Interior $C^2$ estimates to hessian-type equations on $\mathbb{S}^n$

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Abstract. In this article, we study global $C^2$ estimates to smooth admissible solutions of the Hessian-type equations on $\mathbb{S}^n$ and then obtain a unique smooth admissible solution up to a constant. As a special case, the Hessian-type equations contain a Monge-Ampère equation arising in designing a reflecting surface in geometric optics.

1. Introduction

In this paper, we consider interior $C^2$ estimates the following problem

$$\sigma_k(K_{ij}) = f(x), \quad \text{on} \quad \mathbb{S}^n, \quad (1.1)$$

where

$$K_{ij} = \frac{2uu_{ij} + (u^2 - |\nabla u|^2)\delta_{ij}}{u^2 + |\nabla u|^2}, \quad (1.2)$$

$\sigma_k$ is the $k$-th elementary symmetric function which will be defined in section 2, $u_{ij}$ is the second-order derivative with orthogonal frames on $\mathbb{S}^n$. The quantity $K_{ij}$ is referred to as intensity density by Oliker [1]. Equations (1.1) and (1.2) have intimate relation with the reflector designing problem in geometric optics. Many mathematicians have done excellent works in this field. For $k = n$, it is an equation of Monge-Ampère type. Wang [2] and Guan-Wang [3] obtained the existence and uniqueness of smooth solutions up to multiplication of positive constants under the condition

$$\int_{S^n} f(x)dx = \int_{S^n} g(x)dx.$$ 

When $k = 1$, it is an equation of quasi-linear type. Oliker [1] obtained the existence and uniqueness of this special case.

In this article, we consider the general $\sigma_k$ case of this reflector designing problem. As for the counterpart of this problem, Christoffel-Minkowski problems have been considered by many mathematicians and received considerable results, such as in [4], [5], [6] and the references therein.
In [4], Guan-Lin-Ma considered the existence of convex Weingarten hypersurfaces. In [5], Caffarelli-Nirenberg-Spruck studied the problem of finding embedded Weingarten hypersurfaces by using the continuity method.

In this paper, we establish the $C^2$ regularity, existence and uniqueness of the admissible solution to equation (1.1) and (1.2). A solution $u$ to the equation (1.1) and (1.2) is called admissible if the eigenvalues of the intensity density $K_\sigma$ are in the convex cone $\Gamma_\sigma$ which the definition can be seen in section 2. Admissible solutions are natural since they guarantees the ellipticity of the equation. The problem (1.1) and (1.2) has been studied in [7] where logarithmic gradient estimates were obtained. It is proved in [7] that if $0<u \in C^1(S^n)$ is an admissible solution to equation (1.1) and (1.2), where $f \in C^1(S^n)$ is a positive function, then there holds the logarithmic gradient estimates

$$\sup_{x \in B_r} |\nabla \log u| \leq C_0, \text{ where } C_0 = C_0(n,k, \sup_{x \in \overline{B}} f \sup_{x \in \overline{B}} |\nabla f|).$$

This article is a continuation of the paper [7] which further establish the $C^{1,\alpha}$ estimates and existence and uniqueness of problem (1.1) and (1.2). Our main result of this article is the following

**Theorem 1.1.** (Interior $C^2$ estimates) Let $u \in C^2(S^n)$ be an admissible solution to equation (1.1) and (1.2), $f(x) \in C^2(S^n)$ be a nonnegative function. Then we have the interior $C^2$ estimates

$$\sup_{x \in B_r} |\nabla^2 u| \leq C_0, \text{ where } C_0 = C_0(n,k,r, |f|, |\nabla f|, |\nabla^2 f|).$$

**Remark 1.2.** Since $S^n$ is compact, by using the finite covering theorem and Theorem 1.1, we indeed obtain global $C^2$ estimates.

By using the logarithmic gradient estimates in [7] and Theorem 1.1, we have the following theorem concerning the existence and uniqueness of the $\sigma_\sigma$ - reflector designing problem.

**Theorem 1.3.** Suppose $f(x)$ is a $C^\infty$ positive function on $S^n$. Then there exists a $C^\infty$ admissible solution to equation (1.1) and (1.2), and the solution is unique up to multiplication of positive constants.

The approach of proving Theorem 1.3 is the continuity method as illustrated in Gilbarg-Trudinger [8]. The key step here is to establish a-priori estimates for solution to (1.1) and (1.2) up to the second order derivatives and then obtain $C^{2,\alpha}$ estimates by Evans-Krylov Theorem [8].

The article is organized as follows. In section 2, we recall definition and some basic properties of the $k$ -th elementary symmetric functions. We derive the interior $C^2$ apriori estimates in section 3. In the last section, we obtain the existence and uniqueness of the smooth admissible solution to equation (1.1) and (1.2) by using the continuity method.
2. Preliminaries
In this section, we give some preliminaries on the definitions and properties of elementary symmetric functions which will be frequently used in the following sections.

Definition 2.1. For any \( k = 1, \ldots, n \), and for any \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \), we define the \( k \)-th elementary symmetric function as

\[
\sigma_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k < n} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_k}.
\] (2.1)

We also set \( \sigma_0 = 1 \) and \( \sigma_k = 0 \) for \( k > n \). For \( 1 \leq k \leq n \), we define \( \Gamma_k \) is a cone in \( \mathbb{R}^n \) by

\[
\Gamma_k = \{ \lambda \in \mathbb{R}^n : S_k(\lambda) > 0, \ldots, \sigma_k(\lambda) > 0 \}.
\] (2.2)

Obviously \( \Gamma_k \) contains the positive cone \( \Gamma_k = \{ \lambda \in \mathbb{R}^n : \lambda_i > 0, i = 1, 2, \ldots, n \} \) and \( \Gamma_k \) is symmetric in the sense that if \( \lambda \in \Gamma_k \), then any permutation of \( \lambda \) also lies in \( \Gamma_k \). In fact, \( \Gamma_k \) is a convex cone.

Let us denote by \( \sigma_i(\lambda | i) \) the sum of the terms of \( \sigma_i(\lambda) \) not containing the factor \( \lambda_i \). Then the following identities hold.

Proposition 2.2. For any \( k = 0, 1, \ldots, n \), \( i = 1, 2, \ldots, n \) and \( \lambda \in \mathbb{R}^n \), there hold the following identities.

\[
\frac{\partial \sigma_{k+1}}{\partial \lambda_i}(\lambda) = \sigma_k(\lambda | i),
\] (2.3)

\[
\sigma_{k+1}(\lambda) = \sigma_{k+1}(\lambda | i) + \lambda_i \sigma_k(\lambda | i),
\] (2.4)

\[
\sum_{i=1}^{n} \sigma_k(\lambda | i) = (n-k)\sigma_k(\lambda),
\] (2.5)

\[
\sum_{i=1}^{n} \lambda_i \sigma_k(\lambda | i) = (k+1)\sigma_{k+1}(\lambda).
\] (2.6)

Proposition 2.3. If \( \lambda \in \Gamma_k \) for \( k \in \{1, 2, \cdots, n\} \), then we have \( \sigma_k(\lambda | i) > 0 \), for any \( h \in \{0, 1, \cdots, k-1\} \) and \( i \in \{1, 2, \ldots, n\} \).

Proposition 2.4. Set \( F(W) = (\sigma_n(W))^{\frac{1}{n}}(W) \), where \( W = \{W_i\} \) is a symmetric matrix. If the eigenvalues of \( W \) belong to \( \Gamma_1 \), then the matrix \( \left( \frac{\partial F}{\partial W_i} \right) \) is positive definite and

\[
\sum_{i=1}^{n} \frac{\partial F}{\partial W_i} \geq 1.
\] (2.7)

Moreover, \( F \) is concave on the convex cone \( \Gamma_k \), that is for any \( (\eta_{ij}) \in \mathbb{R}^{n \times n} \).
\[
\sum_{i,j,k,l=1}^{n} \frac{\partial^2 F}{\partial W_{ij} \partial W_{kl}} (W) \eta_{ij} \eta_{kl} \leq 0
\]  

(2.8)

**Proposition 2.2** is standard which can be checked directly. For the proof of **Proposition 2.3** and **Proposition 2.4**, the readers can refer to [9] for example.

3. Interior $C^2$ estimates

In this section, we prove the Interior $C^2$ estimates of Theorem 1.1. We follow the method in [10].

Set $v = \log u$, that is, $u = e^v$.

By direct calculation, we obtain

\[
u_i = e^v v_i,
\]

(3.1)

\[
u_j = e^v (v_j + v_i v_j),
\]

(3.2)

\[
u^2 + |\nabla u|^2 = e^{2v}(1 + |\nabla v|^2).
\]

(3.3)

Therefore the equation (1.1) and (1.2) can be written as $\sigma_\epsilon(W_\xi) = \left(1 + \frac{|\nabla v|^2}{2}\right) f(\xi)$, where

\[
W_\xi = v_\xi + v_i v_j + \frac{1 - |\nabla \xi|^2}{2} \delta_{ij}.
\]

(3.4)

Let $F(W_\xi) = \left(\frac{\sigma_\epsilon(W_\xi)}{C_\epsilon}\right)^{\frac{1}{2}}$, $F^\eta = \frac{\partial F}{\partial W_{\xi}}$, then we get

\[
F(W_\xi) = (C_\epsilon)^{\frac{1}{2}} \left(1 + |\nabla v|^2\right)^{\frac{1}{2}} f(\xi)^{\frac{1}{2}}.
\]

(3.5)

To obtain interior $C^2$ estimates, let us consider an auxiliary function

\[
G(\xi, \eta) = \eta^2 (W_\xi + \frac{1}{2} |\nabla \eta|^2),
\]

(3.6)

where $0 \leq \eta \leq 1$ is a cut-off function such that $\eta = 1$ in $B_1$ and $\eta = 0$ outside $B_r$, and also $|\nabla \eta| < C \frac{\sqrt{\eta}}{r}$ and $|\nabla^2 \eta| < C \frac{1}{r^2}$. Without loss of generality, we assume $r = 1$, since we can make a scaling for general $r$ and the proof is similar.

Suppose $(\xi_0, \xi_0)$ is the maximal point of the auxiliary function $G$, by rotating the coordinate system suitably, we may assume that $\{W_\xi(\xi_0)\}$ is diagonal and $\xi_0 = 1$. Therefore $\{F^\eta(\xi_0)\}$ is diagonal. At $\xi_0$, by taking the first derivative of $\log G$, we have
(log \( G \)) = \frac{2\eta_i}{\eta} + \frac{W_{ij} + v_i v_{ji}}{W_{ii} + \frac{1}{2}|\nabla v|^2} = 0. \tag{3.7}

By taking the second derivative of (3.6) once more and using (3.7), we yield

\[
(\log G)_i = \frac{2\eta_i}{\eta} - \frac{\eta^2}{\eta^2} + \frac{W_{ij} + 2v_i v_{ji} + v_{ji} - (W_{ij} + v_i v_{ji})(W_{ii} + v_i v_{ji})}{W_{ii} + \frac{1}{2}|\nabla v|^2} - \frac{W_{ii} v_{ii} + v_{ii}^2}{W_{ii} + \frac{1}{2}|\nabla v|^2} - \frac{2\eta_i}{\eta} \geq 0.
\tag{3.8}

Using the positivity of \( F'' \), we multiplying by \( F'' \) on the both sides of the above inequality and get

\[
0 \geq F''(\log G)_i = F''(\log G)_i
\]

\[
= \frac{1}{W_{ii} + \frac{1}{2}|\nabla v|^2} (F''W_{ii} + F''v^2_{ji} + F''v^2_{ji}) + F''\left(\frac{2\eta_i}{\eta} - \frac{6\eta_i^2}{\eta^2}\right)
\]

\[
= \frac{1}{W_{ii} + \frac{1}{2}|\nabla v|^2} I + II. \tag{3.9}
\]

To compute the term \( I \), we first recall the following formulas on \( S^n \) which can be referred to [1] or [4] for example:

\[
v_{ij} = v_{ji}, \tag{3.10}
\]

\[
v_{ik} = v_{ki} - v_{kj}\delta_{ij} + v_j\delta_{ik}, \tag{3.11}
\]

\[
v_{jk} = v_{kj} - v_{lj}\delta_{ij} + v_i\delta_{jk}. \tag{3.12}
\]

By using these formulas, we deduce that

\[
W_{ij} = W_{ji} \tag{3.13}
\]

and

\[
W_{i,j} = W_{a,i} - 2v_i v_{ai} + v_i v_{p1i} + 2v_i v_{ai} - p v_{p1i} + 2v_{i}^2 - 2v_i - v_{p1}^2 \tag{3.14}
\]

Therefore, we obtain

\[
I = F''W_{i,j} + F''v_{p1i} + F''v_{p1i}
\]

\[
= F''W_{i,j} - 2F''v_{p1i} + \sum_i F''v_{p1i} + 2v_{i}F''v_{ai} + 2v_i + F''v_{p1i} + 2v_i + 2v_i \sum_i F''v_{p1i}
\]

\[
= F''W_{i,j} - 2F''v_{p1i} + \sum_i F''v_{p1i} + 2v_i F''v_{ai} + 2v_i + 2v_i \sum_i F''v_{p1i}
\]

\[
= F''W_{i,j} - 2F''v_{p1i} + \sum_i F''v_{p1i} + 2v_i F''v_{ai} + 2v_i + 2v_i \sum_i F''v_{p1i}
\]
\[-2F''v_{ii} + \sum_i F''v_{pi}^2 - 2F''v_i^2 + v_i^2 \sum_i F'' + |\nabla v|^2 \sum_i F''.\] (3.15)

By (3.4), it follows that
\[v_{jk} = W_{j,k} - v_j v_{ik} - v_i v_{jk} + v_p v_{pk} \delta_{ij}.\] (3.16)

Then we calculate the term \(I\) as follows:
\[I = F''W_{1,11} - 2F''v_i W_{1i} + F''v_i W_{1i} + 2v_i F''W_{ii}\]
\[-2F''v_i v_p v_{ip} + \sum_i F''v_p v_i v_{pi} + 2v_i \sum_i F'' - 2F''v_i + \sum_i F''v_i^2 + |\nabla v|^2 \sum_i F''.\] (3.17)

Taking first derivative of equation (3.5), we obtain
\[F''W_{ii} = [(C_n^i)^\frac{1}{2} (1 + |\nabla v|^2) f^{\frac{1}{2}}]_1.\] (3.18)

Taking second derivative of equation (3.5) once more and using the concavity property of \(F(W_i)\) which is the inequality (2.8) in Proposition 2.4, we obtain
\[F''W_{1,11} \geq [(C_n^1)^\frac{1}{2} (1 + |\nabla v|^2) f^{\frac{1}{2}}]_1.\] (3.19)

By (3.7), (3.18) and (3.19), it follows that
\[I \geq (C_n^1)^\frac{1}{2} \left[\frac{1}{2} (1 + |\nabla v|^2) f^{\frac{1}{2}}\right]_1 - 2F''v_i [\frac{\eta}{2} (W_{1i} + |\nabla v|^2) - v_p v_{ip}]\]
\[+ F''v_i \frac{\eta}{2} (W_{1i} + |\nabla v|^2) - v_p v_{ip}] + 2(C_n^1)^\frac{1}{2} \left[\frac{1}{2} (1 + |\nabla v|^2) f^{\frac{1}{2}}\right]_1\]
\[-2F''v_i v_p v_{ip} + \sum_i F''v_p v_i v_{pi} + 2v_i \sum_i F'' - 2F''v_i + \sum_i F''v_i^2 + |\nabla v|^2 \sum_i F''\]
\[= [(C_n^1)^\frac{1}{2} f^{\frac{1}{2}} + \sum_i F''] v_{pi}^2 + \sum_i F''v_{pi}^2 + \eta v_p v_{ip} - 4F''v_i v_p v_{ip} + 2\sum_i F'' v_p v_{pi} (W_{1i} + |\nabla v|^2)\]
\[+[(C_n^1)^\frac{1}{2} f^{\frac{1}{2}} + \sum_i F''] |\nabla v|^2 - [(C_n^1)^\frac{1}{2} f^{\frac{1}{2}} + \sum_i F''] v_i^2 + (C_n^1)^\frac{1}{2} \frac{1}{2} (1 + |\nabla v|^2) f^{\frac{1}{2}} (f^{\frac{1}{2}})]_1.\] (3.20)

Using the condition on the cut-off function \(\eta\) and the inequality (2.7) in Proposition 2.4, we get
\[I \geq \frac{1}{2} \sum_i F''v_{pi}^2 - \frac{C}{\sqrt{\eta}} (\sum_i F'' + 1)W_{1i} - \frac{C}{\sqrt{\eta}} (\sum_i F'' + 1)\]
\[\geq \frac{1}{2} \sum_i F''v_{pi}^2 - \frac{C}{\sqrt{\eta}} (\sum_i F'' + 1)W_{1i} - \frac{C}{\sqrt{\eta}} (\sum_i F'' + 1)\]
\[
\frac{1}{2} \sum_i F^{v^2} W_{i1} - v_i^2 - \frac{1}{2} (1-|\nabla v|)^2
\geq \frac{1}{2} \sum_i F^{w^2} W_{i1} - \frac{C}{\sqrt{\eta}} \left( \sum_i F^{w^2} + 1 \right) W_{i1} - \frac{C}{\sqrt{\eta}} \left( \sum_i F^{w^2} + 1 \right).
\]  

(3.21)

For the term \( II \), we use the condition on the cut-off function \( \eta \) and the inequality (2.7) in Proposition 2.4 to obtain
\[
II = F^{v^2} \left( \frac{2\eta}{\eta} - \frac{6\eta^2}{\eta^2} \right) \geq -C \sum_i F^{w^2}.
\]  

(3.22)

Now multiplying \( \eta^4 (W_{i1} + \frac{1}{2} |\nabla v|^2) \) on the both sides of (3.9) and using the expressions of I and \( II \), we derive that
\[
0 \geq \eta^4 I + \eta^4 (W_{i1} + \frac{1}{2} |\nabla v|^2) II
\geq \frac{1}{2} \eta^4 \sum_i F^{w^2} W_{i1}^2 - C \eta^2 \left( \sum_i F^{w^2} + 1 \right) W_{i1} - C \eta^4 \left( \sum_i F^{w^2} + 1 \right) W_{i1} + C \sum_i F^{w^2}
\geq \frac{1}{4} \sum_i F^{w^2} (\eta^2 W_{i1})^2 - C \sum_i F^{w^2} + 1.
\]  

(3.23)

That is
\[
C \geq \sum_i F^{w^2} (\eta^2 W_{i1})^2 - C.
\]  

(3.24)

Noting that \( \sum_i F^{w^2} \geq 1 \) by inequality (2.7) in Proposition 2.4, we deduce that \( \eta^2 W_{i1} \leq C \) and then \( G \leq C \). Therefore, \( |\nabla^2 v| \) and \( |\nabla^2 u| \) is bounded. We thus complete the proof of Theorem 1.1.

4. Proof of Theorem 1.3

In this section, we turn to prove the existence and uniqueness of the smooth admissible solutions to equations (1.1) and (1.2).

By setting \( v = \log u \), we can normalize the equations (1.1) as in section 3
\[
\left( \frac{C}{\sigma_i} \right)^{\frac{1}{2}} (v_i + v_j + \frac{(1-|\nabla v|^2)\delta_{ij}}{2}) = \left( C_i \right)^{\frac{1}{2}} (1+|\nabla v|^2) f_i^{\frac{1}{2}},
\]  

(4.1)

which can also be written in the form
\[
F(W_i) = g(x, \nabla v).
\]  

(4.2)

We first get \( C^{2,\alpha} \) estimates. By using the logarithmic gradient estimate in [7], we can get the Harnack estimates by the path integral
\[
\sup_{x \in S^n} v(x) \leq C \inf_{x \in S^n} v(x).
\]  

(4.3)

Since the equation (1.1) is homogeneous, without loss of generality, we may assume the solution \( v \) satisfies \( \inf_{x \in S^n} v(x) = 1 \). Hence we know that \( \sup_{x \in S^n} v(x) \leq C \). Since we can get gradient estimate by
interpolation, we obtain \( \| v(x) \|_{C^{2\alpha}} \leq C \). Since the operator \( F \) is concave with respect to the variable \( D^2v \) which is by (2.8) in Proposition 2.4, we know the Hölder norms of second order derivatives of solutions are bounded by the \( C^2 \) norm of solutions by the Evans-Krylov estimates (see Gilbarg-Trudinger [8]). Hence we know that
\[
\| v(x) \|_{C^{2\alpha}} \leq C \quad \text{for some } 0 < \alpha < 1.
\]  

4.1. Existence of admissible solutions

For \( 0 \leq t \leq 1 \), consider the family of equations
\[
F(W_t) = \frac{1-t}{2} + t(C^1_\alpha)^{\frac{1}{\alpha}} \frac{(1 + \| \nabla v \|^2)}{2} f^\frac{1}{\alpha}.
\]  

Let us define a set
\[
\Theta = \{ t \mid 0 \leq t \leq 1, \quad \text{the equation (4.5) has positive admissible solution} \}
\]

Obviously, \( u = 1 \) is the solution of equation (4.5) when \( t = 0 \). Therefore the set \( \Theta \neq \emptyset \).

We derive the linearized operator of the Equation (4.5) at \( v_0 \). Consider the equation
\[
G(x,v,\nabla v, \nabla^2 v) = \sigma^k_\alpha (W_t) - \frac{1-t}{2} - t(C^1_\alpha)^{\frac{1}{\alpha}} \frac{(1 + \| \nabla v \|^2)}{2} f^\frac{1}{\alpha}.
\]  

If we set \( v = v_0 + sW \), then we see that \( v \) is admissible when \( s \) is small enough.

Taking derivative with respect to \( s \), we derive the linearized operator of equation (4.5) is:
\[
F^k W_y + 2 F^k v_j W_j - F^k v_j W_j - t(C^k_\alpha)^{\frac{1}{\alpha}} v^k W_k f^\frac{1}{\alpha} = 0.
\]  

Since the linearized operator is surjective, the openness of \( \Theta \) comes from the implicit function Theorem, Schauder theory and the Fredholm alternative [8]. The set \( \Theta \) is also closed since we have established \( C^{2\alpha} \) estimates (4.4) of the admissible solution to equation (4.5). By using the smoothness of \( f(x) \), we easily derive that the admissible solution is smooth by using the bootstrap argument.

So we get the existence part of Theorem 1.3.

4.2. Uniqueness of admissible solution

As the equation (1.1) and (4.1) are equivalent, we prove the uniqueness of equation (1.1) up to multiplications of constants. We prove this fact by using a contradiction argument. Suppose \( u_1 \) and \( u_2 \) are two smooth admissible solutions to equation (1.1) and (1.2). Suppose \( u_2 > u_1 \) somewhere on \( \mathbb{S}^n \).

We can take \( t \geq 1 \), such that
\[
tu_1 \geq u_2 \quad \text{on } \mathbb{S}^n ; \quad tu_2 = u_2 \quad \text{at some point } P \in \mathbb{S}^n.
\]  

From the homogeneity of the equation, we get
\[
F(tu_i) = F\left( \frac{2u_i + (u^2 - |\nabla u|^2)\delta_i}{u^2 + |\nabla u|^2} \right)_{u_1 = u_2} = F(u_1) = f(x) = F(u_2).
\]  

Hence
\[ \dot{L}(u_1 - u_2) = 0 \]  

Here, \( L \) is the linearized operator of equation (1.1) and (1.2), that is

\[
\dot{L}(W) = \frac{F^0}{(u^2 + |\nabla u|^2)^2} \left[ (u^2 + |\nabla u|^2)[2u_y W + 2u W_y + (2u W - 2u_y W_y)\delta_y] \right]
- [2uu_y + (u^2 - |\nabla u|^2)\delta_y](2u W + 2u_y W_y),
\]

where \( u \) takes some admissible function between \( u_1 \) and \( u_2 \) and \( W = u_1 - u_2 \). The strong maximum principle implies that \( u_1 - u_2 = 0 \) on \( \mathbb{S}^n \). We hence obtain the uniqueness and finish the proof of

**Theorem 1.3.**

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