Universal arbitrage aggregator in discrete-time markets under uncertainty

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Abstract In a model-independent discrete-time financial market, we discuss the richness of the family of martingale measures in relation to different notions of arbitrage, generated by a class $S$ of significant sets, which we call arbitrage de la classe $S$. The choice of $S$ reflects the intrinsic properties of the class of polar sets of martingale measures. In particular, for $S = \{\Omega\}$, absence of model-independent arbitrage is equivalent to the existence of a martingale measure; for $S$ being the open sets, absence of open arbitrage is equivalent to the existence of full support martingale measures. These results are obtained by adopting a technical filtration enlargement and by constructing a universal aggregator of all arbitrage opportunities. We further introduce the notion of market feasibility and provide its characterization via arbitrage conditions. We conclude providing a dual representation of open arbitrage in terms of weakly open sets of probability measures, which highlights the robust nature of this concept.

Keywords Model uncertainty · First fundamental theorem of asset pricing · Feasible market · Open arbitrage · Full support martingale measure

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1 Introduction: No arbitrage under uncertainty

The introduction of Knightian uncertainty in mathematical models for finance has recently renewed the attention on foundational issues such as option pricing rules, superhedging, and arbitrage conditions. We can distinguish two extreme cases:

1) We are completely sure about the reference probability measure $P$. In this case, the classical notion of no arbitrage or NFLVR can be successfully applied (as in [12, 15, 16]).

2) We face complete uncertainty about any probabilistic model, and therefore we must describe our model independently of any probability. In this case, we might adopt a model-independent (weak) notion of no arbitrage.

In the second case, a pioneering contribution was given in the paper by Hobson [22], where the problem of pricing exotic options is tackled under model mis-specification. In his approach, the key assumption is the existence of a martingale measure for the market, consistent with the prices of some observed vanilla options (see also [7, 11, 14] for further developments). In [13], Davis and Hobson relate the previous problem to the absence of model-independent arbitrages by means of semistatic strategies. A step forward toward a model-free version of the first fundamental theorem of asset pricing in discrete time was achieved by Riedel [29] in a one-period market and by Acciaio et al. [1] in a more general setup.

Between cases 1 and 2, there is the possibility to accept that the model could be described in a probabilistic setting, but we cannot assume the knowledge of a specific reference probability measure, but at most of a set of priors, which leads to the new theory of quasi-sure stochastic analysis as in [5, 10, 17, 18, 25, 30, 31]. The idea is that classical probability theory can be reformulated when the single reference probability $P$ is replaced by a class $\mathcal{P}'$ of (possibly nondominated) probability measures. This is the case, for example, of uncertain volatility (e.g. [31]) where in a general continuous-time market model, the volatility is only known to lie in a certain interval $[\sigma_m, \sigma_M]$.

In the theory of arbitrage for nondominated sets of priors, important results were provided by Bouchard and Nutz [6] in discrete time. A suitable notion of no arbitrage with respect to a class $\mathcal{P}'$, named $NA(\mathcal{P}')$, was introduced, and it was shown that the no-arbitrage condition is equivalent to the existence of a family $\mathcal{Q}'$ of martingale measures having the same polar sets as $\mathcal{P}'$. In continuous-time markets, a similar topic has been recently investigated also by Biagini et al. [4].

Bouchard and Nutz [6] answer the following question: What is a good notion of arbitrage opportunity for all admissible probabilistic models $P \in \mathcal{P}'$ (i.e., one single $H$ that works as an arbitrage for all admissible models)? To pose this question, one has to know a priori which are the admissible models, i.e., we have to exhibit a subset of probabilities $\mathcal{P}'$.

In this paper, our aim is to investigate arbitrage conditions and robustness properties of markets that are described independently of any reference probability or set of priors.

We consider a financial market described by a discrete-time adapted stochastic process $S := (S_t)_{t \in I}$, $I = \{0, \ldots, T\}$, defined on $(\Omega, \mathcal{F}, \mathbb{F})$, $\mathbb{F} := (\mathcal{F}_t)_{t \in I}$ with
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Let $T < \infty$, and taking values in $\mathbb{R}^d$ (see Sect. 2). Note that we are not imposing any restriction on $S$, so that it may describe generic financial securities (for example, stocks and/or options). Differently from previous approaches in the literature, in our setting, the measurable space $(\Omega, \mathcal{F})$ and the price process $S$ defined on it are given, and we investigate the properties of martingale measures for $S$ induced by no-arbitrage conditions. The class $\mathcal{H}$ of admissible trading strategies is formed by all $\mathcal{F}$-predictable $d$-dimensional stochastic processes, and we denote by $\mathcal{M}$ the set of all probability measures under which $S$ is an $\mathcal{F}$-martingale and by $\mathcal{P}$ the set of all probability measures on $(\Omega, \mathcal{F})$. We introduce a flexible definition of arbitrage, which allows us to characterize the richness of the set $\mathcal{M}$ in a unified framework.

1.1 Arbitrage de la classe $\mathcal{S}$

Let

$$\mathcal{V}_H^+ = \{ \omega \in \Omega \mid V_T(H)(\omega) > 0 \},$$

where $V_T(H) = \sum_{t=1}^{T} H_t \cdot (S_t - S_{t-1})$ is the final value of the strategy $H$. It is natural to introduce several notions of arbitrage according to the properties of the set $\mathcal{V}_H^+$.

**Definition 1.1** Let $\mathcal{S}$ be a class of measurable subsets of $\Omega$ such that $\emptyset \notin \mathcal{S}$. A trading strategy $H \in \mathcal{H}$ is an arbitrage de la classe $\mathcal{S}$ if

$$V_0(H) = 0, V_T(H)(\omega) \geq 0, \forall \omega \in \Omega,$$

and $\mathcal{V}_H^+$ contains a set in $\mathcal{S}$.

The class $\mathcal{S}$ has the role to translate mathematically the meaning of a “true gain.” When a probability $P$ is given (the “reference probability”), we agree on representing a true gain as $P[V_T(H) > 0] > 0$, and therefore the classical no-arbitrage condition can be expressed as follows: no losses, $P[V_T(H) < 0] = 0$, implies no true gain, $P[V_T(H) > 0] = 0$. In a similar fashion, when a subset $\mathcal{P}'$ of probability measures is given, one may replace the $P$-a.s. conditions above with $\mathcal{P}'$-q.s. conditions, as in [6]. However, if we cannot or do not want to rely on a set of a priori assigned probability measures, then we may well use another concept: there is a true gain if $\mathcal{V}_H^+$ contains a set considered significant. This is exactly the role attributed to the class $\mathcal{S}$, which is the core of Sect. 3. Families of sets, not determined by some probability measures, have been already used in the context of the first and second fundamental theorems of asset pricing, respectively, by Bättig and Jarrow [3] and Cassese [9] (see Sect. 4.2 for a more specific comparison).

In order to investigate the properties of the martingale measures induced by no-arbitrage conditions of this kind, we first study (see Sect. 4) the structural properties of the market adopting a geometric approach in the spirit of [26], but with $\Omega$ being a general Polish space instead of a finite sample space. In particular, we characterize the class $\mathcal{N}$ of $\mathcal{M}$-polar sets, that is, the class of $N \subset \Omega$ such that there is no martingale measure that can assign a positive measure to $N$. We stress that in the model-independent framework, the set of martingale measures $\mathcal{M}$ and the $\mathcal{M}$-polar set $\mathcal{N}$ do not depend on any reference probability $P$ and are induced only by the market $(\Omega, \mathcal{F}, \mathcal{P}, S)$. Once these polar sets are identified, we explicitly build in Sect. 4.7 a process $H^\bullet$ that depends only on the price process $S$ and satisfies
• $V_T(H^\bullet)(\omega) \geq 0 \, \forall \omega \in \Omega$.
• $N \subseteq V^+_H$ for every $N \in \mathcal{N}$.

This strategy is a measurable selection of a set-valued process $\mathbb{H}$, and we baptize the latter \emph{universal arbitrage aggregator} since for any $P$ that is not absolutely continuous with respect to $\mathcal{M}$, an arbitrage opportunity $H^P$ (in the classical sense) can be found among the values of $\mathbb{H}$. All the inefficiencies of the market are captured by the process $H^\bullet$, but in general, it fails to be $\mathbb{R}^S$-predictable. However, we recover predictability with respect to a filtration $\tilde{\mathbb{F}}$, which is an enlargement of the natural filtration of the process $S$ and does not affect the set of martingale measures, that is, any martingale measure $Q \in \mathcal{M}$ can be uniquely extended to a martingale measure $\tilde{Q}$ in the enlarged filtration. This allows us to prove in Sect. 4.7 the main result of the paper.

\textbf{Theorem 1.2} Let $(\Omega, \tilde{\mathbb{F}}_T, \tilde{\mathbb{F}})$ be the enlarged filtered space as in Sect. 4.6, and let $\tilde{\mathcal{H}}$ be the set of $d$-dimensional discrete-time $\tilde{\mathbb{F}}$-predictable stochastic processes. Then:

\begin{equation*}
\text{No arbitrage de la classe } S \text{ in } \tilde{\mathcal{H}} \iff \mathcal{M} \neq \emptyset \text{ and } \mathcal{N} \text{ does not contain sets of } S.
\end{equation*}

In other words, properties of the family $S$ have a dual counterpart in terms of polar sets of the pricing functional.

In Sect. 4.7, we further provide our version of the fundamental theorem of asset pricing: the equivalence between absence of arbitrage de la classe $S$ in $\tilde{\mathcal{H}}$ and the existence of martingale measures $Q \in \mathcal{M}$ with the property that $Q[C] > 0$ for all $C \in S$.

\subsection*{1.2 Model-independent arbitrage}

When $S := \{\Omega\}$, arbitrage de la classe $S$ corresponds to the notion of a model-independent arbitrage. Since $\Omega$ never belongs to the class of polar sets $\mathcal{N}$, we directly obtain from Theorem 1.2 the following result.

\textbf{Theorem 1.3}

No model-independent arbitrage in $\tilde{\mathcal{H}} \iff \mathcal{M} \neq \emptyset$.

An analogous result has been obtained in [1] when considering a single risky asset $S$ as the canonical process on the path space $\Omega = \mathbb{R}^T_+$, a possibly uncountable collection of options $(\varphi_\alpha)_{\alpha \in A}$ whose prices are known at time 0, and when trading is possible through semistatic strategies (see also [23] for a detailed discussion). Assuming the existence of an option $\varphi_0$ with a specific payoff, the equivalence in Theorem 1.3 is achieved in the original measurable space $(\Omega, \mathcal{F}, \mathbb{F}, \mathcal{H})$. In our setup, although we are free to choose a $(d + k)$-dimensional process $S$ for modeling a finite number $k$ of options on possibly different $(d)$ underlyings, the admissible strategies in the class $\tilde{\mathcal{H}}$ are dynamic in every $S^i$ for $i = 1, \ldots, d + k$. In order to incorporate the case of semistatic strategies, we should need to consider restrictions on $\tilde{\mathcal{H}}$, and for this reason, the two results are not directly comparable.
1.3 Arbitrage with respect to open sets

In a topological context, in order to obtain full support martingale measures, the suitable choice for $\mathcal{S}$ is the class of open sets. This selection determines the notion of arbitrage with respect to open sets, which we shorten as “open arbitrage”:

- An open arbitrage is an admissible trading strategy $H$ such that $V_0(H) = 0$, $V_T(H)(\omega) \geq 0 \forall \omega \in \Omega$, and $V_H^+$ contains an open set.

This concept admits the following dual reformulation (see Sect. 6, Proposition 6.2). An open arbitrage consists of a trading strategy $H \in \mathcal{H}$ and a nonempty weakly open set $\mathcal{U} \subseteq \mathcal{P}$ such that

$$\text{for all } P \in \mathcal{U}, V_T(H) \geq 0 \text{ P-a.s. and } P[V_H^+] > 0.$$  \hspace{1cm} (1.1)

The robust feature of an open arbitrage is therefore evident from this dual formulation since a certain strategy $H$ satisfies (1.1) if it represents an arbitrage in the classical sense for a whole open set of probabilities. In addition, if $H$ is such a strategy and we disregard any finite subset of probabilities, then $H$ remains an open arbitrage. Moreover, every weakly open subset of $\mathcal{U}$ contains a full support probability $P$ (see Lemma 5.6) under which $H$ is a $P$-arbitrage in the classical sense. Full support martingale measures can be efficiently used whenever we face model mis-specification since they have a well-spread support that captures the features of the sample space of events without neglecting significantly large parts. Dolinsky and Soner [19] proved the equivalence of a local version of NA and the existence of full support martingale measures (see [19, Sect. 2.5]) in a continuous-time market determined by one risky asset with proportional transaction costs.

1.4 Feasibility and approximating measures

In Sect. 5, we answer the following question: Which are the markets that are feasible in the sense that the properties of the market are nice for “most” probabilistic models? Clearly, this problem depends on the choice of the feasibility criterion, but for this aim, we do not need to exhibit a priori a subset of probabilities. On the contrary, given a market (described without reference probability), the induced set of no-arbitrage models (probabilities) for that market will determine if the market itself is feasible or not. What is needed here is a good notion of “most” probabilistic models. More precisely, given the price process $S$ defined on $(\Omega, \mathcal{F})$, we introduce the set $\mathcal{P}_0$ of probability measures that exhibit no arbitrage in the classical sense as

$$\mathcal{P}_0 = \{ P \in \mathcal{P} \mid \text{no arbitrage with respect to } P \}. \hspace{1cm} (1.2)$$

When

$$\overline{\mathcal{P}_0}^\tau = \mathcal{P}$$

with respect to some topology $\tau$, the market is feasible in the sense that any “bad” reference probability can be approximated by no-arbitrage probability models. We
show in Proposition 5.7 that this property is equivalent to the existence of a full support martingale measure if we choose \( \tau \) as the weak* topology.

One other contribution of the paper, proved in Sect. 5, is the following characterization of feasible markets and absence of open arbitrage in terms of existence of full support martingale measures. We denote by \( \mathcal{P}_+ \subset \mathcal{P} \) the set of full support probability measures.

**Theorem 1.4** The following are equivalent:

1. The market is feasible, that is, \( \overline{\mathcal{P}}_0^{\sigma(\mathcal{P},C_b)} = \mathcal{P} \).
2. There exists \( P \in \mathcal{P}_+ \) such that no arbitrage with respect to \( P \) (in the classical sense) holds.
3. \( \mathcal{M} \cap \mathcal{P}_+ \neq \emptyset \).
4. No open arbitrage holds with respect to admissible strategies \( \tilde{\mathcal{H}} \).

Riedel [29] already pointed out the relevance of the concept of full support martingale measures in a probability-free setup. Indeed, in a one-period market model and under the assumption that the price process is continuous with respect to the state variable, he showed that the absence of a one-point arbitrage (nonnegative payoff with strict positivity in at least one point) is equivalent to the existence of a full support martingale measure. As shown in Sect. 6.1, this equivalence is no longer true in a multiperiod model (or in a single-period model with nontrivial initial \( \sigma \)-algebra) even for price processes continuous in \( \omega \). In this paper, we consider a multiasset multiperiod model without \( \omega \)-continuity assumptions on the price process, and we develop the concept of open arbitrage and its dual reformulation that yields the equivalence stated in the theorem.

Finally, we present a number of simple examples that point out: the differences between single-period and multiperiod models (Examples 3.5, 6.4, 6.5); the geometric approach to absence of arbitrage and existence of martingale measures (Sect. 4.1); the need in the multiperiod setting for disintegration of atoms (Example 4.10); the need for one-period anticipation of some polar sets (Example 4.16). The consequences of our version of the fundamental theorem of asset pricing for the robust formulation of the superhedging duality is analyzed in the paper [8].

2 Financial markets

We assume that \((\Omega, d)\) is a Polish space and \( \mathcal{F} = B(\Omega) \) is the Borel \( \sigma \)-algebra induced by the metric \( d \). The requirement that \( \Omega \) is Polish is used in Sect. 4.4 to guarantee the existence of a regular conditional probability; see Theorem 4.12. We fix a finite time horizon \( T \geq 1 \), a finite set of time indices \( I := \{0, \ldots, T\} \), and we set \( I_1 := \{1, \ldots, T\} \). Let \( \mathbb{F} := (\mathcal{F}_t)_{t \in I} \) be a filtration with \( \mathcal{F}_0 = \{\emptyset, \Omega\} \) and \( \mathcal{F}_T \subseteq \mathcal{F} \). We denote by \( \mathcal{L}(\Omega, \mathcal{F}_t; \mathbb{R}^d) \) the set of \( \mathcal{F}_t \)-measurable random variables \( X : \Omega \rightarrow \mathbb{R}^d \) and by \( \mathcal{L}(\Omega, \mathbb{F}; \mathbb{R}^d) \) the set of adapted processes \( X = (X_t)_{t \in I} \) with \( X_t \in \mathcal{L}(\Omega, \mathcal{F}_t; \mathbb{R}^d) \).

The market consists of one nonrisky asset \( S_0^t = 1 \) for all \( t \in I \) and \( d \geq 1 \) risky assets \( S^j = (S^j_t)_{t \in I}, j = 1, \ldots, d, \) that are real-valued adapted stochastic processes.
Let \( S = (S^1, \ldots, S^d) \in \mathcal{L}(\Omega, \mathbb{P}; \mathbb{R}^d) \) be the \( d \)-dimensional vector of the (discounted) price processes.

In this paper, we focus on arbitrage conditions and therefore without loss of generality restrict our attention to self-financing trading strategies of zero initial cost. Therefore, we may assume that a trading strategy \( H = (H_t)_{t \in I} \) is an \( \mathbb{R}^d \)-valued predictable stochastic process \( H = (H^1, \ldots, H^d) \) with \( H_t \in \mathcal{L}(\Omega, \mathcal{F}_{t-1}; \mathbb{R}^d) \), and we denote by \( \mathcal{H} \) the class of all trading strategies. The (discounted) value process \( V(H) = (V_t(H))_{t \in I} \) is defined by

\[
V_0(H) := 0, \quad V_t(H) := \sum_{i=1}^{t} H_i \cdot (S_i - S_{i-1}), \quad t \geq 1.
\]

A (discrete-time) financial market is therefore determined, without any reference probability measure, by the quadruple \((\Omega, d); (B(\Omega), \mathcal{F}); S; \mathcal{H})\) satisfying the previous conditions.

**Notation 2.1** For \( \mathcal{F} \)-measurable random variables \( X \) and \( Y \), we write \( X > Y \) (resp. \( X \geq Y \), \( X = Y \)) if \( X(\omega) > Y(\omega) \) for all \( \omega \in \Omega \) (resp. \( X(\omega) \geq Y(\omega) \), \( X(\omega) = Y(\omega) \)) for all \( \omega \in \Omega \).

### 2.1 Probability and martingale measures

Let \( \mathcal{P} := \mathcal{P}(\Omega) \) be the set of all probabilities on \((\Omega, \mathcal{F})\), and \( C_b := C_b(\Omega) \) the space of continuous and bounded functions on \( \Omega \). Except when explicitly stated, we endow \( \mathcal{P} \) with the weak* topology \( \sigma(\mathcal{P}, C_b) \), so that \((\mathcal{P}, \sigma(\mathcal{P}, C_b))\) is a Polish space (see [2, Chap. 15] for further details). The convergence of \( P_n \) to \( P \) in the topology \( \sigma(\mathcal{P}, C_b) \) is denoted by \( P_n \xrightarrow{w} P \) and the \( \sigma(\mathcal{P}, C_b) \)-closure of a set \( Q \subseteq \mathcal{P} \) by \( \overline{Q} \).

We define the **support** of an element \( P \in \mathcal{P} \) as

\[
supp(P) = \bigcap \{ C \in \mathcal{C} \mid P[C] = 1 \},
\]

where \( \mathcal{C} \) are the closed sets in \((\Omega, d)\). Under our assumptions, the support is given by

\[
supp(P) = \{ \omega \in \Omega \mid P[B_\varepsilon(\omega)] > 0 \text{ for all } \varepsilon > 0 \},
\]

where \( B_\varepsilon(\omega) \) is the open ball with radius \( \varepsilon \) centered in \( \omega \).

**Definition 2.2** We say \( P \in \mathcal{P} \) has **full support** if \( supp(P) = \Omega \), and we denote by

\[
\mathcal{P}^+ := \{ P \in \mathcal{P} \mid supp(P) = \Omega \}
\]

the set of all probability measures having full support.

Observe that \( P \in \mathcal{P}^+ \) if and only if \( P[A] > 0 \) for every open set \( A \). Full support measures are therefore important from a topological point of view since they assign positive probability to all open sets.
Definition 2.3 The set of $\mathbb{F}$-martingale measures is
\[
\mathcal{M}(\mathbb{F}) = \{ Q \in \mathcal{P} | S \text{ is a } (Q, \mathbb{F})\text{-martingale} \},
\]
and we set: $\mathcal{M} := \mathcal{M}(\mathbb{F})$, when the filtration is not ambiguous, and
\[
\mathcal{M}_+ = \mathcal{M} \cap \mathcal{P}_+.
\]

Definition 2.4 Let $P \in \mathcal{P}$ and $\mathcal{G} \subseteq \mathcal{F}$ be a sub-$\sigma$-algebra of $\mathcal{F}$. The generalized conditional expectation of a nonnegative $X \in L(\Omega, \mathcal{F}, \mathbb{R})$ is defined by
\[
E_P[X | \mathcal{G}] := \lim_{n \to +\infty} E_P[X \wedge n | \mathcal{G}],
\]
and for $X \in L(\Omega, \mathcal{F}, \mathbb{R})$, we set $E_P[X | \mathcal{G}] := E_P[X^+ | \mathcal{G}] - E_P[X^- | \mathcal{G}]$, where we adopt the convention $\infty - \infty = -\infty$. All basic properties of the conditional expectation still hold (see e.g. [20]). In particular, if $Q \in \mathcal{M}$ and $H \in \mathcal{H}$, then $E_Q[H_t \cdot (S_t - S_{t-1}) | \mathcal{F}_{t-1}] = H_t \cdot E_Q[S_t - S_{t-1} | \mathcal{F}_{t-1}] = 0$ $Q$-a.s., so that $E_Q[V_T(H)) = 0$ $Q$-a.s.

3 Arbitrage de la classe $\mathcal{S}$

Let $H \in \mathcal{H}$ and recall that $\mathcal{V}^+_H := \{ \omega \in \Omega | V_T(H)(\omega) > 0 \}$ and that $V_0(H) = 0$.

Definition 3.1 Let $P \in \mathcal{P}$. A $P$-classical arbitrage is a trading strategy $H \in \mathcal{H}$ with $V_T(H) \geq 0$ $P$-a.s. and $P[\mathcal{V}^+_H] > 0$.

We denote by $\text{NA}(P)$ the absence of $P$-classical arbitrage.

Recall the definition of arbitrage de la classe $\mathcal{S}$ stated in the introduction.

Example 3.2 Some examples of arbitrage de la classe $\mathcal{S}$:

1) $H$ is a $1p$-arbitrage when $\mathcal{S} = \{ C \in \mathcal{F} | C \neq \emptyset \}$. This is the weakest notion of arbitrage since $\mathcal{V}^+_H$ might reduce to a single point. The $1p$-arbitrage corresponds to the definition given in [29]. This can be easily generalized to the following notion of $n$-point-arbitrage: $H$ is an $np$-arbitrage when
\[
\mathcal{S} = \{ C \in \mathcal{F} | C \text{ has at least } n \text{ elements} \}.
\]

2) $H$ is an open arbitrage when $\mathcal{S} = \{ C \in \mathcal{B}(\Omega) | C \text{ open nonempty} \}$.

3) $H$ is a $\mathcal{P}'$-q.s. arbitrage when $\mathcal{S} = \{ C \in \mathcal{F} | P[C] > 0 \text{ for some } P \in \mathcal{P}' \}$ for a fixed family $\mathcal{P}' \subseteq \mathcal{P}$. Notice that $\mathcal{S} = (\mathcal{N}(\mathcal{P}'))^c$, the complements of the polar sets of $\mathcal{P}'$. Then there is no $\mathcal{P}'$-q.s. arbitrage if \[ H \in \mathcal{H} \text{ such that } V_T(H)(\omega) \geq 0 \forall \omega \in \Omega \implies V_T(H) = 0 \mathcal{P}'\text{-q.s.} \]
This definition is similar to the no-arbitrage condition in [6], the only difference being that here we require $V_T(H)(\omega) \geq 0 \forall \omega \in \Omega$, whereas [6] only requires $V_T(H) \geq 0$ $\mathcal{P}'$-q.s. Hence, no $\mathcal{P}'$-q.s. arbitrage is a condition weaker than no arbitrage in [6].

4) $H$ is a $P$-a.s. arbitrage when $\mathcal{S} = \{ C \in \mathcal{F} | P[C] > 0 \}$ for a fixed $P \in \mathcal{P}$. As in the previous example, no $P$-a.s. arbitrage is a weaker condition than no $P$-classical arbitrage, the only difference being that here we require $V_T(H)(\omega) \geq 0 \forall \omega \in \Omega$, whereas the classical definition only requires $V_T(H) \geq 0$ $P$-a.s.

5) $H$ is a model-independent arbitrage when $\mathcal{S} = \{ \omega \}$ in the spirit of [1, 13, 11].

6) $H$ is an $\varepsilon$-arbitrage when $\mathcal{S} = \{ C_{\varepsilon}(\omega) | \omega \in \Omega \}$, where $\varepsilon > 0$ is fixed and $C_{\varepsilon}(\omega)$ is the closed ball in $(\Omega, d)$ of radius $\varepsilon$ and centered in $\omega$.

Obviously, for any class $\mathcal{S}$,

$$\text{no } 1p\text{-arbitrage} \implies \text{no arbitrage de la classe } \mathcal{S} \implies \text{no model-independent arbitrage}, \quad (3.1)$$

and these notions depend only on the properties of the financial market and are not necessarily related to any probabilistic models.

Remark 3.3 The no-arbitrage concepts defined above and the possible generalization of no free lunch de la classe $\mathcal{S}$, can be considered also in more general, continuous-time financial market models. We choose to present our theory in the discrete-time framework since the subsequent results in the next sections rely crucially on the discrete-time setting.

Example 3.4 The flexibility of our approach relies on the arbitrary choice of the class $\mathcal{S}$. Let $\Omega = C^0([0, T]; \mathbb{R})$ be the set of real-valued continuous functions defined on the interval $[0, T]$. It is a Polish space when endowed with the supremum norm $\| \cdot \|_{\infty}$. We may consider the two classes

$$\mathcal{S}^\infty = \{ \text{open balls in } \| \cdot \|_{\infty} \} \quad \text{and} \quad \mathcal{S}^1 = \{ \text{open balls in } \| \cdot \|_1 \},$$

where $\| \omega \|_1 = \int_0^T |\omega(t)| dt$. Since the integral operator $\int_0^T | \cdot | dt : C^0([0, T]; \mathbb{R}) \to \mathbb{R}$ is $\| \cdot \|_{\infty}$-continuous, every open ball in $\| \cdot \|_1$ is also open in $\| \cdot \|_{\infty}$. Hence, every arbitrage de la classe $\mathcal{S}^1$ is also an arbitrage de la classe $\mathcal{S}^\infty$, but not conversely. For instance, consider a market described by an underlying process $\mathcal{S}^1$ and a digital option $\mathcal{S}^2$, where trading is allowed only in a finite set of times $\{0, 1, \ldots, T - 1\}$. Define $S_0^1(\omega) = s_0$ for every $\omega \in \Omega$, $S_t^1(\omega) = \omega(t)$, and $S_t^2(\omega) = 1_B(\omega)1_{\{T\}}(t)$, where $B := \{ \omega | S_t^1(\omega) \in (s_0 - \varepsilon, s_0 + \varepsilon) \forall t \in [0, T] \}$. A long position in the option at time $T - 1$ is an arbitrage de la classe $\mathcal{S}^\infty$ even though there does not exist any arbitrage de la classe $\mathcal{S}^1$.

3.1 Defragmentation

When the reference probability $P \in \mathcal{P}$ is fixed, the market admits a $P$-classical arbitrage if and only if there exist $t \in \{1, \ldots, T\}$ and $\eta \in L^0(\Omega, \mathcal{F}_{t-1}, P; \mathbb{R}^d)$
such that \( \eta \cdot (S_t - S_{t-1}) \geq 0 \) \( P \)-a.s. and \( P[\eta \cdot (S_t - S_{t-1}) > 0] > 0 \) (see [12] or [21, Proposition 5.11]). In our context, the existence of an arbitrage de la classe \( S \) over a certain time interval \([0, T]\) does not necessarily imply the existence of a single time step where the arbitrage is realized on a set in \( S \). It might happen instead that the agent needs to implement a strategy over multiple time steps to achieve an arbitrage de la classe \( S \). The following example shows a simple case in which this phenomenon occurs. Recall that \( \mathcal{L}(\Omega, \mathcal{F}; \mathbb{R}^d) \) is the set of \( \mathbb{R}^d \)-valued \( \mathcal{F} \)-measurable random variables on \( \Omega \).

**Example 3.5** Consider a 2-period market model composed by two risky assets \( S^1, S^2 \) on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\) described by the trajectories

\[
\begin{array}{c|c|c}
3 & 3 & \omega \in A_1 \\
3 & 5 & \omega \in A_2 \\
S^1 & 2 & \omega \in A_3 \\
1 & 1 & \omega \in A_4 \\
7 & 7 & \omega \in A_1 \\
7 & 3 & \omega \in A_2 \\
S^2 & 2 & \omega \in A_3 \\
1 & 1 & \omega \in A_4 \\
\end{array}
\]

Consider \( H_1 = (-1, +1) \) and \( H_2 = (1_{A_2 \cup A_3}, -1_{A_2 \cup A_3}) \). Then \( H_1 \cdot (S_1 - S_0) = 41_{A_1} \) and \( H_2 \cdot (S_2 - S_1) = 21_{A_2} \). Choosing \( A_1 = \emptyset \cap (0, 1) \), \( A_2 = (\mathbb{R} \setminus \emptyset) \cap (0, 1) \) and \( A_3 = [1, +\infty) \), \( A_4 = (-\infty, 0] \), we observe that an open arbitrage can be obtained only by a two-step strategy, whereas in each step we have only 1p-arbitrages.

In general, a multistep strategy can realize an arbitrage de la classe \( S \) at time \( T \) even if it does not necessarily yield a positive gain at each time, that is, there might exist \( t < T \) such that \( \{V_t(H) < 0\} \neq \emptyset \). This is the case in Example 4.16.

In the remainder of this section, \( \Delta S_t = (S^1_t - S^1_{t-1}, \ldots, S^d_t - S^d_{t-1}) \).

**Lemma 3.6** The strategy \( H \in \mathcal{H} \) is a 1p-arbitrage if and only if there exist a time \( t \in I_1 \), an \( \alpha \in \mathcal{L}(\Omega, \mathcal{F}_{t-1}; \mathbb{R}^d) \), and a nonempty \( A \in \mathcal{F}_t \) such that

\[
\begin{align*}
\alpha(\omega) \cdot \Delta S_t(\omega) &\geq 0 & \forall \omega \in \Omega, \\
\alpha(\omega) \cdot \Delta S_t(\omega) &> 0 & \text{on } A.
\end{align*}
\]

**Proof** (\( \Rightarrow \)) Let \( H \in \mathcal{H} \) be a 1p-arbitrage. Set

\[
\overline{t} = \min \{t \in \{1, \ldots, T\} \mid V_t(H) \geq 0 \text{ with } V_t(H)(\omega) > 0 \text{ for some } \omega \in \Omega\}.
\]

If \( \overline{t} = 1 \), then \( \alpha = H_1 \) satisfies the requirements. If \( \overline{t} > 1 \), then either \( \{V_{\overline{t}-1}(H) < 0\} \neq \emptyset \) or \( \{V_{\overline{t}-1}(H) = 0\} = \Omega \). In the first case, for \( \alpha = H_{\overline{t}}1_{\{V_{\overline{t}-1}(H) < 0\}} \), we have \( \alpha \cdot \Delta S_{\overline{t}} \geq 0 \) with strict inequality on \( \{V_{\overline{t}-1}(H) < 0\} \). In the latter case, \( \alpha = H_T \) satisfies the requirements.

(\( \Leftarrow \)) Take \( \alpha \in \mathcal{L}(\Omega, \mathcal{F}_{t-1}; \mathbb{R}^d) \) as by assumption and define \( H \in \mathcal{H} \) by \( H_s = 0 \) for every \( s \neq t \) and \( H_t = \alpha \). Hence, \( V_T(H) = V_t(H) \), so that \( V_T(H) \geq 0 \). Note that \( \{\omega \in \Omega \mid V_T(H)(\omega) > 0\} = \{\omega \in \Omega \mid \alpha \cdot \Delta S_t(\omega) > 0\} \), and the proof is complete. \( \square \)
Remark 3.7 Notice that only the implication \((\Leftarrow)\) of the previous lemma holds for open arbitrage. This means that there exists an open arbitrage if we can find a time \(t \in I_1\), an \(\alpha \in L(\Omega, \mathcal{F}_{t-1}; \mathbb{R}^d)\), and a set \(A \in \mathcal{F}_t\) containing an open set such that (3.2) holds. A similar statement holds for arbitrage de la classe \(\mathcal{S}\). On the other hand, the converse is false in general as shown by Example 3.5.

The following lemma provides a full characterization of arbitrages de la classe \(\mathcal{S}\) by means of a multistep decomposition of the strategy.

Lemma 3.8 (Defragmentation) The strategy \(H \in \mathcal{H}\) is an arbitrage de la classe \(\mathcal{S}\) if and only if there exist

- a finite family \((U_t)_{t \in I}\) with \(U_t \in \mathcal{F}_t\), \(U_t \cap U_s = \emptyset\) for every \(t \neq s\) and such that \(\bigcup_{t \in I} U_t\) contains a set in \(\mathcal{S}\); and
- a strategy \(\tilde{H} \in \mathcal{H}\) such that \(V_T(\tilde{H}) \geq 0\) on \(\Omega\) and \(\tilde{H}_t \cdot \Delta S_t > 0\) on \(U_t\) for any \(U_t \neq \emptyset\).

Proof \((\Rightarrow)\) Let \(H \in \mathcal{H}\) be an arbitrage de la classe \(\mathcal{S}\). Define \(B_t = \{V_t(H) > 0\}\) and observe that

\[
U_1 := B_1 \implies H_1 \cdot \Delta S_1(\omega) > 0 \quad \forall \omega \in U_1;
\]

\[
U_2 := B_1^c \cap B_2 \implies H_2 \cdot \Delta S_2(\omega) > 0 \quad \forall \omega \in U_2;
\]

\[
U_{T-1} := B_1^c \cap \cdots \cap B_{T-2}^c \cap B_{T-1} \implies H_{T-1} \cdot \Delta S_{T-1}(\omega) > 0 \quad \forall \omega \in U_{T-1};
\]

\[
U_T := B_1^c \cap \cdots \cap B_{T-2}^c \cap B_{T-1}^c \cap V_H^+ \implies H_T \cdot \Delta S_T(\omega) > 0 \quad \forall \omega \in U_T.
\]

From the definition of \(\{U_1, U_2, \ldots, U_T\}\) we have that \(V_H^+ \subseteq \bigcup_{i=1}^T U_i\). Set \(\tilde{H}_1 = H_1\) and consider the strategy given, for every \(2 \leq t \leq T\), by

\[
\tilde{H}_t(\omega) = H_t(\omega)1_{D_{t-1}}(\omega),
\]

where \(D_{t-1} = \left( \bigcup_{s=1}^{t-1} U_s \right)^c\).

By construction, \(\tilde{H} \in \mathcal{H}\) and \(\tilde{H}_t \cdot \Delta S_t(\omega) > 0\) for every \(\omega \in U_t\).

\((\Leftarrow)\) This implication is trivial. \(\Box\)

4 Arbitrage de la classe \(\mathcal{S}\) and martingale measures

Before addressing this topic in full generality, we provide some insights into the problem and introduce some examples that will help to develop the intuition on the approach we adopt. The required technical tools are then stated in Sects. 4.3 and 4.4.

Consider the family of polar sets of \(\mathcal{M}\),

\[
\mathcal{N} := \{A \subseteq A' \in \mathcal{F} \mid Q[A'] = 0 \forall Q \in \mathcal{M}\}.
\]

Nutz and Bouchard [6] define the notion of \(\text{NA}(\mathcal{P}')\) for any fixed family \(\mathcal{P}' \subseteq \mathcal{P}\) by

\[
V_T(H) \geq 0 \mathcal{P}'\text{-q.s.} \implies V_T(H) = 0 \mathcal{P}'\text{-q.s.},
\]

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where $H$ is a process predictable with respect to the universal completion of $\mathbb{F}$ (see [6]). One of the main results in [6] asserts that under NA($P'$), there exists a class $Q'$ of martingale measures that has the same polar sets as $P'$. If we take $P' = P$, then NA($P$) is equivalent to no (universally measurable) $1p$-arbitrage since $P$ contains all Dirac measures. In addition, the class of polar sets of $P$ is empty. In Sect. 4.5, we show that the same result is also true in our setting as a consequence of Proposition 4.18. The existence of a class of martingale measures with no polar sets implies that for each $\omega \in \Omega$, there exists $Q \in \mathcal{M}$ such that $Q(\{\omega\}) > 0$, and since $\Omega$ is separable, we can find a dense set $D := \{\omega_n \mid n \in \mathbb{N}\}$ with associated $Q^n \in \mathcal{M}$ such that $\sum_{n=1}^{\infty} \frac{1}{n} Q^n$ is a full support martingale measure (see Lemma A.9).

**Proposition 4.1** We have the following implications:

1) No $1p$-arbitrage $\implies$ $\mathcal{M}_+ \neq \emptyset$.

2) $\mathcal{M}_+ \neq \emptyset \implies$ no open arbitrage.

**Proof** The proof of 1) is postponed to Sect. 4.5. We prove 2) by observing that for any open set $O$ and $Q \in \mathcal{M}_+$, we have $Q(O) > 0$. Since for any $H \in \mathcal{H}$ with $V_T(H) \geq 0$, we have $Q[V_T^+] = 0$, it follows that $V_T^+$ does not contain any open set. □

**Example 4.2** Note that the existence of a full support martingale measure is compatible with $1p$-arbitrage, so that the converse implication of 1) in Proposition 4.1 does not hold. Indeed, let $(\Omega, \mathcal{F}) = (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. Consider the market with one risky asset with $S_0 = 2$ and

$$S_1 = \begin{cases} 3, & \omega \in \mathbb{R}_+ \setminus \mathbb{Q}, \\ 2, & \omega \in \mathbb{Q}_+. \end{cases}$$

Then obviously there exists a $1p$-arbitrage even though there exist full support martingale measures (those probabilities assigning positive mass only to each rational).

As soon as we weaken no $1p$-arbitrage, by adopting any other no-arbitrage conditions in Example 3.2, there is no guarantee of the existence of martingale measures, as will be shown in Sect. 4.1. In order to obtain the equivalence between $\mathcal{M} \neq \emptyset$ and no model-independent arbitrage (the weakest among the no-arbitrage conditions de la classe $\mathcal{S}$), we enlarge the filtration, as explained in Sect. 4.6.

### 4.1 Examples

This section provides a variety of counterexamples to many possible conjectures on the formulation of the fundamental theorem of asset pricing in the model-free framework. A financially meaningful example is the one of two call options with the same spot price $p_1 = p_2$ but strike prices $K_1 > K_2$, formulated in [13], which already highlights that the equivalence between absence of model-independent arbitrage and existence of martingale measures is not possible.

We consider a one-period market (i.e., $T = 1$) with $(\Omega, \mathcal{F}) = (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ and with $d = 2$ risky assets $S = (S^1, S^2)$, in addition to the riskless asset $S^0 = 1$. Admissible trading strategies are represented by vectors $H = (\alpha, \beta) \in \mathbb{R}^2$, so that

$$V_T(H) = \alpha \Delta S^1 + \beta \Delta S^2,$$
where $\Delta S^i = S^i_1 - S^i_0$ for $i = 1, 2$. Let $S_0 = (S^1_0, S^2_0) = (2, 2)$,

$$
S^1_1 = \begin{cases} 3, & \omega \in \mathbb{R}_+ \setminus \mathbb{Q}, \\ 2, & \omega \in \mathbb{Q}_+, 
\end{cases}
S^2_1 = \begin{cases} 1 + \exp(\omega), & \omega \in \mathbb{R}_+ \setminus \mathbb{Q}, \\ 1, & \omega = 0, \\ 1 + \exp(-\omega), & \omega \in \mathbb{Q}_+ \setminus \{0\}. 
\end{cases}
$$

(4.1)

and $\mathcal{F} = \mathcal{F}^S$. We notice the following simple facts.

1) There are no martingale measures,

$$
\mathcal{M} = \emptyset.
$$

Indeed, if we denote by $\mathcal{M}_i$ the set of martingale measures for the $i$th asset, then we have $\mathcal{M}_1 = \{Q \in \mathcal{P} \mid Q[\mathbb{R}_+ \setminus \mathbb{Q}] = 0\}$ and $Q[\mathbb{R}_+ \setminus \mathbb{Q}] > 0$ for all $Q \in \mathcal{M}_2$.

2) The final value of the strategy $H = (\alpha, \beta) \in \mathbb{R}^2$ is

$$
V_T(H) = \begin{cases} \alpha + \beta(\exp(\omega) - 1), & \omega \in \mathbb{R}_+ \setminus \mathbb{Q}, \\ -\beta, & \omega = 0, \\ \beta(\exp(-\omega) - 1), & \omega \in \mathbb{Q}_+ \setminus \{0\}. 
\end{cases}
$$

Only the strategies $H \in \mathbb{R}^2$ having $\beta = 0$ and $\alpha \geq 0$ satisfy $V_T(H)(\omega) \geq 0$ for all $\omega \in \Omega$. For $\beta = 0$ and $\alpha > 0$, $V_H^+ = \mathbb{R}_+ \setminus \mathbb{Q}$, and therefore there are no open arbitrage and no model-independent arbitrage (but $\mathcal{M} = \emptyset$). This fact persists even if we impose boundedness restrictions on the process $S$ or on the admissible strategies $H$, as the following modification of the example shows: let $\tilde{S}_0 = (2, 2)$ and take

$$
\tilde{S}^1_1 = (2 + \exp(-\omega)) 1_{\mathbb{R}_+ \setminus \mathbb{Q}} + 2 1_{\mathbb{Q}_+},
\tilde{S}^2_1 = (1 + \exp(\omega) \land 4) 1_{\mathbb{R}_+ \setminus \mathbb{Q}} + 1_{\{0\}} + (1 + \exp(-\omega)) 1_{\mathbb{Q}_+ \setminus \{0\}}.
$$

3) Define $\mathcal{H}^+ := \{H \in \mathcal{H} \mid V_T(H) \geq 0 \text{ and } V_0(H) = 0\}$, so that we obtain $\bigcup_{H \in \mathcal{H}^+} V_H^+ = \mathbb{R}_+ \setminus \mathbb{Q} \not\subseteq \Omega$. This shows that the condition $\mathcal{M} = \emptyset$ is not equivalent to $\bigcup_{H \in \mathcal{H}^+} V_H^+ = \Omega$, that is, it is not true that the set of martingale measures is empty iff for every $\omega$, there exists a strategy $H$ that gives positive wealth on $\omega$ and $V_0(H) = 0$. In order to recover the equivalence between these two concepts (as in Proposition 4.27), we need to enlarge the filtration in the way explained in Sect. 4.6.

4) By fixing any probability $P$, there exists a $P$-classical arbitrage since the (probabilistic) fundamental theorem of asset pricing holds and $\mathcal{M} = \emptyset$. Indeed:

(a) If $P[\mathbb{R}_+ \setminus \mathbb{Q}] = 0$, then $\beta = -1$ and $\alpha = 0$ yield a $P$-classical arbitrage since $V_H^+ = \mathbb{Q}_+$ and $P[V_H^+] = 1$.

(b) If $P[\mathbb{R}_+ \setminus \mathbb{Q}] > 0$, then $\beta = 0$ and $\alpha = 1$ yield a $P$-classical arbitrage since $V_H^+ = \mathbb{R}_+ \setminus \mathbb{Q}$ and $P[V_H^+] > 0$.

5) Instead, by adopting the definition of a $P$-a.s. arbitrage ($V_T(H)(\omega) \geq 0$ for all $\omega \in \Omega$ and $P[V_H^+] > 0$), there are two possibilities:

(a) If $P[\mathbb{R}_+ \setminus \mathbb{Q}] = 0$, no $P$-a.s. arbitrage holds since only the strategies $H \in \mathbb{R}^2$ having $\beta = 0$ and $\alpha \geq 0$ satisfy $V_T(H)(\omega) \geq 0$ for all $\omega \in \Omega$ and $V_H^+ = \mathbb{R}_+ \setminus \mathbb{Q}$.

(b) If $P[\mathbb{R}_+ \setminus \mathbb{Q}] > 0$, then $\beta = 0$ and $\alpha = 1$ yields a $P$-a.s. arbitrage, since $V_H^+ = \mathbb{R}_+ \setminus \mathbb{Q}$ and $P[V_H^+] > 0$. 

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In Examples (4.1) and (4.2), 0 does not belong to the relative interior of the convex set generated by the points \(\{(\Delta S^1(\omega), \Delta S^2(\omega))\}_{\omega \in \Omega^1}\), and hence there exists a hyperplane which separates the points.

6) Geometric approach: If we plot the vector \((\Delta S^1, \Delta S^2)\) on the real plane (see Fig. 1), then we see that there exists a unique separating hyperplane given by the vertical axis. As a consequence, \(1p\)-arbitrage can arise only by investment in the first asset \((\beta = 0)\). By a separating hyperplane we mean a hyperplane in \(\mathbb{R}^d\) passing through the origin and such that one of the associated half-spaces contains (not necessarily strictly) all the image points of the random vector \((\Delta S^1, \Delta S^2)\).

Let us now consider this other example on \((\mathbb{R}_+, B(\mathbb{R}_+))\). Let \(S_0 = (2, 2)\) and

\[
S_1^1 = \begin{cases} 
3, & \omega \in \mathbb{R}_+ \setminus \mathbb{Q}, \\
2, & \omega = 0, \\
1, & \omega \in \mathbb{Q}_+ \setminus \{0\},
\end{cases} \quad S_1^2 = \begin{cases} 
7, & \omega \in \mathbb{R}_+ \setminus \mathbb{Q}, \\
2, & \omega = 0, \\
0, & \omega \in \mathbb{Q}_+ \setminus \{0\}.
\end{cases} \tag{4.2}
\]

In both Examples (4.1) and (4.2), there exist separating hyperplanes, so that, a \(1p\)-arbitrage can be obtained (see Fig. 1). In Example (4.1), \(\mathcal{M}\) is empty, and we find a unique separating hyperplane; this hyperplane cannot give a strict separation of the set \(\{(\Delta S^1(\omega), \Delta S^2(\omega))\}_{\omega \in \mathbb{Q}_+}\) even though \(\mathbb{Q}_+\) does not support any martingale measure. In Example (4.2), \(\mathcal{M} = \{\delta_{\omega=0}\}\), only the event \(\{\omega = 0\}\) supports a martingale measure, and there exist an infinite number of hyperplanes that strictly separate the image of both polar sets \(\mathbb{R}_+ \setminus \mathbb{Q}\) and \(\mathbb{Q}_+ \setminus \{0\}\), namely those separating the convex grey region in Fig. 1.

In conclusion, the previous examples show that in a model-free environment, the existence of a martingale measure cannot be implied by arbitrage conditions—at least of the type considered so far. This is an important difference between the model-free and quasi-sure analysis approach (see e.g. [6]):

- Model-free approach: We deduce the “richness” of the set \(\mathcal{M}\) of martingale measures starting directly from the underlying market structure \((\Omega, \mathcal{F}, S)\) and analyze the class of polar sets with respect to \(\mathcal{M}\).
- Quasi-sure approach: The class of priors \(\mathcal{P}' \subseteq \mathcal{P}\) and its polar sets are given, and one formulates a no-arbitrage type condition to guarantee the existence of a class of martingale measures that has the same polar sets as the set of priors.
4.2 An alternative definition of arbitrage

The hypothesis of no \( P \)-classical arbitrage, namely, that \( P[V_T(H) < 0] = 0 \) implies \( P[V_T(H) > 0] = 0 \), can be rephrased as \( V_0(H) = 0 \) and

\[
\{ V_T(H) < 0 \} \text{ is negligible} \iff \{ V_T(H) > 0 \} \text{ is negligible,}
\]

or in our setting as

\[
\mathcal{V}_H^- \text{ does not contain sets in } \mathcal{S} \implies \mathcal{V}_H^+ \text{ does not contain sets in } \mathcal{S},
\]

where \( \mathcal{V}_H^- := \{ \omega \in \Omega \mid V_T(H)(\omega) < 0 \} \). In definition (4.4), we are giving up the requirement \( V_T(H) \geq 0 \), and so the differences with respect to the existence of arbitrage opportunities shown in (5) of example (4.1) disappear. However, this alternative definition of arbitrage does not work well, as shown by the following example. Consider \((\Omega, \mathcal{F}) = (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))\), a one-period market with one risky asset with \( S_0 = 2 \) and

\[
S_1 = \begin{cases} 3, & \omega \in [1, \infty), \\ 2, & \omega = [0, 1) \setminus \mathbb{Q}, \\ 1, & \omega \in [0, 1) \cap \mathbb{Q}. \end{cases}
\]

Consider the strategy of buying the risky asset, \( H = 1 \). Then \( \mathcal{V}_H^- = [0, 1) \cap \mathbb{Q} \) does not contain any open set, but \( \mathcal{V}_H^+ = [1, \infty) \) contains open sets. Therefore, there is an open arbitrage (in the modified definition obtained from (4.4)), but there are full support martingale measures, for example, \( Q([0, 1) \cap \mathbb{Q}) = Q([1, \infty)) = \frac{1}{2} \).

A concept of no arbitrage similar to (4.3) was introduced by Cassese [9] by adopting an ideal \( \mathcal{N} \) of “negligible” sets—not necessarily derived from probability measures. In a continuous-time setting, he proves that the absence of such an arbitrage is equivalent to the existence of a finitely additive “martingale measure.” Our results are not comparable with those in [9] since the markets are clearly different, we do not require any structure on the family \( \mathcal{S} \), and [9] works with finitely additive measures. In addition, example (4.5) just discussed shows the limitation in our setting of definition (4.3) for finding martingale probability measures with the appropriate properties.

4.3 Technical lemmata

Recall that \( S = (S_t)_{t \in I} \) is an \( \mathbb{R}^d \)-valued stochastic process defined on a Polish space \( \Omega \) endowed with its Borel \( \sigma \)-algebra \( \mathcal{F} = \mathcal{B}(\Omega) \) and \( I_1 := \{1, \ldots, T\} \).

Throughout the rest of the paper, we use the natural filtration \( \mathcal{F}^S = (\mathcal{F}^S_t)_{t \in I} \) of the process \( S \), and for ease of notation, we do not indicate \( S \), but simply write \( \mathcal{F}_t \) for \( \mathcal{F}^S_t \).

For simplicity, we denote by \( Z := \text{Mat}(d \times (T + 1); \mathbb{R}) \) the space of \( d \times (T + 1) \) matrices with real entries representing the space of all possible trajectories of the price process. Namely, for \( \omega \in \Omega \), we have

\[
(S_0(\omega), S_1(\omega), \ldots, S_T(\omega)) = (z_0, z_1, \ldots, z_T) =: z \in Z.
\]
Fix $s \leq t$; for any $z \in \mathbb{Z}$, we denote the components from $s$ to $t$ by $z_{s:t} = (z_s, \ldots, z_t)$ and $z_{t:t} = z_t$. Similarly, $S_{s:t} = (S_s, S_{s+1}, \ldots, S_t)$ represents the process from time $s$ to $t$.

We denote by $\text{ri}(K)$ the relative interior of a set $K \subseteq \mathbb{R}^d$. In this section, we make extensive use of the geometric properties of the image in $\mathbb{R}^d$ of the increments $\Delta S_t := S_t - S_{t-1}$ of the price process relative to a set $\Gamma \subseteq \Omega$. The typical sets that we consider are the level sets $\Gamma = \Sigma_{i-1}^\circ$,

$$
\Sigma_{i-1}^\circ := \{ \omega \in \Omega \mid S_{0:t-1}(\omega) = z_{0:t-1} \} \in \mathcal{F}_{t-1}, \quad z \in \mathbb{Z}, \; t \in I_1, \quad (4.6)
$$

and $\Gamma = A_{i-1}^\circ$, the intersection of the level set $\Sigma_{i-1}^\circ$ with a set $A \in \mathcal{F}_{t-1}$,

$$
A_{i-1}^\circ := \{ \omega \in A \mid S_{0:t-1}(\omega) = z_{0:t-1} \} \in \mathcal{F}_{t-1}. \quad (4.7)
$$

For any $\Gamma \subseteq \Omega$, define the convex cone

$$
(\Delta S_t(\Gamma))^{cc} := \text{co}\left(\text{conv}(\Delta S_t(\Gamma))\right) \cup \{0\} \subseteq \mathbb{R}^d. \quad (4.8)
$$

If $0 \in \text{ri}(\Delta S_t(\Gamma))^{cc}$, then we cannot apply the separating hyperplane theorem to the convex sets $\{0\}$ and $\text{ri}(\Delta S_t(\Gamma))^{cc}$, meaning that there is no $H \in \mathbb{R}^d$ that satisfies $H \cdot \Delta S_t(\omega) \geq 0$ for all $\omega \in \Gamma$ with strict inequality for some of them. As is intuitively evident, and is further shown in Corollary 4.5, $0 \in \text{ri}(\Delta S_t(\Gamma))^{cc}$ if and only if no $1p$-arbitrage is possible on the set $\Gamma$ since a trading strategy on $\Gamma$ with a nonzero payoff always yields both positive and negative outcomes.

In this situation, for $\Gamma = \Sigma_{i-1}^\circ$, the level set is not suitable for the construction of a $1p$-arbitrage opportunity, and such sets are naturally important for the construction of a martingale measure. We wish then to identify, for $\Gamma = \Sigma_{i-1}^\circ$, those subsets of $\Sigma_{i-1}^\circ$ that are not suitable for the construction of a $1p$-arbitrage opportunity. This result is contained in the following key Lemma 4.4.

Observe first that for a convex cone $K \subseteq \mathbb{R}^d$ such that $0 \notin \text{ri}(K)$, we can consider the family $V = \{ v \in \mathbb{R}^d \mid \|v\| = 1 \text{ and } v \cdot y \geq 0 \; \forall \; y \in K \}$ so that

$$
\overline{K} = \bigcap_{v \in V} \{ y \in \mathbb{R}^d \mid v \cdot y \geq 0 \} = \bigcap_{n \in \mathbb{N}} \{ y \in \mathbb{R}^d \mid v_n \cdot y \geq 0 \},
$$

where $\{v_n \mid n \in \mathbb{N}\} = (\mathbb{Q}^d \cap V) \setminus \{0\}$.

**Definition 4.3** Adopting the above notation, we call $\sum_{n=1}^{\infty} \frac{1}{2^n} v_n \in V$ the standard separator.

**Lemma 4.4** Fix $t \in I_1$ and $\Gamma \neq \emptyset$. If $0 \notin \text{ri}(\Delta S_t(\Gamma))^{cc}$, then there exist $\beta \in \{1, \ldots, d\}$, $H^1, \ldots, H^\beta$, $B^1, \ldots, B^\beta$, $B^*$ with $H^i \in \mathbb{R}^d$, $B^i \subseteq \Gamma$, and $B^* := \Gamma \setminus (\bigcup_{j=1}^\beta B^j)$ such that

1) $B^i \neq \emptyset$ for all $i = 1, \ldots, \beta$, and $\{ \omega \in \Gamma \mid \Delta S_t(\omega) = 0 \} \subseteq B^*$, which may be empty;
2) $B^i \cap B^j = \emptyset$ if $i \neq j$;
3) for every \( i \leq \beta \), \( H^i \cdot \Delta S_t(\omega) > 0 \) for all \( \omega \in B^i \) and \( H^i \cdot \Delta S_t(\omega) \geq 0 \) for all \( \omega \in \bigcup_{j=i}^\beta B^j \cup B^* \).

4) for all \( H \in \mathbb{R}^d \) such that \( H \cdot \Delta S_t \geq 0 \) on \( B^* \), we have \( H \cdot \Delta S_t = 0 \) on \( B^* \).

Moreover, for \( z \in \mathbb{Z} \), \( A \in \mathcal{F}_{t-1} \), and \( \Gamma = A_{t-1} \) (or \( \Gamma = \Sigma_{t-1}^z \)), we have \( B^i, B^* \in \mathcal{F}_t \) and

\[
H(\omega) := \sum_{i=1}^\beta H^i 1_{B^i}(\omega) \tag{4.9}
\]

is an \( \mathcal{F}_t \)-measurable random separator that is uniquely determined when we adopt for each \( H^i \) the standard separator. Clearly in these cases, \( \beta, H^i, H, B^i, \) and \( B^* \) depend on \( t \) and \( z \), and whenever necessary, they are denoted by \( \beta_{t,z}, H^i_{t,z}, H_{t,z}, B^i_{t,z}, \) and \( B^*_{t,z} \).

**Proof** Set \( A^0 := \Gamma, \quad K^0 := (\Delta S_t(\Gamma))^{cc} \subseteq \mathbb{R}^d \), and \( \Delta_0 := \{ \omega \in A^0 \mid \Delta S_t(\omega) = 0 \} \), which may be empty.

**Step 1:** The set \( K^0 \subseteq \mathbb{R}^d \) is nonempty and convex, and so \( \text{ri}(K^0) \neq \emptyset \). Since \( 0 \notin \text{ri}(K^0) \), there exists a standard separator \( H^1 \in \mathbb{R}^d \) with the two properties (i) \( H^1 \cdot \Delta S_t(\omega) \geq 0 \) for all \( \omega \in A^0 \); (ii) \( B^1 := \{ \omega \in A^0 \mid H^1 \cdot \Delta S_t(\omega) > 0 \} \neq \emptyset \). Set \( A^1 := A^0 \setminus B^1 = \{ \omega \in A^0 \mid H^1 \cdot \Delta S_t(\omega) = 0 \} \) and let \( K^1 := (\Delta S_t(A^1))^{cc} \), which is a nonempty convex set with \( \text{dim}(K^1) \leq d - 1 \). If \( 0 \in \text{ri}(K^1) \) (this includes the case \( K^1 = \{0\} \)), then the procedure is complete: one cannot separate \( \{0\} \) from the relative interior of \( K^1 \). The conclusion is that \( \beta = 1 \), \( B^* = A^1 = A^0 \setminus B^1 \), which may be empty, and \( \Delta_0 \subseteq B^* \). Notice that if \( K^1 = \{0\} \), then \( B^* = \Delta_0 \), which may be empty. Otherwise:

**Step 2:** If \( 0 \notin \text{ri}(K^1) \), then we find a standard separator \( H^2 \in \mathbb{R}^d \) with \( H^2 \cdot \Delta S_t(\omega) \geq 0 \) for all \( \omega \in A^1 \), and the set \( B^2 := \{ \omega \in A^1 \mid H^2 \cdot \Delta S_t(\omega) > 0 \} \) is nonempty. Set \( A^2 := A^1 \setminus B^2 \) and let \( K^2 := (\Delta S_t(A^2))^{cc} \), which has \( \text{dim}(K^2) \leq d - 2 \). If \( 0 \in \text{ri}(K^2) \) (this includes \( K^2 = \{0\} \)), then the procedure is complete, and we have the conclusions with \( \beta = 2 \), \( B^* = A^1 \setminus B^2 = A^0 \setminus (B^1 \cup B^2) \), and \( \Delta_0 \subseteq B^* \). Notice that if \( K^1 = \{0\} \), then \( B^* = \Delta_0 \). Otherwise:

**Step \( d - 1 \):** If \( 0 \notin \text{ri}(K^{d-2}) \). Define as previously the sets \( B^{d-1} \neq \emptyset \), \( A^{d-1} := (A^{d-2} \setminus B^{d-1}) \), \( K^{d-1} := (\Delta S_t(A^{d-1}))^{cc} \) with \( \text{dim}(K^{d-1}) \leq 1 \). If we have \( 0 \in \text{ri}(K^{d-1}) \), then the procedure is complete. Otherwise:

**Step \( d \):** We necessarily have \( 0 \notin \text{ri}(K^{d-1}) \), so that \( \text{dim}(K^{d-1}) = 1 \), and the convex cone \( K^{d-1} \) necessarily coincides with a half-line with origin in \( 0 \). We find a separator \( H^d \in \mathbb{R}^d \) with \( B^d := \{ \omega \in A^{d-1} \mid H^d \cdot \Delta S_t(\omega) > 0 \} \neq \emptyset \), and the set

\[
B^* := \{ \omega \in A^{d-1} \mid \Delta S_t(\omega) = 0 \} = \{ \omega \in A^0 \mid \Delta S_t(\omega) = 0 \} = \Delta_0
\]

satisfies \( B^* = A^{d-1} \setminus B^d \). Set \( A^d := A^{d-1} \setminus B^d = B^* = \Delta_0 \) and \( K^d := (\Delta S_t(A^d))^{cc} \). Then \( K^d = \{0\} \).

Since \( \text{dim}(\Delta S_t(\Gamma))^{cc} \leq d \), we have at most \( d \) steps. In case \( \beta = d \), we have \( \Gamma = A^0 = \bigcup_{i=1}^d B^i \cup \Delta_0 \). To prove the last assertion, we note that for any fixed \( t \)

and $z$, the $B^i$ are $\mathcal{F}_t$-measurable since $B^i = A^z_{t-1} \cap (f \circ S_t)^{-1}((0, \infty))$, where $f : \mathbb{R}^d \to \mathbb{R}$ is the continuous function $f(x) = H^i \cdot (x - z_{t-1})$ with $H^i \in \mathbb{R}^d$ fixed. □

**Corollary 4.5** Let $t \in I_1$, $z \in \mathbb{Z}$, $A \in \mathcal{F}_{t-1}$, $\Gamma = A^z_{t-1}$. Then $0 \in \mathfrak{r}(\Delta S_t(\Gamma))^{cc}$ if and only if there is no $1p$-arbitrage on $\Gamma$, that is,

$$\forall H \in \mathbb{R}^d \text{ with } H \cdot (S_t - z_{t-1}) \geq 0 \text{ on } \Gamma, \text{ we have } H \cdot (S_t - z_{t-1}) = 0 \text{ on } \Gamma.$$ (4.10)

**Proof** Let $0 \notin \mathfrak{r}(\Delta S_t(\Gamma))^{cc}$. Then Lemma 4.4, 3) with $i = 1$ gives a $1p$-arbitrage $H^1$ on $\Gamma = \bigcup_{j=1}^\beta B^j \cup B^*$ since $B^1 \neq \emptyset$. Conversely, if $0 \in \mathfrak{r}(\Delta S_t(\Gamma))^{cc}$, then we obtain (4.10) from the argument following (4.8). □

**Definition 4.6** For $A \in \mathcal{F}_{t-1}$ and $\Gamma = A^z_{t-1}$, we naturally extend the definition of $\beta_{t,z}$ in Lemma 4.4 to the case of $0 \in \mathfrak{r}(\Delta S_t(\Gamma))^{cc}$ by using

$$\beta_{t,z} = 0 \iff 0 \in \mathfrak{r}(\Delta S_t(\Gamma))^{cc}$$

with $B^0_{t,z} = \emptyset$ and $B^*_{t,z} = A^z_{t-1} \in \mathcal{F}_{t-1}$. In this case, we also extend the definition of the random variable in (4.9) as $H_{t,z}(\omega) \equiv 0$.

**Corollary 4.7** Let $t \in I_1$, $z \in \mathbb{Z}$, $A \in \mathcal{F}_{t-1}$, and $\Gamma = A^z_{t-1}$ with $0 \notin \mathfrak{r}(\Delta S_t(\Gamma))^{cc}$. For any $P \in \mathcal{P}$ such that $P[\Gamma] > 0$, let

$$j := \inf\{1 \leq i \leq \beta \mid P[B^i_{t,z}] > 0\}.$$ 

If $j < \infty$, then the trading strategy $H(s, \omega) := H^j 1_{\Gamma}(\omega) 1_{[t]}(s)$ is a $P$-classical arbitrage.

**Proof** From Lemma 4.4 we obtain that $H^j \cdot \Delta S_t(\omega) > 0$ on $B^i_{t,z}$ with $P[B^i_{t,z}] > 0$, $H^j \cdot \Delta S_t(\omega) \geq 0$ on $\bigcup_{i=j}^{\beta} B^i_{t,z} \cup B^*_{t,z}$, and $P[B^k_{t,z}] = 0$ for $1 \leq k < j$. □

**Remark 4.8** Let $D \subseteq \mathbb{R}^d$ and $C := (D)^{cc} \subseteq \mathbb{R}^d$ be the convex cone generated by $D$. If $0 \in \mathfrak{r}(C)$, then for any $x \in D$, there exist a finite number of elements $x_j \in D$ such that $0$ is a convex combination of $\{x, x_1, \ldots, x_m\}$ with a strictly positive coefficient of $x$. Indeed, fix $x \in D$ and recall that for any convex set $C \subseteq \mathbb{R}^d$, we have

$$\mathfrak{r}(C) := \{z \in C \mid \forall x \in C \exists \varepsilon > 0 \text{ such that } z - \varepsilon(x - z) \in C\}.$$ 

Since $0 \in \mathfrak{r}(C)$ and $x \in D \subseteq C$, we obtain $-\varepsilon x \in C$ for some $\varepsilon > 0$, and thus

$$\frac{\varepsilon}{1+\varepsilon} x + \frac{1}{1+\varepsilon} (-\varepsilon x) = 0.$$ 

Since $-\varepsilon x \in C$, it is then a linear combination with nonnegative coefficients of elements of $D$, and we obtain

$$\frac{\varepsilon}{1+\varepsilon} x + \frac{1}{1+\varepsilon} \sum_{j=1}^{m} \alpha_j x_j = 0,$$

which can be rewritten as

$$\lambda x + \sum_{j=1}^{m} \lambda_j x_j = 0,$$

with $x_j \in D$, $\lambda > 0$, and $\lambda_j \geq 0$. When the set $D \subseteq \mathbb{R}^d$ is the set $\{(\Delta S_t(\omega))_{t \in \Gamma}\}$ of the image points of the increment of the price process for a fixed time $t$, this observation shows that no matter how we choose $\omega \in \Gamma$, we can always construct conditional martingale measure, ◇ Springer
Corollary 4.9 Let $z, t, \Gamma = A_{t-1}$, and $B_{t,z}^*$ be as in Lemma 4.4. Then:

For all $U \subseteq B_{t,z}^*$, $U \in \mathcal{F}$, there exists $Q \in \mathcal{M}(B_{t,z}^*)$ such that $Q[U] > 0$, where $\mathcal{M}(B) = \{Q \in \mathcal{P} | Q[B] = 1 \text{ and } E_Q[S_t | \mathcal{F}_{t-1}] = S_{t-1} \text{ Q-a.s.} \}$ for $B \in \mathcal{F}$.

Proof By Lemma 4.4, 4), there is no 1p-arbitrage restricted to $\Gamma = B_{t,z}^*$. Applying Corollary 4.5, this implies that $0 \in \text{ri}(\Delta S_t(B_{t,z}^*))^{cc}$. Take any $\omega \in U \subseteq B_{t,z}^*$. Applying Remark 4.8 to the set $D := \Delta S_t(B_{t,z}^*)$ and to $x := \Delta S_t(\omega) \in D$, we deduce the existence of $\{\omega_1, \ldots, \omega_m\} \subseteq B_{t,z}^*$ and nonnegative coefficients $\{\lambda_t(\omega_1), \ldots, \lambda_t(\omega_m)\}$ and $\lambda_t(\omega) > 0$ such that

$$\lambda_t(\omega) + \sum_{j=1}^{m} \lambda_t(\omega_j) = 1 \quad \text{and} \quad 0 = \lambda_t(\omega) \Delta S_t(\omega) + \sum_{j=1}^{m} \lambda_t(\omega_j) \Delta S_t(\omega_j).$$

Because $\{\omega_1, \ldots, \omega_m\} \subseteq B_{t,z}^*$ and $\omega \in B_{t,z}^*$, we obtain that $S_{t-1}(\omega_j) = z_{t-1}$ and $S_{t-1}(\omega) = z_{t-1}$. Therefore,

$$0 = \lambda_t(\omega)(S_t(\omega) - z_{t-1}) + \sum_{j=1}^{m} \lambda_t(\omega_j)(S_t(\omega_j) - z_{t-1}),$$

so that $Q[\{\omega\}] = \lambda_t(\omega)$ and $Q[\{\omega_j\}] = \lambda_t(\omega_j)$ for all $j$ give the desired probability. □

Example 4.10 Let $(\Omega, \mathcal{F}) = (\mathbb{R}_+, \mathcal{B}(\mathbb{R}^+))$ and consider a single-period market with $d = 3$ risky assets $S_t = (S_1^t, S_2^t, S_3^t)$ with $t = 0, 1$ and $S_0 = (2, 2, 2)$. Let

$$S_1^t(\omega) = \begin{cases} 1, & \omega \in \mathbb{R}_+ \setminus \mathbb{Q}, \\ 2, & \omega \in \mathbb{Q} \cap [1/2, +\infty), \\ 3, & \omega \in \mathbb{Q} \cap [0, 1/2), \end{cases}$$

$$S_2^t(\omega) = \begin{cases} 2 + \omega^2, & \omega \in \mathbb{R}_+ \setminus \mathbb{Q}, \\ 2, & \omega \in \mathbb{Q} \cap [1/2, +\infty), \end{cases}$$

$$S_3^t(\omega) = \begin{cases} 2, & \omega \in \mathbb{Q} \cap [0, 1/2). \end{cases}$$

Fix $t = 1$ and $z \in \mathbb{Z}$ with $z_0 = S_0$. It is easy to check that in this case, $\beta_{t,z} = 2$ with $B_{t,z}^1 = \mathbb{R}_+ \setminus \mathbb{Q}$, $B_{t,z}^2 = \mathbb{Q} \cap [0, 1/2)$, $B_{t,z}^3 = \mathbb{Q} \cap [1/2, +\infty)$ (cf. Fig. 2). The corresponding strategies $H \in \mathbb{R}^3$ (standard in the sense of Lemma 4.4) are given by $H_{t,z}^1 = (0, 0, 1)$ and $H_{t,z}^2 = (1, 0, 0)$. Note that $H_{t,z}^1$ is a 1p-arbitrage with $\psi_{H_{t,z}^1}^+ = B_{t,z}^1$.

We have therefore that $B_{t,z}^1$ is a nullset with respect to any martingale measure. The
strategy $H^2_{t,z}$ satisfies $V_T(H^2_{t,z}) \geq 0$ on $(B^1_{t,z})^c$ with $H^2_{t,z} = B^2_{t,z}$; hence, $B^2_{t,z}$ is also an $\mathcal{M}$-polar set. This example shows the need of a multiple separation argument, as it is not possible to find a single separating hyperplane in $\mathbb{R}^d$ such that the image points of $B^1_{t,z} \cup B^2_{t,z}$ (which is $\mathcal{M}$-polar) through the random vector $\Delta S$ are strictly contained in one of the associated half-spaces. We have indeed that $B^2_{t,z} \subseteq \{\omega \in \Omega_1 | H^1_{t,z} \cdot (S_1 - S_0) = 0\}$, where $H^1_{t,z}$ is the only 1p-arbitrage in this market.

Corollaries 4.7 and 4.9 show the difference between the sets $B_i$ and $B^*_i$. Restricted to the time interval $[t-1, t]$, a probability measure whose mass is concentrated on $B^*_i$ admits an equivalent martingale measure, whereas for those probabilities that assign positive mass to at least one $B_i$, an arbitrage opportunity can be constructed. We can summarize the possible situations as follows.

**Corollary 4.11** Let $t \in I_1$. For $\Gamma = A^z_{t-1}$ with $A \in \mathcal{F}_{t-1}$ and $\mathcal{M}(B)$ defined in Corollary 4.9, we have:

1) $B^*_i = A^z_{t-1} \iff$ no 1p-arbitrage on $A^z_{t-1} \iff 0 \in \text{ri}(\Delta S_t(A^z_{t-1}))^{cc}$.
2) $B^*_i = \emptyset \iff 0 \notin \text{conv}(\Delta S_t(A^z_{t-1}))$.
3) $\beta_{t,z} = 1$ and $B^*_i \neq \emptyset \implies \exists H \in \mathbb{R}^d \setminus \{0\}$ such that $B^*_i = \{\omega \in A^z_{t-1} | H \cdot (S_t(\omega) - z_{t-1}) = 0\}$ is “martingalizable,”

that is, for all $U \subset B^*_i, U \in \mathcal{F}$, there exists $Q \in \mathcal{M}(B^*_i)$ such that $Q[U] > 0$.

**Proof** The equivalence 1) immediately follows from Corollary 4.5 and Definition 4.6. To show 2), we use the sets $K^i$ for $i = 1, \ldots, \beta_{t,z}$ and the other notations from the proof of Lemma 4.4. Suppose first that $0 \notin \text{conv}(\Delta S_t(\Gamma))$, which implies $0 \notin \text{ri}(\Delta S_t(\Gamma))^{cc}$ and $\Delta_0 = \emptyset$. From the assumption we have $0 \notin \text{conv}(\Delta S_t(C))$ for any subset $C \subseteq \Gamma$; so in particular $0 \notin \text{ri}(K^1)$ unless $K^1 = \{0\}$. This implies $B^*_i = \Delta_0 = \emptyset$.

Suppose now that $0 \in \text{conv}(\Delta S_t(\Gamma))$. If $0 \in \text{ri}(\Delta S_t(\Gamma))^{cc}$, Definition 4.6 gives $B^*_i = \Gamma$, which is nonempty. Suppose then that $0 \notin \text{ri}(\Delta S_t(\Gamma))^{cc}$. Since $0 \in \text{conv}(\Delta S_t(\Gamma))$, there exists $n \geq 1$ such that $0 = \sum_{j=1}^n \lambda_j (S_t(\omega_j) - z_{t-1})$
with \( \sum_{j=1}^{n} \lambda_j = 1, \lambda_j > 0, \) and \( \omega_j \in \Gamma \) for all \( j \). If 0 is an extremal point, then \( n = 1, S_t(\omega_1) - z_{t-1} = 0, \) and \( \omega_1 \in \Delta_0 \subseteq B^*_t, \) for \( n \geq 2, \) then we have \( 0 \in \text{conv}(\Delta S_t(\omega_1, \ldots, \omega_n)), \) so that for any \( H \in \mathbb{R}^d \) that satisfies \( H \cdot \Delta S_t(\omega_i) \geq 0 \) for any \( i = 1, \ldots, n, \) we have \( H \cdot \Delta S_t(\omega_i) = 0. \) Hence, \( \{\omega_1, \ldots, \omega_n\} \subseteq B^*_t, \) by the definition of \( B^*_t, \).

We conclude by showing 3). By Lemma 4.4, 3) and 4), if we select \( H = H^1, \) then

\[
\{\omega \in \Gamma \mid H^1 \cdot (S_t(\omega) - z_{t-1}) = 0\} = \Gamma \setminus B^*_t, \neq \emptyset,
\]

and on \( B^*_t \) we may apply Corollary 4.9. \( \square \)

### 4.4 On \( \mathcal{M} \)-polar sets

We consider for any \( t \in I \) the \( \sigma \)-algebra \( \mathfrak{F}_t := \bigcap_{Q \in \mathcal{M}} \mathcal{F}_t^Q, \) where \( \mathcal{F}_t^Q \) is the \( Q \)-completion of \( \mathcal{F}_t. \) This \( \mathfrak{F}_t \) is the so-called universal completion of \( \mathcal{F}_t \) with respect to \( \mathcal{M} = \mathcal{M}(\mathfrak{F}). \) Notice that the introduction of this enlarged filtration needs the knowledge a priori of the whole class \( \mathcal{M} \) of martingale measures. Recall that any measure \( Q \in \mathcal{M} \) can be uniquely extended to a measure \( \overline{Q} \) on the enlarged \( \sigma \)-algebra \( \mathfrak{F}_T \) so that we can write with slight abuse of notation \( \mathcal{M}(\mathfrak{F}) = \mathcal{M}(\mathfrak{F}), \) where \( \mathfrak{F} := (\mathfrak{F}_t)_{t \in I}. \)

We wish to show now that under any martingale measure, the sets \( B^i_t \) (and their arbitrary unions) introduced in Lemma 4.4 must be nullsets. To this end, we need to recall some properties of a regular conditional probability (see Theorems 1.1.6–1.1.8 in Stroock and Varadhan [32]).

**Theorem 4.12** Let \( (\Omega, \mathcal{F}, Q) \) be a probability space, where \( \Omega \) is a Polish space, \( \mathcal{F} \) is the Borel \( \sigma \)-algebra, \( Q \in \mathcal{P}. \) Let \( \mathcal{A} \subseteq \mathcal{F} \) be a countably generated sub-\( \sigma \)-algebra of \( \mathcal{F}. \) Then there exists a regular conditional probability given \( \mathcal{A}, \) i.e., a function \( Q_A(\cdot, \cdot) : (\Omega, \mathcal{F}) \to [0, 1] \) such that

- a) for all \( \omega \in \Omega, Q_A(\omega, \cdot) \) is a probability measure on \( \mathcal{F}; \)
- b) for each \( B \in \mathcal{F}, \) the function \( Q_A(\cdot, B) \) is a version of \( Q[B | \mathcal{A}](\cdot); \)
- c) \( \forall N \subseteq \mathcal{A} \text{ with } Q[N] = 0 \text{ such that } Q_A(\omega, B) = 1_B(\omega) \text{ for } \omega \in \Omega \setminus N \text{ and } B \subseteq \mathcal{A}; \)
- d) in addition, if \( X \in L^1(\Omega, \mathcal{F}, Q), \) then \( E_Q[X | \mathcal{A}](\omega) = \int_{\Omega} X(\tilde{\omega}) Q_A(\omega, d\tilde{\omega}) \)

\( Q \)-a.s.

Since \( \mathbb{R}^S \) is generated by \( S, \) which has values in the separable space \( \mathbb{R}^d, \) the filtration \( \mathbb{F} = \mathbb{F}^S \) is countably generated.

**Lemma 4.13** Fix \( t \in I_1 = \{1, \ldots, T\}, A \in \mathcal{F}_{t-1}, Q \in \mathcal{M}, \) and for \( z \in \mathbb{Z}, \) consider the set \( A^*_t := \{\omega \in A \mid S_{0:t-1}(\omega) = z_{0:t-1}\}. \) Then

\[
\bigcup_{z \in \mathbb{Z}} \left\{ \omega \in A^*_t \text{ such that } Q_{\mathcal{F}_{t-1}}(\omega, \bigcup_{i=1}^n B^i_t, z) > 0 \right\}
\]

is a subset of an \( \mathcal{F}_{t-1} \)-measurable \( Q \)-nullset.
Proof If $Q[A] = 0$, then there is nothing to show. Suppose now that $Q[A] > 0$. In this proof, we set for the sake of simplicity $X := S_t$, $Y := E_Q[X | \mathcal{F}_{t-1}] = S_{t-1} \ Q$-a.s., $\beta := \beta_{t,z}$, and $\mathcal{A} := \mathcal{F}_{t-1} = \mathcal{F}_{t-1}^S$. Set

$$D_i^\gamma := \{ \omega \in A_{t-1}^\gamma \ such \ that \ Q_A(\omega, \bigcup_{i=1}^{\beta} B_{i,z}^i) > 0 \}.$$ 

If $z \in \mathbb{Z}$ is such that $0 \in \text{ri}(\Delta S_t(A_{t-1}^\gamma))^{cc}$, then $\bigcup_{i=1}^{\beta} B_{i,z}^i = \emptyset$ and $D_i^\gamma = \emptyset$. So we can consider only those $\omega \in \mathbb{Z}$ such that $0 \not\in \text{ri}(\Delta S_t(A_{t-1}^\gamma))^{cc}$. Fix such a $\omega$. Since $A = \mathcal{F}_{t-1}^S$ is countably generated, $Q$ admits a regular conditional probability $Q_A$. From Theorem 4.12 d) we obtain

$$Y(\omega) = \int_{\Omega} X(\tilde{\omega}) Q_A(\omega, d\tilde{\omega}) \quad Q\text{-a.s.}$$

Since $A_{t-1}^\gamma \in \mathcal{A}$, by Theorem 4.12 c) there exists a set $N \in \mathcal{A}$ with $Q[N] = 0$ such that $Q_A(\omega, A_{t-1}^\gamma) = 1$ on $A_{t-1}^\gamma \setminus N$, and therefore we have

$$\int_{A_{t-1}^\gamma} X(\tilde{\omega}) Q_A(\omega, d\tilde{\omega}) = \int_{A_{t-1}^\gamma} X(\tilde{\omega}) Q_A(\omega, d\tilde{\omega}) \quad \forall \omega \in A_{t-1}^\gamma \setminus N. \quad (4.11)$$

Since $0 \not\in \text{ri}(\Delta S_t(\Gamma))^{cc}$, we may apply Lemma 4.4. For any $i = 1, \ldots, \beta$, there exists $H^i \in \mathbb{R}^d$ such that $H^i \cdot (X(\tilde{\omega}) - z_{t-1}) \geq 0$ for all $\tilde{\omega} \in \bigcup_{i=1}^{\beta} B_{i,z}^i \cup B_{i,z}^*$ and $H^i \cdot (X(\tilde{\omega}) - z_{t-1}) > 0$ for every $\tilde{\omega} \in B_{i,z}^i$. Now we fix $\omega \in D_i^\gamma \setminus N \subseteq A_{t-1}^\gamma \setminus N$. Then the index $j := \min\{1 \leq i \leq \beta | Q_A(\omega, B_{i,z}^i) > 0\}$ is well defined, and we have

1. $H^j \cdot (X(\tilde{\omega}) - z_{t-1}) > 0$ on $B_{i,z}^i$;
2. $Q_A(\omega, B_{i,z}^i) > 0$;
3. $H^j \cdot (X(\tilde{\omega}) - z_{t-1}) \geq 0$ on $\bigcup_{i=j}^{\beta} B_{i,z}^i \cup B_{i,z}^*$;
4. $Q_A(\omega, B_{i,z}^i) = 0$ for $i < j$.

From (i) and (ii) we obtain

$$Q_A(\omega, A_{t-1}^\gamma \cap \{H^j \cdot (X - z_{t-1}) > 0\}) \geq Q_A(\omega, B_{i,z}^i) > 0.$$ 

From (iii) and (iv) we obtain

$$Q_A(\omega, \{H^j \cdot (X - z_{t-1}) \geq 0\}) \geq Q_A(\omega, \bigcup_{i=j}^{\beta} B_{i,z}^i \cup B_{i,z}^*) \geq Q_A(\omega, A_{t-1}^\gamma) - Q_A(\omega, \bigcup_{i<j} B_{i,z}^i) = 1.$$ 

Hence,

$$H^j \cdot \left( \int_{A_{t-1}^\gamma} X(\tilde{\omega}) Q_A(\omega, d\tilde{\omega}) - z_{t-1} \right) = \int_{A_{t-1}^\gamma} H^j \cdot (X(\tilde{\omega}) - z_{t-1}) Q_A(\omega, d\tilde{\omega}) > 0,$$
and therefore, from (4.11) and \(z_{t-1} = Y(\omega)\) we have

\[
H^j \cdot \left( \int_\Omega X(\tilde{\omega})Q_A(\omega, d\tilde{\omega}) - Y(\omega) \right) > 0.
\]

Since this holds for any \(\omega \in D_i^z \setminus N\), we obtain

\[
D_i^z \setminus N \subseteq \left\{ \omega \in \Omega \mid Y(\omega) \neq \int_\Omega X(\tilde{\omega})Q_A(\omega, d\tilde{\omega}) \right\} =: N^* \in \mathcal{F}_{t-1}
\]

with \(Q[N^*] = 0\). Hence, \(D_i^z \subseteq N \cup N^* := N_0\) with \(Q[N_0] = 0\) and \(N_0\) not dependent on \(z\). Since this holds for every \(z \in \mathbb{Z}\), we conclude that \(\bigcup_{z \in \mathbb{Z}} D_i^z \subseteq N_0\). \(\square\)

**Corollary 4.14** Fix \(t \in I_1\) and \(Q \in \mathcal{M}\). If

\[
\mathcal{B}_t := \bigcup_{z \in \mathbb{Z}} \bigcup_{i=1}^{\beta_{t,z}} B_{i,z}^i
\]

for \(B_{i,z}^i\) given in Lemma 4.4 with \(\Gamma = \Sigma_{t-1}^z\) or \(\Gamma = A_{t-1}^z\) (defined in (4.6) and (4.7)), then \(\mathcal{B}_t\) is a subset of an \(\mathcal{F}_t\)-measurable \(Q\)-nullset.

**Proof** First, we consider the case \(\Gamma = \Sigma_{t-1}^z\) and \(B_{i,z}^i\) given in Lemma 4.4 with \(\Gamma = \Sigma_{t-1}^z\). As in the previous proof, we denote the \(\sigma\)-algebra \(\mathcal{F}_{t-1}\) with \(\mathcal{A} := \mathcal{F}_{t-1}\).

Notice that if \(z \in \mathbb{Z}\) is such that \(0 \in \text{ri}(\Delta S_t(\Gamma))^c\), then \(\bigcup_{i=1}^{\beta_{t,z}} B_{i,z}^i = \emptyset\); hence, we may assume that \(0 \notin \text{ri}(\Delta S_t(\Gamma))^c\). From the proof of Lemma 4.13 we have

\[
\bigcup_{z \in \mathbb{Z}} D_i^z \subseteq N_0 = N \cup N^*
\]

with \(Q[N_0] = 0\). Notice that if \(\omega \in \Omega \setminus N_0\), then for all \(z \in \mathbb{Z}\), either \(\omega \notin \Sigma_{t-1}^z\) or \(Q_A(\omega, \bigcup_{i=1}^{\beta_{t,z}} B_{i,z}^i) = 0\). Hence, \(\omega \in \Sigma_{t-1}^z \setminus N_0\) implies \(Q_A(\omega, \bigcup_{i=1}^{\beta_{t,z}} B_{i,z}^i) = 0\). By Theorem 4.12 c) we have \(Q_A(\omega, (\Sigma_{t-1}^z)^c) = 0\) for all \(\omega \in \Sigma_{t-1}^z \setminus N_0\).

Fix now \(\omega \in \Sigma_{t-1}^z \setminus N_0\) and consider the completion \(\mathcal{F}_t^{Q_A(\omega, \cdot)}\) of \(\mathcal{F}_t\) and the unique extension on \(\mathcal{F}_t^{Q_A(\omega, \cdot)}\) of \(Q_A(\omega, \cdot)\), which we call \(\widehat{Q}_A(\omega, \cdot) : \mathcal{F}_t^{Q_A(\omega, \cdot)} \to [0, 1]\). Since we have \(Q_A(\omega, (\Sigma_{t-1}^z)^c) = 0\), we deduce that \(\mathcal{B}_t \cap (\Sigma_{t-1}^z)^c \in \mathcal{F}_t^{Q_A(\omega, \cdot)}\) and \(\widehat{Q}_A(\omega, \mathcal{B}_t \cap (\Sigma_{t-1}^z)^c) = 0\). Since \(\mathcal{B}_t \cap (\Sigma_{t-1}^z)^c = \bigcup_{i=1}^{\beta_{t,z}} B_{i,z}^i\) and \(Q_A(\omega, \bigcup_{i=1}^{\beta_{t,z}} B_{i,z}^i) = 0\), we deduce \(\mathcal{B}_t \cap (\Sigma_{t-1}^z) \in \mathcal{F}_t^{Q_A(\omega, \cdot)}\) and \(\widehat{Q}_A(\omega, \mathcal{B}_t \cap (\Sigma_{t-1}^z) = 0\). Then

\[
\mathcal{B}_t = (\mathcal{B}_t \cap (\Sigma_{t-1}^z) \cup (\mathcal{B}_t \cap (\Sigma_{t-1}^z)^c) \in \mathcal{F}_t^{Q_A(\omega, \cdot)}
\]

and \(\widehat{Q}_A(\omega, \mathcal{B}_t) = 0\). Since \(\omega \in \Sigma_{t-1}^z \setminus N_0\) was arbitrary, we have shown that \(\widehat{Q}_A(\omega, \mathcal{B}_t) = 0\) for all \(\omega \in \Sigma_{t-1}^z \setminus N_0\) and \(z \in \mathbb{Z}\). Since \(\bigcup_{z \in \mathbb{Z}} (\Sigma_{t-1}^z \setminus N_0) = \Omega \setminus N_0\),

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we have
\[ \mathcal{B}_t \in \mathcal{F}_t^Q A(\omega, \cdot) \quad \text{and} \quad \tilde{Q}_A(\omega, \mathcal{B}_t) = 0 \quad \text{for all} \quad \omega \in \Omega \setminus N_0 \quad \text{with} \quad Q[N_0] = 0. \]

Now consider the \( \sigma \)-algebra
\[ \tilde{\mathcal{F}}_t = \bigcap_{\omega \in \Omega \setminus N_0} \mathcal{F}_t^Q A(\omega, \cdot) \]
and observe that \( \mathcal{B}_t \in \tilde{\mathcal{F}}_t \). Notice that if a subset \( B \subseteq \Omega \) satisfies \( B \subseteq C \) for some \( C \in \mathcal{F}_t \) with \( Q_A(\omega, C) = 0 \) for all \( \omega \in \Omega \setminus N_0 \), then
\[ Q[C] = \int_{\Omega} Q_A(\omega, C) Q(d\omega) = \int_{\Omega \setminus N_0} Q_A(\omega, C) Q(d\omega) = 0, \]
so that \( B \in \mathcal{F}_t^Q \). This shows that \( \mathcal{F}_t \subseteq \tilde{\mathcal{F}}_t \subseteq \mathcal{F}_t^Q \). Hence, we get \( \mathcal{B}_t \in \mathcal{F}_t^Q \). Let \( \tilde{Q} : \tilde{\mathcal{F}}_t \rightarrow [0, 1] \) be defined by \( \tilde{Q}[: \omega) := \int_{\Omega} Q_A(\omega, \cdot) Q(d\omega) \). Then \( \tilde{Q} \) is a probability that satisfies \( \tilde{Q}[B] = Q[B] \) for every \( B \in \mathcal{F}_t \) and therefore is an extension on \( \tilde{\mathcal{F}}_t \) of \( Q \). Since \( \tilde{Q} : \mathcal{F}_t^Q \rightarrow [0, 1] \) is the unique extension on \( \mathcal{F}_t^Q \) of \( Q \) and \( \mathcal{F}_t \subseteq \tilde{\mathcal{F}}_t \subseteq \mathcal{F}_t^Q \), then \( \tilde{Q} \) is the restriction of \( \tilde{Q} \) on \( \tilde{\mathcal{F}}_t \), and
\[ \tilde{Q}[\mathcal{B}_t] = \tilde{Q}[\mathcal{B}_t] = \int_{\Omega} \tilde{Q}_A(\omega, \mathcal{B}_t) Q(d\omega) = \int_{\Omega \setminus N_0} \tilde{Q}_A(\omega, \mathcal{B}_t) Q(d\omega) = 0. \]

Suppose now that \( A \in \mathcal{F}_{t-1} \), \( \Gamma = A^z_{t-1} \) and set \( \mathcal{C}_t := \bigcup_{i \in \mathbb{Z}} \bigcup_{\beta_{t,z}} B^i_{t,z} \), where \( B^i_{t,z} \) is given in Lemma 4.4 with \( \Gamma = A^z_{t-1} \). Fix any \( \omega \in A \). Then \( \Sigma^t_0 S_{t}^{\phi} (\omega) \subseteq A \) since \( A \in \mathcal{F}_{t-1} \). As a consequence, \( \mathcal{C}_t \subseteq \mathcal{B}_t \).

**Corollary 4.15** Fix \( t \in I_1 = \{1, \ldots, T\} \) and for any set \( A \in \mathcal{F}_{t-1} \), consider the set \( A^z_{t-1} = \{ \omega \in A_1 \mid S_{t-1}(\omega) = z_{t-1} \} \neq \emptyset \). Then for any \( Q \in \mathcal{M} \), the set
\[ \bigcup_{\beta_{t,z}} A^z_{t-1} \big[ 0 \not\in \text{conv}(\Delta S_t(A^z_{t-1})) \big] \] is a subset of an \( \mathcal{F}_{t-1} \)-measurable \( Q \)-nullset and, as a consequence, is an \( \mathcal{M} \)-polar set.

**Proof** By Corollary 4.11, \( 0 \not\in \text{conv}(\Delta S_t(A^z_{t-1})) \) implies that \( \bigcup_{\beta_{t,z}} B^i_{t,z} \cap A^z_{t-1} \) is a subset of an \( \mathcal{F}_{t-1} \)-measurable \( Q \)-nullset.

From Theorem 4.12 we know that we have \( Q_A(\omega, A^z_{t-1}) = 1 \) on \( A^z_{t-1} \setminus N \) and \( D^z_t = \{ \omega \in A^z_{t-1} \mid Q_{F_{t-1}}(\omega, A^z_{t-1}) > 0 \} \supseteq A^z_{t-1} \setminus N \) and
\[ \bigcup_{\beta_{t,z}} A^z_{t-1} \big[ 0 \not\in \text{conv}(\Delta S_t(A^z_{t-1})) \big] \setminus N \subseteq \bigcup_{\beta_{t,z}} D^z_t \subseteq N \in \mathcal{F}_{t-1}. \]

**4.4.1 Backward effect in the multiperiod case**

The following example shows that additional care is required in the multiperiod setting.
Example 4.16 Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and consider a single risky asset $S_t$ with $t = 0, 1, 2$, where

$$S_0 = 7, \quad S_1(\omega) = \begin{cases} 8, & \omega \in \{\omega_1, \omega_2\}, \\ 3, & \omega \in \{\omega_3, \omega_4\}, \end{cases} \quad S_2(\omega) = \begin{cases} 9, & \omega = \omega_1, \\ 6, & \omega = \omega_2, \\ 5, & \omega = \omega_3, \\ 4, & \omega = \omega_4. \end{cases}$$

Fix $z \in \mathbb{Z}$ with the first two components $(z_0, z_1)$ equal to $(7, 3)$.

First period: $\Sigma^z_0 = \Omega$ and $0 \in \text{ri}(\text{conv}(\Delta S_1(\Sigma^z_0))) = (-4, 1)$, there exists $Q_1$ such that $Q_1[[\omega_i]] > 0$ for $i = 1, 2, 3, 4$, and $S_0 = E_{Q_1}[S_1]$. If we restrict the problem to the first period only, then there exists a full support martingale measure for $(S_0, S_1)$, and there are no $\mathcal{M}$-polar sets.

Second period: $\Sigma^z_1 = \{\omega_3, \omega_4\}$, $0 \notin \text{conv}(\Delta S_2(\Sigma^z_1)) = [1, 2]$, and hence $\Sigma^z_1$ is not supported by any martingale measure for $S$, that is, if $Q \in \mathcal{M}$, then $Q[[\omega_3, \omega_4]] = 0$.

Backward: Since $\{\omega_3, \omega_4\}$ is a $Q$-nullset for any $Q \in \mathcal{M}$, $Q[[\omega_1, \omega_2]] = 1$. This reflects into the first period by means of $0 \notin \text{conv}(\Delta S_1(\{\omega_1, \omega_2\})) = \emptyset$, and we deduce that also $\{\omega_1, \omega_2\}$ is not supported by any martingale measure, implying $\mathcal{M} = \emptyset$.

Example 4.16 shows that new $\mathcal{M}$-polar sets (as $\{\omega_3, \omega_4\}$) can arise at later times, creating a backward effect on the existence of martingale measures. In order to detect these situations at time $t$, we need to anticipate certain polar sets at posterior times. More formally, we need to consider the following iterative procedure. Let

$$\Omega_T := \Omega, \quad \Omega_{t-1} := \Omega_t \setminus \bigcup_{z \in \mathbb{Z}} \{\Sigma^z_{t-1} \mid 0 \notin \text{conv}(\Delta S_t(\Sigma^z_{t-1}))\}, \quad t \in I_1, \quad (4.12)$$

$$\Gamma = \Sigma_t^z.$$ 

We show that the set $B^i_{t,z}$ obtained from Lemma 4.4 with $\Gamma = \Sigma_t^z$ belongs to the family of polar sets of $\mathcal{M}(\mathcal{F})$, that is, to

$$N := \{A \subseteq A' \in \mathcal{F} \mid Q[A'] = 0 \ \forall \ Q \in \mathcal{M}(\mathcal{F})\}.$$ 

More precisely:

**Lemma 4.17** For all $t \in I_1$ and $z \in \mathbb{Z}$, consider the sets $B^i_{t,z}$ from Lemma 4.4 with $\Gamma = \Sigma_t^z$. Let

$$\Sigma_{t-1} := \bigcup_{z \in \mathbb{Z}} \{\Sigma^z_{t-1} \mid 0 \notin \text{conv}(\Delta S_t(\Sigma^z_{t-1}))\}.$$

For any $Q \in \mathcal{M}$, $\tilde{\mathcal{B}}_t$ is a subset of an $\mathcal{F}_t$-measurable $Q$-null set, and $\mathcal{D}_{t-1}$ is a subset of an $\mathcal{F}_{t-1}$-measurable $Q$-null set.
Proof We prove this by backward induction. For \( t = T \), the assertion is true from Corollaries 4.14 and 4.15. Suppose now the claim holds for any \( k + 1 \leq t \leq T \). By the inductive hypothesis there exists \( N^Q_k \in \mathcal{F}_k \) such that \( \mathcal{D}_k \subseteq N^Q_k \) with \( Q[N^Q_k] = 0 \). Introduce the auxiliary \( \mathcal{F}_k \)-measurable random variable

\[
X^Q_k := S_{k-1} 1_{N^Q_k} + S_k 1_{(N^Q_k)^c} \tag{4.14}
\]

and notice that \( E_Q[X^Q_k | \mathcal{F}_{k-1}] = S_{k-1} \) Q-a.s. From \( \Delta X^Q_k := X^Q_k - S_{k-1} = 0 \) on \( N^Q_k \) and \( \Omega \setminus N^Q_k \subseteq \Omega \setminus \mathcal{D}_k \) we can deduce that

\[
0 \notin \text{ri}(\Delta S_k(\Sigma^z_{k-1}))^c \implies 0 \notin \text{ri}(\Delta X^Q_k(\Sigma^z_{k-1}))^c, \tag{4.15}
\]

which implies \( \widehat{\mathcal{B}}_k \subseteq \mathcal{B}_k(X^Q_k) \cup N^Q_k \), where we denote by \( \mathcal{B}_k(X^Q_k) \) the set obtained from Corollary 4.14 with \( \Gamma = \Sigma^z_{k-1} \) and \( X^Q_k \) that replaces \( S_k \). Due to Corollary 4.14, we find \( M^Q_k \in \mathcal{F}_k \) with \( Q[M^Q_k] = 0 \), so that \( \widehat{\mathcal{B}}_k \subseteq \mathcal{B}_k(X^Q_k) \cup N^Q_k \subseteq M^Q_k \cup N^Q_k \). Since \( Q \) is arbitrary, we have the thesis.

We now show the second assertion. For any \( Q \in \mathcal{M} \) and \( \xi = (\varepsilon, \ldots, \varepsilon) \in \mathbb{R}^d \) with \( \varepsilon > 0 \), define

\[
S^Q_k = (S_{k-1} + \varepsilon) 1_{N^Q_k \cup M^Q_k} + S_k 1_{(N^Q_k \cup M^Q_k)^c} \tag{4.16}
\]

and note that \( E_Q[S^Q_k | \mathcal{F}_{k-1}] = S_{k-1} \). With \( \Delta S^Q_k := S^Q_k - S_{k-1} \), we claim that

\[
\mathcal{D}_{k-1} \subseteq \bigcup_{z \in \mathbb{Z}} \{ \Sigma^z_{k-1} | 0 \notin \text{conv}(\Delta S^Q_k(\Sigma^z_{k-1})) \}.
\]

Indeed, let \( z \in \mathbb{Z} \) be such that \( \Sigma^z_{k-1} \subseteq \mathcal{D}_{k-1} \) and observe that

\[
0 \notin \text{conv}(\Delta S_k(\Sigma^z_{k-1})) \iff 0 \notin \text{conv}(\Delta S^Q_k(\Sigma^z_{k-1}) \setminus \mathcal{D}_k)).
\]

Since \( \Sigma^z_{k-1} \setminus N^Q_k \subseteq \Sigma^z_{k-1} \setminus \mathcal{D}_k \subseteq \widehat{\mathcal{B}}_k \subseteq N^Q_k \cup M^Q_k \), we have

\[
\Sigma^z_{k-1} = (\Sigma^z_{k-1} \cap N^Q_k) \cup (\Sigma^z_{k-1} \setminus N^Q_k) \subseteq N^Q_k \cup M^Q_k
\]

\[
\subseteq \bigcup_{z \in \mathbb{Z}} \{ \Sigma^z_{k-1} | 0 \notin \text{conv}(\Delta S^Q_k(\Sigma^z_{k-1})) \}
\]

for any \( \Sigma^z_{k-1} \subseteq \mathcal{D}_{k-1} \). So the claim follows since

\[
\bigcup_z \{ \Sigma^z_{k-1} | 0 \notin \text{conv}(\Delta S^Q_k(\Sigma^z_{k-1})) \}
\]

is a subset of an \( \mathcal{F}_{k-1} \)-measurable \( Q \)-nullset. \( \square \)
4.5 On the maximal $\mathcal{M}$-polar set and the support of martingale measures

The sets introduced in Sects. 4.3 and 4.4.1 provide a geometric decomposition of $\Omega$ into two parts, $\Omega = \Omega_\ast \cup \Omega_\ast^c$, as specified further in Proposition 4.18. The set $\Omega_\ast$ contains the events $\omega$ supported by martingale measures, namely, for any of those events, it is possible to construct a martingale measure (even with finite support) that assigns positive probability to $\{\omega\}$. Observe that this decomposition is induced by $S$ and determined prior to arbitrage considerations.

**Proposition 4.18** Let $(\Omega_t)_{t \in I}$ be as in (4.12) and (4.13), and for any $z \in \mathbb{Z}$, let $\beta_{t,z}$ and $B_{t,z}^*$ be the index $\beta$ and the set $B^*$ from Lemma 4.4 with $\Gamma = \tilde{\Sigma}_{t-1}^z$. Define

$$
\Omega_\ast := \bigcap_{t=1}^{T} \bigcup_{z \in \mathbb{Z}} B_{t,z}^*.
$$

Then we have

$$
\mathcal{M} \neq \emptyset \iff \Omega_\ast \neq \emptyset \iff \mathcal{M} \cap \mathcal{P}_f \neq \emptyset,
$$

where

$$
\mathcal{P}_f := \{ P \in \mathcal{P} | \text{supp}(P) \text{ is finite} \}
$$

is the set of probability measures whose support is a finite number of points $\omega \in \Omega$. If $\mathcal{M} \neq \emptyset$, then for any $\omega_\ast \in \Omega_\ast$, there exists $Q \in \mathcal{M}$ such that $Q[\{\omega_\ast\}] > 0$, so that $\Omega_\ast^c$ is the maximal $\mathcal{M}$-polar set, that is, $\Omega_\ast^c$ is an $\mathcal{M}$-polar set, and

for all $N \in \mathcal{N}$, we have $N \subseteq \Omega_\ast^c$. \hfill (4.17)

**Proof** Observe first that

$$
\Omega_\ast^c = \bigcup_{t=1}^{T} \tilde{\mathcal{B}}_t.
$$

By Lemma 4.17, $\tilde{\mathcal{B}}_t$ is an $\mathcal{M}$-polar set for any $t \in I_1$, which implies that $\Omega_\ast^c$ is an $\mathcal{M}$-polar set. Suppose first that $\Omega_\ast = \emptyset$, so that $\Omega = \bigcup_{t=1}^{T} \tilde{\mathcal{B}}_t$ is a polar set. We can conclude that $\mathcal{M} = \emptyset$.

Suppose now that $\Omega_\ast \neq \emptyset$. We show that for every $\omega_\ast \in \Omega_\ast$, there exists $Q \in \mathcal{M}$ such that $Q[\{\omega_\ast\}] > 0$. Observe that for any $t \in I_1$ and $\omega \in \Omega_\ast$, $0 \in \text{ri}(\Delta S_t(B_{t,z}^*))^c$ with $z = S_{0:T}(\omega)$. As in Corollary 4.9, we apply Remark 4.8 and conclude that there exist a finite number of elements of $B_{t,z}^*$, named $C_t(\omega) := \{\omega, \omega_1, \ldots, \omega_m\} \subseteq B_{t,z}^*$, such that

$$
S_{t-1}(\omega) = \lambda_t(\omega)S_t(\omega) + \sum_{j=1}^{m} \lambda_t(\omega_j)S_t(\omega_j), \hfill (4.18)
$$

where $\lambda_t(\omega) > 0$ and $\lambda_t(\omega) + \sum_{j=1}^{m} \lambda_t(\omega_j) = 1.$
Fix now \( \omega_* \in \Omega_* \). We iteratively build a set \( \Omega^T_f \) suitable for being the finite support of a discrete martingale measure (and containing \( \omega_* \)). Start with \( \Omega^1_f = C_1(\omega_*) \), which satisfies (4.18) for \( t = 1 \). For any \( t > 1 \), given \( \Omega^{t-1}_f := \{ C_t(\omega) \mid \omega \in \Omega^{t-1}_f \} \). Once \( \Omega^T_f \) is settled, it is easy to construct a martingale measure via (4.18) by

\[
Q[\{\omega\}] = \prod_{t=1}^{T} \lambda_t(\omega) \forall \omega \in \Omega^T_f.
\]

Since \( \lambda_t(\omega_*) > 0 \) for any \( t \in I_1 \) by construction, we have \( Q[\{\omega_*\}] > 0 \) and \( Q \in \mathcal{M} \cap \mathcal{P}_f \).

To show (4.17), notice that for every \( \omega_* \in \Omega_* \), there exists \( Q \in \mathcal{M} \) such that \( Q[\{\omega_*\}] > 0 \), and therefore \( \Omega_* \) is not \( \mathcal{M} \)-polar, whereas \( \Omega^c_* = \bigcup_{t=1}^T \tilde{B}_t \) is \( \mathcal{M} \)-polar thanks to Lemma 4.17.

**Proof of Proposition 4.1** The absence of 1p-arbitrages readily implies that \( \Omega_* = \Omega \) (see Corollary 4.11). Take a dense subset \( \{ \omega_n \mid n \in \mathbb{N} \} \) of \( \Omega_* \); by Proposition 4.18, for any \( \omega_n \), there exists a martingale measure \( Q^n \in \mathcal{M} \) such that \( Q^n[\{\omega_n\}] > 0 \). By Lemma A.9 in Appendix A we obtain \( Q := \sum_{i=1}^{\infty} \frac{1}{2^i} Q^i \in \mathcal{M} \), and moreover \( Q[\{\omega_n\}] > 0 \forall n \in \mathbb{N} \). Since \( \{ \omega_n \mid n \in \mathbb{N} \} \) is dense, \( Q \) is a full support martingale measure.

### 4.6 Enlarged filtration and universal arbitrage aggregator

In Sects. 4.3 and 4.4, we solve the problem of characterizing the \( \mathcal{M} \)-polar sets of a certain market model on a fixed time interval \([t-1, t]\) for \( t \in I_1 = \{1, \ldots, T\} \). In particular, if we look at the level sets \( \Sigma^z_{t-1} \) of the price process \((S_t)_{t \in I}\), then we can identify the component of these sets that must be polar (Corollary 4.14), which coincides with the whole level set when \( 0 \not\in \text{conv}(\Delta S_t(\Sigma^z_{t-1})) \) (Corollary 4.15). Further care is required in the multiperiod case due to the backward effects (see Sect. 4.4.1), but nevertheless a full characterization of \( \mathcal{M} \)-polar sets is obtained in Sect. 4.5.

In this section, we build a predictable strategy that embraces all the inefficiencies of the market. Unfortunately, even on a single time step, the polar set given by Corollary 4.14 belongs in general to \( \mathcal{F}_t \) (the universal \( \mathcal{M} \)-completion); hence, the trading strategies suggested by (4.9) in Lemma 4.4 fail to be predictable. This reflects the necessity of enlargement of the original filtration by anticipating some one-step-ahead information. Under this filtration enlargement, which depends only on the underlying structure of the market, the set of martingale measures will not change (see Lemma 4.25).

**Definition 4.19** We call **universal arbitrage aggregator** the strategy

\[
H^*_t(\omega)1_{\Sigma^z_{t-1}} := \begin{cases} 
H_{t,z}(\omega) & \text{on } \bigcup_{i=1}^{\beta_{t,z}} B^i_{t,z}, \\
0 & \text{on } \Sigma^z_{t-1} \setminus \bigcup_{i=1}^{\beta_{t,z}} B^i_{t,z},
\end{cases}
\]

for \( t \in I_1 = \{1, \ldots, T\} \), where \( z \in \mathbb{Z} \) satisfies \( z_0:t-1 = S_0:t-1(\omega) \), and \( H_{t,z}, B^i_{t,z}, B^*_{t,z} \) come from (4.9) and Lemma 4.4 with \( \Gamma = \tilde{\Sigma}^z_{t-1} \).
This strategy is predictable with respect to the enlarged filtration \( \tilde{F} = (\tilde{F}_t)_{t \in I} \) given by
\[
\tilde{F}_t := F_t \cup \sigma(H^*_1, \ldots, H^*_{t+1}), \quad t \in \{0, \ldots, T - 1\},
\]
\[
\tilde{F}_T := F_T \cup \sigma(H^*_1, \ldots, H^*_T).
\]

**Remark 4.20** The strategy \( H^* \) in (4.19) satisfies \( V_T(H^*) \geq 0 \) and
\[
V^+_{H^*} = \bigcup_{t=1}^T \tilde{B}_t.
\]

Indeed, by Lemma 4.4, \( H_{t,z} \cdot \Delta S_t > 0 \) on \( \bigcup_{i=1}^{\beta_{t,z}} B^i_{t,z} \), so that \( \bigcup_{i=1}^{T} \tilde{B}_t \subseteq V^+_{H^*} \). On the other hand, \( V^+_{H^*} \subseteq \{H^*_t \neq 0 \text{ for some } t\} \subseteq \bigcup_{i=1}^{T} \tilde{B}_t \). For \( t < T \), we therefore conclude that \( \tilde{F}_t \subseteq F_t \cup \bigcup_{s=1}^{t+1} \mathcal{N}_s \subseteq \mathcal{F}_t \), where
\[
\mathcal{N}_t := \left\{ A = \bigcup_{z \in V, i \in J(z)} B^i_{t,z} \mid V \subseteq \mathbb{Z}, J(z) \subseteq \{1, \ldots, \beta_{t,z}\}, A \right\} \cup \mathcal{F}_t,
\]
whereas for \( t = T \), \( \tilde{F}_T \subseteq F_T \cup \bigcup_{s=1}^{T} \mathcal{N}_s \subseteq \mathcal{F}_T \). For any \( Q \in \mathcal{M} \) and \( t \in I \), any element of \( \mathcal{N}_t \) is a subset of an \( F_t \)-measurable \( Q \)-nullset.

**From now on, we assume that the class \( \tilde{H} \) of admissible trading strategies is given by all \( \tilde{F} \)-predictable processes.** We can rewrite the definition of arbitrage de la classe \( S \) using strategies predictable with respect to \( \tilde{F} \). Namely, an arbitrage de la classe \( S \) with respect to \( \tilde{H} \) is an \( \tilde{F} \)-predictable process \( H = (H^1, \ldots, H^d) \) such that \( V_T(H) \geq 0 \) and \( \{V_T(H) > 0\} \) contains a set in \( S \).

**Remark 4.21** No arbitrage de la classe \( S \) with respect to \( \tilde{H} \) implies no arbitrage de la classe \( S \) with respect to \( \mathcal{H} \).

**Remark 4.22** (Financial interpretation of the filtration enlargement) Fix \( t \in I_1, z \in \mathbb{Z} \), the event \( \Sigma^z_{i-1} = \{S_{0:t-1} = z_{0:t-1}\} \) and suppose that \( 0 \notin \pi(\Delta S_t(\Sigma^z_{i-1}))^c \). Consider two probabilities \( P_k \in \mathcal{P}, k = 1, 2 \), for which \( P_k[\Sigma^z_{i-1}] > 0 \). Following Lemma 4.4, if \( j_k := \inf\{i = 1, \ldots, \beta \mid P_k[B^i_{t,z}] > 0\} < \infty \), then the rational choice for the arbitrage strategy is \( H^{j_k} \), as shown in Corollary 4.7. Thus, it is possible that \( j_k \) is finite for both probabilities, so that the two agents represented by \( P_1 \) and \( P_2 \) agree that \( \Sigma^z_{i-1} \) is a nonefficient level set of the market, although it is possible that \( j_1 \neq j_2 \), so that they might not agree on the trading strategy \( H^{j_k} \) that establishes the \( P_k \)-classical arbitrage on \( \Sigma^z_{i-1} \). In such a case, these two arbitrages are realized on different subsets of \( \Sigma^z_{i-1} \) and generate different payoffs. Nevertheless, note that any of these agents is able to find an arbitrage opportunity among the finite number of trading strategies \( H^i_{t,z}, i = 1, \ldots, \beta_{t,z} \), given by Lemma 4.4 (recall that \( \beta_{t,z} \leq d \)). The filtration enlargement allows one to embrace them all. This can be compared to the analogous discussion in [13]: “A weak arbitrage opportunity is a situation where we know there must be
an arbitrage, but we cannot tell, without further information, what strategy will realize it.”

We expand on the above argument more formally. Recall that Lemma 4.4 provides a partition of any level set $\Sigma_t$ with the following property: For any $\omega \in \Omega^c$, there exists a unique set $B_{t,z}$ identified by $i = i(\omega)$, such that $\omega \in B_{t,z}$ with $z = S_0:T(\omega)$. Define therefore, for any $t \in I_1$ the multifunction

$$H_t(\omega) := \left\{ H \in \mathbb{R}^d \mid H \cdot \Delta S_t(\hat{\omega}) \geq 0 \text{ for any } \hat{\omega} \in \bigcup_{j = i(\omega)} B_{t,z} \right\}$$

(4.21)

if $\omega \in \Omega^c$ and $H_t(\omega) = \{0\}$ otherwise. Observe that for any $t \in I_1$, if $\omega_1, \omega_2$ satisfy $S_0:T-1(\omega_1) = S_0:T-1(\omega_2)$ and $i(\omega_1) = i(\omega_2)$, then they belong to the same $B_{t,z}$, and $H_t(\omega_1) = H_t(\omega_2)$. In other words, $H_t$ is constant on any $B_{t,z}$, and therefore for any open set $V \subseteq \mathbb{R}^d$, we have

$$\{ \omega \in \Omega \mid H_t(\omega) \cap V \neq \emptyset \} = \bigcup_{z \in Z} \bigcup_{i = 1} B_{t,z} \bigcup \{ H_t(B_{t,z}) \cap V \neq \emptyset \},$$

from which it follows that $H_t$ is measurable with respect to $\mathcal{F}_t \vee \bigcup_{t = 1} \mathcal{N}_s$. Note that since $H_t^*(\omega) \in H_t(\omega)$ for any $\omega \in \Omega$, we have that $H_t^*$ is a selection of $H_t$ with the same measurability. We now show how the process $H := (H_t)_{t \in I_1}$ provides $P$-classical arbitrage as soon as we choose a probabilistic model $P \in \mathcal{P}$ that is not absolutely continuous with respect to the capacity $v(A) := \sup_{Q \in \mathcal{M}} Q[A], A \in \mathcal{F}$ (see Lemma A.1 for more details on the properties of $v$). The case of $P \ll v$ is discussed in Remark 4.24.

**Theorem 4.23** Let $H$ be defined in (4.21). If $P \in \mathcal{P}$ is not absolutely continuous with respect to $v$, then there exists an $\mathbb{P}^P$-predictable trading strategy $H^P$ that is a $P$-classical arbitrage and such that

$$H^P(\omega) \in H(\omega) \quad P\text{-a.s.},$$

where $\mathcal{F}^P_t$ denote the $P$-completion of $\mathcal{F}_t$, and $\mathbb{P}^P := (\mathcal{F}^P_t)_{t \in I}$. 

**Proof** See Appendix A.1. □

By Lemma A.1, if $P \in \mathcal{P}$ fulfills the hypothesis of Theorem 4.23, then there exists an $\mathcal{F}$-measurable set $F \subseteq (\Omega^c)^c$ with $P[F] > 0$. Note that by Remark A.2 such a $P$ always exists if $\Omega^c \neq \emptyset$. Theorem 4.23 asserts therefore that for any probabilistic model that supports $\Omega^c$, an $\mathbb{P}^P$-predictable arbitrage opportunity can be found among the values of the set-valued process $H$. This property suggested us to baptize $H$ as the universal arbitrage aggregator and thus $H^*$ as a (standard) selection of the universal arbitrage aggregator. Note that we could consider a different selection of $H$ satisfying the essential requirement (4.20). Since this choice does not affect any of our results, we simply take $H^*$.  

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Remark 4.24 Recall from (1.2) that any \( P \in (\mathcal{P}_0)^c \) admits a \( P \)-classical arbitrage opportunity. We can distinguish between two different classes in \((\mathcal{P}_0)^c\).

The first one is \( \mathcal{P}_\mathcal{M} := \{ P (\mathcal{P}_0)^c \mid P \ll \nu \} \), or, in other words, an element \( P \in (\mathcal{P}_0)^c \) belongs to \( \mathcal{P}_\mathcal{M} \) iff any subset of \( \Omega^*_{\mathcal{M}} \) is \( P \)-null. Then for each probability \( P \) in this class, there exists a probability \( P' \) with larger support that annihilates any \( P \)-classical arbitrage opportunity. Recall Example 4.10, where \( \Omega_{\mathcal{M}} = \mathbb{Q} \cap [1/2, +\infty) \).

By choosing \( P = \delta_{\mathbb{Q},1} \in \mathcal{P}_\mathcal{M} \) we clearly have \( P \)-classical arbitrages. Nevertheless, by simply taking \( P' = \lambda \delta_{\mathbb{Q},1} + (1 - \lambda) \delta_{\mathbb{Q}} \) for some \( 0 < \lambda < 1 \), this market is arbitrage-free. From a model-independent point of view, these situations must not be considered as market inefficiencies since they vanish as soon as more trajectories are considered. This feature is captured by the universal arbitrage aggregator by means of the property \( H^* = 0 \) on \( \Omega_{\mathcal{M}} \).

On the other hand, when \( P \in (\mathcal{P}_0)^c \setminus \mathcal{P}_\mathcal{M} \), \( P \) assigns a positive measure to some \( \mathcal{M} \)-polar \( \mathcal{F} \)-measurable set \( F \in \mathcal{N} \). Therefore, any other \( P' \in \mathcal{P} \) with larger support will satisfy \( P'[F] > 0 \), and the probabilistic model \((\Omega, \mathcal{F}, \mathcal{P}, \mathcal{S}, P')\) will also exhibit \( P' \)-classical arbitrages. In the case of Example 4.10, \( \Omega^*_{\mathcal{M}} = B^1 \cup B^2 \) where \( B^1 = \mathbb{R}_+ \setminus \mathbb{Q} \) and \( B^2 = \mathbb{Q} \cap [0, 1/2) \). If \( P|\Omega^*_{\mathcal{M}} > 0 \), then the market exhibits a \( P \)-classical arbitrage, but this is still valid for any probabilistic model given by \( P' \) with \( P \ll P' \). In particular, if \( P'[B^1] > 0 \), then \( H^1 := (0, 0, 1) \) is a \( P' \)-classical arbitrage, whereas if \( P'[B^1] = 0 \) and \( P'[B^2] > 0 \), then \( H^2 := (1, 0, 0) \) is the desired strategy. In this example, \( H^*_1 = H^11_{\mathbb{R}_+ \setminus \mathbb{Q}} + H^21_{\mathbb{Q} \cap [0, 1/2)} \).

Lemma 4.25 \( \mathcal{M}(\mathcal{F}) \Rightarrow M(\mathcal{\tilde{F}}) \) with the following meaning:

- the restriction to \( \mathcal{F}_\mathcal{T} \) of any \( \mathcal{\tilde{Q}} \in M(\mathcal{\tilde{F}}) \) belongs to \( \mathcal{M}(\mathcal{F}) \);
- any \( Q \in \mathcal{M}(\mathcal{F}) \) can be uniquely extended to an element of \( \mathcal{M}(\mathcal{\tilde{F}}) \).

Proof Let \( \mathcal{\tilde{Q}} \in \mathcal{M}(\mathcal{\tilde{F}}) \), and let \( Q \in \mathcal{P}(\Omega) \) be the restriction to \( \mathcal{F}_\mathcal{T} \). For any \( t \in I_1 \) and \( A \in \mathcal{F}_{\mathcal{T},-1} \), we have \( E_{\mathcal{\tilde{Q}}}[ (S_t - S_{t-1})1_A] = E_{\mathcal{\tilde{Q}}}[ (S_t - S_{t-1})1_A] = 0 \). Let now \( Q \in \mathcal{M}(\mathcal{F}) \). There exists a unique extension of \( Q \) to \( \mathcal{\tilde{F}}_\mathcal{T} \), which we denote \( \mathcal{\tilde{Q}} \). For any \( \mathcal{\tilde{A}} \in \mathcal{\tilde{F}}_{\mathcal{T},-1} \) with \( t \in I_1 \), there exists \( A \in \mathcal{F}_{\mathcal{T},-1} \) such that \( \mathcal{\tilde{Q}}[\mathcal{\tilde{A}}] = \mathcal{\tilde{Q}}[A] = Q[A] \). Hence, \( E_{\mathcal{\tilde{Q}}}[ (S_t - S_{t-1})1_A] = E_{\mathcal{\tilde{Q}}}[ (S_t - S_{t-1})1_A] = Q[ (S_t - S_{t-1})1_A] = 0 \), where the first equality follows from \( \mathcal{\tilde{Q}}[\mathcal{\tilde{A}} \setminus A] = 0 \) and the second one from the \( \mathcal{F}_\mathcal{T} \)-measurability of \( (S_t - S_{t-1})1_A \). We conclude that \( E_{\mathcal{\tilde{Q}}}[ (S_t | \mathcal{\tilde{F}}_{\mathcal{T},-1}] = S_{t-1} \) and hence \( \mathcal{\tilde{Q}} \in \mathcal{M}(\mathcal{F}) \).

Remark 4.26 The filtration enlargement \( \mathcal{\tilde{F}} \) has been introduced to guarantee the aggregation of \( 1 \)-arbitrages on the sets \( B^i_{t,z} \) obtained from Lemma 4.4 with \( \Gamma = \mathcal{\tilde{S}}_{t-1}^{\mathcal{V}} \). If indeed we follow [10], then we can consider any collection of probability measures \( \Theta_t := (P^i_{t,z}) \) on \( (\mathcal{F}, \mathcal{\tilde{F}}) \) such that \( P^i_{t,z}[B^i_{t,z}] = 1 \). Observe first that

\[
\mathcal{F}^{\Theta_t}_t \supseteq \mathcal{\tilde{F}}^{P^i_{t,z}}_t \cap \bigcup\left\{ B^i_{t,z} \mid z \in \mathcal{V}, i \in J(z) \right\}
\]

with \( \mathcal{V} \) and \( J(z) \) arbitrary. For any \( P^i_{t,z} \), we have indeed that \( \mathcal{\tilde{F}}^{P^i_{t,z}}_t \) contains any subset of \( (B^i_{t,z})^c \). Therefore, if \( A = \bigcup\left\{ B^i_{t,z} \mid z \in \mathcal{V}, i \in J(z) \right\} \), we have
• if \( z \notin V \) or \( i \notin J(z) \), then \( A \in \mathcal{F}_t^{P_i^{i,z}} \) trivially because \( A \subset (B_i^{i,z})^c \);
• if \( z \in V \) and \( i \in J(z) \), then \( A \in \mathcal{F}_t^{P_i^{i,z}} \) because \( A = B_i^{i,z} \cup \bar{A} \) with \( \bar{A} \subset (B_i^{i,z})^c \).

It is easy to check that \( \Theta_t \) has the Hahn property on \( \mathcal{F}_t \) as defined in [10, Definition 3.2], with \( \Phi_t := \Theta_t|_{\mathcal{F}_t} \). We can therefore apply Theorem 3.16 in [10] to find an \( \mathcal{F}_t^{\Theta_t} \)-measurable function \( H_t \) such that \( H_t = H_i^{i,z} P_i^{i,z} \)-a.s., which means that \( H_t(\omega) = H_i^{i,z} \) for every \( \omega \in B_i^{i,z} \).

### 4.7 Main results

Our aim now is to show how the absence of arbitrage de la classe \( S \) provides a pricing functional via the existence of a martingale measure with nice properties. Clearly, the “no 1p-arbitrage” condition is the strongest that one can assume in this model-independent framework, and we have shown in Proposition 4.1 that it automatically implies the existence of a full support martingale measure. On the other hand, we are interested in characterizing those markets that can exhibit 1p-arbitrages but nevertheless admit a rational system of pricing rules.

The set \( \Omega_s \) introduced in Sect. 4.5 has a clear financial interpretation since it represents the set of events for which no 1p-arbitrage can be found. This is the content of the following proposition. Let \( (\Omega, \mathcal{F}_t, \mathcal{P}) \), \( \mathcal{H} \) be as in Sect. 4.6 and define

\[
\mathcal{H}^+ := \{ H \in \mathcal{H} | V_T(H)(\omega) \geq 0 \ \forall \ \omega \in \Omega, \text{ and } V_0(H) = 0 \}.
\]

**Proposition 4.27**

1. \( \mathcal{V}_{H^+} = \bigcup_{H \in \mathcal{H}^+} \mathcal{V}_H = \Omega^c_S \).
2. \( \mathcal{M} \neq \emptyset \) if and only if \( \bigcup_{H \in \mathcal{H}^+} \mathcal{V}_H^+ \) is strictly contained in \( \Omega \).

**Proof** 2) follows from 1) and Proposition 4.18. Indeed, \( \mathcal{M} \neq \emptyset \) iff \( \Omega_s \neq \emptyset \) iff \( \Omega^c_s \subset \Omega \) iff \( \bigcup_{H \in \mathcal{H}^+} \mathcal{V}_H^+ \subset \Omega \). Now we prove 1). Given (4.20), we only need to show the inclusion \( \bigcup_{H \in \mathcal{H}^+} \mathcal{V}_H^+ \subset \Omega^c_S \). Let \( \bar{w} \in \bigcup_{H \in \mathcal{H}^+} \mathcal{V}_H^+ \) then there exist \( \bar{H} \in \mathcal{H}^+ \) and \( t \in I_1 \) such that \( \bar{H}_t(\omega) \cdot \Delta S_t(\omega) \geq 0 \ \forall \ \omega \in \Omega \) and \( \bar{H}_t(\bar{\omega}) \cdot \Delta S_t(\bar{\omega}) > 0 \). Let \( z = S_{0,T}(\bar{\omega}) \). By Lemma 4.4 there exists \( i \in \{1, \ldots, \beta t, z\} \) such that \( \bar{w} \in B_i^{i,z} \); hence, we conclude that \( \bar{w} \in \mathcal{H}_t \) and therefore \( \bar{w} \in \Omega^c_S \). \( \square \)

**Proof of Theorem 1.2** We prove that

\[
\exists \ \text{an arbitrage de la classe } S \text{ in } \mathcal{H} \iff \mathcal{M} = \emptyset \text{ or } \mathcal{N} \text{ contains sets of } S.
\]

Notice that if \( H \in \mathcal{H} \) satisfies \( V_T(H)(\omega) \geq 0 \ \forall \ \omega \in \Omega, \) then \( \mathcal{V}_H^+ \in \mathcal{N} \) if \( \mathcal{M} \neq \emptyset \) since otherwise \( 0 < \mathbb{E}_Q[V_T(H)] = V_0(H) = 0 \) for \( Q \in \mathcal{M} \). If there exists an \( \mathcal{H} \)-arbitrage de la classe \( S \), then \( \mathcal{V}_H^+ \) contains a set in \( S \), and therefore \( \mathcal{N} \) contains a set in \( S \). If instead \( \mathcal{M} = \emptyset \), then we already have the thesis. For the opposite implication, we exploit the universal arbitrage aggregator \( H^* \in \mathcal{H} \) as defined in (4.19) satisfying \( V_T(H^*)(\omega) \geq 0 \ \forall \ \omega \in \Omega \) and \( \mathcal{V}_{H^*}^+ = \bigcup_{t=1}^T \mathcal{B}_t = \Omega^c_S \). If \( \mathcal{M} = \emptyset \), then by Proposition 4.18, \( \Omega^c_S = \Omega \), and \( H^* \) is an \( \mathcal{H} \)-model-independent arbitrage and hence (from
Also an arbitrage de la classe $S$. If $\mathcal{M} \neq \emptyset$ and $\mathcal{N}$ contains a set $C$ in $S$, then $C \subseteq \Omega^n_+ = \mathcal{V}^+_H$ from (4.17) and Proposition 4.27, 1). Therefore, $H^\bullet$ is an $\tilde{H}$-arbitrage de la classe $S$. 

**Definition 4.28** Define the convex subset of $\mathcal{P}$

$$\mathcal{R}_S := \{Q \in \mathcal{P} \mid Q[C] > 0 \text{ for all } C \in S\}. \quad (4.22)$$

We look at conditions ensuring that the existence of martingale measures in the class $R_S$ is equivalent to absence of arbitrage de la classe $S$.

**Example 4.29** We consider the examples introduced in Example 3.2. Suppose that there is no model-independent arbitrage in $\tilde{H}$. By Theorem 1.2 we obtain:

1) 1p-arbitrage: $S = \{C \in \mathcal{F} \mid C \neq \emptyset\}$.
- No 1p-arbitrage in $\tilde{H}$ iff $N = \emptyset$;
- $R_S = \mathcal{P}_+$ if $\Omega$ is finite or countable; otherwise, $R_S = \emptyset$.

In the case of $n$p-arbitrage, we have

$$R_S = \{Q \in \mathcal{P} \mid Q[A] > 0 \text{ for all } A \subseteq \Omega^n \text{ having at least } n \text{ elements}\},$$

and we have no $n$p-arbitrage in $\tilde{H}$ iff $\mathcal{N}$ does not contain elements having more than $n - 1$ elements.

2) Open arbitrage: $S = \{C \in B(\Omega) \mid C \text{ open nonempty}\}$.
- No open arbitrage in $\tilde{H}$ iff $\mathcal{N}$ does not contain nonempty open sets;
- $R_S = \mathcal{P}_+$.

3) $\mathcal{P}'$-q.s. arbitrage: $S = \{C \in \mathcal{F} \mid P[C] > 0 \text{ for some } P \in \mathcal{P}'\}, \mathcal{P}' \subseteq \mathcal{P}$.
- No $\mathcal{P}'$-q.s. arbitrage in $\tilde{H}$ iff $\mathcal{N}$ may contain only $\mathcal{P}'$-polar sets;
- $R_S = \{Q \in \mathcal{P} \mid P' \ll Q \text{ for all } P' \in \mathcal{P}'\}$.

4) $P$-a.s. arbitrage: $S = \{C \in \mathcal{F} \mid P[C] > 0\}, P \in \mathcal{P}$.
- No $P$-a.s. arbitrage in $\tilde{H}$ iff $\mathcal{N}$ may contain only $P$-nullsets;
- $R_S = \{Q \in \mathcal{P} \mid P \ll Q\}$.

5) Model-independent arbitrage: $S = \{\Omega\}$.
- $R_S = \mathcal{P}$.

6) $\varepsilon$-arbitrage: $S = \{C_\varepsilon(\omega) \mid \omega \in \Omega\}$, where $\varepsilon > 0$ is fixed, and $C_\varepsilon(\omega)$ is the closed ball in $(\Omega, d)$ of radius $\varepsilon$ and centered in $\omega$.
- No $\varepsilon$-arbitrage in $\tilde{H}$ iff $\mathcal{N}$ does not contain closed balls of radius $\varepsilon$;
- $R_S = \{Q \in \mathcal{P} \mid Q[C_\varepsilon(\omega)] > 0 \text{ for all } \omega \in \Omega\}$.

**Corollary 4.30** Suppose that the class $S$ has the property

$$\exists\{C_n \mid n \in \mathbb{N}\} \subseteq S \text{ such that } \forall C \in S \exists \pi \text{ satisfying } C_\pi \subseteq C. \quad (4.23)$$

Then we have

$$\text{No arbitrage de la classe } S \text{ in } \tilde{H} \iff \mathcal{M} \cap R_S \neq \emptyset. \quad (4.24)$$

**Proof** Suppose $Q \in \mathcal{M} \cap R_S \neq \emptyset$. Then any polar set $N \in \mathcal{N}$ does not contain sets in $S$ (otherwise, if $C \in S$ and $C \subseteq N$, then $Q[C] > 0$ and $Q[N] = 0$, a contradiction).
Then by Theorem 1.2, no arbitrage de la classe $S$ holds. Conversely, suppose that no arbitrage de la classe $S$ holds, so that $\mathcal{M} \neq \emptyset$, and let $\{C_n \mid n \in \mathbb{N}\} \subseteq S$ be the collection of sets in the assumption. From Theorem 1.2 we obtain that $N \in \mathcal{N}$ does not contain any set in $S$, and so each set $C_n$ is not a polar set; hence, for each $n$, there exists $Q_n \in \mathcal{M}$ such that $Q_n[C_n] > 0$. Set $Q := \sum_{n=1}^{\infty} \frac{1}{2^n} Q_n \in \mathcal{M}$ (see Lemma A.9). Take any $C \in S$ and let $C_n \subseteq C$. Then $Q[C] \geq \frac{1}{2^n} Q[C_n] \geq \frac{1}{2^n} Q_n[C_n] > 0$ and $Q \in \mathcal{M} \cap \mathcal{R}_S$. □

**Corollary 4.31** Let $S$ be the class of nonempty open sets. Then condition (4.23) is satisfied, and therefore

$$\text{No open arbitrage in } \tilde{\mathcal{H}} \iff \mathcal{M}_+ \neq \emptyset. \quad (4.25)$$

**Proof** Consider a dense countable subset $\{\omega_n \mid n \in \mathbb{N}\}$ of $\Omega$, as $\Omega$ is Polish. Consider the open balls

$$B^m(\omega_n) := \left\{ \omega \in \Omega \mid d(\omega, \omega_n) < \frac{1}{m} \right\}, \ m \in \mathbb{N}.$$ 

The density of $\{\omega_n \mid n \in \mathbb{N}\}$ implies that $\Omega = \bigcup_{n \in \mathbb{N}} B^m(\omega_n)$ for any $m \in \mathbb{N}$. Take any open set $C \subseteq \Omega$. Then there exists some $\tilde{n}$ such that $\omega_{\tilde{n}} \in C$. Take $\tilde{m} \in \mathbb{N}$ sufficiently large so that $B^{\tilde{m}}(\omega_{\tilde{n}}) \subseteq C$. □

**Corollary 4.32** Suppose that $\Omega$ is finite or countable. Then condition (4.23) is fulfilled, and therefore

$$\text{No arbitrage de la classe } S \text{ in } \tilde{\mathcal{H}} \iff \mathcal{M} \cap \mathcal{R}_S \neq \emptyset.$$ 

In particular:

$$\text{No } 1p\text{-arbitrage in } \tilde{\mathcal{H}} \iff \mathcal{M}_+ \neq \emptyset, \quad (4.26)$$

$$\text{No } P\text{-a.s. arbitrage in } \tilde{\mathcal{H}} \iff \exists \ Q \in \mathcal{M} \text{ such that } P \ll Q, \quad (4.27)$$

$$\text{No } \mathcal{P}'\text{-q.s. arbitrage in } \tilde{\mathcal{H}} \iff \exists \ Q \in \mathcal{M} \text{ such that } P' \ll Q \forall P' \in \mathcal{P}'. \quad (4.28)$$

**Proof** Define $S_0 := \{\omega \mid \omega \in \Omega \text{ such that there exists } C \in S \text{ with } \omega \in C\}$. Then $S_0$ is an at most countable set and satisfies condition (4.23). □

**Remark 4.33** Whereas (4.26) holds also for $1p$-arbitrage in $\mathcal{H}$ (see Proposition 6.3), (4.27) and (4.28) cannot be improved. Indeed, by replacing in Example (4.1) $\mathbb{R}_+$ with $\mathbb{Q}_+$ and $\mathbb{Q}_+$ with $\mathbb{N}$, $\Omega$ is countable, we still have $\mathcal{M} = \emptyset$, but there is no $P\text{-a.s. arbitrage in } \mathcal{H}$ if $P[\mathbb{Q}_+ \setminus \mathbb{N}] = 0$ (see Sect. 4.1, item 5 (a)).

**Remark 4.34** There are other families of sets satisfying condition (4.23). For example, in a topological setting, nowhere dense subsets of $\Omega$ (those having a closure with

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empty interior) are often considered “negligible” sets. Then the class of sets that are complements of nowhere dense sets satisfies condition (4.23).

**Remark 4.35** Condition (4.23) is not necessary to obtain the desired equivalence (4.24). Consider, for example, the class $S$ defining $\varepsilon$-arbitrage in Example 4.29,6. In this case condition (4.23) fails as soon as $\Omega$ is uncountable. However, we now prove that (4.24) holds when $\Omega = \mathbb{R}$. From the first part of the proof of Corollary 4.30 we already know that $M \cap R_S \neq \emptyset$ implies no arbitrage de la classe $S$ in $\tilde{H}$. For the converse, from no arbitrage de la classe $S$ in $\tilde{H}$ we know that each element in $S := \{[r - \varepsilon, r + \varepsilon] | r \in \mathbb{R}\}$ is not a polar set. Consider the countable class $G := \{[q - \varepsilon, q + \varepsilon] | q \in \mathbb{Q}\} \subseteq S$.

Each set $G_n \in G$ is not a polar set; hence, for each $n$, there exists $Q_n \in M$ such that $Q_n[G_n] > 0$. Set $\overline{Q} := \sum_{n=1}^{\infty} \frac{1}{2^n} Q_n \in M$ (see Lemma A.9). The set $D := \{r \in \mathbb{R} | \overline{Q}[\{[r - \varepsilon, r + \varepsilon]\}] = 0\}$ is at most countable. Indeed, any two distinct intervals $J := [r - \varepsilon, r + \varepsilon]$ and $J' := [r' - \varepsilon, r' + \varepsilon]$ with $r, r' \in D$ must be disjoint since otherwise for a rational $q$ between $r$ and $r'$, we should have $[q - \varepsilon, q + \varepsilon] \subseteq J \cup J'$ and thus $\overline{Q}[[q - \varepsilon, q + \varepsilon]] = 0$, which is impossible by the construction of $\overline{Q}$. For each $r_n \in D$, the set $[r_n - \varepsilon, r_n + \varepsilon] \in S$ is not a polar set; hence, for each $n$, there exists $\hat{Q}_n \in M$ such that $\hat{Q}_n[[r_n - \varepsilon, r_n + \varepsilon]] > 0$. Set $\hat{Q} := \sum_{n=1}^{\infty} \frac{1}{2^n} \hat{Q}_n \in M$. Then $Q := \frac{1}{2} \overline{Q} + \frac{1}{2} \hat{Q} \in M \cap R_S$ is the desired measure.

## 5 Feasible markets

We extend the classical notion of arbitrage with respect to a single probability measure $P \in \mathcal{P}$ to a class of probabilities $\mathcal{R} \subseteq \mathcal{P}$ as follows.

**Definition 5.1** The market admits $\mathcal{R}$-arbitrage if for all $P \in \mathcal{R}$, there exists a $P$-classical arbitrage. We call no $\mathcal{R}$-arbitrage the property that for some $P \in \mathcal{R}$, $\text{NA}(P)$ holds.

**Remark 5.2** (Financial interpretation of $\mathcal{R}$-arbitrage) If a market model admits an $\mathcal{R}$-arbitrage, then the agent will not be able to find a fair pricing rule, whatever model $P \in \mathcal{R}$ he chooses. However, the presence of an $\mathcal{R}$-arbitrage only implies that for each $P$, there exists a trading strategy $H^P$ that is a $P$-classical arbitrage, and this is a different concept to the existence of one single trading strategy $H$ that realizes an arbitrage for all $P \in \mathcal{R}$. In the particular case of $\mathcal{R} = \mathcal{P}$, the notion of $\mathcal{R}$-Arbitrage was first introduced in [13] as “weak arbitrage opportunity” and further studied in [11, 14] and the references therein. The no $\mathcal{R}$-arbitrage property above should not be confused with the condition $\text{NA}(\mathcal{R})$ introduced by Bouchard and Nutz [6] and recalled in Sect. 4 and in Example 3.2, 3).
We set
\[ \mathcal{P}_e(P) = \{ P' \in \mathcal{P} | P' \sim P \}, \quad \mathcal{M}_e(P) = \{ Q \in \mathcal{M} | Q \sim P \}. \]

In discrete-time financial markets, the Dalang–Morton–Willinger theorem applies, so that \( \text{NA}(P) \iff \mathcal{M}_e(P) \neq \emptyset \).

**Proposition 5.3** Suppose that \( \mathcal{R} \subseteq \mathcal{P} \) has the property that \( P \in \mathcal{R} \) implies \( \mathcal{P}_e(P) \subseteq \mathcal{R} \). Then we have

\[ \text{No } \mathcal{R}\text{-arbitrage iff } \mathcal{M} \cap \mathcal{R} \neq \emptyset. \]

In particular,

\[ \text{No } \mathcal{R}_S\text{-arbitrage iff } \mathcal{M} \cap \mathcal{R}_S \neq \emptyset, \]
\[ \text{No } \mathcal{P}_+\text{-arbitrage iff } \mathcal{M}_+ \neq \emptyset, \]
\[ \text{No } \mathcal{P}\text{-arbitrage iff } \mathcal{M} \neq \emptyset, \]

where \( \mathcal{R}_S \) is defined in (4.22), and all arbitrage conditions here are with respect to \( \mathcal{H} \).

**Proof** Suppose \( Q \in \mathcal{M} \cap \mathcal{R} \neq \emptyset \). Since \( Q \in \mathcal{R} \) and \( \text{NA}(Q) \) holds, we have no \( \mathcal{R}\text{-arbitrage} \). Conversely, suppose that no \( \mathcal{R}\text{-arbitrage} \) holds. Then there exists \( P \in \mathcal{R} \) for which \( \text{NA}(P) \) holds and therefore there exists \( Q \in \mathcal{M}_e(P) \). The assumption \( \mathcal{P}_e(P) \subseteq \mathcal{R} \) implies \( Q \in \mathcal{M}_e(P) = \mathcal{M} \cap \mathcal{P}_e(P) \subseteq \mathcal{M} \cap \mathcal{R} \). The particular cases follow from the fact that \( \mathcal{R}_S \) has the property that \( P \in \mathcal{R}_S \) implies \( \mathcal{P}_e(P) \subseteq \mathcal{R}_S \). \( \square \)

**Remark 5.4** As a result of the previous proposition, whenever the condition (4.24) holds, we also obtain

\[ \text{No arbitrage de la classe } \mathcal{S} \text{ in } \tilde{\mathcal{H}} \iff \mathcal{M} \cap \mathcal{R}_S \neq \emptyset \iff \text{no } \mathcal{R}_S\text{-arbitrage}. \]

Similarly, when condition (4.25) holds, we obtain

\[ \text{No open arbitrage in } \tilde{\mathcal{H}} \iff \mathcal{M}_+ \neq \emptyset \iff \text{no } \mathcal{P}_+\text{-arbitrage}. \]

Given a measurable space \((\Omega, \mathcal{F})\) and a price process \( \mathcal{S} \) defined on it, we investigate in this section the properties of the set of arbitrage-free (for \( \mathcal{S} \)) probabilities on \((\Omega, \mathcal{F})\). A minimal reasonable requirement on the financial market is the existence of at least one probability \( P \in \mathcal{P} \) that does not allow any \( P\text{-classical arbitrage} \). Recall from the introduction the definition of the set

\[ \mathcal{P}_0 = \{ P \in \mathcal{P} | \mathcal{M}_e(P) \neq \emptyset \}. \]

By Proposition 5.3 and the definition of \( \mathcal{P}_0 \) it is clear that

\[ \text{No } \mathcal{P}\text{-arbitrage } \iff \mathcal{M} \neq \emptyset \iff \mathcal{P}_0 \neq \emptyset, \]
and each of these conditions is equivalent to no model-independent arbitrage with respect to $\tilde{H}$ (Theorem 1.3). When $P_0 \neq \emptyset$, it is possible that only very few models (i.e., a “small” set of probability measures, the extreme case being $|P_0| = 1$) are arbitrage-free. On the other hand, the financial market could be very “well posed,” so that for “most” models, no arbitrage is assured, the extreme case being $P_0 = P$.

To distinguish these two possibilities, we analyze the conditions under which the set $P_0$ is dense in $P$. In this case, even if there could be some particular models allowing arbitrage opportunities, the financial market is well posed for most models.

**Definition 5.5** The market is feasible if $P_0 = P$. (Recall that we are here considering the $\sigma(P, C_b)$-closure.)

In Proposition 5.7, we characterize feasibility with the existence of a full support martingale measure, a condition independent of any a priori fixed probability. Recall that $P_+^+$ denotes the set of full support probability measures.

**Lemma 5.6** For all $P \in P_+^+$,

$$\overline{P_e(P)} = P, \quad \text{and} \quad P_+^+ \text{ is } \sigma(P, C_b)\text{-dense in } P.$$

**Proof** It is well known that under the assumption that $(\Omega, d)$ is separable, $P_+^+ \neq \emptyset$. Let us first show that for all $a \in \Omega$, we have that $\delta_a \in \overline{P_e(P)}$, where $P \in P_+$, and $\delta_a$ is the point mass in $a$. Let $A_n$ be the open set

$$A_n := \left\{ \omega \in \Omega \mid d(a, \omega) < \frac{1}{n} \right\}.$$

Since $P$ has full support, $0 < P[A_n] < 1$. Define the conditional probability measure $P_n := P[\cdot | A_n]$. For all $0 < \lambda < 1$, the measure $P_\lambda := \lambda P[\cdot | A_n^\lambda] + (1 - \lambda) P[\cdot | A_n]$ has full support and is equivalent to $P$, and $P_\lambda$ converges weakly to $P[\cdot | A_n]$ as $\lambda \downarrow 0$. Hence, $P_n \in \overline{P_e(P)}$. In order to show that $P_n \xrightarrow{w} \delta_a$, we prove that for all open $G$, $\liminf P_n[G] \geq \delta_a(G)$. If $a \in G$, then $\delta_a(G) = 1$ and $P[G \cap A_n] = P[A_n]$ eventually, so that we have $\liminf P_n[G] = 1 = \delta_a(G)$. If $a \notin G$, then $\delta_a(G) = 0$, and the inequality is obvious.

Since $\delta_a \in \overline{P_e(P)}$ for all $a \in \Omega$, then $\text{co}(|\delta_a | a \in \Omega|) \subseteq \overline{P_e(P)}$, and from the density in $P$ with respect to the weak topology it follows that $\overline{P_e(P)} = P$. The last assertion is obvious since $P_e(P) \subseteq P_+$ for each $P \in P_+$. \qed}

**Proposition 5.7** The following assertions are equivalent:

1) $M_+ \neq \emptyset$;
2) no $P_+$-arbitrage;
3) $P_0 \cap P_+ \neq \emptyset$;
4) $\overline{P_0} \cap P_+ = P$;
5) $\overline{P_0} = P$.

**Proof** Because we have $M_+ \neq \emptyset \Leftrightarrow$ no $P_+$-arbitrage according to Proposition 5.3, and no $P_+$-arbitrage $\Leftrightarrow P_0 \cap P_+ \neq \emptyset$ by definition, 1)–3) are clearly equivalent.
Let us show that 3) ⇒ 4). Assume that \( \mathcal{P}_0 \cap \mathcal{P}_+ \neq \emptyset \) and observe that if \( P \in \mathcal{P}_0 \cap \mathcal{P}_+ \), then \( P_e(P) \subseteq \mathcal{P}_0 \cap \mathcal{P}_+ \), which implies that \( P_e(P) \subseteq \overline{\mathcal{P}_0 \cap \mathcal{P}_+} \subseteq \mathcal{P} \). From Lemma 5.6 we conclude that 4) holds.

Observe now that the implication 4) ⇒ 5) holds trivially; so we just need to show that 5) ⇒ 3). Let \( P \in \mathcal{P}_+ \). If \( P \) satisfies NA(P), then there is nothing to show. Otherwise, by 5) there exists a sequence of probabilities \( P_n \in \mathcal{P}_0 \) such that \( P_n \xrightarrow{w} P \) and the condition NA(\( P_n \)) holds for all \( n \in \mathbb{N} \). Define \( P^* := \sum_{n=1}^{+\infty} \frac{1}{2^n} P_n \) and note that for this probability, the condition NA(\( P^* \)) holds; so we just need to show that \( P^* \) has full support. Assume by way of contradiction that \( \text{supp}(P^*) \subseteq \Omega \setminus \Omega_1 \). Then there exists an open set \( O \) such that \( P^*[O] = 0 \) and \( P[O] > 0 \) since \( P \) has full support. Since \( P_n[O] = 0 \) for all \( n \) and \( P_n \xrightarrow{w} P \), we obtain \( 0 = \lim \inf P_n[O] \geq P[O] > 0 \), a contradiction. □

Remark 5.8 From the previous proof we observe that if the market is feasible, then \( \bigcup_{P \in \mathcal{P}_0} \text{supp}(P) = \Omega \), and no “significantly large parts” of \( \Omega \) are neglected by no-arbitrage models \( P \in \mathcal{P}_0 \).

Proof of Theorem 1.4 Proposition 5.7 guarantees that 1) ⇔ 2) ⇔ 3), and Corollary 4.31 assures that 3) ⇔ 4). □

5.1 The case of a countable space \( \Omega \)

When \( \Omega = \{\omega_n \mid n \in \mathbb{N}\} \) is countable, it is possible to provide another characterization of feasibility using the norm topology instead of the weak topology on \( \mathcal{P} \). More precisely, we consider the topology induced by the total-variation norm. A sequence \((P_n)\) of probabilities converges in variation to \( P \) if \( \| P_n - P \| \to 0 \), where the variation norm of a signed measure \( \mu \) is defined by

\[
\| \mu \| = \sup_{A_1, \ldots, A_n \in \mathcal{F}} \sum_{i=1}^{n} |\mu(A_i)|, \tag{5.1}
\]

where \( A_1, \ldots, A_n \) is a finite partition of \( \Omega \).

Lemma 5.9 Let \( \Omega \) be a countable space. Then for all \( P \in \mathcal{P}_+ \),

\[
\overline{\mathcal{P}_e(P)} = \overline{\mathcal{P}_+} = \mathcal{P}.
\]

Proof Since \( \Omega \) is countable, we have that

\[
\mathcal{P} = \{ P := (p_n)_{n \in \mathbb{N}} \in \ell^1 \mid p_n \geq 0 \forall n \in \mathbb{N}, \| P \|_1 = 1 \},
\]

\[
\mathcal{P}_+ = \{ P \in \mathcal{P} \mid p_n > 0 \forall n \in \mathbb{N} \},
\]

with \( \| \cdot \|_1 \) being the \( \ell^1 \)-norm. Observe that in the countable case \( \mathcal{P}_e(P) = \mathcal{P}_+ \) for every \( P \in \mathcal{P}_+ \). So we only need to show that for any \( P \in \mathcal{P} \) and any \( \varepsilon > 0 \), there exists \( P' \in \mathcal{P}_+ \) such that \( \| P - P' \|_1 \leq \varepsilon \).
Let $P \in \mathcal{P} \setminus \mathcal{P}_+$. Then $(p_n)_{n \in \mathbb{N}} \in \ell^1$, and there exists at least one index $n$ for which $p_n = 0$. Let $\alpha > 0$ be the constant satisfying

$$\sum_{n \in \mathbb{N}} \frac{\alpha}{2^n} = 1.$$ 

There also exists one index $n$, say $n_1$, for which $1 \geq p_{n_1} > 0$. Let $p := p_{n_1} > 0$. For any positive $\varepsilon < p$, define $P' = (p'_n)$ by $p'_n = p - \frac{\varepsilon}{2^n}$, $p'_n = p_n$ for all $n \neq n_1$ such that $p_n > 0$, and $p'_n = \frac{\alpha}{2^n} \frac{\varepsilon}{2}$ for all $n$ such that $p_n = 0$. Then $p'_n > 0$ for all $n$ and $\sum_{n=1}^{\infty} p'_n = \sum_{n}^{\infty} p_n$ such that $p_n > 0$ $p_n = 1$, so that $P' \in \mathcal{P}_+$ and $\|P - P'\|_1 = \varepsilon$. □

**Remark 5.10** In the general case where $\Omega$ is uncountable, although it is still true that $\mathcal{P}_+ = \mathcal{P}$, it is no longer true that $\mathcal{P}_e(P)$ $\mathcal{P}$ for any $P \in \mathcal{P}_+$. To see this, take $\Omega = [0, 1]$ and $\mathcal{P}_e(\lambda)$ the set of probability measures equivalent to Lebesgue measure. It is easy to see that $\delta_0 \not\in \mathcal{P}_e(\lambda)$ since $\|P - \delta_0\| \geq P[(0, 1)] = 1$.

**Proposition 5.11** If $\Omega$ is countable, then the following conditions are equivalent:

1) $\mathcal{M}_+ \neq \emptyset$;
2) no $\mathcal{P}_+$-arbitrage;
3) $\mathcal{P}_0 \cap \mathcal{P}_+ \neq \emptyset$;
4) $\mathcal{P}_0 \mathcal{P} = \mathcal{P}$,

where $\| \cdot \|$ is the total-variation norm on $\mathcal{P}$.

**Proof** Using Lemma 5.9, the proof is straightforward by using the same techniques as in Proposition 5.7. □

6 **On open arbitrage**

In the introduction, we have already illustrated the interpretation and robust features of the dual formulation of open arbitrage. In order to prove the equivalence between open arbitrage and (1.1), recall that $\mathcal{V}_H^+ := \{\omega \in \Omega \mid V_T(H)(\omega) > 0\}$ and consider the following definition.

**Definition 6.1** Let $\tau$ be a topology on $\mathcal{P}$, and $\mathcal{H}$ be a class of trading strategies. Set

$$W(\tau, \mathcal{H}) = \{H \in \mathcal{H} \mid \text{there exists a nonempty $\tau$-open set } U \subseteq \mathcal{P} \text{ such that}$$

$$\forall P \in U, V_T(H) \geq 0 \text{ P-a.s. and } P[\mathcal{V}_H^+] > 0\}.$$ 

Clearly, $W(\tau, \mathcal{H})$ consists of the trading strategies satisfying condition (1.1) with respect to the appropriate topology and with the appropriate measurability requirement. The first item in the next proposition is the announced equivalence. The second item shows that the analogous equivalence is true also with respect to the class $\mathcal{H}$. Therefore, in Theorem 1.4 we could add to the four equivalent conditions also the dual formulation of open arbitrage with respect to $\mathcal{H}$.
Proposition 6.2 1) Let $\sigma := \sigma(P, C_b)$, and $\| \cdot \|$ be the variation norm defined in (5.1). Then:

$$H \in W(\| \cdot \|, \mathcal{H}) \iff H \in \mathcal{H} \text{ is a 1p-arbitrage}$$

and

$$H \in W(\sigma, \mathcal{H}) \iff H \in \mathcal{H} \text{ is an open arbitrage.}$$

In addition, if $H \in W(\sigma, \mathcal{H})$, then $V_T(H)(\omega) \geq 0$ for all $\omega \in \Omega$.

2) Let $\mathcal{F} = \mathcal{B}(\Omega)$ be the Borel $\sigma$-algebra, and $\tilde{\mathcal{F}}$ a $\sigma$-algebra such that $\mathcal{F} \subseteq \tilde{\mathcal{F}}$. Define the set

$$\tilde{\mathcal{P}} := \{ \tilde{P} : \tilde{\mathcal{F}} \to [0, 1] | \tilde{P} \text{ is a probability} \}$$

and endow $\tilde{\mathcal{P}}$ with the topology $\tilde{\sigma} := \sigma(\tilde{\mathcal{P}}, C_b)$. The class $\tilde{\mathcal{H}}$ of admissible trading strategies is given by all $\tilde{\mathcal{F}}$-predictable processes. Then

$$H \in W(\tilde{\sigma}, \tilde{\mathcal{H}}) \iff H \in \tilde{\mathcal{H}} \text{ is an open arbitrage in } \tilde{\mathcal{H}}.$$

In addition, if $H \in W(\tilde{\sigma}, \tilde{\mathcal{H}})$, then $V_T(H)(\omega) \geq 0$ for all $\omega \in \Omega$.

Proof We prove 1) and postpone the proof of 2) to Appendix A.

a) $H$ is a 1p-arbitrage $\Rightarrow$ $H \in W(\| \cdot \|, \mathcal{H})$: Let $H \in \mathcal{H}$ be a 1p-arbitrage. Then $V_T(H)(\omega) \geq 0$ for all $\omega \in \Omega$, and there exists a probability $P$ such that $P[\lambda^+] > \varepsilon > 0$. From the implication $\| P - Q \| < \varepsilon \Rightarrow |P[C] - Q[C]| < \varepsilon$ for every $C \in \mathcal{F}$ we obtain $\tilde{P}[\lambda^+] > 0$ for every $\tilde{P} \in B_\varepsilon(P)$, where $B_\varepsilon(P)$ is the ball of radius $\varepsilon$ centered in $P$. Hence, $H \in W(\| \cdot \|, \mathcal{H})$.

b) $H \in W(\| \cdot \|, \mathcal{H})$ $\Rightarrow$ $H$ is a 1p-arbitrage: If $H \in W(\| \cdot \|, \mathcal{H})$, then $V_T(H) \geq 0$ $P$-a.s. for all $P$ in the set $\mathcal{U}$. We need to show that $B := \{ \omega \in \Omega | V_T(H)(\omega) < 0 \}$ is empty. By way of contradiction, let $\omega \in B$, take any $P \in \mathcal{U}$, and define the probability $P_\lambda := \lambda \delta_\omega + (1 - \lambda)P$. Since $V_T(H) \geq 0$ $P$-a.s., we must have $P[[\omega]] = 0$ since otherwise $P[B] > 0$. However, $P_\lambda[B] \geq P[[\omega]] = \lambda > 0$ for all positive $\lambda$, and $P_\lambda$ eventually belongs to $\mathcal{U}$ as $\lambda \downarrow 0$, which contradicts $V_T(H) \geq 0$ $P$-a.s. for any $P \in \mathcal{U}$.

c) $H \in W(\sigma, \mathcal{H})$ $\Rightarrow$ $H \in W(\| \cdot \|, \mathcal{H})$: This claim is trivial because every weakly open set is also open in the norm topology.

d) $H \in W(\sigma, \mathcal{H})$ $\Rightarrow$ $V_T(H)(\omega) \geq 0$ for all $\omega \in \Omega$: This follows from c) and b).

e) $H \in W(\sigma, \mathcal{H})$ $\Rightarrow$ $H$ is an open arbitrage: Suppose $H \in W(\sigma, \mathcal{H})$, so that $V_T(H)(\omega) \geq 0$ for all $\omega \in \Omega$. We claim that $\{ \omega \in \Omega | V_T(H) = 0 \}$ is not dense in $\Omega$. This will imply the thesis as int(\{\omega | V_T(H)^+ (\omega) \}) is then a nonempty open set on which $V_T(H) > 0$. Suppose by way of contradiction that $\{ \omega \in \Omega | V_T(H)^+ (\omega) \}$ is not in $\sigma(\mathcal{P}, C_b)$.

f) $H$ is an open arbitrage $\Rightarrow$ $H \in W(\sigma, \mathcal{H})$: Note first that if $F$ is a closed subset of $\Omega$, then $\mathcal{T}(F) := \{ P \in \mathcal{P} | \supp(P) \subseteq F \}$ is a $\sigma(\mathcal{P}, C_b)$-closed face of $\mathcal{P}$ from [2, Theorem 15.19]. If $H$ is an open arbitrage, then $V_T(H)^+$ contains an open set, and
in particular \( G := (\mathcal{V}^+_H)^c \) is a closed set strictly contained in \( \Omega \). Observe then that \( \mathcal{U} := (\mathcal{P}(G))^c \) is a nonempty open set of probabilities that fulfils the properties in the definition of \( W(\sigma, H) \).

The following proposition is an improvement of (4.26), as the 1p-arbitrage is defined with respect to \( H \).

**Proposition 6.3** For \( \Omega \) countable, we have

\[
\text{No 1p-arbitrage in } H \iff M_+ \neq \emptyset.
\]

**Proof** As a consequence of Propositions 4.1 and 6.2, we only need to prove that \( M_+ \neq \emptyset \) implies \( W(\| \cdot \|, H) = \emptyset \). From Proposition 5.11, we have that \( M_+ \neq \emptyset \) implies \( P_0 = P \), and so for every (norm-)open set \( U \subseteq P \), there exists \( P \in P_0 \cap U \) for which \( \text{NA}(P) \) holds, which implies \( W(\| \cdot \|, H) = \emptyset \).

**6.1 On the continuity of \( S \) with respect to \( \omega \)**

Consider first a one-period market \( I = \{0, 1\} \) with \( S_0 = s_0 \in \mathbb{R}^d \) and \( S_1 \) a random outcome continuous in \( \omega \). Then every 1p-arbitrage generates an open arbitrage (this was shown by [29] and is intuitively clear). By Proposition 4.1 no 1p-arbitrage implies \( M_+ \neq \emptyset \) and therefore no open arbitrage. We then conclude that in this particular case, the three conditions are all equivalent, and Theorem 1.4 holds without the enlargement of the natural filtration, so that we recover in particular the result stated in [29].

Differently from the one-period case, it is no longer true in the multiperiod setting that no open arbitrage and no 1p-arbitrage (with respect to admissible strategies \( H \)) are equivalent, as shown by the following examples. Moreover, even with \( S \) continuous in \( \omega \), no open arbitrage is not equivalent to \( M_+ \neq \emptyset \) as long as we do not enlarge the filtration as in Sect. 4.6.

**Example 6.4** Consider \( \Omega = [0, 1] \times [0, 1] \), \( \mathcal{F} = \mathcal{B}([0, 1]) \otimes \mathcal{B}([0, 1]) \), and the canonical process given by \( S_1(\omega) = \omega_1 \) and \( S_2(\omega) = \omega_2 \). Clearly, for any \( \omega = (\omega_1, \omega_2) \) such that \( \omega_1 \in (0, 1) \), we have that \( 0 \in \text{ri}(\Delta S_2(\Sigma^0_1))^{c} \). On the other hand, for \( \omega = (1, \omega_2) \) or \( \hat{\omega} = (0, \omega_2) \), we have 1p-arbitrage since \( S_2(\omega) \leq S_1(\omega) \) with \( < \) for any \( \omega_2 \neq 1 \) and \( S_2(\hat{\omega}) \geq S_1(\hat{\omega}) \) with \( > \) for any \( \omega_2 \neq 0 \). Define \( \Sigma^1 = \{S_1 = 1\} \) and \( \Sigma^0 = \{S_1 = 0\} \); then \( \alpha(\omega) = -1_{\Sigma^1} + 1_{\Sigma^0} \) is a 1p-arbitrage that does not admit any open arbitrage since neither \( \Sigma^1 \) nor \( \Sigma^0 \) are open sets, and any strategy which is not zero on \( (\Sigma^1 \cup \Sigma^0)^c \) gives both positive and negative payoffs.

**Example 6.5** We show an example of a market with \( S \) continuous in \( \omega \), with no open arbitrage in \( H \), and \( M_+ = \emptyset \). Let us first introduce on \( \Omega = [0, +\infty) \) the continuous functions

\[
\varphi_{a,b}^m(\omega) := \begin{cases} m(\omega - a), & \omega \in [a, \frac{a+b}{2}], \\ -m(\omega - b), & \omega \in [\frac{a+b}{2}, b], \\ 0, & \text{otherwise}, \end{cases}
\]

where \( a \leq b \).
\( \phi_{m,a,b}(\omega) := \begin{cases} 
 m(\omega - a), & \omega \in [a, a + 1], \\
 m, & \omega \in [a + 1, b - 1], \\
 -m(\omega - b), & \omega \in [b - 1, b], \\
 0 & \text{otherwise}, 
\end{cases} \)

with \( a, b, m \in \mathbb{R} \). Define the continuous (in \( \omega \)) stochastic process \((S_t)_{t=0,1,2,3}\) by

\[
S_0(\omega) := \frac{1}{2},
\]

\[
S_1(\omega) := \phi_{[0,3]}(\omega) + \phi_{[3,6]}(\omega) + \sum_{k=3}^{\infty} \phi_{[2k,2k+2]}(\omega),
\]

\[
S_2(\omega) := \phi_{[0,3]}(\omega) + \phi_{[3,6]}(\omega) + \sum_{k=3}^{\infty} \phi_{[2k,2k+2]}(\omega),
\]

\[
S_3(\omega) := \phi_{[0,3]}(\omega) + \phi_{[3,6]}(\omega) + \phi_{[6,8]}(\omega) + \sum_{k=4}^{\infty} \phi_{[2k-1,2k+1]}(\omega).
\]

It is easy to check that for \( z \in \mathbb{Z} \) with \( z_0 := (\frac{1}{2}, 1, 2) \), we have \( \Sigma^z_2 = \{2k + 1 \mid k \geq 3\} \), and \( H := 1_{\Sigma^z_2} \) is the only \( 1 \)-p-arbitrage opportunity in the market. One can also check that \( \nu_{H}^+ = \Sigma^z_2 \), as a consequence, \( H \) is not an open arbitrage, and

\[
Q[[2k + 1 \mid k \geq 3]] = 0 \quad \text{for any } Q \in \mathcal{M}. \tag{6.1}
\]

Consider now \( \hat{z} \in \mathbb{Z} \) with \( \hat{z}_0 := (\frac{1}{2}, 1, \frac{1}{2}) \) and the corresponding level set \( \Sigma^\hat{z}_2 \). It is easy to check that

\[
\Sigma^\hat{z}_2 = [1, 2] \cup [4, 5] \quad \text{and} \quad \Delta S_2 < 0 \text{ on } \Sigma^\hat{z}_2. \tag{6.2}
\]

Observe now that \( z_{0:1} = \hat{z}_{0:1} \) and that \( \Sigma^z_1 = [1, 2] \cup [4, 5] \cup \{2k + 1 \mid k \geq 3\} \). We therefore have

\[
S_2(\omega) = \begin{cases} 
2, & \omega \in \{2k + 1 \mid k \geq 3\}, \\
\frac{1}{2}, & \omega \in [1, 2] \cup [4, 5], 
\end{cases} \quad \text{for } \omega \in \Sigma^z_1.
\]

From \( S_1(\omega) = 1 \) on \( \Sigma^z_1 \), (6.1), and (6.2) we get that any martingale satisfies \( Q[[1, 2] \cup [4, 5]] = 0 \). In other words, there exist polar sets with nonempty interior, which implies \( \mathcal{M}_+ = \emptyset \).

**Appendix A**

A.1 Proof of Theorem 4.23

**Lemma A.1** (Lebesgue decomposition of \( P \)) Let \( v := \sup_{Q \in \mathcal{M}} Q \). For any \( P \in \mathcal{P} \), there exists a set \( F \in \mathcal{F} \) such that \( F \subseteq \Omega^c \) and the measures \( P_c[\cdot] := P[\cdot \cap F^c] \) and \( P_s[\cdot] := P[\cdot \cap F] \) satisfy

\[
P_c \ll v, \quad P_s \perp v \quad \text{and} \quad P = P_c + P_s. \tag{A.1}
\]
Proof We wish to apply Theorem 4.1 in [24] to \( \mu = P \in \mathcal{P} \) and \( \nu = \sup_{Q \in \mathcal{M}} Q \). It is easy to check that 1) \( \mu \) and \( \nu \) are monotone \([0, 1]\)-valued set functions on \( \mathcal{F} \) satisfying \( \mu(\emptyset) = 0 \) and \( \nu(\emptyset) = 0 \); 2) \( P \) is exhaustive, that is, if \( (A_n)_{n \in \mathbb{N}} \) is a disjoint sequence, then \( P[A_n] \to 0 \) (indeed, \( 1 \geq P[\bigcup_n A_n] = \sum_n P[A_n] \geq 0 \) implies that \( P[A_n] \to 0 \)); 3) \( \nu \) is weakly null additive, that is, if \( A, B \in \mathcal{F} \) with \( \nu(A) = \nu(B) = 0 \), then \( \nu(A \cup B) = 0 \) (indeed, if \( \nu(A) = \nu(B) = 0 \), then for any \( Q \in \mathcal{M} \), we have \( Q[A] = Q[B] = 0 \), which implies \( Q[A \cup B] = 0 \) and \( \nu(A \cup B) = 0 \)); and 4) \( \nu \) is continuous from below. Indeed, if \( A_n \nrightarrow A \), then \( Q[A_n] \uparrow Q[A] \), \( Q[A] = \sup_n Q[A_n] \), and

\[
\lim_{n \to \infty} \nu(A_n) = \sup_n \nu(A_n) = \sup_n \sup_{Q \in \mathcal{M}} Q[A_n] = \sup_{Q \in \mathcal{M}} \sup_n Q[A_n] = \nu(A).
\]

So \( \mu \) and \( \nu \) satisfy all the assumptions of Theorem 4.1 in [24] and hence we obtain the existence of \( F \in \mathcal{F} \) such that \( \nu(F) = 0 \), and the decomposition in (A.1) holds. By Proposition 4.18, for all \( A \in \mathcal{F} \) such that \( A \subseteq \Omega^c \), we have \( \nu(A) > 0 \). Therefore, \( F \subseteq \Omega^c \), and this concludes the proof. \( \square \)

Remark A.2 Observe that if \( \Omega^c_1 \neq \emptyset \), then the set of probability measures with non-trivial singular part \( \mathcal{P} \) is nonempty; simply take, for instance, any convex combination of \( \{\delta_\omega | \omega \in \Omega^c_1\} \).

A.2 Preliminary considerations

We want to consider now the probabilistic model specified by \( (\Omega, (\mathcal{F}^P_t)_{t \in I}, S, P) \), and we therefore need to pass from \( \omega \)-wise considerations to \( P \)-a.s. considerations. For this reason, we first need to construct an auxiliary process \( S^P_t \) with the property \( S^P_t = S_t \) \( P \)-a.s. for any \( t \in I \) in the spirit of Lemma 4.17.

Let \( P_{\Delta S_T}(\cdot, \cdot) : \Omega \times \mathcal{B}(\mathbb{R}^d) \to [0, 1] \) be the conditional distribution of \( \Delta S_T \) and denote by \( \gamma_{\Delta S_T} \) its random support. Define the set \( A_{\Delta S_T} := \{0 \notin ri(\text{conv } \gamma_{\Delta S_T})\} \) as in [28]. It may happen that \( P[A_{\Delta S_T}] = 0 \); in this case, \( \mathbb{B}_T \) and \( \Omega_{T-1} \) as in Lemma 4.17 are subsets of \( P \)\(-nullsets (respectively in \( \mathcal{F}_T \) and \( \mathcal{F}_{T-1} \)). Construct iteratively \( X^P_t \) and \( S^P_t \) as in (4.14) and (4.16). Denote \( \Delta X^P_t := X^P_t - S_{t-1} \) and let

\[
\tau := \min\{t \in I_1 | P[A_{\Delta X^P_t}] > 0\}.
\]

Observe that \( \tau \) is well defined since by Lemma 4.17, if \( P[A_{\Delta X^P_t}] = 0 \) for any \( t \geq 1 \), then we have that \( \bigcup_{t \in I_1} \overline{B}_t = \Omega^c_1 \) is a subset of a \( P \)-nullset (cf. (4.15)). This is a contradiction since \( P \) is not absolutely continuous with respect to \( \nu \), and so the set \( F \) from Lemma A.1 satisfies \( F \subseteq \Omega^c_1 \) and \( P[F] > 0 \). From now on, we still denote by \( \{S_t\}_{t \in I} \) the \( P \)-a.s. version of the process given by \( \{S_t 1_{[t<\tau]} + X^P_t 1_{[t \geq \tau]}\}_{t \in I} \).

Remark A.3 For any \( t \in I_1 \), denote by \( P_{t-1}(\cdot, \cdot) : (\Omega, \mathcal{F}) \to [0, 1] \) the conditional probability of \( P \) given \( \mathcal{F}_{t-1} \). Recall from Theorem 4.12 that there exists \( N_1 \in \mathcal{F}_{t-1} \) with \( P[N_1] = 0 \) such that for any \( \omega \in \Omega \setminus N_1 \), we have \( P_{t-1}(\omega, \Sigma_{t-1}) = 1 \), where \( z(\omega) = S_0|\omega \).
Recall that \( \tau \) is defined in (A.2) and denote \( A_\tau := A_{\Delta S_\tau} \). For any \( \omega \in \Omega \), the level set \( \Sigma_{\tau-1}^z \) can be decomposed as \( \Sigma_{\tau-1}^z = \bigcup_{i=1}^{\beta_{\tau,z}} B_{i,\tau,z}^j \cup B_{\tau,z}^\ast \). Define for any \( z \in \mathbb{Z} \),
\[
j_z := \inf \{ j \in \{1, \ldots, \beta_{\tau,z} \} \mid P(\omega, B_{i,\tau,z}^j) > 0 \ \forall \omega \in \Sigma_{\tau-1}^z \}
\]
and recall that \( P(\cdot, B_{i,\tau,z}^j) \) is constant on \( \Sigma_{\tau-1}^z \) due to Theorem 4.12, b). Define the set \( N_2 := \bigcup_{z \in \mathbb{Z} \cup \{j \mid j_z < \infty\}} B_{i,\tau,z}^i \), where \( \mathbb{Z}_f := \{ z \in \mathbb{Z} \mid j_z < \infty \} \). Then \( N_2 \) is a \( \bar{P} \)-nullset because for any \( \omega \in N_1^c \), we have \( \bar{P}(\omega, N_2) = \bar{P}(\omega, \bigcup_{i=1}^{j_z-1} B_{i,\tau,z}^i) = 0 \), and hence \( \bar{P}(N_2) = \bar{P}(N_1 \cap N_2) + \bar{P}(N_1^c \cap N_2) = 0 \) (see also Lemma A.6). Recall that \( \bar{P}[\cdot] \) and \( \bar{P}(\omega, \cdot) \) denote the completions of \( P[\cdot] \) and \( P(\omega, \cdot) \) respectively.

Denote \( N := N_1 \cup N_2 \). We are now able to define the following multifunction \( \Psi : \Omega \to 2^{\mathbb{R}^d} \) with values in the power set of \( \mathbb{R}^d \):
\[
\Psi(\omega) := \begin{cases} \Delta S_\tau(\Sigma_{\tau-1}^z \cap N_1^c), & \omega \in N_1^c, \\ \emptyset, & \text{otherwise} \end{cases}
\]

In Lemma A.4, we show that \( \Psi \) is \( \mathcal{F}_p^{P_{\tau-1}} \)-measurable. We apply now an argument similar to [28, Lemma 2]. Denote by \( S^d_1 \) the closed unit ball in \( \mathbb{R}^d \), by \( \text{lin}(\chi) \) the linear space generated by \( \chi \), and by \( \chi^\circ \) the polar cone of \( \chi \). By preservation of measurability (see Proposition A.8), the (closed-valued) multifunction
\[
\omega \mapsto G_0(\omega) := \text{lin}(\Psi(\omega)) \cap (- \text{cone}(\Psi(\omega)))^\circ \cap S^d_1
\]
is also \( \mathcal{F}_p^{P_{\tau-1}} \)-measurable, and \( G_0(\omega) \neq \emptyset \) if and only if \( \omega \in A_\tau \cap N_1^c \); therefore, \( A_\tau = \{ 0 \notin \text{ri}(\text{conv}(Y_{\Delta S_\tau}^P)) \} \) is \( \mathcal{F}_p^{P_{\tau-1}} \)-measurable. Note that we already have \( G_0(\omega) \subseteq H_\tau(\omega) \) for \( P \)-a.e. \( \omega \in \Omega \).

Nevertheless, the random set \( G_0(\omega) \) contains \( g \in S^d_1 \) with \( g \cdot \Delta S_\tau(\omega) = 0 \). Thus, we do not extract a measurable selection from \( G_0 \), but rather consider for any \( n \in \mathbb{N} \) the closed-valued multifunction
\[
\omega \mapsto G_n(\omega) := \text{lin}(\Psi(\omega)) \cap \left\{ v \in \mathbb{R}^d \mid v \cdot s \geq -\frac{1}{n} \ \forall s \in \Psi(\omega) \setminus \{0\} \right\} \cap S^d_1, \quad n \geq 1,
\]
and look for a measurable selection of \( G := \bigcup_{n=0}^\infty G_n \). By Lemma A.7 all the random sets \( G_n \) are \( \mathcal{F}_p^{P_{\tau-1}} \)-measurable, and therefore the same is true for \( G \). Now for any \( n \geq 1 \), let \( \tilde{H}_n \) be a measurable selection of \( G_n \) on \( \{ G_n \neq \emptyset \} \), which always exists for a (measurable) closed-valued multifunction with \( \tilde{H}_n(\omega) = 0 \) if \( G_n(\omega) = \emptyset \). Define therefore
\[
H_k := \sum_{n=0}^k \tilde{H}_n \quad \text{and} \quad B_k := V_{H_k}^+.
\]
By construction, \((B_k)\) is an increasing sequence of sets converging to \(\bigcup_z B^{j^k}_{\tau,z}\), which is therefore measurable and satisfies

\[
P\left[ \bigcup_z B^{j^k}_{\tau,z} \right] = \int_{\Omega \setminus \mathcal{N}} P(\omega, B^{j^k}_{\tau,z}) P(d\omega) \geq \int_{\mathcal{A}_\tau \setminus \mathcal{N}} P(\omega, B^{j^k}_{\tau,z}) P(d\omega) > 0,
\]

which follows from the definition of conditional probability, from \(P[A_\tau] > 0\), and from \(P(\omega, B^{j^k}_{\tau,z}) > 0\) for every \(\omega \in A_\tau \setminus \mathcal{N}\). We can therefore conclude that there exists \(m \geq 0\) such that \(P[B_m] > 0\), and since obviously \(H_m \cdot \Delta S_\tau \geq 0\), we have that \(H_m\) is a \(P\)-arbitrage. The normalized random variable \(H^P_\tau \equiv \frac{H_m(\omega)}{\|H_m(\omega)\|}\) is a measurable selector of the multifunction \(G_0\) since it satisfies

\[
H^P_\tau[\omega] \in \bigcup_{n=1}^m G_n(\omega) \subseteq G(\omega) \subseteq H_\tau(\omega) \quad P\text{-a.s.,}
\]

and thus the desired strategy is given by \(H^P_\tau^1 = H^P_\tau 1_{[\tau]}(s)\).

**Lemma A.4** The multifunction \(\Psi\) defined in (A.3) is \(\mathcal{F}^P_{\tau-1}\)-measurable.

**Proof** Recall that by definition the multifunction \(\Psi\) is measurable iff for any open set \(V \subseteq \mathbb{R}^d\), we have that \(\{\omega \mid \Psi(\omega) \cap V \neq \emptyset\}\) is a measurable set. Observe that

\[
\Psi^{-1}(V) := \{\omega \mid \Psi(\omega) \cap V \neq \emptyset\} = S^{-1}_{\tau-1}(S_{\tau-1}(\Delta S^{-1}_\tau(V) \cap N^c) \cap N^c).
\]

Let us show that the complement of this set is \(\mathcal{F}^P_{\tau-1}\)-measurable from which the thesis will follow. Observe that for any function \(f\) and set \(A\), we have \((f^{-1}(A))^c = f^{-1}(A^c)\), so that

\[
(\Psi^{-1}(V))^c = S^{-1}_{\tau-1}\left(S_{\tau-1}(\Delta S^{-1}_\tau(V) \cap N^c)^c \cup N\right) \cup N
= S^{-1}_{\tau-1}\left(S_{\tau-1}\left((\Delta S^{-1}_\tau(V))^c \cup N\right)\right) \cup N
= S^{-1}_{\tau-1}\left(S_{\tau-1}(\Delta S^{-1}_\tau(V^c) \cup N)\right) \cup N.
\]

Note now that \(A_1 := \Delta S^{-1}_\tau(V^c) \cup N\) is an analytic set since it is the union of a Borel set and a \(\tilde{P}\)-null set. The set \(B_1 := S^{-1}_{\tau-1}(A_1)\) is an analytic subset of \(\mathbb{R}^d\) since \(S\) is a Borel function and the image of an analytic set through a Borel-measurable function is analytic. Finally, \(A_2 := S^{-1}_{\tau-1}(B_1)\) is an analytic subset of \(\Omega\) since the preimage of an analytic set through a Borel-measurable function is analytic. Since the \(P\)-completion of \(\mathcal{F}\) contains any analytic set, \(A_2 \cup N\) is also analytic and belongs to \(\mathcal{F}^P_{\tau-1}\). \(\square\)

**Remark A.5** For sure, \(A_2 \cup N\) is analytic and belongs to \(\mathcal{F}^P\). The heuristic for \(A_2 \cup N\) belonging to \(\mathcal{F}^P_{\tau-1}\) is that this set is a union of atoms of \(\mathcal{F}^P_{\tau-1}\). More formally, since \(B_1\) is analytic in \(\mathbb{R}^d\), for any measure \(\mu\), there exist \(F, G\) such that \(B_1 = F \cup G\) with
$F$ a Borel set and $G$ a subset of $\mu$-null measure (because analytic sets are in the completion of $B$ with respect to any measure $\mu$). Taking $\mu$ as the distribution of $S_{T-1}$ under $P$, we have $A_2 = S_{T-1}^{-1}(F) \cup S_{T-1}^{-1}(G)$. Since $S_{T-1}^{-1}(F) \in \mathcal{F}_{T-1}$ and $S_{T-1}^{-1}(G)$ is a subset of a $\mathcal{F}_{T-1}$-measurable $P$-nullset, we have $A_2 \in \mathcal{F}_{T-1}$ and hence the same for $A_2 \cup N$.

**Lemma A.6** Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\mathcal{G}$ a sub-$\sigma$-algebra of $\mathcal{F}$. Let $P_G(\omega, \cdot)$ be the conditional probability of $P$ given $\mathcal{G}$. Then

$$
\tilde{P}[A] = \int_{\Omega \setminus N(A)} \tilde{P}_G(\omega, A) P(d\omega), \quad A \in \mathcal{F}^P,
$$

(A.4)

where $\tilde{P}_G(\omega, \cdot)$ is the completion of $P_G(\omega, \cdot)$, and $N(A) \in \mathcal{G}$ is a $P$-nullset, which depends on $A$.

**Proof** It is easy to see that every set in $\mathcal{F}^P$ is the union of a set $F \in \mathcal{F}$ and a subset of a $P$-nullset. For any $F \in \mathcal{F}$, $\tilde{P}[F] = P[F]$ and $P_G(\omega, F) = \tilde{P}_G(\omega, F)$, so that (A.4) is obvious from the definition of conditional probability (with $N(F) = \emptyset$). If $A$ is a subset of a $P$-nullset $A_1$, then we have $0 = P[A_1] = \int_{\Omega} P_G(\omega, A_1) P(d\omega)$, which means that $P_G(\omega, A_1) = 0$ $P$-a.s. Thus, we also have $\tilde{P}_G(\omega, A) = 0$ $P$-a.s., and (A.4) follows by taking $N(A) = \{\omega \in \Omega | P_G(\omega, A_1) > 0\} \in \mathcal{G}$. □

**A.4 Measurable selection results**

**Lemma A.7** Let $(\Omega, A)$ be a measurable space, and $\Psi : \Omega \mapsto 2^{\mathbb{R}^d}$ an $A$-measurable multifunction. Let $\varepsilon > 0$. Then

$$
\Psi^\varepsilon : \omega \mapsto \{v \in \mathbb{R}^d | v \cdot s \geq \varepsilon \forall s \in \Psi(\omega) \setminus \{0\}\}
$$

is an $A$-measurable multifunction.

**Proof** Observe first that for $v \in \mathbb{R}^d$,

$$
v \cdot s \geq \varepsilon, \forall s \in \Psi(\omega) \setminus \{0\} \iff v \cdot s \geq \varepsilon, \forall s \in \overline{\Psi}(\omega) \setminus \{0\}
$$

(A.5)

where $D(\omega)$ is a dense subset of $\Psi(\omega)$. This is obvious by the continuity of the scalar product. With no loss of generality, we can then consider $\Psi$ closed-valued, and we denote by $(\psi_n)$ its Castaing representation (see [27, Theorem 14.5] for details). For any $n \in \mathbb{N}$, consider the closed-valued multifunction

$$
\Lambda_n(\omega) = \begin{cases} 
\{v \in \mathbb{R}^d | v \cdot \psi_n(\omega) \geq \varepsilon\} & \text{if } \omega \in \text{dom} \Psi, \psi_n(\omega) \neq 0, \\
\mathbb{R}^d & \text{if } \omega \in \text{dom} \Psi, \psi_n(\omega) = 0, \\
\emptyset & \text{otherwise.}
\end{cases}
$$

We claim that $\Lambda_n$ is measurable for all $n \in \mathbb{N}$, from which it follows that the map $\omega \mapsto \bigcap_{n \in \mathbb{N}} \Lambda_n(\omega)$ is also measurable (cf. Proposition A.8). From (A.5) we thus conclude that $\Psi^\varepsilon$ is measurable.
We are only left to show the claim. To this end, observe that \( \Lambda_n(\omega) \) has a nonempty interior on \( \{ \Lambda_n \neq \emptyset \} \). Therefore, for any open set \( V \subseteq \mathbb{R}^d \), we have

\[
\{ \omega \in \Omega \mid \Lambda_n(\omega) \cap V \neq \emptyset \} = \{ \omega \in \Omega \mid \text{int}(\Lambda_n(\omega)) \cap V \neq \emptyset \}.
\]

Note now that

\[
\{ \omega \in \Omega \mid \text{int}(\Lambda_n(\omega)) \cap V \neq \emptyset \} = \psi_n^{-1}\left( \Pi_y(\Pi_x^{-1}(V) \cap \langle \cdot, \cdot \rangle^{-1}(\mathbb{R}^{d} \times [\varepsilon, \infty))) \right) \cup \psi_n^{-1}(0),
\]

which is measurable (when \( \psi_n \) is measurable) by the continuity of the scalar product \( \langle \cdot, \cdot \rangle \) and the open mapping property of the projections \( \Pi_x, \Pi_y : \mathbb{R}^{d} \times \mathbb{R}^{d} \to \mathbb{R}^{d} \).

**Proposition A.8** (From Propositions 14.2, 14.11, and 14.12 in [27]) Consider a family of \( \mathcal{A} \)-measurable set-valued functions. The following operations preserve \( \mathcal{A} \)-measurability: countable union, countable intersection (if the functions are closed-valued), finite linear combination, convex/linear/affine hull, generated cone, polar set, closure.

A.5 Complementary results

Recall that we are assuming that \( \Omega \) is a Polish space.

**Lemma A.9** Let \( Q_i \in \mathcal{M} \) for any \( i \in \mathbb{N} \). Then

\[
Q := \sum_{i \in \mathbb{N}} \frac{1}{2^i} Q_i \in \mathcal{M}.
\]

**Proof** We first observe that \( Q \in \mathcal{P} \); hence, we just need to show that it is a martingale measure. Consider the measures \( Q_k := \sum_{i=1}^{k} \frac{1}{2^i} Q_i \), which are not probabilities, and note that for each \( k \), we have \( \int_{\Omega} 1_B 1_{\mathcal{F}_{i-1}} dQ_k = 0 \) if \( B \in \mathcal{F}_{i-1} \). We observe that

\[
\|Q_k - Q\| \to 0 \text{ as } k \to \infty,
\]

where \( \| \cdot \| \) is the total-variation norm. We have indeed that

\[
\sup_{A \in \mathcal{F}} |Q_k[A] - Q[A]| = \sup_{A \in \mathcal{F}} \sum_{i=k+1}^{\infty} \frac{1}{2^i} Q_i[A] = \sum_{i=k+1}^{\infty} \frac{1}{2^i} \to 0 \text{ as } k \to \infty.
\]

In particular, we have \( Q_k[A] \uparrow Q[A] \) for any \( A \in \mathcal{F} \). Representing any simple function \( f \) as \( \sum_{j=1}^{n(f)} a_j(f)1_{A_j} \), we obtain, for a nonnegative random variable \( X \),

\[
\lim_{k \to \infty} \int_{\Omega} X dQ_k = \lim_{k \to \infty} \sup_{f \in \mathcal{S}} \sum_{j=1}^{n(f)} a_j(f)Q_k[A_j] = \sup_{f \in \mathcal{S}} \sup_{k} \sum_{j=1}^{n(f)} a_j(f)Q_k[A_j]
\]

\[
= \sup_{f \in \mathcal{S}} \sum_{j=1}^{n(f)} a_j(f)Q_k[A_j] = \sup_{f \in \mathcal{S}} \sum_{j=1}^{n(f)} a_j(f)Q[A_j] = \int_{\Omega} X dQ,
\]
where $\mathcal{S}$ is the class of simple functions less than or equal to $X$. For any $B \in \mathcal{F}_{t-1}$, we then have

$$E_Q[1_B \Delta S_t] = \int_{\Omega} (1_B \Delta S_t)^+ dQ - \int_{\Omega} (1_B \Delta S_t)^- dQ$$

$$= \lim_{k \to \infty} \int_{\Omega} (1_B \Delta S_t)^+ dQ_k - \lim_{k \to \infty} \int_{\Omega} (1_B \Delta S_t)^- dQ_k = \lim_{k \to \infty} \int_{\Omega} 1_B \Delta S_t dQ_k = 0. \quad \square$$

**Lemma A.10** For any dense set $D \subseteq \Omega$, the set of probabilities $\text{co}([\delta_\omega | \omega \in D])$ is $\sigma(\mathcal{P}, C_b)$-dense in $\mathcal{P}$.

**Proof** Take $\omega^* \notin D$ and let $\omega_n \to \omega^*$. Note that for every open set $G$, we have $\liminf \delta_{\omega_n}(G) \geq \delta_{\omega^*}(G)$, and this is equivalent to the weak convergence $\delta_{\omega_n} \overset{w}{\to} \delta_{\omega^*}$. Observe that for every set $X$, we have

$$\text{co}(X) = \overline{\text{co}}(X) := \bigcap \{C | C \text{ convex closed containing } X\} = \overline{\text{co}}(X).$$

Hence, by taking $X = ([\delta_\omega | \omega \in D])$ and by the $\sigma(\mathcal{P}, C_b)$-density in $\mathcal{P}$ of the set of measures with finite support, we obtain the thesis. \square

**Lemma A.11** Let $\mathcal{F} = \mathcal{B}(\Omega)$ be the Borel $\sigma$-algebra, and $\tilde{\mathcal{F}}$ a $\sigma$-algebra such that $\mathcal{F} \subseteq \tilde{\mathcal{F}}$. The set $\tilde{\mathcal{P}} := \{\tilde{P} : \tilde{\mathcal{F}} \to [0, 1] | \tilde{P} \text{ is a probability}\}$ is endowed with the topology $\sigma(\tilde{\mathcal{P}}, C_b)$. Then:

1. If $A \subseteq \Omega$ is dense in $\Omega$, then $\text{co}([\delta_\omega | \omega \in A])$ is $\sigma(\tilde{\mathcal{P}}, C_b)$-dense in $\tilde{\mathcal{P}}$. Notice that any element $Q \in \text{co}([\delta_\omega | \omega \in A])$ can be extended to $\tilde{\mathcal{F}}$.

2. If $D \subseteq \Omega$ is closed, then

$$\tilde{\mathcal{P}}(D) := \{\tilde{P} \in \tilde{\mathcal{P}} | \text{supp}(\tilde{P}) \subseteq D\}$$

is $\sigma(\tilde{\mathcal{P}}, C_b)$-closed, where the support is well defined by

$$\text{supp}(\tilde{P}) := \bigcap \{C \in \mathcal{C} | \tilde{P}[C] = 1\},$$

and $\mathcal{C}$ are the closed sets in $(\Omega, d)$.

**Proof** By construction, for any $\tilde{P} \in \tilde{\mathcal{P}}$, we have $\int f d\tilde{P} = \int f d\tilde{P}$ for any $f \in C_b$, where $P \in \mathcal{P}$ is the restriction of $\tilde{P}$ to $\mathcal{F}$.

To show the first claim, we choose any $\tilde{P} \in \tilde{\mathcal{P}}$. Consider $P \in \mathcal{P}$, the restriction of $\tilde{P}$ to $\mathcal{F}$. Then by Lemma A.10 there exists a sequence $(Q_n)$ in $\text{co}([\delta_\omega | \omega \in A])$ such that $\int f dQ_n \to \int f dP$ for every $f \in C_b$. As a consequence, $\int f dQ_n \to \int f d\tilde{P}$ for every $f \in C_b$.

To show the second claim, consider any net $(\tilde{P}_\alpha)_{\alpha} \subset \tilde{\mathcal{P}}(D)$ such that $\tilde{P}_\alpha \overset{w}{\to} \tilde{P}$. We want to show that $\tilde{P} \in \tilde{\mathcal{P}}(D)$. Consider $P_\alpha, P$, the restrictions to $\mathcal{F}$ of $\tilde{P}_\alpha, \tilde{P}$, respectively. Then $P_\alpha \overset{w}{\to} P$. By definition we have $\text{supp}(P_\alpha) = \text{supp}(\tilde{P}_\alpha) \subseteq D$ and $\square$
supp(\(P\)) = supp(\(\tilde{P}\)). Moreover, we know that \(\mathcal{P}(D) = \{P \in \mathcal{P} | \text{supp}(P) \subseteq D\}\) is \(\sigma(\mathcal{P}, C_b)\)-closed by [2, Theorem 15.19], so that \(D \supseteq \text{supp}(P) = \text{supp}(\tilde{P})\).

**Proof of Proposition 6.2, 2)** Recall that an open arbitrage in \(\tilde{\mathcal{H}}\) is an \(\mathbb{F}\)-predictable process \(H\) such that \(V_T(H) \geq 0\) and \(V_H^+ = \{V_T(H) > 0\}\) contains an open set. First, we show that \(H \in W(\tilde{\sigma}, \mathcal{H})\) implies \(V_T(H)(\omega) \geq 0\) for all \(\omega \in \Omega\). We only need to show that the set \(B := \{\omega \in \Omega | V_T(H)(\omega) < 0\}\) is empty. By way of contradiction, let \(\omega \in B\), take any \(P \in \mathcal{U}\), and define the probability \(P_\lambda := \lambda \delta_\omega + (1 - \lambda)P\). Since \(V_T(H) \geq 0\) \(P\)-a.s., we must have \(P[\{\omega\}] = 0\), otherwise \(P[B] > 0\). However, \(P_\lambda[B] \geq P_\lambda[\{\omega\}] = \lambda > 0\) for all positive \(\lambda\), and since \(P_\lambda\) weakly converges to \(P \in \mathcal{U}\), there exists \(\tilde{\lambda}\) such that \(P_{\tilde{\lambda}} \in \mathcal{U}\). This contradicts \(V_T(H) \geq 0\) \(P\)-a.s. for any \(P \in \mathcal{U}\). To prove the equivalence, assume first that \(H \in W(\tilde{\sigma}, \mathcal{H})\). We claim that \((V_H^+)^c = \{\omega \in \Omega | V_T(H) = 0\}\) is not dense in \(\Omega\). This will imply the thesis as int\((V_H^+)^c\) will then be a nonempty open set on which \(V_T(H) > 0\). Suppose by way of contradiction that \((V_H^+)^c = \Omega\). We know by Lemma A.11 that the corresponding set \(\mathcal{Q}\) of embedded probabilities \(\text{co}(\{\delta_\omega | \omega \in (V_H^+)^c\})\) is weakly dense in \(\mathcal{P}\), and hence it intersects in particular the weakly open set \(\mathcal{U}\). However, for every \(P \in \mathcal{Q}\), we have \(V_T(H) = 0\) \(P\)-a.s., and so this contradicts the assumption. Suppose now that \(H \in \mathcal{H}\) is an open arbitrage. Note that by Lemma A.11, if \(F\) is a closed subset of \(\Omega\), then \(\tilde{P}(F) := \{P \in \mathcal{P} | \text{supp}(P) \subseteq F\}\) is \(\sigma(\tilde{\mathcal{P}}, C_b)\)-closed. Since \(H\) is an open arbitrage, \(V_H^+\) contains an open set, and in particular \(G := (V_H^+)^c\) is a closed set strictly contained in \(\Omega\). Observe then that \((\tilde{P}(G))^c\) is a nonempty \(\sigma(\tilde{\mathcal{P}}, C_b)\)-open set of probabilities such that for all \(P \in \mathcal{U}\), we have \(V_T(H) \geq 0\) \(P\)-a.s. and \(P[V_H^+] > 0\). \(\square\)

**References**

1. Acciaio, B., Beiglböck, M., Penkner, F., Schachermayer, W.: A model-free version of the fundamental theorem of asset pricing and the super-replication theorem. Math. Finance (2013, forthcoming). Available online: http://onlinelibrary.wiley.com/doi/10.1111/mafi.12060/abstract
2. Aliprantis, C.D., Border, K.C.: Infinite Dimensional Analysis. Springer, Berlin (2006)
3. Bättig, R.I., Jarrow, R.A.: The second fundamental theorem of asset pricing: a new approach. Rev. Financ. Stud. **12**, 1219–1235 (1999)
4. Biagini, S., Bouchard, B., Kardaras, C., Nutz, M.: Robust fundamental theorem for continuous processes (2014). Available online: http://onlinelibrary.wiley.com/doi/10.12110/mafi.12060/abstract
5. Bion-Nadal, J., Kervarec, M.: Risk measuring under model uncertainty. Ann. Appl. Probab. **22**, 213–238 (2012)
6. Bouchard, B., Nutz, M.: Arbitrage and duality in nondominated discrete-time models. Ann. Appl. Probab. **25**, 823–859 (2015)
7. Brown, H.M., Hobson, D.G., Rogers, L.C.G.: Robust hedging of barrier options. Math. Finance **11**, 285–314 (2001)
8. Burzoni, M., Frittelli, M., Maggis, M.: Model-free superhedging duality. Preprint (2015). arXiv:1506.06068
9. Cassese, G.: Asset pricing with no exogenous probability measure. Math. Finance **18**, 23–54 (2008)
10. Cohen, S.N.: Quasi-sure analysis, aggregation and dual representations of sublinear expectations in general spaces. Electron. J. Probab. **17**, 1–15 (2012)
11. Cox, A.M.G., Obloj, J.: Robust pricing and hedging of double no-touch options. Finance Stoch. **15**, 573–605 (2011)
12. Dalang, R.C., Morton, A., Willinger, W.: Equivalent martingale measures and no-arbitrage in stochastic securities market models. Stoch. Stoch. Rep. **29**, 185–201 (1990)
13. Davis, M.H.A., Hobson, D.G.: The range of traded option prices. Math. Finance 17, 1–14 (2007)
14. Davis, M.H.A., Obłoj, J., Raval, V.: Arbitrage bounds for weighted variance swap prices. Math. Finance 24, 821–854 (2014)
15. Delbaen, F., Schachermayer, W.: A general version of the fundamental theorem of asset pricing. Math. Ann. 300, 463–520 (1994)
16. Delbaen, F., Schachermayer, W.: The fundamental theorem of asset pricing for unbounded stochastic processes. Math. Ann. 312, 215–250 (1998)
17. Denis, L., Hu, M., Peng, S.: Function spaces and capacity related to a sublinear expectation: application to G-Brownian motion paths. Potential Anal. 34, 139–161 (2011)
18. Denis, L., Martini, C.: A theoretical framework for the pricing of contingent claims in the presence of model uncertainty. Ann. Appl. Probab. 16, 827–852 (2006)
19. Dolinsky, Y., Soner, H.M.: Robust hedging with proportional transaction costs. Finance Stoch. 18, 327–347 (2014)
20. Filipović, D., Kupper, M., Vogelpoth, N.: Separation and duality in locally $L^0$-convex modules. J. Funct. Anal. 256, 3996–4029 (2009)
21. Föllmer, H., Schied, A.: Stochastic Finance. An Introduction in Discrete Time, 2nd edn. de Gruyter Studies in Mathematics, vol. 27 (2004)
22. Hobson, D.G.: Robust hedging of the lookback option. Finance Stoch. 2, 329–347 (1998)
23. Hobson, D.G.: The Skorokhod embedding problem and model-independent bounds for option prices. In: Carmona, R.A., et al. (eds.) Paris–Princeton Lectures on Math. Fin. 2010. Lecture Notes in Math., vol. 2003, pp. 267–318. Springer, Berlin (2011)
24. Li, Ju., Yasuda, M., Li, Ji.: A version of Lebesgue decomposition theorem for non-additive measure. In: Torra, V., et al. (eds.) Modeling Decisions for Artificial Intelligence. Lecture Notes in Computer Science, vol. 4617, pp. 168–173 (2007)
25. Peng, S.: Nonlinear expectations and stochastic calculus under Knightian uncertainty. In: Bensoussan, A., et al. (eds.) Real Options, Ambiguity, Risk and Insurance. Studies in Probability, Optimization and Statistics, vol. 5, pp. 144–184 (2013)
26. Pliska, S.: Introduction to Mathematical Finance: Discrete Time Models. Wiley, New York (1997)
27. Rockafellar, T., Wets, R.: Variational Analysis. Springer, Berlin (1998)
28. Rokhlin, D.: A proof of the Dalang–Morton–Willinger theorem (2008). Available online: http://arxiv.org/abs/0804.3308
29. Riedel, F.: Financial economics without probabilistic prior assumptions. Decis. Econ. Finance 38, 75–91 (2015)
30. Soner, H.M., Touzi, N., Zhang, J.: Martingale representation theorem for the G-expectation. Stoch. Process. Appl. 121, 265–287 (2011)
31. Soner, H.M., Touzi, N., Zhang, J.: Quasi-sure stochastic analysis through aggregation. Electron. J. Probab. 16, 1844–1879 (2011)
32. Stroock, D.W., Varadhan, S.R.S.: Multidimensional Diffusion Processes. Springer, Berlin (2006)