CAMERAL DATA FOR SU(p + 1, p)-HIGGS BUNDLES

ANA PEÓN-NIETO

ABSTRACT. We study the cameral and spectral data for the moduli space of polystable SU(p + 1, p)-Higgs bundles and deduce the latter from the former. As an application, we obtain that the Toledo invariant classifies the connected components of the regular fibers of the Hitchin map.

1. Introduction

Higgs bundle theory has experienced an enormous development since its origins in [Hit87b], due to the rich geometry of these objects.

An instance of this is the so called non-abelian Hodge correspondence, which given a reductive Lie group $G$, establishes a homeomorphism between the moduli space of $G$-Higgs bundles on a Riemann surface $X$, and the moduli space of representations $\rho: \pi_1(X) \to G$ [Hit87b, Don83, Cor88, Sim97, GGM]. A $G$-Higgs bundle is thus naturally seen to be a pair $(E, \phi)$, where $E \to X$ is a holomorphic principal $H^\mathbb{C}$-bundle, for $H \leq G$ maximally compact, and $\phi \in H^0(X, E(m^\mathbb{C}) \otimes K)$ is the Higgs field. In the former, $K \to X$ denotes the canonical bundle, $m$ is the non compact part of the Cartan decomposition $g = h \oplus m$ with complexification $m^\mathbb{C}$ and $E(m^\mathbb{C})$ is the associated bundle via the isotropy representation (38).

The moduli space of polystable $G$-Higgs bundles $\text{Higgs}(G)$ is equipped with extra structure determined by the Hitchin map

$$h_G: \text{Higgs}(G) \to B_G = \bigoplus_{i=1}^{r_{ks}^G} H^0(X, K^{d_i})$$

which sends a pair $(E, \phi)$ to the characteristic coefficients of $\phi$; the numbers $d_i - 1$ are the exponents of $G$, and the space $B_G$ is called the Hitchin base.

The study of the Hitchin map is essential for understanding the geometry of $\text{Higgs}(G)$. In the case of complex groups it defines an algebraically completely integrable system [Hit87a, Don], whose description

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Key words and phrases. Higgs bundles, Hitchin map, cameral/spectral cover, cameral/spectral data.

This work was partially funded by the Mathematics Center Heidelberg.
has been applied to many interesting problems [Hit87a, BNR89, KP95, HT03, DP12].

For real groups, (1) is not yet well understood. The tool to undertake this problem in full generality is Donagi’s cameral construction [Don93, Don], and Donagi-Gaitsgory’s [DG01], which was adapted to real groups in the author’s thesis [PN13] (see also [GPR14, GP]). When \( G \) is a matrix group, Hitchin’s spectral techniques [Hit87a] can also be used to understand the fibers of the Hitchin map. This is the approach found in [FG12, FGN, HS14, Sch13b, Sch13a], all of which deal with some classical groups. Just as in the case of complex groups, the Hitchin map for real groups has been applied to a range of problems, such as the description of a generalization of Teichmüller space [Hit92] and a test of Kapustin-Witten’s approach to mirror symmetry [Hit13].

In the present paper, we study the case \( G = SU(p + 1, p) \) from both the cameral and the spectral construction, and illustrate how to deduce the latter from the former. The choice of \( SU(p + 1, p) \) is not arbitrary, but is due to the particular characteristics of the groups \( SU(p, q) \) as \( q \) varies. Indeed, both \( SU(p, p) \) and \( SU(p + 1, p) \) are quasi-split real forms, (which implies in particular that the fibers of the Hitchin map (1) are generically subvarieties of line bundles only in these two cases [PN13]), but quite different from one another (\( SU(p, p) \) being of Hermitian tube type and \( SU(p + 1, p) \) of Hermitian non-tube type). With respect to the Hitchin map, this causes for the fibers of the Hitchin map to be generically regular and stable in the first case and never completely so in the second.

The case \( p = q \) is studied in [Sch13b] from the spectral data point of view. The cameral and spectral approaches being equivalent for classical groups [Don93], we explain in here how to recover the spectral picture from the cameral one for \( SU(p + 1, p) \), the case of \( SU(p, p) \) following in a similar way.

We describe the generic fibers of the Hitchin map (1) in two ways. Firstly, in terms of principal \((\mathbb{C}^\times)^{2p}\) bundles \( P \) (called cameral data) over the cameral cover \( \tilde{X} \to X \). This is a ramified \( S_{2p+1} \)-Galois cover of \( X \) parametrizing ordered eigenvalues of Higgs fields. Hence, to each Higgs bundle \((E, \phi)\) one can associate a cameral cover, which only depends on the characteristic polynomial of \( \phi \), or equivalently, on the image of \((E, \phi)\) via (1). Using some equivariance properties of cameral data, one can prove it to be determined by \((\mathbb{C}^\times)^p\)-principal bundles over a \( \mathbb{Z}_2 \)-quotient of \( \tilde{X} \), where the action of \( \mathbb{Z}_2 \) is determined by the involution defining \( SU(p + 1, p) \) inside of \( SL(2p + 1, \mathbb{C}) \). See Theorem 4.3 for details.
Secondly, we show in Theorem 5.7 that the fibers are isomorphic to varieties of line bundles over the spectral cover $\tilde{X} \to X$, which parametrizes (unordered) eigenvalues, and is thus a quotient of the cameral cover [Don93, DG01]. In these terms, the points of the fiber are line bundles over a quotient $\tilde{X}/\mathbb{Z}_2$, plus some extra data.

The equivalence of both the cameral and spectral approach is proved in Proposition 6.2.

The paper is structured as follows: Sections 2 and 3 establish the basic notions and results from [PN13, GP], which we use to obtain the description of the generic regular fibers in Section 4.2 in terms of cameral data. We compute the spectral data in Section 5, from which we see in Corollary 5.11 that there is only one connected component of the fibers per invariant. Next, we explain in Section 6 how the spectral picture can be deduced from the cameral one. Section 7 contains a brief discussion on how to complete the results to non regular bundles; we observe the existence of a bundle of toric varieties surjecting onto a full dimensional subset of the Hitchin fiber. A geometric discussion of the algebraic notion of regularity is included in Section 8.

We were notified that Baraglia and Schaposnik have obtained related results.

Acknowledgements. The author wishes to thank Óscar García-Prada, Christian Pauly and Anna Wienhard for useful comments on a first draft of this paper.

2. QUASI-SPLIT REAL FORMS AND CAMERAL DATA

We briefly explain in this section the main theorem in [PN13, GP] that we will apply to our particular case $\text{SU}(p+1, p)$.

**Definition 2.1.** A quasi-split real form $G < G^\mathbb{C}$ is a real form containing a subgroup $B < G$ whose complexification $B^\mathbb{C} < G^\mathbb{C}$ is a Borel subgroup.

**Remark 2.2.** An alternative definition is the following: a real form is quasi-split if and only if regular elements (cf. Definition A.4) have abelian centralisers. The consequence of this is that automorphism preserving the characteristic polynomial have abelian connected component.

Quasi-split real forms include split real forms, and in the simple group case, groups whose Lie algebras are $\text{su}(p, p)$, $\text{su}(p+1, p)$, $\text{so}(p, p+2)$ or $\text{e}_6(2)$.

We will assume that the involution $\sigma$ defining $G$ inside $G^\mathbb{C}$ commutes with a compact involution $\tau$ in $G^\mathbb{C}$. In that case, we can always take $\tau$
such that $G^\tau = H$. In particular, the Cartan involution $\theta$, extended by complex linearity to $\mathfrak{g}^\mathbb{C}$, lifts to a holomorphic involution on $G^\mathbb{C}$ given by $\theta := \sigma \tau$. As a consequence $H^\mathbb{C} = (G^\mathbb{C})^\theta$. This always holds in the connected group case (see [Kna05], Proposition 7.21, or Proposition 3.20 [GPR14] for milder conditions on the group). Let $\mathfrak{d}^\mathbb{C} \subset \mathfrak{g}^\mathbb{C}$ be a $\theta$ and $\sigma$ invariant Cartan subalgebra. We may choose it in such a way that $\mathfrak{a}^\mathbb{C} = \mathfrak{d}^\mathbb{C} \cap \mathfrak{m}^\mathbb{C}$ is maximal. Fix $D^\mathbb{C} = A^\mathbb{C}T^\mathbb{C}$ the corresponding maximal torus, and let $D^\mathbb{C} < B^\mathbb{C}$ be a Borel subgroup obtained from $B < G$. We let $S \subset \Delta(\mathfrak{g}^\mathbb{C}, \mathfrak{d}^\mathbb{C})$ be the corresponding sets of simple roots and roots. The roots can be proven to belong to $\mathfrak{a}^* \oplus i \mathfrak{t}^*$ (suitably extended by complex linearity), and so we may choose $S$ in such a way that

$$\mathfrak{a}^* > i \mathfrak{t}^*.$$ 

Quasi-splitness is equivalent to $\Delta \cap i \mathfrak{t}^* = \{0\}$.

**Definition 2.3.** Let $X$ be a connected smooth complex projective curve, and $K$ its canonical bundle. A $G$-Higgs bundle on $X$ is a pair $(E, \phi)$ where $E \to X$ is a holomorphic principal $H^\mathbb{C}$-bundle, and $\phi \in H^0(X, E(\mathfrak{m}^\mathbb{C}) \otimes K)$, where $E(\mathfrak{m}^\mathbb{C})$ is the associated bundle via the isotropy representation $\iota : H^\mathbb{C} \to \text{GL}(\mathfrak{m}^\mathbb{C})$.

We have two meaningful Hitchin maps. To construct them, consider the Chevalley morphisms

$$\chi_G : \mathfrak{m}^\mathbb{C} \to \mathfrak{m}^\mathbb{C} // H^\mathbb{C} \cong \mathfrak{a}^\mathbb{C} / W(\mathfrak{a})$$

(where $W(\mathfrak{a})$ is the restricted Weyl group [11] and the second isomorphism follows from Theorem [A.2]) and

$$\chi_{G^\mathbb{C}} : \mathfrak{g}^\mathbb{C} \to \mathfrak{g}^\mathbb{C} // G^\mathbb{C} \cong \mathfrak{d}^\mathbb{C} / W,$$

where the second isomorphism is again Theorem [A.2] applied to the real form $(\mathfrak{g}^\mathbb{C})_\mathbb{R} \subset \mathfrak{g}^\mathbb{C} \otimes_\mathbb{C} \mathbb{R}$. These induce

$$h_G : \text{Higgs}(G) \to B_G = H^0(\mathfrak{a}^\mathbb{C} \otimes K / W(\mathfrak{a})), \quad (E, \phi) \mapsto \chi_G(\phi)$$

and

$$h_{G^\mathbb{C}} : \text{Higgs}(G^\mathbb{C}) \to B_{G^\mathbb{C}} = H^0(\mathfrak{d}^\mathbb{C} \otimes K / W), \quad (E, \phi) \mapsto \chi_{G^\mathbb{C}}(\phi).$$

Note that evaluation of (3) and (4) on the Higgs field is well defined by $H^\mathbb{C} \times \mathbb{C}^\times$-equivariance of $\chi_G$ (respectively, $G^\mathbb{C} \times \mathbb{C}^\times$-equivariance of $\chi_{G^\mathbb{C}}$).

Let $\kappa : \text{Higgs}(G) \to \text{Higgs}(G^\mathbb{C})$ be given by $\kappa(E, \phi) = (E(G^\mathbb{C}), d\iota(\phi))$, where $\iota : G \hookrightarrow G^\mathbb{C}$ is the inclusion. This map induces a commutative
diagram
\[
\begin{array}{ccc}
\text{Higgs}(G) & \xrightarrow{\kappa} & \text{Higgs}(G^C) \\
\downarrow & & \downarrow \\
B_G & \rightarrow & B_{G^C}.
\end{array}
\]

In particular, fibers of \( h_G \) are taken to fibers of \( h_{G^C} \).

**Definition 2.4.** Given \( b \in B_{G^C} \), we define the associated cameral cover \( \hat{X}_b \) to be the fibered product fitting in the Cartesian diagram
\[
\begin{array}{ccc}
\hat{X}_b & \rightarrow & \mathfrak{d}^C \otimes K \\
\downarrow & & \downarrow \\
X & \rightarrow & \mathfrak{d}^C \otimes K/W.
\end{array}
\]

Now, assume \( b : X \rightarrow \mathfrak{d}^C \otimes K/W \) splits through \( \mathfrak{a}^C \otimes K/W(\mathfrak{a}) \); that is, \( b \in B_G \). Then we have a subcover
\[
\begin{array}{ccc}
\hat{X}_G & \rightarrow & \mathfrak{a}^C \otimes K \\
\downarrow & & \downarrow \\
X & \rightarrow & r_K,
\end{array}
\]

where \( r_K \) is the image of \( \mathfrak{a}^C \otimes K/W(\mathfrak{a}) \) in \( \mathfrak{d}^C \otimes K/W \). A \( G \)-Higgs bundle \( (E, \phi) \) is said to be **regular** if for all \( x \in X \) we have that \( \phi(x) \in \mathfrak{m}_{\text{reg}} \) is a regular element of \( \mathfrak{m}^C \) (see Definition A.4).

The following theorem follows from Theorem 4.14 in [PN13] (see also [GP]):

**Theorem 2.5.** There is a one to one correspondence between isomorphism classes of
1. Regular \( G^C \)-Higgs bundles \( (E, \phi) \) (not necessarily polystable) such that \( h_{GC}(E, \phi) = b \).
2. Principal \( D^C \) bundles \( P \rightarrow \hat{X}_b \) satisfying
   - **CD1** For all \( \alpha \in S \), \( \gamma_\alpha : s^*_\alpha P^{s_\alpha} \otimes R_\alpha \cong P \), where \( s_\alpha \in W \) is the reflection with respect to \( \alpha \), whose action on the principal bundle is defined by \( P^s = P \times_s D^C \). As for \( R_\alpha \), it is the principal \( D^C \) bundle obtained by pulling back the divisor \( D_\alpha = \{ s_\alpha = 0 \} \subset \mathfrak{d}^C \otimes K \) to \( X \) (thus getting \( D^X_\alpha \), see Section 5. in [DG01]) and taking \( R_\alpha = \hat{\alpha}(\mathcal{O}(D^X_\alpha)) \).
   - **CD2** \((-\theta)^*P|_{\hat{X}_G} \otimes R_{w_0} \cong P \). Here \( w_0 \in W \) is the element operating on \( \mathfrak{d}^C \) as \( \theta \), and \(-\theta\) is the involution on \( \hat{X}_b \) induced by the
action $-\theta \sim \mathfrak{d}^C$. The ramification $R_w = \bigotimes_{\alpha \in \Delta^+ \cap w^{-1}\Delta} R_\alpha$. By definition $R_{w_0} = \bigotimes_{\alpha \in \Delta_r} E_\alpha$, where $\Delta_r$ denotes the real roots, namely, roots $\lambda \in \Delta$ which are obtained from elements in $\mathfrak{a}^*$ by extension by complex linearity.

All isomorphisms should be elements of $N$, the normaliser of $D^C$ in $G^C$.

In particular, this theorem allows to study $\text{Higgs}(G)$ by means of much simpler moduli spaces of principal $D^C$-bundles.

**Remark 2.6.** The $R_w$'s define a cocycle $W \to \hat{X} \times BD^C$, that is, an assignation of a principal $D^C$-bundle to each $w \in W$ satisfying

$$R_{ww'} \cong w' \cdot R_w \otimes R_{w'}$$

canonically. This allows to extend the equivariance properties specified in CD1 of Theorem 2.5 to all elements of the Weyl group and not just reflections associated to simple roots.

**Definition 2.7.** A principal $D^C$-bundle as specified in point 2. of Theorem 2.5 is called a **cameral datum**.

For the convenience of the reader, we include a discussion of the main elements in the proof of Theorem 2.5 which is based on the study of the gerbe the Hitchin map defines on the level of the moduli stack of regular Higgs bundles $\text{Higgs}(G) \to \mathfrak{a}^C \otimes K/W(\mathfrak{a})$. When the real form is quasi split, this gerbe has abelian band $\mathcal{C}$, that is, locally $\text{Higgs}(G)(X) \cong \mathfrak{a}^C \otimes K/W(\mathfrak{a}) \times BC$, a $BC$-torsor over $\mathfrak{a}^C \otimes K/W(\mathfrak{a})$. In particular, pullback allows to identify the fibers to varieties of coherent sheaves of groups on $X$. In order to give the cocyclic description of Proposition 4.3, one checks that the stack of cameral data is also a $BC$-torsor admitting a morphism from $\text{Higgs}(G)$, so they are isomorphic. The latter can be done using [DG01] and identifying the right conditions in order for cameral data to be induced from a $G$-Higgs bundle.

To construct a cameral datum from a $G^C$-Higgs bundle $(E, \phi)$, we reinterpret $\phi$ as a $G^C \times \mathbb{C}^\times$ equivariant map

$$\phi : E \times K \to \mathfrak{g}_{\text{reg}},$$

where $\mathfrak{g}_{\text{reg}}$ denotes the subset of regular elements of $\mathfrak{g}$ and the action of $(g, z) \in G^C \times \mathbb{C}^\times$ on $x \in \mathfrak{g}_{\text{reg}}$ is given by $(g, z) \cdot x = z \text{Ad}(g)x$. Now, the Grothendieck-Springer resolution of $\mathfrak{g}_{\text{reg}}$ can be obtained by pullback.
of two $W$-covers:

\[
\begin{array}{c}
\overline{G^C/D^C} \quad \hat{\mathfrak{g}}_{reg} \quad \overline{\phi^C} \\
\downarrow \quad \downarrow \quad \downarrow \\
\overline{G^C/N} \quad \mathfrak{g}_{reg} \quad \chi_{G^C} \overline{\phi^C/W}.
\end{array}
\]

Here, $\overline{G^C/N}$ is the variety of regular centralisers (a partial compactification of $G^C/N$ inside of $Gr(r, g^C)$, see [DG01]), and $\overline{G^C/D^C}$ is the incidence variety of $\overline{G^C/N} \times G^C/B^C$, with $B^C$ a Borel subgroup containing $D^C$.

Note that $b := \chi_{G^C} \circ \phi$ is a point of the Hitchin base $B_{G^C}$. Let $\hat{E} \times \hat{K} := b^*\hat{\mathfrak{g}}_{reg}$. This is easily seen to descend to a cameral cover $\hat{X} \to X$, so that we have a cartesian diagram:

\[
\begin{array}{c}
\hat{E} \times \hat{K} \quad \hat{X} \\
\downarrow \quad \downarrow \\
E \times K \quad X.
\end{array}
\]

As for the cameral datum, $\overline{G^C/D^C}$ is equipped with a universal principal $D^C$-bundle $D \to \overline{G^C/D^C}$, the pullback of $G^C/U^C \to G^C/B^C$ via the map $\overline{G^C/D^C} \to \overline{G^C/N} \times G^C/B^C \to G^C/B^C$. The same descent arguments yield a principal bundle on $X$, which can be checked to satisfy invariance conditions with respect to the action of the Weyl group. See [DG01, Ngo10] for details on this.

The last step is to establish the extra conditions for cameral data coming from $b \in H^0(X, a^C \otimes K/W(a))$, which is given by condition CD2 in Theorem 2.5.

3. SU($p + 1, p$)-Higgs bundles and the Hitchin map

For the Lie theory required in this section we refer to Appendix A.

**Definition 3.1.** An SU($p + 1, p$)-Higgs bundle is a pair $(E = V \oplus W, \phi)$ consisting of a rank $p + 1$ vector bundle $V$ and a rank $p$ vector bundle $W$ such that $\det(V \oplus W) = O_X$, and a Higgs field $\phi \in H^0(X, \text{End}(E) \otimes K)$ with

\[
\phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix},
\]

where $\beta \in H^0(X, W^* \otimes V \otimes K)$ and $\gamma \in H^0(X, V^* \otimes W \otimes K)$. 


Definition 3.2. An SU\((p+1, p)\)-Higgs bundle is semistable if for every pair of subbundles \(V' \subseteq V, W' \subseteq V \oplus W\) and \(\phi : V' \oplus W' \to (V' \oplus W') \otimes K\), it holds that \(\text{deg } V' \oplus W' \leq 0\). It is stable if it is semistable and the inequality is strict. It is said to be polystable if it is semistable and it decomposes as a direct sum of stable \(U(p_i, q_i)\)-Higgs bundles of degree 0 for suitable integers \(p_i\) and \(q_i\).

Let \(\text{Higgs}(SU(p+1, p))\) be the moduli space of \(SU(p+1, p)\)-Higgs bundles, which is defined to be the space of isomorphism classes of polystable \(SU(p+1, p)\)-Higgs bundles [BGG03].

The degree \(\text{deg } W\) is a topological invariant of \(\text{Higgs}(SU(p+1, p))\) which hence identifies connected components of the moduli space. By Theorem 6.1 in [BGG03] the following Milnor-Wood inequality is satisfied by polystable Higgs bundles:

\[
0 \leq |\text{deg}(W)| \leq p(g-1).
\]

Remark 3.3. This invariant is half the Toledo invariant defined in [BGG03], as in the fixed determinant case \(\text{deg}(W) = -\text{deg } V\).

Choose a maximally anisotropic Cartan subalgebra \((\mathfrak{a'})^C \cong \mathbb{C}^p\) (cf. Defin A.1), and let \((\mathfrak{v'})^C \cong \mathbb{C}^{2p}\) be a \(\theta'\)-invariant Cartan subalgebra containing \((\mathfrak{a'})^C\), where \(\theta'\) is defined as in (43). By Lemma A.3 imply that Hitchin map (5) specifies to

\[
h : \text{Higgs}(SU(p+1, p)) \to H^0(X, \oplus_{k=1}^{2p} K^{2k})
\]

which maps each pair \((E, \phi)\) to the characteristic coefficients of \(\phi\), \((a_2, \ldots, a_{2p})\). More specifically,

\[
a_{2i} = \text{tr}(\wedge^{2i} \phi) = 2^i \text{tr} \left(\wedge^i (\beta \wedge \gamma)\right).
\]

On the other hand, the complex Hitchin map (6) reads

\[
h_C : \text{Higgs}(SL(2p+1, \mathbb{C})) \to H^0(X, \oplus_{i=1}^{2p} K^i), \ (E, \phi) \mapsto (\text{tr } \wedge^i \phi)^{2p}_{i=1}.
\]

Note that \(\text{tr } \wedge^i \phi = 0\) if \(2 \nmid i\) for \(\phi\) as in (11), so we have indeed that \(\kappa\) commutes with the respective Hitchin maps. In what follows, we describe the fibers of the restriction of \(h_C\) to \(\kappa(\text{Higgs}(SU(p+1,p)))\) by means of cameral techniques, then recovering the spectral curve construction.

4. Cameral data for \(SU(p+1, p)\)-Higgs bundles

In this section we compute the cameral data for \(SU(p+1, p)\)-Higgs bundles using the results in Section 2 to \(SU(p+1, p)\). For simplicity, we realize \(SU(p+1, p)\) as the subgroup of fixed points whose Cartan involution is \(\theta\) (cf. Appendix A).
4.1. Cameral covers. Let \( \omega \in B_{\text{SU}(p+1,p)} \). To define the associated cameral cover \([14]\), let

\[
\mathcal{O}_C \otimes K \cong K^{\otimes 2p} \cong \{(l_1, \ldots, l_{2p+1}) \in K^{\otimes 2p+1} : \sum l_i = 0\};
\]

then, we have the projection

\[
\mathcal{O}_C \otimes K \rightarrow \mathcal{O}_C \otimes K/W, \quad (l_1, \ldots, l_{2p+1}) \mapsto (\sigma_1(l_i), \ldots, \sigma_{2p+1}(l_i))
\]

where \( \sigma_i \) denotes the \( i \)-th symmetric polynomial in \( 2p + 1 \) variables.

In particular, the **cameral cover** associated to \( \omega \in B_{\text{SU}(p+1,p)} \) is

\[
\hat{X}_\omega = \left\{(\lambda_1, \ldots, \lambda_{2p+1}) \in K^{\otimes 2p+1} \bigg| \begin{array}{l}
\sum_k \lambda_k = 0, \\
\sigma_{2i}(\lambda_j) = \omega_i, \\
\sigma_{2i+1}(\lambda_j) = 0.
\end{array} \right\}
\]

As for \( \hat{X}_{\text{SU}(p+1,p)} \) \([9]\), it corresponds to the subscheme

\[
\hat{X}_{\text{SU}(p+1,p)} = \left\{(\lambda_1, \ldots, \lambda_{2p+1}) \in \hat{X}_\omega \bigg| \begin{array}{l}
\lambda_i = -\lambda_{p+1+i}, \\
\lambda_{p+1} = 0.
\end{array} \right\}
\]

Note that the vanishing of \( \sigma_{2p+1}(\lambda_1, \ldots, \lambda_{2p+1}) \) distinguishes \( 2p + 1 \) Galois subcovers determined by \( \hat{X}_i = \{\lambda_j = 0\} \) whose Galois group is \( S_{2p} \subset W = S_{2p+1} \). In particular, \( \hat{X}_{\text{SU}(p+1,p)} \subset \hat{X}_{p+1} \).

**Remark 4.1.** For any \( (l_i) \in \hat{X}_\omega, \lambda \in \{l_i : i\} \) if and only if \( -\lambda \in \{l_i : i\} \), so it follows that generically over \( B_{\text{SU}(p+1,p)} \), \( \hat{X}_{\text{SU}(p+1,p)} \) is irreducible and all other irreducible components in \( \hat{X}_{p+1} \) are obtained by translating \( \hat{X}_{\text{SU}(p+1,p)} \) by elements of \( W/W(\mathfrak{a}^C) = S_{2p+1}/S_p \ltimes \mathbb{Z}_2^2 \).

**Example 4.2.** In the rank one case, \( \text{SU}(2,1) \), the projection \( K \oplus K \rightarrow K^3 \oplus K^2 \cong \mathcal{O}_C \otimes K/W \) reads

\[
(l, l') \mapsto \begin{pmatrix}
l + l' & 0 & 0 \\
0 & -2l & 0 \\
0 & 0 & l - l'
\end{pmatrix} \mapsto (l^2 - (l')^2 - 4l'l, l'((l')^2 - l^2))
\]

Thus, any cameral cover corresponding to a real Higgs bundle (with corresponding point of the Hitchin base \( \omega \in K^2 \)) satisfies

\[l'((l')^2 - l^2) = 0.\]

Namely, we have three subcovers

\[
\hat{X}_1 = \{l^2 = \omega, l' = 0\}, \quad \hat{X}_2 = \{l' = l, -4l^2 = \omega\}, \quad \hat{X}_3 = \{l' = -l, 4l^2 = \omega\}.
\]

Note that \( \hat{X}_1 = \hat{X}_{\text{SU}(2,1)} \). All three are double covers, with involutions induced by elements of the Weyl group: \( (1,3) \in S_3 \) restricts to the cover involution on \( \hat{X}_1 \), as so do \( (1,2) \) on \( \hat{X}_2 \) and \( (2,3) \) on \( \hat{X}_3 \).
4.2. Cameral data. Applying Theorem 2.5 we have a correspondence between everywhere regular SU($p + 1, p$)-Higgs bundles $(E, \phi)$ whose image via $h_{SU(p+1,p)}$ is $\omega \in B_{SU(p+1,p)}$ (note that no polystability condition is assumed) and $(\mathbb{C}^*)^{2p}$-principal bundles $P \to \hat{X}_\omega$ satisfying the equivariance conditions:

$$(i, j)^* (P \otimes R_{ij})^{(i,j)} \simeq P, \quad P|_{\hat{X}_{SU(p+1,p)}} \simeq P|_{\hat{X}_{SU(p+1,p)}}^{w_0} \otimes R_{w_0}.$$ 

In the case under consideration, regularity just means that eigenspaces for the standard representation have dimension one. In the semisimple case, this is just the fact that eigenvalues are different. As for non semisimple elements, it means that the nilpotent part is SL(2+$p$, $\mathbb{C}$)-conjugate to an element with 1’s over the diagonal on a block of the matrix. As for the semisimple part, it should have different eigenvalues.

By definition, the involution $\theta$ acts as multiplication by $-1$ on $m^C$ and $+1$ on $h^C$, so that the element $w_0 \in W$ defined as in condition CD2 in Theorem 2.5 has the form

$$(17) \quad w_0 = \prod_{i=1}^{p} (i, p + 1 + i).$$

**Theorem 4.3.** Let $\omega \in B_{SU(p+1,p)}$ be such that $\hat{X}_{SU(p+1,p)}$ is smooth. Then, the choice of a cameral datum $P_0$ establishes a correspondence between regular SU($p + 1, p$)-Higgs bundles mapping to $\omega$ and the subvariety of $H^1(\hat{X}/\theta, T^C)$ consisting of elements $Q$ such that $w^* Q(D^C)^w \simeq Q(D^C)$ for all $w \in W$. Here $\hat{X} \to \hat{X}/\theta$ is the quotient curve, and $Q(D^C)^w = Q(D^C) \times wD^C$.

**Proof.** First note that the assigment

$$P \mapsto PP^{-1}$$

establishes a morphism $Cam \to H^1(\hat{X}, D^C)^W$, where the action of $W$ on $H^1(\hat{X}, D^C)$ is given by $w \cdot Q = w^* Q \times_w D^C$.

Denote by $\hat{X}_w = w \cdot \hat{X}_{SU(p+1,p)}$ (15) $Q_w := Q|_{\hat{X}_w}$. Then:

$$\theta^* Q_w = \theta^* w^* Q_{w}^w = \theta^* w^* Q_{w^\theta}^w = Q_{w^\theta}.$$ 

Given that $\theta$ exchanges $\hat{X}_w$ and $\hat{X}_{w^\theta}$, it follows that $\theta^* Q \simeq Q$. Moreover, since $w_0^\theta Q = \theta^* Q \simeq Q^{w_0}$, it follows that the structure group reduces to $T^C$, as also $w_0^\theta Q^{w_0} \simeq Q$. This finishes the proof, as the action of $\theta$ on $T^C = (D^C)^\theta$ is by definition the identity on the fixed locus of $w_0$, and so Kempf’s descent Lemma (Theorem 2.3 [DN89]) applies. \qed
5. Spectral data

In the case of matrix groups, Hitchin’s spectral techniques are available for abelianization of Higgs bundles. In this section, we compute spectral data for SU($p+1, p$).

Let $(E, \phi) \in \text{Higgs}(SU(p+1, p))$ be such that $h(\phi) = (\omega_2, \ldots, \omega_{2p})$. The characteristic polynomial of $\phi$ over the total space of the canonical bundle $\pi : |K| \to X$ produces a section

$$s_\omega := \lambda(\lambda^{2p} + \pi^*\omega_2\lambda^{2p-2} + \cdots + \pi^*\omega_{2p}) \in H^0(|K|, \pi^*K^{2p+1})$$

vanishing over the spectral curve

$$\tilde{X} := \text{Spec} \left( \text{Sym}^\bullet \left( K^*/\lambda(\lambda^{2p} + \sum_i \lambda^{2(p-i)}\pi^*\omega_{2i}) \right) \right).$$

The generic spectral curve $\tilde{X}$ is reduced and consists of two smooth irreducible components

$$\tilde{X}_0 \cong \text{Spec} \left( \text{Sym}^\bullet(K^*/\lambda) \right) \cong X,$n

$$\tilde{X}_1 := \text{Spec} \left( \text{Sym}^\bullet(K^*/\lambda^{2p} + \sum_i \lambda^{2(p-i)}\omega_{2i}) \right).$$

Remark 5.1. Remark 4.1 implies that the genericity hypothesis for $\tilde{X}$ to be of the above form is the same as the one for $\tilde{X}_{SU(p+1, p)}$ being smooth, and also for generic regularity of $(E, \phi)$.

By Remark 5.1, the kernel of the Higgs field has rank one, so $V \subset E$ is an extension

$$0 \to E_0 \to V \to V_1 \to 0$$

where $E_0 = \text{Ker}(\phi) \in \text{Pic}(X)$; the Higgs field induces one on $E_1 = V_1 \oplus W$, so that we obtain $(E_1, \phi_1)$ an induced $U(p, p)$-Higgs bundle, which is regular by Remark 5.1 and a general result of Ngô’s and Arinkin (private communication). Moreover, regularity implies that $\phi_1 = (\beta_1, \gamma_1)$ induces generic isomorphisms

$$\beta_1 : W \to V_1 \otimes K, \quad \gamma_1 : V_1 \to W \otimes K.$$

Taking determinants, we obtain subdivisors of the branching locus $B = \{\omega_{2p} = 0\}$

$$B_\beta = (s_\beta), \quad B_\gamma = (s_\gamma)$$

given by vanishing of

$$s_\gamma := \wedge^p\gamma_1 \in H^0(X, \wedge^pV_1^{-1} \otimes \wedge^pW \otimes K^p)$$

and

$$s_\beta := \wedge^p\beta_1 \in H^0(X, \wedge^pW^{-1} \otimes \wedge^pV_1 \otimes K^p).$$
Remark 5.2. The ramification divisor consists of points over which the Higgs field $\phi$ is not semisimple. Let $B^\text{reg}_\beta \subset B_\beta$ denote the subdivisor over which $\phi$ is regular. That is, $\beta|_{B^\text{reg}_\beta} : W \mapsto E_0K$, as otherwise the kernel would increase its dimension by one. We let $B^0_\gamma := B_\gamma \setminus B^\text{reg}_\beta$.

As for $B_\gamma$, the genericity hypothesis implies it is disjoint from $B_\beta$. Denote by $B^\text{reg}_\gamma \subset B_\gamma$ the subset over which $\phi$ remains regular. This implies that $E_0 \otimes \mathcal{O}_{B^\text{reg}_\gamma} \hookrightarrow V_1 \otimes \mathcal{O}_{B^\text{reg}_\gamma}$, or, in other words, so that the kernel has still rank 1. Let $B^0_\gamma := B_\gamma \setminus B^\text{reg}_\gamma$.

From now on we will assume that
\begin{equation}
B_{3/\gamma} = B^\text{reg}_\beta,
\end{equation}
so that the Higgs field is everywhere regular and $\beta|_{B_\beta} : W|_{B_\beta} \mapsto E_0K|_{B_\beta}$ over the whole $B_\beta$ and $i : E_0|_{B_\beta} \hookrightarrow V_1|_{B_\gamma}$ over $B_\gamma$. See Section 7 for remarks on the general case. By Remark 5.2 we have that the extension $[V] \in H^1(X, E_0^{-1}V_1)$ defined by (20) can be recovered as the image via the Bockstein map of the inclusion $f \in H^0(X, E_0^{-1}V_1 \otimes \mathcal{O}_{B_\gamma})$.

We recall that the Bockstein operator
\begin{equation}
b : H^0(B_\gamma, E_0^{-1}V_1) \to H^1(X, E_0^{-1}V_1), \quad f \mapsto [V],
\end{equation}
is obtained by considering the long exact sequence induced from the short exact sequence
\[0 \rightarrow E_0^{-1}V_1 \rightarrow E_0^{-1}V_1(B_\gamma) \rightarrow E_0^{-1}V_1 \otimes \mathcal{O}_{B_\gamma} \rightarrow 0.\]

We next give an alternative approach to $\phi$. Consider the commutative diagram of extensions:
\[\begin{array}{cccccc}
0 & \rightarrow & W^{-1}E_0K(-B_\beta) & \rightarrow & W^{-1}E_0K & \rightarrow & W^{-1}E_0K \otimes \mathcal{O}_{B_\beta} & \rightarrow & 0 \\
0 & \rightarrow & W^{-1}VK(-B_\beta) & \rightarrow & W^{-1}VK & \rightarrow & W^{-1}VK \otimes \mathcal{O}_{B_\beta} & \rightarrow & 0 \\
0 & \rightarrow & W^{-1}V_1K(-B_\beta) & \rightarrow & W^{-1}V_1K & \rightarrow & W^{-1}V_1K \otimes \mathcal{O}_{B_\beta} & \rightarrow & 0.
\end{array}\]

Studying the corresponding long exact sequences, we see that a necessary and sufficient condition for $\beta_1 \in H^0(X, W^{-1}V_1K)$ and $p \in H^0(X, W^{-1}E_0K \mathcal{O}_{B_\beta})$ to determine $\beta \in H^0(X, W^{-1}VK)$ is
\[p \in \text{Ker} \left(H^0(X, W^{-1}E_0K \otimes \mathcal{O}_{B_\beta}) \rightarrow H^1(X, W^{-1}E_0K(-B_\beta))\right)\]
and
\[\beta_1 \in \text{Ker} \left(H^0(X, W^{-1}V_1K) \rightarrow H^1(X, W^{-1}E_0K)\right).\]

This proves the following
Proposition 5.3. An everywhere regular SU($p+1,p$)-Higgs bundle $(E,\phi)$ is equivalent to the following piece of data:

1. An everywhere regular SU($p+1,p$)-Higgs bundle $(E_1,\phi_1)$, where
$$E_1 = V_1 \oplus W$$
and
$$\phi_1 = (\beta_1, \gamma_1), \quad \beta_1 : W \to V_1K, \quad \gamma_1 : V_1 \to WK.$$

2. An embedding $i : H^0(B_\gamma, E_0^{-1}V_1)$, where $E_0 = (\det W \det V_1)^{-1}$ and $B_\gamma$ is given by \[21\] determining an extension $V \in H^1(X, E_0^{-1}V_1)$.

3. A map
$$p \in \text{Ker} \left( H^0(X, W^{-1}E_0K \otimes O_{B_\gamma}) \to H^1(X, W^{-1}E_0K(-B_\gamma)) \right)$$

4. The Higgs field $\phi_1$ should satisfy
$$\phi_1 \in \text{Ker} \left( H^0(X, \text{End}(E_1) \otimes K) \to H^1(X, \text{Hom}(E_1, E_0)K) \right).$$

Now, the involution $\theta'$ (see \[37\]) induces an involution on $\tilde{X}$ sending $\lambda \mapsto -\lambda$. Let
$$\overline{X} = \tilde{X}/\theta,$$
whose irreducible components read
$$\overline{X}_0 \cong X, \quad \overline{X}_1 \cong \tilde{X}_1/\theta.$$
We have a diagram
$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{\pi}} & \overline{X} \\
\downarrow{\pi} & & \downarrow{\pi} \\
X & & \\
\end{array}$$
The ramification divisor of $\tilde{\pi} : \tilde{X} \to \overline{X}$ is given by $R = \{\pi^*\omega_{2p} = 0\}$ and generically equals $\tilde{X}_0 \cap \tilde{X}_1$. The divisor $R$ is also the ramification divisor of $\tilde{\pi}|_{\tilde{X}_1}$. Let $F \in \text{Pic}(\tilde{X})$ be the line bundle defined by:
$$0 \to F(-R) \xrightarrow{\tilde{\pi}^*E^{\lambda(Md-\tilde{\pi}^*\phi)}} \tilde{\pi}^*(E \otimes K) \to F \otimes \tilde{\pi}^*K \to 0.$$

Remark 5.4. We need only remark that the torsor structure of the fibers as described in Theorem \[5.7\] comes from the decomposition of Pic($\tilde{X}$) as a torsor over Pic($\tilde{X}^n$), the Picard variety of its normalization, which in this case is just $\tilde{X}^n = \tilde{X}_0 \sqcup \tilde{X}_1 \xrightarrow{p} \tilde{X}$. To see this, consider the short exact sequence
$$0 \to \mathcal{O}_X^\times \xrightarrow{p_*} \mathcal{O}_{\tilde{X}^n}^\times \to p_*\mathcal{O}_{\tilde{X}^n}^\times / \mathcal{O}_X^\times \to 0.$$
which induces a long exact sequence in cohomology

\[ 0 \to H^0(\tilde{X}, O^\times_{\tilde{X}}) \to H^0(\tilde{X}, p_*O^\times_{\tilde{X}_n}) \to H^0(\tilde{X}^{\text{sing}}, p_*O^\times_{\tilde{X}_n}/O^\times_{\tilde{X}}) \to \]

\[ H^1(\tilde{X}, O^\times_{\tilde{X}}) \to H^1(\tilde{X}, p_*O^\times_{\tilde{X}_n}) \to 0 \]

See Section 9.2 in [BLR90] for details.

Geometrically, line bundles on $\tilde{X}$ are given by line bundles on its normalization with suitable automorphisms over the singular points. In our case, the $O_{\tilde{X}}$-module structure of the line bundle $F \to \tilde{X}$ consists of an $O_{\tilde{X}_i}$-module structure on $F_i$ on the respective irreducible components together with an isomorphism $f : F_0|_{\tilde{X}_0} \cong F_1|_{\tilde{X}_1}$. Note that $R = \tilde{X}_0$ with our genericity hypothesis.

Note that $F_0 = F|_{\tilde{X}_0}$ is the kernel of $\pi^*\phi$, and $F_1 = F|_{\tilde{X}_1}$ is the spectral datum for the induced $U(p, p)$-Higgs bundle $(E_1, \phi_1)$ (see discussion following Remark 5.4).

It is easy to see that the spectral datum $F$ satisfies $\theta^*F \cong F$.

We next summarize some results of [Sch13b] for the group $U(p, p)$, as the line bundle $F_1 \to \tilde{X}_1$ is the spectral datum corresponding to the $U(p, p)$-Higgs bundle $(E_1, \phi_1)$. Define the subdivisors $R_+$ and $R_-$, where $R_+$ is the set of points over which $\theta$ acts on the fibers of $F_1$ via the identity and $R_-$ the set of points where it lifts as multiplication by $-1$. This determines $\tilde{\pi}_*F_1 \cong F_+ \oplus F_-$, with $F_-^1 \otimes F_+ \otimes \pi^*K \cong O(R_+)$. As for $\phi_1$, the $O_{\tilde{X}_1} = \tilde{\pi}^*O_{\tilde{X}_1}$-module structure of $F_1$ induces a $\tilde{\pi}_*O_{\tilde{X}_1} = O \oplus \pi^*K^{-1}$-module structure on $\tilde{\pi}_*F_1$, totally determined by the action of $\pi^*K^{-1}$. Since the involution $\theta$ acts by $-1$ on $\pi^*K$ (as by definition $\lambda \mapsto -\lambda$) we get $s_\pm : \pi^*K^{-1} \otimes F_ \to F_\mp$. We have that

$W = \pi^*F, V_+ = \pi^*F_+, \phi_1 = (\pi^*s_+, \pi^*s_-)$.

We will use this to recover the $SU(p+1, p)$-Higgs bundle. By Remark 5.4 the $O_{\tilde{X}}$ structure of the line bundle $F \to \tilde{X}$ pushes forward to a $O_{X}$-module structure on $F_0$, and a $O_{\tilde{X}_1} \oplus \pi^*K^{-1} = \tilde{\pi}_*O_{\tilde{X}_1}$-module structure on $\tilde{\pi}_*F_1 = F_+ \oplus F_-$. Away from ramification, the latter is generated by the action of $\pi^*K^{-1}$, yielding $s_\pm$. Denote $\tilde{R} := \tilde{\pi}_*R$; since the involution is trivial on the irreducible component $\tilde{X}_0$ over which the kernel lives and it becomes trivial on the quotient, we get the following $\tilde{\pi}_*O_R = O_{\tilde{R}} \oplus \pi^*K^{-1}$-module structure on $\tilde{\pi}_*F|_{\tilde{R}}$; on the one hand, the usual action of $O_{\tilde{R}}$, and the restriction of $s_\pm$ to ramification

$s_\pm : \pi^*K^{-1} \otimes F_ \to F_\mp$.

On the other hand, isomorphisms induced from $f$ as in Remark 5.4

$f_- : \pi^*K^{-1} \otimes F_- \cong F_0$
on $\overline{R}_{-}$, and
\[ f_+: F_+ \cong F_0. \]
on $\overline{R}_{+}$.

**Remark 5.5.** By Remark 5.2, we could also consider just morphisms, corresponding to generically (but not everywhere) regular Higgs bundles.

**Remark 5.6.** Clearly $\pi^*R_{+} = \pi^*B_{\gamma} = \pi^*B_{\beta}$.

The above discussion suggests the following.

**Theorem 5.7.** Let $\omega \in BSU(p+1, p)$ be generic. Let $\pi: \tilde{X}_\omega \to X$ be the corresponding spectral curve, and $\tilde{\pi}: \tilde{X}_\omega/\theta \to X$. Denote by $\tilde{X}_0/\tilde{X}_1$ the intersection of both irreducible components (27).

Let
\[ \text{Pic}(\tilde{X}_1)^\theta = \{ F_1 \in \text{Pic}(\tilde{X}_1) : \theta^*F_1 \cong F_1 \}, \]
and let $\mathcal{P}$ be the $(\mathbb{C}^*)^{4p(g-1)}$-torsor over with fiber over $F_1$ equal to $\text{Isom}(\tilde{X}_0/L^0_0F_1)$, with
\[ L_0 = Nm(F_1)^{-1} \otimes K^{-p(2p+1)} \in \text{Pic}(X). \]

With the notation of the preceding paragraph:

1. There exists a correspondence between isomorphism classes of everywhere regular $SU(p+1, p)$-Higgs bundles mapping via $h_{SU(p+1, p)}$ to $\omega$ (note that we assume no stability condition) and the elements of the quotient $\mathcal{P}/C^\times$ where the action is given by
\[ \mu(L, z) \mapsto (L, \mu z), \quad (L, z) \in \mathcal{P}. \]

2. Let $\tilde{\pi}_*F_1 = F_+ \oplus F_-$. Then, the topological invariant (12) is
\[ \text{deg } W = \text{deg } F_- - 2p(p-1)(g-1). \]

3. The corresponding Higgs bundle is stable, thus
\[ (2p^2 - 3p)(g-1) < | \text{deg } F_- | < (2p^2 - p)(g-1), \]
and extremal values of the Milnor-Wood inequality (12) $\text{deg } W = \pm p(g-1)$ are not met by regular points.

**Proof.** To prove 1., given $(E, \phi)$, we assign to it $F$ as in (29). Generically, the spectral curve has smooth irreducible components, and so restriction induces on its Picard variety a structure of a $(\mathbb{C}^*)^{4p(g-1)}$- torsor over $\text{Pic}(\tilde{X}_0) \times \text{Pic}(\tilde{X}_1)$. See Remark 5.4. The topological restriction that $\det \pi_* F = 0$ implies (32), by Corollary 3.12 and Lemma 3.5 in [HP].
As for the inverse map, pushforward induces the remaining structure as explained in the discussion preceding this theorem. There remains to check (a) that the construction descends to the quotient by the action of $\mathbb{C}^\times$, and (b) that conditions (25) and (24) in Proposition 5.3 are satisfied for

$$\beta_1 := \pi_\ast s_- \in H^0(X, \text{Hom}(\pi_\ast F_- \otimes \pi_\ast F_+ \otimes K))$$

and

$$p := \pi_\ast f_- \in H^0(X, \text{Hom}(\pi_\ast F_-, E_0 \otimes K))$$

Statement (b) follows by the fact that $\text{Hom}(F_0, F_1)$ is a sheaf supported on $\tilde{X}_{01}$, thus $H^1(\tilde{X}_{01}, \text{Hom}(F_0, F_1)) = 0$. As for (a), an $SU(p + 1, p)$-Higgs bundle is determined by point $L := (L_0, L_1, f) \in \text{Jac}(\tilde{X})$, with the restriction that the degrees of $L_0$ and $L_1$ are such that the pushforward has degree zero. Now, for $\lambda \in \mathbb{C}^\times$ note that the morphism given by multiplication of $\lambda^{-1/2}$ on $L_0$ and $\lambda^{1/2}$ on $L_1$ send $(L_0, L_1, f) \mapsto (L_0, L_1, \lambda f)$; this yields isomorphic Higgs bundles.

Statement 2. follows from $\det \pi_\ast F_\ast = Nm(F_-) \otimes Nm(\mathbb{X}/X)$ together with $Nm(\mathbb{X}/X) = \bigotimes_{i=0}^{p-1} K^{-2i} = K^{-p(p-1)}$.

Finally, 3. follows because by irreducibility of $\tilde{X}_1$ and regularity, the only non-trivial $\phi$-stable subbundle is the kernel; by definition, for $B_\gamma$ as in (21), $O(B_\gamma) \cong E_0^{-1} \det W^{-2} K^p$, so it follows that $\deg E_0 = 2\deg W - 2p(g-1) - \deg B_\gamma$, which is strictly smaller than 0 by point 2. above and because $B_\gamma \neq \emptyset$ by regularity. As for the strictness of the inequality, it follows from Theorem 6.7 in [BG03].

Remark 5.8. We note that by reducibility of the spectral curve, there are always unstable Higgs bundles mapping to a point of the Hitchin base. Indeed, take any direct sum $(E_0, 0) \oplus (E_1, \phi_1)$, where $(E_1, \phi_1)$ is a $U(p, p)$-Higgs bundle of degree $\deg E_1 = -\deg E_0$ mapping to $\omega \in \mathcal{A}_{U(p, p)}$. Making $\deg E_0 > 0$ suffices to get an unstable point.

An interesting consequence of this is the lack of intrinsiciy of the Milnor-Wood inequality with respect to the spectral data, unlike what happens for $U(p, p)$-Higgs bundles [Sch13], which is why point 2. in Theorem 5.7 is necessary.

The same phenomenon will show for all forms of Hermitian non-tube type, as they all contain a maximal tube-type subgroup of the same rank.

Remark 5.9. The set we recover has the expected dimension. By the preceding discussion, using the sequence (31) we have that the generic fiber $F_\omega$ has dimension $\dim F_\omega = \dim \text{Pic}(\tilde{X}_1)^\theta + 4(p(g-1) - 1)$. Given that $\text{Pic}(\tilde{X}_1)^\theta$ is a smooth fiber of the Hitchin map for $U(p, p)$, and by
Theorem 5.7

\[ \dim F_\omega = 4p(g - 1) + (4p^2(g - 1) + 1 - \dim B_{SU(p,p)}) - 1, \]
so that the set of all generic fibers has dimension

\[ \dim B_{SU(p+1,p)} + \dim F_\omega = 4p(p + 1)(g - 1) = \dim SU(p + 1, p)(g - 1). \]

We can give another characterization in terms of data over \( X \):

Corollary 5.10. There is a one to one correspondence between isomorphism classes of regular Higgs bundles \((E, \phi)\) mapping to \( \omega \) via \( h_{SU(p+1,p)} \) and the quotient by the action of \( \mathbb{C}^* \) on tuples

\[
(F_{-}, s_{+}, s_{-}, f_{+}, f_{-})
\]

where

1. A line bundle \( F_{+} \in \text{Pic}(\overline{X}_1) \), subject to condition in 4. below.
2. Morphisms

\[
(33) \quad s_{+} \in \text{Ker}(H^0(\overline{X}, F_{+}^{-1}F_{-}\pi^*K) \to H^1(X, F_{+}^{-1}\pi^*(E_0K))),
\]

\[
(34) \quad s_{-} \in \text{Ker}(H^0(\overline{X}, F_{-}^{-1}F_{+}\pi^*K) \to H^1(X, F_{-}^{-1}\pi^*(E_0K))).
\]

where \( R_{\pm} = (s_{\pm}), F_{-} := O(-R_{-})F_{+}\pi^*K \) and \( E_0 = NmF_{+}^{-2}\otimes O(-\pi_{+}R_{+}K)^{-p(2p+1)} \in \text{Pic}(X) \).
3. Isomorphisms

\[
(34) \quad f_{-} \in \text{Isom}(E_0|_{R_{-}}, F_{+}|_{R_{-}}),
\]

\[
(34) \quad f_{+} \in \text{Isom}(F_{+}|_{R_{+}}, E_0|_{R_{+}} \otimes \pi^*K).
\]

4. The bundle \( F_{+} \) should satisfy that \( F_{+} \oplus F_{+} \otimes O(-R_{-}) \) be a \( \pi_{+}\mathcal{O}_{\overline{X}} \) module.
5. Moreover

\[ \deg W = \deg F_{+} - (2p^2 - 2p + 1)(g - 1). \]

The corresponding Higgs bundle is stable and

\[ (2p^2 - 3p)(g - 1) < |\deg F_{-}| < (2p^2 - p)(g - 1). \]

Proof. It follows from Theorem 5.7 observing that pushforward to \( \overline{X} \) produces the objects in this corollary. \( \square \)

Regarding connected components, we have

Corollary 5.11. The generic regular fiber has connected components classified by values of \( \deg W \) other than \( \pm p(g - 1) \).
6. FROM CAMERAL TO SPECTRAL DATA

As the name of the section suggest, we next explain how the spectral data can be obtained from the cameral data.

Following [Don93], we associate to the cameral cover \( \hat{\mathcal{X}} \) an intermediate cover (isomorphic to the spectral cover) associated to the standard representation.

To do this, we consider the system of simple roots

\[
S = \{ \alpha_i = L_i - L_{i+1}, \ i = 1, \ldots, 2p-1 \},
\]

where where \( L_i \) applied to a diagonal matrix returns the \( i \)-th entry and \( \alpha_i > \alpha_{i+1} \). This ordering satisfies condition (2) and so we may choose a compatible Borel subgroup \( B < \text{SU}(p+1, p) \).

Let \( \delta_1 \) be the first fundamental weight, which is the highest weight of the standard representation, and its associated maximal parabolic \( P_{\delta_1} \). Let \( W_{\delta_1} < W \) be the corresponding Weyl group. Then \( W_{\delta_1} \cong S_{2p} \).

We have maps

\[
\hat{\mathcal{X}} \xrightarrow{\pi} \hat{\mathcal{X}}_{\delta_1} := \hat{\mathcal{X}}/W_{\delta_1} \xrightarrow{\pi} X
\]

and moreover, by the discussion in [Don93] §4, \( \hat{\mathcal{X}}_{\delta_1} \) is isomorphic to the spectral cover (18) via the morphism:

\[
(35) \quad X \times_{\mathfrak{o}_C \otimes K/W} \mathfrak{o}_C \otimes K \to K
\]

\[
(x, q) \mapsto (x, \delta_1(q))
\]

Identifying \( \hat{\mathcal{X}}_{\delta_1} \) and \( \hat{\mathcal{X}} \), we have a diagram

\[
(36) \quad \hat{\mathcal{X}} \xrightarrow{\pi} \hat{\mathcal{X}} \xrightarrow{\pi} X.
\]

**Lemma 6.1.** The action of \( \theta \) on \( \hat{\mathcal{X}} \) descends to the involution on \( \hat{\mathcal{X}}_{\delta_1} \) sending \( (x, \lambda) \mapsto (x, -\lambda) \).

**Proof.** Let \( s_i \) denote the reflection associated to the simple root \( \alpha_i \). From (17) we check that \( (1, 2) \circ \theta \in W_{\delta_1} \), so that both maps induce the same one on \( \hat{\mathcal{X}}_{\delta_1} \). Since \( \delta_1 = \alpha_1^* \in (\mathfrak{o}_C)^* \), it follows that \( -\delta_1 = \delta_1 \circ s_1 \), whence the result. \( \square \)

**Proposition 6.2.** Let \( \omega \in B_{\text{SU}(p+1, p)} \) be generic, and let \( P_0 \to \hat{\mathcal{X}}_\omega \) be a cameral datum. Let \( \text{Cam}(\hat{\mathcal{X}}) \) denote the variety of isomorphism classes
of cameral data. Then \( \text{Cam}(\hat{X}) \cong \{ L \in \text{Pic}(\tilde{X}) : \theta^* L \cong L, \deg L_0 = -\deg L_1 - 2p(2p + 1)(g - 1) \} \).

**Proof.** Proceeding as in Theorem 4.3, we assign

\[ P \mapsto PP^{-1} \]

thus establishing an isomorphism \( \text{Cam}(\hat{X}) \to H^1(\hat{X}, D^C)^{W, \theta} \), with the induced action (and conditions on it) of \( W \) and \( \theta \), that is \( w \cdot Q = w^* Q \times_w D^C \), and \( \theta^* Q|_{\tilde{X}_{SU}} \cong Q|_{\tilde{X}_{SU}} \).

Now, given \( Q \in H^1(\hat{X}, D^C) \), then \( L_Q := Q \times_{\delta_1^C} \mathbb{C}^\times \) is a line bundle satisfying \( w^* L_Q \cong L_Q \) for all \( w \in W_{\delta_1} \) (by Proposition 5.5 in [DG01] and the fact that the action of \( w \in W_{\delta_1} \) on the fibers is trivialised by composing with \( \delta_1 \)). The same reasons imply that the action of \( w \) over its associated ramification divisor is trivial. So by Kempf’s Descent Lemma, \( L_Q \) descends to \( \tilde{L}_Q \to \hat{X} \). As for equivariance \( \theta^* \tilde{L}_Q \cong \tilde{L}_Q \), it follows from Lemma 6.1.

The map is an isomorphism by Proposition 9.5 in [DG01] and Theorem 4.3. \( \square \)

### 7. Non-regular Higgs bundles

Reducibility of the spectral curve causes for the existence of non-regular Higgs bundles over any given point of the base, which are not captured by the cameral construction, as in fact they aren’t intrinsic to the point of the base.

The explanation lies in Theorem 17.5 in [DG01], according to which a Higgs bundle within a Hitchin fiber is equivalent to a cameral datum together with the extra data of a \( W \)-equivariant morphism \( F : \hat{X} \to \mathfrak{d}^C \otimes K \), regular fields are given by embeddings, or projections onto the second factor \( \hat{X} := X \times_{\mathfrak{d}^C \otimes K/W} (\mathfrak{d}^C \otimes K) \), and hence are intrinsic to the point of the base.

Now, such a morphism \( F \) can be proved to be equivalent to fixing a Higgs field with values in a subsheaf of regular centralisers totally determined by the point of the Hitchin base [DG01]. In Theorem 5.7 the hypotheses on the point of the Hitchin base ensure that the Higgs field is completely determined away from ramification. Hence, it suffices to determine it over ramification. This is precisely the information encoded in \( f_{\pm} \) in (34) and requirement (33) in Corollary 5.10.

When the Higgs field is not regular, \( f_{\pm} \) are not isomorphisms anymore, but morphisms. The locus over which they vanish is \( B_{\beta/\gamma}^0 \) defined in Remark 5.2. A part of these non-regular bundles can be produced by considering the a bundle of toric varieties over \( \text{Pic}(\tilde{X}_1)^6 \) with
fiber \( \text{Hom}(\tilde{X}_{01}, L^{-1}F_1) \) over \( F_1 \) (and dense torus \( \text{Isom}(\tilde{X}_{01}, L^{-1}F_1) \subset \text{Hom}(\tilde{X}_{01}, L^{-1}F_1) \)), cf. Theorem 5.7). These are coherent sheaves over \( \tilde{X} \) whose restrictions to the irreducible components are locally free away from the singular locus. The pushforward of these is again a well-defined Higgs bundle; nonetheless, the correspondence fails to be unique at this level, as the Higgs bundles thus produced will be those having zero nilpotent part over the ramification locus of the curve. In order to describe all of them it is necessary to introduce a stratification by “degree of regularity” (measured by the dimension of the centralisers of the Higgs field over \( B^0_{\beta/\gamma} \)). We hope to address this questions in the near future.

8. On regularity

The relation between regularity (cf. Definition [A.4]) and smoothness of points of the complex Hitchin fiber essentially goes back to Kostant’s [Kos63], as it is proved by Biswas and Ramanan ([BR94], Theorem 5.9). Their proof applies to the real group case, so we have:

**Proposition 8.1.** If a \( G \)-Higgs bundle \((E, \phi)\) is a smooth point of \( h^{-1}_{G}(\omega) \), then \( \phi(x) \in m_{\text{reg}} \) for all \( x \in X \).

**Proof.** Let \( x \in X \). We have that \( \text{ev}_x \circ h_G(E, \phi) = \chi \phi_x \), where \( \chi: m^{C} \to a^{C}/W(a) \) is the Chevalley map. At a smooth point of the fiber, \( dh_G \) is surjective, and since \( \text{ev}_x \) is surjective too, it follows that \( d(\chi \circ \text{ev}_x) \) is itself surjective. Since \( \text{dev}_x : H^0(X, E(m^{C} \otimes K)) \to m^{C} \otimes K_x \) is surjective, and is itself evaluation at \( x \), this implies that \( d_{\phi_x} \chi \) is surjective. But Kostant–Rallis’ work [KR71] implies this happens if and only if \( \phi_x \) is regular. \( \square \)

**Appendix A. Lie theory**

The subgroup \( SU(p + 1, p) \leq SL(2p + 1, \mathbb{C}) \) is defined as the locus of fixed points of the involution

\[
\sigma(X) = \text{Ad}(I_{p+1,p})^t X^{-1}
\]

where

\[
I_{p+1,p} = \begin{pmatrix} I_{p+1} & 0 \\ 0 & -I_p \end{pmatrix}.
\]

By composing with the compatible compact involution \( \tau(X) =^t X^{-1} \), we obtain a linear involution

\[
\theta^\prime = \text{Ad}(I_{p+1,p}).
\]
whose restriction to $\text{SU}(p+1, p)$ is $\tau$. The differential of this involution (denoted also by $\theta'$) induces the Cartan decomposition

$$\mathfrak{su}(p + 1, p) = \mathfrak{s}(\mathfrak{u}(p + 1) \oplus \mathfrak{u}(p)) \oplus \mathfrak{m}.$$ 

Here, $\mathfrak{m}$ is defined by

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \in \mathfrak{sl}(2p + 1, \mathbb{C}) \mid B \in \text{Mat}_{p+1 \times p}(\mathbb{C}), \ C = \begin{pmatrix} t \\ B \end{pmatrix} \right\}.$$ 

To this decomposition it corresponds a polar decomposition on the level of the group. Indeed, $\text{SU}(p+1, p) = \text{He}^\mathfrak{m}$, where $H = S(\text{U}(p+1) \times \text{U}(p))$ is realised as the subgroup of matrices of $\text{SL}(2p + 1, \mathbb{C})$ of the form

$$\begin{pmatrix} A & 0 \\ 0 & B^{-1} \end{pmatrix}$$

where $A \in \text{U}(p)$, $B \in \text{U}(p + 1)$, $\det A \det B = 1$.

We next revise the theory of the isotropy representation necessary for this article. Recall that this representation

$$\iota : H^\mathbb{C} \rightarrow \text{GL}(\mathfrak{m}^\mathbb{C}),$$

is obtained by restriction of the adjoint representation.

**Definition A.1.** A maximal anisotropic Cartan subalgebra of $\mathfrak{sl}(2p + 1, \mathbb{C})$ (associated with $\mathfrak{su}(p+1, p)$) is the complexification of a maximal abelian subspace $\mathfrak{a}' \subset \mathfrak{m}$.

One calculates easily (cf. [Kna05], Chapter VI) that the maximal anisotropic Cartan subalgebra $(\mathfrak{a}')^\mathbb{C}$ consists of the matrices of the form

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & X \\ 0 & tX & 0 \end{pmatrix}$$

with $X \in \text{Mat}_{p \times p}(\mathbb{C})$ antidiagonal.

The centraliser of $(\mathfrak{a}')^\mathbb{C}$ in $H^\mathbb{C}$ is defined as the subgroup $C_H((\mathfrak{a}')^\mathbb{C}) \subset H^\mathbb{C}$ defined as

$$C_H((\mathfrak{a}')^\mathbb{C}) = \{ h \in H^\mathbb{C} \mid \text{Ad}(h)A = A \text{ for any } A \in (\mathfrak{a}')^\mathbb{C} \}.$$ 

One readily checks that

$$C_H((\mathfrak{a}')^\mathbb{C}) = \left\{ \begin{pmatrix} \det A^{-2} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix} \mid A \in \text{GL}_{p \times p}(\mathbb{C}) \text{ diagonal} \right\}.$$
Note that the direct sum \( \mathfrak{d}^C = (\mathfrak{a'})^C \oplus \mathfrak{c}_h((\mathfrak{a'})^C) \), where \( \mathfrak{c}_h((\mathfrak{a'})^C) := \text{Lie}(C_H(a)) \), is a \( \theta' \)-stable Cartan subalgebra. Let \( W \) be the corresponding Weyl group. Recall that we define the restricted Weyl group
\[
(W'^{\prime})_C := N_W((\mathfrak{a'})^C)/C_W((\mathfrak{a'})^C),
\]
where \( N_W((\mathfrak{a'})^C) \) is the normaliser of \( (\mathfrak{a'})^C \) in \( W \) and \( C_W((\mathfrak{a'})^C) \) its stabiliser. This is isomorphic to the quotient \( N_H(a)/C_H(a) \). Chevalley’s restriction theorem reduces the problem of studying the quotient \( C^C//H^C \) to the much simpler \( (\mathfrak{a'})^C/W((\mathfrak{a'})^C) \):

**Theorem A.2** (Chevalley’s Restriction Theorem). Restriction \( C^[[m^C]] \to \mathbb{C}[[\mathfrak{a'}^C]] \) induces an isomorphism \( C^[[m^C]] \cong \mathbb{C}[[\mathfrak{a'}^C]^W((\mathfrak{a'})^C)] \).

**Lemma A.3.** The algebra of polynomial invariants \( \mathbb{C}[[\mathfrak{a'}^C]^W((\mathfrak{a'})^C)] \) is isomorphic to \( \bigoplus_i \text{Sym}^i(\mathbb{C}) \).

**Definition A.4.** We define the subset \( m_{reg} \subset m^C \) of regular elements by
\[
m_{reg} = \{ y \in m^C : \dim H^C \cdot y \geq \dim H^C \cdot z \ \forall z \in m^C \}.
\]
Equivalently, \( \dim \mathfrak{c}_m(x) = \dim(\mathfrak{a'})^C \), where \( \mathfrak{c}_m(x) = \{ y \in m^C : [y, x] = 0 \} \).

**Lemma A.5.** \( SU(2p + 1, \mathbb{C}) < SL(2p + 1, \mathbb{C}) \) is a quasi-split form. In particular,
\[
m_{reg} = m^C \cap \mathfrak{g}_{reg}.
\]

**Proof.** A form is quasi-split if and only if \( \mathfrak{c}_h((\mathfrak{a'})^C) \) is abelian (cf. [Kna05]), which follows from [40]. In particular an element is regular if and only if its centraliser in \( \mathfrak{g} \) has the dimension of a Cartan subalgebra, as this is the case for semisimple elements. Thus it is regular in \( \mathfrak{g} \) if and only if it is regular in \( m \).

It is useful to consider the following realization of \( \mathfrak{su}(p+1, p) \). Define the involution
\[
\theta = \text{Ad}(J_{p+1, p})
\]
where \( J_{p+1, p} \) is the matrix with 0 entries everywhere except for the antidiagonal entries, which are \( j_{2p-i+1, 1} = 1 \) if \( i \neq p+1 \) and \( j_{p+1, p+1} = -1 \). Then the subalgebra \( \mathfrak{d}^C = \mathfrak{a}^C \oplus \mathfrak{t}^C \), with
\[
\mathfrak{a} = \left\{ \begin{pmatrix}
A & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -A 
\end{pmatrix} : A \in \text{Mat}_{p \times p}(\mathbb{C}) \text{ diagonal} \right\},
\]
\begin{equation}
t = \left\{ \begin{pmatrix} B & 0 & 0 \\ 0 & -2\text{tr}(B) & 0 \\ 0 & 0 & B \end{pmatrix} : B \in \text{Mat}_{p \times p}(\mathbb{C}) \text{ diagonal} \right\}.
\end{equation}

is a $\theta$-invariant Cartan subalgebra which is also maximally anisotropic, as explained in [GW09], Section 12.3.2.

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Mathematik des Institut, Ruprecht-Karls Universität, Im Neuenheimer Feld 288, 69120 Heidelberg

E-mail address: apexonettomath.uni-heidelberg.de