N=1 Heterotic/M-theory Duality and Joyce Manifolds

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Abstract

We present an ansatz which enables us to construct heterotic/M-theory dual pairs in four dimensions. It is checked that this ansatz reproduces previous results and that the massless spectra of the proposed new dual pairs agree. The new dual pairs consist of M-theory compactifications on Joyce manifolds of \(G_2\) holonomy and Calabi-Yau compactifications of heterotic strings. These results are further evidence that M-theory is consistent on orbifolds. Finally, we interpret these results in terms of M-theory geometries which are \(K3\) fibrations and heterotic geometries which are conjectured to be \(T^3\) fibrations. Even though the new dual pairs are constructed as non-freely acting orbifolds of existing dual pairs, the adiabatic argument is apparently not violated.

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1 Orbifolds and M-theory.

The predictions of string dualities \cite{1, 2} have given rise to a fascinating web of interconnections between our most promising candidate descriptions of nature. An underlying structure is slowly emerging. In particular, it has now become apparent \cite{2, 21} that the strong coupling dynamics of both the TypeIIa and $E_8 \times E_8$ heterotic string theories can be understood in terms of certain one-dimensional compactifications of $M$-theory. Specifically the TypeIIa theory is related to $M$-theory on a circle, $S^1$, and the heterotic string to $M$-theory on $S^1/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ acts as reflection on the coordinate of the $S^1$.

A precise definition of $M$-theory is yet to be made; however consistency with the remarkable evidence that string theories in various dimensions are connected, not only to each other but to a supersymmetric theory in eleven dimensions ($M$-theory) gives us some information about some of the properties of $M$-theory. In particular, it seems clear that the low energy dynamics of $M$-theory are described by eleven-dimensional supergravity theory. Secondly, $M$-theory must share certain properties with string theory. For example, $M$-theory on an orbifold must be a consistent quantum theory \cite{21, 22}. This is certainly not a property shared by its low energy cousin. Finally, $M$-theory must contain higher dimensional objects, $p$-branes which play a fundamental role in the duality conjectures. One viewpoint is that fundamental strings in $d < 11$ arise from closed $p$-branes in $M$-theory, and that Dirichlet-branes (D-branes) in string theory \cite{23} arise from open $p$-branes in $M$-theory \cite{24}.

One hope is that one may be able to derive all connections between various string theories in lower dimensions from $M$-theory.

In \cite{2} evidence was presented that the strong coupling limit of the TypeIIa string theory in ten dimensions is effectively described by eleven dimensional supergravity compactified on a circle. The arguments leading to this conclusion are well known so we do not review them here and instead concentrate on the heterotic string. In \cite{21} evidence was presented that $M$-theory on an $S^1/\mathbb{Z}_2$ orbifold gives a description of the strongly coupled $E_8 \times E_8$ heterotic string. The arguments were as follows: The $\mathbb{Z}_2$ kills one of the two supersymmetries present in $M$-theory on a circle. The two fixed points which arise in the orbifold of the theory define two fixed ten dimensional hyperplanes, on which anomaly cancellation requires $E_8 \times E_8$ gauge symmetry to be present in the theory. This gauge symmetry was understood to have arisen from the
twisted sectors of the orbifold, in analogy with string theory.

One is then led to the picture that $M$-theory on $X \times S^1$ is associated to the TypeIIa theory on $X$, with $X$ any space. Similarly $M$-theory on $X \times S^1/Z_2$ is related to the $E_8 \times E_8$ heterotic string on $X$.

Consider then, $M$-theory on $T^3 \times S^1/Z_2$. One expects that this theory is related to the heterotic string on $T^3$. On the other hand, one expects that the strong coupling limit of the heterotic string on $T^3$ is related to eleven-dimensional supergravity on $K3[2]$. This implies various possibilities; one is that whatever $M$-theory may be, on $T^3 \times S^1/Z_2$ it is a theory which at low energies looks like eleven dimensional supergravity on $K3$. A second possibility is that $M$-theory on two different spaces, $K3$ and $T^3 \times S^1/Z_2$ are both related to the $T^3$ compactified heterotic string. We will show that a strikingly similar result arises in lower dimensions for dual $M$-theory/heterotic compactifications which have less supersymmetry. Specifically, for $N = 2$ and $N = 1$ heterotic string compactifications to four dimensions it will become apparent that there are again two $M$-theory compactifications which arise as the duals of these heterotic theories.

The general ambition of this paper is to find a heterotic dual for compactifications of $M$-theory on various Joyce 7-manifolds [13, 14]. We do this by means of an ansatz, which we present shortly. It will later transpire that (at least locally), these Joyce manifolds are $K3$ fibrations. It is thus natural to expect that, if the heterotic dual is the correct one, then we are discussing a fibration of the seven-dimensional duality between the $T^3$ compactified heterotic string and $M$-theory on $K3$ by the fibrewise application of the seven dimensional duality in the adiabatic limit [3]. However, if the heterotic dual is on some space $X$, then we also expect that we have an $M$-theory dual compactification on $X \times S^1/Z_2 [21]$. We can naturally interpret this as a fibration of the other $M$-theory/heterotic duality in seven dimensions.

The aim of the remainder of this section is to use an ansatz to rederive some of the previously constructed dual pairs, where apart from the $Z_2$ orbifold which defines the $K3$, all other elements of the orbifold group act freely.

Consider then $M$-theory on $T^4$. We can take a $Z_2$ orbifold of this theory in the following way.

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2A related discussion of these compactifications appeared in [31], where $M$-theory on $K3 \times S^1/Z_2$ was considered. From one point of view this ‘should’ give the heterotic string on $T^3 \times S^1/Z_2$ which is inconsistent. It was therefore argued that this $Z_2$ acts also on the $T^3$ giving the heterotic string on $K3$. 

3
\[
\alpha : (x_1, x_2, x_3, x_4) = (-x_1, -x_2, -x_3, -x_4)
\] (1)

where \((x_1,...,x_4)\) are the coordinates of the four torus.

This orbifold has sixteen fixed points and defines a particular orbifold limit of \(K3\). From our preceding discussions, one would expect that \(M\)-theory on such an orbifold has certain twisted sectors, associated with these fixed points, at which extra massless particles may arise. Given the lack of a definition of \(M\)-theory it is difficult to make any precise statements about such twisted sectors. However, one can draw an analogy again with string theory whereby blowing up the fixed points, we should recover the results of the theory on the resulting smooth manifold. In this case, the smooth manifold is \(K3\), and we expect that in this limit, the theory is dual to the heterotic string on \(T^3\) [4].

We now come to the first and most crucial part of the ansatz which will enable us to construct dual pairs of heterotic/\(M\)-theory compactifications in lower dimensions. The \textit{first} part of the ansatz is to label the heterotic \(T^3\) coordinates with the \textit{same} labels as three of the \(M\)-theory \(K3\) coordinates. We will then toroidally compactify both theories to four dimensions on a further \(T^3\) with coordinates \((x_5, x_6, x_7)\). We thus have \(M\)-theory on \(K3 \times T^3\) and the heterotic string on \(T^6\). The \textit{second} part of the ansatz is the following: we will take further \(Z_2\) orbifolds of this \(M\)-theory background giving vacua with \(N = 2\) and \(N = 1\) supersymmetry in four dimensions. The isometries defining these orbifolds will act on the \(M\)-theory coordinates, \((x_1,...,x_7)\). The crucial point is that, because of the labelling choice in the first part of the ansatz, the six coordinate labels of the heterotic string on \(T^6\) are a subset of the seven coordinate labels of the \(M\)-theory compactification. Thus the definition of an orbifold isometry in the \(M\)-theory geometry \textit{also} defines an orbifold isometry in the heterotic geometry. In general, if we begin with \(M\)-theory on a \(K3 \times T^3\) orbifold defined by \(\alpha\), as in eq.(1) and take a further orbifold of the theory, generated by a group of isometries denoted by \(\Theta\), then because of the labelling choice in the first part of our ansatz, \(M\)-theory on this \(T^7/\langle \alpha, \Theta \rangle\) orbifold should be equivalent to the heterotic string on a \(T^6/\Theta\) orbifold.

A priori this labelling choice may seem like a rather bizarre thing to do; however, given this as an ansatz and nothing more we will show that such a choice of labelling always gives rise to the correct heterotic/\(M\)-theory spectra
when we orbifold to produce new dual pairs in lower dimensions. This only appears to work when the only non-freely acting members of the orbifold group possess $K3$ orbifold singularities. More precisely, this construction only works when the singular set of $\Theta$, consists solely of $K3$ orbifold singularities. For the rest of the paper we thus choose the heterotic $T^3$ coordinates as $(x_1, x_2, x_3)$, where these are the same labels used to define the $K3$ in $M$-theory. A clue that this is the correct choice to make was given in [31] and we refer the reader there and to the previous footnote for details. The fact that this strategy appears to work in all cases strongly suggests that there is some as yet underlying structure to the way one can construct $M$-theory/heterotic dual pairs in dimensions less than seven. Further evidence of this has emerged in [31].

Now let us toroidally compactify the $x_5, x_6$ and $x_7$ directions. On $K3 \times T^3$, we thus relate $M$-theory to the heterotic string on $T^6$ and to the TypeII theory on $K3 \times T^2$. This four dimensional theory has $N = 4$ supersymmetry.

We can further orbifold this theory by a $Z_2$ isometry which gives $N = 2$ supersymmetry in four dimensions. This example was considered in [8]. The action of this isometry is defined on the seven coordinates on which $M$-theory is compactified as follows:

$$\beta : (x_1, \ldots, x_7) = (-x_1 + 1/2, -x_2, x_3, x_4 + 1/2, -x_5, -x_6, x_7 + 1/2) \quad (2)$$

Because of the half shifts on the torus defined by $(x_4, x_7)$, this $Z_2$ acts freely. When combined with the $Z_2$ defined by $\alpha$, the blown up $Z_2 \times Z_2$ orbifold will give $M$-theory on $CY_{11} \times S^1$; where the Calabi-Yau manifold $CY_{11}$ is self-mirror and has $h_{11} = 11$ [3]. This $M$-theory compactification is then related to the TypeIIa theory on $CY_{11}$. This thus gives rise to an $N=2$ theory whose massless spectrum at generic points in the moduli space of the vector multiplets is $(12, 12)$, where following [8] $(M, N)$ denotes an $N=2$ theory with $N$ vector multiplets and $M$ hypermultiplets.

In [8] and all known examples to date, the action of orbifold isometry groups (which act on the TypeIIa theory on $K3 \times T^2$) on the heterotic string on $T^6$ were calculated using the connection between the lattice of integral cohomology of $K3$ and the Narain lattice for the $T^4$ compactified heterotic string. In other words, all existing dual pairs, constructed as orbifolds, have been derived from string-string duality in six dimensions.

Now, let us suppose that we know nothing about the cohomology of $K3$, but that we know how to construct consistent heterotic string orbifolds.
Then we can ask how can we reproduce the \( N = 2 \) \((12, 12)\) spectrum from a heterotic \( Z_2 \) orbifold? Because of our ansatz, the action of \( \beta \) on the \( M \)-theory geometry also defines its action on the \( T^6 \) of the heterotic theory, because the \( T^6 \) coordinates we have chosen are \((x_1, x_2, x_3, x_5, x_6, x_7)\). Thus, all that remains is to specify the action on the gauge degrees of freedom i.e. the sixteen left movers. Given that \( \beta \) acts freely on \((x_3, x_7)\), we have an invariant two-torus, which will give rise in general to four vector multiplets. Thus, all that remains is to project out eight of the sixteen possible additional vectors which are associated with the Cartan subalgebra of the ten dimensional heterotic gauge group. Because the orbifold is of order two, we know that essentially the only possible action is exchanging the two \( E_8 \) factors in the gauge group. Finally, in order to achieve modular invariance we are forced to include an asymmetric \( Z_2 \) shift\(^3\) which is related to the shift on \( x_7 \). This reproduces the model of \([8]\) without any knowledge of the cohomology of \( K3 \).

In a similar manner, one can consider a further freely acting \( Z_2 \) orbifold of the above \( N = 2 \) theories which was considered in \([7]\) to produce dual theories with \( N = 1 \) supersymmetry in four dimensions. Again without using any knowledge about the cohomology of \( K3 \) one can reproduce the result of \([7]\). We, of course, are not suggesting that the identification of the lattice of integral cohomology of \( K3 \) with the Narain lattice for the heterotic string on \( T^4 \) is incorrect. For the examples considered in \([6, 7]\), which we reproduced above, the \( K3 \) orbifold defined by \( \alpha \) in equation (1) is the only element of the orbifold group of this type. We make the supposition that we know nothing of the cohomology of \( K3 \), in order to proceed further and construct dual pairs when two or more elements of the orbifold group are of the \( K3 \) orbifold type. The identification between these lattices will be seen to hold in the adiabatic limit \([6]\) when, in section four, we interpret our results in terms of the ‘fibration picture’ of \([6]\).

In the examples we have just considered we resolved all singularities because it is unclear at present how to deal with twisted sectors in \( M \)-theory.

\(^3\)Specifically, the vacuum energy in the left moving twisted sector is \(-1/4\) which does not lead to a modular invariant orbifold. However if we translate the shift on \( x_7 \) in the \( M \)-theory background to an asymmetric shift of the \( x_7 \) in the heterotic background, then we can achieve modular invariance in the following way \([8]\): the \( \Gamma^{1,1} \) which corresponds to \( x_7 \) can be orbifolded by a shift vector \( \delta \) of the form \( \delta = (p_l, p_r)/2 \) with \( p^2 = 2 \). Because \( \delta^2/2 = 1/4 \), the difference between left and right moving vacuum energies is zero, and hence the orbifold is modular invariant.
Luckily there were no singularities associated with $\beta$, so there were none to resolve. However in the more general cases we will consider, we will orbifold the heterotic theory with isometries that do have singularities and it is natural to resolve ie blow up these as well. The blowing up modes may naturally be associated with the twisted sectors of the heterotic theory.

This suggests the following strategy: (I) Take $M$-theory on an orbifold. By analogy with string theory, we can naturally identify the twisted sector states with the blowing up modes of the orbifold. (II) The action on the $T^6$ of the heterotic string will already be specified by the action of the orbifolds on the geometry of $M$-theory by our ansatz. (III) Then simply project out the necessary number of vectors from the heterotic string spectrum. (IV) If the heterotic orbifold is a $(2, 2)$ superconformal field theory, in which case the blowing up modes are truly moduli [25] then proceed to the smooth limit and include the blowing up moduli in the spectrum. In this case one must choose an embedding of the spin connection in the gauge connection such that the resulting spectrum is correct. In fact, if the orbifold group acts left-right symmetrically on the $\Gamma^6, 6$ of the $T^6$ compactified heterotic string, then the orbifold has a classical geometric interpretation and one can study the heterotic string on the blown up orbifold and the moduli of the smooth manifold will in any case appear as scalar fields of the theory. Because such a theory is a string theory on a blown up orbifold, the massless spectrum is easy to determine: it is just the untwisted sector at the orbifold limit plus the moduli associated with blowing up. (V) Check if the theory is consistent with modular invariance or can be made so by adding appropriate shift vectors.

In fact, it may be possible to go further than just considering the theories on blown up orbifolds. We will give strong evidence in some four dimensional $N = 1$ examples that both the untwisted and twisted sector spectra coincide for $M$-theory on Joyce orbifolds and the heterotic string on Calabi-Yau orbifolds. In the following sections, we will apply our presented strategy to propose new dual pairs. These constitute examples of dual pairs constructed as non-freely acting, supersymmetry breaking (ie less than $N = 4$ in 4d) orbifolds of existing dual pairs. Section two discusses an $N = 2$ example in detail. In section three we construct some $N = 1$ examples. In section four we interpret our results in terms of the fibration picture in the adiabatic limit [3]. Remarkably, even though our examples are constructed as non-freely acting orbifolds of existing dual pairs, the adiabatic argument of [3] is apparently not violated. This is a consequence of the fact that the orbifolds
we restrict ourselves to are precisely the ones which preserve the fibration structure. Following this we end with some conclusions and comments.

2 An \(N = 2\) Example.

In this section we will consider an \(N = 2\) example first following the analysis given in [3] and then following the strategy presented in the last section. In [3] several examples of potential dual pairs of \(N = 2\) theories in four dimensions were constructed. The dual pairs in question were Calabi-Yau compactifications of TypeII strings and \(K3 \times T^2\) compactifications of heterotic strings. The massless spectrum of the heterotic string on \(K3 \times T^2\) is determined from the expectation value of the gauge fields on \(K3\) and index theory [3, 26]. An \(N = 2\) theory in four dimensions is characterised by vector multiplets and hypermultiplets. The vector multiplets contain adjoint scalar fields, which in addition to the moduli hypermultiplets of the theory are also moduli. At special points in the moduli space of these scalars, the theory contains massless charged hypermultiplets. These become massive at generic points in the moduli space of the adjoint scalars which correspond to the Cartan subalgebra of the gauge group. Neutral massless fields will always remain massless as one moves through the moduli space of these scalars. After the spin connection has been embedded in the gauge connection in such a way that the theory is anomaly free, the theory is then characterised by the neutral fields (the rest of the moduli) and \(N\) vector multiplets, where \(N\) is the rank of the gauge group which survives the embedding. Kachru and Vafa denoted the theories at these generic points\(^4\) by \((M, N)\), where \(M\) is the number of massless hypermultiplets. There is a universal contribution of 20 to \(M\) coming from the moduli of \(K3\) [13], hence \(M \geq 20\). Further, because we have a rank sixteen gauge group in ten dimensions and a further four \(U(1)\)'s coming from the torus at generic points, \(N \leq 20\).

We wish to consider an example where both these inequalities are saturated. This places two constraints on possible embeddings of the gauge bundle on \(K3\): (i) we are forced to consider giving expectation values to \(U(1)\) or products of \(U(1)\) gauge fields on \(K3\). Fortunately, the spectra of many of these embeddings has been calculated in [26], although we know not

\(^4\) We assume we are at generic points in the torus.
of any full classification; (ii) in order that we obtain 20 moduli hypermultiplets we need to find an example in which all hypermultiplets are charged except the 20 gravitational moduli.

Examining the spectra given in [26], it is not too difficult to convince oneself that many of the examples satisfy these criteria, giving a (20,20) spectrum at generic points. For definiteness we consider the model with a single $U(1)$ embedded in $E_8 \times E_8$, in such a way that one $E_8$ is broken to $E_7 \times U(1)$. This gives a spectrum containing 10 $56$'s of $E_7$ (half with one $U(1)$ charge and half with the opposite charge) and 46 $E_7$ singlets (23 each with opposite $U(1)$ charges). This is the example given in equation (6.1) of [26]. Because all matter is charged, we have a (20,20) spectrum at generic points.

As an aside, it is interesting to note that we can connect this example to the chain of examples considered in section 3 of [8]. Namely, by higgsing the $U(1)$, we find 45 additional gauge neutral hypermultiplets giving a (65,19) spectrum. This model can be similarly higgsed several times to give the chain: (20,20) → (65,19) → (84,18) → (101,17) → (116,16).\footnote{While this work was in progress, we realised that many of the examples of [28] are connected via Higgs’ and Coulomb branches and which have TypeII dual candidates on $K3$-fibrations [3].}

We would now like to find a TypeII dual for this (20,20) model. According to [8], the Calabi-Yau space describing the background of the TypeII theory should be a $K3$ fibration. Further it must be a self-mirror Calabi-Yau with $h_{11}=h_{21}=19$ because a $(M,N)$ model arises from a TypeIIA compactification on a Calabi-Yau with $(M = h_{11}+1, N = h_{21}+1)$. Luckily there is a manifold which fulfills these requirements. We denote this Calabi-Yau by $CY_{19}$. It can be constructed as a blown up orbifold as we now describe, following Joyce [14].

In [14], Joyce constructed $CY_{19} \times S^1$ as a $Z_2 \times Z_2$ blown-up orbifold of the seven torus. We repeat the construction here:

Define the seven-torus coordinates as $(x_1, \ldots, x_7)$. Two $Z_2$ isometries of $T^7$ are defined by:

\[ \alpha(x_1, \ldots, x_7) = (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7) \] (3)

\[ \beta(x_1, \ldots, x_7) = (-x_1, 1/2 - x_2, x_3, x_4, -x_5, -x_6, x_7) \] (4)

Let the $Z_2 \times Z_2$ isometry group generated by $\alpha$ and $\beta$ be denoted by $\Gamma$. In fact it is easy to see that this is precisely the construction of [8], without the extra
$Z_2$ shifts which made that construction freely acting. If considered separately, each of these $Z_2$’s has 16 fixed $T^3$’s and each one defines an orbifold limit of a particular $K3 \times T^3$. Hence, the singular set of $T^7/\alpha$ contains 16 $T^3$ components, as does the singular set of $T^7/\beta$. However, as $\beta$ acts freely on the 16 fixed three tori of $\alpha$, $\alpha$ contributes eight three tori to the singular set of $T^7/\Gamma$. Similarly, $\beta$ also contributes eight three tori to the singular set of $T^7/\Gamma$.

The betti numbers of the original torus which survive the orbifold projection ie the betti numbers of $T^7/\Gamma$ are $b_1 = 1$, $b_2 = 3$ and $b_3 = 11$. The blowing up procedure is carried out by inserting non-compact Eguchi-Hanson geometries($\times T^3$) in each of the singular regions. Each of these adds 1 to $b_2$ and 3 to $b_3$, giving a seven manifold of $SU(3) \times 1$ holonomy with betti numbers: $b_1 = 1$, $b_2 = 19$ and $b_3 = 59$. In particular, if we consider the six-torus defined by the coordinates $x_1$ through $x_6$, then the holomorphic three form is preserved by the $Z_2 \times Z_2$. The seven manifold thus obtained has the form $CY_{19} \times S^1$. Compactification of eleven dimensional supergravity on this manifold yields $N = 2$ supergravity with 20 hypermultiplets and 19 vector multiplets (not including the graviphoton). The counting goes as follows:

- In eleven dimensions, the massless bosonic fields of $M$-theory are the metric, $G^{\mu\nu}$, and antisymmetric three-form tensor, $A^{\mu\nu\rho}$. On compactification to four dimensions on $CY_{19} \times S^1$, the three form gives rise to $b_3$ scalars, $b_2$ vectors and $b_1$ two forms.
- In general a higher dimensional metric yields $n$ scalars, where $n$ is the dimension of the moduli space of the compactification metric. In our case, this is $59 - 1 = 58$. The metric tensor will also yield a vector in the lower dimensional theory for every continuous isometry of the compactifying manifold. In our case, the $S^1$ has a $U(1)$ isometry yielding a $U(1)$ gauge field in four dimensions.

All in all, for this example we get 118 scalars and 20 vectors (including graviphoton) plus the graviton. The fermion spectrum is implied by supersymmetry and we thus have the $(20, 20)$ model as required. This is the same spectrum as the Type IIa/IIb string on $CY_{19}$.

Now let us apply the strategy suggested in the last section and see if it

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6In general we define the singular set $S'$ of $M$ to be the set of points, surfaces and submanifolds of the manifold $M$, which are fixed under the action of some finite group $G$. The singular set, $S$, of $M/G$ is then the image of $S'$ in $M/G$.

7See [18] for a review.

8In four dimensions, two forms are dual to scalars and so may be counted as scalars.
gives the correct results. Firstly, the action of $\beta$ on the heterotic string $T^6$ does indeed give an orbifold limit of $K3 \times T^2$ as before. Now, however, we do not have the extra half shift on $x_7$, so this orbifold is not freely acting. Secondly, we would like a rank 20 gauge group, which means that the action on the gauge degrees of freedom is trivial. But, this is not the whole story, for we would certainly like to preserve modular invariance in this orbifold and the natural choice we make is the standard embedding. Away from the orbifold limit ie on the smooth $K3 \times T^2$ we must also specify an anomaly free background; and, as mentioned at the beginning of this section this will limit us to $U(1)$ embeddings of the spin connection in the gauge connection to give the required $(20, 20)$ spectrum as before. Thus it appears that our strategy is consistent at least for the first example we have considered.

We can provide a further check on whether we have indeed produced a dual pair, by considering a freely acting orbifold of this dual pair. If the spectra again agree, we will have also produced another dual pair. A simple freely acting orbifold which does not break any supersymmetry is the following:

$$\sigma(x_1, x_2, ..., x_7) = (x_1, x_2 + 1/2, x_3, x_4, x_5, x_6, x_7)$$

On the M-theory geometry, this isometry has the effect of halving the number of elements of the singular set of $T^7/\Gamma$. This produces an example with a $(12, 12)$ spectrum at generic points.

On the heterotic side, the action of $\sigma$ corresponds to exchanging the two $E_8$ factors of the gauge group, plus identifying, in eight pairs of two, the sixteen fixed points associated with $\beta$ which defines the $K3$ orbifold heterotic background. The shift on $x_2$ also eliminates massless modes coming from the $\sigma$ twisted sector. In fact the twist by $\sigma$ is precisely the one which was considered in [28]. Because the sixteen fixed points of $\beta$ on the heterotic background are associated with sixteen neutral moduli hypermultiplets in the blowing up limit, the $\sigma$ action reduces this number to eight. The resulting spectrum is therefore precisely $(12, 12)$ at generic points in accord with the expectations of string-string duality.

As a concluding remark to this section, it is useful to point out that because the respective moduli spaces of these conjectured dual pair of compactifications are constrained by $N = 2$ supersymmetry, many of the important results of [8] also apply here.
3 \ N = 1 \ Examples.

Compactification of \( M \)-theory on a seven-manifold of \( G_2 \) holonomy gives rise to \( N = 1 \) supergravity with \( b_2 \) vector multiplets and \( b_3 \) chiral multiplets. We denote these manifolds by \( J_{b_2}^{b_3} \). Many examples of such manifolds were recently constructed in [3, 4]. A duality between the eleven dimensional theory on \( J_{39}^{16} \) and the heterotic string on a Calabi-Yau with precisely the same Hodge diamond as \( CY_{19} \) was conjectured in [12], on the basis of counting Betti numbers and matching the spectra. We will see later in this section that we can \textit{derive} this result utilising our presented ansatz.

In this section, we proceed to apply the strategy of the preceding sections to produce dual pairs with \( N = 1 \) supersymmetry in four dimensions. The first example we consider has \( b_2=8 \) and \( b_3=31 \). This example will be constructed by considering a freely acting orbifold of the \((20,20)\) \( N = 2 \) example of the previous section. Evidence for the existence of the \( N = 1 \) dual pair, is then also evidence for the \( N = 2 \) dual pair.

Consider then the \( \Gamma \equiv Z_2 \) orbifold of the seven torus defined by \( \alpha \) and \( \beta \) of equations (3) and (4); and the third \( Z_2 \) defined as follows:

\[
\gamma(x_1, x_2, \ldots, x_7) = (1/2 - x_1, x_2 + 1/2, -x_3, x_4, 1/2 - x_5, x_6, -x_7)
\]

(6)

Because of the half shift on \( x_2 \), \( \gamma \) acts freely. In fact, \( \gamma \) takes the sixteen elements of the singular set of \( T^7/(\alpha, \beta) \) and identifies them in eight pairs of two. The betti numbers of \( T^7/\Gamma \) are \( b_2=0 \) and \( b_3=7 \). The singular set contains eight elements, the resolution of each of which adds 1 to \( b_2 \) and 3 to \( b_3 \), giving a Joyce manifold of \( G_2 \) holonomy, \( J_{8}^{31} \).

Thus far, we have said little about the possible gauge groups allowed by string/string/M-theory duality. The mechanism for gauge symmetry enhancement in the TypeII theories is a generalisation of that considered in [20], where p-brane solitons wrap around p-cycles of the compactification space and give rise to massless gauge multiplets (and matter multiplets) when the cycles degenerate to zero volume. In general, because the singularities corresponding to the vanishing cycles are of A-D-E type, one expects A-D-E symmetries [2, 9]. The singularities we have been considering are all \( SU(2) \) orbifold singularities, thus we can at least expect an \( SU(2) \) factor in

\[9\text{To the best of our knowledge, this Joyce manifold has not been constructed previously, even though a manifold with the same betti numbers appeared in [14].}\]
the gauge group for each element of the singular set that we blow up. For example, in the \( N = 2 \) \((20, 20)\) example that we constructed, we should expect an \( SU(2) \)^{16} factor in the gauge group if we consider \( M \)-theory at the orbifold limit defined in the previous section, because we resolved sixteen \( SU(2) \) singularities on the TypeII side to construct \( CY_{19} \). So let us break the \( E_8 \times E_8 \) gauge symmetry of the heterotic string to \( SU(2) \)^{16} by orbifolding the \( \Gamma^{22,6} \) Narain lattice for toroidal compactification to four dimensions. This is equivalent to the turning on of Wilson lines. We will orbifold the theory by three \( Z_2 \) shift vectors given by:

\[
\delta_1 = (1, 0^7; 1, 0^7; 1/2, 0^5)(1/2, 0^5);
\]

\[
\delta_2 = ((1/2)^4, 0^4; (1/2)^4, (0)^4; 0, 1/2, (0)^4)(0, 1/2, (0)^4)
\]

and

\[
\delta_3 = ((1/2)^2, (0)^2, (1/2)^2, (0)^2; (1/2)^2, (0)^2, (1/2)^2, (0)^2; (0)^2, 1/2, (0)^3)
\]

\((0)^2, 1/2, (0)^3)\).

This then breaks the \( E_8 \times E_8 \) symmetry to \( SU(2) \)^{16} as required. The action of \( \beta \) from equation (4) on the heterotic string is as in the given previous section, but now \( \beta \) acts on the theory with reduced symmetry.

Because each \( SU(2) \) factor in the gauge group is associated with an orbifold singularity on the TypeII/M-theory side of the duality map, the action of \( \gamma \) on the heterotic string gauge group is easily seen from the discussion above to be the exchange of the two \( SU(2)^8 \) factors. The following action on the \( T^6 \) coordinates is given by the ansatz of the preceeding sections.

\[
\gamma(x_1, ..., x_6) = (1/2 - x_1, x_2 + 1/2, -x_3, 1/2 - x_5, x_6, -x_7) \quad (7)
\]

In the untwisted sector of the theory the massless spectrum is given by eight \( N = 1 \) vector multiplets and 15 chiral multiplets. Of the chiral multiplets, seven are singlet untwisted moduli multiplets and the other eight are adjoint multiplets of \( SU(2) \). The \( \gamma \) and \( \beta \gamma \) twisted sectors produce no massless states. The \( \beta \) twisted sector produces 64 chiral multiplet \( SU(2) \) doublets, of which only 16 are \( \gamma \) invariant. Hence the resulting massless spectrum is precisely that of \( M \)-theory on \( J_{31}^8 \).

We can modify this example slightly and produce nine more potential \( N = 1 \) dual examples. This is done as follows:

\[
\gamma(x_1, ..., x_7) = (1/2 - x_1, x_2, -x_3, x_4, -x_5, x_6, -x_7) \quad (8)
\]

\footnote{This choice of symmetry breaking vectors was considered in [3].}
This modification has the following significance: (i): $\gamma$ is no longer freely acting on the geometry; (ii): the element $\alpha\beta$ acts trivially on the fixed three tori of $\gamma$. This means that the presence of $\gamma$ in this form removes four elements of the singular set of $\alpha$ and $\beta$ of the original $N = 2$ model. So we definitely have eight vectors surviving the $\gamma$ projection. However, we still need to consider the elements of the singular set induced by $\gamma$. Because the element $\alpha\beta$ acts trivially on the singular set from $\gamma$, $\gamma$ must contribute eight additional elements. However, these elements are different to the other eight, because when the blowup is performed, the additional action of $\alpha\beta$ must be considered on the blowup itself. It turns out that there are two topologically distinct ways of considering this action on the blowing up modes. These two ways differ by the fact that one preserves the generator of $H^2(X, R)$ and the other changes its sign, (where $X$ is the Eguchi-Hanson blowing up mode)\footnote{\[13, 14\] can be consulted for further details.}

It follows that these two blowups contribute different betti numbers to the Joyce manifold. If the extra $Z_2$ action was not present then each blowup would add one to $b_2$ and three to $b_3$. When the $Z_2$ is present, the two choices in defining its action on the blowup has the effect of splitting these original betti numbers, so that the first type of resolution adds one to $b_2$ and one to $b_3$; and the second adds zero to $b_2$ and two to $b_3$. So all in all we have eight ‘standard’ blowups and eight for which there are two choices. The betti numbers from the original seven torus which survive the $Z_2^3$ isometries are $b_1=b_2=0$ and $b_3=7$. The eight ‘standard’ blowups add one to $b_2$ and three to $b_3$, giving $b_2 = 8$ and $b_3 = 31$. Of the remaining eight ‘nonstandard’ blowups, if we choose $l$ of them to be of the first type, then this adds $l$ to both $b_2$ and $b_3$. The remaining $8 - l$ add zero to $b_2$ and $16 - 2l$ to $b_3$. This means that the Joyce manifold has

$$b_2 = 8 + l, \quad b_3 = 47 - l, \quad l = 0, 1, \ldots 8 \quad (9)$$

Let us now see if we can find a family of heterotic duals for this family of Joyce manifolds.

The first point to note is that if we consider the $Z_2 \times Z_2$ orbifold of $M$-theory on $T^7$ which is generated by $(\alpha, \gamma)$ then the resulting $N = 2$ spectrum is precisely $(20, 20)$ ie the manifold is of the form $CY_{19} \times S^1$. This is the same spectrum we obtained using $(\alpha, \beta)$ as the orbifold generators. Hence, the
action of $\gamma$ is identical to that of $\beta$. This symmetry between the generators should be preserved when we consider the action of $\beta$ and $\gamma$ on the heterotic theory. Because we have already found that the action of $\beta$ preserved the rank of the gauge group originating in $E_8 \times E_8$ in our $(20, 20)$ $N = 2$ model, we expect $\gamma$ to do so also. Further, because the heterotic model will have $N = 1$ supersymmetry in four dimensions, we can expect a rank 16 gauge group. We therefore may expect on these general grounds that the heterotic model will be dual to $M$-theory on the Joyce manifold with $l = 8$, above, which has a spectrum of 16 vector multiplets and 39 chiral multiplets. We now construct the heterotic background.

Before considering the action on the gauge degrees of freedom, we first specify the action of $\beta$ and $\gamma$ on the six-torus coordinates of the heterotic string. According to our ansatz, these are as follows:

\begin{align}
\beta(x_1, \ldots, x_6) &= (-x_1, 1/2 - x_2, x_3, -x_5, -x_6, x_7) \quad (10) \\
\gamma(x_1, \ldots, x_6) &= (1/2 - x_1, x_2, -x_3, -x_5, x_6, -x_7) \quad (11)
\end{align}

$\beta$ leaves invariant a two-torus which is inverted by $\gamma$; and $\gamma$ leaves invariant a two-torus inverted by $\beta$. Thus the heterotic background has $N = 1$ spacetime supersymmetry as expected by duality. In fact it is interesting to note that if this orbifold is blown up, the resulting smooth manifold is none other than the $CY_{19}$ that appeared on the TypeII side in our $N = 2$ example. If we now consider the heterotic theory on the manifold $CY_{19}$ we will find a non-chiral $N = 1$ theory with 39 moduli multiplets. If we also specify a $U(1)^n$ embedding of the spin connection in the gauge connection, then we arrive at the rank sixteen model conjectured previously in [12]!

However, we can go one stage further and actually give evidence that $M$-theory on the orbifold defined by $(\alpha, \beta, \gamma)$ is equivalent to the heterotic string on the orbifold defined by $(\beta, \gamma)$. The action of the orbifold on the $T^6$ piece of the heterotic theory has been given. It remains to specify the action on the gauge degrees of freedom. As already noted, we wish to preserve a symmetry between $\beta$ and $\gamma$, ie they should give rise to identical spectra when considered separately. This can be achieved by considering identical embeddings of the spin connection in the gauge connection, with the $\beta$ connection in one...
SU(2)\(^8\) factor and the \(\gamma\) connection in the other. The choices are restricted by modular invariance. Further, because we expect 32 chiral multiplets in \(M\)-theory to arise from the twisted sectors (because 32 harmonic three-forms arise from blowing up), we can expect the same in the heterotic theory. We find there are essentially two inequivalent choices of abelian embeddings which give rise to massless states in the twisted sector. Only one choice, corresponding to the standard embedding in each \(SU(2)\(^8\) factor, gives rise to the correct number of twisted sector multiplets. These are the following:

\[
\delta_\beta = ((1/2)^2, (0)^6)((0)^8) \quad (12)
\]

\[
\delta_\gamma = ((0)^8)((1/2)^2, (0)^8) \quad (13)
\]

where the first(second) bracket denotes the shift in the first(second) \(SU(2)\(^8\) factor. Let us consider the spectrum. In the untwisted sector, the spectrum contains 16 \(N = 1\) vector multiplets and seven moduli chiral multiplets (including dilaton). In fact, with the ansatz we have alluded to one will always find seven moduli multiplets in the untwisted sector. The analogue of this statement from the \(M\)-theory point of view is that any Joyce manifold of \(G_2\) holonomy constructed as a blown up \(Z_2^3\) orbifold of the seven torus has \(b_3(T^7/Z_2^3)=7\), corresponding to seven chiral moduli multiplets in the untwisted sector of \(M\)-theory!

Now consider the twisted sectors. We find 16 \(SU(2)\) doublet multiplets from each of the \(\beta\) and \(\gamma\) sectors respectively. This gives a total spectrum of 16 vector multiplets and 39 chiral multiplets. This is the same spectrum as the example with \(l = 8\) above, as we initially expected. Before commenting on the examples with \(l = 0, 1..7\) we would like to make some observations.

Firstly, if we examine the Calabi-Yau orbifold of the heterotic theory defined by equations (13),(14), we note that if we resolved all the orbifold singularities then the resulting Calabi-Yau manifold is none other than \(CY_{19}\)! We have thus derived the result presented in [12].

Secondly, we have seen that the untwisted matter content will always agree for heterotic/\(M\)-theory duals if the \(M\)-theory background is a \((Z_2)^3\) orbifold of \(T^7\). In the example we have just considered, we have further observed that the twisted sector spectrum in the heterotic theory precisely reproduces the spectrum which arises in \(M\)-theory from the blowing up procedure. This is compelling evidence that we have again constructed the correct heterotic background, dual to \(M\)-theory on a Joyce manifold. It is
also further evidence that orbifold backgrounds are consistent in \( M \)-theory. In fact similar reasoning also applies to \( N = 2 \) dual pairs. The \( M \)-theory background for \( N = 2 \) supersymmetry in four dimensions will be of the form \( CY \times S^1 \), for \( CY \) any Calabi-Yau space. This background is equivalent to the TypeIIa theory on \( CY \). If \( CY \) is constructed as a \( Z_2 \times Z_2 \) orbifold of \( T^6 \), then the untwisted matter spectrum at generic points will \textit{always} contain four massless hypermultiplets. The heterotic dual background will then be of the form \( T^4 / Z_2 \times T^2 \) where the \( Z_2 \) defines a \( K3 \) orbifold. This background also contains four hypermultiplets in the untwisted sector massless spectrum, and it is yet again tempting to postulate that the twisted sector of \( M \)-theory on the (\( Z_2 \))^2 orbifold is identical to that of the heterotic \( Z_2 \) orbifold. Of course, the heterotic spectrum depends strongly on the choice of discrete Wilson lines or shift vectors required for modular invariance and it would be interesting to identify such degrees of freedom in \( M \)-theory. Such an identification was made for the TypeII theory recently \cite{27} where it took the form of generalised discrete torsion.

We have identified a heterotic dual theory for the compactification of \( M \)-theory on one of a family of nine Joyce manifolds, parametrised by \( l \). What can we say about the other members of the family?

Consider first \( M \)-theory on the example with \( l = 8 \) ie \( J^{16}_{39} \). To make the transition to the next member of the family, \( J^{15}_{40} \), we must blow down a two-cycle and blow up a three-cycle; in other words this is precisely an example of an \( M \)-theory conifold type transition, and we have reasonable grounds to suspect that such a transition is physically non-singular \cite{20}. In fact this is nothing but the Higgs mechanism in an \( N = 1 \) \( M \)-theory background. From the heterotic string point of view, we would need a field content which along certain Higgs directions reproduces the field content of all these Joyce compactifications of \( M \)-theory. However, we have chosen the most simple breaking of \( E_8 \times E_8 \) and have restricted ourselves to abelian embeddings of the spin connection in the gauge connection. There are of course many other consistent possibilities that one may consider and it may certainly be the case that we can reproduce the required field content from the heterotic compactification. We are investigating such possibilities. Thus the heterotic analogue of these topological transitions between different Joyce manifolds remains a mystery. This is also a consequence of the fact that the singularities which allow transitions between the different Joyce manifolds are certainly not \( SU(2) \) singularities, but orbifolds of them. One would certainly need to
identify the physical implications of this statement from the $M$-theory point of view, before the heterotic description of the transition could be made.

4 Interpretation.

Given the apparent success of our ansatz, the question arises as to whether these results have a more satisfying explanation. It is natural to expect that this should come from some substructure in the Joyce manifolds considered in this paper. A clue comes from the ubiquity of $K3$ fibrations in string duality [1, 3, 5, 29]. Specifically, given a $K3$ for the five brane of $M$-theory to wrap around and an $S^1$ for the dual two-brane to wrap around, one can derive connections between string theories in lower dimensions and $M$-theory by fibering the $K3$ over another space [6, 7, 29]. For example $N = 2$ string-string duality in four dimensions has an interpretation in terms of $N = 2$ string-string duality in six dimensions, whereby the geometries of the TypeIIa theory ($K3$) and heterotic theory ($T^4$) respectively are fibered over $CP^1$ [4, 5]. We will now show that (at least locally) a similar interpretation holds here for the results of the preceding sections. More precisely, if the $M$-theory compactification is a $K3$ fibration over some three manifold and the heterotic dual compactification is a $T^3$ fibration over the same three manifold, then in the adiabatic approximation of [6], we can expect the duality to hold between the two theories in four dimensions. The following analysis relies heavily on that of [13, 14].

A seven manifold of $G_2$ holonomy has two classes of special submanifolds. This is essentially a consequence of the fact that the torsion-free $G_2$ structure of a Joyce seven-manifold is defined by specifying a particular closed three form [13, 14]. This is analogous to the Kahler form in Calabi-Yau spaces. The three form has a four form Hodge dual. The three form is naturally identified with the volume form of a special class of three manifolds which are submanifolds of the Joyce manifold. These are known as associative submanifolds. They minimize volume in their homology class. The three cycle dual to the (minimum) volume form is known as a supersymmetric cycle [17].

Similarly the four form which is dual to the three form can be used to define special four dimensional submanifolds of the Joyce manifold. These are known as coassociative submanifolds. It was proven in [16] that the di-
mension of the moduli space of such manifolds is given by the number of self-dual, harmonic two-forms of the submanifold. Therefore, if this number is non-zero, then the Joyce manifold is (at least locally) fibered by the coassociative submanifold.

Joyce [14] has given a prescription for proving whether or not such submanifolds exist for the manifolds he constructs. The argument proceeds in the following way: given a Joyce manifold, one can consider further orbifolds by isometries which preserve the holonomy structure. If the singular set of the isometry consists of \( n \) elements of dimension three (eg three-tori) then the Joyce manifold contains \( n \) associative submanifolds (given by these three-tori). If the singular set contains \( n \) dimension four elements (eg K3) then these are coassociative submanifolds of the Joyce manifold.

If we now begin with the duality in seven dimensions between \( M \)-theory on \( K3 \) and the heterotic string on \( T^3 \), which was our original starting point, then we would expect that fibering both of these manifolds over the same three manifold will lead to a duality in four dimensions between these two theories, at least in the adiabatic approximation [3]. We will show, following Joyce [13, 14], that the seven manifolds used for \( M \)-theory compactification in this paper are (at least locally) fibred by \( K3 \). Further, we will similarly show that the seven manifolds contain \( T^3 \) submanifolds, which “pull back” to the six manifold used for the heterotic compactification. Unfortunately, it is not yet known whether or not these three-tori fiber the \( M \)-theory and heterotic geometries, so we cannot say conclusively that the fibration picture [3] holds. However, given our preceding evidence for dual pairs in four dimensions it is natural to expect that the \( M \)-theory (heterotic) compactification space is globally a \( K3 \) (\( T^3 \)) fibration.

Let us consider as an example our \( N = 2, (20, 20) \) model of section 2, which is defined on the \( M \)-theory background by equations (3) and (4). This gave us an \( M \)-theory compactification on \( CY_{19} \times S^1 \). Consider now the following isometry of this manifold:

\[
\sigma(x_1,...,x_7) = (x_1, x_2, x_3, x_4, 1/2 - x_5, 1/2 - x_6, 1/2 - x_7)
\]  

The singular set of \( T^7/\sigma \) contains eight copies of \( T^4 \). Now let us consider the action on these of \( \alpha \) and \( \beta \) which define the manifold. Firstly \( \beta \) exchanges these eight four-tori in four pairs of two, leaving four independent elements. Thus, the singular set of \( T^7/(\sigma, \beta) \) contains four copies of \( T^3 \) associated with
However, the fixed points of $\sigma$ intersect those of $\alpha$, hence the singular set of $T^7/(\alpha, \beta, \sigma)$ contains four copies of $T^4/\mathbb{Z}_2$ which are associated with $\sigma$, where the $\mathbb{Z}_2$ is the action of $\alpha$ on the four-tori fixed by $\sigma$. It is easily seen that the action of $\alpha$ on these four-tori defines a $K3$ orbifold metric for each one. Finally, we can say that the fixed set of $\sigma$ in $CY_{19} \times S^1$ is four copies of $K3$. $CY_{19} \times S^1$ thus contains at least four $K3$ submanifolds, all of which are coassociative.

From our discussion above it follows that the dimension of the moduli space of each of these is $b_2(K3) = 3$. It is therefore true that the manifold $CY_{19} \times S^1$ is at least locally fibered by $K3$ \[14\]. In fact it is not difficult to see along similar lines that all of the seven manifolds used for $M$-theory compactification in this paper admit local $K3$ fibrations.

Using the same example as we have just discussed, consider the following isometry of $CY_{19} \times S^1$:

$$\sigma(x_1, ... x_7) = \left(\frac{1}{2} - x_1, \frac{1}{2} - x_2, -x_3, -x_4, x_5, x_6, x_7\right)$$ (15)

It is not too difficult to convince oneself that the fixed set of $\sigma$ in $CY_{19} \times S^1$ is four copies of $T^3$. It follows that these are all associative submanifolds of the seven manifold. Now, because these three tori are labelled by $(x_5, x_6, x_7)$, which are also coordinates of the heterotic background, the above isometry also fixes precisely four three-tori of the heterotic background. Unfortunately little appears to be known about the moduli space of such submanifolds and so we cannot conclusively say that the six manifold admits a $T^3$ fibration. However consistency with the dualities presented here and with the general picture of fibering existing dual pairs to obtain more dual pairs \[8\] suggests the following:

(i) That the Joyce seven manifolds used for $M$-theory compactification in this paper admit global $K3$ fibrations over some three manifold $Y$.

(ii) That the six manifolds, used to produce the heterotic duals to the above $M$-theory compactifications, admit global $T^3$ fibrations over the same three manifold $Y$.

We have thus far given examples of $M$-theory compactifications on Joyce manifolds which admit local $K3$ fibrations. The dual heterotic compactifications were found to be on particular Calabi-Yau spaces, which we generically denote by $CY$. However, it is also possible to interpret other $M$-theory duals of these heterotic compactifications, as compactifications of the eleven
dimensional theory on $CY \times S^1/Z_2$ \cite{21}. This is analogous to the two $M$-theory duals of the heterotic string in seven dimensions, which we discussed in section (1).

It thus appears that if the fibration picture \cite{6} does hold for the examples presented here, then even though we have constructed new dual pairs with non-freely acting orbifolds of existing dual pairs (namely heterotic/$M$-theory duality in seven dimensions), the adiabatic argument is not violated. This is only true because the orbifolds to which we have restricted our attention are precisely those which preserve (at least locally) the fibration of the seven manifold. It is thus presumably true that if one can find examples of orbifolds with higher order isometry groups which are also of this type, then one may be able to construct more examples, some of which may be phenomenologically appealing.

5 Summary and Conclusions

Summarizing the key points of this paper: For the conjectured dualities between the various string theories and $M$-theory, the theories should yield identical massless spectra. We began with the 11d-theory on a $K^3 \times T^3$ orbifold and the heterotic string on $T^6$. We then orbifolded the seven manifold on the $M$-theory side and resolved the singularities by blowing up, first for an $N = 2$ example and then for several $N = 1$ examples. We presented an ansatz mapping six of the $M$-theory coordinates to the six heterotic coordinates. Because of this ansatz, the action of the orbifold isometry group on the $M$-theory geometry also specified the action on the heterotic geometry. Thus all that remained to specify the heterotic background was the embedding of the spin connection in the gauge connection. This was derived by requiring modular invariance at the orbifold limit. Using this ansatz we rederived previous results in the literature and produced new dual pairs in four dimensions. It therefore appears that this ansatz is consistent for producing dual pairs by orbifolding whenever the non-freely acting elements of the orbifold group have singularities which are of the $K3$ orbifold type. This is in accord with the underlying structure of $K3$ fibrations, \cite{1, 2, 3}, because, as was demonstrated in \cite{14}, and as we showed in the last section, manifolds which are constructed with such orbifold elements admit (at least locally) a $K3$ fibration. The duality is then expected to hold on general grounds if the
heterotic geometry is a $T^3$ fibration $[3]$. Moreover, we also gave further evidence that $M$-theory is consistent on orbifolds by comparing the untwisted and twisted matter content in such a theory with its heterotic dual. The untwisted matter contents for $N = 2$ and $N = 1$ dual pairs always agree for the dual pairs constructed according to our ansatz. Specifically, if the four dimensional $M$-theory background is of the form $CY \times S^1$, where $CY$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of $T^6$, then the dual heterotic background will, according to our ansatz, necessarily be of the form $T^4/\mathbb{Z}_2 \times \mathbb{Z}_2$. Both of these $N = 2$ theories have an untwisted matter content of four hypermultiplets. Similarly, $M$-theory on a Joyce $G_2$ orbifold of the form $T^7/\mathbb{Z}_2^3$ will always contain an untwisted sector of seven $N = 1$ moduli multiplets. Then, according to our ansatz, the dual heterotic background will be of the form $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$, and will also have seven $N = 1$ moduli multiplets in the untwisted matter sector. What is even more compelling is that we were able to show that for some simple choices of orbifold gauge embeddings, the number of twisted sector multiplets agree also.

Finally we wish to add that this method of constructing dual pairs has been successfully applied to the construction of $N = 1$ dual pairs in three dimensions $[30]$. In fact, using this construction we have constructed heterotic duals (on Joyce $G_2$ manifolds) for $M$-theory compactifications on all known Joyce manifolds of Spin(7) holonomy $[15]$. This may shed some light on the mysterious supersymmetric theory in twelve dimensions which appears to be required to complete the duality picture $[32,33]$. As noted in $[33]$, these theories may be an explicit realisation of the beautiful ideas of Witten $[34]$ which may solve some of the long standing problems of theoretical physics and possibly take duality towards reality.

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References

[1] C.M.Hull and P. Townsend, ‘Unity of Superstring Dualities,’ Nucl. Phys. B438 (1995) 109.
[2] E. Witten, ‘String Theory Dynamics in Various Dimensions,’ Nucl.Phys. B443 (95) 85.

[3] S. Kachru and C. Vafa, ‘Exact Results for N=2 Compactifications of Heterotic Strings,’ Nucl.Phys.B450 (95) 69.

[4] A.Klemm, W.Lerche and P.Mayr, ‘K3 Fibrations and Heterotic- Type II duality,’ Phys.Lett. B357 (95) 313.

[5] P.Aspinwall, J.Louis, ‘On the Ubiquity of K3 Fibrations in String Duality,’ hep-th 9510234.

[6] C. Vafa and E. Witten, ‘Dual String Pairs With N=1 and N=2 Supersymmetry in Four Dimensions,’ hep-th/9507050.

[7] J.Harvey, D.Lowe and A.Strominger, ‘N=1 String Duality,’ Phys.Lett. B362 (95) 65.

[8] S. Ferrara, J.Harvey, A.Strominger and C.Vafa, ‘Second Quantized Mirror Symmetry,’ Phys.Lett. B361 (95) 59.

[9] P.Aspinwall, ‘Enhanced Gauge Symmetries and K3 surfaces,’ Phys.Lett. B357 (1995) 329.

[10] J. Schwarz and A. Sen, ‘The Type IIA Dual of the CHL Compactification,’ Phys.Lett. B357 (1995) 323.

[11] C.M.Hull and P. Townsend, ‘Enhanced Gauge Symmetries In Superstring Theory,’ Nucl.Phys.B455 (1995) 525.

[12] G.Papadopoulos and P.Townsend, ‘Compactification of d=11 supergravity on spaces of exceptional holonomy.’ Phys.Lett.B356 (1995) 300.

[13] D.D.Joyce, ‘Compact Riemannian 7-Manifolds with G_2 Holonomy:1’ Oxford 1994 Preprint, to appear in J.Diff.Geom.

[14] D.D.Joyce, ‘Compact Riemannian 7-manifolds with G_2 Holonomy:2’ Oxford 1994 Preprint, to appear in J.Diff.Geom.

[15] D.D.Joyce, ‘Compact Riemannian 8-manifolds with Holonomy Spin(7)’ Oxford 1994 Preprint, to appear in Inv.Math.
[16] R.McLean, ‘Deformations and Moduli of Calibrated Submanifolds.’ Ph.D thesis, Duke University, (1990).

[17] K.Becker,M.Becker,A.Strominger, ‘Fivebranes, Membranes and Nonperturbative String Theory.’ Nucl.Phys. B456 (1995) 130.

[18] M.Duff,B.Nilsson and C.Pope, ‘Kaluza-Klein Supergravity.’ Phys.Rep 130, 1986, 1-142.

[19] P.Townsend, ‘A New Anomaly Free Chiral Supergravity from Compactification on K3’, Phys.Lett.139B (84) 283.

[20] A.Strominger, ‘Massless Black holes and Conifolds in String Theory’ Nucl.Phys.B451 (1995) 96, B.Greene,D.Morrison and A.Strominger, ‘Black Hole Condensation and the Unification of String Vacua’ Nucl. Phys. B451 (1995) 109.

[21] P.Horava, E.Witten, ‘Heterotic and TypeI Dynamics from Eleven Dimensions.’ hepth/9510209.

[22] E.Witten, ‘Five-branes and M-theory on an Orbifold’ hepth 9512219, K.Dasgupta, S.Mukhi, ‘Orbifolds of M-theory’ hepth 9512196.

[23] J.Polchinski, ‘Dirichlet Branes and RR charges.’ Phys.Rev.Lett.75 (1995) 4724.

[24] P.Townsend, ‘D branes from M branes’ hepth 9512062, A.Strominger, ‘Open P-branes’ hepth 9512059.

[25] L.Dixon, ‘World Sheet Aspects of String Compactification’. Proceedings of 1987 Summer Workshop ICTP, World Scientific.

[26] M.Green, J.Schwarz, P.West, ‘Anomaly Free chiral Theories in Six Dimensions’ Nucl.Phys.B254 (1985) 327.

[27] P.Aspinwall, ‘An N=2 Dual Pair and a Phase Transition’ hepth 9510142.

[28] S.Chaudhuri, J.Polchinski, ‘Moduli Space of CHL Strings’ Phys.Rev/D52 (1995) 7168.
[29] John Schwarz, ‘The Power of M Theory’ Phys.Lett. B367 (1996) 97.

[30] B.S. Acharya, in preparation.

[31] M. Duff, R. Minasian, E. Witten, ‘Evidence for heterotic-heterotic duality’, hep-th/9601036.

[32] C. M. Hull, ‘String Dynamics at Strong Coupling’, hep-th/9512181.

[33] C. Vafa, ‘Evidence for F-theory’, hep-th/9602022.

[34] E. Witten, ‘Is Supersymmetry Really Broken?’ Int.J.Mod.Phys.A10 (1995) 1247, ‘Strong Coupling and The Cosmological Constant.’ Mod.Phys.Lett. A10 (1995) 2153.