Finiteness Conditions for Light-Front Hamiltonians

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In the context of simple models, it is shown that demanding finiteness for physical masses with respect to a longitudinal cutoff, can be used to fix the ambiguity in the renormalization of fermions masses in the Hamiltonian light-front formulation. Difficulties that arise in applications of finiteness conditions to discrete light-cone quantization are discussed.

I. INTRODUCTION

Many advantages of the light-front (LF) formulation for bound state problems arise from the manifest boost invariance in the longitudinal direction \[\mathbb{R}^3\]. The price for this advantage is that other symmetries, such as parity or rotational invariance (for rotations around a transverse axis) are no longer manifest. From the technical point of view, the loss of manifest parity and full rotational invariance implies that LF Hamiltonians allow for a richer set of counter-terms in the renormalization procedure, i.e. in general LF Hamiltonians contain more parameters than the underlying Lagrangian.

Of course, even though parity and full rotational invariance are not manifest symmetries in the LF formulation, a consistent calculation should still give rise to physical observables which are consistent with these symmetries. In Ref. [4] this fact has been used to determine one of these additional parameters by imposing parity covariance on the vector form factor of mesons. While such a procedure is practical, it is nevertheless desirable to have alternative procedures available for determining these “additional” parameters in the Hamiltonian. In this paper, finiteness conditions are exploited to develop algorithms for determining seemingly independent parameters in LF Hamiltonians.

As a specific example, let us consider a Yukawa model in 1+1 dimensions

\[
L = \bar{\psi} \left( i \partial^+ - m - g \phi \right) \psi - \frac{1}{2} \phi \left( \Box + \lambda^2 \right) \phi. \tag{1.1}
\]

In order to simplify the analysis further, we will in the following consider the Yukawa model in a planar approximation (formally this can easily be achieved by introducing “color” degrees of freedom and by assuming an infinite number of “colors”. However, while a planar approximation will in the following always be implicitly used, explicit color degrees of freedom will not be shown in order to keep the notation simple.

The main difference between scalar and Dirac fields in the LF formulation is that not all components of the Dirac field are dynamical: multiplying the Dirac equation

\[
(i \partial - m - g \phi) \psi = 0 \tag{1.2}
\]

by \(\frac{1}{2} \gamma^+\) yields a constraint equation (i.e. an “equation of motion” without a time derivative)

\[
i \partial_- \psi^- = (m + g \phi) \gamma^+ \psi^+, \tag{1.3}
\]

where

\[
\psi_\pm \equiv \frac{1}{2} \gamma^\pm \gamma^\pm \psi. \tag{1.4}
\]

For the quantization procedure, it is convenient to eliminate \(\psi^-\) from the classical Lagrangian before imposing quantization conditions, yielding

\[
L = \sqrt{2} \psi_+^\dagger \partial_+ \psi_+ - \frac{1}{2} \phi \left( \Box + \lambda^2 \right) \phi - \psi_+^\dagger \frac{m^2}{\sqrt{2} \imath \partial_-} \psi_+ \tag{1.5}
\]

\[
- \psi_+^\dagger \left( \phi \frac{g m}{\sqrt{2} \imath \partial_-} + \frac{g m}{\sqrt{2} \imath \partial_-} \phi \right) \psi_+ - \psi_+^\dagger \phi \frac{g^2}{\sqrt{2} \imath \partial_-} \psi_+.
\]

The rest of the quantization procedure very much resembles the procedure for self-interacting scalar fields. In particular, we must be careful about generalized tadpoles, which might cause additional counter-terms in the LF Hamiltonian. In the Yukawa model one usually (i.e. in a covariant formulation) does not think about tadpoles. However, after eliminating \(\psi_-\), one is left with a four-point interaction in the Lagrangian, which does give rise to time-ordered diagrams that resemble tadpole diagrams. In fact, the four-point interaction gives rise to diagrams where a fermion emits a boson, which may or may not self-interact, and then re-absorb the boson at the same LF-time. \[\Box\] such interactions cannot be generated by a LF Hamiltonian, i.e. the LF formalism generally defines such tadpoles to be zero. An exception are the so-called self-induced inertias, which arise from normal ordering the LF Hamiltonian. These terms, which are \(O(g^2)\), are usually kept.

\[\Box\] There are also tadpoles, where the fermions get contracted. But those only give rise to an additional boson mass counter-term, but not to the non-covariant fermion mass counter-term that is investigated here.
II. PERTURBATIVE COUNTER-TERM
ANALYSIS

At tree level, i.e. at order $g^0$, the kinetic mass and the vertex mass have to be the same. In order to see this, let us consider the two $O(g^2)$ Compton scattering diagrams in Fig. 1. For simplicity we consider only forward scattering and we consider only diagrams which are singular.

The amplitude with an on shell fermion intermediate state diverges as the $p^+\rightarrow 0$ momentum of its intermediate fermion line goes to zero

$$T_o = \frac{g^2}{q^+ - p^+} \left( \frac{m_q}{q^+} + \frac{m_p}{p^+} \right)^2$$

(2.1)

(the subscript o stands for on-shell). This divergence is canceled exactly by the amplitude with an instantaneous fermion line

$$T_i = \frac{g^2}{q^+ - p^+} \frac{1}{p^+}$$

(2.2)

The subscript i stands for “instantaneous”. Note that this cancellation occurs if and only if the mass in the numerator (the “vertex mass”) and the mass in the denominator (the “kinetic mass”) are the same in Eq. (2.1). This is also the only choice of parameters that is consistent with parity invariance for Compton scattering at $O(g^2)$.

Choosing the vertex mass equal to the kinetic mass is also crucial for a cancellation between the (momentum dependent!) self-induced inertia (kinetic mass) counter term

$$\Delta m^2 = \frac{g^2}{4\pi} \int_0^{p^+} \frac{dk^+}{k^+}$$

and the divergent piece of the $O(g^2)$ self-energy

$$\Delta^{(2)} p^- = \frac{g^2}{4\pi} \int_0^{p^+} \frac{dk^+}{p^+ - k^+} \frac{\left( \frac{m_q}{p^+} + \frac{m_p}{k^+} \right)^2}{p^+-\frac{m_q^2}{k^+}-\frac{m_p^2}{k^+}-\lambda^2}.$$  

(2.4)

This well known result has recently also been obtained using so-called ladder relations [9], by investigating divergences in the non-perturbative coupled Fock space equations for bound states.

While the self-induced inertia certainly cancels the divergent part of the $O(g^2)$ self-energy, it has been questioned whether it also contains the correct finite part. In fact, in Ref. [7], parity invariance for physical observables has been used to determine the finite piece of the kinetic mass counter-term non-perturbatively.

However, the above analysis shows that the cancellation of divergences may also be used to determine the finite piece: if the tree level cancellation between instantaneous and on shell amplitudes is spoiled by a wrong choice for the kinetic mass then higher order diagrams will contain a divergence of integrals over longitudinal momenta as a result of the incomplete cancellation. The question is — and this will be subject of the rest of this paper — whether such “finiteness conditions” also arise at higher orders in the coupling constants and whether they can be used to determine the finite part of the kinetic mass counter-term.

For this purpose, let us consider the one-loop $O(g^4)$ corrections to the Compton amplitude. Again we restrict ourselves to planar diagrams. Since we are interested only in corrections to the $p^+\rightarrow 0$ singular contributions, it is also sufficient to consider only loop corrections to the fermion line which propagates between the two vertices. In LF-perturbation theory, we thus have to consider the four diagrams in Figure 2.
All three amplitudes diverge like $1/p^+$ where $p^+$ (with instantaneous.

Fig. 2. $\mathcal{O}(g^4)$ contributions to the forward Compton amplitude. (a) All fermion lines on mass shell. (b) same as (a), but the loop replaced by the self-induced inertia. (c) One of the two diagrams with an instantaneous fermion interaction (denoted by a slashed line) adjacent to the self-energy insertion. (d) Both fermion propagators adjacent to the loop instantaneous.

(a) and (b) together are finite (for finite $p^+$) and contribute

$$T_{oo} = \frac{g^4}{4\pi(q^+ - p^+)} D_1^2 \int_0^p dk^+ \left( \frac{(m^+ + m^2)}{(p^+ - k^+)} D_2 + \frac{1}{k^+} \right),$$

(2.5)

where

$$D_1 = q^+ - \frac{m^2}{p^+} - \frac{\lambda^2}{q^+ - p^+},$$

$$D_2 = p^+ - \frac{m^2}{k^+} - \frac{\lambda^2}{p^+ - k^+},$$

(2.6)

(with $p^- = q^- - \frac{\lambda^2}{q^+ - p^+}$) are the energy denominators for the intermediate states. The diagrams with one or two instantaneous lines are finite without counter-terms (for finite $p^+$) and yield, respectively

$$T_{oi} = \frac{2g^4}{4\pi(q^+ - p^+)} p^+ D_1 \int_0^p dk^+ \frac{(m^+ + m^2)}{(p^+ - k^+) D_2}$$

$$T_{ii} = \frac{g^4}{(q^+ - p^+)^2} \frac{1}{4\pi(p^+ - k^+)} D_2.$$  

(2.7)

All three amplitudes diverge like $1/p^+$ as $p^+ \to 0$! One finds

$$\lim_{p^+ \to 0} p^+ T_{oo} = \frac{g^4}{4\pi q^+} \left[ \frac{1}{m^2} \ln \frac{\lambda^2}{m^2} - \int_0^1 dx \frac{2 + x}{x} \right]$$

$$\lim_{p^+ \to 0} p^+ T_{oi} = \frac{g^4}{4\pi q^+} \int_0^1 dx \frac{2 + 2x}{m^2(1 - x) + \lambda^2 x}$$

$$\lim_{p^+ \to 0} p^+ T_{ii} = -\frac{g^4}{4\pi q^+} \int_0^1 dx \frac{x}{m^2(1 - x) + \lambda^2 x}. \quad (2.8)$$

The divergence at small $p^+$ does not cancel when one sums up the three terms. In fact, what one finds is

$$\lim_{p^+ \to 0} p^+ (T_{oo} + T_{oi} + T_{ii}) = \frac{g^4}{4\pi m^2 q^+} \ln \frac{\lambda^2}{m^2}. \quad (2.9)$$

Since there are no diagrams other than the ones listed in Fig. 2 which are singular at $\mathcal{O}(g^4)$, this implies that there is a problem: The $\mathcal{O}(g^4)$ self-energy of a fermion (Fig. 3) is obtained by integrating the $\mathcal{O}(g^4)$ forward Compton amplitude over $p^+$ and one obtains a logarithmic divergence! This divergence should not be there since Yukawa model on the LF with only the self-induced inertias added as counter-terms does not lead to finite answers.

Surprisingly, the resolution to this problem does not require to add another infinite counter-term. In Ref. 3 a finite kinetic mass counter-term (in addition to the infinite self-induced inertias) was introduced and it was found to be necessary in order to obtain parity invariant form-factors. The effect of a $\mathcal{O}(g^2)$ kinetic mass counter-term is an additional $\mathcal{O}(g^4)$ term in the forward Compton amplitude

$$T_{\Delta m^2} = \frac{g^2}{4\pi(q^+ - p^+)} \frac{(m^2 + m^2)}{(q^+ - p^+) D_1^2 \frac{\Delta m^2_{kin}}{p^+}}.$$ 

(2.10)

It can easily be verified that the choice

$$\Delta m^2_{kin} = \frac{g^2}{4\pi} \ln \frac{m^2}{\lambda^2}$$

(2.11)

$^2$An exception is the "supersymmetric" case $m^2 = \lambda^2$. 

\[ FIG. 3. \mathcal{O}(g^4) contributions to the fermion self-energy, which is sensitive to the small $p^+$ behavior of the $\mathcal{O}(g^2)$ fermion self-energy. \]
leads to
\[
\lim_{p^+ \to 0} p^+ (T_{oo} + T_{oi} + T_{iu} + T_{\Delta m^2}) = 0
\] (2.12)
and hence the $\mathcal{O}(g^4)$ self-energy of a fermion is finite with this (and only this) particular choice for the kinetic mass counter-term. Note that exactly the same values for the $\mathcal{O}(g^2)$ kinetic mass counter-term also lead to parity invariant scattering amplitudes.

Note that while the calculations presented above had been done for a scalar Yukawa theory, very similar results hold for models with similar interactions, such as pseudoscalar Yukawa of fermions coupled to the $\perp$ component of a vector field.

### III. A NON-PERTURBATIVE EXAMPLE

For a non-perturbative example, let us consider the model introduced in Ref. [10]: fermions in 3+1 dimensions coupled to the $\perp$ components of a massive vector field in planar approximation.

The non-perturbative Green’s function for a fermion in this model can be written in the form
\[
G(p^+) = \gamma^+ p^+ G_+ (2p^+ p^-, \vec{p}^2_\perp) + \gamma^- p^+ G_- (2p^+ p^-, \vec{p}^2_\perp) + \gamma_\perp G_\perp (2p^+ p^-, \vec{p}^2_\perp) + G_0 (2p^+ p^-, \vec{p}^2_\perp),
\] (3.1)
where each of the $G_i$ has a spectral representation
\[
G_i (2p^+ p^-, \vec{p}^2_\perp) = \int_0^\infty dM^2 \frac{\rho_i^{LF}(M^2, \vec{p}^2_\perp)}{2p^+ p^- - M^2 + i\varepsilon}.
\] (3.2)
Note that $\text{tr} (\gamma^- G)$ cannot contain a term proportional to $\frac{1}{M^2-p^+-M^2+i\varepsilon}$ because this would lead to severe small $p^+$ divergences, which are not canceled by the self-induced inertias.

From the fermion Green’s function, one computes the self-energy self-consistently via
\[
G^{-1} = \gamma^+ \Sigma^+ + \gamma^- \Sigma^- + \gamma_\perp \Sigma_\perp + \Sigma_0,
\] (3.3)
where
\[
\Sigma = \gamma^+ \Sigma^+ + \gamma^- \Sigma^- + \Sigma_0.
\] (3.4)
For the LF components of the self-energy one finds
\[
\Sigma_i = g^2 \int_0^\infty dM^2 \int_0^{p^+} \frac{dk^+}{k^+} \int \frac{d^2 k_\perp}{16\pi^3} f_i,
\] (3.5)
where
\[
f_+ = \left[ \frac{k^-}{D(p^+ - k^+)} - \frac{1}{p^+} \right] \rho_+ \left( M^2, \vec{k}^2_\perp \right)
\]
\[
f_- = \frac{k^+ \rho_- \left( M^2, \vec{k}^2_\perp \right)}{D(p^+ - k^+)}
\]
\[
f_\perp = -\rho_0 \left( M^2, \vec{k}^2_\perp \right) \frac{k^+}{D(p^+ - k^+)}
\] (3.6)
and
\[
D = p^+ - M^2 - \frac{\lambda^2}{2k^+} - \frac{\lambda^2 \left( \vec{p}^2_\perp - \vec{k}^2_\perp \right)^2}{2(p^+ - k^+)}
\]
\[
\vec{k}^- = p^+ - \frac{\lambda^2 \left( \vec{p}^2_\perp - \vec{k}^2_\perp \right)^2}{2(p^+ - k^+)}.
\] (3.7)
Note that in this toy model from Ref. [10], such a truncation of the Schwinger-Dyson equations is exact.

In order to be able to investigate whether the self-induced inertias cancel the infinite part of the self-energy one needs to know the small $p^+$ behavior of $G$ and thus the small $p^+$ behavior of $\Sigma$.

As an example, let us suppose that $\Sigma = c \gamma^+ / 2p^+$, in which case one finds for $p^+ \to 0$
\[
G = \frac{1}{\gamma^+ - m + \Sigma} \rightarrow \frac{c}{c + m^2 + \vec{p}^2_\perp} \frac{\gamma^+}{p^+},
\] (3.8)
which diverges, while the propagator for $c = 0$ remains finite in this limit. The self induced inertias cancel the infinite part of the self-energy in the case where the fermion propagator inside the loop is a free propagator. If one wants that the same cancellation occurs with the full propagator, it is thus necessary that the self-energy which modifies the propagator remains finite as $p^+ \to 0$.

We will now investigate the consequences of this fact for the model studied in Ref. [10]. In particular, we will focus on the $\gamma^+$ component of $\Sigma$, which is the most singular term as $p^+ \to 0$. Including only the self-induced inertia counter-term, one finds [10]
\[
\Sigma^+ = g^2 p^+ \int_0^{p^+} \frac{dk^+}{p^+} \int \frac{d^2 k_\perp}{8\pi^3} G_+ (2k^+ \vec{k}^-, \vec{k}^2_\perp)
\]
\[
- \frac{g^2}{2p^+} \int_0^{\infty} dM^2 \int \frac{d^2 k_\perp}{8\pi^3} b_+ (M^2, \vec{k}^2_\perp) \ln \frac{M^2}{\lambda^2 + \vec{k}^2_\perp}.
\] (3.9)
The first term on the r.h.s. of Eq. (3.9) is finite as $p^+ \to 0$, but the second term diverges in this limit. This second term is the non-perturbative analog of the $\ln \lambda^2 / m^2$ term in the one-loop self-energy, which we had to cancel by introducing a finite kinetic mass counter-term in order to avoid divergences in the two-loop self-energy. Here we also need to cancel the second term by means of a kinetic mass counter-term in order to obtain finite solutions to the self-consistent LF version of the Schwinger-Dyson equations.

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3 For details and definitions the reader is referred to this paper.

4 Below this assumption will be shown to be self-consistent.
In summary, what we have found is that the two-loop result generalizes directly to all orders in this non-perturbative example. In fact, one can show that the result generalizes to an entire class of models with Yukawa (scalar and pseudo-scalar) interactions as well as models with couplings to transverse components of vector fields. However, I was not able to show that the result generalizes to all orders to models with couplings to longitudinal components of a vector field (i.e. gauge theories). Semi-perturbative considerations suggest that the results also apply to dimensionally reduced models for QCD \(^5\) as well as \(\perp\) lattice QCD, but I could not find a general proof (beyond perturbative calculations). Nevertheless, let us in the following conjecture that the result generalizes at least to models for QCD which have only a two-dimensional continuum (such as dimensionally reduced models and the \(\perp\) lattice) and let us discuss the consequences.

First of all, at least in principle, this means that one can use the dependence of physical masses on the longitudinal cutoff to fine-tune the finite part of the kinetic mass counter term. However, finite physical masses do not necessarily imply that one has the correct kinetic mass. To illustrate this point, consider simple quantum mechanical scattering in two or more space dimensions from a \(\delta\)-function potential. Regardless of the sign of the potential, higher order Born terms in the scattering amplitude all diverge. Nevertheless, non-perturbative energies for physical states only diverge in the attractive case, but not in the repulsive case. What this implies for the fine-tuning procedure of the kinetic mass in the LF-Yukawa model is the following: If one wants to use finiteness conditions to determine the correct value of the kinetic mass term, one needs to vary the physical mass and study the cutoff dependence of physical masses for each kinetic mass. If the kinetic mass is smaller than the correct value, physical masses will become tachyonic as the cutoff is removed. On the other hand, for any value of the kinetic mass which is larger than the correct value the spectrum will not necessarily diverge. \(^6\) The correct kinetic mass is thus obtained by working right at (i.e. infinitesimally above) the critical point where the spectrum becomes tachyonic.

While this algorithm seems to be quite easy to use, there are several reasons why one should be very careful in its application to practical problems.

First of all, in many LF calculations higher Fock components typically contribute only small corrections to physical masses at a given cutoff. What this means for the practical applicability of finiteness criteria is that any log-dependencies on a cutoff (which one needs to identify in order to apply finiteness conditions) may enter with a very small coefficient so that they might be practically invisible.

Secondly, it is very important to discuss the cutoff scheme dependence! So far, we have on purpose avoided to specify a cutoff procedure — which one always has to do when dealing with divergent (or potentially divergent) quantities. The reason we did not have to specify the cutoff procedure is that the one-loop divergence is canceled locally (the singularities of the integrand cancel) by the self-induced inertia and higher order divergences are also canceled locally by the finite kinetic mass counter terms. However, we still assumed implicitly that the result for the inner loop was (apart from trivial kinematical factors) momentum independent — otherwise it would not have been sufficient to add merely a number (not a function) as a counter-term.

It is easily possible to introduce cutoffs which have this property, for example an invariant mass difference cutoff at each 3-point vertex and a cutoff for the instantaneous fermion exchange diagrams which is consistent with cutoffs on iterated 3-point vertices. However, one of the most popular cutoffs used in non-perturbative LF-calculations is DLCQ, where all momenta are discretized and thus a cutoff on the longitudinal momenta is provided by the spacing of the grid in momentum space. With such a cutoff procedure the self energy of a fermion does depend on its momentum (beyond the trivial \(1/p^+\) dependence). This point will be elaborated in Section \(^5\). However, before we discuss numerical implications in DLCQ, let us first consider finiteness relations derived by using perturbative relations between Fock space components in non-perturbative bound state problems.

IV. FINITENESS CONDITIONS AND LADDER RELATIONS

In bound state problems it is often possible to relate Fock space components which are highly off energy shell to lower Fock components using perturbation theory. This fact has been used within a dimensionally reduced model for QCD in Ref. \(^7\) to relate the end-point behavior of Fock space amplitudes with \(n+1\) quanta to Fock space amplitudes with \(n\) quanta, via

\[
ψ_{n+1}(x_1, x_2, ..., x_{n-1}, 0) \propto \frac{1}{m\sqrt{x_{n-1}}} ψ_n(x_1, x_2, ..., x_{n-1}).
\]

\[\text{(4.1)}\]

\(^5\) One can construct examples where the whole spectrum diverges as a divergent positive term is added to the Hamiltonian, but also examples where part of the spectrum remains finite.

\(^6\) Working near a critical point is a frightening prospect for practitioners, but the critical point is only of first order.

\(^7\) No distinction between vertex and kinetic masses has been made in Ref. \(\text{[9]}\).
Eq. (4.1) shows that wave functions in higher Fock components do not vanish near the end-point (i.e. for vanishing fermion momenta), which leads to divergent matrix elements of the kinetic energy as well as the interaction. The divergence that arises when only the fermion momentum goes to zero is canceled exactly by the self-induced inertias [Eq. (2.3)] if and only if the vertex mass \( m_V \) and the kinetic mass \( m_{\text{kin}} \) are the same.

In Ref. [5] it is thus claimed that the bound state equation (with \( m_V = m_{\text{kin}} \)) is finite. This claim is false: the Hamiltonian studied in Ref. [5] is in general not finite! The point is that both Eq. (4.1) as well as the cancellation conditions require to be modified when two momenta go to zero simultaneously. The best way to see that without going into too much detail is to consider the matrix element which connects states which differ by one boson. Such a matrix element involves the inverse of the momenta of both the incoming and outgoing fermion. If only the outgoing momentum goes to zero, then the term with the inverse of the momentum of the incoming fermion can obviously be neglected. However, this is not the case if both incoming and outgoing momentum go to zero simultaneously. Since the vanishing of both incoming and outgoing fermion momenta also implies that the momentum of the emitted boson also vanishes, one can therefore conclude that the end-point behavior gets modified if the momenta of both the fermion and a boson vanish simultaneously.

Furthermore, the cancellation conditions also get modified when proper care is taken for the case where several momenta go to zero simultaneously. In particular, in order for the Hamiltonian to give finite results one does in general need to keep \( m_V \neq m_{\text{kin}} \).

The two-loop example considered above can be considered a formal proof (by counter-example!) for these intuitively obvious facts.

In Ref. [3], numerical evidence is offered for the finiteness claim made in the same paper. Below, in Section V, it will be demonstrated that the (logarithmic) divergence arising from the two-loop diagram shows up only for very large values of the DLCQ parameter \( K \). This is probably the main reason why the divergence did not show up in the numerical results presented in Ref. [3].

V. FINITENESS CONDITIONS IN DLCQ

It is very easy to see that discretization in momentum space leads to a momentum dependent self-mass. Compared to a continuum calculation, integrals are approximated by sums and the number of points over which the summation is performed is determined by the total momentum. In this section, we will investigate the implications of this obvious fact for finiteness conditions.

In order to simplify the discussion, let us consider a cutoff which is very similar to the DLCQ cutoff, namely a sharp momentum cutoff (in the continuum) on all momenta that are smaller than an arbitrary constant \( \varepsilon \).

The point is that since the cutoff acts both on the boson and on the fermion line, self-energy corrections to the \( O(g^4) \) Compton amplitude are absent for \( p^+ < 2\varepsilon \) and they are suppressed for \( p^+ \approx \varepsilon \). On the other hand, a (momentum independent!) kinetic mass counter-term would contribute all the way down to the cutoff, namely \( p^+ = \varepsilon \). For the self-energy this implies that there is an incomplete cancellation between terms that would cancel if the cutoff on the inner loop would be sent to zero before the outer loop integration is performed.

In order to illustrate what consequences this might have, let us consider a simple mathematical model which has the right qualitative features: let us assume that the sum of amplitudes in Fig. 2 in the presence of a cutoff is given by

\[
p^+ T = \frac{c}{q^+} \Theta(p^+ - 2\varepsilon). \tag{5.1}
\]

Including a kinetic mass counter-term \( \Delta m_{\text{kin}}^2 \), the two loop self-energy is then given by

\[
\Delta^{(4)} q^- \propto \int_{\varepsilon} q^+ \frac{dp^+}{p^+} \left[ c \Theta(p^+ + 2\varepsilon) - \Delta m_{\text{kin}}^2 \right]. \tag{5.2}
\]

Despite the fact that the integral over the self-energy piece starts at \( p^+ = 2\varepsilon \), while the integral over the mass counter-term contribution starts at \( p^+ = \varepsilon \), the unique choice for \( \Delta m_{\text{kin}}^2 \) which yields a finite two loop self-energy as \( \varepsilon \rightarrow 0 \) is \( \Delta m_{\text{kin}}^2 = c \). And the result of the integral in this case is \( -c \ln 2 \) (independent of \( \varepsilon \)). Had we taken the limit \( \varepsilon \rightarrow 0 \) in the integrand, then the integrand would identically vanish and the integral would be zero.

In other words, the finiteness condition would have given us the correct value for the kinetic mass counter-term at \( O(g^2) \), but the wrong result for the physical mass at \( O(g^4) \).

In order to demonstrate that this problem does indeed occur in DLCQ, let us consider a concrete problem, namely the \( O(g^4) \) self-mass \( \Delta M^2 = q^+ \delta^{(4)} q^- \) resulting from the rainbow diagram (Fig. 3). Even though we know the correct kinetic mass counter-term for this case from Eq. (2.1), let us pretend here that we do not know it and let us consider the two loop self-energy both as a function of the momentum \( q^+ \) (in discrete units) and the kinetic mass counter-term. The coupling constant is set to \( g = \sqrt{4\pi} \), and for the masses we choose \( \lambda^2 = 1 \) and \( m^2 = 2 \). Figure 4 shows \( 4\pi q^+ \) times the self-energy (including the kinetic mass counter-term) of the fermion as a function of \( q^+ \) for different values of the parameter \( \Delta m_{\text{kin}}^2 \). There are several things one can learn from this calculation.
First of all, Fig. 4 clearly shows that a kinetic mass counter-term $\Delta m_{\text{kin}}^2$ (in addition to the self-induced inertia) is necessary in order to obtain finite results: the two-loop result for $\Delta M^2$ obviously diverges when one sets $\Delta m_{\text{kin}}^2 = 0$.

Secondly, the procedure is not very sensitive since the divergence is only logarithmic and the coefficient of the divergent piece is not very large. In order to obtain a precise picture about which value for the kinetic mass parameter leads to a convergent one has to go to values of $q^+ > 1000$, which is forbiddingly large for a non-perturbative calculation, but a reasonable estimate can already be obtained at lower values.

Thirdly, the finiteness condition does give the correct value for the kinetic mass counter-term. Only for $\delta m_{\text{kin}}^2 \approx \ln 2$ (for $\delta^2 = 2$ and $\lambda^2 = 1$ one finds no noticeable $q^+$ dependence of the self-energy for large $q^+$. Even small deviations lead to a log $q^+$ divergence proportional to that deviation.

Finally, and this is very important, despite the fact that the finiteness condition yields the correct value for $\delta m_{\text{kin}}^2$, the final result of the $O(q^4)$ differs from the covariant result: For $\Delta m_{\text{kin}}^2 \approx \ln 2$ one finds $\lim_{q^+ \to \infty} \Delta M^2 \approx -1.204$, while the correct (covariant) result for the two loop diagram (Fig. 3) is given by $\Delta M^2 \approx -2.112$ for the same masses and couplings. As we discussed above, this is because in DLCQ the momentum of a line that enters a sub-loop is not necessarily high above the cutoff inside that sub-loop. Therefore, the sensitivity to the cutoff never goes away — not even when the overall momentum is sent to infinity. Another way to look at this result is to conclude that in DLCQ one cannot introduce just one kinetic mass counter-term, but instead one needs to introduce a kinetic mass which depends on on the momentum. Formally, this should not come as a surprise, since the boost invariance (which is normally manifest in LF quantization) is broken by the DLCQ regulator. However, in a number of examples, such as 1+1 dimensional QED/QCD and theories with only self-interacting scalar fields, momentum dependent counter terms are not necessary and DLCQ workers have become accustomed to assume momentum independence of all counter-terms as a starting point. Unfortunately, the Yukawa model that we have considered here is a clear counter-example to this simplified picture.

Of course, for a perturbative diagram one can always calculate the proper momentum dependence, but this seems impossible to do analytically in a non-perturbative context. An alternative procedure is the one employed in Refs. [8,9], where a momentum dependent kinetic mass is introduced such that the physical mass of the lightest states is independent of the momentum. The physical mass then replaces the bare kinetic mass as a renormalization parameter. In Refs. [7,10] the new parameters were determined by imposing parity invariance on physical amplitudes or by comparison with a covariant calculation. However, it is not obvious how to translate the finiteness condition for kinetic masses into a condition for the physical masses.

The fact that a simple (i.e. momentum independent) kinetic mass counter-term yields incorrect results also means that the ansatz for the LF Hamiltonian in theories with fermions and Yukawa type interactions (this includes QED/QCD!) used by DLCQ workers (see for example Refs. [3,4]) is insufficient.

There are several obvious patches that one can apply to the DLCQ calculations, but they all seem to have one feature in common: one needs to introduce another cutoff — beyond DLCQ — which has the feature that it gives momentum independent results. Typical examples are a Pauli-Villars regulator [3,4] or a cutoff on the invariant energy transfer. Of course, even with a cutoff that gives momentum independent results, one still needs to keep the kinetic mass as an “independent parameter”.\(^8\)

\(^8\) An exception is Pauli-Villars regularization with sufficiently many regulator particles [3,4].
VI. SUMMARY

We have investigated the conditions under which light-front Hamiltonians with fermions interacting via Yukawa type interactions (including interactions to the transverse component of a vector field) lead to convergent loop integrals at small values of the LF momentum \( p^\perp \equiv p^0 + p^3 \).

In the continuum, it was found that it is both necessary and sufficient to add a kinetic mass counter-term (in addition to the self-induced inertias) to the Hamiltonian in order to obtain finite results w.r.t. the small LF momentum cutoffs.) for higher order diagrams. That additional parameter is determined by demanding finiteness for the \( p^\perp \) integrals. Imposing such a finiteness condition makes sense, since the small \( p^\perp \) divergence is an artifact of the LF approach. It turns out that the kinetic mass counter-term obtained is identical to the one determined by imposing parity invariance for physical observables. In a non-perturbative calculation, one obtains tachyonic behavior if the kinetic mass is smaller than its correct value. Above its correct value no tachyonic behavior is observed, but the spectrum may or may not diverge if the kinetic mass is too large. This “critical” behavior at the correct value can be used as a signature for non-perturbative determinations of the kinetic mass counter-term.

Unfortunately, there are several obstacles before one can apply “finiteness conditions” in practical calculations — particularly in DLCQ. One reason is that the divergences that one needs to look for are only logarithmic, which makes them hard to detect numerically. Furthermore, the situation in DLCQ is not quite as simple as it is in the continuum. DLCQ breaks manifest boost invariance, and the results in this paper show that a simple Ansatz, where the kinetic mass counter-term is not a function of the momentum, is inconsistent in DLCQ already for perturbative calculations within a super-renormalizable model. However, it is conceivable that a DLCQ calculation with additional cutoffs (such that momentum independence of the results is achieved) can be based on Hamiltonians with momentum independent mass counter-terms. These counter-terms can then, at least in principle, be determined using the finiteness condition that was derived in this paper.

The results in this paper were based on perturbatively analyzing Yukawa type interactions in 1+1 dimensions, and on non-perturbative results involving fermions coupled to the \( \perp \) component of a vector field in 3+1 dimensions. It would be interesting to know what these results imply for QCD in 3+1 dimensions. First of all, the limitation to 1+1 dimensions can be easily overcome by introducing a lattice in the transverse space coordinates. That way one obtains a 3+1 dimensional theory which is formally equivalent to coupled 1+1 dimensional theories and the results of this paper immediately translate. The real limitation of the results in this paper is that while QCD contains interactions (couplings to the transverse components of the gauge field) which resemble Yukawa interactions, QCD also contains also couplings to the longitudinal components of the gauge fields and those are much more singular for \( p^\perp \to 0 \) than the Yukawa-type couplings. It is not clear whether renormalizing the kinetic mass will be sufficient to compensate divergences arising from the couplings to the longitudinal components of the gauge field as well. However, while it is not clear whether independent renormalization of the kinetic mass will be sufficient in QCD (most likely it is not), the mere fact that QCD contains interactions which resemble Yukawa interactions means that kinetic mass renormalization will be necessary. Another result of this paper, namely that using only a DLCQ regulator is inconsistent with a momentum independent mass term translates to QCD as well. This comment also applies to dimensionally reduced models for QCD.

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