Rigidity and Volume Preserving Deformation on Degenerate Simplices

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Abstract Given a degenerate \((n + 1)-\)simplex in a \(d\)-dimensional space \(M^d\) (Euclidean, spherical or hyperbolic space, and \(d \geq n\)), for each \(k\), \(1 \leq k \leq n\), Radon’s theorem induces a partition of the set of \(k\)-faces into two subsets. We prove that if the vertices of the simplex vary smoothly in \(M^d\) for \(d = n\), and the volumes of \(k\)-faces in one subset are constrained only to decrease while in the other subset only to increase, then any sufficiently small motion must preserve the volumes of all \(k\)-faces; and this property still holds in \(M^d\) for \(d \geq n + 1\) if an invariant \(c_{k-1}(\alpha^{k-1})\) of the degenerate simplex has the desired sign. This answers a question posed by the author, and the proof relies on an invariant \(c_k(\omega)\) we discovered for any \(k\)-stress \(\omega\) on a cell complex in \(M^d\). We introduce a characteristic polynomial of the degenerate simplex by defining \(f(x) = \sum_{i=0}^{n+1} (-1)^i c_i(\alpha^i)x^{n+1-i}\), and prove that the roots of \(f(x)\) are real for the Euclidean case. Some evidence suggests the same conjecture for the hyperbolic case.

Keywords Rigidity · Cell complex · \(k\)-Stress · Schlöfli differential formula · Characteristic polynomial

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1 Introduction

1.1 Main Results and Motivations

Let $M^d$ of dimension $d \geq n$ be the Euclidean, spherical or hyperbolic space of constant curvature $\kappa$, and $A = \{A_1, \ldots, A_{n+2}\}$ be a set of vertices of a degenerate $(n+1)$-dimensional simplex in $M^d$, where by degenerate we mean the vertices are confined in a lower dimensional $M^n$. Assume further that all $n$-faces of $A$ are non-degenerate. By Radon’s theorem the vertices of $A$ can be partitioned into two subsets whose convex hulls in $M^d$ intersect. The only trivial exception is for the spherical case when the vertices are not confined in any open half sphere, then in this case one subset of vertices should be the empty set. For each $k, 1 \leq k \leq n$, counting each $k$-face’s number of vertices mod 2 in each subset induces a partition of the set of $k$-faces into two subsets $X_{1,k}$ and $X_{2,k}$. The author asked the following question in [17]:

**Question 1.1** If $A$ varies smoothly in $M^d$, and the volumes of $k$-faces in one subset ($X_{1,k}$ or $X_{2,k}$) are constrained only to decrease while in the other subset only to increase, does the motion preserve the volumes of all $k$-faces of $A$?

The purpose of this paper is twofold. First, we prove a rigidity theorem which gives an affirmative answer to Question 1.1 for $d = n$, and shows that it still holds for $d \geq n + 1$ if an invariant $c_{k-1}(\alpha^{k-1})$ we obtained from $A$ has the desired sign. Second, under the motivation of Question 1.1, we develop a theory to link $k$-stress (a notion introduced by Lee [8], see also [13, 16]) with the volume deformation on cell complexes (not necessarily simplicial) in $M^d$, discover a geometric invariant $c_k(\omega)$ for any $k$-stress $\omega$ on a cell complex in $M^d$, and introduce a notion of characteristic polynomial of a degenerate simplex, which is also of interest by its own right. These two topics are strongly related. To some extent, Question 1.1 serves the purpose of storytelling, which leads to the development of the theory of the second topic above.

To state our results, we first introduce some basic notions. Let the spherical space $S^d$ be the standard unit sphere centered at the origin in a Euclidean space $\mathbb{R}^{d+1}$, and the hyperbolic space $\mathbb{H}^d$ be described by the hyperboloid model: Let $\mathbb{R}^{d,1}$ be a $(d+1)$-dimensional vector space endowed with a metric

$$x \cdot y = -x_0 y_0 + x_1 y_1 + \cdots + x_d y_d,$$

then $\mathbb{H}^d$ is defined by

$$\{x \in \mathbb{R}^{d,1} : x \cdot x = -1, \ x_0 > 0\},$$

which is the upper sheet of a two-sheeted hyperboloid. Under this embedding, we can use the vector space to discuss the linear relations between points in $S^d$ or $\mathbb{H}^d$.

Since every $n$-face of $A$ is non-degenerate, so up to a constant factor, there is a unique affine dependence among the vertices of $A$ for the Euclidean case, or a linear dependence for the non-Euclidean case. Namely, there is a sequence of non-zero coefficients $\alpha_1, \ldots, \alpha_{n+2} \in \mathbb{R}$, such that
\[ \sum \alpha_i A_i = 0 \quad \text{and} \quad \sum \alpha_i = 0 \quad \text{(for the Euclidean case)}, \]
\[ \sum \alpha_i A_i = 0 \quad \text{(for the spherical or hyperbolic case)}. \]

We call \( \alpha := \{ \alpha_1, \ldots, \alpha_{n+2} \} \) a 1-stress on \( A \). We reserve the notations \( A \) and \( \alpha \), or simply \( (A, \alpha) \), as well as \( G_{n,k} \) and \( G'_{n,k} \) defined next, for the rest of this paper.

**Definition 1.2** Let \( (A, \alpha) \) be as in (1.1) where \( \alpha \) is a 1-stress on \( A \). For each \( k, 1 \leq k \leq n \), define \( G_{n,k} \) to be a framework equipped with the following volume constraints on \( k \)-faces of \( A \): the volume of a \( k \)-face \( F \) is constrained only to decrease (under tension) if \( \prod_{\alpha_i \in F} \alpha_i < 0 \), and only to increase (under compression) if \( \prod_{\alpha_i \in F} \alpha_i > 0 \). And define \( G'_{n,k} \) by flipping the tension-compression volume constraints in \( G_{n,k} \).

Let \( A(t) \) be a smooth motion of \( A \) in \( M^d \), and \( A(0) = A \) be the initial position. Then our rigidity theorem can be formulated as follows.

**Theorem 1.3** (Main Theorem 1) If \( A(t) \) varies smoothly over \( t \) in \( M^n \), then for both \( G_{n,k} \) and \( G'_{n,k} \) that equipped with the volume constraints on \( A \), the motion must preserve the volumes of all \( k \)-faces of \( A(t) \) for small \( t \geq 0 \).

The case \( d \geq n + 1 \) is much harder and very different from the case \( d = n \). One of the most important results of this paper is an invariant \( c_k(\omega) \) (Theorem 2.13) we obtained from any \( k \)-stress \( \omega \) on a cell complex in \( M^d \). Particularly for \( A \) we derive a sequence of invariants \( c_0(\alpha^0), \ldots, c_{n+1}(\alpha^{n+1}) \) (Definition 2.16), which plays a key role in both the formulation and proof of the following theorem.

**Theorem 1.4** (Main Theorem 2) For \( d \geq n + 1 \), if \( A(t) \) varies smoothly over \( t \) in \( M^d \) and \( c_{k-1}(\alpha^{k-1}) > 0 \) (resp. \( c_{k-1}(\alpha^{k-1}) < 0 \)), then for \( G_{n,k} \) (resp. \( G'_{n,k} \)) that equipped with the volume constraints on \( A \), the motion must preserve the volumes of all \( k \)-faces of \( A(t) \), and the vertices are confined in a lower dimensional \( M^n \) for small \( t \geq 0 \).

For \( k = n \) with \( n \geq 2 \), the statement that “the vertices are confined in a lower dimensional \( M^n \) for small \( t \geq 0 \)” is somewhat surprising, because the number of volume constraints \( n + 2 \) is far less than the degree of freedom of \( A \) in \( M^d \) up to congruence, which is \((n+2)(n+1)/2 \), or subtract by 1 if \( A \) is restricted in \( M^n \). Note that in both Theorems 1.3 and 1.4, except for \( k = 1 \) we do not prove that the motion is rigid, which is a stronger notion than the type of volume rigidity we proved. This can be a potential improvement to our results, and will be addressed in Sect. 2.12 along with some related questions.

As remarked above, a key tool we use to prove the rigidity theorem is \( k \)-stress, a notion first introduced by Lee on simplicial complexes with vertices chosen in the Euclidean space \([8]\). The introduction of the notion was partly inspired by Kalai’s proof of the lower bound theorem using classical stresses \([7]\), and motivated to give a geometric understanding of Stanley’s proof of the necessity of the \( g \)-theorem for simplicial convex polytopes \([15]\), which used algebraic geometry. A notable property of \( k \)-stress is that for a simplicial \((d-1)\)-sphere \( \Delta \) with vertices chosen generically in \( S^{d-1} \), according to Lee \([8]\), the dimension of the space of \( k \)-stresses on \( \Delta \) in \( S^{d-1} \) is \( h_k \), where \((h_0, \ldots, h_d)\) is the \( h \)-vector of \( \Delta \). What remains open, which if true
proves the g-conjecture for simplicial spheres, is to show that for $\Delta$ with vertices chosen generically in $\mathbb{R}^d$, the dimension of the space of $k$-stresses on $\Delta$ in $\mathbb{R}^d$ is $g_k$ for $k \leq \lfloor d/2 \rfloor$, where $g_0 = h_0$, $g_k = h_k - h_{k-1}$, $k = 1, \ldots, \lfloor d/2 \rfloor$. As $k$-stresses are the central theme of this paper and non-degenerate simplices do not admit any $k$-stress, that is why the $(n+1)$-simplices we looked at in this paper are degenerate.

For $k = 1$, where $G_{n,1}$ and $G'_{n,1}$ are tensegrity frameworks, Bezdek and Connelly proved the Euclidean case of Theorems 1.3 and 1.4 [3]. They actually proved a stronger version: $G_{n,1}$ is globally rigid in $\mathbb{R}^d$ for any $d \geq n$. Rigidity and flexibility of tensegrity frameworks, which analyze geometric structures equipped with distance constraints on edges, have been extensively investigated in the past (see [5]). However, the analogue for $k \geq 2$ with volume constraints on $k$-faces, especially for the non-Euclidean case, has been much less studied in the literature.

For $k = n$, recall the notation $X_{1,n}$ and $X_{2,n}$ as in Question 1.1. Note that in $\mathbb{R}^n$ both $G_{n,n}$ and $G'_{n,n}$ must preserve the volumes of all $n$-faces, as under a continuous motion the sum of the volumes of $n$-faces in $X_{1,n}$ is equal to those in $X_{2,n}$. Also note that $G_{n,n}$ and $G'_{n,n}$ are not rigid in $\mathbb{R}^n$, as under affine motions they can change the shapes smoothly while preserving the volumes of all $n$-faces. However, it is far from trivial to tell if $G_{n,n}$ or $G'_{n,n}$ will still preserve the volumes of $n$-faces in $\mathbb{R}^{n+1}$, as potentially the vertices can be lifted in $\mathbb{R}^{n+1}$ to form a non-degenerate $(n+1)$-simplex, therefore the sum of the volumes in $X_{1,n}$ is no longer necessarily equal to those in $X_{2,n}$. For $n = 2$, from our results we come up with a particularly interesting example of “four points on a circle” to address this phenomenon (Example 2.24), which we present here as well:

In $\mathbb{R}^3$, given four points that are initially in convex position in a 2-dimensional plane. If we allow the four points to move smoothly in $\mathbb{R}^3$ but constrain all the triangles formed by any three points to preserve the areas during the motion, then in order for the four points to form a non-degenerate 3-simplex in $\mathbb{R}^3$, they have to be confined in a plane first until they move on to a common circle. And only from this common circle they can be lifted to form a non-degenerate 3-simplex.

A similar analogue for the non-Euclidean case is also given in Example 2.24. In fact, these examples were part of the motivations for the author to pose Question 1.1 and formulate the rigidity theorem in the first place.

1.2 Strategy Overview

Our strategy to prove the rigidity theorem is as follows. Using the Schläfli differential formula, we develop techniques for $k$-stresses on cell complexes in $M^d$ (Theorems 2.2 and 2.13). Applying them we obtain a differential equality (Proposition 2.7) for the $k$-faces of $A$ if $d = n$, and a differential inequality (Proposition 2.19) if $d \geq n + 1$, which directly lead to the proofs of Theorems 1.3 and 1.4 respectively. We also obtain a new version of Schläfli differential formula on simplices based on edge lengths.

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1 It will be interesting to see if this phenomenon can be demonstrated in the “real” world by using some physical material, e.g., just as the minimal surface can be visualized by using soap film.
(Proposition 2.11). Some remarks on the history of the Schl"afli differential formula can be found in Milnor's paper [11].

To analyze the interrelation between the rigidity properties of different dimensions \(k\), we introduce a notion of characteristic polynomial of the degenerate \((n+1)\)-simplex by defining

\[
f(x) = \sum_{i=0}^{n+1} (-1)^i c_i (\alpha_i) x^{n+1-i}.
\]

For the Euclidean case, we prove that the roots of \(f(x)\) are real and give a way to count the number of positive roots (Theorem 3.4). Some evidence suggests the same conjecture for the hyperbolic case (Conjecture 3.6). And in Sect. 3.2, we naturally generalize the notion of characteristic polynomial \(f(x)\) to a set of points (continuous distribution allowed) in \(M^n\) associated with a 1-stress on the points.

1.3 Historical Works

Rigidity and deformation of geometric structures have attracted the attention of mathematicians for a long time. One of the first substantial mathematical results concerning rigidity is Cauchy’s rigidity theorem, which proved that all convex polyhedra with solid faces and flexible dihedral angles are rigid. It was widely believed and conjectured that the same held true for non-convex polyhedron as well. However, Connelly disproved the rigidity conjecture by constructing a flexible polyhedron in \(\mathbb{R}^3\) [4], and with D. Sullivan, they conjectured that the volume bounded by a flexible polyhedron is constant during the flex. Sabitov proved the conjecture of Connelly and Sullivan for flexible polyhedra homeomorphic to a sphere [14]; and Connelly, Sabitov, and Walz proved it for general polyhedral surfaces in “The bellows conjecture” [6]. The same conjecture in the spherical space is not true though. Alexandrov constructed a flexible polyhedron in an open half sphere in \(S^3\) which does not conserve the volume [2].

Motivated by the historical works on rigidity, we bring a different view to the field. Instead of analyzing geometric structures equipped with distance constraints between vertices (as in the above works), we analyze volume constraints on \(k\)-faces of the underlying geometric structure, as well as the interrelation between the rigidity properties of different dimensions \(k\). We also generalize our main results to the Euclidean, spherical and hyperbolic space together, so the validity of our rigidity theorem is independent of the constant curvature value of the underlying space.

2 Volume Preserving Deformation

To prove the main results, our approach emphasizes on the non-Euclidean case, and treats the Euclidean case as a limit of the spherical case.

2.1 Basic Terminology

The following terminology and definitions are intended to clarify the meaning of terms used in this paper. Some terms are new.

By a \(k\)-dimensional convex polytope in \(M^k\) we mean a compact subset which can be expressed as a finite intersection of closed half spaces. A cell complex in \(M^d\) is
a finite set of convex polytopes (called cells) in $M^d$, such that every face (empty set included) of a cell is also a cell in the set, and any two cells share a unique maximal common face, the intersection. However, in this paper we do not worry about the self-intersections between the cells in $M^d$. For the spherical case, we also require that each cell of a cell complex lies strictly in an open half sphere, so $S^d$ is not a cell and a 0-cell always contains only one point. Also a half circle is not a cell.

We want to point out that the convexity of the polytopes above plays almost no role in the context of this paper. However, for simplicity, we content ourselves with only considering convex polytopes in $M^d$.

For a cell complex in $M^d$, we call it a $k$-tensegrity framework if it is equipped with volume constraints on $k$-faces (equalities and inequalities, as tension and compression), e.g., $G_{n,k}$ and $G'_{n,k}$. A $k$-tensegrity framework $p$ is rigid in $M^d$, if any continuous motion in $M^d$ that satisfies the volume constraints is also a rigid motion; and it is globally rigid in $M^d$, if for any other configuration $q$ in $M^d$ satisfying the volume constraints, $q$ is congruent to $p$.

The notion of $k$-tensegrity framework, a new term introduced in this paper and [17], is a natural higher dimensional generalization of the notion of tensegrity frameworks (see [5]), which is a finite graph with vertices in $M^d$ and equipped with length constraints on edges.

### 2.2 Stresses on Cell Complex

The notion of $k$-stresses on cell complexes in $M^d$ plays an important role in proving the main rigidity theorem. While our rigidity theorem concerns the boundary complex of a degenerate simplex in $M^d$, our results about $k$-stresses are much more general, which can be extended to cell complexes (not necessarily simplicial) in $M^d$ without much extra effort.

If $K$ is a cell complex in $M^d$, with a slight abuse of notation we simply denote by $K$ as well the set of all its cells, and by $K^r$ the subset of its $r$-cells.

**Definition 2.1** Consider a cell complex $K$ (not necessarily of dimension $d - 1$ or $d$) in $M^d$. A $k$-stress ($2 \leq k \leq d + 1$) on $K$ is a real-valued function $\omega$ on the $(k - 1)$-cells of $K$, such that for each $(k - 2)$-cell $F$ of $K$,

$$\sum_{G \in K^{k-1}, F \subset G} \omega(G)u_{F,G} = 0,$$

where the sum is taken over all $(k - 1)$-cells $G$ of $K$ that contain $F$, and $u_{F,G}$ is the inward unit normal to $G$ at its facet $F$. For $k = 1$, a 1-stress is an affine dependence among the vertices for the Euclidean case, or a linear dependence for the non-Euclidean case.

The notion of $k$-stress was first introduced by Lee [8] on simplicial complexes with vertices chosen in the Euclidean space with a slightly different setting. Lee considered two types of $k$-stresses, affine and linear. For a simplicial complex $K$ with vertices chosen in $\mathbb{R}^d$, the space of affine $k$-stresses is isomorphic to the space of our notion.
of \(k\)-stresses. Denote by \(b_a\) (resp. \(b_l\)) the affine (resp. linear) \(k\)-stress on \(K\) in \(\mathbb{R}^d\), then \(\omega(G) = (k - 1)!V_{k-1}(G)b_a(G)\) for each \((k - 1)\)-face \(G\) of \(K\), where \(\omega\) is our notion of \(k\)-stress and \(V_{k-1}(G)\) denotes the \((k - 1)\)-dimensional volume of \(G\). If \(K\) is a simplicial complex with vertices chosen in \(\mathbb{S}^{d-1}\), as \(\mathbb{S}^{d-1}\) is embedded in \(\mathbb{R}^d\), we can also loosely treat \(K\) as an Euclidean simplicial complex in \(\mathbb{R}^d\) in the sense of Lee. Under this interpretation, the space of our notion of \(k\)-stresses on \(K\) in \(\mathbb{S}^{d-1}\) is isomorphic to the space of linear \(k\)-stresses on \(K\) in \(\mathbb{R}^d\). For a \((k - 1)\)-face \(G\) of \(K\) in \(\mathbb{S}^{d-1}\), let \(\|G\|\) be \(k!\) times the volume of the Euclidean \(k\)-simplex formed by the vertices of \(G\) and the origin \(O\), then \(\omega(G) = \|G\|b_l(G)\). Note that unlike \(\omega\), the linear \(k\)-stress \(b_l\) cannot be extended to non-simplicial cell complexes in \(\mathbb{S}^{d-1}\), as \(\|G\|\) cannot be properly defined for spherical cells \(G\) that are not simplicial.

Rybnikov in \([13]\) extended the notion of \(k\)-stress to cell complexes in Euclidean and spherical spaces, and our terminology agrees with that terminology. Similar notions were also considered in \([16]\). McMullen also considered weights on simple polytopes \([9]\), a notion dual to stresses. The relationship between \(k\)-stresses and volumes of simplicial or simple polytopes in the Euclidean case was discussed in \([8]\) and \([9]\). However, it seems that our work is the first to give a systematic discussion of the relationship between \(k\)-stresses and the volumes of faces of cell complexes in the non-Euclidean case.

### 2.3 A Differential Formula

As a first step to proving Theorem 1.3, we obtain a differential formula for \((k + 1)\)-stresses in Theorem 2.2, which also establishes a correspondence between the signs of volume constraints and the signs of \((k + 1)\)-stresses on \(k\)-faces of a cell complex. It generalizes the well established correspondence of classical stresses on 1-dimensional faces of a framework.

For a \(k\)-polytope \(G\) in \(M^d\), denote by \(V_k(G)\) the \(k\)-dimensional volume of \(G\). To compute the differential of the volumes of \(k\)-dimensional polytopes in \(\mathbb{S}^d\) or \(\mathbb{H}^d\), Schl"afli’s differential formula plays a central role. Some remarks on the history of the Schl"afli differential formula can be found in Milnor’s paper \([11]\). Consider a family of \(k\)-dimensional convex polytopes \(P\) which vary smoothly in a space of constant curvature \(\kappa\). For each \((k - 2)\)-dimensional face \(F\) let \(\theta_F\) be the dihedral angle at \(F\). Then the Schl"afli differential formula states that

\[
\kappa \cdot dV_k(P) = \frac{1}{k - 1} \sum_F V_{k-2}(F) d\theta_F,
\]

where the sum is taken over all \((k - 2)\)-faces \(F\) of \(P\). When \(k - 2 = 0\), \(V_0(F)\) is the number of points in \(F\).

For each \((k - 2)\)-face \(F\) of \(P\), it can be uniquely described as an intersection \(F = E \cap E'\) of two \((k - 1)\)-faces \(E\) and \(E'\) of \(P\). Let \(u_{E,F}\) be the inward unit normal to \(P\) at its facet \(E\), \(u_{F,E}\) be the inward unit normal to \(E\) at its facet \(F\), and so on. Note that \(u_{E,F}, u_{E',F}, u_{F,E}, u_{F,E'}\) are all in a single 2-dimensional plane; the angle between \(u_{E,F}\) and \(u_{E',F}\) is \(\pi - \theta_F\); the angle between \(u_{F,E}\) and \(u_{F,E'}\) is \(\theta_F\); the angle between

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$u_{E',P}$ and $u_{F,E}$ is $\pi/2$; and the angle between $u_{E',P}$ and $u_{F,E'}$ is $\pi/2$ as well. It is easy to check that
\begin{equation}
    d\theta_F = -u_{E,P} \cdot du_{F,E} - u_{E',P} \cdot du_{F,E'},
\end{equation}
which was employed by Alexander to give a direct proof of the Schläfli differential formula in the Euclidean case \[1\]. Plug (2.2) into (2.1), then
\begin{equation}
    \kappa \cdot dV_k(P) = -\frac{1}{k-1} \sum_{F \subset E \subset P} V_{k-2}(F) u_{E,P} \cdot du_{F,E},
\end{equation}
which will be useful in the proof of the following theorem.

**Theorem 2.2** (Main Theorem 3) Let $K(t)$ be a family of cell complexes in $M^d$ depending smoothly on a parameter $t$ and $K(0) = K$, and $\omega$ be a $(k + 1)$-stress $(k \geq 1)$ on $K$. Then
\begin{equation}
    \sum_{G \in K^k} \omega(G) dV_k(G) = 0
\end{equation}
at $t = 0$, where the sum is taken over all $k$-cells $G$ of $K$.

**Proof** For $k = 1$, if $G$ is a 1-cell of $K$ and $B$ is a vertex of $G$, let $u_{B,G}$ be the inward unit normal to $G$ at $B$. Then we have $dV_1(G) = -\sum_{B \in G} u_{B,G} \cdot dB$. Taking the sum over all 1-cells $G$ of $K$, we have
\begin{equation}
    \sum_{G \in K^1} \omega(G) dV_1(G) = -\sum_{G \in K^1} \omega(G) \sum_{B \in G} u_{B,G} \cdot dB
    = -\sum_{\{B\} \in K^0} \left( \sum_{G \in K^1, B \in G} \omega(G) u_{B,G} \right) \cdot dB.
\end{equation}
As $\omega$ is a 2-stress on $K$, by Definition 2.1, $\sum_{G \in K^1, B \in G} \omega(G) u_{B,G}$ is 0 for each vertex $B$, so the above formula is 0 at $t = 0$.

For $k \geq 2$ and $\omega$ is a $(k + 1)$-stress on $K$ in $M^d$, we first consider the non-Euclidean case. Applying (2.3) on each $G \in K^k$,
\begin{equation}
    \kappa \cdot (k - 1) \sum_{G \in K^k} \omega(G) dV_k(G)
    = -\sum_{G \in K^k} \omega(G) \sum_{F \subset E \subset G} V_{k-2}(F) u_{E,G} \cdot du_{F,E}
    = -\sum_{\{F,E\} \subset E \in K^k} V_{k-2}(F) \left( \sum_{G \in K^k, E \subset G} \omega(G) u_{E,G} \right) \cdot du_{F,E}.
\end{equation}
As $\omega$ is a $(k + 1)$-stress on $K$, by Definition 2.1, $\sum_{G \in K^k, E \subset G} \omega(G) u_{E,G} = 0$ for each $(k - 1)$-cell $E$ of $K$, so the above formula is 0 at $t = 0$. Since $\kappa \neq 0$ for the non-Euclidean case, therefore $\sum_{G \in K^k} \omega(G) dV_k(G) = 0$ at $t = 0$. 

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For the Euclidean case, we show that the $(k+1)$-stress $\omega$ can be treated as a “limit” of some spherical $(k+1)$-stresses. For any $r > 0$, embed $\mathbb{R}^d$ into $\mathbb{R}^{d+1}$, $x \mapsto (x, r)$, and let $S^d_r$ be a $d$-dimensional sphere in $\mathbb{R}^{d+1}$ with radius $r$ and centered at the origin $O$ of $\mathbb{R}^{d+1}$. By a radial projection (from the center of $S^d_r$) of $\mathbb{R}^d$ onto $S^d_r$, we obtain a family (with respect to $t$) of spherical cell complexes $K_r(t)$ in $S^d_r$ from the Euclidean cell complexes $K(t)$ in $\mathbb{R}^d$. Denote $K_r(0)$ by $K_r$. For each $k$-cell $G$ of $K$, denote by $G_r$ the corresponding spherical cell of $K_r$, and by $v_{G,r}$ the altitude vector for the point $O$ with respect to the affine span of $G$. We define a real-valued function $\omega_r$ on all the $k$-cells $G_r$ of $K_r$ by

$$\omega_r(G_r) := \omega(G) \cdot \frac{\|v_{G,r}\|}{r}.$$ (2.4)

By Definition 2.1, $\sum_{G \in K^k, E \subseteq G} \omega(G)u_{E,G} = 0$ for each $(k-1)$-cell $E$ of $K$. Let $u'_{E,G}$ be the orthogonal component of $u_{E,G}$ that is perpendicular to the linear span of $E$ under the embedding, then $\sum_{G \in K^k, E \subseteq G} \omega(G)u'_{E,G} = 0$ for each $(k-1)$-cell $E$ of $K$.

It is easy to see that

$$u'_{E,G} = \frac{\|v_{G,r}\|}{\|v_{E,r}\|} \cdot u_{E_r,G},$$

so

$$\sum_{G \in K^k, E \subseteq G} \omega(G)\|v_{G,r}\| \cdot u_{E_r,G} = 0,$$

thus by (2.4) we have $\sum_{G \in K^k, E \subseteq G} \omega_r(G_r)u_{E_r,G} = 0$ for each $(k-1)$-cell $E$ of $K$.

By Definition 2.1, $\omega_r$ is a $(k+1)$-stress on $K_r$. Therefore, for any fixed $r$, we have $\sum_{G \in K^k} \omega_r(G_r) dV_k(G_r) = 0$ at $t = 0$. As when $r \to \infty$, we have $\frac{\|v_{G,r}\|}{r} \to 1$, so by (2.4), $\omega_r(G_r)$ converges to $\omega(G)$, and $K_r(t)$ converges uniformly to $K(t)$ with respect to small $t \geq 0$. Thus $\sum_{G \in K^k} \omega(G) dV_k(G) = 0$ at $t = 0$. This completes the proof.

Particularly in the Euclidean case, but not in the non-Euclidean case, Theorem 2.2 implies the following property. Let $\omega$ be a $(k+1)$-stress $(k \geq 1)$ on a cell complex $K$ in $\mathbb{R}^d$, then $K$ can be proportionally scaled with a factor $t$, with the same $\omega$ as a $(k+1)$-stress. As the volumes of all $k$-faces of $K$ are scaled with a factor $t^k$, then by Theorem 2.2 and taking the derivative at $t = 1$, we have

$$\sum_{G \in K^k} \omega(G) V_k(G) = 0.$$

Remark 2.3 While Theorem 2.2 was initially developed as a tool to prove Theorem 1.3, it is a much more general result. Theorem 2.2 establishes a correspondence between the signs of tension-compression constraints and the signs of $(k+1)$-stresses on $k$-faces of a cell complex, which generalizes the well-established correspondence of $k = 1$. Namely, if the signs of volume constraints agree with the signs of any $(k+1)$-stress on $k$-faces, then the volumes of all $k$-faces are instantaneously preserved at $t = 0$.\(\Box\) Springer
If \((k + 1)\)-stresses can also be assigned in a continuous manner over \(t\) on the family of cell complexes, then the volumes of all \(k\)-faces are preserved for small \(t \geq 0\). So to some extent, it justifies the physical meaning of \((k + 1)\)-stresses, which was first introduced more of a mathematical concept for \(k > 1\) by Lee.

It can be summarized as follows.

**Corollary 2.4** Let \(K(t)\) be a family of cell complexes in \(M^d\) depending smoothly on a parameter \(t\) and \(K(0) = K\), and \(\omega_t\) be \((k + 1)\)-stresses on \(K(t)\) in a continuous manner over \(t \geq 0\) and \(\omega_0 = \omega\). Then \(K(t)\) cannot be a non-trivial deformation for small \(t \geq 0\) under which the volumes of \(k\)-faces with negative signs of \(\omega\) only decrease (resp. increase), and the volumes of \(k\)-faces with positive signs of \(\omega\) only increase (resp. decrease). Here by non-trivial it means that the volume of at least one \(k\)-face is non-constant.

### 2.4 Proof of Theorem 1.3

We begin with some basic notions. Let \(\Lambda(\mathbb{R}^{d+1})\) be the exterior algebra of \(\mathbb{R}^{d+1}\). An inner product on \(\Lambda^k(\mathbb{R}^{d+1})\), induced by the standard inner product on \(\mathbb{R}^{d+1}\), can be well defined by

\[
(r_1 \wedge \cdots \wedge r_k) \cdot (s_1 \wedge \cdots \wedge s_k) := \det(r_i \cdot s_j)_{1 \leq i, j \leq k},
\]

with extension by bilinearity, where \(r_i\) and \(s_i\) are any \(2k\) elements in \(\mathbb{R}^{d+1}\).

The notions above can be extended to \(\Lambda(\mathbb{R}^{d,1})\) in parallel, with the exception that the inner product on \(\Lambda^k(\mathbb{R}^{d,1})\) is not positive definite, but it is not a concern of this paper. Particularly if \(F\) is a \(k\)-simplex in \(M^d\) or \(H^d\) (recall that they are embedded in \(\mathbb{R}^{d+1}\) and \(\mathbb{R}^{d,1}\) respectively) and \(B_1, \ldots, B_{k+1}\) are the vertices, for convenience we introduce a new notation

\[
\|F\| := |\det(B_i \cdot B_j)_{1 \leq i, j \leq k+1}|^{1/2}.
\]

For the spherical case \(\|F\|\) is simply \((k + 1)!\) times the volume of the Euclidean \((k + 1)\)-simplex whose vertices are \(O, B_1, \ldots, B_{k+1}\); for the hyperbolic case, pseudovolume. With the volume interpretation of \(\|F\|\) in mind, it will be very helpful for understanding the calculations involving \(\|F\|\) for the rest of this paper.

With the new notation, we have the following definition for a more general \((A, \alpha)\).

**Definition 2.5** Let \((A, \alpha)\) be as in (1.1) where \(\alpha\) is a 1-stress on \(A\), but \(A\) is more general and may contain \(m \geq n+2\) points in general position in \(M^n\). For a given \(k\) (\(1 \leq k \leq n\)) and each simplicial \(k\)-face \(F\) of \(A\), define a \((k + 1)\)-stress \(\alpha^{k+1}\) by \(\alpha^{k+1}(F) := (\prod_{A_s \in F} \alpha_s)! V_k(F)\) for the Euclidean case, and \(\alpha^{k+1}(F) := (\prod_{A_s \in F} \alpha_s) \|F\|\) for the non-Euclidean case.

**Remark 2.6** Recall the discussion in Sect. 2.2 about the relationship between our notion of \((k + 1)\)-stresses and Lee’s affine and linear \((k + 1)\)-stresses, it is not hard to...
see the following general fact: For a simplicial $k$-face $F$ of $A$, $\prod_{A_s \in F} \alpha_s$ corresponds to Lee’s affine $(k + 1)$-stress in the Euclidean case, or to Lee’s linear $(k + 1)$-stress in the non-Euclidean case. For notational reasons, we use $\alpha^{k+1}$ to denote the $(k + 1)$-stress obtained by multiplying $\alpha$ with itself for $k + 1$ times and then normalized by a volume factor, rather than taking the value of $\prod_{A_s \in F} \alpha_s$ directly.

Then by Theorem 2.2, we immediately have the following fact.

**Proposition 2.7** Let $(A, \alpha)$ be as in (1.1) where $\alpha$ is a $1$-stress on $A$, and $\alpha^{k+1}$ be a $(k + 1)$-stress on $A$ as in Definition 2.5. Then by Theorem 2.2

$$\sum_{F \subset A, \dim(F) = k} \alpha^{k+1}(F) dV_k(F) = 0 \tag{2.6}$$

holds at $t = 0$.

Then it leads to the proof of Theorem 1.3.

**Proof of Theorem 1.3** For a given $k$, $1 \leq k \leq n$, let $\alpha^{k+1}$ be the $(k + 1)$-stress on $A$ as above in Proposition 2.7. As $A(t)$ is confined in $M^n$, so $A(t)$ is degenerate for $t \geq 0$. This allows us to assign $1$-stresses $\alpha_t$ on $A(t)$ in a continuous manner over $t$. Therefore by Definition 2.5 we can assign $(k + 1)$-stresses $\alpha^{k+1}_t$ on $A(t)$ continuously over $t$ as well. As the signs of volume constraints of $G_{n,k}$ and $G'_{n,k}$ agree with the signs of $\alpha^{k+1}_t$ (including the opposite of) for small $t \geq 0$, Theorem 1.3 is just a special case of Corollary 2.4. This completes the proof. \hfill $\square$

To see if Theorem 1.3 can be improved to claim that $G_{n,k}$ and $G'_{n,k}$ are rigid in $M^n$, check Remark 2.31.

### 2.5 A Key Definition $g_F(P, Q)$

As Question 1.1 is settled for case $d = n$ primarily using a new property of $k$-stresses (Theorem 2.2), we plan to apply similar techniques for the more general case $d \geq n + 1$. However, unlike the case $d = n$, for $d \geq n + 1$ when $A(t)$ moves in $M^d$ and is not confined in a lower dimensional $M^n$, there is no $1$-stress on $A(t)$, and therefore no $(k + 1)$-stress on $A(t)$ for $t > 0$. By applying Theorem 2.2, though we can still show that for both $G_{n,k}$ and $G'_{n,k}$ the volumes of all $k$-faces are instantaneously preserved at $t = 0$, it is a weaker result than what we are looking for, i.e., like Theorem 1.4. Thus Theorem 2.2 alone is not enough for our purposes.

To fix this issue, in Sect. 2.6, for each $k$-stress $\omega$ on a cell complex in $M^d$, we discover an invariant $c_k(\omega)$ associated with $\omega$. This is one of the most important results of this paper, and this invariant leads to both the formulation and proof of Theorem 1.4. In this section we first introduce a notion $g_F(P, Q)$ in Definition 2.8, an important step for introducing the invariant $c_k(\omega)$. We also address the properties of $g_F(P, Q)$ in detail, which is of interest by its own right.

Consider a $k$-dimensional simplex $F$ and two points $P$ and $Q$ in $M^d$, and denote by $\hat{F}$ the $(k + 2)$-dimensional simplex in $M^d$ which is the join of $F$ with the segment
$P Q$. Also let $\theta_F$ be the dihedral angle of $\hat{F}$ at face $F$. Assume $\hat{F}$ is non-degenerate, then all edge lengths of $\hat{F}$ can vary independently of each other, thus $\theta_F$ can vary in such a manner that the distances between any pair of vertices of $\hat{F}$ are preserved except between $P$ and $Q$. It follows that $V_{k+2}(\hat{F})$ can be treated as a function of a single variable $\theta_F$, and we write the differential as $dV_{k+2}(\hat{F})/d\theta_F$.\footnote{It should not be confused with another similar notion that treats all the dihedral angles of $\hat{F}$ as independent variables in the non-Euclidean case.}

For the non-Euclidean case, let $P'$ (resp. $Q'$) be the vertical projection of point $P$ (resp. $Q$) on the linear span of $F$. Then $(P - P') \cdot (Q' - P') = 0$ and $(Q - Q') \cdot (Q' - P') = 0$. So if $\theta_F$ varies while all edge lengths of $\hat{F}$ are fixed except between $P$ and $Q$, then

$$
\begin{align*}
\overrightarrow{PQ}^2 &= d((Q - Q') + (Q' - P') - (P - P'))^2 = -2d((P - P') \cdot (Q - Q')) \\
&= -2\|P - P'\| \cdot \|Q - Q'\| d\cos \theta_F = 2\|P - P'\| \cdot \|Q - Q'\| \cdot \sin \theta_F d\theta_F \\
&= 2 \cdot \|\hat{F}\| \cdot d\theta_F,
\end{align*}
$$

where the second step is because the squared terms are constants when $\theta_F$ varies, and the last step uses the volume interpretation of $\|F\|$ and $\|\hat{F}\|$. Therefore we obtain

$$\frac{dV_{k+2}(\hat{F})}{d\theta_F} = 2 \cdot \|\hat{F}\| \cdot \frac{\partial_{\overrightarrow{PQ}^2} V_{k+2}(\hat{F})}{\|F\|}, \quad (2.7)$$

where $\partial_{\overrightarrow{PQ}^2}$ is the partial derivative with respect to $\overrightarrow{PQ}^2$ with all other edge lengths of $\hat{F}$ fixed.

This interpretation of $dV_{k+2}(\hat{F})/d\theta_F$ can be easily extended to $k$-dimensional convex polytope $F$ that is not necessarily simplicial. Consider two points $P$ and $Q$ in $M^d$ such that the segment $PQ$ is in general position with respect to $F$, denote by $\hat{F}$ the $(k+2)$-dimensional polytope in $M^d$ which is the join of $F$ with the segment $PQ$, and by $\theta_F$ the dihedral angle of $\hat{F}$ at face $F$. Here it is not crucial for $\hat{F}$ to be a convex polytope in the strict sense, and some degeneracy is allowed as long as $V_{k+2}(\hat{F})$ and $\theta_F$ can be properly defined. Same as the simplicial case above, $V_{k+2}(\hat{F})$ can be treated as a function of a single variable $\theta_F$.

Now we give a key definition, a new definition introduced in this paper.

**Definition 2.8** Let $F$ be a $k$-dimensional convex polytope in $M^d$ and $\hat{F}$, $\theta_F$ be as above. If $\theta_F$ varies while all edge lengths of $\hat{F}$ are fixed except between $P$ and $Q$, then define $g_F : M^d \times M^d \to \mathbb{R}$ by

$$g_F(P, Q) := (k + 2)! \frac{dV_{k+2}(\hat{F})}{d\theta_F}. \quad (2.8)$$

Also set $g_F(P, Q) = 1$.\footnote{\copyright{} Springer}
For a $k$-polytope $F$, note that if we decompose it into simplices $F_1, \ldots, F_m$, then it induces a decomposition of $\hat{F}$ into $\hat{F}_1, \ldots, \hat{F}_m$. As $\theta_{F_i} = \theta_F$ for each $F_i$, so by (2.8) we immediately have the following fact.

**Lemma 2.9** If $F$ is decomposed into simplices $F_1, \ldots, F_m$, then $g_F = \sum_i g_{F_i}$.

For a $k$-dimensional simplex $F$ in $\mathbb{S}^d$ or $\mathbb{H}^d$, by (2.8) and (2.7) we have

$$g_F(P, Q) = 2 \cdot (k + 2)! \frac{\|\hat{F}\|}{\|F\|} \cdot \frac{\partial - \rightarrow PQ}{2V_{k+2}(\hat{F})}.$$  

To give an explicit formula for $g_F(P, Q)$, we introduce the following notation. Let $G$ be a $k$-simplex in $\mathbb{S}^d$ or $\mathbb{H}^d$ and $B_1, \ldots, B_{k+1}$ be the vertices, then define

$$R_{ij}^k(G) := (-1)^{i+j} \frac{(B_1 \wedge \cdots \wedge \hat{B}_i \wedge \cdots \wedge B_{k+1}) \cdot (B_1 \wedge \cdots \wedge \hat{B}_j \wedge \cdots \wedge B_{k+1})}{(B_1 \wedge \cdots \wedge \hat{B}_i \wedge \cdots \wedge B_{k+1})^2}.$$  

Roughly speaking, for $i \neq j$, if $B_i$ is projected onto the linear span of $B_1, \ldots, \hat{B}_i, \ldots, B_{k+1}$, and expressed as $\sum_{s \neq i} \beta_s B_s$, then $-\beta_j$ is $R_{ij}^k(G)$; and $R_{ii}^k(G) = 1$. For notational reasons, if $E_j$ is the $(k-1)$-face $G \setminus \{B_j\}$, for $i \neq j$ we also define $R_{ij}^k(E_j) := 0$.

We now give the explicit formula for $g_F(P, Q)$ in $\mathbb{S}^d$ or $\mathbb{H}^d$ when $F$ is simplicial.

**Lemma 2.10** Let $G$ be a $k$-simplex in $\mathbb{S}^d$ or $\mathbb{H}^d$ of constant curvature $\kappa$ and $B_1, \ldots, B_{k+1}$ be the vertices. Also let $E_i$ be the $(k-1)$-face $G \setminus \{B_i\}$, and $F_{ij}$ be the $(k-2)$-face $G \setminus \{B_i, B_j\}$. Then for $k \geq 2$

$$\kappa \cdot \|F_{12}\| \cdot g_{F_{12}}(B_1, B_2) = k(k-2)! \sum_{i < j} R_{12}^{ij}(G) \|F_{ij}\| V_{k-2}(F_{ij}),$$

where for $i \neq j$ and $s \neq t$,

$$R_{st}^{ij}(G) := R_s^i(G) R_t^j(E_i) + R_s^j(G) R_t^i(E_j).$$  

Particularly, $R_{st}^{st}(G) = 1$, and $R_{st}^{ij}(G) = R_s^i(G)$.

We want to point out that as $g_{F_{st}}$ is symmetric on $B_s$ and $B_t$, we have $R_{st}^{ij}(G) = R_{ts}^{ij}(G)$, although it is not so obvious to see from (2.11) itself.

While we defer the proof of Lemma 2.10 to Sect. 2.7, we give a direct consequence of Lemma 2.10 here, a new version of Schlafli differential formula on simplices based on edge lengths.

**Proposition 2.11** (Schlafli differential formula on simplices based on edge lengths) Let $G$ be a $k$-simplex in $\mathbb{S}^d$ or $\mathbb{H}^d$ and $B_1, \ldots, B_{k+1}$ be the vertices, and $F_{ij}$ be the $(k-2)$-face $G \setminus \{B_i, B_j\}$. Then
\[ 2 \cdot k! \| G \| dV_k(G) = \sum_{i<j} \| F_{ij} \| g_{F_{ij}}(B_i, B_j) d\overrightarrow{B_iB_j}, \]

where the explicit formula of \( g_{F_{ij}}(B_i, B_j) \) is given in Lemma 2.10.

**Proof** Apply (2.9) and chain rule. \( \square \)

Particularly for \( g_F \) when \( F \) is a single point \( B \), we have the following.

**Corollary 2.12** Let \( B, P, Q \) be three points in \( S^d \) or \( \mathbb{H}^d \) of constant curvature \( \kappa \), then

\[ g_B(P, Q) = \frac{2}{1 + \kappa \cdot \overrightarrow{P\overrightarrow{B}} \cdot \overrightarrow{Q\overrightarrow{B}}}. \tag{2.12} \]

**Proof** Let \( G \) be a 2-simplex and \( B_1 = P, B_2 = Q, B_3 = B \) be the vertices. Then by Lemma 2.10 we have

\[ \kappa \cdot g_{B_3}(B_1, B_2) = 2(1 + R_1^3(G) + R_2^3(G)). \]

Multiplying \( (B_1 \wedge B_2)^2 \) on both sides, and applying (2.10) and (2.5), we have

\[
\begin{align*}
\kappa \cdot g_{B_3}(B_1, B_2)(B_1 \wedge B_2)^2 &= 2((B_1 \wedge B_2)^2 + (B_1 \wedge B_2) \cdot (B_2 \wedge B_3) - (B_1 \wedge B_2) \cdot (B_1 \wedge B_3)) \\
&= 2((B_1 \wedge B_2)^2 + (B_1 \cdot B_2)(B_2 \cdot B_3) - B_2^2(B_1 \cdot B_3) \\
&\quad - B_1^2(B_2 \cdot B_3) + (B_1 \cdot B_2)(B_1 \cdot B_3)) \\
&= 2\left((B_1 \wedge B_2)^2 - \left(\frac{1}{\kappa} - B_1 \cdot B_2\right)(B_1 \cdot B_3 + B_2 \cdot B_3)\right).
\end{align*}
\]

As \( (B_1 \wedge B_2)^2 = B_1^2 B_2^2 - (B_1 \cdot B_2)^2 = \left(\frac{1}{\kappa} - B_1 \cdot B_2\right)(\frac{1}{\kappa} + B_1 \cdot B_2) \), factor out \( \left(\frac{1}{\kappa} - B_1 \cdot B_2\right) \) from above we have

\[ (1 + \kappa B_1 \cdot B_2)g_{B_3}(B_1, B_2) = 2\left(\frac{1}{\kappa} + B_1 \cdot B_2 - B_1 \cdot B_3 - B_2 \cdot B_3\right) = \overrightarrow{B_1B_3} \cdot \overrightarrow{B_2B_3}, \]

which finishes the proof. \( \square \)

Note that as \( P, Q \to B \) and \( \kappa P \cdot Q \to 1 \), \( g_B(P, Q) \sim \overrightarrow{P\overrightarrow{B}} \cdot \overrightarrow{Q\overrightarrow{B}} \) is approximately the Riemannian metric at \( B \), with the difference that \( g_B \) is defined on the whole \( S^d \) or \( \mathbb{H}^d \) instead of on the tangent space at \( B \). The positive definiteness of \( g_B \) will be addressed in Sect. 2.11.

If \( B, P, Q \) are three points in \( \mathbb{R}^d \), by Definition 2.8 we have

\[ g_B(P, Q) = \| \overrightarrow{P\overrightarrow{B}} \| \cdot \| \overrightarrow{Q\overrightarrow{B}} \| \cdot \frac{d \sin \theta_B}{d\theta_B} = \| \overrightarrow{P\overrightarrow{B}} \| \cdot \| \overrightarrow{Q\overrightarrow{B}} \| \cdot \cos \theta_B = \overrightarrow{P\overrightarrow{B}} \cdot \overrightarrow{Q\overrightarrow{B}}, \tag{2.13} \]

which can also be viewed as a limit of the spherical case (2.12).
2.6 An Invariant of $k$-Stress

Now we are ready to state a key result of this paper, which leads to both the formulation and proof of Theorem 1.4.

**Theorem 2.13** (Main Theorem 4) Let $K$ be a cell complex in $M^d$ of constant curvature $\kappa$ and $\omega$ be a $k$-stress on $(k-1)$-faces of $K$ for $k \geq 1$. Then as long as $g_F(P, Q)$ is properly defined for each $F \in K^{k-1}$, we have

$$\sum_{F \in K^{k-1}} \omega(F) g_F(P, Q) = c_k(\omega),$$

(2.14)

where $c_k(\omega)$ is an invariant independent of the choice of points $P, Q \in M^d$. And for the non-Euclidean case,

$$c_k(\omega) = \kappa (k+1)(k-1)! \sum_{F \in K^{k-1}} \omega(F) V_{k-1}(F).$$

(2.15)

We first give some examples to illustrate Theorem 2.13.

**Example 2.14** Let $K$ be a cell complex in $S^n$ whose top dimensional cells form a decomposition (not necessarily simplicial) of $S^n$. Then the canonical $(n+1)$-stress $\omega$ on $K$ can be defined by $\omega(F) = 1$ for all $n$-dimensional faces $F$ of $K$. Then by (2.15) we have $c_{n+1}(\omega) = (n+2)n! V_n(S^n)$, which is always positive no matter how the vertices of $K$ are positioned.

**Example 2.15** Let $\omega$ be a 1-stress on a finite points $x_0, \ldots, x_m$ in $\mathbb{R}^1$, and $\omega(x_i) > 0$ for $i > 0$ and $\omega(x_0) = -1$. First, if we treat $x_i$ for $i > 0$ as a value taken by a discrete random variable $X$ with probability $\omega(x_i)$, then $x_0$ can be treated as the mean of $X$. And second, in (2.14) set $k = 1$ and $P = Q = 0$, then by applying (2.13) we have $c_1(\omega) = \sum_{i > 0} \omega(x_i)x_i^2 - x_0^2$, which is the variance $\text{var}(X)$ of random variable $X$. A continuous analogue can be easily generalized for continuous random variables.

Now we are ready to prove Theorem 2.13.

**Proof of Theorem 2.13** We first consider the case that $K$ is simplicial in $S^d$ or $\mathbb{H}^d$, which is the most important step of the proof. For a $(k-1)$-face $F$ of $K$, denote by $\widehat{F}$ the $(k+1)$-dimensional simplex which is the join of $F$ with segment $PQ$. By Lemma 2.10, we can view $g_F(P, Q)$ as a weighted sum of the volumes of all $(k-1)$-faces $F'$ of $\widehat{F}$, and particularly, the weight on $F$ is independent of the choice of $P$ and $Q$. There are four types of $F'$: $\{P, Q\} \subset F'$; $P \in F'$, $Q \notin F'$; $P \notin F'$, $Q \in F'$; and $F' = F$. When summing over all $(k-1)$-faces $F$ of $K$ on the left side of (2.14), applying Lemma 2.10 and (2.10) with some linear algebra, it can be shown that for any given $F'$ of the first three types, the sum of the weights on $F'$ is 0. So only the 4-th type of terms are left, and (2.15) immediately follows.

For the more general case that $K$ is not necessarily simplicial in $S^d$ or $\mathbb{H}^d$, by a barycentric subdivision of all the cells of $K$ with dimension $k-1$ and lower, and
ignoring all the cells with dimension \( k \) and above, we obtain a simplicial cell complex \( K' \) with dimension \( k - 1 \). Note that any \((k - 1)\)-simplicial face \( F' \) of \( K' \) is obtained from the decomposition of a \((k - 1)\)-cell \( F \) of \( K \). We define a real-valued function \( \omega' \) on \( F' \) of \( K' \) by \( \omega'(F') := \omega(F) \). It can be shown that \( \omega' \) is a \( k \)-stress on \( K' \) by using Definition 2.1 to verify the following two types of \((k - 2)\)-simplicial faces \( G' \) of \( K' \): the first type of \( G' \) is part of a \((k - 2)\)-cell \( G \) of \( K \), so it automatically satisfies the condition in Definition 2.1; the second type of \( G' \) is introduced by the decomposition of a \((k - 1)\)-cell \( F \) of \( K \) but not on the boundary of \( F \), so \( G' \) is shared by exactly two \((k - 1)\)-simplicial faces of \( K' \) who have the opposite inward unit normals at their common facet \( G' \), and therefore satisfies the condition in Definition 2.1 as well.

Applying the facts (1) \( \omega' \) is a \( k \)-stress on \( K' \), (2) the simplicial version of (2.14) and (2.15) we just proved above, and (3) the formula \( s = \sum_i g_i \) from Lemma 2.9, we prove that (2.14) and (2.15) still hold for the case that when \( k \) is not necessarily simplicial in \( S^d \) or \( \mathbb{H}^d \).

Finally, for the Euclidean case, let \( \omega \) be a \( k \)-stress on \( K \). For any \( r > 0 \), embed \( \mathbb{R}^d \) into \( \mathbb{R}^{d+1} \), \( x \mapsto (x, r) \), and let \( S^d_r \) be a \( d \)-dimensional sphere in \( \mathbb{R}^{d+1} \) with radius \( r \) and centered at the origin \( O \) of \( \mathbb{R}^{d+1} \). By a radial projection (from the center of \( S^d_r \) of \( \mathbb{R}^d \) onto \( S^d_r \) (see (2.4) and the nearby discussion), it induces a \( k \)-stress in \( S^d_r \). Taking \( r \to \infty \) in the spherical case, we prove that \( \sum_{F \in K^{k-1}} \omega(F) g_F(P, Q) \) of (2.14) is independent of \( P \) and \( Q \) in the Euclidean case as well. This completes the proof. \( \square \)

Particularly for a general \((\mathbf{A}, \alpha)\), we have the following definition.

**Definition 2.16** Let \((\mathbf{A}, \alpha)\) be as in (1.1) where \( \alpha \) is a 1-stress on \( \mathbf{A} \), but \( \mathbf{A} \) is more general and may contain \( m \geq n + 2 \) points in general position in \( M^n \). Also let \( \alpha^k \) be a \( k \)-stress on \( \mathbf{A} \) as in Definition 2.5. Then by Theorem 2.13 we define a sequence of invariants \( c_1(\alpha^1), \ldots, c_{n+1}(\alpha^{n+1}) \) for \((\mathbf{A}, \alpha)\). Also set \( c_0(\alpha^0) = 1 \). For the non-Euclidean case by Theorem 2.13 we have

\[
c_k(\alpha^k) = \kappa(k + 1)(k - 1)! \sum_{F \subseteq \mathbf{A}, \dim(F) = k-1} \left( \prod_{A_s \in F} \alpha_s \right) \|F\| V_{k-1}(F).
\] (2.16)

**Remark 2.17** When \( \mathbf{A} \) contains exactly \( m = n + 2 \) points, for the non-Euclidean case, by (2.16), \( c_{n+1}(\alpha^{n+1}) \) vanishes unless \( \mathbf{A} \) is not confined in any open half sphere in the spherical case (see Example 2.14 for a case that it does not vanish); \( c_{n+1}(\alpha^{n+1}) \) also vanishes in the Euclidean case as a limit of the spherical case. Using the special case of \( m = n + 2 \), with the proof skipped, the same conclusion can be proved for \( m > n + 2 \) or even when \((\mathbf{A}, \alpha)\) is distributed in a continuous manner.

Note that if \( \alpha_i > 0 \) for all \( i \), which can only happen in the spherical case, then all \( c_k(\alpha^k) \) are positive, including \( c_{n+1}(\alpha^{n+1}) \).

**Corollary 2.18** Let \( \alpha^k \) be the \( k \)-stress on \( \mathbf{A} \) as above in Definition 2.16. Then for \( k \leq n \) and any \( i \neq j \), we also have

\[
\sum_{F \subseteq \mathbf{A} \setminus \{A_i, A_j\}, \dim(F) = k-1} \alpha^k(F) g_F(A_i, A_j) = c_k(\alpha^k).
\] (2.17)
Proof Following essentially the same proof of Theorem 2.13, the non-Euclidean case can be proved by applying Lemma 2.10 and (2.10), and the Euclidean case can be proved by treating it as a limit of the spherical case.

2.7 Proof of Lemma 2.10

The proof of Lemma 2.10 is mainly computational, the reader not interested in technicalities can skip this section for now without missing the flow of the paper.

Let $G$ be a $k$-simplex in $S^d$ or $H^d$ of constant curvature $\kappa$ and $B_1, \ldots, B_{k+1}$ be the vertices. The main idea to prove Lemma 2.10 is to compute $dV_k(G)$ in two different ways, and to compare the coefficients of the outcomes. One way is to expand $dV_k(G)$ as a linear sum of $\vec{B}_i B_j$ by using (2.9), and the other is to expand $dV_k(G)$ using the Schlafli differential formula (2.1).

Let $E_i$ be the $k-1$-face $\{B_i\}$ of $G$, $F_{ij}$ the $(k-2)$-face that can be described as an intersection $E_i \cap E_j$, and $\theta_{ij}$ the dihedral angle between $E_i$ and $E_j$. Also let $e_i$ be the inward unit normal to $G$ along the $(k-1)$-face $E_i$, and $f_{ij}$ the inward unit normal to $E_i$ along the $(k-2)$-face $F_{ij}$. It is obvious that the angle between $e_i$ and $f_{ij}$ is $\pi/2$, and therefore $e_i \cdot f_{ij} = 0$. For $i \neq j$, recall (2.2) that $d\theta_{ij} = -e_i \cdot df_{ij} - e_j \cdot df_{ji}$. As $e_i \cdot f_{ij} = 0$, we have $e_i \cdot df_{ij} + f_{ij} \cdot de_i = 0$, therefore

$$d\theta_{ij} = f_{ij} \cdot de_i + f_{ji} \cdot de_j,$$

which will be very useful in the proof of Lemma 2.10.

Recall (2.10) and the nearby interpretation of $R^j_i(G)$, it is easy to see that

$$\sum_s R^j_i(G)B_s$$

is the altitude vector for point $B_i$ with respect to the linear span of $B_1, \ldots, \hat{B}_i, \ldots, B_{k+1}$. As the norm of the altitude vector is $\|G\|\|E_i\|$, normalizing the vector we have

$$e_i = \frac{\|E_i\|}{\|G\|} \sum_s R^j_i(G)B_s.$$

And similarly, because by definition for $i \neq j$ we have $R^j_i(E_i) = 0$, thus

$$f_{ij} = \frac{\|F_{ij}\|}{\|E_i\|} \sum_s R^j_s(E_i)B_s.$$

Proof of Lemma 2.10 For $k \geq 2$, by (2.11), Lemma 2.10 is equivalent to proving

$$\kappa \cdot \|F_{12}\| g_{F_{12}}(B_1, B_2) = k(k-2)! \sum_{i \neq j} R^j_1(G) R^j_2(E_i) \|F_{ij}\| V_{k-2}(F_{ij}).$$

In the rest of the proof, we assume the vertices of $G$ are moving in such a manner that $B_2$ is the only vertex that is moving, and $\vec{B}_i B_j^2$ are preserved for $3 \leq i \leq k + 1$. Under this assumption, $B_1 \cdot dB_2$ is the only non-zero term of the form $B_i \cdot dB_j$ for $1 \leq i, j \leq k + 1$. 

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2.8 Proof of Theorem 1.4

The invariant $c_k(\alpha^k)$ (Definition 2.16) of $(A, \alpha)$ plays a role in both the formulation and proof of Theorem 1.4.

Assume $A(t)$ is in $M^d$ with $d \geq n + 1$. Let $A_0(t)$ in $M^d$ be the mirror reflection of $A_1(t)$ through a lower dimensional $M^n$ that contains points $A_2(t), \ldots, A_{n+2}(t)$. It is not hard to see that if $A(t)$ varies smoothly over $t$, then $A_0(t)$ varies smoothly as well. We denote $A(t) \cup \{A_0(t)\}$ by $A^*(t)$. By adding $A_0(t)$, we treat $A^*(t)$ as a degenerate $(n+2)$-simplex in $M^d$. So for each $t \geq 1$, $\alpha_i$ can be extended to a continuous function $\alpha_i(t)$ with $\alpha_i(0) = \alpha_i$, such that $\{\beta_0(t), \ldots, \beta_{n+2}(t)\}$ is a 1-stress on $A^*(t)$, where $\beta_0(t) = \beta_1(t) = \frac{1}{2} \alpha_i(t)$ and $\beta_i(t) = \alpha_i(t)$ for $2 \leq i \leq n+2$.

Denote $\{\alpha_1(t), \ldots, \alpha_{n+2}(t)\}$ by $\alpha_t$ and $\{\beta_0(t), \ldots, \beta_{n+2}(t)\}$ by $\beta_t$. So Proposition 2.7 can be applied on $(A^*(t), \beta_t)$ for $t \geq 0$. Since $A_0(0)$ is the reflection of $A_1(t)$, so
for each $k$-face of $A^*(t)$ that contains point $A_1(t)$ but not $A_0(t)$, there is a congruent $k$-face that contains the same set of vertices except with $A_1(t)$ replaced by $A_0(t)$. These are the key ideas to prove the following result, a final step before proving Theorem 1.4.

**Proposition 2.19** Let $A(t)$, $\alpha_t$ and $(A^*(t), \beta_t)$ be as above. Assume $A(t)$ varies smoothly for $t \geq 0$ in $M^d$ with $d \geq n + 1$. If $c_{k-1}(\alpha^{k-1}) \neq 0$ and $A_0(t) \neq A_1(t)$ for small $t > 0$, then for small $t > 0$, for the non-Euclidean case

$$2 \cdot k! \sum_{G \subseteq A(t), \dim(G) = k} \left( \prod_{A_s(t) \subseteq G} \alpha_s(t) \right) \|G\| dV_k(G) \sim -\frac{1}{4} \alpha_t^2 c_{k-1}(\alpha^{k-1}) d\overrightarrow{A_0A_1}^2, \quad (2.23)$$

and for the Euclidean case

$$2 \cdot (k!)^2 \sum_{G \subseteq A^*(t), \dim(G) = k} \left( \prod_{A_s(t) \subseteq G} \alpha_s(t) \right) V_k(G) dV_k(G) \sim -\frac{1}{4} \alpha_t^2 c_{k-1}(\alpha^{k-1}) d\overrightarrow{A_0A_1}^2. \quad (2.24)$$

**Proof** For the non-Euclidean case, for $(A^*(t), \beta_t)$, by Definition 2.5 let $\beta_t^{k+1}$ be the $(k+1)$-stress on the $k$-faces $G$ of $A^*(t)$ for $k \geq 0$ such that

$$\beta_t^{k+1}(G) := \left( \prod_{A_s(t) \subseteq G} \beta_s(t) \right) \|G\|.$$ 

Also set $\beta_t^0(\emptyset) = 1$. Now switch the index from $k + 1$ to $k - 1$ if $k \geq 1$, by Theorem 2.13, $\beta_t^{k-1}$ has an associated invariant $c_{k-1}(\alpha^{k-1})$. By applying (2.15), one sees that as $t \to 0$, $c_{k-1}(\beta_t^{k-1})$ converges to $c_{k-1}(\alpha^{k-1})$. So if $c_{k-1}(\alpha^{k-1}) \neq 0$, then for small $t > 0$,

$$2 \cdot k! \sum_{G \subseteq A(t), \dim(G) = k} \left( \prod_{A_s(t) \subseteq G} \alpha_s(t) \right) \|G\| dV_k(G)$$

$$= 2 \cdot k! \sum_{G \subseteq A^*(t), \dim(G) = k} \beta_t^{k+1}(G) dV_k(G)$$

$$= -2 \cdot k! \sum_{G \subseteq A^*(t), \dim(G) = k} \beta_t^{k+1}(G) dV_k(G)$$

$$\sim -2 \cdot k! \sum_{G \subseteq A^*(t), \dim(G) = k} \beta_t^{k+1}(G) dV_k(G)$$

$$= -\beta_t^{0}(A_0, A_1) \sum_{F \subseteq A^*(t) \setminus \{A_0(t), A_1(t)\}, \dim(F) = k - 2} \beta_t^{k-1}(F) g_F(A_0(t), A_1(t)) d\overrightarrow{A_0A_1}^2$$
\[ \begin{align*}
\beta_0(t) & = -\beta_0(t)\beta_1(t)c_{k-1}(\beta_t^{k-1}) \| A_0A_1 \|^2 \\
& \sim -\frac{1}{4} \alpha_t^2 c_{k-1}(\alpha_t^{k-1}) \| A_0A_1 \|^2,
\end{align*} \]

where the second step is by applying Proposition 2.7 on \( A^*(t) \); the third step is because \( \| G \| dV_k(G) \sim \| G \| \| A_0A_1 \|^2 \) for small \( t > 0 \) when \( \{ A_0(t), A_1(t) \} \subset G \); the fourth step is because (2.9); the fifth step is by applying Corollary 2.18 on \( A^*(t) \); and the last step is because \( \beta_0(t) = \beta_1(t) = \frac{1}{2} \alpha_1(t) \) and \( c_{k-1}(\beta_t^{k-1}) \) converges to \( c_{k-1}(\alpha_t^{k-1}) \). This completes the proof of (2.23).

For the Euclidean case, the only change we need to make, is by Definition 2.5 let \( \beta_t^{k-1} \) be the \((k+1)\)-stress on the \(k\)-faces \( G \) of \( A^*(t) \) such that

\[ \beta_t^{k-1}(G) := \left( \prod_{A_s(t) \in G} \beta_s(t) \right) k! V_k(G). \]

Also set \( \beta_0^0(\emptyset) = 1 \). Following the same steps above, we then prove (2.24).

Note that the proof of Proposition 2.19 starts with a \((k+1)\)-stress on \( A \), but it is the invariant \( c_{k-1}(\alpha_t^{k-1}) \) of a \((k-1)\)-stress, rather than the invariant \( c_{k+1}(\alpha_t^{k+1}) \), plays a role in both the formulation and proof. The same applies to the following proof of Theorem 1.4, which is a direct consequence of Proposition 2.19.

**Proof of Theorem 1.4** We need only to prove the non-Euclidean case, as the Euclidean case can be proved similarly. Also assume \( c_{k-1}(\alpha_t^{k-1}) > 0 \), as the case \( c_{k-1}(\alpha_t^{k-1}) < 0 \) is similar.

Let \( A_0(t) \) and \( \alpha_t = \{ \alpha_1(t), \ldots, \alpha_{n+2}(t) \} \) be the same as in Proposition 2.19. For \( G_{n,k} \), now assume that the vertices are not always confined in a lower dimensional \( S^n \) or \( \mathbb{H}^n \) for small \( t > 0 \), then there exists arbitrarily small \( t > 0 \) such that \( A_0(t) \neq A_1(t) \). Therefore there exist arbitrarily small \( t_1 \) and \( t_2 \) with \( 0 \leq t_1 < t_2 \), such that \( A_0(t_1) = A_1(t_1) \), but \( A_0(t) \neq A_1(t) \) for any \( t \) with \( t_1 < t < t_2 \). So \( \alpha_t \) is a \( 1\)-stress on \( A(t_1) \), then \( c_{k-1}(\alpha_t^{k-1}) \) can be properly defined by Definition 2.16. As \( c_{k-1}(\alpha_t^{k-1}) > 0 \), so \( t_1 \) can be small enough such that \( c_{k-1}(\alpha_t^{k-1}) > 0 \) as well.

Then we can apply Proposition 2.19 to \( A(t) \) near \( t = t_1 \), so for small \( t - t_1 > 0 \),

\[ 2 \cdot k! \sum_{F \subseteq A(t)} \left( \prod_{A_s(t) \in F} \alpha_s(t) \right) \| F \| dV_k(F) \sim -\frac{1}{4} \alpha_1(t_1)^2 \cdot c_{k-1}(\alpha_t^{k-1}) \| A_0A_1 \|^2. \]

However, for each \( k \)-face \( F \) of \( G_{n,k} \), by Definition 1.2 we have

\[ \left( \prod_{A_s(t) \in F} \alpha_s \right) dV_k(F) \geq 0 \]

---

3 It is possible to still have infinitely many small \( t > 0 \) such that \( A_0(t) = A_1(t) \), e.g., at the zeros of the function \( e^{-1/t^2} \sin(1/t) \) near \( t = 0 \), so we should be cautious about this kind of scenario. However, this is not a concern if \( A(t) \) is real analytic.
for $t \geq t_1 \geq 0$, which is a contradiction to (2.25). So the vertices of $G_{n,k}$ must be confined in a lower dimensional $\mathbb{S}^n$ or $\mathbb{H}^n$ for small $t \geq 0$. Applying Theorem 1.3, we then show that $V_k(F)$ must be preserved for small $t \geq 0$. This completes the proof. □

To see if Theorem 1.4 can be improved to claim that $G_{n,k}$ or $G'_{n,k}$ is rigid in $M^d$, check Remark 2.31. More results and examples for cases $k = 1$ and 2 are given next.

### 2.9 Tensegrity Framework $G_{n,1}$

Theorem 1.4 for $k = 1$ states that $G_{n,1}$ is rigid in $\mathbb{R}^d$ for $d \geq n + 1$. While this result is not new, our theorem provides a new interpretation by using $c_0(\alpha^0) = 1 > 0$. Bezdek and Connelly [3] proved a stronger result that $G_{n,1}$ is globally rigid in $\mathbb{R}^d$ for any $d \geq n$, and with a little modification the spherical case can be proved as well. However, the methodology they used cannot be directly applied to prove the hyperbolic case, which mainly because the metric in $\mathbb{R}^{d,1}$ is not positive definite. To our knowledge, the following result we obtained in hyperbolic space is new.

**Theorem 2.20** $G_{n,1}$ is globally rigid in $\mathbb{H}^d$ for any $d \geq n$.

**Proof** Recall (1.1) that $\sum_{i=1}^{n+2} \alpha_i A_i = 0$. Assume $\alpha_1, \ldots, \alpha_m > 0$ and $\alpha_{m+1}, \ldots, \alpha_{n+2} < 0$. Let $B_1, \ldots, B_{n+2}$ be $n + 2$ points in $\mathbb{H}^d$ that satisfy the constraints of $G_{n,1}$. Namely,$\alpha_i\alpha_j B_i B_j^{-2} \leq 0$ for $i \neq j$, which is the same as $\alpha_i \alpha_j B_i \cdot B_j \leq 0$, and so there is $f_1 > 0$ such that $f_1 \sum_{i=1}^{m} \alpha_i B_i$ is a point in $\mathbb{H}^d$; similarly, there is $f_2 > 0$ such that $-f_2 \sum_{i=m+1}^{n+2} \alpha_i B_i$ is a point in $\mathbb{H}^d$. Denote these two points by $D_1$ and $D_2$, and let $\beta_i = f_1 \alpha_i$ if $1 \leq i \leq m$ and $\beta_i = f_2 \alpha_i$ if $m + 1 \leq i \leq n + 2$. As $D_2 D_1^{-2} \geq 0$, so

$$0 \leq (D_1 - D_2)^2 = \left( \sum \beta_i B_i \right)^2 \leq \left( \sum \beta_i A_i \right)^2 = \left( f_1 \sum_{i=1}^{m} \alpha_i A_i + f_2 \sum_{i=m+1}^{n+2} \alpha_i A_i \right)^2 \leq 0,$$

where the third step is because $\alpha_i \alpha_j B_i \cdot B_j \leq \alpha_i \alpha_j A_i \cdot A_j$ and $f_1, f_2 > 0$, so $\beta_i \beta_j B_i \cdot B_j \leq \beta_i \beta_j A_i \cdot A_j$; the fifth step is because $\sum_{i=1}^{m+2} \alpha_i A_i = 0$; the last step is because $\sum_{i=1}^{m} \alpha_i A_i$ is a multiple of a point in $\mathbb{H}^d$, so $(\sum_{i=1}^{m} \alpha_i A_i)^2 < 0$.

Then $B_i \cdot B_j = A_i \cdot A_j$ holds for any $i \neq j$, and so $G_{n,1}$ is globally rigid in $\mathbb{H}^d$. □

### 2.10 2-Tensegrity Frameworks $G_{n,2}$ and $G'_{n,2}$

In Theorem 1.4 the sign of $c_1(\alpha^1)$ plays an important role in the case $k = 2$. In this section, we give a geometric interpretation of $c_1(\alpha^1) = 0$, which is amazingly simple as shown below.
Proposition 2.21 For the spherical (resp. hyperbolic) case, \( c_1(\alpha^1) = 0 \) if and only if \( A_1, \ldots, A_{n+2} \) are affinely dependent in \( \mathbb{R}^{n+1} \) (resp. \( \mathbb{R}^n \)). For the Euclidean case, \( c_1(\alpha^1) = 0 \) if and only if \( A_1, \ldots, A_{n+2} \) lie on an \( (n-1) \)-dimensional sphere in \( \mathbb{R}^n \).

Proof For the spherical (resp. hyperbolic) case, by (2.16) we have \( c_1(\alpha^1) = \kappa \cdot 2 \sum \alpha_i \). Since \( \sum \alpha_i A_i = 0 \), so \( c_1(\alpha^1) = 0 \) (the same as \( \sum \alpha_i = 0 \)) if and only if \( A_1, \ldots, A_{n+2} \) are affinely dependent.

For the Euclidean case, let \( S_1^{n-1} \) be an \( (n-1) \)-dimensional sphere in \( \mathbb{R}^n \) that contains points \( A_2, \ldots, A_{n+2} \); \( O_1 \) be the center of the sphere and \( r \) be the radius. From (2.13) we have \( g_{A_i}(P, Q) = \overrightarrow{PA_i} \cdot \overrightarrow{QA_i} \) in \( \mathbb{R}^n \), then by choosing \( P = Q = O_1 \) in (2.14) we have \( c_1(\alpha^1) = \sum \alpha_i O_1 A_i^2 \). Since \( \sum \alpha_i = 0 \), so \( c_1(\alpha^1) = \alpha_1(O_1 A_1^2 - r^2) \). Therefore \( c_1(\alpha^1) = 0 \) if and only if \( A_1 \) is on \( S_1^{n-1} \).

To show the geometric properties of \( G_{n,2} \) and \( G'_{n,2} \), we give some examples for \( n = 2 \). Without loss of generality, assume \( \alpha_1 > 0 \) in the following examples.

Example 2.22 In Fig. 1, assume \( A_1, A_2, A_3 \) and \( A_4 \) are the vertices of a convex quadrilateral in a Euclidean plane in \( \mathbb{R}^3 \). Topologically, it is hard to tell apart \( G_{2,2} \) from \( G'_{2,2} \), because for both frameworks, the quadrilateral is double covered by volume constraints with opposite signs. So how to determine that which one of them (\( G_{2,2} \) or \( G'_{2,2} \)) preserves the volumes of 2-faces in \( \mathbb{R}^3 \) for small \( t \geq 0 \)? In Fig. 1(a), \( A_1 \) is inside the dotted circle where points \( A_2, A_3 \) and \( A_4 \) lie on; and in Fig. 1(b) \( A_1 \) is outside.

Let \( O_1 \) be the center of the circle and \( r \) be the radius, then \( c_1(\alpha^1) = \sum \alpha_i O_1 A_i^2 = \alpha_1(O_1 A_1^2 - r^2) \). As \( \alpha_1 > 0 \), so in (a), \( c_1(\alpha^1) < 0 \) and therefore by Theorem 1.4 \( G_{2,2} \) preserves the volumes of 2-faces in \( \mathbb{R}^3 \) for small \( t \geq 0 \); in (b), \( c_1(\alpha^1) > 0 \) and therefore \( G_{2,2} \) preserves the volumes of 2-faces in \( \mathbb{R}^3 \) for small \( t \geq 0 \).

Example 2.23 Figure 2 is the hyperbolic version, where the dotted circle is the intersection between \( \mathbb{H}^2 \) and a 2-dimensional plane in \( \mathbb{H}^{2,1} \) where \( A_2, A_3 \) and \( A_4 \) lie on, with \( A_1 \) “outside” of the dotted circle (but in general the intersection need not be a closed “circle”). This implies \( A_1 \) and the origin \( O \) are on the opposite sides of the 2-dimensional plane. As \( \kappa < 0, \alpha_1 > 0 \) and \( c_1(\alpha^1) = \kappa \cdot 2 \sum \alpha_i \) by (2.16), so \( c_1(\alpha^1) > 0 \), and therefore \( G_{2,2} \) preserves the volumes of 2-faces in \( \mathbb{H}^3 \) for small \( t \geq 0 \).
Fig. 2 A hyperbolic 2-tensegrity framework in $H^2$, where the dotted circle is the intersection between $H^2$ and a 2-dimensional plane in $R^2$, where $A_2, A_3$ and $A_4$ lie on, with $A_1$ outside of the dotted circle, which implies $c_1(\alpha^1) > 0$.

From Theorem 1.4 and Proposition 2.21 we come up with the following example of “four points on a circle”, which is rather interesting.

Example 2.24 Still in Fig. 1, with four points that are initially in convex position in a 2-dimensional plane in $R^3$, but now we constrain all four 2-faces to preserve the volumes while the vertices vary smoothly in $R^3$. In order for the four points to form a non-degenerate 3-simplex in $R^3$, by Theorem 1.4 this can only happen when $c_1(\alpha^1) = 0$. Namely, they have to be confined in a plane first until they move on to a common circle, and only from this circle they can be lifted to form a non-degenerate 3-simplex in $R^3$. For the same set up in $H^3$ (Fig. 2), the “critical position” for the four points to be lifted from $H^2$ to form a non-degenerate 3-simplex in $H^3$ is also when $c_1(\alpha^1) = 0$, namely, when the points are affinely dependent. The spherical case is also similar, where the “critical position” of $c_1(\alpha^1) = 0$ is when the four points are on a small circle in $S^2$.

2.11 A Positive Definite Kernel on the Hyperbolic Space

In this section we discuss the positive definiteness of $g_F$ (Definition 2.8) on the hyperbolic space $H^d$. Recall that for a set $X$, a symmetric function $L : X \times X \to R$ is a positive definite kernel on $X$ if for any $m \in N$ and $x_1, \ldots, x_m \in X$, the matrix $(L(x_i, x_j))_{1 \leq i,j \leq m}$ is positive semi-definite. This is also equivalent to having all the principal minors of the matrix $(L(x_i, x_j))_{1 \leq i,j \leq m}$ non-negative.

In $R^d$, by (2.13), $g_B(P, Q) = \overrightarrow{PB} \cdot \overrightarrow{QB}$, thus $g_B$ is a positive definite kernel on $R^d$. In $S^d$ or $H^d$ of constant curvature $\kappa$, by Corollary 2.12,

$$g_B(P, Q) = \frac{2}{1 + \kappa P \cdot Q} \overrightarrow{PB} \cdot \overrightarrow{QB},$$

and we have the following analogue for hyperbolic space $H^1$.

Theorem 2.25 Let $B \in H^1$, then $g_B$ is a positive definite kernel on $H^1$.

Proof To prove $g_B$ is a positive definite kernel on $H^1$, it suffices to show that for any $m \in N$ and $P_1, \ldots, P_m \in H^1$,

$$\det(g_B(P_i, P_j))_{1 \leq i,j \leq m} \geq 0.$$
Pick up a direction in $\mathbb{H}^1$, denote the geodesic distance between $P_i$ and $B$ by $r_i$ if $P_i$ is at the “right” side of $B$, and by $-r_i$ if $P_i$ is at the “left” side of $B$. Then by (2.26),

$$\det(g_B(P_i, P_j))_{1 \leq i, j \leq m} = \det\left(\frac{2}{1 - P_i \cdot P_j \overrightarrow{P_i B} \cdot \overrightarrow{P_j B}}\right)$$

$$= \det\left(\frac{2}{1 + \cosh(r_i - r_j)}(\cosh r_i + \cosh r_j - \cosh(r_i - r_j) - 1)\right)$$

$$= \det\left(\frac{2}{2 \cosh^2 \frac{r_i - r_j}{2}}\left(2 \cosh\frac{r_i + r_j}{2} \cosh\frac{r_i - r_j}{2} - 2 \cosh^2\frac{r_i - r_j}{2}\right)\right)$$

$$= \det\left(\frac{4}{\cosh^2 \frac{r_i - r_j}{2}} \sinh\frac{r_i}{2} \sinh\frac{r_j}{2}\right) = 2^m\left(\prod_i (e^{r_i} - 1)\right)^2 \det\left(\frac{1}{e^{r_i} + e^{r_j}}\right)_{1 \leq i, j \leq m}.$$

By Cauchy’s determinant identity, which states that

$$\det\left(\frac{1}{x_i + y_j}\right)_{1 \leq i, j \leq n} = \frac{\prod_{i < j}(x_j - x_i)(y_j - y_i)}{\prod_{i,j}(x_i + y_j)},$$

we have

$$\det(g_B(P_i, P_j))_{1 \leq i, j \leq m} = 2^m\left(\prod_i (e^{r_i} - 1)\right)^2 \cdot \frac{\left(\prod_{i < j}(e^{r_j} - e^{r_i})\right)^2}{\prod_{i,j}(e^{r_i} + e^{r_j})} \geq 0,$$

which proves (2.27) and finishes the proof. \hfill\qed

**Remark 2.26** Following a similar proof, we can show that $g_B$ is not a positive definite kernel on $\mathbb{S}^1$ for any point $B$ in $\mathbb{S}^1$.

With Theorem 2.25 proved, it seems natural for us to conjecture the following.

**Conjecture 2.27** Let $B \in \mathbb{H}^d$ and $d \geq 2$, then $g_B$ is a positive definite kernel on $\mathbb{H}^d$.

Next we show that if $F$ is a $k$-polytope in $\mathbb{R}^d$, then $g_F$ is a positive definite kernel on $\mathbb{R}^d$. Let $w_P$ (resp. $w_Q$) be the altitude vector for the point $P$ (resp. $Q$) with respect to the affine span of $F$, then it is not hard to show that $g_F(P, Q) = k!V_k(F)w_P \cdot w_Q$, and the positive definiteness of $g_F$ immediately follows.

For the hyperbolic case, we have the following conjecture.

**Conjecture 2.28** Let $F$ be a $k$-polytope in $\mathbb{H}^d$, then $g_F$ is a positive definite kernel on $\mathbb{H}^d$.

### 2.12 Related Questions and a Counterexample

Once Theorems 1.3 and 1.4 are proved, one question naturally arises: Under the same condition, is the motion also rigid?

This relates to a question raised by Connelly and others:
Question 2.29 For \( r \geq 2 \), do the volumes of all \( r \)-faces of an \( n \)-simplex in \( M^n \) determine the \( n \)-simplex up to congruence?

Question 2.29 was initially posed in the Euclidean space only, but in the context of this paper, we are also interested in the spherical and hyperbolic case, particularly when continuous motion is involved. The case \( r = n - 2 \) must be classical, and various counterexamples were constructed for the Euclidean case (see [10,12]). And following an idea in Mohar and Rivin [12], we give a construction for all \( r \geq 2 \) at the end of this section, for both Euclidean and non-Euclidean cases.

However, to our knowledge the following continuous analogue of Question 2.29 for case \( r = n - 2 \) is still open, and may very likely to have an affirmative answer.

Question 2.30 For an \( n \)-simplex in \( M^n \) with \( n \geq 4 \), if a continuous motion preserves the volumes of all \((n - 2)\)-faces of the \( n \)-simplex, then is the motion rigid?

Note that an \( n \)-simplex has the same number of edges and \((n - 2)\)-faces, and up to congruence is determined by its edge lengths, so the question is natural. As the volumes of \((n - 2)\)-faces are algebraically independent over the edge lengths (see, for example, [12]), Question 2.30 should hold an affirmative answer for “almost all” configurations. While in this paper we do not try to solve Question 2.30, which is mutually independent of our rigidity theorem, an affirmative answer to Question 2.30 will further improve our main theorems.

Remark 2.31 If Question 2.30 holds an affirmative answer for all non-degenerate simplices, then Theorems 1.3 and 1.4 can be improved to claim that \( G_{n,k} \) and \( G'_{n,k} \) are rigid for \( k \leq n - 2 \). If Question 2.30 also holds an affirmative answer for degenerate simplices with non-degenerate codimension 1 faces, then \( G_{n,n-1} \) and \( G'_{n,n-1} \) are rigid as well. However, when \( n \geq 2 \), \( G_{n,n} \) and \( G'_{n,n} \) are never rigid, as the number of volume constraints is less than the degree of freedom of \( A \) up to congruence.

Now we give our construction of a counterexample to Question 2.29 for general \( r \geq 2 \), essentially following an idea in [12]. Let \( \Delta_\epsilon(t) \) be an \( n \)-simplex in \( M^n \) whose all sides are equal to a small \( \epsilon \) except for one side whose length is \( \epsilon \cdot t \). It can be shown that for the Euclidean case: first, \( t \) may take any positive value smaller than \( \sqrt{\frac{2n}{n - 1}} \); and second, for any of the \( r \)-faces that contains the edge with length \( \epsilon \cdot t \), the square of its volume is a quadratic function of \( t^2 \), and it peaks when \( t = t_0 := \sqrt{\frac{r}{r - 1}} \).

As \( t_0 \leq \sqrt{2} \leq \sqrt{\frac{2n}{n - 1}} \) for \( r \geq 2 \), so for sufficiently small \( \epsilon \), \( \Delta_\epsilon(t_0) \) is obtainable for both Euclidean and non-Euclidean cases. By properly choosing two close values \( t_1 \) and \( t_2 \) satisfying \( t_1 < t_0 < t_2 \), \( \Delta_\epsilon(t_1) \) and \( \Delta_\epsilon(t_2) \) can have the same volumes on all the corresponding \( r \)-faces.

3 Characteristic Polynomial of \((A, \alpha)\)

For the degenerate \((n + 1)\) simplex \( A \) and a 1-stress \( \alpha \) (see (1.1)), recall that an invariant \( c_{k-1}(\alpha^{k-1}) \) (Definition 2.16) plays an important role in a rigidity property of the \( k \)-faces of \( A \) in Theorem 1.4. To analyze the relationship between these rigidity properties
of different dimensions \( k \), we introduce a notion of characteristic polynomial of \((\mathbf{A}, \alpha)\) by defining

\[
    f(x) = \sum_{i=0}^{n+1} (-1)^i c_i(\alpha^i) x^{n+1-i}.
\]

Our main result of \( f(x) \) is Theorem 3.4, which shows that the roots of \( f(x) \) are real for the Euclidean case, and gives a way to count the number of positive roots.

### 3.1 Properties of the Characteristic Polynomial

In this section, let \((\mathbf{A}, \alpha)\) be as in (1.1), but for the spherical case we assume \( \mathbf{A} \) is confined in an open half sphere. By Remark 2.17, we have \( c_{n+1}(\alpha^{n+1}) = 0 \). Let \( \{\lambda_1, \ldots, \lambda_n\} \), with no particular order, be the rest roots of \( f(x) \) besides a 0.

For a \( k \)-simplex \( F \) (as opposed to a more general \( k \)-polytope) and two points \( P \) and \( Q \) in \( M^d \), instead of using \( g_F(P, Q) \) (Definition 2.8) sometimes it is more convenient to use \( d_F(P, Q) \), also a new notation introduced in this paper.

**Definition 3.1** For a \( k \)-simplex \( F \) in \( M^d \), define

\[
d_F(P, Q) = \frac{k!}{(k+2)!} V_k(\mathbf{F}) \left( \prod_{\mathbf{A}_s \in \mathbf{F}} \alpha_s \right) d_F(P, Q), \quad (3.1)
\]

which is independent of the choice of \( P \) and \( Q \).

In \( \mathbb{R}^d \), we give a useful formula for \( d_F(P, Q) \): Let \( B_1, \ldots, B_{k+1} \) be the vertices of \( F \) in \( \mathbb{R}^d \), then

\[
d_F(P, Q) = (PB_1 \wedge \cdots \wedge PB_{k+1}) \cdot (QB_1 \wedge \cdots \wedge QB_{k+1}). \quad (3.2)
\]

When \( F \) is degenerate, the right side of the formula is always well defined, so we can also extend the definition of \( d_F \) accordingly.

**Proof of (3.2)** Denote by \( \widehat{F} \) the \((k+2)\)-dimensional simplex, which is the join of \( F \) with a line segment \( PQ \), and let \( \theta_F \) be the dihedral angle at face \( F \). Also let \( w_P \) (resp. \( w_Q \)) be the altitude vector for point \( P \) (resp. \( Q \)) with respect to the affine span of \( F \). Combine Definitions 3.1 and 2.8, we have

\[
d_F(P, Q) = k! V_k(F) (k+2)! \frac{dV_{k+2}({\widehat{F}})}{d\theta_F}.
\]

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Note that \((k + 2)! V_{k+2}(\hat{F}) = k!V_k(F) \|w_P\| \cdot \|w_Q\| \cdot \sin \theta_F\), therefore

\[
d_f(P, Q) = (k!V_k(F))^2 \|w_P\| \cdot \|w_Q\| \cdot \cos \theta_F = (k!V_k(F))^2 w_P \cdot w_Q
\]

\[
= (PB_1 \wedge \cdots \wedge PB_{k+1}) \cdot (QB_1 \wedge \cdots \wedge QB_{k+1}). \tag{3.2}
\]

For the Euclidean case, without loss of generality, we coordinate the vector \(\mathbb{R}^n\) for \(A\) in the following. Let \(B\) be an \((n + 1) \times n\) matrix whose \(i\)-th row is the row vector \(\overline{A_{n+2}A_i}\) for \(i \leq n + 1\), and \(D = \text{diag}(\alpha_1, \ldots, \alpha_{n+1})\) be a diagonal matrix.

**Lemma 3.3** The characteristic polynomials of matrix \(BB^T D\) and \(B^T DB\) are \(f(x)\) and \(f(x)/x\) respectively.

**Proof** The coefficient of \(x^{n+1-k}\) in the characteristic polynomial of \(BB^T D\) is \((-1)^k\) times the sum of all principal minors of \(BB^T D\) of order \(k\), which can be shown to be \((-1)^k c_k(a^n)\) by choosing \(P = Q = A_{n+2}\) in (3.1) and then applying (3.2). Therefore \(f(x)\) is the characteristic polynomial of \(BB^T D\). Let \(B_1\) be \(B\) and \(B_2\) be \(B^T D\). A well known property in linear algebra states that: If \(B_1\) is an \(m \times n\) matrix and \(B_2\) is an \(n \times m\) matrix, then the characteristic polynomial of \(B_1 B_2\) is \(x^{m-n}\) times the characteristic polynomial of \(B_2 B_1\). Therefore \(f(x)/x\) is the characteristic polynomial of \(B^T DB\). \(\Box\)

Now we have the following main property for \(f(x)\).

**Theorem 3.4** In the Euclidean case, the roots of \(f(x)\) are real. And if \(\{\alpha_1, \ldots, \alpha_{n+2}\}\) has \(s\) positive numbers, then \(f(x)\) has \(s - 1\) positive and \(n + 1 - s\) negative roots.

**Proof** As \(\{\lambda_1, \ldots, \lambda_n\}\) are the roots of \(f(x)/x\), which by Lemma 3.3 is the characteristic polynomial of a symmetric matrix \(B^T DB\), so all \(\lambda_i\) are real. In (3.1) by choosing \(P = A_1\) and \(Q = A_2\) for \(k = n\), we have \(c_n(a^n) = \left(\prod_{A_j \in F_{12}} a_s\right) d_{f_{12}}(A_1, A_2)\) where \(F_{12}\) is the \((n - 1)\)-face of \(A\) that without the vertices \(A_1\) and \(A_2\). Then by (3.2) we have \(c_n(a^n) \neq 0\), and thus all \(\lambda_i\) are also non-zero. So \(\{\lambda_1, \ldots, \lambda_n\}\) must have the same signs as an \(n\)-subset of the diagonal entries of \(D = \text{diag}(\alpha_1, \ldots, \alpha_{n+1})\). By symmetry, \(\{\lambda_1, \ldots, \lambda_n\}\) should have the same signs as an \(n+2\)-subset of \(\{\alpha_1, \ldots, \alpha_2, \ldots, \alpha_{n+2}\}\) for any \(j\) with \(1 \leq j \leq n + 2\). So \(\{\lambda_1, \ldots, \lambda_n\}\) must have \(s - 1\) positive and \(n + 1 - s\) negative roots. \(\Box\)

**Theorem 3.5** In the Euclidean case, \(f(x)/x\) has \(n\)-repeated roots if and only if \(\overline{A_i A_j} \cdot \overline{A_k A_l} = 0\) for all distinct numbers \(i, j, k\) and \(l\).

**Proof** The “only if” part. If \(\lambda_1 = \cdots = \lambda_n\), denote it by \(\lambda\). Then by Lemma 3.3, \(B^T DB = \lambda I_n\) where \(I_n\) is the \(n \times n\) identity matrix. Let \(B_1\) be an \((n + 1) \times (n + 1)\) matrix, whose first \(n\) columns are \(B\), and every entry on the last column is \(\sqrt{-\lambda/\alpha_{n+2}}\) (it is ok if it is not real). Easy to see that \(B_1^T DB_1 = \lambda I_{n+1}\). So \(B_1^T = \lambda B_1^{-1} D^{-1}\) and therefore \(B_1 B_1^T = \lambda D^{-1}\). So if \(i \neq j\), then the \((i, j)\)-th entry of \(B_1 B_1^T\) is 0, and therefore

\[
\overline{A_{n+2}A_i} \cdot \overline{A_{n+2}A_j} - \frac{\lambda}{\alpha_{n+2}} = 0. \tag{3.3}
\]
If \( i, j, k \) and \( n + 2 \) are distinct, replace “\( j \)” with “\( k \)” in (3.3) and subtract it from (3.3), then \( \overrightarrow{A_{n+2}A_i} \cdot \overrightarrow{A_jA_k} = 0 \). By symmetry, \( \overrightarrow{A_iA_j} \cdot \overrightarrow{A_kA_l} = 0 \) for all distinct \( i, j, k \) and \( l \).

The “if” part. Assume \( \overrightarrow{A_iA_j} \cdot \overrightarrow{A_kA_l} = 0 \) for all distinct \( i, j, k \) and \( l \). Then

\[
\overrightarrow{A_iA_j} \cdot \overrightarrow{A_iA_k} = \overrightarrow{A_iA_j} \cdot \overrightarrow{A_iA_l}.
\]

(3.4)

So for a fixed \( i \), \( \overrightarrow{A_iA_j} \cdot \overrightarrow{A_iA_k} \) is independent of \( j \) and \( k \), as long as \( i \), \( j \) and \( k \) are distinct.

We denote it by \( b_i \). Let \( B_2 \) be an \( n \times n \) matrix whose \( i \)-th row is vector \( \overrightarrow{A_{n+1}A_i} \) for \( i \leq n \), \( B_3 \) be an \( n \times n \) matrix whose \( i \)-th row is vector \( \overrightarrow{A_{n+2}A_i} \), and \( D_1 = \text{diag}(\alpha_1, \ldots, \alpha_n) \) be a diagonal matrix. By choosing \( P = A_{n+1} \) and \( Q = A_{n+2} \) in (3.1) and then applying (3.2), \( f(x)/x \) is the characteristic polynomial of \( B_2B_3^T D_1 \). Since \( \overrightarrow{A_{n+1}A_i} \cdot \overrightarrow{A_{n+2}A_j} \) is \( b_i \) when \( i = j \) and 0 when \( i \neq j \), then \( B_2B_3^T D_1 = \text{diag}(\alpha_i b_i)_{i \leq n} \). So \( \{\lambda_1, \ldots, \lambda_n\} \) is \( \{\alpha_i b_i\}_{i \leq n} \) in some order. By symmetry, any \( n \)-subset of \( \{\alpha_i b_i\}_{i \leq n+2} \) is \( \{\lambda_1, \ldots, \lambda_n\} \) in some order as well. So all \( \alpha_i b_i \) are equal, and therefore \( \lambda_1 = \cdots = \lambda_n \).

It is natural to ask if \( f(x) \) still has real roots in the non-Euclidean case, and some evidence suggests that the hyperbolic version of Theorem 3.4 might still hold. We numerically computed some examples for \( n = 2 \), and our tests for the hyperbolic space all have real roots, but the same test shows that the spherical version of Theorem 3.4 is not true.

**Conjecture 3.6** In the hyperbolic case, the roots of \( f(x) \) are real. And if \( \{\alpha_1, \ldots, \alpha_{n+2}\} \) has \( s \) positive numbers, then \( f(x) \) has 1 zero, \( s - 1 \) positive and \( n + 1 - s \) negative roots.

### 3.2 Generalization of Characteristic Polynomial

So far the characteristic polynomial \( f(x) \) is defined on degenerate \((n+1)\)-simplices only. To complete the discussion of \( f(x) \), with the proofs skipped, we loosely discuss a generalization of \( f(x) \) by showing that it can be naturally generalized to a general \((A, \alpha)\) (not necessarily finite) in \( M^n \). We start with a finite set.

Abuse of notation: Let \( A = \{A_1, \ldots, A_m\} \) be a set of \( m \) \((m \geq n + 2)\) points in \( M^n \) in general position, and \( \alpha = \{\alpha_1, \ldots, \alpha_m\} \) be a \( 1 \)-stress on \( A \). For this new \((A, \alpha)\) and each \( k \leq n + 1 \), by Definition 2.16 and Theorem 2.13 we have

\[
\sum_{F \subseteq A, \text{dim}(F) = k-1} \left( \prod_{A_s \in F} \alpha_s \right) d_F(P, Q) = c_k(\alpha^k),
\]

(3.5)

where the \( k \)-stress \( \alpha^k \) is by Definition 2.5 and \( c_k(\alpha^k) \) is an invariant independent of the choice of \( P \) and \( Q \); and for the non-Euclidean case,

\[
c_k(\alpha^k) = \kappa(k + 1)(k - 1)! \sum_{F \subseteq A, \text{dim}(F) = k-1} \left( \prod_{A_s \in F} \alpha_s \right) \|F\| V_{k-1}(F).
\]

(3.6)
Remark 3.7 The right-hand side of (3.6) is well defined when \( F \) is allowed to be degenerate. Even if the term \( V_{k-1}(F) \) is not well defined for some reason, say, \( F \) contains a pair of antipodal points in the spherical case, then the term \( \|F\| \) is always zero and will make their product zero. This suggests the possibility to define \( c_k(\alpha^k) \) in a much more general sense, say, when \((A, \alpha)\) is distributed in a continuous manner.

Define \( f(x) = \sum_{i=0}^{n+1} (-1)^i c_i(\alpha^i) x^{n+1-i} \) as the characteristic polynomial of \((A, \alpha)\) as before. We want to point out that \( c_{n+1}(\alpha^{n+1}) \) still vanishes unless \( A \) is not confined in any open half sphere in the spherical case.

Slightly modifying Lemma 3.3 and Theorem 3.4, we can prove that the roots of this newly generalized \( f(x) \) are still real in the Euclidean case. And again, we conjecture the same for the hyperbolic case.

Conjecture 3.8 In the hyperbolic case, the roots of the generalized \( f(x) \) are real.

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