Research Article

Residual Symmetries and Bäcklund Transformations of (2 + 1)-Dimensional Strongly Coupled Burgers System

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1. Introduction

Nonlinear partial differential equations have wide applications in the field of physical science, engineering, and other applied disciplines, e.g., nonlinear optics [1–4], fluid flows [5–7], plasma physics [8, 9], excitable media, and so on [10–14]. Burgers equation \( p_t = 2pp_x + p_{xx} \) is a very important nonlinear partial differential equation occurring in various areas of applied sciences, such as fluid mechanics [15], nonlinear acoustics [15], gas dynamics, and traffic flow [16]. The equation was first introduced by Harry Bateman in 1915 and later studied by Johannes Martinus Burgers in 1948 [17, 18]. The study of symmetries plays an important role in branches of some natural sciences especially in integrable systems [19, 20]. In [21, 22], the authors proposed a residual symmetry in the process of the residue of the truncated Painlevé expansion for the bosonized super symmetric KdV equation which is a nonlocal symmetry [23–33]. In [34, 35], the authors concerned with the application of the nonlocal residual symmetry analysis to (2 + 1)-dimensional Burgers system, which has the form as follows:

\[
\begin{align*}
    p_t &= pp_y + arp_x + bp_{yy} + abp_{xx}, \\
    p_x &= r_y, \\
    q_x &= s_y,
\end{align*}
\]  

where \( a \) and \( b \) are arbitrary constants.

In [36, 37], a hierarchy called the Frobenius-valued Kakomtsev–Petviashvili hierarchy which takes values in a maximal commutative subalgebra of \( \text{gl}(m, \mathbb{C}) \) was constructed. Then, in [38], the authors considered the Hirota quadratic equation of the commutative version of extended multicomponent Toda hierarchy, which should be useful in Frobenius manifold theory [39, 40]. Recently, we studied \( Z_2 \)-Painlevé IV equation, Frobenius Painlevé I equation, and Frobenius Painlevé III equation [41]. In this paper, we consider a new (2 + 1)-dimensional strongly coupled Burgers system which is defined by us through taking values in a commutative subalgebra \( Z_2 = \mathbb{C}[\Gamma]/(\Gamma^2) \). We replace the \( p \) and \( r \) of (1) with the commutative matrix

\[
\begin{pmatrix}
    p & q \\
    q & p
\end{pmatrix},
\begin{pmatrix}
    r & s \\
    s & r
\end{pmatrix}.
\]  

Then, we can get

\[
\begin{align*}
    p_t &= pp_y + q p_y + ar p_x + asq_x + bp_{yy} + abp_{xx}, \\
    q_t &= pq_y + qp_y + arq_x + asp_x + bq_{yy} + abq_{xx}, \\
    p_x &= r_y, \\
    q_x &= s_y,
\end{align*}
\]  

which is called (2 + 1)-dimensional strongly coupled Burgers system.
The aim of this paper is to promote the (2+1)-dimensional Burgers system to a Frobenius integrable systems which is called (2+1)-dimensional strongly coupled Burgers system. A suitable enlarged system is given to localize the nonlocal residual symmetry. It follows that a Bäcklund transformation is derived by solving an initial value problem. It means that one can find various solutions of the (2+1)-dimensional strongly coupled Burgers system from a seed solution.

2. Residual Symmetries of (2+1)-Dimensional Strongly Coupled Burgers System

We first introduce the truncated Painlevé expansion:

\[ p = \sum_{i=0}^{n} p_i (\psi + \phi)^i + q_i (\psi - \phi)^i , \]
\[ q = \sum_{i=0}^{n} q_i (\psi + \phi)^i + q_i (\psi - \phi)^i , \]
\[ r = \sum_{i=0}^{n} r_i (\psi + \phi)^i + r_i (\psi - \phi)^i , \]
\[ s = \sum_{i=0}^{n} s_i (\psi + \phi)^i + s_i (\psi - \phi)^i . \]

where \( p_0, q_0 \) are a set of arbitrary solutions of the equation, and \( p_{n+1}, p_{n+2}, \ldots, q_{n+1}, q_{n+2}, \ldots, q_0 \) to be expressed by derivatives of \( \psi \) and \( \phi \). By balancing the dispersion and nonlinear terms according to the leader order analysis to the system (3a and 3b), the truncated Painlevé expansion has the following form:

\[ p = \frac{p_0 \psi - q_0 \phi}{\psi^2 - \phi^2} + p_1 , \quad q = \frac{q_0 \psi - p_0 \phi}{\psi^2 - \phi^2} + q_1 , \]
\[ r = \frac{r_0 \psi - s_0 \phi}{\psi^2 - \phi^2} + r_1 , \quad s = \frac{s_0 \psi - r_0 \phi}{\psi^2 - \phi^2} + s_1 . \]

Then, plugging (5a and 5b) into (3a and 3b) and vanishing all the coefficients of each power of \( (\psi + \phi)^i \) and \( (\psi - \phi)^i \), we obtain

\[ p_0 = 2b \psi \phi , \quad q_0 = 2b \phi \psi , \]
\[ r_0 = 2b \phi \phi , \quad s_0 = 2b \phi \phi . \]

\[ a \psi \phi r_1 + a \phi \phi s_1 + a \psi \psi r_1 + a \psi \phi s_1 + a \phi \psi r_1 + a \phi \phi s_1 = 0 , \]
\[ a \phi \phi r_1 + a \psi \psi s_1 + a \phi \psi s_1 + a \phi \phi s_1 + a \phi \phi s_1 + a \phi \phi s_1 = 0 , \]
\[ a \psi \psi r_1 + a \phi \phi r_1 + a \phi \psi r_1 + a \phi \phi s_1 + a \phi \phi s_1 + a \phi \phi s_1 = 0 . \]

\[ P_{1\mu} - P_{1\nu} \psi - q_0 \phi \psi = \alpha r_1 \psi + a s_1 q_1 \phi - b p_{1\psi \psi} - a b p_{1\phi \phi} = 0 , \]
\[ q_{1\mu} - P_{1\nu} q_1 \psi - q_0 \phi q_1 + a s_1 q_1 \phi - b q_{1\psi \psi} - a b q_{1\phi \phi} = 0 , \]
\[ r_{1\mu} - P_{1\nu} r_1 \psi - q_0 \phi r_1 = 0 , \quad s_{1\mu} - P_{1\nu} s_1 \psi - q_0 \phi s_1 = 0 . \]

It is easy to find that (8) and (9) are just the (2+1)-dimensional strongly coupled Burgers system (3) with \( p_0, q_0, r_0 \), and \( s_0 \) of (6) into the linearized form of (8) and (9) with (7) that one can find the \( p_0, q_0, r_0 \) and \( s_0 \) in (6) are the symmetries of the strongly coupled Burgers system.

According to the theorem of the residual symmetry [22], the strongly coupled Burgers system has a residual symmetry:

\[ \sigma^\psi = 2b \psi \phi , \quad \sigma^\phi = 2b \phi \psi , \quad \sigma^\psi = 2b \phi \phi , \quad \sigma^\phi = 2b \phi \phi , \]

which is nonlocal for \( \psi \) and \( \phi \) related to \( p_0, q_0, r_0 \), and \( s_0 \) by (7). Then, we introduce auxiliary variables \( f, g, h, \) and \( k \) with the relations \( f = \psi \phi, g = \phi \phi, h = \psi \psi \), and \( k = \phi \phi \) to obtain a local symmetry in the following enlarged system:

\[ p_1 - p \psi \psi - q \phi \psi - a \phi \phi r - a s_1 \phi \psi - b p_{\psi \psi \psi} - a b p_{\psi \phi \phi} = 0 , \]
\[ q_1 - q \psi \psi - q \phi \psi - a \phi \phi r - a s_1 \phi \psi - b q_{\psi \psi \psi} - a b q_{\psi \phi \phi} = 0 , \]
\[ r_1 - r \psi \psi - q \phi \psi - a \phi \phi r - a s_1 \phi \psi - b r_{\psi \psi \psi} - a b r_{\phi \phi \phi} = 0 , \]
\[ s_1 - s \psi \psi - q \phi \psi - a \phi \phi r - a s_1 \phi \psi - b s_{\psi \psi \psi} - a b s_{\phi \phi \phi} = 0 . \]

Further, the residual symmetry can be localized into the Lie point symmetry

\[ \sigma^h = h , \quad \sigma^k = k , \]
\[ \sigma^f = f , \quad \sigma^g = g , \]
\[ \sigma^\phi = -\frac{1}{b} (f \psi + g \phi) , \quad \sigma^\phi = -\frac{1}{b} (g \psi + f \phi) , \]
\[ \sigma^g = h , \quad \sigma^k = k , \]
\[ \sigma^\psi = -\frac{1}{2b} (\psi^2 + \phi^2) , \quad \sigma^\phi = -\frac{1}{b} \psi \phi , \]

which satisfy

\[ \sigma_{\phi}^h = \sigma_{\phi}^g = \sigma_{\phi}^f = \sigma_{\phi}^k = 0 , \]
\[ \sigma_{\phi}^\psi = \sigma_{\phi}^\phi = \sigma_{\phi}^\phi = \sigma_{\phi}^\phi = 0 . \]
\[ \sigma_y^p = \sigma^h, \quad \sigma_y^e = \sigma^k. \]  \hfill (13e)

Therefore, the enlarged system (14) has the Lie point symmetry vector:

\[
\begin{align*}
\vec{V}' &= h \partial_h + f \partial_f - \frac{1}{b} (f \psi g + \phi \partial_g), \\
\vec{W}' &= k \partial_k + g \partial_g - \frac{1}{b} (g \psi h + \phi \partial_h).
\end{align*}
\]  \hfill (14)

Next we will give the Bäcklund symmetry theorem, which is obtained by using a finite transformation of the Lie point symmetry (14).

**Theorem 1.** If \( \{ p, q, r, s, f, g, h, k, \psi, \phi \} \) is a solution of the coupled system (11a–11e), then so is \( \{ \vec{p}, \vec{q}, \vec{r}, \vec{f}, \vec{h}, \vec{k}, \vec{\psi}, \vec{\phi} \} \) with

\[
\begin{align*}
\vec{p} &= p + \frac{2eb(chy - c\phi + 2bh)}{(e\psi + 2b)^2 - e^2 \phi^2}, \\
\vec{q} &= q + \frac{2eb(chy - c\phi + 2bh)}{(e\psi + 2b)^2 - e^2 \phi^2}, \\
\vec{r} &= r + \frac{2eb(c\psi - ey + 2bf)}{(e\psi + 2b)^2 - e^2 \phi^2}, \\
\vec{s} &= s + \frac{2eb(c\psi - ey + 2bf)}{(e\psi + 2b)^2 - e^2 \phi^2}, \\
\vec{f} &= \frac{4b\phi}{(e\psi + 2b)^2 - e^2 \phi^2}, \\
\vec{h} &= \frac{4b(C - Dg)}{C^2 - D^2}, \\
\vec{\psi} &= \frac{4b(C - Dk)}{C^2 - D^2}, \\n\vec{\phi} &= \frac{4b(C - Dh)}{C^2 - D^2}. 
\end{align*}
\]  \hfill (15a)

where

\[ C = e^2 (\psi^2 + \phi^2) + 4be\psi + 4b\phi, \quad D = 2e^2 \phi \psi + 4b e\phi, \]  \hfill (16)

and \( \epsilon \) is an arbitrary group parameter.

**Proof.** According to Lie’s first theorem on vector (14), it is not difficult to find that the key to prove this theorem is to solve the following initial value problem:

\[
\begin{align*}
d\vec{p}/de &= \vec{h}, \\
d\vec{q}/de &= \vec{k}, \\
d\vec{r}/de &= \vec{f}, \\
d\vec{s}/de &= \vec{g}, \\
d\vec{f}/de &= -\frac{1}{b} (\vec{\psi} f + \vec{\phi} g), \\
d\vec{h}/de &= -\frac{1}{b} (\vec{\psi} h + \vec{\phi} k), \\
d\vec{k}/de &= -\frac{1}{b} (\vec{\psi} k + \vec{\phi} h), \\
d\vec{\psi}/de &= -\frac{1}{2b} (\vec{\psi}^2 + \vec{\phi}^2), \\
d\vec{\phi}/de &= -\frac{1}{b} \vec{\psi} \vec{\phi}. 
\end{align*}
\]  \hfill (17a)

And (15a–15e) is the solution to the above system. Therefore, we have completed the proof of Theorem 1.

\[ \square \]

3. Bäcklund Transformations of Strongly Coupled Burgers System Related to Multiple Residual Symmetries

For the \((2 + 1)\)-dimensional strongly coupled Burgers system (3a and 3b), the original residual symmetry

\[ \sigma^o = 2b \psi_x, \quad \sigma^o = 2b \phi_x, \quad \sigma^o = 2b \psi_x, \quad \sigma^o = 2b \phi_x, \]  \hfill (18)

is related to the solution of Equation (11c). Then, according to the linear property of symmetry equations, the multiple residual symmetries are expressed in terms of any linear superposition of symmetry

\[ \sigma_n = \sum_{i=1}^n \sigma_i, \quad \sigma_n = \sum_{i=1}^n \sigma_i, \]  \hfill (19)

where \( \psi \) and \( \phi \) are different solutions of (11c). And the symmetry (19) should be localized to a Lie point symmetry by introducing more variables. In this way, one can find the finite transformation group of the symmetry (19).

**Theorem 2.** If \( \{ p, q, r, s, f, g, h, k, \psi, \phi \} \) \( (i = 1, 2, \ldots, n) \) is a solution of the enlarged system

\[ p_i - pp_y - q \phi_x - ar \phi_y - bs \psi_x - bp \psi_y - ab p_{xx} = 0, \quad q_i - pq_y - q \phi_x - ar \phi_y - bs \psi_x - bp \psi_y - ab p_{xx} = 0, \]  \hfill (20a)

\[ p_y = r_y, \quad q_y = s_y, \]  \hfill (20b)

\[ \psi_{xx} + r \psi_{xx} + s \psi_{xx} + p \psi_y + q \phi_y + b \psi_{yy} - \psi_{xy} = 0, \]  \hfill (20c)

\[ \phi_{xx} + \psi_{xx} + s \phi_{xx} + q \psi_y + p \phi_y + b \phi_{yy} - \phi_{xy} = 0, \]  \hfill (20d)

\[ f_i = \psi_{xx}, \quad g_i = \phi_{xx}, \]  \hfill (20e)

\( (i = 1, 2, \ldots, n), \)

then the symmetry (19) is localized to the Lie point symmetry

\[ \sigma_n^p = \sum_{i=1}^n c_i h_i, \quad \sigma_n^o = \sum_{i=1}^n c_i k_i, \]  \hfill (21a)

\[ \sigma_n^p = \sum_{i=1}^n c_i f_i, \quad \sigma_n^o = \sum_{i=1}^n c_i g_i, \]  \hfill (21b)
\[\sigma^v = -\frac{c_i}{2b}(\psi_i^2 + \phi_i^2) - \sum_{j \neq i} \frac{c_j}{2b}(\psi_j \psi_i + \phi_i \phi_j),\]  
(21c)
\[\sigma^h = \frac{c_i}{b} \psi_i \phi_i - \sum_{j \neq i} \frac{c_j}{2b}(\psi_j \psi_i + \phi_i \phi_j),\]  
(21d)
\[\sigma^j = -\frac{c_i}{2b}(f_i \psi_i + g_i \phi_i) - \sum_{j \neq i} \frac{c_j}{2b}(f_j \psi_j + g_j \phi_j + f_i \psi_i + g_i \phi_i),\]  
(21e)

Proof. The extended system (20a)–(20e) has the following linearized form:
\[\sigma^i - ab \sigma^r - a \sigma^s - ar \sigma^t - as \sigma^u - a_s \sigma^v - a_r \sigma^w = 0,\]  
(22a)
\[\sigma^v = -\frac{c_m}{2b}(\psi_m^2 + \phi_m^2), \quad \sigma^h = -\frac{c_m}{2b}(\psi_m \phi_m),\]  
(22b)

Let us first consider the special case: for any fixed \(m\), \(c_m \neq 0\), while \(c_i = 0\), \(j \neq m\) in (19), from (12a)–(12e), the localized symmetry for \(p, q, r, s, f, g, h, k, m, \psi, \phi\) can be obtained as follows:
\[\sigma^g = c_m h_m, \quad \sigma^d = c_m k_m,\]  
(23a)
\[\sigma^r = c_m f_m, \quad \sigma^s = c_m g_m,\]  
(23b)
\[\sigma^u = -\frac{c_m}{2b}(\psi_m^2 + \phi_m^2), \quad \sigma^v = -\frac{c_m}{2b}(f_m \phi_m + g_m \psi_m),\]  
(23c)

In (20c), we let \(i = m\) and \(i = j\), and then we obtain
\[a \psi_j \psi_j + a \phi_j \phi_j + b \psi_j + b \phi_j - \psi_j = 0,\]  
(24)
\[a \psi_m \psi_m + a \phi_m \phi_m + b \psi_m + b \phi_m - \psi_m = 0.\]  
(25)

Then, we substitute (23a) and (23b) into (22c) with \(i = j\) yield
\[\sigma^i - ab \sigma^r - a \sigma^s - ar \sigma^t - as \sigma^u - a_s \sigma^v - a_r \sigma^w = 0,\]  
(26)

Further, one can obtain a solution of Equation (26):
\[\sigma^v = -\frac{c_m}{2b}(\psi_m \phi_m), \quad \sigma^h = -\frac{c_m}{2b}(\psi_m \phi_m + \phi_m \psi_m),\]  
(27)

which can be verified by using (20b), (24), and (25). From (22d) and (22e) with \(i = j\), the symmetry for \(f, g, h, k\) can be given by:
\[\sigma^i = -\frac{c_m}{2b}(f \psi_m + g \phi_m + h \phi_m + k \psi_m),\]  
(28)
\[\sigma^j = -\frac{c_m}{2b}(f \phi_m + g \psi_m + h \phi_m + k \psi_m),\]  
(29)

Further, (21a)–(21e) can be obtained by taking a linear combination of the above results for \(m = 1, 2, \ldots, n\). Therefore, we have completed the proof of Theorem 2. \(\square\)

According to Lie’s first theorem, the initial value problem of the Lie point symmetry (21a)–(21e) has the following form:
\[\frac{dP(e)}{de} = \sum_{j=1}^{n} c_j \psi_j(e), \quad \frac{dQ(e)}{de} = \sum_{j=1}^{n} c_j \phi_j(e),\]  
(30a)
\[\frac{dR(e)}{de} = \sum_{j=1}^{n} c_j \psi_j(e), \quad \frac{dS(e)}{de} = \sum_{j=1}^{n} c_j \phi_j(e),\]  
(30b)
\[\frac{d\Psi(e)}{de} = -\frac{c_i}{2b}(\psi_i^2 + \phi_i^2) - \sum_{j \neq i} \frac{c_j}{2b}(\psi_j \psi_i + \phi_i \phi_j),\]  
(30c)
\[\frac{d\Phi(e)}{de} = -\frac{c_i}{2b}(\psi_i \phi_i) - \sum_{j \neq i} \frac{c_j}{2b}(\psi_j \phi_i + \phi_i \psi_j).\]  
(30d)
\[
\frac{dF_i(c)}{dc} = -\frac{c}{b} (F_i(c)\Psi_i(c) + G_i(c)\Phi_i(c)) \\
- \sum_{j \neq i} \frac{c}{b} (F_j(c)\Psi_j(c) + G_j(c)\Phi_j(c) + F_j(c)\Psi_j(c) + G_j(c)\Phi_j(c)),
\]
\[
\frac{dG_i(c)}{dc} = -\frac{c}{b} (F_i(c)\Phi_i(c) + G_i(c)\Psi_i(c)) \\
- \sum_{j \neq i} \frac{c}{2b} (F_j(c)\Phi_j(c) + G_j(c)\Psi_j(c) + F_j(c)\Phi_j(c) + G_j(c)\Psi_j(c)),
\]
\[
\frac{dH_i(c)}{dc} = -\frac{c}{b} (H_i(c)\Phi_i(c) + K_i(c)\Phi_i(c)) \\
- \sum_{j \neq i} \frac{c}{2b} (H_j(c)\Phi_j(c) + K_j(c)\Phi_j(c) + H_j(c)\Phi_j(c) + K_j(c)\Phi_j(c)),
\]
\[
\frac{dK_i(c)}{dc} = -\frac{c}{b} (H_i(c)\Phi_i(c) + K_i(c)\Psi_i(c)) \\
- \sum_{j \neq i} \frac{c}{2b} (H_j(c)\Phi_j(c) + K_j(c)\Phi_j(c) + H_j(c)\Phi_j(c) + K_j(c)\Psi_j(c)),
\]

\[
P(0) = p, \quad Q(0) = q, \quad R(0) = r, \quad S(0) = s, \quad \Psi_i(0) = \epsilon_i, \quad \Phi_i(0) = \phi_i, \quad F_i(0) = f_i, \quad G_i(0) = g_i,
\]

\[
H_i(0) = h_i, \quad K_i(0) = k_i, \quad i = 1, 2, \ldots, n.
\]

Then, one can get the following N-th Bäcklund theorem for the extended system (20a)–(20e) by solving (30a)–(30f).

**Theorem 3.** If \( \{p, q, r, s, f_i, g_i, h_i, k_i, \psi_i, \phi_i\} \) is a solution of the coupled system (20a)–(20e), then so is \( \{P(e), Q(e), R(e), S(e), F_i(e), G_i(e)H_i(e), K_i(e), \Psi_i(e), \Phi_i(e)\}, \) (i = 1, 2, \ldots, n), where

\[
P(e) = p + b(\ln(A + B) + \ln(A - B)),
\]
\[
Q(e) = q + b(\ln(A + B) - \ln(A - B)),
\]
\[
R(e) = r + b(\ln(A + B) + \ln(A - B)),
\]
\[
S(e) = s + b(\ln(A + B) - \ln(A - B)),
\]
\[
\Psi_i(e) = \frac{2b}{A^2 - B^2} (A, A - B, B), \quad \Phi_i(e) = \frac{2b}{A^2 - B^2} (B, A, A - B),
\]

where

\[
A = \begin{bmatrix}
\frac{-\frac{c}{2b} \epsilon_1}{-\frac{c}{2b} \epsilon_1 - 1} & c_1 e_{M_1,2} & \cdots & c_1 e_{M_1,i-1} & c_1 e_{M_1,i} & c_1 e_{M_1,i+1} & \cdots & c_1 e_{M_1,n} \\
\frac{-\frac{c}{2b} \epsilon_2}{-\frac{c}{2b} \epsilon_2 - 1} & c_2 e_{M_2,2} & \cdots & c_2 e_{M_2,i-1} & c_2 e_{M_2,i} & c_2 e_{M_2,i+1} & \cdots & c_2 e_{M_2,n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{-\frac{c}{2b} \epsilon_{i-1}}{-\frac{c}{2b} \epsilon_{i-1} - 1} & c_{i-1} e_{M_{i-1,2}} & \cdots & c_{i-1} e_{M_{i-1,i-1}} & c_{i-1} e_{M_{i-1,i}} & c_{i-1} e_{M_{i-1,i+1}} & \cdots & c_{i-1} e_{M_{i-1,n}} \\
\frac{-\frac{c}{2b} \epsilon_{i}}{-\frac{c}{2b} \epsilon_{i} - 1} & c_{i} e_{M_{i,2}} & \cdots & c_{i} e_{M_{i,i-1}} & c_{i} e_{M_{i,i}} & c_{i} e_{M_{i,i+1}} & \cdots & c_{i} e_{M_{i,n}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{-\frac{c}{2b} \epsilon_{n}}{-\frac{c}{2b} \epsilon_{n} - 1} & c_{n} e_{M_{n,2}} & \cdots & c_{n} e_{M_{n,n-1}} & c_{n} e_{M_{n,n}} & c_{n} e_{M_{n,n+1}} & \cdots & c_{n} e_{M_{n,n}} \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
\frac{-\frac{c}{2b} \phi_1}{-\frac{c}{2b} \phi_1 - 1} & c_1 e_{N_1,2} & \cdots & c_1 e_{N_1,i-1} & c_1 e_{N_1,i} & c_1 e_{N_1,i+1} & \cdots & c_1 e_{N_1,n} \\
\frac{-\frac{c}{2b} \phi_2}{-\frac{c}{2b} \phi_2 - 1} & c_2 e_{N_2,2} & \cdots & c_2 e_{N_2,i-1} & c_2 e_{N_2,i} & c_2 e_{N_2,i+1} & \cdots & c_2 e_{N_2,n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{-\frac{c}{2b} \phi_{i-1}}{-\frac{c}{2b} \phi_{i-1} - 1} & c_{i-1} e_{N_{i-1,2}} & \cdots & c_{i-1} e_{N_{i-1,i-1}} & c_{i-1} e_{N_{i-1,i}} & c_{i-1} e_{N_{i-1,i+1}} & \cdots & c_{i-1} e_{N_{i-1,n}} \\
\frac{-\frac{c}{2b} \phi_{i}}{-\frac{c}{2b} \phi_{i} - 1} & c_{i} e_{N_i,2} & \cdots & c_{i} e_{N_i,i-1} & c_{i} e_{N_i,i} & c_{i} e_{N_i,i+1} & \cdots & c_{i} e_{N_i,n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{-\frac{c}{2b} \phi_{n}}{-\frac{c}{2b} \phi_{n} - 1} & c_{n} e_{N_n,2} & \cdots & c_{n} e_{N_n,n-1} & c_{n} e_{N_n,n} & c_{n} e_{N_n,n+1} & \cdots & c_{n} e_{N_n,n} \\
\end{bmatrix}
\]
with

\[ M_{ij} = -\frac{1}{4b}\left(\psi_i\psi_j + \psi_j\psi_i + \psi_i\psi_j + \psi_j\psi_i\right)^{1/2} + \left(\psi_i\psi_j - \psi_j\psi_i\right)^{1/2}, \]
\[ N_{ij} = -\frac{1}{4b}\left(\psi_i\psi_j + \psi_j\psi_i + \psi_i\psi_j + \psi_j\psi_i\right)^{1/2} - \left(\psi_i\psi_j - \psi_j\psi_i\right)^{1/2}. \]

(34)

From any seed solution of the \((2+1)\)-dimensional strongly coupled Burgers system, one can get an infinite number of new solutions because \(n\) is an arbitrary positive integer. We consider a special case about the solution of the \((2+1)\)-dimensional strongly coupled Burgers system. The \((2+1)\)-dimensional strongly coupled Burgers system has a soliton solution:

\[ p = a_1 + \frac{2b}{(e^{2k+2l} + 1)(e^{2k+2l} + 1)} (e^{2k+2l} + 1) (e^{2k+2l} + 1), \]
\[ q = a_1 + \frac{2b}{(e^{2k+2l} + 1)(e^{2k+2l} + 1)} (e^{2k+2l} + 1) (e^{2k+2l} + 1), \]
\[ r = s = 0, \]

where

\[ k_1 = (b_1a_1 + b_1a_1)t + b_1y, \]
\[ k_2 = (b_1a_1 + b_1a_1)t + b_1y, \]

and \(a_0, a_1, b_0\) and \(b_1\) are arbitrary constants.

Then, the solution of (20c) can be obtained

\[ \psi_i = \frac{d^2 (e^{4k} - 1)}{(e^{2k+2l} + 1)(e^{2k+2l} + 1)} + e_i (e^{2k+2l} - e^{2k+2l}), \]
\[ \phi_i = \frac{d^2 (e^{4k} - 1)}{(e^{2k+2l} + 1)(e^{2k+2l} + 1)} + e_i (e^{4k} - 1), \]

where \(d_1\) and \(e_i\) are arbitrary constants.

Further, one can obtain the \(M_{ij}\) and \(N_{ij}\) of Theorem 3

\[ M_{ij} = -\frac{1}{4b}\left(\psi_i\psi_j + \psi_j\psi_i + \psi_i\psi_j + \psi_j\psi_i\right)^{1/2} + \left(\psi_i\psi_j - \psi_j\psi_i\right)^{1/2}, \]
\[ N_{ij} = -\frac{1}{4b}\left(\psi_i\psi_j + \psi_j\psi_i + \psi_i\psi_j + \psi_j\psi_i\right)^{1/2} - \left(\psi_i\psi_j - \psi_j\psi_i\right)^{1/2}, \]

where

\[ C_0 = (d_1d_1 + e_1e_1 + d_1e_1 + e_1d_1)^{1/2} + (d_1d_1 + e_1e_1 - d_1e_1 - e_1d_1)^{1/2}, \]
\[ D_0 = (d_1d_1 + e_1e_1 + d_1e_1 + e_1d_1)^{1/2} - (d_1d_1 + e_1e_1 - d_1e_1 - e_1d_1)^{1/2}. \]

(39)

We write \(C_1\) and \(D_1\) as

\[ C_1 = \frac{e^{4k} - 1}{(e^{2k+2l} + 1)(e^{2k+2l} + 1)}, \]
\[ D_1 = \frac{e^{2k+2l} - e^{2k+2l}}{(e^{2k+2l} + 1)(e^{2k+2l} + 1)}. \]

(40)

Finally, consider the case of \(n = 2\) in Theorem 3, one can get a new soliton solution

\[ p = C_1\left(c_0d_1 + a_1e_1 + a_1d_1 + a_1e_1 + a_1d_1\right) + D_1\left(c_0d_1 + a_1e_1 + a_1d_1 + a_1e_1\right), \]
\[ q = c_0d_1 + a_1e_1 + a_1d_1 + a_1e_1 + a_1d_1 + a_1e_1, \]
\[ r = s = 0. \]

(41)

\[ C_2 = C_1\left(c_0d_1 + a_1e_1 + a_1d_1 + a_1e_1 + a_1d_1\right) + 4b^2b_0, \]
\[ + D_1\left(c_0d_1 + a_1e_1 + a_1d_1 + a_1e_1\right) + 4b^2b_1, \]
\[ + 2bc_0\left(b_0d_1 + b_1e_1 + b_1d_1 + b_1e_1\right) + 2b_a, \]
\[ D_2 = D_1\left(c_0d_1 + a_1e_1 + a_1d_1 + a_1e_1\right) + 4b^2b_0, \]
\[ + C_1\left(c_0d_1 + a_1e_1 + a_1d_1 + a_1e_1\right) + 4b^2b_1, \]
\[ + 2bc_0\left(b_0d_1 + b_1e_1 + b_1d_1 + b_1e_1\right) + 2b_a, \]

(42)

4. Conclusion and Discussion

In this paper, we first defined a new \((2+1)\)-dimensional strongly coupled Burgers system which takes values in a two component commutative subalgebra \(Z_h\). Then, the residual symmetry of the strongly coupled Burgers system was obtained by using the truncated Painlevé expansion, and the corresponding residual system was just the nonlocal symmetry. To localize the residual symmetry, we introduced a suitable enlarged system. According to Lie’s first theorem, the finite Bäcklund transformation was derived. Further, the \(N\)-th Bäcklund transformation of the strongly coupled Burgers system was obtained by localizing the linear superposition of multiple residual symmetries, and the \(N\)-th Bäcklund transformation was expressed by determinants in a compact form.

Data Availability

This work does not have any experimental data.

Ethical Approval

This work did not involve any active collection of human data.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

Yufeng Zhang contribute the idea and did some calculation. Haifeng Wang get computational results, and wrote the paper. All authors gave final approval for publication.

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