Wavelet Design with Optimally Localized Ambiguity Function: a Variational Approach

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Abstract

In this paper, we design mother wavelets for the 1D continuous wavelet transform with some optimality properties. An optimal mother wavelet here is one that has an ambiguity function with minimal spread in the continuous coefficient space (also called phase space). Since the ambiguity function is the reproducing kernel of the coefficient space, optimal windows lead to phase space representations which are "optimally sharp." Namely, the wavelet coefficients have minimal correlations with each other. Such a construction also promotes sparsity in phase space. The spread of the ambiguity function is modeled as the sum of variances along the axes in phase space. In order to optimize the mother wavelet directly as a 1D signal, we pull-back the variances, defined on the 2D phase space, to the so called window-signal space. This is done using the recently developed wavelet-Plancharel theory. The approach allows formulating the optimization problem of the 2D ambiguity function as a minimization problem of the 1D mother wavelet. The resulting 1D formulation is more efficient and does not involve complicated constraints on the 2D ambiguity function. We optimize the mother wavelet using gradient descent, which yields a locally optimal mother wavelet.

Keywords. Continuous wavelet, mother wavelet design, uncertainty principle, Plancherel theorem, variational method

1 Introduction

In this paper, we consider the 1D continuous wavelet transform (1D CWT in short). The 1D CWT is a special case of a generalized wavelet transform – a signal transform which is based on taking the inner product of the input signal with a set of transformations of a window function. Some examples of generalized wavelet transforms are the short time Fourier transform (STFT)\textsuperscript{15}, the 1D CWT \textsuperscript{10,11}, the Shearlet transform \textsuperscript{17}, the Curvelet transform \textsuperscript{4}, and the dyadic wavelet transform \textsuperscript{11}. The first three examples are based on square integrable representations of a group. Such transforms are called continuous wavelet transforms (see Subsection 2.1). A continuous wavelet transform is defined by the choice of the set of transformations, and the choice of the window function. In this paper, we introduce a systematic approach for choosing the window function.

There are various approaches for window design in the literature. One line of work for window design is the classical method introduced by Daubechies in \textsuperscript{10}, in which an orthogonal wavelet basis is designed, with a compactly supported window having some degree of vanishing moments. A high order of vanishing moments is linked with a sparser approximation of a (sufficiently regular) signal in phase space (\textsuperscript{28} Ch.7.1). Another family of approaches are adaptive methods, which also
take into account the signal at hand and aim to maximize the correlation between the analyzing window and the signal (see, for example, [5]).

In this paper, we develop a method for choosing a window for the 1D continuous wavelet transform, based on an uncertainty minimization approach. The motivation comes from the use of wavelet transforms as means for measuring physical quantities of signals. For example, the STFT measures the content of signals at different times and frequencies, and the 1D CWT measures times and scales. For the measurements to be as accurate as possible, it is desirable to choose a window with minimal uncertainty with respect to the different physical quantities.

In the following subsection, we review the evolution of the uncertainty minimization approach to window design in previous work. These past works will lead naturally to our proposed approach.

### 1.1 Window Design via Uncertainty Minimization

Consider a signal transform based on a square integrable representation $\pi(g)$ of a locally compact group $G$ in the Hilbert space $\mathcal{H}$. Given an admissible window $f \in \mathcal{H}$, the corresponding wavelet transform $V_f : \mathcal{H} \rightarrow L^2(G)$ reads, for $s \in \mathcal{H}$ and $g \in G$,

$$V_f(s)(g) = \langle s, \pi(g)f \rangle .$$

#### 1.1.1 The Classical Approach to Wavelet Localization

For certain transforms, e.g., the 1D CWT and the Shearlet transform, there is a classical approach to window design. The idea is to generalize the classical localization framework of the STFT, and specifically to generalize the Heisenberg uncertainty principle. The general scheme can be described as follows. Consider a set of linearly independent infinitesimal self-adjoint generators $T_1, \ldots, T_n$ of the group of transformations $\pi(G)$. Each $T_j$ generates a one-parameter group of transformations $e^{i t T_j} = \{ e^{i t T_j} \mid t \in \mathbb{R} \}$, which is a subgroup of $\pi(G)$. Each $e^{i t T_j}$ is interpreted as a set of transformations that translates a certain physical quantity. For example, in the STFT, the one parameter transformation groups are translations, which change the physical quantity time, and modulations, which change the quantity frequency. Now, take the generators $T_1, \ldots, T_n$ as the observables of their respective physical quantities. Namely, use the variance of $f$ with respect to each $T_j$, $v_f(T_j)$ (see Definition [5]), as a measure of the localization of $f$ with respect to the physical quantity underlying $e^{i t T_j}$. The overall uncertainty of the window function is then defined as the multiplication of the above variances $\prod_{j=1}^n v_f(T_j)$. This approach can be found in the literature, e.g., [3, 9], and in papers, e.g., [3, 9]. In the classical case of the STFT, the generators are the frequency observable $T_1 : f(x) \mapsto \frac{d}{dx} f(x)$ and the time observable $T_2 : f(x) \mapsto x f(x)$, and $v_f(T_1)$ and $v_f(T_2)$ measure the spread of $f$ in frequency and time, respectively.

For any pair of observables $T_k$ and $T_l$, the uncertainty principle poses a lower bound on the simultaneous concentration of $f$ with respect to the physical quantities underlying $T_k$ and $T_l$. Namely,

$$v_f(T_k)v_f(T_l) \geq \frac{1}{4} \langle f, [T_k, T_l]f \rangle ,$$

where $[T_k, T_l] := T_k T_l - T_l T_k$ denotes the commutator of $T_k$ and $T_l$. The conventional approach to window design is to choose two observables of interest, and to find an equalizer of the above inequality – an $f$ for which the inequality in (1) becomes an equality. Such an $f$ is then declared as an uncertainty minimizers. In [22], it was shown that this approach is erroneous, and that windows admitting equality are not generally minimizers of the left-hand-side of (1). Instead, in order to find a window with minimal uncertainty, one should minimize the uncertainty of $f$ using some variational method.

#### 1.1.2 Towards a Coherent Approach to Wavelet Uncertainty

Even after applying variational methods to find the minimizer of the left-hand-side of (1), the above approach yields rather strange and counter-intuitive results (see e.g., [21]). The reason for that, as suggested in [20], is that the group generators $T_j$ are not appropriate for defining localization...
of the physical quantities underlying each transformation group \( e^{iRT_j} \). The mistake comes from the fact that defining the observables as the generators works for the STFT “by accident”, but this accident does not repeat in other transforms. For the STFT case, the generator of time translations is the frequency observable \( T_1 : f(x) \mapsto i \frac{\partial}{\partial x} f(x) \), and the generator of modulations is the time observable \( T_2 : f(x) \mapsto xf(x) \). Hence, each of these two observables happen to be appropriate as a localization operator for the other transformation group. When combined in an uncertainty, the two generators happen to measure together the correct pair of physical quantities. However, this lucky accident does not generalize to other transforms, like the 1D CWT. Hence, the generators cannot be taken as the observables in the general case. Instead, [21, 20] suggested a distinction between the group generators and the operators that measure localization, i.e., the observables.

In [20], an interpretation of the uncertainty minimizer from the point of view of sparsity in phase space was proposed. Consider a signal \( s \) for which a sparse representation in the 1D CWT time-scale phase space exists. Namely, there exists a “sparse phase space function” \( F = \sum_{k=1}^{K} c_k \delta_{g_k} \) such that \( s = V_f^* (F) = \sum_{k=1}^{K} c_k \delta_{g_k} f \), where \( \delta_{g_k} \) is the delta functional concentrated at \( g_k \in G \). Finding \( F \) given only knowledge of \( s \) is a difficult problem. Conversely, calculating the continuous wavelet transform of \( s \) is direct and simple, but does not reconstruct the sparse function \( F \). To see this, consider the ambiguity function \( V_f (f) \in L^2(G) \). It is known that synthesizing any function \( H \in L^2(G) \), and immediately after analyzing it, \( V_f V_f^* H \), is given by the convolution of \( H \) with the ambiguity function \( V_f (f) \) in phase space. Similarly, \( V_f (s) \) is given by convolving \( F \) with the ambiguity function \( V_f (f) \). Therefore, finding an ambiguity function with minimal spread is desirable, as it results in a minimal blurring of \( F \). For a well localized ambiguity function, the peaks \( g_k \) of \( F \) are well separated in \( V_f (s) = V_f V_f^* F \), so they can be easily extracted from \( V_f (s) \) without direct knowledge of \( F \).

The above analysis was the motivation for the definition of the observables in [20]. However, the localization measures in [20] were not directly defined in phase space, not measuring the concentration of \( V_f (f) \) directly, but rather based on surrogate localization measures of \( f \) in the signal domain. The signal localization measures of \( f \) were only shown to pose loose bounds on the decay of \( V_f (f) \). Our goal, instead, is to define the uncertainty as the spread of \( V_f (f) \) directly in phase space, and to pull-back this phase space-based definition to a formulation in the signal domain. The problem in pulling-back the uncertainty of \( V_f (f) \) to a signal domain formulation in terms of \( f \), is that the 1D CWT is not an isomorphism, but only an isometric embedding of the signal domain to \( L^2(G) \).

In a subsequent paper [19], the wavelet-Plancherel theory was developed. The wavelet-Plancherel theorem establishes an isometric isomorphism between \( L^2(G) \) and the so called window-signal space – the tensor product of the space of admissible windows with the space of signals. The wavelet-Plancherel theory introduces closed form formulas for pulling back phase space operations to window-signal space operations. In some cases, the theory results in formulations of operations applied on \( V_f (s) \), as a combination of operations applied on \( s \) and \( f \) separately. This allows implementing 2D operations in phase space efficiently as 1D computations in the window and signal spaces. The wavelet-Plancherel theory is at the core of the current work, as it allows us to exactly formulate the localization measures of the ambiguity function \( V_f (f) \) as 1D operations on \( f \).

1.2 Related Work in Ambiguity Function Localization

The idea for localizing the ambiguity function was also studied in other papers. In the context of the STFT, the ambiguity function is known to play an important role in the estimation of optimal localization properties in many different areas, e.g., for operator approximation by Gabor multipliers [12] or for RADAR and coding applications [7, 2, 18]. In [13], Feichtinger et al. presented a method for designing optimally localized windows in the time-frequency plane of the STFT. This is done by maximizing some measure of the concentration of the ambiguity function. Under general hypotheses and shape constraints, using a variational method (recursive quadratizations), the algorithm converges to a window whose ambiguity function has approximate (locally) optimal properties in the time-frequency domain. Our approach differs from this in
several ways. First, our method is generic, and can be easily generalized to other continuous
wavelet transforms based on square integrable representations. In addition, the computations in
our method are performed in the 1D window and signal spaces, in contrast to the 2D phase space,
and are thus more efficient.

1.3 Our Contribution
We summarize our contribution as follows.

- We define the localization/uncertainty of the ambiguity function in the 2D phase space using
  variances along the axes.
- We utilize the wavelet-Plancherel theory to pull-back the 2D formulation of uncertainty to
  a 1D formulation. The 1D formulation is more efficient and does not require complicated
  constraints for the 2D ambiguity function, since we optimize the window directly.
- We show how to compute a locally optimal mother wavelet using calculus of variations and
  gradient descent.

The technique presented in this paper can be applied to other general wavelet transforms such
as the Shearlet transform. The 1D wavelet transform can be regarded as a prototype to the
localization machinery, that can be generalized to every transform for which the wavelet-Plancherel
theory is applicable – the so called semi-direct product wavelet transforms (see [19, Definition 8]).

1.4 Outline
In Section 2, we recall the 1D continuous wavelet transform and its wavelet-Plancherel theory. In
Section 3 we discuss the observable-based localization theory of the 1D CWT, and define our
notion of wavelet uncertainty as the localization of the ambiguity function in phase space. In
Section 4 we develop formulas for the pull-back of the the phase space uncertainty to the window-
signal space. We use these pull-back formulas to obtain closed-form-formulas of the phase space
uncertainty as a combination of window and signal localization measures. In order to find a local
uncertainty minimizer via a variational method, in Section 5 we develop a general theory for
calculus of variations of observables. In Section 6 we use this theory to derive the Euler-Lagrange
equations for the uncertainty minimizer. Finally, in Section 7 we implement a gradient descent
method to numerically estimate an optimal window of the 1D wavelet transform.

2 The 1D Wavelet-Plancherel Theory
The wavelet-Plancherel framework was developed in [20, 19]. This theory encompasses a broad
class of wavelet transforms which are based on square integrable representations. In this section,
we recall briefly some of the fundamentals of the theory for the special case of the 1D continuous
wavelet transform. We begin by recalling general continuous wavelet transforms.

2.1 Continuous Wavelet Transforms
Let $G$ be a locally compact group that we call phase space, and let $d\mu$ denote the left Haar measure
in $G$. Consider a Hilbert space $\mathcal{H}$ that we call the signal space, and let $\pi : G \to U(\mathcal{H})$
be a square integrable representation of $G$. Here, $U(\mathcal{H})$ denotes the space of unitary operators on $\mathcal{H}$. The
square integrable assumption means that there is a signal $f \in \mathcal{H}$, that we call a window function
or a mother wavelet, such that the mapping

$$V_f : s \mapsto \langle s, \pi(\cdot)f \rangle_{\mathcal{H}}$$
The group law is 

\[ G \times G \to G, \quad (\alpha, \beta) \mapsto \alpha \alpha' + \beta \alpha' \beta'. \]

The wavelet system is generated by the representation \( \pi \) acting on \( L^2(\mathbb{R}) \) via

\[ (\pi(\alpha, \beta)f)(x) = |\alpha|^{-\frac{1}{2}}f(\alpha(x - \beta)). \]

To use the wavelet-Plancherel theory of semi-direct product wavelet transforms, the subgroups of \( G \) corresponding to the coordinates \( \alpha \) and \( \beta \) should be represented as translation groups. We thus choose a different parameterization of \( G \), substituting \( \alpha \) with \( e^\alpha \). Note that this parameterization only contains positive dilations for \( \alpha > 0 \) to \( (e^{ln(\alpha)}, 1) = (e^\alpha, 1) \) and \( \alpha < 0 \) to \( (e^{ln(-\alpha)}, -1) = (e^\alpha, -1) \). The group rule of this new parameterization of \( G \) is given by

\[ (a, b, c) \cdot (a', b', c') = (a + a', b + ce^a b', cc'). \]

With this parameterization of \( G \), the representation, which by abuse of notation is still denoted by \( \pi \), takes the form

\[ (\pi(a, b, c)f)(x) = e^{-\frac{2}{a}}f(ec^{-a}(x - b)). \]  

\[ \text{(2)} \]

The window \( f \in L^2(\mathbb{R}) \) is assumed to meet the admissibility condition, which takes the following form in the frequency domain

\[ \int_{\mathbb{R}} \frac{1}{|\omega|} |\hat{f}(\omega)|^2 d\omega < \infty. \]  

\[ \text{(3)} \]

The wavelet transform associated to the window \( f \) maps signals \( s \) in \( L^2(\mathbb{R}) \) to functions \( V_f(s) \) defined over \( G \) by

\[ V_f(s)(a, b, c) = \langle s, \pi(a, b, c)f \rangle, \]

where \( \langle \cdot, \cdot \rangle \) denotes the standard inner product in \( L^2(\mathbb{R}) \). The wavelet transform is also called the analysis operator.

The admissibility condition ensures that the mapping \( V_f \) is an isometric embedding of \( L^2(\mathbb{R}) \) to \( L^2(G, d\mu_G(a, b, c)) \) up to a constant, where the weighted Lebesgue measure \( d\mu_G(a, b, c) = e^{-a}dadb \) is the left Haar measure of \( G \). More accurately, integrating measurable functions \( F : G \to \mathbb{C} \), under the \( (a, b, c) \) parameterization, is given by

\[ \int_G F(a, b, c)d\mu_G(a, b, c) = \sum_{c \in \{-1, 1\}} \int_{\mathbb{R}^2} F(a, b, c)e^{-a}dadb. \]

We write in short \( L^2(G) \) instead of \( L^2(G, d\mu_G(a, b, c)) \).

The wavelet synthesis operator is defined as the adjoint \( V_f^* \) of the wavelet transform. We have, for every \( F \in L^2(G) \),

\[ V_f^* F = \int_G F(a, b, c) \pi(a, b, c)f \ d\mu_G(a, b, c), \]  

\[ \text{(4)} \]
where the integration in (4) is a weak integral. The synthesis operator is the pseudo inverse of the analysis operator, up to constant, and we have the reconstruction formula

\[ s = \left( \int_{\mathbb{R}} |\hat{f}(\omega)|^2 \frac{1}{|\omega|} d\omega \right)^{-1} V_f^* V_f(s) \]

The isometric embedding property can be derived from the more general orthogonality relation, given as follows. For every pair of signals \( s_1, s_2 \in L^2(\mathbb{R}) \) and admissible vectors \( f_1, f_2 \in L^2(\mathbb{R}) \),

\[ \langle V_{f_1}(s_1), V_{f_2}(s_2) \rangle = \int_{\mathbb{R}} \hat{f}_1(\omega) \hat{f}_2(\omega) \frac{1}{|\omega|} d\omega \langle s_1, s_2 \rangle_{L^2(\mathbb{R})}. \] (5)

2.3 The Wavelet-Plancherel Theory

The wavelet transform \( V_f : L^2(\mathbb{R}) \mapsto L^2(G) \) is an isometric embedding of \( L^2(\mathbb{R}) \) into \( L^2(G) \), and is not surjective. Consequently, it is impossible to pull-back most phase space operators isometrically. The wavelet-Plancherel theorem aims to remedy this problem. This is done by embedding the signal space \( L^2(\mathbb{R}) \) in a larger space, called the window-signal space, and canonically extending the wavelet transform to an isometric isomorphism between the larger signal space and \( L^2(G) \). In this subsection we recall the wavelet-Plancherel theory for the special case of the 1D CWT \[19\].

2.3.1 The Window-Signal Space

We begin by defining the window-signal space in the case of the 1D CWT. To avoid carrying the Fourier transform in our formulas, we formulate the theory directly in the frequency domain. The window-signal space is defined by combining two spaces, called the Fourier transform in our formulas, we formulate the theory directly in the frequency domain. The wavelet-Plancherel theorem aims to remedy this problem. This is done by embedding the signal space \( L^2(\mathbb{R}) \) in a larger space, called the window-signal space, and canonically extending the wavelet transform to an isometric isomorphism between the larger signal space and \( L^2(G) \). In this subsection we recall the wavelet-Plancherel theory for the special case of the 1D CWT \[19\].

**Definition 1.**

- The window-signal space \( W \otimes S \) is defined as the tensor product of \( W \) with \( S \). Namely,
  \[ W \otimes S := L^2(\mathbb{R}^2; \frac{1}{|\omega'|} d\omega' d\omega), \]
  where \( \omega' \) denotes the window variable and \( \omega \) denotes the signal variable.

- The tensor product operator \((\hat{f}, \hat{s}) \mapsto \hat{f} \otimes \hat{s}\) maps window-signal pairs from \( W \times S \) to \( W \otimes S \), where the simple function \( \hat{f} \otimes \hat{s} \) is defined as
  \[ \hat{f} \otimes \hat{s}(\omega', \omega) = \hat{f}(\omega') \hat{s}(\omega). \] (6)

Note that not every function in \( W \otimes S \) is simple. The window-signal space \( W \otimes S \) is the linear closure of the space of simple functions. Namely, every \( F \in W \otimes S \) can be approximated by a finite sum of simple functions. Also, note that for any two simple functions, the inner product in \( W \otimes S \) satisfies

\[ \langle \hat{f}_1 \otimes \hat{s}_1, \hat{f}_2 \otimes \hat{s}_2 \rangle = \langle \hat{f}_1, \hat{f}_2 \rangle_W \langle \hat{s}_1, \hat{s}_2 \rangle_S. \] (7)
2.3.2 The Wavelet-Plancharel Transform

The wavelet-Plancharel transform extends the wavelet transform to an isometric isomorphism \( V : \mathcal{W} \otimes \mathcal{S} \rightarrow L^2(G) \) as follows. For simple functions, the wavelet-Plancharel transform is defined by

\[
V(\hat{f} \otimes \hat{s}) = V\hat{f}(\hat{s}),
\]

and this definition extends by linear closure to any \( F \in \mathcal{W} \otimes \mathcal{S} \). From (7) and the orthogonality relation (5), it follows that \( V \) is an isometric embedding into \( L^2(G) \). The next theorem from [19, Example 24] states that \( V \) is in fact an isometric isomorphism for the 1D continuous wavelet transform.

**Theorem 2 (The Wavelet-Plancharel Theorem).** The wavelet-Plancherel transform \( V \) is an isometric isomorphism between \( \mathcal{W} \otimes \mathcal{S} \) and \( L^2(G) \).

In [19, Equation (13)] it was shown that the wavelet-Plancherel transform can be written by the following explicit formula. For any \( F \in \mathcal{W} \otimes \mathcal{S} \),

\[
V(F)(a,b,c) = \int_{\mathbb{R}} e^{2\pi ib\omega} e^{i\frac{c}{2} a} F(ce^a,\omega) d\omega.
\]

The inverse wavelet-Plancherel transform, \( V^* \), also has a closed form formula, presented in [19, Section 2.2.4]. We skip this explicit inversion formula as we do not use it in this paper.

3 Localization in Wavelet Analysis

In this section, we recall the localization theory of wavelet transforms, presented in [21, 20, 19], for the specific case of the 1D CWT. We use the approach to define our notion of wavelet uncertainty.

A well known fact from wavelet theory (see, e.g., [14]) is that the image space \( V_f(L^2(\mathbb{R})) \) of the wavelet transform is a reproducing kernel Hilbert space. The kernels of the image space are the translations of the ambiguity function \( V_f(f) \in L^2(G) \). Namely, for a phase space function \( F \in V_f(L^2(\mathbb{R})) \), it holds, for every \((a,b,c) \in G\),

\[
F(a,b,c) = \langle F, \lambda(a,b,c)V_f(f) \rangle.
\]

Here, \( \lambda(a,b,c) : L^2(G) \rightarrow L^2(G) \) is the left translation in \( L^2(G) \), defined for functions \( H \in L^2(G) \) by \( \lambda(a,b,c)H(a',b',c') = H((a,c)-1(a',b',c')) \). We can view \( V_f(f) \) as the “point spread function” of the space \( V_f(L^2(\mathbb{R})) \), whose spread describes the “blurriness” of the space. This motivates the main goal in our paper – to make the ambiguity function as localized as possible. In order to define precisely how to measure the locality of the ambiguity function, we need to introduce special operators called observables.

3.1 Observables

An observable in a separable Hilbert space \( \mathcal{H} \) is a self-adjoint or unitary operator. Observables are seen as entities that define and measure physical quantities. One way in which observable measure their underlying quantities is through expected values and variances.

**Definition 3.** Let \( \hat{T} \) be an observable (i.e., a self-adjoint or unitary operator) in the Hilbert space \( \mathcal{H} \). The expected value and the variance of a normalized vector \( f \in \text{Dom}(\hat{T}) \), with respect to \( \hat{T} \), are defined to be, respectively,

\[
e_f(\hat{T}) = \langle \hat{T} f, f \rangle,
\]

\[
v_f(\hat{T}) = \left\| (\hat{T} - e_f(\hat{T})) f \right\|^2.
\]

The expected value and variance are also called the (first order and second order) moments of \( f \) with respect to \( \hat{T} \).
The simplest examples of observables are multiplicative operators. The phase space scale observable is the operator $A$, defined for functions $F : G \to \mathbb{C}$ by

$$AF(a, b, c) = aF(a, b, c).$$

The operator $A$ is self-adjoint in the domain

$$\text{Dom}(A) = \{ F \in L^2(G) \mid \text{the function } (a, b, c) \mapsto aF(a, b, c) \text{ is in } L^2(G) \}.$$  

We interpret $A$ as an entity that measures scale as follows. For a normalized $F \in L^2(G)$, we can think about $|F(a, b, c)|^2$ as the weight of the point $(a, b, c)$, for each $(a, b, c) \in G$. Computing $e_F(A) = \langle AF, F \rangle$ weights the scale value $a$ of every point $(a, b, c)$ by $|F(a, b, c)|^2$, and computes the weighted average of the scale values $a$. Thus, $e_F(A)$ is seen as the center of mass of $F$ along the axis $a$ in phase space. Similarly, the variance $v_F(A)$ is seen as the spread of $F$ along $a$ about its center of mass. Equivalently, the phase space time observable $B$, is defined by

$$BF(a, b, c) = bF(a, b, c),$$

and is self-adjoint over the domain

$$\text{Dom}(B) = \{ F \in L^2(G) \mid \text{the function } (a, b, c) \mapsto bF(a, b, c) \text{ is in } L^2(G) \}.$$  

Now, we combine the moments of the ambiguity function $V_f(f)$ to define one uncertainty measure of mother wavelets. Two natural ways to combine $v_{V_f(f)}(A)$ and $v_{V_f(f)}(B)$ are by the multiplication $v_{V_f(f)}(A)v_{V_f(f)}(B)$ or by the sum $v_{V_f(f)}(A) + v_{V_f(f)}(B)$. Since minimizing the sum of the variances ensures both small area and small radius of the domain that contains most of the energy of the ambiguity function, we base the uncertainty on sum.

**Definition 4.** The phase space uncertainty associated to the window $\hat{f}$ is defined to be

$$\mathcal{L}(\hat{f}) = v_{K_f}(A) + v_{K_f}(B),$$

where $K_f := V_f^{\hat{f}} \left( \frac{\hat{f}}{\|\hat{f}\|_s} \right)$ is the normalized ambiguity function. The domain of $\mathcal{L}$ is defined to be

$$\text{Dom}(\mathcal{L}) = \{ \hat{f} \in \mathcal{W} \cap \mathcal{S} \mid \hat{f} \neq 0 \text{ and } K_f \in \text{Dom}(A) \cap \text{Dom}(B) \}.$$  

For an observable $\hat{T}$, when the vector $f \in \mathcal{H}$ is not normalized, we still use the notations $e_f(\hat{T})$ and $v_f(\hat{T})$ defined in (8) and (9). In this case, $e_f(T)$ and $v_f(T)$ are no longer interpreted as center of mass and spread. In the context of the window and signal spaces, we denote by $e_f^W(\hat{T})$ and $v_f^W(\hat{T})$ the expected value and variance of $\hat{f}$ with respect to an observable $\hat{T}$ in the space $W$, and similarly denote by $e_f^S(\hat{T})$ and $v_f^S(\hat{T})$ the localization measures in the signal domain. For example

$$e_f^W(\hat{T}) = \langle T\hat{f}, \hat{f} \rangle_W.$$  

Our next goal is to formulate the pull-back of the moments of $A$ and $B$ to the window-signal space, so $\mathcal{L}(\hat{f})$ can be computed efficiently and directly using signal and window operations on the window function $\hat{f}$. For the pull-back formulas of $\mathcal{L}(\hat{f})$, we first define observables corresponding to localization in wavelet analysis directly in the signal domain. For that, we need to define rigorously what is meant by the statements “the dilation group changes scale,” and “translations change time.”

### 3.2 Transforms Associated with the CWT

In this subsection, we recall the transformation subgroups of $\pi(G)$ that are associate with the physical quantities underlying the 1D CWT: time and scale. We also recall two useful signal transforms in the context of wavelet analysis.
We identify the frequency domain $L$ negatively supported signals where $L$ space, we can call points signal space $L$ warping transforms two

When we define these spaces directly in the frequency domain, Reflection in the frequency domain stays reflection, a

We call the parameter $b \in \mathbb{R}$ time, call $a \in \mathbb{R}$ scale, and $c \in \{-1,1\}$ direction.

Since the window and signal spaces, $W$ and $S$, are defined directly as function spaces over the frequency line, we next formulate $\pi$ in the frequency domain. For an operator $T$ in the time domain, we denote the pull-back to the frequency domain by $\hat{T} := \mathcal{F}TF^{-1}$. Here, $\mathcal{F}$ denotes the Fourier transform. Translation by $b$ in the time domain takes the form of modulation in frequency,

$$\hat{L}(b)\hat{f}(\omega) := e^{-ib\omega}\hat{f}(\omega).$$

Dilation by scale $a$ in frequency is

$$\hat{D}(a)\hat{f}(\omega) = e^{i\omega a}(e^{a\omega}).$$

Reflection in the frequency domain stays reflection,

$$\hat{R}(c)\hat{f}(\omega) = \hat{f}(c\omega).$$

Overall,

$$(\hat{\pi}(a,b,c)\hat{f})(\omega) = e^{-ib\omega}e^{i\omega a}(e^{c\omega}).$$

(13)

One way to formulate rigorously the statement "dilations change scale," is to transform the signal space $L^2(\mathbb{R})$ into another space $L^2(Y)$, where dilations operate as translations. In this space, we can call points $\sigma \in Y$ scales. Next, we recall the transform from the frequency domain $L^2(\mathbb{R})$ to the so called scale space. Define the subspaces of positively supported (i.e. analytic) and negatively supported signals,

$$L^2_{A^+}(\mathbb{R}) = \{f \in L^2(\mathbb{R}) | \text{support} (\hat{f}) \subseteq \mathbb{R}_+ \},$$

$$L^2_{A^-}(\mathbb{R}) = \{f \in L^2(\mathbb{R}) | \text{support} (\hat{f}) \subseteq \mathbb{R}_- \}.$$ When we define these spaces directly in the frequency domain, $L^2_{A^+}(\mathbb{R})$ and $L^2_{A^-}(\mathbb{R})$ take the forms $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$ respectively, where

$$L^2(\mathbb{R}_+) := \{\hat{f} \in L^2(\mathbb{R}) | \text{support}(\hat{f}) \subseteq \mathbb{R}_+ \},$$

$$L^2(\mathbb{R}_-) := \{\hat{f} \in L^2(\mathbb{R}) | \text{support}(\hat{f}) \subseteq \mathbb{R}_- \}.$$ We identify the frequency domain $L^2(\mathbb{R})$ as the direct sum $L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_-)$, and consider the two warping transforms

$$W_+ : L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}), \quad W_- : L^2(\mathbb{R}_-) \rightarrow L^2(\mathbb{R})$$

$$W_+\hat{f}_+(\sigma) = \hat{f}_+(\sigma) = e^{-\sigma/2}\hat{f}_+(e^{-\sigma}), \quad W_-\hat{f}_-(\sigma) = \hat{f}_-(\sigma) = e^{-\sigma/2}\hat{f}_-(e^{-\sigma}),$$

(14)

defined separately on the positively supported and negatively supported components, as in [20].

The inverse warping transforms are given by

$$W_+^{-1}\hat{f}_+(\omega) = \begin{cases} \omega^{-\frac{1}{2}}\hat{f}_+(\ln(\omega)) & \text{if } \omega > 0 \\ 0 & \text{if } \omega \leq 0 \end{cases} = \hat{f}_+(\omega),$$

$$W_-^{-1}\hat{f}_-(\omega) = \begin{cases} -(\omega)^{-\frac{1}{2}}\hat{f}_-(\ln(-\omega)) & \text{if } \omega < 0 \\ 0 & \text{if } \omega \geq 0 \end{cases} = \hat{f}_-(\omega).$$

The scale transform is defined as the application of the two warping transforms on the two frequency components of signals $\hat{f}$. 

9
Definition 5.

- We define the scale space as $L^2(\mathbb{R})^2$, and parameterize its variable as $(\sigma, \text{sign})$. Namely, for $\tilde{f} = (\tilde{f}_+, \tilde{f}_-) \in L^2(\mathbb{R})^2$, we define $\tilde{f}(\sigma, \text{sign}) = \tilde{f}_{\text{sign}}(\sigma)$. We call the variable $(\sigma, \text{sign})$ scale. The inner product in $(L^2(\mathbb{R}))^2$ is defined as

$$\langle (\tilde{f}_+, \tilde{f}_-), (\tilde{g}_+, \tilde{g}_-) \rangle_{(L^2(\mathbb{R}))^2} := \langle \tilde{f}_+, \tilde{g}_+ \rangle_{L^2(\mathbb{R})} + \langle \tilde{f}_-, \tilde{g}_- \rangle_{L^2(\mathbb{R})},$$

for every $(\tilde{g}_+, \tilde{g}_-), (\tilde{f}_+, \tilde{f}_-) \in (L^2(\mathbb{R}))^2$.

- The scale transform $U : L^2(\mathbb{R}) \to L^2(\mathbb{R})^2$ is defined by its frequency pull-back as $UFU^{-1} = W_+ \oplus W_-$, where $W_+$ and $W_-$ are the warping transforms $[14]$. Namely, for a time signal $f \in L^2(\mathbb{R})$, with $\tilde{f} = \tilde{f}_+ + \tilde{f}_-$ as the decomposition of $\tilde{f}$ to its positive and negative supports, we have

$$UFU^{-1}(\tilde{f}_+ + \tilde{f}_-)(\sigma, \text{sign}) = \tilde{f}_{\text{sign}}(\sigma).$$

We denote the image of $f \in L^2(\mathbb{R})$ under the scale transform in short by $Uf = \tilde{f}$. It is easy to see that the scale transform is a unitary operator. Namely,

$$\|f\|_{L^2(\mathbb{R})} = \|\tilde{f}\|_{(L^2(\mathbb{R}))^2}$$

(15)

Translation and dilation in scale space take the forms

$$UL(b)U^{-1}f(\sigma, \text{sign}) = \tilde{L}(b)f(\sigma, \text{sign}) = e^{-ib\sigma} \tilde{f}(\sigma, \text{sign}),$$

$$UD(a)U^{-1}f(\sigma, \text{sign}) = \tilde{D}(a)f(\sigma, \text{sign}) = \tilde{f}(\sigma - a, \text{sign}).$$

(16)

Reflection in scale space is

$$UR(c)U^{-1}f(\sigma, \text{sign}) = \tilde{R}(c)f(\sigma, \text{sign}) = \tilde{f}(\sigma, c \cdot \text{sign}).$$

Overall,

$$U\pi(a, b, c)U^{-1}f(\sigma, \text{sign}) = \tilde{\pi}(a, b, c)\tilde{f}(\sigma, \text{sign}) = e^{-iae\sigma} \tilde{f}(\sigma - a, c \cdot \text{sign})$$

(17)

From [16], we see that dilations operate as translations in the scale space under the scale transform, which justifies the terms scale space and scale transform. Namely, given a normalized function $\tilde{f} \in L^2(\mathbb{R})^2$, where $|\tilde{f}(\sigma, \text{sign})|^2$ is interpreted as the weight or probability of the scale $(\sigma, \text{sign})$, $UD(a)U^{-1}$ translates the scale distribution by $a$.

Since both the Fourier transform and the scale transform are isometries, we have

$$V_f(s) = V_f(\hat{s}) = V_f(\tilde{s}),$$

where $V_f(s)$ and $V_f(\hat{s})$ denote by abuse of notation

$$V_f(\hat{s})(a, b, c) = \langle \hat{s}, \tilde{\pi}(a, b, c)f \rangle_{L^2(\mathbb{R})}, \quad V_f(\tilde{s})(a, b, c) = \langle \tilde{s}, \tilde{\pi}(a, b, c)f \rangle_{(L^2(\mathbb{R}))^2}.$$

Lastly, the *time transform* is defined as the identity in the time domain $L^2(\mathbb{R})$, as $L(b)$ is already represented as translations in $L^2(\mathbb{R})$.\[10\]
3.3 Observables in the Signal Domain

Next, we recall two observables of scale and time, defined in the signal space. These signal space observables will be used in the explicit formula of the pull-back of the uncertainty $L(\hat{f})$. The signal space time and scale observables are defined as multiplicative operators in the time and the scale domains, as defined in the previous subsection. Since we focus in this paper on frequency space formulations, we define all operators directly in the frequency domain, assuming that $S = L^2(\mathbb{R})$ is the signal space in the frequency domain. Here, the time transform is given by $F^{-1}$. We define in this setting the scale transform by $\hat{U} = FU F^{-1} = W_+ \oplus W_-$ (see Definition 5). In the following, we denote by $\tilde{Y}$ the self-adjoint operator in $L^2(\mathbb{R})$ defined by

$$\tilde{Y} : f(y) \mapsto yf(y),$$
onumber

on the domain

$$\text{Dom}(\tilde{Y}) = \{ f \in L^2(\mathbb{R}) \mid y \mapsto yf(y) \in L^2(\mathbb{R}) \}.$$ 

**Definition 6.** Let $S = L^2(\mathbb{R})$ be the frequency domain.

1. The time signal space observable, $\hat{T}_x$, is the operator that multiplies the signal in the time domain by its $x$ coordinate, i.e.,

$$\hat{T}_x = F\tilde{Y} F^{-1}.$$

2. The scale signal space observable, $\hat{T}_\sigma$, is the operator that multiplies the signal in the scale space by its scale coordinate $\sigma$

$$\hat{T}_\sigma = U^{-1}(\tilde{Y} \oplus \tilde{Y}) U.$$

Here,

$$(\tilde{Y} \oplus \tilde{Y}) \tilde{f}(\sigma, \text{sign}) = \sigma \tilde{f}(\sigma, \text{sign}).$$

The next claim is direct.

**Proposition 7.** 1. The time observable is given by $\hat{T}_x = i \frac{\partial}{\partial \omega}$ on the domain $\text{Dom}(\hat{T}_x)$ of all $\hat{f} \in L^2(\mathbb{R})$ such that $\hat{f}$ is absolutely continuous in $[\alpha, \beta]$ for every $\beta > \alpha$, and $\omega \mapsto i \hat{f}'(\omega)$ is in $L^2(\mathbb{R})$. Moreover, $\hat{T}_x$ is self-adjoint.

2. The scale observable is given by the multiplicative operator

$$\hat{T}_\sigma \hat{f}(\omega) = -\ln(|\omega|) \hat{f}(\omega),$$

on the domain

$$\text{Dom}(\hat{T}_x) = \{ \hat{f} \in L^2(\mathbb{R}) \mid \omega \mapsto -\ln(|\omega|) \hat{f}(\omega) \in L^2(\mathbb{R}) \}.$$ 

The scale observable is self-adjoint.

The two signal space observables $\hat{T}_x$ and $\hat{T}_\sigma$ were used to define the uncertainty measure in $L(\hat{f})$ as a combination of $v_f(\hat{T}_x)$ and $v_f(\hat{T}_\sigma)$. In our theory, the uncertainty is defined via the phase space observables, instead of the signal space observables. The signal space observables appear in our formulations only as a result of using the wavelet-Plancherel theory to pull-back $L(\hat{f})$.

3.4 The Uncertainty Minimization Problem

The signal space observables $\hat{T}_x$ and $\hat{T}_\sigma$ are also used to restrict the search space of the window, when minimizing the uncertainty $L(\hat{f})$ of Definition 4. Precisely, the window $\hat{f}$ is assumed to satisfy $e_f(\hat{T}_\sigma) = e_f(\hat{T}_x) = 0$. For motivation, consider the role of $\hat{f}$ as an analyzing function, i.e., as a mean for probing the signal content at different times and scales. From this point of view, $e_f(\hat{T}_\sigma)$ is interpreted as the scale location of the window $\hat{f}$, and $e_f(\hat{T}_x)$ as its time location. A window $\hat{f}$
with $e_f(\check{T}_\sigma) = e_f(\check{T}_x) = 0$ will be localized in time about 0, and have a rate of one oscillation per time unit. Now, consider the following transformation formulas from [21, Proposition 21]

$$e_{D(a)}f(\check{T}_\sigma) = e_{D(a)}f(\check{T}_x) = 0 \quad (18)$$

$$e_{L(b)}f(\check{T}_x) = e_{L(b)}f(\check{T}_\sigma) + b \quad (19)$$

As a result of (18) and (19), by picking a window with zero scale and time expected values, we assure that the transformed window $\check{D}(a)f$ measures the scale $a$, and $\check{L}(b)f$ measures the time $b$. Similarly, $\check{\pi}(a,b,c)f$ is a scale-time atom that probes the scale-time pair $(a,b)$ (see [19, Formula (58)]).

To conclude this section, we formulate the minimization problem, following the above discussion.

**Minimization Problem 8.** Let $\mathcal{L}$ be the uncertainty from definition 4. Find

$$\text{Arg} \min_{\hat{f} \in \text{Dom}(\mathcal{L})} \mathcal{L}(\hat{f}),$$

subject to $e_f(\check{T}_x) = e_f(\check{T}_\sigma) = 0$.

### 4 The Pull-Back of the Phase Space Uncertainty

Let $K f = V f_{\check{T}_\sigma} \left( \frac{f}{\|f\|_S} \right)$ be the normalized ambiguity function. In this section, we formulate the variances of the 2D ambiguity function $v_{K f}(A)$ and $v_{K f}(B)$ in terms of the 1D window function $\hat{f}$, and hence formulate the phase space uncertainty via signal and window space computations. Note that $\hat{f}$ plays two roles in the ambiguity function, both as the window and as the signal. In addition, observe that windows and signals are treated differently in the wavelet-Plancherel theory. Therefore, it is beneficial to first study more general variances of the form $v_{V f(\check{s})}(A)$ and $v_{V f(\check{s})}(B)$ for general normalized signals $\check{s} \in S = L^2(\mathbb{R})$ and $\hat{f} \in W = L^2(\mathbb{R}, \frac{1}{|\omega|} d\omega)$, which need not be identical.

#### 4.1 The Pull-Back of Phase Space Observables

In this section, we recall formulas for the pull-back of the time and scale observables in the window-signal space. In [19, Propositions 27 and 33], there is a general formula for calculating the pull-back of a broad class of multiplicative operators for diffeomorphism the so called geometric wavelet transforms. Using this general formula we can compute our case of the 1D CWT. For completeness, we provide direct computations for the pull-back of the time and scale phase space observables in $A$.

**Definition 9.** The pull-backs of the phase space observables $A$ and $B$ of (10) and (11), to the window-signal space, are defined as follows.

1. The pull-back $\check{T}_a := V^*AV$ is defined over the domain

$$\text{Dom}(\check{T}_a) = \{ F \in W \otimes S \mid VF \in \text{Dom}(A) \}.$$

2. The pull-back $\check{T}_b := V^*BV$ is defined defined over

$$\text{Dom}(\check{T}_b) = \{ F \in W \otimes S \mid VF \in \text{Dom}(B) \}.$$
Proposition 10. 1. The operator $\tilde{T}_a$ is self-adjoint, and defined by
\[
\tilde{T}_a F(\omega, \omega') = (\ln(|\omega'|) - \ln(|\omega|)) F(\omega, \omega')
\]
over the domain
\[
\text{Dom}(\tilde{T}_a) = \{ F \in \mathcal{W} \otimes \mathcal{S} \mid (\ln(|\omega'|) - \ln(|\omega|)) F(\omega, \omega') \in L^2(\mathbb{R}^2; \frac{1}{|\omega'|} d\omega' d\omega) \}.
\]

2. The operator $\tilde{T}_b$ is self-adjoint, and defined by
\[
\tilde{T}_b F(\omega, \omega') = (i \frac{\partial}{\partial \omega} + i \frac{\omega'}{\omega} \frac{\partial}{\partial \omega'}) F(\omega, \omega')
\]
over the domain $\text{Dom}(\tilde{T}_b)$ of all functions $F \in \mathcal{W} \otimes \mathcal{S}$ such that their restriction to every compact interval in almost every (with respect to $\kappa \in \mathbb{R}$) line of the form
\[
l_\kappa = \{(\omega, \kappa \omega) \mid \omega \in \mathbb{R}\}
\]
is absolutely continuous, and
\[
(\omega', \omega) \mapsto (i \frac{\partial}{\partial \omega} + i \frac{\omega'}{\omega} \frac{\partial}{\partial \omega'}) F(\omega, \omega') \in L^2(\mathbb{R}^2; \frac{1}{|\omega'|} d\omega' d\omega).
\]

In the explicit formula of $\tilde{T}_b$, note that the lines $l_\kappa$ are the integral lines of the translation group $\{e^{it\tilde{T}_b}\}_{t \in \mathbb{R}}$, and hence for absolutely continuous functions $F$ in every interval along these integral lines, $(i \frac{\partial}{\partial \omega} + i \frac{\omega'}{\omega} \frac{\partial}{\partial \omega'}) F(\omega, \omega')$ is well defined.

Next, we consider the pull-back formulas of Proposition 10 for simple functions $F = \hat{f} \otimes \hat{s}$. To be able to derive closed-form formulas that disentangle the computations on $\hat{s}$ from those on $\hat{f}$, we restrict the domain of $\tilde{T}_b$.

For the rest of this paper, we follow the following convention when writing operators. By abuse of notation, we write multiplicative operators $Q$ that operate on function $f$ by
\[
Q f(x) = q(x) f(x),
\]
where $q$ is some function, by $q(x)$. We write operators $W$ that operate of function $f$ by
\[
W f(x) = q(x) \frac{\partial}{\partial x} f(x),
\]
where $q$ is some function, by $q(x) \frac{\partial}{\partial x}$.

Corollary 11. Let $\hat{s} \in \mathcal{S}$ and $\hat{f} \in \mathcal{W}$.

1. Suppose that $\hat{s}$ is in the domain
\[
\text{Dom}_\mathcal{S}(\tilde{T}_a) = \{ \hat{s} \in L^2(\mathbb{R}) \mid \omega \mapsto \ln(|\omega|) \hat{s}(\omega) \in L^2(\mathbb{R}) \},
\]
and $\hat{f}$ is in the domain
\[
\text{Dom}_\mathcal{W}(\tilde{T}_a) = \{ \hat{f} \in L^2(\mathbb{R}) \mid \omega' \mapsto \ln(|\omega'|) \hat{f}(\omega') \in L^2(\mathbb{R}; \frac{1}{|\omega'|} d\omega') \}.
\]

Then $\hat{f} \otimes \hat{s} \in \text{Dom}(\tilde{T}_a)$, and
\[
\tilde{T}_a (\hat{f} \otimes \hat{s}) = -\hat{f} \otimes (\ln(|\omega|) \hat{s}) + (\ln(|\omega'|) \hat{f}) \otimes \hat{s}.
\]
In this subsection we formulate it is enough to make the windows orthogonal. Hence, we reformulate (20) and (21) as

\[ \hat{T}_b(\hat{f} \otimes \hat{s}) = \hat{f} \otimes i \frac{\partial}{\partial \omega} \hat{s} - i\omega' \frac{\partial}{\partial \omega'} \hat{f} \otimes \frac{1}{\omega} \hat{s}. \]

(21)

For future calculations, it is beneficial to formulate \( \hat{T}_a \) and \( \hat{T}_b \) for simple vectors as sums of orthogonal simple vectors. By the inner product formula of simple functions in \( \mathcal{W} \otimes \mathcal{S} \) (see (7)), it is enough to make the windows orthogonal. Hence, we reformulate (20) and (21) as

\[ \hat{T}_a(\hat{f} \otimes \hat{s}) = \hat{f} \otimes \left( -\ln(|\omega|) + e^W_j(\ln(|\omega'|)) \right) \hat{s} + \left( \ln(|\omega'|) - e^W_j(\ln(|\omega'|)) \right) \hat{f} \otimes \hat{s}, \]

(22)

\[ \hat{T}_b(\hat{f} \otimes \hat{s}) = \hat{f} \otimes \left( i \frac{\partial}{\partial \omega} - e^W_j(i\omega' \frac{\partial}{\partial \omega'}) \right) \frac{1}{\omega} \hat{s} + \left( -i\omega' \frac{\partial}{\partial \omega'} + e^W_j(i\omega' \frac{\partial}{\partial \omega'}) \right) \hat{f} \otimes \left( \frac{1}{\omega} \hat{s} \right). \]

(23)

Here, recall that \( e^W_j(T) = \langle T\hat{f}, \hat{f} \rangle_\mathcal{W} \) denotes the expected value of \( \hat{f} \) with respect to an observable \( T \) in the space \( \mathcal{W} \).

4.2 The Pull-Back of Scale Localization Measures

In this subsection we formulate \( e_{\hat{f} \otimes \hat{s}}(\hat{T}_a) \) and \( v_{\hat{f} \otimes \hat{s}}(\hat{T}_a) \) as a combination of signal and window expected values and variances.

**Proposition 12.** Let \( \hat{s} \in \text{Dom}_S(\hat{T}_a) \) and \( \hat{f} \in \text{Dom}_W(\hat{T}_a) \) (see Proposition 11) such that \( \|\hat{f}\|_\mathcal{W} = \|\hat{s}\|_\mathcal{S} = 1 \). Then the following holds.

1. The expected value of \( \hat{f} \otimes \hat{s} \) with respect to \( \hat{T}_a \) is

\[ e_{\hat{f} \otimes \hat{s}}(\hat{T}_a) = -e^S_\hat{s}(\ln(|\omega|)) + e^W_j(\ln(|\omega'|)). \]

2. The variance of \( \hat{f} \otimes \hat{s} \) with respect to \( \hat{T}_a \) is

\[ v_{\hat{f} \otimes \hat{s}}(\hat{T}_a) = v^S_\hat{s}(\ln(|\omega|)) + e^W_j(\ln(|\omega'|)). \]

**Proof.**

1. Calculate

\[ e_{\hat{f} \otimes \hat{s}}(\hat{T}_a) = \langle \hat{T}_a(\hat{f} \otimes \hat{s}), \hat{f} \otimes \hat{s} \rangle_{\mathcal{W} \otimes \mathcal{S}} \]

\[ = \langle \hat{f} \otimes \left( -\ln(|\omega|) + e^W_j(\ln(|\omega'|)) \right) \hat{s}, \hat{f} \otimes \hat{s} \rangle_{\mathcal{W} \otimes \mathcal{S}} \]

\[ + \langle \left( \ln(|\omega'|) - e^W_j(\ln(|\omega'|)) \right) \hat{f} \otimes \hat{s}, \hat{f} \otimes \hat{s} \rangle_{\mathcal{W} \otimes \mathcal{S}} \]

\[ = \langle \hat{f}, \hat{f} \rangle_{\mathcal{W}} \langle \left( -\ln(|\omega|) + e^W_j(\ln(|\omega'|)) \right) \hat{s}, \hat{s} \rangle_{\mathcal{S}} \]

\[ + \langle \left( \ln(|\omega'|) - e^W_j(\ln(|\omega'|)) \right) \hat{f}, \hat{f} \rangle_{\mathcal{W}} \langle \hat{s}, \hat{s} \rangle_{\mathcal{S}} \]

\[ = -e^S_\hat{s}(\ln(|\omega|)) + e^W_j(\ln(|\omega'|)). \]
Proof. 1. Compute Let Proposition 13. The expected values and variances.

2. We start by computing

\[
(\tilde{T}_a - e_{f\otimes S}(\tilde{T}_a))(\hat{f} \otimes \hat{s}) = \hat{f} \otimes \left( -\ln(|\omega|) + e_S^W(\ln(|\omega|)) \right) \hat{s} + \left( \ln(|\omega'|) - e_f^W(\ln(|\omega'|)) \right) \hat{f} \otimes \hat{s}.
\]

Hence, the variance is

\[
v_{f\otimes S}(\tilde{T}_a) = \left\| (\tilde{T}_a - e_{f\otimes S}(\tilde{T}_a))(\hat{f} \otimes \hat{s}) \right\|_{W \otimes S}^2 = \left\| \hat{f} \right\|_W^2 \left\| \left( -\ln(|\omega|) + e_S^W(\ln(|\omega|)) \right) \hat{s} \right\|_S^2 + \left\| \left( \ln(|\omega'|) - e_f^W(\ln(|\omega'|)) \right) \hat{f} \right\|_W^2 \left\| \hat{s} \right\|_S^2 = e_S^W(\ln(|\omega|)) + v_f^W(\ln(|\omega'|)),
\]

where the second equality in (25) follows from linearity, the inner product formula of simple functions in \( W \otimes S \) (7), and by the orthogonality of the two windows \( \hat{f} \) and \( \left( \ln(|\omega'|) - e_f^W(\ln(|\omega'|)) \right) \hat{f} \).

\[\square\]

4.3 The Pull-Back of Time Localization measures

In this subsection we formulate \( e_{f\otimes S}(\tilde{T}_b) \) and \( v_{f\otimes S}(\tilde{T}_b) \) as a combination of signal and window expected values and variances.

**Proposition 13.** Let \( \hat{s} \in \text{Dom}_S(\tilde{T}_b) \) and \( \hat{f} \in \text{Dom}_W(\tilde{T}_b) \) (see Proposition 11) such that \( \left\| \hat{f} \right\|_W = \left\| \hat{s} \right\|_S = 1 \). Then the following holds.

1. The expected value of \( \hat{f} \otimes \hat{s} \) with respect to \( \tilde{T}_b \) is

\[
e_{f\otimes S}(\tilde{T}_b) = e_S^W(i \frac{\partial}{\partial \omega}) - e_f^W(i \omega' \frac{\partial}{\partial \omega}) e_S^W(\frac{1}{\omega}).
\]

2. The variance of \( \hat{f} \otimes \hat{s} \) with respect to \( \tilde{T}_b \) is

\[
v_{f\otimes S}(\tilde{T}_b) = v_S^W(i \frac{\partial}{\partial \omega} - e_f^W(i \omega' \frac{\partial}{\partial \omega}) \frac{1}{\omega}) + v_f^W(i \omega' \frac{\partial}{\partial \omega}) \left\| \frac{1}{\omega} \hat{s} \right\|_S^2.
\]

**Proof.** 1. Compute \( e_{f\otimes S}(\tilde{T}_b) \)

\[
= \left\langle \hat{f} \otimes \left( i \frac{\partial}{\partial \omega} \hat{s} - e_f^W(i \omega' \frac{\partial}{\partial \omega}) \frac{1}{\omega} \hat{s} \right) , \hat{f} \otimes \hat{s} \right\rangle + \left\langle \left( -i \omega' \frac{\partial}{\partial \omega} \hat{f} + e_f^W(i \omega' \frac{\partial}{\partial \omega}) \hat{f} \right) \otimes \left( \frac{1}{\omega} \hat{s} \right) , \hat{f} \otimes \hat{s} \right\rangle
\]

\[
= e_S^W(i \frac{\partial}{\partial \omega}) - e_f^W(i \omega' \frac{\partial}{\partial \omega}) e_S^W(\frac{1}{\omega}).
\]
2. Compute

\[(\hat{T}_b - e_{f \otimes s}(\hat{T}_b))(\hat{f} \otimes \hat{s})\]

\[= \hat{f} \otimes \left( i \frac{\partial}{\partial \omega} \hat{s} - e_{f}^{W}(i\omega') \frac{\partial}{\partial \omega'}\frac{1}{\omega} \hat{s} \right) + \left( -i \omega' \frac{\partial}{\partial \omega} \hat{f} + v_{f}^{W}(i\omega') \frac{\partial}{\partial \omega'}\frac{1}{\omega} \hat{f} \right) \otimes 1_{\omega} \hat{s}(\omega) \]

\[= \hat{f} \otimes \left( i \frac{\partial}{\partial \omega} \hat{s} - e_{f}^{W}(i\omega') \frac{1}{\omega} \hat{s} - (e_{s}^{S}(i \frac{\partial}{\partial \omega}) - e_{f}^{W}(i\omega') \frac{\partial}{\partial \omega'}\frac{1}{\omega} \hat{s}) \right) + \left( -i \omega' \frac{\partial}{\partial \omega} \hat{f} + e_{f}^{W}(i\omega') \frac{\partial}{\partial \omega'}\frac{1}{\omega} \hat{f} \right) \otimes 1_{\omega} \hat{s} \]
5 Calculus of Variations of Observables

Our goal is to develop a calculus of variations approach for solving the Minimization Problem with \( L(f) \) given in the form of Proposition 15. In this section, we start by developing general calculus of variations for localization measures based on observables.

5.1 General Calculus of Variations

To formulate the general calculus of variations framework, we consider (non-linear) functionals \( S \) which are defined in separable Hilbert spaces \( \mathcal{H} \). A functional is simply a function from a domain in \( \mathcal{H} \) to the scalar field of \( \mathcal{H} \). The domain of \( S \), denoted by \( \text{Dom}(S) \), is assumed to be a dense linear subspace of \( \mathcal{H} \), possibly excluding zero. Note that our uncertainty functional \( L \) of (31), defined on \( \text{Dom}_{W+S}(L) \), is such a functional. We start by presenting the definition of Gateaux differential and the variation in Hilbert spaces.

**Definition 16.** Let \( \mathcal{H} \) be a separable Hilbert space over the field \( \mathbb{F} \), which is \( \mathbb{R} \) or \( \mathbb{C} \), and let \( S : \text{Dom}(S) \to \mathbb{F} \) be a (generally non-linear) functional. Suppose that \( \text{Dom}(S) \cup \{0\} \subset \mathcal{H} \) is a dense linear subspace of \( \mathcal{H} \).

- The Gateaux differential of \( S \) at \( f \in \text{Dom}(S) \) in the direction \( h \in \text{Dom}(S) \) is defined to be
  \[
  dS(f; h) := \lim_{t \to 0} \frac{S(f + th) - S(f)}{t},
  \]
  if the limit exists.

- If there exists a vector \( \frac{\partial S}{\partial f} \in \mathcal{H} \) such that
  \[
  dS(f; h) = \left\langle \frac{\partial S}{\partial f}, h \right\rangle_{\mathcal{H}},
  \]
  for every \( h \in \text{Dom}(S) \), \( \frac{\partial S}{\partial f} \) is called the variation of \( S \) at \( f \).

Next, we define the \( o \)-notation for functions, functionals, and operators.

**Definition 17.** Let \( \mathcal{H} \) be a Hilbert space over the field \( \mathbb{F} \), where \( \mathbb{F} \) is \( \mathbb{R} \) or \( \mathbb{C} \).

1. For functions \( e_1, e_2 : \mathbb{F} \to \mathbb{F} \), we say \( e_1(h) = o(e_2(h)) \) as \( h \to 0 \) if
   \[
   \lim_{h \to 0} \frac{|e_1(h)|}{|e_2(h)|} = 0.
   \]

2. Let \( S_1 : \text{Dom}(S_1) \to \mathbb{C} \) and \( S_2 : \text{Dom}(S_2) \to \mathbb{C} \) be (non-linear) functionals in \( \mathcal{H} \), such that \( \text{Dom}(S_j) \cup \{0\} \) is a dense linear subspace of \( \mathcal{H} \), for \( j = 1, 2 \). Suppose that \( \text{Dom}(S_1) \subseteq \text{Dom}(S_2) \). We say \( S_1(h) = o(S_2(h)) \) as \( h \to 0 \) if
   \[
   \lim_{t \to 0} \frac{|S_1(th)|}{|S_2(th)|} = 0
   \]
   for every \( h \in \text{Dom}(S_1) \).

3. Let \( E : \text{Dom}(E) \to \mathcal{H} \) be (non-linear) operator in \( \mathcal{H} \), where \( \text{Dom}(E) \cup \{0\} \) is a dense subspace of \( \mathcal{H} \). Let \( S : \text{Dom}(S) \to \mathbb{C} \) be a functional with \( \text{Dom}(S) \cup \{0\} \) a dense subspace of \( \mathcal{H} \), such that \( \text{Dom}(E) \subset \text{Dom}(S) \). We say \( E(h) = o(S(h)) \) as \( h \to 0 \) if
   \[
   \lim_{t \to 0} \frac{\|E(th)\|}{|S(th)|} = 0
   \]
   for every \( h \in \text{Dom}(E) \).
The following proposition is a useful tool when computing and proving the existence of variations.

**Proposition 18.** Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{F}$, where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$. Let $S : \text{Dom}(S) \rightarrow \mathbb{F}$ be a functional over the domain $\text{Dom}(S) \subset \mathcal{H}$. Suppose that $\text{Dom}(S) \cup \{0\}$ is a dense linear subspace of $\mathcal{H}$. Let $f \in \text{Dom}(S)$ and $g \in \mathcal{H}$. The following conditions are equivalent.

- There exists a variation of $S$ at $f$ and $\frac{\delta}{\delta f} S(f) = g$.
- It holds
  \[ S(f + h) - S(f) = \langle g, h \rangle_{\mathcal{H}} + o(\|h\|), \]
  as $\text{Dom}(S) \ni h \to 0$.

Calculus of variations helps us find local minima of functionals by locating their stationary points. The next theorem is a version of Fermat’s extreme value theorem for calculus of variations.

**Theorem 19** (Fermat’s Theorem for Stationary Points). Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{F}$, where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$. Let $S : \text{Dom}(S) \rightarrow \mathbb{R}$ be a functional. Suppose that $\text{Dom}(S) \cup \{0\}$ is a dense linear subspace of $\mathcal{H}$. If $f_0$ is a local minimum of $S$, and $S$ has a variation at $f_0$, then $\frac{\delta S}{\delta f}(f_0) = 0$.

### 5.2 Realification

Our goal is to minimize $L$ using calculus of variations. Note that a global minimum is a set theoretic notion that does not depend on additional structure endowed upon the set, like a vector space structure or inner product. Hence, we have the freedom to choose any vector space and inner product structure on the domain of the uncertainty $L$. To allow the use of calculus of variations tools for minimization, we require a vector space structure for which the variations of the uncertainty $L$ exist. A natural choice is to take either $S$ or $W$ as the Hilbert space structure of the domain of $L$. However, it can be shown that the variations of $L$ do not exist with respect to neither of these two Hilbert spaces. Fortunately, as we show constructively in the next sections, the variations do exist with respect to the realifications of the spaces $W$ and $S$.

**Definition 20** ([6]). The realification of a complex Hilbert space $\mathcal{H}$ (with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$) is the real Hilbert space $\mathcal{H}^R$, which is defined as follows.

- The space $\mathcal{H}^R$ is defined to be equal to $\mathcal{H}$ as a set.
- The space $\mathcal{H}^R$ is a vector space over the field $\mathbb{R}$, with the same vector addition as in $\mathcal{H}$, and the multiplication by scalars in $\mathcal{H}^R$ is defined as the multiplication in $\mathcal{H}$, restricted to real scalars.
- The inner product in $\mathcal{H}^R$ is defined to be
  \[ \langle \cdot, \cdot \rangle_{\mathcal{H}^R} := \text{Re}(\langle \cdot, \cdot \rangle_{\mathcal{H}}), \] (32)
  where $\text{Re}(\cdot)$ is the real part of a complex number.

Equation (32) indeed defines an inner product in $\mathcal{H}^R$, and $\mathcal{H}^R$ is a Hilbert space.

**Remark 21.** The realified signal space $\mathcal{S}^R$ is the space of measurable complex valued functions $\hat{s} : \mathbb{R} \rightarrow \mathbb{C}$ with the inner product

\[ \langle \hat{s}_1, \hat{s}_2 \rangle_{\mathcal{S}^R} = \text{Re}\left( \int_{\mathbb{R}} \hat{s}_1(x) \overline{\hat{s}_2(x)} dx \right). \]

Note that $\mathcal{S}^R$ is not a space of real valued functions.
5.3 Variations with Normalized Vectors

The moments in the uncertainty $\mathcal{L}$ are computed with respect to normalized windows $\frac{f}{\|f\|}$. Namely, for each moment $S$ that appears in $\mathcal{L}$, there exists a functional $H$ (densely defined in $S$), such that $S(\hat{f}) = H(\frac{f}{\|f\|})$, where the normalization is with respect to either $S$ or $W$. In this subsection, we derive useful formulas that aid in computing variations of functionals that involve normalized windows.

We start by giving a definition for the variation of (possibly) non-linear operators that map vectors to vectors in general Hilbert spaces.

**Definition 22.** Let $\mathcal{H}$ be a separable Hilbert space. Let $\text{Dom}(g) \subseteq \mathcal{H}$ be a set such that $\text{Dom}(g) \cup \{0\}$ is a dense linear subspace of $\mathcal{H}$. Let $g : \text{Dom}(g) \rightarrow \mathcal{H}$ be a (generally non-linear) operator. If there exists a bounded linear operator $T$ in $\mathcal{H}$ such that

$$g(f + h) - g(f) = T(h) + o(h),$$

where the $o$ notation is with respect to $\text{Dom}(g) \ni h \rightarrow 0$, then $T$ is called the variation of $g$ at $f$. In this case, we denote $\delta g(f) = T$.

The following lemma shows the existence of a variation for the normalizing operator $f \mapsto \frac{f}{\|f\|}$, and gives an explicit formula for $\frac{\delta f}{\delta f}$.

**Lemma 23.** Let $\mathcal{H}$ be a real separable Hilbert space, with the inner product $\langle \cdot , \cdot \rangle$ and norm $\|\cdot\|$. The variation of the normalizing operator $f \mapsto \frac{f}{\|f\|}$, with respect to $\mathcal{H}$, exists at every non-zero $f$, and satisfies

$$\frac{\delta f}{\delta f} = \frac{1}{\|f\|}I - \frac{f}{\|f\|^2} \otimes f,$$  \hspace{1cm} (33)

where $I$ is the identity operator, and $f \otimes \frac{f}{\|f\|^2}$ is the rank-one operator $v \mapsto \langle v, f \rangle \frac{f}{\|f\|^2}$.

**Proof.** In this proof, $\approx$ denotes equality up to $o(h)$. If the following, we use the fact that the scalar field of $\mathcal{H}$ is $\mathbb{R}$. Since $h$ is taken asymptotically small, we may assume $\|h\| < \|f\|$, and

$$\frac{f + h}{\|f + h\|} \approx \frac{f + h}{\sqrt{\langle f, f \rangle + 2 \langle f, h \rangle}} = \frac{f + h}{\|f\| \sqrt{1 + 2 \frac{\langle f, h \rangle}{\|f\|^2}}} \approx (f + h)(\frac{1}{\|f\|} - \frac{\langle f, h \rangle}{\|f\|^2} h) \approx \frac{f}{\|f\|} + \frac{h}{\|f\|} - \frac{\langle f, h \rangle}{\|f\|^2} h f.$$

By Definition 22 the lemma follows. \hfill \Box

To be able to compute the variation of a functional composed on the normalization operator, in the next lemma we first formulate a version of the chain rule.

**Lemma 24.** Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{F}$, where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$. Let $L : \text{Dom}(S) \rightarrow \mathbb{F}$ be a functional over the domain $\text{Dom}(S) \subset \mathcal{H}$. Suppose that $\text{Dom}(S) \cup \{0\}$ is a dense linear subspace of $\mathcal{H}$. Suppose that there exist a functional $H : \text{Dom}(H) \rightarrow \mathbb{F}$, where $\text{Dom}(H) \cup \{0\}$ is a dense linear subspace of $\mathcal{H}$, and an operator $T : \text{Dom}(S) \rightarrow \text{Dom}(H)$, such that for every $f \in \text{Dom}(S)$

$$S(f) = H(T(f)).$$

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In addition, suppose that the variations $\frac{\delta H}{\delta f}(T(f))$ and $\frac{\delta T}{\delta f}(f)$ exist for some $f \in \text{Dom}(S)$. Then, the variation $\frac{\delta S}{\delta f}(f)$ exists and satisfies

$$\frac{\delta S}{\delta f}(f) = \left(\frac{\delta T}{\delta f}(f)\right)^* \left(\frac{\delta H}{\delta t}(T(f))\right),$$

where the $\left(\frac{\delta T}{\delta f}(f)\right)^*$ is the adjoint operator of $\frac{\delta T}{\delta f}(f)$.

Proof. Compute

$$S(f + h) - S(h) = H(T(f + h)) - H(T(f))$$

$$= \left\langle \frac{\delta H}{\delta t}(T(f)), T(f + h) - T(f) \right\rangle + \epsilon(h)$$

$$= \left\langle \frac{\delta H}{\delta t}(T(f)), \frac{\delta T}{\delta f}(f)h + E(h) \right\rangle + \epsilon(h)$$

$$= \left\langle \frac{\delta T^*}{\delta f}(f) \frac{\delta H}{\delta t}(T(f)), h \right\rangle + \left\langle \frac{\delta H}{\delta t}(T(f)), E(h) \right\rangle + \epsilon(h),$$

where the error-terms $E(h) = o(||h||)$, and $\epsilon(h) = o(||T(f + h) - T(f)||)$. Then, by Cauchy–Schwarz inequality we have $\frac{\delta H}{\delta t}(T(f)), E(h) = o(||h||)$. Thus, it remains to show that $o(||T(f + h) - T(f)||)$ implies $o(||h||)$. Since the variation $\frac{\delta T}{\delta f}$ exists, it follows that $T(f + h) - T(f) = \frac{\delta T}{\delta f}(f)h + o(||h||)$.

Now, since by definition $\frac{\delta T}{\delta f}$ is a bounded operator, we have

$$||T(f + h) - T(f)|| \leq \left\|\frac{\delta T}{\delta f}\right\| ||h|| + o(||h||),$$

so $o(||T(f + h) - T(f)||) = o(||h||)$.

\[\Box\]

Corollary 25. Let $\mathcal{H}$ be a real separable Hilbert space. Let $S : \text{Dom}(S) \rightarrow \mathbb{R}$ be a functional over the domain $\text{Dom}(S) \subset \mathcal{H}$. Suppose that there exists a functional $H : \text{Dom}(H) \rightarrow \mathbb{R}$ such that $S(f) = H(f/f\|f\|)$, where $\text{Dom}(H) \cup \{0\}$ is a dense linear subspace of $\mathcal{H}$. Then, for every non-zero $f \in \text{Dom}(H)$ such that the variation $\frac{\delta H}{\delta y}(f/f\|f\|)$ exists, the variation of $S$ exists at $f$ and is equal to

$$\frac{\delta S}{\delta f} = \frac{1}{\|f\|} \frac{\delta H}{\delta y}(f/f\|f\|) = \left\langle \frac{\delta H}{\delta y}(f/f\|f\|), f/f\|f\| \right\rangle \frac{f}{\|f\|}.$$

Proof. By Lemma 23, the variation $\frac{\delta T}{\delta f}$ exists for every non-zero $f$, and is a self-adjoint operator. Thus, by Lemma 24

$$\frac{\delta S}{\delta f} = \frac{\delta T}{\delta f} \left(\frac{\delta H}{\delta y} \frac{f}{\|f\|}\right).$$

Plugging in the formula (33) for $\frac{\delta T}{\delta f}$ yields the result.

\[\Box\]

5.4 Variations of Localization Measures

We now derive formulas for the variations localization measures based on observables.

Throughout this section, $T$ denotes a symmetric, possibly unbounded, operator in a separable Hilbert space $\mathcal{H}$. The domain of $T$, denoted by $\text{Dom}(T)$, is assumed to be a dense linear subspace of $\mathcal{H}$, and the variations are computed with respect to the realification $\mathcal{H}^\mathbb{R}$. First, we compute the variation of the expected value of a (generally) non-normalized vector with respect to $T$. 

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Lemma 26. Consider a Hilbert space $\mathcal{H}$ and its realification $\mathcal{H}^\mathbb{R}$. Let $T$ be a self-adjoint operator on $\mathcal{H}$, and consider the functional $f \mapsto e_f(T)$. Then, the variation $\frac{\delta e_f(T)}{\delta f}$ with respect to $\mathcal{H}^\mathbb{R}$ exists in the domain of $T$, and satisfies

$$\frac{\delta e_f(T)}{\delta f} = 2Tf.$$  \hspace{1cm} (34)

**Proof.** Compute,

$$e_{f+h}(T) = \langle (f + h, f + h) = e_f(T) + 2\operatorname{Re} \langle Tf, h \rangle + o(h),$$

where the $o$ notation is with respect to $\operatorname{Dom}(T) \ni h \to 0$. \hfill $\square$

Next, we compute the variation of the variance with respect to (generally) non-normalized vectors. For that, for an operator $T$ with domain $\operatorname{Dom}(T)$ in the Hilbert space $H$, the operator $T^2$ is defined as the composition of $T$ with itself over the domain

$$\operatorname{Dom}(T^2) = \{ f \in \operatorname{Dom}(T) \mid Tf \in \operatorname{Dom}(T) \}.$$  

**Lemma 27.** Consider a Hilbert space $\mathcal{H}$ and its realification $\mathcal{H}^\mathbb{R}$. Let $T$ be a self-adjoint operator on $\mathcal{H}$, and consider the functional $f \mapsto v_f(T)$ defined over $\operatorname{Dom}(T)$. Then, the variation $\frac{\delta v_f(T)}{\delta f}$ with respect to $\mathcal{H}^\mathbb{R}$ exists for every $f \in \operatorname{Dom}(T^2)$, and satisfies

$$\frac{\delta v_f(T)}{\delta f} = 2(T - e_f(T))^2 f.$$  

**Proof.** By Definition 9 and the fact $T$ is symmetric, for $f \in \operatorname{Dom}(T^2)$

$$v_f(T) = \langle (T - e_f(T))^2 f, f \rangle.$$  

Substituting $f \mapsto f + h$ we get,

$$v_{f+h}(T) = \langle (T - e_{f+h}(T))^2 (f + h), f + h \rangle$$

$$= \langle (T - e_{f+h}(T))^2 f, f \rangle$$

$$+ 2\operatorname{Re} \left( \langle (T - e_{f+h}(T))^2 f, h \rangle + o(h) \right)$$

$$= \left( T - e_f(T) - 2 \langle Tf, h \rangle \right)^2 f, f \right)$$

$$+ 2\operatorname{Re} \left( \left( T - e_f(T) - 2 \langle Tf, h \rangle \right)^2 f, h \right) + o(h)$$

$$= v_f(T) + 2 \langle Tf, h \rangle + 2\operatorname{Re} \left( \left( T - e_f(T) \right)^2 f, h \right) + o(h)$$

$$= v_f(T) + 2 \left( T - e_f(T) \right)^2 f, h \rangle + o(h).$$

\hfill $\square$

From Corollary 25 and Lemmas 26 and 27, we directly deduce the following.

**Proposition 28.** Let $\mathcal{H}$ be a Hilbert space with realification $\mathcal{H}^\mathbb{R}$. Let $T$ be a self-adjoint operator in $\mathcal{H}$, and consider the functionals $f \mapsto e_{\frac{T}{\|T\|}}(T)$ and $f \mapsto v_{\frac{T}{\|T\|}}(T)$ defined on $\operatorname{Dom}(T) \setminus \{0\}$.

1. The variation of $e_{\frac{T}{\|T\|}}(T)$ with respect to $\mathcal{H}^\mathbb{R}$ exists for every $f \in \operatorname{Dom}(T) \setminus \{0\}$ and

$$\frac{\delta e_{\frac{T}{\|T\|}}(T)}{\delta f} = 2T - \frac{f}{\|f\|^2}$$

$$- 2e_{\frac{T}{\|T\|}}(T) \frac{f}{\|f\|^2}.$$  

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2. The variation of $\frac{\delta v}{\delta f}$ with respect to $\mathcal{H}^R$ exists for every $f \in \text{Dom}(T^2) \setminus \{0\}$ and

$$\frac{\delta v}{\delta f} = 2(T - e^{-Tf})^2 f \frac{f}{\|f\|^2} - 2v_{eff}(T) \frac{f}{\|f\|^2}.$$ 

Last, the following lemma is proved similarly to Lemma 27.

**Lemma 29.** Let $\mathcal{H}$ be a separable Hilbert space with realization $\mathcal{H}^R$, and $T$ a self-adjoint operator in $\mathcal{H}$ over the domain $\text{Dom}(T)$. The variation of $\|Tf\|^2$ with respect to $\mathcal{H}^R$ exists at every $f \in \text{Dom}(T^2)$, and satisfies

$$\frac{\delta \|Tf\|^2}{\delta f} = 2T^2f. \quad (35)$$

## 6 Optimizing the Uncertainty Minimization

In this section we use the calculus of variation results from Section 5 for finding a local minimum of the Minimization Problem 8.

### 6.1 Variations in Windows and Signals

Some of the terms in the uncertainty $\mathcal{L}(\hat{f})$ are defined with the inner product of $\mathcal{S}$, and some with that of $\mathcal{W}$. It is thus useful to understand the relation between the variation of a functional $S$ with respect to the realificated signal space $\mathcal{S}$, and its variation with respect to $\mathcal{W}$ (see Definition 20). We denote the variations of $S$ with respect to $\mathcal{S}$ and $\mathcal{W}$ by $\frac{\delta S}{\delta f}$ and $\frac{\delta W}{\delta f}$ respectively.

**Lemma 30.** Let $S : \text{Dom}(S) \to \mathbb{C}$ be a functional, where $\text{Dom}(S) \cup \{0\}$ is a dense linear subspace of $\mathcal{W} \cap \mathcal{S}$. Let $f \in \text{Dom}(S)$. Then, if the variation $\frac{\delta S}{\delta f}$ exists at $f$, and satisfies $\omega \frac{\delta S}{\delta f} \in \mathcal{W}$, then $\frac{\delta W}{\delta f}$ exists at $f$ and

$$\frac{\delta W}{\delta f} = \omega \frac{\delta S}{\delta f}. \quad (36)$$

**Proof.** From Proposition 18 $\frac{\delta S}{\delta f}$ exists if and only if

$$S(f + \hat{h}) - S(f) = \text{Re} \left\langle \frac{\delta S}{\delta f}, \hat{h} \right\rangle_S + o(\|\hat{h}\|_S), \quad (37)$$

where the $o$ notation is with respect to $\text{Dom}(S) \ni h \to 0$. Note that for a function $\hat{g} \in \mathcal{W} \cap \mathcal{S}$, such that $\omega \hat{g}$ is in $\mathcal{W}$, it holds

$$\left\langle \hat{g}, \hat{h} \right\rangle_{W^W} = \text{Re} \left( \int_{\mathbb{R}} \hat{g}(\omega) \hat{h}(\omega) d\omega \right) = \left\langle \omega \hat{g}, \hat{h} \right\rangle_{W^W}. \quad (36)$$

Thus, (36) is equivalent to

$$S(f + h) - S(f) = \text{Re} \left\langle \omega \frac{\delta S}{\delta f}, \hat{h} \right\rangle_W + o(\|\hat{h}\|_W), \quad (37)$$

which, by Proposition 18 is equivalent to the existence of $\frac{\delta W}{\delta f}$ and its equality to $\omega \frac{\delta S}{\delta f}$. Here, note that $o(\|\hat{h}\|_W) = o(\|\hat{h}\|_S)$ for any $h \in \mathcal{W} \cap \mathcal{S}$, since the denominator in the $o$-notation of Definition 17 (2) satisfies $\|h\|_W \geq \|h\|_S \frac{\|h\|_W}{\|h\|_S}$. 

The above lemma can be viewed as a formula for computing the variation of the functional $S$ with respect to the window $f$, given a formula for its variation with respect to $\hat{f}$ as a signal.
6.2 The Unconstrained Variation of the Uncertainty

In this section, we compute the variations of the functional \( \mathcal{L} \). We present the results under the assumptions that \( e_f(i \frac{\partial}{\partial \omega}) = e_f(\ln(|\omega|)) = 0 \), where in the mapping \( f \mapsto f + h \) underlying the definition of the variation, \( f + h \) is not restricted to lie in the constraint \( e_f(i \frac{\partial}{\partial \omega}) = e_f(\ln(|\omega|)) = 0 \).

We first define a subset of \( \text{Dom}_{W \cap S}(\mathcal{L}) \) in which the variation of \( \mathcal{L} \) exists. This is done by combining the requirements from Proposition 28 and 30 applied on the different components of (31).

**Definition 31.** The domain \( \text{Dom}_{W \cap S}(\mathcal{L}) \subset W \cap S \) is defined as the set of all \( 0 \neq \hat{f} \in L^2(\mathbb{R}) \) such that

1. \( \hat{f} \) is differentiable in every compact interval with absolutely continuous derivative, and
   \[
   \frac{\partial^2}{\partial \omega^2} \hat{f} \in L^2(\mathbb{R}), \quad \sqrt{|\omega|} \frac{\partial^2}{\partial \omega^2} \hat{f} \in L^2(\mathbb{R}) \quad \text{and} \quad \left( \sqrt{|\omega|} \frac{\partial^2}{\partial \omega^2} + \sqrt{|\omega|} \frac{\partial}{\partial \omega} \right) \hat{f} \in L^2(\mathbb{R})
   \]
2. \( \omega \mapsto \frac{1}{\omega} \hat{f}(\omega) \in L^2(\mathbb{R}) \)
3. \( \omega \mapsto \sqrt{|\omega|} \ln(|\omega|) \hat{f}(\omega) \in L^2(\mathbb{R}) \).

**Theorem 32.** The variation of the functional \( \mathcal{L} \) with respect to \( W^R \) exists at every \( \hat{f} \in \text{Dom}_{W \cap S}(\mathcal{L}) \), and for \( \hat{f} \) satisfying \( e_f(i \frac{\partial}{\partial \omega}) = e_f(\ln(|\omega|)) = 0 \), is given by

\[
\frac{\delta \mathcal{L}}{\delta \hat{f}} = \frac{\delta \mathcal{L}}{\delta \hat{f}} \left( \frac{v_V}{\|\hat{f}\|_W} \right) (A) + \frac{\delta \mathcal{L}}{\delta \hat{f}} \left( \frac{v_V}{\|\hat{f}\|_W} \right) (B),
\]

where,

\[
\frac{\delta \mathcal{L}}{\delta \hat{f}} \left( \frac{v_V}{\|\hat{f}\|_W} \right) (A) = 2\omega \left( \ln(|\omega|)^2 - v^S \frac{\partial}{\|\hat{f}\|_W} \ln(|\omega|) \right) \frac{\hat{f}}{\|\hat{f}\|_W}^2
\]

and

\[
\frac{\delta \mathcal{L}}{\delta \hat{f}} \left( \frac{v_V}{\|\hat{f}\|_W} \right) (B) = 2\omega \left( -\omega^2 \frac{\partial^2}{\partial \omega^2} - v^S \frac{\partial}{\|\hat{f}\|_W} \right) \frac{\hat{f}}{\|\hat{f}\|_W}^2 + 2 \left( -\omega^2 \frac{\partial^2}{\partial \omega^2} - \omega \frac{\partial}{\|\hat{f}\|_W} \left( i\omega \frac{\partial}{\partial \omega} \right) \right) \frac{\|\hat{f}\|_W^2}{\|\hat{f}\|_W} \frac{\hat{f}}{\|\hat{f}\|_W}^2.
\]

**Proof.** The formulas are proven directly by applying Proposition 28, Lemma 29, Lemma 30, the product rule, and the fact that \( e_f(i \frac{\partial}{\partial \omega}) = 0 \), which follows from \( e^S_f \left( i \frac{\partial}{\partial \omega} \right) = 0 \).

\[\square\]
6.3 Variations Under Expected Value Constraints

Our next goal is to add the constraints \( e_f^\ast(\ln(\omega)) = e_f^\ast(i \frac{\partial}{\partial x}) = 0 \) to the variational analysis. In the previous section, we calculated the unconstrained variations of \( \mathcal{L} \). Informally, adding the constraints to the variation involves formulating the Lagrange multipliers, namely, projecting the unconstrained variation \( \delta \mathcal{S}^\ast \) to the tangent space of the constraint surface. Intuitively, taking an infinitesimal step along the constrained variation preserves the constraints.

The general Lagrange multipliers formulation is given by the following definition and theorem.

**Definition 33.** Let \( S, c_1 \) and \( c_2 \) be densely defined functionals in the Hilbert space \( \mathcal{H} \). Let \( f \) be in the domains of \( S, c_1 \) and \( c_2 \), and suppose that the variations of \( S, c_1 \) and \( c_2 \) exist at \( f \), and that \( c_1(f) = c_2(f) = 0 \). The constrained variation of \( S \) at \( f \), under the constraints \( c_1(f) = c_2(f) = 0 \), is defined to be

\[
\delta S^\text{cons} \quad \frac{\delta S}{\delta f} := \frac{\delta S}{\delta f} - \lambda \frac{\delta c_1}{\delta f} - \mu \frac{\delta c_2}{\delta f}
\]

where \( \lambda \) and \( \mu \), called the Lagrange multipliers associated to the constraints, are scalars that satisfy

\[
\langle \frac{\delta S^\text{cons}}{\delta f}, \frac{\delta c_1}{\delta f} \rangle_{\mathcal{H}} = 0, \quad \langle \frac{\delta S^\text{cons}}{\delta f}, \frac{\delta c_2}{\delta f} \rangle_{\mathcal{H}} = 0.
\]

In the above definition, if \( \frac{\delta c_1}{\delta f} \) and \( \frac{\delta c_2}{\delta f} \) are not co-linear, equation (41) has a unique solution. If \( \frac{\delta c_1}{\delta f} \) and \( \frac{\delta c_2}{\delta f} \) are co-linear, then \( \lambda \) and \( \mu \) are not unique, but \( \delta S^\text{cons} \) is. Also, note that \( \lambda \) and \( \mu \) depend on \( f \), as \( \frac{\delta c_1}{\delta f}, \frac{\delta c_2}{\delta f} \) and \( \frac{\delta S}{\delta f} \) do. One motivation for the above definition comes from the following extreme value theorem, which is restricted to expected value constraints. For the theorem, given self-adjoint operators \( T_1 \) and \( T_2 \) with domains \( \text{Dom}(T_1) \) and \( \text{Dom}(T_2) \) respectively, we denote by \( \text{Dom}(T_1 T_2) \) the set of all \( f \in \text{Dom}(T_2) \) such that \( T_2 f \in \text{Dom}(T_1) \).

**Theorem 34** (Fermat’s Extreme Value Theorem Under Constraints). Let \( S \) be a densely defined functional in the real Hilbert space \( \mathcal{H} \). Let \( T_1 \) and \( T_2 \) be self-adjoint operators in \( \mathcal{H} \), and consider the expected value constraints \( e_f(T_1) = e_f(T_2) = 0 \). Denote the domains of \( S, T_1 \) and \( T_2 \) respectively by \( \text{Dom}(S), \text{Dom}(T_1) \) and \( \text{Dom}(T_2) \). Let \( f_0 \in \text{Dom}(S) \cap \text{Dom}(T_1^2) \cap \text{Dom}(T_1 T_2) \cap \text{Dom}(T_2^2) \) and suppose that the variation of \( S \) exists at \( f_0 \), \( e_f(T_1) = e_f(T_2) = 0 \), and \( T_1 f_0 \) and \( T_2 f_0 \) are not co-linear.

If \( S \) attains a local minimum in the domain \( e_f(T_1) = e_f(T_2) = 0 \) at \( f_0 \), then there exist scalars \( \lambda \) and \( \mu \) such that

\[
\frac{\delta S}{\delta f}(f_0) - \lambda T_1 f_0 - \mu T_2 f_0 = 0.
\]

It is trivial to see that \( \lambda \) and \( \mu \) in the above theorem are Lagrange multipliers (satisfying (41)). In the following, we offer a proof of the one constraint counterpart of Theorem 34. Namely, under the constraint \( e_f(T) = 0 \), we show that any local minimum point of \( S \) at \( f \), where \( f \in \text{Dom}(S) \cap \text{Dom}(T^2) \), \( T f \neq 0 \), and where \( \frac{\delta S}{\delta f}(f) \) exists, must satisfy

\[
\frac{\delta S}{\delta f}(f) - \lambda T f = 0
\]

for some \( \lambda \). The general case is shown similarly.

**Proof.** We denote in short the expected value functional by \( e : f \mapsto e_f(T) \), and note that \( \frac{\delta}{\delta f} e(f) \) exists and is equal to \( 2T f \) by Lemma 26.

Let \( f \) be as in the theorem. Suppose that there is no \( \lambda \) that satisfies (43). In this proof, we build a new point \( q \neq f \) on the constraint \( e_q(T) = 0 \) for which \( S(q) < S(f) \), which contradicts the assumption. In the following, we construct such a \( q \) of the form

\[
q = f - \epsilon \frac{\delta}{\delta f} S + (\lambda(\epsilon) + k)2T f,
\]

for some \( \lambda(\epsilon) \) and \( k \) depending on \( \epsilon \). The details of this construction are similar to the proof of Lemma 26.

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with

\[ \lambda(\epsilon) = \frac{\epsilon \left( \frac{\delta S}{\delta f} S, 2Tf \right)}{\|2Tf\|^2} \]

for small enough nonzero \( \epsilon \) and \( k = o(\epsilon) \). Here, indeed \( q \neq f \), since (44) has \( q - f \) of the form \( \epsilon\left( \frac{\delta S}{\delta f} + \lambda \frac{\delta S}{\delta f} \right) \), which must be nonzero by assumption.

First, we show the existence of \( k \) such that \( q \) is on the constraint \( e(q) = 0 \).

\[ e(q) = \left\langle T\left(f - \epsilon \frac{\delta}{\delta f} S + (\lambda(\epsilon) + k)2Tf\right), f - \epsilon \frac{\delta}{\delta f} S + (\lambda(\epsilon) + k)2Tf \right\rangle \\
= e(f) + 2 \left\langle Tf, -\epsilon \frac{\delta}{\delta f} S + (\lambda(\epsilon) + k)2Tf \right\rangle \\
+ \left\langle T\left(-\epsilon \frac{\delta}{\delta f} S + (\lambda(\epsilon) + k)2Tf\right), -\epsilon \frac{\delta}{\delta f} S + (\lambda(\epsilon) + k)2Tf \right\rangle. \tag{45} \]

Let us compute the different terms in the right-hand-side of (45).

\[ 2 \left\langle Tf, -\epsilon \frac{\delta}{\delta f} S + (\lambda(\epsilon) + k)2Tf \right\rangle \\
= 2 \left\langle Tf, -\epsilon \frac{\delta}{\delta f} S + \frac{\epsilon \left( \frac{\delta S}{\delta f} S, 2Tf \right)}{\|2Tf\|^2} \right\rangle + 2 \left\langle Tf, 2k2Tf \right\rangle \\
= 4k \left\langle Tf, Tf \right\rangle. \tag{46} \]

Now, the second term of (45) satisfies

\[ \left\langle T\left(-\epsilon \frac{\delta}{\delta f} S + (\lambda(\epsilon) + k)2Tf\right), -\epsilon \frac{\delta}{\delta f} S + (\lambda(\epsilon) + k)2Tf \right\rangle \\
= \left\langle T\left(-\epsilon \frac{\delta}{\delta f} S + \lambda(\epsilon)2Tf\right), -\epsilon \frac{\delta}{\delta f} S + \lambda(\epsilon)2Tf \right\rangle \\
+ 2 \left\langle T\left(-\epsilon \frac{\delta}{\delta f} S + \lambda(\epsilon)2Tf\right), 2k2Tf \right\rangle + \left\langle 2kT^2 f, 2kTf \right\rangle. \tag{47} \]

Thus, combining (46) with (47), we can write

\[ e(q) = ak^2 + (b + \epsilon c)k + de^2. \tag{48} \]

where \( a, b \) and \( c \) are constants that do not depend on \( \epsilon \). By the fact that \( f \) is not in the kernel of \( T \), we must have \( b \neq 0 \). For small enough \( \epsilon \), there is hence a solution \( k \) to the equation \( e(q) = ak^2 + (b + \epsilon c)k + de^2 = 0 \) with \( k = O(\epsilon^2) \). Hence \( q \) satisfies the constraint \( e(q) = 0 \) for any small enough \( \epsilon \) with a corresponding choice of \( k = o(\epsilon) \).

We next show that \( S(q) < S(f) \) for some choice of \( \epsilon \). For small enough \( \epsilon \), the \( k \)-term in the definiton (44) of \( q \) cannot cancel the \( \epsilon \) and \( \lambda(\epsilon) \) terms, since \( k = o(\epsilon) \). We hence we get, by Proposition 18,

\[ S(q) = S(f) + \left\langle \frac{\delta S}{\delta f}, -\epsilon \frac{\delta}{\delta f} S + \lambda(\epsilon)2Tf \right\rangle + \left\langle \frac{\delta}{\delta f} S, 2Tf \right\rangle + o(\epsilon). \]

where first term does not vanish by the non-colinearity assumption of \( Tf \) and \( \frac{\delta S}{\delta f} S(f) \), and the second and third terms can be made small enough relative to the first term by choosing small enough \( \epsilon \). Hence, there exists a small as we wish \( \epsilon \) such that \( S(q) < S(f) \) and \( e(q = 0) \), so \( f \) is not a local minimum.

Similarly to the above proof, we can also show the following. Given a point on the constraint, by taking a small enough step in the direction of \( \frac{\delta S}{\delta f} \), and projecting the result to the constraint \( e_f(T_1) = e_f(T_2) = 0 \), we can decrease the value of the functional \( S \), unless \( \frac{\delta S}{\delta f} \) is zero. This is the basis of the gradient descent approach for finding local minima.
6.4 The Variation of the Uncertainty Under the Expected Value Constraints

Next, we give an explicit formula for \( \frac{\delta W L^{\text{cons}}}{\delta W f} \). This is a direct result of Lemma 26 and 41.

**Proposition 35.** The constrained variation of \( L \) with respect to \( W^R \) exists at every \( \hat{f} \in \text{Dom}^{4}_{W \cap S}(L) \), and satisfies

\[
\frac{\delta W L^{\text{cons}}}{\delta W f} = \frac{\delta W L}{\delta W f} - \lambda \ln(\omega)|\hat{f} - \mu i \frac{\partial}{\partial \omega}\hat{f},
\]

where the Lagrange multipliers \( \mu \) and \( \lambda \) satisfy

\[
\begin{pmatrix}
\lambda \\
\mu
\end{pmatrix} = 
\begin{pmatrix}
\|\ln(\omega)|\hat{f}\|^2_W & \Re \langle i \frac{\partial}{\partial \omega}\hat{f}, \ln(\omega)|\hat{f}\rangle_W \\
\Re \langle \ln(\omega)|\hat{f}, i \frac{\partial}{\partial \omega}\hat{f}\rangle_W & \|\frac{\partial}{\partial \omega}\hat{f}\|^2_W
\end{pmatrix}^{-1}
\begin{pmatrix}
\Re \langle \frac{\delta W L}{\delta W f}, \ln(\omega)|\hat{f}\rangle_W \\
\Re \langle \frac{\delta W L}{\delta W f}, i \frac{\partial}{\partial \omega}\hat{f}\rangle_W
\end{pmatrix}.
\]  

(50)

In the special case, where \( \hat{f} \) is real valued, formula (50) can be simplified.

**Corollary 36.** The constrained variation of \( L \) with respect to \( W^R \) at a real valued \( \hat{f} \in \text{Dom}^{4}_{W \cap S}(L) \) is given by

\[
\frac{\delta W L^{\text{cons}}}{\delta W f} = \frac{\delta W L}{\delta W f} - \frac{\Re \langle \frac{\delta W L}{\delta W f}, \ln(\omega)|\hat{f}\rangle_W}{\|\ln(\omega)|\hat{f}\|^2_W} \ln(\omega)|\hat{f}.
\]

In particular, \( \frac{\delta W L^{\text{cons}}}{\delta W f} \) is real valued as well.

**Proof.** By Theorem 32 for a real valued \( \hat{f} \), both \( \frac{\delta L}{\delta f} \) and \( \ln(\omega)|\hat{f} \) are real valued. In addition, \( i \frac{\partial}{\partial \omega}\hat{f} \) takes values in \( i\mathbb{R} \). Thus,

\[
\Re \langle i \frac{\partial}{\partial \omega}\hat{f}, \ln(\omega)|\hat{f}\rangle_W = \Re \left\langle \frac{\delta L}{\delta f}, i \frac{\partial}{\partial \omega}\hat{f}\right\rangle_W = 0.
\]

As a result, by (50), \( \mu = 0 \) and \( \lambda = \frac{\Re \langle \frac{\delta W L}{\delta W f}, \ln(\omega)|\hat{f}\rangle_W}{\|\ln(\omega)|\hat{f}\|^2_W} \). \( \square \)

7 Numerical Results

We illustrate the uncertainty minimization problem numerically by implementing a discrete gradient descent algorithm. The scheme is based on discretizing the frequency line on a grid, and implementing a discrete version of the constrained variation of Corollary 36. The derivatives are discretized via central difference. We initialize the window as a Gaussian \( f_0 \), translated in the \( y \) axis to satisfy \( f_0(0) = 0 \), and zeroed out for negative values. The initial \( f_0 \) is chosen with expected values of time and scale equal to 0. We call such an \( f_0 \) a truncated Gaussian.

We choose the variance of the initial Gaussian \( f_0 \) optimally – to minimize the uncertainty over the family of truncated Gaussians. This means that the optimization process only changes the shape of the Gaussian, and not its first and second moments. In Figure 1 we show the initial condition, and in Figure 2 the optimal window. The uncertainty 11.427 of the initial condition improves to 8.0619 in the optimal window (see Figure 3). Note that the shape of the optimal window is similar to the shape of the initial condition. Hence, from a signal-processing/feature-extraction point of view, both windows are reasonable and roughly measure the same features. However, the shape of the uncertainty minimizing window is optimized for best phase space localization.
Figure 1: Initial window in time, frequency, and phase space. To make the localization more visible in phase space, we show $|\langle B-e_{K(B)} \rangle^2 K(a,b,c)|$ in addition to $|K(a,b,c)|$, where $K := V_f(f)$ is the ambiguity function.

8 Summary

In this work, we approached the task of constructing windows for the 1D wavelet transform from the uncertainty minimization point of view. We based the approach on localization via observables, as in [20]. In contrast to [20], we defined an observable-based uncertainty directly in phase space. Our proposed optimal windows promote sparsity in phase space in a non-asymptotic manner. Basing the computations on the wavelet-Plancherel theorem enabled us to formulate the 2D phase space variances as a combination of 1D signal and window localization measures. This allowed us to optimize the 1D window directly, as opposed to optimizing a general 2D function in phase space, which would require complicated constraints for restricting the function to be an ambiguity function. While we studied the 1D wavelet transform in this paper, our technique can be seen as a step-by-step guide for computing optimal windows for every generalized wavelet transform based on a semi-direct product of physical quantities (see [20, Section 3.3]), like, for example, the Shearlet transform.

References

[1] S.T Ali, J.P. Antoine, and J.P. Gazeau. Canonical Coherent States, pages 13–32. Springer New York, New York, NY, 2000.

[2] W. Alltop. Complex sequences with low periodic correlations. Information Theory, IEEE Transactions on, 26:350–354, 06 1980.

[3] J.P. Antoine, R. Murenzi, P. Vanderheyden, and S.T. Ali. Two-Dimensional Wavelets and their Relatives. Cambridge University Press, 2004.
Figure 2: **Final window** in time, frequency, and phase space. To make the localization more visible in phase space, we show $|(B-e_{K}(B))^2 K(a,b,c)|$ in addition to $|K(a,b,c)|$, where $K := V_f(f)$ is the ambiguity function.

Figure 3: Phase space uncertainty per gradient descent iteration.
[4] E.J. Candès and D.L. Donoho. Continuous curvelet transform: I. resolution of the wavefront set. *Applied and Computational Harmonic Analysis*, 19(2):162–197, 2005.

[5] J. Chapa and R. Rao. Algorithms for designing wavelets to match a specified signal. *Signal Processing, IEEE Transactions on*, 48:3395–3406, 01 2001.

[6] A. Constantin. *Fourier Analysis*, volume 1 of *London Mathematical Society Student Texts*. Cambridge University Press, 2016.

[7] J. Costas. A study of a class of detection waveforms having nearly ideal range—doppler ambiguity properties. *Proceedings of the IEEE*, 72:996–1009, 09 1984.

[8] S. Dahlke, G. Kutyniok, P. Maass, C. Sagiv, and H.G. Stark. The uncertainty principle associated with the continuous shearlet transform. In *International Journal of Wavelets, Multiresolution and Information Processing*, 2008.

[9] S. Dahlke and P. Maass. The affine uncertainty principle in one and two dimensions. In *Computers and Mathematics with Applications*, pages 30:293–305, 1995.

[10] I. Daubechies. Orthonormal bases of compactly supported wavelets. *Communications on Pure and Applied Mathematics*, 41(7):909–996, 1988.

[11] I. Daubechies. *Ten Lectures on Wavelets*. Society for Industrial and Applied Mathematics, USA, 1992.

[12] M. Doerfler and B. Torrésani. Representation of operators in the time-frequency domain and generalized gabor multipliers. *Journal of Fourier Analysis and Applications*, 16:261–293, 04 2010.

[13] H.G. Feichtinger, D. Onchis-Moaca, B. Ricaud, B. Torrésani, and C. Wiesmeyr. A method for optimizing the ambiguity function concentration. In *EUSIPCO 2012, European Signal Processing Conference*, pages 804–808, Bucarest, Romania, August 2012.

[14] H. Führ. *Abstract Harmonic Analysis of Continuous Wavelet Transforms*. Springer, 2005.

[15] K. Gröchenig. *Foundations of Time-Frequency Analysis*. Birkhäuser Basel, 2001.

[16] A. Grossmann and J. Morlet. Decomposition of hardy functions into square integrable wavelets of constant shape. *SIAM Journal on Mathematical Analysis*, 15:723–736, 07 1984.

[17] K. Guo, G. Kutyniok, and D. Labate. Sparse multidimensional representations using anisotropic dilation and shear operators. In *Wavelets and Splines*, 05 2005.

[18] M. Herman and T. Strohmer. High-resolution radar via compressed sensing. *Ieee Transactions on Signal Processing*, 57:2275–2284, 07 2009.

[19] R. Levie and N. Sochen. A wavelet plancherel theory with application to sparse continuous wavelet transform. *ArXiv*, abs/1712.02770, 2017.

[20] R. Levie and N. Sochen. Uncertainty principles and optimally sparse wavelet transforms. *Applied and Computational Harmonic Analysis*, 48(3):811–867, 2020.

[21] R. Levie, H.G. Stark, F. Lieb, and N. Sochen. Adjoint translation, adjoint observable and uncertainty principles. *Advances in Computational Mathematics*, 40, 06 2014.

[22] P. Maass, C. Sagiv, N. Sochen, and H. G. Stark. Do uncertainty minimizers attain minimal uncertainty? *Journal of Fourier Analysis and Applications*, 16(3):448–469, 2010.

[23] S. Mallat. *A Wavelet Tour of Signal Processing, Third Edition: The Sparse Way*. Academic Press, Inc., USA, 3rd edition, 2008.

[24] H.G. Stark and N. Sochen. Square integrable group representations and the uncertainty principle. *Journal of Fourier analysis and applications (JFAA)*, 17:916–931, 2011.
A Direct Computation of the Pull-Backs of Phase Space Observables

In this appendix we compute directly the pull-back formulas of the phase space observables for simple vectors $\hat{f} \otimes \hat{s}$.

A.1 Pull-Back of the Scale Window-Signal Observable

In this section we proof the formula for $\hat{T}_a(\hat{f} \otimes \hat{s})$ given in Corollary 11 in a direct manner. To that end, we formulate $aV_f(s)(a, b, c)$ as a linear combination of simple wavelet functions. From (17), we can deduce the wavelet transform in scale space

$$V_f(\hat{s})(\sigma, s) = \langle \hat{\pi}(\sigma, s) \hat{f}, \hat{s} \rangle$$

From (52) we can compute

$$aV_f(\hat{s})(\sigma, s) = \int_{-\infty}^{\infty} e^{-ibc} \left( \hat{f}(\sigma - a, \text{sign}(c)) \hat{s}(\sigma, 1) + \hat{f}(\sigma - a, \text{sign}(-c)) \hat{s}(\sigma, -1) \right) d\sigma.$$

From the definition of the scale parameter $\sigma$ we get,

$$bV_f(\hat{s})(\sigma, s) = V_f(\hat{x}s)(\sigma, b) - V_{\sigma f}(\hat{s})(\sigma, b,c).$$

Thus,

$$aV_f(\hat{s})(\sigma, s) = V_f(\hat{x}s)(\sigma, b) - V_{\sigma f}(\hat{s})(\sigma, b,c).$$

Next, we show that $ce^{-a}V_{\sigma f}(\hat{s})(\sigma, b,c)$ is a simple function.

$$ce^{-a}V_{\sigma f}(\hat{s})(\sigma, b,c) = \int_{-\infty}^{\infty} s(x)e^{-a/2}e^{-a(x-b)}f(ce^{-a}(x-b))dx$$

Thus,

$$bV_f(s)(\sigma, s) = V_f(xs)(\sigma, b,c) - ce^{-a}V_{\sigma f}(s)(\sigma, b,c).$$

Next, we show that $ce^{-a}V_{\sigma f}(s)(\sigma, b,c)$ is a simple function.

$$ce^{-a}V_{\sigma f}(s)(\sigma, b,c) = \int_{-\infty}^{\infty} s(x)e^{-a}f(ce^{-a}(x-b))dx.$$
Integrating by parts, setting $v'(x) := s(x)$, $u(x) := e^{-\frac{a}{2} (x-b)f(ce^{-a}(x-b))}$, where the prime
sign indicates derivative with respect to $x$, we get $v(x) := \int_{-\infty}^{x} s(t)dt$, i.e. $v = \int s$, which is the
anti-derivative of $s$, and $u'(x) = e^{-\frac{a}{2} \left( f(ce^{-a}(x-b)) + (x-b)ce^{-a}f'(ce^{-a}(x-b)) \right)}$.

Thus,
\[
\int_{-\infty}^{\infty} v'(x)u(x)dx = vu|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v(x)u'(x)dx = -\int_{-\infty}^{\infty} \left( \int_{-\infty}^{x} s(t)dt \right) e^{-\frac{a}{2} \left( f(ce^{-a}(x-b)) + (x-b)ce^{-a}f'(ce^{-a}(x-b)) \right)} dx.
\]

Note that since $\frac{1}{w} \hat{s} \in S$, and since $\hat{f} \in \text{Dom}_W(\hat{T}_b)$ implies $xf \in S$, the first term vanishes. The
second term satisfies
\[ e^{-\frac{a}{2} \left( f(ce^{-a}(x-b)) + (x-b)ce^{-a}f'(ce^{-a}(x-b)) \right)} = \pi(a,b,c) \frac{\partial}{\partial x}xf. \]

Therefore, the above equals $V_{\frac{a}{x}xf}(\int s)$, and the formula in the frequency domain is
\[ bV_f(s)(a,b,c) = V_f(i \frac{\partial}{\partial \omega})a,b,c - V_{i\omega} \frac{a}{x} f(1) \hat{s}(a,b,c). \]

Hence, the pull-back to the window-signal space reads
\[ \hat{T}_b(\hat{f} \otimes \hat{s}) = \hat{f} \otimes i \frac{\partial}{\partial \omega} \hat{s} - i\omega' \frac{a}{x} \hat{f} \otimes \frac{1}{\omega} \hat{s}. \]

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