BRAIDED STRUCTURE OF FRACTIONAL
$Z_3$-SUPERSYMMETRY

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It is shown that fractional $Z_3$-superspace is isomorphic to the $q \to \exp(2\pi i/3)$ limit of the
braided line. $Z_3$-supersymmetry is identified as translational invariance along this line.
The fractional translation generator and its associated covariant derivative emerge as the
$q \to \exp(2\pi i/3)$ limits of the left and right derivatives from the calculus on the braided
line.

1 Brackets and $q$-grading

Our aim here is to reformulate some results of a previous paper [1], where the
structure of fractional supersymmetry was investigated from a group theoretical
point of view, from a braided Hopf algebra approach. We shall not be concerned
here with the possible applications of fractional supersymmetry and will refer instead to [2,3] for references on this aspect.

We begin by defining the bracket

$$[A, B]_{q^r} := AB - q^r BA,$$  \hspace{1cm} (1.1)

where $q$ and $r$ are just arbitrary complex numbers. If we assign an integer grading
$g(X)$ to each element $X$ of some algebra, such that $g(1)=0$ and $g(XY) = g(X) + g(Y)$,
for any $X$ and $Y$, we can define a bilinear graded $z$-bracket as follows,

$$[A, B]_z = AB - q^{g(A)g(B)} BA , \quad z = q^{-g(A)g(B)} .$$  \hspace{1cm} (1.2)

Here $A$ and $B$ are elements of the algebra, and of pure grade. The definition may
be extended to mixed grade terms using the bilinearity. We also have

$$[r]_{q} := \frac{1 - q^r}{1 - q} , \quad [r]_q! := [r]_q[r-1]_q[r-2]_q...[2]_q[1]_q ,$$  \hspace{1cm} (1.3)

supplemented by $[0]_q! = 1$. When $q$ is $n$-root of unity the previous grading scheme
becomes degenerate, so that in effect the grading of an element is only defined
modulo $n$. In this case also have $[r]_{q} = 0$ when $r$ modulo $n$ is zero ($r \neq 0$).
2 $q$-calculus and the braided line

Consider the braided line $[3, 4]$, a simple deformation of the ordinary line characterized by a single parameter $q$. Our braided line Hopf algebra will consist of a single variable $\theta$, of grade 1, upon which no additional conditions are placed for generic $q$, by which we mean that $q$ is not a root of unity.

The braided Hopf structure of this deformed line is as follows. It has braided a coproduct,

$$\Delta \theta = \theta \otimes 1 + 1 \otimes \theta \; , \tag{2.1}$$

$$\theta \otimes (\theta \otimes 1) = q\theta \otimes \theta \; , \quad (\theta \otimes 1)(1 \otimes \theta) = \theta \otimes \theta \; , \tag{2.2}$$

the braiding being given by $B(\theta \otimes \theta) = q\theta \otimes \theta$. There are also a counit and antipode,

$$\varepsilon(\theta) = 0 \; , \quad S(\theta^r) = q^{\frac{r(r-1)}{2}}(-\theta)^r \; , \tag{2.3}$$

which satisfy all the usual Hopf algebraic relations, as long as the braiding is remembered. From the braided Hopf algebra perspective, the coproduct generates a shift along the braided line. To bring this out more clearly we use a group-like notation for the coproduct and write

$$\theta = 1 \otimes \theta \; , \quad \delta \theta = \epsilon = \theta \otimes 1 \; , \tag{2.4}$$

so that (2.3) leads to

$$[\epsilon, \theta]_{q^{-1}} = 0 \; . \tag{2.5}$$

In this form, the coproduct (2.1) expresses the additive group law,

$$\Delta \theta = \epsilon + \theta \; \tag{2.6}$$

which on the fractional Grassmann variable $\theta$ corresponds [1] to the action of the left translation $L_\epsilon$, $L_\epsilon \theta \equiv \theta' = \epsilon + \theta$.

The above expressions provide a basis upon which to construct a differential (and integral [2]) calculus on the braided line. We can introduce an algebraic left differentiation operator $D_L$, in analogy with the undeformed case, via the requirement $[\epsilon D_L, \theta] = \epsilon$, which implies that

$$[D_L, \theta] = 1 \; , \quad \frac{d}{d\theta} \theta = 1 \; . \tag{2.7}$$

This corresponds to defining [cf. (1.2)] the left derivative $D_L$ by

$$[D_L, \theta]_z := 1 \; , \quad z = q \; . \tag{2.8}$$

Regarding (2.6) as the definition the left translation by $\epsilon$, and identifying $D \equiv D_L$, we can go on considering right shifts $R_\eta : \theta \mapsto \theta + \eta$ of parameter $\eta$ where $[\theta, \eta]_{q^{-1}} =
Braided structure of fractional $Z_3$-supersymmetry

$[\eta, \theta]_q = 0$. Reasonings similar to the above lead us to a right derivative operator $\mathcal{D}_R$, which satisfies

$$[\theta, \mathcal{D}_R]_z = [\theta, \mathcal{D}_R]_q := 1 \quad , \quad ([\mathcal{D}_R, \theta]_{q^{-1}} = -q^{-1}) \quad . \quad (2.9)$$

It may be shown that the left and right derivative operators are in general related by

$$\mathcal{D}_R = -q^{-(1+N)} \mathcal{D}_L \quad , \quad (2.10)$$

where $N$ is a number-like operator satisfying,

$$[N, \theta] = \theta \quad , \quad [N, \mathcal{D}_L] = -\mathcal{D}_L \quad , \quad (2.11)$$

and consequently $[N, \mathcal{D}_R] = -\mathcal{D}_R$. This implies that $[\mathcal{D}_L, \mathcal{D}_R]_{q^{-1}} = 0$ or, alternatively, $[\epsilon \mathcal{D}_L, \eta \mathcal{D}_R] = 0$ (commutation of the left $L_\epsilon$ and right $R_\eta$ shifts).

Let $f(\theta)$ be a function of $\theta$ defined by the positive power series expansion,

$$f(\theta) = \sum_{m=0}^{\infty} C_m \theta^m [m]_q ! \quad , \quad (2.12)$$

where the $C_m$ are ordinary numbers. The derivative of $f(\theta)$ is generated by the graded bracket (2.7) as follows,

$$\frac{d}{d\theta} f(\theta) := [\mathcal{D}, f(\theta)]_z = \sum_{m=0}^{\infty} C_m \left[ \mathcal{D}, \frac{\theta^m}{[m]_q !} \right]_z = \sum_{m=0}^{\infty} C_m \left[ \mathcal{D}, \frac{\theta^m}{[m]_q !} \right]_{q^m} \quad (2.13)$$

This clearly reduces to the calculus on the undeformed line when $q = 1$. The left translation i.e., the coproduct (2.6) is given by a deformed exponentiation (see (see 3, 4, 5)) of $\epsilon \mathcal{D}_L$. We shall refer to the differential calculus defined by (2.7), (2.9) and (2.10) as $q$-calculus.

3 $q$-calculus in the $q$ root of unity limit and ($Z_3$) fractional supersymmetry

When $q^m = 1$, $[m]_q = 0 \ (m \neq 1)$ and expressions such as $\frac{\theta^m}{[m]_q !}$ can be made meaningful only by assuming that $\theta^m$ is also zero. In that case, we may identify the limit $\lim_{q \to \exp(2\pi i/m)} \frac{\theta^m}{[m]_q !}$ with a degree zero (‘bosonic’) variable $t$. It was shown in [8] that for the $m = 2$ case this procedure leads to a braided interpretation of supersymmetry, the $Z_2$-graded group structure of which was discussed in [1]. We shall consider here in detail the case of $Z_3$ fractional supersymmetry, the group analysis of which may be found in [4] (the general $Z_n$ case will be discussed in [5]).
The limit relevant here is the \( q \to \exp(2\pi i/3) \) limit. To take this limit we note that for \( q \) not a root of unity we have in general the relationships

\[
\left[ D_L, \frac{\theta^m}{[m]_q!} \right] = \left[ D_L, \frac{\theta^m}{[m]_q!} \right]_{q=q_0} = \frac{\theta^{m-1}}{[m-1]_q!} = \left[ \frac{\theta^m}{[m]_q!}, D_R \right]_{q=q_0} = \left[ \frac{\theta^m}{[m]_q!}, D_R \right]_{q=q_0} .
\]

In taking the \( q \to \exp(2\pi i/3) \) limit of the above formulae we encounter difficulties when \( m = 3 \) since \([3]_q = 0\). But it is possible to retain \([3]_q \) by requiring that the \( q \to \exp(2\pi i/3) \) limit of \( \frac{\theta^3}{[3]_q!} \) be finite and nonzero. This in turn requires \( \theta^3 \to 0 \) as \( q \to \exp(2\pi i/3) \). This is preserved under the left shift \( \theta \to \epsilon + \theta \), since \((\epsilon + \theta)^3 = 0 \) follows from \((2.5) \) when \( q \to \exp(2\pi i/3) \) provided that \( \theta^3 = 0 = \epsilon^3 \). We now note that under complex conjugation we have \([3]_q! = q^{-3}[3]_q! \) (along the circle of radius 1). Then, in the \( q \to \exp(2\pi i/3) \) limit the \( q \)-factorial \([3]_{\exp(2\pi i/3)}! \) is real. As a result, we define

\[
t := -\lim_{q \to \exp(2\pi i/3)} \frac{\theta^3}{[3]_q!} ,
\]

where the - sign is introduced to compare easily with \([1] \). Since \( \theta \) is assumed real, \( t \) will also be real. By using the identities

\[
\lim_{q \to \exp(2\pi i/3)} \frac{[3r]_q}{[3]_q!} = \lim_{q \to \exp(2\pi i/3)} \left( \frac{1 - q^{3r}}{1 - q^3} \right) = r ,
\]

\[
\lim_{q \to \exp(2\pi i/3)} \frac{[3r + 1]_q}{[1]_q!} = \lim_{q \to \exp(2\pi i/3)} \left( \frac{1 - q^{3r+1}}{1 - q} \right) = 1 ,
\]

and

\[
\lim_{q \to \exp(2\pi i/3)} \frac{[3r + 2]_q}{[2]_q!} = \lim_{q \to \exp(2\pi i/3)} \left( \frac{1 - q^{3r+2}}{1 - q^2} \right) = 1 ,
\]

we have, for \( p = 0, 1, 2 \),

\[
\lim_{q \to \exp(2\pi i/3)} \frac{\theta^{3r+p}}{[3r+p]_q!} = \frac{\theta^p}{[p]_{\exp(2\pi i/3)}!} .
\]

As mentioned, the finite limit in \([3.2] \) denoted by \( t \) was introduced in order to make possible the \( q \to \exp(2\pi i/3) \) limit of \([3.1] \) at \( m = 3 \). Similar problems arise for all \( m \geq 3 \) \([3] \), and the importance of \([3.6] \) is that it shows that these can also be handled in terms of \( t \). Thus, any function \( f(\theta) \) on the braided line at generic \( q \) leads in the \( q \to \exp(2\pi i/3) \) limit to a function of the form \( f(t, \theta) \) (or ‘fractional superfield’ on fractional superspace \((t, \theta)\)). To investigate further the properties of \( t \), and to see how it fits into our \( q \)-calculus, let us now consider

\[
\left[ D_L, \left[ D_L, \frac{\theta^3}{[3]_q!} \right]_{q=q_0} \right] = \left[ \frac{\theta^3}{[3]_q!}, D_R \right]_{q=q_0} = \left[ \frac{\theta^3}{[3]_q!}, D_R \right]_{q=q_0} = \left[ \frac{\theta^3}{[3]_q!}, D_R \right]_{q=q_0} .
\]

4
Braided structure of fractional $Z_3$-supersymmetry

(notice the appropriate $q$-factor in each bracket depending on the grading of its components, cf. (3.1), (1.2)) valid for all $q \neq \exp(2\pi i/3)$. Taking the $q \to \exp(2\pi i/3)$ limit we see that (3.7) reduces to $[D^3_L, t] = -1 = [t, D^3_R]$, so that by identifying

$$\partial_t = -D^3_L = D^3_R \ ,$$

we have

$$[\partial_t, t] = 1 \ ,$$

which is just the defining relation of the algebra associated with ordinary calculus.

Let us consider the left calculus. Using $\partial_t$ given by (3.8) to induce differentiation with respect to $t$, the full $q$ calculus for $q = \exp(2\pi i/3)$ obtained from (2.13) and (3.2) is given by,

$$\frac{d}{d\theta} \theta = [D_L, \theta]_{\exp(2\pi i/3)} = 1 \ ,$$

$$\frac{d}{dt} t = [D_L, t] = \lim_{q \to \exp(2\pi i/3)} [D_L, \frac{\theta^3}{[3]_q}]_{q^a} = -\frac{\theta^2}{[2]_{\exp(2\pi i/3)}} = \exp(2\pi i/3)\theta^2 \ ,$$

$$\frac{\partial}{\partial t} t = [\partial_t, t] = 1 \ , \quad \frac{\partial}{\partial t} \theta = [\partial_t, \theta] = -[D^3_L, \theta] = -[D_L, [D_L, \theta]_{q^a}]_{q^{-1}} = 0 \ .$$

Since $\frac{\partial}{\partial \theta} \theta = 0$ and $\frac{d}{d\theta} t \neq 0$, we can only avoid a contradiction by interpreting $\frac{\partial}{\partial \theta}$ as a partial derivative, and $\frac{d}{d\theta}$ as a total derivative, a result which we implicitly took into account when choosing our notation. We can also define partial differentiation with respect to $\theta$. We do this as follows

$$\frac{\partial}{\partial \theta} \theta := [\partial_\theta, \theta]_{\exp(2\pi i/3)} = 1 \ , \quad \frac{\partial}{\partial \theta} t := [\partial_\theta, t] = 0 \ .$$

Using this definition we are able to perform a chain rule expansion of the total derivative, so that

$$\frac{d}{d\theta} = \frac{d}{d\theta} \frac{\partial}{\partial \theta} + \frac{dt}{d\theta} \frac{\partial}{\partial t} = \frac{\partial}{\partial \theta} - \frac{\theta^2}{[2]_{\exp(2\pi i/3)}} \frac{\partial}{\partial t} = \frac{\partial}{\partial \theta} + \exp(2\pi i/3)\theta^2 \frac{\partial}{\partial t} \ .$$

By substituting (3.12) into the definition (3.8) we obtain an additional but expected condition,

$$\frac{\partial^3}{\partial \theta^3} = 0 \ .$$

This can all be put into the algebraic form $D_L = \partial_\theta + \exp(2\pi i/3)\theta^2 \partial_t$, where

$$[\partial_\theta, \theta]_{\exp(2\pi i/3)} = 1 \quad ([\theta, \partial_\theta]_{\exp(-2\pi i/3)} = -\exp(-2\pi i/3))$$

and $\partial^3_\theta = 0$. 

5
The right calculus may be introduced similarly. Besides \( \partial_t = D_R^2 \), we have

\[
\frac{d_{Rt}}{dR} = [\theta, D_R]_{\exp(2\pi i/3)} = 1, \quad \frac{d_{Rt}}{dR} t = -\lim_{q \to \exp(2\pi i/3)} \left[ \frac{\theta^3}{[3]^q}; D_R \right]_{q^3} = \exp(2\pi i/3)\theta^2, \\
\frac{\partial}{\partial t} t = [\partial_t, t] = 1, \quad \frac{\partial}{\partial t} \theta = [D_R^2, \theta] = 0.
\]  

Introducing a partial right derivative \( \delta_{\theta} \) by

\[
[\theta, \delta_{\theta}]_{\exp(2\pi i/3)} := 1 \quad ([\delta_{\theta}, \theta]_{\exp(-2\pi i/3)} = -\exp(-2\pi i/3)) ,
\]

the expression of \( D_R \) differs by a sign from that of \( D \equiv D_L \), namely

\[
\frac{d_{Rt}}{dR} = \delta_{\theta} + \frac{\theta^2}{[2]}_{\exp(2\pi i/3)} \frac{\partial}{\partial t} = \delta_{\theta} - \exp(2\pi i/3)\theta^2 \frac{\partial}{\partial t} , \quad D_R = \delta_{\theta} - \exp(2\pi i/3)\theta^2 \partial_t.
\]

The \( \delta_{\theta} \) introduced above differs from that found in Ref. \([1]\) in a \(-\exp(2\pi i/3)\) factor, \( \delta_{\theta}(\text{here}) = -\exp(-2\pi i/3)\delta_{\theta}(\text{ref. \([1]\)}) \).

\[\text{(3.15)}\]

4. \( Z_3 \)-fractional supersymmetry from a braided point of view

If follows from the above that \( D_L = Q \) and \(- \exp(2\pi i/3)D_R = D \) are just the supercharge and the corresponding covariant derivative encountered in \((Z_3)\) fractional supersymmetry \([1]\). Hence

\[
Q^3 = -\partial_t , \quad D^3 = -\partial_t.
\]

\[\text{(4.1)}\]

We may identify \( Q \) and \( D \) as, respectively, the generators of left and right shifts along the braided line at \( q = \exp(2\pi i/3) \). These were shown in Ref. \([1]\) to correspond to the right- \([Q]\) and left-invariant \([D]\) ‘fractional translation’ generators.

To further investigate this point of view we examine the Hopf structure on the braided line in the \( q \to \exp(2\pi i/3) \) limit. When \( q = \exp(2\pi i/3) \), \((2.2)\) reduces to

\[
(1 \otimes \theta)(\theta \otimes 1) = \exp(2\pi i/3)\theta \otimes \theta , \quad (\theta \otimes 1)(1 \otimes \theta) = \theta \otimes \theta ,
\]

so that from \((2.1)\), we find

\[
\Delta \theta^3 = \theta^3 \otimes 1 + 1 \otimes \theta^3 + (1 + \exp(2\pi i/3) + [\exp(2\pi i/3)]^2)(\theta \otimes \theta^2 + \theta^2 \otimes \theta) = 0.
\]

\[\text{(4.2)}\]

as required by the homomorphism property of the coproduct. The counit and antipode take the following form

\[
\varepsilon(\theta) = 0 , \quad S(\theta) = -\theta .
\]

\[\text{(4.4)}\]

The braided structure \((2.1),(4.2)\) is the standard one, see for example Ref. \([1,2]\) for a discussion of super and anyonic quantum groups. The new structure appears when
the variable $t$ defined by the limit (3.2) is introduced [8, 2]. From the definition (3.2) it follows that $\theta$ and $t$ commute. From (3.2) and (2.1)-(2.2) we compute
\[
\lim_{q \to \exp(2\pi i/3)} \Delta \frac{\theta^3}{[3]_q^3} \cdot q!
\]
using that $[n]_q = 1 + q + \ldots + q^{n-1}$ for generic $q$. This, together with $[t, \theta] = 0$ and (2.1)-(2.3), shows that the algebra generated by $(t, \theta)$ has a braided Hopf algebra structure, with
\[
\Delta t = t \otimes 1 + 1 \otimes t + \exp(2\pi i/3)(\theta \otimes \theta^2 + \theta^2 \otimes \theta) \quad , \quad \epsilon(t) = 0 \quad , \quad S(t) = -t \quad , \quad (4.5)
\]
(for instance $\Delta([t, \theta]) = 0 = [\Delta t, \Delta \theta]$). This means that although $t$ and $\partial_t$ satisfy the algebra associated with ordinary (undeformed) calculus, $t$ has non primitive coproduct: the coaddition no longer corresponds to a time translation. Considered along with the chain rule expansion of the $q$-calculus derivative (3.12), it is clear that in the $q \to \exp(2\pi i/3)$ limit we cannot decompose the $q$-calculus algebra into unrelated $t$ and $\theta$ parts. Indeed, from (4.5) we see that when the braided group is considered, the appearance of $\theta$ in the coproduct of $t$ means that no such decomposition can be performed. The fact that we cannot regard this braided Hopf algebra as a product entity is an essential feature of fractional supersymmetry in general. To see this for the present $Z_3$-case we rewrite the coproducts of $\theta$ and $t$ using the notation (2.4). Using (2.1), (4.3) and definition (3.1) we obtain
\[
\theta \to \Delta \theta = \epsilon + \theta \quad , \quad t \to \Delta t = t + \tau + q(\epsilon \theta^2 + \epsilon^2 \theta) \quad , \quad (4.6)
\]
where now $q \equiv \exp(2\pi i/3)$ (so that $-1/[2]_q = q$) and
\[
\tau = \lim_{q \to \exp(2\pi i/3)} = -\frac{\epsilon^3}{[3]_q^4} \quad (4.7)
\]
is a time translation independent of $t$. This is just the form of the finite $Z_3$-supersymmetry transformation of [1]; the transformation of $t$ follows from that of $\theta$ via the relationship (3.2) and the coproduct $\Delta$.

5 Final remarks

To conclude, let us summarize our results [8, 2] and outline the new point of view which they provide. For generic $q$ the braided line described in section 3 is well defined. In the $q \to \exp(2\pi i/3)$ limit the nilpotency of $\theta$ prevents us from having a complete description of the braided line and its associated differential calculus. A convenient way in which we can obtain such a description is to introduce an additional variable $t$, defined as in (3.2). From (3.3) this is seen to carry a structure which, for generic $q$, is related to $\theta^3$ and higher powers of $\theta$. So in the $q \to \exp(2\pi i/3)$ limit the braided line is made up of the two variables $\theta$ and $t$, which span the one-dimensional fractional superspace. Furthermore, under a shift along this braided line $\theta$ and $t$ transform exactly as in $Z_3$ fractional supersymmetry. Thus we are able to identify $Z_3$-superspace with the $q = \exp(2\pi i/3)$ limit of the braided
line, and \(Z_3\)-supersymmetry as translational invariance along this line. Clearly, (fractional) superspace cannot be regarded as the tensor product of independent \(t\) and \(\theta\) parts; it is instead a single braided geometric entity. This provides a braided interpretation of the central extension character of the \(Z_3\)-graded group aspect of fractional supersymmetry discussed in \cite{1}. To conclude, we wish to stress that the above results are not restricted to the \(Z_3\) case. Similar results also hold for supersymmetry \cite{8} (cf. \cite{9}) and in the \(Z_n\) case \cite{2}.

Acknowledgements

This paper describes research supported in part by E.P.S.R.C and P.P.A.R.C. (UK) and by the C.I.C.Y.T (Spain). J.C.P.B. wishes to acknowledge an FPI grant from the CSIC and the Spanish Ministry of Education and Science.

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