Some odd spectra

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Abstract

This text do not provide new great ideas, but fills a somptuous gap between the state of the art in Operator Theory and the available documentation ([DUSC]) of PDE specialists (see [Au11], [Mc110]).

The spectrum for an element of an unitary Banach Algebra is non-void, closed and bounded (i.e. compact). This result applies naturally to the Banach Algebra of bounded operators on a Banach Space.

In the theory of Partial Derivate Equations appear some “unbouded operators” which are in fact partial operators defined on a dense sub-space of the involved Banach Space. These operators can have some odd spectra. The property allowing to build a resolvent and a spectrum is the closeness.

We want to show the problem is not in the fact that these operators are unbounded, as suggested by [DUSC] (VII § 1 N. 9), but only in the fact they are partial.

We give briefly the methods for building resolvent ans spectra for such operators and explain why the classical proofs do not work in this case. After that we give some counterexamples, void or unbou nded spectra.

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Notations

- I write $\langle x, y \rangle$ for the ordered pair built with $x$ and $y$, because I hold that the $(x, y)$ notation is far too much employed.

On the same basis, I write $[a, b]$, french notation, rather than $(a, b)$, the open interval $\{x \in \mathbb{R} / a < x < b\}$. In the same way, I write $[a, b] = \{x \in \mathbb{R} / a \leq x < b\}$ and $]a, b] = \{x \in \mathbb{R} / a < x \leq b\}$.

- if $f$ is a linear mapping from the vector space $F$ to another vector space $E$, I write $\ker f$ for the kernel of $f$, $R(f)$ for its range.

- $E$ and $F$ being vector spaces, I write $L_0(F, E)$ for the set of linear mappings from $F$ to $E$. If $F = E$, I write $L_0(E)$ for $L_0(E, E)$. $I$ is written for an identical map, context must be clear enough to choose which identity.

- if $E$ and $F$ are endowed with vector space topologies, the continuous linear mapping (which are called “bounded” in the case of Banach Spaces) are by definition the elements of $L(F, E) = L(E)$ if $F = E$.

- Banach spaces norms are generally written as $x \mapsto \|x\|$, context ought to make clear which norm (i.e. which space) is in question.

This is the case, among others, for operator norms, always defined, for $T \in L_0(F, E)$, $G(T) = \{\langle u, v \rangle \in F \times E / Tu = v\}$ (graph of $T$).

1 Introduction: classical formulas

Let $\mathcal{A}$ a Banach Algebra with unit $e$. The series expansion, for $x \in \mathcal{A}$ such that $\|x\| < 1$, $(e - x)^{-1} = \sum_{n \in \mathbb{N}} x^n$ leads to the following result:

Lemma 1.1 Let $x \in \mathcal{A}$ such that $\|x\| < 1$.

Then $e - x$ is invertible in $\mathcal{A}$ and:

$$\|(e - x)^{-1}\| \leq \frac{1}{1 - \|x\|}$$

$$\|(e - x)^{-1} - e\| \leq \frac{\|x\|}{1 - \|x\|}$$

$$\|(e - x)^{-1} - e - x\| \leq \frac{\|x\|^2}{1 - \|x\|}$$
1 INTRODUCTION: CLASSICAL FORMULAS

The proof uses the power expansion $(e - x)^{-1} = \sum_{k \in \mathbb{N}} x^k$, hence, for $n \in \mathbb{N}$,

$(e - x)^{-1} - \sum_{k=0}^{n} x^k = \sum_{k>n} x^k = x^{n+1}(e - x)^{-1}.$

Note that the third inequality is not quite classical, but the proof is fully similar to the proof of the second inequality.

By applying lemma 1.1 to $xa^{-1}$, we obtain the following inversion theorem:

**Inversion theorem 1** Let $a \in A$ invertible.

Let $x \in A$ such that $\|x\| < \|a^{-1}\|^{-1}.$

Then $a - x$ is invertible in $A$ and:

$$
\|(a - x)^{-1}\| \leq \frac{\|a^{-1}\|}{1 - \|x\| \cdot \|a^{-1}\|}
$$

$$
\|(a - x)^{-1} - a^{-1}\| \leq \frac{\|a^{-1}\|^2 \cdot \|x\|}{1 - \|x\| \cdot \|a^{-1}\|}
$$

$$
\|(a - x)^{-1} - a^{-1} - a^{-1}xa^{-1}\| \leq \frac{\|a^{-1}\|^3 \cdot \|x\|^2}{1 - \|x\| \cdot \|a^{-1}\|}
$$

This inversion theorem is the startpoint of definition of resolvents and spectra in an unitary Banach Algebra, among others in $L(\mathbb{E})$ for a complex Banach space $\mathbb{E}$.

Another inversion theorem, still straightforwardly proved with the help of lemma 1.1, is less usual. I have seen it only in [Sch70], a book that Laurent Schwartz has extracted from his course at « l’X » (Ecole Polytechnique), theorem T.2, XIV, 7; 1, p. 178 and in [Mci10], prop. 3.1.14 p.35. The wordings of these two authors give only the first of the three inequalities.

**Inversion theorem 2** Let $\mathbb{E}$ and $\mathbb{F}$ two Banach Spaces (both real or both complex). Let $S \in L(\mathbb{F}, \mathbb{E})$ having a inverse $S^{-1}$ in $L(\mathbb{E}, \mathbb{F})$.

Let $T \in L(\mathbb{F}, \mathbb{E})$ such that $\|T\| < \|S^{-1}\|^{-1}.$

Then $S - T \in L(\mathbb{E}, \mathbb{F})$ is invertible in $L(\mathbb{E}, \mathbb{F})$ and:

$$
\|(S - T)^{-1}\| \leq \frac{\|S^{-1}\|}{1 - \|T\| \cdot \|S^{-1}\|}
$$

$$
\|(S - T)^{-1} - S^{-1}\| \leq \frac{\|S^{-1}\|^2 \cdot \|T\|}{1 - \|T\| \cdot \|S^{-1}\|}
$$

$$
\|(S - T)^{-1} - S^{-1} - S^{-1}TS^{-1}\| \leq \frac{\|S^{-1}\|^3 \cdot \|T\|^2}{1 - \|T\| \cdot \|S^{-1}\|}
$$
The proof applies the lemma 1.1 in $\mathcal{L}(F)$ to $S^{-1}T$.
Like for classical inversion in a Banach algebra, these inequalities show that the set of invertible $S \in \mathcal{L}(F, E)$ is an open set $\mathcal{U}(F, E)$ and the inverse mapping from $\mathcal{U}(F, E)$ to $\mathcal{U}(E, F)$ is a differentiable homeomorphism. Its differential at $S$ is $T \mapsto -S^{-1}TS^{-1}$ — element of $\mathcal{L}(\mathcal{L}(F, E), \mathcal{L}(E, F))$!

2 Resolution between two Banach spaces

In this section, we fix $E$ and $F$ to be two complex Banach spaces, $J$ a one-one continuous mapping from $F$ in $E$, such that $\|J\| \leq 1$. We explicitly assume that $J$ is not onto — i.e. $J$ is not an isomorphism. The notation $I$ is for the identitiy (unit) of $E$, $I'$ for the one of $F$.

Remark: we may identify $F$ with a subspace of $E$, $J$ being the canonical injection. It will be done in section 3. Until then, the "$J$" is maybe somewhat heavy in the notations, but gives more reliability when computing and is responsible for the failure of the traditional proofs that the spectrum is nonvoid and bounded.

Definition 2.1 Let $T \in \mathcal{L}(F, E)$ .

$\zeta \in \mathbb{C}$ is resolved if $\zeta J - T$ has an inverse $U \in \mathcal{L}(E, F)$. $U$ will be called inverse (or reciprocal) of $\zeta J - T$ and denoted $R_T(\zeta)$.

The set of resolved elements in $\mathbb{C}$ will be denoted $\rho(T)$ and called resolvent set of $T$, $\mathbb{C} \setminus \rho(T)$ is the spectrum of $T$, denoted $\sigma(T)$.

The mapping assigning to $\zeta \in \rho(T)$ the operator $(\zeta J - T)^{-1} \in \mathcal{L}(E, F)$ will be denoted $R_T : \zeta \mapsto R_T(\zeta)$ and called resolvent operator (or in brief resolvent) of $T$.

N.B.: if, for $\zeta \in \mathbb{C}$, $\zeta J - T$ has an algebraic inverse $U$ (i.e. in $\mathcal{L}_0(F, E)$), then by Banach’s closed graph theorem, $U$ is bounded, so $\zeta$ is resolved.

One can prove as usual that $\rho(T)$ is open in $\mathbb{C}$ — so $\sigma(T)$ is closed — and, when $\rho(T) \neq \emptyset$, the resolvent function is continuous and holomorphic from $\rho(T)$ to $\mathcal{L}(E, F)$.

if $J$ is a linear isomorphism from $F$ to $E$, we can prove that $\sigma(T)$ is nonvoid and bounded. This property is lost when $J$ is not invertible.

in fact, to prove that $\sigma(T)$ is nonvoid and bounded, we use the invertibility of $J$ to show that $J - \zeta^{-1}T$ is invertible if $\zeta$ is great enough. So these properties cannot be proved by the usual means for $J$ not invertible. The counterexamples in the sections 4 and 5 will prove they are false.
3 Spectral theory of the closed operators

In this section, $E$ is a complex Banach space, $F$ a subspace (a priori not closed) of $E$. $J$ is the canonical injection from $F$ into $E$.

Because $F$ is endowed with the induced topology, an unbounded operator is nothing else than an element $T$ of $L_0(F,E) \setminus L(F,E)$. We will omit in fact the restriction $T \notin L(F,E)!$ We fix a $T \in L_0(F,E)$.

Now, the announced renorming.

**Proposition 3.1** For $p \in [1, +\infty]$, the mapping $N^T_p$:

$$u \mapsto (\|u\|^p + \|Tu\|^p)^{\frac{1}{p}} \text{ if } p < +\infty,$$

$$u \mapsto \max(\|u\|, \|Tu\|) \text{ if } p = +\infty,$$

is a norm. Moreover:

(i) the norms $N^T_p$ are equivalent each other.

(ii) $T$ is continuous from $F$ endowed with one of the normes $N^T_p$ to $E$.

(iii) the identity mapping from $F$ endowed with the topology $T_2$ defined by the equivalent norms $N^T_p$ onto $F$ endowed with the induced topology $T_1$ is continuous. In other words, $T_2$ is stronger than $T_1$.

(iv) if $T$ is continuous from $F$ with the induced topology to $E$, then the $N^T_p$ norms are equivalent to the induced norm.

**Proof:**

The relations $N^T_p(\alpha u) = |\alpha| N^T_p(u)$, $N^T_p(u) = 0 \iff u = 0$ and the triangular inequality are straightforward, with the help of the Minkowski inequality for $p \in ]1, +\infty[$. Other immediate relations: $\|u\| \leq N^T_p(u)$ (hence (iii)), $\|Tu\| \leq N^T_p(u)$ (hence (ii)).

For $a$ and $b$ in $\mathbb{R}_+$ and $p \in [1, +\infty[$, $(a^p + b^p)^{\frac{1}{p}} \leq a + b$ follows from Minkowski in $\mathbb{R}^2$, with the two vectors $\langle a, 0 \rangle$ and $\langle 0, b \rangle$. So, for $p \in [1, +\infty]$ and $u \in F$, $N^T_p(u) \leq N^T_1(u) \leq N^T_p(u) \leq 2N^T_\infty(u)$, which proves all the norms $N^T_p$ are equivalent to $N^T_\infty$. Hence (i).

Finally, if $T$ is continuous as soon as $F$ is endowed with the induced norm, we have $C \in \mathbb{R}_+$ such that $\|Tu\| \leq C \|u\|$ for every $u \in F$. Then, for $u \in F$, $\|u\| \leq N^T_1(u) \leq (1 + C) \|u\|$. That proves (iv).
Definition 3.2 For $p \in [1, +\infty]$, the norm $N_p^T$ defined in prop. 3.1 is called norm associated to $T$ with exponent $p$.

The topology defined by the $N_p^T$ norms will be called topology associated to $T$. We shall write $\mathcal{T}_1$ for the induced topology on $\mathcal{F}$, $\mathcal{T}_2$ for the topology defined by the equivalent norms $N_p^T$.

Remark: it would have been a few easier to use merely $p = 1$ or $p = +\infty$. But $p = 2$ will be usefull with Hilbert spaces! Hence this little expensive generalization.

From $\mathcal{F}$ with this new norm into $\mathcal{E}$, $T$ is continuous (prop. 3.1 (ii)), and we have another fact:

Proposition 3.3 Let $\zeta \in \mathbb{C}$ such that $\zeta J - T$ is algebraically invertible, $S$ its inverse. Then:

(i) $S$ is continuous from $\mathcal{E}$ to $\mathcal{F}$ endowed with $\mathcal{T}_2$.

(ii) $S$ is continuous from $\mathcal{E}$ to $\mathcal{F}$ endowed with $\mathcal{T}_1$.

Proof:

(i) is an immediate consequence of Banach’s closed graph theorem (cf. the remark after def. 2.1).

(ii) follows because $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$.

\[\square\]

To apply the results of section 2, it suffices that $\mathcal{F}$ with $N_1^T$ be a Banach space.

The following theorem 3.4 is very near prop. 3.1.4, p. 34 of [Mci10], given without proof.

Theorem 3.4 Let $T \in \mathcal{L}_0(\mathcal{F}, \mathcal{E})$, $G(T)$ its graph. The following assertions are equivalent:

(i) $G(T)$ is closed in $\mathcal{F} \times \mathcal{E}$.

(ii) for every sequence $(u_n)_{n \in \mathbb{N}}$ with values in $\mathcal{F}$, if $(u_n)$ converges to some $u \in \mathcal{E}$ and if the sequence $(Tu_n)_{n \in \mathbb{N}}$ converges to some $v \in \mathcal{E}$, then $u \in \mathcal{F}$ et $v = Tu$.

(iii) $\mathcal{F}$ endowed with the topology associated to $T$ is a Banach space with the norm $N_p^T$ for any $p \in [1, +\infty]$.

Proof:

If $G(T)$ is closed and if the sequence $(u_n)_{n \in \mathbb{N}}$ verifies the assumptions of (ii), then the sequence $(\langle u_n, Tu_n \rangle)_{n \in \mathbb{N}}$ is in $G(T)$ and converges to $\langle u, v \rangle$, hence, since $G(T)$ is closed, $\langle u, v \rangle \in G(T)$.

In other words, $u \in \mathcal{F}$ and $v = Tu$. So (i) $\Rightarrow$ (ii).
Suppose (ii). Let \((u_n)_{n \in \mathbb{N}}\) a Cauchy sequence in \(\mathbb{F}\) for, say, the norm \(N_1^T\).
\[
\|u_p - u_q\| \leq N_1^T \|u_p - u_q\| \text{ and } \|Tu_p - Tu_q\| \leq N_1^T \|u_p - u_q\|
\]
for \(p\) and \(q\) in \(\mathbb{N}\), so the sequences \((u_n)_{n \in \mathbb{N}}\) and \((Tu_n)_{n \in \mathbb{N}}\) are Cauchy in \(\mathbb{E}\) and converge, the first to a \(u \in \mathbb{E}\), the second to a \(v \in \mathbb{E}\).
\(u \in \mathbb{F}\) and \(v = Tu\) by (ii).

Then let \(\epsilon \in \mathbb{R}_+\), \(\eta = \frac{\epsilon}{2}\). We have \(N_1 \in \mathbb{N}\) such that, for \(n \in \mathbb{N}\) verifying \(n \geq N_1\), \(\|u_n - u\| < \eta\) and \(N_2 \in \mathbb{N}\) such that, for \(n \in \mathbb{N}\) verifying \(n \geq N_2\), \(\|Tu_n - Tu\| < \eta\). If \(n \geq \max(N_1, N_2)\), then:
\[
N_1^T \|u - u_n\| = \|u - u_n\| + \|Tu - Tu_n\| < 2\eta = \epsilon.
\]
Which proves that \((u_n)_{n \in \mathbb{N}}\) converges to \(u\) in \(\mathbb{F}\) with \(N_1^T\). So, since the norms \(N_p^T\) for \(p \in [1, +\infty]\) are all uniformly equivalent, \(\mathbb{F}\) with an associated norm is complete. Hence (ii) \(\Rightarrow\) (iii).

Suppose now (iii): \(\mathbb{F}\) endowed with \(N_1^T\) is therefore a Banach space. Let \((\langle u_n, Tu_n \rangle)_{n \in \mathbb{N}}\) in \(G(T)\) converging to \(\langle u, v \rangle\) in \(\mathbb{E} \times \mathbb{E}\) endowed with product topology. Among others, the sequence is Cauchy for this topology. One of the norms defining it is \(\nu : \langle u, v \rangle \mapsto \|u\| + \|v\|\).

if \(\epsilon \in \mathbb{R}_+\), then we have \(N \in \mathbb{N}\) such that, for every \(p\) and \(q\) in \(\mathbb{N}\) verifying \(p > N\) and \(q > N\), \(\nu(\langle u_p, Tu_p \rangle - \langle u_q, Tu_q \rangle) < \epsilon\). in other words, \(\nu(\langle u_p - u_q, Tu_p - Tu_q \rangle) < \epsilon\).

But that is still equivalent to \(\|u_p - u_q\| + \|Tu_p - Tu_q\| < \epsilon\), i.e. \(N_1^T \|u_p - u_q\| < \epsilon\). So the sequence \((u_n)_{n \in \mathbb{N}}\) is Cauchy in \(\mathbb{F}\) with \(N_1^T\), supposed complete. \((u_n)_{n \in \mathbb{N}}\) has here a limit \(u'\).

Moreover, since \(\|u' - u_n\| \leq N_1^T \|u' - u_n\|\), \(u' = \lim_{n \to \infty} u_n = u\).

And, since \(\|Tu' - Tu_n\| \leq N_1^T \|Tu' - Tu_n\|\), \(Tu = Tu' = \lim_{n \to \infty} Tu_n\) in \(\mathbb{E}\), hence \(Tu = v\).

Which proves that \(\langle u, v \rangle \in G(T)\). So \(G(T)\) is closed. \(\square\)

**Definition 3.5** \(T \in \mathcal{L}_0(\mathbb{F}, \mathbb{E})\) is a closed operator if it verifies the equivalent conditions of th. 3.4.

If \(T\) is a closed operator, we can define its spectrum, its resolvent because \(\mathbb{F}\) is a Banach space with the topology \(\mathcal{T}_2\) and \(T\) is bounded for this topology. Like the icing of the cake, the continuity of the resolvent does not depend on endowing \(\mathbb{F}\) with the induced norm or the new norm \(N_1^T\) (prop. B.3.3). Elementary computation shows that \(R_T(\zeta)\) is holomorphic for both topologies.
4 Three correlated examples

Some examples will shed the light on the fickleness of spectra for partial operators. With $T$ closed unbounded operator, we may have $\rho(T) = \emptyset$ ([McI10] p. 36, at the beginning of section 3.2). So I’m keen on clearing out the ground. These examples are very near of examples given in exercises in [DUSC]. The examples will be handled like propositions, because their assertions have to be proved. Some computations preparing them come before.

In this section, $X = [0, 1]$, $\mathcal{C} = \mathcal{C}(X, \mathbb{C})$, the set of continuous functions from $X$ to $\mathbb{C}$, a well-known $\mathbb{C}$-based separable Banach space. $\mathcal{D}$ is the vector subspace of $\mathcal{C}$ whose elements are the continuously derivable mappings on $X$ (right-derivable at $0$, left-derivable at $1$). $\mathcal{C}$ is endowed with the usual norm, $f \mapsto \max_{x \in X} |f(x)|$, restriction to $\mathcal{C}$ of the $\mathcal{L}\infty$-seminorm. It will be denoted $f \mapsto \|f\|_{\infty}$.

$T$ will be, for these three examples, the derivation operator from $\mathcal{D}$ to $\mathcal{C}$, or its restriction to a subspace $\mathcal{F}$ of $\mathcal{D}$.

Lemma 4.1 Let $k \in \mathbb{C}^*$, $a$ and $b$ in $\mathbb{C}$.

The mapping $t \mapsto \left(\frac{at + b}{k} - \frac{a}{k^2}\right) e^{kt}$ is a primitive of the mapping from $\mathbb{R}$ to $\mathbb{C}$ $t \mapsto (at + b) e^{kt}$.

Proof: elementary calculus.

Definition 4.2 Let $\zeta \in \mathbb{C}$ A mapping and a mapping family, all these in $\mathcal{D}$, are associated to $\zeta$:

(i) $h_\zeta : x \mapsto e^{\zeta x}$.

(ii) for each $f \in \mathcal{C}$, $K_\zeta f : x \mapsto e^{\zeta x} \int_0^x e^{-\zeta t} f(t) dt$.

Lemma 4.3 Let $\zeta \in \mathbb{C}$, $f \in \mathcal{C}$, $a = \Re \zeta$.

(1) if $a \neq 0$, $\|K_\zeta\| = \frac{e^a - 1}{a} > 0$.

(2) if $a = 0$ $\|K_\zeta\| = 1$.

(3) For every $u \in \mathcal{D}$, the following assertions are equivalent:

(i) $\zeta u - u' = f$.

(ii) there exists a constant $\gamma \in \mathbb{C}$ such that $u = \gamma h_\zeta - K_\zeta f$.

Proof:

$$|K_\zeta f(x)| = \left| e^{\zeta x} \int_0^x e^{-\zeta t} f(t) dt \right|,$$

so:

$$|K_\zeta f(x)| \leq \left| e^{\zeta x} \right| \int_0^x |e^{-\zeta t}| |f(t)| dt \leq e^{ax} \|f\|_{\infty} \int_0^x e^{-at} dt.$$
If $a \neq 0$, the mapping $t \mapsto e^{-at}$ is the derivative of $t \mapsto \frac{e^{-at}}{-a}$, so 
$$
\int_{0}^{x} e^{-at} dt = \frac{e^{-ax} - 1}{-a} = \frac{1 - e^{-ax}}{a} .
$$
Notice that, for $a > 0$, $1 - e^{-ax} \geq 0$ and, for $a < 0$, $1 - e^{-ax} \leq 0$: the quotient is nonnegative.

It follows that $|K_\zeta f(x)| \leq e^{ax} \frac{1 - e^{-ax}}{a} \|f\|_\infty = e^{ax} \frac{e^a - 1}{a} \|f\|_\infty$ .

If $a > 0$, then $e^{ax} \leq e^a$ for every $x \in X$, so $\frac{e^{ax} - 1}{a} \leq \frac{e^a - 1}{a}$, hence $\|K_\zeta f\|_\infty \leq e^a \frac{1}{a} \|f\|_\infty$.

If $a < 0$, then $-e^{ax} \leq -e^a$ for every $x \in X$ and consequently $\frac{e^{ax} - 1}{a} = \frac{1 - e^{ax}}{-a} \leq \frac{1 - e^a}{-a} = e^a \frac{e^a - 1}{a}$, hence, like in the previous case, $\|K_\zeta f\|_\infty \leq e^a \frac{1}{a} \|f\|_\infty$.

If $a = 0$, $\int_{0}^{x} e^{-at} dt = \int_{0}^{x} 1 dt = x$ and $e^{ax} = 1$, hence straight 
$$
|K_\zeta f(x)| \leq x \|f\|_\infty ,
$$
with implies $\|K_\zeta f\|_\infty \leq \|f\|_\infty$ .

So we have $\|K_\zeta\| \leq \frac{e^a - 1}{a}$ if $a \neq 0$, $\|K_\zeta\| \leq 1$ if $a = 0$.

Now, let $b = \Im \zeta$, so that $\zeta = a + bi$. Let $f \in \mathcal{C}$ defined by $f(x) = e^{bx}$ . Since $|f(x)| = 1$ for every $x \in X$, $\|f\|_\infty = 1$. For $x \in X$:

$$
K_\zeta f(x) = e^{\zeta x} \int_{0}^{x} e^{-\zeta t} e^{bt} dt = e^{\zeta x} \int_{0}^{x} e^{-at} dt .
$$

If $a = 0$, $K_\zeta f(x) = xe^{bx}$, so $|K_\zeta f(x)| = x$ and $\|K_\zeta f\|_\infty = 1$.

Which proves $\|K_\zeta\| = 1$ .

If $a \neq 0$:

$$
K_\zeta f(x) = e^{ax} e^{bx} \frac{e^{-ax} - 1}{-a} = e^{bx} \frac{e^{ax} - 1}{a} \|f\|_\infty ,
$$

$$
|K_\zeta f(x)| = \frac{e^{ax} - 1}{a} .
$$

Hence (see the above computations of upper bounds) we have $\|K_\zeta f\|_\infty = \frac{e^a - 1}{a}$ and $\|K_\zeta\| = \frac{e^a - 1}{a}$ .

(3) is classical calculus. □
Proposition 4.4 Let $\mathcal{E}$ a closed subspace of $\mathcal{C}$ (therefore a Banach space with induced norm and topology), $\mathcal{F} = \{ u \in \mathcal{E} \cap \mathcal{D} / u' \in \mathcal{E} \}$ . The operator $T$ from $\mathcal{F}$ to $\mathcal{E}$ is closed.

Proof:

Suppose $(u_n)_{n \in \mathbb{N}}$ sequence in $\mathcal{F}$ converging in $\mathcal{E}$ to $u$, suppose more that $u_n'$ converges in $\mathcal{E}$ to $v$.

A classical calculus result show that the sequence $(w_n)_{n \in \mathbb{N}}$, defined by $w_n(x) = \int_0^x u_n'(t)dt$ , converges in $\mathcal{C}$ to $w(x) = \int_0^x v(t)dt$ .

Now, for $x \in \mathcal{X}$, $\int_0^x u_n'(t)dt = u_n(x) - u_n(0)$ . Since $(u_n)_{n \in \mathbb{N}}$ converges uniformly to $u$, $\lim_{n \to \infty} (u_n(x) - u_n(0)) = u(x) - u(0)$ .

Therefore $u(x) - u(0) = \int_0^x v(t)dt$ , hence $u \in \mathcal{D}$ and $u' = v$.

Moreover, since $u_n' \in \mathcal{E}$ for every $n \in \mathbb{N}$ and $\mathcal{E}$ is closed in $\mathcal{C}$, $v = u' \in \mathcal{E}$. Which proves that the graph $G(T)$ of $T$ is closed, so $T$ is closed. \hfill $\square$

Proposition 4.5 Let $\Lambda$ a continuous linear functional on $\mathcal{C}$. Let $\mathcal{E} = \ker \Lambda$, $\mathcal{F} = \{ u \in \mathcal{E} \cap \mathcal{D} / u' \in \mathcal{E} \} = \{ u \in \mathcal{D} / \Lambda u = 0 \text{ and } \Lambda u' = 0 \}$ (cf. prop. 4.4). Then $\mathcal{E}$ is closed, $T$ is a closed operator from $\mathcal{F}$ to $\mathcal{E}$ and, for every $\zeta \in \mathcal{C}$, the following assertions are equivalent:

(i) $\zeta \in \sigma(T)$.

(ii) $h_\zeta \in \mathcal{E}$, i.e. $\Lambda (h_\zeta) = 0$.

Moreover, if $\zeta \notin \sigma(T)$, for every $f \in \mathcal{E}$, we have $R_T(\zeta)f = \gamma h_\zeta - K_{\zeta}f$ with $\gamma = \frac{\Lambda (K_{\zeta}f)}{\Lambda (h_\zeta)}$.

Proof:

$\mathcal{E}$, kernel of the continuous linear functional $\Lambda$, is closed and then prop. 4.4 proves that $T$ is closed.

Let $\zeta \in \mathcal{C}$ . We have, for given $f \in \mathcal{E}$, to resolve the equation in $u \in \mathcal{F}$ $\zeta u' - u = f$ . By lemma 4.3 for $u \in \mathcal{D}$, $\zeta u - u' = f$ if and only if there exists $\gamma \in \mathcal{C}$ such that $u(x) = \gamma e^{\zeta x} - K_{\zeta}f(x)$ . Can we have $u \in \mathcal{F}$?
First, if \( u \in \mathbb{E} \), then \( u \in \mathbb{F} \) since, from \( \zeta u - u' = f \), we deduce \( \Lambda (u') = \zeta \Lambda (u) - \Lambda (f) = 0 \) for \( f \in \mathbb{E} \) and \( u \in \mathbb{E} \cap \mathbb{D} \).

Let \( A(\zeta) = \Lambda (h_\zeta) \) and \( B(\zeta) = \Lambda (K_\zeta f) \).

\( \Lambda (u) = \gamma A(\zeta) - B(\zeta) \), so everything depends on \( A(\zeta) \):

\( \diamond \) if \( A(\zeta) \neq 0 \), there exists one and only one \( \gamma = \frac{B(\zeta)}{A(\zeta)} \) such that \( u \in \mathbb{E} \), so \( u \in \mathbb{F} \).

And we have \( M \in \mathbb{R}_+^* \) such that \( |\Lambda (K_\zeta f)| \leq M \| f \|_{\infty} \) by continuity of \( \Lambda \) and lemma 4.3, hence:

\[
\| u \|_{\infty} \leq \left( 1 + \frac{1}{|A(\zeta)|} \right) M \| f \|_{\infty} .
\]

Which proves that \( \zeta \in \rho(T) \).

\( \diamond \) if \( A(\zeta) = 0 \), following the value of \( B(\zeta) \), there is no value of an infinity of values for \( \gamma \) such that \( u \in \mathbb{E} \) (and \( u \in \mathbb{F} \)). So \( \zeta \in \sigma(T) \).

Hence \( \sigma(T) = \{ \zeta \in \mathbb{C} | A(\zeta) = 0 \} \). \( \square \)

Continuous linear functionals on \( \mathcal{C} \) are nothing else than functional Radon measures on \([0,1]\). We will write \( \delta_x \) for the Dirac measure on \( x \in [0,1] \) (\( \delta_x(f) = f(x) \) for \( f \in \mathcal{C} \)).

The three examples thereafter use prop. 4.5. Except for the first one (where \( \rho(T) = \emptyset \)), we consider the behaviour when \( |\zeta| \to \infty \) of \( \| R_T(\zeta) \| \).

**Example 1** Let us take for \( \Lambda \) the null functional.

\( \mathbb{E} = \mathcal{C} \) (hence \( \mathbb{F} = \mathbb{D} \)), \( T \) is closed, \( \sigma(T) = \mathbb{C} \) and \( \rho(T) = \emptyset \).

**Proof:**

By prop. 4.5 \( h_\zeta \in \mathcal{E} = \ker \Lambda \) for every \( \zeta \in \mathbb{C} \), so \( \sigma(T) = \mathbb{C} \). \( \square \)

**N.B:** in this example, \( T \) is onto; for \( f \in \mathcal{C} \), \( u : x \mapsto \int_0^x f(t)dt \) is in \( \mathbb{D} \) and \( T u = f \).

**Example 2** Let us take \( \Lambda = \delta_0 \).

\( \mathbb{E} = \{ f \in \mathcal{C} | f(0) = 0 \} \) is closed, \( \mathbb{F} = \{ u \in \mathbb{D} | u(0) = 0 \) and \( u'(0) = 0 \}) \) and, for every \( \zeta \in \mathbb{C} \), \( \zeta J - T \) is a bijection with reciprocal \( R_T(\zeta) = -K_\zeta \), continuous mapping.

\( \rho(T) = \mathbb{C} \) and \( \sigma(T) = \emptyset \).
For $\zeta \in \mathbb{R}^*_+, \frac{e^\zeta - 1 - \zeta}{\zeta^2} \leq \|R_T(\zeta)\| \leq \frac{e^\zeta - 1}{\zeta}$.

So $\lim_{\zeta \to +\infty} \|R_T(\zeta)\| = +\infty$

Proof:

We apply prop. 4.5. Since $\Lambda(h_\zeta) = 1$ for each $\zeta \in \mathbb{C}$, $\sigma(T) = \emptyset$.

$K_\zeta f(0) = e^{\zeta x} \int_0^x e^{-\zeta t} f(t) dt = 0$, so $\Lambda(K_\zeta f) = 0$, and, using the formulas of prop. 4.5, we obtain $\gamma = 0$ and $R_T(\zeta) f = -K_\zeta f$.

So $\|R_T(\zeta)\| = \|K_\zeta\|$ (norms in $\mathcal{L}(\mathbb{E}, \mathbb{F})$). We deduce, by lemma 4.3 (since $\mathbb{E} \subseteq \mathbb{C}$) $\|K_\zeta\| \leq \frac{e^\zeta - 1}{\zeta}$. — notice that the operator norm of the derivation operator from $\mathcal{C}$ has been computed by using the mapping $1 : x \mapsto 1$, which is not in $\mathbb{E}$.

To get a lesser bound of $\|K_\zeta\|$, we will compute $\|K_\zeta\|_\infty$ for some $f \in \mathbb{E}$ such that $\|f\|_\infty = 1$. We merely take for $f$ the identity of $X$, $f(x) = x$.

By lemma 4.1, the mapping $x \mapsto xe^{-\zeta x}$, which is involved in the computation, admits the primitive $x \mapsto (-\zeta^{-1} x - \zeta^{-2}) e^{-\zeta x}$.

For $x \in X$:

$$K_\zeta f(x) = e^{\zeta x} \int_0^x te^{-\zeta t} dt$$

$$= e^{\zeta x} \left[ (-\zeta^{-1} t - \zeta^{-2}) e^{-\zeta t} \right]_0^x$$

$$= -\frac{e^{\zeta x}}{\zeta^2} \left[ (\zeta t + 1) e^{-\zeta t} \right]_0^x$$

$$= -\frac{e^{\zeta x}}{\zeta^2} \left[ (\zeta x + 1) e^{-\zeta x} - 1 \right]$$

$$= \frac{e^{\zeta x} - 1 - \zeta x}{\zeta^2}.$$

The mapping $g : x \mapsto e^{\zeta x} - 1 - \zeta x$ is nondecreasing (see it like partial summation of the exponential series, or use the derivate, $g'(x) = \zeta e^{\zeta x} - \zeta = \zeta (e^{\zeta x} - 1) \geq 0$).

Since $g(0) = 0$, $\|g\|_\infty = g(1) = e^\zeta - 1 - \zeta$.

Because $K_\zeta f = \frac{1}{\zeta^2} g$. 
\[ \|K_\zeta f\|_\infty = \frac{1}{\zeta^2} \|g\|_\infty = \frac{e^\zeta - 1 - \zeta}{\zeta^2}. \]

Since \( \|f\|_\infty = 1 \), we obtain \( \|R_T(\zeta)\| = \|K_\zeta\| \geq \frac{e^\zeta - 1 - \zeta}{\zeta^2}. \)

It suffices now to observe that \( \lim_{\zeta \to +\infty} \frac{e^\zeta - 1 - \zeta}{\zeta^2} = +\infty \quad \square \)

In the following example \( R_T(\zeta) \neq -K_\zeta \).

**Example 3** Let \( \Lambda \) the bounded linear functional on \( \mathbb{C} \delta_1 - \delta_0 \), such that, for \( f \in \mathbb{C} \), \( \Lambda f = f \left(\frac{1}{2}\right) - f(0) \), \( E = \left\{ f \in \mathbb{C} / f(0) = f \left(\frac{1}{2}\right) \right\} \), kernel of \( \Lambda \).

\( E \) is closed and \( \sigma(T) = 4i\pi \mathbb{Z} = \left\{ 4i\pi n / n \in \mathbb{Z} \right\} \).

For \( f \in E \), \( \zeta \in \rho(T) \) and \( x \in \mathbb{X} \):
\[ R_T(\zeta)f(x) = \frac{e^{\zeta x}}{e^\zeta - 1} K_\zeta f \left(\frac{1}{2}\right) - K_\zeta f(x). \]

Let \( \zeta \in \mathbb{R}_+^* \) (hence \( \zeta \in \rho(T) \)). Then there exists a mapping \( f \in E \) such that:

- (i) \( f(0) = f \left(\frac{1}{2}\right) = 0 \).
- (ii) \( \|f\|_\infty = e^{\zeta^2} \).
- (iii) \( K_\zeta f \left(\frac{1}{2}\right) = 0 \).
- (iv) for every \( x \in \mathbb{X} \), \( R_T(\zeta)f(x) = -K_\zeta f(x) \).
- (v) \( R_T(\zeta)f(1) = e^{\zeta^2} \left(\frac{1}{\zeta} + \frac{2}{\zeta^2}\right) - \frac{2e^\zeta}{\zeta^2} \).

We have therefore:
\[ \|R_T(\zeta)\| \geq \frac{2e^\zeta}{\zeta^2} - \frac{1}{\zeta} - \frac{2}{\zeta^2}, \]
\[ \lim_{\zeta \to +\infty} \|R_T(\zeta)\| = +\infty. \]

**Proof:**

Since \( E \) is the kernel of \( \Lambda \), prop. [45] applies.
Let \( \zeta \in \mathbb{C} \). \( \Lambda(h_{\zeta}) = e^{\frac{\zeta}{2}} - 1 \). So \( \zeta \in \sigma(T) \) if and only if \( e^{\frac{\zeta}{2}} = 1 \), i.e. if and only if there exists \( n \in \mathbb{Z} \) such that \( \frac{\zeta}{2} = 2i\pi n \), i.e. \( \zeta = 4i\pi n \).

To compute \( \gamma \) by using prop. 4.5 formula, for \( f \in \mathbb{E} \), since \( K_{\zeta}f(0) = 0 \), \( \Lambda(K_{\zeta}f) = K_{\zeta}f \left( \frac{1}{2} \right) - K_{\zeta}f(0) = K_{\zeta}f \left( \frac{1}{2} \right) \).

Hence, for \( \zeta \in \rho(T) = \mathbb{C} \setminus 4i\pi \mathbb{Z}, x \in \mathbb{X} \):

\[
\gamma = \frac{K_{\zeta}f \left( \frac{1}{2} \right)}{e^{\frac{\zeta}{2}} - 1},
\]

\[
R_T(\zeta)f(x) = \frac{e^{cx}}{e^{\frac{\zeta}{2}} - 1}K_{\zeta}f \left( \frac{1}{2} \right) - K_{\zeta}f(x).
\]

Let \( \zeta \in \mathbb{R}_+^* \) fixed and \( f \) the mapping defined from \( \mathbb{X} = [0, 1] \) in \( \mathbb{R} \) by:

\( \circ \) if \( x \in \left[ 0, \frac{1}{2} \right] \), \( f(x) = e^{cx} \sin 4\pi x \).

\( \circ \) if \( x \in \left[ \frac{1}{2}, 1 \right] \), \( f(x) = e^{\frac{\zeta}{2}}(2x - 1) \).

So \( f \) takes the value of \( f_1(x) = e^{cx} \sin 4\pi x \) for \( x \leq \frac{1}{2} \) and the value of \( f_2(x) = e^{\frac{\zeta}{2}}(2x - 1) \) for \( x > \frac{1}{2} \). \( f_1 \) and \( f_2 \) are real analytic, thus continuous. To establish the continuity of \( f \), it suffices to show \( f_1 \left( \frac{1}{2} \right) = f_2 \left( \frac{1}{2} \right) \). Now:

\[
f_1 \left( \frac{1}{2} \right) = e^{\frac{\zeta}{2}} \sin 2\pi = 0,
\]

\[
f_2 \left( \frac{1}{2} \right) = e^{\frac{\zeta}{2}} \left( 2 \frac{1}{2} - 1 \right) = 0.
\]

Hence moreover \( f \left( \frac{1}{2} \right) = 0 \) and \( f(x) = f_2(x) \) on \( \left[ \frac{1}{2}, 1 \right] \).

To fulfill (i), it remains to compute \( f(0) = \sin 4\pi 0 = \sin 0 = 0 \).

By the way, \( f \left( \frac{1}{2} \right) = f(0) \), so \( f \in \mathbb{E} \).

For \( x \in \left[ 0, \frac{1}{2} \right] \), \( |f(x)| = e^{cx} |\sin 4\pi x| \leq e^{cx} \leq e^{\frac{\zeta}{2}} \).
For $x \in \left[\frac{1}{2}, 1\right]$, $0 \leq 2x - 1 \leq 1$, so $0 \leq f(x) \leq e^{\frac{2}{\zeta}}$. Thus, $\|f\|_\infty \leq e^{\frac{2}{\zeta}}$.

But $f(1) = e^{\frac{2}{\zeta}}$, so $\|f\|_\infty = e^{\frac{2}{\zeta}}$.

We compute $I(x) = \int_0^x e^{-\zeta t} f(t) dt$ and $K_\zeta f(x) = e^{\zeta x} I(x)$ in two steps:

\begin{itemize}
  \item If $x \in \left[0, \frac{1}{2}\right]$:
    \[
    I(x) = \int_0^x e^{-\zeta t} e^{\zeta t} \sin 4\pi t dt = \int_0^x \sin 4\pi t dt = \left[-\frac{\cos 4\pi t}{4\pi}\right]^x_0 = \frac{1 - \cos 4\pi x}{4\pi}.
    \]
    \[
    K_\zeta f(x) = e^{\zeta x} \frac{1 - \cos 4\pi x}{4\pi}.
    \]

    Observe by the way that $I\left(\frac{1}{2}\right) = \frac{1 - \cos 2\pi}{4\pi} = 0$ and thus $K_\zeta f\left(\frac{1}{2}\right) = 0$.

  \item If $x \in \left[\frac{1}{2}, 1\right]$:
    \[
    I(x) = \int_0^{\frac{1}{2}} e^{-\zeta t} f(t) dt + \int_0^x e^{-\zeta t} f(t) dt = I\left(\frac{1}{2}\right) + \int_0^x e^{-\zeta t} f(t) dt = e^{\zeta x} \int_0^x (2t - 1)e^{-\zeta t} f(t) dt = e^{\zeta x} \left[\left(-\frac{2t - 1}{-\zeta} - \frac{2}{\zeta^2}\right) e^{-\zeta t}\right]_{\frac{1}{2}}^x.
    \]
\end{itemize}
\[ K_\zeta f(x) = e^{\zeta x} \left( \frac{1 - 2x}{\zeta} - \frac{2}{\zeta^2} \right) + \frac{2e^{\zeta x}}{\zeta^2} . \]

Among other values, \( K_\zeta f(1) = e^{\zeta} \left( \frac{-1}{\zeta} - \frac{2}{\zeta^2} \right) + \frac{2e^{\zeta}}{\zeta^2} . \)

Since \( K_\zeta f \left( \frac{1}{2} \right) = 0 \), the formula for computing the resolvent becomes \( R_T(\zeta) f(x) = -K_\zeta f(x) \), so:

- if \( x \in \left[ 0, \frac{1}{2} \right] \), \( R_T(\zeta) f(x) = -e^{\zeta x} \frac{1 - \cos 4\pi x}{4\pi} . \)
- if \( x \in \left[ \frac{1}{2}, 1 \right] \), \( R_T(\zeta) f(x) = e^{\zeta x} \left( \frac{2x - 1}{\zeta} + \frac{2}{\zeta^2} \right) - \frac{2e^{\zeta x}}{\zeta^2} . \)

We deduce \( R_T(\zeta) f(1) = e^{\zeta} \left( \frac{1}{\zeta} + \frac{2}{\zeta^2} \right) - \frac{2e^{\zeta}}{\zeta^2} . \) What remains to prove is deduced from:

\[
\begin{align*}
-R_T(\zeta) f(1) &\leq |R_T(\zeta) f(1)| \leq \|R_T(\zeta) f\|_{\infty} \\
\|R_T(\zeta)\| &\geq \frac{\|R_T(\zeta) f\|_{\infty}}{\|f\|_{\infty}} = \frac{\|R_T(\zeta) f\|_{\infty}}{e^{\zeta/2}} .
\end{align*}
\]

\[ \Box \]

5 Example with the shift

In the primitive text wherefrom this paper is derived, the fourth example was again with a subspace \( E \) of \( \mathcal{C} = \mathcal{C}([0, 1], \mathbb{C}) \). But my impression was that this example was basically a shift.

So, in this section, we fix a \( p \in [1, +\infty] \) and work with the complex Banach space \( E = \ell^p \). An element \( x \) of \( \ell^p \) will be systematically written \( x = (x_n)_{n \in \mathbb{N}} \).

**Definition 5.1** \( S \) is the mapping from \( E \) to \( E \) such that, for every \( x \in \mathcal{C} \), \( Sx = y \) with \( y_n = x_{n+1} \).
Proposition 5.2  \(S\) from def. 5.1 has the following properties:

(i) \(S \in \mathcal{L}(\ell^p)\) and \(\|S\|_p = 1\).

(ii) \(S\) is onto.

(iii) \(\sigma(S) = \mathbb{D}\), with \(\mathbb{D} = \{\zeta \in \mathbb{C} / |\zeta| \leq 1\}\) (unitary disk of \(\mathbb{C}\)).

(iv) for \(\zeta \in \rho(S)\) (i.e. \(\zeta \in \mathbb{C}\) such that \(|\zeta| > 1\)), \(R_T(\zeta)(x) = y\), with \(y\) such that, for every \(n \in \mathbb{N}\):

\[
y_n = \sum_{k=0}^{\infty} \zeta^{-k-1} x_{n+k}.
\]

Among others, \(y \in \ell^p\).

Proof:

The linearity of \(S\) is obvious.

Let \(x \in \ell^p\)

- if \(p < +\infty\), \(\|Sx\|_p^p = \sum_{n \in \mathbb{N}} |x_{n+1}|^p = \sum_{n=1}^{\infty} |x_n|^p\).
  
  Since \(\|Sx\|_p^p + |x_0|^p = \|x\|_p^p\), \(\|Sx\|_p \leq \|x\|_p\). Hence \(\|S\| \leq 1\).
  
- if \(p = +\infty\), for every \(n \in \mathbb{N}\), \(|(Sx)_n| = |x_{n+1}| \leq \|x\|_\infty\), so \(\|Sx\|_\infty \leq \|x\|_\infty\).

Let \(\alpha \in ]0,1[.\) \(x\) defined by \(x_n = \alpha^n\) is in \(\ell^p\) and \(\|x\|_p > 0\). \(Sx = \alpha x\), so \(\|S\| \geq \alpha\). \(\alpha\) is as close as wanted of 1, so \(\|S\| = 1\).

It remains to show that \(S\) is onto. Now, if \(y \in \ell^p\), for \(k \in \mathbb{C}\) \(x\) defined by \(x_0 = k\) and \(x_n = y_{n-1}\) if \(n \geq 1\) is clearly in \(\ell^p\), and at once \(Sf = g\). Moreover \(S\) is not one-one, which proves that \(0 \in \sigma(S)\).

For \(\alpha \in \mathbb{C}\) such that \(0 < |\alpha| < 1\), \(x\) defined by \(x_n = \alpha^n\) is in \(\ell^p\) and \(x \neq 0\). \(Sx = \alpha x\), so \(\alpha\) is an eigenvalue of \(S\), a fortiori \(\alpha \in \sigma(S)\).

Since \(\sigma(S)\) is closed, \(\sigma(S) \supseteq \mathbb{D}\).

For \(\zeta > 1\), \(\zeta I\) is invertible, its inverse is \(\zeta^{-1} I\) and \(\|((\zeta I)^{-1})^{-1}\| = \zeta\).

Now, \(\|S\| = 1 < \zeta\), so, by the usual inversion theorem, \(\zeta I - S\) is invertible in \(\mathcal{L}(\mathbb{E})\), which proves \(\zeta \in \rho(S)\) and \(\zeta \notin \sigma(S)\).

Therefore \(\sigma(S) = \mathbb{D}\).
For $\zeta \in \rho(S) = \mathbb{C} \setminus \mathbb{D}$, we can apply the series expansion of $I - \zeta^{-1}S$ which is the basis of the inversion theorem 1:

$$(I - \zeta^{-1}S)^{-1} = \sum_{k=0}^{\infty} \zeta^{-k}S^k,$$

$$R_T(\zeta) = \sum_{k=0}^{\infty} \zeta^{-k-1}S^k.$$ 

$R_T(\zeta) = \zeta^{-1}(I - \zeta^{-1}S)^{-1} \in \mathcal{L}(\ell^p)$, so $R_T(\zeta)x \in \ell^p$. We can compute, for $n \in \mathbb{N}$, since $(S^kx)_n = x_{n+k}$:

$$(R_T(\zeta)x)_n = \sum_{k=0}^{\infty} \zeta^{-k-1}x_{n+k}.$$ 

\[\square\]

**Example 4** With $E = \ell^p$, let $F$ be the set of $x \in E$ such that $x_0 = 0$. $F$ is a closed vector subspace of $E$ as kernel of the continuous linear functional $x \mapsto x_0$. Let $J$ be the canonical injection from $F$ to $E$. Let $T$ be the mapping from $F$ to $E$ restriction of $S$ to $F$.

1. $T$ is a linear isometry from $F$ onto $E$.
2. $0 \in \rho(T)$.
3. If $\zeta \in \mathbb{C}$ and $|\zeta| > 1$, then $\zeta \in \sigma(T)$.

**Proof:**

$T$ is the restriction to $F$ of $S$, hence directly linearity and continuity of $T$ and since, for $x \in F$, $\|Tx\|_p = \|Sx\|_p \leq \|x\|_p$, $\|T\| \leq 1$.

$T$ is onto because, among the reciprocal images of $x \in \ell^p$ built in the proof of prop. 5.2, one (and only one), for $k = 0$, is in $F$: $y$ with $y_0 = 0$ and, for $n > 1$, $y_n = x_{n-1}$. Let $x \in F$.

- if $p < +\infty$, since $x_0 = 0$:

  $$\|Tx\|_p^p = \sum_{n \in \mathbb{N}} |x_{n+1}|^p$$

  $$= \sum_{n=1}^{\infty} |x_n|^p$$

  $$= \sum_{n=0}^{\infty} |x_n|^p - |x_0|^p$$

  $$= \|x\|_p^p.$$ 

  So $\|Tx\|_p = \|x\|_p$. 

if \( p = +\infty \), for every \( n \in \mathbb{N} \), \( |(Tn)x| = |x_{n+1}| \leq \|x\|_\infty \), so
\( \|Tx\|_\infty \leq \|x\|_\infty \).

But, \( |x_0| = 0 \leq \|Tx\|_\infty \) and \( |x_n| = |(Tn)x_{n-1}| \leq \|Tx\|_\infty \) for
\( n \in \mathbb{N}^* \), so \( \|Tx\|_\infty = \|x\|_\infty \).

Hence \( T \) is an isometry and \( \|T\| = 1 \). Therefore \( T^{-1} \) is a linear
isometry from \( E \) to \( F \), which implies (ii) since \( -T^{-1} = R_T(0) \).
Since \( F \), closed subspace of \( E \), is a Banach space endowed with
the induced norm, \( T \) is a closed operator.

Let \( \zeta \in \mathbb{C} \) such that \( |\zeta| > 1 \). We will search, for \( y \in E \), if there
exists \( x \in F \) such that \((\zeta J - T)x = y \), i.e. \( \zeta x - Sx = y \). Now,
there exists an only solution of this equation in \( \ell^p \), \( x = R_S(\zeta)y \),
verifying the equality, for each \( n \in \mathbb{N} \):

\[
x_n = \sum_{k=0}^{\infty} \zeta^{-k-1}y_{n+k},
\]

But have we \( x \in F \)? Take \( y \) defined by \( y_0 = 1 \) and \( y_n = 0 \) for
\( n > 0 \) (clearly in \( \ell^p \)). We have \( x_0 = \zeta^{-1} \) and \( x_n = 0 \) for \( n > 0 \) But
the so computed \( x \) is not element of \( F \), so \( y \) nas no antecedent by
\( \zeta J - T \), which proves \( \zeta \in \sigma(T) \).

\[\Box\]

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