Marginal Trapped Surfaces in the Nonsymmetric Gravitational Theory

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We consider a simple, physical approach to the problem of marginally trapped surfaces in the Nonsymmetric Gravitational Theory (NGT). We apply this approach to a particular spherically symmetric, Wyman sector gravitational field, consisting of a pulse in the antisymmetric field variable. We demonstrate that marginally trapped surfaces do exist for this choice of initial data.

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I. THE METRICS OF SPHERICALLY SYMMETRIC, WYMAN SECTOR NGT

In general, the causal structure of NGT is not known. However, for the special case of spherically symmetric, Wyman sector data, the causal structure of NGT, whether old (see [14]), new (see [15]), or dynamically-constrained (see [16]) collapses back to its Unified Field Theory (UFT) form (see [17,18,19]). The causal structure of UFT was investigated by F. Maurer-Tison (see [20]); there it was found that the field equations possessed three (most likely distinct) characteristic signatures. This definition can be put into 3+1 form by means of a simple argument due to York (see [21], p. 100). Us-

In spherically symmetric, Wyman sector NGT

We use greek letters for spacetime indices and reserve latin letters for spatial indices. The $g_{\mu\nu}$ are the components of the fundamental tensor, and $h$ and $g$ are the determinants of the matrices formed by $h_{\mu\nu}$ and $g_{\mu\nu}$, respectively. The fact that there are three metrics implies an ambiguity in the measurement of geometrical quantities. We see no a priori reason to choose any one metric over another, and hence we will consider all three.

In spherically symmetric, Wyman sector NGT, the fundamental tensor takes the form $g^{-1} = e_\perp \otimes e_\perp - \phi^{-4} \gamma^{ij} e_i \otimes e_j$, where the surface fundamental tensor is

$$\gamma^{ij} = \begin{pmatrix} \gamma_{11} & \gamma_{22} & \gamma_{23} \sin \theta/\gamma_{23} \\
-\gamma_{23} \sin \theta & -\gamma_{23} \sin^2 \theta/\gamma_{23} \end{pmatrix}$$

and $\phi$ is the conformal factor, solution to the Hamiltonian constraint (see [22] for a discussion on the solvability of the Hamiltonian constraint in NGT). The basis vectors form a surface-adapted frame, with the $e_i$ forming a basis in a spatial hypersurface $\Sigma$, and $e_\perp$ being normal to this hypersurface. These basis vectors are related to the holonomic coordinate frame by $\partial_i = N e_\perp + N^i e_i$ and $\partial_\perp = e_\perp$, where $N$ is the lapse function and $N^i e_i$ is the shift vector. We take $\gamma_{22} = r^{-2} \cos \psi$ and $\gamma_{23} = r^{-2} \sin \psi$, where $r$ is the radial coordinate and $\psi$ is an arbitrary function characterizing the strength of the antisymmetric variables versus the symmetric variables; in particular, $\gamma_{23}^2/\gamma_{22}^2 = \tan \psi$.

For the details of the Hamiltonian formulation of NGT and its initial-value problem, we refer the reader to the literature (see [20] in addition to [17,18,19]).

In this parametrization, it can be verified that $h_{\mu\nu} = \text{diag}[1, -\phi^4/\gamma_{11}, -\phi^4 r^2 \cos \psi, -\phi^4 r^2 \cos \psi \sin^2 \theta]$, $g_{\mu\nu} = \text{diag}[1, -\phi^4/\gamma_{11}, -\phi^4 r^2 \sec \psi, -\phi^4 r^2 \sec \psi \sin^2 \theta]$, and $h_{\mu\nu} = \text{diag}[(\cos^2 \psi - \sin^2 \psi) \gamma_{11}^{-1}, -\phi^4 \gamma_{11}^{-1} \gamma_{23} \cos \psi, -\phi^4 \gamma_{23} \cos \psi \sin^2 \theta]$. We note in passing that if $\psi = \pi/4$, then this last metric is degenerate, while in a region $\pi/4 - \epsilon < \psi < \pi/4 + \epsilon$ it changes signature. These pathologies occur despite the fact that $\psi \sim \pi/4$ does not correspond to a particularly strong field region in NGT, since $\gamma_{22} \sim \gamma_{23}$.

II. MARGINALLY TRAPPED SURFACES IN NGT

The standard definition of a marginally trapped surface in General Relativity is as follows. Consider a spatial slice $\Sigma$ into which is embedded a 2-surface $S$ (see Fig. 1, where the 2-surface $S$ is represented as a circle). Let $n = n^\mu e_\mu$ be the (timelike) normal vector field to $S$, and let $s = s^\mu e_\mu$ be the (spacelike) normal vector field to $S$. We therefore have $n \cdot n = 1$ and $s \cdot s = -1$, by proper normalization. It follows that $l = n + s$ is a null vector field: $l \cdot l = 0$. The 2-surface $S$ is a marginally trapped surface if the hypersurface generated by $l$ has vanishing expansion, i.e., if the trace of its extrinsic curvature vanishes. This definition can be put into 3+1 form by means of a simple argument due to York (see [23], p. 100). Using the projection operator $P$ that projects a spacetime quantity onto the surface generated by $l$,

$$P = (1 - n \otimes n) \cdot (1 + s \otimes s) = 1 - n \otimes n + s \otimes s$$

or

$$P_{\beta}^\alpha = \delta_{\beta}^\alpha - n^\alpha n_\beta + s^\alpha s_\beta,$$
then the extrinsic curvature of interest is \( \kappa = -(P \otimes P') \cdot (\nabla |l|) \) or \( \kappa_{ij} = -P_{\mu} P_{\nu} \nabla_{\alpha} |l|^2 \), according to the usual definition. The trace of \( \kappa \) is the expansion of the surface generated by \( l \); making this trace vanish yields the apparent horizon equation:

\[
(3) \nabla \cdot s - \text{Tr}[K] + K(s, s) = 0 \tag{2a}
\]

or

\[
(3) \nabla[s]^i - K_i^i + K_{ij}s^is^j = 0. \tag{2b}
\]

FIG. 1. A spatial slice \( \Sigma \) of a spacetime is shown, with an embedded surface \( S \), represented here as a circle on \( \Sigma \). The timelike vector \( n \) is normal to \( \Sigma \), while the spacelike vector \( s \) is normal to \( S \), and is thus tangent to \( \Sigma \). The null vector \( l \) generates the evolution of \( S \). The surface \( S \) is marginally trapped if the rate of change of its area vanishes along \( l \): \( \partial_l [A] = 0 \).

It is difficult to generalize the above derivation to NGT: it relies on the definitions of both the extrinsic curvature and the covariant derivative. Presumably, some generalization can be given, however since both of these concepts are ambiguously defined in NGT, it is preferable to approach the problem from a different perspective.

We again consider a spatial slice \( \Sigma \) into which is embedded a 2-surface \( S \). Consider a null vector field \( l \), where null is defined with respect to one of the three metrics in [10]. This vector field generates a one-parameter congruence off the surface \( \Sigma \); this parameter, which we denote by \( \tau \), represents the time for “observers” flowing along the curves drawn out by \( l \). Let \( A \) be the area of the 2-surface \( S \). Suppose we measure the area \( A(\tau_0) \) at some \( \tau = \tau_0 \). We then travel along the flow to some \( \tau = \tau_0 + d\tau \), where \( d\tau \ll \tau_0 \), and re-measure the area \( A(\tau_0 + d\tau) \). If \( A(\tau_0 + d\tau) - A(\tau_0) = 0 \), then the surface \( S \) is defined to be marginally trapped. In other words, for infinitesimal \( d\tau \),

\[
A(\tau_0 + d\tau) - A(\tau_0) \rightarrow \frac{dA(d\tau)}{d\tau} \bigg|_{\tau=\tau_0} = \partial_l [A] \equiv 0. \tag{3}
\]

Therefore, a marginally trapped surface has the property that its area is unchanging along the flow generated by a null vector field \( l \). Note that [10] contains a partial derivative, since the area \( A \) is, strictly speaking, a 2-form: \( \partial_l [A] = (dA, l) \), where \( A \) is the area 2-form of \( S \) and \( dA \) is its exterior derivative.

III. A SIMPLE ARGUMENT FOR THE EXISTENCE OF MARGINALLY TRAPPED SURFACES IN NGT

For a spherically symmetric, Wyman sector gravitational field it is a simple matter to demonstrate that the area of a surface of constant \( r \) is given by \( A(r) = 4\pi g_{22} \), where \( g_{22} \) is the 22-component of one of the three metrics given above. This gives us three possible values for the surface area of a sphere of radius \( r \) centred on the origin, two of which are equal:

\[
A_h(r) = 4\pi r^2 \phi^4 \cos \psi
\]

and

\[
A_l(r) = A_h(r) = 4\pi r^2 \phi^4 \sec \psi.
\]

Here, \( \phi(r) \) is the conformal factor, solution to the Hamiltonian constraint. For a moment of time symmetry, [3] simplifies considerably and we find that a marginally trapped surface obtains if \( \partial_t [A] = 0 \). We thus have three equations, two of which are identical, whose solution gives the location of a marginally trapped surface:

\[
f_h(r) = 1 - \frac{M}{2r} + \partial_t [M] - \frac{r \tan \psi \partial_r \psi}{2} \left[ 1 + \frac{M}{2r} \right] = 0 \tag{4a}
\]

and

\[
f_l(r) = f_h(r) = 1 - \frac{M}{2r} + \partial_r [M] + \frac{r \tan \psi \partial_r \psi}{2} \left[ 1 + \frac{M}{2r} \right] = 0, \tag{4b}
\]

where we have introduced the mass function \( M(r) \), defined by \( \phi(r) = 1 + M(r)/2r \). For an asymptotically flat spatial slice, the mass function converges to a constant, \( \mathcal{M}_{ADM} \), the ADM mass of the system. Setting \( \psi = 0 \) demonstrates that these equations indeed reduce to their GR counterpart: \( f_{GR}(r) = 1 - M/2r + \partial_r [M] = 0 \).

Consider an initial data set consisting of some arbitrary function \( \gamma^{11} \) that falls off asymptotically (see [17], p. 116, for the precise requirement), a function \( \psi \), and a vanishing extrinsic curvature. We assume that this initial data set forms a well-posed initial-value problem, and hence that there exists a solution to the Hamiltonian constraint, \( \phi \). The function \( \psi \) is taken to be more or less localized about a point \( r_0 \) of the initial slice. Thus, \( \psi \) can be characterized by three parameters: an overall amplitude factor \( A \) that essentially serves to fix the size of the ADM mass of the system, the position of its peak \( r_0 \), and its width \( \sigma \). The exact form of \( \psi \) is irrelevant: all that is required is that \( \sigma \ll r_0 \). In the regions \( r \lesssim r_0 - \sigma \) and \( r \gtrsim r_0 + \sigma \), we assume that \( \psi \) is negligible relative to its peak value at \( r = r_0 \). For such a data set, it is
a relatively simple form to demonstrate that the mass function takes the form (see [16])

\[ M(r) \approx \begin{cases} M_{\text{ADM}} r / r_0 & \text{for } r \lesssim r_0 - \sigma, \text{ while} \\
M_{\text{ADM}} & \text{for } r \gtrsim r_0 + \sigma. \end{cases} \]

The corrections are of the order \( \sigma / r_0 \), so that we require that \( \sigma / r_0 \ll 1 \). This approximation is most inaccurate in the region \( r_0 - \sigma < r < r_0 + \sigma \); however, since we assume that \( \sigma / r_0 \ll 1 \), this region is of negligible importance. Placing ourselves in a region where \( \psi \) is small, \( \ldots, \) somewhere in the regions \( r \lesssim r_0 - \sigma \) or \( r \gtrsim r_0 + \sigma \), we conclude that both marginally trapped surface \( f(r) \approx \begin{cases} 1 + M_{\text{ADM}} / 2 r_0 & \text{for } r \lesssim r_0 - \sigma, \text{ while} \\
1 - M_{\text{ADM}} / 2 r & \text{for } r \gtrsim r_0 + \sigma. \end{cases} \) (5)

We have neglected terms of order \( \psi^2 \). We conclude from (3) that if the ADM mass of the gravitational system is sufficiently small, that is, if \( M_{\text{ADM}} < 2 r_0 \), then \( f(r) \) is strictly positive, and no marginally trapped surface exists. On the other hand, if \( M_{\text{ADM}} > 2 r_0 \), then \( f(r) \) crosses the axis at \( r = M_{\text{ADM}} / 2 \), and there exists a marginally trapped surface. In [16], it was found that this inequality could be satisfied by a large range of (physically admissible) values of the parameters \( \{A, r_0, \sigma\} \). However, note that this analysis is not valid for \( M_{\text{ADM}} \approx 2 r_0 \), for then the marginally trapped surface would reside at \( r \sim r_0 \), while we have stated from the outset that our approximations are not valid in such a régime.

Recently, there has been some controversy on the formation of black holes in NGT (see [21,22]). Although our results will eventually help in resolving these issues, we should point out that they do not comprise a proof that black holes indeed form in a generic stellar collapse. We expect that our current numerical studies will answer these questions.

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