Proofs of Some Conjectures of Chan on Appell-Lerch Sums

Nayandeep Deka Baruah and Nilufar Mana Begum

Abstract. On page 3 of his lost notebook, Ramanujan defines the Appell-Lerch sum

\[ \phi(q) := \sum_{n=0}^{\infty} \frac{(-q;q)_{2n} q^{n+1}}{(q;q^2)_n^2}, \]

which is connected to some of his sixth order mock theta functions. Let \( \sum_{n=1}^{\infty} a(n)q^n := \phi(q) \). In this paper, we find a representation of the generating function of \( a(10n+9) \) in terms of \( q \)-products. As corollaries, we deduce the congruences \( a(50n+19) \equiv a(50n+39) \equiv a(50n+49) \equiv 0 \) (mod 25) as well as \( a(1250n+250r+219) \equiv 0 \) (mod 125), where \( r = 1, 3, \) and \( 4 \). The first three congruences were conjectured by Chan in 2012, whereas the congruences modulo 125 are new. We also prove two more conjectural congruences of Chan for the coefficients of two Appell-Lerch sums.

Key Words: Appell-Lerch sum; Theta function; Mock theta function; Congruence

2010 Mathematical Reviews Classification Numbers: Primary 11P83; Secondary 33D15.

1. Introduction

Throughout the paper, we use the customary \( q \)-series notation:

\[ \begin{align*}
(a; q)_0 & := 1, \\
(a; q)_n & := \prod_{k=0}^{n-1} (1 - aq^k), \quad n \geq 1, \\
(a; q)_{\infty} & := \lim_{n \to \infty} (a; q)_n, \quad |q| < 1,
\end{align*} \]

and

\[ (a_1, a_2, \ldots, a_k; q)_{\infty} := (a_1; q)_{\infty}(a_2; q)_{\infty} \cdots (a_k; q)_{\infty}. \]

For any positive integer \( j \), for brevity, we also use \( E_j := (q^j; q^j)_\infty \).

Let \( x, z \in \mathbb{C}^* \) with neither \( z \) nor \( xz \) an integral power of \( q \). Following the definition given by Hickerson and Mortenson in [12, Definition 1.1], an Appell-Lerch sum \( m(x, q, z) \) is a series of the form

\[ m(x, q, z) := \frac{1}{(q, q/z, q; q)_\infty} \sum_{r=-\infty}^{\infty} \frac{(-1)^{n+1} q^{n(n+1)/2} z^{n+1}}{1 - xzq^n}. \]

These sums were first studied in the nineteenth century by Appell [2, 3, 4] and then by Lerch [15]. But, in recent years, there has been considerable work on these sums and their connections to mock theta functions. We refer the readers to [1, 8, 12, 13, 16, 17, 21, 22].

In his lost notebook [19, pp. 2, 4, 13, 17], Ramanujan recorded seven mock theta functions and eleven identities involving them. Andrews and Hickerson [1] proved these
eleven identities and called the seven functions sixth order mock theta functions. Three of the sixth order mock theta functions are

\[ \rho(q) := \sum_{n=0}^{\infty} \frac{(-q; q)_n q^{n(n+1)/2}}{(q; q^2)_n}, \]

\[ \mu(q) := \sum_{n=0}^{\infty} \frac{(-1)^n (q^2; q)_n q^{n(n+1)^2}}{(-q; q)_{2n+1}}, \]

and

\[ \lambda(q) := \sum_{n=0}^{\infty} \frac{(-1)^n (q^2; q)_n q^n}{(-q; q)_n}. \]

On page 3 of his lost notebook [19], Ramanujan defines the function

\[ \phi(q) := \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n+1}}{(q; q^2)_{n+1}}, \]

and then states that

\[ \rho(q) = 2q^{-1} \phi(q^3) + \frac{(q^2; q^2)_\infty (-q^3; q^3)_\infty}{(q; q^2)_{\infty}^2 (q^3; q^3)_{\infty}}. \]

Choi [11] proved two analogous identities involving \( \phi \) and the two functions \( \mu \) and \( \lambda \). The function \( \phi(q) \) was also studied by Hikami [14].

Now, let \( \sum_{n=1}^{\infty} a(n) q^n := \phi(q) \). Chan [8] proved several congruences for the coefficients \( a(n) \) of the function \( \phi \) modulo 2, 3, 4, 5, 7, and 27. In particular, Chan [8] proved the congruence

\[ a(10n + 9) \equiv 0 \pmod{5} \tag{1.1} \]

and conjectured ([8] Conjecture 7.1) that, for any nonnegative integer \( n \),

\[ a(50n + 19) \equiv a(50n + 39) \equiv a(50n + 49) \equiv 0 \pmod{25}. \tag{1.2} \]

In this paper, we find a representation of the generating function of \( a(10n + 9) \) so that [11] follows trivially. We then prove [1.2] from that representation of the generating function of \( a(10n + 9) \). Furthermore, we find the following new congruences:

For any nonnegative integer \( n \), we have

\[ a(1250n + 250r + 219) \equiv 0 \pmod{125}, \quad \text{for } r = 1, 3, 4. \tag{1.3} \]

In [8], Chan studied some other functions similar to \( \phi \) and found congruences for them. In particular, he considered, for any integer \( p \geq 2 \) and \( 1 \leq j \leq p - 1 \) with \( p \) and \( j \) coprime, the Appell-Lerch sum

\[ \sum_{n=0}^{\infty} a_{j,p}(n) q^n = \frac{1}{(q^j, q^{p-j}, q^p, q^p)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{pn+1}/2+jn+j}{1 - q^{pn+j}}, \]
and proved that
$$\sum_{n=0}^{\infty} a_{j,p}(pn + (p - j)j)q^n = p \frac{E_4^j}{E_4^j(q^p, q^{p-j}; q^p)_\infty},$$
which readily implies the congruence
$$a_{j,p}(pn + (p - j)j) \equiv 0 \pmod{p}.$$

It is to be noted that $2a(n) = a_{1,2}(n)$.

In [9], Chan and Mao gave a generalization of $a_{j,p}$.

Chan [8, Conjecture 7.1] also presented the following conjectural congruences:

$$a_{1,6}(2n) \equiv 0 \pmod{2},$$

$$a_{1,10}(2n) \equiv a_{3,10}(2n) \equiv 0 \pmod{2},$$

$$a_{1,6}(6n + 3) \equiv 0 \pmod{3},$$

$$a_{1,3}(5n + 3) \equiv a_{1,3}(5n + 4) \equiv 0 \pmod{5},$$

$$a_{1,10}(10n + 5) \equiv 0 \pmod{5},$$

$$a_{3,10}(10n + 5) \equiv 0 \pmod{5}.$$  

(1.4)

(1.5)

(1.6)

(1.7)

(1.8)

(1.9)

Recently, Qu, Wang, and Yao [20] proved (1.4) and (1.5) by finding the following general congruence:

If $j$ and $k$ are positive integers with $1 \leq j \leq k - 1$ and $j$ odd, then for any nonnegative integer $n$,

$$a_{j,2k}(2n) \equiv 0 \pmod{2}.$$

They also proved (1.6) by finding the following identity:

$$\sum_{n=0}^{\infty} a_{1,6}(6n + 3)q^n = 3 \frac{E_5^3 E_6^5}{E_1^5 E_6^3},$$

which is analogous to Ramanujan’s [18] so-called “most” beautiful identity for the partition function $p(n)$, namely,

$$\sum_{n=0}^{\infty} p(5n + 4)q^n = 5 \frac{E_5^5}{E_1^5},$$

(1.10)

that immediately implies one of his three famous partition congruences, namely,

$$p(5n + 4) \equiv 0 \pmod{5}.$$

Congruences in (1.7) were proved by Ding and Xia [10].

In this paper, we prove the remaining conjectural congruences (1.8) and (1.9) of Chan [8].

We organize the paper in the following way. In Section 3, we find an exact generating function of $a(10n + 9)$ analogous to (1.10) and deduce the congruences in (1.2) as well as the new congruences in (1.3). In Section 4, we prove congruences (1.8) and (1.9).
We employ Ramanujan’s simple theta function identities and some other known identities for the Rogers-Ramanujan continued fraction, which is defined by
\[ R(q) := \frac{q^{1/5}}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \cdots}}}} = q^{1/5} \frac{(q; q^5)_\infty (q^4; q^5)_\infty}{(q^2; q^5)_\infty (q^3; q^5)_\infty}, \quad |q| < 1. \]

In the next section, we present the background material on Ramanujan’s theta functions and some lemmas that will be used in the subsequent sections.

2. Background material on Ramanujan’s theta functions and some useful lemmas

Ramanujan’s general theta function \( f(a, b) \) is defined by
\[ f(a, b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \quad |ab| < 1. \]

We require the following special cases of \( f(a, b) \):
\[ \varphi(q) := f(q, q) = \sum_{j=-\infty}^{\infty} q^{j^2} = (-q; q^2)_\infty (q^2; q^2)_\infty = \frac{E_2^3}{E_1^2 E_4^2}, \quad (2.1) \]
and
\[ \psi(q) := f(q, q^3) = \sum_{j=0}^{\infty} q^{j(j+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} = \frac{E_2^3}{E_1^2}, \quad (2.2) \]

where the product representations arise from Jacobi’s famous triple product identity \[ p. 35, \text{Entry 19} \]
\[ f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty. \quad (2.3) \]

Among many beautiful properties satisfied by \( f(a, b) \) we recall the following from \[ 5 \].

**Lemma 2.1.** (Berndt \[ 5 \] p. 45, Entry 29) If \( ab = cd \), then
\[ f(a, b) f(c, d) + f(-a, -b) f(-c, -d) = 2 f(ac, bd) f(ad, bc) \quad (2.4) \]
and
\[ f(a, b) f(c, d) - f(-a, -b) f(-c, -d) = 2af \left( \frac{b}{c} \right) \left( \frac{ac^2d}{d} \right) f \left( \frac{b}{d} \right) \left( \frac{acd^2}{d} \right). \quad (2.5) \]

In the next lemma, we state Jacobi’s identity and an analogous identity.

**Lemma 2.2.** (Berndt \[ 6 \] Theorem 1.3.9 and Corollary 1.3.22) We have
\[ E_1^3 = \sum_{k=0}^{\infty} (-1)^k (2k + 1) q^{k(k+1)/2} \quad (2.6) \]
and
\[ \frac{E_2^2}{E_1^2} = \sum_{n=-\infty}^{\infty} (3n + 1) q^{3n^2+2n} \quad (2.7) \]
In the following well-known results, the first two are 5-dissections of \( E_1 \) and \( 1/E_1 \), respectively.

**Lemma 2.3.** (Berndt [6, p. 165]) If \( T(q) := \frac{q^{1/5}}{R(q)} = \frac{(q^2; q^5)_\infty (q^3; q^5)_\infty}{(q; q^5)_\infty (q^4; q^5)_\infty} \), then

\[
E_1 = E_{25} \left( T(q^5) - q - \frac{q^2}{T(q^5)} \right) \quad (2.8)
\]

\[
\frac{1}{E_1} = \frac{E_{25}^5}{E_5^5} \left( T(q^5)^4 + qT(q^5)^3 + 2q^2T(q^5)^2 + 3q^3T(q^5) + 5q^4 - \frac{3q^5}{T(q^5)} \right)
+ \frac{2q^6}{T(q^5)^2} - \frac{q^7}{T(q^5)^3} + \frac{q^8}{T(q^5)^4},
\]

and

\[
11q + \frac{E_1^6}{E_5^6} = T(q^5) - \frac{q^2}{T(q^5)^5}. \quad (2.10)
\]

In the following two lemmas, we recall some useful results from our paper [7].

**Lemma 2.4.** (Baruah and Begum [7, Lemma 1.3]) If \( x = T(q) \) and \( y = T(q^2) \), then

\[
xy^2 - \frac{q^2}{xy^2} = K, \quad (2.11)
\]

\[
\frac{x^2}{y} - \frac{y}{x^2} = \frac{4q}{K}, \quad (2.12)
\]

\[
\frac{y^3}{x} + \frac{q^2x}{y^3} = K + \frac{4q^2}{K} - 2q, \quad (2.13)
\]

\[
x^3y + \frac{q^2}{x^3y} = K + \frac{4q^2}{K} + 2q, \quad (2.14)
\]

where \( K = (E_2E_5^5)/(E_1E_{10}^5) \).

**Lemma 2.5.** (Baruah and Begum [7, Eqs. (2.6), (2.7), (2.29)])

\[
\frac{E_5^5}{E_1^5E_{10}^5} = \frac{E_5}{E_2^2E_{10}} + 4q \frac{E_{10}^2}{E_1^3E_2} , \quad (2.15)
\]

\[
\frac{E_2^3E_5^2}{E_1^3E_{10}^2} = \frac{E_5^5}{E_1^3E_{10}^3} + q \frac{E_{10}^2}{E_2^3} , \quad (2.16)
\]

\[
\frac{E_5^3E_2^3}{E_1^3E_{10}^3} = \frac{E_5}{E_2^2E_{10}^2} + 5q \frac{E_{10}^2}{E_1^3E_2} , \quad (2.17)
\]

3. An exact generating function of $a(10n + 9)$ and proofs of (1.2) and (1.3)

**Theorem 3.1.** The generating function of $a(10n + 9)$ is given by

$$
\sum_{n=0}^{\infty} a(10n + 9)q^n = 5\left(46 \frac{E_5^2 E_{10}^3}{E_2^6} + 60q \frac{E_5^5 E_{10}^9}{E_1^8 E_2} + 1125q^2 \frac{E_5^8 E_{10}^{15}}{E_1^{12} E_5^5} + 1875q^3 \frac{E_5^8 E_{10}^9 E_1^4}{E_1^{16}} + 15625q^4 \frac{E_5^8 E_{10}^{15} E_1^{22}}{E_1^{22}}\right).
$$

(3.1)

Note that the congruence (1.1) immediately follows from (3.1).

**Proof of Theorem 3.1.** From [8, Eq. (5.1)], we have

$$
\sum_{n=0}^{\infty} a(2n + 1)q^n = \frac{E_5^2}{E_1^3},
$$

which, with the aid of (2.16) and (2.17), may be simplified as

$$
\sum_{n=0}^{\infty} a(2n + 1)q^n = \frac{E_5^3 E_{10}^3}{E_1^8 E_5^5} + 5q \frac{E_5^4 E_{10}^4}{E_1^8 E_5^5}
= \left(\frac{E_5^2}{E_1^3} + q \frac{E_{10}^5}{E_2 E_5^3}\right) + 5q \left(\frac{E_{10}^5}{E_2 E_5^3} + 5q \frac{E_{10}^8}{E_1^8 E_5^5}\right)
= \frac{E_5^2}{E_1^3} + 6q \frac{E_{10}^5}{E_2 E_5^3} + 25q^2 \frac{E_{10}^8}{E_1^8 E_5^5}.
$$

(3.2)

Employing (2.9) in the above, extracting the terms involving $q^{5n+4}$, dividing both sides of the resulting identity by $q^4$, and then replacing $q^5$ by $q$, we find that

$$
\sum_{n=0}^{\infty} a(10n + 9)q^n = 5\left(\frac{E_5^5}{E_1^4} + 30q \frac{E_{10}^5}{E_4^3 E_2} + 1775q^2 \frac{E_2^8 E_{10}^{15}}{E_1^{22}} + 4425 \frac{E_2^8 E_{10}^{15}}{E_1^{22}}\right)\left(T(q)^5 - \frac{q^2}{T(q)^3}\right)
+ 225 \frac{E_5^8 E_{10}^{15} E_1^{22}}{E_1^2} \left(T(q)^{10} + \frac{q^4}{T(q)^{10}}\right).
$$

(3.3)

By (2.10), the above can be simplified to

$$
\sum_{n=0}^{\infty} a(10n + 9)q^n = 5\left(\frac{E_5^5}{E_1^4} + 6q \frac{E_{10}^5}{E_4^3 E_2} + 45 \frac{E_2^8 E_{10}^3}{E_1^{10}} + 1875q \frac{E_2^8 E_{10}^9}{E_1^{16}} + 15625q^2 \frac{E_2^8 E_{10}^{15} E_1^{22}}{E_1^{22}}\right).
$$

Employing (2.10) in the above, we arrive at

$$
\sum_{n=0}^{\infty} a(10n + 9)q^n = 5\left(\frac{E_5^3 E_{10}^3}{E_1^8} + 5q \frac{E_{10}^5}{E_1^8 E_2} + 45 \frac{E_2^8 E_{10}^3}{E_1^{10}} + 1875q \frac{E_2^8 E_{10}^9}{E_1^{16}}
+ 15625q^2 \frac{E_2^8 E_{10}^{15} E_1^{22}}{E_1^{22}}\right).
$$

(3.4)
With the aid of (2.17), the above can be further simplified to
\[
\sum_{n=0}^{\infty} a(10n+9)q^n = 5 \left( 45 \left( \frac{E_5^3 E_2^2 E_{10}^4}{E_1^5} + 5q \frac{E_5^4 E_{10}^4}{E_1^8} \right) + \frac{E_5^3 E_2^2 E_{10}^4}{E_1^5} + 5q \frac{E_5^5}{E_1^3 E_2} \right) \\
+ 1875q \frac{E_5^8 E_{10}^9}{E_1^{16}} + 15625q^2 \frac{E_5^8 E_{10}^{15}}{E_1^{22}},
\]
which is equivalent to (3.1).

**Remark 3.2.** If we extract the terms involving \(q^{5n+r}\), \(r = 0, 1, 2, 3\) after employing (2.9) in (3.2), then we arrive at the generating functions of \(a(10n+2r+1)\) as in (3.3). But in these cases, the expressions involving \(T(q)\) and \(T(q^2)\) could not be expressed in terms of \(E_1, E_2, E_5\) and \(E_{10}\) as in the above case because of the non-availability of the expressions similar to those in (2.10) -- (2.14). Therefore, in Theorem 3.1 we considered only the case \(a(10n+9)\) among \(a(10n+2r+1)\) where \(0 \leq r \leq 4\).

As corollaries to the above theorem, we now deduce the congruence in (1.2) originally conjectured by Chan in [8] and the new congruences in (1.3).

**Corollary 3.3.** The congruences in (1.2) hold good.

**Proof.** By the binomial theorem, we have
\[
E_1^5 \equiv E_5 \pmod{5}.
\]
Taking congruences modulo 5 in (3.1) and using the above, we see that
\[
\sum_{n=0}^{\infty} a(10n+9)q^n \equiv 5 \times 46 \ E_5 E_{10} E_2^3 \pmod{25},
\]
which can be rewritten with the aid of (2.6) as
\[
\sum_{n=0}^{\infty} a(10n+9)q^n \equiv 5 \times 46 \ E_5 E_{10} \sum_{k=0}^{\infty} (-1)^k (2k+1)q^{k(k+1)} \pmod{25}.
\]
As \(k(k+1) \equiv 0, 1, \text{ or } 2 \pmod{5}\), equating the coefficients of \(q^{5n+r}\), \(r = 3, 4\) from both sides of the above, we easily arrive at the last two congruences of (1.2). Furthermore, we note that \(k(k+1) \equiv 1 \pmod{5}\) only when \(k \equiv 2 \pmod{5}\), that is, only when \(2k+1 \equiv 0 \pmod{5}\). Therefore, equating the coefficients of \(q^{5n+1}\) from both sides of the above we arrive at the other congruence of (1.2), to complete the proof. \(\square\)

**Corollary 3.4.** The congruences in (1.3) hold good.

**Proof.** From (3.1), we have
\[
\sum_{n=0}^{\infty} a(10n+9)q^n \equiv 5 \times 46 \left( \frac{E_5 E_{10}^3}{E_2^2} + 10q E_{10}^3 \right) \pmod{125}.
\]
Now, let \([q^{5n+r}] \{F(q)\}, r = 0, 1, \ldots, 4\) denotes the terms after extracting the terms involving \(q^{5n+r}\), dividing by \(q^r\) and then replacing \(q^5\) by \(q\).

With the aid of (2.2), we have

\[
[q^{5n+1}] \left\{ \frac{E_5 E_7^{10}}{E_2^2} \right\} = \frac{E_1 E_7^{10}}{E_2^{10}} \left( 15q^3 + 10q \left( T(q^2)^5 - \frac{q^4}{T(q^2)^5} \right) \right),
\]

which, by (2.10), implies

\[
[q^{5n+1}] \left\{ \frac{E_5 E_7^{10}}{E_2^2} \right\} = 10q \frac{E_1 E_7^{10}}{E_2^{10}} + 125q^3 \frac{E_1 E_7^{10}}{E_2^{10}}.
\]

Again, by (2.15) and (2.9), we obtain

\[
[q^{5n+1}] \left\{ 10q \frac{E_7^{10}}{E_1^2} \right\} = [q^{5n+1}] \left\{ \frac{5}{2} \left( \frac{E_5}{E_1} - \frac{E_5 E_7^{10}}{E_2^2} \right) \right\}
\]

\[
= \frac{5}{2} \left( \frac{E_5}{E_1} - \frac{E_5 E_7^{10}}{E_2^2} \right) x^{15} - \frac{q^6}{x^{15}} \right) + 209q \frac{E_5^{20}}{E_1^{19}} \left( x^{10} + \frac{q^4}{x^{10}} \right)

+ 5q \frac{E_1 E_7^{10}}{E_2^{10}} \left( y^5 - \frac{q^4}{y^5} \right) + 920q^2 \frac{E_7^{20}}{E_1^{19}} \left( x^5 - \frac{q^2}{x^5} \right)

+ \frac{q^3}{2} \left( 1015 \frac{E_7^{20}}{E_1^{19}} - 15 \frac{E_1 E_7^{10}}{E_2^{10}} \right).
\]

Employing (2.10), and then simplifying by using the identities in Lemma 2.5, we find that

\[
[q^{5n+1}] \left\{ 10q \frac{E_7^{10}}{E_2 E_5^3} \right\} = 10 \left( \frac{E_7^{14} E_1^{10} E_2^{15}}{E_5^3} + 150q \frac{E_7^{11} E_1^{10} E_2^{16}}{E_5^3} + 5650q^2 \frac{E_7^{9} E_1^{10} E_2^{15}}{E_5^3} + 101825q^3 \frac{E_7^{12} E_1^{10} E_2^{15}}{E_5^4}

+ 1068125q^4 \frac{E_7^{20} E_1^{10} E_2^{15}}{E_5^5} + 7042500q^5 \frac{E_7^{18} E_1^{10} E_2^{15}}{E_5^5}

+ 2980000q^6 \frac{E_7^{21} E_1^{10} E_2^{15}}{E_5^5} + 7900000q^7 \frac{E_7^{20} E_1^{10} E_2^{15}}{E_5^5}

+ 12000000q^8 \frac{E_7^{20} E_1^{10} E_2^{15}}{E_5^5} + 8000000q^9 \frac{E_7^{30} E_1^{10} E_2^{15}}{E_5^5} \right).
\]

Invoking (3.9) and (3.10) in (3.8), we obtain

\[
\sum_{n=0}^{\infty} a(50n + 19)q^n \equiv 5^2 \times 92 \left( \frac{E_1^{14} E_7^{10} E_2^{15}}{E_5^3} + q \frac{E_1 E_7^{10}}{E_2^{10}} \right) \quad \text{(mod 125)},
\]

which, by (3.6), reduces to

\[
\sum_{n=0}^{\infty} a(50n + 19)q^n \equiv 5^2 \times 92 \left( \frac{E_2^5}{E_1} + q E_1 E_2 E_7^{10} \right) \quad \text{(mod 125)}.
\]
Employing once again (2.9) in the above, extracting the terms involving $q^{5n+4}$, dividing both sides by $q^4$, and then replacing $q^5$ by $q$, we arrive at

$$\sum_{n=0}^{\infty} a(250n + 219)q^n \equiv 5^2 \times 92 \ E_5 E_{10} E_2^3 \pmod{125}.$$  

Employing (2.6) in the above and then proceeding as in the proof of the previous corollary, we conclude that

$$a(250(5n + r) + 219) \equiv 0 \pmod{125}, \quad \text{for } r = 1, 3, 4.$$  

Thus, we complete the proof of (1.3). \qed

Remark 3.5. Proceeding as in the proof of Theorem 3.1, we may obtain the exact generating function of $a(50n + 19)$, but the calculations and expressions are too lengthy and tedious even if we use Mathematica. Therefore, we decided not to include that lengthy generating function of $a(50n + 19)$.

### 4. Proofs of (1.8) and (1.9)

In this section, we prove the congruences (1.8) and (1.9) originally conjectured by Chan [8].

At first, setting $k = 5$ and $j = 1$ and 3 in [20, Eq. (2.7)], we have

$$4 \sum_{n=0}^{\infty} a_{1,10}(n)q^n = \frac{(-q; q^{10})_\infty^2 (-q^9; q^{10})_\infty^2 E_5^5}{(q; q^{10})_\infty^2 (q^9; q^{10})_\infty^2 E_{20}^4} - 2 \frac{E_{10}}{E_{20}^2} \sum_{n=-\infty}^{\infty} \frac{q^{5n(n+1)}}{1 + q^{10n}}$$

and

$$4 \sum_{n=0}^{\infty} a_{3,10}(n)q^n = \frac{(-q^3; q^{10})_\infty^2 (-q^7; q^{10})_\infty^2 E_{10}^5}{(q^3; q^{10})_\infty^2 (q^7; q^{10})_\infty^2 E_{20}^4} - 2 \frac{E_{10}}{E_{20}^2} \sum_{n=-\infty}^{\infty} \frac{q^{5n(n+1)}}{1 + q^{10n}},$$

respectively.

In view of (2.3), we can rewrite the above identities as

$$4 \sum_{n=0}^{\infty} a_{1,10}(n)q^n = \frac{f^2(q, q^9) E_{10}^5}{f^2(-q^3, -q^7) E_{20}^4} - 2A(q^2) \quad (4.1)$$

and

$$4 \sum_{n=0}^{\infty} a_{3,10}(n)q^n = \frac{f^2(q^3, q^7) E_{10}^5}{f^2(-q^3, -q^7) E_{20}^4} - 2A(q^2), \quad (4.2)$$

where

$$A(q) := \frac{E_5}{E_{10}^2} \sum_{n=-\infty}^{\infty} \frac{q^{5n(n+1)/2}}{1 + q^{5n}}.$$

Proof of (1.8). To prove (1.8), first we aim to find a generating function of $a_{1,10}(2n+1)$. For that purpose, we need to find a 2-dissection of the first term of the right side of
To that end, we recast (4.3) as

\[
4 \sum_{n=0}^{\infty} a_{1,10}(n)q^n = \frac{f^2(q, q^9)f^2(-q^3, -q^7)E_{10}^5}{f^2(-q, -q^9)f^2(-q^3, -q^7)E_{20}^2} - 2A(q^2). \tag{4.3}
\]

By Jacobi triple product identity, (2.3), we have

\[
f(-q, -q^9)f(-q^3, -q^7) = \frac{(q; q^2)_{\infty}E_{10}^1}{(q^5; q^{10})_{\infty}} = \frac{E_1E_3}{E_2E_5}, \tag{4.4}
\]

and hence, (4.3) reduces to

\[
4 \sum_{n=0}^{\infty} a_{1,10}(n)q^n = \frac{E_2E_5}{E_2E_{10}E_{20}^2} f^2(q, q^9)f^2(-q^3, -q^7) - 2A(q^2)
\]

\[
= \frac{E_2}{E_{20}^4} \cdot \frac{A(-q^5)}{\varphi(-q)} \left( f^2(q, q^9)f^2(-q^3, -q^7) \right) - 2A(q^2), \tag{4.5}
\]

where (2.1) is used to arrive at the last equality.

Now, setting \(a = q, b = q^9, c = -q^3,\) and \(d = -q^7\) in (2.4) and (2.5), and then adding, we find that

\[
f(q, q^9)f(-q^3, -q^7) = f(-q^4, -q^{16})f(-q^8, -q^{12}) + qf(-q^6, -q^{14})f(-q^2, -q^{18}). \tag{4.6}
\]

But, by Jacobi triple product identity, (2.3), we have

\[
f(-q, -q^4)f(-q^2, -q^3) = E_1E_5.
\]

Using the above identity and (4.4) in (4.6), we see that

\[
f(q, q^9)f(-q^3, -q^7) = E_4E_{20} + q \frac{E_2E_{20}^3}{E_4E_{10}}.
\]

Therefore, (4.5) can be rewritten as

\[
4 \sum_{n=0}^{\infty} a_{1,10}(n)q^n = \frac{E_2}{E_{20}^4} \cdot \frac{A(-q^5)}{\varphi(-q)} \left( E_{10}^4E_{20}^2 + q^2 \frac{E_2E_{20}^6}{E_4E_{10}^2} + 2q \frac{E_2E_{20}^4}{E_{10}} \right) - 2A(q^2). \tag{4.7}
\]

Replacing \(q\) by \(-q\) in the above, and then subtracting the resulting identity from (4.7), we find that

\[
4 \sum_{n=0}^{\infty} a_{1,10}(n)q^n - 4 \sum_{n=0}^{\infty} a_{1,10}(n)(-q)^n
\]

\[
= \frac{E_2}{E_{20}^4} \left( E_{10}^4E_{20}^2 + q^2 \frac{E_2E_{20}^6}{E_4E_{10}^2} \left( \frac{A(-q^5)}{\varphi(-q)} - \frac{A(q^5)}{\varphi(q)} \right) + 2q \frac{E_2E_{20}^4}{E_{10}} \left( \frac{A(-q^5)}{\varphi(-q)} + \frac{A(q^5)}{\varphi(q)} \right) \right).
\]
With the aid of the trivial identity $\varphi(q)\varphi(-q) = \varphi^2(-q^2) = E_2^4/E_4^2$, we can rewrite the above as

$$4 \sum_{n=0}^{\infty} a_{1,10}(n)q^n - 4 \sum_{n=0}^{\infty} a_{1,10}(n)(-q)^n$$

$$= \frac{E_4^2}{E_2^2 E_2^{10}} \left( E_4^2 E_2^6 + q^2 \frac{E_2^2 E_2^{10}}{E_2^4 E_2^{10}} \right) \left( \varphi(q)\varphi(-q^5) - \varphi(-q)\varphi(q^5) \right)$$

$$+ 2q \frac{E_4^2}{E_2^2 E_2^{10}} \left( \varphi(q)\varphi(-q^5) + \varphi(-q)\varphi(q^5) \right).$$

(4.8)

Now, recall from [5, p. 278] that

$$\varphi(q)\varphi(-q^5) - \varphi(-q)\varphi(q^5) = 4qE_4 E_2^{10}.$$  

Furthermore, from Entries 25(i) and 25(ii) of [5, p. 40], it is easy to show that

$$\varphi(q)\varphi(-q^5) + \varphi(-q)\varphi(q^5) = 2\varphi(q^4)\varphi(q^{20}) - 8q^6\psi(q^8)\psi(q^{40}).$$

Therefore, (4.8) becomes

$$4 \sum_{n=0}^{\infty} a_{1,10}(n)q^n - 4 \sum_{n=0}^{\infty} a_{1,10}(n)(-q)^n$$

$$= 4q \frac{E_4^2}{E_2^2 E_2^{10}} \left( E_4^2 E_2^6 + q^2 \frac{E_2^2 E_2^{10}}{E_2^4 E_2^{10}} \right) + 2q \frac{E_4^2}{E_2^2 E_2^{10}} \left( 2\varphi(q^4)\varphi(q^{20}) - 8q^6\psi(q^8)\psi(q^{40}) \right).$$

Extracting the terms involving $q^{2n+1}$ from both sides of the above, dividing by $q$, and then replacing $q^2$ by $q$, we find that

$$2 \sum_{n=0}^{\infty} a_{1,10}(2n+1)q^n$$

$$= \frac{E_4^2}{E_2^2 E_2^{10}} \left( E_2^2 E_1^2 + q \frac{E_2^2 E_2^{10}}{E_2^4 E_2^{10}} \right) + \frac{E_2^2}{E_2^4 E_2^{10}} \left( \varphi(q^2)\varphi(q^{10}) - 4q^3\psi(q^4)\psi(q^{20}) \right)$$

$$= \frac{E_1^2}{E_5} + 6q \frac{E_2 E_2^3}{E_1 E_5^2} + \frac{E_2^2}{E_1 E_5^2} \left( \varphi(q^2)\varphi(q^{10}) - 4q^3\psi(q^4)\psi(q^{20}) \right),$$

(4.9)

where we have used (2.17) to arrive at the last equality.

Now, to prove (1.8), we see from the above that it is enough to show that the coefficients of $q^{5n+2}$ of the terms on the right side of the above are multiples of 5. We accomplish this in the remaining part of the proof.

With the aid of (2.8) and (2.9), we find that

$$[q^{5n+2}] \left\{ \frac{E_2^2}{E_5} + 6q \frac{E_2 E_2^3}{E_1 E_5^2} \right\}$$

$$= -\frac{E_2^2}{E_1} + 6 \frac{E_2^2 E_2^5 E_1}{E_5} \left( x^3 y + \frac{q^2}{x^3 y} - 2q \left( x^2 - \frac{y}{x} \right) - 5q \right),$$
where $x$ and $y$ are as defined in Lemma 2.4. Employing (2.12) and (2.14) in the above, and then simplifying by using (2.15), we obtain

$$[q^{5n+2}] \left\{ \frac{E_2^2}{E_5} + 6q \frac{E_2 E_3^3}{E_1 E_5^2} \right\} = 5 \frac{E_5^2}{E_1} + 30q \frac{E_2 E_5 E_3^3}{E_4}. \quad (4.10)$$

Next, by (3.6), we have

$$\frac{E_2^2}{E_1^2 E_5} \varphi(q^2) \varphi(q^{10}) = \varphi(q^{10}) \frac{E_4^2}{E_1^2 E_5} E_8^2 \equiv \frac{E_{20} \varphi(q^{10})}{E_5^2 E_{40}} E_1^3 E_8^3 \pmod{5},$$

which, with the aid of Jacobi’s identity, (2.6), can be written as

$$\frac{E_2^2}{E_1^2 E_5} \varphi(q^2) \varphi(q^{10}) \equiv \frac{E_{20} \varphi(q^{10})}{E_5^2 E_{40}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} (2j+1)(2k+1) q^{j(j+1)/2+4k(k+1)} \pmod{5}. \quad (4.11)$$

We now observe that $j(j+1)/2 + 4k(k+1) \equiv 2 \pmod{5}$ only when $j \equiv 2 \pmod{5}$ and $k \equiv 2 \pmod{5}$; i.e., only when both $2j+1$ and $2k+1$ are multiples of 5. Therefore, from (4.11), we find that

$$[q^{5n+2}] \left\{ \frac{E_2^2}{E_1^2 E_5} \varphi(q^2) \varphi(q^{10}) \right\} \equiv 0 \pmod{5}. \quad (4.12)$$

Finally, by (2.2) and (3.6), we have

$$q^3 \frac{E_2^2}{E_1^2 E_5} \psi(q^4) \psi(q^{20}) = q^3 \frac{E_2^2 E_3^2}{E_1^2 E_4 E_5} \psi(q^{20}) \equiv \frac{\psi(q^{20})}{E_5^2} \frac{E_3^3}{E_1} \cdot \frac{E_2^3 E_3^2}{E_4} \pmod{5},$$

which can be rewritten, with the help of (2.6) and (2.7), as

$$q^3 \frac{E_2^2}{E_1^2 E_5} \psi(q^4) \psi(q^{20}) \equiv \frac{\psi(q^{20})}{E_5^2} \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} (-1)^j (2j+1)(2k+1) q^{j(j+1)/2+2k(3k+2)+3} \pmod{5}. \quad (4.13)$$

We observe that $j(j+1)/2 + 2k(3k+1) + 3 \equiv 2 \pmod{5}$ only when $j \equiv 2 \pmod{5}$ and $k \equiv 3 \pmod{5}$; i.e., only when both $2j+1$ and $3k+1$ are multiples of 5. Therefore, from (4.13), we arrive at

$$[q^{5n+2}] \left\{ q^3 \frac{E_2^2}{E_1^2 E_5} \psi(q^4) \psi(q^{20}) \right\} \equiv 0 \pmod{5}. \quad (4.14)$$

With the aid of (4.10), (4.12) and (4.14), we conclude from (4.9) that

$$a_{1,10}(10n+5) \equiv 0 \pmod{5}.$$
Thus, we complete the proof of (1.8).

Proof of (1.9). We can recast (4.2) as
\[
4 \sum_{n=0}^{\infty} a_{3,10}(n)q^n = \frac{f^2(q^3, q^7)f^2(-q, -q^9)E_{10}^5}{f^2(-q^3, -q^7)f^2(-q, -q^9)E_{20}^5} - 2A(q^2),
\]

Proceeding exactly in the same way as in the proof of (1.8), we find that
\[
2 \sum_{n=0}^{\infty} a_{3,10}(2n + 1)q^n = \frac{E_1^2}{E_5} + 6q \frac{E_2^2}{E_1E_5} - \frac{E_2^2}{E_1^2E_5} \left( \varphi(q^2) \varphi(q^{10}) - 4q^3 \psi(q^4) \psi(q^{20}) \right).
\]

We notice that the right side of the above is almost the same as that of (4.9) except a negative sign. Hence, (1.9) can be deduced as in the previous case.

References

[1] Andrews, G.E., Hickerson, D.: Ramanujan’s ‘lost’ notebook VII: the sixth order mock theta functions. Adv. Math. 89, 60–105 (1991)
[2] Appell, P.: Sur les fonctions doublement périodiques de troisième espèce. Ann. Sci. Éc. Norm. Supér. 1, 135–164 (1884)
[3] ———: Développements en série des fonctions doublement périodiques de troisième espèce. Ann. Sci. Éc. Norm. Supér. 2, 9–36 (1885)
[4] ———: Sur les fonctions doublement périodiques de troisième espèce. Ann. Sci. Éc. Norm. Supér. 3, 9–42 (1886)
[5] Berndt, B.C.: Ramanujan’s Notebooks, Part III. Springer, New York (1991)
[6] Berndt, B.C.: Number Theory in the Spirit of Ramanujan. American Mathematical Society, Providence, RI (2006)
[7] Baruah, N.D., Begum, N.M.: On exact generating functions for the number of partitions into distinct parts. Int. J. Number Theory. To appear.
[8] Chan, S.H.: Congruences for Ramanujan’s \( \varphi \) function. Acta Arith. 153, 161–189 (2012)
[9] Chan, S.H., Mao, R.: Two congruences for Appell-Lerch sums. Int. J. Number Theory 8, 111–123 (2012)
[10] Ding, H., Xia, E.X.W.: Arithmetic properties for Appell-Lerch Sums. Preprint.
[11] Choi, Y.-S.: Identities for Ramanujan’s sixth-order mock theta functions. Quart. J. Math. 53, 147–159 (2002)
[12] Hickerson, D.R., Mortenson, E.T.: Hecke-type double sums, Appell-Lerch sums, and mock theta functions, I. Proc. London Math. Soc. 109, 382–422 (2014)
[13] Hickerson, D.R., Mortenson, E.T.: Dyson’s Ranks and Appell-Lerch sums. Math. Ann. 367, 373–395 (2017)
[14] Hikami, K.: Transformation formula of the “second” order mock theta function. Lett. Math. Phys. 75, 93–98 (2006)
[15] Lerch, M.: Poznámky k teorii funkcí elliptických. Prag. České Ak. Fr. Jos. Rozpr. 24, 465–480 (1892)
[16] Mortenson, E.T.: Ramanujan’s \( 1 \psi_1 \) summation, Hecke-type double sums, and Appell-Lerch sums. Ramanujan J. 29, 121–133 (2012)
[17] Mortenson, E.T.: On the dual nature of partial theta functions and Appell-Lerch sums. Adv. Math. 264, 236–260 (2014)
[18] Ramanujan, S.: Some properties of \( p(n) \), the number of partitions of \( n \). Proc. Camb. Philos. Soc. 19, 207–210 (1919)
[19] Ramanujan, S.: The Lost Notebook and Other Unpublished Papers. Narosa, New Delhi (1988)
[20] Qi, Y.K., Wang, Y.J., Yao, O.X.M.: Generalizations of some conjectures of Chan on congruences for Appell-Lerch sums. J. Math. Anal. Appl. 460, 232–238 (2018)
[21] Waldherr, M.: On certain explicit congruences for mock theta functions. Proc. Amer. Math. Soc. 139, 865–879 (2011)
[22] Zwegers, S.P.: Mock theta functions. Ph. D. Thesis, Universiteit Utrecht (2002)

DEPARTMENT OF MATHEMATICAL SCIENCES, TEZPUR UNIVERSITY, SONITPUR, ASSAM, INDIA, PIN-784028
E-mail address: nayan@tezu.ernet.in

DEPARTMENT OF MATHEMATICAL SCIENCES, TEZPUR UNIVERSITY, SONITPUR, ASSAM, INDIA, PIN-784028
E-mail address: nilufar@tezu.ernet.in