Spin-dependent magnetotransport through a ring
in the presence of spin-orbit interaction

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Abstract

The Schrödinger equation for an electron in a mesoscopic ring, in the presence of the Rashba and linear Dresselhaus terms of the spin-orbit interaction (SOI) and of a magnetic field $B$, is solved exactly. The effective electric fields of these terms as well as $B$ have perpendicular and radial components. The interplay between them and $B$ and their influence on the spectrum is studied. The transmission through such a ring, with two leads connected to it, is evaluated as a function of the SOI strengths and of the orientations of these fields. The Rashba and Dresselhaus terms affect the transmission in different ways. The transmission through a series of rings with different radii and with SOI in both arms of the rings or only in one of them is also evaluated. For weak magnetic fields $B \leq 1$ T the influence of the Zeeman term on the transmission, assessed by perturbation theory, is negligible.

PACS numbers: 72.25.-b, 71.70.Ej, 03.65.Vf, 85.35.-p

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I. INTRODUCTION

The concept of the geometric phase \[1, 2, 3, 4\] has attracted considerable interest since it is established in a general way. In mesoscopic rings, the geometric phase of electrons in magnetic and electric fields can be obtained by solving the time-independent Schrödinger equation. \[4\] From the point of view of the stationary Schrödinger equation, the energy dispersion relation of electrons changes as the configuration of the system varies. As a result, electrons of the same energy have different wavevectors in different systems and may accumulate different phases after passing even the same path in real space. In the presence of external magnetic and electric fields, the one-particle Hamiltonian can be expressed as \[5\]

$$H = (\mathbf{p} - e \mathbf{A} - \mu_B \mathbf{\sigma} \times \mathbf{E}/2c^2)^2/2m$$  \hfill (1)

The contribution from the vector potential \(\mathbf{A}\) corresponds to the Aharonov-Bohm (AB) phase \[6\] and the contribution from the SOI to the Aharonov-Casher (AC) phase. \[7\] In mesoscopic systems of semiconductor heterostructures, however, the macroscopic SOI results from the asymmetry of the microscopic crystal field and may appear in different forms depending on the materials and structures involved. In materials with asymmetric crystal structure, the cubic Dresselhaus SOI term exists in bulk materials while an extra linear Dresselhaus SOI (DSOI) term appears in confined, low-dimensional systems due to the change of the crystal structure along the direction of the confinement. In systems with asymmetric confinement, the Rashba SOI (RSOI) term results from a non-vanishing confining electric field as well as from various other SOI mechanisms such as the one related to differing band discontinuities at the heterostructure interfaces considered in \(\mathbf{k.p}\) models. The aggregate strength of all these SOI mechanisms is denoted by \(\alpha\) \[8\].

By inserting a mesoscopic ring into a circuit, we can study the quantum transport through the ring when the inelastic diffusion length is larger than the size of the ring. In the two-terminal configuration, the transmission depends on the interference between electrons propagating through the ring’s two arms and the transmission properties through the two junctions connecting the ring and the leads. In general, a symmetric junction can be described by a \(3 \times 3\) scattering matrix with the transmission through each arm as a parameter. \[9, 10, 11\] For a ballistic, one-dimensional (1D) ring connected to two leads, the scattering matrix can be determined by imposing the continuity of the wave function and
of the spin flux at each junction. In the presence of SOI the transmission through the ring as well as the geometric phases are spin-dependent and the ring can be used as a spin-interference device. Similar considerations apply to a square loop or arrays of such loops. Theoretically this spin interference was further studied in 1D and 2D rings but only in the presence of the RSOI.

In this paper we study ballistic transport through one or more 1D rings, symmetrically connected to two leads, in the presence of a magnetic field, with components along the radial and perpendicular direction, and of both terms of the SOI, RSOI and DSOI. The corresponding effective electric fields have perpendicular and radial components. For weak magnetic fields, $B \leq 1$ T, the influence of the Zeeman term is validly assessed by perturbation theory. In Sec. II we present the one-electron energy spectrum and formulate the corresponding transfer-matrix transmission problem. In Sec. III we present numerical results for the transmission, through one or more rings, and in Sec. IV concluding remarks.

II. ONE-ELECTRON PROBLEM

A. Hamiltonian

We consider a one-dimensional ring, of radius $a$, in the ($x$-$y$) or ($r$-$\theta$) plane and in a magnetic field with components $B_z = B \cos \gamma_3$ and $B_r = B \sin \gamma_3$.

For the vector potential we choose the gauge $A = (A_r, A_\theta, A_z) = (0, B_z r / 2 - B_r z, 0)$ with $z = 0$ in the plane of the ring. The one-electron Hamiltonian is

$$H = \left( \frac{\hbar^2}{2m^* a^2} \right) (-i \partial/\partial \theta + \Phi/\Phi_0)^2 + g\mu_B \mathbf{\sigma} \cdot \mathbf{B} / 2, \quad (2)$$

where $\mathbf{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \equiv (\sigma_r, \sigma_\theta, \sigma_z)$ are the Pauli matrices, $\Phi = B_z \pi a^2$ the magnetic flux passing through the ring, $\Phi_0 = h/e$ the flux quantum, $\mu_B$ the Bohr magneton, and $g$ the $g$ factor. For a ring fabricated out of a heterostructure the SOI can result from asymmetric confinement along the $z$ direction (RSOI, $H_\alpha$) or from the crystal structure changing along the $z$ direction (DSOI, $H_\beta$). Considering both terms, we have

$$H_\alpha = \frac{\alpha}{\hbar} \sigma_x (\hat{p}_y + eA_y) - \frac{\alpha}{\hbar} \sigma_y (\hat{p}_x + eA_x), \quad (3)$$

$$H_\beta = \frac{\beta}{\hbar} \sigma_x (\hat{p}_x + eA_x) - \frac{\beta}{\hbar} \sigma_y (\hat{p}_y + eA_y). \quad (4)$$
Here $\alpha \propto \langle E_R \rangle$ and $\beta \propto \langle E_D \rangle$ are the usual RSOI and DSOI strengths, respectively, and $\langle E_R \rangle = \langle E_R \rangle e_r$ and $\langle E_D \rangle = \langle E_D \rangle e_z$, the corresponding effective electric fields. However, here we study the more general case in which $\langle E_R \rangle$ and $\langle E_D \rangle$ can have a radial component as well as has already been the case for the RSOI [4, 22]. Accordingly, we consider them in the form

$$E_R = E_R (\sin \gamma_1 e_r + \cos \gamma_1 e_z),$$

$$E_D = E_D (\sin \gamma_2 e_r + \cos \gamma_2 e_z).$$

That is, we assume the same form of Eqs. (3) and (4) but with the effective electric fields having components along the radial and $z$ directions as specified in Eqs. (5) and (6). Then the total Hamiltonian in the ring reads [9, 24]

$$\hat{H} = \hbar \omega_0 [-i \partial / \partial \theta + \phi + \alpha (\sigma_r \cos \gamma_1 - \sigma_z \sin \gamma_1) + \beta \sigma_\theta (\sin \gamma_2 - \cos \gamma_2)]^2$$

$$+ \hbar \omega_B (\sigma_r \sin \gamma_3 + \sigma_z \cos \gamma_3),$$

where $\phi = \Phi / \Phi_0$, $\omega_0 = \hbar / 2m^* a^2$, $\omega_B = g\mu_B B / 2\hbar$, $\alpha = \alpha a^* / \hbar^2$, and $\beta = \beta a^* / \hbar^2$.

In an isolated ring we can expand the wave function $\Psi$ in terms of an orthogonal and complete set of eigenvectors $e^{in\theta} / \sqrt{2\pi}$ of the Hamiltonian $\hat{h} = -[\hbar^2 / 2m^* a^2] \partial^2 / \partial^2 \theta$ with $n$ integer. The expansion takes the form

$$\Psi = \frac{1}{\sqrt{2\pi}} \sum_n \begin{pmatrix} C_n^+ e^{in\theta} \\ C_n^- e^{in\theta} \end{pmatrix},$$

where $C_n^+$ and $C_n^-$ are the coefficient of the spin-up and spin-down eigenstates, respectively. We can then write the secular equation $\hat{H} \Psi = E \Psi$ as

$$\begin{pmatrix} \Omega_n^+ - E & \Sigma_n \\ \Sigma_n^* & \Omega_{n+1}^- - E \end{pmatrix} \begin{pmatrix} C_n^+ \\ C_{n+1}^- \end{pmatrix} = 0. \tag{9}$$

Here $\Omega_n^+ = \omega_0 [(n + \phi)^2 + 2(n + \phi) \alpha \sin \gamma_1 + \alpha^2 + \beta^2 (\sin \gamma_2 - \cos \gamma_2)^2] \pm \omega_B \cos \gamma_3$, $\Sigma_n = \hbar \omega_n (\alpha \cos \gamma_1 - i \beta (\sin \gamma_2 - \cos \gamma_2) + \hbar \omega_B \sin \gamma_3$, and $\omega_n = \omega_0 (n + \phi + 1 / 2)$. In this representation the Hamiltonian is expressed as a matrix composed of a series of $2 \times 2$ blocks and is explicitly Hermitian. The secular equation is solved exactly. The eigenvalues (for $\sigma = \pm$) are

$$E_{n\sigma} = \hbar \omega_0 (n + \phi)^2 + \hbar \omega_0 [\alpha^2 + \bar{\alpha} \sin \gamma_1 + \bar{\beta}^2 (\sin \gamma_2 - \cos \gamma_2)^2]$$

$$+ \hbar \omega_n + \sigma \hbar (-\omega_n - 2\omega_n \alpha \sin \gamma_1 + \omega_B \cos \gamma_3) \cos \delta_{n\sigma}$$

$$+ \sigma \hbar \left[2\omega_n \alpha \cos \gamma_1 + \omega_B \sin \gamma_3 \right]^2 + 4\omega_n^2 \bar{\beta}^2 (\sin \gamma_2 - \cos \gamma_2)^2]^{1 / 2} \sin \delta_{n\sigma} \tag{10}$$
and the corresponding eigenvectors

$$\Psi_{n\sigma} = \frac{e^{i n\theta}}{\sqrt{2\pi}} \chi_{n\sigma}(\theta), \quad \text{with spinors} \quad \chi_{n\sigma}(\theta) = \begin{bmatrix} \cos(\delta_{n\sigma}/2) \\ \sin(\delta_{n\sigma}/2)e^{i\phi_n+\theta} \end{bmatrix}. \quad (11)$$

Here \(\delta_{n-} = \delta_{n+} - \pi\),

$$\cot(\delta_{n+}) = \frac{-\omega_n(1 + 2\bar{\alpha}\sin\gamma_1) + \omega_B\cos\gamma_3}{[(2\omega_n\bar{\alpha}\cos\gamma_1 + \omega_B\sin\gamma_3)^2 + 4\omega_n^2\bar{\beta}^2(\sin\gamma_2 - \cos\gamma_2)^2]^{1/2}}, \quad (12)$$

and

$$\tan\varphi_n = \frac{\omega_n\bar{\beta}(\sin\gamma_2 - \cos\gamma_2)}{\omega_n\bar{\alpha}\cos\gamma_1 + \omega_B\sin\gamma_3/2}. \quad (13)$$

With the help of the Pauli matrices \(\sigma\) in polar coordinates, we easily find that the orientation of the spin of the state \(\Psi_{n+}\) is \((\delta_{n+}, \varphi_n + \theta)\) and opposite to that of the state \(\Psi_{n-}\).

If the ring is not isolated but coupled to outside leads, as shown in Fig. 1, the periodic boundary condition on \(\theta\) is relaxed and \(n\) can be any real number. In this case, we find the above wave functions and eigenvalues are still eigenvectors and eigenvalues of the Hamiltonian (7). However, because \(\delta_{n\sigma}\) and \(\varphi_n\) depend on \(n\) in the presence of the Zeeman term, the spin orientations of different eigenstates depend on \(n\) and the spinors \(\chi_{n\sigma}(\theta)\) given by Eq. (11) for electrons of the same energy are generally not orthogonal to each other. \[9\]

When the Zeeman term is negligible, \(\delta_{n+}\) and \(\varphi_n\) are independent of \(n\) and \(\sigma\) and will be replaced by \(\delta\) and \(\varphi\), respectively. In this case the energy eigenvalues (10) can be written as

$$E_{n\sigma} = \hbar\omega_0[n + \phi - \phi_{AC}/(2\pi)]^2 \quad (14)$$

with the Aharonov-Casher phase \(\phi_{AC}\) given by

$$\phi_{AC} = -\pi\{1 + 2\sigma[\bar{\alpha}^2 + \bar{\alpha}\sin\gamma_1 + \bar{\beta}^2(\sin\gamma_2 - \cos\gamma_2)^2 + 1/4]^{1/2}\}. \quad (15)$$

Since the effects of the RSOI on the energy spectrum have been reported before, here we focus mostly on those on it when the RSOI is replaced by the DSOI of a similar strength. For clarity though we show its dependence on both \(\alpha\) and \(\beta\). When \(B, E_R\), and \(E_D\) have only a \(z\) component, the energy spectrum and the angle \(\delta\) versus the SOI strength \(\alpha\) or \(\beta\) do not change after this replacement, though there is a rotation of the spin orientation in the \((x-y)\) plane by \(\varphi\). This is in line with the unitary equivalence between the Rashba term and the linear Dresselhaus term \[21\] that is often exploited in the literature \[23\]. The radial
FIG. 1: (a) Geometry of an isolated 1D ring and the components of the electric and magnetic fields. (b) The ring of (a) connected to two leads, regions I and II, and the corresponding wave functions.

components of $E_R$ and $E_D$, however, break this equivalence and lead to different effects of the RSOI and DSOI on the energy spectrum as well as on $\delta$.

In Fig. 2 we show the energy spectrum and the absolute value of the angle $\delta$ for the RSOI (solid curves) and the DSOI (dotted curves). $\varphi$ equals 0 or $\pi$ in the former case and $\pi/2$ or $3\pi/2$ in the later.

B. Transfer-matrix formulation

Single ring. Now we consider the quantum transport of electrons with energy $E$ through a ring connected to two leads I and II as shown in Fig. 1 with the local coordinate systems attached to the different regions of the device. Assuming there is no SOI in the leads, the electron wavevector is readily obtained as $k = \pm \sqrt{2m^*E}$. Solving Eq. (10) for $n$, we obtain the angular wavevector $n_\sigma^\mu$ of the electron with $\mu$ the mode index and $\sigma$ the spin branch. In this case it is appropriate to apply a spin-dependent version of Griffith’s boundary conditions [12, 25] at the intersections as specified below. This reduces the electron transport through the ring to an exactly solvable, 1D scattering problem. The conditions at each junction are: (i) the wave function must be continuous, and (ii) the spin probability current density must be conserved. Notice that we consider the case where the magnetic field is weak and the Zeeman term is negligible. When the Zeeman term is taken into account exactly, the resulting spinors are not orthogonal; this renders the transmission through each junction and the ring uncertain and a phenomenological parameter is required to solve the problem.

The incident wavefunction $\Psi_I$ and the outgoing one $\Psi_{II}$ can be expanded in terms of
FIG. 2: (a) Energy spectrum and (b) the angle between the z axis and the spin of the eigenstates of a 1D ring vs $\alpha$ or $\beta$ in the presence of only a radial SOI field with $E_R = E_R e_r$ (solid curves) and $E_D = E_D e_r$ (dotted curves). The solid curves in panel (a) also describe the energy spectrum of the 1D ring as a function of the combined SOI strength $\sqrt{\alpha^2 + \beta^2}$ for $E_R = E_R e_z$ and $E_D = E_D e_z$. The magnetic field is perpendicular to the plane of the ring and the Zeeman term is neglected. The parameters used are $n + \phi = 4$, $g = 0$, $m^* = 0.023$, $a = 250\text{nm}$, $\gamma_1 = \gamma_2 = \pi/2$, and the strength unit $\alpha_0 = 10^{-11} \text{eV} \text{m}$.

spinors $\chi_{n\sigma}$

$$\Psi_I(x) = \sum_{\sigma} \Psi_I^\sigma(x) = \sum_{\sigma} (A^\sigma e^{ikx} + B^\sigma e^{-ikx}) \chi^\sigma(\pi), \quad x \in [-\infty, 0], \quad (16)$$

$$\Psi_{II}(x') = \sum_{\sigma} \Psi_{II}^\sigma(x') = \sum_{\sigma} (C^\sigma e^{ikx'} + G^\sigma e^{-ikx'}) \chi^\sigma(0), \quad x' \in [0, \infty], \quad (17)$$

see Fig. 1 for the local coordinates $x$ and $x'$. In a similar way the wave functions corresponding to the upper and lower arms of the ring can be written as

$$\Psi_u(\theta) = \sum_{\sigma} \Psi_u^\sigma(\theta) = \sum_{\sigma, \mu} D_\mu e^{in_\mu^\sigma \theta} \chi^\sigma(\theta), \quad \theta \in [0, \pi], \quad (18)$$

$$\Psi_l(\theta) = \sum_{\sigma} \Psi_l^\sigma(\theta) = \sum_{\sigma, \mu} F_\mu e^{in_\mu^\sigma \theta} \chi^\sigma(\theta), \quad \theta \in [\pi, 2\pi]. \quad (19)$$

In the absence of the Zeeman term, energy conservation leads to $n_\mu^\sigma = \mu ka - \phi + \phi_{AC}^\sigma/(2\pi)$ with $\mu = \pm 1$.

Using Griffith’s boundary conditions described above and the spin current operator

$$J^\sigma = \frac{\hbar}{m^*} \text{Re}\{(\Psi^\sigma)^\dagger (-i\partial/\partial \theta + \hat{\sigma}) \Psi^\sigma\} \quad (20)$$
with \( \hat{\sigma} = a[\bar{\alpha}\cos \gamma_1\sigma_r - \bar{\alpha}\sin \gamma_1\sigma_z + \beta(\sin \gamma_2 - \cos \gamma_2)\sigma_\theta]/\hbar + \phi \), we have

\[
\Psi^\sigma_I(0) = \Psi^\sigma_u(\pi) = \Psi^\sigma_I(\pi),
\]

\[
\Psi^\sigma_{II}(0) = \Psi^\sigma_u(0) = \Psi^\sigma_I(2\pi),
\]

\[
-ia \frac{\partial}{\partial x} \Psi^\sigma_I(x)|_{x=0} + (-i \frac{\partial}{\partial \theta} + \hat{\sigma}) \Psi^\sigma_u(\theta)|_{\theta=\pi} - (-i \frac{\partial}{\partial \theta} + \hat{\sigma}) \Psi^\sigma_I(\theta)|_{\theta=\pi} = 0,
\]

\[
-ia \frac{\partial}{\partial x} \Psi^\sigma_{II}(x')|_{x'=0} + (-i \frac{\partial}{\partial \theta} + \hat{\sigma}) \Psi^\sigma_u(\theta)|_{\theta=0} - (-i \frac{\partial}{\partial \theta} + \hat{\sigma}) \Psi^\sigma_I(\theta)|_{\theta=2\pi} = 0.
\]

Then the coefficients of the wave functions in regions I and II are related by

\[
\begin{pmatrix}
A^\sigma \\
B^\sigma
\end{pmatrix} = \frac{1}{2ka} \begin{bmatrix}
k_1 e^{in^1_1 \pi} & k_2 e^{in^2_1 \pi} & n_1^e e^{-in^1_1 \pi} & n_2^e e^{-in^2_1 \pi} \\
k_1 e^{in^1_2 \pi} & k_2 e^{in^2_2 \pi} & -n_1^e e^{-in^1_2 \pi} & -n_2^e e^{-in^2_2 \pi}
\end{bmatrix} \begin{pmatrix}
D^\sigma_1 \\
D^\sigma_2 \\
F^\sigma_1 \\
F^\sigma_2
\end{pmatrix} \tag{25}
\]

and

\[
\hat{P}^\sigma \begin{pmatrix}
D^\sigma_1 \\
D^\sigma_2 \\
F^\sigma_1 \\
F^\sigma_2
\end{pmatrix} = \begin{bmatrix}
0 & 0 \\
1 & 1 \\
1 & 1 \\
-ak & ak
\end{bmatrix} \begin{pmatrix}
C^\sigma \\
G^\sigma
\end{pmatrix} \tag{26}
\]

with

\[
\hat{P}^\sigma = \begin{bmatrix}
e^{in^1_1 \pi} & e^{in^2_1 \pi} & -e^{-in^1_1 \pi} & -e^{-in^2_1 \pi} \\
1 & 1 & 0 & 0 \\
0 & 0 & e^{2\pi n^1_1} & e^{2\pi n^2_1} \\
n_1^e & n_2^e & -n_1^e e^{2\pi n^1_2} & -n_2^e e^{2\pi n^2_2}
\end{bmatrix}, \tag{27}
\]

where \( k_{s\pm} = (ak \pm n^\sigma_s), s = 1, 2 \). Now we can write the transfer matrix \( M^\sigma \) through a single ring in the representation of the eigenspinors of the ring at \( \theta = \pi \) and \( \theta = 0 \) for the incident and outgoing wave function, respectively.

If the Zeeman term is neglected, the transmission amplitude through a single ring, with both terms of the SOI present, takes the same form as that in the absence of the DSOI [13].

\[
t_\sigma = C^\sigma/A^\sigma = \frac{8i \cos((-\phi/2 + \phi^\sigma_{AC}/2) \sin(kap))}{1 - 5 \cos(2kap) + 4 \cos(-\phi + \phi^\sigma_{AC}) + 4i \sin(2kap)} \tag{28}
\]

but with the phase \( \phi^\sigma_{AC} \) modified, cf. Eq. (15), and accounting for \( \beta \) and the angles \( \gamma_1, \gamma_2 \). Here \( \sigma = \pm \) corresponds to the \( \pm \) spinors at \( \theta = \pi \) and \( \theta = 0 \) of the ring for incident and output electrons respectively. At zero temperature, the conductance of the ring is given by

\[
G = \frac{e^2}{h} \sum_\sigma |t_\sigma|^2 = \frac{e^2}{h} \sum_\sigma g_0(k, \Delta^\sigma_{AC})[1 - \cos(\Delta^\sigma_{AC})]. \tag{29}
\]
The dimensionless coefficient $g_0$ is given by

$$g_0(k, \Delta_{AC}^\sigma) = \frac{32 \sin^2(ka\pi)}{[1 - 5 \cos(2ka\pi) - 4 \cos(\Delta_{AC}^\sigma)]^2 + 16 \sin^2(2ka\pi)}$$  \hspace{1cm} (30)$$

and $\Delta_{AC}^\sigma = -\phi + \sigma(\phi_{AC}^+ - \phi_{AC}^-) / 2$.

A series of rings. For single a ring we used different spinors $\chi^\sigma(\pi)$ and $\chi^\sigma(0)$ for the incident and outgoing electrons. This introduces inconveniences to the description of spins because, e.g., the "+ spinor" may represent different spin orientations for the incident and outgoing electrons. Furthermore, if there are more than one rings in the system, in order to take advantage of the single-ring results, a unitary transformation is necessary between the outgoing (from one ring) and the incident (to the next ring) spinor representations. Accordingly, in the following we will work in the representation of the eigenspinors of $\sigma_z$, $(1, 0)^T$ for spin-up states (branch $+$) and $(0, 1)^T$ for spin-down states (branch $-$). Here the superscript $T$ denotes the transpose of a matrix. We use $U_L$ and $U_R$ to denote the unitary transformation matrices between the $\sigma_z$ representation and that for the incident ($\chi^\sigma(\pi)$) and outgoing ($\chi^\sigma(\pi)$) wave functions, respectively. We express the incident wave functions as

$$\Psi_I(x) = \begin{pmatrix} A^+ \\ A^- \end{pmatrix} e^{ikx} + \begin{pmatrix} B^+ \\ B^- \end{pmatrix} e^{-ikx},$$  \hspace{1cm} (31)$$

the outgoing wave function $\Psi_{II}(x)$ is given by the same expression with $A$ and $B$ replaced by $C$ and $G$, respectively. The coefficients $C$ and $G$ are related to $C$ and $G$ in the manner

$$\begin{pmatrix} D^+ \\ D^- \end{pmatrix} = U_R \begin{pmatrix} D^+ \\ D^- \end{pmatrix}, \quad U_R = \begin{bmatrix} \cos(\delta/2) & e^{i\varphi} \sin(\delta/2) \\ \sin(\delta/2) & -e^{i\varphi} \cos(\delta/2) \end{bmatrix},$$  \hspace{1cm} (32)$$

where $D = C, G$. Similarly, the coefficients $A$ and $B$ are related to $A$ and $B$ in the manner

$$\begin{pmatrix} E^+ \\ E^- \end{pmatrix} = U_L \begin{pmatrix} E^+ \\ E^- \end{pmatrix}, \quad U_L = \begin{bmatrix} \cos(\delta/2) & \sin(\delta/2) \\ -e^{-i\varphi} \sin(\delta/2) & e^{-i\varphi} \cos(\delta/2) \end{bmatrix}. $$  \hspace{1cm} (33)$$

with $E = A, B$. Then the transfer matrix of the $i$th ring in the $\sigma_z$ representation becomes

$$\begin{pmatrix} A^+ \\ A^- \\ B^+ \\ B^- \end{pmatrix} = M_i \begin{pmatrix} C^+ \\ C^- \\ G^+ \\ G^- \end{pmatrix}$$  \hspace{1cm} (34)$$
with

$$\mathcal{M}_i = \begin{bmatrix} U_L & 0 \\ 0 & U_L \end{bmatrix} \begin{bmatrix} M_{11}^+ & 0 & M_{12}^+ & 0 \\ 0 & M_{11}^- & 0 & M_{12}^- \\ M_{21}^+ & 0 & M_{22}^+ & 0 \\ 0 & M_{21}^- & 0 & M_{22}^- \end{bmatrix} \begin{bmatrix} U_R & 0 \\ 0 & U_R \end{bmatrix}$$ (35)

and $M^\sigma_{ij}$ the elements of the transfer matrix $M^\sigma$ determined by Eqs. (25)-(27).

For a system of $n$ rings, the matrix $\mathcal{M}_i$ in Eq. (34) should be replaced by the total transfer matrix of the system

$$\mathcal{M}_T = \prod_i \mathcal{M}_i$$ (36)

If no electrons enter the system from the right lead, i.e., if $G^+ = G^- = 0$, the spin transmission and reflection rates can be calculated by relating the coefficients of the transmitted $(C^+, C^-)^T$ and reflected $(B^+, B^-)^T$ wave function to those of the incident wave function $(A^+, A^-)^T$ in Eq. (33). The total transmission is obtained as

$$T = \sum_{\sigma} T^\sigma = \sum_{\sigma} |C^\sigma|^2 / (|A^+|^2 + |A^-|^2)$$ (37)

with

$$\begin{pmatrix} C^+ \\ C^- \end{pmatrix} = \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{bmatrix}^{-1} \begin{pmatrix} A^+ \\ A^- \end{pmatrix}$$ (38)

and $\mathcal{M}_{ij}$ the elements of the total matrix $\mathcal{M}_T$. In the following numerical calculation, we take the incident spinor as $A^+ = A^- = \sqrt{2}/2$ which is oriented along the $x$ direction. The zero-temperature conductance is then evaluated from the transmission of electrons at the Fermi energy as $G = 2e^2T/h$. Note the partial transmissions $T^\sigma$ in Eq. (37) and the partial transmission amplitude $t_\sigma$ in Eq. (28) are defined in different spinor representations but they give the same conductance through a single ring.

### III. RESULTS AND DISCUSSION

#### A. Single ring

In Fig. 3 we show the transmission as function of the SOI strength and of the SOI electric field orientation, for $\beta = 0$ in (a) and for $\alpha = 0$ in (b), in the absence of a magnetic field. In (a) the transmission is symmetric along the angle $\gamma_1$, with respect to the line $\gamma_1 = \pi/2$. 
FIG. 3: Total transmission through a ring vs $\alpha$ and $\gamma_1$ in (a), for $\beta = 0$, and vs $\beta$ and $\gamma_2$ in (b) for $\alpha = 0$. The other parameters are $B = 0$, $a = 25 nm$, $m^* = 0.023$, and $E = 11.3 \text{meV}$.

In (b) on the contrary, the transmission through the ring depends on the sign of the $z$ component of $E_D$. As illustrated in the Hamiltonian (7), the RSOI affects the system via $\sigma_r$ and $\sigma_z$ but the DSOI via only $\sigma_\theta$. As a result, the energy spectrum is symmetric along $\gamma_1$ with respect to $\gamma_1 = \pi/2$ in the absence of the Zeeman term but asymmetric along $\gamma_2$. This explains the different behavior of the transmission as a function of the RSOI parameters $\alpha$, $\gamma_1$ or of the DSOI parameters $\beta$, $\gamma_2$ shown in Fig. 3.

In Fig. 4, the transmission in the $(\alpha - \beta)$ space is shown for $\gamma_1 = \gamma_2 = 0$ in panel (a) and $\gamma_1 = \gamma_2 = \pi/2$ in panel (b). Similar to the energy spectrum, the transmission is a function of $(\alpha^2 + \beta^2)^{1/2}$ and shows a symmetry along the $\alpha$ and $\beta$ axes if both $E_R$ and $E_D$ are along the $z$ direction. This is in line with the unitary equivalence of the RSOI and DSOI in a 2DEG for $\gamma_1 = \gamma_2 = 0$, as discussed in Ref. [21], and can be deduced from Eq. (7). However, this is not the case for $\gamma_1 \neq \gamma_2 \neq 0$. For instance, if both $E_R$ and $E_D$ are along the radial direction, the transmission along the $\alpha$ axis is different than that along the $\beta$ axis. As a result, the curves in the $(\alpha - \beta)$ plane of equal transmission are circles in Fig. 4(a) and ellipses in Fig. 4(b).

In Fig. 5 we show the transmission vs the azimuthal angle $\gamma_3$ of a weak magnetic field $B = 0.01 T$. The transmission oscillates upon varying this angle. When $\gamma_3 = 0$ the total transmissions for $\alpha = \beta = \alpha_0$ and for $\alpha = \sqrt{2} \alpha_0$ and $\beta = 0$ are identical but there is a
FIG. 4: Total transmission through a ring vs $\alpha$ and $\beta$ for $\gamma_1 = \gamma_2 = 0$ in (a) and $\gamma_1 = \gamma_2 = \pi/2$ in (b). The other parameters are $B = 0, a = 25\text{nm}, m^* = 0.023$, and $E = 11.3\text{meV}$.

small difference between the partial transmissions for the two sets of parameters. In a tilted magnetic field ($\gamma_3 \neq 0$), the rotational symmetry of the transmission in the ($\alpha$-$\beta$) space, shown in Fig. 4(a), is broken and there is a difference between the solid and dotted curves in Fig. 5.

B. Multiple rings

For a series of identical rings the transmission gaps become wider and acquire a square-wave character. This is evident in Fig. 6 where results for one and four rings are shown for the case that the SOI exists everywhere in the rings (a) or only in their lower arms (b). When the SOI exists in both arms, it introduces a phase difference between electrons propagating in the upper and lower arms. This is greatly reduced in a system where the SOI exists only in one of the arms. As a result, the oscillation frequency of the transmission, when the SOI strength is varied, is reduced. In Fig. 6 we show results only for the DSOI but similar results are obtained for the RSOI.

For a series of rings, we can also change the ring radius $a$ from one ring to another. This
FIG. 5: Total transmission (a), branch + transmission $T^+$ (b), and branch − transmission $T^-$ (c) through a ring vs $\gamma_3$ under a cone-shaped magnetic field of magnitude 0.01T. The solid curves are for $\alpha = \sqrt{2} \alpha_0$ and the dotted ones for $\alpha = \beta = \alpha_0$. The other parameters are $a = 25nm$, $m^* = 0.023$, $E = 11.3meV$, and $\gamma_1 = \gamma_2 = 0$.

FIG. 6: Total transmission through ring structures vs $\beta$ when the DSOI exists (a) in both arms of the rings or (b) only in their lower arms. The dotted curves show the result for one ring and the solid ones the result for four rings. The other parameters are $a = 25nm$, $m^* = 0.023$, $E = 11.3meV$, $\alpha = 0$, $B = 0$, and $\gamma_1 = 0$.

leads to a significant change in the transmission pattern and results from the modification of the energy levels, when $a$ changes, cf. Eqs. (7) and (10). In Fig. 7 we show results for two rings of the same radius (solid curve) and of different radii (dotted curve). For a series of many rings, the gaps acquire a more pronounced square-type character, see Ref. 26 for more results when $\beta$, $\gamma_1$, and $\gamma_3$ are zero.

If only the RSOI term is present and $\gamma_1$ vanishes, a previous study 26 showed that for
FIG. 7: Transmission through two rings as a function of the strength $\alpha$ for two sets of radii: $a_1 = a_2 = 0.25 \mu m$ (solid curve) and $a_1 = 0.25 \mu m$, $a_2 = 0.15 \mu m$ (dashed curve).

$k a = (2m + 1)/2$, with $m$ integer, and even number of identical rings, the zero-magnetic-field transmission is a discontinuous function of $\alpha$: it takes the highest constant value and vanishes at some special points $\alpha_m$ given by $\alpha_m = (\hbar^2/2ma) \sqrt{4(m + 1)^2 - 1}$. On the other hand, the transmission through an odd number of rings is identical to that for one ring. The analytical explanation relies on the properties of the corresponding transfer-matrix that connects the expansion coefficients in leads I and II [26]. We verify numerically that the same holds in our much more complex situation.

For a ring of radius $a = 0.25 \mu m$ and with only RSOI, the transmission of an even number of rings vanishes at $\alpha = 1.148\alpha_0$, $2.566\alpha_0$, and $3.92\alpha_0$ corresponding to $m=1$, 2, and 3. If only the DSOI is present, the transmission exhibits the same profile if $\alpha$ is replaced by $\beta$. In a system with both DSOI and RSOI present, similar results hold. If we tilt the SOI fields by increasing $\gamma_1$ and $\gamma_2$ from zero, the separation between these points increases. We show that in Fig. where we plot the transmissions (thick dotted curves) as functions of $\beta$ for $N = 1, 8, 9$ rings. The RSOI term is for $\alpha = \alpha_0$ and $\gamma_1 = 0$ and the tilt angle of the DSOI field is $\gamma_2 = \pi/32$. For $N = 8$ the transmission vanishes at $\beta = 0.628\alpha_0$, $2.634\alpha_0$, and $4.225\alpha_0$. The transmission through $N = 8$ rings vanishes at the same points as that for one
FIG. 8: Transmissions for half-integer $ka = 20.5$ through $N = 1, 8, 9$ rings, as functions of the strength $\beta$ without magnetic field (dotted curves) and with magnetic field of magnitude $B = 0.001T$ (solid curves). The parameters are $a = 0.25\mu m$, $E = 11.138\text{meV}$, $\alpha = \alpha_0$, $\gamma_1 = 0$, and $\gamma_2 = \pi/32$.

ring and otherwise is equal to one. The transmission through $N = 9$ rings is identical to that for one ring. The dotted curves in Fig. 8 make these two statements clear if we compare the curves for $N = 1$ and $N = 8$ and separately for $N = 1$ and $N = 9$. This holds only if the magnetic field is zero. If we apply a finite magnetic field to the system, the above result breaks down and the transmission, as shown by the thin solid curves in Fig. 8, oscillates near the points at which it vanishes when the magnetic field is zero.
C. Influence of the Zeeman term

So far we neglected the effect of the Zeeman term on the transmission. The reason is that in the presence of this term the spin orientations of different eigenstates depend on the orbital quantum number $n$ and for electrons of the same energy the spinors are not orthogonal to each other [9]. This renders the transmission unwieldy or very difficult to solve unless one resorts to the treatment of Ref. [9] at the expense of introducing a phenomenological parameter. However, the question arises to what extent its inclusion would modify the previous results. An exact numerical treatment is beyond the scope of this work but it is probably unnecessary for weak magnetic fields $B \leq 1$ T for which its influence can by assessed by treating it as a perturbation and neglecting the correction to the eigenfunctions (10). The resulting eigenvalues read

$$E_{n\sigma} = \hbar \omega_0 (n+\phi)^2 + \hbar \omega_n + \hbar \omega_B \left( \alpha^2 + \beta^2 \right) + \hbar (\omega_B - \omega_n) + \frac{\sigma \omega_n (\alpha^2 + \beta^2)}{(\alpha^2 + \beta^2 + 1/4)^{1/2}}. \quad (39)$$

For a typical SOI strength $\sqrt{\alpha^2 + \beta^2} = \alpha_0$, the ratio between the spin splitting due to the Zeeman term and that due the SOI is $\omega_B \sqrt{\alpha_0^2 + 1/4}/2\alpha_0^2 \omega_n$. Notice that $\omega_n$ is of the same order as the electron energy $E$. For the parameters used in Fig. 9 and $B = 1$ T this ratio is about 2%. In Fig. 8 the transmission without ($g = 0$) and with ($g = 10$) the Zeeman term is shown for perpendicular magnetic fields $B = 0.1$ T in (a) and $B = 1$ T in (b). Upon increasing the weak magnetic field some of the transmission gaps vary due to the change in the A-B phase. As shown though, the Zeeman term has an overall negligible effect. Notice though that for $\alpha$ close to zero the perturbation treatment is not valid despite the agreement between the results without and with the Zeeman term.

IV. CONCLUDING REMARKS

We studied the electron energy spectrum and transmission through mesoscopic rings, in a tilted magnetic field, in the presence of the Rashba (RSOI) and Dresselhaus (DSOI) terms of the spin-orbit interaction due to the confinement along the perpendicular and radial direction. We solved exactly the one-electron Schrödinger equation and obtained the spectrum and the AC phase including the Zeeman term. This was followed by the formulation of
FIG. 9: Transmission through a ring vs the RSOI strength $\alpha$ when (a) a weak magnetic field $B = 0.1$ T and (b) a medium magnetic field $B = 1$ T is applied. The solid curves are the results without the Zeeman term and the dotted ones with it ($g = 10$). The other parameters are $a = 0.25\mu m$, $m^* = 0.023$, $E = 11.3$meV, $\beta = 0$, and $\gamma_1 = 0$.

the transmission problem, using a spin-dependent version of Griffith’s boundary conditions, with the Zeeman term treated by perturbation theory.

We evaluated the electron transmission through one ring or a series of rings as a function of the SOI strengths $\alpha$, $\beta$, the orientations $\gamma_1$ and $\gamma_2$ of the corresponding fields, and the orientation $\gamma_3$ of the magnetic field for various parameters. As all figures demonstrate, the transmission shows a rich nontrivial structure with well-pronounced gaps as a function of any of these variables. For a series of rings these gaps acquire a square-wave shape, cf. Fig. 6, similar to that reported for $\beta = 0$ and $\gamma_1 = \gamma_3 = 0$ [26]. We also studied the case with the SOI present only in one arm of the ring(s) and saw how the oscillation pattern in the transmission changes due to the changes in the AC phase. If we change the parameters from one ring to another in a series of rings, we can further modulate the transmission versus the SOI strength, cf. Fig. 7 where the radius is changed.

A particular case of interest is that of the transmission, at zero magnetic field, as a function of the SOI strength when the incident energy is such that $ka$ is a half integer. As
elaborated at the end of Sec. III B and shown in Fig. 8, the transmission is identical for any odd number of rings. If the number of rings is even, the transmission vanishes at the same points as that for one ring and otherwise takes the highest constant value. A simple explanation holds for $\beta = 0$ and $\gamma_1 = 0$ [20]. We confirmed numerically that this holds in our much more complicated case. As shown though in Fig. 8, this breaks down when a small magnetic field is present.

For an incident electron initially spin-oriented along the direction of propagation ($x$), the influence on the transmission of the RSOI and DSOI terms, with effective electric field along the $z$ direction, is identical with that when the magnetic field is perpendicular to the ring. Otherwise, the RSOI and DSOI affect the transmission in different ways.

For weak magnetic fields $B \leq 1$ T and realistic values of the SOI strength, the $g$ factor, and the effective mass, we showed the Zeeman term can be treated by perturbation theory and has negligible effect on the transmission.

V. ACKNOWLEDGEMENT

This work was supported by the Canadian NSERC Grant No. OGP0121756.

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