Geometry of the Field-Moment Spaces for Quadratic Bosonic Systems: Diabolically Degenerated Exceptional Points on Complex $k$-Polytopes

Ievgen I. Arkhipov,$^{1,*}$ Adam Miranowicz,$^{2,3}$ Franco Nori,$^{4,5}$ Şahin K. Özdemir,$^{6}$ and Fabrizio Minganti$^{7,8}$

$^1$Joint Laboratory of Optics of Palacký University and Institute of Physics of CAS, Faculty of Science, Palacký University, 17. listopadu 12, 771 46 Olomouc, Czech Republic
$^2$Theoretical Quantum Physics Laboratory, RIKEN Cluster for Pioneering Research, Wako-shi, Saitama 351-0198, Japan
$^3$Institute of Spintronics and Quantum Information, Faculty of Physics, Adam Mickiewicz University, 61-614 Poznań, Poland
$^4$Theoretical Quantum Physics Laboratory, Cluster for Pioneering Research, RIKEN, Wako-shi, Saitama 351-0198, Japan
$^5$Physics Department, The University of Michigan, Ann Arbor, Michigan 48109-1040, USA
$^6$Department of Engineering Science and Mechanics, and Materials Research Institute (MRI), The Pennsylvania State University, University Park, Pennsylvania 16802, USA
$^7$Institute of Physics, Ecole Polytechnique Fédérale de Lausanne (EPFL), CH-1015 Lausanne, Switzerland
$^8$Center for Quantum Science and Engineering, Ecole Polytechnique Fédérale de Lausanne (EPFL), CH-1015 Lausanne, Switzerland

(Dated: October 28, 2022)

$k$-Polytopes are a generalization of polyhedra in $k$ dimensions. Here, we show that complex $k$-polytopes naturally emerge in the higher-order field moments spaces of quadratic bosonic systems, thus revealing their geometric character. In particular, a complex-valued evolution matrix, governing the dynamics of $k$th-order field moments of a bosonic dimer, can describe a complex $k$-dimensional hypercube. The existence of such $k$-polytopes is accompanied by the presence of high-order diabolic points (DPs). Interestingly, when the field-moment space additionally exhibits exceptional points (EPs), the formation of $k$-polytopes may lead to the emergence of diabolically degenerated EPs, due to the interplay between DPs and EPs. Such intriguing spectral properties of complex polytopes may enable constructing photonic lattice systems with similar spectral features in real space. Our results can be exploited in various quantum protocols based on EPs, paving a new direction of research in this field.

I. INTRODUCTION

Hermiticity is not a necessary requirement for Hamiltonians to exhibit real spectra [1]. Non-Hermitian Hamiltonians (NHHs), describing open systems with loss and gain, can also exhibit real spectra if they satisfy parity-time ($\mathcal{PT}$) symmetry [2–5]. Realization of this principle has triggered a very active theoretical and experimental research in various areas of classical and quantum physics. Note that NHHs have been already used since the early 1990s as a key concept in, e.g., the Monte Carlo wave-function (or quantum trajectory) formalisms [6–8].

In addition to their exceptional point (EP) degeneracies, characterized by the coalescence of both the eigenvalues and the corresponding eigenvectors, NHHs can also exhibit diabolic point (DP) degeneracies, which are typically associated with Hermitian Hamiltonians and are characterized by coalescence of eigenvalues only, thus, with eigenmodes remaining different. Both the EPs and DPs possess remarkable features such as chiral transport and acquired Berry phase, which can be revealed by encircling such points in the system parameter space [9–13]. A deeper understanding of the emergence of EPs and DPs, and their physical consequences, are required, e.g., to find and classify new types of phase transitions, to achieve a better control of state switching and Berry’s phase around them, to explore new effects, and, in a longer term, to develop new practical devices for quantum technologies.

Here we focus on the geometry and spectral properties of bosonic quadratic Hamiltonians, expressed in their field-moment (FM) spaces. Such Hamiltonians play a fundamental role in: atom optics [14], quantum optics (see, e.g., [15–19]), quantum information processing [20], quantum metrology [21, 22], and quantum field theory [23]. In special cases, they can describe linear and nonlinear wave mixing, quantum squeezing generation, parametric down and up conversion, the Casimir effect, dynamical phase transitions [24–26], and even black-hole radiation, among various other phenomena. Despite of their seemingly simple mathematical form, such Hamiltonians have surprisingly rich and nontrivial mathematical and physical properties in infinite dimensions. In particular, these Hamiltonians can exhibit pathologies of a particular interest in quantum field theory [27]; in some special cases, e.g., an infinite renormalization is required to compute their eigenvalues [28].

In this article, we reveal the coexistence and merging of DPs and EPs in quadratic bosonic systems. Such a nontrivial effect naturally arises in the synthetic space spanned by high-order FMs. We show that, geometrically, the FMs space of quadratic bosonic systems can be described by a chain of $k$-polytopes, which are a generalization of polyhedra in $k$ dimensions [29]. Namely, one can associate the evolution matrices, governing higher-order FMs, with a certain series of $k$-polytopes. This...
polytope geometry makes the FMs eigenspace simply formed by tensor product states (TPS), comprised by the first-order moment eigenstates. In particular, the eigenspace of such emergent \( k \)-polytopes reveals a non-trivial interplay between EPs and DPs, forming diabolically degenerated exceptional points (DDEPs).

The formation of complex polytopes with exotic DDEPs in the synthetic FMs space of quadratic systems can also be inspiring for constructing similar structures in real space. This implies that our predictions can be tested in various of real-space configurations, ranging from optomechanical resonators and Bose-Einstein condensates to superconducting circuits, by simply mapping the FMs evolution matrices to real-space NHHs. Our work, thus, paves the road to experimental observations of intriguing effects at EPs, and can prompt further research to explore novel phenomena arising from the existence of DDEPs in the spectrum of multimode NHHs.

II. EVOLUTION MATRICES GOVERNING HIGHER-ORDER FIELD MOMENTS

We first study the composition of the evolution matrices governing the dynamics of higher-order operator moments of any quadratic Markovian system. We will reveal that an \( m \)-series of evolution matrices, governing \( t \)-\( i \)-order FMs (\( i = 1, \ldots, m \)), can be associated with a corresponding \( m \)-chain of \( k_i \)-polytopes. Therefore, the corresponding higher-order FM eigenspace becomes simply comprised by the TPS of the first-order moment eigenstates.

We start by considering a general nonlinear \( N \)-mode-coupled quadratic bosonic system, which is not necessarily Hermitian. The bosonic quadratic system can be described by a Nambu operator vector \( \hat{\Psi} = \left[ a_1, a_2, \ldots, a_N, a_1^\dagger, a_2^\dagger, \ldots, a_N^\dagger \right]^T \), where \( a_k \) (\( a_k^\dagger \)) is the annihilation (creation) operator of a mode \( k \), obeying \( [a_k, a_l] = \delta_{kl} \) and \( [a_k, a_l^\dagger] = 0 \). The quadratic (non-)Hermitian Hamiltonian, which determines the dynamics of the \( N \)-mode system takes a general form \( H = \sum H_{mn} \hat{\Psi}_m^\dagger \hat{\Psi}_n \), with \( \hat{\Psi}_j \) being the \( j \)th element of the Nambu vector \( \hat{\Psi} \). From the Heisenberg equations of motions, one can easily write down the equations for the dynamics of the first-order field moments \( \langle \hat{\Psi} \rangle \) as
\[
\frac{d}{dt} \langle \hat{\Psi} \rangle = M_1 \langle \hat{\Psi} \rangle,
\]
where \( M_1 \) is the corresponding evolution matrix for the first-order FMs. Note that the same procedure can be straightforwardly extended to Hermitian operators as well, e.g., to bosonic quadrature operators.

A remarkable feature of quadratic systems is that by knowing the form of the evolution matrix \( M_1 \) for the first-order FMs, one can immediately obtain an analytical form of the evolution matrix governing the dynamics of any higher-order FMs [30]. This can be done by exploiting properties of matrices formed by Kronecker sums

![FIG. 1. Chains of \( k_i \)-polytopes in the FMs space of bosonic systems: (a) dimers and (b) trimers, formed by the Cartesian powers of 1-polytope (line) and 2-polytope (triangle), respectively. (a) 1-polytope (a line) for the first-order FMs of the dimer; 2-polytope (a square) for its second-order FMs; 3-polytope (a cube) for its third-order FMs, and 4-polytope (a hypercube, shown via its 3D Schlegel diagram) for its fourth-order FMs. (b) 2-polytope (a triangle) for the first-order FMs of the trimer, and 4-polytope (a four-dimensional duoprism, shown via its 3D Schlegel diagram) for its second-order FMs. Numbered vertices correspond to the diagonal elements of the respective evolution matrices \( M_i \), and the weights of the edges correspond to their off-diagonal elements.](image-url)
FIG. 2. Emergent tensor-product-structure chain of the eigenvectors of the evolution matrices $M_m$ (with $m = 0, 1, 2, 3$) for a quadratic dimer, according to Eq. (2). The index $m = 0$ corresponds to the identity element of the chain.

of vertices and edges are prescribed, expectively, by the diagonal and off-diagonal elements of the corresponding matrix $M_1$.

For instance, in case of a bosonic dimer, an evolution matrix $M_1$ describes a one-dimensional polytope $P_1$, i.e., a line, formed by two vertices and one edge, provided $M_1$ is symmetric (see Fig. 1). A 2-polytope $P_2$, corresponding to $M_2$, is then a square. As a result, one obtains an $m$-hypercube, describing the $m$th-order FM space, whose associative matrix is $M_m$ (see Fig. 1). This polytope geometry with its spectral properties can, thus, be applied to real-space systems.

IV. TENSOR PRODUCT STATES AND THEIR DEGENERACY IN HIGHER-ORDER OPERATOR-MOMENT EIGENSACES

Here, we briefly discuss the eigenspace structure of the evolution matrices $M_m$ associated with $k$-polytopes. Equation (1) implies that one can determine a complete eigendecomposition of the matrix $M_m$, governing $m$th-order FMs, knowing the eigenvalues and eigenvectors of the matrix $M_1$, and, therefore, the corresponding $k_m$-polytope.

The eigenvectors of the matrix $M_m$ are found via all $(2N)^m$ combinations of the tensor products of the eigenvectors $\psi_j^{(1)}$, $j = 1, \ldots, 2N$ of the matrix $M_1$ (see Fig. 2). The $m$th-order moments eigenspace is, thus, spanned by $(2N)^m$ eigenvectors of the form

$$\psi_{i_1,i_2,\ldots,i_m}^{(m)} = \psi_{i_1}^{(1)} \otimes \psi_{i_2}^{(1)} \otimes \cdots \otimes \psi_{i_m}^{(1)},$$

where each index $i_k = 1, \ldots, 2N$, for each $k = 1, \ldots, m$. The eigenvector $\psi_{i_1,i_2,\ldots,i_m}^{(m)}$ corresponds to the eigenvalue

$$\lambda_{i_1,i_2,\ldots,i_m}^{(m)} = \sum_{k=1}^{m} \lambda_k^{(1)},$$

where $\lambda_k^{(1)}$ are eigenvalues of the matrix $M_1$.

The diabolic degeneracy $D$ of a given eigenvalue $\lambda_{i_1,i_2,\ldots,i_m}^{(m)}$ in Eq. (3) equals $D(\lambda_{i_1,i_2,\ldots,i_m}^{(m)}) = m! / \left[ n_{i_1}! n_{i_2}! \cdots n_{i_m}! \right]$, where $n_{i_k}$ denotes the number of times the index $i_k$ appears in both Eqs. (2) and (3). For example, in the case of second-order moments, the moments vector $\langle \hat{\Psi} \otimes \hat{\Psi} \rangle$ possesses moments $\langle \hat{a}_j \hat{a}_j \rangle$ and $\langle \hat{a}_k \hat{a}_j \rangle$, as well as $\langle \hat{a}_j \hat{a}_j \rangle$ and $\langle \hat{a}_j^\dagger \hat{a}_j \rangle$, which result in a degeneracy. In total, there are $(2N)^m - S_m(2N)$ degenerate eigenvalues of the system, where $S_m(2N)$ denotes the effective (i.e., non-degenerate) dimension of the matrix $M_m$. The latter can be calculated via the formula for combination with repetitions, i.e., $S_m(2N) = (2N + m - 1)! / [m!(2N - 1)!]$. Methods, for calculating eigenvectors of the effectively reduced evolution matrices, can be found in [32] and in Appendices B and C.

V. DIABOLICALLY DEGENERATED EXCEPTIONAL POINTS ON COMPLEX $k$-POLYTOPES

The aforementioned degeneracy, originating in a given FM polytope eigenspace, can lead to a nontrivial interplay between DPs and EPs, resulting in the formation of DDEPs. Because of this, high-order FM space can possess, in general, two types of EPs. Namely, non-degenerate EPs (NDEPs) and DDEPs. Compared to the former, the DDEPs can only exist in the degenerate eigenspace of the moments of TPS. Once such a diabolic degeneracy is removed, e.g., by the effective reduction of the FM space [30], there can only remain NDEPs in the FM dynamics.

VI. EXAMPLE: QUANTUM PARAMETRIC SUBHARMONIC GENERATION PROCESSES

To explicitly illustrate the existence of polytopes and the TPS structure with two types of EPs in the synthetic space of quadratic systems we now consider the FM dynamics (up to third order) for parametric subharmonic generation processes. Previous works [24, 33, 34] have already revealed the existence of EPs in such non-dissipative quadratic Hamiltonians. Here, we show that such non-Hermitian systems can possess additional non-trivial spectral features in the FM space, i.e., DDEPs. Let us first start from a quadratic Hermitian Hamiltonian $\hat{H}$ describing a second subharmonic generation with a classical pump, and working in the reference frame rotating at the pump frequency $\omega_p$: $\hat{H} = \Delta \hat{a}^\dagger \hat{a} + (ig/2) (\hat{a}^2 - \hat{a}^2)$, where $\Delta = \omega_p - \omega$ is the resonance detuning, i.e., the difference between the frequencies of the pump ($\omega_p$) and quantum field ($\omega$), the parameter $g$ is assumed to be a real-valued coupling constant, which involves the amplitude of the pump field [35].

The dynamics of the first-order moments of the Nambu operator vector $\hat{\Psi} = [\hat{a}, \hat{a}^\dagger]^T$ is $\frac{d}{dt} \langle \hat{\Psi} \rangle = M_1 \langle \hat{\Psi} \rangle$, where the evolution matrix $M_1 = \begin{pmatrix} -g & i\Delta \\ -i\Delta & -g \end{pmatrix}$ is $\mathcal{PT}$-symmetric [3, 36]. $M_1$ is invariant under action
$P$TM$_1$(P$T$)$^{-1}$, of the parity $P$ and time-reversal ($T$) operators, which action is defined as $P$TM = $\sigma_z K$, where the operator $K$ accounts for the complex conjugate operation. The symmetric matrix $M_1$ describes a complex 1-polytope, shown in Fig. 1(a), with two vertices having complex weights $\pm i\Delta$ and a single edge with weight $-g$. The eigenvalues of $M_1$ are $\lambda_{1,2} = \pm \Lambda$, where $\Lambda = \sqrt{g^2 - \Delta^2}$. The corresponding first-order moment eigenvectors become

$$\psi_{1,2} = \left(\mp \exp(\mp i\phi)\right)/1, \quad \phi = \arctan(\Delta/\Lambda),$$

which satisfy biorthogonality [37] (see also Appendix E). Both the eigenvalues and eigenvectors coalesce at the second-order EP, defined by the condition: $g_{EP} = \Delta$, with the phase $\phi_{EP} = \pi(2k + 1)/2$, $k \in \mathbb{Z}$, in Eq. (4).

According to Eq. (1), the evolution matrix $M_2$, governing the second-order FM vector $\langle \hat{\Psi} \otimes \hat{\Psi} \rangle = [(\hat{a}^2), (\hat{a}\hat{a}^\dagger), (\hat{a}^\dagger\hat{a})]^T$, and describing a 2-polytope in the from of a square [see Fig. 1(a)] reads

$$M_2 = \begin{pmatrix} -2i\Delta & -g & -g & 0 \\ -g & 0 & 0 & -g \\ -g & 0 & 0 & -g \\ 0 & -g & -g & 2i\Delta \end{pmatrix}.$$ 

(5)

Eigenvalues of $M_2$ are found as $\lambda^{(2)}_{j,k} = 2\Delta \text{diag}[1,-1]$, where all four eigenvalues are listed in the matrix form, with $j, k = 1, 2$.

The eigenvectors of $M_2$ are then found via all $2^2 = 4$ combinations of tensor products of the eigenvectors of $\psi_{1,2}$ in Eq. (4), according to Eq. (2): $\psi^{(2)}_{i_1,i_2} = \psi_{i_1} \otimes \psi_{i_2}$, where each index $i_k = 1, 2$ for each $k = 1, 2$. The matrix $M_2$ possesses only a single NDEP of third-order, i.e., there is no DDEP in the second-order FM space.

More interesting features emerge in the space of third-order FMs, which is associated with a complex 3-dimensional cube [see Fig. 1(a)]. The matrix $M_3$ can, thus, be cast to a certain NHH describing an eight-mode system. The corresponding $8 \times 8$ evolution matrix $M_3$, describing the dynamics of the vector of moments $\langle \otimes \hat{\Psi} \rangle$ is found according to Eq. (1) (see Appendix D).

The eigenvalues of $M_3$ are: $\lambda_{111,222} = \pm 3\Lambda$, $\lambda_{112} = \lambda_{121} = \lambda_{211} = \Lambda$, $\lambda_{122} = \lambda_{212} = \lambda_{221} = -\Lambda$. The corresponding eigenvectors are found accordingly to Eq. (3) (see Appendix D). The effective dimension of $M_3$ is four $[S_3(2) = 4]$, and, therefore, there is a single NDEP of the fourth order. However, apart from the NDEP, also one DDEP of second order emerges, due to the triply degenerated pairs of eigenvalues. In order to resolve the presence of such a DDEP in the degenerate eigenspace of $M_3$, one can perturb the matrix $M_3$, as $M_3 \rightarrow M_3 + \epsilon P$, where $\epsilon$ is the perturbation strength (see also Appendix D for details). The effect of such a perturbation on the resolution between the NDEP and DDEP is shown in Fig. 3. One can see that the DDEP splits into two second-order EPs under the perturbation.

Importantly, the revealed spectral features of complex polytopes, which emerge in the space of quadratic systems, can also be exploited in the construction of similar NHHs in real space. Such NHHs can describe, e.g., photonic lattices [38]. For instance, one of the nontrivial outcomes of the presence of a DDEP in the spectrum of real multimode NHHs can be the implementation of a programmable multimode switch by dynamically encircling the DDEP [39]. A direct detection of DDEPs in the FMs space of a bosonic system can be performed by measuring bosonic commutators [40] and/or anomalous FMs [41]. The TPS structure of a FM space can also be recovered from $s$-ordered FMs, which can be obtained from the measured normally-ordered moments using standard photon-detection schemes [35, 42, 43]. Note that our conclusions remain valid also for the quadrature field moments, which could be easier to access experimentally, e.g., via standard balanced homodyne detection [24, 44].

VII. CONCLUSIONS

In this work, we have revealed the geometry of the high-order FM space for bosonic quadratic systems which can be described by a chain of $k$-polytopes. The eigenspace of these polytopes is simply formed by the TPS formed by the first-order FM eigenstates. The emergent polytope and TPS structures in the FM eigenspace can lead to the existence of DDEPs. Our results, thus, reveal the nontrivial nature of the spectral degeneracy in quadratic bosonic systems, which can be exploited in various classical and quantum protocols based on EPs or
DPs, and can ignite further research in this field.

ACKNOWLEDGMENTS

I.A. acknowledges funding by the Ministry of Education, Youth and Sports of the Czech Republic Project no. CZ.02.1.01/0.0/0.0/16_019/0000754. A.M. is supported by the Polish National Science Centre (NCN) under the Maestro Grant No. DEC-2019/34/A/ST2/00081. F.N. is supported in part by: Nippon Telegraph and Telephone Corporation (NTT) Research, the Japan Science and Technology Agency (JST) [via the Quantum Leap Flagship Program (Q-LEAP), and the Moonshot R&D Grant Number JPMJMS2061], the Japan Society for the Promotion of Science (JSPS) [via the Grants-in-Aid for Scientific Research (KAKENHI) Grant No. JP20H00134], the Army Research Office (ARO) (Grant No. W911NF-18-1-0358), the Asian Office of Aerospace Research and Development (AOARD) (via Grant No. FA2386-20-1-4069), and the Foundational Questions Institute Fund (FQXi) via Grant No. FQXi-IAF19-06. S.K.O. acknowledges support from Air Force Office of Scientific Research (AFOSR) Multidisciplinary University Research Initiative (MURI) Award on Programmable systems with non-Hermitian quantum dynamics (Award No. FA9550-21-1-0202).

Appendix A: Tensor product states in the higher-order field moments dynamics of quadratic systems: Kronecker products and sums

In this section we describe in detail the construction of the evolution matrices for higher-order field moments based on the Kronecker sum operations. For completeness, we repeat some of the lines from in the main text of the Article.

The bosonic quadratic system can be described by a Nambu operator vector \( \hat{\Psi} = \left[ \hat{\Psi}_1, \hat{\Psi}_2, \ldots, \hat{\Psi}_N, \right] \), where \( \hat{\Psi}_k \) is the annihilation (creation) operator of a mode \( k \), obeying: \( \left[ \hat{\Psi}_k, \hat{\Psi}_l^\dagger \right] = \delta_{kl} \) and \( \left[ \hat{\Psi}_k, \hat{\Psi}_l \right] = 0 \). The quadratic (non-)Hermitian Hamiltonian, which determines the dynamics of an \( N \)-mode system takes the general form

\[
\hat{H} = \sum_{mn} H_{mn} \hat{\Psi}_m^\dagger \hat{\Psi}_n, \tag{A1}
\]

with \( \hat{\Psi}_j \) being the \( j \)th element of the Nambu vector \( \hat{\Psi} \).

From the Heisenberg equations of motions, one can easily write down the equations for the dynamics of first-order field moments \( \langle \hat{\Psi} \rangle \) as

\[
\frac{d}{dt} \langle \hat{\Psi} \rangle = M_1 \langle \hat{\Psi} \rangle, \tag{A2}
\]

where \( M_1 \) is the corresponding evolution matrix for the first-order field moments.

For open systems, the benefit of the moments space (compared to the operators space) is that it allows to discard quantum noise for odd-order field moments, which is presented via Langevin forces in the Heisenberg equations of motion \cite{15, 16, 35}. This is always true for linear systems, when dealing with moments comprised by only annihilation or creation operators. However, for general Markovian systems, there exists a noise vector, stemming from quantum fluctuations \cite{45}, thus, rendering the equation of motion for higher-order field moments inhomogeneous. Nevertheless, it is natural to assume that quantum noise has no effect of the very spectrum of the FM dynamics.

One of the remarkable features of quadratic systems, is that one can obtain the analytical form of an evolution matrix ruling the dynamics of any higher-order field moments from the form of the evolution matrix \( M_1 \) for the first-order field moments in Eq. (A2). This can be done by exploiting some properties of matrices formed by Kronecker sums. Below we summarize the most important spectral features of Kronecker sum matrices in the following theorem.

**Theorem.** A square complex matrix \( C \in \mathbb{C}^{(m+n) \times (m+n)} \), which is obtained as a Kronecker sum of two square complex matrices \( A \in \mathbb{C}^{m \times m} \) and \( B \in \mathbb{C}^{n \times n} \), i.e., \( C = A \oplus B = A \otimes I_B + I_A \otimes B \), has the eigenvalues, which are sums of the eigenvalues of \( A \) and \( B \), and the corresponding right eigenvectors are the tensor products of the right eigenvectors of \( A \) and \( B \), i.e.,

\[
\lambda(C) = \lambda(A) + \lambda(B), \quad \psi_C = \psi_A \otimes \psi_B. \tag{A3}
\]

**Proof.** The proof is straightforward. We start from the eigenvalue-eigenvector equation for the matrix \( C \), by feeding into the equation the right eigenvector, which is a tensor product of two eigenvectors \( \psi_A \) and \( \psi_B \), with eigenvalues \( \lambda(A) \) and \( \lambda(B) \), respectively. Namely,

\[
C(\psi_A \otimes \psi_B) = (A \oplus B)(\psi_A \otimes \psi_B) = (A \otimes I_B)(\psi_A \otimes \psi_B),
\]

\[
= (A \psi_A \otimes I_B \psi_B) + (I_A \otimes B \psi_B) = (\lambda_A \psi_A \otimes \psi_B),
\]

\[
= (\lambda_A \psi_A \otimes \psi_B) + (\psi_A \otimes B \lambda_B \psi_B) = (\lambda_A + \lambda_B) \psi_A \otimes \psi_B, \tag{A4}
\]

where we used the tensor and dot product properties of matrices and vectors. In other words, the eigenvector of the matrix \( C = A \oplus B \), corresponding to the eigenvalue \( \lambda(C) \), is indeed just the tensor product of the two eigenvectors of the matrices \( A \) and \( B \) with eigenvalues \( \lambda(A) \) and \( \lambda(B) \).

According to Eq. (A4), the matrix, whose eigenvectors are formed by the tensor products of eigenvectors of two matrices can be utilized in the construction of the evolution matrices of any higher order. To reveal how this works in practice, let us consider the second-order field moments. The various combinations of second-order field moments are obtained from the tensor product of the Nambu vector on itself, i.e., from the \( 4N^2 \) dimensional...
vector $\langle \hat{\Psi} \otimes \Psi \rangle$. According to Eq. (A4), the evolution matrix $M_2$, governing the vector of the second-order field moments

$$\frac{d}{dt}\langle \hat{\Psi} \otimes \Psi \rangle = M_2\langle \hat{\Psi} \otimes \hat{\Psi} \rangle,$$  

attains the form

$$M_2 = M_1 \otimes M_1 = M_1 \otimes I_{2N} + I_{2N} \otimes M_1.$$  

The form of the evolution matrix $M_2$ coincides, as it should, with that derived from the Lyapunov equation for the covariance matrix, when the latter is presented as a column vector [46]. This procedure can, thus, be iteratively continued to any $n$th order field moment vectors \[\langle \hat{\Psi} \otimes \Psi \rangle\], thus obtaining the Eq. (1) in the main text.

**Appendix B: Reduction of the TPS structure and diabolic degeneracy in the field-moment space of bosonic quadratic systems**

In this section we discuss the effective elimination of diabolic degeneracy in the FM space.

The described degeneracy of the eigenspace formed by TPS in the field-moment space occurs due to the non-commutative nature of the Kronecker sum operation, when constructing evolution matrices for any higher-order field moments. Evidently, this degeneracy can be removed by properly collecting the field moment elements in the FM vector. This also means that the corresponding eigenstates are also reduced to effective eigenstates which are no longer the TPS. The $(2N)^m \times (2N)^m$-dimensional eigenspace of the matrix $M_m$ can be reduced to the size $S_m(2N) \times S_m(2N)$ with $S_m(2N) < (2N)^m$, where

$$S_m(2N) = \binom{2N + m - 1}{m, 2N - 1} = \frac{(2N + m - 1)!}{m!(2N - 1)!}. \quad (B1)$$

In particular, for linear dimers, the initial $N^m$-dimension of the eigenspace of $m$-order field moments can always be reduced to $(m + 1)$, as expected [30] (note that for the linear case, the annihilation operators are decoupled from the creation operators, as such, a general $(2N)^m \times (2N)^m$ dimension of evolution matrices automatically reduces to $N^m \times N^m$). That is, the TPS structure of the eigenstates is always eliminated in that case. Indeed, for the classical FM, the two moments $\langle \hat{a}\hat{a}^\dagger \rangle$ and $\langle \hat{a}^\dagger\hat{a} \rangle$ refer to the same second-order classical field moment $|\alpha|^2$. As a result, these two moments can be combined into one, i.e., their average $(\langle \hat{a}\hat{a}^\dagger \rangle + \langle \hat{a}^\dagger\hat{a} \rangle)/2 = |\alpha|^2$.

This effective reduction of the initial degenerate eigenspace of higher-order field moments can be performed in the following way. First, the $D(\lambda)$ degenerate eigenvectors, in Eq. (2) of the main text, corresponding to the same eigenvalue $\lambda$, are merged into a symmetric superposition of the degenerate eigenmodes

$$\psi^{(m)}(\lambda) = \frac{1}{D(\lambda)} \sum_{i_1, i_2, \ldots, i_m} \psi^{(m)}_{i_1, i_2, \ldots, i_m}. \quad (B2)$$

As such, there remain $S_m(2N)$ non-degenerate eigenvectors of size $(2N)^m$. The subsequent reduction of the $(2N)^m$ elements of those vectors to $S_m(2N)$ elements is performed by merging all equivalent moment elements also into their average. In particular, for linear dimers, in the case of the second-order field moments, after the effective reduction of the four eigenvectors into three, the two elements $\langle \hat{a}\hat{a}^2 \rangle$ and $\langle \hat{a}^2\hat{a} \rangle$ of the four-dimensional moment of three-left eigenvectors are substituted by the single element $\frac{1}{2}(\langle \hat{a}\hat{a}^2 \rangle + \langle \hat{a}^2\hat{a} \rangle)$. That is, the moments $\langle \hat{a}\hat{a}^2 \rangle$ and $\langle \hat{a}^2\hat{a} \rangle$ are removed from the moment vectors, and are substituted by a single moment which is their symmetrical sum. This leads to the reduction of the four-dimensional eigenvectors to three-dimensional eigenvectors. This procedure ensures the reduction of all $(N)^m$ eigenvectors to the $S_m(N)$ effective ones. We elaborate in detail on such a procedure in Appendix C. Interestingly, when dealing with dimer systems, that reduction always leads to the formation of the effective evolution matrices with a Sylvester matrix shape [30, 32, 38]. The Sylvester matrix is a tridiagonal matrix, whose elements obey certain relations (see Ref. [32] for details). Only recently a formula for the eigendecomposition of such matrices has been proposed [32]. A such, our results offer an alternative solution to the eigendecomposition problem of Sylvester matrices and highlight their possible physical origin.

**Appendix C: Example of degeneracy reduction in the second-order field-moment space generated in parametric subharmonic processes**

In this section, we elaborate on the removal of the diabolic degeneracy for the case of fields generated in the parametric subharmonic processes.

The two-fold degenerate eigenvalue $\lambda_{12} = \lambda_{21} = 0$, below Eq. (5) in the main text, arises because of the redundant moment elements $\langle \hat{a}\hat{a}^2 \rangle$ and $\langle \hat{a}^2\hat{a} \rangle$ in the moments vector $\langle \hat{\Psi} \otimes \hat{\Psi} \rangle$. As such, the $(2^2 = 4)$-dimensional eigenspace of the second-order moments can be decreased to a $[2 + 1 = 3]$-dimensional one, according to Eq. (B1). Following the procedure, described at the end of Appendix B, the effective moments space can be obtained by substituting those two redundant elements by a single one, namely, by their symmetrical sum. That is, the second and third rows in the moments vector and evolution matrix are merged into one:

$$\langle \hat{\Psi} \otimes \hat{\Psi} \rangle \rightarrow \langle \hat{\Psi} \otimes \hat{\Psi} \rangle_{\text{eff}} = \begin{bmatrix} \langle \hat{a}^2 \rangle, \langle \hat{a}\hat{a}^2 \rangle, \langle \hat{a}^2\hat{a} \rangle \end{bmatrix}^T, \quad (C1)$$

where the symmetrically-ordering moment is defined as

$$\langle \hat{a}\hat{a}^2 \rangle_s = \frac{1}{2}(\langle \hat{a}\hat{a}^2 \rangle + \langle \hat{a}^2\hat{a} \rangle). \quad (C2)$$

This reduction directly corresponds to the classical limit of the fields, i.e., with the fields vector $\langle \hat{\Psi} \otimes \hat{\Psi} \rangle_{\text{eff}} = [\alpha^2, |\alpha|^2, \alpha^*\alpha]^T$. The corresponding effective evolution
matrix attains the following Sylvester form:

\[ M_2 \rightarrow M_2^{\text{eff}} = \begin{pmatrix} -2i\Delta & -2g & 0 \\ -g & 0 & -g \\ 0 & -2g & 2i\Delta \end{pmatrix}. \] (C3)

The corresponding reduction of the eigenvectors \( \psi^{(2)} \) is obtained similarly

\[ \psi^{(2)}_{\text{eff}} = \begin{pmatrix} e^{-2i\phi} & -1 & e^{2i\phi} \\ -e^{-i\phi} \sin(\phi) & e^{i\phi} \\ 1 & 1 & 1 \end{pmatrix}, \] (C4)

by combining together the second and third eigenvectors, due to the two-fold degeneracy of the eigenvalue \( \lambda = 0 \), and accompanied by merging everywhere the second and third elements of the initial eigenvectors in Eq. (C1). All the three eigenvectors in Eq. (C4) coalesce at the EP \( g_{\text{EP}} = \Delta \), to the singular vector \( \psi^{(2)}_{\text{EP}} = [-1, i, 1] \), implying that the order of the EP is three, and no other eigenstate exists, in drastic contrast with the genuine quantum regime.

---

Appendix D: Diabolic degeneracy in the eigenspace of the third-order field moments: Hybrid Diabolic-Exceptional Points

The vector of third-order field moments reads

\[ \left\langle \hat{\Psi} \right| \hat{a}^2 \right| \hat{a}^1 \rangle = \left[ \langle \hat{a}^3 \rangle, \langle \hat{a}^2 \rangle, \langle \hat{a}^1 \rangle, \langle \hat{a}^{12} \rangle, \langle \hat{a}^3 \rangle \right]^T. \] (D1)

The evolution matrix \( M_3 \) for the vector \( \langle \hat{\Psi} \rangle \) is obtained as follows

\[ M_3 = M_2 \oplus I_2 + I_4 \oplus M_1, \] (D2)

which results in

\[ M_3 = \begin{pmatrix} -3i\Delta & -g & -g & 0 & -g & 0 & 0 & 0 \\ -g & -i\Delta & 0 & -g & 0 & 0 & 0 & 0 \\ -g & 0 & -i\Delta & -g & 0 & 0 & 0 & 0 \\ 0 & -g & -g & i\Delta & 0 & 0 & 0 & -g \\ -g & 0 & 0 & 0 & -i\Delta & -g & 0 & 0 \\ 0 & -g & 0 & 0 & -g & i\Delta & 0 & -g \\ 0 & 0 & -g & 0 & -g & 0 & i\Delta & -g \\ 0 & 0 & 0 & -g & 0 & -g & 0 & 3i\Delta \end{pmatrix}. \] (D3)

Its eigenvalues \( \lambda_{ijk} \), according to Eq. (3), can be listed as

\[ \lambda^{(3)} = \begin{pmatrix} \lambda_{111} & \lambda_{112} & \lambda_{121} & \lambda_{122} & \lambda_{211} & \lambda_{212} & \lambda_{221} & \lambda_{222} \\ 3s & s & -s & s & -s & -s & -s & -3s \end{pmatrix}, \] (D4)

where \( s = \sqrt{g^2 - \Delta^2} \). The corresponding eigenvectors are easily found, according to Eq. (2):

\[ \psi^{(3)} = \begin{pmatrix} -e^{-3i\phi} & e^{-i\phi} & e^{-i\phi} & e^{i\phi} & e^{i\phi} & e^{-i\phi} & e^{-i\phi} & e^{i\phi} & e^{3i\phi} \\ e^{-2i\phi} & e^{-2i\phi} & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ e^{-2i\phi} & -1 & e^{-2i\phi} & 1 & 1 & e^{2i\phi} & e^{2i\phi} & e^{2i\phi} & 1 \\ -e^{-i\phi} & -e^{-i\phi} & -e^{-i\phi} & e^{i\phi} & e^{i\phi} & e^{i\phi} & e^{i\phi} & e^{i\phi} & e^{i\phi} \\ e^{-2i\phi} & -1 & -1 & e^{2i\phi} & e^{-2i\phi} & 1 & 1 & 1 & 1 \\ -e^{-i\phi} & -e^{-i\phi} & e^{i\phi} & e^{i\phi} & e^{i\phi} & e^{i\phi} & e^{i\phi} & e^{i\phi} & e^{i\phi} \\ -e^{-i\phi} & e^{i\phi} & e^{-i\phi} & e^{i\phi} & e^{i\phi} & e^{i\phi} & e^{i\phi} & e^{i\phi} & e^{i\phi} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \] (D5)

It is seen, that two eigenvalues \( \pm \sqrt{g^2 - \Delta^2} \) are triply degenerate. At the EP \( g = \Delta \), the Jordan form of \( M_3 \)
which implies the existence of one NDEP of fourth order, and one DDEP, which consists of two diabolically degenerate EPs of second-order.

In order to lift the degeneracy of DDEP, one can induce a specific perturbation to the matrix $M_3$, i.e., $M_3 \rightarrow M_3 + \epsilon P$, where $\epsilon$ denotes the perturbation strength. The matrix $P$ can read:

$$ P = \text{diag}[1, 0, 0, 1, 0, 0, 0, 0]. $$  \hspace{1cm} (D7)

The result of such a perturbation on the eigenvalues of $M_3$ is shown in Fig. 3 in the main text. Other choices of perturbation do not necessarily lead to the DDEP detection. Note that the perturbation is actually fictitious because of the symmetry protection of the evolution matrix $M_3$ in the FM space. Indeed, by unfolding the evolution matrices $M_3$ from the first-order moments $2 \times 2$ matrix $M_1$, it is impossible to attain such a perturbed matrix $M_3$. However, the formation of such DDEP in the spectrum can be checked by mapping the evolution matrix $M_3$ to a certain real-space photonic-lattice Hamiltonian.

**Appendix E: Biorthogonality of the eigenvectors**

In order to make the solution of the eigendecomposition of the studied complex matrices complete, we briefly mention the notion of the biorthogonality. Because of the complex nature of the evolution matrices $M_m \in \mathbb{C}$, the right eigenvectors $\psi^{(m)}_j$ in Eq. (2) are, in general, not orthogonal [37]. However, by introducing the notion of biorthogonality, i.e., when the inner product is defined by means of both left and right eigenvectors of a complex matrix, one may overcome the difficulty. The left eigenvectors of a complex matrix $M_m$ are defined as

$$ M_m^\dagger \xi^{(m)}_j = \lambda^{(m)}_j \xi^{(m)}_j. $$  \hspace{1cm} (E1)

And the biorthogonality condition reads

$$ \sum_p \xi_j^{(m)*}(p) \psi_k^{(m)}(p) = \delta_{jk}. $$  \hspace{1cm} (E2)

The biorthogonality condition in Eq. (E2) is necessary when determining the decomposition of arbitrary vector on right eigenvectors. However, in our study, we exclusively focus on the eigenspace spanned by the right eigenvectors of the studied quadratic systems.

---

[1] C. M. Bender and S. Boettcher, “Real spectra in non-Hermitian Hamiltonians having $PT$ symmetry,” Phys. Rev. Lett. 80, 5243 (1998).
[2] C. M. Bender, “Making sense of non-Hermitian Hamiltonians,” Rep. Prog. Phys. 70, 947–1018 (2007).
[3] R. El-Ganainy, K. G. Makris, M. Khajavikhan, Z. H. Musslimani, S. Rotter, and D. N. Christodoulides, “Non-Hermitian physics and $PT$ symmetry,” Nat. Phys. 14, 11 (2018).
[4] Ş. K. Özdemir, S. Rotter, F. Nori, and L. Yang, “Parity-time symmetry and exceptional points in photonics,” Nat. Mater. 18, 783 (2019).
[5] Y. Ashida, Z. Gong, and M. Ueda, “Non-Hermitian physics,” Adv. Phys. 69, 249–435 (2020).
[6] J. Dalibard, Y. Castin, and K. Mølmer, “Wave-function approach to dissipative processes in quantum optics,” Phys. Rev. Lett. 68, 580–583 (1992).
[7] H. J. Carmichael, “Quantum trajectory theory for cascaded open systems,” Phys. Rev. Lett. 70, 2273–2276 (1993).
[8] K. Mølmer, Y. Castin, and J. Dalibard, “Monte Carlo wave-function method in quantum optics,” J. Opt. Soc. Am. B 10, 524–538 (1993).
[9] J. Doppler, A. A. Mailybaev, J. Böhm, U. Kuhl, A. Girschik, F. Libisch, T. J. Milburn, P. Rabl, N. Moiseyev, and S. Rotter, “Dynamically encircling an exceptional point for asymmetric mode switching,” Nature 537, 76 (2016).
[10] W. Liu, Y. Wu, C.-K. Duan, X. Rong, and J. Du, “Dynamically encircling an exceptional point in a real quantum system,” Phys. Rev. Lett. 126, 170506 (2021).
[11] Q. Zhong, M. Khajavikhan, D. N. Christodoulides, and R. El-Ganainy, “Winding around non-Hermitian singularities,” Nat. Commun. 9, 4808 (2018).
[12] R. S. Nikam and P. Ring, “Manifestation of the Berry phase in diabolic pair transfer in rotating nuclei,” Phys. Rev. Lett. 58, 980–983 (1987).
[13] P. Bruno, “Berry phase, topology, and degeneracies in quantum nanomagnets,” Phys. Rev. Lett. 96, 117208 (2006).
[14] P. Meystre, Atom Optics (Springer, Berlin, 2001).
[15] G. Agarwal, Quantum Optics (Cambridge University Press, Cambridge, UK, 2013).
[16] H. J. Carmichael, Statistical Methods in Quantum Optics 1 (Springer, Berlin, 2010).
[17] C. Gerry and P. Knight, Introductory Quantum Optics (Cambridge Univ. Press, Cambridge, 2004).
[18] J. Peřina, Quantum Statistics of Linear and Nonlinear Optical Phenomena (Kluwer, Dordrecht, 1991).
[19] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge Univ. Press, Cambridge, 1997).
[20] H.-S. Zhong *et al.*, “Quantum computational advantage using photons,” *Science* **370**, 1460–1463 (2020).
[21] F. Acernese *et al.*, “Increasing the astrophysical reach of the advanced Virgo detector via the application of squeezed vacuum states of light,” *Phys. Rev. Lett.* **123**, 231108 (2019).
[22] M. Tse *et al.*, “Quantum-enhanced advanced LIGO detectors in the era of gravitational-wave astronomy,” *Phys. Rev. Lett.* **123**, 231107 (2019).
[23] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw Hill, New York, 1980).
[24] Y.-X. Wang and A. A. Clerk, “Non-Hermitian dynamics without dissipation in quantum systems,” *Phys. Rev. A* **99**, 063834 (2019).
[25] V. P. Flynn, E. Cobanera, and L. Viola, “Topology by dissipation: Majorana bosons in metastable quadratic Markovian dynamics,” *Phys. Rev. Lett.* **127**, 245701 (2021).
[26] A. Roy, S. Jahani, C. Langrock, M. Fejer, and A. Marandi, “Spectral phase transitions in optical parametric oscillators,” *Nat. Commun.* **12** (2021).
[27] J. Dereziński and C. Gérard, *Mathematics of Quantization and Quantum Fields* (Cambridge Univ. Press, Cambridge, 2013).
[28] J. Dereziński, “Bosonic quadratic Hamiltonians,” *J. Math. Phys.* **58**, 121101 (2017).
[29] B. Grünbaum, *Convex polytopes* (Springer; 2nd edition, New York & London, 2003).
[30] I. I. Arkhipov, F. Minganti, A. Miranowicz, and F. Nori, “Generating high-order quantum exceptional points in synthetic dimensions,” *Phys. Rev. A* **104**, 012205 (2021).
[31] The Kronecker sum of two matrices $A$ and $B$ is defined as $A \bigoplus B = A \bigotimes I_B + I_A \bigotimes B$, where $I_A, I_B$ is the identity matrix. For more details see Refs. [46].
[32] Z. Hu, “Eigenvalues and eigenvectors of a class of irreducible tridiagonal matrices,” *Linear Algebra Its Appl.* **619**, 328–337 (2021).
[33] A. Roy, S. Jahani, Q. Guo, A. Dutt, S. Fan, M.-A. Miri, and A. Marandi, “Nondissipative non-Hermitian dynamics and exceptional points in coupled optical parametric oscillators,” *Optica* **8**, 415 (2021).
[34] V. P. Flynn, E. Cobanera, and L. Viola, “Deconstructing effective non-Hermitian dynamics in quadratic bosonic Hamiltonians,” *New J. Phys.* **22**, 083004 (2020).
[35] J. Peřina, *Quantum Statistics of Linear and Nonlinear Optical Phenomena* (Kluwer, Dordrecht, 1991).
[36] D. Christodoulides and J. Yang, eds., *Parity-Time Symmetry and Its Applications* (Springer Singapore, 2018).
[37] A. Mostafazadeh, “Pseudo-Hermitian representation of quantum mechanics,” *Int. J. Geom. Meth. Mod. Phys.* **07**, 1191–1306 (2010).
[38] I. I. Arkhipov and F. Minganti, “Emergent non-Hermitian skin effect in the synthetic space of (anti-)PT-symmetric dimers,” (2022), arXiv:2110.15286.
[39] I. I. Arkhipov, A. Miranowicz, Ş. K. Özdemir, and F. Nori, “Dynamically encircling an exceptional curve by crossing diabolic points: A programmable multimode switch,” (to be submitted).
[40] A. Zavatta, V. Parigi, M. S. Kim, H. Jeong, and M. Bellini, “Experimental demonstration of the bosonic commutation relation via superpositions of quantum operations on thermal light fields,” *Phys. Rev. Lett.* **103**, 140406 (2009).
[41] B. Kühn, W. Vogel, M. Mraz, S. Köhnke, and B. Hage, “Anomalous quantum correlations of squeezed light,” *Phys. Rev. Lett.* **118**, 153601 (2017).
[42] J. Peřina, I. I. Arkhipov, V. Michálek, and O. Haderka, “Nonclassicality and entanglement criteria for bipartite optical fields characterized by quadratic detectors,” *Phys. Rev. A* **96**, 043845 (2017).
[43] E. V. Shechukin and W. Vogel, “Nonclassical moments and their measurement,”*Phys. Rev. A* **72**, 043808 (2005).
[44] F. Roccati, S. Lorenzo, G. M. Palma, G. T. Landi, M. Brunelli, and F. Ciccarello, “Quantum correlations in PT-symmetric systems,” *Quantum Sci. Technol.* **6**, 025005 (2021).
[45] J. Peřina, A. Miranowicz, G. Chimczak, and A. Kowalewska-Kudlaszyk, “Quantum Liouvillian exceptional and diabolical points for bosonic fields with quadratic Hamiltonians: The Heisenberg-Langevin equation approach,” to be published (2022).
[46] S.V. Lototsky, “Simple spectral bounds for sums of certain Kronecker products,” *Linear Algebra and its Applications* **469**, 114 – 129 (2015).