EXISTENCE OF RICHARDSON ELEMENTS IN SEAWEED LIE ALGEBRAS OF TYPE $\mathcal{B}$, $\mathcal{C}$ AND $\mathcal{D}$

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Abstract. Seaweed Lie algebras are a natural generalisation of parabolic subalgebras of Lie algebras. A well-known result of Richardson says that for any parabolic subalgebra of a reductive Lie algebra, there always exists a dense open orbit in its nilpotent radical under the adjoint action \cite{14}. Elements in the open orbit are called Richardson elements.

In \cite{9} Jensen, Su and Yu showed that Richardson elements exist for seaweed Lie algebras of type $\mathcal{A}$. In this paper, we complete the story on Richardson elements for seaweeds, by showing that Richardson elements exist for any seaweed Lie algebra of type $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ into a sum of subalgebras, two of which are a seaweed of type $\mathcal{A}$ and a parabolic of classical type. We then make use of the existence of Richardson elements for the two subalgebras and analyze how their stabilizers act on the remaining part. This can then be used to prove directly the existence for a special case (Lemma \ref{5.3}) in type $\mathcal{D}$ and to obtain a sufficient condition on the existence of Richardson elements for the other cases (Lemma \ref{2.11}). Properties of endomorphisms deduced from the categorical construction of Richardson elements for type $\mathcal{A}$ play an important role in the proof of the existence of Richardson elements for the other types.

Keywords: seaweed Lie algebras, Richardson elements, stabilizers and endomorphisms of representations of quivers.

1. Introduction

Throughout we work over $k = \mathbb{C}$. Let $\mathfrak{g}$ be a reductive Lie algebra and let $G$ be a connected reductive algebraic group with Lie algebra $\mathfrak{g}$.

Definition 1.1. \cite{5, 13} A Lie subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ is called a seaweed subalgebra if there exists a pair $(\mathfrak{p}, \mathfrak{p}')$ of parabolic subalgebras of $\mathfrak{g}$ such that $\mathfrak{q} = \mathfrak{p} \cap \mathfrak{p}'$ and $\mathfrak{p} + \mathfrak{p}' = \mathfrak{g}$. Two such parabolic subalgebras are said to be opposite.

We are interested in the adjoint action of a seaweed Lie algebra on its nilpotent radical and the density of the action, continuing the work in \cite{8} and \cite{9}. Note that seaweed Lie algebras are also called biparabolic subalgebras by Joseph \cite{10} and that the coadjoint action and the index of seaweed Lie algebra were considered by Dergachev-Kirilov \cite{5}. A number of subsequent papers have studied the coadjoint action for these algebras, for instance \cite{6} by Duflo and Yu, \cite{10, 11, 12} by Joseph, \cite{13} by Panyushev and \cite{16} by Tauvel and Yu.

Definition 1.2. An element $x$ in the nilpotent radical $\mathfrak{n}$ of a seaweed Lie algebra $\mathfrak{q}$ is called a Richardson element if $[\mathfrak{q}, x] = \mathfrak{n}$.

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In the case where \( q \) is parabolic, a well known theorem of Richardson \([14]\) (see also \([15, \text{Chapter 33}]\)) says that Richardson elements exist. The following natural question was raised independently by Duflo and Panyushev.

**Question 1.3.** \([9]\) Does a seaweed Lie algebra have a Richardson element?

Inspired by earlier work of Brüstle-Hille-Ringel-Röhrle \([4]\) for parabolics in type \( A \), Jensen-Su-Yu \([9]\) give a positive answer to this question for all seaweeds of type \( A \). In this paper we prove the existence of Richardson elements for seaweed Lie algebras of type \( B, C \) and \( D \).

**Theorem 1.4.** Let \( g \) be a Lie algebra of type \( B, C \) or \( D \). Then any seaweed Lie algebra in \( g \) has a Richardson element.

We should mention that the method of proving the theorem is different from Richardson’s \([14]\). Although the fact that the parabolic group is the normaliser of the nilpotent radical plays a crucial role in Richardson’s proof, it is not a necessary condition. Evidently, abelian seaweeds have Richardson elements, but the corresponding seaweed groups are often not normalisers of the nilpotent radicals. Also, Brüstle-Hille-Ringel-Röhrle’s \([4]\) constructive proof of Richardson Theorem does not use this fact. For seaweed Lie algebras of type \( A \), Jensen-Su-Yu \([9]\) adapted the quiver approach in \([4]\) and proved constructively the existence of Richardson elements. See Example 3.5 for an explicit construction in type \( A \). The key ingredient of the quiver approach is the interplay between Lie algebras and quiver representation theory. For instance, endomorphism rings of representations that give us Richardson elements in seaweeds of type \( A \) correspond to stabilisers of the Richardson elements. In this paper we exclusively analyse properties of the endomorphisms at vertices in the quiver and apply these properties to prove the main theorem.

As a consequence of Theorem 1.4, Richardson elements exist for any seaweed Lie algebra of classical type. It is possible to show (using GAP) that Richardson elements exist in all exceptional types except \( E_8 \) (see \([9]\)). We provide more detail on the counterexample at the end of this paper, including the GAP source code. At the moment, there is no conceptual explanation as to why \( E_8 \) is different. Also, no categorical realisation of Richardson elements similar to \([4]\) and \([9]\) in type \( A \) is known in the other classical types. See however \([2]\) for partial results in the parabolic case.

The remainder of this paper is organised as follows. In Section 2, we recall the notion of a standard seaweed Lie algebras and prove some technical lemmas. We decompose standard seaweeds, including standard parabolics, as sums of two Lie subalgebra and consider how one subalgebra interacts with elements in the nilpotent radical of the others. Further, we show that a local property of stabilisers of Richardson elements of seaweeds of type \( A \) is sufficient for the existence of Richardson elements in seaweeds of other types. This condition can be verified using the categorical construction of Richardson elements \([3, 4]\) in type \( A \). This is the key to the proof of the existence of Richardson elements in this paper. As such, we recall the categorical construction in type \( A \) in Section 3, and prove essential results on stabilisers in Section 4. In Section 5, we prove the main result. The proof is split into two cases. The first case deals with algebras with a decomposition as in Section 2. A completely different argument is needed for the remaining case, which is of type \( D \). In the last section, we discuss the counterexample in \( E_8 \).
2. Richardson elements and decomposition of seaweeds

2.1. Standard seaweeds and parabolics. We fix a Borel subalgebra $b$ of $g$ and a Cartan subalgebra $h$ contained in $b$. Denote by $\Phi$, $\Phi^+$, $\Phi^-$ and $\Pi$, respectively, the root system, the set of positive roots, the set of negative roots and the set of positive simple roots, determined by $h$, $b$ and $g$. For $\alpha \in \Phi$, denote by $g_\alpha$ the root space corresponding to $\alpha$. We write

$$\alpha = \sum_{\alpha_i \in \Pi} x_i \alpha_i.$$ 

We say that $\alpha$ is supported at a positive simple root $\alpha_i$ if $x_i \neq 0$ and call the set of all such simple roots the support of $\alpha$. For $S, T \subset \Pi$, let $\Phi_S$ be the set of roots with support in $S$,

$$\Phi^\pm_S = \Phi^\pm \cap \Phi_S,$$ 

and $q_{S,T} = p_S^- \cap p_T^+$. Note that $p_{S,T}^\pm$ are parabolic subalgebras and $q_{S,T}$ is a seaweed Lie algebra. Parabolics and seaweeds constructed in this way are said to be standard with respect to the choice of $h$ and $b$.

**Proposition 2.1.** [13, 15] Any seaweed Lie algebra in $g$ is $G$-conjugate to a standard seaweed Lie algebra.

As a consequence, it suffices to consider standard seaweeds when proving the existence of Richardson elements.

Let $\Phi_{S,T}^+ = \Phi_S^+ \setminus \Phi_{S \cap T}^-$, $\Phi_{S,T}^- = \Phi_T^- \setminus \Phi_{S \cap T}^+$ and $\Phi_{S,T} = \Phi_{S,T}^+ \cup \Phi_{S,T}^-$. We have

$$q_{S,T} = n_{S,T}^- \oplus l_{S,T} \oplus n_{S,T}^+,$$

where $n_{S,T}^\pm = \bigoplus_{\alpha \in \Phi^\pm_{S,T}} g_\alpha$ and $l_{S,T} = h \oplus \bigoplus_{\alpha \in \Phi_{S \cap T}} g_\alpha$. Then $l_{S,T}$ is the Levi-subalgebra of $q_{S,T}$ and $n_{S,T} = n_{S,T}^\pm \oplus n_{S,T}^-$ is the nilpotent radical of $q_{S,T}$.

In the sequel, we will assume that neither $S$ or $T$ is equal to $\emptyset$ or $\Pi$. Note that two algebras of the same type may have different rank and we view $B_1 = C_1 = D_1 = A_1$.

By the type of a seaweed $q \subseteq g$ we mean the type of $g$ and thus the type of $q$ is not well-defined without an embedding $q \subseteq g$.

We note the following symmetry with respect to the choice of $S$ and $T$.

**Lemma 2.2.** The seaweed $q_{S,T}$ has a Richardson element if and only if so does the seaweed $q_{T,S}$.

*Proof.* The lemma follows from the involution $g \to g$ mapping $g_\alpha$ onto $g_{-\alpha}$. □

2.2. A decomposition of seaweed subalgebras and Richardson elements. Let $q_{S,T}$ be a standard seaweed in a simple Lie algebra $g$ of type $B$, $C$ or $D$. Note that

$$g = \bigoplus_{\alpha \in \Phi} g_\alpha \oplus \bigoplus_{\alpha \in \Pi} [g_\alpha, g_{-\alpha}],$$

and

$$[g_\alpha, g_\beta] = 0 \text{ if } \alpha + \beta \notin \Phi \cup \{0\}. (*)$$

Denote the positive simple roots of $g$ by $\alpha_1, \ldots, \alpha_n$, with the corresponding Dynkin graph numbered as follows.
Let $C = (c_{ij})$ be the Cartan matrix of $g$ and $e_i, f_i, h_i$ be the Chevalley generators of $g$ with $g_{\alpha_i} = \text{span}\{e_i\}$. That is, $g$ is generated by the generators subject to the relations.

(1) $[e_i, f_j] = \delta_{ij} h_i$;
(2) $[h_i, e_j] = c_{ij} e_j$;
(3) $[h_i, f_j] = -c_{ij} f_j$.

We choose a new basis $h_1, \ldots, h_n$ of the Cartan subalgebra of $g$ as follows. When $g$ is of type $B$, $h_1 = h_1$; when $g$ is of type $C$, $h_1 = \frac{1}{2} h_1$; when $g$ is of type $D$, $h_1 = \frac{1}{2} (h_1 - h_2)$. For $i \geq 2$,

$$h_i = h_i + h_{i-1}.$$

**Lemma 2.3.** If $i = 2$ and $g$ is of type $D$, then

$$[h_i, g_{\pm \alpha_j}] = \begin{cases} g_{\pm \alpha_j} & \text{if } j = 1, 2, 3, \\ 0 & \text{otherwise.} \end{cases}$$

For all other types and for all other $i$ when $g$ is of type $D$,

$$[h_i, g_{\pm \alpha_j}] = \begin{cases} g_{\pm \alpha_j} & \text{if } j = i, i+1, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** Direct computation gives the following,

$$[h_i, e_j] = \begin{cases} e_j & \text{if } j = i; \\ -e_j & \text{if } j = i + 1; \\ 0 & \text{otherwise,} \end{cases}$$

except when $i = 2$ and $g$ is of type $D$, where we have

$$[h_i, e_j] = \begin{cases} e_j & \text{if } j = 1 \text{ or } 2; \\ -e_j & \text{if } j = 3; \\ 0 & \text{otherwise.} \end{cases}$$

So the lemma follows. 

Let $h_i$ be the subspace spanned by $h_i$. Let

$$\epsilon = \min\{i \mid \alpha_i \not\in S\} \text{ and } \eta = \min\{i \mid \alpha_i \not\in T\}.$$  

By Lemma 2.2 we may assume that $\epsilon \geq \eta \geq 1$. Let

$$\omega = \max\{i \mid i \leq \epsilon, \alpha_i \not\in T\}.$$
We define two subspaces of \( g \),
\[
\mathfrak{g}_1 = \bigoplus_{\alpha \in \Phi_{(\alpha, i < \epsilon)}} \mathfrak{g}_{\alpha} \oplus \bigoplus_{i < \epsilon} \mathfrak{h}_i,
\]
and
\[
\mathfrak{g}_2 = \bigoplus_{\alpha \in \Phi_{(\alpha, i > \omega)}} \mathfrak{g}_{\alpha} \oplus \bigoplus_{i \geq \omega} \mathfrak{h}_i.
\]

**Lemma 2.4.** (1) If \( \epsilon > 2 \) when \( g \) is of type \( \mathbb{D} \) or \( \epsilon > 1 \) for other types, then \( \mathfrak{g}_1 \) is a Lie subalgebra of the same type as \( g \).
(2) The subspace \( \mathfrak{g}_2 \) is Lie subalgebra isomorphic to \( \mathfrak{gl}_{n-\omega+1} \).

**Proof.** By the definition of \( \mathfrak{h}_i \), we have
\[
\bigoplus_{i < \epsilon} \mathfrak{h}_i = \bigoplus_{i < \epsilon} [\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{-\alpha_i}],
\]
and
\[
\bigoplus_{i \geq \omega} \mathfrak{h}_i = \bigoplus_{i > \omega} [\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{-\alpha_i}] \oplus \mathfrak{h}_\omega.
\]
So the lemma follows. \( \square \)

**Lemma 2.5.** If \( \epsilon = \omega \), then \( q_{S,T} \) has a Richardson element.

**Proof.** If \( \omega = \epsilon = 1 \), then the root spaces in \( q_{S,T} \) are supported on a subgraph of type \( \mathbb{A} \), and so \( q_{S,T} \) is isomorphic to a seaweed of type \( \mathbb{A} \). Similarly, if \( \omega = \epsilon = 2 \) and \( g \) is of type \( \mathbb{D} \), then \( q_{S,T} \) is also isomorphic to a seaweed of type \( \mathbb{A} \). If \( \omega = \epsilon = n \), then \( q_{S,T} \) is isomorphic to a parabolic of the same type as \( g \). In all three cases, by Theorem 1.2 in \([9]\) or Richardson’s Theorem \([14]\), \( q_{S,T} \) has a Richardson element.

Otherwise,
\[
q_{S,T} = q_2 \oplus q_1,
\]
where \( q_1 \subseteq \mathfrak{g}_1 \) is parabolic of the same type as \( g \), and \( q_2 \subseteq \mathfrak{g}_2 \) is a seaweed of type \( \mathbb{A} \). Further, by (\( \ast \)),
\[
[q_1, q_2] = 0 \quad \text{and} \quad [q_1, q_2] = 0.
\]
Since both \( q_1 \) and \( q_2 \) have Richardson elements, we can conclude that \( q_{S,T} \) has a Richardson element. \( \square \)

**Consequently we assume that \( \epsilon > \omega \) for the remainder of the paper.** We also assume that \( (\epsilon, \omega) \neq (2, 1) \), when \( g \) is of type \( \mathbb{D} \). Let
\[
S'' = \{ \alpha_i \in S \mid i > \omega \}, \quad T'' = \{ \alpha_i \in T \mid i > \omega \},
\]
\[
S' = \{ \alpha_i \in S \mid i < \epsilon \} \quad \text{and} \quad T' = \{ \alpha_i \in T \mid i < \epsilon \}.
\]
Note that \( S' = \{ \alpha_i \mid i < \epsilon \} \). These subsets determine two subalgebras of \( q_{S,T} \), the positive parabolic subalgebra \( c_{S,T} \) of \( \mathfrak{g}_1 \) determined by \( T' \) and the seaweed Lie subalgebra \( a_{S,T} \) of \( \mathfrak{g}_2 \) determined by \( S'' \) and \( T'' \).

**Example 2.6.** Let \( g \) be a Lie algebra of type \( \mathbb{D}_6 \), \( S = \{ 5, 3, 2, 1 \} \) and \( T = \{ 6, 4, 2, 1 \} \). Then \( \epsilon = 4 \), \( \omega = 3 \). The subalgebras \( a_{S,T} \) and \( c_{S,T} \) can for instance be described using matrices as follows, where \( a_{S,T} \) is marked by \( * \) and \( \dagger \), and \( c_{S,T} \) is marked by \( - \) and \( \dagger \). The one dimensional intersection is marked by \( \dagger \), and there is an anti-symmetry to the anti-diagonal.
Let \( n = \{1, 2, \ldots, \omega \} \), and so

\[
\begin{pmatrix}
* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & * & \dagger & - & - & - & - & 0 & 0 \\
0 & 0 & 0 & 0 & - & - & - & - & 0 & 0 \\
0 & 0 & 0 & - & - & - & - & - & 0 & 0 \\
0 & 0 & 0 & 0 & - & - & - & - & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dagger & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Let \( l = g_2 \cap g_1 \). Then

\[
I = ( \bigoplus_{\alpha \in \Phi_{\{\alpha_i \mid i > \omega\}}} g_{\alpha} ) \oplus \bigoplus_{i \geq \omega} h_i.
\]

Let \( n_a \) and \( n_c \) be the nilpotent radicals of \( a_{S,T} \) and \( c_{S,T} \), respectively.

**Lemma 2.7.** Assume \( (\epsilon, \omega) \neq (2, 1) \) if \( g \) is of type \( \mathbb{D} \). Then

1. \( q_{S,T} = a_{S,T} + c_{S,T} \)
2. \( a_{S,T} \cap c_{S,T} = I \) is a block in the Levi subalgebra of \( q_{S,T} \).
3. \( n_{S,T} = n_a \oplus n_c \)

**Proof.** (1) Let \( \alpha \) be a positive root such that \( g_{\alpha} \subseteq q_{S,T} \). Then \( \alpha \) is not supported at \( \alpha_i \), since \( \epsilon \not\in S \). By the assumption on the root system, \( \alpha \) must be supported on simple roots \( \alpha_i \) with all \( i < \epsilon \) or all \( i > \epsilon \). So \( g_{\alpha} \subseteq a_{S,T} \) or \( g_{\alpha} \subseteq c_{S,T} \). Similarly, a negative root \( \beta \) with \( g_{\beta} \subseteq q_{S,T} \) is supported on simple roots \( -\alpha_i \) for all \( i < \omega \) or all \( i > \omega \), and so \( g_{\beta} \subseteq a_{S,T} \) or \( g_{\beta} \subseteq c_{S,T} \). By construction, \( [g_{\alpha_i}, g_{-\alpha_i}] \subseteq a_{S,T} + c_{S,T} \) for all simple roots \( \alpha_i \), and so \( q_{S,T} = a_{S,T} + c_{S,T} \).

(2) follows from the construction and (3) follows from (1) and (2). \( \square \)

By Theorem 1.2 in [9], Richardson elements exist in \( a_{S,T} \). Let \( r_2 \in a_{S,T} \) be a Richardson element and denote by

\[
\text{stab}_{a_{S,T}}(r_2) = \{ x \in a_{S,T} \mid [x, r_2] = 0 \},
\]

the stabiliser of \( r_2 \) in \( a_{S,T} \). For a subalgebra \( u \subseteq g \) given as a direct sum of root spaces and subspaces \( [g_{\alpha_i}, g_{-\alpha_i}] \), let \( x|_u \) be the canonical projection of \( x \in g \) onto \( u \). Let

\[
c_r = \{ x \in c_{S,T} \mid x|_l = y|_l \text{ for some } y \in \text{stab}_{a_{S,T}}(r_2) \}.
\]

**Lemma 2.8.** Assume \( (\epsilon, \omega) \neq (2, 1) \) if \( g \) is of type \( \mathbb{D} \) and \( l = n_c \). If \( [c_r, r_1] = n_c \), then \( r_1 + r_2 \) is a Richardson element of the seaweed \( q_{S,T} \).

**Proof.** Assume \( [c_r, r_1] = n_c \). Take any \( (x_a, x_c) \in n_a \oplus n_c \). There exists \( y_a \in a_{S,T} \) such that

\[
[y_a, r_2] = x_a.
\]

Write \( y_a = y'_a + (y_a)|_l \). Note that \( r_1|_{g_a} \neq 0 \) can occur only for positive roots \( \alpha \) with support contained in \( \{\alpha_{\epsilon-1}, \ldots, \alpha_1\} \) and \( y'_a|_{g_a} \neq 0 \) can occur only for positive roots \( \beta \) with support contained in \( \{\alpha_n, \ldots, \alpha_{\epsilon+1}\} \), or negative roots with support contained in \( \{\alpha_n, \ldots, \alpha_{\omega+1}\} \) and containing at least one \( \alpha_j \) for some \( j \geq \epsilon \). So by the property
Let $z$ and $y$.

Therefore, we have

$\begin{align*}
[y_a, r_1] &= 0
\end{align*}$

and so

$\begin{align*}
[y_a, r_1] &= [(y_a), l, r_1] \in n_c.
\end{align*}$

Let $y_c \in c_{r_2}$ be such that

$\begin{align*}
[y_c, r_1] &= x_c - [y_a, r_1].
\end{align*}$

Let $z \in \text{stab}_{S,T}(r_2)$ with $z_l = (y_c), l$ and $z_\alpha = z - z_l$. Then similar to (1), we have

$\begin{align*}
[z', r_1] &= 0
\end{align*}$

and

$\begin{align*}
[y_c - (y_c), l, r_2] &= 0.
\end{align*}$

Therefore

$\begin{align*}
[y_a + z' + y_c, r_1 + r_2] &= [y_a, r_1 + r_2] + [z', r_1 + r_2] + [y_c, r_1 + r_2] \\
&= x_a + [y_a, r_1] + [z', r_2] + [z_l, r_2] + x_c - [y_a, r_1] \\
&= x_a + x_c + [z, r_2] \\
&= x_a + x_c.
\end{align*}$

This completes the proof of the lemma.

2.3. A decomposition of parabolic subalgebras and Richardson elements.

Let $S$, $T$, $\epsilon$, $\omega$, $g_1$, $g_2$ and $I$ be defined as in Section 2.2 with $\omega < \epsilon$. We also continue the assumption that $(\epsilon, \omega) \neq (2, 1)$ if $g$ is of type $D$. Let $g'$ be a Lie algebra of the same type as $g$, with rank at least $\epsilon$ and root system denoted by $\Phi'$. We may assume that both $g$ and $g'$ are subalgebras of a Lie algebra of the same type as $g$ such that $g \subseteq g'$ or $g' \subseteq g$. Here all inclusions are induced by inclusions of Dynkin diagrams.

Let $p_U^+ \subseteq g'$ be the standard parabolic subalgebra determined by $U$ with $\epsilon \not\in U$ and

$\{\alpha_i| i < \epsilon\} \cap U = \{\alpha_i| i < \epsilon\} \cap T.$

We choose a basis $\{h_i\}; i$ in the same manner as we did for the basis $\{h_i\};$ of the Cartan subalgebra of $g$. Let $g_1' = g_1$ and let $g_2' \subseteq g'$ be defined similarly to $g_2 \subseteq g$, i.e.

$\begin{align*}
\left( \bigoplus_{\alpha \in \Phi(\alpha_i| i > \omega)} g_\alpha' \bigoplus \bigoplus_{i \geq \omega} h_i', \right)
\end{align*}$

which is of type $A$. Further, let $U'' = \{\alpha_i \in U| i > \omega\}$, $U' = \{\alpha_i \in U| i < \epsilon\}$. These two sets determine the following standard parabolic subalgebras of $g_2'$ and $g_1'$,

$\begin{align*}
a_U = \bigoplus_{\alpha \in \Phi(\alpha_i| i > \omega)} g_\alpha' \bigoplus \bigoplus_{i \geq \omega} h_i' \subseteq g_2'
\end{align*}$

and

$\begin{align*}
c_U = \bigoplus_{\alpha \in \Phi(\alpha_i| i < \epsilon)} g_\alpha' \bigoplus \bigoplus_{i < \epsilon} h_i' \subseteq g_1'.
\end{align*}$

Note that $c_U = c_{S,T}$.

Let $d_U \subseteq p_U^+$ be the direct sum of all root spaces $g_\alpha$ with $\alpha$ a positive root such that $g_\alpha$ is neither contained in $a_U$ nor in $c_U$. Let $n_a'$ be the nilpotent radical of $a_U$. Recall that $n_a$ is the nilpotent radical of $c_U = c_{S,T}$.

Lemma 2.9. The following are true.
(1) $[p^+ U, d_U] \subseteq d_U$.
(2) $[p^+ U, n_a] \subseteq n_a + d_U$.
(3) $p^+ U = (a_U + c_U) \oplus d_U$.
(4) $a_U \cap c_U = I$.
(5) $n_U = n'_a \oplus n_c \oplus d_U$.

Proof. By the construction, $d_U$ is the direct sum of the root spaces $g_\alpha$ with $\alpha$ positive and supported at both simple roots $\alpha_\epsilon$ and $\alpha_\omega$. For any $g_{-\alpha} \subseteq p^+ U$ with $-\alpha$ a negative root, the root $-\alpha$ not supported at $\alpha_\epsilon$ and $\alpha_\omega$. So (1) follows. Similar, (2) holds.

(3) and (4) follow from the construction. (5) follows from (3) and (4).

By Richardson’s theorem, there exists

$$r = r_1 + r_2' + r_d$$

with $(r_1, r_2', r_d) \in n'_a \oplus n_c \oplus d_U$ such that $[p^+ U, r] = n_U$. By Lemma 2.9 (1) (2), we may assume that $r_1$ is the Richardson element for $c_{S,T}$ from Section 2.2. Again by Lemma 2.9 we can identify

$$(a_U + c_U) = p^+_U/d_U \quad \text{and} \quad n'_a \oplus n_c = n_U/d_U.$$ 

So we have a well-defined action $p^+_U$ on $n'_a \oplus n_c$ and

$$n'_a \oplus n_c = [p^+_U, r_1 + r_2'] = [a_U + c_U, r_1 + r_2']$$

Let $c_{r_2} = \{x \in c_U | x|| = y|| \text{ for some } y \in \text{stab}_{a_U}(r_2')\}$.

Lemma 2.10. We have $[c_{r_2}, r_1] = n_c$.

Proof. Let $x \in n_c$. There exists $y \in a_U + c_U$ such that $[y, r_1 + r_2'] = x$. We write

$$y = y'' + y'| + y'',$

where

$$y'' + y|| \in a_U \quad \text{and} \quad y|| + y' \in c_U.$$ 

Then similar to (1) in the proof of Lemma 2.8

$$[y'', r_1] = 0, \quad [y', r_2'] = 0$$

and so

$$[y, r_2'] = [y'' + y||, r_2'] \in n'_a \quad \text{and} \quad [y, r_1] = [y|| + y', r_1] \in n_c.$$

Since $n'_a \cap n_c = 0$ and $[y, r_2' + r_1] = x \in n_c$, we have

$$[y, r_2'] = [y'' + y||, r_2'] = 0 \quad \text{and} \quad [y, r_1] = [y|| + y', r_1] = x.$$ 

It follows that $y|| + y' \in c_{r_2}$ and so $[c_{r_2}, r_1] = n_c$.

Recall that $r_2$ is a Richardson element for $a_{S,T}$. We have the following observation.

Lemma 2.11. If $\text{stab}_{a_U}(r_2')|| = \text{stab}_{a_{S,T}}(r_2)||$, then $q_{S,T}$ has a Richardson element.

Proof. Assume $\text{stab}_{a_U}(r_2')|| = \text{stab}_{a_{S,T}}(r_2)||$. Then $c_{r_2} = c_{r_2}$ and so $[c_{r_2}, r_1] = n_c$ by Lemma 2.10. Then $q_{S,T}$ has a Richardson element, by Lemma 2.8.

The condition in the statement is actually true and can be verified using properties deduced from the categorical construction of Richardson elements [9]. Thus verifying the condition is a key step in the proof of the existence of Richardson elements in the seaweed $q_{S,T}$ and we recall the construction in next section.
3. Construction of Richardson elements in type A

Let $Q$ be a quiver of type $A_m$ with vertices $Q_0 = \{1, \ldots, m\}$ and arrows

$Q_1 = \{\alpha_i \mid i \to i + 1 \text{ or } \alpha_i : i \leftarrow i + 1 \text{ for } i = 1, \ldots, m - 1\}$.

Let $A = kQ$, the path algebra of $Q$. We denote the projective indecomposable $A$-module associated to vertex $i$ by $P_i$. Let

$$P(d) = \bigoplus_{i=1}^{m} P_i^{d_i}$$

for any $d \in \mathbb{Z}_{\geq 0}^m$. Note that $\text{End}_A P(d)$ is a seaweed in a Lie algebra of type $\mathbb{A}$, and a Richardson elements in $\text{rad}\text{End}_A P(d)$ can be constructed from a good rigid representation $X(d)$ [8, 9] of a double quiver of $Q$ with relations. We recall the double quiver with relations from [3, 7] and the construction of $X(d)$.

Let $\tilde{Q}$ be the double quiver of $Q$, i.e. $\tilde{Q}_0 = Q_0$ and $\tilde{Q}_1 = Q_1 \cup Q_1^*$ with

$$Q_1^* = \{\alpha^* : i \to j \mid \alpha : j \to i \in Q_1\}.$$

Let $\mathcal{I}$ be the ideal of $k\tilde{Q}$ generated by

$$\alpha^* \alpha - \sum_{\beta \in Q_1, t(\beta) = s(\alpha)} \beta \beta^*$$

for any arrow $\alpha \in Q_1$, and

$$\alpha^* \beta$$

for pairs of arrows $\alpha \neq \beta$ in $Q_1$ terminating at the same vertex. Let

$$D = k\tilde{Q}/\mathcal{I}.$$

Any $D$-module is an $A$-module via the inclusion $A \subseteq D$ and any $A$-module is a $D$-module via the surjection $D \twoheadrightarrow A$ mapping all arrows in $Q_1^*$ to zero. We use the notation $AX$ to indicate the $A$-module structure of a $D$-module $X$ and note that

$$\text{Hom}_A(M, N) = \text{Hom}_D(M, N)$$

for two $A$-modules $M$ and $N$.

The algebra $D$ is quasi-hereditary with Verma modules $P_1, \ldots, P_m$ (see [3]). The modules filtered by the Verma modules are called good modules. So for any good $D$-module $M$, we have

$$A_{\Delta} M \cong P(d)$$

as $A$-modules for some $d \in \mathbb{Z}_{\geq 0}^m$. We call $d$ the $\Delta$-dimension vector of $M$, denoted by $\text{dim}_A M$, and the set $\text{supp}_\Delta(M) = \{i \mid d_i \neq 0\}$ the $\Delta$-support of $M$. This definition is similar to the support of a module, which is defined using the usual dimension vector.

We identify modules with the corresponding quiver representations. So a $D$-module $M$ is a collection of vector spaces $M_i, i \in Q_0$ and linear maps $M_\beta, \beta \in Q_1 \cup Q_1^*$, satisfying the relations $\mathcal{I}$, and a homomorphism $f : M \to N$ of $D$-modules is a collection of linear maps $(f_i)_{i \in Q_0}$ commuting with the module structure on $M$ and $N$. 
3.1. The linear case \cite{4}. Let $Q$ be a linear quiver with $m$ the unique sink vertex. Then $\mathcal{I}$ is generated by commutative relations at $2, \ldots, m-1$, and a zero relation at 1. In this case the indecomposable projective $D$-module $Q_m$, at vertex $m$, is injective. A submodule $X$ of $Q_m$ is uniquely determined by its $A$-structure $AX \cong \oplus_{i=1}^{m} P_i$ with $d_i \in \{0, 1\}$. Thus there is a natural bijection between subsets $I \subseteq Q_0$ and submodules of $Q_m$. More precisely, under this bijection a subset $I$ corresponds to the unique submodule $X(I) \subseteq Q_m$ with $\Delta$-support $I$. For any vector $d \in \mathbb{Z}_{\geq 0}^m$, define

$$X(d) = \sum_{i=1}^{t} X(I_i),$$

with $\dim_{\Delta} X(d) = d$ and $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_t$. We give an example to illustrate the construction. See \cite{4} for more details.

Example 3.1. Let $m = 3$ and $d = (2, 1, 2)$. The algebra $D$ is given by the quiver

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3,$$

with the ideal $\mathcal{I}$ generated by $\alpha_1^* \alpha_1$ and $\alpha_1 \alpha_1^* - \alpha_2^* \alpha_2$. The projective-injective $D$-module $Q_3$ has the following seven nonzero submodules with the first one $Q_3$,

![Diagram of a linear quiver with arrows indicating nonzero action of the arrows in $Q_1$.](attachment:diagram.png)

corresponding to the subsets $\{1, 2, 3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1\}$, $\{2\}$, $\{3\}$, respectively. In the picture a number $i$ indicates a one dimensional basis element at vertex $i$ and the arrows indicate the nonzero action of the arrows in $Q_1$. We have

$$X(d) = X(\{1, 2, 3\}) \oplus X(\{1, 3\}).$$

3.2. The general case \cite{8, 9}. Now suppose that $Q$ has an arbitrary orientation. Recall that a vertex is admissible if it is a source or a sink. Let

$$i_1 < i_2 < \cdots < i_{t-1} < i_t$$

be the complete list of interior admissible vertices in $Q$ and let $i_0 = 1$ and $i_{t+1} = m$. Each interval $[i_s, \ldots, i_{s+1}]$ has a unique sink and a unique source. Similar to the linear case, each subset $I \subseteq \{i_s, \ldots, i_{s+1}\}$ determines a unique (up to isomorphism) indecomposable rigid good $D$-module, which has $\Delta$-support $I$ and is a submodule of the indecomposable projective module at the sink in this interval.

Two indecomposable rigid good $D$-modules $M$ and $N$ with

$$\text{supp}_{\Delta}(M) \cap \text{supp}_{\Delta}(N) = \{i_j\},$$

$$\text{supp}_{\Delta}(M) \subseteq \{i|i \leq i_j\} \text{ and } \text{supp}_{\Delta}(N) \subseteq \{i|i \geq i_j\},$$
can be glued by identifying $P_{i_j}$ to obtain a new indecomposable rigid good $D$-module.
Definition 3.2. Let $u$ be a vertex with $i_u < u \leq i_{v+1}$. Suppose that two indecomposable rigid $D$-modules $M$ and $N$, glued from $X(I_s)$s and $X(J_s)$s, respectively, are supported (but not necessarily $\Delta$-supported) at $u$. We define $M \leq_u N$ if for any $s$ with both $I_s$ and $J_s$ nonempty, $I_s \subseteq J_s$ when $s$ is even and $I_s \supseteq J_s$ when $s - v$ is odd.

In general, using the order $\leq_u$ we can construct rigid good $D$-modules as follows. Let $M$ and $N$ be two good rigid $D$-modules with $\dim_{\Delta}(M)_i = 0$ for $i > i_J$, $\dim_{\Delta}(N)_i = 0$ for $i < i_J$, and $\dim_{\Delta}(N)_{i_J} = \dim_{\Delta}(M)_{i_J}$. With respect to $\leq_{i_J}$, we glue the $i_J$th biggest summand of $M$ to the $i_J$th biggest summand of $N$. In this way, we obtain a rigid good $D$-module $X(d)$ for any $\Delta$-dimension vector $d$. We illustrate the construction by an example. See [8, 9] for more details.

Example 3.3. Let $Q$ be the quiver
$$1 \leftarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5.$$ Let $d = (2, 1, 2, 1, 2)$. Let $d^1 = (2, 1, 2, 0, 0)$ and $d^2 = (0, 0, 2, 1, 2)$. Then $X(d^1) = M^1 \oplus M^2$ and $X(d^2) = N^1 \oplus N^2$ with $M^1 = X(\{1, 3\})$, $M^2 = X(\{1, 2, 3\})$, $N^1 = X(\{3, 4, 5\})$ and $N^2 = X(\{3, 5\})$ as follows,

We have $M^1 \leq_3 M^2$ and $N^1 \leq_3 N^2$. So $X(d)$ is the direct sum of the gluings of $M^1$, $M^2$ with $N^1$ and $N^2$, respectively, i.e.,

By the construction of rigid good modules, we have the following lemma.

Lemma 3.4. Let $u$ be a source or sink vertex, the indecomposable summands of a rigid good $D$-modules $X(d)$ which are supported at $u$ are totally ordered by $\leq_u$.

3.3. Construction of Richardson elements. We briefly describe how to construct a Richardson element $r(d)$ from $X(d)$. Let
$$X(d) = \bigoplus_i X^i$$
be a decomposition of $X(d)$ into indecomposable summands and let $n = \sum_i d_i$. For each summand $X^i$ that is $\Delta$-supported at $j$, choose a number $x_{ij}$, where
$$\sum_{i < j} d_i < x_{ij} \leq \sum_{i \leq j} d_i$$
such that \( x_{ij} \neq x_{lj} \) for two different summands \( X^i \) and \( X^l \). If \( X^i \) is \( \Delta \)-supported at both \( s \) and \( t \) with \( s < t \), but not at \( s + 1, \ldots, t - 1 \), then as a matrix \( r(d) \in \mathfrak{gl}_n \) has a 1 at either \((x_{is}, x_{it})\) or \((x_{it}, x_{is})\), depending on which root-space belongs to \( q_{S,T} = \text{End}_A(P(d)) \). All other entries in \( r(d) \) are equal to 0. In this way, we construct a Richardson element \( r(d) \) in \( q_{S,T} \). Note that 

\[
\text{End}_D(X(d)) \cong \text{stab}_{q_{S,T}}(r(d)).
\]

The matrix \( r(d) \) is also the adjacency matrix of an oriented graph with components corresponding to indecomposable summands of \( X(d) \). See the example below for an illustration and [1, 9, 4] for more detail.

**Example 3.5.** The rigid modules \( X(d) \) in Example 3.1 and 3.3 correspond, respectively, to the parabolic subalgebra of \( \mathfrak{gl}_5 \) with Richardson element \( r_1 \) and the seaweed subalgebra of \( \mathfrak{gl}_8 \) with Richardson element \( r_2 \), where

\[
r_1 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
r_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

The two subalgebras are direct sum of the Cartan subalgebras and root spaces in the bold faces, respectively. The corresponding oriented graphs are.

\[
P_1 \quad P_2 \quad P_3 \\
\quad P_4 \quad P_5
\]

\[
P_1 \quad P_3 \\
\quad P_2 \quad P_3 \quad P_5
\]

4. **Stabilisers of Richardson elements in type \( A \)**

Consider the quiver \( Q \) of type \( A_m \) of arbitrary orientation, defined in Section 3. For \( D \)-modules \( M \) and \( N \), let

\[
\text{Hom}_D(N, M)_i = \{ f_i | f \in \text{Hom}_D(N, M) \}
\]

be the space of homomorphisms from \( N \) to \( M \) restricted to vertex \( i \). Let \( \text{End}_D(M)_i \) and \( \text{Aut}_D(M)_i \) be defined similarly. We study the structure of the endomorphism algebra of a good rigid \( D \)-module and its restriction to a vertex. Three special cases are needed in the proof of the main result in the last section.

Let

\[
X(d) = \bigoplus_i (X^i)^{n_i}
\]

be a good rigid \( D \)-module with \( A X(d) = P(d) \). We order the summands such that \( X^i \prec_m X^{i+1} \) for all indecomposable summands \( X^i \).

4.1. **Restriction to \( m \), when \( m \) is a source.** Let \( V = X(d)_m \). Then

\[
\text{End}_D(X(d))_m \subseteq \mathfrak{gl}(V).
\]

**Lemma 4.1.** The subalgebra \( \text{End}_D(X(d))_m \subseteq \mathfrak{gl}(V) \) is parabolic.
Lemma 4.2. The subalgebra $\text{End}_{D}(X(d), X(d))_{m}$ is one dimensional for $i \leq j$ and zero otherwise. Further we may choose basis elements $r_{ji} \in \text{Hom}_{D}(X^{i}, X^{j})_{m}$ for all $i$ and $j$, such that $r_{ji}r_{kl} = r_{jl}$ if $i = k$ and zero otherwise. The lemma follows.

4.2. Restriction to $m$, when $m$ is a sink. Recall that $\alpha_{m-1} : m-1 \to m$ is the arrow that ends at $m$. For $D$-modules $M$ and $N$, let

$$\text{Hom}_{D}(M, N)_{m}^{0} = \{ f : M_{m}/\alpha_{m-1}(M_{m-1}) \to N_{m}/\alpha_{m-1}(N_{m-1}) | f \in \text{Hom}_{D}(M, N) \}. $$

Let

$$V = X(d)_{m}/\alpha_{m-1}(X(d)_{m-1}). $$

Then $\text{End}_{D}(X(d))_{m}^{0} \subseteq \mathfrak{gl}(V)$.

Lemma 4.2. The subalgebra $\text{End}_{D}(X(d))_{m}^{0} \subseteq \mathfrak{gl}(V)$ is parabolic.

Proof. The proof is similar to the proof of Lemma 4.1.

4.3. Stabilisers of indecomposable rigid modules. There is an obvious embedding of endomorphism rings

$$\prod_{i} \text{End}_{D}(X^{i})^{n_{i}} \subseteq \text{End}_{D}(X(d)).$$

As before let $1 = i_{0} < i_{1} < \cdots < i_{t+1} = m$ be the admissible vertices of $Q$.

Let $v$ be a sink in $Q$ and let $\alpha$ be an arrow ending at $v$. Then $\alpha \alpha^{*}$ induces a nilpotent endomorphism $x$ of $X^{i}$. So we have such an endomorphism $x_{s} : X^{i} \to X^{i}$ for each interval $[i_{s-1}, i_{s}]$ with $s = 1, \cdots, t+1$. Note that $x_{s}$ is zero on vertices that are not in the interval $[i_{s-1}, i_{s}]$. Let $m_{s} \geq 1$ be the smallest number such that $x_{s}^{m_{s}} = 0$. In fact,

$$m_{s} = |\supp_{\Delta}(X^{i}) \cap [i_{s-1}, i_{s}]| - 1. $$

Lemma 4.3. The map $y_{s} \mapsto x_{s}$ induces an isomorphism

$$\frac{k[y_{1}, \cdots, y_{t+1}]}{< y_{s}y_{t} = 0 \text{ for } s \neq t, y_{s}^{m_{s}} = 0>} \cong \text{End}_{D}(X^{i}).$$

Proof. Let $x_{s}$ and $x_{t}$ be two arbitrary endomorphisms, induced by $\alpha \alpha^{*}$ and $\beta \beta^{*}$, respectively, where $\alpha$ and $\beta$ are two different arrows. Clearly $x_{s}x_{t} = 0$ if $\alpha$ and $\beta$ end at different sinks. Otherwise, $x_{s}x_{t} = 0$ follows from the relation $\alpha^{*} \beta = 0$. By definition, $x_{s}^{m_{s}} = 0$. So the map is well-defined.

By the construction of $X^{i}$, $\text{End}_{D}(X^{i})$ is generated by the $x_{s}$ and thus the map is surjective. The intersection of the images of $x_{s}$ and $x_{t}$ is zero if $s \neq t$, and so the injectivity follows.

By the embedding

$$\text{End}_{D}(X(d)) \subseteq \text{End}_{A}(P(d)) \subseteq \mathfrak{gl}_{n},$$

and the construction of Richardson elements discussed before Example 3.3, each element of $\prod_{i} \text{End}_{D}(X^{i})^{n_{i}}$ can be explicitly described in terms of matrices. They can also be described in terms of the oriented graphs constructed from $X^{i}$.

Example 4.4. We use the two Richardson elements from Example 3.3. In both cases, there are two indecomposable summands $X^{1}$ and $X^{2}$ in $X(d)$. We use two different
colours to describe \( \text{End}_D(X^1) \) and \( \text{End}_D(X^2) \) in each case,

\[
\begin{pmatrix}
a & 0 & b & c & 0 \\
0 & d & 0 & 0 & e \\
0 & 0 & a & b & 0 \\
0 & 0 & 0 & a & 0 \\
0 & 0 & 0 & 0 & d
\end{pmatrix}
\quad \text{stab}(r_1)
\begin{pmatrix}
a & 0 & 0 & 0 & 0 & 0 \\
0 & e & 0 & 0 & 0 & 0 \\
0 & f & e & 0 & 0 & 0 \\
b & 0 & 0 & a & 0 & c \\
0 & g & f & e & 0 & 0 \\
0 & 0 & 0 & 0 & a & 0 \\
0 & 0 & 0 & 0 & 0 & e
\end{pmatrix}
\quad \text{stab}(r_2)
\]

where for instance in the first matrix, the \( a \)-entries are \( a \cdot x_1 \), the \( b \)-entries are \( b \cdot x_1 \) with \( x_1 : X^1 \to X^1 \) and the \( c \)-entries are \( c \cdot x_1^2 \).

The non-zero off-diagonal entries in the matrices correspond to non-trivial paths in the oriented graphs as follows.

\[
P_1 \xrightarrow{c} P_2 \xrightarrow{b} P_3 \quad P_1 \xrightarrow{d} P_3 \xrightarrow{c} P_4 \xrightarrow{c} P_5
\]

\[
P_1 \xrightarrow{e} P_3 \quad P_1 \xrightarrow{g} P_2 \xrightarrow{f} P_3 \xrightarrow{h} P_5.
\]

5. The main result

**Theorem 5.1.** Let \( \mathfrak{g} \) be a simple Lie algebra of type \( \mathbb{B}, \mathbb{C} \) or \( \mathbb{D} \). Then any seaweed in \( \mathfrak{g} \) has a Richardson element.

We continue to use the notation from Section 2. The proof of the theorem is split into two lemmas. The first lemma deals with type \( \mathbb{B}, \mathbb{C} \), and type \( \mathbb{D} \) when \((\epsilon, \omega) \neq (2, 1)\). We will use quiver representations and results from Section 4 to verify that the condition in Lemma \([2.1]\) holds, and so Richardson elements exist. The second lemma deals with the situation, where \( \mathfrak{g} \) is of type \( \mathbb{D} \) and \((\epsilon, \omega) = (2, 1)\). In this case, unlike the first one, there can be root spaces that are not contained in \( \mathfrak{a}_{S,T} + \mathfrak{c}_{S,T} \), that is, \( \mathfrak{q}_{S,T} \) is not necessarily equal to \( \mathfrak{a}_{S,T} + \mathfrak{c}_{S,T} \).

**Lemma 5.2.** Let \( \mathfrak{q}_{S,T} \subseteq \mathfrak{g} \) be a seaweed, where \( \mathfrak{g} \) is of type \( \mathbb{B} \) or \( \mathbb{C} \), or of type \( \mathbb{D} \) with \((\epsilon, \omega) \neq (2, 1)\). Then \( \mathfrak{q}_{S,T} \) has a Richardson element.

**Proof.** Note that we only need to consider the situation where \( \epsilon > \omega \) and neither \( S \) nor \( T \) is equal to \( \Pi \) or \( \emptyset \).

Let \( P = P(d) \) be a projective \( A \)-module such that \( \text{End}_A(P(d)) \simeq \mathfrak{a}_{S,T} \). Let \( r \in \mathfrak{n}_a \) be a Richardson element constructed from the good rigid \( D \)-module \( X(d) \) as in Section 3.3. Fix an embedding

\[
\text{End}_A(P(d)) \subseteq \mathfrak{g}
\]

such that \( \text{End}_A(P(d)) = \mathfrak{a}_{S,T} \) and \( \text{End}_A(P(d))_m = \mathfrak{l} \), where \( \mathfrak{l} = \mathfrak{a}_{S,T} \cap \mathfrak{c}_{S,T} \) is as in Section 2.2. By the condition \( \epsilon > \omega \) on \( \mathfrak{a}_{S,T} \), the vertex \( m \) is a source. So

\[
\text{stab}_{\mathfrak{a}_{S,T}}(r)|_{\mathfrak{l}} = \text{End}_D(X(d))_m
\]

is a parabolic in \( \mathfrak{l} \), by Lemma \([4.1]\).

We order the summands \( X(d) \) from big to small with respect to the order \( \leq_m \), so that \( \text{End}_D(X(d))_m \) is standard upper triangular. Suppose that the sizes of the blocks
in the Levi subalgebra of $\text{End}_D(X(d))_m$ are $c_1, \ldots, c_l$ and let

$$\hat{c} = (c_l, c_{l-1} + c_l, \ldots, \sum_{j \geq i} c_j, \ldots, \sum_{j \geq 1} c_j).$$

Let $B$ be the path algebra of the linearly oriented quiver $A_l$ with the unique sink $l$ and let

$$P(\hat{c}) = P_{l_1}^{c_1} \oplus \cdots \oplus P_{l}^{c_l},$$

a projective representation of this quiver. Denote by $F$ the algebra of the associated double quiver with relations, defined as $D$ in Section 3, and let $X(\hat{c})$ be the rigid good $F$-module.

Choose $\mathfrak{g}'$ and $U$ (see Section 2.3), and an embedding

$$\text{End}_B(P(\hat{c})) \subseteq \mathfrak{g}'$$

such that

$$\text{End}_B(P(\hat{c})) = \mathfrak{a}_U \text{ and } \text{End}_B(P(\hat{c}))^0 = 1.$$

Let $r' \in \mathfrak{a}_U$ be a Richardson element corresponding to $X(\hat{c})$, where the summands of $X(\hat{c})$ are ordered from small to big with respect to the order $\leq l$, so that

$$\text{End}_F(X(\hat{c}))^0_{l} = \text{stab}_{\mathfrak{a}_U}(r')_{l}$$

is standard upper triangular. Both $\text{End}_F(X(\hat{c}))^0_{l} \subseteq 1$ and $\text{End}_D(X(d))^0_{m} \subseteq 1$ are standard upper triangular with Levi blocks of equal sizes, and so

$$\text{stab}_{\mathfrak{g}_{S,T}}(r')_{l} = \text{stab}_{\mathfrak{g}_{S,T}}(r')_{l}.$$

Then $q_{S,T}$ has a Richardson element by Lemma 2.11.

Denote by $E_{ij}$ the elementary matrix with 1 at $(i, j)$-entry and 0 elsewhere.

**Lemma 5.3.** If $\mathfrak{g}$ has type $D$ and $(\epsilon, \omega) = (2, 1)$, then the seaweed $q_{S,T} \subseteq \mathfrak{g}$ has a Richardson element.

**Proof.** Let $W = S \backslash \{1\}$ and

$$q_{S,T}^1 = \bigoplus_{\alpha \text{ supported at } \alpha_1, \alpha_2} \mathfrak{g}_\alpha.$$ 

Then

$$q_{S,T} = q_{S,T}^1 \oplus q_{S,T}^2 \text{ and } n_{S,T} = n_{S,T}^1 \oplus q_{S,T}^1.$$

Note that $q_{W,T}$ is a seaweed of type $A$ and so it has a Richardson element $r \in n_{W,T}$.

Since $\alpha_1 \not\in T$, we have $[q_{S,T}, q_{S,T}^1] \subseteq q_{S,T}^1$. Then $r + r'$ with $r' \in q_{S,T}^1$ is a Richardson element in $q_{S,T}$ if $[\text{stab}_{q_{W,T}}(r), r'] = q_{S,T}^1$. Indeed, let $x + x' \in n_{W,T} \oplus q_{S,T}^1$ and let $y \in q_{W,T}$ and $y' \in \text{stab}_{q_{W,T}}(r)$ such that

$$[y, r] = x \text{ and } [y', r'] = x' - [y, r'].$$

Then

$$[y + y', r + r'] = x + x'$$

and so $r + r'$ is a Richardson element. Hence, to prove the lemma, it suffices to show

$$[\text{stab}_{q_{W,T}}(r), r'] = q_{S,T}^1.$$

We choose the representation of $\mathfrak{g}$ given by $2n \times 2n$-matrices anti-symmetric to the anti-diagonal,

$$\mathfrak{g}_{\alpha_1} = k \cdot (E_{n-1,n+1} - E_{n,n+2}) \text{ and } \mathfrak{g}_{\alpha_2} = k \cdot (E_{n-1,n} - E_{n+1,n+3}).$$
Then \( q_{S,T}^1 \) has a basis \( E_{l,n+1} - E_{n,l+1} \) for \( l = n - 1, \ldots, n - \alpha + 2 \), where
\[
\alpha = \begin{cases} 
\min \{ p | p > \epsilon, p \notin S \} & \text{if such a } p \text{ exists,} \\
n + 1 & \text{otherwise.} 
\end{cases}
\]

Fix an embedding \( gl_n \subseteq g \) where \( E_{ij} \mapsto E_{ij} - E_{2n-j+1,2n-i+1} \). We may assume a Richardson element \( r(d) \) is constructed from a rigid good module \( X(d) = \oplus_i X^i \), as in Section 3.3. Choose an embedding \( \text{End}_A(P(d)) \subseteq g \) so that \( \text{End}_A(P(d)) = q_{W,T} \) and \( \text{End}_D(X(d)) = \text{stab}_{q_{W,T}}(r) \). Then
\[
\text{End}_D(X^i) \subseteq \text{stab}_{q_{W,T}}(r),
\]
with
\[
1_{X^i} = \sum_{j \in \text{Supp}_\Delta(X^i)} (E_{x_{ij},x_{ij}} - E_{2n-x_{ij}+1,2n-x_{ij}+1}),
\]
where the \( x_{ij} \) are constructed from the summand \( X^i \) as in Section 3.3. Let
\[
V_i = q_{S,T}^1 \cap \bigoplus_{[1_{X^i},g_\alpha] \neq 0} g_\alpha
\]
By the construction of \( X(d) \) and \( r \),
\[
q_{S,T}^1 = \bigoplus_i V_i,
\]
since \( x_{ij} \neq x_{ls} \) for \( i \neq l \) or \( j \neq s \).

As \( \epsilon > \omega \), the vertex \( m \) is a source. There are now two cases to consider. First, when \( m - 1 \) is non-admissible, then each indecomposable summand \( P_{m-1} \) in \( P(d) \) is contained in a different indecomposable summand of \( X(d) \). Therefore each \( V_i \) is at most one-dimensional and \( [k \cdot 1_{X^i}, r_i] = V_i \) for any non-zero \( r_i \in V_i \).

Second, when \( m - 1 \) is admissible, then it is a sink, and we let \( \beta_i \) be the smallest root with \( [1_{X^i}, g_{\beta_i}] \neq 0 \). Then
\[
V_i = \sum_j [k \cdot x^j, r_i]
\]
for any non-zero \( r_i \in g_{\beta_i} \), where \( x \) is the endomorphism induced by \( \alpha \alpha^* \) with \( \alpha \) the arrow from vertex \( m - 2 \) to vertex \( m - 1 \) (see Lemma 4.3).

In both cases,
\[
[\text{stab}_{q_{W,T}}(r), \sum_i r_i] = \bigoplus_i V_i = q_{S,T}^1.
\]
This proves that \( q_{S,T} \) has a Richardson element. \( \square \)

**Remark 5.4.**

a) The Richardson elements and their stabilisers can be explicitly constructed using results from [9], the proofs of Lemma 5.2 and Lemma 5.3.

The work of Baur [1] on parabolic Lie algebras is also needed in the case Lemma 5.2.

b) The method of Lemma 5.3 can be generalised to Lie algebras of exceptional types and therefore provide an explanation why Richardson elements do not exist for some seaweed Lie algebras of type \( E_8 \).

We end this paper with an example of constructing Richardson elements, using the method discussed in Lemma 5.3.
Example 5.5. Let \( g = so_{10} \), a Lie algebra of type \( D_5 \).

(1) Consider the seaweed Lie algebras \( q_{S,T} \) and \( q_{K,L} \) with \( T = \{2, 4, 5\} \), \( S = \{1, 3, 4\} \), \( L = \{2, 3, 4\} \) and \( K = \{1, 3, 5\} \). These two seaweeds are like those discussed in the proof of Lemma 5.3 and have the following shapes. The matrices are anti-symmetric to the anti-diagonal.

\[
\begin{pmatrix}
* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & 0 & * & 0 & 0 & 0 & 0 \\
* & * & * & 0 & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & * & 0 & * & 0 & 0 & 0 \\
0 & 0 & * & * & 0 & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & * & *
\end{pmatrix}
\]

(2) The seaweeds \( q_{S\setminus\{1\},T} \) and \( q_{K\setminus\{1\},L} \) are of type \( A \). They are isomorphic to \( \text{End}_A(P(c)) \) and \( \text{End}_A(P(d)) \) of the following quivers, respectively, where \( c = (1, 2, 1, 1) \) and \( d = (1, 1, 2, 1) \),

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\]

In both cases, the vertex \( m \) in the proof of Lemma 5.3 is 4. The vertex 3 (= \( m - 1 \)) is a sink for \( q_{S\setminus\{1\},T} \) and is non-admissible for \( q_{K\setminus\{1\},L} \). We have

\[
q_{S,T}^1 = V_1 \oplus V_2 \quad \text{and} \quad q_{K,L}^1 = W_1 \oplus W_2
\]

with

\[
V_1 = g_{\alpha_1 + \alpha_3 + \alpha_4}, \quad V_2 = g_{\alpha_1} \oplus g_{\alpha_3}, \quad W_1 = g_{\alpha_1 + \alpha_3} \quad \text{and} \quad W_2 = g_{\alpha_1}.
\]

(3) The Richardson elements of \( q_{S\setminus\{1\},T} \) and \( q_{K\setminus\{1\},L} \) are as below. Entries of the same colour come from the same indecomposable summand. Note that for \( X(d) \), one of the indecomposable summands is a Verma module, so both the 4th column and row are zero.

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

(4) The stabilizers of the two Richardson elements are as follows.
There are two indecomposable direct summands in each of $X(a)$ and $X(d)$. The different colours indicate homomorphisms between different pairs of summands.

(5) Denote the two Richardson element in (3) by $r_{S \setminus \{1\}, T}$ and $r_{K \setminus \{1\}, L}$. The action of $\text{stab}_{S \setminus \{1\}, T}(r_{S \setminus \{1\}, T})$ on $q_{S \setminus \{1\}, T}^1$ is equivalent to the natural action of

$$\begin{pmatrix}
a & 0 & g \\
0 & c & d \\
a & 0 & 0
\end{pmatrix},$$

on $k^3$, although in the proof of Lemma 5.3 we only use the action of

$$\begin{pmatrix}
a & 0 & 0 \\
0 & c & d \\
0 & 0 & 0
\end{pmatrix}$$

with $r' = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \in q_{S,T}^1$. The action of $\text{stab}_{K \setminus \{1\}, L}(r_{K \setminus \{1\}, L})$ on $q_{K \setminus \{1\}, L}^1$ is equivalent to the natural action of $\begin{pmatrix} a & 0 \\ 0 & e \end{pmatrix}$ on $k^2$, and $r' = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in q_{K,L}^1$. So we have the following Richardson elements for the two seaweed $q_{S,T}$ and $q_{K,L}$,

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

6. A COUNTEREXAMPLE

In our joint work with Yu [9], we gave a counterexample to the existence of dense orbits for a seaweed of type $E_8$, but the detail of the computation did not appear in the published version [9]. In this section, we add explanations, including a GAP source file.

Recall that if $r$ is a Richardson element for a seaweed $q_{S,T}$, then

$$r = r^+ + r^-$$

with

$$r^+ \in n^+_{S,T}, r^- \in n^-_{S,T}, [q_{S,T}, r^+] = n^+_{S,T} \text{ and } [q_{S,T}, r^-] = n^-_{S,T}.$$
Lemma 6.1. Let \( r^- \in n^-_{S,T} \) with \([q_{S,T}, r^-] = n^-_{S,T} \). Then \( r^- + r^+ \) for \( r^+ \in n^+_{S,T} \) is a Richardson element if and only if \([\text{stab}_{q_{S,T}}(r^-), r^+] = n^+_{S,T} \).

Proof. Assume that \( r^- + r^+ \) is a Richardson element. Then for any \( y^+ \in n^+_{S,T} \), there exists a \( q \in q_{S,T} \) such that \([q, r^- + r^+] = y^+ \). As \([q, r^-] \in n^-_{S,T} \) and \([q, r^+] \in n^+_{S,T} \), we have
\[
[q, r^-] = 0 \text{ and } [q, r^+] = y^+.
\]
So
\[
[\text{stab}_{q_{S,T}}(r^-), r^+] = y^+.
\]
For the converse, assume that \( r^+ \in n^+_{S,T} \) such that
\[
[\text{stab}_{q_{S,T}}(r^-), r^+] = n^+_{S,T}.
\]
Let \( y = y^- + y^+ \in n_{S,T} \) with \( y^+ \in n^+_{S,T} \) and \( y^- \in n^-_{S,T} \). By our assumption, there exist \( q \in q_{S,T} \) and \( q' \in \text{stab}_{q_{S,T}}(r^-) \) such that
\[
[q, r^-] = y^- \text{ and } [q', r^+] = -[q, r^+] + y^+.
\]
Then
\[
[q + q', r^- + r^+] = y^- + 0 + [q, r^+] - [q, r^+] + y^+ = y.
\]
Therefore \( r^- + r^+ \) is a Richardson element. \( \square \)

Lemma 6.2. Suppose that \( r^- \in n_{S,T} \) with \([q_{S,T}, r^-] = n^-_{S,T} \) and \( q_{S,T} \) has a Richardson element. Then there exists \( r^+ \in n^+_{S,T} \) such that \( r^- + r^+ \) is a Richardson element.

Proof. Assume that \( x = x^- + x^+ \) is a Richardson element. Then there exists group element \( g \in Q_{S,T} \), where \( Q_{S,T} \) is a Lie group corresponding to \( q_{S,T} \) such that
\[
\text{Ad}_g(x^-) = r^-.
\]
Let \( r^+ = \text{Ad}_g(x^+) \in n^+_{S,T} \). Note that
\[
[\text{Ad}_g(q), \text{Ad}_g(x^-)] = 0
\]
if and only if
\[
[q, r^-] = 0.
\]
Therefore
\[
\text{Ad}_g(\text{stab}_{q_{S,T}}(x^-)) = \text{stab}_{q_{S,T}}(r^-).
\]
As \( \text{Ad}_g(n_{S,T}) = n_{S,T} \), we have
\[
n_{S,T} = \text{Ad}_g[q_{S,T}, x] = [q_{S,T}, r].
\]
Therefore \( r^- + r^- \) is a Richardson element. \( \square \)

The following example was briefly mentioned in [9].

Example 6.3 (Jensen-Su-Yu). Let \( g \) be the simple Lie algebra of type \( E_8 \). We use the following numbering of the Dynkin diagram.

```
 3
1 -- 2 -- 4 -- 5 -- 6 -- 7 -- 8
```
Let $S = \Pi \setminus \{ \alpha_4, \alpha_5 \}$ and $T = \Pi \setminus \{ \alpha_8 \}$. Then a direct computation shows that
\[
\dim q_{S,T} = 81, \dim n_{S,T} = 59 \text{ and } l_{S,T} = 22.
\]
Moreover,
\[
\dim n_{S,T}^- = 56
\]
and
\[
n_{S,T}^+ = \mathfrak{g}_{\alpha_6 + \alpha_7 + \alpha_8} \oplus \mathfrak{g}_{\alpha_7 + \alpha_8} \oplus \mathfrak{g}_{\alpha_8}
\]
is abelian and 3-dimensional.

We need an element $r^-$ with a dense open orbit in $n_{S,T}^-$, i.e. $[q_{S,T}, r^-] = n_{S,T}^-$. We choose
\[
r^- = \sum_{i=1}^{56} p_i \cdot v_i
\]
where $p_i$ is the $i$’th prime and $v_i$ is a basis element of the $i$’th root space of $n_{S,T}^-$. The numbering of root spaces follows that of GAP. Much of the calculation below is done, using GAP \[17\]. We have
\[
\dim \text{stab}_{q_{S,T}}(r^-) = 25
\]
and so
\[
\dim [q_{S,T}, r^-] = 81 - 25 = \dim n_{S,T}^-.
\]
Hence $[q_{S,T}, r^-] = n_{S,T}^-$. Next we produce a basis of the centralizer of $r^-$ and compute $[x, n_{S,T}^+]$ for all the basis elements $x$. We have
\begin{enumerate}
  \item $[x, -]_{n_{S,T}^+} = \text{Id}_{n_{S,T}}$ for exactly one basis element $x$.
  \item $[y, n_{S,T}] = [z, n_{S,T}] = \text{span}\{ v \}$ is 1-dimensional for exactly two basis elements $y, z$ and $v = 89 \cdot w_1 + 67 \cdot w_2 + 3 \cdot w_3$, where $w_i$ denotes the basis of the $i$’th root space in $n_{s,T}^+$ produced by GAP.
  \item $[-, n_{S,T}] = 0$ for the other basis elements.
\end{enumerate}
This shows that for any $r^+ \in n_{S,T}^+$, we will have
\[
[\text{stab}_{q_{S,T}}(r^-), r^+] \subseteq \text{span}\{ r^+, v \},
\]
which is at most 2-dimensional. So
\[
\dim [\text{stab}_{q_{S,T}}(r^-), r^+] < \dim n_{S,T}^+
\]
for any $r^+ \in n_{S,T}^+$. Thus $r^-$ cannot be completed to a Richardson element. Now by Lemma \[6.2\], we conclude that $q_{S,T}$ does not have a Richardson element.

Below we give the full source code in GAP \[17\] for the computation in the example.
\begin{verbatim}
g := SimpleLieAlgebra("E",8,Rationals);
PHI := RootSystem(g);
Geng := CanonicalGenerators(PHI);
PIp := Geng[1];
PIm := Geng[2];
h := Geng[3];

S := [1,2,3,6,7,8];
T := [1,2,3,4,5,6,7];
Genq := List(S, x->PIp[x]);
Append(Genq, List(T, x->PIm[x]));
Append(Genq, List([1..8], x->h[x]));
q := Subalgebra(g,Genq);
nm := Ideal(q, List (Difference(T,S), x->PIm[x]));
np := Ideal(q, List (Difference(S,T), x->PIp[x]));

Dimension(q);
Dimension(nm);
Dimension(np);
IsAbelian(np);
nmVec := BasisVectors(Basis(nm));
npVec := BasisVectors(Basis(np));

Rm := Sum(List( [1..Dimension(nm)], x->Primes[x]*nmVec[x]));
RmCent := LieCentralizer(q, Subalgebra(q, [Rm]));
Dimension(RmCent);

LO := LieObject;

idaction := function(x)
  local ret;
  ret := LO(x)*LO(npVec[1])=LO(npVec[1]);
  ret := ret and (LO(x)*LO(npVec[2])=LO(npVec[2]));
  ret := ret and (LO(x)*LO(npVec[3])=LO(npVec[3]));
  return ret;
end;

RmCentVec := BasisVectors(Basis(RmCent));
RmCentVec:=Difference(RmCentVec, Filtered(RmCentVec, x->idaction(x)));

v := List( RmCentVec, x -> LO(x)*LO(npVec[1]));
Append(v, List( RmCentVec, x -> LO(x)*LO(npVec[2])));
Append(v, List( RmCentVec, x -> LO(x)*LO(npVec[3])));
Apply(v,UnderlyingRingElement);

Dimension(Subalgebra(q,v));
\end{verbatim}
References

[1] Baur, K., Richardson elements for classical Lie algebras, Journal of Algebra 297 (2006), 168–185.
[2] Baur, K., Erdmann, K. and Parker, A., $\Delta$-filtered modules and nilpotent orbits of a parabolic subgroup in $O_N$, Journal of Pure and Applied Algebra 215 (2011), 885–901.
[3] Brüstle, T. and Hille, L., Matrices over upper triangular bimodules and $\Delta$-filtered modules over quasi-hereditary algebras, Colloquium Mathematicum 83 (2000), 295–303.
[4] Brüstle, T., Hille, L., Ringel, C. M. and Röhrl, G., The $\Delta$-filtered modules without self-extensions for the Auslander algebra of $k[T]/(T^n)$, Algebras and Representation Theory 2 (1999), 295–312.
[5] Dergachev, V. and Kirillov, A. Index of Lie algebras of seaweed type, Journal of Lie Theory 10 (2000), 331–343.
[6] Duflo, M. and Yu, R. W. T., On compositions associated to Frobenius parabolic and seaweed subalgebras of $\mathfrak{sl}_n(k)$, Journal of Lie Theory 25 (2015), 191–1213.
[7] Hille, L. and Vossieck, D., The quasi-hereditary algebra associated to the radical bimodule over a hereditary algebra. Colloquium Mathematicum 98 (2003), no. 2, 201–211.
[8] Jensen, B. T. and Su, X., Adjoint action of automorphism groups on radical endomorphisms, generic equivalence and Dynkin quivers, Algebras and Representation Theory, 17 (4), 1095–1136.
[9] Jensen, B. T., Su, X. and Yu, W. T. R., Rigid representations of a double quiver of type $\mathbb{A}$, and Richardson elements in seaweed Lie algebras, Bulletin of the London Mathematics Society, 42 (2009), 1–15.
[10] Joseph, A., On semi-invariants and index for biparabolic (seaweed) algebras I, Journal of Algebra 305 (2006), 487–515.
[11] Joseph, A., Slices for biparabolic coadjoint actions in type $A$, Journal of Algebra 319 (2008), no. 12, 5060–5100.
[12] Joseph, A., An algebraic slice in the coadjoint space of the Borel and the Coxeter element, Advances in Mathematics 227 (2011), no. 1, 522–585.
[13] Panyshev, D., Inductive formulas for the index of seaweed Lie algebras, Moscow Mathematical Journal (2001), 221–241.
[14] Richardson, R. W., Conjugacy classes in parabolic subgroups of semisimple algebraic groups, Bulletin of the London Mathematics Society (1974), 21–24.
[15] Tauvel, P., and Yu, W. T. R., Lie algebras and algebraic groups, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005. xvi+653 pp.
[16] Tauvel, P., and Yu, W. T. R., Affine slice for the coadjoint action of a class of biparabolic subalgebras of a semisimple Lie algebra, Algebras and Representation Theory 16 (2013), 859–872.
[17] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.8.7; 2017. [http://www.gap-system.org]

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