DUAL GRAPHS AND MODIFIED BARLOW–BASS RESISTANCE ESTIMATES FOR REPEATED BARYCENTRIC SUBDIVISIONS

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Abstract. We prove Barlow–Bass type resistance estimates for two random walks associated with repeated barycentric subdivisions of a triangle. If the random walk jumps between the centers of triangles in the subdivision that have common sides, the resistance scales as a power of a constant $\rho$ which is theoretically estimated to be in the interval $5/4 \leq \rho \leq 3/2$, with a numerical estimate $\rho \approx 1.306$. This corresponds to the theoretical estimate of spectral dimension $d_S$ between 1.63 and 1.77, with a numerical estimate $d_S \approx 1.74$. On the other hand, if the random walk jumps between the corners of triangles in the subdivision, then the resistance scales as a power of a constant $\rho^T = 1/\rho$, which is theoretically estimated to be in the interval $2/3 \leq \rho^T \leq 4/5$. This corresponds to the spectral dimension between 2.28 and 2.38. The difference between $\rho$ and $\rho^T$ implies that the the limiting behavior of random walks on the repeated barycentric subdivisions is more delicate than on the generalized Sierpinski Carpets, and suggests interesting possibilities for further research, including possible non-uniqueness of self-similar Dirichlet forms.

1. Introduction. There has been an wide interest in studying analysis and random processes on various metric measure spaces that satisfy either the volume doubling property, or curvature bounds, or both. One part of this very large literature deals with spaces where Lipschitz functions can be analyzed. Without even attempting to suggest a representative sample of relevant papers, we briefly mention such recent works as [1,10,29,33]. Another kind of more probabilistically inspired analysis deals with spaces that are more fractal in nature and have sub-Gaussian heat kernel estimates (see [2–4,6,7,9] and references therein). Typically for such fractal examples, Lipschitz functions play little or no role, as intrinsically smooth functions are only Hölder continuous. In some sense all these results are related to the Nash-Moser theory of uniformly elliptic operators. However, there are natural spaces that have no volume doubling, no curvature bounds, and no heat kernel estimates. Analysis

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Figure 1. On the left: barycentric subdivision of a 2-simplex, the graphs $G_0^T$, $G_1^T$ and $G_2^T$. On the right: adjacency (dual) graph $G_2$, in bold, pictured together with the thin image of $G_2^T$.

of such spaces is in its infancy, and considering even simplest examples is very challenging. After laying some of the initial framework for this model, our aim is to connect to a series of other works, such as [11,27,40–42,55,62].

Figure 2. On the left: the graph $G_4^T$ for barycentric subdivision of a 2-simplex. On the right: the adjacency (dual) graph $G_4$.

The repeated barycentric subdivision of a simplex is a classical and fundamental notion from algebraic topology, see [28, and references therein]. Recently it was considered from a probabilistic point of view in [17–20,66] and graph theory point of view in [48,49]. Understanding how resistance scales on finite approximating graphs is the first step to developing analysis on fractals and fractal-like structures, such as self-similar graphs and groups, see [8,23,37,59,60, and references therein]. For finitely ramified post-critically finite fractals, including nested fractals, the resistance scales by the same factor between any two levels of approximating graphs (see [43,44,54, and references therein]), and this fact can be used to prove the existence and uniqueness of a Dirichlet form on the limiting fractal structures. In the infinitely ramified case, resistance estimates are more difficult to obtain, but are just as important to understanding diffusions on fractals. Barlow and Bass [2–4,6,7,9] proved such estimates for the Sierpinski carpet and its generalizations. These techniques were extended to understanding resistance estimates between more complicated regions of the Sierpinski carpet, see [57]. The paper [50] provides another technique for proving the existence of Dirichlet forms on non-finitely ramified self-similar fractals, which estimates the parameter $\rho$ by studying the Poincaré inequalities on the approximating graphs of the fractals. The long term motivation for our work comes from probability and analysis on fractals [5,13,14,58,61,63], vector
analysis for Dirichlet forms \([30–32,34–36,51]\), and especially from the works on the heat kernel estimates \([3,6,7,24,25,38,39,45–47,52,53,64]\).

In general terms, a Dirichlet form on a fractal is a bilinear form which is analogous to the classic Dirichlet energy on \(\mathbb{R}^d\) given by \(\mathcal{E}(f) = \int |\nabla f|^2 \, dx\). Dirichlet forms have many applications in geometry, analysis and probability. The theory of Dirichlet forms is equivalent, in a certain sense, to the theory of symmetric Markov processes, see \([15,16,22]\). The potential theoretic properties of the Dirichlet form have implications for this stochastic process. In particular, the resistance between two boundary sets is related to the crossing times. In the discrete setting, the Dirichlet form is the graph energy. In this case the resistance between two sets is determined using Kirchhoff’s laws. For a more thorough introduction to these topics, one can see, for example, \([21,56]\).

Although the results of Barlow and Bass et al are applicable to a large class of fractals, we concentrate on one prototypical but difficult to analyze generalization of the classical Sierpinski carpet. Our work further develops existing techniques to obtain resistance scaling estimates for the 1-skeleton of \(n\)-times iterated barycentric subdivisions of a triangle which we will denote \(G_T^n\), and its weak dual the hexacarpet (introduced in \([12]\)), which we will denote \(G_H^n\). In our case, on the 1-skeleton \(G_T^n\), the Markov process jumps between corners of the triangles in the subdivision. Our theoretical estimates correspond to a process with limiting spectral dimension between 2.28 and 2.38. The Markov process on the hexacarpet graphs (which are denoted by \(G\) and \(G^H\) later on) corresponds to a random walk which jumps between the centers of these triangles, with spectral dimension between 1.63 and 1.77 (\(\approx 1.74\) using the numerical estimates in \([12]\)). This is a substantial difference implying, in particular, that one Markov process is not recurrent, while the other is recurrent. From the point of view of fractal analysis, our results suggest that the corresponding self-similar diffusion is not unique, unlike \([7,26]\).

If \(R_T^n\) and \(R_n\) is the resistance between the appropriate boundaries in \(G_T^n\) and \(G_H^n\) respectively (see Figures 1, 2, 3), then we prove that the resistance \(R_T^n\) and \(R_n\) scale by constants \(\rho_T\) and \(\rho\) respectively, and obtain estimates on these constants. Note that in the current work the hexacarpet graph \(G_H^n\) is a modification of \(G^n\) by adding a set of “boundary” vertexes. Our main result is the following theorem.

**Theorem 1.1.** The resistances across graphs \(G_T^n\) and \(G_H^n\) (defined in Subsection 2.2) are reciprocals, that is \(R_T^n = 1/R_n\), and the asymptotic limits

\[
\log \rho_T = \lim_{n \to \infty} \frac{1}{n} \log R_T^n \quad \text{and} \quad \log \rho = \lim_{n \to \infty} \frac{1}{n} \log R_n
\]

exist (and \(\rho_T = 1/\rho\)). Furthermore, \(2/3 \leq \rho_T \leq 4/5\) and \(5/4 \leq \rho \leq 3/2\).

These estimates agree with the numerical experiments from \([12]\), which suggest that there exists a limiting Dirichlet form on these fractals and estimates \(\rho \approx 1.306\), and hence \(\rho_T \approx 0.7655\).

This paper is organized as follows. Subsection 2.1 defines general graph energy. Subsection 2.2 lays out the definitions of \(G_H^n\) and \(G_T^n\) and shows how to take advantage of the duality to prove that \(R_n = 1/R_T^n\). In Section 3 we prove sub-multiplicative estimates \(R_{m+n} \leq c R_m R_n\) for some constant \(c\) independent of \(m\) and \(n\) in a fashion generalized from \([3]\). Then Fekete’s theorem implies that the limits \(\rho\) and \(\rho_T\) exist. To show that these limits are finite, in Subsections 3.3 and 3.4 we prove upper and lower estimates on the constants \(\rho\) and \(\rho_T\) by establishing
upper and lower estimates on $R_n^T$ and $R_n$. This is done by comparing $G_n^T$ and $G_n^H$ to subgraphs and quotient graphs respectively.

2. Energy, resistance and duality.

2.1. Energy, potentials, and flows on graphs. This subsection collects the basic definitions and facts about graph energies and resistances, which will be used in the current work. For more detailed expositions on this subject see, for instance, [3, 21, 56]. Let $G = (V,E)$ be a finite graph with vertex set $V$, and edge set $E$, which is a symmetric subset of $V \times V$. We define the set $\ell(V) = \{ f : V \to \mathbb{R} \}$ to be the set of real valued functions on $V$, which we will sometimes refer to as functions or potentials on the graph $G$. For all $p, q \in V$ we define conductances (weights) $c_{p,q} = c_{q,p}$ such that $c_{p,q} > 0$ if $(p,q) \in E$ and $c_{p,q} = 0$ when $(p,q) \notin E$. Resistances between points $p$ and $q$ for any $(p,q) \in E$ will be defined $r_{p,q} := 1/c_{p,q}$. For a graph with associated conductances, we define the graph energy

$$\mathcal{E} : \ell(V) \times \ell(V) \to \mathbb{R}, \quad \mathcal{E}(f,g) = \frac{1}{2} \sum_{(p,q) \in E} c_{p,q} (f(p) - f(q)) (g(p) - g(q)),$$

and write $\mathcal{E}(f) := \mathcal{E}(f,f)$.

Antisymmetric functions (with orientation) on the edge set will be denoted $\ell^a(E) = \{ J : V \times V \to \mathbb{R} \mid J(p,q) = -J(q,p), \text{ and } J(p,q) = 0 \text{ if } (p,q) \notin E \}$, and we define the energy dissipation $E : \ell^a(E) \times \ell^a(E) \to \mathbb{R}$ to be the inner product

$$E(J,K) = \frac{1}{2} \sum_{(p,q) \in E} r_{p,q} J(p,q) K(p,q)$$
on $\ell^a(E)$. The discrete gradient $\nabla : \ell(V) \to \ell^a(E)$ is given by

$$\nabla f(p,q) = c_{p,q}(f(p) - f(q)),$$

the discrete divergence $\text{div} : \ell^a(E) \to \ell(V)$ is defined by

$$\text{div} J(p) = -\sum_{q : (p,q) \in E} J(p,q),$$

and the associated Laplace operator $\Delta : \ell(V) \to \ell(V)$ is defined by

$$-\Delta f(p) = -\text{div} \nabla f(p) = \sum_{q : (p,q) \in E} c_{p,q} (f(q) - f(p)).$$

We follow the usual probabilistically and physically inspired convention where $\Delta$ is a non-positive operator. This is analogous to the classical second derivative Laplace operator $\Delta = \frac{\partial^2}{\partial x^2}$ on $\mathbb{R}^1$, which is also non-positive, and sometimes is also denoted as “$\nabla^2$”. In the more combinatorially and algebraically oriented literature $L = -\Delta$ is sometimes called the weighted graph Laplacian.

For $A,B \subset V$, $J \in \ell^a(E)$ is called a flow from $A$ to $B$, if $\text{div} J(p) = 0$ for $p \notin A \cup B$. The flux of a flow $J$ from $A$ to $B$ is defined by

$$\text{flux}(A,B,J) = \sum_{p \in A} \text{div} J(p) = -\sum_{p \in B} \text{div} J(p).$$

The effective resistance between sets $A$ and $B$ is defined by

$$R(A,B)^{-1} = \inf \{ \mathcal{E}(f) \mid f|_A \equiv 0, \ f|_B \equiv 1 \}.$$
Energy is minimized by the function \( \phi \) such that \( \phi|_A \equiv 0 \), \( \phi|_B \equiv 1 \) and \( \Delta \phi(p) = 0 \) for \( p \notin A \cup B \), and thus \( \mathcal{E}(\phi) = \frac{1}{R(A,B)} \). We refer to such a function \( \phi \) as the harmonic function with boundary \( A \) and \( B \). The only functions which satisfy \( \Delta f \equiv 0 \) (i.e., harmonic without boundary) and \( f|_{A \cup B} \equiv 0 \) is the constant 0 function, thus the \( \phi \) is unique. Similarly \( I = R(A,B)^{-1} \nabla \phi \) is the unique energy minimizing flow from \( A \) to \( B \) with \( E(I) = 1 \). The following four characterizations of \( R(A,B) \) which are equivalent to the original, as seen in Section 2 of [3].

1. \( R(A,B) = \text{sup} \{1/\mathcal{E}(f) : f|_A \equiv 0, f|_B \equiv 1\} \)
2. \( R(A,B) = 1/\mathcal{E}(\phi) \) where \( \phi \in \ell(V) \) is the the unique function with \( \phi|_A \equiv 0, \phi|_B \equiv 1, \Delta \phi(p) = 0 \) for \( p \notin A \cup B \).
3. \( R(A,B) = \inf \{E(J) : \text{flux}(A,B,J) = 1\} \).
4. \( R(A,B) = E(I) \) where \( I \in \ell^\varnothing(E) \) is the unique energy dissipation minimizing unit flow from \( A \) to \( B \).

2.2. Barycentric subdivision, the hexacarpet and the resistance problem.

We define the 2-simplicial complexes \( \mathbb{T}_n = (E^0_n, E^1_n, E^2_n) \) where \( E^i_n \) are the \( i \)-simplices of the complex, starting with a 2-simplex (a triangle) \( \mathbb{T}_0 \) with 0-simplices (vertices) \( E^0_0 = \{p_0, p_1, p_2\} \), 1-simplices (edges) \( E^1_0 = \{[p_i, p_j] \}_{i \neq j} \), and 2-simplex (triangle) \( E^2_0 = \{[p_0, p_1, p_2]\} \). Here, if \( q_0, q_1, q_2 \in E^0_0 \), then \([q_0, q_1]\) will refer to the 1-simplex with \( q_0 \) and \( q_1 \) as endpoints (which may or may not be in \( E^1_1 \)), and similarly for \([q_0, q_1, q_2]\). We will only be considering simple simplicial complexes (without multiple edges/triangles). Thus \([q_0, q_1] = [q_0, q_1]\) determines a unique 1-simplex for example. We also use the notation \( \prec \) to denote containment of simplices, i.e. \( q_0 \prec [q_0, q_1] \prec [q_0, q_1, q_2] \). \( \mathbb{T}_{n+1} \) is defined inductively from \( \mathbb{T}_n \) by barycentric subdivision, pictured in Figure 1. That is \( E^i_{n+1} \) is \( E^i_n \) along with the barycenters of simplices in \( E^i_n \), \( i = 1,2 \), which we refer to as \( b(e) \) for \( e \in E^i_n \), \( i = 1,2 \). 1- and 2-simplices of \( \mathbb{T}_{n+1} \) are formed from barycenters of nested simplices. More concretely put, elements of \( E^1_{n+1} \) are either of the form \([q_0, q_1, q_2]\) where \( q_0, q_1 \in E^0_n \) with \([q_0, q_1] \in E^1_n \), \([q_0, b([q_0, q_1, q_2])]\), or \([b([q_0, q_1]), b([q_0, q_1, q_2])]\) where \( q_0, q_1, q_2 \in E^0_n \) with \([q_0, q_1, q_2] \in E^2_n \), and elements of \( E^2_{n+1} \) are of the form \([q_0, b([q_0, q_1]), b([q_0, q_1, q_2])]\) where \( q_0, q_1, q_2 \in E^0_n \) with \([q_0, q_1, q_2] \in E^2_n \) and \([b([q_0, q_1]), b([q_0, q_1, q_2])]\).

Definition 2.1. We define the graph \( G^T_n = (V^T_n, E^T_n) \) with vertex set \( V^T_n = E^0_n \) and edge relation \( q \sim^T q' \) if \([q, q'] \in E^1_n \). We will refer to this as the 1-skeleton of the \( \mathbb{T}_n \).

Definition 2.2. Following [12], we define the graph \( G_n = (V_n, E_n) \) with vertex set \( V_n = E^2_n \) and edge relation \([q_0, q_1, q] \sim [q_0, q_1, q']\). That is, the vertexes of \( G_n \) are the 2-simplices of \( \mathbb{T}_n \) and they are connected by an edge if these simplices share a 1-simplex.

Remark 1. \( G^T_n \) is classically known to be a planar graph, as seen in Figure 1, although throughout this work we will refer to the hexagonal embedding from Figure 3 more often. With either of these embeddings, \( G_n \) is the weak planar dual, that is each of its vertexes correspond to a plane region carved out by the embedding of \( G^T_n \) with the exception of the unbounded component.

We will need explicit names for the elements of \( E^i_n = \{p_0, p_1, p_2, p_0', p_1', p_2', p'\} \) where \( \{p_0, p_1, p_2\} = E^0_n \) as above, \( p'_i = b([p_i, p_{i'}]) \), \( i' \equiv i + 1 \mod 3 \), and \( p' = b([p_0, p_1, p_2]) \).

This is convenient for recursively defining functions on \( \mathbb{T}_n \). Define self-similarity maps \( F_i : \mathbb{T}_n \rightarrow \mathbb{T}_{n+1} \) for \( i = 0, 1, \ldots, 5 \), which are defined on \( \mathbb{T}_0 \) by \( F_i(p_0) = p'_i \),
$F_i(p_1) = p_{[i/2]}$, and $F_i(p_2) = p'_{[i/2]}$, where the index is taken mod 3. $F_i$ is extended to $T_n$ by the relations $F_i([q_0, q_1]) = [F_i(q_0), F_i(q_1)]$, $F_i([q_0, q_1, q_2]) = [F_i(q_0), F_i(q_1), F_i(q_2)]$ and $b \circ F_i = F_i \circ b$. If $w = w_1w_2 \cdots w_k$ is a word in \{0, 1, \ldots, 5\}$^k$, then we define $F_w : T_n^k \to T_{n+k}^k$ by $F_w := F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_k}$. $F_i$ as a function from $G^T_n \to G^T_{n+1}$ or $G_n \to G_{n+1}$ is a graph homomorphism, and $G^T_{n+1} = \cup_{i=0}^5 F_i(G^T_n)$. However, this is not true for $G_n$ and $G_{n+1}$, since not every edge of $G_{n+1}$ is covered by $F_i(G_n)$ for some $i = 0, 1, \ldots, 5$. We want to take advantage of this self-similarity throughout the current work, thus we define a modified hexacarpet graph.

**Definition 2.3.** The (modified) hexacarpet graph $G^H_n$ is defined to have vertex set

$$V^H_n := E^2_n \cup E^1_n$$

where adjacency is determined by $[q_0, q_1, q] \sim_H [q_0, q_1]$, i.e., $e \in E^1_n$ is connected to $f \in E^2_n$ if $e < f$.

For $G^T_n$, define the conductance of edges

$$c^T_{q,q'} = \begin{cases} 0 & \text{if } [q, q'] \not\in E^1_n \\ 1 & \text{if there exists } q_0 \neq q'_0 \text{ with } [q, q', q_0], [q, q', q'_0] \in E^2_n \\ 2 & \text{if there exist only one } q_0 \text{ such that } [q, q', q_0] \in E^2_n. \end{cases}$$

We take $\mathcal{E}^T_n$ to be the graph energy defined with the above conductance. The advantage of these conductance values is the resulting self-similarity relation

$$\mathcal{E}^T_{n+1}(f) = \sum_{i=0}^5 0 \mathcal{E}^T_n(f \circ F_i).$$

Similarly if we define $c^H_{q,q'} = 1/2$ if $q \sim_H q'$ and 0 otherwise, then the resulting energy function $\mathcal{E}^H_n$ satisfies the following relation

$$\mathcal{E}^H_{n+1}(f) = \sum_{i=0}^5 0 \mathcal{E}^H_n(f \circ F_i).$$

Both of these relations are also true for energy dissipation of functions on edges of these graphs.

It will often be useful to think of these graphs as embedded in the plane $\mathbb{R}^2$. It is typical to think of $T_n$ as a subdivided triangle in the plane, but we embed it as a hexagon, as $T_n$, $n > 0$, has symmetry group $D_6$, the dihedral group on 6 elements. As such, for $n > 0$ we define a map $F^T : E^0_n = V^T_n \to \mathbb{R}^2$ by $F^T(p') = (0, 0)$, $F^T(p_k) = (\cos(2k\pi/3), \sin(2k\pi/3))$, $F^T(p'_k) = (\cos((2k+1)\pi/6), \sin((2k+1)\pi/6))$ for $k = 0, 1, 2$. Thus $E^0_n$ is mapped to the corners and midpoint of a regular hexagon centered at $(0, 0)$, see Figure 3. We extend to $E^1_n$ by taking averages:

$$F^T \circ b([q_0, q_1]) = \frac{F^T(q_0) + F^T(q_1)}{2}, \quad F^T \circ b([q_0, q_1, q_2]) = \frac{F^T(q_0) + F^T(q_1) + F^T(q_2)}{3}.$$

We embed $G^H_n$ by the map $F^H : V^H_n \to \mathbb{R}^2$ by $F^H = F^T \circ b$. Thus the vertexes of the embedded $G^H_n$ are the centers of the triangles and edges of the embedding of $G^T_n$. 
**Definition 2.4.** Define when there is no danger of confusion.

where $j$

The resistances

**Theorem 2.1.**

Similarly, if $f = [q_0, q_1, q_2] \in E_n^2$, then $|f|$ is defined to be the convex hull of $F(T)$.

To define the resistance problem on these graphs, we need to define the boundary of these graphs. $|e| \subset |f|$, for $e \in E_n^1$ and $f \in E_n^2$, if the geometric realization of $e$ is a subset of the geometric realization of $f$, e.g. $[b(e)] \subset |e|$ or $|[q_0, b([q_0, q_1])]| \subset |[q_0, q_1, q_2]|$. We define $L_i^{(n,T)}$, resp. $L_i^{(n,H)}$, $i = 0, 1, \ldots, 5$ to be the vertexes $q \in V_n^T$, resp. $V_n^H$, such that $|q| \subset |[p_{i/2}, p_{i/2}']|$, if $i$ is even, and the set of $|q| \subset |[p_{i-1/2}, p_{i-1/2}]|$ where $j \equiv (i+1)/2 \mod 3$ if $i$ is odd. We will suppress arguments of the superscript when there is no danger of confusion.

**Definition 2.4.** Define $A_{(n)}^H = L_0^{(n,H)} \cup L_1^{(n,H)}$ and $B_{(n)}^H = L_3^{(n,H)} \cup L_4^{(n,H)}$. Further, define $R_n$ to be the effective resistance with respect to $\varepsilon_n^H$ between $A_{(n)}^H$ and $B_{(n)}^H$.

**Definition 2.5.** Define $A_{(n)}^T = L_2^{(n,T)}$ and $B_{(n)}^T = L_5^{(n,T)}$. Further, define $R_n^T$ to be the effective resistance with respect to $\varepsilon_n^T$ between $A_{(n)}^T$ and $B_{(n)}^T$.

**Theorem 2.1.** The resistances $R_n$ and $R_n^T$ are related by $R_n^T = 1/R_n$.

**Proof.** The main tool of this proof is proposition 9.4 from [56]. Define the weighted graph $\bar{G}_{(n)}^T = (\bar{V}_{(n)}^T, \bar{E}_{(n)}^T)$ where $\bar{V}_{(n)}^T$ is $V_{(n)}^T$ modulo the relation which identifies all elements of $A_{(n)}^T$ and $B_{(n)}^T$ into two vertexes called $a^T$ and $b^T$ respectively. The effective resistance between $a^T$ and $b^T$ with respect to $\varepsilon_{(n)}^T$ is $R_n^T$.

Further define $\bar{G}_{(n)}^H$ by, not only identifying $A_{(n)}^H$ and $B_{(n)}^H$ into single vertexes $a^H$ and $b^H$ respectively, but also to replace the sequential edges of the form $[q_0, q_1, q] \sim_H$.

**Figure 3.** $A_{(2)}^T$ and $B_{(2)}^T$ on the hexagonal embedding of $G_{(2)}^T$. 

If $e = [q_0, q_1] \in E_n^1$ then we define the geometric realization of $e$, $|e|$, to be the convex hull of $F(T)$ and $F(T)$. That is $|e| = \{ \theta_0 F(T) + \theta_1 F(T) : \theta_0 + \theta_1 = 1 \}$.

Similarly, if $f = [q_0, q_1, q_2] \in E_n^2$, then $|f|$ is defined to be the convex hull of $F(T)$, $F(T)$ and $F(T)$. 

Further define $|e| \subset |f|$ for $e \in E_n^1$ and $f \in E_n^2$, if the geometric realization of $e$ is a subset of the geometric realization of $f$, e.g. $|b(e)| \subset |e|$ or $|[q_0, b([q_0, q_1])]| \subset |[q_0, q_1, q_2]|$.

We define $L_i^{(n,T)}$, resp. $L_i^{(n,H)}$, $i = 0, 1, \ldots, 5$ to be the vertexes $q \in V_n^T$, resp. $V_n^H$, such that $|q| \subset |[p_{i/2}, p_{i/2}']|$, if $i$ is even, and the set of $|q| \subset |[p_{i-1/2}, p_{i-1/2}]|$, where $j \equiv (i+1)/2 \mod 3$ if $i$ is odd. We will suppress arguments of the superscript when there is no danger of confusion.

**Definition 2.4.** Define $A_{(n)}^H = L_0^{(n,H)} \cup L_1^{(n,H)}$ and $B_{(n)}^H = L_3^{(n,H)} \cup L_4^{(n,H)}$. Further, define $R_n$ to be the effective resistance with respect to $\varepsilon_n^H$ between $A_{(n)}^H$ and $B_{(n)}^H$.

**Definition 2.5.** Define $A_{(n)}^T = L_2^{(n,T)}$ and $B_{(n)}^T = L_5^{(n,T)}$. Further, define $R_n^T$ to be the effective resistance with respect to $\varepsilon_n^T$ between $A_{(n)}^T$ and $B_{(n)}^T$.

**Theorem 2.1.** The resistances $R_n$ and $R_n^T$ are related by $R_n^T = 1/R_n$.

**Proof.** The main tool of this proof is proposition 9.4 from [56]. Define the weighted graph $\bar{G}_{(n)}^T = (\bar{V}_{(n)}^T, \bar{E}_{(n)}^T)$ where $\bar{V}_{(n)}^T$ is $V_n^T$ modulo the relation which identifies all elements of $A_{(n)}^T$ and $B_{(n)}^T$ into two vertexes called $a^T$ and $b^T$ respectively. The effective resistance between $a^T$ and $b^T$ with respect to $\varepsilon_{(n)}^T$ is $R_n^T$.

Further define $\bar{G}_{(n)}^H$ by, not only identifying $A_{(n)}^H$ and $B_{(n)}^H$ into single vertexes $a^H$ and $b^H$ respectively, but also to replace the sequential edges of the form $[q_0, q_1, q] \sim_H$.
Figure 4. $\tilde{G}_2^H$ and $\tilde{G}_2^T$ without the additional edges.

$[q_0, q_1] \sim_H [q_0, q_1, q']$ using Kirchoff’s laws. Thus the sequential connections with resistance $1/2$ are replaced with one connection $[q_0, q_1] \sim_H [q_0, q_1, q']$ with resistance $1$. It is easy to see that the resistance between $a^H$ and $b^H$ is $R_n$. Also, remove the vertexes contained in $A^T_{(n)}$ and $B^T_{(n)}$ and associated edges — since these vertexes are connected to the graph by only 1 edge, removing them has no impact on the resistance.

If we define the graphs $(\tilde{G}_n^H)\dagger$ and $(\tilde{G}_n^T)\dagger$ to be the $\tilde{G}_n^H$ and $\tilde{G}_n^T$ with an additional edge connecting $a^H$ to $b^H$ and $a^T$ to $b^T$ respectively, then $(\tilde{G}_n^H)\dagger$ is the planar dual of $(\tilde{G}_n^T)\dagger$ (see Figure 4), and thus by Proposition 9.4 in [56], $R_n = 1/R_n^T$.

3. Multiplicative estimates and other proofs.

**Theorem 3.1.** $R_{m+n} \leq \frac{4}{3} R_n R_m$, and equivalently $R_{m+n}^T \geq \frac{3}{4} R_m^T R_n^T$.

We give two proofs of this theorem. The constants differ in the proofs, but the exact value of the constant does not affect the existence of $\rho$ and $\rho^T$. This establishes the result independently of the duality of the graphs (variations on these proofs work for different choices of boundary). One version proves the upper estimate on $R_{m+n}$ directly, and uses flows on $G_n^H$. The direct proof of the lower estimate on $R_{m+n}^T$ is proven using potentials on $G_n^T$. The two proofs mirror the upper and lower bounds for the resistance of the pre-carpet approximations for the Sierpinski carpet in [3]. The two versions of our proof highlight the importance of duality in proving Barlow–Bass style resistance estimates, and suggest possible generalizations.

Theorem 3.1 implies that there are constants $c_0$ and $c_1$ such that $c_0 + \log R_n^T$ and $c_1 + \log R_n$ are superadditive/subadditive positive sequences and thus we have the following.

**Corollary 3.1.** The limits $\lim_{n \to \infty} \frac{1}{n} \log R_n^T = \log \rho^T$ and $\lim_{n \to \infty} \frac{1}{n} \log R_n = \log \rho$ exist.
the Y-network associated to a sign then is arbitrary and a rearrangement of those in $H$ applying the appropriate symmetry. Since the values of which is built from $\rho$ Subsections 3.3 and 3.4 we establish positive upper and lower estimates on $\rho$ and $\rho^T$. 

3.1. Upper estimate and flows on $G_n^H$. Consider the flow $I^n$ which is the minimizing flow on $G_n^H$ from $A(\omega)$ to $B(\omega)$. We will construct a flow on $G_{n+m}$ which has global structure resembling $I^n$ but on $F_\omega G_n^H$ for $\omega \in \{0,1,\ldots,5\}$ has structure which is built from $I^n$.

Define $H_{02}^n = I^n$ when restricted to the half of the $G_n^H$ which connects vertexes contained in $[p_0, p'_0, p'], [p'_0, p_1, p'], [p_1, p'_1, p']$. On the other half, $H_{02}^n$ is $I^n$ composed with the symmetry which exchanges $p_0$ with $p'_1$, $p'_0$ with $p_1$, and $p_2$ with $p'_2$ and is extended to the rest of $V_n^H$ by convex combinations. The construction of $H_{02}^n$ is depicted in Figure 5. Using the embedding $F^H$, this symmetry is the reflection through the line which makes a $\pi/2$ angle with the x-axis. Because $I^n$ subjected to a $\pi$-rotation is $-I^n$, $H_{02}^n$ is a flow.

Thus $H_{02}^n$ is a flow on $G_n^H$ between $L_0^{(n)} \cup L_1^{(n)}$ and $L_4^{(n)} \cup L_5^{(n)}$, with flux 1. $H_{01}^n$ is the flow from $L_0^{(n)} \cup L_1^{(n)}$ and $L_2^{(n)} \cup L_3^{(n)}$ with flux 1 obtained from $H_{02}^n$ by applying the appropriate symmetry. Since the values of $H_{01}^n$ (resp. $H_{02}^n$) are just a rearrangement of those in $I^n$, then $E(H_{01}^n) = E(H_{02}^n) = E(I^n) = R_m$ for all $i \neq j$.

We consider $G_m^H$ as the set of Y-networks, centered at $x \in E_m^2 \subset V_m^H$. Take $a_0(x), a_1(x), a_2(x)$ to refer to the outward flow of $I^n$ restricted to each of these edges in the Y-network associated to $x$ with orientation such that $a_0(x)$ is of a different sign then $a_1(x)$ and $a_2(x)$. I.e. if $a_0(x) < 0$, then $a_1(x), a_2(x) \geq 0$, if $a_0(x) > 0$, $a_1(x), a_2(x) \leq 0$, and in the case when $a_0(x) = a_1(x) = a_2(x) = 0$ then the choice is arbitrary and

$$E(I^n) = \frac{1}{2} \sum_{x \in E_m^2} (a_0(x)^2 + a_1(x)^2 + a_2(x)^2) = R_m.$$ 

Take $F_\omega G_n^H \subset G_{n+m}^H$ such that $F_\omega([p_0, p_1, p_2]) = x$ to be the subgraph of $G_{n+m}^H$ isomorphic to $G_n^H$ corresponding to $x \in E_m^2$. We assume that the labeling of the sides $F_\omega(L_2^p \cup L_3^p)$ is contained in the edge which corresponds to the values $a_1(x)$, and $F_\omega(L_1^p \cup L_3^p)$ corresponds to $a_2(x)$. The set of subgraphs $\{F_\omega G_n^H\}_{x \in E_m^2, \omega \in \{0,1,\ldots,5\}}$ 

Note that this corollary does not rule out the possibility that $\rho = 0$ or $\infty$. In Subsections 3.3 and 3.4 we establish positive upper and lower estimates on $\rho$ and $\rho^T$.

Figure 5. The transformation from the flow $I^n$ (left) to the flow $H_{02}^n$ (right).
cover $G_{m+n}^H$ and define the flow $J$ on $G_{m+n}^H$ by its values on these subgraphs

$$J \circ F_0 = a_1(x)H_{01}^n + a_2(x)H_{02}^n.$$ 

$J$ is a well defined flow because $H_{02}^n$ is obtained by a reflection of $H_{01}^n$ which is symmetric with respect to $I^n$, so $aH_{01}^n(y, z) + bH_{02}^n(y, z) = (a + b)I^n(y, z)$ for all $y \in L^n_0 \cup L^n_1$.

Now we see that

$$E(J) = \sum_{x \in E_m^2} E(a_1(x)H_{01}^n + a_2(x)H_{02}^n)$$

$$= \sum_{x \in E_m^2} \left( a_1(x)^2 E(H_{01}^n) + a_2(x)^2 E(H_{02}^n) + 2a_1(x)a_2(x) E(H_{01}^n, H_{02}^n) \right)$$

$$\leq \sum_{x \in E_m^2} \left( a_1(x)^2 E(H_{01}^n) + a_2(x)^2 E(H_{02}^n) + 2a_1(x)a_2(x) E(H_{01}^n)^{1/2} E(H_{02}^n)^{1/2} \right)$$

$$= \sum_{x \in E_m^2} (a_1(x) + a_2(x))^2 E(I^n) = R_n \sum_{x \in E_m^2} a_0(x)^2$$

$$\leq R_n \frac{2}{3} \sum_{x \in E_m^2} (a_0(x)^2 + a_1(x)^2 + a_2(x)^2) = \frac{4}{3} R_m R_n.$$ 

Note that the first inequality holds because of Cauchy-Schwartz inequality and the fact that, by our labeling convention, $a_1(x) + a_2(x) \geq 0$, and the last inequality holds because $a_1^2(x) + a_2^2(x) \geq (a_1(x) + a_2(x))^2 / 2 = a_0^2(x) / 2$.

### 3.2. Lower estimate and potentials on $G_n^{T_1}$

Let $\phi_n$ be the harmonic potential on $G_n^{T_1}$ with boundary values 0 on $A_{m+n}^T$ and 1 on $B_{m+n}^T$. On $G_n^{T_1}$, define $u = \phi_n \circ F_0$, $v = \phi_n \circ F_1$ and $w = \phi_n \circ F_3$, $\phi_n \circ F_2$, $\phi_n \circ F_3$, and $\phi_n \circ F_4$ can be written in terms of $u, v, w$ and the constant function as illustrated in Figure 6.

Notice that the function $w$ is $v \circ \sigma$ where $\sigma : G_n^{T_1} \to G_n^{T_1}$ is the symmetry which exchanges $p_0'$ and $p_2$, fixing $p_0$, and is extended to the rest of $G_n^{T_1}$ by averages (with respect to $F^T$, $\sigma$ is the flip about the horizontal axis). Also,

$$R_n^{-1} = \varepsilon_n^{T_1}(\phi_n) = 2(\varepsilon_n^{T_1}(u) + \varepsilon_n^{T_1}(v) + \varepsilon_n^{T_1}(w)) = 2\varepsilon_n^{T_1}(u) + 4\varepsilon_n^{T_1}(v).$$

![Figure 6. The function $u, v$ and $w$.](image-url)
Lemma 3.2. $\varepsilon_{n-1}^T(u, v - w) = 0$

Proof. On one hand, $\varepsilon_{n-1}^T(f) = \varepsilon_{n-1}^T(f \circ \sigma)$ for all $f$ because $\sigma$ is a graph isometry with respect to the conductances. However, the function $u$ symmetric about the horizontal axis, i.e. $u(x) = u \circ \sigma(x)$, and $v$ and $w$ is anti-symmetric about this axis, i.e. $(v \circ \sigma(x) - w \circ \sigma(x)) = (w(x) - v(x))$. Thus $\varepsilon_{n-1}^T(u, v - w) = \varepsilon_{n-1}^T(u \circ \sigma, v \circ \sigma - w \circ \sigma) = -\varepsilon_{n-1}^T(u, v - w)$. □

For each $x \in E_{n-1}^2$, $i' \equiv i + 1 \mod 6$ define $a^x = b(x)$ to be the barycenter of $x$. Also, define $a_i^x$, $i = 0, 1, \ldots, 5$ to be vertexes and barycenters of edges contained in $x$ ordered in such way that $[a_i^x, a_i^{x+1}]$ is an edge in $E_i^n$, and that the vectors

$$(u \circ F_\omega(a^x), v \circ F_\omega(a^x)) = (1/2, 0, 1/2),$$

$$(w \circ F_\omega(a^x), w \circ F_\omega(a^x)) = (1/2, 1/2, 0),$$

$$(u \circ F_\omega(a^x), u \circ F_\omega(a^x)) = (1/2, 0, 0),$$

for $\omega \in \{0, 1, \ldots, 5\}^{n-1}$ such that $F_\omega([p_0, p_1, p_2]) = x$. Notice that for all $x \in E_{n-1}^2$, $a^x$ and $a_i^x$ are contained in $E_0^n$, and that, for any function $f$ on $G_n^T$, then

$$\varepsilon_n^T(f) = \sum_{x \in E_{n-1}^2} \sum_{i=0}^5 \left( (f(a^x) - f(a_i^x))^2 - \frac{1}{2}(f(a^x) - f(a_i^x))^2 \right)$$

where $i' \equiv i + 1 \mod 6$. We now define a function $f_{n+m}$ on $G_n^{T+}$ such that $f_{n+m}|G_n^T = \phi_m$ as follows: if $F_\omega : G_n^T \rightarrow G_{n+m}^T$ is the contraction mapping which takes $G_n^T$ to the $[a^x, a_i^x, a_j^x]$, then $f_{n+m} \circ F_\omega$ is equal to

$$(2\phi_m(a^x) - \phi_m(a_i^x) - \phi_m(a_j^x))u + (\phi_m(a_i^x) - \phi_m(a_j^x))(v - w) + \frac{1}{2}(\phi_m(a_i^x) + \phi_m(a_j^x))z,$$

and so

$$\varepsilon_{n+m}^T(f_{n+m}) = \sum_{\omega \in \{0, \ldots, 5\}^{n+1}} \varepsilon_{n-1}^T(f_{n+m} \circ F_\omega)$$

$$= \sum_{x \in E_{n-1}^2} \sum_{i=0}^5 \left( 2(\phi_m(a^x) - \phi_m(a_i^x) - \phi_m(a_j^x))^2 \varepsilon_{n-1}^T(u) \right.$$}

$$\left. + (\phi_m(a_i^x) - \phi_m(a_j^x))^2 \varepsilon_{n-1}^T(v) \right)$$

$$\leq \sum_{x \in E_{n-1}^2} \sum_{i=0}^5 \left( 2(\phi_m(a^x) - \phi_m(a_i^x) - \phi_m(a_j^x))^2 \varepsilon_{n-1}^T(u) \right.$$}

$$\left. + 4(\phi_m(a_i^x) - \phi_m(a_j^x))^2 \varepsilon_{n-1}^T(v) \right)$$

$$\leq \sum_{x \in E_{n-1}^2} \sum_{i=0}^5 \left( 2((\phi_m(a^x) - \phi_m(a_i^x))^2 + (\phi_m(a_i^x) - \phi_m(a_j^x))^2) \varepsilon_{n-1}^T(u) \right.$$}

$$\left. + 4(\phi_m(a_i^x) - \phi_m(a_j^x))^2 \varepsilon_{n-1}^T(v) \leq \right)$$

$$2\left( 2\varepsilon_{n-1}^T(u) + 4\varepsilon_{n-1}^T(w) \right) \sum_{x \in E_{n-1}^2} \sum_{i=0}^5 \left( (\phi_m(a^x) - \phi_m(a_i^x))^2 + \frac{1}{2}(\phi_m(a_i^x) - \phi_m(a_j^x))^2 \right).$$

This is less or equal to $2R_m^{-1}R_n^{-1}$, which implies that $R_{m+n} \geq R_mR_n/2$. 

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3.3. Upper bound on $\rho$ by removing edges.

**Proposition 3.2.** $R_n \leq (3/2)^n$, and thus $\rho \leq 3/2$.

*Proof.* Figure 7 shows $\hat{G}_1 = (V_1^H, E_1)$, which is obtained from $G_1^H$ by removing all edges which have the vertexes contained in $[p'_0, p'_1]$ and $[p'_2, p'_3]$. Figure 7 also illustrates how six $\hat{G}_n$ graphs can be glued together to form one $\hat{G}_{n+1}$ graph. By induction, we see that each $\hat{G}_n$ is made up of $2^n$ paths between $L_n(0) \cup L_n(1)$ and $L_n(5) \cup L_n(4)$.

The lengths of all the paths from $L_n(0) \cup L_n(1)$ and $L_n(5) \cup L_n(4)$ in $\hat{G}_n$ can be encoded in a sequence of $2^n$ integers. We will call this sequence $l_n = \{l_{n,1}, l_{n,2}, \ldots, l_{n,2^n}\}$ where $l_{n,j}$ is the length of the path which has initial (or terminal) point $j^{th}$ closest to $p_0$ (in graph or Euclidean distance with our embedding). Since $\hat{G}_{n+1} = 6$ copies of $\hat{G}_n$ glued together, $\sum_{k=1}^{2^n} l_{n,k} = 6 \sum_{k=1}^{2^n-1} l_{n-1,k} = 6^n$, because $l_1 = \{2, 4\}$.

The corresponding resistance $\hat{R}_n$ between $L_0(0) \cup L_1(1)$ and $L_5(5) \cup L_4(4)$ of the $\hat{G}_n$ graph is the resistance of $2^n$ paths connected in parallel so, by Kirchhoff’s laws, $\hat{R}_n = \left(\sum_{j=1}^{2^n} \frac{1}{l_{n,j}}\right)^{-1}$. Using Jensen’s inequality for the convex function $x \mapsto 1/x$ on $(0, \infty)$, we have

$$\hat{R}_n = \frac{1}{2^n} \left(\frac{1}{2^n} \sum_{j=1}^{2^n} \frac{1}{l_{n,j}}\right)^{-1} \leq \frac{1}{2^n} \frac{1}{2^n} \sum_{j=1}^{2^n} l_{n,j} = \left(\frac{3}{2}\right)^n.$$  

To obtain an upper bound on $R_n$, we glue together 6 copies of $\hat{G}_{n-1}$ to produce a graph which has an edge set contained in $E_n^H$ consisting of $2^n$ paths from $A_n^H$ to $B_n^H$. Then, using Kirchhoff’s laws and the above argument, we obtain that we are connecting $A_n^H$ to $B_n^H$ with 2 parallel connections of three sequential wires of resistance $\hat{R}_n$. Thus we have that $R_n \leq 3\hat{R}_{n-1}/2 = (3/2)^n$. \hfill $\square$
This also implies a lower bound on $\rho^T \geq 2/3$, which can also be obtained by considering the graph $\tilde{G}_n^T$ with vertex set $\tilde{V}_n^T$ where $x, y \in V_n^T$ (the vertex set of $G_n^T$) are identified as one vertex if an edge connecting $x$ and $y$ was deleted in the construction of $\hat{G}_n$ with “end” points $P$ and $Q$, which correspond to the points in $A^T$ and $B^T$, Figure 8:

![Figure 8. Short-circuited graphs $\tilde{G}_1^T$ and $\tilde{G}_2^T$.](image-url)

### 3.4. Lower bound on $\rho$ by shorting graph.

In this subsection we define a graph $\tilde{G}_n^H$, such that $\tilde{V}_n^H$ is $V_n^H$ modulo an equivalence relation. Thus, resistance in $\tilde{G}_n^H$ between two sets is less than the resistance between the fibers of these sets.

**Proposition 3.3.** $R_n \geq c(5/4)^n$ for a constant $c$ independent of $n$, and so $\rho \geq 5/4$.

**Proof.** We define that two vertexes in $\tilde{V}_n^H$ to be equivalent if they are both contained in $F_\omega([p_i, p_j])$ where $j \equiv i + 1 \mod 3$ for some $i$ and some $\omega \in \{0, 1, \ldots, 5\}$.

Figure 9 shows $\tilde{G}_2^H$. These graphs appeared in [65], as an example of non-p.c.f. Sierpinski gaskets, where it was determined that their resistance scaling factor is $5/4$.

This implies that $\tilde{R}_{n+1} = \frac{5}{4} \tilde{R}_n$ and subsequently $\tilde{R}_n = \tilde{R}_1 (\frac{5}{4})^{n-1}$. From this, and a gluing argument as in the previous subsection, it follows that $R_n \geq c(\frac{5}{4})^n$. \qed

Using a similar argument, $G_n^T$ can be obtained by identifying points in graph approximations of another non-p.c.f. Sierpinski gasket which appears at the end of [65], and which is pictured in Figure 9. The resistance between the corner points of these graphs is some constant times $(4/5)^n$, and thus the resistance between the corner points of $G^T_n$ is less than this value. This proves that $\rho^T \leq 4/5$ because including more points in the boundary decreases resistance, so the resistance between the corner points is greater than the resistance between $A$ and $B$. Alternatively, if we connect the boundary points of the graph in the left of Figure 9 to a $\triangle$-network, it is dual to the network attained by connecting the corner points in the graph in the right of Figure 9 to a $Y$-network. This also explains why the resistances are reciprocal.

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**REFERENCES**

[1] L. Ambrosio, M. Erbar and G. Savaré, Optimal transport, Cheeger energies and contractivity of dynamic transport distances in extended spaces, *Nonlinear Anal.*, 137 (2016), 77–134.

[2] M. T. Barlow, Analysis on the Sierpinski carpet, in *Analysis and Geometry of Metric Measure Spaces*, vol. 56 of CRM Proc. Lecture Notes, Amer. Math. Soc., Providence, RI, 2013, 27–53.

[3] M. T. Barlow and R. F. Bass, On the resistance of the Sierpiński carpet, *Proc. Roy. Soc. London Ser. A.*, 431 (1990), 345–360.

[4] M. T. Barlow, R. F. Bass and J. D. Sherwood, Resistance and spectral dimension of Sierpiński carpets, *J. Phys. A.*, 23 (1990), L253–L258.
FIGURE 9. Left: Graph $\tilde{G}^H_2$ with short circuits. Right: Non-p.c.f Sierpinski gasket.

[5] M. Barlow, Diffusions on fractals, in Lectures on Probability Theory and Statistics (Saint-Flour, 1995), vol. 1690 of Lecture Notes in Math., Springer, Berlin, 1998, 1–121.

[6] M. Barlow and R. Bass, Brownian motion and harmonic analysis on Sierpinski carpets, Canad. J. Math., 51 (1999), 673–744.

[7] M. Barlow, R. F. Bass, T. Kumagai and A. Teplyaev, Uniqueness of Brownian motion on Sierpiński carpets, J. Eur. Math. Soc. (JEMS), 12 (2010), 655–701.

[8] L. Bartholdi, R. Grigorchuk and V. Nekrashevych, From fractal groups to fractal sets, in Fractals in Graz 2001, Trends Math., Birkhäuser, Basel, 2003, 25–118.

[9] R. Bass, Diffusions on the Sierpinski carpet, in Trends in Probability and Related Analysis (Taipei, 1996), World Sci. Publ., River Edge, NJ, 1997, 1–34.

[10] F. Baudoin and D. J. Kelleher, Differential one-forms on dirichlet spaces and bakry-emery estimates on metric graphs, arXiv:1604.02820, Trans. AMS, to appear.

[11] F. Bauer, M. Keller and R. K. Wojciechowski, Cheeger inequalities for unbounded graph Laplacians, J. Eur. Math. Soc. (JEMS), 17 (2015), 259–271.

[12] M. Begue, D. Kelleher, A. Nelson, H. Panzo, R. Pellico and A. Teplyaev, Random walks on barycentric subdivisions and the Strichartz hexacarpet, Exp. Math., 21 (2012), 402–417.

[13] R. Bell, C.-W. Ho and R. S. Strichartz, Energy measures of harmonic functions on the Sierpiński gasket, Indiana Univ. Math. J., 63 (2014), 831–868.

[14] J. Bello, Y. Li and R. S. Strichartz, Hodge–de Rham theory of K-forms on carpet type fractals, in Excursions in Harmonic Analysis, Appl. Numer. Harmon. Anal., Birkhäuser/Springer, Cham, 3 (2015), 23–62.

[15] N. Bouleau and F. Hirsch, Dirichlet Forms and Analysis on Wiener Space, vol. 14 of de Gruyter Studies in Mathematics, Walter de Gruyter & Co., Berlin, 1991.

[16] Z.-Q. Chen and M. Fukushima, Symmetric Markov Processes, Time Change, and Boundary Theory, vol. 35 of London Mathematical Society Monographs Series, Princeton Univ. Press, 2012.

[17] P. Diaconis and D. Freedman, Iterated random functions, SIAM Rev., 41 (1999), 45–76.

[18] P. Diaconis and C. McMullen, Barycentric Subdivision, Unpublished, 2008.

[19] P. Diaconis and L. Miclo, On barycentric partitions, with simulations, https://hal.archives-ouvertes.fr/hal-00353842.

[20] P. Diaconis and L. Miclo, On barycentric subdivision, Combin. Probab. Comput., 20 (2011), 213–237.

[21] P. G. Doyle and J. L. Snell, Random Walks and Electric Networks, vol. 22 of Carus Mathematical Monographs, Mathematical Association of America, Washington, DC, 1984.

[22] M. Fukushima, Y. Oshima and M. Takeda, Dirichlet Forms and Symmetric Markov Processes, vol. 19 of de Gruyter Studies in Mathematics, extended edition, Walter de Gruyter & Co., Berlin, 2011.
[22] R. Grigorchuk and V. Nekrashevych, Self-similar groups, operator algebras and Schur complement, J. Mod. Dyn., 1 (2007), 323–370.
[23] A. Grigoryan and J. Hu, Heat kernels and Green functions on metric measure spaces, Canad. J. Math., 66 (2014), 641–699.
[24] A. Grigoryan and A. Telcs, Two-sided estimates of heat kernels on metric measure spaces, Ann. Probab., 40 (2012), 1212–1284.
[25] B. M. Hambly, V. Metz and A. Teplyaev, Self-similar energies on post-critically finite self-similar fractals, J. London Math. Soc. (2), 74 (2006), 93–112.
[26] K. E. Hare, B. A. Steinhurst, A. Teplyaev and D. Zhou, Disconnected Julia sets and gaps in the spectrum of Laplacians on symmetric finitely ramified fractals, Math. Res. Lett., 19 (2012), 537–553.
[27] A. Grigor’yan and J. Hu, Heat kernels and Green functions on metric measure spaces, Canad. J. Math., 66 (2014), 641–699.
[28] A. Grigor’yan and A. Telcs, Two-sided estimates of heat kernels on metric measure spaces, Ann. Probab., 40 (2012), 1212–1284.
[29] B. M. Hambly, V. Metz and A. Teplyaev, Self-similar energies on post-critically finite self-similar fractals, J. London Math. Soc. (2), 74 (2006), 93–112.
[30] K. E. Hare, B. A. Steinhurst, A. Teplyaev and D. Zhou, Disconnected Julia sets and gaps in the spectrum of Laplacians on symmetric finitely ramified fractals, Math. Res. Lett., 19 (2012), 537–553.
[31] A. Hatcher, Algebraic Topology, Cambridge University Press, Cambridge, 2002.
[32] J. Heinonen, P. Koskela, N. Shanmugalingam and J. T. Tyson, Sobolev Spaces on Metric Measure Spaces, vol. 27 of New Mathematical Monographs, Cambridge University Press, Cambridge, 2015, An approach based on upper gradients.
[33] M. Hinz, Dirac and magnetic Schrödinger operators on fractals, J. Funct. Anal., 265 (2013), 2830–2854.
[34] M. Ionescu, L. Rogers and A. Teplyaev, Derivations and Dirichlet forms on fractals, J. Funct. Anal., 263 (2012), 2141–2169.
[35] V. A. Kaimanovich, “Münchhausen trick” and amenability of self-similar groups, Internat. J. Algebra Comput., 15 (2005), 907–937.
[36] N. Kajino, Heat kernel asymptotics for the measurable Riemannian structure on the Sierpinski gasket, Potential Anal., 36 (2012), 67–115.
[37] N. Kajino, Analysis and geometry of the measurable Riemannian structure on the Sierpiński gasket, in Fractal Geometry and Dynamical Systems in Pure and Applied Mathematics. I. Fractals in Pure Mathematics, vol. 600 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2013, 91–133.
[38] C. J. Kauffman, R. M. Kesler, A. G. Parshall, E. A. Stamey and B. A. Steinhurst, Quantum mechanics on Laakso spaces, J. Math. Phys., 53 (2012), 042102, 18pp.
[39] D. J. Kelleher, B. A. Steinhurst and C.-M. M. Wong, From self-similar structures to self-similar groups, Internat. J. Algebra Comput., 22 (2012), 1250056, 16pp.
[40] M. Keller, D. Lenz and R. K. Wojciechowski, Volume growth, spectrum and stochastic completeness of infinite graphs, Math. Z., 274 (2013), 905–932.
[41] J. Kigami, Harmonic calculus on p.c.f. self-similar sets, Trans. Amer. Math. Soc., 335 (1993), 721–755.
[42] J. Kigami, Analysis on Fractals, vol. 143 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 2001.
[43] O. Knill, The graph spectrum of barycentric refinements, arXiv:1508.02027.
[44] O. Knill, Universality for Barycentric subdivision, arXiv:1609.06092.
[50] S. Kusuoka and X. Y. Zhou, Dirichlet forms on fractals: Poincaré constant and resistance, *Probab. Theory Related Fields*, **93** (1992), 169–196.

[51] M. Lapidus and J. Sarhad, Dirac operators and geodesic metric on the harmonic Sierpinski gasket and other fractal sets, *J. Noncommut. Geom.*, **8** (2014), 947–985.

[52] P. Li, Large time behavior of the heat equation on complete manifolds with nonnegative Ricci curvature, *Ann. of Math. (2)*, **124** (1986), 1–21.

[53] P. Li and S.-T. Yau, On the parabolic kernel of the Schrödinger operator, *Acta Math.*, **156** (1986), 153–201.

[54] T. Lindstrom, Brownian motion on nested fractals, *Mem. Amer. Math. Soc.*, **83** (1990), iv+128pp.

[55] D. Lougee and B. Steinhurst, Bond percolation on a non-P.C.F. Sierpiński gasket, iterated barycentric subdivision of a triangle, and hexacarpet, *Fractals*, **23** (2016), 1650023, 12pp.

[56] R. Lyons and Y. Peres, *Probability on Trees and Networks*, vol. 42 of Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, New York, 2016, Available at [http://pages.iu.edu/~rdlyons/](http://pages.iu.edu/~rdlyons/).

[57] I. McGillivray, Resistance in higher-dimensional Sierpiński carpets, *Potential Anal.*, **16** (2002), 289–303.

[58] D. Molitor, N. Ott and R. Strichartz, Using Peano curves to construct Laplacians on fractals, *Fractals*, **23** (2015), 1550048, 29pp.

[59] V. Nekrashevych, *Self-similar Groups*, vol. 117 of Mathematical Surveys and Monographs, Amer. Math. Soc., 2005.

[60] V. Nekrashevych and A. Teplyaev, Groups and analysis on fractals, in *Analysis on Graphs and Its Applications*, vol. 77 of Proc. Sympos. Pure Math., Amer. Math. Soc., 2008, 143–180.

[61] L. Rogers and A. Teplyaev, Laplacians on the basilica Julia sets, *Commun. Pure Appl. Anal.*, **9** (2010), 211–231.

[62] B. Steinhurst, Uniqueness of locally symmetric Brownian motion on Laakso spaces, *Potential Anal.*, **38** (2013), 281–298.

[63] R. S. Strichartz, *Differential Equations on Fractals. A Tutorial*, Princeton Univ. Press, 2006.

[64] A. Télcs and V. Vespri, Resolvent metric and the heat kernel estimate for random walks, *Stochastic Process. Appl.*, **124** (2014), 3965–3985.

[65] A. Teplyaev, Harmonic coordinates on fractals with finitely ramified cell structure, *Canad. J. Math.*, **60** (2008), 457–480.

[66] S. Volkov, Random geometric subdivisions, *Random Structures Algorithms*, **43** (2013), 115–130.

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