Is Angular Momentum in an Accretion Disk Transported Inwards?

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The derivation of the viscosity formula of accretion disks appearing in a number of textbooks is based on a mean free path theory of gas particles. However, this procedure, when followed precisely, leads to the incorrect conclusion that the angular momentum of an accretion disk flows from the outer part to the inner part.

Introduction The standard model of accretion disks, also called the $\alpha$ disk model, proposed by Shakura and Sunyaev $^{1)}$ states that the angular momentum is transported from the inner part of the disk to the outer part, due to the action of some kind of viscosity. The Reynolds number of the flow in accretion disks is estimated to be as high as $10^{11}$, which means, according to the accepted hydrodynamical wisdom, that the flow is turbulent (Spruit $^{2)}$). It has been assumed that in calculating the viscous stress due to the turbulence one can employ the usual molecular viscosity formula.

There are many textbooks and reviews about the formation of accretion disks and the mechanism involved in the transport of angular momentum (e.g., Pringle, $^{3)}$, Frank, King and Raine, $^{4)}$ Spruit, $^{2)}$ Hartmann, $^{5)}$ and Kato, Fukue and Mineshige, $^{6)}$). Some of these textbooks, $^{4)}$, $^{5)}$ however, give elementary explanations about the phenomenon of angular momentum transport based on physical arguments, i.e. a mean free path theory, rather than with mathematical rigor. This is perhaps because the authors of such textbooks intend to make the explanation easier for beginners to understand. We find that these explanations in fact do not lead to the results intended by the authors but to the incorrect conclusion that the angular momentum in an accretion disk is transported inwards. To the contrary, the angular momentum must be transported outwards, or accretion will not occur. There are, however, some other textbooks and reviews that give rigorous mathematical derivations of formulas describing the angular momentum transport, starting from the formula for the molecular viscosity (e.g., Spruit, $^{2)}$ Kato et al. $^{6)}$).

The above mentioned “elementary explanations” based on the mean free path theory, however, seemed very plausible to us. We therefore began to doubt if the “molecular viscosity formula” itself is applicable to gas in an accretion disk rotating with Keplerian motion in the gravitational field of a central compact star. Eventually, we found that these “elementary explanations” lead to the incorrect conclusion and that the “molecular viscosity formula” is basically correct.
Is mean free path theory applicable to the present problem? In the standard disk model, the equation for angular momentum conservation is

$$\frac{\partial}{\partial t}(R\Sigma \Omega R^2) + \frac{\partial}{\partial R}(R\Sigma v_r \Omega R^2) = \frac{\partial}{\partial R} \left( \Sigma \nu R^3 \frac{\partial \Omega}{\partial R} \right),$$

(1)

where

$R =$ distance from the center of the central star,
$\Sigma =$ surface density,
$\Omega = v_\phi / R,$ and $\nu =$ kinematic viscosity.

How can this equation be derived? It can be obtained using the formula of molecular viscosity for incompressible gas (e.g., Spruit\(^2\))

$$\sigma_{ik} = \mu \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right),$$

(2)

where $\sigma_{ik}$ is the stress tensor, $\mu$ is the viscosity coefficient, and $v_i$ and $v_k$ are averages of the molecular velocity in the $i$- and $k$-direction, respectively. The viscous force is

$$F_i = \frac{\partial \sigma_{ik}}{\partial x_k} = \frac{\partial \mu}{\partial x_k} \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) + \mu \Delta v_i.$$

(3)

A problem emerged when we were learning with a rather elementary textbook, which gives explanations for beginners. Since the situation is best described in the textbook, we make a rather lengthy quotation from Accretion Processes in Star Formation, by Hartmann,\(^5\) p. 30 (the latter part of §5.1):

Following Frank et al. (1992), we can calculate the magnitude of the angular momentum transfer in terms of a kinematic viscosity. The basic picture is one in which turbulent elements of the gas moving at a typical random velocity $w$ travel a mean free path $\lambda$ before mixing with other material. Thus, the net torques or angular momentum transfer at cylindrical radius $R$ (Figure 5.1) will be produced (schematically) by the differing angular momenta of two streams of material; one from material originating at $R - \lambda/2$ and moving outward across $R$ to mix with annular material centered at $R + \lambda/2$; and the other starting at $R + \lambda/2$ and moving inward across $R$ to mix with the inner annulus at $R - \lambda/2$.

In this kinematic viscosity model, no net angular momentum is transported unless there is shearing orbital motion, $d\Omega / dR \equiv \Omega' \neq 0$. The net angular momentum fluxes can be calculated in the following way. Material originating at $R - \lambda/2$ has an angular momentum of

$$J_m = (R - \lambda/2)^2 \Omega (R - \lambda/2) = (R - \lambda/2)^2 [\Omega(R) - (\lambda/2)(d\Omega / dR)],$$

(5.9)

where we have approximated the difference in the angular velocities in terms of the first derivative with respect to $R$. A similar expression with the negative values changed to positive applies to the material at $R + \lambda/2$. The inner material diffuses outwards at velocity $w$ and the outer material...
diffuses inward at \( w \) across \( R \). (The net inward motion of material in the accretion disk is assumed to be small in comparison with the turbulent velocity \( w \)). For simplicity we integrate or average the disk structure in the direction perpendicular to the disk plane; then the net outward transfer of angular momentum across \( R \) per unit length for a disk with mass density per unit area \( \Sigma \) is

\[
\Sigma w[(R-\lambda/2)^2(-\lambda/2)d\Omega/dR -(R+\lambda/2)^2(\lambda/2)d\Omega/dR] = -\Sigma w \lambda R^2 d\Omega/dR, \tag{5.10}
\]

where we have assumed that \( \lambda \) is a short distance compared with the scale over which \( \Omega \) varies significantly. With this result, the total angular momentum flux outward across \( R \), i.e. the torque of the inner annulus on the outer annulus, can be written as

\[
g = -2\pi R \Sigma \nu_v R^2 d\Omega/dR, \tag{5.11}
\]

where the viscosity \( \nu_v \) is

\[
\nu_v = \lambda w. \tag{5.12}
\]

Note that a negative gradient of angular velocity (i.e. \( \Omega \) decreasing outward) leads to a positive outward flux of angular momentum, as predicted by the qualitative picture of friction between neighboring annuli discussed above.

Hereafter, we explain why the above derivation of (5.11) is not correct. In Eq. (5.9), \( (R-\lambda/2) \) in the original textbook should be \( (R-\lambda/2)^2 \), and is so corrected in the present text. From (5.9) and a similar expression for \( \dot{J}_{\text{out}} \), we have

\[
J_{\text{in}} - J_{\text{out}} = \Omega(R)[(R-\lambda/2)^2 - (R+\lambda/2)^2] \\
- (\lambda/2) d\Omega/dR[(R-\lambda/2)^2 + (R+\lambda/2)^2] \\
= \Omega(R)(-2R\lambda) - (\lambda/2)d\Omega/dR(2R^2) = -\lambda d(R^2 \Omega)/dR. \tag{4}
\]

Then, this gives, not Eq. (5.11), but

\[
g = -2\pi R \Sigma \nu_v d(R^2 \Omega)/dR. \tag{5}
\]

The difference between Eqs. (5.11) and (5) is that the factor \( R^2 \) is operated on by the \( d/dR \) operator in Eq. (5) and not operated on by it in Eq. (5.11). If Eq. (5.11) is correct, the angular momentum flows in such a direction as to realize \( \partial \Omega/\partial R = 0 \), that is, to produce rigid body rotation. In that case, the angular momentum flows outwards, which is what the standard theory predicts.

Attention should be paid here to the following. That is, with a Keplerian disk, we have \( \Omega \propto R^{-3/2}, v_\phi = R\Omega \propto R^{-1/2} \) and \( L = Rv_\phi \propto R^{1/2} \), where \( v_\phi \) is the circumferential velocity and \( L \) is the angular momentum. An accretion disk has the characteristic features that \( \Omega \) and \( v_\phi \) decrease with increasing \( R \) and that \( L \), in contrast, increases with increasing \( R \). (See Fig. 1 to better understand the above description.)
If, in contrast, Eq. (5) holds, the angular momentum flows in the direction to realize \( d(R^2 \Omega)/dR = 0 \), that is, to a constant angular momentum. In this case the angular momentum flows inward. This means that even an originally disk-shaped gas will end up as a ring. We conclude here that the correct manipulation of Hartmann’s procedure leads to the conclusion that he did not intend.

There is another famous textbook entitled *Accretion Power in Astrophysics* written by Frank, King and Raine.\(^4\) There, the authors give a similar elementary argument, with a similar confusion, while referring to a similar figure, as follows (on pages 58 and 59 of the second edition)*:

... As these elements of fluid are exchanged, they carry slightly different amounts of angular momentum: elements such as A will on average carry an angular momentum corresponding to the location \( R - \lambda/2 \), while elements such as B will be representative of a radial location \( R + \lambda/2 \).

\( ^* \) Interestingly, this description has been revised and differs from the corresponding one in the first edition (1985). There, the “correct” conclusion is derived from the incorrect assumption that a gas element in a disk would, in chaotic motion, move radially a distance of order \( \lambda \) away, while carrying its momentum (not its angular momentum). Since the rotating gas element should carry its angular momentum on that occasion, this assumption is wrong. This may have been the reason for the revision in the second edition.
the same result as from Hartmann’s “correct” (i.e. corrected in the present paper) manipulation. They, instead, continue as follows:

... As seen by an observer corotating with the fluid at $P$ [at $R$] with angular velocity $\Omega(R)$, the fluid at $R - \lambda/2$ will appear to move with velocity $(R - \lambda/2)\Omega(R - \lambda/2) + \Omega(R)\lambda/2$. Thus the average angular momentum flux per unit arc length through $R = \text{const}$ in the outward direction is

\[ \rho\tilde{v}H(R - \lambda/2)[(R - \lambda/2)\Omega(R - \lambda/2) + \Omega(R)\lambda/2] \quad (a) \]

[where $\tilde{v}$ is the speed of gas elements in a chaotic motion, and the formula numbers (a) and (b) are temporarily put by the present authors]. An analogous expression, changing the sign of $\lambda$, gives the average inward angular momentum flux per unit arc length. The torque exerted on the outer ring by the inner ring is given by the net outward angular momentum flux. Since the mass flux due to chaotic motions is the same in both directions, one obtains to first order in $\lambda$ the torque per unit arc length as

\[ -\rho\tilde{v}H\lambda R^2\Omega', \quad (b) \]

where $\Omega' = d\Omega/dr$ [and $H$ is the half thickness of the disk]...

Although the meaning of (a) is not clear and is discussed below, let us accept (a) for the moment. We denote the formula (a) by $L_{in}$ and make a linear approximation, leading to:

\[ L_{in} = \rho\tilde{v}H(R - \lambda/2)[(R - \lambda/2)(\Omega(R) - (\lambda/2)\Omega(R') + \Omega(R)\lambda/2] \]
\[ = \rho\tilde{v}H(R - \lambda/2)[R\Omega - R(\lambda/2)\Omega']. \quad (6) \]

Similarly,

\[ L_{out} = \rho\tilde{v}H(R + \lambda/2)[R\Omega + R(\lambda/2)\Omega']. \quad (7) \]

Then, the torque should not be, $-\rho\tilde{v}H\lambda R^2\Omega'$, but rather

\[ L_{in} - L_{out} = \rho\tilde{v}H[(R - \lambda/2)(R\Omega - R(\lambda/2)\Omega') - (R + \lambda/2)(R\Omega + R(\lambda/2)\Omega')] \]
\[ = -\lambda\rho\tilde{v}HR(\Omega + R\Omega') = -\nu \Sigma R d(R\Omega)/dR. \quad (8) \]

If this is true, the angular momentum flows in a direction to realize $d(R\Omega)/dR = 0$, which implies a constant azimuthal velocity and is unrealistic in a rotating gas, although in this case the angular momentum flows outward.

Now we return to the problem of (a). When seen from $P$, the fluid at $R - \lambda/2$ should appear to move with velocity $(R - \lambda/2)\Omega(R - \lambda/2) - R\Omega(R) + \Omega(R)\lambda/2$, which completes the transformation from the inertial frame to the corotational frame with the origin at $P$. Eq. (a) then should be

\[ \rho\tilde{v}H(R - \lambda/2)[(R - \lambda/2)\Omega(R - \lambda/2) - R\Omega(R) + \Omega(R)\lambda/2], \quad (a)' \]

so that

\[ L_{in} = \rho\tilde{v}H[(R - \lambda/2)(-R(\lambda/2)\Omega') = -\rho\tilde{v}HR^2(\lambda/2)\Omega', \quad (b)' \]
and

\[ L_{in} - L_{out} = -\rho \dot{v} HR^2 \lambda \Omega'. \]  (8)'

This is the correct result. We then, however, notice that the procedure to obtain (8)' is simply that to obtain the well-known viscosity formula (2), thus being no longer based on the mean free path theory.

**Discussion** Having pointed out the confusion in the derivations given in the above cited textbooks, we now clarify how naturally the wrong conclusion comes from application of the mean free path theory (for discussion of this theory see, for example, Vincenti and Kruger\(^7\) Chap. 1, §4.) to a rotating gas in an accretion disk.

A gas element rotating in a Keplerian orbit possesses an angular momentum, linear momentum and angular velocity. The element is then assumed to travel across \( R \) over a distance of \( \lambda \), the mean free path. If we denote the quantity to be transported then by the element as \( Q \), which is the angular momentum, linear momentum or angular velocity, then, by linear approximation,

\[ Q(R + \lambda) - Q(R) = \lambda dQ(R)/dR. \]  (9)

Thus, if we want to obtain the correct result, i.e. to introduce the angular velocity alone in \( d/dR \), we have to make the impossible assumption that \( Q \) is the angular velocity. This argument clearly shows the failure of the application of the mean free path theory.

Do we then have to use the molecular viscosity formula to consider the present problem? This is the next question. In a strict sense, the molecular viscosity formula (2) is derived for a non-rotating gas. A viscosity formula for a rotating gas should be derived using the Boltzmann equation with the Coriolis force taken into account. For a rigidly rotating gas, this was formulated by Chapman and Cowling,\(^8\) who considered the effect of the Lorentz force rather than the Coriolis force, with the result that the viscosity coefficient along the direction parallel to the rotation axis is the same as that in a non-rotating gas, while that perpendicular to the axis is suppressed in a manner that depends on the Knudsen number. The viscosity formula for a Keplerian rotating gas is much more complicated, and an exact formula cannot be obtained. Approximate formulas obtained by a few authors are summarized in the textbook of Fridman and Gorkavyi.\(^9\)

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