Dimer scattering in the \( \varepsilon \) expansion

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The atom-dimer and dimer-dimer scattering lengths are analytically calculated in an expansion around four spatial dimensions for fermions with a large 2-body scattering length \( a \). We find the atom-dimer scattering length \( a_{ad}/a = \frac{1}{3} - \frac{3}{2} \varepsilon + O(\varepsilon^2) \) and the dimer-dimer scattering length \( a_{dd}/a = \varepsilon - 0.344 \varepsilon^2 + O(\varepsilon^3) \), where \( \varepsilon = 4 - d \) and \( d \) is the number of spatial dimensions. These ratios \( a_{ad}/a = 1.11 \) and \( a_{dd}/a = 0.656 \) at \( \varepsilon = 1 \) are to be compared with the non-perturbatively calculated numerical results 1.18 and 0.6 respectively. The neutron-deuteron scattering length in the quartet channel \( a_{DD} \approx 4.78 \text{ fm} \) is in reasonable agreement with the experimental value \( a_{exp} = 6.35 \pm 0.02 \text{ fm} \), considering 2-body effective range corrections \( r_0/a \sim 0.4 \) were excluded. The deuteron-deuteron scattering length \( a_{DD} \approx 3.15 \text{ fm} \) in the spin-2 channel.

In \( d = 3 \) spatial dimensions, it is possible to establish a systematic perturbation in \( R/a \) where the range of the interaction \( R \ll a \) requires resumming an infinite set of diagrams all of which are leading order in the perturbation. This has been quite successful in the 2 and 3-nucleon systems, see for example Refs. [10, 12, 14]. The leading order atom-dimer amplitude is again \( O(a) \) and requires resumming the indicated diagrams at leading order [13]. In this paper, we consider the calculation of the atom-dimer and dimer-dimer scattering amplitude at large but finite 2-body \( S \)-wave scattering length \( a \) such that the interaction is completely determined by 2-body physics. Besides providing a perturbative method for few-body calculations in the vacuum, this work would allow study of many-body properties at finite \( a \) such as the Bose-Einstein condensation (BEC) in atomic systems near the Feshbach resonance. The relevant theory for large scattering length \( a \) is described by the following Lagrangian density:

\[
\mathcal{L} = \psi^\dagger \left( i \partial_0 + \frac{\nabla^2}{2M} \right) \psi - \frac{c_0}{4} \left( \psi^\dagger \begin{pmatrix} \sigma_2 \end{pmatrix} \psi \right)^\dagger \left( \psi^\dagger \begin{pmatrix} \sigma_2 \end{pmatrix} \psi \right),
\]

where \( \psi \) are spin-\( \frac{1}{2} \) fermionic fields and the pauli matrix \( \sigma_2 \) projects the interaction onto the singlet \( S \)-wave. An equivalent form for the Lagrangian density after a Hubbard-Stratonovich transformation is

\[
\mathcal{L} = \psi^\dagger \left( i \partial_0 + \frac{\nabla^2}{2M} \right) \psi + \frac{\varepsilon}{2} \left[ \phi^\dagger \left( \psi^\dagger \begin{pmatrix} \sigma_2 \end{pmatrix} \psi \right) + h.c. \right] + \frac{\varepsilon^2}{c_0} \phi^\dagger \phi.
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![FIG. 1: The leading order scattering amplitudes for atoms and dimers in \( d = 3 \) spatial dimensions. Single lines: atoms/fermions, double lines: fully dressed dimers/boasons.](image-url)

Working in \( d = 4 \) spatial dimensions will avoid the non-perturbative treatment of the dimer mentioned above [13]. We
will formulate an ε expansion for atom-dimer and dimer-dimer scattering in $d = 4 - ε$ dimensions. First we define the coupling $c_0$ in arbitrary $d$ spatial dimensions using the pole in the 2-body scattering amplitude, Fig. 1

$$i\mathcal{T}_{aa} = \frac{-ic_0}{1 + ic_0L(p_0, \bar{p})}$$ \hspace{1cm} (3)

located at the 2-body binding energy $B = 1/(a^2M)$:

$$\frac{1}{c_0} = -iL(p_0, \bar{p})\bigg|_{p_0 = -B + p^2/(4M)}$$ \hspace{1cm} (4)

$$= -\frac{M}{(4\pi)^{d/2}} (MB)^{d/2} \Gamma\left(1 - \frac{d}{2}\right),$$

where

$$L(p_0, \bar{p}) = -iM \int \frac{d^dq}{(2\pi)^d} \frac{1}{q^2 - Mp_0 + p^2/4 - i0^+}$$ \hspace{1cm} (5)

$$= -i\frac{M}{(4\pi)^{d/2}} \left(-Mp_0 + \frac{p^2}{4} - i0^+\right)^{d/2} \Gamma\left(1 - \frac{d}{2}\right),$$

and then write the Lagrangian density in terms of the kinetic and interaction piece as

$$\mathcal{L} = \psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2M}\right)\psi + \phi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2M} + B\right)\phi + \mathcal{L}_I,$$ \hspace{1cm} (6)

$$\mathcal{L}_I = -\phi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2M} + B\right)\phi + \frac{g^2}{c_0} \left[\phi^\dagger (\psi^\dagger \sigma_2 \psi) + h.c.\right]$$

The original Lagrangian density in Eq. (11) had only one coupling $c_0$ that was determined by the 2-body scattering length $a$, the only parameter in the theory. In Eq. (4), after the Hubbard-Stratonovich transformation, we have an additional coupling $g$. However, its value is arbitrary and only the ratio of the yukawa coupling squared $g^2/c_0$ and the “mass” term $\sqrt{8\pi^2\varepsilon}$, i.e. $c_0$, contributes after the dimer wavefunction renormalization. In $d = 3$ dimensions, the wavefunction renormalization for the dimer $Z_d \sim 1/a$ \cite{11}. In $d = 4 - ε$ dimensions, where we expect a non-interacting theory of dimers, the coupling $g$ is chosen such that using the Lagrangian density in Eq. (6) gives $Z_d \sim 1 + O(ε)$. This then requires the dimer self-energy $iΣ(p_0, \bar{p})$ in Fig. 2 to vanish on-shell and only contribute to $Z_d$ through a derivative term at $O(ε)$:

$$iΣ(p_0, \bar{p}) \approx ε \left|p_0 + B - \frac{p^2}{4M}\right| \frac{\partial Σ}{\partial p_0}\bigg|_{p_0 = -B + p^2/(4M)}$$ \hspace{1cm} (7)

$$Z_d = 1 - ε \frac{\partial Σ}{\partial p_0}\bigg|_{p_0 = -B + p^2/(4M)} + \cdots,$$

where “…” indicates terms higher order in the $ε$ expansion.

In $ε = 4 - d$ dimensions, for Eq. (7) to hold, the coupling $g$ has to be

$$g = \frac{\sqrt{8\pi^2\varepsilon}}{a^2M} \text{ arbitrary scale}$$ \hspace{1cm} (8)

where only the factor $\sqrt{8\pi^2\varepsilon}$ is uniquely determined from the dimer wavefunction renormalization. With this choice of the coupling $g$, it is straightforward to estimate the relative sizes of diagrams. For elastic scattering, diagrams involve even powers of $g$ and we associate a power of $ε$ for every factor of $g^2$. In effect, the $ε$ expansion is a loop expansion.

For the wavefunction renormalization $Z_d$, we get:

$$iΣ(p_0, \bar{p}) = -g^2L(p_0, \bar{p}) + i\frac{g^2}{c_0} - i \left(p_0 + B - \frac{p^2}{4M}\right),$$ \hspace{1cm} (9)

$$Z_d \approx 1 + \frac{ε}{2} \left[γ_ε - \log(4πa^2M^2)\right],$$

where $γ_ε \approx 0.57722$ is the Euler-Mascheroni constant.

The atom-atom scattering amplitude is given by the diagrams in Fig. 3. We find at threshold:

$$\frac{iΣ^{aa}}{a^{3-ε}} = -i\frac{8\pi^2}{aM}ε + i\frac{4π^2}{aM} \left[1 - γ_ε + \log (4π)\right]ε^2 + \cdots,$$ \hspace{1cm} (10)

where we divided the amplitude $iΣ^{aa}$ by factors $a^{3-ε}$ to look at dimensionless ratios for convenience. Unlike in $d = 3$ dimensions, the atom-atom scattering amplitude $iΣ^{aa}$ scales as $O(ε^2)$ in $d = 4$ dimensions.

A non-trivial check of the $ε$ expansion with finite $a$ would be the atom-dimer scattering length. Like the atom-atom scattering amplitude, the atom-dimer amplitude $iΣ^{ad}$ also starts at
\( O(\varepsilon) \), see Fig. 4 At \( O(\varepsilon^2) \), there is a contribution from a 1-loop diagram shown in Fig. 4 in addition to the wavefunction renormalization factor \( Z_d \) calculated in Eq. (9). The contribution from the tree diagram (including a factor of \( a^{d-3} \) in Fig. 4) is:

\[
-i \frac{g^2 a^2 M Z_d}{a^{d-3}} = -i \frac{8 \pi^2}{a M} - i \frac{4 \pi^2}{a M} \left[ \gamma E - \log(4\pi) \right] \varepsilon^2 + \cdots . \tag{11}
\]

Together with the contribution from the 1-loop diagram, we get at next-to-leading order:

\[
\frac{\mathcal{T}_{ad}}{a^{d-3}} = -i \frac{8 \pi^2}{a M} - i \frac{4 \pi^2}{3 a M} \left[ \gamma E - 4 - 3 \log(4\pi) \right] \varepsilon^2 . \tag{12}
\]

To compare with numerical results calculated in 3 spatial dimensions, we consider the dimensionless ratio

\[
\frac{\mathcal{T}_{ad}}{\mathcal{T}_{aa}} = 1 - \frac{\varepsilon}{6} + O(\varepsilon^2) \approx 0.83 \quad \varepsilon = 1 . \tag{13}
\]

In \( d = 3 \) spatial dimensions, \( \mathcal{T}_{ad}/\mathcal{T}_{aa} = 3 a_{ad}/(4a) \). Numerical evaluation by solving the 3-body Faddeev equation 16 12

\[
\mathcal{T}_{ad} = \frac{3 \pi a}{M} b(3x^2/4 - 1, x) \bigg|_{x=0} , \tag{14}
\]

\[
b(\eta, x) = -K(\eta, x) - \frac{2}{\pi} \int_0^\infty dy \frac{y^2 b(\eta, y)}{y^2 - \frac{3}{4} (\eta + 1)} K(\eta, y) ,
\]

\[
K(\eta, x) = \left. \frac{2}{5} \log \left( 1 + \frac{x^2 - \eta}{x^2} \frac{x^2 + z^2 + xz - \eta}{x^2 + z^2 + xz - \eta} \right) \right|_{x=0} ,
\]

gives \( \mathcal{T}_{ad}/\mathcal{T}_{aa} \approx 0.885 \ (a_{ad} \approx 1.18a) \). The \( \varepsilon \) expansion result is perturbatively close to the non-perturbatively calculated numerical value, and it is a tremendous simplification over the non-perturbative calculation in \( d = 3 \) dimensions.

\[\mathcal{T}_{ad}/\mathcal{T}_{aa} \approx 0.885 \quad (a_{ad} \approx 1.18a) . \tag{15}\]

The 2-loop contribution gives

\[
\frac{2 M^2 Z_d g^4}{a^{3-\varepsilon}} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + 1/a^2)^3} \tag{16}
\]

\[
= \frac{2 M^2 Z_d g^4}{(4\pi)^d/2} \frac{\Gamma(3 - d/2)}{\Gamma(3)} \frac{\gamma E - \log(4\pi)}{a M} \varepsilon^3 + \cdots .
\]

Thus, for the next-to-leading order dimer-dimer amplitude, we get the analytic result:

\[
\frac{i \mathcal{T}_{dd}}{a^{d-3}} = -i \frac{4 \pi^2}{a M} \varepsilon^2 + i \frac{2 \pi^2}{3a M} \left[ 8 - 3 \gamma E + 224 \log(2) \right] \varepsilon^3 , \tag{17}
\]

\[
-144 \log(3) \left[ 3 \log(4\pi) \right] \varepsilon^3 , \tag{18}
\]

\[
\mathcal{T}_{dd}/\mathcal{T}_{aa} = a_{dd}/2a \quad d = 3 \quad \text{spatial dimensions and numerical calculation gives} \quad a_{dd} = 0.6a \quad \text{[21], a value perturbatively reproduced in the epsilon expansion. From the dimer-dimer calculation, the deuteron-deuteron scattering length in channels with all the proton and neutron spins in the same direction is} \quad a_{dd} \approx 3.15 \text{ fm in the } \varepsilon \text{ expansion, ignoring 2-body effective range } r_0 \text{ effects.}
\]

Systems with unnaturally large 2-body scattering lengths are common in nature and typically necessitate non-perturbative calculations that are computationally difficult and expensive. Study and development of perturbative techniques
are always useful. Many times perturbative calculations can be done analytically that allow insights otherwise not possible in a numerical calculation. In this paper, we considered a fermi system with a large but finite 2-body scattering length $a$ in an expansion around $d = 4$ spatial dimensions. The effect of the 2-body scattering length was systematically incorporated and the atom-dimer and dimer-dimer scattering length was calculated to next-to-leading order in perturbation. The results seem to converge to the non-perturbatively calculated numerical solutions at this order of the calculation, and the $\varepsilon$ expansion should be an useful tool for studying strongly interacting systems with large 2-body scattering length $a$. For example, the current work would be important for research in the BEC-BCS crossover region in atomic systems as one varies the scattering length $a$ across the Feshbach resonance. As shown, the dimer calculation can be generalized to deuterons in nuclear physics in certain spin channels. However, it is not clear from this calculation how the Efimov effect in bosons and fermions with isospin (more than one fermion species as in the proton-neutron system) associated with non-perturbative renormalization emerges in the $\varepsilon$ expansion. It could be that the Efimov effect cannot be described by an analytic continuation from $d = 4$ spatial dimensions. It is known that the Efimov effect occur only in spatial dimensions $2 < d < 3.8$, with $d = 3$ being the only integer dimension.

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