Isoperimetric profile of radial probability measures on Euclidean spaces

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Abstract

We derive the isoperimetric profile of Gaussian type for an absolutely continuous probability measure on Euclidean spaces with respect to the Lebesgue measure, whose density is a radial function. The key is a generalization of the Poincaré limit which asserts that the $n$-dimensional Gaussian measure is approximated by the projections of the uniform probability measure on the Euclidean sphere of appropriate radius to the first $n$-coordinates as the dimension diverges to infinity. The generalization is done by replacing the projections with certain maps.

1 Introduction

The isoperimetric profile of a Borel probability measure $\mu$ on $\mathbb{R}^n$ describes a relation between the volume $\mu[A]$ and the boundary measure $\mu^+[A] := \lim_{\varepsilon \to 0}(\mu[A^\varepsilon] - \mu[A])/\varepsilon$ of $A \subset \mathbb{R}^n$, where $A^\varepsilon := \{x \in \mathbb{R}^n \mid \inf_{a \in A} |x - a| < \varepsilon \}$ denotes the $\varepsilon$-neighborhood of $A$ with respect to the standard Euclidean norm $| \cdot |$. Throughout this note, any subset of $\mathbb{R}^n$ is assumed to be Borel. Precisely, the isoperimetric profile $I[\mu]$ of $\mu$ is a function on $[0, 1]$ defined by

$$I[\mu](a) := \inf \{ \mu^+[A] \mid A \subset \mathbb{R}^n \text{ with } \mu[A] = a \}.$$ 

Let $A_n$ denote the boundary measure of the unit ball in $\mathbb{R}^n$ with respect to the Lebesgue measure. For a measurable, nonnegative function $f$ on $(0, \infty)$ satisfying

$$M_n^f := \frac{1}{A_n} \int_0^\infty f(r)r^{n-1}dr < \infty,$$

the $n$-dimensional radial probability measure $\mu_n^f$ with density $f$ is the absolutely continuous probability measure on $\mathbb{R}^n$ with density

$$\frac{d\mu_n^f}{dx}(x) = \frac{1}{M_n^f}|x|$$

with respect to the $n$-dimensional Lebesgue measure. For example, the $n$-dimensional Gaussian measure $\gamma_n$ is the radial probability measure with density $g(r) := \exp(-r^2/2)$,

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and its isoperimetric profile was provided by Borell [1] and Sudakov–Tsirel’son [5] independently of the form

$$I[\gamma_n](a) = I[\gamma_1](a) = G'(G^{-1}(a)), \quad G(r) := \int_{-\infty}^{r} (2\pi)^{-1/2} g(s) ds = \gamma_1([-\infty, r]).$$

The proof relies on the approximation procedure, so-called Poincaré limit: let $S_N$ be the $(N - 1)$-dimensional Euclidean sphere of radius $N^{1/2}$ and $v_N$ be the uniform probability measure on $S_N$. We consider the orthogonal projection from $\mathbb{R}^N$ to the first $n$-coordinates, and denote by $P_{n,N}$ the restriction of it on $S_N$. Then $\gamma_n$ is obtained as the limit of $(P_{n,N})_x v_N$ as $N \to \infty$, where $(P_{n,N})_x v_N$ denotes the push-forward measure of $v_N$ by $P_{n,N}$, namely $(P_{n,N})_x v_N[A] = v_N[P^{-1}_{n,N}(A)]$ for any $A \subset \mathbb{R}^n$.

The aim of this note is to derive the isoperimetric profile of Gaussian type for $\mu_n^f$, that is, estimate $I[\mu_n^f]$ below by $I[\gamma_1]$. To do this, let us generalize the Poincaré limit by replacing $P_{n,N}$ with $P_{n,N}^\rho := s_n^\rho \circ P_{n,N}$, where $s_n^\rho$ is the map on $\mathbb{R}^n$ defined as

$$s_n^\rho(x) := \begin{cases} \rho(|x|)x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

for a function $\rho$ on $(0, \infty)$ satisfying the following condition.

(C) $\rho$ is a $C^1$, positive function on $(0, \infty)$ and $s_1^\rho$ is strictly increasing.

**Theorem 1.1** For a function $\rho$ satisfying (C), let $\sigma$ be the inverse function of $s_1^\rho$. For any $x \in \mathbb{R}^n \setminus \{0\}$, $\{f_{n,N}^\rho(x) := d(P_{n,N}^\rho v_N)(x)/dx\}_{N \in \mathbb{N}}$ converges to

$$f_n^\rho(x) := \begin{cases} (2\pi)^{-n/2} \exp \left(-\frac{\sigma(|x|)^2}{2}\right) \left\{ \frac{\sigma(|x|)}{|x|} \right\}^{n-1} \sigma'(|x|) & \text{if } x \in s_n^\rho(\mathbb{R}^n \setminus \{0\}), \\ 0 & \text{otherwise} \end{cases}$$

as $N \to \infty$. The function $f_n^\rho$ has unit mass on $\mathbb{R}^n$ with respect to the Lebesgue measure and hence $\{P_{n,N}^\rho v_N\}_{N \in \mathbb{N}}$ converges weakly to the absolutely continuous probability measure $\nu_n^\rho$ on $\mathbb{R}^n$ such that $d\nu_n^\rho/dx = f_n^\rho$ as $N \to \infty$.

Theorem [1.1] for the case of $\rho \equiv 1$ recovers the original Poincaré limit. A radial probability measure $\mu_n^f$ is said to be a generalized Poincaré limit if there exists a function $\rho$ satisfying (C) such that $\mu_n^f = \nu_n^\rho$. To estimate the isoperimetric profile of $\mu_n^f = \nu_n^\rho$, we impose an additional condition on $\rho$.

(Cn) The map $s_n^\rho$ is Lipschitz continuous.

**Theorem 1.2** For $m = 1$ and $n$, let $\mu_n^f$ be the generalized Poincaré limit with $\rho_m$ satisfying (Cm). Then it holds for any $a \in [0, 1]$ with $a \neq 1/2$ that

$$I[\mu_n^f](a) \geq \frac{1}{L_n} I[\gamma_1](a),$$

where $L_n$ is the smallest Lipschitz constant of $s_n^\rho$. Moreover, if $\lim_{r \to 0} f(r) \in (0, \infty)$, then the above inequality also holds true for $a = 1/2$. 

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Theorem 1.2 for the case of \( f(r) = \exp(-r^2/2) \) with \( \rho_m \equiv 1 \) corresponds to the result of the Gaussian measure.

This note is organized as follows. Section 2 concerns Theorem 1.1 which is a generalization of the Poincaré limit. In Section 3, we prove Theorem 1.2, namely derive the isoperimetric profile of Gaussian type for a radial probability measure. Section 4 provides criteria and examples of \( \mu_f^n \) which is applicable to Theorems 1.1, 1.2.

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2 Generalized Poincaré limit

In this section, we always assume that \( \rho \) satisfies (C) and \( \sigma \) is the inverse function of \( s_1^\rho \).

We moreover define the map \( \Sigma \) on \( \mathbb{R}^n \) by

\[
\Sigma(x) = \begin{cases} 
\frac{\sigma(|x|)}{|x|} x & \text{if } x \in s_1^\rho(\mathbb{R}^n \setminus \{0\}), \\
0 & \text{otherwise.}
\end{cases}
\]

We then have \( \Sigma \circ s_1^\rho(x) = x \) for any \( x \in \mathbb{R}^n \) and

\[ s_1^\rho(\mathbb{R}^n \setminus \{0\}) = s_1^\rho(\mathbb{R}^n) \setminus \{0\} = \{x \in \mathbb{R}^n \mid |x| \in s_1^\rho((0, \infty))\}. \]

Let \( V_n \) denote the volume of the unit ball in \( \mathbb{R}^n \) with respect to the Lebesgue measure.

For any \( x \in P_{n,N}(S_N) \setminus \{0\} \), the Lebesgue differentiation theorem yields

\[
f_{n,N}^\rho(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^n V_n} \int_{B_\varepsilon(x)} f_{n,N}^\rho(x') dx' = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^n V_n} v_N([P_{n,N}^\rho]^{-1}(B_\varepsilon(x))],
\]

where \( B_\varepsilon(x) \) is the open ball in \( \mathbb{R}^n \) with center \( x \) and radius \( \varepsilon \). We compute the right-hand side in \( (2.1) \).

Lemma 2.1 For any \( x \in P_{n,N}(S_N) \setminus \{0\} \), we have

\[
f_{n,N}^\rho(x) = \frac{A_{N-n}}{N^{n/2} A_N} \left( 1 - \frac{\sigma(|x|)^2}{N} \right)^{(N-n-2)/2} \left\{ \frac{\sigma(|x|)}{|x|} \right\}^{n-1} \sigma'(|x|).
\]

Proof. We prove only the case of \( n \geq 2 \), however a similar argument works for the case of \( n = 1 \). By symmetry, we may assume that \( x \) lies in the positive first coordinate axis.

Let us consider the orthogonal projection \( p_m \) from \( \mathbb{R}^m \) to the last \( (m-1) \) coordinates. We define the functions \( r_+^\varepsilon(x) \) and \( r_-^\varepsilon(x) \) on \( p_n(\Sigma(B_\varepsilon(x))) \) by

\[
r_+^\varepsilon(y) := \sup\{|x'| \mid x' \in B_\varepsilon^y(x)\}, \quad r_-^\varepsilon(y) := \inf\{|x'| \mid x' \in B_\varepsilon^y(x)\},
\]

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where $B^y_\varepsilon(x) := (p_n \circ \Sigma)^{-1}(y) \cap B_\varepsilon(x)$. Since $p_N$ is a chart on a set containing

$$(P_{n,N}^\varepsilon)^{-1}(B_\varepsilon(x)) = \{(\Sigma(x'), \xi) \in \mathbb{R}^n \times \mathbb{R}^{N-n} \mid |\xi|^2 = N - \sigma(|x'|)^2, x' \in B_\varepsilon(x)\} \subset S_N$$

for $\varepsilon > 0$ small enough, we directly compute

$$v_N[(P_{n,N}^\varepsilon)^{-1}(B_\varepsilon(x))] = \frac{1}{N^{(N-1)/2}A_N} \int_{p_N((P_{n,N}^\varepsilon)^{-1}(B_\varepsilon(x)))} \left( \frac{N}{N - |u|^2} \right)^{1/2} du$$

$$= \frac{1}{N^{(N-1)/2}A_N} \int_{\rho_n(\Sigma(B_\varepsilon(x)))} \left( |\xi|^2 \leq N - \sigma(|x'|)^2, x' \in B_\varepsilon(x) \right) \left( \frac{N}{N - |\xi|^2} \right)^{1/2} d\xi dy$$

$$= \frac{A_{N-n}}{N^{(N-1)/2}A_N} \int_{\rho_n(\Sigma(B_\varepsilon(x)))} \left( \frac{N}{N - |\xi|^2} \right)^{1/2} s^{N-n-1} d\xi dy$$

where we set

$$U_\varepsilon(x) := \{(p_n(\Sigma(x')), s) \in \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0} \mid s^2 = N - \sigma(|x'|)^2, x' \in B_\varepsilon(x)\}.$$ 

According to the assumption that $x$ lies in the first axis, $U_\varepsilon(x)$ converges to the point $(0, \{N - \sigma(|x'|)^2\}^{1/2})$ as $\varepsilon \searrow 0$. If the volume $|U_\varepsilon(x)|_n$ of $U_\varepsilon(x)$ with respect to the $n$-dimensional Lebesgue measure satisfies

$$\lim_{\varepsilon \downarrow 0} \frac{|U_\varepsilon(x)|_n}{\varepsilon^n V_n} = \frac{\sigma(|x|)\sigma'(|x|)}{N - \sigma(|x'|)^2} \left( \frac{\sigma(|x|)}{|x|} \right)^{n-1}, \quad (2.2)$$

then we find that

$$f^p_{n,N}(x) = \frac{A_{N-n}}{N^{(N-1)/2}A_N} \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^n V_n} \int_{U_\varepsilon(x)} \left( \frac{N}{N - |y|^2 - s^2} \right)^{1/2} s^{N-n-1} dy ds$$

$$= \frac{A_{N-n}}{N^{(N-1)/2}A_N} \cdot \frac{\sigma(|x|)\sigma'(|x|)}{\{N - \sigma(|x'|)^2\}^{1/2}} \left( \frac{\sigma(|x|)}{|x|} \right)^{n-1} \cdot \left( \frac{N}{\sigma(|x'|)^2} \right)^{1/2} \left( \{N - \sigma(|x'|)^2\}^{(N-n)/2} \right)$$

as desired. To prove (2.2), we need several claims.

**Claim 2.2** The functions $r^{\pm}_\varepsilon(y)$ depend only on $|y|$ not on $y$ itself.

**Proof.** For any $y \in p_n(\Sigma(B_\varepsilon(x))) \setminus \{0\}$, it turns out that

$$B^y_\varepsilon(x) = \left\{ x(a, b) := \left| x \right| + a, \frac{b}{|y|} \right\} \mid (a, b) \in D_\varepsilon, y = p_n(\Sigma(x(a, b))) = \frac{\sigma(|x(a, b)|)}{|x(a, b)|} \frac{b}{|y|} \}$$

$$D_\varepsilon := \{(a, b) \in \mathbb{R}^2 \mid a^2 + b^2 < \varepsilon^2, b \geq 0\}.$$
This means that $r_\varepsilon^+(y)$ (resp. $r_\varepsilon^-(y)$) is equal to the supremum (resp. infimum) of
\[
|x(a, b)| = \left\{ (|x| + a)^2 + b^2 \right\}^{1/2}
onumber
\]
on D_\varepsilon subject to
\[
|y| = Y(a, b) := \frac{\sigma(|x(a, b)|)}{|x(a, b)|} b = |p_n(\Sigma(x(a, b)))|,
\]
which concludes the proof of the claim. ◊

We sometimes denote $r_\varepsilon^\pm(y)$ by $r_\varepsilon^\pm(|y|)$ as functions on $[0, \eta_\varepsilon)$, where $\eta_\varepsilon = \eta_\varepsilon(x)$ given by
\[
\eta_\varepsilon := \sup\{|y| \mid y \in p_n(\Sigma(B_\varepsilon(x)))\} = \sup\{Y(a, b) \mid (a, b) \in D_\varepsilon\}.
\]

Note that $\lim_{\eta_\varepsilon \to 0} \eta_\varepsilon = 0$. It follows from Claim 2.2 that
\[
\lim_{\varepsilon \to 0} \frac{|U_\varepsilon(x)|}{V_n} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n V_n} \int_{p_n(\Sigma(B_\varepsilon(x)))} \int_{\{|s| \geq 0 \mid N - \sigma(r_\varepsilon^-(|y|)) \leq s^2 \leq N - \sigma(r_\varepsilon^+(|y|))\}} \frac{ds}{dy} \
\]
\[
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n V_n} \int_{p_n(\Sigma(B_\varepsilon(x)))} \left[ \left\{ N - \sigma(r_\varepsilon^-(|y|)) \right\}^{1/2} - \left\{ N - \sigma(r_\varepsilon^+(|y|)) \right\}^{1/2} \right] \frac{dy}{\sigma} \
\]
\[
= \lim_{\varepsilon \to 0} \frac{A_{n-1}}{V_n} \frac{(\eta_\varepsilon)^n}{n} - \frac{1}{\varepsilon} \int_0^1 u_\varepsilon(t) dt, \quad (2.3)
\]
where, in the last equality, we substitute $\eta = t\eta_\varepsilon$ and set
\[
u_\varepsilon(t) := \frac{t^{n-2}}{\varepsilon} \left[ \left\{ N - \sigma(r_\varepsilon^-(t\eta_\varepsilon)) \right\}^{1/2} - \left\{ N - \sigma(r_\varepsilon^+(t\eta_\varepsilon)) \right\}^{1/2} \right].
\]

We will investigate the limits of $\eta_\varepsilon/\varepsilon$ and $u_\varepsilon(t)$ as $\varepsilon \searrow 0$.

It is easy to check that $|x(a, b)|$ does not have extrema on $D_\varepsilon$ by using Lagrange multipliers, and $y \neq 0$ leads to $b \neq 0$. In other words, for any $\eta \in (0, \eta_\varepsilon)$, there exist $a_\varepsilon^\pm(\eta) \in I_\varepsilon := [-\varepsilon, \varepsilon]$ such that
\[
r_\varepsilon^\pm(\eta) = \left| x \left( a_\varepsilon^\pm(\eta), \left\{ \varepsilon^2 - a_\varepsilon^\pm(\eta)^2 \right\}^{1/2} \right) \right|, \quad \eta = Y \left( a_\varepsilon^\pm(\eta), \left\{ \varepsilon^2 - a_\varepsilon^\pm(\eta)^2 \right\}^{1/2} \right). \quad (2.4)
\]
The monotonicity of $|x(a, (\varepsilon^2 - a^2)^{1/2})|$ in $a$ and the definition of $r_\varepsilon^\pm$ imply
\[
a_\varepsilon^+(\eta) = \max \left\{ a \in I_\varepsilon \mid \eta = Y \left( a, (\varepsilon^2 - a^2)^{1/2} \right) \right\}, \quad (2.5)
a_\varepsilon^-(\eta) = \min \left\{ a \in I_\varepsilon \mid \eta = Y \left( a, (\varepsilon^2 - a^2)^{1/2} \right) \right\}. \quad (2.6)
\]
Due to the fact $r_\varepsilon^\pm(0) = |x| \pm \varepsilon, a_\varepsilon^\pm(\eta)$ are extended to $[0, \eta_\varepsilon)$ by putting $a_\varepsilon^\pm(0) = \pm \varepsilon$.

Claim 2.3 The functions $a_\varepsilon^\pm(\eta)$ are monotone and $|a_\varepsilon^\pm(\eta)| \leq \varepsilon$ on $\eta \in (0, \eta_\varepsilon)$. 

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Proof. From the monotonicity
\[
\frac{\partial}{\partial b} Y(a, b) = \frac{\sigma'(|x(a, b)|) b^2 + \sigma(|x(a, b)|)}{|x(a, b)|^2} (|x| + a)^2 \geq 0,
\]
where we use the nonnegativity of \( \sigma' \), we deduce
\[
\eta_{\varepsilon} = \sup \{ Y(a, b) \mid (a, b) \in D_{\varepsilon} \} = \max \left\{ Y \left( a, (\varepsilon^2 - a^2)^{1/2} \right) \mid a \in I_{\varepsilon} \right\}. \quad (2.7)
\]
Since the function \( Y(a, (\varepsilon^2 - a^2)^{1/2}) \) is continuous on \( a \in I_{\varepsilon} \) and takes the value 0 at the boundary, the intermediate value theorem yields that, for any \( \eta_1, \eta_2 \in (0, \eta_{\varepsilon}) \) with \( \eta_1 < \eta_2 \), there exist \( a^\pm(\eta_i) \in I_{\varepsilon} \) for \( i = 1, 2 \) such that
\[
Y = Y \left( a^\pm(\eta_i), \{ \varepsilon^2 - a^\pm(\eta_i)^2 \}^{1/2} \right), \quad a^-(\eta_1) < a^-(\eta_2) < a^+(\eta_2) < a^+(\eta_1).
\]
This with \( 2.5, 2.6 \) leads to
\[
-\varepsilon = a^-_\varepsilon(0) < a^-_\varepsilon(\eta_1) < a^-_\varepsilon(\eta_2) < a^+_\varepsilon(\eta_2) < a^+_\varepsilon(\eta_1) < a^+_\varepsilon(0) = \varepsilon.
\]
Since Claim 2.3 implies
\[
r^\pm_\varepsilon(\eta) = \left| x \left( a^\pm_\varepsilon(\eta), \{ \varepsilon^2 - a^\pm_\varepsilon(\eta)^2 \}^{1/2} \right) \right| = \left\{ |x|^2 + \varepsilon^2 + 2a^\pm_\varepsilon(\eta)|x| \right\}^{1/2} \in [\varepsilon - |x|, |x| + \varepsilon]
\]
for any \( \eta \in (0, \eta_{\varepsilon}) \), the mean value theorem yields
\[
0 \leq u_\varepsilon(t) \leq \frac{t^{n-2}}{\varepsilon} \left[ \max_{r \in [\varepsilon - |x|, |x| + \varepsilon]} -\frac{\sigma'(r) \sigma(r)}{N - \sigma(r)^2} \left\{ r^-_\varepsilon(t \varepsilon) - r^+_\varepsilon(t \varepsilon) \right\} \right]
\leq \frac{t^{n-2}}{\varepsilon} \max_{r \in [\varepsilon - |x|, |x| + \varepsilon]} \left\{ \frac{2\sigma'(r) \sigma(r)}{N - \sigma(r)^2} \right\},
\]
which ensures that \( u_\varepsilon(t) \) is dominated by an integrable function on \( t \in (0, 1) \).

Claim 2.4 For any \( t \in (0, 1) \), the limits
\[
\eta_\sigma = \eta_\sigma(x) := \lim_{\varepsilon \downarrow 0} \frac{\eta_{\varepsilon}}{\varepsilon}, \quad a^+_\sigma(t) := \lim_{\varepsilon \downarrow 0} \frac{a^+_\varepsilon(t \varepsilon)}{\varepsilon}
\]
exist and satisfy \( \eta_\sigma = \sigma(|x|)/|x|, \ a^+_\sigma(t) = \pm(1 - t^2)^{1/2} \).

Proof. Claim 2.3 leads to \( a^+_2(t) := \lim_{\varepsilon \downarrow 0} a^+_\varepsilon(t \varepsilon)/\varepsilon \in [0, 1] \) and combining (2.4) with the fact \( |x(a, \varepsilon^2 - a^2)^{1/2}| \rightarrow |x| \) for any \( a \in I \) as \( \varepsilon \rightarrow 0 \) yields
\[
\lim_{\varepsilon \downarrow 0} \frac{t \varepsilon}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} Y \left( a^+_\varepsilon(t \varepsilon), \{ \varepsilon^2 - a^+_\varepsilon(t \varepsilon)^2 \}^{1/2} \right) = \sigma(|x|) \left\{ 1 - a^+_\sigma(t) \right\}^{1/2} \leq \frac{\sigma(|x|)}{|x|}
\]
for any \( t \in (0, 1) \). Letting \( t \nearrow 1 \), we have \( \lim_{\varepsilon \downarrow 0} \eta_{\varepsilon}/\varepsilon \leq \sigma(|x|)/|x| \). On the other hand, it holds by (2.7) that
\[
\lim_{\varepsilon \downarrow 0} \frac{\eta_{\varepsilon}}{\varepsilon} \geq \lim_{\varepsilon \downarrow 0} \frac{Y(0, \varepsilon)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{\sigma(|x(0, \varepsilon)|)}{|x(0, \varepsilon)|} \frac{\varepsilon}{\varepsilon} = \frac{\sigma(|x|)}{|x|},
\]
meaning \( \eta_\sigma = \sigma(|x|)/|x| \) and hence \( a^+_\sigma(t)^2 = a^+_2(t) = 1 - t^2 \). If there exists \( t_0 \in (0, 1) \) such that \( a^+_\sigma(t_0) = a^-_\sigma(t_0) \), then Claim 2.3 with the squeeze lemma implies \( a^+_\sigma(t) = a^+_\sigma(t_0) \) for any \( t \in (t_0, 1) \), which contradicts \( a^+_\sigma(t)^2 = 1 - t^2 \). We thus have \( a^+_\sigma(t) = \pm(1 - t^2)^{1/2} \).
Since \( u_\varepsilon(t) \) is dominated by an integrable function on \( t \in (0, 1) \) and Claim 2.4 implies

\[
\lim_{\varepsilon \downarrow 0} u_\varepsilon(t) = \lim_{\varepsilon \downarrow 0} \frac{t^{n-2}}{\varepsilon} \left[ \left\{ N - \sigma(r_\varepsilon(t_\varepsilon)) \right\}^{1/2} - \left\{ N - \sigma(t_\varepsilon(s_\varepsilon)) \right\}^{1/2} \right]
\]

\[
= \sigma'(|x|)\sigma(|x|) \frac{t^{n-2}}{\varepsilon} \left\{ r_\varepsilon(t_\varepsilon) - r_\varepsilon(t_\varepsilon) \right\}
\]

\[
= \sigma'(|x|)\sigma(|x|) \frac{t^{n-2}}{\varepsilon} \left\{ (|x|^2 + \varepsilon^2 + 2a_\varepsilon(t_\varepsilon)|x|)^{1/2} - (|x|^2 + \varepsilon^2 + 2a_\varepsilon(t_\varepsilon)|x|)^{1/2} \right\}
\]

\[
= \frac{2\sigma'(|x|)\sigma(|x|)}{\{N - \sigma(|x|^2)\}^{1/2}} t^{n-2}(1 - t^2)^{1/2}
\]

for any \( t \in (0, 1) \), Lebesgue’s dominated convergence theorem yields

\[
\lim_{\varepsilon \downarrow 0} \int_{0}^{1} u_\varepsilon(t) dt = \int_{0}^{1} \frac{2\sigma'(|x|)\sigma(|x|)}{\{N - \sigma(|x|^2)\}^{1/2}} t^{n-2}(1 - t^2)^{1/2} \frac{1}{A_{n-1}} \frac{V_n}{V_n} = \frac{\sigma'(|x|)\sigma(|x|)}{\{N - \sigma(|x|^2)\}^{1/2}} A_{n-1}
\]

where \( B(\cdot, \cdot) \) is the beta function. According to the relation \( V_n/A_{n-1} = B(3/2, (n-1)/2) \) and (2.3), we compute

\[
\lim_{\varepsilon \downarrow 0} \frac{U_\varepsilon(x)}{\varepsilon^n V_n} = A_{n-1} \frac{V_n}{V_n} \frac{1}{\varepsilon^n} \frac{\eta_\varepsilon}{\varepsilon} \frac{\sigma'(|x|)\sigma(|x|)}{\{N - \sigma(|x|^2)\}^{1/2}} A_{n-1}
\]

which is (2.2). This completes the proof of the lemma.

Let us now generalize the Poincaré limit.

**Proof.** (Theorem 1.1) Given any \( x \notin s_n^0(\mathbb{R}^n) \), we find that \( x \notin P_{n,N}^0(S_N) \) for any \( N \in \mathbb{N} \) hence \( f_{n,N}^0(x) = 0 = f_n^0(x) \). For any \( x \in s_n^0(\mathbb{R}^n) \setminus \{0\} \), Lemma 2.1 with the relation \( \lim_{N \rightarrow \infty} A_{n,N}^{-n/2}/A_N = (2\pi)^{-n/2} \) yields \( f_{n,N}^0(x) \rightarrow f_n^0(x) \) as \( N \rightarrow \infty \). We thus have the pointwise convergence of \( f_{n,N}^0 \) to \( f_n^0 \) on \( \mathbb{R}^n \setminus \{0\} \) as \( N \rightarrow \infty \).

It is easy to check that \( f_n^0 \) has unit mass on \( \mathbb{R}^n \) with respect to the Lebesgue measure, additionally, for any \( R \in \mathbb{R} \) satisfying \( \sigma(R)^2 = 2(n+2) \) and any \( N \geq 2(n+2) \), we find

\[
\left[ 1 - \frac{\sigma(|x|^2)}{N} \right]^{(N-n-2)/2} \leq 1_{B_R(0)}(x) + \exp \left( -\frac{\sigma(|x|^2)^2}{2} \right),
\]

where \([t]_+ := \max\{t, 0\}\) for \( t \in \mathbb{R} \) and \( 1_{B_R(0)} \) is the characteristic function on \( B_R(0) \). This ensures that \( f_{n,N}^0 \) is dominated by an integrable function and hence Lebesgue’s dominated convergence theorem yields the weak convergence of \( P_{n,N}^0 v_N \) to \( v_n^0 \) as \( N \rightarrow \infty \).

We give a necessary and sufficient condition for \( \mu_n^0 \) to be a generalized Poincaré limit in terms of \( f \). In what follows, we denote by \( \text{supp}(f) \) the support of \( f \) and set

\[
r_f := \inf\{r \mid r \in \text{supp}(f)\}, \quad R_f := \sup\{r \mid r \in \text{supp}(f)\}.
\]
Claim 3.2 For any $t \in (0, (R_f - \alpha)/2)$, there exists $N_0 \in \mathbb{N}$, independent of $\beta$, such that
\[
\left( (P_{1,N}^\alpha)^{-1}((-\infty, \beta]) \right)^{L(\beta,t)} \supset (P_{1,N}^\alpha)^{-1}((-\infty, \beta + t]), \quad L(\beta,t) := \sup_{r \in [\beta,\beta+t]} \sigma'(r)t + t^2
\]
holds for any $N \geq N_0$. 

**Proposition 2.5** A radial probability measure $\mu_n^f$ is a generalized Poincaré limit if and only if $\text{supp}(f)$ is connected, on the interior of which $f$ is continuous. In the case of $\mu_n^f = \nu_n^\rho$, $\text{supp}(f)$ coincides with the closure of $s_n^\rho((0, \infty))$.

**Proof.** The “only if” part and the claim on the support follow immediately from Theorem 1.1. To prove the “if” part, let $\sigma$ be the function on $(r_f, R_f)$ solving the equation
\[
\int_{0}^{\sigma(r)} (2\pi)^{-n/2} \exp \left(-\frac{s^2}{2}\right) s^{n-1} ds = \frac{1}{M_n} \int_{r_f}^{e} f(s)s^{n-1} ds. \tag{2.8}
\]
Then $\sigma$ is $C^1$, strictly increasing and $\sigma((r_f, R_f)) = (0, \infty)$, which ensures the existence of the function $\rho$ satisfying (C) such that $s_n^\rho \circ \sigma(r) = r$ holds for any $r \in (r_f, R_f)$. For $f_n^\rho = d\nu_n^\rho/dx$, Theorem 1.1 and differentiating (2.8) yield that
\[
f_n^\rho(x) = (2\pi)^{-n/2} \exp \left(-\frac{\sigma(|x|)^2}{2}\right) \left\{ \frac{\sigma(|x|)}{|x|} \right\}^{n-1} \frac{\sigma'(|x|)}{M_n} = \frac{f(|x|)}{M_n}
\]
for almost every $x \in \text{supp}(\mu_n^f) = \text{supp}(\nu_n^\rho)$, that is, $\mu_n^f = \nu_n^\rho$. \hfill \Box

**3 Isoperimetric profile of Gaussian type**

We derive the isoperimetric profile of Gaussian type for the generalized Poincaré limit with $\rho$ satisfying $(C_n)$ by using Lévy's isoperimetric inequality for Euclidean spheres. In analogy with $\mathbb{R}^n$, we denote by $X^\varepsilon$ the $\varepsilon$-neighborhood of $X \subset S_N$ with respect to the spherical distance function $d_{SN}$.

**Proposition 3.1** (Lévy’s isoperimetric inequality [14]) For any $X \subset S_N$, take a closed metric ball $B \subset S_N$ with $\nu_N[B] = \nu_N[X]$. Then we have $\nu_N[X^\varepsilon] \geq \nu_N[B^\varepsilon]$ for any $\varepsilon > 0$.

**Proof.** (Theorem 1.2) Let $\sigma$ be the inverse function of $s_1^{\mu_1}$ and $F(r) := \mu_1^f[(-\infty, r])$ be the cumulative distribution function of $\mu_1^f$, which is differentiable on $(r_f, R_f)$. Since $\rho_1$ satisfies $(C_1)$, we find $r_f = 0$ and $\inf_{r \in (0, R_f)} \sigma'(r) > 0$ (see Lemma 4.4 below). Moreover, by (2.8), it holds for any $r \in (0, R_f)$ that $G(\sigma(r)) = F(r)$ and
\[
G'(\sigma(r)) = \frac{1}{\sigma'(r)} F'(r). \tag{3.1}
\]

The claim is trivial for the case of $a = 0$ and 1 since the right-hand side is equal to 0. We first consider the case of $a \in (1/2, 1)$. For any $A \subset \mathbb{R}^n$ with $\mu_1^f[A] \in (1/2, 1)$, there exists a unique $\alpha \in (0, R_f)$ such that $\mu_1^f[A] = \mu_1^f[(-\infty, \alpha]] = F(\alpha)$. Fix $\beta \in (0, \alpha)$.

**Claim 3.2** For any $t \in (0, (R_f - \alpha)/2)$, there exists $N_0 \in \mathbb{N}$, independent of $\beta$, such that
\[
\left( (P_{1,N}^\alpha)^{-1}((-\infty, \beta]) \right)^{L(\beta,t)} \supset (P_{1,N}^\alpha)^{-1}((-\infty, \beta + t]), \quad L(\beta,t) := \sup_{r \in [\beta,\beta+t]} \sigma'(r)t + t^2
\]
Proof. We first remark that $L(\beta, t)$ is positive finite by $[\beta, \beta + t] \subset (0, R_f)$. Without loss of generality, we may assume that $N \in \mathbb{N}$ satisfies $\alpha + t < s_0^n(N^{1/2})$, which ensures the unique existence of $x_0 \in (0, N^{1/2})$ such that $s_0^n(x_0) = \beta$. Moreover, for any point $(x, \xi) \in (P_{1,N}^n)^{-1}((-\infty, \beta + t])$ with $P_{1,N}^n = s_0^n(x) \in (\beta, \beta + t]$, we find $\xi \neq 0$. This implies that, for $\xi_0 := (N - x_0^2)^{1/2} \xi / |\xi|$, it holds that $(x_0, \xi_0) \in (P_{1,N}^n)^{-1}((-\infty, \beta])$ and $|(x, \xi) - (x_0, \xi_0)|^2 = 2N \left(1 - \cos \left(\frac{d_{SN}((x, \xi), (x_0, \xi_0))}{N^{1/2}}\right)\right) = d_{SN}((x, \xi), (x_0, \xi_0))^2 + O(N^{-1}).$ Since the direct computation provides $|(x, \xi) - (x_0, \xi_0)|^2 = 2N \left(1 - \frac{xx_0}{N} - \left(1 - \frac{x_0^2}{N}\right)^{1/2} \left(1 - \frac{x_0^2}{N}\right)^{1/2}\right) = \frac{|x - x_0|^2}{2N} + O(N^{-1})$ for any $t > 0$, there exists $N_0 \in \mathbb{N}$ such that if $N \geq N_0$, then we have $d_{SN}((x, \xi), (x_0, \xi_0)) \leq |x - x_0| + t^2 = |\sigma(s_0^n(x)) - \sigma(s_0^n(x_0))| + t^2 \leq \left(\sup_{r \in [s_0^n(x), s_0^n(x_0)]} \sigma'(r)\right) |s_0^n(x) - s_0^n(x_0)| + t^2 \leq L(\beta, t),$ which concludes the proof of the claim. \hfill \Box

For $N \in \mathbb{N}$ large enough, we deduce from Theorem [11] that $v_N[(P_{n,N}^n)^{-1}(A)] = (P_{n,N}^n)_* v_N[A] > (P_{1,N}^n)_* v_N[(-\infty, \beta)] = v_N[(P_{1,N}^n)^{-1}((-\infty, \beta])].$
The Lipschitz continuity of $P_{n,N}^n$ with Lipschitz constant $L_n$ deduced from (C_n) provides $v_N \left[\left(P_{n,N}^n\right)^{-1}\left(A^{L_n(\beta, t)}\right)\right] \geq v_N \left[\left((P_{n,n}^n)^{-1}(A)\right)^{L(\beta, t)}\right] \geq v_1 \left[\left(P_{1,n}^n\right)^{-1}((-\infty, \beta + t])\right]$, where the last inequality follows from the fact that $(P_{1,n}^n)^{-1}((-\infty, \beta])$ is a closed metric ball with Proposition [3.1] and Claim [3.2]. Letting $N \to \infty$ in the above inequality together with Theorem [11] implies $v_n \left[A^{L_n(\beta, t)}\right] \geq \nu_1^n \left[(-\infty, \beta + t]\right]$ for any $t \in (0, (R_f - \alpha)/2).$
Since $\beta < \alpha$ is arbitrary and $L(\beta, t)$ is continuous in $\beta$, this also holds for $\alpha$, namely $\mu_1^n \left[A^{L_n(\alpha, t)}\right] = \nu_1^n \left[A^{L_n(\alpha, t)}\right] \geq \nu_1^n \left[(-\infty, \alpha + t]\right] = F(\alpha + t).$ \hfill (3.2)

Claim 3.3 $\mu_1^n[A] \geq I[\gamma_1]([\mu_1^n[A]])/L_n.$

Proof. For $\varepsilon > 0$ small enough, the continuity and monotonicity of $L(\alpha, t)$ in $t > 0$ guarantees the existence of $t(\varepsilon) > 0$ such that $L_n(\alpha, t(\varepsilon)) = \varepsilon$ and

$$\lim_{\varepsilon \downarrow 0} \frac{t(\varepsilon)}{\varepsilon} = \lim_{t \downarrow 0} \frac{t}{L_n(\alpha, t)} = \frac{1}{\sigma'(\alpha)L_n}.$$

We then have

$$\mu_1^n[A] = \lim_{\varepsilon \downarrow 0} \frac{\mu_1^n[A_{\varepsilon}] - \mu_1^n[A]}{\varepsilon} \geq \lim_{\varepsilon \downarrow 0} \frac{F(\alpha + t(\varepsilon)) - F(\alpha)}{\varepsilon} = \frac{F'(\alpha)}{\sigma'(\alpha)L_n} = I[\gamma_1]([\mu_1^n[A]])/L_n,$$

where the last equality follows from $G(\sigma(\alpha)) = F(\alpha) = \mu_1^n[A]$, that is, $\sigma(\alpha) = G^{-1}(\mu_1^n[A])$ and (3.1). \hfill \Box

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The arbitrary choice of $A \subset \mathbb{R}^n$ with $\mu'_n[A] \in (1/2, 1)$ implies $I[\mu'_n](a) \geq I[\gamma_1](a)/L_n$ for $a \in (1/2, 1)$. Since the similar argument works for the case of $a \in (0, 1/2)$ and moreover $a = 1/2$ which is equivalent to $\alpha = 0$ if $\lim_{r \to 0} \sigma'(r)/(2\pi)^{1/2} = \lim_{r \to 0} f(r)/M'_f \in (0, \infty)$, we have the desired result.

\[ \square \]

**Remark 3.4** In the case of $f(r) = \exp(-r^2/2)$, Theorem 1.2 corresponds to the case of finite dimensional Gaussian measures in [1, 5], where the case of an infinite dimensional Gaussian measure $\gamma_\infty$ was also proved. However we may not expect to extend Theorem 1.2 for infinite dimensional cases since $I[\gamma_\infty]$ is obtained through the fact that $\{\gamma_n\}_{n \in \mathbb{N}}$ is a cylinder set measure but $\{\mu'_n\}_{n \in \mathbb{N}}$ is generally not a cylinder set measure.

## 4 Condition and Example

In order to provide criteria for $\mu'_n = \nu^n_r$ such that $\rho$ satisfies (C$_n$) in terms of $f$, we first prepare the following lemma.

**Lemma 4.1** For a function $\rho$ satisfying (C) and $L > 0$, the following (i), (ii) and (iii) are equivalent to each other.

(i) $\rho$ satisfies (C$_1$) and the smallest Lipschitz constant of $s'_1$ is $L$.

(ii) $\rho$ satisfies (C$_n$) and the smallest Lipschitz constant of $s'_n$ is $L$ for any $n \in \mathbb{N}$.

(iii) $\lim_{r \to 0} s'_n(r) = s'_n(0) = 0$ and $\sup_{r > 0}(s'_1)'(r) = L$.

**Proof.** Since (iii) $\Rightarrow$ (i) is trivial, we show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). Note that, for any $r \in \mathbb{R} \setminus \{0\}$, $(s'_n)'(r) = \rho'(\|r\|)\|r\| + \rho(\|r\|) = (s'_1)'(\|r\|)$ holds.

If (ii), then we have $\lim_{r \to 0} s'_1(r) = s'_1(0) = 0$ by continuity of $s'_n$ and $\sup_{r > 0}(s'_1)'(r) = L$ by the differentiability with the Lipschitz continuity of $s'_n$. We thus find (ii) $\Rightarrow$ (iii).

Assume (iii). Since the mean value theorem yields $s'_n(r) - s'_n(\varepsilon) \leq L(r - \varepsilon)$ for any $r \geq \varepsilon > 0$, letting $\varepsilon \searrow 0$ and dividing by $r$ provide $\sup_{r > 0} \rho(r) \leq L$. Let $\{\lambda^x_m\}_{m = 1}^n$ be the eigenvalues of the Jacobi matrix $(J^x_{ij})_{1 \leq i, j \leq n}$ of $s' _n$ at $x$. By the differentiability of $s'_n$, (iii) is equivalent to $\max_{1 \leq m \leq n} \sup_{x \in \mathbb{R}^n \setminus \{0\}} |\lambda^x_m| = L$. Fix $x = (x^i)_{i = 1}^n \in \mathbb{R}^n \setminus \{0\}$ and let $\{v^x_m\}_{m = 1}^n$ be an orthogonal basis such that $v_1 = x/\|x\|$. Then, for each $1 \leq m \leq n$, $\rho'(|x|)|x|\delta_m + \rho(|x|)$ is an eigenvalue $\lambda^x_m$ with eigenvector $v^x_m$ since we compute

$$J^x_{ij} = \rho'(|x|) \frac{x^ix^j}{\|x\|} + \rho(|x|)\delta_{ij},$$

where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$. This provides

$$\max_{1 \leq m \leq n} \sup_{x \in \mathbb{R}^n \setminus \{0\}} |\lambda^x_m| = \max_{1 \leq m \leq n} \sup_{x \in \mathbb{R}^n \setminus \{0\}} |\rho'(|x|)|x|\delta_m + \rho(|x|)| = \sup_{r > 0} \max_{r > 0} \rho(r), (s'_1)'(r)\} = L.$$

\[ \square \]

Let us next consider the following conditions on $f$.

(a) $f$ is positive on $(0, R_f)$ and $\lim_{r \to 0} f(r) > 0$. 

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(b) There exist a continuous, positive function $\psi$ on $(R, R_f)$ for some $R > 0$ such that

$$\lambda f(r) \psi(r) r^{n-1} \leq \int_r^{R_f} f(s) s^{n-1} ds \leq \frac{1}{\lambda} f(r) \psi(r) r^{n-1}$$

for some $\lambda \in (0, 1)$.

(b2) $\lim_{r \uparrow R_f} \{ \psi(r)^2 \ln (f(r) \psi(r) r^{n-1}) \} > -\infty$.

In the case of $f(r) = \exp(-r^2/2)$, (b2) holds for $\psi(r) = 1/r$, especially, for any $\lambda \in (0, 1)$

$$\lambda \exp\left(\frac{-r^2}{2}\right) r^{n-2} \leq \int_r^{\infty} \exp\left(-\frac{s^2}{2}\right) s^{n-1} - \frac{1}{\lambda} \exp\left(-\frac{r^2}{2}\right) r^{n-2}$$

(4.1)

holds for $r$ large enough.

**Proposition 4.2** For the generalized Poincaré limit $\mu_n^f$ with $\rho$, assume $\lim_{r \uparrow 0} f(r) < \infty$. Then $(C_n)$ is equivalent to the combination of (a) and (b).

**Proof.** Let $\sigma$ be the inverse function of $s_n^f$, then

$$\frac{M_n^f}{(2\pi)^{n/2}} \sigma'(r) = f(r) \exp\left(\frac{\sigma(r)^2}{2}\right) \left\{ \frac{r}{\sigma(r)} \right\}^{n-1}$$

(4.2)

holds on $(r_f, R_f)$ by differentiating (4.1). We also have $\lim_{r \uparrow r_f} \sigma(r) = 0$, $\lim_{r \uparrow R_f} \sigma(r) = \infty$ and the equivalence between $\inf_{r > 0} (s_n^f)'(r) < \infty$ and $\inf_{r \in (r_f, R_f)} \sigma'(r) > 0$.

Assume (a) and (b). Since we deduce $r_f = 0$ from (a), $\lim_{r \downarrow 0} s_n^f(r) = 0$ holds. Hence it is enough for $(C_n)$ to show $\lim_{r \downarrow 0} \sigma'(r)$, $\lim_{r \uparrow R_f} \sigma'(r) > 0$ by Lemma 4.1(iii) together with (4.2) and (a).

The conditions (2.8) and $\lim_{r \downarrow 0} f(r) < \infty$ yield

$$\lim_{r \downarrow 0} \left[ \sigma'(r) \left\{ \frac{\sigma(r)}{r} \right\}^{n-1} \right] = \frac{(2\pi)^{n/2}}{M_n^f} \lim_{r \downarrow 0} \left\{ f(r) \exp\left(\frac{\sigma(r)^2}{2}\right) \right\} \leq \frac{(2\pi)^{n/2}}{M_n^f} \lim_{r \downarrow 0} f(r) < \infty,$$

which ensures the existences of $c > 0$ and $r_0 > 0$ such that $\sigma'(r) \sigma(r)^{n-1} < c r^{n-1}$ on $(0, r_0)$. Integrating it with the condition $\sigma(0) = 0$ implies $\sigma(r)^n < c r^n$ on $(0, r_0)$. We thus have

$$0 < \frac{(2\pi)^{n/2}}{M_n^f} \lim_{r \downarrow 0} f(r) = \lim_{r \downarrow 0} \left[ \sigma'(r) \left\{ \frac{\sigma(r)}{r} \right\}^{n-1} \right] \leq c^{(n-1)/n} \lim_{r \downarrow 0} \sigma'(r).$$

It follows from (b1), (4.1) and (2.8) that

$$\lambda \exp\left(-\frac{\sigma(r)^2}{4}\right) \geq \frac{M_n^f}{(2\pi)^{n/2}} \int_{\sigma(r)}^{\infty} \exp\left(-\frac{s^2}{2}\right) s^{n-1} ds = \int_r^{R_f} f(s) s^{n-1} ds \geq \lambda f(r) \psi(r) r^{n-1}$$

for $r < R_f$ large enough, which implies

$$0 \leq \lim_{r \uparrow R_f} \left\{ \psi(r)^2 \sigma(r)^2 \right\} \leq -4 \lim_{r \uparrow R_f} \left\{ \psi(r)^2 \ln(f(r) \psi(r) r^{n-1}) \right\} < \infty.$$
Similarly, we deduce from (2.8) with the converse inequalities of \( (h1) \) and (4.1) that
\[
\frac{M^f_n}{(2\pi)^{n/2}} \lim_{r \uparrow R_f} \sigma'(r) = \lim_{r \uparrow R_f} \left[ f(r) \exp \left( \frac{\sigma(r)^2}{2} \right) \left\{ \frac{r}{\sigma(r)} \right\}^{n-1} \right] \geq \lim_{r \uparrow R_f} \frac{\lambda^2}{\psi(r) \sigma(r)} > 0.
\]
We thus have \( \lim_{r \uparrow 0} \sigma'(r), \lim_{r \uparrow R_f} \sigma'(r) > 0 \).

Conversely, suppose \((C_n)\). Lemma 4.1(iii) and Proposition 2.5 yield \( r_f = 0 \), hence \( f(r) > 0 \) on \((0, R_f)\) holds by \((4.2)\). It moreover follows from the mean value theorem that \( \lim_{r \uparrow 0} \sigma'(r) \leq \lim_{r \uparrow R_f} \sigma(r)/r \), which means \((h1)\). This with Lemma 4.3 below concludes the proof of Proposition 4.2.

\[\square\]

**Lemma 4.3** Given a generalized Poincaré limit \( \mu^f_n = \nu^\sigma_n \), if \( \rho \) satisfies \((C_n)\) and a continuous, positive function \( \psi \) on \((R, R_f)\) for some \( R > 0 \) satisfies \((h1)\), then \((h2)\) holds.

**Proof.** For the inverse function \( \sigma \) of \( s_1^\rho \), we deduce from \((h1), (4.1)\) and \((2.8)\) that
\[
\frac{1}{\lambda} \exp \left( -\sigma(r)^2 \right) \leq \frac{M^f_n}{(2\pi)^{n/2}} \int_{\sigma(r)}^\infty \exp \left( -\frac{s^2}{2} \right) s^{n-1} ds = \int_{r}^{R_f} f(s)s^{n-1} ds \leq \frac{1}{\lambda} f(r) \psi(r) r^{n-1},
\]
hence \( \psi(r)^2 \sigma(r)^2 \geq \psi(r)^2 \ln(f(r) \psi(r) r^{n-1}) > 0 \). Since \((C_n)\) leads to \( \lim_{r \uparrow R_f} \sigma'(r) > 0 \), \((2.8)\) with the converse inequalities of \((h1)\) and \((4.1)\) yields
\[
0 < \frac{M^f_n}{(2\pi)^{n/2}} \lim_{r \uparrow R_f} \sigma'(r) = \lim_{r \uparrow R_f} \left[ f(r) \exp \left( \frac{\sigma(r)^2}{2} \right) \left\{ \frac{r}{\sigma(r)} \right\}^{n-1} \right] \leq \lim_{r \uparrow R_f} \frac{1}{\lambda^2} \psi(r) \sigma(r).
\]
Combining the two inequalities, we have \((h2)\). \[\square\]

We give a simple criterion \( f \) to satisfy \((h1)\), which concerns the logarithmic concavity (see Remark 4.5 below for the definition of logarithmic concavity).

**Lemma 4.4** For a radial probability measure \( \mu^f_n \), suppose that \( f \) is \( C^2 \), positive on \((R, R_f)\) for some \( R > 0 \) and \( \lim_{r \uparrow R_f} f(r) = 0 \). Moreover if the function \( \Phi := -\ln f \) satisfies \( \lim_{r \uparrow R_f} \Phi''(r) \in (0, \infty) \), \( \lim_{r \uparrow R_f} \Phi'(r) = \infty \) and \( \lim_{r \uparrow R_f} \Phi'(r)/\Phi''(r)^2 < \infty \), then \((h1)\) holds.
Proof. Let us prove that (b) holds for $\psi := 1/\Phi'$. For any $c \in \mathbb{R}$, define the function by

$$
\Psi_c(r) := f(r)\psi(r)r^{n-1} - c \int_r^{R_f} f(s)s^{n-1}ds.
$$

It turns out that $\lim_{r \uparrow R_f} \Psi_c(r) = 0$ and

$$
\Psi'_c(r) = f(r)r^{n-1}\left(-1 - \frac{\Phi''(r)}{\Phi'(r)^2} + \frac{n-1}{r}\psi + c\right).
$$

This means that, for $\lambda \in (0, 1)$ satisfying

$$
\lambda < \lim_{r \uparrow R_f} \left(1 + \frac{\Phi''(r)}{\Phi'(r)^2} - \frac{n-1}{r}\psi\right) \leq \lim_{r \uparrow R_f} \left(1 + \frac{\Phi''(r)}{\Phi'(r)^2} - \frac{n-1}{r}\psi\right) < \frac{1}{\lambda},
$$

$\Psi_{1/\lambda}(r) \leq 0 \leq \Psi_{\lambda}(r)$ holds for $r < R_f$ large enough, which is equivalent to (b1). Note that the existence of $\lambda$ is guaranteed by the assumptions.

Since we compute directly

$$
\lim_{r \uparrow R_f} \left\{\psi(r)^2\ln \left(f(r)\psi(r)r^{n-1}\right)\right\} = \lim_{r \uparrow R_f} \left\{-\frac{\Phi(r)}{\Phi'(r)^2} - \frac{\ln \Phi(r)}{\Phi'(r)^2} + (n-1)\frac{\ln r}{\Phi'(r)^2}\right\}
$$

$$
\geq -\lim_{r \uparrow R_f} \frac{\Phi(r)}{\Phi'(r)^2} + (n-1)\lim_{r \uparrow R_f} \frac{r}{\Phi'(r)^2}
$$

and find $\lim_{r \uparrow R_f} \ln r/\Phi'(r)^2 \in (-\infty, \infty]$ (note that $R_f < 1$ may happen), we only need to show $\lim_{r \uparrow R_f} \Phi(r)/\Phi'(r)^2 < \infty$ for (b2). This follows from L’Hôpital’s rule, that is,

$$
\lim_{r \uparrow R_f} \frac{\Phi(r)}{\Phi'(r)^2} = \lim_{r \uparrow R_f} \frac{\Phi'(r)}{2\Phi''(r)} = \lim_{r \uparrow R_f} \frac{1}{2\Phi''(r)} < \infty.
$$

\[\square\]

Remark 4.5 An absolutely continuous probability measure $\mu$ on $\mathbb{R}^n$ with respect to the Lebesgue measure is said to be logarithmic concave if $-\log(d\mu/dx) : \mathbb{R}^n \to (-\infty, \infty]$ is convex. For such probability measures, Bobkov [2] derived the isoperimetric profile of Gaussian type. The conditions in Lemma 4.3 also concern the convexity of $-\log(d\mu/dx)$, however, we only consider the behavior of $f$ around $R_f$, not on whole $\mathbb{R}$.

Let us show that some probability measures characterized by $\varphi$-exponential functions satisfy the conditions in Theorems 4.1, 4.2. See [4] and references therein for details of $\varphi$-exponential functions. In what follows, $\varphi$ is always assumed to be a continuous, positive, non-decreasing function on $(0, \infty)$.

The $\varphi$-logarithmic function is defined for $t \in (0, \infty)$ by $\ln_\varphi(t) := \int_t^1 1/\varphi(s)ds$, which is strictly increasing. Set $l_\varphi := \lim_{t \downarrow 0} \ln_\varphi(t)$ and $L_\varphi := \lim_{t \uparrow \infty} \ln_\varphi(t)$. The inverse function of $\ln_\varphi$ is called the $\varphi$-exponential function and is naturally extended to $\mathbb{R}$ as

$$
\exp_\varphi(\tau) :=
\begin{cases}
0 & \text{if } \tau \leq l_\varphi,
\ln_\varphi^{-1}(\tau) & \text{if } \tau \in (l_\varphi, L_\varphi),
\infty & \text{if } \tau \geq L_\varphi.
\end{cases}
$$
Define a kind of differentiable coefficients of \( \varphi \) by

\[
\theta_\varphi := \sup_{s > 0} \left\{ \frac{s}{\varphi(s)} \cdot \lim_{\varepsilon \downarrow 0} \frac{\varphi(s + \varepsilon) - \varphi(s)}{\varepsilon} \right\}, \quad \delta_\varphi := \inf_{s > 0} \left\{ \frac{s}{\varphi(s)} \cdot \lim_{\varepsilon \downarrow 0} \frac{\varphi(s + \varepsilon) - \varphi(s)}{\varepsilon} \right\}.
\]

**Remark 4.6** The case of \( \varphi(s) = s \) is the crucial case, where the \( \varphi \)-exponential function coincides with the usual exponential function and \( \theta_\varphi = \delta_\varphi = 1 \). Another significant case is that \( \varphi(s) = s^q \) for \( q > 0 \) and \( q \neq 1 \). In this case, \( \theta_\varphi = \delta_\varphi = q \) and the \( \varphi \)-exponential function is the power function given by

\[
\exp_q(\tau) := [1 + (1 - q)\tau]_+^{1/q},
\]

where \( 0^a := \infty \) for \( a < 0 \). Since we have \( \exp_q(\tau) \to \exp(\tau) \) as \( q \to 1 \), we regard \( \exp_1(\tau) := \exp(\tau) \) for the convenience.

We recall the following lemma.

**Lemma 4.7** (\cite{[4]} Lemmas 2.10, 2.12, Propositions 2.13, 2.14) Suppose \( \theta_\varphi < \infty \).

1. \( s^{\theta_\varphi}/\varphi(s) \) (resp. \( s^{\delta_\varphi}/\varphi(s) \)) is non-decreasing (resp. non-increasing) in \( s \in (0, \infty) \).
2. We have \( \exp_\varphi(r) \leq \exp_{\varphi}(\varphi(1)r) \) for any \( r \in \mathbb{R} \).
3. \( \delta_\varphi \geq 1 \Rightarrow l_\varphi = -\infty \Rightarrow \theta_\varphi \geq 1 \) (or equivalently \( \theta_\varphi < 1 \Rightarrow l_\varphi > -\infty \Rightarrow \delta_\varphi < 1 \)).

For \( p > 0 \), let us consider the function of the form \( \phi_p(r) := \exp_{\varphi}(-r^p/p) \) and set

\[
R_\phi := \sup \{ r \mid r \in \text{supp}(\phi_p) \} = (-pl_\varphi)^{1/p} \in (0, \infty].
\]

**Proposition 4.8** (1) \( \phi_p \) satisfies \( \lim_{r \downarrow 0} \phi_p(r) = 1 \) and \( \text{(A)} \), hence \( \text{supp}(\phi_p) \) is connected.

If moreover we assume either \( l_\varphi > -\infty \) or \( \theta_\varphi < (n + p)/n \) for \( n \in \mathbb{N} \), then

\[
\int_0^{-l_\varphi} \phi_p(r)r^{n-1}dr < \infty
\]

holds. In this case, there exists a function \( \rho \) satisfying \( \text{(C)} \) such that \( \mu^\rho_{n,p} = \mu^n_\varphi \).

(2) Suppose that \( \varphi \) is \( C^1 \) around 0.

(i) If \( \theta_\varphi < 1 \), then \( \phi_p \) satisfies \( \text{(A)}, \) implying \( \text{(C)} \).

(ii) If \( 1 \leq \delta_\varphi, 1 < \theta_\varphi < (n + p)/n \) and \( (\theta_\varphi - \delta_\varphi)(\theta_\varphi - 1) \leq 1/p \), then \( \text{(C)} \) does not hold.

**Proof.** (1) Obviously the first claim holds and the above integral is finite if \( l_\varphi \) is finite.

In the case of \( l_\varphi = -\infty \), Lemma \([4, 7, 3]\) implies \( \theta_\varphi \geq 1 \) and we compute for \( \theta_\varphi = 1 \) that

\[
\int_0^\infty \phi_p(r)r^{n-1}dr \leq \int_0^\infty \exp \left( -\varphi(1)\frac{r^p}{p} \right) r^{n-1}dr < \infty
\]

and for \( \theta_\varphi > 1 \) that

\[
\int_0^\infty \phi_p(r)r^{n-1}dr \leq \int_0^\infty \left\{ 1 - \varphi(1)(1 - \theta_\varphi)\frac{r^p}{p} \right\}^{1/(1-\theta_\varphi)} r^{n-1}dr
\]

\[
= \frac{1}{p} \left\{ \frac{p}{\varphi(1)(\theta_\varphi - 1)} \right\}^{n/p} B \left( \frac{n}{p}, \frac{1}{\theta_\varphi - 1} - \frac{n}{p} \right) < \infty.
\]
(2) We first remark that $\phi_p$ is $C^2$ on $(R, R_\phi)$ for some $R > 0$.

(i) Let us show that $\Phi := -\ln \phi_p$ satisfies the conditions in Lemma 4.4. We mention that Lemma 4.7 with $\theta_\varphi < 1$ yields $0 \leq \lim_{s \downarrow 0} s/\varphi(s) \leq \lim_{s \downarrow 0} s^{1-\theta_\varphi}/\varphi(1) = 0$, $R_\phi < \infty$ and by definition $\delta_\varphi \leq \varphi'(s)/\varphi(s) \leq \theta_\varphi$ if $\varphi$ is differentiable at $s$. We directly compute that

$$\Phi'(r) = \frac{\varphi(\phi_p(r))}{\phi_p(r)} r^{p-1}, \quad \Phi''(r) = \Phi'(r)^2 \left( \frac{p-1}{r \Phi'(r)} + 1 - \frac{\varphi'(\phi_p(r)) \phi_p(r)}{\varphi(\phi_p(r))} \right),$$

which implies

$$\lim_{r \uparrow R_\phi} \Phi'(r) = \lim_{s \downarrow 0} \frac{\varphi(s)}{s} R_\phi^{p-1} = \infty,$$

$$\lim_{r \uparrow R_\phi} \Phi''(r) \geq \lim_{r \uparrow R_\phi} \Phi'(r)^2 \left( \frac{p-1}{r \Phi'(r)} + 1 - \theta_\varphi \right) = \infty,$$

$$\lim_{r \uparrow R_\phi} \frac{\Phi''(r)}{(\Phi'(r))^2} \leq \lim_{r \uparrow R_\phi} \left( \frac{p-1}{r \Phi'(r)} + (1 - \delta_\varphi) \right) = 1 - \delta_\varphi$$

as desired.

(ii) Let $\sigma$ be the inverse function of $s^\rho$ and consider the function

$$\Psi(r) := 2\phi_p(r) r^{n-1} - \int_r^{R_\phi} \phi_p(s) s^{n-1} ds.$$

Note that Lemma 4.7 yields $l_\varphi = -\infty$, that is, $R_\phi = \infty$, and

$$\lim_{r \uparrow \infty} \Psi(r) \leq \lim_{r \uparrow \infty} 2 \left\{ 1 - \varphi(1) \left( 1 - \theta_\varphi \right) r^p \right\}^{1/(1-\theta_\varphi)} r^{n-1} = 0,$$

where we use the assumption $p/(\theta_\varphi - 1) > n$. If $\Psi$ is nonpositive around $\infty$, then we have

$$\frac{2M_n}{(2\pi)^{n/2}} \frac{1}{\sigma(r)^{n-2}} \sigma(r)^{n-2} \geq \frac{M_n}{(2\pi)^{n/2}} \int_0^\infty e^{-s^2/2} s^{n-1} ds = \int_r^{R_\phi} \phi_p(s) s^{n-1} ds \geq 2\phi_p(r) r^{n-1}$$

around $\infty$, which provides

$$\lim_{r \uparrow \infty} \sigma'(r) = \frac{(2\pi)^{n/2}}{M_n} \lim_{r \uparrow \infty} \left[ \phi_p(r) \exp \left( \frac{\sigma(r)^2}{2} \right) \left\{ \frac{r}{\sigma(r)} \right\}^{n-1} \right] \leq \lim_{r \uparrow \infty} \frac{1}{\sigma(r)} = 0.$$

Thus (C$_n$) does not hold by Lemma 4.1.

The rest is to prove the nonpositivity of $\Psi$ around $\infty$, which is equivalent to the nonnegativity of $\Psi'$ around $\infty$. By Lemma 4.7, there exists $c > 0$ depending on only $p$ and $\varphi$ such that

$$\frac{\varphi(\phi_p(r))}{\phi_p(r)} \leq \varphi(1) \frac{\phi_p(r)^{\delta_\varphi - 1} \varphi(1) \left\{ 1 - \varphi(1) \left( 1 - \theta_\varphi \right) \right\}^{(\delta_\varphi - 1)/(1-\theta_\varphi)}}{c r^{p(\delta_\varphi - 1)/(1-\theta_\varphi)}}$$

holds around $\infty$, which with the assumption $(\theta_\varphi - \delta_\varphi)/(\theta_\varphi - 1) \leq 1/p$ implies

$$\Psi'(r) = 2\phi_p(r) r^{n-1} \left( -\frac{\varphi(\phi_p(r))}{\phi_p(r)} r^{p-1} + \frac{n-1}{r} + \frac{1}{2} \right) \geq 2\phi_p(r) r^{n-1} \left( -cr^{p(\theta_\varphi - \delta_\varphi)/(\theta_\varphi - 1) - 1} + \frac{n-1}{r} + \frac{1}{2} \right) \geq 0.$$

$\square$
Remark 4.9 Let $\rho$ be the function such that $\mu_n^p = \nu_n^p$. In the case of $\theta_\varphi = \delta_\varphi > 1$, Proposition 4.8(2-ii) implies that $(C_n)$ does not hold for any $p > 0$. However in the case of $\theta_\varphi = \delta_\varphi = 1$, that is $\phi_p(r) = \exp(-r^p/p)$, $(C_n)$ can or cannot hold depending on $p$. Indeed, for $\psi(r) := r^{1-p}, \ (b1)$ holds for any $\lambda \in (0,1)$ and $p > 0$, but $(b2)$ holds only for $p \geq 2$ according to $\lim_{r \to \infty} \{ \psi(r)^2 \ln (f(r) \psi(r) r^{\alpha-1}) \} = -\lim_{r \to \infty} r^{2-p}/p$. Then by Lemma 4.3, $(C_n)$ holds if and only if $p \geq 2$.

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