Quartic graphs with minimum spectral gap

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Abstract
Aldous and Fill conjectured that the maximum relaxation time for the random walk on a connected regular graph with \( n \) vertices is \( (1 + o(1)) \frac{3n^2}{2\pi^2} \). This conjecture can be rephrased in terms of the spectral gap as follows: the spectral gap (algebraic connectivity) of a connected \( k \)-regular graph on \( n \) vertices is at least \( (1 + o(1)) \frac{2k\pi^2}{3n^2} \), and the bound is attained for at least one value of \( k \). We determine the structure of connected quartic graphs on \( n \) vertices with a minimum spectral gap which enables us to show that the minimum spectral gap of connected quartic graphs on \( n \) vertices is \( (1 + o(1)) \frac{4\pi^2}{n^2} \). From this result, the Aldous–Fill conjecture follows for \( k = 4 \).

KEYWORDS
algebraic connectivity, quartic graph, relaxation time, spectral gap

MATHEMATICAL SUBJECT CLASSIFICATION
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1 | INTRODUCTION

All graphs we consider are simple, that is, undirected graphs without loops or multiple edges. The difference between the two largest eigenvalues of the adjacency matrix of a graph \( G \) is called the spectral gap of \( G \). If \( G \) is a regular graph, then its spectral gap is equal to the second smallest eigenvalue of its Laplacian matrix and is known as algebraic connectivity.

In 1976, Bussemaker, Čobelić, Cvetković, and Seidel ([5], see also [6]), by means of a computer search, found all nonisomorphic connected cubic graphs with \( n \leq 14 \) vertices. They observed that when the algebraic connectivity is small, the graph is “long” (meaning roughly that their diameter is as large as possible; this was later on justified in [4]). Indeed, as the algebraic connectivity decreases, both connectivity and girth decrease and diameter increases.
On the basis of these results, L. Babai (see [10]) made a conjecture that described the structure of the connected cubic graph with minimum algebraic connectivity. Guiduli [10] (see also [9]) proved that the cubic graph with minimum algebraic connectivity must look like a path, built from specific blocks. The result of Guiduli was improved as follows confirming Babai’s conjecture.

**Theorem 1.1** (Brand, Guiduli, and Imrich [4]). Among all connected cubic graphs on $n$ vertices, $n \geq 10$, the cubic graph depicted in Figure 1 is the unique graph with minimum algebraic connectivity.

The relaxation time of the random walk on a graph $G$ is defined by $\tau = 1/(1 - \eta_2)$, where $\eta_2$ is the second largest eigenvalue of the transition matrix of $G$, that is, the matrix $\Delta^{-1}A$ in which $\Delta$ and $A$ are the diagonal matrix of vertex degrees and the adjacency matrix of $G$, respectively. A central problem in the study of random walks is to determine the mixing time, a measure of how fast the random walk converges to the stationary distribution. As seen throughout the literature [3, 7], the relaxation time is the primary term for controlling mixing time. Therefore, relaxation time is directly associated with the rate of convergence of the random walk.

Our main motivation in this study is the following conjecture on the maximum relaxation time of the random walk in regular graphs.

**Conjecture 1.2** (Aldous and Fill [3, p. 217]). Over all connected regular graphs on $n$ vertices, $\max \tau = (1 + o(1)) \frac{3n^2}{2\pi^2}$.

For a graph $G$, $L(G) = \Delta - A$ is its Laplacian matrix. The second smallest eigenvalue of $L(G)$, that is, the algebraic connectivity of $G$ is denoted by $\mu(G)$. When $G$ is regular, of degree $k$ say, then its transition matrix is $\frac{1}{k}A$ and its Laplacian is $kI - A$. It is then seen that the relaxation time of $G$ is equal to $k/\mu(G)$. Also as $G$ is regular, $\mu(G)$ is the same as its spectral gap. So within the family of $k$-regular graphs, maximizing the relaxation time is equivalent to minimizing the spectral gap. More precisely, we have the following rephrasement of the Aldous–Fill conjecture.

**Conjecture 1.3.** The spectral gap (algebraic connectivity) of a connected $k$-regular graph on $n$ vertices is at least $\left(1 + o(1)\right) \frac{2k\pi^2}{3n^2}$, and the bound is attained at least for one value of $k$.

**FIGURE 1** Cubic graph with minimum spectral gap on $n \equiv 2 \pmod{4}$ and $n \equiv 0 \pmod{4}$ vertices, respectively
It is worth mentioning that in [2], it is proved that the maximum relaxation time for the random walk on a connected graph on \( n \) vertices is \((1 + o(1)) \frac{n^4}{54}\), settling another conjecture by Aldous and Fill ([3, p. 216]).

Abdi, Ghobani, and Imrich [1] proved that the algebraic connectivity of the graphs of Theorem 1.1 is \((1 + o(1)) \frac{2\pi^2}{n^2}\), implying that this is indeed the minimum spectral gap of connected cubic graphs of order \( n \). This settled the Aldous–Fill conjecture for \( k = 3 \). As the next case of the Aldous–Fill conjecture and as a continuation of Babai’s conjecture, we consider quartic, that is, 4-regular graphs and investigate minimal quartic graphs of a given order \( n \), that is, the graphs which attain the minimum spectral gap among all connected quartic graphs of order \( n \). In [1], it was shown that similar to the cubic case, minimal quartic graphs have a path-like structure (see Figure 2) with specified blocks (see Theorem 3.1). However, the precise description of minimal quartic graphs was left as a conjecture given below.

**Definition 1.4.** For any \( n \geq 11 \), define a quartic graph \( G_n \) as follows. Let \( m \) and \( r \leq 4 \) be nonnegative integers such that \( n - 11 = 5m + r \). Then \( G_n \) consists of \( m \) middle blocks \( M_0 \) and each end block is one of \( D_0, D_1, D_2, D_4 \) of Figure 3. If \( r = 0 \), then both end blocks are \( D_0 \). If \( r = 1 \), then the end blocks are \( D_0 \) and \( D_1 \). If \( r = 2 \), then both end blocks are \( D_1 \). If \( r = 3 \), then the end blocks are \( D_1 \) and \( D_2 \). Finally, if \( r = 4 \), then the end blocks are \( D_0 \) and \( D_4 \). (Note that the right end blocks are the mirror images of one of \( D_0, D_1, D_2, D_4 \).)

**Conjecture 1.5 (Abdi et al. [1]).** For any \( n \geq 11 \), the graph \( G_n \) is the unique minimal quartic graph of order \( n \).

We “almost” prove Conjecture 1.5 (see Theorem 3.2) by showing that in a minimal quartic graph any middle block is \( M_0 \) and any end block is one of \( D_0, D_1, D_2, D_4 \), or possibly one additional block \( D_3 \); see Figure 4. This result is enough to prove that the minimum algebraic

![Path-like structure](image1)

**Figure 2** Path-like structure

![Blocks of the quartic graphs](image2)

**Figure 3** Blocks of the quartic graphs \( G_n \)

![Block D3](image3)

**Figure 4** Block \( D_3 \)
connectivity of connected quartic graphs of order \( n \) is \( \left( 1 + o(1) \right) \frac{4\pi^2}{n^2} \) (see Corollary 3.3). This shows that, although the Aldous–Fill conjecture does hold for \( k = 4 \), the bound given by it (i.e., \( \left( 1 + o(1) \right) \frac{8\pi^2}{3n^2} \)) is not tight for 4-regular graphs (in contrast to 3-regular graphs where the bound is tight).

The outline of the paper is as follows. In Section 2, we determine the algebraic connectivity of the graphs \( \mathcal{G}_n \) as well as those with end block \( D_3 \), and in Section 3, we prove that indeed the minimal quartic graphs belong to this family.

## 2 | MINIMUM ALGEBRAIC CONNECTIVITY OF QUARTIC GRAPHS

In this section, we show that the algebraic connectivity of the quartic graphs of order \( n \) with a path-like structure whose middle blocks are all \( M_0 \) and end blocks are from \( D_0, \ldots, D_4 \) is \( \left( 1 + o(1) \right) \frac{4\pi^2}{n^2} \). In Section 3, we will prove that minimal quartic graphs belong to the family described above. From these two results, it readily follows that \( \left( 1 + o(1) \right) \frac{4\pi^2}{n^2} \) is indeed the minimum algebraic connectivity of connected quartic graphs of order \( n \).

Let \( G \) be a graph of order \( n \) and \( L(G) = \Delta - A \) be its Laplacian matrix. It is well known that \( L(G) \mathbf{1}^T = 0^T \), where \( \mathbf{1} \) is the all-1 vector,\(^1\) and 0 is in fact the smallest eigenvalue of \( L(G) \). The second smallest eigenvalue \( \mu(G) \) of \( L(G) \) is the algebraic connectivity of \( G \). The reason for this name is the well-known fact that \( \mu(G) > 0 \) if and only if \( G \) is connected. It is also known that

\[
\mu(G) = \min_{x \neq 0, x \perp \mathbf{1}} \frac{xL(G)x^T}{xx^T}. \tag{1}
\]

An eigenvector corresponding to \( \mu(G) \) is known as a Fiedler vector of \( G \). In passing, we note that if \( x = (x_1, \ldots, x_n) \), then

\[
xL(G)x^T = \sum_{ij \in E(G)} (x_i - x_j)^2, \tag{2}
\]

where \( E(G) \) is the edge set of \( G \). Recall that if \( x \) is an eigenvector corresponding to \( \mu \), then for any vertex \( i \) with degree \( d_i \),

\[
\mu x_i = d_i x_i - \sum_{j:ij \in E(G)} x_j. \tag{3}
\]

We refer to (3) as the eigenequation.

By (1), \( \mu(G) \) is determined by the vectors orthogonal to \( \mathbf{1} \). The other vectors, however, can also be used to obtain potentially good upper bounds for \( \mu(G) \). This is formulated in the following key lemma.

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\(^1\)We treat vectors as “row vectors.”
Lemma 2.1. Let $G$ be a graph of order $n$, $\mathbf{x}$ an arbitrary vector of length $n$ which is not a multiple of $\mathbf{1}$ and $\delta := \mathbf{x}^\top$. Then

$$\mu(G) \leq \frac{\mathbf{x}L(G)x^\top}{\|\mathbf{x}\|^2 - \frac{\delta^2}{n}}.$$ 

Proof. Let $\mathbf{y} = \mathbf{x} - \frac{\delta}{n} \mathbf{1}$. Then $\mathbf{y} \perp \mathbf{1}$ and so $\mu(G) \leq \frac{\mathbf{y}L(G)y^\top}{\|\mathbf{y}\|^2}$. As $L(G)\mathbf{1}^\top = 0^\top$, we have $\mathbf{y}L(G)y^\top = \mathbf{x}L(G)x^\top$. Furthermore,

$$\|\mathbf{y}\|^2 = \|\mathbf{x}\|^2 - \frac{2\delta}{n} \mathbf{x}^\top + \frac{\delta^2}{n} = \|\mathbf{x}\|^2 - \frac{\delta^2}{n},$$

and thus the result follows. □

Let $\Pi = \{C_1, \ldots, C_p\}$ be a partition of the vertex set $V(G)$. Then $\Pi$ is called an equitable partition of $G$ if for all $i, j$ (possibly $i = j$), the number of neighbors of any vertex of $C_i$ in $C_j$ only depends on $i$ and $j$.

Definition 2.2. Let $D$ and $D'$ be two graphs. We say that $D'$ fits $D$ if

(i) $D$ has an equitable partition $\Pi = \{C_1, \ldots, C_p\}$ and $V(D')$ has a partition $\Pi' = \{C_1', \ldots, C_p'\}$ (not necessarily equitable);

(ii) $|C_i| \leq |C_i'|$ for $i = 1, \ldots, p$;

(iii) for $1 \leq i < j \leq p$, the number of edges between $C_i$ and $C_j$ in $D$ is the same as the number of edges between $C_i'$ and $C_j'$ in $D'$.

In Table 1, we illustrate the blocks from $D_1, \ldots, D_4$ which fit each of $D_0, \ldots, D_3$. For instance, the first row shows that each of $D_1, \ldots, D_4$ fit $D_0$. The corresponding partitions $\Pi$ and $\Pi'$ are specified by dashed lines.

The next lemma captures the effect of replacing a block $D$ by another block $D'$ (which fits $D$) on the algebraic connectivity.

Lemma 2.3. Let $G$ be a graph and $D$ be an end block of $G$ attaching to the rest of the graph through a cut vertex $v$. Let $D$ have an equitable partition $\Pi$ such that the components of a Fiedler vector $\mathbf{x}$ of $G$ on each cell of $\Pi$ are equal. If we replace $D$ by another block $D'$ which fits $D$ and $v \in C_p \cap C_p'$, then for the resulting graph $G'$, we have $\mu(G') \leq \mu(G)$.

Proof. Let $\Pi$ and $\Pi'$ be as in Definition 2.2 and $d_{ij}$ be the number of edges between $C_i$ and $C_j$ for $1 \leq i < j \leq p$ which is the same for $C_i'$ and $C_j'$. Let $\mathbf{x} = (x_1, \ldots, x_n)$. By the assumption, the components of $\mathbf{x}$ on each $C_i$ are equal. Suppose that $a_1, \ldots, a_p$ are the values taken by $\mathbf{x}$ on these components. It turns out that

$$\sum_{ij \in E(D)} (x_i - x_j)^2 = \sum_{1 \leq i < j \leq p} d_{ij} (a_i - a_j)^2.$$
Let $H = G \setminus (D - v)$. Now, we define a vector $x'$ (with length $n'$) on $G'$ as follows: for $i = 1, \ldots, p$, on each $C'_i$, all the components of $x'$ are equal to $a_i$, and on $H$, $x'$ is the same as $x$. Although $v$ belongs to both $D'$ and $H$, the component of $x'$ on $v$ is well defined and equals to $a_p$ (which is guaranteed by the assumption that $v$ belongs to both $C_p$ and $C'_p$). It follows that

$$
\sum_{ij \in E(D')} (x'_i - x'_j)^2 = \sum_{1 \leq i < j \leq p} d_{ij}(a_i - a_j)^2,
$$

which in turn implies that

$$
x'L(G')x'^T = \sum_{ij \in E(G')} (x'_i - x'_j)^2 = \sum_{ij \in E(G)} (x_i - x_j)^2 = xL(G)x^T.
$$

Let $c_i = |C_i|$ and $c'_i = |C'_i|$. By the property (ii) of Definition 2.2, $c'_i \geq c_i$. Since $x1^T = 0$, it is seen that $\delta = x1^T = \sum_{i=1}^p (c'_i - c_i)a_i$. Moreover,

$$
\|x'\|^2 - \|x\|^2 = \sum_{i=1}^p (c'_i - c_i)a_i^2,
$$

and so

$$
\|x'\|^2 - \frac{\delta^2}{n'} - \|x\|^2 = \sum_{i=1}^p (c'_i - c_i)a_i^2 - \frac{1}{n'} \left( \sum_{i=1}^p (c'_i - c_i)a_i \right)^2.
$$
The right-hand side is nonnegative because by the Cauchy–Schwarz inequality,
\[
\left( \sum_{i=1}^{p} (c'_i - c_i) a_i \right)^2 \leq \left( \sum_{i=1}^{p} (c'_i - c_i) \right) \left( \sum_{i=1}^{p} (c'_i - c_i) a_i^2 \right) \leq n' \sum_{i=1}^{p} (c'_i - c_i) a_i^2.
\]

It follows that \( \|x\|^2 - \frac{e^2}{n'} \geq \|x\|^2 \). Therefore, by Lemma 2.1,
\[
\mu(G') \leq \frac{x' L(G') x^T}{\|x\|^2 - \frac{e^2}{n'}} \leq \frac{x L(G) x^T}{\|x\|^2} = \mu(G).
\]

\[\square\]

**Remark 2.4.** In [1], it was proved that minimal quartic graphs belong to a family of graphs with path-like structure whose blocks, among others, include \( M_0 \) as middle blocks and \( D_0, ..., D_4 \) as end blocks (see Theorem 3.1 for the precise description of such graphs). On the basis of a result of [8], it was also shown that any graph of this family has an equitable partition \( \Pi \) (whose cells, among others, include the pairs of vertices drawn vertically above each other) and a Fiedler vector \( x \) such that (i) the components of \( x \) on each cell of \( \Pi \) are equal; (ii) the components of \( x \) on the cells of \( \Pi \) form a strictly decreasing sequence changing sign once.

We now present the main result of this section.

**Theorem 2.5.** Let \( G \) be a quartic graph of order \( n \) with a path-like structure whose middle blocks are all \( M_0 \) and end blocks are from \( D_0, ..., D_4 \). Then \( \mu(G) = (1 + o(1)) \frac{4\pi^2}{n^2} \).

**Proof.** We denote the quartic graph with \( m \) middle blocks \( M_0 \) and end blocks \( D_i \) and \( D_j \) by \( \mathcal{H}_{i,j} \), for \( 0 \leq i, j \leq 4 \). The order of \( \mathcal{H}_{i,j} \) is between \( 5m + 11 \) (the order of \( \mathcal{H}_{0,0} \)) and \( 5m + 19 \) (the order of \( \mathcal{H}_{4,4} \)). So it is enough to prove that \( \mu(\mathcal{H}_{i,j}) = (1 + o(1)) \frac{4\pi^2}{25m^2} \).

We first consider the graph \( \mathcal{H}_{0,0} \). Define the vector \( x = (x_1, ..., x_{m+1}) \) with
\[
x_i = \cos\left( \frac{(2i - 1)\pi}{2m + 2} \right), \quad i = 1, ..., m + 1.
\]

Note that \( x \) is a skew-symmetric vector, that is, \( x_i = -x_{m+2-i} \) for \( 1 \leq i \leq \lfloor (m + 1)/2 \rfloor \).

We extend \( x \) to define a vector \( \bar{x} \) on \( \mathcal{H}_{0,0} \) whose components are as given in Figure 5.
As \( \mathbf{x} \) is skew-symmetric, \( \mathbf{x}^\top \) is also skew-symmetric. It follows that \( \mathbf{x} \perp \mathbf{1} \). Therefore, by (2) we have

\[
\frac{\mathbf{x}^\top (\mathcal{H}_{0,0}) \mathbf{x}}{\mathbf{x}^\top \mathbf{x}} = \frac{20}{25} \sum_{i=1}^{m+1} (x_i - x_{i+1})^2 \sum_{i=1}^{m} x_i^2 + \frac{2}{25} \sum_{i=1}^{m} (3x_i + 2x_{i+1})^2 + \frac{2}{25} \sum_{i=1}^{m} (2x_i + 3x_{i+1})^2 + 10x_i^2
\]

\[
= \frac{20}{25} \sum_{i=1}^{m+1} (x_i - x_{i+1})^2 \sum_{i=1}^{m} (13x_i^2 + 24x_i x_{i+1} + 13x_{i+1}^2) + 10x_i^2
\]

\[
\leq \frac{20}{25} \sum_{i=1}^{m} (x_i - x_{i+1})^2 \sum_{i=1}^{m} x_i^2 + \frac{48}{25} \sum_{i=1}^{m} x_i x_{i+1}
\]

\[
= \frac{80 \sin^2\left(\frac{\pi}{2m+2}\right) \sum_{i=1}^{m} \sin^2\left(\frac{\pi i}{m+1}\right)}{77 \sum_{i=1}^{m} \cos^2\left(\frac{(2i-1)\pi}{2m+2}\right) + 24 \left( m \cos\left(\frac{\pi}{m+1}\right) + \sum_{i=1}^{m} \cos\left(\frac{2\pi i}{m+1}\right) \right)}
\]

\[
= \frac{40(m+1)\sin^2\left(\frac{\pi}{2m+2}\right)}{77\left(\frac{m+1}{2} - \cos^2\left(\frac{\pi}{2m+2}\right)\right) + 24 \left( m \cos\left(\frac{\pi}{m+1}\right) - 1 \right)}
\]

\[
= \frac{m+1}{2} - \frac{\cos\left(\frac{2\pi}{2m+2}\right)}{25m^2}.
\]

The last equality is obtained using Taylor’s series for sine and cosine. Note that (4) is obtained using the identities

\[
\cos \alpha - \cos \beta = 2 \sin \frac{\alpha + \beta}{2} \sin \frac{\beta - \alpha}{2},
\]

\[
\cos \alpha \cos \beta = \frac{1}{2} (\cos(\alpha - \beta) + \cos(\alpha + \beta)),
\]

and (5) is deduced from the identities

\[
\sum_{i=1}^{m} \sin^2\left(\frac{\pi i}{m+1}\right) = \frac{m+1}{2},
\]

\[
\sum_{i=1}^{m} \cos\left(\frac{2\pi i}{m+1}\right) = -1,
\]

\[
\sum_{i=1}^{m} \cos^2\left(\frac{(2i-1)\pi}{2m+2}\right) = \frac{m+1}{2} - \cos^2\left(\frac{\pi}{2m+2}\right).
\]

Therefore, we conclude that
\[
\mu(\mathcal{H}_{0,0}) \leq (1 + o(1)) \frac{4\pi^2}{25m^2}.
\]  

(6)

We now prove that \((1 + o(1)) \frac{4\pi^2}{25m^2}\) is a lower bound for \(\mu(\mathcal{H}_{4,4})\). Let \(y = (y_1, ..., y_n)\) be a Fiedler vector of \(\mathcal{H}_{4,4}\). The graph \(\mathcal{H}_{4,4}\) has \(m + 1\) cut vertices and \(m + 2\) blocks, say \(B_1, ..., B_{m+2}\). Consider the components of \(y\) on the cut vertices and on the end blocks of \(\mathcal{H}_{4,4}\) which give rise to a vector \(z\) consisting of \(m + 11\) components as depicted in Figure 6. Note that \(y\) is skew-symmetric. To verify this, observe that by the symmetry of \(\mathcal{H}_{4,4}\), \(y' = (y_n, y_{n-1}, ..., y_1)\) is also an eigenvector for \(\mu(\mathcal{H}_{4,4})\). It follows that \(y - y'\) itself is a skew-symmetric eigenvector for \(\mu(\mathcal{H}_{4,4})\) (note that from Remark 2.4, it is seen that \(y - y' \neq 0\), so that we may replace \(y - y'\) for \(y\). Now, from Remark 2.4 it follows that \(z = (z_1, z_2, ..., z_{m+11}) \neq 0\). As \(y\) is skew-symmetric, it follows that \(z\) is also skew-symmetric and thus \(z \perp 1\). Let \(B_r\) be one of the middle blocks of \(\mathcal{H}_{4,4}\) and the components of \(y\) on the left vertex and the right vertex of \(B_r\) be \(z_k\) and \(z_{k+1}\), respectively. Let \(s\) and \(t\) be the components of \(y\) on the two middle vertices of \(B_r\) (which are equal by Remark 2.4) as shown in Figure 7.

We have

\[
\sum_{ij \in E(B_r)} (y_i - y_j)^2 = 2(z_k - s)^2 + 4(s - t)^2 + 2(t - z_{k+1})^2.
\]

The right-hand side, considered as a function of \(s\) and \(t\), is minimized at \(s = \frac{1}{5} (3z_k + 2z_{k+1})\) and \(t = \frac{1}{5} (2z_k + 3z_{k+1})\). This implies that

\[
\sum_{ij \in E(B_r)} (y_i - y_j)^2 \geq \frac{4}{5} (z_k - z_{k+1})^2.
\]

It follows that

\[
\sum_{ij \in E(\mathcal{H}_{4,4})} (y_i - y_j)^2 = \sum_{ij \in E(B_1)} (y_i - y_j)^2 + \sum_{r=2}^{m+1} \sum_{ij \in E(B_r)} (y_i - y_j)^2 + \sum_{ij \in E(B_{m+2})} (y_i - y_j)^2 \\
\geq \frac{4}{5} \sum_{k=1}^{m+10} (z_k - z_{k+1})^2.
\]

(7)

**Figure 6** Graph \(\mathcal{H}_{4,4}\) and components of \(z\)

**Figure 7** Middle \(B_r\) and components of \(y\) on its vertices
Since $y$ is skew-symmetric and its components are decreasing (by Remark 2.4), for the middle block $B_r$ we have $z_k^2 \geq s^2 \geq t^2$ if $2 \leq r \leq (m + 3)/2$, and $s^2 \leq t^2 \leq z_{k+1}^2$ if $(m + 3)/2 \leq r \leq m$. Furthermore, $\sum_{i=1}^{9}y_i^2 \leq 2\sum_{i=1}^{5}z_i^2$ and $\sum_{i=n-8}^{n}y_i^2 \leq 2\sum_{i=m+7}^{m+11}z_i^2$. It turns out that
\[ \sum_{i=1}^{n}y_i^2 \leq 5 \sum_{i=1}^{m+11}z_i^2. \] (8)

Now, from (7) and (8), we infer that
\[ \mu(H_{4,4}) = \frac{yL(H_{4,4})y^T}{yy^T} \geq \frac{4\sum_{i=1}^{m+10}(z_i - z_{i+1})^2}{25\sum_{i=1}^{m+11}z_i^2}. \] (9)

Note that the right-hand side of (9) is the same as $\frac{4}{25} \frac{zL(P_{m+11})z^T}{zz^T}$, where $P_{m+11}$ is the path of order $m + 11$. Thus, by the fact that $\mu(P_h) = 2(1 - \cos \frac{\pi}{h})$ (see [8]), it follows that
\[ \frac{\sum_{i=1}^{m+10}(z_i - z_{i+1})^2}{\sum_{i=1}^{m+11}z_i^2} \geq \mu(P_{m+11}) = (1 + o(1)) \frac{\pi^2}{m^2}. \]

Therefore, by (9)
\[ \mu(H_{4,4}) \geq \frac{4}{25}\mu(P_{m+11}) = (1 + o(1)) \frac{4\pi^2}{25m^2}. \] (10)

From Table 1 it is evident that all $D_i$’s fit $D_0$. Also, by Remark 2.4, they fulfill the conditions of Lemma 2.3. So by applying twice Lemma 2.3, we obtain
\[ \mu(H_{i,j}) \leq \mu(H_{0,j}) \leq \mu(H_{0,0}). \]

Also $D_4$ fits $D_3$ and $D_3$ fits each of $D_0$, $D_1$, and $D_2$. Therefore, again by Lemma 2.3,
\[ \mu(H_{i,j}) \geq \mu(H_{4,i}) \geq \mu(H_{4,4}). \]

The result now follows from (6) and (10).

\[ \square \]

3 | STRUCTURE OF MINIMAL QUARTIC GRAPHS

Motivated by the Aldous–Fill conjecture and also as an analogue to Babai’s conjecture on minimal cubic graphs, we consider the problem of determining the structure of minimal quartic graphs. In [1], it was proved that minimal quartic graphs have a path-like structure with specified blocks. We start this section by quoting this result.

The possible blocks of minimal quartic graphs are of two types: “short” and “long.” By short blocks we mean the blocks $M_0$, $D_0$, $D_1$, $D_2$, $D_3$ and those given in Figure 8. The long blocks, roughly speaking, are constructed by putting some short blocks together with the general structure given in Figure 9. More precisely, the building “bricks” of long blocks are the graphs
"M" obtained by removing the right degree 2 vertex of the corresponding short blocks, as well as the graphs \(M_1', M_2'\), obtained by removing both degree 2 vertices of \(M_0, M_1\) for any of these graphs, say \(B\), we denote its mirror image by \(\hat{B}\).

A long block is constructed from some \(s \geq 2\) bricks \(B_1, ..., B_s\), where each \(B_i\) is joined by two edges to \(B_{i+1}\) (as shown in Figure 9) and \(B_2, ..., B_{s-1} \in \{M_0', M_1'\}\). There are three types of long blocks:

(i) **long end block**: if \(B_1 \in \{D_0', D_3'\}\) and \(B_s \in \{\tilde{M}_0', \tilde{M}_1', \tilde{M}_2'\}\);
(ii) **long middle block**: if \(B_1 \in \{M_0', M_1', M_2'\}\) and \(B_s \in \{\tilde{M}_0', \tilde{M}_1', \tilde{M}_2'\}\);
(iii) **long complete block**: if \(B_1 \in \{D_0', D_3'\}\) and \(B_s \in \{\tilde{D}_0', \tilde{D}_3'\}\).

In passing, we remark that in a quartic graph, any cut vertex belongs to exactly two blocks and moreover has degree 2 in each of them. Therefore, in the quartic graphs having a path-like structure, the middle and end blocks have exactly two and one vertices of degree 2, respectively.

**Theorem 3.1** (Abdi, Ghorbani, and Imrich [1]). Let \(G\) be a graph with the minimum spectral gap in the family of connected quartic graphs on \(n \geq 11\) vertices. If \(G\) is a block, then it is a long complete block. If \(G\) itself is not a block, then it has a path-like structure in which each left end block is either one of \(D_0, ..., D_3\) or a long end block, and each middle block is either one of \(M_0, M_1, M_2, \tilde{M}_2, M_3\) or a long middle block. Each right end block is the mirror image of some left end block.

As the main result of this section, we improve considerably Theorem 3.1 by giving a much more precise description of the minimal quartic graphs.

**Theorem 3.2.** Let \(G\) be a graph with minimum algebraic connectivity among connected quartic graphs on \(n \geq 11\) vertices. Then \(G\) has a path-like structure, any middle block of \(G\) is \(M_0\), the left end block of \(G\) is one of \(D_0, ..., D_4\), and the right end block is the mirror image of one of these five blocks.

Theorems 2.5 and 3.2 imply the Aldous–Fill conjecture for \(k = 4\):

**Corollary 3.3.** The minimum algebraic connectivity of connected quartic graphs of order \(n\) is \((1 + o(1)) \frac{4\pi^2}{n^2}\).
The rest of this section is devoted to the proof of Theorem 3.2. More precisely, the assertion on the end blocks will be established in Theorem 3.8 and on the middle block in Theorem 3.13.

In the remainder of this section, we suppose that \( n_1 \geq 0 \), \( \Gamma \) is a minimal quartic graph of order \( n \) and \( \mu(\Gamma) \). By Theorem 3.1, the equitable partition of \( \Gamma \) mentioned in Remark 2.4 has cells of size 1 or 2 consisting of the vertices drawn vertically above each other (with the exceptions for the first three vertices of \( D_0' \) and the first four vertices of \( D_1, D_3' \) which make a cell too). So by Remark 2.4, \( \Gamma \) has a decreasing unit Fiedler vector \( \mathbf{x} \) which is constant on the cells.

In our arguments in this section, we will need upper bounds on \( \mu \).

**Lemma 3.4.** On the basis of the value of \( n \), we have the upper bounds given in Table 2 for \( \mu \).

**Proof.** Note that \( \mu \leq \mu(\mathcal{G}_n) \). So it suffices to find the upper bound for \( \mu(\mathcal{G}_n) \). Let \( n - 11 = 5m + r \) for some \( 0 \leq r \leq 4 \), and \( \mathcal{H}_{0,0} \) (with \( m \) middle blocks) be as in the proof of Theorem 2.5. Therefore, \( \mu(\mathcal{G}_n) \leq \mu(\mathcal{H}_{0,0}) \). Let us denote the expression given in (5) by \( f(m) \) for which \( \mu(\mathcal{H}_{0,0}) \leq f(m) \). By taking the derivative of \( f \) with respect to \( m \) it is seen that \( f \) is decreasing for \( m \geq 1 \). So for \( m \geq 6 \), we have \( f(m) \leq f(6) < 0.046 \). This implies that for \( n \geq 41 \), \( \mu(\mathcal{G}_n) < 0.046 \). For \( 11 \leq n \leq 40 \), we compute \( \mu(\mathcal{G}_n) \) directly by a computer. It turns out that \( \mu(\mathcal{G}_n) \) is strictly decreasing for \( 11 \leq n \leq 40 \). The rounded-up values of \( \mu(\mathcal{G}_n) \) for \( n = 11, 13, 18, 21, 26 \) are exactly those given in the second row of Table 2. Thus the result follows. \( \square \)

3.1 | End blocks

By Theorem 3.1, the left end block of \( \Gamma \) is either a short block, that is, it is one of \( D_0, ..., D_3 \), or it is a long block starting with \( D_0' \) or \( D_3' \). In this subsection, we show that the end blocks of \( \Gamma \) should be short blocks or one exceptional long block that is \( D_4 \).

Our first lemma concerns long blocks starting with \( D_3' \).

**Lemma 3.5.** The graph \( \Gamma \) does not contain a long end block starting with \( D_3' \).

**Proof.** For a contradiction suppose that \( \Gamma \) contains \( D_3' \) with the components of \( \mathbf{x} \) as depicted in Figure 10.

**TABLE 2** Upper bounds for \( \mu \) based on the values of \( n \)

| \( n \geq \) | 11 | 13 | 18 | 21 | 26 |
|---|---|---|---|---|---|
| \( \mu < \) | 0.355 | 0.268 | 0.129 | 0.091 | 0.059 |

**FIGURE 10** \( D_3' \) in a long block (the last two vertices have different neighbors on their right)
We may assume that $x_1 > 0$ (since otherwise we consider $-x$). By using the eigenequation (3), we have

$$x_3 = (\mu^2 - 5\mu + 2) \frac{x_1}{2}.$$ 

As $\mu < 0.355$ (by Lemma 3.4), we have $\mu^2 - 5\mu + 2 > 0$ which implies that $x_3 > 0$.

Now, we replace $D'_3$ by $D_2$ to obtain $\Gamma'$. We define a vector $x'$ on $\Gamma'$ such that its components on the new end block $D_2$ are as given in Figure 11, and on the rest of the vertices agree with $x$. We observe that $x'T - xT = x_2 - x_3$ and since $x \perp 1$, $\delta = x'T = x_2 - x_3$. Also $x'L(\Gamma')x'T = xL(\Gamma)T - 2(x_2 - x_3)^2 = \mu - 2(x_2 - x_3)^2$, and

$$\|x'|^2 - \frac{\delta^2}{n} = \|x|^2 + x_2^2 - x_3^2 - \frac{\delta^2}{n}$$
$$= 1 + (x_2 - x_3)\left(x_2 + x_3 - \frac{x_2 - x_3}{n}\right)$$
$$\geq 1 + (x_2 - x_3)^2 \left(1 - \frac{1}{n}\right),$$

where the last inequality follows from the fact that $x_3 > 0$. Therefore, we have $x'L(\Gamma')x'T < \mu$ and $\|x'|^2 - \frac{\delta^2}{n} > 1$. So by Lemma 2.1, $\mu(\Gamma') \leq \frac{x'L(\Gamma')x'T}{\|x'|^2 - \frac{\delta^2}{n}} < \mu$, a contradiction. \hfill \Box

We now deal with the long blocks starting with $D'_0$ in the next two lemmas. Suppose that $\Gamma$ contains a long end block starting with $D'_0$. Then, by Theorem 3.1, it contains either $D'_0 + M''_0$, $D'_0 + M''_1$, or $D'_0 + M''_2$ (these are the graphs obtained by joining the two degree 3 vertices of $D'_0$ by two parallel edges to $M''_0$, $M''_1$, or $M''_2$, respectively; see Figure 12). If $\Gamma$ contains $D'_0 + M''_0$, it is either $D_4$ (as desired) or it contains the subgraph of Figure 12 that we will show is not possible in Lemma 3.6. If $\Gamma$ contains $D'_0 + M''_1$ or $D'_0 + M''_2$, then it must contain the subgraph of Figure 13 that we shall show is not possible in Lemma 3.7.

**Lemma 3.6.** The graph $\Gamma$ does not contain the graph $H$ of Figure 12 as a subgraph.

![Figure 11](image1)

**Figure 11** Block $D_2$ in $\Gamma'$ and components of $x'$

![Figure 12](image2)

**Figure 12** $D'_0 + M''_0$ in a long block (the last two vertices have different neighbors on their right)
Proof. If \( 11 \leq n \leq 17 \), the only graphs of Theorem 3.1 which contain \( H \) as a subgraph are the top two graphs of Figure 14 for which the two graphs on the bottom, respectively, have smaller algebraic connectives. This means that we are done for \( n \leq 17 \). So we assume that \( n \geq 18 \), and by Lemma 3.4, \( \mu < 0.129 \). Now, for a contradiction assume that \( \Gamma \) contains \( H \). By using the eigenequation (3) on the first seven vertices of \( H \), we obtain

\[
x_2 = \left( -\frac{\mu}{2} + 1 \right) x_1, \quad x_3 = \left( \frac{\mu^2}{2} - 3\mu + 1 \right) x_1, \quad x_4 = \left( -\frac{\mu^3 + 9\mu^2 - 19\mu}{4} + 1 \right) x_1.
\] (11)

We replace \( H \) by \( D_3 \) to obtain \( \Gamma' \). We define a vector \( \mathbf{x}' \) on \( V(\Gamma') \) such that its components on the new end block \( D_3 \) are as given in Figure 15 (with \( z_4, z_2, z_3 \) to be specified later), and on the rest of the vertices of \( \Gamma' \) agree with \( \mathbf{x} \).

Although \( \mu \) need not to be an eigenvalue of \( \Gamma' \), we may choose \( z_4, z_2, z_3 \) to satisfy the following equations which resemble the eigenequation (3) for \( \mu \):

\[
(1 - \mu)z_4 - z_2 = 0, \\
(4 - \mu)z_2 - 2z_4 - 2z_3 = 0, \\
(3 - \mu)z_3 - 2z_2 - x_4 = 0.
\]

By plugging in the value of \( x_4 \) from (11) and solving the equations in \( x_3 \) and \( \mu \), we obtain that
\[ z_4 = \omega x_1, \quad z_2 = \omega (1 - \mu) x_1, \quad z_3 = \frac{\omega}{2} (\mu^2 - 5\mu + 2) x_1, \quad (12) \]

where

\[ \omega = \frac{\mu^3 - 9\mu^2 + 19\mu - 4}{2(\mu^3 - 8\mu^2 + 13\mu - 2)}. \]

Taking into account that \( \mathbf{x} \perp \mathbf{1} \), we see that

\[ \delta = \mathbf{x}' \mathbf{1}^T = 4z_1 + 2z_2 + 2z_3 - 3x_1 - 2x_2 - 2x_3 - x_4. \]

Also, as \( \|\mathbf{x}\| = 1 \), we obtain that

\[ \|\mathbf{x}'\|^2 = 1 + 4z_1^2 + 2z_2^2 + 2z_3^2 - 3x_1^2 - 2x_2^2 - 2x_3^2 - x_4^2. \]

Further, we have

\[ \mathbf{x}' \mathbf{L}(\Gamma') \mathbf{x}'^T - \mu = 4(z_1 - z_2)^2 + 4(z_2 - z_3)^2 + 2(z_3 - x_4)^2 - 6(x_1 - x_2)^2 - 2(x_2 - x_3)^2 - 4(x_3 - x_4)^2. \]

As \( \|\mathbf{x}'\|^2 - \frac{\delta^2}{n} > 0 \), we have \( \frac{\mathbf{x}' \mathbf{L}(\Gamma') \mathbf{x}'^T - \mu \|\mathbf{x}'\|^2 + \mu \frac{\delta^2}{n}}{\|\mathbf{x}'\|^2 - \frac{\delta^2}{n}} < \mu \) if and only if \( \mathbf{x}' \mathbf{L}(\Gamma') \mathbf{x}'^T - \mu \|\mathbf{x}'\|^2 + \mu \frac{\delta^2}{n} < 0 \).

We have

\[ \mathbf{x}' \mathbf{L}(\Gamma') \mathbf{x}'^T - \mu \|\mathbf{x}'\|^2 + \mu \frac{\delta^2}{n} = 12x_1x_2 + 4(x_2x_3 + 2x_3x_4) - 4(z_1x_4 + 2z_2z_1 + 2z_2z_3)
\]
\[ + (\mu - 2)(3x_1^2 + x_2^2) + (2\mu - 8)(x_2^2 - z_2^2)
\]
\[ + (2\mu - 6)(x_3^2 - z_3^2) - (4\mu - 4)z_4^2 + \mu \frac{\delta^2}{n}. \]

By substituting the values of \( x_2, x_3, x_4, z_1, z_2, z_3, \delta \) in terms of \( x_1 \) and \( \mu \), we can write each term in the above as follows:

\[ 12x_1x_2 = 6(2 - \mu)x_1^2, \]
\[ 4(x_2x_3 + 2x_3x_4) = (3 - \mu)(\mu^2 - 6\mu + 2)x_1^2, \]
\[ -4(z_1x_4 + 2z_2z_1 + 2z_2z_3) = (\mu^5 - 13\mu^4 + 59\mu^3 - 107\mu^2 + 72\mu - 20)\omega^2x_1^2, \]
\[ 3x_1^2 + x_2^2 = (\mu^6 - 18\mu^5 + 119\mu^4 - 350\mu^3 + 433\mu^2 - 152\mu + 64)\frac{x_1^2}{16}, \]
\[ x_2^2 - z_2^2 = \mu(2\mu^5 - 30\mu^4 + 157\mu^3 - 336\mu^2 + 263\mu - 40)\frac{x_1^2}{p(\mu)}, \]
\[ x_3^2 - z_3^2 = (\mu - 6)\mu(\mu^3 - 8\mu^2 + 12\mu - 3)(3\mu^5 - 42\mu^4
\]
\[ + 192\mu^3 - 309\mu^2 + 134\mu - 16)\frac{x_1^2}{4p(\mu)}, \]
\[ z_4^2 = \omega^2x_1^2, \]
\[ \delta^2 = \mu^2(\mu - 5)^2(\mu - 6)^2\frac{(\mu^3 - 8\mu^2 + 14\mu - 5)^2}{16(\mu^3 - 8\mu^2 + 13\mu - 2)^2}x_1^2. \]
in which  

\[ p(t) = 4(t - 2)^2(t^2 - 6t + 1)^2. \]

Plugging in all these terms and simplifying, it follows that  
\[
\frac{x' L(\Gamma') x^T}{\|x'\|^2 - \frac{\mu^2}{n}} < \mu \text{ if and only if } 2
\]

\[
\frac{(p_1(\mu)n + p_2(\mu))p_3(\mu)x_1^2}{4np(\mu)} < 0, \quad (13)
\]

where  

\[
\begin{align*}
p_1(t) &= t^6 - 17t^5 + 104t^4 - 275t^3 + 297t^2 - 90t + 8, \\
p_2(t) &= t^6 - 19t^5 + 132t^4 - 399t^3 + 475t^2 - 150t, \\
p_3(t) &= (t - 6)(t - 5)t^2(t^3 - 8t^2 + 14t - 5).
\end{align*}
\]

The smallest root of \( t^3 - 8t^2 + 14t - 5 \) is about 0.481. As \( \mu < 0.129 \), we have that \( p_3(\mu) < 0 \). The smallest root of \( p_1(t) \) is about 0.171 and so \( p_1(\mu) > 0 \). Since \( n \geq 18 \) we also have  
\[ p_1(\mu)n + p_2(\mu) \geq 18p_1(\mu) + p_2(\mu) = p_4(\mu), \]

where  
\[ p_4(t) = 19t^6 - 325t^5 + 2004t^4 - 5349t^3 + 5821t^2 - 1770t + 144. \]

The smallest root of \( p_4(t) \) is about 0.132 which means that \( p_4(\mu) > 0 \). Therefore, \((p_1(\mu)n + p_2(\mu))p_3(\mu) < 0\), and so (13) is established. Hence, by Lemma 2.1, \( \mu(\Gamma') < \mu \) which is a contradiction.  

\[ \square \]

**Lemma 3.7.** The graph \( \Gamma \) does not contain the graph \( H \) of Figure 13 as a subgraph.

**Proof.** For a contradiction suppose that \( \Gamma \) contains \( H \). So \( n \geq 13 \) (otherwise \( \Gamma \) cannot contain \( H \)). We may assume that \( x_1 > 0 \) (since otherwise we consider \(-x\)). By using the eigenequation (3), we have  
\[ x_3 = (\mu^2 - 6\mu + 2) \frac{x_1}{2}. \]

As \( \mu < 0.268 \) (by Lemma 3.4), we have \( \mu^2 - 6\mu + 2 > 0 \) which implies that \( x_3 > 0 \).

\[ \frac{x' L(\Gamma') x^T}{\|x'\|^2 - \frac{\mu^2}{n}} < \mu \text{ if and only if } 2 \]

\[ \frac{(p_1(\mu)n + p_2(\mu))p_3(\mu)x_1^2}{4np(\mu)} < 0, \quad (13) \]

where  

\[
\begin{align*}
p_1(t) &= t^6 - 17t^5 + 104t^4 - 275t^3 + 297t^2 - 90t + 8, \\
p_2(t) &= t^6 - 19t^5 + 132t^4 - 399t^3 + 475t^2 - 150t, \\
p_3(t) &= (t - 6)(t - 5)t^2(t^3 - 8t^2 + 14t - 5).
\end{align*}
\]

The smallest root of \( t^3 - 8t^2 + 14t - 5 \) is about 0.481. As \( \mu < 0.129 \), we have that \( p_3(\mu) < 0 \). The smallest root of \( p_1(t) \) is about 0.171 and so \( p_1(\mu) > 0 \). Since \( n \geq 18 \) we also have  
\[ p_1(\mu)n + p_2(\mu) \geq 18p_1(\mu) + p_2(\mu) = p_4(\mu), \]

where  
\[ p_4(t) = 19t^6 - 325t^5 + 2004t^4 - 5349t^3 + 5821t^2 - 1770t + 144. \]

The smallest root of \( p_4(t) \) is about 0.132 which means that \( p_4(\mu) > 0 \). Therefore, \((p_1(\mu)n + p_2(\mu))p_3(\mu) < 0\), and so (13) is established. Hence, by Lemma 2.1, \( \mu(\Gamma') < \mu \) which is a contradiction.  

\[ \square \]

**Lemma 3.7.** The graph \( \Gamma \) does not contain the graph \( H \) of Figure 13 as a subgraph.

**Proof.** For a contradiction suppose that \( \Gamma \) contains \( H \). So \( n \geq 13 \) (otherwise \( \Gamma \) cannot contain \( H \)). We may assume that \( x_1 > 0 \) (since otherwise we consider \(-x\)). By using the eigenequation (3), we have  
\[ x_3 = (\mu^2 - 6\mu + 2) \frac{x_1}{2}. \]

As \( \mu < 0.268 \) (by Lemma 3.4), we have \( \mu^2 - 6\mu + 2 > 0 \) which implies that \( x_3 > 0 \).

\[ \frac{x' L(\Gamma') x^T}{\|x'\|^2 - \frac{\mu^2}{n}} < \mu \text{ if and only if } 2 \]

\[ \frac{(p_1(\mu)n + p_2(\mu))p_3(\mu)x_1^2}{4np(\mu)} < 0, \quad (13) \]

where  

\[
\begin{align*}
p_1(t) &= t^6 - 17t^5 + 104t^4 - 275t^3 + 297t^2 - 90t + 8, \\
p_2(t) &= t^6 - 19t^5 + 132t^4 - 399t^3 + 475t^2 - 150t, \\
p_3(t) &= (t - 6)(t - 5)t^2(t^3 - 8t^2 + 14t - 5).
\end{align*}
\]

The smallest root of \( t^3 - 8t^2 + 14t - 5 \) is about 0.481. As \( \mu < 0.129 \), we have that \( p_3(\mu) < 0 \). The smallest root of \( p_1(t) \) is about 0.171 and so \( p_1(\mu) > 0 \). Since \( n \geq 18 \) we also have  
\[ p_1(\mu)n + p_2(\mu) \geq 18p_1(\mu) + p_2(\mu) = p_4(\mu), \]

where  
\[ p_4(t) = 19t^6 - 325t^5 + 2004t^4 - 5349t^3 + 5821t^2 - 1770t + 144. \]

The smallest root of \( p_4(t) \) is about 0.132 which means that \( p_4(\mu) > 0 \). Therefore, \((p_1(\mu)n + p_2(\mu))p_3(\mu) < 0\), and so (13) is established. Hence, by Lemma 2.1, \( \mu(\Gamma') < \mu \) which is a contradiction.  

\[ \square \]

**Lemma 3.7.** The graph \( \Gamma \) does not contain the graph \( H \) of Figure 13 as a subgraph.

**Proof.** For a contradiction suppose that \( \Gamma \) contains \( H \). So \( n \geq 13 \) (otherwise \( \Gamma \) cannot contain \( H \)). We may assume that \( x_1 > 0 \) (since otherwise we consider \(-x\)). By using the eigenequation (3), we have  
\[ x_3 = (\mu^2 - 6\mu + 2) \frac{x_1}{2}. \]

As \( \mu < 0.268 \) (by Lemma 3.4), we have \( \mu^2 - 6\mu + 2 > 0 \) which implies that \( x_3 > 0 \).
Now, we replace $H$ by $H'$ to obtain $\Gamma'$ and define a vector $x'$ on $V(\Gamma')$ such that its components on $H'$ are as in Figure 16, and on the rest of the vertices agree with $x$.

By the fact that $x$ is a unit Fiedler vector of $\Gamma$, we see that

$$
\delta = x' \mathbf{1}^T = x_1 - x_3, \quad \|x'\|^2 = 1 + x_1^2 - x_3^2,
$$

$$
x' L(\Gamma') x'^T = \mu - 2(x_1 - x_2)^2 - 2(x_3 - x_4)^2.
$$

The situation here is similar to the proof of Lemma 3.5. Therefore, we can deduce that $\mu(\Gamma') < \mu$ which is a contradiction. $$\square$$

All in all, we have proved the following theorem:

**Theorem 3.8.** The end blocks of $\Gamma$ belong to $D_0, ..., D_4$.

We remark that Theorem 3.8, in particular, implies that $\Gamma$ cannot be a long complete block.

### 3.2 Middle blocks

Our goal is to show that beside $M_0$, $\Gamma$ contains no other types of middle blocks. The following lemma provides a key tool for our arguments.

**Lemma 3.9.** Let $\rho$ be a unit Fiedler vector of a quartic graph $G$ of order $n$, and $H$ be a subgraph of $G$ such that $G - H$ has two connected components $G_1$ and $G_2$. Let $H'$ be a graph with the same number of vertices and edges as $H$. In $G$ we replace $H$ by $H'$ to obtain a new graph $G'$ such that $G'$ is also a quartic graph. Suppose that there is a vector $\rho'$ on $V(G')$ such that

$$
\ell := \sum_{ij \in E(G) \setminus E(H)} (\rho_i - \rho_j)^2 = \sum_{ij \in E(G') \setminus E(H')} (\rho'_i - \rho'_j)^2.
$$

Set

$$
h := \sum_{ij \in E(H)} (\rho_i - \rho_j)^2, \quad h' := \sum_{ij \in E(H')} (\rho'_i - \rho'_j)^2, \quad \epsilon := \|\rho'\|^2 - \frac{\delta^2}{n} - 1,
$$

where $\delta = \rho' \mathbf{1}^T$. If $h' - h - \epsilon \mu(G) < 0$, then $\mu(G') < \mu(G)$. 
Proof. We have $\mu(G) = \rho L(G) \rho^T = \ell + h$, and by Lemma 2.1,

$$
\mu(G') \leq \frac{\rho' L(G') \rho'^T}{\|\rho'\|^2} = \frac{\ell + h'}{1 + \epsilon}.
$$

It follows that if $h' - h - \epsilon \mu(G) < 0$, then $\mu(G') < \mu(G)$. \(\blacksquare\)

**Lemma 3.10.** The graph $\Gamma$ does not contain the graphs of Figure 17 as induced subgraphs with the given conditions on the components of a Fielder vector:

(i) $H_1$ such that $x_{r+3} \geq 0$,
(ii) $H_2$ such that $x_{r+4} \geq 0$,
(iii) $H_3$ such that $x_{r+3} \geq 0$.

Proof. (i) By contradiction, let $H_1$ be a subgraph of $\Gamma$. By the eigenequation (3) considered on $H_1$, it can be seen that $x_{r+1} = f(x_r, x_{r+3})$ and $x_{r+2} = g(x_r, x_{r+3})$, where

$$
f(x, y) = \frac{2(3x - \mu x + 2y)}{\mu^2 - 7\mu + 10},
$$

$$
g(x, y) = \frac{2(x - \mu y + 4y)}{\mu^2 - 7\mu + 10}.
$$

We replace $H_1$ by $H'_1$ to obtain $\Gamma'$ and define a vector $x'$ on $V(\Gamma')$ such that its components on $H'_1$ are as given in Figure 18 (with $z_{r+1}, z_{r+2}$ to be specified later) and on the rest of the vertices of $\Gamma'$ agree with $x$.

We set

$$
z_{r+1} = g(x_{r+3}, x_r), \quad z_{r+2} = f(x_{r+3}, x_r).
$$

We have $\delta = x' 1^T = 2z_{r+1} + z_{r+2} - x_{r+1} - 2x_{r+2}$. By substituting the values of $x_{r+1}, x_{r+2}, z_{r+1}, z_{r+2}$ in terms of $x_r, x_{r+3}, \mu$ we obtain that

**Figure 17** Some forbidden subgraphs for $\Gamma$

**Figure 18** Subgraph $H'_1$ in $\Gamma'$ and components of $x'$
Now, in the notation of Lemma 3.9, we have

\[
\begin{align*}
\delta &= \frac{2(x_r - x_{r+3})}{2 - \mu}, \\
\end{align*}
\]

Moreover, since \( \epsilon = \frac{\delta^2}{n} \), we have

\[
\begin{align*}
\epsilon &= \frac{\delta^2}{n} = 2z_{r+1}^2 + z_{r+2}^2 - x_{r+1}^2 - 2x_{r+2}^2 - \delta^2. \\
\end{align*}
\]

It follows that

\[
\begin{align*}
h' - h - \epsilon \mu &= 2(x_r + 2x_{r+1} - 4z_{r+1})x_r + 4\left(2z_{r+1}^2 - x_{r+1}^2\right) + 4(x_{r+1}x_{r+2} - z_{r+1}z_{r+2}) \\
&\quad - 6\left(x_{r+2}^2 - z_{r+1}^2\right) + 2x_{r+3}(4x_{r+2} - x_{r+3} - 2z_{r+2}) - \epsilon \mu \\
&= \frac{2\mu + 4x_r^2}{\mu - 2} + \frac{16(1 - \mu)(x_r^2 - x_{r+3}^2)}{(\mu - 2)^2(\mu - 5)} + \frac{16(x_r^2 - x_{r+3}^2)}{(\mu - 2)(\mu - 5)} \\
&\quad + \frac{24(\mu - 3)(x_r^2 - x_{r+3}^2)}{(\mu - 2)^2(\mu - 5)} - 2\mu + 4x_{r+3}^2 - \frac{4\mu(x_r^2 - x_{r+3}^2)}{(\mu - 2)^2} + \frac{\mu \delta^2}{n} \\
&= \frac{2\mu(x_r - x_{r+3})}{(\mu - 2)^2n}(x_r + x_{r+3}) + 2(x_r - x_{r+3}),
\end{align*}
\]

which is negative because \( x_r > x_{r+3} \geq 0, \ n \geq 11, \) and \( \mu < 0.355. \) Therefore, from Lemma 3.9 it follows that \( \mu(\Gamma') < \mu, \) a contradiction.

(ii) For a contradiction, assume that \( \Gamma \) contains \( H_2. \) By the eigenequation (3) considered on the vertices of \( H_2, \) it can be seen that \( x_{r+1} = f(x_r, x_{r+4}), \ x_{r+2} = g(x_r, x_{r+4}), \) and \( x_{r+3} = l(x_r, x_{r+4}), \) where

\[
\begin{align*}
f(x, y) &= \frac{2(-\mu^2x + 7\mu x + \mu y - 10x - 6y)}{\mu^3 - 11\mu^2 + 36\mu - 32}, \\
g(x, y) &= \frac{-\mu y + 2x + 6y}{\mu^2 - 7\mu + 8}, \\
l(x, y) &= \frac{2(-\mu^2y + 8\mu y - 2x - 14y)}{\mu^3 - 11\mu^2 + 36\mu - 32}.
\end{align*}
\]

We replace \( H_2 \) by \( H_2' \) to obtain \( \Gamma' \) and define a vector \( x' \) on \( V(\Gamma') \) such that its components on \( H_2' \) are as given in Figure 19 and on the rest of the vertices agree with \( x. \) We set

\[
z_{r+1} = l(x_{r+4}, x_r), \quad z_{r+2} = g(x_{r+4}, x_r), \quad z_{r+3} = f(x_{r+4}, x_r).
\]
We have \( \delta = \mathbf{x}' \mathbf{T}' = z_{r+1} + 2z_{r+2} + z_{r+3} - x_{r+1} - 2x_{r+2} - x_{r+3} \). By substituting the values of \( x_{r+1}, x_{r+2}, x_{r+3}, z_{r+1}, z_{r+2}, z_{r+3} \) in terms of \( x_r, x_{r+4}, \mu \), we obtain that
\[
\delta = \frac{2(6 - \mu)(x_r - x_{r+4})}{\mu^2 - 7\mu + 8}.
\]

Now, in the notations of Lemma 3.9, we have
\[
h = 2 \sum_{i=r}^{r+3} (x_i - x_{i+1})^2 + 2(x_{r+2} - x_{r+4})^2,
\]
\[
h' = 2(x_r - z_{r+1})^2 + 2(x_r - z_{r+2})^2 + 2(z_{r+1} - z_{r+2})^2 + 2(z_{r+2} - z_{r+3})^2
+ 2(z_{r+3} - x_{r+4})^2.
\]

Moreover,
\[
\epsilon = ||\mathbf{x}'||^2 - 1 - \frac{\delta^2}{n} = z_{r+1}^2 + 2z_{r+2}^2 + z_{r+3}^2 - x_{r+1}^2 - 2x_{r+2}^2 - x_{r+3}^2 - \frac{\delta^2}{n}.
\]

It follows that
\[
h' - h - \epsilon\mu = 2x_r(x_r + 2x_{r+1} - 2z_{r+1} - 2z_{r+2}) - 6\left(x_{r+2}^2 - z_{r+2}^2\right)
- 4\left(x_{r+1}^2 + x_{r+3}^2 - z_{r+1}^2 - z_{r+3}^2\right) + 2x_{r+4}(2x_{r+3} - x_{r+4} - 2z_{r+3})
+ 4x_{r+2}(x_{r+1} + x_{r+3} + x_{r+4}) - 4z_{r+2}(x_{r+1} + z_{r+3}) - \epsilon\mu
= \frac{2}{\omega}(\mu^2 - 5\mu - 8)x_r^2 + \frac{6}{\omega^2}(\mu^2 - 12\mu + 32)\left(x_r^2 - x_{r+4}^2\right)
- \frac{32}{\omega^2}(\mu - 4)\left(x_r^2 - x_{r+4}^2\right)
- \frac{2}{\omega}(\mu^2 - 7\mu + 4)x_{r+4} + 4x_r x_{r+4} - \frac{4}{\omega^2}(\mu - 6)x_{r+4} - 2x_r^2
(\mu - 3)
- \frac{8}{\omega^2}(\mu x_r - 6x_r - 2x_{r+4})(x_r + x_{r+4})\mu - 5x_r - 3x_{r+4})
- \frac{2}{\omega^2}(\mu^2 - 16\mu + 48)(x_r^2 - x_{r+4}^2) + \frac{\mu\delta^2}{n}
= \Phi \frac{2\mu(\mu - 6)(x_r - x_{r+4})}{\omega^2 n},
\]
where $\omega = \mu^2 - 7\mu + 8$ and $\Phi = (\mu^2 - 7\mu + 8)n(x_r + x_{r+4}) + (2\mu - 12)(x_r - x_{r+4})$. It is easy to check that $\Phi > 0$ since $x_r > x_{r+4} \geq 0$, $n \geq 11$, and $\mu < 0.355$. It follows that $h' - h - \varepsilon\mu < 0$ and so by Lemma 3.9, $\mu(\Gamma') < \mu$, a contradiction.

(iii) For a contradiction, assume that $\Gamma$ contains $H_3$. It should have at least 21 vertices to contain $H_3$. So $n \geq 21$, and by Lemma 3.4, $\mu < 0.091$. By the eigenequation (3) considered on the vertices of $H_3$, it can be seen that $x_{r+1} = f(x_r, x_{r+4}), x_{r+2} = g(x_r, x_{r+4})$, and $x_{r+3} = l(x_r, x_{r+4})$, where

$$
\begin{align*}
f(x, y) &= \frac{2(-\mu^2 x + 6\mu x - 5x - 2y)}{\mu^3 - 10\mu^2 + 27\mu - 14}, \\
g(x, y) &= \frac{2(\mu x + \mu y - 3x - 4y)}{\mu^3 - 10\mu^2 + 27\mu - 14}, \\
l(x, y) &= \frac{-\mu^2 y + 7\mu y - 4x - 10y}{\mu^3 - 10\mu^2 + 27\mu - 14}.
\end{align*}
$$

We have that $x_r > x_{r+4}$ (by Remark 2.4). We further claim that

$$3(x_r + x_{r+4}) > x_r - x_{r+4} > 0. \quad (14)$$

If $x_{r+4} \geq 0$, this trivially holds. If $x_{r+4} < 0$, it holds by the following argument. Note that $l(x_r, x_{r+4}) = x_{r+3} \geq 0$ which means $-\mu^2 x_{r+4} + 7\mu x_{r+4} - 4x_r - 10x_{r+4} \leq 0$. Therefore, $4x_r \geq (-\mu^2 + 7\mu - 10)x_{r+4} > -8x_{r+4}$, that is, $x_r > -2x_{r+4}$, from which (14) follows.

We now replace $H_3$ by $H'_3$ to obtain $\Gamma'$ and define a vector $x'$ on $V(\Gamma')$ such that its components on $H'_3$ are as given in Figure 20 and on the rest of the vertices agree with $x$.

We set $z_{r+1} = l(x_{r+4}, x_r), z_{r+2} = g(x_{r+4}, x_r), z_{r+3} = f(x_{r+4}, x_r)$.

We have $\delta = x'T' = 2z_{r+1} + 2z_{r+2} + z_{r+3} - x_{r+1} - 2x_{r+2} - 2x_{r+3}$. By substituting the values of $x_{r+1}, x_{r+2}, x_{r+3}, z_{r+1}, z_{r+2}, z_{r+3}$ in terms of $x_r, x_{r+4}, \mu$ we obtain that

$$\delta = \frac{2(\mu - 5)(x_k - x_{k+4})}{\mu^3 - 10\mu^2 + 27\mu - 14}.$$ 

Now, in the notations of Lemma 3.9, we have

$$
\begin{align*}
h &= 2 \sum_{i=r}^{r+3} (x_i - x_{i+1})^2 + 2(x_{r+2} - x_{r+3})^2, \\
h' &= 2(x_r - z_{r+1})^2 + 4(z_{r+1} - z_{r+2})^2 + 2(z_{r+2} - z_{r+3})^2 + 2(z_{r+3} - x_{r+4})^2.
\end{align*}
$$

![Figure 20](https://example.com/figure20.png) Subgraph $H'_3$ in $\Gamma'$ and components of $x'$
Moreover,

$$\epsilon = \|x\|^2 - 1 - \frac{\delta^2}{n} = 2z_{r+1}^2 + 2x_{r+2}^2 + 2x_{r+3}^2 - x_{r+1}^2 - 2x_{r+2}^2 - 2x_{r+3}^2 - \frac{\delta^2}{n}.$$ 

It follows that

$$h' - h - \epsilon \mu = 4(x_{r+1} - z_{r+1})x_r - 6(x_{r+2}^2 + x_{r+3}^2 - z_{r+1}^2 - z_{r+2}^2)$$

$$- 4z_{r+3}(x_{r+4} + z_{r+2} - z_{r+3}) + 8(x_{r+2}x_{r+3} - z_{r+1}z_{r+2})$$

$$+ 4(x_{r+1}x_{r+2} + x_{r+3}x_{r+4} - x_{r+1}^2) - \epsilon \mu$$

$$= \Phi \frac{-2\mu(\mu - 5)(x_r - x_{r+4})}{(\mu^3 - 10\mu^2 + 27\mu - 14)^2n},$$

in which

$$\Phi = (\mu^3 - 10\mu^2 + 27\mu - 14)n(x_r + x_{r+4}) + (10 - 2\mu)(x_r - x_{r+4}).$$

The only real zero of \(t^3 - 10t^2 + 27t - 4\) is about 0.157. Since \(\mu < 0.091\), we have \(\mu^3 - 10\mu^2 + 27\mu - 4 < 0\). It follows that

$$\Phi < -10n(x_r + x_{r+4}) + 10(x_r - x_{r+4}).$$  \(15\)

In view of (14), the right-hand side of (15) is negative, and thus \(h' - h - \epsilon \mu < 0\). The result now follows from Lemma 3.9.

**Lemma 3.11.** Any middle block of \(\Gamma\) is either \(M_0\) or \(M_3\).

**Proof:** Let \(B\) be an arbitrary middle block of \(\Gamma\). According to Theorem 3.1, \(B\) is either one of the short blocks \(M_0, M_1, M_2, \tilde{M}_2, M_3\), or it is a long block starting with either \(M'_0, M'_1, \text{ or } M'_2\).

First assume that \(B\) is either a long block starting with \(M'_1\), or it is one of the short blocks \(M'_1\), \(M_2, \text{ or } M_3\). Then \(\Gamma\) contains \(H_1\) (of Lemma 3.10(i)) as an induced subgraph. If all the components of \(x\) on the first five vertices of \(B\) are nonnegative, then the required condition of Lemma 3.10(i) is satisfied, and so we are done. Otherwise, \(x\) is negative on \(B\) except possibly for its first three vertices. Recall that if \(B\) is a long block, then its last brick \(B_s\) is either \(\tilde{M}'_0, \tilde{M}'_1, \text{ or } M'_2\). Now consider the mirror image \(\tilde{\Gamma}\) and \(-x\) as its Fiedler vector. It turns out that in \(\tilde{\Gamma}\), \(-x\) is nonnegative on \(\tilde{B}\) except possibly for its last three vertices. If \(B = M_1\) or \(B_s = \tilde{M}'_1\), then \(\tilde{\Gamma}\) contains \(H_1\); if \(B = \tilde{M}_2\) or \(B_s = \tilde{M}'_2\), then \(\tilde{\Gamma}\) contains \(H_2\); and if \(B_s = \tilde{M}'_2\), then \(\tilde{\Gamma}\) contains \(H_3\). Furthermore, the required conditions on the sign of the components of the Fiedler vector in Lemma 3.10 holds in all these three cases. This leads again to a contradiction by Lemma 3.10.

If \(B\) is long block starting with an \(M'_2\) or \(B = M_2\), we obtain a contradiction similarly. So far we have proved that if \(B\) is a middle block, then it must start and end with \(M'_0\) and \(\tilde{M}'_0\), respectively. Such a block must contain \(H_3\). If the components of \(x\) satisfy the
condition (iii) of Lemma 3.10, then we are done. Otherwise, \( x_{r+3} < 0 \) and so \( \tilde{B} \) in \( \hat{\Gamma} \) fulfills the condition (iii), and again we reach a contradiction.

It follows that \( B \) must be one of the short blocks \( M_0 \) or \( M_3 \), as desired. \( \square \)

It remains to show that no middle block in \( \Gamma \) can be \( M_3 \). This will be essentially done by the next lemma. By \( D_iM_3 \) we denote the graph obtained by identifying the degree 2 vertices of \( D_i \) and \( M_3 \).

**Lemma 3.12.** The graph \( \Gamma \) does not contain the following subgraphs:

(i) \( H_4 \) (of Figure 21) such that \( x_{r+4} \geq 0 \) and for the two left-most vertices, all their neighbors on their left lie in a single cell of the equitable partition of \( \Gamma \),

(ii) \( D_0M_3 \),

(iii) \( D_2M_3 \).

**Proof.** (i) For a contradiction, assume that \( \Gamma \) contains \( H_4 \). By the assumption, we can assume that for the two left-most vertices, all their neighbors on their left have the weight \( x_r \). Applying the eigenequation (3) to the vertices of \( H_4 \) in \( \Gamma \), we see that

\[
\begin{align*}
\mu_{x, r+4} &= \mu_{x, r+1} - 2(\mu x - 11 \mu x + 36 \mu x + \mu y - 32 x - 6 y), \\
\mu_{x, r+3} &= \mu_{x, r+2} - 2(2 \mu x + \mu y - 14 \mu x - 9 \mu y + 20 x + 18 y), \\
\mu_{x, r+2} &= \mu_{x, r+1} - \mu^3 y - 13 \mu^3 y + 4 \mu x + 52 \mu y - 16 x - 60 y, \\
\mu_{x, r+1} &= \mu_{x, r} - 2(\mu^3 y - 11 \mu^3 y + 36 \mu y - 34 y - 4 x).
\end{align*}
\]

We have that \( x_r > x_{r+5} \) (by Remark 2.4). We further claim that

\[
2(x_r + x_{r+5}) > x_r - x_{r+5} > 0. \tag{16}
\]

If \( x_{r+5} \geq 0 \), this trivially holds. If \( x_{r+5} < 0 \), it holds by the following argument. Note that

\[
p(x_r, x_{r+5}) = x_{r+4} \geq 0 \quad \text{which means} \quad \mu^3 x_{r+5} - 11 \mu^2 x_{r+5} + 36 \mu x_{r+5} - 34 x_{r+5} - 4 x_r \leq 0.
\]

From this and the fact that \( \mu < 0.355 \) (by Lemma 3.4), we have \( 4x_r \geq (\mu^3 - 11 \mu^2 + 36 \mu - 34)x_{r+5} > -12x_{r+5} \), that is, \( x_r > -3x_{r+5} \), and thus (16) follows.

**FIGURE 21** Subgraph \( H_4 \) and components of \( x \)
We now replace \( H_4 \) by \( H'_4 \) to obtain \( \Gamma' \) and define a vector \( \mathbf{x}' \) on \( V(\Gamma') \) such that its components on \( H'_4 \) are as given in Figure 22 and on the rest of the vertices agree with \( \mathbf{x} \).

We set

\[
z_{r+1} = p(x_{r+5}, x_r), \quad z_{r+2} = l(x_{r+5}, x_r), \quad z_{r+3} = g(x_{r+5}, x_r), \quad z_{r+4} = f(x_{r+5}, x_r).
\]

We have \( \delta = \mathbf{x}' \mathbf{1}' = z_{r+1} + 2z_{r+2} + z_{r+3} + 2z_{r+4} - 2x_{r+1} - x_{r+2} - 2x_{r+3} - x_{r+4} \). By substituting the values of \( x_{r+1}, \ldots, x_{r+4} \), and \( z_{r+1}, \ldots, z_{r+4} \) in terms of \( x_r, x_{r+5}, \mu \) we obtain that

\[
\delta = \frac{2(\mu - 4)(\mu - 5)(x_r - x_{r+5})}{\mu^4 - 14\mu^3 + 67\mu^2 - 126\mu + 76}.
\]

Now, in the notation of Lemma 3.9, we have

\[
h = 4(x_r - x_{r+1})^2 + 2 \sum_{i=r+1}^{r+4} (x_i - x_{i+1})^2 + 2(x_{r+3} - x_{r+5})^2,
\]

\[
h' = 2(x_r - z_{r+1})^2 + 2(x_r - z_{r+2})^2 + 2 \sum_{i=r+1}^{r+3} (z_i - z_{i+1})^2 + 4(z_{r+4} - x_{r+5})^2.
\]

Moreover,

\[
\varepsilon = \|\mathbf{x}'\|^2 - 1 - \frac{\delta^2}{n} = z_{r+1}^2 + 2z_{r+2}^2 + z_{r+3}^2 + 2z_{r+4}^2 - 2x_{r+1}^2 - 2x_{r+2}^2 - 2x_{r+3}^2 - x_{r+4}^2 - \frac{\delta^2}{n}.
\]

It follows that

\[
h' - h - \varepsilon \mu = 8(x_r x_{r+1} - 8z_{r+4}x_{r+5}) + 4((x_{r+4} + x_{r+5})x_{r+3} - z_{r+2}(x_r + x_{r+1}))
\]
\[
- 4(x_{r+2}^2 + x_{r+4}^2 - z_{r+1}^2 - z_{r+3}^2) - 6(x_{r+1}^2 + x_{r+3}^2 - z_{r+2}^2 - z_{r+4}^2)
\]
\[
- 4(x_r z_{r+1} - x_{r+2}x_{r+1} - x_{r+4}x_{r+5} + z_{r+3}z_{r+4})
\]
\[
+ 4(x_{r+2}x_{r+3} - z_{r+2}z_{r+3}) - \varepsilon \mu
\]
\[
= \Phi \frac{-2(x_r - x_{r+5})(\mu - 4)(\mu - 5)\mu}{(\mu^4 - 14\mu^3 + 67\mu^2 - 126\mu + 76)^2 n},
\]

where

![Figure 22](attachment:image.png) Subgraph \( H'_4 \) in \( \Gamma' \) and components of \( \mathbf{x}' \)
The smallest zero of 
\[ t^4 - 14t^3 + 67t^2 - 126t + 46 \] is about 0.472. Hence
\[ \mu^4 - 14\mu^3 + 67\mu^2 - 126\mu + 46 > 0 \]
because \( \mu < 0.355 \), and thus
\[ \Phi > 30n(x_k + x_{k+5}) - 40(x_k - x_{k+5}). \]  

In view of (16), the right-hand side of (17) is positive which means \( h' - h - \varepsilon \mu < 0 \). The result now follows from Lemma 3.9.

(ii) If \( 17 \leq n \leq 20 \), and \( \Gamma \) contains \( D_0M_3 \), then the right end block must be \( \tilde{D}_i \), \( 0 \leq i \leq 3 \), respectively, which leads to a contradiction as \( \mu (G_n) < \mu (D_0M_3\tilde{D}_i) \) for \( 17 \leq n \leq 20 \). So we assume that \( n \geq 21 \) and thus from Lemma 3.4, \( \mu < 0.091 \).

For a contradiction, assume that \( \Gamma \) contains \( H_5 = D_0M_3 \) with the components of \( x \) as depicted in Figure 23. By using the eigenequation (3), we can write \( x_2, ..., x_6 \) in terms of \( x_1 \) and \( \mu \) as follows:

\[
\begin{align*}
x_2 &= \left( -\frac{\mu}{2} + 1 \right) x_1, \\
x_3 &= \left( \frac{\mu^2}{2} - 3\mu + 1 \right) x_1, \\
x_4 &= \frac{-1}{4} (\mu^3 - 10\mu^2 + 24\mu - 4) x_1, \\
x_5 &= \left( \frac{\mu^4 - 14\mu^3 + 62\mu^2 - 88\mu + 12}{2(6 - \mu)} \right) x_1, \\
x_6 &= \left( \frac{\mu^5 - 17\mu^4 + 102\mu^3 - 252\mu^2 + 216\mu - 24}{4(\mu - 6)} \right) x_1.
\end{align*}
\]

We obtain a contradiction by showing that if we replace \( H_5 \) by \( H'_5 \) (of Figure 23) to obtain \( \Gamma' \), then \( \mu (\Gamma') < \mu \). We define a vector \( x' \) on \( V(\Gamma') \) such that its components on \( H'_5 \) are as given in Figure 23 and on the rest of the vertices agree with \( x \). We set
\[
\begin{align*}
z_1 &= \omega x_1, \\
z_2 &= \omega (1 - \mu) x_1, \\
z_3 &= \omega (\mu^2 - 4\mu + 1) x_1, \\
z_4 &= \frac{-\omega}{2} (\mu^3 - 8\mu^2 + 15\mu - 2) x_1,
\end{align*}
\]

where

\[ FIGURE 23 \] Subgraphs \( H_5 \) (left) and \( H'_5 \) (right) and components of \( x \) and \( x' \)
\[ \omega = \frac{\mu^5 - 17\mu^4 + 102\mu^3 - 252\mu^2 + 216\mu - 24}{\mu^5 - 17\mu^4 + 103\mu^3 - 261\mu^2 + 238\mu - 24}. \]

Taking into account that \( x \perp 1 \), we see that
\[ \delta = x'1^T = 4z_4 + 2z_2 + z_3 + 2z_4 - 3x_1 - 2x_2 - x_3 - 2x_4 - x_5. \]

Also, since \( ||x|| = 1 \), we obtain that
\[ ||x'||^2 = 1 + 4z_4^2 + 2z_2^2 + z_3^2 + 2z_4^2 - 3x_1^2 - 2x_2^2 - x_3^2 - 2x_4^2 - x_5^2. \]

Further we have
\[ x'L'(\Gamma')x^T = \mu + 2(z_4 - z_2)^2 + 2\sum_{i=1}^{3} (z_i - z_{i+1})^2 + 4(z_4 - x_6)^2 - 4(x_1 - x_2)^2 - 2\sum_{i=1}^{5} (x_i - x_{i+1})^2 - 2(x_4 - x_6)^2. \]

As \( ||x'||^2 - \frac{\delta^2}{n} > 0 \), then \( \frac{x' L'(\Gamma') x^T}{||x'||^2 - \frac{\delta^2}{n}} < \mu \) if and only if \( x' L'(\Gamma') x^T - \mu||x'||^2 + \frac{\delta^2}{n} < 0 \).

So, by substituting the values of \( x_2, ..., x_6, z_1, ..., z_4 \), and \( \delta \) in terms of \( x_1 \) and \( \mu \), we must show that
\[ ((\mu^9 - 28\mu^8 + 326\mu^7 - 2042\mu^6 + 7429\mu^5 - 15,770\mu^4 + 18,492\mu^3 - 10,320\mu^2 \\
+ 1800\mu - 96)n \\
+ (2\mu^7 - 46\mu^6 + 418\mu^5 - 1886\mu^4 + 4296\mu^3 - 4280\mu^2 + 1000\mu)) \\
(\mu^5 - 18\mu^4 + 119\mu^3 - 348\mu^2 \\
+ 408\mu - 100)(\mu - 5)x_1^2\mu < 0. \]

Given that \( n \geq 21 \) and \( \mu < 0.091 \), this is equivalent to
\[ 21\mu^9 - 588\mu^8 + 6848\mu^7 - 42,928\mu^6 + 156,427\mu^5 - 333,056\mu^4 + 392,628\mu^3 \\
- 221,000\mu^2 + 38,800\mu - 2016 < 0. \]

This holds as \( \mu < 0.091 \), and so by Lemma 2.1, \( \mu(\Gamma') < \mu \), a contradiction.

(iii) For a contradiction, assume that \( \Gamma \) contains \( H_6 = D_2M_3 \). If \( \Gamma \) has only one middle block, and its right end block is different from \( D_2 \), then we are done by the previous lemmas. The right end block also cannot be \( \tilde{D}_2 \), since \( \mu(G_{21}) < \mu(D_2M_3\tilde{D}_2) \). It follows that \( \Gamma \) has at least two middle blocks, which in turn implies that \( n \geq 26 \) and so by Lemma 3.4, \( \mu < 0.059 \).

Let the components of \( x \) on \( H_6 \) be as depicted in Figure 24. By using the eigenequation (3), we can write the weights of the vertices of \( H_6 \) in terms of \( \mu \) and \( x_8 \) as follows:
\[ x_1 = -4(2\mu^2 - 19\mu + 42)\omega x_8, \]
\[ x_2 = 2(\mu - 6)(\mu^2 - 7\mu + 14)\omega x_8, \]
\[ x_3 = 4(\mu^3 - 12\mu^2 + 43\mu - 42)\omega x_8, \]
\[ x_4 = -2(\mu^4 - 16\mu^3 + 87\mu^2 - 176\mu + 84)\omega x_8, \]
\[ x_5 = 2(\mu^5 - 19\mu^4 + 132\mu^3 - 400\mu^2 + 470\mu - 84)\omega x_8, \]
\[ x_6 = -(\mu^6 - 23\mu^5 + 206\mu^4 - 896\mu^3 + 1896\mu^2 - 1612\mu + 168)\omega x_8, \]
\[ x_7 = -2(\mu^6 - 21\mu^5 + 170\mu^4 - 662\mu^3 + 1244\mu^2 - 932\mu + 84)\omega x_8, \]
where \( \omega = (\mu^7 - 24\mu^6 + 231\mu^5 - 1136\mu^4 + 2996\mu^3 - 4012\mu^2 + 2200\mu - 168)^{-1}. \)

Now, we replace \( H_6 \) by \( H_6' \) (of Figure 24) to obtain \( \Gamma' \). We define a vector \( x' \) on \( V(\Gamma') \) such that its components on \( H_6' \) are as given in Figure 24 and on the rest of the vertices \( \Gamma' \) agree with \( x \). We specify the new components as follows:

\[ z_1 = -8\omega' x_8, \quad z_2 = 8(\mu - 1)\omega' x_8, \quad z_3 = -4(\mu^2 - 5\mu + 2)\omega' x_8, \]
\[ z_4 = 4(\mu^3 - 8\mu^2 + 13\mu - 2)\omega' x_8, \quad z_5 = -2(\mu^4 - 12\mu^3 + 43\mu^2 - 44\mu + 4)\omega' x_8, \]

where \( \omega' = (\mu^5 - 15\mu^4 + 77\mu^3 - 157\mu^2 + 110\mu - 8)^{-1} \). Taking into account that \( x \perp 1 \), we see that
\[ \delta = x' \top = 4z_4 + 2z_2 + 2z_3 + z_4 + 2z_5 - 2x_1 - 2x_2 - x_3 - 2x_4 - x_5 - 2x_6 - x_7. \]

Also, since \( \|x'\| = 1 \), we obtain
\[ \|x'\|^2 = 1 + 4z_4^2 + 2z_2^2 + 2z_3^2 + z_4^2 + 2z_5^2 - 2x_1^2 - 2x_2^2 - x_3^2 - 2x_4^2 - x_5^2 - 2x_6^2 - x_7^2. \]

Further we have
\[ x'L(\Gamma')x'\top = \mu + 4(z_1 - z_2)^2 + 4(z_2 - z_3)^2 + 2(z_3 - z_4)^2 + 2(z_4 - z_5)^2 + 4(z_5 - x_8)^2 \]
\[ - 4(x_1 - x_2)^2 - 2(x_1 - x_3)^2 - 2(x_2 - x_4)^2 - 2 \sum_{i=3}^7 (x_i - x_{i+1})^2 - 2(x_6 - x_8)^2. \]

As \( \|x'\|^2 - \frac{\delta}{n} > 0 \), we have
\[ \frac{x'L(\Gamma')x'\top}{\|x'\|^2 - \frac{\delta}{n}} < \mu \text{ if and only if } x'L(\Gamma')x'\top - \mu\|x'\|^2 + \mu\frac{\delta}{n} < 0. \]

So, by substituting the values of \( x_1, ..., x_7, z_3, ..., z_5, \) and \( \delta \) in terms of \( x_8 \) and \( \mu \), we need to show that
in which
\[
p_1(t) = t^{12} - 39t^{11} + 668t^{10} - 6606t^9 + 41,701t^8 - 175,339t^7 + 497,026t^6 - 939,272t^5 \\
+ 1,140,452t^4 - 823,624t^3 + 300,472t^2 - 36,080t + 1,140,452,
\]
\[
p_2(t) = 2t^{10} - 68t^9 + 992t^8 - 8100t^7 + 40,446t^6 - 126,432t^5 + 242,312t^4 - 264,520t^3 \\
+ 137,816t^2 - 20,208,
\]
\[
p_3(t) = t^8 - 28t^7 + 328t^6 - 2082t^5 + 7731t^4 - 16,830t^3 + 20,176t^2 - 11,204t + 1684.
\]

The smallest root of \( p_3(t) \) is about 0.227. As \( \mu < 0.059 \), we have that \( p_3(\mu) > 0 \). The smallest root of \( p_1(t) \) is about 0.081 and so \( p_1(\mu) > 0 \). Since \( n \geq 26 \) we also have
\[
(p_1(\mu)n + p_2(\mu))p_3(\mu)^2(\mu - 6)x_8^2 < 0,
\]
in which
\[
(p_1(\mu)n + p_2(\mu))p_3(\mu)^2(\mu - 6)x_8^2 < 0,
\]
where
\[
p_4(t) = 26t^{12} - 1014t^{11} + 17,370t^{10} - 171,824t^9 + 1,085,218t^8 - 4,566,914t^7 + 12,963,122t^6 \\
- 24,547,504t^5 + 29,894,064t^4 - 21,678,744t^3 + 7,950,088t^2 - 958,288t + 34,944.
\]

The smallest root of \( p_4(t) \) is about 0.070 which means that \( p_4(\mu) > 0 \). Therefore, \( (p_1(\mu)n + p_2(\mu))p_3(\mu) < 0 \), and so (18) is established. Hence, by Lemma 2.1, \( \mu(\Gamma') < \mu \) which is a contradiction.

Now we are prepared to prove the desired result on the middle blocks of \( \Gamma \).

**Theorem 3.13.** Any middle block of \( \Gamma \) is \( M_0 \).

**Proof.** By Lemma 3.11, any middle block of \( \Gamma \) is either \( M_0 \) or \( M_3 \). For a contradiction, assume that some middle block \( B \) of \( \Gamma \) is \( M_3 \).

First, assume that \( B \) is a block next to an end block. By Theorem 3.8, either of the end blocks of \( \Gamma \) is one of \( D_0, D_1, D_2, D_3 \), or \( D_4 \). By Lemma 3.12(ii),(iii), \( D_0M_3 \) and \( D_2M_5 \) are not possible. Let \( H \in \{D_1M_3, D_3M_3, D_4M_3\} \). Then \( H \) contains an induced subgraph isomorphic to \( H_4 \) possessing the additional condition of Lemma 3.12(i) on the neighbors of the left-most vertices. If the components of \( x \) on the first four vertices of \( M_3 \) in \( H \) are nonnegative, then we are done. Otherwise, \( x \) is negative on the vertices of \( \Gamma \) starting from the middle vertex of \( M_3 \) in \( H \). If all the middle blocks of \( \Gamma \) are \( M_3 \), then we have a right end block \( M_3\bar{D}_1 \) which is not possible by the above argument. Otherwise, we have the subgraph \( M_3M_0 \) in \( \Gamma \). Under the above condition, \( \bar{\Gamma} \) contains \( H_4 \) satisfying the conditions of Lemma 3.12(i), which leads again to a contradiction.

Now, assume that \( B \) is not a block next to an end block. So the block on its left must be an \( M_0 \) which means that \( \Gamma \) has an \( M_0M_3 \) subgraph. However, \( M_0M_3 \) also contains an induced subgraph isomorphic to \( H_4 \). If the components of \( x \) on the first three vertices of \( M_3 \) in \( B \) are nonnegative, then we are done. Otherwise, similar to the above argument, we obtain a contradiction. \( \square \)
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REFERENCES
1. M. Abdi, E. Ghorbani, and W. Imrich, Regular graphs with minimum spectral gap, Eur. J. Combin. 95 (2021), 103328, 18pp.
2. S. G. Aksoy, F. R. Chung, M. Tait, and J. Tobin, The maximum relaxation time of a random walk, Adv. in Appl. Math. 101 (2018), 1–14.
3. D. Aldous, and J. Fill, Reversible Markov chains and random walks on graphs, University of California, Berkeley, 2002. http://www.stat.berkeley.edu/~aldous/RWG/book.html
4. C. Brand, B. Guiduli, and W. Imrich, The characterization of cubic graphs with minimal eigenvalue gap, Croat. Chem. Acta. 80 (2007), 193–201.
5. F. C. Bussemaker, S. Čobeljić, D. M. Cvetković, and J. J. Seidel, Computer investigation of cubic graph, Technical Report No. 76-WSK-01, Technological University Eindhoven, 1976.
6. F. C. Bussemaker, S. Čobeljić, D. M. Cvetković, and J. J. Seidel, Cubic graphs on \( \leq 14 \) vertices, J. Combin. Theory Ser. B. 23 (1977), 234–235.
7. F. R. Chung, Spectral graph theory, 92, American Mathematical Society, Providence, RI, 1997.
8. M. Fiedler, Algebraic connectivity of graphs, Czechoslovak Math. J. 23 (1973), 298–305.
9. B. Guiduli, Spectral extrema for graphs, Ph.D. Thesis, University of Chicago, 1996.
10. B. Guiduli, The structure of trivalent graphs with minimal eigenvalue gap, J. Algebraic Combin. 6 (1997), 321–329.

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