Vortex dynamics and specific heat of type II superconductor with quasi-periodic geometry

S. Acharjee and U. D. Goswami

Department of Physics, Dibrugarh University, Dibrugarh 786 004, Assam, India,

The vortex dynamics and the specific heat of a type II superconducting system with quasi-periodic geometry is studied theoretically for different values of interaction parameters using the numerical simulation technique, where the vortex-vortex interaction potential is considered in the form of the modified Bessel’s function of first kind. The dynamics of the system is analysed by phase space trajectories of the vortex for both high and low values as well as for both high and low mismatch of vortex-vortex and vortex-pinning interaction parameters. The specific heat variation with temperature is analysed statistically for different values of interaction parameters. It is observed that for low values and lower mismatch of interaction parameters, the system is highly chaotic and shows a bifurcation pattern similar to Hopf bifurcation. The specific heat also shows a highly divergent character in this situation. However for high values and higher mismatch, the superconducting system tends to be a very regular one. The trajectory of the vortices will also be very stable in this situation. Similar situations are also observed respectively for low and high values of the quasi-periodic parameter.

I. INTRODUCTION

Recently the models of periodic and quasi-periodic geometry of the pinning sites of a superconductor attract a lot of attention from both the experimental and theoretical point of view. The basic reason behind this is that, the regularity and periodicity in the position of the pinning sites are highly useful to provide an idea about various properties of the superconducting system. The vortex motion not only controls the electrical and magnetic properties of a superconductor, but it provides a useful information regarding its thermal properties such as specific heat, entropy, Gibb’s free energy etc. Several ideas have been given on the periodic geometry in theoretical and on the thermodynamics of superconducting system experimentally in. While the dynamics of the vortex motion for d-wave superconductors in periodic array is analysed theoretically in. On the other hand in experimentally analysed the specific heat for different types of superconducting samples. A study of specific heat by path integral formalism is done in.

Moreover, the geometry of the pinning sites depends on the vortex pinning interaction parameters. If the pinning potential is strong enough then the vortices will arrange themselves in a square array by pinning themselves at the pinning sites. In this phase it forms a square lattice. But if the potential of the pinning becomes of the order of the elastic energy of the flux line lattice, then the vortex-vortex interaction becomes important. In this phase triangular geometry of the vortex array is achieved. However, if in the vortex lattice system, the perturbing potential is intermediate then the vortices slightly depin them from the pinning sites. In this phase the system is not exactly periodic but it is quasi-periodic. So, it is important to study theoretically the vortex motion and corresponding thermodynamics of a superconducting sample with interaction parameters. In this paper we study the vortex dynamics using the quasi-periodic geometry and the consequent thermodynamical property such as specific heat by using quantum statistical mechanics. The variation of the specific heat is studied for both stable and divergent motion of the vortex-pinning system at moderate temperatures using a quasi-periodic potential.

To study the dynamics of the vortices we consider a pure superconducting sample composed of homogeneous and isotropic lattice points with filling factor. For simplicity, here we consider that the system is free from all types of defects. At a very low temperature the vortices are arranged in a very regular manner and pinned themselves at the lattice points and attain square geometry. The system is highly periodic in this phase as shown in Fig.(a). As the system is slightly perturbed by increasing the temperature of the system slightly, the vortices started to depin themselves from the pinning sites and attains the quasi-periodic phase, which is shown in Fig.(b). With the further rise in temperature the vortices completely depinned themselves from the lattice points and attain triangular geometry. As shown in the Fig.(c), the system is highly aperiodic in such a phase. In this study, we strictly focused on the thermodynamics of the superconducting sample in this quasi-periodic phase only.

The paper is organised as follows. In the Section II, a theoretical model is developed based on earlier ideas by considering quasi-periodic geometry with a potential of the form of modified Bessel’s function of first kind. The results of our work is discussed in the Section III, where the phase portraits and specific heat variations are drawn using numerical simulation. Finally we conclude our work in the Section V.

II. THEORETICAL MODEL

In the quasi-periodic phase, where the vortices are not completely depinned themselves from the pinning sites of the system, various types of interactions present in the vortex lattice system of the superconductor are as follows:

A. Interaction of a vortex with the nearby pinning sites

For simplicity if we consider that the vortices have interactions only with the nearest pinning sites, then interaction of
FIG. 1: Three different phases of pure type II superconductor, where the blue circles represents the position of the pinning sites and black dots represents the vortices. (a) represents the periodic phase in which the vortices forms a square array. (b) represents the quasi-periodic phase in which the vortices slightly depinned themselves, and (c) represents the aperiodic phase in which the vortex completely depinned and achieved triangular geometry.

the $i^{th}$ vortex with $k^{th}$ pinning site can be written as

$$V(r_i, r_k) = C_{vp}[\sin(r_i^v - r_k^p) + \alpha],$$

(1)

where $C_{vp}$ is the vortex-pinning interaction parameter, $\alpha$ is a constant factor which gives the order of quasi periodicity. $r_i^v$ and $r_k^p$ are respectively be the positions of $i^{th}$ vortex and $k^{th}$ pinning site.

B. Interaction of two nearby vortices

Two nearby vortices are interacting in such a way that the interaction is rapidly decreasing as we move away from the vortex point. So we may represent it by using the modified Bessel’s Functions of 1$^{st}$ kind. The interaction of the $i^{th}$ vortex with the nearby $j^{th}$ vortex is thus given by

$$V(r_i, r_j) = C_{vv}K_0(r_i^v - r_j^v),$$

(2)

where $r_i$ and $r_j$ are the position of $i^{th}$ and $j^{th}$ vortices respectively. $C_{vv}$ is the vortex-vortex coupling parameter and $K_0$ is the modified Bessel’s function of first kind. A plot of such potential is shown in the Fig.2.

C. Magnetic interaction and Lorentz term

The Lorentz interaction term arises due to spin motion of the charged vortices and also due to the applied field. However, at moderate temperature the contribution due to spin term is very small in comparison to the Lorentz term and hence can be neglected. The Lorentz interaction term can be written as

$$V_L = V_{LO} \sin \omega t,$$

(3)

where $V_{LO}$ is a factor which depends on the applied field, velocity of the vortices and also on its charge.

D. Temperature contribution

Due to thermal agitation there must have some temperature contribution on the vortex motion, which can be written by some stochastic term of the form as

$$<f^T_i(t)f^T_j(t')> = 2\eta K_B T \delta_{ij} \delta(t - t'),$$

(4)

which obeys the condition

$$<f^T_i(t)> = 0,$$

where $\eta$ is the vortex viscosity and responsible for dissipation in the vortex-pinning system. Here for simplicity we consider the value of $\eta$ as unity. If $N_v/p$ is the total number of vortices or pinning sites of the system then the temperature contribution on the vortex motion of the system may be approximated as $2N_v/p K_B T$. Thus the Hamiltonian of the system can be written as
where $r_{ik} = r_i - r_k$. The last two terms represents the kinetic energies of the vortex-pinning and vortex-vortex systems respectively. $\mu_{pv}$ and $\mu_{vv}$ are their respective reduced masses. However, for the superconducting state of a system the kinetic terms are negligible in magnitude, so we may ignore them in the derivation of other measurable quantities. The equation of motion of the $i^{th}$ vortex in the background of all possible interactions is

$$
\frac{d^2 \psi}{dt^2} + \eta \frac{d\psi}{dt} = -\sum_{k=1}^{N_{v/p}} C_{vp}[\cos(r_{ik}^p) + \alpha] - \sum_{i \neq j}^{N_{v/p}-1} C_{vv}K_0(r_{ij}^v) + 2N_{v/p}K_BT + V_{LO}\sin\omega t - \frac{\hbar^2}{2\mu_{pv}} \sum_{i,k=1}^{N_{v/p}} \nabla^2_{pv} - \frac{\hbar^2}{2\mu_{vv}} \sum_{i \neq j}^{N_{v/p}-1} \nabla^2_{vv},
$$

(5)

where $\psi$ is the displacement of $i^{th}$ vortex from its balanced position, $K_1 = \nabla K_0$ and $f^T$ is the temperature contribution term. The term $f_{LO} = qv\omega B$ depends on velocity of the vortices and on the applied field. It should be noted that the contribution from the temperature term $f^T$ has negligible con-

tribution to the dynamics of the system at temperature of the superconducting phase.

With the form of equation (5) the partition function of this system takes the form

$$
Q = \sum \exp \left[ -\beta \left\{ \sum_{i,k=1}^{N_{v/p}} C_{vp}[\sin(r_{ik}^p) + \alpha] + \sum_{i \neq j}^{N_{v/p}-1} C_{vv}K_0(r_{ij}^v) + 2N_{v/p}K_BT + V_{LO}\sin\omega t - \frac{\hbar^2}{2\mu_{pv}} \sum_{i,k=1}^{N_{v/p}} \nabla^2_{pv} - \frac{\hbar^2}{2\mu_{vv}} \sum_{i \neq j}^{N_{v/p}-1} \nabla^2_{vv} \right\} \right],
$$

(7)

where $\beta = \frac{1}{k_BT}$ is the Boltzmann factor.

We consider that when two vortices of opposite spin interact with each other at moderate temperature they form a fermionic pair similar to Cooper pairs. Again, when an atom lose an electron, the escaping electron imparts an opposite spin to the resulting ion to conserved the total angular momentum of the parent atom. As the pinning sites consist basically with ions, so in that sense the vortex-pinning pair may be also considered as a boson. Under these consideration the vortex-lattice system can be studied by using Bose-Einstein Statistics. Hence the equation of state for the vortex-lattice system using Bose-Einstein statistics can be written as

$$
U = \frac{\partial}{\partial \beta} \left[ \sum_n \log \{ 1 - \exp(-\beta E_n) \} \right] = \sum_n \frac{E_n \exp(-\beta E_n)}{1 - \exp(-\beta E_n)},
$$

(9)

Thus the specific heat of the vortex-lattice system is given by

$$
C_V = K_B\beta^2 \sum_n \frac{E_n^2 \exp(\beta E_n)}{[\exp(\beta E_n) - 1]^2},
$$

(10)

In general the total specific heat of this system is

$$
C_V = C_{\text{vortex-pinning}} + C_{\text{vortex-vortex}} + C_{\text{magnetic}} + C_{\text{defects}} + C_{\text{hyperfine}}.
$$

(11)
The specific heat contributions are analysed individually for various values of interaction parameters $C_{vp}$ and $C_{vv}$ by numerical simulation process. Analysis have been made for a very stable vortex and also for divergent motion. For convenience we ignore the contributions due to defects and spin as they are very small at moderate temperatures. Moreover, in this work we have not considered the magnetic contribution to specific heat of the system. We plan to take into account the magnetic contribution separately in future. The contribution due to vortex-pinning interaction is

$$C_{\text{vortex-pinning}} = \frac{K_B \beta^2 \sum_{i,k=1}^{N_{v/p}} [C_{vp}[\sin(r_{ik}^p) + \alpha]]^2 \exp{\{\beta C_{vv}[\sin(r_{ik}^p) + \alpha]\}}}{\sum_{i,k=1}^{N_{v/p}} \left[\exp{\{\beta C_{vv}[\sin(r_{ik}^p) + \alpha]\}} - 1\right]^2}. \quad (12)$$

Similarly the contribution due to vortex-vortex interaction is

$$C_{\text{vortex-vortex}} = \frac{K_B \beta^2 \sum_{i\neq j}^{N_{v/p}-1} C_{vv} K_0(r_{ij}^v)^2 \exp{\{\beta C_{vv}K_0(r_{ij}^v)\}}}{\sum_{i\neq j}^{N_{v/p}-1} \left[\exp{\{\beta C_{vv}K_0(r_{ij}^v)\}} - 1\right]^2}. \quad (13)$$

Due to the lack of mathematical techniques available to evaluate equations (12) and (13) analytically, they are calculated numerically. The contribution in specific heat due to the interaction of a vortex with the nearest pinning sites can be calculated for various values of quasi-periodic parameter $\alpha$ and also for the vortex-pinning interaction parameter $C_{vp}$. Since with the increase of $\alpha$, the system tends to become aperiodic, so for simplicity we here consider that $\alpha = 0.1$ and with this assumption the vortex pinning interaction term takes the form for the 100 iteration as

$$\sum_{i,k=1}^{100} [\sin(r_{ik}^p) + \alpha] = 1.832943108 \approx 1.83.$$

With this value, the contribution term in specific heat due to vortex-pinning interactions takes the form

$$C_{\text{vortex-pinning}} = \frac{(1.83 C_{vp} \beta)^2 K_B \exp(1.83 \beta C_{vp})}{\exp(1.83 \beta C_{vp}) - 1^2}. \quad (14)$$

Similarly, the contribution in specific heat due to the interaction of a vortex with the nearest vortex can be calculated for various values of the vortex-vortex interaction parameter $C_{vv}$. In the numerical calculation, we have found that the Bessel’s function of first kind terminates after a certain iteration with a value

$$\sum_{i,k=1}^{N_{v/p}-1} K_0(x) = 0.58640216 \approx 0.59.$$

In view of this, the specific heat contribution to the system from the vortex-vortex interaction term can be expressed as

$$C_{\text{vortex-vortex}} = \frac{(0.59 C_{vv} \beta)^2 K_B \exp(0.59 \beta C_{vv})}{\exp(0.59 \beta C_{vv}) - 1^2}. \quad (15)$$

Thus the specific heat of the system under our consideration can numerically be calculated by using the equations (14) and (15).

III. RESULTS AND DISCUSSIONS

A. Behaviour of Vortex trajectory

We have analysed the dynamics of the $i^{th}$ vortex for different values of the vortex-vortex and vortex-pinning interaction parameters by numerically solving the equation (6) without magnetic field and after neglecting the temperature contribution term. This analysis is made in terms of the phase portraits of the vortices, which are drawn for different values of $C_{vp}$ keeping $C_{vv}$ constant and for vise verse.

1. Phase portraits of $i^{th}$ vortex for different values of $C_{vp}$ keeping $C_{vv}$ constant

We have studied the phase portraits for different values of $C_{vp}$ keeping $C_{vv}$ fixed at 0.01. Some results of this study are
FIG. 3: Phase space trajectories of the vortex for different values of vortex-pinning interaction parameter $C_{vp}$ keeping the vortex-vortex interaction parameter fixed at $C_{vv} = 0.01$.

shown in the Fig[3]. It is observed from the figure that with a slight variation of the vortex-pinning interaction parameter the vortex motion is highly affected, which is the basic signature of chaos in quasi-periodic system. For $C_{vv} = 0.01$ and $C_{vp} = 0.45$, it is seen that the phase portraits oscillates nearly with two frequency as seen from the left top panel. As the $C_{vp}$ is slightly increased to 0.9, the motion tends to be slightly stable but still oscillating with more than one frequency, which is clear from the right top panel. Thus in both cases the system shows a bifurcation pattern similar to Hopf bifurcation. When the parameter $C_{vp}$ is increased further to 1.9, the system shows a very small divergence as shown in the bottom left panel. In this phase the system tends to be a stable one but still have some oscillatory motion. However, if the parameter $C_{vp}$ is increased further to 4.5 the trajectories oscillates only with one frequency as shown in bottom right panel. The system is highly stable in this situation.

2. **Phase portraits of $i^{th}$ vortex for different values of $C_{vv}$ keeping $C_{vp}$ constant**

Fig[4] shows some important phase portraits of the $i^{th}$ vortex for different values of $C_{vv}$ when we keep $C_{vp}$ constant at 0.45. For vortex-vortex parameter $C_{vv} = 0.6$, the system shows highly oscillatory motion and oscillating with more than one frequency. The system is highly unstable in this phase. For $C_{vv} = 0.65$ the system is still oscillating with more than one frequency. For $C_{vv} = 0.85$ the system is stable and phase portrait is oscillating with only one frequency. It is observed that for low values and lower mismatch of $C_{vv}$ and $C_{vp}$ the system is highly unstable with shrinking area which is also observed earlier in the Fig[3]. This signifies that the vortex motion is highly chaotic for low values and lower mismatch. However, when one interaction parameter is increased the system becomes stable with the phase trajectory oscillating nearly with one frequency as observed from the Figs[3] and [4]. If the parameter $C_{vp}$ is increased to 0.9 or more, the system shows a open trajectory implying that the superconducting system is highly stable and is not influenced any more by the vortex-pinning interactions.

B. **Behaviour of Specific heat**

1. **Variation of Specific heat with temperature for different values of interaction parameters $C_{vv}$ and $C_{vp}$**

The specific heat of the system is calculated using the equations (14) and (15) at different temperatures for different values of interaction parameters $C_{vv}$ and $C_{vp}$. A plot of specific heat with temperature is shown for different values of vortex-pinning parameter $C_{vp}$ keeping vortex-vortex parameter $C_{vv}$ constant at 0.01 in the Fig[5]. While in the Fig[7] the variation
FIG. 4: Phase space trajectories of the vortex for different values of vortex-vortex interaction parameters \( C_{vv} \) keeping the vortex-pinning interaction parameter fixed at \( C_{vp} = 0.45 \).

FIG. 5: The variation of specific heat with temperature for different values of interaction parameter \( C_{vp} \) keeping \( C_{vv} = 0.01 \).

FIG. 6: The variation of specific heat with interaction parameters \( C_{vv} = 0.01 \) and \( C_{vp} = 45 \) for the low temperature regime.

of specific heat is shown with temperature for different values of \( C_{vv} \) keeping \( C_{vp} \) constant at 0.45.

It is observed from the Fig.5(a) that for \( C_{vv} = 0.01 \) and \( C_{vp} = 0.45 \) the system shows a highly divergent character where specific heat suddenly increases with slight increase in temperature. Exactly similar behaviour is also observed in the Figs.5(a) and 5(b) for \( C_{vv} = 0.02 \) and 0.1 respectively keeping \( C_{vp} = 0.45 \) fixed. As the parameter \( C_{vp} \) is increased to 0.9 in Fig.5(b) and to 1.9 in the Fig.5(c), the variation is not that much rapid but still very fast. For further increase in \( C_{vp} \) to 45, the superconducting system tends to behave like a normal conductor as shown in the Fig.5(d). For clarity a plot of this behaviour is also shown in Fig.6 at low temperature regime. Almost similar behaviour is also observed in
the Fig. 7(d) for \( C_{vv} = 45 \) and \( C_{vp} = 0.45 \). Thus it is clear from the Figs. 5 and 7 that the lower values of interaction parameters \( C_{vv} \) and \( C_{vp} \) lead the system to the superconducting phase, whereas the higher values of these parameters lead the system to a normal conductor.

2. Variation of Specific heat with temperature for different values of quasi-periodic factor \( \alpha \)

We have also studied the behaviour of specific heat of the vortex-lattice system for different values of the quasi-periodic factor \( \alpha \) using the general equation (12) together with the simplified equation (15). For this purpose we consider the interaction parameters \( C_{vv} \) and \( C_{vp} \) at superconducting phase. A plot of specific heat with temperature for different values of quasi-periodic parameter \( \alpha \) for \( C_{vv} = 0.01 \) and \( C_{vp} = 0.45 \) is shown in the Fig. 8. For a very low value of the parameter \( \alpha \) the specific heat rapidly increases with slight increase in temperature as observed for \( \alpha = 0.1 \) and \( \alpha = 0.2 \), which is basically observed in periodic and quasi-periodic phases. It becomes little stable for slight increase of the parameter as observed for \( \alpha = 0.5 \). With further increase of the parameter to 2.5 or more the system becomes highly stable with the specific heat curve behaving like for a normal conductor as observed in the Fig. 8 similar to that we observed from earlier Figs. 5 and 7.

IV. CONCLUSIONS

To understand the various properties of superconducting systems it is useful to study vortex dynamics and specific heat of such a system with different vortex-pinning geometries. Our study with a quasi-periodic geometry leads many interesting results. We found that the superconducting system is highly dependant on the interaction parameters. With a slight change in parameters the system shows a very rapid change in its phase portraits which is basically arises due to quasi-periodicity and is the basic signature of chaos in this type of systems. Another direct route to chaos is the area shrinking and a bifurcation pattern seen in the Fig. 8 similar to Hopf bifurcation observed in quasi periodic system. It is seen that the system is highly chaotic for low values and lower mismatch of the interaction parameters \( C_{vv} \) and \( C_{vp} \). Whereas the systems tends to be a stable and regular one as their difference as well as values are increased. It is also observed that the specific heat of the system also depends on vortex-vortex and vortex-pinning interaction parameters. For low values and lower mismatch of them, the specific heat shows a highly divergent characteristics with sudden rise in specific heat with a slight increase in temperature as shown in Figs. 5 and 7. However, it is observed from these figures that the system becomes regular for high values and higher mismatch of interaction parameters. For low values and lower mismatch, the system behaves like a superconducting one. On the other hand it tends to behave like a normal metal for high values and higher mismatch of the interaction parameters. It can also be conclude that the quasi-periodic parameter also play a very vital role in vortex motion. As observed from the Fig. 8 for low values of quasi-periodic parameter \( \alpha \) the system shows at first periodic and then quasi-periodic geometry. In this phase the sample behaves like a superconductor as observed from specific heat curve in the Fig. 8. As the parameter \( \alpha \) is increased further the system tends to behave like a normal conductor.

1 W. V. Pogosov, V. R. Misko, H. J. Zhao, and F. M. Peeters, Phys. Rev. B 79, 014504 (2009).
2 H.-T. Lin, M. Pan, C. H. Cheng, Y. J. Cui, Y. Zhao, Physica C 468, 1325 (2008).
3 H. T. Lin, C. Ke, C. H. Cheng, Physica C 470, 1118 (2008).
4 A. Junod, A. Erb, C. Renner, Physica C 317-318, 333 (1999).
5 A. Junod, M. Roulin, B. Revaz, A. Erb, Physica B 280, 214 (2000).
6 Y. Nakajima, G. J. Li, T. Tamegai, Physica C 468, 1138 (2008).
7 E. J. Calegari, S. G. Magalhaes, C. M. Chaves, A. Troper, Solid State Communications 158, 20 (2013).
8 H. Takeyaa, S. Kasahara, M. E. Massalami, T. Mochiku, K. Hirata, K. Togano, Physica B 403, 1078 (2008).
9 Q. Li, Z.D. Wang, Physica C 364-365, 495 (2001).
10 L. Xie, T. S. Su, X. G. Li, Physica C 480, 14 (2012).
11 J. Kačmarčík, Z. Pribulová, C. Marcenat, T. Klein, P. Rodière, L. Cario, and P. Samuely, Physical Review B 82, 014518 (2010).
12 V. Grinenko, D. V. Efremov, S.-L. Drechsler, S. Aswartham, D. Gruner, M. Roslova, I. Morozov, K. Neenko, S. Wurmehl, A. U. B. Wolter, B. Holzapfel, and B. Büchner, Physical Review B 89, 060504(R) (2014).
13 J. Kačmarčík, Z. Pribulová, V. Paľuchová, P. Husaníková, G. Karapetrov, V. Komanický, P. Samuely, ACTA Physica Polonica A 126, 322 (2014).
14 Y. Bang, [arXiv:1410.1244v1 (2014).
15 A. T. Hirsheeld, H. A. Leupqld, H. A. Boorse, Physical Review 127, 1501 (1962)
16 E. Gnelin and K. Guckelsberger, J. Phys. C: Solid St. Phys. 13, L269 (1980).
17 M Sato, H Fujishita and S Hoshino, J. Phys. C:Solid State Phys. 16, L417 (1983).
18 A. Kumar, R. P. Tandon, J. Wang, R. Zeng, V. P. S. Awana, J. Appl. Phys. 111, 07E140 (2012).
19 J. P. Neirotti and D. L. Freeman, J. Chem. Phys. 112, 3990 (2000).
20 J. Schulte, M. C. Böhm, Phys. Rev. B 53, 15385 (1996).