Status of Average-\(x\) from Lattice QCD

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Abstract. As algorithms and computing power have advanced, lattice QCD has become a precision technique for many QCD observables. However, the calculation of nucleon matrix elements remains an open challenge. I summarize the status of the lattice effort by examining one observable that has come to represent this challenge, average-\(x\): the fraction of the nucleon’s momentum carried by its quark constituents. Recent results confirm a long standing tendency to overshoot the experimentally measured value. Understanding this puzzle is essential to not only the lattice calculation of nucleon properties but also the broader effort to determine hadron structure from QCD.

Keywords: moments of parton distribution functions, hadron structure, lattice QCD

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INTRODUCTION

Understanding hadron structure from first principles is a fundamental goal of lattice QCD. The nucleon plays a special role as a benchmark for lattice calculations due to the extensive experimental effort to measure its properties. Successfully reproducing the measured values of basic observables, like the axial charge measured in neutron beta decay or the charge radius measured in elastic electron scattering, would provide a strong validation of the lattice technique. This would not only give confidence in the calculations of the many other properties of the nucleon but it would also bolster the lattice effort to calculate hadron structure more generally. Furthermore, there is a burgeoning program to calculate nuclear properties using lattice QCD. It is clearly essential to have a well-controlled calculation of the single nucleon state in order to trust future computations of multi-nucleon systems. Thus in many ways the nucleon is the keystone for a much broader lattice QCD program to understand the properties of hadrons as predicted from the underlying theory of QCD.

I have chosen to illustrate the status of the lattice effort to understand nucleon structure by focusing on average-\(x\). This quantity has persistently come out too high from lattice calculations. A variety of explanations have been offered over the years, and I’ll mention a few, but recent calculations have dramatically confirmed this trend. The apparent disagreement with experiment is a real puzzle and its resolution will likely require a concentrated effort to carefully examine all sources of error in the lattice calculation.

The advantage of using lattice QCD, as opposed to any other technique, to calculate average-\(x\) is that the list of possible errors is finite and each source of error can be systematically removed. This is both a challenge and an opportunity for lattice QCD.
AVERAGE-$x$

By average-$x$, I mean specifically the difference of the up and down contributions. Written as a moment of the nucleon parton distributions, average-$x$ is

$$\langle x \rangle_u^u - d\mu = \int_0^1 dx x (u(x, \mu) - d(x, \mu)) + \int_0^1 dx x(\bar{u}(x, \mu) - \bar{d}(x, \mu)).$$

(1)

The unpolarized quark and anti-quark distribution functions $q(x, \mu)$ for $q = u, d, \bar{u}$ and $\bar{d}$ are extracted from the results of many experiments, particularly deeply inelastic electron-nucleon scattering. It is important to remember that the various $q(x, \mu)$ are indirectly determined by performing global fits to the measured cross-sections and that the values of $x$ are limited by the kinematics of each experiment. For this reason, it would be preferable to calculate $q(x, \mu)$ directly as a function of $x$, but this is not possible with the lattice QCD methods that we currently have. This is because lattice computations are performed in Euclidean space whereas the distributions $q(x, \mu)$ are related to the square of the light-cone wave function, which are not easily accessible outside of Minkowski space-time. However, moments of the quark distributions can be related to matrix elements of local operators, and these are calculable in Euclidean space. Thus lattice computations determine $\langle x \rangle_u^u - d\mu$ from

$$\langle p, s | (\pi \gamma_\mu iD_\nu u - \bar{d}\gamma_\mu iD_\nu \bar{d}) | p, s \rangle_{\mu} = 2 \langle x \rangle_u^u - d\mu p_{\mu} p_\nu.$$ 

(2)

As we begin to contemplate what calculations are required for a definitive determination of average-$x$, it is important to keep in mind that there is a significant difference between what is computed and how that is measured. Currently, the burden is on the lattice community to nail down the calculations of the nucleon matrix elements, but it is not inconceivable that there may ultimately be some subtlety in the comparison of Eq. (1) and Eq. (2). For simplicity, in the following the renormalization scale $\mu$ will be dropped and $\langle x \rangle_u^u - d\mu$ will be understood as evaluated in the $\overline{MS}$-scheme with $\mu = 2$ GeV.

PERSISTENT PUZZLE

The puzzle with $\langle x \rangle_u^u - d\mu$ began with the earliest quenched lattice calculations. As an example, in Fig. 1 I show a quenched calculation of average-$x$ from [1]. As seen there, $\langle x \rangle_u^u - d\mu$ has a mild, nearly flat, pion mass dependence. Naively, this is not entirely unexpected. Dimensionless quantities like $\langle x \rangle_u^u - d\mu$ tend to have a weaker dependence on the quark mass than dimensionful quantities like the nucleon mass. In fact, this sort of behavior would normally be welcomed, except in this case the lattice calculation of $\langle x \rangle_u^u - d\mu$ clearly overshoots the phenomenologically determined value.

At the time it was natural to dismiss this problem as simply being an artifact of the quenched approximation that drops all contributions from the so-called sea-quark loops. However, the results from the earliest full QCD calculation [3], which included two dynamical quark flavors, were found to agree with the quenched calculations. The two-flavor results from [3] are shown in Fig. 2, where again the lattice calculations came out
FIGURE 1. Example quenched results for $\langle x \rangle^{u-d}$. The quenched results for $\langle x \rangle^{u-d}$ from [1] are plotted against the square of the pion mass $m_{PS}$ in units of the inverse Sommer scale $r_0^{-1}$. The combination $(r_0 m_{PS})^2$ is proportional to the quark mass in some units as the chiral limit is approached. The quenched results for $\langle x \rangle^{u-d}$ have a quite mild quark mass, equivalently pion mass, dependence over a large range of pion masses. Additionally, the linear extrapolation of this pion mass dependence results in a substantial overestimate of the phenomenologically determined value for $\langle x \rangle^{u-d}$. The quenched approximation was a potential source of this discrepancy that has since been eliminated. This plot was taken from [1].

Staying with Fig. 2, another explanation for the puzzling behavior of $\langle x \rangle^{u-d}$ was put forth. The pion mass dependence of average-$x$ was calculated in chiral perturbation theory [4, 5] and combined with a phenomenological regulator [6] that was capable of accommodating both the lattice calculation and the physical value of $\langle x \rangle^{u-d}$. It was understood that the pion masses were too heavy to apply chiral perturbation but [6] offered a plausibility argument that chiral dynamics may lead to a strong quark mass dependence for yet lighter quark masses while producing a mild quark mass dependence for the range of quark masses used in contemporary lattice computations. This was put on a slightly stronger footing with calculations to higher-order in the chiral expansion [7]. It was shown that appropriate choices of the undetermined counterterms in the resulting functional form could lead to a flat pion mass dependence for heavy pion masses [7, 8].

This line of reasoning has dominated the lattice effort on nucleon structure for much of the last decade. It was understood that physically motivated regulators would introduce model dependence to the extrapolation of the lattice calculations. It also seemed that higher-order calculations would require the determination of too many extra counterterms and low energy constants to be practically useful. Thus the hope was to push to light enough pion masses to directly observe the missing chiral logarithms. These are the contributions to $\langle x \rangle^{u-d}$ of the form $m_{\pi}^2 \ln m_{\pi}^2$ that are, more or less, uniquely predicted by chiral perturbation theory.

Fully dynamical lattice calculations of $\langle x \rangle^{u-d}$ have continued to lighter quark masses in search of these missing logarithms. The initial two-flavor calculations [3, 9, 10, 11] have been extend to include the strange quark [12, 8, 13] and extended further to even include the charm quark [14]. The basic observation from the earliest quenched calculations remains correct: average-$x$ appears to have a mild pion mass dependence,
FIGURE 2. First full QCD results for $\langle x \rangle^{u-d}$. The results for $\langle x \rangle^{u-d}$ from [3] are shown. They are plotted against the square of the pion mass. Similar to the quenched results, average-$x$ from full QCD calculations, like the one shown here, have a mild pion mass dependence and overshoot the phenomenological value for $\langle x \rangle^{u-d}$. Chiral perturbation theory predicts the leading pion mass dependence of $\langle x \rangle^{u-d}$ [4, 5]. When combined with a physically motivated regulator [6], the resulting functional form was capable of smoothly matching the lattice computation to the expected pion mass dependence in the chiral limit. Thus it was hypothesized that chiral dynamics might help explain the seemingly strong quark mass dependence that would be required to make the lattice results agree with the physical value of $\langle x \rangle^{u-d}$. This explanation is being challenged by current calculations. This plot was taken from [3].

the extrapolation of which is higher than the physical value. The lightest pion mass used in the calculations referenced so far was approximately 250 MeV. Understanding that finite-size effects and lattice artifacts are seldom checked at the lightest pion mass used in a calculation, we could argue that the lightest reliable pion mass was closer to 300 MeV. This left some room for the rapid pion mass dependence that would be required to reconcile the lattice computation with the experimental measurement of $\langle x \rangle^{u-d}$, but recent calculations have begun to challenge this scenario.

RECENT RESULTS

Many of the most recent results were summarized in [15], but rather than showing all the calculations of average-$x$, I focus on the results from the QCDSF collaboration [11]. Their calculation of $\langle x \rangle^{u-d}$ illustrates the most recent trend in lattice calculations: several collaborations are now calculating at or near the physical pion mass [16, 17, 18]. There are a variety of compromises that are made to accomplish this, but it is still a very important advance. The calculation of $\langle x \rangle^{u-d}$ from [11], with pion masses approaching
FIGURE 3. Recent results from the QCDSF collaboration for $\langle \chi \rangle^{u-d}$. Preliminary results from the QCDSF collaboration [11] for $\langle \chi \rangle^{u-d}$ are plotted versus the square of the pion mass. These results rather dramatically continue the long trend of lattice calculations of $\langle \chi \rangle^{u-d}$ with quite mild pion mass dependence that extrapolates to values noticeably higher than the experimental measurement. These results are challenging the prevailing view that chiral dynamics would cause sufficient curvature in $m_{\pi}^2$ to reconcile the lattice calculations at heavy pion masses with the value of $\langle \chi \rangle^{u-d}$ at the physical point. Also shown are a set of the most recent results for $\langle \chi \rangle^{u-d}$ from global analyses. These were collected in [19] using results from [20, 21, 22, 23, 24, 25]. Note that all these results are N$^2$LO except for the one explicitly marked as N$^3$LO. The results in this plot were communicated privately by the QCDSF collaboration.

the physical pion mass, is shown in Fig. 3. It is very plain to see that these latest results for average-$\chi$ confirm the nearly flat pion mass dependence of $\langle \chi \rangle^{u-d}$ down to essentially the physical point. This calculation achieves a long sought milestone, but the conclusion is far from clear.

If the results from [11] are taken at face value, then it is hard to escape the obvious conclusion that one would draw from Fig. 3. Either there are unaccounted for sources of error in the lattice computation (or the global fits) or there is a sizable discrepancy between the lattice determination of $\langle \chi \rangle^{u-d}$ and the experimental measurement of it. This later option seems unlikely, so the current view among those doing the lattice calculations is that one or more of the systematic errors that must be checked for in lattice calculations is currently underestimated.

Regarding the possibility of underestimated errors in the value of $\langle \chi \rangle^{u-d}$ extracted from the experimental measurements, I have shown the results from six recent analyses of average-$\chi$ in Fig. 3. There is some spread beyond that expected by the quoted errors on $\langle \chi \rangle^{u-d}$, but it is certainly not large enough to account for the difference between the lattice results and the global fits. Thus it seems unlikely that there is a significant problem in the phenomenological results for $\langle \chi \rangle^{u-d}$, but it is useful to keep in mind that
an extrapolation in $x$ is required to evaluate the integral in Eq. (1). Additionally, lattice calculations often fail to specify the order to which the matching to $\overline{MS}$ is done, but the difference between the experimental $N^2$LO and $N^3$LO results in Fig. 3 suggests that this effect is small.

**SYSTEMATIC ERRORS**

The resolution of the puzzle in Fig. 3 will likely hinge on a careful examination of all the systematic errors present in the lattice calculation of $\langle x \rangle^{u-d}$. Most of these sources of uncertainty have been checked, to some extent, in previous calculations, so it was believed that the dominant source of error in $\langle x \rangle^{u-d}$ was due to the poorly constrained extrapolation in the pion mass. However, the results in Fig. 3 now suggest that this might not be the case. Certainly, the chiral extrapolation is no longer the single stand out systematic error. This raises the possibility that other errors were underestimated or that the discrepancy in Fig. 3 could be a combination of several smaller uncertainties.

The advantage of using a renormalizable description of the fundamental theory is that we know with confidence that the list of potential errors is very limited. First, we have to reliably calculate the basic nucleon matrix element in Eq. (2). This involves the underlying algorithms used to stochastically evaluate the functional integrals that define the matrix element. Since these methods are used in many successful lattice computations, it seems unlikely that there is a special algorithmic problem for the nucleon. Calculating the matrix element in Eq. (2) also requires isolating the ground state, corresponding to the nucleon, using Euclidean space methods. Several ongoing investigations [26, 27] suggest that this may be responsible for some, but not all, of the discrepancy in $\langle x \rangle^{u-d}$. This can be called the *plateau problem* because most calculations of nucleon matrix elements rely on finding a plateau as a function of Euclidean time in appropriately chosen correlation functions. This issue has been examined off and on, most recently in [28], and a variety of new methods have been developed to address it [29, 30].

Once we have calculated the so-to-speak bare matrix element, the operator renormalization must be accounted for. This is now regularly calculated using nonperturbative methods, thus eliminating one potential source of error. However, the method for renormalizing composite operators nonperturbatively has its own set of potential systematic errors. Most of these are quite technical in nature and go well beyond the level of presentation in these proceedings, but the overarching concern regards the separation of scales that is necessary to nonperturbatively match the lattice operator to the continuum $\overline{MS}$ operator needed in Eq. (2). Because $\overline{MS}$ is a perturbative scheme, the matching must ultimately involve some form of perturbation theory. To reduce the error from this, the matching is done, ideally, at large renormalization scales $\mu$, but this runs afoul of the constraint $\mu \ll 1/a$ that must be maintained to control lattice cut-off effects.

There is some indirect evidence for a problem in the renormalization. In [19] it was pointed out that there does appear to be some variation in the relative normalization of the results for $\langle x \rangle^{u-d}$ from different actions. Additionally, it was be found that ratios of observables that cancel the renormalization lead to results in agreement with experimental measurements [11, 16]. Additionally, the renormalization is an interesting
potential culprit because it would lead to a simple rescaling of $\langle x \rangle^{u-d}$. This is because $\overline{MS}$ is a mass independent scheme, so the renormalization of the operator depends only on the lattice spacing and not the quark masses. Thus a mistake in the renormalization would correspond to a multiplicative rescaling of Fig. 3, for example. It seems unlikely that such an effect would account for the entire discrepancy, but it may be one piece of the puzzle. As a separate cross check, there are attempts to eliminate the renormalization issue entirely and calculate moments of structure functions directly [31].

Having reliably calculated the properly renormalized matrix element, then the only remaining systematic errors are the extrapolation of the heavier-than-physical pion mass to the physical pion mass and the continuum and infinite-volume limits. Collectively, the results from all the calculations of $\langle x \rangle^{u-d}$ suggest that these errors are small within the range of pion mass, lattice spacing and volumes that have been used. The concern, though, is that the asymptotic values of each of the three limits may not have been seen in the currently used ranges. The size of lattice artifacts can be checked by establishing not just weak lattice spacing dependence, as is customary, but by demonstrating the expected scaling as the continuum limit is approached. For the $L$ dependence, one could explicitly check the generically expected exponential suppression at large $L$, rather than the current standard of simply demonstrating an apparent convergence to within the errors. It is hard to point to any compelling indications of finite-size or cut-off effects in the current calculations of $\langle x \rangle^{u-d}$, but that may be so because these issues have never been pursued with the high precision likely required to detect such effects.

The pion mass dependence is harder to check because the only expectations come from chiral perturbation theory and that may simply not be applicable for the physical pion mass or heavier. But as Fig. 3 demonstrates, current calculations are quickly reducing the extent of the required extrapolation in the pion mass and not-so-far-off calculations will be able to bridge the physical pion mass, thus converting an extrapolation into an interpolation.

We also must keep in mind that each of these possible systematics can interfere with each other. For example, failing to reliably determine the matrix element may produce results that erroneously have a flat pion mass dependence or failing to properly renormalize the needed operator may obscure the cut-off dependence. And of course, all the calculations must ultimately be done with sufficient statistical precision to be capable of clearly checking the, relatively short, list of systematic errors.

CONCLUSIONS

Lattice calculations are proceeding steadily down to the physical pion mass. This has facilitated the precision calculation of many QCD observables, however, it has also produced some puzzles. In particular, recent calculations of average-$x$ have continued a long established trend of overshooting the value for $\langle x \rangle^{u-d}$ determined by global analyses. For quite some time, this had been assumed to be caused by a suppression of chiral dynamics due to the use of heavier-than-physical pion masses, but this explanation is much less compelling in the light of recent results. Clarification of this situation will likely require a careful study of all the possible uncertainties in the computation of $\langle x \rangle^{u-d}$. The use of a well-defined nonperturbative regulator, namely lattice QCD,
ensures that the errors in the calculation of $\langle x \rangle^{u-d}$ are identifiable and all systematically improvable. Controlling the chiral limit is still, of course, one source of error in the lattice calculations, but it is now just one among several potentially comparable errors. This marks a milestone in the lattice effort to determine hadron structure directly from QCD.

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