On the nonasymptotic prime number distribution

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Abstract

The objective of this paper is to introduce an approach to the study of the nonasymptotic distribution of prime numbers. The natural numbers are represented by theorem 1 in the matrix form $2N$. The first column of the infinite matrix $2N$ starts with the unit and contains all composite numbers in ascending order. The infinite rows of this matrix except for the first elements contain prime numbers only, which are determined by an uniform recurrence law. At least one of the elements of the twin pairs of prime numbers is an element of the second column of the matrix $2N$ (theorem 3). The basic information on the nonasymptotic prime number distribution is contained in the distribution of the elements of the second column of the matrix $2N$.

1 Introduction

The multiplicative and the additive structure of natural numbers is based on prime numbers. The derivation of results on the distribution of prime numbers in the set of natural numbers can affect almost all mathematical theories and their applications. The objective of this paper is to introduce an approach to the study of the nonasymptotic distribution of prime numbers. The results obtained could be applicable in quantum physics, quantum chemistry and molecular biology.

The Euclid theorem on the existence of an infinite number of prime numbers

$$
\pi(x) \to \infty \quad \text{as} \; x \to \infty, \quad (1)
$$

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(where $\pi(x)$ is the number of primes not exceeding $x$), the Eratosthenes sieve formula

$$\pi(x) = \pi(\sqrt{x}) - 1 + \sum_{d} (-1)^{\nu(d)} \left[ \frac{x}{d} \right],$$

(2)

(where $d$ runs over the divisors of the products of all primes not exceeding $\sqrt{x}$, $\nu(d)$ is the number of the prime divisors of $d$ and $[u]$ is the integer part of $u$) as well as the asymptotic distribution law of prime numbers (see f.i. [1] - [4])

$$\pi(x) = li x + O(xe^{-c\sqrt{\ln x}}) \quad as \quad x \to \infty, \quad c = \text{const} > 1,$$

(3)

where

$$li x = \int_{2}^{x} \frac{dt}{\ln t} = \frac{x}{\ln x} + \frac{1!x}{\ln^2 x} + \ldots + \frac{(k-1)!x}{\ln^k x} + O\left( \frac{x}{\ln^{k-1} x} \right)$$

(4)

do not answer the question how often the prime numbers are encountered and how they are distributed amidst the natural numbers when $x < \infty$. Even if the Riemann fifth hypothesis [5] stating that the nontrivial solutions of the equation

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 0, \quad s = \sigma + it$$

(5)

lie on the complex straight line ($\sigma = 1/2$, $t$) was proved it could not give an answer to that question too. The reason is that the fourth Riemann hypothesis [5] (proved by H. von Mangoldt) stating that the expression

$$P_0(x) = li x - \sum_{\rho} li x^\rho + \int_{x}^{\infty} \frac{du}{(u^2 - 1) \ln u} - \ln 2$$

where $\rho$ runs over all nontrivial solutions of eq. (5) and

$$P_0(x) = \frac{1}{2} (P(x + 0) + P(x - 0)), \quad P(x) = \sum_{n \leq x} \frac{\Lambda(n)}{\ln x},$$

$$\Lambda(n) = \begin{cases} \ln \rho, & \text{when } n = p^m \text{ and } p \text{ is a prime}, \quad m \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

only gives a connection between the distribution of the nontrivial zeroes of eq. (5) and the problem of the prime numbers distribution.

In the present paper an approach based on the separation of subsets of prime numbers for which explicit distribution law exists is used in the search for an answer to the question is there any exact law for $\pi(x)$, $2 \leq x < \infty$ or the prime numbers are spread amidst the natural numbers in total disorder. The separation of these subsets
is made by means of a new sieve ("the multiple Eratosthenes sieve" [6]). The sieve and its generalizations will be published in a separate paper. In the presentation of the results on the nonasymptotic distribution of prime numbers the sieve is present only implicitly which simplifies the text. In the present paper all results are obtained by elementary methods except the use of the inequality from [7].

2 The prime number inner distribution law

Let the set of prime numbers is enumerated in incremental order of elements according to (1)

\[ P = \{2, 3, 5, 7, 11, 13, \ldots \} = \{p_n\}_{n \in \mathbb{N}}, \]

where \( N \) is the set of natural numbers. Thus two reciprocal number-theoretic functions are introduced

\[ \pi(p) : P \to \mathbb{N} \quad (i.e. \quad \pi(p_n) = n) \]

"the number \( \pi(p) \) of the prime number \( p \)" and

\[ \pi^{-1}(n) : \mathbb{N} \to P \quad (i.e. \quad \pi^{-1}(n) = p_n) \]

"the prime number \( \pi^{-1}(n) \) of number \( n \).

It is evident that \( \pi(p) \) and \( \pi^{-1}(n) \) satisfy the identities

\[ \pi(\pi^{-1}(n)) = n \quad (6) \]

and

\[ \pi^{-1}(\pi(p)) = p. \]

These functions are also strictly monotonous

\[ p' < p'' \implies \pi(p') < \pi(p'') , \quad p', p'' \in P, \]

\[ n' < n'' \implies \pi^{-1}(n') < \pi^{-1}(n'') , \quad n', n'' \in \mathbb{N}. \quad (7) \]

The following auxiliary proposition is a corollary from the inequality \( p_n > n \ln n \) from [7].

Lemma 1 For any prime number \( n \geq 2 \) the following estimates are correct:

\[ n > \pi(n) \ln \pi(n) \quad (8) \]

and

\[ n < \frac{\pi^{-1}(n)}{\ln n}. \quad (9) \]
Using the function $\pi^{-1}(n)$ two basic entities of the present paper are introduced: the sequence $\epsilon_{p_0}$ and the aggregate $r_{p_0}$.

**Definition 1** The prime number sequence

$$\epsilon_{p_0} : p_0 \in N, \ p_{k+1} = \pi^{-1}(p_k), \ k = 0, 1, 2, \ldots ,$$

is called "Eratosthenes progression of base $p_0$". The infinite prime number aggregate

$$r_{p_0} = \{p_{k+1} : p_0 \in N, \ p_{k+1} = \pi^{-1}(p_k), \ k = 0, 1, 2, \ldots \}$$

is called "Eratosthenes ray of base $p_0$".

The following two auxiliary propositions are true for the Eratosthenes rays:

**Lemma 2** Let two rays $r'_{p_0}$ and $r''_{p_0}$ of bases

$$p'_0 < p''_0$$

are given. Then if:

1. $p''_0 \in r'_{p_0}$ the ray $r''_{p_0}$ is contained in the ray $r'_{p_0}$;

2. $p''_0 \not\subseteq r'_{p_0}$ the ray $r''_{p_0}$ does not contain common elements with the ray $r'_{p_0}$ and the inequality (12) implies the inequalities

$$p'_k < p''_k, \ k = 0, 1, 2, \ldots .$$

**Proof**

1. The inequality (12) gives the possibility to make the assumption that $p''_0 \in r'_{p_0}$. That means that a number $k^* > 0$ exists so that $p''_0 \equiv p'_k$. Now the inclusion $r_{p_0} \subseteq r_{p_0}$ follows from the recursion $p''_k = \pi^{-1}(p'_{k+k^*})$, $k = 0, 1, 2, \ldots$, given in (11).

2. Let $p''_0 \not\subseteq r'_{p_0}$. Now suppose a number $k^* > 0$ exists so that $p''_{k^*} \in r'_{p_0}$. This implies the existence of a finite reverse sequence

$$p''_k = \pi^{-1}(p''_{k+1}) \in r'_{p_0}, \ \text{where} \ k = k^* - 1, k^* - 2, k^* - 3, \ldots , 0,$$

that terminates on the inclusion $p'_0 \in r'_{p_0}$ which contradicts the initial assumption $p''_0 \not\subseteq r'_{p_0}$. Thus the ray $r''_{p_0}$ does not have common elements with the ray $r'_{p_0}$.
The inequality (12) gives the possibility to use the implication (7) together with the recursion (11) that defines the elements of $r_{p_0}'$ and $r_{p_0}''$ and this leads to the following chain of implications

\[
p_0' < p_0'' \implies p'_1 = \pi^{-1}(p_0') < p''_1 = \pi^{-1}(p_0''), \\
p'_1 < p''_1 \implies p'_2 = \pi^{-1}(p_1') < p''_2 = \pi^{-1}(p_1''), \\
\vdots \\
p'_{k-1} < p''_{k-1} \implies p'_k = \pi^{-1}(p_{k-1}) < p''_k = \pi^{-1}(p_k').
\]

\[
\text{Lemma 3} \quad \text{The principal Eratosthenes rays do not intersect}
\]

\[
\bigcap_{p_0 \in \mathcal{C}} r_{p_0} = \emptyset \tag{14}
\]

where

\[
\mathcal{C} \equiv N \setminus P = \{1, 4, 6, 8, 9, 10, 12, \ldots \}.
\]

The union of all principal rays contains the union of all rays

\[
\bigcup_{p_0 \in P} r_{p_0} \subset \bigcup_{p_0 \in \mathcal{C}} r_{p_0}. \tag{15}
\]

\textbf{Proof}

The proving method is by induction.

The first induction step starts considering the following 6 cases:

- \textbf{Case} $p_0 = 1 \in \mathcal{C}$. The progression $\epsilon_1$ from (10) generates the first principal prime numbers ray $r_1$;

- \textbf{Case} $p_0 = 2 \in r_1$ and $p_0 = 3 \in r_1$. The progressions $\epsilon_2$ and $\epsilon_3$ from (10) determine the rays $r_2$ and $r_3$, which according to lemma 2(1) do not contain any new elements compared to the principal ray that is generated already. So the following inclusion is true:

\[
r_2 \cup r_3 \subset r_1; \tag{16}
\]

- \textbf{Case} $p_0 = 4 \in \mathcal{C}$. The progression $\epsilon_4$ from (10) generates the second principal prime numbers ray $r_4$, which according to lemma 2(2) does not have common elements with the ray $r_1$. Thus their intersection is void:

\[
r_1 \cap r_4 = \emptyset; \tag{17}
\]

\[
5
\]
• Case $p_0 = 5 \in r_1$. According to lemma 2(1) the Eratosthenes progression $\epsilon_5$ generates the ray $r_5 \subset r_1$. Together with (16) this inclusion implies the new inclusion:

$$r_2 \cup r_3 \cup r_5 \subset r_1; \quad (18)$$

• Case $p_0 = 6 \in \overline{C}$. According to lemma 2(2) the Eratosthenes progression $\epsilon_6$ generates the third principal prime numbers ray, which does not contain common elements with the already existing principal rays $r_1$ and $r_4$. Together with the relation (17) this leads to a new particular case of the equality (14)

$$r_1 \cap r_4 \cap r_6 = \emptyset;$$

• Case $p_0 = 7 \in r_4$. According to lemma 2(1) the Eratosthenes progression $\epsilon_7$ generates the ray $r_7 \subset r_4$. This inclusion together with the inclusion (18) imply the extended inclusion

$$(r_2 \cup r_3 \cup r_5 \cup r_7) \subset (r_1 \cup r_4),$$

On its turn this inclusion appears to be a particular case of the inclusion (15).

The second induction step. Suppose the relations

$$\bigcap_{p_0 \in C_n} r_{p_0} = \emptyset, \quad (19)$$

$$R_n \equiv \bigcup_{p_0 \in P_{C_n}} r_{p_0} \subset \bigcup_{p_0 \in C_n} r_{p_0}, \quad (20)$$

where

$$C_n = \bigcup_{i=1,2,3,...,n} c_i, \quad P_{C_n} = \{p \in P : p < c_n\},$$

are satisfied up to the n-th arbitrary element $c_n > 7$ of the set $\overline{C}$.

We shall prove that the relations (19) and (20) remain true for $c_n = c_n + 1$. Let us consider the following two cases:

• Case $p_0 = c_n + 1 \in P$. We shall proceed starting from the inequality

$$c_{n+1} > \pi(c_n + 1) \quad (21)$$

(the prime number $c_{n+1}$ is greater than its ordinal number $\pi(c_n + 1)$, which follows directly from inequality (8).
The right hand side of inequality (21) $\pi(c_n + 1)$ is either a composite or a prime number. Assume that $\pi(c_n + 1)$ is a prime number. Then applying the operation $\pi$ on both sides of (21) we obtain a double chain of inequalities

$$c_{n+1} > \pi(c_n + 1) > \pi^2(c_n + 1).$$

Now we assume that the "second number" $\pi^2(c_n + 1)$ is again a prime number and apply once more the operation $\pi$ on every element of the double chain of inequalities (22). Thus we obtain a triple chain of inequalities analogous to (22). The process can go further $\alpha$-times obtaining $\alpha$-multiple chain of inequalities

$$c_{n+1} > \pi(c_n + 1) > \pi^2(c_n + 1) > \ldots > \pi^{\alpha}(c_n + 1)$$

until the "$\alpha$-number" $\pi^{\alpha}(c_n + 1) \equiv p^{*}_{0} \in \mathcal{C}$. Applying the operation $\pi$ on both sides of the recurrence formula (11) for the ray $\epsilon_{p^{*}_{0}}$ and taking into account the identity (6) we obtain the generator of the finite reverse sequence

$$p^{*}_{i} = \pi(p^{*}_{i+1}), \quad i = \alpha, \alpha - 1, \alpha - 2, \ldots, 0.$$  

The terms of the inequalities (23) are generated by the (24): $c_n + 1 = p^{*}_{\alpha}$, $\pi(c_n + 1) = p^{*}_{\alpha - 1}$, $\pi^2(c_n + 1) = p^{*}_{\alpha - 2}$, $\ldots$, $\pi^{\alpha}(c_n + 1) = p^{*}_{0}$. Thus the inequality $c_n + 1 > \pi^{\alpha}(c_n + 1)$ and the fact that $c_n + 1$ belongs to the principal ray of base $\pi^{\alpha}(c_n + 1)$ (the latter is shown by the sequence (24)) allows for the use of the lemma 2(1). This implies the inclusion $r_{c_n+1} \subset R_n$, which together with eq. (20) allows for the needed extension of the inclusion (20)

$$r_{c_n+1} \cup \left( \bigcup_{p_0 \in \mathcal{P}_n} r_{p_0} \right) = \bigcup_{p_0 \in \mathcal{P}_{n+1}} r_{p_0} \subset \bigcup_{p_0 \in \mathcal{C}_n} r_{p_0}.$$ 

• Case $c_n+1 = c_{n+1} \in \mathcal{C}$. According to lemma 2(2) the Eratosthenes progression $\epsilon_{c_{n+1}}$ generates the new principal ray $r_{c_{n+1}}$, which is not contained in the union $R_n$. So that due to the extension of relation (19)

$$\bigcap_{p_0 \in \mathcal{C}_{n+1}} r_{p_0} = \emptyset.$$ 

The third induction step takes into account the limits $C_n \rightarrow \mathcal{C}$ for $n \rightarrow \infty$ and $P_{c_{n+1}} \rightarrow P$ for $n \rightarrow \infty$. Thus we find out that the relations (19) and (20) in
Lemma 3 makes possible to show that the following basic theorem concerning
the separating of prime numbers into subsets with explicit law for the determination
of their elements actually takes place (Prime Number Separating Theorem - PNST
[6]).

**Theorem 1** The set of prime numbers has a two-dimensional representation labeled
by the index \( k \) and the base \( p_0 \)

\[
P = \bigcup_{p_0 \in N \setminus P} \left\{ r_{p_0} = \{ p_{k+1} : p_{k+1} = \pi^{-1}(p_k), \; k = 0, 1, 2, \ldots \} \right\}. \tag{25}
\]

**Proof**

The union of all Eratosthenes rays of bases covering the set of all natural numbers

\[
Q = \bigcup_{p_0 \in N} \{ p_{k+1} : p_{k+1} = \pi^{-1}(p_k), \; k = 0, 1, 2, \ldots \} \tag{26}
\]

can be represented in the form

\[
Q = Q_1 \bigcup Q_2,
\]

where

\[
Q_1 = \{ p_1 : p_1 = \pi^{-1}(p_0), \; p_0 \in N \},
\]

\[
Q_2 = \bigcup_{p_1 \in P} \{ p_{k+1} : p_{k+1} = \pi^{-1}(p_k), \; k = 1, 2, 3, \ldots \}.
\]

\( Q_1 \) coincides with \( P \) according to the Euclid theorem (1). On the other hand \( Q_2 \) is
contained in \( P \). Thus the right hand side of (26) coincides with \( P \) (i.e. \( P \equiv Q \)).

The set \( Q \) can be also represented in the form

\[
P \equiv Q = Q_3 \bigcup Q_4, \tag{27}
\]

where

\[
Q_3 = \bigcup_{p_0 \in N \setminus P} r_{p_0} and \; Q_4 = \bigcup_{p_0 \in P} r_{p_0}.
\]

Owing to relation (15) from lemma 3 it follows that \( Q_4 \) appears to be a fraction of
\( Q_3 \) (\( Q_4 \subset Q_3 \)). Now it follows from (27) that \( P \equiv Q_3 \).
3 Implications of the prime number separating theorem

Since \( N \setminus P = C \cup \{1\} \equiv \overline{C} \) where \( C \) is the set of composite numbers it appears that PNST is mapping the elements of the extended set of composite numbers \( p_0 \in \overline{C} \) into the infinite prime number rays \( r_{p_0} \). So that we come to

**Corollary 1** There is an unique reciprocal mapping

\[
\varphi_1(p_0) : \overline{C} \rightarrow \ 2P,
\]

which maps the elements \( p_0 \) of the set \( \overline{C} \) into the principal infinite prime number rays \( r_{p_0} \).

Let us note that

\[
2P = \{r_{p_0}\}_{p_0 \in N \setminus P} = \{p_{\mu \nu}\} \quad \mu = 1, 2, 3, \ldots
\]

\[
\nu = 1, 2, 3, \ldots
\]
denotes the prime number representation matrix introduced by eq. (25).

The corollary 1 on its turn is showing that an analogous matrix representation exists for the natural numbers too:

\[
2N = \{\overline{C}, \ 2P\}.
\]

The upper left hand side of the infinite matrix \( 2N \) is given in the Appendix.

Theorem 1 has diverse implications. Here we shall mention just one more. Let us denote the following classes of Eratosthenes rays:

\[
K_1 = r_1,
\]
\[
K_2 = \{r_i\}_{i=2^j}, \ j = 1, 2, 3, \ldots
\]
\[
K_3 = \{r_i\}_{i=3^j} \bigcup r_{2,3}, \ j = 1, 2, 3, \ldots
\]
\[
K_5 = \{r_i\}_{i=5^j} \bigcup r_{2,5} \bigcup r_{3,5}, \ j = 1, 2, 3, \ldots
\]
\[
\vdots
\]
\[
K_p = \{r_i\}_{i=p^j} \bigcup \left( \bigcup_{\alpha \leq p, \alpha \in P} r_{\alpha, i} \right), \ j = 1, 2, 3, \ldots
\]

**Corollary 2** There is an unique reciprocal mapping

\[
\varphi_2(p) : \overline{P} \rightarrow \{K_p\}_{p \in \overline{P}},
\]

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which maps the elements of the set \( \mathcal{P} = \{1\} \cup P \) into the elements of the set of classes \( \{K_p\}_{p \in \mathcal{P}} \). For the equivalence classes \( K_p \) the following set-theoretic equalities
\[
\bigcap_{p \in \mathcal{P}} K_p = \emptyset, \quad \bigcup_{p \in \mathcal{P}} K_p = P.
\]
take place.

The following two propositions for the elements of the matrix \( ^2P \) rows take place.

**Theorem 2** The series
\[
s(p_0) = \sum_{i=1}^{\infty} \frac{1}{p_i(p_0)}, \quad p_0 \in N \setminus P
\]
are convergent.

**Proof**

Following the inequalities (13) from lemma 2(2) the series \( s(1) \) majorates all the series \( s(p_0) \), for \( p_0 \in C \). On its turn for \( i \geq 5 \) the series \( s(1) \) is majorated by the series \( \sum_{i=1}^{\infty} 1/i^2 \). Indeed the inequalities (7), (9) and \( 5^2 < 31 = p_5(1) \), imply the following chain of inequalities
\[
i^2 < 2i^2 < \pi^{-1}(i^2) < \pi^{-1}(p_{i-1}(1)), \quad \text{for all } i = 6, 7, 8, \ldots.
\]

The estimate (8) from lemma 1 implies

**Lemma 4** The spacing between two adjacent elements of a ray from \( \{r_{p_0}\}_{p_0 \in C} \) except for \( p_{1,1} = 2 \in r_1 \) only is estimated by
\[
p_{(k+1)p_0} - pk_{p_0} > pk_{p_0}(\ln pk_{p_0} - 1), \quad \text{for } k = 1, 2, 3, \ldots.
\]  

(28)

Theorem 1 and its implications are applicable to all mathematical constructions which include countable sets, or even sets containing finite segments of \( N \) or \( P \) only.

### 4 An analytic form of the prime number inner distribution law

The right hand side of relation (25) gives the definition for the recurrent element of any ray \( r_{p_0}, \quad p_0 \in C \) in terms of the preceding element of the same ray only:
\[
p_0 \in C, \quad p_{j+1} = \pi^{-1}(p_j), \quad j = 0, 1, 2, 3, \ldots.
\]  

(29)
This rule we comprehend to be the prime number inner distribution law — PNIDL.

The disadvantage of the law (29) is that the function \( \pi^{-1}(x) \) (as well as the function \( \pi(x) \)) can be realized by means of the Eratosthenes sieve only. The Legendre formula (2) and its generalizations (see [8], p.343]) can not be used for the purpose.

So far the attempts to derive analytic formulae for \( \pi^{-1}(p_j) \) for all \( j \times p_0 \) led to formulae which do not allow for a new information on prime numbers. All these formulae represent \( \pi^{-1}(x) \) as a discrete function. Using these formulae it is only possible to determine these prime numbers which are initially presupposed by their construction. These formulae can not be extrapolated so as to increase the amount of prime numbers (formula (2) is an example of that type of formulae).

We shall show here that the Eratosthenes rays can be approximated by continuous functions which have extrapolation properties.

For an arbitrary row of the matrix \( 2P \) from the Appendix we consider the solution of the quadratic system of \( m = 2n \) equations
\[
\left\{ \sum_{k=1}^{n} \alpha_{k,\lambda} q_{k,\lambda}(j) e^{-\beta_{k,\lambda} j} = \frac{\ln \ln \ln p_{1p_0}}{\ln \ln p_{j,\lambda}} \right\}_{j=1,2,3,...,m},
\] respective to the \( n \) unknown pairs \( \{\alpha_{k,\lambda}, \beta_{k,\lambda}\}_{k=1,2,3,...,n} \), where

\[
\begin{align*}
n &= 4 \text{ in case of } 1 \leq p_0 \leq 18 \text{ (the first 11 rays from Appendix)} \\
n &= 3 \text{ in case } 20 \leq p_0 \leq 64 \text{ (the succeeding 32 rays from Appendix)} \\
and \\
n &= 2 \text{ in case of } 65 \leq p_0 \leq 132 \text{ (the last 57 rays from Appendix)}
\end{align*}
\]

The polynomials \( q_{k,\lambda} \) in system (30) take the values
\[
\begin{align*}
q_{k,\lambda}(j) &\equiv 1 \text{ in case } 1 \leq p_0 \leq 16, 20 \leq p_0 \leq 28, 32 \leq p_0 \leq 132; \\
q_{1,18}(j) &\equiv 1, q_{3,18}(j) \equiv q_{4,18}(j) \equiv j \text{ for } p_0 = 18; \\
and \\
q_{1,30}(j) &\equiv 1, q_{3,30}(j) \equiv j, \text{ for } p_0 = 30.
\end{align*}
\]

**Remark 1** There are only two exceptions appearing for the first 100 rays from the Appendix for which \( q_{k,\lambda}(j) \neq 1 \). These are \( p_0 = 18 \) and \( p_0 = 30 \).

**Remark 2** Constituting the system (30) for \( p_0 = 1, 4, 6 \) the first prime number \( p_{1p_0} \) is taken to be that for which the inequality \( \ln \ln \ln (p_{1p_0}) > 0 \) takes place for the first time. So that \( p_{1,1} = 31, p_{1,4} = 17, p_{1,6} = 41 \).
The principle feature of the system (30) is that it is exactly soluble. The coefficients
\[
\{\alpha_{kp}, \beta_{kp}\}_{k=1,2,3,\ldots,n},
\]
\[
p_0 = 1,4,6,\ldots,132
\]
satisfy the system (30) with residuals \(< 10^{-16}\). The solutions of the system (30) have also the following properties:

1. the amplitudes \(\alpha_{kp}\) and the decrements \(\beta_{kp}\) are positive numbers;
2. \(\alpha_{kp}\) and \(\beta_{kp}\) decrease when the index \(k\) is increasing;
3. the increase of the amplitudes \(\alpha_{kp_0}\) corresponds to an increase of the decrements \(\beta_{kp_0}\) (this trend is strictly manifested for the first 26 rays);
4. each pair of two consequent elements of the ray \(r_{p_0}\) determine a new term of the sum (30).

The nonlinear systems (30) were analyzed by means of the program AFXY [9], which determines the number of solutions and their accuracy according to the method developed in [10] – [14]. It was established that all systems (30) – (31) are uniquely soluble, while the systems (30) – (32), (30) – (33) have triple solutions. In average the coefficients \(\{\alpha_{kp_0}, \beta_{kp_0}\}\) have 8 – 10 correct decimal signs.

The solution of the system (30) led to the following approximation for the function \(\tilde{\pi}^{-1}(x)\) which covers all the elements contained in the first 100 Eratosthenes rays from the Appendix:
\[
\tilde{\pi}^{-1}(x; p_0) = \exp \exp \frac{\ln \ln \ln p_1p_0}{\eta(x; p_0, n)}, \quad x \in [1, m] \subset R^1,
\] (34)

where
\[
\eta(x; p_0, n) = \sum_{k=1}^{n} \alpha_{kp_0}q_{kp_0}(x)e^{-\beta_{kp_0}x}.
\]

The formula
\[
p_{j_{p_0}} = \text{round-off} \left(\tilde{\pi}^{-1}(j; p_0)\right)
\] (35)

exactly reproduces the prime numbers from the Appendix up to the even number \(j^* \leq m\).

The function \(\tilde{\pi}^{-1}(x)\) predicts the values of the new prime numbers \(p_{2n+1}^f\). As seen from Table where the mean accuracy values in respect to \(p_0\) are given
\[
\delta_n = \frac{|p_{2n+1} - p_{2n+1}^f|}{p_{2n+1}} \times 100
\]
its prediction accuracy increases with the increase of the number $n$.

**Table**

| $n$ | 2       | 3       | 4       |
|-----|---------|---------|---------|
| $\delta_n$ | $21\% - 16\%$ | $5\% - 1.7\%$ | $2.5\% - 0.19\%$ |

To compare with we point out that for $p_0 = 4$ the relative accuracy of the solution $p_{9,4}^f$ obtained from the equation

$$li \ x = p_{j p_0}$$

when the right hand side is $p_{j p_0} = p_{8,4}$ equals to $0.004\%$.

Let $l$ is the number of the correct decimal signs (the length of a computer word) for which the arithmetic floating point operations in a given computing environment (the computer) are produced. The above found approximation for $\tilde{\pi}^{-1}(x)$, which is limited with respect to its domain of definition, suggests to check the hypothesis:

**Conjecture 1**  
For any ray $r_{p_0}$, $p_0 \in \mathbb{C}$ there exist finite numbers $l$ and $n^*(l)$ such that the natural numbers $\tilde{p}_{mp_0}$ with $m = 2n^*(l) + 1$ predicted by formulae (34), (35) lie closer to the prime numbers $p_{mp_0}$ than the solutions $x$ of equation (36) when its right hand sides are $p_{(m-1)p_0}$.

5  The set of origins of the Eratosthenes rays and coagulates of prime numbers

The set

$$P_1 = \{p_1(p_0) : p_1(p_0) = \pi^{-1}(p_0), \quad p_0 \in \mathbb{C}\} = \{p_{\mu 1}\}_{\mu = 1, 2, 3, \ldots}$$

consists from the start-points (origins) of the principal rays $\{r_{p_0}\}_{p_0 \in \mathbb{C}}$.

The set of prime numbers $^2P = \{p_{\mu \nu}\}_{\mu = 1, 2, 3, \ldots}$ is separated according to $\nu = 1, 2, 3, \ldots$

$^2P = \{P_1, P_{erat}\} \quad where \quad P_{erat} = \{p_{\mu \nu}\}_{\mu = 1, 2, 3, \ldots, \nu = 2, 3, 4, \ldots}$

The set $P_1$ takes a special place amidst the prime numbers. It seems unlikely that a nonasymptotic distribution law similar to the PNIDL (29), accounting for the rows of the matrix $P_{erat}$ too, can be derived for it. Studying the nonasymptotic
distribution of the elements of the subset $P_1$ amidst the prime numbers one can encounter its main peculiarity: $P_1$ contains all possible coagulates of prime numbers such as twin pairs and other closely disposed sequences of prime numbers.

Let the sets of twin-pairs, twin-triples, twin-quadruples, twin-quintuples of prime numbers greater than 5 are denoted respectively by

$$T_1 = \{\overline{p}_i, \overline{p}_i + 2\}_{i=1,2,3,...,l_1},$$
$$T_2 = \{\overline{p}_i, \overline{p}_i + 2, \overline{p}_i + 6\}_{i=1,2,3,...,l_2},$$
$$T_3 = \{\overline{p}_i, \overline{p}_i + 2, \overline{p}_i + 6, \overline{p}_i + 8\}_{i=1,2,3,...,l_3},$$
$$T_4 = \{\overline{p}_i, \overline{p}_i + 2, \overline{p}_i + 6, \overline{p}_i + 8, \overline{p}_i + 12\}_{i=1,2,3,...,l_4}.$$  

Following from lemma 4 is the basic proposition which binds the sets $T_1$, $T_2$, $T_3$, and $T_4$ with the set $P_1$:

**Theorem 3** None of the elements of the subsets $T_1$, $T_2$, $T_3$ and $T_4$ is not contained in the Eratosthenes rays $\{r_{p_0}\}_{p_0} \in \mathbb{C}$. For any $i = 1, 2, 3, \ldots, l_k$, ($k = 1, 2, 3, 4$) only one element from pairs $T_1$, only two elements from the triplets $T_2$ and the quadruplets $T_3$, and only three elements from the quintuplets $T_4$ could not be origins of certain principal ray $\{r_{p_0}\}_{p_0} \in \mathbb{C}$.

In theorem 3 the words "could not be origins" mean a probability whose amount among the first 104683 elements of the set $T_1$ does not exceed the value 0.15 (1270 twin-pairs are contained among these first elements). The elements of the sets $T_2$, $T_3$, and $T_4$ occur much more rarely amidst the natural numbers compared to these of the set $T_1$. Thus theorem 3 is showing that the essential part of the prime numbers included in the sets $T_1$, $T_2$, $T_3$ and $T_4$ appear to be origins of principal Eratosthenes rays.

A twin-pair assigns a characteristic "arhythmicity of condensation" among the coagulates from $T_2$, $T_3$ and $T_4$ as well as among all elements of the set $P_1$. Let us note that twin-pairs occur twice in $T_3$ and $T_4$. For instance in the elements $t_{4,1} = \{11, 13, 17, 19, 23\} \in T_4$ and $t_{4,2} = \{101, 103, 107, 109, 113\} \in T_4$ these are the pairs $\{11, 13\}$, $\{17, 19\}$, and $\{101, 103\}$, $\{107, 109\}$ respectively.

Theorem 1 and theorem 3 together show that the Eratosthenes rays "coagulate between themselves" through the twin-pairs only and that according to the character of the "ray coagulates" these twin-pairs are classified into two new types:

1. pairs such as $\{\overline{p}_{iu}, \overline{p}_{iu} + 2\}$ which simultaneously give origin of two new rays;
2. and the pairs $\{\overline{p}_{ib}, \overline{p}_{ib} + 2\}$ for which one of the elements is the origin of a new ray while the other "coagulates a new ray" branching with an already existing ray.
Thus

$$T_1 = T_{1u} \bigcup T_{1b} ,$$

(37)

where

$$T_{1u} = \{ p_{iu}, p_{iu} + 2 \}_{i=1,2,3,...,l_5} \subset P_1$$

and

$$T_{1b} = \{ p_{ib}, p_{ib} + 2 \}_{i=1,2,3,...,l_6} = T_1 \setminus T_{2u} .$$

As seen from the Appendix \{71, 73\}, \{101, 103\}, \{137, 139\} and \{149, 151\} are examples of u-pairs while \{11, 13\}, \{17, 19\}, \{59, 61\} and \{107, 109\} are examples of b-pairs.

The collective divisibility coefficient – CDC

$$D(l, s) = \frac{(d(l-s+1) + d(l-s+2) + \ldots + d(l))}{s+2} ,$$

(38)

where \(l \geq 13\), \(s \geq 3\) are natural numbers, \(d(\lambda)\) is the number of the divisors of the natural \(\lambda\) except the unit, when applied to the consequent triples \((s = 3)\) of natural numbers \(p, p+1, p+2\) leads to a new class of twin-pairs — those for which \(D(l,3) = 1\) (the Strongly Related Twins – \(T_{sr}\)):

$$T_{sr} = \{ \{11, 13\}, \{17, 19\}, \{29, 31\}, \{41, 43\}, \{101, 103\}, \{137, 139\}, \ldots \} = \{p_i, p_i + 2\}_{i=1,2,3,...,l_7} .$$

(39)

The even number \(p+1\) from the twin-triple \(\{p, p+1, p+2\}\) is factorized as follows

$$p + 1 = 2 \cdot 3 \cdot \sigma , \quad \sigma \in P .$$

(40)

The numbers \(\sigma = 2, 3, 5, 7, 17, 23\) from the factorization (40) correspond to the twin-pairs (39) and form the origin of a new subset \(\Sigma \subset P\) which has a curious property: the last decimal digit of all \(\sigma \in \Sigma\) is either 3 or 7 (never 1 or 9; this property is checked up to the \(T_{sr}\) – pair \(\{47777, 47779\}\)).

**Remark 3**

$$\min_{l \geq 13, s \geq 3} D(l, s) = D(l, 3) = 1 .$$

**Remark 4** The exceptions mentioned in Remark 1 are the even numbers 18 and 30 corresponded to \(T_{sr}\) \{17, 19\} and \{29, 31\}.
The decomposition (39) is carried over the elements of the set $T_{sr}$ too

$$T_{sr} = T_{(sr)u} \cup T_{(sr)b},$$

where

$$T_{(sr)u} = \{\overline{p}_{iu}, \overline{p}_{iu} + 2\}_{i=1,2,3,...,l_8}$$

and

$$T_{(sr)b} = \{\overline{p}_{ib}, \overline{p}_{ib} + 2\}_{i=1,2,3,...,l_9}.$$  

The existence of a potential infinity $l_k = \infty$ for the lengths $l_k$, $k = 1, 2, 3, \ldots, 9$ of the introduced sets of prime numbers is still not proved even for the simplest case $k = 1$.

The following extension of the V. Brun [15] theorem is possible:

**Theorem 4** If $l_k = \infty$ for $k = 1, 5, 6, 7, 8, 9$ the series

$$\sum_{i=1}^{\infty} \left( \frac{1}{v_i} + \frac{1}{v_i + 2} \right), \quad \text{where } v_i = \overline{p}_i, \overline{p}_{iu}, \overline{p}_{ib}, \overline{p}_i, \overline{p}_{iu}, \overline{p}_{ib}$$

are convergent.

In case of $l_2 = l_3 = l_4 = \infty$ a statement analogous to theorem 4 will take place for the elements of the sets $T_2, T_3,$ and $T_4$.

Besides the sets $T_2$ and $T_3$ one can consider the sets of coagulates of the type

$\{p_i, p_i + 4, p_i + 6\}, \{p_i, p_i + 2, p_i + 8\}, \{p_i, p_i + 4, p_i + 6, p_i + 10\}$ and $\{p_i, p_i + 2, p_i + 8, p_i + 12\}$ too. For these sets the analogues of theorems 3 and 4 are also correct.

Let us introduce the set

$$\tilde{P}_1 = P_1 \setminus (T_1 \cup T_2 \cup T_3 \cup T_4) \equiv \{\tilde{p}_{i1}\}_{i=1,2,3,...}.$$  

Treating numerically the origins of the partial sums of the reciprocal values of the elements of the columns of the matrix $2P$ we come to a hypothesis concerning the set of origins of the principal rays of $P_1$ which is opposite in since of the V. Brun theorem:

**Conjecture 2** The series

$$\sum_{i=1}^{\infty} \frac{1}{w_i}, \quad \text{where } w_i = p_{i1}, \tilde{p}_{i1}, \ p_{i\nu} \quad (\nu = 2, 3, 4, \ldots)$$  \hspace{1cm} (41)

is divergent.
Remark 5  The divergency of the series (41) is very slow for $w_i = p_{i1}$, $\tilde{p}_{i1}$ and even more slow for $w_i = p_{i\nu}$. What is more the slowness is growing with $\nu \to \infty$ (?).

The coagulates of primes elements of the sets $T_1, T_2, T_3$ and $T_4$ considered so far should be generalized as

$$
\text{coag} \left( p, \{u_i\}_{i=1,2,3,...,l} \right) = \{p, p + u_1, p + u_2, \ldots, p + u_l\}
$$

where the steps $\{u_i\}$ should be smaller than the steps $p_{(k+1)p_0} - p_{kp_0} = \Delta\epsilon_{kp_0}$ in the nearest rays while the rays and their steps themselves (i.e. the particular $p_0$ and $k$) are determined by the initial prime number $p$ and the length $l$ of the generalized coagulate. The pointed out relativity of the values of $p, l$ and $\Delta\epsilon_{kp_0}$ in $\text{coag} \left( p, \{u_i\}_{i=1,2,3,...,l} \right)$ is used in order to obtain the conditions under which the set of generalized coags satisfy theorems analogous to theorems 3 and 4.

The general meaning of theorems 3 and 4, and conjecture 2 is that the main information on the originality of the nonasymptotic prime-numbers distribution is concentrated in the set of start-points $P_1$; by its nature the set $P_1$ contains "enough amount" of prime-numbers and they are disposed "enough closely" (conjecture 2), however only the set of the type of $T_1, T_2, T_3$ and $T_4$, which contain "relatively small amount of elements" (theorem 4) introduce the characteristic, unique arrhythmia of concentration of elements of the set $P_1$. 

section*{Appendix}

The left upper corner of the infinite matrix $^{2}N$.

|   | 2   | 3   | 5   | 11  | 31  |
|---|-----|-----|-----|-----|-----|
| 1 | 127 | 709 | 5381| 52711| 648391|
|   | 973733 | 174440041... |
| 4 | 7   | 17  | 59  | 277 | 1787|
|   | 15299 | 167449 | 2269733 | 37139213 | 718064159... |
| 6 | 13  | 41  | 179 | 1063| 8527|
|   | 87803 | 1128889 | 17624813 | 326851121... |
| 8 | 19  | 67  | 331 | 2221| 19577|
|   | 219613 | 3042161 | 50728129... |
| 9 | 23  | 83  | 431 | 3001| 27457|
|   | 318211 | 4535189 | 77557187... |
| 10| 29  | 109 | 599 | 4397| 42043|
|   | 506683 | 7474967 | 131807699... |
| 12| 37  | 157 | 919 | 7193| 72727|
|   | 919913 | 14161729 | 259336153... |
| 14| 43  | 191 | 1153| 9319| 96797|
|   | 1254739 | 19734581 | 368345293... |
| 15| 47  | 211 | 1297| 10631| 112129|
|   | 1471343 | 23391799 | 440817757... |
| 16| 53  | 241 | 1523| 12763| 137077|
|   | 1828669 | 29499439 | 563167303... |
| 18| 61  | 283 | 1847| 15823| 173867|
|   | 2364361 | 38790341 | 751783477... |
| 20| 71  | 353 | 2381| 21179| 239489|
|   | 3338989 | 56011909... |
| 21| 73  | 367 | 2477| 22093| 250751|
|   | 3509299 | 59053067... |
| 22| 79  | 401 | 2749| 24859| 285191|
|   | 4030889 | 68425619... |
| 24| 89  | 461 | 3259| 30133| 352007|
|   | 5054303 | 87019979... |
| 25| 97  | 509 | 3637| 33967| 401519|
|   | 5823667 | 101146501... |
| 26| 101 | 547 | 3943| 37217| 443419|
|   | 6478961 | 113256643... |
The left upper ... Continuation

|   | 27  | 28  | 30  | 32  | 33  | 34  | 35  | 36  | 38  | 39  | 40  | 42  | 44  | 45  | 46  | 48  | 49  | 50  | 51  |
|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|   | 103 | 107 | 113 | 131 | 137 | 139 | 149 | 151 | 163 | 167 | 173 | 181 | 193 | 197 | 199 | 223 | 227 | 229 | 233 |
|   | 6816631 | 587 | 617 | 739 | 773 | 797 | 859 | 877 | 967 | 991 | 1031 | 1087 | 1171 | 1201 | 1217 | 1409 | 1433 | 1447 | 1471 | 109535373 |...|
|   | 119535373 | 4273 | 4549 | 5623 | 5869 | 6113 | 6661 | 6823 | 7607 | 8441 | 8221 | 8719 | 9461 | 9739 | 9859 | 11743 | 11953 | 12097 | 12301 | 119535373 |...|
|   | 4091 | 40819 | 43651 | 55351 | 57943 | 60647 | 66851 | 68639 | 77431 | 80071 | 84347 | 90023 | 98519 | 101701 | 103069 | 125113 | 127643 | 129229 | 131707 | 119535373 |...|
|   | 38833 | 490643 | 527623 | 683873 | 718807 | 755387 | 839483 | 864013 | 985151 | 1021271 | 1080923 | 1159901 | 1278779 | 1323503 | 1342907 | 1656649 | 1693031 | 1715761 | 1751411 | 119535373 |...|
|   | 464939 | 7220981 | 7807321 | 10311439 | 10875147 | 11469013 | 12393937 | 13243033 | 15239333 | 1751411 | 1656649 | 1693031 |...|
|   |   |   |   |   |   |
|---|---|---|---|---|---|
| 52 | 239 | 1499 | 12547 | 134597 | 1793237 |
|   | 28889363... |
| 54 | 251 | 1597 | 13469 | 145547 | 1950629 |
|   | 31599859... |
| 55 | 257 | 1621 | 13709 | 148439 | 1993039 |
|   | 32332763... |
| 56 | 263 | 1669 | 14177 | 153877 | 2071583 |
|   | 33691309... |
| 57 | 269 | 1723 | 14723 | 160483 | 2167937 |
|   | 35368547... |
| 58 | 271 | 1741 | 14867 | 162257 | 2193689 |
|   | 35815873... |
| 60 | 281 | 1823 | 15641 | 171697 | 2332537 |
|   | 38235377... |
| 62 | 293 | 1913 | 16519 | 182261 | 2487943... |
| 63 | 307 | 2027 | 17627 | 195677 | 2685911... |
| 64 | 311 | 2063 | 17987 | 200017 | 2750357... |
| 65 | 313 | 2081 | 18149 | 202001 | 2779781... |
| 66 | 317 | 2099 | 18311 | 204067 | 2810191... |
| 68 | 337 | 2269 | 20063 | 225503 | 3129913... |
| 69 | 347 | 2341 | 20773 | 234293 | 3260657... |
| 70 | 349 | 2351 | 20899 | 235891 | 3284657... |
| 72 | 359 | 2417 | 21529 | 243781 | 3403457... |
| 74 | 373 | 2549 | 22811 | 259657 | 3643579... |
| 75 | 379 | 2609 | 23431 | 267439 | 3760921... |
| 76 | 383 | 2647 | 23801 | 271939 | 3829223... |
| 77 | 389 | 2683 | 24107 | 275837 | 3888551... |
| 78 | 397 | 2719 | 24509 | 280913 | 3965483... |
| 80 | 409 | 2803 | 25423 | 292489 | 4142053... |
| 81 | 419 | 2897 | 26371 | 304553 | 4326473... |
| 82 | 421 | 2909 | 26489 | 305999 | 4348681... |
| 84 | 433 | 3019 | 27689 | 321017 | 4578163... |
| 85 | 439 | 3067 | 28109 | 326203 | 4658099... |
| 86 | 443 | 3109 | 28573 | 332099 | 4748047... |
| 87 | 449 | 3169 | 29153 | 339601 | 4863959... |
| 88 | 457 | 3229 | 29803 | 347849 | 4989697... |
| 90 | 463 | 3299 | 30557 | 357473 | 5138719... |
The left upper ... End.

|    |    |    |    |    |    |
|----|----|----|----|----|----|
| 91 | 467| 3319| 30781| 360293| 5182717...|
| 92 | 479| 3407| 31667| 371981| 5363167...|
| 93 | 487| 3469| 32341| 380557| 5496349...|
| 94 | 491| 3517| 32797| 386401| 5587537...|
| 95 | 499| 3559| 33203| 391711| 5670851...|
| 96 | 503| 3593| 33569| 396269| 5741453...|
| 98 | 521| 3733| 35023| 415253| 6037513...|
| 99 | 523| 3761| 35311| 418961| 6095731...|
|100 | 541| 3911| 36887| 439357| 6415081...|
|102 | 557| 4027| 38153| 455849| 6673993...|
|104 | 569| 4133| 39239| 470207| 6898807...|
|105 | 571| 4153| 39451| 472837| 6940103...|
|106 | 577| 4217| 40151| 481847| 7081709...|
|108 | 593| 4339| 41491| 499403| 7359427...|
|110 | 601| 4421| 42293| 510031| 7528669...|
|111 | 607| 4463| 42697| 515401| 7612799...|
|112 | 613| 4517| 43283| 522829| 7730539...|
|114 | 619| 4567| 43889| 530773| 7856939...|
|115 | 631| 4663| 44879| 543967| 8066533...|
|116 | 641| 4759| 45071| 558643| 8300687...|
|117 | 643| 4787| 46279| 562711| 8365481...|
|118 | 647| 4801| 46451| 565069| 8402833...|
|119 | 653| 4877| 47297| 576203| 8580151...|
|120 | 659| 4933| 47857| 583523| 8699617...|
|121 | 661| 4943| 47963| 584999| 8720227...|
|122 | 673| 5021| 48821| 596243| 8900383...|
|123 | 677| 5059| 49207| 601397| 8982923...|
|124 | 683| 5107| 49739| 608459| 9096533...|
|125 | 691| 5189| 50591| 619739| 9276991...|
|126 | 701| 5281| 51599| 633467| 9498161...|
|128 | 719| 5441| 53353| 657121| 9878657...|
|129 | 727| 5503| 54013| 665843|...|
|130 | 733| 5557| 54601| 673793|...|
|132 | 743| 5651| 55681| 688249|...|

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