EXISTENCE, UNIQUENESS AND REGULARITY OF VISCOSITY SOLUTIONS TO NON-MONOTONE WEAKLY COUPLED SYSTEMS OF HAMILTON-JACOBI EQUATIONS

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Abstract. We show the existence and uniqueness of the viscosity solution to Cauchy problem of certain non-monotone weakly coupled systems of first order evolutionary Hamilton-Jacobi equations. Moreover, for a typical model problem, we obtain the locally Lipschitz continuity of the viscosity solution with the continuous initial data and a series of corollaries from this property.

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1. INTRODUCTION AND MAIN RESULTS

In this paper, we shall consider the following Cauchy problem of the weakly coupled system of evolutionary Hamilton-Jacobi equations

\[ \begin{aligned}
\partial_t u_i + H_i(x, \partial_x u_i, u) &= 0, \quad (x,t) \in M \times [0, \infty); \\
u(x,0) &= \varphi(x), \quad x \in M, 1 \leq i \leq m.
\end{aligned} \] (1.1)

where \( u \) (resp. \( \varphi \)) are continuous vector-valued functions from \( M \times [0, \infty) \) (resp. \( M \)) to \( \mathbb{R}^m \). Hereinafter, the notation \( u_i \) always denotes the \( i \)-th component of the vector \( u \in \mathbb{R}^m \).

To the best of our knowledge, the existence and uniqueness of viscosity solutions for weakly coupled systems (1.1) of Hamilton-Jacobi equations have been firstly established by [8, 11] under certain monotonicity assumptions at about 90s. In recent years, motivated by optimal switching problems, there have been many studies on the properties of viscosity solutions of weakly coupled systems of Hamilton-Jacobi equations, including the corresponding extensions of the weak KAM and Aubry-Mather theories [10, 7, 14], the large-time behavior of solutions [15, 16, 18], and homogenization problems [5, 17]. Most of the studies focus on the linearly coupled case, i.e. Hamiltonians \( H_i \) linearly depend on \( u \in \mathbb{R}^m \) (the linear coefficients may depend on \( x \) variable). In this case, the well posedness results in [8, 11] can be applied since the monotonicity assumption is satisfied.

Generally speaking, the linearly coupled case could be seen as a direct generalization of classical Hamilton-Jacobi equation independent of \( u \)-variable when \( m = 1 \). From this point of view, the nonlinear weakly coupled system (1.1) could be considered as a natural generalization of contact Hamilton-Jacobi equations studied in the series works [19, 20, 21]. The contact Hamilton-Jacobi equations are functional dual of contact Hamiltonian

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systems, which have various applications in different areas, for instance, geometric optics and wave propagations, see [1], thus our research begins with non-monotonicity assumptions (H1)-(H4) (see Section 2.2 below).

First of all, we establish an implicit variational principle, from which we introduce the notions of the variational solution and the solution semigroup for (1.1). Moreover, by showing the equivalence between variational solutions and viscosity solutions, we obtain the existence and uniqueness of the viscosity solution of (1.1). More precisely, we have

**Theorem 1.1.** The Cauchy problem of the weakly coupled system (1.1) of evolutionary Hamilton-Jacobi equations has a unique viscosity solution \( u(x,t) \in C(M \times [0,\infty), \mathbb{R}^m) \), which can be represented by

\[
  u_i(x,t) = \inf_{\gamma \in C^1([0,t],M)} \left\{ \varphi_i(\gamma(0)) + \int_0^t L_i(\gamma(s),\dot{\gamma}(s), u_l(\gamma(s), s)) ds \right\},
\]

where \( L_i : TM \times \mathbb{R}^m \to \mathbb{R} \) denotes the convex dual associated to \( H_i \).

The regularity property of viscosity solutions plays a fundamental role in applying the dynamical approach to the study of both classical and contact Hamilton-Jacobi equations, for example, weak KAM theory, see [9, 19]. With no doubt, the same situation occurs for the study of weakly coupled systems (1.1).

Note that if the initial data \( \varphi(x) \) is just continuous instead of being Lipschitz continuous, it is difficult and subtle to obtain more regularity (Lipschitz continuity or semiconcavity for instance) of the viscosity solution to (1.1) beyond uniform continuity by using PDE approaches. From this view of calculus of variations, a standard technique to achieve that is formulated as Fleming lemma, see [9, Lemma 4.6.3, Page 148]. Unfortunately, it can not be directly applied to the case with the systems of Hamilton-Jacobi equations. Hence, a new method has to be developed to handle this issue. In order to avoid technical difficulties, we restrict ourselves to a typical model. More precisely, under (H1)-(H4) and the additional assumption:

(A) For \( 1 \leq i \leq m, H_i \) can be divided into two parts, i.e.

\[
  H_i(x,p,u) = h_i(x,p) + P_i(x,u),
\]

and there exists \( 0 < a < 1 < A \) such that for any \( 1 \leq i \leq m, \)

\[
  a|p|^2 \leq h_i(x,p) \leq A|p|^2,
\]

we obtained the Lipschitz regularity result:

**Theorem 1.2.** Let \( u \) be the unique viscosity solution associated to the Cauchy problem (1.1), then \( u \) is locally Lipschitz on \( M \times (0,\infty) \).

One notice that the Hamiltonians satisfying (A) can be viewed as a generalization of classical mechanical system, where \( h_i(x,p) \) denotes the kinetic energy, always represented by a Riemannian metric, and \( P_i \) denotes the potential. Under the additional assumption (A), the 1/2-Hölder regularity can obtained by a direct calculation. An estimate for the linearly coupled case can be found in [16, Lemma 3.2]. It is tricky to increase from 1/2-Hölder continuity to Lipschitz continuity, which will be achieved by the inductive method (see Section 5 below).

Based on Theorem 1.2, we obtain some further properties, including the semiconcavity of the viscosity solutions and the regularity of the action minimizing curves.

**Corollary 1.1.** Let \( u \) be the unique viscosity solution associated to the Cauchy problem (1.1) and for each \( 1 \leq i \leq m, \xi_i : [0,t] \to M \) be an absolutely continuous curve with \( \xi_i(t) = x \) such that

\[
  u_i(x,t) = \varphi_i(\xi_i(0)) + \int_0^t L_i(\xi_i(s),\xi_i(s), u_l(\xi_i(s), s)) ds,
\]

then there hold

(i) \( u_i \) is locally semiconcave on \( M \times (0,\infty) \).

(ii) \( \xi_i \) is locally Lipschitz on \( (0,t] \).

(iii) let \( (x_0,t_0) \in M \times (0,\infty) \) be a differentiable point of \( u_i \), then \( \xi_i \) is differentiable at \( t_0 \) from the left side. Moreover, denote by \( V_i \) the left derivative of \( \xi_i \) at \( t_0 \) and let \( P_i = \partial_xu_i(x_0,t_0) \), we have

\[
  \langle P_i, V_i \rangle = L_i(x_0,V_i, u(x_0,t_0)) + H_i(x_0,P_i, u(x_0,t_0)).
\]
Or equivalently,

\[ V_i = \frac{\partial H}{\partial p}(x_0, P_i, u(x_0, t_0)). \]

The last statement of this corollary asserts that the action minimizing curves passing through the differentiable points of \( u \) satisfy certain ODEs. Comparably, for the contact Hamilton-Jacobi equation corresponding to \( m = 1 \) in (1.1), we know that the action minimizing curve \( \xi \) is smooth enough and it is the \( x \)-component of the characteristics (contact Hamilton equations) if \( H \) is of class \( C^3 \) [20]:

\[
\begin{align*}
\dot{x} &= \frac{\partial H}{\partial p}(x, u, p), \\
\dot{p} &= -\frac{\partial H}{\partial u}(x, u, p) - \frac{\partial H}{\partial x}(x, u, p)p, \\
\dot{u} &= \frac{\partial H}{\partial p}(x, u, p) \cdot p - H(x, u, p).
\end{align*}
\]

Unfortunately, there does not exist characteristics for the case with \( m \geq 2 \) in this generality. Consequently, it seems that the Lipschitz regularity of \( \xi \) can not be improved.

This paper is outlined as follows. In Section 2, we will introduce the notations and general settings as preliminaries. In Section 3, we will define the variational solution and the solution semigroup in an implicit way. In Section 4, by showing the equivalence between viscosity solutions and variational solutions, we will complete the proof of Theorem 1.1. In Section 5 and Section 6, we will discuss the locally Lipschitz continuity of the viscosity solution and further benefits from this property. Moreover, the proofs of Theorem 1.2 and Corollary 1.1 will be completed.

2. Preliminaries

2.1. Notions and Notations. This part devotes to fixing some notations that is convenient for the presentation of our results.

Once and for all, let \( N \) and \( m > 1 \) be two fixed positive integers, \( M = \mathbb{T}^N \) be the \( N \)-dimensional flat torus, the quotient space of Euclidean space \( \mathbb{R}^N \) under the usual \( \mathbb{Z}^N \)-action.

We shall consider, in the following, continuous maps from \( M \) or \( M \times [0, \infty) \) to \( \mathbb{R}^m, m \geq 1 \). Denoting by \( | \cdot | \) the Euclidean norm on \( \mathbb{R}^N \) and \( \mathbb{R} \), we would like to first choose metric structures on the domains and ranges of these continuous maps. For domains, we

- let \( x, y \) be two points on \( M \), we use \( |x - y| \) to denote the distance on \( M \) induced by Euclidean metric on \( \mathbb{R}^N \);
- define a metric \( d \) on \( M \times [0, \infty) \) (and on its subspaces) by

\[ d((x, t), (y, s)) = |x - y| + |t - s|, \]

where \((x, t), (y, s)\) are two points in \( M \times [0, \infty) \).

For the range space, we choose the \( L^\infty \)-norm \( \| \cdot \| \) for vectors in \( \mathbb{R}^m \), i.e. for any \( v \in \mathbb{R}^m \),

\[ \|v\| = \max_{1 \leq i \leq m} |v_i|, \]

and the induced metric.

Denote by \( TM \) the tangent bundle of \( M \) and \((x, \dot{x})\) denotes a point of \( TM \), with \( x \in M \) and \( \dot{x} \in T_x M = \mathbb{R}^N \); denote by \( T^*M \) the cotangent bundle of \( M \) and \((x, p)\) denotes a point of \( T^*M \), with \( x \in \mathbb{T}^N \) and \( p \in \mathbb{R}^N \) a linear form on \( T_x M \). The latter will be identified with the vector \( p \in \mathbb{R}^N \) such that

\[ p(\dot{x}) = \langle p, \dot{x} \rangle, \]

for all \( \dot{x} \in T_x M = \mathbb{R}^N \),

where \( \langle \cdot, \cdot \rangle \) denotes the Euclidean scalar product in \( \mathbb{R}^N \). For every \( x \in M \), the fibers \( T_x M \) and \( T^*_x M \) are also endowed with the Euclidean norm.

Given an integer \( m > 0 \), denote by \( C(M, \mathbb{R}^m) \) the Banach space of continuous vector-valued functions \( u \) from \( M \) to \( \mathbb{R}^m \), endowed with the norm

\[ \|u\|_{\infty, M} = \max_{x \in M} \|u(x)\|. \]

Similarly, given number \( T > 0 \), denote by \( C(M \times [0, T], \mathbb{R}^m) \) the Banach space of continuous vector-valued functions \( u \) from \( M \times [0, T] \) to \( \mathbb{R}^m \), endowed with the norm

\[ \|u\|_{\infty, M \times [0, T]} = \max_{(x, t) \in M \times [0, T]} \|u(x, t)\|. \]

Since a solution to the weakly coupled system (1.1) is a vector-valued continuous function, the notion of viscosity solution should be reformulated. We shall use use the following definition given in [5]:
Definition 2.1. Let $u: M \times [0, \infty) \to \mathbb{R}^m$ be a continuous function,

(i) it is called a viscosity subsolution of (1.1) if for each $1 \leq i \leq m$,

- $u_i(\cdot, 0) \leq \phi_i$ on $M$,
- whenever $\phi$ is a real-valued $C^1$ function on a neighborhood of $(x, t), t > 0$ such that $u_i - \phi$ attains a local maximum at $(x, t),$ 
$$\partial_t \phi(x, t) + H_i(x, \partial_x \phi(x, t), u(x, t)) \leq 0.$$ 

(ii) it is called a viscosity supersolution of (1.1) if for each $1 \leq i \leq m$,

- $u_i(\cdot, 0) \geq \phi_i$ on $M$,
- whenever $\phi$ is a real-valued $C^1$ function on a neighborhood of $(x, t), t > 0$ such that $u_i - \phi$ attains a local minimum at $(x, t),$ 
$$\partial_t \phi(x, t) + H_i(x, \partial_x \phi(x, t), u(x, t)) \geq 0.$$ 

(iii) it is called a viscosity solution if it is both a viscosity sub and supersolution of (1.1).

Now we shall focus on the notion of regularity for functions in $C(M, \mathbb{R}^m)$ or $C(M \times [0, T], \mathbb{R}^m)$. Since we have chosen metrics on the domain and range of $u$, the notion of Hölder continuity is naturally defined. Given $\kappa > 0$ and $\alpha \in (0, 1]$, $u \in C(M, \mathbb{R}^m)$ is called $\kappa$-Hölder continuous if 
$$\|u(x) - u(x')\| \leq \kappa|x - x'|^\alpha,$$
and by the definition of $\| \cdot \|$, it is equivalent to that each component of $u$ is $(\kappa, \alpha)$-Hölder continuous. We call the constant $\kappa$ a $\alpha$-Hölder constant for $u$. In particular, a $(\kappa, 1)$-Hölder continuous function $u$ is called $\kappa$-Lipschitz continuous and $\kappa$ is called a Lipschitz constant for $u$. The same definition applies for functions in $C(M \times [0, \infty), \mathbb{R}^m)$.

Given $\alpha \in (0, 1]$, a function $u \in C(M \times [0, \infty), \mathbb{R}^m)$ is locally $\alpha$-Hölder continuous on $M \times (0, \infty)$ if for every $(x_0, t_0) \in M \times (0, \infty)$, there exist positive numbers $r, \kappa$ depending on $(x_0, t_0)$ such that $u$ is $(\kappa, \alpha)$-Hölder continuous on the open set $\{(x, t) \in M \times (0, \infty) | \|x - x_0\| + |t - t_0| < r\}$. In particular, a function $u \in C(M \times [0, \infty), \mathbb{R}^m)$ is locally Lipschitz continuous on $M \times (0, \infty)$ if it is locally 1-Hölder continuous on $M \times (0, \infty)$. To simplify the statement, we say a function $u \in C(M \times [0, \infty), \mathbb{R}^m)$ is locally Lipschitz continuous if it is locally Lipschitz continuous on $M \times (0, \infty)$.

From now on, a function is defined naturally on the subsets of its domain by restriction and we shall use the same notation for any of its restriction and itself if there is no confusion. Let $\delta > 0$, we shall use $O(h)$ ($o(h)$) to denote some function $f : [-\delta, \delta] \to \mathbb{R}$ such that $\frac{f(h)}{h}$ is bounded (goes to 0) as $h$ goes to 0 respectively.

2.2. General settings. In this part, we introduce some general assumptions on the Hamiltonians arose in weakly coupled systems 1.1 on which Theorem 1.1 could be established.

For $1 \leq i \leq m$, let $H_i : T^* M \times \mathbb{R}^m \to \mathbb{R}$ be a Hamiltonian such that the map $(x, p, u) \mapsto H_i(x, p, u)$ is

(H1) $C^2$ with respect to $(x, p, u),$
(H2) strictly convex with respect to $p$ for every $(x, u) \in M \times \mathbb{R}^m,$
(H3) superlinear with respect to $p$ for every $(x, u) \in M \times \mathbb{R}^m,$ i.e. fix any $(x, u),$
$$\frac{H_i(x, p, u)}{|p|} \to \infty \text{ as } |p| \to \infty,$$
(H4) uniformly Lipschitz with respect to $u$, i.e. there exists $\Theta_i > 0$ such that
$$|H_i(x, p, u) - H_i(x, p, v)| \leq \Theta_i \| u - v \| \text{ holds for all } x, p, u, v.$$

For $1 \leq i \leq m$, let $L_i : TM \times \mathbb{R}^m \to \mathbb{R}$ be the convex dual associated to $H_i$, that is,
$$L_i(x, \dot{x}, u) := \sup_{p \in T_x^* M} \{ (p, \dot{x}) - H(x, p, u) \}.$$ 

(2.1)

The conditions (H1)-(H4) on $H_i$ are easily translated to the conditions on $L_i$: (L1) $C^2$ with respect to $(x, \dot{x}, u),$
(L2) strictly convex with respect to $v$ for every $(x, u) \in M \times \mathbb{R}^m,$
(L3) superlinear with respect to $\dot{x}$ for every $(x, u) \in M \times \mathbb{R}^m,$ i.e. fix any $(x, u),$
$$\frac{L_i(x, \dot{x}, u)}{|\dot{x}|} \to \infty \text{ as } |\dot{x}| \to \infty,$$
For later use, we define $\Theta = \max_{1 \leq i \leq m}\{\Theta_i\}$.
For later use, we define $\Theta_i > 0$ such that
\[ |L_i(x, \dot{x}, u) - L_i(x, \dot{x}, v)| \leq \Theta_i\|u - v\| \text{ holds for all } x, \dot{x}, u, v, \]
where $\Theta_i$ denotes the same constant arises in (H4).

Remark 2.1. Technically, combining conditions (H1)-(H3), we know that: for any $K > 0$ and any compact set $C \subset \mathbb{R}^m$, there exist constants $R(K, C), D(K, C)$ such that for every $u \in C$ and every $p \in \{p \in T_x^\ast M||p|| \geq R(K, C)\}$,
\[ H(x, p, u) \geq K||p|| - D(K, C). \] (2.2)
By the dual property of $L$ and $H$, the above proposition is also valid for $L$ with $p$ replaced by $\dot{x}$.

We end this section by adding a brief and historical introduction on generality of (H1)-(H4) or (L1)-(L4).

1. If the Hamiltonian $H = H(x, p, t)$ depends on time variable $t$ periodically but contains no $u$-variable, conditions (H1)-(H3) and the additional assumption that the associated Hamilton flow is complete constitute the so-called Tonelli conditions, which are almost the most general conditions known to us for developing global variational methods for positive definite Hamiltonian systems. Tonelli conditions are firstly proposed by J. Mather when he was developing the celebrated Aubry-Mather theory for time periodic Hamiltonian system with arbitrary dimension, see [12], [13].

In the case that $H$ does not depend on $t$, (H1)-(H3) implies the completeness of the Hamilton flow.

2. If $m = 1$, such a Hamiltonian $H = H(x, p, u)$ is called contact Hamiltonian. Recent study shows that (H1)-(H3) is also a suitable setting for generalizing the global variational methods including Aubry-Mather theory and weak KAM theory to contact Hamiltonian systems. For such a topic, we refer to the series of works by K. Wang, the second and third author, see [19], [20], [21]. The condition (H4) is crucial in establishing the implicit variational principle in the contact case.

3. Implicit variational principle, variational solution and solution semigroup

In this section, we construct an implicit variational principle corresponding to system (1.1), which is a multidimensional analogy of the one constructed for contact Hamilton-Jacobi equations, see [19], [20], [21]. Based on this variational principle, we define the notion of variational solution and solution semigroup associated to system (1.1), necessary properties of the variational solution are proved in this procedure.

Given $T > 0$ and $\varphi \in C(M, \mathbb{R}^m)$, we define the operator $A_{\varphi, T}$ from $C(M \times [0, T], \mathbb{R}^m)$ to itself as the following, for any $u \in C(M \times [0, T], \mathbb{R}^m)$,
\[ A_{\varphi, T}[u](x, t) = \inf_{\gamma \in C^{\alpha_{\varphi}(0)}([0, T], M)} \left\{ \varphi(\gamma(0)) + \int_0^T L_i(\gamma(s), \dot{\gamma}(s), u(\gamma(s), s))ds \right\}, \] (3.1)
where $(x, t) \in M \times [0, T]$. By Tonelli theorem (see for instance [4]), for each $1 \leq i \leq m$, the above infimum can be achieved.

Lemma 3.1. Given $\varphi \in C(M, \mathbb{R}^m)$, $A_{\varphi, T}$ admits a unique fixed point $u_{\varphi, T} \in C(M \times [0, T], \mathbb{R}^m)$.

Proof. For any $u \in C(M \times [0, T], \mathbb{R}^m)$ and any $(x, t) \in M \times [0, T]$, by Tonelli theorem, for each $1 \leq i \leq m$, there exists $\gamma_{i, 1} : [0, t] \to M$ with $\gamma_{i, 1}(t) = x$ such that
\[ A_{\varphi, T}[u](x, t) = \varphi(\gamma_{i, 1}(0)) + \int_0^T L_i(\gamma_{i, 1}(s), \dot{\gamma}_{i, 1}(s), u(\gamma_{i, 1}(s), s))ds. \]
For any $v \in C(M \times [0, T], \mathbb{R}^m)$, from (L4) we have
\[ A_{\varphi, T}[u](x, t) - A_{\varphi, T}[v](x, t) \leq \int_0^t |L_i(\gamma_{i, 1}(s), \dot{\gamma}_{i, 1}(s), v(\gamma_{i, 1}(s), s)) - L_i(\gamma_{i, 1}(s), \dot{\gamma}_{i, 1}(s), u(\gamma_{i, 1}(s), s))|ds \leq \Theta\|u - v\|_{\infty, M \times [0, T]} \cdot t. \]
By exchanging the position of $u$ and $v$, we obtain
\[ |A_{\varphi, T}[u](x, t) - A_{\varphi, T}[v](x, t)| \leq t\Theta\|u - v\|_{\infty, M \times [0, T]}, \]
\[ A_{\varphi, T}[u](x, t) = \varphi(\gamma_{i, 1}(0)) + \int_0^T L_i(\gamma_{i, 1}(s), \dot{\gamma}_{i, 1}(s), u(\gamma_{i, 1}(s), s))ds. \]
which is equivalent to
\[ \|A_{\phi,T}[u](x,t) - A_{\phi,T}[v](x,t)\| \leq t\Theta\|u - v\|_{\infty,M \times [0,T]}. \] (3.2)

Let \( \gamma_{i,2} : [0,t] \to M \) be a curve such that
\[ A_{\phi,T}[v](x,t) = \varphi(\gamma_{i,2}(0)) + \int_0^t L_i(\gamma_{i,2}(s), \gamma_{i,2}(s), A_{\phi,T}[v](\gamma_{i,2}(s), s))ds. \]

It follows from (3.2) that for \( s \in [0, t] \), we have
\[ \|A_{\phi,T}[u](\gamma_{i,2}(s), s) - A_{\phi,T}[v](\gamma_{i,2}(s), s)\| \leq s\Theta\|u - v\|_{\infty,M \times [0,T]}. \]

Thus we have the following estimates
\[
\begin{align*}
&\|A_{\phi,T}[v](x,t) - A_{\phi,T}[u](x,t)\| \\
&\leq \int_0^t \Theta\|A_{\phi,T}[u](\gamma_{i,2}(s), s) - A_{\phi,T}[v](\gamma_{i,2}(s), s)\|ds \\
&\leq \Theta^2\|u - v\|_{\infty,M \times [0,T]}\int_0^t sds \\
&\leq \frac{(t\Theta)^2}{2}\|u - v\|_{\infty,M \times [0,T]};
\end{align*}
\]

By exchanging \( u \) and \( v \), we obtain
\[ \|A_{\phi,T}^n[u](x,t) - A_{\phi,T}^n[v](x,t)\| \leq \frac{(t\Theta)^n}{n!}\|u - v\|_{\infty,M \times [0,T]}, \]

which implies
\[ \|A_{\phi,T}[u] - A_{\phi,T}^n[v]\|_{\infty,M \times [0,T]} \leq \frac{(T\Theta)^n}{n!}\|u - v\|_{\infty,M \times [0,T]}. \]

Therefore, there exists \( N \in \mathbb{N} \) large enough such that \( A_{\phi,T}^N \) is a contraction mapping. Since \( C(M \times [0,T], \mathbb{R}^m) \) is complete, by Banach fixed point theorem, there exists a unique \( u_{\phi,T} \in C(M \times [0,T], \mathbb{R}^m) \) such that
\[ A_{\phi,T}^N[u_{\phi,T}] = u_{\phi,T}. \]

Since
\[ A_{\phi,T}[u_{\phi,T}] = A_{\phi,T} \circ A_{\phi,T}^N[u_{\phi,T}] = A_{\phi,T}^N[u_{\phi,T}], \]
\( A_{\phi,T}[u_{\phi,T}] \) is also a fixed point of \( A_{\phi,T}^N \). By the uniqueness of fixed point of \( A_{\phi,T}^N \), we have
\[ A_{\phi,T}[u_{\phi,T}] = u_{\phi,T}. \]

This completes the proof of Lemma 3.1. \( \square \)

For any \( 0 < T < T' \), denote by \( u_{\phi,T} \) the fixed point of \( A_{\phi,T'} \). From the definition of \( A_{\phi,T} \), \( u_{\phi,T\prime}|_{M \times [0,T]} \) is also a fixed point of \( A_{\phi,T} \), thus by uniqueness of fixed point of \( A_{\phi,T} \), \( u_{\phi,T\prime}|_{M \times [0,T]} = u_{\phi,T} \). So we could define
\[ u(x,t) = u_{\phi,T}(x,t) \quad \text{for } t \leq T. \] (3.3)

\( u \) coincides with the notion of variational solution associated to (1.1) defined as the following:

**Definition 3.1.** Let \( u : M \times [0,\infty) \to \mathbb{R} \) be a continuous function, \( u \) is called a variational solution to the system (1.1) if for any \( (x,t) \in M \times [0,\infty) \) and any \( 1 \leq i \leq m \),
\[
u_i(x,t) = \inf_{\gamma(t)=x, \gamma \in C^m([0,t],M)} \left\{ \varphi_i(\gamma(0)) + \int_0^t L_i(\gamma(s), \dot{\gamma}(s), u(\gamma(s), s))ds \right\}. \]

(3.4)

One could translate the above definition into calibration properties:

**Proposition 3.1.** Let \( u \) be a variational solution associated to system (1.1), then
Proof. (i) is just a restatement of the definition of variational solution. In fact, there exists an absolutely continuous curve \( \eta : [t_1, t_2] \to M \) such that

\[
\int_{t_1}^{t_2} L_i(\gamma(s), \gamma(s), u(\gamma(s), s))ds.
\]

Let \( \xi \) be a curve with \( \xi(t_1) = \eta(t_1) \) such that

\[
u_i(\eta(t_2), t_2) - u_i(\eta(t_1), t_1) > \int_{t_1}^{t_2} L_i(\eta(s), \eta(s), u(\eta(s), s))ds,
\]

then \( \gamma := \eta \ast \xi \in C^{ac}([0, t_2], M) \) and

\[
u_i(\gamma(t_2), t_2) - \varphi_i(\gamma(0)) = u_i(\eta(t_2), t_2) - \varphi_i(\gamma(0))
\]

\[
> \int_{t_1}^{t_2} L_i(\eta(s), \eta(s), u(\eta(s), s))ds + \int_{t_1}^{t_2} L_i(\xi(s), \xi(s), u(\xi(s), s))ds
\]

which is a contradiction to Definition 3.1.

For (ii), let \( \xi \in C^{ac}([0, t_2], M) \) be a curve with \( \xi(t_2) = x \) such that

\[
u_i(\xi(t_2), t_2) = \varphi_i(\xi(0)) + \int_{0}^{t_2} L_i(\xi(s), \xi(s), u(\xi(s), s))ds.
\]

Now by (i), for any \( 0 \leq t_1 < t_2 \),

\[
u_i(\xi(t_2), t_2) - u_i(\xi(t_1), t_1) \leq \int_{t_1}^{t_2} L_i(\xi(s), \xi(s), u(\xi(s), s))ds,
\]

\[
u_i(\xi(t_1), t_1) - \varphi_i(\xi(0), 0) \leq \int_{0}^{t_1} L_i(\xi(s), \xi(s), u(\xi(s), s))ds.
\]

We add the above two inequalities and use (3.7) to find that the inequalities are actually equalities. So we define \( \gamma_i = \xi|[t_1, t_2] \) to complete the proof.

Now let us define a family of operators \( \{ T^-_t \}_{t \in \mathbb{R}} \) from \( C(M, \mathbb{R}^m) \) to itself.

**Definition 3.2.** For each \( \varphi \in C(M, \mathbb{R}^m) \), let \( u \in C(M \times [0, \infty), \mathbb{R}) \) be the variational solution associated to system (1.1), define

\[
T^{-}_t \varphi(x) = u(x, t), \quad \forall (x, t) \in M \times [0, \infty)
\]

Due to the following proposition, the operator family \( \{ T^{-}_t \}_{t \geq 0} \) really constitutes a semigroup, we call such an operator family the solution semigroup associated to system (1.1).

**Proposition 3.2.** For any \( t, s \geq 0 \), \( T^-_{t+s} = T^-_t \circ T^-_s \).

**Proof.** For every fixed \( s \geq 0 \), we define

\[
u(x, t) = T^{-}_t \circ T^{-}_s \varphi(x)
\]

On one hand, by definition of \( T^-_t \) and \( u \), for each \( 1 \leq i \leq m \) and \( 0 < t \leq T \),

\[
u_i(x, t) = T^{-}_t \circ T^{-}_s \varphi_i(x)
\]
where $\gamma \in C^{ac}([0,t], M)$, so
\[ u = A_{T^-} \varphi \cdot [u]. \]

On the other hand, by definition of $T^-_t$ and $\mathbf{v}$, for each $1 \leq i \leq m$ and $0 < t \leq T$,
\[ v(x, t) = \left[ T_{T^-_t} \varphi \cdot \right]_i(x) = \inf_{\gamma(t)=x} \left\{ \varphi_i(\gamma(0)) + \int_0^t L_i(\gamma(\tau), \dot{\gamma}(\tau), T^-_t \varphi(\gamma(\tau))) d\tau \right\} = \inf_{\gamma(t)=x} \left\{ T^-_t \varphi_i(\gamma(0)) + \int_0^t L_i(\gamma(\tau), \dot{\gamma}(\tau), T^-_t \varphi(\gamma(\tau))) d\tau \right\}
\]
\[ = \inf_{\tilde{\gamma}(t)=x} \left\{ \tilde{T}^-_t \varphi_i(\tilde{\gamma}(0)) + \int_0^t L_i(\tilde{\gamma}(\tau), \dot{\tilde{\gamma}}(\tau), \mathbf{v}(\tilde{\gamma}(\tau), \tau)) d\tau \right\}
\]
\[ = A_{T^-} \varphi \cdot [v]_i(x), \]
where $\gamma \in C^{ac}([0, t + s], M)$ and $\tilde{\gamma} \in C^{ac}([0, t], M)$, so
\[ v = A_{T^-} \varphi \cdot [v]. \]

Since both $u$, $v$ are fixed points of $A_{T^-} \varphi \cdot T$ for any $T > 0$, they must be equal, i.e.
\[ T^-_t \circ T^-_s \varphi(x) = u(x, t) = v(x, t) = T^{t+s}_t \varphi(x), \quad (x, t) \in M \times [0, \infty), \]
this completes the proof. $\square$

4. Viscosity solution and variational solution

This section devotes to a proof of Theorem 1.1 under assumptions (H1)-(H4). First of all, let us recall a comparison result that is used in this section without proof, it is just an adaption of [3, Corollary 5.1, Page 66] to our case. For readers that are interested in the PDE aspects of the theory of viscosity solution, we recommend [3] as an excellent survey.

Lemma 4.1. Let $T > 0$ and $G \in C(T^* M \times \mathbb{R}, \mathbb{R})$, if $u, v \in C(M \times [0, T], \mathbb{R})$ are respectively sub and supersolution of
\[ \partial_t u + G(x, \partial_x u, t) = 0, \quad (x, t) \in M \times (0, T) \]
and either $u$ or $v$ is Lipschitz continuous in $x$, uniformly with respect to $t$, then
\[ \max_{M \times [0,T]} (u - v)^+ \leq \max_{M} (u(x, 0) - v(x, 0))^+, \]
where $(\cdot)^+ := \max\{\cdot, 0\}$.

By Lemma 4.1, we have

Lemma 4.2. Let $T > 0$ and $H : T^* M \times \mathbb{R}^m \to \mathbb{R}$ be a function satisfying (H1)-(H4), if $\bar{u} \in C^2(M \times [0, T], \mathbb{R}^m)$, then for any $\varphi \in C(M, \mathbb{R})$, the viscosity solution to the Cauchy problem
\[ \begin{cases} \partial_t u + H(x, \partial_x u, \bar{u}(x, t)) = 0, & (x, t) \in M \times [0, T]; \\ u(0, x) = \varphi(x), & x \in M. \end{cases} \]
\[ u(x, t) = \inf_{\gamma \in C^{ac}([0,t], M)} \left\{ \varphi(\gamma(0)) + \int_0^t L(\gamma(\tau), \dot{\gamma}(\tau), \bar{u}(\gamma(\tau), \tau)) d\tau \right\}, \]
and it is locally Lipschitz on $M \times (0,T]$, where $L$ is the convex dual associated to $H$. In particular, the viscosity solution to (4.1) is unique.

Proof. Let $G(x,p,t) := H(x,\partial_x u, \bar{u}(x,t))$, $G^*(x,\dot{x},t) := L(x,\dot{x},\bar{u}(x,t))$ the convex dual of $G$ and $\Lambda := \|\bar{u}\|_{C^2(M \times [0,T],\mathbb{R})} = \max_{1 \leq i \leq m} |\bar{u}_i|_{C^2(M \times [0,T],\mathbb{R})}$. We denote by $\phi_G^t$ and $\phi_G^{t*}$ the local Hamiltonian flow induced by $G$ and the corresponding local Lagrangian flow. Denote by $L : TM \times [0,T] \to T^*M \times [0,T]$, $(x,\dot{x},t) \mapsto (x,\partial_\tau L(x,\dot{x}),t)$ the Legendre transformation, since $G^*$ is a $C^2$ Tonelli Lagrangian, by [9, Theorem 3.4.2, Page 98], we have

$$\phi_G^t \circ L = L \circ \phi_G^{t*}.$$  

(4.3)

First, we observe that along $\phi_G^t$,

$$\frac{d}{dt} G \circ \phi_G^t = \frac{\partial}{\partial t} G \circ \phi_G^t \leq \Theta \|\partial_t \bar{u}(x(t),t)\| \leq \Theta \Lambda \quad \text{for } 0 \leq t \leq T,$$

which implies the local Hamiltonian flow $\phi_G^t$ is defined on $[0,T]$. By the conjugate relation (4.3), $\phi_G^{t*}$ is also defined on $[0,T]$ and [9, Theorem 3.7.2, Page 108] implies that every absolutely continuous minimal curve $\gamma$ that achieves the infimum in (4.2) is a $C^2$ extremal. By the way, it is easy to deduce that $u$ is a viscosity solution to (4.1) by directly verifying the definition.

Next, by almost the same proof of [9, Proposition 4.4.4, Page 138], for every $0 < t \leq T$, there is a constant $C_t > 0$ such that for any $\gamma : [0,t] \to M$ achieving the infimum in (4.2),

$$\int_0^t G^*(\gamma(s),\dot{\gamma}(s),s)ds \leq C_t.$$  

Then by continuity, there is $t_0 \in [0,t]$ such that $G^*(\gamma(t_0),\dot{\gamma}(t_0),t_0) = L(\gamma(t_0),\dot{\gamma}(t_0),\bar{u}(\gamma(t_0),t_0)) \leq \frac{C_t}{t}$, so

$$\{\gamma(t_0),\dot{\gamma}(t_0),t_0\} \in C_t := \{(x,\dot{x},t) \in TM \times [0,T] | L(x,\dot{x},0) \leq \frac{C_t}{t} + \Theta \Lambda\},$$

it is clear that $C_t$ is a compact subset of $TM$. We define a larger compact subset $C'_t$ of $TM$ as

$$C'_t = L^{-1}\{(x,p,s) \in T^*M \times [0,T] | H(x,p,0) \leq \max_{(x,p,\tau) \in C_t} G(x,p,\tau) + \Theta \Lambda(1 + T)\},$$

then $0 \leq s \leq t$, $(\gamma(s),\dot{\gamma}(s),s) \in C'_t$. For each $t > 0$, by setting $A_t = \max_{(x,\dot{x},s) \in C'_t} \|\dot{x}\| < \infty$, we obtain that any minimal curve $\gamma : [0,t] \to M$ that attains the infimum in (4.2) is $A_t$-Lipschitz continuous. Using the same proof of [9, Lemma 4.6.3, Page 148], we show that $u$ defined by (4.2) is locally Lipschitz continuous on $M \times (0,T]$.

Finally, we only need to show the uniqueness of viscosity solution to (4.1): for any $0 < \delta < T$, $u$ is Lipschitz continuous in $x$, uniformly in $t$ on $M \times [\delta,T]$. Assuming that there is another viscosity solution $v \in C(M \times [0,T],\mathbb{R})$, by Lemma 4.1, we conclude that

$$\max_{M \times [0,T]} (u - v)_{+} \leq \max_{M \times [0,T]} (u(\cdot,\delta) - v(\cdot,\delta))_{+}.$$  

(4.4)

Let $\delta$ in (4.4) goes to 0, the continuity of $u$ and $v$ implies that $u \equiv v$ on $M \times [0,T]$, this completes the proof. \[\square\]

Remark 4.1. It is worth mentioning that in general, for a given $L(x,\dot{x},t) \in C^2(TM \times [0,T],\mathbb{R})$, the action minimizing curves may not be of class $C^2$ and fail to be an extremal of the corresponding Euler-Lagrangeian equation, see [2].

Theorem 4.1. The variational solution associated to system (1.1) is a viscosity solution to (1.1).

Proof. Let $u \in C(M \times [0,\infty),\mathbb{R}^m)$ be the variational solution associated to (1.1), by Definition 2.1, to prove that $u$ is a viscosity solution to the system

$$\begin{cases} \partial_t u + H_i(x,\partial_x u, u) = 0, (x,t) \in M \times [0,\infty); \\
\quad u(x,0) = \varphi(x), \\
\quad x \in M, 1 \leq i \leq m, \end{cases}$$

we only need to show that, for any $1 \leq i \leq m$, $u_i$ is a viscosity solution to the equation

$$\begin{cases} \partial_t u + H_i(x,\partial_x u, u(x,t)) = 0, (x,t) \in M \times [0,\infty); \\
u(x,0) = \varphi_i(x), \\
x \in M. \end{cases}$$  

(4.5)
We define two functions
\[ G_\epsilon(x, p, t) = H_i(x, p, u_\epsilon(x, t)) \]
\[ G(x, p, t) = H_i(x, p, u(x, t)) \]
(4.6)
on \( T^* \times [0, T] \), then it is clear that \( G_\epsilon(x, p, t) \) satisfies the condition of Lemma 4.2.

Denote by \( u^\epsilon : M \times [0, T] \to \mathbb{R} \) be the viscosity solution of
\[
\begin{cases}
\partial_t u + G_\epsilon(x, \partial_x u, t) = 0, & (x, t) \in M \times [0, T];
\end{cases}
\]
then by Lemma 4.2, for \( (x, t) \in M \times [0, T] \),
\[
u^\epsilon(x, t) = \inf_{\gamma \in \gamma([0, t], M)} \left\{ \varphi(\gamma(0)) + \int_0^t L_i(\gamma(s), \dot{\gamma}(s), u_\epsilon(\gamma(s), s))ds \right\}.
\]
(4.7)
By Definition 3.1 and (4.7), for any \( 0 \leq t \leq T \) and \( \gamma \in C^{ac}([0, t], M) \) with \( \gamma(t) = x \), we have
\[
|u_i(x, t) - u^\epsilon(x, t)| \leq \sup_{\gamma} \left| \int_0^t L_i(\gamma(s), \dot{\gamma}(s), u(\gamma(s), s))ds - \int_0^t L_i(\gamma(s), \dot{\gamma}(s), u_\epsilon(\gamma(s), s))ds \right|
\]
\[
\leq \sup_{\gamma} \int_0^t |L_i(\gamma(s), \dot{\gamma}(s), u(\gamma(s), s)) - L_i(\gamma(s), \dot{\gamma}(s), u_\epsilon(\gamma(s), s))|ds
\]
\[
\leq \Theta \sup_{\gamma} \int_0^t \|u(\gamma(s), s) - u_\epsilon(\gamma(s), s)\|ds
\]
\[
\leq \Theta T \|u_i - u_\epsilon\|_{\infty, M \times [0, T]} \leq \Theta T \epsilon,
\]
which implies that \( u^\epsilon \) converges uniformly to \( u_i \) on \( M \times [0, T] \) as \( \epsilon \to 0 \).

Now we have
- \( G_\epsilon(x, p, t) \) converges uniformly to \( G(x, p, t) \) on \( T^* \times [0, T] \) as \( \epsilon \to 0 \), more precisely,
\[
|G_\epsilon(x, p, t) - G(x, p, t)| \leq \Theta \|u_\epsilon - u\|_{\infty, M \times [0, T]} \leq \Theta \epsilon,
\]
- \( u^\epsilon \) is the viscosity solution to \( \partial_t u + G_\epsilon(x, \partial_x u, t) = 0 \) with \( u^\epsilon(x, 0) = \varphi_i(x) \),
- \( u^\epsilon \) converges uniformly to \( u_i \) on \( M \times [0, T] \) as \( \epsilon \to 0 \).

we conclude, by using the stability theorem for viscosity solutions, that \( u_i \) is a viscosity solution to (4.5) on \( M \times [0, T] \). Since \( T > 0 \) is arbitrary, we complete the proof.

\[ \square \]

\textbf{Theorem 4.2.} Any viscosity solution to the system (1.1) is identical with the variational solution to system (1.1). In particular, the viscosity solution to system (1.1) is unique.

\textbf{Proof.} Let \( u \) be a viscosity solution to the system (1.1), by Definition 2.1, for any \( 1 \leq i \leq m \), \( u_i \) is a viscosity solution to the Cauchy problem (4.5). We shall prove that for any \( i \) and \( (x, t) \in M \times [0, \infty) \), equation (3.4) holds.

Once again we fix \( i \) and \( T > 0 \), for any \( \epsilon > 0 \), there exist \( u_\epsilon \in C^2(M \times [0, T], \mathbb{R}^m) \) and \( c_\epsilon := \Theta \epsilon > 0 \) such that, by using the notation introduced in (4.6),
- \( \|u_\epsilon - u\|_{\infty, M \times [0, T]} < \epsilon \),
- \( c_\epsilon \to 0 \) as \( \epsilon \to 0 \),
- the inequalities
\[
G_\epsilon(x, p, t) - c_\epsilon < G(x, p, t) < G_\epsilon(x, p, t) + c_\epsilon
\]
(4.8)
hold.
The first two items also imply that \( G_\epsilon(x, p, t) \pm c_\epsilon \) converges uniformly to \( G(x, p, t) \) on \( T^* \times [0, T] \) as \( \epsilon \to 0 \).

Let \( u_\pm^\epsilon : M \times [0, T] \to \mathbb{R} \) be the unique viscosity solution to
\[
\begin{cases}
\partial_t u + G_\epsilon(x, \partial_x u, t) \pm c_\epsilon = 0, & (x, t) \in M \times [0, T];
\end{cases}
\]
\[
u(x, 0) = \varphi_i(x), \quad x \in M,
\]
respectively.
The conditions of Lemma 4.1 are satisfied since
by Lemma 4.2, \( u^\pm_\delta \) is locally Lipschitz continuous on \( M \times (0, T) \), so for any \( 0 < \delta < T \), \( u^\pm_\delta \) is Lipschitz continuous in \( x \), uniformly in \( t \) on \( M \times [\delta, T] \),

- for \( 0 < \delta < T \), by (4.8), \( u_i|_{M \times ([\delta, T])} \) is supersolution and subsolution to equations
  \[
  \partial_t u + G_i(x, \partial_x u, t) + c_i = 0, \quad \text{for} \ (x, t) \in M \times (\delta, T),
  \]
  \[
  \partial_t u + G_i(x, \partial_x u, t) - c_i = 0, \quad \text{for} \ (x, t) \in M \times (\delta, T),
  \]
  respectively,

so we conclude that
  \[
  \max_{M \times [\delta, T]} (u^+_\delta - u_i)^+ \leq \max_M (u^+_\delta(\cdot, \delta) - u_i(\cdot, \delta))^+ \]
  \[
  \max_{M \times [\delta, T]} (u_i - u^-_\delta)^+ \leq \max_M (u_i(\cdot, \delta) - u^-_\delta(\cdot, \delta))^+.
  \]

By letting \( \delta \) in (4.9) goes to 0, we obtain
  \[
  u^+_i(x, t) \leq u_i(x, t) \leq u^-_i(x, t), \quad \text{for} \ (x, t) \in M \times [0, T].
  \]

We know that by Lemma 4.2,
  \[
  u^+_i(x, t) = \inf_{\gamma \in C^{ac}([0, t], M)} \left\{ \varphi_\gamma(0) + \int_0^t [L_i(\gamma(s), \dot{\gamma}(s), u_i(\gamma(s), s)) \mp c_i]ds \right\},
  \]

It is easy to use the above formula to obtain that \( ||u^+_i - u^-_i||_{M \times [0, T]} \leq 2c_o T \), and by (4.10), \( ||u_i - u^\mp_i||_{M \times [0, T]} \leq 2c_o T \).

For any \( 0 < t < T \) and any \( \gamma \in C^{ac}([0, t], M) \) with \( \gamma(t) = x \),
  \[
  D_o := \sup_{\gamma} \left| \int_0^t [L_i(\gamma(s), \dot{\gamma}(s), u(\gamma(s), s))ds - \int_0^t [L_i(\gamma(s), \dot{\gamma}(s), u_i(\gamma(s), s)) + c_i]ds \right|
  \]
  \[
  \leq \sup_{\gamma} \int_0^t [L_i(\gamma(s), \dot{\gamma}(s), u(\gamma(s), s)) - L_i(\gamma(s), \dot{\gamma}(s), u_i(\gamma(s), s))]ds + c_o t
  \]
  \[
  \leq \Theta ||u_i - u||_{M \times [0, T]} + c_o t \leq \Theta \epsilon T + c_o T = (\Theta \epsilon + c_o) T.
  \]

So now we have, for \( (x, t) \in M \times [0, T] \),
  \[
  |u_i(x, t) - \inf_{\gamma \in C^{ac}([0, t], M)} \left\{ \varphi_\gamma(0) + \int_0^t [L_i(\gamma(s), \dot{\gamma}(s), u(\gamma(s), s))ds \right\}| \leq |u_i(x, t) - u^-_i(x, t)| + D_o \leq 2c_o T + (\Theta \epsilon + c_o) T \leq (\Theta \epsilon + 3c_o) T.
  \]

We complete the proof by letting \( \epsilon \) goes to 0 and noticing that \( T > 0 \) is arbitrary.

So far, we complete the proof of Theorem 1.1.

5. Locally Lipschitz continuity in a model problem

In this section, we shall first prove Theorem 1.2 and then verify the results in Corollary 1.1 by directly apply the regularity property we have obtained.

More precisely, for the proof of Theorem 1.2, we shall consider the weakly coupled system (1.1) with Hamiltonians satisfying (H1)-(H4) and additional assumption (A). Thus Lagrangians \( L_i \) must satisfy (L1)-(L4) and

(A') For \( 1 \leq i \leq m \), \( L_i \) can be divided into two parts, i.e.
  \[
  L_i(x, \dot{x}, u) = l_i(x, \dot{x}) - P_i(x, u),
  \]
  \[
  \text{and there exists } 0 < a < A \text{ such that for any } 1 \leq i \leq m,
  \]
  \[
  a|x|^2 \leq l_i(x, \dot{x}) \leq A|\dot{x}|^2, \\
  |\partial_x l_i(x, \dot{x})| \leq A|\dot{x}|^2, \\
  |\partial_{\dot{x}} l_i(x, \dot{x})| \leq 2A|\dot{x}|.
  \]
We note that $|\partial_x l_i(x, \dot{x})| \leq 2A|\dot{x}|$ is a direct consequence of $a|\dot{x}|^2 \leq l_i(x, \dot{x}) \leq A|\dot{x}|^2$. By (L4) and (A'), $P_t : M \times \mathbb{R}^m$ is uniformly Lipschitz continuous in $u$ with Lipschitz constant $\Theta > 0$.

Let $T \geq 0$ and $u$ be the viscosity solution associated to (1.1), we list some constants that will be used later:

- $C := \frac{A}{a}$,
- $t_\Theta := \min\{1, \frac{1}{10\Theta^2}\}$,
- $Q_T := \|u\|_{\infty, M \times [0, T]}$,
- $B_T := \max_{i=1,...,m}\{|P_t|_{\infty, M \times [-Q_T, Q_T]^m}\}$,
- $F_T := 2Q_T + T \cdot B_T$,
- $\sigma_T := \max_{i=1,...,m}\{\|\partial_x P_t|_{\infty, M \times [-Q_T, Q_T]^m}\}$.

**Remark 5.1.** We note that $Q_T, B_T, F_T, \sigma_T$ are monotone increasing continuous positive functions in $T \in [0, \infty)$.

Let us define:

- $\beta_n = 2^{-n}, \alpha_n = 1 - \beta_n, n \in \mathbb{N}$.

It is clear that $\beta_0 = 1, \alpha_0 = 0$ and $\alpha_{n+1} = \frac{1+\alpha_n}{2}$.

We have the following lemma:

**Lemma 5.1.** Let $u$ be the unique viscosity solution to (1.1) and $n \geq 1$, there exists constant $\kappa_n > 0$ such that for every $0 < t < t_\Theta$ and any two points $x, y$ on $M$ with $|x - y| \leq t$,

$$\|u(x, t) - u(y, t)\| \leq \frac{\kappa_n}{\sqrt{t}} |x - y|^{\alpha_n},$$

where $\kappa_n$ is determined by a recursion formula

$$\kappa_n(C, B_1, F_1, \sigma_1) := \frac{1}{2}\kappa_{n-1}(C, B_1, F_1, \sigma_1) + C(F_1 + \sqrt{2(F_1 + 1)}) + \frac{\sigma_1}{2}.$$ (5.4)

**Remark 5.2.** Since all $\kappa_n, n \in \mathbb{N}$ depends on $C, B_1, F_1, \sigma_1$, the Hölder constants in our estimate depend on the initial data $\varphi$.

We will prove Lemma 5.1 by inductive method.

**1st-Step:** $\frac{1}{2}$-Hölder continuity

Let $u$ be the unique viscosity solution to (1.1), then there exists constant $\kappa_1(C, B_1, F_1) > 0$ such that for every $t \in (0, 1]$ and any two points $x, y$ on $M$ with $|x - y| \leq t$,

$$\|u(x, t) - u(y, t)\| \leq \frac{\kappa_1}{\sqrt{t}} |x - y|^{\alpha_1}.$$ (5.5)

*Proof of the $\frac{1}{2}$-Hölder continuity*:

Let $\Delta := y - x$ and $\xi : [0, t] \to M$ be an absolute continuous curve with $\xi(t) = x$ such that

$$u_1(x, t) = \varphi_1(\xi(0)) + \int_0^t L_1(\xi(s), \dot{\xi}(s), u(\xi(s), s))ds.$$ (5.6)

Let $t_0 := t(1 - |\Delta|)$, then $t - t_0 = t|\Delta|$, we define $\xi_0 : [0, t] \to M$ as

$$\xi_0(s) = \begin{cases} \xi(s), & s \in [0, t_0]; \\ \xi(s) + \frac{\Delta}{|\Delta|}(s - t_0), & s \in [t_0, t]. \end{cases}$$

From the above construction and $|\Delta| \leq t$, we have

(i) $\xi_0(0) = \xi(0) = x, \xi_0(t) = \xi(t) + \Delta = y$,

(ii) $|\xi_0(s) - \xi(s)| = \frac{|\Delta|s}{t} \leq \frac{\alpha_1}{t} |\Delta| \leq \min(s, |\Delta|),$

(iii) $\dot{\xi}_0(s) = \dot{\xi}(s) = \frac{\Delta}{|\Delta|}.$

By (A') and a direct calculation,

$$u_1(y, t) - u_1(x, t) = u_1(x + \Delta, t) - u_1(x, t) \leq D_1 + D_2 + D_3,$$
where

\begin{align*}
D_1 &= \int_{t_0}^{t} |l_1(\xi_0(x), \dot{\xi}_0(s)) - l_1(\xi_0(s), \dot{\xi}(s))| ds, \\
D_2 &= \int_{t_0}^{t} |l_1(\xi_0(s), \dot{\xi}(s)) - l_1(\xi(s), \dot{\xi}(s))| ds, \\
D_3 &= \int_{t_0}^{t} |P_t(\xi_0(s), u(\xi_0(s), s)) - P_t(\xi(s), u(\xi(s), s))| ds.
\end{align*}

First, we observe that

\[ D_3 \leq 2B_t \cdot t|\Delta| \]

and

\[ \left| \int_{t_0}^{t} l_1(\xi(s), \dot{\xi}(s)) ds \right| = |u_1(x, t) - u_1(\xi(t_0), t_0) + \int_{t_0}^{t} P_t(\xi(s), u(\xi(s), s)) ds| \leq 2Q_t + B_t \cdot t|\Delta| \leq F_t, \]

since \(|\Delta| \leq t \leq 1\).

Now using (5.6), assumption (A’) and above inequalities, we estimate

\begin{align*}
D_1 &\leq \frac{1}{t} \int_{t_0}^{t} |\partial_x l_1(\xi_0(s), \dot{\xi}(s) + \theta \frac{\Delta}{t|\Delta|})| ds \\
&\leq \frac{A}{t} \int_{t_0}^{t} |\dot{\xi}(s) + \frac{\Delta}{t|\Delta|}| ds \leq \frac{A}{t} \left[ \int_{t_0}^{t} |\dot{\xi}(s) + \frac{\Delta}{t|\Delta|}|^2 ds \cdot \int_{t_0}^{t} 1 ds \right]^\frac{1}{2} \\
&\leq \frac{A}{t} \left[ \int_{t_0}^{t} (2|\dot{\xi}(s)|^2 + 2\frac{\theta^2}{t^2}) ds \cdot t|\Delta| \right]^\frac{1}{2} \leq \frac{A}{t} \left[ \frac{2t}{a} |\Delta| \int_{t_0}^{t} |\dot{\xi}(s), \xi(s)| ds + 2|\Delta|^2 \theta^2 \right]^\frac{1}{2} \\
&\leq \frac{A}{a \sqrt{t}} \left[ 2a|\Delta|F_t + 2a^2 \frac{|\Delta|^2}{t} \right]^\frac{1}{2} \leq \frac{A}{a \sqrt{t}} \left[ 2(F_t + 1) \right]^\frac{1}{2} \cdot |\Delta|^\frac{1}{2} \\
&= \sqrt{\frac{2(F_t + 1)}{t}} C \cdot |\Delta|^\frac{1}{2},
\end{align*}

where the first inequality uses mean value theorem; the second and fourth inequality use (A’); the third one is the classical Cauchy-Schwarz inequality; the sixth inequality uses (5.7) and \( \theta \in [0, 1] \); the last inequality use the fact that \( 0 < a < 1 \) and \(|\Delta| \leq t\).

\begin{align*}
D_2 &\leq \int_{t_0}^{t} |\partial_x l_1(\xi(s) + \theta \frac{\Delta}{t|\Delta|}(s - t_0), \dot{\xi}(s))| \cdot \frac{2}{t} |\Delta| ds \\
&\leq \left[ \int_{t_0}^{t} |\partial_x l_1(\xi(s) + \theta \frac{\Delta}{t|\Delta|}(s - t_0), \dot{\xi}(s))| ds \right] \cdot |\Delta| \\
&\leq \left[ \int_{t_0}^{t} A|\dot{\xi}(s)|^2 ds \right] \cdot |\Delta| \\
&\leq \frac{A}{a} |\Delta| \int_{t_0}^{t} l_1(\xi(s), \dot{\xi}(s)) ds \\
&\leq CF_t |\Delta|,
\end{align*}

where the first inequality uses mean value theorem; the second one uses \( s \leq t \); the third and fourth one use (A’); the last inequality uses (5.7).

So combining the estimates above, we obtain that

\[ u_1(y, t) - u_1(x, t) \leq (2tB_t + CF_t) \cdot |y - x| + \sqrt{\frac{2(F_t + 1)}{t}} \cdot |y - x|^\frac{1}{2} \]
Moreover, then one can find with
\[ \kappa_1 \frac{L}{\sqrt{t}} |y - x|^{\frac{1}{2}}, \]
where \( \kappa_1(C, B_1, F_1) := 2B_1 + C(F_1 + \sqrt{2F_1 + 1}) \).

Exchanging the role of \( x \) and \( y \), we have
\[ |u_1(y, t) - u_1(x, t)| \leq \frac{\kappa_1}{\sqrt{t}} |y - x|^{\frac{1}{2}} = \kappa_1 \frac{L}{\sqrt{t}} |y - x|^\alpha_1. \quad (5.8) \]

To complete the proof, we note that the estimation holds for \( u_2, \ldots, u_m \) by the same procedure as above. \( \square \)

\((n + 1)\text{-th Step: From } \alpha_n\text{-Hölder continuity to } \alpha_{n+1}\text{-Hölder continuity}\)

Assume that there exists constant \( \kappa_n > 0 \) such that for every \( 0 < t < t_\Theta \) and any two points \( x, y \) on \( M \) with \( |x - y| \leq t \),
\[ \| u(x, t) - u(y, t) \| \leq \frac{\kappa_n}{\sqrt{t}} |x - y|^{\alpha_n}, \quad (5.9) \]
then one can find \( \kappa_{n+1} > 0 \) such that for every \( 0 < t < t_\Theta \) and any two points \( x, y \) on \( M \) with \( |x - y| \leq t \),
\[ \| u(x, t) - u(y, t) \| \leq \frac{\kappa_{n+1}}{\sqrt{t}} |x - y|^{\alpha_{n+1}}. \]

Moreover,
\[ \kappa_{n+1}(C, B_1, F_1, \sigma_1) := \frac{1}{2} \kappa_n(C, B_1, F_1, \sigma_1) + C(F_1 + \sqrt{2F_1 + 1}) + \frac{\sigma_1}{2}. \]

**Proof.** Let \( \Delta = y - x \) and \( \xi : [0, t] \to M \) be an absolute continuous curve with \( \xi(t) = x \) such that
\[ u_1(x, t) = \varphi_1(\xi(0)) + \int_0^t L_1(\xi(s), \dot{\xi}(s), u(\xi(s), s)) ds. \]

Let \( t_n = t(1 - |\Delta|^{\beta_n}) \), then \( t - t_n = t|\Delta|^{\beta_n} \), we define \( \xi_n : [0, t] \to M \) as
\[ \xi_n(s) = \begin{cases} \xi(s), & s \in [0, t_n]; \\ \xi(s) + \frac{\Delta}{t|\Delta|^{\beta_n}}(s - t_n), & s \in [t_n, t]. \end{cases} \quad (5.10) \]

From the above construction and \( |\Delta| \leq t \), we have
(i) \( \xi_n(0) = \xi(0) = x \), \( \xi_n(t) = \xi(t) = y \),
(ii) \( |\xi_n(s) - \xi(s)| = \frac{t - t_n}{t} \Delta|^{\alpha_n} \leq \frac{\alpha}{t} |\Delta| \leq \min\{s, |\Delta|\} \),
(iii) \( \dot{\xi}_n(s) - \dot{\xi}(s) = \frac{\Delta}{t|\Delta|^{\beta_n}} \).

Again by (A') and a direct calculation,
\[ u_1(x + \Delta, t) - u_1(x, t) \leq D_1 + D_2 + D_3, \]
where
\[ D_1 = \int_{t_n}^t |l_1(\xi_n(s), \dot{\xi}_n(s)) - l_1(\xi_n(s), \dot{\xi}(s))| ds, \]
\[ D_2 = \int_{t_n}^t |l_1(\xi_n(s), \dot{\xi}(s)) - l_1(\xi(s), \dot{\xi}(s))| ds, \]
\[ D_3 = \int_{t_n}^t |P_1(\xi_n(s), u(\xi_n(s), s)) - P_1(\xi(s), u(\xi(s), s))| ds. \]

Now by the same estimates of \( D_1, D_2 \) in 1-step, we have
\[ D_1 \leq \frac{|\Delta|^{\alpha_n}}{t} \int_{t_n}^t |\partial_x l_1(\xi_n(s), \dot{\xi}(s)) + \theta \frac{\Delta}{t|\Delta|^{\beta_n}}| ds \]
\[ \leq \frac{4|\Delta|^{\alpha_n}}{t} \int_{t_n}^t |\dot{\xi}(s) + \theta \frac{\Delta}{t|\Delta|^{\beta_n}}| ds \leq \frac{4|\Delta|^{\alpha_n}}{t} \left[ \int_{t_n}^t |\dot{\xi}(s) + \theta \frac{\Delta}{t|\Delta|^{\beta_n}}|^2 ds \cdot \int_{t_n}^t ds \right]^{\frac{1}{2}} \]
\[ \leq \frac{4|\Delta|^{\alpha_n}}{t} \left[ \int_{t_n}^t (2|\dot{\xi}(s)|^2 + 2 \frac{\Theta^2|\Delta|^{2\alpha_n}}{t^2} ds \cdot t|\Delta|^{\beta_n} \right]^{\frac{1}{2}} \]
\[ \leq \frac{4|\Delta|^{\alpha_n}}{t} \left[ \int_{t_n}^t (2|\dot{\xi}(s)|^2 + 2 \frac{\Theta^2|\Delta|^{2\alpha_n}}{t^2} ds \cdot t|\Delta|^{\beta_n} \right]^{\frac{1}{2}} \]
\[
\leq \frac{A|\Delta|^{\alpha_n}}{t} \left[ \frac{2|\Delta|^{\beta_n}a}{t} \int_{t_n}^{t} l_1(\xi(s), \dot{\xi}(s)) ds + 2|\Delta|^2 \right]^{\frac{1}{2}} \\
\leq \frac{A}{a\sqrt{t}} \left[ 2aF_t|\Delta|^{1+\alpha_n} + \frac{2a^2}{t} |\Delta|^{2+2\alpha_n} \right]^{\frac{1}{2}} \leq \frac{A}{a\sqrt{t}} \left[ 2(F_t + 1) \right]^{\frac{1}{2}} |\Delta|^{\alpha_n+1} \\
= \frac{2(F_t + 1)}{t} C \cdot |\Delta|^{\alpha_n+1}
\]

and

\[
D_2 \leq \int_{t_n}^{t} |\partial_l l_1(\xi(s) + \theta \frac{\Delta}{|\Delta|^{\alpha_n}} (s-t_n), \dot{\xi}(s))| \cdot \frac{s}{t} |\Delta| ds \\
\leq \int_{t_n}^{t} |\partial_l l_1(\xi(s) + \theta \frac{\Delta}{|\Delta|^{\beta_n}} (s-t_n), \dot{\xi}(s))| ds \cdot |\Delta| \\
\leq \int_{t_n}^{t} A|\xi(s)|^2 ds \cdot |\Delta| \\
\leq \frac{A}{a} |\Delta| \int_{t_n}^{t} l_1(\xi(s), \dot{\xi}(s)) ds \\
\leq CF_t |\Delta|.
\]

The main difference between 1-Step and this step is the estimate on \(D_3\) which we now start to do:

\[
D_3 \leq \Theta \int_{t_n}^{t} \|u(\xi_n(s), s) - u(\xi(s), s)\| ds + \int_{t_n}^{t} \sigma_t |\xi_n(s) - \xi(s)| ds.
\]

By \(|\xi_n(s) - \xi(s)| \leq s\) and the inductive assumption (5.9), we have:

\[
\int_{t_n}^{t} \|u(\xi_n(s), s) - u(\xi(s), s)\| ds \leq \int_{t_n}^{t} \frac{K_n}{\sqrt{s}} |\xi_n(s) - \xi(s)|^{\alpha_n} ds \\
= \int_{t_n}^{t} \frac{K_n}{\sqrt{s}} \left| \frac{s-t_n}{t} |\Delta|^{\alpha_n} \right| ds \\
= \kappa_n \cdot |\Delta| \int_{0}^{1} \frac{z^{\alpha_n}}{\sqrt{t_n + t|\Delta|^{\beta_n} z}} dz \\
\leq \sqrt{\kappa_n} \cdot |\Delta|^{1-\frac{\alpha_n}{2}} \int_{0}^{1} z^{\alpha_n-\frac{1}{2}} dz \\
= \frac{\sqrt{\kappa_n}}{\alpha_n + \frac{1}{2}} |\Delta|^{\alpha_n+1} \leq 2\sqrt{\kappa_n} |\Delta|^{\alpha_n+1}.
\]

where we set \(s = t_n + t|\Delta|^{\beta_n} z\) in the second equality above. The second inequality above uses \(|\Delta| < t \leq 1\); the last inequality uses \(\alpha_n \geq 0\).

The estimate of last part of \(D_3\) reduce to the estimate of

\[
\int_{t_n}^{t} |\xi_n(s) - \xi(s)| ds \\
= \int_{t_n}^{t} \left| \frac{s-t_n}{t} |\Delta|^{\alpha_n} \right| ds = t|\Delta| \int_{0}^{1} |\Delta|^{\beta_n} z dz \\
= \frac{t}{2} |\Delta|^{1+\beta_n}.
\]

So combining the estimates above, for \(t \leq t_\Theta\) and \(x, y\) on \(M\) with \(|x-y| \leq t\), we have

\[
u_1(y, t) - u_1(x, t) \leq CF_t \cdot |y-x| + (C\sqrt{\frac{2(F_t + 1)}{t}} + 2\Theta\sqrt{\kappa_n}) \cdot |y-x|^{\alpha_n+1} + \frac{\sigma_t}{2} |y-x|^{1+\beta_n}
\]
\[ \leq (CF_t + C \sqrt{\frac{2(F_1 + 1)}{t}} + 2\Theta \sqrt{t\kappa_n} + \frac{\sigma_1}{2}) \cdot |y - x|^{\alpha_{n+1}} \]
\[ \leq (CF_t + C \sqrt{\frac{2(F_1 + 1)}{t}} + \frac{1}{2} \kappa_n + \frac{\sigma_1}{2}) \cdot |y - x|^{\alpha_{n+1}} \]
\[ = \frac{\kappa_{n+1}}{\sqrt{t}} \cdot |y - x|^{\alpha_{n+1}}, \]

where \( \kappa_{n+1} \) is defined by (5.4).

Exchanging the role of \( x \) and \( y \), we have
\[ |u_1(y, t) - u_1(x, t)| \leq \frac{\kappa_{n+1}}{\sqrt{t}} |y - x|^{\alpha_{n+1}}. \]

To complete the proof, we note that the estimation holds for \( u_2, \ldots, u_m \) by the same procedure as above. \( \square \)

**Proof of Theorem 1.2:** First, by Lemma 5.1, for any \( n \in \mathbb{N} \), any \( t \in (0, t_\Theta] \) and any two points \( x, y \) on \( M \) with \( |x - y| \leq t \), (5.3) holds. By the recursion formula (5.4), \( \kappa_\Theta \) converges and we define
\[ \kappa_\infty(C, F_1, \sigma_1) := \lim_{n \to \infty} \kappa_n = 2(C(F_1 + \sqrt{2(F_1 + 1)}) + \frac{\sigma_1}{2}). \] (5.11)

Let \( n \) in (5.3) goes to infinity, we obtain that for any \( t \in (0, t_\Theta] \) and any two points \( x, y \) on \( M \) with \( |x - y| \leq t \)
\[ ||u(x, t) - u(y, t)|| \leq \frac{\kappa_\infty}{\sqrt{t}} |x - y|. \] (5.12)

Now we can easily delete the restriction that two points \( x, y \) must keep close enough: for a given \( t \in (0, t_\Theta] \), let \( x, y \) be any points on \( M \), there exists \( k \in \mathbb{N} \) and \( x = x_0, x_1, \ldots, x_k = y \) such that \( |x_j - x_{j+1}| \leq t, j = 1, \ldots, k \) and \( \sum_{j=0}^{k-1} |x_j - x_{j+1}| = |x - y| \). Then by (5.12), for each \( i \),
\[ |u_i(x, t) - u_i(y, t)| \leq \sum_{j=0}^{k-1} |u_j(x_j, t) - u_j(x_{j+1}, t)| \]
\[ \leq \sum_{j=0}^{k-1} \|u(x_j, t) - u(x_{j+1}, t)\| \]
\[ \leq \frac{\kappa_\infty}{\sqrt{t}} \sum_{j=0}^{k-1} |x_j - x_{j+1}| = \frac{\kappa_\infty}{\sqrt{t}} |x - y|. \]

We reformulate the above \( x \)-locally Lipschitz continuity result as the following: let \( \varphi \in C(M, \mathbb{R}^m) \) be any initial data, for any \( t \in (0, t_\Theta] \) and any two points \( x, y \) on \( M \),
\[ \|T_t^{-1} \varphi(x) - T_t^{-1} \varphi(y)\| \leq \frac{\kappa_\infty}{\sqrt{t}} |x - y|, \] (5.13)

where the Lipschitz constant \( \kappa_\infty = \kappa_\infty(C, F_1, \sigma_1) \) depends on \( Q_1 \) and thus on the initial data \( \varphi \). So for any \( t \geq t_\Theta \), we use Proposition 3.2 and (5.13) to obtain that
\[ \|u(x, t) - u(y, t)\| = \|T_{t_\Theta}^{-1} u(x, t - t_\Theta) - T_{t_\Theta}^{-1} u(y, t - t_\Theta)\| \leq \frac{\kappa_t}{\sqrt{t_\Theta}} |x - y|, \]
where
\[ \kappa_t := \kappa_\infty(C, F_{t+1-t_\Theta}, \sigma_{t+1-t_\Theta}). \]

By (5.11) and Remark 5.1, \( \kappa_t \) is a monotone increasing continuous positive function for \( t \in [t_\Theta, \infty) \). In particular, \( \kappa_t \) is locally bounded on \([t_\Theta, \infty)\) and we complete the proof of locally Lipschitz continuity in \( x \).

Next, we turn to the proof of \( t \)-local Lipschitz continuity. For any \( 0 < t < \tau \) and each \( i \), by (3.5),
\[ u_i(x, \tau) - u_i(x, t) \leq - \int_t^\tau P_t(x, u(x, s))ds \leq B_\gamma |\tau - t|, \] (5.14)
where we use the constant curve \( \gamma(s) \equiv x, s \in [t, \tau] \) to connect \((x, t)\) with \((x, \tau)\).

The other side of the inequality needs a bit more work. Let \( \xi : [0, \tau] \to M \) be an absolutely continuous curve with \( \xi(\tau) = x \) such that
\[ u_1(x, \tau) = \varphi_1(\xi(0)) + \int_0^\tau L_1(\xi(s), \dot{\xi}(s), u_1(\xi(s), s))ds. \]
By locally Lipschitz continuity in $x$, we have
\[
|u_1(x, \tau) - u_1(x, t)| \leq \kappa_x |x - \xi(t)| + B_x|\tau - t|,
\]
where
\[
\kappa_x = \begin{cases} \frac{a}{\sqrt{\Theta}}, & \tau \leq t_\Theta; \\ \frac{a}{\sqrt{\Theta}} - \frac{\kappa_x}{\sqrt{\Theta}}, & \tau > t_\Theta.
\end{cases}
\]
By the above estimate and Cauchy-Schwarz inequality, we have
\[
|\xi(s)|t^2 \leq \left[ \int_t^\tau |\xi(s)|s \right]^2 \\
\leq \int_t^\tau |\xi(s)|^2s \cdot (\tau - t) \\
\leq \frac{1}{a} \int_t^\tau l_1(\xi(s), x, \xi(s))ds \cdot (\tau - t) \\
\leq \frac{1}{a} \left[ |u_1(x, \tau) - u_1(\xi(t), t)| + \int_t^\tau |P_1(\xi(s), u(\xi(s), s))|ds \right] \cdot (\tau - t) \\
\leq \frac{1}{a} [\kappa_x |x - \xi(t)| + 2B_x|\tau - t|] \cdot (\tau - t).
\]
Thus we obtain
\[
a|\xi(s)|t^2 \leq \kappa_x |x - \xi(t)| \cdot |\tau - t| + 2B_x|\tau - t|^2,
\]
which implies that there exists $c > 0$ such that $|\xi(s)|t^2 \leq c|\tau - t|$, where $c := c(a, \kappa_x, B_x) = \frac{a}{\sqrt{\Theta}} + \sqrt{\frac{2B_x}{a}}$.

By using the above estimate and locally Lipschitz continuity in $x$, we obtain
\[
|u_1(x, \tau) - u_1(x, t)| = \int_t^\tau L_1(\xi(s), x, u(x, s))ds - \kappa_x c \cdot |\tau - t| \quad (5.15)
\]
Combining (5.14) and (5.15), we complete the proof of $t$-locally Lipschitz continuity for $u_1$ and the same estimate also holds for $u_2, ..., u_m$.

Finally, we note that
\[
|u_i(x, t) - u_i(y, \tau)| \leq |u_i(x, t) - u_i(x, \tau)| + |u_i(x, \tau) - u_i(y, \tau)|,
\]
which reduces the proof of locally Lipschitz continuity to the proof of both $x$-locally Lipschitz continuous and $t$-locally Lipschitz continuous.

\section{Further results}

Now we could verify the results listed in Corollary 1.1 one by one.

\textbf{Proof of (i) of Corollary 1.1:} First, we note that, for each $1 \leq i \leq m$, $u_i : M \times [0, \infty) \to \mathbb{R}$ is a viscosity solution associated to the Hamilton-Jacobi equation
\[
\begin{cases}
\begin{align*}
\partial_t u + H_i(x, \partial_x u, u(x, t)) &= 0, & (x, t) \in M \times [0, \infty); \\
u(0, x) &= \varphi_i(x), & x \in M.
\end{align*}
\end{cases}
\] (6.1)

By locally Lipschitz continuity of $u$ on $M \times (0, \infty)$, $H_i(x, p, u(x, t))$ is locally Lipschitz continuous on $M \times (0, \infty)$; by condition (H2), $H_i(x, p, u(x, t))$ is strictly convex with respect to $p$. Now we use Theorem [6, Theorem 5.3.8] to complete the proof.
Proof of (ii) of Corollary 1.1: We fix $i$, for any $0 < \tau \leq t$, by locally Lipschitz continuity of $u_i$, there exists $\kappa > 0$ and $h > 0$ small enough such that for any $\tau - h \leq t_1 < t_2 \leq \tau + h$,

$$\left| u_i(\xi(t_2), t_2) - u_i(\xi(t_1), t_1) \right| \leq \kappa \left( |\xi(t_2) - \xi(t_1)| + |t_2 - t_1| \right).$$

(6.2)

On the other hand, $u(x, t) \in C = [-Q, Q]^m$ for any $(x, t) \in M \times [0, t]$ (by definition of $Q_i$). So for $K > 2\kappa$, by (L3), there exists $D > 0$ such that for any $(x, t) \in M \times [0, t]$,

$$L(x, \dot{x}, u(x, t)) \geq K|\dot{x}| - D.$$

By Proposition 3.1 and the above inequality,

$$u_i(\xi(t_2), t_2) - u_i(\xi(t_1), t_1) = \int_{t_1}^{t_2} L_i(\xi(s), \dot{s}, u(\xi(s), s))ds \geq \int_{t_1}^{t_2} K|\dot{s}| - D ds \geq K|\xi(t_2) - \xi(t_1)| - D|t_2 - t_1|.$$

Combining the above inequality and (6.2), we obtain that

$$\kappa|\xi(t_2) - \xi(t_1)| \leq (K - \kappa)|\xi(t_2) - \xi(t_1)| \leq (\kappa + D)|t_2 - t_1|,$$

(6.3)

this implies the locally Lipschitz continuity near $\tau$. \qed

Proof of (iii) of Corollary 1.1: Again we fix $i$, by (ii), for any $0 < h < t$, $\dot{\xi}_{[t-h,t]} \in \mathcal{L}^\infty([t-h, t], TM)$, i.e. there exists $R > 0$ such that $|\dot{\xi}(s)| \leq R$ for any $s \in [t-h, t]$. Thus the difference $\frac{\dot{\xi}(t-h) - \dot{\xi}(t)}{h}$ is bounded and we denote the limit points of this difference by $\Omega$. It is clear that $\Omega$ is a bounded subset of $T_x M$.

Since $u_i$ is differentiable at $(x, t)$ and $\xi$ is locally Lipschitz continuous at $t$, we have

$$u_i(x, t) - u_i(\xi(t-h), t-h) = \partial_t u_i(x, t) + P \cdot \frac{\xi(t) - \xi(t-h)}{h} + o(h + |\xi(t) - \xi(t-h)|)$$

(6.4)

$$= \partial_t u_i(x, t) + P \cdot \frac{\xi(t) - \xi(t-h)}{h} + o(h).$$

On the other hand, by the boundedness of $\dot{\xi}$ near $t$ and the locally Lipschitz continuity of $u$ at $(x, t)$, there exists $\kappa > 0$ independent of $h$ such that

$$u_i(x, t) - u_i(\xi(t-h), t-h) = \int_{t-h}^{t} L_i(\xi(s), \dot{\xi}(s), u(\xi(s), s))ds \geq \int_{t-h}^{t} \left[ L_i(x, \dot{\xi}(s), u(x, t)) - \kappa(|\dot{\xi}(s)| + |s|) \right] ds$$

(6.5)

$$= \int_{t-h}^{t} L_i(x, \dot{\xi}(s), u(x, t))ds - \frac{O(h^2)}{h}$$

$$\geq L_i(x, \frac{1}{h} \int_{t-h}^{t} \dot{\xi}(s)ds, u(x, t)) + O(h)$$

$$= L_i(x, \frac{\dot{\xi}(t-h)}{h}, u(x, t)) + O(h),$$

where in the first inequality, we use that $\frac{\partial L_i}{\partial s}$ is bounded on compact sets of $TM \times \mathbb{R}^m$ and in the second inequality, we use the convexity of $L_i$ in $\dot{s}$. For any $v \in \Omega$, we choose $h_n \rightarrow 0$ such that $v = \frac{\dot{\xi}(t) - \dot{\xi}(t-h_n)}{h_n}$. Now take $h = h_n$ in (6.4) and (6.5) and let $n$ goes to infinity, we obtain

$$P \cdot v - L_1(x, v, u(x, t)) \geq H_1(x, P, u(x, t)).$$

By (H2),(L2) and the definition of Legendre transformation, $\Omega$ is a singleton $\{V\}$ and the equation (1.5) holds. \qed

We end this section by the following two remarks:
Remark 6.1. Even if at some differentiable point \((x_0, t_0)\) of the solution \(u\), we could not obtain the uniqueness of the minimal curve passing through it.

Remark 6.2. From the proofs above, if we obtain the locally Lipschitz continuity of the viscosity solution, then \((L1)-(L4)\) is sufficient for the validity of Corollary 1.1. Namely, the additional assumption \((A')\) is not necessary for this corollary.

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