THE MODULI SPACE OF GENERALIZED MORSE FUNCTIONS

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ABSTRACT. We study the moduli and determine a homotopy type of the space of all generalized Morse functions on $d$-manifolds for given $d$. This moduli space is closely connected to the moduli space of all Morse functions studied in [11] and the classifying space of the corresponding cobordism category.

1. Introduction

Given a smooth compact manifold $M^d$ and a fixed smooth function $\varphi : M^d \to \mathbb{R}$, let $\mathcal{G}(M^d, \varphi)$ denote the space of generalized Morse functions $f : M \to \mathbb{R}$ which agrees with $\varphi$ in a neighbourhood of the boundary $\partial M$. This space (in the Whitney topology) satisfies an $h$-principle in the sense of Gromov, [5]. Here precisely we define $h\mathcal{G}(M^d, \varphi)$ to be the space of sections of the bundle $J^3_{\text{gmf}}(M)$ of generalized Morse 3-jets that agrees with $j^3\varphi$ near $\partial M$. Taking the 3-jet of a generalized Morse function defines a map

$$j^3 : \mathcal{G}(M^d, \varphi) \to h\mathcal{G}(M^d, \varphi).$$

This map was first considered by Igusa in [6]. He proved that the map $j^3$ in (1) is $d$-connected and in [7] he calculated the “$d$-homotopy type” of $h\mathcal{G}(M^d, \varphi)$ by exhibiting a $d$-connected map

$$h\mathcal{G}(M^d, \varphi) \to \Omega^\infty S^\infty (BO_+ \wedge M^d),$$

thus determining the $d$-homotopy type of the space $\mathcal{G}(M^d, \varphi)$.

Eliashberg and Mishachev [2, 3] and Vassiliev [13] showed that the map in (1) is actually a homotopy equivalence rather than just being $d$-connected. This is the starting point for this paper.

We study the moduli space of all generalized Morse functions on $d$-manifolds, i.e. the space of $\mathcal{G}(M^d, \varphi)$ as $(M^d, \varphi)$ varies. There can be several candidates for such a moduli space. The one we present below is closely connected to the “moduli space” of all Morse functions considered in Section 4 of [11]. Indeed, the present note can be viewed as an addition to [11].

In Section 2 below we give the precise definition of our moduli space, and in Section 3 we determine its homotopy type, following the argument from [11].
2. Definitions and results

2.1. The moduli space. Let \( J^3(\mathbb{R}^d) \) be the space of 3-jets of smooth functions on \( \mathbb{R}^d \),

\[
p(x) = c + \ell(x) + q(x) + r(x),
\]

where \( c \) is a constant, \( \ell(x) \) is linear, \( q(x) \) quadratic and \( r(x) \) cubic,

\[
\ell(x) = \sum_i a_i x_i, \quad q(x) = \sum_{ij} a_{ij} x_i x_j, \quad r(x) = \sum_{ijk} a_{ijk} x_i x_j x_k,
\]

with the coefficients \( a_{ij}, a_{ijk} \) symmetric in the indices.

Let \( J^3_{\text{gmf}}(\mathbb{R}^d) \subset J^3(\mathbb{R}^d) \) be the subspace of \( p \in J^3(\mathbb{R}^d) \) such that one the following holds:

1. \( 0 \in \mathbb{R}^d \) is not a critical point of \( p \) (\( \ell \neq 0 \));
2. \( 0 \in \mathbb{R}^d \) is a non-degenerate critical point of \( p \) (\( \ell = 0 \) and \( q \) is non-degenerate);
3. \( 0 \in \mathbb{R}^d \) is a birth-death singularity of \( p \) (\( q: \mathbb{R}^d \to \text{Hom}_{\mathbb{R}}(\mathbb{R}^d, \mathbb{R}) \) has 1-dimensional kernel on which \( r(x) \) is non-trivial).

The space \( J^3_{\text{gmf}}(\mathbb{R}^d) \) is invariant under the \( O(d) \)-action on the space \( J^3(\mathbb{R}^d) \). Given a smooth manifold \( M^d \) with a metric, let \( \mathcal{P}(M^d) \to M \) be the principal \( O(d) \)-bundle of orthogonal frames in the tangent bundle \( TM^d \). Then

\[
J^3_{\text{gmf}}(TM^d) = \mathcal{P}(M^d) \times_{O(d)} J^3_{\text{gmf}}(\mathbb{R}^d)
\]

is a smooth fiber bundle on \( M \), a subbundle of

\[
J^3(TM^d) = \mathcal{P}(M^d) \times_{O(d)} J^3(\mathbb{R}^d).
\]

Remark 2.1. Up to change of coordinates, a birth-death singularity is of the form

\[
p(x) = x_1^{i+1} - \sum_{j=2}^{i+1} x_j^2 + \sum_{k=i+2}^d x_k^2.
\]

The integer \( i \) is the Morse index of the quadratic form \( q(x) \). In general, a quadratic form \( q: \mathbb{R}^d \to \mathbb{R} \) induces a canonical decomposition

\[
\mathbb{R}^d = V_-(q) \oplus V_0(q) \oplus V_+(q)
\]

into negative eigenspace, the zero eigenspace and the positive eigenspace. In the case of generalized Morse jets, \( \dim V_0(q) \) is either 0 or 1, and in the latter case the cubic term \( r(x) \) restricts non-trivially to \( V_0(q) \). The dimension of \( V_-(q) \) is the index of the gmf-jet.

For a smooth manifold \( M^d \), we have the 3-jet bundle \( J^3(M, \mathbb{R}) \to M \) whose fiber \( J^3(M, \mathbb{R})_x \) is the germ of 3-jets \( (M, x) \to \mathbb{R} \), and the associated subbundle \( J^3_{\text{gmf}}(M, \mathbb{R}) \to M \). A choice of exponential function induces a fiber bundle isomorphism

\[
J^3_{\text{gmf}}(M, \mathbb{R}) \cong J^3_{\text{gmf}}(TM),
\]
and \( f : M \to \mathbb{R} \) is a generalized Morse function precisely if \( j^3(f) \in \Gamma(J_{gmf}^3(M, \mathbb{R})) \), where we use \( \Gamma(E) \) to denote the space of smooth sections of a vector bundle \( E \).

**Definition 2.1.** Let \( X \) be a \( k \)-dimensional manifold without boundary. Let \( J_d(X) \) be the set of 4-tuples \((E, \pi, f, j)\) of a \((k + d)\)-manifold \( E \) with maps
\[
(\pi, f, j) : E \longrightarrow X \times \mathbb{R} \times \mathbb{R}^{d-1+\infty}
\]
subject to the conditions

1. \( (\pi, f) : E \longrightarrow X \times \mathbb{R} \) is a proper map;
2. \( (f, j) : E \longrightarrow \mathbb{R} \times \mathbb{R}^{d-1+\infty} \) is an embedding;
3. \( \pi : E \longrightarrow X \) is a submersion;
4. for any \( x \in X \), the restriction \( f_x = f|_{E_x} : E_x \longrightarrow \mathbb{R} \) to each fiber \( E_x = \pi^{-1}(x) \) is a generalized Morse function.

In (ii) above \( \mathbb{R}^{d-1+\infty} \) is the union or colimit of \( \mathbb{R}^{d-1+N} \) as \( N \to \infty \) and (ii) means that \( (f, j) \) embeds \( E \) into \( \mathbb{R} \times \mathbb{R}^{d-1+\infty} \) for sufficiently large \( N \). The definition above is the obvious analogue of Definition 2.7 of [11].

A smooth map \( \phi : Y \longrightarrow X \) induces a pull-back
\[
\phi^* : J_d(X) \longrightarrow J_d(Y), \quad \phi^* : (E, \pi, f, j) \mapsto (\phi^*E, \phi^*\pi, \phi^*f, \phi^*j)
\]
where
\[
\phi^*E = \{ (y, z) \in Y \times \mathbb{R}^{d+\infty} \mid (\phi(y), z) \in E \subset X \times \mathbb{R}^{d+\infty} \}
\]
and the maps \( \phi^*\pi, \phi^*f \) and \( \phi^*j \) are given by corresponding projections from
\[
\phi^*E \subset Y \times \mathbb{R} \times \mathbb{R}^{d-1+\infty}
\]
on the factors \( Y, \mathbb{R} \) and \( \mathbb{R}^{d-1+\infty} \), respectively. In particular, we have \( (\psi \circ \phi)^* = \phi^* \circ \psi^* \) (rather than just being naturally equivalent), so the correspondence \( X \mapsto J_d(X) \) is a set-valued sheaf \( J_d \) on the category \( \mathcal{X} \) of smooth manifolds and smooth maps.

A set-valued sheaf on \( \mathcal{X} \) gives rise to a simplicial set \( N \bullet J_d \) with
\[
N_k J_d = J_d(\Delta^k), \quad \Delta^k = \{ (x_0, \ldots, x_k) \in \mathbb{R}^{k+1} \mid \sum x_i = 1 \}.
\]
The geometric realization of \( N \bullet J_d \) will be denoted by \(|J_d|\), and we make the following definition.

**Definition 2.2.** The moduli space of generalized Morse functions of \( d \) variables is the loop space \( \Omega|J_d| \).

**Remark 2.2.** If in Definition 2.1 we drop the assumption that \( f : E \longrightarrow \mathbb{R} \) is a generalized Morse function, then \( J_d \) reduces to the sheaf \( D_d = D_d(-, \infty) \) of [4] Definition 3.3] associated to the space of embedded \( d \)-manifolds, and \( \Omega|D_d| \cong \Omega^\infty MT(d) \) by Theorem 3.4 of [4]. □
Associated with the set valued sheaf $\mathcal{J}_d(X)$, $X \in \mathcal{X}$, we have a sheaf $\mathcal{J}^A_d(X)$ of partially ordered sets, i.e. a category valued sheaf, cf. [11, Section 4.2]. For connected $X$, an object of $\mathcal{J}^A_d(X)$ consists of an element $(E, \pi, f, j) \in \mathcal{J}_d(X)$ together with an interval $A = [a_0, a_1] \subset \mathbb{R}$ subject to the condition that $f : E \to \mathbb{R}$ be fiberwise transverse to $\partial A$ (i.e. $\{a_0, a_1\}$ are regular values for each $f_x : E_x \to \mathbb{R}$, $x \in X$). The partial ordering is given by

$$(E, \pi, f, j) \leq (E', \pi', f', j') \quad \text{if}$$

$$(E, \pi, f, j) = (E', \pi', f', j') \quad \text{and} \quad A \subset A'.$$

If $X$ is not connected, $\mathcal{J}^A_d(X)$ is the product of $\mathcal{J}^A_d(X_j)$ over the connected components $X_j$.

We notice that each element $(E, \pi, f, j; A)$ restricts to a family of generalized Morse functions on a compact manifolds

$$(\pi, f, j) : f^{-1}(A) \hookrightarrow X \times [a_0, a_1] \times \mathbb{R}^{d-1+\infty},$$

where $A = [a_0, a_1]$. On the other hand, given such a family

$$(\pi, f, j) : E(A) \hookrightarrow X \times [a_0, a_1] \times \mathbb{R}^{d-1+\infty},$$

we can extend it to an element $(\hat{E}(A), \hat{\pi}, \hat{f}, \hat{j})$ by adding long collars:

$$\hat{E}(A) = (-\infty, a_0] \times f^{-1}(a_0) \cup E(A) \cup [a_1, \infty) \times f^{-1}(a_1).$$

If $(E(A)\pi, f, j)$ is a restriction of $(E, \pi, f, j) \in \mathcal{J}_d(X)$, then $(\hat{E}(A), \hat{\pi}, \hat{f}, \hat{j}; A)$ is concordant to $(E, \pi, f, j)$ by [11, Lemma 2.19].

The forgetful map $\mathcal{J}_d^A(X) \to \mathcal{J}_d(X)$ is a map of category valued sheaves when we give $\mathcal{J}_d^A(X)$ the trivial category structure (with only identity morphisms). It induces a map

$$|\mathcal{J}_d^A| \to |\mathcal{J}_d|$$

of topological categories, and hence a map of their classifying spaces:

$$B|\mathcal{J}_d^A| \to B|\mathcal{J}_d| = |\mathcal{J}_d|,$$

where $B|\mathcal{J}| = |N_\bullet \mathcal{J}|$.

**Theorem 2.1.** The map $B|\mathcal{J}_d^A| \to |\mathcal{J}_d|$ is a weak homotopy equivalence.

**Proof.** This follows from [11, Theorem 4.2], which identifies $|B\mathcal{J}_d^A|$ with $|\beta \mathcal{J}_d^A|$, where $\beta \mathcal{J}_d^A$ is a set-valued sheaf of [11, Definition 4.1], together with the analogue of [11, Proposition 4.10]: the map

$$|\beta \mathcal{J}_d^A| \to |\mathcal{J}_d^A|$$

is a weak homotopy equivalence. Indeed, the proof of Proposition 4.10, which treats the case where $f : E \to \mathbb{R}$ is a fiberwise Morse function (in a neighbourhood of $f^{-1}(0)$) carries over word by word to the situation of generalized Morse functions. \qed
2.2. **The $h$-principle.** For a submersion $\pi : E \to X$, $T^\pi E$ denotes the tangent bundle along the fibers. We can form the bundle

$$J^3_{\text{gmf}}(T^\pi E) \to E$$

of gmf-jets. Sections of this bundle will be denoted by $\hat{f}$, $\hat{g}$, etc. For given $z \in E$, the restriction $\hat{f}(z) : T(E_{\pi(z)}) \to \mathbb{R}$ is a gmf-jet; its constant term will be denoted $f(z)$.

**Definition 2.3.** For a smooth manifold $X$, let $h\mathcal{J}_d(X)$ consists of maps

$$(\pi, f, j) : E \to X \times \mathbb{R} \times \mathbb{R}^{d-1+\infty}$$

satisfying (i), (ii), and (iii) of Definition 2.1 together with a jet $\hat{f} \in \Gamma(J^3_{\text{gmf}}(T^\pi E))$ having constant term $f$.

A metric on $T^\pi E$ and an associated exponential map induces an isomorphism

$$(4) \quad J^3_\pi(E, \mathbb{R}) \xrightarrow{\cong} J^3(T^\pi E)$$

that sends gmf-jets to gmf-jets. Here $J^3_\pi(E, \mathbb{R}) \to E$ is the fiberwise 3-jet bundle. Differentiation in the fiber direction only, defines a map

$$j^3_\pi : C^\infty(E, \mathbb{R}) \to J^3_\pi(E, \mathbb{R})$$

which sends fiberwise generalized Morse functions into gmf-jets. This induces a map of sheaves

$$j^3_\pi : \mathcal{J}_d(X) \to h\mathcal{J}_d(X),$$

and hence a map of their nerves

$$j^3_\pi : |\mathcal{J}_d| \to |h\mathcal{J}_d|.$$

Using the sheaves $\mathcal{J}_d^A$ and the associated $h\mathcal{J}_d^A$, together with the $h$-principle of [3, 13], the argument of [11, Proposition 4.17] proves that the induced map

$$|\beta \mathcal{J}_d^A| \to |\beta h\mathcal{J}_d^A|$$

is a weak homotopy equivalence. Finally, since the forgetful maps

$$|\beta \mathcal{J}_d^A| \to |\mathcal{J}_d^A|, \quad |\beta h\mathcal{J}_d^A| \to |h\mathcal{J}_d^A|$$

are weak homotopy equivalences by [11, Proposition 4.10], we get

**Theorem 2.2.** The map $j^3_\pi : |\mathcal{J}_d| \to |h\mathcal{J}_d|$ is weak homotopy equivalence.
Homotopy type of the moduli space

3.1. The space $|hJ_d|$. For any set-valued sheaf $\mathcal{F} : \mathfrak{X} \to \text{Sets}$, let $\mathcal{F}[X]$ denote the set of concordance classes: $s_0, s_1 \in \mathcal{F}(X)$ are concordant ($s_0 \sim s_1$) if there exists an element $s \in \mathcal{F}(X \times \mathbb{R})$ such that $pr_X^*(s_0)$, $pr_X : X \times \mathbb{R} \to X$, agrees with $s$ on an open neighborhood of $X \times (-\infty, 0]$ and $pr_X^*(s_1)$ agrees with $s$ on an open neighborhood of $X \times [1, \infty)$. The relation to the space $|\mathcal{F}|$ is given by

$$[X, |\mathcal{F}|] \cong \mathcal{F}[X].$$

Let $G(d, n)$ denote the Grassmannian of $d$-planes in $\mathbb{R}^{d+n}$, and $G^{gmf}(d, n)$ the space of pairs $(V, f)$ with $V \in G(d, n)$ and $f : V \to \mathbb{R}$ a generalized Morse function with $f(0) = 0$. The space

$$U_{d,n}^\perp = \{ (v, V) \in \mathbb{R}^{d+n} \times G(d, n) \mid v \perp V \}$$

is an $n$-dimensional vector bundle on $G(d, n)$. Let $V_{d,n}^\perp$ be its pull-back along the forgetful map $G^{gmf}(d, n) \to G(d, n)$:

$$
\begin{array}{ccc}
V_{d,n}^\perp & \longrightarrow & U_{d,n}^\perp \\
\downarrow & & \downarrow \\
G^{gmf}(d, n) & \longrightarrow & G(d, n)
\end{array}
$$

Similarly, we have canonical $d$-dimensional vector bundles $U_{d,n}$ and $V_{d,n}$ on $G(d, n)$ and $G^{gmf}(d, n)$ respectively. The Thom spaces of the bundles $U_{d,n}^\perp$ and $V_{d,n}^\perp$ give rise to spectra $MT(d)$ and $MT^{gmf}(d)$ which in degrees $(d + n)$ are

$$MT(d)_{d+n} = \mathbb{Th}(U_{d,n}^\perp), \quad MT^{gmf}(d)_{d+n} = \mathbb{Th}(V_{d,n}^\perp).$$

The infinite loop space of the spectrum $MT^{gmf}(d)$ is defined to be

$$\Omega^\infty MT^{gmf}(d) = \colim_n \Omega^{d+n} \mathbb{Th}(V_{d,n}^\perp).$$

**Theorem 3.1.** There is a weak homotopy equivalence

$$\Omega |hJ_d| \simeq \Omega^\infty MT^{gmf}(d).$$

**Proof.** This is completely similar to the proof of [11, Theorem 3.5] for the case of Morse functions using only transversality and the submersion theorem, [12].

We have left to examine the right-hand side of the equivalence (8). The results are similar in spirit to ones in [11, Section 3.1].
Lemma 3.2. There is a homeomorphism
\[ \hat{G}_{gmf}(d, n) \cong O(d + n)/ (O(d - 1) \times O(n)). \]

Proof. For a pair \((V, f) \in \hat{G}_{gmf}(d, n)\) we have \(V \in G(d, n)\) and \(f = \ell\) with \(|\ell| = 1\). We may think of \(\ell\) as a linear projection on the first coordinate, which is the same as to say that the space \(V\) contains a subspace
\[ \mathbb{R} \times \{0\} \times \{0\} \subset \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}^n \]
with \(\ell\) being a projection on it. This identifies \(\hat{G}_{gmf}(d, n)\) with the homogeneous space \(O(d + n)/ (O(d - 1) \times O(n))\).

Since \(G(d - 1, n) = O(d - 1 + n)/(O(d - 1) \times O(n))\), we observe that the map
\[ i_{d,n} : G(d - 1, n) \to \hat{G}_{gmf}(d, n) \]
is \((d + n - 2)\)-connected and that \(i_{d,n}^*(V_{d,n}^\perp|_{\hat{G}_{gmf}(d, n)}) \cong U_{d-1,n}^+\).

On the other hand, \(\Sigma_{gmf}(d, n) \subset G_{gmf}(d, n)\) has normal bundle \(V_{d,n}^* \cong V_{d,n}\) and the inclusion
\[ (D(V_{d,n}), S(V_{d,n})) \to (G_{gmf}(d, n), G_{gmf}(d, n) \setminus \Sigma_{gmf}(d, n)) \]
is an excision map. This leads to the cofibration
\[ \text{Th}(j^* V_{d,n}^+) \to \text{Th}(V_{d,n}^+) \to \text{Th}((V_{d,n}^+ \oplus V_{d,n})|_{\Sigma_{gmf}(d, n)}), \]
where \(j\) is the inclusion
\[ j : G_{gmf}(d, n) \setminus \Sigma_{gmf}(d, n) \to G_{gmf}(d, n). \]

By the above, there is \((2n + d - 2)\)-connected map
\[ i_{d,n} : \text{Th}(U_{d-1,n}^+) \to \text{Th}(V_{d,n}^+). \]

With the notation of [4], we get from (10) a cofibration of spectra
\[ \Sigma^{-1}MT(d - 1) \to MT_{gmf}(d) \to \Sigma^\infty(\Sigma_{gmf}(d, \infty)_+) \]
and the corresponding homotopy fibration sequence of infinite loop spaces
\[ \Omega^\infty\Sigma^{-1}MT(d - 1) \to \Omega^\infty MT_{gmf}(d) \to \Omega^\infty\Sigma^\infty(\Sigma_{gmf}(d, \infty)_+) \]

Remark 3.1. The main theorem of [4] asserts a homotopy equivalence
\[ \Omega^\infty MT(d) \simeq \Omega BC_{\phi_d}, \]
where \(C_{\phi_d}\) is the cobordism category of embedded manifolds: the objects are \((M^{d-1}, a)\) of a closed \((d - 1)\)-submanifold of \(\{a\} \times \mathbb{R}^{\infty + d - 1}\) and the morphisms are embedded cobordisms \(W^d \subset [a_0, a_1] \times \mathbb{R}^{\infty + d - 1}\) transversal at \(\{a_i\} \times \mathbb{R}^{\infty + d - 1}\). In particular, we have weak homotopy equivalence
\[ \Omega^\infty\Sigma^{-1}MT(d - 1) \simeq \Omega^2 BC_{\phi_{d-1}}. \]
3.2. The singularity space. In [7], Igusa analyzed the singularity space \( \Sigma_{gmf}(d) \subset J^3_{gmf}(\mathbb{R}^d) \) by decomposing it with respect to the Morse index. The result, stated in [7, Proposition 3.4], is as follows. Consider the homogeneous spaces

\[
X^1(i) = O(d) \big/ O(i) \times O(1) \times O(d - i - 1),
\]

\[
X(i) = O(d) \big/ O(i) \times O(d - i).
\]

and note that there are quotient maps

\[
f_i : X^1(i) \to X(i), \quad g_i : X^1(i) \to X(i + 1),
\]

upon embedding \( O(i) \times O(1) \times O(d - i - 1) \) in \( O(i) \times O(d - i) \) and in \( O(i + 1) \times O(d - i - 1) \), respectively. These maps fit into the diagram

\[
\mathcal{D}(d) = \begin{pmatrix}
X^1(0) & X^1(1) & \cdots & X^1(d - 1) \\
\vdots & \ddots & \ddots & \ddots \\
X(0) & X(1) & X(2) & X(d - 1) & X(d)
\end{pmatrix}
\]

and [7, Proposition 3.4] states that the homotopy colimit of the diagram \( \mathcal{D}(d) \) is homotopy equivalent to \( \Sigma_{gmf}(d) \)

\[
(13) \quad \Sigma_{gmf}(d) \simeq \hocolim \mathcal{D}(d).
\]

It is easy to see that there are homeomorphisms

\[
G_{gmf}(d, n) = \left( O(d + n) \big/ O(n) \right) \times_{O(d)} J^3_{gmf}(\mathbb{R}^d),
\]

\[
\Sigma_{gmf}(d, n) = \left( O(d + n) \big/ O(n) \right) \times_{O(d)} \Sigma_{gmf}(d),
\]

and (13) implies that \( \Sigma_{gmf}(d, n) \) is homotopy equivalent to the homotopy colimit of the diagram

\[
\mathcal{D}(d, n) = \left( O(d + n) \big/ O(n) \right) \times_{O(d)} \mathcal{D}(d).
\]

For \( n \to \infty \), the Stiefel manifold \( O(n + d) / O(n) \) becomes contractible, and \( \mathcal{D}(d, \infty) \) is the diagram

\[
(14) \quad \begin{split}
&Y^1(0) & Y^1(1) & \cdots & Y^1(d - 1) \\
\vec{f}_0 & \vec{g}_0 & f_1 & \vec{g}_1 & \vec{f}_2 & \vec{g}_{d-2} f_{d-1} & \vec{g}_{d-1} \\
Y(0) & Y(1) & Y(2) & Y(d - 1) & Y(d)
\end{split}
\]
with
\[ Y^1(i) = BO(i) \times BO(1) \times BO(d - i - 1), \]
\[ Y(i) = BO(i) \times BO(d - i), \]
and \( \bar{f}_i \) and \( \bar{g}_i \) the obvious maps. So \( \Sigma_{gmf}(d, \infty) \) is the homotopy colimit of \( \Sigma \).

We want to compare this to the singular set \( \Sigma_{mf}(d, \infty) \) which appears when one considers
the moduli space of Morse functions rather than generalized Morse functions was calculated
in [11, Lemma 3.1]:
\[ \Sigma_{mf}(d, n) \cong \prod_{i=0}^{d} \left[ \left( O(d + n) \big/ O(n) \right) \times O(d) \big/ O(i) \times O(d - i) \right] \]
so that
\[ \Sigma_{mf}(d, \infty) \cong \prod_{i=0}^{d} BO(i) \times BO(d - i) = \prod_{i=0}^{d} Y(i). \]
The cofiber of the map \( \Sigma_{mf}(d, \infty) \to \Sigma_{gmf}(d, \infty) \) is by [11] equal to the homotopy colimit of the diagram:
\[
\begin{array}{c c c c}
Y^1(0) & Y^1(1) & \cdots & Y^1(d-1) \\
\ast & \ast & \ast & \ast
\end{array}
\]
But this homotopy colimit is easy calculated to be
\[ \bigvee_{i=0}^{d-1} S^1 \wedge Y^1(i)_+ \cong \bigvee_{i=0}^{d-1} S^1 \wedge (BO(i) \times BO(d - i - 1))_+. \]

We get a cofibration of suspension spectra:
\[ \Sigma^\infty(\Sigma_{mf}(d, \infty)_+) \to \Sigma^\infty(\Sigma_{gmf}(d, \infty)_+) \to \bigvee_{i=0}^{d-1} \Sigma^\infty(S^1 \wedge (BO(i) \times BO(d - i - 1))_+). \]
Taking the associated infinite loop spaces we get

**Proposition 3.3.** There is a homotopy fibration:
\[ \prod_{i=0}^{d-1} \Omega^\infty \Sigma^\infty(BO(i) \times BO(d - i - 1)_+) \to \prod_{i=0}^{d} \Omega^\infty \Sigma^\infty(\Sigma_{gmf}(d, \infty)_+) \to \Omega^\infty \Sigma^\infty(\Sigma_{gmf}(d, \infty)_+). \]

The constant map \( \mathcal{D}(d) \to \ast \) into the constant diagram induces the map
\[ \mathcal{D}(d, n) \to \left( O(d + n) \big/ O(n) \right) \times_{O(d)} \ast , \]
where the target space is homotopy equivalent to \(G(d,n)\). For \(n \to \infty\), this induces the fiber bundle

\[
p : \Sigma^{\text{gmf}}(d, \infty) \to BO(d)
\]

with the fiber \(\Sigma^{\text{gmf}}(d)\). We obtain the commutative diagram of cofibrations:

\[
\begin{array}{ccc}
\Sigma^{-1}MT(d-1) & \longrightarrow & MT^{\text{gmf}}(d) \\
\downarrow \text{Id} & & \downarrow f \\
\Sigma^{-1}MT(d-1) & \longrightarrow & \Sigma^{\infty}(\Sigma^{\text{gmf}}(d, \infty)_+)
\end{array}
\]

and a corresponding diagram of homotopy fibrations:

\[
\begin{array}{ccc}
\Omega^{\infty}\Sigma^{-1}MT(d-1) & \longrightarrow & \Omega^{\infty}MT^{\text{gmf}}(d) \\
\downarrow \text{Id} & & \downarrow \Omega^{\infty}f \\
\Omega^{\infty}\Sigma^{-1}MT(d-1) & \longrightarrow & \Omega^{\infty}\Sigma^{\infty}(BO(d)_+)
\end{array}
\]

Since \(\Sigma^{\infty}(\Sigma^{\text{gmf}}(d, \infty)_+)\) and \(\Sigma^{\infty}(BO(d)_+)\) are \((-1)\)-connected, we obtain that the forgetful map \(F : MT^{\text{gmf}}(d) \to MT(d)\) induces isomorphism

\[
\pi_{-i}MT^{\text{gmf}}(d) \cong \pi_{-i}MT(d), \quad i \geq 0.
\]

We consider the forgetful map \(\theta^{\text{gmf}} : G^{\text{gmf}}(d, \infty) \to G(d, \infty)\) as a structure on \(d\)-dimensional bundles. Then we denote by \(C\theta^{\text{gmf}}_d\) the category \(C\theta^{\text{gmf}}_d\) (see (5.3) and (5.4) of [4]) of manifolds (objects) and cobordisms (morphisms) equipped with a tangential \(\theta^{\text{gmf}}\)-structure. Then the main theorem of [4] gives the following result:

**Corollary 3.4.** There is weak homotopy equivalence

\[
BC\theta^{\text{gmf}}_d \cong \Omega^{\infty-1}MT^{\text{gmf}}(d),
\]

and the forgetting map \(BC\theta^{\text{gmf}}_d \to BC\theta_d\) induces isomorphism

\[
\Omega^{\text{gmf}}_d = \pi_0BC\theta^{\text{gmf}}_d \cong \pi_0BC\theta_d = \Omega_d,
\]

where \(\Omega^{\text{gmf}}_d\) and \(\Omega_d\) are corresponding cobordism groups.

### 3.3. Remarks on the moduli space of Morse functions.

The paper [11] studied the moduli space of fiberwise Morse functions. The fibers are the space of functions which locally has 2-jets of the form \(f : \mathbb{R}^d \to \mathbb{R}, f = f(0) + \ell(x) + q(x)\) subject to the conditions:

1. \(f(0) \neq 0\) or
2. \(f(0) = 0\) and \(\ell(x) \neq 0\) or
3. \(f(0) = 0, \ell(x) = 0\) and \(q(x)\) is non-singular quadratic form.
The associated sheaf $J^{mf}(X)$, denoted by $\mathcal{W}(X)$ in [11], consists of maps
\[(\pi, f, j) : E \longrightarrow X \times \mathbb{R} \times \mathbb{R}^{d-1+\infty} \text{ with} \]
(a) $(\pi, f)$ is proper map,
(b) $(f, j)$ is an embedding,
(c) $\pi : E \longrightarrow X$ is a submersion of relative dimension $d$,
(d) for $x \in X$, $f_x : E_x \rightarrow \mathbb{R}$ is “Morse”, i.e. its 2-jet satisfies the conditions (i), (ii) and (iii).

The space $|J^{mf}| = |\mathcal{W}|$ was determined up to homotopy in Theorems 1.2 and 3.5 of [11]. We recall the results. Let $G^{mf}(d, n)$ be the space of pairs
\[(V, f) \in G(d, n) \times J^2(V) \]
with $f$ satisfying the above conditions (i), (ii) and (iii) and $f(0) = 0$. Let $\hat{U}^\perp_{d,n}$ be the canonical $n$-dimensional bundle over $G^{mf}(d, n)$ and $MT^{{mf}}(d)$ be the spectrum with
\[MT^{{mf}}(d)_{d+n} = \text{Th}(\hat{U}^\perp_{d,n}) \, .\]

**Theorem 3.5.** ([MW]) There is a homotopy equivalence
\[\Omega|J^{mf}| \cong \Omega^{\infty}MT^{{mf}}(d) := \colim_n \Omega^{d+n}\text{Th}(\hat{U}^\perp_{d,n}) \, .\]

Analogous to [11], there is the cofibration of spectra
\[\Sigma^{-1}MT(d-1) \longrightarrow MT^{{mf}}(d) \longrightarrow \Sigma^{\infty}(\Sigma^{mf}(d, \infty)_+) \]
The inclusion $J^{mf}_d(X) \longrightarrow J^{gmf}_d(X)$ induces a map
\[\Omega|J^{mf}_d| \longrightarrow \Omega|J^{gmf}_d| \]
of moduli spaces which can be examined upon comparing the [11] and [17] We have the homotopy commutative diagram of homotopy fibrations
\[
\begin{array}{ccc}
\Omega^{\infty}\Sigma^{-1}MT(d-1) & \longrightarrow & \Omega^{\infty}MT^{mf}(d) \\
\upsilon & 
\cong
& 
\Omega^{\infty}\Sigma^{\infty}(\Sigma^{mf}(d, n)_+) \\
\Omega^{\infty}\Sigma^{-1}MT(d-1) & \longrightarrow & \Omega^{\infty}MT^{gmf}(d) \\
\Omega^{\infty}\Sigma^{\infty}(\Sigma^{gmf}(d, n)_+) & & \\
\end{array}
\]
The middle vertical row can be identified with the map
\[\Omega|J^{mf}_d| \longrightarrow \Omega|J^{gmf}_d| \]
and the right-hand vertical row corresponds to the right-hand arrow of Proposition 3.3. This gives
Corollary 3.6. There is a homotopy fibration
\[ \prod_{i=0}^{d-1} \Omega^\infty \Sigma^{-1}(BO(i) \times BO(d-i-1))_+ \longrightarrow \Omega|J^\text{mf}_d| \longrightarrow \Omega|J^\text{gmf}_d|. \]

3.4. Generalization to tangential structures. Let \( \theta : B \to BO(d) \) be a Serre fibration thought as a structure on \( d \)-dimensional vector bundles: If \( f : X \to BO(d) \) is a map classifying a vector bundle over \( X \), then a map \( \ell : X \to B \) such that \( f = \theta \circ \ell \). For a given \( n \), we define the space \( G^{\theta, \text{gmf}}(d, n) \) as the pull-back:

\[
\begin{array}{ccc}
G^{\theta, \text{gmf}}(d, n) & \longrightarrow & B \\
\downarrow^{\theta_{d,n}} & & \downarrow^{\theta} \\
G^{\text{gmf}}(d, n) & \xrightarrow{i_n} & BO(d)
\end{array}
\]

where \( i_n \) is the composition of the forgetful map \( G^{\text{gmf}}(d, n) \to G(d, n) \) and the canonical embedding \( i^*_n : G(d, n) \hookrightarrow G(d, \infty) = BO(d) \). To define a corresponding sheaf \( J^\theta_d \) of generalized Morse function on manifolds with tangential structure \( \theta \), we use Definition 2.1 but adding the requirement that the manifold \( E \) and corresponding fibers \( E_x = \pi^{-1}(x) \) are equipped with the compatible tangential structures. Similarly the sheaf \( hJ^\theta_d \) is well-defined and there is the corresponding map \( j^3_\pi(\theta) : |J^\theta_d| \longrightarrow |hJ^\theta_d| \). The following result is a generalization of Theorem 2.2 providing the \( h \)-principle:

Theorem 3.7. The map
\[ j^3_\pi(\theta) : |J^\theta_d| \longrightarrow |hJ^\theta_d| \]
is weak homotopy equivalence.

To describe the homotopy type of the moduli space \( \Omega|hJ^\theta_d| \), we consider the bundle diagram:

\[
\begin{array}{ccc}
V^{\theta, \perp}_{d,n} & \longrightarrow & V^\perp_{d,n} \\
\downarrow & & \downarrow \\
G^{\theta, \text{gmf}}(d, n) & \xrightarrow{\theta_{d,n}} & G^{\text{gmf}}(d, n)
\end{array}
\]

where \( V^{\theta, \perp}_{d,n} \) is a pull-back of \( V^\perp_{d,n} \). Similarly, let \( V^\theta_{d,n} \to G^{\theta, \text{gmf}}(d, n) \) be the pull-back of the bundle \( V^\perp_{d,n} \to G^{\text{gmf}}(d, n) \). The Thom space of the bundle \( V^\theta_{d,n} \) gives rise to the spectrum \( MT^{\theta, \text{gmf}}(d) \) which in degrees \( (d + n) \) is

\[
MT^{\theta, \text{gmf}}(d)_{d+n} = \Theta(V^\theta_{d,n}).
\]

We have the following version of Theorem 3.1:

Theorem 3.8. There is weak homotopy equivalence
\[ \Omega|hJ^\theta_d| \simeq \Omega^\infty MT^{\theta, \text{gmf}}(d). \]
We next examine the homotopy type of $\Omega^\infty MT^\theta,\text{gmf}(d)$ in terms of the corresponding singular sets $\Sigma^\theta,\text{gmf}(d,n) \subset G^\theta,\text{gmf}(d,n)$. Define the $\theta$-Grassmannian $G^\theta(d,n)$ as the pull-back:

$$
\begin{array}{ccc}
G^\theta(d,n) & \longrightarrow & B \\
\theta_n \downarrow & & \downarrow \theta \\
G(d,n) & \longrightarrow & BO(d)
\end{array}
$$

where $i_n^\theta : G(d,n) \hookrightarrow BO(n)$ is the canonical embedding,

$$
G^\theta(d,n) = \{ (V,b) \mid i_n^\theta(V) = \theta(b) \} \subset G(d,n) \times BO(d).
$$

Then it is easy to identify $G^\theta,\text{gmf}(d,n)$ with the subspace

$$
G^\theta,\text{gmf}(d,n) = \{ (V,b,f) \mid (V,b) \in G^\theta(d,n), \ f \in J^3_{\text{gmf}}(V) \}.
$$

Let $\Sigma^\theta,\text{gmf}(d,n)$ be the singular set of triples $(V,b,f)$, where $f : V \to \mathbb{R}$ has vanishing linear term. The non-singular subspace $G^\theta,\text{gmf}(d,n) \setminus \Sigma^\theta,\text{gmf}(d,n)$ is the set of triples $(V,b,f)$ with $f = \ell + q + r$, where the linear part $\ell \neq 0$. By analogy with the space $\hat{G}^\theta,\text{gmf}(d,n)$ from Lemma 3.2, we define

$$
\hat{G}^\theta,\text{gmf}(d,n) = \{ (V,b,f) \mid f = \ell + q + r, \ |\ell| = 1, \ q = 0, \ r = 0 \}
$$

and notice that the space $G^\theta,\text{gmf}(d,n) \setminus \Sigma^\theta,\text{gmf}(d,n)$ retracts to $\hat{G}^\theta,\text{gmf}(d,n)$. By construction, the space $\hat{G}^\theta,\text{gmf}(d,n)$ is the pull-back in the diagram:

$$
\begin{array}{ccc}
\hat{G}^\theta,\text{gmf}(d,n) & \longrightarrow & \hat{G}^{\text{gmf}}(d,n) \\
\downarrow & & \downarrow \ g_n \\
G^\theta(d,n) & \longrightarrow & G(d,n)
\end{array}
$$

where $\theta_n : G^\theta(d,n) \to G(d,n)$ is from (21) and $g_n$ is a composition of the forgetting map and the inclusion:

$$
g_n : \hat{G}^{\text{gmf}}(d,n) \hookrightarrow G^{\text{gmf}}(d,n) \to G(d,n)
$$

Similarly to the above case, the normal bundle of the inclusion $\Sigma^\theta,\text{gmf}(d,n) \hookrightarrow G^\theta,\text{gmf}(d,n)$ coincides with $(V^\theta_{d,n})^* \cong V^\theta_{d,n}$ restricted to $\Sigma^\theta,\text{gmf}(d,n)$, and again the inclusion

$$
(D(V^\theta_{d,n}), S(V^\theta_{d,n})) \longrightarrow (G^\theta,\text{gmf}(d,n), G^\theta,\text{gmf}(d,n) \setminus \Sigma^\theta,\text{gmf}(d,n))
$$

is an excision map. Let $j_\theta : G^\theta,\text{gmf}(d,n) \setminus \Sigma^\theta,\text{gmf}(d,n) \to G^\theta,\text{gmf}(d,n)$ be the inclusion. This leads to the cofibration

$$
\mathcal{Th}(j_\theta V^\theta_{d,n}) \to \mathcal{Th}(V^\theta_{d,n}) \to \mathcal{Th}((V^\theta_{d,n} \oplus V^\theta_{d,n})_{\Sigma^\theta,\text{gmf}(d,n)})
$$

and to the cofibration of spectra

$$
\Sigma^{-1}MT^\theta(d - 1) \to MT^\theta,\text{gmf}(d) \to \Sigma^\infty(\Sigma^\theta,\text{gmf}(d, \infty)_+).
$$
with corresponding homotopy fibration of infinite loop spaces:

\[ \Omega^\infty \Sigma^{-1} MT^\theta (d - 1) \rightarrow \Omega^\infty MT^\theta, \text{gmf}(d) \rightarrow \Omega^\infty \Sigma^\infty \left( \Sigma^\theta, \text{gmf}(d, \infty) \right)_+. \]

We denote by \( \mathcal{C}^{\theta} \) the corresponding cobordism category of manifolds equipped with tangential structure \( \theta \) and by \( \mathcal{C}^{\theta, \text{gmf}} \) the category with the condition that each morphism be equipped with a generalized Morse function as above.

**Corollary 3.9.** There is weak homotopy equivalence

\[ BC\mathcal{C}^{\theta, \text{gmf}} \simeq \Omega^\infty MT^\theta, \text{gmf}(d), \]

and the forgetful map \( BC\mathcal{C}^{\theta, \text{gmf}} \rightarrow BC\mathcal{C}^{\theta} \) induces isomorphism

\[ \Omega^\theta, \text{gmf} \simeq \pi_0 BC\mathcal{C}^{\theta, \text{gmf}} \simeq \pi_0 BC\mathcal{C}^{\theta} = \Omega^\theta, \]

where \( \Omega^\theta, \text{gmf} \) and \( \Omega^\theta \) are corresponding cobordism groups.

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