DOUBLE SCALING LIMIT OF
THE SUPERVIRASORO CONSTRAINTS

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ABSTRACT

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We obtain the double scaling limit of a set of superloop equations recently proposed to describe the coupling of two-dimensional supergravity to minimal superconformal matter of type $(2, 4m)$. The continuum loop equations are described in terms of a $\hat{c} = 1$ theory with a $Z_2$-twisted scalar field and a Weyl-Majorana fermion in the Ramond sector. We have computed correlation functions in genus zero, one and partially in genus two. An integrable supersymmetric hierarchy describing our model has not yet been found. We present a heuristic argument showing that the purely bosonic part of our model is described by the KdV-hierarchy.
1. INTRODUCTION

A description of the coupling of $N = 1$ superconformal field theories [1] to two-dimensional supergravity in terms of discrete triangulations of super-Riemann surfaces is an interesting problem still unsolved. The continuum analysis carried out in [2][3] has not yet found a complete discrete counterpart. The use of the KdV-Petviashvili (KP) hierarchy initiated by Douglas [4] in the analysis of the double scaling limit of the purely bosonic theories [5] was not fully successfully pursued for the unitary superconformal $(m, m+2)$ models in [6] in terms of the Manin-Radul [7] supersymmetric extension of the KP-equations. The root of the problem was the incompatibility between the string equation and the fermionic flows. In [8] a proposal was made to describe the coupling between $(2, 4m)$ superconformal models and world-sheet supergravity. The model is based on a set of superloop equations which are motivated by analogy with the bosonic case [9]. The planar solution to this model allowed the construction of bosonic and fermionic operators whose dressed gravitational dimensions are in agreement with the continuum limit result for the $(2, 4m)$-models [2,3]. The planar solution was constructed for simplicity starting with even bosonic potentials. This as expected generated a doubling of degrees of freedom in the Neveu-Schwarz (NS) and Ramond (R) sectors of the theory. It was also possible to obtain the correlation functions of an arbitrary number of planar superloops.

The guiding principle in [8] was a set of super-Virasoro constraints satisfied by the partition function, which code the superloop equations. Since the Virasoro constraints [10][11] played a prominent role in the Witten-Kontsevich theory (the intersection theory of certain line bundles on the moduli space $\mathcal{M}_{g,n}$ of genus $g$-surfaces with $n$ distinguished points) [12], it is reasonable to expect that a similar set of super-Virasoro constraints should capture some important features of the supermoduli space of super-Riemann surfaces. The model obtained in this way is an “eigenvalue” model. It is formulated in terms of a collection of $N$ even and $N$ odd eigenvalues $(\lambda_i, \theta_i)$. We have still not succeeded in finding a description of the
model in terms of generalized matrices, whose large-$N$ expansion may provide a geometrical interpretation in terms of triangulated super-surfaces. The aim of this paper is to continue the analysis of this eigenvalue model. In section two we briefly formulate the model and its loop equations, in section three we solve it in the planar limit for general potentials. We identify the scaling limit, the critical points and see how the doubling of degrees of freedom is lifted for general potentials in section four. In section five we obtain the continuum limit for bosonic and fermionic loops. We prove, that in the continuum limit they become respectively a $Z_2$-twisted scalar and a Weyl-Majorana fermion in the Ramond sector. Thus the super-Virasoro constraints in the double scaling limit are described by the super-energy-momentum tensor of a $\hat{c} = 1$ superconformal field theory. In section six we obtain the solution to the continuum loop equations in genus zero, one and partially in genus two. For genus zero this provides a good verification of previous arguments. We find that the terms independent of fermionic couplings in the free energy coincides with the result of the bosonic model. This result is in slight conflict with the genus one calculation of [13]. However, there was no clear way of normalizing the supermoduli integration in [13], and therefore this may be the origin of the discrepancy. Motivated by the agreement up to genus two of the purely bosonic partition function of our model with the one appearing in the Kazakov multicritical points, we present a heuristic proof that this equivalence works to all orders of string perturbation theory. On the basis of our computations plus the heuristic proof, we conjecture that the KdV hierarchy describes the even flows of our model when all fermionic couplings are set to zero. This is indeed the hallmark for a supersymmetric extension of the KdV hierarchy, although at present we have not been able to identify it fully. In section seven we briefly review the difficulties we find in identifying our bosonic and fermionic flows with known supersymmetric extensions of the KdV- or KP-hierarchies [7,14,15,16]. In [8] the Mulase-Rabin [14] extension of KP seemed a promising candidate to describe our model. Recently some work in this direction has been done [17]*. The difficulties

* In this paper the double scaling limit of the super-Virasoro constraints is also obtained.
we find may be related to the fact pointed out in [15] that the algebro-geometric solutions of the Mulase-Rabin or Manin-Radul hierarchies do not describe the moduli space of super-Riemann surfaces, but rather the moduli space of algebraic curves with genus $g$ and a generic line bundle of degree $g - 1$. In this sense, our difficulties can be interpreted positively. It may be disappointing to find that the part of the free energy without fermion couplings coincides with the results of one-matrix models. Perhaps one should look at this result more positively, because it indicates that our model does describe geometric objects which extend fermionically Riemann surfaces, even though we have not formulated the model in terms of triangulations. The kinds of geometries described by the Super-loop Equations depend very much on the type of supersymmetric extension of the KdV hierarchy embodied by our model. Section eight contains the conclusions and outlook.

2. SUPERLOOP EQUATIONS

We first review some properties of the purely bosonic one-matrix models and see how they are generalized in the supersymmetric case.

The general one-matrix model partition function is

$$Z = \int d^{N^2} \Phi \exp \left[ - \frac{N}{\Lambda} \text{tr} V(\Phi) \right],$$

$$V(\Phi) = \sum_{k \geq 0} g_k \Phi^k \quad \Lambda = e^{-\mu_B} \quad (2.1)$$

where $\Phi$ is a Hermitian $N \times N$ matrix, and $\mu_B$ is the bare cosmological constant. The starting point of Kazakov’s analysis of multicritical points [18] was a set of planar loop equations:
\[ \sum_{k \geq 1} k g_k \frac{\partial^{k-1}}{\partial l^{k-1}} w(l) = \int_0^l dl' w(l - l') w(l') , \quad (2.2) \]

\( w(l) \) describes a loop of length \( l \) bounding a surface with the topology of a disk. Equation (2.2) is the planar limit of the Schwinger-Dyson equations satisfied by the loop operator

\[ w(l) = \frac{\Lambda}{N} \text{tre}^{t \Phi} = \sum_{n=0}^{\infty} \frac{l^n}{n!} w^{(n)} \quad (2.3) \]

in the general one-matrix model (2.1). Writing the partition function in terms of the free energy \( Z = e^{N^2 F} \), \( F = F_0 + N^{-2} F_1 + \ldots \) the moments \( w^{(n)} \) can be represented as:

\[ w^{(0)} = \Lambda \quad w^{(n)} = -\Lambda^2 \frac{\partial F}{\partial g_n} , \quad (2.4) \]

and the loop equations are equivalent to a set of Virasoro constraints satisfied by (2.1) [19]. They are obtained by implementing invariance of the partition function (2.1) under the change \( \Phi \to \Phi + \epsilon \Phi^{n+1} \), \( n \geq -1 \) in (2.1):

\[ L_n Z = 0 \quad n \geq -1 , \quad (2.5) \]

\[ L_n = \frac{\Lambda^2}{N^2} \sum_{k=0}^{n} \frac{\partial^2}{\partial g_{n-k} \partial g_k} + \sum_{k \geq 0} k g_k \frac{\partial}{\partial g_{k+n}} . \quad (2.6) \]

Using the Laplace transform of the loop operator (2.3),

\[ w(p) = \int_0^\infty e^{-pl} w(l) dl = \sum_{k=0}^{\infty} w^{(k)} \frac{1}{p^{k+1}} , \quad (2.7) \]

and defining:

\[ \chi(p, q) = \sum_{k,l \geq 0} \frac{\chi_{k,l}}{p^{k+1} q^{l+1}} \quad \chi_{k,l} = \Lambda^4 \frac{\partial^2 F}{\partial g_k \partial g_l} \quad (2.8) \]

the loop equation, which is equivalent to the Virasoro constraints (2.5) and (2.6)
is
\[ w(p)^2 - V'(p)w(p) + \frac{1}{N^2}\chi(p, p) = \text{Polynomial}(p) \]  
(2.9)

It is easy to check that (2.9) in the limit \( N^2 \rightarrow \infty \) becomes (2.2). The explicit form of the polynomial on the right hand side is irrelevant. A useful way of rewriting (2.9) is obtained after introducing an infinite set of creation and annihilation operators:
\[ \alpha_{-n} = -\frac{N}{\Lambda\sqrt{2}}ng_n, \quad n > 0 \quad ; \quad \alpha_n = -\frac{\Lambda\sqrt{2}}{N}\frac{\partial}{\partial g_n}, \quad n \geq 0 \]  
(2.10)

and a scalar field:
\[ \partial\varphi(p) = \sum_{n \in \mathbb{Z}} \alpha_n p^{-n-1} \]  
(2.11)

with energy-momentum tensor:
\[ T(p) = \frac{1}{2} : \partial\varphi(p)\partial\varphi(p) : \]  
(2.12)

Then, (2.9) becomes
\[ Z^{-1}T(p)Z = \text{Polynomial}(p) \]  
(2.13)

Furthermore, if we write the partition function (2.1) in terms of the eigenvalues of the \( \Phi \)-matrix \( (\lambda_1, \lambda_2, \ldots, \lambda_N) \), but leaving the measure \( \Delta^2(\lambda) \) undetermined:
\[ Z = \int \prod_i d\lambda_i \Delta^2(\lambda)e^{-\frac{\chi}{N} \sum_i V(\lambda_i)} \]  
(2.14)

the constraints (2.5), (2.6) yield a differential equation satisfied by \( \Delta \)
\[ \sum_i \lambda_i^{n+1}\frac{\partial\Delta}{\partial\lambda_i} = \Delta \sum_{i \neq j} \frac{\lambda_i^{n+1}}{\lambda_i - \lambda_j} \]  
(2.15)

whose solution up to a constant is the expected Van-der-Monde determinant
\[ \Delta = \prod_{i<j}(\lambda_i - \lambda_j) \]  
(2.16)

In the supersymmetric case we proceed by analogy [8] with the above arguments
to obtain the corresponding loop equations. The loop operator depends on two variables \((l, \theta)\) where \(l\) is even and \(\theta\) is odd. We can define the Laplace transform as well
\[
\begin{align*}
   w(p, \Pi) & \equiv v(p) + \Pi u(p) = \int_0^\infty dl \int d\theta e^{-pl-\Pi\theta} w(l, \theta) .
\end{align*}
\]
(2.17)
In particular, the Laplace transform of the operator \(D = \frac{\partial}{\partial \Pi} + \Pi \frac{\partial}{\partial p}\) is \(P = \theta - z \frac{\partial}{\partial \Pi}\). Assuming the loop \(w(l, \theta)\) to behave well at \(l = 0, \infty\), we can expand \(w(p, \Pi)\) in inverse powers of \(p\):
\[
\begin{align*}
   v(p) &= \sum_{k \geq 0} v^{(k)} p^{k+1} ; \quad u(p) = \sum_{k \geq 0} u^{(k)} p^{k+1} ,
\end{align*}
\]
(2.18)
\(v(p)\) and \(u(p)\) are respectively the fermionic and bosonic loops. To define the moments \(v^{(k)}\), \(u^{(k)}\) in terms of the free energy \(F = \frac{\ln Z}{N}\) we introduce bosonic and fermionic oscillators:
\[
\begin{align*}
   \alpha_p &= -\frac{\Lambda}{N} \frac{\partial}{\partial g_p} ; \quad \alpha_{-p} = -\frac{N}{\Lambda} pg_p , \quad p = 0, 1, 2, \ldots
\end{align*}
\]
(2.19)
\[
\begin{align*}
   b_{p+1/2} &= -\frac{\Lambda}{N} \frac{\partial}{\partial \xi_{p+1/2}} ; \quad b_{-p-1/2} = -\frac{N}{\Lambda} \xi_{p+1/2} , \quad p = 0, 1, 2, \ldots
\end{align*}
\]
(2.20)
together with a free massless superfield:
\[
X(p, \Pi) = x(p) + \Pi \psi(p)
\]
(2.21)
\[
\partial X(p) = \sum_{n \in \mathbb{Z}} \alpha_n p^{-n-1} \quad \psi(p) = \sum_{r \in \mathbb{Z}+1/2} b_r p^{-r-1/2} ,
\]
(2.22)
with the energy-momentum tensor:
\[
T(p, \Pi) \propto DX \partial X = \psi \partial_p x + \Pi : (\partial_p x \partial_p x + \partial_p \psi \psi) :
\]
(2.23)
The basic postulate in [8] is to take the superloop equations to be

\[ Z^{-1}T(p, \Pi)Z = \text{Polynomial}(p) \quad (2.24) \]

In terms of \( u(p), v(p) \) these equations become:

\[ (u(p) - V'(p))^2 + (v(p) - \xi(p))^2 + \frac{\chi_{BB}(p,p)}{N^2} + \frac{\chi_{FF}(p,p)}{N^2} = Q_0 \quad (2.25) \]

\[ u(p)v(p) - V'(p)v(p) - \xi(p)u(p) + \frac{\chi_{BB}(p,p)}{N^2} = Q_1 \quad (2.26) \]

where:

\[ V(p) = \sum_{k \geq 0} g_k p^k \quad \xi(p) = \sum_{k \geq 0} \frac{\xi_{k+1/2}}{2} p^k \]

\[ \chi_{BF}(p,q) = \sum_{k,l \geq 0} \Lambda_4 \frac{\partial^2 F}{p^{k+1} q^{l+1} \partial \xi_k \partial g_l} \]

\[ \chi_{BB}(p,q) = \sum_{k,l \geq 0} \Lambda_4 \frac{\partial^2 F}{p^{k+1} q^{l+1} \partial g_k \partial g_l} \]

\[ \chi_{FF}(p,p) = \sum_{n \geq 1} \sum_{r=1/2}^{n-1/2} \Lambda_4 \frac{p^{n+2}(n-r)}{2} \frac{\partial^2 F}{\partial \xi_r \partial \xi_{n-r}} \]

The quantities \( Q_0, Q_1 \) are polynomials in \( p \), and although their explicit form can be computed, they will not be needed throughout this paper. In terms of the original loop variables \( w(l, \theta) \), the equations (2.25), (2.26) take a form similar to (2.2)

\[ \mathcal{P} \mathcal{K} w(l, \theta) + 2\mathcal{K} \mathcal{P} w(l, \theta) = (w \circ \mathcal{P} w)(l, \theta) \quad (2.29) \]

with

\[ \mathcal{P} = \theta - l \frac{\partial}{\partial \theta} \quad (2.30) \]

\[ \mathcal{K} \equiv \sum_{k \geq 1} (kg_k - \xi_{k-1/2}) \frac{\partial^{k-1}}{\partial l_{k-1}} \quad (2.31) \]

where the convolution between two superfunctions \( f_1(z, \theta), f_2(z, \theta) \) is defined ac-
According to:

\[
(f \circ g)(z, \theta) \equiv \int d\theta' \int_0^z f(z', \theta') g(z - z', \theta - \theta') dz',
\]

(2.32)

(see [8] for more details). The previous arguments suggest, in analogy with the one-matrix model, the introduction of a "superpotential"

\[
V(\lambda, \theta) = \sum_{k \geq 0} \sum_{i=1}^N (g_k \lambda_i^k + \xi_{k+1/2} \theta_i^k).
\]

(2.33)

The moments \(u^{(k)}\) and \(v^{(k)}\) (2.18) can thus be identified with derivatives of the free energy

\[
u(0) = -\Lambda \frac{\partial F}{\partial g}, \quad v(n) = -\Lambda \frac{\partial F}{\xi_{n+1/2}}.
\]

(2.34)

Writing the partition function as:

\[
Z = \int \prod_{i=1}^N d\lambda_i d\theta_i \Delta(\lambda, \theta) e^{-\frac{N}{\Lambda^2} V(\lambda, \theta)},
\]

(2.35)

we can determine the explicit form of the measure \(\Delta(\lambda, \theta)\) by imposing the super-Virasoro constraints (2.24). The explicit representation of the super-Virasoro operators using (2.19), (2.20) as differential operators is:

\[
G_{n-1/2} = \sum_{k=0}^{\infty} \xi_{k+1/2} \frac{\partial}{\partial g_{k+n}} + \sum_{k=0}^{\infty} k g_k \frac{\partial}{\partial \xi_{k+n-1/2}} + \frac{\Lambda^2}{N^2} \sum_{k=0}^{n-1} \frac{\partial}{\partial \xi_{k+1/2}} \frac{\partial}{\partial g_{n-1-k}} \quad n \geq 0,
\]

(2.36)

\[
L_n = \frac{\Lambda^2}{2N^2} \sum_{k=0}^{n} \frac{\partial^2}{\partial g_k \partial g_{n-k}} + \sum_{k=1}^{\infty} k g_k \frac{\partial}{\partial g_{n+k}}
\]

\[
+ \frac{\Lambda^2}{2N^2} \sum_{r=1/2}^{n-1/2} (n - r) \frac{\partial}{\partial \xi_r} \frac{\partial}{\partial \xi_{n-r}} + \sum_{r=1/2}^{\infty} (n + r) \xi_r \frac{\partial}{\partial \xi_{r+n}} \quad n \geq -1.
\]

(2.37)

Since \(\{G_{n-1/2}, G_{m-1/2}\} \propto L_{n+m-1}\), it suffices to implement \(G_{n-1/2} Z = 0\). This
leads to a set of equations:

\[
\sum_i \lambda_i^n \left( -\frac{\partial}{\partial \theta_i} + \theta_i \frac{\partial}{\partial \lambda_i} \right) \Delta = \Delta \sum_{i \neq j} \theta_i \frac{\lambda_i^n - \lambda_j^n}{\lambda_i - \lambda_j},
\]

(2.38)

whose unique solution, up to a multiplicative constant is:

\[
\Delta(\lambda, \theta) = \prod_{i < j} (\lambda_i - \lambda_j - \theta_i \theta_j).
\]

(2.39)

Hence the model we would like to solve in the large-\(N\) limit is:

\[
Z = \int \prod_{i=1}^N d\lambda_i d\theta_i \prod_{i < j} (\lambda_i - \lambda_j - \theta_i \theta_j) e^{-\frac{\Lambda}{N} V(\lambda, \theta)}.
\]

(2.40)

The loop operator can be explicitly written as:

\[
w(l, \theta) \equiv \frac{\Lambda}{N} \sum_i e^{l \lambda_i + \theta_i}.
\]

(2.41)

From (2.40) and (2.41) one can derive (2.29). The simplifying assumption made in [8] was \(g_{2k+1} = 0, \ k \geq 0\); i.e. the bosonic part of the potential (2.33) was taken to be even. In the next section we begin the analysis of (2.40) without this restriction.
3. SOLUTION TO THE PLANAR LOOP EQUATIONS: GENERAL POTENTIAL

In this section we study the loop equation (2.25), (2.26) in genus zero for an arbitrary bosonic part of the super-potential. The simplifying assumption $g_{2k+1} = 0$ made in [8] generated a doubling of degrees of freedom in the Neveu-Schwarz and Ramond sectors of the theory that is not present in the continuum super-Liouville theory [20]. The planar loop equations follow from (2.25),(2.26):

\[(u(p) - V'(p))^2 + (v(p) - \xi(p))(v(p) - \xi(p)) = Q_0(p),\]  

\[(v(p) - \xi(p))(u(p) - V'(p)) = Q_1(p).\]  

(3.1)  

(3.2)

Using the fact that $Q_1$ is fermionic, the solution to (3.1), (3.2) is:

\[u(p) - V'(p) = \sqrt{Q_0(p)} - \frac{Q'_1(p)Q_1(p)}{2Q_0(p)^{3/2}}\]  

\[v(p) - \xi(p) = \frac{Q_1(p)}{\sqrt{Q_0(p)}}.\]  

(3.3)  

(3.4)

As in the pure gravity case we look for the one-cut solution. Since we make no assumptions concerning the parity of $V(p)$, the one-cut solution takes the form:

\[u(p) = u_0(p) + u_2(p) = V'(p) - M(p)\sqrt{\Delta} - \frac{A(p)}{\Delta^{3/2}}\]  

\[v(p) = \xi(p) - \frac{N(p)}{\sqrt{\Delta}}\]  

(3.5)  

(3.6)

with $\Delta = (p - x)(p - y)$. The subindex in (3.5) indicates the order in fermionic couplings. We can also introduce variables $R, S$:

\[x = S + \sqrt{R}\]  

\[y = S - \sqrt{R} .\]  

(3.7)

For $V(p) = V(-p)$ the cut is symmetric and $S = 0$. Since $u(p) \sim O(1/p)$, $v(p) \sim$
\(O(1/p)\) as \(|p| \to \infty\), \(M(p), N(p)\) are determined as functions of \(V'(p)\) and \(\xi(p)\) respectively directly from this requirement. After \(M\) and \(N\) are determined, the form of \(A(p)\) follows from demanding that the left hand sides of (3.1) and (3.2) must be polynomials. To write down the explicit form of \(M\), \(N\) and \(A\) we note, that any analytic function \(f(p)\) can be written in the form:

\[
f(p) = f_0(\Delta) + \Delta' f_1(\Delta) ,
\]

\[
\Delta' = \frac{d\Delta}{dp} = 2p - x - y = 2(p - S) .
\] (3.8)

We split \(f(p)\) into two terms with opposite parity with respect to the change \((p - S) \to (S - p)\). Hence:

\[
M(p) = M^- (\Delta) + \Delta' M^+ (\Delta) ,
\]

\[
N(p) = \Delta N^- (\Delta) + \Delta' N^+ (\Delta) ,
\]

\[
A(p) = A^- (\Delta) + \Delta' A^+ (\Delta) .
\] (3.9)

In \(N(p)\) we have used the fact that our solution should agree with the result obtained in \([8]\), when \(V(p) = V(-p)^*\). The expansions of \(M\), \(N\) and \(A\) in powers of \(\Delta\) are given by:

\[
M^\pm (\Delta) = \sum_{k \geq 0} m_k^\pm \Delta^k , \quad N^\pm = \sum_{k \geq 0} n_k^\pm \Delta^k , \quad A^\pm (\Delta) = \sum_{k \geq 0} a_k^\pm \Delta^k .
\] (3.10)

To determine \(A\), we substitute these expressions in (3.5), (3.6) and require the left-hand side of (3.1) and (3.2) to be polynomials in \(p\). After some computations,

\* To obtain the even case a factor of two must be taken into account as a normalization of the \(n_k^+\)'s in (3.10)
we obtain that $A^-(\Delta)$ and $A^+(\Delta)$ are completely given by $a_0^-, a_0^+$. The results are

$$A^- = -\frac{2R}{(m_0^-)^2 - 4R(m_0^+)^2}(m_0^-n_0^-n_0^+ - 4Rm_0^+n_1^+n_0^+) , \quad (3.11)$$

$$A^+ = -\frac{2R}{(m_0^-)^2 - 4R(m_0^+)^2}(m_0^-n_1^+n_0^+ - m_0^+n_0^-n_0^+) . \quad (3.12)$$

Once we determine $M(p)$ and $N(p)$ we will have the complete solution to the planar model. Notice that $N(p)$ will be linear in the fermionic couplings. This implies that the non-vanishing planar correlators contain at most two fermionic operators. This also occurs in the even case $V(p) = V(-p)$, so that this rather surprising phenomenon does not depend on the type of potential chosen. We refer the reader to [8] for further discussions.

The solution to the planar model given by eqs. (3.5) and (3.6) is parametrized by $x$ and $y$. Since there is only one single physical parameter $\Lambda$, we should be able to express both $x$ and $y$ as functions of $\Lambda$. Let $u_0(p)$ be the purely bosonic part of the loop operator (no dependence on the $\xi_{k+1/2}$ couplings). Then the one-cut solution is:

$$u_0(p) = V'(p) - M(p)\sqrt{(p-x)(p-y)} . \quad (3.13)$$

Since $V'(p)$ and $M(p)$ are polynomials in $p$, $u_0(p)$ admits the following expansion

$$u_0(p) = C(x, y) + \frac{\Lambda(x, y)}{p} + O\left(\frac{1}{p^2}\right) \quad (3.14)$$

Dividing this expression by $\Delta^{1/2}$ gives

$$\frac{u_0}{\sqrt{(p-x)(p-y)}} = \frac{V'(p)}{\sqrt{(p-x)(p-y)}} - M(p)$$

$$= \frac{C(x, y)}{p} + \frac{\Lambda(x, y) + \frac{x+y}{2}C(x, y)}{p^2} + O\left(\frac{1}{p^3}\right) \quad (3.15)$$
which implies
\[ C(x, y) = -\oint_{\infty} dp \frac{V'(p)}{2\pi i \sqrt{(p - x)(p - y)}} \] (3.16)

\[ \Lambda(x, y) = -\oint_{\infty} dp \frac{V'(p)}{2\pi i \sqrt{(p - x)(p - y)}} \left( p - \frac{x + y}{2} \right) \] (3.17)

But according to (2.18), \( u_0(p) \sim 0(1/p) \). Thus we have the following constraint on \( x \) and \( y \)

\[ C(x, y) = 0 \] (3.18)

Equations (3.17) and (3.18) can be used, in principle, to rewrite the solution in terms of the single physical parameter \( \Lambda \). These equations have appeared previously in the matrix model literature (see for instance [21] and references therein) in a slightly different form. It is important to notice that the derivation is not based on the method of orthogonal polynomials [22] since this formalism is still missing in the supersymmetric case.

Note that the piece proportional to \( \frac{1}{2}(x + y) \) in (3.17) vanishes when (3.18) is enforced. Then one recovers the expression for \( \Lambda \) which is usually found in the literature. However, in order to compute partial derivatives it is essential to use the complete form (3.17).

Expanding \( M \) in powers of \( p \)

\[ M(p) = \sum_{n \geq 0} M_n p^n \] (3.19)

we find

\[ M_n = -\oint_{\infty} dq \frac{q^{-n-1}V'(q)}{2\pi i \sqrt{(q - x)(q - y)}} \] (3.20)

and

\[ M(p) = -\oint_{\infty} dq \frac{V'(q)}{2\pi i (q - p) \sqrt{(q - x)(q - y)}} \] (3.21)
In order to evaluate (3.11), (3.12), we also need explicit formulae for \( m_0^\pm \). These can be obtained as follows. Differentiating (3.17) and comparing the result with (3.21) gives
\[
\partial_x \Lambda = \frac{1}{4}(x - y)M(x), \quad \partial_y \Lambda = \frac{1}{4}(y - x)M(y)
\] (3.22)

On the other hand, by (3.10)
\[
M(x) = m_0^- + (x - y)m_0^+ \\
M(y) = m_0^- + (y - x)m_0^+
\] (3.23)

This gives \( m_0^\pm \) in terms of the single function \( \Lambda(x, y) \)
\[
m_0^- = \frac{2}{(x - y)}(\partial_x - \partial_y)\Lambda \\
m_0^+ = \frac{2}{(x - y)^2}(\partial_x + \partial_y)\Lambda
\] (3.24)

We conclude our analysis of the purely bosonic part of the solution by obtaining a simple expression for the derivative of \( u_0 \) with respect to the cosmological constant \( \Lambda \). Under a variation \((dx, dy)\) compatible with the constraint (3.16), \( du_0 \) is given by
\[
du_0(p) = \frac{1}{\sqrt{\Delta}}[-\frac{1}{2}M(p)d\Delta - \Delta dM(p)] \\
= \frac{d\Lambda}{p} + O\left(\frac{1}{p^2}\right)
\] (3.25)

Since the expression multiplied by \( \Delta^{-1/2} \) is a polynomial, we must have
\[
d\Lambda = \frac{1}{2}M(p)d\Delta - \Delta dM(p)
\] (3.26)

Therefore:
\[
\frac{\partial u_0(p)}{\partial \Lambda} = \frac{1}{\sqrt{(p - x)(p - y)}}
\] (3.27)

When we discuss the continuum limit, the following loop operator will be rel-
evant
\[ \tilde{u}_0(p) \equiv u_0(p) - V'(p) \] (3.28)

Note that this also satisfies
\[ \frac{\partial \tilde{u}_0(p)}{\partial \Lambda} = \frac{1}{\sqrt{(p-x)(p-y)}} \] (3.29)

We now turn to the fermionic part of the solution. It is convenient to rewrite (3.6) in a slightly different form. Defining
\[ \tilde{N} \equiv \sum_{k \geq 0} (n_k^- + \Delta' n_{k+1}^+) \Delta^k \] (3.30)

eq (3.6) becomes
\[ v(p) + \frac{\Delta'}{\sqrt{\Delta}} n_0^+ = \xi(p) - \tilde{N} \sqrt{\Delta} \] (3.31)

where
\[ n_0^+ = -\frac{1}{2} \oint dq \frac{\xi(q)}{2\pi i(q-p)\sqrt{(q-x)(q-y)}} \] (3.32)

and
\[ \tilde{N}(p) = -\oint dq \frac{\xi(q)}{2\pi i(q-p)\sqrt{(q-x)(q-y)}} \] (3.33)

We can also find an expression analogous to (3.27). Differentiating (3.31) with respect to \( x \) gives
\[ \partial_x (v(p) + \frac{\Delta'}{\sqrt{\Delta}} n_0^+) = \frac{1}{\sqrt{\Delta}} (\frac{1}{2} (p-y) \tilde{N} - \Delta \partial_x \tilde{N}) \] (3.34)

On the other hand, since by (2.18) \( v(p) \sim 0(1/p) \),
\[ \partial_x (v(p) + \frac{\Delta'}{\sqrt{\Delta}} n_0^+) = 2\partial_x n_0^+ + O(\frac{1}{p}) \] (3.35)
This implies
\[ \frac{1}{2}(p - y)\tilde{N} - \Delta \partial_x \tilde{N} = a + b\Delta' \]  
(3.36)
and if we set \( p = y \) we find \( a = (x - y)b \). Since
\[ \frac{1}{\sqrt{\Delta}}(a + b\Delta') = 2b + O\left(\frac{1}{p}\right) \]  
(3.37)
comparison with (3.35) yields \( b = \partial_x n_0^+ \).

We finally obtain
\[ \partial_x(v(p) + \frac{\Delta'}{\sqrt{\Delta}}n_0^+) = \frac{2(p - y)\partial_x n_0^+}{\Delta^\perp} \]  
(3.38)
Interchanging \( x \) and \( y \) gives
\[ \partial_y(v(p) + \frac{\Delta'}{\sqrt{\Delta}}n_0^+) = \frac{2(p - x)\partial_y n_0^+}{\Delta^\perp} \]  
(3.39)
In order to complete the planar solution, we need explicit formulae for the functions \( n_0^- \) and \( n_1^+ \) appearing in (3.11),(3.12). First, note that eq. (3.36) can be written
\[ \frac{1}{2}(p - y)\tilde{N} - \Delta \partial_x \tilde{N} = 2(p - y)\partial_x n_0^+ \]  
(3.40)
Setting \( p = x \) and using (3.30) gives
\[ \frac{1}{2}(n_0^- + (x - y)n_1^+) = 2\partial_x n_0^+ \]  
(3.41)
This, together with
\[ \frac{1}{2}(n_0^- + (y - x)n_1^+) = 2\partial_y n_0^+ \]  
(3.42)
(obtained from (3.39)) can be used to determine \( n_0^- \) and \( n_1^+ \) as functions of the single function \( n_0^+ \)
\[ n_0^- = 2(\partial_x + \partial_y)n_0^+ \]
\[ n_1^+ = \frac{2}{x - y}(\partial_x - \partial_y)n_0^+ \]  
(3.43)
As in the bosonic case, we may define \( \hat{v}(p) \equiv v(p) - \xi(p) \), which satisfies relations identical to (3.38) and (3.39). At this point we have determined the complete
solution to the planar model. In the next section we will consider the corresponding continuum limit.

4. THE SCALING LIMIT

The definition of the double scaling limit with an arbitrary potential involves some subtle considerations. These can be best understood by studying first the purely bosonic part of the theory. We take as the fundamental geometric ‘observable’ the macroscopic loop, defined by

\[ \hat{u}(l) \equiv \lim_{n \to \infty} \frac{1}{2 \pi i} \sum_{i=1}^{N} \lambda_i^n = \lim_{n \to \infty} \frac{N}{\Lambda} u^{(n)} \] (4.1)

with \( na^{2/m} = l \) fixed. Here \( m \) is a positive integer related to the order of criticality. The continuum limit should be taken in such a way that (4.1) makes sense. This will be our guiding principle. Eq. (3.29) implies

\[ \partial_{\Lambda} \hat{u}_0(l) = \lim_{n \to \infty} \frac{1}{2 \pi i} \frac{N}{\Lambda} \oint_C \frac{dp}{\sqrt{(p-x)(p-y)}} \] (4.2)

where the contour \( C \) encloses the cut \((y, x)\). We may assume \(|y| < |x| \leq 1\) without loss of generality. We will comment on the symmetric case \( y = -x \) later. Taking \( C \) as the unit circle around the origin with \( p = e^{i\theta} \),

\[ \frac{1}{2 \pi i} \oint_C \frac{dp}{\sqrt{(p-x)(p-y)}} = \frac{1}{2 \pi} \oint_C \frac{d\theta e^{i\frac{2}{m}a^2\theta}}{(e^{i\theta} - x)(e^{i\theta} - y)} \] (4.3)

We set \( \Lambda_c = 1 \), with \( \Lambda = 1 - a^2 t \), where \( t \) is the renormalized cosmological constant. \( x \) and \( y \) will approach the critical values \( x_c \) and \( y_c \), and \( N \) will be related to the
renormalized string coupling constant $\kappa$ by

$$N = a^{-2-1/m}(x_c - y_c)^{1/2}\frac{1}{\kappa} \quad (4.4)$$

(The factor $(x_c - y_c)^{1/2}$ is introduced for later convenience). Changing to the new variable $z = a^{-2/m}\theta$, eq. (4.2) becomes

$$-\partial_t \tilde{u}_0(l) = \lim_{a \to 0} a^{1/m} \frac{\sqrt{(x_c - y_c)}}{\kappa} \frac{1}{2\pi} \oint_C \frac{dz e^{ilz}}{\sqrt{(e^{ia2/m}z - x)(e^{ia2/m}z - y)}} \quad (4.5)$$

This will vanish unless the integral itself is of order $a^{-1/m}$, i.e., if $x$ approaches 1 as $a^{2/m}$,

$$x = 1 - a^{2/m}u_+ \quad (4.6)$$

Since $x$ and $y$ are not independent variables, $y$ will approach its critical value $y_c$ ($|y_c| < 1$) at the same time

$$y = y_c + a^{2/n}u_- \quad (4.7)$$

Here $n$ is another positive integer, which may be different from $m$. The integral in (4.5) is dominated by the region $p \sim x_c = 1$, and the contour can be deformed into a straight line

$$\kappa \partial_t \tilde{u}_0(l) = -\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \frac{e^{iz}}{\sqrt{z + u_+}} \quad (4.8)$$

Here we recognize the definition of the inverse Laplace transform. Thus, if we define

$$\tilde{u}_0(z) \equiv \kappa \mathcal{L}[\tilde{u}_0(l)] \quad (4.9)$$

we have

$$\partial_t \tilde{u}_0(z) = -\frac{1}{\sqrt{z + u_+}} \quad (4.10)$$

It is interesting to note that this result can also be obtained from (3.27) by a scaling $p = 1 + a^{2/m}z$ in $\tilde{u}_0(p)$, together with (4.6) and (4.7).
There are two different ways of viewing $\hat{u}_0(z)$: as the Laplace transform of the macroscopic loop $\hat{u}_0(l)$, or as the continuum limit of the loop operator $\hat{u}_0(p)$.

From (4.8) one gets the usual expression for the macroscopic loop

$$
\partial_t \hat{u}_0(l) = -\frac{1}{\kappa} \frac{e^{-lu_+}}{\sqrt{\pi l}} \quad (4.11)
$$

Note that $\hat{u}_0(l)$ is independent of the scaling variable $u_-$. This result holds as long as $|y_c| < |x_c|$, independently of the values of $m$ and $n$. It is easy to see that, for $|y_c| = |x_c|$, the dominant endpoint will be the one with the highest order of criticality. For a symmetric potential both endpoints contribute, and we get the familiar phenomenon of ‘doubling’. As we shall see below, the situation is not so simple when one considers the fermionic contributions to the loop operators.

In order to complete our description of the continuum limit for the purely bosonic part of the theory, we must consider the scaling of equations (3.16) and (3.17). Comparing first derivatives of $\Lambda(x, y)$ and $C(x, y)$, we find that they are not independent. Instead,

$$
\begin{align*}
\partial_x \Lambda &= \frac{1}{2} (x - y) \partial_x C \\
\partial_y \Lambda &= \frac{1}{2} (y - x) \partial_y C
\end{align*} \quad (4.12)
$$

Similarly, we have the following identities

$$
\begin{align*}
(x - y) \partial^2_{xy} \Lambda &= -\frac{1}{2} (\partial_x \Lambda - \partial_y \Lambda) \\
(x - y) \partial^2_{xy} C &= \frac{1}{2} (\partial_x C - \partial_y C)
\end{align*} \quad (4.13)
$$

Scaling (4.13) according to (4.6) and (4.7) gives

$$
\partial^2_{++} t \sim -\frac{1}{2(1 - y_c)} (a^{2/n} \partial_+ t + a^{2/m} \partial_- t) \quad (4.14)
$$

with an identical expression for $C$. For $a \to 0$ the RHS vanishes, and we find

$$
\partial^2_{++} t = 0 , \quad \partial^2_{+-} c = 0 \quad (4.15)
$$

where we have defined $\partial_\pm \equiv \frac{\partial}{\partial u_\pm}$ and $c \equiv -\frac{1}{2} a^{-2}(1 - y_c)C$. Moreover, (4.12)
implies
\[ \partial_+ c = \partial_+ t, \quad \partial_- c = -\partial_- t \] (4.16)

The general solution to (4.15) and (4.16) can be written
\[
\begin{align*}
  t &= -\frac{1}{2} \sum_{p \geq 0} (\tilde{t}_p^+ u_+^p + \tilde{t}_p^- u_-^p) \\
  c &= -\frac{1}{2} \sum_{p \geq 0} (\tilde{t}_p^+ u_+^p - \tilde{t}_p^- u_-^p) = 0
\end{align*}
\] (4.17)

where \( \tilde{t}_p^\pm \) are the renormalized couplings. Eqs. (4.17) are the continuum version of (3.16) and (3.17). Adding and subtracting the equations in (4.17) give rise to two decoupled string equations
\[
\begin{align*}
  t &= - \sum_{p \geq 0} \tilde{t}_p^+ u_+^p , \quad t &= - \sum_{p \geq 0} \tilde{t}_p^- u_-^p \quad (4.18)
\end{align*}
\]

Comparing (4.17) and (4.18) we see that
\[
\begin{align*}
  d_\pm^p t &= 2 \partial_\pm^p t \\
  \tilde{t}_p^\pm &= -\frac{1}{p!} d_\pm^p t|_{u_\pm=0} \quad (4.20)
\end{align*}
\]

The total derivatives in (4.20) are computed for variations \( du_\pm \) consistent with the constraint (4.17). The relative factor of 2 between total and partial derivatives is important. It means that we can not set \( \partial_- t = 0 \) consistently.

The \( m \)-multicritical point (at \( x \)) is defined by
\[
\begin{align*}
  d_+ t|_0 = \ldots = d_+^{m-1} t|_0 = 0 \quad (4.21)
\end{align*}
\]

One can impose similar constraints at \( y \), with a different index \( n \). But, as mentioned above, for \( |y| < |x| \) the continuum limit is controled by \( m \) (the converse is of course true for \( |y| > |x| \)).
Eq. (4.18) can be used to compute the derivatives of $u_+$ with respect to the renormalized couplings
\[
\frac{\partial u_+}{\partial \tilde{t}_k} = u_+^k \partial_t u_+ , \quad \frac{\partial u_+}{\partial \tilde{t}_k} = 0 \quad (4.22)
\]
These expressions will be useful in connection with the definition of the free energy.

We now turn our attention to the fermionic contributions. The following identity can be derived for $n_0^+$
\[
(x - y)\partial_{xy}^2 n_0^+ = \frac{1}{2}(\partial_x n_0^+ - \partial_y n_0^+) \quad (4.23)
\]
and, if $\tau$ is the scaling function corresponding to $n_0^+$, we have
\[
\partial_{+-}^2 \tau = 0 \quad (4.24)
\]
which implies
\[
\tau = \tilde{\tau}_0 + \frac{1}{2} \sum_{p>0} (\tilde{\tau}_p^+ u_+^p + \tilde{\tau}_p^- u_-^p) \quad (4.25)
\]
This defines the renormalized fermionic couplings $\tilde{\tau}_n^\pm$. Using eqs. (3.5), (3.11), (3.12), (3.24), (3.43) we find the following expression for $u_2$
\[
u_2(p) \sim \frac{1}{2}(1 - y_c)^{3/2} a^{-2-3/m} n_0^+ \partial_x u_+ \partial_x u_+ \quad (4.26)
\]
The fact that $u_0$ and $u_2$ must scale in the same way fixes the scaling for $n_0^+$
\[
n_0^+ = a^{2+1/m} \frac{\sqrt{2}\tau}{1 - y_c} \quad (4.27)
\]
Then
\[
\hat{u}_2(z) = \frac{\tau \partial_+ \tau \partial_+ u_+}{(z + u_+)^{3/2}} \quad (4.28)
\]
and we get for the bosonic loop
\[
\partial_t(\hat{u}(z) - \frac{\tau \partial_+ \tau \partial_+ u_+}{(z + u_+)^{3/2}}) = -\frac{1}{\sqrt{z + u_+}} \quad (4.29)
\]
In order to obtain the fermionic loop we first note that, in the continuum limit,
the RHS of (3.39) vanishes. This turns the partial derivative in (3.38) into a total derivative \(d_x\), and from (3.38) we can write

\[
\partial_t (\hat{v}(z) + \frac{\tau}{\sqrt{z + u_+}}) = 2 \frac{\partial_+ \tau \partial_t u_+}{\sqrt{z + u_+}}
\]

(4.30)

where we have defined

\[
\hat{v}(z) \equiv a^{-2} \sqrt{1 - y_c} \hat{v}(p) \frac{\sqrt{2}}{\sqrt{2}}
\]

(4.31)

The different powers of \(a\) in the definitions of the continuum bosonic and fermionic loops are a consequence of their different scaling dimension. The macroscopic loops are obtained by the inverse Laplace transform. The result is

\[
\partial_t (\hat{u}(l) - \frac{2}{\kappa} \sqrt{\frac{l}{\pi}} e^{-lu_+} \tau \partial_+ \tau \partial_t u_+) = -\frac{e^{-lu_+}}{\kappa \sqrt{\pi l}}
\]

(4.32)

\[
\partial_t (\hat{v}(l) + \frac{1}{\kappa \sqrt{\pi l}} \tau e^{-lu_+}) = 2 \frac{e^{-lu_+}}{\kappa \sqrt{\pi l}} \partial_+ \tau \partial_t u_+
\]

(4.33)

Note that the fermionic contributions to the loops are not independent of \(u_-\). In other words, even though the loops are dominated by the contribution from \(p \sim x_c\), their expectation values depend on all the couplings, even those defining the behaviour of the potentials at \(y\). This peculiarity of the supersymmetric theory implies that one has to be careful when trying to obtain the double scaling limit of the SuperVirasoro constraints. This will be our main concern in section 5.

Multiloop correlators are obtained by acting on \(\hat{u}(l)\) and \(\hat{v}(l)\) with the appropriate ‘loop insertion operators’. We consider first the bosonic case. Since

\[
\frac{d}{dg_k} \langle \ldots \rangle = -\frac{N}{\Lambda} \left( \sum_{i=1}^{N} \lambda_i^n \ldots \right)
\]

(4.34)

we have

\[
\langle \hat{u}(l) \ldots \rangle = -\frac{\Lambda}{N} \frac{d}{dg_k} \langle \ldots \rangle \equiv J^B(l) \langle \ldots \rangle
\]

(4.35)

We need to know the derivatives of \(x\) and \(y\) with respect to \(g_k\). These are obtained by differentiating of (3.16) and (3.17). Since we are interested in the limit \(k \to \infty\),
we need the asymptotic behavior of integrals like (4.2). Converting the integral
around the cut into an integral along the real axis, and using Laplace method for
integrals dominated by endpoint contributions, we find
\[ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dp}{\sqrt{(x-p)(p-y)}} \sim \frac{1}{\sqrt{x-y} \sqrt{\pi k}}, \quad |x| > |y| \] (4.36)
and
\[ \frac{\partial x}{\partial g_k} \sim -\frac{\sqrt{x-y}}{4} \sqrt{\frac{k}{\pi \partial_x \Lambda}}, \quad \frac{\partial y}{\partial g_k} \sim 0, \quad |x| > |y| \] (4.37)
These formulae can be used for an alternative derivation of the macroscopic loop.
Scaling this expression gives the ‘macroscopic loop insertion operator’
\[ \mathcal{J}_B(l) = -\Lambda \frac{\partial u_+}{\partial g_k} \frac{\delta}{\delta u_+} \sim -\kappa \sqrt{\frac{l}{\pi}} e^{-lu_+} \partial_t u_+ \frac{\delta}{\delta u_+} \] (4.38)
It is important to note that this expression assumes that all the dependence on the
couplings is given implicitly through \( u_+ \) and \( u_- \). But one can always rewrite the
expressions so that this is actually true. The Laplace transform of (4.38) is
\[ \mathcal{J}_B(z) = -\frac{1}{2} \frac{\partial_t u_+}{(z + u_+)^{3/2}} \frac{\delta}{\delta u_+} \] (4.39)
This operator inserts a loop \( \hat{u}(z) \).

The fermionic insertion operators are obtained along the same lines. The result
is
\[ \mathcal{J}_F(l) = -a^{-1/m} \Lambda \frac{\partial \tau}{\partial \xi_{k+1}} \frac{\delta}{\delta \tau} \sim -\kappa e^{-lu_+} \frac{\delta}{4 \sqrt{\pi l} \delta \tau} \] (4.40)
and
\[ \mathcal{J}_F(z) = \frac{1}{4} \frac{\delta}{\sqrt{z + u_+} \delta \tau} \] (4.41)
It is easy to see that the macroscopic loops can be obtained by acting with \( \mathcal{J}_B(l) \)
and \( \mathcal{J}_F(l) \) on the following free energy
\[ \kappa^2 \partial_t^2 F = -u_+ + 2\partial_t (\tau \partial_+ \tau \partial_t u_+ - \tau \partial_- \tau \partial_t u_-) \] (4.42)
5. CONTINUUM SUPER-VIRASORO CONSTRAINTS

We will now see that, in the continuum limit, the bosonic and fermionic loop operators become a $Z_2$-twisted scalar bosonic field and a Weyl-Majorana fermion in the Ramond sector. This is the supersymmetric generalization of the well known result for the bosonic matrix model. However, proving this statement turns out to be rather subtle in our case. The reason is that, as mentionned in the last section, the correlators depend on all the couplings, not just those describing the continuum limit at the dominant endpoint. It is not a priori clear that one can use a single $Z_2$-twisted bosonic field and a single Weyl-Majorana fermion. However, this turns out to be true.

First we will show that $\hat{u}_0(z)$ can be identified with a free bosonic scalar field $\varphi(z)$ in two dimensions with antiperiodic boundary conditions

$$\varphi(z) = \sum_{n \in \mathbb{Z}} \alpha_{n+1/2} z^{n-3/2}, \quad \varphi(e^{2\pi i} z) = -\varphi(z). \quad (5.1)$$

The Laplace transform of the bosonic loop can be decomposed into two pieces

$$\hat{u}(z) \equiv t(z) + u(z) \quad (5.2)$$

where

$$u(z) = \kappa^2 \sum_{k \geq 0} \frac{\langle \sigma_k^+ \rangle}{z^{k+3/2}}, \quad \langle \sigma_k^+ \rangle = \frac{\partial F}{\partial t_k^+} \quad (5.3)$$

and $t(z)$ is the non-universal part. $t(z)$ admits an expansion in powers $z^{n-1/2}$, for $n \geq 0$.

Eq. (5.3) is in fact a definition of the scaling operators $\sigma_k^+$. Since all the contributions to the loop come from $p \sim x_c = 1$, it makes sense to expand in terms
of operators associated with the $x$-endpoint. Comparing (4.29) with (5.3) implies
\[
\partial_t \left( u(z) - \frac{\tau \partial_+ \tau \partial_t u_+}{(z + u_+)^{3/2}} \right) = -\frac{1}{\sqrt{z + u_+}} + \frac{1}{\sqrt{z}} \tag{5.4}
\]

With the Laurent expansion of (5.4) for $z \to \infty$:
\[
\frac{\partial u(z)}{\partial t} = -\sum_{k \geq 0} \frac{(-1)^{k+1} \Gamma(k + \frac{3}{2})}{k! \Gamma(\frac{1}{2})} \left( \frac{u_+^{k+1}}{k+1} + 2\partial_t (u_+^k \tau \partial_+ \tau \partial_t u_+) \right) z^{-k-3/2}.
\]

the scaling operators $\langle \sigma_k^+ \rangle$ are given by:
\[
\partial_t \langle \sigma_k^+ \rangle = \frac{1}{\kappa^2} \frac{(-1)^{k+1} \Gamma(k + \frac{3}{2})}{k! \Gamma(\frac{1}{2})} \left( \frac{u_+^{k+1}}{k+1} + 2\partial_t (u_+^k \tau \partial_+ \tau \partial_t u_+) \right), \tag{5.5}
\]

This result should also be obtained from the free energy (4.42) by differentiation with respect to $t_k^+$. Then (5.5) implies the following relation between $t_k$ and $\tilde{t}_k$
\[
\tilde{t}_k^+ = \frac{(-1)^{k+1} \Gamma(k + \frac{3}{2})}{k! \Gamma(\frac{1}{2})} t_k^+ \tag{5.6}
\]

We now turn to the computation of $t(z)$. Since $(z + u_+)^{-3/2}$ can not contribute to the non-universal part, it is obvious that $t(z)$ is a function of $u_+$, independent of the fermionic couplings. Then, subtracting (5.4) from (4.29) gives
\[
\partial_t t(z) = \frac{\partial t(z)}{\partial \tilde{t}_0^+} = -\frac{1}{\sqrt{z}} \tag{5.7}
\]

where we have used
\[
\partial_t u_+ = \frac{\partial u_+}{\partial \tilde{t}_0^+} \tag{5.8}
\]

Thus $t(z)$ satisfies
\[
t(z) = -\frac{\tilde{t}_0^+}{\sqrt{z}} + \text{(indep.of } u_+) \tag{5.9}
\]
On the other hand, according to (3.28), \( \hat{u}_0(z) \) has the following form

\[
\hat{u}_0(z) = -M(z)\sqrt{\Delta} \equiv \sum_{n \geq 0} \hat{\alpha}_n^0 \Delta^{n+\frac{1}{2}}
\]  

(5.10)

where \( M(z) \) is defined as the scaling limit of \( M(p) \), and \( \Delta = z + u_+ \). For \( u_+ = 0 \), \( \hat{u}_0(z) \) contains only positive half integral powers of \( z \). This implies

\[
t(z) = \hat{u}_0(z)|_{u_+ = 0} - \frac{\hat{t}_0^+}{\sqrt{z}}
\]

(5.11)

Differentiating (5.10) with respect to \( t \) and taking into account that \( \partial_t \hat{u}_0 = -1/\sqrt{\Delta} \), and using (5.7) yields:

\[
\hat{u}_0^0 = 2d_+ t, \quad d_+ \hat{\alpha}_0^{n-1} = -(n + \frac{1}{2}) \hat{\alpha}_0^n
\]

(5.12)

and we find the following expression for \( \hat{u}_0(z) \)

\[
\hat{u}_0(z) = \sum_{n \geq 0} (-1)^{n+1} \frac{\Gamma(\frac{1}{2})}{\Gamma(n + \frac{1}{2})} (d_+ t)(z + u_+)^{n+\frac{1}{2}}
\]

(5.13)

Setting \( u_+ = 0 \) gives

\[
t(z) = -\frac{\hat{t}_0^+}{\sqrt{z}} + \sum_{n \geq 0} (-1)^{n+1} \frac{\Gamma(\frac{1}{2})}{\Gamma(n + \frac{1}{2})} (d_+ t)|_{u_+ = 0} z^{n+\frac{1}{2}}
\]

(5.14)

and using (4.20), (5.6) we finally get

\[
t(z) = \sum_{k \geq 0} (k + \frac{1}{2}) t_k^+ z^{-1/2} \quad .
\]

(5.15)

In total

\[
\hat{u}(z) = \sum_{k \geq 0} (k + \frac{1}{2}) t_k^+ z^{-1/2} + \kappa^2 \sum_{k \geq 0} \frac{\langle \sigma_k^+ \rangle}{z^{k+3/2}} \quad .
\]

(5.16)

The relations between the coupling constants of the model and the modes of the
bosonic field are
\[
\alpha_{n+1/2} = \kappa \frac{\partial}{\partial t_n^+}, \quad \alpha_{-n-1/2} = (n + \frac{1}{2}) \frac{t_n^+}{\kappa} \quad n \geq 0 \quad (5.17)
\]

so that we can write
\[
\eta(z) = \kappa Z^{-1} \partial \varphi(z) Z \quad (5.18)
\]

We proceed analogously with the fermionic loop. First, decompose into universal and non-universal parts
\[
\hat{v}(z) \equiv \eta(z) + v(z) \quad (5.19)
\]

with
\[
v(z) = \kappa^2 \sum_{n \geq 0} \frac{1}{z^{n+1/2}} \langle \nu_n^+ \rangle, \quad \langle \nu_n^+ \rangle = \frac{\partial F}{\partial \tau_n^+} \quad (5.20)
\]

\[
\hat{v}(z) \] will be identified with a Weyl-Majorana fermion in the Ramond sector. The Laurent expansion of (4.30) determines the correlator of the scaling operators
\[
\partial_t \left( (-1)^{k+1} \frac{k! \Gamma\left(\frac{1}{2}\right) \langle \nu_k^+ \rangle + \frac{1}{\kappa^2} u_k^+ \tau}{\Gamma(k + \frac{1}{2})} \right) = \frac{2}{\kappa^2} u_k^+ \partial_+ \tau \partial_t u_+ \quad (5.21)
\]

and the relation between \( \tilde{\tau}_k \) and \( \tau_k \) is
\[
\tilde{\tau}_k^+ = \frac{(-1)^{k+1} \Gamma(k + \frac{1}{2})}{k! \Gamma\left(\frac{1}{2}\right)} \tau_k^+. \quad (5.22)
\]

The computation of \( \eta(z) \) follows closely the method used with \( t(z) \), and we simply quote the result
\[
\eta(z) = \frac{\tau_0}{2\sqrt{z}} + \sum_{k \geq 0} \tau_{k+1}^+ z^{k+1/2}. \quad (5.23)
\]

This makes it possible to identify \( \hat{v}(z) \) as a Weyl-Majorana fermion in the Ramond
sector. We can write

\[ \hat{v}(z) = \kappa Z^{-1} \psi(z) Z \] (5.24)

In terms of the mode expansions (5.20) and (5.23),

\[ \psi(z) = \sum_{n \in \mathbb{Z}} \psi_n z^{-n-1/2} , \]
\[ \psi_n = \kappa \frac{\partial}{\partial \tau_n^+} , \quad \psi_{-n} = \frac{\tau_n^+}{\kappa} ; \quad n > 0 \] (5.25)

while for the zero mode we have,

\[ \psi_0 = \frac{\tau_0}{2\kappa} + \kappa \frac{\partial}{\partial \tau_0} , \] (5.26)

guaranteeing \( \psi_0^2 = \frac{1}{2} \).

We have thus succeeded in writing \( \hat{u}(z) \) and \( \hat{v}(z) \) entirely in terms of the couplings associated with a single endpoint. One can easily check that the loop insertion operators constructed in the last section can be written as

\[ \mathcal{J}^B(z) = \sum_{n \geq 0} z^{-n-\frac{3}{2}} \frac{\partial}{\partial \tau_n^+} \] (5.27)

and

\[ \mathcal{J}^F(z) = \sum_{n \geq 0} z^{-n-\frac{1}{2}} \frac{\partial}{\partial \tau_n^+} \] (5.28)

This means that the couplings \( t_n^- \) and \( \tau_n^- \) merely parametrize the loop correlators, but do not contribute to the ‘dynamics’.
The super-Virasoro constraints in the continuum are therefore described by
the super-energy momentum tensor of a single $\hat{c} = 1$ superconformal field theory,

$$ T_F(z) = \frac{1}{2} \partial \varphi(z) \psi(z) , \quad (5.29) $$

$$ T_B(z) = \frac{1}{2} : \partial \varphi(z) \partial \varphi(z) : + \frac{1}{2} : \partial \psi(z) \psi(z) : + \frac{1}{8 z^2} . \quad (5.30) $$

With the mode expansion

$$ T_F(z) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \frac{G_{n+1/2}}{z^{n+2} } \, , \, \quad T_B(z) = \sum_{n \in \mathbb{Z}} \frac{L_n}{z^{n+2} } \, , \quad (5.31) $$

we obtain, in terms of the coupling constants:

$$ G_{n+1/2} = \frac{t_0^+ \tau_0}{4 \kappa^2} \delta_{n,-1} + \sum_{k \geq 0} (k + \frac{1}{2}) t_k^+ \frac{\partial}{\partial t_{n+k+1}^+} + \frac{\tau_0}{2} \frac{\partial}{\partial t_n^+} + \sum_{k \geq 0} \tau_{k+1}^+ \frac{\partial}{\partial t_{n+k+1}^+} $$

$$ + \kappa^2 \sum_{k=0}^n \frac{\partial^2}{\partial t_k^+ \partial \tau_n^{+}_{-k}} , \quad (5.32) $$

$$ L_n = \frac{t_0^+ - 2 \tau_0 \tau_+}{8 \kappa^2} \delta_{n,-1} + \sum_{k \geq 0} (k + \frac{1}{2}) t_k^+ \frac{\partial}{\partial t_{n+k}^+} + \frac{\kappa^2}{2} \sum_{k=1}^n \frac{\partial^2}{\partial t_{k-1}^+ \partial t_{n-k}^+} $$

$$ + \frac{n}{4} \tau_0 \frac{\partial}{\partial \tau_n^+} + \sum_{k \geq 0} (\frac{n}{2} + k + 1) \tau_{k+1}^+ \frac{\partial}{\partial \tau_{n+k+1}^+} - \frac{\kappa^2}{2} \sum_{k=0}^n (k + \frac{1}{2}) \frac{\partial}{\partial \tau_k^+} \frac{\partial}{\partial \tau_{n-k}^+} + \frac{1}{8} \delta_{n,0} \quad (5.33) $$

They satisfy the commutation relations

$$ \{ G_m, G_n \} = 2 L_{m+n} + \frac{1}{2} (m^2 - \frac{1}{4}) \delta_{m+n,0} \quad (5.34) $$

In fact, the non-universal term $\frac{1}{8} \delta_{n,0}$ is fixed by these relations.
The basic postulate of [8] was to take the discrete superloop equations equivalent to:

\[ Z^{-1}T(p, \Pi)Z = \text{Polynomial}(p, \Pi) \quad (5.35) \]

After the double scaling limit we obtain that the superloop equations in the continuum are equivalent to:

\[ Z^{-1}T(z)Z = \text{Polynomial}(z) \quad (5.36) \]

with \( T(z, \theta) = T_F(z) + \theta T_B(z) \) given by (5.29) and (5.30) This proves one of the main results of this paper. The continuum limit of our superloop equations are described by a \( Z_2 \)-twisted massless scalar field, and a Weyl-Majorana fermion in the Ramond sector.

We would like to remark that the superVirasoro operators act on \( Z \) instead of \( \sqrt{Z} \), as happens for cases with doubling. This fact was conjectured in [10], but we do not know of any previous complete proof of this statement.

We finish this section with a comment about the dimensions of scaling operators. With a similar calculation as in [8] we obtain for \( \langle \sigma^+_k \rangle \), \( d_k = k/m \) and for \( \langle \nu^+_k \rangle \), \( d_k = k/m - 1/2m \). These are the gravitational scaling dimensions of operators in the NS- resp. R- sector of \((2, 4m) N = 1\) superconformal minimal models coupled to two dimensional gravity. Since we have considered generic potentials there appears no doubling of operators as it was in [8].
6. SUPERLOOP EQUATIONS IN THE CONTINUUM

In the following we solve the continuum loop equations determined by (5.36). Here the formulae get more transparent than in the discrete theory. It is interesting to consider also the planar case and see how the results for the loop operators and the string equation found in section four appear in a simple way. For higher genera, i.e. for the torus, the double-torus, ... we obtain a systematic expansion determining all correlators beyond the planar limit. In the arguments of this section we are going to take for simplicity $\tau(u_+,u_-)$ as a function only of $u_+$. In other words, we set $\tau_k^- = 0$ for all $k$. We first show how the loop equations are solved up to genus two, and then present a general heuristic argument showing that the purely bosonic part of our model is equivalent to the KdV-hierarchy.

The two superloop equations in the double scaling limit, which are equivalent to the continuum super-Virasoro constraints are:

\begin{align}
\hat{u}(z)\hat{v}(z) + \kappa^2 \chi^{BF}(z) &= \text{Polynomial}(z), \\
\hat{u}(z)^2 - \hat{v}(z)\partial\hat{v}(z) + \kappa^2 (\chi^{BB}(z) + \chi^{FF}(z) + \frac{1}{4z^2}) &= \text{Polynomial}(z),
\end{align}

where the two-loop operators are defined by:

\begin{align}
\chi^{BF}(z) &= \sum_{k,l \geq 0} \frac{1}{z^{k+l+2}} \frac{\partial^2 \kappa^2 F}{\partial t_k \partial \tau_l}, \\
\chi^{BB}(z) &= \sum_{k,l \geq 0} \frac{1}{z^{k+l+3}} \frac{\partial^2 \kappa^2 F}{\partial t_k \partial t_l}, \\
\chi^{FF}(z) &= \left( \partial_z \sum_{k \geq 0} \frac{1}{z^{k+1/2}} \frac{\partial}{\partial \tau_k} \right) \sum_{l \geq 0} \frac{1}{z^{l+1/2}} \frac{\partial \kappa^2 F}{\partial \tau_l}.
\end{align}
The loop operators and the free energy have the following genus expansion

\[ u(z) = u_0(z) + \kappa^2 u_1(z) + \ldots = \sum_{k \geq 0} u_0^{(2k)}(z) + \kappa^2 u_1^{(2k)}(z) + \ldots , \]

\[ v(z) = v_0(z) + \kappa^2 v_1(z) + \ldots = \sum_{k \geq 0} v_0^{(2k+1)}(z) + \kappa^2 v_1^{(2k+1)}(z) + \ldots , \]

\[ \chi^{(a)}(z) = \chi_0^{(a)}(z) + \kappa^2 \chi_1^{(a)}(z) + \ldots \quad a = BF, BB, FF , \]

\[ F = F_0 + \kappa^2 F_1 + \ldots . \]

The subindices in our notation indicate the genus, while for the order in fermionic couplings we introduce upper indices.

**Planar solution**

The leading terms in the genus expansion of equations (6.1) and (6.2) are

\[ \hat{u}_0(z)\hat{v}_0(z) = \text{Polynomial}(z) , \]

\[ \hat{u}_0(z)^2 - \hat{v}_0(z)\partial_z \hat{v}_0(z) = \text{Polynomial}(z) . \]

We follow closely the steps of the discrete case with a one-cut ansatz for \( \hat{u}_0^{(0)}(z) \)

\[ \hat{u}_0(z) = M(z)\sqrt{z + u} + \frac{A(z)}{(z + u)^{3/2}} , \]

\[ \hat{v}_0(z) = \frac{N(z)}{\sqrt{z + u}} . \]

Expanding in powers of \( (z + u) \)

\[ N(z) = \sum_{k \geq 0} n_k(z + u)^k , \]

\[ M(z) = \sum_{k \geq 0} m_k(z + u)^k , \]

we see that \( A(z) \) is determined by demanding the right hand side of (6.5) to be
polynomial in $z$

$$\tilde{u}_0(z) = M(z)\sqrt{z+u} - \frac{1}{2m_0(z+u)^{3/2}} n_1n_0 . \quad (6.10)$$

$M(z)$ is determined from $w_0^{(0)} \sim O(z^{-3/2})$. We have:

$$\frac{\partial u_0^{(0)}}{\partial u} = \frac{1}{\sqrt{z+u}} \left( \frac{1}{2} M(z) + (z+u)\partial_u M(z) \right) - \frac{1}{2\sqrt{z}} \frac{\partial t_0}{\partial u} , \quad (6.11)$$

which holds only when

$$\frac{1}{2} M(z) + (z+u)\partial_u M(z) = \frac{1}{2} \frac{\partial t_0}{\partial u} . \quad (6.12)$$

Inserting (6.9) in this equation we obtain a relation between the coefficients $m_k$ and the renormalized cosmological constant $t_0$:

$$m_0 = \frac{\partial t_0}{\partial u}$$

$$m_k = \frac{(-1)^k}{2} \frac{\Gamma(\frac{1}{2})}{\Gamma(k+\frac{3}{2})} \frac{\partial^{k+1} t_0}{\partial u^{k+1}} , \quad k \geq 1 . \quad (6.13)$$

The modes of $N(z)$ are determined by demanding

$$v_0(z) = \frac{N(z)}{\sqrt{z+u}} - \eta(z) = O(z^{-1/2}) . \quad (6.14)$$

This implies $\frac{\partial v_0(z)}{\partial u} \sim O(z^{-1/2})$, leading to

$$\frac{\partial n_k}{\partial u} = -(k + \frac{1}{2})n_{k+1} \quad k \geq 1 . \quad (6.15)$$

Fermi-statistics and compatibility between the bosonic and fermionic loop gives

$$n_1 = -4\partial_u n_0 .$$
Thus
\[ \hat{u}_0(z) = M(z)\sqrt{z + u} - \frac{2}{m_0(\sqrt{z + u})^{3/2}} n_0 \partial u n_0, \] (6.16)
\[ \partial_t v_0(z) = -\frac{1}{\sqrt{z + u}} \hat{\tau} n_0. \] (6.17)

With the ansatz
\[ n_0 = \sum_{n \geq 0} \beta_n u^n, \]
and (6.14) we obtain
\[ \beta_n = \frac{(-1)^n \Gamma(n + \frac{1}{2})}{2n! \Gamma(\frac{1}{2})} \hat{\tau}_n. \]

Thus
\[ n_0 = -\tau(u), \quad \tau(u) = \sum \hat{\tau}_n u^n. \] (6.18)

The final result coming from (6.16) and (6.17) is then
\[ \hat{u}_0(z) = M(z)\sqrt{z + u} + \frac{\tau \partial_t \tau}{(z + u)^{3/2}}, \]
\[ \partial_t v_0(z) = \frac{1}{\sqrt{z + u}} \hat{\partial} \tau(u), \]
and this coincides with the results of previous sections.

It is nice to see that from the purely bosonic part of \( \hat{u}_0(z) \) we obtain the planar string equation, since from:
\[ \frac{u_0^{(0)}}{\sqrt{z + u}} = M(z) - \frac{t(z)}{\sqrt{z + u}} = O(z^{-2}), \] (6.19)
we obtain the expected result from the vanishing of the terms proportional to \( z^{-1} \):
\[ \sum_{k \geq 0} \frac{(-1)^k \Gamma(k + \frac{3}{2})}{k! \Gamma(\frac{1}{2})} u^k t_k = 0. \] (6.20)

Genus one solution
The two superloop equations obtained from (6.1) and (6.2) are:

\[
2\hat{u}_0 u_1 - v_1 \partial \hat{v}_0 - \hat{v}_0 \partial v_1 + \chi_0^{BB} + \chi_0^{FF} + \frac{1}{4z^2} = \text{Polynomial}(z),
\]

\[
v_1 \hat{u}_0 + \hat{v}_0 u_1 + \chi_0^{BF} = \text{Polynomial}(z).
\]

\(\hat{w}_0\) and \(\hat{v}_0\) are already determined. The two-loop correlators are obtained from (6.3) for \(F = F_0\)

\[
\chi_0^{BF} = -\frac{1}{2(z + u)^{3/2}} \left( \frac{1}{\sqrt{z + u}} \partial_t \tau(u) \right),
\]

\[
\chi_0^{BB} = \frac{1}{8} \left( \frac{1}{(z + u)^2} - \frac{1}{z^2} \right) - \frac{1}{2} \partial_t \tau \partial_t \tau,
\]

\[
\chi_0^{FF} = \frac{1}{8} \left( \frac{1}{(z + u)^2} - \frac{1}{z^2} \right).
\]

We solve now the equations (6.21) and (6.22) according to the order of fermionic couplings

\[
u_1 = \sum_{k \geq 0} u_1^{(2k)} \quad v_1 = \sum_{k \geq 0} v_1^{(2k+1)}. \tag{6.24}
\]

For the fermionic loop operator we obtain

\[
v_1 = \sum_{k=0}^{3} \frac{V_{k+1/2}}{(z + u)^{k+1/2}},
\]

with

\[
V_{1/2} = -\frac{1}{3} D \left( \frac{D^2 \tau}{Du} \right) \quad \quad V_{5/2} = -\frac{D u \tau}{2} \]

\[
V_{3/2} = \frac{2}{3} D \left( \frac{V_{5/2}}{Du} \right) - \tau D \left( \frac{D^2 u}{2Du} \right) \quad \quad V_{7/2} = -\frac{5}{8} (Du)^2 \tau.
\]

\* In the following we will omit the \(z\) dependences
Where we have introduced the notation $D = \partial/\partial t_0$. The bosonic loop operator is

$$u_1(z) = \sum_{k=1}^{4} \frac{W_{(k+1/2)}}{(z + u)^{k+1/2}}, \quad (6.25)$$

with the coefficients

$$
\begin{align*}
W_{3/2} &= \frac{D^2 u}{12 D u} - 2\tau \overset{\leftrightarrow}{D V_{1/2}} \\
W_{5/2} &= -\frac{D u}{8} + 6V_{3/2} D\tau - 2\tau \overset{\leftrightarrow}{D^2 \tau} \\
W_{7/2} &= 5\tau D(Du D\tau) \\
W_{9/2} &= 7V_{7/2} \overset{\leftrightarrow}{D \tau}.
\end{align*}
\quad (6.26)
$$

The value of $V_{1/2}$ cannot be determined by the requirement that the left hand side of (6.22) is a polynomial in $z$. It follows from the consistency between $v_1(z)$ and $u_1(z)$. This condition is given by the requirement of commuting derivatives

$$\frac{\partial^2 F_1}{\partial t_0 \partial \tau_k} = \frac{\partial^2 F_1}{\partial \tau_k \partial t_0}, \quad (6.27)$$

where $F_1$ denotes the genus one contribution to the free energy.

For the purely bosonic part of $u_1(z)$ (no presence of fermionic couplings) one easily sees that this is the same result as for the one-matrix model. The equation to be solved in this case is

$$\hat{u}_0(z) u_1(z) + \chi_0(z) + \frac{1}{8z^2} = \text{Polynomial}(z), \quad (6.28)$$

$$\chi_0(z) = \frac{1}{8} \left( \frac{1}{(z + u)^2} - \frac{1}{z^2} \right), \quad (6.29)$$

where $\hat{u}_0(z) = M(z) \sqrt{z + u}$ is the solution for genus zero. Equation (6.28) coincides with the order zero equation in fermionic couplings coming from (6.21) just because the bosonic two-point correlators (6.23) and (6.29) coincide in the bosonic part. Our result does not agree with the continuum Liouville calculation for genus one of [13]. The origin of this discrepancy may be the fact, that there was no simple way to normalize the supermoduli integration in [13].
The reader may have noticed, that the solution presented for $v_1(z)$ is only linear in the fermionic couplings, as it was for genus zero. This comes from the fact that, the third order equation in fermionic couplings

$$v_1^{(3)} u_0^{(0)} + \tilde{v}_0 u_1^{(2)} + v_1^{(1)} \tau \frac{\tau \partial \tau}{\Delta^{3/2}} = \text{Polynomial}(z) \quad (6.30)$$

is solved by $v_1^{(3)} = 0$ which means that also on the torus we have a maximal coupling of two fermions.

**Genus two solution**

We consider now the situation in genus two to see if the free energy changes with respect to the one-matrix model. The two superloop equations to be solved are

$$2 \tilde{u}_0 u_2 + u_1^2 - v_2 \tilde{v}_0 - \tilde{v}_0 \partial v_2 - v_1 \partial v_1 + \chi_{1}^{BB} + \chi_{1}^{FF} = \text{Polynomial}(z) \quad , \quad (6.31)$$

$$\tilde{v}_0 u_2 + \tilde{u}_0 v_2 + u_1 v_1 + \chi_{1}^{BF} = \text{Polynomial}(z) \quad . \quad (6.32)$$

To calculate the free energy, we are only interested in the purely bosonic piece of the two loop correlators, which are given by (take (6.3) for $F = F_1$)

$$\chi_{1}^{FF} = \chi_{1}^{BB^{(0)}} = \frac{13}{16m_0^2} \frac{1}{\Delta^5} - \frac{3 m_1}{4 m_0^2} \frac{1}{\Delta^4} + \left( \frac{3 m_1^2}{8 m_0^4} - \frac{5 m_2}{16 m_0^4} \right) \frac{1}{\Delta^3} . \quad (6.33)$$

We see, that again these two loop correlators coincide with the one-matrix model values. This immediately implies, that also in genus two we have the same value in the purely bosonic part of the free energy as in the one-matrix model. The
complete solution to order zero in fermionic couplings is

\[
\begin{align*}
\frac{u_2^{(0)}}{\Delta^{3/2}} &= \frac{a}{\Delta^{5/2}} + \frac{b}{\Delta^{7/2}} + \frac{c}{\Delta^{9/2}} + \frac{d}{\Delta^{11/2}} + \frac{e}{\Delta^{13/2}} \\
\frac{a}{32m_0^4} &= \frac{63m_1^2}{16m_0^6} - \frac{75m_2m_2}{128m_0^6} - \frac{145m_3^2}{32m_0^5} - \frac{77m_1m_3}{128m_0^5} + \frac{105m_4}{128m_0^4} \\
\frac{b}{32m_0^4} &= \frac{63m_1^3}{32m_0^5} - \frac{87m_1m_2}{32m_0^5} + \frac{105m_3}{128m_0^4} \\
\frac{c}{32m_0^5} &= -\frac{63m_1^2}{128m_0^4} + \frac{145m_2}{128m_0^4} \\
d &= \frac{203m_1}{128m_0^4}, \\
e &= -\frac{105}{128m_0^3}.
\end{align*}
\]

We now continue the calculation for the case of pure gravity \((m_k = 0, \ k \geq 2)\). Our results reproduce, up to genus two the values expected for the Painlevé-I equation. Since

\[
m_0 = \frac{1}{Du} \quad m_1 = \frac{2}{3} m_0^3 D^2 u
\]

we get for the second derivative of the free energy the expansion

\[
\langle \sigma_0 \sigma_0 \rangle = -\frac{u}{4} + \frac{\kappa^2}{12} D \left( \frac{D^2 u}{Du} \right) - \kappa^4 \frac{63}{162} D \left( \frac{(D^2 u)^4}{(Du)^5} \right) + \ldots .
\]

For pure gravity we take \(u = \sqrt{t}\), to obtain

\[
\langle \sigma_0 \sigma_0 \rangle = -\frac{1}{4}(\sqrt{t} - \frac{\kappa^2}{24} t^2 - \frac{49}{1152} \kappa^4 t^{9/2} + \ldots)
\]

which agrees with the first three terms of the solution to the Painlevé-I equation appearing in pure gravity [5].

**General properties of the Free Energy**

We now give the heuristic argument showing that the bosonic loop operator \(u(z)\) coincides with the pure gravity case with an even potential when all the fermion couplings are set to zero. First we write the loop equations for the pure...
gravity case derived from an even potential. They are given by

\[ \frac{1}{2} \hat{u}^2 + \kappa^2 (\chi(z) + \frac{1}{8z^2}) = \text{Polynomial}(z). \] (6.38)

Note that equation (6.29) implies \( \chi(z) + \frac{1}{8z^2} = \frac{1}{8} \Delta^{-2} \), where \( \Delta = z + u \). The correlator \( \chi(z) \) can be written in terms of the loop creation operator as in the supersymmetric case

\[ \chi(z) = \mathcal{J}_F(z)u(z). \] (6.39)

The explicit form of the loop insertion operators can be found in (4.41),(5.28). Since \( \hat{u}_0 = M(z)\Delta^{\frac{1}{2}} \), \( u_n(z) \) for \( n > 0 \) will be obtained in terms of the negative half-integer powers of \( \Delta \),

\[ u_n(z) = \sum_{k \geq 1} u_n^k \Delta^{-k-1/2}. \] (6.40)

The expansion in (6.40) starts with \( \Delta^{-3/2} \) because of the general structure of the loop operator discussed previously. Similarly, by (4.34) \( \mathcal{J}_B \sim O(\Delta^{-3/2}) \), and the first power of \( \Delta \) in \( \chi(z) \) is \( \Delta^{-3} \). These general remarks about the power series representation of \( \chi(z,0,u(z)) \) apply beyond genus zero. On the sphere there are non-universal contributions like \( 1/z^2 \) in (6.29) and we do not have such simple expansions in \( \Delta \). These considerations will be important in the following.

The supersymmetric loop equations are given by (6.1,2). Since we are only interested in the leading contribution \( u^{(0)}(z) \) i.e. to order zero in fermionic couplings, it suffices to keep \( \chi^{BB} \) and \( \chi^{FF} \) to order zero and \( v(z) \) to first order in fermionic couplings. Similarly, we can drop the term \( v\partial v \) in (6.2) which becomes,

\[ \hat{u}(z)^2 + \kappa^2 (\chi^{BB}(z) + \chi^{FF}(z) + \frac{1}{4z^2}) = \text{Polynomial}(z). \] (6.41)

We showed by direct computation that \( \chi^{BB}_n(z) = \chi^{FF}_n(z) \) at least for \( n = 0,1 \), the
sphere and the torus. If this were true to all orders, then (6.41) would become:

\[
\frac{1}{2} \hat{u}(z)^2 + \kappa^2 (\chi_{BB}^n(z) + \frac{1}{8z^2}) = \text{Polynomial}(z),
\]

which is identical to (6.38), and our proposition would be proved. We will argue by induction that indeed \( \chi_{BB}^n(z) = \chi_{FF}^n(z) \). Assume this to be true up to order \( n-1 \). Expand (6.41) in powers of \( \kappa^2 \). This yields,

\[
2\hat{u}_0 u_n + \sum_{k=1}^{n-1} u_k u_{n-k} + 2 \chi_{BB}^{n-1} = \mathcal{O}(1),
\]

where we have used \( \chi_{BB}^{n-1}(z) = \chi_{FF}^{n-1}(z) \), and we have labeled a polynomial in \( z \) as \( \mathcal{O}(1) \) in \( \Delta \). We have included the non-universal factor \( t(z) \) in \( \hat{u}_0 \). Expanding (6.2) in powers of \( \kappa^2 \) we obtain

\[
\hat{u}_0 v_n + u_n \hat{v}_0 + \sum_{k=1}^{n-1} u_k v_{n-k} + \chi_{BB}^{n-1} = \mathcal{O}(1).
\]

Acting with \( J_B \) on (6.43) yields

\[
2\hat{u}_0 \chi_{BB}^n + 2 \chi_{BB}^0 u_n + 2 \sum_{k=1}^{n-1} u_{n-k} \chi_{BB}^n + 2 J_B \chi_{BB}^{n-1} = \mathcal{O}(\Delta^{-3/2}),
\]

next, acting with \( \partial_z J_F \) on (6.44) we obtain

\[
\hat{u}_0 \chi_{FF}^n(z) + u_n \hat{\chi}_0^{BB} + \sum_{k=1}^{n-1} u_k \chi_{BB}^{n-k} + \partial_z J_F \chi_{BB}^{n-1} = \mathcal{O}(\Delta^{-3/2}),
\]

where we have used the induction hypothesis. For the same reason, we have the equalities

\[
\partial_z J_F \chi_{BB}^{n-1} = \partial_z J_J \partial_z J_B v_{n-1} = J_B \chi_{BB}^{n-1} = J_B \chi_{BB}^{n-1},
\]

and we find that (6.45) and (6.46) are identical. Since \( \hat{u}_0 = \mathcal{O}(\Delta^{-1/2}) \), this would imply that \( \chi_{BB}^n = \chi_{FF}^n \), as was to be shown. The basic subtlety in this argument
has to do with the equality between $\tilde{\chi}_0^{BB}$ and $\tilde{\chi}_0^{FF}$. This is because naively these quantities contain infinities. To regulate them we have to point-split the variable in the loop creation operator with respect to the variable $z$ in which we express the loop equations. This makes the count of powers of $\Delta$ rather subtle, and to assure that no mistakes are made we have to study in detail the two-loop equations. Nevertheless, we believe that after the appropriate subtleties of the mathematical details are clarified, the conclusion of this heuristic proof will remain true.

7. RELATION TO SUPERINTEGRABLE HIERARCHIES

Up to this point we have been able to calculate correlation functions for the first cases in the genus expansion. Although this method is straightforward to apply for any genus, the procedure is cumbersome and of course it is essential to find the relation to supersymmetric integrable hierarchies.

This relation is well established for the one-matrix model [4, 5, 23]. If we denote the specific heat with $U = \langle PP \rangle$ where $P$ is the puncture operator, then the different multicritical points are characterized by the string equation

$$-x = \sum_{n=1}^{\infty} t_n R_n[U],$$

(7.1)

here $R_n[U]$ are the Gelfand-Dikii polynomials of the KdV-hierarchy, defined through the recursion relations

$$DR_{n+1}[U] = (\kappa^2 D^3 + 4UD + 2(DU)) R_n[U]$$

(7.2)

$R_0 = \frac{1}{2}$, $R_1 = U$, $R_2 = (3U^2 + \kappa^2 U'')$, $\ldots$

(7.2) implies therefore recursion relations between the flows. Equations (7.1) and (7.2) contain all the information about the correlation functions of the model.
Through the recent work of Witten and Kontsevich [12] we can directly relate the Virasoro constraints $L_n \tau = 0, \ n \geq -1$ with the $\tau$–function for the KdV hierarchy with initial condition $L_{-1} \tau = 0$. Since in the supercase we do not yet have a formulation in terms of generalized matrices, it is useful to take a more pedestrian approach and see how the KdV-hierarchy emerges from the explicit solution of the loop equations. In the purely bosonic model, the one-point functions are given by:

$$D \langle \sigma_n \rangle_0 = \frac{(-1)^{n+1} \Gamma(n+\frac{3}{2}) u^{n+1}}{2n! \Gamma(\frac{1}{2}) n + 1}$$

(7.4)

The definition of $U$ as the two-point function of the puncture operator implies:

$$D \langle \sigma_0 \rangle_0 = U^{(0)} = -\frac{1}{4} u$$

(7.5)

Using the planar string equation and (7.5) we learn that

$$D^2 \langle \sigma_n \rangle = \frac{\partial U^{(0)}}{\partial t_n} = DR^{(0)}_{n+1}, \quad R^{(0)}_{n+1} = \frac{2^{2n+1} \Gamma(n+\frac{3}{2})}{(n+1)! \Gamma(\frac{1}{2})} U^{n+1}$$

(7.6)

At genus one,

$$w^{(1)}(z) = \frac{m_1}{8m_0^2} \frac{1}{(z+u)^{3/2}} - \frac{1}{8m_0} \frac{1}{(z+u)^{5/2}}$$

(7.7)

Therefore,

$$\langle \sigma_n \rangle_1 = \kappa^2 \frac{(-1)^n \Gamma(n+\frac{3}{2})}{2m_0 n! \Gamma(\frac{1}{2})} \left( \frac{m_1}{2m_0} u^n + \frac{n}{3} u^{n-1} \right)$$

(7.8)

Hence the “heat capacity” $U$ is given to genus one by:

$$U = U^{(0)} + \kappa^2 U^{(1)} = -\frac{1}{4} u + \frac{1}{12} \kappa^2 D \left( \frac{D^2 u}{Du} \right)$$

(7.9)

Similarly, to this order we can compute $\partial U/\partial t_1$, and obtain the well known KdV
equation

\[ \frac{\partial U}{\partial t_1} = \kappa^2 D^3 U + 3DU^2 \]  \hspace{1cm} (7.10)

Hence, to this order:

\[ R_2[U] = \kappa^2 D^2 U + 3U^2, \]  \hspace{1cm} (7.11)

and

\[ R_{n+1}[U] = \frac{2^{2n+1}\Gamma(n+\frac{3}{2})}{n!\Gamma(\frac{1}{2})} \left( \frac{U^{n+1}}{n+1} + \frac{n}{12}\kappa^2 U^{n-1}D^2 U + \frac{\kappa^2}{12}D^2 U^n \right) + O(\kappa^4) \]  \hspace{1cm} (7.12)

Including now the genus two correction to \( U \) which was computed in the previous section:

\[ U = U^{(0)} + \kappa^2 U^{(1)} + \kappa^4 U^{(2)} \]  \hspace{1cm} (7.13)

it is easy to see that at genus two there is no correction to (7.10), and furthermore, that to this order we have agreement with (7.2). Although this is by no means the cleanest way to exhibit the equivalence between KdV and the loop equations, it is clear that to determine the differential operator appearing on the right hand side of (7.2) (assuming its existence) we need to know explicit correlators only for genera zero, one and two. Equation (7.10) together with the fact that it is not corrected in genus two and the perturbative understanding of (7.2) provide strong hints that the flows in the one-matrix model are governed by the KdV-hierarchy. In the supersymmetric case, we could proceed by analogy with the previous arguments. Since we have computed all correlation functions for genus zero and one, and we have partial results in genus two, we can try to explore what type of differential relations allow us to express the flows \( \partial/\partial t_n, \partial/\partial \tau_n \) in terms of the basic flows \( \partial/\partial t_0, \partial/\partial \tau_0 \). We can also examine the supersymmetric extensions of the KdV- or KP-hierarchies. For KdV, the papers [16] find a one parameter family of
supersymmetric extensions given by:

\[ \dot{u} = -u''' + 6uu' + 3ia\xi'', \quad (7.14) \]

\[ \dot{\xi} = -a\xi''' + (2 + a)u\xi' + 3u'\xi, \quad (7.15) \]

\[ f = \frac{\partial f}{\partial t} \quad f' = \frac{\partial f}{\partial x} \]

\( u \) (resp. \( \xi \)) is a bosonic (resp. fermionic) function. This system is integrable only for \( a = 1, 4 \). Only the case \( a = 1 \) is really supersymmetric. If we formally set \( \xi = 0 \), we obtain KdV. Equations (7.14) and (7.15) can be therefore considered as supersymmetric extensions to the KdV equation. If on the other hand, we are also interested in fermionic flows \((\partial/\partial \tau_n)\), we need to extend these systems to incorporate odd flows. This has been done in [7, 14, 15]. In reference [7] the KP-hierarchy is extended using a supersymmetric Lax-pair formulation. A different approach is taken in [14, 15], where only the even (i.e. bosonic) flows admit a Lax pair representation. The fermionic flows have a simpler expression than those of [7]. A common feature of all these supersymmetric hierarchies is that if we consider only even flows, and we set all the fermionic variables to zero, we recover the standard KdV- or KP-hierarchies. In terms of loop equations this is similar to formally considering only the bosonic part of the energy momentum tensor \( T_B(z) \) and ignore the fermion field dependence. This would generate the KdV-hierarchy as in the bosonic case. In this respect our loop equations are a good starting point to construct another supersymmetric extension of KdV.

If we take the fermionic dependence into account, we have not yet been able to identify the corresponding differential relations which should lead to a local integrable hierarchy. Our analysis does not seem to make our one- or higher-point functions compatible with known extensions (7.14) and (7.15) or those in [16, 14, 15]. This is perhaps not so bad as it sounds. If our model indeed describes a non-critical superstring, the alleged hierarchy describing the continuum limit should
capture the geometry of the supermoduli space of super-Riemann surfaces. From the work in [15] we know that the algebro-geometric data needed to construct quasi-periodic solutions to the Manin-Radul or Mulase-Rabin hierarchies does not include super-Riemann surfaces. We continue looking for an integrable supersymmetric hierarchy compatible with our loop equations.

8. CONCLUSIONS AND OUTLOOK

In this paper we have taken the double scaling limit of the superloop equations proposed in [8]. Working with general potentials, we have shown that the spectrum of anomalous dimensions coincides with those which follow from [2,3], when one couples two-dimensional supergravity to minimal superconformal matter of type \((2,4m)\). The continuum limit of these theories is described by a \(Z_2\)-twisted scalar representing bosonic loops, and a Weyl-Majorana fermion in the Ramond sector representing fermionic loops. We have solved the continuum superloop equations in genus zero and one completely, and partially in genus two. The piece of the free energy independent of the fermionic couplings is given in the pure supergravity case by the same expression as for the one-matrix models [5] i.e. by the same solution to the Painlevé-I equation. This conclusion was shown to hold in general by a heuristic argument in Section seven. So far we have not been able to identify an integrable superhierarchy, which reproduces our correlation functions. As explained in the text, this is not necessarily negative. No superhierarchy is yet known, which incorporates fully the geometry of super-Riemann surfaces. Finally, the corresponding generalization of multimatrix models in our context is still missing. Work in these directions is in progress, and we hope to report on the results elsewhere.

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