Revisiting factorability and indeterminism

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Perhaps it is not completely superfluous to remind that Clauser-Horne factorability, introduced in [1], is only necessary when λ, the hidden variable (HV), is sufficiently deterministic: for \{M_i\} a set of possible measurements (isolated or not by space-like intervals) on a given system, the most general sufficient condition for factorability on λ is obtained by finding a set of expressions \(M_i = M_i(\lambda, \xi_i)\), with \{\xi_i\} a set of HV’s, all independent from one another and from λ. Otherwise, factorability can be recovered on γ = λ ⊕ μ, with μ another additional HV, so that now \(M_i = M_i(\gamma, \xi_i)\): conceptually, this is always possible; experimentally, it may not: μ may be inaccessible or even its existence unknown (and so, too, from the point of view of a phenomenallegical theory). Results here may help clarify our recent post in [6].

In relation to Bell inequalities, and maybe caused by the lack of a common perspective, factorability and indeterminism are sometimes a subject of prejudiced argumentation (at least at the informal level; that is my experience): let us for that reason revisit those two concepts here. We will try to settle a simple, completely abstract approach; not necessary orthodox, we must warn.

Definitions:

We will say a measurement \(M\) upon a certain physical system, with \(k\) possible outcomes \(m_k\), is deterministic on a hidden variable \(HV\) \(\lambda\) (summarizing the state of that system), if (and only if)

\[ P(M = m_k | \lambda) \in \{0, 1\}, \forall k, \lambda, \]  

which allows us to write

\[ M \equiv M(\lambda), \]  

and indeterministic iff, for some \(\lambda\), some \(k'\),

\[ P(M = m_{k'} | \lambda) \neq \{0, 1\}, \]  

i.e., at least for some (at least two) of the results for at least one (physically meaningful) value of \(\lambda\).

Let also \(M = \{M_i\}\) be a set of possible measurements, each with a set \(\{m_{i,k}\}\) of possible outcomes, not necessarily isolated from each other by a space-like interval.

Now, indeterminism can be turn into determinism, i.e., [3] can into [1], by defining a new hidden variable \(\mu\), so that now, with \(\gamma \equiv \lambda \oplus \mu\):

\[ P(M_i = m_{i,k} | \gamma) \in \{0, 1\}, \forall i, \forall k, \gamma, \]  

which means we can write, for any of the \(M_i\)'s,

\[ M_i \equiv M_i(\lambda, \mu), \]  

a proof that such a new hidden variable \(\mu\) can always be found (or built) given in [3].

So far, then, our determinism and indeterminism are conceptually equivalent, though of course they may correspond to different physical situations, depending for instance on whether \(\gamma\) is experimentally accesible or not. Nevertheless, for us there is still another natural step to take, introducing the following distinction: we will say indeterminism is

(a) \(\lambda\)-factorizable, iff we can find a set \{\xi_i\} of random variables, independent from each other and from \(\lambda\) too, such that

\[ \mu = \bigoplus_i \xi_i, \]  

and [1] holds again for each \(M_i\) on \(\gamma_i \equiv \lambda \oplus \xi_i\):

\[ P(M_i = m_{i,k} | \gamma_i) \in \{0, 1\}, \forall i, \forall k, \gamma_i, \]  

this last expression meaning of course that we can write, again for any of the \(M_i\)'s,

\[ M_i \equiv M_i(\lambda, \xi_i). \]  

(b) non \(\lambda\)-factorizable, iff [1] is not possible for any set of statistically independent \(\xi_i\)'s.

Now let us, for simplicity, restrict our reasonings to \(A, B \in M\), with two possible outcomes, \(A, B \in \{+1, -1\}\), all without loss of generality. We have seen that, as the more general formulation, we can always write something like \(A = A(\lambda, \xi_A), B = B(\lambda, \xi_B)\).

Lemma:

(i) If \(A\) and \(B\) are deterministic on \(\lambda\), i.e., [1] holds for \(A\) and \(B\), then they are also \(\lambda\)-factorizable, i.e.,

\[ P(A = a, B = b | \lambda) = P(A = a | \lambda) \cdot P(B = b | \lambda), \]  

for any \(a, b \in \{+1, -1\}\). Eq. [9] is nothing but the so-called Clauser-Horne factorability condition [1].
(ii) If $A$ and $B$ are indeterministic on $\lambda$, i.e., if (1) does not hold for $\lambda$, then: for some $\mu$ (always possible to find $\lambda$) such that now (4) holds for $\gamma \equiv \lambda \oplus \mu$, $A, B$ are $\gamma$-factorizable,

$$P(A = a, B = b|\gamma) = P(A = a|\gamma) \cdot P(B = b|\gamma), \quad (10)$$
i.e., (9) holds for $\gamma$, but this time not necessarily for $\lambda$.

(iii) Let (7) hold for $A, B$, on $\lambda, \xi_A, \xi_B$: if $\lambda, \xi_A, \xi_B$ are statistically independent, (hence, $A$ and $B$ are what we have called $\lambda$-factorizable), then (9) holds for $\lambda$, not necessarily on the contrary.

Proof:

(i) When (1) holds, for any $\lambda$ and any $a, b \in \{+1, -1\}$, $P(A = a|\lambda), P(B = b|\lambda) \in \{0, 1\}$, from where we can, trivially, get to (9).

(ii) It is also trivial that, if (9) holds, (1) can be recovered for $\gamma$. That the same is not necessary for $\lambda$ can be seen with the following counterexample: suppose, for instance, that for $\lambda = \lambda_0$, either $A = B = 1$ or $A = B = -1$ with equal probability. It is easy to see that

$$P(A = B = 1|\lambda_0) \neq P(A = 1|\lambda_0) \cdot P(B = 1|\lambda_0), \quad (11)$$
numerically: $\frac{1}{2} \neq \frac{1}{4}$.

(iii) We have, from independence of $\lambda, \xi_A, \xi_B$, and working with probability densities $\rho$’s: $\rho_A(\lambda, \xi_A, \xi_B) = \rho_A(\lambda) \cdot \rho_A(\xi_A) \cdot \rho_B(\xi_B)$, which we can use to write

$$P(A = a, B = b|\lambda) = \int P(A = a, B = b|\lambda, \xi_A, \xi_B) \cdot \rho_A(\xi_A) \cdot \rho_B(\xi_B) \ d\xi_A d\xi_B. \quad (12)$$

Those conditioned probabilities should be defined also as densities but for simplicity we leave that aside.

Using now the fact that we can recover (4) for $A$ ($B$) on $\gamma_A = \lambda \oplus \xi_A \ (\gamma_B = \lambda \oplus \xi_B)$,

$$P(A = a, B = b|\lambda) = \int P(A = a|\lambda, \xi_A) \cdot P(B = b|\lambda, \xi_B) \cdot \rho_A(\xi_A) \cdot \rho_B(\xi_B) \ d\xi_A d\xi_B = \int P(A = a|\lambda) \cdot \rho_B(\xi_B) \ d\xi_B = P(A = a|\lambda) \cdot P(B = b|\lambda). \quad (13)$$

On the other hand, let $\lambda, \xi_A, \xi_B$ be not statistically independent: we can set for instance, as a particular case, $\xi_i \equiv \mu, \ \forall i$, therefore reducing our case to that of (4). Once this is done, our previous counterexample in (ii) is also valid to show that factorability is not necessary for $\lambda$ here.

Conclusions: In a bipartite (multipartite) Bell experiment, assuming information is not degraded on its way from the source to the measurement devices, $\xi_A, \xi_B$ ($\xi_i$) can be interpreted as the state of devices $A, B$ (device $i$-th) and their surrounding, their independency guaranteed by a space-like separation between observers. Given a theory that predicts results for the set $\mathcal{M}$ of measurements, $\mathcal{M}$ will be necessarily $\lambda$-factorizable only whenever all relevant physical variables are actually included in the vector of hidden variables $\lambda$.

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[1] J.F. Clauser, M.A. Horne. “Experimental consequences of objective local theories”, Phys. Rev. D 10, 526 (1974).
[2] P. Selleri, G. Tarozzi. “Is Clauser and Horne’s factorability a necessary requirement for a probabilistic local theory?”, Lett. Nuovo Cimento (1971 – 1985), 29, N. 16, 533-536 (1980).
[3] P. Selleri, G. Tarozzi. “Quantum mechanics reality and separability”, La Rivista del Nuovo Cimento (1978-1999), 4, N. 2, 1-53 (1981).
[4] R. Risco-Delgado. “Bell’s inequalities and indeterminism”, arXiv:quant-ph/0202099v1 (2002).
[5] A possible (not unique) procedure to build $\mu$ is this: for each $\mathcal{M}_i$, we define a new random variable $\sigma_i$ and assign values for each pair $(\lambda, m_{i,k})$: $\sigma_i \equiv \sigma_i(\lambda, m_{i,k})$, and now simply do

$$\mu \equiv \bigoplus_i \sigma_i. \quad (14)$$

As built, $\sigma_i$’s are not necessarily independent from one another, nor are they necessarily independent from $\lambda$.
[6] D. Rodríguez, “Wigner-PDC description of photon entanglement can still be made completely local”. Arxiv.