THE PARTIAL UNIFORM ELLIPTICITY AND PRESCRIBED PROBLEMS ON THE CONFORMAL CLASSES OF COMPLETE METRICS

RIRONG YUAN

ABSTRACT. We clarify how close a second order fully nonlinear equation can come to uniform ellipticity, through counting large eigenvalues of the linearized operator. This suggests an effective and novel way to understand the structure of fully nonlinear equations of elliptic and parabolic type. As applications, we solve a fully nonlinear version of the Loewner-Nirenberg problem and a noncompact complete version of fully nonlinear Yamabe problem. Our method is delicate as shown by a topological obstruction.

1. Introduction

1.1. Partial uniform ellipticity. In their pioneering paper [8], Caffarelli-Nirenberg-Spruck initiated the study of a fully nonlinear equation

\[ F(D^2u) = \psi \text{ in } \Omega \subset \mathbb{R}^n. \]  

This equation has the form \( F(D^2u) = f(\lambda(D^2u)) \), where \( \lambda(D^2u) = (\lambda_1, \cdots, \lambda_n) \) are the eigenvalues of \( D^2u \). As suggested by Caffarelli-Nirenberg-Spruck, \( f \) is a smooth symmetric function of \( n \) real variables defined in an open symmetric convex cone \( \Gamma \subset \mathbb{R}^n \), with vertex at the origin and boundary \( \partial \Gamma \neq \emptyset \), containing the positive cone \( \Gamma^+ \). In addition, \( f \) satisfies some fundamental structural conditions including

\[ \begin{align*}
  f & \text{ is concave in } \Gamma, \\
  f_i(\lambda) := \frac{\partial f}{\partial \lambda_i}(\lambda) & > 0 \text{ in } \Gamma, \quad \forall 1 \leq i \leq n. 
\end{align*} \]  

In some cases, one may replace (1.3) by the following weaker condition

\[ f_i(\lambda) \geq 0, \quad \forall 1 \leq i \leq n, \quad \sum_{i=1}^n f_i(\lambda) > 0, \text{ in } \Gamma. \]  

This includes among others Poisson equation and Monge-Ampère equation, on which there are substantial literature. Other typical examples are as follows:

\[ f = (\sigma_k/\sigma_l)^{1/(k-l)}, \quad \Gamma = \Gamma_k, \quad \text{or} \quad f = (\sigma_k/\sigma_l)^{1/(k-l)} \circ P_{n-1}, \quad \Gamma = P_k \]

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for $0 \leq l < k \leq n$, where $\sigma_k$ is the $k$-th elementary symmetric function, $\Gamma_k$ is the $k$-th Gårding cone, and $P_{n-1}(\lambda) = \sum_{i=1}^n \lambda_i \mathbf{1} - \lambda$, $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^n$, $\mathcal{P}_k = \{ \lambda : P_{n-1}(\lambda) \in \Gamma_k \}$.

We first look at the structure of equation (1.1). The eigenvalues of coefficient matrix of the linearized operator at $u$ are precisely given by $f_1(\lambda), \cdots, f_n(\lambda)$, for $\lambda = \lambda(D^2 u)$.

As a consequence, (1.3) yields that (1.1) is elliptic at $u \in C^2(\bar{\Omega})$ satisfying $\lambda(D^2 u) \in \Gamma$, while (1.2) implies that the operator $F(D^2 u) = f(\lambda(D^2 u))$ is concave with respect to $D^2 u$ when $\lambda(D^2 u) \in \Gamma$, thereby Evans-Krylov theorem \cite{15,36} applies once the bound of real Hessian has been proved.

As is well known, the uniform ellipticity plays central roles in the study of second order elliptic equations, especially in the theory of a priori estimates, see e.g. \cite{19}. However, the uniform ellipticity of (1.1) breaks down in general, thereby causing various serious difficulties. Consequently, it is of importance to determine when a fully nonlinear equation becomes uniformly elliptic.

This leads to the concept of partial uniform ellipticity.

**Definition 1.1.** We say that $f$ is of $m$-uniform ellipticity in $\Gamma$, if $f$ satisfies (1.4) and there is a uniform positive constant $\vartheta$ such that for any $\lambda \in \Gamma$ with $\lambda_1 \leq \cdots \leq \lambda_n$,

$$f_i(\lambda) \geq \vartheta \sum_{j=1}^n f_j(\lambda), \ \forall 1 \leq i \leq m. \quad (1.5)$$

In particular, $n$-uniform ellipticity is also called fully uniform ellipticity.

Accordingly, we have a similar notion of partial uniform ellipticity for a second order equation, if its linearized operator satisfies a similar condition.

There naturally arises a problem:

**Problem 1.2.** To determine the integer $m$ from (1.5) in Definition 1.1.

The first part of this paper is devoted to this problem. For any $f$ satisfying

$$\lim_{t \to +\infty} f(t \lambda) > f(\mu) \text{ for any } \lambda, \mu \in \Gamma, \quad (1.6)$$

we observe that a criterion, extending the one proposed initially in \cite{62} leads to bridging partial uniform ellipticity and the maximum count of zero components of vectors contained in $\Gamma$. To this end, we introduce an integer for $\Gamma$.

**Definition 1.3.** For the cone $\Gamma$, we define

$$\kappa_\Gamma = \max \left\{ k : (0, \cdots, 0, 1, \cdots, 1) \in \Gamma \right\}.$$ 

Below we state our main result on the partial uniform ellipticity.

**Theorem 1.4.** Suppose $(f, \Gamma)$ satisfies (1.2) and (1.6). Then we have (1.4) and that there exists a universal positive constant $\vartheta_\Gamma$ depending only on $\Gamma$, such that for any
\[ \lambda \in \Gamma \text{ with } \lambda_1 \leq \cdots \leq \lambda_n, \]
\[ f_i(\lambda) \geq \vartheta_\Gamma \sum_{j=1}^n f_j(\lambda), \quad \forall 1 \leq i \leq \kappa_\Gamma + 1. \]

**In particular, \( f \) is of fully uniform ellipticity in the type 2 cone.**

**Remark 1.5.** A choice of \( \vartheta_\Gamma \) can be found in Remark 2.7. In Proposition 2.5 we prove \( \kappa_\Gamma \) is in effect the maximum count of negative components of vectors in \( \Gamma \).

This theorem asserts that \( f \) is of \((\kappa_\Gamma + 1)\)-uniform ellipticity in \( \Gamma \), which is in effect sharp as shown in Corollary 2.10 below. This is one of the main contributions we make in this paper. The theorem applied to the case \( (f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k) \) gives back [40, Theorem 1.1] of Lin-Trudinger with a different method, essentially using specific properties of \( \sigma_k^{1/k} \) which cannot be adapted to general functions.

Theorem 1.4 suggests an effective and novel way to understand the structure of fully nonlinear equations of elliptic and parabolic type. As a by-product, we can confirm the following inequality in general context
\[ f_i(\lambda) \geq \vartheta_\Gamma \sum_{j=1}^n f_j(\lambda) \text{ if } \lambda_i \leq 0. \]

This inequality was imposed as a key assumption by Li [39], subsequently by Trudinger [58], Guan-Spruck [25] and Sheng-Urbas-Wang [53] to study W eingarten equations. Also, it can be applied to various geometric PDEs from conformal geometry and real Hessian type fully nonlinear equations, see e.g. [10, 11, 59, 23].

Another remarkable consequence is that a type 2 cone (see Definition 3.1) guarantees the uniform ellipticity of the corresponding equations. This allows us to study a class of fully nonlinear equations, with applications to a fully nonlinear version of Loewner-Nirenberg problem and a complete noncompact version of fully nonlinear Yamabe problem.

### 1.2. Fully nonlinear equations on Riemannian manifolds.

Let \( (M, g) \) be a connected Riemannian manifold of dimension \( n \) with Levi-Civita connection \( \nabla \). Let \( \partial M \) denote the boundary, and \( \bar{M} = M \cup \partial M \). Notice \( \bar{M} = M \) if \( \partial M = \emptyset \).

We denote \( \Delta u, \nabla^2 u \) and \( \nabla u \) the Laplacian, Hessian and gradient of \( u \) with respect to \( g \), respectively. For \( U \) a symmetric \((0,2)\)-tensor, \( \lambda(g^{-1}U) \) denote the eigenvalues of \( U \) with respect to \( g \), and the meaning of \( \lambda(\tilde{g}^{-1}U) \) is obvious.

We apply partial uniform ellipticity to study a fully nonlinear equation
\[ f(\lambda(g^{-1}(\Delta u \cdot g - \varrho \nabla^2 u + A(x, \nabla u)))), \quad \psi(x, u), \]
with a constant \( \varrho \) subject to
\[ \varrho < \frac{1}{1 - \kappa_\Gamma \vartheta_\Gamma} \text{ and } \varrho \neq 0, \]
where the right-hand side is a smooth positive function such that
\[ \psi(x, z) > 0, \quad \lim_{z \to -\infty} \psi(x, z) = 0, \quad \psi(x, z) \geq h(x) e^{\alpha(x)z}, \quad (x, z) \in \bar{M} \times \mathbb{R} \]
holds with positive continuous functions $h$ and $a$. In addition,

$$\sup_{\Gamma} f = +\infty,$$

(1.10) 

$$f > 0 \text{ in } \Gamma, \quad f = 0 \text{ on } \partial \Gamma,$$

(1.11) 

and $A(x, p)$ is a smooth symmetric $(0, 2)$-tensor satisfying for any $(x, p) \in T\tilde{M}$,

$$\lim_{|p|\to +\infty} \left( \frac{|A(x, p)|}{|p|^2} + \frac{|D_p A(x, p)|}{|p|} \right) \leq \eta(x), \quad \lim_{|p|\to +\infty} \frac{\nabla A(x, p)}{|p|^3} = 0,$$

(1.12) 

where $\eta$ is a positive continuous function, $\nabla A$ denotes partial covariant derivative of $A$, the meaning of $D_p A$ is obvious when viewed as depending on $x \in \tilde{M}$.

The equations of this form closely connect with prescribed curvature equations from conformal geometry. An analogue of equation (1.7) with $\varphi = 1$ is also called a $(n - 1)$ type fully nonlinear equation. In recent years, $(n - 1)$ type Monge-Ampère equation has been studied extensively in complex geometry, due to its close connection to the Calabi-Yau theorem for Gauduchon and balanced metrics [17, 56].

First, we recall the concept of admissible and maximal solution.

**Definition 1.6.** We say $u$ is an admissible function for the equation (1.7) if

$$\lambda(g^{-1}(\Delta u - g\nabla^2 u + A(x, \nabla u))) \in \Gamma$$

pointwisely. Meanwhile, we say $u$ is a maximal solution of an equation, if $u \geq w$ in $M$ for any admissible solution $w$ to the same equation.

Similarly, we have notions of admissible and maximal conformal metrics.

Below, we solve the Dirichlet problem with infinite boundary value condition.

**Theorem 1.7.** Let $(M, g)$ be a compact connected Riemannian manifold of dimension $n \geq 3$ with smooth boundary. Suppose, in addition to (1.2), (1.8), (1.9), (1.10), (1.11) and (1.12), that there is a $C^2$-admissible function on $\tilde{M}$. Then the equation (1.7) admits a smooth admissible solution $u$ with \( \lim_{x \to \partial \tilde{M}} u(x) = +\infty. \)

When the background manifold is complete and noncompact, we can solve the equation, given an asymptotic condition at infinity which is indeed sharp

$$f(\lambda(g^{-1}(\Delta u - g\nabla^2 u + A(x, \nabla u)))) \geq \psi(x, u), \quad \forall x \in M.$$

(1.13) 

**Theorem 1.8.** Assume $(M, g)$ is a complete noncompact Riemannian manifold of dimension $n \geq 3$ and suppose a $C^2$-admissible function satisfying (1.13). In addition to (1.8), (1.9) and (1.12), we assume $(f, \Gamma)$ satisfies (1.2), (1.10) and (1.11). Then the equation (1.7) possesses a unique smooth maximal admissible solution $u$. Moreover, $u \geq u$ in $M$.

To prove the above results, the primary problem is to derive local estimates for first and second derivatives. The key ingredient is that, in the presence of (1.2), (1.8), (1.10) and (1.11), we can apply Theorem 1.4 to confirm the fully uniform ellipticity of equation (1.7) in Proposition 3.2. This is the most important step for our approach. With this crucial procedure at hand, we can prove the above theorems.
To be brief, we apply singularity solutions of a class of semi-linear equations obtained by McOwen [46] to construct supersolutions. Then we complete the proof of Theorem 1.7. In an attempt to show Theorem 1.8, we set up appropriate approximate problems on an exhaustion series of domains, building on Theorem 1.7. The approximation is based on a maximum principle: the approximate Dirichlet problems with infinite boundary value condition produce a decreasing sequence of approximate solutions (see Proposition 4.4).

1.3. **Prescribed problems for complete conformal metrics.** Let $\text{Sec}_g$, $\text{Ric}_g$ and $R_g$ denote the sectional, Ricci and scalar curvature of $g$, respectively. When $n \geq 3$ we denote for $g$

$$A^{\tau,\alpha}_g = \frac{\alpha}{n-2} \left( \text{Ric}_g - \frac{\tau}{2(n-1)} R_g \cdot g \right), \quad \alpha = \pm 1, \: \tau \in \mathbb{R}.$$  

When $\alpha = 1$, it is the modified Schouten tensor $A^\tau_g = \frac{1}{n-2} (\text{Ric}_g - \frac{\tau}{2(n-1)} R_g \cdot g)$. When $\tau = \alpha = 1$, it is the Schouten tensor $A^1_g = \frac{1}{n-2} (\text{Ric}_g - \frac{1}{2(n-1)} R_g \cdot g)$. For $\tau = n - 1$ and $\alpha = 1$, it corresponds to the Einstein tensor $G^\tau_g = \text{Ric}_g - \frac{1}{2} R_g \cdot g$.

One natural question to ask is: given a smooth positive function $\psi$, is there a smooth complete metric $\tilde{g} = e^{2u} g$ satisfying

$$f(\lambda(\tilde{g}^{-1} A^{\tau,\alpha}_{\tilde{g}})) = \psi \text{ in } M.$$  

The equations of this type include many important equations as special cases. When $f = \sigma_1$, $\tau = 0$ and $\psi$ is a proper constant, it is closely related to the well-known Yamabe problem, proved by Schoen [50] combining the important work of Aubin [1] and Trudinger [57]. For $f = \sigma_k^{1/k}$, $\psi = 1$ and $\tau = \alpha = 1$, the equation on closed Riemannian manifolds was proposed by Viaclovsky [60], known as $k$-Yamabe problem. From then on it has received much attention in [6, 9, 18, 27, 37, 31, 52]. When $M$ has boundary, the prescribed scalar curvature problem with certain boundary properties was studied by [12, 13, 14], and further complemented in [32, 7, 44, 45], where the obtained metrics are not complete. The case is fairly different when the metric is complete. A deep result of Aviles-McOwen [4], extending the celebrated theorem of Loewner-Nirenberg [41], yields that every compact Riemannian manifold of dimension $n \geq 3$ with smooth boundary admits a complete conformal metric with negative constant scalar curvature. Since then Loewner-Nirenberg and Aviles-McOwen’s results were extended by many experts to prescribed $\sigma_k$ curvature equation

$$\sigma_k^{1/k}(\lambda(\tilde{g}^{-1} A^{\tau,\alpha}_{\tilde{g}})) = \psi \text{ in } M,$$  

when imposing restrictions to $(\alpha, \tau)$

$$\begin{cases} 
\tau < 1 & \text{if } \alpha = -1, \\
\tau > n - 1 & \text{if } \alpha = 1.
\end{cases}$$

Under this assumption, the equation is automatically of fully uniform ellipticity according to the formula (3.3) below. Below we list part of related literature. On compact manifolds with boundary, the existence of solutions to prescribed $\sigma_k$-curvature
equation for \(-Ric_g\) was obtained by Guan [22] and Gursky-Streets-Warren [29], and the case \(\tau > n - 1, \alpha = 1\) was considered by Li-Sheng [38]. On the other hand, the prescribed \(\sigma_k\)-curvature equation on a closed manifold was studied by Gursky-Viaclovsky [30] with \(\tau < 1, \alpha = -1\), and by Sheng-Zhang [54] for \(\tau > n - 1, \alpha = 1\) based on a flow approach.

Nevertheless, most known results on fully nonlinear Loewner-Nirenberg problem for modified Schouten tensors require condition (1.16), and it is rarely known under a broader assumption. To close this gap, one challenge we face is the critical case \(\tau = n - 1\), in which the uniform ellipticity has to break down when \(\Gamma = \Gamma_n\). This critical case coincides with prescribed curvature equation for the Einstein tensor (1.17)

\[
f(\lambda(\tilde{g}^{-1}G_\tilde{g})) = \psi \quad \text{in } M.
\]

Unfortunately, the topological obstruction as described in Remark 1.13 below indicates that when \(\Gamma = \Gamma_n\), at least for \(n = 3\), the fully nonlinear Loewner-Nirenberg problem for positive Einstein tensor is unsolvable in general. This manifests that the obstruction to the solvability of (1.14) in the conformal class of complete metrics arises primarily from the lack of fully uniform ellipticity.

### 1.3.1. Prescribed problem I: compact manifolds with boundary.

When restricted to the equation (1.14), the assumption (1.8) reads as follows

\[
\begin{align*}
\tau < 1 & \quad \text{if } \alpha = -1, \\
\tau > 1 + (n - 2)(1 - \kappa_F \vartheta_F) & \quad \text{if } \alpha = 1.
\end{align*}
\]

It would be worthwhile to note that the assumption (1.18) is much more broader than (1.16), and it allows the critical case \(\tau = n - 1\) whenever \(\Gamma \neq \Gamma_n\). As a corollary of Theorem 1.7, we solve (1.14) in the conformal class of complete metrics.

**Theorem 1.9.** In addition to (1.2), (1.11) and (1.18), we assume

\[
f(t \lambda) = t^\varsigma f(\lambda), \quad \forall \lambda \in \Gamma, \forall t > 0, \text{ for some constant } 0 < \varsigma \leq 1.
\]

Assume that \((M, g)\) is a compact connected Riemannian manifold of dimension \(n \geq 3\) with smooth boundary and support a \(C^2\) compact conformal admissible metric. Then for any \(0 < \psi \in C^\infty(\bar{M})\), there exists at least one smooth complete metric \(\tilde{g} = e^{2u} g\) satisfying (1.14).

We construct \(C^2\) compact conformal admissible metrics, using Morse functions.

**Theorem 1.10.** In Theorem 1.9 the assumption on the existence of a compact conformal admissible metric can be dropped if \((\alpha, \tau)\) further satisfies

\[
\begin{align*}
\tau \leq 0 & \quad \text{if } \alpha = -1, \\
\tau \geq 2 & \quad \text{if } \alpha = 1.
\end{align*}
\]

We apply this construction to draw geometric conclusions. For the Ricci tensor case, together with Theorem 7.1, Theorem 1.10 deduces the following theorem.

**Theorem 1.11.** Suppose \((M, g)\) is a compact connected Riemannian manifold of dimension \(n \geq 3\) with smooth boundary. Then there is a unique complete conformal metric \(\tilde{g}\) with \(\text{Ric}_{\tilde{g}} < -1\) and \(\sigma_n / \sigma_{n-1}(\lambda(\text{Ric}_{\tilde{g}})) = 1\).
This gives a new proof of [43, Theorem 1] when \( n \geq 3 \). Notice also that the complete metric is obtained in each conformal class. This is in contrast with [42, Theorems A and C] of Lohkamp.

In addition, we can deform the Einstein tensor.

**Theorem 1.12.** Let \((M, g)\) be a compact connected Riemannian manifold of dimension \( n \geq 3 \) with smooth boundary. Let \((f, \Gamma)\) satisfy (1.2), (1.11) and (1.19). Suppose in addition that \( \Gamma \neq \Gamma_{\infty} \). Then for each \( 0 < \psi \in C^\infty(M) \), there exists a smooth admissible complete metric \( \bar{g} = e^{2u} g \) satisfying (1.17).

Note that this theorem is significantly interesting in dimension three, since the Einstein tensor is closely connected with the sectional curvature. Based on this, we present a topological obstruction to reveal that the condition \( \Gamma \neq \Gamma_{\infty} \) imposed in Theorem 1.12 is crucial and cannot be further dropped.

**Remark 1.13 (Topological obstruction).** Denote \( B_i(a) = \{ x \in \mathbb{R}^3 : |x - a|^2 < r^2 \} \). Let \( \bar{B}_i(a_1), \ldots, \bar{B}_i(a_m) \) be pairwise disjoint. For \( r \gg 1 \) with \( \bigcup_{i=1}^{m} B_i(a_i) \subset B_i(0) \), we denote \( \Omega = B_{r+1}(0) \setminus \bigcup_{i=1}^{m} \bar{B}_i(a_i) \), \( g \) a Riemannian metric on \( \Omega \). Fix \( x \in \Omega \), let \( \Sigma \subset T_x \Omega \) be a tangent 2-plane, \( \vec{n} \in T_x \Omega \) the unit normal vector to \( \Sigma \), then

\[
G_g(\vec{n}, \vec{n}) = -\text{Sec}_g(\Sigma),
\]

see [28, Section 2] or (3.2) below. If Theorem 1.12 holds in the positive cone case, then the solution on \( \Omega \) is a complete conformal metric with negative sectional curvature. This contradicts to the Cartan-Hadamard theorem.

1.3.2. **Prescribed problem II: complete noncompact manifolds.** In contrast with the resolution of Yamabe problem on closed Riemannian manifolds, the complete noncompact version of Yamabe problem is not always solvable as shown by Jin [34]. Consequently, one could only expect the solvability of prescribed curvature problem in the conformal class of complete noncompact metrics under proper additional assumptions. Indeed prior to Jin’s work, Ni [48, 49] obtained the existence and nonexistence of solutions to prescribed scalar curvature equation on standard Euclidean spaces, which was partially extended by Sui [55] to the prescribed \( \sigma_k \)-curvature equation for negative Ricci curvature. When imposed fairly strong restrictions to the asymptotic ratio of prescribed functions and curvature, or even to the topology of background manifolds, Aviles-McOwen [2, 3] and Jin [35] investigated prescribed scalar curvature equation on negatively curved complete noncompact Riemannian manifolds, followed by Fu-Sheng-Yuan [16] recently, who studied prescribed \( \sigma_k \)-curvature equation (1.15) for \( \alpha = -1 \) and \( \tau < 1 \). Unfortunately, the restrictions imposed there are not optimal, even for conformal prescribed scalar curvature equation.

It still remains widely open to determine under which conditions the conformal prescribed curvature equation is solvable on complete noncompact manifolds. We close this gap as a consequence of Theorem 1.8. That is, we can solve fully nonlinear version of Yamabe problem on a complete noncompact Riemannian manifold, assuming existence of a \( C^2 \) complete admissible metric \( \bar{g} = e^{2u} g \) with

\[
(1.21) \quad f(\bar{\lambda}(g^{-1}A^{(r, \alpha)})) \geq \lambda_0 \psi \text{ holds uniformly in } M \setminus K_0.
\]
Here $\Lambda_0$ is a uniform positive constant, and $K_0$ is a compact subset of $M$.

**Theorem 1.14.** Let $(M, g)$ be a complete noncompact Riemannian manifold of dimension $n \geq 3$. Suppose $(f, \Gamma)$ satisfies (1.2), (1.11) and (1.19). Given a smooth positive function $\psi$ and $(a, \tau)$ obeying (1.18), we assume that $(M, g)$ carries a $C^2$ complete admissible conformal metric subject to (1.21). Then there exists a unique smooth complete maximal conformal admissible metric satisfying (1.14).

**Remark 1.15.** Theorems 1.8 and 1.14 reveal that all of geometric and analytic obstructions to solvability of the corresponding equations are embodied in the assumption of asymptotic condition at infinity, which is sufficient and necessary:

- The solution we expect to obtain tautologically satisfies the asymptotic condition at infinity, which is therefore necessary for the existence.
- Prescribed curvature problem (1.14) is not always solvable. Please refer to [35, 49, 55] for some nonexistence results on conformal prescribed scalar and Ricci curvature equations.

In conclusion, except the asymptotic assumption at infinity, it is not required to impose further restrictions to curvatures, prescribed functions and the topology of background manifolds. This is in contrast to related works cited above. This is new even in Euclidean spaces. An analogue of Theorem 1.14 also holds for conformal prescribed scalar curvature equation, see Theorem 4.9 below.

**Remark 1.16.** A somewhat surprising fact to us is that we impose neither (1.3) nor (1.4) in Theorems 1.4, 1.7, 1.8, 1.9, 1.10, 1.12 and 1.14. This is in contrast with huge literature on fully nonlinear elliptic equations.

In conclusion, in this paper we first prove partial uniform ellipticity for fully nonlinear equations and then use it to confirm the uniform ellipticity of a special class of fully nonlinear equations. As an application, we solve a fully nonlinear Loewner-Nirenberg problem, given a compact conformal admissible metric. This assumption is partially confirmed, utilizing Morse theory. As a result, we can deduce various geometric conclusions. Furthermore, we examine asymptotic behavior and uniqueness of solutions to fully nonlinear Loewner-Nirenberg problem. Building on these solutions, we prove the existence and uniqueness of complete maximal conformal metric solving a complete noncompact version of fully nonlinear Yamabe problem, under an asymptotic condition at infinity which is in effect sharp. Our approach also works for conformal scalar curvature equation. Moreover, the topological obstruction indicates that our strategy is delicate.

The paper is organized as follows. In Section 2 we investigate the partial uniform ellipticity. This is one of the most important parts of the paper. In Section 3 we confirm the uniform ellipticity of (1.7). This is crucial for our approach. In Section 4 we solve a class of equations of fully uniform ellipticity on complete noncompact Riemannian manifolds, using an approximate argument. The key ingredient is the solution of the Dirichlet problem with infinite boundary value condition. The proof of existence of such solutions is left to Section 5. Moreover, in Section 7 the approximation method is also used to analyze asymptotic behavior and uniqueness.
of complete conformal metrics. In Section 6 we construct conformal admissible metrics, based on a result on Morse functions. In the final section, we derive local and boundary estimates for equations of fully uniform ellipticity.

2. The Partial Uniform Ellipticity

We start with the following lemma.

**Lemma 2.1.** For \((f, \Gamma)\) satisfying (1.2), the following statements are equivalent.

1. \((f, \Gamma)\) satisfies (1.6).
2. For each \(\lambda, \mu \in \Gamma\), \(\sum_{i=1}^{n} f_i(\lambda)\mu_i > 0\).
3. \(f(\lambda + \mu) > f(\lambda), \forall \lambda, \mu \in \Gamma\).

**Proof.** It follows from the concavity of \(f\) that

\[
f(\lambda) \geq f(\mu) + \sum_{i=1}^{n} f_i(\lambda)(\lambda_i - \mu_i), \quad \forall \lambda, \mu \in \Gamma.
\]

(1) \(\Rightarrow\) (2): Fix \(\lambda \in \Gamma\). The condition (1.6) implies that for any \(\mu \in \Gamma\), there is \(T \geq 1\) (may depend on \(\mu\)) such that for each \(t > T\), \(f(t\mu) > f(\lambda)\). Together with (2.1), one gets \(\sum_{i=1}^{n} f_i(\lambda)(t\mu_i - \lambda_i) > 0\). Thus, \(\sum_{i=1}^{n} f_i(\lambda)\lambda_i > 0\) (if one takes \(\mu = \lambda\)) and moreover \(\sum_{i=1}^{n} f_i(\lambda)\mu_i > 0\).

(2) \(\Rightarrow\) (3): The proof uses (2.1).

(3) \(\Rightarrow\) (1): For any \(\lambda, \mu \in \Gamma\), \(t\lambda - \mu \in \Gamma\) for \(t > t_{\lambda, \mu}\), depending only on \(\lambda\) and \(\mu\). So \(f(t\lambda) > f(\mu)\) for such \(t\).

**Remark 2.2.** In the presence of (1.2) and (1.3), this lemma was initially proposed by the author in [62, Lemma 3.2] to set up quantitative boundary estimate of the form

\[
\sup_{\partial M} |\bar{\partial} \bar{u}| \lesssim C(1 + \sup_{M} |\bar{\partial} \bar{u}|^2)
\]

for the Dirichlet problem of fully nonlinear elliptic equations on complex manifolds. In Lemma 2.1, the assumption (1.3) has been removed.

**Corollary 2.3.** If \(f\) satisfies (1.2) and (1.6), then (1.4) holds.

**Proof.** Fix \(\lambda \in \Gamma\). According to Lemma 2.1, \(\sum_{i=1}^{n} f_i(\lambda)\mu_i > 0\), \(\forall \mu \in \Gamma\). Note that \(\Gamma_n \subseteq \Gamma\), then \(f_i(\lambda) > 0\), \(\forall 1 \leq i \leq n\). We have \(\sum_{i=1}^{n} f_i(\lambda) > 0\) by setting \(\mu = \bar{1}\).

In the following proposition, we relate \(\kappa_{\Gamma}\) to the maximal count of negative components of vectors in \(\Gamma\). Let’s denote

**Definition 2.4.**

\[
\bar{\kappa}_{\Gamma} = \max \left\{ k : (-\alpha_1, \cdots, -\alpha_k, \alpha_{k+1}, \cdots, \alpha_n) \in \Gamma, \text{ where } \alpha_j > 0, \forall 1 \leq j \leq n \right\}.
\]

**Proposition 2.5.** \(\kappa_{\Gamma} = \bar{\kappa}_{\Gamma}\).
Proof: The case $\Gamma = \Gamma_n$ is true since $\kappa_{\Gamma_n} = 0$ and $\overline{\kappa}_{\Gamma_n} = 0$. Next, we consider the case $\Gamma \neq \Gamma_n$. Assume $(-\alpha_1, \cdots, -\alpha_{\kappa_f}, \alpha_{\kappa_f+1}, \cdots, \alpha_n) \in \Gamma$, for $\alpha_i > 0$, $\forall 1 \leq i \leq n$. Then $(0, \cdots, 0, \alpha_{\kappa_f+1}, \cdots, \alpha_n) \in \Gamma$, which implies $\kappa_{\Gamma} \geq \kappa_{\Gamma_f}$. Conversely, if

$$(0, \cdots, 0, 1, \cdots, 1) \in \Gamma,$$

then by the openness of $\Gamma$, we have for some $0 < \epsilon \ll 1$,

$$(\epsilon, \cdots, \epsilon, 1, \cdots, 1) \in \Gamma.$$

Consequently, $\overline{\kappa}_{\Gamma} \geq \kappa_{\Gamma}$. □

Corollary 2.3 is a part of Theorem 1.4. Below we complete the proof. Fix $\lambda \in \Gamma$. It follows from the concavity and symmetry of $f$ that

$$f_i(\lambda) \geq f_j(\lambda) \text{ for } \lambda_i \leq \lambda_j.$$ 

In particular

$$f_i(\lambda) \geq \frac{1}{n} \sum_{i=1}^{n} f_i(\lambda) \text{ if } \lambda_1 \leq \cdots \leq \lambda_n.$$ 

Therefore, for the case $\Gamma = \Gamma_n$ (i.e. $\kappa_{\Gamma} = 0$), we immediately obtain Theorem 1.4. For general $\Gamma$, Theorem 1.4 is a consequence of Proposition 2.5 and the following proposition.

Proposition 2.6. Assume $\Gamma \neq \Gamma_n$ and $f$ satisfies (1.2) and (1.6) in $\Gamma$. For the $\kappa_{\Gamma}$ as defined in Definition 1.3, let $\alpha_1, \cdots, \alpha_n$ be $n$ strictly positive constants such that

$$(-\alpha_1, \cdots, -\alpha_{\kappa_f}, \alpha_{\kappa_f+1}, \cdots, \alpha_n) \in \Gamma.$$ 

In addition, assume $\alpha_1 \geq \cdots \geq \alpha_{\kappa_f}$. Then for each $\lambda \in \Gamma$ with order $\lambda_1 \leq \cdots \leq \lambda_n$, (2.2)

$$f_{\kappa_f+1}(\lambda) \geq \frac{\alpha_1}{\sum_{i=\kappa_f+1}^{n} \alpha_i - \sum_{i=2}^{\kappa_f} \alpha_i} f_1(\lambda).$$

Proof: Fix $\lambda \in \Gamma$ and we assume $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. From Corollary 2.3, $f_i(\lambda) \geq 0$ and $\sum_{i=1}^{n} f_i(\lambda) > 0$. By Lemma 2.1,

$$-\sum_{i=1}^{\kappa_f} \alpha_i f_i(\lambda) + \sum_{i=\kappa_f+1}^{n} \alpha_i f_i(\lambda) > 0$$

which simply yields $f_{\kappa_f+1}(\lambda) \geq \frac{\alpha_1}{\sum_{i=\kappa_f+1}^{n} \alpha_i} f_1(\lambda)$. In addition, one can derive (2.2) by using the iteration. □

Remark 2.7. When $\Gamma \neq \Gamma_n$ the constant $\theta_{\Gamma}$ in Theorem 1.4 can be achieved as follows:

$$\theta_{\Gamma} = \sup_{(-\alpha_1, \cdots, -\alpha_{\kappa_f}, \alpha_{\kappa_f+1}, \cdots, \alpha_n) \in \Gamma, \alpha_i > 0} \frac{\alpha_1/n}{\sum_{i=\kappa_f+1}^{n} \alpha_i - \sum_{i=2}^{\kappa_f} \alpha_i}.$$ 

A somewhat remarkable fact to us is that in the description of partial uniform ellipticity, $\kappa_{\Gamma}$ and $\theta_{\Gamma}$ depend only on $\Gamma$ instead of specifically on $f$. Consequently, it has great advantages in applications to PDEs and geometry. For instance, Theorem
1.4 allows us to compute the partial uniform ellipticity of fully nonlinear equations in connection with the notions of $p$-convexity [51, 61], $\mathbb{R}$-subharmonic and $\mathbb{G}$-plurisubharmonic functions in the sense of Harvey-Lawson [33].

The following proposition precisely reveals the interaction between $\kappa_{\Gamma}$ and partial uniform ellipticity. Denote

$$(2.4) \quad \Gamma_{R}^{\infty} := \{ \lambda' \in \mathbb{R}^{k} : (\lambda', c, \cdots, c) \in \Gamma \text{ for some } c > 0 \}.$$  

**Proposition 2.8.** In addition to (1.2), we assume for some $1 \leq k \leq n - 1$ that $f$ is of $(k + 1)$-uniform ellipticity in $\Gamma$. Then $\kappa_{\Gamma} \geq k$ and $\Gamma_{R}^{\infty} = \mathbb{R}^{k}$.

**Proof.** Let $\emptyset$ be as in (1.5). As above, $\emptyset = (1, \cdots, 1)$. For the $(f, \Gamma)$, we denote

$$\Gamma^{\emptyset} = \{ \lambda \in \Gamma : f(\lambda) > a \}.$$

Let $c_{0}$ be the positive constant with $f(c_{0} \emptyset) > \sup_{\partial \emptyset} f$, and then we set $a = 1 + c_{0}$. Thus $f(a \emptyset) > f(c_{0} \emptyset)$ by $\sum_{i=1}^{k} f(\lambda_{i}) > 0$. For $0 < \epsilon < a$ and $R = \frac{\epsilon}{\emptyset}$, we denote

$$\lambda_{\epsilon,R} = (\epsilon, \cdots, \epsilon, R, \cdots, R).$$

We can deduce from (2.1) that

$$f(\lambda_{\epsilon,R}) \geq f(a \emptyset) + \epsilon \sum_{i=1}^{k} f(\lambda_{\epsilon,R}) + R \sum_{i=k+1}^{n} f(\lambda_{\epsilon,R}) - a \sum_{i=1}^{n} f(\lambda_{\epsilon,R})$$

$$\geq f(a \emptyset) + (R \emptyset - a) \sum_{i=1}^{n} f(\lambda_{\epsilon,R}) \text{ (using $(k + 1)$-uniform ellipticity)}$$

$$= f(a \emptyset) \text{ (noticing } R = \frac{\epsilon}{\emptyset})$$

$$> f(c_{0} \emptyset).$$

So $\lambda_{\epsilon,R} = (\epsilon, \cdots, \epsilon, R, \cdots, R) \in \Gamma(f(a \emptyset))$. Notice that $R = \frac{\epsilon}{\emptyset}$ does not depend on $\epsilon$.

Taking $\epsilon \to 0^{+}$, we get $(0, \cdots, 0, R, \cdots, R) \in \Gamma(f(a \emptyset)) \subset \Gamma(f(c_{0} \emptyset)) \subset \Gamma$. Thus $\kappa_{\Gamma} \geq k$. $\square$

**Remark 2.9.** The proof uses the simple fact $\Gamma_{R} \subseteq \Gamma$. This is different from the proof of Proposition 3.3 below.

**Corollary 2.10.** The $(\kappa_{\Gamma} + 1)$-uniform ellipticity asserted in Theorem 1.4 cannot be improved.

The following two corollaries are key ingredients in the construction of local barriers in Section 5.

**Corollary 2.11.** In addition to (1.2), we assume that $f$ is of fully uniform ellipticity in $\Gamma$. Then the corresponding cone $\Gamma$ is of type 2.

**Corollary 2.12.** If $\Gamma$ carries a smooth symmetric concave $f$ satisfying (1.2) and (1.6), then the following statements are equivalent to each other.

(1) $f$ is of fully uniform ellipticity in $\Gamma$.

(2) $\Gamma$ is of type 2. That is $\Gamma_{\mathbb{R}^{n-1}}^{\infty} = \mathbb{R}^{n-1}$, where $\Gamma_{\mathbb{R}^{n-1}}^{\infty}$ is as we denoted in (2.4).
Lemma 2.13. Let \( \kappa_\Gamma \) be as in Definition 1.3, \( \vartheta_\Gamma \) be as in Theorem 1.4, then

1. \( \kappa_\Gamma \) is an integer with \( 0 \leq \kappa_\Gamma \leq n - 1 \).
2. For \( \Gamma = \Gamma_k \), \( \kappa_\Gamma = n - k \).
3. \( \kappa_\Gamma = 0 \) if and only if \( \Gamma = \Gamma_n \).
4. \( \kappa_\Gamma = n - 1 \) if and only if \( \Gamma \) is of type 2.
5. \( 0 < \vartheta_\Gamma \leq \frac{1}{n} \).

Moreover, if \( \vartheta_\Gamma = \frac{1}{n} \) and \( \kappa_\Gamma = n - 1 \) occur simultaneously then

\[
f_i(\lambda) = \frac{1}{n} \sum_{j=1}^{n} f_j(\lambda), \quad \forall \lambda \in \Gamma, \forall 1 \leq i \leq n.
\]

Proof. Note that \( \Gamma \subseteq \Gamma_1 \), we know \( \kappa_\Gamma \leq n - 1 \). Next we prove the last statement.

Set \( \lambda = \vec{1} \), then \( f_i(\vec{1}) = \frac{1}{n} \sum_{j=1}^{n} f_j(\vec{1}) \), \( 1 \leq i \leq n \). This simply yields \( \vartheta_\Gamma \leq \frac{1}{n} \). The proofs of the other three statements are obvious, and we omit them here. \( \square \)

As consequences of Lemma 2.13, we obtain the following lemmas.

Lemma 2.14. Suppose \( \varrho \) obeys (1.8). Then \( \varrho < n \) and \( \varrho \neq 0 \).

Lemma 2.15. Suppose \( (\alpha, \tau) \) satisfies (1.18). Then

\[
\alpha(n\tau + 2 - 2n) > 0, \quad (\tau - 1)(n\tau + 2 - 2n) > 0.
\]

It seems very interesting to answer whether Gårding’s cone is the largest one to share same \( \kappa_\Gamma \).

Problem 2.16. Fix \( 1 < k < n \). For the cone \( \Gamma \) with \( \kappa_\Gamma = n - k \), is \( \Gamma \subseteq \Gamma_k \) true?

Problem 2.17. For any \( \Gamma, \hat{\Gamma} \) with \( \kappa_\Gamma \geq \kappa_{\hat{\Gamma}} + 1 \), is \( \hat{\Gamma} \subset \Gamma \) true?

Finally, we confirm (1.6) under the assumptions (1.10) and (1.11).

Lemma 2.18. Assume, in addition to (1.2), that \( \sup_{\Gamma} f = +\infty \) and

\[
(2.5) \quad \lim_{t \to +\infty} f(t\lambda) > -\infty, \quad \forall \lambda \in \Gamma.
\]

Then \( (f, \Gamma) \) satisfies (1.6).

Proof. We have by (2.1) the following inequality

\[
\sum_{i=1}^{n} f_i(\lambda) \mu_i \geq \lim_{t \to +\infty} \sup_{t} f(\mu)/t, \quad \forall \lambda, \mu \in \Gamma.
\]

Thus

\[
(2.6) \quad \sum_{i=1}^{n} f_i(\lambda) \mu_i \geq 0, \quad \forall \lambda, \mu \in \Gamma.
\]

So \( f_i(\lambda) \geq 0, \forall \lambda \in \Gamma, 1 \leq i \leq n \). Together with \( \sup_{\Gamma} f = +\infty \) and (2.1), we derive

\[
(2.7) \quad \sum_{i=1}^{n} f_i(\lambda) > 0 \text{ in } \Gamma.
\]

By (2.6), (2.7) and the openness of \( \Gamma \), we get (2) from Lemma 2.1, as required. \( \square \)
3. Problem reduction

3.1. Preliminaries. On a Riemannian manifold \((M, g)\), one defines the curvature tensor by

\[
R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.
\]

Let \(e_1, \ldots, e_n\) be a local frame on \(M\). From now on we denote

\[
\langle X, Y \rangle = g(X, Y), \quad g_{ij} = \langle e_i, e_j \rangle, \quad \{g^{ij}\} = \{g_{ij}\}^{-1}.
\]

Under Levi-Civita connection of \((M, g)\), \(\nabla_{e_i} e_j = \Gamma^k_{ij} e_k\), and \(\Gamma^k_{ij}\) denote the Christoffel symbols. The curvature coefficients are given by \(R_{ijkl} = \langle e_i, R(e_k, e_l)e_j \rangle\). For simplicity we write

\[
\nabla_i = \nabla_{e_i}, \quad \nabla_{ij} = \nabla_i \nabla_j - \Gamma^k_{ij} \nabla_k, \quad \nabla_{ijk} = \nabla_i \nabla_{jk} - \Gamma^l_{ij} \nabla_{lk} - \Gamma^l_{ik} \nabla_{jl}, \text{ etc.}
\]

Under the local unit orthogonal frame

\[
\text{Ric}_g(e_i, e_i) = \sum_{j=1}^{n} \langle R(e_i, e_j)e_i, e_j \rangle, \quad \text{Ric}_g = \sum_{i=1}^{n} \text{Ric}_g(e_i, e_i),
\]

\[
\text{Ric}_g(e_i, e_i) = \frac{\text{Ric}_g}{2} - \frac{1}{2} \sum_{k,l,i,j} \langle R(e_k, e_l)e_i, e_j \rangle, \quad \forall 1 \leq i \leq n.
\]

Under the conformal change \(\tilde{g} = e^{2\varphi} g\), one has (see e.g. [5])

\[
\text{Ric}_{\tilde{g}} = \text{Ric}_g - \Delta \varphi g - (n - 2)\nabla^2 \varphi u - (n - 2)|\nabla u|^2 g + (n - 2) \varphi du \otimes du.
\]

Thus

\[
A^\tau_{\alpha g} = A^\tau_{\alpha g} + \frac{\alpha(\tau - 1)}{n - 2} \Delta \varphi g - \alpha \nabla^2 \varphi u + \frac{\alpha(n - 2)}{2} |\nabla u|^2 g + \alpha du \otimes du.
\]

In particular, the Schouten tensor obeys

\[
A_{\tilde{g}} = A_g - \nabla^2 \varphi u - \frac{1}{2} |\nabla u|^2 g + du \otimes du.
\]

3.2. Fully uniform ellipticity and construction of type 2 cones. First, we shall summarize the definition of type 2 cones.

**Definition 3.1** ([8]). \(\Gamma\) is said to be of type 1 if the positive \(\lambda_i\) axes belong to \(\partial \Gamma\); otherwise it is called of type 2.

Except \(\Gamma_1\), it seems not easy to find general type 2 cones. Based on Theorem 1.4, we can construct some examples of type 2 cones.

Given a cone \(\Gamma\), we take a constant \(\varphi\) satisfying (1.8). Note that \(\varphi < n\) and \(\varphi \neq 0\) by Lemma 2.14. We let

\[
\mu_i = \frac{1}{n - \varphi} \left( \sum_{j=1}^{n} \lambda_j - \varphi \lambda_i \right), \quad \text{i.e.} \quad \lambda_i = \frac{1}{\varphi} \left( \sum_{j=1}^{n} \mu_j - (n - \varphi) \mu_i \right),
\]

\[
\bar{\Gamma} = \left\{ (\lambda_1, \cdots, \lambda_n) : \lambda_i = \frac{1}{\varphi} \left( \sum_{j=1}^{n} \mu_j - (n - \varphi) \mu_i \right), \langle \mu_1, \cdots, \mu_n \rangle \in \Gamma \right\}.
\]
So $\tilde{\Gamma}$ is also an open symmetric convex cone of $\mathbb{R}^n$ with $\tilde{\Gamma} \subseteq \Gamma_1$. One has a symmetric concave function $\tilde{f}$ on $\tilde{\Gamma}$ as follows:

$$\tilde{f}(\lambda) = f(\mu).$$

**Proposition 3.2** (Fully uniform ellipticity). Let $(f, \Gamma)$ satisfy (1.2) and (1.6). Let $(\tilde{f}, \tilde{\Gamma})$ be as above. Suppose in addition that $\varrho$ satisfies (1.8). Then there is a uniform positive constant $\theta$ such that

$$\frac{\partial \tilde{f}}{\partial \lambda_i}(\lambda) \geq \theta \sum_{j=1}^{n} \frac{\partial \tilde{f}}{\partial \lambda_j}(\lambda) > 0 \text{ in } \tilde{\Gamma}, \quad \forall 1 \leq i \leq n.$$  

Moreover, $(\tilde{f}, \tilde{\Gamma})$ also satisfies (1.2) and (1.6).

**Proof.** **Case 1:** $\varrho < 0$. A straightforward computation shows

$$\frac{\partial \tilde{f}}{\partial \lambda_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial \mu_j} \frac{\partial \mu_j}{\partial \lambda_i} = \frac{1}{n-\varrho} \left( \sum_{j=1}^{n} \frac{\partial f}{\partial \mu_j} - \varrho \frac{\partial f}{\partial \mu_i} \right) \geq \frac{1}{n-\varrho} \sum_{j=1}^{n} \frac{\partial f}{\partial \mu_j} = \frac{1}{n-\varrho} \sum_{j=1}^{n} \frac{\partial \tilde{f}}{\partial \lambda_j}.$$  

**Case 2:** $0 < \varrho < \frac{1}{1-\kappa_\Gamma}$. By Theorem 1.4, $(1-\kappa_\Gamma)^n \sum_{j=1}^{n} \frac{\partial f}{\partial \mu_j} > \sigma, \forall 1 \leq i \leq n$. Thus

$$\frac{\partial \tilde{f}}{\partial \lambda_i} > \frac{1-\varrho(1-\kappa_\Gamma)}{n-\varrho} \sum_{j=1}^{n} \frac{\partial f}{\partial \mu_j} = \frac{1-\varrho(1-\kappa_\Gamma)}{n-\varrho} \sum_{j=1}^{n} \frac{\partial \tilde{f}}{\partial \lambda_j}.$$  

$\square$

**Proposition 3.3** (Construction of type 2 cones). Let $\varrho$ satisfy (1.8), and let $\tilde{\Gamma}$ be the cone as defined in (3.6). Then $\Gamma_n \subset \tilde{\Gamma}$ and $\tilde{\Gamma}$ is a type 2 cone.

**Proof.** Given $(f, \Gamma)$, as in (3.6) and (3.7), we obtain $(\tilde{f}, \tilde{\Gamma})$. It only requires to verify

$$(0, \ldots, 0, 1) \in \tilde{\Gamma}.$$  

Denote $\lambda_t = (t \cdots, t, 1)$ and $I = \{t \in [0, 1] : \lambda_t \in \tilde{\Gamma}\}$. Clearly, $1 \in I$, and $I$ is an open subset of $I$ by the openness of $\tilde{\Gamma}$. It is sufficient to prove the closeness. Let $\{t_i\} \subset I, 0 < t_i \leq 1$ and

$$t_0 = \lim_{i \rightarrow +\infty} t_i.$$  

It only requires to prove $t_0 \in I$. We denote $\tilde{\Gamma}^\sigma = \{\lambda \in \tilde{\Gamma} : \tilde{f}(\mu) > \sigma\}$. Let $\theta$ be as in (3.8), and let $\tilde{f}(\theta \bar{A}) > \sup_{\partial \tilde{\Gamma}} \tilde{f}$. We deduce from (3.8) and the concavity of $\tilde{f}$ that

$$\tilde{f}\left(\frac{a}{\theta} \lambda_{t_0}\right) \geq \tilde{f}(\theta \bar{A}) + \frac{a}{\theta} \sum_{j=1}^{n-1} \frac{\partial \tilde{f}}{\partial \lambda_j}(\frac{a}{\theta} \lambda_{t_0}) + \frac{a}{\theta} \cdot \frac{\partial \tilde{f}}{\partial \lambda_n}(\frac{a}{\theta} \lambda_{t_0}) - a \sum_{j=1}^{n} \frac{\partial \tilde{f}}{\partial \lambda_j}(\frac{a}{\theta} \lambda_{t_0}) > \tilde{f}(\theta \bar{A}).$$  

Thus $\frac{a}{\theta} \lambda_{t_0} \in \tilde{\Gamma}^{\tilde{f}(\theta \bar{A})}$. As required, $\lambda_{t_0} \in \tilde{\Gamma}$. This gives $I = [0, 1]$. $\square$

**Corollary 3.4.** For $\varrho$ satisfying (1.8), $(1, \cdots, 1, 1 - \varrho) \in \Gamma$.

**Remark 3.5.** From the construction of $\tilde{\Gamma}$, it is not easy to see $\Gamma_n \subset \tilde{\Gamma}$. 

3.3. **Reduction to Schouten tensor case.** We reduce the prescribed curvature equation (1.14) to a uniform elliptic equation for conformal deformation of Schouten tensor. To do this, we can check
\[
\text{tr} \left( g^{-1}(-A_g) \right) g - \varrho (-A_g) = \frac{n - 2}{\alpha (\tau - 1)} A_g^{\tau, \alpha},
\]
where we take
\[
\varrho = \frac{n - 2}{\tau - 1}.
\]

**Lemma 3.6.** Let \((\alpha, \tau)\) satisfy (1.18), let \(\varrho = \frac{n - 2}{\tau - 1}\). Then
\[
\varrho < \frac{1}{1 - \kappa \Gamma \vartheta} \text{ and } \varrho \neq 0.
\]
Therefore, when \(f\) is homogeneous of degree \(\varsigma\), the equation (1.14) is equivalent to
\[
(3.9) \quad \tilde{f}(\lambda(-g^{-1}A_g)) = \left( \frac{n - 2}{\alpha (n\tau + 2 - 2n)} \right)^\varsigma \psi,
\]
in which \(\tilde{f}\) is of fully uniform ellipticity in \(\tilde{\Gamma}\) according to Proposition 3.2. Here \((\tilde{f}, \tilde{\Gamma})\) is as in (3.6)-(3.7). To study this equation, according to the formula (3.4), it requires to consider
\[
(3.10) \quad \tilde{f}(\lambda(g^{-1}(\nabla^2u + \frac{1}{2}|\nabla u|^2 g - du \otimes du - A_g))) = \left( \frac{n - 2}{\alpha (n\tau + 2 - 2n)} \right)^\varsigma \psi e^{2\varsigma u}.
\]

**Remark 3.7.** In what follows, we replace \((\tilde{f}, \tilde{\Gamma})\) by \((f, \Gamma)\) if necessary.

### 4. The equations on complete noncompact manifolds and applications to prescribed curvature problem

From Proposition 3.2 and Lemma 2.18, (1.7) falls into the equation of the form
\[
(4.1) \quad f \left( A(g^{-1}(\nabla^2u + A(x, \nabla u))) \right) = \psi(x, u),
\]
in which \(f\) is of fully uniform ellipticity in \(\Gamma\), i.e., there is a uniform constant \(\theta\) such that
\[
(4.2) \quad f^i(\lambda) \geq \theta \sum_{j=1}^n f_j(\lambda) > 0 \text{ in } \Gamma, \quad \forall 1 \leq i \leq n.
\]
In addition, \(\psi(x, t)\) is a smooth positive function on \(M \times \mathbb{R}\) satisfying (1.9), and \(A(x, p)\) is a smooth symmetric \((0, 2)\)-tensor satisfying (1.12).

In this section we solve the equation (4.1) on a complete noncompact Riemannian manifold, given an asymptotic condition at infinity
\[
(4.3) \quad f(\lambda(g^{-1}(\nabla^2u + A(x, \nabla u)))) \geq \psi(x, u), \quad \forall x \in M.
\]
Also, for the equation (4.1), we say a \(C^2\) function \(w\) is an *admissible* function, if
\[
\lambda(g^{-1}(\nabla^2w + A(x, \nabla w))) \in \Gamma.
\]
Theorem 4.1. Assume \((M, g)\) is a complete noncompact Riemannian manifold of dimension \(n \geq 3\) and suppose a \(C^2\)-admissible function \(u\) satisfying the asymptotic condition (4.3). Suppose in addition that (1.2), (1.10), (1.11), (1.9), (1.12) and (4.2) hold. Then the equation (4.1) has a unique smooth maximal admissible solution \(u\). Moreover, \(u \geq u\) in \(M\).

As a consequence, we obtain Theorem 1.8.

Remark 4.2. Suppose for any positive constant \(\Lambda\) that
\[\Delta \psi(x, t) \geq \psi(x, t + C_{\Lambda}), \quad \forall (x, t) \in M \times \mathbb{R}\]
holds for a constant \(C_{\Lambda}\). Then the asymptotic condition (4.3) can be replaced by
\[f(\lambda(\nabla^2 \psi + A(x, \nabla \psi))) \geq \Lambda \psi(x, \psi).
\]

Comparison principle. First of all, we deduce a comparison principle.

Lemma 4.3. Let \((\Omega, g)\) be a compact Riemannian manifold with boundary \(\partial \Omega\). Suppose (1.2), (1.10), (1.11) and (1.9) hold. Let \(w, v \in C^2(\Omega) \cap C(\bar{\Omega})\) be admissible functions subject to
\[f(\lambda(\nabla^2 v + A(x, \nabla v))) \geq \psi(x, v), \quad f(\lambda(\nabla^2 w + A(x, \nabla w))) \leq \psi(x, v)\text{ in } \Omega,
\]
\[w - v \leq 0 \text{ on } \partial \Omega.
\]

Then
\[w - v \leq 0 \text{ in } \Omega.
\]

Proof. Assume by contradiction that there is an interior point \(x_0 \in \Omega\) such that
\[0 < (w - v)(x_0) = \sup_{\Omega}(w - v).
\]
Therefore at \(x_0\),
\[\nabla^2 v \geq \nabla^2 w, \quad \nabla v = \nabla w.
\]

By Lemma 2.18 and Corollary 2.3, one has (1.4). Thus at \(x_0\)
\[f(\lambda(\nabla^2 v + A(x, \nabla v))) \geq f(\lambda(\nabla^2 w + A(x, \nabla w))),
\]
which further yields \(v(x_0) \geq w(x_0)\) by (1.9). This contradicts to \(w(x_0) > v(x_0)\). \(\Box\)

Approximation and the proof of Theorem 4.1. We employ an approximate method, based on Theorem 5.3 below. Let \(\{M_k\}_{k=1}^{+\infty}\) be an exhaustion series of domains with
- \(M = \bigcup_{k=1}^{+\infty} M_k\), \(\bar{M}_k = M_k \cup \partial M_k\), \(\bar{M}_k \subset M_{k+1}\).
- \(\bar{M}_k\) is a compact \(n\)-manifold with smooth boundary \(\partial M_k\).

According to Theorem 5.3, for each \(k \geq 1\), there is an admissible function \(u_k \in C^\infty(M_k)\) satisfying
\[f(\lambda(\nabla^2 u_k + A(x, \nabla u_k))) = \psi(x, u_k)\text{ in } M_k,
\]
(4.4)
\[\lim_{x \to \partial M_k} u_k(x) = +\infty.
\]
(4.5)

By the comparison principle (Lemma 4.3), we immediately derive

Proposition 4.4. Let \(u_k\) be the admissible solution to problem (4.4)-(4.5), then
\[u_{k+1} \leq u_k \text{ in } M_k, \quad \forall k \geq 1.
\]
Proposition 4.5. For any admissible solution $u_k$ to (4.4)-(4.5), we have

$$u_k \geq u \quad \text{in} \quad M_k, \quad \forall k \geq 1.$$  

Proposition 4.4 states that $\{u_k\}_{k \geq m}$ is a decreasing sequence of functions on $M_k$; while such a decreasing sequence is uniformly bounded from below according to Proposition 4.5. Thus we can take the limit

$$u_\infty = \lim_{k \to +\infty} u_k.$$  

Moreover, $u_\infty \geq u$ in $M$. In Propositions 4.4-4.5, and Theorems 8.1-8.2 below, we establish the local estimates up to second order derivatives for each $u_k$. Combining with Evans-Krylov theorem \cite{15, 36} and classical Schauder theory, we know that $u_\infty$ is in fact a smooth admissible solution to the equation (4.1).

Let $w_\infty$ be another admissible solution to the equation (4.1), then by the comparison principle we have $u_k \geq w_\infty$ in $M_k$ for all $k \geq 1$. Thus $u_\infty(x) \geq w_\infty(x)$, $\forall x \in M$. This means that the solution given by (4.6) is the maximal solution. The uniqueness of maximal solution is obvious.

Prescribed curvature problem on complete noncompact manifolds. Note that the equation (3.10) is a special case of (4.1). As a consequence of Theorem 4.1, we obtain the following result and Theorem 1.14.

Theorem 4.6. Suppose $(f, \Gamma)$ satisfies (1.2), (1.11), (1.19) and (4.2). Let $(M, g)$ be a complete noncompact Riemannian manifold of dimension $n \geq 3$ and with a $C^2$ complete conformal metric $g$ subject to

$$\lambda(-g^{-1}A_g) \in \Gamma \quad \text{in} \quad M$$

and

$$f(\lambda(-g^{-1}A_g)) \geq \Lambda_1 \psi \quad \text{in} \quad M$$

where $0 < \psi \in C^\infty(M)$ and $\Lambda_1$ is a uniform positive constant. Then there exists a unique smooth maximal complete metric $\tilde{g} = e^{2u}g$ satisfying $f(\lambda(-\tilde{g}^{-1}A_{\tilde{g}})) = \psi$ and $f(\lambda(-g^{-1}A_g)) \in \Gamma$.

Remark 4.7. When $M \subset \mathbb{R}^n$ is a bounded domain, the continuous viscosity solution for $f(\lambda(-\tilde{g}^{-1}A_{\tilde{g}})) = 1$ was studied by González-Li-Nguyen \cite{20} using Perron’s method. The Dirichlet problem for $\sigma_k^{1/k}(\tilde{g}^{-1}A_{\tilde{g}}) = \psi$ was studied by Guan \cite{21}, while the obtained metric is not complete.

Our method is fairly different, which is based on partial uniform ellipticity. In addition, the metrics obtained in this paper are smooth and complete.

As a consequence of Theorem 1.14, together with the Maclaurin inequalities

$$\left(\sigma_k(\lambda)/C_n^k\right)^{1/k} \leq \left(\sigma_l(\lambda)/C_n^l\right)^{1/l}, \quad \forall \lambda \in \Gamma_k, \quad 1 \leq l < k \leq n$$

we obtain the existence of solutions to prescribed $\sigma_k$-curvature equations.
**Theorem 4.8.** Fix $\Gamma = \Gamma_k$ and $(\alpha, \tau)$ satisfying (1.18), we assume $(M, g)$ is a complete noncompact Riemannian manifold of dimension $n \geq 3$ with $\lambda(g^{-1}A^{\tau,\alpha}_g) \in \Gamma_k$ and

$$c^{1/k}_k(\lambda(g^{-1}A^{\tau,\alpha}_g)) \geq \delta$$

for some positive constant $\delta$. Then there is a unique $(\tilde{\mathbf{g}}, \tilde{\mathbf{g}}_1, \cdots, \tilde{\mathbf{g}}_k)$, which consists of smooth maximal complete conformal metrics, to satisfy

$$\begin{cases}
-R_{\tilde{g}_1} = \frac{m(n-2)}{n(n+2-2\eta)}, \\
\sigma_2(\lambda(\tilde{g}_2^{-1}A^{\tau,\alpha}_{\tilde{g}_2})) = \frac{n(n-1)}{2}, \lambda(g^{-1}A^{\tau,\alpha}_{\tilde{g}_2}) \in \Gamma_2, \\
& \vdots \\
\sigma_k(\lambda(\tilde{g}_k^{-1}A^{\tau,\alpha}_{\tilde{g}_k})) = \frac{n^k}{k!(n-k)!}, \lambda(g^{-1}A^{\tau,\alpha}_{\tilde{g}_k}) \in \Gamma_k.
\end{cases}$$

The approximate method also works for conformal scalar curvature equation.

**Theorem 4.9.** Let $(M, g)$ be a complete noncompact Riemannian manifold of dimension $n \geq 3$ with negative scalar curvature. Let $\psi$ be a smooth positive function which is bounded from above in terms of $-R_g$, i.e. there is a positive constant $\delta$ such that $0 < \psi \leq -\delta R_g$. Then there exists a unique smooth maximal complete conformal metric $\tilde{g}$ with prescribed scalar curvature $-R_{\tilde{g}} = \psi$.

**Remark 4.10.** One can apply Theorem 4.9 to improve [3, Theorem A] of Aviles-McOwen. More precisely, let $(M, g)$ be a complete noncompact Riemannian manifold with nonpositive scalar curvature subject to $R_g(x) < -\delta < 0$ outside a compact subset $K$, then for any smooth positive function $\psi$ with $\text{sup}_M \psi < +\infty$, there is a unique smooth maximal complete conformal metric $\tilde{g}$ with $-R_{\tilde{g}} = \psi$.

5. **The Dirichlet problem**

As described in Section 4, it only requires to solve the equation (4.4) on $M$ with infinite boundary value condition (4.5). Throughout this section, we assume for simplicity that $(M, g)$ is a compact connected Riemannian manifold with smooth boundary $\partial M$, and $\varphi \in C^\infty(\partial M)$.

5.1. **The Dirichlet problem with finite boundary value condition.** First we solve the Dirichlet problem with finite boundary value condition.

**Theorem 5.1.** Suppose, in addition to (1.2), (1.10), (1.11), (1.9), (1.12) and (4.2), there is a $C^2$ admissible function. Then the equation (4.1) admits a unique smooth admissible solution $u$ with $u|_{\partial M} = \varphi$.

**$C^0$-estimate.** By the maximum principle, we obtain $C^0$-estimate.

**Lemma 5.2.** In addition to (1.2), (1.4), (1.9), (1.10) and (1.11), we assume there is an admissible function $w$. Let $u \in C^2(\bar{M})$ be an admissible solution to the equation (4.1) with $u = \varphi$ on $\partial M$, then

$$\min_{\partial M} \left\{ \inf_{\partial M} (\varphi - w), A_1 - \sup_{M} w \right\} \leq u - w \leq \max_{\partial M} \left\{ \sup_{\partial M} (\varphi - w), A_2 - \inf_{M} w \right\},$$
where
\[ \sup_{x \in M} \psi(x, A_1) \leq \inf_{x \in M} f(\lambda(g^{-1}(\nabla^2 w + A(x, \nabla w)))), \]
\[ \inf_{x \in M} \psi(x, A_2) \geq \sup_{x \in M} f(\lambda(g^{-1}(\nabla^2 w + A(x, \nabla w))). \]

**Boundary gradient estimate.** We denote the distance from \( x \) to \( \partial M \) with respect to \( g \) by
\[ \sigma(x) = \text{dist}_g(x, \partial M). \]
Then \( \sigma(x) \) is smooth near the boundary. We use \( \sigma \) to construct local barriers. Denote
\[ \Omega_\delta := \{ x \in M : \sigma(x) < \delta \}. \]

The assumption (1.12) on \( A(x, p) \) yields that there exists a positive uniform constant \( B \) such that
\[ |\text{tr}(g^{-1}A(x, p))| \leq B(1 + |p|^2). \]

**Local upper barrier.** Take \( \tilde{w} = \varepsilon \log(1 + \frac{\sigma}{\delta}) + \varphi \), where \( \varepsilon \) is a positive constant to be determined later. By (5.3), one can see that there is a uniform constant \( C_B \) such that
\[ \text{tr}(g^{-1}A(x, \tilde{w})) + \Delta \tilde{w} \leq \frac{2(\delta^2 + \sigma)\Delta \sigma - |\nabla \sigma|^2}{4C_B'(\delta^2 + \sigma)^2} + C_B' |\nabla \varphi|^2 + \Delta \varphi \leq 0 \]
provided \( 0 < \delta \ll 1 \). Notice also that \( \lim_{\delta \to 0^+} \tilde{w}|_{\sigma=\delta} = +\infty \). Combining with the \( C^0 \)-estimate in Lemma 5.2, when \( \delta \) is sufficiently small, we know
\[ u \leq \tilde{w} \text{ on } \Omega_\delta. \]

**Local lower barrier.** Fix \( k \geq 1 \). Pick \( \varepsilon \) a positive constant to be determined later. Inspired by [22] we take
\[ w = \varepsilon \log \frac{\delta^2}{k\sigma + \delta^2} + \varphi. \]
The straightforward computation gives that for any \( k \geq 1 \),
\[ \nabla w = -\frac{k\varepsilon \nabla \sigma}{\delta^2 + k\sigma} + \nabla \varphi, \quad \nabla^2 w = \frac{k^2 \varepsilon \rho \nabla \sigma}{(\delta^2 + k\sigma)^2} - \frac{k\varepsilon \nabla^2 \sigma}{(\delta^2 + k\sigma)^2} + \nabla^2 \varphi. \]

Note that \( |\nabla \sigma| = 1 \) on \( \partial M \). By (5.3) again, there exist positive constants \( C_B \) and \( \delta' \) such that
\[ \left| \nabla^2 \varphi + A \left( x, \nabla \varphi - \frac{k\varepsilon \nabla \sigma}{\delta^2 + k\sigma} \right) \right| \leq \frac{k^2 \varepsilon C_B |\nabla \sigma|^2}{(\delta^2 + k\sigma)^2} \text{ in } \Omega_\delta, \forall 0 < \delta < \delta'. \]

By Lemma 2.18, \( f \) satisfies (1.6). According to Corollary 2.11 or 2.12,
\[ \Gamma^\infty_{\mathbb{R}^{n-1}} = \mathbb{R}^{n-1} \text{ and } (0, \ldots, 0, 1) \in \Gamma. \]
Theorem 5.3. Let together with Evans-Krylov theorem and Schauder theory, one can derive higher
It follows from Lemmas 5.4, we obtain estimates up to second order
Combining with local estimates in Theorems 8.1-8.2, and boundary estimate in
Theorem 8.4, we obtain estimates up to second order
Together with Evans-Krylov theorem and Schauder theory, one can derive higher
estimates. By the standard continuity method, we obtain Theorem 5.1.

5.2. The Dirichlet problem with infinite boundary value condition.

Theorem 5.3. Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 3\) with smooth boundary. Suppose in addition that (1.2), (1.9), (1.12), (1.10), (1.11) and (4.2) hold. Then the equation (4.1) admits a smooth admissible solution \(u\) with

\[
\lim_{x \to \partial M} u(x) = +\infty,
\]

provided that there is a \(C^2\) admissible function on \(\bar{M}\).

The following lemma is useful. It is a consequence of the concavity.

Lemma 5.4. Let \(f\) satisfy (1.2) and \(f(\bar{1}) < \sup_{\Gamma} f\), then

\[
\sum_{i=1}^{n} \lambda_i \geq n + A_f \left(f(\lambda) - f(\bar{1})\right), \quad \forall \lambda \in \Gamma,
\]

where \(A_f = n \left(\sum_{i=1}^{n} f_i(\bar{1})\right)^{-1}\). In particular, if \(f\) satisfies in addition

\[
f(t\lambda) = tf(\lambda), \quad \forall \lambda \in \Gamma, \quad t > 0,
\]

with normalization \(f(\bar{1}) = 1\), then

\[
\sum_{i=1}^{n} \lambda_i \geq nf(\lambda), \quad \forall \lambda \in \Gamma.
\]
Proof of Theorem 5.3. Let $u^{(k)}$ be an admissible solution to
\begin{equation}
(5.10) \quad f \left( \lambda_1 g^{-1}(\nabla^2 u^{(k)} + A(x, \nabla u^{(k)})) \right) = \psi(x, u^{(k)}), \quad u^{(k)}|_{\partial M} = \varepsilon \log k,
\end{equation}
where the constant $\varepsilon$ comes from (5.5). The comparison principle (Lemma 4.3) simply yields
\begin{equation}
(5.11) \quad u^{(k+1)} \geq u^{(k)}, \quad \forall k \geq 1.
\end{equation}
Namely, $\{u^{(k)}\}$ is an increasing sequence. From Lemma 5.4 we get
\begin{equation}
A_f \psi(x, u^{(k)}) \leq \Delta u^{(k)} + \text{tr}(g^{-1}A(x, \nabla u^{(k)})) + A_f \bar{f}(\bar{1}) - n.
\end{equation}
On the other hand, from (5.3)
\begin{equation}
(5.12) \quad \text{tr}(g^{-1}A(x, \nabla u^{(k)})) \leq B(1 + |\nabla u^{(k)}|^2).
\end{equation}

By the assumption (1.9) on $\psi(x, t)$ and the compactness of $\bar{M}$, there are constants $\Lambda_0$ and $\Theta$ such that
\begin{equation}
\psi(x, u^{(k)}) \geq \Lambda_0 e^{\Theta u^{(k)}}.
\end{equation}
Fix $\Theta$. Take $q$ a constant bigger than 1 but sufficiently close to 1, such that
\begin{equation}
\frac{\Theta}{q - 1} \geq B,
\end{equation}
where $B$ is as in (5.12). Fix such $q$. As a corollary of [46, Theorem 2.2], there is $\bar{u} \in C^\infty(M)$ such that
\begin{equation}
(5.13) \quad \Delta \bar{u} + \frac{\Theta}{q - 1} |\nabla \bar{u}|^2 = \frac{q - 1}{\Theta} e^{\Theta \bar{u}} \text{ in } M, \quad \lim_{x \to \partial M} \bar{u}(x) \to +\infty.
\end{equation}

We know that $u^{(k)} - \bar{u}$ achieves its maximum at an interior point $x_0 \in M$. At $x_0$,
\begin{equation}
\nabla u^{(k)} = \nabla \bar{u}, \quad \nabla^2 u^{(k)} \leq \nabla^2 \bar{u}.
\end{equation}
Thus
\begin{equation}
A_f \Lambda_0 e^{\Theta (u^{(k)} - \bar{u})} \leq \frac{q - 1}{\Theta} + \left( A_f \bar{f}(\bar{1}) + B \right) e^{-\Theta \bar{u}}.
\end{equation}
Notice also that $\inf_M \bar{u} > -\infty$. We conclude
\begin{equation}
(5.14) \quad u^{(k)} \leq \bar{u} + C
\end{equation}
for a uniform positive constant $C$, independent of $k$.

We have derived the (local) $C^0$-estimate in (5.11) and (5.14). We then have
\begin{equation}
(5.15) \quad u(x) = \lim_{k \to +\infty} u^{(k)}(x), \quad \forall x \in M.
\end{equation}

Combining with Evans-Krylov theorem, Schauder theory as well as the local estimates up to second derivatives established in Theorems 8.1-8.2 below, we can conclude that $u$ is a smooth admissible solution to (4.1) with $\lim_{x \to \partial M} u(x) = +\infty$.

Moreover, the inequality (5.6) gives (note $\varphi = \varepsilon \log k$)
\begin{equation}
(5.16) \quad u^{(k)} \geq \varepsilon \log \frac{k\delta^2}{k\sigma + \delta^2} \text{ on } \Omega_{\delta},
\end{equation}
which implies $u \geq -\varepsilon \log \sigma - C_0$ for some $C_0$ near the boundary. □
Proposition 5.5. Suppose that \( A(x, du) \) in the equation (4.1) is of the form
\[
A(x, \nabla u) = U + \alpha(x) |\nabla u|^2 g + \beta(x) du \otimes du + R(x, du)
\]
where \( U \) is a smooth symmetric \((0, 2)\) tensor, \( R(x, p) \) is a smooth symmetric \((0, 2)\) tensor depending linearly on \( p \). Assume in addition that \( \alpha(x) \) and \( \beta(x) \) are smooth functions with
\[
(\alpha(x), \cdots, \alpha(x), \alpha(x) + \beta(x) + 1) \in \Gamma \text{ in } \mathcal{M}.
\]
Then in (5.5) we can choose \( \varepsilon = 1 \), hence the solution \( u \) (defined by (5.15)) in Theorem 5.3 satisfies
\[
u \geq -\log \sigma - C_0
\]
for some \( C_0 \) near the boundary. Moreover, the metric \( \tilde{g} = e^{2u}g \) is complete.

Proof. We take \( \varepsilon = 1 \) in (5.5), i.e. \( w = \log \frac{k\delta^2}{\kappa \sigma + \delta^2} \). By (5.6),
\[
\nabla^2 w + A(x, \nabla w) = U + \frac{k^2 \alpha |\nabla \sigma|^2 g + k^2 (1 + \beta) d\sigma \otimes d\sigma}{(\delta^2 + k\sigma)^2} - \frac{k^2 \sigma}{\delta^2 + k\sigma} + R \left( x, \frac{-k d\sigma}{\delta^2 + k\sigma} \right)
\]
Note that
\[
\lambda \left( g^{-1} \left( \frac{k^2 \alpha |\nabla \sigma|^2 g + k^2 (1 + \beta) d\sigma \otimes d\sigma}{(\delta^2 + k\sigma)^2} \right) \right) = \frac{k^2 |\nabla \sigma|^2}{(\delta^2 + k\sigma)^2} (\alpha, \cdots, \alpha, \alpha + \beta + 1).
\]
So we can prove by Lemma 2.1, that for \( 0 < \delta \ll 1 \),
\[
f \left( \lambda (g^{-1} (\nabla^2 w + A(x, \nabla w))) \right) \geq \psi(x, w) \text{ on } \Omega_\delta.
\]
Similarly, we obtain an inequality analogous to (5.6) (or (5.16)) near boundary
\[
u^{(k)} \geq \log \frac{k\delta^2}{\kappa \sigma + \delta^2}
\]
as required. \( \square \)

Remark 5.6. Since \( \Gamma \) is of type 2 (by Corollary 2.12 or 2.11), the assumption (5.18) is automatically satisfied, provided
\[
\alpha(x) \geq 0, \ \alpha(x) + \beta(x) + 1 \geq 0, \ n\alpha(x) + \beta(x) + 1 > 0.
\]

As a corollary, we get the following result and Theorem 1.9.

Theorem 5.7. Suppose \((f, \Gamma)\) satisfies (1.2), (1.11), (1.19) and (4.2). Let \((M, g)\) be a compact connected Riemannian manifold of dimension \( n \geq 3 \) with smooth boundary and support a \( C^2 \) conformal metric satisfying (4.7). Then for any \( 0 < \psi \in C^\infty(\bar{M}) \), there exists at least one smooth complete metric \( \bar{g} = e^{2u}g \) satisfying \( f(\lambda(-\bar{g}^{-1}A_{\bar{g}})) = \psi \) and \( \lambda(-\bar{g}^{-1}A_{\bar{g}}) \in \Gamma \) in \( \mathcal{M} \).
6. Construction of admissible metrics

Now we focus on the equation (1.14). Throughout this section and Section 7, the parameters in $A^{\tau,\alpha}_g$ obey (1.20), i.e.

\[
\begin{cases}
\tau \leq 0 & \text{if } \alpha = -1, \\
\tau \geq 2 & \text{if } \alpha = 1.
\end{cases}
\]

And we denote

\[
V[u] = \Delta u g - \varrho \nabla^2 u + \gamma |\nabla u|^2 g + \varrho du \otimes du + A,
\]

\[
\varrho = \frac{n - 2}{\tau - 1}, \quad \gamma = \frac{(\tau - 2)(n - 2)}{2(\tau - 1)}, \quad A = \frac{n - 2}{\alpha(\tau - 1)} A^{\tau,\alpha}_g.
\]

Here one can check $V[u] = \frac{n - 2}{\alpha(\tau - 1)} A^{\tau,\alpha}_g \tilde{g} = e^{2u} g$. Under the assumption (1.20), we have the following lemma.

**Lemma 6.1.** Under the assumption (1.20), we have

\[
(6.1) \quad \gamma = \frac{(n - 2)(\tau - 2)}{2(\tau - 1)} \geq 0, \quad \gamma + \varrho = \frac{\tau(n - 2)}{2(\tau - 1)} \geq 0.
\]

The following lemma asserts that a compact manifold with boundary carries a Morse function without critical points. The construction of such a Morse function is standard in differential topology.

**Lemma 6.2.** Let $(M, g)$ be a compact connected Riemannian manifold of dimension $n \geq 2$ with smooth boundary. Then there is a smooth Morse function $v$ without critical points, that is $dv \neq 0$ in $\bar{M}$.

**Proof.** Let $X$ be the double of $M$. Let $w$ be a smooth Morse function on $X$ with the critical set $\{p_i\}_{i=1}^{m+k}$, among which $p_1, \ldots, p_m$ are all the critical points being in $\bar{M}$. Pick $q_1, \ldots, q_m \in X \setminus M$ but not the critical point of $w$. By homogeneity lemma (see e.g. [47]), one can find a diffeomorphism $h : X \to X$, which is smoothly isotopic to the identity, such that

- $h(p_i) = q_i$, $1 \leq i \leq m$,
- $h(p_i) = p_i$, $m + 1 \leq i \leq m + k$.

Then $v = w \circ h^{-1}|_M$ is the desired Morse function. \qed

**Proposition 6.3.** For $(\alpha, \tau)$ satisfying (1.20), there exists a smooth conformal admissible metric on $\bar{M}$.

**Proof.** According to Lemma 6.2, there exists a smooth function $v$ with

\[
|\nabla v|^2 \geq a_0 \text{ in } \bar{M}
\]

for some positive constant $a_0$. Without loss of generality, $v \geq 0$.

Set $\varrho = e^{Nv}, \quad g = e^{2u} g$, then

\[
V[u] = N^2 e^{Nv} \left( (\Delta v - \varrho \nabla^2 v)/N + (1 + \gamma e^{Nv}) |\nabla v|^2 g + \varrho (e^{Nv} - 1)dv \otimes dv \right) + A.
\]

The discussion consists of two cases:
Case 1: \( \varrho > 0 \). In this case
\[
(1 + \gamma e^{Nv})|\nabla v|^2 g + \varrho (e^{Nv} - 1)dv \otimes dv \geq |\nabla v|^2 g.
\]

Case 2: \( \varrho < 0 \). In this case
\[
(1 + \gamma e^{Nv})|\nabla v|^2 g + \varrho (e^{Nv} - 1)dv \otimes dv \geq \left((1 - \varrho) + (\gamma + \varrho) e^{Nv}\right)|\nabla v|^2 g \geq |\nabla v|^2 g.
\]

Here we use (6.1). Therefore, \( \lambda(g^{-1}V[u]) \in \Gamma \) in \( \bar{M} \) for \( N \gg 1 \). That is, \( g = e^{2u}g \) is an admissible metric.

\[\square\]

Remark 6.4. By Corollary 3.4, the construction also works if \( (\gamma, \cdots, \gamma, \gamma + \varrho) \in \Gamma \).

7. Asymptotic Behavior and Uniqueness of Solutions

In this section we further assume \( f \) satisfies (5.8), i.e.,
\[f(t\lambda) = tf(\lambda), \quad \forall \lambda \in \Gamma, \quad t > 0, \quad \text{with normalization} \quad f(\bar{I}) = 1.
\]

Then the equation (1.14) reads as follows
\[f(\lambda(g^{-1}V[u])) = \frac{(n-2)\psi}{\alpha(\tau - 1)} e^{2u}.
\]

For the complete conformal metrics satisfying (1.14), we prove the following asymptotic behavior and uniqueness of metrics, when the prescribed function is a constant when restricted to boundary. Let \( \sigma \) denote as in (5.1) the distance function.

**Theorem 7.1.** Let \( (M, g) \) be a compact connected Riemannian manifold of dimension \( n \gg 3 \) with smooth boundary. Suppose in addition that (1.2), (1.11), (5.8), (1.18) and (1.20) hold. Then for any positive smooth function with \( \psi_{\partial M} \equiv 1 \), there is a unique smooth complete admissible metric \( g = e^{2u}g \) satisfying (1.14). Moreover,
\[
\lim_{x \to \partial M} (\bar{u}(x) + \log \sigma(x)) = \frac{1}{2} \log \frac{\alpha(n\tau + 2 - 2n)}{2(n-2)}.
\]

We now summarize a theorem due to Aviles-McOwen, extending a famous result of Loewner-Nirenberg to general Riemannian manifolds.

**Theorem 7.2 ([4]).** Let \( (M, g) \) be a compact connected Riemannian manifold of dimension \( n \gg 3 \) with smooth boundary. There is a smooth complete conformal metric with scalar curvature \(-1\).

The theorem yields that there is a \( \bar{u} \in C^\infty(M) \) such that
\[
2(n-1)\Delta \bar{u} + (n-1)(n-2)|\nabla \bar{u}|^2 - R_g = e^{2\bar{u}} \quad \text{in} \quad M, \quad \lim_{x \to \partial M} \bar{u}(x) = +\infty.
\]

Geometrically, \( e^{2\bar{u}}g \) is a smooth complete metric with constant scalar curvature \(-1\).

For conformal scalar curvature equation (7.2) on \( M = \Omega \) a smooth bounded domain of Euclidean spaces, Loewner-Nirenberg [41] proved the asymptotic ratio
\[
\lim_{x \to \partial M} (\bar{u}(x) + \log \sigma(x)) = \frac{1}{2} \log n(n-1).
\]
This asymptotic property can be extended to general Riemannian manifolds, as pointed out by McOwen in [46]

Theorem 7.1 follows from the propositions below.

**Proposition 7.3.** Let \( \overline{g}_{\infty} = e^{2\overline{w}_{\infty}} g \) be a complete metric obeying the equation (1.14) with (1.2), (1.11), (5.8), (1.18) and (1.20), then

\[
\lim_{x \to \partial M} (\overline{u}_{\infty}(x) + \log \sigma(x)) \leq \frac{1}{2} \log \frac{\alpha(n \tau + 2 - 2n)}{2(n - 2) \inf_{\partial M} \psi}.
\]

**Proof.** We utilize an approximation in analogy with that used in proof of Theorem 4.1. Let \( \Omega_\delta \) be as in (5.2), and let

\[
\Omega_{\delta,\delta'} = \{ x \in M : 0 < \delta' < \sigma(x) < \delta - \delta' \}.
\]

We choose \( 0 < \delta' < \frac{\delta}{2} \ll 1 \) such that \( \Omega_\delta \) and \( \Omega_{\delta,\delta'} \) are both smooth.

We use Theorem 7.2 repeatedly. By Theorem 7.2, \( \Omega_\delta \) admits a smooth complete conformal metric \( g_{\infty}^\delta = e^{2w_{\infty}^\delta} g \) with constant scalar curvature \(-1\). Again, there is a smooth complete metric \( g_{\delta,\delta'}^\delta = e^{2w_{\infty}^\delta} g \) on \( \Omega_{\delta,\delta'} \) with constant scalar curvature \(-1\), that is,

\[
\tr(g^{-1}V[u_{\infty}^\delta]) = \frac{n \tau + 2 - 2n}{2(n - 1)(\tau - 1)} e^{2u_{\infty}^\delta} \text{ in } \Omega_{\delta,\delta'}, \quad u_{\infty}^\delta|_{\partial \Omega_{\delta,\delta'}} = +\infty.
\]

By Lemma 2.15, \( \frac{n \tau + 2 - 2n}{2(n - 1)(\tau - 1)} > 0 \). By maximum principle

\[
u_{\infty}^\delta \geq w_{\infty}^\delta \text{ in } \Omega_{\delta,\delta'}.
\]

Furthermore, for any \( 0 < \delta' < \delta_0' < \frac{\delta}{2} \), we have \( \Omega_{\delta,\delta'_0} \subset \Omega_{\delta,\delta'} \) and

\[
u_{\infty}^\delta \leq u_{\infty}^{\delta_0'} \text{ in } \Omega_{\delta,\delta'_0}.
\]

According to (5.9), the solution \( \overline{u}_{\infty} \) satisfies

\[
\tr(g^{-1}V[\overline{u}_{\infty}]) \geq \frac{n(n - 2)\psi}{\alpha(\tau - 1)} e^{2\overline{w}_{\infty}}.
\]

The maximum principle implies

\[
u_{\infty}^\delta \geq \overline{u}_{\infty} + \frac{1}{2} \log \left( \frac{2n(n - 1)(n - 2)}{\alpha(n \tau + 2 - 2n)} \inf_{\Omega_{\delta,\delta'}} \psi \right) \text{ in } \Omega_{\delta,\delta'}.
\]

Take \( u_{\infty}^\delta(x) = \lim_{\delta' \to 0} u_{\infty}^{\delta_0'}(x) \) (such a limit exists by (7.5)-(7.6)), then

\[
u_{\infty}^\delta \geq w_{\infty}^\delta, \quad u_{\infty}^\delta \geq \overline{u}_{\infty} + \frac{1}{2} \log \left( \frac{2n(n - 1)(n - 2)}{\alpha(n \tau + 2 - 2n)} \inf_{\Omega_{\delta}} \psi \right) \text{ in } \Omega_{\delta}
\]

and \( e^{2u_{\infty}^\delta} g \) is a smooth complete conformal metric on \( \Omega_{\delta} \) of scalar curvature \(-1\). (We have local estimates since it is of uniform ellipticity). Furthermore, (7.3) gives

\[
\lim_{x \to \partial M} (u_{\infty}^\delta(x) + \log \sigma(x)) = \frac{1}{2} \log[n(n - 1)].
\]

Putting them together, we get (7.4). \( \square \)
Proposition 7.4. Assume (1.2), (1.11), (5.8), (1.18) and (1.20) hold. Then any admissible complete metric \( g_\infty = e^{2u_\infty} g \) satisfying (1.14) obeys

\[
\lim_{x \to \partial M} (\bar{u}_\infty(x) + \log \sigma(x)) \geq \frac{1}{2} \log \frac{\alpha(n \tau + 2 - 2n)}{2(n-2) \sup_{\partial M} \psi}.
\]

Proof: The proof is different from that presented in [41, 29]. The key ingredients in the proof are (5.9) and \( |\nabla \sigma| = 1 \) on \( \partial M \). Fix a constant \( 0 < \epsilon < 1 \). We consider

\[
\begin{align*}
\begin{cases}
    f(\lambda(g^{-1} V(u_{k,\epsilon}))) = \frac{(n-2) \psi}{\alpha(\tau-1)} e^{2u_{\epsilon}} \text{ in } M, \\
    u_{k,\epsilon} = \log k + \frac{1}{2} \log \frac{(1-\epsilon)^2 \alpha(n \tau + 2 - 2n)}{2(n-2) \sup_{\partial M} \psi + \epsilon} \text{ on } \partial M.
\end{cases}
\end{align*}
\]

The solvability is obtained by Theorem 5.1. The maximum principle implies

\[
\inf_M u_{k,\epsilon} \geq \frac{1}{2} \min \left\{ \inf_M \frac{f(\lambda(g^{-1} A_0^\alpha))}{\psi}, \log \frac{k^2 (1-\epsilon)^2 \alpha(n \tau + 2 - 2n)}{2(n-2) \sup_{\partial M} \psi + \epsilon} \right\}.
\]

The local subsolution is given by

\[
h_{k,\epsilon,\delta}(\sigma) = \log \frac{k}{k \sigma + 1} + \frac{1}{2} \log \frac{(1-\epsilon)^2 \alpha(n \tau + 2 - 2n)}{2(n-2) \sup_{\partial M} \psi + \epsilon} + \frac{1}{\sigma + \delta} - \frac{1}{\delta}.
\]

The part \( \frac{1}{\sigma + \delta} - \frac{1}{\delta} \) is inspired by [29]. Straightforward computation tells

\[
\begin{align*}
h_{k,\epsilon,\delta}' &= -\frac{k}{k \sigma + 1} - \frac{1}{(\sigma + \delta)^2}, \\
h_{k,\epsilon,\delta}'' &= \frac{k^2}{(k \sigma + 1)^2} + \frac{2}{(\sigma + \delta)^3}, \\
h_{k,\epsilon,\delta}' &= \frac{k^2}{(k \sigma + 1)^2} + \frac{2k}{(k \sigma + 1)(\sigma + \delta)^2}, \\
h_{k,\epsilon,\delta}'' + \gamma h_{k,\epsilon,\delta}^2 &= \frac{(1+\gamma)k^2}{(k \sigma + 1)^2} + \frac{2}{(\sigma + \delta)^3} + \frac{\gamma}{(\sigma + \delta)^4} + \frac{2k \gamma}{(k \sigma + 1)(\sigma + \delta)^2}, \\
V[h_{k,\epsilon,\delta}] &= (h_{k,\epsilon,\delta}'' + \gamma h_{k,\epsilon,\delta}^2) |\nabla \sigma|^2 g + g(h_{k,\epsilon,\delta}^2 - h_{k,\epsilon,\delta}'') d\sigma \otimes d\sigma \\
&+ h_{k,\epsilon,\delta}' \partial (\Delta \sigma \cdot g - g \nabla^2 \sigma) + A.
\end{align*}
\]

By (6.1) we have \( \varphi + \gamma \geq 0 \), then

\[
\begin{align*}
h_{k,\epsilon,\delta}'' + \gamma h_{k,\epsilon,\delta}^2 + \varphi (h_{k,\epsilon,\delta}^2 - h_{k,\epsilon,\delta}'') &\geq \frac{(1+\gamma)k^2}{(k \sigma + 1)^2} + \frac{2(1-\varphi)}{(\sigma + \delta)^2}.
\end{align*}
\]

Next we are going to verify that

Claim. If \( \delta \leq \frac{1}{4} \) or \( k \geq \frac{1}{\delta} \) then

\[
V[h_{k,\epsilon,\delta}] \geq \left( \frac{(1+\gamma)k^2}{(k \sigma + 1)^2} + \frac{2}{(\sigma + \delta)^3} \right) |\nabla \sigma|^2 g - \left( \frac{k}{k \sigma + 1} + \frac{1}{(\sigma + \delta)^2} \right) (\Delta \sigma g - g \nabla^2 \sigma) + A.
\]
First, it is easy to see $h_{k,\varepsilon,\delta}'^2 - h_{k,\varepsilon,\delta}'' > 0$ provided $\delta < \frac{1}{\alpha}$ or $k > \frac{1}{\delta}$. So the case $\varepsilon > 0$ is trivial. When $\varepsilon < 0$, together with (7.12), we get
\[(h_{k,\varepsilon,\delta}'' + y h_{k,\varepsilon,\delta}'^2)\|\nabla \sigma\|^2 \geq g(h_{k,\varepsilon,\delta}'^2 - h_{k,\varepsilon,\delta}'')d\sigma \otimes d\sigma \geq \frac{(1 + \gamma) k^2}{(k\sigma + 1)^2} + \frac{2}{(\sigma + \delta)^3} |\nabla \sigma|^2 g.
\]
This completes the proof of the claim.

Note that $|\nabla \sigma|^2 = 1$ on $\partial M$, $\Delta \sigma \cdot g - \varphi^2 \nabla \sigma$ is bounded in $\tilde{M}$, $\frac{k}{\sigma + \delta} \leq \frac{1}{\sigma + \delta}$ for $k > \frac{1}{\delta}$, and $\frac{1}{\sigma + \delta} \geq 1$ in $\Omega_\delta$ if $0 < \delta < 1$. For $\varepsilon$ fixed, there is a $\delta_{\varepsilon}$ depending on $\varepsilon$ and other known data such that, in $\Omega_{\delta}$ for $0 < \delta < \delta_{\varepsilon}$,
\[
(1) \quad \sigma(x) \text{ is smooth, and } |\nabla \sigma|^2 \geq 1 - \varepsilon,
\]
\[
(2) \quad \frac{2}{(\sigma + \delta)^3} |\nabla \sigma|^2 \geq \frac{1}{\sigma + \delta} (\Delta \sigma \cdot g - \varphi^2 \nabla \sigma) + A > 0.
\]
\[
(3) \quad k^2 \sigma^2 (1 + \gamma) |\nabla \sigma|^2 \geq \frac{k}{\sigma + \delta} (\Delta \sigma \cdot g - \varphi^2 \nabla \sigma) \geq 0.
\]
\[
(4) \quad \sup_{\partial M} \psi + \varepsilon \geq \sup_{\Omega_{\delta}} \psi.
\]
From now on we fix $0 < \delta < \delta_{\varepsilon}$ (for example $\delta = \frac{\delta_{\varepsilon}}{2}$) and $k > \frac{1}{\delta}$. Putting the discussion above together, in $\Omega_{\delta}$, we have
\[
(7.13) \quad f(\lambda(g^{-1}V[h_{k,\varepsilon,\delta}]))) \geq f\left(\frac{k^2 (1 + \gamma)(1 - \varepsilon)}{(k\sigma + 1)^2} \cdot |\nabla \sigma|^2 \right) \geq \frac{(n - 2)\varepsilon}{\alpha(\tau - 1)} e^{2h_{k,\varepsilon,\delta}}.
\]
Note that $u_{k,\varepsilon} = h_{k,\varepsilon,\delta}$ on $\partial M$, and $u_{k,\varepsilon}|_{\tau = \delta} \geq h_{k,\varepsilon,\delta}(\delta)$ (according to (7.11)). Here we also use $\frac{\log \lambda_M}{\lambda_M} \leq \frac{1}{\delta} \to 0$ as $\delta \to 0^+$. Then the maximum principle yields
\[u_{k,\varepsilon} \geq h_{k,\varepsilon,\delta} = \log \frac{k}{k\sigma + \delta} + \frac{1}{2} \log \frac{1}{2(n - 2)(\sup_{\partial M} \psi + \varepsilon)} + \frac{1}{\sigma + \delta - 1}\delta \]
on $\Omega_{\delta}$. From (7.10), let $k \to +\infty$, we know that on $\Omega_{\delta}$,
\[\overline{u}_{\infty}(x) + \log \sigma(x) \geq \frac{1}{2} \log \frac{1}{2(n - 2)(\sup_{\partial M} \psi + \varepsilon)} + \frac{1}{\sigma + \delta - 1}\delta.
\]
Thus (7.8) follows.

\[\square\]

**Proposition 7.5.** Suppose (1.2), (1.11), (1.18), (5.8) and (1.20) hold. Let $u_{\infty}(x) = \lim_{k \to +\infty} u^{(k)}(x)$, where
\[f(\lambda(g^{-1}V[u^{(k)}]))) = \frac{(n - 2)\varepsilon}{\alpha(\tau - 1)} e^{2u^{(k)}} \text{ in } M, \quad u^{(k)} = \log k \text{ on } \partial M.
\]
For any admissible solution $\overline{u}_{\infty}$ to the Dirichlet problem for equation (7.1) with infinite boundary value condition, then
\[u_{\infty} \leq \overline{u}_{\infty} \leq u_{\infty} + \frac{1}{2} (\sup_{\partial M} \psi - \inf_{\partial M} \log \psi) \text{ in } M.
\]

**Proof.** The comparison principle yields $\overline{u}_{\infty} \geq u^{(k)}$ ($\forall k > 1$). Then $\overline{u}_{\infty} \geq u_{\infty}$.

Next, we prove the second inequality. Note that (7.4) and (7.8) imply
\[\lim_{x \to \partial M} (\overline{u}_{\infty}(x) - u_{\infty}(x)) \leq \frac{1}{2} (\sup_{\partial M} \psi - \inf_{\partial M} \log \psi).
\]
Assume there is an interior point \( x_0 \in M \) such that
\[
(\overline{u}_\infty - u_\infty)(x_0) = \sup_M (\overline{u}_\infty - u_\infty) > \frac{1}{2} (\sup M \log \psi - \inf M \log \psi).
\]
This is a contradiction since the maximum principle yields \( \overline{u}_\infty(x_0) \leq u_\infty(x_0) \). \( \Box \)

8. Local and boundary estimates for nonlinear uniformly elliptic equations

This section is devoted to deriving the local and boundary estimates for a more general equation
\[
(8.1) \quad F(\nabla^2 u + A(x, u, \nabla u)) := f \left( \lambda (g^{-1} \nabla^2 u + A(x, u, \nabla u)) \right) = \psi(x, u, \nabla u)
\]
which is of fully uniform ellipticity, i.e.,
\[
f_i(\lambda) \geq \theta \sum_{j=1}^n f_j(\lambda) > 0 \text{ in } \Gamma, \quad \forall 1 \leq i \leq n,
\]
where \( \psi(x, z, p) \) and \( A(x, z, p) \) denote respectively smooth function and symmetric (0, 2)-type tensor of variables \( (x, z, p) \), \( (x, p) \in TM, z \in \mathbb{R} \).

Again, we say \( u \) is an admissible function for (8.1) if
\[
\lambda (g^{-1} (\nabla^2 u + A(x, u, \nabla u))) \in \Gamma.
\]
Throughout this section, for \( u \) we denote
\[
g[u] = \nabla^2 u + A(x, u, \nabla u), \quad \psi[u] = \psi(x, u, \nabla u).
\]
Moreover, for any admissible solution \( u \) to (8.1), \( g = g[u] \), \( \lambda = \lambda (g^{-1} g) \), \( F^{ij} = \frac{\partial F}{\partial u_{ij}}(g) \). Then the matrix \( \{F^{ij}\} \) (with respect to \( \{g^{ij}\} \)) has eigenvalues \( f_1, \cdots, f_n \).

Moreover, \( \sum_{i=1}^n f_i = F^{ij} g_{ij}, \sum_{i=1}^n f_i \lambda_i = F^{ij} g_{ij} \).

The local estimates for second derivatives can be stated as follows.

**Theorem 8.1.** Let \( B_r \subset M \) be a geodesic ball of radius \( r \). Let \( u \in C^4(B_r) \) be an admissible solution to equation (8.1) in \( B_r \). Assume (1.2), (1.6) and (4.2) hold. Then
\[
\sup_{B_{r/2}} |\nabla^2 u| \leq C,
\]
where \( C \) depends on \( |u|_{C^1(B_r)} \), \( r^{-1} \), \( \theta^{-1} \) and other known data.

We will prove local gradient estimate under asymptotic assumptions:
\[
|D_pA| \leq \gamma(x, z)|p|, \quad D_zA + \frac{1}{|p|^2} g(\nabla' A, p) \leq \beta(x, z, |p|)|p|^2 g,
\]
\[
|D_p\psi| \leq \gamma(x, z)|p|, \quad -D_z\psi - \frac{1}{|p|^2} g(\nabla' \psi, p) \leq \beta(x, z, |p|)|p|^2,
\]
where \( \gamma = \gamma(x, z) \) and \( \beta = \beta(x, z, r) \) are positive continuous functions with
\[
\lim_{r \to +\infty} \beta(x, z, r) = 0, \quad (x, z, r) \in \bar{M} \times \mathbb{R} \times [0, +\infty).
\]
Here \( \nabla' \psi, \nabla' A \) and \( D_pA \) are as in Introduction, and the meanings of \( D_zA, D_z\psi \) and \( D_p\psi \) are similar.
Theorem 8.2. Let $B_r \subset M$ as above and $u \in C^3(B_r)$ be an admissible solution to (8.1) in $B_r$. In addition to (1.2), we assume (4.2) and (8.2) hold. Suppose there is a positive constant $\kappa_0$ depending not on $\nabla u$ such that

$$F^{ij}g_{ij} \geq \kappa_0.$$  

Then there is a positive constant $C$ depending on $|u|_{C^0(B_r)}$, $\theta^{-1}$ and other known data such that

$$\sup_{B_{r/2}} |\nabla u| \leq C/r.$$  

Remark 8.3. By Lemma 2.1 and (2.1), if (1.2) and (1.6) hold then

$$\sum_{i=1}^n f_i(\lambda) > \kappa := (f(R_0 \bar{1}) - \psi[u])/R_0 > 0 \text{ for some } R_0 > 0.$$  

Moreover, if $f$ is homogeneous of degree one, then $\sum_{i=1}^n f_i(\lambda) > f(\bar{1}) > 0$.

As a result, the assumption (8.3) holds when $f$ is a homogeneous function of degree one, or $f$ satisfies (1.6) and $\psi[u] = \psi(x, u)$.

The boundary estimates for second derivatives are also obtained.

Theorem 8.4. Let $(M, g)$ be a compact Riemannian manifold with smooth boundary. Suppose (1.2), (1.6) and (4.2) hold. Let $u \in C^3(M) \cap C^2(\bar{M})$ be an admissible solution to (8.1) with $u = \varphi$ on $\partial M$ for $\varphi \in C^3(\bar{M})$, then

$$\sup_{\partial M} |\nabla^2 u| \leq C$$  

holds for a uniform constant depending on $\theta^{-1}$, $|u|_{C^1(\bar{M})}$, $|\varphi|_{C^3(\bar{M})}$ and other known data.

8.1. Useful computation. Let $e_1, ..., e_n$ be the local frame as above. We denote $A_{ij}(x, u, \nabla u) = A(x, u, \nabla u)(e_i, e_j)$, and $g_{ij} = \nabla_i u + A_{ij}(x, u, \nabla u)$. The linearized operator of (8.1) is given by

$$\mathcal{L}v = F^{ij}\nabla_{ij}v + (F^{ij}A_{ij,kl} - \psi_{kl})\nabla kl v \text{ for } v \in C^2(\bar{M}).$$

Similar computation can be found in [24]. We have

$$\nabla_k g_{ij} = \nabla_k \nabla_{ij} u + \nabla_k A_{ij} + A_{ij,kl} \nabla_k u + A_{ijkl} \nabla_{kl} u.$$  

By differentiating equation (8.1),

$$F^{ij}\nabla_k g_{ij} = \nabla_k \psi + \psi_{kl} \nabla_k u + \psi_{kl} \nabla u,$$  

$$F^{ij}\nabla_{11} g_{ii} = \nabla_{11} \psi - F^{ijkl} \nabla_{11} g_{ij} + 2\nabla_{11} \psi \nabla_{11} u + \psi_{12} \nabla_{11} u + 2\nabla_{11} \psi_{kl} \nabla_{11} u$$  

$$+ 2\psi_{kl} \nabla_{11} u \nabla_{11} u + \psi_{zz} |\nabla_{11} u|^2 + \psi_{kl} |\nabla_{11} u|^2 + \psi_{11} \nabla_{11} u \nabla_{11} u + \psi_{11} \nabla_{11} u.$$  

Using (8.6), $\nabla_{ij} u = \nabla_{ij} u$ and $\nabla_{ijk} u = \nabla_{ijk} u - R_{ijk} \nabla_{ij} u$,

$$F^{ij}\nabla_{ij} u + (F^{ij}A_{ij,kl} - \psi_{kl})\nabla_{ij} u$$  

$$= \nabla_k \psi + \psi_{kk} \nabla_k u - F^{ij}(R_{ijk} \nabla_{ij} u + \nabla_k A_{ij} + A_{ijk} \nabla_{ij} u).$$
Let \( w = |\nabla u|^2 \). The straightforward computation yields
\[
\nabla_i w = 2 \nabla_k u \nabla_{jk} u, \quad \nabla_{ij} w = 2 \nabla_{ik} u \nabla_{jk} u + 2 \nabla_k u \nabla_{ijk} u, \tag{8.9}
\]
\[
\mathcal{L} w = 2 F^{ij} \nabla_{ik} u \nabla_{jk} u - 2 F^{ij} R_{ijk} \nabla_k u \nabla_{jk} u + 2 \nabla_k \psi \nabla_{jk} u + 2 \phi \nabla^2 \nabla u + 2 \psi \nabla |\nabla u|^2 - 2 F^{ij}(\nabla'_i A_{ij} \nabla_k u + A_{ijk} |\nabla u|^2). \tag{8.10}
\]

8.2. **Local gradient estimate.** Note that the equation (8.1) is of fully uniform ellipticity, the proof of local estimates is standard. For convenience, we present the details below. As above we denote \( w = |\nabla u|^2 \). We consider the quantity
\[
Q := \eta w e^\phi
\]
where \( \phi \) is determined later. Following [26] let \( \eta \) be a smooth function with compact support in \( B_r \subset M \) and
\[
0 \leq \eta \leq 1, \quad \eta|_{B_{2r}} \equiv 1, \quad |\nabla \eta| \leq C \sqrt{\eta}/r, \quad |\nabla^2 \eta| \leq C/r^2. \tag{8.11}
\]
The quantity \( Q \) attains its maximum at an interior point \( x_0 \in M \). We may assume \( |\nabla u|(x_0) \geq 1 \). By maximum principle at \( x_0 \)
\[
\nabla_i \eta/\eta + \nabla_i w + \nabla_i \phi = 0, \quad \mathcal{L}(\log \eta + \log w + \phi) \leq 0. \tag{8.12}
\]
Around \( x_0 \) we choose a local orthonormal frame \( e_1, \ldots, e_n \); for simplicity, we further assume \( e_1, \ldots, e_{n} \) have been chosen such that at \( x_0 \), \( g_{ij} \) is diagonal (so is \( F^{ij} \)).

**Step 1.** Computation and estimation for \( \mathcal{L}(\log w) \). By (8.9),
\[
F^{ij} \nabla_i w \nabla_j w \leq 4 |\nabla u|^2 F^{ij} \nabla_{ik} u \nabla_{jk} u. \tag{8.13}
\]
Using Cauchy-Schwarz inequality, one derives by (8.12)
\[
F^{ij} \frac{\nabla_i w \nabla_j w}{w^2} \leq (1 + \epsilon) \left( \frac{1}{\epsilon} F^{ij} \nabla_i \eta \nabla_j \eta \eta^2 + F^{ij} \nabla_i \phi \nabla_j \phi \right),
\]
which, together with (8.13), yields
\[
F^{ij} \frac{\nabla_i w \nabla_j w}{w^2} \leq (1 - \epsilon^2) \left( \frac{1}{\epsilon} F^{ij} \nabla_i \eta \nabla_j \eta \eta^2 + F^{ij} \nabla_i \phi \nabla_j \phi \right) + \frac{4 \epsilon}{w} F^{ij} \nabla_{ik} u \nabla_{jk} u.
\]
Set \( 0 < \epsilon \leq \frac{1}{4} \). We now obtain
\[
\mathcal{L}(\log w) \geq \frac{1}{w} F^{ij} \nabla_{ik} u \nabla_{jk} u + 2(\psi + \frac{1}{w} \nabla'_k \psi \nabla_k u) - 2 F^{ij}(A_{ij} + \frac{1}{w} \nabla' k A_{ij} \nabla_k u) - \frac{1 - \epsilon^2}{\epsilon} F^{ij} \nabla_i \eta \nabla_j \eta \eta^2 - (1 - \epsilon^2) F^{ij} \nabla_i \phi \nabla_j \phi - C_0 \sum F^{ii}. \tag{8.14}
\]
**Step 2.** Construction and computation of \( \phi \). As in [22], let \( \phi = v^{-N} \) where \( v = u - \inf_{B_r} u + 2 (v \geq 2 \text{ in } B_r) \) and \( N \geq 1 \) to be chosen later. By direct computation,
\[
\mathcal{L} \phi = N(N + 1) v^{-N-2} F^{ij} \nabla_i u \nabla_j u - N v^{-N-1} F^{ij} \nabla_{ij} u - N v^{-N-1} (F^{ij} A_{ij} - \psi_{ij}) \nabla_i u. \tag{8.15}
\]
Step 3. Completion of the proof. By (4.2),
\[
F^{ij} \nabla u \nabla_{jk} u \geq \theta |\nabla u|^2 \sum F^{ii}, \quad F^{ij} \nabla_{iu} u \geq \theta w \sum F^{ii}.
\]
By Cauchy-Schwarz inequality again, one derives
\[
\frac{1}{8} N^2 v^{-N-2} \theta w + \frac{\theta}{2w} |\nabla u|^2 \geq \frac{1}{2} N \theta |\nabla u|^v v^{-N-1}.
\]
We choose \( N \gg 1 \) so that
\[
\frac{1}{4} N \theta v^{-N-1} = N v^{-N-1}, \quad N(N + 1) v^{-N-2} - N^2 v^{-2N-2} \geq \frac{N^2}{2} v^{-N-2}.
\]
Suppose furthermore that \( \frac{1}{4} N \theta v^{-N-2} \theta w \geq C_0 \) where \( C_0 \) is as in (8.14) (otherwise we are done). Together with (8.12), (8.14), (8.15) and (8.16), we derive
\[
0 \geq \frac{1}{4} \theta N^2 w^v v^{-N-2} \sum F^{ii} + \frac{1}{4} N \theta |\nabla u|^v v^{-N-1} \sum F^{ii} + 2(\psi \zeta + \frac{1}{w} |\nabla \psi \nabla u|)
- 2F^{ij} (A_{ij} + \frac{1}{w} |\nabla \psi \nabla_{jk} u|) - N v^{-N-1} (F^{ij} A_{ij} \cdot p_i - \psi \cdot p_i) |\nabla u|
+ (F^{ij} A_{ij} \cdot p_i - \psi \cdot p_i) \frac{|\nabla \psi|}{\eta} + \frac{F^{ij} \nabla_{ij} \eta}{\eta} - \frac{1 + \varepsilon - \varepsilon^2}{\frac{1}{2} F^{ij} \nabla_{ij} \eta}.
\]
Using (8.11) and the asymptotic assumption (8.2), we obtain
\[
0 \geq \frac{1}{4} \theta N^2 w^v v^{-N-2} \sum F^{ii} + \frac{1}{4} N \theta |\nabla u|^v v^{-N-1} \sum F^{ii} - \frac{C}{r^2 \eta} \sum F^{ii}
- \left( \beta(x, u, |\nabla u|)w + CNv^{-N-1}w + C \sqrt{w \over r^2} \right) \left( 1 + \sum F^{ii} \right),
\]
which gives \( \eta w \leq \frac{C}{r^2} \), as required.

8.3. Local estimate for second derivatives. Let’s consider
\[
P(x) = \max_{\xi \in T_\delta(x_0), \|\xi\| = 1} \eta \varphi(\xi, \xi)e^\varphi,
\]
where \( \eta \) is the cutoff function as given by (8.11) and \( \varphi \) is a function to be chosen later. One knows \( P \) achieves maximum at an interior point \( x_0 \in B \), and for \( \xi \in T_{x_0} M \). Around \( x_0 \) we choose a smooth orthonormal local frame \( e_1, \cdots, e_n \) such that \( e_1(x_0) = \xi \), \( \Gamma_k \) \( (x_0) = 0 \) and \( \{g_{ij}(x_0)\} \) is diagonal (so is \( \{F^{ij}(x_0)\} \)). We may assume \( g_{11}(x_0) = 1 \). At \( x_0 \) one has
\[
\nabla \varphi + \frac{\nabla \varphi_{11} + \nabla \eta}{\eta}, \quad \mathcal{L}(\varphi + \log \eta + \log g_{11}) \leq 0.
\]
Step 1. Estimation for \( \mathcal{L}(\varphi_{11}) \). From (8.7), we get
\[
F^{ii} \nabla_{11} \varphi_{1i} \geq \psi \cdot \psi_{1i} \nabla_{11} \varphi_{1i} - C_0 g_{11}^2.
\]
The straightforward computation shows
\[
F^{ii}(\nabla_{1i} \varphi_{1i} - \nabla_{11} \varphi_{i}) \geq -F^{ii} A_{1i} \psi \cdot \psi_{1i} - C_2 g_{11}^2 \sum_{i=1}^n F^{ii} - C_2 g_{11}.
\]
Here we use (8.6). Combining (8.18) and (8.19),

\[ \mathcal{L}(g_{11}) = F^{ii} \nabla_i g_{11} + (F^{ii} A_{ii,\rho i} - \psi_i) \nabla_i g_{11} \geq -C \beta_{11}^2 (1 + \sum_{i=1}^n F^{ii}) \text{.} \]

Using (8.17) and Cauchy-Schwarz inequality,

\[ F^{ii} \frac{\nabla_i g_{11}}{g_{11}^{3/2}} \leq \frac{3}{2} F^{ii} |\nabla_i \varphi|^2 + 3 F^{ii} \frac{|\nabla_i \varphi|^2}{\eta^2} \text{.} \]

Hence

\[ \mathcal{L}(\log g_{11}) \geq -C \beta_{11}^2 (1 + \sum_{i=1}^n F^{ii}) - \frac{3}{2} F^{ii} |\nabla_i \varphi|^2 - 3 F^{ii} \frac{|\nabla_i \varphi|^2}{\eta^2} \text{.} \]

**Step 2. Construction and estimate for \( L \varphi \).** Following [22] we set

\[ \varphi = \varphi(w) = (1 - \frac{w}{2N})^{-\frac{1}{2}} \text{ where } w = |\nabla u|^2 \text{ and } N = \sup_{|\eta| > 0} |\nabla u|^2 \text{.} \]

One can check \( \varphi' = \frac{3 \varphi^3}{4N^2} \), \( \varphi'' = \frac{3 \varphi^3}{16N^2} \) and \( 1 \leq \varphi \leq \sqrt{2} \). By (8.10), (4.2) and Cauchy-Schwarz inequality, we obtain

\[ \mathcal{L} \varphi \geq \frac{3 \varphi^3}{16N^2} F^{ii} |\nabla \varphi|^2 + \frac{\varphi^3}{4N} (\frac{\theta}{2 \tilde{g}_{11}} - C_4) \sum_{i=1}^n F^{ii} \text{.} \]

Consequently,

\[ \mathcal{L} \varphi \geq \frac{3 \varphi^3}{16N^2} F^{ii} |\nabla \varphi|^2 + \frac{\varphi^3}{4N} (\frac{\theta}{2 \tilde{g}_{11}} - C_4) \sum_{i=1}^n F^{ii} \text{.} \]

**Step 3. Completion of the proof.** From (8.11),

\[ F^{ii} \frac{|\nabla \varphi|^2}{\eta^2} \leq \frac{C}{r^2 \eta} \sum_{i=1}^n F^{ii}, \quad \mathcal{L}(\log \eta) \geq -\frac{C}{r^2 \eta} \sum_{i=1}^n F^{ii} - \frac{C}{r^2 \eta} \text{.} \]

Next,

\[ \frac{3}{2} F^{ii} |\nabla \varphi|^2 = \frac{3 \varphi^6}{32N^2} F^{ii} |\nabla \varphi|^2 \leq \frac{3 \varphi^6}{16N^2} F^{ii} |\nabla \varphi|^2 \text{.} \]

Finally, at \( x_0 \)

\[ 0 \geq \frac{3}{4N} \left( \frac{\theta}{2 \tilde{g}_{11}} - \frac{4NC_3}{3} \tilde{g}_{11} - C_4 \right) \sum_{i=1}^n F^{ii} - \frac{C}{r^2 \eta} \sum_{i=1}^n F^{ii} - \frac{C}{r^2 \eta} - C \beta_{11} \text{.} \]

Combining with (8.4) we have \( \eta_{11} \leq \frac{C}{r} \) at \( x_0 \). Therefore the proof is complete.

8.4. **Boundary estimate.** For a point \( x_0 \in \partial M \) we choose a smooth orthonormal local frame \( e_1, \cdots, e_n \) around \( x_0 \) such that \( e_n \) is unit outer normal vector field when restricted to \( \partial M \). We denote \( \rho(x) := \text{dist}_g(x, x_0) \) the distance function from \( x \) to \( x_0 \) with respect to \( g \), and \( M_\delta := \{ x \in M : \rho(x) < \delta \} \). As in (5.1), \( \sigma(x) \) is the distance from \( x \) to \( \partial M \). We know that \( \sigma(x) \) is smooth and \( |\nabla \sigma| \geq \frac{1}{\sigma} \) in \( M_\delta \) for small \( \delta \).
Case 1. Pure tangential derivatives. From the boundary value condition,
\[ \nabla u = \nabla \varphi, \quad \nabla_{\alpha \beta} u = \nabla_{\alpha \beta} \varphi + \nabla_n (u - \varphi) \Pi(e_\alpha, e_\beta) \] on \( \partial M \)
for \( 1 \leq \alpha, \beta < n \), where \( \Pi \) is the fundamental form of \( \partial M \). This gives the bound of second estimates for pure tangential derivatives
\[ (8.20) \quad |\varphi_{\alpha \beta}(x_0)| \leq C. \]

Case 2. Mixed derivatives. For \( 1 \leq \alpha < n \), the local barrier function is given by
\[ \Psi = \pm \nabla_\alpha (u - \varphi) + A_1 \left( \frac{N \sigma^2}{2} - t \sigma \right) - A_2 \rho + A_3 \sum_{k=1}^{n-1} |\nabla_k (u - \varphi)|^2 \] in \( M_\delta \),
where \( A_1, A_2, A_3, N, t \) are all positive constants to be determined, and \( N \delta - 2t \leq 0 \). The construction of local barrier is standard.

To deal with local barrier functions, we need the following formula
\[ \nabla_{ij} (\nabla_k u) = \nabla_{ijk} u + \Gamma^i_{jk} \nabla_j u + \Gamma^i_{jk} \nabla_i u + \nabla_{ij} \nabla_k u, \]
see e.g. [23]. Combining with \( (8.8) \) one derives
\[ |\mathcal{L}(\nabla_k (u - \varphi))| \leq C \left( 1 + \sum_{i=1}^{n} f_i + \sum_{i=1}^{n} f_i |\lambda_i| \right). \]
As in \( (8.10) \), we can prove
\[ \mathcal{L}(|\nabla_k (u - \varphi)|^2) \geq F^{ij} \varphi_{i j k} \varphi_{j k} - C \left( 1 + \sum_{i=1}^{n} f_i + \sum_{i=1}^{n} f_i |\lambda_i| \right). \]

By [23, Proposition 2.19], there exists an index \( 1 \leq r \leq n \) with
\[ \sum_{k=1}^{n-1} F^{ij} \varphi_{i j k} \varphi_{j k} \geq \frac{1}{2} \sum_{i \neq r} f_i \lambda_i^2. \]
It follows from \( (2.1) \) and Lemma 2.1 that
\[ 0 < \sum_{i=1}^{n} f_i \lambda_i \leq \psi(x, u, \nabla u) - f(1) + \sum_{i=1}^{n} f_i. \]
Combining with Cauchy-Schwarz inequality and
\[ \sum_{i=1}^{n} f_i |\lambda_i| = 2 \sum_{\lambda_i > 0} f_i \lambda_i - \sum_{\lambda_i = 0} f_i \lambda_i = \sum_{\lambda_i = 0} f_i \lambda_i + 2 \sum_{\lambda_i < 0} f_i \lambda_i, \]
we have
\[ \sum_{i=1}^{n} f_i |\lambda_i| \leq \varepsilon \sum_{i \neq r} f_i \lambda_i^2 + \frac{C}{\varepsilon} \sum_{i=1}^{n} f_i + C. \]
Let \( \kappa \) be as in \( (8.4) \). If \( \delta \) and \( t \) are chosen small enough such that
\[ \mathcal{L} \left( \frac{N \sigma^2}{2} - t \sigma \right) \geq \frac{N \kappa \theta}{8(1 + \kappa)} \left( 1 + \sum_{i=1}^{n} f_i \right). \]
Putting those inequalities together, if $A_1 \gg A_2, A_1 \gg A_3 > 1$ then

$$L(\Psi) > 0 \text{ in } M_\delta.$$  

On the other hand $\Psi|_{\partial M} = -A_2 \rho^2 \leq 0$; while on $\partial M_\delta \setminus \partial M, \Psi \leq 0$ if $N\delta - 2t \leq 0, A_2 \gg A_1$. Note that $\Psi(x_0) = 0$. Thus

$$|g_{\alpha\beta}(x_0)| \leq C \text{ for } 1 \leq \alpha \leq n - 1.$$  

(8.21)  

Case 3. Double normal derivatives. Fix $x_0 \in \partial M$. Since $\text{tr}(g^{-1}g) > 0$, (8.20) and (8.21), we have $g_{\alpha\beta} \gg -C$. We assume $g_{\alpha\beta}(x_0) \geq 1$ (otherwise we are done). According to (4.2), $F^{\alpha\beta} \geq \theta \sum f_i$. By the concavity of equation, (8.20) and (8.21),

$$\psi(u)(x_0) - F(g) \geq F^{ij}(g_{ij} - \delta_{ij}) \geq -C' \sum_{i=1}^n f_i(\lambda) + (\theta g_{\alpha\beta} - 1) \sum_{i=1}^n f_i(\lambda).$$

This gives $g_{\alpha\beta}(x_0) \leq C$. Here we also use (8.4).

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RIRONG YUAN

School of Mathematics, South China University of Technology, Guangzhou 510641, China
Email address: yuanrr@scut.edu.cn