SOME PROBLEMS OF GUARANTEED CONTROL OF THE
SCHLÖGL AND FITZHugh—NAGUMO SYSTEMS

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Abstract. A game control problems of the Schlögl and FitzHugh—Nagumo
equations are considered. The problems are investigated both from the view-
point of the first player (the partner) and of the second player (the opponent).
For both players, their own procedures for forming feedback controls are spec-
ified.

1. Introduction. In the recent years, a part of mathematical control theory,
namely, the theory of control for distributed systems, has been intensively de-
veloped. To a considerable degree, this is stimulated by the fact that a rather wide set
of applied problems is described by such systems. At present, there exists a number
of monographs devoted to control problems for distributed systems [8, 9, 5, 20, 1, 6].
In all these works, the emphasis is on the problems of open-loop or feedback control
in the case when all system’s parameters are precisely specified. But, the investi-
gation of control problems for systems with uncontrollable disturbances (game or
robust control problems) is also natural. Similar problems have been insufficiently
investigated; in our opinion, this is connected with the fact that the well-known
Pontryagin maximum principle is not really suitable for solving such problems.
In the early 1970es, N.N. Krasovskii [7] suggested an effective approach to solving
guaranteed control problems. This approach is based on the formalism of positional
strategies.

In these paper, some problems of guaranteed (feedback) control for a parabolic
equation with memory are investigated. Such an equation includes the Schlögl and
FitzHugh—Nagumo equations. The fundamental theory of guaranteed control for
some equations (variational inequalities) with distributed parameters within the
framework of the formalization suggested in [7] was presented in [18, 10, 11, 12, 13].
In all the works cited above, the cases when equations (variational inequalities)
do not contain a previous history were considered. In the present work, from the
position of the approach [7, 18, 10, 11, 12, 13], the problems of guaranteed control
(for the partner and opponent) in the case of an equation containing the previous
history of its phase state are investigated. To solve these problems, we use the
known method of stable paths. Note that optimal control problems for the system
in question were discussed in [3, 4, 19, 17, 2], see also their bibliography. For

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systems.
example, in [3, 17] different numerical and theoretical aspects of optimal control of the Schlögl and FitzHugh–Nagumo equations are discussed. The work [19] is devoted to the positional (feedback) control for such equations. In this paper three different approach are considered. First, an analytical solution is proposed. Second, an appropriate optimal control procedure is applied. The third approach extends the standard optimal control to the so-called spare optimal control; this results in very localized control signals and allows to analyze second order optimality conditions. The game control problem for systems with distributed parameters has been investigated by many authors (see, for example, [8, 14, 15, 16]).

2. Problem statement. Solution method. We discuss the game control problem for an equation of the following form

\[
\frac{\partial}{\partial t} x - \Delta_L x + e^{-\eta t} R(e^\eta t x) + \eta x + \alpha K_\eta(t) x_{0,t}(\cdot) = u - v + f \quad \text{in } Q_\vartheta
\]

\[
\frac{\partial}{\partial t} x(\nu, t) = 0 \quad \text{in } \Sigma_\vartheta
\]

\[
x(\nu, 0) = x_0(\nu) \quad \text{in } \Omega.
\]

(1)

In this setting, \(f(\cdot) \in L_\infty([0, \vartheta]; L_2(\Omega))\) is a given function, \(\Delta_L\) is the Laplace operator, \(u\) and \(v\) (controls) are elements of the space \(H^1(\Omega)\), \(\alpha\) is a real number, \(R\) is a cubic polynomial

\[R(y) = k(y - y_1)(y - y_2)(y - y_3)\]

with given real numbers \(k > 0\) and \(y_1 < y_2 < y_3\), \(\Omega\) is a bounded open Lipschitz domain in \(\mathbb{R}^n\) with boundary \(\partial\Omega\), \(n \in \{1, 2, 3\}\), \(\vartheta > 0\) is a fixed time, \(\eta\) is a sufficiently large real parameter. We use the notation \(Q_\vartheta := \Omega \times (0, \vartheta)\) and \(\Sigma_\vartheta := \partial\Omega \times (0, \vartheta)\). Moreover, an initial function \(x_0 \in L_\infty(\Omega)\) is given. By \(\mu\) and \(\partial_\mu\), we denote the outward unit normal vector and the associated outward normal derivative on \(\partial\Omega\), respectively.

The family of operators \(K_\eta(t)\) is defined by the relations

\[(K_\eta(t)y_{0,t}(\cdot))(\nu) = \int_0^t \gamma ds e^{-(\beta + \eta)(t-s)} y(\nu, s)\]

for a.a. \(\nu \in \Omega\). Here \(\beta\) and \(\gamma\) are real numbers and the symbol \(x_{0,t}(\cdot)\) stands for the function \(x(s), s \in [0, t]\).

Note that equation (1) was introduced in [3]. This equation arises from the system

\[
\frac{\partial}{\partial t} y(\nu, t) - \Delta_L y(\nu, t) + R(y(\nu, t)) + \alpha z(\nu, t) = u(\nu, t) - v(\nu, t) + f(\nu, t) \quad \text{in } Q_\vartheta
\]

\[
\frac{\partial}{\partial t} y(\nu, t) = 0 \quad \text{in } \Sigma_\vartheta
\]

\[
y(\nu, 0) = y_0(\nu) \quad \text{in } \Omega
\]

\[
\frac{\partial}{\partial t} z(\nu, t) + \beta z(\nu, t) - \gamma y(\nu, t) + \delta^* = 0 \quad \text{in } Q_\vartheta
\]

\[
z(\nu, 0) = z_0(\nu) \quad \text{in } \Omega.
\]

(2)

Here, some real constant \(\delta^*\) and initial state \(z_0 \in L_\infty(\Omega)\) are given. Indeed, the latter two equations in (2) can be resolved by

\[
z(\nu, t) = e^{-\beta t} z_0(\nu) + \int_0^t e^{-\beta(t-s)}(\gamma y(\nu, s) - \delta^*) \, ds
\]
which is given by
\[
\int_\Omega \varphi \, d\nu + \frac{\delta^*}{\beta} (e^{-\beta t} - 1) + (K_\eta(t)y_0(t)) \langle \nu, \varphi \rangle
\]
Inserting \(z(\nu, t)\) into the first equation of (2), we derive the equation
\[
\frac{\partial}{\partial t} y(\nu, t) - \Delta_L y(\nu, t) + R(y(\nu, t)) + \alpha(K_\eta(t)y_0(t)) (\nu)
\]
\[
= u(\nu, t) - v(\nu, t) + f(\nu, t) - \alpha(e^{\beta t} z_0(\nu) + \delta^* \beta^{-1} (e^{-\beta t} - 1)) \text{ in } Q_\vartheta.
\]
Following [3] and substituting \(y(\nu, t) = e^{\mu t} x(\nu, t)\), we come to the equation for \(x\) of form (1).

For the choice \(\alpha = 0\), both equations in (2) are decoupled; the state function \(y\) is a solution of the Schlögl equation. The latter has the form
\[
\frac{\partial}{\partial t} y(\nu, t) - \Delta_L y(\nu, t) + R(y(\nu, t)) + z(\nu, t) = u(\nu, t) - v(\nu, t) + f(\nu, t) \text{ in } Q_\vartheta
\]
\[
\partial_h y(\nu, t) = 0 \text{ in } \Sigma_\vartheta
\]
\[
y(\nu, 0) = y_0(\nu) \text{ in } \Omega.
\]
For the choice \(\alpha = 1\), system (2) converts into the FitzHugh—Nagumo system, which is given by
\[
\frac{\partial}{\partial t} y(\nu, t) - \Delta_L y(\nu, t) + R(y(\nu, t)) + z(\nu, t) = u(\nu, t) - v(\nu, t) + f(\nu, t) \text{ in } Q_\vartheta
\]
\[
\partial_h y(\nu, t) = 0 \text{ in } \Sigma_\vartheta
\]
\[
y(\nu, 0) = y_0(\nu) \text{ in } \Omega.
\]

A function \(x(\cdot) = x(\cdot; 0, x_0, u(\cdot), v(\cdot)) \in W(0, \vartheta) \cap L_\infty(Q_\vartheta)\) is said to be a (weak) solution of (1), if the equality
\[
\int_0^\vartheta \langle x'(t), \varphi(t) \rangle \, dt + \int_0^\vartheta \left\{ \nabla x(u(t)) \cdot \nabla \varphi(t) + [e^{-\eta t} R(e^{\eta t} x(u(t))) + \eta x(u(t))] \varphi(t) \right\} \, d\nu \, dt
\]
\[
+ \int_0^\vartheta \alpha(K_\eta(t)x_0(t)) \varphi(t) \, d\nu = \int_0^\vartheta \left\{ u(\nu, t) - v(\nu, t) + f(\nu, t) \right\} \varphi(t) \, d\nu \, dt
\]
holds for all \(\varphi \in W(0, \vartheta)\) and the initial condition \(x(0) = x_0\) is satisfied in \(\Omega\). Here, \(\langle \cdot, \cdot \rangle\) denotes the duality between \(H^1(\Omega)\) and \((H^1(\Omega))^*\), \(W(0, \vartheta) = \{y(\cdot) \in L_2(T; H^1(\Omega)) : y(\cdot) \in L_2(T; H^1(\Omega))^*\}, T = [0, \vartheta]\). By virtue of the corresponding embedding theorem, without loss of generality, one can assume that the space \(W(0, \vartheta)\) is embedded into the space \(C(T; H)\). Here, \(H = L_2(\Omega)\). Therefore, the element \(x(t)\) (the phase state of equation (1)) is defined at the time \(t \in T\). As is known (see [4], theorem 2.1), under our conditions and for appropriate values of \(\eta\) (a namely, for \(\eta\) satisfying Condition 1 from Section 3), there exists a unique solution of (1) for any \(u(\cdot) \in L_\infty(T; H^1(\Omega))\) and \(v(\cdot) \in L_\infty(T; H^1(\Omega))\).

The game control problems under consideration in the paper consists in the following. Let a target set \(N\) or some quality criterion \(I = I(x(\cdot), u(\cdot), v(\cdot))\) depending on the solution \(x(\cdot)\) of equation (1) and controls \(u(\cdot)\) and \(v(\cdot)\) and its prescribed value \(I_0\) be given. At discrete times \(\tau_1 \in \Delta = \{\tau_i\}_{i=0}^m, \tau_0 = 0, \tau_{i+1} = \tau_i + \delta, \tau_m = \vartheta,\)
the phase state $x(\tau_i)$ is measured. The results of these measurements are functions $\xi_i^h \in (H^1(\Omega))^* \ (\xi_i^0 \in H)$, $i \in [1 : m - 1]$, $(\xi_0^h = x_0)$, satisfying the inequalities

$$|x(\tau_i) - \xi_i^h|_{(H^1(\Omega))^*} \leq h \quad (|x(\tau_i) - \xi_i^h|_H \leq h).$$

Here, $h \in (0, 1)$ stands for a level of informational noise. There are two players—antagonists controlling equation (1) by means of various input actions. One of them is called a partner, another one is called an opponent. Assume $P \subset H^1(\Omega)$ and $V \subset H^1(\Omega)$ are given bounded and closed sets. The problem undertaken by the partner is as follows. It is necessary to construct a law (a strategy) for forming the control $u$ (with values from $P$) by a certain feedback principle (on the base of measuring the state $x(\tau_i)$) in such a way that this control provides the attainment of the target set $N$ by the solution (1) at the time $t = \vartheta$ or, in the case when the quality criterion is specified, guarantees a value of the criterion not exceeding $I_*$ for any admissible control $v(\cdot)$ chosen by the opponent. The problem undertaken by the opponent is “inverse”: it consists in the choice of a law (a strategy) for forming the control $v$ (with values from $V$) also by the feedback principle (on the base of measuring the state $x(\tau_i)$) in such a way that this control provides the evasion of the solution of equation (1) from the target set $N$ at the time $t = \vartheta$ or guarantees a value of the quality criterion exceeding $I_*$ for any admissible control $u(\cdot)$ chosen by the partner. This is the description of the problems considered in the paper.

The scheme of the algorithm for solving the problem undertaken by the partner, in the case when the target set is given, is as follows. In the beginning, an auxiliary equation $M_1$ is introduced. The equation $M_1$ has an input $u^*(\cdot)$ and an output $w(\cdot)$ being the solution of this equation. Here, $u^*(\cdot)$ is the solution of some auxiliary optimal control problem. Before the algorithm starts, a value $h$ and a partition $\Delta$ with diameter $\delta = \delta(\Delta)$, as well as some open-loop control $u^*(\cdot)$ serving as an input of $M_1$ are fixed. The process of synchronous feedback control of systems (1) and $M_1$ is organized on the interval $T$. This process is decomposed into $m - 1$ identical steps. At the $i$th step carried out during the time interval $\delta_i = [\tau_i, \tau_{i+1})$, the following actions are fulfilled. First, at the time $\tau_i$, according to some a priori chosen rule $U$, elements

$$u_i^h \in U(\tau_i, \xi_i^h, w(\tau_i)) \subset P$$

are calculated. Then (till the time $\tau_{i+1}$), the control $u_i^h(t) = u_i^h$ is fed onto the input of equation (1). Under the action of this control, as well as of the given control $u^*(t)$, $\tau_i \leq t < \tau_{i+1}$, and an unknown control of the opponent $v(t)$, $\tau_i \leq t < \tau_{i+1}$, the states $x(\tau_{i+1})$ and $w(\tau_{i+1})$ are realized at the time $\tau_{i+1}$. The procedure stops at the moment $\vartheta$.

Remark 1. Below, four problems are investigated. The first two problems are solved in the case when $\xi_i^h$, the results of measuring the states $x(\tau_i)$, satisfy inequalities (3) in the $(H^1(\Omega))^*$-metric (whereas the values $\xi_i^h$ may be elements of the space $L_2(\Omega)$). The remaining two problems deal with the metric of the space $L_2(\Omega)$. As is know, the first metric is weaker. Therefore, when solving the first two problems, we weaken the requirements on the measurement results in comparison with the requirements imposed for solving the later two problems.

The scheme of the algorithm for solving the problem undertaken by the opponent is analogous to the scheme of the algorithm for solving the problem by the partner. In the beginning, an auxiliary equation $M_2$ is introduced. The equation $M_2$ has an input $v^*(\cdot)$ and an output $z(\cdot)$ being the solution of this equation. Before the
algorithm starts, the value \( h \) and the partition \( \Delta \) with diameter \( \delta \) are fixed. The process of synchronous feedback control of equations (1) and \( M_2 \) is organized on the interval \( T \). This process is decomposed into \( m - 1 \) identical steps. At the \( i \)th step carried out during the time interval \( \tau_i = [\tau_i, \tau_{i+1}) \), the following actions are fulfilled. First, at the time \( \tau_i \), according to some chosen rules \( V_1 \) and \( V \), elements

\[
v_i^h \in V(\tau_i, \zeta_i^h, z(\tau_i)) \subset V, \quad v_i^s \in V_1(\tau_i, \xi_i^h, z(\tau_i)) \subset V
\]

are calculated. Then (till the moment \( \tau_{i+1} \)), the control \( v^*(t) = v_i^s \), \( \tau_i \leq t < \tau_{i+1} \), is fed onto the input of \( M_2 \) and the control \( v^h(t) = v_i^h \), \( \tau_i \leq t < \tau_{i+1} \), onto the input of equation (1).

In the case when the quality criterion is given, corresponding systems are used instead of equations \( M_1 \) or \( M_2 \).

Let us give the strict formulation of the considered problems. Before this, we give some definitions. The sets of all open-loop controls of the partner and opponent are denoted by the symbols \( P_T(\cdot) \) and \( V_T(\cdot) \), respectively:

\[
P_T(\cdot) = \{ u(\cdot) \in L_2(T; H^1(\Omega)) : u(t) \in P \text{ a.e. } t \in T \},
\]

\[
V_T(\cdot) = \{ v(\cdot) \in L_2(T; H^1(\Omega)) : v(t) \in V \text{ a.e. } t \in T \}.
\]

The symbols \( u_{a,b}(\cdot) \) and \( u_{a,b}(\cdot) \) stand for the restrictions of the set \( P_T(\cdot) \) and function \( u(t), t \in T \) onto the segment \([a,b] \subset T \). Any strongly measurable functions \( u(\cdot) : T \to P \) and \( v(\cdot) : T \to V \) are called open-loop controls of the partner and opponent, respectively. Elements of the product \( T \times \mathcal{H} \) are called positions, where the space

\[
\mathcal{H} = (H^1(\Omega))^* \times H
\]

is called the space of positions. A unique solution of equation (1) with the properties

\[
x(t_*) = x_* \in L_\infty(\Omega), \quad x(\cdot) = x(\cdot; t_*, x_*, u_{t_*} , \vartheta(\cdot), v_{t_*} , \vartheta(\cdot)) \in W(t_*, \vartheta),
\]

is called a trajectory of equation (1) from the position \((t_*, x_*)\) corresponding to the controls \( u_{t_*} , \vartheta(\cdot) \in P_{t_*} , \vartheta(\cdot) \) and \( v_{t_*} , \vartheta(\cdot) \in V_{t_*} , \vartheta(\cdot) \). Any function (perhaps, multifunction)

\[
\mathcal{U} : T \times \mathcal{H} \to P
\]

is said to be a positional strategy of the partner. A positional strategy of the opponent is defined by analogy:

\[
\mathcal{V} : T \times \mathcal{H} \to V.
\]

Positional strategies correct controls at discrete times given by some partition of the interval \( T \).

Let the partition of \( T \) be any finite family \( \Delta = \{ \tau_i \}_{i=0}^m \), where \( \tau_0 = 0 \), \( \tau_m = \vartheta \), \( \tau_{i+1} = \tau_i + \delta \); \( \delta = \delta(\Delta) \) is the diameter of \( \Delta \). Let an equation \( M_1 \) with a phase trajectory \( w(\cdot) \) be fixed. The trajectory \( w(\cdot) \) is generated by some open-loop control \( u^*(\cdot) \). A trajectory \( x(\cdot) \) of equation (1) starting from an initial state \((t_*, x_*)\) and corresponding to a piecewise constant control \( u^h(\cdot) \) (formed by the feedback principle) and to a control \( v_{t_*} , \vartheta(\cdot) \in V_{t_*} , \vartheta(\cdot) \) is called an \( (h, \Delta, w, \mathcal{U}) \)-motion \( x_{\Delta,w}^h(\cdot) = x_{\Delta,w}^h(\cdot; t_*, x_*, \mathcal{U}, v_{t_*} , \vartheta(\cdot)) \) generated by the positional strategy \( \mathcal{U} \) on the partition \( \Delta \). Thus, the process of forming the motions \( x_{\Delta,w}^h(\cdot) \) and \( w(\cdot) \) is realized simultaneously. For \( t \in [\tau_i, \tau_{i+1}] \), these functions are specified as follows:

\[
x_{\Delta,w}^h(t) = x(t; \tau_i, x_{\Delta,w}^h(\tau_i), u_{\tau_i, \tau_{i+1}}^h(\cdot), v_{\tau_i, \tau_{i+1}}(\cdot)), \quad w(t) = w(t; \tau_i, w(\tau_i), u_{\tau_i, \tau_{i+1}}^s(\cdot)), \quad \text{(6)}
\]
where
\[ u^h(t) = u^h_i \in \mathcal{U}(\tau_i, \xi^h_i, w(\tau_i)) \quad \text{for} \quad t \in [\tau_i, \tau_{i+1}), \quad i \in [i(t_s) : m - 1], \quad (7) \]
\[ |\xi^h_i - x^h_{\Delta,w}(\tau_i)(H^1(\Omega))^*| \leq h, \quad i(t_s) = \min\{i : \tau_i > t_s\}, \quad (8) \]
\[ u^h(t) = u^h_i \in P \quad \text{for} \quad t \in [t_s, \tau_{i(t_s)}). \]

Thus, under \( x^h_{\Delta,w}() \) we understand the solution of equation (1) constructed by the feedback principle described above; i.e., the control \( u^h(\cdot) \) is formed by formula (7) and \( x^h_{\Delta,w}(\cdot) \), by formula (6). The set of all \((h, \Delta, w, \mathcal{U})\)-motions is denoted by \( X_h(t_s, x_s, \mathcal{U}, \Delta, w) \). By the results of [12], the set \( X_h(t_s, x_s, \mathcal{U}, \Delta, w) \) is not empty for \((t_s, x_s) \in T \times L_\infty(\Omega)\).

The task of the partner can be formulated in the following way. Let equation (1) be considered on the given time interval \( T \). Its trajectory \( x(\cdot) \) depends on two controls \( u = u(t) \in P \) and \( v = v(t) \in V \). One takes a uniform partition \( \Delta = \{\tau_i\}_{i=0}^m \), \( \tau_0 = 0, \tau_m = \tau \) with the diameter \( \delta = \delta(\Delta) = \tau_{i+1} - \tau_i \). The phase states \( x(\tau_i) \) are measured inaccurately at the times \( \tau_i \). The results of measurements \( \xi^h_i \in (H^1(\Omega))^* \) satisfy inequalities (8).

**Problem 1.** It is necessary to specify an equation \( M_1 \), a control \( u^*() \) for this equation, as well as a positional strategy of the partner \( \mathcal{U} : T \times \mathcal{H} \to P \) with the following properties: whatever a value \( \varepsilon > 0 \), one can specify (explicitly) numbers \( h_* > 0 \) and \( \delta_* > 0 \) such that the inclusion
\[ x^h_{\Delta,w}(\varnothing) \in N^\varepsilon \quad \text{for all} \quad x^h_{\Delta,w}(\cdot) \in X_h(0, x_0, \mathcal{U}, \Delta, w) \]
is fulfilled uniformly with respect to all measurements \( \xi^h_i \) with properties (8) if \( h \leq h_* \) and the diameter \( \delta = \delta(\Delta) \leq \delta_* \).

In what follows, the symbol \( N^\varepsilon \) stands for a closed \( \varepsilon \)-neighborhood of the set \( N \) in \( H \).

**Remark 2.** Taking into account the rule of constructing the set \( X_h(0, x_0, \mathcal{U}, \Delta, w) \), one can make the following conclusion. Whatever the open-loop control \( v_T(\cdot) \in V_T(\cdot) \) maybe, the solution \( x^h_{\Delta,w}(\cdot; 0, x_0, \mathcal{U}, v_T(\cdot)) \) belong to an \( \varepsilon \)-neighborhood of the set \( N \) at the time \( t = \varnothing \).

**Remark 3.** As is seen from the proof of Theorem 3.1 presented below, the rule for constructing the strategy \( \mathcal{U} \) (see (4), (14)) is based on the following simple idea. Namely, the strategy \( \mathcal{U} \) is chosen in such a way that the \((h, \Delta, w, \mathcal{U})\)-motion \( x^h_{\Delta,w}(\cdot) \) (see (6)–(8)), generated by this strategy slightly deviates from the solution \( w(\cdot; 0, x_0, u^*(\cdot)) \) for the whole time interval \( T \). The last property should be valid for any open-loop control \( v^*(\cdot) \) chosen by the opponent and unknown for the partner. In essence, the strategy \( \mathcal{U} \) means the choice of a control according to the principle of extremal shift [7], which is well-known in the theory of guaranteed control. As to this principle applied to distributed systems, see, for example, [18, 10, 11, 12, 13].

By analogy with the motion \( x^h_{\Delta,w}(\cdot) = x^h_{\Delta,w}(\cdot; t_s, x_s, \mathcal{U}, u_{t_s}, \varnothing(\cdot)) \), we define a motion \( x^h_{\Delta,z}(\cdot) = x^h_{\Delta,z}(\cdot; t_s, x_s, \mathcal{V}, \mathcal{V}_1, u_{t_s}, \varnothing(\cdot)) \) corresponding to piecewise constant controls \( u^h(\cdot) \) and \( v^*(\cdot) \) (formed by the feedback principle) and to a control \( u_{t_s}, \varnothing(\cdot) \in \mathcal{P}_{t_s}, \varnothing(\cdot) \). This motion is called an \((h, \Delta, \mathcal{V}, \mathcal{V}_1)\)-motion generated by the positional strategies \( \mathcal{V} \) and \( \mathcal{V}_1 \) on the partition \( \Delta \). The set of all \((h, \Delta, \mathcal{V}, \mathcal{V}_1)\)-motions is denoted by \( X_h(t_s, x_s, \mathcal{V}, \mathcal{V}_1, \Delta) \). It is clear that the set \( X_h(t_s, x_s, \mathcal{V}, \mathcal{V}_1, \Delta) \) is not empty for \((t_s, x_s) \in T \times L_\infty(\Omega)\).
Note that the trajectory \( x_{\Delta,z}^h(\cdot; t_*, x_*, \mathcal{V}, \mathcal{V}_1, u_{t_*}, \phi(\cdot)) \) is formed simultaneously with another trajectory \( z(\cdot) \). All these two trajectories are formed by the feedback principle, i.e., for \( t \in [\tau_i, \tau_{i+1}] \), it is assumed that

\[
x_{\Delta,z}^h(t) = x_{\Delta,z}^h(t; \tau_i, x_{\Delta,z}^h(\tau_i), u_{\tau_i, \tau_{i+1}}(\cdot), v_{\tau_i, \tau_{i+1}}(\cdot)),
\]

\[
z(t) = z(t; \tau_i, z(\tau_i), v_{\tau_i, \tau_{i+1}}^*(\cdot)),
\]

where

\[
v^h(t) = v^h(\tau_i, \xi^h_i, z(\tau_i)), \quad v^*(t) = v^*_i \in \mathcal{V}_i(\tau_i, \xi^h_i, z(\tau_i)) \quad (11)
\]

for \( t \in [\tau_i, \tau_{i+1}] \), \( i \in \{t_\tau : m-1\} \),

\[
|\xi^h_i - x_{\Delta,z}^h(\tau_i)|_{(H^1(\Omega))^*} \leq h, \quad i(t_\tau) = \min\{i : \tau_i > t_\tau\}, \quad (12)
\]

\[
v^*(t) = v^* \in \mathcal{V}, \quad v^h(t) = v^*_i \in \mathcal{V} \quad \text{for} \quad t \in [t_\tau, \tau_{i(t_\tau)}].
\]

Under \( x_{\Delta,z}^h(\cdot) \) we understand the trajectory of (1) constructed by the feedback principle described above; i.e., the controls \( v^h(\cdot) \) and \( v^*(\cdot) \) are formed by formula (11) and \( x_{\Delta,z}^h(\cdot) \), by formula (9). Here, the function \( z(\cdot) \) (see (10)) is an auxiliary function, which is analogous to the function \( w(\cdot) \) in the problem of the partner.

The problem undertaken by the opponent is inverse with respect to the problem undertaken by the partner. Its essence is as follows.

**Problem 2.** It is necessary to specify an equation \( M_2 \), as well as a positional strategy of the opponent \( \mathcal{V} : \mathcal{T} \times \mathcal{H} \rightarrow \mathcal{V} \), and a positional strategy \( \mathcal{V}_1 : \mathcal{T} \times \mathcal{H} \rightarrow \mathcal{V} \) with the following properties: whatever a value \( \varepsilon > 0 \), one can specify (explicitly) numbers \( h_* > 0 \) and \( \delta_* > 0 \) such that the condition

\[
x_{\Delta,z}^h(\theta) \notin N^\varepsilon \quad \forall x_{\Delta,z}^h(\cdot) \in X_h(0, x_0, \mathcal{V}, \mathcal{V}_1, \Delta)
\]

is fulfilled uniformly with respect to all measurements \( \xi^h_i \) with properties (12) if \( h \leq h_* \) and the diameter \( \delta = \delta(\Delta) \leq \delta_* \).

Along with Problems 1 and 2 formulated above, we consider another two problems (Problems 3 and 4). Let a cost functional

\[
I(x(\cdot), u_T(\cdot), v_T(\cdot)) = \int_0^\vartheta \phi(t, x(t), u(t), v(t)) dt + \sigma(x(\vartheta))
\]

be given. Here, \( \phi(t, x, u, v) = \phi_1(t, x, u) + \phi_2(t, x, v) : \mathcal{T} \times (H^1(\Omega))^* \times \mathcal{P} \times \mathcal{V} \in \mathbb{R} \) and \( \sigma : \mathcal{H} \in \mathbb{R} \) are given functions satisfying the local Lipschitz condition with respect to the set of variables. For example, the cost functional may be a quadratic function, i.e.

\[
\phi(t, x, u, v) = |x|^2_{(H^1(\Omega))^*} + |u|^2_{H^1(\Omega)} - |v|^2_{H^1(\Omega)}, \quad \sigma(x) = |x|^2_H.
\]

To investigate Problems 3 and 4 formulated below, we need to extend the space of positions. Namely, instead of the space \( \mathcal{H} \), we take the space \( \mathcal{H}_1 = \mathcal{H} \times \mathbb{R}^2 \) as the space of positions. The reason of such a choice will be clear after the description of the algorithm for solving Problem 3.

So, let a prescribed value of the criterion, a number \( I_\varepsilon \), be fixed.

Introduce the set of triples \( \phi_{\Delta,g}^h(\cdot) = \{x_{\Delta,g}^h(\cdot), u^h(\cdot), v(\cdot)\} \), where \( x_{\Delta,g}^h(\cdot) \) is the solution of equation (1) generated by controls \( u^h(\cdot) \) and \( v(\cdot) \in \mathcal{V}_T(\cdot) \). In the process, the control \( u^h(\cdot) \) is formed by some strategy \( \mathcal{U} \). The set of triples \( \phi_{\Delta,g}^h(\cdot) \) is denoted
Problem 3. It is necessary to specify a system $M_1$, a control $u_\ast(\cdot)$ for this system, as well as a positional strategy of the partner $U : T \times \mathcal{H}_1 \to P$ with the following properties: whatever a value $\varepsilon > 0$, one can specify (explicitly) number $h_\ast > 0$ such that the inequality

$$I(\phi^h_{\Delta,g}(\cdot)) \leq I_\ast + \varepsilon \quad \text{for all} \quad \phi^h_{\Delta,g}(\cdot) \in \Phi_h(0,x_0,\mathcal{U},\Delta,g)$$

is fulfilled uniformly with respect to all measurements $\xi^h_i \in (H^1(\Omega))^\ast$ with the properties $|\xi^h_i - x^h_{\Delta,g}(\tau_i)|_H \leq h$ if $h \leq h_\ast$ and the diameter $\delta = \delta(\Delta) = C_1^* h^{\mu_1}$.

Here $C_1^* > 0$ and $\mu_1$ are fixed constants and $\delta$ is the diameter of the partition $\Delta$.

Remark 4. It is clear that Problem 3 is solvable not for all $I_\ast$. The number providing the solvability of this problem is specified in Condition 4 presented in the section devoted to the algorithm of the solvability of Problem 3. In the case when the function $\phi$ does not depend on $x$, this condition means the solvability of the corresponding problem of optimal open-loop control.

The problem undertaken by the opponent is inverse with respect to the problem undertaken by the partner. Its essence is as follows.

Introduce the set of functions $\psi^h_{\Delta,g}(\cdot) = \{x^h_{\Delta,g,v}(\cdot), u(\cdot), v^h(\cdot)\}$, where $x^h_{\Delta,g,v}(\cdot)$ is the solution of equation (1) generated by controls $u(\cdot) \in P_T(\cdot)$ and $v^h(\cdot)$. In the process, the control $v^h(\cdot)$ is formed by some strategy $\mathcal{V}$. The set of triples $\psi^h_{\Delta,g}(\cdot)$ is denoted by the symbol $\Psi_h(0,x_0,\Delta,\mathcal{V})$.

Problem 4. It is necessary to specify a system $M_2$, as well as a positional strategy of the opponent $\mathcal{V} : T \times \mathcal{H}_1 \to V$ with the following properties: whatever a value $\varepsilon > 0$, one can specify (explicitly) number $h_\ast > 0$ such that the inequality

$$I(\psi^h_{\Delta,g}(\cdot)) \geq I_\ast - \varepsilon \quad \forall \psi^h_{\Delta,g}(\cdot) \in \Psi_h(0,x_0,\Delta,\mathcal{V})$$

is fulfilled uniformly with respect to all measurements $\xi^h_i \in (H^1(\Omega))^\ast$ with the properties $|\xi^h_i - x^h_{\Delta,g,v}(\tau_i)|_H \leq h$ if $h \leq h_\ast$ and the diameter $\delta = \delta(\Delta) = C_2^* h^{\mu_2}$.

Here $C_2^* > 0$ and $\mu_2$ are fixed constants.

The solution of Problem 4 is omitted here, since it is analogous to the solution of Problem 3 (see the remark in the end of Section 5).

3. Algorithm for solving Problem 1. We specify an algorithm for solving Problem 1. In this paper, we assume that the following condition is fulfilled.

Condition 1 ([4]). The parameter $\eta$ in equation (1) satisfies the condition

$$\max\{3|c_R|, 3|\alpha\gamma|/\theta^{1/2}, 0.5 - \beta\} \leq \eta,$$

where $c_R \leq dR/dv$ for all $v \in \mathbb{R}$.

Here and below, $|a|$ stands for the module of the number $a$. In this and the next sections, we assume that the following condition is also fulfilled.

Condition 2. There exists a closed set $D \subset H^1(\Omega)$ such that $P = V + D$.

Here, the symbol $V + D$ stand for the algebraic sum of sets $V$ and $D$, i.e. $V + D = \{u : u = u_1 + u_2, \ u_1 \in V, \ u_2 \in D\}$.
Condition 3. There exists an optimal open-loop control \( u^*(\cdot) \) with the property:

\[
w(\cdot; 0, x_0, u^*(\cdot)) \in N,
\]

where

\[
u^*(\cdot) \in DT(\cdot) = \{ u(\cdot) \in L_2(T; H^1(\Omega)) : u(t) \in D \text{ for a. a. } t \in T \}.
\]

Here, the symbol \( w(\cdot) = w(\cdot; 0, x_0, u(\cdot)) \), \( (u(\cdot) \in DT(\cdot)) \) denotes the solution of the equation

\[
\frac{\partial}{\partial t} w(\nu, t) - \Delta_L w(\nu, t) + e^{-\nu t} R(e^{\nu t} w(\nu, t)) + \eta w(\nu, t) + \alpha(K_{\eta}(t)u_{0,t}(\cdot))(\nu) = u(\nu, t) + f(\nu, t) \quad \text{in } Q_{\delta}
\]

\[
\frac{\partial \mu}{\partial t} w(\nu, t) = 0 \quad \text{in } \Sigma_{\delta}
\]

\[
w(\nu, 0) = x_0(\nu) \quad \text{in } \Omega.
\]

As the equation \( M_1 \), we take equation (13) with the program control \( u(\cdot) = u^*(\cdot) \).

The strategy \( U \) (see (4)) is defined in such a way:

\[
U(t, \xi, w) = \{ u^* \in P : (\xi - w, u^*) \leq \inf \{ (\xi - w, u) : u \in P \} + h \}, \quad \text{if } t > 0, (14)
\]

\[
U(0, \xi, w) = P.
\]

Let us pass to the description of the algorithm for solving Problem 1. Namely, we describe the procedure of forming the \((h, \Delta, w, U)-motion \) \( x^h_{\Delta, w}(\cdot) \) corresponding to some fixed partition \( \Delta \) and the strategy \( U \) of form (4), (14).

Before the algorithm starts, we fix a value \( h \in (0, 1) \), and a partition \( \Delta = \{ \tau_i \}_{i=0}^{m} \) with the diameter \( \delta = \tau_{i+1} - \tau_i \). The work of the algorithm is decomposed into \( m - 1 \) identical steps. We assume that

\[
u^h(t) = u^h_0 \in P \quad (15)
\]
on the interval \([0, \tau_1]\). Under the action of this constant control as well as of the unknown disturbance \( v_{0, \tau_1}(\cdot) \), the \((h, \Delta, w, U)-motion \)

\[
\{ x^h_{\Delta, w}(\cdot) \}_{0, \tau_1} = \{ x^h_{\Delta, w}(\cdot; 0, x_0, U, v_{0, \tau_1}(\cdot)) \}_{0, \tau_1}
\]
of equation (1) and the trajectory \( \{ w(\cdot) \}_{0, \tau_1} = \{ w(\cdot; 0, x_0, u^*_0(\cdot)) \}_{0, \tau_1} \) of the equation \( M_1 \) are realized.

At the time \( t = \tau_1 \), we determine \( u^h(\tau_1) \) from the condition

\[
u^h_1 \in U(\tau_1, \xi^h_1, w(\tau_1)), \quad |\xi^h_1 - x^h_{\Delta, w}(\tau_1)|_{(H^1(\Omega))} \leq h; \quad (16)
\]

i.e., we assume that

\[
u^h(t) = u^h_1 \quad \text{for } t \in [\tau_1, \tau_2).
\]

Then, we calculate the realization of the \((h, \Delta, w, U)-motion \)

\[
\{ x^h_{\Delta, w}(\cdot) \}_{\tau_1, \tau_2} = \{ x^h_{\Delta, w}(\cdot; \tau_1, x^h_{\Delta, w}(\tau_1), v_{\tau_1, \tau_2}(\cdot)) \}_{\tau_1, \tau_2},
\]
and the trajectory

\[
\{ w(\cdot) \}_{\tau_1, \tau_2} = \{ w(\cdot; \tau_1, w(\tau_1), u^*_\tau_1, \tau_2(\cdot)) \}_{\tau_1, \tau_2}.
\]
of the equation \( M_1 \).

Let the \((h, \Delta, w, U)-motion \) \( x^h_{\Delta, w}(\cdot) \) and the trajectory \( w(\cdot) \) of the equation \( M_1 \) be defined on the interval \([0, \tau_i]\). At the time \( t = \tau_i \), we assume that

\[
u^h_i \in U(\tau_i, \xi^h_i, w(\tau_i)), \quad |\xi^h_i - x^h_{\Delta, w}(\tau_i)|_{(H^1(\Omega))} \leq h; \quad (17)
\]
i.e., we set

\[
u^h(t) = u^h_i \quad \text{for } t \in [\tau_i, \tau_{i+1}).
\]
As a result of the action of this control and of the unknown disturbance \( v_{\tau_i, \tau_{i+1}}(\cdot) \), the \((h, \Delta, w, \mathcal{U})\)-motion \( \{x_{\Delta, w}^h(\cdot)\}_{\tau_i, \tau_{i+1}} = \{x_{\Delta, w}^h(\cdot; \tau_i, x_{\Delta, w}^h(\tau_i), \mathcal{U}, v_{\tau_i, \tau_{i+1}}(\cdot))\}_{\tau_i, \tau_{i+1}} \), and the trajectory

\[
\{w(\cdot)\}_{\tau_i, \tau_{i+1}} = \{w(\cdot; \tau_i, w(\tau_i), u_{\tau_i, \tau_{i+1}}^*(\cdot))\}_{\tau_i, \tau_{i+1}}
\]

of the equation \( M_1 \) are realized on the interval \([\tau_i, \tau_{i+1}]\). The above procedure of forming the \((h, \Delta, w, \mathcal{U})\)-motion and the trajectory of equation \( M_1 \) stops at the moment \( \vartheta \).

**Theorem 3.1.** Let the equation \( M_1 \) be specified by relation (13). Then the strategy \( \mathcal{U} \) of form (4), (14) solves Problem 1.

**Proof.** Define \( R_\eta(t, v) = e^{-\eta t}R(e^{\eta t}v) + \eta/3v \). Then equations (1) and (13) take the following form:

\[
\begin{align}
\frac{\partial}{\partial t} x_{\Delta, w}^h(\nu, t) &- \Delta L x_{\Delta, w}^h(\nu, t) + R_\eta(t, x_{\Delta, w}^h(\nu, t)) + \frac{2}{3}\eta x_{\Delta, w}^h(\nu, t) \\
+ \alpha(K_\eta(t)(x_{\Delta, w}^h(\cdot, t))(\cdot))(\nu) &= w^h(\nu, t) - v(\nu, t) + f(\nu, t) \quad \text{in } Q_\vartheta \\
\partial_\mu x_{\Delta, w}^h(\nu, t) &= 0 \quad \text{in } \Sigma_\vartheta \\
x_{\Delta, w}^h(\nu, 0) &= x_0(\nu) \quad \text{in } \Omega
\end{align}
\]  

(18)

and

\[
\begin{align}
\frac{\partial}{\partial t} w(\nu, t) &- \Delta L w(\nu, t) + R_\eta(t, w(\nu, t)) + \frac{2}{3}qw(\nu, t) \\
+ \alpha(K_\eta(t)w(\cdot, t))(\cdot))(\nu) &= u^*(\nu, t) + f(\nu, t) \quad \text{in } Q_\vartheta \\
\partial_\mu w(\nu, t) &= 0 \quad \text{in } \Sigma_\vartheta \\
w(\nu, 0) &= x_0(\nu) \quad \text{in } \Omega
\end{align}
\]  

(19)

To prove the theorem, we estimate the variation of the function

\[
\varepsilon(t) = |x_{\Delta, w}^h(t) - w(t)|_H^2, \quad t \in T.
\]

Subtracting (19) from (18) and multiplying scalarly the difference by \( x_{\Delta, w}^h(t) - w(t) \) (in \( H \)), then integrating (for \( t \in [\tau_i, \tau_{i+1}], i = 0, \ldots, m - 1 \)) and taking into account the monotonicity of the mapping \( v \to R_\eta(t, v) \), we derive

\[
\mu_i(t) \equiv \varepsilon(t) + 2\eta \int_{\tau_i}^{t} |\varepsilon(s)| ds + \int_{\tau_i}^{t} |x_{\Delta, w}^h(\tau) - w(\tau)|_{H_1(\Omega)}^2 d\tau
\]

(20)

\[
\leq \varepsilon(\tau_i) + \nu_i(t) + \int_{\tau_i}^{t} x_{\Delta, w}^h(s) - w(s), u^*(s) - v(s) - u^*(s) \rangle ds,
\]

where

\[
\nu_i(t) = |\alpha| \int_{\tau_i}^{t} \varepsilon^{1/2}(s)|K_\eta(s)(x_{\Delta, w}^h)(\cdot) - w(\cdot)|_H ds.
\]

From the results of [4] we obtain \((0 \leq a < b \leq \vartheta)\)

\[
\int_{a}^{b} |K_\eta(s)v_0, s(\cdot)|_H^2 ds = \int_{a}^{b} \left( \int_{\Omega} |K_\eta(\tau)v_0, \tau(\cdot))(\cdot)| d\rho \right)^2 d\tau
\]
\[
\begin{align*}
&\int_a^b \int_0^\tau e^{-(\beta+\eta)(\tau-s)}v(\varrho, \tau) \, d\varrho \, d\tau \leq \frac{d_1}{\varepsilon} \int_a^b \int_0^\tau v^2(\varrho, s) \, d\varrho \, d\tau \\
&= \frac{d_1}{\varepsilon} \int_a^b \int_0^\tau |v(s)|^2 \, d\varrho \, d\tau \quad \text{for any } v(\cdot) \in L^2([a, b]; H),
\end{align*}
\]
where \( d_1 = \frac{\gamma^2}{\sqrt{2(\beta+\eta)}} \). Moreover,

\[
\nu_i(t) \leq \frac{2}{3\eta} \int_{\tau_i}^t \varepsilon(s) \, ds + \frac{3|\alpha|^2}{(8\eta)} \pi_i(t),
\]

where

\[
\pi_i(t) = \int_{\tau_i}^t \left| K_{\eta}(s) \left( (x^h_{\Delta,w})(\cdot) - w_{0,s}(\cdot) \right) \right|^2_H \, ds.
\]

By (21), the inequality \((t \in [\tau_i, \tau_{i+1}]\))

\[
\pi_i(t) \leq \int_{\tau_i}^t \left| x^h_{\Delta,w}(s) - w(s) \right|^2_H \, ds \, d\tau = \int_{\tau_i}^t \int_0^s \varepsilon(s) \, ds \, d\tau + \int_{\tau_i}^t \int_0^s \varepsilon(s) \, ds \, d\tau
\]

\[
= \int_{\tau_i}^t \int_0^s \varepsilon(s) \, ds + \int_{\tau_i}^t \int_{\tau_i}^s \varepsilon(s) \, ds
\]

\[
= \int_{\tau_i}^t (t - \tau_i) \varepsilon(s) \, ds + \int_{\tau_i}^t \int_0^s \varepsilon(s) \, ds
\]

\[
(22)
\]

takes place. Therefore, by virtue of (22), we have

\[
\pi_i(t) \leq d_1 \delta \int_{\tau_i}^t \varepsilon(s) \, ds \quad \text{for } t \in [\tau_i, \tau_{i+1}].
\]

(23)

In this case, for \( t \in [\tau_i, \tau_{i+1}] \), using (2) and (23), we obtain

\[
\varepsilon(t) \leq \mu_i(t) \leq \varepsilon(\tau_i) + 3|\alpha|^2 d_1 \delta/(8\eta) \int_{\tau_i}^t \varepsilon(s) \, ds
\]

\[
+ \int_{\tau_i}^t \int_0^s \left| x^h_{\Delta,w}(s) - \dot{w}(s) \right|_{(H^1(\Omega))^*} \, ds \, d\tau + s_i^t,
\]

where

\[
s_i^t = \int_{\tau_i}^t \int_0^s \left| x^h_{\Delta,w}(\tau_i) - w(\tau_i), u^h(s) - v(s) - u^*(s) \right| \, ds.
\]

Then, using (8), we get

\[
s_i^t \leq \int_{\tau_i}^t \int_0^s \left| x^h_i - w(\tau_i), u^h(s) - v(s) - u^*(s) \right| \, ds + d_2 h(t - \tau_i).
\]

(25)
Here, \( d_2 = d(P) + d(V) + d(D) \), the symbol \( d(A) \) stands for the diameter of the set \( A \subset H^1(\Omega) \), i.e., \( d(A) = \sup \{ |z|_{H^1(\Omega)} : z \in A \} \). In its turn, Condition 2 implies the inclusion
\[
v(s) + u^*(s) \in P \quad \text{for a.a. } s \in T.
\]
In this case, the first term in the right-hand part of inequality (25), by virtue of the rule of finding the control \( u^h(\cdot) \) (see (15)–(17)), does not exceed the value \( (2 + d_2)h(t - \tau_i) \) for \( i \in [1 : m] \). It is easily seen that the inequality
\[
\sum_{j=1}^{i} \int_0^{\tau_j} \varepsilon(s) \, ds \leq \int_0^{t} \varepsilon(s) \, ds
\]
is fulfilled for \( t \in [\tau_i, \tau_{i+1}] \). Consequently, using (24) and (26), we obtain (for \( t \in [0, \vartheta] \))
\[
\varepsilon(t) \leq \varepsilon(0) + 3|\alpha|^2d_1\vartheta/(8\eta) \int_0^{t} \varepsilon(s) \, ds + d_2\vartheta \int_0^{t} |x^h_{\Delta,w}(s) - \dot{w}(s)|_{(H^1(\Omega))^*} \, ds + (2 + d_2)\vartheta h.
\]
From Theorem 2.1 [4], it follows that
\[
\int_0^{\vartheta} |x^h_{\Delta,w}(s) - \dot{w}(s)|_{(H^1(\Omega))^*} \, ds \leq d_3.
\]
The latter inequality does not depend on \( \Delta, u^h(\cdot), v(\cdot) \), and \( u^*(\cdot) \) but depends on the parameters of equation (1), namely, on the initial state \( x_0 \) and the sets \( P \) and \( V \). As well, \( \varepsilon(0) = 0 \). Therefore (for \( t \in T \)), the estimate
\[
\varepsilon(t) \leq (d_2d_3 + d_4)d^{1/2} + (2 + d_2)\vartheta h + 3|\alpha|^2d_1\vartheta/(8\eta) \int_0^{t} \varepsilon(s) \, ds
\]
is fulfilled. By virtue of the Gronwall lemma, we obtain
\[
\varepsilon(t) \leq d_5(h + \delta^{1/2}), \quad t \in T.
\]
The statement of the theorem follows from the latter inequality; i.e. \( w(\vartheta) = w(\vartheta; 0, x_0, u^*(\cdot)) \in N \). The theorem is proved.

Remark 5. Let the equation for the partner have the form (1) with \( v(t) = 0 \), i.e. the disturbance is absent in this equation. Assuming that, along with equation (1) we have the equation for the opponent
\[
\frac{\partial}{\partial t} x^{(1)}(\nu, t) - \Delta_L x^{(1)}(\nu, t) + e^{-\eta t} R(e^{\eta t} x^{(1)}(\nu, t)) + \eta x^{(1)}(\nu, t) + \alpha(K_0(t)x^{(1)}_{0,t}(\cdot))(\nu) = v(\nu, t) + f(\nu, t) \text{ in } Q_{\vartheta}
\]
\[
\partial \nu x^{(1)}(\nu, t) = 0 \text{ in } \Sigma_{\vartheta}
\]
\[
x^{(1)}(\nu, 0) = x^{(1)}_{0}(\nu) \text{ in } \Omega.
\]
Here, \( v = v(t) \) is an unknown disturbance subject to the restrictions \( v(t) \in V \). Let also
\[
|x^{(1)}_{0} - x_{0}|_{H} \leq h.
\]
and the control $u$ satisfies the condition: $u \in P$. At the moments $\tau_i \in \Delta$, the phase states $x(\tau_i)$ and $x^{(1)}(\tau_i)$ are measured with errors. The results of the measurements $\xi^{h}_i \in (H^1(\Omega))^*$ and $\psi^{h}_i \in (H^1(\Omega))^*$, satisfy the inequalities
\[
|\mathbf{w}(\tau_i) - \xi^{h}_i|_{(H^1(\Omega))^*} \leq h, \quad |\mathbf{w}^{(1)}(\tau_i) - \psi^{h}_i|_{(H^1(\Omega))^*} \leq h.
\]

It is necessary to construct a law (a strategy) for forming the control $u = u^h = \mathcal{U}(\tau_i, \xi^{h}_i, \psi^{h}_i)$ for $t \in [\tau_i, \tau_{i+1}]$ in equation (1) with the properties: whatever a value $\varepsilon > 0$, one can specify (explicitly) $h_1 > 0$ and $\delta_1 > 0$ such that the inequality
\[
|x(\partial_1 t, 0, x^h(u)) - x^{(1)}(0, 0, x^h(u), v(\cdot))|_{H} \leq \varepsilon
\]
is fulfilled uniformly with respect to all measurements $\xi^h_i$ and $\psi^h_i$ with the properties listed above if $h \leq h_1$ and $\delta = \delta(\Delta) \leq \delta_1$. If Condition 2 is valid, then the algorithm described above is applied to solve the problem in question. In this case, it is necessary to use $\psi^h_i$ and $\psi^h_i$ instead $w(\tau_i)$ and $w(\tau_i)$ in formulas (14), (16) and (17) respectively.

4. Algorithm for solving Problem 2. We specify an algorithm for solving Problem 2. Assume that, as everywhere above, Condition 1 is fulfilled.

As the equation (27), we take the equation
\[
\frac{\partial}{\partial t} w(\nu, t) = -\Delta_L w(\nu, t) + e^{-\nu t} R(e^{\nu t} z(\nu, t)) + \eta w(\nu, t) + \alpha (K_0(t) z_0(t))(\nu) = w^*(\nu, t) + f(\nu, t) \quad \text{in } Q_\delta
\]
\[
\frac{\partial}{\partial t} w(\nu, t) = 0 \quad \text{in } \Sigma_\delta
\]
\[
w(\nu, 0) = x(\nu) \quad \text{in } \Omega.
\]

Let the symbol $w(\cdot) = w(\cdot; 0, x_0, u(\cdot))$, $u(\cdot) \in D_T(\cdot)$ denote the solution of the equation
\[
\frac{\partial}{\partial t} w(\nu, t) = -\Delta_L w(\nu, t) + e^{-\nu t} R(e^{\nu t} w(\nu, t)) + \eta w(\nu, t) + \alpha (K_0(t) w_{0, t}(\cdot))(\nu) = u(\nu, t) + f(\nu, t) \quad \text{in } Q_\delta
\]
\[
\frac{\partial}{\partial t} w(\nu, t) = 0 \quad \text{in } \Sigma_\delta
\]
\[
w(\nu, 0) = x(\nu) \quad \text{in } \Omega.
\]
The symbol $W_T(\cdot) = W_T(\cdot; 0, x_0)$ stands for the bundle of solutions of equation (27); i.e., $W_T(\cdot; 0, x_0) = \{ w(\cdot; 0, x_0, u(\cdot)) : u(\cdot) \in D_T(\cdot) \}$, whereas $W_T(t)$ denotes the section of this bundle at the moment $t$.

The strategies $\mathcal{V}$ and $\mathcal{V}_1$ (see (5)) are defined as follows:
\[
\mathcal{V}(t, \xi, z) = \left\{ \mathbf{v}^e \in V : \langle z - \xi, \mathbf{v}^e \rangle \leq \inf \{ \langle z - \xi, v \rangle : v \in V \} + h \right\}, \quad \text{if } t > 0,
\]
\[
\mathcal{V}(0, \xi, z) = V,
\]
\[
\mathcal{V}_1(t, \xi, z) = \tilde{u} - \tilde{v}, \quad \text{if } t > 0, \quad \mathcal{V}_1(0, \xi, z) = D,
\]
where
\[
\tilde{u} \in \left\{ \tilde{u}^e \in P : \langle z - \xi, \tilde{u}^e \rangle \leq \inf \{ \langle z - \xi, u \rangle : u \in P \} + h \right\},
\]
and
\[
\tilde{v} \in \left\{ \tilde{v}^e \in P : \langle z - \xi, \tilde{v}^e \rangle \leq \inf \{ \langle z - \xi, v \rangle : v \in V \} + h \right\}.
\]
\[ \hat{v} = \hat{v}(\hat{u}) \] is an arbitrary element from the set \( V \) such that \( \hat{u} - \hat{v} \in D \). Note that equation (1) in this case takes the form
\[
\begin{align*}
\frac{\partial}{\partial t} x^h_{\Delta,z}(\nu, t) &- \Delta x^h_{\Delta,z}(\nu, t) + R_{\eta}(t, x^h_{\Delta,z}(\nu, t)) + 2/3\pi x^h_{\Delta,z}(\nu, t) \\
+ \alpha(\varphi(t)(x^h_{\Delta,z}(\nu, t), 0, t))(\nu) &= u(\nu, t) - v^h(\nu, t) + f(\nu, t) \quad \text{in } Q_0 \\
\sigma_t x^h_{\Delta,z}(\nu, t) &= 0 \quad \text{in } \Sigma_0 \\
x^h_{\Delta,z}(0) &= x_0(\nu) \quad \text{in } \Omega.
\end{align*}
\] (31)

Let us pass to the description of the algorithm for solving Problem 2. Namely, we describe the procedure of forming the \((h, \Delta, \mathcal{V}, \mathcal{V}_1)\)-motion \( x^h_{\Delta,z}(\cdot) \) corresponding to some fixed partition \( \Delta \) and strategies \( \mathcal{V} \) and \( \mathcal{V}_1 \) of form (29), (30).

Before the algorithm starts, we fix a value \( h \in (0, 1) \) and a partition \( \Delta = \{\tau_i\}_{i=0}^n \) with diameter \( \delta = \tau_{i+1} - \tau_i \). The work of the algorithm is decomposed into \( m - 1 \) identical steps. We assume that
\[
(v^h(t) = v^h_0 \in \mathcal{V}, \quad v^*(t) = v^*_0 \in D)
\] (32)
on the interval \([0, \tau_1]\). Under the action of this constant controls as well as of the unknown control \( u_{0,\tau_1}(\cdot) \), the \((h, \Delta, \mathcal{V}, \mathcal{V}_1)\)-motion \( x^h_{\Delta,z}(\cdot) \) (the solution of equation (31)) and the trajectory \( \{z(\cdot)\}_{0,\tau_1} = \{z(\cdot, 0, v^*_0, \tau_1(\cdot))\}_{0,\tau_1} \) of the equation \( M_2 \) (see (27)) are realized. At the time \( t = \tau_1 \), we determine \( v^h_1 \) and \( v^*_1 \) from the condition
\[
v^h_1 \in \mathcal{V}(\tau_1, \xi^h_1, z(\tau_1)), \quad v^*_1 \in \mathcal{V}_1(\tau_1, \xi^h_1, z(\tau_1)), \quad |\xi^h_1 - x^h_{\Delta,z}(\tau_1)|_{(H^1(\Omega))} \leq h,
\] (33)
where (see (30))
\[
v^*_1 = \hat{u}^h - \hat{v}^h_1,
\] (34)
\[
\hat{u}^h_1 = \{\hat{u}^\in \in P : \langle z(\tau_1) - \xi^h_1, \hat{u}^\in \rangle \leq \inf\{(z(\tau_1) - \xi^h_1, u) : u \in P\} + h\},
\]
\[
\hat{v}^h_1 = \hat{v}^h_1(\hat{u}^h_1) \] is an arbitrary element from the set \( V \) such that \( \hat{u}^h_1 - \hat{v}^h_1 \in D \). We assume that
\[
v^h(t) = v^h_1 \quad \text{and} \quad v^*(t) = v^*_1 \quad \text{for } t \in [\tau_1, \tau_2].
\] (35)
Then, we calculate the realization of the \((h, \Delta, \mathcal{V}, \mathcal{V}_1)\)-motion \( x^h_{\Delta,z}(\cdot) \) \( \tau_1, \tau_2 = \{x^h_{\Delta,z}(\cdot; \tau_1, x^h_{\Delta,z}(\tau_1), u_{\tau_1,\tau_2}(\cdot), v^h_{\tau_1,\tau_2}(\cdot))\}_{\tau_1,\tau_2} \) and the trajectory \( z_{\tau_1,\tau_2}(\cdot) = \{z(\cdot; \tau_1, z(\tau_1), u_{\tau_1,\tau_2}(\cdot), v^h_{\tau_1,\tau_2}(\cdot))\}_{\tau_1,\tau_2} \) of the equation \( M_2 \).

Let the \((h, \Delta, \mathcal{V}, \mathcal{V}_1)\)-motion \( x^h_{\Delta,z}(\cdot) \) and the trajectory \( z(\cdot) \) of the equation \( M_2 \) be defined on the interval \([0, \tau_1]\). At the time \( t = \tau_i \), we assume that
\[
v^h_1 \in \mathcal{V}(\tau_i, \xi^h_i, z(\tau_i)), \quad v^*_1 \in \mathcal{V}_2(\tau_i, \xi^h_i, z(\tau_i)), \quad |\xi^h_i - x^h_{\Delta,z}(\tau_i)|_{(H^1(\Omega))} \leq h,
\] (36)
where
\[
v^*_i = \hat{u}^h - \hat{v}^h_1,
\] (37)
\[
\hat{u}^h_i = \{\hat{u}^\in \in P : \langle z(\tau_i) - \xi^h_i, \hat{u}^\in \rangle \leq \inf\{(z(\tau_i) - \xi^h_i, u) : u \in P\} + h\},
\]
\[
\hat{v}^h_i = \hat{v}^h_i(\hat{u}^h_i) \] is an arbitrary element from the set \( V \) such that \( \hat{u}^h_i - \hat{v}^h_i \in D \). We assume that
\[
v^h(t) = v^h_i \quad \text{and} \quad v^*(t) = v^*_i \quad \text{for } t \in [\tau_i, \tau_{i+1}].
\] (38)
As a result of the action of these controls and of the unknown control \( u_{\tau_i,\tau_{i+1}}(\cdot) \), the \((h, \Delta, \mathcal{V}, \mathcal{V}_1)\)-motion \( \{x^h_{\Delta,z}(\cdot)\}_{\tau_i,\tau_{i+1}} = \{x^h_{\Delta,z}(\cdot; \tau_i, x^h_{\Delta,z}(\tau_i), u_{\tau_i,\tau_{i+1}}(\cdot), v^h_{\tau_i,\tau_{i+1}}(\cdot))\}_{\tau_i,\tau_{i+1}} \) and the trajectory \( \{z(\cdot)\}_{\tau_i,\tau_{i+1}} = \{z(\cdot; \tau_i, z(\tau_i), u_{\tau_i,\tau_{i+1}}(\cdot), v^h_{\tau_i,\tau_{i+1}}(\cdot))\}_{\tau_i,\tau_{i+1}} \)
of the equation $M_2$ are realized on the interval $[\tau, \tau_{i+1}]$. The described above procedure of forming the $(h, \Delta, \mathcal{V}, \mathcal{V}_1)$-motion and the trajectories $z(\cdot)$ of the equation $M_2$ stops at the moment $\theta$.

Theorem 4.1. Let $W_T(\vartheta) \cap M = \emptyset$ and let the equation $M_2$ be described by relations (27). Then the strategies $\mathcal{V}$ and $\mathcal{V}_1$ of form (5), (29), and (30) solve Problem 2.

Proof. Note that the inclusions

$$z(t) \in W_T(t)$$

take place for all $t \in T$. Therefore, to prove the theorem, it is sufficiently to show that, for any $\varepsilon > 0$, there exist $h_1 > 0$ and $\delta > 0$ such that the inequality $\lambda(\vartheta) \leq \varepsilon$ is fulfilled for $h \in (0, h_1)$ and $\delta \in (0, \delta_1)$. Here,

$$\lambda(t) = \lambda(x_{\Delta,z}^h(t), z(t)) = |x_{\Delta,z}^h(t) - z(t)|^2_H.$$

Let us estimate the variation of the value $\lambda(t)$. Subtracting (27) from (31) and multiplying scalarly both parts of this relation by $x_{\Delta,z}^h(t) - z(t)$, then integrating (for $t \in [\tau, \tau_{i+1}]$), we obtain

$$\lambda(t) \leq \lambda(\tau_i) + 2/3 \eta \int_{\tau_i}^t \lambda(s) ds + |\alpha| \int_{\tau_i}^t \lambda^{1/2}(s) K_\eta(s)(x_{\Delta,z}^h(s) - z_0, s)\rangle^2_H ds$$

$$+ \int_{\tau_i}^t \langle x_{\Delta,z}^h(s) - z(s), u(s) - v^h(s) - v^*(s) \rangle ds. \quad (39)$$

Now (see (23)), we have for $t \in [\tau_i, \tau_{i+1}]$

$$\int_{\tau_i}^t |K_\eta(s)((x_{\Delta,z}^h)_0, z(s) - z_0, s)\rangle^2_H ds \leq d_1 \delta \int_0^t \lambda(s) ds \text{ for } t \in [\tau_i, \tau_{i+1}]. \quad (40)$$

In this case, for $t \in [\tau_i, \tau_{i+1}]$, from (39) and (40), by analogy with (24), we get

$$\lambda(t) \leq \lambda(\tau_i) + 3|\alpha|^2 d_1 \delta/(8\eta) \int_0^t \lambda(s) ds \quad (41)$$

$$+ d_2(t - \tau_i) \int_{\tau_i}^t |x_{\Delta,z}^h(s) - \dot{z}(s)|_{H^1(\Omega)}^2 ds + \mu^{(i)}(t),$$

where

$$\mu^{(i)}(t) = \int_{\tau_i}^t \langle x_{\Delta,z}^h(\tau_i) - z(\tau_i), u(s) - v^h(s) - v^*(s) \rangle ds.$$

Note that the inequality

$$\mu^{(i)}(t) \leq \int_{\tau_i}^t \langle \xi_i - z(\tau_i), u(s) - v^h(s) - v^*(s) \rangle ds + d_2 h(t - \tau_i)$$

$$\leq \int_{\tau_i}^t \langle \xi_i - z(\tau_i), u(s) - \bar{u}_i \rangle ds + \int_{\tau_i}^t \langle \xi_i - z(\tau_i), \ddot{v}_i - v^h(s) \rangle ds + d_2 h(t - \tau_i) \quad (42)$$
holds. Using the rule of forming the functions $v^h(\cdot)$, $v^*(\cdot)$, $\tilde{u}(\cdot)$ and $\tilde{v}(\cdot)$ (see (33)–(38)), we conclude that the right-hand part of inequality (42) does not exceed $(2 + d_2)h(t - \tau)$. Consequently, from (41) we derive (for $t \in [0, \vartheta]$)

$$\lambda(t) \leq \lambda(0) + 3|\alpha|^2d_1\vartheta/(8\eta) \int_0^t \lambda(s) \, ds + d_2\vartheta \int_0^t |\dot{\sigma}_h^\varphi(s) - \dot{z}(s)|_{(H^1(\Omega))'} \, ds + (2 + d_2)d_3\vartheta h.$$  

The statement of the theorem follows from the latter inequality. The theorem is proved.

5. Algorithm for solving Problem 3. Let us specify an algorithm for solving Problem 3. We assume that there is the function $F(t, u, v) = F_1(t, u) + F_2(t, v)$ instead of the difference $u - v + f$ in the right-hand part of equation (1); i.e., the equation takes the form

$$\frac{\partial}{\partial t}x(\nu, t) - \Delta_L x(\nu, t) + e^{-\eta t}R(e^{\eta t}x(\nu, t)) + \eta x(\nu, t) + \alpha(K_\eta(t)x_0(\cdot))(\nu) = F(t, u(\nu, t), v(\nu, t)) \quad \text{in } Q_\vartheta$$

$$\partial_\nu x(\nu, t) = 0 \quad \text{in } \Sigma_\vartheta$$

$$x(\nu, 0) = x_0(\nu) \quad \text{in } \Omega.$$  

Here, $F_j : T \times H^1(\Omega) \to H^1(\Omega), j = 1, 2$, are given functions satisfying the Lipschitz conditions in the metric of the space $H$.

Consider the ordinary differential equation

$$\dot{g}(t) = \phi(t, x(t), u(t), v(t)), \quad g(0) = 0, \quad t \in T, \quad g \in \mathbb{R}. \quad (44)$$

Introducing this new variable $g$, we reduce the robust control problem of Bolza type to a control problem with a terminal quality criterion of the form $I_1(x(\cdot), g(\cdot)) = \sigma(x(\vartheta)) + g(\vartheta)$. In this case, the controlled system consists of equation (43) in the space $H$ and ordinary differential equation (44) in the space $\mathbb{R}$.

As the system $M_1$, we take the system described by the equation of form (1); i.e., the equation

$$\frac{\partial}{\partial t}g^{(1)}(\nu, t) - \Delta_L g^{(1)}(\nu, t) + e^{-\eta t}R(e^{\eta t}g^{(1)}(\nu, t)) + \eta g^{(1)}(\nu, t) + \alpha(K_\eta(t)g_0^{(1)}(\cdot))(\nu) = u^{(1)}(\nu, t) \quad \text{in } Q_\vartheta$$

$$\partial_\nu g^{(1)}(\nu, t) = 0 \quad \text{in } \Sigma_\vartheta$$

$$g^{(1)}(\nu, 0) = x_0(\nu) \quad \text{in } \Omega.$$  

and the ordinary differential equation

$$\dot{g}^{(2)}(t) = v^{(2)}(t), \quad g^{(2)}(0) = 0, \quad g^{(2)} \in \mathbb{R}, \quad t \in T. \quad (46)$$

Let

$$\Phi(t, x, u, v) = \{F(t, u, v), \phi(t, x, u, v)\} \in H \times \mathbb{R},,$$

$$\Phi_u(t, x, v) = \bigcup_{u \in P} \Phi(t, x, u, v), \quad H_u(t; x) = \bigcap_{v \in V} \Phi_u(t, x, v),$$

$$H_u(x; t) = \{u(\cdot) \in L^2(T; H^1(\Omega) \times \mathbb{R}) : u(t) \in H_u(t; x) \text{ for a. a. } t \in T\}.$$  

In this section we assume that the following condition is fulfilled.

Condition 4. There exists a control $u_*(\cdot) = \{u_*^{(1)}(\cdot), u_*^{(2)}(\cdot)\}, u_*(t) \in H_u(t; g^{(1)}(t))$ for a.a. $t \in T$, such that $I_* = \sigma(g^{(1)}(\vartheta)) + g^{(2)}(\vartheta)$.
Here, \( g^{(1)}(\cdot) = g^{(1)}(\cdot; 0, x_0, u^{(1)}_*) \) and \( g^{(2)}(\cdot) = g^{(2)}(\cdot; 0, 0, u^{(2)}_*) \) are the solutions of equations (45) and (46) for \( u^{(1)}(\cdot) = u^{(1)}_*(\cdot) \) and \( u^{(2)}(\cdot) = u^{(2)}_*(\cdot) \) respectively.

Let, for example, \( F(t, u, v) = u - v \), and there exists a closed set \( D \subset H^1(\Omega) \) such that \( P = V + D \). Assume also that \( \phi = \phi(t, u) \). Then

\[
H_*(t; x) = H_*(t) = D \times \{ \bigcup_{u \in P} \phi(t, u) \} \subset H^1(\Omega) \times \mathbb{R}.
\]

In this case, Condition 4 implies the existence of a pair of open-loop controls \( \{ u^{(1)}_*, u^{(2)}_* \} \) \((P) \in P, v \in V \) such that \( \phi \in P, v \in V \) holds, where \( 1 + 2 \leq \lambda \). Hence, Condition 4 provides the existence of a solution of the corresponding problem of optimal open-loop control.

Denote by the symbol \( X(\cdot; 0, x_0) \) the set of all solutions \( x(\cdot; 0, x_0, u(\cdot), v(\cdot), u(\cdot) \in P_T(\cdot), v(\cdot) \in V_T(\cdot) \) of equation (43). Let \( Z = \{(t, x, u, v) : t \in T, x \in X(t; 0, x_0), u \in P, v \in V \} \), \( \lambda = L(Z) \) be a Lipschitz constant of the function \( \Phi \) in \( Z, \tilde{X} = \{ x : x \in X(t; 0, x_0), t \in T \} \), and \( \lambda_\sigma \) be a Lipschitz constant of the function \( \sigma \) in \( \tilde{X} \).

By virtue of [4], the following lemma takes place.

**Lemma 5.1.** It is possible to specify a number \( d^{(1)} > 0 \) such that the inequality

\[
2 \int_0^T |\dot{x}(s)|^2_{(H^1(\Omega))} \, ds \leq d^{(1)} \text{ holds for all } x(\cdot) \in X(\cdot; 0, x_0).
\]

Let numbers \( d^{(2)} > 0 \) and \( d^{(3)} > 0 \) be such that, for any \( x(\cdot) \in X(\cdot; 0, x_0) \), the inequalities

\[
2 \sup\{|F(t, u, v)|_{H^1(\Omega)} : u \in P, v \in V, t \in T \} \leq d^{(2)},
\]

\[
2 \sup\{|\phi(t, x(t), u, v, )| : u \in P, v \in V, t \in T \} \leq d^{(2)}
\]

take place. Note that, by virtue of the continuity of the embedding of corresponding spaces, the inequality

\[
|x|_{(H^1(\Omega))} \leq d^{(3)}|x|_H \text{ for all } x \in H
\]

holds, where \( 1 \leq d^{(3)} \) is a constant.

In this section, we assume that \( \xi_i^h \in H \),

\[
|\xi \tau_i - \xi_i^h|_H \leq h,
\]

and the strategy \( \mathcal{U} \) is defined on some partition \( \Delta \).

Let us fix the function

\[
\mu(\delta, h) = c_1 \delta^{1/2} + c_0 h
\]

and the number \( \lambda \) such that \( 1 + 2k + \beta_* \leq \lambda \). Here,

\[
k = 3|\alpha| d_1/(8 \eta), \quad \beta_* = 1 + 2L(d^{(3)})^2, \quad c_0 = 3 + 4L(d^{(3)})^2 + d^{(2)},
\]

\[
c_1 = L(d^{(1)} + 2L^2)(d^{(3)})^2 + 2d^{(2)}(d^{(1)})^{1/2} + d^{(2)}
\]

For any quadruple \( \{ \tau_i, \xi, g^{(1)}, g^{(2)} \} \), we choose a maximal number \( \tilde{g}(\tau_i)(\tau_i \in \Delta, i = 0, \ldots, m - 1) \) with the property

\[
|\Phi(\xi, g^{(1)}, \tilde{g}(\tau_i), g^{(2)}) = |\xi - g^{(1)}|^2 + |\tilde{g}(\tau_i) - g^{(2)}|^2
\]

\[
= (\mu(\delta, h) + 2(i - 1)ch) \exp\{\lambda \tau_i\} + ch \text{ if } i \in [1: m - 1],
\]

\[
|\Phi(\xi, g^{(1)}, \tilde{g}(\tau_i), g^{(2)}) = |\tilde{g}(0)|^2 \text{ if } i = 0.
\]
Here, $|a|$ stands for the module of the number $a$, $c = 1 + 2\hat{g}^{1/2}(d^{(1)})^{1/2}$. If there does not exist an approximate number $\hat{g}$ for the given quadruple, then we take an arbitrary number as $\hat{g}(\tau_i)$.

The strategy $\mathcal{U}$ is defined in such a way:

$$\mathcal{U}(t, \xi, g^{(1)}, \hat{g}(t), g^{(2)}) = \left\{ u^e \in P : (\hat{g}(\tau_i) - g^{(2)})\phi_1(\tau_i, \xi, u^e) + (\xi - g^{(1)}, F_1(\tau_i, u^e)) \right\}$$

$$\leq \inf \left\{ (\hat{g}(\tau_i) - g^{(2)})\phi_1(\tau_i, \xi, u) + (\xi - g^{(1)}, F_1(\tau_i, u)) : u \in P \right\} + \hat{h} \quad \text{if } t \in [\tau_i, \tau_{i+1}).$$

Below, under the $(h, \Delta, g, \mathcal{U})$-motion we understand the pair \{$x^{(h)}_{\Delta, g}(\cdot), g_{\Delta}(\cdot)$\}. Here, $x^{(h)}_{\Delta, g}(\cdot)$ is the solution of equation (43) and $g_{\Delta}(\cdot)$ is the solution of equation (44) for $u(\cdot) = u^{(h)}(\cdot)$. So,

$$x^{(h)}_{\Delta, g}(t) = x(t; \tau_i, x^{(h)}_{\Delta, g}(\tau_i), u^{(h)}_{\tau_i, \tau_{i+1}}(\cdot), v_{\tau_i, \tau_{i+1}}(\cdot)), \quad x^{(h)}_{\Delta, g}(0) = x_0,$$

$$g_{\Delta}(t) = g(t; \tau_i, g_{\Delta}(\tau_i), u^{(h)}_{\tau_i, \tau_{i+1}}(\cdot), v_{\tau_i, \tau_{i+1}}(\cdot)), \quad g_{\Delta}(0) = 0,$$

$$u^{(h)}(t) = u^{(h)}_i \in \mathcal{U}(\tau_i, \xi^{(h)}, (g^{(1)})(\tau_i), (\hat{g})(\tau_i), (g^{(2)})(\tau_i))$$

for $t \in [\tau_i, \tau_{i+1}]$.

Let us pass to the description of the algorithm for solving Problem 3. Namely, we describe the procedure of forming the $(h, \Delta, g, \mathcal{U})$-motion \{$x^{(h)}_{\Delta, g}(\cdot), g_{\Delta}(\cdot)$\} corresponding to some fixed partition $\Delta$ and the strategy $\mathcal{U}$ of form (47).

Before the algorithm starts, we fix a value $h \in (0, 1)$ and a partition $\Delta = \{\tau_i\}_{i=0}^m$ with diameter $\delta = \tau_{i+1} - \tau_i$. The work of the algorithm is decomposed into $m - 1$ identical steps. We assume that

$$u^{(h)}(t) = u^{(h)}_0 \in \mathcal{U}(0, x_0, x_0, \hat{g}(0), 0) \quad (48)$$
on the interval $[0, \tau_1]$. Under the action of these constant controls as well as of the unknown disturbance $v_{0, \tau_1}(\cdot)$, the $(h, \Delta, g, \mathcal{U})$-motion

$$\{x^{(h)}_{\Delta, g}(\cdot)\}_{0, \tau_1} = \{x^{(h)}_{\Delta, g}(\cdot; 0, x_0, v^{(h)}_{0, \tau_1}(\cdot), v_{0, \tau_1}(\cdot))\}_{0, \tau_1},$$

$$\{g_{\Delta}(\cdot)\}_{0, \tau_1} = \{g_{\Delta}(\cdot; 0, 0, u^{(h)}_{0, \tau_1}(\cdot), v_{0, \tau_1}(\cdot))\}_{0, \tau_1},$$

of system (43), (44) and the trajectory

$$\{g^{(1)}(\cdot)\}_{0, \tau_1} = \{g^{(1)}(\cdot; 0, x_0, (u^{(1)}_{0, \tau_1}(\cdot))\}_{0, \tau_1},$$

$$\{g^{(2)}(\cdot)\}_{0, \tau_1} = \{g^{(2)}(\cdot; 0, 0, (u^{(2)}_{0, \tau_1}(\cdot))\}_{0, \tau_1},$$

of the system $M_1$ of form (45), (46) are realized.

At the time $t = \tau_1$, we determine $u^{(h)}_1$ from the condition

$$u^{(h)}_1 \in \mathcal{U}(\tau_1, \xi^{(h)}, (g^{(1)})(\tau_1), (\hat{g})(\tau_1), (g^{(2)})(\tau_1)), \quad |\xi^{(h)} - x^{(h)}_{\Delta, g}(\tau_1)|u \leq h; \quad (49)$$

i.e., we assume that

$$u^{(h)}(t) = u^{(h)}_1 \quad \text{for } t \in [\tau_1, \tau_2].$$

Then, we calculate the realization of the $(h, \Delta, w, \mathcal{U})$-motion

$$\{x^{(h)}_{\Delta, g}(\cdot)\}_{\tau_1, \tau_2} = \{x^{(h)}_{\Delta, g}(\cdot; \tau_1, x^{(h)}_{\Delta, g}(\tau_1), u^{(h)}_{\tau_1, \tau_2}(\cdot), v_{\tau_1, \tau_2}(\cdot))\}_{\tau_1, \tau_2},$$

$$\{g_{\Delta}(\cdot)\}_{\tau_1, \tau_2} = \{g_{\Delta}(\cdot; \tau_1, g_{\Delta}(\tau_1), u^{(h)}_{\tau_1, \tau_2}(\cdot), v_{\tau_1, \tau_2}(\cdot))\}_{\tau_1, \tau_2}$$

and the trajectory

$$\{g^{(1)}(\cdot)\}_{\tau_1, \tau_2} = \{g^{(1)}(\cdot; \tau_1, g^{(1)}(\tau_1), (u^{(1)}_{\tau_1, \tau_2}(\cdot))\}_{\tau_1, \tau_2},$$

$$\{g^{(2)}(\cdot)\}_{\tau_1, \tau_2} = \{g^{(2)}(\cdot; \tau_1, g^{(2)}(\tau_1), (u^{(2)}_{\tau_1, \tau_2}(\cdot))\}_{\tau_1, \tau_2}.$$
of the system $M_1$.

Let the $(h, \Delta, w, U)$-motion $\{x_{\Delta, g}^h(\cdot)\}$ and the trajectory $\{g^{(1)}(\cdot), g^{(2)}(\cdot)\}$ of the system $M_1$ be defined on the interval $[\tau_0, \tau_i]$. At the time $t = \tau_i$, we assume that

$$u_i^h \in U(\tau_i, \xi_i^h, g^{(1)}(\tau_i), g^{(2)}(\tau_i)),$$

i.e., we set

$$|\xi_i^h - x_{\Delta, g}^h(\tau_i)|_H \leq h;$$

As a result of the action of this control and of the unknown disturbance $v$, the $(h, \Delta, w, U)$-motion

$$\{x_{\Delta, g}^h(\cdot)\}_{\tau_i, \tau_{i+1}} = \{x_{\Delta, g}^h(\cdot; \tau_i, x_{\Delta, g}^h(\tau_i), u_{\tau_i, \tau_{i+1}}^h(\cdot), v_{\tau_i, \tau_{i+1}}(\cdot))\}_{\tau_i, \tau_{i+1}},$$

and the trajectory

$$\{g^{(1)}(\cdot)\}_{\tau_i, \tau_{i+1}} = \{g^{(1)}(\cdot; \tau_i, g_{\Delta, g}^h(\tau_i), u_{\tau_i, \tau_{i+1}}^h(\cdot), v_{\tau_i, \tau_{i+1}}(\cdot))\}_{\tau_i, \tau_{i+1}},$$

and

$$\{g^{(2)}(\cdot)\}_{\tau_i, \tau_{i+1}} = \{g^{(2)}(\cdot; \tau_i, g_{\Delta, g}^h(\tau_i), u_{\tau_i, \tau_{i+1}}^h(\cdot), v_{\tau_i, \tau_{i+1}}(\cdot))\}_{\tau_i, \tau_{i+1}}$$

of the system $M_1$ are realized on the interval $[\tau_i, \tau_{i+1}]$. The above procedure of forming the $(h, \Delta, w, U)$-motion and the trajectories of system $M_1$ stops at the moment $\vartheta$.

**Theorem 5.2.** Let the system $M_1$ be specified by relations (45) and (46). Then the strategy $U : T \times H_1$ of form (4), (47) solves Problem 3. To be exact, the strategy $U$ guarantees the fulfillment of the inequality

$$I(x_{\Delta, g}^h(\cdot), u^h(\cdot), v(\cdot)) \leq I_* + (1 + L_\alpha)\{\mu(\delta, h) + 2\delta h\delta^{-1}\}^{1/2}\exp\{0.5\lambda\vartheta\}.$$

**Proof.** We fix a partition $\Delta$ and a value $h$. Let us rewrite equations (43) for $u = u^h$ and (45) for $u_i = u_i^\alpha$ in the form

$$\frac{\partial}{\partial t} x_{\Delta, g}^h(\nu, t) - \Delta_L x_{\Delta, g}^h(\nu, t) + R_\eta(t, x_{\Delta, g}^h(\nu, t)) + 2/3\eta x_{\Delta, g}^h(\nu, t) + \alpha(K_\eta(t)(x_{\Delta, g}^h(\nu, t))) = F(t, u^h(\nu, t), v(\nu, t))$$

in $Q_\vartheta$

$$\frac{\partial}{\partial \nu} x_{\Delta, g}^h(\nu, t) = 0$$

in $\Sigma_\vartheta$

$$x_{\Delta, g}^h(0) = x_0(\nu)$$

in $\Omega$ (51)

and

$$\frac{\partial}{\partial t} g^{(1)}(\nu, t) - \Delta_L g^{(2)}(\nu, t) + R_\eta(t, g^{(2)}(\nu, t)) + 2/3\eta g^{(2)}(\nu, t) + \alpha(K_\eta(t)g^{(2)}(\nu, t)) = u^{(1)}(\nu, t)$$

in $Q_\vartheta$

$$\frac{\partial}{\partial \nu} g^{(2)}(\nu, t) = 0$$

in $\Sigma_\vartheta$

$$g^{(2)}(\nu, 0) = x_0(\nu)$$

in $\Omega$. (52)

To prove the theorem, we estimate the variation of the functional

$$\varepsilon_1(t) = \varepsilon_2(t) + |g_i(t) - g^{(2)}(t)|^2, \quad t \in [\tau_i, \tau_{i+1}], \quad i = 0, \ldots, m - 1.$$

Here,

$$\varepsilon_2(t) = |x_{\Delta, g}^h(t) - g^{(1)}(t)|^2_H,$$
Let us estimate the variation of the value $\varepsilon_1(t)$ on the interval $[\tau_i, \tau_{i+1}]$ assuming that $\tilde{g}_i(\tau_i)$ satisfies the equality above. Let us subtract (52) from (51) and multiply the difference scalarly (in $H$) by $x_{\Delta,g}^h(t) - g^{1}(t)$. Analogously, we subtract (46) from (44) and multiply the difference by $g(t) - g^{(2)}(t)$. Integrating (for $t \in [\tau_i, \tau_{i+1}]$) and taking into account the monotonicity of the mapping $v \rightarrow R_{0}(t, v)$, analogously to (20), we obtain

$$\varepsilon_1(t) + 2/3\eta \int_{\tau_i}^{t} \varepsilon_2(s) \, ds + \int_{\tau_i}^{t} |x_{\Delta,g}^h(\tau) - g^{1}(\tau)|_{H^1(\Omega)} \, d\tau$$

$$\leq \varepsilon_1(\tau_i) + |\alpha| \int_{\tau_i}^{t} \varepsilon_{1/2}^2(s) \left| K_{\eta}(s) \left( x_{\Delta,g}^h(\cdot, \cdot) - g_{0,s}^{(1)}(\cdot) \right) \right|_H \, ds$$

$$+ \int_{\tau_i}^{t} \left\{ (g_i(s) - g^{(2)}(s)) \left( \phi(s, x_{\Delta,g}^h(s), u^h(s), v(s)) - u_*^{(2)}(s) \right) + \left( x_{\Delta,g}^h(s) - g^{(1)}(s), F(s, u^h(s), v(s)) - u_*^{(1)}(s) \right) \right\} \, ds. \quad (53)$$

Now, we have

$$\int_{\tau_i}^{t} \left| K_{\eta}(s) \left( x_{\Delta,g}^h(\cdot, \cdot) - g_{0,s}^{(1)}(\cdot) \right) \right|_H^2 \, ds \leq d_1(t - \tau_i) \int_{0}^{t} \varepsilon_2(s) \, ds \quad \text{for} \quad t \in [\tau_i, \tau_{i+1}]. \quad (54)$$

In this case, for $t \in [\tau_i, \tau_{i+1}]$, from (53) and (54), it follows that

$$\varepsilon_1(t) \leq \varepsilon_1(\tau_i) + d_1(t - \tau_i) \int_{0}^{t} \varepsilon_2(s) \, ds + d^{(2)}(t - \tau_i) \int_{\tau_i}^{t} \left| x_{\Delta,g}^h(s) - g^{(1)}(s) \right|_{(H^1(\Omega))} \, ds$$

$$+ s_1(t) + d^{(2)}(t - \tau_i) \int_{\tau_i}^{t} \left| \tilde{g}_i(s) - \tilde{g}_i^{(2)}(s) \right| \, ds, \quad (55)$$

where

$$s_1(t) = \int_{\tau_i}^{t} \left\{ \left( x_{\Delta,g}^h(\tau_i) - g^{(1)}(\tau_i) \right) \left( \phi(s, x_{\Delta,g}^h(s), u^h(s), v(s)) - u_*^{(1)}(s) \right) + \left( \tilde{g}_i(\tau_i) - g^{(2)}(\tau_i) \right) \left( \phi(s, x_{\Delta,g}^h(s), u^h(s), v(s)) - u_*^{(2)}(s) \right) \right\} \, ds.$$

By virtue of the inequalities $|x_{\Delta,g}^h(\tau_i) - \xi_i^h|_H \leq h$, we have

$$s_1(t) \leq \int_{\tau_i}^{t} \left\{ \left( \xi_i^h - g^{(1)}(\tau_i) \right) \left( \phi(s, x_{\Delta,g}^h(s), u^h(s), v(s)) - u_*^{(1)}(s) \right) + \left( \tilde{g}_i(\tau_i) - g^{(2)}(\tau_i) \phi(s, x_{\Delta,g}^h(s), u^h(s), v(s)) - u_*^{(2)}(s) \right) \right\} \, ds + d^{(2)}d^{(3)}h(t - \tau_i). \quad (56)$$
Let the symbol $(\cdot, \cdot)_{H^1 \times \mathbb{R}}$ denote the sum of duality of $H^1(\Omega)$ and $(H^1(\Omega))^*$ as well the product of two numbers. Then
\[
(s_i, \Phi(t, \xi^h_i, u, v))_{H^1 \times \mathbb{R}} = (\xi^h_i - g(1)(\tau_i), F(t, u, v)) + (\tilde{g}(\tau_i) - g(2)(\tau_i))\phi(t, \xi^h_i, u, v),
\]
where
\[
s_i = \{\xi^h_i - g(1)(\tau_i), \tilde{g}(\tau_i) - g(2)(\tau_i)\} \in (H^1(\Omega))^* \times \mathbb{R}.
\]
Let us define elements $v^e_i$ from the conditions
\[
\inf_{u \in P} (s_i, \Phi(\tau_i, \xi^h_i, u, v^e_i))_{H^1 \times \mathbb{R}} \geq \sup_{v \in V} \inf_{u \in P} (s_i, \Phi(\tau_i, \xi^h_i, u, v))_{H^1 \times \mathbb{R}} - h. \tag{57}
\]
It is obvious (see Condition 4) that
\[
u_* (t) \in H(t, g(1)(t)) \subset \bigcup_{u \in P} \Phi(t, g(1)(t), u, v^e_i) \quad \text{for a. a.} \quad t \in [\tau_i, \tau_{i+1}].
\]
Then there exists a control $u^{(s)}(t) \in P$, $t \in [\tau_i, \tau_{i+1})$, such that
\[
\Phi(t, g(1)(t), u^{(s)}(t), v^e_i) = u_*(t) \quad \text{for a. a.} \quad t \in [\tau_i, \tau_{i+1}] \tag{58}.
\]
Using the rule of definition of the strategy $U$, we deduce that
\[
(s_i, \Phi(\tau_i, \xi^h_i, u^{(s)}(t), v(t)))_{H^1 \times \mathbb{R}} \leq \sup_{v \in V} (s_i, \Phi(\tau_i, \xi^h_i, u^{(s)}(t), v))_{H^1 \times \mathbb{R}} \leq \inf_{u \in P} \sup_{v \in V} (s_i, \Phi(\tau_i, \xi^h_i, u, v))_{H^1 \times \mathbb{R}} + h. \tag{59}
\]
In turn, from (57) we have
\[
\sup_{v \in V} \inf_{u \in P} (s_i, \Phi(\tau_i, \xi^h_i, u, v))_{H^1 \times \mathbb{R}} \leq \inf_{u \in P} (s_i, \Phi(\tau_i, \xi^h_i, u, v^e_i))_{H^1 \times \mathbb{R}} + h. \tag{60}
\]
Moreover, it is evident that the equality
\[
\inf_{u \in P} \sup_{v \in V} (s_i, \Phi(\tau_i, \xi^h_i, u, v))_{H^1 \times \mathbb{R}} = \sup_{v \in V} \inf_{u \in P} (s_i, \Phi(\tau_i, \xi^h_i, u, v))_{H^1 \times \mathbb{R}} \tag{61}
\]
is valid. From (59)–(61) we have
\[
(s_i, \Phi(\tau_i, \xi^h_i, u^{(s)}(t), v(t)))_{H^1 \times \mathbb{R}} \leq \inf_{u \in P} (s_i, \Phi(\tau_i, \xi^h_i, u, v^e_i))_{H^1 \times \mathbb{R}} + 2h
\]
\[
\leq (s_i, \Phi(t, \xi^h, u^{(s)}(t), v^e_i))_{H^1 \times \mathbb{R}} + 2h + L\pi_i (t - \tau_i), \tag{62}
\]
where
\[
\pi_i = |\xi^h_i - g(1)(\tau_i)|_{(H^1(\Omega))^*} + |\tilde{g}(\tau_i) - g(2)(\tau_i)|.
\]
In this case, it follows from (58) and (62) that
\[
(s_i, \Phi(\tau_i, \xi^h_i, u^e_i, v(t)))_{H^1 \times \mathbb{R}} \leq 2h + L\pi_i (t - \tau_i) + L\pi_i |\xi^h_i - g(1)(\tau_i)|_{(H^1(\Omega))^*} \tag{63}
\]
for a.a $t \in [\tau_i, \tau_{i+1}]$. Therefore, using (63), for $t \in [\tau_i, \tau_{i+1}]$, we deduce that
\[
(s_i, \Phi(t, \xi^h_i, u^e_i, v(t)) - u_*(t))_{H^1 \times \mathbb{R}} \leq 2h + 2L\pi_i (t - \tau_i) + L\pi_i |\xi^h_i - g(1)(\tau_i)|_{(H^1(\Omega))^*}. \tag{64}
\]
It is easily seen that the following estimate
\[
\pi_i \leq d^{(3)}(\xi^h_i(1/2)(\tau_i) + h), \tag{65}
\]
\[
|g(1)(\tau_i)|_{(H^1(\Omega))^*} \leq \int_{\tau_i}^t |g(1)(s)|_{(H^1(\Omega))^*} ds \leq (0.5d^{(1)})^{1/2}(t - \tau_i)^{1/2}
\]
takes place. In this case, by virtue of (65), the inequalities
\[ |\xi^h_i - g^{(1)}(t)|_{(H^1(\Omega))} \leq |\xi^h_i - g^{(1)}(\tau_i)|_{(H^1(\Omega))} + \int_{\tau_i}^t |g^{(1)}(\tau)|_{(H^1(\Omega))} \, d\tau \]
\[ \leq d^{(3)}h + |x^h_{A,g}(\tau_i) - g^{(1)}(\tau_i)|_{(H^1(\Omega))} + (0.5 \, d^{(1)})^{1/2} (t - \tau_i)^{1/2} \]
\[ \leq d^{(3)}h + d^{(3)}\varepsilon_2^2/2(\tau_i) + (0.5 \, d^{(1)})^{1/2} (t - \tau_i)^{1/2} \quad \text{for a.a. } t \in [\tau_i, \tau_{i+1}] \]
hold. In addition, the inequality
\[ \varepsilon_2(t) \leq \varepsilon_1(t), \quad t \in T, \]
takes place. Therefore,
\[ \pi_i |\xi^h_i - g^{(1)}(t)|_{(H^1(\Omega))} \leq \{d^{(3)}(\varepsilon_1(\tau_i)^{1/2} + h)\} \{d^{(3)}(\varepsilon_2(\tau_i)^{1/2} + h)\} + (t - \tau_i)^{1/2} (0.5 \, d^{(1)})^{1/2} \leq 2(d^{(3)})^2 \varepsilon_1(\tau_i) + 4(d^{(3)})^2 h^2 + d^{(1)}(t - \tau_i). \]
Here, we use the inequality $ab \leq a^2/2 + b^2/2$. Consequently, see (64),
\[ (s_i, \Phi(t, \xi^h_i, u^h_i, v(t)) - u_*(t))_{H \times R} \leq 2h + 2L(t - \tau_i) \{d^{(3)}(\varepsilon_1^{1/2}(\tau_i) + h)\} + 4(d^{(3)})^2 Lh^2 + Ld^{(1)}(t - \tau_i) + 2L(d^{(3)})^2 \varepsilon_1(\tau_i). \] (66)
In addition, for $s \in [\tau_i, \tau_{i+1}]$, the inequality
\[ 2L(t - \tau_i) \{d^{(3)}(\varepsilon_1^{1/2}(\tau_i) + h)\} \leq \varepsilon_1(\tau_i) + h^2 + 2L^2(d^{(3)})^2 (t - \tau_i)^2 \]
is fulfilled. In this case, for $t \in [\tau_i, \tau_{i+1}]$, $\tau_{i+1} - \tau_i \leq 1$, we deduce from (56) that
\[ s_i(t) \leq \int_{\tau_i}^t (s_i, \Phi(s, \xi^h_i, u_i, v(s)) - u_*(s))_{H \times R} \, ds + d^{(2)}h(t - \tau_i) \leq (1 + 4L)(d^{(3)})^2 h^2 (t - \tau_i) + (1 + 2L(d^{(3)})^2)(t - \tau_i)\varepsilon_1(\tau_i) + L^2(d^{(1)})^2 (t - \tau_i)^2 + (2 + d^{(2)})h(t - \tau_i). \] (67)
By virtue of Lemma 1,
\[ \int_{\tau_i}^t |\dot{\xi}^h_{A,g} - \dot{g}^{(1)}(s)|_{(H^1(\Omega))} \, ds \leq (t - \tau_i)^{1/2}(d^{(1)})^{1/2}. \]
Moreover, the inequality
\[ \int_{\tau_i}^t |\dot{g}_i(s) - \dot{g}^{(2)}(s)| \, ds \leq (t - \tau_i)d^{(2)} \]
is valid. Therefore,
\[ d^{(2)}(t - \tau_i) \int_{\tau_i}^t |\dot{\xi}^h_{A,g} - \dot{g}^{(1)}(s)|_{(H^1(\Omega))} \, ds \]
\[ + d^{(2)}(t - \tau_i) \int_{\tau_i}^t |\dot{g}_i(s) - \dot{g}^{(2)}(s)| \, ds \leq 2d^{(2)}(t - \tau_i)^{3/2}((d^{(1)})^{1/2} + d^{(2)}). \]
Consequently, using (55) and (67), we get for $t \in [\tau_i, \tau_{i+1})$
\[ \varepsilon_1(t) \leq \varepsilon_1(\tau_i) + k(t - \tau_i) \int_0^t \varepsilon_2(s) \, ds + c_0 h(t - \tau_i) \]
To prove the theorem, it is sufficient to establish that the inequality
\[ \varepsilon_1(t) \leq (\mu(\delta, h) + 2\epsilon t) \exp{\lambda t} \]
takes place for \( t \in [\tau_j, \tau_{j+1}] \), \( j \in [0 : m - 1] \). Now, we apply the method of induction. Note that
\[ \varepsilon_1(0) = |\tilde{g}_0(0) - g^{(2)}(0)|^2 = |\tilde{g}_0(0)|^2 = \mu(\delta, h). \]
The existence of \( \tilde{g}_0(0) \) satisfying this equality is evident. For \( t \in [0, \tau_1] \) from (68) we deduce that
\[ \varepsilon_1(t) \leq \mu(\delta, h)(1 + \beta_s t + t) + kt \int_0^t \varepsilon_1(s) \, ds. \]
Using the Gronwall lemma and this inequality, we obtain
\[ \varepsilon_1(t) \leq \mu(\delta, h)\{1 + \beta_s t + t\} \exp(k\delta t) \]
for \( t \in [0, \tau_1] \). It is easily seen that
\[ 1 + \beta_s t + t \leq \exp\{(\lambda - k\delta)\} \]
for \( t \in [0, \delta] \) and \( \delta \in (0, 1) \). Therefore,
\[ \varepsilon_1(t) \leq \mu(\delta, h) \exp(\lambda t) \text{ for } t \in [0, \tau_1]. \]
Inequality (69) takes place for \( t \in [0, \tau_1] \). Let this inequality fulfill for \( j = 1, \ldots, i - 1 \), i.e.
\[ \varepsilon_1(\tau_{i-1} + s) \leq (\mu(\delta, h) + 2(i - 1)ch) \exp{\lambda(\tau_{i-1} + s)} \text{ for } s \in [0, \delta). \]
Note that
\[ \lim_{\tau \to +\infty} \varepsilon_1(\tau_{i-1} + s) = |x^{(1)}_{\Delta, g}(\tau_i) - g^{(1)}(\tau_i)|^2_H \]
\[ + |\tilde{g}_{i-1}(\tau_i) - g^{(2)}(\tau_i)|^2 \leq (\mu(\delta, h) + 2(i - 1)ch) \exp{\lambda(\tau_i)}. \]
Moreover, by virtue of Lemma 1 and inequalities \( |\xi^h - x^{(1)}_{\Delta, g}(\tau_i)|^2 \leq h, \)
we conclude that
\[ |\xi^h - g^{(1)}(\tau_i)|^2_H \leq |x^{(1)}_{\Delta, g}(\tau_i) - g^{(1)}(\tau_i)|^2_H \]
\[ + 2|x^{(1)}_{\Delta, g}(\tau_i) - g^{(1)}(\tau_i)|^2_H h + h^2 \leq |x^{(1)}_{\Delta, g}(\tau_i) - g^{(1)}(\tau_i)|^2_H + 2\delta^{1/2}(\xi^h)^{1/2} h + h^2 \leq ch. \]
Therefore,
\[ \Phi(\xi^h, g^{(1)}(\tau_i), \tilde{g}_{i-1}(\tau_i), g^{(2)}(\tau_i)) = |\xi^h - g^{(1)}(\tau_i)|^2_H \]
\[ + |\tilde{g}_{i-1}(\tau_i) - g^{(2)}(\tau_i)|^2 \leq (\mu(\delta, h) + 2(i - 1)ch) \exp{\lambda(\tau_i)} + ch. \]
In this case, there exists \( \tilde{g}(\tau_i) \) satisfying the relations
\[ \Phi(\xi^h, g^{(1)}(\tau_i), \tilde{g}(\tau_i), g^{(2)}(\tau_i)) = (\mu(\delta, h) + 2(i - 1)ch) \exp{\lambda(\tau_i)} + ch. \]
Therefore,
\[ \varepsilon_1(\tau_i) \leq (\mu(\delta, h) + 2(i - 1)ch) \exp{\lambda(\tau_i)} + ch + ch \]
\[ \leq (\mu(\delta, h) + 2cch) \exp{\lambda(\tau_i)}. \]
To prove the theorem, it is sufficient to establish that inequality (69) (for \( j = i \)) is valid for \( t \in [\tau_i, \tau_{i+1}] \). Note that \( \varepsilon_2(t) \leq \varepsilon_1(t) \). Therefore
\[ \int_0^{\tau_i} \varepsilon_2(s) \, ds \leq \{\mu(\delta, h) + 2(i - 1)ch\} \int_0^{\tau_i} \exp{\lambda s} \, ds \]
\[ = \{\mu(\delta, h) + 2(i - 1)ch\} \lambda^{-1}(\exp{\lambda \tau_i} - 1). \]
From this inequality and (68), for $t \in [\tau_i, \tau_{i+1})$ we derive the estimate
\[
\varepsilon_1(t) \leq (1 + \beta_s(t - \tau_i))\varepsilon_1(t_i) + \{\mu(\delta, h) + 2(i - 1)ch\} \lambda^{-1}k(t - \tau_i)(\exp\{\lambda\tau_i\} - 1) + \mu(\delta, h)(t - \tau_i) + k\delta \int_{\tau_i}^t \varepsilon_1(s) \, ds.
\] (71)

By virtue of (70) and (71), we obtain
\[
\varepsilon_1(t) \leq \{\mu(\delta, h) + 2ich\}\{1 + \beta_s(t - \tau_i) + \lambda^{-1}k(t\tau_i)\} \exp\{\lambda\tau_i\} + \mu(\delta, h)(t - \tau_i) + k\delta \int_{\tau_i}^t \varepsilon_1(s) \, ds
\] (72)

Using the Gronwall lemma and (72), for $t \in [\tau_i, \tau_{i+1}]$ we get the inequality
\[
\varepsilon_1(t) \leq \left[\{\mu(\delta, h) + 2ich\}\{1 + \beta_s(t - \tau_i) + \lambda^{-1}k(t - \tau_i)\} \exp\{\lambda\tau_i\} + \mu(\delta, h)(t - \tau_i) \exp\{k\delta(t - \tau_i)\}\right];
\]
i.e.,
\[
\varepsilon_1(t) \leq \{\mu(\delta, h) + 2ich\}\{1 + \beta_s(t - \tau_i) + \lambda^{-1}k(t - \tau_i) \exp\{\lambda\tau_i + k\delta(t - \tau_i)\} + \mu(\delta, h)(t - \tau_i) \exp\{k\delta(t - \tau_i)\}\}.
\]

To prove (69), it is sufficient to show that the right-hand part of this inequality is no greater than $\{\mu(\delta, h) + 2ich\} \exp\{\lambda\} \text{ for } t \in [\tau_i, \tau_{i+1})$. Obviously, this is true if
\[
1 + \beta_s(t - \tau_i) + \lambda^{-1}k(t - \tau_i) + t - \tau_i \leq \exp\{(\lambda - k\delta)(t - \tau_i)\}
\]
and
\[
1 + \beta_s(t - \tau_i) + \lambda^{-1}k(t - \tau_i) + t - \tau_i \leq \exp\{(\lambda - k\delta)(t - \tau_i)\}.
\]

Therefore, inequality (73) is true if
\[
1 + \beta_s + k\lambda^{-1} \leq \lambda - k\delta \quad \text{for } \delta \in [0, 1].
\]

However, the latter inequality holds in virtue of the choice of $\lambda$. Inequality (69) is verified. Thus,
\[
\varepsilon_1(\vartheta) \leq \{\mu(\delta, h) + 2mch\} \exp\{\lambda\vartheta\}.
\]

From this inequality, it follows that
\[
|x_{\Delta,g}^{h}(\vartheta) - g^{(1)}(\vartheta)||_H \leq \{\mu(\delta, h) + 2mch\}^{1/2} \exp\{0.5\lambda\vartheta\},
\] (74)
\[
|g_{m-1}(\vartheta) - g^{(2)}(\vartheta)| \leq \{\mu(\delta, h) + 2mch\}^{1/2} \exp\{0.5\lambda\vartheta\}.
\]

Therefore, using (74), we obtain
\[
|\sigma(g^{(1)}(\vartheta)) - \sigma(x_{\Delta,g}^{h}(\vartheta))| \leq L_\sigma\{\mu(\delta, h) + 2mch\}^{1/2} \exp\{0.5\lambda\vartheta\}.
\] (75)

Note that $m = \vartheta\delta^{-1}$. To complete the proof of theorem, it remains to show that the inequalities
\[
g(t) \leq g_i(t) \text{ for } t \in [\tau_i, \tau_{i+1}), \quad i \in [0 : m - 1],
\] (76)

are valid. In this case, taking into account (75) and (76), we derive
\[
\sigma(x_{\Delta,g}^{h}(\vartheta)) + g(\vartheta) \leq \sigma(x_{\Delta,g}^{h}(\vartheta)) + g_{m-1}(\vartheta) \leq \sigma(g^{(1)}(\vartheta)) + g^{(2)}(\vartheta) + (1 + L_\sigma)\{\mu(\delta, h) + 2\vartheta ch\delta^{-1}\}^{1/2} \exp\{0.5\lambda\vartheta\} = I_\vartheta + (1 + L_\sigma)\{\mu(\delta, h) + 2\vartheta ch\delta^{-1}\}^{1/2} \exp\{0.5\lambda\vartheta\}. \] (77)
Here, we use the equality
\[ \sigma(g^{(1)}(\vartheta)) + g^{(2)}(\vartheta) = I_\ast \]
(see Condition 4). The statement of the theorem follows from (77). The theorem is proved.

**Remark 6.** The solution of Problem 4 is analogous to the solution of Problem 3. In this case, instead of the set \( H_\ast(t, g) \), we apply the set
\[ H_1(t; x) = \bigcap_{u \in P} \Phi_v(t, x, u), \]
where
\[ \Phi_v(t, x, u) = \bigcup_{v \in V} \Phi(t, x, u, v), \]
Condition 4 is replaced by the following condition: there exists a control \( \tilde{u}_\ast(\cdot) = \{\tilde{u}_\ast^{(1)}(\cdot), \tilde{u}_\ast^{(2)}(\cdot)\} \),
\[ \tilde{u}_\ast(t) \in H_1(t; g^{(1)}(t)) \quad \text{for a.a.} \ t \in T, \]
such that
\[ I_\ast = \sigma(g^{(1)}(\vartheta)) + g^{(2)}(\vartheta). \]
Here, \( g^{(1)}(\cdot) \) and \( g^{(2)}(\cdot) \) are the solutions of equations (45) and (46) for \( u^{(1)}(\cdot) = \tilde{u}_\ast^{(1)}(\cdot) \) and \( u^{(2)}(\cdot) = \tilde{u}_\ast^{(2)}(\cdot) \). The system \( M_2 \) coincides with the system \( M_1 \) of form (45), (46). The strategy \( \mathcal{V} \) is defined by the rule
\[ \mathcal{V}(\tau_1, \xi, g^{(1)}, \tilde{g}(\tau_1), g^{(2)}) = \left\{ v^c \in V : (\tilde{g}(\tau_1) - g^{(2)})\phi_2(\tau_1, \xi, v^c) + (\xi - g^{(1)}, F_2(\tau_1, v^c)) \leq \inf \left\{ (\tilde{g}(\tau_i) - g^{(2)})\phi_1(\tau_i, \xi, v) + (\xi - g^{(1)}, F_2(\tau_i, v)) : v \in V \right\} + h \right\} \quad \text{if } t \in [\tau_i, \tau_{i+1}), \]
where \( \tilde{g}(\tau_i) \) is the minimal number with the property
\[ \Phi(\xi, g^{(1)}, \tilde{g}(\tau_i), g^{(2)}) = |\xi - g^{(1)}|^2 + |\tilde{g}(\tau_i) - g^{(2)}|^2 \]
\[ = \{\mu(\vartheta, h) + 2(i - 1)ch\} \exp\{\lambda t\} + ch \quad \text{if } i \in [1 : m - 1], \]
\[ \Phi(\xi, g^{(1)}, \tilde{g}(0), g^{(2)}) = |\tilde{g}(0)|^2 = \mu(\vartheta, h) \quad \text{if } i = 0. \]
This choice of the strategy \( \mathcal{V} \) guarantees the fulfilment of the inequality
\[ I_\ast - (1 + L_\vartheta )\{\mu(\vartheta, h) + 2\vartheta ch\delta^{-1}\}^{1/2} \exp\{0.5\lambda \vartheta\} \leq I(x_{\Delta, g, v}^h(\cdot), u(\cdot), v^h(\cdot)). \]

**Remark 7.** Let problems 1 and 3 be complexified by the presence of hard constraints on the state dynamics. Namely, we have to solve problem 1 (or 3) under the condition that the solution \( x_\ast(\cdot) \) of equation (1) does not leave a given set \( N_\ast \subset H \).
Formally, it means that the additional conditions:
\[ x_{\Delta, w}^h(t) \in N_\ast \quad \text{for all } t \in T, \]
\[ x_{\Delta, q}^h(t) \in N_\ast \quad \text{for all } t \in T \]
appear in the formulations of problem 1 and 3, respectively.

Let, instead of Condition 3, we have *Condition 3′*: There exists an optimal control \( u_\ast(\cdot) \) with the properties:
\[ w(\vartheta; 0, x_0, u_\ast(\cdot)) \in N, \]
\[ w(t; 0, x_0, u_\ast(\cdot)) \in N_\ast \quad \forall t \in T. \]
In its turn, instead of Condition 4, we consider Condition 4': There exists a control \( u_\ast(\cdot) = \{u_\ast^{(1)}(\cdot), u_\ast^{(2)}(\cdot)\} \subset H_\ast(t; g^{(1)}(t)) \) for a.a. \( t \in T \) such that \( g^{(1)}(t) \in N_\ast \forall t \in T \) and \( I_\ast = \sigma(g^{(1)}(\vartheta)) + g^{(2)}(\vartheta) \).

Then, the algorithms for solving problems 1 and 3 described in Section 3 and 5 can be applied to solving corresponding problems in the case of additional constraints specified above.

6. Numerical example. In this section we present a numerical example for the Schlögl model. The problem described in Remark 5 is solved. This problem consists in the following. There is the equation for the partner

\[
\frac{\partial}{\partial t} x(\nu, t) - \Delta_L x(\nu, t) + R(x(\nu, t)) = u(\nu, t) \quad \text{in} \quad Q_\vartheta
\]

\[
\partial_\mu x(\nu, t) = 0 \quad \text{in} \quad \Sigma_\vartheta
\]

\[x(\nu, 0) = x_0(s) \quad \text{in} \quad \Omega.\]  

(78)

It is assumed that we have the second equation for the opponent

\[
\frac{\partial}{\partial t} x^{(1)}(\nu, t) - \Delta_L x^{(1)}(\nu, t) + R(x^{(1)}(\nu, t)) = v(\nu, t) \quad \text{in} \quad Q_\vartheta
\]

\[
\partial_\mu x^{(1)}(\nu, t) = 0 \quad \text{in} \quad \Sigma_\vartheta
\]

\[x^{(1)}(\nu, 0) = x^{(1)}_0(s) \quad \text{in} \quad \Omega = (0, 1),\]  

(79)

influenced by the cation of an unknown disturbance \( v = v(\nu, t) \in V \). The problem is to design a control

\[u(t) = u^h(t) = U(\tau_i, \xi^h_i, \psi^h_i) \subset P \quad \text{for} \quad t \in [\tau_i, \tau_{i+1})\]

providing the validness of the inequality

\[|x(\vartheta, 0, x^h_0(\cdot)) - x^{(1)}(\vartheta, 0, x^{(1)}_0(\cdot), v(\cdot))|_H \leq \varepsilon.\]

As is noted in Remark 5, to solve this problem, one can use a strategy \( U \) of the form (14), (16) and (17), namely,

\[U(\tau_i, \xi^h_i, \psi^h_i) = \{u^c \in P : (\xi^h_i - \psi^h_i, u^c) \leq \inf \{(\xi^h_i - \psi^h_i, u) : u \in P\} + h\},\]

\[\text{if} \ i \in [1 : m - 1], \quad U(0, \xi^0_i, \psi^0_i) = 0, \quad \text{if} \ i = 0.\]

Thus, as a control in equation (78), one can take the value

\[u^h(t) = u^h_i \quad \text{for a.a.} \quad t \in [\tau_i, \tau_{i+1}),\]

where \( u^h_i \) satisfies the relation

\[u^h_i = \arg \min \{(\xi^h_i - \psi^h_i, u) : u \in P\}.\]  

(80)

In the experiment, we set

\[P = \{\pi(\nu)u : |u| \leq 2\}, \quad V = \{\pi(\nu)v : |v| \leq 1\},\]  

(81)

where \( \pi(\nu) = 0.5(\nu^2 - 1), \nu \in [0, 1] \), i.e. \( \pi(\cdot) \in H^1(\Omega) \). The control in the right-hand part of equation (78) is calculated by formula (80). In the case when \( \Omega = (0, 1) \) and the set \( P \) has the form (81), we have

\[u^h_i = \begin{cases} 2, & \text{if} \ (\xi^h_i - \psi^h_i, \pi)_H \leq 0, \\ -2, & \text{if} \ (\xi^h_i - \psi^h_i, \pi)_H > 0. \end{cases}\]

Equations (78) and (79) are solved by the set method with the step \( \Delta \omega \) in the domain \( \Omega \). In Figs. 1, 2 the results of computer modeling are presented for the
following case:

\[ y_1 = 0.1; \quad y_2 = 0.2; \quad y_3 = 0.3; \]
\[ \vartheta = 2, \quad \Delta \omega = 1/15, \quad \delta = 0.02, \]
\[ x_0(\nu) = 25j\Delta \omega(1 - j\Delta \omega), \quad x_0^{(1)}(\nu) = 25j\Delta \omega(1 - j\Delta \omega) + h, \]
\[ j = 0, \ldots, n; \quad n = 15. \]

During the experiment, we assume

\[ \xi_h^{(b)}(\nu_j) = x(\nu_j, \tau_i) + h, \quad \psi_h^{(b)}(\nu_j) = x^{(1)}(\nu_j, \tau_i) + h, \]

where \( \nu_j = j\Delta \omega, \quad j = 0, \ldots, \frac{1}{\Delta \omega} \). Fig. 1 corresponds to the case when

\[ v(\nu, t) = 0.5\nu(\nu - 1)v(t), \quad v(t) = -1, \]

and Fig. 2, to the case when

\[ v(\nu, t) = 0.5\nu(\nu - 1)v(t), \]
\[ v(t) = \begin{cases} 1, & t \in [0, 1] \\ |\sin(10t)|, & t \in [1, 2]. \end{cases} \]

In these figures the solid line represents the function \( x^{(1)}(2; 0, x_0^{(1)}, v(\cdot)) \), the dotted line, the dashed line and the chain line represent \( x(2; 0, x_0, u^b(\cdot)) \) for \( h = 0.01 \),
$h = 0.1$, $h = 0.5$, respectively. As is seen from the figures, the curves $x^{(1)}(2; 0, x_0^{(1)}, v(\cdot))$ and $x(2; 0, x_0, u^h(\cdot))$ actually coincide for $h = 0.01$.

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