The definition of Finsler spacetime

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Abstract

In recent works the author developed the local and global causality theory of Finsler spacetimes. Subsequently other authors restated these results for a modified theory in which the Finsler Lagrangian is defined only over a conic subbundle, thus apparently departing from John Beem’s classical definition of Finsler spacetime for which the Lagrangian is defined over the whole slit tangent bundle. It is shown here that this modified theory is no more general, since the ‘conic’ Finsler Lagrangian is the restriction of a Beem’s Lagrangian. Since causality theory depends on curves defined through the future cone, this work establishes the essential uniqueness of Finsler spacetime theories and Finsler causality.

1 Introduction

The Finslerian generalization of general relativity and the mathematical theory of Lorentz-Finsler manifolds have seen considerable progress quite recently. In this work I shall consider the most mathematical aspects of the theory while for some physical considerations, including dynamical equations the reader is referred to [12, 13, 15, 19, 25–27].

Let \( M \) be a paracompact, Hausdorff, connected, \( n + 1 \)-dimensional manifold. Let \( \{ x^\mu \} \) denote a local chart on \( M \) and let \( \{ x^\mu, v^\nu \} \) be the induced local chart on \( TM \). The Finsler Lagrangian is a function on the slit tangent bundle \( \mathcal{L} : TM \setminus 0 \to \mathbb{R} \) positive homogeneous of degree two in the velocities, \( \mathcal{L}(s, v) = s^2 \mathcal{L}(x, v) \) for every \( s > 0 \). The metric is defined as the Hessian of \( \mathcal{L} \) with respect to the velocities

\[
g_{\mu\nu}(x, v) = \frac{\partial^2 \mathcal{L}}{\partial v^\mu \partial v^\nu},
\]

and in index free notation will be denoted with \( g_v \) to stress the dependence on the velocity. Lorentz-Finsler geometry is obtained whenever \( g_v \) is Lorentzian, namely of signature \((-\cdot, +\cdot, \cdots, +)\). The just given definition of Lorentz-Finsler manifold is due to John Beem [4].

By positive homogeneity we have \( \mathcal{L} = \frac{1}{2} g_v(v, v) \) and \( d\mathcal{L} = g_v(v, \cdot) \). The usual Lorentzian case is obtained for \( \mathcal{L} \) quadratic in the velocities.

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A vector $v \in T_x M \setminus \{0\}$ is said to be timelike, lightlike, or spacelike depending on the sign of $\mathcal{L}(x, v)$, respectively negative, zero, or positive. Since the work by Beem [4] and Perlick [23] it is known that the set of timelike vectors is made by the union of disjoint convex cones, so introducing a time orientation by terming one of the cones *future* one can hope to develop a causality theory for these spaces. A Lorentz-Finsler manifold so time oriented is called *Finsler spacetime* (there is always an time oriented double covering).

In two recent works the author has indeed developed the local and global causality theory of Finsler spacetimes [18, 20] while in a third I proposed some dynamical equations [19]. In [18] Sect. 1.4 I showed how to extend global causality theory. Indeed, I proved:

(a) existence of convex normal neighborhoods (Theor. 1.16), and
(b) that curves that move pointwisely slower than light move also locally slower than light (Theor. 1.23), in other words that lightlike geodesics are locally achronal.

From here standard causality theory follows [6, 21] since proofs generalize word by word from the Lorentzian domain as long as they do not involve tricky aspects related to curvature (see [18]). For instance, the chronological relation is open, two points connected by a causal curve are either chronologically related or the curve is an achronal lightlike geodesic, globally hyperbolic spacetimes are defined in the usual way and split as a product, the causal hierarchy of spacetime holds unaltered, and so on. Actually, many of these causality results hold under low differentiability assumptions since the metric need only be $C^{1,1}$ (under stronger differentiability conditions (a) was already obtained by Whitehead [28]). In fact the correspondence is so complete that it has not yet been found a significative instance where it fails. For instance, in causality theory the Finslerian curvature most often appears through the Ricci tensor, so there is really little ambiguity even when curvature is involved.

The reader can also find some of these claims in a posterior somewhat sketchy work by Javaloyes and Sánchez [10] where, however, no proof of (a) or (b) is given.

Subsequently, in [20] I proved another analogy with the usual Lorentz spacetime theory, namely that any reversible (i.e. $\mathcal{L}(-v) = \mathcal{L}(v)$) Lorentz-Finsler Lagrangian has precisely two timelike cones as in general relativity provided the spacetime dimension is larger than two. This work proves that the Legendre map $v \rightarrow g_{\nu}(v, \cdot)$ is bijective and obtains for the first time two inequalities that I announced in [18], namely the Lorentz-Finsler reverse Cauchy-Schwarz inequality and the reverse triangle inequality. These inequalities had passed unnoticed in previous investigations, e.g. in the study of Finsler norms by Javaloyes and Sánchez [11], but are really fundamental for the development of causality theory, and in particular for the proof of (b) above.

Finally, two months ago Aazami and Javaloyes [1] used these inequalities and reobtained some of the results on Finsler causality given by the author in [18], but unfortunately, as in [10], without referring to [18]. In fact, Javaloyes
explained that they regard their theory as independent since the definition of Finsler spacetime they use is different.

Javaloyes and collaborators use a definition according to which:

(i) $\mathcal{L}$ is defined just inside a closed sharp cone subbundle $J^+$ of $TM$,

(ii) $\mathcal{L}$ is positive homogeneous of degree two,

(iii) its Hessian with respect to the velocities is Lorentzian on $J^+$,

(iv) $\mathcal{L} = 0$ at the boundary $E^+$ while $\mathcal{L} < 0$ on the interior $I^+$ (I modify the sign to facilitate comparison with my conventions),

(v) $d\mathcal{L} \neq 0$ at the boundary.

Observe that Beem’s theory satisfies these properties. This definition is a refinement over Asanov’s [2] who defined the Lagrangian just over an open cone (related approaches are [3,24]) and is motivated by the necessity of including lightlike vectors and lightlike geodesics in order to develop causality theory. This definition is more involved than Beem’s as spacelike geodesics do not make sense and so there is no notion of convex neighborhood. This fact complicates matters and so, according to Javaloyes and collaborators, several known results of Lorentzian causality theory should be checked again.

While I understand these authors’ concerns I think that they did not handle these difficulties in the best way. On the one hand their conditions on $\mathcal{L}$ imply that $\mathcal{L}$ can be extended in a neighborhood of the causal cone while keeping the Lorentzianity of the Hessian[1] and also that the geodesic spray can be extended on the whole slit tangent bundle. This fact is fairly easy to check (the spray does not need to be induced from a Lagrangian outside the causal cone) and shows at once that convex neighborhoods can be introduced in their theory, and so that the whole study of [18] passes through.

But one can do much more, namely prove that there are no two different Lorentz-Finsler theories, to be developed independently, since a definition resting on (i)-(v) is actually no more general than Beem’s. In this work we shall prove that any such Lagrangian $\mathcal{L}$ can be smoothly extended beyond the light cone over the whole slit tangent bundle while preserving positive homogeneity and Lorentzianity of the Hessian. This fact proves that Javaloyes and collaborators have been working with Beem’s theory in disguise, namely with a Beem’s Lagrangian restricted to its future causal cone. In other words Beem’s theory retains all the physical content that might be expressed by this alternative theory with the advantage of some good mathematical properties, as the existence of convex neighborhoods, spacelike geodesics, and the surjectivity of the Legendre map.

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[1] Differentiability over a closed set should be understood in Whitney’s sense [29]. Then by Whitney extension theorem $\mathcal{L}$ can be extended in a neighborhood, (v) states that the boundary $E^+$ is an embedded manifold, and by continuity $\mathcal{L}$ has Lorentzian Hessian near $E^+$. 

3
Physically, our result establishes the essential uniqueness of Finsler spacetime theories and so supports considerably their physical interests.

Mathematically, proofs extend easily from their Lorentzian analog where the concept of convex neighborhood is often used. For instance, in [11] the authors state that they do not know whether globally hyperbolic Finsler spacetimes, defined through the compactness of causal diamonds admit Cauchy hypersurfaces and split as a product. As their theory is nothing but Beem’s, the conclusion is no different from the one I reached in [18]: they do. Just apply Gerocchi’s proof word by word [8] [9, Prop. 6.6.8] (or Fathi and Siconolfi’s [7] for the smooth case).

Remark 1.1. It can be observed that limit curve theorems [5,17] can be applied safely. One could rework and modify the Lorentzian proofs, but there is a simple fast trick to see this fact. Any sharp convex cone is the intersection of a countable number of round (elliptic) cones. As a consequence on any Finsler spacetime we can find Lorentzian metrics $g_k$ such that the future causal cone of the Finsler spacetime at any point is the intersection of the future Lorentzian cones. Thus if a sequence of inextendible Finsler causal curves $\gamma_i$ accumulates to a point $p$, it converges, by the Lorentzian limit curve theorem, to a limit curve $\gamma$ passing through $p$ which is $g_k$-casual for every $k$ and hence Finsler causal.

2 Extending the Finsler Lagrangian

In order to obtain the extension we have to study some properties of $C^2$ Finsler Lagrangians defined on the slit tangent bundle, so let $\mathcal{L}$ be defined all over $TM\setminus 0$. Let us define the subsets of $TM\setminus 0$, of timelike vectors $I = \mathcal{L}^{-1}((\infty, 0))$, lightlike vectors $E = \mathcal{L}^{-1}(0)$ and causal vectors $J = I \cup E$. Let $\mathcal{I} \subset (TM\setminus 0)\setminus J$ be the subbundle obtained through the condition $2\mathcal{L} = 1$. This is called space-time indicatrix and $\mathcal{I}_x = \mathcal{I} \cap T_x M$ is the spacetime indicatrix at $x$. By positive homogeneity the Finsler Lagrangian is determined outside $J$ by the spacetime indicatrix, indeed

$$\mathcal{L}(x,v) = s^2/2, \text{ where } v/s \in \mathcal{I}_x.$$  

Let us recall a few well known facts [4,14] on the relationship between Finsler metric at a given point $x \in M$ and second fundamental form (affine metric) of $\mathcal{I}_x \subset T_x M$.

The Liouville vector field on $L : T_x M \to TT_x M$ is given by $v \in TT_x M$ at $v \in T_x M$ and is transverse to the indicatrix because $\frac{\partial \mathcal{L}}{\partial v} v = 2\mathcal{L} = 1 \neq 0$. Let $D$ denote the usual covariant derivative on $T_x M$ due to its affine structure, and let $X$ and $Y$ be two vectors tangent to the indicatrix at $v \in \mathcal{I}_x$. Let us extend them in a neighborhood $U \subset T_x M\setminus 0$ of $v$ to two vector fields which are tangent to $\mathcal{I}_x$. Since $Y$ is tangent to the indicatrix we have over $U \cap \mathcal{I}_x$, $\frac{\partial \mathcal{L}}{\partial v} Y = 0$. Thus we can define the affine metric $h$ (second fundamental form) in the sense of affine differential geometry [22] with the Gauss equation for the minus-Liouville transverse field

$$D_X Y = \nabla_X Y - h(X,Y)v, \quad (Gauss).$$  

(1)
Here $\nabla$ is a covariant derivative induced over the indicatrix which is not necessarily the Levi-Civita connection of $h$ (the difference between the two connections is the cubic form $[22]$). Contracting with $d\mathcal{L}$ and using $\frac{\partial}{\partial v^\alpha} v^\alpha = 2\mathcal{L}$ and

$$d\mathcal{L}(D_XY) = \frac{\partial}{\partial v^\alpha}(X^\beta \frac{\partial Y^\alpha}{\partial v^\beta}) = X^\beta \frac{\partial}{\partial v^\beta}(\frac{\partial \mathcal{L}}{\partial v^\alpha} Y^\alpha) - X^\beta Y^\alpha \frac{\partial^2 \mathcal{L}}{\partial v^\alpha \partial v^\beta} = -g_v(X,Y)$$

we arrive at

$$g_v(X,Y) = h(X,Y)2\mathcal{L}. \tag{2}$$

Summarizing, since on the spacetime indicatrix $2\mathcal{L} = 1$ we have:

**Proposition 2.1.** The Finsler metric $g_v$ induces on the spacetime indicatrix $\mathcal{I}_x$ a metric which coincides with the affine metric of the indicatrix, where the indicatrix is regarded as a hypersurface of the affine space $T_xM$ with centro-affine transverse field ($v$ is the Liouville vector field).

Observe that the tangent space to the indicatrix at $v \in \mathcal{I}_x$ is $\ker g_v(v,\cdot)$, and since the transverse field $v$ is $g_v$-spacelike, $g_v(v,v) = 2\mathcal{L}(x,v) = 1$, the affine metric on the spacetime indicatrix is Lorentzian. Thus if we are given a Finsler Lagrangian $\mathcal{L}$ on a sharp cone $J^+_x$, its extension on the slit tangent bundle can be accomplished through the construction of a spacetime indicatrix with Lorentzian affine metric.

The next result is related to a known convenient parametrization of the indicatrix $[16]$.

**Theorem 2.2.** Let $t$ be projective coordinates on $T_xM \cap \{v : v^0 > 0\}$ so that $v = (v^0,v)^T = -\frac{1}{u(t)} (1,t)$. The spacetime indicatrix $\mathcal{I}_x$, regarded has an embedding $\sigma : t \to -\frac{1}{u(t)} (1,t)$ has affine metric relative to the transverse field $-v$ given by

$$h_{ij} = \frac{u_{ij}}{u}.$$

**Proof.** Let us observe that

$$\sigma_*(e_j) = D_{\sigma_*(e_j)}v = \partial_j\{-\frac{1}{u(t)} (1,t)\} = -\frac{u_j}{u} v - \frac{1}{u(t)} \partial_j (1,t),$$

where $\partial_j$ is a shorthand for $\partial/\partial t^j$. Thus

$$D_{\sigma_*(e_i)}\sigma_*(e_j) = D_{\sigma_*(e_i)}D_{\sigma_*(e_j)}v = \partial_i \partial_j\{-\frac{1}{u(t)} (1,t)\}$$

$$= \frac{u_{ij}}{u} (-v) + \frac{u_i}{u^2} \partial_i (1,t) + \frac{u_j}{u^2} \partial_j (1,t) + 2 \frac{u_i u_j}{u^2} v$$

$$= -\frac{u_i}{u} \sigma_*(e_j) - \frac{u_j}{u} \sigma_*(e_i) + \frac{u_{ij}}{u} (-v).$$

The first two terms are tangent to the indicatrix, thus the last one gives the affine metric. \hfill \Box
Remark 2.3. Of course the coordinate $v^0$ does not play any privileged role here. For instance an analogous statement holds on the region $v^1 > 0$ for similarly introduced projective coordinates. This result tells us that the Lorentzianity of the affine metric of the indicatrix can be read from a certain Hessian related to the projective radius.

The Lorentzianity of the affine metric on the indicatrix will be obtained passing through the convexity of the level sets of its graph. For this reason we shall need a lemma on convexity. We recall that for a $C^2$ function (strict) convexity coincides with the non-negativity (resp. positivity) of the Hessian. A quasi-convex function is a function for which the sublevel sets are convex. With $B(p, r) \subset \mathbb{R}^n$ we denote the open centered at $p$ of radius $r$.

**Lemma 2.4.** Let $\Omega \subset \mathbb{R}^n$ be a compact convex neighborhood of the origin and let $A$ be a closed (possibly empty) set such that $A \subset \text{Int}\Omega$. Let $f$ be a $C^k$, $2 \leq k \leq \infty$, function on $\Omega$ which is strictly convex on $\Omega - A$. Then there are constants $a, b, r > 0$, $\Omega \subset B(0, r)$, such that $f$ can be extended to a $C^k$ function on $\mathbb{R}^n$, strictly convex on $\mathbb{R}^n - A$, in such a way that $f(x) = -a + b\|x\|^2$ for $\|x\| \geq r$.

The proof is based on the elaboration of some nice ideas on the extension of convex $C^2$ functions due to Min Yan [30].

**Proof.** By Whitney’s extension theorem [29, Theor. I] $f$ can be extended to $\mathbb{R}^n$ preserving its differentiability properties. Let $r_1 > 0$ be such that $\Omega_1 \subset B(0, r_1)$. According to a recent result by Min Yan [30, Theor. 4.4] $f$ can actually be extended to a $C^k$ function on $\mathbb{R}^n$, strictly convex on $\mathbb{R}^n - A$. From now on we denote with $f$ this extension.

Let $0 < \epsilon < r_1$ and let $\alpha: \mathbb{R}^n \rightarrow [0, 1]$ be a smooth function such that $\alpha = 1$ on $B(0, 2r_1)$ and $\alpha = 0$ outside $B(0, 2r_1 + \epsilon)$. Let $\beta: [0, +\infty) \rightarrow [0, 1]$ be a smooth function such that $\beta(s) = 0$, for $s \in [0, 2r_1]$, $\beta(s), \beta'(s) > 0$ for $2r_1 < s < 3r_1$, and $\beta = 1$ in a neighborhood of $[4r_1, \infty)$. Let $\gamma: [0, +\infty) \rightarrow \mathbb{R}$ be defined by

$$\gamma(t) = \int_0^t s\beta(s) \, ds.$$  

Finally, let

$$g(x) = \gamma(\|x\|),$$

then

$$h = \alpha f + 2bg,$$

is the searched $C^k$ extension, where $b > 0$ is a sufficiently large constant. Indeed, observe that $g = 0$ on $B(0, 2r_1)$ and for $v \in T_x\mathbb{R}^n$

$$Hg(v, v) = \frac{1}{\|x\|} \left\{ \gamma'(||x||)||v||^2 + \beta'(||x||)(x \cdot v)^2 \right\},$$

$^2$The $C^k$, $k > 2$, case is not mentioned in [30] but follows immediately from the proof.
thus the Hessian of \( g \) is positive definite outside \( \bar{B}(0,2r_1) \). Now, \( h \) has positive definite Hessian on \( \bar{B}(0,2r_1) - A \) since it coincides with \( f \) on \( \bar{B}(0,2r_1) \). It has positive definite Hessian on the compact set \( \bar{B}(0,2r_1 + \epsilon) - \bar{B}(0,2r_1) \) for sufficiently large \( \epsilon \), since the Hessian of \( \alpha f \) is bounded from below there. Finally, it has positive Hessian outside \( \bar{B}(0,2r_1 + \epsilon) \) since it coincides with \( bg \) there. Observe that \( \gamma(t) = \frac{1}{2}t^2 - \int_0^t s(1 - \beta(s)) \, ds \) where the last integral is non-negative. Thus since \( \beta = 1 \) in a neighborhood of \([4r_1, +\infty)\) we have that \( \gamma(t) = \frac{1}{2}t^2 - a/(2b) \) there for some constant \( a > 0 \). As a consequence, \( h = b\|x\|^2 - a \) for \( \|x\| \geq r := 4r_1 \) and renaming \( h \to f \) the theorem is proved. \( \square \)

We are ready to state our equivalence theorem.

**Theorem 2.5.** Let \( \mathcal{L} \) be a \( C^k \), \( 2 \leq k \leq \infty \), function defined on a conic subbundle \( J^+ \subset TM \setminus 0 \) with the properties (i)-(v) mentioned in the introduction, then there is an extension \( \tilde{\mathcal{L}} : TM \setminus 0 \to \mathbb{R} \) to a \( C^k \) Finsler Lagrangian in Beem’s sense, namely \( \tilde{\mathcal{L}} \) is positive homogeneous of degree two and its Hessian with respect to the velocities has Lorentzian signature. Furthermore, the locus \( \mathcal{L} < 0 \) is at each point \( x \in M \) the union of two open convex sharp cones whose closures do not intersect and whose union span \( T_x M \) (to be interpreted as the future and past timelike cones).

I have not been able to accomplish reversibility of the extended Lagrangian so \( J^- \) might be different from \( -J^+ \).

**Proof.** Let \( u \in I^+ = \text{Int} J^+ \) be a future-directed timelike vector field, and let \( N \subset TM \) be a linear subbundle of the tangent bundle such that \( N_x \) does not intersect \( J_x \). The affine space \( u + N_x \) intersects \( J^+ \) in a convex set \( C_x \) with \( C^k \) boundary. Let us introduce an auxiliary Riemannian metric on \( M \) in such a way that \( u \) has unit norm. Thus we can regard \( T_x M \) as a Euclidean space. From now one we shall focus on a single tangent space \( T_x M \) and on its spacetime indicatrix. In order to get the extension over the whole manifold several constant introduced in the following argument should be regarded as smooth functions of \( x \) although the dependence on \( x \) will not be mentioned.

Let us introduce coordinates \((v^0,v^1,\cdots,v^n)\) over \( T_x M \) so that \( u = \partial_0 \) and \( N_x = \text{span}\{\partial_1,\cdots,\partial_n\} \). In this way \( u + N_x = (x^0)^{-1}(1) \). Let \( \tilde{\mathcal{L}} = \mathcal{L}|_{v^0=1} \) so that

\[
\mathcal{L}(v) = (v^0)^2 \tilde{\mathcal{L}}(v/v^0). \tag{3}
\]

Since \( \mathcal{L} \) is defined so far only over \( J_x \), \( \tilde{\mathcal{L}} \) is defined so far only over \( C_x \) but we are now going to extend it over \( N_x \).

We know that \( \mathcal{L} \) can be extended over a cone neighborhood of \( J^+ \) preserving the Lorentzianity of the Hessian, so \( \tilde{\mathcal{L}} \) can be also extended in a neighborhood of \( C_x \).

If \( v \in \partial C_x \) then \( d\mathcal{L} = g_v(v,\cdot) \) where \( v \) is a lightlike vector, the ker of this one form being \( T_xE^+_x \). Since \( u + N_x \) is transverse to \( E^+_x \), \( d\mathcal{L} \neq 0 \) on \( \partial C_x \). Since \( E^+_x \) is a null hypersurface in the spacetime \((T_x M, g_v)\) its induced space metric must be positive definite, thus if \( X \in T_x \partial C_x, X \neq 0, \) as \( X \not\parallel v \), we have

\[
0 < g_v(X,X) = H \mathcal{L}(X,X) = H \tilde{\mathcal{L}}(X,X).
\]
In other words the Hessian of \( \tilde{\mathcal{L}} \) is positive definite when restricted to the tangent space to \( \partial C_x \). Let \( 0 < \varepsilon < 1 \) be so small that \( \mathcal{L} \) has Lorentzian Hessian over \( C^{3\varepsilon}_x := \mathcal{L}^{-1}((\varepsilon, 3\varepsilon]) \), and \( \tilde{\mathcal{L}} \) has strictly convex compact sublevel sets. In particular, \( \tilde{\mathcal{L}} \) is quasi-convex on \( C^{3\varepsilon}_x \). Let \( h: [2\varepsilon, 3\varepsilon] \to \mathbb{R} \) be a smooth increasing convex function interpolating the function \( x \) in a neighborhood of \( 2\varepsilon \) and a quadratic function \( cx^2 + d \) in a neighborhood of \( 3\varepsilon \) (for some \( d \)), then for sufficiently large \( c > 0 \), \( h(\tilde{\mathcal{L}}) \) is convex in a compact inner neighborhood \( C_x - A, A = C^{3\varepsilon}_x \), of \( \partial C^{3\varepsilon}_x \).

Applying Lemma 2.4 with \( \Omega = C^{3\varepsilon}_x \) we get that \( \tilde{\mathcal{L}} \) on \( C^{3\varepsilon}_x \) can be extended over \( u + N_x \) to a quasi-convex function which coincides with \( bv^2 - a \) for sufficiently large \( \|v\| \). Let \( \mathbf{t} \) be projective coordinates on \( T_x M \) so that \( v = (v^0, \mathbf{v}) = v^0(1, \mathbf{t}) \). The graphing function \( v^0(t) \) determining the indicatrix \( 2\tilde{\mathcal{L}} = 1 \) on the region \( v^0 > 1 \) is obtained from Eq. (3), \( v^0(t) = \sqrt{2\tilde{\mathcal{L}}(t)} \). In the notations of Theorem 2.2 we have \( u = -\sqrt{2\tilde{\mathcal{L}}} \) thus we have only to show that \( \sqrt{2\tilde{\mathcal{L}}} \) has Lorentzian Hessian.

For some \( r > 0 \), and \( \|\mathbf{t}\| > r \) this is so because \( \tilde{\mathcal{L}} = bt^2 - a \) (or because \( \mathcal{L} = -a(v^0)^2 + bv^2 \)). However we shall need to change \( \tilde{\mathcal{L}} \) in this region.

For \( \mathbf{t} \in C^{3\varepsilon}_x - C_x \) Lorentzianity follows from the fact that \( \mathcal{L} \) is Lorentzian on the corresponding convex cone neighborhood of \( J^+_x \). Observe that \( \partial C^0_x, 0 \leq \delta \leq 2\varepsilon \), is the boundary of a strictly convex set, thus as they are level sets for \( \sqrt{\mathcal{L}} \), they are spacelike hypersurfaces on \( u + N_x \) for the metric given by the Hessian of \( \sqrt{\mathcal{L}} \), and so \( \sqrt{\mathcal{L}} \) plays the role of a time function there.

As \( \mathcal{L} \) is increasing and convex outside \( C^{3\varepsilon}_x \) it goes to infinity on every radial line of \( u + N_x \) starting from \( \mathbf{t} = 0 \). We wish to prove the Lorentzianity of \( H\sqrt{\mathcal{L}} \) on a compact set \( \mathcal{L}^{-1}([\varepsilon^2, R^2]) \supset \{\|\mathbf{t}\| \leq r\} - C^{3\varepsilon}_x \). The idea is to modify \( \sqrt{\mathcal{L}} \) through composition with an increasing function \( F \) so as to preserve its convex sublevel sets but adjust its Hessian. Let \( X \) be a vector field on \( (u + N_x) \cap C_x \) transverse to \( \partial C^{3\varepsilon}_x \), such that \( \partial_X \mathcal{L} > 0 \), \( X \) is timelike with respect to \( H\sqrt{\mathcal{L}} \) on \( C^{2\varepsilon}_x \cap C_x^{\varepsilon} \) and \( X \) coincides with \( \mathbf{t} \) for \( \|\mathbf{t}\| \geq r \). Let

\[
\beta = \sup_{\mathcal{L}^{-1}([\varepsilon^2, R^2])} \frac{H\sqrt{\mathcal{L}}(X,X)}{\left(\partial_X \sqrt{\mathcal{L}}\right)^2}
\]

Let \( \alpha : [0, +\infty) \to [0, +\infty) \) be a smooth function which vanishes in a neighborhood of \( 0 \) and in a neighborhood of \( [2R, +\infty) \), and which is larger than \( \beta \) on \( [\varepsilon, R] \). Let

\[
F(x) = \int_0^x e^{-\int_0^y \alpha(z)dz}dy.
\]

Note that \( F' > 0, F = x \) on \( [0, \varepsilon/2] \), while \( F = A + Bx \) with \( A, B > 0 \) in a neighborhood of \( [2R, +\infty) \). The inequality \( A > 0 \) is easily checked from

\[
\int_0^{2R} (e^{\int_0^y \alpha dz} - \int_0^y \alpha dz - 1)dy > 0 \Rightarrow \int_0^{2R} e^{-\int_0^y \alpha dz}dy > 2Re^{-\int_0^{2R} \alpha dz} \Rightarrow A > 0.
\]
The equality \( F = x \) assures that \( F \circ \sqrt{L} \) coincides with \( \sqrt{L} \) in a neighborhood of \( C_x \). Observe that
\[
H(F \circ \sqrt{L})(Y, Y) = F'(\partial_Y \sqrt{L})^2 + F''(\partial_Y \sqrt{L})^2,
\]
so when \( Y \) is tangent to the level sets of \( F \circ \sqrt{L} \) (or \( \tilde{L} \)) we have by their strict convexity
\[
H(F \circ \sqrt{L})(Y, Y) > 0,
\]
while when \( Y = X \)
\[
H(F \circ \sqrt{L})(X, X) = F'(\partial_X \sqrt{L})^2 - \alpha(\partial_X \sqrt{L})^2.
\]
Thus the Hessian is Lorentzian on the whole region \( L \geq 0 \). Redefining \( F \circ \sqrt{L} \rightarrow \sqrt{L} \) we get Lorentzianity of \( L \) all over the half-space \( v^0 > 0 \).

Observe that outside a cone containing the positive \( v^0 \)-axis \( L \) is rotationally symmetric around that axis. As its level sets are round cones, in order to infer the Lorentzianity of the indicatrix it is sufficient to check that the intersection between the indicatrix and the \( v^0 - v^1 \) plane is strictly convex when expressed as a function \( v^1(v^0) \). Outside the mentioned cone
\[
L = (v^0)^2(A + B\sqrt{-a + b(v/v^0)^2})^2,
\]
thus
\[
v^1(v^0) = \left[ \frac{a}{b} + \frac{1}{B^2b}(\frac{1}{\sqrt{2}} - Av^0)^2 \right]^{1/2}, \quad v^0 > 0.
\]
This function is convex for \( v^0 > 0 \) but it does not match with that obtained for \( v^0 < 0 \) through reflection. However, since \( \partial v^1/\partial v^0 |_{v^0=0} < 0 \) it is possible to continue the function for \( v^0 \leq 0 \) (and modify it slightly near \( v^0 = 0 \) if needed) while preserving strict convexity so as to match \( v^1 = \sqrt{d + e(v^0)^2} \), \( d > 0 \), for some sufficiently large \( e > 0 \), on \( v^0 < 0 \) for sufficiently large \( |v^0| \). In this way \( L \) has Lorentzian Hessian everywhere and coincides with a quadratic function
\[
L = [-e(v^0)^2 + v^2] / (2d)
\]
in a neighborhood of the past causal cone.

3 Conclusion

I have shown that the modified Finsler spacetime definition by Javaloyes and Sánchez is no more general than Beem’s, and thus that including lightlike geodesics into Asanov’s conic subbundle approach leads to Beem’s slit tangent bundle Lorentz-Finsler theory. This result proves the essential uniqueness of Finsler causality and the mathematical convenience of Beem’s proposal. In fact, one can safely use the concept of convex neighborhood and space geodesic, and so translate almost effortlessly several Lorentzian results into their Finsler-Lorentzian analog.

Physically, it can well be that most of the interesting information be contained in the future cone since causality theory seems to depend just on it. It is probably too early to tell since we are just beginning to investigate Finsler gravity. Still, one should keep in mind that, mathematically speaking, it could be inconvenient to force oneself into a cone subdomain.
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