Preface to the Japanese version

The purpose of this book is to give an introduction to algebraic combinatorics.

There is no explicit definition of algebraic combinatorics. In Algebraic Combinatorics I (1984), written by Eiichi Bannai and Tatsuro Ito, algebraic combinatorics is described as “a group theory without groups” or “a character theoretical study of combinatorial objects.” Specifically speaking, we pursue the study of combinatorics as an extension or a generalization of the study of finite permutation groups. In this book, we keep this direction in our mind. This is also the approach, initiated by Philippe Delsarte, which enables us to look at a wide range of combinatorial objects such as graphs, codes, and designs from a unified viewpoint. Based on these thoughts, we explain Delsarte’s theory and its various extensions.

When we finished writing Algebraic Combinatorics I, we wanted to write the sequel Algebraic Combinatorics II soon. We regret we did not write it up at that time, but due to various reasons, we could not. We would not say that it has nothing to do with our laziness, but if we could use an excuse, it seems that the developments in the field are not enough to complete a book, and the range of mathematical objects that we are interested in has expanded too widely to handle. Therefore, we thought we could and should write on algebraic combinatorics on spheres, and in 1999, Eiichi Bannai and Etsuko Bannai published Algebraic Combinatorics on Spheres, written in Japanese. We did not prepare an English edition because we planned to write Algebraic Combinatorics II in the future, and we regard the above book as a preparation for it. In the present book, we want to present another subject, Delsarte’s theory in association schemes. Besides, we would like to start writing on the classification of P- and Q-polynomial association schemes, which is the most important problem in Algebraic Combinatorics I, through the study of Terwilliger algebras by Terwilliger and Ito. In this sense, writing up this book enabled us to work seriously on Algebraic Combinatorics II. Before we know it, we get older. Counting the remaining time in our lives, we would like to write one more work.

We give an overview on the contents of this book in the following. Chapter 1 is the introduction to classical combinatorics. The contents are suitable for undergraduate courses. In Japan, there are not so many universities offering lectures on combinatorics for undergraduates. Therefore, we selected basic subjects which beginners in combinatorics should learn. The contents are slightly long for a one-semester course. Chapter 2 is the introduction to association schemes. Some contents overlap with Algebraic Combinatorics I (and with Algebraic Combinatorics on Spheres). We start with the basics, so the chapter is comprehensive for readers who are not familiar with this area. Chapter 3 is the introduction to Delsarte’s theory, which is the theory of codes and designs in association schemes. The description is faithful to the original paper by Delsarte (1973). The contents up to this chapter are understandable to undergraduate students. Chapter 4 is the extension of Delsarte’s theory. Our aim is to introduce
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several papers by Delsarte. The contents are not so difficult compared to the previous chapter, but there remain several parts where we wonder if we make the best use of our “ingredients.” In Chapter 5, away from association schemes, we study finite sets (and algebraic combinatorics) on spheres or on other spaces. As for the case of spheres, Delsarte–Goethals–Seidel (1977) is the starting point of the study in this direction, and the main content of this chapter is to introduce this theory. Of course, some contents of this chapter overlap with Algebraic Combinatorics on Spheres. In some parts, we refer to the above book to avoid duplication. Chapter 5 also contains a survey of the recent works by Eiichi Bannai and Etsuko Bannai and the topics they are recently interested in. The reason to include this subject is to show the similarity of the progress in the study of algebraic combinatorics on association schemes with that on spheres, which is what we think is the essence of algebraic combinatorics.

In Chapter 6, we return to association schemes. As is seen in Algebraic Combinatorics I and also in the previous chapters, Bose–Mesner algebras play an important role in the study of association schemes. We explain the importance of Terwilliger algebras, which are the refined concept of Bose–Mesner algebras, and the role they play in the study of association schemes in the past and the future. This chapter is written by Tatsuro Ito himself on years of research by Terwilliger and him, and the contents are very original. The main theme of Chapter 6 is the classification of P- and Q-polynomial schemes, which overlaps with Chapter 3 of Algebraic Combinatorics I in many parts. We revised the representation theory of Bose–Mesner algebras, which was discussed in the above book, by using principal representations of Terwilliger algebras. Therefore, Leonard’s theorem appearing in Algebraic Combinatorics I, which claims that dual systems of orthogonal polynomials are Askey–Wilson polynomials, is reformulated into the theorem on the classification of Leonard pairs (L-pairs). While the proof of the theorem requires many tools and becomes longer than the direct proof in Algebraic Combinatorics I, the logic used here is much clearer. The concept of L-pairs is obtained by generalizing principal representations of Terwilliger algebras, and a much wider concept called tridiagonal pairs (TD-pairs) arises from general irreducible representations of Terwilliger algebras. In fact, the classification of TD-pairs is completed as they become some sort of tensor products of L-pairs. (Some of the results are still unpublished.) In this book, however, we do not cover this topic because it requires preparations for the representation theory of the quantum affine algebra $U_q(\widehat{sl}_2)$ and the contents would be too long. If the reader is interested in this topic, we recommend the original paper [260]. We introduce basic facts with proofs on TD-pairs which will be needed to read the original paper. In Section 6.4 of Chapter 6, we give the list of known P- and Q-polynomial schemes, where we add Hemmeter schemes, Ustimenko schemes, and twisted Grassmann schemes as new families. In the last section, the author (T. Ito) states his personal opinion on the present situation and the prospect toward the classification of P- and Q-polynomial schemes.

As we mentioned above, the writing styles of Chapter 6 and the other chapters are quite different. The authors are sure that algebraic combinatorics is located at the
junction of the two streams. In this sense, with confidence, we put them together in this book.

Finally, we would like to thank many people who supported us during the preparation of this book. Actually, it was about 20 years ago when we were requested to write this book. While we thought we could write anytime, two decades have passed. We apologize to the publisher and editors (especially Kei Akagi) for the inconvenience. We could have written this book several years ago, and we recognize that we should have completed the book earlier. It may sound like an excuse, but the environments around the authors have drastically changed during this period. In 2008 Etsuko Bannai and in 2009 Eiichi Bannai retired from Kyushu University. In 2011, Eiichi Bannai got a position at Shanghai Jiao Tong University and moved to Shanghai with Etsuko Bannai. Tatsuro Ito retired from Kanazawa University in 2014, and got a position at Anhui University in China. We are trying to develop algebraic combinatorics in new environments. We would like to thank many people who gave us the opportunity to study in China, especially the following people: Zhexian Wan, Hao Shen, Yaokun Wu, Yangxian Wang, Suogang Gao, Michel Deza, Paul Terwilliger, Xiaodong Zhang, Yuehui Zhang, and Yizheng Fan.

It has been a long time since we started writing this book. During this period, in 2011 Nagayoshi Iwahori and in 2015 Noboru Ito passed away. We are sorry that we cannot show them this book. We do not know for how long we can continue, but we would like to study mathematics for as long as possible.

Finally, we would like to thank the people who read the draft and made various comments. Especially, we are grateful to Ryuzaburo Noda, Masaaki Harada, Makoto Tagami, Hajime Tanaka, and Tsuyoshi Miezaki. We would also like to thank Takamichi Ookoshi, the editor, for helping with proofreading.

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