) - 3) + \sum a_i + 3|E(\Gamma)| = 3g - 3 + \sum a_i.
\]
This completes the proof of Theorem 5.1.
5.5. **Yamaguchi–Yau’s prediction.** In this subsection, we prove the second (numerical) part of Theorem 1.4 which implies Yamaguchi–Yau’s prediction.

For Calabi–Yau 3-folds, all nonzero-degree Gromov–Witten invariants vanish unless all insertions are divisor classes. Let

\[ \bar{F}_{g,n}^{c} := \int_{\mathcal{M}_{g,n}} \bar{\Omega}_{g,n}^{c}(\phi_{1}^{\otimes n}) = 5^{g-1} \frac{L_{2g-2}}{L_{2g}^{2g-2}} F_{g,n}^{c}(\phi_{1}^{\otimes n}). \]

Notice that by the divisor equation, we have

\[ \bar{F}_{g,n+1} = (\partial_n - n(\mathcal{Y} - \mathcal{X}) + (2g - 2)\mathcal{X}) \bar{F}_{g,n}. \quad (25) \]

**Corollary 5.10.** We have

\[ \bar{F}_{g,n} \in \mathbb{R}_{3g - 3 + n} \]

**Example 5.11.** We have the following initial data and low genus formulae

\[
\begin{align*}
\bar{F}_{0,3} &= 1 & \bar{F}_{0,4} &= \mathcal{X} - 3\mathcal{Y} & \bar{F}_{1,1} &= -\frac{59}{6} \mathcal{X} - \frac{1}{2} \mathcal{Y} - \frac{125}{24} \mathcal{Z} \\
\bar{F}_{1,2} &= -\frac{25}{3} \mathcal{X} + \frac{28}{3} \mathcal{Y} - \mathcal{X} + \mathcal{Y}^2 + \frac{125}{2} (\mathcal{Y} - \mathcal{X}) \mathcal{Z} - \frac{205}{24} \mathcal{Z}_2 \\
\bar{F}_{2,0} &= 5 \cdot \left( \frac{70}{9} \mathcal{X} + \frac{575}{18} \mathcal{X}^2 + \frac{5}{6} \mathcal{Y} \mathcal{X} + \frac{557}{72} \mathcal{X}^3 - \frac{629}{72} \mathcal{Y} \mathcal{X}^2 - \frac{23}{24} \mathcal{Y}^2 \mathcal{X} - \frac{\mathcal{Y}^3}{24} \right) \\
&\quad + \frac{625}{36} \mathcal{Z} \mathcal{X} - \frac{175}{48} \mathcal{Z} \mathcal{Y} \mathcal{X} + \frac{1441}{144} \mathcal{Z} \mathcal{X}^2 - \frac{25}{24} \mathcal{Z} (\mathcal{X}^2 + \mathcal{Y}^2) \\
&\quad - \frac{3125}{288} \mathcal{Z}^2 (\mathcal{X} + \mathcal{Y}) + \frac{41}{48} \mathcal{Z} \mathcal{Y} + \frac{625}{144} \mathcal{Z}^3 - \frac{2233}{128} \mathcal{Z} \mathcal{Z}_2 + \frac{547}{72} \mathcal{Z}_3 \right)
\end{align*}
\]

where we have used the genus one [36] and genus two mirror theorems [19].

5.6. **Grading and orbifold regularity.** In this section, we will explain how the “orbifold regularity” follows from the “graded finite generation property”.

The original “orbifold regularity” is a global property for the Gromov–Witten potential. In the paper [1], the authors constructed a non-holomorphic completion \( \mathcal{F}_g(q, \bar{q}) \) of the Gromov–Witten potential \( F_g(q) \) for each \( g \), such that

\[ \lim_{\bar{q} \to 0} \mathcal{F}_g(q, \bar{q}) = F_g(Q(q)). \]

Further, they conjecture that there exist an “LG” theory with the potential function \( F_{g}^{\text{orb}}(\tau^{\text{orb}}) \) such that

\[ \lim_{\bar{q} \to \infty} \mathcal{F}_g(q, \bar{q}) = F_{g}^{\text{orb}}(\tau^{\text{orb}}) \bigg|_{\tau^{\text{orb}} = \tau^{\text{orb}}(q - 1/5)}. \]

Nowadays it is known that such an LG theory is the FJRW theory. Then the “orbifold regularity” in physics can be translated as follows:

**Theorem 5.12.** Let

\[ \frac{L_{2g-2}}{L_{2g}^{2g-2}} F_g = P_g(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}, L^{-1}, \mathcal{Z}) \]

be the polynomial of the six generators. Then the limit

\[ \lim_{L \to 0} P_g(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}, L^{-1}, L^5) \]
exists.

Proof. The theorem follows from the degree bounds in the definition of \(R\) (see Equation (2)). \(\square\)

6. Holomorphic anomaly equations (HAE)

6.1. Derivations acting on the \(R\)-matrix. With the finite generation property for the \(R\)-matrix (see Lemma [5.5]), we can consider vector fields in \(\mathfrak{D}_R\) acting on the \(R\)-matrix. The key to the proof of the holomorphic anomaly equations of Theorem [1.11] are the following differential equations satisfied by the generalized \(Q\)-operator:

\[
\Theta R^{-1}(z) = R^{-1}(z) \cdot \Lambda_{\Theta}(z) \quad \text{and} \quad \Theta S_\delta = S_\delta \cdot \Lambda_{\Theta}(-H_\delta),
\]

where \(R^{-1}(z) = R^*(-z)\) (see Section [5.2]), and where \(H_\delta = -5H/\delta\). Furthermore, for any \(\Theta, \Theta_1\) and \(\Theta_2\), we have the following properties:

1. \(\Lambda_{\Theta}\) is strictly lower triangular: \(\Lambda_{\Theta} \bar{\phi}_j \in \text{span}_Q\{\bar{\phi}_0, \ldots, \bar{\phi}_{j-1}\}[z]\);
2. \(\Lambda_{\Theta}\) is skew adjoint: \(\Lambda_{\Theta} + \Lambda_{\Theta}^*|_{z \mapsto -z} = 0\); and
3. \(\Lambda_{[\Theta_1, \Theta_2]} = [\Lambda_{\Theta_1}, \Lambda_{\Theta_2}] + \Theta_1 \Lambda_{\Theta_2} - \Theta_2 \Lambda_{\Theta_1}\).

Remark 6.2. The proof of the above Lemma will give an algorithm for computing the map \(\Lambda\) via the QDE. Furthermore, it also works for the fully equivariant (without specialization) twisted theory. In a sequel [18], we will use it in the proof of the LG/CY correspondence.

Remark 6.3. The map can be extended to the ring \(\mathfrak{D}_R\), if we enlarge the codomain of the map and if we only consider the \(R\)-matrix. To be precise, there exist a map

\[
\Lambda: \mathfrak{D}_R \to \text{End}H \otimes R[[z]]
\]

such that the first equality in (27) holds.

Corollary 6.4. There exists a map \(\Delta: \mathfrak{D}_R \to H[z_1] \otimes H[z_2]\) such that for any \(\Theta \in \mathfrak{D}_R\):

\[
\Theta \left[ \sum_\alpha e_\alpha \otimes e_\alpha - R^{-1}(z)e_\alpha \otimes R^{-1}(z)e_\alpha \right]_{z_1 + z_2}^{z_1 - H_\delta_2} = -(\bar{R}^{-1}(z_1) \otimes \bar{R}^{-1}(z_2)) \Delta_{\Theta}(z_1, z_2),
\]

\[
\Theta \left[ \sum_\alpha \bar{R}^{-1}(z_1)e_\alpha \otimes \bar{S}_{\delta_2}e_\alpha \right]_{z_1 - H_\delta_2}^{z_1} = -(\bar{R}^{-1}(z_1) \otimes \bar{S}_{\delta_2}) \Delta_{\Theta}(z_1, -H_\delta_2),
\]

\[
\Theta \left[ \sum_\alpha -\bar{S}_{\delta_1}e_\alpha \otimes \bar{S}_{\delta_2}e_\alpha \right]_{-H_\delta_1 - H_\delta_2}^{-H_\delta_1} = -(\bar{S}_{\delta_1} \otimes \bar{S}_{\delta_2}) \Delta_{\Theta}(-H_\delta_1, -H_\delta_2).
\]

Proof. We can define the map as follows:

\[
\Delta_{\Theta}(z_1, z_2) := \sum_\alpha \Lambda_{\Theta}(z_1)e_\alpha \otimes e_\alpha + e_\alpha \otimes \Lambda_{\Theta}(z_2)e_\alpha
\]

Note that the skew-adjoint property makes (29) a polynomial of \(z_1\) and \(z_2\). \(\square\)

\textit{We sometimes abuse notation and will identify End}_H with \text{End span}_Q\{\bar{\phi}_0, \ldots, \bar{\phi}_4\}.
Proof of Lemma 6.1. The differential operator $\partial_u = \frac{d}{du}$ can be considered as the following vector field

$$D_u = \sum_{G \in \{X_1, X_2, X_3, Y, L\}} (\partial_u G) \cdot \partial_G = \mathcal{Y}_2 \partial_y + \sum_{i=1}^{3} x_i \partial x_i + \frac{1}{3} (L^5 - 1) \partial L \in \mathcal{D}_R,$$

where $\mathcal{Y}_2, x_i \in \mathbb{R}$ are given by

$$\mathcal{Y}_2 = -3x_2 - y^2 - x^2 - \frac{15}{4} z_2,$$

$$x_i = -4x_1 x_2 - 3x_2^2 - 6x_2^2 x_2 - x^4 - \frac{15}{4} (z_2 x^2 + z_2 x_2 + z_3 x) - \frac{25}{8} z_4 + \frac{25}{9} z_2^2 z_2 - \frac{65}{72} z_1 z_3 - \frac{3}{4} z_2^4.$$

We obtain for $\Theta \in \mathcal{D}_R$

$$[\Theta, \partial_u] = [\Theta, \mathcal{Y}_2 \partial y + x_2 \partial x_1 + x_3 \partial x_2 + x_4 \partial x_3] \in \mathcal{D}_R^*.$$

Now we inductively construct the map $\Lambda$ and prove that it satisfies Property (1). First, by the first statement in Lemma 7.2, we have

$$\Theta \bar{R}^{-1}(z) \bar{\phi}_0 = 0,$$

so that we can define $\Lambda_\Theta \bar{\phi}_0 = 0$.

Assume that for any $k \leq i$, we have defined $\Lambda_\Theta$ such that $\Theta \bar{R}^{-1}(z) \bar{\phi}_k = \bar{R}^{-1}(z) \Lambda_\Theta \bar{\phi}_k$ and that Property (1) holds. By using the ODE (18) for $R^*(z) = R^{-1}(-z)$, we first get for $k < i$:

$$\left(-z \partial_z + \Lambda_{-1}\right) \bar{R}^{-1}(z) \bar{\phi}_k = \bar{R}^{-1}(z) \Lambda_{-1} \bar{\phi}_k$$

$$\left(H_{\delta} \partial_z + 5H \bar{z}\right) \bar{S}_\delta \bar{\phi}_k = \bar{S}_\delta \Lambda_{-1} \bar{\phi}_k$$

The next equation gives

$$\Theta \bar{R}^{-1}(z) \bar{\phi}_{i+1} = -z \Theta \left[D_{C} \bar{R}^{-1}(z) \bar{\phi}_i\right] + \Lambda_{-1} \Theta \left[\bar{R}^{-1}(z) \bar{\phi}_i\right]$$

$$\left(-z \partial_z + \Lambda_{-1}\right) \bar{R}^{-1}(z) \bar{\phi}_i + z \bar{R}^{-1}(z) \left(\Theta C + [\Lambda_\Theta, C]\right) \bar{\phi}_i$$

$$+ \left(-z \partial_z + \Lambda_{-1}\right) \bar{R}^{-1}(z) \Lambda_{\Theta} \bar{\phi}_i$$

$$= \bar{R}^{-1}(z) \left[z(-\Lambda_{[\Theta, \partial_z]} + \Theta C + [\Lambda_\Theta, C]) + (-z \partial_z + \Lambda_{-1}) \Lambda_{\Theta}\right] \bar{\phi}_i$$

$$\Theta \bar{S}_\delta \bar{\phi}_{i+1} = H_{\delta} \Theta \left[D_{C} \bar{S}_\delta \bar{\phi}_i\right] + 5 \bar{z} \cdot H \Theta [\bar{S}_\delta \bar{\phi}_i]$$

$$= H_{\delta}\left([\Theta, \partial_z]\right) [\bar{S}_\delta \bar{\phi}_i] - H_{\delta} \bar{S}_\delta \left(\Theta C + [\Lambda_\Theta, C]\right) \bar{\phi}_i$$

$$+ \left(H_{\delta} \partial_z + 5 \bar{z} \cdot H\right) \bar{S}_\delta \Lambda_{\Theta} \bar{\phi}_i$$

$$= \bar{S}_\delta \left[z\left(-\Lambda_{[\Theta, \partial_z]} + \Theta C + [\Lambda_\Theta, C]\right) + (-z \partial_z + \Lambda_{-1}) \Lambda_{\Theta}\right] \bar{\phi}_i$$

where in the last equality we have again used the ODE (18) and the induction assumption. We define

$$\Lambda_\Theta(z) \bar{\phi}_{i+1} := [z(-\Lambda_{[\Theta, \partial_z]}(z) + \Theta C + [\Lambda_\Theta(z), C]) + (-z \partial_z + \Lambda_{-1}) \Lambda_{\Theta}(z)] \bar{\phi}_i$$

and hence finish the inductive definition.

Property (2) follows from the symplectic property of $R(z)$. Property (3) follows from a direct computation via (27).
6.2. Explicit formulae of PDEs for the $R$-matrix. In this section, we set $R' = \mathbb{Q}[X_1, X_2, X_3, Y]$. We introduce the following derivations, which generate $\mathcal{D}_{R'}$.\footnote{One can directly check that for $\partial_0$ and $\partial_1$, the definition matches with the one given in \cite{citation}.}

\begin{equation*}
\begin{aligned}
\partial_1 & := -\partial_Y + \partial_X - (X - Y) \partial_{X_2} - (2X Y + 4X_2 + \frac{15}{4} Z_2) \partial_{X_3} \\
\partial_2 & := \partial_{X_2} - 2X \partial_{X_3}, \quad \partial_3 := \partial_{X_3}, \quad \partial_6 := \partial_Y
\end{aligned}
\end{equation*}

By a direct calculation via the algorithm in the proof of Lemma \ref{lemma}, we have

**Lemma 6.5.** For $\Theta = g_0 \partial_0 + g_1 \partial_1 + g_2 \partial_2 + g_3 \partial_3 \in \mathcal{D}_{R'}$ the maps $\Lambda$ and $\Delta$ are given by

\begin{equation*}
\begin{aligned}
\Lambda_{\Theta} &= (g_0 - g_1) \Lambda_0 \psi + g_1 \Lambda_1 \psi - g_2 \Lambda_2 \psi^2 + g_3 \Lambda_3 \psi^3 \in \text{End} \mathbb{H}[[\psi]], \\
\Delta_{\Theta} &= (g_0 - g_1) \frac{1}{5} \bar{\phi}_1 \otimes \bar{\phi}_1 + g_1 \Delta_2 + g_2 \Delta_1 + g_3 \Delta_0 \in \mathbb{H}[[\psi_1]] \otimes \mathbb{H}[[\psi_2]],
\end{aligned}
\end{equation*}

where we define the following lower triangular maps (where we, by abuse of notation, set $\bar{\phi}_k := 0$ for $k < 0$)

\begin{equation*}
\forall j, \quad \Lambda_0(\bar{\phi}_j) := \delta_{j,2} \bar{\phi}_1, \quad \Lambda_1(\bar{\phi}_j) := \bar{\phi}_{j-1}, \quad \Lambda_2(\bar{\phi}_j) := (-1)^j \bar{\phi}_{j-2}, \quad \Lambda_3(\bar{\phi}_j) := \bar{\phi}_{j-3};
\end{equation*}

and we define certain “diagonal” classes in $\mathbb{H}[z_1] \otimes \mathbb{H}[z_2]$:

\begin{equation*}
\begin{aligned}
\Delta_0 & := \frac{1}{5} \left( z_1^2 \bar{\phi}_0 \otimes \bar{\phi}_0 - z_1 \bar{\phi}_0 \otimes z_2 \bar{\phi}_0 + \bar{\phi}_0 \otimes z_2^2 \bar{\phi}_0 \right) \\
\Delta_1 & := -\frac{1}{5} \left( z_1 - z_2 \right) \left( 1 \otimes \bar{\phi}_1 - \bar{\phi}_1 \otimes 1 \right), \\
\Delta_2 & := \frac{1}{5} \left( \bar{\phi}_0 \otimes \bar{\phi}_2 + \bar{\phi}_1 \otimes \bar{\phi}_1 + \bar{\phi}_2 \otimes \bar{\phi}_0 \right)
\end{aligned}
\end{equation*}

**Example 6.6.** We have the following explicit forms of \cite{citation}

\begin{equation*}
\begin{aligned}
\partial_Y R^{-1}(z) \bar{\phi}_j &= \delta_{j,2} \cdot z R^{-1}(z) \bar{\phi}_1 \\
\partial_X R^{-1}(z) \bar{\phi}_j &= z R^{-1}(z) \bar{\phi}_{j-1} + \delta_{j,2} \cdot (Y - X) z^2 R^{-1}(z) \bar{\phi}_0 \quad (\bar{\phi}_{-1} := 0) \\
&\quad + \delta_{j,3} \cdot \left[ -(Y - X) z^2 R^{-1}(z) \bar{\phi}_1 + (2X_2 + 4X_2 + \frac{15}{4} Z_2) z^3 R^{-1}(z) \bar{\phi}_0 \right] \\
\partial_{X_2} R^{-1}(z) \bar{\phi}_j &= -\delta_{j,2} \cdot z^2 R^{-1}(z) \bar{\phi}_0 + \delta_{j,3} \cdot (z^2 R^{-1}(z) \bar{\phi}_1 + 2X z^3 R^{-1}(z) \bar{\phi}_0) \\
\partial_{X_3} R^{-1}(z) \bar{\phi}_j &= \delta_{j,3} \cdot z^3 R^{-1}(z) \bar{\phi}_0
\end{aligned}
\end{equation*}

and we have the following explicit forms of \cite{citation}

\begin{equation*}
\begin{aligned}
-\partial_Y V(z_1, z_2) &= R^{-1}(z_1) \otimes R^{-1}(z_2) \left( \frac{1}{5} \bar{\phi}_1 \otimes \bar{\phi}_1 \right) \\
-\partial_Y V(z_1, z_2) &= R^{-1}(z_1) \otimes R^{-1}(z_2) \left( \Delta_2 + (Y - X) \Delta_1 + (2X_2 + 4X_2 + \frac{15}{4} Z_2) \Delta_0 \right) \\
-\partial_{X_2} V(z_1, z_2) &= R^{-1}(z_1) \otimes R^{-1}(z_2) \left( \Delta_1 + 2X \Delta_0 \right) \\
-\partial_{X_3} V(z_1, z_2) &= R^{-1}(z_1) \otimes R^{-1}(z_2) \left( \Delta_0 \right).
\end{aligned}
\end{equation*}
where $V(z_1, z_2) := \sum a_n e_\alpha \otimes e^\alpha - R^{-1}(z) e_\alpha \otimes R^{-1}(z) e^\alpha$. If we replace $R$ by $S$ and set $z = 5H/\delta$, the same equalities hold for the specialized $S$-matrix.

**Corollary 6.7.** For the operators $\partial_0, \partial_1 \in \mathcal{D}_R$ defined in Equation (5), under Yamaguchi-Yau’s generators, we simply have

$$\Delta_{\partial_0} = \frac{1}{5} \tilde{\phi}_1 \otimes \tilde{\phi}_1,$$

$$\Delta_{\partial_1} = \frac{1}{5} \tilde{\phi}_0 \otimes \tilde{\phi}_2 + \frac{1}{5} \tilde{\phi}_2 \otimes \tilde{\phi}_0.$$  

In particular, $\Delta_{\partial_0}$ and $\Delta_{\partial_1}$ do not depend on $z_1$ and $z_2$.

**Remark 6.8** (Yamaguchi–Yau’s generators). Indeed, by using the generators defined in [4] the QDE for $\mathcal{R}$ can be written in a much simpler form:

$$\bar{R}_{k+1;1} = (\partial_u - \mathcal{U}) \bar{R}_{k;0} + \bar{R}_{k+1;0}$$

$$\bar{R}_{k+1;2} = (\partial_u + \mathcal{U} - \mathcal{V}) \bar{R}_{k;1} + (\partial_u - \mathcal{U}) \bar{R}_{k;0} + \bar{R}_{k+1;0}$$

$$= (\partial_u^2 - \mathcal{V} \cdot \partial_u - \mathcal{V}) \bar{R}_{k-1;0} + (2\partial_u - \mathcal{V}) \bar{R}_{k;0} + \bar{R}_{k+1;0}$$

$$\bar{R}_{k+1;3} = \bar{R}_{k+1;2} + (\partial_u + \mathcal{U}) \bar{R}_{k;2}$$

$$\bar{R}_{k+1;4} = \bar{R}_{k+1;0} - \partial_u \bar{R}_{k;4}.$$  

Then by using the differential relations

$$\partial_u \mathcal{V} = -2\mathcal{V}_2 - \mathcal{V}^2 - \frac{15}{4} \mathcal{Z}_2, \quad \partial_u \mathcal{V}_2 = \mathcal{V}_3 - \mathcal{V} \mathcal{V}_2,$$

$$\partial_u \mathcal{V}_3 = \mathcal{V}_2^2 - \frac{23}{24} \mathcal{Z}_1 + \frac{29}{9} \mathcal{Z}^2 \mathcal{Z}_2 - \frac{65}{72} \mathcal{Z} \mathcal{Z}_3 - \frac{3}{4} \mathcal{Z}_2^2,$$

the PDE for $R$ with respect to the differential operator $\partial_1 = \partial_\mathcal{U}$ can be written in a much simpler way.

### 6.3. Proof of the holomorphic anomaly equations

In the following section, we prove the holomorphic anomaly equations of Theorem 1.11. They will be a direct consequence of:

**Theorem 6.9.** For any $\Theta \in \{\partial_0, \partial_1\}$, we have

$$-\Theta \tilde{\Omega}^c_g = \frac{1}{2} r^* \tilde{\Omega}^c_{g-1,2} (5\Delta_\Theta) + \frac{1}{2} \sum \mathcal{L}_s(\tilde{\Omega}^c_{g_1,1} \otimes \tilde{\Omega}^c_{g_2,1}) (5\Delta_\Theta),$$

where we recall that $\tilde{\Omega}^c_{g,n} := 5^{g-1}(L/I_0)^{2g-2}\Omega^c_{g,n}$, and where $r: \bar{\mathcal{M}}_{g-1,2} \to \bar{\mathcal{M}}_g$ and $s: \bar{\mathcal{M}}_{g_1,1} \times \bar{\mathcal{M}}_{g_2,1} \to \bar{\mathcal{M}}_g$ are the gluing maps.

**Proof.** This is similar to [27] and [26].

We consider the action of $\Theta$ on the contribution of a decorated graph $\Gamma$ in the generalized $\mathcal{R}$-matrix action for $\tilde{\Omega}^c_g$. Since we have no markings, only the edge contribution $\mathcal{V}$ depends on non-holomorphic generators (see Section 5.4). Therefore the contribution of $\Gamma$ to $\Theta \tilde{\Omega}^c_g$ naturally splits into a sum over edges $e$ of $\Gamma$. We can consider the contribution to $2\Theta \tilde{\Omega}^c_g$ instead as split into a sum over half-edges $h$ of $\Gamma$. When the edge $e$ is non-disconnecting, the half-edge $h$ determines a decorated genus $g - 1$ graph with two (ordered) legs. This corresponds to the first term in the HAE. When $e$ is disconnecting,
the half-edge \( h \) determines a decorated genus \( g_1 \) graph with a leg corresponding to \( h \) and a decorated genus \( g_2 \) graph with one leg, where \( g_1 + g_2 = g \). This corresponds to the second term in the HAE.

The holomorphic anomaly equation therefore follows from Corollary 6.4. \( \square \)

### 6.4. Examples of HAEs.

**Theorem 6.10.** Recall

\[
\bar{F}_{g,n}^e := \int_{\bar{\mathcal{M}}_{g,n}} \bar{\Omega}_{g,n}^e(\phi_1, \ldots, \phi_n).
\]

The following HAEs for the extended quintic family hold

\[
-\partial_0 \bar{F}_g^e = \frac{1}{2} \bar{F}_{g-1,2}^e + \frac{1}{2} \sum_{g_1+g_2=g} \bar{F}_{g_1,1}^e \cdot \bar{F}_{g_2,1}^e, \quad -\partial_1 \bar{F}_g^e = 0.
\]

**Proof.** The first HAE directly follows from Theorem 6.9 and (31). For the second HAE, in addition to Theorem 6.9 and (31), we need to use that by pullback and dimension considerations both \( \Omega_{g,1}^c(\phi_2) \) and \( \Omega_{g,2}^c(\phi_0, \phi_2) \) vanish. \( \square \)

**Remark 6.11.** Under the Yamaguchi–Yau generators \( \{\mathcal{U}, \mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{L}\} \) (see (4)), the divisor equation (25) becomes

\[
\bar{F}_{g,n+1} = (\partial_u - n(\mathcal{V} - 2\mathcal{U}) + (2g - 2)\mathcal{U}) \bar{F}_{g,n}.
\]  

(33)

Then we have

\[
- (\partial_{\mathcal{V}_1} - \partial_{\mathcal{V}_2} - \partial_{\mathcal{V}_3}) \bar{F}_g = \frac{1}{2} \left( \partial_u^2 + (2(2g - 1)\mathcal{U} - \mathcal{V}) \partial_u + (2g - 2)(\mathcal{V}_2 + (2g - 1)\mathcal{U}^2) \right) \bar{F}_{g-1} + \frac{1}{2} \sum_{g_1 + g_2 = g} (\partial_u + (2g_1 - 2)\mathcal{U}) \bar{F}_{g_1} \cdot (\partial_u + (2g_2 - 2)\mathcal{U}) \bar{F}_{g_2}
\]

Since \( \partial_\mathcal{U} \bar{F}_g = 0 \) and \( \partial_u \mathcal{V}_i \in \mathbb{Q}[\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{L}] \), we have \( \partial_\mathcal{U}(\partial_u \bar{F}_g) = 0 \). Hence, we obtain

\[
\partial_{\mathcal{V}_1} \bar{F}_g = - \frac{1}{2} \left( \partial_u^2 - \mathcal{V} \partial_u + (2g - 2)\mathcal{V}_2 \right) \bar{F}_{g-1} - \frac{1}{2} \sum_{g_1 + g_2 = g} \partial_u \bar{F}_{g_1} \cdot \partial_u \bar{F}_{g_2}
\]

\[
\partial_{\mathcal{V}_2} \bar{F}_g = (2g - 1) \partial_u \bar{F}_{g-1} + \sum_{g_1 + g_2 = g} \partial_u \bar{F}_{g_1} \cdot (2g_2 - 2) \bar{F}_{g_2}
\]

\[
\partial_{\mathcal{V}_3} \bar{F}_g = (g - 1)(2g - 1) \bar{F}_{g-1} + \sum_{g_1 + g_2 = g} 2(g_1 - 1)(g_2 - 1) \bar{F}_{g_1} \cdot \bar{F}_{g_2}
\]

We can see that up to a polynomial of \( \mathcal{L} \) (which we consider as a global meromorphic function in the moduli space), \( \bar{F}_g \) is determined from the initial data by the holomorphic anomaly equations.

**Example 6.12.** We have the following HAEs for the genus two cases

\[
\partial_y \bar{F}_{2,0} = - \frac{1}{2} \bar{F}_{1,2} - \frac{1}{2} \bar{F}_{1,1}^2
\]

\[
= - \frac{1}{2} \left( \frac{5^6 \mathcal{Z}^2}{144} + \frac{1750 \mathcal{X} \mathcal{Z}}{9} + \frac{125 \mathcal{Y} \mathcal{Z}}{6} + \frac{3145 \mathcal{X}^2}{36} + \frac{115 \mathcal{X} \mathcal{Y}}{6} + \frac{5 \mathcal{Y}^2}{24} - \frac{205 \mathcal{Z}_2}{24} - \frac{25 \mathcal{X}_2}{3} \right)
\]
and
\[ -\partial_{\gamma} + \partial_{\chi} - (\mathcal{X} - \mathcal{Y}) \partial_{\chi_2} - (2\mathcal{X}\mathcal{Y} + 4\mathcal{X}_2 + \frac{15}{4} \mathcal{Z}_2) \partial_{\chi_3} \] \( \bar{F}_{2,0} = 0 \)

This matches with the following mirror formula proved in [19]

\[ F^\text{GW}_2(\tau(q)) = \frac{\tau^2}{L^2} \left( \frac{70 \mathcal{X}^3}{9} + \frac{557 \mathcal{X}^3}{72} - \frac{629 \mathcal{Y} \mathcal{X}^2}{72} - \frac{23 \mathcal{Y}^2 \mathcal{X}}{24} - \frac{\mathcal{Y}^3}{24} \right) \]
\[ + \frac{255 \mathcal{Z} \mathcal{X}^2}{36} - \frac{175 \mathcal{Y} \mathcal{X}^2}{9} + \frac{1441 \mathcal{Z} \mathcal{X}}{48} - \frac{25 \mathcal{Z} (\mathcal{X}^2 + \mathcal{Y}^2)}{24} - \frac{3125 \mathcal{Z}^2 (\mathcal{X} + \mathcal{Y})}{288} \]
\[ + \frac{41 \mathcal{Z} \mathcal{Y}}{48} - \frac{625 \mathcal{Z}^3}{144} + \frac{2233 \mathcal{Z} \mathcal{Z}_2}{128} + \frac{547 \mathcal{Z}_3}{72} \].

7. A TECHNICAL RESULT OF THE FORMAL QUINTIC THEORY

In this Section we prove a somewhat technical result about the formal quintic theory whose results we used to prove Lemma 5.5. As a by-product, we prove a conjecture raised in [35].

For future applications, we consider more generally the \((\mathbb{C}^*)^m\)-equivariant \(\mathcal{O}(m)\)-twisted GW theory of \(\mathbb{P}^{m-1}\), and we use \(\lambda_0, \ldots, \lambda_{m-1}\) to denote the equivariant parameters. Following Givental (see [15, 12]), for any \(\alpha = 0, \ldots, m - 1\), let \(\mathcal{I}_\alpha\) be the following oscillatory integrals in the Landau–Ginzburg (LG) model

\[ \mathcal{I}_\alpha(q, z) := \int_{\gamma_\alpha \subset (\mathbb{C}^*)^m} e^{W(x_0, \ldots, x_{m-1})/z} dx_0 \wedge \cdots \wedge dx_{m-1}, \]

where \(\gamma_\alpha\) are the \(m\) Lefschetz thimbles and \(W\) is the LG potential

\[ W(x_0, \ldots, x_{m-1}) = \sum_{i=0}^{m-1} (x_i + \lambda_i \ln x_i) - (q^{-1} \prod_{i=0}^{m-1} x_i)^{1/m}. \]

**Theorem 7.1.** When \(\lambda_i = \zeta_m^i \cdot \lambda\), where \(\zeta_m\) is a primitive \(m\)th root of unity, we have the following asymptotic expansion

\[ \mathcal{I}_\alpha(q, z) \asymp e^{\mu_{\alpha}/z} (-2\pi z)^{m/2} \left( 1 + \sum_{k>0} r_{k\alpha} \cdot (-z)^k \right) \quad (34) \]

as \(z \to 0^-\), such that \(L_{k\alpha}^k \cdot r_{k\alpha}\) where \(L_{\alpha} = \zeta^\alpha \lambda L = \zeta^\alpha \lambda (1 - m^m q)^{-1/m}\) satisfies

(1) \(L_{k\alpha}^k \cdot r_{k\alpha} \in L^k \mathbb{Q}[L] \cap \mathbb{Q}[L^m]\) (Regularity)

(2) \(\deg_{L^m}(L_{k\alpha}^k \cdot r_{k\alpha}) = k\) (Degree bound)

(3) \((mL_{\alpha})^k \cdot r_{k\alpha} \in \mathbb{Q}[m, L^m]\) (Polynomiality in \(m\))

This theorem will follow from Proposition 7.5, Lemma 7.8 and Lemma 7.9.

**Remark 7.2.** When \(\lambda_i = \zeta_m^i \cdot \lambda\), the oscillatory integral \(\mathcal{I}_\alpha(q, z)\) is related to the \(\tilde{I}\)-function defined in [35] (where it is denoted by \(z\mathcal{F}_{-1}(h_\alpha/z, q)\), and defined in Equation (17)) by

\[ (-2\pi z)^{-m/2} e^{-h_\alpha \log q/z} C_\alpha(z) \cdot \mathcal{I}_\alpha(q, z) = \tilde{I}_\alpha(q, z) := \sum_{d \geq 0} q^d \frac{\prod_{k=0}^{md-1} (mh_\alpha + kz)}{\prod_{k=1}^{d} (h_\alpha + kz)^m - \lambda^m} \]

where \(h_\alpha = \zeta_m^\alpha \cdot \lambda\) is the restriction of the hyperplane class \(H \in H^2_{(\mathbb{C}^*)^m}(\mathbb{P}^{m-1})\) at the \(\alpha\)th fixed point, and where \(C_\alpha(z)\) is defined as in (17).
Then as a direct consequence of the above theorem, we prove a conjecture originally proposed in [35, Equation (39)]. We now present the conjecture in a slightly stronger form:

**Corollary 7.3.** When \( \lambda_i = c_i^\alpha \cdot \lambda \), we have the following asymptotic expansion

\[
 z q \frac{d}{dq} I_\alpha(q, z) \asymp e^{u_\alpha/z} (-2\pi z)^{\frac{m}{2}} \cdot L_\alpha \left( 1 + \sum_{k > 0} (R_k)_{0\bar{\alpha}} \cdot (-z)^k \right)
\]

as \( z \to 0^- \), such that

\[
 (mL_\alpha)^k (R_k)_{0\bar{\alpha}} \in L^k Q[L] \cap Q[L^m]
\]

is a polynomial of degree \( k \) with coefficients in \( Q[m] \).

**Proof.** This follows from Theorem 7.1 since, as we will see below,

\[
 q \frac{d}{dq} u_\alpha = L_\alpha
\]

and

\[
 q \frac{d}{dq} L_\alpha = -\frac{m}{L_\alpha}.
\]

\( \square \)

In particular, when \( m = 5 \) the oscillatory integral is related to our quintic \( I \)-function by (recall the definition of \( \tilde{I}_\alpha(z) \) in (23))

\[
 C_{\alpha}^{-1}(z) e^{h_\alpha \log q/z} \tilde{I}_\alpha(q, z) = I_\alpha(q, z),
\]

so that

\[
 C_{\alpha}^{-1}(z) e^{h_\alpha \log q/z} I_\alpha(q, z) = q \frac{d}{dq} I_\alpha(q, z),
\]

and we have that

\[
 [z^l] R_{\alpha\bar{\alpha}}(z) = r_{\alpha,0,l}(L_\alpha), \quad [z^l] R_{0\bar{\alpha}}(z) = r_{\alpha,1,l}(L_\alpha),
\]

are regular at \( L_\alpha = 0 \). In the case \( m = 5 \), the corollary specializes to:

**Corollary 7.4.** For the entries in the first column of the \( R \)-matrix, we have

1. \( [z^k] \tilde{R}_{0\bar{\alpha}}(z) \in Q[L] \cap L^{-k} Q[L^5] \)
2. \( [z^k] \tilde{R}_{0\bar{\alpha}}(z) = 0 \) if \( k + j \notin 5 \mathbb{Z} \).

### 7.1. Computing the asymptotic expansion by using stationary phase method.

In this subsection, we study the oscillatory integral using the stationary phase method in order to prove the regularity in Theorem 7.1.

We first note that the \( m \) critical points of \( W \) can be computed directly

\[
 (x_i)_\alpha = L_\alpha - \lambda_i, \quad y_\alpha = mL_\alpha
\]

with critical value \( u_\alpha = \int L_\alpha \frac{dq}{q} \), where for \( \alpha = 0, \ldots, m-1 \)

\[
 L_\alpha = \lambda_\alpha + O(q)
\]

are the \( m \) solutions of the indicial equation \( \prod_i (L - \lambda_i) = (mL)^m q \).

**Proposition 7.5.** The oscillatory integral has an asymptotic expansion of the form

\[
 I_\alpha(q, z) \asymp e^{u_\alpha/z} (-2\pi z)^{\frac{m}{2}} \Psi_{-1\bar{\alpha}} \cdot \left( 1 + \sum_{l > 0} r_l \cdot (-z)^l \right)
\]
as \( z \to 0 \) from the negative real axis, where \( Q \) is the Hessian bilinear form of \( W \),
\[
\Psi_{-1,\alpha} := \left( m^{-1} \sum_{j=0}^{m-1} (-1)^{m-j} (m-j) \epsilon_{m-j} L_\alpha^j \right)^{-1/2}
\]  
(35)

with \( e_j \) being the \( j \)th elementary symmetric polynomial in the \( \lambda_i \), and where \( r_l \) are rational functions of \( L_\alpha \). Furthermore, \( r_l(L_\alpha) \) is regular at \( L_\alpha = 0 \).

**Proof.** We can Taylor-expand
\[
W((x)_{\alpha} + \xi_0, \ldots, (x_{m-1})_{\alpha} + \xi_{m-1}) = u_\alpha + \frac{1}{2} Q(\vec{\xi}) + W'(\xi_0, \ldots, \xi_{m-1}),
\]
where \( W' \) consists of terms of order \( \geq 3 \) in the \( \xi_i \). With this, we can write
\[
\mathcal{I}_\alpha(q, z) \asymp \frac{e^{u_\alpha/z}}{\prod_i (x_{i,\alpha})} \int_{\gamma'_\alpha \subset (C^*)^m} e^{\frac{1}{2} Q(\vec{\xi})} \frac{e^{W'(\xi_0, \ldots, \xi_{m-1})/z}}{\prod_i (1 + (x_{i,\alpha})^{-1} \xi_i)} d\xi_0 \wedge \ldots \wedge d\xi_{m-1}
\]
\[
= \frac{e^{u_\alpha/z}}{\prod_i (x_{i,\alpha})} (-z)^m \int_{\gamma'_\alpha} e^{-\frac{1}{2} Q(\vec{\xi})} d\xi_0 \cdots d\xi_{m-1} \cdot \left( 1 + \sum_{l>0} r_l \cdot (-z)^l \right),
\]
where \( \gamma'_\alpha \) is \( \gamma_\alpha \) shifted by \( ((x_0)_{\alpha}, \ldots, (x_{m-1})_{\alpha}) \). By Wick’s (or Isserlis’) theorem, for each \( l > 0 \), \( r_k \) is polynomial in \( (Q^{-1})_{ij}, (x_i)_{\alpha}^{-1} \) and the higher order derivatives \( W_l := \frac{\partial^{m|W}}{\partial x_1 \cdots \partial x_l}|_{x_i = (x_i)_{\alpha}} \) It therefore remains to show that
\[
\Psi_{-1,\alpha} = (2\pi)^{-\frac{m}{2}} \prod_i (x_i)_{\alpha}^{-1} \int_{\gamma'_\alpha} e^{-\frac{1}{2} Q(\vec{\xi})} d\xi_0 \cdots d\xi_{m-1},
\]  
(36)

and that \( \det(Q)^{-1}, (x_i)^{-1}_{\alpha} \) and the higher order derivatives of \( W \) at \( (x_i)_{\alpha} \) are rational functions in \( L_\alpha \) well-defined at \( L_\alpha = 0 \).

First, note that
\[
(x_i)^{-1}_{\alpha} = \frac{1}{L_\alpha - \lambda_i}
\]
is a rational function of \( L_\alpha \) regular at \( L_\alpha \). Therefore, also the derivatives
\[
\frac{d^k W}{dx_i^k}((x_0)_{\alpha}, \ldots, (x_{m-1})_{\alpha}) = (x_i)^{-k}_{\alpha} \left( \lambda_i (-1)^{k-1}(k-1)! - mL_\alpha \prod_{i=0}^{k-1} \left( \frac{1}{m} - i \right) \right),
\]
for \( k = \sum_i k_i \geq 2 \) are rational functions of \( L_\alpha \) regular at \( L_\alpha = 0 \).

Note that
\[
(2\pi)^{-\frac{m}{2}} \prod_i (x_i)^{-1}_{\alpha} \int_{\gamma'_\alpha} e^{-\frac{1}{2} Q(\vec{\xi})} d\xi_0 \cdots d\xi_{m-1} = \prod_i (x_i)^{-1}_{\alpha} \cdot (\det Q)^{-1/2},
\]
and that
\[
\det Q = \frac{\sum_{j=0}^{m-1} (-1)^{m-j} (m-j) \epsilon_{m-j} L_\alpha^j}{m \prod_i (L_\alpha - \lambda_i)^2},
\]
so that \( \det(Q^{-1}) \) is rational in \( L_\alpha \), well-defined at \( L_\alpha = 0 \), and that (36) holds. \( \square \)
Corollary 7.6. For any $k > 0$, the $k$th derivative of $I_\alpha$ has an asymptotic expansion
\[
(zq \frac{d}{dq})^k I_\alpha(q, z) \simeq e^{u_\alpha/z}(-2\pi z)^{\frac{\alpha}{m}} \Psi_{k-1, \alpha} \cdot \left(1 + \sum_{l>0} r_{\alpha, k;l} \cdot (-z)^l\right),
\]
were
\[
\Psi_{k-1, \alpha} = L^k_{\alpha} \Psi_{-1, \alpha}
\]
such that for any $k, l > 0$
\[
r_{\alpha, k;l}(L_\alpha) \text{ is a rational function of } L_\alpha \text{ which is regular at } L_\alpha = 0.
\]

Proof. This follows from the following two observations: First, we have
\[
zq \frac{d}{dq} e^{u_\alpha/z} \Psi_{k-1, \alpha} = e^{u_\alpha/z} \Psi_{k, \alpha} \left(1 + zL_\alpha^{-1}q \frac{d}{dq} \log \Psi_{k-1, \alpha} + zL_\alpha^{-1}q \frac{d}{dq}\right)
\]
Second, the operator $L_\alpha^{-1}q \frac{d}{dq}$ preserves regularity, in the sense that for any rational function $f(x)$ that is regular at $x = 0$, the function
\[
L_\alpha^{-1}q \frac{d}{dq} f(L_\alpha) = -\prod_{i=1}^{m-1} (L_\alpha - \lambda_i) \frac{1}{\sum_{j=0}^{m-1} (-1)^{m-j} (m-j) e_{m-j} L_\alpha} f'(L_\alpha),
\]
is regular at $L_\alpha = 0$. □

By the above results, we can set $L_\alpha = 0$ in $I_\alpha$ and its derivatives. We have the following explicit formulae which will be crucial in the proof of the LG/CY correspondence[18]:

Lemma 7.7. The rational functions $r_{\alpha, k;l}$ take the following values at $L_\alpha = 0$,
\[
\left(1 + \sum_{l>0} r_{\alpha, k;l} \cdot z^l\right) \bigg|_{L_\alpha=0} = \exp \left[\sum_{i=0}^{m-1} \sum_{j=0}^{\infty} (-1)^{j+1} B_j(z) \frac{z^j}{j+1} \right],
\]
where $B_j(x)$ is the $j$th Bernoulli polynomial. In particular, if we set $\lambda_i = \zeta^i \lambda$, we have
\[
\left(1 + \sum_{l>0} r_{\alpha, k;l} \cdot z^l\right) \bigg|_{L_\alpha=0} = \exp \left[m \sum_{j=1}^{\infty} (-1)^{mj+1} B_{mj+1}(\frac{k}{m}) \frac{z^{mj}}{mj(mj+1)} \right]. \tag{37}
\]

Proof. For any $\alpha$, if $L_\alpha = 0$, we have $q^{-1} = 0$, and furthermore that $q^{-1} L_\alpha^{-1}$ becomes
\[
m \prod_{i=0}^{m-1} (-\lambda_i)^{-\frac{1}{m}}.
\]

The $k$th derivative $L_\alpha^{-k}(zq \frac{d}{dq})^k I_\alpha$ therefore is
\[
\int_{C^*} \prod_{i=0}^{m-1} (-\lambda_i)^{-\frac{1}{m}}(x_i)^{\frac{k}{m}} e^{\sum_{i=0}^{m-1} z^{-1}(x_i + \lambda_i \ln x_i) dx_0 \wedge \cdots \wedge dx_{m-1}} \frac{dx_0 \wedge \cdots \wedge dx_{m-1}}{x_0 \cdots x_{m-1}}
\]
at $L_\alpha = 0$, which agrees with the asymptotic expansion of the product of Gamma functions
\[
\prod_{i=0}^{m-1} (-\lambda_i)^{-\frac{1}{m}}(-z)^{\frac{k}{m}} + \frac{k}{m} \Gamma \left(\frac{\lambda_i}{z} + \frac{k}{m}\right).
\]
By a direct computation using the asymptotic expansion [22] (see also [30, (1.8)])
\[ \ln \Gamma(z + a) \approx \left( z + a - \frac{1}{2} \right) \ln(z) - z + \frac{1}{2} \ln(2\pi) + \sum_{j=1}^{\infty} \frac{(-1)^{j+1} B_{j+1}(a)}{j(j + 1)z^j}, \]
we conclude that \( L_{\alpha}^{-k}(zq \frac{d}{dq})^k I_{\alpha} \) has the asymptotic expansion
\[ \prod_i e^{-\lambda_i L_{\alpha}} \sqrt{-2\pi z/(-\lambda_i)} \exp \left[ \sum_{j=1}^{\infty} \frac{(-1)^{j+1} B_{j+1} \left( \frac{k}{m} \right) z^j}{j(j + 1) \lambda_i^j} \right]. \]
Unwrapping the definition of \( r_{\alpha,k,i} \), we conclude the lemma. □

**Lemma 7.8.** If we take \( \lambda_i = \zeta_m^i \lambda \), then
\[ L_{\alpha}^k \cdot r_{\alpha} \in Q[L_{\alpha}^m / \lambda^m] \]

**Proof.** Notice that the coefficient of \( L_{\alpha} \) in \( r_i \) are symmetric polynomials in \( \lambda_i \), and that
\[ \det Q^{-1} = \frac{(L_{\alpha}^m - \lambda^m)^2}{\lambda^m} \in \lambda^{-m}Q[L_{\alpha}, \lambda]_{2m} \]
Then by using \( Q_{ij} = (L_{\alpha} - \lambda_i)^{-1}(L_{\alpha} - \lambda_j)^{-1}Q[L_{\alpha}, L_{\alpha}]_1 \), we obtain
\[ (Q^{-1})_{ij} \in \lambda^{-m}Q[L_{\alpha}, \lambda]_{m+1}. \]
Here we consider both \( L_{\alpha} \) and \( \lambda \) as homogeneous degree 1 elements. Furthermore,
\[ (x_i)_\alpha^{-1} \cdot (x_j)_\alpha^{-1} \cdot (Q^{-1})_{ij} \in \lambda^{-m}Q[L_{\alpha}, \lambda]_{m-1}. \]
□

7.2. **Strengthening using the Picard–Fuchs equations.** In this section we will give the degree bound of the \( r_k \) for each \( k \), and hence give the degree bound for the \( R \)-matrix.

**Lemma 7.9.** Recall that we have proved
\[ L_{\alpha}^k \cdot r_{\alpha} \in Q[L_{\alpha}^m] \]
In addition, we have the degree bound
\[ \deg_{L_{\alpha}^m}(L_{\alpha}^k \cdot r_{\alpha}) = k, \] (38)
and the polynomiality in \( m \):
\[ (mL_{\alpha})^k \cdot r_{\alpha} \in Q[m, L_{\alpha}^m] \]

**Proof.** We first illustrate the proof of the degree bound for the formal quintic case: \( m = 5 \). Recall that the \( I \)-function satisfies the following Picard–Fuchs equation
\[ \left[ D^5 - \lambda^5/z^5 - q \cdot 5D(5D + 1) \cdots (5D + 4) \right] I_{\alpha}(q, z) = 0 \]
where \( D := qd/dq \). By using the asymptotic expansion (34), we see
\[ \left[ D_{L_{\alpha}}^5 - \lambda^5 - q \cdot 5D_{L_{\alpha}}(5D_{L_{\alpha}} + z) \cdots (5D_{L_{\alpha}} + 4z) \right](1 + \sum r_{\alpha}(-z)^k) = 0 \]
where (recall \( L_{\alpha} = \zeta^\alpha \lambda \cdot L \))
\[ D_{L_{\alpha}} := zD + L_{\alpha}. \]
This equation is equivalent to the following recursive relations for \( r_{ka} \):

\[-Dr_{ka} = D_1 r_{k-1,a} + D_2 r_{k-2,a} + D_3 r_{k-3,a} + D_4 r_{k-4,a} \]

where \( X = 1 - L^5 \) and

\[
D_1 := \frac{1}{25L_\alpha} (3X^2 - 3X + 10XD^1 + 50D^2)
\]

\[
D_2 := \frac{1}{125L^2_\alpha} (-24X^3 + 39X^2 - 15X + (15X^2 + 5X)D + 150XD^2 + 250D^3)
\]

\[
D_3 := \frac{1}{3125L^3_\alpha} (396X^4 - 870X^3 + 575X^2 - 101X + (-450X^3 + 725X^2 - 125X)D + 1375XD^2 + 3750XD^3 + 3125D^4)
\]

\[
D_4 := \frac{1}{3125L^4_\alpha} (X \cdot (24D + 250D^2 + 875D^3 + 1250D^4) + 625D^5)
\]

Notice that \( r_{0a} = 1 \), and that the differential operator \( D_i \) satisfies

\[
D_i \colon L^{-\langle k-i \rangle}_\alpha \mathbb{Q}[X]_{\leq (k-i)} \rightarrow L^{-k}_\alpha X \mathbb{Q}[X]_{\leq k},
\]

where we have used \( DL_\alpha = -\frac{1}{5}XL_\alpha \) and \( DX = X(1-X) \). Then together with the initial data \( I \) of \( L^k_\alpha \cdot r_{ka} \) at \( L_\alpha = 0 \) \( (X = 1) \), we see that Equation (38) follows by induction.

In the general case, the \( I \)-function satisfies the following Picard–Fuchs equation

\[
\left[ D^m - \lambda^m/z^m - q \cdot mD(mD + 1) \cdots (mD + m - 1) \right]I_\alpha(q, z) = 0
\]

Letting \( X := 1 - L^m \), the equation for \( r_{ka} \) follows

\[
\left[ L^{-m}_\alpha (1-X) \cdot D^m_{L_\alpha} - 1 + (mL_\alpha)^{-m}X \cdot mD_{L_\alpha}(mD_{L_\alpha} + z) \cdots (mD_{L_\alpha} + (m-1)z) \right] (1 + \sum r_{ka} (-z)^k) = 0
\]

Note that \( DL_\alpha = -\frac{1}{m}XL_\alpha \) and that \( DX = X(1-X) \). We have for \( s > 0 \),

\[
L^{-s}_\alpha DL^s_\alpha = 1 + \sum_{j=1}^{s} \frac{mz^j}{m^jL^j_\alpha} \cdot \sum_{i=0}^{j} c_{j,i}(m, X)D^{j-i},
\]

in which

\[
c_{j,0} = \binom{s}{j}, \quad c_{j,i} \in X \mathbb{Q}[m][X]_{i-1} \text{ for } i > 0.
\]

Then the above equation gives the following relation for \( r_{ka} \)

\[-Dr_{ka} = D_1 r_{k-1,a} + D_2 r_{k-2,a} + \cdots + D_{m-1} r_{k-m+1,a} \]

such that \( D_i \in (mL_\alpha)^{-i} \mathbb{Q}[m, X, mD] \) satisfies

\[
D_i \colon L^{-\langle k-i \rangle}_\alpha \mathbb{Q}[X]_{\leq (k-i)} \rightarrow L^{-k}_\alpha X \mathbb{Q}[X]_{\leq k}.
\]
For example

\[ D_1 = \frac{(m+1)(m-1)(m-2)}{24m^2 L_\alpha} X (X-1) + \frac{m-1}{2mL_\alpha} XD + \frac{m-1}{2L_\alpha} D^2 \]

\[ D_2 = \frac{(m+1)(m-1)(m-2)(m-3)}{24m^3 L_\alpha^2} \left( m \left( X - \frac{1}{2} \right) - X \right) (1-X) \]

\[ + \frac{(m-1)(m-2)(m^2(X-1)-5m(X-1)+6X+2)}{24m^2L_\alpha^2} XD \]

\[ + \frac{(m-1)(m-2)}{2mL_\alpha^2} XD^2 + \frac{(m-1)(m-2)}{6L_\alpha^2} D^3. \]

The rest is similar to the \( m = 5 \) case. Together with the regularity \( (37) \) of \( L_\alpha^k \cdot r_{k\alpha} \) at \( L_\alpha = 0 \), we see that Equation \( (38) \) follows by induction.

The polynomiality in \( m \) also follows from the polynomiality of \( (37) \) and \( D_1 \) by induction.

\[ \square \]

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