Interplay of intra- and inter-dependence affects the robustness of network of networks

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Abstract. The existence of inter-dependence between multiple networks imparts an additional scale of complexity to such systems often referred to as “network of networks” (NON). We have investigated the robustness of NONs to random breakdown of their components, as well as targeted attacks, as a function of the relative proportion of intra- and inter-dependence among the constituent networks. We focus on bi-layer networks with the two layers comprising different number of nodes in general and where the ratio of intra-layer to inter-layer connections, \(r\), can be varied, keeping the total number of nodes and overall connection density invariant. We observe that while the responses of the different networks to random breakdown of nodes are similar, dominantly intra-dependent networks \((r \ll 1)\) are robust with respect to attacks that target nodes having highest degree but when nodes are removed on the basis of highest betweenness centrality (CB), they exhibit a sharp decrease in the size of the largest connected component (resembling a first order phase transition) followed by a more gradual decrease as more nodes are removed (akin to a second order transition). As \(r\) is increased resulting in the network becoming strongly inter-dependent \((r \gg 1)\), we observe that this hybrid nature of the transition in the size of the largest connected component in response to targeted node removal (based on highest CB) changes to a purely continuous or second-order transition. We also explore the role of layer size heterogeneity on robustness, finding that for a given \(r\) having layers comprising very different number of nodes results in a bimodal degree distribution. For dominantly inter-dependent networks, this results in the nodes of the smaller layer becoming structurally central. Selective removal of these nodes, which constitute a relatively small fraction of the network, leads to breakdown of the entire system - making the inter-dependent networks even more fragile to targeted attacks than scale-free networks having power-law degree distribution.

Keywords. Networks, Inter-dependence, Robustness, Percolation transition

1. Introduction

Robustness, a property often attributed to complex systems occurring in nature, refers to their ability to maintain most of their vital functions even when subjected to noise or perturbations, both extrinsic and intrinsic, that may result in loss or damage of a significant fraction of their components [1]. The investigation of robust systems, especially those that occur in biology and ecology, with the aim of identifying the features that contribute to their ability to withstand component failures or attacks on parts thereof, have obvious implications in terms of applications. These include designing robust man-made systems, as well as, arriving at fail-safe strategies to reduce vulnerabilities of existing systems such as the electrical power grid, where an initially small local perturbation (such as shorting caused by a branch falling on a transmission line) can occasionally trigger a massive system-wide breakdown resulting in power blackouts over entire regions [2]. As many complex systems can be represented as networks, with the components represented as nodes while the interactions between them are represented as links, robustness can also be measured in terms of the ability of a system to maintain its integrity even after a specified fraction of its nodes and/or links have been removed [3]. An oft-cited example is the internet, comprising servers (nodes) connected by data cables (links), whose functioning should not be affected significantly by temporary loss of components through failures occurring randomly, as well as, malicious denial-of-service attacks that may target specific nodes [6]. Following the 2007-9 financial crisis, the robustness of the network of financial institutions has also been the subject of intense investigation by scientists who seek to understand factors contributing to systemic risk that can cause credit default by a few firms to eventually result in an overall economic catastrophe [7].

Complicating the already difficult question of what factors lead to robustness of complex networks is the
fact that in reality, most networks do not operate completely in isolation but often are seen to interact with other equally complex networks. Moreover, inter-dependent between multiple networks could be a crucial feature underlying the proper functioning of each of them. An example is the coupled system of the electrical power grid and the communication network of computers [8]. While the network of computers control the functioning of the power grid, the computers are dependent on the grid for their power. Failure in nodes of one of the networks (e.g., shutting down of a power generation unit) would affect nodes in the other network (e.g., disrupting the communication between computers), which in turn will lead to further breakdowns of both the networks in a recursive fashion [9, 10, 11]. In general, inter-dependent networks can be seen as comprising different layers in a composite network of networks (NON).

A strikingly novel aspect of inter-dependent networks is that they typically respond very differently to structural perturbations such as removal of a fraction of their nodes when compared to the behavior of the component networks in isolation. In particular, inter-dependent networks exhibit a first order phase transition in the size of the largest connected component when nodes are gradually removed, which changes to a continuous transition when the fraction of inter-dependent nodes is reduced [12]. Assuming that only nodes belonging to the largest connected component remain functional, this would suggest that inter-dependent networks are more vulnerable to node failure and targeted attacks than the individual systems that they comprise [8, 13]. While a few earlier studies have considered the role of intra-, as well as, inter-network dependences in determining the robustness of NONs [14, 15, 16], it is important to keep the average degree of the nodes invariant when comparing systems with different ratios of intra- to inter-network connections (as otherwise we cannot disambiguate the contribution of the overall number of connections from that specifically of the inter-dependent links). In addition, the different networks have often been chosen to be of the same size. However, in reality, NONs can comprise component networks comprising widely differing number of nodes. In this paper we report the results of a systematic investigation of the robustness of NONs to different types of node removal strategies, incorporating the different aspects mentioned above.

2. Model

The model system we consider for our investigation is a NON of two networks comprising \( N_1 \) and \( N_2 \) nodes, respectively. In order to analyze the relative contributions of intra- and inter-dependence in this system, we alter the probabilities of a connection between nodes belonging to the same layer (\( p_{\text{intra}} \)) and those belonging to different layers (\( p_{\text{inter}} \)). This is done by assigning different values to the ratio \( r = \frac{p_{\text{intra}}}{p_{\text{inter}}} \) while keeping the total size of the NON \( N = N_1 + N_2 \) and the average degree of the network \( \langle k \rangle \) invariant [17]. For \( r \ll 1 \), the NON is dominantly intra-dependent [Fig. 1(a)], while it is dominantly inter-dependent if \( r \gg 1 \) [Fig. 1(c)]. The special case of \( r = 1 \) corresponds to a homogeneous Erdős-Rényi network [Fig. 1(b)]. Thus, as \( r \) is increased from 0, the NON changes gradually from being completely intra-dependent (consisting of two isolated modules) in one limit to completely inter-dependent (corresponding to a bipartite network, which can be viewed as a hierarchical network consisting of two levels) in the other limit.

Randomly connected bi-layer networks where the two layers are of the same size \( (N_1 = N_2) \) have Poisson degree distributions regardless of \( r \) [Fig. 1(d-e)],
solid curves]. However, if $N_1$ and $N_2$ are very different, this results in the two layers having very different average degrees (even though average degree of the NON, $\langle k \rangle$, remains unchanged) with the overall degree distribution exhibiting a bimodal form [broken curves in Fig. 1(d-e)]. The exact profile of the bimodal distribution depends on the value of $r$, with the lower peak corresponding to the smaller (larger) layer for dominantly intra-dependent (inter-dependent) networks.

We have considered the robustness of the model bi-layer networks described above using a standard percolation theoretic approach [3]. Specifically, we remove nodes one at a time using different strategies, e.g., at random or choosing nodes having the highest degree or betweenness centrality (CB). After removing a fraction $f$ of the $N$ nodes in the NON, we measure the probability that a randomly chosen node is still part of the largest connected component (LCC) of the NON after these removals [$P_{LCC}(f)$], by expressing it in terms of the probability that the node was part of the LCC of the NON before any nodes were removed [$P_{LCC}(0)$]. Note that, for a homogeneous Erdős-Renyi random network (for $r = 1$), it is well-known that even after removal of a fraction $f$ of the nodes [such that the effective size of the network is now $N_{eff} = (1 - f)N$], the Poisson character of the degree distribution is preserved with only the effective average degree reducing to $k_{eff} = (1 - f)k$.

As the condition for a Erdős-Renyi network to possess a giant component is $\langle k^2 \rangle / \langle k \rangle \geq 2$ [18], the critical value of fraction of nodes removed beyond which the network exhibits a transition to isolated fragments is given by $f_c = 1 - (1/\langle k \rangle)$. This provides a natural benchmark against which to compare the robustness of the random bi-layer networks in response to removal of a fraction of their nodes. We have also compared the results with that of the Price-Barabasi-Albert scale-free network that has been shown to be more robust with respect to random removal of nodes compared to Erdős-Renyi networks, but extremely vulnerable to attacks targeted at nodes having highest degree or CB [5].

3. Results

We first consider the response of bi-layer networks to removal of nodes chosen at random for NONs characterized by different ratios of intra- and inter-dependence and where the layers are of same size [Fig. 2(a)]. We observe that regardless of $r$, the networks exhibit a similar response profile to removal of nodes. A second-order transition is seen to occur at a critical value $f_c \sim 0.9$ of the fraction of nodes removed, where the system reduces to several disconnected fragments. Introducing

layer size heterogeneity does not appreciably alter the results as can be seen from panels (d) and (g) of Fig. 2 that correspond to $N_1 = 2N_2/3$ and $N_1 = N_2/4$, respec-

![Figure 2](Color online) Robustness of random networks comprising two layers (with $N_1$ and $N_2$ nodes, respectively) that have different proportions of intra- and inter-layer dependence shown for different types of node failures and layer size heterogeneity. On removing a fraction $f$ of the nodes in the network, the probability $P_{LCC}(f)$ that a node will be part of the largest connected component (LCC) is expressed relative to the probability $P_{LCC}(0)$ that it was part of the LCC in the original network. The first row (panels a-c) shows the situation where the two layers possess an identical number of nodes (i.e., $N_1 = N_2$), while the second row (panels d-f) and third row (panels g-i) considers layers of unequal size, viz., $N_1 = 2N_2/3$ and $N_1 = N_2/4$, respectively. In all cases the total size of the network $N = N_1 + N_2 = 500$ and average degree $\langle k \rangle = 10$. In each row, different panels show the robustness of a network against different types of node failure protocols, corresponding to removal of nodes at random (left), according to highest degree (center) and according to highest betweenness centrality (right). Each panel shows the response to successive removal of nodes for networks that are (i) dominantly intra-dependent ($r = 0.01$, blue squares), (ii) homogeneous ($r = 1$, maroon triangles) and (iii) dominantly inter-dependent ($r = 100$, black circles). For comparison, we also show the response of Price-Barabasi-Albert scale-free networks (red curve). Each data point is obtained by averaging over 10 network realizations. We note that while for random breakdown of nodes, the response of the different networks have similar profiles, with respect to attacks that target nodes having highest degree or betweenness centrality, the dominantly inter-dependent networks are relatively more vulnerable than the other types of networks when the layers have very different sizes. In addition, the dominantly intra-dependent networks show a sharp decrease in the fraction remaining in LCC for low $f$ when the attacks target nodes having highest betweenness centrality.
We next consider robustness of the NON against targeted attacks aimed at structurally important nodes. These could either be the hubs, i.e., nodes having the highest degree, or may be connecting a large number of nodes to each other through shortest paths that pass through them, i.e., nodes with highest CB [4]. We observe that dominantly intra-dependent networks are almost as robust as Erdős-Rényi networks against attacks targeted at highest degree nodes, while the dominantly inter-dependent networks are only marginally less robust [Fig.2(b)]. We observe that at around \( f_e \sim 0.5 \), the networks exhibit a smooth transition to fragmentation. Note that, the Price-Barabasi-Albert scale-free network is much less robust against degree-based attacks and collapses at \( f_e \sim 0.4 \). With increasing layer size heterogeneity however, the dominantly inter-dependent networks become increasingly fragile with the transition to fragmented state occurring at critical values of \( f \) that may be even lower than that for scale-free networks [see panels (e) and (h) of Fig. 2]. By contrast, intra-dependent networks do not show any variation with respect to changing sizes of the layers.

Dominantly inter-dependent networks show a similar behavior when instead of targeting highest degree nodes, highest BC nodes are removed preferentially [panels (c), (f) and (i) of Fig. 2]. However, the dominantly intra-dependent networks exhibit a strikingly different response, with the size of the LCC showing a very sharp decrease (resembling a first-order phase transition) from \( N \) to \( N_1 \) upon removing only about 3% of the nodes. This suggests that at this value of \( f \) (\( \sim 0.03 \), the layers of the NON become isolated from each other. Following this, the effect of removing additional nodes according to highest CB is similar to that for Erdős-Rényi networks and consequently, we observe a continuous transition to the fragmented state, explaining the hybrid phase transition seen for the case of dominantly intra-dependent networks. Increasing layer size heterogeneity only changes this picture by decreasing the critical value of \( f \) at which the initial sharp decrease in the LCC size occurs, as well as the magnitude of the decrease.

The response of the dominantly inter-dependent networks with respect to targeted attacks on nodes (based either on highest degree or highest CB) as layer size heterogeneity increases can be understood in terms of the changing connectivity profile as revealed by the degree distribution [Fig. 1(e)]. When the two layers are similar in terms of size, almost all nodes are equivalent in terms of their degree. Thus, the response of the network to attacks will be almost identical to that seen for Erdős-Rényi networks. However, when the sizes of the two layers are very different, the nodes of the smaller layer typically would have much higher degree than the average degree of the NON [as revealed by the bimodal degree distribution shown in panel (e) of Fig. 1]. Thus, these will function as hubs of the network. Targeting these relatively fewer number of nodes will severely damage the network in terms of connectivity. However, identifying such nodes in dominantly inter-dependent NONs and providing them additional protection will be an efficient procedure for increasing the robustness of the entire system.

4. Conclusion

In this paper we have reported the results of our investigation on the role played by intra- and inter-dependence in imparting robustness to NONs by considering an ensemble of model random bi-layer networks. By systematically varying the relative density of intra- and inter-layer connections we show that increasing inter-dependence can make such NONs vulnerable to targeted attacks on nodes, especially when different layers are populated by very different numbers of nodes. This can be related to the very different connectivity profiles of the nodes in the two layers, manifested in a bimodal degree distribution for the NON. We also observe that when faced with attacks targeted at nodes having highest CB, increased dominant intra-dependence results in a hybrid transition. This corresponds to an initially sharp decrease in the size of the LCC (resembling a first-order phase transition) followed by a continuous or second-order transition with increasing fraction of nodes removed. As in NONs occurring in nature the sizes of the different component networks can be quite different, our results may provide insights into their robustness and help in suggesting guidelines for constructing more robust artificial NONs.

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