Non Linear Lorentz Transformation and Doubly Special Relativity

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We generate non-linear representations of the Lorentz Group by unitary transformation over the Lorentz generators. To do that we use deformed scale transformations by introducing momentum-depending parameters. The momentum operator transformation is found to be equivalent to a particle momentum transformation. The configuration space transformation is found to depend on the old momentum operator and we show that this transformation generates models with two scales, one for the velocity (c) and another one for the energy. A Lagrangian formalism is proposed for these models and an effective metric for the deformed Minkowski space is found. We show that the Smolin model is one in a family of doubly special relativity. Finally we construct an ansatz for the quantization of such theories.

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I. INTRODUCTION

There are several theoretical reasons to establish a new physical and fundamental scale of nature that would enable us to explore deep into the transition threshold between general relativity and quantum mechanics [1], [2]. One of those reasons is to come closer to a theory of quantum gravity QG [3]. Traditionally, the Planck’s length $L_p = \sqrt{\hbar G/c^3}$ has been postulated as a fundamental scale. However, there are other ways to explore this second invariant scale. These are called deformed special relativity (DSR) models and can considered a theoretical limit of some larger theories of Quantum Gravity (QG) [4]. DSR models are being outlined as a phenomenological description of presently unknown quantum gravity effects [5]. Those models are initially thought as a way of modifying the dispersion relations of relativistic particles without introducing a preferred frame. This modifications have been used to look into CPT/Lorentz symmetry violations [6] and provide possible explanations of resulting baryon asymmetry in cosmology without using the Sakharov’s conditions [7].

In recent years many aspects of DSR models have been studied and some of them are still the subject of research [8], however there is no consensus about which is the position space associated with these theories.

In this work we present a method to construct some DRS models, applying deformed scale transformations to the Lorentz generators. In doing this, a non-linear representation of the Lorentz group arises which, in turn, leaves the Heisenberg algebra unchanged. The scale transformations are introduced by deforming a momentum-dependent scale parameter. These transformations are carried out on the quantum operator instead of on the eigenvalues, as it is usually presented in the literature [9]. We show the equivalence of these two approximations.

This work is organized as follows: in Sec. II some generalities about the Lorentz group are presented. In section III the deformed scale transformations of the momentum and position operators are calculated and in section IV the non linear Lorentz transformation is applied to both the momentum and position operators. The new dispersion relations are found and the two scales are shown explicitly for these theories. Finally, in section V the equivalence between the transformation on the momentum operators and on the momentum of the particles is proven and some particular cases are analyzed.

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II. GENERAL CONSIDERATIONS

It is well known that the angular momentum operators can be written as a function of the momentum and position operators as follows

\[ M^\alpha\beta = i(p^\alpha x^\beta - p^\beta x^\alpha). \]  

(1)

They are the boosts and rotations generators and satisfy the Lorentz algebra

\[ [M_{\alpha\beta}, M_{\mu\nu}] = -i(g_{\alpha\nu}M_{\beta\mu} + g_{\beta\mu}M_{\alpha\nu} - g_{\alpha\mu}M_{\beta\nu} - g_{\beta\nu}M_{\alpha\mu}), \]  

(2)

this algebra is a consequence of the more fundamental Heisenberg algebra

\[ [x^\alpha, x^\beta] = 0, \quad [p^\alpha, p^\beta] = 0, \quad [p^\alpha, x^\beta] = ig^{\alpha\beta}. \]  

(3)

The set of \( M \) matrices in (1) are the generators of an unitary representation of the Lorentz group.

\[ \Lambda = \exp(i\omega_{\alpha\beta}M^{\alpha\beta}/2). \]  

(4)

We can find other unitary representations for the Lorentz group applying an unitary transformation to the group elements

\[ \Lambda \rightarrow \tilde{\Lambda} = U\Lambda U^\dagger. \]  

(5)

which implies the transformation on the generators

\[ M \rightarrow \tilde{M}^{\alpha\beta} = U M^{\alpha\beta} U^\dagger. \]  

(6)

these new generators satisfy the same Lorentz algebra (2).

It is clear that if \( U \) and the \( M \)'s commute, these representations are actually the same. Thus, in order to find other non trivial equivalent representations, \( U \) must not be a Lorentz scalar. That transformation on the generators is carried out by the transformation on the basic operators

\[ p^\alpha \rightarrow \tilde{p}^\alpha = Up^\alpha U^\dagger, \]  

(7)

\[ x^\alpha \rightarrow \tilde{x}^\alpha = Ux^\alpha U^\dagger, \]  

(8)

which, as stated, preserve the Heisenberg algebra. We can build non standard representation of the Lorentz group using an adequate operator \( U \). In this paper we will construct such representation, starting with the ordinary scale transformation on the basic operators but using a parameter that depends on some components of the momentum operator in order to prevent \( U \) from being a Lorentz scalar. This is not a new idea; in fact the Magueijo-Smolin model [1] is generated in this way, but constructed in the momentum representation. We will extend that procedure for the quantum field operators.

III. DEFORMED SCALE TRANSFORMATIONS

Let us now build the ordinary scale transformations in order to gain the necessary insight and learn how they can be deformed. The finite scale transformation is given in equation (7) with \( U = \exp(eD) \) are

\[ \tilde{p}^\alpha = e^{D}p^\alpha e^{-eD}, \]  

(9)

where \( e \) is a parameter and \( D \) is the dilatation operator given by

\[ D = ip_\alpha x^\alpha + c, \]  

(10)

with \( c \) some constant. The Heisenberg algebra (3) implies

\[ [D, p^\alpha] = p^\alpha, \quad [D, x^\alpha] = -x^\alpha. \]  

(11)

The ordinary scale transformations mean that \( e \) is a constant and the transformation (11) can be reduced to \( \tilde{p}^\alpha = e^{p^\alpha} \); in this case and since \( M^{\alpha\beta} \) commutes with \( D \), we have \( \tilde{\Lambda} = \Lambda \).

In order to avoid the usual scale transformation one can propose that \( e \) does not commute with the \( M^{\alpha\beta} \), although it should indeed commute with \( p^\alpha \). This can be done by choosing \( e \) as a function of \( p \) and not Lorentz invariant. In this case we obtain one class of deformed scale transformations.

We will work with the scale parameter \( e \) as an homogeneous function of degree \( s \) in \( p \), that is

\[ e(ap) = a^s e(p), \]  

(12)

where \( a \) is a constant. Now \( e \) has dimension \( s \) and we have

\[ [D, e] = se, \]  

(13)

where the operator \( D \) is given in (10) with \( \text{Re} \ c = 2+s/2 \), in order to make \( eD \) anti Hermitian or equivalently, \( U \) unitary.
A. Deformed Scale Transformation of $p$ and $x$

The expression (9) can be written, using a Hausdorff expansion, as

$$\tilde{p}^\alpha = \sum_{n=0}^{\infty} \frac{[(\epsilon D)^{(n)}], p^\alpha]}{n!},$$

where $[\ldots]$ is the multiple commutator defined by the recurrence relation

$$[A^{(n+1)}, B] = [A, [A^{(n)}, B]],$$

with the initial condition $[A^{(0)}, B] = B$. Proposing $[(\epsilon D)^{(n)}, p^\alpha] = \theta_n \epsilon^n p^\alpha$ and using (15) we find $\theta_{n+1} = (ns + \epsilon) \theta_n$, with the initial condition $\theta_0 = 1$, this gives

$$\theta_n = (-1)^n s^n \frac{(-1/s)!}{(-1/s - n)!}. \tag{16}$$

Introducing (16) in (14) and adding all the terms, we have

$$\tilde{p}^\alpha = (1 - s\epsilon)^{-1/s} p^\alpha. \tag{17}$$

In the simplest case, $s = 0$, we obtain $\tilde{p}^\alpha = e^\epsilon p^\alpha$ as expected when $\epsilon$ is a constant; nevertheless in (17) with $s = 0 \epsilon$ can be a function of $p$.

Similarly, the transformation of $x$ is performed via a Hausdorff expansion for (8) which now reads

$$\tilde{x}^\alpha = \sum_{n=0}^{\infty} \frac{[(\epsilon D)^{(n)}, x^\alpha]}{n!} \tag{18}$$

In general the $n$-th commutator can be parametrized as

$$[(\epsilon D)^{(n)}, x^\alpha] = \alpha_n \epsilon^n x^\alpha + i \beta_n \epsilon^{n-1} \epsilon^\alpha D, \tag{19}$$

where

$$\epsilon^\alpha = i[x^\alpha, \epsilon] = \frac{\partial \epsilon(p)}{\partial p_\alpha}. \tag{20}$$

From (15) in (19) we obtain the recurrence relation

$$\alpha_{n+1} = (ns - \epsilon) \alpha_n,$$

$$\beta_{n+1} = [(n - 1)s - \epsilon] \beta_n + \alpha_n,$$

to finally find

$$\alpha_n = s^n \frac{(1/s)!}{(1/s - n)!} (-1)^n, \tag{21}$$

$$\beta_n = n \alpha_{n-1}, \tag{22}$$

where the initial conditions $\alpha_0 = 1$ and $\beta_0 = 0$ were considered.

Adding all the terms in (18) and considering (21) and (22) we obtain

$$\tilde{x}^\alpha = (1 - s\epsilon)^{1/s} [x^\alpha + i\epsilon^\alpha D]. \tag{23}$$

One can check from equations (17) and (23) that the canonical relations

$$[\tilde{p}^\alpha, \tilde{x}^\beta] = ig^{\alpha\beta}$$

still hold.

B. Non Linear Lorentz Transformations.

Using the expressions for the scaled momenta (17) and coordinates (23), one can construct explicitly the new Lorentz generators from (8),

$$\tilde{M}^{\alpha\beta} = M^{\alpha\beta} - i(p^\alpha \epsilon^\beta - p^\beta \epsilon^\alpha) D. \tag{24}$$

The non-linear Lorentz transformations over the momentum operators are therefore

$$p^\alpha \rightarrow \tilde{p}^\alpha = \tilde{\Lambda}^\alpha \tilde{p}^\alpha \tilde{\Lambda}. \tag{25}$$

Associativity of this transformation can be implemented in steps, $\tilde{p}^\alpha = U(\tilde{\Lambda}^1(U^1 p^\alpha U^1)\Lambda(U^1)^1$, to give

$$\tilde{p}^\alpha = [1 + s(\epsilon' - \epsilon)]^{-1/s} p^\alpha', \tag{26}$$

where $\epsilon'$ is the $\epsilon$ function applied over $p^\alpha' = \Lambda^\alpha'_{\beta} p^\beta$. From (26) it can be seen that if $\epsilon$ is a Lorentz scalar, then $\epsilon = \epsilon'$ and $\tilde{p}^\alpha = p^\alpha$.

In the same fashion, as in the case of $p$, under non-linear Lorentz transformation, $x$ transforms as

$$\tilde{x}^\alpha = [1 + s(\epsilon' - \epsilon)]^{1/s} [x^\alpha' - i(\epsilon^\alpha' - \epsilon^\alpha) D], \tag{27}$$

where $\epsilon^\alpha' = \Lambda^\alpha'_{\beta} \partial \epsilon'(p) / \partial p_\beta$ and $\epsilon^\alpha' = \partial \epsilon'/\partial \beta^\alpha'$. Once again, from (26) and (27), it follows that the new operators satisfy the canonical commutation relations

$$[\tilde{p}^\alpha, \tilde{x}^\beta] = ig^{\alpha\beta}. \tag{28}$$
IV. GENERAL PROPERTIES OF THE NON-LINEAR LORENTZ TRANSFORMATION.

A simultaneous unitary transformation over the momentum operators \( p \) and coordinates \( x \) can be called a canonical transformation, just like in classical mechanics. This is because it preserves the canonical commutation relations (3). This type of transformation can be seen as a passive canonical transformation since the coordinate operators in the phase space are transformed and the coordinates themselves occur are not. The corresponding active transformations occur when only the states are directly transformed.

Let the state \(|p_e\rangle\) be an eigenstate of \( p^\alpha \) with eigenvalue \( p^\alpha_e \),
\[
p^\alpha |p_e\rangle = p^\alpha_e |p_e\rangle.
\]
(28)
under non-linear Lorentz transformations, it changes as
\[
|p_e\rangle \rightarrow \Lambda |p_e\rangle.
\]
(29)
To find out how a new state appears we apply \( p^\alpha \) as in
\[
p^\alpha \Lambda |p_e\rangle = \Lambda (\Lambda^\dagger p^\alpha \Lambda) |p_e\rangle = \frac{p^\alpha_e}{[1 + s\epsilon(p^\alpha_e) - \epsilon(p_e)]^{1/2}} \Lambda |p_e\rangle,
\]
(30)
which means that \( \Lambda |p_e\rangle \) is an eigenstate of \( p^\alpha \) with eigenvalue \( p^\alpha_e / [1 + s\epsilon(p^\alpha_e) - \epsilon(p_e)]^{1/2} \). Thus one can write
\[
\Lambda |p_e\rangle = |\tilde{p}_e\rangle.
\]
(31)
The equivalence of active and passive transformations is realized as the invariance of the mean value of \( p \) when the transformation is carried out,
\[
\langle \psi | p | \psi \rangle \rightarrow \langle \psi | \tilde{p} | \psi \rangle = \langle \tilde{\psi} | p | \tilde{\psi} \rangle.
\]
(32)
According to (27) we can see that it is not possible to proceed similarly for the coordinates because the operator transforms mixing the coordinate and momentum operators.

An apparent paradox is found here since coordinate transformations can be written in such a way that the particle spacial coordinates are now a function of its energy. This can be explained by considering that, from the operator point of view, the new spacial coordinates depend on the old momenta, so the new and old coordinates do not actually commute, although the new momenta do commute with their untransformed partners.

A point in space time is the eigenvalue of the \( x \) eigenstate, which describes a particle with a well defined position. This state, as seen by any other observer, is a superposition of the \(|\tilde{p}\rangle\) eigenstates, so it does not have a well defined position in the new system.

A. Velocity Scale.

From now on we will work with the eigenvalues of the momentum operators instead of the operators themselves and we will omit the subindex \( e \). From (26) we conclude that the moment eigenvalues satisfy
\[
\frac{\hat{\mathbf{p}}}{\mathbf{p}^0} = \frac{\mathbf{p}'}{\mathbf{p}^0'}.
\]
(33)
Calling \( \mathbf{p}/\mathbf{p}^0 \) the Lorentz velocity \( \mathbf{v} \), we have \( \mathbf{v} = \mathbf{v}' \), from the linear Lorentz transformation (33) can be written in terms of the velocities as
\[
\hat{\mathbf{v}}_\perp = \frac{\beta - v_\parallel}{1 - \beta v_\parallel}, \quad \hat{\mathbf{v}}_\parallel = \frac{v_\parallel}{\Gamma(1 - \beta v_\parallel)},
\]
(34)
where \( \perp \) and \( \parallel \) stand for the parallel and perpendicular components of the Lorentz velocities with respect to the relative velocity \( \beta \), and
\[
\Gamma = \frac{1}{\sqrt{1 - \beta^2}}.
\]
(35)
This transformation satisfies the addition rule as in the conventional relativity, thus the speed of light \( v = 1 \) is still a natural scale of the theory. To see that more clearly one can write (34) as
\[
1 - \hat{v}^2 = \frac{1}{\Gamma^2(1 - \beta v_\parallel)^2}(1 - \beta^2).
\]
(36)
From (36) we see that if \( \beta < 1 \) and \( v \leq 1 \) then \( \hat{v} \leq 1 \). But the Lorentz velocity is not the real particle velocity, it is only the velocity of the particle in the limit \( \epsilon \rightarrow 0 \). The connection between Lorentz velocity and particle velocity will be developed in the next subsection.

B. Dispersion Relations.

The function \( \epsilon(\hat{p}) \) is given by
\[
\hat{\epsilon} \equiv \epsilon(\hat{p}) = \frac{\epsilon'}{1 + s(\epsilon' - \epsilon)},
\]
(37)
where the homogeneity of $\epsilon$ \cite{12} was used. From this expression and \cite{26} one can obtain
\[ \frac{\tilde{p}^\alpha}{\tilde{\epsilon}^{1/2}} = \frac{p^\alpha'}{\epsilon^{1/2}}. \] (38)

Moreover, from \cite{37} we get
\[ \frac{\tilde{\epsilon}}{\epsilon} = \frac{1 - s\tilde{\epsilon}}{1 - s\epsilon}, \] (39)

therefore \cite{38} is written as
\[ \frac{\tilde{p}^\alpha}{(1 - s\tilde{\epsilon})^{1/2}} = \frac{p^\alpha'}{(1 - s\epsilon)^{1/2}}. \] (40)

Squaring both sides of \cite{40} we find an invariant quantity, which can be identified as the invariant mass of the particle
\[ \frac{p^2}{(1 - s\epsilon)^{2/2}} = m^2. \] (41)

This equation also gives us the new dispersion relations; that is, a new relation between momentum and energy.

Despite some discussions in the literature on a proper definition of particle speed in the DSR theories, \cite{3, 10}, in this paper we take the particle velocity as the group velocity $\mathbf{u} = \partial p_0/\partial \mathbf{p}$ which is different from the Lorentz velocity $\mathbf{v}$. Starting from \cite{41} and taking the $\mathbf{p}$ derivative we find
\[ \mathbf{u} = \mathbf{v} \left( \frac{1 - s\epsilon + p^i\epsilon_i/(\gamma v)^2}{1 - s\epsilon - p^i\epsilon_i/\gamma^2} \right), \] (42)

where $i = 1, 2, 3$ and $\epsilon_i = \partial \mathbf{v}/\partial p^i$. One can see that if $|\mathbf{v}| \to 1$ then $1/\gamma^2 \to 0$ and $|\mathbf{u}| \to 1$. This means that the particle velocity has the same limit that the Lorentz velocity.

At this point we can calculate all the dynamics of the free particle starting from the Hamiltonian, which is given by $H = \int \mathbf{u} \cdot d\mathbf{p} = p^0$; nevertheless, in what follows we will use the Lagrangian formalism in covariant form, which in principle is equivalent.

\section{Energy Scale}

We will now show that these type of models have both a momentum and an energy scale. Let us start analyzing first the massless particles, that is $|\mathbf{v}| = 1$ or $|\mathbf{p}| = p_0$. With this in mind and reversing \cite{40} we can compute
\[ \frac{1}{(\tilde{p}^0)^s} - \frac{s\tilde{\epsilon}}{(\tilde{p}^0)^s} = \frac{1}{\Gamma^s(1 - \beta v^0_\parallel)^s} \left( \frac{1}{(p^0)^s} - \frac{s\epsilon}{(p^0)^s} \right) \] (43)

and because $\epsilon' = \epsilon(\Lambda p)$ and a Lorentz transformation with $p^0 = |\mathbf{p}|$ we have
\[ \epsilon' = \epsilon \left( \Gamma(1 - \beta v^0_\parallel) \right) = \Gamma^s(1 - \beta v^0_\parallel)^s \epsilon, \]

we obtain
\[ \frac{\epsilon^{1/2}}{p^0} = \frac{\epsilon^{1/2}}{p^0}. \] (44)

According to \cite{38} and \cite{41} one can see that $\frac{\epsilon^{1/2}}{p^0}$ is an invariant for a massless particle; this quantity has length units and therefore can be equated to some length $l_p$ that could be the same order of the Planck length
\[ l_p = \frac{(s\epsilon)^{1/2}}{p^0}, \] (45)

thus \cite{43} is written as
\[ \frac{1}{(\tilde{p}^0)^s} - \frac{1}{\Gamma^s(1 - \beta v^0_\parallel)^s} = \frac{1}{(p^0)^s - l_p^s}. \] (46)

If the particle energy in some system satisfies $p_0 < 1/l_p$ then, in any other system, the energy will satisfy $\tilde{p}^0 < 1/l_p$, because $\Gamma(1 - \beta v^0_\parallel) > 0$.

For massive particles the energy scale is the same. It suffices to note that when the particle energy is increasing, the Lorentz velocity limit is 1. In this limit the massive particle behaves like a massless particle. Then the analysis for the massless particle applies in this limit as well. From \cite{45} we can see that for high energy massive particles the quantity $s\epsilon$ behaves like $(l_pl_{p0})^s$. Hence it is natural to think that, for low energy, $s\epsilon$ has the form $(l_pl_{p0})^s f(v)$ where $f(v) \to 1$ when $v \to 1$.

\section{COVARIANT LAGRANGIAN FORMULATIONS.}

In this section we will find the Lagrangian for free particles in some of these models. From the Lagrangian it could be possible to make and educated guess about the way in which the coordinates transform in those models. Actually, the dispersion relations allow us to find the particle energy as a function of the particle momentum, the only obstacle is that we must deal with a not easily solvable algebraic equation. Nevertheless when $s$ is small it becomes easier. In particular, we will concentrate in low values $s = 1$ and $s = 2$. For each value of $s$ we still can
choose the function of the Lorentz speed magnitude $v$. For simplicity we only consider functions of the type $v^r$ where $0 < r < s$ then for fixed values of $s$ and $r$ we have

$$se = \frac{m^s}{p^0}p^r |p|^r,$$

and the dispersion relation is

$$p^2 = m^2(1 - \frac{m^s}{p^0}p^r |p|^r)^{2/s}. $$

Therefore for $s = 1, 2$ we have five different models: (1,0), (1,1), (2,0), (2,1), (2,2) according to $(s, r)$ as in Table I. Here we can identify the model (1,0) with the Magueijo-Smolin Model [9].

For each of these models we need to find the conjugate momentum coordinates and the Lagrangian based on the dispersion relations associated with each model, following the procedure given in [10]. The idea is to construct the Lagrangian linearly in the velocities, imposing the dispersion relation as a constraint through a Lagrange multiplier in the following way

$$L = \dot{x} \cdot p - \frac{e}{2} p^2 - m^2(1 - 8\epsilon)^{2/s},$$

then applying the Euler-Lagrange equations for $p$ and using the constraint we finally find the Lagrangian as a function of the velocities. The results of this procedure for some models are shown in Table II.

**Table I: Lagrangian of the different models.**

| $(s, r)$ | Lagrangian |
|----------|------------|
| (1,0)    | $L = \frac{m}{1 - m^2 p^2} \left( \sqrt{\dot{x}^2_0 - (1 - m^2 p^2)\ddot{x}^2_0} - m_l \dot{x}_0 \right)$ |
| (1,1)    | $L = \frac{m}{1 + m^2 p^2} \left( \sqrt{(1 + m^2 p^2)\ddot{x}^2_0 - \ddot{x}^2_0} - m_l \dot{x} \right)$ |
| (2,0)    | $L = \frac{m}{\sqrt{1 + m^2 p^2}} \sqrt{\ddot{x}^2_0 - (1 + m^2 p^2)\ddot{x}^2_0}$ |
| (2,1)    | $L = \frac{m}{1 + m^2 p^2} \sqrt{\ddot{x}^2_0 - m^2 p^2 |x| \dot{x}_0 - \ddot{x}^2}$ |
| (2,2)    | $L = \frac{m}{\sqrt{1 + m^2 p^2}} \sqrt{(1 + m^2 p^2)\ddot{x}^2_0 - \ddot{x}^2}$ |

**A. Effective Metric.**

Except in the case $(1,1)$ these results suggest that the Lagrangians of those models can be written as a function of the effective metric $\tilde{g}_{\mu\nu}$, to give

$$L = m' \sqrt{\tilde{x}^0 \tilde{g}_{\mu\nu} \tilde{x}^\nu},$$

where $m'$ is chosen in such a way that $\tilde{g}_{ij} = -\delta_{ij}$.

In general $\tilde{g}_{\mu\nu}$ will depend on the mass of the particle and could depend on the direction of the vector over which it acts. We can propose a Lorentz transformation that leaves invariant this metric, $\tilde{\Lambda}^T \tilde{g} \tilde{\Lambda} = \tilde{g}$, writing $\tilde{g} = \Gamma^T g \Gamma$ we find that $\tilde{\Lambda} = \Gamma^{-1} \Lambda \Gamma$.

For Example in the models (1,0), (2,0) and (2,2) the metric and the $\Gamma$ matrix can be written as

$$\tilde{g} = \begin{pmatrix} 1/\alpha^2 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 1/\alpha & 0 \\ 0 & -1 \end{pmatrix}$$

where $m'$ and $\alpha$ are given in the Table II.

**Table II: Constants of the different models.**

| $(s, r)$ | $m'$ | $\alpha$ |
|----------|------|----------|
| (1,0)    | $m(1 - m^2 p^2)^{1/2}$ | $(1 + m^2 p^2)^{1/2} - (1 - m^2 p^2)^{-1/2}$ |
| (2,0)    | $m$ | $(1 + m^2 p^2)^{1/2} - (1 - m^2 p^2)^{-1/2}$ |
| (2,2)    | $m(1 + m^2 p^2)^{-1/2}$ | $(1 + m^2 p^2)^{1/2} - (1 - m^2 p^2)^{-1/2}$ |

The representation in the coordinate space of a Lorentz transformation is

$$\tilde{\Lambda} = \begin{pmatrix} \gamma & -\gamma \beta \alpha \\ -\gamma \beta / \alpha & \gamma P_\parallel + P_\perp \end{pmatrix},$$

where $P_\parallel = \beta \beta^T$ is parallel to the velocity projection operator and $P_\perp = 1 - P_\parallel$ is the corresponding perpendicular projection operator. This leads to the following velocity addition formula (in two dimension for simplicity)

$$u' = \frac{u + \beta / \alpha}{1 + u \beta / \alpha},$$

this implies that $u'$ depends on the mass of the particle. Note that for a photon in any system $u = 1$, this is because the photon is massless and $\alpha = 1$. In the other hand [8] implies that “together” is a relative concept because the coordinates of particles of different mass transform differently.
Finally the particle momentum $p_\mu = \partial L/\partial \dot{x}^\mu$ satisfy the constraint
\[ p_\mu g^{\mu \nu} p_\nu = \alpha^2 p_0^2 - p^2 = m'^2. \] (54)

It is easy to prove that this relation is equivalent to the dispersion relations given in [53] for the three models with the respective constants shown in Table [11].

**B. Quantum Field Theory.**

Something interesting in this kind of theories is the fact that there exists a natural Lorentz invariant cutoff in the loop integrals which appear to higher order in the quantum corrections. Let us first consider a real field scalar associated with a free particle. In order to quantize the theory the replacement $p_0 \to i\partial/\partial t$ and $p \to -i\nabla$, $(m' \to m)$ are made in [53] and the modified Klein-Gordon equation reads
\[ (\alpha^2 \partial_0^2 - \nabla^2 + m^2) \phi = 0. \] (55)
The Lagrangian for free particle associated to this equation would be
\[ L = \frac{1}{2} \left( \alpha^2 \dot{\phi}^2 - (\nabla \phi)^2 + m^2 \phi^2 \right), \] (56)
then momentum of the associated field $\pi(x) = \alpha^2 \dot{\phi}(x)$ which satisfy the canonical equal time commutations relations $[\phi(x), \pi(y)]_{x_0=y_0} = i\delta^3(x-y)$.

The Fourier expansion of the Klein-Gordon field is therefore
\[ \phi(x) = \int \frac{d^3 k}{(2\pi)^3 2\omega_k} \left( a(k)e^{-i\hat{k} \cdot x} + a^\dagger(k)e^{i\hat{k} \cdot x} \right), \] (57)
with $\hat{k} = (\omega_k/\alpha, k)$ and $\omega_k = \sqrt{k^2 + m^2}$, where the $a$ operator algebra is given by
\[ [a(k), a(k')] = 2\alpha \omega_k \delta^3(k-k'). \] (58)
Finally the propagator, in the momentum space has the form
\[ \Delta(k) = \frac{i}{\alpha^2 \omega_k^2 - k^2 - m^2}, \] (59)
where $\text{Im } m^2 \to 0^-$. When the interaction $\lambda \phi^4$ is introduced the one loop correction to the scalar propagator is
\[- i\Sigma(k) = \frac{\lambda}{8\pi^3} \int_0^{E_p} k^2 d|k| \int_{-E_p}^{E_p} dk_0 \frac{1}{\alpha^2 \omega_k^2 - k^2 - m^2}. \] (60)

When this integral is carried out the leading terms will contain terms proportional to $\sim E_p^2$ and $\sim \ln(E_p/m)$. Here $E_p$ acts a cutoff. This is somehow natural because the theory is invariant under this deformed Lorentz transformations. Nevertheless we would need a finite renormalization term in order to remove the quadratic and logarithmic terms in $E_p$. At this point we can not proceed any further with the radiative corrections. This is because it is not known how to add two momenta in order to find the vertex correction in this new relativity. This kind of problem is highly non trivial and is characteristic of all of these theories with two scales.

**VI. CONCLUSIONS.**

An alternative approach to the non-linear representations of Lorentz transformations has been introduced; in this approach one changes the generators of a scale deformed transformation as in equation (59). The deformation of the transformation is obtained using momentum operator-dependent parameters. When the transformation of the operator is calculated in terms of momentum eigenstates it becomes clear that this type of representations of the Lorentz Group correspond to a double scaled invariant models. We have found Smolin’s model is one of a family of such models, described here. Moreover, this type of transformations can be applied to the position operator and it was found that it transforms mixing both momentum and position operators, very much like a typical canonical transformation in classical mechanics. From this one can conclude that it is not possible to find a coordinate transformation for a particle in this model which does not contain the momentum operator. The interpretation is that having a precise position in space depends on the frame of the observer. If a particle appears at a definite position for one observer (occupying an eigenstate of the position operator in that frame) it appears for another observer, in relative movement, as having a position which is the combination of different eigenstates of the position operator. A covariant La-
grangian formulation of this relativistic models has been presented as well. This allows us to study this models introducing an effective metric which has a dependency on the mass of the particle. Advancing in this program we present a second quantization of a spinless particle in this conditions, invariant under the transformations found above. This theory, in principle, should be finite since the integrals in the momenta can be properly re-normalized. This is possible because the second scale of the theory implies a cut-off for the momenta of the particles in the quantized version of the model. Although there appear some quadratic terms on the energy scale they could be removed as well using renormalization again. Vertex corrections are a problem since the addition rules for the momenta in these models is not well understood and therefore vertex corrections are yet to be calculated. Double scaled Lorentz transformations are an interesting approach to understand a fundamental length scale in nature. This work advances that understanding and proposes a whole family which systematically introduces those models, although several problems persist. More research would be needed to find for example non-linear Lorentzian transformations extended to include more parameters in the scale transformations that can depend on the coordinates. With this one would obtain a length, instead of an energy, cut-off.

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