On quasi-Hopf smash products and twisted tensor products of quasialgebras

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Abstract

We analyze some relations between quasi-Hopf smash products and certain twisted tensor products of quasialgebras. Along the way we obtain also some results of independent interest, such as a duality theorem for finite dimensional quasi-Hopf algebras and a universal property for generalized diagonal crossed products.

Introduction

If \((A, \mu_A, u_A)\) and \((B, \mu_B, u_B)\) are algebras in a monoidal category \(C\) and \(R : B \otimes A \rightarrow A \otimes B\) is a morphism in \(C\) satisfying a certain list of axioms, then \(A \otimes B\) becomes also an algebra in \(C\), with a multiplication defined in terms of \(\mu_A, \mu_B\) and \(R\). This construction appeared in a number of contexts and under different names. Following [15] we call such an \(R\) a twisting map and the algebra structure on \(A \otimes B\) afforded by it the twisted tensor product of \(A\) and \(B\) and denote it by \(A \otimes_R B\) (if \(A\) and \(B\) are ordinary associative algebras and \(R\) is the usual flip, then \(A \otimes_R B\) coincides with the usual tensor product of algebras). Analogues of twisting maps for monads and operads are known as distributive laws, see for instance [4], [25], [30]. The twisted tensor product of associative algebras can be regarded as a representative for the cartesian product of noncommutative spaces, better suited than the ordinary tensor product, see [15], [18], [20] for a detailed discussion and references. Prominent examples of twisted tensor products of algebras are the so-called braided tensor products, which are part of the "braided geometry" developed by Majid in the early 1990's; namely, a braiding on a monoidal category provides a twisting map between any two algebras in the category. Hopf algebra theory provides also plenty of examples of twisted tensor products, in particular the usual smash product \(A \# H\) is the twisted tensor product \(A \otimes_R H\), with \(R : H \otimes A \rightarrow A \otimes H, R(h \otimes a) = h_1 \cdot a \otimes h_2\).

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The term *quasialgebra* was introduced in [2], to designate an algebra in a monoidal category. We restrict here the term to algebras in monoidal categories associated to a quasi-bialgebra or quasi-Hopf algebra $H$, such as $H\mathcal{M}$, $H\mathcal{M}_H$, $H\mathcal{YD}$. Since these categories have nontrivial associators, such quasialgebras are usually *nonassociative* as algebras, but their lack of associativity is (well) controlled by the associators of the category. Important examples of quasialgebras are the octonions and the other Cayley algebras, see [2]. Another class of examples of interest for us here is obtained as follows: if $H$ is a quasi-Hopf algebra, $B$ an associative algebra and $v : H \to B$ an algebra map, then on $B$ one can introduce a structure of algebra in $H\mathcal{M}$, denoted by $B^v$, see [12]. For the particular case $B = H$, $v = id_H$, the quasialgebra $B^v$ is denoted by $H_0$, and by [11] it follows that it is actually an algebra in the Yetter-Drinfeld category $H\mathcal{YD}$.

A feature of quasialgebras in this sense, not available for algebras in arbitrary monoidal categories (unless very special conditions hold, cf. [29]) is the fact that there exist several crossed products associated to them, which are usually associative algebras. For instance, if $H$ is a quasi-bialgebra and $A$ is an algebra in $H\mathcal{M}$, one can consider the smash product $A\# H$ introduced in [12], an associative algebra generalizing the classical Hopf smash product and sharing many of its properties. Other examples are diagonal crossed products, L-R-smash products, two-sided crossed products, see [13], [17], [28].

These quasialgebras are also part of an emerging *nonassociative geometry* (cf. [1], [5], [24]), regarded as a further extension of noncommutative geometry, with the “coordinate algebra” allowed to be nonassociative. By analogy with the associative situation, the twisted tensor product of quasialgebras might be regarded as a representative for the cartesian product of ”nonassociative spaces”.

The aim of this paper is to study certain classes of twisted tensor products of quasialgebras and their relations with quasi-Hopf smash products. If $H$ is a quasi-Hopf algebra, $A$ a left $H$-module algebra and $C$ an algebra in the Yetter-Drinfeld category $H\mathcal{YD}$, we consider an object denoted by $A \odot C$, which is a certain twisted tensor product in the category $H\mathcal{M}$ and is defined in such a way that if $A$ is also an algebra in $H\mathcal{YD}$ then $A \odot C$ coincides with the braided tensor product $A \otimes C$ in $H\mathcal{YD}$. We are mainly interested in the case $C = H_0$, so we have the left $H$-module algebra $A \odot H_0$ and we want to see how it is related to the smash product $A \# H$. For Hopf algebras this question has a trivial answer: $A \# H$ and $A \odot H_0$ coincide. This cannot be the case for proper quasi-Hopf algebras, because $A \# H$ is associative while $A \odot H_0$ is not. The answer is the following: $A \odot H_0$ is isomorphic as left $H$-module algebras to $(A \# H)^j$, where $j : H \to A \# H$, $j(h) = 1 \# h$. If $A$ is moreover an algebra in $H\mathcal{YD}$, we prove that $(A \# H)^j$ becomes also an algebra in $H\mathcal{YD}$ in a natural way and we have $A \odot H_0 \simeq (A \# H)^j$ as algebras in $H\mathcal{YD}$.

More can be said if $A$ is $H_0$ itself. For this, we prove a result of independent interest: an algebra isomorphism $B^v \# H \simeq B \otimes H$, the quasi-Hopf analogue of a well-known result from Hopf algebra theory (see [26]). As a consequence, we obtain $\tilde{H}_0 \odot H_0 \simeq (H \otimes H)^\Delta$ as algebras in $H\mathcal{YD}$. As another consequence, also of independent interest, we obtain a duality theorem, stating that if $H$ is a finite dimensional quasi-Hopf algebra then the two-sided crossed product $H \bowtie H^* \bowtie H$ introduced by Hausser and Nill in [17] is isomorphic to $\text{End}(H) \otimes H$ as algebras.

In the last two sections we analyze some iterated products (in particular in the sense of [18]) involving quasialgebras, and some universal properties. For instance, we find a relation between the universal property of $A \# H$ and the universal property of $A \odot H_0$ regarded as a twisted tensor product of quasialgebras, and we find a new kind of universal property of the smash product $A \# H$, obtained as an immediate consequence of a very general universal property for generalized diagonal crossed products (as in [13]), which we obtain inspired by the universal property of the diagonal crossed product $H^* \bowtie \mathbb{A}$, formulated by Hausser and Nill in [17].
1 Preliminaries

In this section we recall some definitions and results and we fix some notation that will be used throughout the paper.

We work over a base field \( k \). All algebras, linear spaces etc. are over \( k \); unadorned \( \otimes \) means \( \otimes_k \). Following Drinfeld [16], a quasi-bialgebra is a fourtuple \((H, \Delta, \varepsilon, \Phi)\), where \( H \) is an associative algebra with unit 1, \( \Phi \) is an invertible element in \( H \otimes H \otimes H \), and \( \Delta : H \rightarrow H \otimes H \) and \( \varepsilon : H \rightarrow k \) are algebra homomorphisms satisfying the identities (for all \( h \in H \)):

\[
(id \otimes \Delta)(\Delta(h)) = \Phi(\Delta \otimes id)(\Delta(h))\Phi^{-1},
\]

\[
(id \otimes \varepsilon)(\Delta(h)) = h \otimes 1, \quad (\varepsilon \otimes id)(\Delta(h)) = 1 \otimes h,
\]

\[
(1 \otimes \Phi)(id \otimes \Delta \otimes id)\Phi(1) = (id \otimes id \otimes \Delta)\Phi(\Delta \otimes id \otimes id)\Phi,
\]

\[
(\varepsilon \otimes id \otimes id)\Phi = (id \otimes \varepsilon \otimes id)\Phi = (id \otimes id \otimes \varepsilon)\Phi = 1 \otimes 1 \otimes 1.
\]

The map \( \Delta \) is called the coproduct or the comultiplication, \( \varepsilon \) the counit and \( \Phi \) the reassociator.

We use the version of Sweedler's sigma notation: \( \Delta(h) = h_{(1)} \otimes h_{(2)} \otimes h_{(3)} \) and \( (id \otimes \Delta)(\Delta(h)) = h_{(1)} \otimes h_{(2,1)} \otimes h_{(2,2)} \). We denote the tensor components of \( \Phi \) by capital letters and those of \( \Phi^{-1} \) by small letters:

\[
\Phi = X^1 \otimes X^2 \otimes X^3 = T^1 \otimes T^2 \otimes T^3 = Y^1 \otimes Y^2 \otimes Y^3 = \ldots
\]

\[
\Phi^{-1} = x^1 \otimes x^2 \otimes x^3 = t^1 \otimes t^2 \otimes t^3 = y^1 \otimes y^2 \otimes y^3 = \ldots
\]

The quasi-bialgebra \( H \) is called a quasi-Hopf algebra if there exists an anti-automorphism \( S \) of the algebra \( H \) and elements \( \alpha, \beta \in H \) such that, for all \( h \in H \), we have:

\[
S(h_1)\alpha h_2 = \varepsilon(h)\alpha \quad \text{and} \quad h_1 \beta S(h_2) = \varepsilon(h)\beta,
\]

\[
X^1 \beta S(X^2)\alpha X^3 = 1 \quad \text{and} \quad S(x^1)\alpha x^2 \beta S(x^3) = 1.
\]

The axioms for a quasi-Hopf algebra imply that \( \varepsilon(\alpha)\varepsilon(\beta) = 1 \), so, by rescaling \( \alpha \) and \( \beta \), we may assume without loss of generality that \( \varepsilon(\alpha) = \varepsilon(\beta) = 1 \) and \( \varepsilon \circ S = \varepsilon \).

We recall that the definition of a quasi-bialgebra or quasi-Hopf algebra is "twist covariant" in the following sense. An invertible element \( F \in H \otimes H \) is called a gauge transformation or twist if \( (\varepsilon \otimes id)(F) = (id \otimes \varepsilon)(F) = 1 \). If \( H \) is a quasi-bialgebra or a quasi-Hopf algebra and \( F = F^1 \otimes F^2 \in H \otimes H \) is a gauge transformation with inverse \( F^{-1} = G^1 \otimes G^2 \), then we can define a new quasi-bialgebra (respectively quasi-Hopf algebra) \( H_F \) by keeping the multiplication, unit, counit (and antipode in the case of a quasi-Hopf algebra) of \( H \) and replacing the reassociator, comultiplication and the elements \( \alpha \) and \( \beta \) by

\[
\Phi_F = (1 \otimes F)(id \otimes \Delta)(F)\Phi(\Delta \otimes id)(F^{-1})(F^{-1} \otimes 1),
\]

\[
\Delta_F(h) = F\Delta(h)F^{-1}, \quad \alpha_F = S(G^1)\alpha G^2, \quad \beta_F = F^1 \beta S(F^2).
\]

The antipode of a Hopf algebra is an anti-coalgebra morphism. For a quasi-Hopf algebra \( H \), we have the following: there exists a gauge transformation \( f \in H \otimes H \) such that

\[
f\Delta(S(h))f^{-1} = (S \otimes S)(\Delta^{cop}(h)), \quad \text{for all} \ h \in H.
\]

The element \( f \) can be computed explicitly. First set

\[
A^1 \otimes A^2 \otimes A^3 \otimes A^4 = (\Phi \otimes 1)(\Delta \otimes id \otimes id)(\Phi^{-1}),
\]
\[B^1 \otimes B^2 \otimes B^3 \otimes B^4 = (\Delta \otimes \text{id} \otimes \text{id})(\Phi^{-1} \otimes 1),\] (1.11)

and then define \(\gamma, \delta \in H \otimes H\) by

\[\gamma = S(A^2)\alpha A^3 \otimes S(A^1)\alpha A^4 \quad \text{and} \quad \delta = B^1 \beta S(B^4) \otimes B^2 \beta S(B^3).\] (1.12)

Then \(f\) and \(f^{-1}\) are given by the formulae

\[
\begin{align*}
f &= (S \otimes S)(\Delta^{\text{cop}}(x^1))\gamma \Delta(x^2 \beta S(x^3)), \\
f^{-1} &= \Delta(S(x^1)\alpha x^2)\delta(S \otimes S)(\Delta^{\text{cop}}(x^3)).
\end{align*}
\] (1.13) (1.14)

Moreover, \(f\) satisfies the following relations:

\[
f\Delta(\alpha) = \gamma, \quad \Delta(\beta)f^{-1} = \delta,
\] (1.15)

and the corresponding twisted reassociator (cf. (1.7)) is given by

\[\Phi_f = S(X^3) \otimes S(X^2) \otimes S(X^1).\] (1.16)

We record also the following relation from [10], for \(f^{-1} = g^1 \otimes g^2\) given by (1.14):

\[S^{-1}(\alpha g^2)g^1 = \beta.\] (1.17)

Suppose that \((H, \Delta, \varepsilon, \Phi)\) is a quasi-bialgebra. If \(U, V, W\) are left (right) \(H\)-modules, define \(a_{U,V,W}, a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)\) by \(a_{U,V,W}((u \otimes v) \otimes w) = \Phi \cdot (u \otimes (v \otimes w))\) and \(a_{U,V,W}((u \otimes v) \otimes w) = (u \otimes (v \otimes w)) \cdot \Phi^{-1}\). The category \(H\mathcal{M}(\mathcal{M}_H)\) of left (right) \(H\)-modules becomes a monoidal category (see [19], [21] for terminology) with tensor product \(\otimes\) given via \(\Delta\), associativity constraints \(a_{U,V,W}\) (\(a_{U,V,W}\)), unit \(k\) as a trivial \(H\)-module and the usual left and right unit constraints.

Let again \(H\) be a quasi-bialgebra. We say that a \(k\)-vector space \(A\) is a left \(H\)-module algebra if it is an algebra in the monoidal category \(H\mathcal{M}\), that is \(A\) has a multiplication and a usual unit \(1_A\) satisfying the following conditions:

\[
(aa')a'' = (X^1 \cdot a)[(X^2 \cdot a')(X^3 \cdot a'')],
\] (1.18)

\[
h \cdot (aa') = (h_1 \cdot a)(h_2 \cdot a'), \quad h \cdot 1_A = \varepsilon(h)1_A,
\] (1.19)

for all \(a, a', a'' \in A\) and \(h \in H\), where \(h \otimes a \to h \cdot a\) is the left \(H\)-module structure of \(A\). Following [12] we define the smash product \(A \# H\) as follows: as vector space \(A \# H\) is \(A \otimes H\) (elements \(a \otimes h\) will be written \(a \# h\)) with multiplication given by

\[a \# h)(a' \# h') = (x^1 \cdot a)(x^2 h_1 \cdot a') \# x^3 h_2 h'.\] (1.20)

The smash product \(A \# H\) is an associative algebra with unit \(1_A\# 1_H\).

If \(H\) is a quasi-Hopf algebra, \(B\) an associative algebra and \(\nu : H \to B\) an algebra map, then, following [12], we can introduce on the vector space \(B\) a left \(H\)-module algebra structure, denoted by \(B^\nu\) in what follows, for which the multiplication, unit and left \(H\)-action are:

\[b \star b' = v(X^1)bV(S(x^1)\alpha x^2 X_1^3)b'v(S(x^3 X_2^3)), \quad \forall b, b' \in B,\] (1.21)

\[1_{B^\nu} = v(\beta), \quad h \triangleright_v b = v(h_1)bv(S(h_2)), \quad \forall h \in H, b \in B.\] (1.22)
If $H$ is a quasi-Hopf algebra and $A$ is a left $H$-module algebra, define the following maps:

\[ j : H \to A#H, \quad j(h) = 1 \cdot h, \quad \forall h \in H, \quad (1.23) \]

\[ i_0 : A \to A#H, \quad i_0(a) = x^1 \cdot a#x^2 \beta S(x^3), \quad \forall a \in A. \quad (1.24) \]

Then, by [12], $j$ is an algebra morphism and $i_0$ is a morphism of left $H$-module algebras from $A$ to $(A#H)^j$. Moreover, the following universal property of the smash product $A#H$ holds (see [12, Proposition 2.9]): if $B$ is an associative algebra, $v : H \to B$ is an algebra map and $u : A \to B^v$ is a morphism of left $H$-module algebras, then there exists a unique algebra map $u#v : A#H \to B$ such that $(u#v) \circ i_0 = u$ and $(u#v) \circ j = v$; this map may be described explicitly as follows:

\[ (u#v)(a#h) = v(X^1)u(a)v(S(X^2)\alpha X^3 h), \quad \forall a \in A, \ h \in H. \quad (1.25) \]

We record the following relation from [12], which holds in $(A#H)^j$ for all $a \in A, \ h \in H$:

\[ i_0(a) \ast j(h) = x^1 \cdot a#x^2 hS(x^3). \quad (1.26) \]

We recall now the invariance under twisting of the smash product (see for instance [10, 12, 22]). Let $H$ be a quasi-bialgebra, $F \in \mathcal{H} \otimes H$ a gauge transformation and $A$ a left $H$-module algebra. Then we can define a new multiplication on $A$, by $a \ast a' = (G^1 \cdot a)(G^2 \cdot a')$, for all $a, a' \in A$, where $F^{-1} = G^1 \otimes G^2$. If we denote by $A_{F^{-1}}$ the resulting structure, then $A_{F^{-1}}$ becomes a left $H_{F}$-module algebra, with the same unit and $H$-action as for $A$, and moreover the map $\pi : A#H \to A_{F^{-1}}#H_{F}$. $\pi(a#h) = F^1 \cdot a#F^2 h$, is an algebra isomorphism.

For further use we need also the notion of right $H$-module algebra. Let $H$ be a quasi-bialgebra. We say that a $k$-linear space $B$ is a right $H$-module algebra if $B$ is an algebra in the monoidal category $\mathcal{M}_H$, i.e. $B$ has a multiplication and a usual unit $1_B$ satisfying the following conditions:

\[ (bb')b'' = (b \cdot x^1)(b' \cdot x^2)(b'' \cdot x^3), \quad (1.27) \]

\[ (bb') \cdot h = (b \cdot h_1)(b' \cdot h_2), \quad 1_B \cdot h = \varepsilon(h)1_B, \quad (1.28) \]

for all $b, b', b'' \in B$ and $h \in H$, where $b \otimes h \rightarrow b \cdot h$ is the right $H$-module structure of $B$.

Recall from [17] the notion of comodule algebra over a quasi-bialgebra.

**Definition 1.1** Let $H$ be a quasi-bialgebra. A unital associative algebra $\mathfrak{A}$ is called a right $H$-comodule algebra if there exist an algebra morphism $\rho : \mathfrak{A} \to \mathfrak{A} \otimes H$ and an invertible element $\Phi_\rho \in \mathfrak{A} \otimes H \otimes H$ such that:

\[ \Phi_\rho(\rho \otimes id)(\rho(a)) = (id \otimes \Delta)(\rho(a))\Phi_\rho, \quad \forall a \in \mathfrak{A}, \quad (1.29) \]

\[ (1_\mathfrak{A} \otimes \Phi)(id \otimes \Delta \otimes id)(\Phi_\rho)(\Phi_\rho \otimes 1_H) = (id \otimes id \otimes \Delta)(\Phi_\rho)(\rho \otimes id \otimes id)(\Phi_\rho), \quad (1.30) \]

\[ (id \otimes \varepsilon) \circ \rho = id, \quad (1.31) \]

\[ (id \otimes \varepsilon \otimes id)(\Phi_\rho) = (id \otimes id \otimes \varepsilon)(\Phi_\rho) = 1_\mathfrak{A} \otimes 1_H. \quad (1.32) \]

Similarly, a unital associative algebra $\mathfrak{B}$ is called a left $H$-comodule algebra if there exist an algebra morphism $\lambda : \mathfrak{B} \to H \otimes \mathfrak{B}$ and an invertible element $\Phi_\lambda \in H \otimes H \otimes \mathfrak{B}$ such that:

\[ (id \otimes \lambda)(\lambda(b))\Phi_\lambda = \Phi_\lambda(\Delta \otimes id)(\lambda(b)), \quad \forall b \in \mathfrak{B}, \quad (1.33) \]

\[ (1_H \otimes \Phi_\lambda)(id \otimes \Delta \otimes id)(\Phi_\lambda)(\Phi \otimes 1_\mathfrak{B}) = (id \otimes id \otimes \lambda)(\Phi_\lambda)(\Delta \otimes id \otimes id)(\Phi_\lambda), \quad (1.34) \]

\[ (\varepsilon \otimes id) \circ \lambda = id, \quad (1.35) \]

\[ (id \otimes \varepsilon \otimes id)(\Phi_\lambda) = (\varepsilon \otimes id \otimes id)(\Phi_\lambda) = 1_H \otimes 1_\mathfrak{B}. \quad (1.36) \]
When $H$ is a quasi-bialgebra, particular examples of left and right $H$-comodule algebras are given by $\mathfrak{A} = \mathfrak{B} = H$ and $\rho = \lambda = \Delta$, $\Phi_\rho = \Phi_\lambda = \Phi$. Another basic example of a comodule algebra is provided by the smash product. Namely, if $A$ is a left $H$-module algebra, then $(A\# H, \rho, \Phi_\rho)$ is a right $H$-comodule algebra with structures (cf. [12]):

$$
\rho : A\# H \to (A\# H) \otimes H, \quad \rho(a\# h) = (x^1 \cdot a\# x^2 h_1) \otimes x^3 h_2,
$$

$$
\Phi_\rho = (1\# X^1) \otimes X^2 \otimes X^3 \in (A\# H) \otimes H \otimes H.
$$

For a right $H$-comodule algebra $(\mathfrak{A}, \rho, \Phi_\rho)$ we will denote, for any $a \in \mathfrak{A}$, by $\rho(a) = a_{<0_>} \otimes a_{<1_>}$, $(\rho \otimes \text{id})(\rho(a)) = a_{<0,0>} \otimes a_{<0,1_>} \otimes a_{<1_>}$ etc. Similarly, for a left $H$-comodule algebra $(\mathfrak{B}, \lambda, \Phi_\lambda)$, if $b \in \mathfrak{B}$ then we denote $\lambda(b) = b_{[-1]} \otimes b_{[0]}$, $(\text{id} \otimes \lambda)(\lambda(b)) = b_{[-1]} \otimes b_{[0,-1]} \otimes b_{[0,0]}$ etc. In analogy with the notation for the reassociator $\Phi$ of $H$, we will write

$$
\Phi_\rho = \tilde{X}_\rho \otimes \tilde{X}_\rho^2 \otimes \tilde{X}_\rho^3 = \tilde{Y}_\rho \otimes \tilde{Y}_\rho^2 \otimes \tilde{Y}_\rho^3 = \cdots
$$

$$
\Phi_\rho^{-1} = \tilde{x}_\rho \otimes \tilde{x}_\rho^2 \otimes \tilde{x}_\rho^3 = \tilde{y}_\rho \otimes \tilde{y}_\rho^2 \otimes \tilde{y}_\rho^3 = \cdots
$$

and similarly for the element $\Phi_\lambda$ of a left $H$-comodule algebra $\mathfrak{B}$.

If $\mathfrak{A}$ is a right $H$-comodule algebra then we define the elements $\tilde{p}_\rho, \tilde{q}_\rho \in \mathfrak{A} \otimes H$ as follows:

$$
\tilde{p}_\rho = \tilde{p}_\rho^1 \otimes \tilde{p}_\rho^2 = \tilde{x}_\rho \otimes \tilde{x}_\rho^2 \beta S(\tilde{x}_\rho^3), \quad \tilde{q}_\rho = \tilde{q}_\rho^1 \otimes \tilde{q}_\rho^2 = \tilde{X}_\rho \otimes S^{-1}(\alpha \tilde{X}_\rho^3) \tilde{X}_\rho^2.
$$

By [17] Lemma 9.1, we have the following relations, for all $a \in \mathfrak{A}$:

$$
\rho(a_{<0_>})(\tilde{p}_\rho)[1_\mathfrak{A} \otimes S(a_{<1_>})] = \tilde{p}_\rho[a \otimes 1_H],
$$

$$
[1_\mathfrak{A} \otimes S^{-1}(a_{<1_>})](\tilde{q}_\rho \rho(a_{<0_>})) = [a \otimes 1_H](\tilde{q}_\rho),
$$

$$
\rho(\tilde{q}_\rho^1) \rho([1_\mathfrak{A} \otimes S(\tilde{q}_\rho^2)]) = [1_\mathfrak{A} \otimes 1_H],
$$

$$
[1_\mathfrak{A} \otimes S^{-1}(\tilde{p}_\rho^2)](\tilde{q}_\rho \rho(\tilde{p}_\rho^1)) = [1_\mathfrak{A} \otimes 1_H],
$$

$$
\Phi_\rho(\rho \otimes \text{id}_H)(\tilde{p}_\rho \otimes \rho \otimes \text{id}_H) = (\text{id} \otimes \Delta)(\rho(\tilde{x}_\rho))(1_\mathfrak{A} \otimes g^1 S(\tilde{x}_\rho^3) \otimes g^2 S(\tilde{x}_\rho^2)),
$$

$$
(\tilde{q}_\rho \otimes 1_H)(\rho \otimes \text{id}_H)(\tilde{q}_\rho \rho(\tilde{x}_\rho))(\Phi_\rho)^{-1} = [1_\mathfrak{A} \otimes S^{-1}(f^1 \tilde{X}_\rho^3) \otimes S^{-1}(f^1 \tilde{X}_\rho^2)](\text{id} \otimes \Delta)(\tilde{q}_\rho \rho(\tilde{x}_\rho))(\Phi_\rho)^{-1},
$$

where $f^{-1} = g^1 \otimes g^2$ is given by (1.11). If $\mathfrak{A}$ is $H$ itself, the elements $\tilde{p}_\rho, \tilde{q}_\rho \in H \otimes H$ are denoted by $p_R$ and $q_R$ and are given by the formulae

$$
p_R = p^1 \otimes p^2 = x^1 \otimes x^2 \beta S(x^3), \quad q_R = q^1 \otimes q^2 = X^1 \otimes S^{-1}(\alpha X^3) X^2.
$$

The next definition appeared in [17] under the name ”quasi-commuting pair of $H$-coactions”.

**Definition 1.2** Let $H$ be a quasi-bialgebra. By an $H$-bicomodule algebra $\mathfrak{A}$ we mean a quintuple $(\lambda, \rho, \Phi_\lambda, \Phi_\rho, \Phi_{\lambda,\rho})$, where $\lambda$ and $\rho$ are left and right $H$-coactions on $\mathfrak{A}$, respectively, and where $\Phi_\lambda \in H \otimes H \otimes \mathfrak{A}$, $\Phi_\rho \in \mathfrak{A} \otimes H \otimes H$ and $\Phi_{\lambda,\rho} \in H \otimes \mathfrak{A} \otimes H$ are invertible elements, such that $(\mathfrak{A}, \lambda, \Phi_\lambda)$ is a left $H$-comodule algebra, $(\mathfrak{A}, \rho, \Phi_\rho)$ is a right $H$-comodule algebra and the following compatibility relations hold:

$$
\Phi_{\lambda,\rho}(\lambda \otimes \text{id})(\rho(u)) = (\text{id} \otimes \rho)(\lambda(u))\Phi_{\lambda,\rho}, \quad \forall \ u \in \mathfrak{A},
$$

$$
(1_H \otimes \Phi_{\lambda,\rho})(\text{id} \otimes \lambda \otimes \text{id})(\Phi_{\lambda,\rho} \otimes 1_H) = (\text{id} \otimes \text{id} \otimes \rho)(\Phi_\lambda)(\Delta \otimes \text{id} \otimes \text{id})(\Phi_{\lambda,\rho}),
$$

$$
(1_H \otimes \Phi_{\rho})(\text{id} \otimes \rho \otimes \text{id})(\Phi_{\lambda,\rho} \otimes 1_H) = (\text{id} \otimes \text{id} \otimes \Delta)(\Phi_{\lambda,\rho})(\lambda \otimes \text{id} \otimes \text{id})(\Phi_\rho).
$$
As pointed out in [17], if \( A \) is a bicomodule algebra then, in addition, we have that
\[
(id_H \otimes id_A \otimes \varepsilon)(\Phi_{\lambda,\rho}) = 1_H \otimes 1_A, \quad (\varepsilon \otimes id_A \otimes id_H)(\Phi_{\lambda,\rho}) = 1_A \otimes 1_H. \tag{1.48}
\]

An example of a bicomodule algebra is \( A = H; \lambda = \rho = \Delta \) and \( \Phi_\lambda = \Phi_\rho = \Phi_{\lambda,\rho} = \Phi \). For the left and right comodule algebra structures of \( A \) we will keep the same notation as above. We also denote \( \Phi_{\lambda,\rho} = \Theta^1 \otimes \Theta^2 \otimes \Theta^3 = \mathfrak{G} \otimes \mathfrak{G}^2 \otimes \mathfrak{G}^3 \) and \( \Phi_{\lambda,\rho}^{-1} = \theta^1 \otimes \theta^2 \otimes \theta^3 = \mathfrak{G}^{-1} \otimes \mathfrak{G}^2 \otimes \mathfrak{G}^3 \).

Let us denote by \( {}_H \mathcal{M}_H \) the category of \( H \)-bimodules; it is also a monoidal category, the associativity constraints being given by \( \Phi_{U,V,W}(\Phi_{V,W}((u \otimes v) \otimes w)) = \Phi \cdot (u \otimes (v \otimes w)) \cdot \Phi^{-1} \), for \( U,V,W \in {}_H \mathcal{M}_H \) and \( u \in U, v \in V, w \in W \). Therefore, we can define algebras in the category of \( H \)-bimodules, which will be called \( H \)-module algebras.

Namely, a \( k \)-vector space \( A \) is an \( H \)-bimodule algebra if \( A \) is an \( H \)-module (denote the actions by \( h \cdot \varphi \) and \( \varphi \cdot h \), for \( h \in H \) and \( \varphi \in A \)) with a multiplication and a usual unit \( 1_A \) such that:
\[
(\varphi \psi)(x) = (X^1 \cdot \varphi \cdot x^2)(X^2 \cdot x^3), \quad \forall \varphi, x \in A, \tag{1.49}
\]
\[
h \cdot (\varphi \psi) = (h_1 \cdot \varphi)(h_2 \cdot \psi), \quad h \cdot (\varphi \cdot h_1)(\psi \cdot h_2), \quad \forall h, \varphi, \psi \in A, h \in H, \tag{1.50}
\]
\[
h \cdot 1_A = \varepsilon(h)1_A, \quad 1_A \cdot h = \varepsilon(h)1_A, \quad \forall h \in H. \tag{1.51}
\]

If \( H \) is a quasi-bialgebra then \( H^* \), the linear dual of \( H \), is an \( H \)-bimodule with \( H \)-actions \( h \mapsto h', h' = \varphi(h'h) \) and \( h \mapsto h', h' = \varphi(h'h) \), for all \( \varphi \in H^* \), \( h, h' \in H \). The convolution \( \varphi \psi, h := \sum \varphi(h_1) \psi(h_2) \), for \( \psi, \varphi \in H^* \) and \( h \in H \), is a multiplication on \( H^* \), and with this multiplication \( H^* \) becomes an \( H \)-bimodule algebra. Note that a left (right) \( H \)-module algebra becomes also an \( H \)-bimodule algebra, with trivial right (left) \( H \)-action. Unlike the case of ordinary bialgebras, an \( H \)-bimodule algebra is not necessarily a left or right \( H \)-module algebra, for instance there is no visible left or right \( H \)-module algebra structure on \( H^* \).

We recall from [23] the definition of (left) Yetter-Drinfeld modules over a quasi-bialgebra \( H \).

**Definition 1.3** A \( k \)-linear space \( M \) is called a left Yetter-Drinfeld module over \( H \) if \( M \) is a left \( H \)-module (with action denoted by \( h \odot m \mapsto h \cdot m \)) and \( \Delta \) coacts on \( M \) to the left (the coaction is denoted by \( \lambda_M : M \to H \otimes M, \lambda_M(m) = m_{(-1)} \otimes m_{(0)} \)) such that
\[
X^1 m_{(-1)} \otimes (X^2 \cdot m_{(0)})_{(-1)} X^3 \otimes (X^2 \cdot m_{(0)})_{(0)}
= X^1 (Y^1 \cdot m)_{(-1)} Y^2 \otimes X^2 (Y^1 \cdot m)_{(-1)} Y^3 \otimes X^3 \cdot (Y^1 \cdot m)_{(0)}, \tag{1.52}
\]
\[
\varepsilon(m_{(-1)})m_{(0)} = m, \tag{1.53}
\]
\[
h_1 m_{(-1)} \otimes h_2 \cdot m_{(0)} = (h_1 \cdot m)_{(-1)} h_2 \otimes (h_1 \cdot m)_{(0)}, \tag{1.54}
\]
for all \( m \in M \) and \( h \in H \). The category \( {}^H_H \mathcal{YD} \) consists of such objects, the morphisms in the category being the \( H \)-linear maps intertwining the \( H \)-coactions.

The category \( {}^H_H \mathcal{YD} \) is (pre) braided monoidal; explicitly, if \( (M, \lambda_M) \) and \( (N, \lambda_N) \) are objects in \( {}^H_H \mathcal{YD} \), then \( (M \otimes N, \lambda_{M \otimes N}) \) is also object in \( {}^H_H \mathcal{YD} \), where \( M \otimes N \) is a left \( H \)-module with action \( h \cdot (m \otimes n) = h_1 \cdot m \otimes h_2 \cdot n \), and the coaction \( \lambda_{M \otimes N} \) is given by
\[
\lambda_{M \otimes N}(m \otimes n) = X^1 (x^1 Y^1 \cdot m)_{(-1)} x^2 (Y^2 \cdot n)_{(-1)} Y^3 \otimes X^2 \cdot (x^1 Y^1 \cdot m)_{(0)} \otimes X^3 x^3 \cdot (Y^2 \cdot n)_{(0)}.
\]

The associativity constraints are the same as in \( H \mathcal{M} \), and the (pre) braiding is given by
\[
c_{M,N} : M \otimes N \to N \otimes M, \quad c_{M,N}(m \otimes n) = m_{(-1)} \cdot n \otimes m_{(0)}.
\]
Since $H^\otimes YD$ is a monoidal category, we can speak about algebras in $H^\otimes YD$. Namely, if $A$ is an object in $H^\otimes YD$, then $A$ is an algebra in $H^\otimes YD$ if and only if $A$ is a left $H$-module algebra and $A$ is a left quasi-comodule algebra, that is its unit and multiplication intertwine the $H$-coaction $\lambda_A$, namely (for all $a, a' \in A$):

\[
\lambda_A(1_A) = 1_H \otimes 1_A, \\
\lambda_A(aa') = X^1(x^1Y^1 \cdot a)(-1)x^2(Y^2 \cdot a')(0)Y^3
\otimes [X^2 \cdot (x^1Y^1 \cdot a)[X^3 \cdot (Y^2 \cdot a')(0)].
\] (1.56)

If $H$ is a quasi-Hopf algebra, we can consider the algebra map $id_H : H \rightarrow H$, and then the left $H$-module algebra $H^\otimes$, which was denoted by $H^\otimes$ in [12]; its unit is $\beta$, the $H$-action is $h \triangleright h' = h_1h'S(h_2)$, and the multiplication is $h \triangleright h' = X^1hS(x^1X^2)\alpha x^2X^3h'S(x^3X^5)$. Moreover, it was proved in [11] that $H^\otimes$ becomes an algebra in $H^\otimes YD$, with coaction $\lambda_{H^\otimes} : H^\otimes \rightarrow H \otimes H^\otimes$, where $f^{-1} = q_1 \otimes q_2$ is given by (1.14) and $q_{R} = q_1 \otimes q_2$ is given by (1.44).

Let $H$ be a quasi-bialgebra, $A$ a left $H$-module algebra and $B$ a left $H$-module algebra. Denote by $A \bowtie B$ the $k$-vector space $A \otimes B$ with multiplication:

$$((a \bowtie b)(a' \bowtie b')) = (x_1 \cdot a)(x_2 \cdot b_{(-1)} \cdot a') \bowtie x_3 \cdot b_{(0)} \cdot b', \quad \forall a, a' \in A, \ b, b' \in B.$$ By $\bowtie$, $A \bowtie B$ is an associative algebra with unit $1_A \bowtie 1_B$, called the generalized smash product of $A$ and $B$ (it coincides with the usual smash product if $B = H$).

We recall the so-called L-R-smash product, introduced in [28] as a generalization of the cocommutative case from [7], [8]. Let $H$ be a quasi-bialgebra, $A$ an $H$-bimodule algebra and $A$ an $H$-bicomodule algebra. Define on $A \otimes A$ the product

$$((\varphi \triangleleft u)(\varphi' \triangleleft u')) = (\bar{x}_1 \cdot \varphi \cdot \theta^1 \cdot u_{(0)} \cdot \bar{x}_2 \cdot \varphi' \cdot \bar{x}_3 \cdot u'_{(0)} \cdot \bar{x}_4) \triangleleft \bar{x}_5 \cdot u_{(0)} \cdot \theta^2 \cdot u'_{(0)} \cdot \bar{x}_6$$ (1.57)

for $\varphi, \varphi' \in A$ and $u, u' \in A$, where we write $\varphi \triangleleft u$ in place of $\varphi \otimes u$ to distinguish the new algebraic structure. Then this product defines on $A \otimes A$ a structure of associative algebra with unit $1_A \otimes 1_A$, denoted by $A \bowtie A$ and called the L-R-smash product. Note that if the right $H$-action on $A$ is trivial then $A \bowtie A$ coincides with the generalized smash product $A \bowtie \bowtie A$.

If $H$ is a quasi-Hopf algebra, $A$ an $H$-bimodule algebra and $A$ an $H$-bicomodule algebra, there exists another associative algebra structure built on $A \otimes A$, which was introduced in [13] under the name generalized diagonal crossed product (for $A = H^\ast$ it gives the diagonal crossed product introduced by Hauser and Nill in [17]). It is constructed as follows. Define the element

$$\Omega = (\bar{x}_1 \cdot \varphi \cdot \theta^1 \cdot u_{(0)} \cdot \bar{x}_2 \cdot \varphi' \cdot \bar{x}_3 \cdot u'_{(0)} \cdot \bar{x}_4 \cdot u_{(0)} \cdot \theta^2 \cdot u'_{(0)} \cdot \bar{x}_5 \cdot u_{(0)} \cdot \theta^3) \triangleleft \bar{x}_6$$ (1.58)

in $H^\otimes \bowtie \bowtie H^\otimes$, where $\Phi^1 = X^1 \otimes X^2 \otimes X^3, \Phi^2 = \bar{x}_1 \otimes \bar{x}_2 \otimes \bar{x}_3, \Phi^3 = \bar{x}_1 \otimes \bar{x}_2 \otimes \bar{x}_3$. Define the multiplication on $A \otimes A$, by:

$$(\varphi \bowtie u)(\varphi' \bowtie u') = (\Omega^1 \cdot \varphi \cdot \Omega^2())(\Omega^2 u_{(0)} \cdot \varphi' \cdot S^{-1}(u_{(1)} \cdot \Omega^3) \bowtie \Omega^3 u_{(0)} \cdot u'_{(0)} \cdot \Omega^4)$$ (1.59)

for all $\varphi, \varphi' \in A$ and $u, u' \in A$, where we denoted $\Omega = \Omega^1 \bowtie \cdots \bowtie \Omega^5$. Then this multiplication defines an associative algebra structure with unit $1_A \bowtie 1_A$, which will be denoted by $A \bowtie A$. Note that, as in the case of the L-R-smash product, if the right $H$-action on $A$ is trivial then
\[ A \bowtie A \text{ coincides with the generalized smash product } A \triangleright A. \]

It was proved in [28] that actually \( A \cong A \) and \( A \bowtie A \) are isomorphic as algebras, a pair of algebra isomorphisms \( \nu : A \bowtie A \to A \cong A \) and \( \nu^{-1} : A \cong A \to A \bowtie A \) being given by

\[
\nu(\varphi \bowtie u) = \Theta^1 \cdot \varphi \cdot S^{-1}(\Theta^3 \delta^2)_{u<1>^1} u_{<1>} \otimes \tilde{\delta}^1_0 \Theta^2_{u<0>}, \tag{1.60}
\]

\[
\nu^{-1}(\varphi \bowtie u) = \Theta^1 \cdot \varphi \cdot S^{-1}(\Theta^3 \delta^2_{u<1>^1} \tilde{\delta}^1_{u<0>^1} \otimes \Theta^2_{u<0>}). \tag{1.61}
\]

for all \( \varphi \in A, u \in A \), where \( \tilde{\delta}^1_0 = \tilde{\delta}^1_0 \otimes \tilde{\delta}^1_0 \) and \( \tilde{\delta}^1_0 = \tilde{\delta}^1_0 \otimes \tilde{\delta}^1_0 \) are the elements given by (1.37).

We recall now several facts about twisted tensor products of algebras in monoidal categories (see for instance [15], [31]). Let \( \mathcal{C} \) be a monoidal category with associativity constraints \( a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W) \) and unit \( I \). If \( A \) is an algebra in \( \mathcal{C} \) we denote its multiplication by \( \mu_A : A \otimes A \to A \) and its unit by \( \eta_A : I \to A \). Let \( (A, \mu_A, \eta_A) \) and \( (B, \mu_B, \eta_B) \) be two algebras in \( \mathcal{C} \) and \( R : B \otimes A \to A \otimes B \) a morphism in \( \mathcal{C} \). We call \( R \) a twisting map between \( A \) and \( B \) if the following conditions hold:

\[
R \circ (id_B \otimes \eta_A) = \eta_A \otimes id_B, \quad R \circ (\eta_B \otimes id_A) = id_A \otimes \eta_B, \tag{1.62}
\]

\[
R \circ (\mu_B \otimes id_A) = (id_A \otimes \mu_B) \circ a_{A,B,B} \circ (R \otimes id_B) \circ a^{-1}_{B,A,B}, \tag{1.63}
\]

\[
R \circ (id_B \otimes \mu_A) = (\mu_B \otimes id_A) \circ a^{-1}_{A,B,A} \circ (id_A \otimes R) \circ a_{A,B,A}, \tag{1.64}
\]

Given such a twisting map, \( A \otimes B \) becomes an algebra in \( \mathcal{C} \), with multiplication

\[
\mu = (\mu_A \otimes \mu_B) \circ a^{-1}_{A,B,B \otimes B} \circ (id_A \otimes a_{A,B,B}) \circ (id_A \otimes R \otimes id_B) \circ (id_A \otimes a_{A,B,B} \otimes B), \tag{1.65}
\]

and unit \( \eta = \eta_A \otimes \eta_B \). This algebra structure on \( A \otimes B \) will be denoted by \( A \otimes_R B \) and will be called the twisted tensor product of \( A \) and \( B \) afforded by the twisting map \( R \). It has moreover the property that the morphisms \( i_A := id_A \otimes \eta_B : A \to A \otimes R \) and \( i_B := \eta_A \otimes id_B : B \to A \otimes R \) are morphisms of algebras in \( \mathcal{C} \).

If \( c_{U,V} : U \otimes V \to V \otimes U \) is a braiding on \( \mathcal{C} \), then for any two algebras \( A \) and \( B \) in \( \mathcal{C} \), the map \( R = c_{B,A} : B \otimes A \to A \otimes B \) is a twisting map, and in this case \( A \otimes_R B \) is denoted by \( B \) and is called the braided tensor product of \( A \) and \( B \) (relative to the braiding \( c \)).

If \( A \otimes_R B \) is a twisted tensor product of algebras in \( \mathcal{C} \), it has the following universal property (see [6], [14]): if \( (X, \mu_X, \eta_X) \) is an algebra in \( \mathcal{C} \) and \( u : A \to X \) and \( v : B \to X \) are morphisms of algebras in \( \mathcal{C} \) such that

\[
\mu_X \circ (u \otimes v) \circ R = \mu_X \circ (v \otimes u), \tag{1.66}
\]

then there exists a unique morphism \( w : A \otimes_R B \to X \) of algebras in \( \mathcal{C} \) such that \( w \circ i_A = u \) and \( w \circ i_B = v \). This morphism \( w \) is given explicitly by \( w = \mu_X \circ (u \otimes v) \).

2 Twisted tensor products of quasialgebras

**Proposition 2.1** Let \( H \) be a quasi-bialgebra, \( A \) an \( H \)-bimodule algebra and \( A \) an algebra in \( H \mathcal{YD} \). We regard \( A \) as an \( H \)-bimodule algebra with trivial right \( H \)-action. Define the map

\[
R : A \otimes A \to A \otimes A, \quad R(a \otimes \varphi) = a_{(-)} \cdot \varphi \otimes a_{(0)}, \quad \forall a \in A, \varphi \in A. \tag{2.1}
\]
Then $R$ is a twisting map between $A$ and $A$ in the monoidal category $\mathcal{H}\mathcal{M}_H$. We will denote by $A \otimes A$ the $H$-bimodule algebra $A \otimes_R A$, the twisted tensor product of $A$ and $A$ in $\mathcal{H}\mathcal{M}_H$.

**Proof.** The fact that $R$ is right $H$-linear is obvious, and the fact that it is left $H$-linear follows immediately from (1.54), so indeed $R$ is a morphism in $\mathcal{H}\mathcal{M}_H$. The relations (1.62) follow immediately from (1.53) and (1.55). The relation (1.63) reduces to (1.56), while the relation (1.64) reduces to (1.52). □

**Remark 2.2** The explicit structure of $A \otimes A$ is the following: the unit is $1_A \otimes 1_A$, the left $H$-action is $h \cdot (\varphi \otimes a) = h_1 \cdot \varphi \otimes h_2 \cdot a$, the right $H$-action is $(\varphi \otimes a) \cdot h = \varphi \cdot h \otimes a$, and the multiplication is (by (1.65)):

$$(\varphi \otimes a)(\varphi' \otimes a') = (y^1X^1 \cdot \varphi)(y^2Y^1(x^1X^2 \cdot a)(-1)x^2X^3 \cdot \varphi') \\ \otimes (y^3Y^2 \cdot (x^1X^2 \cdot a)(0))(y^4Y^3x^3X^3 \cdot a').$$

We will be mainly interested in the following particular case of Proposition 2.1.

**Corollary 2.3** Let $H$ be a quasi-Hopf algebra, $A$ a left $H$-module algebra and $H_0$ the algebra in $\mathcal{H}_H^YD$ as in the Preliminaries. Define the map $R : H_0 \otimes A \to A \otimes H_0$,

$$R(h \otimes a) = h_{(-1)} \cdot a \otimes h_{(0)} = X^1Y^1h_1g^1S(q^2Y_2^2)Y^3 \cdot a \otimes X^2Y^2h_2g^2S(X^3q^1Y_1^2). \quad (2.2)$$

Then $R$ is a twisting map between $A$ and $H_0$ in the monoidal category $\mathcal{H}\mathcal{M}_H$. We will denote by $A \otimes H_0$ the left $H$-module algebra $A \otimes_R H_0$, the twisted tensor product of $A$ and $H_0$ in $\mathcal{H}\mathcal{M}_H$. Its unit is $1_A \otimes \beta$, the $H$-action is $h \cdot (a \otimes h') = h_1 \cdot a \otimes h_2 \triangleright h'$, and the multiplication is

$$(a \otimes h)(a' \otimes h') = (y^1X^1 \cdot a)(y^2Y^1(x^1X^2 \triangleright h)(-1)x^2X^3 \cdot a') \\ \otimes (y^3Y^2 \triangleright (x^1X^2 \triangleright h)(0)) \ast (y^4Y^3x^3X^3 \triangleright h').$$

More generally, if $C$ is a left $H$-module algebra and $A$ is an algebra in $\mathcal{H}_H^YD$, then $C \otimes A$ is a left $H$-module algebra, which will be denoted by $C \circ A$.

Since the braiding in $\mathcal{H}_H^YD$ is given by $m \otimes n \mapsto m_{(-1)} \cdot n \otimes m_{(0)}$, we obtain:

**Corollary 2.4** Let $H$ be a quasi-Hopf algebra and $A$ an algebra in $\mathcal{H}_H^YD$. Then the left $H$-module algebra $A \circ H_0$ is actually an algebra in $\mathcal{H}_H^YD$ and it coincides with the braided tensor product $A \otimes H_0$ in $\mathcal{H}_H^YD$.

As another class of examples of twisted tensor products of quaasialgebras, we present the quasialgebra analogue of the so-called Clifford process, cf. [3], [32]. Let $A$ be a (not necessarily associative) algebra over $k$ with unit 1, let $q \in k$ be a nonzero fixed element and $\sigma : A \to A$ an involutive unital algebra automorphism. We denote by $C(k, q)$ the 2-dimensional associative algebra $k[v]/(v^2 = q)$. We define the linear map

$$R : C(k, q) \otimes A \to A \otimes C(k, q), \quad R(1 \otimes a) = a \otimes 1, \quad R(v \otimes a) = \sigma(a) \otimes v, \quad (2.3)$$

for all $a \in A$. The Clifford process associates to the pair $(A, \sigma)$ a (not necessarily associative) algebra structure built on $A \otimes C(k, q)$, with multiplication

$$(a \otimes 1 + b \otimes v)(c \otimes 1 + d \otimes v) = (ac + qb\sigma(d)) \otimes 1 + (ad + b\sigma(c)) \otimes v, \quad (2.4)$$
for all $a, b, c, d \in A$. This algebra structure is denoted by $\overline{A}$. Moreover, the linear map
$$\sigma : \overline{A} \to \overline{A}, \quad \sigma(a \otimes 1 + b \otimes v) = \sigma(a) \otimes 1 - \sigma(b) \otimes v, \quad \forall a, b \in A,$$
is an involutive unital algebra automorphism for $\overline{A}$. It is clear (cf. [3]) that if $A$ is associative then $\overline{A}$ is also associative, as in this case $R$ is a twisting map and $\overline{A}$ is the twisted tensor product $\overline{A} = A \otimes_R C(k, q)$.

Consider now the initial data for a Clifford process, and assume that $A$ is a left module algebra over a quasi-bialgebra $H$ and $\sigma$ is $H$-linear. We regard $C(k, q)$ as a left $H$-module algebra with trivial $H$-action. Then one can see that the map $R$ given by (2.3) is a twisting map between $A$ and $C(k, q)$ in the monoidal category $\mathcal{H}_A$, and so $\overline{A} = A \otimes_R C(k, q)$ is also a left $H$-module algebra. It is also clear that the extended involutive automorphism $\sigma$ is $H$-linear too.

Dually, a similar result holds if $A$ is a comodule algebra over a coquasi-bialgebra. In particular, if $G$ is a group and $A$ is a $G$-graded quasi-algebra as in [2] and $\sigma$ is a graded involutive automorphism of $A$, then $\overline{A}$ is also a $G$-graded quasi-algebra.

We emphasize an important conceptual difference between the associative and quasiassociative versions of the Clifford process. In the associative case (and assuming $\text{char}(k) \neq 2$), as noted in [2], the algebra $A$ becomes a $\mathbb{Z}_2$-graded algebra and the twisted tensor product $A \otimes_R C(k, q)$ is actually a braided tensor product, i.e. the twisting map $R$ is obtained from the canonical braiding $c(x \otimes y) = (-1)^{|x||y|} y \otimes x$ of the category of $\mathbb{Z}_2$-graded vector spaces. In the quasiassociative case, the twisting map $R$ given by (2.3) does not seem to come from any braiding.

### 3 An isomorphism $A \triangleright H_0 \simeq (A\#H)^j$ of algebras in $\mathcal{H}_A$

**Lemma 3.1** Let $H$ be a quasi-Hopf algebra and $A$ a left $H$-module algebra. Then for the maps $j$ and $i_0$ given by (1.23) and (1.24) the following relation holds, for all $h \in H$ and $a \in A$:
$$j(h) \ast i_0(a) = i_0(Y^1X_1^1h_1g^1S(T^2X_2^2)\alpha T^3X^3 \cdot a) \ast j(Y^2X_1^2h_2g^2S(Y^3T^1X_1^2)), \quad (3.1)$$

where $f^{-1} = g^1 \otimes g^2$ is given by (1.14) and $\ast$ is the multiplication in $(A\#H)^j$.

**Proof.** We compute:

$$j(h) \ast i_0(a) = (1\#h) \ast (x^1 \cdot a \# x^2 \beta S(x^3))$$
$$= (1\#X^1)(1\#h)(1\#S(y^1X^2)\alpha g^2X^3)$$
$$= X^1_1h_1S(y^1X^2)_1a1y^2X^3_1x^1 \cdot a$$
$$\# X^1_2h_2S(y^1X^2_2)\alpha g^2X^3_1x^2 \beta S(y^3X^3_2x^3)$$

$$1.1 \ 1.5 \ 1.9$$
$$= X^1_1h_1g^1S(y^1X^2_2)\gamma^1y^1x_1X^3_1 \cdot a$$
$$\# X^1_2h_2g^2S(y^1X^2_1)\gamma^2y^2x_2X^3_1\beta S(X^3_2x_1)S(y^3x^3)$$

$$1.5 \ 1.12$$
$$= X^1_1h_1g^1S(T^2t^3y^1X^2_2)\alpha T^3t^2y^2x_1X^3 \cdot a$$
$$\# X^1_2h_2g^2S(T^1t^3y^1X^2_1)\alpha g^2x^2 \beta S(y^3x^3)$$

$$1.3 \ 1.15$$
$$= X^1_1h_1g^1S(T^2t^3y^1X^2_2)\alpha T^3X^3 \cdot a \# X^1_2h_2g^2S(T^1X^2_1)S(y^1)\alpha g^2\beta S(y^3)$$

$$1.6$$
$$= X^1_1h_1g^1S(T^2X_2^2)\alpha T^3X^3 \cdot a \# X^1_2h_2g^2S(T^1X_1^2)$$

$$1.26$$
$$= i_0(Y^1X_1^1h_1g^1S(T^2X_2^2)\alpha T^3X^3 \cdot a) \ast j(Y^2X_1^2h_2g^2S(Y^3T^1X_1^2)), \quad (3.1)$$
Proposition 3.2 Let $H$ be a quasi-Hopf algebra and $A$ a left $H$-module algebra. Then the map $\Pi : A \otimes H_0 \rightarrow (A \# H)^\lambda$, $\Pi(a \otimes h) = x^1 \cdot a \# x^2 hS(x^3)$, for all $a \in A$ and $h \in H$, is an isomorphism of left $H$-module algebras.

Proof. Note first that $\Pi$ is bijective, with inverse given by $\Pi^{-1}(a \# h) = X^1 \cdot a \otimes X^2 hS(X^3)$. For proving that $\Pi$ is a morphism of left $H$-module algebras, we will use the universal property of $A \otimes H_0$ as a twisted tensor product of algebras in $H \mathcal{M}$. Namely, we know that $i_0 : A \rightarrow (A \# H)^\lambda$ is a morphism of left $H$-module algebras, and it is easy to see that $j : H_0 \rightarrow (A \# H)^\lambda$ is also a morphism of left $H$-module algebras. Moreover, the relation (3.1) reduces in this case exactly to (1.66). We can thus use the universal property of $A \otimes H_0$, which provides a morphism $w : A \otimes H_0 \rightarrow (A \# H)^\lambda$ of left $H$-module algebras, which is moreover given by $w(a \otimes h) = i_0(a) \ast j(h)$. Relation (1.26) shows that actually we have $w = \Pi$, finishing the proof. □

As an application of Proposition 3.2 we will prove a certain kind of invariance under Drinfeld twisting for the $H$-module algebra $A \otimes H_0$. We recall first the following results from [27]:

Lemma 3.3 ([27]) Let $H$ be a quasi-Hopf algebra, $B$, $C$ associative algebras, $\eta : B \rightarrow C$, $j : H \rightarrow B$, $v : H \rightarrow C$ algebra maps such that $\eta \circ j = v$. Then the map $\eta : B^j \rightarrow C^v$ is a morphism of left $H$-module algebras.

Proposition 3.4 ([27]) Let $H$ be a quasi-Hopf algebra, $F \in H \otimes H$ a gauge transformation, $B$ an associative algebra, $v : H \rightarrow B$ an algebra map, which will be denoted by $v_F$ when considered as a map from $H_F$ to $B$. Then the map

$$\psi : (B^v)_F \rightarrow B^{v_F}, \quad \psi(b) = v(F^1)bv(S(F^2)), \quad \forall \ b \in B,$$

is an isomorphism of left $H_F$-module algebras.

We can state now the desired result.

Proposition 3.5 Let $H$ be a quasi-Hopf algebra, $F \in H \otimes H$ a gauge transformation and $A$ a left $H$-module algebra. Then $(A \otimes H_0)_F \simeq A_{F^{-1}} \ast (H_F)_0$ as left $H_F$-module algebras.

Proof. We denote as before by $j : H \rightarrow A \# H$, $j(h) = 1 \# h$, and by $j_F$ the same map when considered as a map from $H_F$ to $A \# H$ or to $A_{F^{-1}} \# H_F$. Since obviously the algebra isomorphism $\pi : A \# H \simeq A_{F^{-1}} \# H_F$, $\pi(a \# h) = F^1 \cdot a \# F^2 h$, satisfies $\pi \circ j_F = j_F$, we obtain that $(A \# H)^{j_F} \simeq (A_{F^{-1}} \# H_F)^{j_F}$ as left $H_F$-module algebras, by using Lemma 3.3. On the other hand, by Proposition 3.4 we obtain that $((A \# H)^F)_{F^{-1}} \simeq (A \# H)^{j_F}$ as left $H_F$-module algebras. Now by Proposition 3.2 we have $A \otimes H_0 \simeq (A \# H)^j$ as left $H$-module algebras and $A_{F^{-1}} \ast (H_F)_0 \simeq (A_{F^{-1}} \# H_F)^{j_F}$ as left $H_F$-module algebras. It is also easy to see that if $B$ and $C$ are isomorphic left $H$-module algebras then $B_{F^{-1}}$ and $C_{F^{-1}}$ are isomorphic left $H_F$-module algebras. Using all these facts we finally obtain that $(A \otimes H_0)_F \simeq A_{F^{-1}} \ast (H_F)_0$ are isomorphic as left $H_F$-module algebras. □
4 An isomorphism $A \otimes H_0 \simeq (A \# H)^j$ of algebras in $H^j_H \mathcal{YD}$

We have seen in the previous section that, if $H$ is a quasi-Hopf algebra and $A$ is a left $H$-module algebra, then the map $\Pi : A \circ H_0 \to (A \# H)^j$, $\Pi(a \otimes h) = x^1 \cdot a \# x^2 hS(x^3)$, is an isomorphism of left $H$-module algebras. If we assume that $A$ is moreover an algebra in $H^j_H \mathcal{YD}$, in which case $A \circ H_0$ becomes the braided tensor product $A \otimes H_0$, an algebra in $H^j_H \mathcal{YD}$, then we intend to show that $(A \# H)^j$ becomes also an algebra in $H^j_H \mathcal{YD}$ in a natural way and $\Pi$ becomes an isomorphism of algebras in $H^j_H \mathcal{YD}$. We begin with a result of independent interest.

**Proposition 4.1** Let $H$ be a quasi-bialgebra and $A$ an algebra in $H^j_H \mathcal{YD}$. Then $(A \# H, \lambda, \Phi_\lambda)$ is a left $H$-comodule algebra, with structures:

\[
\lambda : A \# H \to H \otimes (A \# H), \quad \lambda(a \# h) = T^1(t^1 \cdot a)(-1)t^2 h_1 \otimes (T^2 \cdot (t^1 \cdot a)(0) \# T^3 t^3 h_2),
\]

\[
\Phi_\lambda = X^1 \otimes X^2 \otimes (1 \# X^3) \in H \otimes H \otimes (A \# H).
\]

**Proof.** We first check that $\lambda$ is an algebra map. It is easy to see that $\lambda$ is unital, so we only check that it is multiplicative. We compute:

\[
\lambda((a \# h)(a' \# h')) = \lambda((z^1 \cdot a)(z^2 h_1 \cdot a') \# z^3 h_2 h')
\]

\[= T^1([t^1z^1 \cdot a](t^2 z^2 h_1 \cdot a'))(-1)t^2 z^3 h_{(2,1)} h'_1 \]

\[\otimes T^2 \cdot [(t^1z^1 \cdot a)(t^2 z^2 h_1 \cdot a')](0) \# T^3 t^3 z^3 h_{(2,2)} h'_2
\]

\[= T^1((t^1y^1 \cdot a)(t^2 y^2 x^1 h_1 \cdot a'))(-1)t^3 y^3 x^2 h_{(2,1)} h'_1 \]

\[\otimes T^2 \cdot [(t^1y^1 \cdot a)(t^2 y^2 x^1 h_1 \cdot a')](0) \# T^3 y^3 x^3 h_{(2,2)} h'_2
\]

\[= T^1 X^1((z^1 \cdot a)(-1) z^2 (y^2 z h_{(1,1)} x^1 \cdot a')(-1)y^2 h_{(1,2)} x^2 h'_1
\]

\[\otimes T^2 X^2 \cdot (z^1 \cdot a)(0)] [T^2 X^3 z^3 (y^2 h_{(1,1)} x^1 \cdot a')(0) \# T^3 y^3 h_{2 x^3} h'_2
\]

\[= T^1 X^1((z^1 \cdot a)(-1) z^2 y^2 z h_{(1,1)} x^1 \cdot a'(-1)x^2 h'_1
\]

\[\otimes T^2 X^2 \cdot (z^1 \cdot a)(0)] [T^2 X^3 z^3 y^2 h_{(1,2)} \cdot (x^1 \cdot a')(0) \# T^3 y^3 h_{2 x^3} h'_2
\]

\[= T^1 X^1((y^1 z \cdot a)(-1) y^2 z^2 Z h_{(1,1)} x^1 \cdot a'(-1)x^2 h'_1
\]

\[\otimes T^2 X^2 \cdot (z^1 \cdot a)(0)] [T^2 X^3 y^2 z^3 Z h_{(1,2)} \cdot (x^1 \cdot a')(0) \# T^3 y^3 z^3 h_{2 x^3} h'_2
\]

\[= T^1 X^1 y^1 (z^1 \cdot a)(-1) z^2 Z h_{(1,1)} x^1 \cdot a'(-1)x^2 h'_1
\]

\[\otimes T^2 X^2 y^2 \cdot (z^1 \cdot a)(0)] [T^2 X^3 y^2 z^3 Z h_{(1,2)} \cdot (x^1 \cdot a')(0) \# T^3 y^3 z^3 h_{2 x^3} h'_2
\]

\[= T^1 z^1 \cdot a((-1) z^2 h_1 Z^1 x^1 \cdot a'(-1)x^2 h'_1 \otimes [y^1 T^2 \cdot (z^1 \cdot a)(0)]
\]

\[= [y^2 T^1 z^2 Z h_{(2,1)} Z^2 \cdot (x^1 \cdot a')(0) \# y^3 T^2 z^3 Z h_{(2,2)} Z^3 h^2_2
\]

\[= [T^1(z^1 \cdot a) \otimes (T^2 \cdot (z^1 \cdot a)(0) \# T^3 z^3 h_2)]
\]

\[= Z^1(x^1 \cdot a'(-1) x^2 h'_1 \otimes (Z^2 \cdot (x^1 \cdot a')(0) \# Z^3 x^3 h'_2)
\]

\[= \lambda(a \# h) \lambda(a' \# h'), \quad q.e.d.
\]

Now we check (1.33). We compute:

\[(id \otimes \lambda)(\lambda(a \# h)) \Phi_\lambda
\]

\[= [T^1(t^1 \cdot a)(-1)t^2 h_1 \otimes X^1(t^1 T^2 \cdot (t^1 \cdot a)(0)(-1)x^2 T^3 t^3 h_{(2,1)}]
\]
The conditions (1.34), (1.35) and (1.36) are easy to check and are left to the reader.

**Definition 4.2** Let \(\Phi, \mu, \lambda \in \mathbf{C} \) be a left \(H\)-comodule algebra. A morphism of comodule algebras from \(\mathfrak{B}\) to \(\mathfrak{C}\) is an algebra map \(\nu : \mathfrak{B} \to \mathfrak{C}\) such that \(\Phi = (id_H \otimes \mu H \otimes \nu)(\Phi_\lambda)\) and \(\mu \circ \nu = (id_H \otimes \nu) \circ \lambda\).

**Remark 4.3** If \(H\) is a quasi-bialgebra, \(A\) is an algebra in \(\mathcal{H}YD\) and we consider the left \(H\)-comodule algebra \(A\#H\) as in Proposition 4.1, then one can easily see that the map \(j : H \to A\#H\), \(j(h) = 1\#h\), is a morphism of left \(H\)-comodule algebras.

The next result is a generalization of the fact, proved in [11], Proposition 2.5, that \(H_0\) is an algebra in \(\mathcal{H}YD\); the proof is similar to the one in [11] and will be omitted.

**Proposition 4.4** If \(H\) is a quasi-Hopf algebra, \((\mathfrak{B}, \lambda, \Phi_\lambda)\) a left \(H\)-comodule algebra and \(\nu : H \to \mathfrak{B}\) a morphism of \(H\)-comodule algebras, then \(\mathfrak{B}^\nu\) becomes an algebra in \(\mathcal{H}YD\) with coaction

\[
\mathfrak{B}^\nu \to H \otimes \mathfrak{B}^\nu, \quad b \mapsto X_1^1 Y_1^1 b_{(-1)} g_1 S(q_2 Y_2^2) Y_3 \otimes v(X_2^2 Y_2^1) b_{(0)} v(g_2 S(q_3 Y_2^1 Y_1^2)),
\]

where \(f^{-1} = g_1 \otimes g_2\) is given by (1.14) and \(q_R = q_1 \otimes q_2 = Z_1 \otimes S^{-1}(\alpha Z_2) Z_2\). Moreover, the map \(\nu : H_0 \to \mathfrak{B}^\nu\) is a morphism of algebras in \(\mathcal{H}YD\).

As a consequence of these results, we obtain:

**Corollary 4.5** Let \(H\) be a quasi-Hopf algebra and \(A\) an algebra in \(\mathcal{H}YD\). Then \((A\#H)^j\) becomes an algebra in \(\mathcal{H}YD\), with coaction \(\lambda_{(A\#H)^j} : (A\#H)^j \to H \otimes (A\#H)^j\),

\[
\lambda_{(A\#H)^j}(a\#h) = X_1^1 Y_1^1 T^1 (t_1 \cdot a)(-1)_1 t_2 h_1 g_1 S(q_2 Y_2^2) Y_3 \otimes [X_2^2 Y_2^1 Y_1^2 T^2 \cdot (t_1 \cdot a)(0) \\
# X_3^2 (T_3 h_2 g_2 S(q_3 Y_2^1 Y_1^2)),
\]

and the map \(j : H_0 \to (A\#H)^j\) is a morphism of algebras in \(\mathcal{H}YD\).
Lemma 4.6 Let $H$ be a quasi-Hopf algebra and $A$ an algebra in $^H_H\mathcal{YD}$. Then $i_0$ is a morphism of algebras in $^H_H\mathcal{YD}$ from $A$ to $(A\#H)^i$.

Proof. We already know that $i_0$ is a morphism of left $H$-module algebras from $A$ to $(A\#H)^i$, so the only thing left to prove is that $\lambda_{(A\#H)^i} \circ i_0 = (id_H \otimes i_0) \circ \lambda_A$. We first record the following relations, whose proofs are easy and left to the reader, which hold for the elements $p_R = p^1 \otimes p^2$ and $q_Z = q^1 \otimes q^2$ given by (1.44):

\[
\begin{align*}
T^1 p_1^1 \otimes T^2 p_2^1 \otimes T^3 p^2 &= y^1 \otimes y^2_1 p^1 \otimes y^2_2 p^2 S(y^3), \\
Y^1 Z^1 \otimes q^1 Y^2 Z^2 \otimes S(q^2 Y^2_2 Z^3) Y^3 &= q^1 \otimes q^2 \otimes S(q^2).
\end{align*}
\]

Now, by denoting $p_R = P^1 \otimes P^2$ another copy of $p_R$, we compute:

\[
\begin{align*}
(\lambda_{(A\#H)^i} \circ i_0)(a) &= \lambda_{(A\#H)^i}(p^1 \cdot a#p^2) \\
&= X^1 Y^1 \cdot T^1 ((t^1 p^1 \cdot a)_{(-1)} t^2 p^2 q^1 S(q^2 Y^2_2) Y^3 \\
&\quad \otimes [X^2 Y^2_{(1,2)} T^2 (t^1 p^1 \cdot a)_{(0)} \# X^3 Y^2_{(2,2)} T^3 t^2 p^2 q^1 S(X^3 q^1 Y^3_1)] \\
&= X^1 Y^1 \cdot T^1 (Z^1_{(1,1)} p^1 \cdot a)_{(-1)} Z^1_{(1,2)} p^2 S(q^2 Y^2_2 Z^3) Y^3 \\
&\quad \otimes [X^2 Y^2_{(1,2)} T^2 (Z^1_{(1,1)} p^1 \cdot a)_{(0)} \# X^2 Y^2_{(2,2)} T^3 Z^1_2 p^2 S(X^3 q^1 Y^2_1 Z^2)] \\
&= X^1 T^1 q^1 (P^1 \cdot a)_{(-1)} P^2 S(q^2) \\
&\quad \otimes [X^2 T^2 q^1_{(1,2)} p^2 (P^1 \cdot a)_{(0)} \# X^2 T^3 q^1_{(1,2)} p^2 S(q^2) S(X^3)] \\
&= X^1 T^1 q^1 (P^1 \cdot a)_{(-1)} P^2 S(q^2) \\
&\quad \otimes [X^2 T^2 p^2_2 q^1 (P^1 \cdot a)_{(0)} \# X^2 T^3 p^2 S(X^3)] \\
&= X^1 y^1 \cdot (q^1 P^1 \cdot a)_{(-1)} q^2 p^2 S(q^2) \\
&\quad \otimes [X^2 y^2 p^1 (q^1 P^1 \cdot a)_{(0)} \# X^2 y^2 p^2 S(X^3 y^3)] \\
&= (id_H \otimes i_0) \circ \lambda_A(a),
\end{align*}
\]

finishing the proof. 

Theorem 4.7 Let $H$ be a quasi-Hopf algebra and $A$ an algebra in $^H_H\mathcal{YD}$. Then the map $\Pi : A \otimes H_0 \to (A\#H)^i$, $\Pi(a \otimes h) = x^1 \cdot a#x^2 H S(x^3)$, is an isomorphism of algebras in $^H_H\mathcal{YD}$.

Proof. We proved that $j : H_0 \to (A\#H)^i$ and $i_0 : A \to (A\#H)^i$ are morphisms of algebras in $^H_H\mathcal{YD}$, and together with the commutation relation (3.1) this allows to apply the universal property of the twisted tensor product $A \otimes H_0 = A \otimes_R H_0$ in the category $^H_H\mathcal{YD}$, obtaining thus a morphism of algebras in $^H_H\mathcal{YD}$ between $\underline{\otimes} H_0$ and $(A\#H)^i$, $\omega$, say, which has to be given by $\omega(a \otimes h) = i_0(a) \ast j(h)$, that is $\omega$ coincides with the map $\Pi$, which finishes the proof. \qed

15
5 An algebra isomorphism $B^u \# H \simeq B \otimes H$ and applications

**Proposition 5.1** Let $H$ be a quasi-Hopf algebra, $B$ an associative algebra and $v : H \to B$ an algebra map. Denote by $\eta$ the algebra map $\eta : H \to B \otimes H$, $\eta(h) = v(h_1) \otimes h_2$. Define the map

$$u : B \to B \otimes H, \quad u(b) = v(x_1) bv(S(x_1^3 X^3) f^1) \otimes x_2 X_1 \beta S(x_1^3 X^2) f^2,$$

(5.1)

where $f = f^1 \otimes f^2$ is the Drinfeld twist given by (1.13). Then $u$ is a morphism of left $H$-module algebras from $B^u$ to $(B \otimes H)^n$.

**Proof.** The fact that $u(v(\beta)) = \eta(\beta)$ follows immediately from (1.15). We check now that $u$ is a morphism of left $H$-modules:

$$h \triangleright u(b) = \eta(h_1) u(b) \eta(S(h_2)) = v(h_{1(1)}) v(x_1) bv(S(x_1^3 X^3) f^1) v(S(h_{2(1)})) \otimes h_{1(2)} x_2 X_1 \beta S(x_1^3 X^2) f^2 S(h_{2(2)})$$

(1.9)

$$= v(h_{1(1)}) x_1 bv(S(h_{2(2)} x_1^3 X^3 f^1) \otimes h_{1(2)} x_2 X_1 \beta S(h_{2(1)} x_1^3 X^2) f^2$$

(1.1)

$$= v(x \triangleright_{h_1} x_1) bv(S(h_{2(2)} x_1^3 X^3 f^1) \otimes x_2 h_{2(1)} X_1 \beta S(x_1^3 h_{2(2)} X^2) f^2$$

(1.5)

$$= v(x_1 h_1) bv(S(x_1^3 X^3 h_2 f^1) \otimes x_2 X_1 \beta S(x_1^3 X^2) f^2$$

$$= u(h \triangleright v, b), \quad q.e.d.$$

Now we check that $u$ is multiplicative (we denote by $F^1 \otimes F^2$ another copy of $f$):

$$u(b) \ast u(b') = \eta(Y) u(b) \eta(S(y^3 Y_2 \alpha Y^2 Y_1)) = v(Y_1 x_1) bv(S(x_1^3 X^3 f^1) S(y^3 Y_2 \alpha Y^2 Y_1)) \otimes x_2 X_1 \beta S(y^3 Y_2 \alpha Y^2 Y_1) f^2$$

(1.9) (1.10)

$$= v(Y_1 x_1) bv(S(T^2 y_2 x_1^3 X^3 \alpha T^3 y_2 x_1^3 X^3) \alpha T^3 y_2 x_1^3 X^3) \beta S(y^3 Y_2 \alpha Y^2 Y_1) f^2$$

(1.1) (1.5)

$$= v(Y_1 x_1) bv(S(T^2 y_2 x_1^3 X^3 \alpha T^3 y_2 x_1^3 X^3) \alpha T^3 y_2 x_1^3 X^3) \beta S(y^3 Y_2 \alpha Y^2 Y_1) f^2$$

(1.3) (1.1) (1.5)

$$= v(Y_1 x_1) bv(S(T^2 y_2 x_1^3 X^3 \alpha T^3 y_2 x_1^3 X^3) \alpha T^3 y_2 x_1^3 X^3) \beta S(y^3 Y_2 \alpha Y^2 Y_1) f^2$$

(1.3) (1.1) (1.5)

$$= v(Y_1 x_1) bv(S(T^2 y_2 x_1^3 X^3 \alpha T^3 y_2 x_1^3 X^3) \alpha T^3 y_2 x_1^3 X^3) \beta S(y^3 Y_2 \alpha Y^2 Y_1) f^2$$

(1.3) (1.1) (1.5)

$$= v(Y_1 x_1) bv(S(T^2 y_2 x_1^3 X^3 \alpha T^3 y_2 x_1^3 X^3) \alpha T^3 y_2 x_1^3 X^3) \beta S(y^3 Y_2 \alpha Y^2 Y_1) f^2$$

(1.3) (1.1) (1.5)

$$= v(Y_1 x_1) bv(S(T^2 y_2 x_1^3 X^3 \alpha T^3 y_2 x_1^3 X^3) \alpha T^3 y_2 x_1^3 X^3) \beta S(y^3 Y_2 \alpha Y^2 Y_1) f^2$$

(1.3) (1.1) (1.5)

$$= v(Y_1 x_1) bv(S(T^2 y_2 x_1^3 X^3 \alpha T^3 y_2 x_1^3 X^3) \alpha T^3 y_2 x_1^3 X^3) \beta S(y^3 Y_2 \alpha Y^2 Y_1) f^2$$

(1.3) (1.1) (1.5)
$$\otimes x^2 X^1 \beta S(y^1 x_1^1 T_{1,1}^1 z_1^1 X^2) \alpha y^2 Z^1 \beta S(y_1^1 Z^2)^2$$

$$= v(x^1 Y^1) b v(S(T^2 z_2^2 X_3^3 Y^2) \alpha T^3 z_1^1 X^2) y^2 Z^1 \beta S(y_1^1 Z^2)^2$$

$$= v(S(x_3^3 x_2^2 X_3^3 Y^2)^2) \otimes x^2 X^1 \beta S(x_3^3 \alpha T_{1,1}^1 z_1^1 X^2) y^2 Z^1 \beta S(y_1^1 Z^2)^2$$

$$= v(x^1 Y^1) b v(S(y_1^1 t_1^1 X_3^3 Y^2) y^2 t_2^2 X_3^3 Y^2) y^2 Z^1 \beta S(y_1^1 Z^2)^2$$

$$= v(x^1 Y^1) b v(S(y_1^1 t_1^1 X_3^3 Y^2) y^2 t_2^2 X_3^3 Y^2) y^2 Z^1 \beta S(y_1^1 Z^2)^2$$

$$(= u(v(Y^1) b v(S(t_1^1 Y^2) y^2 t_1^1 Y^2))$$

finishing the proof. \(\square\)

**Corollary 5.2** The map \(\psi : B^v \# H \to B \otimes H\), given by

$$\psi(b \# h) = v(X^1 x_1^1) b v(S(X^2 x_2^1) \alpha X^3 x^2 h_1) \otimes x^3 h_2,$$  \hspace{1cm} (5.2)

is an algebra map.

**Proof.** The universal property of the smash product, applied to the maps \(\eta\) and \(u\) from Proposition 5.1 provides an algebra map \(B^v \# H \to B \otimes H\), which turns out to be exactly the map \(\psi\) given by (5.2), we leave the details to the reader. \(\square\)

**Proposition 5.3** If \(H\) is a quasi-Hopf algebra, \(B\) an algebra, \(v : H \to B\) an algebra map, then:

(i) The map \(\theta : B \to B^v \# H\), \(\theta(b) = v(z^1) b v(Z^1 \beta S(z^2 Z^2)) \# z^3 h^3\), is an algebra map.

(ii) The map \(\mu : H \to B^v \# H\), \(\mu(h) = v(z^1 Z^1 \beta S(z^2 h_1 Z^2)) \# z^3 h_2 Z^3\), is an algebra map.

(iii) For all \(h \in H\) and \(b \in B\), the following relation holds:

$$\theta(b) \mu(h) = \mu(h) \theta(b) = v(z^1) b v(Z^1 \beta S(z^2 h_1 Z^2)) \# z^3 h_2 Z^3.$$

(iv) Consequently, the map

$$\xi : B \otimes H \to B^v \# H, \quad \xi(b \otimes h) = v(z^1) b v(Z^1 \beta S(z^2 h_1 Z^2)) \# z^3 h_2 Z^3,$$  \hspace{1cm} (5.3)

is an algebra map.

**Proof.** We only prove (i) and leave the rest to the reader. Obviously \(\theta(1) = v(\beta) \# 1\), so we only have to check that \(\theta\) is multiplicative. We compute:

$$\theta(b) \theta(b') = [x^1 z_1^1] b v(z^1) b v(Z^1 \beta S(z^2 Z^2)) \# [x^2 z_2^2 Z_3^3 \beta T^3]$$

$$= v(X^1 x_1^1 z_1^1) b v(Z^1 \beta S(y^1 X^2 x_2^1 z_2^2 Z^2) \alpha y_1^1 z_1^1 x_1^1 Z^3_{(1,1)} t_2^1)$$

$$b v(T^1 \beta S(w^3 x_2^2 z_3^3 (1,2) Z^3_{(1,2)} t^2 T^2)) \# [x^3 z_2^2 Z_3^3 \beta T^3]$$

$$= v(x^1 w^1 z_1^1 y^1) b v(Z^1 \beta S(y^1 X^2 z_2^2 y_1^1 x_1^1 Z^2) \alpha X^3 x_1^1 z_1^1 y_2^2 (1,2) Z^3_{(1,1)} t_1^1)$$

$$b v(T \beta S(w^3 x_2^2 z_3^3 (1,2) Z^3_{(1,2)} t^2 T^2)) \# [y^3 z_3^3 x_3^3 \beta T^3]$$

$$= v(y^1) b v(Z^1 \beta S(z^1 y^1 x_1^1 Z^2) \alpha z_2^2 y_2^2 (1,2) Z^3_{(1,1)} t_1^1)$$

17
Moreover, with structures provided thus an algebra isomorphism $B'\#H \simeq B \otimes H$.

**Proof.** We compute:

\[
\begin{align*}
\xi(\psi(b\#h)) &= v(z^1 X^1 x_1^1) b v(S(X^2 x_2^1) \alpha X^3 x_3^1 Z^2 \beta S(z^2 x_1^1 h_{(2,1)} Z^2)) \# z^3 x_2^1 h_{(2,2)} Z^3 \\
&\quad + v(z^1 X^1 x_1^1) b v(S(X^2 x_2^1) \alpha X^3 x_3^1 Z^2 \beta S(z^2 x_1^1 Z^2)) \# z^3 x_2^1 Z^3 h \\
&\quad + v(z^1 X^1 Y_{(1,1)} t_1^1) b v(S(X^2 Y_{(1,2)} t_2^1) \alpha X^3 Y_{(2,1)} t_2^2 \beta S(z^2 Y_{(2,2)} t^2)) \# z^3 Y^3 h \\
&\quad + v(X^1 t_1^1) b v(S(X^2 t_2^1) \alpha X^3 t_3^2 \beta S(t^3)) \# h \\
&\quad = b \# h, \\
\psi(\xi(b \otimes h)) &= v(X^1 x_1^1 z^1) b v(Z^1 \beta S(X^2 x_2^1 z^2 h_1 Z^2) \alpha X^3 x_3^1 Z^3 h_{(2,2)} Z^3) \otimes x^1 z_2^1 h_{(2,2)} Z^3 \\
&\quad + b v(Z^1 \beta S(h_1 Z^2) \alpha t_2^2 h_{(2,1)} Z^1) \otimes t^3 h_{(2,2)} Z_2^3 \\
&\quad + b v(Z^1 \beta S(h_1 Z^2) \alpha t_2^2 Z^1) \otimes h t^3 Z_2^3 \\
&\quad + b v(Z^1 \beta S(Z^2) \alpha Z^3) \otimes h \\
&\quad = b \otimes h,
\end{align*}
\]

finishing the proof. \qed

**Theorem 5.4** The maps $\psi$ and $\xi$ given by (5.2) and respectively (5.3) are inverse to each other, providing thus an algebra isomorphism $B'\#H \simeq B \otimes H$.

**Proof.** We compute:

\[
\begin{align*}
\xi(\psi(b\#h)) &= v(z^1 X^1 x_1^1) b v(S(X^2 x_2^1) \alpha X^3 x_3^1 Z^2 \beta S(z^2 x_1^1 h_{(2,1)} Z^2)) \# z^3 x_2^1 h_{(2,2)} Z^3 \\
&\quad + v(z^1 X^1 x_1^1) b v(S(X^2 x_2^1) \alpha X^3 x_3^1 Z^2 \beta S(z^2 x_1^1 Z^2)) \# z^3 x_2^1 Z^3 h \\
&\quad + v(z^1 X^1 Y_{(1,1)} t_1^1) b v(S(X^2 Y_{(1,2)} t_2^1) \alpha X^3 Y_{(2,1)} t_2^2 \beta S(z^2 Y_{(2,2)} t^2)) \# z^3 Y^3 h \\
&\quad + v(X^1 t_1^1) b v(S(X^2 t_2^1) \alpha X^3 t_3^2 \beta S(t^3)) \# h \\
&\quad = b \# h, \\
\psi(\xi(b \otimes h)) &= v(X^1 x_1^1 z^1) b v(Z^1 \beta S(X^2 x_2^1 z^2 h_1 Z^2) \alpha X^3 x_3^1 Z^3 h_{(2,2)} Z^3) \otimes x^1 z_2^1 h_{(2,2)} Z^3 \\
&\quad + b v(Z^1 \beta S(h_1 Z^2) \alpha t_2^2 h_{(2,1)} Z^1) \otimes t^3 h_{(2,2)} Z_2^3 \\
&\quad + b v(Z^1 \beta S(h_1 Z^2) \alpha t_2^2 Z^1) \otimes h t^3 Z_2^3 \\
&\quad + b v(Z^1 \beta S(Z^2) \alpha Z^3) \otimes h \\
&\quad = b \otimes h,
\end{align*}
\]

finishing the proof. \qed

We present now some applications of Theorem 5.4. We first record two results of independent interest, whose proofs are straightforward and will be omitted.

**Lemma 5.5** If $H$ is a quasi-Hopf algebra, then $(H \otimes H, \lambda, \Phi_\lambda)$ is a left $H$-comodule algebra, with structures

\[
\lambda : H \otimes H \to H \otimes (H \otimes H), \quad \lambda(h' \otimes h) = \Phi \cdot ((\Delta \otimes id_H)(h' \otimes h)) \cdot \Phi^{-1},
\]

\[
\Phi_\lambda = (id_H \otimes id_H \otimes \Delta)(\Phi) \in H \otimes H \otimes (H \otimes H).
\]

Moreover, $\Delta : H \to H \otimes H$ is a morphism of left $H$-comodule algebras.
Lemma 5.6 If $H$ is a quasi-Hopf algebra, $\nu : \mathcal{B} \to \mathcal{C}$ is a morphism of left $H$-comodule algebras and $v : H \to \mathcal{B}$, $w : H \to \mathcal{C}$ are morphisms of left $H$-comodule algebras such that $\nu \circ v = w$, then $\nu$ is a morphism of algebras in $\hat{H}^\mathcal{YD}$ between $\mathcal{B}^v$ and $\mathcal{C}^w$, where $\mathcal{B}^v$ and $\mathcal{C}^w$ become algebras in $\hat{H}^\mathcal{YD}$ as in Proposition 4.4.

We consider now the particular case $B = H$ and $v = id_H$ in Theorem 5.4 so we have an algebra isomorphism

$$\Psi : H_0 \# H \to H \otimes H, \quad \Psi(h' \otimes h) = X^1 x_1 h' h_S(X^2 x_1) a X^3 x_2 h_1 \otimes x^3 h_2. \quad (5.4)$$

By a slightly longer but also straightforward computation, one can prove the following result:

Lemma 5.7 If we consider $H_0 \# H$ a left $H$-comodule algebra as in Proposition 4.4 and $H \otimes H$ a left $H$-comodule algebra as in Lemma 5.5, then the map $\Psi$ given by (5.4) is an isomorphism of left $H$-comodule algebras.

By using (1.5) and (1.8), one can see that $\Psi \circ j = \Delta$, where $j : H \to H_0 \# H$, $j(h) = 1_{H_0} \# h$. Thus, as a consequence of the above results and of Theorem 4.7 we finally obtain:

Proposition 5.8 If $H$ is a quasi-Hopf algebra, then $H_0 \otimes H_0 \simeq (H \otimes H)^\Delta$ as algebras in $\hat{H}^\mathcal{YD}$.

As a second application of Theorem 5.4, we will obtain a duality theorem for quasi-Hopf algebras. We recall from [17] the construction of the two-sided crossed product $H \bowtie H^* \bowtie H$ associated to a finite dimensional quasi-Hopf algebra $H$, which is an associative algebra structure on $H \otimes H^* \otimes H$ with multiplication

$$(h \otimes \varphi \otimes l)(h' \otimes \varphi' \otimes l') = hh' x_1 \otimes (y_1 \varphi h_2 x_2 (y_2 l_1 \varphi h_3 x_3) \otimes y^3 l_2 l'),$$

for $h, h', l, l' \in H$ and $\varphi, \varphi' \in H^*$, where $\varphi$ and $\varphi'$ are the regular actions of $H$ on $H^*$. We also recall from [9] the construction of the so-called quasi-smash product, denoted by $H \# H^*$, which is a left $H$-module algebra structure on $H \otimes H^*$, with multiplication and $H$-action given by

$$(h \otimes \varphi)(h' \otimes \varphi') = hh' x_1 \otimes (\varphi h_2 x_2 (\varphi' h_3 x_3),$$

$$h \cdot (h' \otimes \varphi) = h' \otimes h \varphi,$$

for all $h, h' \in H$ and $\varphi, \varphi' \in H^*$. By [9], the following identification of algebras holds:

$$H \bowtie H^* \bowtie H \equiv (H \# H^*) \# H.$$

By [27], $H \# H^*$ is isomorphic, as left $H$-module algebras, to $End(H)^*$, where $End(H)$ is regarded as an associative algebra in the usual way and $v : H \to End(H)$ is a certain algebra map.

All these facts combined with Theorem 5.4 yield the desired duality theorem:

Theorem 5.9 If $H$ is a finite dimensional quasi-Hopf algebra, then the two-sided crossed product $H \bowtie H^* \bowtie H$ is isomorphic to $End(H) \otimes H$ as associative algebras.

6 Some iterated products

Let $H$ be a quasi-bialgebra and $A$ an algebra in $\hat{H}^\mathcal{YD}$. By Proposition 4.1, $(A \# H, \lambda, \Phi_{\lambda})$ is a left $H$-comodule algebra, and by the Preliminaries $(A \# H, \rho, \Phi_{\rho})$ is a right $H$-comodule algebra.
Proposition 6.1 (A#H, λ, ρ, Φγ, Φλρ) is an H-bicomodule algebra, with Φλρ = X1 ⊗ (1#X2) ⊗ X3 ∈ H ⊗ (A#H) ⊗ H.

Proof. We first check the relation (1.45). We compute:

\[ \Phi_\lambda\rho(\lambda \otimes \text{id})(\rho(a \# h)) \]
\[ = \Phi_\lambda\rho(\lambda \otimes \text{id})(x^1 \cdot a \# x^2 h_1) \otimes x^3 h_2 \]
\[ = \Phi_\lambda\rho(T^1(t^1 x^1 \cdot a)(-1) t^2 x^2 h_{(1,1)} \otimes (T^2 \cdot (t^1 x^1 \cdot a)(0) \# T^3 x^3 h_{(1,2)}) \otimes x^3 h_2) \]
\[ = X^1 T^1(t^1 x^1 \cdot a)(-1) t^2 x^2 Y^1 h_{(1,1)} \otimes (X^2 T^2 \cdot (t^1 x^1 \cdot a)(0) \# X^3 T^3 t^3 x^3 h_{(2,1)}) \]
\[ \otimes X^3 t^3 x^3 h_2 \]
\[ = X^1 T^1 t^1(x^1 \cdot a)(-1) t^2 x^2 h_1 Y^1 \otimes (X^2 T^2 t^2 \cdot (x^1 \cdot a)(0) \# X^3 T^3 t^3 h_{(2,1)}) \]
\[ \otimes x^3 h_2 \]
\[ = (id \otimes \rho)(X^1(x^1 \cdot a)(-1) t^2 x^2 h_1 \otimes (X^2 \cdot (x^1 \cdot a)(0) \# X^3 h_{(2,1)}) \Phi_\lambda\rho \]
\[ = (id \otimes \rho)(\lambda(\lambda \# h))\Phi_\lambda\rho, \text{ q.e.d.} \]

Then, using the fact that λ(1#h) = h_1 \otimes (1#h_2) and ρ(1#h) = (1#h_1) \otimes h_2, one can see immediately that the conditions (1.46) and (1.47) reduce to the condition (1.3) for Φ.

Let now again H be a quasi-bialgebra, A an algebra in \( \mathcal{H} \mathcal{D} \) and A an H-bimodule algebra. We can consider the H-bicomodule algebra A#H as above, and then the L-R-smash product A \( \bowtie \) (A#H). On the other hand, we can first consider the H-bimodule algebra A \( \bowtie \) A, and then the L-R-smash product (A \( \bowtie \) A) \( \bowtie \) H.

Proposition 6.2 There is an algebra isomorphism Ψ : A \( \bowtie \) (A#H) \( \simeq \) (A \( \bowtie \) A) \( \bowtie \) H, given by

\[ \Psi(\varphi \bowtie (a \# h)) = (x^1 \cdot \varphi \otimes x^2 \cdot a) \bowtie x^3 h, \forall \varphi \in A, a \in A, h \in H. \]

Proof. Obviously Ψ is bijective and unital, so we only have to check that it is multiplicative. We first write down the multiplication rule in A \( \bowtie \) (A#H):

\[ [\varphi \bowtie (a \# h)][\varphi' \bowtie (a' \# h')] \]
\[ = (\tilde{x}_1^\lambda \cdot \varphi \cdot \theta^3(a' \# h'))_{<0} x^2_\rho^2(\tilde{x}_1^\lambda(a \# h)_{[-1]} \theta^1 \cdot \varphi' \cdot \tilde{x}_\rho^3) \bowtie \tilde{x}_1^\lambda(a \# h)_{[0]} \theta^2(a' \# h')_{<0} \tilde{x}_\rho^1 \]
\[ = (x^1 \cdot \varphi \cdot t^3 w^3 h'_2)_{<0} (x^2 Y^1(z^1 \cdot a)(-1) z^2 x^1 h_1 \cdot \varphi' \cdot \tilde{y}^3) \]
\[ \bowtie (1#x^2)(1#y^1) \bowtie (1#z^3 h_2)(1#w^3 h'_1) \bowtie (1#y^1) \]
\[ = (x^1 \cdot \varphi \cdot t^3 w^3 h'_2)_{<0} (x^2 Y^1(z^1 \cdot a)(-1) z^2 x^1 h_1 \cdot \varphi' \cdot \tilde{y}^3) \]
\[ \bowtie (v^1 x^2 Y^2 \cdot (z^1 \cdot a)(0))(v^2 x^3_{(2)})(z^3 h_{(2,1)} t^2 w^3 h'_1) \bowtie (v^3 x^3_{(2,2)} Y^2 z^3 h_{(2,2)} t^2 w^3 h'_1), \]

where for the second equality we have used the formulae, presented before, giving the H-bicomodule algebra structure of A#H. Now we compute:

\[ \Psi(\varphi \bowtie (a \# h)) \Psi(\varphi' \bowtie (a' \# h')) \]
\[ = [(z^1 \cdot \varphi \otimes z^2 \cdot a) \bowtie z^3 h][w^1 \cdot \varphi' \otimes w^2 \cdot a') \bowtie w^3 h'] \]
Proposition 6.4

Note first that finishing the proof.

We recall from [13] that, if $\phi \otimes (a \# h) \simeq (C \circ A) \# h$, given by $\psi (c \triangleright (a \# h)) = (x^1 \cdot c \otimes x^2 \cdot a) \# x^3 \cdot h$, for all $c \in C$, $a \in A$, $h \in H$.

We recall from [13] that, if $H$ is a quasi-bialgebra, $C$ a left $H$-module algebra and $A$ an $H$-bicomodule algebra, then $C \triangleright A$ becomes a right $H$-comodule algebra, with structure defined for all $c \in C$ and $u \in A$ by:

$$\rho : C \triangleright A \rightarrow (C \triangleright A) \otimes H, \quad \rho (c \triangleright u) = (\theta^1 \cdot c \triangleright \theta^2 u) \otimes \theta^3 u_{<1>}.$$

$$\Phi_\rho = (1_C \triangleright \hat{X}_\rho^1) \otimes \hat{X}_\rho^2 \otimes \hat{X}_\rho^3 \in (C \triangleright A) \otimes H \otimes H.$$

Corollary 6.3 If $H$ is a quasi-bialgebra, $A$ an algebra in $H \triangleright \mathcal{D}$ and $C$ a left $H$-module algebra, then we have an algebra isomorphism $\psi : C \triangleright (A \# H) \simeq (C \circ A) \# H$, given by $\psi (c \triangleright (a \# h)) = (x^1 \cdot c \otimes x^2 \cdot a) \# x^3 \cdot h$, for all $c \in C$, $a \in A$, $h \in H$.

Proposition 6.4 The map $\psi$ in Corollary 6.3 is an isomorphism of right $H$-comodule algebras.

Proof. Note first that $C \triangleright (A \# H)$ and $(C \circ A) \# H$ are indeed right $H$-comodule algebras, since $C$ and $C \circ A$ are left $H$-module algebras and $A \# H$ and $H$ are $H$-bicomodule algebras. It is easy to see that $\psi$ respects the reassociators, so we only prove that it intertwines the coactions:
Proposition 6.5 Let \( H \) be a quasi-Hopf algebra, \( A \) a left \( H \)-module algebra, \( \mathcal{B} \) a left \( H \)-comodule algebra and \( v : H \to \mathcal{B} \) a morphism of left \( H \)-comodule algebras. Then the map 
\[
\tilde{v} : H \to A \triangleright \mathcal{B}, \quad \tilde{v}(h) = 1_A \triangleright v(h),
\]
is an algebra map and the map \( \Pi : A \triangleright \mathcal{B} \to (A \triangleright \mathcal{B})^\triangleright \), 
\( \Pi(a \otimes b) = x^1 \cdot a \triangleright v(x^2) \triangleright v(S(x^3)) \),
is an isomorphism of left \( H \)-module algebras.

Proposition 6.6 Let \( H \) be a quasi-Hopf algebra, \( A \) an algebra in \( \mathcal{H}_H \mathcal{YD} \) and \( C \) a left \( H \)-module algebra. Then we have an isomorphism of algebras in \( H \mathcal{M} \):
\[
(C \odot A) \triangleright H_0 \simeq C \odot (A \triangleright H_0), \quad (c \otimes a) \triangleright h \mapsto X^1 \cdot c \otimes (X^2 \cdot a \otimes X^3 \triangleright h). \tag{6.1}
\]

Proof. We have \((C \odot A) \triangleright H_0 \simeq ((C \odot A) \triangleright H)^\triangleright j\) by Proposition 3.2, then \((C \triangleright (A \triangleright H))^{\triangleright j} \simeq (C \triangleright (A \triangleright H))^{\triangleright j}\) by using Corollary 6.3 and finally \(C \odot (A \triangleright H)^\triangleright j \simeq C \odot (A \triangleright H)^\triangleright j\) by Proposition 6.5.

We recall now some more facts from [13]. Let \( H \) be a quasi-bialgebra, \( A \) a left \( H \)-module algebra and \( B \) a right \( H \)-module algebra. The two-sided smash product \( A \# H \# B \) is the following associative algebra structure on \( A \otimes H \otimes B \):
\[
(a \# h \# b)(a' \# h' \# b') = (x^1 \cdot a)(x^2 h_1 y^1 \cdot a') \# x^3 h_2 y^2 h'_1 z^1 \# (b \cdot y^3 h'_2 z^2)(b' \cdot z^3),
\]
for all \( a, a' \in A, h, h' \in H \) and \( b, b' \in B \). For the particular case when \( A = k \), this reduces to the (right-handed) smash product \( H \# B \), whose multiplication is
\[
(h \# b)(h' \# b') = h h'_1 x^1 \# (b \cdot h'_2 x^2)(b' \cdot x^3),
\]
for all \( b, b' \in B, h, h' \in H \). Moreover, \( H \# B \) is a left \( H \)-comodule algebra, with structures
\[
\lambda : H \# B \to H \otimes (H \# B), \quad \lambda(h \# b) = h_1 x^1 \otimes (h_2 x^2 \# b \cdot x^3), \quad \forall h \in H, b \in B, \quad \Phi_\lambda = X^1 \otimes X^2 \otimes (X^3 \# 1_B) \in H \otimes H \otimes (H \# B),
\]
and the map \( v : H \to H \# B, v(h) = h \# 1 \), is a morphism of left \( H \)-comodule algebras. Moreover, it was proved in [13] that with respect to this comodule algebra structure of \( H \# B \) we have \( A \# H \# B = A \triangleright (H \# B) \).

Now, if we assume that \( H \) is a quasi-Hopf algebra and if we denote by \( j : H \to A \# H \# B, j(h) = 1_A \# h \# 1_B \), which is an algebra map, as a consequence of Proposition 6.5 and of the above facts we obtain the following description of the left \( H \)-module algebra \((A \# H \# B)^\triangleright j\) as a twisted tensor product:
Proposition 6.7 \((A\# H \# B)^{j} \simeq A \circ (H \# B)^{v}\) as left \(H\)-module algebras.

We discuss now some different kinds of iterated products, namely in the sense of [18]. We begin by recalling the framework in [18], extended to the case of nontrivial associators. Let \(\mathcal{C}\) be a monoidal category, \(A, B, C\) algebras in \(\mathcal{C}\) and \(R_{1} : B \otimes A \to A \otimes B, R_{2} : C \otimes B \to B \otimes C, R_{3} : C \otimes A \to A \otimes C\) twisting maps in \(\mathcal{C}\). Define the maps

\[
T_{1} : C \otimes (A \otimes B) \to (A \otimes B) \otimes C, \quad T_{2} : (B \otimes C) \otimes A \to A \otimes (B \otimes C),
\]

\[
T_{1} = a_{A,B,C}^{-1} \circ (id_{A} \otimes R_{2}) \circ a_{A,C,B} \circ (R_{3} \otimes id_{B}) \circ a_{C,A,B}^{-1}, \quad T_{2} = a_{A,B,C} \circ (R_{1} \otimes id_{C}) \circ a_{B,A,C}^{-1} \circ (id_{B} \otimes R_{3}) \circ a_{B,C,A}.
\]

Assume that \(R_{1}, R_{2}, R_{3}\) satisfy the braid equation in \(\mathcal{C}\):

\[
(id_{A} \otimes R_{2}) \circ a_{A,C,B} \circ (R_{3} \otimes id_{B}) \circ a_{C,A,B}^{-1} \circ (id_{C} \otimes R_{1}) \circ a_{C,B,A} = a_{A,B,C} \circ (R_{1} \otimes id_{C}) \circ a_{B,A,C}^{-1} \circ (id_{B} \otimes R_{3}) \circ a_{B,C,A} \circ (R_{2} \otimes id_{A}).
\] (6.2)

Then \(T_{1}\) is a twisting map between \(A \otimes R_{1} B\) and \(C\), \(T_{2}\) is a twisting map between \(A\) and \(B \otimes R_{2} C\), and moreover \(a_{A,B,C}\) is an algebra isomorphism between \((A \otimes R_{1} B) \otimes T_{1} C\) and \(A \otimes T_{2} (B \otimes R_{2} C)\).

We present first a quaialgebra analogue of a result in [18].

Proposition 6.8 Let \(H\) be a quasi-Hopf algebra, \(A, B\) two left \(H\)-module algebras and \(C\) an algebra in \(\mathcal{H}^{YD}\). Assume that \(R_{1} : B \otimes A \to A \otimes B\) is a twisting map in \(\mathcal{H}\) and denote by \(R_{2} : C \otimes B \to B \otimes C\) and \(R_{3} : C \otimes A \to A \otimes C\) the twisting maps in \(\mathcal{H}\) given by \(R_{2}(c \otimes b) = c_{(-1)} \cdot b \otimes c_{(0)}\) and \(R_{3}(c \otimes a) = c_{(-1)} \cdot a \otimes c_{(0)}\), for all \(a \in A, b \in B, c \in C\). Then the twisting maps \(R_{1}, R_{2}, R_{3}\) satisfy the braid equation (6.2) in \(\mathcal{H}\) and the iterated product \((A \otimes R_{1} B) \otimes T_{1} C\) coincides with \((A \otimes R_{1} B) \otimes C\).

Proof. For \(a \in A\) and \(b \in B\), we denote \(R_{1}(b \otimes a) = a_{R_{1}} \otimes b_{R_{1}}\). We check (6.2):

\[
(id_{A} \otimes R_{2}) \circ a_{A,C,B} \circ (R_{3} \otimes id_{B}) \circ a_{C,A,B}^{-1} \circ (id_{C} \otimes R_{1}) \circ a_{C,B,A}(c \otimes b \otimes a) = Z^{1}(x^{1} Y^{1} \cdot c_{(-1)} x^{2} \cdot (Y^{3} \cdot a)_{R_{1}} \otimes [Z^{2} \cdot (x^{1} Y^{1} \cdot c_{(0)})_{(-1)} Z^{3} x^{3} \cdot (Y^{2} \cdot b)_{R_{1}} \otimes [Z^{2} \cdot (x^{1} Y^{1} \cdot c_{(0)})_{0}]
\]

\[
= X^{1}(Y^{1} \cdot c_{(-1)} X^{2} (Y^{3} \cdot a)_{R_{1}} \otimes X^{3} (Y^{1} \cdot c_{(0)})_{R_{1}} = a_{A,B,C}(Y^{1} \cdot c_{(-1)} \cdot R_{1}(Y^{2} \cdot b \otimes Y^{3} \cdot a) \otimes (Y^{1} \cdot c_{(0)}))
\]

\[
= a_{A,B,C}(R_{1}(Y^{1} \cdot c_{(-1)} X^{2} \cdot Y^{3} \cdot a) \otimes (Y^{1} \cdot c_{(0)}) = a_{A,B,C}(R_{1}(X^{1} c_{(-1)} \cdot b \otimes x^{2} (X^{3} \cdot c_{(0)})(X^{3} \cdot a) \otimes x^{3} (X^{2} \cdot c_{(0)}))
\]

\[
= a_{A,B,C} \circ (R_{1} \otimes id_{C}) \circ a_{B,A,C}^{-1} \circ (id_{B} \otimes R_{3}) \circ a_{B,C,A} \circ (R_{2} \otimes id_{A})(c \otimes b \otimes a).
\]

A similar computation shows that in this case the map \(T_{1}\) is given by \(T_{1}(c \otimes a \otimes b) = c_{(-1)} \cdot a \otimes c_{(-1)} b \otimes c_{(0)}\), which implies immediately that \((A \otimes R_{1} B) \otimes T_{1} C = (A \otimes R_{1} B) \otimes C\).

Second, we will prove that iterating the quaialgebra version of the Clifford process fits inside the theory developed in [18]. Let \(A\) be a left module algebra over a quasi-bialgebra \(H\), let \(\sigma\) be an \(H\)-linear involutive unital automorphism of \(A\), let \(q, s\) be two nonzero scalars and denote \(C(k, q) = k[v]/(v^{2} = q)\) and \(C(k, s) = k[w]/(w^{2} = s)\). Consider the linear maps \(R_{1} : C(k, q) \otimes A \to A \otimes C(k, q), R_{2} : C(k, s) \otimes C(k, q) \to C(k, q) \otimes C(k, s), R_{3} : C(k, s) \otimes C(k, q) \to C(k, q) \otimes C(k, s)\).
A \rightarrow A \otimes C(k, s)

by \(R_1(1 \otimes a) = a \otimes 1
\)

\(R_2(v \otimes a) = \sigma(a) \otimes v,
\)

\(R_2(1 \otimes 1) = 1 \otimes 1,
\)

\(R_3(v \otimes v) = -v \otimes w,
\)

\(R_3(w \otimes a) = a \otimes 1,
\)

\(R_3(w \otimes v) = \sigma(a) \otimes w.
\)

We consider \(C(k, q)\) and \(C(k, s)\) as left \(H\)-module algebras with trivial \(H\)-action,

and so \(R_1, R_2, R_3\) become twisting maps in the category \(H \mathcal{M}\). One can easily check that they satisfy the braid equation \((6.2)\), so we have the two twisting maps \(T_1\) and \(T_2\) and moreover

\(A \otimes R_1 C(k, q)) \otimes T_1 C(k, s) \equiv A \otimes T_2 (C(k, q) \otimes R_2 C(k, s)),\)

because \(a_{A,C(k,q),C(k,s)}\) is the identity.

One can also check that, if we regard \(\overline{A} = A \otimes R_1 C(k, q)\) as the Clifford process applied to the pair \((A, \sigma)\), then \((A \otimes R_1 C(k, q)) \otimes T_1 C(k, s)\) is the Clifford process applied to the pair \((\overline{A}, \overline{\sigma})\),

where \(\overline{\sigma}\) is the extended automorphism given by \(\overline{\sigma}(a \otimes 1 + b \otimes v) = \sigma(a) \otimes 1 - \sigma(b) \otimes v.
\)

Note that in the strictly associative case all these facts were obtained in \([3]\) as an immediate consequence of the fact that the associative Clifford process is given (unlike the quasialgebra one) by a braided tensor product of algebras.

## 7 Universal properties

We have seen in Proposition \((3.2)\) a relation between the algebra \(A \# H\) and the quasialgebra \(A \circ H_0\). We analyze now the relation between the universal property of \(A \# H\) and the universal property of \(A \circ H_0\) regarded as a twisted tensor product of algebras in \(H \mathcal{M}\).

### Proposition 7.1

Let \(H\) be a quasi-Hopf algebra, \(A\) a left \(H\)-module algebra, \(B\) an associative algebra, \(v : H \rightarrow B\) an algebra map and \(u : A \rightarrow B^\circ\) a morphism of left \(H\)-module algebras.

Then we have

\[
\mu_{B^\circ} \circ (u \otimes v) \circ R = \mu_{B^\circ} \circ (v \circ u),
\]

\[(7.1)\]

where \(R\) is given by \((2.2)\), that is, the input data for the universal property of \(A \# H\) is also an input data for the universal property of the twisted tensor product \(A \otimes_R H_0\) in the category \(H \mathcal{M}\). Moreover, if we denote by \(u \# v : A \# H \rightarrow B\) the algebra map provided by the universal property of \(A \# H\) and with \(v : A \circ H_0 \rightarrow B^\circ\) the morphism of left \(H\)-module algebras provided by the universal property of \(A \circ H_0 = A \otimes_R H_0\), then we have \((u \# v) \circ \Pi = w\), where \(\Pi\) is the isomorphism from Proposition \((3.2)\).

### Proof.

We denote by \(f^{-1} = g^1 \otimes g^2 = G^1 \otimes G^2\) two copies of the element \(f\) given by \((1.14)\).

We first check \((7.1)\) :

\[
(\mu_{B^\circ} \circ (u \otimes v) \circ R)(h \otimes a)\]

\[
= v(T^1)u(Z^1X^1h_1g^1S(Y^2X^2)\alpha Y^3X^3 \cdot a)v(S(x^1T^2)\alpha x^2T^3)\]

\[
v(Z^2X^1h_2g^2S(Z^3Y^1X^3))v(S(x^3T^2))\]

\[
= v(T^1Z^1h_1(1,1)g^1S(Y^2X^2)\alpha Y^3X^3)u(a)\]

\[
v(S(x^1T^2Z^2X^1h_2g^2S(Y^2X^2)\alpha Y^3X^3)\alpha x^2T^3Z^2X^1h_2g^2S(x^3T^2Z^3Y^1X^3))\]

\[
(1.15)\]

\[
= v(T^1Z^1h_1(1,1)g^1S(W^2t_2Y^2X^2)\alpha W^3t^2Y^3X^3)u(a)\]

\[
v(S(x^1T^2Z^2h_2g^2S(W^1t_1Y^1X^1)\alpha Y^3X^3)\alpha x^2T^3Z^2X^1h_2g^2S(x^3T^2Z^3Y^1X^3))\]

\[
(1.3)\]

\[
= v(T^1h_1Z^1g^1S(W^2t_2Y^2X^2)\alpha W^3t^2Y^3X^3)u(a)\]
of the quasi-Hopf smash product

\[ H \] quasi-Hopf algebra by both the universal property of the diagonal crossed product

property of the generalized diagonal crossed product

not a twisted tensor product, neither of associative algebras nor of quasialgebras.

twisted tensor product between the associative algebras

\[ u \]

Now we check the relation (1.17)

\[ \text{(1.17)} \]

Our strategy to arrive at such a formulation is the following. We obtain first a universal

Let \( A \) be an ordinary Hopf algebra and \( H \)

\[ \text{from } \frac{\mathcal{A}^\circ}{\approx} \text{as a} \]

Now we check the relation \((u\#v) \circ \Pi = \mu_B^v \circ (u \otimes v)\). We compute:

\[ ((u\#v) \circ \Pi)(a \otimes h) = (u\#v)(x^1 \cdot a \# x^2 h S(x^3)) \]

\[ \text{(1.25)} \]

\[ v(X^1)u(x^1 \cdot a)v(S(X^2)\alpha X^3 x^2 h S(x^3)) = v(X^1)u(a)v(S(X^2 x^3)\alpha X^3 x^2 h S(x^3)) \]

\[ \text{(1.3)} \]

\[ v(X^1)u(a)v(S(x^1 X^2)\alpha x^2 X^3 h S(x^3 X^2)) = (\mu_B^v \circ (u \otimes v))(a \otimes h), \]

finishing the proof.

\[ \square \]

Let \( H \) be an ordinary Hopf algebra and \( A \) a left \( H \)-module algebra. We can regard \( A\#H \) as a twisted tensor product between the associative algebras \( A \) and \( H \), and thus it has a corresponding universal property, which is easily seen to be equivalent to the usual universal property of \( A\#H \).

We would like to have a quasi-Hopf analogue of this situation, that is, a universal property of the quasi-Hopf smash product \( A\#H \) resembling the universal property of a twisted tensor product. It is not clear a priori how to formulate such a universal property, since \( A\#H \) itself is not a twisted tensor product, neither of associative algebras nor quasialgebras.

Our strategy to arrive at such a formulation is the following. We obtain first a universal property of the generalized diagonal crossed product \( A \bowtie \mathcal{A} \), which generalizes (and is inspired by) both the universal property of the diagonal crossed product \( H^* \bowtie \mathcal{A} \) over a finite dimensional quasi-Hopf algebra \( H \) from [17] and the universal property of a generalized diagonal crossed
product over a Hopf algebra, regarded as a twisted tensor product of associative algebras. Then, since $A \# H$ is a particular case of $A \bowtie A$, we will obtain the desired universal property of $A \# H$.

If $H$ is a quasi-Hopf algebra, $A$ an $H$-bimodule algebra and $\mathbb{A}$ an $H$-bicomodule algebra, we denote by $\Gamma: A \to A \bowtie \mathbb{A}$ and $j: \mathbb{A} \to A \bowtie \mathbb{A}$ the linear maps given by

$$\Gamma(\varphi) = (\tilde{p}_\rho^1)_{[-1]} \cdot \varphi \cdot S^{-1}(\tilde{p}_{\rho}^2) \bowtie (\tilde{p}_{\rho}^1)_{[0]}, \quad j(u) = 1_A \bowtie u,$$

for all $\varphi \in A$, $u \in \mathbb{A}$, where $\tilde{p}_\rho^1 \otimes \tilde{p}_{\rho}^2$ is given by (1.37). By (1.3), the map $\Gamma$ has the property that $A \bowtie \mathbb{A}$ is generated as algebra by $A$ and $\Gamma(A)$, while $j$ is obviously an algebra map.

**Proposition 7.2** Let $H$ be a quasi-Hopf algebra, $A$ an $H$-bimodule algebra, $\mathbb{A}$ an $H$-bicomodule algebra, $B$ an associative algebra, $\gamma: \mathbb{A} \to B$ an algebra map and $\upsilon: A \to B$ a linear map such that the following conditions are satisfied:

$$\gamma(u_{>0})v(\varphi \cdot u_{<1}) = v(u_{1)} \cdot \varphi)(u_{[0]}),$$

$$v(\varphi\varphi') = \gamma(\tilde{X}_\rho^1)\upsilon(\theta^1 \tilde{X}_\rho^1 \cdot \varphi \cdot \tilde{X}_\rho^2)\gamma(\theta^2)\upsilon(\tilde{X}_\rho^1 \cdot \varphi \cdot \tilde{X}_\rho^3)\gamma(\tilde{X}_\rho^2),$$

$$v(1_A) = 1_B,$$

for all $\varphi, \varphi' \in A$ and $u \in \mathbb{A}$. Then there exists a unique algebra map $w: A \bowtie \mathbb{A} \to B$ such that $w \circ \Gamma = \upsilon$ and $w \circ j = \gamma$. Moreover, $w$ is given by the formula

$$w(\varphi \bowtie u) = \gamma(\tilde{q}_\rho^1)\upsilon(\varphi \cdot \tilde{q}_\rho^2)\gamma(u),$$

for all $\varphi \in A$ and $u \in \mathbb{A}$, where $\tilde{q}_\rho = \tilde{q}_\rho^1 \otimes \tilde{q}_\rho^2$ is the element given by the formula (1.37).

**Proof.** We first prove the uniqueness of $w$. By (1.3), the map $\Gamma$ satisfies the relation

$$\varphi \bowtie u = (1_A \bowtie \tilde{q}_\rho^1)\Gamma(\varphi \cdot \tilde{q}_\rho^2)(1_A \bowtie u),$$

for all $\varphi \in A$ and $u \in \mathbb{A}$, hence we can write

$$w(\varphi \bowtie u) = w(j(\tilde{q}_\rho^1))\Gamma(\varphi \cdot \tilde{q}_\rho^2)j(u)) = w(j(\tilde{q}_\rho^1))w(\Gamma(\varphi \cdot \tilde{q}_\rho^2))w(j(u)) = \gamma(\tilde{q}_\rho^1)\upsilon(\varphi \cdot \tilde{q}_\rho^2)\gamma(u),$$

showing that $w$ is indeed unique. We now prove the existence part. Define $w$ by formula (7.5); it is obvious that $w$ is unital and satisfies $w \circ j = \gamma$. We check now that $w \circ \Gamma = \upsilon$:

$$(w \circ \Gamma)(\varphi) = w((\tilde{p}_\rho^1)[_{[-1]}] \cdot \varphi \cdot S^{-1}(\tilde{p}_{\rho}^2) \bowtie (\tilde{p}_{\rho}^1)_{[0]}) = \gamma(\tilde{q}_\rho^1)\upsilon((\tilde{p}_\rho^1)_{[-1]} \cdot \varphi \cdot S^{-1}(\tilde{p}_{\rho}^2)\tilde{q}_{\rho}^2)\gamma((\tilde{p}_{\rho}^1)_{[0]}) = \gamma(\tilde{q}_\rho^1)\gamma((\tilde{p}_\rho^1)_{<0>})\upsilon(\varphi \cdot S^{-1}(\tilde{p}_{\rho}^2)\tilde{q}_{\rho}^2(\tilde{p}_{\rho})_{<1>}) = (w \circ \Gamma)(\varphi).$$

Thus, the only thing left to prove is that $w$ is multiplicative. We denote by $\tilde{Q}_\rho^1 \otimes \tilde{Q}_\rho^2$ another copy of the element $\tilde{q}_\rho$, and we record the obvious relation

$$\tilde{Q}_\rho^1\bar{x}_\rho^1 \otimes S^{-1}(\bar{x}_\rho^3)\tilde{Q}_\rho^2\bar{x}_\rho^2 = 1 \otimes S^{-1}(\alpha).$$

Now we compute:
This universal property of \( A \cong \mathbb{A} \) looks quite different from the universal property of \( H^* \cong \mathbb{A} \) given in [17], Theorem II, which is formulated in terms of so-called *normal coherent intertwiners*, but for finite dimensional \( H \) and \( \mathcal{A} = H^* \) the two universal properties are actually equivalent. Indeed, if \( T = T^* \otimes T^2 \in H \otimes B \) is a normal coherent intertwiner as in [17], then we can define the map \( v : H^* \to B \), \( v(\varphi) = \varphi(T^1)T^2 \), and the three conditions (7.2), (7.3) and (7.4) for \( v \) follow from the three conditions in [17] defining a normal coherent intertwiner.
We intend now to prove that the algebra isomorphism \( \nu : A \rtimes A \rightarrow A \rtimes A \) given by (7.60) may be naturally reobtained by using Proposition 7.2. We define the map

\[
\lambda : A \rightarrow A \rtimes A, \quad \lambda(\varphi) = \theta^1 \cdot \varphi \cdot \theta^3 \rtimes \varphi^2.
\] (7.7)

We claim that the conditions in Proposition 7.2 are satisfied for \( B = A \rtimes A, \gamma = j \) and \( v = \lambda \). Indeed, condition (7.2) follows easily from (1.45) and condition (7.4) is trivial, so we only have to check (7.3). We denote by \( \theta^1 \rtimes \theta^2 \rtimes \theta^3 \) one more copy of \( \Phi_{\lambda,\rho} \) and we compute:

\[
\begin{align*}
j(\tilde{X}_1^\lambda)\lambda(\theta^1 \tilde{X}_1^\lambda \cdot \varphi \cdot \tilde{X}_2^\rho)j(\theta^2)\lambda(\theta^3 \tilde{X}_3^\rho \cdot \varphi^2 \cdot \tilde{X}_3^\rho)j(\tilde{X}_3^\rho) &= \left((\tilde{X}_1^\lambda)[-1]\theta^1 \tilde{X}_1^\lambda \cdot \varphi \cdot \tilde{X}_2^\rho \theta^2_{\varphi^2} \theta^3_{\varphi^3} \th_0(\tilde{X}_1^\lambda) \rtimes \theta^2_{\varphi^2} \theta^3_{\varphi^3} \rtimes \varphi^2 \cdot \tilde{X}_3^\rho \theta^3 \tilde{X}_3^\rho \theta^3 \tilde{X}_3^\rho)j(\theta^2)\right) \lambda(\tilde{X}_3^\rho) \\
&= (\theta^1 \tilde{X}_1^\lambda \cdot \varphi \cdot \theta^1 \tilde{X}_2^\rho \theta^2_{\varphi^2} \theta^3 \tilde{X}_3^\rho \theta^3 \tilde{X}_3^\rho)j(\theta^2) \lambda(\tilde{X}_3^\rho) \\
&= (\theta^1 \tilde{X}_1^\lambda \cdot \varphi \cdot \theta^3 \tilde{X}_3^\rho \theta^3 \tilde{X}_3^\rho)j(\theta^2) \lambda(\tilde{X}_3^\rho) \\
&= \lambda(\theta^3 \varphi^2 \cdot \tilde{X}_3^\rho \theta^3 \tilde{X}_3^\rho)j(\theta^2) \lambda(\tilde{X}_3^\rho) \\
&= \lambda(\varphi^2).
\end{align*}
\]

finishing the proof of (7.3). We can thus apply Proposition 7.2 obtaining an algebra map \( w : A \rtimes A \rightarrow A \rtimes A \) satisfying \( w \circ \Gamma = \Lambda \) and \( w \circ j = j \), and given by the formula

\[
w(\varphi \rtimes u) = j(\tilde{q}_\rho^1)\lambda(\varphi \cdot \tilde{q}_\rho^2)j(u) = (\tilde{q}_\rho^1)[-1] \theta^1 \cdot \varphi \cdot \tilde{q}_\rho^2 \theta^3 u_{\varphi^2} \rtimes (\tilde{q}_\rho^2)u_{\varphi^2}.\]

On the other hand, by using (1.47) and (1.48), one can easily check the following relation:

\[
(\tilde{q}_\rho^1)[-1] \theta^1 \cdot (\tilde{q}_\rho^2)u_{\varphi^2} \rtimes (\tilde{q}_\rho^2)u_{\varphi^2} \theta^3 = \Theta^1 \rtimes \tilde{q}_\rho^2 \Theta^2_{\varphi^2} \rtimes S^{-1}(\Theta^3 \tilde{q}_\rho^2 \Theta^3_{\varphi^2}),
\]

which implies that the map \( w \) above coincides with the algebra isomorphism \( \nu \) given by (7.60).

As a consequence of Proposition 7.2, we immediately obtain the following new kind of universal property for the quasi-Hopf smash product:

**Proposition 7.4** Let \( H \) be a quasi-Hopf algebra and \( A \) a left \( H \)-module algebra. Denote by \( i : A \rightarrow A \# H, i(a) = a \# 1 \) and \( j : H \rightarrow A \# H, j(h) = 1 \# h \). Let \( B \) be an associative algebra, \( \gamma : H \rightarrow B \) an algebra map and \( v : A \rightarrow B \) a linear map satisfying the following conditions:

\[
\begin{align*}
\gamma(h) v(a) &= v(h_1 \cdot a) \gamma(h_2), \\
v(aa') &= v(X^1 \cdot a) v(X^2 \cdot a') \gamma(X^3), \\
v(1_A) &= 1_B,
\end{align*}
\]

for \( a, a' \in A, h \in H \). Then there exists a unique algebra map \( w : A \# H \rightarrow B \) such that \( w \circ i = v \) and \( w \circ j = \gamma \). Moreover, \( w \) is given by the formula \( w(a \# h) = v(a) \gamma(h) \), for \( a \in A, h \in H \).
The relation between the two universal properties of $A\#H$ is the following: Proposition 7.4 implies the usual universal property (unlike the case of ordinary Hopf algebras, the converse does not seem to hold). Indeed, if $B$ is an associative algebra, $\gamma : H \to B$ an algebra map and $u : A \to B^*$ a morphism of left $H$-module algebras, define the map

$$v : A \to B, \quad v(a) = \gamma(q^1)u(a)\gamma(S(q^2)),$$

where $q_R = q^1 \otimes q^2$ is given by (1.44). Then one can check that the maps $\gamma$ and $v$ satisfy the hypotheses of Proposition 7.4 and obviously the unique algebra map $w : A\#H \to B$ provided by Proposition 7.4 coincides with the map $u\#\gamma$ given by (1.25).

Proposition 7.5 Let $H$ be a quasi-Hopf algebra, $A$ a left $H$-module algebra and $B$ a right $H$-module algebra. Denote by $i_A$, $i_B$, $j$ the standard inclusions of $A$, $B$, $H$ respectively into $A\#H\#B$. Let $X$ be an associative algebra, $\gamma : H \to X$ an algebra map and $v_A : A \to X$, $v_B : B \to X$ two linear maps satisfying the following conditions:

$$\gamma(h)v_A(a) = v_A(h_1 \cdot a)\gamma(h_2), \quad v_A(aa') = v_A(X^1 \cdot a)v_A(X^2 \cdot a')\gamma(X^3),$$

$$v_B(b)\gamma(h) = \gamma(h_1)v_B(b \cdot h_2), \quad v_B(bb') = \gamma(X^1)v_B(b \cdot X^2)v_B(b' \cdot X^3),$$

$$v_A(1_A) = 1_X = v_B(1_B), \quad v_B(b)v_A(a) = v_A(x^1 \cdot a)\gamma(x^2)v_B(b \cdot x^3),$$

for all $a, a' \in A$, $b, b' \in B$, $h \in H$. Then there exists a unique algebra map $w : A\#H\#B \to X$ such that $w \circ i_A = v_A$, $w \circ i_B = v_B$ and $w \circ j = \gamma$. Moreover, $w$ is given by the formula $w(a\#h\#b) = v_A(a)\gamma(h)v_B(b)$, for all $a \in A$, $h \in H$, $b \in B$.

For instance, the algebra isomorphism $A\#H\#B \simeq (A \otimes B)\#H$ from [28] may be easily reobtained by using this universal property.

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