Statistical physics of independent component analysis

R. Urbanczik

Institut für theoretische Physik - Universität Würzburg, Am Hubland, D-97074 Würzburg, Germany

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Abstract. – Statistical physics is used to investigate independent component analysis with polynomial contrast functions. While the replica method fails, an adapted cavity approach yields valid results. The learning curves, obtained in a suitable thermodynamic limit, display a first order phase transition from poor to perfect generalization.

During the last decade, independent component analysis (ICA) has emerged as one of the most powerful unsupervised learning procedure for many signal processing tasks\cite{1,2}. It assumes that the observed, often high dimensional signal, is a linear mixture of independent source signals and aims to recover these sources just from observing the mixed up signal. Hence, ICA is sometimes also called blind signal deconvolution. An illustrative scenario is the cocktail party problem where, to understand any single speaker, we first need to identify her voice amidst the jumble of sounds reaching our ears.

The basic finding in ICA is that the distribution of the observed signal will be similar to a Gaussian, especially when many independent sources contribute to the linear mixture. The source signals, however, will often be highly structured, and non-Gaussian. ICA thus searches for a linear transformation of the observations which maximizes non-Gaussianity by evaluating a suitable contrast function. To detect this, the contrast function used must compute a higher than quadratic statistics of the transformed data.

In a principled way, ICA can be derived by considering the mutual information of the transformed data, which is a natural measure of statistical dependence. To avoid the problem of density estimation, which arises in a direct evaluation of the mutual information, one then uses expansions (Edgeworth, Gram-Charlier) around Gaussianity to approximate the mutual information\cite{3,4}. This leads to contrast functions which are related to the higher order cumulants of the transformed data.

This Letter provides a first analysis of ICA for polynomial contrast functions using the statistical physics of disordered systems. Surprisingly, the replica method, one of the most powerful tools in analyzing quenched disorder, fails since it cannot control the contributions to the contrast function in the large deviations regime. However, a physically valid analysis is obtained by adapting the cavity method, showing that the scale of the learning curve depends on the degree of the polynomial. Unusually, for a system with continuous couplings, the curve itself is a step function, jumping from poor to perfect generalization. But a badly generalizing
state is always metastable and it is remarkable that we can nevertheless find polynomial time algorithms which generalize well.

In formal terms, we assume that the observable signal $\xi$ can be written as $\xi = M\hat{\xi}$, where the source $\xi$ is an $N$-dimensional random variable with independent components and $M$ is the $N \times N$ mixing matrix. Learning is based on a training set $D$ of $P$ independent observations $\xi^\mu$ of the signal $\xi$, obtained for a fixed, if unknown, mixing matrix $M$. The deconvolution problem (finding $\hat{\xi}$) can be decomposed by first finding just one independent component, subtracting it from the mixture, and reapplying the procedure to the remaining $N - 1$ dimensional task. Hence, I shall just deal with finding the first component $\xi_1$ and assume that it is non-Gaussian whereas all other components of $\xi$ are Gaussian.

Normally, the first step in ICA is to whiten the data, so that it has zero mean and its covariance matrix is the identity. So, I shall further assume that the source components have zero mean and unit variance and that $M$ is orthogonal, $M^T M = 1$. In short, the ICA task now is to find, based on the training set $D$, a vector $J$ such that $J^T \xi = \pm \xi_1$. For this, one picks a suitable non-quadratic contrast function $g$, computes the empirical contrast

$$c_D(J) = P^{-1} \sum_{\mu=1}^P g(J^T M \hat{\xi}^\mu),$$

and chooses $J$ to maximize $c_D(J)$ under the constraint $|J| = 1$. To analyze this problem, one will first consider the Gibbs weight $\exp(\beta N c_D(J))$ at some finite inverse temperature $\beta$ and calculate the typical value of the logarithm of its partition function $Z_D = \int dJ \exp(\beta N c_D(J))$, where the integration is over the uniform density on the unit sphere in $\mathbb{R}^N$. Since, via a gauge, the partition function is independent of the mixing matrix $M$, we set $M = 1$ for the analysis.

I shall first consider the replica approach to this calculation and for brevity assume that the contrast function is $g(x) = x^3$. We are then immediately faced with the problem that the moments $\langle Z^n_D \rangle$ do not exist, indeed $Z_D$ does not even have a mean $\langle 1 \rangle$. A second issue arises since $c_D(J)$ is $O(N^{3/2}/P)$ for $J = \xi^\mu/|\xi^\mu|$. So, if we have just $P = \alpha N$ examples, $\ln Z_D$ is not an extensive quantity for large $N$.

To address the first problem, we introduce a cutoff $K_N > 0$, replacing $g(x) = x^3$ by $g_N(x) = \max\{x^3, K_N^3\}$ in Eq. (1). Since we want to ultimately recover the $g(x) = x^3$ case, we assume that $K_N$ diverges with increasing $N$. Nevertheless, due to the cutoff, the moments of $Z_D$ now exist for any finite $N$. Further, we assume that the training set has $P = \alpha L_N N$ and not just $\alpha N$ patterns. Then, if $L_N$ diverges sufficiently quickly w.r.t. $N$ and $K_N$, in $Z_D$ will be an extensive quantity. Finally, we should find that for the purpose of calculating $\ln Z_D$ for large $N$, choosing $K_N = \sqrt[3]{N}$ is equivalent to not cutting off at all. The reason for this quite simply is that for $N \to \infty$ the fields $J^T \xi^\mu$ are bounded by $\sqrt{N}$ for almost all training sets.

In this setting, standard arguments yield the exact finite $N$ result

$$\langle Z^n_D \rangle = \lambda_{N,n} \int dR dQ \det(Q - RR^T)^{\frac{N-n-1}{2}} \mathcal{G}_N(R, Q)^N$$

$$\mathcal{G}_N(R, Q) = \left\langle \prod_{\alpha=1}^n \exp \left( \frac{\beta \max\{R^a \xi_1 + X^a\}^3, K_N^3\}}{\alpha L_N} \right) \right\rangle_{\xi_1, X}$$

Here $R$ is an $n$-vector, $Q$ a symmetric $n \times n$ matrix with $Q^{aa} = 1$, and the domain of integration is such that the matrix $Q - RR^T$ is positive definite. The $X^a$ are zero mean.

\(^{(1)}\)In a sense, this problem already crops up for principal component analysis where $g(x) = x^2$. Then $\langle Z^n_D \rangle$ diverges, if $n$ or $\beta$ are large enough. So, using replicas, one is in effect computing a continuation from small $\beta$ and large $n$ to large $\beta$ and small $n$. 

\(\)
Gaussian with covariances \( \langle X^a X^b \rangle = Q^{ab} - R^a R^b \), and \( \lambda_{N,n} \) is obtained using that the moments equal 1 for \( \beta = 0 \). Now, given any sequence of cutoffs \( K_N \), we can certainly find \( L_N \) so that \( G_N(R,Q) \) stays finite for large \( N \). Then, we should be able to use Laplace’s method of the maximum point to find that in the large \( N \) limit
\[
\frac{1}{N} \ln \langle Z_D \rangle = \sup_{R,Q} \ln G_N(R,Q) + \frac{1}{2} \ln \det(Q - RR^T).
\] (2)

But at this point, at the latest, it is clear that something is amiss. The limiting value of the above RHS depends only on the relative scalings of \( K_N \) and \( L_N \) and not on the relationship of these scalings to the system size \( N \). So \( 2 \) implies that the scale of learning curve can be arbitrarily stretched by using cutoffs which diverge quickly with \( N \). This problem arises regardless of assumptions about replica symmetry.

We proceed anyway and, using the replica symmetric parameterization of \( 2 \), find for \( N \to \infty \)
\[
\frac{1}{N} \ln \langle Z_D \rangle = \inf_q G_r(q,R) + G_s(q,r)
\]
\[
G_r(q,R) = \alpha L_N \left\langle \ln \left( \exp \left( \frac{\beta}{\alpha L_N} g_N(\xi_1, \xi_2, y_1, y_2) \right) \right) \right\rangle_{y_1, y_2}
\]
\[
G_s(q,r) = \frac{1}{2} \frac{q - r^2}{1 - q} + \frac{1}{2} \ln(1 - q)
\] (3)

where \( y_0, y_1 \) are standard Gaussians, i.e. with zero mean, unit variance. The extremal \( r \) is just the typical value of the first component of a weight vector picked from the Gibbs density and measures to which extent the structure in the data is recognized. Using \( 3 \), we relate the scalings of \( K_N \) and \( L_N \). For \( L_N \gg K_N \) the energy term converges to \( G_r(q,R) = r^3 \langle \xi_1^3 \rangle \). This is the limit of many examples where \( r = 1 \) for all \( \alpha \). In contrast, for \( L_N \ll K_N \) there are too few examples and \( G_r(q,R) \) diverges.

So, the scale of the learning curve is given by setting \( L_N = K_N \). On this scale, we find that \( G_r(q,R) \) converges to \( r^3 \langle \xi_1^3 \rangle \) as in the limit of many examples if \( q \) exceeds a critical value \( q_c(\alpha, \beta) \), whereas \( G_r(q,R) \) diverges for \( q < q_c(\alpha, \beta) \). Solving the extremal problem for \( q \) by taking the limit \( q \to q_c(\alpha, \beta) \) from above, then taking the \( \beta \to \infty \) limit, we finally find the simple result for the ground state: \( c(\alpha) = \sup_q r^3 \langle \xi_1^3 \rangle \xi_1 + (1 - r^2)/\alpha \). Here \( c(\alpha) \) is the typical value of the highest achievable empirical contrast, \( \max_{|J|=1} c_D(J) \). The learning curve for \( r \) thus obtained, is a step function showing a first order phase transition at \( \alpha_c = 1/\langle \xi_1^3 \rangle \xi_1 \) from no learning \( (r = 0) \) to perfect learning \( (r = 1) \). But the \( r = 0 \) state is metastable for all values \( \alpha > \alpha_c \).

The replica theory predicts that for any divergent sequence of cutoffs \( K_N \), e.g. \( K_N = e^N \), we need \( P > \alpha_c N \) examples for good generalization when \( N \) is large. While this is ridiculous, I have argued above that choosing \( K_N = \sqrt{N} \) is, for \( N \to \infty \), equivalent to not cutting off at all. To compare the replica result for this choice of \( K_N \) to numerical simulations, let us consider actually finding a weight vector maximizing \( c_D(J) \). It turns out that a rather simple discrete dynamics can be used since \( g(x) = x^3 \). Starting with a random vector of unit length \( J^0 \), at the \( k \)-th time step we first compute the matrix \( A(J^k) = \sum_{\mu=1}^P \xi^\mu (J^k)^T \xi^\mu \xi^\mu^T \) and then choose \( J^{k+1} \) to maximize \( |J^T A(J^k) J| \) under the constraint \( |J| = 1 \). So, \( J^{k+1} \) is an eigenvector to the eigenvalue of largest magnitude of \( A(J^k) \). Standard results on quadratic forms imply that \( |J^{k+1} A(J^k) | \geq |J^T A(J^k) J^k| \), and the inequality is strict unless we are at a fixed
Fig. 1 – Prediction of $K_N = \sqrt{N}$ replica theory (bold line) compared to simulation results. The non-Gaussian source is $\xi_1 = (y^2 - 1)/\sqrt{2}$, where $y$ is a standard Gaussian. The empty symbols show the results for the algorithm finding local maxima of the empirical contrast. The full symbols, denoting results for the iterated version of the procedure described in the main text, show that the agreement with the replica theory improves quickly with increasing system size $N$ for this algorithm. The error bars estimate the standard deviation of the sample to sample fluctuations.

point. Hence, the iteration converges to a vector $J^\infty$ which is a local maximum or minimum of $c_D(J)$. In the latter case, we just flip the sign of $J^\infty$ to obtain a local maximum.

Simulation results for the procedure, compared to the $K_N = \sqrt{N}$ replica theory in Fig. 1, show that the performance of the algorithm is rather poor. This is in line with the theoretical findings, since these predict that $r = 0$ is metastable, and the algorithm is only finding a local maximum. Figure 1 also shows result for an iterated version of the algorithm. There the algorithm is rerun with $m = 0.1N$ different random initial conditions, and the weight vector maximizing $c_D(J)$ among the $m$ outcomes is chosen. These result are in good agreement with the $K_N = \sqrt{N}$ replica theory, indicating that beyond the phase transition the basin of attraction of the global maximum is quite large.

Even if the simulations indicate that the replica approach is saved by in the end plugging in the correct scaling of the cutoff $K_N$, the theoretical situation is highly unsatisfactory. I shall next show that a physically reasonable analysis can be provided by adapting the cavity method. This is much simplified if make some major changes to the notation. From now on the non-Gaussian source will be denoted by $\gamma$, whereas all of the $N$ components of $\xi$ are assumed independent standard Gaussian. Our primary goal is to calculate the typical value of $C_r = \max_{|J|=1} C_r(J)$ with

$$C_r(J) = \frac{1}{P} \sum_{\mu=1}^{P} g(r\gamma^\mu + J^T \xi^\mu)$$

where $J$ is an $N$-dimensional vector. So $C_r$ is the maximal value of the empirical contrast achievable on an $r$-shell. For generality, we shall now longer assume that $g(x)$ must be cubic but consider any super-quadratic function which does not diverge too quickly. In particular, for some $k > 0$, $\lim_{x \to \infty} g(x)/x^{2+k} = \psi$ should exist and be positive. Without loss of generality, we may then assume $\psi = 1$. 

We still have \( P = \alpha L_N N \) examples and consider the random variable \( J_D \) with the Gibbs density

\[
p_D(J) = \frac{1}{Z_D(\beta)} \frac{e^{-\frac{1}{2}J^2}}{(2\pi)^{\frac{N}{2}}} \prod_{\mu=1}^P e^{\frac{1}{2}g(\gamma^\mu)(J^\mu)}
\]

\[
g(\gamma^\mu, [J]^T\xi^\mu) = g(r\gamma^\mu + \sqrt{1-r^2}[J]^T\xi^\mu).
\]

Here \( [J] = J/|J| \) and \( Z_D(\beta) \) is given by the normalization \( \int dJ P_D(J) = 1 \). Note, that we are now using a factoring Gaussian prior on \( J \) and, to compensate for this, the normalized vector \( [J] \) is used to calculate the field in (3).

A key task in the cavity approach is obtain the field distribution by calculating the thermal average \( \langle \phi(\gamma^\mu, [J_D]^T\xi^\mu) \rangle_{J_0} \) for any function \( \phi \). One finds

\[
\langle \phi(\gamma^\mu, [J_D]^T\xi^\mu) \rangle_{J_0} = \frac{Z_{D/\mu}(\beta)}{Z_D(\beta)} \left| e^{\frac{1}{2}g(\gamma^\mu, [J_D/\mu]^T\xi^\mu)} \phi(\gamma^\mu, [J_D/\mu]^T\xi^\mu) \right|_{J_{D/\mu}}.
\]

where \( J_{D/\mu} \) is the random variable with the Gibbs density obtained when pattern \( \mu \) is removed from the system, i.e. omitting the \( \mu \)-th factor of the product in (3) and adjusting the partition function to \( Z_{D/\mu}(\beta) \). The variance of the cavity field \( [J_{D/\mu}]^T\xi^\mu \) is a self averaging quantity and it must then equal \( 1-q^2 \) for large \( N \), where \( q = |\langle [J_{D/\mu}]_{J_{D/\mu}} \rangle|^2 \). Normally, one would further argue that \( [J_{D/\mu}]^T\xi^\mu \) becomes Gaussian in the thermodynamic limit. But if we assume this, the \( J_{D/\mu} \) average in (4) diverges even when \( \phi \) is a simple bounded function. This highlights the fact that the cavity field is not Gaussian in the large deviations regime because \( [J_{D/\mu}]^T\xi^\mu \) cannot be larger than \( |\xi^\mu| \).

Hence, I rephrase the cavity argument as follows: For the purpose of calculating overlaps with a random vector such as \( \xi^\mu \), the not normalized \( J_{D/\mu} \) can for large \( N \) be treated as a Gaussian (with covariance matrix \( (1-q)1 \)). Then, the fluctuations of the cavity field obtained using the normalized \( [J_{D/\mu}] \),

\[
P_{N,q}(h) = \langle \delta \left( h - \left( [J_{D/\mu}]^T - \langle [J_{D/\mu}]^T \rangle_{J_{D/\mu}} \right) \xi^\mu \right) \rangle_{J_{D/\mu}}
\]

can be explicitly calculated. This yields the important fact that there are just two relevant scales for the cavity fluctuations. For large \( N \), \( P_{N,q}(h) \) converges to \( e^{-\frac{1}{2}h^2/(1-q)} / \sqrt{2\pi(1-q)} \) if \( h \ll 1/N \), but in the large deviations regime, for \( h = d\sqrt{N} \),

\[
\lim_{N \to \infty} N^{-1} \ln P_{N,d\sqrt{N}}(d\sqrt{N}) = -\frac{1}{2} \frac{q d^2}{1-q} + \frac{1}{2} \ln(1-d^2)
\]

if \( 0 \leq d \leq 1 \). Now, in terms of the functional

\[
\mathcal{L}^q_{y^\gamma,\gamma^\mu}(\phi) = \int_{-\sqrt{N}}^{\sqrt{N}} dh P_{N,q}(h) \phi(\gamma^\mu, \sqrt{q} y + h) e^{\frac{\gamma^\mu}{\sqrt{N}} g(\gamma^\mu, \sqrt{q} y + h)}
\]

the average in Eq. (3) can in the limit of large \( N \) be rewritten as \( \langle \phi(\gamma^\mu, [J_D]^T\xi^\mu) \rangle_{J_0} = \mathcal{L}^q_{y^\gamma,\gamma^\mu}(\phi)/\mathcal{L}^q_{y^\gamma,\gamma^\mu}(1) \) with \( y^\mu = q^{-\frac{1}{2}} [J_{D/\mu}]^T \xi^\mu \). So the quenched averages are

\[
\langle \phi(\gamma^\mu, [J_D]^T\xi^\mu) \rangle_{J_0} = \left( \mathcal{L}^q_{y^\gamma,\gamma^\mu}(\phi)/\mathcal{L}^q_{y^\gamma,\gamma^\mu}(1) \right)_{y^\gamma}
\]

\[
\langle \ln Z_D(\beta) \rangle = \langle \ln \mathcal{L}^q_{y^\gamma,\gamma^\mu}(1) \rangle_{y^\gamma}
\]
where $y$ is standard Gaussian. The last equation is obtained by setting $\phi = 1$ in (9).

We can now consider whether the large deviations regime contributes to the averages in (8) for a polynomially bounded $\phi$. Using that for large arguments $g(x) \sim x^{\frac{1}{\beta}}$ and referring to Eq. (7), we find that it will contribute if the maximum of

$$
u(d) = \beta d^{k+2} \frac{N \frac{1}{2}}{L_N} - \frac{1}{2} \ln(1-d^2)$$

is positive for large $N$. This won’t happen if $L_N \gg N^{\frac{1}{2}}$ and Eq. (8) then implies that $\langle \langle \phi(\gamma^{\mu}, [J_D]^T, \xi^{\mu}) \rangle \rangle_{D} = \langle \phi(\gamma, y) \rangle_{y, \gamma}$. The empirical mean equals the expectation value and so the learning curve is trivial. Henceforth, we focus on the relevant scale, setting $L_N = N^{\frac{1}{2}}$.

Our next task is to calculate the response when a new coupling $J_0$ is added to the system and each pattern $\xi^\mu$ is augmented by a new component $\xi^\mu_0$. We denote the augmented training set by $\tilde{D}$ and use (8) to define the partition function $Z_D(\beta)$ of the $N + 1$ dimensional system. Due to the $N$-dependence of the Gibbs weight $e^{-\frac{N}{\beta} \langle g(\gamma^\mu, [J_D]^T, \xi^\mu) \rangle}$, it is simplest to assume a slightly different temperature $\tilde{\beta}_N = \beta L_{N+1}/L_N$ in the augmented system. Then, when considering the ratio $Z_D(\tilde{\beta}_N)/Z_D(\beta)$, the two systems have the same Gibbs weight per pattern. Standard arguments (5) thus apply and yield that $\langle \ln Z_D(\tilde{\beta}_N)/Z_D(\beta) \rangle_D = G_s(q, 0)$ for large $N$. Here $G_s(q, 0)$ is the entropy term of the replica theory (Eq. (6)), but evaluated at $r = 0$ because we are calculating the partition function for each $r$-shell individually.

Having identified, via $L_N = N^{\frac{1}{2}}$, the scale of the learning curve, $N^{-1} \langle \ln Z_D(\beta) \rangle_D$ will converge to a finite quantity $z(\alpha, \beta)$ in the thermodynamic limit. We then have

$$\langle \ln Z_D(\tilde{\beta}_N)/Z_D(\beta) \rangle_D = z(\alpha, \beta) - \frac{\beta k + 2}{\partial \alpha} z(\alpha, \beta) + \frac{\beta k}{\partial \beta} z(\alpha, \beta).$$

The derivative of $z$ with respect to $\alpha$ is obtained from Eq. (9), and the thermal derivative is found from (8) using $\phi = g$.

Putting things together, we finally find for large $N$

$$z(\alpha, \beta) = \left\{ \frac{\beta k + 2}{2} N^{\frac{1}{2}} \ln L_{q, \gamma}^{y, \gamma}(1) - \frac{\beta k}{2} \frac{L_{q, \gamma}^{y, \gamma}(g)}{L_{q, \gamma}^{y, \gamma}(1)} \right\}_{y, \gamma} + G_s(q, 0),$$

where the value of $q$ still has to be determined.

For this, let us reconsider when the large deviations regime contributes to the value of $L_{q, \gamma}^{y, \gamma}(1)$. Going back to Eq. (10), with $L_N = N^{\frac{1}{2}}$, we see that as in the replica theory this is governed by a critical value $q_c(\beta)$ of $q$. For $q < q_c(\beta)$, $\max_q u(d)$ is positive in the large $N$ limit, so (10) diverges. The possible range for $q$ is thus $q_c(\beta) \leq q \leq 1$. But, if we assume $q > q_c(\beta)$, the large $N$ limit yields the very simple result $z(\alpha, \beta) = G_s(q) + \alpha \beta \langle g(\gamma, y) \rangle_{y, \gamma}$. Now, on one hand, the empirical contrast is found by differentiating $z(\alpha, \beta)$ w.r.t to $\beta$. This yields $\langle g(\gamma, y) \rangle_{y, \gamma} + \frac{1}{\alpha} G_s(q) \frac{\partial}{\partial q} q$. But computing the same quantity using (8) yields $\langle g(\gamma, y) \rangle_{y, \gamma}$.

So $q$ must stay constant when $\beta$ varies, but this is impossible since $q_c(\beta) \rightarrow 1$ for $\beta \rightarrow \infty$.

Hence, the only possible value for $q$ is $q_c(\beta)$. Evaluating (11) by taking the limit $q \rightarrow q_c(\beta)$ from above, leads to the same result as in the $K_N = \sqrt{N}$ replica theory. But, of course, this has the same inconsistencies as found for the $q > q_c(\beta)$ assumption. It also makes no physical sense to use (11) at the point of discontinuity since the cavity derivation neglects fluctuations of $q$. Even if these vanish with increasing $N$, at the point of discontinuity, $q = q_c(\beta)$, the true result will nevertheless depend on the unknown fluctuations.
But some conclusions can be drawn, knowing that \( q \) has the critical value. Let \( d_\beta \) be the unique positive value such that \( u(d_\beta) = 0 \) for \( q = q_c(\beta) \). Then arguments analogous to the derivation of (8) show that the probability of the posterior field \( [J_D] \xi^\mu \) exceeding \( d\sqrt{N} \) is not exponentially small if \( d \) is lower than \( d_\beta \). More precisely, one finds for \( N \to \infty \) and \( d < d_\beta \)

\[
\langle N^{-1} \ln(\Theta([J_D]^T \xi^\mu - d\sqrt{N}]) \rangle_D = \langle N^{-1} \ln L_{y,\gamma}^{\alpha,\beta}(\Theta(h - d\sqrt{N}))/L_{y,\gamma}^{\alpha,\beta}(1) \rangle_{\gamma,\gamma} = 0.
\]

Further, \( d_\beta \) approaches 1 with increasing \( \beta \). But this is only possible if simply aligning the weight vector with the pattern \( \xi^\mu \) maximizes the empirical contrast, at least up to sub-extensive corrections. So, in the notation of Eq. 4 we have \( C_r = C_r([\xi^\mu]) \) for large \( N \), and thus finally

\[
C_r = (1 - r^2)^{2+n}/\alpha + \langle g(r\gamma + \sqrt{1 - r^2 y}) \rangle_{\gamma,\gamma}.
\]

Maximizing this in \( r \), the same learning curve is obtained for the cubic case, \( g(x) = x^3 \), as in the \( K_N = \sqrt{N} \) replica theory (\(^2\)). It is important to note that we have in essence just used the standard weak correlation assumptions of the cavity method in deriving (12). In view of the good agreement with numerical simulations (Fig. 1), this strongly suggests that the cavity result is indeed exact in the thermodynamic limit.

From an analytical point of view, it is intriguing that the present problem reveals a difference in the scope of the replica and the cavity method. The latter can be transparently adapted to take into account that the cavity field is not Gaussian in the large deviations regime. But, commuting the thermal average with the disorder average, at the expense of considering moments, is part and parcel of using replicas. As a consequence, all the relevant fields become truly Gaussian. This points to implicit assumptions in the replica method, which need to be taken care of in any program to put the approach on a solid mathematical footing [6].

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REFERENCES

[1] A. Hyvärinen, J. Karhunen, and E. Oja. *Independent Component Analysis*. Wiley-Interscience, 2001.

[2] A. Cichoki and S. Amari. *Adaptive Blind Signal and Image Processing*. John Wiley & Sons, 2002.

[3] P. Comon. Independent component analysis, a new concept? *Signal Processing*, 36:287 – 314, 1994.

[4] A. Amari, A. Cichoki, and H.H. Yang. A new learning algorithm for blind source separation. In *NIPS 8*, pages 757–763. MIT Press, 1996.

[5] M. Mézard. The space of interactions in neural networks: Gardner’s computation with the cavity method. *J. Phys. A.*, 22:2181–2190, 1989.

[6] G. Parisi. http://arxiv.org/abs/cond-mat/0207334, 2002.

\(^2\)For \( g(x) = x^4 \), the curve depends on whether \( \langle \gamma^4 \rangle_\gamma > 3 \), since the fourth moment of a standard Gaussian is 3. If so, the value of \( r \) jumps from 0 to 1 at \( \alpha_c = 1/(\langle \gamma^4 \rangle_\gamma - 3) \). The \( \langle \gamma^4 \rangle_\gamma < 3 \) case, where one will use \( g(x) = -x^4 \), shall be described elsewhere. It is much simpler since the large deviations regime does not contribute.