Estimating occupation time functionals

Randolf Altmeyer

Institute of Mathematics
Humboldt-Universität zu Berlin
altmeyrx@math.hu-berlin.de

Abstract

We study the estimation of integral type functionals \( \int_0^t f(X_r) \, dr \) for a function \( f \) and a \( d \)-dimensional càdlàg process \( X \) with respect to discrete observations by a Riemann-sum estimator. Based on novel semimartingale approximations in the Fourier domain, central limit theorems are proved for \( L^2 \)-Sobolev functions \( f \) with fractional smoothness and continuous Itô semimartingales \( X \). General \( L^2(P) \)-upper bounds on the error for càdlàg processes are given under weak assumptions. These bounds combine and generalize all previously obtained results in the literature and apply also to non-Markovian processes. Several detailed examples are discussed. As application the approximation of local times for fractional Brownian motion is studied. The optimality of the \( L^2(P) \)-upper bounds is shown by proving the corresponding lower bounds in case of Brownian motion.

1 Introduction

Let \( X = (X_t)_{0 \leq t \leq T} \) be an \( \mathbb{R}^d \)-valued stochastic process with càdlàg paths on a filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P}) \). The goal of this paper is to estimate occupation time functionals

\[
\Gamma_t(f) = \int_0^t f(X_r) \, dr, \quad 0 \leq t \leq T,
\]

for a function \( f \) from discrete observations of \( X \) at \( t_k = k\Delta_n \), where \( \Delta_n = T/n \) and \( k = 0, \ldots, n \). Integral-type functionals of this form are important tools for studying the properties of \( X \) and appear therefore in many fields (see e.g. Chesney et al. (1997), Hugonnier (1999), Mattingly et al. (2010), Catellier and Gubinelli (2016)). The most important case for applications is the occupation time \( \Gamma_T(1_A) \) for a Borel

*Many thanks to Jakub Chorowski for helpful comments on an early draft of this manuscript. Support by the DFG Research Training Group 1845 “Stochastic Analysis with Applications in Biology, Finance and Physics” is gratefully acknowledged.

Key words and Phrases: Markov processes; integral functionals; occupation time; semimartingale; Sobolev spaces; fractional Brownian motion; lower bound.

AMS subject classification: Primary 60G99; 62M99; Secondary 60F05.
set $A$, which measures the time that the process spends in $A$. From a statistical point of view, occupation time functionals are also used to study functionals with respect to the invariant measure $\mu$ of an ergodic process $X$, because $T^{-1}\Gamma_T(f) \to \int f d\mu$ as $T \to \infty$ by the ergodic theorem under appropriate regularity assumptions (Dalalyan (2005), Mattingly et al. (2010)).

The natural estimator for discrete observations is the Riemann-sum estimator

$$\hat{\Gamma}_{n,t}(f) = \Delta_n \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} f(X_{t_k}).$$

It has been applied in the statistics literature, for instance, in order to estimate the occupation time (Chorowski (2013)) or functionals of the local time of a diffusion process (Florens-Zmirou (1993), Jacod (1998)). The obtained error bounds for $\Gamma_t(f) - \hat{\Gamma}_{n,t}(f)$ are often suboptimal and very specific to the problem at hand. The approximation error has to be determined also, if $\hat{\Gamma}_{n,t}(f)$ is used for simulating from the law of $\Gamma_t(f)$. For this, the $X_{t_k}$ actually have to be approximated by some $X^r_{t_k}$, obtained for example by an Euler-scheme (Mattingly et al. (2010)). The increasing availability of exact simulation methods, however, alleviates this problem to some extent (Beskos and Roberts (2005)).

Jacod et al. (2003) considered the Riemann-sum estimator for $f(x) = x$ in order to find the rate of convergence of the integrated error $\int_0^t (X_r - X_{[r/\Delta_n]}) dr$ for semimartingales with jump discontinuities, because in this case the error $X_t - X_{[t/\Delta_n]}$ does not converge to zero in the Skorokhod sense. Estimation of occupation time functionals, where the process is not observed directly, has been considered for example by Li et al. (2013), when $X$ is the volatility of an Itô semimartingale.

The theoretical properties of $\hat{\Gamma}_{n,t}(f)$ have been studied systematically only in few works and only for rather specific processes $X$ and functions $f$. Consistency as $\Delta_n \to 0$ follows from Riemann approximation already under weak assumptions. A central limit theorem for Itô semimartingales and $f \in C^2(\mathbb{R}^d)$ was proven in the monograph of Jacod and Protter (2011, Chapter 6) with rate of convergence $\Delta_n$. This is much faster than the $\Delta_n^{1/2}$-rate when approximating $f(X_t)$ by $f(X_{[t/\Delta_n]})$ for continuous $X$. Interestingly, the weak limit depends only on $\nabla f$ and therefore it seems that the CLT might also hold for $C^1(\mathbb{R}^d)$-functions. The proof, however, works only for $f \in C^2(\mathbb{R}^d)$, using Itô's formula.

For less smooth functions no CLT has been obtained so far. Instead, several authors considered $L^2(\mathbb{P})$-bounds for the estimation error $\Gamma_t(f) - \hat{\Gamma}_{n,t}(f)$. For $\alpha$-Hölder functions $f$ and $0 \leq \alpha \leq 1$ the rate of convergence $\Delta_n^{(1+\alpha)/2}$, up to log factors, has been obtained by Malliavin calculus for one dimensional diffusions (Kohatsu-Higa et al. (2014)) and by assuming heat kernel bounds on the transition densities for Markov processes in $\mathbb{R}^d$ (Ganychenko (2015); Ganychenko and Kulik (2014)). The only result for indicator functions, which is of high importance for applications, is the surprising rate $\Delta_n^{3/4}$ for one-dimensional Brownian motion and indicators $f = 1_{(a,b)}$, $a < b$ (see Ngo and Ogawa (2011)). Interestingly, this corresponds to the Hölder-rate for $\alpha = 1/2$. A partial explanation combining the different rates was given by Altmeyer and Chorowski (2016) which considered $f$ in fractional $L^2$-Sobolev spaces using a specific analysis with respect to stationary Markov pro-
cesses. It is not clear if similar results hold generally in higher dimensions or for different processes. Note that all studied processes until now are Markov processes.

In this work we study the estimation of occupation time functionals from several different points of views. Related to the classical work of Geman and Horowitz (1980) on occupation densities, a central idea is to rewrite the error $\Gamma_n(f) - \Gamma_n,T(f)$ as

$$(2\pi)^{-d} \int \mathcal{F} f(u) \left( \sum_{k=1}^{(t/\Delta_n)} \int_{t_{k-1}}^{t_k} \left( e^{-i\langle u,X_t \rangle} - e^{-i\langle u,X_{t_{k-1}} \rangle} \right) d\tau \right) du$$

by inverse Fourier transform under suitable regularity assumptions. Together with a pathwise analysis of the exponentials $e^{-i\langle u,X_t \rangle}$ and with functions $f$ having sufficiently regular Fourier transforms this is just the right idea to control the estimation error. The pathwise analysis is inspired by the one-step Euler approximations of Fournier and Printems (2008). These ideas allow us in Section 2 to extend the central limit theorem of Jacod and Protter (2011) to $L^2$-Sobolev functions $f \in H^1(\mathbb{R}^d)$ and non-degenerate continuous Itô semimartingales with the same rate of convergence $\Delta_n$. The proof is based on tight bounds for the Itô-correction term in Itô’s formula. Note that a function $f \in H^1(\mathbb{R}^d)$ is not necessarily continuous for $d > 1$.

For less smooth functions it is in general not possible to prove central limit theorems, because the bias becomes degenerate asymptotically. Instead, Section 3 provides non-asymptotic upper bounds for the $L^2(\mathbb{P})$-error $\Gamma_n(f) - \hat{\Gamma}_n,T(f)$ and general $d$-dimensional càdlàg processes $X$ under weak assumptions. Only the smoothness of the bivariate distributions of $(X_h, X_r)$ in $0 \leq h < r \leq T$ is required, i.e. either the joint densities or the characteristic functions are differentiable in $h$ and $r$. This allows us to prove the rate $\Delta_n^{(3+s)/2}$ for a large class of $d$-dimensional processes and $L^2$-Sobolev functions with fractional smoothness $0 \leq s \leq 1$. In particular, this covers the previous results for Höllder and indicator functions. We therefore obtain a unifying mathematical explanation for the different rates. Several examples demonstrate the wide applicability of these upper bounds, for example to Markov processes, but also to fractional Brownian motion. These results are used to prove, to the best of our knowledge, unknown rates of convergence for approximating the local times of fractional Brownian motion. Note that the $L^2(\mathbb{P})$-bounds also yield improved bounds for the so-called weak approximations $\mathbb{E}[\Gamma_n(f) - \hat{\Gamma}_n,T(f)]$, which are of key importance in Monte-Carlo simulations (cf. Gobet and Labart (2008)).

Rate optimality is addressed in Section 4. We prove the corresponding lower bounds for the $L^2(\mathbb{P})$-error in case of $L^2$-Sobolev functions and $d$-dimensional Brownian motion. In this case we can even conclude the efficiency of the Riemann-sum estimator in terms of its asymptotic variance.

We want to emphasize that the $L^2(\mathbb{P})$-bounds are not only optimal and explicit with respect to their dependence on $\Delta_n$, but also with respect to $T$. This allows for approximating functionals $\int fd\mu$ in an ergodic setting with respect to the invariant measure $\mu$ at the optimal rate $T^{-1/2}$ by the estimator $T^{-1}\hat{\Gamma}_n,T(f)$, independent of $\Delta_n$ being fixed or $\Delta_n \to 0$. We therefore believe that our results may be instrumental in bridging the gap between results in statistics obtained for high-frequency and low-frequency observations. In fact, the results in Section 3 have been crucial for approximating $\int_0^T 1_{(a,b)}(X_t)dr$, $a < b$, with respect to a one-dimensional stationary
diffusion $X$ in an effort to find a universal estimator for the volatility process which is minimax optimal at high and low frequency (cf. Chorowski (2015)). Moreover, it is well-known that, under suitable regularity assumptions, $T^{-1}\Gamma_T(f)$ converges to $\int f\,d\mu$ at the rate $T^{-1/2}$. This is the same rate as for $T^{-1}\hat{\Gamma}_{n,T}(f)$. This suggests that our results can also be applied to transfer results obtained in statistics for continuous observations to discrete observations by approximating the corresponding integral functionals.

Proofs can be found in the appendix. Let us first introduce some notation. $\|\cdot\|$ and $\|\cdot\|_{\infty}$ always denote the Euclidean norm and allows for standardizing estimators when the parameter of interest is random (cf. Remark 2.2). If $F$ for all $U$ with respect to stochastic processes is generally quite difficult. Our main tool will be central limit theorems for the error $\Delta_n \rightarrow 0$ with $0 \leq t \leq T$ and $T$ fixed. We assume that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ satisfies the usual conditions and that $X$ is a $d$-dimensional continuous Itô semimartingale of the form

$$X_t = X_0 + \int_0^t b_s \, ds + \int_0^t \sigma_s \, dW_s, \quad 0 \leq t \leq T,$$

(2.1)

where $X_0$ is $\mathcal{F}_0$-measurable, $(W_t)_{0 \leq t \leq T}$ is a standard $d$-dimensional Brownian motion, $b = (b_t)_{0 \leq t \leq T}$ is a locally bounded $\mathbb{R}^d$-valued process and $\sigma = (\sigma_t)_{0 \leq t \leq T}$ is a càdlàg $\mathbb{R}^{d \times d}$-valued process, all adapted to $(\mathcal{F}_t)_{0 \leq t \leq T}$.

The central limit theorems are based on the concept of stable convergence (Rényi (1963)), which we recall now. For more details and examples refer to Jacod and Shiryaev (2013) or Podolskij and Vetter (2010). Let $(Y_n)_{n \geq 1}$ be a sequence of random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a Polish space $(E, \mathcal{E})$. We say that $Y_n$ converges stably to $Y$, written $Y_n \xrightarrow{st} Y$, if $Y$ is defined on an extension $(\Omega', \mathcal{F}', \mathbb{P}')$ of the original probability space and if $(Y_n, U) \xrightarrow{d} (Y, U)$ for all $\mathcal{F}$-measurable random variables $U$. Stable convergence implies convergence in distribution and allows for standardizing estimators when the parameter of interest is random (cf. Remark 2.2). If $Z_n$ and $Z$ are stochastic processes on $[0, T]$, we further write $(Z_n)_t \xrightarrow{ucp} Z_t$ for $\sup_{0 \leq t \leq T} \| (Z_n)_t - Z_t \| \xrightarrow{p} 0$. Proving stable convergence with respect to stochastic processes is generally quite difficult. Our main tool will be Theorem 7.28 of Jacod and Shiryaev (2013).
2.1 CLT for $C^2$-functions

We first review the basic situation when $f \in C^2(\mathbb{R}^d)$. The following is a special case of Theorem 6.1.2 of [Jacod and Protter (2011)] for continuous $X$.

**Theorem 2.1.** Let $f \in C^2(\mathbb{R}^d)$. Then we have the stable convergence

\[
\Delta_n^{-1} \left( \Gamma_n(f) - \hat{\Gamma}_{n,t}(f) \right) \xrightarrow{st} \frac{f(X_t) - f(X_0)}{2} + \frac{1}{\sqrt{12}} \int_0^t \left\langle \nabla f(X_r), \sigma_r d\tilde{W}_r \right\rangle
\]

as processes on $\mathcal{D}([0,T], \mathbb{R}^d)$, where $\tilde{W}$ is a d-dimensional Brownian motion defined on an independent extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$.

In order to explain the main ideas of the proof consider the decomposition $\Gamma_n(f) - \hat{\Gamma}_{n,t}(f) = M_{n,t}(f) + D_{n,t}(f)$, where

\[
M_{n,t}(f) = \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} \int_{t_{k-1}}^{t_k} (f(X_r) - \mathbb{E} \left[ f(X_r) \mid \mathcal{F}_{t_{k-1}} \right]) \, dr,
\]

\[
D_{n,t}(f) = \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} \int_{t_{k-1}}^{t_k} \mathbb{E} \left[ f(X_r) - f(X_{t_{k-1}}) \mid \mathcal{F}_{t_{k-1}} \right] \, dr.
\]

By the martingale structure of $M_{n,t}(f)$ and Itô’s formula it is easy to check using Theorem 7.28 of [Jacod and Shiryaev (2013)] that

\[
\Delta_n^{-1} M_{n,t}(f) \xrightarrow{st} \frac{1}{2} \int_0^t \left\langle \nabla f(X_r), \sigma_r dW_r \right\rangle + \frac{1}{\sqrt{12}} \int_0^t \left\langle \nabla f(X_r), \sigma_r d\tilde{W}_r \right\rangle
\]

holds for $n \to \infty$ as processes on $\mathcal{D}([0,T], \mathbb{R}^d)$, where $\tilde{W}$ is a d-dimensional Brownian motion defined on an independent extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$. In fact, here $f \in C^1(\mathbb{R}^d)$ is sufficient (for a proof see Proposition A.4). With respect to $D_{n,t}(f)$ it can be shown by Itô’s formula that

\[
\Delta_n^{-1} D_{n,t}(f) \xrightarrow{ucp} \frac{f(X_t) - f(X_0)}{2} - \frac{1}{2} \int_0^t \left\langle \nabla f(X_r), \sigma_r dW_r \right\rangle.
\]

In particular, $\Delta_n^{-1} D_{n,t}(f)$ is not negligible asymptotically. Summing up $\Delta_n^{-1} M_{n,t}(f)$ and $\Delta_n^{-1} D_{n,t}(f)$ as well as the corresponding limits yields the theorem. It is interesting to note that the CLT implies the stable convergence of $\Delta_n^{-1}(\Gamma_n(f) - \hat{\Theta}_{n,t}(f))$ to $1/\sqrt{12} \int_0^t \left\langle \nabla f(X_r), \sigma_r d\tilde{W}_r \right\rangle$, where

\[
\hat{\Theta}_{n,t}(f) = \Delta_n \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} \frac{f(X_{t_{k-1}}) + f(X_{t_k})}{2}
\]

is the trapezoid rule estimator. Therefore $\hat{\Theta}_{n,t}(f)$ is actually the more natural estimator for $\Gamma_n(f)$. In particular, $\hat{\Gamma}_{n,t}(f)$ and $\hat{\Theta}_{n,t}(f)$ have the same rate of convergence. This is not true generally for deterministic integrands. We will see in Section 4 that both estimators are rate optimal and that the asymptotic variance in (2.2) is efficient.
Remark 2.2. From a statistical point of view Theorem 2.1 can be exploited to obtain a feasible central limit theorem. More precisely, the estimator $\hat{AVAR}_T(f) = 1/12 \sum_{k=1}^{n} (\nabla f(X_{t_k}) - X_{t_k} - X_{t_{k-1}})^2$ converges in probability to $1/12 \int_0^T \| \sigma_t \nabla f(X_t) \|^2 \, dt$, which is equal to $\text{Var}(1/\sqrt{12} \int_0^T (\nabla f(X_t) \cdot \sigma_t \, d\tilde{W}_t))$. The stable convergence and the continuous mapping theorem therefore yield $\Delta_n^{-1}(\hat{AVAR}_T(f))^{-1/2}(\Gamma_T(f) - \hat{\Theta}_{n,T}(f)) \overset{d}{\to} N(0,1)$. This can be used to derive asymptotic confidence intervals for $\hat{\Theta}_{n,T}(f)$.

2.2 CLT for Fourier-Lebesgue functions

Interestingly, the weak limit in (2.2) is also well-defined for less smooth functions. The argument above, however, cannot be applied, since it relies on Itô’s formula. In order to study the limit of $\Delta_n^{-1}D_{n,t}(f)$ for more general $f$, note that we can write

$$f(X_t) - f(X_{t_{k-1}}) = (2\pi)^{-d} \int \mathcal{F}f(u) \left( e^{-i(u,X_t)} - e^{-i(u,X_{t_{k-1}})} \right) \, du \quad (2.7)$$

for sufficiently regular $f$, where $\mathcal{F}f(u) = \int f(x)e^{i(u,x)} \, dx$ is the Fourier transform of $f$. In principle, we can now study $e^{-i(u,X_t)} - e^{-i(u,X_{t_{k-1}})}$ instead of $f(X_t) - f(X_{t_{k-1}})$. The error can be calculated exactly, if the characteristic functions of the marginals $X_t$ are known. For the general Itô semimartingale $X$ in (2.1), however, this is a difficult issue. Instead, the key idea is to replace the marginals $X_r$ for some $\varepsilon = \varepsilon(u,n)$ by the close approximations $X_{r-\varepsilon} + b_{r-\varepsilon}(r-\varepsilon) + \sigma_{r-\varepsilon}(W_r - W_{r-\varepsilon})$, whose distributions are Gaussian conditional on $\mathcal{F}_{r-\varepsilon}$. This idea is inspired by the one-step Euler approximation of Fournier and Printems (2008). For this $\sigma$ needs to be non-degenerate and the approximation error has to be sufficiently small. We therefore work under the following Assumption.

Assumption 2.3 (SM-$\alpha$-$\beta$). Let $0 \leq \alpha, \beta \leq 1$. There exists a constant $C$ and a sequence of stopping times $(\tau_K)_{K \geq 1}$ with $\tau_K \to \infty$ as $K \to \infty$ such that

$$\mathbb{E} \left[ \sup_{0 \leq r \leq t} \| \sigma_{(s+r)\wedge \tau_K} - \sigma_{s\wedge \tau_K} \|^2 \right] \leq Ct^{2\alpha}, \quad \mathbb{E} \left[ \sup_{0 \leq r \leq t} \| b_{(s+r)\wedge \tau_K} - b_{s\wedge \tau_K} \|^2 \right] \leq Ct^{2\beta}$$

for all $0 \leq s, t \leq T$ with $s + t \leq T$. Moreover, the process $((\sigma_t \sigma_t^\top)^{-1})_{0 \leq t \leq T}$ is almost surely bounded.

The smoothness assumptions on $\sigma$ and $b$ are rather general and appear frequently in the literature (see e.g. Jacod and Mykland (2015), Jacod and Protter (2011, Section 2.1.5)). They exclude fixed times of discontinuities, but allow for non-predictable jumps. The assumptions are satisfied, if $\sigma$ and $b$ are themselves Itô semimartingales (with $\alpha = 1/2$ or $\beta = 1/2$) or if their paths are Hölder continuous with regularity $\alpha$ or $\beta$. In particular, they hold with $\alpha = \beta = 1/2$ if $X$ is a diffusion process such that $\sigma_t = \bar{\sigma}(X_t)$, $b_t = \bar{b}(X_t)$ with Lipschitz continuous functions $\bar{\sigma}$, $\bar{b}$.

The right hand side in (2.7) shows that it is natural to assume that the Fourier transform of $f$ is integrable, which leads to the the Fourier-Lebesgue spaces. They appear in the form below for example in Catellier and Gubinelli (2016).
Definition 2.4. Let $s \in \mathbb{R}$, $p \geq 1$ and denote by $FL^{s,p}(\mathbb{R}^d) := \{f \in L^p(\mathbb{R}^d) : \|f\|_{FL^{s,p}} < \infty\}$ the Fourier-Lebesgue spaces of order $(s, p)$ with norm $\|f\|_{FL^{s,p}} = (\int |\mathcal{F}f(u)|^p(1 + \|u\|)^{sp}du)^{1/p}$. Denote by $FL^{s,p}_{loc}(\mathbb{R}^d)$ the localized Fourier-Lebesgue spaces which contain all functions $f$ such that $f \varphi \in FL^{s,p}(\mathbb{R}^d)$ for all $\varphi \in C_c^\infty(\mathbb{R}^d)$.

This definition assumes implicitly for $f \in FL^{s,p}(\mathbb{R}^d)$ that the Fourier transform $\mathcal{F}f$ exists as a function in $L^p(\mathbb{R}^d)$. For $p = 1$ we just write $FL^s(\mathbb{R}^d)$ (or $FL^s_{loc}(\mathbb{R}^d)$) and $\|f\|_{FL^s}$. For $p = 2$ the spaces $H^s(\mathbb{R}^d) := FL^{s,2}(\mathbb{R}^d)$ (or $H^s_{loc}(\mathbb{R}^d) := FL_{loc}^{s,2}(\mathbb{R}^d)$) are the fractional $L^2$-Sobolev spaces of order $s$ with norm $\|\cdot\|_{H^s} := \|\cdot\|_{FL^{s,2}}$. In particular, a function $f \in H^s(\mathbb{R}^d)$ is $[s]$-times weakly differentiable. By properties of the Fourier transform it can be shown for $s \geq 0$ that $FL^s_{loc}(\mathbb{R}^d) \subset C^s(\mathbb{R}^d)$, $C^s(\mathbb{R}^d) \subset FL^{s-\varepsilon}_{loc}(\mathbb{R}^d)$ for any $\varepsilon > 0$ and $H^s_{loc}(\mathbb{R}^d) \subset FL^{s}_{loc}(\mathbb{R}^d)$ if $s > s' + d/2$.

Note that we can gain in regularity for some functions by considering larger $p$. For example, the Fourier transforms of the indicator functions $1_{[a,b]}$, $a < b$, decay as $|u|^{-1}$ for $|u| \to \infty$ and thus $1_{[a,b]} \in FL^0(\mathbb{R})$, but also $1_{[a,b]} \in H^{1/2}(\mathbb{R})$. Similarly, $x \mapsto e^{-|x|}$ lies in $FL^{1-}(\mathbb{R})$ and in $H^{3/2-}(\mathbb{R})$. For another example of negative regularity see Theorem 3.14. More details on these spaces can be found in Adams and Fournier (2003), Di et al. (2012) and Triebel (2010).

If $f \in FL^s_{loc}(\mathbb{R}^d)$ for $s \geq 1$, then $f \in C^1(\mathbb{R}^d)$ such that (2.5) remains true. Moreover, we will prove for sufficiently smooth $\sigma$ and $b$ that also the limit for $\Delta_n^{-1}D_{t,n}(f)$ in (2.6) remains valid. This yields the wanted CLT. For a concise statement we use the trapezoid rule estimator from the last section.

Theorem 2.5. Assume (SM-$\alpha$-$\beta$) for $0 \leq \alpha, \beta \leq 1$. Let $s > 2-2\alpha$, $s \geq 1$, $s+\beta > 1$. Then we have for $f \in FL^s_{loc}(\mathbb{R}^d)$ the stable convergence

$$\Delta_n^{-1}\left(\Gamma_t(f) - \hat{\Theta}_{n,t}(f)\right) \overset{s.t.}{\longrightarrow} \frac{1}{\sqrt{12}} \int_0^t \left\langle \nabla f(X_r), \sigma_r d\tilde{W}_r \right\rangle$$

as processes on $\mathcal{D}([0, T], \mathbb{R}^d)$, where $\tilde{W}$ is a $d$-dimensional Brownian motion defined on an independent extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$. The feasible central limit theorem of Remark 2.6 remains valid.

This result is remarkable since it is only based on regularity assumptions for $f$, $\sigma$ and $b$. In particular, for smoother coefficients the conditions on $f$ can be relaxed. For $\alpha > 1/2$, $f \in FL^s_{loc}(\mathbb{R}^d)$ is allowed. For $\alpha \leq 1/2$ there is a trade-off between the regularities of $f$ and $\sigma$. The theorem also extends to $L^2$-Sobolev functions for sufficiently large regularity, because $H^s_{loc}(\mathbb{R}^d) \subset FL^s_{loc}(\mathbb{R}^d)$, if $s > s' + d/2$.

Remark 2.6. We want to emphasize that, as the proof of Theorem 2.5 reveals, it is not possible to argue as in Section 2.1 by using a more general Itô formula for $f \in C^1(\mathbb{R}^d)$, for example by Russo and Vallois (1996).

2.3 CLT for $L^2$-Sobolev functions

The proof of Theorem 2.5 does not apply to all $C^1(\mathbb{R}^d)$-functions. The weak limit, however, is also well-defined for $f \in H^s_{loc}(\mathbb{R}^d)$. A minor issue in this case is that the random variables $f(X_t)$ depend on the version of $f$ that we choose in its equivalence
class in $L^2_{loc}(\mathbb{R}^d)$. This problem disappears if $f$ is continuous or if $X_r$ has a density. Note that $H^1(\mathbb{R}^d) \subset C(\mathbb{R}^d)$ only for $d = 1$. Interestingly, it can be shown by the methods of Romito (2017), which are in turn also inspired by Fournier and Printems (2008), under Assumption (SM-$\alpha,\beta$) that the marginals $X_r$ have Lebesgue densities $p_r$ for $r > 0$.

In order to extend the central limit theorem to $f \in H^1_{loc}(\mathbb{R}^d)$, we need to make the following stronger assumption.

**Assumption (X0).** $X_0$ is independent of $(X_t - X_0)_{0 \leq t \leq T}$ and Lebesgue density $\mu$. Either, $F\mu \in L^1(\mathbb{R}^d)$, or $F\mu$ is non-negative and $\mu$ is bounded.

This assumption can be understood in two ways. First, the independence and the boundedness of $\mu$ imply that the marginals $X_r$ have uniformly bounded Lebesgue densities. Second, $f$ itself becomes more regular, as by independence $\mathbb{E}[\Gamma_t(f)|(X_r - X_0)_{0 \leq t \leq s}] = \int_0^1 (f * \tilde{\mu})(X_r - X_0) dr$ with $\tilde{\mu}(x) = \mu(-x)$. Unfortunately, this property cannot be used directly in the proof.

We can show under this assumption that (2.5) remains true for $f \in H^1_{loc}(\mathbb{R}^d)$. Moreover, for $f \in H^s_{loc}(\mathbb{R}^d)$ and sufficiently large $s \geq 1$ we can prove that $\Delta_n^{-1}D_{n,T}(f)$ converges to (2.6) in probability. This convergence is not uniform in $0 \leq t \leq T$ anymore. Therefore the weak convergence is not functional and holds only at the fixed time $T$.

**Theorem 2.7.** Assume (SM-$\alpha,\beta$) for $0 \leq \alpha, \beta \leq 1$ and (X0) Let $s > 2 - 2\alpha$, $s \geq 1$, $s + \beta > 1$. Then we have for $f \in H^s_{loc}(\mathbb{R}^d)$ the stable convergence

$$\Delta_n^{-1} \left( \Gamma_T(f) - \Theta_n,T(f) \right) \overset{s.t.}{\to} \frac{1}{\sqrt{12}} \int_0^T \left\langle \nabla f(X_r), \sigma_r d\tilde{W}_r \right\rangle,$$

where $\tilde{W}$ is a $d$-dimensional Brownian motion defined on an independent extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$. The feasible central limit theorem of Remark 2.2 remains valid.

Because of independence, Assumption (X0) can be relaxed by randomizing the initial condition and a coupling argument. This yields the following corollary.

**Corollary 2.8.** Assume (SM-$\alpha,\beta$) for $0 \leq \alpha, \beta \leq 1$. Let $s > 2 - 2\alpha$, $s \geq 1$, $s + \beta > 1$. For any function $f \in H^s_{loc}(\mathbb{R}^d)$ there exists a set $E \subset \mathbb{R}^d$ such that $\mathbb{R}^d \setminus E$ has Lebesgue measure 0 and such that the stable convergence in Theorem 2.7 holds for all $X_0 = x_0 \in E$.

This result generalizes Theorem 2.1 considerably. The set $E$ depends in general on the function $f$, i.e. it can change if we consider a different function $\tilde{f}$ with $f = \tilde{f}$ almost everywhere. If $f$ has a bit more regularity, then the CLT holds for all initial values.

**Corollary 2.9.** Assume (SM-$\alpha,\beta$) for $0 \leq \alpha, \beta \leq 1$. Let $s > 2 - 2\alpha$, $s > 1$. Then we have the stable convergence in Theorem 2.7 for any $f \in C^s(\mathbb{R}^d)$ and all initial values $X_0 = x_0 \in \mathbb{R}^d$. 

8
Note that here $s$ is strictly larger than 1. In a way this generalizes Theorem 2.5 because $FL^s(\mathbb{R}^d) \subset C^s(\mathbb{R}^d)$ for $s \geq 1$. On the other hand, the stable convergence in Theorem 2.5 is functional, while Corollary 2.9 proves stable convergence at a fixed time.

Remark 2.10. In some cases it is possible to derive similar CLTs for $f \in H^{1/2}_0(\mathbb{R})$ with $0 \leq s < 1$. For example, we have $f = 1_{[a,\infty)} \in H^{1/2}_0(\mathbb{R})$ and the proof of Theorem 2.7 implies a CLT for $\Delta_n^{-3/4}(\Gamma_T(f) - \hat{\Gamma}_{n,T}(f))$, where $f_\varepsilon = f * \varphi_\varepsilon$ with $\varphi \in C^\infty_c(\mathbb{R}^d)$, $\varphi_\varepsilon = \varepsilon^{-1}\varphi(\varepsilon^{-1}(\cdot))$ and $\varepsilon = \Delta_n^{1/2}$. The limiting distribution is similar to Corollary 3.4 of [Ngo and Ogawa (2011)] and involves local times of $X$. The rate $\Delta_n^{-3/4}$ will be explained in the next section. It is not possible to extend this to a CLT for $\Delta_n^{-3/4}(\Gamma_T(f) - \hat{\Gamma}_{n,T}(f))$, as the error $\Gamma_T(f - f_\varepsilon) - \hat{\Gamma}_{n,T}(f - f_\varepsilon)$ is only of order $O_p(\Delta_n^{-1/4})$.

3 Upper bounds for less smooth functions

The aim of this section is to derive finite sample upper bounds on $\|\Gamma_T(f) - \hat{\Gamma}_{n,T}(f)\|_{L^2(\mathbb{P})}$ with explicit dependence on $\Delta_n$, $T$ and $f$. The function $f$ is possibly much rougher than in the last section. It is therefore not possible to use arguments based on Taylor’s theorem such as Itô’s formula. Except for special cases, it is impossible to prove central limit theorems for $\Gamma_T(f) - \hat{\Gamma}_{n,T}(f)$ in this case (cf. Remark 2.10). Instead of using martingale arguments, the results here are based on direct calculations with respect to the distribution of $X$. The following is inspired by the proof of [Ganychenko (2015), Theorem 1].

We always assume that $X = (X_t)_{0 \leq t \leq T}$ is a càdlàg process with respect to $(\Omega, \mathcal{F}, (\mathcal{F})_{0 \leq t \leq T}, \mathbb{P})$, not necessarily a semimartingale or a Markov process. Then

$$
\|\Gamma_T(f) - \hat{\Gamma}_{n,T}(f)\|_{L^2(\mathbb{P})}^2 = \sum_{k,j=1}^n \int_{t_{k-1}}^{t_k} \int_{t_{j-1}}^{t_j} \mathbb{E} \left[ (f(X_r) - f(X_{t_{k-1}}))(f(X_h) - f(X_{t_{j-1}})) \right] \, dh \, dr.
$$

Assume that the bivariate distributions of $(X_a, X_b)$, $a < b$, have Lebesgue densities $p_{a,b}$. Under suitable regularity assumptions the expectation in the last display can be written as

$$
\int_{t_{k-1}}^{t_k} \left( \int f(x) f(y) \partial_b p_{h,b}(x,y) - \partial_b p_{t_{k-1},b}(x,y) \right) \, dx \, dy \, db
$$

$$
= \int_{t_{k-1}}^{t_k} \int_{t_{j-1}}^{t_j} \left( \int f(x) f(y) \partial^2_{ab} p_{a,b}(x,y) \, dx \, dy \right) \, db \, da.
$$

From this we can obtain general upper bounds on $\|\Gamma_T(f) - \hat{\Gamma}_{n,T}(f)\|_{L^2(\mathbb{P})}^2$. Their structure reflects that the distributions of $(X_a, X_b)$ degenerate for $a = b$, therefore requiring a different argument.

Proposition 3.1. Assume that the joint densities $p_{a,b}$ of $(X_a, X_b)$ exist for all $0 < a < b \leq T$. 

9
(i) Assume that \( b \mapsto p_{a,b}(x,y) \) is differentiable for all \( x,y \in \mathbb{R}^d \), \( 0 < a < b < T \), with locally bounded derivatives \( \partial_b p_{a,b} \). Then there exists a constant \( C \) such that for all bounded \( f \) with compact support

\[
\| \Gamma_T (f) - \hat{\Gamma}_{n,T} (f) \|_{L^2(\mathbb{P})}^2 \leq C \Delta_n \int (f(y) - f(x))^2 \left( \sum_{k=1}^n \int_{t_{k-1}}^{t_k} p_{t_{k-1},r} (x,y) \, dr \right)
+ \sum_{k-1 > j \geq 2} \int_{t_{k-1}}^{t_k} \int_{t_{j-1}}^{t_j} \left( |\partial_r p_{h,r} (x,y)| + |\partial_r p_{t_{j-1},r} (x,y)| \right) \, dh \, dr \, d(x,y).
\]

(ii) In addition, assume that \( a \mapsto \partial_a p_{a,b}(x,y) \) is differentiable for all \( x,y \in \mathbb{R}^d \) and \( 0 < a < b < T \), with locally bounded derivatives \( \partial_a^2 p_{a,b} \). Then we also have

\[
\| \Gamma_T (f) - \hat{\Gamma}_{n,T} (f) \|_{L^2(\mathbb{P})}^2 \leq C \Delta_n \int (f(y) - f(x))^2 \left( \Delta_n^{-1} \sum_{k=1}^n \int_{t_{k-1}}^{t_k} p_{t_{k-1},r} (x,y) \, dr \right)
+ \sum_{k-1 > j \geq 2} \int_{t_{k-1}}^{t_k} \int_{t_{j-1}}^{t_j} |\partial^2_r p_{h,r} (x,y)| \, dh \, dr \, d(x,y).
\]

Concrete upper bounds can be obtained from this by combining the smoothness of \( f \) with bounds on \( \partial_b p_{a,b} \) and \( \partial_a^2 p_{a,b} \). Another way for getting upper bounds comes from formally applying the Plancherel theorem to \((3.1)\). Denote by \( \varphi_{a,b} = F p_{a,b} \) the characteristic function of \((X_a,X_b)\). Under sufficient regularity conditions \((3.1)\) is equal to

\[
(2\pi)^{-2d} \int_{t_{k-1}}^{t_k} \int_{t_{j-1}}^{t_j} \left( \int \mathcal{F} f(u) \mathcal{F} f(v) \overline{\partial^2_{ab} \varphi_{a,b}} (u,v) d(u,v) \right) da db.
\]

This yields the following version of the last proposition.

**Proposition 3.2.** Let \( \varphi_{a,b} \) be the characteristic functions of \((X_a,X_b)\) for \( 0 \leq a,b \leq T \) with \( \varphi_{a,a}(u,v) = \varphi_a(u+v) \) for \( u,v \in \mathbb{R}^d \).

(i) Assume that \( b \mapsto \varphi_{a,b}(u,v) \) is differentiable for \( 0 < a < b < T \), \( u,v \in \mathbb{R}^d \), with locally bounded derivatives \( \partial_b \varphi_{a,b} \). Then there exists a constant \( C \) such that for all \( f \in \mathcal{S}(\mathbb{R}^d) \)

\[
\| \Gamma_T (f) - \hat{\Gamma}_{n,T} (f) \|_{L^2(\mathbb{P})}^2 \leq C \Delta_n \int |\mathcal{F} f(u)| |\mathcal{F} f(v)| \left( \sum_{k=1}^n \int_{t_{k-1}}^{t_k} g_{t_{k-1},r} (u,v) \, dr \right)
+ \sum_{k-1 > j \geq 2} \int_{t_{k-1}}^{t_k} \int_{t_{j-1}}^{t_j} \left( |\partial_r \varphi_{h,r} (u,v)| + |\partial_r \varphi_{t_{j-1},r} (u,v)| \right) \, dh \, dr \, d(u,v),
\]

with \( g_{t_{k-1},r} (u,v) = |\varphi_{r,r}(u,v)| + |\varphi_{t_{k-1},r}(u,v)| + |\varphi_{t_{k-1},t_{k-1}}(u,v)| \).
(ii) In addition, assume that \( a \mapsto \partial_v \varphi_{a,b}(u,v) \) is differentiable for all \( u, v \in \mathbb{R}^d \) and \( 0 < a < b < T \), with locally bounded derivatives \( \partial^2_v \varphi_{a,b} \). Then we also have

\[
||\Gamma_T (f) - \hat{\Gamma}_{n,T} (f)||_{L^2(\mathbb{P})}^2 \\
\leq C \Delta_n^2 \int |\mathcal{F} f (u)| |\mathcal{F} f (v)| \left( \Delta_n^{-1} \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_{0}^{t_k} \left| (\partial_v \varphi_{h,r} (u,v)) \right| \right) dhr + \sum_{k-1 \leq j \leq 2} \int_{t_{k-1}}^{t_k} \int_{t_j}^{t_k} \left| \partial^2_v \varphi_{h,r} (u,v) \right| dhr \right) d(u,v).
\]

The second proposition is useful if the characteristic functions \( \varphi_{a,b} \) are explicitly known, while the densities \( p_{a,b} \) are not. This is true for many Lévy or affine processes. Moreover, it can be easier to find upper bounds on characteristic functions than for the respective densities. Note that the second proposition does not require the joint densities \( p_{a,b} \) to exist. This is relevant, for instance, when studying jump processes without marginal densities (cf. Example 3.12). In some cases both propositions apply and the results can differ as we will see in the next section.

We will now study several concrete examples of processes \( X \) and function spaces for \( f \) and derive explicit upper bounds.

### 3.1 Markov processes

Let \( X \) be a continuous-time Markov process on \( \mathbb{R}^d \) with transition densities \( \xi_{h,r} \), \( 0 \leq h < r < T \), such that \( \mathbb{E}[g(X_r)|X_h = x] = \int g(y) \xi_{h,r}(x,y) dy \) for \( x \in \mathbb{R}^d \) and all continuous, bounded functions \( g \). Denote by \( \mathbb{P}_{x_0} \) the law of \( X \) conditional on \( X_0 = x_0 \). The joint density of \( (X_h, X_r) \), conditional on \( X_0 = x_0 \), is \( p_{h,r}(x,y|x_0) = \xi_{0,h}(x_0,x) \xi_{h,r}(x,y) \). The necessary differentiability conditions on \( p_{h,r} \) from Proposition 3.1 translate to assumptions on \( \xi_{h,r} \). The following heat kernel bounds are similar to the ones in Ganychenko (2013).

**Assumption 3.3.** The transition densities \( \xi_{h,r} \) for \( 0 \leq h < r < T \) satisfy one of the following conditions:

(A) The function \( r \mapsto \xi_{h,r}(x,y) \) is continuously differentiable for all \( x, y \in \mathbb{R}^d \) and there exist probability densities \( q_r \) on \( \mathbb{R}^d \) satisfying

\[
\sup_{x,y \in \mathbb{R}^d} |\xi_{h,r}(x,y)| q_{r-h} (y-x) \leq 1, \quad \sup_{x,y \in \mathbb{R}^d} |\partial_r \xi_{h,r}(x,y)| q_{r-h} (y-x) \leq \frac{1}{h-r}. \tag{3.2}
\]

(B-γ) Let \( 0 < \gamma \leq 2 \). In addition to (A), the function \( h \mapsto \partial_r \xi_{h,r}(x,y) \) is continuously differentiable for all \( x, y \in \mathbb{R}^d \) and the \( q_h \) satisfy

\[
\sup_{x,y \in \mathbb{R}^d} |\partial^2_{r} \xi_{h,r}(x,y)| q_{r-h} (y-x) \leq \frac{1}{(r-h)^2}. \tag{3.3}
\]

Moreover, if \( \gamma < 2 \), then \( \sup_{x \in \mathbb{R}^d} (||x||^{2s+d} q_h(x)) \leq h^{2s/\gamma} \) for \( 0 < s \leq \gamma/2 \), while \( \int ||x||^{2s} q_h(x) dx \leq h^s \) for \( 0 < s \leq \gamma/2 = 2 \).
These conditions are satisfied in case of elliptic diffusions with Hölder continuous coefficients with \( q_h(x) = c_1 h^{-d/2} e^{-c_2 h^{-1/2}} \) and \( \gamma = 2 \) for some constants \( c_1, c_2 \). They are also satisfied for many Lévy driven diffusions with \( q_h(x) = c_1 h^{-d/\gamma} (1 + \|xh^{-1/\gamma}\|^b\gamma d)^{-1} \) and \( 0 < \gamma < 2 \) (Ganychenko et al. (2015)). Different upper bounds in (3.2), (3.3) are possible yielding different results below.

Based on Proposition 3.1 we recover the main results of Ganychenko (2015) and Ganychenko et al. (2015). For \( 0 \leq s \leq 1 \) denote by \( \|f\|_{C^s} \) the Hölder seminorm \( \|f(x)-f(y)\|_{x,y} \).

**Theorem 3.4.** Let \( n \geq 2 \) and \( x_0 \in \mathbb{R}^d \). Let \( X \) be a Markov process with transition densities \( \xi_{a,b} \).

(i) Assume \([A]\). There exists a constant \( C \) such that for every bounded \( f \)

\[
\|\Gamma_T(f) - \hat{\Gamma}_{n,T}(f)\|_{L^2(\mathbb{R}^d)} \leq C \|f\|_{\infty} T^{1/2} \Delta_n^{1/2} (\log n)^{1/2}.
\]

(ii) Assume \([B-\gamma]\) for \( 0 < \gamma \leq 2 \). There exists a constant \( C \) such that for \( f \in C^s(\mathbb{R}^d) \) with \( 0 \leq s \leq \gamma/2 \)

\[
\|\Gamma_T(f) - \hat{\Gamma}_{n,T}(f)\|_{L^2(\mathbb{R}^d)} \leq C \|f\|_{C^s} T^{1/2} \begin{cases} \Delta_n^{1+2s/\gamma}, & 2s/\gamma < 1, \\ \Delta_n (\log n)^{1/2}, & 2s/\gamma = 1. \end{cases}
\]

Up to log factors the rate of convergence (for fixed \( T \)) is \( \Delta_n^{(1+2s/\gamma)/2} \) for \( f \in C^s(\mathbb{R}^d) \), interpolating between the worst-case rates \( \Delta_n^{1/2} \) and the “best” rate \( \Delta_n \). Interestingly, smaller \( \gamma \) means faster convergence for the same smoothness \( s \).

**Remark 3.5.** The \( T^{1/2}\)-term in the upper bound is optimal and appears in almost every other example below (observe however Theorem 3.13). If \( X \) is ergodic with invariant measure \( \mu \), then this can be used to estimate functionals \( \int f d\mu \) with respect to \( \mu \) by the estimator \( T^{-1} \hat{\Gamma}_{n,T}(f) \) with optimal rate \( T^{-1/2} \), independent of any condition on the discretization order \( \Delta_n \), i.e. there is essentially no difference between the high and the low frequency setting. This generalizes Theorem 2.4 of Altmeyer and Cherowski (2016) considerably, since stationarity is not required.

Theorem 3.4 yields for \( f = 1_{[a,b]} \), \( a < b \), only the rate \( \Delta_n^{1/2} (\log n)^{1/2} \). This cannot explain the \( \Delta_n^{3/4} \)-rate obtained for Brownian motion in Ngo and Ogawa (2011). In order to find a unifying view consider now \( f \in H^s(\mathbb{R}^d) \), \( 0 \leq s \leq 1 \).

**Theorem 3.6.** Let \( X \) be a Markov process with transition densities \( \xi_{a,b} \) and bounded initial density \( \mu \).

(i) Assume \([A]\). There exists a constant \( C \) such that for \( f \in L^2(\mathbb{R}^d) \)

\[
\|\Gamma_T(f) - \hat{\Gamma}_{n,T}(f)\|_{L^2(\mathbb{R}^d)} \leq C \|\mu\|_{\infty}^{1/2} \|f\|_{L^2} T^{1/2} \Delta_n^{1/2} (\log n)^{1/2}.
\]

(ii) Assume \([B-\gamma]\) for \( 0 < \gamma \leq 2 \). There exists a constant \( C \) such that for \( f \in H^s(\mathbb{R}^d) \) with \( 0 \leq s \leq \gamma/2 \)

\[
\|\Gamma_T(f) - \hat{\Gamma}_{n,T}(f)\|_{L^2(\mathbb{R}^d)} \leq C \|\mu\|_{\infty}^{1/2} \|f\|_{H^s} T^{1/2} \begin{cases} \Delta_n^{1+2s/\gamma}, & \gamma < 2, \\ \Delta_n^{2s/\gamma} (\log n)^{1/2}, & \gamma = 2, \\ \Delta_n (\log n)^{1/2}, & 2s/\gamma = 1. \end{cases}
\]
While the regularity of $f$ is now measured in the $L^2$-Sobolev sense, we still obtain the interpolating rate $\Delta_n^{(1+2s/\gamma)}/2$ up to log factors. Since $C^s(K) \subset H^{s-}(\mathbb{R}^d)$ for compacts $K \subset \mathbb{R}^d$ and because $f = 1_{[a,b]} \in H^{1/2-}(\mathbb{R})$, this theorem also yields the rates $\Delta_n^{(1+2s/\gamma)/2}$ for $s$-Hölder functions on compacts and $\Delta_n^{3/4}$ (up to log factors) for indicators. By an explicit interpolation as in Theorems 3.5 and 3.6 of Altmeier and Chorowski (2016) this can be improved to $\Delta_n^{(1+2s/\gamma)/2}$ and $\Delta_n^{3/4}$, respectively. By considering $L^2$-Sobolev spaces we therefore unify the different rates obtained for Markov processes. The log factors in Theorem 3.6 can be removed in many cases (cf. Section 3.2).

Remark 3.7. (i) The role of $\mu$ in the proof of Theorem 3.6 is to ensure that the marginals have uniformly bounded densities $p_h$, i.e. $\sup_{0 \leq t \leq T} \|p_h\|_\infty \leq \|\mu\|_\infty$. This is necessary, because the bounds in Assumption 3.3 degenerate at 0. Otherwise it is not even clear that $\|\Gamma_T(f)\|_{L^2(\mathbb{R})} < \infty$ for $f \in L^2(\mathbb{R}^d)$. If $\sup_{x \in \mathbb{R}^d} \int_0^T \xi_{0,r}(x)dr < \infty$, then the initial distribution can be arbitrary. This holds, for instance, when $d = 1$ and $q_h(x) = c_1 h^{-1/2} e^{-c_2 \|x h^{-1/2}\|^2}$.

(ii) A different possibility for removing the initial condition is to wait until $T_0 > 0$ such that $X_{T_0}$ has dispersed enough to have a bounded Lebesgue density. The proof of Theorem 3.6 can then be applied to $\|\int_{T_0}^T f(X_r)dr - \hat{\Gamma}_{n,T_0,T}(f)\|_{L^2}$, where $\hat{\Gamma}_{n,T_0,T}(f)$ is a Riemann-sum estimator taking only observations in $[T_0,T]$ into account.

(iii) A similar argument as in the proof of Corollary 2.8 shows $\Gamma_T(f) - \hat{\Gamma}_{n,T}(f) = O_{\varepsilon_0}(a_n)$ for almost all initial conditions $X_0 = x_0 \in \mathbb{R}^d$, where $a_n$ corresponds to the rates in Theorem 3.6 up to an additional log factor.

3.2 Additive processes

Let $Y = (Y_t)_{0 \leq t \leq T}$ be an additive process on $\mathbb{R}^d$ with $Y_0 = 0$ and local characteristics $(\sigma^2_t, F_t, b_t)$, where $\sigma^2_t = (\sigma^2_t)_{0 \leq t \leq T}$ is a continuous $\mathbb{R}^{d \times d}$-valued function such that $\sigma^2_t$ is symmetric for all $t$, $b = (b_t)_{0 \leq t \leq T}$ is a locally integrable $\mathbb{R}^d$-valued function and $(F_t)_{0 \leq t \leq T}$ is a family of positive measures on $\mathbb{R}^d$ with $F_t(\{0\}) = 0$ and $\sup_{0 \leq t \leq T} \{f(||x||^2 \wedge 1) dF_t(x)\} < \infty$ (cf. Tankov (2003, Section 14.1)). The increments $Y_r - Y_h$, $0 \leq h < r \leq T$, are independent and have infinitely divisible distributions. In particular, the corresponding characteristic functions are $e^{\psi_{h,r}(u)}$, $u \in \mathbb{R}^d$, by the Lévy-Khintchine representation (Tankov (2003, Theorem 14.1)), where the characteristic exponents $\psi_{h,r}(u)$ are given by

$$i \int_0^r \langle u, b_t \rangle dt - \frac{1}{2} \int_0^r \|\sigma_t^u\|^2 dt + \int_0^r \int_h^r (e^{i(u,x)} - 1 - i \langle u, x \rangle 1_{\{||x|| \leq 1\}}) dF_t(x) dt.$$ 

Applying Proposition 3.2 yields the following result. The independence in (X0) is always satisfied, because $Y$ has independent increments.

Theorem 3.8. Let $T \geq 1$. Consider the process $X_t = X_0 + Y_t$, where $Y = (Y_t)_{0 \leq t \leq T}$ is an additive process with local characteristics $(\sigma^2_t, F_t, b_t)$ as above and such that $X_0$ satisfies (X0).
(i) Let $0 < \gamma \leq 2$ and assume that $|\partial_t \psi_{h,r}(v)| \leq c(1 + \|v\|)^{\gamma + \beta r}$ and $|e^{\psi_{h,r}(v)}| \leq ce^{-c\|v\|^{\gamma + (r-h)}}$ for a constant $c$ and all $0 \leq h < r \leq T$, $v \in \mathbb{R}^d$ and some $0 \leq \beta_v \leq \beta^* \leq \gamma/2$ with $0 < \gamma + \beta_v \leq 2$. Then there exists a constant $C_\mu$ such that for $f \in H^s(\mathbb{R}^d)$ with $\beta^*/2 \leq s \leq \gamma/2 + \beta^*$\n\n$$\|\Gamma_T(f) - \Gamma_{n,T}(f)\|_{L^2(\mathbb{R}^d)} \leq C_\mu \|f\|_{H^s(\mathbb{R}^d)} \Delta_n^{1/2 - \frac{s}{\gamma + \beta^*}}.$$\n\nIf $F_\mu \in L^1(\mathbb{R}^d)$, then $C_\mu = C\|F_\mu\|_{L^1(\mathbb{R}^d)}^{1/2}$ and otherwise $C_\mu = C\|\mu\|_{\mathcal{P}}^{1/2}$. If even $|\partial_t \psi_{h,r}(v)| \leq c\|v\|^{\gamma + \beta r}$, then the same upper bound holds with $T^{1/2}$ instead of $T$.

(ii) If $|\partial_t \psi_{h,r}(v)| \leq c$, then we have for $f \in L^2(\mathbb{R}^d)$\n\n$$\|\Gamma_T(f) - \Gamma_{n,T}(f)\|_{L^2(\mathbb{R}^d)} \leq C_\mu \|f\|_{L^2(\mathbb{R}^d)}.$$\n\nThe same upper bound holds with $T^{1/2}$ instead of $T$, if $c_1 \leq \rho(v) \leq \partial_t \psi_{h,r}(v) \leq c_2\rho(v) \leq 0$ for a bounded function $v \mapsto \rho(v)$ and constants $c_1 \leq c_2$.

By the comments before Remark 3.7 we can obtain from this upper bounds for Hölder and indicator functions. The condition $|\partial_t \psi_{h,r}(v)| \leq c(1 + \|v\|)^{\gamma + \beta r}$ gives an additional degree of freedom in order to account for time-inhomogeneity (cf. Example 3.11). Note that there are no log terms as compared to Theorem 3.6. The smaller $\gamma/2 + \beta^*$, the less smoothness is necessary for $f$ to achieve a $\Delta_n$ rate.

Remark 3.9. In some situations it is sufficient to consider directly $X_t = Y_t$. This is true, for instance, if $d = 1$ and $\gamma > 1$ (cf. Remark 3.3). For different $d$ or $\gamma$ it follows in (i) that $Y_{T_0}$ for any $T_0 > 0$ has a density $p_{T_0}$ with $\mathcal{F}_{p_{T_0}} \in L^1(\mathbb{R}^d)$. Similarly to Remark 3.7(ii) the proof of Theorem 3.8 can then be applied to $\|f_{T_0}^T f(X_s)dr - \Gamma_{n,T_0}(f)\|_{L^2}$. For $O_\gamma$ bounds and almost all initial values $X_0 = x_0 \in \mathbb{R}^d$ refer to Remark 3.7(iii).

We study now a few examples.

Example 3.10 (Non-vanishing Gaussian part). Assume that $Y$ has local characteristics $(\sigma_t^2, F_t, 0)$ with $\sup_{t < s \leq T} \|\sigma_t\| < \infty$. Then $\gamma = 2$, $\beta^* = 0$ and $|\partial_t \psi_{h,r}(v)| \lesssim \|v\|^2$ (cf. Sato (1999, Equation (8.9))). Part (i) of Theorem 3.8 therefore yields up to a constant the upper bound $\|f\|_{H^s(\mathbb{R}^d)} \Delta_n^{(1+s)/2}$ for $f \in H^s(\mathbb{R}^d)$ with $0 \leq s \leq 1$, thus improving on Theorem 3.6.

Example 3.11 ($\gamma$-stable processes). Let $\psi_{h,r}(v) = -c \int_T^t \|v\|^{\gamma + \beta_r}dt$ with $c, \gamma, \beta_r$ as in Theorem 3.8. A process with these characteristic exponents exists if $\beta_r$ is continuous. $X$ is a generalized symmetric $\gamma$-stable process with stability index $\gamma + \beta_r$ changing in time. For $d = 1$ it is a multistable Lévy motion (cf. Example 4.1 in Falconer and Liu (2012)). If $\beta^* = 0$, then $X$ is just a symmetric $\gamma$-stable process and Theorem 3.8 yields the upper bound $\|f\|_{H^s(\mathbb{R}^d)} \Delta_n^{(1+s)/2}$ for $f \in H^s(\mathbb{R}^d)$ and $0 \leq s \leq \gamma/2$. Again, this improves on Theorem 3.6.

Example 3.12 (Compound Poisson process). Let $X$ be a compound Poisson process. Then $\psi_{h,r}(v) = (r-h) \int e^{(\psi(x)-1)}dF(x)$ for all $0 \leq h < r \leq T$ and a finite
are independent and centered Gaussian processes with covariance function \( H < \)
any \( X \) similar processes with self-similarity index \( \alpha \), bounded function \( g \).

Let \( H < 1 \), we can directly consider \( X \). Then there exists a constant \( C \mu \) as in Theorem 3.8 such that for any \( f \in H^s(\mathbb{R}^d) \) and \( 0 < \alpha \) 

\[
\| \Gamma_T (f) - \widehat{\Gamma}_{n,T} (f) \|_{L^2(\mathbb{P})} \leq C \mu \| f \|_{H^s} \begin{cases} T^H \Delta_n^{1+\alpha}, & H \geq 1/2, \\ T^{1/2} \Delta_n^{1+2H}, & H < 1/2. \end{cases}
\]

Again, from this we can obtain upper bounds for Hölder and indicator functions (cf. comments before Remark 3.7). It is interesting that the rate remains unchanged but the dependency on \( T \) differs for \( H > 1/2 \), while this effect is reversed for \( H < 1/2 \). The dependency on \( H \) is optimal. Indeed, if \( f \) is the identity, then for some constant \( C \) 

\[
\| \Gamma_T (f) - \widehat{\Gamma}_{n,T} (f) \|_{L^2(\mathbb{P})} \geq C \begin{cases} T^H \Delta_n, & H > 1/2, \\ T^{1/2} \Delta_n^{1+2H}, & H < 1/2. \end{cases}
\]  

(3.4)

Remark 3.9 applies here as well in order to relax the assumption on \( X_0 \). In particular, we can directly consider \( X_t = B_t \) if \( d = 1 \). Comparing the theorem (at least for \( H < 1/2 \)) to Example 3.11 suggests that there is a more general result for self-similar processes with self-similarity index \( \alpha \) and upper bound \( \| f \|_{H^s} T^{1/2} \Delta_n^{1/2+\alpha s} \).

The key idea in the proof is that fractional Brownian motion is locally nondeterministic. There are many more processes (and random fields) with this property. In principle, the proof of the theorem will apply in these cases as well, as long as the time derivatives of \( \Phi_{h,r}(u,v) \) can be controlled. This holds, for instance, for multifractional Brownian motion with time varying Hurst index \( H(t) \) (cf. Boufoussi et al. (2007)) and stochastic differential equations driven by fractional Brownian motion (cf. Lou and Ouyang (2017)).

We will now apply Theorem 3.13 to approximate local times from discrete data. Let \( d = 1 \) and let \( (L_T^a)_{a \in \mathbb{R}} \) be the family of local times of \( B \) until \( T \) which satisfies the occupation time formula \( \int_0^T g(B_r) \, dr = \int_0^T g(x) L_T^a \, dx \) for every continuous and bounded function \( g \) (cf. Nualart (1995, Chapter 5)). We can write \( L_T^a = \delta_a (L_T) \) for
\( a \in \mathbb{R} \), where \( \delta_a \) is the Dirac delta function. Note that \( \delta_a \in H^{-1/2}_-(\mathbb{R}) \) has negative regularity. Theorem 3.13 therefore suggests the rate \( T^{1/2}\Delta_n^{1/4} \) (for \( H = 1/2 \)). This turns out to be almost correct.

**Theorem 3.14.** Let \( T \geq 1, \ n \geq 2, \ d = 1 \). Let \( X_t = B_t \), where \( (B_t)_{0 \leq t \leq T} \) is a fractional Brownian motion with Hurst index \( 0 < H < 1 \). Consider \( f_{a,\varepsilon}(x) = (2\varepsilon)^{-1}\mathbf{1}_{(a-\varepsilon, a+\varepsilon]}(x) \) for \( x, a \in \mathbb{R} \) and \( \varepsilon = \Delta_n^a \) with \( \alpha = \frac{3}{2} \cdot \frac{H}{1+H} - \rho \) when \( H \geq 1/2 \) and \( \alpha = H - \rho \) when \( H < 1/2 \) for any small \( \rho > 0 \). Then we have for some constant \( C \), independent of \( a \), that

\[
\|L_T^a - \tilde{\Gamma}_{n,T}(f_{a,\varepsilon})\|_{L^2(\mathbb{P})} \leq C \begin{cases} 
T^H \Delta_n^{\frac{3}{2} \cdot \frac{H}{1+H} - \rho}, & H \geq 1/2, \\
T^{1/2}\Delta_n^{\frac{3}{2} \cdot \frac{H}{1+H} - \rho}, & H < 1/2.
\end{cases}
\]

For Brownian motion the rate \( \Delta_n^{1/4} \) is already contained in \cite{jacob1998} and the corresponding \( L^2(\mathbb{P}) \)-bound in \cite{kohatsu-higa2014} Theorem 2.6). For \( H \) close to 1 the rate of convergence becomes arbitrarily slow, because the paths of \( B \) are almost differentiable and the occupation measure becomes more and more singular with respect to the Lebesgue measure.

### 4 Lower bounds

We will now address the important question if the upper bounds for \( \|\Gamma_T(f) - \tilde{\Gamma}_{n,T}(f)\|_{L^2(\mathbb{P})} \) derived in the last two sections are optimal. Optimality here means that the upper bounds cannot be improved uniformly for all \( f \) belonging to a given class of functions. For this it is sufficient to find a candidate \( f \) where the error \( \|\Gamma_T(f) - \tilde{\Gamma}_{n,T}(f)\|_{L^2(\mathbb{P})} \) matches the upper bound up to an absolute constant. The only explicit lower bound in the literature has been established by \cite{ngo2011} for Brownian motion in \( d = 1 \) and indicator functions \( f = \mathbf{1}_{[a,b]} \), matching the upper bound \( \Delta_n^{3/4} \).

Apart from optimality with respect to the Riemann-sum estimator, it is interesting from a statistical point of view to ask for optimality across all possible estimators. Note that \( \|\Gamma_T(f) - \tilde{\Gamma}_{n,T}(f)\|_{L^2(\mathbb{P})} \) is bounded from below by

\[
\inf_g \|\Gamma_T(f) - g\|_{L^2(\mathbb{P})} = \|\Gamma_T(f) - \mathbb{E}[\hat{\Gamma}_T(f)|G_n]\|_{L^2(\mathbb{P})}, \tag{4.1}
\]

where \( G_n = \sigma(X_{t_k} : 0 \leq k \leq n) \) and where the infimum is taken over all \( G_n \)-measurable random variables. If \( f \) is the identity, then it is well-known that \( \mathbb{E}[\Gamma_T(f)|G_n] = \hat{\Theta}_{n,T}(f) \), where \( \hat{\Theta}_{n,T}(f) \) is the trapezoid rule estimator from Section 2.1 (see e.g. \cite{diaconis1988}). If \( f \in H^1(\mathbb{R}^d) \), then this still holds approximately.

The methods from Section 2 allow for identifying the limit of the right hand side in (4.1) as \( n \to \infty \), yielding an explicit lower bound valid for all \( f \in H^1(\mathbb{R}^d) \). For the \( L^2 \)-Sobolev spaces \( H^s(\mathbb{R}^d) \) with \( 0 < s < 1 \) such a universal result is not possible. Instead, we derive a lower bound for an explicit candidate matching the upper bound established in Example 3.10.

**Theorem 4.1.** Let \( T \geq 1 \) and let \( X_t = X_0 + W_t \), where \( (W_t)_{0 \leq t \leq T} \) is a \( d \)-dimensional Brownian motion and where \( X_0 \) satisfies \( (X0) \).
(i) We have for any $f \in H^1(\mathbb{R}^d)$ the asymptotic lower bound
\[
\liminf_{n \to \infty} \left( \Delta_n^{-1} \| \Gamma_T (f) - \hat{\Gamma}_{n,T} (f) \|_{L^2(\mathbb{P})} \right) \\
\geq \liminf_{n \to \infty} \left( \Delta_n^{-1} \inf_g \| \Gamma_T (f) - g \|_{L^2(\mathbb{P})} \right) \\
= \mathbb{E} \left[ \frac{1}{12} \int_0^T \| \nabla f (X_t) \|^2 dt \right]^{1/2},
\]
where the infimum is taken over all $\mathcal{G}_n$-measurable random variables.

(ii) Let $f_\alpha \in L^2(\mathbb{R}^d)$, $0 < \alpha < 1$, be the $L^2(\mathbb{R}^d)$ function with Fourier transform $\mathcal{F} f_\alpha (u) = (1 + \| u \|)^{-\alpha - d/2}$, $u \in \mathbb{R}^d$. Then $f_\alpha \in H^s(\mathbb{R}^d)$ for all $0 \leq s < \alpha$, but $f_\alpha \notin H^{\alpha}(\mathbb{R}^d)$. Moreover, $f_\alpha$ satisfies for all $0 \leq s < \alpha$ the asymptotic lower bound
\[
\liminf_{n \to \infty} \left( \Delta_n^{-\frac{s+1}{2}} \| \Gamma_T (f_\alpha) - \hat{\Gamma}_{n,T} (f_\alpha) \|_{L^2(\mathbb{P})} \right) \\
\geq \liminf_{n \to \infty} \left( \Delta_n^{-\frac{s+1}{2}} \inf_g \| \Gamma_T (f_\alpha) - \hat{\Gamma}_{n,T} (f_\alpha) \|_{L^2(\mathbb{P})} \right) > 0.
\]

For $d = 1$ the lower bounds also hold for $X_t = W_t$ (cf. Remark 3.9). Interestingly, the asymptotic lower bound in (i) corresponds exactly to the asymptotic variance obtained for the CLTs in Section 2. This proves the asymptotic efficiency of $\hat{\Gamma}_{n,T}(f)$ and $\hat{\Theta}_{n,T}(f)$ for $f \in H^1(\mathbb{R}^d)$. Note that Brownian motion is a major example for the upper bounds derived in the last section.

The key step in the proof is to calculate the conditional expectation $\mathbb{E}[\Gamma_T (f) | \mathcal{G}_n]$, which reduces to Brownian bridges interpolating between the observations. The same calculations hold when $X$ is a Lévy process with finite first moments (cf. Jacod and Protter (1988, Theorem 2.6)) and similarly when $X$ belongs to a certain class of Markov processes (cf. Chaumont and Uribe Bravo (2011)).

**Appendix A: Proofs of Section 2**

In the following, $T$ is fixed and $\Delta_n \to 0$ as $n \to \infty$. Consider first the following preliminary observations.

### A.1 Localization

By a well-known localization procedure (cf. Jacod and Protter (2011, Lemma 4.4.9)) and Assumption ([SM-α-β]) it is sufficient to prove the central limit theorems under the following stronger Assumption.

**Assumption (H-α-β).** Let $0 \leq \alpha, \beta \leq 1$. There exists a constant $K$ such that almost surely
\[
\sup_{0 \leq t \leq T} \left( \| X_t \| + \| b_t \| + \| \sigma_t \| + \| (\sigma_t \sigma_t^T)^{-1} \| \right) \leq K
\]
and such that for all $0 \leq s, t \leq T$ with $s + t \leq T$

$$\mathbb{E} \left[ \sup_{0 \leq r \leq t} \| \sigma_{s+r} - \sigma_s \|^2 \right] \leq C t^{2\alpha}, \quad \mathbb{E} \left[ \sup_{0 \leq r \leq t} \| b_{s+r} - b_s \|^2 \right] \leq C t^{2\beta}.$$

In this case we only have to consider $f$ with compact support. Indeed, if $f \in FL^s_{\text{loc}}(\mathbb{R}^d)$ ($f \in H^s_{\text{loc}}(\mathbb{R}^d)$), is replaced by $\tilde{f} = f \varphi$ for a smooth cutoff function $\varphi$ with compact support in a ball $B_{K+\varepsilon} = \{ x \in \mathbb{R}^d : \| x \| \leq K + \varepsilon \}$ of radius $K + \varepsilon$, $\varepsilon > 0$, and $\varphi = 1$ on $B_K$, then $\tilde{f} = f$ on $B_K$ and $\tilde{f} \in FL^s(\mathbb{R}^d)$ ($\tilde{f} \in H^s(\mathbb{R}^d)$).

### A.2 Preliminary estimates

We will use different approximations for $X$. For $\varepsilon > 0$ and $t \geq 0$ let $t_\varepsilon = \max([t/\varepsilon] \varepsilon - 2\varepsilon, 0)$ and define the processes

$$X_t(\varepsilon) = \int_0^t b_{\lfloor r/\varepsilon \rfloor} \varepsilon dr + \int_0^t \sigma_{\lfloor r/\varepsilon \rfloor} \varepsilon dW_r,$$

$$\tilde{X}_t(\varepsilon) = X_{t_\varepsilon} + b_{t_\varepsilon}(t - t_\varepsilon) + \sigma_{t_\varepsilon} (W_t - W_{t_\varepsilon}).$$

Then the following estimates hold by the Burkholder-Davis-Gundy inequality. The reason for introducing $\tilde{X}(\varepsilon)$ instead of $X(\varepsilon)$ is the first inequality in (iii) which improves on the second.

**Proposition A.1.** Let $p \geq 1$. Assume $[H_{\alpha-\beta}]$ for $0 \leq \alpha, \beta \leq 1$. Then the following holds for some absolute constant $C_p$ and all $0 \leq s, t \leq T$ with $s + t \leq T$:

(i) $\mathbb{E} \left[ \| Z_{s+t} - Z_s \|^p \right] \leq C_p t^{p/2}$ for $Z = X, X(\varepsilon), \tilde{X}(\varepsilon)$

(ii) $\mathbb{E}[\| X_{s+t} - X_s - (X_{s+t}(\varepsilon) - X_s(\varepsilon)) \|^p] \leq C_p t^{p/2} \varepsilon^{\alpha p}$,

(iii) $\mathbb{E}[\| X_t - \tilde{X}_t(\varepsilon) \|^p] \leq C_p (\varepsilon^{(1+\beta)p} + \varepsilon^{(\alpha+1/2)p})$, $\mathbb{E}[\| X_t - X(t) \|^p] \leq C_p (\varepsilon^{\beta p} + \varepsilon^{\alpha p})$,

(iv) $\mathbb{E}[\| X_{s+t} - X_s - (\tilde{X}_{s+t}(\varepsilon) - \tilde{X}_s(\varepsilon)) \|^p] \leq C_p t^{p/2} (\varepsilon^{(3+1/2)p} + \varepsilon^{\alpha p})$.

The main estimates for the proofs of Theorems 2.5 and 2.7 are collected in the following lemma.

**Lemma A.2.** Assume $[H_{\alpha-\beta}]$ for $0 \leq \alpha, \beta \leq 1$ and let either $f \in C^1(\mathbb{R}^d)$ have compact support or assume $[X0]$ in addition and let $f \in H^1(\mathbb{R}^d)$ have compact support. Then it follows with $\kappa_f = \| \nabla f \|_{\infty}$ or $\kappa_f = \| f \|_{H^1}$ for $k = 1, \ldots, n$ and $t_{k-1} \leq r \leq t_k$, uniformly in $r$ and $k$:

(i) $\mathbb{E}[\| \nabla f(X_t) \|^2] = O(\kappa_f^2)$,

(ii) $\mathbb{E}[\| \nabla f(X_{t_{k-1}}), X_r - X_r(\Delta_n) \|^2] = o(\Delta_n^2 \kappa_f^2)$,

(iii) $\mathbb{E}[\| f(X_r) - f(X_{t_{k-1}}) - \langle \nabla f(X_{t_{k-1}}), X_r - X_{t_{k-1}} \rangle \|^2] = o(\Delta_n^2 \kappa_f^2)$,

(iv) $\mathbb{E}[\| \nabla f(X_r) - \nabla f(X_{t_{k-1}}) \|^2] = o(\kappa_f^2)$,

(v) $\mathbb{E}[\sup_t | \sum_{k=1}^{[t/\Delta_n]} \int_{t_{k-1}}^{t_k} (t_k - r - \Delta_n/2) \mathbb{E}[\langle \nabla f(X_r), b_r \rangle | F_{t_{k-1}}] dr |] = o(\Delta_n \kappa_f^2)$. 

18
Proof. For \( f \in C^1(\mathbb{R}^d) \) we only prove (v). The other statements follow from the boundedness of \( \nabla f \) and Proposition A.1 (v) follows immediately for bounded and continuous \( b \), because \( \langle \nabla f (X_t), b_t \rangle \) can be approximated uniformly at the left end \( \langle \nabla f (X_{t_k}), b_{t_k} \rangle \). For bounded \( b \) let \( g_e \) be continuous and adapted processes such that \( \sup_{0 \leq t \leq T} \| g_{e,t} \| < \infty \) uniformly in \( \varepsilon \) and \( \mathbb{E} [ \int_0^T \| b_t - g_{e,h} \| dh ] \to 0 \) as \( \varepsilon \to 0 \). Then (v) holds for \( g_e \) and by approximation also for \( b \).

For \( f \in H^1(\mathbb{R}^d) \) we argue differently. Under (X0) the marginals \( X_t \) have uniformly bounded Lebesgue densities \( p_t \). Hence (i) follows from

\[
\mathbb{E} [ \| \nabla f (X_t) \|^2 ] = \sum_{m=1}^d \int (\partial_m f (x))^2 p_t (x) \, dx \lesssim \| f \|^2_{H^1}. \tag{A.1}
\]

With respect to (ii) consider first \( f \in \mathcal{S}(\mathbb{R}^d) \). By inverse Fourier transform and \( \mathcal{F}(\nabla f)(u) = iu \mathcal{F}(f)(u) \), \( u \in \mathbb{R}^d \), it follows that \( \langle \nabla f (X_{t_k}), X_t - X_t(\Delta_n) \rangle \) is equal to

\[
(2\pi)^{-2d} \left( \int \mathcal{F}f(u) \langle u, X_t - X_t(\Delta_n) \rangle e^{-i \langle u, X_{t_k} - x_0 \rangle} e^{-i \langle u, x_0 \rangle} du \right)^2 \]
\[
= -(2\pi)^{-2d} \int \mathcal{F}f(u) \mathcal{F}f(v) \langle u, X_t - X_t(\Delta_n) \rangle \cdot \langle v, X_t - X_t(\Delta_n) \rangle e^{-i \langle u + v, X_{t_k} - x_0 \rangle} e^{-i \langle u + v, x_0 \rangle} d(u,v).
\]

As \( X_0 \) and \( (X_t - X_0)_{0 \leq t \leq T} \) are independent, \( \mathbb{E} [ \langle \nabla f (X_{t_k}), X_t - X_t(\Delta_n) \rangle ] \) is up to a constant bounded by

\[
\left( \int |\mathcal{F}f(u)| |\mathcal{F}f(v)| |u||v||\mathcal{F}\mu(u+v)| d(u,v) \right) \mathbb{E} [ \| X_t - X_t(\Delta_n) \|^2 ],
\]

which is of order \( o(\Delta_n \| f \|^2_{H^1}) \) by Lemma A.3 (see below) and Proposition A.1. This yields (ii) for \( f \in \mathcal{S}(\mathbb{R}^d) \). For \( f \in H^1(\mathbb{R}^d) \) consider a sequence \( (f_m)_{m \geq 1} \subset \mathcal{S}(\mathbb{R}^d) \) converging to \( f \) with respect to \( \| \cdot \|_{H^1} \). Then \( \| X_t - X_t(\Delta_n) \| \leq \| X_t \| + \| X_t(\Delta_n) \| \lesssim 1 + \| W_t - W_{t_k} \| \). Independence yields

\[
\| \langle \nabla f (X_{t_k}), X_t - X_t(\Delta_n) \rangle \|_{L^2(\mathbb{P})} \leq \| \langle \nabla f_m (X_{t_k}), X_t - X_t(\Delta_n) \rangle \|_{L^2(\mathbb{P})} \lesssim \mathbb{E} [ \| \nabla (f - f_m) (X_{t_k}) \|^2 ]^{1/2} \mathbb{E} [ (1 + \| W_t - W_{t_k} \|) ]^{1/2} \lesssim \| f - f_m \|_{H^1} \to 0,
\]
as \( m \to \infty \). Hence (ii) also holds for \( f \in H^1(\mathbb{R}^d) \). With respect to (iii) consider again first \( f \in \mathcal{S}(\mathbb{R}^d) \). Arguing by inverse Fourier transform, the left hand side is because of Taylor’s theorem bounded by

\[
\int_0^1 \mathbb{E} \left[ \langle \nabla f (X_{t_k} + t (X_t - X_{t_k})) - \nabla f (X_{t_k}), X_t - X_{t_k} \rangle \right] dt
\]
\[
\lesssim \int |\mathcal{F}f(u)| |\mathcal{F}f(v)| |u||v| \mathbb{E} [ g_n(u) g_n(v) ] |\mathcal{F}\mu(u+v)| d(u,v)
\]
\[
\quad \cdot \mathbb{E} [ \| X_t - X_{t_k} \|^4 ]^{1/2},
\]
where \( g_n(u) = \sup_{r,h:|r-h| \leq \Delta_n} \int_0^1 |1 - e^{-i(u,X_n)}|^2 dt \) and where we applied the Cauchy-Schwarz inequality twice. Lemma A.3 together with \( \mathbb{E} [ \| X_t - X_{t_k} \|^4 ]^{1/2} = \]
O(Δ_n) shows that the left hand side in (iii) is for \( f \in \mathcal{S}(\mathbb{R}^d) \) up to a constant bounded by
\[
\Delta_n \int |\mathcal{F}f(u)|^2 \|u\|^2 \mathbb{E} \left[ g_n^2(u) \right]^{1/2} du.
\]
A similar approximation argument as for (ii) yields the same bound for \( f \in H^1(\mathbb{R}^d) \). \( g_n(u) \) is bounded in \( n \) and \( u \) and converges \( \mathbb{P} \)-almost surely to 0 as \( n \to \infty \) for any \( u \in \mathbb{R}^d \). By dominated convergence the last display is thus of order \( o(Δ_n) \). This yields (iii). (iv) is proved similarly. For (v) and bounded and continuous \( b \) the claim follows from
\[
\langle \nabla f(X_r), b_r \rangle - \langle \nabla f(X_{t_{k-1}}), b_{t_{k-1}} \rangle = \langle \nabla f(X_r) - \nabla f(X_{t_{k-1}}), b_r - b_{t_{k-1}} \rangle.
\]
part (iv) and (A.1). For bounded \( b \) argue as in part (v) for \( f \in C^1(\mathbb{R}^d) \).

Lemma A.3. Let \( \xi \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) and let \( \mu \) be a probability density on \( \mathbb{R}^d \).

(i) If \( \mathcal{F}\mu \in L^1(\mathbb{R}^d) \), then
\[
\int |\xi(u)\xi(v)| |\mathcal{F}\mu(u + v)| d(u,v) \lesssim \|\mathcal{F}\mu\|_{L^1} \|\xi\||^2_{L^2}.
\]

(ii) If \( \mathcal{F}\mu \) is non-negative and \( \mu \) is bounded, then the upper bound is instead
\[
\|\mu\|_\infty \|\xi\|_{L^2}^2.
\]

(iii) If \( \mu \) is the density of the \( N(0,I_d) \)-distribution, then
\[
\int \prod_{j=1}^p |\xi(u_j)| \mathcal{F}\mu \left( \sum_{j=1}^p u_j \right) d(u_1,...,u_p) \lesssim \|\xi\|^p_{L^2}.
\]

Proof. By a density argument we can assume that \( \xi, \mu \in \mathcal{S}(\mathbb{R}^d) \) and that \( \mathcal{F}\mu \) is non-negative in (ii). Let \( g,h \in L^2(\mathbb{R}^d) \) with \( \mathcal{F}g(u) = |\xi(u)|, \mathcal{F}h(u) = |\mathcal{F}\mu(u)| \) such that the \( d(u,v) \) integral is equal to
\[
\int \mathcal{F}g(u) \mathcal{F}g(v) \mathcal{F}h(u + v) d(u,v) = \int \mathcal{F}g(u - v) \mathcal{F}g(v) \mathcal{F}h(u) d(u,v)
\]
\[
= \int (\mathcal{F}g * \mathcal{F}g)(u) \mathcal{F}h(u) du = \int \mathcal{F}g^2(u) \mathcal{F}h(u) du = C \int g^2(u) h(u) du,
\]
where we used the Plancherel Theorem in the last line. If \( \mathcal{F}\mu \in L^1(\mathbb{R}^d) \), then the last line is bounded by
\[
C \|g\|_{L^2}^2 \|h\|_\infty \lesssim \|\xi\|_{L^2}^2 \sup_{u \in \mathbb{R}^d} \left| \int \mathcal{F}h(x) e^{i(u,x)} dx \right| \lesssim \|\mathcal{F}\mu\|_{L^1} \|\xi\|_{L^2}^2.
\]
If, on the other hand, $\mathcal{F} \mu$ is non-negative, then $h(u) = \mathcal{F} \mathcal{F} h(-u) = \mu(-u)$ and therefore (A.2) is bounded by
\[
C \| g \|^2_{L^2} \| h \|_\infty \lesssim \| \mu \|_\infty \| \xi \|^2_{L^2}.
\]
This shows (i) and (ii). With respect to (iii) the left hand side of the claimed inequality can be written as $\int (\mathcal{F}g * \cdots * \mathcal{F}g)(u) \mathcal{F} \mu(u) du$, where $\mathcal{F}g * \cdots * \mathcal{F}g$ is the $p$-fold convolution product. Since $\mathcal{F}g * \cdots * \mathcal{F}g = \mathcal{F}g^p$, this is also equal to
\[
\int \mathcal{F} (g^p) (u) \mathcal{F} \mu (u) du = C \int g^p (u) \mu (u) du = \left( \int (\mathcal{F} (\mathcal{F}g)(u) \mathcal{F} (\mathcal{F} \mu)^{1/p} (u))^p \right) du
\]
\[
= \int \left( \mathcal{F} (\mathcal{F}g * \mathcal{F} \mu^{1/p} (u)) \right)^p du \lesssim \left( \int |\mathcal{F}g * \mathcal{F} \mu^{1/p} (u)|^{p/(p-1)} du \right)^{p-1}
\]
\[
\lesssim \| \mathcal{F}g \|^p_{L^2} = \| \xi \|^p_{L^2},
\]
where we applied in the first equality the Plancherel Theorem and for the last two inequalities the Hausdorff-Young and the Young inequalities, because $\mathcal{F}g \in L^2(\mathbb{R}^d)$ and $\mathcal{F} \mu^{1/p} \in L^q(\mathbb{R}^d)$ for any $q > 0$.

**A.3 Proof of Theorem 2.5.**

It is enough to show the CLT in (2.2) for $f \in FL^\alpha_{loc}(\mathbb{R}^d)$, which immediately yields the claim in terms of $\Gamma_t(f) - \hat{\Theta}_{n,t}(f)$. Recall the decomposition $\Gamma_t(f) - \hat{\Theta}_{n,t}(f) = M_{n,t}(f) + D_{n,t}(f)$ with $M_{n,t}(f)$ and $D_{n,t}(f)$ as in (2.3) and (2.4). By the localization argument in A.1 and because $FL^s(\mathbb{R}^d) \subset C^1(\mathbb{R}^d)$ for $s \geq 1$ the proof follows from the following two propositions.

**Proposition A.4.** Assume $[\text{H-} \alpha, - \beta]$ for $0 \leq \alpha, \beta \leq 1$. Let $f \in C^1(\mathbb{R}^d)$ have compact support. Then we have the stable convergence
\[
\Delta_n^{-1} M_{n,t} (f) \overset{s}{\Rightarrow} \frac{1}{2} \int_0^t \langle \nabla f (X_r), \sigma_r dW_r \rangle + \frac{1}{\sqrt{12}} \int_0^t \langle \nabla f (X_r), \sigma_r d\tilde{W}_r \rangle
\]
as processes on $\mathcal{D}([0,T], \mathbb{R}^d)$, where $\tilde{W}$ is a $d$-dimensional Brownian motion defined on an independent extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$.

**Proposition A.5.** Assume $[\text{H-} \alpha, - \beta]$ for $0 \leq \alpha, \beta \leq 1$. Let $s > 2 - 2\alpha$, $s \geq 1$, $s + \beta > 1$. Then we have for $f \in FL^s(\mathbb{R}^d)$ with compact support that
\[
\Delta_n^{-1} D_{n,t} (f) \overset{\text{w.p.}}{\Rightarrow} \frac{1}{2} (f(X_t) - f(X_0)) - \frac{1}{2} \int_0^t \langle \nabla f (X_r), \sigma_r dW_r \rangle.
\]

We note in the proofs precisely where Lemma A.2 is used. This will allow us later to deduce Theorem 2.7 by small modifications.

**Proof of Proposition A.4.** We write $M_{n,t}(f) = \sum_{k=1}^{[t/\Delta_n]} Z_k$ and $\hat{M}_{n,t}(f) = \sum_{k=1}^{[t/\Delta_n]} \hat{Z}_k$ for random variables
\[
Z_k = \int_{t_{k-1}}^{t_k} \left( f(X_r) - \mathbb{E} \left[ f(X_r) \mid \mathcal{F}_{t_{k-1}} \right] \right) dr, \tag{A.4}
\]
\[
\hat{Z}_k = \int_{t_{k-1}}^{t_k} \langle \nabla f(X_{t_{k-1}}), X_r (\Delta_n) - \mathbb{E} \left[ X_r (\Delta_n) \mid \mathcal{F}_{t_{k-1}} \right] \rangle dr. \tag{A.5}
\]
\[ \tilde{Z}_k \text{ "linearizes" } Z_k \text{ with respect to } f. \] The proof is based on the following statements for \( 0 \leq t \leq T \):

\begin{equation}
\Delta_n^{-1} \sup_{0 \leq t \leq T} \left| M_{n,t} (f) - \tilde{M}_{n,t} (f) \right| \overset{P}{\to} 0, \quad (A.6)
\end{equation}

\begin{equation}
\Delta_n^{-2} \sum_{k=1}^{[t/\Delta_n]} \mathbb{E} \left[ \tilde{Z}_k^2 \mathbb{1}_{\{|Z_k| > \epsilon\}} \right| \mathcal{F}_{t_{k-1}} \right] \overset{P}{\to} 0, \quad \text{for all } \epsilon > 0, \quad (A.7)
\end{equation}

\begin{equation}
\Delta_n^{-1} \sum_{k=1}^{[t/\Delta_n]} \mathbb{E} \left[ \tilde{Z}_k (W_t - W_{t_{k-1}})^\top \right| \mathcal{F}_{t_{k-1}} \right] \overset{P}{\to} 0, \quad (A.8)
\end{equation}

\begin{equation}
\Delta_n^{-1} \sum_{k=1}^{[t/\Delta_n]} \mathbb{E} \left[ \tilde{Z}_k (N_{t_k} - N_{t_{k-1}}) \right| \mathcal{F}_{t_{k-1}} \right] \overset{P}{\to} 0, \quad (A.9)
\end{equation}

\begin{equation}
\Delta_n^{-1} \sum_{k=1}^{[t/\Delta_n]} \mathbb{E} \left[ \tilde{Z}_k (N_{t_k} - N_{t_{k-1}}) \right| \mathcal{F}_{t_{k-1}} \right] \overset{P}{\to} 0, \quad (A.10)
\end{equation}

where \( (A.10) \) holds for all bounded \((\mathbb{R}\text{-valued})\) martingales \( N \) which are orthogonal to all components of \( W \). \( (A.6) \) yields \( M_{n,t}(f) = \tilde{M}_{n,t}(f) + o_{ucp}(\Delta_n) \). The claim follows thus from the remaining statements \((A.7)\) through \((A.10)\) and Theorem 7.28 of Jacod and Shiryaev (2013).

We prove now the five statements above. \( M_{n,t}(f) - \tilde{M}_{n,t}(f) \) is a discrete martingale such that by the Burkholder-Davis-Gundy inequality

\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} (M_{n,t}(f) - \tilde{M}_{n,t}(f))^2 \right] \leq \sum_{k=1}^{n} \mathbb{E} \left[ (Z_k - \tilde{Z}_k)^2 \right]. \]

Decompose any such \( Z_k - \tilde{Z}_k \) into

\begin{equation}
\int_{t_{k-1}}^{t_k} (A_{k,r} - \mathbb{E} [A_{k,r} \mathcal{F}_{t_{k-1}}]) \, dr \quad (A.11)
\end{equation}

\begin{equation}
+ \int_{t_{k-1}}^{t_k} \langle \nabla f (X_{t_{k-1}}), X_r - X_r(\Delta_n) - \mathbb{E} [X_r - X_r(\Delta_n) \mathcal{F}_{t_{k-1}}] \rangle \, dr, \quad (A.12)
\end{equation}

where \( A_{k,r} = f(X_r) - f(X_{t_{k-1}}) - \langle \nabla f(X_{t_{k-1}}), X_r - X_{t_{k-1}} \rangle \). The second moment of \((A.12)\) is bounded by \( 2\Delta_n \int_{t_{k-1}}^{t_k} \mathbb{E} (\langle \nabla f(X_{t_{k-1}}), X_r - X_r(\Delta_n) \rangle^2) \, dr = o(\Delta_n^2) \) using Lemma \((A.2)(ii)\). The same order follows for the second moment of \((A.11)\) from Lemma \((A.2)(iii)\). This yields \((A.6)\). In order to prove the remaining statements observe first by the (stochastic) Fubini theorem that \( \tilde{Z}_k \) is equal to

\[ \langle \nabla f(X_{t_{k-1}}), \int_{t_{k-1}}^{t_k} (t_k - r)(b_r - \mathbb{E}[b_r \mathcal{F}_{t_{k-1}}]) \rangle \, dr \]

\[ + \langle \nabla f(X_{t_{k-1}}), \sigma_{t_{k-1}} \int_{t_{k-1}}^{t_k} (t_k - r) \, dW_r \rangle. \]

The first term is of smaller order than the second one. By Itô isometry, because \( \sigma \) is càdlàg and from Lemma \((A.2)(i)\), \((iv)\) it therefore follows that the left hand side in
The first sum is just \( \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} \sigma_{tk-1}^\top \nabla f (X_{tk-1}) \|^2 + o_P(\Delta_n) = \frac{1}{3} \int_0^t \| \sigma_r^\top \nabla f (X_r) \|^2 dr + o_P(1) \).

With respect to (A.8), note that \( \tilde{Z}_k > \varepsilon \) implies \( \int_{tk}^{tk+1} (t_k-r) dW_r \geq \varepsilon' \) for some \( \varepsilon' > 0 \) and sufficiently large \( n \) because of the Cauchy-Schwarz inequality. Consequently, it follows from Lemma (A.2) and independence that

\[
\mathbb{E} \left[ \tilde{Z}_k^2 1\{|\tilde{Z}_k| > \varepsilon\} \right] \leq \mathbb{E} \left[ \| \nabla f (X_{tk-1}) \|^2 \right] \left( \Delta_n^4 + \mathbb{E} \left[ \int_{tk}^{tk+1} (t_k-r) dW_r \|^4 \right] \right),
\]

which is of order \( O(\Delta_n^4) \), thus implying (A.8). The left hand side of (A.9), on the other hand, is equal to \( R_n + \frac{\Delta_n^2}{2} \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} \nabla f (X_{tk-1}) \sigma_{tk-1} \) with \( \mathbb{E}[\| R_n \|] = o(1) \) by Itô’s isometry (applied coordinatewise). (A.9) follows then from \( \sigma \) being càdlàg and (A.2) (iv). The same argument shows that the left hand side in (A.10) is of order \( o_P(1) \).

**Proof of Proposition A.5** Lemma (A.6) below shows

\[
D_{n,t} (f) = \frac{\Delta_n}{2} \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E} \left[ f(X_{tk}) - f(X_{tk-1}) \right| \mathcal{F}_{tk-1} ] + o_{ucp}(\Delta_n).
\]  

(A.13)

In order to find the limit of this sum, write it as

\[
\frac{\Delta_n}{2} \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E} \left[ A_k \right| \mathcal{F}_{tk-1} ] + \frac{\Delta_n}{2} \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E} \left[ B_k \right| \mathcal{F}_{tk-1} ],
\]  

(A.14)

where \( A_k = f (X_{tk}) - f (X_{tk-1}) - B_k \) and \( B_k = \langle \nabla f (X_{tk-1}), X_{tk} - X_{tk-1} \rangle \). Note that by the Burkholder-Davis-Gundy inequality

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E} \left[ A_k \right| \mathcal{F}_{tk-1} ] - A_k \right) \right] \leq \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E} \left[ A_k^2 \right],
\]

which is of order \( o(\Delta_n) \) by Lemma (A.2) (iii). Therefore, (A.14) is up to an error of order \( o_{ucp}(\Delta_n) \) equal to

\[
\frac{\Delta_n}{2} \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} \left( f (X_{tk}) - f (X_{tk-1}) \right) + \frac{\Delta_n}{2} \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} \left( \mathbb{E} \left[ B_k \right| \mathcal{F}_{tk-1} ] - B_k \right),
\]

The first sum is just \( \frac{\Delta_n}{2} (f (X_{\lfloor t/\Delta_n \rfloor}) - f (X_0)) = \frac{\Delta_n}{2} (f (X_{t}) - f (X_0)) + o_{ucp}(\Delta_n) \), while the second one is equal to

\[
\frac{\Delta_n}{2} \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} \int_{tk-1}^{tk} \langle \nabla f (X_{tk-1}) , \left( \mathbb{E} \left[ b_r \right| \mathcal{F}_{tk-1} ] - b_r \right) \rangle dr
\]

\[
- \frac{\Delta_n}{2} \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} \int_{tk-1}^{tk} \langle \nabla f (X_{tk-1}) , \sigma_r dW_r \rangle.
\]
This is equal to $-\frac{\Delta_n}{2} \int_0^{[t/\Delta_n]} \langle \nabla f(X_r), \sigma_r dW_r \rangle + o_{ucp}(\Delta_n)$ and the claim follows.

In the second line use Lemma A.2 (iv) and for the first line note that it is a discrete martingale of order $o_{ucp}(\Delta_n)$ by the Burkholder-Davis-Gundy inequality and Lemma A.2 (i).

We now state and prove the lemmas used above.

**Lemma A.6.** Assume $(H_{\alpha-\beta})$ for $0 \leq \alpha, \beta \leq 1$. Let $s > 2 - 2\alpha, s \geq 1, s + \beta > 1$. Then we have for $f \in FL^s(\mathbb{R}^d)$ with compact support that

$$D_{n,t}(f) - \frac{\Delta_n}{2} \sum_{k=1}^{[t/\Delta_n]} \mathbb{E}[f(X_{t_k}) - f(X_{t_{k-1}}) | \mathcal{F}_{t_{k-1}}] = o_{ucp}(\Delta_n).$$

**Proof.** Consider first $f \in \mathcal{S}(\mathbb{R}^d)$. By applying Itô’s formula and the Fubini theorem the left hand side in the statement is equal to $D_{n,t}(1, f) + D_{n,t}(2, f)$, where $D_{n,t}(1, f)$ and $D_{n,t}(2, f)$ are defined by

$$\sum_{k=1}^{[t/\Delta_n]} \int_{t_{k-1}}^{t_k} \left( t_k - r - \frac{\Delta_n}{2} \right) \mathbb{E} \left[ \langle \nabla f(X_r), b_r \rangle | \mathcal{F}_{t_{k-1}} \right] dr,$$

$$\sum_{k=1}^{[t/\Delta_n]} \int_{t_{k-1}}^{t_k} \left( t_k - r - \frac{\Delta_n}{2} \right) \mathbb{E} \left[ \frac{1}{2} \sum_{l,m=1}^{d} \partial^2_{lm} f(X_r) (\sigma_r \sigma_r^\top)^{(l,m)} | \mathcal{F}_{t_{k-1}} \right] dr.$$

We will show that

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |D_{n,t}(1, f) + D_{n,t}(2, f)| \right] \lesssim o(\Delta_n \|f\|_{FL^s}) + \Delta_n \int |\mathcal{F} f(u)| (1 + \|u\|)^s g_n(u) du,$$  \hspace{1cm} \text{(A.15)}

with $g_n$ as in Lemma A.7 below. Choose now any sequence $(f_m) \subset \mathcal{S}(\mathbb{R}^d)$ converging to $f \in FL^s(\mathbb{R}^d)$ with respect to $\|\cdot\|_{FL^s}$. This means, in particular, that $f_m$ converges to $f$ uniformly. Therefore (A.15) also holds for $f$. The properties of $g_n$ and dominated convergence therefore imply the claim.

In order to show (A.15) note first that $\mathbb{E} \left[ \sup_{0 \leq t \leq T} |D_{n,t}(1, f)| \right] = o(\Delta_n \|f\|_{FL^s})$ follows already from Lemma A.2 (v). With respect to $D_{n,t}(2, f)$ write $\Sigma_t = \sigma_t \sigma_t^\top$ and fix $l, m = 1, \ldots, d$. For $f \in \mathcal{S}(\mathbb{R}^d)$ it is always justified to exchange integrals in the following calculations. We can write $\partial^2_{lm} f(X_r) = -(2\pi)^{-d} \int \mathcal{F} f(u) u_l u_m e^{-i(u,X_r)} du$ such that

$$D_{n,t}(2, f) = -(2\pi)^{-d} \int \mathcal{F} f(u) u_l u_m Q_{n,t}(u) du,$$

where

$$Q_{n,t}(u) = \sum_{k=1}^{[t/\Delta_n]} \int_{t_{k-1}}^{t_k} \left( t_k - r - \frac{\Delta_n}{2} \right) \mathbb{E} \left[ e^{-i(u,X_r) \Sigma_r^{(l,m)}} | \mathcal{F}_{t_{k-1}} \right] dr.$$
Consequently, because of

\[ E \left[ \sup_{0 \leq t \leq T} \left| \mathcal{F} f (u) u_t u_{\alpha} \right| \right] \leq \int |\mathcal{F} f (u)| ||u||^2 E \left[ \sup_{0 \leq t \leq T} |Q_{n,t} (u)| \right] du, \]

the remaining part of (A.15) follows from Lemma A.7.

The following lemma is stronger than necessary here. This will become useful for Theorem 2.7 and Corollary 2.9.

**Lemma A.7.** Assume (H-$\alpha$-$\beta$) for $0 \leq \alpha, \beta \leq 1$. Let $s > 2 - 2\alpha$, $s \geq 1$, $s + \beta > 1$. Then we have for any $p \geq 1$ and uniformly in $u \in \mathbb{R}^d$ that

\[ \| \sup_{0 \leq t \leq T} Q_{n,t} (u) \|_{L^p (\mathbb{P})} \leq C_p \Delta_n (1 + ||u||)^{s-2} g_n (u), \]

where $\sup_{n \geq 1} \sup_{u \in \mathbb{R}^d} |g_n (u)| < \infty$ and $g_n (u) \to 0$ for all $u \in \mathbb{R}^d$ as $n \to \infty$.

**Proof.** The proof is separated into five steps.

1. **Step 1.** Let $0 < \varepsilon \leq 1$ and define $t_\varepsilon = \max([t/\varepsilon] \varepsilon - 2 \varepsilon, 0)$ for $0 \leq t \leq T$. Define $t_\varepsilon$ as the grid point to the grid $\{0, \varepsilon, 2 \varepsilon, \ldots, [T/\varepsilon] \varepsilon \}$ such that $t - t_\varepsilon \leq 3 \varepsilon$ and $t - t_\varepsilon \geq \min(2 \varepsilon, t)$. Later, we will choose $\varepsilon$ depending on $n$ and $u$, i.e. $\varepsilon = \varepsilon(u, n)$. Define $X_t(\varepsilon) = X_t + b_{t_\varepsilon}(t - t_\varepsilon) + \sigma_t(W_t - W_{t_\varepsilon})$. Assumption (H-$\alpha$-$\beta$) implies $E[(\sum_{l,m} (l,m) - \sum_{l,m} (l,m))|^p] \leq \varepsilon^{\alpha p}$ and Proposition A.1 yields $E[||X_t - \tilde{X}_t(\varepsilon)||]^p] \leq \varepsilon^{(\beta + 1)p} + \varepsilon^{(\alpha + 1/2)p}$. Define

\[ Q_{n,t}(\varepsilon, u) = \sum_{k=1}^{t/\Delta_n} \int_{t_{k-1}}^{t_k} \left( t_k - r - \frac{\Delta_n}{2} \right) E \left[ e^{-i(u, \tilde{X}_r(\varepsilon)) \sum_{r}\epsilon_r} \right] \mathcal{F}_{t_{k-1}} dr. \]

The Lipschitz-continuity of $x \mapsto e^{ix}$ therefore yields

\[ \| \sup_{0 \leq t \leq T} (Q_{n,t}(u) - Q_{n,t}(\varepsilon, u)) \|_{L^p(\mathbb{P})} \leq \Delta_n \left( \int_0^T E \left[ \left| e^{-i(u, \tilde{X}_r(\varepsilon)) \sum_{r}\epsilon_r} - e^{-i(u, \tilde{X}_r(\varepsilon)) \sum_{r}\epsilon_r} \right|^p \right] dr \right)^{1/p} \]

\[ \leq \Delta_n (1 + ||u||)^{s-2} g_{n,1}(u), \]

with $g_{n,1}(u) = (1 + ||u||)^{2-s}(\varepsilon^{\alpha} + ||u||^{\varepsilon+\beta} + ||u||^{\varepsilon+1/2}).$ We study now $Q_{n,t}(\varepsilon, u)$.

2. **Step 2.** With respect to the new grid $\{0, \varepsilon, 2 \varepsilon, \ldots, [T/\varepsilon] \varepsilon \}$ and $0 \leq t \leq T$ let

\[ I_j(t) = \{ k = 1, \ldots, [t/\Delta_n] : (j-1) \varepsilon < t_k \leq j \varepsilon \}, \quad 1 \leq j \leq [T/\varepsilon], \]

be the set of blocks $k \leq [t/\Delta_n]$ with right endpoints $t_k \leq t$ inside the interval $(j-1)\varepsilon, j\varepsilon]$. Then $Q_{n,t}(\varepsilon, u) = \sum_{j=1}^{\lceil T/\varepsilon \rceil} R_{j,t}(u) + \sum_{j=1}^{\lceil T/\varepsilon \rceil} E[A_{j,t}(u) | \mathcal{F}_{(j-1)\varepsilon}]$ for $R_{j,t}(u) = A_{j,t}(u) - E[A_{j,t}(u) | \mathcal{F}_{(j-1)\varepsilon}]$ and where

\[ A_{j,t}(u) = \sum_{k \in I_j(t)} \int_{t_{k-1}}^{t_k} \left( t_k - r - \frac{\Delta_n}{2} \right) \xi_{r,k} dr, \quad \xi_{r,k} = E \left[ e^{-i(u, \tilde{X}_r(\varepsilon)) \sum_{r}\epsilon_r} \right] \mathcal{F}_{t_{k-1}}. \]
such that \( A_{j,t}(u) \) is \( \mathcal{F}_{je} \)-measurable for fixed \( u \). We want to show that 
\[
\sup_{0 \leq t \leq T} \left| \sum_{j=1}^{\lceil T/\varepsilon \rceil} R_{j,t}(u) \right| \text{ is negligible. Note first that } I_j(t) = 0 \text{ for } t \leq (j - 1)\varepsilon \text{ and } I_j(t) = I_j(T) \text{ for } t > j\varepsilon. \text{ Therefore, } A_{j,t}(u) = 0 \text{ for } t \leq (j - 1)\varepsilon \text{ and } A_{j,t}(u) = A_{j,T}(u) \text{ for } t > j\varepsilon. \text{ Denote by } j^* \text{ the unique } j \in \{1, \ldots, \lceil T/\varepsilon \rceil \} \text{ with } (j - 1)\varepsilon < t \leq j\varepsilon. \text{ Then } \sum_{j=1}^{\lceil T/\varepsilon \rceil} R_{j,t}(u) = B_{j-1}(u) + R_{j^*,t}(u), \text{ where } B_m(u) = \sum_{j=1}^{m} R_{j,T}(u) \text{ defines a complex-valued martingale } (B_m(u))_{m=0, \ldots, \lceil T/\varepsilon \rceil} \text{ with respect to the filtration } (\mathcal{F}_{me})_{m=0, \ldots, \lceil T/\varepsilon \rceil}. \text{ The Burkholder-Davis-Gundy inequality then yields}
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \sum_{j=1}^{\lceil T/\varepsilon \rceil} R_{j,t}(u) \right|^p \right] \leq \mathbb{E} \left[ \sup_{m \in \{0, \ldots, \lceil T/\varepsilon \rceil \}} |B_m(u)|^p + \sup_{0 \leq t \leq T} |R_{j^*,t}(u)|^p \right]
\]
\[
\leq \mathbb{E} \left[ \left( \sum_{j=1}^{\lceil T/\varepsilon \rceil} |A_{j,T}(u)|^2 \right)^{p/2} \right] + \mathbb{E} \left[ \sup_{0 \leq t \leq T} |A_{j^*,t}(u)|^p \right].
\]

If \( \varepsilon < \Delta_n \), then each \( I_j(t) \) contains at most one block \( k \) and for \( t_{k-1} \leq r \leq t_k \leq j\varepsilon \) we have necessarily \( t_{k-1} \leq (j - 1)\varepsilon = r. \text{ Hence, } |\xi_{r,k}| \lesssim |\mathbb{E}[e^{-(u, \sigma_r(W_r - W_{r_n}))}|\mathcal{F}_{r_n}]| \leq e^{-\frac{|u|^2}{4}} \text{ by Assumption } [\text{H-}\alpha-\beta] \text{ and thus } |A_{j,t}(u)| \lesssim \Delta_n^p e^{-\frac{|u|^2}{4}}. \text{ Moreover, there are clearly at most } \Delta_n^{-1} \text{ many non-empty } I_j(t). \text{ Consequently in this case the last display is up to a constant bounded by } \Delta_n^{2p/2} e^{-\frac{|u|^2}{4}}. \text{ Assume in the following that } \varepsilon \geq \Delta_n. \text{ Then } I_j(t) \text{ contains at most } C\varepsilon \Delta_n^{-1} \text{ many blocks } k \text{ and therefore}
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |A_{j^*,t}(u)|^p \right] \lesssim \Delta_n^p \varepsilon^p.
\]
(A.16)

Moreover,
\[
\mathbb{E} \left[ \left( \sum_{j=1}^{\lceil T/\varepsilon \rceil} |A_{j,T}(u)|^2 \right)^{p/2} \right] \leq \mathbb{E} \left[ \left( \sum_{j=1}^{\lceil T/\varepsilon \rceil} |A_{j,T}(u)|^2 \right)^p \right]
\]
\[
\lesssim \frac{\Delta_n^{2p}}{\Delta_n} \sum_{j_1, \ldots, j_p=1}^{\lceil T/\varepsilon \rceil} \sum_{k_1, k_2} \cdots \sum_{k_{p-1}, k_p} \int_{t_{k-1}}^{t_{k+1}} \int_{t_{k-1}}^{t_{k+1}} \cdots \int_{t_{k_{p-1}}}^{t_{k_p}} \left| \mathbb{E} \left[ \xi_{r_1 k_1} \xi_{r_1 k_2} \cdots \xi_{r_p k_p} \xi_{r_p k_p} \right] \right| d (r_1, r_1', \ldots, r_p, r_p').
\]

Fix \( j \) and \( k_1, k_1', \ldots, k_p, k_p' \), \( r_1, r_1', \ldots, r_p, r_p' \). Let \( r \) and \( h \) be the largest and second largest indices in the set \( \{r_l, r_{l'} : 1 \leq l \leq p \} \) with corresponding blocks \( k, k' \) such that \( t_{k-1} \leq r \leq t_k, t_{k'-1} \leq h \leq t_{k'} \). Without loss of generality assume \( h \leq r. \text{ If } t_{k-1} \leq r < t_k \), then
\[
\left| \mathbb{E} \left[ \xi_{r_1 k_1} \xi_{r_1 k_1'} \cdots \xi_{r_p k_p} \xi_{r_p k_p'} \right] \right| \lesssim \mathbb{E} \left[ |\xi_{r,k}| \right] \lesssim e^{-\frac{|u|^2}{4}}.
\]

If, on the other hand, \( h \leq r < t_{k-1} \leq r < t_k \), then
\[
\left| \mathbb{E} \left[ \xi_{r_1 k_1} \xi_{r_1 k_1'} \cdots \xi_{r_p k_p} \xi_{r_p k_p'} \right] \right| \lesssim \mathbb{E} \left[ |\mathbb{E} [\xi_{r,k}|\mathcal{F}_{r_n}]| \right] \lesssim e^{-\frac{|u|^2}{24}}.
\]

26
In the two cases $r_\varepsilon < t_{k-1} \leq h \leq r < t_k$ and $r_\varepsilon < h < t_{k-1} \leq r < t_k$ conditioning on $\mathcal{F}_h$ instead gives

$$|\mathbb{E} [\xi_{r_\varepsilon/k_1} \xi_{r_\varepsilon/k_1'} \cdots \xi_{r_\varepsilon/k_p} \xi_{r_\varepsilon/k_p'}]| \lesssim \mathbb{E} \left[ \mathbb{E} \left[ e^{-i(u, \sigma_r(W_r-W_h))} \bigg| \mathcal{F}_h \right] \right] \lesssim e^{-\frac{\|u\|^2}{2K}|r-h|}.$$

As $\varepsilon \geq \Delta_n$, it follows that $\sum_{k \in I_j(T)} \int_{t_{k-1}}^{t_k} 1 dr \leq \varepsilon$. In all, we conclude that $\mathbb{E}((\sum_{j=1}^{[T/\varepsilon]} |A_j,T(u)|^2)^{p/2})^2$ is up to a constant bounded by

$$\Delta_n^{2p} \left( e^p e^{-\frac{\|u\|^2}{2K}\varepsilon} + e^{p-1} \sum_{j=1}^{[T/\varepsilon]} \sum_{k \in I_j(T)} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} e^{-\frac{\|u\|^2}{2K}(|r-h|) drdh} \right).$$

By symmetry in $r, h$ we find for $u \neq 0$ that

$$\sum_{j=1}^{[T/\varepsilon]} \sum_{k \in I_j(T)} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} e^{-\frac{\|u\|^2}{2K}(|r-h|) drdh} \leq 2 \sum_{j=1}^{[T/\varepsilon]} \sum_{k \in I_j(T)} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} e^{-\frac{\|u\|^2}{2K}(\varepsilon-h) drdh} \lesssim \sum_{j=1}^{[T/\varepsilon]} \sum_{k \in I_j(T)} \int_{t_{k-1}}^{t_k} 1 dh \|u\|^{-2} \left( 1 - e^{-\frac{\|u\|^2}{2}(\varepsilon+\Delta_n)} \right) \lesssim \|u\|^{-2} \left( 1 - e^{-\frac{\|u\|^2}{2}(\varepsilon+\Delta_n)} \right),$$

because $1 - e^{-\frac{\|u\|^2}{2}(\varepsilon-h)} \leq 1 - e^{-\frac{\|u\|^2}{2}(\varepsilon+\Delta_n)}$ for $t_{k-1} \leq h \leq j\varepsilon$ and $k \in I_j(T)$. Combining the estimates for $\varepsilon < \Delta_n$ and $\varepsilon \geq \Delta_n$ in all we have shown in this step that

$$\| \sup_{0 \leq t \leq T} Q_{n,t}(\varepsilon, u) \|_{L^p(\mathbb{P})} \lesssim \Delta_n (1 + \|u\|)^{s-2} g_{n,2}(u) + \| \sup_{0 \leq t \leq T} \sum_{j=1}^{[T/\varepsilon]} \mathbb{E} \left[ A_{j,t}(u) \big| \mathcal{F}_{(j-1)\varepsilon} \right] \|_{L^p(\mathbb{P})}$$

with

$$g_{n,2}(u) = (1 + \|u\|)^{2-s} (\Delta_n^{1/2} e^{-\frac{\|u\|^2}{2K}\varepsilon})^{1/(2p)} + \varepsilon^{1/2-1/(2p)} \|u\|^{-1/p} \left( 1 - e^{-\frac{\|u\|^2}{2}(\varepsilon+\Delta_n)} \right)^{1/(2p)} + \varepsilon).$$

**Step 3.** We need to use two martingale decompositions. Write

$$\sum_{j=1}^{[T/\varepsilon]} \mathbb{E} \left[ A_{j,t}(u) \big| \mathcal{F}_{(j-1)\varepsilon} \right] = \sum_{j=1}^{[T/\varepsilon]} F_{j,t}^{(1)}(u) + \sum_{j=1}^{[T/\varepsilon]} R_{j,t}^{(2)}(u) + \sum_{j=1}^{[T/\varepsilon]} \mathbb{E} \left[ A_{j,t}(u) \big| \mathcal{F}_{(j-3)\varepsilon} \right],$$

27
where \( R_{j,t}^{(1)}(u) = \mathbb{E}[A_{j,t}(u)|\mathcal{F}_{(j-1)\varepsilon}] - \mathbb{E}[A_{j,t}(u)|\mathcal{F}_{(j-2)\varepsilon}] \), \( R_{j,t}^{(2)} = \mathbb{E}[A_{j,t}(u)|\mathcal{F}_{(j-3)\varepsilon}] - \mathbb{E}[A_{j,t}(u)|\mathcal{F}_{(j-4)\varepsilon}] \). The arguments in step 2 can be applied to \( \sum_{j=1}^{[T/\varepsilon]} R_{j,t}^{(1)}(u) \) and \( \sum_{j=1}^{[T/\varepsilon]} R_{j,t}^{(2)}(u) \) instead of \( \sum_{j=1}^{[T/\varepsilon]} R_{j,t}(u) \). Moreover, for \( r \leq 3\varepsilon \) observe that \( r_\varepsilon = 0 \). Hence \( \mathbb{E}[A_{j,t}(u)|\mathcal{F}_{(j-3)\varepsilon}] \) is for \( j \in \{1, 2, 3\} \) up to a constant bounded by

\[
\sum_{k \in I_j(t)} \int_{t_{k-1}}^{t_k} \left( t_k - r - \Delta_n \right) e^{-\frac{\|u_0\|^2}{2}} dr \lesssim \Delta_n \int_0^{\varepsilon} e^{-\frac{\|u\|^2}{2K}} dr \leq \Delta_n \varepsilon.
\]

We conclude that

\[
\| \sup_{0 \leq t \leq T} \sum_{j=1}^{[T/\varepsilon]} \mathbb{E} \left[ A_{j,t}(u)|\mathcal{F}_{(j-1)\varepsilon} \right] \|_{L^p(\mathbb{P})} \lesssim \Delta_n (1 + \|u\|)^{s-2} g_{n,2}(u) + \| \sup_{0 \leq t \leq T} \sum_{j=1}^{[T/\varepsilon]} \mathbb{E} \left[ A_{j,t}(u)|\mathcal{F}_{(j-3)\varepsilon} \right] \|_{L^p(\mathbb{P})}.
\]

**Step 4.** For \( t_{k-1} \leq r \leq t_k \) and \( k \in I_j(t) \), \( j \geq 4 \), note that \( r_\varepsilon = (j-3)\varepsilon \). Hence \( \mathbb{E}[\xi_{r,k} | \mathcal{F}_{(j-3)\varepsilon}] = Y_k V_{r,k} \), where

\[
V_{r,k} = e^{-i(u,b_{j-3}\varepsilon (r-t_{k-1})) - \frac{\varepsilon}{2}(r-t_{k-1})},
\]

\[
Y_k = e^{-i(u,X_{(j-3)\varepsilon} + b_{(j-3)\varepsilon}(t_{k-1} - (j-3)\varepsilon)) - \frac{\|u\|^2}{2}(t_{k-1} - (j-3)\varepsilon)} \sigma_{(j-3)\varepsilon}.
\]

Since also \( t_{k-1} - (j-3)\varepsilon > \varepsilon \), it follows that \( |Y_k| \lesssim e^{-\frac{\|u\|^2}{2K}} \). Moreover, \( \int_{t_{k-1}}^{t_k} (t_k - r) e^{-\frac{\|u\|^2}{2K}} dr = 0 \). We therefore conclude that \( \sum_{j=1}^{[T/\varepsilon]} \mathbb{E}[A_{j,t}(u)|\mathcal{F}_{(j-3)\varepsilon}] \) is bounded by

\[
\Delta_n \left( \sum_{j=4}^{[T/\varepsilon]} \sum_{k \in I_j(t)} \int_{t_{k-1}}^{t_k} |Y_k| |V_{r,k} - V_{t,k}| dr \right) \lesssim \Delta_n^2 (1 + \|u\|)^2 e^{-\frac{\|u\|^2}{2K}}.
\]

Consequently, it follows with \( g_{n,3}(u) = \Delta_n (1 + \|u\|)^{s-2} e^{-\frac{\|u\|^2}{2K}} \) that

\[
\| \sup_{0 \leq t \leq T} \sum_{j=1}^{[T/\varepsilon]} \mathbb{E} \left[ A_{j,t}(u)|\mathcal{F}_{(j-3)\varepsilon} \right] \|_{L^p(\mathbb{P})} \lesssim \Delta_n (1 + \|u\|)^{s-2} g_{n,3}(u).
\]

**Step 5.** The four previous steps combined show that \( \sup_{0 \leq t \leq T} Q_n(t)(u) \) is up to a constant bounded by \( \Delta_n (1 + \|u\|)^{s-2} g_n(u) \) with \( g_n(u) = g_{n,1}(u) + g_{n,2}(u) + g_{n,3}(u) \). Set \( \varepsilon = \varepsilon(u, n) := \min(\nu_n\|u\|^{-2}, 1) \) for \( \nu_n = 2K \log(1 + \|u\|^3 \Delta_n^{1/2}) \). Hence \( 0 < \varepsilon \leq 1 \) and \( \varepsilon \to 0 \) for fixed \( u \). Choose \( C \geq 1 \) large enough to ensure that \( \varepsilon(u, n) < 1 \) for \( \|u\| > C \) and \( n = 1 \) (and thus for all \( n \)). For \( \|u\| \leq C \) this means \( \varepsilon \leq \nu_n\|u\|^{-2} \lesssim \Delta_n^{1/2} \).
and \( \sup_{u: \|u\| \leq C} g_n(u) = o(1) \). For \( \|u\| > C \), on the other hand, it follows that
\[
\begin{align*}
g_{n,1}(u) &\lesssim \|u\|^{2-s} \left( \|u\|^{-1-2s} \nu_n^{1+s} + \|u\|^{-2s} \nu_n^{1/2} \right), \\
g_{n,2}(u) &\lesssim \|u\|^{2-s} \left( \Delta_n^{1/2} \left( 1 + \|u\|^3 \Delta_n^{-1/2} \right) + \|u\|^{-2s} \nu_n \right), \\
g_{n,3}(u) &\lesssim \|u\|^{4-s} \Delta_n \left( 1 + \|u\|^3 \Delta_n^{-1} \right).
\end{align*}
\]

The assumptions that \( 2 - s - 2\alpha < 0 \), \( \alpha \geq 1 \), \( \alpha + \beta > 1 \) and the fact that \( \nu_n \) grows in \( u \) only logarithmically imply that \( \sup_{\|u\| \leq C} g_n(u) \) is bounded in \( n \). Consequently, \( \sup_{n \geq 1} \sup_{u \in \mathbb{R}^d} g_n(u) \) is bounded. Moreover, for fixed \( u \) with \( \|u\| > C \) it follows that \( g_n(u) \to 0 \) as \( n \to \infty \). This proves the claim.

**A.4 Proof of Theorem 2.7**

Similar to Theorem 2.5 it is sufficient to prove the following two propositions for \( f \in H^s(\mathbb{R}^d) \).

**Proposition A.8.** Assume \([H-\alpha-\beta]\) for \( 0 \leq \alpha, \beta \leq 1 \) and \([X0]\). Then we have for \( f \in H^1(\mathbb{R}^d) \) the stable convergence
\[
\Delta_n^{-1} M_{t,n}(f) \xrightarrow{\mathbb{P}} \frac{1}{2} \int_0^t \langle \nabla f(X_r), \sigma_r dW_r \rangle + \frac{1}{\sqrt{12}} \int_0^t \langle \nabla f(X_r), \sigma_r d\tilde{W}_r \rangle 
\]
as processes on \( \mathcal{D}([0,T], \mathbb{R}^d) \), where \( \tilde{W} \) is a \( d \)-dimensional Brownian motion defined on an independent extension of \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P}) \).

**Proposition A.9.** Assume \([H-\alpha-\beta]\) for \( 0 \leq \alpha, \beta \leq 1 \) and \([X0]\). Let \( s > 2 - 2\alpha \), \( s \geq 1 \), \( s + \beta > 1 \). Then we have for \( f \in H^s(\mathbb{R}^d) \) that
\[
\Delta_n^{-1} D_{T,n}(f) \xrightarrow{\mathbb{P}} \frac{1}{2} \left( f(X_T) - f(X_0) \right) - \frac{1}{2} \int_0^T \langle \nabla f(X_r), \sigma_r dW_r \rangle. \tag{A.18}
\]

Note that the convergence in the second proposition is not functional as compared to Proposition A.5. Since the weak limit in (A.17) is a continuous process, convergence with respect to the Skorokhod topology and thus the stable convergence also hold at \( t = T \) (Billingsley (2013)). This yields the CLT in (2.2) for \( f \in H^s(\mathbb{R}^d) \) and at the fixed time \( T \).

**Proof of Proposition A.8** The proof of Proposition A.4 can be repeated word by word, since Lemma A.2 applies also to \( f \in H^1(\mathbb{R}^d) \). We only have to argue differently for (A.8), because \( \nabla f(X_r) \) may not be bounded.

As \( f_{t_k-1}(t_k - r) dW_r \) is independent of \( \mathcal{F}_{t_{k-1}} \), it follows from the Cauchy-Schwarz inequality that \( E[\tilde{Z}_2^2 1_{\{|Z_2| > \varepsilon\}} | \mathcal{F}_{t_{k-1}}] \) is up to a constant bounded by \( \|\nabla f(X_{t_{k-1}}) \Delta_n\|^2 E[(\Delta_n^4 + \Delta_n^3 Y_k^2) 1_{\{|\nabla f(X_{t_{k-1}}) \Delta_n| \|\Delta_n^3 (1 + |Y_k|) \| \varepsilon\}} | \mathcal{F}_{t_{k-1}}] \) for \( \varepsilon > 0 \) and with \( Y_k \sim N(0, 1) \) independent of \( \mathcal{F}_{t_{k-1}} \). Since the marginals have uniformly bounded
Lebesgue densities (uniform in time), it follows that the first moment of the left hand side in (A.8) is up to a constant bounded by

\[ \int \| \nabla f(x) \|^2 \mathbb{E} \left[ \left( \Delta_n + Y_1^2 \right) 1_{\{\|\nabla f(x)\| \geq \frac{1}{2} \}} \right] dx. \]

This converges to 0 by dominated convergence, implying (A.8). □

**Proof of Proposition A.9.** The proof follows as the one of Proposition A.5 because Lemma A.2 applies also to \( f \in H^1(\mathbb{R}^d) \). We only have to use Lemma A.10 instead of Lemma A.6, while also replacing all \( o_{ucp} \)-expressions by the respective \( o_p \)-terms. □

**Lemma A.10.** Assume \( (H_{-\alpha, \beta}) \) for \( 0 \leq \alpha, \beta \leq 1 \) and (X0). Let \( s > 2 - 2\alpha, s \geq 1 \), \( s + \beta > 1 \). Then we have for \( f \in H^s(\mathbb{R}^d) \) with compact support, \( s \geq 1 \) and \( s > 2 - 2\alpha \), that

\[ D_{n,T}(f) - \frac{\Delta_n}{2} \sum_{k=1}^{n} \mathbb{E}[f(X_{t_k}) - f(X_{t_{k-1}})]|\mathcal{F}_{t_{k-1}}] = o_p(\Delta_n). \]

**Proof.** Using the notation from Lemma A.6 we only have to show for \( f \in \mathcal{S}(\mathbb{R}^d) \) that

\[ \mathbb{E}[\|D_{n,T}(1, f) + D_{n,T}(2, f)\|] \leq o(\Delta_n\|f\|_{H^s}) + \Delta_n \left( \int |\mathcal{F}f(u)|^2 \left( 1 + \|u\| \right)^{2s} g_n^2(u) du \right)^{1/2}, \] \hspace{1cm} (A.19)

with \( g_n \) as in Lemma A.7. This can be extended to \( f \in H^s(\mathbb{R}^d) \) by an approximation argument as in Lemma A.6.

\[ \mathbb{E}[\|D_{n,T}(1, f)\|] = o(\Delta_n\|f\|_{H^s}) \] follows from Lemma A.2(v). With respect to \( D_{n,T}(2, f) \) write

\[ D_{n,T}(2, f) = -(2\pi)^{-d} \int \mathcal{F}f(u) u_t u_m e^{-i(u,X_0)} \tilde{Q}_{n,T}(u) du \]

with

\[ \tilde{Q}_{n,T}(u) = \sum_{k=1}^{\lfloor t/\Delta_n \rfloor} \int_{t_{k-1}}^{t_k} \left( t_k - r - \frac{\Delta_n}{2} \right) \mathbb{E} \left[ e^{-i(u,X_0-\Sigma_r^{(l,m)})} |\mathcal{F}r_{t_{k-1}} \right] dr. \]

This corresponds to \( Q_{n,T}(u) \) in Lemma A.7 with \( X_r - X_0 \) instead of \( X_r \). Consequently, the independence from (X0) shows that \( \mathbb{E}[\|D_{n,T}(2, f)\|^2] \) is equal to

\[ (2\pi)^{-2d} \int \mathcal{F}f(u) \mathcal{F}f(v) \mathcal{F}u(u+v) u_t u_m v_t v_m \mathbb{E} \left[ \tilde{Q}_{n,T}(u) \tilde{Q}_{n,T}(v) \right] d(u,v) \]

\[ \leq \int \|\mathcal{F}f(u)\|^2 \|u\|^4 \mathbb{E} \left[ \|\tilde{Q}_{n,T}(u)\|^2 \right] du, \]

by Lemma A.3. The remaining part of (A.19) follows therefore from Lemma A.7. □
A.5 Proof of Corollary 2.8

Proof. Without loss of generality we can assume in the following that \( \mathcal{F} \) and the corresponding extensions are separable. In fact, it is enough to prove stable convergence for separable \( \mathcal{F} \), essentially because the \( \sigma \)-fields generated by \( X, b \) and \( \sigma \) are separable (see Jakov and Shiryaev (2013, Theorem IX.7.3) for details). Assume first that \( X_0 = 0 \). On a suitable extension as in Theorem 2.7 denoted by \((\Omega', \mathcal{F}', (\mathcal{F}')_{0 \leq t \leq T}, \mathbb{P}')\), let \( F_n(X, x_0) \) be defined as the random variables

\[
\Delta_n^{-1} \left( \int_0^T f(X_r + x_0) \, dr - \Delta_n \sum_{k=1}^n \frac{1}{2} (f(X_{t_{k-1}} + x_0) + f(X_{t_k} + x_0)) \right)
\]

and let \( F(X, \sigma, \tilde{W}, x_0) = \sqrt{1/12} \int_0^T \nabla f(X_r + x_0), \sigma, d\tilde{W}_r \), where \( F_n \) and \( F \) are measurable functions and \( x_0 \in \mathbb{R}^d \). The stable convergence in the claim is equivalent to \( \mathbb{E}[UG(F_n(X, x_0))] \to \mathbb{E}[UG(F(X, \sigma, \tilde{W}, x_0))] \) for any continuous bounded function \( g : \mathbb{R} \to \mathbb{R} \) and any bounded \( \mathcal{F} \)-measurable random variable \( U \) (cf. Podolskij and Vetter (2010)). We have to show that this holds for almost all \( x_0 \in \mathbb{R}^d \).

Let \((\Omega'', \mathcal{F}'', (\mathcal{F}'')_{0 \leq t \leq T}, \mathbb{P}'')\) be another extension of \((\Omega, \mathcal{F}, (\mathcal{F})_{0 \leq t \leq T}, \mathbb{P})\) such that there is a random variable \( Y \overset{d}{\sim} N(0, I_d) \), with the \( d \)-dimensional identity matrix \( I_d \), which is independent of \( \mathcal{F} \) and such that \( Y \) is \( \mathcal{F}_0'' \)-measurable. On this space the process \((X_t + Y)_{0 \leq t \leq T} \) satisfies Assumption \([X0]\). Without loss of generality \((\Omega', \mathcal{F}', (\mathcal{F}')_{0 \leq t \leq T}, \mathbb{P}')\) also extends this space. Theorem 2.7 yields \( \mathbb{E}[UG(F_n(X, Y))] \to \mathbb{E}[UG(F(X, \sigma, \tilde{W}, Y))] \) for all continuous and bounded \( g \) and all \( \mathcal{F}_n \)-measurable random variables \( U \). By independence of \( \tilde{Y} \) and \( \mathcal{F} \) this holds in particular for all \( \mathcal{F} \)-measurable \( U \) independent of \( Y \).

By a coupling argument (cf. Kallenberg (2002, Corollary 6.12)) there are (again on another extension of the probability space) \( \tilde{X}, \tilde{Y}, \tilde{\sigma}, \tilde{B}, \tilde{U} \) with \((X, \sigma, \tilde{W}, Y, U) \overset{d}{\sim} (\tilde{X}, \tilde{\sigma}, \tilde{B}, \tilde{Y}, \tilde{U}) \) such that \( \tilde{Y} \) is independent of \((\tilde{X}, \tilde{\sigma}, \tilde{B}, \tilde{U}) \) and \((F_n(\tilde{X}, \tilde{Y}, \tilde{U}) \to (\tilde{F}(\tilde{X}, \tilde{\sigma}, \tilde{B}, \tilde{Y}, \tilde{U}) \) almost surely. By conditioning on \( \tilde{Y} = x_0 \) and using independence this implies that \( \mathbb{E}[UG(F_n(X, x_0))] \to \mathbb{E}[UG(F(X, \sigma, \tilde{W}, x_0))] \) for \( \mathbb{P}' \)-almost all \( x_0 \) (by dominated convergence for conditional expectations, cf. Kallenberg (2002, Theorem 6.1)). Since \( \tilde{Y} \overset{d}{\sim} Y \overset{d}{\sim} N(0, 1) \), this holds for almost all \( x_0 \). In particular, this holds for all \( g \in C_c(\mathbb{R}^d) \), i.e. all continuous functions with compact support. Since this space is separable and because \( \mathcal{F} \) is separable, this implies the claim (cf. Theorem Kallenberg (2002, 5.19)).

□

A.6 Proof of Corollary 2.9

As in Theorem 2.7 we only have to consider the CLT for the Riemann-sum estimator. Let \( S_n(f, x_0) = \Delta_n^{-1} \left( \int_0^T f(X_r + x_0) \, dr - \Delta_n \sum_{k=1}^n \frac{1}{2} (f(X_{t_{k-1}} + x_0) + f(X_{t_k} + x_0)) \right) \) for \( x_0 \in \mathbb{R}^d \) and \( S(f, x_0) = 1/2(f(X_T) - f(X_0)) + \sqrt{1/12} \int_0^T (\nabla f(X_r + x_0), \sigma, d\tilde{W}_r). \) The dependence on \( X, \sigma \) and \( \tilde{W} \) is suppressed. Consider first the following lemma.

Lemma A.11. Assume \([H_{\alpha, \beta}]\) for \( 0 \leq \alpha, \beta \leq 1 \) and \( X_0 \sim N(0, I_d) \) independent of \((X_t - X_0)_{0 \leq t \leq T} \) such that \([X0]\) holds. Let \( s > 2 - 2\alpha, \ s \geq 1. \) Then we have for \( f \in C^s(\mathbb{R}^d) \) with compact support that \( \|S_n(f, X)\|_{L^p(\mathbb{P})} \leq C_p (\|f\|_{\infty} + \|\nabla f\|_{\infty}). \)
**Proof.** Recall the decomposition $S_n(f, x_0) = \Delta_n^{-1} M_{n,T}(f) + \Delta_n^{-1} D_{n,T}(f)$ from Theorem 2.7. Similar as in the proof of Proposition A.3 it follows that

$$
\mathbb{E} \left[ M_{n,T}^p(f) \right] \lesssim \mathbb{E} \left[ \left( M_{n,T}(f) - \tilde{M}_{n,T}(f) \right)^p \right] + \mathbb{E} \left[ \tilde{M}_{n,T}^p(f) \right] \lesssim \Delta_n^p \| \nabla f \|_\infty^p.
$$

For this use the bounds on (A.11), (A.12) and use slightly modified statements of Lemma A.2. With respect to $D_{n,T}(f)$ we argue as in Lemma A.10 (note that $f \in H^{1+\varepsilon}(\mathbb{R}^d)$ for some small $\varepsilon > 0$).

Write $D_{n,T}(f)$ as $\tilde{D}_{n,T}(1, f) + \tilde{D}_{n,T}(2, f)$, where $\tilde{D}_{n,T}(1, f)$ and $\tilde{D}_{n,T}(2, f)$ are defined just as $D_{n,T}(1, f)$ and $D_{n,T}(2, f)$ in Lemma A.6, but with $t_k - r - \frac{\Delta n}{2}$ replaced by $t_k - r$. It follows similar to Lemma A.2(v) that $\| \tilde{D}_{n,T}(1, f) \|_{L^p(\mathbb{P})} \lesssim \Delta_n \| \nabla f \|_{\infty}$. Moreover, if $f \in \mathcal{S}(\mathbb{R}^d)$, then

$$
\tilde{D}_{n,T}(2, f) = -(2\pi)^{-d} \int \mathcal{F} f(u) u_i u_m e^{-i(u,X_0)} \tilde{Q}_{n,T}(u) \, du
$$

with $\tilde{Q}_{n,T}(u)$ as in Lemma A.10, but also with $t_k - r - \frac{\Delta n}{2}$ instead of $t_k - r$. Assume first that $p \geq 2$ is even. Then we find from independence via (X0) with $\mu$ being the density of $N(0, I_d)$ that $\| \tilde{D}_{n,T}(2, f) \|_{L^p(\mathbb{P})}$ is bounded by

$$(2\pi)^{-pd} \int \prod_{j=1}^p \left( |\mathcal{F} f(u_j)| \|u_j\|^2 \right) \mathcal{F} \mu \left( \sum_{j=1}^p u_j \right) \mathbb{E} \left[ \prod_{j=1}^p \tilde{Q}_{n,T}(u_j) \right] \, d(u_1, \ldots, u_p).$$

Because of $|\mathbb{E}[\prod_{j=1}^p \tilde{Q}_{n,T}(u_j)| \leq \prod_{j=1}^p \| \tilde{Q}_{n,T}(u_j) \|_{L^p(\mathbb{P})}$, Lemmas A.7 and A.3 this is up to a constant bounded by $\| f \|_{H^1} \lesssim \| f \|_{\infty} + \| \nabla f \|_{\infty}$. If $p$ is not even or $p = 1$, then we have instead $\| \tilde{D}_{n,T}(2, f) \|_{L^p(\mathbb{P})} \leq \| \tilde{D}_{n,T}(2, f) \|_{L^{2p}(\mathbb{P})} \lesssim \Delta_n (\| f \|_{\infty} + \| \nabla f \|_{\infty})$. This is the claimed bound for $\| \tilde{D}_{n,T}(2, f) \|_{L^p(\mathbb{P})}$ if $f \in \mathcal{S}(\mathbb{R}^d)$. For $f \in C^s(\mathbb{R}^d)$ use a density argument. Together with the bound on $\| M_{n,T}^p(f) \|_{L^p(\mathbb{P})}$ and $\| \tilde{D}_{n,T}(1, f) \|_{L^p(\mathbb{P})}$ this yields the claim.

**Proof.** As in the proof of Proposition 2.8 assume $X_0 = 0$ such that $X + x_0$ has initial value $x_0 \in \mathbb{R}^d$. Observe further that $f \in C^s(\mathbb{R}^d) \subset H^1_{\text{loc}}(\mathbb{R}^d)$ and $S_n(f, X + x_0) = S_n(f, X) = f(x + x_0)$. It is sufficient to prove the claim under Assumption $\text{(II-\alpha-\beta)}$ for $f$ with compact support (cf. Section A.1).

The claim is equivalent to $(S_n(f, x_0), U) \overset{d}{\rightarrow} (S(f, x_0), U)$ for all $x_0 \in \mathbb{R}^d$ and any $\mathcal{F}$-measurable real-valued random variable $U$. For this note that, since $f$ is continuous, $g_n(x_0) = (S_n(f, X + x_0), U)$ defines a sequence of continuous stochastic processes $(g_n(x_0))_{x_0 \in \mathbb{R}^d}$. Similarly, $g(x_0) = (S(f, Y + x_0), U)$ defines a continuous stochastic process $(g(x_0))_{x_0 \in \mathbb{R}^d}$. We will show below that $(g_n(x_0))_{x_0 \in \mathbb{R}^d} \overset{d}{\rightarrow} (g(x_0))_{x_0 \in \mathbb{R}^d}$ with respect to the sup norm on $\mathbb{R}^d$. By a coupling argument as in the proof of Corollary 2.8 this means that $(S_n(f, y + x_0), U)_{x_0 \in \mathbb{R}^d} \overset{d}{\rightarrow} (S(f, y + x_0), U)_{x_0 \in \mathbb{R}^d}$ for almost all $y \in \mathbb{R}^d$. Since point evaluations are continuous with respect to the sup norm, and because $y + x_0$ runs through all of $\mathbb{R}^d$, this implies the claim of the corollary.

In order to show $(g_n(x_0))_{x_0 \in \mathbb{R}^d} \overset{d}{\rightarrow} (g(x_0))_{x_0 \in \mathbb{R}^d}$ let $Y \overset{d}{\rightarrow} N(0, I_d)$ be defined on an appropriate extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ as in Corollary 2.8. The process $X + Y + \ldots$
x₀ then satisfies Assumption \([X0]\) for any \(x₀ ∈ \mathbb{R}^d\). By linearity of \(f → Sₙ(f, Y + x₀)\), the convergence of the finite dimensional distributions of \(∈ \mathbb{R}^d\) follows from Theorem \([27]\) and the Cramér-Wold Theorem (Kallenberg (2002, Corollary 5.5)). With respect to tightness, observe for any \(x₀, y₀ ∈ \mathbb{R}^d\) by linearity and the last lemma that

\[
\|gₙ(x₀) - gₙ(y₀)\|_{L^p(\mathbb{P})} ≤ \|Sₙ(f, Y + x₀) - Sₙ(f, Y + y₀)\|_{L^p(\mathbb{P})}
\]

\[
≤ \|fₓ₀ - fᵧ₀\|_∞ + \|∇fₓ₀ - ∇fᵧ₀\|_∞ ≤ \|x₀ - y₀\|^α,
\]

because \(∇f\) is \((1 - s)\)-Hölder continuous and has compact support. Choose \(p ≥ 1\) such that \(pˢ > d\). From the Kolmogorov-Chentsov criterion for tightness on \(C(\mathbb{R}^d)\) (Kallenberg (2002, Corollary 16.9)) we therefore obtain the tightness of \(gₙ(x₀)\) and thus the claimed weak convergence \(gₙ(x₀) → g(x₀)\).

\[\square\]

### Appendix B: Proofs of Section 3

Observe first the following lemma, which will be used frequently.

**Lemma B.1.** Let \(0 < a < b ≤ T\) and \(α, β ≤ 2\). It follows that

\[
\sum_{k-1 > j ≥ 2}^{n} \int_{t_{k-1}}^{t_k} \int_{t_{j-1}}^{t_j} (b - a) - α a^β da \lesssim \begin{cases} 
\log^2 n, & α = 1, β = 1, \\
T^{1-α} \log n, & α < 1, β = 1, \\
T^{1-α} Δ₁^{-β}, & α < 1, β > 1, \\
T^{2-α-β}, & α < 1, β < 1.
\end{cases}
\]

The same holds when \(α\) and \(β\) are switched.

**Proof.** The sum is equal to \(Δ₂^{-α-β} \sum_{k-1 > j ≥ 2} (k - 1 - j)^-α \int_{j-1}^{j} a^β da\), which is bounded by \(Δ₂^{-α-β} (\sum_{k-1}^{n} a^{α}) (\int_{j-1}^{j} a^{-β} da)\). If \(α = 1\), then the sum is of order \(log n\), while it is of order \(n^{1-α}\) when \(α < 1\) and just finite when \(α > 1\). The same statements hold for the integral, depending on \(β\). Considering all possible combinations yields the claim. \(\square\)

#### B.1 Proof of Proposition 3.1

**Proof.** Write \(\|Γₜ(f) - Γₜ,ₙ(f)\|^2_{L^2(\mathbb{P})} = A₁ + A₂ + A₃\), where \(A₁ = \sum_{k-j ≤ 1} M_{k,j}\), \(A₂ = 2 \sum_{k-j ≥ 2} M_{k,j}\) and \(A₃ = 2 \sum_{k ≥ 2} M_{k,1}\) and where

\[
M_{k,j} = \int_{t_{k-1}}^{t_k} \int_{t_{j-1}}^{t_j} \mathbb{E} \left[ (f(Xₜ) - f(Xₜ₀₋₁)) (f(Xₜ₁) - f(Xₜ₋₁₋₁)) \right] dhdr.
\]

Applying the Cauchy-Schwarz inequality several times yields \(A₁ + A₂ + A₃ ≤ S₁ + S₂\), where \(S₁ = Δₙ \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \mathbb{E}|f(Xₜ) - f(Xₜ₀₋₁)|^2 \right| dh\) and \(S₂ = \sum_{k-j ≥ 2} |M_{k,j}|\). It follows that

\[
S₁ = Δₙ \int (f(y) - f(x))^2 \left( \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} p_{t_{k-1},r}(x, y) dr \right) d(x, y).
\]

33
The following idea generalizes Equation (8) of Ganychenko (2015) to arbitrary processes. For (i) consider \( t_{j-1} < h < t_j < t_{k-1} < r < t_k \) and let \( g_{h,t_{j-1},b}(x,y) = p_{h,b}(x,y) - p_{t_{j-1},b}(x,y) \). The Fubini theorem implies for bounded \( f \) with compact support that \( M_{k,j} \) is equal to

\[
\int_{t_{k-1}}^{t_k} \int_{t_{j-1}}^{t_j} \int_{t_{k-1}}^{r} f(x) f(y) \partial_r g_{h,t_{j-1},b}(x,y) d(x,y) \, db dhdr.
\]

By interchanging integration and differentiation the inner integral is equal to \( \partial_b (\int f(x)f(y) g_{h,t_{j-1},b}(x,y) d(x,y)) \). Observe that \( \int g_{h,t_{j-1},b}(x,y) dy \) is independent of \( b \). Consequently, \( \partial_b (\int f^2(x)g_{h,t_{j-1},b}(x,y) d(x,y)) = 0 \). This holds similarly if \( f^2(x) \) is replaced by \( f^2(y) \), because \( \int g_{h,t_{j-1},b}(x,y) dx = 0 \). It follows that \( M_{k,j} \) is equal to

\[
-\frac{1}{2} \int_{t_{k-1}}^{t_k} \int_{t_{j-1}}^{t_j} \int_{t_{k-1}}^{r} (f(y) - f(x))^2 \partial_b g_{h,t_{j-1},b}(x,y) d(x,y) \, db dhdr
\]

and \( S_2 \) is up to a constant bounded by

\[
\Delta_n \int (f(y) - f(x))^2 \left( \sum_{k-1 > j \geq 2} \int_{t_{k-1}}^{t_k} \left( \partial_r g_{h,t_{j-1},r}(x,y) \right) dhdr \right) d(x,y).
\]

Together with the bound for \( S_1 \) this yields (i). For (ii) it follows similarly that \( M_{k,j} \) is equal to

\[
-\frac{1}{2} \int_{t_{k-1}}^{t_k} \int_{t_{j-1}}^{t_j} \int_{t_{k-1}}^{r} \left( \int (f(y) - f(x))^2 \partial^2_{bb} p_{h,b}(x,y) d(x,y) \right) \, dadbdbdhdr.
\]

(ii) follows from the bound on \( S_1 \) and because \( S_2 \) is up to a constant bounded by

\[
\Delta_n^2 \int (f(y) - f(x))^2 \left( \sum_{k-1 > j \geq 2} \int_{t_{k-1}}^{t_k} \left( \partial^2_{rr} p_{h,r}(x,y) \right) dhdr \right) d(x,y).
\]

\[\square\]

### B.2 Proof of Proposition 3.2

**Proof.** As in the proof of Proposition 3.1 it is sufficient to bound \( S_1 + S_2 \). For \( f \in S(\mathbb{R}^d) \) we can write \( f(X_r) = (2\pi)^{-d} \int f(u) e^{-i(u,X_r)} du \) for all \( 0 < r < T \). It follows that \( \mathbb{E}[(f(X_r) - f(X_{t_{k-1}}))(f(X_h) - f(X_{t_{j-1}}))] \) is equal to

\[
(2\pi)^{-2d} \int \mathcal{F} f(u) \mathcal{F} f(v) \mathbb{E} \left[ e^{-i(v,X_r)} - e^{-i(v,X_{t_{k-1}})} \right] \left( e^{-i(u,X_h)} - e^{-i(u,X_{t_{j-1}})} \right) d(u,v).
\]

With \( \varphi_{h,h}(u,v) = \mathbb{E}[e^{i(u+v,X_h)}] \) the expectation is for all \( h, r, t_{k-1}, t_{j-1} \) equal to

\[
\varphi_{h,r}(u,v) - \varphi_{t_{j-1},r}(u,v) - \varphi_{h,t_{k-1}}(u,v) + \varphi_{t_{j-1},t_{k-1}}(u,v).
\]

(B.1)
For (i) this implies by symmetry in $u,v$ that $S_1$ is up to a constant bounded by

$$
\Delta_n \int |\mathcal{F}f(u)||\mathcal{F}f(v)| \left( \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} g_{t_{k-1},r}(u,v) \, dr \right) \, d(u,v) \quad (B.2)
$$

with $g_{t_{j-1},r}(u,v)$ as in the statement. Let $\tilde{g}_{h,t_{j-1},b}(u,v) = \partial_b \varphi_{h,b}(u,v) - \partial_b \varphi_{t_{j-1},b}(u,v)$. Then (B.1) is for $t_{j-1} < h < t_j < t_{k-1} < r < t_k$ equal to $\int_{t_{k-1}}^{r} \tilde{g}_{h,t_{j-1},b}(u,v) \, db$. Therefore $S_2$ is up to a constant bounded by

$$
\Delta_n \int |\mathcal{F}f(u)||\mathcal{F}f(v)| \left( \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \int_{t_{j-1}}^{t_j} |\tilde{g}_{h,t_{j-1},r}(u,v)| \, dhdr \right) \, d(u,v).
$$

This yields (i). With respect to (ii) note that the last argument also applies to $r = h$, $k = j$ such that (B.2) is bounded by

$$
\Delta_n^2 \int |\mathcal{F}f(u)||\mathcal{F}f(v)| \left( \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \int_{t_{j-1}}^{t_j} |\partial_{br}^2 \varphi_{h,r}(x,y)| \, dhdr \right) \, d(u,v).
$$

giving a bound on $S_1$. For $S_2$ note that (B.1) is equal to $\int_{t_{k-1}}^{r} \int_{t_{j-1}}^{h} \partial_{ab}^2 \varphi_{a,b}(u,v) \, dadb$. This yields (ii), because $S_2$ is up to a constant bounded by

$$
\Delta_n \int |\mathcal{F}f(u)||\mathcal{F}f(v)| \left( \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \int_{t_{j-1}}^{t_j} |\partial_{br}^2 \varphi_{h,r}(x,y)| \, dhdr \right) \, d(u,v).
$$

\[\square\]

### B.3 Proof of Theorem 3.4

**Proof.** If $f$ is bounded, then $f_m(x) = f(x)1_{\{ |x| \leq m \}}$ defines a sequence of bounded functions with compact support converging to $f$ pointwise with $\|f_m\|_{\infty} \leq \|f\|_{\infty}$ for all $m$. If $f$ is Hölder-continuous, then we can similarly find a sequence $(f_m)_{m \geq 1} \subset C^\infty_c(\mathbb{R}^d)$ converging to $f$ pointwise with $\|f_m\|_{C^s} \leq \|f\|_{C^s}$. In both cases it follows $P_x$ almost surely that $\Gamma_T(f_m) - \widehat{\Gamma}_{n,T}(f_m) \to \Gamma_T(f) - \widehat{\Gamma}_{n,T}(f)$ as $m \to \infty$ by dominated convergence. The lemma of Fatou implies

$$
\|\Gamma_T(f) - \widehat{\Gamma}_{n,T}(f)\|_{L^2(P_x)}^2 \leq \liminf_{m \to \infty} \|\Gamma_T(f_m) - \widehat{\Gamma}_{n,T}(f_m)\|_{L^2(P_x)}^2.
$$

It is therefore sufficient to prove the theorem for bounded $f$ with compact support.

Conditional on $x_0$ the random variables $(X_h, X_r)$, $h \neq r$, have the joint densities $p_{h,r}(x, y; x_0) = \xi_{0,r}(x_0, x)\xi_{h,r}(x, y)$, $x, y \in \mathbb{R}^d$. Moreover, the heat kernel bounds in Assumption 3.3 imply

$$
|p_{h,r}(x, y; x_0)| \leq q_{r-h}(y-x) q_h(x-x_0),
$$
$$
|\partial_r p_{h,r}(x, y; x_0)| \leq \frac{1}{r-h} q_{r-h}(y-x) q_h(x-x_0),
$$
$$
|\partial_{hr} p_{h,r}(x, y; x_0)| \leq \left( \frac{1}{(r-h)^2} + \frac{1}{(r-h)h} \right) q_h(x-x_0) q_{r-h}(y-x).
$$
Then \( \int (\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} p_{t_{k-1},r}(x,y;x_0) dr) d(x,y) = T \) and Lemma \([3.1]\) yields
\[
\int \left( \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \int_{t_{j-1}}^{t_j} \left( |\partial_r p_{h,r}(x,y;x_0)| + |\partial_r p_{t_{j-1},r}(x,y;x_0)| \right) dh \right) d(x,y)
\]
\[
\lesssim \int \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \int_{t_{j-1}}^{t_j} (r-h)^{-1} dh dr \lesssim T \log n.
\]

Applying Proposition \([3.1] (i)\) to \( p_{h,r}(\cdot;x_0) \) yields the claim in \((i)\) for bounded \( f \). For \((ii)\), on the other hand, the moment conditions on \( q_a \) imply that \( \int \|y-x\|^{2s} q_a(x-x_0) q_{b-a}(y-x) d(x,y) \lesssim (b-a)^{2s/\gamma} \) for \( 0 < s \leq \gamma/2 \). Consequently, Lemma \([3.1]\) yields for \( \Delta_n^{-1} \int \|y-x\|^{2s} \left( \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} p_{t_{k-1},r}(x,y;x_0) dr \right) d(x,y) \) up to a constant the upper bound \( \Delta_n^{-1} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (r-t_{k-2})^{2s/\gamma} dr \) and also
\[
\int \|y-x\|^{2s} \left( \sum_{k=1}^{n} \sum_{j=2}^{n} \int_{t_{k-1}}^{t_k} \int_{t_{j-1}}^{t_j} |\partial_{hr} p_{h,r}(x,y;x_0)| dh dr \right) d(x,y)
\]
\[
\lesssim \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \int_{t_{j-1}}^{t_j} \left( (r-h)^{2s/\gamma - 2} + (r-h)^{2s/\gamma - 1} \right) dh dr.
\]

For \( 2s/\gamma < 1 \) Lemma \([3.1]\) implies for the sum of these two upper bounds the order \( O(T \Delta_n^{2s/\gamma - 1} + T^{2s/\gamma} \log n) \), while it is \( O(T \log n) \) for \( 2s/\gamma = 1 \). In the first case note that
\[
T^{2s/\gamma} \log n = T \Delta_n^{2s/\gamma - 1} (T^{2s/\gamma - 1} \Delta_n^{1-2s/\gamma}) \log n \leq T \Delta_n^{2s/\gamma - 1} \frac{\log n}{n^{1-2s/\gamma}},
\]
which is of order \( O(T \Delta_n^{1+2s/\gamma}) \), i.e. there is no \( \log n \)-term. This implies \((ii)\) for \( f \in C^s(\mathbb{R}^d) \).

\[\square\]

**B.4 Proof of Theorem 3.6**

**Proof.** Note that \( L^2(\mathbb{R}^d) = H^0(\mathbb{R}^d) \). For \( f \in H^s(\mathbb{R}^d) \), \( 0 \leq s \leq 1 \), let \( (f_m)_{m \geq 1} \subset C_c^\infty(\mathbb{R}^d) \) be a sequence of functions converging to \( f \) with respect to \( \|\cdot\|_{H^s} \), with \( \|f_m\|_{H^s} \leq \|f\|_{H^s} \). Then \( \|\Gamma_T(f) - \Gamma_{n,T}(f)\|_{L^2(\mathbb{P})}^2 \) is bounded by
\[
2 \|\Gamma_T(f - f_m) - \Gamma_{n,T}(f - f_m)\|_{L^2(\mathbb{P})}^2 + 2 \|\Gamma_T(f_m) - \Gamma_{n,T}(f_m)\|_{L^2(\mathbb{P})}^2.
\]

Then \( \|\Gamma_T(f - f_m)\|_{L^2(\mathbb{P})} \lesssim \int_0^T E[\int (f(x) - f_m(x))^2 p_r(x) dx]^{1/2} dr \), where the marginal densities \( p_r \) satisfy \( \sup_{0 \leq r \leq T} \|p_r\| = \sup_{0 \leq r \leq T} \|\xi_{0,h}(x_0,r) \mu(x_0) dx_0\| \leq \|\mu\|_\infty \). It follows that \( \|\Gamma_T(f - f_m)\|_{L^2(\mathbb{P})} \) is up to a constant bounded by \( \|f - f_m\|_{L^2} \), which converges to 0 as \( m \to \infty \). A similar argument shows \( \|\Gamma_{n,T}(f - f_m)\|_{L^2(\mathbb{P})} \to 0 \) as \( m \to \infty \). It is therefore sufficient to prove the theorem for \( f \in C^\infty(\mathbb{R}^d) \).

The random variables \( (X_n, X_r) \), \( h \neq r \), have the joint densities \( p_{h,r}(x,y) = \ldots \)
\( p_r(x) \xi_{h,r}(x, y) \), \( x, y \in \mathbb{R}^d \) and the heat kernel bounds in Assumption 3.3 imply
\[
|p_{h,r}(x, y)| \leq \| \mu \|_{\infty} q_{r-h}(y - x),
\]
\[
|\partial_r p_{h,r}(x, y)| \leq \| \mu \|_{\infty} \frac{1}{r - h} q_{r-h}(y - x),
\]
\[
|\partial^2_{rr} p_{h,r}(x, y)| \leq \| \mu \|_{\infty} \left( \frac{1}{(r - h)^2} + \frac{1}{(r - h)^3} \right) q_{r-h}(y - x).
\]

Then \( \int f^2(x) (\sum_{k=1}^n \int_{t_k}^{t_k} p_{t_{k-1},r}(x, y)d\tau) d(x, y) \leq \| \mu \|_{\infty} \| f \|_{L^2}^2 T \) and it follows by Lemma B.1 that
\[
\int f^2(x) \left( \sum_{k-1 > j \geq 2} \int_{t_k}^{t_j} \int_{t_{j-1}}^{t_k} \left( |\partial_r p_{h,r}(x, y)| + |\partial_r p_{t_{j-1},r}(x, y)| \right) d\tau d\tau \right) d(x, y)
\]
\[
\lesssim \| \mu \|_{\infty} \| f \|_{L^2}^2 \sum_{k-1 > j \geq 2} \int_{t_k}^{t_j} (r - h)^{-1} d\tau \lesssim \| \mu \|_{\infty} \| f \|_{L^2}^2 T \log n.
\]

By symmetry the same holds with \( f^2(y) \) instead of \( f^2(x) \). Applying Proposition 3.1(i) along with the trivial bound \( (f(x) - f(y))^2 \leq 2f(x)^2 + 2f(y)^2 \) therefore yields (i). For (ii) we distinguish the cases \( \gamma < 2 \) and \( \gamma = 2 \). Let first \( 0 < s \leq \gamma/2 < 1 \). In this case, the \( L^2 \)-Sobolev norm defined via the Fourier transform is equivalent to the Sobolev norm
\[
\| f \|_{H^s} = \left( \| f \|_{L^2}^2 + \int \frac{(f(x) - f(y))^2}{\| x - y \|^{2s + d}} d(x, y) \right)^{\frac{1}{2}},
\]
i.e. \( \| f \|_{H^s} \lesssim \| f \|_{H^s} \) (cf. Di et al. (2012) for more details). Similar to the proof of Theorem 3.6 the moment conditions on \( q_a \) imply for \( 0 < s \leq \gamma/2 \) that
\[
\Delta_n^{-1} \int (f(y) - f(x))^2 \left( \sum_{k=1}^n \int_{t_k}^{t_k} p_{t_{k-1},r}(x, y)d\tau \right) d(x, y)
\]
\[
\leq \| f \|_{H^s}^2 \Delta_n^{-1} \sup_{x, y \in \mathbb{R}^d} \left( \sum_{k=1}^n \int_{t_k}^{t_k} \| y - x \|^{2s + d} p_{t_{k-1},r}(x, y)d\tau \right)
\]
\[
\lesssim \| f \|_{H^s}^2 \Delta_n^{-1} \left( \sum_{k=1}^n \int_{t_k}^{t_k} (r - t_{k-1})^{2s/\gamma} d\tau \right),
\]
\[
\int (f(y) - f(x))^2 \left( \sum_{k-1 > j \geq 2} \int_{t_k}^{t_j} \int_{t_{j-1}}^{t_k} |\partial^2_{rr} p_{h,r}(x, y)| d\tau d\tau \right) d(x, y)
\]
\[
\leq \| f \|_{H^s}^2 \sup_{x, y \in \mathbb{R}^d} \left( \| y - x \|^{2s + d} \sum_{k-1 > j \geq 2} \int_{t_k}^{t_j} \int_{t_{j-1}}^{t_k} |\partial^2_{rr} p_{h,r}(x, y)| d\tau d\tau \right)
\]
\[
\lesssim \| f \|_{H^s}^2 \left( \sum_{k-1 > j \geq 2} \int_{t_k}^{t_j} (r - h)^{2s/\gamma - 1} h^{-1} d\tau d\tau \right).
\]

We surprisingly recover the same upper bounds as in the proof of Theorem 3.6. This yields the claim in (ii) for \( 0 < s \leq \gamma/2 < 1 \). Consider now \( \gamma = 2 \) and \( 0 < s \leq 1 \).
Unfortunately, the Slobodeckij-norm is not equivalent to the $\|\cdot\|_{H^s}$-norm when $s = 1$. We already know from (i) that the operator $\Gamma_T - \widehat{\Gamma}_{n,T}$ is a continuous linear operator from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{P})$. It is therefore sufficient to show that it is also a continuous linear operator from $H^1(\mathbb{R}^d)$ to $L^2(\mathbb{P})$. Indeed, as the Sobolev spaces $H^s(\mathbb{R}^d)$ for $0 \leq s \leq 1$ form interpolation spaces, the general claim is obtained by interpolating the operator norms of $\Gamma_T - \widehat{\Gamma}_{n,T}$ for $s = 0$ and $s = 1$ (cf. Adams and Fournier 2003, Theorem 7.23)). For $s = 1$ we have $f(y) - f(x) = \int_0^1 (\nabla f(x + t(y - x), y - x) \, dt$. It follows for any $0 < h < r < T$ that

$$
\int (f(y) - f(x))^2 q_{r-h}(y-x) \, d(x,y)
$$

$$
\leq \int_0^1 \left( \int \left( \|\nabla f(x + t(y - x))\|^2 \|y - x\|^2 q_{r-h}(y-x) \, d(x,y) \, dt \right) \right.
$$

$$
= \int \|\nabla f(x + tz)\|^2 \|z\|^2 q_{r-h}(z) \, d(x,z) \leq \|f\|^2_{H^1}(r-h),
$$

using $\int \|z\|^2 q_a(x) \, dx \lesssim a$. Proposition 3.1(ii) therefore implies

$$
\|\Gamma_T(f) - \widehat{\Gamma}_{n,T}(f)\|^2_{L^2(\mathbb{P})} \lesssim \|\mu\|_{\infty} \|f\|^2_{H^1} \left( \Delta_n \sum_{k=1}^n \int_{t_{k-1}}^{t_k} (r-t_{k-1}) \, dr \right.
$$

$$
+ \Delta_n^2 \sum_{k-1 > j \geq 2} \int_{t_{k-1}}^{t_k} \int_{t_{j-1}}^{t_j} ((r-h)^{-2} + h^{-1}) \, dh \, dr \right).
$$

Using the bounds from above yields the claim in (ii) for $s = 1$.

**B.5 Proof of Theorem 3.8**

$Y$ is independent of $\mathcal{F}_0$ and thus of $X_0$. Therefore the characteristic function of $(X_h, X_r)$ at $(u,v) \in \mathbb{R}^{2d}$ for $0 \leq h < r \leq T$ is $\varphi_{h,r}(u,v) = \tilde{\varphi}_{h,r}(u,v) \mathcal{F}\mu(u+v)$, where

$$
\tilde{\varphi}_{h,r}(u,v) = e^{\psi_{h,r}(v)+\psi_{0,h}(u+v)}
$$

is the characteristic function of $(Y_h, Y_r)$. $\psi_{h,r}(u)$ is for almost all $r$ differentiable with

$$
\partial_r \psi_{h,r}(u) = i \langle u, b_r \rangle - \frac{1}{2} \|\sigma_r u\|^2 + \int \left( e^{i(u,x)} - 1 - i \langle u, x \rangle 1_{\{\|x\| \leq 1\}} \right) \, dF_r(x),
$$

and also $\partial^2_{rr} \psi_{h,r}(u) = 0$. Hence

$$
\partial_r \varphi_{h,r}(u,v) = \partial_r \psi_{h,r}(v) \tilde{\varphi}_{h,r}(u,v) \mathcal{F}\mu(u+v),
$$

$$
\partial^2_{rr} \varphi_{h,r}(u,v) = (\partial_r \psi_{h,r}(v) + \partial_k \psi_{0,h}(u+v)) \partial_r \psi_{h,r}(v) \tilde{\varphi}_{h,r}(u,v) \mathcal{F}\mu(u+v).
$$

$\varphi_{h,r}$ as well as the derivatives $\partial_r \varphi_{h,r}$ and $\partial^2_{rr} \varphi_{h,r}$ satisfy the assumptions of Proposition 3.2(i) and (ii). Consider first the following lemma.

**Lemma B.2.** Fix $u, v \in \mathbb{R}^d$ such that $v \neq 0$ and $\|u + v\| \neq 0$ and let

$$
U_n = \sum_{k-1 > j \geq 2} \int_{t_{k-1}}^{t_k} \int_{t_{j-1}}^{t_j} \left( |\tilde{\varphi}_{h,r}(u,v)| + |\tilde{\varphi}_{t_{j-1},r}(u,v)| \right) \, dh dr,
$$

$$
V_n = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} \left( |\tilde{\varphi}_{h,r}(u,v)| + |\tilde{\varphi}_{t_{k-1},r}(u,v)| \right) \, dh dr.
$$

Then we have the following under the assumptions of Theorem 3.3(i):
(i) \((1 + \|v\|)^{\gamma + \beta^*} U_n \lesssim T^2 (1 + \|v\|)^{\beta^*/2} (1 + \|u\|)^{\beta^*/2}\).

(ii) \((1 + \|v\|)^{\gamma + \beta^*} V_n \lesssim T \Delta_n (1 + \|v\|)^{\gamma/2 + \beta^*} (1 + \|u\|)^{\gamma/2 + \beta^*}\).

(iii) \((1 + \|v\|)^{2\gamma + 2\beta^*} + (1 + \|v\|)^{\gamma + \beta^*} (1 + \|u + v\|)^{\gamma + \beta^*} U_n \lesssim T^2 (1 + \|v\|)^{\gamma/2 + \beta^*} (1 + \|u\|)^{\gamma/2 + \beta^*}\).

Proof. Observe first the following estimates:

Proof of Theorem 3.8.

Consider first the claim in (i). We only have to show it for \(s = \beta^*/2\) and \(s = \gamma/2 + \beta^*\). As in the proof of Theorem 3.6 the general claim for \(\beta^*/2 \leq s \leq \gamma/2 + \beta^*\) follows by interpolation. Let \(u, v \in \mathbb{R}^d\). Then for any \(0 \leq h, r \leq T\) it holds

\[|g_{h,r}(u, v)| \lesssim |\mathcal{F} \mu(u + v)|\]

with \(g\) from Proposition 3.2(i). Moreover, by assumption
\[ |\partial_v \varphi_{h,v}(v)| \leq c(1 + \|v\|)^{\gamma + \beta^*}. \] Lemma \[ \text{B.2}(i) \] and Proposition \[ \text{B.2}(i) \] therefore imply that \[ \|\Gamma_T(f) - \tilde{\Gamma}_{n,T}(f)\|_{L^2(\mathbb{P})}^2 \] is up to a constant bounded by
\[
T^2 \Delta_n \int \|Ff(u)\| \|Ff(v)\| (1 + \|u\|)^{\beta^*/2} (1 + \|v\|)^{\beta^*/2} |F\mu(u+v)| \, d(u,v). \tag{B.7}
\]

Lemma \[ \text{A.3} \] shows for this the upper bound \[ T^2 \Delta_n \|f\|_{H^s}^2, \] implying the claim for \[ s = \beta^*/2. \] With respect to \[ s = \gamma/2 + \beta^* \] it follows similarly by Lemma \[ \text{B.2}(ii) \] and \[ (iii) \], Proposition \[ \text{B.2}(ii) \] and Lemma \[ \text{A.3} \] that \[ \|\Gamma_T(f) - \tilde{\Gamma}_{n,T}(f)\|_{L^2(\mathbb{P})}^2 \] is up to a constant bounded by \[ T^2 \Delta_n \|f\|_{H^s}^2. \] This is the claimed bound for \[ s = \gamma/2 + \beta^*. \] To see that the improved bound holds note that \[ |\partial_v \varphi_{h,v}(v)| \leq c\|v\|^{\gamma + \beta^*} \] simplifies the calculations in Lemma \[ \text{B.2} \], since there is no need to distinguish the cases \[ \|v\| \geq 1 \] or \[ \|v\| < 1. \]

At last, consider \[ (ii) \]. From \[ |\partial_v \varphi_{h,v}(v)| \leq 1 \] it follows immediately that \[ \varphi_{h,v}(u,v) \] and the time derivatives \[ \partial_t \varphi_{h,v}(u,v), \partial_{hr} \varphi_{h,v}(u,v) \] are bounded by \[ T^2 |F\mu(u+v)|. \] As \[ T \geq 1, \] Proposition \[ \text{B.2}(ii) \] and Lemma \[ \text{A.3} \] imply the claim. If \[ c_1 \rho(v) \leq \partial_t \varphi_{h,v}(v) \leq c_2 \rho(v) \leq 0 \] for all \( h, r \leq T \), then \[ |\tilde{\varphi}_{h,v}(u,v)| \leq e^{-c_2 \rho(v)(r-h)} |F\mu(u+v)| \] and \[ \sum_{k-1 \geq 2} k \int_{t_{k-1}}^{t_k} |\partial_t \tilde{\varphi}_{h,v}(u,v)| \, dh \] is up to a constant bounded by
\[
\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (-\rho(v)) e^{-c_2 \rho(v)(r-h)} \, dh \lesssim \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left( e^{-c_2 \rho(v)(t_k-h)} - 1 \right) \, dh,
\]
and similarly for \[ \partial_r \tilde{\varphi}_{k-1,r}(u,v), \] while \[ \sum_{k-1 \geq 2} k \int_{t_{k-1}}^{t_k} |\partial_{hr} \tilde{\varphi}_{h,v}(u,v)| \, dh \] is up to a constant bounded by \[ \int_{h}^{T} \int_{h}^{T} (-\rho(v)) e^{-c_2 \rho(v)(r-h)} \, drdh. \] The first expression is of order \( O(T \Delta_n) \) and the second one of order \( O(T) \). Again, the claim follows from Proposition \[ \text{B.2}(ii) \] and Lemma \[ \text{A.3} \].

\[ \square \]

Remark \[ \text{B.3} \]. If \[ d = 1 \] and \[ \gamma > 1, \beta^* = 0, \] then the proof applies to \[ X_t = Y_t. \] Indeed, replace \[ T \] by \[ \int_{0}^{T} e^{-c\|u+v\|^2} \, dh \] in \[ \text{B.5} \] and \[ \text{B.6}. \] Together with a slightly different argument for \[ \|v\| < 1 \] this yields e.g. instead of \[ (B.7) \] the bound
\[
T \Delta_n \int_{0}^{T} \int |Ff(u)| |Ff(v)| e^{-c\|u+v\|^2} \, d(u,v) dh \leq T \Delta_n \int_{0}^{T} \int |Ff(u)|^2 e^{-c\|u+v\|^2} \, d(u,v) dh \lesssim \|f\|^2_{H^s} T^2 \Delta_n.
\]
This works, because \( u \mapsto e^{-c\|u\|^\gamma} \) is integrable and because \( \int_{0}^{T} h^{-1/\gamma} \, dh \) is finite.

\[ \text{B.6} \quad \text{Proof of Theorem 3.13} \]

The characteristic function of \( (X_h, X_r) \) at \( (u,v) \in \mathbb{R}^d \) for \( 0 \leq h < r \leq T \) is \( \varphi_{h,r}(u,v) = \tilde{\varphi}_{h,r}(u,v) \mathcal{F}\mu(u+v) \), where \( \tilde{\varphi}_{h,r}(u,v) \) is the characteristic function of \( (B_h, B_r) \). As \( B \) is a Gaussian process, it follows that \( \varphi_{h,r}(u,v) \) is equal to \( e^{-\frac{1}{2} \Phi_{h,r}(u,v)} \) with \( \Phi_{h,r}(u,v) = \|u\|^2 h^{2H} + \|v\|^2 r^{2H} + 2 \langle u, v \rangle c(h, r) \). Since fractional Brownian
motion is \textit{locally nondeterministic} (cf. \cite{Pitt1978} Proposition 7.2)), there exists a constant $c > 0$ independent of $u, v, r, h$ such that

$$
\Phi_{h,r}(u, v) = \text{Var}(\langle v, B_r \rangle + \langle u, B_h \rangle) = \text{Var}(\langle v, B_r - B_h \rangle + \langle u + v, B_h \rangle) \\
\geq c \left( \|v\|^2 (r-h)^{2H} + \|u + v\|^2 h^{2H} \right)
$$

Consequently, $\tilde{\varphi}_{h,r}(u, v) \leq e^{-c\|v\|^2(r-h)^{2H} - c\|u + v\|^2 h^{2H}}$. Moreover,

$$
\partial_r \varphi_{h,r}(u, v) = -\frac{1}{2} \partial_r \Phi_{h,r}(u, v) \varphi_{h,r}(u, v), \\
\partial^2_{r,r} \varphi_{h,r}(u, v) = \left(-\frac{1}{2} \partial^2_{r,r} \Phi_{h,r}(u, v) + \frac{1}{4} \partial^2_r \Phi_{h,r}(u, v) \partial_r \Phi_{h,r}(u, v) \right) \varphi_{h,r}(u, v), \\
\partial_r \Phi_{h,r}(u, v) = 2H(\|v\|^2 + \langle u, v \rangle)(r - h)^{2H-1} - 2H \langle u, v \rangle (r - h)^{2H-1}, \\
\partial^2_{r,r} \Phi_{h,r}(u, v) = 2H (2H - 1) \langle u, v \rangle (r - h)^{2H-2}.
$$

We first prove a lemma. Denote for any function $(r, h) \mapsto g(r, h)$ and fixed $u, v \in \mathbb{R}^d$ by $U_n(g)$ the sum $\sum_{k-1 > \cdots > k \geq 2} \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}^d} g(r, h) \varphi_{h,r}(u, v) dh dr$.

\textbf{Lemma B.4.} Let $T \geq 1$ and assume $[X0]$ Fix $u, v \in \mathbb{R}^d \setminus \{0\}$ and let $0 < H < 1$, $H \neq 1/2$. Consider for $0 < h < r < T$ the functions $g_1(r, h) = (r-h)^{2H-1}$, $g_2(r, h) = h^{2H-1}$, $g_3(v, h) = (r-h)^{2H-2}$, $g_4(r, h) = (r-h)^{2H-2}$, $g_5(r, h) = (r-h)^{2H-1}$ and $g_6(r, h) = v^{2H-1} h^{2H-1}$. Then we have the following estimates with absolute constants:

\begin{enumerate}[(i)]
\item $\mathbb{E}(\|v\|^2 + \langle v, u \rangle + \|u\|)(U_n(g_1) + U_n(g_2)) \lesssim T,$
\item $\mathbb{E}(\|v\|^2 + \langle v, u \rangle + \|u\|)(U_n(g_3) + U_n(g_4)) \lesssim T^{2H} \text{ or } \lesssim T \Delta^2 \text{ when } H > 1/2 \text{ or } H < 1/2,$
\item $\mathbb{E}(\|v\| + \|u + v\|)^2(U_n(g_5) + U_n(g_6)) \lesssim T^{2H} \text{ or } \lesssim T \Delta^2 \text{ when } H > 1/2 \text{ or } H < 1/2,$
\item $\sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_h \left((r^{2H-1} + (r-h)^{2H-1}) \mathbb{E}\varphi_{h,r}(u, v) dr dh \lesssim T^{2H} \Delta_n.\right.$
\end{enumerate}

\textbf{Proof.} We need to bound the integrals in $U_n(g_i)$ in several different ways. Observe for $0 \leq a < b \leq T$ and $q = 2H - 1, 4H - 2, 1$ the following estimates for $R^{(q)}_{a,b,v} := \int_a^b r^q e^{-\frac{1}{2}\|v\|^2 r^{2H}} dr$:

\begin{align}
R^{(2H-1)}_{a,b,v} & \lesssim \|v\|^{-2} \left(e^{-\frac{1}{2}\|v\|^2 a^{2H}} - e^{-\frac{1}{2}\|v\|^2 b^{2H}} \right) \lesssim \left\{ \begin{array}{ll}
\|v\|^{-2}, \\
\|v\|^{-1}(b^{2H} - a^{2H})^{1/2}, \\
(b^{2H} - a^{2H}),
\end{array} \right. \quad (B.8) \\
R^{(4H-2)}_{a,b,v} & \lesssim \frac{\|v\|^2}{\|v\|} \int_a^b r^{2H-2} dr \lesssim \left\{ \begin{array}{ll}
\|v\|^{-2} \left(b^{2H-1} - a^{2H-1} \right), \\
\|v\|^{-1}(b^{3H-1} - a^{3H-1}),
\end{array} \right. \quad (B.9) \\
R^{(1)}_{a,b,v} & \lesssim \frac{\|v\|}{b-a} \int_a^b r^{-H} dr \lesssim \left\{ \begin{array}{ll}
\|v\|^{-1}(b^{1-H} - a^{1-H}), \\
b-a,
\end{array} \right. \quad (B.10)
\end{align}
where we used that \( \sup_{v \in \mathbb{R}^d} \|v\|^p r^{pH} e^{-\frac{1}{2} \|v\|^2 r^{2H}} = \sup_{x \geq 0} x e^{-\frac{1}{2} x^2} < \infty \) for any \( p \geq 0 \).

It follows from (B.8) and (B.10) that \( U_n(g_1) \) is bounded by

\[
\int_0^T \int_0^T (r - h)^{2H-1} e^{-c_2 \|v\|^2 (r-h)^{2H} + \|u+v\|^2 h^{2H}} dr dh \leq T \left\{ \frac{\|v\|^2}{\|u+v\|}, \frac{\|v\|^2}{\|u+v\|^2}, \frac{\|v\|}{\|u+v\|}, \frac{\|v\|}{\|u+v\|^2} \right\}.
\]

The estimate for \( g_2 \) follows in the same way. For \( g_3 \) and \( H > 1/2 \) it follows similarly from (B.9), (B.10), \( T \geq 1 \) and Lemma [B.1] that

\[
U_n(g_3) \leq \sum_{k-1 < j \geq 2} \int_{t_{k-1}}^{t_k} \int_{t_{j-1}}^{t_j} \left\{ \frac{\|v\|^2}{\|u+v\|}, \frac{\|v\|^2}{\|u+v\|^2}, \frac{\|v\|}{\|u+v\|}, \frac{\|v\|}{\|u+v\|^2} \right\} dh dr.
\]

while for \( H < 1/2 \)

\[
U_n(g_3) \leq T \Delta_n^{2H-1} \left\{ \frac{\|v\|^2}{\|u+v\|}, \frac{\|v\|^2}{\|u+v\|^2}, \frac{\|v\|}{\|u+v\|}, \frac{\|v\|}{\|u+v\|^2} \right\}.
\]

The estimates for \( g_4 \) follow similarly (they are even easier). With respect to \( g_5 \) the integrals decompose and (B.8) and (B.10) yield for \( U_n(g_5) \) the bound

\[
R_{0, T,v}^{(2H-1)} R_{0, T,u+v}^{(2H-1)} \leq T^{2H} \left\{ \frac{\|v\|^2}{\|u+v\|}, \frac{\|v\|^2}{\|u+v\|^2}, \frac{\|v\|}{\|u+v\|}, \frac{\|v\|}{\|u+v\|^2} \right\}.
\]

For \( U_n(g_6) \), on the other hand, the same equations imply for \( H > 1/2 \) the upper bound

\[
\int_0^T \int_0^T R_{h,T,v}^{(2H-1)} e^{-c_2 \|u+v\|^2 h^{2H}} dh \leq T^{2H} \left\{ \frac{\|u+v\|^2}{\|u+v\|^2}, \frac{\|u+v\|^2}{\|u+v\|^2}, \frac{\|u+v\|}{\|u+v\|^2}, \frac{\|u+v\|}{\|u+v\|^2} \right\},
\]

and for \( H < 1/2 \) by \( r^{2H-1} h^{2H-1} \leq h^{4H-2} \) and Lemma [B.1]

\[
\sum_{k-1 > j \geq 2} \int_{t_{k-1}}^{t_k} \int_{t_{j-1}}^{t_j} \left\{ \frac{\|u+v\|^2}{\|u+v\|^2}, \frac{\|u+v\|^2}{\|u+v\|^2}, \frac{\|u+v\|}{\|u+v\|^2}, \frac{\|u+v\|}{\|u+v\|^2} \right\} dh dr dh dr.
\]

because \( T \geq 1 \) and because \( 1 \leq \log n \leq \Delta_n^{2H-1} \). Observe that we did not prove any bound on \( \|v\|^2 U_n(g_6) \) for \( H > 1/2 \). For this, we need a different upper bound on \( \varphi_{h,r}(u, v) \). If \( \|u+v\| \geq \|v\| \), then \( \varphi_{h,r}(u, v) \leq e^{-c_2 \|v\|^2 (r-h)^{2H} - c_2 \|u+v\|^2 h^{2H}} \) is clearly
bounded by $e^{-c_2 \|v\|^2 h^{2H}}$. As $r^{2H-1}h^{2H-1} \lesssim (r-h)^{2H-1}h^{2H-1} + h^{4H-2}$ for $H > 1/2$, it thus follows from (B.11) and Lemma B.4 that

$$U_n (g_6) \lesssim U_n (g_5) + \|v\|^{-2} \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_{t_{j-1}}^{t_j} h^{4H-2} dh dr \lesssim T^{2H} \|v\|^{-2}.$$

If $\|u + v\| < \|v\|$, however, then $\tilde{\varphi}_{h,r} (u, v) \leq e^{-c_2 (\|v\|^2 r^{2H} + \|u\|^2 h^{2H})}$. To see why this holds note that in this case necessarily $\langle u, v \rangle \geq 0$ by elementary geometrical considerations. But then $\Phi_{h,r} (u, v) \geq \|u\|^2 h^{2H} + \|v\|^2 r^{2H}$, since also $c(h, r) = \mathbb{E}[(Y_r - Y_h)Y_h] + h^{2H} \geq 0$ (recall that increments of fractional Brownian motion are positively correlated when $H > 1/2$). From the new bound and (B.8) follows immediately that

$$U_n (g_6) \lesssim \int_0^T \int_h (r^{2H-1} + (r-h)^{2H-1}) \tilde{\varphi}_{h,r} (u, v) dh dr \lesssim T^{2H} \Delta_n.$$

Arguing as for $U_n (g_6)$ with the different upper bounds for $\tilde{\varphi}_{h,r} (u, v)$, it follows that the left hand side is bounded by $\|v\|^{-1} T^{2H} \Delta_n$. This yields (iv). \hfill \Box

**Proof of Theorem 3.13.** As in the proof of Theorem 3.8 it is sufficient to prove the claim for $f \in \mathcal{S} (\mathbb{R}^d)$ and $s \in \{0, 1\}$. The conclusion follows by interpolation. We consider only $H \neq 1/2$, since the case $H = 1/2$ corresponds to Brownian motion and is already covered by Example 3.10.

Let $0 \leq h < r \leq T$ and $u, v \in \mathbb{R}^d$. From $\|u\| \leq \|v\| + \|u + v\|$ it follows that $|\partial_r \Phi_{h,r} (u, v)| \lesssim (\|v\|^2 + \|v\|^2 \|u + v\|)(r-h)^{2H-1} + h^{4H-1})$. Lemma B.3(i) therefore implies that $\sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_{t_{j-1}}^{t_j} (|\partial_r \varphi_{h,r} (u, v)| + |\partial_r \varphi_{t_{j-1},r} (u, v)|) dh dr$ is of order $O(T)$. Moreover, $|g_{t_{k-1},r} (u, v)| \lesssim |\Phi_{h,r} (u, v)|$ for all $1 \leq k \leq n$ and $t_{k-1} \leq r < t_k$ with $q$ from Proposition 3.2(i). Applying Proposition 3.2(i) and Lemma A.3 shows that $\|\Gamma_T (f) - \hat{f}_{n,T} (f)\|_{L_2}^2$ is up to a constant bounded by $C_{\mu} T \Delta_n \|f\|_{L_2}^2$. With $T \leq T^{2H}$ for $H > 1/2$ this yields the claimed bound for $s = 0$. With respect to $s = 1$ note first that

$$|\partial_r \Phi_{h,r} (u, v)| \lesssim (1 + \|u\|)(1 + \|v\|)(\|v\|^2 + (r-h)^{2H-1} + (r-h)^{2H-1}),$$

$$|\partial_r \Phi_{h,r} (u, v) \partial_r \Phi_{h,r} (u, v)| \lesssim (1 + \|u\|)(1 + \|v\|) (\|v\|^2 + \|u + v\|)^2$$

$$\cdot (r^{2H-1}h^{2H-1} + (r-h)^{2H-1}h^{2H-1} + (r-h)^{2H-1}h^{2H-1})$$

$$+ (1 + \|u\|)(1 + \|v\|)(\|v\|^2 + \|u + v\|)(r-h)^{4H-2},$$

$$|\partial_{rr} \Phi_{h,r} (u, v)| \lesssim (1 + \|u\|)(1 + \|v\|)(r-h)^{4H-2}.$$
Lemma [3.4](ii), (iii) and (iv) imply
\[
\Delta_n^{-1} \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} (|\partial_r \varphi_{h,r}(u,v)| + |\partial_r \varphi_{t_{k-1},r}(u,v)|) dh dr
\leq (1 + \|u\|) (1 + \|v\|) (\|v\| + 1) T^{2H} |\mathcal{F}_\mu (u + v)|,
\]
\[
\sum_{k-1 > j > 1} \int_{t_{k-1}}^{t_k} |\partial^2_{hr} \varphi_{h,r}(u,v)| dh dr
\leq (1 + \|u\|) (1 + \|v\|) |\mathcal{F}_\mu (u + v)| \begin{cases} T^{2H}, H > 1/2, \\ T \Delta_n^{2H-1}, H < 1/2. \end{cases}
\]
This yields the claim for \(s = 1\) by applying Proposition 3.2(ii) and Lemma A.3 as above.

\section*{B.7 Proof of Theorem 3.14}

Proof. \(f_{a,\varepsilon} \in H^{1/2-\rho}(\mathbb{R})\) for any small \(\rho > 0\) with \(\|f_{a,\varepsilon}\|_{H^{1/2-\rho}} \lesssim \varepsilon^{-1+\rho}\). By the triangle inequality and Theorem 3.13 (Assumption (X0) can be removed for \(d = 1\), cf. Remark 3.9) \(\|L_n^a - \tilde{\Gamma}_n(f_{a,\varepsilon})\|_{L^2(\mathbb{P})}\) is bounded by
\[
\|L_n^a - \Gamma_T(f_{a,\varepsilon})\|_{L^2(\mathbb{P})} + \|\Gamma_T(f_{a,\varepsilon}) - \tilde{\Gamma}_n(f_{a,\varepsilon})\|_{L^2(\mathbb{P})}
\lesssim \|L_n^a - \Gamma_T(f_{a,\varepsilon})\|_{L^2(\mathbb{P})} + \varepsilon^{-1+\rho} \begin{cases} T^{H}\Delta_n^{\frac{3}{2} - \rho}, H \geq 1/2, \\ T^{1/2}\Delta_n^{\frac{1}{2} - \rho}H, H < 1/2. \end{cases}
\]
By the occupation time formula (cf. Geman and Horowitz (1980)) and \(\int f_{a,\varepsilon}(x) dx = 1\) it follows that \(\|L_n^a - \Gamma_T(f_{a,\varepsilon})\|_{L^2(\mathbb{P})}^2\) is equal to \(\mathbb{E}[(L_n^a - \int f_{a,\varepsilon}(x) L_n^a dx)^2] = \mathbb{E}[(\frac{1}{T} \int_0^T (L_n^a - L_T^a)^2 dx)^2]\). Equation Pitt (1978, (4.1)) implies (together with the proof of Pitt (1978, Theorem 4)) that \(\mathbb{E}[(L_n^a - L_T^a)^2] \lesssim (a - b)^2 \varepsilon\) for all \(0 < \varepsilon < \frac{1}{2\pi}(1 - H)\). Consequently, \(\|L_n^a - \Gamma_T(f_{a,\varepsilon})\|_{L^2(\mathbb{P})} \lesssim \varepsilon^{2H/(1-H)-\rho}\). Optimizing in \(\varepsilon\) yields the claim.

\section*{Appendix C: Proof of Theorem 4.1}

Consider first the following two lemmas.

\textbf{Lemma C.1. Assume [X0].} For \(f \in H^1(\mathbb{R}^d)\) we have
\[
\|\Gamma_T(f) - \mathbb{E} [\Gamma_T (f) \mid \mathcal{G}_n]\|_{L^2(\mathbb{P})}^2 = \Delta_n^2 \mathbb{E} \left[ \frac{1}{12^2} \int_0^T \|\nabla f(X_t)\|^2 \, dt \right] + o(\Delta_n^2 \|f\|_{H^1}).
\]
In particular, \(\Delta_n^2 \|\Gamma_T(f) - \mathbb{E}[\Gamma_T (f) \mid \mathcal{G}_n]\|_{L^2(\mathbb{P})}^2\) converges to \(\mathbb{E}[(\frac{1}{12^2} \int_0^T \|\nabla f(X_t)\|^2 \, dt)]\) as \(n \to \infty\).

\textbf{Proof.} By independence of \(X_0\) and \((X_t - X_0)_{0 \leq t \leq T}\) the \(\sigma\)-algebra \(\mathcal{G}_n\) is also generated by \(X_0\) and the increments \(X_{t_k} - X_{t_{k-1}}, 1 \leq k \leq n\). The independence
of increments and the Markov property then imply for \( t_{k-1} \leq r \leq t_k \) that 
\[ \mathbb{E}[f(X_r) | \mathcal{G}_n] = \mathbb{E}[f(X_r) | X_{t_{k-1}}, X_{t_k}] \]
The same argument shows that the random variables 
\[ Y_k = \int_{t_{k-1}}^{t_k} (f(X_r) - \mathbb{E}[f(X_r) | \mathcal{G}_n]) dr \]
are uncorrelated. Therefore 
\[ \| \Gamma_T(f) - \mathbb{E}[\Gamma_T(f) | \mathcal{G}_n] \|_{L^2(\mathbb{P})}^2 = \sum_{k=1}^{n} \mathbb{E} \left[ Y_k^2 \right] = \sum_{k=1}^{n} \mathbb{E} \left[ \text{Var}_k \left( \int_{t_{k-1}}^{t_k} f(X_r) dr \right) \right], \]
where \( \text{Var}_k(Z) \) is the conditional variance of a random variable \( Z \) with respect to the 
\( \sigma \)-algebra generated by \( X_{t_{k-1}} \) and \( X_{t_k} \). In order to linearize \( f \), note that the random 
variable \( \text{Var}_k \left( f_{t_{k-1}}^t (f(X_r) - f(X_{t_{k-1}})) dr \right) \) can be written as 
\[ \text{Var}_k \left( \int_{t_{k-1}}^{t_k} (\nabla f(X_{t_{k-1}}), X_r - X_{t_{k-1}}) dr \right) + \kappa_n \]
\[ + \text{Var}_k \left( \int_{t_{k-1}}^{t_k} \left( f(X_r) - f(X_{t_{k-1}}) - \langle \nabla f(X_{t_{k-1}}), X_r - X_{t_{k-1}} \rangle \right) dr \right), \]
where \( \kappa_n \) is the corresponding crossterm of the decomposition. From Lemma A.2(ii) 
and (iii) it follows that the first and the last term are of order \( o(\Delta_n^3 \| f \|_{H^1}^2) \) and 
\( O(\Delta_n^3 a_n(f)) = O(\Delta_n^3 \| f \|_{H^1}^2) \), respectively, and thus by the Cauchy-Schwarz inequality 
\( \kappa_n = o(\Delta_n^3 \| f \|_{H^1}^2) \). Hence, \( \| \Gamma_T(f) - \mathbb{E}[\Gamma_T(f) | \mathcal{G}_n] \|_{L^2(\mathbb{P})}^2 \) is equal to 
\[ \sum_{k=1}^{n} \mathbb{E} \left[ \text{Var}_k \left( \int_{t_{k-1}}^{t_k} \langle \nabla f(X_{t_{k-1}}), X_r \rangle dr \right) \right] + o \left( \Delta_n^2 \| f \|_{H^1}^2 \right). \]

Conditional on \( X_{t_{k-1}}, X_{t_k} \), the process \( (X_r)_{t_{k-1} \leq r \leq t_k} \) is a Brownian bridge starting 
from \( X_{t_{k-1}} \) and ending at \( X_{t_k} \). In particular, \( \mathbb{E}[X_r | X_{t_{k-1}}, X_{t_k}] = X_{t_{k-1}} + \frac{r - t_{k-1}}{2\Delta_n} (X_{t_k} - X_{t_{k-1}}) \) (see e.g. [Karatzas and Shreve (1991, 6.10)]). The stochastic Fubini theorem 
and Itô isometry thus imply that the last display is equal to 
\[ \sum_{k=1}^{n} \mathbb{E} \left[ \left\langle \nabla f(X_{t_{k-1}}), \int_{t_{k-1}}^{t_k} \left( t_k - r - \frac{\Delta_n}{2} \right) dX_r \right\rangle \right]^2 + o \left( \Delta_n^2 \| f \|_{H^1}^2 \right) \]
\[ = \frac{\Delta_n^3}{12} \sum_{k=1}^{n} \mathbb{E} \left[ \| \nabla f(X_{t_{k-1}}) \|^2 \right] + o \left( \Delta_n^2 \| f \|_{H^1}^2 \right) \]
\[ = \frac{\Delta_n^2}{12} \int_0^T \| \nabla f(X_r) \|^2 dr + o \left( \Delta_n^2 \| f \|_{H^1}^2 \right), \]
where the last line follows from Lemma A.2(iv).

**Lemma C.2. Assume \( (X0) \).** Fix \( 0 \leq s < \alpha \) and let \( \varphi(x) = (2\pi)^{-d/2} e^{-\|x\|^2/2} \) for 
\( x \in \mathbb{R}^d \). Consider the approximations \( f_{\alpha, \varepsilon} = f_\alpha * \varphi_\varepsilon \), where \( \varphi_\varepsilon = \varepsilon^{-d} \varphi(\varepsilon^{-1} \cdot) \) and 
\( \varepsilon = \Delta_n^{\frac{1}{2d}}. \) Then the following statements hold as \( n \to \infty \):

(i) \( \| \Gamma_T(f_\alpha - f_{\alpha, \varepsilon}) - \mathbb{E}[\Gamma_T(f_\alpha - f_{\alpha, \varepsilon}) | \mathcal{G}_n] \|_{L^2(\mathbb{P})} = o(\Delta_n^{1+s}) \),

(ii) \( \| \Gamma_T(f_{\alpha, \varepsilon}) - \mathbb{E}[\Gamma_T(f_{\alpha, \varepsilon}) | \mathcal{G}_n] \|_{L^2(\mathbb{P})} = O(\Delta_n^2 \| f_{\alpha, \varepsilon} \|_{H^1}^2) = O(\Delta_n^{1+s}) \),

45
(iii) \( \liminf_{n \to \infty} (\varepsilon^{2-2\alpha} E[\frac{1}{T} \int_0^T \|\nabla f_{a,\varepsilon}(X_r)\|^2 dr]) > 0 \).

**Proof.** Applying (4.1) from right to left and Theorem 3.13 for the function \( f = f_{a,\varepsilon} \in L^2(\mathbb{R}^d) \) shows that the left hand side of the equation in (i) is up to a constant bounded by \( \Delta_n \| f_a - f_{a,\varepsilon} \|_{L^2}^2 \). The Plancherel theorem and \( \mathcal{F} \varphi \varepsilon (u) = \mathcal{F}(\varphi(\varepsilon u)) \) yield that this is equal to

\[
(2\pi)^{-d} \Delta_n \| \mathcal{F} f_a (1 - \mathcal{F} \varphi \varepsilon) \|_{L^2}^2 \lesssim \Delta_n \varepsilon^{2\alpha} \int \| u \|^{-2\alpha - d} \left( 1 - e^{-\|u\|^2/2} \right) du.
\]

The \( du \)-integral is finite and therefore the last line is of order \( O(\Delta_n \varepsilon^{2\alpha}) = o(\Delta_n^{1+s}) \), because \( \alpha > s \), implying (i). Similarly, applying (4.1) from right to left and Theorem 3.13 for the function \( f = f_{a,\varepsilon} \in H^1(\mathbb{R}^d) \), the left hand side of the equation in (ii) is up to a constant bounded by \( \Delta_n^2 \| f_{a,\varepsilon} \|_{H^1}^2 \). As above this can be bounded from the Plancherel theorem by

\[
(2\pi)^{-d} \Delta_n^2 \int |\mathcal{F} f_a(u)|^2 |\mathcal{F} \varphi \varepsilon(u)|^2 (1 + \| u \|)^2 du \\
\lesssim \Delta_n^2 \varepsilon^{2\alpha - 2} \int (\varepsilon + \| u \|)^{-2\alpha - d} e^{-\| u \|^2/2} du \\
\lesssim \Delta_n^2 \varepsilon^{2\alpha - 2} \int_0^\infty (\varepsilon + \| r \|)^{-1 - 2\alpha} e^{-\frac{r^2}{2}} dr.
\]

As \( \alpha < 1 \), the \( dr \)-integral is finite for \( \varepsilon = 0 \) and thus the last line is of order \( O(\Delta_n \varepsilon^{2\alpha - 2}) = O(\Delta_n^{1+s}) \). This is the claimed order in (ii). Finally, with respect to (iii), denote by \( p_r \) the marginal density of \( X_r \). Then we have by the Plancherel theorem, applied componentwise, for any \( T_0 > 0 \) that \( E[\frac{1}{T_0} \int_0^T \|\nabla f_{a,\varepsilon}(X_r)\|^2 dr] \) is bounded from below up to a constant by

\[
\int_{T_0}^T \left( \int \|\nabla f_{a,\varepsilon}(x)p_r^{1/2}(x)\|^2 dx \right) dr \\
= (2\pi)^{-2d} \int_{T_0}^T \left( \int \|\mathcal{F} f_a(u - y) \mathcal{F} \varphi(\varepsilon(u - y))(u - y) h_r(y) dy\|^2 du \right) dr.
\]

where \( h_r(y) = 2^{d/2}(2\pi)^{d/4} r^{d/4} e^{-\| y \|^2/2} \) is the Fourier transform of \( p_r^{1/2} \). The substitution \( \varepsilon u \to u \) then yields that the \( du \)-integral above is equal to

\[
\varepsilon^{2\alpha - 2} \int \|\nu_\varepsilon(u - \varepsilon y) h_r(y) dy\|^2 du = \varepsilon^{2\alpha - 2} \int \| (\nu_\varepsilon * h_{r,\varepsilon})(u) \|^2 du,
\]

for \( h_{r,\varepsilon}(u) = \varepsilon^{-d} h_r(\varepsilon^{-1} y) \) and \( \nu_\varepsilon(u) = u(\varepsilon + \| u \|)^{-\alpha/2} e^{-\| u \|^2/2} \). Interestingly, \( \nu_\varepsilon \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) for all \( \varepsilon \geq 0 \) as \( \alpha < 1 \). As also \( h_{r,\varepsilon} \in L^1(\mathbb{R}^d) \), Young’s inequality, also applied componentwise, implies that

\[
\int \| (\nu_\varepsilon - \nu_0) * h_{r,\varepsilon} \|(u)\|^2 du \leq \| h_{r,\varepsilon} \|^2_{L^1} \left( \int \| (\nu_\varepsilon - \nu_0)(u) \|^2 du \right).
\]

Since \( \| h_{r,\varepsilon} \|^2_{L^1} \lesssim r^{-d/2}, \| \nu_\varepsilon(u) \| \leq \| \nu_0(u) \| \) and \( \nu_\varepsilon(u) \to \nu_0(u) \) for any \( u \in \mathbb{R}^d \) we therefore conclude by dominated convergence that the last line is of order \( o(r^{-d/2}) \).
Moreover, it follows again by the Plancherel theorem with \( \mathcal{F} h_{r, \varepsilon}(x) = (2\pi)^d p_r^{1/2}(\varepsilon x) \) that \( \int \| (\nu_0 * h_{r, \varepsilon}) (u) \|^2 du = (2\pi)^d \int \| \mathcal{F} \nu_0 (x) \|^2 p_r (\varepsilon x) dx \). Letting \( \varepsilon \to 0 \) yields the convergence to \( (2\pi)^d r^{-d/2} \int \| \mathcal{F} \nu_0 (x) \|^2 dx \). By Pythagoras we thus find for any \( r > T_0 > 0 \) that also \( \int \| (\nu_\varepsilon * h_{r, \varepsilon}) (u) \|^2 du \to cr^{-d/2} \) for some constant \( 0 < c < \infty \). Consequently,

\[
\liminf_{n \to \infty} \left( \varepsilon^{2-2a} \mathbb{E} \left[ \frac{1}{12} \int_0^T \| \nabla f_{a, \varepsilon} (X_r) \|^2 dr \right] \right) \gtrsim \int_{T_0}^T r^{-d/2} dr,
\]

which is bounded from below as \( T_0 > 0 \).

Now we prove the theorem.

**Proof of Theorem 4.1.** The first and the second inequality in (i) are clear. The limit in the last equality follows from Lemma C.1 with respect to (ii) observe that

\[
\| \Gamma_T (f_\alpha) - \mathbb{E} [\Gamma_T (f_\alpha)] \|_{L^2(\mathbb{P})}^2 = \| \Gamma_T (f_\alpha - f_{a, \varepsilon}) - \mathbb{E} [\Gamma_T (f_\alpha - f_{a, \varepsilon})] \|_{L^2(\mathbb{P})}^2 + \kappa_n + \| \Gamma_T (f_{a, \varepsilon}) - \mathbb{E} [\Gamma_T (f_{a, \varepsilon})] \|_{L^2(\mathbb{P})}^2,
\]

where \( \kappa_n \) is the crossterm of the expansion. From Lemma C.2 it follows that the first term is of order \( o(\Delta_n^{1+s}) \), while the third one is of order \( O(\Delta_n^{1+s}) \). Therefore, the crossterm is via the Cauchy-Schwarz inequality itself of order \( o(\Delta_n^{1+s}) \). Hence, Lemma C.1 implies that \( \liminf_{n \to \infty} \Delta_n^{-(1+s)} \| \Gamma_T (f_\alpha) - \mathbb{E} [\Gamma_T (f_\alpha)] \|_{L^2(\mathbb{P})}^2 \) is equal to

\[
\liminf_{n \to \infty} \Delta_n^{-(1+s)} \| \Gamma_T (f_{a, \varepsilon}) - \mathbb{E} [\Gamma_T (f_{a, \varepsilon})] \|_{L^2(\mathbb{P})}^2 = \liminf_{n \to \infty} \left( \Delta_n^{1-s} \mathbb{E} \left[ \frac{1}{12} \int_0^T \| \nabla f_{a, \varepsilon} (X_r) \|^2 dr \right] \right) + \liminf_{n \to \infty} \left( o(\Delta_n^{-(1+s)} \Delta_n^2 \| f_{a, \varepsilon} \|_{H^1}) \right).
\]

From part (ii) of Lemma C.2 it follows that the last term is 0, while part (iii) implies the wanted lower bound for the first term, as \( \Delta_n^{1-s} \varepsilon^{2a-2} = 1 \).

**References**

Adams, R. and Fournier, J. (2003). *Sobolev Spaces*. Pure and Applied Mathematics. Elsevier Science.

Altmeyer, R. and Chorowski, J. (2016). Estimation error for occupation time functionals of stationary Markov processes. *arXiv preprint arXiv:1610.05225*.

Beskos, A. and Roberts, G. O. (2005). Exact simulation of diffusions. *The Annals of Applied Probability*, 15(4):2422–2444.

Billingsley, P. (2013). *Convergence of Probability Measures*. Wiley Series in Probability and Statistics. Wiley.

Boufoussi, B., Dozzi, M., and Guerbaz, R. (2007). Sample path properties of the local time of multifractional Brownian motion. *Bernoulli*, 13(3):849–867.
Catellier, R. and Gubinelli, M. (2016). Averaging along irregular curves and regularisation of ODEs. *Stochastic Processes and their Applications*, 126(8):2323–2366.

Chaumont, L. and Uribe Bravo, G. (2011). Markovian bridges: Weak continuity and pathwise constructions. *The Annals of Probability*, 39(2):609–647.

Chesney, M., Jeanblanc-Picqué, M., and Yor, M. (1997). Brownian Excursions and Parisian Barrier Options. *Advances in Applied Probability*, 29(29).

Chorowski, J. (2015). Nonparametric volatility estimation in scalar diffusions: Optimality across observation frequencies. *arXiv preprint arXiv:1507.07139*.

Dalalyan, A. (2005). Sharp adaptive estimation of the drift function for ergodic diffusions. *The Annals of Statistics*, 33(6):2507–2528.

Di, N., Palatucci, G., and Valdinoci, E. (2012). Hitchhiker’s guide to the fractional Sobolev spaces. *Bulletin des Sciences Mathématiques*, 136:521–573.

Diaconis, P. (1988). Bayesian numerical analysis. *Statistical decision theory and related topics IV*, 1:163–175.

Falconer, K. and Liu, L. (2012). Multistable Processes and Localizability. *Stochastic Models*, 28(3):503–526.

Florens-Zmirou, D. (1993). On Estimating the Diffusion Coefficient from Discrete Observations. *Journal of Applied Probability*, 30(4):790.

Fournier, N. and Printems, J. (2008). Absolute continuity for some one-dimensional processes. *Bernoulli*, 16(2):343–360.

Ganychenko, I. (2015). Fast $L_2$-approximation of integral-type functionals of Markov processes. *Modern Stochastics: Theory and Applications*, 2:165–171.

Ganychenko, I., Knopova, V., and Kulik, A. (2015). Accuracy of discrete approximation for integral functionals of Markov processes. *Modern Stochastics: Theory and Applications*, 2(4):401–420.

Ganychenko, I. and Kulik, A. (2014). Rates of approximation of non-smooth integral type functionals of Markov processes. *Modern Stochastics: Theory and Applications*, 1(2):117–126.

Geman, D. and Horowitz, J. (1980). Occupation Densities. *The Annals of Probability*, 8(1):1–67.

Gobet, E. and Labart, C. (2008). Sharp estimates for the convergence of the density of the Euler scheme in small time. *Electronic Communications in Probability*, 13:352–363.

Hugonnier, J.-N. (1999). The Feynman–Kac formula and pricing occupation time derivatives. *International Journal of Theoretical and Applied Finance*, 2(02):153–178.
Jacod, J. (1998). Rates of convergence to the local time of a diffusion. *Annales de l’Institut Henri Poincare (B) Probability and Statistics*, 34(4):505–544.

Jacod, J., Jakubowski, A., and Mémin, J. (2003). On asymptotic errors in discretization of processes. *The Annals of Probability*, 31(2):592–608.

Jacod, J. and Mykland, P. (2015). Microstructure noise in the continuous case: Approximate efficiency of the adaptive pre-averaging method. *Stochastic Processes and their Applications*.

Jacod, J. and Protter, P. (1988). Time Reversal on Levy Processes. *The Annals of Probability*, 16(2):620–641.

Jacod, J. and Protter, P. (2011). *Discretization of Processes*. Stochastic Modelling and Applied Probability. Springer Berlin Heidelberg.

Jacod, J. and Shiryaev, A. (2013). *Limit Theorems for Stochastic Processes*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg.

Kallenberg, O. (2002). *Foundations of Modern Probability*. Probability and Its Applications. Springer New York.

Karatzas, I. and Shreve, S. (1991). *Brownian Motion and Stochastic Calculus*. Springer.

Kohatsu-Higa, A., Makhlouf, R., and Ngo, H.-L. (2014). Approximations of non-smooth integral type functionals of one dimensional diffusion processes. *Stochastic Processes and their Applications*, 124(5):1881–1909.

Li, J., Todorov, V., and Tauchen, G. (2013). Volatility occupation times. *The Annals of Statistics*, 41(4):1865–1891.

Lou, S. and Ouyang, C. (2017). Local times of stochastic differential equations driven by fractional Brownian motions. *Stochastic Processes and their Applications*.

Mattingly, J. C., Stuart, A. M., and Tretyakov, M. V. (2010). Convergence of Numerical Time-Averaging and Stationary Measures via Poisson Equations. *SIAM Journal on Numerical Analysis*, 48(2):552–577.

Ngo, H.-L. and Ogawa, S. (2011). On the discrete approximation of occupation time of diffusion processes. *Electronic Journal of Statistics*, 5:1374–1393.

Nualart, D. (1995). *The Malliavin Calculus and Related Topics*. Probability and its applications : a series of the applied probability trust. Springer-Verlag.

Pitt, L. (1978). Local times for Gaussian vector fields. *Indiana University Mathematics Journal*, 27(2):309–330.

Podolskij, M. and Vetter, M. (2010). Understanding limit theorems for semimartingales: a short survey. *Statistica Neerlandica*, 64(3):329–351.
Rényi, A. (1963). On Stable Sequences of Events. *Sankhya: The Indian Journal of Statistics, Series A*, 25(3):293–302.

Romito, M. (2017). A simple method for the existence of a density for stochastic evolutions with rough coefficients. *arXiv preprint arXiv:1707.05042*.

Russo, F. and Vallois, P. (1996). Ito formula for C1-functions of semimartingales. *Probability Theory and Related Fields*, 104(1):27–41.

Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge Studies in Advanced Mathematics. Cambridge University Press.

Tankov, P. (2003). *Financial Modelling with Jump Processes*. Chapman and Hall/CRC Financial Mathematics Series. CRC Press.

Triebel, H. (2010). *Theory of Function Spaces*. Modern Birkhäuser Classics. Springer Basel.