Consensus Control for Heterogeneous Multi-Agent Systems
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Abstract

We study distributed output feedback control for a heterogeneous multi-agent system (MAS), consisting of \( N \) different continuous-time linear dynamical systems. For achieving output consensus, a virtual reference model is assumed to generate the desired trajectory that the MAS is required to track and synchronize. A distinct feature of our results lies in the local optimality and robustness achieved by our proposed consensus control algorithm. In addition our study is focused on the case when the available output measurements contain only relative information from the neighboring agents and reference signal. Indeed by exploiting properties of strictly positive real (SPR) transfer matrices, conditions are derived for the existence of distributed output feedback control protocols, and solutions are proposed to synthesize the stabilizing and consensus control protocol over a given connected digraph. It is shown that design techniques based on the LQG, LQG/LTR and \( H_\infty \) loop shaping can all be directly applied to synthesize the consensus output feedback control protocol, thereby ensuring the local optimality and stability robustness. Finally the reference trajectory is required to be transmitted to only one or a few agents and no local reference models are employed in the feedback controllers thereby eliminating synchronization of the local reference models. Our results complement the existing ones, and are illustrated by a numerical example.

1 Introduction

In the last decade, research on cooperative MASs has intensified mainly due to the wide variety of applications that make use of the MAS framework, cf. \[7, 10, 21, 24\]. Early work focused on agents with scalar integrator dynamics. This includes \[7, 23\] where the agents’ dynamics are represented by linear switched systems or by changing interaction topologies, and \[19, 22\], where the dynamics correspond to time-varying systems. Scenarios that are more realistic, such as systems with communication noises, have also been considered \[12, 25\]. More recently the majority of the existing work has been concerned with homogeneous agents (i.e. agents with identical dynamics) represented by the state-space model. See for example \[13, 17, 26, 31\]. Such MASs are more general and include integrator dynamics as a special case. Solutions to the problem of achieving consensus through distributed control protocols are presented in these investigations. Since the separation principle holds for MASs of homogeneous agents, the control protocols are mostly observer-based, including Luenberger observers.

Motivated by the recent developments, we study consensus control of heterogeneous MASs which is in general more difficult than that of homogeneous MASs. Some recent results include \[3, 9, 16, 27, 28\] and references therein. The authors of \[3, 9, 28\] smartly bypass the difficulty posed by heterogeneity by embedding a homogeneous reference model into each agent’s local controller. The state vectors of the local reference models are synchronized prior to achieving output consensus of the heterogeneous MAS. In addition, the results in \[28\] make use of those in \[20\] and \[26\] to prove asymptotic synchronization over time-varying directed graphs that satisfy a uniform connectivity...
condition. The authors of [3] study the same consensus control problem over a fixed or time-invariant graph topology. However it differs from [9, 28] in that it considers heterogeneous non-introspective agents. Roughly speaking, non-introspective agents refer to those agents that make use of only relative information with respect to neighbors or reference inputs. The controller design is hinged on an observer that depends on a high gain parameter. It is important to highlight that in [3] the network graph is only present at the output but not in the input of the MAS. The case when the graph is present at the input introduces more difficulties. It is worth mentioning that analysis results are also available for heterogeneous MASs. For instance, [16] provides a theoretical condition, stated in terms of “intersection dynamics”, for synchronization that requires inclusion of the internal modes of the root agent or reference model. The work [27] provides a consensusability condition in the presence of unknown communication delays. Finally, nonlinear heterogeneous MASs are also studied in, for instance, [2, 5, 32], where the focus is on passive and dissipative systems. Such nonlinear agents exclude unstable dynamical systems, yet the results are instrumental to future work on nonlinear MASs.

In spite of the recent developments, the design of distributed and local control protocols to achieve not only feedback stability but also output consensus in tracking reference trajectories remains a major challenge. For instance local optimality and robustness are not addressed thus far in the existing literature for heterogeneous MASs. This paper is aimed at developing a more accessible method for consensus control and deriving a consensusability condition for heterogeneous MASs, taking local optimal and robust control into consideration. It will be shown that similar results to the ones found in [13, 17, 31] for homogeneous MASs are available for heterogeneous MASs. It will also be shown that existing design methods, such as linear quadratic regulator (LQR), linear quadratic Gaussian (LQG) and loop transfer recovery (LTR) [1], and $H_{\infty}$ loop shaping [18], developed for multi-input/multi-output (MIMO) feedback control systems can be employed to synthesize consensus controllers for heterogeneous MASs. Consequently, the consensus method as proposed in this paper ensures the local optimality and stability robustness. To show that our controller design achieves output consensus, we exploit a property of SPR transfer matrices in a feedback connection based on which optimal and robust control can be applied. Another distinction of our work compared to other investigations is that we provide a solution to the consensus problem for the case when not all agents have access to the reference trajectory. In fact, if the communication graph is connected (or contains a spanning tree), then it is sufficient for one agent to have access to the reference output for the heterogeneous MAS to synchronize and to achieve consensus. Moreover our proposed design method does not require duplication of the reference model in each of the local controllers thereby eliminating synchronization of the local reference models commonly adopted in the existing work. Both features lower significantly the communication overhead between agents by avoiding the need to communicate the reference trajectory to all agents and by removing additional synchronization between local reference models. Furthermore we focus on non-introspective agents as in [3] and use only relative information for both state feedback and state estimation.

The notation in this paper is more or less standard. The $N$-dimensional real/complex space is denoted by $\mathbb{R}^N/\mathbb{C}^N$. The space of all $p \times m$ real/complex matrices is denoted by $\mathbb{R}^{p\times m}/\mathbb{C}^{p\times m}$. Let $M = [\mu_{ij}]$ be a matrix with $\mu_{ij}$ the $(i,j)$th entry. Its $i$th singular value is denoted by $\sigma_i(M)$ arranged in descending order with $\sigma(M) = \sigma_1(M)$. For square $M$, its $i$th eigenvalue is denoted by $\lambda_i(M)$. The symbol $\otimes$ represents the Kronecker product. A real square matrix $M$ is called row dominant if $|\mu_{ii}| \geq \sum_{j \neq i} |\mu_{ij}|$, column dominant if $|\mu_{jj}| \geq \sum_{i \neq j} |\mu_{ij}|$, and doubly dominant if it is both row and column dominant. If the inequalities are strict then one calls such matrices strictly row or column or doubly dominant. The rest of the notation will be made clear as we proceed.
2 Preliminaries

This section prepares the results in later sections by reviewing some graph theory, formulating the consensus problem, and providing an important preliminary result.

2.1 Graph and Its Associated Matrices

We focus on directed graphs (digraphs), although our results also hold for MASs over an undirected feedback graph. Consider a weighted digraph specified by $\mathcal{G} = (V, E)$, where $V = \{v_i\}_{i=1}^N$ is the set of nodes and $E \subset V \times V$ is the set of edges or arcs, where an edge starting at node $i$ and ending at node $j$ is denoted by $(v_i, v_j) \in E$. The node index set is denoted by $N = \{1, \ldots, N\}$. The neighborhood of node $i$ is denoted by the set $N_i = \{j \mid (v_i, v_j) \in E\}$. A path on the digraph is an ordered set of distinct nodes $\{v_{i_1}, \ldots, v_{i_K}\}$ such that $(v_{i_{j-1}}, v_{i_j}) \in E$. If there is a path in $\mathcal{G}$ from node $v_i$ to node $v_j$, then $v_j$ is said to be reachable from $v_i$, denoted as $v_i \rightarrow v_j$. The digraph is called strongly connected if $v_i \rightarrow v_j$ and $v_j \rightarrow v_i \forall i, j \in N$. The set of nodes which can reach node $v_k$ is denoted as $S_k = \{v_j \in V : \exists \text{ a path } v_j \rightarrow v_k\}$. The digraph is called connected if there exists a node $v_k$ such that $v_j \in S_k$ for $j = 1, \ldots, N$, $j \neq k$.

Let $A = [a_{ij}] \in \mathbb{R}^{N \times N}$ be the weighted adjacency matrix. The value of $a_{ij} \geq 0$ represents the coupling strength of edge $(v_i, v_j)$. Self edges are not allowed, i.e., $a_{ii} = 0 \forall i \in N$. Denote the degree matrix for $A$ by $D = \text{diag}\{\deg_1, \ldots, \deg_N\}$ with $\deg_i = \sum_{j \in N_i} a_{ij}$ and the Laplacian matrix as $L = D - A$. Let $1_N \in \mathbb{R}^N$ be a vector of 1’s. It is clear that $L1_N = 0$ and thus it has at least one zero eigenvalue. It is also known that $\text{Re}\{\lambda_i(L)\} \geq 0 \forall i$. In fact the only eigenvalues of the Laplacian matrix on the imaginary axis are zero in light of the Gershgorin circle theorem. In addition, zero is a simple eigenvalue of $L$, if and only if $\mathcal{G}$ is a connected digraph. We would like to call attention to the fact that similar conditions on the eigenvalues of $L$ can be obtained through other properties of $\mathcal{G}$. For example, in [23], it is stated that $L$ has one zero eigenvalue if and only if $\mathcal{G}$ has a spanning tree.

An $M$-matrix has all its off-diagonal elements being either negative or zero, and all its principal minors being strictly positive. A semi $M$-matrix differs from the $M$-matrix in that it has all its principal minors being nonnegative. Clearly the Laplacian matrix is a semi $M$-matrix. An $M$-matrix is said to be row/column (strictly) dominant, if each of its rows/columns sums to a (strictly) positive number. More properties on $M$-matrices may be found in [29].

2.2 Problem Formulation

We consider $N$ heterogeneous agents with the dynamics of the $i$th agent described by

$$
\dot{x}_i(t) = A_ix_i(t) + B_iu_i(t), \quad y_i(t) = C_ix_i(t)
$$

where $x_i(t) \in \mathbb{R}^{n_i}$ is the state, $u_i(t) \in \mathbb{R}^{m_i}$ is the control input, and $y_i(t) \in \mathbb{R}^p$ is the controlled output. Note that the state dimension $n_i$ and input dimension $m_i$ can be different from each other. However, all agents have the same number of outputs, and we assume that $p \leq m_i$ for all $i$. Let $I_n$ be the $n \times n$ identity matrix. The $i$th agent admits transfer matrix $P_s = C_i(sI_{n_i} - A_i)^{-1}B_i$.

For heterogeneous MASs, we are concerned with output consensus aimed at achieving

$$
\lim_{t \to \infty} [y_i(t) - y_j(t)] = 0 \forall i, j \in N.
$$

(2)
In order to take performance into consideration, our focus will be synchronization of all the agents’ output to the desired trajectories generated by an exosystem or reference model described by

\[
\dot{x}_0(t) = A_0 x_0(t), \quad y_0(t) = C_0 x_0(t)
\]  

with zero steady-state error. This is a virtual reference generator where \( x_0(t) \in \mathbb{R}^{n_0} \) and \( y_0(t) \in \mathbb{R}^{p} \), and all eigenvalues of \( A_0 \) are restricted to lie on the imaginary axis. A real-time reference trajectory may not be actually from this exosystem, but consists of piece-wise step, ramp, sinusoidal signals, etc. whose poles coincide with eigenvalues of \( A_0 \). To reduce the communication overhead, the reference signal is often transmitted to only one or a few of the \( N \) agents. Following \[31\], we call these agents controlled agents.

Assume that the realizations of \( N \) agents are all stabilizable and detectable. In this paper we will study under what condition for the feedback graph, there exist distributed stabilizing controllers and consensus control protocols such that the outputs of \( N \) agents satisfy not only (2) but also

\[
\lim_{t \to \infty} [y_i(t) - y_0(t)] = 0 \quad \forall \ i \in \mathcal{N}.
\]  

Moreover we will study how to synthesize the required distributed and local controllers in order to achieve output consensus, taking performance into account.

2.3 A Fundamental Lemma

Let \( e_i \in \mathbb{R}^N \) be a vector with 1 in the \( i \)th entry and zeros elsewhere. We state the following lemma that is instrumental to the main results of this paper.

**Lemma 1** Suppose that \( L \) is the Laplacian matrix associated with the directed graph (digraph) \( G \). The following statements are equivalent:

(i) There exists an index \( i_R \in \mathcal{N} \) such that

\[
\text{rank} \{ L + e_{iR}e_{iR}' \} = N;
\]  

(ii) There exist diagonal \( D > 0 \) and an index \( i_R \in \mathcal{N} \) such that

\[
M_D + M_D' > 0, \quad M_D = D(L + e_{iR}e_{iR}');
\]  

(iii) The digraph \( G \) is connected.

Proof: Let \( \Rightarrow \) stand for “implies”. We will show that (iii) \( \Rightarrow \) (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) in order to establish the equivalence of the three statements. For (iii) \( \Rightarrow \) (i), assume that \( G \) is connected. Then there exists a reachable node \( v_{i_R} \in \mathcal{V} \) for some index \( i_R \in \mathcal{N} \). We adapt a result in \[5\] (Lemma 2) to construct an augmented graph \( \overline{G} \) by adding a node \( v_0 \), and adding an edge from \( v_{i_R} \) to \( v_0 \) with weight 1. The augmented graph is again connected with \( v_0 \) as the only reachable node. It follows that the Laplacian matrix associated with the augmented graph \( \overline{G} \) is given by

\[
\overline{L} = \begin{bmatrix} 0 & \cdots & 0 \\ -e_{iR} & L + e_{iR}e_{iR}' \end{bmatrix}.
\]  

Since the augmented graph is connected, the Laplacian matrix \( \overline{L} \) has only one zero eigenvalue, implying the rank condition (5), and thus (i) is true.
For (i) ⇒ (ii), assume that the rank condition \([5]\) is true. Then \(L + e_i R e'_i R\) is an M-matrix, because it is not only a semi M-matrix but also has all its eigenvalues on strict right half plane, in light of the Gershgorin circle theorem. Using the properties of M-matrices in \([29]\), we conclude the existence of a diagonal matrix \(D\) such that

\[
\mathcal{M}_D = D(L + e_i R e'_i R)
\]

is strictly column dominant. Since \(\mathcal{M}_D\) is row dominant, although not strictly, \(\mathcal{M}_D + \mathcal{M}'_D\) is both strictly row and column dominant, thereby concluding (ii).

For (ii) ⇒ (iii), assume that \([\ref{eq:rank}]\) is true. Then \(\mathcal{M}_D\) is an M-matrix, by the fact that all its eigenvalues lie on strict right half plane. Hence there holds

\[
N = \text{rank}\{D^{-1} \mathcal{M}_D\} = \text{rank}\{L + e_i R e'_i R\} \leq \text{rank}\{L\} + 1,
\]

by the rank inequality and rank \(\{e_i R e'_i R\} = 1\). The above implies that rank \(\{L\} \geq N - 1\). It follows that the Laplacian matrix \(L\) has only one zero eigenvalue, concluding that the graph \(G\) is connected, and thus (iii) is true. The proof is now complete. \(\square\)

**Remark 1** For any \(i_R \in N\) corresponding to a reachable node, there exists a diagonal matrix \(D > 0\) such that \(\mathcal{M}_D + \mathcal{M}'_D > 2I\), i.e.,

\[
D L + L'D + 2D e_i R e'_i R > 2I
\]  

(7)

Efficient algorithms for linear matrix inequality (LMI) can be used to search for \(D\). Such a \(D\) helps to design control protocol achieving not only the MAS feedback stability but also optimizing local performance for each agent. However the existence of the stabilizing control protocols does not depend on \(D\). For MIMO agents with \(m\)-input/p-output, a commonly adopted graph has the weighted adjacency matrix in the form of \(A = \{a_{ij} I_{q_i}\}\) with \(q_i \equiv m_i\) or \(q_i \equiv p\). Thus \(D\) is modified to \(D = \text{diag}(d_1 I_{q_1}, \ldots, d_N I_{q_N})\) and \(\mathcal{M}_D\) has a similar modification.

### 3 Distributed Stabilization

This section is focused on the distributed control protocol over the connected graph \(G\), represented by its Laplacian matrix \(L\). Distributed stabilization will be studied and a stabilizability condition will be derived for both the case of state feedback and output feedback.

Let \(v_i(t) = F_{0i} x_0(t) - F_i x_i(t)\) be the full information control signal for the \(i\)th agent with \(F_i\) the state feedback gain for agent \(i\), and \(F_{0i}\) the state feedforward gain from the reference model. Denote \(r_i(t) = F_{0i} x_0(t)\) and \(\delta_K(\cdot)\) as the Kronecker delta function. Consider the full information (FI) control protocol for the \(i\)th agent specified by

\[
u_i(t) = d_i \delta_K(i - i_R) v_i(t) + d_i \sum_{j=1}^N a_{ij} [v_i(t) - v_j(t)]
\]  

(8)

with \(i_R\) corresponding to one of the reachable nodes in order to minimize the communication overhead. This is why Lemma \([3]\) becomes useful. By denoting \(x(t)\) as the collective state and \(r(t)\) as the collective reference, i.e., the stacked vector of \(\{x_i(t)\}_{i=1}^N\) and \(\{r_i(t)\}_{i=1}^N\), respectively, the closed loop dynamics with protocol \([8]\) can be written as

\[
\dot{x} = [A - B \mathcal{M}_D F] x + B \mathcal{M}_D r.
\]  

(9)

where \(A = \text{diag}(A_1, \ldots, A_N)\), \(B = \text{diag}(B_1, \ldots, B_N)\), and \(F = \text{diag}(F_1, \ldots, F_N)\). The following result is concerned with distributed stabilization under state feedback.
Theorem 1 Suppose that \((A_i, B_i)\) is stabilizable for each \(i \in \mathcal{N}\). There exist stabilizing FI control protocols in the form of \([6]\) for the feedback MAS over the directed graph \(G\), if \(G\) is connected.

Proof: Since \(G\) is connected, Lemma \([1]\) and Remark \([1]\) imply the existence of a reachable node with index \(i_R \in \mathcal{N}\) and \(D = \text{diag}(d_1I_{m_1}, \ldots, d_NI_{m_N})\) such that \(M_D + M_D' > 2I\) holds. Feedback stability of the underlying MAS requires the existence of \(F\) such that

\[
\det(sI - A + B M_D F) \neq 0 \quad \forall \ \text{Re}\{s\} \geq 0. \tag{10}
\]

Let \(Z = M_D - I\). Then \(Z + Z' > 0\), and thus the inequality \([10]\) is equivalent to

\[
\det(sI - A + BF + BZF) \neq 0 \quad \forall \ \text{Re}\{s\} \geq 0.
\]

Denote \(T_F(s) = F(sI - A + BF)^{-1}B\). The above inequality is in turn equivalent to

\[
\det[I + T_F(s)Z] \neq 0 \quad \forall \ \text{Re}\{s\} \geq 0. \tag{11}
\]

To show the existence of the stabilizing \(F = \text{diag}(F_1, \cdots, F_N)\) that satisfies inequality \([11]\), consider the algebraic Riccati equation (ARE)

\[
A_i'X_i + X_iA_i - XB_iB_i'X_i + Q_i = 0 \tag{12}
\]

for an arbitrary index \(i \in \mathcal{N}\). By the stabilizability of \((A_i, B_i)\), such an ARE admits the stabilizing solution \(X_i \geq 0\), provided that \(Q_i \geq 0\) and \((Q_i^{1/2}, A_i)\) is observable for any unstable modes of \(\dot{x}_i(t) = A_i x_i(t)\) corresponding to the imaginary eigenvalues of \(A_i\). The matrix \(Q_i\) should be chosen to optimize the local control performance for each \(i\). It follows that \(F_i = B_i'X_i\) is stabilizing, and more importantly, both

\[
T_{F_i}(s) = F_i(sI - A_i + B_iF_i)^{-1}B_i, \quad T_F(s) = \text{diag}\{T_{F_1}(s), \cdots, T_{F_N}(s)\}
\]

are strictly positive real (SPR) \([1]\) (page 106). An application of Theorem 6.3 in \([8]\) (page 250) concludes inequality \([11]\), i.e., \([10]\), and thus the stability of the feedback MAS. \(\Box\)

Remark 2 If the knowledge on \(D\) is not available, then the following FI control protocol

\[
u_i(t) = \delta_K(i - i_R)v_i(t) + \sum_{j=1}^N a_{ij} [v_i(t) - v_j(t)]
\]

will have to be used. The above results in \(\dot{x}(t) = [A - BMF]x + BMr\) for the feedback MAS. In this case \(F_\epsilon = \epsilon^{-1}B_i'X_i\) can be used as an alternative for each \(i \in \mathcal{N}\) where \(X_i \geq 0\) is the stabilizing solution to \([12]\). We claim that there exists an \(\epsilon \in (0, 1)\) such that \((A - BMF_\epsilon)\) is a Hurwitz matrix where \(F_\epsilon = \text{diag}(F_1\epsilon, \cdots, F_N\epsilon)\). Indeed inequality \([10]\) is now replaced by

\[
\lambda(s) := \det(sI - A + BMF_\epsilon) \neq 0 \quad \forall \ \text{Re}\{s\} \geq 0.
\]

Denote \(D = \text{diag}(d_1I_{n_1}, \cdots, d_NI_{n_N})\). By the block diagonal form of \(A, B,\) and \(F_\epsilon\), there holds

\[
\lambda(s) = \det[D(sI - A + BMF_\epsilon)D^{-1}] = \det(sI - A + BM_D D^{-1}F_\epsilon).
\]

Since a diagonal \(D > 0\) exists such that inequality \([7]\) holds, \(Z = M_D - I\) satisfies \(Z' + Z > 0\). The above inequality is now equivalent to

\[
\det(sI - A + BD^{-1}F_\epsilon + BZD^{-1}F_\epsilon) \neq 0 \quad \forall \ \text{Re}\{s\} \geq 0,
\]
that is in turn equivalent to \( \det[I + T_{F_i}(s)Z] \neq 0 \forall \Re\{s\} \geq 0 \) where
\[
T_{F_i}(s) = \text{diag}[T_{F_{i1}}(s), \ldots, T_{F_{iN_i}}(s)], \quad T_{F_{ie}}(s) = d_i^{-1}F_{ie}(sI - A_i + B_id_i^{-1}F_{ie})^{-1}B_i.
\] (14)

Specifically choosing \( \epsilon < 2/\max\{d_i\} \) ensures that
\[
A_{F_i} = A_i - B_i d_i^{-1}F_{ie} = A_i - B_i (\epsilon d_i)^{-1}B_i'X_i
\]
is a Hurwitz matrix in light of the fact that ARE (12) can be rewritten as
\[
A'_{F_i}X_i + X_iA_{F_i} + X_iB_i[2(\epsilon d_i)^{-1} - 1]B_i'X_i + Q_i = 0
\]
and \( 2(\epsilon d_i)^{-1} - 1 > 0 \) for all \( i \in \mathcal{N} \). The same argument as in the proof of Theorem 1 can be used to conclude the SPR of \( T_{F_{ie}}(s) \) and thus \( T_{F_i}(s) \) in (13), implying that \( (A - BMF) \) is indeed a Hurwitz matrix. The above shows that distributed stabilization can be achieved without knowledge of \( D \) satisfying (7). However the local optimality is lost due to the scaling of \( \epsilon \) in \( F_{ie} = \epsilon^{-1}B_i'X_i \), and it is debatable if the search for \( \epsilon > 0 \) to achieve the stability of \( (A - BMF_i) \) is any simpler than the search for \( D \) satisfying (7). There are strong incentives to search for \( D \) rather than \( \epsilon \).

\[ \square \]

Theorem 1 provides a sufficient condition for stabilizability under the distributed state feedback control. This sufficient condition becomes necessary for two special cases as shown next.

**Corollary 1** Consider state feedback control for the MAS over the directed graph \( \mathcal{G} \). If feedback stability holds for the MAS consisting of either (i) homogeneous multi-input unstable agents or (ii) heterogeneous single input unstable agents with \( \{A_i\}_{i=1}^{N} \) having a common unstable eigenvalue, then the directed graph \( \mathcal{G} \) is connected.

**Proof:** For case (i), the homogeneous hypothesis implies
\[
F(sI - A)^{-1}B = I_N \otimes F_a(sI - A_a)^{-1}B_a
\]
where \( (A_i, B_i, F_i) = (A_a, B_a, F_a) \forall i \in \mathcal{N} \). Using the same procedure as in [13, 22], the feedback stability condition in (10) can be shown to be equivalent to
\[
\det[I + F_a(sI - A_a)^{-1}B_a\lambda_i(\mathcal{M}_D)] \neq 0 \forall \Re[s] \geq 0.
\]
Since \( A_a \) has unstable eigenvalues by the hypothesis, the above inequality implies \( \lambda_i(\mathcal{M}_D) \neq 0 \) for all \( i \), concluding that the graph \( \mathcal{G} \) is connected. For case (ii), feedback stability implies stability of \( (A - BMDF) \) for some \( F \). Thus
\[
\text{rank}\left\{ \left[ \begin{array}{c} \text{sI}_{n_i} - A \\ BMDF \end{array} \right] \right\} = n \forall \Re[s] \geq 0
\]
where \( n = n_1 + \cdots + n_N \). Recall \( A_i \) has dimension \( n_i \times n_i \). For single input agents, \( BMDF \) has \( N \) columns. Taking \( s \) to be the common unstable eigenvalue of \( \{A_i\}_{i=1}^{N} \) implies that \( (sI_{n_i} - A) \) has rank \( (n - N) \), and thus \( BMDF \) has rank \( N \), leading to the conclusion of nonsingular \( \mathcal{M}_D \) that concludes the proof for (ii).

\[ \square \]

When the states of the MAS are not available for feedback, a distributed observer can be designed to estimate the state of each agent, which can then be used for feedback control. See [13, 31] for homogeneous MASs. We will modify the neighborhood observers in [31] for designing distributed output feedback controllers in the case of heterogeneous MASs to incorporate the relative information of the output measurements. Let \( \hat{x}_i(t) \) be estimate of \( x_i(t) \), and
\[
e_{x_i}(t) = x_i(t) - \hat{x}_i(t), \quad e_{y_i}(t) = y_i(t) - C\hat{x}_i(t),
\] (15)
be the estimation error for $x_i(t)$ and $y_i(t)$, respectively. There holds

$$\[ y_i(t) - y_j(t) \] - [\hat{y}_i(t) - \hat{y}_j(t)] = e_{y_i}(t) - e_{y_j}(t) = C_ie_{x_i}(t) - C_je_{x_j}(t). \tag{16}$$

The neighborhood observer modified from \[31\] is proposed as follows:

$$\dot{x}_i = A_i\hat{x}_i + B_iu_i + d_i\delta_K(i - i_R)L_ie_{x_i} + d_iL_i\sum_{j=1}^{N}a_{ij}[C_ie_{x_i} - C_je_{x_j}] \tag{17}$$

for $1 \leq i \leq N$. The following result holds.

**Theorem 2** Suppose that $D$ satisfying $\square$ in Remark $\square$ is known, and $(A_i, B_i, C_i)$ is both stabilizable and detectable for all $i \in \mathcal{N}$. Then there exist distributed output feedback stabilizing controllers for the underlying heterogeneous MAS, if the feedback graph is connected.

Proof: In light of Theorem $\square$ there exists a FI control protocol $u(t) = M_Dr(t) - M_Fx(t)$ that stabilizes the feedback MAS, i.e., $(A - BMDF)$ is a Hurwitz matrix. Since $x(t)$ is unavailable, a distributed estimator in $\square$ can be designed by synthesizing $L_i = Y_iC_i'$ with $Y_i \geq 0$ being the stabilizing solution to ARE

$$A_iY_i + Y_iA_i^T - Y_iC_iC_iY_i + \hat{Q}_i = 0 \tag{18}$$

for some $\hat{Q}_i \geq 0$ that can be used to optimize the local estimation performance. In addition the state estimation gain $L_i$ not only stabilizes the local estimation error dynamics, but also satisfies the SPR property for the resulting

$$T_{L_i}(s) = C_i(sI - A_i + L_iC_i)^{-1}L_i \quad \forall \ i \in \mathcal{N} \tag{19}$$

which is dual to $T_F(s)$ in $\square$ for the ease of state feedback. Taking difference between the state space equation $\dot{x}_i = A_i\hat{x}_i + B_iu_i$ and that in $\square$ leads to

$$\dot{e}_{x_i} = A_ie_{x_i} - d_iL_i\sum_{j=1}^{N}a_{ij}[C_ie_{x_i} - C_je_{x_j}] - d_i\delta_K(i - i_R)L_iC_ie_{x_i},$$

The above results in the collective error dynamics described by

$$\dot{e}_x(t) = [A - LMDC]e_x(t), \quad e_x(t) = x(t) - \hat{x}(t),$$

where $\mathcal{M}_D$ is the same as that in $\square$. The proof for Theorem $\square$ can be adapted to show the Hurwitz stability for $A - LMDC$. Using $\ddot{x}(t)$ in place of $x(t)$ for the FI control protocol results in

$$\ddot{x}(t) = Ax(t) + BMDF[r(t) - F\dot{x}(t)] = [A - BMDF]x(t) + BMDFe_x(t) + BMDr(t).$$

The overall MAS thus admits the state space description:

$$\begin{bmatrix} \ddot{x}(t) \\ \dot{e}_x(t) \end{bmatrix} = \begin{bmatrix} A - BMDF & BMDF \\ 0 & A - LMDC \end{bmatrix} \begin{bmatrix} x(t) \\ e_x(t) \end{bmatrix} + \begin{bmatrix} BMDF \\ 0 \end{bmatrix} r(t). \tag{20}$$

The separation principle for stabilization holds true as manifested in the collective dynamics $\square$ that concludes the proof.
If $D$ satisfying inequality (7) in Remark 1 is unknown, then the neighborhood observer

$$\dot{x}_i = A_i \dot{x}_i + B_i u_i + \delta_K(i - i_R)L_i C_i e_{x_i} + L_i \sum_{j=1}^{N} a_{ij} (C_i e_{x_i} - C_j e_{x_j})$$

(21)

can be employed, resulting in the same error dynamics described in (20), except that $\mathcal{M}_D$ is replaced by $\mathcal{M}$. Remark 2 can be used to synthesize both the stabilizing state feedback gain and state estimation gain. The detail is omitted.

In light of Corollary 1, the sufficient condition in Theorem 2 for distributed output stabilizability condition can be made necessary for SISO heterogeneous MASs when all agents share some common unstable poles. We would like to point out that Theorem 2 assumes the same communication graph at both input and output. In practice they can be different from each other which can give more design freedom. In case that the graphs differ, then both need to be connected. Moreover each agent may have different measurement output from the consensus output. For simplicity, our paper considers only the case when the measurement output is the same as the consensus output; however our design method can be easily adapted to fit to the case when they differ by replacing $C_i$ in the state estimator by a different $\hat{C}_i$ corresponding to the measurement output.

**Remark 3 (Design of control/estimation gains)**

If $D$ satisfying inequality (7) in Remark 1 is known, the protocols in (8) and (17) can be used, which achieve not only local optimality but also local robustness. Indeed many known output feedback controllers, including the controllers designed using LQG, $\mathcal{H}_\infty$ loop shaping, and LQG/LTR methods, are observer based and satisfy the required SPR property for

$$T_{F_i}(s) = F_i(sI - A_i + B_i F_i)^{-1} B_i, \quad T_{L_i}(s) = C_i(sI - A_i + L_i C_i)^{-1} L_i.$$ (22)

While LQG is obvious due to its optimality, the SPR property is kind of obscure for the other two design methods, which will be clarified next.

For $\mathcal{H}_\infty$ loop shaping based on a right coprime factorization (a dual result is presented in [18], page 69-72), $F_i = B_i' X_i$ and $L_i = Y_{\infty} C_i'$ with $X_i \geq 0$ the stabilizing solution to the control ARE

$$A_i' X_i + X_i A_i - X_i B_i B_i' X_i + C_i' C_i = 0,$$ (23)

and $Y_{\infty} \geq 0$ the stabilizing solution to the filtering ARE

$$A_i Y_{\infty} + Y_{\infty} A_i' - Y_{\infty} C_i' C_i Y_{\infty} + B_i B_i' + \Delta_i = 0$$ (24)

where $\Delta_i = (\gamma_i^2 - 1)(I + Y_{\infty} X_i) B_i B_i'(I + X_i Y_{\infty})$. Since $\gamma_i > \gamma_{\text{opt}} = \sqrt{1 + \lambda_{\text{max}}(X_i Y_i)} \geq 1$, $\Delta_i \geq 0$ is true. Both $T_{F_i}(s)$ and $T_{L_i}(s)$ are SPR in light of [18] 8.

For LQG/LTR, $F_i = B_i' X_i$ with $X_i \geq 0$ the stabilizing solution to the ARE (23), and $L_i = Y_i C_i'$ with $Y_i \geq 0$ the stabilizing solution to the ARE

$$A_i Y_i + Y_i A_i' - Y_i C_i' C_i Y_i + q_i^2 \bar{Q}_i = 0$$

for some design parameter $q_i > 0$ sufficiently large. The matrix $\bar{Q}_i = B_i B_i'$ if $P_i(s) = C_i(sI - A_i)^{-1} B_i$ is minimum phase. Otherwise $\bar{Q}_i = B_{im} B_{im}'$ where $P_i(s) = P_{im}(s) B_{ia}(s)$ with $P_{im}(s) = C_i(sI - A_i)^{-1} B_{im}$ being the minimum phase part of $P_i(s)$ and $B_{ia}(s)$ satisfying $B_{ia}(-s)' B_{ia}(s) = I$ and containing all unstable zeros of $P_i(s)$ [30]. Hence both $T_{F_i}(s)$ and $T_{L_i}(s)$ in (22) are again SPR. However if $D$ in Remark 1 is unknown, and $N$ is too large to search for it, then the protocols in Remark 2 for distributed control and in (21) for distributed estimation will have to be used, that may destroy the local optimality and robustness of the feedback MAS. So there are incentives to search for $D$ whenever it is possible. □
Remark 4 indicates that the required SPR property helps to render not only the local optimality, but also local robustness for the heterogeneous MAS, in light of the fact that LQG/LTR design improves the gain/phase margin and $\mathcal{H}_\infty$ loop shaping provides the robustness against coprime factor uncertainties. However it remains unclear for the collective robustness. The next result is concerned with the existence of static output stabilizing control law for heterogeneous MASs.

**Corollary 2** Suppose that $(A_i, B_i, C_i)$ is both stabilizable and detectable satisfying $\det(C_i B_i) \neq 0$ and $(A_i, B_i, C_i)$ is strictly minimum phase for all $i \in \mathcal{N}$. If the feedback graph exists only at the MAS input or output but not at both, then there exist distributed static output stabilizing controllers for the underlying heterogeneous MAS, if the feedback graph is connected.

Proof: It is shown in [4] that under the hypotheses for $(A_i, B_i, C_i)$, there exists $K_i$ such that

$$u_i(t) = K_i y_i(t) = K_i C_i x_i(t)$$

is an LQR control law for the system $\dot{x}_i = A_i x_i + B_i u_i$ with $R = I$. Since this is true for each $i \in \mathcal{N}$, the result for distributed state feedback control can be applied, if the feedback graph exists at the MAS input. Specifically $T_F(s)$ in the proof of Theorem 1 with $F_i = K_i C_i$ for each $i$ can be made SPR. The corollary is thus true. \qed

## 4 Output Consensus

This section is focused on output consensus, assuming that the feedback MAS is stabilized by the observer-based distributed controllers developed in the previous section. Two cases will be investigated. The first considers the case when the state vector $x(t)$ of the reference model in (3) is available for feedforward control. We will show how the feedforward gain $F_{0i}$ can be designed for $r_i(t) = F_{0i} x_0(t)$ to achieve the output synchronization as required in [4] for each $i \in \mathcal{N}$. The second assumes $x_0(t)$ unavailable. We will develop distributed cooperative observers to estimate the feedforward control signal $r_i(t) = F_{0i} x_0(t)$ based on relative output measurements of the neighboring agents, and the desired output trajectory $y_0(t) = C_0 x_0(t)$ that is transmitted to only one agent. The following assumption is crucial.

**Assumption (a):** Agent $i$ has input dimension $m_i$ no smaller than its output dimension $p$, and each eigenvalue of $A_0$ is a pole of each column of $P_i(s)$ for all $i \in \mathcal{N}$.

The above assumption ensures the right invertibility of $P_i(s)$ at each eigenvalue of $A_0$. If this assumption fails, weighting functions $\{W_i(s)\}$ can be used so that $P_i(s) W_i(s)$ satisfies Assumption (a), and consensus control design can be carried out for the weighted agents. Such a method is used widely in control system design [18]. The next result is instrumental.

**Lemma 2** Suppose that Assumption (a) holds, $(A_i, B_i)$ is stabilizable for all $i$, and the feedback graph is connected. Then $\{F_i\}_{i=1}^N$ exist such that $\det[s I - A + B M_D F] = \text{Hurwitz}$. Denote

$$T_{M_D}(s) = C [s I - A + B M_D F]^{-1} B M_D, \quad T(s) = C (s I - A + B F)^{-1} B,$$

and $\Delta(s) = T_{M_D}(s) - T(s)$. There holds

$$\lim_{s \to s_\kappa} \frac{\Delta(s)}{(s - s_\kappa)^{\mu_\kappa - 1}} = 0.$$  \hspace{1cm} (26)

at each eigenvalue $\lambda_\kappa$ of $A_0$ with multiplicity $\mu_\kappa$. 

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Proof: Recall \( P(s) = C(sI - A)^{-1}B \) and denote \( P_F(s) = F(sI - A)^{-1}B \). The hypotheses imply
\[
\Delta(s) = C(sI - A + B\mathcal{M}_D F)^{-1}B\mathcal{M}_D - C(sI - A + BF)^{-1}B = P(s)[I + \mathcal{M}_D P_F(s)]^{-1}\mathcal{M}_D - P(s)[I + P_F(s)]^{-1}
\]
\[
= P(s)P_F(s)^{-1}[I + \mathcal{M}_D^{-1}P_F(s)^{-1}]^{-1} - P(s)P_F(s)^{-1}[I + P_F(s)^{-1}]^{-1}
\]
\[
= P(s)P_F(s)^{-1}\{[I + \mathcal{M}_D^{-1}P_F(s)^{-1}]^{-1} - [I + P_F(s)^{-1}]^{-1}\}.
\]
Since \( P(s) \) and \( P_F(s) \) are block diagonal transfer matrices and each of their columns has \( s_{\kappa} \) as pole with multiplicity \( \mu_{\kappa} \), \( P_F(s)^{-1} \to 0 \), \( [(s - s_{\kappa})^{\mu_{\kappa}} P_F(s)^{-1}] \to 0 \), and \( P(s)P_F(s)^{-1} \) approaches a finite block diagonal matrix as \( s \to s_{\kappa} \). Moreover
\[
[I + P_F(s)^{-1}]^{-1} = I - P_F(s)^{-1} + o(\{P_F(s)^{-1}\}^2),
\]
\[
[I + \mathcal{M}_D^{-1}P_F(s)^{-1}]^{-1} = I - \mathcal{M}_D^{-1}P_F(s)^{-1} + o(\{P_F(s)^{-1}\}^2),
\]
with \( o(\{P_F(s)^{-1}\}^2) \) indicating that each of its terms approaches zero in the order of \( (s - s_{\kappa})^{2\mu_{\kappa}} \) as \( s \to s_{\kappa} \). Consequently there holds
\[
\Delta(s) \to P(s)P_F(s)^{-1}\{I - \mathcal{M}_D^{-1}\} P_F(s)^{-1} + o(\{P_F(s)^{-1}\}^2)
\]
as \( s \to s_{\kappa} \). Substituting the above into the left hand side of (26) yields
\[
\frac{\Delta(s)}{(s - s_{\kappa})^{\mu_{\kappa}-1}} \to o(\{P_F(s)^{-1}\}^2) + \frac{P(s)P_F(s)^{-1}\{I - \mathcal{M}_D^{-1}\} P_F(s)^{-1}}{(s - s_{\kappa})^{\mu_{\kappa}-1}} \to 0
\]
as \( s \to s_{\kappa} \), that concludes the proof.

Lemma 2 indicates that \( \Delta(s)R(s) = T\mathcal{M}_D(s)R(s) - T(s)R(s) \) has no pole at \( s_{\kappa} \). Otherwise its partial fraction in computing the term with pole at \( s_{\kappa} \) would contradict the limit in (26). Since \( s_{\kappa} \) is an arbitrary eigenvalue of \( A_0 \), no eigenvalue of \( A_0 \) is a pole of \( \Delta(s)R(s) \), implying the stability of \( \Delta(s)R(s) \). The next result is concerned with the full information (FI) control protocol.

**Theorem 3** Suppose that Assumption (a) holds, the feedback graph is connected, and the realization \( \{A_i,B_i,C_i\} \) is both stabilizable and detectable for all \( i \in N \). Then there exist FI control protocols in the form of (8) such that not only \( A - B\mathcal{M}_D F \) is a Hurwitz matrix, but also the output consensus as required in (4) is achieved.

Proof: Under the assumption of connectivity for the feedback graph and stabilizability for \( (A_i,B_i) \forall i \), stabilizing \( \{F_i\}_{i=1}^N \) can be synthesized according to the proof of Theorem 1 such that not only each \( (A_i - B_i F_i) \), but also \( A - B\mathcal{M}_D F \) is a Hurwitz matrix. Hence both \( T(s) \) and \( T\mathcal{M}_D(s) \) as in (25) are stable where
\[
T(s) = \text{diag}[T_1(s), \cdots, T_N(s)], \quad T_i(s) = C_i(sI - A_i + B_i F_i)^{-1}B_i.
\]
(27)

It is clear that \( T\mathcal{M}_D(s) \) is the transfer matrix from \( r(t) \) to \( y(t) \) for the feedback MAS under the FI control protocol, and it becomes equal to \( T(s) \), if \( \mathcal{M}_D = I \), corresponding to \( N \) decoupled feedback systems under the FI control. In addition the hypotheses on stabilizable and detectable realization for \( P_i(s) \) and on eigenvalues of \( A_0 \) being poles of each column of \( P_i(s) \) imply that none of the eigenvalues of \( A_0 \) is a transmission zero of \( P_i(s) \) for all \( i \). Consequently the two equations
\[
(A_i - B_i F_i)\Pi_i + B_i F_0i = \Pi_i A_0, \quad C_i\Pi_i - C_0 = 0,
\]
(28)
admit solutions \((\Pi_i, F_{0i})\) for each \(i\), in light of the internal model principle \([3]\). Let \(R_i(s)\) and \(Y_0(s)\) be the Laplace transform of \(r_i(t) = F_{0i}x_0(t)\) and \(y_0(t) = C_0x_0(t)\), respectively. It is easy to see that \(T_i(s)R_i(s) - Y_0(s)\) is stable, and the output of \(T_i(s)\) tracks \(y_0(t) = C_0x_0(t)\) with zero steady-state error for all \(i\). To show the output consensus for the feedback MAS under the FI control protocol, we examine the synchronization error. Recall \(\Delta(s) = T_{MD}(s) - T(s)\). Denote \(Y_0(s)\) and \(R(s)\) as the Laplace transform of \(y_0(t) = 1_N \otimes y_0(t)\), and \(r(t) = F_0\bar{z}_0\), respectively with \(\bar{z}_0(t) = 1_N \otimes x_0(t)\). Then the synchronization error in the \(s\)-domain is given by

\[
E_y(s) = T_{MD}(s)R(s) - Y_0(s) = \Delta(s)R(s) + [T(s)R(s) - Y_0(s)]
\]

(29)

for the feedback MAS under the FI control protocol \([9]\). Since the first term on the right hand side is stable by Lemma \([2]\) and the second term on right of (29) is also stable, \(e_y(t) = y(t) - y_0(t) \to 0\) as \(t \to \infty\), thereby concluding the proof.

Remark 4 If the state vector of each agent is not available for feedback control, then the state estimation results in the previous section can be employed. Specifically the estimator in (17) can be employed to result in the error dynamics described by

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{e}_x(t)
\end{bmatrix} = \begin{bmatrix}
A - BM_{DF} & BM_{DF} \\
0 & A - LM_{DC}
\end{bmatrix}
\begin{bmatrix}
x(t) \\
e_x(t)
\end{bmatrix} + \begin{bmatrix}
BM_D \\
0
\end{bmatrix} r(t)
\]

The output consensus can be achieved under the same hypotheses as those in Theorem \([3]\) in light of the fact that \(e_x(t) \to 0\) as \(t \to \infty\), provided that \((A_i - L_iC_i)\) is Hurwitz for all \(i \in N\). If the state vector of the reference model is not available for feedforward control, then the estimator design is more involved, which will be tackled in the later part of the section.

□

Assumption (a) is not only important for achieving output consensus, but also has an important implication that is valuable to output estimation in the case when full information is not available for output consensus. This is presented in the next result.

Corollary 3 Suppose that \(T_F(s)\) in (13) is not only internally stable but also SPR for all \(i \in N\), \(M_D\) satisfies inequality \([6]\), and the rest of the hypothesis of Theorem \([3]\) holds. Then the feedforward gains \(\{F_{0i}\}_{i=1}^N\) achieving the output consensus satisfy \(A_i\Pi_i = \Pi_iA_0\) and \(F_{0i} = F_i\Pi_i\) for all \(i \in N\).

Proof: In light of Theorem \([3]\), there exists a FI control protocol that achieves not only feedback stability but also output consensus. Hence there exists a solution pair \((\Pi_i, F_{0i})\) to (28) for each \(i \in N\). Denote \(C_0 = I_N \otimes C_0\), \(A_0 = I_N \otimes A_0\), and \(\Pi = \text{diag}(\Pi_1, \cdots, \Pi_N)\). Then (28) for all \(i \in N\) can be put together, leading to

\[
(A - BF)\Pi + BF_0 = \Pi A_0, \quad CPI - C_0 = 0.
\]

(30)

Recall \(E_y(s)\) in (29) under the FI control protocol as in the proof of Theorem \([3]\). Its dynamic behaviors in the time domain are described by

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{e}_x(t) \\
\end{bmatrix} = \begin{bmatrix}
A - BM_{DF} & BM_{DF} \\
0 & A_0
\end{bmatrix}
\begin{bmatrix}
x(t) \\
e_x(t)
\end{bmatrix} + \begin{bmatrix}
BM_D \\
0
\end{bmatrix} r(t)
\]

\[
E_y(t) = C x(t) - C_0 \bar{z}_0(t)
\]

(31)

where \(\bar{z}_0(t) = 1_N \otimes x_0(t)\), and \(e_y(t)\) and \(E_y(s)\) are a Laplace transform pair. In light of Theorem \([3]\) we can show that \(e_y(t) \to 0\) as \(t \to \infty\). By the hypothesis on \(M_D\), \(M_D = I + Z\) and \(Z + Z' > 0\). The above state equation can now be written as

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{e}_x(t)
\end{bmatrix} = \begin{bmatrix}
A - BF & BF_0 \\
0 & A_0
\end{bmatrix}
\begin{bmatrix}
x(t) \\
e_x(t)
\end{bmatrix} + \begin{bmatrix}
BZ \\
0
\end{bmatrix} [r(t) - Fx(t)].
\]

(32)
Applying the similarity transform
\[
\begin{bmatrix}
\hat{x}(t) \\
\hat{x}_0(t)
\end{bmatrix} =
\begin{bmatrix}
I & -\Pi \\
0 & I
\end{bmatrix}
\begin{bmatrix}
x(t) \\
x_0(t)
\end{bmatrix}
\]
to state equation \([32]\) and error equation \([31]\), and utilizing the relation \(C\Pi - C_0 = 0\) yield
\[
\dot{\hat{x}}(t) = (A - BF)\hat{x}(t) + BZ[r(t) - Fx(t)], \quad \dot{\hat{x}}_0(t) = A_0\hat{x}_0(t),
\]
and \(e_y(t) = C\hat{x}(t)\). By \(r(t) = F_0\hat{x}_0(t)\), there holds
\[
v(t) := r(t) - Fx(t) = F_0\hat{x}_0(t) - Fx(t) = (F_0 - F\Pi)x_0(t) - F\hat{x}(t).
\]
As a result, the state equation for \(\hat{x}(t)\) can be written as
\[
\dot{\hat{x}}(t) = (A - BMDF)\hat{x}(t) + BZ(F_0 - F\Pi)x_0(t).
\]
Since \(e_y(t) = C\hat{x}(t) \to 0\) as \(t \to \infty\) and \((A - BMDF)\) is internally stable, it holds that
\[
(F_0 - F\Pi)x_0(t) \to 0
\]
as \(t \to \infty\) in light of \(Z + Z' > 0\). The persistency or divergence of \(x_0(t)\) imply that \((F_0 - F\Pi) = 0\), concluding \(F_0i = F_i\Pi_i \forall i \in \mathcal{N}\). Upon substituting \(F_0 = F\Pi\) into \([30]\) yields \(A\Pi = \Pi A_0\) and thus \(A_i\Pi_i = \Pi_i A_0\) for all \(i \in \mathcal{N}\), that concludes the proof. \(\square\)

**Remark 5** The solution \(\Pi_i\) to \(A_i\Pi_i = \Pi_i A_0\) and \(F_0i = F_i\Pi_i\) for each \(i\) does not have to be solved from \([30]\). Once \(F_0i\) and \(F_i\) are available, \(\Pi_i\) can be computed via
\[
\begin{bmatrix}
C_i \\
F_i
\end{bmatrix} \Pi_i = \begin{bmatrix}
C_0 \\
F_{0i}
\end{bmatrix} \implies \Pi_i = \begin{bmatrix}
C_i' \\
F_i'
\end{bmatrix}^{-1} \begin{bmatrix}
C_0 \\
F_{0i}
\end{bmatrix},
\]
provided that the inverse exists. If the inverse does not exist, pseudo-inverse can be used in place of the inverse. It is also important to point out that \(\{F_0i\}\) can be synthesized rather easily which will be demonstrated in the example section. \(\square\)

In practice it is possible that the states of the MAS are not available for feedback control and the state of the reference model is not available for feed-forward control, giving rise to the problem of distributed observer. By Corollary \([3]\) and \([33]\), there holds
\[
v(t) = -F\hat{x}(t), \quad \hat{x}(t) = x(t) - \Pi x_0(t).
\]
Consequently the FI control protocol \([8]\) for \(1 \leq i \leq N\) yields
\[
u(t) = M_Dv(t) = -M_DF\hat{x}(t).
\]
In addition the result on \(A\Pi = \Pi A_0\) in Corollary \([3]\) shows that
\[
\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t), \quad e_y(t) = C\hat{x}(t),
\]
by \(\dot{\hat{x}}(t) = Ax(t) + Bu(t) - A\Pi x_0(t) = Ax(t) + Bu(t) - A\Pi x_0(t)\). Hence we have an estimation problem for \(\hat{x}(t)\), governed by the state space system in \([35]\), rather than for both \(x(t)\) and \(x_0(t)\). As such the relative output measurement for agent \(i\) in \([16]\) can be expressed as
\[
\epsilon_{i,j}(t) = y_i(t) - y_j(t) = e_{yi}(t) - e_{yj}(t) = C_i\hat{x}_i(t) - C_j\hat{x}_j(t).
\]
Recall that $e_{yi}(t) = y_i(t) - y_0(t)$ is the tracking error for the $i$th agent, i.e., the $i$th component of $e_y(t)$. There is at least one agent that has the access to

$$e_{yiR} = y_i(t) - C_0x_0(t) = C_iR_i x_i(t) - C_0x_0(t) = C_iR_i \tilde{x}_iR(t)$$

with index $i_R$ corresponding to a reachable node. The above leads to the distributed estimator:

$$\hat{x}_i = A_i\hat{x}_i + B_iu_i + d_i\delta K(i - i_R)L_iC_ie_{\tilde{x}_i} + d_iL_i\sum_{j=1}^{N} a_{ij}[C_i e_{\tilde{x}_i} - C_j e_{\tilde{x}_j}]$$

modified from \[17\] with $e_{\tilde{x}_k}(t) = \tilde{x}_k(t) - \hat{x}_k(t)$. The next result is parallel to Theorem 2.

**Theorem 4** Suppose that the feedback graph $G$ is connected, and each agent admits a stabilizable and detectable realization. Let $D$ satisfying \[7\] be known. Then there exist $F_i$ and $L_i$ such that both $T_{F_i}(s)$ and $T_{L_i}(s)$ in \[22\] are SPR, and the distributed estimator \[38\] and protocol $u(t) = -MDF\tilde{x}(t)$ achieve output consensus.

Proof: Taking difference between $\hat{x}_i(t) = A_i\hat{x}_i(t) + B_iu_i(t)$ and local estimator \[38\] yields

$$\dot{e}_{\tilde{x}_i} = A_i e_{\tilde{x}_i} + d_i\delta K(i - i_R)L_iC_i e_{\tilde{x}_i} + d_iL_i\sum_{j=1}^{N} a_{ij}[C_i e_{\tilde{x}_i} - C_j e_{\tilde{x}_j}]$$

for each $i \in \mathcal{N}$. Packing together leads to the collective error equation of

$$\dot{e}_x(t) = (A - LMDF)e_x(t).$$

It is now easy to verify that with $u(t) = -MDF\tilde{x}(t)$, the overall feedback MAS admits the state space equation

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ \dot{e}_{\tilde{x}}(t) \end{bmatrix} = \begin{bmatrix} A - BMDF & BMDF \\ 0 & A - LMDF \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ e_{\tilde{x}}(t) \end{bmatrix}.$$ \[39\]

In light of the hypothesis on $T_{F_i}(s)$ and $T_{L_i}(s)$ being SPR, both $A - BMDF$ and $A - LMDF$ are Hurwitz. Recall the proof of Theorem 1. It follows that $\tilde{x}(t) \to 0$ and $e_{\tilde{x}}(t) \to 0$ as $t \to \infty$. Consequently $\tilde{x}(t) \to 0$ as $t \to \infty$. In light of the relations $C_0 = C_i\Pi_i$ in \[30\] and $\tilde{x}_i(t) = x_i(t) - \Pi_i x_0(t)$, there holds

$$C_i\tilde{x}_i(t) = C_i[x_i(t) - \Pi_i x_0(t)] = C_i x_i(t) - C_0 x_0(t) = y_i(t) - y_0(t) \to 0$$

as $t \to \infty$. Hence the output consensus is achieved. \hfill \square

**Remark 6** It is important to note that due to equation \[37\], the reference trajectory $y_0(t) = C_0 x_0(t)$ is transmitted to only one node, i.e., one of the reachable nodes with index $i_R$. This has the advantage of lowering the communication overhead. However, there is a tradeoff between consensus robustness and communication overhead. If $y_0(t)$ is transmitted to more than one nodes, then the single point of failure scenario is avoided. This way the closed loop system is more robust in the presence of broken communication links. There exists a tradeoff between the communication overhead and consensus robustness. \hfill \square
5 An Illustrative Example

In this section we will demonstrate with a numerical example the output consensus results presented in this paper. Following [11], consider a system of 8 point masses moving in one spatial dimension. Dynamics are governed by

\[
\dot{x}_i = A_i x_i + B_i u_i, \quad y_i = C_i x_i
\]

where \( A_i = \begin{bmatrix} 0 & 1 \\ 0 & -f_{d_i} \end{bmatrix} \), \( B_i = \begin{bmatrix} 0 \\ \frac{1}{m_i} \end{bmatrix} \), and \( C_i = [1 \ 0] \) for \( i = 1, 2, 3, 7, 8 \); and

\[
\dot{x}_i = \frac{1}{m_i} u_i, \quad y_i = x_i
\]

for \( i = 4, 5, 6 \). The first set of dynamics represents agents that experience drag forces and whose acceleration is directly controlled, while the second set represents agents whose velocity is directly controlled. The output signal corresponds to the position of the point mass. Figure 1 shows the interconnection graph for the network of 8 agents, with parameters \( \{m_i\} = \{2.5, 3, 4, 1, 2, 5, 7, 6\} \) and \( \{f_{d_i}\} = \{0.5, 0.9, 1.9, 0, 0, 0, 1.1, 3.9\} \).

![Graph for N = 8 point masses.](image)

It is important to note that the network graph is connected with only one reachable node at node 1. Consequently condition (6) is satisfied with \( i_R = 1 \). This means that only agent 1 has direct access to its own full information control signal, every other agent only has access to the difference between its and its neighbor’s full information control signal. The consensus problem for this example is to synchronize all agents to the signal \( y_0(t) = \cos(\omega_0 t) \) generated by the reference model (3) with \( \omega_0 = 3.77 \). To satisfy assumption (a), we use a weighting function, \( W(s) \), with poles at \( \pm j\omega_0 \) so that \( P_i(s)W(s) \) (where \( P_i(s) \) is the transfer function of agent \( i \)) contains the eigenvalues of the reference model, i.e., \( \pm j\omega_0 \). In addition, \( W(s) \) includes a proportional-derivative (PD) compensator by adding a finite stable zero to its dynamics which helps to enlarge the bandwidth and thus enhance the transient performance of the control system. The state feedback and estimation gains are computed for each agent individually by solving ARE (23) and its dual, respectively. The feedforward gain for each agent could be computed by solving (28) for each agent. However, we take this opportunity to present a simple method that may be used in place of solving (28): The feedforward gain has the form \( F_0 = [F_{0_1} \ F_{0_2}] \). Denoting \( X_0(s) \) as the Laplace transform of \( x_0(t) \), it is easy to see that for each agent in closed loop,

\[
Y_i(s) = T_i(s)F_0 X_0(s) = T_i(s) \left( \frac{\omega_0 F_{0_1}}{s^2 + \omega_0^2} + \frac{s F_{0_2}}{s^2 + \omega_0^2} \right) = \frac{K}{s - j\omega_0} + \frac{K^*}{s + j\omega_0}
\]

by partial fraction expansion. Thus, to enforce \( y_i(t) = \cos(\omega_0 t) \) and achieve tracking we must choose \( K \) such that \( |K| = 1 \) and \( \angle K = 0 \). In other words, \( F_0 \) must be chosen such that

\[
\sqrt{F_{0_1}^2 + F_{0_2}^2} = \frac{1}{T_i(j\omega_0)}, \quad \angle \left[ F_{0_1} + jF_{0_2} \right] = \frac{\pi}{2} - \angle T_i(j\omega_0)
\]
admit a solution $F_{0_1}$ and $F_{0_2}$. Indeed, it is always possible to find a unique solution to both equations. It is important to remember that control design, either using our proposed method or by solving (28), can be carried out in a fully distributed manner. Furthermore, the simple method we have proposed to compute $F_{0_1}$ is applicable to tracking step and ramp functions, and sinusoids with arbitrary amplitudes and phase angles. In fact, it is applicable when the signal to track consists of any linear combination of the previously mentioned functions. Figure 2 shows that output consensus is achieved when the state vector of each agent and the reference model is available. For the case when the distributed estimator in (38) is required, Figure 3 illustrates that consensus is also achieved.

![Figure 2: Output consensus using a full information control law.](image1)

![Figure 3: Output consensus using a full information control law with distributed estimator.](image2)

The simulation study is also carried out for the unit ramp reference signal. Figure 4 shows that output consensus is achieved under the FI control protocol when the state vector of each agent and the reference model is available. When the states of the $N$ agents and the reference model are unavailable, the FI control signal can be estimated using the distributed estimator in (38). Figure 5 shows that output consensus is achieved again.
For both sinusoidal and ramp reference signals, the consensus control based on the estimator in [38] takes a longer time to synchronize the output signals. In addition, there are greater fluctuations in the initial stage of the consensus process than that based on the FI control protocol. The reason lies in the fact that the estimation error affects the tracking performance negatively, and full synchronization is not possible until the estimation error is negligibly small. Certainly, there are incentives to design faster estimators for consensus control.

6 Conclusion

Output consensus control for continuous-time heterogeneous MASs is studied in this paper, aimed at synchronizing all the agents’ output to the desired trajectory generated by a reference model. Our contribution includes the use of the positive real real property of transfer matrices in achieving the distributed stability for the feedback MAS, and the synthesis algorithm developed for design of the consensus protocol to achieve output consensus. We have demonstrated that output consensus
control can be decomposed into tracking regulation for each individual agent control system, provided that the graph is connected and each agent’s dynamics contain the reference dynamics as its internal modes. In establishing our main results, Lemma 1 and the positivity of the graph matrix $M_D$ in Section 2 play a fundamental role. In order to achieve consensus to a reference trajectory, it is sufficient for one agent to have access to the reference signal, which lowers the communication overhead for the MAS. In addition it is not necessary to duplicate the reference model in each of the $N$ local and distributed feedback controllers, thereby eliminating synchronization of the local reference models commonly required in the existing work for consensus control. Thus the communication cost can be lowered further. Furthermore the communication graph can be different at the input and output. Our controller synthesis is based on $H_\infty$ loop shaping and LQG/LTR methods, and therefore can accommodate performance and robustness requirements. How these local properties translate to the robustness and performance of the collective dynamics is currently under study.

References

[1] B.D.O. Anderson and J.B. Moore, *Optimal Control – Linear Quadratic Methods*, Prentice-Hall, Englewoods Cliffs, NJ, 1990.

[2] Chopra, N., Spong, M., 2008, “Output synchronization of nonlinear systems with relative degree one,” in: Blondel, V., Boyd, S., Kimura, H. (Eds.), *Recent Advances in Learning and Control*, vol. 371 of Lecture Notes in Control and Information Sciences, Springer-Verlag, pp. 51-64, 2008.

[3] H. Grip, T. Yang, A. Saberi, and A. Stoorvogel “Output synchronization for heterogeneous networks of non-introspective agents,” *Automatica*, vol. 48, pp. 2444-2453, 2012.

[4] G. Gu, “On the existence of linear optimal control with output feedback,” *SIAM J. Contr. and Optimiz.*., vol. 28, pp. 711-719, May 1990.

[5] J. Hu and Y. Hong, “Leader-following coordinaton of multi-agent systems with coupling time delays,” *Physica A*, 374, pp. 853-863, 2007.

[6] J. Huang, *Nonlinear Output Regulation: Theory and Applications*, SIAM Publisher, 2004.

[7] A. Jadbabaie, J. Lin, and A. Morse, “Coordination of groups of mobile autonomous agents using nearest neighbor rules,” *IEEE Trans. Automat. Contr.*, vol. 48, no. 6, pp. 988-1001, 2003.

[8] H. Khalil, *Nonlinear Systems*, 3rd edition, Prentice Hall, 2001.

[9] H. Kim, H. Shim, and J. H. Seo, “Output consensus of heterogeneous uncertain linear multi-agent systems,” *IEEE Trans. Automat. Contr.*, vol. 56, no. 1, pp. 200-206, 2011.

[10] G. Lafferriere, A. Williams, J. Caughman, and J. Veerman, “Decentralized control of vehicle formations,” *Syst. & Contr. Lett.*, vol. 54, no. 9, pp. 899-910, 2005.

[11] D. Lee and M. W. Spong, “Stable flocking of multiple inertial agents on balanced graphs,” *IEEE Trans. Automat. Contr.*, vol. 52, no. 8, pp. 1469-1475, 2007.
[12] T. Li and J. Zhang, “Consensus conditions of multi-agent systems with time-varying topologies and stochastic communication noises,” *IEEE Trans. Automat. Contr.*, vol. 55, pp. 2043-2057, Sept. 2010.

[13] Z. Li, Z. Duan, G. Chen, and L. Huang, “Consensus of multi-agent systems and synchronization of complex networks: A unified viewpoint,” *IEEE Trans. on Circuits Syst. I*, vol. 57, no. 1, pp. 213-224, 2010.

[14] X. Li, X. Wang, and G. Chen, “Pinning a complex dynamical network to its equilibrium,” *IEEE Trans. on Circuits Syst. I*, vol. 51, no. 10, pp. 2074-2087, 2004.

[15] Z. Lin, B. Francis, and M. Maggiore, “Necessary and sufficient graphical conditions for formation control of unicycles,” Dept. ECE, Univ. Toronto, Toronto, ON, Canada.

[16] J. Lunze, “Synchronization of heterogeneous agents,” *IEEE Trans. Automat. Contr.*, vol. 57, no. 11, pp. 2885-2890, 2012.

[17] C. Ma and J. Zhang, “Necessary and sufficient conditions for consensusability of linear multi-agent systems,” *IEEE Trans. Automat. Contr.*, vol. 55, no. 5, pp. 1263-1268, 2010.

[18] D.C. McFarlane and K. Glover, *Robust Controller Design using Normalised Coprime Factor Plant Descriptions*, Springer Verlag, Lecture Notes in Control and Information Sciences, vol. 138, 1989.

[19] L. Moreau, “Stability of continuous-time distributed consensus algorithms,” in *Proc. IEEE Conf. Decision and Control*, pp. 3998-4003, Nassau, 2004.

[20] L. Moreau, “Stability of continuous-time distributed consensus algorithms,” arXiv:math.OC/0409010v1, Sept. 2004.

[21] R. Olfati-Saber, J. Fax, and R. Murray , “Consensus and cooperation in networked multi-agent systems,” *Proc. IEEE*, vol. 95, no. 1, pp. 215-233, 2007.

[22] R. Olfati-Saber, and R. Murray , “Consensus problems in networks of agents with switching topology and time-delays,” *IEEE Trans. Automat. Contr.*, vol. 49, no. 9, pp. 1520-1533, 2004.

[23] W. Ren and R. Beard “Consensus seeking in multiagent systems under dynamically changing interaction topologies,” *IEEE Trans. Automat. Contr.*, vol. 50, no. 5, pp. 655-661, 2005.

[24] W. Ren, R.W. Beard, E.M. Atkins, “Information consensus in multivehicle cooperative control,” *IEEE Control Systems Magazine*, vol. 27, no. 2, pp. 71-82, 2007.

[25] W. Ren, R. Beard, and D. Kingston “Multi-agent Kalman consensus with relative uncertainty,” in *2005 American Control Conference*, pp. 1865-1870, Portland, OR, 2005.

[26] L. Scardovi, R. Sepulchre, “Synchronization in networks of identical linear systems,” *Automatica*, vol. 45, no. 11, pp. 2557-2562, 2009.

[27] Y. Tian and Y. Zhang, “High-order consensus of heterogeneous multi-agent systems with unknown communication delays,” *Automatica*, vol. 48, pp. 1205-1212, 2012.

[28] P. Wieland, R. Sepulchre, and F. Allgöwer, “An internal model principle is necessary and sufficient for linear output synchronization,” *Automatica*, vol. 47, pp. 1068-1074, 2011.
[29] J.C. Willems, “Lyapunov functions for diagonally dominant systems,” *Automatica*, vol. 12, pp. 519-523, 1976.

[30] Z. Zhang and J.S. Freudenberg, “Loop transfer recovery for nonminimum phase plants,” *IEEE Trans. Automat. Contr.*, AC-35:547-553, May 1990.

[31] H. Zhang, F.L. Lewis, and A. Das, “Optimal design for synchronization of cooperative systems: State feedback, observer and output feedback,” *IEEE Trans. Automat. Contr.*, vol. 56, no. 9, pp. 1948-1952, 2011.

[32] J. Zhao, D.J., Hill, and T. Liu, “Synchronization of dynamical networks with nonidentical nodes: Criteria and control,” *IEEE Trans. Circ. Sys. - I. Reg.*, vol. 58, no. 3, pp 584-594, 2011.