Information under Lorentz Transformation

N. Metwally, H. Eleuch, and M. Abdel-Aty

1 Department of Mathematics, Faculty of Science, Aswan University, Aswan, Egypt
2 Mathematics Department, College of Science, Bahrain University, Bahrain
3 DIRO, Université de Montréal, H3T 1J4, Montréal, Canada
4 Scientific Publishing Center, Bahrain University, Bahrain
5 Department of Mathematics, Faculty of Science, Sohag University, Sohag, Egypt

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Abstract In the context of quantum information, we investigate extensively some important classes of a general form of a two-qubit system under Lorentz transformation. It is shown Lorentz transformation causes a decay of entanglement and consequently information loses. On the other hand, it generates entangled states between systems prepared initially in a separable states. The partial entangled states are more robust under Lorentz transformation than maximally entangled states. Therefore the rate of information lose is larger for maximum entangled states compared with that for partially entangled states.

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1 Introduction

It is now well known that the increase of the classical computers speed has physical limitations. These limitations are fundamentally encrypted in the quantum mechanical effects. Since few decades several groups are developing a new concept of information processing, quantum information processing, in order to overcome the classical information processing limitations. One of the most powerful tools for the quantum computing is the entanglement, a pure quantum effect allowing to speed up the quantum algorithms and to exchange the information in non classical way. In the last two decades a large number of works has been done to study the entanglement in physical systems. However almost all these contributions were limited to the non-relativistic effects. Since the development of the special relativity the way of looking to the dynamical systems with high speed is drastically changed. Dirac introduced the relativistic effect in quantum mechanics just few years of the establishing its concepts and foundations.

Recently there are some achievements on relativistic quantum information. For example, Saldanha and Vedral showed that a massive 2-qubit particles prepared in a maximally entangled state is not capable of maximally violating the Clouser–Horne Shimony–Holt inequality. A scheme for storing quantum information in the field modes of moving cavities in non-inertial frames was reported. Choi investigated the relativistic effects on the spin of entanglement of two massive Dirac particles. The effect of the special relativity on the entanglement between spins and momenta of two-qubit system is investigated by Cafaro et al. Some properties of a system of two-spin-1 particles under Lornetz transformation have investigated by Ruiz and Achar.

In this work we study, in the relativistic context, the dynamical behavior of the entanglement for some particular and important classes of initial states of two-qubit system, namely Werner, X-and generic pure states. We analyze the dynamical evolution of the entropy and the information loses for this system. We show that non-intuitive results are emerging from the relativistic transformation.

The paper is organized as follows. In Sec. 2, a short review of the effect of Lorentz transformation on a two qubits state is presented. In Sec. 3, we describe the couplings between the two qubits, and then we obtain the Bloch vectors under the effect of the lorentz transformation. In Sec. 4, we discuss how to characterize the quantum entanglement by using the concurrence, in contrast to the Werner state and Bell states. In Sec. 5, the basic principle of information loss is discussed. In particular, we discuss the effect of Lorentz transformation on the local and non local information via calculating the entropy of both subsystems and the total state. Finally, we summary the main results of the paper in Sec. 6.

2 Model Description and Lorentz Transformation

In this section we review the effect of Lorentz transformation on a two qubits state given in the rest frame
where $|\psi_{pa}\rangle$ and $|\psi_{pb}\rangle$ are the momentum states for the first and the second qubit respectively, while $|\psi_s\rangle$ represents the spin state vector for both particles.\[19\] A Lorentz transformation $\Lambda$ changes $|\psi_p\rangle$ to $|\Lambda \psi_p\rangle$, where $i = a, b$. This transformation represents a unitary operator on the space of momenta space.\[19\] Therefore the Lorentz transformation changes the state (1) as:

$$|\Psi \rangle = |\Lambda |\psi_p\rangle \rangle |\Lambda \psi_p\rangle \rangle |\Lambda \psi_p\rangle \rangle$$

where, $U_a$ and $U_b$ are operators on the spin states for both particles. In the computational basis $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$, the operator $U = U_a \otimes U_b$ can be written as

$$U = |00\rangle \langle 00| + |11\rangle \langle 11| + e^{-i\phi}|01\rangle \langle 10| + e^{i\phi}|10\rangle \langle 10| \quad \text{(3)}$$

Assume that a source supplies two users Alice and Bob with a two qubits of massive Dirac particles. The spin part of this system is defined by 15 parameters: 6 of them represent Bloch vectors for the first and the second qubits respectively. The other nine parameters represent the component of the correlation tensor.\[20\]

$$\rho_{ab} = \frac{1}{4} \left( 1 + \sum_{i=1}^{3} s_i \sigma_i + \sum_{j=1}^{3} t_j \tau_j + \sum_{ij=1}^{3} c_{ij} \sigma_i \tau_j \right) \quad \text{(4)}$$

where $\sigma_i, \tau_j, i, j = 1, 2, 3$ are the Pauli matrices for the first and second’s qubit respectively, $s_i, t_j$ with $s_i = \text{tr} \{ \rho_{ab} \sigma_i \}$ and $t_j = \text{tr} \{ \rho_{ab} \tau_j \}$ are Bloch vectors for both qubits respectively. The tensor correlation is defined by a $3 \times 3$ matrix with the following elements $c_{ij} = \text{tr} \{ \rho_{ab} \sigma_i \tau_j \}$. For example, $c_{11} = \text{tr} \{ \rho_{ab} \sigma_1 \tau_1 \}$, $c_{12} = \text{tr} \{ \rho_{ab} \sigma_1 \tau_2 \}$, $c_{13} = \text{tr} \{ \rho_{ab} \sigma_1 \tau_3 \}$, and so on.

From the general form (4), one can obtain different classes.\[21\] The dynamics of the state (4) under the effect of the lorentz transformation is characterized by its new Bloch vectors,

$$\tilde{s}_1 = s_1 \cos \phi - s_2 \sin \phi \quad \tilde{s}_2 = s_2 \cos \phi - s_1 \sin \phi \quad \tilde{s}_3 = \frac{s_3}{2} (1 + \cos 2\phi) + \frac{t_3}{2} (1 - \cos 2\phi)$$

$$\tilde{t}_1 = t_1 \cos \phi - t_2 \sin \phi \quad \tilde{t}_2 = t_1 \sin \phi + t_2 \cos \phi$$

$$\tilde{t}_3 = \frac{t_3}{2} (1 - \cos 2\phi) + \frac{t_3}{2} (1 + \cos 2\phi)$$

and the 9 elements of the cross dyadic which are defined as

$${\tilde{c}}_{xx} = \frac{1}{2} (1 + \cos 2\phi) c_{xx} + \frac{1}{2} (1 - \cos 2\phi) c_{yy}$$

$${\tilde{c}}_{xy} = \frac{1}{2} (1 + \cos 2\phi) c_{xy} + \frac{1}{2} (1 - \cos 2\phi) c_{yx}$$

$${\tilde{c}}_{xz} = \frac{1}{2} (1 + \cos 2\phi) c_{xz} - \frac{1}{2} (1 - \cos 2\phi) c_{zy}$$

and $${\tilde{c}}_{yz} = \frac{1}{2} (1 + \cos 2\phi) c_{yz} - \frac{1}{2} (1 - \cos 2\phi) c_{zx}$$

$${\tilde{c}}_{zz} = \frac{1}{2} (1 + \cos 2\phi) c_{zz}$$

Figure 1 shows the behavior of entanglement of Werner state.\[28\] where this state is initially defined by its zero Bloch vectors, i.e., $\tilde{s} = \tilde{t} = 0$ and $c_{xx} = c_{yy} = c_{zz} = -x$. It has been shown that this state is separable for $x \in [-1/3, 1/3]$ and nonseparable for $1/3 < x \leq 1$ (see Fig. 1).

For entangled state classes i.e., $x \in [1/3, 1]$, the entanglement decreases to reach its minimum value as $\varphi$ in-
creases. Then the entanglement re-birthed again to reach its maximum bound. This maximum bound depends on the value of the parameter $x$, where Werner state represents a singlet state (Bell state) at $x = 1$, which is a maximum entangled state.

On the other hand, for separable classes namely for any $x \in [-1/3, 1/3]$, the separable states turn into entangled states as shown in Fig. 1(b). However the degree of entanglement is smaller than those depicted for initially entangled classes.

![Fig. 1](image1.png)

**Fig. 1** The dynamics of entanglement under Lorentz transformation for a system is initially prepared in different classes of Werner states.

Figure 2 shows the effect of Lorentz transformation on the degree of entanglement for maximum entangled state, where $c_{xx} = c_{yy} = c_{zz} = -1$, $X$-state which defined by $c_{xx} = -0.9 \neq c_{yy} = -0.8 \neq c_{zz} = -0.7$ and Werner state with $c_{xx} = c_{yy} = c_{zz} = 0.7$. As a general behavior, the entanglement $\mathcal{E}$ decreases as $\phi$ increases. The decreasing rate depends on the degree of entanglement for the initial state. However for maximum entangled states, MES the entanglement decreases very fast to completely vanish and then re-birth again to reach its maximum value ($\mathcal{E} = 1$). For less entangled states, a similar behavior is depicted as MES, but the entanglement is completely death for longer interval of $\phi$. Also, this figure shows the effect of Lorentz transformation on systems prepared initially in a separable states, where we set $x = -0.6$. It is clear that the initial entanglement is zero. However as $\phi$ increases an entangled state is generated and its maximum value is $\mathcal{E} \approx 0.44$.

![Fig. 2](image2.png)

**Fig. 2** The dynamics of entanglement under Lorentz transformation for a system is initially prepared in maximum entangled states (dash-dotted curve), Werner state with $x = -0.6$ (solid curves) and $X$-states with (dotted curves) $c_{xx} = -0.9$, $c_{yy} = -0.8$, and $c_{zz} = -0.7$.

(ii) Generic pure state

This state is described by the density operator,\(^{[20]}\)

$$\rho_p = \frac{1}{4} \left(1 + p(\sigma_x - \tau_x) - \sigma_x \tau_x - q(\sigma_y \tau_y + \sigma_z \tau_z)\right), \quad (10)$$

where, $q = \sqrt{1 - p^2}$. Under the Lorentz transformation this state is transformed into

$$\tilde{\rho}_p = \frac{1}{4} \left\{1 + p \cos \phi (\sigma_x - \tau_x) - \frac{1}{2} \left[(1 + q) + (1 - q) \cos 2\phi\right] \sigma_x \tau_x \right.$$

$$+ \frac{1}{2} (1 + q) \sin 2\phi \sigma_x \tau_y - \frac{1}{2} (1 - q) \sin 2\phi \sigma_y \tau_x$$

$$+ \frac{1}{2} \left[(1 - q) - (1 + q) \cos 2\phi\right] \sigma_y \tau_y - q \sigma_z \tau_z \right\}. \quad (11)$$

![Fig. 3](image3.png)

**Fig. 3** The same as Fig. 1 but for a system is prepared initially in a pure state.

4 Information Loss

In this section, we investigate the effect of Lorentz transformation on the local and non local information via calculating the entropy of both subsystems and the total state. The entropy of a density operator $\rho$ is defined by Von Numman entropy $\mathcal{P}_n = -\rho \ln \rho$. This value indicates how much information that the state $\rho$ is lost.

In Fig. 4, we investigate the dynamics of entropy of different classes of Werner type under the effect of Lorentz
transformation. It is clear that starting from entangled classes i.e., $1/3 < x \leq 1$, the initial entropy is not maximum. This means that the state contains some quantum information. As $x$ increases the entropy $P_n$ decreases to reach its minimum values at $x = 1$ i.e., the initial state is maximum entangled state. On the other hand, as one increases $\varphi$, the entropy increases faster for larger values of $x$, namely classes with larger degree of entanglement. However for less entangled states the entropy is increased. This shows that the less entangled classes of states are more robust under Lorentz transformation. These results are shown in Fig. 4(a). Starting from a separable classes, the initial entropy $P_n$ is larger than that for entangled states. However, the entropy reaches its maximum value for less entangled states as shown in Fig. 4(b). Also, as one increases $\varphi$, the entropy $P_n$ oscillates between its maximum and minimum values.

Figure 5 shows the behavior of the entropy of different classes of the $X$-state (8). It is clear that, starting from a maximum entangled class, i.e., we set $c_{xx} = c_{yy} = c_{zz} = -1$, the entropy $P_n = 0$ at $\varphi = 0$. This means that the amount of information on this state is maximum. However as one increases $\varphi$, the entropy increases as a result of a non-local information to reach its maximum values ($P_n = 2$). This maximum value is reached at $\varphi \approx \pi/3$ i.e., at the same value of the minimum amount of entanglement (see Fig. 2). Also, this figure depicts the behavior of entropy for a class of Werner state, where we set $c_{xx} = c_{yy} = c_{zz} = x = 0.7$ i.e., the initial state represents a partial entangled state with small degree of entanglement, the initial value of entropy is larger than that depicted for maximum entangled state. As one increases the Lorentz transformation’s parameter $\varphi$, the entropy decreases to reach its minimum value and increases again. The behavior of entropy starting from a separable class of Werner type, where we set $x = -0.6$, the initial entropy is maximum, i.e., $P_n = 2$. However this entropy oscillates between minimum and maximum values as $\varphi$ increases. This shows that there is an entangled state which is generated for different values of $\varphi$. Finally this figure depicts the behavior of entropy under the effect of Lorentz transformation for a class of $X$-states i.e., $c_{xx} \neq c_{yy} \neq c_{zz}$. It is clear that a similar behavior is shown as the previous classes, but the entropy does not reach its maximum value. So, this class is more robust than the previous class under the effect of Lorentz transformation.
5 Conclusion

In this contribution we obtain an analytical form for the spin part of the general two-qubit systems. Some classes as Werner, $X$- and a generic pure states are investigated extensively. The behavior of entanglement as well as the entropy, which measures the information loses are investigated. It is shown that, the degree of entanglement decreases faster to completely vanish for system $s$ prepared initially in maximum entangled states. Starting from less entangled states, the entanglement decreases faster to completely vanish for systems prepared initially in maximum entangled states. Therefore partially entangled states are more robust than maximum entangled states and consequently the rate of information loss is larger for maximum entangled states compared with that for partially entangled states.

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