Quantum Dynamics of Magnetic Skyrmions: Consistent Path Integral Formulation

Sergey A. Nikolaev\textsuperscript{1} and Akihiro Tanaka\textsuperscript{2}

\textsuperscript{1}Laboratory for Materials and Structures, Tokyo Institute of Technology, 4259 Nagatsuta, Midori-Ku, Yokohama, Kanagawa, 226-8503, Japan\textsuperscript{2}International Center for Materials Nanoarchitectonics, National Institute for Materials Science 1-1 Namiki, Tsukuba 305-0044, Japan

We present a path integral formalism for the intrinsic quantum dynamics of magnetic skyrmions coupled to a thermal background of magnetic fluctuations. Upon promoting the skyrmion’s collective coordinate $\mathbf{R}$ to a dynamic variable and integrating out the magnonic heat bath, we derive the generalized equation of motion for $\mathbf{R}$ with a non-local damping term that describes a steady-state skyrmion dynamics at finite temperatures. Being essentially temperature dependent, the intrinsic damping is shown to originate from the coupling of thermally activated magnon modes to the adiabatic potential driven by a rigid skyrmion motion, which can be regarded as another manifestation of emergent electrodynamics inherent to topological magnetic textures. We further argue that the diagonal components of the damping term act as the source of dissipation and inertia, while its off-diagonal components modify the gyrotropic motion of a magnetic skyrmion. By means of numerical calculations for the lattice spin model of chiral ferromagnets, we study the temperature behavior of the intrinsic damping as a function of magnetic field in periodic and confined geometries. The intrinsic damping is demonstrated to be highly non-local, revealing its quantum-mechanical nature, that becomes more pronounced with increasing temperature. At high temperatures when the magnon occupation factors are large, the intrinsic damping is shown to yield a modified Thiele’s equation with the additional non-local dissipative and mass terms that exhibit an almost linear temperature behavior. Our results provide a microscopic background for semiclassical magnetization dynamics and establish a framework for understanding spin caloritronics effects in topological magnetic textures.

I. INTRODUCTION

Magnetic skyrmions constitute a special class of topologically protected defects in magnetic materials that have been in the focus of active research for the past several decades after their prediction \[1\] \[3\]. Following the success in the search for materials that host skyrmionic textures \[4\] \[7\], the area of their application has drastically expanded into a rapidly growing subfield of spintronics, where controlling and manipulating the spin of an electron is a central goal for building next-generation electronic devices \[8\]. Owing to their high mobility at ultralow current densities, magnetic skyrmions hold potential as a non-volatile information carrier in magnetic media and can offer diverse functionalities in memory, logic, and neuro-inspired technologies \[9\] \[10\].

Skyrmion configurations are primarily characterised by a nontrivial spatial wrapping of $\mathbf{n}(\mathbf{r})$, the unit vector parallel to the local magnetization:

$$Q = \frac{1}{4\pi} \int d\mathbf{r} \mathbf{n}(\mathbf{r}) \cdot (\partial_x n_x(\mathbf{r}) \times \partial_y n_y(\mathbf{r})), \quad (1)$$

that can be continuously mapped onto a sphere, yielding an integer value which reveals the topological nature of these objects. The intrinsic topology of a smoothly varying magnetic texture, such as domain walls, vortices and skyrmions, is known to largely determine its motion and dynamic properties and is regarded to be the origin of the topological Hall effect \[11\] \[12\], skyrmion Hall effect \[13\] \[14\], and emergent electrodynamics \[15\] \[18\].

The dynamics of topological magnetic defects has been a fast-paced topic in magnetism due to its broad connections to technological applications. Historically, the motion of a single defect has been described as that of a rigid magnetic configuration that behaves like a massless particle with an electric (topological) charge $Q$ in an applied magnetic field. From magnetization dynamics governed by the Landau-Lifshitz-Gilbert equation, the motion of a solid profile $\mathbf{n}(\mathbf{r})$ can be parametrized by its center-of-mass coordinate $\mathbf{R}$ and described by a modified Newton’s equation of motion or the so-called Thiele’s equation \[19\]:

$$\mathbf{G} \times \dot{\mathbf{R}} + \mathbf{D} \dot{\mathbf{R}} - \mathbf{F} = 0, \quad (2)$$

where $\mathbf{G} \sim Q$ is a gyromagnetic coupling vector oriented perpendicular to the plane, $\mathbf{D}$ describes the dissipation, and $\mathbf{F} = -\partial U/\partial \mathbf{R}$ is a generalized force acting on the magnetic profile and incorporating the effects of electric currents, magnetic field gradients, or confining potentials. The equation above is found to be a good approximation for the dynamics of magnetic vortices in thin ferromagnetic films \[20\] \[21\] and has proven successful in describing the current-driven motion of magnetic skyrmions \[22\] \[23\]. Moreover, stochastic magnetization dynamics validates the form of Eq. (2) for the diffusive behavior of chiral domain walls and skyrmions at finite temperatures with the modified diffusion law as a function of the Gilbert damping parameter \[24\] \[25\]. However, simulations of skyrmion dynamics induced by an external magnetic field gradient suggested that Eq. (2) may be insufficient to fully describe the motion of topological defects \[26\]. In addition,
the gigahertz dynamics of magnetic skyrmions probed by a pulsed magnetic field revealed the presence of strong inertia and advocated for an additional skyrmion mass term $\mathcal{M}\dot{R}$ in Eq. (2) in order to reproduce the skyrmion trajectory [27].

In a general context, it was proposed that the dynamics of solitons at finite temperatures can be affected by the low-lying excitations that are thermally activated and act as a source of damping and thermal diffusion [28]. In magnetic systems, collective excitations or magnons were shown to experience a scattering potential induced by a skyrmion configuration that mediates the momentum transfer and exerts influence on the skyrmion motion [29, 30]. Skyrmion dynamics simulated by the stochastic Landau-Lifshitz-Gilbert equation demonstrated that the presence of thermal diffusion can induce both the effect of damping and mass, thus implying an intrinsic connection to magnetic excitations [31, 32]. In particular, the origin of the skyrmion mass was attributed to the transverse fluctuations of the circular domain wall next to the skyrmion core [33]. On the other hand, recent simulations of the current-driven and Brownian motion of ferromagnetic skyrmions suggested that the overall effect of the magnonic heat bath can be accounted by an additional dissipative term in Eq. (2) that is linear in temperature [34].

Theoretical studies within semiclassical approaches clearly point out that the coupling to a thermal background of magnon excitations strongly affects dynamical properties of a moving skyrmion. Nevertheless, a rigorous consideration should require a quantum-mechanical treatment of the problem and is still lacking. An attempt to provide a microscopic description of skyrmion dynamics coupled to the magnon modes was given in Ref. [35] and delivered the damping term at equilibrium that was interpreted as the skyrmion mass and predicted to be finite even at zero temperature in the presence of translational symmetry breaking. This approach has further been used to study various aspects of the skyrmion motion [36, 37]. However, the case of a temperature-independent skyrmion mass would seem to be in conflict with the simple observation that in the absence of external forces, magnons are excited through thermal effects. Furthermore, the authors limited their focus on temperatures much lower than the characteristic magnon frequency. On the contrary, one expects that incorporating possible dissipation effects driven by the magnonic heat bath would be necessary to account for the results of semiclassical simulations.

The main goal of this study is to formulate a microscopic description for the dynamics of magnetic skyrmions that properly takes into account the effect of magnetic fluctuations. Starting with a micromagnetic model for chiral ferromagnets, we construct a consistent field-theoretical formalism for the fast magnetization dynamics and the skyrmion’s collective motion based on the path integral approach in the imaginary time domain. By integrating out the magnonic heat bath, we derive an intrinsic damping term whose origin is attributed to an emergent electrodynamic potential inherent to topological magnetic textures. In contrast to previous studies, we demonstrate that the intrinsic damping is essentially temperature dependent, as it comes from thermal excitations of the magnon modes, and is highly non-local, reflecting its quantum mechanical nature. We further formulate the generalized equation of motion for a steady-state skyrmion dynamics in the real time domain and argue that the diagonal components of the damping term encode the effects of thermal dissipation and inertia, while its off-diagonal components yield corrections to the gyrotrropic motion. Based on extensive numerical calculations for the lattice spin model, we analyze the temperature behaviour of the intrinsic damping as a function of magnetic field in periodic and confined geometries. We argue that at sufficiently high temperatures the quantum effects are smeared out, and the damping term leads to a modified Thiele’s equation Eq. (2) with the additional non-local friction and mass terms, supporting previous results from semiclassical simulations [31] and a recent phenomenological prediction based on stochastic magnetization dynamics [34].

The paper is organized as follows. In Section II, we present a detailed path integral approach for magnetic fluctuations in a moving magnetic texture by using the framework of collective coordinates. In Section III, we derive the generalized equation of motion for the skyrmion’s collective coordinates by integrating out the magnonic heat bath and discuss the properties of the intrinsic damping term. Section IV is devoted to the numerical analysis of the intrinsic damping in chiral ferromagnets based on the lattice spin model. In Section V, we discuss our results in comparison with previous theoretical studies. Comprehensive technical details on the formalism are given in the Supplementary Material [39], and the computational details are explained in Appendix.

## II. PATH INTEGRAL FOR MAGNETIC FLUCTUATIONS

We start by formulating a path integral approach for magnetic fluctuations in the presence of a moving skyrmion configuration. For the sake of concreteness, we consider a micromagnetic model for chiral ferromagnets on a two-dimensional lattice:

$$H_S = \int \text{d}r \left( -\frac{JS^2}{2} \mathbf{n} \cdot \partial_\alpha \mathbf{n} - \frac{DS^2}{\alpha} e_\alpha \cdot (\mathbf{n} \times \partial_\alpha \mathbf{n}) - \frac{KS^2}{a^2} (n_z^2 - \frac{HS}{a^2} n_z^2) \right),$$

where $\mathbf{n} \equiv \mathbf{n}(r) = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)^T$ is the unit magnetization vector of spin $S$ written in terms

$$\begin{align*}
\mathbf{n} \equiv \mathbf{n}(r) &= (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)^T, \\
\mathbf{R}(r) &= (r_x, r_y, r_z), \\
\mathbf{M}(r) &= \mathbf{n}(r) \times \mathbf{R}(r), \\
\mathbf{H}(r) &= \mathbf{M}(r) \times \mathbf{H},
\end{align*}$$
of spherical angles φ and θ, a is the lattice constant, α = x, y, and the summation over repeated indices is implied. The model includes the isotropic exchange coupling J, Dzyaloshinskii-Moriya interaction D, on-site anisotropy K, and an applied magnetic field H perpendicular to the lattice. The direction of D is chosen along the bond connecting two neighbouring spins, so that a chiral or Bloch-type skyrmion configuration is stabilized under an applied magnetic field, as depicted in Fig. 1.

It is worth noting that the formalism described below is valid for an arbitrary magnetic system, and the details of the microscopic model can be omitted until the last moment. We should also remark that this study primarily concerns insulating materials whose magnetic properties can be described by localized spins and magnons as magnetic fluctuations. Thereby, we neglect any possible coupling to itinerant electrons and discard from consideration the lattice degrees of freedom. Hereafter, we refer the reader to the Supplementary Material where the formalism is discussed in more detail [39].

A. Euclidian action

Magnetization dynamics can be derived from a standard path integral approach for spin S with the Euclidean action in the imaginary time domain τ [40]:

$$S_E[n] = S_{WZ}[n] + S_S[n]$$

$$= \frac{iS}{a^2} \int dx n(x) \cdot \mathbf{A}_S(x) + \int dx H_S[n(x)],$$

(4)

where \(x = (r, \tau)\), \(\int dx = \int d\mathbf{r} \int d\tau\), and \(\beta = 1/T\) is the inverse temperature \((k_B = 1)\). The magnetization \(n(r, \tau)\) is considered a dynamic variable, and the periodic boundary condition \(n(r, 0) = n(r, \beta)\) is implied. The first term in the action is the Wess-Zumino or Berry phase term with the vector potential \(\mathbf{A}_S[n]\) corresponding to a magnetic monopole centered at each spin. By choosing \(\mathcal{A}_S[n]\) so that \(\partial_r \mathbf{n} \cdot \mathbf{A}_S = (1 - \cos \theta) \partial_\tau \mathbf{v}\), \(S_{WZ}[n]\) can be shown to give a solid angle on the unit sphere swept by the evolution of \(n\). The magnetization dynamics is described by the equation of motion \(\delta S_E[n] / \delta n = 0\) that gives the Landau-Lifshitz equation:

$$-iS \frac{\beta}{a^2} \partial_\tau \mathbf{n} = \mathbf{n} \times \frac{\delta S_S[n]}{\delta \mathbf{n}}.$$  

(5)

The thermodynamic properties of the magnetic system are extracted from the partition function:

$$Z = \int D\mathbf{n} e^{-S_E[n]},$$

(6)

with \(D\mathbf{n} = \lim_{N,M \to \infty} \prod_{n=1}^N \prod_{m=1}^M 2\pi a^2 / 4\pi d\mathbf{n}(r_i, \tau_m)\), where the space and imaginary time domains are discretized into \(N\) and \(M\) pieces, respectively.

Drawing on the path integral machinery, one can separate the initial magnetization dynamics into slow and fast magnetization variables. For the former, let us consider a magnetic skyrmion \(n_0 \equiv n_0(r)\) characterized by a non-zero winding number \(Q\) as a metastable stationary solution of \(H_S\), or \(\delta S_S[n] / \delta \mathbf{n} = 0\). Quantum fluctuations over a static skyrmion configuration describing magnon excitations can be introduced by fast magnetization variables \(m \equiv m(r, \tau)\) as \(n = n_0 \sqrt{1 - m^2} + m\) with \(n_0 \cdot m = 0\) for all \(\tau\) [41]. Assuming that the magnetic fluctuations are small, the action Eq. (4) can be decomposed within a saddle-point approximation:

$$S_E[n] = S_E[n_0]$$

$$+ \frac{1}{2} \int dx dx' m^T(x) \left( \frac{\delta^2 S_E[n]}{\delta n(x) \delta n(x')} \right)_{m=n_0} m'(x),$$

(7)

where the first variation vanishes due to the equation of motion. The resulting partition function is rewritten as \(Z = Z_0 \int Dm e^{-S[m]}\), where \(Z_0 = e^{-S[n_0]}\) is the static contribution, and \(S[m]\) comes form the second variation of Eq. (4) and specifies the fast magnetization dynamics:

$$S[m] = \int dx \frac{iS}{2a^2} m \cdot (\partial_\tau m \times n_0) + \frac{1}{2} m^T H_m m.$$  

(8)

The corresponding equation of motion for magnetic fluctuations is obtained from \(\delta S[m] / \delta m = 0\):

$$-iS \frac{\beta}{a^2} \partial_\tau m = n_0 \times H_m m,$$

(9)

where \(H_m = \delta^2 S_S / \delta n(x) \delta n(x')\) is the magnonic Hamiltonian, and the eigenvalue problem Eq. (9) yields the normal magnon modes with the pseudo-orthogonality relation \(\int d\tau m_n \cdot (n_0 \times m_{n'}) = i\delta_{n,n'}\), where \(\delta_{n,n'}\) is the Kronecker symbol [39].
B. Collective coordinates

The decomposition of the magnetization variables outlined above is the starting point for our further analysis of magnetic fluctuations in a moving skyrmion. Being a rigid soliton solution, the dynamics of a skyrmion configuration \( \mathbf{n}_0 \) can be described by its collective (center-of-mass) coordinate \( \mathbf{R} = (X, Y) \) \cite{20, 12}:

\[
\mathbf{R} = \frac{\int d\mathbf{r} n_0 \cdot (\partial_0 \mathbf{n}_0 \times \partial_0 \mathbf{n}_0)}{\int d\mathbf{r} n_0 \cdot (\partial_0 \mathbf{n}_0 \times \partial_0 \mathbf{n}_0)},
\] (10)

which gives a good approximation provided that a skyrmion is not strongly distorted by confinement or external forces. To incorporate the skyrmion motion into the path integral approach, the collective coordinate is elevated to a dynamic variable \( \mathbf{R} \equiv \mathbf{R}(\tau) \), and the magnetization profile can be represented by \( \mathbf{n}(\mathbf{r}, \tau) \approx \mathbf{n}_0(\mathbf{r} - \mathbf{R}) + \mathbf{m}(\mathbf{r} - \mathbf{R}, \tau) \).

Described as a whole by its collective coordinates, a moving configuration \( \mathbf{n}_0 \) should be invariant with respect to rigid translations, or \( \mathbf{n}_0(\mathbf{r}) \equiv \mathbf{n}_0(\mathbf{r} - \mathbf{R}) \approx \mathbf{n}_0(\mathbf{r}) - \partial_\mathbf{r} \mathbf{n}_0 \mathbf{R}_\alpha \), which can also be accounted for by adding up a Goldstone or zero mode \( \mathbf{m} \sim \partial_\mathbf{r} \mathbf{n}_0 \). To get rid of such redundancy within the framework of collective coordinates, one can require that the magnon modes be (pseudo-)orthogonal to the zero modes:

\[
\int d\mathbf{r} \mathbf{m}(\mathbf{r}, \tau) \cdot (\mathbf{n}_0(\mathbf{r}) \times \partial_\mathbf{r} \mathbf{n}_0(\mathbf{r})) = 0. \tag{11}
\]

This constraint can be included in the path integral formalism by using the Faddeev-Popov technique \cite{43, 45}:

\[
Z = \int \mathcal{D}\mathbf{n} \mathcal{D}\mathbf{R} \delta(\mathbf{R}) \det \frac{\delta \mathbf{A}}{\delta \mathbf{R}} e^{-S_{WZ}[\mathbf{n}]}, \tag{12}
\]

where \( \mathbf{A}[\mathbf{R}] = \int d\mathbf{r} \mathbf{n}(\mathbf{r} - \mathbf{R}, \tau) \cdot (\mathbf{n}_0(\mathbf{r}) \times \partial_\mathbf{r} \mathbf{n}_0(\mathbf{r} - \mathbf{R})) \) is the functional specifying the orthogonality condition in Eq. (11). Since \( \int d\mathbf{r} \mathbf{n}_0(\mathbf{r} - \mathbf{R}, \tau) \cdot (\mathbf{n}_0(\mathbf{r}) \times \partial_\mathbf{r} \mathbf{n}_0(\mathbf{r} - \mathbf{R})) = 0 \) for \( \mathbf{R} = \mathbf{0} \), the delta-function \( \delta(\mathbf{A}_\alpha) \) enforces the constraint Eq. (11). To leading order in \( \mathbf{R} \) and \( \mathbf{m} \), the determinant in Eq. (12) is \( \sim O^2 \) and can be absorbed into the integral measure \( 39 \).

At this juncture, we decompose the magnetic action Eq. (4) over magnetic fluctuations in connection with Eq. (12). While \( \delta S_{WZ}[\mathbf{n}] / \delta \mathbf{n} = 0 \) vanishes for a stationary configuration, the Wess-Zumino term should be treated with care:

\[
S_{WZ}[\mathbf{n}] = S_{WZ}[\mathbf{n}_0] + \int dx \left. \left( \frac{\delta S_{WZ}[\mathbf{n}]}{\delta \mathbf{n}(x)} \right) \right|_{\mathbf{n}=\mathbf{n}_0} \mathbf{m}(x) + \frac{1}{2} \int dxdx' \mathbf{m}^T(x) \left. \left( \frac{\delta^2 S_{WZ}[\mathbf{n}]}{\delta \mathbf{n}(x) \delta \mathbf{n}(x')} \right) \right|_{\mathbf{n}=\mathbf{n}_0} \mathbf{m}(x'), \tag{13}
\]

where the first variation \( \delta S_{WZ}[\mathbf{n}] / \delta \mathbf{m} \big|_{\mathbf{n}=\mathbf{n}_0} = \frac{i \mathbf{S}}{a} \partial_\mathbf{r} \mathbf{n}_0 \times \mathbf{n}_0 \) with \( \partial_\mathbf{r} \mathbf{n}_0 = -\partial_\mathbf{r} \mathbf{n}_0 \mathbf{R}_\alpha \) can generally survive, as the configuration \( \mathbf{n}_0(\mathbf{r} - \mathbf{R}) \) does not necessarily remain to be a saddle point during its motion. An essential part in our formalism is that the Faddeev-Popov constraint allows us to eliminate this obstacle, and the imposed pseudo-orthogonality condition Eq. (11) removes \( \delta S_{WZ}[\mathbf{n}] / \delta \mathbf{n} \) from the decomposition. One should note that this approximation is valid in the limit when the skyrmion motion is slow enough so that the time evolution of \( \mathbf{n}_0 \) preserves its rigidity and validates the orthogonality condition, decoupling magnetic fluctuations and zero modes.

The resulting partition function as an integral over collective coordinates and magnetic fluctuations can be written as:

\[
Z = Z_0 \int \mathcal{D}\mathbf{R} e^{-S_0[\mathbf{R}]} \left[ \int \mathcal{D}\mathbf{m} \delta(\mathbf{A}) e^{-S[m, \mathbf{R}]} \right], \tag{14}
\]

with \( S[m, \mathbf{R}] = S[m(\mathbf{r} - \mathbf{R}, \tau)] \). In the left part of the path integral, \( S_0[\mathbf{R}] \) originates from the Wess-Zumino term and does not depend on magnetic fluctuations, yielding the first intrinsic contribution to the collective motion of a single skyrmion in the absence of external forces and dissipation \[46]:

\[
S_{WZ}[\mathbf{n}_0] = \frac{2\pi i S Q}{a^2} \int_0^\beta d\tau \left( X \dot{Y} - Y \dot{X} \right) \equiv S_0[\mathbf{R}], \tag{15}
\]

where a dot symbol stands for the partial imaginary time derivative. In a nutshell, this term produces a Magnus force that pushes a moving skyrmion perpendicular to its direction of motion giving rise to the skyrmion Hall effect \cite{13, 14}.

C. Magnon modes in a moving frame

For the fast magnetization dynamics in a moving magnetic profile defined by \( \delta S[m, \mathbf{R}] / \delta \mathbf{m} = 0 \), one can reformulate the problem by rotating the magnetization to a local quantization axis \( \mathbf{n} \rightarrow \mathbf{n}' = \mathbf{R}^{-1} \mathbf{n} \), where it is understood that the rotation matrix \( \mathbf{R} = [\mathbf{e}_\theta, \mathbf{e}_\phi, \mathbf{n}_0] \), when acted on a vector, rotates it with respect to the local \( \mathbf{n}_0(\mathbf{r}) \), with \( \mathbf{e}_\theta = (\cos \phi \sin \theta, \cos \phi \cos \theta, -\sin \theta)^T \) and \( \mathbf{e}_\phi = (-\sin \phi, \cos \phi, 0)^T \). This transformation introduces a covariant derivative \( \partial_\mathbf{r} \mathbf{n} \rightarrow \partial_\mathbf{r} \mathbf{n} + \mathbf{A}_\alpha \times \mathbf{n} \) with an induced connection in the adjoint representation of the SO(3) rotation group, \( \mathbf{A}_\alpha = (-\sin \theta \partial_\mathbf{r} \phi, \partial_\mathbf{r} \theta, \cos \theta \partial_\mathbf{r} \phi)^T \). In a moving coordinate frame along \( \mathbf{n}_0(\mathbf{r} - \mathbf{R}) \) defined as the local z axis, one can express \( \partial_\mathbf{r} \mathbf{m} = \hat{\mathbf{m}} = \partial_\mathbf{r} \mathbf{m}_\mathbf{R}_\alpha - \mathbf{A}_\alpha \times \mathbf{m} \hat{\mathbf{R}}_\alpha \), and the fast magnetization dynamics in Eq. (9) can be reformulated from \( \delta S[m, \mathbf{R}] / \delta \mathbf{m} = 0 \) \cite{17, 45}:

\[
\frac{S}{a^2} \left( \hat{m}_+ - \hat{R}_\alpha \partial_\mathbf{r} m_+ - i \hat{R}_\alpha \mathbf{A}_\alpha^0 m_+ \right) + \mathcal{H}_m m_+ = 0, \tag{16}
\]
where $m_\pm = (m_x \pm im_y)/\sqrt{2}$ is a chiral magnetization, and $A_{\alpha}^\parallel = \cos \theta \partial_\tau \phi = A_{\alpha} \cdot n_0$ is the longitudinal component of $A_{\alpha}$. Driven by a moving magnetic profile, the potential $A_{\alpha}^\parallel$ defines emergent electrodynamics with the effective magnetic and electric fields, $E_z = \partial_z A_{\alpha}^\parallel - \partial_\theta A_{\alpha}^\parallel = -n_0 \cdot (\partial_\tau n_0 \times \partial_\theta n_0)$ and $E_n = \partial_n A_{\alpha}^\parallel - \partial_\theta A_{\alpha}^\parallel = -n_0 \cdot (\partial_\tau n_0 \times \partial_\theta n_0)$, respectively, with $A_{\alpha}^\parallel = \cos \theta \partial_\tau \phi$.

The fluctuation modes can further be parametrized by using the Polyakov decomposition and introducing a dimensionless complex field $\psi(r, \tau)$ [51, 52]:

$$m = e_+(r - R, \tau) \psi^\dagger(r - R, \tau) + e_-(r - R, \tau) \psi(r - R, \tau),$$

with $e_\pm = (e_0 \pm i e_\phi)/\sqrt{2}$ satisfying $e_+ \cdot e_- = 1$ and $e_\pm \cdot e_\pm = 0$. By this substitution, the time-dependent term in Eq. (8) can be neatly recast in terms of the two component variable $\hat{\psi}^\dagger = (\psi^* \psi)$:

$$m \cdot (\partial_\tau m \times n_0) = -i \hat{\psi}^\dagger \left( \hat{\sigma}_z \partial_\tau - \hat{K}_\alpha \hat{R}_\alpha \right) \hat{\psi},$$

where $\hat{\sigma}_0$ is the unity matrix, $\hat{\sigma}_k$ is the vector of the Pauli matrices acting in the particle-hole space $(k = x, y, z)$, and $\hat{K}_\alpha = \hat{\sigma}_z \partial_\alpha + i \hat{\sigma}_0 A_{\alpha}^\parallel$ (here, a hat symbol refers to the two component representation $\hat{\psi}$). One can see that $\hat{K}_\alpha$ is nothing else but a standard Berry connection term of the conventional adiabatic perturbation theory that describes the adiabatic change of a quantum system in some parameter space [33]. Similar to Eq. (16), the magnetic fluctuations adjust adiabatically to a slowly varying magnetization profile of the moving skyrmion, and their dynamics couples to the emergent electrodynamical potential of the skyrmion $A_{\alpha}^\parallel$.

The resulting action Eq. (8) describing the fast magnetization dynamics in terms of the complex field variables can be written as:

$$S[\hat{\psi}, \hat{\psi}^\dagger, R] = \int dx \hat{\psi}^\dagger \left( \hat{G}_0^{-1} - \frac{S}{a^2} \hat{K}_\alpha \hat{R}_\alpha \right) \hat{\psi},$$

where a (bare) bosonic Green’s function in the absence of the skyrmion motion is defined as:

$$\hat{G}_0(\tau) = \left( \frac{S}{2a^2} \hat{\sigma}_z \partial_\tau + \frac{1}{2} \hat{H}_{\text{BdG}} \right)^{-1},$$

and $\hat{H}_{\text{BdG}}$ is a (bosonic) Bogoliubov-de Gennes Hamiltonian that describes the magnon spectrum [54]. An explicit form of $\hat{H}_{\text{BdG}}$ obtained from the decomposition Eq. (17) and including both longitudinal and transverse magnetic fluctuations is given in the Supplementary Material [39]. The magnon wavefunctions can be found by solving the eigenvalue problem $\hat{H}_{\text{BdG}} \Psi_n^s(\tau) = e_n^\pm \hat{\sigma}_z \Psi_n^s(\tau)$ with $s = \pm$, satisfying the paraunitary orthogonality condition $\int dr \hat{\Psi}_n^s(\tau) \hat{\sigma}_y \hat{\Psi}_n^s(\tau) = s \delta_{nn'} \delta_{ss'}$. The spectrum of $\hat{H}_{\text{BdG}}$ is characterized by pairs $e_n^\pm = \pm |\varepsilon_n| \epsilon$ as a consequence of the particle-hole symmetry $\hat{C} \hat{\Psi}_n^\pm(\tau) = \hat{C} \hat{\Psi}_n^\mp(\tau)$, where $\tau$ is complex conjugation. From the spectral theorem and orthogonality condition, the magnonic Green’s function $\hat{G}_0(\tau) = \frac{1}{\pi} \sum_{\nu} \hat{G}_0(\nu | \nu) e^{-i \omega_{\nu} \tau}$ in Matsubara frequencies $i \omega_{\nu} = 2 \pi \nu / \beta$ can be written as:

$$\hat{G}_0(\tau, i \omega_{\nu}) = \frac{2 a^2}{S} \sum_{sn} s \delta \Psi_n^s(\tau) \Psi_n^s(\tau) \hat{\sigma}_z - i \omega_{\nu} + \bar{\varepsilon}_n$$

with $\bar{\varepsilon}_n = a^2 \varepsilon_n / S$ and $s = \text{sgn} \bar{\varepsilon}_n$.

Following the analysis of [29], the magnonic Hamiltonian can be represented as $\hat{H}_{\text{BdG}} = \hat{H}_0 + \hat{V}_{\text{SK}}$, where $\hat{H}_0$ describes the spin-wave modes of a collinear ferromagnet and $\hat{V}_{\text{SK}}$ accounts for the magnonic scattering off the skyrmionic profile. Thus, the magnonic modes are not only modified in the presence of a skyrmion configuration but also experience an emergent potential driven by the skyrmion motion. As will be shown below, the latter gives an intrinsic damping kernel for the dynamics of the skyrmion collective coordinates.

### III. SKYRMION DYNAMICS

Having defined the dynamics of magnetic fluctuations in the presence of the skyrmion motion, the fast magnetization variables can be integrated out leaving the action formulated entirely for the collective coordinate [39]:

$$\int \mathcal{D}(\hat{\psi}, \hat{\psi}^\dagger) e^{-S[\hat{\psi}, \hat{\psi}^\dagger, R]} = \frac{1}{\det'(\hat{G}_0^{-1} - \frac{S}{a^2} \hat{K}_\alpha \hat{R}_\alpha)}$$

$$\times \text{Tr}' \left( \frac{S}{a^2} \sum_{\alpha} \hat{K}_\alpha \hat{R}_\alpha + \frac{S^2}{8 a^4} \text{Tr} \hat{K}_\alpha \hat{R}_\alpha \hat{R}_\alpha \right)$$

$$= Z_\psi e^{-S_M[R]},$$

where $Z_\psi = \det' \hat{G}_0$ is the partition function of the fluctuation modes, and the prime in the trace and determinant indicates the exclusion of zero modes. Given the smallness of the adiabatic potential, the determinant is decomposed to second order in $\hat{R}$. The linear term vanishes due to the periodic boundary conditions in imaginary time, and the second order term gives the second intrinsic contribution to the skyrmion dynamics:

$$S_M[R] = - \frac{S^2}{8 a^2} \text{Tr} \left[ \hat{G}_0 \hat{K}_\alpha \hat{G}_0 \hat{K}_\alpha \hat{R}_\alpha \hat{R}_\alpha \right]$$

$$= \frac{1}{2} \int d\tau d\tau' \hat{R}_\alpha(\tau) M_{\alpha\alpha'}(\tau - \tau') \hat{R}_{\alpha'}(\tau').$$
The integral kernel defines the intrinsic damping term:

\[ M_{\alpha\alpha'}(\tau) = -\frac{1}{\beta^2} \sum_{n, n'} \sum_{\nu, \nu'} \int \frac{d\nu}{\nu} \frac{s s' K_{ss'}^{\alpha, n, n'} K_{s's'}^\nu n e^{i(\omega\nu - \omega\nu')\tau}}{(i\omega\nu - \vec{\varepsilon}_n)(i\omega\nu' - \vec{\varepsilon}_{n'})} \]

\[ = \sum_{s s', s' n} s s' K_{ss'}^{\alpha, n, n'} K_{s's'}^\nu n \mathcal{P}_{nn'}(\tau), \]  

for \( n \leq 0 \).

Combining Eqs. (15) and (23), the resulting action for the magnon modes and is essentially a massless particle-hole symmetry, the following identities are satisfied:

\[ \mathcal{K}_{s s'}^{\alpha, n, n'} = -[\mathcal{K}_{s s'}^{\alpha, n, n'}]^\dagger, \]

and (25) remains the same and can merely be calculated for the skyrmion dynamics at thermal equilibrium.

It should be pointed out that both the skyrmion configuration and magnon spectrum can be altered in the presence of a confining potential, whose effect can be incorporated in the equation of motion for \( \mathcal{G}(t) \) and in the magnonic Hamiltonian as an additional potential term \( \mathcal{V} \).

To analyze the structure of the damping kernel, let us consider a moving skyrmion specified by the micromagnetic model Eq. (3) in a translationally invariant geometry. A chiral skyrmion configuration shown in Fig. [1] can be parametrized as \( \phi = \chi + \frac{\theta}{2} \) and \( \theta = \theta(\rho), \) where \( \chi \) and \( \rho \) are polar coordinates centered at the skyrmion core [39]. Owing to the rotational symmetry, the magnon modes can be found by solving the eigenvalue problem for \( \mathcal{H}_{\text{BDG}} \) in the form \( \hat{\Psi}_m(\chi, \rho) = e^{im\chi} \hat{\delta}_m(\rho) \) with \( m \in Z \) [28, 39]. By using \( \partial_x = \cos \chi \partial_\rho - \frac{1}{2} \sin \chi \partial_\chi \) and \( \partial_\theta = \sin \chi \partial_\rho + \frac{1}{2} \cos \chi \partial_\chi \) integrating over the polar angle, it is straightforward to show that \( \mathcal{K}_x \) and \( \mathcal{K}_y \) have purely real and imaginary matrix elements, respectively. From the paramunitary orthogonality and particle-hole symmetry, the following identities are satisfied:

\[ \mathcal{K}_{s s'}^{\alpha, n, n'} = -[\mathcal{K}_{s s'}^{\alpha, n, n'}]^\dagger, \]

\[ \mathcal{K}_{s s'}^{\alpha, n, n'} = -[\mathcal{K}_{s s'}^{\alpha, n, n'}]^\dagger. \]

FIG. 2. Diagrammatic representation of the intrinsic damping kernel that originates from the magnonic heat bath coupled to a moving skyrmion.
be written as [55 56]:

$$Z = \int D^{R^1}D^{R^2}e^{iS[R^1,R^2]}$$  \hspace{1cm} (26)$$

with the action:

$$\omega \tilde{G}(\omega)Y_{c1}(\omega) + (\omega^2M_x(\omega) - i\omega D_x(\omega))X_{c1}(\omega) = 0, \hspace{1cm} (30)$$

where \( \tilde{G}(\omega) = \frac{4\pi SQ}{a^2} + \omega \text{Re} M_{xy}^{R}(\omega) \) is the modified gyo-
magnetic coupling (the imaginary part of \( M_{xy}^{R}(\omega) \) defines the gyro-
magnetic damping and is omitted). As can be seen, the inertial and friction terms enter the dynam-
ics in Eq. (30) as essentially non-local quantities. If one goes beyond the classical saddle point approx-
imation and keeps higher order terms in \( \tilde{R}_\alpha \), the Keldysh component \( M_{xx}^{R}(\omega) \) is not dropped out and Eq. (30) can be modified to a Langevin-type equation [56].

**IV. NUMERICAL RESULTS**

Analytical results in the previous sections are derived for the micromagnetic model. The details of the model itself were not important, and the formalism presented above is valid in the general case. In order to make a connection to realistic simulations, the numerical analysis is performed for the lattice counterpart of Eq. (3) on the square lattice:
theory by using the Holstein-Primakoff transformation for Eq. (31). Choosing a sufficiently large size for the supercell as compared to the skyrmion radius, the skyrmion-skyrmion interactions caused by the periodicity can be considered small, and the magnon spectra can be calculated for the $\Gamma$ point only. This approach allows us to compute the intrinsic damping kernel for a single skyrmion fully taking into account the effect of magnetic fluctuations.

The calculated magnon spectra as a function of magnetic field are presented in Fig. 3 for the periodic and confined geometries. For example, the periodic and open boundary conditions are imposed along the $x$ and $y$ axes, respectively. From the low-lying magnonic wavefunctions, the modes corresponding to the breathing, clockwise (CW) and counterclockwise (CCW) fluctuations, as well as the polyhedron modes can be well identified in the magnon spectrum. These spin-wave modes have been both predicted theoretically and recognized experimentally as intrinsic deformations of the skyrmionic profile. Below the magnon continuum gap $\sim H + 2K$ of the ferromagnetic background, one can see the formation of the magnon bound states that correspond to the (breathing) $m = 0$ and (polyhedron) $m = 2$ modes (the $m = 3$ mode is also found at low magnetic fields in a thermodynamically metastable regime). Above the magnon gap starting with the rotational modes, there follows a continuum of the magnon scattering states deformed by the skyrmionic potential. The calculated magnon spectra are in good agreement with the magnon mode analysis in terms of the micromagnetic model.

Based on the magnon spectra, the intrinsic damping kernel can be calculated as a function of temperature and magnetic field. The results for the periodic geometry are presented in Fig. 4. In agreement with the analysis above, the calculations confirm that the diagonal and off-diagonal components of $M_{\alpha\alpha'}(\tau)$ are, respectively, real and imaginary quantities. We find that $M_{xx}(\tau) = M_{yy}(\tau)$ as satisfied for a translationally invariant system, and, being inversely proportional to the magnon energy, $M_{\alpha\alpha'}(\tau)$ expectedly decreases with increasing magnetic field. The temperature dependence of the intrinsic damping, in turn, is found to be non-trivial. At temperatures within the magnon gap, $M_{\alpha\alpha'}(\tau)$ reveals a non-local behavior in imaginary time that becomes more noticeable as the temperature rises. This feature may be considered substantial when quantum fluc-
tations play a role. In this context, it is worth noting that the damping term to the conjugate variables, in the form of Eq. (23), alongside with a conventional dissipative damping for the coordinate variables was studied for several quantum systems [61, 62], where the momentum coupling was shown to increase quantum fluctuations in position, thus competing with the classical localization and enhancing delocalization effects at low temperatures. Such effects could be relevant for skyrmion dynamics at low temperatures and sufficiently high magnetic fields when the skyrmion size is small. Nevertheless, the damping kernel in this regime is found to be small, and its effect on the skyrmion motion can turn out to be minor.

At temperatures higher than the characteristic magnon gap, the non-local behavior of the intrinsic damping kernel becomes more pronounced as more magnon modes are thermally activated. As shown in Fig. 4b, the off-diagonal components reach a steady linear behavior in imaginary time. From \( \mathcal{M}_{xy}(\tau) = \mathcal{M}^*(\tau) \), one can show that the off-diagonal components give a thermal correction to the Magnus force, thus renormalizing the skyrmion’s gyrotropic motion. On the other hand, the diagonal components are dominated by a zero Matsubara frequency that grows almost linearly as a function of temperature (Fig. 4c). Such situation can be regarded as the classical limit when the occupation factors of the magnon modes become large and quantum fluctuations are smeared out by temperature. As was discussed above, this problem can be reformulated on the real-time contour where the intrinsic damping kernel is obtained by analytic continuation of \( \mathcal{M}^{\alpha\alpha'}(i\omega_{\nu}) \). The diagonal component of the retarded damping kernel in the real frequency domain is shown in Fig. 5. One can see that both real and imaginary parts of \( \mathcal{M}^{\alpha\alpha'}_R(\omega) \) are non-local and grow almost linearly with increasing temperature. The real part of \( \mathcal{M}^{\alpha\alpha'}_R(\omega) \) is dominated by a zero-frequency component, and if the non-zero frequency components are discarded the related mass term does not enter the dynamics of collective coordinates [63]. On the contrary, the friction term defined by the imaginary part of \( \mathcal{M}^{\alpha\alpha'}_R(\omega) \) is given by non-zero frequencies and is necessarily included in the modified equation of motion. As was argued above, all the intrinsic terms entering the dynamics in Eq. (30) are essentially non-local quantities.

The case of the confined geometry is included for illustration purposes in order to outline the idea of possible deformations in the magnon spectrum. When the motion of a skyrmion is confined, its profile and the magnon bound modes are found to be distorted. If the confining potential is sufficiently small, the character of the magnon bound modes is well preserved, as shown in Fig. 3d. Given the same model parameters used in the calculations, the intrinsic damping kernel in the confined case closely follows the one of the translationally invariant system with a small difference in the diagonal components, \( \mathcal{M}_{xx}(\tau) \neq \mathcal{M}_{yy}(\tau) \). Importantly, the intrinsic
Our work establishes a path integral formalism for the intrinsic dynamics of magnetic skyrmions coupled to the magnonic heat bath, where a moving skyrmion is shown to acquire the non-local damping acting as the source of dissipation and inertia.

A laborious attempt to describe the dynamics of magnetic skyrmions by taking into account magnetic fluctuations was done in Ref. [35]. Following a similar path integral approach, the authors of Ref. [35] derived two contributions for the skyrmion mass, the temperature dependent term coming from the thermal excitations of the magnon modes and the temperature independent term that arises in the presence of translational symmetry breaking. Aside from analytical discrepancies for the temperature dependent damping with the one obtained in Section III, the presence of its temperature independent counterpart stemming from magnetic fluctuations does not conform with the formalism presented above and appears to look unusual, as the magnon excitations in equilibrium are justified as a thermal effect. To trace the origin of the temperature independent mass term in Ref. [35], one can assume that the first variation \( \delta S_{\text{WZ}}/\delta \mathbf{n} \) is now included in the decomposition of the Euclidean action. By using \( \delta S_{\text{WZ}} = \frac{iS}{\alpha^2} \int dx \delta \mathbf{n} \cdot (\partial_\alpha \mathbf{n} \times \mathbf{n}) \), Eq. (7) would be modified by a linear term \( \int dx (\delta S_{\text{WZ}}/\delta \mathbf{n}(x))(n=n_0) \mathbf{m}(x) = -\frac{\lambda_s}{\alpha^2} \int dx \mathbf{m} \cdot (\partial_\alpha n_0 \mathbf{n}_0 \times \mathbf{n}_0) \mathbf{R}_\alpha \), and the action for magnetic fluctuations would have the following form:

\[
S[\hat{\psi}, \hat{\psi}^\dagger, \mathbf{R}] \rightarrow S[\hat{\psi}, \hat{\psi}^\dagger, \mathbf{R}] + \int dx \hat{J}^\dagger \hat{\psi} + \hat{\psi}^\dagger \hat{J} \tag{32}
\]

with \( \hat{J}^\dagger = \frac{iS}{\alpha^2} (A_\alpha - A_\alpha^\dagger) \) and \( A_\alpha^\dagger = (A_\alpha^\dagger \pm iA_\alpha^\dagger)/\sqrt{2} \), from which the temperature independent damping could be reproduced by integrating out the complex field variables. By contrast, it is straightforward to show that the equation of motion Eq. (16) for the fast magnetization variables would be modified as:

\[
\frac{iS}{\alpha^2} \left( \hat{m} - \partial_\alpha \mathbf{R}_\alpha - A_\alpha^\dagger n_0 \times \mathbf{m} \mathbf{R}_\alpha - \partial_\alpha n_0 \mathbf{R}_\alpha \right) + n_0 \times \mathbf{m} = 0. \tag{33}
\]

On the one hand, one can see that \( \delta S_{\text{WZ}}/\delta \mathbf{n} \) changes the dynamics of the magnon modes up to an addition of zero modes \( \partial_\alpha n_0 \). On the other hand, the Faddeev-Popov constraint Eq. (11) was introduced into the path integral to ensure that the zero modes and magnetic fluctuations are mutually orthogonal. Thus, the inconsistent description of skyrmion dynamics in Ref. [35] is seen to come from the redundant inclusion of the first variation \( \delta S_{\text{WZ}}/\delta \mathbf{n} \), and the temperature independent damping does not appear in the resulting action for collective coordinates. In contrast, our formalism restores the role of the Faddeev-Popov technique in the description of collective coordinates. In this regard, we reiterate that the saddle point approximation is valid in the limit of sufficiently small skyrmion velocities, so that a stationary configuration \( n_0 \) retains its rigidity and \( \delta S_{\text{WZ}}/\delta \mathbf{n} \) can be disposed via the orthogonality condition. As an aside, the orthogonality of the magnon and zero modes is justified does not appear in the resulting driving term of collective coordinates. In this regard, we reiterate that the saddle point approximation is valid in the limit of sufficiently small skyrmion velocities, so that a stationary configuration \( n_0 \) retains its rigidity and \( \delta S_{\text{WZ}}/\delta \mathbf{n} \) can be disposed via the orthogonality condition. As an aside, the orthogonality of the magnon and zero modes is justified does not appear in the resulting driving term of collective coordinates. In this regard, we reiterate that the saddle point approximation is valid in the limit of sufficiently small skyrmion velocities, so that a stationary configuration \( n_0 \) retains its rigidity and \( \delta S_{\text{WZ}}/\delta \mathbf{n} \) can be disposed via the orthogonality condition. As an aside, the orthogonality of the magnon and zero modes is justified does not appear in the resulting driving term of collective coordinates. In this regard, we reiterate that the saddle point approximation is valid in the limit of sufficiently small skyrmion velocities, so that a stationary configuration \( n_0 \) retains its rigidity and \( \delta S_{\text{WZ}}/\delta \mathbf{n} \) can be disposed via the orthogonality condition. As an aside, the orthogonality of the magnon and zero modes is justified.
alyzed numerically by solving the stochastic Landau-Lifshitz-Gilbert equation. In particular, a large skyrmion mass was shown to arise in response to thermal diffusion, and the friction term was found to survive even in the limit of zero Gilbert damping. When the thermal effects are neglected, a vanishing mass was obtained for an electric current-driven motion, and a non-zero mass at high frequencies was shown to be induced by an oscillating magnetic field. These results were misinterpreted in Ref. [35] to occur as a consequence of translational symmetry breaking. As a matter of fact, the analysis of Ref. [31] evidently points out to an intrinsic coupling of the skyrmion motion to magnetic fluctuations regardless of the symmetry and geometry, where the effects of friction and inertia can be explained by the magnon bound modes excited within the magnon gap. Despite different approaches and numerical methods used in our study, the calculated features of the intrinsic damping in real frequencies are found to be similar to the case of thermal diffusion reported in Ref. [31].

A recent study on the current-driven and Brownian motion of magnetic skyrmions within stochastic magnetization dynamics suggested a phenomenological description of the magnonic heat bath as an additional dissipative term in the Thiele’s equation [34]. Although the inertial effects were abandoned in the analysis of Ref. [34], our calculations clearly support the assumption for the magnon driven friction with a nearly linear temperature dependence, and can in principle be regarded as a microscopic rationale of the semiclassical phenomenology.

Finally, it is worth noting that while our study is primarily focused on the thermal effect of magnetic fluctuations in the absence of time-dependent forces, the magnon modes can be excited per se by a pulsed magnetic field or alternating currents leading to their strong coupling with the magnetization profile. The former was experimentally conjectured to produce the inertia in magnetic skyrmions [27], and the latter was shown to bring the mass of magnetic domain walls [66, 67]. Based on the Keldysh-Schwinger formalism outlined above for a steady-state motion, non-equilibrium dynamics can be addressed in a similar manner by introducing an external time-dependent source field coupled to magnetic fluctuations, which will lead to the excitation of a resonance magnon mode. Regardless of the problem, this analysis can be applied to an arbitrary magnetic configuration that permits description by collective coordinates. We believe that it can consolidate previous studies where the origin of a mass term was attributed to magnetic excitations, as was done for magnetic vortices [68–70] and skyrmions [27, 33]. Nevertheless, as far as the thermal excitations are concerned, it should be emphasized that an overall effect of magnetic fluctuations is not simply reduced to a local mass term.

VI. CONCLUDING REMARKS

In this study, we addressed the problem of intrinsic quantum dynamics of magnetic skyrmions coupled to a thermal background of magnetic fluctuations. Based on the micromagnetic model and path integral approach, the magnonic heat bath is shown to generate the intrinsic damping that comes from the magnon scattering off the adiabatic potential driven by a moving skyrmion, revealing its connection to the emergent electrodynamic properties. We derived the generalized equation for the skyrmion’s collective motion where the thermally excited magnon modes give rise to the non-local dissipation and inertial terms, as well as the correction to the gyrotrropic motion. Finally, we presented a numerical scheme for calculating the properties of the intrinsic damping based on linear spin-wave theory for the lattice spin models, upon which an almost linear temperature behavior of the damping was shown for chiral ferromagnets, supporting the results of semiclassical simulations.

The established formalism can be generalized to an arbitrary magnetic system driven out-of-equilibrium, and its applicability can provide a microscopic analysis to a variety of practical situations and support experimental studies on the dynamics and spin caloritronics effects in topological magnetic textures. We believe that this study will stimulate further endeavors to tailor the dynamical properties of magnetic skyrmions and promote their application in future devices.

ACKNOWLEDGEMENT

A.T. was supported in part by JSPS KAKENHI grant no.19K03662.

APPENDIX: NUMERICAL DETAILS

Starting with the micromagnetic model Eq. [3], we adopt \( \theta \equiv \theta(\rho) \) and \( \phi = \chi + \frac{\pi}{2} \) for a chiral skyrmion configuration, where \( \rho \) and \( \chi \) denote a cylindrical coordinate system centered at the skyrmionic core. The profile \( \theta(\rho) \) is obtained by minimizing the micromagnetic Hamiltonian:

\[
J \theta'' + J \frac{\theta'}{\rho} - J \sin \theta \cos \theta \frac{\sin \theta}{\rho^2} + 2D \sin^2 \theta \frac{\theta'}{\rho} - 2K \sin \theta \cos \theta - \frac{\rho}{\rho^2} \sin \theta = 0 , \tag{34}
\]

with \( \theta' \equiv \partial_\rho \theta \). The resulting magnetization profile is placed on the lattice and is further relaxed under an applied magnetic field by the Landau-Lifshitz equation and the lattice spin model.
Magnon spectra are calculated by using linear spin-wave theory. Taking the spin Hamiltonian in a general form on a lattice with $N$ spins:

$$\mathcal{H}_S = \sum_{(ij)} S_i^T A_{ij} S_j - \sum_i H \cdot S_i,$$  

(35)

where the intersite exchange interactions are incorporated into the matrix $A_{ij}$, we employ the Holstein-Primakoff transformation for the spin operators, which are to leading order \[57\]:

$$S_i^z \approx \sqrt{2Sb_i^+},$$  

(36)

$$S_i^- \approx \sqrt{2Sb_i^+},$$

with $S_i^z = S_i^z$ being the magnonic mode energies and $\hat{\Psi}$, and the bosonic operators satisfy $[b_i^+, b_j^+] = \delta_{ij}$. The operators $S_i^z$ are obtained through rotations to the local coordinate frame $S_i = R_i S_i^z$ with:

$$R_i = \begin{pmatrix}
\cos \phi_i & \sin \phi_i & 0 \\
-\sin \phi_i & \cos \phi_i & 0 \\
0 & 0 & \cos \phi_i
\end{pmatrix},$$

or equivalently:

$$S_i = \sqrt{S}(e_- b_i + e_+ b_i^+) + n_{0i}(S - b_i^+ b_i),$$

(38)

where $e_{\pm}^k = \frac{1}{\sqrt{2}}(R_i^{k1} \pm iR_i^{k2})$ and $n_{0i} = R_i^{k3}$ with $k = 1, 2, 3$. The resulting Bogoliubov-de Gennes Hamiltonian for magnonic variables is written as \[53\]:

$$\hat{\mathcal{H}}_{BdG} = b^\dagger \begin{pmatrix} 
\Lambda & \Sigma \\
\Sigma^* & -\Lambda^*
\end{pmatrix} b + \text{const}$$

(39)

with $b^\dagger = (b_1^+ b_2^+ ... b_N^+ b_1 b_2 ... b_N)^T$ and the matrix elements obtained as:

$$\Lambda_{ij} = S e_{-i}^T A_{ij} e_{-j} - S \delta_{ij} \sum_l n_{0l} A_{i+l} n_{0l},$$

$$\Sigma_{ij} = S e_{+i}^T A_{ij} e_{+j},$$

(40)

where $l$ runs over all neighbours. The magnonic Hamiltonian is diagonalized as $\hat{\mathcal{H}}_{BdG} = \hat{\Psi} \tilde{\mathcal{E}} \hat{\Psi}^\dagger$, where $\hat{\Psi} = [\hat{\Psi}_1 ... \hat{\Psi}_N \hat{\Psi}_{N+1} ... \hat{\Psi}_{2N}]$ is a paramatry matrix, $\hat{\Psi}^\dagger \tilde{\mathcal{E}} \hat{\Psi} = \hat{\Psi} \hat{\Psi}^\dagger = \hat{\sigma}_z$, and $\tilde{\mathcal{E}} = \text{diag}(\varepsilon_1, ..., \varepsilon_N, -\varepsilon_1, ..., -\varepsilon_N)$ with $\varepsilon_n$ being the magnonic mode energies and $\hat{\Psi}_n$ being the corresponding wavefunctions ($n = 1, ..., N$) \[71\]. From the particle-hole symmetry of $\hat{\mathcal{H}}_{BdG}$, $\hat{\Psi}^n = C \hat{\sigma}_z \hat{\Psi}_n$ is satisfied. It should be stressed that the Holstein-Primakoff transformation gives an equivalent description of the magnon modes to the one defined by Eq. \[17\], where the magnonic operators and the complex field variables are related by a factor of $\sqrt{S}$, namely $b \rightarrow \sqrt{S} \hat{\Psi}$, $b^\dagger \rightarrow \sqrt{S} \hat{\Psi}^\dagger$, and $\varepsilon_n \rightarrow S \varepsilon_n$ \[52\]. Note that there is only one zero mode in the numerical spectrum of $\hat{\mathcal{H}}_{BdG}$, while translational symmetry implies the existence of two distinct zero modes. For the sake of comparison, Fig. \[9\] shows the calculated magnon spectra for the spin model Eq. \[31\] on a triangular lattice in the periodic and confined geometries.

The magnon Green’s function in Eq. \[21\] can be redefined as:

$$\hat{G}_0(\omega_n) = 2 \sum_{n=1}^{2N} s \hat{\sigma}_n^+ \hat{\Psi}_n^\dagger \hat{\Psi}_n \hat{\sigma}_n \bigg( -i \omega_n + \varepsilon_n \bigg),$$

(41)

with $s = \text{sgn} \varepsilon_n$. The damping kernel is calculated as:

$$M_{aa'}(\tau) = \sum_{n,n'=1}^{2N} s \hat{\xi}_{n,a,n',a'}(\varepsilon_n - \varepsilon_{n'}) P_{nn'}(\tau),$$

(42)

and

$$M_{aa'}(i\omega_n) = \sum_{n,n'=1}^{2N} s \hat{\xi}_{n,a,n',a'}(\varepsilon_n - \varepsilon_{n'}) \bigg( -i \omega_n + \varepsilon_n - \varepsilon_{n'} \bigg),$$

(43)

where the prime in the sum indicates the exclusion of zero modes ($n = 1$ and $N + 1$), $P_{nn'}(\tau) = \delta(\varepsilon_{n'} - \varepsilon_n)(i\tau) n_B(\varepsilon_{n'}) n_B(\varepsilon_n)$, and $\hat{\xi}_{n,a,n',a'} = \hat{\Psi}_n^\dagger K_{n,a}\hat{\Psi}_{n'}$. is calculated as a sum over the lattice sites within the skyrmion radius. Spatial derivatives in $K_{n,a}$ are calculated by finite differences between neighbouring sites.

It should be pointed out that in order to calculate the matrix elements of $K_{n,a}$, one has to eliminate the ferromagnetic background surrounding the skyrmionic profile. We resolve this problem by reducing the integration region to the skyrmion radius. This can be regarded as a good approximation because the magnon bound states are not mixed with the continuum states and are localized within the skyrmion profile, and the magnon scattering states are primarily deformed within the skyrmion radius (as can be seen from the form of the scattering potentials \[39\]). That being said, the relations for the matrix elements $K_{n,n'}$ discussed in Section III are not strictly satisfied. Nevertheless, the calculated intrinsic damping in imaginary time have the same properties discussed in Section III. In addition, the matrix elements converge fast with respect to the number of magnon modes, and given a large size of the supercell our calculations can be considered valid in the continuum limit within a numerical accuracy.

Analytic continuation from the Matsubara to real frequencies was performed by using the maximum entropy method \[72\].

[nikolaev.s.aa@m.titech.ac.jp]
FIG. 6. (a) Magnon spectrum of the chiral ferromagnet with a single skyrmion on the triangular lattice as a function of magnetic field, and (b) the corresponding wavefunctions (real part) of the first low-lying modes at $H = 0.086$ (blue line in the spectrum). The model parameters are $J = 1.0$, $D = 0.3$, $K = 0$, and the calculations are performed on a $51 \times 51$ lattice with periodic boundary conditions. (c) Magnon spectra for the periodic and confined geometries (confined along the $y$ axis). The model parameters are $J = 1.0$, $D = 0.3$, $K = 0.03$.

[1] A. N. Bogdanov, D. A. Yablonskii, Thermodynamically stable “vortices” in magnetically ordered crystals. The mixed state of magnets, Sov. Phys. JETP 68, 101–103 (1989).
[2] A. N. Bogdanov and A. Hubert, Thermodynamically Stable Magnetic Vortex States in Magnetic Crystals, J. Magn. Magn. Mater. 138, 255–269 (1994).
[3] U. K. Rößler, N. Bogdanov, C. Pfleiderer, Spontaneous skyrmion ground states in magnetic metals, Nature 442, 797–801 (2006).
[4] S. Mühlbauer, B. Binz, F. Jonietz, C. Pfleiderer, A. Rosch, A. Neubauer, R. Georgii, P. Böni, Skyrmion Lattice in a Chiral Magnet, Science 323, 915–919 (2009).
[5] X. Z. Yu, Y. Onose, N. Kanazawa, J. H. Park, J. H. Han, Y. Matsui, N. Nagaosa, and Y. Tokura, Real-space observation of a two-dimensional skyrmion crystal, Nature 465, 901–904 (2010).
[6] X. Z. Yu, N. Kanazawa, Y. Onose, K. Kimoto, W. Z. Zhang, S. Ishiwata, Y. Matsui, Y. Tokura, Near room-temperature formation of a skyrmion crystal in thin-films of the helimagnet FeGe, Nature Mater. 10, 106–109 (2010).
[7] Stefan Heinze, Kirsten von Bergmann, Matthias Menzel, Jens Brede, André Kubetzka, Roland Wiesendanger, Gustav Bihlmayer, Stefan Blügel, Spontaneous atomic-scale magnetic skyrmion lattice in two dimensions, Nature Phys. 7, 713–718 (2011).
[8] A. Fert, N. Reyren, and V. Cros, Magnetic skyrmions: advances in physics and potential applications, Nat. Rev. Mater. 2, 17031 (2017).
[9] A. Fert, V. Cros, and J. Sampaio, Skyrmions on the track, Nat. Nanotechnol. 8, 152–156 (2013).
[10] J. Sampaio et al. Nucleation, stability and current-induced motion of isolated magnetic skyrmions in nanostructures, Nature Nanotechnol. 8, 839–844 (2013).
[11] A. Neubauer, C. Pfleiderer, B. Binz, A. Rosch, R. Ritz, P. G. Niklowski, and P. Böni, Topological Hall Effect in the A Phase of MnSi, Phys. Rev. Lett. 102, 186602 (2009).
[12] Minhyea Lee, W. Kang, Y. Onose, Y. Tokura, and N. P. Ong, Unusual Hall Effect Anomaly in MnSi under Pressure, Phys. Rev. Lett. 102, 186601 (2009).
[13] Wanjun Jiang, Xichao Zhang, Guoqiang Yu, Wei Zhang, Xiao Wang, M. Benjamin Jungfleisch, John E. Pearson, Xuemei Cheng, Olle Heinonen, Kang L. Wang, Yan Zhou, Axel Hoffmann, Suzanne G. E. te Velthuis, Direct observation of the skyrmion Hall effect, Nat. Phys. 13, 162–169 (2017).
[14] Kai Litzius, et al., Skyrmion Hall effect revealed by direct time-resolved X-ray microscopy, Nat. Phys. 13, 170–175 (2017).
[15] G. Volovik, Linear momentum in ferromagnets, J. Phys. C 20, L83–87 (1987).
[16] S. E. Barnes, S. Maekawa, Generalization of Faraday’s law to include nonconservative spin forces, Phys. Rev. Lett. 98, 246601 (2007).
[17] T. Schulz, R. Ritz, A. Bauer, M. Halder, M. Wagner, C. Franz, C. Pfleiderer, K. Everschor, M. Garst, A. Rosch, Emergent electrodynamics of skyrmions in a chiral magnet, Nature Phys. 8, 301–304 (2012).
[18] N. Nagaosa, Y. Tokura, Topological properties and dynamics of magnetic skyrmions, Nature Nanotechnology 8, 899–911 (2013).
[19] A. A. Thiele, Steady-State Motion of Magnetic Domains, Phys. Rev. Lett. 30, 230–233 (1973).
[20] J. P. Park and P. A. Crowell, Interactions of Spin Waves with a Magnetic Vortex, Phys. Rev. Lett. 95, 167201 (2005).
[21] K. Yu. Guslienko, X. F. Han, D. J. Keavney, R. Divan, and S. D. Bader, Magnetic Vortex Core Dynamics in Cylindrical Ferromagnetic Dots, Phys. Rev. Lett. 96, 067205 (2006).
[22] J. Iwasaki, M. Moehl, and N. Nagaosa, Universal current-velocity relation of skyrmion motion in chiral magnets, Nat. Commun. 4, 1463 (2013).
[23] J. Sampaio, V. Cros, S. Rohart, A. Thiaville, and A. Fert, Nucleation, stability and current-induced motion of isolated magnetic skyrmions in nanostructures, Nat. Nanotechnol. 8, 839–844 (2013).
[24] Jacques Militat, Stanislav Rohart, and André Thiaville, Brownian motion of magnetic domain walls and skyrmions, and their diffusion constants, Phys. Rev. B
Here, we neglect longitudinal fluctuations $\sim \sqrt{1 - m^2}$ that can be included by expanding $S_S[\mathbf{m}_0]$. Both longitudinal and transverse fluctuations are explicitly taken into account in the final form of the magnonic Hamiltonian and numerical results.

Volodymyr P. Kravchuk, Denis D. Shela, Ulrich K. Röfler, Jeroen van den Brink, and Yuri Gaididei, Spin eigenmodes of magnetic skyrmions and the problem of the effective skyrmion mass, Phys. Rev. B 97, 064403 (2018).

L. D. Faddeev and V. N. Popov, Feynman Diagrams for the Yang-Mills Field, Phys. Lett. B 25, 29–30 (1967).

B. Sakita, Quantum Theory of Many Variable Systems and Fields (World Scientific, Singapore, 1985).

Hans-Benjamin Braun and Daniel Loss, Berry’s phase and quantum dynamics of ferromagnetic solitons, Phys. Rev. B 53, 3237–3255 (1996).

A. M. Polyakov, Interaction of goldstone particles in two dimensions. Applications to ferromagnets and massive Yang-Mills fields, Phys. Lett. B 59, 79–81 (1975).

A. Auerbach, Interacting Electrons and Quantum Magnetism (Springer, 1994).

J. J. Sakurai, Jim Napolitano, Modern Quantum Mechanics (Cambridge University Press, 2020).

For the sake of convenience, we keep an explicit factor of $1/2$ in front the magnonic Hamiltonian, which is meant to yield $\frac{1}{2} \varepsilon (\mathbf{b} \cdot \mathbf{b} + \mathbf{b}^* \cdot \mathbf{b}^* - \mathbf{b} \cdot \mathbf{b}^* - \mathbf{b}^* \cdot \mathbf{b}) = \varepsilon (\mathbf{b} \cdot \mathbf{b} + \frac{1}{2})$.

A. Kamenev and A. Levchenko, Keldysh technique and non-linear $\sigma$-model: basic principles and applications, Advances in Physics 58, 197–319 (2009).

Alex Kamenev, Field Theory of Non-Equilibrium Systems (Cambridge University Press, 2011).

T. Holstein and H. Primakoff, Field Dependence of the Intrinsic Domain Magnetization of a Ferromagnet, Phys. Rev. 58, 1098–1113 (1940).

Olga Petrova and Oleg Tchernyshyov, Spin waves in a skyrmion crystal, Phys. Rev. B 84, 214433 (2011).

M. Mochizuki, Spin-Wave Modes and Their Intense Excitation Effects in Skyrmion Crystals, Phys. Rev. Lett. 108, 017601 (2012).

Y. Onose, Y. O Kamura, S. Seki, S. Ishiwata, and Y. Tokura, Observation of Magnetic Excitations of Skyrmion Crystal in a Helimagnetic Insulator Cu$_2$OSeO$_3$, Phys. Rev. Lett. 109, 037603 (2012).

Joachim Ankerhold and Eli Pollak, Dissipation can enhance quantum effects, Phys. Rev. E 75, 041103 (2007).

D. Maile, S. Andergassen, and G. Rastelli, Effects of a dissipative coupling to the momentum of a particle in a double well potential, Phys. Rev. Research 2, 013226 (2020).

A conventional mass term $M \mathbf{R}$ can be shown to come from the action with the damping term $M_{\alpha\alpha}(\tau) = M\delta(\tau)$, $M_{\alpha\alpha}(\tau) = \omega_\alpha$. Likewise, $M_{\alpha\alpha}(\tau) \sim \omega_\alpha$ corresponds to $M_{\alpha\alpha}(\tau) \sim \delta(\omega_\alpha)$ and can be removed from the equation of motion.

Junichi Iwasaki, Wataru Koshikai, and Naoto Nagaosa, Colossal Spin Transfer Torque Effect on Skyrmion along the Edge, Nano Lett. 14, 4432–4437 (2014).

D. J. Clarke, O. A. Tretiakov, G.-W. Chern, Ya. B. Bazaliy, and O. Tchernyshyov, Dynamics of a vortex domain wall in a magnetic nanostric: Application of the collective-coordinate approach, Phys. Rev. B 78, 134412 (2008).
[66] E. Saitoh, H. Miyajima, T. Yamaoka, G. Tatara, Current-induced resonance and mass determination of a single magnetic domain wall, Nature 432, 203-2006 (2004).

[67] Gen Tatara, Hiroshi Kohno, Junya Shibata, Microscopic approach to current-driven domain wall dynamics, Phys. Rep. 468, 213–301 (2008).

[68] G. M. Wysin, Magnetic vortex mass in two-dimensional easy-plane magnets, Phys. Rev. B 54, 15156–15162 (1996).

[69] K. Y. Guslienko, G. R. Aranda, J. Gonzalez, Topological gauge field in nanomagnets: Spin-wave excitations over a slowly moving magnetization background, Phys. Rev. B 81, 014414 (2010).

[70] K. Y. Guslienko, G. N. Kakazei, J. Ding, X. M. Liu, A. O. Adeyeye, Giant moving vortex mass in thick magnetic nanodots, Sci. Rep. 5, 13881 (2015).

[71] J. H. P. Colpa, Diagonalization of the quadratic boson hamiltonian, Physica A 93, 1978, 327–353 (1978).

[72] Ryan Levy, J. P. F. Le Blanc, EmanuelGull, Implementation of the maximum entropy method for analytic continuation, Computer Physics Communications 215, 149–155 (2017).