Combining Weak Distributive Laws: Application to Up-To Techniques

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Abstract

The coalgebraic modelling of alternating automata and of probabilistic automata has long been obstructed by the absence of distributive laws of the powerset monad over itself, respectively of the powerset monad over the finite distribution monad. This can be fixed using the framework of weak distributive laws. We extend this framework to the case when one of the monads is only a functor. We provide abstract compositionality results, a generalized determinization procedure, and systematic soundness of up-to techniques. Along the way, we apply these results to alternating automata as a motivating example. Another example is given by probabilistic automata, for which our results yield soundness of bisimulation up-to convex hull.

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1 Introduction

Coalgebras have known great success in the abstract modelling of a wide range of systems originating from computer science. The theory is parametric and modular with respect to the base category and the type functor. Compositionality concerns as well as generalization of major landmarks such as the determinization constructions for automata [24] led the coalgebra community to make heavy use of Beck’s theory of distributive laws, which can be seen as a way of composing monads, see for e.g. [16, 15, 17]. Given two monads $S$ $T$ modelling two branching types for a given system, a distributive law is a natural transformation of type $ST \Rightarrow ST$ that satisfies four coherence diagrams. This simple swapping operation suffices to produce a monad on the composite functor $ST$ for multiplication can now be defined as $STST \Rightarrow SSTT \Rightarrow ST$. Notably, distributive laws have been extended to the case when one monad structure and the two corresponding coherence diagrams are suppressed. This slight alteration produces a powerful tool to model the interplay between branching behaviour (represented by a monad) and machine-like behaviour (represented by a plain functor).

The powerset monad $P$ and the finite distribution monad $D$ on the category of sets are amongst the most commonly used, as they constitute the basic bricks for representing respectively non-deterministic and probabilistic behaviour. In the recent years, the community stumbled over the seemingly surprising fact that there is neither a distributive law of type $DP \Rightarrow PD$ see [27], nor one of type $PP \Rightarrow PP$ see [18]. Acknowledging that further such impossibilities may arise, Zwart and Marsden recently provided general algebraic conditions that make distributive laws unattainable [28]. One unpleasant impact of all these negative results is that the coalgebraic study of alternating automata and probabilistic automata must
Alternating automata are systems for which a transition consists of an existential step and then a universal step — making the composed $\mathbb{P} \times (\mathbb{P})^A$ a functor of choice to store transitions. Unfortunately, there is no distributive law of powerset over itself and even no monad structure on $\mathbb{P} \times (\mathbb{P})^A$ at all. Some substitute modellings have been built to reason coalgebraically about alternating automata, for example, by going back and forth to the category of posets. Similarly, probabilistic automata send a state to a set of distributions — making them a seemingly easy target for the composed $\mathbb{P} \times (\mathbb{P})^A$. But again, the lack of distributive law and even of possible monad structures on $\mathbb{P} \times (\mathbb{P})^A$ make their analysis much harder than expected.

It is not only generalized determinization that poses problems for these systems. Another aspect where the theory is not very smooth concerns the so called up-to techniques. Computing a bisimulation between two systems can be tedious or even require an infinite number of steps. Up-to techniques’ stated objective is to prune branches in the exploration of the state space. Originating from a lattice-theoretic mindset, see e.g. [21] for a comprehensive account, they have proved hugely popular in the last few years, partly on account of the impressive results of Bonchi and Pous [7] to accelerate bisimulation computation on determinized automata. From a category-theoretic perspective, generalized determinization, distributive laws, and the soundness of up-to techniques are intrinsically linked, as explored in [5]. In the absence of a distributive law, we cannot reuse this compositionality results for alternating or probabilistic automata. A breakthrough occurred in [8], where Bonchi et al. do obtain a form of coalgebraic determinization of probabilistic automata and use it to prove the soundness of the up-to convex hull technique. However, some of the required constructions have to be redone by hand, in a way that is disappointingly close to the usual framework relying on distributive laws. Things almost work well, but not quite.

Coincidentally, a not-quite theory of distributive laws has been brought to light by Garner in a very recent paper [13]. This theory originated in the work of Street [25] and Böhm [4], from which Garner picked a particular set of axioms to the purpose of exhibiting the Vietoris monad as a canonical ‘almost’ lifting of the powerset monad. The basic observation is that in many cases, a not-quite distributive law $\mathbb{T}S \Rightarrow \mathbb{S}T$ fails to be one because of one specific coherence diagram, namely the one that states that the unit of $\mathbb{T}$ is compatible with $\mathbb{S}$. A weak distributive law is defined as making the three other coherence diagrams commute. By a well-rounded category-theoretic analysis, such laws are proved to produce a distributive law-lifting-extension trinity similar to the standard theory, as well as a monad structure that combines $\mathbb{S}$ and $\mathbb{T}$ but whose functor is not $\mathbb{S}T$. In a previous paper [14], we prove that there is a weak distributive law of type $\mathbb{D}P \Rightarrow \mathbb{E}D$ and thus exhibit the convex powerset monad presented in [8] as a canonical weak lifting of the powerset monad to $\mathbb{D}$-algebras.

In the present paper we continue our exploration of other applications of weak distributive laws to the theory of systems modelled as coalgebras. Our contributions are three-fold:

Coalgebraic semantics for alternating automata. Adapting an example from Garner, we point out that there is a weak distributive law of type $\mathbb{P}P \Rightarrow \mathbb{P}P$. We use this to enlighten that the procedure for determination of alternating automata of [17] is canonical in the sense of weak distributive laws. Notice however that alternating automata are coalgebras for the functor $2 \times (\mathbb{P})^A$, and not just $\mathbb{P}P$ which leads us to our next contribution.

Generalized determinization. Secondly, we extend the theory of generalized determinization via weak distributive laws to the setting where one monad is replaced by the composition of a monad and a functor — only one coherence diagram is left in this case. In this context, we provide a compositionality result inspired by the work of Cheng on iterated distributive laws.
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Soundness of up-to techniques. Once the category-theoretic understanding of the
determinization of alternating automata, respectively of probabilistic automata is settled,
we are ready to tackle another application of weak distributive laws, namely to proving the
soundness of up-to techniques and exploiting the compostionality approach of [5, 6]. We
show that up-to techniques obtained via weak distributive laws are sound, on an equal basis
with the ones of distributive laws. If in [14] the focus was set on the weak distributive law
\[ DP \Rightarrow PD \] and probabilistic automata, we now concentrate on the weak distributive law
\[ PP \Rightarrow PP \] and alternating automata and the compatibility of the associated up-to technique.
We also retrieve compatibility of up-to convex hull as stated in [8].

Synopsis. Sections 2 and 3 consist of reminders about the standard and weak theory
of distributive laws, respectively. Section 3 ends with a compositionality result for weak
distributive laws. In Section 4, we perform generalized determinization with respect to
multiple weak distributive laws, at once from a theoretical viewpoint and on the case study of
alternating automata. Section 5 deals with compatibility of up-to techniques: we prove that
the standard (bialgebraic) method remains valid in our case and derive the up-to techniques
arising from generalized determinization for alternating automata and probabilistic automata.

2 Prerequisites

2.1 Functors and Monads

We hereby recall a few popular functors and monads on the category \textbf{Set} of sets and functions.

1. The finite powerset functor \( P : \textbf{Set} \rightarrow \textbf{Set} \) maps a set \( X \) to the set all of subsets of \( X \) and
   maps a function \( f : X \rightarrow Y \) to its direct image. It can be extended into a monad \( P = (P, \eta_P, \mu_P) \)
   where unit is \( \eta_P X(x) = \{x\} \) and multiplication is given by union.

2. The finite distribution functor \( D : \textbf{Set} \rightarrow \textbf{Set} \) maps a set \( X \) to the set of finitely supported
   probability distributions on \( X \). Given a function \( f : X \rightarrow Y \), the function \( Df \) maps a
   probability distribution \( \varphi \in DX \) to its pushforward measure with respect to \( f \):
   \[
   Df(\varphi) = y \mapsto \varphi[f^{-1}(\{y\})] = \sum_{x \in f^{-1}(\{y\})} \varphi(x)
   \]
   (1)
   The functor \( D \) can be extended as well into a monad \( D = (D, \eta_D, \mu_D) \) where unit is taking
   the Dirac distribution \( \eta_D X(x) = \delta_x \) and multiplication is given by distribution flattening:
   \[
   \mu_D \Phi = x \mapsto \sum_{\varphi \in DX} \Phi(\varphi) \varphi(x)
   \]
   (2)

3. Assume \( A \) is a fixed finite alphabet. The machine functor \( M : \textbf{Set} \rightarrow \textbf{Set} \) maps a set \( X \)
to the set \( 2 \times X^A \) whose elements are pairs \( (o, t) \) with \( o \in \{0, 1\} \) and \( t : A \rightarrow X \). It maps
a function \( f : X \rightarrow Y \) to the function \( Mf : (o, t) \mapsto (o, a \mapsto f(t(a))) \)

2.2 Distributive Laws

In this section we recall Beck’s framework of distributive laws, liftings and extensions [2].
Let \( T = (T, \eta_T, \mu_T) \) and \( S = (S, \eta_S, \mu_S) \) be monads on a category \( C \).
Definition 1 (Distributive Law). A distributive law of type $\text{T}S \Rightarrow \text{ST}$ is a natural transformation $\delta : \text{T}S \Rightarrow \text{ST}$ making the following diagrams commute:

\[
\begin{align*}
\text{TST} & \xrightarrow{\text{ST}} \text{STT} & \text{TSS} & \xrightarrow{\text{STS}} \text{SST} \\
\text{TS} & \xrightarrow{\text{ST}} \text{TS} & \text{ST} & \xrightarrow{\text{TS}} \text{ST} \\
\text{TS} & \xrightarrow{\text{ST}} \text{TS} & \text{ST} & \xrightarrow{\text{TS}} \text{ST}
\end{align*}
\]

A lifting of $S$ on $T$ is a monad $S : \text{EM}(\text{T}) \rightarrow \text{EM}(\text{T})$ in the Eilenberg-Moore category of $T$ such that $U_T S = S U_T$ (i.e. $U_T$ commutes with functor, unit and multiplication). An extension of $T$ on $S$ is a monad $T : \text{Kl}(S) \rightarrow \text{Kl}(S)$ in the Kleisli category of $S$ such that $F T S = T F S$. In contrast to the weaker notions seen in the next section, these ones will sometimes be called strong.

Proposition 2. There is a bijective correspondence between distributive laws of type $\text{T}S \Rightarrow \text{ST}$, liftings of $S$ on $T$ and extensions of $T$ on $S$.

Additionally, the existence of a distributive law of type $\text{T}S \Rightarrow \text{ST}$ yields a monad structure on the functor $\text{ST}$ defined by $\text{ST} = (\text{ST}, \eta_T S, \mu_T S, \delta_T S \circ \mu_T S)$. Note that in order to distinguish this composite monad $\text{ST}$ from the mere distributive-law-type notation $\text{ST}$, characters $S$ and $T$ are glued together. This framework can be restricted to the case where only one of the involved functors is a monad, with the required commutative diagrams adapting accordingly.

A distributive law of type $\text{G}S \Rightarrow \text{S}G$ is a natural transformation of the obvious type such that the $(\mu_T)$ and $(\eta_T)$ diagrams commute. Such distributive laws are also called EM-laws because they correspond to liftings of $G$ on $T$, i.e. functors $G : \text{EM}(\text{T}) \rightarrow \text{EM}(\text{T})$ such that $U_T G = G U_T$. A distributive law of type $\text{G}S \Rightarrow \text{S}G$ is a natural transformation of the obvious type such that the $(\mu_S)$ and $(\eta_S)$ diagrams commute. Such distributive laws are also called Kl-laws because they correspond to extensions of $G$ on $S$, i.e. functors $G : \text{Kl}(S) \rightarrow \text{Kl}(S)$ such that $F G S = G F S$.

In the case when $S$ is the powerset monad $\text{P : Set} \rightarrow \text{Set}$, it is a natural requirement to ask that an extension of $T$ on $P$ preserves the order structure obtained by identifying $\text{Kl}(P)$ as the category Rel of sets and relations. We have the following result from Barr [11]:

Proposition 3. There is a (necessarily unique) extension of $T$ on $P$ whose functor is a 2-functor $\text{Rel} \rightarrow \text{Rel}$ if and only if both following facts hold:

- the functor $T$ is weakly cartesian, meaning that it preserves weak pullbacks, and
- the natural transformations $\mu_L$ and $\mu_U$ are weakly cartesian, meaning that their naturality squares are weak pullbacks.

The unique extension of Proposition 3, along with the corresponding lifting and distributive law, will be called canonical.
3 The Weak Framework

In various cases, there simply does not exist any distributive law between two monads. More importantly, the uncomfortable situation when there is no monad structure at all on the composite functor happens quite frequently: examples in the category \( \text{Set} \) include \( \mathbb{P} \) \(^{[19]} \) and \( \mathbb{PD} \) \(^{[27]} \). Recently, Zwart and Marsden \(^{[28]} \) provided general theorems that forbid certain distributive laws to exist.

3.1 Weak Distributive Laws

Böhm \(^{[4]} \) and Street \(^{[26]} \) observed that weakening the distributive law axioms in a clever way actually still make \( \mathbb{S} \mathbb{T} \) almost a monad. A particular way of weakening axioms was then picked out by Garner \(^{[13]} \), who proved that the Vietoris functor on \( \text{KHaus} \) is almost a lifting of \( \mathbb{P} \) on the ultrafilter monad \( \mathbb{B} \). Let us be more precise by introducing properly the notions.

The cheeky basic idea is to simply drop the axiom that is often causing trouble, namely the \( \eta \mathbb{T} \) diagram.

\[ \text{Definition 4 (Weak Distributive Law).} \quad \text{A weak distributive law of type} \mathbb{S} \mathbb{T} \Rightarrow \mathbb{S} \mathbb{T} \text{ is a natural transformation} \delta : \mathbb{S} \mathbb{T} \Rightarrow \mathbb{S} \mathbb{T} \text{ such that diagrams} \mu \mathbb{T}, \mu \mathbb{S} \text{ and} \eta \mathbb{S} \text{ commute.} \]

\[ \text{Definition 5 (Weak Lifting).} \quad \text{A weak lifting of} \mathbb{S} \text{ on} \mathbb{T} \text{ is a monad} \mathbb{T} \mathbb{S} : \text{EM}(\mathbb{T}) \rightarrow \text{EM}(\mathbb{T}) \text{ along with two natural transformations} \pi : \mathbb{S} \mathbb{T} \Rightarrow \mathbb{T} \mathbb{S}, \iota : \mathbb{T} \mathbb{S} \Rightarrow \mathbb{S} \mathbb{T} \text{ such that} \pi \circ \iota = 1 \text{ and the following diagrams commute:} \]

\[ \text{Definition 6 (Weak Extension).} \quad \text{A weak extension of} \mathbb{T} \text{ on} \mathbb{S} \text{ is a functor} \mathbb{T} : \text{Kl}(\mathbb{S}) \rightarrow \text{Kl}(\mathbb{S}) \text{ along with a natural} \mu : \mathbb{T} \mathbb{S} \Rightarrow \mathbb{T} \mathbb{S} \text{ such that} \mathbb{T} \mathbb{F} = \mathbb{F} \mathbb{T} \text{ and} \mu \mathbb{T} = \mu \mathbb{T}. \]

Proposition 2 has a counterpart in the weak framework:

\[ \text{Proposition 7.} \quad \text{There is a bijective correspondence between weak distributive laws of type} \mathbb{T} \mathbb{S} \Rightarrow \mathbb{S} \mathbb{T} \text{ weak extensions of} \mathbb{T} \text{ on} \mathbb{S} \text{ and, whenever idempotents split in the base category} \text{C, weak liftings of} \mathbb{S} \text{ on} \mathbb{T}. \]

Proof. As this result is proved in \(^{[13]} \), we only consider the bit of the proof that will be useful in the remainder of the paper. Let \( \mathbb{S} \) be a weak lifting of \( \mathbb{T} \) on \( \mathbb{S} \). The corresponding weak distributive law is given by \( \mathbb{T} \mathbb{S} \Rightarrow \mathbb{S} \mathbb{T} \), where \( \mu : \mathbb{T} \mathbb{S} \Rightarrow 1 \) is the counit of the adjunction \( \mathbb{T} \mathbb{F} = \mathbb{F} \mathbb{T} \).

Similarly to the strong framework, the existence of a weak distributive law of type \( \mathbb{T} \mathbb{S} \Rightarrow \mathbb{S} \mathbb{T} \) allows to build a monad structure mixing \( \mathbb{S} \) and \( \mathbb{T} \) from the composite adjunction...
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As in the strong framework, this monad will be denoted by \( S \circ T \). Again, characters \( S \) and \( T \) are glued together to denote the composite monad. Note that, unlike the strong case, the functor of \( S \circ T \) is not \( ST \). We will often abuse notation and identify \( S \circ T \) with its underlying functor \( UT \). Note that there are natural transformations \( \iota = \iota F \) and \( \pi = \pi F \) such that \( \pi \circ \iota = 1 \). We use as much as possible the notation \( S \circ T \) to stress the fact that this monad is a kind of composition of \( S \) and \( T \).

Again, one can consider the case when one of the monads is a plain functor. A weak distributive law of type \( T \Rightarrow G T \) only has to make the sole \( \mu T \) diagram commute. If idempotents split in \( C \), such laws are in bijection with weak liftings of \( G \) on \( T \), i.e., functors \( \Xi : \text{EM}(T) \rightarrow \text{EM}(T) \) coming with natural \( \pi : G U \Xi : \Xi \rightarrow U \Xi G \) such that \( \pi \circ \iota = 1 \). Concerning the other type \( G S \Rightarrow S G \), we refrain to define a weak notion of distributive law, as it would require \( \mu S \) and \( \eta S \) to commute, hence bring nothing new in comparison with the strong framework.

As idempotents split in \( \text{Set} \), Barr’s result fits very well into the weak framework:

▶ Proposition 8. There is a (unique) weak extension of \( T \) on \( P \) whose functor is a 2-functor \( \text{Rel} \rightarrow \text{Rel} \) if and only if both following facts hold:

- the functor \( T \) is weakly cartesian, and
- the natural transformation \( \mu T \) is weakly cartesian.

Whenever it exists, this weak extension and the corresponding weak distributive law and weak lifting will be described as canonical.

3.2 Examples

We now give three important examples where the weak framework proves useful. In all three cases, there is no canonical distributive law because the unit \( \eta T \) is not weakly cartesian. The gap is even bigger concerning (at least) the second and third example, because as said previously, there is no distributive law of type \( PP \Rightarrow PP \) or \( DP \Rightarrow PD \) at all.

▶ Example 9. The category of algebras for the ultrafilter monad \( \text{EM}(B) \) is isomorphic to the category of compact Hausdorff spaces \( \text{KHaus} \). There is a canonical weak distributive law of type \( BP \Rightarrow PB \) whose weak lifting is the Vietoris monad on \( \text{KHaus} \) and such that \( PB \) is the filter monad on \( \text{Set} \). This result is the original motivation of Garner’s paper [13].

Garner also shows in Lemma 17 of [13] that there is a canonical weak distributive law of type \( PP \Rightarrow PP \) where \( PP \) is the finite powerset monad. For symmetry purposes we make use of the variant he also mentions:

▶ Example 10. There is a canonical weak distributive law of type \( PP \Rightarrow PP \) defined by

\[
\delta_X(A) = \left\{ B \subseteq X \mid B \subseteq \bigcup A \quad \text{and} \quad \forall A \in A, \ A \cap B \neq \emptyset \right\}
\]

Let us give an expression of the corresponding weak lifting of \( PP \) on \( P \). The category \( \text{EM}(P) \) is isomorphic to the category of complete join semi-lattices. Let \( (X, \sqcup) \) be an object. The
underlying set of $(\mathcal{P}(X, \sqcup))$ is $\Omega = \{A \in \mathcal{P}X | A \text{ is stable under non-empty } \sqcup\}$, and its join is given for every $A \in \mathcal{P}\Omega$ by
\[
\bigsqcup \mathcal{U} = \{\sqcup \{x_U \mid U \in \mathcal{U}\} | \forall U \in \mathcal{U}, x_U \in U\}
\] (4)
We also have a monad $\mathcal{P} \mathcal{P}$ on $\text{Set}$ whose functor maps a set $X$ to the set of all subsets of $X$ closed under non-empty union. On functions, it takes direct images twice.

\textbf{Remark 11.} From the point of view of logic, the transformation performed by the $\delta$ of Example 10 amounts to transforming a conjunctive normal form into an equivalent disjunctive normal form. Indeed, consider that $X$ is a set of propositional variables. Seeing $U \in \mathcal{P} \mathcal{P}X$ as a CNF and $\delta_X(U)$ as a DNF, straightforward computations show that
\[
\bigwedge \bigvee_{U \in \mathcal{U}} x \equiv \bigvee \bigwedge_{V \in \delta_X(U)} x
\]
(5)

\textbf{Example 12.} There is a canonical \textit{weak distributive law} of type $\mathcal{D} \mathcal{P} \Rightarrow \mathcal{P} \mathcal{D}$ defined by
\[
\delta_X \left( \sum_i p_i A_i \right) = \left\{ \sum_i p_i \varphi_i \mid \forall i, \text{supp}(\varphi_i) \subseteq A_i \right\}
\] (6)
where we use the formal sum notation with distinct $A_i$ and positive $p_i$. The corresponding weak lifting [14] is the \textit{convex powerset monad} $\mathcal{P}c$ described in detail in [8] and in a slightly different way in [9]. The monad $\mathcal{P} \mathcal{D}$ on $\text{Set}$ is the convex sets of distributions monad, denoted by $C$ in [8].

3.3 Compositionality

Results about distributive law composition [12] can be adapted to the weak framework to a limited extent.

\textbf{Theorem 13.} Let $\mathcal{T}, \mathcal{S}$ be monads on a category in which idempotents split. Let $G$ be an endofunctor on $\mathcal{C}$. Let $\delta: \mathcal{T} \mathcal{S} \Rightarrow \mathcal{S} \mathcal{T}$ and $\sigma: \mathcal{T} G \Rightarrow G \mathcal{T}$, $\tau: \mathcal{S} G \Rightarrow G \mathcal{S}$ be \textit{weak distributive laws}. Assume the so-called Yang-Baxter diagram commutes:

Then the composite $\lambda = \mathcal{S} \mathcal{T} G \Rightarrow \mathcal{S} G \mathcal{T}$ is a \textit{weak distributive law} of type $\mathcal{S} \mathcal{T} G \Rightarrow \mathcal{G} \mathcal{S} \mathcal{T}$. Moreover, if $\sigma$ and $\tau$ are strong, then $\lambda$ is strong.

A similar result holds if types are $\sigma: \mathcal{G} \Rightarrow \mathcal{G} \mathcal{T}$ and $\tau: \mathcal{G} \mathcal{S} \Rightarrow \mathcal{S} \mathcal{G}$. However, we believe that the results of [12] can not be adapted to the case where $G$ is replaced with a monad $\mathcal{R}$ because the diagram for $\mathcal{R}$ is unlikely to commute.
Example 14. Let $C = \text{Set}$, $T = S = P$ and let $G = M$ be the machine functor $2 \times (-)^A$ with respect to a finite alphabet $A$. Consider the weak distributive law $\delta : PP \Rightarrow PP$ given in Example 10 and the renowned (strong) distributive laws $\sigma, \tau : PM \Rightarrow MP$ defined as

$$
\sigma_X(S) = \left( \bigwedge_{(o,f) \in S} o, a \mapsto \bigcup_{(o,f) \in S} f(a) \right) \quad \tau_X(S) = \left( \bigvee_{(o,f) \in S} o, a \mapsto \bigcup_{(o,f) \in S} f(a) \right)
$$

These laws satisfy the Yang-Baxter condition. Hence, there is a (strong) distributive law $\lambda : PP \Rightarrow PP$ given for any $U \in PP$ by

$$
\lambda_X(U) = \left( \bigvee_{S \in U} \bigwedge_{(o,f) \in S} o, a \mapsto \text{unions} \left( \{ f(a) | (o,f) \in S \} | S \in U \} \right) \right)
$$

where $\text{unions} = \pi : PP \Rightarrow PP$ denotes closure of a set of sets under non-empty unions.

4 Generalized Determinization

In [17], Klin and Rot perform a powerset construction on alternating automata, turning them into non-deterministic automata. To this purpose, a first version of the paper used to introduce a transformation that was wrongly identified as a distributive law. Spotting the mistake sparked the chase of a correct distributive law of type $PP \Rightarrow PP$ which ended brutally with the result of Klin and Salamanca [19] that there can be no such law. In the corrected version of [17], the authors introduce another natural transformation of the same type which happens to be the $\delta$ of Example 10. Thanks to this $\delta$, they manage to correctly turn alternating automata into equivalent non-deterministic automata. However, this construction is not standard and it is still unclear how it would relate to the generalized determinization of coalgebras that heavily relies on distributive laws. In this section, we show that the powerset construction of [17] is an instance of a mild extension of the generalized determinization procedure described in [14] with respect to weak distributive laws.

4.1 Determinization Procedure

In our recent paper [14], the generalized determinization process for coalgebras is adapted to case of monad-monad weak distributive laws. It is immediate that this generalized determinization stills works fine with monad-functor laws. Moreover, compositionality plays it role and modelling systems involving more than two monads and functors does not raise any difficulties. To keep things simple, and having in mind that we aim at modelling alternating automata, we will focus on a special case involving two monads and one functor. The following result can be easily inferred from the constructions of Lemma 5.1 of [14].

Proposition 15. Let $T, S$ be monads on a category $C$ in which idempotents split. Let $G$ be an endofunctor on $C$. Let $\delta : TS \Rightarrow ST$, $\sigma : TG \Rightarrow GT$ be weak distributive laws and $\tau : MG \Rightarrow GM$ the corresponding strong liftings. Then we have the following determinization diagram.
Moreover, it is interesting (and was not already remarked in [14]) to note that in the above diagram, the functor $\text{Coalg}(\mathcal{G}ST) \to \text{Coalg}(\mathcal{G}\mathcal{S})$ can be expressed with the very same formula as in the case of distributive laws.

$\text{Lemma 16.}$ Let $(X, c)$ be a $\mathcal{G}ST$-coalgebra. Then

$$\hat{U}^T (X, c) = T X \xrightarrow{\sigma} T \mathcal{T} X \xrightarrow{\delta} \mathcal{T} \mathcal{G} \mathcal{S} \mathcal{T} X \xrightarrow{\mu} \mathcal{G} \mathcal{S} \mathcal{T} X$$

$\text{(9)}$

$4.2$ Application to Alternating Automata

The spirit of alternating automata originates in [11]. It is well-suited to make systems deal with $\forall/\exists$ alternation — a leitmotiv in logic. In this section, we give a coalgebraic modelling to alternating automata and provide their generalized determinization. This construction could have been equally performed on probabilistic automata to retrieve their determinized belief-state transformer. For this reason, notations and examples are kept close to the ones of [8].

$\text{Definition 17}$ (Alternating automaton). An alternating automaton is a tuple $(X, A, F, \rightarrow)$ where $X$ is a set of states, $A$ is a set of action labels, $F \subseteq X$ is a set of final states and $\rightarrow \subseteq X \times A \times \mathcal{P}(X)$ is the transition relation. We will denote $(x, a, U) \in \rightarrow$ by $x \xrightarrow{a} U$.

As often happens, the structure of alternating automata can be rephrased coalgebraically.

$\text{Proposition 18.}$ An alternating automaton $(X, A, F, \rightarrow)$ can be identified with a coalgebra $c = \langle o, t \rangle: X \rightarrow \mathcal{2} \times \mathcal{P}(\mathcal{P}(X))$ on $\text{Set}$, where $o(x) = 1$ iff $x \in F$ and $U \in t(x)(a)$ iff $x \xrightarrow{a} U$.

The alternating automaton of Figure 1 is directly inspired from the example given in [8] in order to highlight the vivid similarities between determinization of a probabilistic automaton into a belief-state transformer and determinization of alternating automaton into a non-deterministic automaton.

The language map $[-]_{aa}: X \rightarrow 2^A^*$ of an alternating automaton $c = (o, t)$ is

$$[x]_{aa}(\varepsilon) = o(x) \quad \quad \quad [x]_{aa}(aw) = \bigvee_{U \in t_a(x)} \bigwedge_{y \in U} [y]_{aa}(w) \quad \quad \quad (10)$$

Because of the lack of distributive law of type $\mathcal{P}\mathcal{P} \Rightarrow \mathcal{P}\mathcal{P}$ in the past few years this modelling has been neglected in favor of workarounds e.g. using the category of posets [3]. Using the canonical weak distributive law $\delta: \mathcal{P}\mathcal{P} \Rightarrow \mathcal{P}\mathcal{P}$ and the determinization procedure of Section 4.1 we can take a fresh look at determinization of alternating automata. Note that performing once a powerset construction on $c$ yields a non-deterministic automaton $c^+$. We will still call $c^+$ the determinized of $c$.

Let us apply Proposition 15 with $C = \text{Set}$, $S = T = \mathcal{T}$, $G = M$, $\mathcal{M}$ and $[\tau, \sigma]$ the two distributive laws of type $\mathcal{P}\mathcal{P} \Rightarrow \mathcal{P}\mathcal{P}$ defined in Example 14. Let $c: X \rightarrow \mathcal{M} \mathcal{P} \mathcal{P} X$ be an alternating automaton. Remind that $c = (o, t)$ with $o: X \rightarrow 2$ and $t: X \rightarrow \mathcal{P}(\mathcal{P}(X))^A$. We use the convenient notation $t_a(x) = t(x)(a)$ and hereby determinize $c$ with respect to the inner powerset. Thanks to Lemma 16 this amounts to computing

$$c^+ = T X \xrightarrow{\sigma} T \mathcal{T} \mathcal{P} \mathcal{P} X \xrightarrow{\delta} \mathcal{T} \mathcal{M} \mathcal{P} \mathcal{P} X$$

$\text{(11)}$
Figure 1 An alternating automaton $c_0$ on the alphabet $A = \{a\}$ with no final states. Solid lines denote existential transitions and dotted lines denote universal transitions. In other terms, any $U \in t_a(x)$ gives rise to one solid line starting from $x$, from the end of which starts one dotted line per element $t \in U$. For instance, $t_a(x_1) = \{\{x_1, x_2\}, \{x_2, x_3\}\}$.

Figure 2 A portion of the non-deterministic automaton $c_0^+$ obtained by determinizing once the alternating automaton $c_0$ of Figure 1.

Then $c^+ = \langle o^+, t^+ \rangle$ maps $U$ to $(\bigwedge_{x \in U} o(x), a \mapsto \{\bigcup_{x \in U} K_x \mid \forall x \in U, K_x \in \text{unions} \, t_a(x)\})$. As noted in [19], this non-standard determinization of alternating automata may not be efficient because many states are added on account of unions. However, our generic framework also comes with systematic compatibility of up-to techniques that will fix this issue. Let $[\text{ind}^{-}] P X \rightarrow 2^A$ be the usual non-deterministic semantics of the determined $c^+ : P X \rightarrow M P P X$. As remarked previously in [19]:

**Proposition 19.** For all $(w, U) \in A^* \times P X$, $[U]_{\text{ind}^{-}} w = \bigwedge_{x \in U} [x]_{\text{aa}}(w)$. In particular, equation $[\text{ind}^{-}] [P X] = [\text{aa}]$ holds.

**Remark 20.** Note that by applying multiple times Proposition 15-like results, one can actually chain determinizations. For instance, provided a weak distributive law of type $S \Rightarrow \epsilon S$ one can further determinize $c^+ : T X \Rightarrow C[S] X$ by applying Proposition 15 by formally replacing $(S \Rightarrow \epsilon S)$ with $\langle I \Rightarrow S, I \rangle$ with $I$ the identity monad. In the case of an alternating automaton with $n$ states and the $C$ of Example 14, this amounts to performing an additional standard powerset construction, resulting in a standard deterministic automaton with $2^n$ states.
5 Up-To Techniques

This section is dedicated to derive and use abstract results about up-to techniques in the case of weak distributive laws. Such techniques have already been obtained in some cases where there is no distributive law for the convex powerset monad. By identifying the presence of a weak distributive law we are able to easily adapt the known results concerning up-to techniques.

5.1 Context and Congruence Closure

Let be endofunctors on . We consider a system given by an -coalgebra . For any relation let

\[ \text{Rel}(F)\xi(R) = \{(u,v) \in X \times X \mid \exists t \in FR, (\xi(u),\xi(v)) = (F\pi_1(t), F\pi_2(t))\} \]  

(12)

where \( \pi_1, \pi_2 : R \to X \) are the canonical projections. A relation is a bisimulation if \( R \subseteq \text{Rel}(F)\xi(R) \). Bisimilarity is the greatest fixpoint of \( \text{Rel}(F)\xi \) on the lattice of relations. The coinduction principle states that in order to prove \( x \sim y \), it is sufficient to produce a bisimulation \( R \) that contains the pair \( (x,y) \). However, bisimulations may be hard to compute or too large to be computed. In the last years, a great interest has been given to the theory of up-to techniques. Such techniques allow to prove bisimilarity of two states \( x \sim y \) by exhibiting a relation \( R \) such that \( (x,y) \in R \) and \( R \) is a bisimulation up-to some other operator on the lattice \( X \times X \). More precisely, let \( \text{Tch}(T)X \times X \to \text{Tch}(T)X \times X \) be a monotone function called up-to technique. The technique \( \text{Tch} \) is sound (with respect to \( \xi \)) if for all \( R \subseteq X \times X \),

\[ R \subseteq (\text{Rel}(F)\xi \circ \text{Tch})(R) \Rightarrow R \subseteq \sim \]  

(13)

Sound techniques are not closed under composition, which led the community to introduce the subclass of compatible techniques \([20]\). The function \( \text{Tch} \) is \( \text{Rel}(F)\xi \)-compatible if \( \text{Tch} \circ \text{Rel}(F)\xi \subseteq \text{Rel}(F)\xi \circ \text{Tch} \). Compatibility entails soundness and enjoys very good compositional properties. A standard approach to derive compatible up-to techniques is to identify a so-called bialgebra. We can e.g. use Theorem 4 of \([23]\):

\[ \text{Proposition 21.} \] Let \( F[T] \) be \( \text{Set} \)-functors, \( \lambda \) a natural transformation, \( (X,\alpha) \) a \( T \)-algebra and \( (X,\xi) \) an \( F \)-coalgebra. Contextual closure \( \text{Ctx}_n \) is defined by

\[ \text{Ctx}_n(R) = \{ (\alpha \circ T_1(t), \alpha \circ T_2(t)) \mid t \in TR \} \]  

(14)

Confluence closure is the coupling of equivalence closure and context closure, formally:

\[ \text{Cgr}_n(R) = \bigcup_{n \geq 0} (\text{Tra} \cup \text{Sym} \cup \text{Ctx}_n \cup \text{Rfl})^n \]  

(15)

If \( F \alpha \circ T_1 = \xi \circ \alpha \), meaning that \( (X,\alpha,\xi) \) is a \( \lambda \)-bialgebra, \( \text{Ctx}_n \) is \( \text{Rel}(F)\xi \)-compatible. Under the additional assumption that \( F \) preserves weak pullbacks, \( \text{Cgr}_n \) is \( \text{Rel}(F)\xi \)-compatible.

We place ourselves the situation of Proposition 13. Let \( G \) be a \( \text{Set} \)-functor and there are two weak distributive laws \( \text{Set} \leftarrow G \leftarrow \text{Set} \). Let \( (X,c) \) be a \( G\text{Set} \)-coalgebra. In accordance with Lemma 16 let the determinized \( c \) be \( c^+ = (G^S\xi + G\text{Set} \circ G\text{Set} \circ T)X \to G\text{Set}X \). In the case of distributive laws the proof that \( c^+ \) is part of a bialgebra would use only multiplication diagrams. Hence, the result still holds in the weak case:
Combining Weak Distributive Laws: Application to Up-To Techniques

Lemma 22. The triple \( (\mathcal{T}X, \mathcal{T}X, \mu^+) \) is a \( C\delta + C\mathcal{P} \)-bialgebra.

Proposition 21 and Lemma 22 together yield

Theorem 23. Contextual closure \( C\mathcal{P} \) is \( \mathcal{R}(\mathcal{P}) \)-compatible. If \( \mathcal{G} \mathcal{P} \) preserves weak pullbacks, then \( C\mathcal{P} \mathcal{G} \) is \( \mathcal{R}(\mathcal{P}) \)-compatible.

Note that Proposition 8, in which \( \mathcal{S} = \mathcal{P} \) plays a prominent role in obtaining concrete weak distributive laws, Indeed, all three Examples 9, 10 and 12 are instances of canonical laws. Thus, it is interesting to note what happens to \( \mathcal{R}(\mathcal{P}) \)-compatible. Straightforward computations show that we retrieve the classical Milner-Park bisimulations of transition systems:

\[ \mathcal{R}(\mathcal{P}) \mathcal{G} = \{(u, v) \in \mathcal{T}X \times \mathcal{T}X \mid \forall u' \in c^+(u), \exists v' \in c^+(v) \text{ such that } (u', v') \in R\} \]  

5.2 Application to Alternating Automata

Consider the concrete case of alternating automata with the data presented in Section 4.2. We call the corresponding context closure union closure because of the expression:

\[ C\mathcal{P}(I)(R) = \left\{ \left( \bigcup_{i \in I} A_i \cup B_i \right) \mid \forall i \in I, (A_i, B_i) \in R \right\} \]  

Both \( \mathcal{M} \) and \( \mathcal{P} \) preserve weak pullbacks, see e.g. Proposition 4.2.6 in [16]. Using Theorem 23 we get that

Proposition 24. Union closure and its corresponding congruence are compatible with respect to alternating automata determined once.

Example 25. Consider the alternating automaton \( \mathcal{G} \) of Figure 1 and its determined \( c^+ \) pictured in Figure 2. Following the steps of [8], we consider various bisimulations on \( c^+ \).

First, one can see that \( \{x_2\} \cup \{y_2\} \) because \( \{x_2\} \cup \{y_2\} \) is a bisimulation. Indeed, there is exactly one arrow going out of these states, and this is a loop. Similarly, \( \{x_3\} \cup \{y_3\} \) is a bisimulation because there is no arrow going out of these states. Now let us try to prove that \( \{x_0\} \cup \{y_0\} \). By exploring every possible transition, one can easily see that a bisimulation relating \( \{x_0\} \cup \{y_0\} \) of minimal cardinality is

\[ R = \{(\{x_0\}, \{y_0\}), (\{x_1\}, \{y_1, y_2\}), (\{x_2\}, \{y_2, y_3\}), (\{x_3\}, \{y_3\})\} \]  

However, the following smaller relation is a bisimulation up-to congruence witnessing the fact that \( \{x_0\} \cup \{y_0\} \):

\[ R_0 = \{(\{x_0\}, \{y_0\}), (\{x_1\}, \{y_1, y_2\}), (\{x_2\}, \{y_2\}), (\{x_3\}, \{y_3\})\} \]  

The only non-trivial verifications concern the pair \( (\{x_1\}, \{y_1, y_2\}) \):
Remark 26. For finite systems, the congruence obtained from union closure is the same thing as the up-to technique presented in [7]. Note however that in loc. cit. the authors use compatibility of congruence closure to compute bisimulations on deterministic automata i.e. $M$-coalgebras, whereas we use it to compute Milner-Park bisimulations on non-deterministic automata i.e. $MP$-coalgebras. In particular, in the setting of loc. cit., bisimilarity and behavioural equivalence coincide because there is a final $M$-coalgebra — whereas Milner-Park bisimilarity strictly entails behavioural equivalence on $MP$-coalgebras.

5.3 Application to Probabilistic Automata

Let the functor $-^A: \textbf{Set} \rightarrow \textbf{Set}$ be the second component of the machine functor $M: \textbf{Set} \rightarrow \textbf{Set}$ of Example 12 along with the following distributive law $\sigma: D(\textbf{Set}) \Rightarrow D(\textbf{Set})$:

$$\sigma_X(\Phi)(a)(x) = \sum_{f \in X^A, f(a) = x} \Phi(f)$$  \hspace{1cm} (23)

Note that given a weak distributive law of type $TS \Rightarrow ST$ and using the terminology of [8], $S$ is a $(S,T)$-quasi-lax lifting because of the equality $ST = U(TS)$ and the monic $\iota: U(TS) \Rightarrow S$. Using this observation, and going through the construction of Section 6 in [8], one can see that it amounts exactly to the left half of the generalized determinization of our Proposition 15. Hence, determinizing a probabilistic automaton $c: X \rightarrow (PDX)^A$ into a $c^+: DX \rightarrow (PDX)^A$ yields the same belief-state transformer as in [8]. As this fact was already remarked in [14] — modulo the set of labels $A$ — we are not going to detail this more. Back to up-to techniques, note that context closure with respect to the free $D$-algebra is convex hull:

$$\text{Ctx} \mu_D(R) = \left\{ \left( \sum_{i \in I} p_i \varphi_i, \sum_{i \in I} p_i \psi_i \right) \mid I \text{ finite, } \sum_{i \in I} p_i = 1, (\varphi_i, \psi_i) \in R, p_i \in (0,1) \right\}$$  \hspace{1cm} (24)

Both $(-)^A$ and $P$ preserve weak pullbacks [16], hence Theorem 23 yields a result already stated in [8].

Proposition 27. Convex hull and its corresponding congruence are compatible with respect to the belief-state transformer determinization of a probabilistic automaton.

Remark 28. As previously mentioned, the structure of the alternating automaton of Figure 1 is the same as the one of the probabilistic automaton example in [8] — and determination of both systems remain strikingly similar. This is due to the fact that the natural transformation $\text{supp} T \Rightarrow T$ is a morphism of weak distributive laws in the sense that the following diagram commutes, where vertical arrows are the canonical weak distributive laws described in Example 12 and Example 10:

\[
\begin{array}{ccc}
\text{supp} T & \Rightarrow & T \\
\downarrow & & \downarrow \\
\text{supp} T & \Rightarrow & T
\end{array}
\]
Conclusion

Summary of Results

The slogan of this article might be: if there is almost a distributive law, then everything goes almost smoothly. By paying the light price of removing one unit diagram, or equivalently dislocating one identity natural transformation into two that are pseudo-inverses, one retrieves a lot of the results that have drawn interest from the community over the last few decades.

First, we noticed that suppressing one of the monad structures leads to a notion of weak distributive law of a monad over a functor. As weak distributive laws can give rise to new monads, we provided an abstract compositionality result (Theorem 13) to derive a weak distributive law of the composite monad. We also fully extended generalized determinization to the setting of weak distributive laws (Proposition 15) — this is the logical continuation of [14] where only one (monad-monad) weak distributive law was involved. Finally, we identified that the bialgebraic approach to up-to techniques could be adapted effortlessly to the weak framework (Theorem 23), hence providing a full batch of sound up-to techniques that were out of the reach of plain distributive laws.

Our two primary examples, alternating automata and probabilistic automata, come directly from the two black sheep that have caused much ink to flow in the recent years: powerset over powerset, and distribution over powerset. Our framework explains how generalized determinization of alternating automata modelled with double covariant powerset monad occurs in a distributive law-like manner. To our knowledge, this is the first time that this is performed. This determinization is sound in the sense that semantics is preserved, and canonical in the sense that it corresponds to the canonical powerset-powerset weak distributive law. Our results also yield that union closure is compatible with Milner-Park bisimulation for alternating automata, and we retrieve back that convex hull is compatible with Milner-Park bisimulation for probabilistic automata.

Future Work

Exciting insights are lying ahead. Can we tell more about the weak distributive law morphisms of Remark 28, e.g. can they be used as a source of new examples of weak distributive laws? Such morphisms have already been studied in the strong framework e.g. in [22], [10]. This might be further investigated to provide answers to the more general question: is there something more beyond, once the case of $\text{DP} \Rightarrow \text{PD}$ and $\text{PP} \Rightarrow \text{PP}$ is settled? Our framework introduces monad-functor weak distributive laws, but what are instances of this that do not arise directly from forgetting a monad structure?

There is a variety of other ways of performing generalized determinization with composed weak distributive laws. One example in the case when Yang-Baxter holds is to start from a system in $\text{Coalg}(\mathcal{G} \mathcal{S} \mathcal{T})$, use $\pi^*$ to go into $\text{Coalg}(\mathcal{G} \mathcal{S} \mathcal{T})$, and then perform determinization with respect to the $\mathcal{S} \mathcal{T} \Rightarrow \mathcal{G} \mathcal{S} \mathcal{T}$ obtained in Theorem 13. For the moment it is unclear how this construction would relate to the double determinization mentioned in Remark 20. A further point is we did not prove that replacing the functor with a third monad in the compositionality theorem makes diagram $\mu_R$ with respect to $\mathcal{S} \mathcal{T}$ fail. This would be an interesting result, as this would entail that in Theorem 13 one can not add further monads as in [12].
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A Proofs of Section 3

A.1 Proof of Theorem 13

Let us prove Theorem 13 i.e. \( \lambda = G \pi \circ T \circ S \sigma \circ \iota \) is a weak distributive law of type \( STG \Rightarrow GST \). Recall that notation \( \pi^* \) (resp. \( \iota^* \)) stands for \( \pi F \) (resp. \( \iota F \)).

The proof that the \((\mu S T)\) diagram commutes begins as follows:

\[
\begin{array}{cccccc}
S & S & T & S & T & G \\
\text{STSTG} & \text{STSGT} & \text{STGST} & \text{SGTST} & \text{GSTST} & \text{SSTTG} \\
\end{array}
\]

The three external curved diagrams commute by definition of \( \lambda \). Diagrams (1), (2), (3) commute respectively because of naturality of \( \iota^* \), naturality of \( \iota^* \), naturality of \( \tau T \circ S \sigma \). Diagram (5) commutes as in [12] by making use of the Yang-Baxter hypothesis. Hence only diagrams (4) and (6) remain to be proved commutative. Note that as \( ST \) is obtained from the composite adjunction \( EM(T) \times EM(S) \), we have

\[
\begin{array}{c}
\mu S T = U F \pi F U \times U F \sigma U \times \iota F \tau U
\end{array}
\]

where \( \varepsilon U \) is the counit of adjunction \( F U \dashv U F \). Recall also that according to the proof of Proposition 7, \( \delta = TS T ^* S F T \times T S T ^* S F T \times \tau F T \times \sigma F T \times \varepsilon F T \). Denote \( \alpha = U F \pi F U \times U F \sigma U \times \iota F \tau U \) for convenience.

Let us first prove that diagram (4) commutes. It suffices to prove that the following diagram commutes (we just suppressed all the \( G \) on the right):
The three external triangles with bent arrows commute by definition. The two internal triangles commute because of a monad property and the fact $\pi \circ \iota = 1$. Diagrams (a), (b) commute by naturality of $\iota$. Diagram (c) commutes by definition of $\alpha$. Diagram (e) commute because of the $(\iota \mu)$ diagram of weak liftings. The diagram (d) is a little bit harder. By monad properties, $\mu_T X : TX \Rightarrow TX$ is a actually a morphism $[\mu_T] : (TX, \mu_T) \Rightarrow (TX, \mu_T)$ in $EM(T)$. This yield a natural transformation $[\mu] : F \Rightarrow F$ such that $U[\mu] = [\mu]$ Diagram (d) then commutes by replacing $[\mu]$ with $U[\mu]$ and using naturality of $\alpha : [\iota \iota] \Rightarrow [\iota \iota]$. 

Let us finally prove that diagram (6) commutes. It amounts to the following:
The three external triangles with bent arrows commute by definition. The two internal triangles commute because of a monad property and the fact $\pi \circ \iota = 1$. Diagram (i) commutes for the same reasons as for diagram (d) in the preceding diagram. Diagram (ii) commutes by definition of $\alpha$, diagram (iii) by naturality of $\pi$, and diagram (iv) by using the $(\pi \mu)$ diagram of weak liftings. This achieves the proof that the $(\mu \Sigma \Gamma)$ diagram commutes.

Now assume that weak distributive laws $\sigma$ and $\tau$ are actually strong. The following diagram proves that the $(\eta \Sigma \Gamma)$ diagram commutes i.e. that $\lambda$ is strong:

The two external triangles commute by definition of $\eta \Sigma \Gamma$. The leftmost and rightmost triangles commute by weak liftings properties $(\pi \eta)$ and $(\iota \eta)$. The two squares commute by naturality. Finally, the two remaining triangles use the $(\eta \Sigma \Gamma)$ and $(\eta \Gamma \Sigma)$ diagrams of weak distributive laws with respect to $G$.

This achieves the proof of Theorem 13.

A.2 Proof of Example 14

We prove that the following Yang-Baxter diagram related to Example 14 commutes.

Let us first compute both sides of the equation. Let $S \in \mathcal{F}^{\mathcal{M}}\mathcal{M}X$. The pair $(\mathcal{M}^\mathcal{M} \mathcal{F}^\mathcal{M} \mathcal{F}^\mathcal{M})X(S)$ has output

$$\bigwedge_{S \in S} \bigvee_{(o,f) \in S} o$$  \hspace{1cm} (25)$$

and $\sigma$-transition

$$\{ V \subseteq X \mid V \subseteq \{ f(a) \mid (o,f) \in S, S \in S \} \text{ and } \forall S \in S, \{ f(a) \mid (o,f) \in S \} \cap V \neq \emptyset \}$$ \hspace{1cm} (26)$$

The pair $(\mathcal{F}^\mathcal{M} \mathcal{F}^\mathcal{M} \mathcal{M}^\mathcal{M})X(S)$ has output

$$\bigvee_{(W \subseteq \mathcal{M}X \text{ s.t. } \forall V \subseteq \bigcup_{S} S \text{ and } \forall S \in S, W \neq \emptyset \text{ and } (o,f) \in W} \bigwedge_{(o,f) \in W} o$$ \hspace{1cm} (27)$$
and \( a \)-transition

\[
\{ \{ f(a) \mid (o, f) \in W \} \mid W \subseteq M^X \text{ s.t. } W \subseteq \bigcup S \text{ and } \forall S \in S, S \cap W \neq \emptyset \}
\]  \hspace{1cm} (28)

Let us first prove that outputs are equal. As they equal either 0 or 1, this amounts to showing the following lemma.

\begin{itemize}
  \item \textbf{Lemma 29.} The two following propositions are equivalent:
    \begin{enumerate}
      \item For all \( S \in S \), there is \( f \in X^A \) such that \((1, f) \in S\).
      \item There is \( W \subseteq M^X \) such that \( W \subseteq \bigcup S \), for all \( S \in S \) one has \( S \cap W \neq \emptyset \), and for all \((o, f) \in W, o = 1\).
    \end{enumerate}
  \end{itemize}

\textbf{Proof.} We begin by the implication \((i) \Rightarrow (ii)\). For every \( S \in S \) we can fix \( f_S \in X^A \) such that \((1, f) \in S\). Let \( W = \{(1, f_S) \mid S \in S\} \subseteq M^X\). As \((1, f_S) \in S \in S\) we have \( W \subseteq \bigcup S \) as well. Let \( S \in S \), then \((1, f_S) \in S \cap W\) so that \( S \cap W \neq \emptyset\). Finally, let \((o, f) \in W\). By construction, \((o, f) = (1, f_S)\) for some \( S \in S\); hence \( o = 1\). Focus now on the converse implication \((ii) \Rightarrow (i)\). Assume there is a \( W \subseteq M^X \) that satisfies the three conditions stated in \((ii)\). Let \( S \in S \). By the second condition, there is some \((o, f) \in S \cap W\). By the third condition, as \((o, f) \in W\) we have \( o = 1\). Hence there is indeed a \( f \in X^A \) such that \((1, f) \in S\). \hfill \endproof

Second step, let us prove that transitions are equal.

\begin{itemize}
  \item \textbf{Lemma 30.} The two following sets are equal:
    \begin{enumerate}
      \item \( \{ V \subseteq X \mid V \subseteq \{ f(a) \mid (o, f) \in S, S \in S \} \text{ and } \forall S \in S, \{ f(a) \mid (o, f) \in S \} \cap V \neq \emptyset \} \)
      \item \( \{ \{ f(a) \mid (o, f) \in W \} \mid W \subseteq M^X \text{ s.t. } W \subseteq \bigcup S \text{ and } \forall S \in S, S \cap W \neq \emptyset \} \)
    \end{enumerate}
  \end{itemize}

\textbf{Proof.} Begin with the converse inclusion. Consider \( V = \{ f(a) \mid (o, f) \in W \} \) where \( W \subseteq M^X \) is such that \( W \subseteq \bigcup S \) and for all \( S \in S, S \cap W \neq \emptyset \). First we must prove that \( V \subseteq \{ f(a) \mid (o, f) \in S, S \in S \} \). Indeed, for any \((o, f) \in W\), as \( W \subseteq \bigcup S \) there is some \( S \in S \) such that \((o, f) \in S\), whence the inclusion. Second, take \( S \in S \) and prove that \( \{ f(a) \mid (o, f) \in S \} \cap V \neq \emptyset \). This is directly obtained from the hypothesis \( S \cap W \neq \emptyset \). This achieves the proof of the converse inclusion. Let us now prove that the direct inclusion holds as well. Let \( V \subseteq X \) be such that \( V \subseteq \{ f(a) \mid (o, f) \in S, S \in S \} \) and for all \( S \in S \), \( \{ f(a) \mid (o, f) \in S \} \cap V \neq \emptyset \). We have to find some \( W \subseteq M^X \) such that the three following facts hold true:

\begin{enumerate}
  \item \( W \subseteq \bigcup S \)
  \item for all \( S \in S, S \cap W \neq \emptyset \)
  \item \( V = \{ f(a) \mid (o, f) \in W \} \)
\end{enumerate}

By using the hypothesis we can make the following constructions. For any \( x \in V\), find some \( S_x \in S \) and some \((o_x, f_x) \in S_x\) such that \( x = f_x(a)\). For any \( S \in S \), find some \((o_S, f_S) \in S\) such that \( f_S(a) \in V\). Now define \( W = \{(o_x, f_x) \mid x \in V\} \cup \{(o_S, f_S) \mid S \in S\} \) and prove that it satisfies the three conditions above. For \((i)\), let \((o, f) \in W\). Either \((o, f) = (o_x, f_x)\) for some \( x \in V\), and then \((o, f) \in S_x \in S\) so that \((o, f) \in \bigcup S\); or \((o, f) = (o_S, f_S)\) and then \((o, f) \in S \in S \) from what \( (o, f) \in \bigcup S \) follows. This proves that \((i)\) holds true. Consider now \((ii)\). Let \( S \in S \). By construction, \((o_S, f_S) \in S \cap W\). Finally prove property \((iii)\) by double inclusion. Let \( x \in V\). Then \((o_x, f_x) \in W\) and \( f_x(a) = x\) whence \( x \in \{ f(a) \mid (o, f) \in W\}\). Lastly, let \((o, f) \in W\) and prove that \( f(a) \in V\). Either \((o, f) = (o_x, f_x) \in S_x \in S\) for some
The proof is elementary and as we said, already sketched in [17]. Let us recall all necessary ingredients, in particular the usual semantics of non-deterministic automata.

These lemma jointly prove the following proposition:

**Proposition 31.** The Yang-Baxter diagram A.2 commutes.

### B.1 Proof of Proposition 15

This proof consists in a slight extension of the proof of Lemma 5.1 of [14]. For the sake of completeness, we explicit here the full details of the proof.

We will denote the weak liftings natural transformations by \( \eta \) for \( \delta \) and \( \varepsilon \) for \( \pi \).

Let \((X, c)\) be a \( G_S \) coalgebra. Consider the morphism

\[
X \xrightarrow{c} G_S X \xrightarrow{U} U G_S X \xrightarrow{u} U S U X
\]

and take its adjoint transpose with respect to the adjunction \( F : G_S \rightleftarrows U : EM \) to get a coalgebra \( c^* : EM X \rightarrow G_S x \). The functor \( F \) maps \((X, c)\) to \((F X, c^*)\). Let \((X, d)\) be a \( G_S \) coalgebra in \( EM(X) \). The functor \( U \) maps \((X, d)\) to \( G_S \) coalgebra

\[
X \xrightarrow{U} U G_S (X, x) \xrightarrow{U, \xi} G_S (X, x) \xrightarrow{\eta} G_S X
\]

### B.2 Proof of Lemma 16

The proof consists in the following diagram. Starting from the top-left node of the diagram, the path going right then down is expression 9. The path going down then right equals the construction given in the proof of Proposition 15.

On top of the diagram, the two pentagons commute because of the expressions of \( \eta \) and \( \varepsilon \) obtained via the proof of Proposition 7. The top-left triangle commutes trivially. All the other polygons commute because of naturality, except two triangles that use the adjunction property \( U T \xi = 1 \).

### B.3 Proof of Proposition 19

The proof is elementary and as we said, already sketched in [17]. Let us recall all necessary ingredients, in particular the usual semantics of non-deterministic automata.
\[ [x]_{aa}(\varepsilon) = o(x) \quad [x]_{aa}(aw) = \bigvee_{U \in t_a(x), y \in U} [y]_{aa}(w) \]  
\[ [U]_{nda}(\varepsilon) = o^+(U) \quad [U]_{nda}(aw) = \bigvee_{W \in t_a^+(x)} [W]_{nda}(w) \]  

where

\[ o^+(U) = \bigwedge_{x \in U} o(x) \]  
\[ t_a^+(U) = \left\{ \bigcup_{x \in U} K_x \mid \forall x \in U, K_x \in \text{unions}(t_a(x)) \right\} \]

The proof is by induction on \( w \). The basic case \( w = \varepsilon \) is trivial given the above definitions.

Assume \( \forall U \in PX, [U]_{nda}(w) = \bigwedge_{y \in U} [y]_{aa}(w) \)  

Then fix a \( U \in PX \) and compute the two quantities for \( aw \):

\[ [U]_{nda}(aw) = \bigvee_{V \in t_a^+(U)} [V]_{nda}(w) = \bigvee_{V \in t_a^+(U), x \in V} [x]_{aa}(w) \]  
\[ \bigwedge_{y \in U} [y]_{aa}(aw) = \bigwedge_{y \in U} \bigvee_{W \in t_a(y), x \in W} [x]_{aa}(w) \]

These quantities are equal.

Assume the first quantity equals 1. There is \( V \in t_a^+(U) \) such that for all \( x \in V, [x]_{aa}(w) = 1 \). One can find for every \( y \in U \) a \( K_y \in \text{unions}(t_a(y)) \) such that \( V = \bigcup_{y \in U} K_y \). Fix some \( y \in U \). By the definition of \( t_a(y) \), there is a \( W \in t_a(y) \) such that \( W \subseteq K_y \). Let \( x \in W \), then \( x \in K_y \) so \( x \in V \), which yields \( [x]_{aa}(w) = 1 \) by hypothesis.

Assume the second quantity equals 1. For every \( y \in U \) there is a \( W \in t_a(y) \) such that for all \( x \in W, [x]_{aa}(w) = 1 \). Take \( K_y \) to be such a \( W \), then \( K_y \in t_a(y) \subseteq \text{unions}(t_a(y)) \) so that we can define \( V = \bigcup_{y \in U} K_y \in t_a^+(U) \). Let \( x \in V \). By construction of \( V \), \( x \in K_y \) for some \( y \in U \). Hence \( [x]_{aa}(w) = 1 \) by hypothesis on \( K_y \). This achieves the proof.

### C Proofs of Section 5

#### C.1 Proof of Proposition 22

The proof that \( (TX, \mu_T, c^+) \) is a \( G \)-\( \sigma \)-bialgebra consists in the following diagram. Actually, this is the same diagram occurring with standard \( \text{distribution laws} \) because the expression of \( c^+ \) is the same in both frameworks and units of monads are not involved.
where the conditions of the first set. Indeed, it is clear that \(B\) and \(\emptyset\) are distinct, \(\emptyset\) satisfies the definition of \(c^+\), naturality of \(\sigma\) and \(\sigma\) monad property of \(\sigma\) definition of \(c^+\), naturality of \(\mu\) diagram of the weak distributive laws. Going left and then counter-clockwise, polygons commute by definition of \(c^+\), naturality of \(\sigma\) and \(\sigma\) monad property of \(\sigma\) definition of \(c^+\), naturality of \(\mu\) diagram of the weak distributive laws.

### C.2 Proof of the diagram in Remark 28

First note that \(\text{supp}\) is natural because for any function \(f : X \to Y\), reminding that \(Df(\varphi)(y) = \sum_{x \in f^{-1}(\{y\})} \varphi(x)\) we have

\[
\text{(supp}_Y (Df)(\varphi) = \{y \mid \text{supp}_X (\varphi) \cap f^{-1}(\{y\}) \neq \emptyset\} = \{f(x) \mid x \in \text{supp}_X (\varphi)\} = (D\circ \text{supp}_X)(\varphi) \tag{38}\n\]

Let \(\text{supp}_Y (Df)(\varphi) = \text{supp}_X (\varphi) \cap f^{-1}(\{y\}) \neq \emptyset\) and \(\text{supp}_X (\varphi) \subseteq \text{supp}_X (\varphi)\) as in Examples 12 and 10. Take \(\Phi = \sum_{i \in I} p_i A_i \in \mathcal{P} X\), where the \(A_i\) are distinct, \(p_i > 0\) and \(\sum_{i \in I} p_i = 1\) (hence \(I \neq \emptyset\)). Let us compute both paths:

\[
(D\circ \text{supp}_X)(\Phi) = \text{supp}_X (A_i \mid i \in I) = \left\{ B \subseteq X \mid B \subseteq \bigcup_{i \in I} A_i \text{ and } \forall i \in I, B \cap A_i \neq \emptyset \right\} \tag{40}
\]

\[
(D \circ \text{supp}_X)(\Phi) = \left(\bigcup_{i \in I} \text{supp}_X (\varphi_i) \mid \forall i \in I, \text{supp}(\varphi_i) \subseteq A_i \right) \tag{41}
\]

Note that in the above expression the \(\varphi_i\) are not necessarily distinct. Let us prove that these two sets are the same. Let \(B \subseteq X\) such that \(B \subseteq \bigcup_{i \in I} A_i\) and for all \(i \in I, B \cap A_i \neq \emptyset\). Denote \(B \cap A_i = \{x_1, \ldots, x_{n_i}\}\) with \(n_i \geq 1\). Define for all \(i \in I\) the distribution \(\varphi_i = \sum_{k=1}^{n_i} \frac{1}{n_i} x_k\), then \(\sum_{i \in I} \frac{1}{n_i} \varphi_i\) is a distribution because \(I \neq \emptyset\). We have \(\text{supp}_X (\varphi_i) = B \cap A_i \subseteq A_i\) and \(\bigcup_{i \in I} \text{supp}_X (\varphi_i) = B \cap \bigcup_{i \in I} A_i = B\). For the converse inclusion, let \(\varphi_i \in DX\) such that \(\text{supp}_X (\varphi_i) \subseteq A_i\). Let us prove that \(\bigcup_{i \in I} \text{supp}_X (\varphi_i)\) satisfies the conditions of the first set. Indeed, it is clear that \(\bigcup_{i \in I} \text{supp}_X (\varphi_i) \subseteq \bigcup_{i \in I} A_i\). Let \(i_0 \in I\), \(\bigcup_{i \in I} \text{supp}_X (\varphi_i) \cap \text{supp}_X (\varphi_{i_0}) = \text{supp}_X (\varphi_{i_0}) \neq \emptyset\). This achieves the proof.