Active Ranking from Pairwise Comparisons and the Futility of Parametric Assumptions

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June 29, 2016

Abstract

We consider sequential or active ranking of a set of \( n \) items based on noisy pairwise comparisons. Items are ranked according to the probability that a given item beats a randomly chosen item, and ranking refers to partitioning the items into sets of pre-specified sizes according to their scores. This notion of ranking includes as special cases the identification of the top-\( k \) items and the total ordering of the items. We first analyze a sequential ranking algorithm that counts the number of comparisons won, and uses these counts to decide whether to stop, or to compare another pair of items, chosen based on confidence intervals specified by the data collected up to that point. We prove that this algorithm succeeds in recovering the ranking using a number of comparisons that is optimal up to logarithmic factors. This guarantee does not require any structural properties of the underlying pairwise probability matrix, unlike a significant body of past work on pairwise ranking based on parametric models such as the Thurstone or Bradley-Terry-Luce models. It has been a long-standing open question as to whether or not imposing these parametric assumptions allow for improved ranking algorithms. Our second contribution settles this issue in the context of the problem of active ranking from pairwise comparisons: by means of tight lower bounds, we prove that perhaps surprisingly, these popular parametric modeling choices offer little statistical advantage.

1 Introduction

Given a collection of \( n \) items, it is frequently of interest to estimate a ranking based on noisy comparisons between pairs of items. Such rank aggregation problems arise across a wide range of applications. Some traditional examples in sports include identifying the best player in a tournament, selecting the top \( k \) teams for playoffs, and finding the full ranking of players. More recently, the internet era has led to a variety of applications involving pairwise comparison data, including recommender systems [Pie+13; Agg16] for rating movies, books, or other consumer items; peer grading [Sha+13] for ranking students in massive open online courses; and online sequential survey sampling [SL15] for assessing the popularity of proposals in a population of voters. In many of these and other such applications, it is possible to make comparisons in an active or adaptive manner—that is, based on the outcomes of comparisons of previously chosen pairs. Motivated by these applications, the focus of this paper is the problem of obtaining statistically sound rankings based on a sequence of actively chosen pairwise comparisons.

We consider a collection of \( n \) items, and our data consists of outcomes between pairs of items in this collection collected actively. We assume that the outcomes of comparisons are stochastic—that is, item \( i \) beats item \( j \) with an unknown probability \( M_{ij} \in (0, 1) \). The outcomes of pairwise
comparisons are furthermore assumed to be statistically mutually independent. We define the ordering of the items in terms of their (unknown) scores, where the score \( \tau_i \) of item \( i \) is defined as the probability that item \( i \) beats an item chosen uniformly at random from all other items:

\[
\tau_i := \frac{1}{n-1} \sum_{j \neq i} M_{ij}.
\] (1)

In the context of social choice theory [DB81], these scores are also known as the Borda scores of the items. Apart from its intuitive appeal, this score is of particular interest as it can be considered as a natural generalization of the assumed orderings in several popular comparison models. Specifically, the parametric Bradley-Terry-Luce (BTL) [BT52; Luc59] and Thurstone [Thu27] models, as well as the non-parametric Strong Stochastic Transitivity (SST) model [TER69], induce a total ordering of the items; this ordering is identical to that given by the scores \( \tau_1, \ldots, \tau_n \). In this paper, we consider the problem of partitioning the items into sets of pre-specified sizes according to their respective scores \( \tau_1, \ldots, \tau_n \). This notion of ranking includes as special cases identification of the top-\( k \) items and the total ordering of the items.

We make two primary contributions. We begin by presenting and analyzing a simple active ranking algorithm for estimating a partial or total ranking of the items. At each round, this algorithm first counts the number of comparisons won, then computes confidence bounds from those counts, which it finally uses to select a subset of pairs to be compared at the next time step. We provide performance guarantees showing that with high probability, the algorithm recovers the desired partial or total ranking from a certain number of comparisons. This number is a function of the (unknown) scores \( \tau_1, \ldots, \tau_n \) and therefore distribution dependent. We prove matching (up to log-factors) distribution dependent lower bounds, showing that the algorithm is near-optimal in the number of comparisons. Our analysis leverages the fact that ranking in terms of the scores \( \tau_1, \ldots, \tau_n \) is related to a particular class of multi-armed bandit problems [ED+06; Bub+13; Urv+13]. This connection has been observed in past work [Yue+12; Jam+15; Urv+13] in the context of finding the top item.

Our second main contribution relates to the popular parametric modeling choices made in the literature. Our analysis of the aforementioned algorithm does not make any assumptions on the pairwise comparison probabilities. Having said this, we note that it is common in the literature—for instance, see the papers [Sz15; Hun04; Neg+12; Haj+14; CS15; Sou+14; Sha+16a; MG15] as well as references therein—to impose various parametric assumptions on the pairwise comparison probabilities, such as those arising from the Bradley-Terry-Luce (BTL) and Thurstone parametric models. There is a long standing debate on whether such assumptions are reasonable—that is, in which situations they (approximately) hold, and in which they fail [Reg+11]. The primary motivation for such assumptions is to leverage the specific parametric structure to enable a (possibly significant) reduction in the sample complexity. In this paper we show that, perhaps surprisingly, none of these aforementioned parametric assumptions reduce the sample complexity of finding a ranking. An important consequence of our result is that, for a given ranking problem, there is no advantage in imposing parametric assumptions on the generative process; in fact, such assumptions may be harmful since estimators tailored to these specific models may perform much worse when the corresponding model is not a good fit to the actual data.

**Related work:** There is a vast literature on ranking and estimation from pairwise comparison data. Most works assume probabilistic comparison outcomes; we refer to the paper [JN11] and references therein for ranking problems assuming deterministic comparison outcomes. Several prior
works [Hun04; Neg+12; Haj+14; Sou+14; Sha+16a; SW15; Che+16] consider settings where pairs to be compared are chosen a priori. In contrast, we consider settings where the pairs may be chosen in an active manner. The recent work [Sz15] assumes the Bradley-Terry-Luce (BTL) parametric model, and considers the problem of finding the top item and the full ranking in an active setup. For certain underlying distributions, the corresponding results [Sz15, Theorem 3 and Theorem 4] are close to what our more general result implies. On the other hand, for several other problem instances the performance guarantees Theorem 3 and Theorem 4 in the work [Sz15] lead to a significantly larger sample complexity. Our work thus offers better guarantees for the BTL model, despite the additional generality of our setting in that we do not restrict ourselves to the BTL model. The paper [MG15] considers the problem of finding a full ranking of items for a BTL pairwise comparison model, and provides a performance analysis for a probabilistic model on the BTL parameter vector. Finally, Eriksson [Eri13] considers the problem of finding the very top items using graph based techniques, Busa-Fekete et al. [BF+13] consider the problem of finding the top-k items, and Ailon [Ail11] considers the problem of linearly ordering the items so as to disagree in as few pairwise preference labels as possible. Our work is also related to the literature on multi-armed bandits, and we revisit these relations later in the paper.

Organization: The remainder of this paper is organized as follows. We begin with background and problem formulation in Section 2. We then present a description and a sharp analysis of our ranking algorithm in Section 3. In Section 4, we then show that parametric assumptions do not reduce the sample complexity. Section 5 presents results from numerical simulations on some additional aspects of this problem and our proposed algorithm. We provide proofs of all our results in Section 6, and conclude with a discussion in Section 7.

2 Problem formulation and background

In this section, we formally state the ranking problem considered in this paper and formalize the notion of an active ranking algorithm. We also formally introduce the class of parametric models in this section.

2.1 Pairwise probabilities, scores, and rankings

Given a collection of items \([n] := \{1, \ldots, n\}\), let us denote by \(M_{ij} \in (0, 1)\) the (unknown) probability that item \(i\) wins a comparison with item \(j\). For all items \(i\) and \(j\), we require that each comparison results in a winner (meaning that \(M_{ij} + M_{ji} = 1\)), and we set \(M_{ii} = 1/2\) for concreteness. For each item \(i \in [n]\), consider the score (1) given as \(\tau_i := \frac{1}{n-1} \sum_{j \in [n] \setminus \{i\}} M_{ij}\). Note that the (unknown) score \(\tau_i \in (0, 1)\) corresponds to the probability that item \(i\) wins a comparison with an item \(j\) chosen uniformly at random from \([n] \setminus \{i\}\).

Assuming that the scores are all distinct, they define a unique ranking of the \(n\) items; this (unknown) ranking corresponds to the permutation \(\pi : [n] \to [n]\) such that

\[
\tau_{\pi(1)} > \tau_{\pi(2)} > \cdots > \tau_{\pi(n)}.
\]

In words, \(\pi(i)\) denotes the \(i^{th}\) ranked item according to the scores. A number of ranking problems can be defined in terms of \(\pi\): at one extreme, finding the best item corresponds to determining the item \(\pi(1)\), whereas at the other extreme, finding a complete ranking is equivalent to estimating \(\pi(j)\) for all \(j \in [n]\). We introduce a general formalism that allows us to handle these and many
other ranking problems. In particular, given an integer \( L \geq 2 \), we let \( \{k_\ell\}_{\ell=1}^L \) be a collection of positive integers such that \( 1 \leq k_1 < k_2 < \ldots < k_{L-1} < k_L = n \). Any such collection of positive integers defines a partition of \([n]\) into \( L \) disjoint sets of the form

\[
S_1 := \{\pi(1), \ldots, \pi(k_1)\}, \quad S_2 := \{\pi(k_1 + 1), \ldots, \pi(k_2)\}, \ldots, \quad S_L := \{\pi(k_{L-1} + 1), \ldots, \pi(n)\}.
\]  

(2)

For instance, if we set \( L = 2 \) and \( k_1 = k \), then the set partition \((S_1, S_2)\) corresponds to splitting \([n]\) into the top \( k \) items and its complement. At the other extreme, if we set \( L = n \) and \((k_1, k_2, \ldots, k_n) = (1, 2, \ldots, n)\), then the partition \( \{S_\ell\}_{\ell=1}^L \) allows us to recover the full ranking of the items, as specified by the permutation \( \pi \).

For future reference, we define

\[
\mathcal{C}_{\text{min}} := \{M \in (0, 1)^{n \times n} \mid M_{ij} = 1 - M_{ji}, M_{ij} \geq M_{\text{min}}, \text{ and } \tau_i \neq \tau_j \text{ for all } (i, j)\},
\]

(3)

corresponding to the set of pairwise comparison matrices with pairwise comparison probabilities lower bounded by \( M_{\text{min}} \), and for which a unique ranking exists.\(^1\)

### 2.2 The active ranking problem

An active ranking algorithm acts on a pairwise comparison model \( M \in \mathcal{C}_0 \). Consider any specified values of \( L \) and \( \{k_\ell\}_{\ell=1}^L \) defining a partition of the form (2) in terms of their latent scores (1). The goal is to obtain a partition of the items \([n]\) into \( L \) disjoint sets of the form (2) from active comparisons. At each time instant, the algorithm can compare two arbitrary items, and the choice of which items to compare may be based on the outcomes of previous comparisons. As a result of comparing two items \( i, j \), the algorithm receives an independent draw of a binary random variable with success probability \( M_{ij} \) in response. After termination dictated by an associated stopping rule, the algorithm returns a ranking \( \hat{S}_1, \ldots, \hat{S}_L \).

For a given tolerance parameter \( \delta \in (0, 1) \), we say that a ranking algorithm \( \mathcal{A} \) is \( \delta \)-accurate for a comparison matrix \( M \) if the ranking it outputs obeys

\[
\mathbb{P}_M \left[ \hat{S}_\ell = S_\ell, \text{ for all } \ell = 1, \ldots, L \right] \geq 1 - \delta.
\]  

(4)

For any set of comparison matrices \( \mathcal{C} \), we say that the algorithm \( \mathcal{A} \) is uniformly \( \delta \)-accurate over \( \mathcal{C} \) if it is \( \delta \)-accurate for each matrix \( M \in \mathcal{C} \). The performance of any algorithm is measured by means of its sample complexity, that is, the number of comparisons required to obtain the desired partition.

### 2.3 Active ranking and multi-armed bandits

The ranking problem we consider is related to multi armed bandits [Kau+16; BCB12] as follows. A multi armed bandit model consists of a collection of \( n \) arms with corresponding unknown distributions. The goal is to maximize the reward obtained by drawing samples from the distributions. The prior works [Yue+12; Jam+15; Urv+13] draw a relation of the pairwise comparison setup to that of multi-armed bandits. To see this relation, notice that \( \tau_i \) can be regarded as the mean of a Bernoulli random variable with mean \( \tau_i \), and comparing item \( i \) to an item chosen uniformly at

\(^1\)We note that our results actually do not require the entire underlying ordering of the scores to be strict; rather, we require strict inequalities only at the boundaries of the sets \( S_1, \ldots, S_L \).
random from all items but \( i \) is equivalent to taking an independent draw of this Bernoulli random variable. Our subsequent analysis in Section 3 relies on this relation.

When cast in the multi-armed bandit setting, the setting of pairwise comparisons is often referred to as that of “dueling bandits”. Prior works in this setting [Yue+12; Jam+15; Urv+13] address the problem of finding the best arm from noisy comparisons between arms—that is, the problem of finding the item with the largest score. In this paper, we consider the more general problem of finding a partial or total ordering of the items.

Although our pairwise ranking problem is closely related to the multi-armed bandit setting, there is an important distinguishing factor between the two settings. If we view our problem as a multi-armed bandit problem with Bernoulli random variables with means \( \{ \tau_i \}_{i=1}^n \), these means are actually coupled together, in the sense that information about any particular mean imposes constraints on all the other means. In particular, any set of scores \( \{ \tau_i \}_{i=1}^n \) must be realized by some valid set of pairwise comparison probabilities \( \{ M_{ij} \}_{i,j \in [n]} \). These pairwise comparison probabilities must obey \( M_{ij} = 1 - M_{ji} \), which means that the scores must also satisfy certain constraints, not all of which are obvious. One straightforward constraint follows from definition (1), which implies that any valid set of scores must satisfy \( \sum_{i=1}^n \tau_i = n/2 \). However, this condition is not sufficient: for instance, there is no set of pairwise comparison probabilities with scores \([1, 1, 0, 0] \), even though those scores sum to 4/2. In order to verify this fact, note that \( \tau_1 = 1 \) implies \( M_{12} = M_{13} = M_{14} = 1 \). Thus, we have \( M_{21} = 0 \), which implies \( \tau_2 \leq 2/3 \) and therefore contradicts \( \tau_2 = 1 \).

### 2.4 Parametric models

In this section, we introduce a family of parametric models that form a basis of several prior works [Sz15; Hun04; Neg+12; Haj+14; Sha+16a]. To be clear, we make no modeling assumptions for our algorithm and its analysis in Section 3. We analyze these parametric models in Section 4, where we show that, perhaps surprisingly, none of these parametric assumptions reduce the sample complexity of finding a ranking.

Any member in this family of models is defined by a strictly increasing and continuous function \( \Phi: \mathbb{R} \to [0, 1] \) obeying \( \Phi(t) = 1 - \Phi(-t) \), for all \( t \in \mathbb{R} \). The function \( \Phi \) is assumed to be known. A pairwise comparison matrix in this family is associated to an unknown vector \( w \in \mathbb{R}^n \), where each entry of \( w \) represents some quality or strength of the corresponding item. The parametric model \( C_{\text{PAR}}(\Phi) \) associated with the function \( \Phi \) is defined as:

\[
C_{\text{PAR}}(\Phi) = \{ M_{ij} = \Phi(w_i - w_j) \text{ for all } i, j \in [n], \text{ for some } w \in \mathbb{R}^n \}.
\]  (5)

Popular examples of models in this family are the Bradley-Terry-Luce (BTL) model, obtained by setting \( \Phi \) equal to the sigmoid function \( (\Phi(t) = \frac{1}{1 + e^{-t}}) \), and the Thurstone model, obtained by setting \( \Phi \) equal to the Gaussian CDF. Note that \( \tau_1 > \tau_2 > \ldots > \tau_n \) is equivalent to \( w_1 > w_2 > \ldots > w_n \), meaning that rankings in terms of the scores and the entries of \( w \) are equivalent.

### 3 Active ranking from pairwise comparisons

In this section, we present our algorithm for obtaining the desired partition of the items as described earlier in Section 2, and a sharp analysis of this algorithm proving its optimality up to logarithmic factors.
latent scores of items in \( S_1 \)

\[ \tau_1 \bullet \]

\[ \tau_2 \bullet \]

latent scores of items in \( S_2 \)

\[ \tau_3 \bullet \]

\[ \tau_4 \bullet \]

Figure 1. Illustration of the AR algorithm applied to the problem of finding the top 2 items out of \( n = 4 \) items total, corresponding to \( S_1 = \{1, 2\}, S_2 = \{3, 4\} \). The figure depicts the estimates \( \hat{\tau}_i(t) \), along with the corresponding confidence intervals \([\hat{\tau}_i(t) - 4\alpha t, \hat{\tau}_i(t) + 4\alpha t]\), at different time steps \( t \). At time \( t = 5 \), the algorithm is not confident about the position of any of the items, and hence it continues to sample further. At time \( t = 10 \), the confidence interval of item (1) indicates that (1) is either the best or the second best item, therefore the AR algorithm assigns (1) to \( \hat{S}_1 \). Likewise, it assigns item (4) to \( \hat{S}_2 \). At time step \( t = 15 \), the AR algorithm assigns items (1) and (2) to \( \hat{S}_1 \) and \( \hat{S}_2 \), respectively, and terminates.

3.1 Active ranking (AR) algorithm

Our active ranking algorithm is based two ingredients:

- Successive estimation of the scores \( \tau_1, \ldots, \tau_n \). We estimate \( \tau_i \) by comparing item \( i \) with items chosen uniformly at random from \([n] \setminus \{i\}\).

- Assigning an item \( i \) to an estimate of the set \( S_\ell \), denoted by \( \hat{S}_\ell \), if the confidence that \( i \) is in \( S_\ell \) is sufficiently large.

This strategy is essentially an adaption of the successive elimination approach from the bandit literature, proposed in the classical paper [Pau64], and studied in a long line of subsequent work (for example, [ED+06; Bub+13; Urv+13; Jam+15]).

More precisely, the inputs to the algorithm are a collection of positive integers \( \{k_\ell\}_{\ell=0}^L \) such that

\[ k_0 = 0 < k_1 < k_2 < \ldots < k_{L-1} < k_L = n \]

that define a desired ranking, and a tolerance parameter \( \delta \in (0, 1) \) that defines the probability with which the algorithm is allowed to fail.

Algorithm 1 (Active Ranking (AR)). At time \( t = 0 \), define and initialize the following quantities:

- \( S = [n] \) (set of items not ranked yet);
- \( \hat{S}_\ell = \emptyset \) for all \( \ell \in [L] \) (estimates of the partition);
- \( \hat{k}_\ell = k_\ell \) for all \( \ell \in \{0, \ldots, L\} \) (borders of the sets);
- \( \hat{\tau}_i(0) = 0 \) for all \( i \in [n] \) (estimates of the scores).
At any time $t \geq 1$:

1. For every $i \in S$: Compare item $i$ to an item chosen uniformly at random from $[n] \setminus \{i\}$, and set
   \[
   \hat{\tau}_i(t) = \begin{cases} 
   \frac{t-1}{t} \hat{\tau}_i(t-1) + \frac{1}{t} & \text{if } i \text{ wins} \\
   \frac{1}{t} \hat{\tau}_i(t-1) & \text{otherwise.}
   \end{cases}
   \] (6)

2. Sort the items in set $S$ by their current estimates of the scores: For any $k \in [|S|]$, let $(k)$ denote the item with the $k$-th largest estimate of the score.

3. With $\alpha_t := \sqrt{\frac{\log(4nt^2)}{\sqrt{t}}}$, do the following for every $j \in S$:
   If the following pair of conditions (7a) and (7b) hold simultaneously for some $\ell \in [L]$,
   \[
   \hat{k}_{\ell-1} = 0 \quad \text{or} \quad \hat{\tau}_j(t) < \hat{\tau}_{(\hat{k}_{\ell-1})}(t) - 4\alpha_t \quad (j \text{ likely is one of the lower } n - k_{\ell-1} - 1 \text{ items})
   \] (7a)
   \[
   \hat{k}_\ell = |S| \quad \text{or} \quad \hat{\tau}_j(t) > \hat{\tau}_{(\hat{k}_{\ell+1})}(t) + 4\alpha_t \quad (j \text{ likely is one of the top } k_\ell \text{ items}),
   \] (7b)
   then add $j$ to $\hat{S}_\ell$, remove $j$ from $S$, and set $\hat{k}_{\ell'} \leftarrow \hat{k}_{\ell} - 1$ for all $\ell' \leq \ell$.

4. If $S = \emptyset$, terminate.

See Figure 1 for an illustration of the progress of this algorithm on a particular instance.

### 3.2 Guarantees and optimality of the AR algorithm

In this section, we establish guarantees on the number of samples for the AR algorithm to succeed. As we show below, the sample complexity is a function of the gaps between the scores, defined as

\[
\bar{\Delta}_{\ell,i} := \tau_{k_{\ell-1}} - \tau_i, \quad \text{and} \quad \Delta_{\ell,i} := \tau_i - \tau_{k_{\ell+1}}.
\] (8)

The dependence on these gaps is controlled via the functions

\[
f_0(x) := \frac{1}{x^2}, \quad \text{and} \quad f_{AR}(x) := \frac{\log(2/x)}{x^2}.
\] (9)

In part (a) of the theorem to follow, we prove an upper bound on the AR algorithm involving $f_{AR}$, and in part (b), we prove a lower bound applicable to any uniformly $\delta$-accurate algorithm involving $f_0$. As one might intuitively expect, the number of comparisons required is lower when the underlying gaps are larger. See Figure 2 for an illustration of the gaps for the particular problem of finding a partitioning of the items $\{1, 2, \ldots, 6\}$ into three sets of cardinality two each.

**Theorem 1.**

There are positive universal constants $(c_{up}, c_{low})$ such that:

(a) For any $M \in C_0$, and any $\delta \in (0, 0.14]$ the AR algorithm is $\delta$-accurate for $M$ using a query size upper bounded by

\[
c_{up} \log \left( \frac{n}{\delta} \right) \left\{ \sum_{i \in S_1} f_{AR}(\bar{\Delta}_{1,i}) + \sum_{\ell=2}^{L-1} \sum_{i \in S_\ell} \max \left\{ f_{AR}(\bar{\Delta}_{\ell,i}), f_{AR}(\Delta_{\ell,i}) \right\} + \sum_{i \in S_L} f_{AR}(\bar{\Delta}_{L,i}) \right\}. \] (10a)
Figure 2. Illustration of the gaps $\bar{\Delta}_{\ell,i}$ and $\Delta_{\ell,i}$ relevant for finding an partitioning of the items $\{1,2\ldots,6\}$ into the set $S_1 = \{1,2\}$, $S_2 = \{3,4\}$, and $S_3 = \{5,6\}$.

(b) For any $\delta \in (0,0.14]$, consider a ranking algorithm that is uniformly $\delta$-accurate over $C_{1/8}$. Then when applied to a given pairwise comparison model $M \in C_{3/8}$, it must make at least

$$c_{\text{low}} \log \left( \frac{1}{2\delta} \right) \left\{ \sum_{i \in S_1} f_0(\Delta_{1,i}) + \sum_{\ell=2}^{L-1} \sum_{i \in S_\ell} \max \left\{ f_0(\Delta_{\ell,i}), f_0(\bar{\Delta}_{\ell,i}) \right\} + \sum_{i \in S_L} f_0(\bar{\Delta}_{L,i}) \right\}$$

(10b)

comparisons on average.

Part (a) of Theorem 1 proves that the AR algorithm is $\delta$-accurate, and characterizes the number of comparisons required to find a ranking as a function of the gaps between scores. In contrast, part (b) shows that, up to logarithmic factors, the AR algorithm is optimal, not only in a minimax sense, but in fact when acting on any given problem instance. The proof of part (b) involves constructing pairs of comparison matrices that are especially hard to distinguish, and makes use of a change of measure lemma [Kau+16, Lem. 1] from the bandit literature, that was also instrumental in developing lower bounds on the sample complexity for identifying the best $k$ arms in a multi armed bandit problem [Kau+16], and the best arm in a dueling bandit problem [Jam+15].

In order to gain intuition on this result, in particular the dependence on the squared gaps, it is useful to specialize to the toy case $n = 2$. In this special case with $n = 2$, we have $\tau_1 = M_{12}$ and $\tau_2 = M_{21} = 1 - M_{12}$. Thus, the ranking problem reduces to testing the hypothesis $\{\tau_1 > \tau_2\}$. One can verify that the hypothesis $\{\tau_1 > \tau_2\}$ is equivalent to $\{M_{12} > \frac{1}{2}\}$. Let $X_i, i = 1,\ldots,N$ be the outcomes of $N$ independent comparisons of items 1 and 2, that is, $P[X_i = 1] = M_{12}$ and $P[X_i = 0] = 1 - M_{12}$. A natural test for $\{M_{12} > \frac{1}{2}\}$ is to test whether $\bar{X} > 1/2$, where $\bar{X} := \frac{1}{N} \sum_{i=1}^{N} X_i$. Supposing without loss of generality that $M_{12} > \frac{1}{2}$, by Hoeffding’s inequality, we can upper bound the corresponding error probability as

$$P[\bar{X} \leq 1/2] = P[\bar{X} - M_{12} \leq 1/2 - M_{12}] \leq e^{-2N(1/2-M_{12})^2} = e^{-2N(\tau_1-\tau_2)^2}.$$

Thus, for $N = \frac{\log(1/\delta)}{2(\tau_1-\tau_2)^2}$ the error probability is less than $\delta$. The bound (10a) in Theorem 1(a) yields an identical result up to constant factors.
More generally, testing for the inclusion $i \in S_\ell$ amounts to testing for $\hat{\Delta}_{\ell,i} > 0$ and $\Delta_{\ell,i} > 0$, where $\hat{\Delta}_{\ell,i} = \tau_{k_{\ell-1}} - \tau_i$ and $\Delta_{\ell,i} = \tau_i - \tau_{k_{\ell+1}}$. These requirements provide some intuition regarding the dependence of our bounds on the inverses of the squared gaps.

### 3.3 Gains due to active estimation

We next briefly compare our (optimal) estimator for the active setting to a minimax optimal estimator for settings that do not allow active (or adaptive) choice of comparisons. We show that the sample complexity of our active estimator is typically significantly lower than that of the optimal non-active estimator. We hasten to add that, in the context of pairwise comparisons, it is well known that active estimators typically significantly improve over non-active estimators.

Recent work by a subset of the current authors [SW15] considers the problem of ranking items from pairwise comparisons in a passive random design setup. Theorem 1 in the paper [SW15] proves that it is possible to recover top $k$ items with high probability based on comparing a total of $n \log n$ items. The procedure to do so is very simple: it ranks items according to the total number of comparisons won. Moreover, Theorem 1 of the paper [SW15] shows that this estimator is optimal in a minimax sense. In contrast, Theorem 1 of the present paper shows that in the active setting, the number of comparisons necessary and sufficient for finding the top $k$ items is of the order $\sum_{i=1}^{k} \frac{1}{(\tau_i - \tau_{k+1})^2} + \sum_{i=k+1}^{n} \frac{1}{(\tau_k - \tau_i)^2}$, up to a logarithmic factor. This shows that the gains in sample complexity are typically large when the pairs may be chosen actively. Specifically, the sample complexity of the non-active estimator is typically significantly larger, except in the special case when the scores satisfy the specific relations $\tau_1 = \ldots = \tau_k$ and $\tau_{k+1} = \ldots = \tau_n$, in which case the two estimators would have similar performance. A similar conclusion holds if we compare the results of the paper [SW15] with those of the present paper for the problem of recovering the full ranking.

The specific gains of active rank estimation over non-active rank estimation depend on the distribution of the scores. As an illustration, Figure 3 shows some real-world examples of this distribution for data collected by Salganik and Levy [SL15]: the left panel shows the scores estimated in the paper [SL15] of a collection of environmental proposals for New York City, whereas the right panel shows a collection of educational proposals for the Organisation for Economic Co-operation and Development (OECD). These data were collected by asking interviewees in corresponding online surveys for preferences between two options. The goal of such online surveys is, for example, to identify the top proposals or a total ranking of the proposals. Our results show that estimation of the top $k$ ideas or another ranking with an active scheme would require a significantly smaller number of queries compared to an non-active estimator.

### 4 Futility of parametric modeling for ranking

The AR algorithm described and analyzed in the previous section applies to any comparison matrix $M$—that is, it neither assumes nor exploits any particular structure in $M$, such as that present in a parametric model. As discussed in the introduction, parametric models are often studied, in the hope of reducing sample complexity, and there is a long standing debate on whether or not such assumptions are reasonable—that is, in which situations do they (approximately) hold, and in which they fail [Reg+11]. Given that the AR algorithm imposes no conditions on the model, one might suspect that when ranking data is actually drawn from a parametric model (for example, of BTL
or Thurstone type), it could be possible to come up with another algorithm with a lower sample complexity. Surprisingly, as we show in this section, this intuition turns out to be false: imposing parametric assumptions does not reduce the sample complexity of ranking in any significant way.

Recall that a parametric model is described by a continuous and strictly increasing CDF $\Phi$; in this section, we prove a lower bound that applies even to algorithms that are given a priori knowledge of the function $\Phi$. For any pair of constants $0 < \phi_{\min} \leq \phi_{\max} < \infty$, we say that a CDF $\Phi$ is $(\phi_{\min}, \phi_{\max}, M_{\min})$-bounded, if it is differentiable, and if its derivative $\Phi'$ satisfies the bounds

$$\phi_{\min} \leq \Phi'(t) \leq \phi_{\max}, \quad \text{for all } t \in [\Phi^{-1}(M_{\min}), \Phi^{-1}(1 - M_{\min})]. \quad (11)$$

Note that these conditions hold for standard parametric models, such as the BTL and Thurstone models.

The following result applies to any parametric model $\mathcal{C}_{\text{PAR}}(\Phi)$ described by a CDF of this type. It also involves the complexity parameter

$$F(\tau(M)) := \sum_{i \in S_1} f_0(\Delta_{1,i}) + \sum_{\ell=2}^{L-1} \sum_{i \in S_\ell} \max \left\{ f_0(\Delta_{\ell,i}), f_0(\bar{\Delta}_{\ell,i}) \right\} + \sum_{i \in S_L} f_0(\bar{\Delta}_{L,i}), \quad (12)$$

which appeared previously in the lower bound from Theorem 1(b).

**Theorem 2.**

(a) Given a tolerance $\delta \in (0, 0.15]$, and a continuous and strictly increasing CDF $\Phi$ whose derivative is $(\phi_{\min}, \phi_{\max}, M_{\min})$-bounded, consider any algorithm $A$ that is uniformly $\delta$-accurate over $\mathcal{C}_{\text{PAR}}(\Phi) \cap \mathcal{C}_{M_{\min}}$. Then, when applied to a given pairwise comparison matrix $M \in \mathcal{C}_{\text{PAR}}(\Phi) \cap \mathcal{C}_{M_{\min}}$, the expected query size of $A$ under the model $\mathbb{P}_M$ is lower bounded as

$$\mathbb{E}_M [N] \geq c_{\text{par}} \log \left( \frac{1}{2\delta} \right) F(\tau(M)), \quad \text{where } c_{\text{par}} := \frac{M_{\min} \phi_{\min}^2}{2 \phi_{\max}^2 \phi_{\min}}. \quad (13)$$
(b) Let $\tau \in (0,1)^n$ be any set of scores that is realizable by some pairwise comparison matrix $M' \in \mathcal{C}_{M_{\min}}$. Then, for any continuous and strictly increasing $\Phi$, there exists a pairwise comparison matrix in $M \in \mathcal{C}_{\text{PAR}}(\Phi) \cap \mathcal{C}_{M_{\min}}$ with scores $\tau$, and in particular $F(\tau(M)) = F(\tau(M'))$.

Let us consider the implications of Theorem 2. First, it should be noted that the lower bound (13) in part (a) is—in a certain sense—stronger than the lower bound from Theorem 1, because it applies to a broader class of algorithms—namely, those that are $\delta$-accurate only over the smaller class of parametric models. On the flip side, one might suspect that the lower bound (13) is also weaker in some sense: more precisely, it is conceivable that there might exist some “difficult” matrix $M' \in \mathcal{C}_{M_{\min}}$ such that $\sup_{M \in \mathcal{C}_{\text{PAR}}(\Phi) \cap \mathcal{C}_{M_{\min}}} F(\tau(M))$ is, possibly significantly, smaller than $F(\tau(M'))$. Part (b) of Theorem 2 rules out this possibility: for any pairwise comparison matrix $M'$—possibly one that is not generated by a parametric model—it guarantees the existence of a parametric matrix $M$ for which the ranking problem is equally hard. This result is surprising because one might think that imposing parametric assumptions might simplify the ranking problem. In fact, the full set $\mathcal{C}_{M_{\min}}$ is substantially larger than the parametric subclass $\mathcal{C}_{\text{PAR}}(\Phi) \cap \mathcal{C}_{M_{\min}}$; in particular, one can demonstrate matrices in $\mathcal{C}_{M_{\min}}$ that cannot be well-approximated by any parametric model (for example, see the paper [Sha+16b] for inapproximability results of this type).

A consequence of Theorem 2 is that up to logarithmic factors, the AR algorithm is again optimal, even if we restrict ourselves only to algorithms that are uniformly $\delta$-accurate over a parametric subclass. This confirms the futility of making parametric assumptions: at least in terms of rank aggregation, they only limit the flexibility while failing to provide any significant reductions in sample complexity.

To put those results in a broader perspective, we note that while assuming parametric models does not improve the sample complexity of ranking, such assumptions can significantly reduce the sample complexity of estimating the values of the pairwise comparison probabilities $M$ themselves, as compared to not making any structural assumptions on the probabilities [Sha+16b].

Finally, a comment on the specific values of the constant $c_{\text{par}}$ in the bound (13): for $M_{\min} = \frac{3}{8}$, we have $c_{\text{par}} = 0.164$ and $c_{\text{par}} = 0.169$ for the BTL and Thurstone models, respectively. Similarly, for $M_{\min} = \frac{1}{4}$, we have $c_{\text{par}} = 0.07$ and $c_{\text{par}} = 0.079$ for the BTL and Thurstone models, respectively.

5 Numerical results

In this section, we study numerically two aspects that are relevant when applying variants of our algorithm in practice. Specifically, we study whether the particular selection of the confidence interval of the AR algorithm (that is $\alpha_t$) is conservative, and demonstrate that a heuristic variant of our algorithm allows to find an approximate ranking with a significant improvement in the sample complexity.

5.1 Selection of confidence interval

Recall that the AR algorithm eliminates an item if the confidence that it belongs to one of the sets $S_1, \ldots, S_L$, is sufficiently large. Our main results show that the AR algorithm succeeds at recovering the ranking with probability at least $1 - \delta$, provided that the length of the confidence interval is chosen proportional to $\alpha_t = \sqrt{2 \beta t}$, with $\beta_t = \log(4nt^2/\delta)$. Here, $\delta$ is the error tolerance input parameter of the AR algorithm. While this result is optimal up to log-factors, the particular
Figure 4. (a) Number of comparisons required to find the top-2 items out of 5 items. (b) Empirical error probability required to find the top-2 items out of 5 items. The three choices of \( \beta_t \) perform qualitatively very similar. The particular choice \( \beta_t = \log(4n t^2)/\delta \) is overly conservative, in the sense that for obtaining a \( \delta \)-accurate ranking, \( \beta_t \) can be chosen smaller, which in turn results in fewer comparisons.

choice of \( \beta_t \) might be overly conservative, and improvements in the (empirical) sample complexity might be obtained by choosing \( \beta_t \) smaller, as we show next. To investigate this claim, we compare the choice \( \beta_t = \log(4 t/\delta) \) from our theory to two alternatives, namely

\[
\beta_t = \log(4/\delta), \quad \text{and} \quad \beta_t = \log(4(\log(t) + 1)/\delta).
\]

The last choice is conjectured to be optimal for certain multi-armed bandits [Kau+16]. We generate a pairwise comparison model with \( n = 5 \), scores \( \tau = (0.9, 0.7, 0.5, 0.3, 0.1) \), and use the AR algorithm to find the top 2 items. The results, depicted in Figure 4, show that for all three choices of \( \beta_t \), the (empirical) error probability is close to linear as a function of the error tolerance \( \delta \) in the log-log plot, and there is little qualitative difference between the three choices of \( \beta_t \), for the values of \( \delta \) considered in our simulations.

5.2 Approximate recovery

In several applications, exact recovery of a rating might not be necessary, or one might not be able to query sufficiently many comparisons in order to find an exact ranking. We leave theoretical results on approximate recovery for a future publication. In this section, we nevertheless show numerically that an adaption of the AR algorithm can be used for approximate ranking, which results in significant savings in the sample complexity over exact ranking.

We begin by defining a notion of an approximate ranking. Let us say that a ranking \( \widehat{S}_1, \ldots, \widehat{S}_L \) is \( \epsilon \)-accurate if

\[
\min_{i \in S_{\ell}, j \in \widehat{S}_{\ell}} |\tau_i - \tau_j| \leq \epsilon, \quad \text{for all } \ell = 1, \ldots, L.
\]
Here $\epsilon \in [0, 1)$ is a user-specified real valued accuracy parameter. We consider an adaptation of the AR algorithm obtained by relaxing the conditions (7a) and (7b) by adding $\epsilon$ and subtracting $\epsilon$, respectively, on the right hand sides of the conditions (7a) and (7b). This yields a heuristic algorithm for $\epsilon$-accurate set recovery, termed approximate AR algorithm for future reference.

In order to evaluate this approach, we set $n = 120$ and consider the problem of partitioning $[n]$ into the sets corresponding to the top 40 items, the middle 40 items, and the bottom 40 items. We generate a random BTL model by choosing the entries of the parameter vector $w$ i.i.d. at random from a standard Gaussian distribution.

The results, plotted in Figure 5, show that it is possible to find an approximate ranking with a significantly smaller number of comparisons, relative to finding an exact ranking. In Figure 6(b), we also plot the scores of the random BTL model on which the simulations are performed on, and note that those scores have some resemblance to the empirical scores observed on real data in Figure 3. We note that for other random pairwise comparison models, we have obtained similar results for approximate ranking.

![Figure 5](image-url)

**Figure 5.** Application of the AR algorithm for the problem of finding the top, middle, and bottom items, out of $n = 120$ for a random BTL model. (a) Number of comparisons until termination as a function of the parameter $\epsilon$ of the approximate AR algorithm. (b) Misclassification ratio, defined as the number of misclassified items over all items (we say an item $i \in \hat{S}_\ell$ is misclassified if $i \notin S_\ell$), as a function of $\epsilon$. The error bars in the figures correspond to one standard deviation. Significant savings in sample complexity can be obtained when one is content with only finding an approximate ranking.

6 Proofs

In this section, we provide the proofs of our two main theorems. For convenience, we assume through this section that the permutation $\pi$ is equal to the identity, such that $\tau_1 > \tau_2 > \ldots > \tau_n$. 
6.1 Proof of Theorem 1(a)

In this section, we provide a proof of the achievable result stated in part (a) of Theorem 1. Our proof consists of three main steps. We begin by showing that the estimate $\hat{\tau}_i(t)$ is guaranteed to be $\alpha_t$-close to $\tau_i$, for all $i \in S$, with high probability. We then use this result to show that the AR algorithm never misclassifies any item, and that it stops with the number of comparisons satisfying the claimed upper bound.

Throughout the paper, we use $S$ to denote the set of items that have not been ranked yet; to be clear, since items are eliminated from $S$ at certain time steps $t$, the set $S$ changes with $t$, but we suppress this dependence for notational simplicity.

Lemma 1. With probability at least $1 - \delta$, the event

$$E_\alpha := \{|\hat{\tau}_i(t) - \tau_i| \leq \alpha_t, \text{ for all } i \in S \text{ and for all } t \geq 1\}$$

occurs.

Our next step is to show that provided that the event $E_\alpha$ occurs, the AR algorithm never misclassifies any item, that is, $\hat{S}_t \subseteq S_\ell$ for all $\ell$ and for all $t \geq 1$. First suppose that, at a given time step $t$, the AR algorithm did not misclassify any item at a previous time step. We show that, at time $t$, conditioned on the event $E_\alpha$, any item $j \in S$ is added to $\check{S}_t$ only if $j \in \hat{S}_t$, which implies that the AR algorithm does not misclassify any item at time $t$. This fact is a consequence of the following lemma.

In order to state the lemma, we begin by introducing some additional notation. Let $\tau_{\{k\}}$ denote the $k$-th largest score among the latent scores $\tau_i$, $i \in S$. Note that we use the notation $\{\cdot\}$ to emphasize that the index $\{k\}$ is not necessarily equal to the index $(k)$, since the latter corresponds to the $k$-th largest score amongst the estimated scores $\hat{\tau}_i(t)$, $i \in S$.

Lemma 2. Suppose that the event $E_\alpha$ occurs. Then the implications

for any $j \in S$, \quad $\hat{\tau}_j(t) < \hat{\tau}_{(\hat{k}_{\ell-1})}(t) - 4\alpha_t$ \quad implies \quad $\tau_j < \tau_{(\hat{k}_{\ell-1})}$, \quad and \quad (16a)

for any $j \in S$, \quad $\hat{\tau}_j(t) > \hat{\tau}_{(\hat{k}_{\ell+1})}(t) + 4\alpha_t$ \quad implies \quad $\tau_j > \tau_{(\hat{k}_{\ell+1})}$; \quad (16b)
hold simultaneously for every \( t \geq 1 \).

Provided that the AR algorithm did not misclassify any item at a previous time step, which holds by assumption, a consequence of implications (16a) and (16b) is that, at time \( t \), an item is added to \( \hat{S} \) only if \( j \in S_\ell \). This implies that at time \( t + 1 \), \( \hat{S}_\ell \subseteq S_\ell \), for all \( \ell \). By induction, this concludes the proof of the claim that the AR algorithm never misclassifies any item.

We next show that, conditioned on the event \( \mathcal{E}_\alpha \), where the AR algorithm does not misclassify any item, all items are eliminated after the number of comparisons stated in equation (10a) has been carried out. Since, by Lemma 1, the event \( \mathcal{E}_\alpha \) holds with probability at least \( 1 - \delta \), this concludes the proof of Theorem 1(a).

In order to establish the former claim, we use the following lemma, in which we made the dependence of the set of candidates \( S \) on \( t \) explicit by writing \( S(t) \).

**Lemma 3.** Suppose that the event \( \mathcal{E}_\alpha \) occurs. For any index \( \ell \in \{2, \ldots, L\} \) and any item \( i \in S_\ell \cap S(t_i) \), we have

\[
\hat{\tau}_{i}(t_i) < \hat{\tau}_{(k_{\ell-1})}(t_i) - 4\alpha_i, \quad \text{where} \quad t_i := \frac{448}{\Delta_2^{\ell,i}} \log(n/(\delta \Delta_\ell,i)), \quad \Delta_\ell,i = \tau_{k_{\ell-1}} - \tau_i; \tag{17a}
\]

and for \( \ell \in \{1, \ldots, L - 1\} \) and any item \( i \in S_\ell \cap S(t_i) \), with probability at least \( 1 - \frac{\delta}{4n} \), we have

\[
\hat{\tau}_{i}(t_i) > \hat{\tau}_{(k_{\ell+1})}(t_i) - 4\alpha_i, \quad \text{where} \quad t_i := \frac{448}{\Delta_2^{\ell,i}} \log(n/(\delta \Delta_\ell,i)), \quad \Delta_\ell,i = \tau_i - \tau_{k_{\ell+1}}. \tag{17b}
\]

Consequently, the index \( i \in S_\ell \) is eliminated from the set of candidates \( S \) after no more than the following number of many time steps (and hence comparisons):

\[
\begin{cases} 
T_i, & \text{if } \ell = 1 \\
\max(T_i, T_i), & \text{if } \ell \in \{2, \ldots, L - 1\} \\
T_i, & \text{if } \ell = L
\end{cases}
\]

Using the relations

\[
\bar{\tau}_i \leq c_{\text{up}} \frac{\log(2/\Delta_\ell,i)}{\Delta_2^{\ell,i}} \log(n/\delta), \quad \text{and} \quad \bar{L} \leq c_{\text{up}} \frac{\log(2/\Delta_\ell,i)}{\Delta_2^{\ell,i}} \log(n/\delta),
\]

where the inequalities hold for some constant \( c_{\text{up}} \), it follows that the AR algorithm terminates after the number of comparisons stated in equation (10a) has been carried out.

It remains to prove Lemmas 1, 2, and 3, and we do so in the following subsections.

### 6.1.1 Proof of Lemma 1

In order to show that the event \( \mathcal{E}_\alpha \) occurs with probability at least \( 1 - \delta \), first recall that comparing item \( i \) to an item chosen uniformly at random from \( [n] \setminus \{i\} \) is equivalent to taking an independent draw from a Bernoulli random variable with mean \( \tau_i \). One can verify from the recursion (6) that
\( \tau_i(t) \) is a sum of \( t \) independent Bernoulli random variables, each of which has mean \( \tau_i/t \). For \( i \in S \), Hoeffding’s inequality yields

\[
P \left[ |\hat{\tau}_i(t) - \tau_i| \geq \alpha_t \right] \leq 2e^{-2\alpha_t^2 t} \leq 2\frac{\delta}{4nt^2},
\]

where the last inequality follows from our choice \( \alpha_t = \sqrt{\log(4nt^2/\delta)} \). Taking the union bound over all indices \( i \in S \subseteq [n] \), and all times \( t \geq 1 \), and noting that \( \sum_{t \geq 1} \frac{1}{t^2} = \pi^2/6 \leq 2 \), we conclude that the event \( E_\alpha \) occurs with probability at least \( 1 - \delta \), as claimed.

### 6.1.2 Proof of Lemma 2

We show that implications (16a) and (16b) follow from the inequality in event \( E_\alpha \). In order to do so, consider any index \( k' \) such that \( \hat{\tau}_{k'}(t) = \tau_{(k)}(t) \), where we have allowed for the possibility that \( (k) \) may not be unique. We start by showing that the inequality in event \( E_\alpha \) implies that

\[
|\tau_{(k)} - \tau_{k'}| \leq 2\alpha_t. \tag{18}
\]

We claim that \( \tau_{(k)} - \tau_{k'} \geq -2\alpha_t \). By definition of \( k' \), there are \( k \) indices \( \{i_1, \ldots, i_k\} \) such that \( \hat{\tau}_{i}(t) \geq \hat{\tau}_{i'}(t) \) for every \( i \in [k] \). In conjunction with the inequality in event \( E_\alpha \), we obtain

\[
\tau_i + \alpha_t \geq \tau_{i'} - \alpha_t.
\]

Since this inequality holds for \( k \) many indices \( \{i_1, \ldots, i_k\} \), one of those indices must be \( \{k\} \), due to \( \tau_1 \geq \tau_2 \geq \ldots \geq \tau_k \). It follows that \( \tau_{(k)} - \tau_{k'} \geq -2\alpha_t \). It remains to establish that \( \tau_{(k)} - \tau_{k'} \leq 2\alpha_t \).

By definition of \( k' \), there are \( k - |S| + 1 \) many indices \( \hat{k} \) obeying

\[
\hat{\tau}_{\hat{k}}(t) \leq \hat{\tau}_{\hat{k}'}(t).
\]

By the inequality in event \( E_\alpha \), this yields \( \tau_{\hat{k}} - \alpha_t \leq \tau_{\hat{k}'} + \alpha_t \). Since this inequality holds for \( |S| - k + 1 \) indices \( \hat{k} \), it must hold for \( \hat{k} = \{k\} \), which implies \( \tau_{(k)} \leq \tau_{k'} + 2\alpha_t \). Thus, we have established that inequality (18) must hold under event \( E_\alpha \).

We are now ready to establish the claim (16a). Let \( k' \) be any index such that \( \hat{\tau}_{k'} = \hat{\tau}_{(k_{\ell-1})} \). As long as \( \hat{\tau}_{j}(t) < \hat{\tau}_{(k_{\ell-1})} - 4\alpha_t \), we have

\[
4\alpha_t < \hat{\tau}_{k'}(t) - \hat{\tau}_{j}(t) \leq \tau_{k'} + \alpha_t - \tau_j + \alpha_t \leq \tau_{(k_{\ell-1})} - \tau_j + 2\alpha_t + 2\alpha_t. \tag{19}
\]

Here step (i) follows by the inequality in event \( E_\alpha \), whereas inequality (19) follows by inequality (18). Noting that inequality (19) is equivalent to \( \tau_j < \tau_{(k_{\ell-1})} \), we have established the claim (16a). The proof of claim (16b) is analogous, so we omit the details.

### 6.1.3 Proof of Lemma 3

We first prove that, if the event \( E_\alpha \) occurs, then for any given index \( i \in S_\ell \), and for all \( \ell > 1 \), inequality (17a) holds. Let \( k' \) be any index satisfying the equality \( \hat{\tau}_{k'}(t) = \hat{\tau}_{(k_{\ell-1})}(t) \), and recall
that $\tau_{(k)}$ is the $k$-th largest score out of the latent scores $\tau_i, i \in S$. On the event $E_\alpha$, we have

$$\hat{\tau}_{k'}(\tilde{t}_i) \geq \tau_{k'} - \alpha \tau_i - \hat{\tau}_i(\tilde{t}_i) \geq \tau_{\{k_{\ell-1}\}} - 3\alpha \tau_i \quad \text{(i)}$$

$$\geq \tau_{k_{\ell-1}} - 3\alpha \tau_i = \bar{\Delta}_{\ell,i} - 3\alpha \tau_i + \tau_i \quad \text{(ii)}$$

$$\geq \bar{\Delta}_{\ell,i} - 4\alpha \tau_i + \hat{\tau}_i(\tilde{t}_i) \quad \text{(iii)}$$

(20)

Here step (i) follows by inequality (18), which holds on the event $E_\alpha$, and inequality (ii) follows from $\tau_{\{k_{\ell-1}\}} \geq \tau_{k_{\ell-1}}$, which is seen as follows. As shown above, on the event $E_\alpha$, the AR algorithm never misclassifies any item, therefore the $k_{\ell-1}$-th largest score among the items in the set $S$ must be larger or equal to the $k_{\ell-1}$-th largest score among all scores. Finally, inequality (iii) follows from $\hat{\tau}_i(\tilde{t}_i) - \tau_i \leq \alpha \tau_i$, which holds on the event $E_\alpha$.

From the definition of $\alpha \tau_i$, some algebra leads to the lower bound

$$\bar{\Delta}_{\ell,i} > 8\alpha \tau_i. \quad \text{(21)}$$

See the end of this subsection for details of this calculation. Application of inequality (21) on the RHS of inequality (20) yields

$$\hat{\tau}_{k'}(\tilde{t}_i) > 4\alpha \tau_i + \hat{\tau}_i(\tilde{t}_i),$$

which concludes the proof of inequality (17a).

Analogously, it follows that inequality (17b) holds for a given item $i$ if the event $E_\alpha$ occurs.

**Proof of the lower bound (21):**

$$\alpha^2 \tau_i = \frac{\log(4n\bar{\Delta}_{\ell,i}^2/\delta)}{\bar{t}_i} = \frac{\log \left( \frac{n}{\delta} \cdot 448^2 \log \left( \frac{n}{\delta\bar{\Delta}_{\ell,i}} \right) \right)}{448 \log \left( \frac{n}{\delta\bar{\Delta}_{\ell,i}} \right)} < \frac{\bar{\Delta}_{\ell,i}^2}{64},$$

where the inequality follows from $4 \cdot 448^2 \log^2(n/\delta) < (n/\delta)^6$, for $n/\delta \geq 14$, which holds by the assumptions $\delta \leq 0.14$ and $n \geq 2$.

### 6.2 Proof of Theorem 1(b)

We now turn to the proof of the lower bound from Theorem 1. We first introduce some notation required to state a useful lemma [Kau+16, Lem. 1] from the bandit literature. Let $\nu = \{\nu_j\}_{j=1}^m$ be a collection of $m$ probability distributions, each supported on the real line $\mathbb{R}$. Consider an algorithm $A$, that, at times $t = 1, 2, \ldots$, selects the index $i_t \in [m]$ and receives an independent draw $X_t$ from the distribution $\nu_{i_t}$ in response. Algorithm $A$ may select $i_t$ only based on past observations, that is, $i_t$ is $\mathcal{F}_{t-1}$ measurable, where $\mathcal{F}_t$ is the $\sigma$-algebra generated by $i_1, X_{i_1}, \ldots, i_t, X_{i_t}$. Algorithm $A$ has a stopping rule $\xi$ that determines the termination of $A$. We assume that $\xi$ is a stopping time measurable with respect to $\mathcal{F}_t$ and obeying $\mathbb{P}[\xi < \infty] = 1$. 

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Let $N_i(\xi)$ denote the total number of times index $i$ has been selected by the algorithm $A$ (until termination). For any pair of distributions $\nu$ and $\nu'$, we let $\text{KL}(\nu, \nu')$ denote their Kullback-Leibler divergence, and for any $p, q \in [0, 1]$, let $d(p, q) := p \log \frac{p}{q} + (1 - p) \log \frac{1 - p}{1 - q}$ denote the Kullback-Leibler divergence between two binary random variables with success probabilities $p, q$.

With this notation, the following lemma relates the cumulative number of comparisons to the uncertainty between the actual distribution $\nu$ and an alternative distribution $\nu'$.

**Lemma 4 ([Kau+16, Lem. 1]).** Let $\nu, \nu'$ be two collections of $m$ probability distributions on $\mathbb{R}$. Then for any $E \in \mathcal{F}_\xi$ with $\mathbb{P}_\nu [E] \in (0, 1)$, we have

$$\sum_{i=1}^{m} \mathbb{E}_\nu [N_i(\xi)] \text{KL}(\nu_i, \nu'_i) \geq d(\mathbb{P}_\nu [E], \mathbb{P}_{\nu'} [E]).$$

(22)

Let us now use Lemma 4 to prove Theorem 1(b). Define the event

$$E := \left\{ \hat{S}_\ell = S_\ell, \text{ for all } \ell = 1, \ldots, L \right\},$$

(23)

corresponding to success of the algorithm $A$. Recalling that $\xi$ is the stopping rule of algorithm $A$, we are guaranteed that $E \in \mathcal{F}_\xi$. Given the linear relations $M_{ij} = 1 - M_{ji}$, the pairwise comparison matrix $M$ is determined by the entries $\{M_{ij}, i = 1, \ldots, n, j = i + 1, \ldots, n\}$. Let $N_{ij}(\xi)$ be the total number of comparisons between items $i$ and $j$ made by $A$. For any other pairwise comparison matrix $M' \in \mathcal{C}_0$, Lemma 4 ensures that

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathbb{E}_M [N_{ij}] d(M_{ij}, M'_{ij}) \geq d(\mathbb{P}_M [E], \mathbb{P}_{M'} [E]).$$

(24)

Consider an item $m \in S_\ell(M), \ell > 1$ (in this proof, we make the dependence of $S_\ell = \{\pi(k_{\ell-1} + 1), \ldots, \pi(k_{\ell})\}$ on the distribution $M$ explicit; note that $\pi$ is a function of $M$). We construct $M' \in \mathcal{C}_{1/8}$ such that $m \notin S_\ell(M')$ under the distribution $M'$. Since we assume algorithm $A$ to be uniformly $\delta$-accurate over $\mathcal{C}_{1/8}$, we have both $\mathbb{P}_M [E] \geq 1 - \delta$ and $\mathbb{P}_{M'} [E] \leq \delta$. It follows that

$$d(\mathbb{P}_M [E], \mathbb{P}_{M'} [E]) \geq d(\delta, 1 - \delta) = (1 - 2\delta) \log \frac{1 - \delta}{\delta} \geq \log \frac{1}{2\delta},$$

(25)

where the last inequality holds for $\delta \leq 0.15$.

It remains to specify the alternative matrix $M' \in \mathcal{C}_0$ for use in the inequality (25). The alternative matrix $M'$ is defined as

$$M'_{ij} = \begin{cases} M_{mj} + (\tau_{k_{\ell-1}} - \tau_m), & \text{if } i = m, j \in [n] \setminus \{m\} \\ M_{im} - (\tau_{k_{\ell-1}} - \tau_m), & \text{if } j = m, i \in [n] \setminus \{m\} \\ M_{ij} & \text{otherwise.} \end{cases}$$

(26)

It follows that

$$\tau'_m = \frac{1}{n - 1} \sum_{j \in [n] \setminus \{m\}} M'_{mj} = \frac{1}{n - 1} \sum_{j \in [n] \setminus \{m\}} (M_{mj} + (\tau_{k_{\ell-1}} - \tau_m)) = \tau_{k_{\ell-1}}.$$

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Similarly, all other scores \( \tau'_i \) are smaller than \( \tau_i \) by a common constant, that is, for \( i \in [n] \setminus \{m\} \)

\[
\tau'_i = \tau_i - \frac{1}{n-1} (\tau_{k_{\ell-1}} - \tau_m).
\]

See Figure 7 for an illustration. It follows that, under the distribution \( M' \), the score of item \( m \) is among the \( k_{\ell-1} \) highest scoring items, therefore \( m \notin S_{\ell}(M') \). Moreover, \( M \in \mathcal{C}_{3/8} \). This follows from the assumption \( M \in \mathcal{C}_{3/8} \), which implies

\[
M'_{mj} \leq \frac{5}{8} + \left( \frac{5}{8} - \frac{3}{8} \right) \leq \frac{7}{8},
\]

and similarity \( M'_{mj} \geq \frac{1}{8} \).

Next consider the total number of comparisons of item \( m \) with all others items, that is, \( N_m = \sum_{j \in [n] \setminus \{m\}} N_{mj} \). By the linearity of expectation, we have

\[
\max_{j \in [n] \setminus \{m\}} d(M_{mj}, M'_{mj}) \mathbb{E}_M [N_m] = \max_{j \in [n] \setminus \{m\}} d(M_{mj}, M'_{mj}) \sum_{j' \in [n] \setminus \{m\}} \mathbb{E}_M [N_{mj'}]
\geq \sum_{j \in [n] \setminus \{m\}} \mathbb{E}_M [N_{mj}] d(M_{mj}, M'_{mj})
\overset{(i)}{=} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathbb{E}_M [N_{ij}] d(M_{ij}, M'_{ij})
\overset{(ii)}{=} d(\mathbb{P}_M [\mathcal{E}], \mathbb{P}_{M'} [\mathcal{E}])
\geq \log \frac{1}{2^8}.
\]

(27)
Here step (i) follows from the fact that \( d(M_{ij}, M'_{ij}) = 0 \) for all \((i, j)\) not in \(\{(m, j) \mid j \in [n] \setminus \{m\}\}\) and not in \(\{(i, m) \mid i \in [n] \setminus \{m\}\}\), by definition of the \(M'_{ij}\) (see equation (26)), and step (ii) follows from inequality (24) (that is, from Lemma 4). Finally, inequality (27) follows from inequality (25).

We next upper bound the KL divergence on the left hand side of inequality (27). Using the inequality \( \log x \leq x - 1 \) valid for \( x > 0 \), we have that
\[
d(M_{mj}, M'_{mj}) \leq \frac{(M_{mj} - M'_{mj})^2}{M'_{mj}(1 - M'_{mj})} \leq 16(\tau_{k_{l-1}} - \tau_m)^2. \tag{28}
\]
Here, the last inequality follows from the definition of \(M'\) in equation (26), for \( j \in [n] \setminus \{m\} \), and from \( \frac{1}{8} \leq M'_{mj} \leq \frac{7}{8} \), which implies \(M_{mj}(1 - M'_{mj}) \leq 16\).

Applying inequality (28) to the left hand side of inequality (27) yields, for \( m \in S_\ell(M), \ell > 1 \),
\[
\mathbb{E}_M [N_m] \geq \frac{\log(1/(2\delta))}{16(\tau_{k_{l-1}} - \tau_m)^2}. \tag{29}
\]

Next, let \( m \in S_\ell(M) \) with \( \ell < L \). We again construct an alternative pairwise comparison matrix \(M'\) under which \( m \notin S_\ell(M')\). Specifically, for notational convenience, we set
\[
M'_{ij} = \begin{cases} 
M_{mj} - (\tau_m - \tau_{k_{l+1}}), & i = m, j \in [n] \setminus \{m\} \\
M_{im} + (\tau_m - \tau_{k_{l+1}}), & j = m, i \in [n] \setminus \{m\} \\
M_{ij} & \text{otherwise.}
\end{cases}
\]

In a similar manner to our earlier argument, we have \( \tau'_i = \tau_i + \frac{1}{n-1}(\tau_m - \tau_{k_{l+1}}) \) for \( i \in [n] \setminus \{m\} \) and \( \tau'_m = \tau_{k_{l+1}} \) (relative to the scores \( \tau_i \), the score of \( m \) is smaller and all others are larger by the same factor). Under \( M'\), item \( m \) is not amongst the \( k_\ell \) items with the largest scores, and therefore \( m \notin S_\ell(M')\). Carrying out the same computations as above yields:
\[
\mathbb{E}_M [N_m] \geq \frac{\log(1/(2\delta))}{16(\tau_m - \tau_{k_{l+1}})^2}. \tag{30}
\]

Combining inequalities (29) and (30) across all items \( m \) yields the bound
\[
\mathbb{E}_M [N] = \sum_{i=1}^{n} \mathbb{E} [N_i] \geq c_{\text{low}} \log(1/(2\delta)) \left[ \sum_{i \in S_1} \Delta_{i,1}^2 + \sum_{\ell=2}^{L-1} \sum_{i \in S_\ell} \max \left\{ \bar{\Delta}_{i,\ell}^{-2}, \bar{\Delta}_{\ell,i}^{-2} \right\} + \sum_{i \in S_L} \bar{\Delta}_{L,i}^{-2} \right],
\]
with \( c_{\text{low}} = 1/16 \), thereby yielding the claimed result.

### 6.3 Proof of Theorem 2(a)

Our goal is to prove that any algorithm \( \mathcal{A} \) that is uniformly \( \delta \)-accurate over \( C_{\text{PAR}}(\Phi) \cap C_{M_{\min}} \), when applied to a given pairwise comparison model \( M \in C_{\text{PAR}}(\Phi) \cap C_{M_{\min}} \), must make at least
\[
\mathbb{E}_M [N] \geq \frac{M_{\min} \phi_{\min}^2}{2.004 \phi_{\max}^2} \log \left( \frac{1}{2\delta} \right) F(\tau(M)) \tag{12}
\]
comparisons on average. Here \( F(\tau(M)) \) is the complexity parameter defined in equation (12).
The proof is similar to that of Theorem 1(b). The primary difference from the proof of Theorem 1 is that the alternative matrix $M'$ must now be constructed such that it lies in the parametric class. In what follows, we show how to modify the proof of Theorem 1 at appropriate positions in order to accommodate this difference.

Consider any parametric pairwise comparison matrix $M \in C_{\text{PAR}}(\Phi) \cap C_{M_{\text{min}}}$. Then there exists a parameter vector $w \in \mathbb{R}^n$ such that $M_{ij} = \Phi(w_i - w_j)$. By the assumption $\tau_1 > \ldots > \tau_n$, this parameter vector obeys $w_1 > \ldots > w_n$. Consider an item $m \in S_\ell(M), \ell > 1$, and set $k := k_{\ell-1}$, for notational convenience. We construct an alternative matrix $M' \in C_{\text{PAR}}(\Phi) \cap C_{M_{\text{min}}}$ as follows. Consider some scalar value $\rho$ that lies in the interval $0 < \rho < w_k - w_{k-1}$. Define a set of alternative parameters as

\[
\begin{align*}
    w'_i & := \begin{cases} 
        w_k & \text{if } i = m, \\
        w_k - \rho & \text{if } i = k, \\
        w_i & \text{otherwise.}
    \end{cases}
\end{align*}
\]

Now let $M'$ be the matrix with pairwise comparison probabilities $M'_{ij} = \Phi(w'_i - w'_j)$. By definition, we have $w_1 \geq w'_1 \geq w_n$ for all $i \in [n]$, which ensures that $M' \in C_{\text{PAR}}(\Phi) \cap C_{M_{\text{min}}}$. Moreover, by definition, item $m$ is among the top $k$ items, so that $m \notin S_\ell(M')$. Since (by assumption) algorithm $A$ is uniformly $\delta$-accurate over $C_{\text{PAR}}(\Phi) \cap C_{M_{\text{min}}}$, we have both $\mathbb{P}_M[\mathcal{E}] \geq 1 - \delta$ and $\mathbb{P}_{M'}[\mathcal{E}] \leq \delta$, which ensures that inequality (25) holds. Here $\mathcal{E}$ denotes the previously defined event (23) that the algorithm $A$ correctly recovers the set structure.

Next consider the total number of comparisons of item $m$ with all others items, denoted by $N_m$. As in inequality (27), we have that

\[
\begin{align*}
    \max_{j \in [n] \setminus \{m\}} d(M_{mj}, M'_{mj}) & \geq \sum_{j \in [n] \setminus \{m\}} \mathbb{E}_M[N_{mj}] d(M_{mj}, M'_{mj}) \\
    & \overset{(i)}{=} \sum_{i=1}^n \sum_{j=i+1}^n \mathbb{E}_M[N_{ij}] d(M_{ij}, M'_{ij}) - \sum_{j \in [n] \setminus \{k,m\}} \mathbb{E}_M[N_{jk}] d(M_{jk}, M'_{jk}) \\
    & \overset{(ii)}{=} d(\mathbb{P}_M[\mathcal{E}], \mathbb{P}_{M'}[\mathcal{E}]) - 0.001 d(\mathbb{P}_M[\mathcal{E}], \mathbb{P}_{M'}[\mathcal{E}]) \\
    & \overset{(iii)}{=} d(\mathbb{P}_M[\mathcal{E}], \mathbb{P}_{M'}[\mathcal{E}]) - \sum_{j \in [n] \setminus \{k,m\}} \mathbb{E}_M[N_{jk}] d(M_{jk}, M'_{jk}) \\
    & \geq 0.999 \log \frac{1}{2\delta}. \quad (31)
\end{align*}
\]

Here inequality (i) follows from the fact that $d(M_{ij}, M'_{ij}) = 0$ for all $(i, j)$ with $i, j \in [n] \setminus \{k, m\}$, by definition of $M'$. Inequality (ii) follows from inequality (24) (that is, from Lemma 4). Inequality (iii) is a result of the fact that $\lim_{\rho \to 0} d(M_{ik}, M'_{ik}) = \lim_{\rho \to 0} d(\Phi(w_i - w_k), \Phi(w_i - w_k + \rho)) = 0$ for every $i \in [n] \setminus \{k, m\}$, where we have also employed the continuous mapping theorem: Due to this relation we can choose $\rho$ sufficiently close to 0 to obtain the bound of (iii). Finally, inequality (31) is a consequence of inequality (25).

We proceed by upper bounding the KL divergence on the LHS of inequality (27). For each
\[ j \in [n] \setminus \{k, m\}, \text{ we have} \]
\[ d(M_{mj}, M'_{mj}) \leq \frac{\left( M'_{mj} - M_{mj} \right)^2}{M_{mj}(1 - M'_{mj})} \]
\[ \quad \leq \frac{\left( M_{kj} - M_{mj} \right)^2}{M_{kj}(1 - M_{kj})} \]
\[ \quad \leq \frac{2}{M_{\text{min}}} \left( \Phi(w_k - w_j) - \Phi(w_m - w_j) \right)^2 \]
\[ \quad \leq \frac{2\phi_{\text{max}}^2}{M_{\text{min}}\phi_{\text{min}}^2} (\tau_k - \tau_m)^2. \]

Here step (i) follows by definition of the parameters \( M'_{ij} \); step (ii) follows because \( M_{ij} \) belongs to the interval \([M_{\text{min}}, 1 - M_{\text{min}}]\); and step (iii) is a consequence of assumption (11). Finally, the last inequality (32) follows from the relations

\[ \tau_k - \tau_m = \frac{1}{n - 1} \left( \Phi(w_k - w_m) - \Phi(w_m - w_k) + \sum_{j \in [n] \setminus \{k, m\}} (\Phi(w_k - w_j) - \Phi(w_m - w_j)) \right) \]
\[ \geq \frac{1}{n - 1} \left( \phi_{\text{min}}(w_k - w_m) - (w_m - w_k) + \sum_{j \in [n] \setminus \{k, m\}} \phi_{\text{min}}(w_k - w_m) \right) \]
\[ = \frac{n}{n - 1} \phi_{\text{min}}(w_k - w_m) \geq \phi_{\text{min}}(w_k - w_m). \]

Here inequality (i) follows from assumption (11) (recall that \( w_k > w_m \), so the difference \( w_k - w_m \) above is positive).

Similarly, we have

\[ d(M_{mk}, M'_{mk}) \leq \frac{2}{M_{\text{min}}} \left( \phi_{\text{max}}(\rho + w_k - w_m) \right)^2 \]
\[ \quad \leq \frac{2.001}{M_{\text{min}}} \left( \phi_{\text{max}}(w_k - w_m) \right)^2 \]
\[ \quad \leq \frac{2.001\phi_{\text{max}}^2}{M_{\text{min}}\phi_{\text{min}}^2} (\tau_k - \tau_m)^2. \]

Here, inequality (i) follows from choosing \( \rho \) sufficiently close to 0, whereas the last inequality follows from the relation (33).

Given an index \( m \) in a set \( S_\ell(M) \) with \( \ell > 1 \), combining inequalities (32) and (34) with inequality (31) yields

\[ \mathbb{E}_M[N_m] \geq \frac{M_{\text{min}}\phi_{\text{min}}^2}{2.004\phi_{\text{max}}^2} \log \left( \frac{1}{2\delta} \right) \left( \tau_{k_{\ell-1}} - \tau_m \right)^2. \]

Similarly, for an index \( m \in S_\ell(M) \) with \( \ell < L \), we define an alternative matrix \( M' \) by defining corresponding parameters as \( w'_m = w_{k_{\ell+1}}, w'_{k_{\ell+1}} = w_{k_{\ell+1}} + \rho \) for \( \rho \in (0, w_k - w_{k_{\ell+1}}) \), and \( w'_i = w_i \),
for all \( i \notin \{m, k_\ell + 1\} \). Under the model specified by \( M' \), item \( m \) is not amongst the \( k_\ell \) items with the largest scores, and therefore \( m \notin \mathcal{S}_\ell(M') \). The same line of arguments as above yields

\[
\mathbb{E}_M[\mathcal{N}_m] \geq \frac{M_{\min} \phi_{\min}^2}{2.004 \phi_{\max}^2 (\tau_m - \tau_{k_\ell+1})^2},
\]

Combining the lower bounds (35a) and (35b) concludes the proof.

6.4 Proof of Theorem 2(b)

Let \( \tau \in (0,1)^n \) be any set of scores that is realizable by some pairwise comparison matrix in \( \mathcal{C}_{\min} \). Theorem 2(b) is proven by showing that for any continuous and strictly increasing \( \Phi \), there exists a pairwise comparison matrix in \( \mathcal{C}_{\text{PAR}}(\Phi) \cap \mathcal{C}_{\min} \) with scores \( \tau \). As mentioned before, the proof of Theorem 2(b) relies on results established by Joe [Joe88] on majorization orderings of pairwise probability matrices. For convenience, we define the set of pairwise probability matrices with scores \( \tau = (\tau_1, \ldots, \tau_n) \) as

\[
\mathcal{C}(\tau) = \left\{ M \in \mathcal{C}_0 \mid \frac{1}{n-1} \sum_{j \neq i} M_{ij} = \tau_i, \text{ for all } i \right\}.
\]

Minimality for pairwise comparison matrices: Our proof requires some background on majorization and a certain notion of minimality for pairwise comparison matrices. We say that a vector \( y \in \mathbb{R}^m \) is non-increasing if its entries satisfy \( y_1 \geq y_2 \geq \ldots \geq y_m \). Given two non-increasing vectors \( y, z \in \mathbb{R}^m \) such that \( \sum_{i=1}^m y_i = \sum_{i=1}^m z_i \), we say \( y \) majorizes \( z \), written \( y \succ z \), if

\[
\sum_{i=1}^k y_i \geq \sum_{i=1}^k z_i, \text{ for all } k = 1, \ldots, m-1.
\]

Given pairwise comparison matrices \( M, Q \in \mathcal{C}(\tau) \), we let \( v(M), v(Q) \in (0,1)^{n(n-1)} \) be vectors with entries corresponding to the off-diagonal elements of \( M \) and \( Q \), respectively, in non-increasing order. We say that \( M \) majorizes \( Q \) if \( v(M) \succ v(Q) \), and we use the shorthand \( M \succ Q \) to denote this relation. Finally, a matrix \( M \in \mathcal{C}(\tau) \) is minimal if any other \( Q \in \mathcal{C}(\tau) \) obeying \( M \succ Q \) satisfies the relation \( v(Q) = v(M) \).

In order to prove Theorem 2(a), we show that there is a minimal \( M \in \mathcal{C}(\tau) \cap \mathcal{C}_{\min} \). We first note that Joe [Joe88, Thm. 2.7] observed that the argument minimizing any Schur convex\(^2\) function over the set \( \mathcal{C}(\tau) \) is a minimal \( M \).

Let us now construct a function that is Schur convex. In particular, we first define a scalar function \( \psi : [0,1] \to [0,\infty) \) as

\[
\psi(u) = \begin{cases} 
\frac{1}{2} \int_{1/2}^u \Phi^{-1}(x)dx, & u \in \left[\frac{1}{2}, 1\right], \\
-\frac{1}{2} \int_0^{1/2} \Phi^{-1}(x)dx, & u \in \left[0, \frac{1}{2}\right].
\end{cases}
\]

The function \( \psi \) is well defined since the inverse \( \Phi^{-1} \) exists due to our assumption that \( \Phi \) is strictly increasing and continuous. Since \( \Phi \) is strictly increasing, so is \( \Phi^{-1} \). It follows that \( \psi \) is strictly

\(^2\)In our context, a function \( f : (0,1)^{n \times n} \to \mathbb{R} \) is Schur convex (or order-preserving) if for all \( M, Q \in \mathcal{C}(\tau) \) such that \( M \) is majorized by \( Q \), we have \( f(M) \leq f(Q) \).
that the optimization problem (equation (38)) is necessary and sufficient for optimality (see, for instance, [BV04, Sec. 5.5]). Thus, the primal and dual optimal solutions $M_{ij}^*$ and $\lambda_{ij}^*, \kappa_{ij}^*, \nu_i^*$ satisfy the KKT conditions:

\begin{align*}
\lambda_{ij}^* \geq 0, \\
\lambda_{ij}^*(M_{ij}^* - 1) = 0, \quad \kappa_{ij}^* M_{ij}^* = 0, \quad \text{and} \quad \psi'(M_{ij}^*) - \psi'(1 - M_{ij}^*) + \lambda_{ij}^* - \kappa_{ij}^* + \nu_i^* - \nu_j^* = 0.
\end{align*}

Thus, by the KKT conditions, we have that $\lambda_{ij}^* = 0$ and $\kappa_{ij}^* = 0$, for all $i, j$. Consequently, equation (38c) takes the simpler form

$$
\nu_j^* - \nu_i^* = \psi'(M_{ij}^*) - \psi'(1 - M_{ij}^*) = \frac{1}{2} \Phi^{-1}(M_{ij}^*) - \frac{1}{2} \Phi^{-1}(1 - M_{ij}^*)
$$

\begin{equation}
\text{(i)} = \Phi^{-1}(M_{ij}^*),
\end{equation}

where step (i) follows because $\Phi(t) = 1 - \Phi(-t)$ for all $t \in \mathbb{R}$ by assumption. It follows that $M_{ij}^* = \Phi(\nu_j^* - \nu_i^*)$ for all $i, j$, meaning that $M^*$ takes a parametric form, as claimed.

7 Discussion

In this paper, we considered the problem of finding a partial or complete ranking from active pairwise comparisons. We proved that a simple and computationally efficient algorithm succeeds
in recovering the ranking with a sample complexity that is optimal up to logarithmic factors. We furthermore proved that this algorithm remains optimal when imposing common parametric assumptions such as the popular BTL or Thurstone models. Finally, we show that, perhaps surprisingly, imposing common parametric assumptions does not reduce the sample complexity of ranking.

There are a number of open and practically relevant questions suggested by our work, a few of which we have partially addressed in the numerical results section. Specifically, in practice, one might only be interested in finding an approximate ranking, or might only be able to find an approximate ranking due to a limited budget of queries. Our numerical results show that a (heuristic) adaption of the AR algorithm allows to trade off accuracy of the ranking and sample complexity. In future work, we will establish analytical results of this form.

Acknowledgements

The work of RH was supported by the Swiss National Science Foundation under grant P2EZP2_159065. This work was partially supported by Office of Naval Research MURI grant DOD-002888, Air Force Office of Scientific Research Grant AFOSR-FA9550-14-1-001, Office of Naval Research grant ONR-N00014, as well as National Science Foundation Grant CIF-31712-23800. The work of NBS was also supported in part by a Microsoft Research PhD fellowship.

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