Abstract

We consider an investor, whose portfolio consists of a single risky asset and a risk free asset, who wants to maximize his expected utility of the portfolio subject to the Value at Risk assuming a heavy tail distribution of the stock prices return. We use Markov Decision Process and dynamic programming principle to get the optimal strategies and the value function which maximize the expected utility for parametric as well as non parametric distributions. Due to lack of explicit solution in the non parametric case, we use numerical integration for optimization.

Highlights

- Used Markov Decision Process and dynamic programming to get recursive relation for optimal strategy of known distribution
- Once it approaches the terminal position we will build up more on the risky asset or will liquidate the risky asset
- In the non-parametric case, we used numerical integration to find the recursive relation for optimal strategy
- In this case the value function we obtained has wide range of fluctuations and the same is also true for the portfolio wealth

*We thank Mrinal K. Ghosh for helpful comments and suggestions which has helped improving the exposition considerably. The usual caveat applies.
Portfolio optimization when returns are heavy tailed

Keywords: Portfolio Optimization, Markov Decision Process, Parametric distribution, Non parametric distribution

AMS Classification: 91G10, 91G80

1 Introduction

1.1 Background and Motivation

Risk management occurs everywhere in the financial world. It is done when an investor buys low-risk government bonds over riskier corporate bonds, bank performing a credit check on an individual before issuing a personal line of credit, stockbrokers buying assets like options & futures in their portfolio and money managers using strategies like portfolio and investment diversification to mitigate or effectively manage risk. Inadequate risk management can result in severe consequences such as the sub prime mortgage meltdown in 2007 that helped trigger the Great Recession stemmed from poor risk-management decisions. In the financial world the performance measure of the portfolio associated with risk and portfolio management is actually risk management. A common definition of investment risk is a deviation from an expected outcome, which we can benchmark with the market parameters. The deviation can be positive or negative. How Do Investors Measure Risk? Investors use a variety of tactics to ascertain risk. One of the most commonly used risk metrics, Value at Risk (VaR), is a statistical measure of the riskiness of financial entities or portfolios of assets. It is defined as the maximum dollar amount expected to be lost over a given time horizon, at a pre-defined confidence level. There are also other risk measure metrics used in the market such as Sharpe’s Ratio or Expected Shortfall. As mentioned before our main focus in this paper will be Value at Risk (VaR).

Consider an investor who is interested to optimize the portfolio return while managing VaR. It is well documented that returns on financial assets often exhibit heavy tails, so much so that even the first moment may not exist for the return distribution (see Campbell et al., 1997). In this paper we consider investment strategies that maximize the median return of the portfolio such that VaR is controlled for a given lower quantile level. We incorporate the effect of transaction cost and do the analysis for a known and an unknown distribution return distribution.

1.2 Literature Review

Interest rate risk immunization is one of the key concerns for fixed income portfolio management. In recent years, risk measures (e.g. value-at-risk and conditional value-at-risk) as tools for the formation of an optimum investment portfolio? The article by Mato (2005) aims to discuss this issue. The paper by Harmantzis et al. (2006) empirically test the performance of different models in measuring VaR and ES in the presence of heavy tails in returns using historical data. Daily returns are modelled with empirical (or historical), Gaussian, Generalized Pareto (peak over threshold (POT) technique of extreme value theory (EVT)). Assessing financial risk and portfolio optimization using a multivariate market model with returns assumed to follow a multivariate normal tempered stable distribution i.e. this distribution is a mixture of the multivariate normal distribution and the tempered stable subordinator can be seen in Kim et al. (2012). Several authors have considered the optimal portfolio problem under drawdown constraint. The first to comprehensively study this problem over infinite time horizon in a market setting with single risky asset modelled as a geometric Brownian motion with constant volatility (log normal model) was done in Grossman and Zhou (1993). Dynamic programming was used to solve the maximization problem of the long term growth rate of the expected utility of the wealth. Cvitanic and Karatzas (1995) streamlined the
analysis of [Grossman and Zhou 1993] and extended the results to the case when there are multiple risky assets whose dynamics. The paper by [Samuelson 1975] gives an idea of portfolio selection by stochastic dynamic programming. The paper by [Aït-Sahalia and Lo 1998] gives an idea of using a non-parametric estimator for the State Price Densities implicit in option prices. Mean-VaR portfolio optimization is studied in the paper by [Lwin et al. 2017] in a non-parametric approach. In this paper the authors have investigated the portfolio optimization problem with six practical constraints widely used in real life trading scenarios. The paper written by [Wozaba 2012] gives an idea of VaR constrained Markowitz style portfolio selection problem. Here the distribution of the returns of considered assets are given in the form of finitely many scenarios. Numerical analysis is used to solve the problem. When the utility function is quadratic how to use the Mean-VaR portfolio optimization is described in the paper by [Sukono et al. 2017]. In the paper by [Chow 2014], the authors have devised an algorithm to solve conditional variance problem by Markov Decision Process. In the book by [Bhatnagar et al. 2013] it gives us the idea of how to stochastic recursive algorithm for optimization problems.

1.3 Our Contribution

In this article we consider an investor who is worried about when to build up on stocks or liquidate the stock when dealing with heavy tail distribution of the return of the stock prices, controlling Value at Risk (Var). The investor’s portfolio has one risky asset and a risk free asset. We consider this problem in discrete time and then apply the general Markov Decision Problem formulation. Markov decision process theory and algorithms for finite horizon problems primarily concerns with determining a policy with the largest expected total reward. We try to determine the policies for two scenarios where the distribution is known and when the distribution is not known. Pareto and Weibull distributions are used for the parametric scenario and for the non-parametric case we have used the kernel density estimator to fit our data and carry out the numerical integration.

1.4 Organization of the paper

In section 2 we outline our approach for portfolio optimization where we discretized the wealth equation and applied the general Markov Decision Problem formulation. So it has become a dynamic programming problem. We tried to find the optimal strategy considering two scenarios where the distribution is known (section 3) and when the distribution is not known (section 4). Pareto and Weibull distributions are used for the parametric scenario and for the non-parametric case we have used the kernel density estimator to fit our data and carry out the numerical integration. We interpret the results under different circumstances, with and without transaction cost. Section 5 concludes the article.

2 Methodology

We work with one risky asset $S_t$ and a risk free bank account with return $r$. Again at a certain point of time we are investing $\pi_t$ of the total investible wealth $M$ in the risky asset and the rest in the bank account (so $\pi_t \in [0,M]$). We can express the equation of wealth as

$$L_t = \pi_t S_t + (M - \pi_t)r. \quad (1)$$

The heavy tail distribution doesn’t allow us to use the properties of Brownian motion and the Hamilton-Jacobi-Bellman (HJB) equation to solve the stochastic differential equations to obtain the optimal strategy and the value function in continuous time. To obtain a solution in such a situation, we convert this problem into a discrete domain and then apply the general Markov Decision Problem (MDP) formulation. Since we have a fixed time horizon, it
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is treated as a Finite Horizon Markov Decision Process. Markov decision process theory and algorithms for finite horizon problems is primarily concerned with determining a policy $\pi^*$ with the largest expected total reward. We seek a policy $\pi^*$ for which (see Puterman, 2005)

$$v_N^\pi(s) \geq v_N^\pi(s).$$

Finite-horizon policy evaluation algorithm is used to compute $v_N^\pi$. To start with let $u_t^\pi$ denote the total expected reward obtained using the policy $\pi$ at decision epochs $t, t + 1, t + 2, \ldots, N - 1$. If the history at decision epochs is $h_t$ then this $u_t^\pi$ defined as

$$u_t^\pi(h_t) = E_h^\pi \left( \sum_{n=1}^{N-1} W_n + W_N \right)$$

$$u_{t-1}^\pi(h_{t-1}) = W_{t-1}(s_{t-1}, d_{t-1}(h_{t-1})) + \sum_{j \in \mathcal{S}} p_{t-1}(j|s_{t-1}, d_{t-1}(h_{t-1})) u_t^\pi(h_{t-1}, d_{t-1}(h_{t-1}), j)$$

(2)

The expected value of policy $\pi$ over periods $t, t + 1, \ldots$ when the history of epoch is $t$ equals the immediate reward received by selecting the action plus the median reward over the time period. The second term contains the product of the probability being in state j at decision epoch $t + 1$ if action $d_t(h_t)$ is used and the median total reward obtained using the policy $\pi$.

If we apply this concept to (1) our optimization problem takes the following form

$$\text{maximize} \quad \pi_t, s \in (t, t + 1, \ldots T - 1) \quad \left\{ u_{t-1}^\pi(h_{t-1}) \right\}$$

subject to

$$P(L_{t-1} > Q_{0.05}|L_t > Q_{0.05}) \geq 0.95,$$

which can be further written as,

$$\text{maximize} \quad \pi_t, s \in (t, t + 1, \ldots T - 1) \quad \left\{ W_{t-1}(s_{t-1}, d_{t-1}(h_{t-1})) + \sum_{j \in \mathcal{S}} p_{t-1}(j|s_{t-1}, d_{t-1}(h_{t-1})) u_t^\pi(h_{t-1}, d_{t-1}(h_{t-1}), j) \right\}$$

subject to

$$P(L_{t-1} > Q_{0.05}|L_t > Q_{0.05}) \geq 0.95,$$

We have considered a quantile of 0.05 for VaR in this paper throughout. The constraint equation will help to identify the recursive relation between each optimal strategy $\pi$ at each time instant and this optimal strategy will help to maximize the value function. Basically we are going to use (2) for the value function as mentioned before and the reward is the change in the wealth from one time period to another time period.

$$W_{t-1} = L_{t-1} - L_t$$

and for each $\pi_{t-1}$ obtained for a period the maximum value of $W_{t-1}$ is calculated which in turn helps to maximize $u_{t-1}^\pi$

$$u_{t-1}^\pi = (L_{t-1} - L_t) + \sum p(t-1,t) u_t^\pi$$

$$u_{t-1}^\pi = (L_{t-1} - L_t) + E(u_t^\pi)$$

(3)
3 Examples and numerical results for known distribution of return

For the numerical illustration, data used are daily closing price of the stock “Entergy Corporation” in the time range 31st August, 2009 till 30th August, 2013 [Quantopian 2018]. The return is calculated for this data.

3.1 Pareto Distribution

We first fit a Pareto distribution to the data-set and estimated the scale and shape parameters of the Pareto distribution. Once we get the parameters we calculate the quantile \( Q_{0.05} \) for the distribution. The constraint is,

\[
P(L_{t-1} > Q_{0.05}, L_t > Q_{0.05}) \geq 0.95
\]

The scale and shape parameters calibrated from data are as follows:

\[
\lambda = 85.34364 \quad \alpha = 10346.37374 \quad Q_{0.05} = 5 \times 10^{-6} \lambda
\]

The rate of interest (daily) is taken to be one of three alternative values. These are 0.00008 (equivalent to an annual rate of 2%), 0.00014 (3.5%) and 0.00024 (6%).

Putting the value of the quantile in equation (4) and putting the value of the wealth from (1) in to the same equation we get,

\[
P(\pi_{t-1} S_{t-1} + (M - \pi_{t-1})r > 5 \times 10^{-6} \lambda, \pi_t S_t + (M - \pi_t)r > 5 \times 10^{-6} \lambda) \geq 0.95
\]

Taking \( M = 1 \)

\[
P(S_{t-1} > \frac{5 \times 10^{-6} \lambda + r(\pi_{t-1}-1)}{\pi_{t-1}}, S_t > \frac{5 \times 10^{-6} \lambda + r(\pi_t-1)}{\pi_t}) \geq 0.95
\]

\[
\frac{\int_{5 \times 10^{-6} \lambda + r(\pi_{t-1}-1)}^{5 \times 10^{-6} \lambda + r(\pi_t-1)} \frac{\alpha \lambda^\alpha}{(x+\lambda)^{\alpha+1}} dx}{\int_{5 \times 10^{-6} \lambda + r(\pi_{t-1}-1)}^{5 \times 10^{-6} \lambda + r(\pi_t-1)} \frac{\alpha \lambda^\alpha}{(x+\lambda)^{\alpha+1}} dx} \geq 0.95
\]

(5)

Solving (5) and putting the values of \( \lambda \) and \( r = 0.00014 \) we get the recursive relation between the optimal strategies for \( t \) to \( t-1 \),

\[
\pi_{t-1} \leq \frac{\pi_t}{(19.1977 \pi_t + 1)}
\]

(6)

The expression for other values of \( r \) are similarly obtained. Plotting (6) for the last 26 days of the historical data we have taken, in R-Studio, we can see how the optimal strategy varies for different values of \( r \). Refer [Figure (1a)] and we note that as the time increases from the initial point the fluctuations in \( \pi \) are present but as it reaches the terminal point it increases steeply with time. The value it reaches is maximum when the rate of interest is least and is minimum when the rate of interest \( r \) is maximum. This is intuitively reasonable as a lower rate of interest would push the investment in the risky asset higher.
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(a) When there is no transaction cost

(b) When there is transaction cost

Figure 1: Change in optimal strategy for different values of $r$ when there is transaction cost and no transaction cost considering Pareto distribution.

The change in the optimal strategy may or may not be associated with transaction cost. We have implemented both the scenarios for all the numerical calculations in R-Studio trying to get some interpretation from the graph.

3.1.1 Without transaction cost

Putting the values of $\pi$ obtained from (6) into the (1) and simultaneously calculating the maximum reward $W_{t-1}$ for the transition from $t$ to $t-1$ we calculate the value function defined in (3). If we plot the value function for the last 26 days of the historical data (Figure (2a)) we can interpret that as it moves towards terminal wealth the fluctuations in the value functions increases more and for least rate of interest the fluctuations are maximum and for maximum rate of interest the fluctuations are least.

Finally when we plot the portfolio wealth for the last 26 days of the historical data it can be interpreted that the wealth gradually increases and decreases as it approaches the terminal wealth but before attaining the terminal wealth the value of the portfolio wealth increases and then decreases. For maximum rate of interest the portfolio wealth is maximum and for least rate on interest the portfolio wealth is minimum (Figure (3a)). This implies that a safer portfolio (when bank rate is higher) is better.

3.1.2 With transaction cost

When transaction cost is taken into account the wealth equation (1) modifies to

$$L_t = \pi_t S_t + (1 - \pi_t) r - (\pi_t - \pi_{t-1}) r_1$$  (7)

and using (7) in the optimisation exercise, the recursive optimal policy modifies to

$$\pi_{t-2} \geq 0.999 \frac{\pi_{t-1}^2}{\pi_t} - 0.0547 \pi_{t-1} - 0.002867 + 0.002868 \frac{\pi_{t-1}}{\pi_t}$$  (8)

we can see that now there is a difference in the order 2 when transaction cost is taken into account, where $r_1$
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Figure 2: Change in value function for different values of $r$ when there is transaction cost and no transaction cost considering Pareto distribution

Figure 3: Change in portfolio wealth for different values of $r$ when there is transaction cost and no transaction cost considering Pareto distribution
Transaction cost rate for a change in optimal strategy = 10%. Putting the values of $\pi$ obtained from (7) in to the (7) and simultaneously calculating the maximum reward $W_{t-1}$ for the transition from $t$ to $t-1$ we calculate the value function defined in (3). When the optimal policy is plotted for the last 26 days of the historical data we can see that for transaction cost scenario the rate of interest doesn’t have any significant impact on the optimal strategy (Figure (1b)). The optimal policy is almost constant when transaction cost is considered and is at a lower level as compared to no transaction cost scenario but when it reaches the terminal position the optimal strategy value increases more than the case of no transaction cost.

If we plot the value function for the last 26 days of the historical data denoted by red line (Figure (2b)) we see that the value function in initial position does not change as compared to the no transaction cost scenario whereas as it closes towards the terminal position the fluctuations are more in the value function and there is no significant effect because of the change in the rate of interest.

Finally when we plot the portfolio wealth over the last 26 days of the historical data it can be interpreted that the wealth in initial position follows the same pattern as the no transaction cost scenario but as it approaches the terminal position the portfolio wealth decreases and then it increases. There is no significant effect because of the change in the rate of interest (Figure (3b)). If we compare this with the case of no transaction cost the fluctuations are more in case of transaction cost scenario.

3.2 Weibull Distribution

We next fit a Weibull distribution to the data-set and estimate the scale and shape parameters of the distribution. Once we get the parameters estimated, we calculate the quantile $Q_{0.05}$ for the distribution. The constraint equation is again similar to (4).

The scale and shape parameters are as follows:

$$\lambda = 1$$
$$\alpha = 1$$
$$Q_{0.05} = 0.0512\lambda$$

The choices for $r$ are as before. Putting the value of the quantile in equation (4) and putting the value of wealth from (1) in to the same equation we get,

$$P\left(S_t + (M - \pi_t) > 0.0512\lambda\right) \geq 0.95$$

Taking $M = 1$

$$P\left(S_t > \frac{0.0512\lambda + r(\pi_t - 1)}{\pi_t}\right) \geq 0.95$$

Putting the value of wealth from (1) in to the same equation we get,

$$\int_{0.0512\lambda + r(\pi_t - 1)}^{0.0512\lambda + r(\pi_t - 1)} e^{-x}dx \geq 0.95$$

Solving (8) and putting the values of $\lambda$ and $r = 0.00014$ we get the recursive relation between the optimal strategies for $t$ to $t-1$.

1In this and all the subsequent cases, we have also studied the effect of a change in the transaction cost rate on our variables of interest. As we could detect no discernible difference, we have not reported those results here.
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(a) When there is no transaction cost

(b) When there is transaction cost

Figure 4: Change in optimal strategy for different values of $r$ when there is transaction cost and no transaction cost considering Weibull distribution

We have assumed the terminal optimal point be 0.00001. Plotting (10) for the last 26 days of the historical data in R-Studio we can see how the optimal policy varies (Figure (4a)) and we notice that as the time increases from the initial point the change in $\pi$ is also linear with time and is same for all the rates of interest $r$.

Again, to check the association between the optimal strategy and transaction cost, we have implemented both the scenarios.

3.2.1 Without transaction cost

Putting the values of $\pi$ obtained from (10) in to the (1) and simultaneously calculating the maximum reward $W_t$ for the transition from $t$ to $t-1$ we calculate the value function defined in (3). If we plot the value function for the last 26 days of the historical data (Figure (5a)) we can interpret that initially as the time period moves the value function is constant but once it is close to terminal position the fluctuation decreases. Overall the value function doesn’t depend on the rate of interest $r$.

Finally when we plot the portfolio wealth over the last 26 days of the historical data it is seen that throughout the time period, portfolio wealth fluctuates very less but when the rate of interest $r$ is maximum then the portfolio wealth is also maximum and when the rate of interest is least then the portfolio wealth is minimum (Figure (6a)).

3.2.2 With transaction cost

When transaction cost is taken into account the wealth equation (1) modifies to

$$L_t = \pi_t S_t + (1 - \pi_t)r - (\pi_t - \pi_{t-1})r_1$$

We have assumed the terminal optimal point be 0.00001. Plotting (10) for the last 26 days of the historical data in R-Studio we can see how the optimal policy varies (Figure (4a)) and we notice that as the time increases from the initial point the change in $\pi$ is also linear with time and is same for all the rates of interest $r$.

Again, to check the association between the optimal strategy and transaction cost, we have implemented both the scenarios.

3.2.1 Without transaction cost

Putting the values of $\pi$ obtained from (10) in to the (1) and simultaneously calculating the maximum reward $W_t$ for the transition from $t$ to $t-1$ we calculate the value function defined in (3). If we plot the value function for the last 26 days of the historical data (Figure (5a)) we can interpret that initially as the time period moves the value function is constant but once it is close to terminal position the fluctuation decreases. Overall the value function doesn’t depend on the rate of interest $r$.

Finally when we plot the portfolio wealth over the last 26 days of the historical data it is seen that throughout the time period, portfolio wealth fluctuates very less but when the rate of interest $r$ is maximum then the portfolio wealth is also maximum and when the rate of interest is least then the portfolio wealth is minimum (Figure (6a)).

3.2.2 With transaction cost

When transaction cost is taken into account the wealth equation (1) modifies to

$$L_t = \pi_t S_t + (1 - \pi_t)r - (\pi_t - \pi_{t-1})r_1$$

We have assumed the terminal optimal point be 0.00001. Plotting (10) for the last 26 days of the historical data in R-Studio we can see how the optimal policy varies (Figure (4a)) and we notice that as the time increases from the initial point the change in $\pi$ is also linear with time and is same for all the rates of interest $r$.

Again, to check the association between the optimal strategy and transaction cost, we have implemented both the scenarios.
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Figure 5: Change in value function for different values of r when there is transaction cost and no transaction cost considering Weibull distribution

Figure 6: Change in portfolio wealth for different values of r when there is transaction cost and no transaction cost considering Weibull distribution
and using (11) the recursive optimal policy modifies to

$$\pi_{t-2} = \frac{\pi_{t-2}^2}{\pi_t} + \frac{\pi_{t-1} - 0.5107 - 6.680\pi_{t-1}}{1.958\pi_t} \quad (12)$$

The recursive equation is of order 2 and $r_1$ (Transaction cost rate for a change in optimal strategy) = 10%. Putting the values of $\pi$ obtained from (12) in to the (11) and simultaneously calculating the maximum reward $W_{t-1}$ for the transition from $t$ to $t-1$ we calculate the value function defined in (3). When the optimal policy is plotted for the last 26 days of the historical data it is seen that the optimal policy remains constant a particular value for a period of time but as it approaches the terminal position the asset is liquidated. Here the rate of interest $r$ has a clear effect on the strategy. For different rate of interest the liquidation of the asset is different. In case of no transaction cost the optimal strategy has no effect of the rate of interest (see Figure (6b)).

If we plot the value function over the last 26 days of the historical data (Figure (5b)) we can see that the value function fluctuates a lot and there is no effect of the rate of interest.

Finally when we plot the portfolio wealth over the last 26 days of the historical data it can be interpreted that the wealth fluctuates throughout the time period and there is no significant difference when we change the rate of interest (Figure (6b)). If we compare this with the case of no transaction cost we can see that it is quite different.

## 4 Example and numerical results for unknown distribution of return

### 4.1 Without transaction cost

When the distribution function is not known for the return of the stock prices we try to fit a distribution using kernel density estimator (KDE). It is a non-parametric way to estimate the probability density function of a random variable. KDE is a fundamental data smoothing problem based on the finite data sample we choose. If we have $(x_1,x_2,\ldots,x_n)$ as the independent univariate samples coming from an unknown distribution $f$ then we can write

$$f_{\hat{h}}(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right)$$

where $K$ is the kernel which is a non-negative function and $h > 0$ is a smoothing parameter called the bandwidth [Markovich (2007)]. Using the KDE function we get the following estimate of the parameters:

- kernel density estimate = log-quadratic fitting
- bandwidth (bw) = 0.00271447

Again we take three different values for $r$. Now, (4) can be used again for finding the conditional probability of the portfolio wealth. Expanding we get,

$$P\left(\begin{array}{c}
\pi_{t-1} - M + (M - \pi_{t-1})r > Q_{0.05}, \pi_{t}S_t + (M - \pi_{t})r > Q_{0.05} \\
\end{array}\right) \geq 0.95$$

$$P\left(\begin{array}{c}
S_{t-1} > Q_{0.05 + r(M - \pi_{t-1})}^\pi, S_t > Q_{0.05 + r(M - \pi_{t})}^\pi \\
\end{array}\right) \geq 0.95$$

$$F\left(\frac{Q_{0.05 + r(M - \pi_{t})}^\pi}{\pi_{t-1}}\right) - F\left(\frac{Q_{0.05 + r(M - \pi_{t-1})}^\pi}{\pi_{t-1}}\right) \geq 0.95$$

$$1 - \left(F\left(\frac{Q_{0.05 + r(M - \pi_{t})}^\pi}{\pi_{t-1}}\right) - F\left(0\right)\right) \geq 0.95 \quad (13)$$
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(a) When there is no transaction cost

(b) When there is transaction cost

Figure 7: Change in optimal strategy for different values of $r$ when there is transaction cost and no transaction cost considering kernel density estimation

Taking $M = 1000$ and now solving (13) with the help of standard numerical integration in R-Studio, we get the relationship as depicted in figure (7a). It can be observed that the change in optimal policy is not so frequent for the last 26 days of the historical data. The investor would keep the asset at a particular state for some time. Also for the least rate of interest the peak value of the fluctuation is minimum and for the maximum rate of interest the peak value of the fluctuation is maximum

If we consider a terminal value as 0.001 and plot the value function [Figure (8a)], it fluctuates but as it approaches the terminal position it decreases and then increases. Changing the terminal value doesn’t create any difference in the behaviour of the value function. Also the rate of interest has little significance in the plot and the significance can be seen only when it is in the state of fluctuation.

Plotting the portfolio wealth [Figure (9a)] it is observed that it increases and decreases but as it approaches the terminal position wealth increases and, in the same way as the value function graph, changing terminal value doesn’t create any difference in the behaviour of the portfolio wealth. Also the change in the rate of interest has very little significance on the plot when the portfolio wealth is constant and during fluctuations it doesn’t have any effect

4.2 With transaction cost

Plotting the effect of transaction cost on optimal strategy we can see that the fluctuations are similar to the no transaction cost scenario but there is no effect of the interest rate $r$ in this case (see Figure (7b)).

When we are considering transaction cost involved in the process and plot the value function (Figure (8b)) we observe that it increases and decreases but as it move towards the terminal position the value function becomes constant. If we compare this with the no transaction cost situation the fluctuations are not affected by the rate of interest $r$.

Similarly when we plot the portfolio wealth (Figure (9b)) we observe that the portfolio wealth fluctuates but there is no effect of the interest rate $r$, as in the case of no transaction cost.
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Figure 8: Change in value function for different values of $r$ when there is transaction cost and no transaction cost considering kernel density estimation

Figure 9: Change in portfolio wealth for different values of $r$ when there is transaction cost and no transaction cost considering kernel density estimation
5 Conclusion

We proposed a new investor objective framework to deal with heavy tail distribution of the return of the stock prices and tried to optimize the portfolio based on managing Value at Risk (Var). As we know that moments do not exist for a heavy tailed distribution, quantiles are the only way to tackle the portfolio optimization problem. In our proposed approach for portfolio optimization we discretized the wealth equation and applied the general Markov Decision Problem formulation. So it has become a dynamic programming problem. We optimize considering two alternative scenarios where the distribution is known and when the distribution is not known. In the known distribution case, we have fitted the Pareto and Weibull distributions to the return of the stock prices and found out the quantiles; and when the distribution is not known we used kernel density estimator to find out the quantiles. In both the cases we have shown the recursive relation between the optimal policy and maximized the total expected reward which is the value function at each of the time points. The optimal policy tells us when to build up on the risky asset and when to liquidate it. In the case where we have transaction cost, it will not allow the the investor to change its policy so frequently and also not by a large margin. Also the value function and portfolio wealth are almost constant but as we approach the terminal position we will build up more on the risky asset or will liquidate the risky asset.

In the situation when the density functions are not known, the value function we obtained has wide range of fluctuations and the same is also true for the portfolio wealth. So the investor should come to know when to liquidate on the risky asset and build up on the risky asset. In this case we have used the standard numerical integration method to solve and get the recursive relation for the optimal policy. We have also shown how the transaction cost will affect the policy implementation. Here also the change in the optimal strategy won’t be as frequent when transaction cost is present. The results also aligns with the general intuitions when the rate of interest increases or decreases when considering the presence of transaction cost or no transaction cost.

As heavy-tailness is quite common in financial returns data, we hope that our proposed methodology will be useful for practitioners. An additional issue that we have not addressed in this paper is how to incorporate multiple risky assets in this type of analysis. In the absence of moments, the challenge would be to replace the covariance function, which will be undefined, with an appropriate measure of association and carry out the portfolio optimization. We aim to take this issue up in our future work.

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