Local conserved charges in principal chiral models

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ABSTRACT

Local conserved charges in principal chiral models in 1+1 dimensions are investigated. There is a classically conserved local charge for each totally symmetric invariant tensor of the underlying group. These local charges are shown to be in involution with the non-local Yangian charges. The Poisson bracket algebra of the local charges is then studied. For each classical algebra, an infinite set of local charges with spins equal to the exponents modulo the Coxeter number is constructed, and it is shown that these commute with one another. Brief comments are made on the evidence for, and implications of, survival of these charges in the quantum theory.
1 Introduction

Integrable lagrangian field theories in 1+1 dimensions exhibit various kinds of higher-spin conserved quantities which constrain their quantum behaviour, forcing their S-matrices to factorize and allowing them to be determined exactly in many instances. Such exotic symmetries can usually be traced to underlying mathematical structures which incorporate Lie algebras in some way. Beyond these broad similarities, however, one encounters many examples with profound physical and mathematical differences. One of the most important distinctions is between conserved charges which are integrals of local functions of the fields, and those which instead involve non-local functions of the fields.

Local charges have been studied extensively in affine Toda theories (ATFTs) for which the defining Lie algebra data are just a set of simple roots (plus the lowest root). Each ATFT possesses infinitely many commuting local charges with spins equal to the exponents of the Lie algebra modulo its Coxeter number \( h \):

\[
\begin{align*}
    a_\ell &= \mathfrak{su}(\ell+1) \quad 1, 2, 3, \ldots, \ell \quad h = \ell + 1 \\
    b_\ell &= \mathfrak{so}(2\ell+1) \quad 1, 3, 5, \ldots, 2\ell - 1 \quad h = 2\ell \\
    c_\ell &= \mathfrak{sp}(2\ell) \quad 1, 3, 5, \ldots, 2\ell - 1 \quad h = 2\ell \\
    d_\ell &= \mathfrak{so}(2\ell) \quad 1, 3, 5, \ldots, 2\ell - 3, \ell - 1 \quad h = 2\ell - 2
\end{align*}
\]

Their local nature means that these charges are additive on asymptotic, multi-particle states, and the fact that they commute means that single-particle states may be chosen to be simultaneous eigenstates of them. Their existence places strong constraints on the possible three-point couplings, and their relationship with the underlying Lie algebra in the theory is crucial in ensuring that there is a consistent solution of the bootstrap equations (see [2] and e.g. [3]) as was made clear in the elegant construction of Dorey [4].

Non-local charges have proved very important in understanding the classical and quantum integrability of certain non-linear sigma-models [5, 6], amongst which are the principal chiral models (PCMs) with target space some compact Lie group. They also occur in ATFTs with imaginary coupling, in connection with the soliton solutions in these theories. In marked distinction to local charges, non-local charges are not additive on multi-particle states and they can have either indefinite or non-integral spin. The non-local charges generate a quantum group structure which underpins the factorizability of the S-matrix by naturally providing solutions of the Yang-Baxter equation (see e.g. [7]). The quantum group is a Yangian [8] in the case of PCMs, whereas it is a quantized affine algebra for
imaginary coupling ATFTs \[8\].

There are some very important physical differences between the types of models we have just mentioned. In ATFTs the coupling constant is small at low energies, so that perturbation theory and semi-classical techniques can be used to determine the spectrum of particles (including solitons for the imaginary coupling theories). One-loop calculations confirm that mass ratios of particles are unchanged for the simply-laced algebras, while they vary with the coupling constant for the nonsimply-laced cases (see e.g. \[3, 2\]). In complete contrast, any sigma-model with a compact target manifold—in particular a PCM—is strongly-coupled in the infra-red, so that information about the quantum theory is much harder to extract from the lagrangian. In fact the classical lagrangians for these theories are scale-invariant, and masses arise through a complicated quantum-mechanical effect at strong coupling.

Despite the disparate properties of these field theories and the very different roles played by the local and non-local charges within them, it turns out that they still have much in common at the quantum level, and in particular the patterns of masses and three-point couplings coincide in a rather remarkable way. The mass ratios which emerge for PCMs (from their exact S-matrices \[10, 11, 12\]) are actually identical to those of the ATFTs \[3, 2\] if one considers theories based on the same, simply-laced Lie algebra. For nonsimply-laced algebras there is a more subtle connection, in which the PCM mass ratios coincide with those of tree-level ATFTs based on twisted affine algebras.

The three-point couplings in ATFTs are given by Dorey’s rule, while the three-point couplings in PCMs arise as S-matrix fusions contained in the tensor product rule for representations of the Yangian (the analogue of the Clebsch-Gordan rule for Lie algebras). There is no reason to suppose \textit{a priori} that these should be related, and yet it has recently been proved \[13\] that the quantum group fusion rule for fundamental representations (derived from the non-local charge algebra) and Dorey’s rule (derived from local charges) are one and the same. Since the proof consists of a complicated case-by-case analysis, however, it remains something of a mystery why this is so.

Our aim in this paper is to study in detail the properties of certain local conserved charges in the principal chiral models. Their existence has been known for some time \[14, 15\], but they have been somewhat neglected in PCMs in favour of the non-local charges discussed above. An obvious possibility is that the PCM local charges can offer a more transparent explanation for the striking common features of PCMs and ATFTs. Another motivation
is that the detailed study of this particular model in which the local and non-local charges co-exist may offer some insights into the way in which these entities are related in general.

The majority of the paper—sections 2, 3 and 4—elucidates the classical properties of the local charges appearing in PCMs. There exists a conserved local charge for each invariant tensor or Casimir of the underlying Lie algebra, and we shall prove that these always commute with the non-local Yangian charges. We then investigate in detail the Poisson bracket algebra of the local charges with one another. Our main result is to find for each PCM an infinite family of local charges which commute classically and which have spins equal to the exponents modulo the Coxeter number, precisely as for the ATFTs. Because of the difficulties inherent in quantizing the PCM, there is only a limited amount which can be said about the quantum behaviour of these charges, which we summarize in section 5. We can, nevertheless, examine the implications of their survival at the quantum level, and we find complete compatibility with the known multiplet structures and exact S-matrices [10, 12].

2 The classical principal chiral model

2.1 The lagrangian, symmetries and currents

The principal chiral model (PCM) is defined by the lagrangian

\[ \mathcal{L} = \frac{\kappa}{2} \text{Tr} \left( \partial_\mu g^{-1} \partial^\mu g \right) \]  

(2.1)

where the field \( g(x^\mu) \) takes values in some compact Lie group \( \mathcal{G} \) and \( \kappa \) is a dimensionless coupling which may be set to any convenient value in the classical theory, without loss of generality. There is a global continuous symmetry

\[ \mathcal{G}_L \times \mathcal{G}_R : \ g \mapsto U_L g U_R^{-1} \]  

(2.2)

with corresponding conserved currents

\[ j^L_\mu = \kappa \partial_\mu g g^{-1}, \quad j^R_\mu = -\kappa g^{-1} \partial_\mu g \]  

(2.3)

which take values in the Lie algebra \( g \) of \( \mathcal{G} \). The equations of motion following from (2.1) correspond to the conservation of these currents:

\[ \partial^\mu j_\mu(x, t) = 0. \]  

(2.4)
They also obey
\[ \partial_\mu j_\nu - \partial_\nu j_\mu - \frac{1}{\kappa} [j_\mu, j_\nu] = 0 \] (2.5)
identically, as a consequence of their definitions in terms of the field \( g \). Here and elsewhere we will adopt the convention that any equation written for a current \( j_\mu \) without a label holds true for both \( L \) and \( R \) currents. It is significant that the last condition above can be interpreted as a zero-curvature condition for a connection with covariant derivative
\[ \nabla_\mu X = \partial_\mu X - \frac{1}{\kappa} [j_\mu, X] \] (2.6)
acting on any \( X \) in \( g \). Here we view \( -\kappa^{-1} j_\mu \) as a two-dimensional gauge field, its definition in terms of \( g \) implying that it is pure-gauge. The two conditions (2.4, 2.5) capture the entire algebraic structure of the PCM.

We shall restrict attention to the classical groups \( \mathcal{G} = SU(\ell), SO(\ell), Sp(\ell) \) (with \( \ell \) even in the last case) with the field \( g(x^\mu) \) in the defining representation. The corresponding Lie algebra \( \mathfrak{g} \) then consists of \( \ell \times \ell \) complex matrices \( X \) which obey
\[ \begin{align*}
\text{su}(\ell) : & \quad X^\dagger = -X, \quad \text{Tr}(X) = 0 \\
\text{so}(\ell) : & \quad X^* = X, \quad X^T = -X \\
\text{sp}(\ell) : & \quad X^\dagger = -X, \quad X^T = -JXJ^{-1}
\end{align*} \] (2.7)
where \( J \) is some chosen symplectic structure. In each case we introduce a basis of anti-hermitian generators \( t^a \) for \( \mathfrak{g} \) with real structure constants \( f^{abc} \) and normalizations given by
\[ [t^a, t^b] = f^{abc} t^c, \quad \text{Tr}(t^a t^b) = -\delta^{ab} . \] (2.8)
(Lie algebra indices will always be taken from the beginning of the alphabet.) For any \( X \in \mathfrak{g} \) we write
\[ X = t^a X^a \quad X^a = -\text{Tr}(t^a X) . \] (2.9)

Spacetime symmetries will also play an important role in what follows. The classical PCM lagrangian is conformally-invariant, and as a result the energy momentum tensor
\[ T_{\mu\nu} = -\frac{1}{2\kappa} \left( \text{Tr}(j_\mu j_\nu) - \frac{1}{2} \eta_{\mu\nu} \text{Tr}(j_\rho j^\rho) \right) \] (2.10)
is not only conserved and symmetric but also traceless. We shall use standard orthonormal coordinates \( x^0 = t \) and \( x^1 = x \) in two dimensions, as well as light-cone coordinates and their derivatives defined by
\[ x^\pm = \frac{1}{2} (t \pm x), \quad \partial_\pm = \partial_t \pm \partial_x . \]
The equations (2.4, 2.5) can then be written
\[ \partial - j_+ = - \partial + j_- = - \frac{1}{2\kappa} [j_+, j_-], \quad (2.11) \]
whilst the energy-momentum tensor takes the familiar form
\[ T_{\pm\pm} = - \frac{1}{2\kappa} \text{Tr}(j_\pm j_\pm), \quad T_{+-} = T_{-+} = 0, \quad (2.12) \]
with
\[ \partial_- T_{++} = \partial_+ T_{--} = 0. \quad (2.13) \]

In addition to the continuous symmetries comprising \( \mathcal{G}_L \) and \( \mathcal{G}_R \), there are also important discrete symmetries of the principal chiral model. For any PCM there is a symmetry
\[ \pi : g \mapsto g^{-1} \Rightarrow j^L \leftrightarrow j^R \quad (2.14) \]
which exchanges \( \mathcal{G}_L \) and \( \mathcal{G}_R \) and which we shall therefore refer to as \( \mathcal{G} \)-parity. (In the PCM effective field theory description of strong interactions in four dimensions, the physical parity operator is our \( \mathcal{G} \)-parity together with spatial reflection.) Additional discrete symmetries arise as outer automorphisms of \( \mathcal{G} \) acting on the field \( g \). Thus we have
\[ \gamma : g \mapsto g^* \Rightarrow j^L \mapsto (j^L)^* = -(j^L)^T, \quad j^R \mapsto (j^R)^* = -(j^R)^T, \quad (2.15) \]
which exchanges complex-conjugate representations. This realizes the outer automorphism of \( g = su(\ell) \). The map is trivial (up to conjugation) if \( g \) has only real (or pseudo-real) representations. For \( g = so(2\ell) \) we also have
\[ \sigma : g \mapsto MgM^{-1} \Rightarrow j^L \mapsto Mj^LM^{-1}, \quad j^R \mapsto Mj^RM^{-1}, \]
where \( M \) is a fixed matrix with determinant \(-1\). This is the outer automorphism which exchanges the inequivalent spinor representations. The maps \( \sigma \) and \( \gamma \) coincide (up to conjugation) for \( g = so(2\ell) \) when \( \ell \) is odd.

2.2 Canonical formalism

The canonical Poisson brackets for the theory are
\[
\begin{align*}
\{ j^a_0(x), j^b_0(y) \} &= f^{abc} j^c_0(x) \delta(x-y) \\
\{ j^a_0(x), j^b_1(y) \} &= f^{abc} j^c_1(x) \delta(x-y) + \kappa \delta^{ab} \delta^i(x-y) \\
\{ j^a_1(x), j^b_1(y) \} &= 0
\end{align*}
\] (2.16)
at equal time. These expressions hold for either of the currents $j^L$ or $j^R$ separately, while the algebra of $j^L$ with $j^R$ involves only $\delta'(x-y)$ terms in the brackets of space- with time-components \cite{16} (we shall not need them here) in keeping with the direct product structure of $G_L \times G_R$. For light-cone current components these brackets become
\begin{align*}
\{ j^a_\pm(x), j^b_\pm(y) \} &= f^{abc} \left( \frac{3}{2} j^c_\pm(x) - \frac{1}{2} j^c_\mp(x) \right) \delta(x-y) \pm 2\kappa \delta^{ab} \delta'(x-y) \\
\{ j^a_+(x), j^b_-(y) \} &= \frac{1}{2} f^{abc} \left( j^c_+(x) + j^c_-(x) \right) \delta(x-y)
\end{align*}
They imply that the energy momentum tensor satisfies the classical, centre-less Virasoro algebra
\begin{equation}
\{ T_\pm(x), T_\pm(y) \} = \pm 2(T_\pm(x) + T_\pm(y)) \delta'(x-y) .
\end{equation}

There are a number of ways to derive the expressions \eqref{2.16}. One approach \cite{16} has the advantage of dealing directly with the currents rather than with the underlying field $g$: we can select $j_1$ as the only independent dynamical variable, since we can regard $j_0$ as a function of it which is determined through the relation \eqref{2.7}. The price to be paid is the introduction of an operator $\nabla^{-1}_1$ which is non-local in space and whose definition assumes suitable boundary conditions for the currents at spatial infinity. This enables us to write $j_0 = \nabla^{-1}_1(\partial_0 j_1)$ and so write the lagrangian as (we set $\kappa = 1$ here for simplicity)
\begin{equation*}
\mathcal{L} = -\frac{1}{2} \text{Tr}(j^2_0 - j^2_1) = -\frac{1}{2} \text{Tr} \left[ (\nabla^{-1}_1 \partial_0 j_1)^2 - j^2_1 \right] 
\end{equation*}
The momentum conjugate to $j_1$ is defined in the usual way: $\pi_1 = \partial \mathcal{L}/\partial(\partial_0 j_1) = -\nabla^{-2}_1 \partial_0 j_1$ and we deduce that $j_0 = -\nabla_1 \pi_1$. (This requires the property $\nabla^{-1}_1(A)B = -A \nabla^{-1}_1(B)$ up to terms which vanish on integrating over space, so again the adoption of suitable boundary conditions on the fields is crucial.) The expressions \eqref{2.16} can now be recovered from the standard equal-time Poisson brackets for $j_1$ and $\pi_1$ after a short calculation. A lengthier but more routine derivation which avoids the use of the operator $\nabla^{-1}_1$ can be found in an appendix.

3 Classical conserved charges

3.1 Non-local charges

There exist infinitely many conserved, Lie algebra-valued, non-local charges in the PCM, which generate a Yangian $Y(g)$ \cite{18}. In fact there are two copies of this structure, constructed either from $j^L_\mu$ or $j^R_\mu$, and so the model has a charge algebra $Y_L(g) \times Y_R(g)$. (It
can be checked that $Y_L$ and $Y_R$ commute.) A full set of non-local charges $Q^{(n)a}$ with $n = 0, 1, 2, \ldots$, can be generated from the obvious local charge

$$Q^{(0)a} = \int_{-\infty}^{\infty} j_0^a \, dx$$

and the first non-local charge

$$Q^{(1)a} = \int_{-\infty}^{\infty} j_1^a \, dx - \frac{1}{2\kappa} f^{abc} \int_{-\infty}^{\infty} j_0^b(x) \int_{-\infty}^{x} j_0^c(y) \, dy \, dx .$$

This set can also be defined by a power series expansion of the transfer matrix in its spectral parameter, or equivalently they can be constructed by the following iterative procedure [6].

Suppose we have Lie algebra-valued currents $j^{(r)}_{\mu}$ defined for $r = 0, 1, \ldots, n$ which are conserved:

$$\partial^- j^{(r)}_+ + \partial^+ j^{(r)}_- = 0 \iff j^{(r)}_\pm = \pm \partial \chi^{(r)}$$

for some scalar Lie algebra-valued functions $\chi^{(r)}$, and that these currents are related to one another by

$$j^{(r+1)}_{\mu} = \nabla \chi^{(r)} = \partial \chi^{(r)} - \kappa^{-1} [j^{(r)}_{\mu}, \chi^{(r)}].$$

Taking $r = n$ defines a new current, $j^{(n+1)}_{\mu}$ which is conserved because

$$\partial^- j^{(n+1)}_+ + \partial^+ j^{(n+1)}_- = (\partial^- \nabla^0 + \partial^0 \nabla^-) \chi^{(n)} = (\nabla^+ \partial^- + \nabla^- \partial^+) \chi^{(n)} = -\nabla^+ j^{(n)}_- + \nabla^- j^{(n)}_+ = -[\nabla^+, \nabla^-] \chi^{(n-1)} = 0$$

for $n \geq 1$. Thus, starting from $j^{(0)}_{\mu} = j_{\mu}$ and $j^{(1)}_{\mu}$ given by

$$j^{(1)a}_0(x) = j^a_1(x) - \frac{1}{2\kappa} f^{abc} j^b_0(x) \int_{-\infty}^{x} j^c_0(y) \, dy , \quad j^{(1)a}_1(x) = j^a_0(x) - \frac{1}{2\kappa} f^{abc} j^b_1(x) \int_{-\infty}^{x} j^c_1(y) \, dy$$

we can define an infinite set of currents which are non-local functions of the original fields for $n > 0$; their conserved charges are $Q^{(n)a} = \int j^{(n)a}_{\mu} \, dx$.

Classically, the non-local charges are Lorentz scalars: applying the boost operator $M$ we obtain $\{ M, Q^{(0)a} \} = \{ M, Q^{(1)a} \} = 0$. (The second of these equations is modified in the quantum theory, however—see below.) Because the charges are non-local they will not generally be additive on products of states. Their action is given by the coproduct:

$$\Delta (Q^{(0)a}) = Q^{(0)a} \otimes 1 + 1 \otimes Q^{(0)a}$$

and

$$\Delta (Q^{(1)a}) = Q^{(1)a} \otimes 1 + 1 \otimes Q^{(1)a} + \frac{1}{2\kappa} f^{abc} Q^{(0)b} \otimes Q^{(0)c} ,$$

(3.5)
which we see is non-trivial in the second case. As well as the usual interpretation in the quantum theory, these equations may also be interpreted classically as giving the values of the charges on widely-separated, localized field configurations \[5\].

### 3.2 Local charges

In any conformally-invariant theory, the conservation of the energy-momentum tensor \((2.13)\) immediately implies a series of higher-spin conservation laws:

\[
\partial_-(T^{n+\pm}_++) = \partial_+(T^{n-\pm}_--) = 0 .
\]

But the PCM has more basic conservation laws which depend on the detailed form of the equations of motion of the currents rather than on conformal invariance alone. The simplest examples are

\[
\partial_- \text{Tr}(j^m_+) = \partial_+ \text{Tr}(j^m_-) = 0 \tag{3.7}
\]

which follow easily from \((2.11)\). More generally, we may consider any rank-\(m\), totally symmetric, invariant tensor \(d_{a_1a_2...a_m}\) associated with a Casimir operator

\[
C_m = d_{a_1a_2...a_m}t^{a_1}t^{a_2}...t^{a_m} \tag{3.8}
\]

where

\[
[C_m, t_b] = 0 \quad \iff \quad d_{c(a_1a_2...a_{m-1}f_{a_m})bc} = 0 \tag{3.9}
\]

(and as usual \((...)\) denotes symmetrization of the enclosed indices). It is then easy to check that invariance of \(d\) ensures the conservation equations

\[
\partial_\pm (d_{a_1a_2...a_m}j^{a_1}_\pm j^{a_2}_\pm ...j^{a_m}_\pm) = 0 . \tag{3.10}
\]

The corresponding conserved charges will be denoted

\[
q_{\pm s} = \int_{-\infty}^{\infty} d_{a_1a_2...a_m} j^{a_1}_\pm(x) j^{a_2}_\pm(x)...j^{a_m}_\pm(x) \, dx \tag{3.11}
\]

and labelled by their spin \(s = m-1\) (the Poisson bracket with the boost generator \(M\) is \(\{M, q_{\pm s}\} = \pm sq_{\pm s}\)). We shall refer to charges \(q_{\pm s}\) with \(s > 0\) as having positive/negative chirality.

Invariance of the \(d\)-tensor implies that the same local conservation laws are obtained using either of the currents \(j^L\) or \(j^R\), so there is just a single copy of these local charges, unlike the two-fold \(L\) and \(R\) copies of the non-local charges. Also in contrast to the non-local
charges, we note that any local charge must be additive on multi-particle states, which we can also express by saying that such a charge has a trivial co-product: $\Delta(q_s) = q_s \otimes 1 + 1 \otimes q_s$.

The currents in (3.6) correspond to even-rank invariant tensors constructed from Kronecker deltas:

$$d_{a_1 a_2 \ldots a_{2n-1} a_{2n}} = \delta(a_1 a_2 \delta a_3 a_4 \ldots \delta a_{2n-1} a_{2n})$$

(3.12)

while those in (3.7) correspond to

$$d_{a_1 a_2 \ldots a_m} = \text{STr}(t^{a_1} t^{a_2} \ldots t^{a_m})$$

(3.13)

with ‘STr’ denoting the trace of a completely symmetrized product of matrices. For $su(\ell)$ this tensor is non-vanishing for each integer $m$, but for $so(\ell)$ or $sp(\ell)$ it is non-zero only when $m$ is even. It is useful to introduce the notation

$$J_m = \text{Tr}(j^m_+)$$

(3.14)

for the corresponding currents. Notice that $J_2$ is proportional to the energy-momentum tensor $T_{++}$, so that the currents in (3.6) can also be written $(J_2)^n$.

There are infinitely many invariant tensors $d_{a_1 \ldots a_m}$ for each algebra $g$, but there are only $\text{rank}(g)$ independent or primitive $d$-tensors and Casimirs (see e.g. [19]) with degrees equal to the exponents of $g$ plus one. All other invariant tensors can be expressed as polynomials in these and the structure constants $f_{abc}$. The choice of these $\text{rank}(g)$ primitive tensors is not unique, the ambiguity being the addition of polynomials in tensors of lower rank. The symmetrized traces in (3.13) are a particular choice for all the primitive $d$-tensors of the classical algebras, with one exception. This exception is the Pfaffian invariant in $so(2\ell)$, which has rank $\ell$ and can be written

$$d_{a_1 \ldots a_\ell} = \epsilon_{i_1 j_1 \ldots i_\ell j_\ell}(t^{a_1})_{i_1 j_1} \ldots (t^{a_\ell})_{i_\ell j_\ell}.$$  

(3.15)

This tensor cannot be expressed as a trace in the defining representation, although it is related to a trace in the spinor representation. We denote the corresponding current by

$$P_\ell = \epsilon_{i_1 j_1 \ldots i_\ell j_\ell}(j^+_{i_1})_{i_1 j_1} \ldots (j^+_{i_\ell})_{i_\ell j_\ell}.$$  

(3.16)

Finally, we should mention that there are infinitely many more conserved currents in the PCM which arise as differential polynomials in those already discussed. For example, $\partial_- \left( \text{Tr}(j^+_{i_+}) \partial_+ \text{Tr}(j^+_{i_+}) \right) = 0$ follows immediately from (3.7). We shall not be directly concerned with the properties of these more general currents.
3.3 Commutation of local with non-local charges

We will now show that all local charges of the general type (3.11) commute with the non-local charges generated by $Q^{(0)a}$ and $Q^{(1)a}$. This means showing that

$$\{ q_s, Q^{(0)b} \} = \{ q_s, Q^{(1)b} \} = 0. \quad (3.17)$$

The vanishing of the first bracket follows immediately from invariance of the $d$-tensor used to define $q_s$; this says simply that the charge $q_s$ is a singlet under the Lie algebra. The calculation of the second bracket is more delicate, and involves a cancellation between contributions originating from ultralocal and non-ultralocal terms.

Consider the expression (3.2) for $Q^{(1)b}$, which involves two terms. The bracket of $q_s$ with the first (local) term is

$$\{ q_s, \int dy j^b_1(y) \} = d_{a_1a_2...a_m} \int dx dy \{ j^{a_1}_+(x) \ldots j^{a_m}_+(x), j^b_1(y) \}$$

$$= -m d_{a_1a_2...a_{m-1}c} f^{bcd} \int dx j^{a_1}_+(x) \ldots j^{a_{m-1}}_+(x) j^d_1(x). \quad (3.18)$$

We have used the fact that $d_{a_1...a_m}$ is totally symmetric and also the boundary conditions $j \to 0$ as $x \to \pm \infty$ which imply that there is no contribution from $\delta'$ terms in the current Poisson bracket. To deal with the second (non-local) term in (3.2) we must handle carefully the limits of the spatial integration: we take these to be $\pm L$ and only afterwards let $L \to \infty$. We must calculate

$$\{ q_s, \int_{-L}^L dx \int_{-L}^L dy \int_{-L}^L dz f^{bcd} j^c_0(y) j^d_0(z) \}$$

$$= m d_{a_1a_2...a_m} f^{bcd} \int_{-L}^L dx \int_{-L}^L dy \int_{-L}^L dz j^{a_1}_+(x) \ldots j^{a_m}_+(x) \{ j^{a_m}_+(x), j^c_0(y) j^d_0(z) \}$$

$$= \kappa m d_{a_1a_2...a_{m-1}c} f^{bcd} \int_{-L}^L dx \int_{-L}^L dy \int_{-L}^L dz j^{a_1}_+(x) \ldots j^{a_{m-1}}_+(x) \left[ j^d_0(z) \delta'(x-y) - j^d_0(y) \delta'(x-z) \right]$$

from (2.16). Introducing a step-function, the multiple integral can be written

$$\int_{-L}^L dx \int_{-L}^L dy \int_{-L}^L dz j^{a_1}_+(x) \ldots j^{a_{m-1}}_+(x) j^d_0(z) \left[ \theta(y-z) - \theta(z-y) \right] \delta'(x-y)$$

$$= \int_{-L}^L dx dz j^{a_1}_+(x) \ldots j^{a_{m-1}}_+(x) j^d_0(z) \left[ \delta(x-z) - \delta(x+L) - \delta(x-L) + \delta(x-z) \right]$$

$$= 2 \int_{-L}^L dx j^{a_1}_+(x) \ldots j^{a_{m-1}}_+(x) j^d_0(x)$$
where we have again made use of the condition $j \to 0$ as $x \to \pm \infty$ which implies that the middle two $\delta$-functions in the penultimate line make no contribution. Thus we find

$$
\{ q_s , \frac{-1}{2\kappa} \int_{-L}^{L} dy \int_{-L}^{y} dz \, f^{bcd} j^c_0(y) j^d_0(z) \} = -m d_{a_1 a_2 \ldots a_{m-1} c} f^{bcd} \int dx \, j^{a_1}_+(x) \ldots j^{a_{m-1}}_+(x) j^d_0(x) 
$$

(3.19)

When we add the terms (3.18) and (3.19) we can combine $j^d_0 + j^d_1 = j^d_+$. The symmetrization on tensor indices contracted with $j^d_+$ currents then implies that the total expression vanishes, by (3.9).

4 Classical Algebra of Local Charges

In this section we discuss in detail the classical Poisson bracket algebra of local charges $q_{\pm s}$ of the type (3.11). In calculating these from (2.17) one finds that the terms involving $\delta(x-y)$ (i.e. the ultra-local terms) always vanish by invariance of the $d$-tensors, leaving just the contributions from the $\delta'(x-y)$ terms. It is clear from (2.17) that these too are absent if we are considering charges of opposite chirality, so that

$$
\{ q_s , q_{-r} \} = 0 , \quad r, s > 0 .
$$

(4.1)

For charges of the same chirality, however, the result is generally non-zero:

$$
\{ q_s , q_r \} = (\text{const}) \int_{-\infty}^{\infty} dx \, d_{ca_1 \ldots a_s d_{cb_1 \ldots b_r}} \partial_x (j^{a_1}_+ \ldots j^{a_s}_+ j^{b_1}_- \ldots j^{b_r}_+) .
$$

(4.2)

Note that the expression on the right is antisymmetric in $s$ and $r$, by integration by parts.

Before proceeding we must pause to prove our assertion that the ultra-local terms do not contribute to (4.1) and (4.2). In (4.1) the ultra-local terms produce an integrand which is a combination of

$$
d_{a_1 \ldots a_s a} d_{b_1 \ldots b_r b} j^{a_1}_+ \ldots j^{a_s}_+ j^{b_1}_- \ldots j^{b_r}_- f^{abc} j^c_+ 
$$

with all fields at the same spacetime argument. These expressions vanish by invariance (3.9) of one or other of the $d$-tensors involved. In (4.2) the ultra-local terms contribute integrands

$$
d_{a_1 \ldots a_s a} d_{b_1 \ldots b_r b} j^{a_1}_+ \ldots j^{a_s}_+ j^{b_1}_+ \ldots j^{b_r}_+ f^{abc} j^c_+ 
$$

The expression with $j^c_+$ vanishes by invariance of either tensor, while the expression with $j^c_-$ can be rearranged using invariance of the second tensor to yield a result proportional to

$$
d_{a_1 \ldots a_s a} d_{b_1 \ldots b_r b} j^{a_1}_+ \ldots j^{a_s}_+ j^{b_1}_- \ldots j^{b_r}_- f^{abc} j^c_+ 
$$
and this vanishes by invariance of the first tensor. This justifies our assertions.

Our aim is to find invariant tensors and conserved currents for which the expression (4.2) vanishes, so that the charges commute. In the special case \( r = 1 \) and \( d_{bc} = \delta_{bc} \), the integrand in (4.2) is clearly a total derivative and hence the Poisson bracket is zero. This simply means that all the local charges (3.11) commute with energy and momentum: they are invariant under translations in space and time. It is also easy to see that the integrand in (4.2) can be written as a total derivative if both currents are of the form (3.6). This is a feature of any classically conformally-invariant theory whose energy-momentum tensor obeys the Virasoro algebra (2.18). It is a simple consequence of this that the charges \( \int (T_{++})^n dx \) all commute. Finding more interesting sets of commuting charges with \( s, r > 1 \) in the PCM is rather more involved, as we shall see.

4.1 Algebra of charges for currents \( J_m \)

The natural currents to consider first are \( J_m \) defined by (3.14), in which case (4.2) can be written

\[
\{ q_s, q_r \} = (const) \int_{-\infty}^{\infty} dx \ Tr(t^c j_s^+ ) \partial_x Tr(t^c j_r^+ ).
\]  

(4.3)

To simplify this we will use the completeness condition \( X = -t^a \ Tr(t^a X) \), which is valid for any \( X \) in the Lie algebra \( \mathfrak{g} \). This can be applied directly to the integrand in (4.3) if we know that a given power \( j^+_{r+s} \) or \( j^s_{r+s} \) lives in \( \mathfrak{g} \) for any \( j_{r+s} \) in \( \mathfrak{g} \). Whenever this is true, the completeness condition implies that the integrand is proportional to \( \partial_x Tr(j^+_{r+s}) \) and hence the charges commute.

This argument applies to the orthogonal and symplectic algebras \( so(\ell) \) and \( sp(\ell) \), so that in these PCMs the charges \( f J_m dx \) always commute. To elaborate on this, consider the orthogonal case. If \( X \) is in \( so(\ell) \) it is a real anti-symmetric matrix, and if \( r \) is odd, \( X^r \) will also be real and anti-symmetric, and hence also in \( so(\ell) \). This implies firstly that \( J_{r+1} \) is zero unless \( r \) is odd, and secondly that under these circumstances we can apply the completeness condition to write the integrand in (4.3) as a total derivative, proportional to \( \partial_x (J_{r+s}) \). For the symplectic algebras it is easy to see that if \( X \) satisfies the defining conditions (2.7) then so does \( X^r \) if \( r \) is odd, so the argument works in precisely the same way.

Although we are concerned in this paper with PCMs based on simple, compact classical groups, it is worth mentioning in passing the non-simple case \( \mathfrak{g} = u(\ell) \). If \( X \) is in \( u(\ell) \),
then so is $X^r$ if $r$ is odd, or $iX^r$ if $r$ is even. Either way, the argument applies in just the same way and the charges $\int J_{s+1} dx$ commute. For $g = su(\ell)$ the situation is more complicated, however. In this case the completeness condition holds only for traceless matrices $X$ and this property is of course spoiled by taking powers. We can still use the completeness condition, but we must do so in a less direct way. Since the generators $t_\pm$ in (4.3) are traceless, we can first replace $j_+^r$ by the traceless quantity $j_+^r - (1/\ell)\text{Tr}(j_+^r)$ and then apply the completeness condition to find

$$\{q_s, q_r\} = (\text{const}) \int_{-\infty}^{\infty} dx \text{Tr}(j_+^r) \partial_x \text{Tr}(j_+^r)$$

(4.4)

This is certainly non-zero in general, and so for $su(\ell)$ the charges $\int J_m dx$ do not commute. Notice that the non-vanishing bracket is nevertheless a conserved quantity that we recognize, namely a differential polynomial in the currents $J_m$.

### 4.2 Commuting charges for the $su(\ell)$ model

It is natural to ask whether we can find more complicated functions of the quantities $J_m$ which will yield commuting charges for the $su(\ell)$ theory. To investigate this we need to know the exact form of their Poisson brackets. It is convenient to set $\kappa = 1/2$, which we assume henceforth. From (2.16), and making appropriate use of the completeness condition as above, we find:

$$\{J_m(x), J_n(y)\} = -mn J_{m+n-2}(x) \delta'(x-y) + \frac{mn}{\ell} J_{m-1}(x) J_{n-1}(x) \delta'(x-y) - \frac{mn(n-1)}{n+m-2} J'_{m+n-2}(x) \delta(x-y) + \frac{mn}{\ell} J_{m-1}(x) J'_{n-1}(x) \delta(x-y)$$

(4.5)

We can use these results to search systematically for conserved currents $K_{s+1}(J_m)$ of homogeneous spin which will give commuting charges. After some laborious calculations we find the following expressions for the first few values of the spin:

$$\begin{align*}
K_2 &= J_2 \\
K_3 &= J_3 \\
K_4 &= J_4 - \frac{3}{2\ell} J_2^2 \\
K_5 &= J_5 - \frac{10}{3\ell} J_3 J_2 \\
K_6 &= J_6 - \frac{5}{3\ell} J_3^2 - \frac{15}{4\ell^2} J_4 J_2 + \frac{25}{8\ell^2} J_2^3
\end{align*}$$

(4.6)

The same argument also applies to the case $g = gl(\ell)$ [21], although this is perhaps of less direct physical interest since the algebra is non-compact.
These are the unique combinations (up to overall constants) for which the corresponding charges commute.

We can extrapolate from these examples to a general formula. To construct a charge of spin $s$ we must define a current of spin $s+1$. In the Lie algebra $su(\ell)$ there is (up to an overall constant) a unique polynomial in the currents $J_2, J_3, \ldots, J_\ell, J_{\ell+1}$ which is homogeneous of spin $\ell+1$ and which vanishes; we write this

$$A_{\ell+1}(J_2, J_3, \ldots, J_\ell, J_{\ell+1}) = 0 \quad \text{in} \quad su(\ell). \quad (4.7)$$

From this we define a current of spin $s+1$ for $g = su(\ell)$, by the formula

$$K_{s+1}(J_m) = A_{s+1}(s\alpha J_m) \quad \text{where} \quad \alpha = \frac{1}{\hbar} = \frac{1}{\ell}. \quad (4.8)$$

It is easily checked that (4.8) reproduces the first five examples listed in (4.6) for any value of $\ell$.

We will prove below that the charges $\int K_{s+1} dx$ always commute. Another important fact about the formula (4.8) is that when $s = \ell$ the current vanishes, by construction. In fact this also happens whenever $s$ is a multiple of $\ell$, which we shall also prove below. Thus, we claim that the formula defines a series of currents whose charges can have spin $s$ equal to any integer which is non-zero mod $h = \ell$. To prove these claims, we need to develop some technology.

Following [17] we consider the generating functions $A(x, \lambda)$ and $F(x, \lambda)$ defined by

$$A(x, \lambda) = \exp F(x, \lambda) = \det(1 - \lambda j_+(x)) \quad (4.9)$$

which implies

$$F(x, \lambda) = \text{Tr} \log(1 - \lambda j_+(x)) = -\sum_{r=2}^{\infty} \frac{\lambda^r}{r} J_r(x). \quad (4.10)$$

Observe that $A(x, \lambda)$ is a polynomial in $\lambda$ of degree $\ell$, with the coefficient of the highest power being $(-1)^{\ell} \det(j_+)$. On substituting the series expansion for $F(x, \lambda)$ into (4.9), we obtain non-trivial identities satisfied by the $J_m$ as the coefficients of $\lambda^r$ must vanish for $r > \ell$ (for details, see e.g. [20]). In particular, the polynomial $A_{\ell+1}$ introduced above is the coefficient of $\lambda^{\ell+1}$. Our definition (4.8) can now be re-written

$$K_{s+1} = A(x, \lambda)^{s/h} \bigg|_{\lambda^{s+1}} = \exp \frac{s}{h} F(x, \lambda) \bigg|_{\lambda^{s+1}} = \exp \left( -\frac{s}{h} \sum_{r=2}^{\infty} \frac{\lambda^r}{r} J_r \right) \bigg|_{\lambda^{s+1}} \quad (4.11)$$
If \( s = ph \) with \( p \) an integer, then \( A(x, \lambda)^{s/h} \) is a a polynomial of degree \( ph = s \). The current \( K_{s+1} \) then vanishes. If \( s/h \) is not an integer, however, then \( A(x, \lambda)^{s/h} \) will be a power series in \( \lambda \) with infinitely many terms, each with a non-vanishing coefficient in general. This shows that the conserved currents and their charges exist precisely when the spin \( s \) is non-zero mod \( h = \ell \).

Finally we are in a position to prove that the charges we have defined commute. This is done by first calculating the Poisson brackets for the generating functions and then extracting the desired charges as the coefficients of particular terms:

\[
\{q_s, q_r\} = \int dx \int dy \{A(x, \mu)^{s/h}, A(y, \nu)^{r/h}\} \bigg|_{\mu^{s+1} \nu^{r+1}}. \tag{4.12}
\]

In the equations that follow, we will suppress the arguments of fields only when there is no possible ambiguity.

From (4.3) it can be shown (after some effort) that

\[
\{F(x, \mu), F(y, \nu)\} = \mu^2 \nu^2 \left[ \frac{1}{\mu - \nu} (\partial _\mu - \partial _\nu) + \frac{1}{h} \partial _\nu \right] A(x, \mu) A(x, \nu) \delta'(x-y) \tag{4.13}
\]

and from this it follows that\textsuperscript{[6]}

\[
\{A(x, \mu), A(y, \nu)\} = \mu^2 \nu^2 \left[ \frac{1}{\mu - \nu} (\partial _\mu - \partial _\nu) + \frac{1}{h} \partial _\nu \right] A(x, \mu) A(x, \nu) \delta'(x-y) \tag{4.14}
\]

It can be checked that this is antisymmetric under exchanging \( x \leftrightarrow y \) and \( \mu \leftrightarrow \nu \). Next we compute

\[
\frac{1}{pq \mu^2 \nu^2} \{A(x, \mu)^p, A(y, \nu)^q\} = \frac{1}{\mu - \nu} \left[ \frac{1}{p} (\partial _\mu - \partial _q) + \frac{1}{h} \frac{\partial _\nu}{pq} \right] A(x, \mu)^p A(x, \nu)^q \delta'(x-y) \tag{4.15}
\]

\[
+ \left[ \frac{1}{\mu - \nu} \left( \frac{1}{p} \frac{\partial _\mu}{p} - \frac{\partial _q}{q} \right) + \frac{1}{h} \frac{\partial _\nu}{pq} \right] A(x, \mu)^p A(x, \nu)^q \delta'(x-y) \tag{4.16}
\]

\[
+ \frac{1}{(\mu - \nu)^2} \left[ \frac{1}{p} (A(x, \mu)^p)^' A(x, \nu)^q - \frac{1}{q} A(x, \mu)^p (A(x, \nu)^q)' \right] \delta(x-y) \tag{4.17}
\]

\textsuperscript{[6]}The corresponding formula in [17], eqn. (4.21), seems to contain a misprint; it is not antisymmetric under the interchange \( x \leftrightarrow y \) and \( \mu \leftrightarrow \nu \).
and this implies

\[
\int dx \int dy \left\{ A(x, \mu)^p, A(y, \nu)^q \right\} = pq \mu^2 \nu^2 \int dx \left[ \left\{ \frac{\partial_{\mu}}{p} - \frac{\partial_{\nu}}{q} \right\} \frac{1}{\mu - \nu} + \frac{1}{h pq} \right] A(\mu)^p (A(\nu)^q)' \]

(4.15)

To find the brackets of charges of spins \(s\) and \(r\) we must extract the terms of degree \(s+1\) in \(\mu\) and \(r+1\) in \(\nu\) from this expression. Provided one is restricting to just these powers, this means that in the formula above we can replace \(\mu \partial_{\mu} \rightarrow s\) and \(\nu \partial_{\nu} \rightarrow r\). The result is

\[
\{ q_s, q_r \} = \int dx \int dy \left\{ A(x, \mu)^p, A(y, \nu)^q \right\} \bigg|_{\mu^{s+1} \nu^{r+1}} = pq \mu \nu \int dx A(\mu)^p (A(\nu)^q)' \left[ \left\{ \frac{s}{p} \nu - \frac{r}{q} \mu \right\} \frac{1}{\mu - \nu} + \frac{1}{h pq} \right] \bigg|_{\mu^{s+1} \nu^{r+1}}
\]

The integrand indeed vanishes when \(p = s/h\) and \(q = r/h\), implying that the charges commute as claimed.

### 4.3 More commuting charges for so(\(\ell\)) and sp(\(\ell\))

A similar approach allows us to construct more general sets of commuting charges for PCMs based on other classical Lie groups. For the orthogonal and symplectic algebras the Poisson brackets of the currents \(J_m\) are somewhat simpler (we again set \(\kappa = 1/2\)):

\[
\{ J_m(x), J_n(y) \} = -mn J_{m+n-2}(x) \delta'(x-y) - \frac{mn(n-1)}{m+n-2} J'_{m+n-2}(x) \delta(x-y) \tag{4.16}
\]

Using these we can again search systematically for polynomials \(K_{s+1}(J_m)\) which produce commuting charges, at least for some low-lying values of the spin. This reveals a family of currents similar to (4.6), except this time there is a single free parameter \(\alpha\) which is allowed to take an arbitrary value. The first few examples are:

\[
\begin{align*}
K_2 &= J_2 \\
K_4 &= J_4 - \frac{1}{2}(3\alpha) J_2^2 \\
K_6 &= J_6 - \frac{3}{4}(5\alpha) J_4 J_2 + \frac{1}{8}(5\alpha)^2 J_2^3 \\
K_8 &= J_8 - \frac{2}{3}(7\alpha) J_6 J_2 - \frac{1}{4}(7\alpha) J_4^2 + \frac{1}{4}(7\alpha)^2 J_4 J_2^2 - \frac{1}{48}(7\alpha)^3 J_2^4 
\end{align*}
\tag{4.17}
\]

The polynomials appearing above actually coincide with those of the same degree in (4.6) if one replaces \(\alpha \rightarrow 1/\ell\), and if one also takes into account the fact that \(J_m\) is non-zero for
the orthogonal and symplectic algebras only if $m$ is even. This immediately suggests the analogous general definition

$$K_{s+1}(\mathcal{J}_m) = A_{s+1}(s\alpha \mathcal{J}_m) \quad (4.18)$$

Once again, it can be shown that the resulting charges $\int K_{s+1} \, dx$ always commute, this time for any value of the parameter $\alpha$. Notice that this new one-parameter family of currents interpolates the two simplest families we found previously for the orthogonal and symplectic algebras. When $\alpha \to 0$ we have $K_{2m} \to J_{2m}$ and in the limit $\alpha \to \infty$ we have (with a suitable rescaling) $K_{2m} \to (\mathcal{J}_2)^m$.

To prove that these new currents give commuting charges we again use generating functions. Since $\mathcal{J}_m$ is now non-zero only for $m$ even, it is convenient to introduce two modified generating functions

$$B(x, \lambda) = \exp G(x, \lambda) \quad (4.19)$$

where

$$B(x, \lambda) = A(x, \sqrt{\lambda}) = \det(1 - \sqrt{\lambda} j_+(x)) \quad (4.20)$$

and

$$G(x, \lambda) = F(x, \sqrt{\lambda}) = \text{Tr} \log(1 - \sqrt{\lambda} j_+(x)) = -\sum_{r=1}^\infty \frac{\lambda^r}{r} J_{2r}(x) \quad (4.21)$$

and to express the Poisson brackets in terms of these. The general definition of the currents for the orthogonal and symplectic algebras can then be written

$$K_{s+1} = B(x, \sqrt{\lambda})^{as} \mid_{\lambda^{(s+1)/2}} \quad (4.22)$$

and we wish to show that

$$\{q_s, q_r\} = \int dx \int dy \{B(x, \mu)^{as}, B(y, \nu)^{ar}\} \mid_{\mu^{(s+1)/2} \nu^{(r+1)/2}} \quad (4.23)$$

vanishes.

From (4.10) we find

$$\{G(x, \mu), G(y, \nu)\} = \frac{4\mu \nu}{\mu - \nu} \left[ \mu \partial_\mu G(x, \mu) - \nu \partial_\nu G(x, \nu) \right] \delta'(x-y) \quad (4.24)$$

$$+ \frac{4\mu \nu}{\mu - \nu} \left[ \frac{1}{2} \frac{\mu + \nu}{\mu - \nu} (G(x, \mu)' - G(x, \nu)') \right] \frac{1}{2} [\mu - \nu] \delta(x-y)$$
which implies

$$\{B(x, \mu), B(y, \nu)\} = \frac{4\mu\nu}{\mu-\nu} \left[ (\mu\partial_\mu - \nu\partial_\nu)B(x, \mu)B(x, \nu) \delta'(x-y) \right. \\
+ (\mu\partial_\mu - \nu\partial_\nu)B(x, \mu)B(x, \nu)' \delta(x-y) \\
+ \left. \frac{\mu+\nu}{2(\mu-\nu)} (B(x, \mu)'B(x, \nu) - B(x, \mu)B(x, \nu)') \delta(x-y) \right]$$

Proceeding as before, we then calculate

$$\frac{1}{4pq\mu\nu} \{B(x, \mu)^p, B(y, \nu)^q\} = \frac{1}{\mu-\nu} \left[ \frac{\mu\partial_\mu}{p} - \frac{\nu\partial_\nu}{q} \right] B(x, \mu)^pB(x, \nu)^q \delta'(x-y) \\
+ \frac{1}{\mu-\nu} \left[ \frac{\mu\partial_\mu}{p} - \frac{\nu\partial_\nu}{q} \right] B(x, \mu)^p(B(x, \nu)^q)' \delta(x-y) \\
+ \frac{\mu+\nu}{2(\mu-\nu)^2} \left[ \frac{1}{p} (B(x, \mu)^p)'B(x, \nu)^q - \frac{1}{q} B(x, \mu)^p(B(x, \nu)^q)' \right] \delta(x-y)$$

and hence

$$\int dx \int dy \{B(x, \mu)^p, B(y, \nu)^q\} = 2pq\mu\nu \int dx \left[ \frac{2\mu\partial_\mu+1}{p} - \frac{2\nu\partial_\nu+1}{q} \right] \frac{1}{\mu-\nu} B(\mu)^p(B(\nu)^q)'$$

(4.26)

When we extract the coefficients of $\mu^{(s+1)/2}$ and $\nu^{(r+1)/2}$ we can replace $\mu\partial_\mu \rightarrow (s-1)/2$ and $\nu\partial_\nu \rightarrow (r-1)/2$ in the expression above. The integrand is then proportional to

$$pq \frac{\mu\nu}{\mu-\nu} \left[ \frac{s}{p} - \frac{r}{q} \right] B(\mu)^p(B(\nu)^q)'$$

(4.27)

This clearly vanishes if $p = \alpha s$ and $q = \alpha r$ for any $\alpha$, as claimed.

### 4.4 The Pfaffian charge and its generalizations

We have shown that any PCM based on a classical algebra has infinitely many commuting local charges constructed from combinations of invariant tensors of type (3.13). Moreover, these charges come in sequences, each associated with an exponent of the algebra, or a primitive invariant tensor, and with the spins in each sequence equal to this exponent modulo the Coxeter number, $h$. But there is one primitive invariant tensor which is not of the type (3.13) and which has therefore been absent from our discussion so far—this is the Pfaffian (3.15) with its associated current (3.16). It is natural to expect that our results can be extended so as to include this last invariant; we now show how this can be done.

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7 As with the $su(l)$ case, the corresponding formula in [7], eqn. (4.22), seems to contain a misprint; it is not antisymmetric under the interchange $x \leftrightarrow y$ and $\mu \leftrightarrow \nu$. 

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We first investigate the behaviour of the Pfaffian charge in $\text{so}(2\ell)$, with current $\mathcal{P}_\ell$, with respect to the other local charges which we have constructed for this algebra, with currents given in (4.22). The relevant Poisson brackets are:

\[
\{\mathcal{P}_\ell(x), J_{2m}(y)\} = -m \mathcal{P}_\ell(x) J_{2m-2}(x) \delta'(x-y) - m \frac{2m-1}{2m-2} \mathcal{P}_{\ell}(x) J_{2m-2}'(x) \delta(x-y) \quad (m \neq 2)
\]

\[
\{\mathcal{P}_\ell(x), J_2(y)\} = -2\ell \mathcal{P}_\ell(x) \delta'(x-y) - 2\mathcal{P}_\ell'(x) \delta(x-y).
\] (4.28)

A derivation of these is described in an appendix. By calculating the brackets directly, we find that for the first few examples listed in (4.17), the charges $\int K_m dx$ commute with the Pfaffian charge $\int \mathcal{P}_\ell dx$ provided we choose $\alpha = 1/h$, where $h = 2\ell - 2$ is the Coxeter number of $\text{so}(2\ell)$. By considering the bracket of the Pfaffian current with the generating function $B(x, \lambda)$, this can be extended to a proof of commutation of the Pfaffian charge with all the charges $\int K_m dx$ defined by (4.17). We omit the details, however, in favour of a more complete treatment from an alternative point of view, as follows.

We are concerned not just with the Pfaffian current but also with finding generalizations $\mathcal{P}_{\ell+ah}$ for all integers $a \geq 0$ (where the subscript denotes the spin, as usual). In other words, we expect it to be just the first member of a sequence of conserved quantities whose spins repeat modulo the Coxeter number. It is far from clear a priori how these generalizations should be defined, but the answer is, rather remarkably, already contained in the generating functions we have been considering. We have shown how to use the generating functions to define the currents $K_m$ as coefficients in an expansion in ascending powers of $\lambda$. It turns out that the Pfaffian and its generalizations naturally emerge from a similar expansion in descending powers of $\lambda$. The formula is then exactly the one already given in (4.22) but with $\alpha = 1/h$:

\[
\mathcal{P}_{\ell+ah} = B(x, \lambda)^{(a+1/2)}|_{\lambda^{(\ell-1)+\ell/2}} \quad a = 0, 1, 2, \ldots.
\] (4.29)

The choice $\alpha = 1/h$ is made so that we recover the Pfaffian current when $a = 0$.

To illustrate how this works it is best to consider the simplest possible example: $g = \text{so}(6)$ with $\ell = 3$ and $h = 4$ in the discussion above. Now

\[
B(x, \lambda) = \exp G(x, \lambda) = \det(1 - \sqrt{\lambda} j_+) = 1 + \lambda Q_2 + \lambda^2 Q_4 + \lambda^3 \mathcal{P}_3^2 \quad (4.30)
\]

where for convenience we have defined

\[
Q_2 = -\frac{1}{2} J_2 \quad \text{and} \quad Q_4 = -\frac{1}{4} (J_4 - \frac{1}{2} J_2^2).
\] (4.31)
To expand in descending powers of $\lambda$ we write

$$B(x, \lambda)^p = \lambda^{3p} P_3^{2p} \left( 1 + \frac{1}{\lambda} Q_1 P_3^{-2} + \frac{1}{\lambda^2} Q_2 P_3^{-2} + \frac{1}{\lambda^3} P_3^{-2} \right)^p$$

(4.32)

and then expand the bracket to any desired order using the binomial theorem. This gives the following results (up to overall constants) for the first few generalizations of the Pfaffian charge:

$$\begin{align*}
\mathcal{P}_7 &= \mathcal{P}_3 Q_4 \\
\mathcal{P}_{11} &= \mathcal{P}_3 Q_4^2 + \frac{4}{3} \mathcal{P}_3^3 Q_2 \\
\mathcal{P}_{15} &= \mathcal{P}_3 Q_4^3 + 4 \mathcal{P}_3^3 Q_4 Q_2 + \frac{8}{5} \mathcal{P}_3^5 \\
\mathcal{P}_{19} &= \mathcal{P}_3 Q_4^4 + 8 \mathcal{P}_3^3 Q_4^2 Q_2 + \frac{32}{5} \mathcal{P}_3^5 Q_4 + \frac{16}{5} \mathcal{P}_3^5 Q_2^2
\end{align*}$$

(4.33)

Since $so(6) = su(4)$ we can compare these results with the non-trivial odd-spin currents predicted by (4.11). Taking due account of normalizations arising from the inequivalence of the defining representations, we find exact agreement (details are given in an appendix).

Returning now to the general case, we re-iterate the important point that our definitions of the currents $\mathcal{K}_m$ and $\mathcal{P}_m$ are both given by the equation (4.22), but with the understanding that the expansions are carried out in ascending and descending powers respectively. Our proof that (4.22) led to commuting charges did not involve any explicit expansion in the parameter $\lambda$; the information about the power of $\lambda$ to be extracted was used only to replace the homogeneous differential operators $\mu \partial_\mu$ and $\nu \partial_\nu$ by appropriate integers. Consequently, our arguments apply equally well if one or both of the charges considered is of Pfaffian type. Thus they ensure that the sets $\int \mathcal{K}_m dx$ and $\int \mathcal{P}_m dx$ all commute with one another when $\alpha = 1/h$.

Finally, it is natural to ask whether new charges could be constructed for the other classical series $a_\ell$, $b_\ell$ and $c_\ell$ by considering expansions in descending powers in a similar way. If we expand $A(x, \lambda)^{s/h}$ in descending powers, it is not difficult to see that there is a term of order $\lambda^{s+1}$ only if the degree of $A(x, \lambda)$ as a polynomial in $\lambda$ exceeds the Coxeter number $h$. For the algebras $a_\ell$, $b_\ell$ and $c_\ell$ the degree of $A(x, \lambda)$ is less than or equal to the Coxeter number, so that no non-trivial charges can be defined in this way. The special feature of $d_\ell$ in this respect is that $A(x, \lambda)$ is of degree $2\ell$ which exceeds $h = 2\ell - 2$, so the construction of charges through an expansion in descending powers works only in this case.
5 Comments on the quantum PCM

It would be very interesting to know whether the classical charges we have discussed are
still conserved in the quantum theory, and whether or not the classical Poisson brackets
we have shown to vanish receive corrections when they are elevated to quantum commu-
tators. Unfortunately the subtle relationship between the classical PCM lagrangian and
its corresponding quantum theory makes these questions extremely difficult to tackle. In
this section we give a brief summary of some relevant facts about the quantum PCM and
explain how these fit together with our classical results.

The non-local charges are conserved in the quantum theory. Their behaviour has been
studied [8, 5] using a point-splitting regularization of $Q^{(1)a}$ and they were found to obey
a quantum version of the classical Yangian symmetry $Y_{L}(g) \times Y_{R}(g)$. An important novel
feature is that the Poisson bracket of the Lorentz boost generator with $Q^{(1)}$, which vanishes
classically, develops a term at order $\hbar^2$. This is essential to the construction of the quantum
S-matrices, which would otherwise be trivial. Point-splitting regularization would be a
very much more complicated procedure for the local charges, since they involve products
of many currents taken at a single point. Another approach which is likely to be rather
cumbrous would be to regularize the model on a lattice. To our knowledge, neither of
these approaches has been developed to study the quantum behaviour of the local charges.
In the absence of a tractable explicit quantization procedure, the only information comes
from less direct arguments.

5.1 Goldschmidt-Witten anomaly counting

In the method of Goldschmidt and Witten [14, 22] (see also [15]) one considers all possible
quantum anomaly terms which might spoil a classical conservation equation $\partial_{-}J = 0$.
The form of this equation reflects the classical conformal invariance of the theory. One
cannot expect it to survive unscathed in the quantum theory, where conformal invariance
is broken, but if the right-hand side gets a quantum correction which can be written as
a derivative, then we will still have a conserved quantity, albeit of a modified form. The
argument is reminiscent of Zamolodchikov’s approach [23] to perturbed conformal field

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8 Although the calculations have not been carried out, it would be surprising if the vanishing commu-
tators between the non-local and local charges received quantum modifications, since it is difficult to see
how the known S-matrices could be consistent if the local charges did not commute with the Yangian.
theory, though it pre-dates it.

To be more precise, suppose we have linearly-independent conservation equations of the form $\partial_- J_i = 0$ with $i = 1, \ldots, n$ which have a certain prescribed behaviour under all global symmetries of the theory. The only quantum modifications which can appear on the right-hand side are operators with the same mass dimension and the same behaviour under continuous and discrete symmetries. Let $A_i$ with $i = 1, \ldots, p$, be a linearly-independent set of such operators. We can also enumerate the linearly-independent total-derivative terms $B_i$ with $i = 1, \ldots, q$, which again have the same behaviour under all symmetries. Since each of the $B$s is expressible as a combination of $A$s, we have $q \leq p$. Now if $n - p + q > 0$, then there are at least this many combinations of the classical conservation equations which survive in the quantum theory, because this is the number of linearly-independent combinations for which the right-hand side is guaranteed to be a spacetime divergence.

Goldschmidt and Witten wrote down lists of $A$ and $B$s for conserved quantities in the PCMs as functions of the field $g$. We have found it much more convenient to use the Lie algebra-valued currents $j_\mu$, particularly in settling the all-important question of which $A$s and $B$s are independent. We shall use only the $L$ currents for definiteness, dropping the label $L$ henceforth. It is important to keep careful track of the behaviour of these currents and their derivatives under the discrete symmetries of the PCM. From (2.14) we see that under $G$-parity $\pi: j_+ \mapsto -g^{-1} j_+ g$ and $j_{++} \equiv \partial_+ j_+ \mapsto -g^{-1} j_{++} g$, but the situation is more complicated for higher derivatives. To overcome this we introduce quantities

$$j_{+++} \equiv \partial_+ j_{++} - \frac{1}{2\kappa} [j_+, j_{++}], \quad j_{++++} \equiv \partial_+ j_{+++} - \frac{1}{2\kappa} [j_+, j_{+++}], \quad \ldots \quad (5.1)$$

which can easily be shown to have the following simple behaviour under all discrete symmetries

$$\begin{align*}
\pi & : \quad j_{++...+} \mapsto -g^{-1} j_{++...+} g \\
\gamma & : \quad j_{++...+} \mapsto j_{++...+}^* = -j_{++...+}^T \\
\sigma & : \quad j_{++...+} \mapsto M j_{++...+} M^{-1} \quad \text{g = so}(2\ell). \quad (5.2)
\end{align*}$$

Note that for $SO(2\ell)$ all the currents are even under $\sigma$ except for the spin-$\ell$ Pfaffian current, which is odd.

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9 In deciding which $A$s are independent, we are free to use the classical equations of motion, because any quantum modifications appearing in the Heisenberg equations will correspond to operators with the correct dimensions and invariance properties to ensure that they will already occur in our list. Similarly, in counting the $B$s, as explained in [14].
The first example of a conserved current is \( J_2 = \text{Tr}(j^2_+) \), the energy-momentum tensor, which we certainly expect to survive quantization. Indeed, there is only one possible anomaly \( A_1 = -\text{Tr}(j_-j_{++}) \), only one derivative \( B_1 = \partial_+ \text{Tr}(j_-j_+) \), and in fact \( A_1 = B_1 \). This modification of the original conservation law reflects the non-vanishing of the trace of the energy-momentum tensor quantum mechanically, corresponding to the breaking of conformal symmetry. The next example is \( J_3 = \text{Tr}(j_3^2) \), which is non-trivial only for \( SU(\ell) \). This current is odd under both \( \pi \) and \( \gamma \). Here again there is one anomaly, \( A_1 = \text{Tr}(j_+ \{j_-,j_+\}) \), and one derivative \( B_1 = \partial_+ \text{Tr}(j_-j^2_+) \), with \( A_1 = B_1 \); the conservation again survives quantization.

The case of currents with spin 4 is the most interesting. There are two classical conserved currents, \( J_4 = \text{Tr}(j_4^2) \), and \( J_2^2 = (\text{Tr}(j_2^2))^2 \). These are even under each of the discrete symmetries. We find

\[
\begin{align*}
A_1 &= \text{Tr}(j_-j_{++}j_{++}) \\
B_1 &= \partial_+ \text{Tr}(j_-j_{++}) \\
A_2 &= \text{Tr}(j_-j_+)\text{Tr}(j_+j_{++}) \\
B_2 &= \partial_+ \left( \text{Tr}(j_-j_+)\text{Tr}(j^2_+) \right) \\
A_3 &= \text{Tr}(j_-j_{++})\text{Tr}(j^2_+) \\
B_3 &= \partial_+ \text{Tr}(j_-j^3_+) \\
A_4 &= \text{Tr}(j_+^2\{j_-,j_{++}\}) \\
B_4 &= \partial_- \text{Tr}(j^2_+) \\
A_5 &= \text{Tr}(j_-j_+j_{++}j_{++})
\end{align*}
\]

(Note that another apparent possibility among the \( B \)s, \( \partial_-\text{Tr}(j_+j_{++}j_+) \), is not allowed; it is not independent of \( B_3 \).) All other possibilities are disallowed: for \( SU(\ell) \) these terms are odd under \( \gamma \), while for other groups the traces vanish. Thus in all cases we have \( p = 5 \), \( q = 4 \) but the number of classical currents is \( n = 2 \). We conclude that there is at least one linear combination of \( \text{Tr}(j_4^2) \) and \( (\text{Tr}(j_2^2))^2 \) which will survive quantization.\(^{10}\)

There is only one other instance in which the counting arguments are known to ensure a quantum conservation law. It was shown in \(^{14}\) that conservation of the Pfaffian current in \( SO(2\ell) \) always generalizes to the quantum theory; we shall not reproduce the details here. For all other currents which we have investigated the Goldschmidt-Witten method

\(^{10}\) We have reached the same conclusions as \(^{14}\) regarding the existence of spin-3 and spin-4 currents. In comparing our lists of anomalies and derivatives with theirs, however, we should point out some discrepancies. There seem to be misprints and/or errors in eqns. (19) and (23) of \(^{14}\): the terms \( A_4 \) in (19) and \( A_2 \) in (23) do not have the correct behaviour under discrete symmetries. Furthermore, the terms \( B_1 \) and \( B_2 \) in eqn. (24) of \(^{14}\) are not independent, since they can be related using the equations of motion. Any obvious modification of the term \( A_2 \) in eqn. (23) to give it the correct symmetry can similarly be related to \( A_1 \), confirming our counting above of one anomaly and one derivative for the spin-3 current, rather than two of each, as claimed in \(^{14}\). This underscores our opinion that it is clearest to work with quantities valued in the Lie algebra – i.e. currents \( j_\mu \) rather than the field \( g \).
is inconclusive. We should emphasize however that these counting criteria, although sometimes sufficient to show quantum conservation, are never necessary. The fact that the counting fails should certainly not be interpreted as meaning that there is no quantum version of a given classical conservation law, but rather that these simple arguments are insufficient to settle the issue either way. Moreover, we have seen that the arguments guarantee the existence of at least one higher-spin conserved charge in each PCM, which is believed to be sufficient for integrability and factorization of the $S$-matrix \cite{30}.

Note also that the counting arguments, when successful, give no information about which combinations of the classical currents might survive. From our earlier work on the classical Poisson brackets of these charges, we have found for the $a_\ell$ and $d_\ell$ series that there were unique preferred sets of charges which all commute with one another. It is natural to anticipate that these are the combinations which generalize to the quantum theory. For the $b_\ell$ and $c_\ell$ series, however, such charges are not unique and we have no means of discriminating amongst the possibilities.

### 5.2 S-matrices, particle multiplets and discrete symmetries

In the exact S-matrix approach to the PCMs, the particle states are assumed to lie in representations $(V, \bar{V})$ of the global symmetry group $G_L \times G_R$ \cite{12, 10}. The representation $V$ of $\mathfrak{g}$ always contains an irreducible component $V_i$ which is one of the fundamental representations (associated to a node $i$ of the Dynkin diagram) of $\mathfrak{g}$. For the $a_\ell$ and $c_\ell$ series the representations are exactly $V = V_i$, but for the $b_\ell$ and $d_\ell$ algebras $V$ may be reducible in general. It is actually more natural to regard $V$ as acted on by the entire Yangian $Y(\mathfrak{g})$, with $(V, \bar{V})$ a representation of $Y_L \times Y_R$; then $V$ is precisely one of the fundamental representations of the Yangian $Y(\mathfrak{g})$ \cite{10, 25, 26}. Based on this assignment of representations, the full $S$-matrices have been determined for the $a_\ell$ and $c_\ell$ PCMs, while for the $b_\ell$ and $d_\ell$ models the scattering amongst the vector and spinor particles has been found (as well as some amplitudes involving second-rank tensor particles—in principle all other amplitudes are determined by the bootstrap procedure, but these have not been calculated).

The action of charge conjugation on $Y_L \times Y_R$ representations is $\gamma : (V, W) \mapsto (\bar{V}, \bar{W})$.

\footnote{For spin-5 currents in the $SU(\ell)$ PCM which are odd under $\pi$ and $\gamma$ we find $n = 2$, $p = 8$ and $q = 6$. For spin-6 currents which are even under $\pi$ and $\gamma$ we find $n = 5$, $p = 25$, $q = 18$, for $SU(\ell)$, and $n = 4$, $p = 23$, $q = 17$ for $SO(\ell)$ or $Sp(\ell)$.}
while the effect of $G$-parity is to exchange $G_L$ and $G_R$, so that $\pi : (V, W) \mapsto (W, V)$. Both discrete symmetries map a particle multiplet $(V, \tilde{V})$ to itself if $V \cong \tilde{V}$. But if $V \not\cong \tilde{V}$ then an implication of either symmetry is that mass-degenerate Yangian representations $(V, \tilde{V})$ and $(\tilde{V}, V)$ must appear together in the spectrum, as proposed in [10].

We have shown that the local charges commute with the Yangian classically. We will now assume that the same holds in the quantum theory, so that each local charge takes a constant value on a particle multiplet $(V, \tilde{V})$. We want to show how this is compatible with the assignment of representations and the effects of the discrete symmetries. Recall in particular that $G$-parity leaves invariant, or commutes with, a local charge whose spin $s$ is an odd integer, but reverses the value of, or anticommutes with, a local charge for which $s$ is an even integer.

The algebras $b_\ell$, $c_\ell$ and $d_{2\ell}$ have only real representations, $V \cong \tilde{V}$. A related fact is that the exponents are always odd integers, and so the associated local charges commute with $G$-parity. This is certainly consistent with the representation content $V \otimes V$, which is the simplest kind of multiplet whose states can be $G$-parity singlets.

For the algebras $a_\ell$ and $d_{2\ell+1}$, however, there is always one exponent which is an even integer, and hence at least one local charge $q$ which anti-commutes with $G$-parity. The only way to have a simultaneous eigenstate of $q$ and $\pi$ is for the eigenvalue of $q$ to vanish, and so on particle multiplets of the form $(V, V)$ we must have $q = 0$. Conversely, Yangian representations on which $q \neq 0$ must appear in $G$-parity pairs. The fact that the algebra has an even exponent is linked to the occurrence of complex representations, $V \not\cong \tilde{V}$, and the multiplets $(V, \tilde{V})$ and $(\tilde{V}, V)$ can indeed be organized into $G$-parity doublets provided they are eigenspaces of $q$ with opposite eigenvalues. Precisely the same phenomenon occurs in ATFTs, where it is the even-spin charges which enable states to be distinguished from their mass-degenerate conjugates [2].

To complete the discussion, we consider the special case $d_\ell = so(2\ell)$ and the additional discrete symmetry $\sigma$ which exchanges the spinor representations $S^\pm$. The representations $S^\pm$ are real for $\ell$ even and complex for $\ell$ odd. For $\ell$ even the Pfaffian charge commutes with $\pi$, but anti-commutes with $\sigma$. The particle multiplets are $(S^+, S^+)$ and $(S^-, S^-)$ which are eigenstates of $\pi$, but these representations are exchanged by $\sigma$, so its eigenstates lie in $(S^+, S^+) \oplus (S^-, S^-)$. For $\ell$ odd, the Pfaffian charge anti-commutes with both $\pi$ and $\sigma$. Now the particle multiplets are $(S^+, S^-)$ and $(S^-, S^+)$, and the eigenstates of both $\pi$ and $\sigma$ lie in $(S^+, S^-) \oplus (S^-, S^+)$. 

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5.3 Dorey’s Rule

The occurrence in affine Toda theories (ATFTs) of local conserved charges with spins equal to the exponents of the underlying Lie algebra modulo the Coxeter number leads to the elegant rule for particle fusings discovered by Dorey [4]. In ATFTs these particle fusings appear both in the tree-level three-point couplings (which can be found easily from the classical lagrangian) and also in the exact S-matrices. For PCMs the corresponding three-point fusings are defined solely by the S-matrices with their Yangian symmetry. Nevertheless, it has been proved [13], by exploiting the technology of Yangian representations in a highly non-trivial fashion, that Dorey’s rule applies to PCM particle fusings too.

We have constructed a set of local conserved charges in each classical PCM with exactly the same patterns of spins as those appearing in ATFTs. We are not at present able to prove that these all survive quantization (though some certainly do), but this fact would provide a very natural explanation of the validity of Dorey’s rule for PCM S-matrices, at least for the simply-laced algebras. For the non-simply-laced cases there are some additional subtleties, which we now briefly discuss.

The non-simply-laced ATFTs appear in dual pairs, typically involving an untwisted and a twisted algebra. Let us consider the example of the pair of algebras \( c_\ell^{(1)} \) and \( d_{\ell+1}^{(2)} \). The charges have spins equal to the exponents of \( c_\ell \), but their values depend on the coupling constant: in the weak-coupling limit they have the \( c_\ell^{(1)} \) tree-level values, while in the strong-coupling limit they are associated in a similar way with \( d_{\ell+1}^{(2)} \). Dorey’s construction does not allow for a coupling constant dependence in the values of the local charges, and gives the tree-level couplings either for \( c_\ell^{(1)} \), the set of which we shall call \( D(c_\ell) \), or (when suitably generalized [27]) for \( d_{\ell+1}^{(2)} \), which we shall call \( D(d_{\ell+1}^{(2)}) \). It is then the intersection of the two sets, \( D'(c_\ell) \equiv D(c_\ell) \cap D(d_{\ell+1}^{(2)}) \), which gives the correct fusings for the bootstrap principle applied to the quantum ATFT S-matrices.

The \( c_\ell \) PCM S-matrices [28] also have \( D'(c_\ell) \) fusings. But in the quantum PCM there is no coupling constant dependence: the classical coupling is replaced by the overall quantum mass-scale (dimensional transmutation). The \( c_\ell \) PCM mass ratios are actually those of the tree-level \( d_{\ell+1}^{(2)} \) ATFT, whilst the values of the other conserved charges must similarly be fixed, with no coupling-dependence. The outstanding issue is whether it is the values taken by these local charges which are sufficient to restrict the PCM fusings to \( D'(c_\ell) \) (rather than, say, \( D(d_{\ell+1}^{(2)}) \)), or whether some more subtle restriction is taking place.
We have carried out a rather thorough investigation of local charges in principal chiral models. We have shown that any local charge constructed from a symmetric invariant tensor commutes with the non-local Yangian charges. We have studied the algebra of these local charges amongst themselves and found that for each classical algebra there is a commuting family with spins equal to the exponents modulo the Coxeter number. These are defined by the universal formula for the currents

\begin{equation}
K_{s+1} = A(x, \lambda)^{s \alpha} \bigg|_{\lambda^{s+1}} \quad \text{where} \quad A(x, \lambda) = \det(1 - \lambda j_+)
\end{equation}

and \( \alpha = 1/\hbar \) (or more generally for the \( b_\ell \) and \( c_\ell \) series, \( \alpha \) can be arbitrary). This formula also defines a current associated with the Pfaffian invariant for the \( d_\ell \) algebras, as well as generalizations of this current, provided we consider an expansion in descending rather than ascending powers of \( \lambda \).

The existence of infinitely many conserved charges in involution is a pre-requisite for the classical integrability of any field theory. It is worth emphasizing that the brackets (2.16) do not apparently allow such a construction via the \( r \)-matrix formalism of classical inverse scattering.\(^\text{12}\) We have shown explicitly how the Poisson brackets (2.16) can lead to infinite sets of charges in involution, irrespective of the applicability of the classical \( r \)-matrix formalism.

Let us now turn to some broader questions surrounding the co-existence of local and non-local charges in integrable field theories, and how these can lead to common conclusions via very different chains of argument. It is worth emphasizing that this happens at the most fundamental level. The celebrated results which constrain the structure of the S-matrix in two \([30]\) (and four \([31]\)) dimensions make use of the local character of conserved quantities in an essential way. Such arguments cannot be applied to non-local charges, and yet these still lead to factorized S-matrices through their quasitriangular Hopf algebra (or quantum group) structure. In the PCMs, the local charges appear to offer a more natural explanation of why the fusings follow Dorey’s rule than is currently available from Yangian

\(^\text{12}\)The quantum current algebra is a different matter: in the quantization method developed by Faddeev and Reshetikhin \([29]\), for instance there is a manifestly Lorentz-covariant quantum current algebra derived from a spin model, with spin-scale \( S \). The classical limit \( \hbar \to 0 \) and the continuum limit \( S \to \infty \) do not commute. If taken in one order, the result is (2.10), but if the order of the limits is reversed, one finds a different set of classical Poisson brackets without non-ultralocal terms, which could lead to charges in involution via an \( r \)-matrix.
representations. It would be interesting to investigate other integrable models with such results in mind, and to determine whether features usually associated with the many exotic properties of non-local charges might also be explained by the existence of local charges, perhaps more simply.

It is also natural to seek some general, model-independent way of relating local and non-local conserved quantities. The Yangian $Y(g)$ has a trivial centre \[32\], so it seems that we cannot hope to construct local charges which commute as ‘Casimirs’ in $Y(g)$ by taking polynomials or series in the Yangian charges. Another avenue would be to consider the transfer matrix which generates the non-local charges, but it is far from clear to us how our local charges might emerge from this (see also e.g. \[33, 34\]). From yet another point of view, there are hints of a connection of the type we seek in recent work of Frenkel and Reshetikhin on deformed $W$-algebras (\[35\] and references therein).

Finally we mention that many of the features we have identified in this paper also appear in the supersymmetric principal chiral model, which we shall deal with in a forthcoming paper (some preliminary results are reported in \[36\]).

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7 Appendix: An alternative derivation of Poisson brackets

The PCM can be regarded as a special case of a general $\sigma$-model with lagrangian

$$\mathcal{L} = \frac{1}{2} g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j$$

where $\phi^i$ are coordinates on some target manifold with metric $g_{ij}(\phi)$. The momenta conjugate to the fields $\phi^i$ are

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^i} = g_{ij}(\phi) \dot{\phi}^j$$
with the standard non-vanishing equal-time PBs
\[ \{ \phi^i(x), \pi_j(y) \} = \delta^i_j \delta(x - y). \]

We may now consider a current of the form
\[ J_\mu^a = E_i^a \partial_\mu \phi^i \]
where \( E_i^a(\phi) \) are vielbeins on the target manifold satisfying
\[ E_i^a E_j^a = g_{ij} \]
(Whether these currents are actually conserved or not is irrelevant for the arguments here.)

In terms of the canonical coordinates \( \phi^i \) and \( \pi_i \) we have
\[ J_0^a = E_i^a g^{ij} \pi_j \quad J_1^a = E_i^a \phi'^i. \]

The PB algebra of these currents can now be calculated routinely, although the general result requires some effort and is not particularly illuminating.

Important simplification occurs for the special case of a group manifold, with currents defined by the vielbeins
\[ E_i^R = \text{Tr}(t^a g^{-1} \partial_i g), \quad E_i^L = -\text{Tr}(t^a \partial_i g g^{-1}) \]
where
\[ E_i^R = -g^{-1} \partial_i g, \quad E_i^L = \partial_i g g^{-1} \]
are the left-invariant and right-invariant forms on the group respectively (so our labels \( L \) and \( R \) signify the symmetries under which the vielbeins transform). To simplify the current algebra calculations it is necessary to use only the properties
\[ \partial_{[i} E_{j]} = E_{[i} E_{j]} \]
(the Maurer-Cartan relations) which are easily verified from the definitions above. The Poisson brackets (2.16) then follow.

8 Appendix: Computing Poisson brackets in \( SO(2\ell) \) PCM

The following method applies to any of the local conserved charges in the \( SO(2\ell) \) PCM, including the Pfaffian and its generalizations.
At each point in space, \( j_+ (x) \) is a real \( 2\ell \times 2\ell \) antisymmetric matrix. Let its skew eigenvalues be \( \lambda_i \), so that there exists an orthogonal \( U \) and block diagonal \( D \) with

\[
Uj_+U^{-1} = D = \sum_{i=1}^{\ell} \lambda_i M_i = \text{diag} \left( \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} \ldots \begin{pmatrix} 0 & \lambda_\ell \\ -\lambda_\ell & 0 \end{pmatrix} \right) .
\]

The block diagonal matrices \( \{ M_i \} \) form a basis for the Cartan subalgebra \( so(2) \oplus \ldots \oplus so(2) \) of \( so(2\ell) \), and are clearly normalized so that \( \text{Tr}(M_iM_j) = -2\delta_{ij} \).

Any function of the current \( j_+ (x) \) which is invariant under the adjoint action of the Lie algebra must depend only on the eigenvalues \( \lambda_i \). For example, we have

\[
\mathcal{J}_{2n} = \text{Tr}(j_+^{2n}) = 2(-1)^n \sum_{i=1}^{\ell} \lambda_i^{2n} \quad \text{and} \quad \mathcal{P}_\ell = \lambda_1 \ldots \lambda_\ell .
\]

To compute the Poisson bracket of two invariants \( P(j_+ (x)) \) and \( Q(j_+ (y)) \) it therefore suffices to know the Poisson brackets between the eigenvalues. There are potential complications from the fact that the orthogonal matrix \( U \) needed to diagonalize the current \( j_+ (x) \) is itself a complicated function of this current. However, this turns out to be irrelevant to the computation of the Poisson brackets, as we now show.

From (2.17) (with \( \kappa = 1/2 \)) and the arguments of section 4, we know that the only contribution to the Poisson bracket is

\[
\{ P(j_+ (x)), Q(j_+ (y)) \} = \frac{\partial P(x)}{\partial j_+^a (x)} \frac{\partial Q(y)}{\partial j_+^a (y)} \delta'(x - y) .
\]

Since \( P \) depends on \( j_+ \) only through its eigenvalues we can write

\[
\frac{\partial P(x)}{\partial j_+^a (x)} = \sum_i \frac{\partial P(x)}{\partial \lambda_i (x)} \frac{\partial \lambda_i (j_+^a (x))}{\partial j_+^a (x)}
\]

(and similarly for \( Q \)) and from the definition of \( \lambda_i \) given above we find

\[
\frac{\partial \lambda_i (j_+^a (x))}{\partial j_+^a (x)} = -\frac{1}{2} \text{Tr} \left( M_i U t_a U^{-1} \right) + \frac{1}{2} \text{Tr} \left( M_i \left[ D, \frac{\partial U}{\partial j_+^a} U^{-1} \right] \right) .
\]

But the second term involving the commutator vanishes by antisymmetry of the matrix \( \partial j_+ UU^{-1} \) (this belongs to the Lie algebra \( so(2\ell) \)) in conjunction with the block-diagonal structure of \( D \). Using the completeness condition for the generators \( \{ t^a \} \) of \( so(2\ell) \), we obtain the simple result

\[
\{ P(j_+ (x)), Q(j_+ (y)) \} = \frac{1}{2} \sum_i \frac{\partial P(x)}{\partial \lambda_i (x)} \frac{\partial Q(y)}{\partial \lambda_i (y)} \delta'(x - y).
\]

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This allows an easy derivation of equations such as (4.28). In fact, all the Poisson bracket calculations for the $so(2\ell)$ PCM can be performed in this way if the formulas involving the generating functions are re-expressed in terms of eigenvalues. This provides an important independent check on our calculations using generating functions.

9 Appendix: Comparing currents in $SU(4)$ and $SO(6)$ PCMs

For $g = su(4)$ the equation (4.9) becomes

$$A(x, \lambda) = \exp(1 - \frac{1}{2} \lambda^2 J_2 - \frac{1}{3} \lambda^3 J_3 - \frac{1}{4} \lambda^4 J_4) = 1 + \lambda^2 \tilde{Q}_2 + \lambda^3 \tilde{Q}_3 + \lambda^4 (\tilde{Q}_4 + \frac{1}{4} \tilde{Q}_2^2)$$

where it is convenient to introduce the quantities

$$\tilde{Q}_2 = -\frac{1}{2} J_2 , \quad \tilde{Q}_3 = -\frac{1}{3} J_3 , \quad \tilde{Q}_4 = -\frac{1}{4} (J_4 - \frac{1}{4} J_2^2) .$$

The definition (4.11) is in this case

$$\mathcal{K}_{s+1} = A(x, \lambda)^{s/4} \bigg|_{s+1}$$

from which we obtain non-trivial, odd-spin currents:

$$\mathcal{K}_3 = \tilde{Q}_3 , \quad \mathcal{K}_7 = \tilde{Q}_3 \tilde{Q}_4 , \quad \mathcal{K}_{11} = \tilde{Q}_3 \tilde{Q}_4^2 - \frac{1}{6} \tilde{Q}_3^3 \tilde{Q}_2 , \quad \mathcal{K}_{15} = \tilde{Q}_3 \tilde{Q}_4^3 - \frac{1}{2} \tilde{Q}_3^2 \tilde{Q}_4 \tilde{Q}_2 - \frac{1}{40} \tilde{Q}_3^5 , \quad \mathcal{K}_{19} = \tilde{Q}_3 \tilde{Q}_4^4 - \tilde{Q}_3^3 \tilde{Q}_4^2 \tilde{Q}_2 - \frac{1}{10} \tilde{Q}_3^5 \tilde{Q}_4 + \frac{1}{20} \tilde{Q}_3^5 \tilde{Q}_2^2$$

With a standard choice of normalization, the relationships between invariants in the four-dimensional and six-dimensional representations of $su(4) = so(6)$ are

$$\text{Tr}_4 X^2 = \frac{1}{2} \text{Tr}_6 X^2 , \quad \text{Tr}_4 X^3 = 3i \text{Pfaff}_6 (X) , \quad \text{Tr}_4 X^4 = -\frac{1}{4} \text{Tr}_6 X^4 + \frac{3}{16} (\text{Tr}_6 X^2)^2 ,$$

which imply

$$\tilde{Q}_2 = \frac{1}{2} Q_2 , \quad \tilde{Q}_3 = -i P_3 , \quad \tilde{Q}_4 = -\frac{1}{4} Q_4 .$$

On substituting into the expressions for the conserved currents written above we find agreement with (4.33) up to overall constants.
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