Computational and Statistical Boundaries for Submatrix Localization in a Large Noisy Matrix

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Abstract

The interplay between computational efficiency and statistical accuracy in high-dimensional inference has drawn increasing attention in the literature. In this paper, we study computational and statistical boundaries for submatrix localization. Given one observation of a signal submatrix (of magnitude \(\lambda\) and size \(k_m \times k_n\)) contaminated with a noise matrix (of size \(m \times n\)), we establish two transition thresholds for the signal to noise \(\lambda/\sigma\) ratio in terms of \(m\), \(n\), \(k_m\), and \(k_n\). The first threshold, \(\text{SNR}_c\), corresponds to the computational boundary. Below this threshold, it is shown that no polynomial time algorithm can succeed in identifying the submatrix, under the hidden clique hypothesis. We introduce an adaptive linear time algorithm that identifies the submatrix with high probability when the signal strength is above the threshold \(\text{SNR}_c\). The second threshold, \(\text{SNR}_s\), captures the statistical boundary, below which no method can succeed with probability going to one in the minimax sense. The exhaustive search method successfully finds the submatrix above this threshold. The results show an interesting phenomenon that \(\text{SNR}_c\) is always significantly larger than \(\text{SNR}_s\), which implies an essential gap between statistical optimality and computational efficiency for submatrix localization.

**Keywords:** Computational boundary, detection, hidden clique hypothesis, lower bounds, minimax, signal to noise ratio, statistical boundary, submatrix localization.

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1 Introduction

The “signal + noise” model

\[ X = M + Z, \]  

where \( M \) is the signal of interest and \( Z \) is noise, is ubiquitous in statistics and is used in a wide range of applications. When \( M \) and \( Z \) are matrices, many interesting problems arise under a variety of structural assumptions on \( M \) and the distribution of \( Z \). Examples include sparse principal component analysis (PCA) (Vu and Lei, 2012; Berthet et al., 2013; Birnbaum et al., 2013; Cai et al., 2013, 2015), non-negative matrix factorization (Lee and Seung, 2001), non-negative PCA (Zass and Shashua, 2006; Montanari and Richard, 2014). Under the conventional statistical framework, one is looking for optimal statistical procedures for recovering the signal or detecting its presence.

As the dimensionality of data becomes large, the computational concerns associated with statistical procedures come to the forefront. In particular, problems with a combinatorial structure or non-convex constraints pose a significant computational challenge because naive methods based on exhaustive search are typically not computationally efficient. Trade-off between computational efficiency and statistical accuracy in high-dimensional inference has drawn increasing attention in the literature. In particular, Chandrasekaran et al. (2012) and Wainwright (2014) considered a general class of linear inverse problems, with different emphasis on convex geometry and decomposition of statistical and computational errors. Chandrasekaran and Jordan (2013) studied an approach for trading off computational demands with statistical accuracy via relaxation hierarchies. Berthet and Rigollet (2013); Ma and Wu (2013); Zhang et al. (2014) focused on computational requirements for various statistical problems, such as detection and regression.

In the present paper, we investigate the question of statistical accuracy and computational efficiency for submatrix localization based on a noisy observation of a large matrix. The problem studied in this paper is formalized as follows.

1.1 Problem Formulation

Consider the matrix \( X \) of the form

\[ X = M + Z, \quad \text{where} \quad M = \lambda \cdot 1_{R_m} 1_{C_n}^T \]  

and \( 1_{R_m} \in \mathbb{R}^m \) with 1 on the index set \( R_m \) and zero otherwise. Here, the entries \( Z_{i,j} \) of the noise matrix are i.i.d. zero-mean sub-Gaussian random vari-
ables with parameter $\sigma$ (defined formally in Equation (4)). Given the parameters $m, n, k_m, k_n, \lambda/\sigma$, the set of all distributions described above—for all possible choices of $R_m$ and $C_n$—forms the submatrix model $\mathcal{M}(m, n, k_m, k_n, \lambda/\sigma)$.

This model can be further extended to the case of multiple submatrices as

$$M = \sum_{s=1}^{r} \lambda_s \cdot 1_{R_s}^T 1_{C_s}$$

where $|R_s| = k_s^{(m)}$ and $|C_s| = k_s^{(n)}$ denote the support set of the $s$-th submatrix. For simplicity, we first focus on the single submatrix and then extend the analysis to the model (3) in Section 4.

There are two fundamental questions associated with the submatrix model (2). One is the detection problem: given one observation of the $X$ matrix, decide whether it is generated from a distribution in the submatrix model or from the pure noise model. Precisely, detection considers the hypothesis testing of the following problem,

$$H_0 : M = 0 \quad \text{v.s.} \quad H_\alpha : M \in \mathcal{M}(m, n, k_m, k_n, \lambda/\sigma).$$

The other is the localization problem, where the goal is to exactly recover the signal index sets $R_m$ and $C_n$ (the support of the mean matrix $M$). It is clear that the localization problem is at least as hard (both computationally and statistically) as the detection problem. As we show in this paper, the localization problem requires larger signal to noise ratio $\lambda/\sigma$, as well as a more detailed exploitation of the submatrix structure.

If the signal to noise ratio is sufficiently large, it is computationally easy to localize the submatrix. On the other hand, if this ratio is small, the localization problem is statistically impossible. To quantify this phenomenon, we identify two distinct thresholds ($\text{SNR}_s$ and $\text{SNR}_c$) for $\lambda/\sigma$ in terms of parameters $m, n, k_m, k_n$. The first threshold, $\text{SNR}_s$, captures the statistical boundary, below which no method (possibly exponential time) can succeed with probability going to one in the minimax sense. The exhaustive search method successfully finds the submatrix above this threshold. The second threshold, $\text{SNR}_c$, corresponds to the computational boundary, above which an adaptive (with respect to the parameters) linear time algorithm finds the signal. Below this threshold, no polynomial time algorithm can succeed, under the hidden clique hypothesis, described later.

### 1.2 Prior Work

There is a growing body of work in statistical literature on submatrix problems. Let us discuss the papers most relevant to the problem studied here. Shabalin
et al. (2009) provided a fast iterative maximization algorithm to solve the submatrix localization problem. However, as with many EM type algorithms, the theoretical result is very sensitive to initialization. Arias-Castro et al. (2011) studied the detection problem for a cluster inside a large matrix. Butucea et al. (2013a,b) formulated the submatrix detection and localization problems under Gaussian noise and determined sharp statistical transition boundaries. For the detection problem, Ma and Wu (2013) provided a computational lower bound result under the assumption that hidden clique detection is computationally difficult.

Balakrishnan et al. (2011); Kolar et al. (2011) focused on statistical and computational trade-offs for the submatrix localization problem. They provided a computationally feasible entry-wise thresholding algorithm, a row/column averaging algorithm, and a convex relaxation for sparse SVD to investigate the minimum signal to noise ratio that is required in order to localize the submatrix. Under the sparse regime $k_m \lesssim m^{1/2}$ and $k_n \lesssim n^{1/2}$, the entry-wise thresholding turns out to be the best polynomial-time algorithm. However, for the dense regime when $k_m \gtrsim m^{1/2}$ and $k_n \gtrsim n^{1/2}$, the algorithms provided in Kolar et al. (2011) are not optimal in the sense that there are other polynomial-time algorithm that can succeed in finding the submatrix with smaller SNR. Concurrently with our work, Chen and Xu (2014) provide a convex relaxation algorithm that improves the SNR boundary of Kolar et al. (2011) in the dense regime. On the downside, the implementation of the method requires a full SVD on each iteration, and therefore does not scale well with the dimensionality of the problem. Furthermore, there is no computational lower bound in the literature to guarantee the optimality of the SNR boundary achieved in Chen and Xu (2014).

A problem similar to submatrix localization is that of clique finding. Deshpande and Montanari (2013) presented an iterative approximate message passing algorithm to solve the latter problem with sharp boundaries on SNR. However, in contrast to submatrix localization, where the signal submatrix can be located anywhere within the matrix, the clique finding problem requires the signal to be centered on the diagonal.

We would like to emphasize the difference between detection and localization problems. When $M$ is a vector, Donoho and Jin (2004) proposed the “higher criticism” approach to solve the detection problem under the Gaussian sequence model. Combining the results in (Donoho and Jin, 2004; Ma and Wu, 2013), in the computationally efficient region, there is no loss in treating $M$ in model (2) as a vector and applying the higher criticism method to the vectorized matrix for the problem of submatrix detection. In fact, the procedure achieves sharper constants in the Gaussian setting. However, in contrast to the detection problem, we will show that for localization, it is crucial to utilize the matrix structure, even in the computationally efficient region.
1.3 Notation

Let \([m]\) denote the index set \([1, 2, \ldots, m]\). For a matrix \(X \in \mathbb{R}^{m \times n}\), \(X_i \in \mathbb{R}^n\) denotes its \(i\)-th row and \(X \cdot j \in \mathbb{R}^m\) denotes its \(j\)-th column. For any \(I \subset [m]\), \(J \subset [n]\), \(X_{IJ}\) denotes the submatrix corresponding to the index set \(I \times J\). We use \(\|\cdot\|_{\ell_p}\) to denote the \(\ell_p\) norm of a vector or the induced \(\ell_p\) norm of a matrix. More precisely, for a vector \(v \in \mathbb{R}^n\), \(\|v\|_{\ell_p} = \left(\sum_{i \in [n]} |v_i|^p\right)^{1/p}\) and for a matrix \(M \in \mathbb{R}^{m \times n}\), \(\|M\|_{\ell_p} = \sup_{v \neq 0} \|Mv\|_{\ell_p}/\|v\|_{\ell_p}\). When \(p = 2\), the latter is the usual operator norm, abbreviated as \(\|M\|_2\). The nuclear norm a matrix \(M\) is defined as a convex surrogate for the rank, with the notation to be \(\|M\|_*\). The Frobenius norm of a matrix \(M\) is defined as \(\|M\|_F = \sqrt{\sum_{i,j} M_{ij}^2}\). The inner product associated with the Frobenius norm is defined as \(\langle A, B \rangle = \text{tr}(A^T B)\).

Denote the asymptotic notation \(a(n) = \Theta(b(n))\) if there exist two universal constants \(c_l, c_u\) such that \(c_l \leq \lim_{n \to \infty} a(n)/b(n) \leq \lim_{n \to \infty} a(n)/b(n) \leq c_u\). \(\Theta^*\) is asymptotic equivalence hiding logarithmic factors in the following sense: \(a(n) = \Theta^*(b(n))\) iff there exists \(c > 0\) such that \(a(n) = \Theta(b(n) \log^c n)\). Additionally, we use the notation \(a(n) \asymp b(n)\) as equivalent to \(a(n) = \Theta(b(n))\), \(a(n) \gtrsim b(n)\) iff \(\lim_{n \to \infty} a(n)/b(n) = \infty\) and \(a(n) \lesssim b(n)\) iff \(\lim_{n \to \infty} a(n)/b(n) = 0\).

We define the zero-mean sub-Gaussian random variable \(z\) with sub-Gaussian parameter \(\sigma\) in terms of its Laplacian. If there exists a universal constant \(c > 0\),

\[ Ee^{\lambda z} \leq \exp(\sigma^2 \lambda^2 / 2c), \quad \text{for all } \lambda > 0 \]  

then we have

\[ P(|z| > \sigma t) \leq 2 \cdot \exp(-c \cdot t^2 / 2). \]

We call a random vector \(Z \in \mathbb{R}^n\) isotropic with parameter \(\sigma\) if

\[ E(v^T Z)^2 = \sigma^2 \|v\|_{\ell_2}^2, \quad \text{for all } v \in \mathbb{R}^n. \]

Clearly, Gaussian and Bernoulli measures, and more general product measures of zero-mean sub-Gaussian random variables satisfy this isotropic definition up to a constant scalar factor.

1.4 Our Contributions

To state our main results, let us first define a hierarchy of algorithms in terms of their worst-case running time on instances of the submatrix localization problem:

\[ \text{LinAlg} \subset \text{PolyAlg} \subset \text{ExpoAlg} \subset \text{AllAlg}. \]
The set LinAlg contains methods $\mathcal{A}$ that produce an answer (in our case, the localization subset $\hat{R}_m^d, \hat{C}_n^d$) in time linear in $m \times n$ (the minimal computation required to read the matrix). The classes PolyAlg and ExpoAlg, respectively, terminate in polynomial and exponential time, while AllAlg has no restriction.

Combining Theorem 3 in Section 2 and Theorem 4 in Section 3, the statistical and computational boundaries for submatrix localization can be summarized as follows.

**Theorem 1** (Computational and Statistical Boundaries). Consider the submatrix localization problem under the model (2). The statistical boundary $\text{SNR}_s$ is

$$\text{SNR}_s \asymp \sqrt{\log n \lor \log m},$$

in the sense that

$$\lim_{m,n,k_m,k_n \to \infty} \inf_{M \in \mathcal{A}} \sup_{(\hat{R}_m^d \neq R_m \text{ or } \hat{C}_n^d \neq C_n)} P\{\hat{R}_m^d \neq R_m \text{ or } \hat{C}_n^d \neq C_n\} = 0, \quad \text{if } \frac{\lambda}{\sigma} \asymp \text{SNR}_s$$

and

$$\lim_{m,n,k_m,k_n \to \infty} \inf_{M \in \mathcal{A}} \sup_{(\hat{R}_m^d \neq R_m \text{ or } \hat{C}_n^d \neq C_n)} P\{\hat{R}_m^d \neq R_m \text{ or } \hat{C}_n^d \neq C_n\} > 0, \quad \text{if } \frac{\lambda}{\sigma} \asymp \text{SNR}_s$$

under the minimal assumption $\max\{k_m, k_n\} \asymp \min\{m, n\}$. The computational boundary $\text{SNR}_c$ is

$$\text{SNR}_c \asymp \sqrt{\frac{m \lor n}{k_m k_n} + \sqrt{\frac{\log n \lor \log m}{k_m k_n}}}$$

in the sense that

$$\lim_{m,n,k_m,k_n \to \infty} \inf_{M \in \mathcal{A}} \sup_{(\hat{R}_m^d \neq R_m \text{ or } \hat{C}_n^d \neq C_n)} P\{\hat{R}_m^d \neq R_m \text{ or } \hat{C}_n^d \neq C_n\} = 0, \quad \text{if } \frac{\lambda}{\sigma} \asymp \text{SNR}_c$$

and

$$\lim_{m,n,k_m,k_n \to \infty} \inf_{M \in \mathcal{A}} \sup_{(\hat{R}_m^d \neq R_m \text{ or } \hat{C}_n^d \neq C_n)} P\{\hat{R}_m^d \neq R_m \text{ or } \hat{C}_n^d \neq C_n\} > 0, \quad \text{if } \frac{\lambda}{\sigma} \asymp \text{SNR}_c$$

where (10) holds under the Hidden Clique hardness hypothesis $\text{HC}_1$ (see Section 2.1). Here we assume in addition $\min\{k_m, k_n\} \asymp \max\{m^{1/2}, n^{1/2}\}$. 


If we parametrize the submatrix model as \( m = n, k_m = k_n = k = \Theta^*(n^\alpha), \lambda/\sigma = \Theta^*(n^{-\beta}) \), for some \( 0 < \alpha, \beta < 1 \), we can summarize the results of Theorem 1 in a phase diagram, as illustrated in Figure 1.

Figure 1: Phase diagram for submatrix localization. Red region (C): statistically impossible. Blue region (B): statistically possible but computationally expensive (under the hidden clique hypothesis). Green region (A): statistically possible and computationally easy.

To explain the diagram, consider the following cases. First, the statistical boundary is

\[
\sqrt{\frac{\log n}{k_m}} \sqrt{\frac{\log m}{k_n}},
\]

which gives the line separating the red and the blue regions. For \( \alpha \geq 1/2 \), the computational boundary given by Theorem 1 is

\[
\sqrt{\frac{m \lor n}{k_m k_n}} + \sqrt{\frac{\log n}{k_m}} \sqrt{\frac{\log m}{k_n}},
\]

which corresponds to the line separating the blue and the green regions. For \( \alpha < 1/2 \), the computational boundary is \( \Theta(1) \lesssim \text{SNR}_c \lesssim \Theta(\sqrt{\log(mn)}) \), since for SNR larger than \( \Theta(\sqrt{\log(mn)}) \) entry-wise thresholding provides exact recovery of the location. In contrast, for SNR smaller than \( \Theta(1) \), the hidden clique hypothesis implies that no polynomial time algorithm will succeed.

As a key part of Theorem 1, we provide a linear time algorithm that will succeed in localizing the submatrix with high probability in the regime above the computational threshold. Furthermore, the method is adaptive: it does not require the prior knowledge of the size of the submatrix. This should be contrasted
with the method of Chen and Xu (2014) which requires the prior knowledge of \( k_m, k_n \); furthermore, the running time of their SDP-based method is superlinear in \( nm \). Under the hidden clique hypothesis, we prove that below the computational threshold there is no polynomial time algorithm that can succeed in localizing the submatrix. This is a new result that has not been established in the literature. We remark that the computational lower bound for localization requires a technique different from the lower bound for detection; the latter has been resolved in Ma and Wu (2013).

As mentioned earlier, the statistical boundary for localization has been investigated by Butucea et al. (2013b) in the Gaussian case. Our result is a generalization to sub-Gaussian noise, at the expense of not providing the exact constants for the thresholds.

Let us finish the discussion by pointing out an interesting phenomenon. The phase transition diagram in Figure 1 for localization should be contrasted with the corresponding result for detection, as shown in (Butucea et al., 2013a; Ma and Wu, 2013). For a large enough submatrix size (as quantified by \( \alpha > 2/3 \)), the computationally-intractable-but-statistically-possible region collapses for the detection problem, but not for localization. In plain words, detecting the presence of a large submatrix becomes both computationally and statistically easy beyond a certain size, while for localization there is always a gap between statistically possible and computationally feasible regions. This phenomenon also appears to be distinct to that of other problems like estimation of sparse principal components (Cai et al., 2013), where computational and statistical easiness coincide with each other over a large region of the parameter spaces.

1.5 Organization of the Paper

The paper is organized as follows. Section 2 establishes the computational boundary, with the computational lower bounds given in Section 2.1 and upper bound results in Section 2.2. Specifically, Section 2.2 introduces the linear-time Algorithm 1 that achieves the upper bound in Theorem 1. The upper and lower bounds for statistical boundary are proved in Section 3. An extension to the case of multiple submatrices is presented in Section 4. Technical proofs are deferred to Section 5. In addition to the spectral method of Section 2.2, we provide a new analysis of the known method that is based on a convex relaxation (Chen and Xu, 2014). Since the method is not as efficient as the spectral approach, we postpone the analysis to the Appendix A. Comparison of computational lower bounds for localization and detection is included in Appendix B.
2 Computational Boundary

We characterize in this section the computational boundaries for the submatrix localization problem. Section 2.1 and 2.2 consider respectively the computational lower bound and upper bound. The computational lower bound given in Theorem 2 is based on the hidden clique hypothesis.

2.1 Algorithmic Reduction and Computational Lower Bound

Theoretical Computer Science identifies a range of problems which are believed to be “hard,” in the sense that in the worst-case the required computation grows exponentially with the size of the problem. Faced with a new computational problem, one might try to reduce any of the “hard” problems to the new problem, and therefore claim that the new problem is as hard as the rest in this family. Since statistical procedures typically deal with a random (rather than worst-case) input, it is natural to seek token problems that are believed to be computationally difficult on average with respect to some distribution on instances. The hidden clique problem is one such example (for recent results on this problem, see Feldman et al. (2013); Deshpande and Montanari (2013)). While there exists a quasi-polynomial algorithm, no polynomial-time method (for the appropriate regime, described below) is known. Following several other works on reductions for statistical problems, we work under the hypothesis that no polynomial-time method exists.

Let us make the discussion more precise. Consider the hidden clique model $G(N, \kappa)$ where $N$ is the total number of nodes and $\kappa$ is the number of clique nodes. In the hidden clique model, a random graph instance is generated in the following way. Choose $\kappa$ clique nodes uniformly at random from all the possible choices, and connect all the edges within the clique. For all the other edges, connect with probability 1/2.

Hidden Clique Hypothesis for Localization ($HC_l$) Consider the random instance of hidden clique model $G(N, \kappa)$. For any sequence $\kappa(N)$ such that $\kappa(N) \leq N^\beta$ for some $0 < \beta < 1/2$, there is no randomized polynomial time algorithm that can find the planted clique with probability tending to 1 as $N \to \infty$. Mathematically, define the randomized polynomial time algorithm class PolyAlg as the class of algorithms $\mathcal{A}$ that satisfies

$$\lim_{N, \kappa(N) \to \infty} \sup_{A \in \text{PolyAlg}} \mathbb{P}_{G(N, \kappa)}(\text{Clique} | \text{runtime of } \mathcal{A} \text{ not polynomial in } N) = 0.$$
Then

\[
\lim_{N, \kappa(N) \to \infty} \inf_{\alpha \in \text{PolyAlg}} \mathbb{E}_{\text{Clique}} \mathbb{P}_{\mathcal{G}(N, \kappa)} \left( \text{clique set returned by } \mathcal{A} \text{ not correct} \right) > 0,
\]

where \( \mathbb{P}_{\mathcal{G}(N, \kappa)} \) is the (possibly more detailed due to randomness of algorithm) \( \sigma \)-field conditioned on the clique location and \( \mathbb{E}_{\text{Clique}} \) is with respect to uniform distribution over all possible clique locations.

**Hidden Clique Hypothesis for Detection** (HC\(_d\)) Consider the hidden clique model \( \mathcal{G}(N, \kappa) \). For any sequence of \( \kappa(N) \) such that \( \kappa(N) \leq N^\beta \) for some \( 0 < \beta < 1/2 \), there is no randomized polynomial time algorithm that can distinguish between

\[
H_0 : \mathcal{P}_{\text{ER}} \quad \text{v.s.} \quad H_\alpha : \mathcal{P}_{\text{HC}}
\]

with probability going to 1 as \( N \to \infty \). Here \( \mathcal{P}_{\text{ER}} \) is the Erdős-Rényi model, while \( \mathcal{P}_{\text{HC}} \) is the hidden clique model with uniform distribution on all the possible locations of the clique. More precisely,

\[
\lim_{N, \kappa(N) \to \infty} \inf_{\alpha \in \text{PolyAlg}} \mathbb{E}_{\text{Clique}} \mathbb{P}_{\mathcal{G}(N, \kappa)} \left( \text{detection decision returned by } \mathcal{A} \text{ wrong} \right) > 0,
\]

where \( \mathbb{P}_{\mathcal{G}(N, \kappa)} \) and \( \mathbb{E}_{\text{Clique}} \) are the same as defined in HC\(_l\).

The hidden clique hypothesis has been used recently by several authors to claim computational intractability of certain statistical problems. In particular, Berthet and Rigollet (2013); Ma and Wu (2013) assumed the hypothesis HC\(_d\) and Wang et al. (2014) used HC\(_l\). Localization is harder than detection, in the sense that if an algorithm \( \mathcal{A} \) solves the localization problem with high probability, it also correctly solves the detection problem. Assuming that no polynomial time algorithm can solve the detection problem implies impossibility results in localization as well. In plain language, HC\(_l\) is a milder hypothesis than HC\(_d\).

We will provide two computational lower bound results, one for localization and the other for detection, in Theorems 2 and 5. The latter one will be deferred to Appendix B to contrast the difference of constructions between localization and detection. The detection computational lower bound was first proved in Ma and Wu (2013). For the localization computational lower bound, to the best of our knowledge, there is no proof in the literature. Theorem 2 ensures the upper bound in Lemma 1 being sharp.
Theorem 2 (Computational Lower Bound for Localization). Consider the sub-matrix model (2) with parameter tuple \((m = n, k_m = k_n = n^\alpha, \lambda/\sigma = n^{-\beta})\), where \(\frac{1}{2} < \alpha < 1, \beta > 0\). Under the computational assumption HC, if

\[
\frac{\lambda}{\sigma} \leq \sqrt{\frac{m+n}{k_m k_n}} \Rightarrow \beta > \alpha - \frac{1}{2},
\]

it is not possible to localize the true support of the submatrix with probability going to 1 within polynomial time.

Our algorithmic reduction for localization relies on a bootstrapping idea based on the matrix structure and a cleaning-up procedure introduced in Lemma 10 given in Section 5. These two key ideas offer new insights in addition to the usual computational lower bound arguments. Bootstrapping introduces an additional randomness on top of the randomness in the hidden clique. Careful examination of these two \(\sigma\)-fields allows us to write the resulting object into mixture of submatrix models. For submatrix localization we need to transform back the submatrix support to the original hidden clique support exactly, with high probability. In plain language, even though we lose track of the exact location of the support when reducing the hidden clique to submatrix model, we can still recover the exact location of the hidden clique with high probability. For technical details of the proof, please refer to Section 5.

2.2 Adaptive Linear Time Algorithm and Computational Upper Bound

In this section, we introduce a linear time algorithm that solves the submatrix localization problem above the computational boundary \(SNR_c\). Our proposed localization Algorithms 1 and 3 are motivated by the spectral algorithm in random graphs (McSherry, 2001; Ng et al., 2002).

**Algorithm 1: Spectral Projection Algorithm**

**Input:** \(X \in \mathbb{R}^{m \times n}\) the data matrix.

**Output:** A subset of the row indexes \(\hat{R}_m\) and a subset of column indexes \(\hat{C}_n\) as the localization sets of the submatrix.

1. Compute top left and top right singular vectors \(U_1\) and \(V_1\), respectively (these correspond to the SVD \(X = U\Sigma V^T\));
2. To compute \(\hat{C}_n\), calculate the inner products \(U_1^T X_{i,j}, 1 \leq j \leq n\). These values form two clusters. Similarly, for the \(\hat{R}_m\), calculate \(X_i V_1, 1 \leq i \leq m\) and obtain two separated clusters. A simple thresholding procedure returns the subsets \(\hat{C}_n\) and \(\hat{R}_m\).
The proposed algorithm has several advantages over the localization algorithms that appeared in literature. First, it is a linear time algorithm (that is, \( \Theta(mn) \) time complexity). The top singular vectors can be evaluated using fast iterative power methods, which is efficient both in terms of space and time. Secondly, this algorithm does not require the prior knowledge of \( k_m \) and \( k_n \) and automatically adapts to the true submatrix size.

Lemma 1 below justifies the effectiveness of the spectral algorithm.

**Lemma 1** (Guarantee for Spectral Algorithm). Consider the submatrix model (2), Algorithm 1 and assume \( \min\{k_m, k_n\} \gtrsim \max\{m^{1/2}, n^{1/2}\} \). There exist a universal \( C > 0 \) such that when

\[
\frac{\lambda}{\sigma} \geq C \left( \sqrt{\frac{m \lor n}{k_m k_n}} + \sqrt{\frac{\log n}{k_m} \lor \frac{\log m}{k_n}} \right),
\]

the spectral method succeeds in the sense that \( \hat{R}_m = R_m, \hat{C}_n = C_n \) with probability at least \( 1 - m^{-c} - n^{-c} - 2 \exp(-c(m + n)) \).

We are now ready to state the SNR boundary for polynomial-time algorithms (under an appropriate computational assumption), thus excluding the exhaustive search procedure.

**Theorem 3** (Computational Boundary). Consider the submatrix model (2) and assume \( \min\{k_m, k_n\} \gtrsim \max\{m^{1/2}, n^{1/2}\} \). There exists a critical rate

\[
\text{SNR}_c = \sqrt{\frac{m \lor n}{k_m k_n}} + \sqrt{\frac{\log n}{k_m} \lor \frac{\log m}{k_n}}
\]

for the signal to noise ratio \( \text{SNR}_c \) such that for \( \lambda/\sigma \gtrsim \text{SNR}_c \), both the adaptive linear time Algorithm 1 and the robust polynomial time Algorithm 4 will succeed in submatrix localization, i.e., \( \hat{R}_m = R_m, \hat{C}_n = C_n \), with high probability. For \( \lambda/\sigma \lesssim \text{SNR}_c \), there is no polynomial time algorithm that will work under the hidden clique hardness hypothesis \( HC_1 \).

The proof of the above theorem is based on the theoretical justification of the spectral Algorithm 1 and convex relaxation Algorithm 4, and the new computational lower bound result for localization in Theorem 2. We remark that the analyses can be extended to multiple, even growing number of submatrices case. We postpone a proof of this fact to Section 4 for simplicity and focus on the case of a single submatrix.
3 Statistical Boundary

In this section we study the statistical boundary. As mentioned in the introduction, in the Gaussian noise setting, the statistical boundary for localization has been established in Butucea et al. (2013b). In this section, we generalize it to sub-Gaussian noise, at the expense of having non-exact constants for the thresholds. We begin with the information theoretic lower bound for the localization accuracy.

**Lemma 2** (Information Theoretic Lower Bound). Consider the submatrix model (2) with Gaussian noise \( Z_{ij} \sim \mathcal{N}(0, \sigma^2) \). For any fixed \( 0 < \alpha < 1 \), there exist a universal constant \( C_\alpha \) such that if

\[
\frac{\lambda}{\sigma} \leq C_\alpha \sqrt{\frac{\log(m/k_m)}{k_n} + \frac{\log(n/k_n)}{k_m}},
\]  

(11)

any algorithm \( \mathcal{A} \) will fail to localize the submatrix with probability at least \( 1 - \alpha - \frac{\log 2}{k_m \log(m/k_m) + k_n \log(n/k_n)} \) in the following minimax sense:

\[
\inf_{\mathcal{A} \in \text{AllAlg}} \sup_{M \in \mathcal{M}} \mathbb{P}\left( \hat{R}_m \neq R_m \text{ or } \hat{C}_n \neq C_n \right) > 1 - \alpha - \frac{\log 2}{k_m \log(m/k_m) + k_n \log(n/k_n)}.
\]

Combinatorial search over all submatrices of size \( k_m \times k_n \) finds the location with the strongest aggregate signal and is statistically optimal (Butucea et al., 2013b,a). Unfortunately, it requires computational complexity \( \Theta \left( \binom{m}{k_m} + \binom{n}{k_n} \right) \), which is exponential in \( k_m, k_n \). The search Algorithm 2 was introduced and analyzed under the Gaussian setting in Butucea et al. (2013a), which naturally extends to sub-Gaussian noise.

**Algorithm 2:** Combinatorial Search Algorithm

**Input:** \( X \in \mathbb{R}^{m \times n} \) the data matrix.

**Output:** A subset of the row indexes \( \hat{R}_m \) and a subset of column indexes \( \hat{C}_n \) as the localization of the submatrix.

For all index subsets \( I \times J \) with \( |I| = k_m \) and \( |J| = k_n \), calculate the sum of the entries in the submatrix \( X_{IJ} \). Report the index subset \( \hat{R}_m \times \hat{C}_n \) with the largest sum.

For the case of multiple submatrices, we can extract out submatrices with the largest sum in a greedy fashion. The analysis is then essentially the same as in the case of a single submatrix.

Lemma 3 below provides a theoretical guarantee for Algorithm 2 to achieve the information theoretic lower bound.
Lemma 3 (Guarantee for Search Algorithm). Consider the submatrix model (2) and Algorithm 2. Assume \( \max\{k_m, k_n\} \gtrsim \min\{m, n\} \). There exists a universal constant \( C > 0 \) such that if

\[
\lambda \sigma \geq C \cdot \sqrt{\log(\frac{em}{km}) \cdot \frac{\log(em/\sqrt{km})}{km}} + \log(\frac{en}{\sqrt{km}}),
\]

then Algorithm 2 will succeed in returning the correct location of the submatrix with probability at least \( 1 - \frac{2k_m k_n}{mn} \).

To complete Theorem 1, we include the following Theorem 4 capturing the statistical boundary. It is proved by exhibiting the information-theoretic lower bound Lemma 2 and analyzing Algorithm 2.

Theorem 4 (Statistical Boundary). Consider the submatrix model (2). There exists a critical rate

\[
\text{SNR}_s = \sqrt{\frac{\log n}{km} \cdot \frac{\log m}{kn}}
\]

for the signal to noise ratio, such that for any problem with \( \lambda / \sigma \gtrsim \text{SNR}_s \), the statistical search Algorithm 2 will succeed in submatrix localization, i.e., \( \hat{R}_m = R_m, \hat{C}_n = C_n \), with high probability. On the other hand, if \( \lambda / \sigma \lesssim \text{SNR}_s \), no algorithm will work (in the minimax sense) with probability tending to 1.

4 Extension to Multiple Submatrices

The computational and statistical boundaries established in the previous sections for a single submatrix can be extended to non-overlapping multiple submatrices model (3). The non-overlapping assumption corresponds to that for any \( 1 \leq s \neq t \leq r, R_s \cap R_t = \emptyset \) and \( C_s \cap C_t = \emptyset \). The Algorithm 3 below is an extension of the spectral projection Algorithm 1 to address the multiple submatrices localization problem.
Algorithm 3: Spectral Algorithm for Multiple Submatrices

**Input:** $X \in \mathbb{R}^{m \times n}$ the data matrix. A pre-specified number of submatrices $r$.

**Output:** A subset of the row indexes $\{\hat{R}_s^r, 1 \leq s \leq r\}$ and a subset of column indexes $\{\hat{C}_s^r, 1 \leq s \leq r\}$ as the localization of the submatrices.

1. Calculate top $r$ left and right singular vectors in the SVD $X = U \Sigma V^T$. Denote these vectors as $U_r \in \mathbb{R}^{m \times r}$ and $V_r \in \mathbb{R}^{n \times r}$, respectively;
2. For the $\hat{C}_s^r, 1 \leq s \leq r$, calculate the projection $U_r(U_r^T U_r)^{-1} U_r^T X_j, 1 \leq j \leq n$, run $k$-means clustering algorithm (with $k = r + 1$) for these $n$ vectors in $\mathbb{R}^m$. For the $\hat{R}_s^r, 1 \leq s \leq r$, calculate $V_r(V_r^T V_r)^{-1} V_r^T X_i^T, 1 \leq i \leq m$, run $k$-means clustering algorithm (with $k = r + 1$) for these $m$ vectors in $\mathbb{R}^n$ (while the effective dimension is $\mathbb{R}^r$).

We emphasize that the following Proposition 1 holds even when the number of submatrices $r$ grows with $m, n$.

**Proposition 1** (Spectral Algorithm for Non-overlapping Submatrices Case). Consider the non-overlapping multiple submatrices model (3) and Algorithm 3. Assume $k_s^{(m)} = k_m, k_s^{(n)} = k_n, \lambda_s = \lambda$ for all $1 \leq s \leq r$ and $\min\{k_m, k_n\} \gtrsim \max\{m^{1/2}, n^{1/2}\}$. There exist a universal $C > 0$ such that when

$$\frac{\lambda}{\sigma} \geq C \left( \sqrt{\frac{r}{k_m \land k_n}} + \sqrt{\frac{\log n}{k_m}} \lor \sqrt{\frac{\log m}{k_n}} + \sqrt{\frac{m \lor n}{k_m k_n}} \right),$$

the spectral method succeeds in the sense that $\hat{R}_s^r = R_s^r, \hat{C}_s^r = C_s^r, 1 \leq s \leq r$ with probability at least $1 - m^{-c} - n^{-c} - 2 \exp(-c(m + n))$.

**Remark 4.1.** Under the non-overlapping assumption, $r k_m \lessapprox m, r k_n \lessapprox n$ hold in most cases. Thus the first term in Equation (12) is dominated by the latter two terms. Thus a growing number $r$ does not affect the bound in Equation (12) as long as the non-overlapping assumption holds.

5 Proofs

**Proof of Lemma 1.** Recall the matrix form of the submatrix model, with the SVD decomposition of the mean signal matrix $M$

$$X = \lambda \sqrt{k_m k_n} U V^T + Z.$$
The largest singular value of \( \lambda UV^T \) is \( \lambda \sqrt{k_m k_n} \), and all the other singular values are 0s. The following Davis-Kahan-Wedin’s perturbation bound tells us how close the singular space of \( X \) is to the singular space of \( M \).

**Lemma 4** (Stewart and Sun (1990) Theorem 4.1). Suppose that \( \tilde{A} = A + E \), all of which are matrices of the same size, and we have the following singular value decomposition

\[
[U_1, U_2, U_3]^T A[V_1, V_2] = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} (13)
\]

and

\[
[	ilde{U}_1, \tilde{U}_2, \tilde{U}_3]^T \tilde{A}[\tilde{V}_1, \tilde{V}_2] = \begin{bmatrix} \tilde{\Sigma}_1 & 0 \\ 0 & \tilde{\Sigma}_2 \end{bmatrix}. (14)
\]

Let \( \Phi \) be the matrix of canonical angles between \( \mathcal{R}(U_1) \) and \( \mathcal{R}(\tilde{U}_1) \), and let \( \Theta \) be the matrix of canonical angles between \( \mathcal{R}(V_1) \) and \( \mathcal{R}(\tilde{V}_1) \) (here \( \mathcal{R} \) denotes the linear space). Define

\[
R = A\tilde{V}_1 - \tilde{U}_1 \tilde{\Sigma}_1
\]

\[
S = A^T \tilde{U}_1 - \tilde{V}_1 \tilde{\Sigma}_1
\]

Then suppose there is a number \( \delta > 0 \) such that

\[
\min |\sigma(\tilde{\Sigma}_1) - \sigma(\Sigma_2)| \geq \delta \quad \text{and} \quad \min \sigma(\tilde{\Sigma}_1) \geq \delta.
\]

Then

\[
\sqrt{\| \sin \Phi \|_F^2 + \| \sin \Theta \|_F^2} \leq \sqrt{\frac{\| R \|_F^2 + \| S \|_F^2}{\delta}}.
\]

Further, suppose there are numbers \( \alpha, \delta \) such that

\[
\min \sigma(\tilde{\Sigma}_1) \geq \delta + \alpha \quad \text{and} \quad \max \sigma(\Sigma_2) \leq \alpha,
\]

then for 2-norm, or any unitarily invariant norm, we have

\[
\max \{ \| \sin \Phi \|_2, \| \sin \Theta \|_2 \} \leq \frac{\max \{ \| R \|_2, \| S \|_2 \}}{\delta}.
\]

Let us use the above version of the perturbation bound to derive a lemma that is particularly useful in our case. Simple algebra tells us that

\[
\tilde{A}[\tilde{V}_1, \tilde{V}_2] = [\tilde{U}_1, \tilde{U}_2, \tilde{U}_3] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} = [\tilde{U}_1 \tilde{\Sigma}_1, \tilde{U}_2 \tilde{\Sigma}_2]. (17)
\]
\[ A[\tilde{V}_1, \tilde{V}_2] = [A\tilde{V}_1, A\tilde{V}_2] \]
\[ (\tilde{A} - A)[\tilde{V}_1, \tilde{V}_2] = [\tilde{O}_1\tilde{\Sigma}_1 - A\tilde{V}_1, \tilde{O}_2\tilde{\Sigma}_2 - A\tilde{V}_2] \]
\[ \| \tilde{A} - A \|^2_F = \text{Tr}((\tilde{A} - A)[\tilde{V}_1, \tilde{V}_2][\tilde{V}_1, \tilde{V}_2]^T(\tilde{A} - A)^T) \]
\[ = \| R \|^2_F + \| \tilde{U}_2\tilde{\Sigma}_2 - A\tilde{V}_2 \|^2_F \geq \| R \|^2_F. \] (21)

Similarly, we have
\[ \| \tilde{A} - A \|^2_F = \text{Tr}((\tilde{A} - A)[\tilde{U}_1, \tilde{U}_2, \tilde{U}_3][\tilde{U}_1, \tilde{U}_2, \tilde{U}_3]^T(\tilde{A} - A)) \]
\[ = \| S \|^2_F + \| \tilde{V}_2\tilde{\Sigma}_2 - A^T\tilde{U}_2 \|^2_F + \| A^T\tilde{U}_3 \|^2_F \geq \| S \|^2_F. \] (24)

Thus, it holds that
\[ \| \tilde{A} - A \|_F \geq \max(\| R \|_F, \| S \|_F) \]
and similarly we have (since the operator norm of a whole matrix is larger than that of the submatrix)
\[ \| \tilde{A} - A \|_2 \geq \max(\| R \|_2, \| S \|_2). \]

Thus the following version of the Wedin’s Theorem holds.

**Lemma 5** (Davis-Kahan-Wedin’s Type Perturbation Bound). It holds that
\[ \sqrt{\| \sin \Phi \|^2_F + \| \sin \Theta \|^2_F} \leq \frac{\sqrt{2}\| E \|_F}{\delta} \]
and also the following holds for 2-norm (or any unitary invariant norm)
\[ \max \{ \| \sin \Phi \|_2, \| \sin \Theta \|_2 \} \leq \frac{\| E \|_2}{\delta}. \]

Let us apply the above bound to \( X = \lambda\sqrt{k_mk_n}UV^T + Z \). Denote the top left and right singular vector of \( X \) as \( \tilde{U} \) and \( \tilde{V} \). One can see that \( \mathbb{E}\| Z \|_2 \leq \sigma(\sqrt{m} + \sqrt{n}) \) under very mild finite fourth moment conditions through a result in (Latała, 2005). However, we require a more explicit probabilisitic bound for the concentration of the largest singular value of i.i.d sub-Gaussian random matrix.

**Lemma 6** (Vershynin (2010), Theorem 39). Let \( Z \in \mathbb{R}^{m \times n} \) be a matrix whose rows \( Z_i \) are independent sub-Gaussian isotropic random vectors in \( \mathbb{R}^n \) with parameter \( \sigma \). Then for every \( t \geq 0 \), with probability at least \( 1 - 2\exp(-ct^2) \) one has
\[ \| Z \|_2 \leq \sigma(\sqrt{m} + C\sqrt{n} + t) \]
where \( C, c > 0 \) are some universal constants.
Because the rows \( Z_i \) are sampled from product measure of mean zero sub-Gaussians, they naturally satisfy the isotropic condition. Hence, with probability at least \( 1 - 2 \exp(-c(m+n)) \), we reach
\[
\|Z\|_2 \leq C \cdot \sigma(\sqrt{m} + \sqrt{n}).
\] (25)

Using Weyl's interlacing inequality, we have
\[
|\sigma_i(X) - \sigma_i(M)| \leq \|Z\|_2
\]
and thus
\[
\sigma_1(X) \geq \lambda \sqrt{k_m k_n} - \|Z\|_2
\]
\[
\sigma_2(X) \leq \|Z\|_2.
\]
Applying Lemma 5, we have
\[
\max \{ |\sin \angle(U, \tilde{U})|, |\sin \angle(V, \tilde{V})| \} \leq \frac{C \sigma(\sqrt{m} + \sqrt{n})}{\lambda \sqrt{k_m k_n}} = \frac{\sigma(\sqrt{m} + \sqrt{n})}{\lambda \sqrt{k_m k_n}}.
\]
In addition
\[
\|U - \tilde{U}\|_{\ell_2} = \sqrt{2 - 2 \cos \angle(U, \tilde{U})} = 2 \sin \frac{1}{2} \angle(U, \tilde{U})
\]
which means
\[
\max \{ \|U - \tilde{U}\|_{\ell_2}, \|V - \tilde{V}\|_{\ell_2} \} \leq C \cdot \frac{\sigma(\sqrt{m} + \sqrt{n})}{\lambda \sqrt{k_m k_n}}.
\]
And according to the definition of the canonical angles, we have
\[
\max \{ \|UU^T - \tilde{U}\tilde{U}^T\|_2, \|VV^T - \tilde{V}\tilde{V}^T\|_2 \} \leq C \cdot \frac{\sigma(\sqrt{m} + \sqrt{n})}{\lambda \sqrt{k_m k_n}}.
\]

Now let's assume we have two observations of \( X \). We use the first observation \( \tilde{X} \) to solve for the singular vectors \( \tilde{U}, \tilde{V} \), we use the second observation \( X \) to project to the singular vectors \( \tilde{U}, \tilde{V} \). We can use Tsybakov's sample cloning argument (Tsybakov (2014), Lemma 2.1) to create two independent observations of \( X \) when noise is Gaussian as follows. Create a pure Gaussian matrix \( Z' \) and define \( X_1 = X + Z' = M + (Z + Z') \) and \( X_2 = X - Z' = M + (Z - Z') \), making \( X_1, X_2 \) independent with the variance being doubled. This step is not essential because we can perform random subsampling as in Vu (2014); having two observations instead of one does not change the picture statistically or computationally. Recall
\[
X = M + Z = \lambda \sqrt{k_m k_n} UV^T + Z.
\]

Define the projection operator to be \( P \), we have the following lemma.
Lemma 7 (Hsu et al. (2012), Projection Lemma). Assume $Z \in \mathbb{R}^n$ is an isotropic sub-Gaussian vector with i.i.d. entries and parameter $\sigma$. $\mathcal{P}$ is a projection operator to a subspace of dimension $r$, then we have the following concentration inequality

$$
\Pr(\|\mathcal{P}Z\|_\ell^2 \geq \sigma^2 (r + 2\sqrt{rt} + 2t)) \leq \exp(-ct),
$$

where $c > 0$ is a universal constant.

The proof of this lemma is a simple application of Theorem 2.1 in the referenced paper for the case that $\mathcal{P}$ is a rank $r$ positive semidefinite projection matrix.

We start the analysis by decomposing

$$
\|\mathcal{P}_j X_j - M_j\|_\ell^2 \leq \|\mathcal{P}_j (X_j - M_j)\|_\ell^2 + \|\mathcal{P}_j - I\| M_j \|_\ell^2
$$

for $1 \leq j \leq n$.

For the first term of (26), note that $X_j - M_j = Z_j \in \mathbb{R}^m$ is an i.i.d. isotropic sub-Gaussian vector, and thus we have through Lemma 7, for $t = (1 + 1/c) \log n$, $Z_j \in \mathbb{R}^m, 1 \leq j \leq n$ and $r = 1$

$$
\Pr \left( \|\mathcal{P}_j (X_j - M_j)\|_\ell^2 \geq \sigma \sqrt{1 + 2\sqrt{1 + 1/c} \cdot \sqrt{\frac{\log n}{r}} + 2(1 + 1/c) \cdot \frac{\log n}{r}} \right) \leq n^{-c-1}.
$$

(27)

We invoke the union bound for all $1 \leq j \leq n$ to obtain

$$
\max_{1 \leq j \leq n} \|\mathcal{P}_j (X_j - M_j)\|_\ell^2 \leq \sigma \sqrt{1 + 2\sqrt{1 + 1/c} \cdot \sqrt{\log n + (1 + 1/c) \cdot \frac{\log n}{\sqrt{r}}}} \leq \sigma + C \cdot \sigma \sqrt{\log n}
$$

(28)

(29)

with probability at least $1 - n^{-c}$.

For the second term $M_j = X_j - Z_j$ of (26), there are two ways of upper bounding it. The first approach is to split

$$
\|\mathcal{P}_j - I\| M_j \|_\ell^2 \leq \|\mathcal{P}_j - I\| \tilde{X} \|_\ell^2 + \|\mathcal{P}_j - I\| \tilde{Z} \|_\ell^2 \leq 2\|\tilde{Z}\|_\ell^2.
$$

(30)

The first term of (30) is $\sigma_2(\tilde{X}) \leq \sigma_2(M) + \|\tilde{Z}\|_\ell^2$ through Weyl's interlacing inequality, while the second term is bounded by $\|\tilde{Z}\|_\ell^2$. We also know that $\|\tilde{Z}\|_\ell^2 \leq C_3 \cdot \sigma(\sqrt{m} + \sqrt{n})$. Recall the definition of the induced $\ell_2$ norm of a matrix $(\mathcal{P}_j - I)M$:

$$
\|\mathcal{P}_j - I\| M_j \|_\ell^2 \geq \frac{\|\mathcal{P}_j - I\| MV \|_\ell^2}{\|V\|_\ell^2} = \|\mathcal{P}_j - I\| \lambda \sqrt{k_m k_n} U \|_\ell^2 \geq \sqrt{k_n} \|\mathcal{P}_j - I\| M_j \|_\ell^2.
$$

(28)
In the second approach, the second term of (26) can be handled through perturbation Sin Theta Theorem 5:

\[ \| (\mathcal{P}_0 - \mathcal{P}_U) M_j \|_{\ell_2} \leq \| \tilde{U} \tilde{U}^T - UU^T \|_{2, \cdot} \cdot \| M_j \|_{\ell_2} \leq C \frac{\sigma \sqrt{m + n}}{\lambda \sqrt{k_m k_n}} \lambda \sqrt{k_m}. \]

This second approach will be used in the multiple submatrices analysis.

Combining all the above, we have with probability at least \( 1 - n^{-c} - m^{-c} \), for all \( 1 \leq j \leq n \)

\[ \| \mathcal{P}_0 X_j - M_j \|_{\ell_2} \leq C \cdot \left( \sigma \sqrt{\log n} + \sigma \sqrt{\frac{m \vee n}{k_n}} \right). \quad (31) \]

Similarly we have for all \( 1 \leq i \leq m \),

\[ \| \mathcal{P}_0 X_i^T - M_i^T \|_{\ell_2} \leq C \cdot \left( \sigma \sqrt{\log m} + \sigma \sqrt{\frac{m \vee n}{k_m}} \right). \quad (32) \]

Clearly we know that for \( i \in R_m \) and \( i' \in [m] \setminus R_m \)

\[ \| M_i^T - M_i^T \|_{\ell_2} = \lambda \sqrt{k_n} \]

and for \( j \in C_n \) and \( j' \in [n] \setminus C_n \)

\[ \| M_j - M_j \|_{\ell_2} = \lambda \sqrt{k_m} \]

Thus if

\[ \lambda \sqrt{k_m} \geq 6C \cdot \left( \sigma \sqrt{\log n} + \sigma \sqrt{\frac{m \vee n}{k_n}} \right) \quad (33) \]

\[ \lambda \sqrt{k_n} \geq 6C \cdot \left( \sigma \sqrt{\log m} + \sigma \sqrt{\frac{m \vee n}{k_m}} \right) \quad (34) \]

hold, then we have learned a metric \( d \) (a one dimensional line) such that on this line, data forms clusters in the sense that

\[ 2 \max_{i,i' \in R_m} |d_i - d_{i'}| \leq \min_{i \in R_m, i' \in [m] \setminus R_m} |d_i - d_{i'}|. \]

In this case, a simple cut-off clustering recovers the nodes exactly.

In summary, if

\[ \lambda \geq C \cdot \sigma \left( \sqrt{\frac{\log n}{k_m}} + \sqrt{\frac{\log m}{k_n}} + \sqrt{\frac{m + n}{k_m k_n}} \right), \]

20
the spectral algorithm succeeds with probability at least
\[ 1 - m^{-c} - n^{-c} - 2 \exp(-c(m+n)). \]

Proof of Theorem 2. Computational lower bound for localization (support recovery) is of different nature than the computational lower bound for detection (two point testing). The idea is to design a randomized polynomial time algorithmic reduction to relate an instance of hidden clique problem to our submatrix localization problem. The proof proceeds in the following way: we will construct a randomized polynomial time transformation \( T \) to map a random instance of \( \mathcal{G}(N, \kappa) \) to a random instance of our submatrix \( \mathcal{M}(m=n, k_m = k_n = k, \lambda/\sigma) \) (abbreviated as \( \mathcal{M}(n, k, \lambda/\sigma) \)). Then we will provide a quantitative computational lower bound by showing that if there is a polynomial time algorithm that pushes below the hypothesized computational boundary for localization in the submatrix model, there will be a polynomial time algorithm that solves hidden clique localization with high probability (a contradiction to \( HC \)).

Let us first introduce the basic lemmas used in this proof. The following two are Chernoff-type bounds for bounded random variables.

**Lemma 8** (Hoeffding (1963), Hoeffding’s Inequality). Let \( X_i, 1 \leq i \leq n \) be independent random variables. Assume \( a_i \leq X_i \leq b_i, 1 \leq i \leq n \). Then for \( S_n = \sum_{i=1}^n X_i \)
\[ P(|S_n - \mathbb{E}S_n| > u) \leq 2 \exp\left(-\frac{2u^2}{\sum_{i=1}^n (b_i - a_i)^2}\right). \] (35)

**Lemma 9** (Bennett (1962), Bernstein’s Inequality). Let \( X_i, 1 \leq i \leq n \) be independent zero-mean random variables. Suppose \( |X_i| \leq M, 1 \leq i \leq n \). Then
\[ P\left(\sum_{i=1}^n X_i > u\right) \leq \exp\left(-\frac{u^2/2}{\sum_{i=1}^n \mathbb{E}X_i^2 + Mu/3}\right). \] (36)

Denote the randomized polynomial time transformation as
\[ \mathcal{T} : \mathcal{G}(N, \kappa(N)) \rightarrow M(n, k = n^a, \lambda/\sigma = n^{-\beta}). \]

There are several stages for the construction of the algorithmic reduction. First we define a graph \( \mathcal{G}^e(N, \kappa(N)) \) that is stochastically equivalent to the hidden clique graph \( \mathcal{G}(N, \kappa(N)) \), but is easier for theoretical analysis. \( \mathcal{G}^e \) has the property: each node independently has the probability \( \kappa(N)/N \) to be a clique node,
and with the remaining probability a non-clique node. Using Bernstein's inequality and the inequality (42) proved below, with probability at least $1 - 2N^{-1}$ the number of clique nodes $\kappa^e$ in $\mathcal{G}^e$

$$\kappa \left(1 - \sqrt{\frac{4\log N}{\kappa}}\right) \leq \kappa^e \leq \kappa \left(1 + \sqrt{\frac{4\log N}{\kappa}}\right) \Rightarrow \kappa^e \approx \kappa$$

as long as $\kappa \gtrsim \log N$.

Consider a hidden clique graph $\mathcal{G}(2N, 2\kappa(N))$ with $N = n$ and $\kappa(N) = \kappa$. Denote the set of clique nodes for $\mathcal{G}(2N, 2\kappa(N))$ to be $C_{N, \kappa}$. Represent the hidden clique graph using the symmetric adjacency matrix $G \in \{-1, 1\}^{2N \times 2N}$, where $G_{ij} = 1$ if $i, j \in C_{N, \kappa}$, otherwise with equal probability to be either $-1$ or $1$. As remarked before, with probability at least $1 - 2N^{-1}$, we have planted $2\kappa(1 \pm o(1))$ clique nodes in graph $\mathcal{G}^e$ with $2N$ nodes. Take out the upper-right submatrix of $G$, denote as $G_{UR}$ where $U$ is the index set $1 \leq i \leq N$ and $R$ is the index set $N + 1 \leq j \leq 2N$. Now $G_{UR}$ has independent entries.

The construction of $\mathcal{F}$ employs the Bootstrapping idea. Generate $l^2$ matrices through bootstrap subsampling as follows. Generate $l - 1$ independent index vectors $\psi^{(s)} \in \mathbb{R}^n, 1 \leq s < l$, where each element $\psi^{(s)}(i), 1 \leq i \leq n$ is a random draw with replacement from the row indices $[n]$. Denote vector $\phi^{(t)}(i) = i, 1 \leq i \leq n$ as the original index set. Similarly, we can define independently the column index vectors $\phi^{(t)}, 1 \leq t < l$. We remark that these bootstrap samples can be generated in polynomial time $\Omega(l^2 n^2)$. The transformation is a weighted average of $l^2$ matrices of size $n \times n$ generated based on the original adjacency matrix $G_{UR}$.

$$\mathcal{F}: M_{ij} = \frac{1}{l} \sum_{0 \leq s, t < l} (G_{UR})_{\psi^{(s)}(i)\phi^{(t)}(j)}, \quad 1 \leq i, j \leq n.$$

Due to the bootstrapping property, the matrices $[(G_{UR})_{\psi^{(s)}(i)\phi^{(t)}(j)}]_{1 \leq i, j \leq n}$, indexed by $0 \leq s, t < l$ are independent of each other. Recall that $C_{N, \kappa}$ stands for the clique set of the hidden clique graph. We define the row candidate set $R_l := \{i \in [n] : \exists 0 \leq s < l, \psi^{(s)}(i) \in C_{N, \kappa}\}$ and column candidate set $C_l := \{j \in [n]: \exists 0 \leq t < l, \phi^{(t)}(j) \in C_{N, \kappa}\}$. Observe that $R_l \times C_l$ are the indices where the matrix $M$ contains signal.

There are two cases for $M_{ij}$, given the candidate set $R_l \times C_l$. If $i \in R_l$ and $j \in C_l$, namely when $(i, j)$ is a clique edge in at least one of the $l^2$ matrices, then $\mathbb{E}M_{ij} \geq l^{-1}$ where the expectation is taken over the bootstrap $\sigma$-field conditioned on the candidate set $R_l \times C_l$ and the original $\sigma$-field of $\mathcal{G}^e$. Otherwise
\( \mathbb{E} M_{i,j} = 0 \) for \((i,j) \in R_l \times C_l\). Note that for the index \(i,j\) that is not a clique for any of the matrices, \(M_{i,j}\) is sub-Gaussian, due to Hoeffding's inequality

\[
P( |M_{i,j} - \mathbb{E} M_{i,j}| \geq u ) \leq 2 \exp(-u^2/2). \tag{39}
\]

For the index \(i,j\) being a clique in at least one of the matrices, we claim the number of matrices has \((i,j)\) being clique is \(O^*\). Due to Bernstein's inequality, we have \(\max_{i} |0 \leq s < l : \psi^{(s)}(i) \in C_{N,k}| \leq \frac{\kappa l}{n} + \frac{8}{3} \log n\) with probability at least \(1 - n^{-1}\). This further implies there are at least \(l^2 - (\frac{\kappa l}{n} + \frac{8}{3} \log n)^2\) many independent Rademacher random variables in each \(i,j\) position, thus

\[
P( |M_{i,j} - \mathbb{E} M_{i,j}| \geq u ) \leq 2 \exp \left( - \frac{u^2}{l^2 - (\frac{\kappa l}{n} + \frac{8}{3} \log n)^2} \right). \tag{40}
\]

Up to now we have proved that when \(i,j\) is a signal node for \(M\), then \(O^*\) \(\geq \mathbb{E} M_{i,j} \geq l^{-1}\). Thus we can take sub-Gaussian parameter to be any \(\sigma < 1\) because \(\kappa n^{-1}, l^{-1} \log n\) are both \(o(1)\). The constructed \(M(n,k,\lambda/\sigma)\) matrix satisfies the submatrix model with \(\lambda/\sigma = l^{-1}\) and sub-Gaussian parameter \(\sigma = 1 - o(1)\).

Let us estimate the corresponding \(k\) in the submatrix model. We need to bound the order of the cardinality of \(R_l\), denoted as \(|R_l|\). The total number of positions with signal (at least one clique node inside) is

\[
\mathbb{E} |R_l| = \mathbb{E} |\{1 \leq i \leq n : i \in R_l\}| = n \left[1 - (1 - \kappa / n)^l\right].
\]

Thus we have the two sided bound

\[
\kappa l \left(1 - \frac{\kappa l}{2n}\right) \leq \mathbb{E} |R_l| \leq \kappa l
\]

which is of the order \(k := \kappa l\). Let us provide a high probability bound on \(|R_l|\). By Bernstein's inequality

\[
P( ||R_l| - \mathbb{E} |R_l|| > u ) \leq 2 \exp \left( - \frac{u^2}{2 \kappa l + u/3} \right). \tag{41}
\]

Thus if we take \(u = \sqrt{4\kappa l \log n}\), as long as \(\log n = o(\kappa l)\),

\[
P( ||R_l| - \mathbb{E} |R_l|| > \sqrt{4\kappa l \log n} ) \leq 2n^{-1}. \tag{42}
\]

So with probability at least \(1 - 2n^{-1}\), the number of positions that contain signal nodes is bounded as

\[
\kappa l \left(1 - \frac{\kappa l}{n}\right) \left(1 - \sqrt{\frac{4 \log n}{\kappa l}}\right) < |R_l| < \kappa l \left(1 + \sqrt{\frac{4 \log n}{\kappa l}}\right) \Rightarrow |R_l| \approx \kappa l. \tag{43}
\]
Equation (43) implies that with high probability
\[ \kappa l (1 - o(1)) \leq |R_l| \leq \kappa l (1 + o(1)), \]
\[ \kappa l (1 - o(1)) \leq |C_l| \leq \kappa l (1 + o(1)). \]

The above means, in the submatrix parametrization, \( k_m = k_n = k \approx n^{\alpha}, \lambda / \sigma \approx l^{-1} \approx n^{-\beta} \), which implies \( \kappa \approx n^{\alpha - \beta} \).

Suppose there exists a polynomial time algorithm \( \mathcal{A}_M \) that pushes below the computational boundary. In other words,
\[ n^{-\beta} \gtrsim \frac{\lambda}{\sigma} \sim \sqrt{\frac{m+n}{k_m k_n}} \approx n^{(1-2\alpha)/2} \Rightarrow \beta > \frac{\alpha}{2} \] (44)
with the last inequality having a slack \( \epsilon > 0 \). More precisely, \( \mathcal{A}_M \) returns two estimated index sets \( \hat{R}_n \) and \( \hat{C}_n \) corresponding to the location of the submatrix (and correct with probability going to 1) under the regime \( \beta = \alpha - 1/2 + \epsilon \). Suppose under some conditions, this algorithm \( \mathcal{A}_M \) can be modified to a randomized polynomial time algorithm \( \mathcal{A}_g \) that correctly identifies the hidden clique nodes with high probability. It means in the corresponding hidden clique graph \( \mathcal{G}(2N, 2\kappa) \), \( \mathcal{A}_g \) also pushes below the computational boundary of hidden clique by the amount \( \epsilon \):
\[ \kappa(N) = 2\kappa \approx (2n)^{a-\beta} \approx n^{1/2-\epsilon} \lesssim n^{1/2} \approx N^{1/2}. \] (45)

In summary, the quantitative computational lower bound implies that if the computational boundary for submatrix localization is pushed below by an amount \( \epsilon \) in the power, the hidden clique boundary is correspondingly improved by \( \epsilon \).

Now let us show that any algorithm \( \mathcal{A}_M \) that localizes the submatrix introduces a randomized algorithm that finds the hidden clique nodes with probability tending to 1. The algorithm relies on the following simple lemma.

**Lemma 10.** For the hidden clique model \( \mathcal{G}(N, \kappa) \), suppose an algorithm provides a candidate set \( S \) of size \( k \) that contains the true clique subset exactly. If
\[ \kappa \gtrsim C \sqrt{k \log N} \]
then by looking at the adjacency matrix restricted to \( S \) we can recover the clique subset exactly with high probability.

The proof of Lemma 10 is immediate. If \( i \) is a clique node, then \[ \min_i \sum_{j \in C} G_{ij} \geq \kappa - C/2 \cdot \sqrt{k \log N}. \] If \( i \) is not a clique node, then \[ \max_i \sum_{j \in C} G_{ij} \leq C/2 \cdot \sqrt{k \log N}. \] The proof is completed.
Algorithm $A_M$ provides candidate sets $R_l, C_l$ of size $k$, inside which $\kappa$ are correct clique nodes, and thus exact recovery can be completed through Lemma 10 since $\kappa \gtrsim (k \log N)^{1/2}$ (since $\kappa \approx n^{1/2-\epsilon} \gtrsim k^{1/2} \approx n^{\alpha/2}$ when $\epsilon$ is small). The algorithm $A_M$ induces another randomized polynomial time algorithm $A_G$ that solves the hidden clique problem $\mathcal{G}(2N, 2\kappa)$ with $\kappa \lesssim N^{1/2}$. The algorithm $A_G$ returns the support $\hat{C}_N, \kappa$ that coincides with the true support $C_N, \kappa$ with probability going to 1 (a contradiction to the hidden clique hardness hypothesis $HC_l$). We conclude that, under the hypothesis, there is no polynomial time algorithm $A_M$ that can push below the computational boundary $\lambda \gtrsim \sqrt{\frac{m+n}{k m k_n}}$.

Proof of Lemma 2. The proof of this lemma uses the well-known Fano’s information inequality.

Lemma 11 (Tsybakov (2009) Corollary 2.6). Let $P_0, P_1, \ldots, P_M$ be probability measures on the same probability space $(\Theta, \mathcal{F}), M \geq 2$. If for some $0 < \alpha < 1$

$$\frac{1}{M+1} \sum_{i=0}^{M} d_{KL}(P_i || \bar{P}) \leq \alpha \cdot \log M \quad (46)$$

where

$$\bar{P} = \frac{1}{M+1} \sum_{i=0}^{M} P_i.$$

Then

$$p_{e,M} \geq \hat{p}_{e,M} \geq \frac{\log(M+1) - \log 2}{\log M} - \alpha \quad (47)$$

where $p_{e,M}$ is the minimax error for the multiple testing problem.

We have that $X = M + Z$, where $M \in \mathbb{R}^{m \times n}$ is the mean matrix. Under the Gaussian noise with parameter $\sigma$, the probability model is

$$P(X|M) \propto \exp \left( - \langle X - M, X - M \rangle / 2\sigma^2 \right) \quad (48)$$

where $M = \lambda \cdot UV^T \in \Theta$. The parameter space $\Theta$ is composed of all $M = \lambda \cdot UV^T$ where $U$ are sampled uniformly on the collection of vectors with $k_m$ ones and other coordinates being zero, and similarly $V$ are sampled uniformly with $k_n$ ones and the rest zero. The cardinality of the parameter space is

$$\text{Card}(\Theta) = \binom{m}{k_m} \binom{n}{k_n}.$$
corresponding to that many probability measures on the same probability space. Put a uniform prior on this parameter space and invoke Fano’s lemma 11. To obtain the lower bound, we need to upper bound the Kullback-Leibler divergence \( d_{\text{KL}}(\mathcal{P}_M||\bar{\mathcal{P}}) \) for any \( M \in \Theta \), where
\[
\bar{\mathcal{P}} = \mathbb{E}_{M' \sim \text{unif}(\Theta)} \mathcal{P}_{M'}.
\]
For any \( M \in \Theta \),
\[
d_{\text{KL}}(\mathcal{P}_M||\bar{\mathcal{P}}) = \mathbb{E}_{\mathcal{P}_M} \log \frac{\mathcal{P}_M}{\bar{\mathcal{P}}}
\leq \mathbb{E}_{X \sim \mathcal{P}_M} \left\{ -\frac{(X - M, X - M)}{2\sigma^2} + \frac{1}{\text{Card}(\Theta)} \sum_{M' \in \Theta} \frac{(X - M', X - M')}{2\sigma^2} \right\}
\leq \mathbb{E}_{X \sim \mathcal{P}_M} \left\{ \frac{1}{\text{Card}(\Theta)} \sum_{M' \in \Theta} \frac{(M - M', X - M)}{2\sigma^2} + \frac{1}{\text{Card}(\Theta)} \sum_{M' \in \Theta} \frac{(M - M', M - M')}{2\sigma^2} \right\}
\leq \frac{1}{\text{Card}(\Theta)} \sum_{M' \in \Theta} \frac{(M - M', M - M')}{2\sigma^2}
\leq \frac{\langle M, M \rangle + \frac{1}{\text{Card}(\Theta)} \sum_{M' \in \Theta} (M', M') - 2 \frac{1}{\text{Card}(\Theta)} \sum_{M' \in \Theta} (M, M')}{2\sigma^2}
\leq \frac{\lambda^2 k_m k_n}{\sigma^2} (1 - \frac{k_m k_n}{mn}).
\]
Thus as long as
\[
\frac{\lambda^2 k_m k_n}{\sigma^2} (1 - \frac{k_m k_n}{mn}) \leq \alpha \log \left( \begin{pmatrix} m \\ k_m \end{pmatrix} \begin{pmatrix} n \\ k_n \end{pmatrix} \right)
\]
we have
\[
\frac{1}{\text{Card}(\Theta)} \sum_{M \in \Theta} d_{\text{KL}}(\mathcal{P}_M||\bar{\mathcal{P}}) \leq \alpha \log(\text{Card}(\Theta)).
\]
Invoke the simple bound on binomial coefficients \( \left( \frac{n}{k} \right)^k \leq \left( \frac{n}{k} \right) \leq \left( \frac{kn}{k} \right)^k \). If we choose
\[
\lambda \leq C_\alpha \cdot \sigma \sqrt{\frac{k_m \log m + k_n \log n}{k_m k_n}}
\]
then the condition (46) holds. Any submatrix localization algorithm translates into a multiple testing procedure that picks a parameter \( M' \in \Theta \). By Fano’s information inequality 11, the minimax error, which is also the localization error, is at least
\[
1 - \alpha \leq \frac{\log^2 k_m \log m + k_n \log n}{k_m k_n}.
\]
Proof of Lemma 3. Recall the definition 4 of a sub-Gaussian random variable. Taking $Z_{i,j}, i \in I, j \in J$, we have the following concentration from the Chernoff’s bound for $\sum_{i \in I, j \in J} Z_{i,j}$

$$\mathbb{E}e^{\lambda \sum_{i \in I, j \in J} Z_{i,j}} \leq \exp\left(\|I\|\|J\| \cdot \sigma^2 \lambda^2 / 2c\right)$$

and

$$\mathbb{P}\left(\left| \sum_{i \in I, j \in J} Z_{i,j} \right| \geq \sqrt{\|I\|\|J\|} \sigma t\right) \leq 2 \exp(-c \cdot t^2 / 2).$$

There are in total

$$\binom{m}{k_m} \binom{n}{k_n}$$

such submatrices, so by a union bound, we have

$$\mathbb{P}\left(\max_{\|I\|=k_m, \|J\|=k_n} \left| \sum_{i \in I, j \in J} Z_{i,j} \right| \geq \sqrt{\|I\|\|J\|} \sigma t\right) \leq \binom{m}{k_m} \binom{n}{k_n} \cdot 2 \exp(-c \cdot t^2 / 2)
\leq 2 \exp\left(k_m \log(\frac{em}{k_m}) + k_n \log(\frac{en}{k_n}) - c t^2 / 2\right).$$

If we take $t = 2\sqrt{\frac{k_m \log(\frac{em}{k_m}) + k_n \log(\frac{en}{k_n})}{c}}$, then with probability at least

$$1 - 2\left(\binom{m}{k_m} \binom{n}{k_n}\right)^{-1}$$

we have

$$\max_{\|I\|=k_m, \|J\|=k_n} \left| \sum_{i \in I, j \in J} Z_{i,j} \right| \leq \frac{2}{\sqrt{c}} \sqrt{\|I\|\|J\|} \sigma \sqrt{k_m \log(\frac{em}{k_m}) + k_n \log(\frac{en}{k_n})}.$$

Thus if

$$k_m k_n \lambda > 2 \max_{\|I\|=k_m, \|J\|=k_n} \left| \sum_{i \in I, j \in J} Z_{i,j} \right|$$

then the maximum submatrix is unique and is the true one. Recollecting terms, we reach

$$\lambda > \frac{4}{\sqrt{c}} \cdot \sigma \sqrt{\frac{\log(\frac{em}{k_m})}{k_n} + \frac{\log(\frac{en}{k_n})}{k_m}}.$$

To make the proof fully rigorous, we need the following monotonicity trick. Consider the submatrix of size $k_m k_n$ with $a$ rows to be in the correct set $R_m$ and
\( b \) columns to be in the correct set \( C_n \), where \( a < k_m \) and \( b < k_n \). The cardinality of the set of such matrices is

\[
\begin{pmatrix}
m - a \\
k_m - a
\end{pmatrix}
\begin{pmatrix}
n - a \\
k_n - b
\end{pmatrix}.
\]

Using the same calculation as before we want

\[
\lambda > \frac{4}{\sqrt{c}} \cdot \sigma \sqrt{\frac{(k_m - a) \log(e(m - a)/(k_m - a)) + (k_n - b) \log(e(n - b)/(k_n - b))}{k_m k_n - ab}}.
\] (57)

By simple algebra,

\[
\frac{k_m - a}{k_m k_n - ab} < \frac{1}{k_n} \quad (58)
\]

\[
\frac{k_n - b}{k_m k_n - ab} < \frac{1}{k_m} \quad (59)
\]

\[
\log(e(m - a)/(k_m - a)) < \log(em) \quad (60)
\]

\[
\log(e(n - b)/(k_n - b)) < \log(en). \quad (61)
\]

Hence, if equation (56) is satisfied, (57) is satisfied up to a universal constant for all \( a < k_m \) and \( b < k_n \). Thus we have proved that if

\[
\lambda \geq C \cdot \sigma \sqrt{\frac{\log(em/k_m)}{k_n} + \frac{\log(en/k_n)}{k_m}}
\]

with a suitable constant \( C \), the statistical search algorithm picks out the correct submatrix. The sum of the probabilities of the bad events is bounded by

\[
\sum_{0 \leq a < k_m, 0 \leq b < k_n} \left( \begin{pmatrix} m - a \\ k_m - a \end{pmatrix} \begin{pmatrix} n - b \\ k_n - b \end{pmatrix} \right)^{-1} \frac{k_m k_n}{(m - k_m)(n - k_n)}.
\]

Proof of Lemma 1 for Multiple Non-overlapping Submatrices Case. We are going to provide theoretical justification to the extension of the submatrix localization algorithm to multiple non-overlapping submatrices case as in Algorithm 3. Write out the matrix form of the submatrix model, with the SVD version of the signal matrix \( M \)

\[
X = M + Z = U \Lambda V^T + Z.
\]
Due to the non-overlapping property, we have $1 \leq s \neq t \leq r$, $1_{R_s}^T 1_{R_t} = 0$, so as to $C_n$. The singular values of $UAV^T$ are $\lambda_s \sqrt{k_s^{(m)} k_s^{(n)}}, 1 \leq s \leq r$, and all the other singular values are 0.

Let's apply the Davis-Kahan-Wedin bound to $X = UAV^T + Z$. Denote the top $r$ left and right singular vector of $X$ as $\tilde{U}$ and $\tilde{V}$. Using Weyl's interlacing inequality, we have

$$|\sigma_s(X) - \sigma_s(M)| \leq \|Z\|_2$$

and

$$\sigma_s(X) \geq \lambda_s \sqrt{k_s^{(m)} k_s^{(n)}} - \|Z\|_2, \; 1 \leq s \leq r;$$

$$\sigma_t(X) \leq \|Z\|_2, \; t > r.$$  

Thus applying Lemma 5, we have

$$\max\{|\sin \angle(U, \tilde{U})|, |\sin \angle(V, \tilde{V})|\} \leq \frac{C \sigma(\sqrt{m} + \sqrt{n})}{\min_{1 \leq s \leq r} \lambda_s \sqrt{k_s^{(m)} k_s^{(n)}} - C \sigma(\sqrt{m} + \sqrt{n})} \min_{1 \leq s \leq r} \lambda_s \sqrt{k_s^{(m)} k_s^{(n)}}$$

According to the definition of the canonical angles, we have

$$\max\{\|UU^T - \tilde{U}\tilde{U}^T\|_2, \|VV^T - \tilde{V}\tilde{V}^T\|_2\} \leq C \cdot \frac{\sigma(\sqrt{m} + \sqrt{n})}{\min_{1 \leq s \leq r} \lambda_s \sqrt{k_s^{(m)} k_s^{(n)}}}.$$  

Now let's assume we have two observation of $X$. We use the first observation $\tilde{X}$ to solve for the singular vectors $\tilde{U}, \tilde{V}$, we use the second observation $X$ to project the to the singular vectors $\tilde{U}, \tilde{V}$. Recall $X = M + Z = UAV^T + Z$. For $1 \leq j \leq n$

$$\|P_{\tilde{U}}X_j - M_j\|_{\ell_2} \leq \|P_{\tilde{U}}(X_j - M_j)\|_{\ell_2} + \|(P_{\tilde{U}} - I)M_j\|_{\ell_2} \quad (62)$$

For the first term of (26) because $X_j - M_j = Z_j \in \mathbb{R}^m$ is an i.i.d. isotropic sub-Gaussian vector, we have through Lemma 7, for $t = (1 + 1/c) \log n$, $Z_j \in \mathbb{R}^m, 1 \leq j \leq n$ and $r > 0$

$$\mathbb{P}\left(\|P_{\tilde{U}}(X_j - M_j)\|_{\ell_2} \geq \sigma \sqrt{T} \sqrt{1 + 2 \sqrt{1 + 1/c \cdot \frac{\log n}{r}} + 2(1 + 1/c) \cdot \frac{\log n}{r}}\right) \leq n^{-c-1}.$$  

(63)
Thus invoke the union bound for all $1 \leq j \leq n$
\[
\max_{1 \leq j \leq n} \| \mathcal{P}_U (X_j - M_j) \|_{\ell_2} \leq \sigma \sqrt{r} + \sqrt{1 + 1/c} \cdot \sigma \sqrt{\log n + (1 + 1/c) \frac{\log n}{\sqrt{r}}} \quad (64)
\]
\[
\leq \sigma \sqrt{r} + C \cdot \sigma \sqrt{\log n} \quad (65)
\]
with probability at least $1 - n^{-c}$.

For the second term $M_j = \hat{X}_j - \hat{Z}_j$ of (26). It can be estimated through perturbation Sin Theta Theorem 5. Basically it is
\[
\| (\mathcal{P}_U - I) M_j \|_{\ell_2} \leq \max_{1 \leq j \leq n} \lambda_s \sqrt{k_s} \quad (66)
\]
\[
\leq C \cdot \frac{\sigma \sqrt{m + n}}{\min_{1 \leq s \leq r} \lambda_s \sqrt{k_s}^{(m)} \sqrt{k_s^{(n)}}} \quad (67)
\]

Combining all the above, we have with probability at least $1 - n^{-c} - m^{-c}$, for all $1 \leq j \leq n$
\[
\| \mathcal{P}_U X_j - M_j \|_{\ell_2} \leq C \cdot \left( \sigma \sqrt{r} + \sigma \sqrt{\log n} + \sigma \sqrt{m + n} \cdot \frac{\max_{1 \leq s \leq r} \lambda_s \sqrt{k_s^{(m)}}}{\min_{1 \leq s \leq r} \lambda_s \sqrt{k_s^{(m)} \sqrt{k_s^{(n)}}}} \right) \quad (68)
\]

Similarly we have for all $1 \leq i \leq m$,
\[
\| \mathcal{P}_U X_i^T - M_i^T \|_{\ell_2} \leq C \cdot \left( \sigma \sqrt{r} + \sigma \sqrt{\log m} + \sigma \sqrt{m + n} \cdot \frac{\max_{1 \leq s \leq r} \lambda_s \sqrt{k_s^{(m)}}}{\min_{1 \leq s \leq r} \lambda_s \sqrt{k_s^{(m)} \sqrt{k_s^{(n)}}}} \right) \quad (69)
\]

Clearly we know for any $1 \leq s \leq r$ and $i \in R_s$ and $i' \in [m] \setminus R_s$
\[
\| M_i^T - M_{i'}^T \|_{\ell_2} \geq \min_{1 \leq s \leq r} \lambda_s \sqrt{k_s^{(n)}}
\]
and for any $1 \leq s \leq r$ and $j \in C_s$ and $j' \in [n] \setminus C_s$
\[
\| M_j - M_{j'} \|_{\ell_2} \geq \min_{1 \leq s \leq r} \lambda_s \sqrt{k_s^{(m)}}.
\]

Thus if $k_s^{(m)} \approx k_m$, $k_s^{(n)} \approx k_n$ and $\lambda_s \approx \lambda$ for all $1 \leq s \leq r$
\[
\lambda \sqrt{k_m} \geq 6C \cdot \left( \sigma \sqrt{r} + \sigma \sqrt{\log n} + \sigma \frac{m + n}{k_n} \right) \quad (70)
\]
\[
\lambda \sqrt{k_n} \geq 6C \cdot \left( \sigma \sqrt{r} + \sigma \sqrt{\log m} + \sigma \frac{m + n}{k_m} \right) \quad (71)
\]
We have learned a metric $d$ (of intrinsic dimension $r$) such that under this metric, data forms into clusters in the sense that

$$2 \max_{i, i' \in R_s} |d_i - d_{i'}| \leq \min_{i \in R_m, i' \in [m] \setminus R_s} |d_i - d_{i'}|.$$ 

Thus it satisfies the geometric separation property.

Thus in summary if

$$\lambda \geq C \cdot \sigma \left( \sqrt{\frac{r}{k_m \wedge k_n}} + \sqrt{\frac{\log n}{k_m}} + \sqrt{\frac{\log m}{k_n}} + \sqrt{\frac{m \lor n}{k_m k_n}} \right),$$

the spectral algorithm succeeds with probability at least

$$1 - m^{-c} - n^{-c} - 2 \exp(-c(m + n)).$$

Due to the fact that

$$\sqrt{\frac{r}{k_m \wedge k_n}} \lesssim \sqrt{\frac{m \lor n}{k_m k_n}}$$

because $r k_m \lesssim m$, $r k_n \lesssim n$ in most cases, the first term does not have an effect in most cases.

### A Convex Relaxation Algorithm

In this section we will investigate a convex relaxation approach to the problem. The same algorithm has also been investigated in a parallel work of Chen and Xu (2014). Our analysis is slightly different, with the explicit construction of the dual certificate using the idea in Gross (2011). For the purposes of comparing to the spectral approach, we include the convex relaxation analysis in this section.

Let us write the optimization problem

$$\min_{u \in \mathbb{R}^m, v \in \mathbb{R}^n} \|X - \lambda u v^T\|_F^2$$

s.t. $\|u\|_{\ell_0} = k_m$, $\|v\|_{\ell_0} = k_n$

$$u \in \{0, 1\}^m, v \in \{0, 1\}^n.$$ 

This problem is non-convex: the feasibility set is non-convex, and so is the optimization function (although it is bi-convex). However, we can relax the problem and transform it into a convex optimization problem. Of course, we need to ensure that the solution to the relaxed problem is the exact solution (with high probability) under appropriate conditions.
The matrix version of the submatrix problem suggests that the signal matrix is of the structure “low rank and sparsity on the singular vectors.” We recall from the low rank matrix recovery literature, see e.g. Candes and Plan (2011) and Cai et al. (2014), that we can utilize the low rank structure and solve relaxed versions as follows.

**Relaxation 1**  Consider the constraint minimization relaxation,

\[
\min_{M \in \mathbb{R}^{m \times n}} \|M\|_* \\
\text{s.t.} \quad \|X - \lambda M\|_2 \leq C \cdot \sigma(\sqrt{m} + \sqrt{n}).
\]

Unfortunately, Relaxation 1 is only good in terms of estimation of the whole matrix. The stronger objective of localization requires simultaneous exploitation of sparsity and low rank-ness, as in Relaxation 2.

**Relaxation 2**  Let us expand the objective of the original non-convex optimization problem, drop the quadratic term to make the procedure adaptive in terms of \(\lambda\), and convexify the feasibility set at the same time.

**Algorithm 4: Convex Relaxation Algorithm**

**Input:** \(X \in \mathbb{R}^{m \times n}\) the data matrix. Size of the submatrix \(k_m \times k_n\).

**Output:** A subset of the row indexes \(\hat{R}_m\) and a subset of column indices \(\hat{C}_n\) as the localization sets of the submatrix.

1. Solve the following convex optimization problem

\[
\hat{M}_0 = \arg\min_{M \in \mathbb{R}^{m \times n}} -\langle X, M \rangle \\
\text{s.t.} \quad 0 \leq M \leq 1_m 1_n^T \\
\quad \|M\|_* \leq 1 \\
\quad \langle M, 1_m 1_n^T \rangle = k_m k_n
\]

2. Perform SVD on \(\hat{M}_0\), denote the set of the non-zero entries on the top left singular vector to be \(\hat{R}_m\) and the set of the non-zero entries on the top right singular vector to be \(\hat{C}_n\).

The time complexity to solve this convex optimization problem is at least \(\Theta((m+n)^3)\) implemented with alternating direction methods of multipliers (ADMM). The disadvantage is that the theoretical guarantee only holds for the exact solution \(\hat{X}_0\); however, in reality we can only approximately find \(\hat{X}_0\) through ADMM or some other optimization methods. We also remark that this algorithm re-
quires the prior knowledge of the submatrix size $k_m, k_n$, which means it is not fully adaptive.

**Lemma 12** (Guarantee for Relaxation Algorithm). Consider the submatrix model (2) and the Algorithm 4. There exists a universal $C > 0$ such that when

$$\frac{\lambda}{\sigma} \geq C \left( \sqrt{\frac{m \lor n}{k_m k_n}} + \sqrt{\frac{\log n}{k_m} \lor \frac{\log m}{k_n}} \right),$$

the convex relaxation succeeds (in the sense that $\hat{R}_m = R_m, \hat{C}_n = C_n$) with probability at least $1 - 2m^{-c} - 2n^{-c} - 2(mn)^{-c} - 2\exp(-c(m + n))$.

**Proof of Lemma 12.** Let us construct the dual certificate to secure that the true solution is the unique solution. If we can construct a pair of a primal certificate $M^* = \sqrt{k_m k_n} U V^T$ and dual certificate $\Delta^*, \Theta^*, \mu^*, \nu^*$ that satisfy

$$X = -\Delta^* + \Theta^* + \mu^* (U V^T + W) + \nu^* 1_n 1_m^T \quad (72)$$

$$\Delta^*_{ij} M^*_{ij} = 0, \Theta^*_{ij} (1 - M^*_{ij}) = 0, \quad 1 \leq i \leq m, 1 \leq j \leq n \quad (73)$$

$$\mu^* > 0. \quad (74)$$

Here $W \in \mathcal{D}_{U^\perp \cap V^\perp}, \|W\|_2 \leq 1$, and $U V^T + W$ denotes the sub differential of $\|\cdot\|_*$ evaluated at $M^*$

$$\partial |_{M=M^*} \|M\|_* = U V^T + W.$$ 

Equation (73) is equivalent to

$$\Delta^*_{ij} > 0, \quad \Theta^*_{ij} = 0, \quad i \notin R_m \cup j \notin C_n \quad (75)$$

$$\Theta^*_{ij} > 0, \quad \Delta^*_{ij} = 0, \quad i \in R_m \cap j \in C_n. \quad (76)$$

We claim that any solution $\hat{M}$ to Relaxation 2 must satisfy $\hat{M} = M^*$. If not, we write $\hat{M} = M^* + H$ and find that

$$0 \leq M^* + H \leq 1_n 1_m^T \quad (77)$$

$$\|M^* + H\|_* \leq \|M^*\|_* \Rightarrow \langle H, U V^T + W \rangle \leq 0 \quad (78)$$

$$\langle H, 1_n 1_m^T \rangle = 0. \quad (79)$$

All of the above equations are due to the primal feasibility, and the second inequality also uses the convexity of $\|\cdot\|_*$. Note that (77) can be written in a more explicit form

$$0 \leq H_{ij} \leq 1, \quad i \notin R_m \cup j \notin C_n \quad (80)$$

$$-1 \leq H_{ij} \leq 0, \quad i \in R_m \cap j \in C_n. \quad (81)$$

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Due to the optimality of $\hat{M}$ in terms of the objective function

$$\langle X, M^* + H \rangle \geq \langle X, M^* \rangle$$

which means

$$0 \leq \langle X, H \rangle \quad (82)$$

$$= \langle -\Delta^* + \Theta^* + \mu^*(U V^T + W) + v^* 1_m 1_n^T, H \rangle \quad (83)$$

$$\leq \langle -\Delta^* + \Theta^*, H \rangle. \quad (84)$$

We can see that if $H_{ij} > 0$, then we must have $M^*_{ij} = 0$, which through complementary slackness implies $\Theta^*_{ij} = 0$. This in turn means $-\Delta_{ij} < 0$. When $H_{ij} < 0$, we must have $M^*_{ij} = 1$, which again means $\Delta_{ij} = 0, \Theta^*_{ij} > 0$, a contradiction. Thus $H_{ij} = 0$ for all $i, j$.

The properties we impose on the dual certificates are motivated from the Karush-Kuhn-Tucker (KKT) conditions. Introduce the dual variables $\Delta, \Theta \in \mathbb{R}^{m \times n}, \mu \in \mathbb{R}_+, v \in \mathbb{R}$ for the four feasibility conditions. Then the Lagrangian is

$$\mathcal{L}(M, \Delta, \Theta, \mu, v) = -\langle X, M \rangle - \langle \Delta, M \rangle + \langle \Theta, M - 1_m 1_n^T \rangle + \mu(\|M\|_* - 1) + v\left(\langle M, 1_m 1_n^T \rangle \right).$$

The associated KKT conditions are

$$X = -\Delta + \Theta + \mu(U M V^T_M + W_M) + v 1_m 1_n^T \quad (85)$$

$$0 \leq M \leq 1_m 1_n^T \quad (86)$$

$$\|M\|_* \leq 1 \quad (87)$$

$$\langle M, 1_m 1_n^T \rangle = k_m k_n \quad (88)$$

$$\Delta \geq 0, \Theta \geq 0, \mu \geq 0 \quad (89)$$

$$\Delta_{ij} M_{ij} = 0, \Theta_{ij}(1 - M_{ij}) = 0. \quad (90)$$

Now let us see how to construct the dual certificate $-\Delta^* + \Theta^*, \mu^*, v^*$ to satisfy the conditions (72) - (74). Expand (85) as

$$-\Delta^* + \Theta^* = \lambda 1_m 1_n^T + \mathcal{P}_{U \cup V}(Z) + \mathcal{P}_{U \cap V^T}(Z) - \mu\left(\frac{1}{\sqrt{k_m k_n}} 1_m 1_n^T C_n + W\right) - v^* 1_m 1_n^T \quad (91)$$

where

$$\mathcal{P}_{U \cap V^T}(Z) = Z - \mathcal{P}_{U \cup T}(Z). \quad (92)$$
Choose
\[ W = \frac{1}{\mu^*} \mathcal{P}_{U \cap V^T}(Z). \] (93)

Thus if we choose \( \mu^* \geq C \cdot \sigma(\sqrt{m} + \sqrt{n}) \), it holds that
\[ \|W\|_2 = \frac{1}{\mu^*} \|\mathcal{P}_{U \cap V^T}(Z)\|_2 \leq \frac{1}{\mu^*} \|Z\|_2 \leq 1 \]
with probability at least \( 1 - 2 \exp(-c(m+n)) \). Thus with the choice of \( W \), Equation (91) becomes
\[ -\Delta^* + \Theta^* = \lambda 1_{R_m} 1_{C_n}^T + \mathcal{P}_{U \cup V}(Z) - \mu^* \frac{1}{\sqrt{k_m k_n}} 1_{R_m} 1_{C_n}^T - \nu^* 1_m 1_n^T. \] (94)

Hence, we need to have
\[ \lambda - \mu^* \frac{1}{\sqrt{k_m k_n}} - \nu^* - \max_{ij} [\mathcal{P}_{U \cup V}(Z)]_{ij} > 0, \quad i \in R_m \cap j \in C_n \]
\[ \max_{ij} [\mathcal{P}_{U \cup V}(Z)]_{ij} - \nu^* < 0, \quad i \in R_m^c \cup j \in C_n^c. \] (95) (96)

Now let us write out the explicit form of the projection \( \mathcal{P}_{U \cup V}(Z) \)
\[ \mathcal{P}_{U \cup V}(Z) = \frac{1}{k_m} 1_{R_m} 1_{R_m}^T Z + \frac{1}{k_n} Z 1_{R_n} 1_{R_n}^T - \frac{1}{k_m k_n} 1_{R_m} 1_{C_n}^T Z 1_{R_n} 1_{R_n}^T. \] (97)

Let us see the concentration property of \( [\mathcal{P}_{U \cup V}(Z)]_{ij} \):
\[ [\mathcal{P}_{U \cup V}(Z)]_{ij} = \frac{1}{k_m} \left( \sum_{k \in R_m} Z_{kj} \right) 1_{i \in R_m} + \frac{1}{k_n} \left( \sum_{l \in C_n} Z_{il} \right) 1_{j \in C_n} - \frac{1}{k_m k_n} \left( \sum_{k \in R_m, l \in C_n} Z_{ij} \right) 1_{i \in R_m, j \in C_n} \] (98)
\[ [\mathcal{P}_{U \cup V}(Z)]_{ij} \leq \left| \frac{1}{k_m} \sum_{k \in R_m} Z_{kj} \right| + \left| \frac{1}{k_n} \sum_{l \in C_n} Z_{il} \right| + \left| \frac{1}{k_m k_n} \sum_{k \in R_m, l \in C_n} Z_{ij} \right|. \] (99)

For all the \( 1 \leq j \leq n \)
\[ \max_{1 \leq j \leq n} \left| \frac{1}{k_m} \sum_{k \in R_m} Z_{kj} \right| \leq \sqrt{2(1+1/c)} \cdot \sigma \sqrt{\frac{\log n}{k_m}} \] (100)
with probability at least \( 1 - 2n^{-c} \). For all the \( 1 \leq i \leq m \)
\[ \max_{1 \leq i \leq m} \left| \frac{1}{k_n} \sum_{l \in C_n} Z_{il} \right| \leq \sqrt{2(1+1/c)} \cdot \sigma \sqrt{\frac{\log m}{k_n}} \] (101)
with probability at least $1 - 2m^{-c}$. For all $i, j$,

$$
\max_{i,j} \left| \frac{1}{k_m k_n} \sum_{k \in R_m, l \in C_n} Z_{i,j} \right| \leq \sqrt{2(1 + 1/c)} \cdot \sigma \sqrt{\log(mn) \over k_m k_n} \tag{102}
$$

with probability at least $1 - 2(mn)^{-c}$. Now, pick the dual certificate variables in the following way

$$
\mu^* = C \cdot \sigma \left( \sqrt{m} + \sqrt{n} \right) \tag{103}
$$

$$
\nu^* = C \cdot \sigma \left( \sqrt{\log m \over k_n} + \sqrt{\log n \over k_m} \right) \tag{104}
$$

$$
\Theta^* > 0, \ i \in R_m \cap j \in C_n \tag{105}
$$

$$
\Delta^* < 0, \ i \in R_m^c \cup j \in C_n^c \tag{106}
$$

where the last two equation follows from (95) and (96). We conclude that the relaxation algorithm succeeds with probability at least

$$
1 - 2m^{-c} - 2n^{-c} - 2(mn)^{-c} - 2 \exp(-c(m + n))
$$

if

$$
\lambda \geq C \cdot \sigma \left( \sqrt{m + n \over k_m k_n} + \sqrt{\log m \over k_n} + \sqrt{\log n \over k_m} \right).
$$

We have achieved the the same boundary as the spectral method upper bound.

\[\square\]

### B Algorithmic Reduction for Detection

**Theorem 5** (Computational Lower Bounds for Detection). Consider the submatrix model (2) with parameter tuple $(m = n, k_m \approx k_n \approx n^{\alpha}, \lambda / \sigma = n^{-\beta})$, where $\frac{1}{2} < \alpha < 1$, $\beta > 0$. Under the hardness assumption $HC_d$, if

$$
\frac{\lambda}{\sigma} \lesssim \frac{m + n}{k_m k_n} \Rightarrow \beta > 2\alpha - 1,
$$

it is not possible to detect the true support of the submatrix with probability going to 1 for any polynomial algorithm.

**Proof of Theorem 5.** We would like to build a randomized polynomial mapping from the hidden clique graph $\mathcal{G}(N, \kappa(N))$ to a matrix $M(m = n, k_m \approx k_n \approx k, \lambda / \sigma)$ for the submatrix model. Denote this transformation as

$$
\mathcal{T} : \mathcal{G}(N, \kappa(N)) \rightarrow M(n, k = n^{\alpha}, \lambda / \sigma = n^{-\beta}).
$$
There are several stages of the construction. First, we define a graph that is stochastically equivalent to the hidden clique graph $\mathcal{G}$, but is easier for the analysis. Let us call it $\mathcal{G}^e$. $\mathcal{G}^e$ has the property: each node independently has the probability $\kappa(N)/N$ to be a clique node. By Bernstein’s inequality, with probability at least $1 - 2^{-N}$, the number of cliques $\kappa^e$ in $\mathcal{G}^e$

$$\kappa \left(1 - \sqrt{\frac{4\log N}{\kappa}}\right) \leq \kappa^e \leq \kappa \left(1 + \sqrt{\frac{4\log N}{\kappa}}\right) \Rightarrow \kappa^e \approx \kappa$$

(107)

as long as $\kappa \gtrsim \log N$.

Consider a double sized hidden clique graph $\mathcal{G}^e(2N, 2\kappa(N))$ with $N = n^{1+\beta}$, and $\kappa(N) = k = n^\alpha$, $\frac{1}{2} < \alpha < 1$. Denote the clique nodes set as $C_{N,k}$. Connect the hidden clique graph to form a symmetric matrix $G \in \{-1, 1\}^{2N \times 2N}$, where $G_{ij} = 1$ if $i, j \in C_{N,k}$, otherwise with equal probability to be either $-1$ or $1$. Take out the upper-right submatrix of $G$, $G_{UR}$ where $U$ is the index set $1 \leq i \leq N$ and $R$ is the index set $N + 1 \leq j \leq 2N$.

Partition the rows of $G_{UR} \in \mathbb{R}^{n^{1+\beta} \times n^{1+\beta}}$, to form $n$ blocks. The $s$’s block, $1 \leq s \leq n$, corresponds to the row index set $I_s = \{i : (s-1)n^\beta + 1 \leq i \leq sn^\beta\}$. Construct the $M \in \mathbb{R}^{n \times n}$ matrix in the following way

$$\mathcal{F} : M_{st} = \frac{1}{n^\beta} \sum_{i \in I_s, j \in I_t} (G_{UR})_{ij}, \ 1 \leq s, t \leq n.$$  (108)

There are two cases, if $I_s \cap C_{N,k} \neq \emptyset$ and $I_t \cap C_{N,k} \neq \emptyset$, namely when $s$’s block contains at least 1 clique node, and so does $t$’s block, then $\mathbb{E}M_{st} \geq n^{-\beta}$. Otherwise $\mathbb{E}M_{st} = 0$. Note that for the index $s, t$ that has no clique inside, $\mathbb{E}M_{st}$ is sub-Gaussian random variable, due to Hoeffding’s inequality

$$\mathbb{P}(|M_{st} - \mathbb{E}M_{st}| \geq u) \leq 2 \exp(-u^2/2).$$  (109)

For the $s, t$ with clique nodes inside, we know the maximum number of clique nodes is $\log n$ (due to Bernstein’s inequality that $\max_{1 \leq s \leq n} |I_s \cap C_{N,k}| \leq \frac{k}{n} + \frac{\sqrt{3}}{2} \log n$ with probability at least $1 - n^{-1}$), which means there are at least $n^\beta - \frac{k}{n} - \frac{\sqrt{3}}{2} \log n$ many independent Rademacher random variables in each $s, t$ block, thus

$$\mathbb{P}(|M_{st} - \mathbb{E}M_{st}| \geq u) \leq 2 \exp\left(- (1 - C \cdot (kn^{-1-\beta} + n^{-\beta} \log n)) u^2/2\right).$$  (110)

Thus we can take sub-Gaussian parameter to be any $\sigma < 1$ because $kn^{-1-\beta}, n^{-\beta} \log n$ are both $o(1)$. Now this constructed $M(n, k)$ matrix satisfies the submatrix model with $\lambda = n^{-\beta}$ and sub-Gaussian parameter $\sigma = 1 - o(1)$.
Let us see how many elements in $1 \leq s \leq n$ are such that $I_s \cap C_{N,k} \neq \emptyset$. Namely, we want to estimate how many clique nodes there exist in the transformed submatrix model. We have the two sided bound

$$k \left( 1 - \frac{1}{2n^{1-\alpha}} \right) \leq \mathbb{E} \left[ |\{s : I_s \cap C_{N,k} \neq \emptyset, 1 \leq s \leq n\}| \right] \leq k,$$

which is of the order $k$. Using Bernstein’s bound, we have with high probability

$$k \left( 1 - \frac{1}{2n^{1-\alpha}} \right) \left( 1 - \sqrt{\frac{4 \log n}{k}} \right) < |\{s : I_s \cap C_{N,k} \neq \emptyset, 1 \leq s \leq n\}| < k \left( 1 + \sqrt{\frac{4 \log n}{k}} \right).$$

Thus the submatrix model $M(m = n, k_m \approx k_n, \lambda / \sigma)$ satisfies $k_m \approx k_n \approx k$.

Suppose there exists a polynomial time algorithm $A_M$ that pushes below the computational boundary quantitatively by a small $\epsilon > 0$ amount

$$n^{-\beta} \approx \frac{\lambda}{\sigma} \approx \frac{m + n}{k_m k_n} \approx n^{1-2\alpha} \Rightarrow \beta > 2\alpha - 1.$$

where $\beta = 2\alpha - 1 + \epsilon$. Namely, $A_M$ detects below the boundary $\sigma \frac{m + n}{k_m k_n}$, then it naturally introduced a polynomial time detection algorithm for hidden clique problem $G(N = n^{1+\beta}, \kappa = n^{\alpha})$, which violates the HCd because

$$\kappa(N) \approx N^{\frac{\beta}{1+\beta}} \approx N^{\frac{\alpha}{1+\alpha}} \ll N^{\frac{1}{2}}.$$  

□

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