Shatter functions with polynomial growth rates

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January 25, 2017

Abstract
We study how a single value of the shatter function of a set system restricts its asymptotic growth. Along the way, we refute a conjecture of Bondy and Hajnal which generalizes Sauer’s Lemma.

1 Introduction
A standard tool in combinatorial and computational geometry is the shatter function $f_F$ of a (geometric) set system $F$. By set system we mean a family of subsets of a ground set $X$. The trace of a set system $F$ on a subset $Y \subset X$ is defined as

$$F|_Y \overset{\text{def}}{=} \{ e \cap Y : e \in F \}$$

and the shatter function of $F$ is

$$f_F(m) \overset{\text{def}}{=} \max \{|F|_Y| : Y \subseteq X, |Y| = m \}$$

where $|S|$ denotes the cardinality of a set $S$. The survey of Matoušek [Mat98] details several geometric and algorithmic applications of shatter functions. The asymptotic growth rate of a shatter function is often its most important feature.

New results. Let $t_k(m)$ denote the largest integer such that every set system $F$ with $f_F(m) \leq t_k(m)$ satisfies $f_F(n) = O(n^k)$. We prove the following bounds.

**Theorem 1.** For any integers $m, k \geq 1$,

$$(2^{k+1} - k - 1)m - 2^{4k} < t_k(m) \leq (2^{k+1} - k - 1)m + 2^{k+1} - k - 2.$$

We also obtain an analogous result for non-integral values of $k$. The inequalities are more cumbersome though, see Corollary 3 and Lemma 4 for the lower and upper bounds respectively.

∗Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA. Supported in part by Sloan Research Fellowship and by U.S. taxpayers through NSF grant DMS-1301548 and through NSF CAREER grant DMS-1555149.

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We establish the upper bound by a probabilistic construction (Section 2). Interestingly, this upper bound already refutes a conjecture of Bondy and Hajnal [Bon72], see also [FP94, Problem 3.3], regarding a generalization of Sauer’s Lemma: they conjectured that if \( f_x(m) \leq g_k(m) \) then \( f_x(n) \leq g_k(n) \) for any large enough\(^1\) \( n \), where
\[
g_k(n) = 1 + n + \binom{n}{2} + \ldots + \binom{n}{k}.
\]
The case \( k = m - 1 \) is Sauer’s Lemma and the conjecture was also proven for \((k, m) = (2, 4)\) [BR95, Theorem 5]. If it were true, the Bondy–Hajnal conjecture would have implied that \( t_k(m) \) grows at least as fast as \( \Omega(m^k) \), which is what our upper bound prevents.

We obtain the lower bound by analyzing the density of certain patterns in simplicial complexes (Section 3). This builds on the argument of Bukh and Conlon [BC15] to bound the number of edges of graphs avoiding certain subgraphs. Theorem 2 below specializes to the lower bound in Theorem 1 for \( s = 2^{k+1} - k - 1 \).

**Theorem 2.** Let \( s \geq 2 \) be a rational number, let \( q \) be its denominator and let \( t = \lfloor \log_2 s \rfloor \). Let \( m \geq 1 \) be an integer. For any set system \( F \),
\[
f_x(m) \leq sm - 3qs^2 \log_2 s \quad \Rightarrow \quad \forall n \geq m, \quad f_x(n) \leq 2^{t+2}m^{2t+2}n^{t+1-(2^{t+1}-t-2)/(s-1)}.
\]

As real numbers can be approximated arbitrarily well by rational numbers, it is easy to extend the preceding theorem to irrational \( s \).

**Corollary 3.** Let \( s \geq 2 \) be a real number, and let \( t = \lfloor \log_2 s \rfloor \). Let \( m \geq s^3 \) be an integer. For any set system \( F \),
\[
f_x(m) < sm - 10\sqrt{m} \cdot s \sqrt{\log_2 s} \quad \Rightarrow \quad \forall n \geq m, \quad f_x(n) \leq 3m^{2t}n^{t+1-(2^{t+1}-t-2)/(s-1)}.
\]

**Proof.** Let \( q = \lfloor (1/s) \sqrt{m/\log_2 s} \rfloor \). Let \( s' \) be the largest rational number of denominator \( q \) such that \( s' \leq s \). Since \( q \leq (2/s)\sqrt{m/\log_2 s} \), we have \( 3q(s')^2 \log_2 s' \leq (6/s)\sqrt{m/\log_2 s} \cdot s^2 \log_2 s = 6\sqrt{m} \cdot s \sqrt{\log_2 s} \). Also since \( s < s' + 1/q \), we have \( sm \leq s'm + \sqrt{m} \cdot s \sqrt{\log_2 s} \). Hence \( sm - 10\sqrt{m} \cdot s \sqrt{\log_2 s} \leq s'm - 3q(s')^2 \log_2 s' \). \( \square \)

**Related work.** The only previous lower bound is due to Cheong et al. [CGN13, Theorem 1], who proved that \( t_k(m) \geq 2^km - (k-1)2^k - 1 \) by adapting the inductive proof of Sauer’s lemma. The only upper bound that we are aware of is the easy \( t_k(m) = O(m^k/k^k) \), for instance we may split \([n]\) into \( k \) almost equal parts, and let \( F \) consist of those \( k \)-sets that contain one vertex from each part.

The argument of Bukh and Conlon was extended from graphs to hypergraphs by Fitch [Fit16]. Both his and our work use generalization of balanced rooted trees from the work of Bukh and Conlon. There are technical differences, though. Fitch works with uniform hypergraphs, whereas we work with simplicial complexes, which results in a slightly different notion of density. Our construction (Proposition 8) uses a different idea from his in [Fit16, Lemma 1].

## 2 Random simplicial complexes

Recall that a simplicial complex is a set system closed under taking subsets.\(^2\) In Lemma 4 we present a construction of random simplicial complexes that implies the upper bound of Theorem 1.

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\(^1\)Here “large enough” depends solely on \( m \) and \( k \), and not on the set system \( F \). The original statement of the conjecture [Bon72] was without this precaution. According to Füredi and Pach [FP94, Problem 3.3], the necessity of allowing exceptional values is due to Frankl. Bollobás and Radcliffe [BR95, Theorem 11] also gave a probabilistic construction showing that \( n_0 \) must be larger than \( 2k \).

\(^2\)This is usually called an abstract simplicial complex but since we consider no embedded simplicial complex in this paper, we do not feel the need to emphasize the distinction.
Lemma 4. For any real number $s \geq 2$ and for each integer $m \geq 1$, for $n$ arbitrarily large there exists a set system $F$ on $n$ vertices, with $f_F(m) \leq sm + (s - 1)$ and $f_F(n) = \Omega \left( n^{t+1-(2^{s+1} - t - 2)/(s-1)} \right)$, where $t = \lfloor \log_2 s \rfloor$.

Proof. Fix $m$. For any $n$ large enough, we build a random $t$-dimensional simplicial complex $C_n$ on $n$ vertices by examining each subset of up to $t + 1$ vertices in the order of increasing size. For each subset $I$, if all $J \subseteq I$ form faces of our complex, we turn $I$ into a face with probability $p = n^{-1/(s-1)}$. All choices are independent, and the complex is initialized with all $n$ vertices.

Adding a $t$-dimensional face to $C_n$ requires to add each of its $2^{t+1} - t - 3$ proper faces of dimension 1 or more, plus the $t$-face itself. The expected number of faces of dimension $t$ of $C_n$ is thus

$$\binom{n}{t+1}p^{2^{t+1} - t - 2}.$$ 

Let $g(n)$ denote the expected number of faces of $C_n$. Note that since $p = n^{-1/(s-1)}$, we have $g(n) = \Omega \left( n^{t+1-(2^{s+1} - t - 2)/(s-1)} \right)$.

Let $z = (s - 1)(m + 1)$. Call an $m$-element set “bad” if the set contains at least $z$ faces of dimension 1 or more. Since there are at most $2^{2m}$ complexes on any given set of $m$ vertices, the expected number of bad $m$-sets is at most

$$2^{2m} \binom{n}{m} p^z = O(n^m n^{-z/(s-1)}) = O(1/n).$$ 

Let $C'$ be the complex obtained from $C_n$ by removing vertices of all bad $m$-sets. Accounting for the traces of size 0 and 1, we have

$$f_{C'}(m) < z + m + 1 = sm + s.$$ 

As each vertex belongs to at most $n^t$ faces of $C_n$, the expected number of faces in $C'$ is at least

$$g(n) - O(n^{t-1}) \geq \frac{1}{2} g(n).$$

So there exists a complex on at most $n^t$ vertices with at least $\frac{1}{2} g(n)$ faces and $f_{C'}(m) \leq sm + (s - 1)$. We can ensure that the complex has exactly $n$ vertices by adding dummy vertices if necessary.

For any $\varepsilon > 0$, Lemma 4 with $s = 2^{k+1} - k - 1 + \varepsilon$ shows that

$$t_k(m) \leq (2^{k+1} - k - 1 + \varepsilon)m + 2^{k+1} - k - 2 + \varepsilon.$$ 

Taking $\varepsilon \to 0$, the upper bound of Theorem 1 follows.

Remark. For most geometric set systems the bound in Sauer’s lemma is not sharp. This includes the family of halfspaces in $\mathbb{R}^d$. In fact, for this family no shatter condition implies the correct bound.

To see this we may make probabilistic construction similar to that of Lemma 4. Start with the complete $(d-1)$-dimensional skeleton of the $(n-1)$-dimensional simplex, add every $d$-simplex randomly and independently, each with probability $p$ with $p = \omega(1/n)$ and $p = o(\log n/n)$, then delete every $d$-simplex supported on a $m$-element subset of vertices that spans at least $m - d + 1$ $d$-simplices. With positive probability, the resulting random simplicial complex $C$ satisfies

$$f_C(m) \leq 1 + m + \ldots + \binom{m}{d} + m - d \quad \text{and} \quad \mathbb{E}[f_C(n)] = \omega(n^d).$$

Considering points on the moment curve shows that the set system of halfspaces in $\mathbb{R}^d$ violates this shatter condition for every $m$. 

3
3 Proof of Theorem 2

We first remark that in proving upper bounds on \( f_F(n) \) we may restrict ourselves to simplicial complexes, since any set system can be “compressed” without changing its number of sets nor increasing its shatter function.

**Lemma 5** (Alon [Alo83] and Frankl [Fra83]). For any finite set system \( F \) there exists an abstract simplicial complex \( C \) with \( |C| = |F| \) and \( f_C \leq f_F \).

We write \( V(C) \) for the set of vertices of a simplicial complex \( C \). Two simplices \( \sigma, \sigma' \in C \) are nonadjacent in \( C \), if they are vertex disjoint, and there is no edge intersecting both \( \sigma \) and \( \sigma' \). A set of pairwise nonadjacent vertices is called an independent set. For a complex \( C \), the degree of a \((d-1)\)-simplex \( \sigma \in C \) is the number of \( d \)-simplices \( \sigma \) is contained in. We denote by \( \delta_d(C) \) the minimum degree of any \((d-1)\)-simplex in \( C \). We define the density of a subset \( S \subseteq V(C) \) of vertices of a simplicial complex \( C \) to be \( \text{dens}_C(S) = \frac{e(S)}{|S|} \), where \( e(S) \) is the number of non-empty simplices in \( C \) with at least one vertex in \( S \).

3.1 Balanced rooted \( d \)-trees and shatter functions

A \( d \)-tree is defined inductively. First, a \( d \)-simplex is a \( d \)-tree. If \( T \) is a \( d \)-tree, and \( \sigma \in T \) is a \((d-1)\)-simplex, then the complex \( T' \) obtained by gluing to \( T \) a \( d \)-simplex formed by \( \sigma \) and a new vertex is also a \( d \)-tree. We say that \( T' \) is obtained by attaching a vertex to \( \sigma \).

![Figure 1: Two examples of 2-trees. The one on the right is obtained by attaching to the one on the left a vertex \( v \) to the edge \( \sigma \) (in thick blue).](image)

A rooted \( d \)-tree \((T, \rho, R)\) consists of a \( d \)-tree \( T \) together with a distinguished \((d-1)\)-simplex \( \rho \in T \) and an independent set \( R \subset V(T) \) such that \( \rho \) is nonadjacent to each of the vertices in \( R \). We call \( R \) vertex roots and \( \rho \) the simplex root of \( T \). The rest of the vertices we call unrooted.

![A rooted 2-tree with three root vertices](image)

The min-density \( \text{min-dens}(T) \) of a rooted \( d \)-tree \((T, \rho, R)\) is the minimum of \( \text{dens}_T(S) \) over all non-empty sets \( S \subseteq V(T) \setminus (\rho \cup R) \) of unrooted vertices. If the minimum is attained by \( V(T) \setminus (\rho \cup R) \), the set of all unrooted vertices, then we call the tree balanced. We use balanced rooted trees to bound from below the shatter function of simplicial complexes as follows.

**Lemma 6.** Let \( d, m, f, r \geq 1 \) be integers. Suppose \((T, \rho, R)\) is a balanced \( d \)-tree with \( f \) facets and \( r \) vertex roots. Every simplicial complex \( C \) on \( n \) vertices with \( \delta_d(C) \geq 2m^{1+d/f}n^{r/f} \) contains \( m \) vertices that span at least \( \text{min-dens}(T)(m - f - d) \) simplices.
Proof. Let \((T, \rho, R)\) be the \(d\)-tree in question, and let \(\mathcal{C}\) be a simplicial complex on \(n\) vertices with \(\delta_d(\mathcal{C}) \geq m^{1+d/f} n^{r/f}\). We assume that \(m \geq f + d\), for otherwise the result is trivially true, and argue that some \(m\)-element set \(V \subset \mathcal{V}(\mathcal{C})\) spans at least \(\min-dens(T)(m - f - d) + 2d + r - 1\) simplices.

Fix an arbitrary \((d - 1)\)-simplex \(\sigma\) of \(\mathcal{C}\). Consider copies of \(T\) such that \(\rho\) is mapped to \(\sigma\) and different \(d\)-simplices of \(T\) are mapped to different \(d\)-simplices of \(\mathcal{C}\). Each such copy can be obtained by embedding facets of \(T\) one-by-one starting with the facet containing the root. Since \(T\) has \(f\) facets, then there are at least \(\delta_d(\mathcal{C}) \delta_d(\mathcal{C}) - 1 \ldots (\delta_d(\mathcal{C}) - f + 1) \geq (\delta_d(\mathcal{C}) - f)^f\) copies of the tree \(T\) such that \(\rho\) is mapped to \(\sigma\). Since \(m \geq f\), it follows that \((\delta_d(\mathcal{C}) - f)^f \geq m^{f+d} n^r\). The pigeonhole principle ensures that some \(\ell \geq m^{f+d}\) of these copies have the same vertex roots; denote them by \(T_1, \ldots, T_\ell\).

Let \(V_i = V(T_1) \cup \cdots \cup V(T_i)\). As \(T\) is balanced, \(T_1\) has at least \(\min-dens(T)(|V_i| - r - d) + 2d + 1 + r\) simplices, and at least \(\min-dens(T)|V_i + 1| \setminus V_i\) of the simplices spanned by \(V_{i+1}\) use a vertex from \(V_i + 1 \setminus V_i\). Thus, by induction on \(i\), each \(V_i\) spans at least \(\min-dens(T)(|V_i| - r - d) + 2d + 1 + r\) simplices in \(\mathcal{C}\).

The set \(V_i\) contains at least \(\ell\) copies of \(T\) with prescribed roots, so \(m^{d+f} \leq \ell \leq \binom{|V_i|}{|V(T)|} = \binom{|V_i|}{d+f}\) and it follows that \(|V_i| \geq m\). Since \(|V_i + 1| \setminus V_i| \leq f - r\), there is \(j\) such that \(m \geq |V_j| \geq m - (f - r)\). Setting \(V = V_j\) we obtain the result.

A similar argument permits us to control the overlap of \(d\)-simplices.

Lemma 7. Suppose \(\mathcal{C}\) is a simplicial complex, and \(\rho\) is a \(d'\)-simplex in \(\mathcal{C}\) which is contained in \(N\) simplices of dimension \(d\). Then \(\mathcal{C}\) contains \(m\) vertices that span at least \(\min(N, \frac{2^{d+1} - 2^{d+1}}{d - d'})\) \(m\)-simplices.

Proof. If these \(N\) simplices are contained in some \(m\)-element set, then we are obviously done. Otherwise, we can find \(d\)-simplices \(\sigma_1, \ldots, \sigma_r\) whose union is of size between \(m - d\) and \(m\). Let \(V_i = \sigma_1 \cup \cdots \cup \sigma_i\).

The number of simplices contained in \(\sigma_{i+1}\) that are not contained in \(V_i\) is

\[
2^{d+1} - 2^{|V_i \cap \sigma_{i+1}|} \geq |V_{i+1} \setminus V_i| \frac{2^{d+1} - 2^{d+1}}{d - d'}.
\]

By induction on \(i\), it follows that the number of simplices spanned by \(V_i\) is at least \(|V_i| \frac{2^{d+1} - 2^{d+1}}{d - d'}\).

3.2 Construction of balanced \(d\)-trees of prescribed rational density

We now prove that balanced \(d\)-trees of every rational density exceeding \(2^d\) exist (Proposition 8). The case \(d = 1\) was previously handled in [BC15], and the following construction borrows some ideas from there.

Our construction starts with a simplicial complex \(T_0\) on the vertex set \([d(Q + 1)]\), whose facets are \(d\)-simplices of the form \(\{i, i + 1, \ldots, i + d\}\) for all \(i = 1, \ldots, dQ\). Alternatively, we can describe \(T_0\) as the complex consisting of all the sets \(\sigma \subset [d(Q + 1)]\) satisfying \(\max \sigma - \min \sigma \leq d\). For \(i = 0, 1, \ldots, Q\) we denote by \(\sigma_i\) the \((d - 1)\)-simplex of \(T_0\) defined by

\[
\sigma_i \coloneqq \{id + 1, id + 2, \ldots, (i + 1)d\}.
\]

Observe that \((T_0, \sigma_0, \emptyset)\) is a rooted \(d\)-tree, and that \(\sigma_1, \ldots, \sigma_Q\) form a partition of unrooted vertices of this tree. With a slight abuse of notation, we denote this rooted \(d\)-tree also by \(T_0\).

If \(r < Q\), we define \(T_r\) to be the rooted \(d\)-tree obtained by attaching to \(T_0\) a rooted vertex to each of the following \(r\) \((d - 1)\)-simplices

\[
\sigma_{[Q/r]}, \sigma_{[2Q/r]}, \ldots, \sigma_Q.
\]

If \(r \geq Q\), we define \(T_r\) recursively to be the rooted \(d\)-tree obtained from \(T_{r-Q}\) by attaching rooted vertices to each of \(\sigma_1, \ldots, \sigma_Q\).

Proposition 8. For every choice of integers \(d, Q \geq 1\), and \(r \geq 0\), \(T_r\) is a balanced \(d\)-tree with \(dQ + r\) facets and \(r\) rooted vertices of min-density \(2^d + \frac{r}{dQ}(2^d - 1)\).
Figure 2: The simplicial complex $T_0$ (top) and the rooted 2-trees $T_3$ (bottom-left) and $T_7$ (bottom-right) for $d = 2$ and $Q = 5$.

Before we prove Proposition 8, we first argue that the min-density of $T_r$ is attained on particularly nice sets of unrooted vertices.

**Lemma 9.** There exist $1 \leq i \leq j \leq Q$ such that $\text{min-dens}(T_r) = \text{dens}_{T_r}(\sigma_i \cup \sigma_{i+1} \cup \ldots \cup \sigma_j)$.

**Proof.** For a set $U$ of unrooted vertices, let the *neighborhood* of $U$ be the set of simplices of $T_r$ that contain at least one vertex from $U$. We denote it by $N(U)$. In particular, $\text{e}(U) = |N(U)|$. Let $S$ denote a set of unrooted vertices that minimizes $\text{dens}_{T_r}(S)$ and is of maximum size among such sets.

We first claim that $S$ is of the form $S = \sigma_i \cup \sigma_{i+1} \cup \ldots \cup \sigma_j$. Suppose, for the sake of contradiction, that $0 < |\sigma_i \cap S| < d$ for some $i$. Pick $a, b \in \sigma_i$ such that $a \in S$, $b \notin S$ and $|a - b| = 1$. By the optimality conditions on $S$, it follows that

$$\text{dens}_{T_r}(S) \leq \text{dens}_{T_r}(S \setminus \{a\}), \quad \text{and} \quad \text{dens}_{T_r}(S) < \text{dens}_{T_r}(S \cup \{b\}),$$

which is equivalent to

$$|N(S) \setminus N(S \setminus \{a\})| \leq \text{dens}_{T_r}(S), \quad \text{and} \quad |N(S \cup \{b\}) \setminus N(S)| > \text{dens}_{T_r}(S)$$

respectively. It follows that $|N(S) \setminus N(S \setminus \{a\})| < |N(S \cup \{b\}) \setminus N(S)|$. To reach a contradiction we now exhibit an injective map from $N(S \cup \{b\}) \setminus N(S)$ to $N(S) \setminus N(S \setminus \{a\})$.

Assume that $b = a + 1$, for the other case $b = a - 1$ is analogous, and consider the map

$$\phi(\sigma) \overset{\text{def}}{=} \begin{cases} (\sigma \cup \{a\}) \setminus \{a+1\} & \text{if } a + d + 1 \notin \sigma, \\ (\sigma \cup \{a\}) \setminus \{a + d + 1\} & \text{if } a + d + 1 \in \sigma. \end{cases}$$

If $\sigma \in T_r$, that is max $\sigma - \text{min} \sigma \leq d$, and $b = a + 1 \in \sigma$ then max $\sigma \leq a + d + 1$ and it follows that $\phi(\sigma) \in T_r$. Moreover, if $\sigma \cap S = \emptyset$ then $\phi(\sigma) \cap S = \{a\}$. This implies that $\phi$ maps $N(S \cup \{b\}) \setminus N(S)$ to $N(S) \setminus N(S \setminus \{a\})$; this map is easily seen to be injective. The existence of $\phi$ contradicts the optimality of $S$, and thus each $|S \cap \sigma_i|$ is $0$ or $d$.

We can now partition $S = S_1 \cup S_2 \cup \ldots \cup S_p'$ where each $S_i$ is a maximal union of consecutive $\sigma_j$’s. Since

$$e(S) = e(S_1 \cup \ldots \cup S_p') = e(S_1) + \cdots + e(S_p') \geq (|S_1| + \cdots + |S_p'|) \min \text{dens}_{T_r}(S_i) = |S| \min \text{dens}_{T_r}(S_i),$$

it follows that $\min_i \text{dens}_{T_r}(S_i) \leq \text{dens}_{T_r}(S) = \text{min-dens}(T_r)$. This completes the proof. \qed
Proof of Proposition 8. In view of Lemma 9, it remains to compute $\text{dens}_{T_r}(\sigma_i \cup \sigma_{i+1} \cup \ldots \cup \sigma_j)$ and show that it is minimal for $i = 1$ and $j = Q$. Let $S = \sigma_i \cup \sigma_{i+1} \cup \ldots \cup \sigma_j$. For $r \geq Q$, 
\[
\text{dens}_{T_r}(S) = \text{dens}_{T_{r-Q}}(S) + (2^d - 1)/d,
\]
so we focus on the cases $r < Q$.

We first express $\text{dens}_{T_r}(S)$ in terms of $L[i,j]$, where for $1 \leq i \leq j \leq Q$
\[
L[i,j] \overset{\text{def}}{=} \{|i,i+1,\ldots,j\} \cap \{[Q/r],[2Q/r],\ldots,Q\}.
\]
The computations are easiest when $j = Q$. In this case, whenever a simplex $\sigma \in T_0$ meets $S$, we necessarily have $\max \sigma \in S$. For each $x \in S$ there are exactly $2^d$ simplices $\sigma \in T_0$ such that $\max \sigma = x$. Therefore the number of simplices of $T_0$ that meet $S$ is $2^d |S|$. Adding the simplices contained in the facets counted by $L[i,j]$ we obtain
\[
\text{dens}_{T_r}(S) = \frac{2^d|S| + (2^d - 1)L[i,j]}{|S|} = 2^d + \frac{(2^d - 1)L[i,Q]}{d(Q - i + 1)} \quad \text{for } S = \sigma_i \cup \sigma_{i+1} \cup \ldots \cup \sigma_Q.
\]

When $j < Q$, the computation is similar except that we also need to count the simplices $\sigma$ of $T_0$ such that $\max \sigma \notin S$. Call such a simplex $\sigma$ dangling. If $\sigma$ is dangling, then $\min \sigma \in \sigma_j$. Furthermore, for each $\ell \in \{1,2,\ldots,d\}$ there are $2^d$ simplices $\sigma$ such that $\min \sigma = jd + \ell$ and exactly $2^d - 2^{d-\ell}$ of them are dangling. The total number of dangling simplices is thus
\[
\sum_{\ell=1}^d (2^d - 2^{d-\ell}) = d2^d - (2^d - 1) = (d - 1)2^d + 1,
\]
yielding
\[
\text{dens}_{T_r}(S) = 2^d + \frac{(2^d - 1)L[i,j] + (d - 1)2^d + 1}{d(j - i + 1)} \quad \text{for } S = \sigma_i \cup \sigma_{i+1} \cup \ldots \cup \sigma_j, \; j < Q.
\]

Next, note that if $i > 1$ and $L[i,j] = L[i-1,j]$, then $\text{dens}_{T_r}(S \cup \sigma_{i-1}) < \text{dens}_{T_r}(S)$. Similarly, if $j \neq Q$, and $L[i,j] = L[i,j+1]$, then $\text{dens}_{T_r}(S \cup \sigma_{j+1}) < \text{dens}_{T_r}(S)$. As we look for the minimum density, we may assume that
\[
i \in \{1, [Q/r] + 1, [2Q/r] + 1, \ldots, [(r-1)Q/r] + 1\}, \quad \text{and} \quad j \in \{[Q/r] - 1, [2Q/r] - 1, \ldots, Q - 1, Q\}.
\]
We can thus assume that $i = \lceil aQ/r \rceil + 1$ with $0 \leq a < r$ and that either $j = Q$ or $j = \lfloor bQ/r \rfloor - 1$ with $a < b \leq r$.

If $j = Q$ then
\[
\text{dens}_{T_r}(S) = 2^d + \frac{(2^d - 1)(r - a)}{d(Q - [aQ/r])} = 2^d + \frac{(2^d - 1)(r - a)}{d((r - a)Q/r)} \geq 2^d + \frac{r(2^d - 1)}{dQ},
\]
with equality if $a = 0$. If $j = \lfloor bQ/r \rfloor - 1$ then
\[
\text{dens}_{T_r}(S) = 2^d + \frac{(2^d - 1)(b - a - 1) + (d - 1)2^d + 1}{d(j - i + 1)}
\]
\[= \quad \frac{2^d + (2^d - 1)(b - a) + (d - 2)2^d + 2}{d(j - i + 1)}
\]
\[> \quad \frac{2^d + (2^d - 1)(b - a) + (d - 2)2^d + 2}{d(bQ/r - aQ/r) + d(j - i + 1)}
\]
\[= \quad \frac{2^d - 1}{dQ/r} + \frac{(d - 2)2^d + 2}{d(j - i + 1)}
\]
As this exceeds $2^d + \frac{2^d-1}{dq/r}$, we conclude that the min-density of $T_r$ is attained for $S = \sigma_1 \cup \sigma_2 \cup \ldots \cup \sigma_Q$ and is min-dens$(T_r) = 2^d + \frac{2^d-1}{dq/r}$.

### 3.3 Wrapping up

We can now prove Theorem 2. Let $s \geq 2$ be a rational number, let $q$ denote its denominator, and let $t = \lfloor \log_2 s \rfloor$. We want to prove that for any simplicial complex $C$, the resulting complex $\delta_2(C')$ contains more than $2^t+2m^{2t+2}n^{t+1-(2^{t+1}-t-2)/(s-1)}$ simplices on $n$ vertices. We will show that some $m$ of its vertices must span more than $sm - 3qs^2 \log_2 s$ simplices. We assume that $m \geq 3s \log_2 s$ as otherwise the statement holds trivially.

For each $0 \leq d \leq t$, let $t_d = (s - 2^d)/(s - 1)$. Let $s_d = 1 + t_1 + t_2 + \cdots + t_d$ and note that $s_d = d + 1 - (2^{d+1} - d - 2)/(s - 1)$. In particular, our assumption states that $C$ contains more than $2^t+2m^{2t+2}n^{s_d}$ simplices.

We note that $s_t = 1 + \frac{2^{d-q}-2}{s-1} > t - 1$.

#### Case 1.

We first consider the case where there is some $d$ with $1 \leq d \leq t$ such that $C$ contains more than $2^d m^{2d} n^{s_d}$ simplices of dimension $d$. Pick the smallest such $d$.

Now, delete from $C$ all $(d-1)$-simplices of degree less than $2m^2 n^{t_d}$, also removing any simplices that contain them. By minimality of $d$, so doing removes fewer than $2^{d-1} m^{2d-2} n^{s_d-1} \cdot 2m^2 n^{t_d}$ of the $d$-simplices. Hence, the resulting complex $C'$ contains at least one $d$-simplex, and satisfies $\delta_2(C') \geq m^{2d} n^{t_d}$.

Write the rational number $(t_d^{-1} - 1)/d = (2d - 1)/d(s - 2^d)$ in the form $Q/r$ with $\gcd(Q, r) = 1$. Let $T$ be a balanced rooted $d$-tree with parameters $Q$ and $r$ as given by Proposition 8. Note that min-dens $T = s$. By Lemma 6 there is set of $m$ vertices on $C'$ (and hence of $C$) that spans at least $s(m - d(Q + 1) - r)$ simplices. The requisite bound follows from $Q \leq 2^d q \leq sq$ and $r \leq dsq$, and from $d \leq t \leq \log_2 s$.

#### Case 2.

In the remaining case, the number of simplices of dimension up to $t$ is at most

$$\sum_{d=0}^{t} 2^d m^{2d} n^{s_d} \leq \sum_{d=0}^{t} 2^d m^{2t} n^{s_t} \leq 2^t+1 m^{2t} n^{s_t}.$$ 

so $C$ contains more than $2^t+1 m^{2t+2} n^{s_t}$ simplices of dimension greater than $t$.

If $C$ contains a $d$-simplex for some $d \geq 2 \log_2 m$, then any $m$-set containing this simplex contains at least $2^d \geq m^2 > sm$ simplices. So assume that $C$ is of dimension at most $2 \log_2 m$.

By the pigeonhole principle, there is a $d$ with $t + 1 \leq d \leq 2 \log_2 m - 1$ such that $C$ contains at least $2^t+1 m^{2t+2} n^{s_t}/(2 \log_2 m - t - 1) \geq 2^t+1 (m^{2t+2} / 2 \log_2 m) n^{s_t}$ simplices of dimension $d$.

Since $C$ contains at most $m^{2d} n^{s_d}$ simplices of dimension $t$, it follows by another application of the pigeonhole principle that there is a $t$-simplex $\rho$ that is contained in at least $2^t+1 m^2 / 2 \log_2 m$ simplices of dimension $d$. Lemma 7 then yields an $m$-element set spanning at least

$$\min(2^t+1 m^2 / 2 \log_2 m, \frac{2^t+1 - 2^{t+1}}{d-t} (m - d))$$

simplices. We have $2^t+1 m^2 / 2 \log_2 m \geq sm$. Also, $\frac{2^t+1 - 2^{t+1}}{d-t} (m - d) \geq 2^t+1 (m - t - 1) \geq sm - s \log_2 s$.

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