FREE PRODUCTS OF ABELIAN GROUPS IN MAPPING CLASS GROUPS

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Abstract. We construct a new family of examples of parabolically geometrically finite subgroups of the mapping class group in the sense of [Dow+20] and prove they are undistorted in Mod(S).

1. Introduction

In [FM02] Farb and Mosher introduce a notion of convex cocompactness for mapping class groups. The original notion of convex cocompactness comes from Kleinian groups, where it is a special case of geometric finiteness. In recent work [Dow+20], Dowdall, Durham, Leininger, and Sisto have introduced a notion of parabolic geometric finiteness for mapping class groups. Examples include convex cocompact groups (as one would hope) and finitely generated Veech groups by work of Tang [Tan21]. In this paper we construct a new example of parabolically geometrically finite groups and prove they are undistorted in Mod(S).

Let $S = S_{g,p}$ be an orientable surface with genus $g(S)$ and $0 \leq p(S) < \infty$ punctures. We measure the complexity of $S$ via $\xi(S) = 3g(S) + p(S) - 3$. Assume $\xi(S) \geq 1$. Let $A, B$ be multicurves in $S$ and consider subgroups $H_A \cong \mathbb{Z}^n, H_B \cong \mathbb{Z}^m$ of Mod(S) generated by multitwists about the components of $A$ and $B$, respectively. Let $G = \langle H_A, H_B \rangle$, and consider the natural homomorphism $\Phi : H_A \ast H_B \to G$. Our first result says that if $A$ and $B$ are "sufficiently far apart" in $C(S)$, then $\Phi$ is injective.

Theorem 1.1. There exists a constant $D_0 \geq 3$, independent of $S$, with the following property. If $d_S(A, B) \geq D_0$, then $\Phi : H_A \ast H_B \to G$ is injective, hence an isomorphism, and $G$ is parabolically geometrically finite. Moreover, any element not conjugate into a factor is pseudo-Anosov.

When $H_A, H_B$ are cyclic, we note the resemblance between Theorem 1.1 and [Güll17, Theorem 1.3]. Notice that obtaining pseudo-Anosovs in this manner is a variation of Thurston’s [Thu88] and Penner’s [Pen88] constructions of pseudo-Anosovs. Whereas Thurston’s and Penner’s constructions only require $A \cup B$ fill $S$ (i.e. $d_S(A, B) \geq 3$), the distance prescribed by Theorem 1.1 is relatively large. It is an interesting question whether $D_0 = 3$ is always sufficient or if there exist counterexamples. If it must be that $D_0 > 3$ in general, what is the optimal bound?

It is known that convex cocompact groups and finitely generated Veech groups are undistorted, i.e. quasiisometrically embedded, in Mod(S). Our second result is that our family of examples are also undistorted in Mod(S). In fact, this result holds for any parabolically geometrically finite $G = H_A \ast H_B$. That is, we needn’t assume $A$ and $B$ are “sufficiently far apart” in $C(S)$ as in Theorem 1.1.

Theorem 1.2. Let $G = H_A \ast H_B \subset \text{Mod}(S)$ be a nontrivial free product which is parabolically geometrically finite with $H_A, H_B$ subgroups generated by multitwists about multicurves $A, B$, respectively. Then $G$ is undistorted in Mod(S).
Our results are complementary to recent results of Runnels [Run]. Given subgroups $H_A, H_B$ of $\text{Mod}(S)$ as described above, without any assumptions on their distance in $C(S)$, [Run, Theorem 1.2] tells you that, if the generators of $H_A, H_B$ are raised to sufficiently large powers, resulting in subgroups say $H'_A, H'_B$, then $\langle H'_A, H'_B \rangle$ is isomorphic to $H_A \ast H_B$ and undistorted in $\text{Mod}(S)$. We note however that this is a special case and [Run, Theorem 1.2] can be used to generate more general, undistorted RAAG’s in $\text{Mod}(S)$. For more on embedding RAAG’s in $\text{Mod}(S)$ see [CLM12; CMM21; Kob12; Seo19].

1.1. Plan of the paper. Section 2 is dedicated to background material. In section 3 we prove Theorem 1.1. In section 4 we introduce a general marking graph and prove a Masur-Minsky style distance formula for it. Using a general marking graph as a model for $\text{Mod}(S)$, we prove in section 5 that our subgroups are undistorted in $\text{Mod}(S)$. In section 6, we discuss another notion of geometric finiteness and how our groups fit into this framework. We note that in each section we reindex the subscripts for variables.

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2. BACKGROUND

2.1. Coarse geometry. Let $(X, d_X), (Y, d_Y)$ be metric spaces. We say that a map $f : X \to Y$ is a $(K, C)$-quasiisometric embedding of $X$ into $Y$, where $K \geq 1, C \geq 0$, if given any two points $a, b \in X$ we have

$$\frac{1}{K} d_X(a, b) - C \leq d_Y(f(a), f(b)) \leq K d_X(a, b) + C.$$ 

If there exists $A > 0$ such that $f(X)$ is $A$-dense in $Y$ we say that $f$ is a $(K, C)$-quasiisometry.

Let $(X, d)$ be a metric space, $I \subset \mathbb{R}$ be an interval, and $f : I \to X$. We say that $f$ is a $(K, C)$-quasigeodesic if it is a $(K, C)$-quasiisometric embedding. We say that $f$ is a $D$-local $(K, C)$-quasigeodesic if $f|_J : J \to X$ is a $(K, C)$-quasiisometric embedding for all subintervals $J \subset I$ of length at most $D$.

Given two nonnegative real numbers $A, B$, we say that $A$ and $B$ are $(K, C)$-comparable for $K \geq 1, C \geq 0$ and write

$$A \asymp_{K, C} B$$

if

$$\frac{1}{K} A - C \leq B \leq KA + C.$$ 

Note that this is not an equivalence relation since symmetry and transitivity fail. However

$$A \asymp_{K, C} B \Rightarrow B \asymp_{K, KC} A,$$

and

$$A \asymp_{K, C} B \asymp_{K', C'} D \Rightarrow A \asymp_{KK', C' + \frac{C}{K}} D.$$ 

We define $A \preceq_{K, C} B$ to mean

$$A \leq KB + C.$$ 

and finally
\[ [A]_B = \begin{cases} A & \text{if } A \geq B \\ 0 & \text{otherwise.} \end{cases} \]

In general, two quantities being comparable does not imply that the truncated quantities are comparable. However, we do have the following:

**Lemma 2.1.** Let \( \{x_i\}_{i=1}^N, \{y_i\}_{i=1}^N \) be two finite sequences of nonnegative numbers such that

\[ x_i \asymp_{K,C} y_i, \]

with \( K \geq 1, C \geq 0 \). If \( \kappa > 2KC \) then

\[ \sum_{i=1}^N [x_i]_{\kappa} \geq 2K,0 \sum_{i=1}^N [y_i]_C. \]

**Proof.** Define

\[ \Omega = \{i \mid x_i \geq \kappa\}. \]

Then for \( i \in \Omega \)

\[ x_i > 2KC > 2C \Rightarrow \frac{1}{2} \geq \frac{1}{2K} > \frac{C}{x_i}. \]

By comparability we have

\[ \frac{x_i}{K} - C \leq y_i \leq Kx_i + C. \]

We divide through by \( x_i \) which gives for \( i \in \Omega \)

\[ \frac{1}{2K} < \frac{1}{K} \frac{C}{x_i} - \frac{y_i}{x_i} \leq K + \frac{C}{x_i} < K + \frac{1}{2} < 2K, \]

i.e.

\[ x_i \asymp_{2K,0} y_i. \]

Using the comparability lower bound we have

\[ \sum_{i=1}^N [x_i]_{\kappa} = \sum_{i \in \Omega} x_i \leq 2K \sum_{i \in \Omega} y_i \leq 2K \sum_{i=1}^N [y_i]_C. \]

\[ \square \]

2.2. \( \delta \)-hyperbolicity. Let \( X \) be a geodesic metric space. We say a geodesic triangle is \( \delta \)-thin if each side is contained in the \( \delta \)-neighborhood of the other two. We say that \( X \) is \( \delta \)-hyperbolic or Gromov hyperbolic if all geodesic triangles are \( \delta \)-thin. \( \delta \)-hyperbolicity is a quasiisometric invariant. See [BH99] for a full treatment.
generated group and relative to is satisfied. Let $G$ be a finitely generated group and $\{H_1, \ldots, H_k\}$ the coned-off Cayley graph $\hat{\Gamma} = \hat{\Gamma}(G, \{H_1, \ldots, H_k\})$ is obtained by adding a vertex $v(gH_i)$ for every left coset and adding an edge of length $\frac{1}{2}$ from every element of $gH_i$ to $v(gH_i)$. We say that $(G, \{H_1, \ldots, H_k\})$ satisfies the bounded coset penetration property (BCP for short) if paths in $\hat{\Gamma}$ that “quotient” to quasigeodesics in $\hat{\Gamma}$ penetrate cosets similarly. See [Far98] for a precise statement. We say that $G$ is hyperbolic relative to $\{H_1, \ldots, H_k\}$ if $\hat{\Gamma}(G, \{H_1, \ldots, H_k\})$ is Gromov hyperbolic and the BCP property is satisfied.

The second definition was proposed by Bowditch in [Bow12]. Again, let $G$ be a finitely generated group and $\{H_1, \ldots, H_k\}$ a finite collection of finitely generated subgroups. Let $H$ denote the set of all $H_i$-conjugates and suppose $G$ acts on a connected graph $T$ such that the following hold:

1. $T$ is hyperbolic and each edge of $T$ is contained in only finitely many circuits of length $n$ for any $n \in \mathbb{N}$. A circuit is a closed path without repeated vertices.
2. There are finitely many edge orbits $Ge$, and each edge stabilizer $S_e$ is finite.
3. The elements of $H$ are precisely the infinite vertex stabilizers $S_v$.

Then we say that $G$ is hyperbolic relative to $\{H_1, \ldots, H_k\}$. Both of the above definitions are known to be equivalent [Bow12].

2.5. Parabolic geometric finiteness. In [Dow+20], Dowdall, Durham, Leininger, and Sisto introduce a notion geometric finiteness for subgroups of $\text{Mod}(S)$, generalizing Farb and Mosher’s notion of convex cocompactness [FM02] (see also [Ham05], KL08). We say $G \subset \text{Mod}(S)$ is parabolically geometrically finite (PGF for short) if the following hold:
(1) $G$ is hyperbolic relative to a (possibly trivial) collection of subgroups $\{H_1, \ldots, H_k\}$ with each $H_i$ an Abelian group virtually generated by multitwists about a multicurve $A_i$.

(2) There is an equivariant quasiisometric embedding $\phi : \hat{\Gamma} \to C(S)$.

Examples include convex cocompact groups by work of Hamenstädt [Ham05] and Kent-Leininger [KL08], and finitely generated Veech groups by work of Tang [Tan21].

2.6. Subsurface projections. Subsurface projections are an important part of our discussion, so we will briefly outline their construction following [MM00]. This construction is originally due to Ivanov. A payoff is the Masur and Minsky’s bounded geodesic image theorem, which is analogous in $C(S)$ to geodesics having uniformly bounded closest-point projections to any disjoint horoball in $\mathbb{H}^3$.

Define a domain $Y$ in $S$ be an isotopy class of a connected, incompressible, non-peripheral, open subsurface. We say $Y$ is a proper subdomain if $Y \neq S$. We first assume $\xi(Y) > 0$, since matters are simpler and the initial intuition is more easily gleaned. A subsurface projection is a map $\pi_Y : C_0(S) \to P(C_0(Y))$ (where $P(X)$ is the set of finite subsets of $X$) constructed as follows.

First, let $\pi'_Y : C_0(S) \to P(C'_0(Y))$ map a simple closed curve to the simplex in $C'_0(Y)$ whose vertices are homotopy classes of arcs which represent the curve’s minimal intersection with $Y$, or just the curve itself if it is contained in $Y$. We say $\pi'_Y(\alpha) = \emptyset$ if $\alpha$ does not meet $Y$ essentially. By [MM00] Lemma 2.2, $C(Y)$ embeds into the $C'(Y)$ as a cobounded set. Specifically, Masur and Minsky show that there is a coarse retraction $\psi_Y : C'_0(Y) \to P(C_0(Y))$ that perturbs points no more than distance 1 and distances between adjacent points no more than 2. Finally, we compose these maps to get a map $\pi_Y = \psi_Y \circ \pi'_Y : C_0(S) \to P(C_0(Y))$. Given $\alpha, \beta \in C_0(S)$ with $\pi_Y(\alpha), \pi_Y(\beta) \neq \emptyset$ we define $d_Y(\alpha, \beta) = \text{diam}_Y(\pi_Y(\alpha), \pi_Y(\beta))$.

Though we haven’t defined subsurface projections for $\xi(Y) = -1$, we go ahead and state the bounded geodesic image theorem.

**Theorem 2.2** (Thm 3.1 of [MM00]). There exists a constant $M(\xi(S))$ with the following property. Let $Y$ be a proper subdomain of $S$ with $\xi(Y) \neq 0$. Let $g$ be a geodesic segment, ray, or bi-infinite line in $C(S)$ such that $\pi_Y(v) \neq \emptyset$ for every vertex $v$ in $g$. Then

$$\text{diam}_Y(g) \leq M$$

In [Web15], Webb proves there exists a uniform $M$ for Theorem 2.2, independent of $S$. An important property of subsurface projections is that they are Lipschitz:

**Theorem 2.3** (Lemma 2.3 of [MM00]). Let $Y \subset S$ be a domain and let $\Delta$ be a simplex in $C(S)$ with $\pi_Y(\Delta) \neq \emptyset$. Then $\text{diam}_Y(\Delta) \leq 2$. If $Y$ is an annulus then $\text{diam}_Y(\Delta) \leq 1$.

2.7. Annular domains and projections. Given an annular subsurface $Y$ in $S$, we say that $Y$ is an annular domain if it has incompressible boundary and is not homotopic to a puncture.

Let $Y$ be an annular domain with core curve $\gamma$ and let $\hat{Y}$ be the compactification of the annular cover $\hat{Y}$ of $S$ corresponding to $\gamma$ (i.e. the quotient of $\mathbb{H}^2 \cup S^{1}_{\infty} - \gamma^{\pm}$ by the action of the isometry corresponding to $\gamma$, where $\gamma^{\pm}$ are the fixed points of said isometry). Let $\hat{\gamma}$ be the core curve of $\hat{Y}$.

We define the vertices of $C(Y)$ to be arcs connecting the two components of $\partial \hat{Y}$ modulo homotopies fixing $\partial \hat{Y}$ pointwise, and we add an edge of length 1 between any two vertices which have representatives with disjoint interiors.
Fix an orientation on $S$ and an ordering on the components of $\partial \hat{Y}$. This allows us to define an algebraic intersection number $\alpha \cdot \beta$ for $\alpha, \beta \in C_0(Y)$. One can show that

\[(1) \quad d_Y(\alpha, \beta) = |\alpha \cdot \beta| + 1,\]

for all $\alpha \neq \beta$ and

\[(2) \quad \alpha \cdot \rho = \alpha \cdot \beta + \beta \cdot \rho + \varepsilon(\alpha, \beta, \rho),\]

with $\varepsilon \in \{-1, 0, 1\}$ depending on the arrangement of the endpoints.

Using (1), (2), we may construct a quasiisometry $f : C(Y) \to \mathbb{Z}$ by fixing $\rho \in C_0(Y)$ and defining $f(\alpha) = \alpha \cdot \rho$. Then the identities give for all $\alpha, \beta \in C_0(Y)$

\[
|f(\alpha) - f(\beta)| \leq d_Y(\alpha, \beta) \leq |f(\alpha) - f(\beta)| + 2.
\]

We now construct subsurface projections $\pi_Y : C_0(S) \to \mathcal{P}(C_0(Y))$ for annuli. If $\alpha \in C_0(S)$ does not intersect $Y$ essentially (notice this includes the core of $Y$) then $\pi_Y(\alpha) = \emptyset$. If $\alpha \in C_0(S)$ intersects $\gamma$ essentially, then at least one lift of $\alpha$ to $\hat{Y}$ connects the components of $\partial \hat{Y}$. Let $\pi_Y(\alpha)$ be the set of such lifts, which has diameter 1 in $C(Y)$.

Let $t\gamma$ be the Dehn twist of $\hat{Y}$ about its core $\hat{\gamma}$. It follows from equation (1) that

\[d_Y(\alpha, t^n\gamma(\alpha)) = |n|,\]

for all $\alpha \in C_0(Y)$ and $n \in \mathbb{Z}$. Now notice that for $\beta \in C_0(S)$ with $\pi_Y(\beta) \neq \emptyset$ we have

\[(3) \quad |n| - 5 \leq d_Y(\beta, t^n\gamma(\beta)) \leq |n| + 5,\]

for all $n \in \mathbb{Z}$. The reason for the coarseness of equation (3) is that the twist $t_\gamma$ of $S$ shifts each intersection of a lift of $\beta$ with all the lifts of $\gamma$. See figure 2.

2.8. **Disconnected domains.** Let $A$ be a multicurve with components $\alpha_1, \ldots, \alpha_k$ and let $Y_i$ be the annular domain corresponding to $\alpha_i$. We define

\[C(A) = C(Y_1) \times \ldots \times C(Y_k),\]

and
We define $d_A$ to be the $L^1$-metric. That is, for $\alpha, \beta \in \mathcal{C}(S)$ such that $\pi_Y(\alpha), \pi_Y(\beta) \neq \emptyset$ for all $1 \leq i \leq k$ we have

$$d_A(\alpha, \beta) = \sum_{i=1}^{k} d_Y(\alpha, \beta).$$

Similarly define $(\mathcal{C}(Z), d_Z)$ with $Z$ a union of disjoint domains $Z_1, ..., Z_k \subset S$.

3. Proving Theorem 1.1

3.1. Relative hyperbolicity of $H_A \ast H_B$. Let $X$ be a space composed of tori $T_A, T_B$ with dimensions equal to the ranks of $H_A, H_B$, respectively, connected by an edge $[0, 1]$ at points $t_A$, $t_B$ on $T_A, T_B$, respectively. Notice $X$ is a Salvetti complex for $H_A \ast H_B$ with a point blown up to an edge. Clearly $\pi_1(X) \cong H_A \ast H_B$. Fix a basepoint $x = \frac{1}{2}$. Let $p : (\tilde{X}, \tilde{x}) \to (X, x)$ denote the universal cover. Notice each component of the preimage of $T_A, T_B$ corresponds to a left coset $gH_A, gH_B$, respectively. Collapsing each of these components in $\tilde{X}$ to points $ga, gb$ gives us a tree $T$. Let $W_A, W_B$ denote the set of all such $ga, gb$, respectively and define $W = W_A \cup W_B$.

Let $\Gamma$ be the Cayley graph of $H_A \ast H_B \cong \mathbb{Z}^n \ast \mathbb{Z}^m$. Notice that $T$ is quasiisometric to $\hat{\Gamma}(H_A \ast H_B, \{H_A, H_B\})$. Using the action of $H_A \ast H_B$ on $T$, it’s easy to see that $H_A \ast H_B$ is hyperbolic relative to $\{H_A, H_B\}$ via Bowditch’s definition. Alternatively, it’s also easy to see directly via Farb’s definition that $H_A \ast H_B$ is hyperbolic relative to $\{H_A, H_B\}$.

3.2. Quasiisometrically embedding $T$ into $\mathcal{C}(S)$. Recall our first theorem:

**Theorem 1.1.** There exists a constant $D_0 \geq 3$, independent of $S$, with the following property. If $d_S(A, B) \geq D_0$, then $\Phi : H_A \ast H_B \to G$ is injective, hence an isomorphism, and $G$ is PGF. Moreover, any element not conjugate into a factor is pseudo-Anosov.

Let $D = d_S(A, B) = \text{diam}_S(A, B)$. Let $\alpha_1, ..., \alpha_{n'}$ and $\beta_1, ..., \beta_{m'}$ denote the components of $A$ and $B$, respectively. Without loss of generality, we may assume that $d_S(\alpha_1, \beta_1) = D$. The choices of $\alpha_1$ and $\beta_1$ are unimportant, we are simply choosing vertices of the simplices $A, B$ in $\mathcal{C}(S)$. We first map $T$ to the scaled up tree $T_D$, isomorphic to $T$ but with edges
of length $D$, with the map defined linearly on edges. Call this map $\phi_D$, which is a $(D,0)$-quasiisometry. $H_A \ast H_B$ also acts on $T_D$ so $\phi_D$ is equivariant. We abuse notation and also let $W$ denote the vertices of $T_D$.

The natural homomorphism $\Phi : H_A \ast H_B \to G = \langle H_A, H_B \rangle \subset \text{Mod}(S)$ induces an action of $H_A \ast H_B$ on $\mathcal{C}(S)$. Define $\phi' : T_D \to \mathcal{C}(S)$ by $\phi'(ga) = \Phi(g)\alpha_1, \phi'(gb) = \Phi(g)\beta_1$, and fixing once and for all a geodesic $[\alpha_1, \beta_1]$, we have isometric embeddings $\phi'(g[a,b]) = \Phi(g)[\alpha_1, \beta_1]$. We claim that $\phi'$ is a quasisymmetric embedding if $D$ is sufficiently large. We show this by showing that unions of adjacent edges in $T_D$ map to $D$-local quasigeodesics.

It is a well-known result of Coornaert, Delzant, Papadopoulos \cite{CDP90} that $D$-local uniform quasigeodesics are global quasigeodesics for $D$ sufficiently large.

**Lemma 3.1.** Let $h \in H_B - \{1\}$. Then

$$d_S(\alpha_1, \Phi(h)(\alpha_1)) \geq \frac{2D - 4}{N},$$

where $N > M + 5$ and $M$ is the constant from Theorem 2.2.

**Proof.** Let $H_B = \langle b_1, ..., b_m \rangle$. Here each $b_i = t_{\beta_1}t_{\beta_2}^i...t_{\beta_m}^m$ is a multitwist about the components $\{\beta_1, \beta_2, ..., \beta_m\}$ of $B$. We have $\Phi(h) = b_1^{N_1}..., b_m^{N_m}$ with $b_i \in \{b_1, ..., b_m\}$ and $N_i \in \mathbb{Z}$. Let $Y_i$ be the corresponding annular subsurface of $S$ corresponding to $\beta_i$. Since $h \neq 1$, $\Phi(h)$ acts with positive translation length on some $Y_i$. Without loss of generality, assume $\Phi(h)$ acts with positive translation length on $Y = Y_2$. One might want to choose $Y = Y_1$, but we choose the annular subsurface corresponding to $\beta_2$ to emphasize that the following argument doesn’t depend on our already chosen $\beta_1 (D = d_S(\alpha_1, \beta_1))$.

Let $x \in [\alpha_1, \beta_1]$ -- st($\beta_2$). We remove st($\beta_2$) to ensure $\pi_Y(x) \neq \emptyset$. It follows from equation (3) and the fact that $t_{\beta_i}$ acts with bounded orbit on $\mathcal{C}(Y)$ for $i \neq 2$ that for $N \neq 0$ we have

$$|NN'| - 5 \leq d_Y(x, \Phi(h)^N(x)) \leq |NN'| + 5,$$

where $N'$ is the net number of $t_{\beta_2}$’s that appear in $\Phi(h)$. Choose $N$ sufficiently large so that $|NN'| - 5 > M$. The contrapositive of Theorem 2.2 gives that any geodesic $[x, \Phi(h)^N(x)]$ must pass close to $\beta_2$ in $\mathcal{C}(S)$. Specifically there is a vertex $\gamma \in [x, \Phi(h)^N(x)]$ such that $\pi_Y(\gamma) = \emptyset$ and hence $d(\beta_2, \gamma) \leq 1$. This gives us

$$d_S(x, \Phi(h)^N(x)) \geq 2d_S(x, \beta_1) - 4$$

**Figure 3.** $d_S(x, \Phi(h)^N(x)) \geq 2d_S(x, \beta_1) - 4$
The triangle inequality gives us
\[
d_S(x, \Phi(h)(x)) \geq \frac{2d_S(x, \beta_1) - 4}{N}.
\]

The same argument shows that taking \( N > M + 5 \) gives the above inequality for any \( h \in H_B - \{1\} \). Taking \( x = \alpha_1 \) gives us
\[
d_S(\alpha_1, \Phi(h)(\alpha_1)) \geq \frac{2D - 4}{N}.
\]

\[ \square \]

We can get a better bound for \( d(\alpha_1, \Phi(h)(\alpha_1)) \) by using hyperbolicity of \( C(S) \).

**Lemma 3.2.** Let \( h \in H_B - \{1\} \). Then
\[
d_S(\alpha_1, \Phi(h)(\alpha_1)) \geq 2D - 2((N + 1)\delta + 2),
\]
where \( N > M + 5 \) and \( M \) is the constant from Theorem 2.2. Here \( \delta \) is the hyperbolicity constant of \( C(S) \).

**Proof.** Since \( C(S) \) is \( \delta \)-hyperbolic, any triangle \( (\alpha_1, \beta_1, \Phi(h)(\alpha_1)) \) is \( \delta \)-thin so we can find points \( x \in [\alpha_1, \beta_1], y \in [\beta_1, \Phi(h)(\alpha_1)], z \in [\alpha_1, \Phi(h)(\alpha_1)] \) with \( d_S(x, y), d_S(x, z) \leq \delta \). We’ll show that \( x \) is uniformly close to \( \beta_1 \) and hence \( d_S(\alpha_1, \Phi(h)(\alpha_1)) \) is roughly \( 2D \). See figure 4.

![Figure 4.](image)

If \( x \in \text{st}(\beta_1) \) then we’re done since this gives
\[
d_S(\alpha_1, \Phi(h)(\alpha_1)) \geq 2d_S(\alpha_1, \beta_1) - 2(1 + \delta),
\]
since this implies \( d_S(z, \beta_1) \leq 1 + \delta \) and \( z \in [\alpha_1, \Phi(h)(\alpha_1)] \). Otherwise, the triangle inequality gives
\[
d_S(x, \beta_1) - \delta \leq d_S(y, \beta_1) \leq d_S(x, \beta_1) + \delta.
\]

Since \( d_S(x, \beta_1) = d_S(\Phi(h)(x), \beta_1) \) and \( y, \Phi(h)(x) \in [\beta_1, \Phi(h)(\alpha_1)] \) we get
\[
d_S(y, \Phi(h)(x)) \leq \delta.
\]

Then by the triangle inequality we have
\[ d_S(x, \Phi(h)(x)) \leq 2\delta. \]

As in the proof of Lemma 3.1 we get

\[ d_S(x, \beta_1) \leq N\delta + 2. \]

Hence

\[ d_S(z, \beta_1) \leq (N + 1)\delta + 2, \]

and since \( z \in [\alpha_1, h(\alpha_1)] \) this gives

\[ d_S(\alpha_1, \Phi(h)(\alpha_1)) \geq 2D - 2((N + 1)\delta + 2). \]

\[ \square \]

So we can indeed ensure that \( d_S(\alpha_1, \Phi(h)(\alpha_1)) \) is as large as we want by making \( D = d_S(A, B) \) large. The same argument gives \( d_S(\beta_1, \Phi(g)(\beta_1)) \geq 2D - 2((N + 1)\delta + 2) \) for any \( g \in H_A - \{1\} \). Hence the adjacent edges are mapped uniformly close to geodesics. We now prove that adjacent edges map to uniform quasigeodesics.

**Lemma 3.3.** The map \( \phi' : T_D \to C(S) \) maps geodesics to \( D \)-local \((1, 13\delta + 2C_0)\)-quasigeodesics where \( C_0 = (N + 1)\delta + 2, N > M + 5, \) and \( M \) is the constant from Theorem 2.2.

**Proof.** Since we are only concerned about local behaviour, given any geodesic in \( T_D \), it suffices to show that any adjacent edges map to a \((1, 13\delta + 2C_0)\)-quasigeodesic segment. By translating and interchanging the roles of \( H_A \) and \( H_B \) it suffices to show that for any \( x_0 \in [a, b], y_0 \in [b, ha] \) with \( h \in H_B - \{1\} \) the following inequality holds

\[ d_{T_D}(x_0, y_0) - 13\delta - 2C_0 \leq d_S(x, y) \leq d_{T_D}(x_0, y_0), \]

where \( x = \phi'(x_0), y = \phi'(y_0) \). The upper bound is automatic by the triangle inequality and the definition of \( \phi' \), so we prove the lower bound. The points \( x, y \) lie on the \( \delta \)-thin triangle \((\alpha_1, \beta_1, \Phi(h)(\alpha_1))\). Specifically \( x \in [\alpha_1, \beta_1], y \in [\beta_1, \Phi(h)(\alpha_1)] \).

**Case 1:** \( d_S(x, \beta_1) \leq C_0 \) or \( d_S(y, \beta_1) \leq C_0 \). Suppose \( d_S(x, \beta_1) \leq C_0 \). Then we have

\[ d_{T_D}(x_0, y_0) = d_{T_D}(x_0, b) + d_{T_D}(b, y_0) \]
\[ = d_S(x, \beta_1) + d_S(\beta_1, x) + d_S(x, y) \]
\[ \leq d_S(x, y) + 2C_0. \]

A similar computation proves the lower bound if \( d_S(y, \beta_1) \leq C_0 \).

**Case 2:** \( d_S(x, \beta_1), d_S(y, \beta_1) > C_0 \). As in the proof of Lemma 3.2 we can find points \( z \in [\alpha_1, \beta_1], z' \in [\beta_1, \Phi(h)(\alpha_1)], z'' \in [\alpha_1, \Phi(h)(\alpha_1)] \) such that \( d_S(z, z'), d_S(z, z'') \leq \delta \) and \( d_S(z, \beta_1), d_S(z', \beta_1) \leq C_0 \). Then \( x \) is on the triangle \((\alpha_1, z, z'')\) and \( y \) is on the triangle \((z', z'', \Phi(h)(\alpha_1))\). Since \((\alpha_1, z, z'')\) is \( \delta \)-thin and \( d_S(z, z'') \leq \delta \) there is a point \( x' \in [\alpha_1, z''] \) such that \( d_S(x, x') \leq 2\delta \). Similarly since \((z', z'', \Phi(h)(\alpha_1))\) is \( \delta \)-thin and \( d_S(z', z'') \leq 2\delta \) there is a point \( y' \in [z'', \Phi(h)(\alpha_1)] \) such that \( d_S(y, y') \leq 3\delta \). Then we have
Proof of Theorem 1.1. It now follows from Théorème 1.4 of Chapter 3 of [CDP90] that there exists some $L_1, K_1, C_1$ such that for all $L > L_1$, any $L$-local $(1, 13\delta + 2C_0)$-quasigeodesic is a global $(K_1, C_1)$-quasigeodesic. Here $L_1, K_1, C_1$ depend only on $\delta, 1, 13\delta + 2C_0$. Hence taking $D_0 > L_1$, all geodesics in $T_D$ map to $(K_1, C_1)$-quasigeodesics in $C(S)$. So $\phi := \phi' \circ \phi_D : T \rightarrow C(S)$ is indeed an equivariant (by construction) $(K_2, C_2)$-quasiisometric embedding. Notice that because there is a uniform bound on $\delta$, a uniform bound on $M$ (and hence on $C_0$), and $L_1$ is uniform in $\delta, 1, 13\delta + 2C_0$, there is a uniform bound on $D_0$ independent of $S$.

This also proves that $H_A * H_B$ injects into $\text{Mod}(S)$ via $\Phi$. To see this, we must show $\ker \Phi$ is trivial. Let $f \in H_A * H_B - \{1\}$. If $f$ is conjugate into a factor of $H_A * H_B$, then $f$ maps to a multitwist. Specifically, if $f = ghg^{-1}$ with $g \in H_A * H_B$ and $h \in H_A \cup H_B$, then $\Phi(f)$ is the multitwist about the $\Phi(g)$ image of the underlying curves of $\Phi(h) \in H_A \cup H_B \subset \text{Mod}(S)$, and so is nontrivial.

Suppose that $f$ is not conjugate into any factor. By conjugating if necessary, we can write $f$ as a word $g$ that starts with a syllable in $H_A$ and ends with a syllable in $H_B$. Let $v$ be the midpoint of the edge $[a, b]$ in $T$. Then the infinite geodesic path $\gamma$ in $T$ connecting $\ldots, g^{-1}(v), v, g(v), g^2(v), \ldots$ maps to a quasigeodesic in $C(S)$ via $\phi$.

Notice that $g$ acts as translations on $\gamma$. Hence by equivariance, $\phi(\gamma)$ is a quasixaxis for the action of $\Phi(g)$ on $C(S)$. Then $\Phi(g)$ is pseudo-Anosov and since being pseudo-Anosov is a conjugacy invariant, so is $\Phi(f)$. Therefore $\ker \Phi = \{1\}$ and $G \cong H_A * H_B$. We also have that $G$ is PGF: (1) is clear and (2) is satisfied since we’ve equivariantly quasigeodesically embedded $T$ into $C(S)$ and $T$ is equivariantly quasiisometric to $\hat{\Gamma}(G, \{H_A, H_B\})$. □

This method of constructing global quasigeodesics from local data is not too dissimilar from the techniques used in the proof of [Gül17, Theorem 1.3].
4. General Marking Graphs

To prove that our groups are quasiisometrically embedded in $\text{Mod}(S)$, we will introduce a model space $\mathcal{M}$ for $\text{Mod}(S)$ which we call a general marking graph. Our definition is a modification of Masur and Minsky’s marking graph, which we denote by $\mathcal{M}_{\text{MM}}$, whose vertices are complete clean markings and with edges between markings that differ by an “elementary move.” See [MM00, Section 2.5] for details. We note that for our purposes, we think of complete clean markings as maximal bases (pants decompositions) together with clean transverse curves. In [MM00], Masur and Minsky define complete clean markings as maximal bases together with projections of clean transverse curves to their respective base curves. But if you have a vertex in an annular complex which is a projection of a curve in $S$, then there is only one curve which it is the projection of. So in either case, a complete clean marking is really the same data.

First we define $R$-markings, which will be the vertices for $\mathcal{M}$. An $R$-marking is the homotopy class of a filling collection of simple closed curves $\mu$ on $S$ such that the curves in $\mu$ pairwise intersect at most $R$ times. For fixed $R$, notice there finitely many $R$-markings up to homeomorphism. That is, there are finitely many $\text{Mod}(S)$-orbits of $R$-markings.

There exists $E \in \mathbb{N}$ such that for all $\mu, \mu' \in \mathcal{M}_0$ there exists mapping classes $g, g' \in \text{Mod}(S)$ such that the curves in $g\mu, g'\mu'$ pairwise intersect at most $E$ times, i.e. $g\mu \cup g'\mu'$ is a $E$-marking.

Consider the 1-marking $\mu_0$ above and let $h_1, ..., h_k \in \text{Mod}(S)$ denote the Humphries generators. We also choose $E$ large enough so that $\mu_0 \cup h_i \mu_0$ is a $E$-marking for all $i = 1, ..., k$. We define the edges of $\mathcal{M}$ to be between $\mu, \mu' \in \mathcal{M}_0$ if $\mu \cup \mu'$ is a $E$-marking.

**Lemma 4.1.** $\mathcal{M}$ is connected.

*Proof.* Let $\mu, \mu' \in \mathcal{M}_0$. By the above, there are mapping classes $g, g' \in \text{Mod}(S)$ such that we have edges $[\mu, g\mu_0], [\mu', g'\mu_0]$ in $\mathcal{M}$. To show connectivity it suffices to show $\mu_0$ is connected to every marking in its orbit. Let $h = h_1 h_2 ... h_i \in \text{Mod}(S)$. Consider the edge path in $\mathcal{M}$ given by the vertices $\mu_0, h_1 \mu_0, h_1 h_2 \mu_0, ..., h_1 h_2 ... h_{i-1} \mu_0, h_i \mu_0$. Hence $\mu_0$ is connected to every marking in its orbit and $\mathcal{M}$ is connected. \hfill $\square$

$\mathcal{M}$ is locally finite since, given a vertex, only finitely many $R$-markings can intersect it at most $E$ times. It then follows from the above that the action of $\text{Mod}(S)$ on $\mathcal{M}$ is cocompact.

Notice that vertex stabilizers are finite. If a mapping class fixes a vertex $\mu$, it is permuting the component curves of $\mu$ and disks of $S - \mu$. Notice that these permutations uniquely determine the mapping class by the Alexander trick. Since there are finitely many permutations, the stabilizer is finite.

It follows that the action of $\text{Mod}(S)$ on $\mathcal{M}$ is properly discontinuous. To see this, let $K$ be a compact set. It is contained in a closed ball of radius say $r$ centered at
some $\mu$. Suppose there exists infinitely many distinct mapping classes $g_1, g_2, \ldots$ such that $g_iK \cap K \neq \emptyset$. Hence for all $i$ we have $d_M(\mu, g_i\mu) \leq 2r$. Since the ball of radius $2r$ around $\mu$ contains finitely many vertices, this means that there is a infinite sequence $i_1, i_2, \ldots$ such that $g_{i_1}\mu = g_{i_2}\mu = \ldots$. But then applying $g_{i_1}^{-1}$ gives $\mu = g_{i_1}^{-1}g_{i_2}\mu = \ldots$ which contradicts finiteness of vertex stabilizers.

It then follows from Milnor-Švarc that $\mathcal{M} \cong \text{Mod}(S) \cong \mathcal{M}_{MM}$. Next, we want to state and prove a Lipschitz result for projections of markings to any subsurface $Y$. It is a well-known fact that for $\alpha, \beta \in C_0(S)$ with $\xi(S) \geq 1$ we have the following bound on $d_S(\alpha, \beta)$ in terms of intersection number

$$d_S(\alpha, \beta) \leq 2\log_2(i(\alpha, \beta)) + 2.$$ 

Define

$$k_Y(A) = \begin{cases} 
2\log_2(4(A + 1)) + 2 & \text{if } \xi(Y) \geq 1 \\
A + 1 & \text{if } Y \text{ is an annulus},
\end{cases}$$

and define

$$k(A) = \max\{2\log_2(4(A + 1)) + 2, A + 1\}.$$ 

**Lemma 4.2.** Let $\mu \in \mathcal{M}_0$ and $Y \subset S$ be a domain with $\xi(Y) \neq 0$. Then $\text{diam}_Y(\mu) \leq k(R)$. Moreover, for any edge $e$ in $\mathcal{M}$ we have $\text{diam}_Y(e) \leq k(E)$.

**Proof.** Given any curves $\alpha, \beta \in \mu$ we have $i(\alpha, \beta) \leq R$. If $Y$ is an annulus, these lift to arcs with at most $R$ intersections, hence $\mu$ must project to a set of diameter at most $R + 1$.

If $\xi(Y) \geq 1$ each intersection point becomes at most 4 intersection points when you project to $Y$. For certain $Y$, you can get 4 new intersections. See [MM00, Lemma 2.2]. Hence we have at most $4R + 4 = 4(R + 1)$ intersections in $Y$ and $\text{diam}_Y(\mu) \leq k(R)$.

The moreover statement follows since edges of $\mathcal{M}$ are $E$-markings. \qed

Notice that by our choice of edges, $R$-markings are always adjacent to complete clean markings in $\mathcal{M}$. Hence for any $\mu, \nu \in \mathcal{M}$ there are complete clean markings $\mu', \nu'$ such that

$$d_M(\mu, \mu'), d_M(\nu, \nu') \leq 1.$$ 

Coupling this with our Lipschitz result, notice that for any $\mu, \nu$ and adjacent clean $\mu', \nu'$ we have

$$|d_Y(\mu, \nu) - d_Y(\mu', \nu')| \leq 2k(E).$$

We are now ready to prove a Masur-Minsky style distance formula for our general marking graph. See [MM00, Theorem 6.12]. Let $d_{Y_{MM}}$ denote distance in $Y$ of markings but via Masur and Minsky’s projections $\pi_{Y_{MM}}$ of markings (see [MM00, Section 2.5]), which we note are different from our projections of $R$-markings, which are simply subsurface projections $\pi_Y$. Let

$$\Omega(\mu, \nu, A) = \{Y \subset S \mid d_Y(\mu, \nu) \geq A\},$$

and

$$\Omega_{MM}(\mu, \nu, A) = \{Y \subset S \mid d_{Y_{MM}}(\mu, \nu) \geq A\}.$$
Theorem 4.3. Given $A_2 > 0$ sufficiently large, there exists constants $K_3(A_2) \geq 1, C_3(A_2) \geq 0$ such that for any $\mu, \nu \in \mathcal{M}_0$ we have

$$d_M(\mu, \nu) \asymp_{K_3, C_3} \sum_{Y \in S} [d_Y(\mu, \nu)] A_2.$$  

Proof. This distance formula is a consequence of the following equation which we explain below. Here $\mu', \nu'$ are complete clean markings adjacent to $\mu, \nu$, respectively.

$$d_M(\mu, \nu) \asymp_{1, 2} d_M(\mu', \nu')$$
$$\asymp d_{MM}(\mu', \nu')$$
$$\asymp K_0(A_0), C_0(A_0) \sum_{Y \in \Omega_{MM}(\mu', \nu', A_0)} d_{MM}(\mu', \nu')$$
$$\asymp_{2, 0} \sum_{Y \in \Omega_{MM}(\mu', \nu', A_0)} d_Y(\mu', \nu')$$
$$\asymp K_1(A_1), C_1(A_1) \sum_{Y \in \Omega(\mu', \nu', A_1)} d_Y(\mu', \nu')$$
$$\asymp_{2, 0} \sum_{Y \in \Omega(\mu', \nu', A_1)} d_Y(\mu, \nu)$$
$$\asymp K_2(A_2), C_2(A_2) \sum_{Y \in \Omega(\mu, \nu, A_2)} d_Y(\mu, \nu).$$

The first comparability is by (4). The second comparability is by Milnor-Švarc. The third is Masur-Minsky distance formula. The last 4 require justification.

If $Y$ is an annulus with core a base curve $\alpha$ of $\mu'$ with transversal $\beta$ then $\pi_Y(\mu') = \pi_{YM}(\mu') = \pi_Y(\beta)$ by “cleanliness.” If instead $Y$ is an annulus with the core a transversal $\beta$ transverse to $\alpha$, then $\pi_{YM}(\mu') = \pi_Y(\text{base}(\mu')) = \pi_Y(\alpha)$. The second equality is because $\alpha$ is the only base curve $\beta$ intersects. But for us the projection $\pi_Y(\mu')$ will include any other curves $\beta$ intersects. By cleanliness this is at most 4 other transversals. But these transversals all lie in the link of $\alpha$ (by cleanliness) hence by Theorem 2.3, they project to a set of diameter at most 2 in $\mathcal{C}(Y)$ with $\alpha$ in the middle of this set. Hence in $\mathcal{C}(Y)$ we have $[d_Y(\mu', \nu') - d_Y(\mu', \nu')] \leq 2 + 2 = 4$ for any other complete clean $\nu'$.

Suppose $Y$ is annulus with core not a base curve or transversal of $\mu'$ or $\xi(Y) > 0$. Again the difference between the projections in $\mathcal{M}$ and ours is that $\pi_{YM}(\mu') = \pi_Y(\text{base}(\mu'))$ and we are projecting all of $\mu'$. In $\mathcal{C}(S)$, base($\mu'$) is a simplex and $\mu'$ is a simplex with adjacent edges and has diameter 3. By Theorem 2.3, $\pi_{YM}(\mu')$ is a set of diameter at most 2 exactly in the middle of $\pi_Y(\mu')$ which has diameter at most 6. It follows that $|d_Y(\mu', \nu') - d_{YM}(\mu', \nu')| \leq 2 + 2 = 4$ for any other complete clean marking $\nu'$.

It follows that for $Y$ with $\xi(Y) \neq 0$, if $d_{YM}(\mu', \nu') \geq A_0$ (i.e. $Y \in \Omega_{YM}(\mu', \nu', A_0)$), then

$$\frac{A_0 - 4}{A_0} \leq \frac{d_Y(\mu', \nu')}{d_{YM}(\mu', \nu')} \leq \frac{A_0 + 4}{A_0},$$

and it follows that
\[
\frac{A_0 - 4}{A_0} \leq \frac{\sum_{Y \in \Omega_{MM}(\mu', \nu', A_0)} d_Y(\mu', \nu')}{\sum_{Y \in \Omega_{MM}(\mu', \nu', A_0)} d_Y(\mu', \nu')} \leq \frac{A_0 + 4}{A_0}.
\]

Choosing \( A_0 > 8 \) gives \((2,0)\)-comparability. For the fifth comparability, letting \( A_1 > A_0 + 4 \) gives

\[
\Omega_{MM}(\mu', \nu', A_1) \subset \Omega(\mu', \nu', A_1) \subset \Omega_{MM}(\mu', \nu', A_0),
\]

where the first inclusion is obvious. Since summing \( d_Y(\mu', \nu') \) over the first and last set is comparable to \( d_{MM}(\mu', \nu') \), so is summing over the middle set, for some constants \( K_1(A_1), C_1(A_1) \). For the sixth comparability, equation (5) gives

\[
\frac{A_1 - 2k(E)}{A_1} \leq \frac{d_Y(\mu, \nu)}{d_Y(\mu', \nu')} \leq \frac{A_1 + 2k(E)}{A_1}.
\]

Choosing \( A_1 > 4k(E) \) gives \((2,0)\)-comparability. For the seventh comparability notice that

\[
\Omega(\mu, \nu, A_1) \subset \Omega(\mu, \nu, A_1 - 2k(E)) \subset \Omega(\mu', \nu', A_1 - 4k(E)).
\]

For \( A_1 \) sufficiently large, the sums of \( d_Y(\mu, \nu) \) over the first and last set are comparable, hence so is summing over the middle set. Letting \( A_2 = A_1 - 2k(E) \) finishes the proof.

5. Quasisimmetrically Embedding into \text{Mod}(S)

Convex cocompact subgroups are quasisisometrically embedded in \text{Mod}(S). This follows from the work of Hamenstädt \cite{Ham05} and Kent-Leininger \cite{KL08}, together with the Masur-Minsky distance formula:

\[
d_{\text{Mod}(S)}(g, h) \leq d_G(g, h) \asymp d_S(gv, hv) \preceq_{1, \kappa} \sum_{Y \in S} [d_Y(gv, hv)]_{1, \kappa} \asymp d_{\text{Mod}(S)}(g, h).
\]

The first inequality follows from the triangle inequality. The next comparability is by \cite{Ham05, KL08}, where \( v \) is some fixed vertex in \( C(S) \). The inequality after that is obvious for say \( \kappa > 0 \). And the last comparability is the Masur-Minsky distance formula with \( \kappa \) sufficiently large. Finitely generated Veech groups are also quasisimmetrically embedded in \text{Mod}(S) by work of Tang \cite{Tan21}. It turns out that our groups are too. Notice that in the statement below, \( A \) and \( B \) needn’t be “sufficiently far apart” in \( C(S) \). That is, it holds for any PGF group \( G = H_A * H_B \), hence applies to more general examples than the groups described in Theorem 1.1.

**Theorem 1.2.** Let \( G = H_A * H_B \subset \text{Mod}(S) \) be a nontrivial free product which is PGF with \( H_A, H_B \) subgroups generated by multitwists about multicurves \( A, B \), respectively. Then \( G \) is undistorted in \text{Mod}(S).

A key ingredient in the convex cocompact and Veech group cases is the Masur-Minsky distance formula. The approach to proving the formula is employing a model space for \text{Mod}(S), what we earlier called \( \mathcal{M}_{MM} \). We prove Theorem 1.2 by quasisimmetrically embedding a model space for \( G \) into the general marking graph \( \mathcal{M} \) which is a model for \text{Mod}(S).

We use the same notation from section 3. Let \( V = \{ p^{-1}(x) \} \) denote the preimage of \( x \) in \( \tilde{X} \). Notice that the collapsing of \( \tilde{X} \) to \( T \) is injective on \( V \), so we also think of \( V \) as a subset of \( T \). Fixing some \( v_0 \in V \), the orbit map \( G \to \tilde{X} \) given by \( g \mapsto gv_0 \) is a
quasiisometry. Define a map \( v : \{ p^{-1}(t_A) \} \cup \{ p^{-1}(t_B) \} \rightarrow V \) that sends points to the closest point of \( V \).

The components of the preimages of \( T_A, T_B \) are copies of \( \mathbb{R}^n, \mathbb{R}^m \), respectively. We call them flats. Recall that each flat corresponds to a coset \( gH_A, gH_B \), and that \( W \) is the set of all vertices in \( T \) that are obtained by collapsing the flats to points \( ga \in W_A, gb \in W_B \). Given \( w \in W \), let \( F(w) \) denote the flat associated to \( w \) and let \( A(w) \) denote the multicurve associated to \( w \).

Define \( \pi_w : X \rightarrow F(w) \) to be a closest point projection. Letting \( d_X \) denote the metric on \( X \), we write \( d_w = d_X|_{F(w)} \), and for \( x_1, x_2 \in X \) we write

\[
d_w(x_1, x_2) = d_w(\pi_w(x_1), \pi_w(x_2)).
\]

Let \( \pi_T : \widetilde{X} \rightarrow T \) denote the collapsing map described above. Letting \( d_T \) denote the metric on \( T \), for \( x_1, x_2 \in \widetilde{X} \) we write

\[
d_T(x_1, x_2) = d_T(\pi_T(x_1), \pi_T(x_2)).
\]

We have the following distance formula in \( \widetilde{X} \).

**Lemma 5.1.** For all \( x_1, x_2 \in \widetilde{X} \) we have

\[
d_{\widetilde{X}}(x_1, x_2) = d_T(x_1, x_2) + \sum_{w \in W} d_w(x_1, x_2).
\]

**Proof.** Let \( \gamma \) denote the geodesic in \( \widetilde{X} \) between \( x_1 \) and \( x_2 \). Then \( \gamma \) decomposes into geodesic segments in the flats \( F(w) \) and the edges connecting the flats, specifically the components of the preimage of the edge \([0, 1]\) in \( X \). The segment of the geodesic in \( F(w) \) starts at \( \pi_w(x_1) \) and ends at \( \pi_w(x_2) \) hence has length exactly \( d_w(x_1, x_2) \). The contribution from traveling across \( p^{-1}([0, 1]) \) is exactly \( d_T(x_1, x_2) \).

Since \( \widetilde{\Gamma} \) is equivariantly quasiisometric to \( T \), parabolic geometric finiteness provides a \( G \)-equivariant \((K_0, C_0)\)-quasiisometric embedding \( \phi : T \rightarrow C(S) \). Hence, any choice of a coarsely Lipschitz \( G \)-equivariant map from \( T \) to \( C(S) \) will be a quasiisometric embedding. So, as in the setup to the proof of Theorem 1.1, we define \( \phi(ga) = g\alpha, \phi(gb) = g\beta \), and \( \phi(g[a, b]) = g[\alpha, \beta] \) with \( \alpha, \beta \) some components of \( A, B \), respectively and \( [\alpha, \beta] \) some choice of a geodesic segment in \( C(S) \). Notice for any \( w \in W \), we have that \( \phi(w) \) is a vertex of the simplex \( A(w) \) in \( C(S) \).

We now fix an \( R \)-marking \( \mu \) that “lies in the middle” of \([\alpha, \beta]\). Specifically, \( \mu \) contains \( \phi(\tilde{x}) \) if \( \phi(\tilde{x}) \in C_0(S) \). If \( \phi(\tilde{x}) \notin C_0(S) \), we perturb \( x \in [0, 1] \) so that \( \phi(\tilde{x}) \in C_0(S) \) and again choose \( \mu \) so that it contains \( \phi(\tilde{x}) \). We define an equivariant map \( \mu : V \rightarrow M \) given by \( \mu(gv) = g\mu \). This gives us a coarse map \( \widetilde{\Gamma} \rightarrow M \). To prove Theorem 1.2, it suffices to show that there exists \( K \geq 1, C \geq 0 \) such that for any given \( v_1, v_2 \in V \) we have

\[
\frac{1}{K} d_M(\mu(v_1), \mu(v_2)) - C \leq d_{\widetilde{X}}(v_1, v_2) \leq K d_M(\mu(v_1), \mu(v_2)) + C.
\]

We will need the following series of lemmas.

**Lemma 5.2.** Let \( \kappa > 0 \). Then for all \( v_1, v_2 \in V \) we have

\[
[d_T(v_1, v_2)]_\kappa + \sum_{w \in W} [d_w(v_1, v_2)]_\kappa \asymp_{\kappa+1, \kappa^2+2} d_T(v_1, v_2) + \sum_{w \in W} d_w(v_1, v_2).
\]
Proof. The comparability lower bound is obvious, so we show the upper bound. Define
\[ \Omega_\geq(\kappa, v_1, v_2) = \{ w \in W | d_w(v_1, v_2) \geq \kappa \}, \]
\[ \Omega_<(\kappa, v_1, v_2) = \{ w \in W | 0 < d_w(v_1, v_2) < \kappa \}. \]
If \( x = \frac{1}{2} \), notice that
\[ d_T(v_1, v_2) = |\Omega_\geq| + |\Omega_<|. \]
If \( x \) was perturbed so that \( \phi(\tilde{x}) \in C_0(S) \), we have
\[ d_T(v_1, v_2) + 1 \geq |\Omega_\geq| + |\Omega_<|. \]
We have
\[
\begin{aligned}
d_T(v_1, v_2) + \sum_{w \in W} d_w(v_1, v_2) &= d_T(v_1, v_2) + \sum_{w \in \Omega_<} d_w(v_1, v_2) + \sum_{w \in \Omega_\geq} d_w(v_1, v_2) \\
&\leq d_T(v_1, v_2) + \kappa|\Omega_<| + \sum_{w \in \Omega_\geq} d_w(v_1, v_2) \\
&\leq d_T(v_1, v_2) + \kappa(d_T(v_1, v_2) + 1) + \sum_{w \in \Omega_\geq} d_w(v_1, v_2) \\
&= (\kappa + 1)d_T(v_1, v_2) + \sum_{w \in W} [d_w(v_1, v_2)]_\kappa + \kappa \\
&\leq (\kappa + 1)[d_T(v_1, v_2)]_\kappa + \sum_{w \in W} [d_w(v_1, v_2)]_\kappa + \kappa + (\kappa + 1)\kappa \\
&< (\kappa + 1)([d_T(v_1, v_2)]_\kappa + \sum_{w \in W} [d_w(v_1, v_2)]_\kappa) + \kappa + (\kappa + 1)\kappa.
\end{aligned}
\]
The second to last line follows because
\[ d_T(v_1, v_2) < \kappa \Rightarrow (\kappa + 1)d_T(v_1, v_2) < (\kappa + 1)\kappa. \]
\[ \square \]

Lemma 5.3. Let \( G \) be PGF with respect to \( H_1, \ldots, H_k \) which are subgroups generated by multitwists about multicurves \( A_1, \ldots, A_k \), respectively. Let \( A = \bigcup_{g \in G} g(A_i) \), i.e. \( A \) is the union of all the \( G \)-orbits of the multicurves. Then for any distinct \( A, A' \in A \), \( A \cup A' \) fills \( S \).

Proof. Suppose not. Let \( H, H' \) denote the cosets corresponding to \( A, A' \). Since \( A \cup A' \) doesn’t fill \( S \), there are some multitwists \( t \in H, t' \in H' \) and a curve \( \gamma \in C_0(S) \) contained in \( S - A \cup A' \) such that \( t(\gamma) = t'(\gamma) = t^k t'^k(\gamma) = \gamma \) for all \( k \in \mathbb{Z} \). If \( [t, t'] = 1 \) then \( t \) and \( t' \) span a \( \mathbb{Z}^2 \) plane in \( \Gamma \), the Cayley graph of \( G \), and the 1-neighborhoods in \( \Gamma \) of \( H \) and \( t' H \) intersect in a finite set which contradicts the BCP property. Hence \( [t, t'] \neq 1 \) and the underlying curves or multicurves of \( t, t' \), say \( \alpha, \alpha' \) intersect. It is a well-known fact (for example see \[ \text{Thur88} \]) that there exists some \( N \) such that \( t^N t'^N \) is pseudo-Anosov on the subsurface filled by \( \alpha \cup \alpha' \). Hence \( t^N t'^N \) is not conjugate into \( H_1, \ldots, H_k \) and has an axis in \( \bar{\Gamma} \). Since \( G \) is PGF, there is a corresponding quasiaxis in \( C(S) \) and the \( t^N t'^N \) acts with positive translation length on \( C(S) \). This contradicts \( t^N t'^N(\gamma) = \gamma \). \[ \square \]

We recall the following well-known theorem and state it without proof. For a reference, see \[ \text{BH99, Theorem III.H.1.7}. \]
Theorem 5.4 (Stability of quasigeodesics). For all \( K \geq 1, C \geq 0, \delta > 0 \) there exists \( R(K, C, \delta) \) with the following property. If \( X \) is a \( \delta \)-hyperbolic metric space, \( \gamma \) is a \((K, C)\)-quasigeodesic segment in \( X \) and \( \gamma' \) is a geodesic segment between the endpoints of \( \gamma \), then the Hausdorff distance between \( \gamma \) and \( \gamma' \) is at most \( R \).

The following lemma is likely a well-known result.

Lemma 5.5. Let \( X \) be a \( \delta \)-hyperbolic space, \( \gamma : [0, L] \to X \) a \((K, C)\)-quasigeodesic, \( R(K, C, \delta) \) the stability constant, and \( \pi : \gamma([0, L]) \to [\gamma(0), \gamma(L)] \) a closest point projection. If \( s, t \in [0, L] \) such that \( s < t \) and \( |t - s| > P := 2K(C + 2R) + K^2 \), then \( \pi(\gamma(s)) < \pi(\gamma(t)) \).

**Proof.** Using the quasigeodesic inequalities and stability constants gives

\[
\frac{1}{K}|t - s| - C - 2R \leq d_X(\pi(\gamma(t)), \pi(\gamma(s))) \leq K|t - s| + C + 2R.
\]

Notice that in particular for all \( t_0 \in [0, L] \)

\[
d_X(\pi(\gamma(t_0 + 1)), \pi(\gamma(t_0))) \leq K + C + 2R.
\]

Suppose \( \pi(\gamma(s)) \geq \pi(\gamma(t)) \). Since the geodesic \([\gamma(0), \gamma(L)]\) must eventually travel from \( \pi(\gamma(t)) \) to \( \gamma(L) \) in bounded increments of length at most \( K + C + 2R \), there exists \( t' \geq t \) such that

\[
d_X(\pi(\gamma(t')), \pi(\gamma(s))) < K + C + 2R.
\]

But then using the lower bound from the quasigeodesic inequality and stability constants we get

\[
d_X(\pi(\gamma(t')), \pi(\gamma(s))) \geq \frac{1}{K}|t' - s| - C - 2R \geq \frac{1}{K}|t - s| - C - 2R > K + C + 2R,
\]

a contradiction. Hence \( \pi(s) < \pi(t) \).

\[\square\]

Let \( R_0(K_0, C_0, \delta) \) be a stability constant for the image of \( \phi \) in \( C(S) \). Since we may take \( R_0 \) as large as we like, assume it is an integer greater than 1. For \( t_1, t_2 \in T \) we write \([t_1, t_2]_T\) to denote the geodesic between them in \( T \).

Lemma 5.6. Let \( v_1, v_2 \in V, t \in T, w \in W \), such that \( t, w \in [v_1, v_2]_T, t < w, d_T(t, w) > P_0 := 2K_0(C_0 + 2R_0) + K_0^2 \), and \( \pi : \phi([v_1, v_2]_T) \to [\phi(v_1), \phi(v_2)] \) is a closest point projection map. If \( s \in [\phi(v_1), \pi(\phi(t))] \) then

\[\pi_{\phi(w)}(s) \neq \emptyset.\]

Moreover for any simplex \( \Delta \) containing \( \phi(w) \) we have

\[\pi_{\Delta}(s) \neq \emptyset.\]

Recall that \( \pi_{\Delta} \) is the product of the projections of its vertices and hence \( \pi_{\Delta}(s) \neq \emptyset \) means that for each vertex \( p \) of \( \Delta \) we have \( \pi_p(s) \neq \emptyset \).

**Proof.** Under the assumptions of the hypothesis, it follows from Lemma 5.5 that

\[\pi(\phi(t)) < \pi(\phi(w)).\]

Notice
Lemma 5.7. There exists a constant $C_1 \geq 0$ with the following property. For all edges $e = [w_1, w_2] \subset T$ and any $Z$ a union of disjoint domains $Z_1, \ldots, Z_N \subset S$ with $\pi_{Z_i}(A(w_1)), \pi_{Z_i}(A(w_2)) \neq \emptyset$ for $i = 1, \ldots, N$ we have

$$d_Z(A(w_1), A(w_2)) \leq C_1.$$  

In particular, for any $w \in W$ with $w \neq w_1, w_2$

$$d_A(w)(A(w_1), A(w_2)) \leq C_1.$$

Proof. Fix an edge $e = [w_1, w_2]$. The number of intersections between $A(w_1)$ and $A(w_2)$ bounds the distance between their projections to any subsurfaces to which they have nonempty projections. Then there is a constant $C'_1(w_1, w_2)$ such that for all $Y \subset S$ with $\pi_Y(A(w_1)), \pi_Y(A(w_2)) \neq \emptyset$

$$d_Y(A(w_1), A(w_2)) < C'_1.$$  

Notice that $C'_1$ does not depend on $Y$. Since $Z$ has at most $\xi(S) = 3g + p - 3$ components there exists $C_1(S, w_1, w_2)$ such that

$$d_Z(A(w_1), A(w_2)) < C_1.$$  

Again, notice $C_1$ does not depend on $Z$. Let $e' = [w_3, w_4]$ be another edge with $\pi_{Z_i}(A(w_3)), \pi_{Z_i}(A(w_4)) \neq \emptyset$ for $i = 1, \ldots, N$. Recall that there is a single edge orbit in $T$. So there is some $g \in G$ such that $ge = e'$. Since for all $g' \in G, w' \in W$ we have $A(g'w') = g'A(w')$ it follows that

$$d_Z(A(w_3), A(w_4)) = d_Z(A(gw_1), A(gw_2)) = d_Z(gA(w_1), gA(w_2)) = d_{g^{-1}Z}(A(w_1), A(w_2)) \leq C_1.$$  

\[\square\]
For the “in particular” statement notice that for any \( w, w' \in W \), \( w \neq w' \) implies \( \pi_{A(w)}(A(w')) \neq \emptyset \) since \( G \) is PGF and hence \( A(w) \cup A(w') \) is a marking by Lemma 5.3.

Define a half-edge to be a segment of length \( \frac{1}{2} \) in \( T \) with one endpoint in \( W \) and the other in \( V \).

**Lemma 5.8.** There exists a constant \( C_2 \geq 0 \) with the following property. For all half-edges \( h = [v, w] \subset T \) and any \( Z \) a union of disjoint domains \( Z_1, ..., Z_N \subset S \) with \( \pi_{Z_i}(A(w)) \neq \emptyset \) for \( i = 1, ..., N \) we have

\[
d_Z(\mu(v), A(w)) \leq C_2.
\]

In particular, for any \( w' \in W \) with \( w' \neq w \)

\[
d_{A(w')}(\mu(v), A(w)) \leq C_2.
\]

**Proof.** The proof is similar to the proof of the preceding lemma. But unlike the preceding lemma, there are two half-edge orbits. So first assume that \( w \in W_A \), run through the argument, and obtain a bound. Using the same argument, find a bound when \( w \in W_B \) and take \( C_2 \) to be the maximum of the two cases. □

**Lemma 5.9.** Let \( v_1, v_2 \in V \). Then for all \( w \in W \) such that \( d_w(v_1, v_2) > 0 \) we have for \( i = 1, 2 \)

\[
d_{A(w)}(\mu(v_i), \mu(v(\pi_w(v_i)))) \leq C_3,
\]

where \( C_3 = P_0 C_1 + 2C_2 + \xi(S)(2M + k(R) + 2) \) and \( M \) is the constant from Theorem 2.2.

![Figure 8. Lemma 5.9](image.png)

**Proof.** There are 4 cases describing whether the points \( v_1, v_2, w \) are \( P_0 \)-close in \( T \). The worst case is when \( d_T(v_i, w) > P_0 \) for \( i = 1, 2 \) in the sense that the bound we derive from it is the largest and works for the other 3 cases. So assume \( d_T(v_i, w) > P_0 \). Notice that by Lemma 5.6 this implies \( \pi_{A(w)}(\phi(v_i)) \neq \emptyset \).

Consider the geodesic \( [v_1, v_2]_T \) in \( T \). Label vertices of \( W \) traversed in order from \( v_1 \) to \( v_2 \) by \( w_1, ..., w_N \). That is

\[
[v_1, v_2]_T = [v_1, w_1]_T \cup [w_1, w_2]_T \cup ... \cup [w_N, v_2]_T.
\]

Suppose \( w_i = w \). Let \( 1 \leq j \leq N \) be the smallest number such that \( d_T(w_j, w) \leq P_0 \). Either \( j = 1 \) or \( d_T(w_{j-1}, w) > P_0 \).

**Case 1:** \( j = 1 \).

We have by the triangle inequality and Lemmas 5.7, 5.8

\[
d_{A(w)}(\mu(v_1), \mu(v(\pi_w(v_1)))) \leq d_{A(w)}(\mu(v_1), A(w_1)) + \sum_{l=1}^{i-2} d_{A(w)}(A(w_l), A(w_{l+1})) + d_{A(w)}(A(w_{i-1}), \mu(v(\pi_w(v_1)))) \leq (P_0 - 1)C_1 + 2C_2.
\]
Case 2: $d_T(w_{j-1}, w) > P_0$.

It follows from Lemma 5.6 that for all $s \in [\phi(v_1), \pi(\phi(w_{j-1}))]$ we have

$$\pi_{A(w)}(s) \neq \emptyset,$$

where $\pi : \phi([v_1, v_2]_T) \to [\phi(v_1), \phi(v_2)]$ is a closest point projection map. Hence it follows from Theorem 2.2, Theorem 2.3, and Lemma 4.2 that

$$d_{A(w)}(\mu(v_1), \pi(A(w_{j-1}))) \leq \xi(S)(M + 2 + k(R)).$$

By the triangle inequality we have

$$d_S(\phi(w_{j-1}), \phi(w)) \geq \frac{1}{K_0} d_T(w_{j-1}, w) - C_0 > \frac{1}{K_0} P_0 - C_0 = 4R_0 + K_0 + C_0 \geq 4R_0 + 1.$$

Since $d_S(\phi(w_{j-1}), \pi(\phi(w_{j-1}))) \leq R_0$ we have for all $s \in [\phi(w_{j-1}), \pi(\phi(w_{j-1}))]$

$$\pi_{A(w)}(s) \neq \emptyset,$$

and hence again by Theorem 2.2, Theorem 2.3, and Lemma 4.2 we have that

$$d_{A(w)}(\mu(v_1), A(w_{j-1})) \leq \xi(S)(2M + 2 + k(R)).$$

By the triangle inequality we have

$$d_{A(w)}(A(w_{j-1}), \mu(v(\pi_w(v_1)))) \leq \sum_{i=1}^{j-1} d_{A(w)}(A(w_i), A(w_{i+1}))) + d_{A(w)}(A(w_{j-1}), \mu(\pi(v(\pi_w(v_1)))))$$

$$\leq P_0 C_1 + C_2.$$
for some constants. Since the inclusion of $H(w)$ into $H_{\text{max}}(w)$ is a quasiisometric embedding, it then follows that

$$d_w(v_1, v_2) \succ_{K_A, C_A} d_{A(w)}(\phi(v(\pi_w(v_1))), \phi(v(\pi_w(v_2)))),$$

for some constants $K_A \geq 1, C_A \geq 0$. Then by Lemma 2.1, assuming $\kappa_A > 2K_A C_A$, we have

$$\sum_{w \in W_A} [d_w(v_1, v_2)]_{\kappa_A} \preceq_{2K_A, 0} \sum_{w \in W_A} [d_{A(w)}(\phi(v(\pi_w(v_1))), \phi(v(\pi_w(v_2))))]_{C_A},$$

and since $\mu(v(\pi_w(v_i)))$ contains $\phi(v(\pi_w(v_i)))$ we have

$$\sum_{w \in W_A} [d_w(v_1, v_2)]_{\kappa_A} \preceq_{2K_A, 0} \sum_{w \in W_A} [d_{A(w)}(\mu(v(\pi_w(v_1))), \mu(v(\pi_w(v_2))))]_{C_A}.$$

A similar inequality holds for $w \in W_B$ with constants say $K_B \geq 1, C_B \geq 1, \kappa_B > K_B C_B$. Taking $\kappa_4 = \max\{\kappa_A, \kappa_B\}, K_4 = \max\{K_A, K_B\}, C_4 = \min\{C_A, C_B\}$ yields

$$\sum_{w \in W} [d_w(v_1, v_2)]_{\kappa_4} \preceq_{2K_4, 0} \sum_{w \in W} [d_{A(w)}(\mu(v(\pi_w(v_1))), \mu(v(\pi_w(v_2))))]_{C_4}. \tag{6}$$

We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Recall that it suffices to show that there exists $K \geq 1, C \geq 0$ such that for any given $v_1, v_2 \in V$ we have

$$\frac{1}{K} d_M(\mu(v_1), \mu(v_2)) - C \leq d_X(v_1, v_2) \leq K d_M(\mu(v_1), \mu(v_2)) + C.$$

The coarse lower bound follows from the triangle inequality so we prove the upper bound.

$$d_X(v_1, v_2) = d_T(v_1, v_2) + \sum_{w \in W} d_w(v_1, v_2)$$

$$\leq_{\kappa_0 + 1, \kappa_3 + 3, \kappa_0} [d_T(v_1, v_2)]_{\kappa_0} + \sum_{w \in W} [d_w(v_1, v_2)]_{\kappa_0}$$

$$\leq_{2K_0, 0} [d_S(\phi(v_1), \phi(v_2))]_{C_0} + \sum_{w \in W} [d_w(v_1, v_2)]_{\kappa_0}$$

$$\leq_{2K_4, 0} [d_S(\phi(v_1), \phi(v_2))]_{C_0} + \sum_{w \in W} [d_{A(w)}(\mu(v(\pi_w(v_1))), \mu(v(\pi_w(v_2))))]_{C_4}$$

$$\leq_{2, 0} [d_S(\phi(v_1), \phi(v_2))]_{C_0} + \sum_{w \in W} [d_{A(w)}(\mu(v_1), \mu(v_2))]_{2C_3}$$

$$\leq \sum_{Y \subset S} [d_Y(\mu(v_1), \mu(v_2))]_{\kappa_1}$$

$$\leq_{K_5(\kappa_1), C_5(\kappa_1)} d_M(\mu(v_1), \mu(v_2)).$$

The second line is by Lemma 5.2 for some $\kappa_0 > 0$. The third is by Lemma 2.1 since $d_T(v_1, v_2) \succ_{K_0, C_0} d_S(\phi(v_1), \phi(v_2))$, assuming $\kappa_0 > 2K_0 C_0$. The fourth is by equation (6) assuming $\kappa_0 > \kappa_4$. The fifth is by Lemmas 2.1 and 5.9 assuming $C_4 > 4C_3$. The sixth is taking $\kappa_1 = \min\{C_0, 2C_3\}$. Finally, the last comparability is by Theorem 4.3. \qed
There exists some $M_1 > 0$ such that for all domains $Y \neq Y_\alpha$, $S$ with $\alpha \in \mathcal{A}$ and for all $\mu_1, \mu_2 \in \mathcal{M}_G$ we have

$$d_Y(\mu_1, \mu_2) \leq M_1.$$  

In full detail the following proof is slightly technical, so we give a brief description of the main idea. Since $\mu_1, \mu_2 \in \mathcal{M}_G$, there are vertices $v_1, v_2 \in T$ such that $\mu(v_i) = \mu_i$ for $i = 1, 2$. The generic case is when the boundary of $Y$ is far from the geodesic $[\phi(v_1), \phi(v_2)]$ and the bound follows from Theorem 2.2. When Theorem 2.2 is not applicable there is some segment $\gamma_Y$ of $[\phi(v_1), \phi(v_2)]$ which has empty projection to $Y$. The general situation to consider here is when $\partial Y$ lies somewhere in the middle of $[\phi(v_1), \phi(v_2)]$.

We describe an explicit path in $\mathcal{C}(S)$ from $\phi(v_1)$ to $\phi(v_2)$ that has nonempty bounded projection to $Y$. See figure 9. Starting at $\phi(v_1)$, we move along $[\phi(v_1), \phi(v_2)]$ and get uniformly close to $\gamma_Y$. Then we move along a geodesic to a vertex $\phi(w_1)$ in $\phi([v_1, v_2])_T$, the quasigeodesic image of the geodesic in $T$ connecting $v_1$ and $v_2$. By Theorem 2.2 the concatenation of these two paths has diameter no more than $2M$ in $\mathcal{C}(Y)$. Similarly there is a path from $\phi(v_2)$ to some vertex $\phi(w_m)$ in $\phi([v_1, v_2])_T$ with $\mathcal{C}(Y)$-diameter no more than $2M$. Finally, we use the $\phi$-quasisymmetric embedding constants to bound the length of $\phi([w_1, w_m])_T$, which is the segment of $\phi([v_1, v_2])_T$ connecting $\phi(w_1)$ and $\phi(w_m)$, and use Lemma 5.7 to bound the projection of this segment to $\mathcal{C}(Y)$.

The concern one should have is whether the path we’ve described has vertices with empty projection to $Y$. By our choice of “close” to $\gamma_Y$, we can assure the paths from $\phi(v_1)$ to $\phi(w_1)$ and from $\phi(w_m)$ to $\phi(v_2)$ have nonempty projection. Finally, for the part connecting $\phi(w_1)$ to $\phi(w_m)$, we describe in the proof why one can simply delete or replace the segments with empty projection from a sum that bounds the distance.

Our last remark is that we make repeated use of the constants $M, 2$, and $k(R)$ from Theorem 2.2, Theorem 2.3, and Lemma 4.2, respectively, throughout the following proof without stating the respective result each time.

![Figure 9. The path](image-url)
Considering the path of the geodesic \([v_1, v_2]_T\) in \(T\), label vertices of \(W\) traversed in order by \(w_1, \ldots, w_N\). That is

\[
[v_1, v_2]_T = [v_1, w_1]_T \cup [w_1, w_2]_T \cup \ldots \cup [w_N, v_2]_T.
\]

Recall that

\[
\phi([v_1, v_2]_T) = \phi(\{v_1, w_1\}) \cup \phi(\{w_1, w_2\}) \cup \ldots \cup \phi(\{w_N, v_2\}).
\]

Either one, two, or three adjacent vertices of \([\phi(v_1), \phi(v_2)]\) have empty projection to \(Y\). Let \(\gamma^*_Y\) denote this segment with empty projection. Going from \(\phi(v_1)\) to \(\phi(v_2)\), let \(\gamma^*_Y, \gamma^*_Y\) denote the first, last point of \(\gamma_Y\), respectively.

Recall that \(R_0(K_0, C_0, \delta)\) is the stability constant and \(D = d_S(A, B) \geq 1\) since \(G\) is a nontrivial free product.

**Case 1:** \(d_S(\phi(v_1), \gamma^*_Y), d_S(\gamma^*_Y, \phi(v_2)) > R_0 + 5D\).

Let \(1 \leq j < k \leq N\) be such that \(d_S(\gamma_j, \gamma_Y) = d_S(\gamma_Y, \gamma_k) = R_0 + 5D\). Let \(\pi: [\phi(v_1), \phi(v_2)] \to [\phi(v_1), v_2]_T\) be a closest point projection. Let \(1 \leq l \leq m \leq N\) be such that \(\pi(\gamma_j) \in [\phi(w_{l-1}), \phi(w_1)], \pi(\gamma_k) \in [\phi(w_m), \phi(w_{m+1})]\). Notice \(d_S(\gamma_j, \phi(w_l)), d_S(\gamma_k, \phi(w_m)) \leq R_0 + D\). By our choice of constants it’s easy to check that for all \(s\) in the connected paths \([\phi(v_1), \gamma_j] \cup [\gamma_j, \phi(w_l)]\) and \([\phi(w_m), \gamma_k] \cup [\gamma_k, \phi(v_2)]\)

\[
\pi_Y(s) \neq \emptyset.
\]

It also follows from our choice of constants that \(\pi_Y(A(\gamma)), \pi_Y(A(w_m)) \neq \emptyset\). Then we have

\[
d_Y(\mu(v_1), A(w_l)), d_Y(A(w_l), \mu(v_2)) \leq 2M + k(R) + 2.
\]

Then we are done if we can show \(d_Y(A(w_l), A(w_m))\) is uniformly bounded. If \(\pi_Y(A(w_n)) \neq \emptyset\) for all \(l < n < m\) then by the triangle inequality we have

\[
(7) \quad \sum_{n=l}^{m-1} d_Y(A(w_n), A(w_{n+1})) \leq C(m - l).
\]

Suppose that there is some \(l < p < m\) such that \(\pi_Y(A(w_p)) = \emptyset\). Then \(Y \subset S - A(w_p)\). Let \(H(w_p)\) denote the coset associated to \(A(w_p)\). Since \(G\) is PGF, \(\pi_Y(A(w_{p+1})) \neq \emptyset\) since \(A(w_p) \cup A(w_{p+1})\) is a marking by Lemma 5.3. Notice that \(A(w_{p-1})\) and \(A(w_{p+1})\) differ by an element of \(H(w_p)\). That is, there is some \(t \in H(w_p)\) such that \(t(A(w_{p-1})) = A(w_{p+1})\). Since \(t\) is some product of twists on \(A(w_p)\) and \(Y \subset S - A(w_p), t\) acts trivially on \(C(Y)\). Hence \(\pi_Y(A(w_{p-1})) = \pi_Y(\gamma_Y)\). Also notice that \(\pi_Y(A(w_p)) = \emptyset\) implies \(\pi_Y(A(w_q)) = \emptyset\) for all other \(l < q < m\) distinct from \(p\). This follows from Lemma 5.3.

By the above, if \(l+1 < p < m-1\) and \(\pi_Y(A(w_p)) = \emptyset\), the following sum is well-defined and bounded \(d_Y(A(w_l), A(w_m))\) by the triangle inequality

\[
d_Y(A(w_l), A(w_{l+1})) + \ldots + d_Y(A(w_{p-2}), A(w_{p-1})) + d_Y(A(w_{p+1}), A(w_{p+2})) + \ldots + d_Y(A(w_{m-1}), A(w_m)) \leq C(m - l).
\]

We’ve simply deleted from (7) the two terms containing \(A(w_p)\). Notice we’ve excluded the cases \(\pi_Y(A(w_{l+1})) = \emptyset\) or \(\pi_Y(A(w_{m+1})) = \emptyset\). Say \(\pi(A(w_{l+1})) = \emptyset\). The two terms in (7) containing \(A(w_{l+1})\) are \(d_Y(A(w_l), A(w_{l+1}))\) and \(d_Y(A(w_{l+1}), A(w_{l+2}))\). Deleting the first term renders the sum useless in bounding \(d_Y(A(w_l), A(w_{l+1}))\). So instead replace these two terms and \(d_Y(A(w_{l+1}), A(w_{l+2}))\) with \(d_Y(A(w_l), A(w_{l+3})) = d_Y(A(w_{l+2}), A(w_{l+3})) \leq C_1\).
It should now be clear that, even if there is \( l < p < m \) with \( \pi_Y(A(w_p)) = \emptyset \), by deleting or replacing terms from (7), there is a well-defined sum showing that

\[
d_Y(A(w_1), A(w_m)) \leq C_1(m - l)
\]

Finally, we note that \( m - l \) is uniformly bounded above by a constant depending only on \( K_0, C_0, R_0, \) and \( D \). Specifically, \( m - l \leq K_0d_S(\phi(w_1), \phi(w_m)) + K_0C_0 \leq K_0(4R_0 + 12D) + K_0C_0 \).

**Case 2:** \( d_S(\phi(v_1), \gamma_Y^1), d_S(\gamma_Y^2, \phi(v_2)) \leq R_0 + 5D \).

If \( \pi_Y(A(w_n)) \neq \emptyset \) for all \( 1 \leq n \leq N' \) then the following sum is well-defined and bounds \( d_Y(\mu, \nu) \)

\[
(8) \quad d_Y(\mu_1, A(w_1)) + \sum_{n=1}^{N' - 1} d_Y(A(w_n), A(w_{n+1})) + d_Y(A(w_{N'}), \mu_2) \leq 2C_2 + C_1(N' - 1)
\]

Similar to case 1, if there is some \( 1 < p < N' \) such that \( \pi_Y(A(w_p)) = \emptyset \), simply delete the two terms in equation (8) that contain \( A(w_p) \) to get a smaller sum.

We’ve excluded two cases: when \( \pi_Y(w_1) = \emptyset \) or \( \pi_Y(w_{N'}) = \emptyset \). Say \( \pi_Y(w_1) = \emptyset \). Then the two terms in (8) containing \( A(w_1) \) are \( d_Y(\mu(v_1), A(w_1)), d_Y(A(w_1), A(w_2)) \). Deleting the first term from (8) renders the sum useless in bounding \( d_Y(\mu(v_1), \mu(v_2)) \). So instead replace the two terms with \( d_Y(\mu(v_1), A(w_2)) = d_Y(\mu(v_1), A(w_0)) \leq C_2 \) where \( w_0 \) is the other endpoint of the edge \([w_0, w_1]\) containing \( v_1 \) in \( T \). Again, \( \pi_Y(A(w_0)) = \pi_Y(A(w_2)) \) since \( \pi_Y(A(w_1)) = \emptyset \).

Similarly define \( w_{N' + 1} \) if \( \pi_Y(w_N) = \emptyset \). It should now be clear that, even if there exists \( 1 \leq p \leq N' \) with \( \pi_Y(A(w_p)) = \emptyset \), by deleting or replacing terms from (8), there is a well-defined sum showing that

\[
d_Y(\mu_1, \mu_2) \leq 2C_2 + C_1(N' - 1)
\]

Finally, we note that \( N' \) is uniformly bounded above by a constant depending only on \( K_0, C_0, R_0, \) and \( D \).

**Case 3:** \( d_S(\phi(v_1), \gamma_Y^1) \leq R_0 + 5D \) and \( d_S(\gamma_Y^2, \phi(v_2)) > R_0 + 5D \), or \( d_S(\phi(v_1), \gamma_Y^1) > R_0 + 5D \) and \( d_S(\gamma_Y^2, \phi(v_2)) \leq R_0 + 5D \).

In words, this is the case when \( \partial Y \) is close to either \( \phi(v_1) \) or \( \phi(v_2) \). It’s easy to see how to combine the techniques of cases 1 and 2 to get a bound for this case.

Finally, take \( M_1 \) to be the maximum bound from all three cases. \( \square \)

### 6. Another Notion of Geometric Finiteness

Since \( G = H_A \ast H_B \cong \mathbb{Z}^n \ast \mathbb{Z}^m \) is a right angled-Artin group, it is a **hierarchically hyperbolic space** \([BHS17]\). Moreover, the inclusion of certain \( G \) into \( \text{Mod}(S) \) is a **hieromorphism** \([DHS17] \) Definition 1.10]. Namely, when the generators of \( H_A, H_B \) are all powers of Dehn twists (that is, they are not multitwists) the inclusion is a hieromorphism. We refer the reader to \([DHS17]\) Definition 1.10 for the terminology and precise definition, and sketch how our groups satisfy the definition.

Using the \( \text{CAT}(0) \) cube complex \( \tilde{X} \) as a model for \( G \) and the general marking graph \( \mathcal{M} \) as a model for \( \text{Mod}(S) \), we let \( f : \tilde{X} \rightarrow \mathcal{M} \) be the coarse map \( \mu : \tilde{X} \rightarrow \mathcal{M} \) defined in section 5. We take \( \tilde{X} \), flats, and “subflats” as our index set (see \([BHS17]\) Proposition 8.3, Remark 13.2]). Given any flat, an axis of the flat corresponds to twisting in a curve in \( \mathcal{A} \), the \( G \)-orbit of \( A \cup B \). This correspondence partially defines the map \( \pi(f) \) between index sets. For flats \( F \) of dimension greater than 1, let \( \pi(f)(F) \) be the disjoint union
of the curves corresponding to its underlying axes. Given an axis \( U \) in \( \tilde{X} \) with points corresponding to elements say \( gt^n \), with \( g \in G \) and \( n \in \mathbb{Z} \), its associated hyperbolic space \( CU \) is a line subdivided into intervals by “midpoints,” say \( gt^{\frac{n+1}{2}} \), corresponding to hyperplanes. A quasiisometric embedding \( \rho(f, U) \) may be given by

\[
\rho(f, U)(gt^{\frac{n+1}{2}}) = \pi_{\pi(f)(U)}(gt^n(\mu)) \subset C_{\pi(f)(U)}.
\]

For flats \( U \subset \tilde{X} \) with dimension greater than 1, the associated hyperbolic space \( CU \) is a join of the associated hyperbolic spaces of its underlying axes, hence has uniformly bounded diameter. So one can simply choose \( \rho(f, U) \) to be a constant map to \( C_{\pi(f)(U)} \).

Coarse commutativity of the diagrams from [DHS17, Definition 1.10] follows from Lemma 5.9.

Theorem 5.10 tells us that these hieromorphisms are in fact an extensible [DHS17, Definition 5.5]. This implies that such \( G = H_A \ast H_B \) are also geometrically finite in the sense of [DHS17, Definition 2].

There is a more general notion of a a slanted hieromorphism [DHS17, Definition 5.1]. This allows us to consider \( H_A, H_B \) with multitwist generators. It is easy to see that if the generators of \( H_A \) have disjoint supports, and similarly for \( H_B \), then the inclusion of \( G \) into \( \text{Mod}(S) \) is an extensible slanted hieromorphism, hence is geometrically finite in the sense of [DHS17].

But in general, when the generators of \( H_A \) (or \( H_B \)) do not have disjoint supports, it is not always clear how define \( \pi(f) \). Defining \( \pi(f) \) in the obvious way generally fails to satisfy item (I) of [DHS17, Definition 5.1]. It is an interesting question whether the definition of a slanted hieromorphism can be relaxed to include these examples yet still lead to geometric finiteness in the sense of [DHS17]. Alternatively, one might look for a different HHS structure on \( \tilde{X} \) and definition of \( \pi(f) \) for these general examples in order to satisfy the current definition of slanted hieromorphisms.

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