The Binomial Transforms of Tribonacci and Tribonacci-Lucas sequences

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Abstract
In this study, we apply the binomial transforms to Tribonacci and Tribonacci-Lucas sequences. Also, the Binet formulas, summations, generating functions of these transforms are found using recurrence relations. Finally, we illustrate the relation between these transforms by deriving new formulas.

Keywords: Tribonacci sequence, Tribonacci-Lucas sequence, binomial transforms.

1 Introduction and Preliminaries
There are so many studies in the literature that concern about the special number sequences such as Fibonacci, Lucas, Tribonacci and Tribonacci-Lucas (see, for example [3, 4, 5, 6, 7]), and the references cited therein). In Fibonacci numbers, there clearly exists the term Golden ratio which is defined as the ratio of two consecutive of Fibonacci numbers that converges to \( \alpha = \frac{1 + \sqrt{5}}{2} \). It is also clear that the ratio has so many applications in, specially, Physics, Engineering, Architecture, etc.[12, 13]. In a similar manner, the ratio of two consecutive Tribonacci and Tribonacci-Lucas numbers converges to

\[ \alpha_T = \frac{1 + \sqrt{19 + 3\sqrt{33}} + \sqrt{19 - 3\sqrt{33}}}{3} \]

that is named as Silver Rectangle and was firstly defined in 1963 by Mark Feinberg.

For \( n \geq 2 \), the Tribonacci sequence \( \{T_n\}_{n \in \mathbb{N}} \) and the Tribonacci-Lucas sequence \( \{K_n\}_{n \in \mathbb{N}} \) are defined by the recursive equations:

\[ T_{n+1} = T_n + T_{n-1} + T_{n-2} \quad (T_0 = 0, \ T_1 = 1, \ T_2 = 1), \quad (1) \]

\[ K_{n+1} = K_n + K_{n-1} + K_{n-2} \quad (K_0 = 3, \ K_1 = 1, \ K_2 = 3). \quad (2) \]
In addition, some matrix based transforms can be introduced for a given sequence. Binomial transform is one of these transforms and there is also other ones such as rising and falling binomial transforms (see \[1, 2, 4, 5, 7, 14\]).

Motivated by \[2, 14\], the goal of this paper is to apply the binomial transforms to the Tribonacci and Tribonacci-Lucas sequences. Also, the generating functions of these transforms are found by recurrence relations. Finally, it is illustrated the relation between of these transforms by deriving new formulas.

Now we give some preliminaries related our study. Given an integer sequence \(X = \{x_0, x_1, x_2, \ldots\}\), the binomial transform \(B\) of the sequence \(X\), \(B(X) = \{b_n\}\), is given by

\[
b_n = \sum_{i=0}^{n} \binom{n}{i} x_i.
\]

## 2 Binomial transform of Tribonacci and Tribonacci-Lucas sequences

In this section, we will mainly focus on binomial transforms of Tribonacci and Tribonacci-Lucas sequences to get some important results. In fact, as a middle step, we will also present the recurrence relations, Binet formulas, summations and generating functions.

**Definition 1** Let \(\{T_n\}\) and \(\{K_n\}\) be the Tribonacci and Tribonacci-Lucas sequences, respectively. The binomial transforms of these sequences can be expressed as follows:

i) the binomial transform of the Tribonacci sequence is \(b_n = \sum_{i=0}^{n} \binom{n}{i} T_i\),

ii) the binomial transform of the Tribonacci-Lucas sequence is \(c_n = \sum_{i=0}^{n} \binom{n}{i} K_i\).

**Proposition 2** For \(n \geq 0\), we note that

\[
b_{3n} = 2^n T_{4n}, \quad b_{3n+1} = 2^n (T_{4n} + T_{4n+1}), \quad b_{3n+2} = 2^n (T_{4n+1} + T_{4n+3}) \quad (3)
\]

and

\[
c_{3n} = 2^n K_{4n}, \quad c_{3n+1} = 2^n (K_{4n} + K_{4n+1}), \quad c_{3n+2} = 2^n (K_{4n+1} + K_{4n+3}). \quad (4)
\]

**Proof.** From Definition 1 and equation (1), for \(n \geq 0\), we obtain

\[
b_0 = T_0, \quad b_1 = T_0 + T_1, \quad b_2 = T_1 + T_3,
\]

\[
b_3 = 2T_4, \quad b_4 = 2T_4 + 2T_5, \quad b_5 = 2T_5 + 2T_7,
\]

\[
b_6 = 4T_8, \quad b_7 = 4T_8 + 4T_9, \quad b_8 = 4T_9 + 4T_{11},
\]

\[\vdots\]

\[
b_{3n} = 2^n T_{4n}, \quad b_{3n+1} = 2^n (T_{4n} + T_{4n+1}), \quad b_{3n+2} = 2^n (T_{4n+1} + T_{4n+3}).
\]
Similarly, from Definition 1 and equation (2), for \( n \geq 0 \), we get

\[
\begin{align*}
c_0 &= K_0, \quad c_1 = K_0 + K_1, \quad c_2 = K_1 + K_3, \
c_3 &= 2K_4, \quad c_4 = 2K_4 + 2K_5, \quad c_5 = 2K_5 + 2K_7, \
c_6 &= 4K_8, \quad c_7 = 4K_8 + 4K_9, \quad c_8 = 4K_9 + 4K_{11}, \\
\vdots \\
c_{3n} &= 2^n K_{4n}, \quad c_{3n+1} = 2^n (K_{4n} + K_{4n+1}), \quad c_{3n+2} = 2^n (K_{4n+1} + K_{4n+3}).
\end{align*}
\]

**Lemma 3** For \( n \geq 0 \), the following equalities are hold:

i) \( b_{n+1} = \sum_{i=0}^{n} \binom{n}{i} (T_i + T_{i+1}) \), 

ii) \( c_{n+1} = \sum_{i=0}^{n} \binom{n}{i} (K_i + K_{i+1}) \).

**Proof.** Firstly, in here we will just prove i), since ii) can be thought in the same manner with i).

i) By using Definition 1 and the well known binomial equality

\[
\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1},
\]

we obtain

\[
\begin{align*}
b_{n+1} &= \sum_{i=1}^{n+1} \binom{n+1}{i} T_i + T_0 \\
&= \sum_{i=1}^{n+1} \binom{n}{i} T_i + \sum_{i=1}^{n+1} \binom{n}{i-1} T_i + T_0 \\
&= \sum_{i=1}^{n} \binom{n}{i} T_i + \sum_{i=0}^{n} \binom{n}{i} T_{i+1} + T_0 \\
&= \sum_{i=0}^{n} \binom{n}{i} (T_i + T_{i+1}).
\end{align*}
\]

which is desired result.

**Theorem 4** For \( n > 0 \),

- \( b_{n+1} \) is also can be written as \( b_{n+1} = b_n + \sum_{i=0}^{n} \binom{n}{i} T_{i+1}, \)
- \( c_{n+1} \) is also can be written as \( c_{n+1} = c_n + \sum_{i=0}^{n} \binom{n}{i} K_{i+1}. \)
i) Recurrence relation of sequences \{b_n\} is
\[ b_{n+2} = 4b_{n+1} - 4b_n + 2b_{n-1}, \]  
with initial conditions \(b_0 = 0, b_1 = 1\) and \(b_2 = 3\).

ii) Recurrence relation of sequences \{c_n\} is
\[ c_{n+2} = 4c_{n+1} - 4c_n + 2c_{n-1}, \]  
with initial conditions \(c_0 = 3, c_1 = 4\) and \(c_2 = 8\).

**Proof.** Similarly the proof of the previous theorem, only the first case \(i)\) will be proved. We will omit the other cases since the proofs will not be different.

\(i)\) By using the equality in \(i)\), let be
\[ b_{n+2} = Xb_{n+1} + Yb_n + Zb_{n-1}, \]  
If we take \(n = 1, 2\) and \(3\), we have the system
\[
\begin{align*}
  b_3 &= Xb_2 + Yb_1 + Zb_0, \\
  b_4 &= Xb_3 + Yb_2 + Zb_1, \\
  b_5 &= Xb_4 + Yb_3 + Zb_2.
\end{align*}
\]
By considering Definition 1 and Cramer rule for the system, we obtain
\[
\begin{align*}
  8 &= 3X + Y, \\
  22 &= 8X + 3Y + Z, \\
  62 &= 22X + 8Y + 3Z,
\end{align*}
\]
and
\[ X = 4, \quad Y = -4, \quad Z = 2 \]
which is completed the proof of this case.

The characteristic equation of sequences \{b_n\} and \{c_n\} in (5) and (6) is \(\lambda^3 - 4\lambda^2 + 4\lambda - 2 = 0\). Let be \(\lambda_1, \lambda_2\) and \(\lambda_3\) the roots of this equation. Then, Binet’s formulas of sequences \{b_n\} and \{c_n\} can be expressed as
\[ b_n = X_1 \lambda_1^n + Y_1 \lambda_2^n + Z_1 \lambda_3^n, \]  
and
\[ c_n = X_2 \lambda_1^n + Y_2 \lambda_2^n + Z_2 \lambda_3^n, \]
where
\[
\begin{align*}
  X_1 &= \frac{\lambda_1 (\lambda_1 - 1)}{2\lambda_1^3 - 4\lambda_1^2 + 2}, \\
  Y_1 &= \frac{\lambda_2 (\lambda_2 - 1)}{2\lambda_2^3 - 4\lambda_2^2 + 2}, \\
  Z_1 &= \frac{\lambda_3 (\lambda_3 - 1)}{2\lambda_3^3 - 4\lambda_3^2 + 2}.
\end{align*}
\]
and

\[
X_2 = \frac{4\lambda_2^2 - 8\lambda_1 + 6}{2\lambda_1^2 - 4\lambda_1^2 + 2},
\]
\[
Y_2 = \frac{4\lambda_2^2 - 8\lambda_2 + 6}{2\lambda_2^2 - 4\lambda_2^2 + 2},
\]
\[
Z_2 = \frac{4\lambda_3^2 - 8\lambda_3 + 6}{2\lambda_3^2 - 4\lambda_3^2 + 2}.
\]

Now, we give the sums of binomial transforms for Tribonacci and Tribonacci-Lucas sequences.

**Theorem 5** Sums of sequences \( \{b_n\} \) and \( \{c_n\} \) are

i) \( \sum_{k=0}^{n-1} b_k = b_{n+1} - 3b_n + 2b_{n-1} \).

ii) \( \sum_{k=0}^{n-1} c_k = c_{n+1} - 3c_n + 2c_{n-1} + 1 \).

**Proof.**

i) By considering equation (7), we have

\[
\sum_{k=0}^{n-1} b_k = \sum_{k=0}^{n-1} \left( X_1\lambda_1^k + Y_1\lambda_2^k + Z_1\lambda_3^k \right).
\]

Then we obtain

\[
\sum_{k=0}^{n-1} b_k = X_1 \left( \frac{\lambda_1^n - 1}{\lambda_1 - 1} \right) + Y_1 \left( \frac{\lambda_2^n - 1}{\lambda_2 - 1} \right) + Z_1 \left( \frac{\lambda_3^n - 1}{\lambda_3 - 1} \right).
\]

Afterward, by taking account equations \( \lambda_1\lambda_2\lambda_3 = 2 \), \( \lambda_1 + \lambda_2 + \lambda_3 = 4 \) and \( b_{-1} = -\frac{1}{2} \) we conclude

\[
\sum_{k=0}^{n-1} b_k = b_{n+1} - 3b_n + 2b_{n-1}.
\]

ii) The proof of the binomial transform of Tribonacci-Lucas sequences can see by taking account equations (8) and \( c_{-1} = 2 \), similarly to proof of i).

**Theorem 6** The generating functions of the binomial transforms for \( \{T_n\} \) and \( \{K_n\} \) are
i) \( b(x) = \frac{x - x^2}{1 - 4x + 4x^2 - 2x^3} \).

ii) \( c(x) = \frac{3 - 8x + 4x^2}{1 - 4x + 4x^2 - 2x^3} \), respectively.

**Proof.** We omit Tribonacci case since the proof be quite similar.

Assume that \( c(x) = \sum_{i=0}^{\infty} c_i x^i \) is the generating function of the binomial transform for \( \{K_n\} \). From Theorem 4, we obtain

\[
c(x) = c_0 + c_1 x + \sum_{i=3}^{\infty} (4c_{i-1} - 4c_{i-2} + 2c_{i-3}) x^i.
\]

\[
= c_0 + c_1 x + 2x^2 + 4x^3 - 4c_0 x - 4c_1 x^2 + 4x \sum_{i=0}^{\infty} c_i x^i
\]

\[
- 4x^2 \sum_{i=0}^{\infty} c_i x^i + 2x^3 \sum_{i=0}^{\infty} c_i x^i
\]

\[
= c_0 + (c_1 - 4c_0) x + (c_2 - 2c_1 + 4c_0) x^2 + 4xc(x) - 4x^2 c(x) + 2x^3 c(x).
\]

Now rearrangement the equation implies that

\[
c(x) = \frac{c_0 + (c_1 - 4c_0) x + (c_2 - 2c_1 + 4c_0) x^2}{1 - 4x + 4x^2 - 2x^3},
\]

which equal to the \( \sum_{i=0}^{\infty} c_i x^i \) in theorem.

Hence the result. ■

We note that, \( b(x) \) and \( c(x) \) may be also be obtained from the generating functions of the Tribonacci and Tribonacci-Lucas sequences, \( f(x) = \frac{x}{1-x-x^2-x^3} \) and \( g(x) = \frac{1-3x+2x^2}{1-x-x^2-x^3} \), respectively [3, 11]. It is seen by using the following result proved by author [4]:

\[
b(x) = \frac{1}{1-x} f \left( \frac{x}{1-x} \right) \quad \text{and} \quad c(x) = \frac{1}{1-x} g \left( \frac{x}{1-x} \right).
\]

In the following theorem, we present the relationship between these binomial transforms.

**Theorem 7** For \( n > 0 \), the relationship between of the transforms \( \{b_n\} \) and \( \{c_n\} \) would be illustrated by following way:

\[
c_n = 2b_{n+1} - 2b_n - 2b_{n-1}.
\]
Proof. The proof will be done by induction steps. First of all, let us consider initial condition for (5) and $n = 1$, then we have

$$2b_2 - 2b_1 - 2b_0 = 4 = c_1.$$ 

Actually, by iterating this procedure and assuming the equation in (9) holds for all $n = k \in \mathbb{Z}^+$, we can end up the proof if we manage to show that the case also holds for $n = k + 1$:

$$c_{k+1} = 4c_k - 4c_{k-1} + 2c_{k-2}$$

$$= 4(2b_{k+1} - 2b_k - 2b_{k-1}) - 4(2b_k - 2b_{k-1} - 2b_{k-2})$$

$$+ 2(2b_{k-1} - 2b_k - 2b_{k-2})$$ 

$$= 8b_{k+1} - 16b_k + 4b_{k-1} + 4b_{k-2} - 4b_{k-3}.$$ 

By taking account equation (5), we obtain

$$c_{k+1} = 8b_{k+2} - 2b_{k+1} - 2b_k + 2b_{k+1} - 6b_k + 4b_{k-2} - 4b_{k-3}$$

$$= 2b_{k+2} - 2b_{k+1} - 2b_k.$$ 

Hence the result.

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