Spherical Functions on 2-adic Ramified Hermitian Spaces

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Abstract
Y. Hironaka introduced the spherical functions on the p-adic space of Hermitian matrices. For the space of $2 \times 2$ Hermitian matrices, we complete Hironaka’s work by also considering the case of a wildly ramified quadratic extension. We compute the spherical functions explicitly and obtain the functional equations.

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1 Notation

- $E$ - Local p-adic field.
- $E^* = E - \{0\}$.
- $O_E$ - The ring of integers of $E$.
- $O_E^*$ - The units of $O_E$.
- $\pi$ - A uniformizer of $E$.
- $E$ - The residue field of the field $E$.
- $|\cdot|_E$ - The p-adic absolute value on $E$ normalized such that $|\pi|_E = |E|^{-1}$.
- $v_E$ - The valuation map corresponding to $|\cdot|_E$.
- $F$ - A quadratic extension of $E$.
- $N = N_{F/E}, Tr = Tr_{F/E}$ - The Norm and Trace maps.
• $\varpi$ - A uniformizer of $F$.
• $q = \# F$. 
• $|| = || F$ - The p-adic absolute value of $F$ normalized by $|\varpi| = q^{-1}$.
• $v_F$ - The valuation map corresponding to $|\cdot|_F$.
• $x \mapsto \overline{x}$ - The conjugation map of the field extension.
• $s = v_F[[(\varpi)\cdot\varpi^{-1} - 1]$ - The number is defined only if $F/E$ is ramified.
• $l = \lfloor \frac{s}{2} \rfloor$ - The largest integer $k$ such that $k \leq \frac{s}{2}$.

2 Introduction

Let $E$ be a p-adic local field and $F$ a quadratic extension of $E$. Let $O_F$ be the ring of integers of $F$, $G = GL_2(F)$, and $K = GL_2(O_F)$. Set $A^* = \overline{A}$. Let $X = \{A \in GL_2(F) | A^* = A\}$. We note that the group $G$ acts on $X$ by $g \cdot x = g x g^\ast$.

We denote by $C^\infty(K\backslash X)$ the space of complex-valued $K$-invariant functions on $X$ and $S(K\backslash X) \subseteq C^\infty(K\backslash X)$ the subspace of all compactly supported $K$ invariant functions on $X$. Let $\mathcal{H}(G,K)$ be the Hecke algebra of $G$ with respect to $K$ : the space of all compactly supported $K$-bi-invariant complex valued functions on $G$.

Let $dg$ be the Haar measure on $G$ normalized by $\int_G 1 dg = 1$. The algebra $\mathcal{H}(G,K)$ acts on $C^\infty(K\backslash X)$ and on $S(K\backslash X)$ by the convolution product:

$$f \cdot \phi(x) = \int_G f(g) \phi(g^{-1} \cdot x) dg.$$ 

Under this action we call $\phi \in C^\infty(K\backslash X)$ a spherical function on $X$ if $\phi$ is a common $\mathcal{H}(G,K)$-eigenfunction.
Recall that in a quadratic extension we have the following maps: \( N : F^* \to E^* \), \( Tr : F \to E \), where:
\[
N(x) = x\bar{x}, \quad Tr(x) = x + \bar{x}.
\]

Let \((s_1, s_2) \in \mathbb{C}^2\), \(\chi_1, \chi_2\) characters of \(E^*/N(F^*)\), \(x \in X\). Hironaka introduced in \(H[1-4]\) the following function:
\[
L(x; \chi_1, \chi_2, s_1, s_2) = \hat{K}'\prod_{i=1}^{2} \chi_i(d_i(k \cdot x))|d_i(k \cdot x)|^{s_i} dk,
\]
where \(dk\) is the Haar measure on \(K\) normalized such that \(\int_k dk = 1\), \(d_1(y) = y_1, d_2(y) = det(y)\) and \(K' = \{ k \in K | \prod_{i=1}^{2} |d_i(k \cdot x)| \neq 0 \}\).

It is known that this integral converges absolutely for \(Re(s_1), Re(s_2) > 0\) and admits a meromorphic continuation to a rational function in \(q^{s_1}, q^{s_2}\).

We transform the variables \(s = (s_1, s_2) \in \mathbb{C}^2\) to new variables \(z = (z_1, z_2)\) by the following equations:
\[
s_1 = z_2 - z_1 - \frac{1}{2}, \quad s_2 = -z_2 + \frac{1}{2}.
\]

It is known that this function is indeed a spherical function \([H]\) and for any \(f \in \mathcal{H}(G, K)\) we have
\[
[f \cdot L(*, \chi_1, \chi_2, z)](x) = \tilde{f}(z) \times L(x, \chi_1, \chi_2, z),
\]
where \(\tilde{f}(z)\) is defined to be the Satake transform:
\[
\tilde{f}(z) = \int_{G} f(g) \Phi_{2^z}(g) dg
\]
and \(\Phi_{2^z}(g) = |a_1|^{2z_1 - \frac{1}{2}} |a_2|^{2z_2 + \frac{1}{2}}\) where \(a_1, a_2\) are determined by the Iwasawa de-composition:
\[
g = k \begin{pmatrix} a_1 & * \\ a_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \ast \\ 0 & 1 \end{pmatrix}, \quad k \in K.
\]

By abuse of notation we denote \(L(x, \chi_1, \chi_2, z) = L(x, \chi_1, \chi_2, s(z))\).

If the field extension is ramified, it is known that \(s = v_F\left[\overline{σ^{-1}(σ)} - 1\right]\) is an invariant of the field extension \(F/E\) (see \([FV]\) Ch3 p. 70 ).
Hironaka computed \( L(x, \chi_1, \chi_2, z) \) for the following cases [H5, H2]:

The case where the field extension \( F/E \) is unramified, the case where the fields extension \( F/E \) is ramified and \( |2| = 1 \) (Tamely Ramified), and the case where the field extension \( F/E \) is ramified , \( |2|_E < 1 \) and \( s = 1, 2 \) (Wildly Ramified). Hironaka also proved that the spherical functions satisfy a functional equation [H3] and used it to compute the spherical functions for general \( G = GL_n \) in Case 1.

In this paper we complete Hironaka’s calculation of Case 3 for the case of general uniformizer \( \pi \) of \( E \) and general \( s \). We compute the spherical functions defined above and provide the general functional equations. Our computation is by brute force and applies properties of the Norm and Trace maps in a quadratic extension of local fields.

Chapter 3 of this work will be dedicated to the summary of the facts we need from the local field theory. In Chapter 4 we compute useful (and interesting) p-adic integrals that we use in Chapters 7 and 8. In Chapter 5 we explicate convenient representatives of \( K \setminus X \), following Jacobowitz [J]. In Chapter 6 we state the main theorems of this work regarding the computation of the spherical functions and the functional equation. In Chapters 7 and 8 we prove the main theorem of Chapter 6.

It is our hope that the functional equations can be used in the future to calculate the spherical functions for general \( GL_n(F) \) case as was shown in [H4, O] with the Casselman-Shalika basis method [CS].

Spherical functions on the Hermitian spaces \( X \) are related to the concept of local densities [H4, H5, HS] and calculation of the spherical functions for \( GL_n \) can be used to calculate the local densities , which are important in several aspects.

Integrals of the form \( \int_{\partial F} |a + Tr(bx) + cN(x)|_x^{i} dx \) appear naturally in many places. It is hoped that their computation will have further applications.

I would like to thank my advisors Dr. Omer Offen and Ass. Prof. Moshe Baruch from the Technion for introducing me to this subject and for their endless help and support for the last 2 years.

### 3 Local Fields Properties:

Let \( E \) be a p-adic local field, and \( F/E \) a quadratic extension. We normalize the absolute value on \( F \) by \( |a| = |\#F|^{-1} \). We denote by \( v_F, v_E \) the corresponding valuation...
maps.

We recall without proofs facts and properties from local class field theory, proofs can be found in ([FV], Chapter 3).

We distinguish between the following cases of extension:

**Case 1:** The unramified case

$F / E$ is an unramified extension. We denote $|F| = q^2$, $|E| = q$. For convenience, we take $\pi = \sigma$.

**Case 2:** The tamely ramified case

$F / E$ is totally ramified and $\text{Char}(\bar{E}) \neq 2$. We denote $q = |E| = |F|$. For convenience, we take $\pi = N(\sigma)$.

**Case 3:** The wildly ramified case

$F / E$ is totally ramified and $\text{Char}(\bar{E}) = 2$. We denote $q = |E| = |F|$. For convenience, we take $\pi = N(\sigma)$.

**Theorem 3.1.** Let $F / E$ be a wildly ramified extension, then:

1. The number $s = v_F[\sigma^{-1}\sigma \pi'] - 1$ does not depend on the choice of the uniformizer $\sigma$.
2. $0 < s \leq 2v_E(2)$.
3. If $s$ is even then $s = 2v_E(2)$.

Proof of Theorem 3.1 could be found in [FV], p. 75.

Theorem 3.1 motivates us to distinguish between two types of extensions:

**Ramified Prime (RP)**

The extension $F / E$ is wildly ramified extension and the invariant $s$ is even. It can be shown [FV] that if $F / E$ is RP then $F = E(\sqrt{\pi'})$, where $\pi'$ is some uniformizer of $E$. 
**Ramified Unit (RU)**

The extension $F/E$ is wildly ramified extension and the invariant $s$ is odd. Again, it can be shown that if $F/E$ is RU then $F = E(\sqrt{1 + \delta \pi^{2k+1}})$ for some $\delta \in O_E^*$. One can take a uniformizer on $F$ to be $\sigma' = 1 + \sqrt{1 + \delta \pi^{2k+1}}$ (note that $|\sigma' \overline{\sigma'}|_E = q^{-1}$). A quick calculation shows that $s = 2v_E(2) - (2k + 1) > 0$.

**Example 3.1.** Consider the extension $F/E$ where $F = \mathbb{Q}_2(\sqrt{2})$, $E = \mathbb{Q}_2$. Since $|2|_E = |\sqrt{2} \cdot \sqrt{2}|_E = 2^{-1} = |\sqrt{2}|_E = \sqrt{2}^{-1}$, but 2 is known to be the uniformizer of $\mathbb{Q}_2$, hence this extension is ramified. (|$\sigma$|_F > |$\pi$|_F). We take $\sigma = \sqrt{2}$.

Note that $s = v_F(\frac{\sigma'(\sqrt{2})}{\sqrt{2}} - 1) = v_F(\sqrt{2} - 1) = v_F(-1) = v_F(-2) = 2$. Therefore this extension is a RP extension.

**Example 3.2.** Consider the extension $F/E$ where $F = \mathbb{Q}_2(\sqrt{-5})$, $E = \mathbb{Q}_2$.

By one of the definitions of the absolute value $| \cdot |_F$:

$$|a + \sqrt{-5}b|_F = \sqrt{|a^2 + 5b^2|_E}, \ a, b \in E.$$  

Therefore: $|1 + \sqrt{-5}|_E = \sqrt{|6|_E} = \sqrt{2}^{-1}$. So $F/E$ is ramified extension. One can take $\sigma = 1 + \sqrt{-5}$. We calculate:

$$s = v_F\left(\frac{1 - \sqrt{-5}}{1 + \sqrt{-5}} - 1\right) = v_F\left(\frac{-2\sqrt{-5}}{1 + \sqrt{-5}}\right) = v_F\left(\frac{-2}{1 + \sqrt{-5}}\right) = v_F(2) - v_F(1 + \sqrt{-5}) = 2 - 1 = 1.$$  

We conclude that $F/E$ is a ramified unit extension. Note that $F = E(\sqrt{1 + (-3) \cdot 2^1})$.

### 3.1 The Norm

For $i > 0$, we denote by $\lambda_{i,F}$ (resp. $\lambda_{i,E}$) the natural map $\lambda_{i,F} : (1 + \pi^i O_F) \to \overline{F}$

$$\lambda_{i,F}(x) = \sigma^{-i}(x - 1) mod \sigma O_F$$

and

$$\lambda_{i,E}(x) = \pi^{-i}(x - 1) mod \pi O_F.$$  

Denote the residue map by $\lambda_{0,K} : O_K \to \overline{K}$, ($K = E, F$).
We summarize the properties of the norm by the following diagrams:

**Case 1-Unramified**:

The following diagrams commute:

\[
\begin{array}{ccc}
F^+ & \overset{\nu_F}{\longrightarrow} & Z \\
\downarrow N_{F/E} & & \downarrow \times 2 \\
E^+ & \overset{\nu_E}{\longrightarrow} & Z \\
\end{array}
\]

\[
\begin{array}{ccc}
O_F & \overset{\lambda_{0,F}}{\longrightarrow} & \tilde{F} \\
\downarrow N_{F/E} & & \downarrow N_{\tau/\tau} \\
O_E & \overset{\lambda_{0,E}}{\longrightarrow} & \tilde{E} \\
\end{array}
\]  \hspace{1cm} (3.1)

\[
1 + \sigma_i O_F \overset{\lambda_i}{\longrightarrow} \tilde{F}, \quad i \geq 1 \\
\downarrow N_{F/E} & & \downarrow \text{Tr}_{\tau/\tau} \\
1 + \pi_i O_E \overset{\lambda_i}{\longrightarrow} \tilde{E} \\
\]

**Case 2- Tamely ramified**

\[
\begin{array}{ccc}
E^* = F^+ & \overset{\nu_F}{\longrightarrow} & Z \\
\downarrow N_{F/E} & & \downarrow \text{id} \\
E^* & \overset{\nu_E}{\longrightarrow} & Z \\
\end{array}
\]

\[
\begin{array}{ccc}
O_F & \overset{\tilde{\lambda}_{0,F}}{\longrightarrow} & \tilde{E} = \tilde{F} \\
\downarrow N_{F/E} & & \downarrow x \mapsto x^2 \\
O_E & \overset{\tilde{\lambda}_{0,E}}{\longrightarrow} & \tilde{E} \\
\end{array}
\]

\[
1 + \sigma_2 O_F \overset{\lambda_{2,F}}{\longrightarrow} \tilde{E} = \tilde{F} \\
\downarrow N_{F/E} & & \downarrow \times 2 \\
1 + \pi_1 O_E \overset{\lambda_i}{\longrightarrow} \tilde{E} \\
\]

And \( N(1 + \sigma^{2i-1} O_F) = N(1 + \sigma^{2i} O_F) \).
Case 3- wildly ramified

Let $\eta \in O_F^*$ to be such that $\frac{\eta}{\mathfrak{m}} = 1 + \eta \varpi^s$.
Denote $\kappa = \lambda_{0,F}(\eta)$.

The following diagrams commute:

\[
\begin{array}{ccc}
N_F/E & \xrightarrow{\lambda_{0,F}} & E = F \\
N_F/E & \xrightarrow{\lambda_{0,E}} & E \\
O_F & \xrightarrow{\lambda_{0,F}} & E = F \\
O_E & \xrightarrow{\lambda_{0,E}} & E
\end{array}
\]

(3.2)

For $i < s$:

\[
\begin{array}{ccc}
1 + \mathfrak{m}^i O_F & \xrightarrow{\lambda_{i,F}} & F = \bar{F} \\
1 + \mathfrak{m}^i O_E & \xrightarrow{\lambda_{i,E}} & \bar{E}
\end{array}
\]

(3.3)

\[
\begin{array}{ccc}
1 + \mathfrak{m}^i O_F & \xrightarrow{\lambda_{i,F}} & F = \bar{F} \\
1 + \mathfrak{m}^i O_E & \xrightarrow{\lambda_{i,E}} & \bar{E}
\end{array}
\]

(3.4)

(Note that the homomorphism $x \mapsto x^2 - \kappa \cdot x$ is an additive homomorphism with kernel of size 2)
For $j > 0$

$$1 + \varpi^{s+2j}O_F \xrightarrow{\lambda_{s+2j,F}} \bar{E} = \mathcal{F}$$  \hspace{1cm} (3.5)

and $N(1 + \varpi^{s+i}O_F) = N(1 + \varpi^{s+i+1}O_F)$ if $i > 0$ and $2 \nmid i$.

Proofs for commutativity could be found in [FV] p. 68-73.

We will use throughout this paper the following corollaries (Case 3):

**Corollary 3.1.** $N(1 + \varpi^{s+i}O_F) = 1 + \pi^{s+i}O_E$.

**Proof.** From (3.5) we conclude that since the map $x \mapsto x \times \kappa$ is a surjective homomorphism between the additive groups $\mathcal{F} \to \bar{E}$ and form the commutativity of diagram (3.5) that for $i > s$, every ball of radius $q^{-i}$ in $1 + \pi^{s+i}O_E$ contains an element that is a norm, thus $N(1 + \varpi^{s+i}O_F)$ is dense in $1 + \pi^{s+i}O_E$. From compactness of $1 + \varpi^{s+i}O_F$ we have $N(1 + \varpi^{s+i}O_F) = 1 + \pi^{s+i}O_E$.

**Corollary 3.2.** $[1 + \pi^sO_E : N(1 + \varpi^{s}O_F)] = 2$.

**Proof.** Follows from commutativity of (3.4) and the last corollary.

**Corollary 3.3.** Let $x, y \in \pi^iO_E^*$. If $|x - y|_E < q^{-s-i}$ then $x \in N(F) \iff y \in N(F)$.

**Proof.** Since $\pi$ is a norm we can assume w.l.o.g that $i = 0$. We have from the last corollary and the commutativity of (3.1) that $N(O_F)$ is a union of cosets of $1 + \pi^{s+i}O_E$, thus if $|x - y|_E < q^{-s}$ then they are in the same coset and we have $x \in N(O_F) \iff y \in N(O_F)$.

**Example.** Consider $F/E$ from example 3.1. Note that $\mathcal{F} = \bar{E} = F_2$ (The field with two elements). Note that $s = 2$, and $\kappa = 1$. We have:

$$1 + 2O_F \xrightarrow{\lambda_{2,F}} F_2$$

$$\xrightarrow{N_{F/E}} \xrightarrow{\lambda_{2,F}} F_2$$

Since the image of the map: $\psi : F_2 \to F_2$, $\psi(x) \mapsto x^2 - x$ is $\{0\}$ we conclude from the commutativity of the diagram that:

$$N(1 + 2O_F) = 1 + 8Z_2.$$
3.2 The Trace

In the wildly ramified case (Case 3) the following holds:

\[ TR_{F/E}(\mathfrak{a}^iO_F) = \pi^{j(i)}O_E, \quad j(i) = s + 1 + [(i - 1 - s)/2]. \]

Proof could be found in [FV], p. 71.

Explicitly,

**RP:**
\[ s = 2l, \ l = v_E(2) \]
\[ TR(\mathfrak{a}^{2i}O_F) = \pi^{i+1}O_E \]
\[ TR(\mathfrak{a}^{2i-1}O_F) = \pi^{i+1}O_E \]

**RU:**
\[ s = 2l + 1 \]
\[ TR(\mathfrak{a}^{2i}O_F) = \pi^{i+1+1}O_E \]
\[ TR(\mathfrak{a}^{2i-1}O_F) = \pi^{i+1}O_E \]

4 Useful Integrals

**Lemma 4.1.** Let \( H \) and \( G \) be compact abelian topological groups with Haar measures \( \mu_H, \mu_G \) and \( \phi : H \rightarrow G \) a surjective (continuous) homomorphism, then the push-forward measure \( \mu_H^\ast = \mu_H \circ \phi^{-1} \) is an invariant Haar measure on \( G \) and \( \mu_H^\ast = \frac{\mu_H(H)}{\mu_G(G)} \mu_G. \)

**Proof.** Because the characters span a dense subset in \( L^1(G, \mu_G) \) it is sufficient to show that \( \int_G \chi d\mu_H^\ast = 0 \) for any non-trivial character \( \chi \) . But \( \int_G \chi d\mu_H^\ast = \int_H \chi(\phi(x))d\mu_H(x) = 0, \) since \( \chi(\phi(x)) \) is a non trivial character of \( H \) . We obtain the multiplicative factor between the measures from integrating over the trivial character.

**Remark 4.1.** The conditions of the previous lemma can be further generalized.

Take \( \mu_E, \mu_F \) to be the Haar measures on \( E, F \) normalized by

\[ \mu_E(O_E) = \mu_F(O_F) = 1. \]

**Lemma 4.2.** For \( n \geq 1 \)

\[ \mu_F[N^{-1}(1 + \pi^nO_F^e)] = \mu_F(N^{-1}[(1 + \pi^nO_E) - (1 + \pi^{n+1}O_E)]) = \]
Case 1. \[ \frac{q^2 - 1}{q^2 - 1} \]

Case 2. \[ \frac{2(q - 1)}{q^2 - 1} \]

Case 3. \[ \begin{cases} (q - 1)q^{-(n+1)} & \text{if } n < s \\ (q - 2)q^{-(s+1)} & \text{if } n = s \\ 2(q - 1)q^{-(n+1)} & \text{if } n > s \end{cases} \]

Proof. In Case 1, since \( N(O_F^*) = O_E^* \) we conclude that the push-forward of the multiplicative Haar measure of \( O_F^* \) is an multiplicative Haar measure of \( O_E^* \). In \( O_F^* \) the additive and multiplicative measures coincide, so we get that

\[
\mu_{F^*} = \frac{\mu_F(O_F^*)}{\mu(E)} = \frac{1 - q^{-2}}{1 - q^{-1}} \mu_E = (1 + q^{-1}) \mu_E.
\]

Since \( \mu_E(1 + p^n O_E^*) = q^n(1 - q^{-1}) \), we get that:

\[
\mu_{F^*}(1 + \pi^n O_E^*) = (1 + q^{-1}) \cdot q^n(1 - q^{-1}) = \frac{q^2 - 1}{q^{n+2}}.
\]

In Case 2, we have \( N(1 + O^*) = 1 + \pi O_E \). Since this case is ramified, we have that \( \mu_F(O_F^*) = \mu_E(O_E^*) \). So \( \mu_F(N^{-1}[1 + \sigma O_F](1 + \pi^n O_E^*)) = q^n(1 - q^{-1}) \) (\( N^{-1}[1 + \sigma O_F] \) is the restriction of the norm to \( 1 + \sigma O_F \)). But since the norm induces the square map between the groups

\[
\tilde{N} : O_F^* \to O_E^*.
\]

The size of the kernel of this map is 2. We have that:

\[
N^{-1}(1 + \pi^n O_E^*) = a \cdot N^{-1}[1 + \sigma O_F](1 + \pi^n O_E^*) \cup N^{-1}[1 + \sigma O_F](1 + \pi^n O_E^*),
\]

where \( a \) is a non trivial representative of the kernel of \( \tilde{N} \). So:

\[
\mu_{F^*}(1 + \pi^n O_E^*) = 2\mu_F(N^{-1}[1 + \sigma O_F](1 + \pi^n O_E^*)) = \frac{2(q - 1)}{q^{n+1}}.
\]

In Case 3 we have \( N(1 + \sigma^{s+1} O_F) = 1 + \pi^{s+1} O_E \). So if we restrict the norm to \( 1 + \sigma^{s+1} O_F \) we have \( \mu_* = \mu_E \).

From the diagrams of Section 3.1 that the norm map induces an homomorphism of the finite groups:

\[
\tilde{N} : \frac{1 + \sigma O_F}{1 + \sigma^{s+1} O_F} \to \frac{1 + \pi O_E}{1 + \pi^{s+1} O_E}.
\]

The size of the kernel of this map is 2.
We have that for \( n > s \):

\[
N^{-1}(1 + \pi^n O_E^i) = a|N|_{1+\varpi^{s+1}O_F}^{-1}(1 + \pi^n O_E^i)\bigcup N|_{1+\varpi^{s+1}O_F}^{-1}(1 + \pi^n O_E^i),
\]

\((N|_{1+\varpi^{s+1}O_F}^{-1}\) is the restriction of the norm to \( 1 + \varpi^{s+1}O_F ), \) where \( a \) is a representative element of the kernel of \( \tilde{N} \). But since \( N|_{1+\varpi^{s+1}O_F}^{-1}\) is surjective on \( 1 + \pi^{s+1}O_E \) we can use Lemma \( 4.1 \) to obtain:

\[
\mu_F[|N^{-1}(1 + \pi^n O_E^i)] = 2\mu_E(1 + \pi^n O_E^i) = 2(1 - q^{-1})q^{-n}.
\]

For \( n = s \), we can conclude from \( (3.5) \) that \( N^{-1}(1 + \pi^n O_E^i) = \cup_{i=1}^{q-2} a_i(1 + \varpi^{s+1}O_F), \) where the \( a_i \) are representatives of the cosets that are the preimages of \( \tilde{N} \). So \( \mu_F(N^{-1}(1 + \pi^n O_E^i)) = (q - 2)q^{-(s+1)} \).

For \( n < s \) the norm induces an isomorphism between the finite groups \( \frac{O_F^i}{1+\varpi^{s+1}O_F} \) and \( \frac{\pi^n O_E^i}{1+\pi^{s+1}O_E}, \) therefore \( \mu_F(N^{-1}(1 + \pi^n O_E^i)) = \mu_E(1 + \pi^n O_E^i) = (q - 1)q^{-(n+1)} \), altogether we obtain the result above.

\( \square \)

**Corollary 4.1.** (case 1)

\[
\int_{x \in O_F} |1 + N(x)|^{s_1} dx = \frac{q^2 - q - 1}{q^2} + \frac{q^2 - 1}{q^2} \left( \frac{q^{-s_1-1}}{1 - q^{-s_1-1}} \right) \quad (4.1)
\]

**Proof.** Note that \(-1\) is a norm, hence \(-1 = \zeta \cdot \overline{\zeta}\) for some \( \zeta \in O_F^i \). We substitute \( x = \zeta y \):

\[
\int_{x \in O_F} |1 + N(x)|^{s_1} dx = \int_{y \in O_F} |1 - N(y)|^{s_1} dy = \int_{x \in O_F} |(-1) + N(y)|^{s_1} dy.
\]

The integrand is equal to 1 for values of \( y \) such that \( N(y) \neq -1 \mod \pi O_E \). From \( (3.1) \) \( q + 1 \) (the size of the kernel of \( N|_{\varpi F} \) cosets of \( O_E^i \)) are map to the coset of \(-1\). It is easy to see that the integrand is equal to 1 in the set \( \varpi O_F \). Therefore:

\[
\mu_F(-1 + O_E^i) = \frac{(q^2 - 1) - q - 1}{q^2} \cdot \frac{1}{\mu_F(\varpi O_F)} = \frac{q^2 - q - 1}{q^2}.
\]
It follows immediately from Lemma 4.2 that the integral is equal to the following geometric sum:

$$\int_{x \in O_F} |1 + N(x)|^s dx = \frac{q^2 - q - 1}{q^2} + \sum_{1 \leq n} \frac{q^2 - q - 1}{q^2} q^{-ns_1 - n} = \frac{q^2 - q - 1}{q^2} \left( \frac{q^{-s_1 - 1}}{1 - q^{-s_1 - 1}} \right)$$

For the next lemmas, let $\chi^*$ be a non-trivial character of $E^*$ that is trivial on $N(F)$. We will define $\chi^*(0) = 0$.

**Lemma 4.3. (Case 3) Let $0 < m < s + 1, \theta \in O_E^+, \text{then } \int_{x \in O_F} \chi^*(1 + \theta \pi^m N(x)) dx = 0**

**Proof.** First, we show that the integrand is constant on the (additive) cosets of group $\frac{O_F}{\theta^s + 1 - m O_F}$:

If $x \in \theta^{s+1-m} O_F$ then $\chi^*(1 + \theta \pi^m N(x)) = 1$ by corollary 3.1.

Let $x, y \in O_F$. Assume that $x = yu$, where $u \in 1 + \theta^{s+1-m} O_F$ (that is, they are in the same coset of $\frac{O_F}{\theta^{s+1-m} O_F}$). We know (see Section 3.1) that $N(u) \in 1 + \pi^{s+1-m} O_F$. So $N(u) = 1 + \pi^{s+1-m} v, v \in O_F$. Then:

$$1 + \theta \pi^m N(x) = 1 + \theta \pi^m N(y) N(u) = 1 + \theta \pi^m N(y) + \delta, \quad \delta = \pi^{s+1} \theta v.$$

Recall from Section 3.1, that for $g_1, g_2 \in O_E$ if $|g_1 - g_2| < q^{-s}$, then $g_1 \in N(F)$ if and only if $g_2 \in N(F)$, since $|\delta| < q^{-s}$ we conclude that:

$$\chi^*(1 + \theta \pi^m N(x)) = \chi^*(1 + \theta \pi^m N(y)).$$

We conclude that the integrand is constant on the cosets of $\frac{O_F}{\theta^{s+1-m} O_F}$.

Since $s + 1 - m < s + 1$, we conclude from the (3.1), (3.3) that the norm map defines a bijection between the groups:

$$\tilde{N}: H = \frac{O_F}{\theta^{s+1-m} O_F} \to \frac{O_E}{\pi^{s+1-m} O_E} = H'.$$

For $x \in O_F$, we denote by $[x]_H$ the coset of $x$ in $H$ (and similarly for the rest of the group in this proof).

Since the integrand is well defined over $H$, integrating over $O_F$ is equivalent to summing over $H$:

$$\int_{x \in O_F} \chi^*(1 + \theta \pi^m N(x)) dx = \mu_F(\theta^{s+1-m} O_F) \sum_{[h]_H \in H} \chi^*(1 + \theta \pi^m N(h))$$
We use the bijection $\tilde{N}$ to calculate the sum over $H'$:

$$
\sum_{[h]_H \in H'} \chi^*(1 + \theta \pi^m h') = \sum_{[h']_H' \in H'} \chi^*(1 + \theta \pi^m h')
$$

The group $\frac{O_F}{\pi^{s-1-m}O_F}$ is isomorphic to the multiplicative group $U = \frac{1 + \pi^m O_E}{1 + \pi^m O_F}$ by the map $\tilde{\psi} : ([x]_{H'}) \mapsto [1 + \pi^m \theta x]_U$, so we get another bijection:

$$
\tilde{\psi} : \frac{O_F}{\pi^{s+1-m}O_F} \to \frac{1 + \pi^m O_F}{1 + \pi^m O_F},
$$

The previous bijection $\tilde{\psi}$ tells us that summing over $H'$ is equivalent to summing over the multiplicative group $U$ by $u = \tilde{\psi}(h)$, so

$$
\sum_{[h']_H' \in H'} \chi^*(1 + \theta \pi^m h') = \sum_{[u]_U \in U} \chi^*(u).
$$

This sum surely vanishes, since it is a summation of a non trivial character of $U$.  

**Lemma 4.4.** (Case 3) Let $0 < m < s$, $\theta \in O_F^*$ then

$$
\int_{x \in O_F} \chi^*(1 + \theta \pi^m N(x)) dx = 0
$$

Proof. From Lemma 4.3

$$
\int_{x \in \mathcal{O}_F} \chi^*(1 + \theta \pi^m N(x)) dx = 0
$$

It is enough to prove that:

$$
\int_{x \in \mathcal{O}_F} \chi^*(1 + \theta \pi^m N(x)) dx = 0.
$$

By substituting $x = \sigma y$, we get:

$$
\int_{x \in \mathcal{O}_F} \chi^*(1 + \theta \pi^m N(x)) dx = q^{-1} \int_{y \in O_F} \chi^*(1 + \theta \pi^{m+1} N(y)) dy.
$$

By Lemma 4.3 we conclude that the last integral vanishes.

**Lemma 4.5.** (Case 3) Let $F/E$ a ramified extension of case 3 and $\eta \in O_F^*$ then if $\text{Re}(s_1) > -\frac{1}{2}$, and $s_1 = -z_1 + z_2 - \frac{1}{2}$ we have

$$
\int_{x \in O_F} |\eta + N(x)|^{s_1} dx = \frac{q^{2z_2} - q^{2z_1 - 1} + \chi^*(-\eta) \cdot (q^{(2z_1 - 2z_2)} + q^{2z_1 - q^{(2z_1 - 2z_2)}} + q^{(2z_1 - 2z_2)} + q^{2z_1})}{q^{2z_2} - q^{2z_1}}
$$

(4.2)
Proof. Suppose \( -\eta \notin N(F) \). Let \( -\nu \in 1 + \pi^i O_E \) be an element that is not a norm, note that \( -\nu = (\mu \cdot e) \in N(F^+) \), that is \( e = \alpha \sigma \) for some \( \alpha \in O_E \). By making coordinate transformation \( \alpha y = x \), we can assume w.l.o.g that \( -\eta \in 1 + \pi^i O_E \).

In the coset \( \sigma O_F \), it is easy to see that the integrand is equal to 1.

The norm map induces an isomorphism \( \tilde{N}_1 : \mathcal{O}_F / \mathcal{O}_F \to \mathcal{O}_F / \mathcal{O}_F \). (The square map \( x \mapsto x^2 \) of the finite group \( F^2 = E^2 \) ) and so \( q - 2 \) cosets of \( \mathcal{O}_F / \mathcal{O}_F \) do not map by \( \tilde{N}_1 \) to the coset \( 1 + \pi^i O_E \) (the coset of \( -\eta \) ) and so \( N(x) - (\eta) \in O_E \) and the integrand on those cosets is equal to 1.

So far we have:
\[
\int_{x \in O_F} |\eta + N(x)|^{s_1} dx = q^{-1} + \int_{x \in O_F} |\eta + N(x)|^{s_1} dx = q^{-1} + \sum_{i=1}^{s_1} \frac{q^{-2} - 2}{q} \int_{x \in 1 + \sigma^i O_F} |\eta + N(x)|^{s_1} dx.
\]

Then for any \( 1 \leq i < s \) we have that the norm map induce a isomorphism
\[
\tilde{N}_2 : \frac{1 + \sigma^i O_F}{1 + \sigma^{i+1} O_F} \to \frac{1 + \pi^i O_E}{1 + \pi^{i+1} O_E}
\]
(The square map of the additive groups \( \mathcal{F} = E \). So, for each \( i \), \( q - 1 \) cosets of \( \frac{1 + \sigma^i O_F}{1 + \sigma^{i+1} O_F} \) do not map to the coset \( 1 + \pi^i O_E \) (the coset of \( -\eta \) ) and the value of the integrand is \( q^{-2s_1} \). We have:
\[
\int_{x \in O_F} |\eta + N(x)|^{s_1} dx = q^{-1} + \frac{q - 2}{q} \sum_{i=1}^{s_1} q^{-2s_1} (q - 1) q^{-i-1} + \int_{x \in 1 + \sigma^O F} |(\eta) - N(x)|^{s_1} dx.
\]

Since \( -\eta \notin N(1 + \sigma^O F) \) and \( [(-\eta) + \pi^O E] \cap N(1 + \sigma^O F) = \phi \), we have that in \( 1 + \sigma^O F \) the integrand is equal to \( q^{-2s_1} \).

Altogether:
\[
\int_{x \in O_F} |\eta + N(x)|^{s_1} dx = (1 - q^{-1}) \sum_{j=0}^{s_1} q^{-2s_1-1} + q^{2s_1-1} = \frac{q^{2s_2} - q^{2s_2-1} + q^{2(s_2-1)}(q^{2s_2-2} - 1 - q^{2(s_2-2) + 2s_1})}{q^{2s_2} - q^{2s_1}}
\]

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Suppose $-\eta \in N(F)$. Then $-\eta = N(\nu)$, $\nu \in O_F^*$ by the substitution $\nu y = x$ and Lemma (4.2):

$$
\int_{x \in O_F} |\eta + N(x)|^{-z_1+z_2-\frac{1}{2}} \, dx = \int_{y \in O_F} |-1 + N(y)|^{-z_1+z_2-\frac{1}{2}} \, dy =
$$

$$(1 - q^{-1}) \sum_{j=0}^{s-1} q^{(2z_1-2z_2)j} + 2 \sum_{j=s+1}^{\infty} (1 - q^{-1})q^{(2z_1-2z_2)j} + q^s(2z_1-2z_2)
$$

$$
= \frac{q^{2z_2} - q^{2z_1-1} + q^{s(2z_1-2z_2)} + q^s(2z_1-2z_2) + 2z_2 - 1}{q^{2z_2} - q^{2z_1}}.
$$

□

**Representative of the K-Orbits (Ramified dyadic)**

**5 Classification of K-Orbits**

We summarize results obtained by Jacobowitz [J] that classified Hermitian lattices over local fields $(O_F$-module equipped with a Hermitian product $\langle \cdot, \cdot \rangle_L$. An Hermitian lattice with basis $\{x_\lambda\}$ can be represented by an Hermitian matrix: $L_{\lambda, \mu} = \langle x_\lambda, x_\mu \rangle_L$ and equivalence classes of lattices correspond to orbits of Hermitian matrices under the action of $K$ given by $k \cdot x = kxk^*$, $k \in K$, $x \in X$.

The following definitions are also presented on [J]. We translate them in term of matrices:

**Definition 5.1.** An hermitian matrix is called $\mathcal{O}^*$ modular if for every primitive vector $x \in M_n(1)$ (that is, a vector $x = (x_i)$, $\exists i_0, 1 \leq i_0 \leq n$ s.t. $x_{i_0} \in O_F^*$) there is a vector $w \in M_{1,n}$ such that $w^*Ax = \mathcal{O}^*$.

**Definition 5.2.** The norm ideal: $nL$ The ideal of $O_F$ generated by elements $\langle v, v \rangle_L, v \in M_{1,n}(O_F)$

**Definition 5.3.** The scalar ideal $sL$ The Ideal generated by $\langle v, w \rangle_L, v, w \in M_{1,n}(O_F)$

**Definition 5.4.** $dL = det(L) \mod N(F)) \in E^*/N(F)$. 

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Definition 5.5. $\varpi^i$-modular hyperbolic matrix: $H(i) \approx \begin{pmatrix} 0 & \varpi^i \\ \overline{\varpi} & 0 \end{pmatrix}$.

The norm ideal, the scalar ideal, $dL$ and $\varpi^i-$modularity are all invariants of the lattice under the action of $K$.

Jacobowitz investigated the ramified non-dyadic case relevant for our problem.

It was shown:

1. Every Hermitian matrix is equivalent to the direct sum of $\varpi^i$ modular $2 \times 2$ and $1 \times 1$ Hermitian matrices.

2. In the case of RP: $n \cdot H(i) = \varpi^{s+2}O_F$
   - In the case of RU: $n \cdot H(i) = \varpi^{s-1+2}O_F$

3. If $L$ is $\varpi^i$ modular then $n \cdot H(i) \subseteq n \cdot L$

We conclude the following representatives of $K \setminus X$ in the case $n \cdot H(i) = n \cdot L$:

**RP**

1-modular matrices:
1. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, 2. $\begin{pmatrix} \pi^2 & 1 \\ 1 & -\pi^2 \rho \end{pmatrix}$

$\varpi$-modular matrices:
3. $\begin{pmatrix} 0 & \varpi \\ \overline{\varpi} & 0 \end{pmatrix}$

**RU**

1-modular matrices:
1. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\varpi$-modular matrices:
2. $\begin{pmatrix} \varpi^{s+1} & \varpi \\ \bar{\varpi} & \varpi^{s+1} \rho \end{pmatrix}$, 3. $\begin{pmatrix} 0 & \varpi \\ \overline{\varpi} & 0 \end{pmatrix}$

Now, suppose $n \cdot H(i) \subset n \cdot L = \varpi^{2m}O_F$:

The other $\varpi^i-$modular planes: (see [J] p. 459)

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RP

1-modular matrices:
0 < 2m < s
4. \( \begin{pmatrix} \pi^m & 1 \\ 1 & 0 \end{pmatrix} \)
5. \( \begin{pmatrix} \pi^m & 1 \\ 1 & \pi^{s-m} \rho \end{pmatrix} \)

σ-modular matrices:
0 < 2m < s + 2
6. \( \begin{pmatrix} \pi^m & \sigma \\ \sigma & 0 \end{pmatrix} \)
7. \( \begin{pmatrix} \pi^m & \sigma \\ \sigma & \pi^{s-m+1} \rho \end{pmatrix} \)

RU

1-modular matrices:
0 < 2m < s − 1
4. \( \begin{pmatrix} \pi^m & 1 \\ 1 & 0 \end{pmatrix} \)
5. \( \begin{pmatrix} \pi^m & 1 \\ 1 & \pi^{s-m} \rho \end{pmatrix} \)

σ-modular matrices:
0 < 2m < s + 1
6. \( \begin{pmatrix} \pi^m & \sigma \\ \sigma & 0 \end{pmatrix} \)
7. \( \begin{pmatrix} \pi^m & \sigma \\ \sigma & \pi^{s-m+1} \rho \end{pmatrix} \)

We now deal with the case when the lattice is a sum of two \( \pi^i \)-modular matrices:

We have the following representatives:
(Both RP and RU)
\( \begin{pmatrix} \pi^{\lambda_1} \varepsilon_1 \\ \pi^{\lambda_2} \varepsilon_2 \end{pmatrix} \), \( \lambda_1 \geq \lambda_2 \), \( \varepsilon_i \in \{1, \Delta\} \), \( \varepsilon_1 = 1 \) if \( \lambda_1 - \lambda_2 \leq s \)
(see [J] p. 463)

The representative we presented (beside the diagonals) are the 1 and σ modular representatives, up to multiplying by the scalar \( \pi^a \) \( (a \in \mathbb{Z}) \), this are all the representatives of \( K \backslash X \).

We denote by \( [K \backslash X] \) to be the set of representatives that were have presented.
6 The spherical functions for Case 3:

\[ L(x, \chi_1, \chi_2, s_1, s_2) = \int_{K'} \hat{\chi}_1(d_1(k \cdot x)) \cdot |d_1(k \cdot x)|^{s_1} \hat{\chi}_2(d_2(k \cdot x)) \cdot |d_2(k \cdot x)|^{s_2} \, dk \]

Substitute:

\[ s_1 = -z_1 + z_2 - \frac{1}{2} \]
\[ s_2 = -z_2 + \frac{1}{2} \]

We denote: \( z = (z_1, z_2) \).

For any \( a_1, a_2, z_1, z_2 \in \mathbb{C} \)

\[ \langle (a_1, a_2), (b_1, b_2) \rangle = \langle (a_1, a_2), z \rangle = a_1 z_1 + a_2 z_2 \]

By abuse of notation we will some times denote:

\[ L(x, \chi_1, \chi_2, s_1, s_2) = L(x, \chi_1, \chi_2, \sigma(z_1), \sigma(z_2)) \]

From now on we will denote by \( \sigma \in \{ \sigma_1, \sigma_2 \} = \Sigma_2 \subset \text{Aut}(\mathbb{C}(q^{z_1}, q^{z_2})) \) where:

\[ \sigma_1 = \text{Id} \]
\[ \sigma_2(q^{z_1}) = q^{z_2}, \quad \sigma_2(q^{z_2}) = q^{z_1} \]

Theorem 6.1. Let \( x \in [K \backslash X] \) and

\[ L(x, \chi_1, \chi_2, s_1, s_2) = \int_{K'} \prod_{i=1}^{2} |d_i(\chi_i(k \cdot x))| d_i(k \cdot x)^{s_i} \, dk \]

as was defined in Section 2 then \( L(x, \chi_1, \chi_2, z_1, z_2) \) is equal to the following:

RP:

1. \( L(x, \chi_1, \chi_2, z_1, z_2) = 0 \) unless \( x = \begin{pmatrix} \pi^{\lambda_1} e_1 \\ \pi^{\lambda_2} e_2 \end{pmatrix} \) with \( \lambda_1 - \lambda_2 > s \)

2. If \( x = \begin{pmatrix} \pi^{\lambda_1} e_1 \\ \pi^{\lambda_2} e_2 \end{pmatrix} \) with \( \lambda_1 - \lambda_2 > s \) then: (Hironaka)

\[ L(x, \chi_1, \chi_2, z) = \frac{q^{\frac{1}{2} \cdot \lambda_1 \cdot \lambda_2}}{1 + q^{-1}} \chi'(e_2) \chi(e_1 e_2) q^{2 z_2 z_2} \left( q^{2 z_2} - q^{2 z_1 - 1} \right) \times \sum_{\sigma \in \Sigma_2} \sigma \left( \frac{q^{\lambda_1 - s \lambda_2 z_2}}{q^{z_2} - q^{z_1 - s \lambda_2 z_2}} \right) \]
3. $L\left(\begin{array}{cc}
\pi^{\lambda_1}E_1 \\
\pi^{\lambda_2}E_2
\end{array}\right), 1, \chi_2, z = \frac{\frac{\lambda_2}{\pi^{\lambda_1}} \lambda_1 (E_1 E_2) q^{-2s_2}}{(q^{m+1})(q^{m+2} - q^{m+1})} \times \left[\chi'(- E_1 E_2) q^{(\lambda_1 + s, \lambda_2, 2s_2)} (q^{2s_2} - q^{2s_2-1}) + q^{((\lambda_2, \lambda_1 + s), 2s_2)} (q^{2s_2} - q^{2s_2-1})\right]$

4. $L\left(\begin{array}{cc}
1 & 1 \\
\sigma & \sigma
\end{array}\right), 1, \chi_2, z = \frac{\chi_2(-1)(1 - q^{-1})q^{-2s_2} q^{s_2(z_1+z_2)}}{q^{m+1} q^{m+2} - q^{m+1}}$

5. $L\left(\begin{array}{cc}
\pi^z & 1 \\
0 & -\pi^z \rho
\end{array}\right), 1, \chi_2, z = \chi_2(-\Delta) q^{-2s_2 + \Delta} q^{s_2(z_1+z_2)}$

6. $L\left(\begin{array}{cc}
\pi^z & 1 \\
1 & -\pi^z \rho
\end{array}\right), 1, \chi_2, z = \chi_2(-\Delta) q^{-2s_2 + \Delta} q^{s_2(z_1+z_2)}$

7. $0 < m < \frac{q}{2}$

$= \frac{\chi_2(-1) q^{m-2s_2}(1 - q^{-1})}{q^{m+1} q^{m+2} - q^{m+1}} \sum_{\sigma} (q^{(m, s, m+1, 2s_2)})$

8. $0 < m < \frac{q}{2} + 1$

$= \frac{\chi_2(-1) q^{m-2s_2}}{q^{m+1} q^{m+2} - q^{m+1}} \sum_{\sigma} (q^{(m, 2s + s, m + 1, 2s_2)} - q^{(s + 1 - m, m + 1, 2s_2) - 1})$

9. $0 < m < \frac{q}{2}$

$= \frac{\chi_2(-\Delta) q^{m-2s_2}}{q^{m+1} q^{m+2} - q^{m+1}} \sum_{\sigma} (q^{(m, 1 - m, s, 2s_2) - 1})$
10. \(0 < m < \frac{3}{2} + 1\)

\[
L\left(\begin{pmatrix} \pi^m & \sigma \\ \sigma & -\pi^s \rho \end{pmatrix} \right), 1, \chi_2, z) = \\
\frac{\chi_2(-\Delta)q^{m+\frac{3}{2}}}{q^{s+1}} \sum_{\sigma} (q^{(m+1)(s+1)+z})
\]

RU

1, 2, 3 are the same as the RP case.

4. \(L\left(\begin{pmatrix} 1 \ 1 \\ 1 \end{pmatrix} \right), 1, \chi_2, z) = q \chi_2(-1)(1 - q^{-1})q^{-2s+2} [q^{(s+1)(z+1)}]
\]

5. \(L\left(\begin{pmatrix} \sigma \\ \sigma \end{pmatrix} \right), 1, \chi_2, z) = \frac{\chi_2(-1)(1 - q^{-1})^{s+\frac{3}{2}}}{1 + q^{-1}} q^{(s+1)(z+1)} [q^{2s+2z+2}]
\]

6. \(L(x, \begin{pmatrix} \pi^{m+1} & \sigma \\ \sigma & -\pi^{s-m} \rho \end{pmatrix} \right), \chi_2, z) = \chi_2(-\Delta)q^{-2s+2z+\frac{3}{2}} [q^{(s+1)(z_1+z_2)}]
\]

For the following representatives:

7. \(\begin{pmatrix} \pi^m & 1 \\ 1 & 0 \end{pmatrix}\), 8. \(\begin{pmatrix} \pi^m & \sigma \\ \sigma & 0 \end{pmatrix}\), 9. \(\begin{pmatrix} \pi^m & 1 \\ 1 & -\pi^{s-m} \rho \end{pmatrix}\), 10. \(\begin{pmatrix} \pi^m & \sigma \\ \sigma & -\pi^{s-m} \rho \end{pmatrix}\)

the result is the same as RP Case.

And any \(a \in E^*\),

\[
L(ax, \chi_1, s_1, s_2) = |a|^{s_1+2s_2} \chi_1(a) \cdot \chi_2(a)^2 L(x, \chi_1, \chi_2, s_1, s_2) \quad (6.1)
\]

So one can calculate the spherical function for any \(K\) orbit in \(X\).

The proof of the theorem will be given in chapters 7 and 8.

6.1 Functional Equations

A corollary that follows from Theorem (6.1) is the following functional equations:
Define \( L(x, \chi_1, \chi_2, z) = q^{2z_2}L(x, \chi_1, \chi_2, z) \)

Then:

\[
L(x, 1, \chi_2, z_2, z_1) = -\chi^*(-1)L(x, 1, \chi^* \chi_2, z_1, z_2)
\]

\[
L(x, \chi^*, \chi_2, z_2, z_1) = q^{4\Delta(z_1 - z_2)} \frac{q^{z_1} - q^{z_2}}{q^{z_2} - q^{z_1}} L(x, \chi^*, \chi_2, z_1, z_2)
\]

And actually if we define:

\[
\chi_1 = \omega_1^{-1} \omega_2 \\
\chi_2 = \omega_2^{-1}
\]

We have the following functional equations:

If \( \omega_1 \omega_2^{-1} = 1 \) then:

\[
L(x, \omega_2, \omega_1, z_2, z_1) = -\chi^*(-1)L(x, \chi^* \omega_1, \chi^* \omega_2, z_1, z_2)
\]

If \( \omega_1 \omega_2^{-1} \neq 1 \) then:

\[
L(x, \omega_2, \omega_1, z_2, z_1) = q^{4\Delta(z_1 - z_2)} \frac{q^{z_1} - q^{z_2}}{q^{z_2} - q^{z_1}} L(x, \chi^* \omega_1, \chi^* \omega_2, z_1, z_2)
\]

**Remark 6.1.** The previous transformation of characters comes form the fact that if we define \( \theta_1(a) = w_1(a)|a|^{z_1} \) and \( \theta_2(a) = w_2(a)|a|^{z_2} \) then for \( p = \left( \begin{array}{cc} a & x \\ c & d \end{array} \right) \) we have

\[
d(p \cdot x) = \chi_1(d_1(p \cdot x)) \cdot |d_1(p \cdot x)|^{z_1} \cdot \chi_2(d_2(p \cdot x)) \cdot |d_2(p \cdot x)|^{z_2} = \theta_1(a) \cdot \theta_2(d) \cdot \delta_F^x(p) \cdot d(x),
\]

where \( \delta_F^x(p) \) is the modular character of the Borel subgroup, this property is related to the principal series representation of \( GL_2(F) \).

**7 Calculation of \( L(x, 1, \chi_2, z) \):**

We will compute the integral over cosets of the Iwahori subgroup, \( B \)

\[
B = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in K, \ b \in \mathfrak{o}O_F.
\]

\( B \) has the factorization:

\[
B = N \cdot A N^+ (\mathfrak{o}O_F) = \left( \begin{array}{cc} 1 & 0 \\ t & 1 \end{array} \right) \left( \begin{array}{cc} a_1 & 0 \\ 0 & a_2 \end{array} \right) \left( \begin{array}{cc} 1 & y \\ 0 & 1 \end{array} \right) = \left( \begin{array}{cc} a_1 & a_1y \\ a_1t & a_1ty + a_2 \end{array} \right),
\]

Where \( a_1, a_2 \in O_F^*, t \in O_F, y \in \mathfrak{o}O_F \).
The Haar measure on $B$, $\mu_B$ is taken to be $dt \times da_1 \times da_2 \times dy$, where:

$dt$—the Haar measure on $O_F$ normalized to be 1,
$da_1, da_2$—the Haar measure of $O_F^*$ normalized to be 1 and $dy$—the Haar measure on $\mathfrak{m}O_F$ normalized to be $q^{-1}$.

Overall we have $\mu_B(B) = q^{-1}$.

Here is a list of coset representatives for $B \setminus K$:

$b_i = \begin{pmatrix} 1 & r_i \\ 0 & 1 \end{pmatrix}$ if $0 \leq i \leq q$ and $r_i$ are representative of $\bar{F}$, $q = |\bar{F}|$

and $b_{q+1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

So:

$Bb_i = \begin{pmatrix} a_1 & a_1y + a_1r_i \\ a_1t & a_1t(y + r_i) + a_2 \end{pmatrix}$ if $1 \leq i \leq q$ and $Bb_{q+1} = \begin{pmatrix} a_1y & a_1 \\ a_1ty + a_2 & a_1t \end{pmatrix}$.

Instead of integrating on $K$, we integrated over the different cosets of $B$ using the Haar measure of $B$.

We want to normalize the Haar measure to be $\mu_G(K) = 1$, so we normalize the measure on each coset by multiplying by $\frac{1}{q^{q+1}}$.

Let $A = \begin{pmatrix} a & c \\ \bar{c} & b \end{pmatrix}$ be a fixed Hermitian matrix and $C = k \cdot A \cdot \mathbf{k}^T$ then using the factorization on $k$ one can parametrize $C$ as follows:

$$d_1(C) = \begin{cases} 
N(a_1) \cdot [a + Tr(\bar{c}(y + r_i)) + bN(y + r_i)] & k \in Bb_i, i < q + 1 \\
N(a_1) \cdot [aN(y) + Tr(\bar{c}y) + b] & k \in Bb_{q+1}
\end{cases}$$

(7.1)

In the first step of the proof we show that we can actually integrate over $K$, from now on we will assume that the characters of $F^*$ are defined on 0, say: $\chi_i(0) = 0$.

Lemma 7.1. Let $A \in X$ be a fixed Hermitian matrix. Then $\mu_K$ measure of the set $\{k \in K | d_1(k \cdot A) = 0\}$ is 0.

Proof. By (7.1), it is sufficient to show that the inverse image of a point of the trace and norm maps is of measure zero. Note that the trace and norm maps: $Tr : O_F \to Tr(O_F)$, $N : O_F^* \to N(O_F^*)$ are surjective homomorphisms, and that the Haar measure of $Tr(O_F), N(O_F^*)$ (being an open sets) is the induced measure of $O_E, O_E^*$ (respectively). In particular the measure of a singleton is 0. We use Lemma 4.1 to deduce that the measure of the inverse images is also 0. \qed
Lemma 7.2.

\[
L(\chi_1, \chi_2; s_1, s_2) = \int_k \chi_1(d_1(k \cdot x)) \cdot |d_1(k \cdot x)|^{s_1} \chi_2(d_2(k \cdot x)) \cdot |d_2(k \cdot x)|^{s_2} \, dk =
\]

\[
\frac{q \chi_2(\det(x)) |\det(x)|^{s_2}}{q + 1} \left[ \int_{\mathcal{O}_F} |a + Tr(\tilde{c}t) + bN(t)|^{s_1} \chi_1(a + Tr(\tilde{c}t) + bN(t)) dt + \int_{\mathcal{O}_F} |aN(y) + Tr(\tilde{c}y) + b|^{s_1} \chi_1(aN(y) + Tr(\tilde{c}y) + b) dy \right].
\]

Proof. For any \(f \in L^1(X)\):

\[
\int_k f(k \cdot x) \, dk = \frac{q}{q + 1} \sum_{i=1}^{q+1} \int_{k=kh \in Bk_i} f(k \cdot x) \, db
\]

Using (7.1) and the Haar measure of \(B\), We have:

\[
L(\chi_1, \chi_2; s_1, s_2) = \frac{q \chi_2(\det(x)) |\det(x)|^{s_2}}{q + 1} \left[ \sum_{i=1}^{q} \int_{\mathcal{O}_F} \chi_1(a + Tr(\tilde{c}(y + r_i)) + bN(y + r_i)) \cdot |a + Tr(\tilde{c}(y + r_i)) + bN(y + r_i)|^{s_1} dy + \int_{\mathcal{O}_F} \chi_1(aN(y) + Tr(\tilde{c}y) + b) \cdot |aN(y) + Tr(\tilde{c}y) + b|^{s_1} dy \right].
\]

The last equation follows from the facts that \(|N(a_1)| = 1\), \(\chi_1(N(a_1)) = 1\) and that \(|\det(k \cdot x)| = |\det(x)|\).

Notice that performing the first sum is equivalent to integrating over \(\mathcal{O}_F\). Altogether we obtain our formula. \(\square\)

7.1 Calculation \(L(x, 1, \chi_2; z)\) on the diagonal elements

We use Lemma 7.2 to calculate the various cases.

On the diagonal representatives:

\[
x \sim \begin{pmatrix} \pi^{\lambda_1} \xi_1 \\ \pi^{\lambda_2} \xi_2 \end{pmatrix}
\]

\[
L(x, 1, \chi_2; s_1, s_2) =
\]
we can calculate this integral easily on this space:

\[
\frac{q^{(\lambda_1 + \lambda_2) - 2s_1} \chi_2(\epsilon_1, \epsilon_2)}{q^{-1} + 1} \left[ \int_{\mathcal{O}_F} |\pi^{\lambda_1} \epsilon_1 + \pi^{\lambda_2} \epsilon_2 N(t)|^{s_1} dt + \int_{\mathcal{O} \mathcal{O}_F} |\epsilon_2 \pi^{\lambda_2} + \epsilon_1 \pi^{\lambda_1} N(y)|^{s_1} dy \right] =
\]

\[
\frac{q^{(\lambda_1 + \lambda_2) - 2s_1} \chi_2(\epsilon_1, \epsilon_2) \cdot q^{-\lambda_2 s_1}}{q^{-1} + 1} \left[ \int_{\mathcal{O}_F} |\pi^{\lambda_1 - \lambda_2} \frac{\epsilon_1}{\epsilon_2} + N(t)|^{s_1} dt + \int_{\mathcal{O} \mathcal{O}_F} \left| 1 + \frac{\epsilon_1}{\epsilon_2} \pi^{\lambda_1 - \lambda_2} N(y) \right|^{s_1} dy \right].
\]

The second integrand is the constant 1 since \( |N(y)| < 1 \) if \( y \in \mathcal{O} \mathcal{O}_F \).

Note that for any \( t \in \mathcal{O}_F - \mathcal{O} \mathcal{O}_F \), we have \( |\pi^{\lambda_1 - \lambda_2} \frac{\epsilon_1}{\epsilon_2} + N(t)|^{s_1} = |t|^{2s_1} \) and so we can calculate this integral easily on this space:

\[
L(x, 1, \chi_2, s_1, s_2) = \frac{q^{(\lambda_1 + \lambda_2) - 2s_1} q^{-\lambda_2 s_1} \chi_2(\epsilon_1, \epsilon_2)}{q^{-1} + 1} \left[ \sum_{j=0}^{\lambda_1 - \lambda_2 - 1} q^{(-2s_1-1)j} (1 - q^{-1})^j + q^{(\lambda_1 - \lambda_2)(-2s_1)} \int_{\mathcal{O} \mathcal{O}_F} \left| \frac{\epsilon_1}{\epsilon_2} + N(t) \right|^{s_1} dt + q^{-1} \right].
\]

After the substitution \( \mathcal{O} \mathcal{O}_F \chi_1 \mathcal{O}_F = t \), we get:

\[
L(x, 1, \chi_2, s_1, s_2) = \frac{q^{(\lambda_1 + \lambda_2) - 2s_1} q^{-\lambda_2 s_1} \chi_2(\epsilon_1, \epsilon_2)}{q^{-1} + 1} \left[ (1 - q^{-1}) \frac{1 - q^{(-2s_1-1)(\lambda_1 - \lambda_2)}}{1 - q^{(-2s_1-1)}} + q^{(\lambda_1 - \lambda_2)(-2s_1-1)} \int_{\mathcal{O}_F} \left| \frac{\epsilon_1}{\epsilon_2} + N(y) \right|^{s_1} dy + \frac{1}{q} \right] + q^{(\lambda_1 - \lambda_2)(-2s_1-1)} \int_{\mathcal{O}_F} \left| \frac{\epsilon_1}{\epsilon_2} + N(y) \right|^{s_1} dy + \frac{1}{q},
\]

\[
L(x, 1, \chi_2, z) = \frac{q^{\lambda_1 - \lambda_2} q^{2s_1 z + 2s_1 z_1} \chi_2(\epsilon_1, \epsilon_2)}{q^{-1} + 1} \left[ q^{2s_2} \frac{1 - q^{(-2s_2-1)(\lambda_1 - \lambda_2) + 2s_2}}{q^{2s_2} - q^{2s_2-1}} + q^{(2s_1 - 2s_2)(\lambda_1 - \lambda_2) + 2s_2 - 1} \right]
\]

\[
+ q^{(\lambda_1 - \lambda_2)(2s_1 - 2s_2)} \int_{\mathcal{O}_F} \left| \frac{\epsilon_1}{\epsilon_2} + N(t) \right|^{s_1} dt + q^{-1}. \tag{7.2}
\]

There are two cases:
Case 1: \(-\frac{\xi_1}{\xi_2} \notin N(F)\): We have by Lemma 4.5:
\[
\int_{\partial F} \left| \frac{\xi_1}{\xi_2} + N(t) \right| dt = \frac{q^2z_2 - q^{2z_2 - 1} + q^{(2z_1 - 2z_2)s + 2z_2 - 1} - q^4(2z_1 - 2z_2) + 2z_1}{q^2z_2 - q^{2z_1}}.
\]
We set this into Eq 7.2:
\[
L(x, 1, \chi_2; z) = \frac{q^{\lambda_2 - \lambda_1} \chi_2(\xi_1 \xi_2)q^{-2z_2}}{q^{-1} + 1} \left[ q^{2z_2} - q^{2z_2 - 1} \right] + q^2z_2 - q^{2z_1}
\]

\[
+ \frac{q^{2z_1 + 2z_2 + 2z_2 - 1} - q^{2z_1 + 2z_2 + 2z_2 - 1} + \lambda_2 \lambda_2 z_2}{q^2z_2 - q^{2z_1}} + \frac{q^{2z_1 - 1 + 2(\lambda_1 + s)z_2 + 2z_1}}{q^2z_2 - q^{2z_1}}.
\]

Case 2: \(-\frac{\xi_1}{\xi_2} \in N(F)\): We have
\[
\int_{\partial F} \left| \frac{\xi_1}{\xi_2} + N(t) \right| dt = \frac{q^2z_2 - q^{2z_2 - 1} + q^{(2z_1 - 2z_2)s + 2z_2 - 1} + q^{(2z_1 - 2z_2)s + 2z_1}}{q^2z_2 - q^{2z_1}}.
\]
Setting this into Eq 7.2:
\[
L(x, 1, \chi_2; z) = \frac{q^{\lambda_2 - \lambda_1} \chi_2(\xi_1 \xi_2)}{q^{-1} + 1} \left[ q^{2z_2} - q^{2z_2 - 1} \right] + q^2z_2 - q^{2z_1}
\]

\[
+ \frac{q^2z_1 + (\lambda_2 - s)z_2 + 2z_2 - 1} + q^{2z_1 + 2z_2 + 2z_2 - 1} - q^{2z_1 + 2z_2 + 2z_2 - 1} + \lambda_2 \lambda_2 z_2}{q^2z_2 - q^{2z_1}} + \frac{q^{2z_1 - 1 + 2(\lambda_1 + s)z_2 + 2z_1}}{q^2z_2 - q^{2z_1}}
\]

\[
\frac{q^{\lambda_2 - \lambda_1} \chi_2(\xi_1 \xi_2)q^{-2z_2}}{q^{-1} + 1} \left[ q^{(\lambda_1 + s)z_2} - q^{2z_2 - 1} \right] + q^{(\lambda_2 \lambda_2 z_2)}q^{2z_2 - q^{2z_1 - 1}}.
\]
7.2 Calculating $L(x, 1, \chi_2, z)$ on the non diagonal elements:

Let $x = \begin{pmatrix} a & \sigma^c \\ \sigma & b \end{pmatrix}$. Using Lemma 7.2:

\[
L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(det(x))|det(x)|^{s_2}}{q + 1} \int_{\mathfrak{o}_F} |a + Tr(\sigma^c t) + bN(t)|^{s_1} dt + \\
\int_{\mathfrak{o}_F} |aN(y) + Tr(\sigma^c y) + b|^{s_1} dy.
\]

In the following section we will make use of Lemma 4.1 and 3.2 (coordinate substitution).

7.2.1 RP case

Recall: $l = \frac{n}{q^2}$, $Tr(\sigma^{2i}O_F) = \pi^{l+i}O_E$, $Tr(\sigma^{2i-1}O_F) = \pi^{l+i}O_E$.

\[
L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-1)}{q^{-1} + 1} \left[ \int_{\mathfrak{o}_F} |Tr(t)|^{s_1} dt + \int_{\mathfrak{m}_O} |Tr(y)|^{s_1} dy \right].
\]

We substitute $u = Tr(t)$, $v = Tr(y)$

\[
L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-1)}{q^{-1} + 1} \left[ \int_{u \in \pi^iO_E} |u|^{s_1} du + \int_{v \in \pi^{i+1}O_E} |v|^{s_1} dv \right].
\]

The integrals above are simple and reduces to geometric sums:

\[
L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-1)}{q^{-1} + 1} \left[ \sum_{i=0}^{\infty} q^{-i}(1 - q^{-1})q^{(l+i)(-2s_1)} + \sum_{i=1}^{\infty} q^{-i}(1 - q^{-1})q^{(l+i)(-2s_1)} \right] =
\]
χ₂(−1)(1−q⁻¹)q−2s₁\frac{q^-1+1}{\sum_{i=0}^{∞}q^{-2s₁-1}i + \sum_{i=1}^{∞}q^{-2s₁-1}i} =

χ₂(−1)(1−q⁻¹)q−2s₁\frac{q^-1+1}{\sum_{i=0}^{∞}q^{-2s₁-1}i + \sum_{i=1}^{∞}q^{-2s₁-1}i}.

After simplifying this expression and performing the transformation \( s \mapsto z \), we obtain:

\[ L(x, 1, χ₂, z) = \frac{χ₂(−1)(1−q⁻¹)q^{−4ls₂}}{q^{−1+1}} \left[ \frac{q^{l(s₁+z₁)}(q^{2s₁}+q^{2s₂})}{q^{z₁}−q^{z₂}} \right]. \]

\[ \chi \sim \left( \begin{array}{c} \Theta \\ \Theta \end{array} \right) \]

\[ L(x, 1, χ₂, s₁, s₂) = \frac{χ₂(−1)q^{−2s₂}}{q^{−1+1}} \left[ \int_{x \in O_F} |Tr(\Theta t)|^{[1]} dt + \int_{y \in \Theta O_F} |Tr(\Theta y)|^{[1]} dy \right]. \]

\[ \frac{χ₂(−1)q^{−2s₂+1}}{q^{−1+1}} \left[ \int_{u \in \Theta O_F} |Tr(u)|^{[1]} du + \int_{u \in \Theta²O_F} |Tr(u)|^{[1]} du \right]. \]

Similarly to the previous calculations, we substitute \( v = Tr(t), y = Tr(u) \), and use Lemma 4.1. Note that the image of both transformations coincides \( Im(\Theta O_F) = \pi^{l+1}O_E \), but the multiplicative factors (The “Jacobians”) are different:

\[ L(x, 1, χ₂, s₁, s₂) = \frac{χ₂(−1)q^{−2s₂+1}}{q^{−1+1}} \left[ q^l \int_{v \in \pi^{l+1}O_E} |v|^{[1]} dv + q^{l⁻¹} \int_{u \in \pi^{l+1}O_E} |y|^{[1]} dy \right] = \]

\[ \frac{χ₂(−1)q^{−2s₂+1}q^{(l⁻¹)}}{q^{−1+1}} \left[ \sum_{j=1}^{∞} q^{−l−i} (1−q⁻¹)q^{(l+i)(−2s₁)} + q^{−1} \sum_{j=1}^{∞} q^{−l−i} (1−q⁻¹)q^{(l+i)(−2s₁)} \right] . \]
Substituting $s \mapsto z$:

$$L(x, 1, \chi_2, z) = \frac{\chi_2(-1)(1 - q^{-1})q^{2z_2 + \frac{1}{2}}q^{((2z_1 - 2z_2)x) + q^{-1}\sum_{j=1}^{\infty} q^{((2z_1 - 2z_2)j)}]}{q^{-1} + 1}$$

$$= \frac{\chi_2(-1)(1 - q^{-1})q^{2z_2 + \frac{1}{2}}q^{((2z_1 - 2z_2)x) + q^{-1}\sum_{j=1}^{\infty} q^{((2z_1 - 2z_2)j)}]}{q^{-1} + 1}$$

After simplifying this expression, we get:

$$L(x, 1, \chi_2, z) = \chi_2(-1)(1 - q^{-1})q^{2z_2 + \frac{1}{2}}q^{((2z_1 - 2z_2)x)} + q^{-1}\sum_{j=1}^{\infty} q^{((2z_1 - 2z_2)j)}.$$

$$3.$$  

$$x \sim \begin{pmatrix} \pi^z & 1 \\ 1 & -\pi^z \rho \end{pmatrix}, \ 1 + \pi^z \rho = \Delta \notin N(F^*)$$

$$L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-\Delta)}{q^{-1} + 1} \int_{F^*} |\pi^z + Tr(t) - \pi^z \rho N(x)|^s dt + \int_{F^*} |\pi^z N(y) + Tr(y) - \pi^z \rho|^s dy.$$  

Because $|Tr(y)|, |\pi^z N(y)| < 1$ (see Sections 3.2 and 3.1), we conclude that the second integrand is constant in $\mathcal{O}_{F^*}$.

$$L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-\Delta)}{q^{-1} + 1} \int_{F^*} |\pi^z + Tr(t) - \pi^z \rho N(t)|^{s_1} dt + q^{-s_2 - 1 + q^{-s_2 - 1}}.$$  

Since $|N(t)| = 1$ if $t \in O_F$, and $N(t^{-1})Tr(t) = Tr(t^{-1})$, one can verify:

$$|\pi^z + Tr(t) - \pi^z \rho N(t)|^{s_1} = |\pi^z N(t^{-1}) + Tr(t^{-1}) - \pi^z \rho|^s.$$  

Setting this into (7.3):
\[ L(x, 1, \chi_2, s_1, s_2) = \frac{X_2(-\Delta)}{q^{-1} + 1} \int_{\mathcal{O}_E^*} |\pi^z N(u(\chi_2 / t) + Tr(1) - \pi^z \rho)^{s_1} dt + q^{-s_1 - 1} + q^{-s_2 - 1}|. \]

We substitute \( u = \frac{1}{t} \)

\[ L(x, 1, \chi_2, s_1, s_2) = \frac{X_2(-\Delta)}{q + 1} \int_{\mathcal{O}_E^*} |\pi^z N(u) + Tr(u) - \pi^z \rho|^{s_1} du + 2q^{-s_1 - 1}. \] (7.4)

We use the following identity:

\[ \pi^z N(u) + Tr(u) = (\bar{\phi}^z u + \bar{\phi}^z \bar{\mu} + \bar{\phi}^z \bar{\pi}) - \pi^z z = N(\bar{\phi}^z u + \bar{\phi}^z \bar{\pi}) - \pi^z z \]

Setting this identity into (7.4):

\[ L(x, 1, \chi_2, s_1, s_2) = \frac{qX_2(-\Delta)}{q^{-1} + 1} \int_{\mathcal{O}_E^*} |N(\bar{\phi}^z u + \bar{\phi}^z \bar{\pi}) - \pi^z z - \pi^z \rho|^{s_1} du + 2q^{-s_1 - 1} = \]

\[ \frac{X_2(-\Delta)}{q^{-1} + 1} \int_{\mathcal{O}_E^*} |\pi^z N(\bar{\phi}^z u + \bar{\phi}^z \bar{\pi}) - \pi^z z - \pi^z \rho|^{s_1} du + 2q^{-s_1 - 1} = \]

\[ \frac{X_2(-\Delta)}{q^{-1} + 1} \int_{\mathcal{O}_E^*} |q^{-s_2} \pi^z [N(\bar{\phi}^z u + 1) - (1 + \pi^z \rho)]|^{s_1} du + 2q^{-s_1 - 1}. \]

Since \( 1 + \pi^z \rho \) is not a norm, we have \( |N(\bar{\phi}^z u + 1) - (1 + \pi^z \rho)| = \pi^s \eta, \eta \in \mathcal{O}_E^* \) (see Section 3.1), we have:

\[ L(x, 1, \chi_2, s_1, s_2) = \frac{X_2(-\Delta)}{q^{-1} + 1} \int_{\mathcal{O}_E^*} q^{-s_2} du + 2q^{-s_1 - 1} = \]

\[ \frac{X_2(-\Delta)}{q^{-1} + 1} [q^{-s_1} (1 - q^{-1}) + 2q^{-s_1 - 1}] = \frac{X_2(-\Delta)}{q^{-1} + 1} [q^{-s_1} + q^{-s_1 - 1}]. \]

Substituting \( s \mapsto z \):

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\[ L(x, 1, \chi_2, z) = \chi_2(-\Delta)q^{2s} + \chi_2^{s1}z. \]

4. 
\[ x = \begin{pmatrix} \pi^m & 1 \\ 1 & 0 \end{pmatrix} \quad 0 < m < \frac{1}{2} \]

\[ L(x, 1, \chi_2; s_1, s_2) = \frac{\chi_2(-1)}{q^{-1} + 1} \left[ \int \pi^m + \text{Tr}(t)^{s1} dt + \int \pi^m N(y) + \text{Tr}(y)^{s1} dy \right]. \]

We have that \(|\text{Tr}(t)| < |\pi^m|\) (see Section 3(2)), and so the first integrand is constant:

\[ L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-1)}{q^{-1} + 1} \left[ q^{-2s1m} + \int \pi^m N(y) + \text{Tr}(y)^{s1} dy \right]. \] (7.5)

We use the following identity:

\[ \pi^m N(y) + \text{Tr}(y) = (\sigma^{-m} + \bar{\sigma}^{-m} y)(\bar{\sigma}^{-m} + \sigma^m y) - \pi^{-m} = N(\sigma^{-m} + \bar{\sigma}^{m} y) - \pi^{-m}. \]

We set it into (7.5):

\[ L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-1)}{q^{-1} + 1} \left[ q^{-2s1m} + q^{2ms1} \int \pi^m N(1 + \pi^m y) - 1^{s1} dy \right]. \]

We substitute \( u = \pi^m y \):

\[ L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-1)}{q^{-1} + 1} \left[ q^{-2s1m} + q^{2ms1 + m} \int \pi^m N(1 + u) - 1^{s1} du \right]. \]

We substitute \( t = 1 + u \):

\[ L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-1)}{q^{-1} + 1} \left[ q^{-2s1m} + q^{2ms1 + 2m} \int \pi^m N(t) - 1^{s1} dt \right]. \]

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We use Lemma\textsuperscript{4.2}. The last integral splits to 3 different sums:

\[
L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-1)}{q^{-1} + 1} \left[ q^{-2s_1m + q^{2ms_1 + 2m} \sum_{i=s+1}^{m} 2(1 - q^{-1})q^{-2is_1 - i} + (q - 2)q^{-1}q^{-2s_1 + \sum_{i=2m+1}^{s-1} q^{-2is_1 - i}(1 - q^{-1})} \right].
\]

After calculating the sums and substituting \( s \mapsto z \), we have:

\[
L(x, 1, \chi_2, z) = \frac{\chi_2(-1)}{q^{-1} + 1} \left[ q^{-m(2z_1 - 2z_2 + 1)} + \left(1 - q^{-1}\right) \left( \frac{2q^{(2z_1 - 2z_2)(s+1)}}{1 - q^{2z_1 - 2z_2}} + \frac{q^{(2m+1)(2z_1 - 2z_2)} - q^{m(2z_1 - 2z_2)}}{1 - q^{2z_1 - 2z_2}} \right) + (q - 2)q^{m(2z_2 - 2z_1 + 1)} \frac{q^{(2z_1 - 2z_2) - 1 + 2z_2} - q^{(2z_1 - 2z_2) - 1 + 2z_1}}{q^{2z_2} - q^{2z_1}} \right] .
\]

Simplifying this expression once more, we get:

\[
L(x, 1, \chi_2, z) = \frac{\chi_2(-1)}{q^{-1} + 1} \frac{q^{-m - 2sz_2}(1 - q^{-1})}{q^{2z_2} - q^{2z_1}} \sum_{\sigma} (q^{(m, s - m + 1, 2z)}).
\]

5.

\[
x = \left( \begin{array}{c} \pi^m \\ \sigma \\ 0 \end{array} \right) \quad 0 < m < \frac{2}{3} + 1
\]

\[
L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-1)}{q^{-1} + 1} \int_{\mathcal{O}_F} \left| \pi^m + \text{Tr}(\sigma \mathfrak{t}) \right|^{s_1} dt + \int_{\mathcal{O}_F} \left| \pi^m N(y) + \text{Tr}(\sigma \mathfrak{t}) \right|^{s_1} dy.
\]

Once again, note that \( |\pi^m + \text{Tr}(\sigma \mathfrak{t})|^{s_1} = q^{-2s_1m} \) for any \( t \in \mathcal{O}_F \). (see Section\textsuperscript{3.2}).

We substitute \( u = \sigma \gamma \):

\[
L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-1)}{q^{-1} + 1} \left[ q^{-2s_1m} + q \int_{\mathcal{O}_F} \left| \pi^m - 1 N(u) + \text{Tr}(u) \right|^{s_1} du \right].
\]

We use the identity: \( \pi^{m-1} N(u) + \text{Tr}(u) = N(\sigma^{1-m} + \sigma^{-m} u) - \pi^{1-m} \) to get:
$$L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-1)q^{-2s_2}}{q^{-1} + 1} \int_{\sigma^2 \Omega_F} \left| q^{2s_1 + q^{1+(m-1)2s_1}} \right| N(1 + \pi^{m-1}u) - 1 |^1 du$$

Substitute $y = \pi^{m-1}u$:

$$L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-1)q^{-2s_2}}{q^{-1} + 1} \int_{\sigma^2 \Omega_F} \left| N(1 + y) - 1 \right|^1 dy,$$

Substitute $t = 1 + y$:

$$L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-1)q^{-2s_2}}{q^{-1} + 1} \int_{1 + \sigma^2 \Omega_F} \left| N(t) - 1 \right|^1 dt.$$

We calculate the last integral by using Lemma 4.2:

$$L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-1)q^{-2s_2}}{q^{-1} + 1} \left( q^{-2s_1 + q^{2m-1+(m-1)2s_1}} \times \sum_{i=s+1}^{\infty} 2(1 - q^{-1})q^{-2is_1-i} + (q - 2)q^{-s-1}q^{-2s_1-s} + \sum_{i=2m}^{s-1} q^{-2is_1-i}(1 - q^{-1}) \right)$$

Substituting $s \mapsto z$ and simplifying the expression, we get:

$$L(x, 1, \chi_2, z) = \frac{\chi_2(-1)q^{-2z_2}}{q^{-1} + 1} \left( q^{-2z_1 + q^{m+(m-1)(2z_2-2z_1)}} \times \right.$$

$$\left. [(1 - q^{-1})(\frac{2q^{(2z_1-2z_2)(s+1)+2z_2}}{q^{2z_2} - q^{2z_1}} + \frac{q^{(2z_1-2z_2)2m+2z_2} - q^{(2z_1-2z_2)s+2z_2}}{q^{2z_2} - q^{2z_1}}) + (q - 2)q^{(2z_1-2z_2)^{-1}}] \right)$$

$$= \frac{\chi_2(-1)q^{2z_2+\frac{1}{2}+m}}{q^{-1} + 1} [q^{2z_1-2z_2} + q^{(1-m)(2z_1-2z_2)}] \frac{2(q - 1)q^{2z_1-2z_2}(s+1)+2z_2-1}{q^{2z_2} - q^{2z_1}} +$$
We extract and simplify the different elements in this expression:

\[
L(x, 1, \chi_2, z) = \frac{\chi_2(-1)q^{-\frac{1}{2}m}}{q^{-1} + 1} \times \frac{q^{m-2z_2} \sum_{\sigma} (q^{(s, m+1-s, m, 2z_2)} - q^{(s, m+1-s, m, 2z_2-1)})}{q^{2z_2} - q^{2z_1}}
\]

Note that \(q^{2z_2}L(x, 1, \chi_2, z)\) is an antisymmetric function. We can express \(L(x, 1, \chi_2, z)\) as follows:

\[
L(x, 1, \chi_2, z) = \frac{\chi_2(-1)q^{-\frac{1}{2}m}}{q^{-1} + 1} \frac{q^{m-2z_2} \sum_{\sigma} (q^{(s, m+1-s, m, 2z_2)} - q^{(s, m+1-s, m, 2z_2-1)})}{q^{2z_2} - q^{2z_1}}
\]

\[6.\]

\[x \sim \begin{pmatrix} \pi^m \\ 1 \\ \pi^{-m} \rho \end{pmatrix}, \quad 0 < m < \frac{s}{2}
\]

\[
L(x, 1, \chi_2, s_1, s_2) = \frac{q\chi_2(-\Delta)}{q + 1} \left( |\pi^m + Tr(t) - \pi^{-m} \rho N(t)|^{s_1} dt + \int_{\omega \Omega y} |\pi^m N(y) + Tr(y) - \pi^{-m} \rho |^{s_1} dy \right).
\]

Because \(|Tr(\Omega)| \leq |\pi^\frac{s}{2}|\) (see Section 3.2) and \(|\pi^{s-m}| < |\pi^m|\), we have that the first integrand is constant:

\[
L(x, 1, \chi_2, s_1, s_2) = \frac{q\chi_2(-\Delta)}{q + 1} \left( |\pi^{m-s_1} + \int_{\omega \Omega y} |\pi^m N(y) + Tr(y) - \pi^{-m} \rho |^{s_1} dy \right). \quad (7.6)
\]

We use the identity \(\pi^m N(y) + Tr(y) = N(\omega^{-m} + \omega^m y) - \pi^{-m}\) and set it into Eq 7.6.
\[ L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-\Delta)}{q^{-1} + 1} [q^{-2m s_1} + \int_{0}^{1} |N(\sigma^{-m} + \sigma^m y) - \pi^{-m} - \pi^{s-m} \rho|^{s_1} dy] = \]

\[ \frac{\chi_2(-\Delta)}{q^{-1} + 1} [q^{-2m s_1} + q^{2ms_1} \int_{0}^{1} |N(1 + \chi^m y) - (1 + \pi^s \rho)|^{s_1} dy]. \]

Substituting \( t = 1 + \pi^m y \):

\[ L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-\Delta)}{q^{-1} + 1} [q^{-2m s_1} + q^{2ms_1 + 2m} \int_{1 + \sigma^{2m+1} F} |N(t) - (1 + \pi^s \rho)|^{s_1} dt]. \]

(7.7)

We calculate the last integral by decomposing \( 1 + \sigma^{2m+1} F = \bigcup_{j=2m+1}^{s-1} (1 + \sigma^j O_F) \cup (1 + \sigma^s O_F) \).

From Section 3.1 we deduce that for any \( t \in 1 + \rho^i O_F^k \) and \( 2m + 1 \leq i < s \) the value of \( |N(t) - (1 + \pi^s \rho)|^{s_1} \) is \( q^{-2s_1} \).

Because \( 1 + \pi^s \rho \notin N(F) \), in the subgroup \( 1 + \sigma^s O_F \) the integrand is simply \( q^{-2s_1} \).

Altogether, we have:

\[ L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-\Delta)}{q^{-1} + 1} [q^{-2m s_1} + q^{2ms_1 + 2m} (\sum_{i=2m+1}^{s-1} (1 - q^{-1}) q^{-i} q^{-2s_1} + q^{-i} q^{-2s_1})] = \]

\[ \frac{\chi_2(-\Delta)}{q^{-1} + 1} [q^{-2m s_1} + q^{2ms_1 + 2m} (\sum_{i=2m+1}^{s-1} q^{(-2s_1-1)i} + q^{(-2s_1-1)})]. \]

We simplify the sum and substitute \( s \mapsto z \) to get:

\[ L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-\Delta)}{q^{-1} + 1} \left[ q^{m(2z_1-2z_2+1)+2z_2} - q^{m(2z_1-2z_2+1)+2z_1} \right] + \]

\[ q^{2ms_1 + 2m} \left( q^{(2z_1-2z_2)(2m+1)} - q^{(2z_1-2z_2)s} \right) \]

\[ \left( \frac{1 - q^{2z_1-2z_2}}{1 - q^{2z_1-2z_2}} + q^{2z_1-2z_2} \right) \]
\[
\frac{\chi_2(-\Delta)q^m q^{-2xz_2}}{q^{-1} + 1} \left( \frac{q^{mz_2 + (1 - m + s)2z_2}}{q^{z_2} - q^{2z_1}} + \frac{q^{(s + 1 - m)2z_1 + m2z_2}}{q^{2z_1} - q^{2z_2}} \right)
\]

And after further manipulations:

\[
L(x, 1, \chi_2, \hat{z}) = \frac{\chi_2(-\Delta)q^{m-2z_2}}{q^{-1} + 1} \sum_\sigma \left[ \frac{q^{(m, 1 - m + s), 2z_2}}{q^{z_2} - q^{2z_1}} \right].
\]

\[
x \sim \left( \frac{\pi^m}{p} - \pi^{s+1-m} \rho \right), \quad 0 < m < \frac{s}{2} + 1.
\]

\[
L(x, 1, \chi_2, \hat{z}) = \frac{\chi_2(-\Delta)q^{-2z_2}}{q^{-1} + 1} \left[ \int \pi^m + Tr(\bar{\theta}t) - \pi^{s+1-m} \rho N(t) \right] dt + \int_{\sigma \partial \Omega} |\pi^m N(y) + Tr(\bar{\theta}y) - \pi^{s+1-m} \rho|^{\sigma_1} dy
\]

Similarly to the previous calculations the first integrand is also constant, so

\[
L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-\Delta)q^{-2z_2}}{q^{-1} + 1} \left[ q^{-2ms_1} + \int_{\sigma \partial \Omega} |\pi^{m-1} N(\bar{\theta}y) + Tr(\bar{\theta}y) - \pi^{s+1-m} \rho|^{\sigma_1} dy \right].
\]

We substitute \( u = \sigma y \):

\[
L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-\Delta)q^{-2z_2}}{q^{-1} + 1} \left[ q^{-2ms_1} + q \int_{\sigma^2 \partial \Omega} |\pi^{m-1} N(u) + Tr(u) - \pi^{s+1-m} \rho|^{\sigma_1} du \right].
\]

We use the identity: \( \pi^{m-1} N(u) + Tr(u) = N(\sigma^{1-m} + \sigma^{m-1}) - \pi^{1-m} \) and set it to Eq\[7.8\]

\[
L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-\Delta)q^{-2z_2}}{q^{-1} + 1} \left[ q^{-2ms_1} + q \int_{\sigma^2 \partial \Omega} |N(\sigma^{1-m} + \sigma^{m-1}) - \pi^{1-m} - \pi^{s+1-m} \rho|^{\sigma_1} du \right] =
\]

\[
\frac{\chi_2(-\Delta)q^{-2z_2}}{q^{-1} + 1} \left[ q^{-2ms_1} + q^{1+(m-1)2s_1} \int_{\sigma^2 \partial \Omega} |N(1 + \sigma^{2m-2}) - (1 + \pi^{\rho})|^{\sigma_1} du \right].
\]

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We substitute \( t = 1 + \sigma^{2m-2} u \):

\[
L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-\Delta)q^{-2s_2}}{q^{-1} + 1} [q^{-2ms_1} + q^{(m-1)2s_1 + m-1} \int_{1+\sigma^{2m}O_F} |N(t) - (1 + \pi \rho)|^{\pi_1} dt].
\]

The calculation of the integral is similar to the calculation of the integral in Eq 7.7.

\[
L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-\Delta)q^{-2s_2}}{q^{-1} + 1} [q^{-2ms_1} + q^{(m-1)(2s_1-2z_1) + m} (\sum_{i=2m}^{s-1} q^{(-2s_1-1)i + q^{-s} q^{-2s_1}})].
\]

We substitute \( s \mapsto z \):

\[
L(x, 1, \chi_2, z) = \frac{\chi_2(-\Delta)q^{-2s_2-\frac{1}{2} + \pi - (s+1)2z_2}}{q^{-1} + 1} [q^{m2z_1 + (2s-m)2z_2 + 2z_2} \left( \frac{q^{2z_2} - q^{2z_1}}{q^{2z_2} - q^{2z_1}} \right) - q^{z_2} q^{2z_1}]
\]

\[
= \frac{\chi_2(-\Delta)q^{-2s_2}}{q^{-1} + 1} [\sum_{\sigma} \left( \frac{q^{(m+2s-m)2z_2}}{q^{2z_2} - q^{2z_1}} \right)].
\]

### 7.2.2 RU case

Recall that \( l = \frac{\pi l^{1+1}}{2} \), \( Tr(\sigma^{2l} O_F) = \pi l^{1+1} O_E \) and \( Tr(\sigma^{2l-1} O_F) = \pi (l+1) O_F \).

\[
x \sim \left( \begin{array}{cc}
\pi l^{1+1} & \sigma \\
\sigma & -\pi^{l+1} \rho
\end{array} \right)
\]

\[
L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-\Delta)q^{-2s_2}}{q^{-1} + 1} |\int_{O_F} \pi l^{1+1} + Tr(\sigma t) - \pi^{l+1} \rho N(t)|^{\pi_1} dt + \int_{\partial O_F} |\pi l^{1+1} N(y) + Tr(\sigma y) - \pi^{l+1} \rho|^{\pi_1} dy|
\]

Similarly to previous calculation, the second integrand is constant because \( |\pi l^{1+1} \rho| > |\pi l^{1+1} N(y) + Tr(\sigma y)| \) (see Section 3.1).
\[ L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-\Delta)q^{-2\nu_2}}{q^{-1}+1} \int_{\mathcal{L}} \left[ \pi^{\frac{\nu_2}{2}} + Tr(\mathfrak{H}t) - \pi^{\frac{\nu_1}{2}} \rho N(t) \right]^{s_1} dt + q^{(s+1)(-s_1)-1}. \]

We have that:

\[ \int_{\mathcal{L}} \left[ \pi^{\frac{\nu_2}{2}} + Tr(\mathfrak{H}t) - \pi^{\frac{\nu_1}{2}} \rho N(t) \right]^{s_1} dt = \int_{\mathcal{L}} \left[ \pi^{\frac{\nu_2}{2}} + Tr(\mathfrak{H}t) - \pi^{\frac{\nu_1}{2}} \rho N(t) \right]^{s_1} dt + . \]

So:

\[ L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-\Delta)q^{-2\nu_2}}{q^{-1}+1} \int_{\mathcal{L}} \left[ \pi^{\frac{\nu_2}{2}} + Tr(\mathfrak{H}y) - \pi^{\frac{\nu_1}{2}} \rho N(y) \right]^{s_1} dy + 2q^{(s+1)(-s_1)-1} = \]

\[ \frac{\chi_2(-\Delta)q^{-2\nu_2}}{q^{-1}+1} \int_{\mathcal{L}} \left[ \pi^{\frac{\nu_2}{2}} N\left(\frac{1}{y}\right) + Tr(\mathfrak{H}y) - \pi^{\frac{\nu_1}{2}} \rho \right]^{s_1} dy + 2q^{(s+1)(-s_1)-1} \]

We substitute \( u = \frac{1}{y} \):

\[ L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-\Delta)q^{-2\nu_2}}{q^{-1}+1} \int_{\mathcal{L}} \left[ \pi^{\frac{\nu_2}{2}} N(u) + Tr(\mathfrak{H}u) - \pi^{\frac{\nu_1}{2}} \rho \right]^{s_1} du + 2q^{(s+1)(-s_1)-1}. \]  

(7.9)

We use the identity: \[ \pi^{\frac{\nu_1}{2}} N(u) + Tr(\mathfrak{H}u) = N((\mathfrak{H}^{\frac{\nu_1}{2}} u + \overline{\mathfrak{H}^{\frac{\nu_1}{2}}}) - \pi^{\frac{\nu_1}{2}} \rho \] and set it to Eq(7.9).

\[ L(x, 1, \chi_2, z) = \frac{q\chi_2(-\Delta)q^{-2\nu_2}}{q+1} \int_{\mathcal{L}} \left[ N((\mathfrak{H}^{\frac{\nu_1}{2}} u + \overline{\mathfrak{H}^{\frac{\nu_1}{2}}}) - \pi^{\frac{\nu_1}{2}} \rho \right]^{s_1} du + 2q^{(s+1)(-s_1)-1} = \]
\[ \frac{q\chi_2(-\Delta)q^{-2s_2}}{q+1} \left\{ \int_{\mathcal{F}} |\pi^{\frac{1}{q}} N((\sigma^* u + 1) - \pi^{\frac{1}{q}} - \pi^{\frac{1}{q}} \rho)|^q du + 2q^{(s+1)(-s_1)-1} \right\} \]

\[ = \frac{\chi_2(-\Delta)q^{-2s_2}}{q^{-1} + 1} \left\{ \int_{\mathcal{F}} |\pi^{\frac{1}{q}} N((\sigma^* u + 1) - (1 + \pi^* \rho))|^q du + 2q^{(s+1)(-s_1)-1} \right\} \]

But, as in the RP case, we have: \( N(\sigma^* u + 1) - (1 + \pi^* \rho) = \pi^* \eta, \eta \in \mathcal{O}_{\text{F}} \)

\[ L(x, 1, \chi_2, z) = \frac{\chi_2(-\Delta)q^{-2s_2}}{q^{-1} + 1} \left\{ \int_{\mathcal{F}} |\pi^{\frac{1}{q}} \eta|^q du + 2q^{(s+1)(-s_1)-1} \right\} = \]

\[ \frac{\chi_2(-\Delta)q^{-2s_2}}{q^{-1} + 1} [q^{(s+1)(-s_1)}(1 - q^{-1}) + 2q^{(s+1)(-s_1)-1}] \]

Substitute \( s \mapsto z \): 

\[ L(x, 1, \chi_2, z) = \chi_2(-\Delta)q^{-2s_2 + \frac{z}{2}} [q^{(s+1)(z_1 + z_2)}]. \]

\[ x \sim \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

\[ L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-1)}{q^{s_1 + 1}} \left\{ \int_{\mathcal{F}} |Tr(t)|^{s_1} dt + \int_{\mathcal{O}_{\text{F}}} |Tr(y)|^{s_1} dy \right\} \]

Similarly to the RP case, we use 3.2 to the substitution \( u = Tr(t), v = Tr(y) \). We have:

\[ L(x, 1, \chi_2, s_1, s_2) = \frac{\chi_2(-1)}{q^{s_1 + 1}} \left\{ \int_{\mathcal{F}} |u|^{s_1} du + q' \int_{\mathcal{O}_{\text{F}}} |v|^{s_1} dv \right\} = \]

Calculation on the integrals above results in geometric sums:

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We substitute $s \mapsto z$:
\[
L(x, 1, \chi_2, z) = q \chi_2(-1)(1 - q^{-1})q^{-2sz_2}q^{l(2z_1 + 2z_2)} \left( \frac{q^{2z_1 + 2z_2}}{q^{2z_2} - q^{2z_1}} \right)
\]

We substitute $u = Tr(i), v = Tr(y)$: (See Section 3.2 and Lemma 4.1)

\[
L(x, 1, \chi_2, s_1, s_2) = \frac{q \chi_2(-1)q^{-2sz_2}}{1 + q^{-1}} \left( \int_{\mathcal{O}_E} |Tr(i)|^{s_1} \, dt + \int_{\mathcal{O}_E} |Tr(y)|^{s_1} \, dy \right)
\]

We compute the geometric sums and simplify:
\[ L(x, 1, x_2, s_1, s_2) = \chi_2(-1)q^{-2s_2+1}q^{-2s_1}(1-q^{-1}) \left[ \frac{q^{-(2s_1-1)}}{1-q^{-2s_1}} + \frac{q^{-(2s_1-1)-2}}{1-q^{-2s_1}} \right] = \]

Substitute \( s \mapsto z \):

\[ L(x, 1, x_2, z) = \chi_2(-1)q^\frac{1}{2}q^{2s_2z_1}(1-q^{-1}) \left[ \frac{q^{2z_2}}{q^{2s_1} - q^{2z_1}} + \frac{q^{2z_1}}{q^{2z_2} - q^{2z_1}} \right] = \]

\[ \chi_2(-1)(1-q^{-1})q^{\frac{1}{2}}q^{-2z_2}q^{(2z_1+2z_2)}q^{2s_1+2z_2} \left[ \frac{q^{2s_1} + q^{2z_2}}{q^{2z_2} - q^{2z_1}} \right] \]

\section{Calculation of \( L(x, x^*, x_2, z) \)}

In the following two subsections we prove that on most of the representatives the spherical function vanish.

For this section, \( A_i, B_i, C_i, \ldots \) will denote constants that are resulted from different coordinate transformation.

\subsection{Calculation on non-diagonal representatives}

Recall that for \( x \sim \left( \begin{array}{cc} a & \sigma^\ast \\ \sigma & b \end{array} \right) \), we have from \(7.2\):

\[ L(x, 1, x_2, s_1, s_2) = \frac{\chi_2(\det(x))|\det(x)|^2}{q+1} \left[ \int_{\partial F} [a + Tr(\sigma t) + bN(t)]^{s_1} dt + \int_{\sigma \partial F} [aN(y) + Tr(\sigma^\ast y) + b]^{s_1} dy \right]. \]

We will denote:

\[ I_1 = \int_{\partial F} [a + Tr(\sigma t) + bN(t)]^{s_1} dt \]
\[ I_2 = \int_{\sigma \partial F} [aN(y) + Tr(\sigma^\ast y) + b]^{s_1} dy. \]
8.1.1 Common Representatives for RP and RU

\[ x \sim \begin{pmatrix} 0 & \sigma^a \\ \sigma^a & 0 \end{pmatrix} \]

\[ L(x, \chi^*, \chi_2, s) = \frac{q \chi_2(1)}{q + 1} \int_{O_F} \chi^*(Tr(\varphi^a t))|Tr(\varphi^a t)|^{s_1} dt + \int_{O_F} \chi^*(Tr(\varphi^a t))|Tr(\varphi^a t)|^{s_1} dt \]

After substitution \( u = Tr(\varphi^a t) \) (see Lemma 3.2, Lemma 4.1) we have:

\[ \hat{\chi}^k O_F \chi^*(Tr(\varphi^a t))|Tr(\varphi^a t)|^{s_1} dt = A_1 \int_{\pi\mathcal{O}_E} \chi^*(u)|u|^{s_1} du = A_1 \sum_{j} \int \chi^*(u)|u|^{s_1} du \]

Where \( h \) is defined by \( Tr(\varphi^{a+k}O_F) = \pi^h \mathcal{O}_E \).

But: \( \int_{\pi\mathcal{O}_E^{s_1}} \chi^*(u)|u|^{s_1} du = q^{2j+1-j} \int \chi^*(y)dy = 0 \), since \( \chi^* \) is a non trivial character on \( \mathcal{O}_E^{s_1} \), the integral on each term vanishes, so \( L(x, \chi^*, \chi_2, s) = 0 \)

\[ 2x = \begin{pmatrix} \pi^m \\ 1 \end{pmatrix} - \pi^{s-m} \rho \]

- We show that \( I_1 = 0 \):

\[ I_1 = \int_{O_F} \chi^*(\pi^m + Tr(t) - \pi^{s-m} \rho N(t))|\pi^m + Tr(t) - \pi^{s-m} \rho N(t)|^{s_1} dt \]

Note that \( |Tr(t) - \pi^{s-m} \rho N(t)| < |\pi^m| \) (see Section 3.3) and so the absolute value is constant:

\[ I_1 = A_1 \int_{O_F} \chi^*(\pi^m + Tr(t) - \pi^{s-m} \rho N(t)) dt \]

We use the identity: \( \pi^m + Tr(t) - \pi^{s-m} \rho N(t) = \pi^m + \frac{\pi^{m-j}}{\rho} - \pi^{s-m} \rho N(t - \frac{2m-j}{\rho}) \) to have:
\[ I_1 = \int_{O_F} \chi^* (\pi^m + \frac{\pi^{m-s}}{\rho} - \pi^{s-m}\rho N(t - \frac{\pi^{m-s}}{\rho})) dt \]

Denote by \( A_2 = \chi^* (\frac{\pi^{m-s}}{\rho}) \), then we have:

\[ I_1 = A_2 \int_{O_F} \chi^* (1 + \pi^s \rho - \pi^{s-m}\rho N(t - \frac{\pi^{m-s}}{\rho})) dt = A_2 \int_{O_F} \chi^* (1 + \pi^s \rho - N(\pi^{s-m}\rho t - 1)) dt. \]

We Substitute \( u = \pi^{s-m}\rho t - 1 \), we have:

\[ I_1 = A_3 \int_{1 + \pi^{2(s-m)}O_F} \chi^* (1 + \pi^s \rho - N(u)) du. \]

We substitute \( v = N(u) \), note that \( N(1 + \sigma^{2(s-m)}O_F) = 1 + \pi^{\frac{s}{2}-m+s}O_E \) (see 3.1). We make use of Lemma 4.1:

we have

\[ I_1 = A_4 \int_{1 + \pi^{2(s-m)}O_F} \chi^* [(1 + \pi^s \rho) - t] dt. \]

We substitute \( s = (1 + \pi^s \rho) - t \). Note that \( \pi^s \rho - \pi^{\frac{s}{2}-m+s}O_E = \pi^s \rho (1 + \pi^{s-m}O_E) \), so for the substitution \( \pi^s \rho y = s \) we get:

\[ I_1 = A_4 \int_{\pi^s \rho (1 + \pi^{s-m}O_E)} \chi^* (s) ds = A_5 \int_{1 + \pi^{\frac{s}{2}-m}O_E} \chi^* (y) dy. \]

Because \( 1 + \pi^{\frac{s}{2}-m}O_F \) contains non norm elements, the character \( \chi^* \) is a non trivial character of the group \( 1 + \pi^{\frac{s}{2}-m}O_E \) and the integral vanish.

• Now we show that:
\[ I_2 = \int_{O_F} ^{O} \chi^*(\pi^m N(y) + Tr(y) - \pi^{s-m} \rho) |\pi^m N(y) + Tr(y) - \pi^{s-m} \rho|^{s_1} dy = 0 \]

We have already shown in our calculation of this representative in Section 7.2 that:

\[ I_2 = \int_{O} \chi^*(\pi^m N(y) + Tr(y) - \pi^{s-m} \rho) |\pi^m N(y) + Tr(y) - \pi^{s-m} \rho|^{s_1} dy = \]

\[ B_1 \int_{1+\mathfrak{m}^{s+1}O_E} \chi^*(N(t) - (1+\pi^s \rho)) |N(t) - (1+\pi^s \rho)|^{s_1} dt \]

The Norm induces an homomorphism:

\[ \tilde{N} : \frac{1 + \mathfrak{m}^{s+1}O_F}{1 + \mathfrak{m}^{s+1}O_F} \rightarrow \frac{1 + \pi^{m+1}O_E}{1 + \pi^{s+1}O_E} \]

On each coset the \(|N(t) - (1+\pi^s \rho)|^{s_1}\) is a constant function. (By our corollaries in Section 3.1)

Note that \(q^{-s} < |N(t) - (1+\pi^s \rho)|_E \leq q^{-m-1}\) since \(N(t) \in 1 + \pi^{m+1}O_E\) and \(1 + \pi^s \rho \notin N(F^*)\).

We integrate on the different coset of the form \(a_i (1 + \mathfrak{m}^{s+1}O_F) \subset 1 + \mathfrak{m}^{s+1}O_F\)

\[ I_2 = \int_{1+\mathfrak{m}^{s+1}O_F} \chi^*(N(t) - (1+\pi^s \rho)) |N(t) - (1+\pi^s \rho)|^{s_1} dy = \sum_i B_i \int_{a_i (1+\mathfrak{m}^{s+1}O_F)} \chi^*(N(t) - (1+\pi^s \rho)) dt \]

Now we show that each integral in the different terms vanish:

\[ I_{2,i} = \int_{a_i (1+\mathfrak{m}^{s+1}O_F)} \chi^*(N(t) - (1+\pi^s \rho)) dt. \]

We Substitute \(u = N(t)\) (note that \(N(1 + \mathfrak{m}^{s+1}O_F) = 1 + \pi^{s+1}O_E\), by making use of Lemma 4.1 we get:

\[ I_{2,i} = C_i \int_{N(a_i)(1+\pi^{s+1}O_E)} \chi^*(u - (1+\pi^s \rho)) du \]
We substitute $\eta = u - (1 + \pi \rho)$. Since we know that $q^{-s} < |\eta|_E \leq q^{-m-1}$, we deduce that we can present the integral in the following form:

$$I_{2,i} = C_2 \int_{\pi^k \zeta_i + \pi^{k+1} O_E} \chi^*(\eta) d\eta,$$

where $\zeta_i \in O_F^*$ and $m + 1 \leq k < s$. We substitute again $y = \pi^{-k} \zeta_i^{-1} \eta$ to get:

$$I_{2,i} = C_3 \int_{1 + \pi^{k+1} O_E} \chi^*(y) dy = 0.$$

Since $1 + \pi^{k+1} O_E$ contains a non-norm elements, $\chi^*$ is a non-trivial character and the integral vanishes.

3. $x = \begin{pmatrix} \pi^m & \alpha \\ \overline{\alpha} & -\pi^{-m} \rho \end{pmatrix}$
   A similar proof to the previous representative show that $L(x, \chi^*, s_1, s_2) = 0$.

4. $x = \begin{pmatrix} \pi^m & 1 \\ 1 & 0 \end{pmatrix}$
   
   - We prove that $I_1 = \int_{O_F} \chi^*(\pi^m + \text{Tr}(t)) |\pi^m + \text{Tr}(t)|^{s_1} dt = 0$

We substitute $u = \text{Tr}(t)$. We know that (see Section [3.3]) $\text{Tr}(O_F) = \pi^h O_F$, $\frac{h}{2} \leq h \leq \frac{s}{2} + 1$:

$$\int_{O_F} \chi^*(\pi^m + \text{Tr}(t)) |\pi^m + \text{Tr}(t)|^{s_1} dt = A_1 \int_{\pi^h O_E} \chi^*(\pi^m + u) |\pi^m + u|^{s_1} du.$$

Since $m < h$ the absolute value is constant. We have:

$$I_1 = A_2 \int_{\pi^h O_E} \chi^*(\pi^m + t) dt = A_3 \int_{1 + \pi^{h-m} O_E} \chi^*(t) dt = 0$$

Since $h - m < s + 1$ we have that the group $1 + \pi^{h-m} O_E$ contains a non norm elements and hence $\chi^*$ is a non-trivial character and the integral vanishes.
• We show that $I_2 = 0$. We have already shown in Section 7.2.1

$$I_2 = \int_{1 + \mathfrak{O}^{2m+1}_F} \chi^*(N(y) + Tr(y)) [\pi^m N(y) + Tr(y)] dy = B_1 \int_{1 + \mathfrak{O}^{2m+1}_F} \chi^*(N(t) - 1)|N(t) - 1| dt.$$

For every $2m + 1 \leq k$ we show that the integral vanish on the set $N^{-1}(1 + \pi^k \mathfrak{O}_E^s) \cap (1 + \mathfrak{O}^{2m+1}_F)$:

For $k < s + 1$ the set $N^{-1}(1 + \pi^k \mathfrak{O}_E^s) \cap (1 + \mathfrak{O}^{2m+1}_F)$ can be represented as a union of cosets:

$$N^{-1}(1 + \pi^k \mathfrak{O}_E^s) \cap (1 + \mathfrak{O}^{2m+1}_F) = \bigcup_i a_i [1 + \mathfrak{O}^{s+1}_F].$$

On each coset the absolute value is constant ans so:

$$\int_{N^{-1}(1 + \pi^k \mathfrak{O}_E^s) \cap (1 + \mathfrak{O}^{2m+1}_F)} \chi^*(N(t) - 1)|N(t) - 1| dt = \sum_i C_i \int_{a_i [1 + \mathfrak{O}^{s+1}_F]} \chi^*(N(t) - 1) dt$$

We substitute $N(t) - 1 = u$ (use Lemma 4.1 and Section 3.1), then on each term in the sum:

$$\int_{a_i [1 + \mathfrak{O}^{s+1}_F]} \chi^*(N(t) - 1) dt = A_i \int_{N(a_i) [1 + \mathfrak{O}^{s+1}_E]} \chi^*(u) du$$

But we know that $|u|_E = q^{-k}$ and so we can represent the domain of integration as:

$$N(a_i) [1 + \mathfrak{O}^{s+1}_E] - 1 = \pi^k \eta_i + \pi^{s+1}_E, \eta_i \in \mathfrak{O}_E^s$$

We have:

$$\int_{N(a_i) [1 + \mathfrak{O}^{s+1}_E]} \chi^*(u - 1) du = \int_{\pi^k \eta_i + \pi^{s+1}_E} \chi^*(u) du$$

We substitute $s = \pi^{-k} \eta_i^{-1} u$ and get:

$$\int_{\pi^k \eta_i + \pi^{s+1}_E} \chi^*(u) du = \int_{1 + \pi^{s+1}_E} \chi^*(s) ds = 0$$

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Since \(1 + \pi^{s+1-k}O_E\) contains a non-norm elements \(\chi^*\) is a non-trivial character and the integral vanish.

For \(s+1 \leq k\) we can represent \(N^{-1}(1 + \pi^kO_E) \cap 1 + \mathfrak{o}^{2m+1}O_F = \cup a_i(1+\mathfrak{o}^{k+1}O_F)\)

Now we integrate on coset of the form \(a_i(1+\mathfrak{o}^{k+1}O_F)\), similarly the absolute value is constant:

\[
\int_{N^{-1}(1+\pi^kO_E) \cap (1+\mathfrak{o}^{2m+1}O_F)} \chi^*(N(t) - 1) dt = \sum_i \int_{a_i(1+\mathfrak{o}^{k+1}O_F)} \chi^*(N(t) - 1) dt.
\]

We substitute \(N(t) - 1 = u\). On each term :

\[
\int_{a_i(1+\mathfrak{o}^{k+1}O_F)} \chi^*(N(t) - 1) dt = \int_{N(a_i)(1+\pi^kO_E)} \chi^*(u) du
\]

But \(|u|_E = q^{-k}\) so one can represent the domain of integration as:

\(N(a_i)(1 + \pi^{k+1}O_E) = \pi^k \eta_i + \pi^{k+1}O_E, \ \eta_i \in O_E^*\)

We have by making the substitution \(y = \pi^{-k} \eta_i^{-1}\):

\[
\int_{\pi^k \eta_i + \pi^{k+1}O_E} \chi^*(u) du = D \int_{1+\pi O_E} \chi^*(y) dy = 0.
\]

We showed that \(\int_{N^{-1}(1+\pi^kO_E) \cap (1+\mathfrak{o}^{2m+1}O_F)} \chi^*(N(t) - 1) |N(t) - 1|^s dt = 0\) for every \(2m+1 \leq k\), and so :

\[
\int_{1+\mathfrak{o}^{2m+1}O_F} \chi^*(N(t) - 1) |N(t) - 1|^s dt = 0 \Rightarrow I_2 = 0.
\]

5.

\(x = \left( \begin{array}{cc} \pi^m & \mathfrak{o} \\ \mathfrak{o} & 0 \end{array} \right)^{-}\)

A similar proof to the previous will show that \(L(x, \chi^*, s_1, s_2) = 0\).
8.1.2 RP representative:

\[ x = \begin{pmatrix} \pi^2 & 1 \\ 1 & -\pi^2 \rho \end{pmatrix} \] (RP)

\[ L(x, 1, \chi_2, z) = \frac{\chi_2(\alpha^2)}{q^{-1} + 1} \int_{O_F} \chi^*(\pi^2 t + Tr(t) - \pi^2 \rho N(t)) [\pi^2 t + Tr(t) - \pi^2 \rho N(t)] \, dt + \int_{\mathcal{S}_0} \chi^*(\pi^2 N(y) + Tr(y) - \pi^2 \rho) [\pi^2 N(y) + Tr(y) - \pi^2 \rho \] \, dy. \]

- We show that \( I_1 = 0 \):

\[ I_1 = \int_{O_F} \chi^*(\pi^2 t + Tr(t) - \pi^2 \rho N(t)) [\pi^2 t + Tr(t) - \pi^2 \rho N(t)] \, dt \quad (8.1) \]

Setting the following identity to (8.1):

\[ -\rho \pi^2 N(t) + Tr(u) = (\sigma^2 \sigma t + \alpha^{-1} \sigma^{-2}) [\sigma^2 \sigma t + \alpha^{-1} \sigma^{-2}] + \frac{\pi^2}{\rho} = -\frac{\pi^2}{\rho} N(\pi^2 \rho t + 1) + \frac{\pi^2}{\rho} \]

\[ I_1 = \int_{O_F} \chi^*(\pi^2 t - \frac{\pi^2}{\rho} N(\pi^2 \rho t + 1) + \frac{\pi^2}{\rho}) [\pi^2 t - \frac{\pi^2}{\rho} N(\pi^2 \rho t + 1) + \frac{\pi^2}{\rho}] \, dt \]

Simplifying this expression, we have:

\[ I_1 = A \int_{O_F} \chi^*(1 + \pi^2 \rho - N(\pi^2 \rho t + 1)) [1 + \pi^2 \rho - N(\pi^2 \rho u + 1)] \, dt. \]

We substitute \( u = 1 + \pi^2 \rho t \):

\[ I_1 = A \int_{1 + \pi^2 O_F} \chi^*(1 + \pi^2 \rho - N(u)) [1 + \pi^2 \rho - N(u)] \, dt. \]

Since \( N(u) \in 1 + \pi^2 O_F \) and \( 1 + \pi^2 \rho \notin N(F^*) \) we have that \( |N(u) - (1 + \pi^2 \rho)| = q^{-2s} \).

\[ I_1 = B \int_{1 + \pi^2 O_F} \chi^*(1 + \pi^2 \rho - N(u)) du. \]
Substitute $ν = N(u)$:

$$I_1 = C \int_{1+\pi^2O_E} \chi^*(1 + \pi^\rho) - ν)dv.$$  

Substitute $y = 1 + \pi^\rho - ν$:

$$I_1 = D \int_{\pi^\rho + \pi^2O_E} \chi^*(y)dy.$$  

Substitute $t = \pi^{-1}\rho^{-1}y$:

$$I_1 = E \int_{1+\pi^\rho} \chi^*(y)dy = 0.$$  

We show that $I_2 = 0$, since the absolute value in the integral is constant, we have that:

$$I_2 = \int_{\partial O} \chi^*(\pi^\frac{z}{2}N(y) + Tr(y) - \pi^\frac{z}{2}\rho)|\pi^\frac{z}{2}N(y) + Tr(y) - \pi^\frac{z}{2}\rho|^s dy =$$

$$A_2 \int_{\partial O} \chi^*(\pi^\frac{z}{2}N(y) + Tr(y) - \pi^\frac{z}{2}\rho) dy$$

We use the identity:

$$\pi^\frac{z}{2}N(y) + Tr(y) = (\sigma^{-\frac{z}{2}} + \sigma^{\frac{z}{2}}y)(\sigma^{-\frac{z}{2}} + \sigma^{\frac{z}{2}}y) - \pi^{-\frac{z}{2}} = N(\sigma^{-\frac{z}{2}} + \sigma^{\frac{z}{2}}y) - \pi^{-\frac{z}{2}}.$$  

Setting it into the integral we get:

$$\int_{\partial O} \chi^*(\pi^\frac{z}{2}N(y) + Tr(y) - \pi^\frac{z}{2}\rho) dy = \int_{\partial O} \chi^*(N(\sigma^{-\frac{z}{2}} + \sigma^{\frac{z}{2}}y) - \pi^{-\frac{z}{2}} - \pi^\frac{z}{2}\rho) dy =$$

$$\int_{\partial O} \chi^*(N(1 + \sigma^{\frac{z}{2}}y) - (1 + \pi^\rho)) dy$$

We substitute $t = 1 + \sigma^{\frac{z}{2}}y$ to get:

$$I_2 = A_3 \int_{1+\sigma^{\frac{z}{2}}O_F} \chi^*(N(t) - (1 + \pi^\rho)) dt$$

We substitute $u = N(x)$ and use Lemma [4.1]:

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\[ I_2 = A_4 \int_{1+\pi^{s_1}O_E} \chi^*(u - (1 + \pi^r \rho))du \] (8.2)

We substitute \( v = u - (1 + \pi^r \rho) \)

\[ I_2 = A_5 \int_{-\pi^r \rho + \pi^{s_1}O_E} \chi^*(v)dv. \]

We substitute \(-\pi^r \rho \cdot \zeta = v\) to get:

\[ I_2 = A_6 \int_{1+\pi O_E} \chi^*(\zeta)d\zeta = 0. \]

Since \(1 + \pi O_E\) contains a non-norm, \(\chi^*\) is a non-trivial character and the integral vanish.

### 8.1.3 RU representative:

\[ x \sim \left( \begin{array}{c|c}
\frac{\pi^{s_1}}{\sigma} & \sigma \\
\hline
\pi^{s_1} \rho & \frac{\pi^{s_1}}{\sigma}
\end{array} \right) \]

Showing that \(L(x, \chi^*, s_1, s_2) = 0\) is similar to the RP equivalent representative in Subsection 8.1.2

### 8.2 Calculating \(L(\chi^*, \chi_2, x, z)\) on the diagonal representatives.

\[ x \sim \left( \begin{array}{c|c}
\pi^{s_1} \epsilon_1 & \\
\hline
\pi^{s_2} \epsilon_2 & 
\end{array} \right) \]

By Lemma 7.2

\[ L(x, \chi^*, \chi_2, s_1, s_2) = \frac{q^{\lambda_1-\lambda_2}q^{\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2} \chi_2(\epsilon_1 \epsilon_2) \chi^*(\epsilon_2) [I_1 + I_2]}{q^{-1} + 1}. \]

Where:

\[ I_1 = \int_{x \in O_F} \chi^*(\pi^{s_1} \epsilon_2 + N(x))|\pi^{s_1} - \lambda_2 \epsilon_2| \epsilon_2 + N(x)|^{s_1}dx \]

\[ I_2 = \int_{y \in \sigma O_F} \chi^*(1 + \epsilon_1 \pi^{s_1} \lambda_2 \epsilon_2 N(y))|1 + \epsilon_1 \pi^{s_1} \lambda_2 N(y)|^{s_1}dy. \]
8.2.1 The representatives with $\lambda_1 - \lambda_2 < s$

Note that

$$I_2 = C \int_{y \in O_F} \chi^*(1 + \frac{\ell_1}{\ell_2} \pi^{\lambda_1 - \lambda_2} N(y)) dy = D \int_{y \in O_F} \chi^*(1 + \frac{\ell_1}{\ell_2} \pi^{\lambda_1 - \lambda_2 + 1} N(y)) dy$$

By Lemma 4.3 this integral vanishes.

Showing $I_1 = 0$:

$$I_1 = \int_{t \in O_F} \chi^*(\pi^{\lambda_1} \ell_1 + \pi^{\lambda_2} \ell_2 N(t)) |\pi^{\lambda_1} \ell_1 + \pi^{\lambda_2} \ell_2 N(t)|^{\leq s} dt =$$

$$q^{-2\lambda_3 t_1} \int_{x \in O_F} \chi^*(\pi^{\lambda_1 - \lambda_2} \frac{\ell_1}{\ell_2} + N(t)) |\pi^{\lambda_1 - \lambda_2} \frac{\ell_1}{\ell_2} + N(t)|^{\leq s} dt$$

This is most complicated and tricky, we prove it with careful steps:

**Step 1:**

We Show that If $0 \leq m < s$

$$I_1 = A \int_{x \in O_F} \chi^*(\pi^{m} \frac{\ell_1}{\ell_2} + N(t)) |\pi^{m} \frac{\ell_1}{\ell_2} + N(t)|^{\leq s} dx = 0$$

Decompose the integral to $\cup_i \pi^{i} O_F$:

$$I_1 = A \sum_{i} \int_{\pi^{i} O_F} \chi^*(\pi^{m} \frac{\ell_1}{\ell_2} + N(t)) |\pi^{m} \frac{\ell_1}{\ell_2} + N(t)|^{\leq s} dt$$

For $i \leq m$ the absolute value is constant, so on each term. We substitute $t = u^{-1}$ to get:

$$\int_{\pi^{i} O_F} \chi^*(\pi^{m-i} \frac{\ell_1}{\ell_2} + N(t)) dt = B \int_{\pi^{i} O_F} \chi^*(1 + \pi^{m-i} \frac{\ell_1}{\ell_2} N(u)) du$$

By Lemma 4.4 this integral vanishes.
So we have:

\[ I_1 = A \sum_{i \geq m} \int \chi^* (\pi^m \frac{E_1}{E_2} + N(t)) |\frac{\pi^m E_1}{E_2} + N(t)|^s dt \]

On the space \( \mathfrak{m}^{m+1} O_F \) the absolute value is constant again, Substitute \( u = \mathfrak{m}^{-m-1} t \):

\[ \int_{\mathfrak{m}^{m-1} O_F} \chi^* (\pi^m \frac{E_1}{E_2} + N(t)) dt = C \int_{O_F} \chi^* (\frac{E_1}{E_2} + \pi N(u)) du = 0 \]

By Lemma 4.3

So we conclude that:

\[ I_1 = A \int_{\mathfrak{m}^m O_F^*} \chi^* (\pi^m \frac{E_1}{E_2} + N(t)) |\frac{\pi^m E_1}{E_2} + N(t)|^s dt \]

Substitute \( u = \mathfrak{m}^{-m-1} t \) to get:

\[ I_1 = C \int_{\mathfrak{m}^m O_F^*} \chi^* (\frac{E_1}{E_2} + N(t)) |\frac{E_1}{E_2} + N(t)|^s dt \quad (8.3) \]

Observe that the integrand is not a locally constant function.

**Step 2:**

We compute the integral in (8.3) on cosets of the group: \( \frac{O_F^*}{\mathfrak{m}^{m+1} O_F} \)

\[ \int_{O_F} \chi^* (\frac{E_1}{E_2} + N(t)) |\frac{E_1}{E_2} + N(t)|^s dt = \sum_{a_i(1+\mathfrak{m} O_F)} \int_{a_i(1+\mathfrak{m} O_F)} \chi^* (\frac{E_1}{E_2} + N(t)) |\frac{E_1}{E_2} + N(t)|^s dt. \]

We substitute on each term \( u = a_i^{-1} t \).

Note that the integrand is invariant to the substitution, it follows that:
\[ I_1 = D \int_{1+\mathfrak{O}_F} \chi^* \left( \frac{\varepsilon_1}{\varepsilon_2} + N(t) \right) \frac{\varepsilon_1}{\varepsilon_2} + N(t) |^{s_1} dt \]

Let \( \zeta \in 1 + \mathfrak{O}_F \), we prove:

\[ J = \int_{1+\mathfrak{O}_F} \chi^* (\zeta + N(t)) |\zeta + N(t)|^{s_1} dx = 0. \]

We decompose the integration on spaces such that the absolute value would be constant.

We have that (up to a measure zero subset):

\[ N(1 + \mathfrak{O}_F) \subseteq 1 + \pi O_E = \bigcup_{j \geq 1} (1 + \pi^j O_E^*). \]

Observe that:

\[ \bigcup_{j \geq 1} (1 + \pi^j O_E^*) = \bigcup_{j \geq 1} (-\zeta + \pi^j O_E^*). \]

Now we calculate \( J \) on the spaces \( N^{-1}(-\zeta + \pi^j O_E^*) \):

\[ J = \sum_{j=1}^{\infty} \int_{N^{-1}(1+\pi/O_E^*)} \chi^* (\zeta + N(t)) |\zeta + N(t)|^{s_1} dt \]

If \( 0 < j \leq s \), the inverse image \( N^{-1}(-\zeta + \pi^j O_E^*) \) is a disjoint union of coset \( \bigcup_k \mathfrak{a}_{k,j}(1+\mathfrak{O}_F) \).

For \( s < j \), \( N^{-1}(-\zeta + \pi^j O_E^*) \) can be represented as union of \( b_{k,j}(1+\mathfrak{O}_F) \).

We compute the integral on each coset:

\[ J = \sum_{j=1}^{s} \int_{N^{-1}(1+\mathfrak{O}_F)} \chi^* (\zeta + N(t)) |\zeta + N(t)|^{s_1} dt + \]

\[ \sum_{j>s} \int_{N^{-1}(-\zeta + \pi/O_E^*)} \chi^* (\zeta + N(t)) |\zeta + N(t)|^{s_1} dt = \]

\[ \sum_{j=1}^{s} \sum_{k} a_{k,j}(1+\mathfrak{O}_F) \int \chi^* (\zeta + N(t)) |\zeta + N(t)|^{s_1} dt + \]

\[ \sum_{j>s} \sum_{k} b_{k,j}(1+\mathfrak{O}_F) \int \chi^* (\zeta + N(t)) |\zeta + N(t)|^{s_1} dt. \]
Since for every \( t \in a_{k,j}(1 + \mathcal{O}^{s+1}O_F) \) we have that \(|\zeta + N(t)|_E = q^{-j}\), after the substitution: \( \nu = \zeta + N(t) \), each one of the domains of integration \( a_{k,j}(1 + \mathcal{O}^{s+1}O_F) \) can be represented:

\[
\pi^j \eta_{i,j} + \pi^{s+1}O_E, \quad \eta_{i,j} \in O_F^*
\]

And so we have that:

\[
\int_{a_{k,j}(1 + \mathcal{O}^{s+1}O_F)} \chi^*(\zeta + N(t))|\zeta + N(t)|_{\mathcal{O}^{s+1}} \, dt = D_{i,j} \int_{\pi^j \eta_{i,j} + \pi^{s+1}O_E} \chi^*(\nu)|\nu|_{\mathcal{O}^{s+1}} \, d\nu.
\]

Substitute \( u = \pi^{-j} \eta_{i,j}^{-1} \nu \) to get:

\[
\int_{\pi^j \eta_{i,j} + \pi^{s+1}O_E} \chi^*(\nu)|\nu|_{\mathcal{O}^{s+1}} \, d\nu = E_{i,j} \int_{x \in 1 + \pi^{s+1-j}O_E} \chi^*(t) \, dt = 0
\]

Because \( \chi^* \) is a non trivial character on \( 1 + \pi^{s+1-j}O_E \).

A similar trick will show that the integral:

\[
\int_{b_{k,j}(1 + \mathcal{O}^{s+1}O_F)} \chi^*(\zeta + N(t))|\zeta + N(t)|_{\mathcal{O}^{s+1}} \, dt = 0.
\]

We deduce that: \( J = 0 \Rightarrow I_1 = 0 \)

In conclusion \( I_1 = I_2 = 0 \Rightarrow L(x, \chi^*, \chi_2, z) = 0 \).

### 8.2.2 The representative with \( \lambda_1 - \lambda_2 = s \):

Suppose \( \lambda_1 - \lambda_2 = s \)

In this case the

\[
I_2 = \int_{y \in \mathcal{O}O_F} \chi^*(1 + \frac{\epsilon_1}{\epsilon_2} \pi^{\lambda_1 - \lambda_2}N(y))|1 + \frac{\epsilon_1}{\epsilon_2} \pi^{\lambda_1 - \lambda_2}N(y)|_{\mathcal{O}^{s+1}} \, dy \neq 0
\]

but we have:

\[
I_1 + I_2 = I_1 - \int_{O_F^*} \chi^*(\pi^{\lambda_1 - \lambda_2} + N(t)) \, dt + \int_{O_F^*} \chi^*(\pi^{\lambda_1 - \lambda_2} \frac{\epsilon_1}{\epsilon_2} + N(t)) \, dt + I_2
\]

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Note that by substituting \( t \mapsto t^{-1} \), we have:

\[
\int_{\Omega_F} \chi^*(\pi^{\lambda_1 - \lambda_2} \frac{\epsilon_1}{\epsilon_2} + N(t)) dt = \int_{\Omega_F} \chi^*(\pi^{\lambda_1 - \lambda_2} \frac{\epsilon_1}{\epsilon_2} N(t) + 1) dt.
\]

By Lemma 4.3, the sum of the integrals vanish:

\[
\int_{\Omega_F} \chi^*(\pi^{\lambda_1 - \lambda_2} \frac{\epsilon_1}{\epsilon_2} N(t) + 1) dt + I_2 = \int_{\Omega_F} \chi^*(1 + \pi^{\lambda_1 - \lambda_2} \frac{\epsilon_1}{\epsilon_2} N(t)) dt = 0.
\]

Hence:

\[
I_1 + I_2 = I_1 - \int_{\Omega_F} \chi^*(\pi^{\lambda_1 - \lambda_2} \frac{\epsilon_1}{\epsilon_2} + N(t)) dt.
\]

But:

\[
I_1 - \int_{\Omega_F} \chi^*(\pi^{\lambda_1 - \lambda_2} \frac{\epsilon_1}{\epsilon_2} + N(t)) dt = C \int_{\omega_{\Omega_F}} |\pi^{\lambda_1 - \lambda_2} \frac{\epsilon_1}{\epsilon_2} + N(t)| \chi^*(\pi^{\lambda_1 - \lambda_2} - 1 \frac{\epsilon_1}{\epsilon_2} + N(t)) dt =
\]

\[
D \int_{\Omega_F} |\pi^{\lambda_1 - \lambda_2 - 1} \frac{\epsilon_1}{\epsilon_2} + N(t)| \chi^*(\pi^{\lambda_1 - \lambda_2 - 1} - \frac{\epsilon_1}{\epsilon_2} + N(t)) dt
\]

The last integral vanish by Subsection 8.2.1 \((\lambda_1 - \lambda_2 < s)\).

8.3 The diagonal representatives with \( \lambda_1 - \lambda_2 > s \)

This is the only case where the spherical function do not vanish and \( \chi_1 = \chi^*:\)

\[
L(x, \chi^*, \chi_2, s_1, s_2) = \frac{q^{\frac{\lambda_1 - \lambda_2}{2}} q^{\lambda_1 \lambda_2 + \frac{\lambda_1 \lambda_2}{2}} \chi_2(\epsilon_1 \epsilon_2) \chi^*(\epsilon_2)}{q^{-1} + 1} [I_1 + I_2]
\]

\[
I_1 = \int_{\Omega_F} \chi^*(\pi^{\lambda_1 - \lambda_2} \frac{\epsilon_1}{\epsilon_2} + N(t)) |\pi^{\lambda_1 - \lambda_2} \frac{\epsilon_1}{\epsilon_2} + N(t)|^{s_1} dt
\]

\[
I_2 = \int_{y \in \omega_{\Omega_F}} \chi^*(1 + \frac{\epsilon_1}{\epsilon_2} \pi^{\lambda_1 - \lambda_2} N(y)) |1 + \frac{\epsilon_1}{\epsilon_2} \pi^{\lambda_1 - \lambda_2} N(y)|^{s_1} dy
\]
Since \(1 + \frac{t^s}{s^2} \pi^{\lambda_1 - \lambda_2} N(y)\) is a norm if \(y \in \sigma O\), it could be shown easily that \(I_2 = q^{-1}\).

We integrate \(I_1\) on the spaces \(\bigcup_{i \geq 0} \sigma_i O^i\): 

- If \(\lambda_1 - \lambda_2 - s > i\) and \(t \in \sigma_i O^i\) then \(\pi^{\lambda_1 - \lambda_2} \frac{t^s}{s^2} + N(t)\) is a norm and

\[
\int_{\sigma_i O^i} \chi^*(\pi^{\lambda_1 - \lambda_2} \frac{t^s}{s^2} + N(t))|\pi^{\lambda_1 - \lambda_2} \frac{t^s}{s^2} + N(t)|^{\sigma_1} dx = q^{-2i-1}(1 - q^{-1})
\]

So

\[
I_1 = \sum_{j=0}^{\lambda_1 - \lambda_2 - s - 1} (1 - q^{-1})q^{-2j+1} + \sum_{i \geq \lambda_1 - \lambda_2 - s} \int_{\sigma_i O^i} \chi^*(\pi^{\lambda_1 - \lambda_2} \frac{t^s}{s^2} + N(t))|\pi^{\lambda_1 - \lambda_2} \frac{t^s}{s^2} + N(t)|^{\sigma_1} dt
\]

- For \(i = \lambda_1 - \lambda_2 - s\), we have that if \(t \in \sigma_i O^i\) then \(|\pi^{\lambda_1 - \lambda_2} \frac{t^s}{s^2} + N(t)|^{\sigma_1} = q^{-2i}\):

\[
\int_{x \in \sigma O^i} \chi^*(\pi^{\lambda_1 - \lambda_2} \frac{t^s}{s^2} + N(t))|\pi^{\lambda_1 - \lambda_2} \frac{t^s}{s^2} + N(t)|^{\sigma_1} dt = q^{-2i+1} \int \chi^*(\pi^{\lambda_1 - \lambda_2} \frac{t^s}{s^2} + N(t)) dt
\]

Substituting \(u = \sigma_i t^{-1}\), we get:

\[
q^{-2i-1} \int_{x \in \sigma O^i} \chi^*(\pi^{\lambda_1 - \lambda_2} \frac{t^s}{s^2} + N(t)) dt = q^{-2i-1} \int \chi^*(1 + \pi^t \frac{t^s}{s^2} N(u)) du. \tag{8.4}
\]

By Lemma 4.3

\[
\int_{O^i O^i} \chi^*(1 + \pi^t \frac{t^s}{s^2} N(t)) dt = 0 \Rightarrow \int \chi^*(1 + \pi^t \frac{t^s}{s^2} N(t)) dt = - \int \chi^*(1 + \pi^t \frac{t^s}{s^2} N(t)) dt = -q^{-1}
\]

(8.5)
Substituting (8.5) to Eq 8.4, we get that for 

\[ i = \lambda_1 - \lambda_2 - s \]:

\[
\int_{\mathcal{O}_F} \mathcal{X}^\ast (\pi^{\lambda_1 - \lambda_2 - s} \mathcal{E}_1 + N(t))|\pi^{\lambda_1 - \lambda_2 - s} \mathcal{E}_1 + N(t)|^s dt = -q^{-2s}i^{-1}. 
\]

So:

\[
I_1 = \sum_{j=0}^{\lambda_1 - \lambda_2 - s - 1} (1 - q^{-1})q^{-2js} - q^{-2(\lambda_1 - \lambda_2 - s)j} - q^{-2(\lambda_1 - \lambda_2 - s)j} = \sum_{j=0}^{\lambda_1 - \lambda_2 - s - 1} (1 - q^{-1})q^{-2js}.
\]

Note that:

\[
C_1 \int_{\mathcal{O}_F} \mathcal{X}^\ast (\pi^{\lambda_1 - \lambda_2 - s} \mathcal{E}_1 + N(t))|\pi^{\lambda_1 - \lambda_2 - s} \mathcal{E}_1 + N(t)|^s dt = 0.
\]

We apply the case of \( \lambda_1 - \lambda_2 < s \) to get that:

\[
C_2 \int_{\mathcal{O}_F} \mathcal{X}^\ast (\pi^{s-1} \mathcal{E}_1 + N(t))|\pi^{s-1} \mathcal{E}_1 + N(t)|^s dt = 0.
\]

Altogether:

\[
I_1 = \int_{\mathcal{O}_F} \mathcal{X}^\ast (\pi^{\lambda_1 - \lambda_2 - s} \mathcal{E}_1 + N(t))|\pi^{\lambda_1 - \lambda_2 - s} \mathcal{E}_1 + N(t)|^s dt = \sum_{j=0}^{\lambda_1 - \lambda_2 - s - 1} (1 - q^{-1})q^{-2js}.
\]

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Recall that:

\[
L(x, \chi^*, \chi_2, s_1, s_2) = \frac{q^{\lambda_1 - \lambda_2} q^{\lambda_1 z_2 + \lambda_2 z_1} \chi_2(e_1 e_2) \chi^*(e_2)}{q^{-1} + 1} I_1
\]

After simplifying:

\[
L(x, \chi^*, \chi_2, s_1, s_2) = \frac{q^{\lambda_1 - \lambda_2} q^{\lambda_1 z_2 + \lambda_2 z_1} \chi_2(e_1 e_2) \chi^*(e_2)}{q^{-1} + 1} \times [q^{-1} + (1 - q^{-1}) \frac{q^{2z_2} - q^{(\lambda_1 - \lambda_2 - s)(2z_1 - 2z_2) + 2z_2}}{q^{2z_1} - q^{2z_2}} - q^{(\lambda_1 - \lambda_2 - s)(2z_1 - 2z_2) - 1}]\]

After further simplification:

\[
L(x, \chi^*, \chi_2, s_1, s_2) = \frac{q^{\lambda_1 - \lambda_2} q^{\lambda_1 z_2 + \lambda_2 z_1} \chi_2(e_1 e_2) \chi^*(e_2)}{q^{-1} + 1} \times \left[ -\frac{q^{2s_1} - q^{2s_2}}{q^{2z_1} - q^{2z_2}} + \frac{q^{2s_2} + q^{(\lambda_1 - \lambda_2 - s)(2z_1 - 2z_2) + 2z_2}}{q^{2z_1} - q^{2z_2}} + \frac{q^{(\lambda_1 - \lambda_2 - s)(2z_1 - 2z_2) - 1 + 2z_1}}{q^{2z_1} - q^{2z_2}} \right] \]

\[
= \frac{q^{\lambda_1 - \lambda_2} \chi_2(e_1 e_2) \chi^*(e_2) q^{2z_2}(q^{2z_2} - q^{2z_1 - 1})}{q^{-1} + 1} \times \frac{(q^{\lambda_1 z_1 + \lambda_2 z_2 - 2z_1 - 2z_2} - q^{\lambda_1 z_2 + \lambda_2 z_1 - 2z_2})}{q^{2z_1} - q^{2z_2}}
\]

We represent the function in a more symmetric form:

\[
L(x, \chi^*, \chi_2, s_1, s_2) = \frac{q^{\lambda_1 - \lambda_2} \chi^*(e_2) \chi_2(e_1 e_2) q^{2z_2}(q^{2z_2} - q^{2z_1 - 1})}{1 + q^{-1}} \times \sum_{\sigma \in \Sigma_2} \sigma(q^{2(\lambda_1 - \lambda_2 - s)\zeta}) \times \frac{1}{q^{2\zeta_1} - q^{2\zeta_2}}.
\]

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