Confining properties of a gas of $Z(2)$ vortices

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Vortex solutions are studied in an SO(3) gauge theory spontaneously broken to SO(2). These vortices have a $Z(2)$ magnetic charge. A dilute gas of $Z(2)$ vortices is studied taking into account vortex-vortex interactions. By going to a dual representation we show that odd charges are confined with a string tension which decreases exponentially with the inverse coupling.

1. Introduction

Topologically non-trivial solutions are known to affect the long distance properties of gauge theories. By expanding around a gas of such solutions the partition function of gauge theories can be rewritten using a dual representation. The case of compact electrodynamics in a spontaneously broken gauge theory was analysed by Polyakov [1] where it was shown that a dilute gas of monopoles produces confinement of charges in the Higgs phase. Recently there has been a lot of discussion on $Z(2)$ vortices as a possible disordering mechanism for Wilson loops. Since $Z(2)$ vortices can emerge as classical solutions in gauge theories it is of interest to see if a gas of such $Z(2)$ vortices can be analysed in the same fashion. Classical solutions with $Z(2)$ magnetic charge are also known in cosmology and are called cosmic strings [2]. However, cosmic strings have the property that a particle going round it becomes its anti-particle. Such cosmic strings are associated with a loss of local charge conservation. The solutions we will consider are not cosmic strings but more like Abrikosov-Nielsen-Olesen vortices in a superconductor [3,4].

2. $Z(2)$ Vortices

The field theory under consideration is an $SO(3)$ gauge invariant theory minimally coupled to a matrix valued Higgs field $M$ in the $(3,3)$ representation of $SO(3)$. By studying an $SO(3)$ theory we would like to highlight the fact that the vortices are not in any way dependent on the center of the gauge group but are a consequence of the topology of the group. The Lagrangian (in 2 + 1 space-time dimensions) is given by

$$L = \int d^2x \, d\tau \left[ \frac{1}{2} \text{tr}(D_\mu M)^4(D_\mu M) - \frac{1}{4} F^\alpha_{\mu\nu} F^\alpha_{\mu\nu} - \text{tr}V(M^4 M) \right],$$

with the usual notations. Under local $SO(3)$ transformations ($V(x)$) the fields transform as

$$M \to VMV^{-1} \quad A_\mu \to VA_\mu V^{-1} - \frac{1}{g}(\partial_\mu V)V^{-1}.$$ 

The Higgs potential is chosen to be

$$V(M) = \lambda(M^4 M - I)^2;$$

this choice breaks the gauge symmetry to $SO(2)$. Vortex solutions have the following form for the gauge fields and the scalar field, $A_0 = 0, A_\mu^1 = A_\mu^2 = 0, A_\theta^3 = A(r), A_\tau^3 = 0, M(r, \theta) = M_c(r)M(\theta)$. The equations of motion become

$$D_{ij\alpha} = g \text{tr}((D_j M)^4[T^\alpha, M])$$

$$\partial_i(D_i M) = -g[A_i, D_i M] + \frac{\partial V}{\partial M}. $$

If we look for solutions in the type II limit ($m_S >> m_V, m_V = \alpha g$) the equation for the
vortices can be written as
\[ \alpha \rho = \alpha^2 (A(r)g^2 - \frac{g}{2r}) \quad , \tag{2} \]
where \( \alpha^2 = \text{tr}([T^3, M(0)]^4[T^3, M(0)]) \).

3. Dual representation

The partition function of a dilute gas of such vortices can be written as
\[ Z = \sum_{N} \mu^{N} \frac{N!}{\prod_{j=1}^{N} dR_{j}} \exp[-\frac{\pi m_{V}^2}{4g^2} \sum_{a \neq b} K_{0}(m_{V}|R_{a} - R_{b}|)] \]
The chemical potential term contains the self-energy of a single vortex. Once we have this expression we can use the method of Polyakov as used in [5] in the analysis of a gas of Abrikosov vortices and go over to the dual field \( \chi(x) \). This again results in a sine-gordon theory
\[ Z = \int D\chi \exp \frac{-A}{2} \int \left[ (\nabla \chi)^2 + m_{V}^2 \chi^2 - 2 M \cos(\chi) \right] d^2 x \]
where \( A = \frac{g^2}{\pi m_{V}^2} \), \( M^2 = \frac{2\pi m_{V}^2}{\sigma} \exp(-\epsilon_1) \), \( \epsilon_1 \) is the energy of a single vortex. The correlation functions in a gas of \( Z_2 \) vortices can be calculated by defining a generating function
\[ < \exp i \int \eta(x) \rho(x) d^2 x > = \frac{Z[\eta]}{Z[0]} \quad , \tag{5} \]
where \( \rho(x) = \sum_{a} \delta(x-x_{a}) \).

The Wilson loop
\[ \exp q \int_{C} A_{\mu} dx_{\mu} \quad , \tag{6} \]
becomes \( \exp \pi q T^3 \) if it encircles a \( Z(2) \) vortex and has trace \( -1 \) for odd multiples of \( g \). In the absence of a vortex the trace is \( +3 \).

There is an old argument which shows that in a gas of randomly distributed magnetic fluxes the Wilson loop will have an area law and this argument can be used to show that a gas of \( Z(2) \) vortices confines charges. However, we can get a more quantitative estimate for the Wilson loop taking into account vortex-vortex interactions by going to the dual representation. Using the properties of \( SU(2) \) and \( SO(3) \) representations we can write
\[ (tr \exp q \int_{C} A_{\mu} T^3 dx_{\mu}) - 1 \]
\[ = tr (\exp i(q \int_{C} A_{\mu} T^3 dx_{\mu})^2) \]
and define the normalized Wilson loop
\[ W_{n}(C) = \frac{1}{2} (tr \exp q \int_{C} A_{\mu} T^3 dx_{\mu} - 1) \quad \tag{7} \]
which takes values from \(-1\) to \(+1\). The expectation value of the normalized Wilson loop can be written as (using (7))
\[ W_{n}(C) = tr < \exp i \int 2gq(x) T^3 \eta(x) d^2 x > \quad , \tag{8} \]
where \( \eta(x) \) is given by
\[ \eta(x) = \frac{m_{V}^2}{2g} \int_{S_{C}} K_{0}(m_{V}|x - x'|) dx' \quad \tag{9} \]
\( S_{C} \) is the minimal surface spanning the loop \( C \). Effectively we have to calculate the average value of a diagonal matrix whose elements are
\[ W_{n}(C) = < \exp i \int q\rho(x) \eta(x) d^2 x > \quad . \tag{10} \]
This can be done by using the method of [1] and [3] and we repeat the main steps here.

Using the dual representation
\[ W_{n}(C) = \frac{Z_{1}}{Z_{2}} \quad , \tag{11} \]
where the numerator and denominator are given by
\[ Z_{1} = \int D\chi \exp -\frac{A}{2} \int \left[ (\nabla(\chi-\eta))^2 + m_{V}^2 (\chi-\eta)^2 - M^2 \cos \chi \right] d^2 x \quad \tag{12} \]
\[ Z_{2} = \int D\chi \exp -\frac{A}{2} \int \left[ (\nabla \chi)^2 + m_{V}^2 \chi^2 - M^2 \cos \chi \right] d^2 x \quad . \tag{13} \]
Since \( \eta \) vanishes outside the surface the numerator and denominator are the same outside the
surface $S_C$ and have to be calculated only on the surface $S_C$. The denominator is
\[
\int D\chi \exp \left[ -\frac{A}{2} \int_{S_C} \left[ -\chi (\nabla^2 \chi) + m^2 \chi^2 - M^2 \cos(\chi) \right] d^2 x \right] .
\]
The numerator can be split into two pieces (after an integration by parts)
\[
\int D\chi \exp \left[ -\frac{A}{2} \int_{S_C} \left[ -\chi (\nabla^2 \chi) + m^2 \chi^2 - M^2 \cos(\chi) \right] d^2 x \right] \exp \left[ -\frac{A}{2} \int_{S_C} \left[ (\nabla^2 \chi - m^2 \chi + \frac{\pi m^2 q}{g} \eta - \frac{\pi m^2 q}{g} \chi) \right] \right].
\]
In the mean field approximation (good in the weak coupling limit) the integrals over the numerator and the denominator yield the following equations
\[
\nabla^2 (\chi_n - \eta) = \frac{M^2}{2} \sin(\chi_n) + m^2 \chi_n - M^2 \cos(\chi_n)
\]
\[
\nabla^2 (\chi_d) = \frac{M^2}{2} \sin(\chi_d) + m^2 \chi_d .
\]
Using
\[
(\nabla^2 - m^2) K_0(m_V r) = -2\pi \delta(r)
\]
and Eq. 11 the first equation becomes
\[
\nabla^2 \chi_n = M^2 \sin(\chi_n) + m^2 (\chi_n - \frac{\pi q \Theta(r, S_C)}{g})
\]
\[
\Theta(r, S_C) \text{ is the surface delta function}
\]
\[
\Theta(r, S_C) = 1 \quad \text{if} \quad r \in S_C
\]
\[
\Theta(r, S_C) = 0 \quad \text{otherwise} .
\]
The mean field equations are solved by the constant solutions $\chi = -(2n + 1) \pi$ for $q = (2n + 1)g$ and $\chi = -(2n) \pi$ for $q = (2n)g$. Substituting these constant solutions in the numerator and the denominator we get
\[
W_n(C) = 1 \quad \text{for} \quad q = (2n)g
\]
\[
W_n(C) = \exp(-\sigma S_C) \quad \text{for} \quad q = (2n + 1)g
\]
The string tension $\sigma$ is exponentially small in the coupling $\sigma = AM^2 \approx \exp(-\frac{1}{g})$.

So far our analysis has been completely classical. We have rewritten the partition function in the Higgs phase as a gas of $Z_2$ vortices. Quantum effects are included by expanding about the classical solution and integrating out the fluctuations. This is a standard procedure and the integration of the zero modes renormalizes the chemical potential to $\mu = \frac{1}{g} f(\lambda) \exp(-\frac{1}{g})$ where $f(\lambda)$ is a determinant function which can in principle be computed. For weak coupling the quantum fluctuations do not change the classical results in any essential way.

4. Conclusions

In the above analysis we have classical $Z(2)$ vortex solutions occurring in certain broken gauge theories. The properties of a gas of such vortices taking into account their mutual interactions can be studied by going to a dual representation using the methods in [1,5]. This analysis yields a quantitative expression for the Wilson loop and odd charges are confined while even charges are not confined. The semi-classical analysis given here presents an instructive example where $Z(2)$ vortices confine charges.

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