Four-body cluster structure of $A = 7 - 10$ double-$\Lambda$ hypernuclei

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Abstract

Energy levels of the double-Λ hypernuclei $^{7}_{\Lambda\Lambda}\text{He}$, $^{7}_{\Lambda\Lambda}\text{Li}$, $^{8}_{\Lambda\Lambda}\text{Li}$, $^{9}_{\Lambda\Lambda}\text{Be}$ and $^{10}_{\Lambda\Lambda}\text{Be}$ are predicted on the basis of the $\alpha+x+\Lambda+\Lambda$ four-body model with $x = n, p, d, t, ^3\text{He}$ and $\alpha$, respectively. Interactions between the constituent particles are determined so as to reproduce reasonably the observed low-energy properties of the $\alpha+x$ nuclei ($^5\text{He}$, $^5\text{Li}$, $^6\text{Li}$, $^7\text{Li}$, $^7\text{Be}$, $^8\text{Be}$) and the existing data of Λ-binding energies of the $x+\Lambda$ and $\alpha+x+\Lambda$ systems ($^3\Lambda\text{H}$, $^4\Lambda\text{H}$, $^5\Lambda\text{He}$, $^6\Lambda\text{He}$, $^6\Lambda\text{Li}$, $^7\Lambda\text{Li}$, $^8\Lambda\text{Li}$, $^8\Lambda\text{Be}$, $^9\Lambda\text{Be}$). Here, an effective ΛΛ interaction is constructed so as to reproduce, within the $\alpha+\Lambda+\Lambda$ model, the $B_{\Lambda\Lambda}$ of $^6_{\Lambda\Lambda}\text{He}$ which was discovered recently in the NAGARA event of the emulsion experiment. With no adjustable parameters for the $\alpha+x+\Lambda+\Lambda$ systems, $B_{\Lambda\Lambda}$ of the ground and bound excited states of the double-Λ hypernuclei with $A = 7–10$ are accurately calculated with the Gaussian-basis coupled-rearrangement-channel method. The Demachi-Yanagi event, observed recently for $^{10}_{\Lambda\Lambda}\text{Be}$, is interpreted as observation of its $2^+$ excited state on the basis of the present calculation. Structure change of the $\alpha+x$ core nuclei due to the participation of the Λ particles is found to be substantially large and it plays an important role in estimating the ΛΛ bond energies of those hypernuclei.
I. INTRODUCTION

A recent finding of the double-Λ hypernucleus $^{\Lambda\Lambda}_6\text{He}$, which is called as NAGARA event in the KEK-E373 experiment [1], has a great impact not only on the study of baryon-baryon interactions in the strangeness $S = -2$ sector but also on the study of dynamics of many-body systems with multi-strangeness. The importance of this event is attributed to the well-defined explanation of the process and the high quality experimental value of the $\Lambda\Lambda$ binding energy $B_{\Lambda\Lambda} = 7.25 \pm 0.19^{+0.18}_{-0.11}$ MeV [1], which leads to a smaller $\Lambda\Lambda$ binding, $\Delta B_{\Lambda\Lambda} = 1.01 \pm 0.20^{+0.18}_{-0.11}$ MeV, than the previous understanding. Sometimes the emulsion events include ambiguities related with serious difficulty of identifying emission of neutral particles such as neutrons and $\gamma$-rays. In the NAGARA event, however, the production of $^{\Lambda\Lambda}_6\text{He}$ has been uniquely identified free from such an ambiguity on the basis of the observation of sequential weak decays.

Historically, in the 1960’s, there appeared two reports on the observation of double-Λ hypernuclei, $^{10}_{\Lambda\Lambda}\text{Be}$ [2] and $^{\Lambda\Lambda}_6\text{He}$ [3], but the reality of the latter case was considered doubtful [4]. Two decades later the modern emulsion-counter hybrid technique has been applied in the KEK-E176 experiment [5], in which a new double-Λ hypernucleus event was found but no unique identification was given so far: One explanation as $^{10}_{\Lambda\Lambda}\text{Be}$ leads to a repulsive $\Lambda\Lambda$ interaction ($\Delta B_{\Lambda\Lambda} < 0$), while the other possibility involving $^{13}_{\Lambda\Lambda}\text{B}$ leads to an attractive $\Lambda\Lambda$ interaction [3, 4]. If the latter is the case, the extracted strength of the $\Lambda\Lambda$ interaction is attractive with $\Delta B_{\Lambda\Lambda} \simeq 4$ MeV. Although the latter option seems consistent with the old data of $^{10}_{\Lambda\Lambda}\text{Be}$ [6], the substantially attractive $\Lambda\Lambda$ interaction has not been convincing.

In the strangeness nuclear physics, the most fundamental problem is to recognize various facets of interactions among octet baryons ($N$, $\Lambda$, $\Sigma$, $\Xi$) in a unified way. Our detailed knowledge for the $S = 0$ $NN$ sector is based on the rich data of $NN$ scatterings as well as nuclear phenomena. Recent studies for $S = -1$ many-body systems such as $\Lambda$ hypernuclei have clarified interesting features of $\Lambda N$ and $\Sigma N$ interactions in spite of scarce data of the free-space scatterings. On the other hand, for baryon-baryon interactions with $S = -2$ sectors, concerned presently, experimental information has been highly limited due to the extreme difficulties of two-body scattering experiments. Therefore the observed $\Lambda\Lambda$ bond energies of double-Λ hypernuclei should be the most reliable source for the $S = -2$
interaction, and such data play a decisive role in determining the strength of underlying ΛΛ interactions.

In view of this relevance and the experimental situation, the NAGARA event is an epoch-making one which provides us with a new and firm basis for understanding the double-Λ hypernuclei. In recent years several experiments to produce $S = -2$ systems (E176 and E373 at KEK, E885 and E906 at BNL) have been performed and some of the data analyses are still in progress to get novel information on the $S = -2$ interactions.

In this exciting situation of the experimental study, it is needed to perform careful theoretical calculations of double-Λ hypernuclei with refreshed viewpoints. As one of the motivations of the present work, we think it necessary and timely to put the NAGARA data of $^6_{\Lambda\Lambda}$He binding energy as a new standard basis for a systematic study of a series of several double-Λ species. Secondly, in order to extract information on the ΛΛ interactions precisely, here we emphasize that hypernuclear calculations should be complete and realistic enough to leave structural ambiguity as negligibly as possible. All the dynamical changes due to successive Λ participation should be also taken faithfully. To meet these requirement we explore light $p$-shell double-Λ hypernuclei ($A = 6-10$) comprehensively using the microscopic three- and four-body models. Thirdly, by these systematic and realistic calculations, we will give reliable prediction of not only the ground-state binding energies but also possible excited-state energies, which encourages double-Λ hypernuclear spectroscopic study in near future.

So far several cluster models have appeared to estimate the ground-state binding energies of double-Λ species: Based on the old data of $^6_{\Lambda\Lambda}$He and $^{10}_{\Lambda\Lambda}$Be, Takaki et al. [8] applied a simplified version of the $\alpha + x + \Lambda + \Lambda$ cluster model to $A = 6-10$ systems in which they put several angular momentum restriction and neglected rearrangement channels. Bodmer et al. [9, 10] performed variational Monte Carlo calculations for $\alpha + \Lambda + \Lambda$ and $\alpha + \alpha + \Lambda + \Lambda$ to investigate consistency between the ΛΛ-binding energies, $B_{\Lambda\Lambda}(^6_{\Lambda\Lambda}\text{He})$ and $B_{\Lambda\Lambda}(^{10}_{\Lambda\Lambda}\text{Be})$, although their old data should be now updated. In the latest stage of this work, we encountered with the Faddeev-Yakubovsky calculations of $^6_{\Lambda\Lambda}$He and $^{10}_{\Lambda\Lambda}$Be by Filikhin and Gal [11] who restricted the equations within the $s$-wave. They compared the results with our previous cluster-model calculation [13] which was performed with wider model space but the stronger ΛΛ interaction strength. In our previous work [13], ΛΛ binding energies have been calculated for $^6_{\Lambda\Lambda}$He and $^{10}_{\Lambda\Lambda}$Be in the framework of the $\alpha + \Lambda + \Lambda$ three-body model and the
\[ \alpha + \alpha + \Lambda + \Lambda \] four-body model, respectively, where the adopted \( \Lambda \Lambda \) interaction is taken to be considerably attractive on the basis of the traditional interpretation for the double-\( \Lambda \) events.

In the present work, by noting the importance of the NAGARA data, we extend this four-body model to more general cases consisting of \( \alpha + x + \Lambda + \Lambda \) systems with \( x = n, p, d, t, ^3\text{He} \) and \( \alpha \left( ^7\Lambda\text{He}, ^7\Lambda\text{Li}, ^8\Lambda\text{Li}, ^9\Lambda\text{Be} \right) \), where nuclear core parts are quite well represented by \( \alpha + x \) cluster models (for example in Ref. 14). Here we remark that the extensive calculations are presented for the first time for \( A = 7 - 9 \) double-\( \Lambda \) species and that the old predictions for \( ^6\Lambda\text{He} \) and \( ^{10}\Lambda\text{Be} \) have been updated in a unified way. The four-body calculations are accurately performed by using the Jacobian-coordinate Gaussian-basis method of Refs. 15, 16, 17, 18, 19, 20, 21 with all the rearrangement channels taken into account. In our model, structure changes of nuclear cores caused by added one and two \( \Lambda \) particles are treated precisely. Namely, we take into account the rearrangement effects on \( \Lambda \Lambda \) bond energies induced by changes of nuclear cores. It is worthwhile to point out that the important effects of core-excitations and core-rearrangement are lacking in the frozen-core approximation used often for calculations of double-\( \Lambda \) hypernuclei.

In our model, it is possible to determine the \( \alpha x \) and \( \Lambda x \) interactions so as to reproduce all the existing binding energies of subsystems (\( \alpha + x, x + \Lambda, \alpha + x + \Lambda \) and \( \alpha + \Lambda + \Lambda \)) in an \( \alpha + x + \Lambda + \Lambda \) system, where that of \( \alpha + \Lambda + \Lambda \) is given by the NAGARA event. This feature is important to discuss the energy levels of the double-\( \Lambda \) hypernuclei and to extract the \( \Lambda \Lambda \) interactions because the ambiguities of \( NN \) and \( \Lambda N \) effective interactions are renormalized by fitting the known binding energies of subsystems phenomenologically. Our analysis is performed systematically for ground and bound excited states of the series of \( \alpha + x + \Lambda + \Lambda \) systems with no more adjustable parameters in this stage, so that these predictions offer an important guidance to interpret coming double-\( \Lambda \) events in the experiments and then to determine the level structure and the \( \Lambda \Lambda \) interaction unambiguously.

In Section II, the calculational method with microscopic \( \alpha + x + \Lambda + \Lambda \) four-body model is described. In Section III, the interactions are introduced. Calculated results are presented and discussed in Section IV. Summary is given in Section V.
II. MODEL AND METHOD

In Ref. [13], the present authors already studied $^6\Lambda_{\Lambda\Lambda}$He and $^{10}\Lambda_{\Lambda\Lambda}$Be with the use of $\alpha + \Lambda + \Lambda$ three-body model and $\alpha + \alpha + \Lambda + \Lambda$ four-body model, respectively. In the same manner, we study in this work the double-$\Lambda$ hypernuclei $^{7}\Lambda_{\Lambda\Lambda}$He, $^{7}\Lambda_{\Lambda\Lambda}$Li, $^{8}\Lambda_{\Lambda\Lambda}$Li, $^{9}\Lambda_{\Lambda\Lambda}$Be and $^{10}\Lambda_{\Lambda\Lambda}$Be on the basis of the $\alpha + x + \Lambda + \Lambda$ four-body model with $x = n, p, d, t,$ $^3$He and $\alpha$, respectively. The $d, t, ^3$He) and $\alpha$ clusters are assumed to be inert having the $(0s)^2, (0s)^3$ and $(0s)^4$ shell-model configurations, respectively, and are denoted by $\Phi_s(x)$ with spin $s (= 1, \frac{1}{2}$ or 0, respectively).

All the nine sets of the Jacobian coordinates of the four-body systems are illustrated in Fig. 1 in which we further take into account the antisymmetrization between two $\Lambda$ particles and the symmetrization between two $\alpha$ clusters when $x = \alpha$. The total Hamiltonian and the Schrödinger equation are given by

$$H = T + \sum_{(a,b)} V_{ab} + V_{\text{Pauli}} \ ,$$

$$\left( H - E \right) \Psi_{JM}^{A_{\Lambda\Lambda}Z} = 0 \ ,$$

where $T$ is the kinetic-energy operator and $V_{ab}$ is the interaction between the constituent particle-pair $a$ and $b$. The Pauli principle between the nucleons belonging to $\alpha$ and $x$ clusters is taken into account by the Pauli projection operator $V_{\text{Pauli}}$ which is explained in the next section as well as $V_{ab}$. The total wave function is described as a sum of amplitudes of the rearrangement channels ($c = 1 − 9$) of Fig. 1 in the LS coupling scheme:

$$\Psi_{JM}^{A_{\Lambda\Lambda}Z} = \sum_{c=1}^9 \sum_{n,N,\nu} \sum_{L,S,\Sigma,I,K} C_{nL}^{(c)} S_{\alpha} \left[ \Phi(\alpha) \left[ \Phi_s(x) \left[ \chi_{\frac{1}{2}}(\Lambda_1)\chi_{\frac{1}{2}}(\Lambda_2) \right]_S \right]_\Sigma \right. \left. \times \left[ \phi_{nlm}^c(r_c) \psi_{NLM}(R_c) \right]_I \right]_K \ ,$$

where the operator $\mathcal{A}_\alpha$ stands for antisymmetrization between the two $\Lambda$ particles, and $S_\alpha$ is the symmetrization operator for exchange between $\alpha$ clusters when $x = \alpha$. $\chi_{\frac{1}{2}}(\Lambda_i)$ is the spin function of the $i$-th $\Lambda$ particle. Following the Jacobian-coordinate coupled-rearrangement-channel Gaussian-basis variational method of Refs. [13, 16, 17, 18, 19, 20, 21], we take the functional form of $\phi_{nlm}(r)$, $\psi_{NLM}(R)$ and $\xi_{\nu\lambda\mu}(\rho_c)$ as

$$\phi_{nlm}(r) = r^l e^{-(r/r_n)^2} Y_{lm}(\hat{r}) \ ,$$

and

$$\psi_{NLM}(R) = Y_{LM}(\hat{R}) \ ,$$

$$\xi_{\nu\lambda\mu}(\rho_c) = \frac{1}{\sqrt{\rho_c}} Y_{\nu\lambda\mu}(\hat{\rho}_c) \ .$$
\[
\psi_{NLM}(\mathbf{R}) = R^L e^{-(R/R_N)^2} Y_{LM}(\hat{\mathbf{R}}),
\]
\[
\xi_{\nu\lambda\mu}(\rho) = \rho^\lambda e^{-(\rho/\rho_\nu)^2} Y_{\lambda\mu}(\hat{\rho}),
\]
\[
\text{(2.4)}
\]
where the Gaussian range parameters are chosen to lie in geometrical progressions:
\[
\begin{align*}
  r_n &= r_1 a^{n-1} \quad (n = 1 - n_{\text{max}}), \\
  R_N &= R_1 A^{N-1} \quad (N = 1 - N_{\text{max}}), \\
  \rho_\nu &= \rho_1 \alpha^{\nu-1} \quad (\nu = 1 - \nu_{\text{max}}).
\end{align*}
\]
\[
\text{(2.5)}
\]
These basis functions have been verified to be suited for describing both short-range correlations and long-range tail behaviour of few-body systems [15, 16, 17, 18, 19, 20, 21]. The eigenenergy \( E \) in Eq.(2.2) and the coefficients \( C \) in Eq.(2.3) are to be determined by the Rayleigh-Litz variational method.

For the angular-momentum space of the wave function, the approximation with \( l, L, \lambda \leq 2 \) was found to be enough in getting satisfactory convergence of the binding energies of the states concerned presently. Note that no truncation is taken of the interactions in the angular-momentum space. As for the numbers of the Gaussian basis, \( n_{\text{max}}, N_{\text{max}} \) and \( \nu_{\text{max}} \), 4 – 10 are enough.

As far as the single \( \Lambda \) hypernuclei \( ^6_\Lambda \text{He}, ^6_\Lambda \text{Li}, ^7_\Lambda \text{Li}, ^8_\Lambda \text{Li}, ^\Lambda \text{Be} \) and \( ^9_\Lambda \text{Be} \) are concerned, the wave functions are described by Eq.(2.3) but with one of \( \Lambda \) particles omitted. As for the core nucleus itself, \( \alpha + x \), the wave function is given by
\[
\Psi_{JM}(\alpha + x) = \sum_{n,l} C_{nl} S_\alpha \Phi(\alpha) \Phi_s(x) \phi_{nl}(\mathbf{r}) J_M.
\]
\[
\text{(2.6)}
\]

III. INTERACTIONS

In the study of double-\( \Lambda \) hypernuclei based on the \( \alpha + x + \Lambda + \Lambda \) four-body model, it is absolutely necessary and important to examine, before the four-body calculation, that the model with the interactions adopted is able to reproduce reasonably well the following observed quantities: (i) energies of the low-lying states and scattering phase shifts of the \( \alpha + x \) nuclear systems, (ii) \( B_\Lambda \) of hypernuclei composed of \( x + \Lambda \), \( x \) being \( d, t, ^3\text{He} \), \( \alpha \), (iii) \( B_\Lambda \) of hypernuclei composed of \( \alpha + x + \Lambda \), \( x \) being \( n, p, d, t, ^3\text{He} \), \( \alpha \) and (iv) \( B_{\Lambda\Lambda} \) of \( ^6_{\Lambda\Lambda} \text{He} = \alpha + \Lambda + \Lambda \). We emphasize that this severe examination were successfully done in the present
model as mentioned below. This encourages us to perform the four-body calculations, with no adjustable parameters at this stage, with high reliability of the results expected.

A. Pauli principle between $\alpha$ and $x$ clusters

The Pauli principle between nucleons belonging to $\alpha$ and $x$ clusters is taken into account by the orthogonality condition model (OCM)\cite{22}. The OCM projection operator $V_{\text{Pauli}}$ is represented by

$$V_{\text{Pauli}} = \lim_{\lambda \to \infty} \lambda \sum_f |\phi_f(r_{\alpha x})\rangle \langle \phi_f(r'_{\alpha x})|$$

(3.1)

which rules out the amplitude of the Pauli-forbidden $\alpha - x$ relative states $\phi_f(r_{\alpha x})$ from the four-body total wave function\cite{23}. The forbidden states are $f = 0S$ for $x = n(p)$, $f = \{0S, 0P\}$ for $x = d$, $f = \{0S, 1S, 0P, 0D\}$ for $x = t(3\text{He})$ and $f = \{0S, 1S, 0D\}$ for $x = \alpha$. The Gaussian range parameter $b$ of the single-particle $0s$ orbit in the $\alpha$ particle is taken to be $b = 1.358$ fm so as to reproduce the size of the $\alpha$ particle. The same size is assumed for clusters $x = d, t$ and $3\text{He}$ to manage the Pauli principle avoiding the calculational difficulty. In the actual calculations, the strength $\lambda$ for $V_{\text{Pauli}}$ is taken to be $10^5$ MeV which is large enough to push up away the unphysical forbidden states in the very high energy region with keeping the physical states unchanged. Usefulness of this Pauli operator method of OCM has been verified in many cluster-model calculations.

In some calculations\cite{9, 10, 11, 12, 24} of three-body systems including two or three $\alpha$ clusters, use is made of an $\alpha\alpha$ potential with a strong repulsive core\cite{25} so as to describe the Pauli exclusion role which prevents the two $\alpha$ cluster from overlapping. But, it is well known\cite{24} that this approximate prescription of the Pauli principle is not suited for the case where the presence of the third particle makes the two $\alpha$ clusters come closer to each other; in other words, the off-energy-shell behaviour of the repulsive potential is not appropriate in the three-body system. Moreover, there is no available potential reported for the $\alpha x$ systems ($x = n, p, d, t$ and $3\text{He}$) of this type. Therefore, we do not employ this prescription in the present systematic study of the structure change of the $\alpha x$ systems due to the addition of $\Lambda$ particles. We take the orthogonality condition model instead which is suited even for the case of heavy overlapping between the two clusters.
B. $\alpha x$ interactions

As for the potentials $V_{\alpha x}$ between the clusters $\alpha$ and $x$, we employ those which have been often used in the OCM-based cluster-model study of light nuclei. Namely, they are the $V_{\alpha N}$ potential introduced in Ref. [27], the $V_{\alpha d}$ and $V_{\alpha t}$ potentials given in Ref. [14] and the $V_{\alpha \alpha}$ potential used in Ref. [28] which reproduce reasonably well the low-lying states and low-energy scattering phase shifts of the $\alpha x$ systems. The potentials are described in the following parity-dependent form with the central and spin-orbit terms:

$$V_{\alpha x}(r) = \sum_{i=1}^{i_{\text{max}}} V_i e^{-\beta_i r^2} + \sum_{i=1}^{i'_{\text{max}}} (-)^l V^p_i e^{-\beta^p_i r^2}$$

$$+ \left[ \sum_{i=1}^{i''_{\text{max}}} V^{ls}_i e^{-\gamma_i r^2} + \sum_{i=1}^{i''_p} (-)^l V^{ls,p}_i e^{-\gamma^p_i r^2} \right] l \cdot s_x ,$$

(3.2)

where $l$ is the relative angular momentum between $\alpha$ and $x$, and $s_x$ is the spin of $x$. In the $\alpha\alpha$ system the spin-orbit term is missing and the odd wave is forbidden by the Pauli principle. The additional Coulomb potentials are constructed by folding the $pp$ Coulomb force into the proton densities of the $\alpha$ and $x$ clusters. The parameters in Eq.(3.2) are listed in Table I (we slightly modified the strength of the central force in $V_{\alpha d}$ and that of the spin-orbit force in $V_{\alpha t}$ to obtain better agreement with the energy levels of $^6\text{Li}$ and $^7\text{Li}$, respectively).

C. $\Lambda x$ interactions

We derive the interaction between the $\Lambda$ particle and the $x$ cluster by folding the G-matrix type $YN$ interaction (the YNG interaction) into the density of the $x$ cluster in the same manner of our previous work on the double-$\Lambda$ hypernuclei [13]. The YNG interactions between $\Lambda$ and $N$ are derived from the $YN$ OBE models as follows: First the G-matrix equation is solved in nuclear matter at each $k_F$, where the so called QTQ prescription is adopted for simplicity. Next the resulting G-matrix is simulated by a three-range Gaussian form with the strengths as a function of $k_F$. Obtained YNG interactions are given in Ref. [29] as

$$v_{\Lambda N}(r; k_F) = \frac{3}{2} \left[ v_{0,\text{even}}(i) + v_{\sigma,\text{even}}(i) \sigma_{\Lambda} \cdot \sigma_N \right] \left( 1 + \frac{P_r}{2} \right)$$
+ (v^{(i)}_{0,\text{odd}} + v^{(i)}_{\sigma\sigma,\text{odd}} \sigma_{\Lambda} \cdot \sigma_{N}) \frac{1 - P_{r}}{2} e^{-\mu_{r}r^{2}}, \quad (3.3)

where $P_{r}$ is the space exchange (Majorana) operator. The strengths $v^{(i)}$ are represented as quadratic functions of $k_{F}$; see Eq.(2.7) of Ref. [29] and Table V of Ref. [13] for various original $YN$ interactions. In the present work, we employ the Nijmegen model D interaction (ND).

The $\Lambda x$ interaction is derived by folding the above $v_{\Lambda N}(r; k_{F})$ interaction into the $x$-cluster wave function. The $k_{F}$ depends on the mass number of the cluster $x$. Because of the operator $P_{r}$ in Eq.(3.3), the resultant $\Lambda x$ potential becomes nonlocal, the explicit form of which is given in Appendix of Ref.[13]. We summarize the functional form of the local and nonlocal parts of the $\Lambda x$ potentials as

$$V_{\Lambda x}(r, r') = \sum_{i=1}^{3} (V_{i} + V^{s}_{i} s_{\Lambda} \cdot s_{x}) e^{-\beta_{i}r^{2}} \delta(r - r') \quad + \sum_{i=1}^{3} (U_{i} + U^{s}_{i} s_{\Lambda} \cdot s_{x}) e^{-\gamma_{i}(r+r')^{2} - \delta_{i}(r-r')^{2}}, \quad (3.4)$$

where $s_{\Lambda} = \sigma_{\Lambda}/2$. Table II lists the parameters in Eq.(3.4) for (a) $\Lambda \alpha$ interaction, (b) $\Lambda t(\Lambda^{3}\text{He})$ interaction and (c) $\Lambda d$ interaction. They were determined in the following manner:

i) $\Lambda \alpha$ interaction: The $\Lambda N$ spin-spin part vanishes by the folding into the $\alpha$ particle. The odd-force contribution is negligible to the $\Lambda$-binding energy of $^{5}_{\Lambda}\text{He}$. We determined the $k_{F}$ parameter as $k_{F} = 0.925 \text{ fm}^{-1}$ in order to reproduce this binding energy (3.12 MeV) within the $\alpha + \Lambda$ two-body model. The $\Lambda N$ odd-force having the same $k_{F}$ was determined by tuning the magnitude of $v^{(3)}_{0,\text{odd}}$, so as to reproduce, within the $\alpha + \alpha + \Lambda$ model, the $\Lambda$-binding energy of the $1/2^{+}$ ground state of $^{6}_{\Lambda}\text{Be}$.

ii) $\Lambda d$ interaction: We determined the value of $k_{F} = 0.84 \text{ fm}^{-1}$ by fitting the experimental $\Lambda$-binding energy of the $1/2^{+}$ ground state of $^{3}_{\Lambda}\text{H}$ within the $d + \Lambda$ model where the $\Lambda N$ odd force plays a negligible role. The odd force was determined, with the same $k_{F}$ kept, by reproducing the $\Lambda$-binding energies of the $1/2^{+}$ and $3/2^{+}$ states of $^{7}_{\Lambda}\text{Li}$ within the $\alpha + d + \Lambda$ model; we tuned $v^{(2)}_{0,\text{odd}}$ and $v^{(2)}_{\sigma\sigma,\text{odd}}$.

iii) $\Lambda t$ interaction: The experimental $\Lambda$-binding energies of the $0^{+}$ and $1^{+}$ states of $^{4}_{\Lambda}\text{H}$ were used to determine the even force of the $\Lambda N$ interaction. The magnitude of $k_{F}$ and $v^{(2)}_{\sigma\sigma,\text{even}}$ were adjusted to reproduce the energies, $k_{F}$ being $0.84 \text{ fm}^{-1}$. This value of $k_{F}$ was substituted into the $k_{F}$ used in the odd force of the $\Lambda N$ interaction of the $\Lambda d$ interaction.
with no other change. The resulting $\Lambda t$ interaction reproduces, by chance, the $\Lambda$-binding energy of the $1^+$ ground state of $^8\Lambda$Li within the $\alpha + t + \Lambda$ model; the calculated energy is 6.80 MeV while the observed one is $6.80 \pm 0.03$ MeV.

D. $\Lambda N$ interaction in $^{\Lambda\Lambda}_7$He ($^{\Lambda\Lambda}_7$Li)

In the study of $^{\Lambda\Lambda}_7$He ($^{\Lambda\Lambda}_7$Li) with the $\alpha + N + \Lambda + \Lambda$ model and of the subsystem $^{\Lambda}_8$He ($^{\Lambda}_8$Li) with the $\alpha + N + \Lambda$ model, it is inadequate to use the G-matrix type $\Lambda N$ interaction because $\Lambda N$ correlations are fully taken into account in our model space. Here, we employ a simple free-space $\Lambda N$ interaction with a three-range Gaussian form, which simulates the Nijmegen model F (NF) $\Lambda N$ interaction. Here, the $\Sigma N$ channel coupling contribution is renormalized into the $\Lambda N$ single channel with the closure approximation. The even- and odd-state parts of our $\Lambda N$ interaction are represented as follows:

$$V_{\Lambda N}(r) = \sum_{i=1}^{3} \left[ (v_i^{\text{even}} + v_i^{\text{even,}\sigma} \sigma_{\Lambda} \cdot \sigma_{N}) \frac{1 + P_r}{2} + (v_i^{\text{odd}} + v_i^{\text{odd,}\sigma} \sigma_{\Lambda} \cdot \sigma_{N}) \frac{1 - P_r}{2} \right] e^{-\mu_i r^2}. \quad (3.5)$$

First, the parameters are determined so as to simulate the $\Lambda N$ scattering phase shifts calculated with NF. Next, the second-range strengths $v_2^{\text{even}}$ and $v_2^{\text{even,}\sigma}$ are adjusted so as to reproduce the $\Lambda$ binding energies of the $0^+$ and $1^+$ states of $^4\Lambda$H with the use of the $N + N + N + \Lambda$ four-body model. Furthermore, strengths $v_2^{\text{odd}}$ and $v_2^{\text{odd,}\sigma}$ are adjusted within the framework of $\alpha + N + n + p$ four-body model so as to reproduce the observed binding energies of the ground-state spin doublet, $1/2^+$ and $3/2^+$ of $^4\Lambda$Li. Our resulting parameters in Eq. (3.5) are listed in Table III. We further found that the energy of the ground state of $^6\Lambda$He ($^6\Lambda$Li) measured from the $^4\Lambda$He $- N$ threshold can be well reproduced with our $\Lambda N$ interaction in the $\alpha + N + \Lambda$ three-body calculation; for $^6\Lambda$He ($^6\Lambda$Li), the calculated energy is $-0.17$ MeV (0.57 MeV), while the observed one is $-0.17$ MeV (0.59 MeV).

E. $\Lambda\Lambda$ interactions

In the present model, since the $\Lambda\Lambda$ relative motion is solved rigorously including the short-range correlations, it is not adequate to use the $\Lambda\Lambda$ G-matrix interaction given in Ref. [29]. However, our $\Lambda\Lambda$ interaction to be used in the present calculation should be
still considered as an effective interaction, since the couplings to ΞN and ΣΣ channels are not treated explicitly. Thus we employ the ΛΛ interaction represented in the following three-range Gaussian form:

\[
v_{\Lambda\Lambda}(r) = \sum_{i=1}^{3} \left( v_i + v_i^\sigma \sigma_\Lambda \cdot \sigma_\Lambda \right) e^{-\mu_i r^2}.
\] (3.6)

It is enlightening here to keep some linkage to the OBE models in determining the interaction parameters \(\mu_i, v_i\) and \(v_i^\sigma (i = 1 - 3)\). In our previous work on \(_{\Lambda\Lambda}^{6}\)He and \(_{\Lambda\Lambda}^{10}\)Be [13], the interaction parameters were chosen so as to simulate the ΛΛ sector of the ND interaction which is a reasonable model for the strong attraction suggested by the old interpretation for double-Λ hypernuclei. The characteristic feature of ND is that there is only a scalar singlet instead of a scalar nonet, which gives strongly attractive contribution in ΛΛ as well as NN.

The other versions of the Nijmegen models with a scalar nonet lead to much weaker ΛΛ attractions, which seems to be appropriate for the weak ΛΛ binding indicated by the NAGARA event. The NF is the simplest among these versions, which is adopted here as a guidance to construct our ΛΛ interaction: The outer two components of the above Gaussian potential \((i = 1, 2)\) are determined so as to simulate the ΛΛ sector of NF, and then the strength of the core part \((i = 3)\) is adjusted so as to reproduce the experimental value of \(B_{\Lambda\Lambda}(\Lambda\Lambda^{6}\)He). The obtained values of parameters are given in Table IV. It is interesting that the resulting ΛΛ interaction is almost equal to the interaction obtained by multiplying a factor 0.5 on the above ND-simulating interaction employed in Ref.[13].

IV. RESULTS AND DISCUSSIONS

Let us show the calculated results for a series of double-Λ hypernuclei with \(\alpha + x + \Lambda + \Lambda\) structures \((x = 0, n, p, d, t, ^3\)He, \(\alpha)\) studied in the microscopic four-body cluster model. In order to understand the role of two Λ particles attached to the core nuclei, it is useful to compare the obtained level structures of the \(\alpha + x + \Lambda + \Lambda\) double-Λ hypernuclei with those of the \(\alpha + x\) nuclei and the \(\alpha + x + \Lambda\) single-Λ hypernuclei. Then, we can see clearly how the ground and excited states of \(\alpha + x\) nuclei are changed due to the participation of Λ particles. It should be noted again here that, in the model description of \(\alpha + x + \Lambda + \Lambda\), the observed low-energy properties of the \(\alpha + x\) nuclei and the existing Λ-binding energies of the \(x + \Lambda\) and \(\alpha + x + \Lambda\) hypernuclei have been reproduced accurately enough to give reliable
predictions for the double-Λ hypernuclei with no adjustable parameters of the interactions in the four-body calculations. It offers the most reliable ground for our cluster model that the threshold energies for every partition into sub-cluster systems are assured to be reproduced.

A. Energy spectra

In Figs. 2 – 7, the calculated level structure of α + x core nuclei, α + x + Λ hypernuclei and α + x + Λ + Λ hypernuclei are illustrated side by side. There are shown all the ground and bound excited states of double-Λ hypernuclei predicted in the present model. In these figures, one sees clearly that injection of one and two Λ particles leads to stronger binding of the whole system and prediction of more bound states. But, there is no bound ‘p-orbit’ of Λ particle in single- and double-Λ hypernuclei with A ≤ 10. In the bound states of double-Λ hypernuclei, two Λ particles are coupled to S = 0 and therefore the spins and parities are the same as those of its nuclear core.

Table V summarizes the calculated ground-state energies for the double-Λ hypernuclei including the 2+ excited state of 10ΛΛ Be. The results are expressed in terms of two quantities: One is the total energy measured from the breakup threshold of α + x + Λ + Λ which is denoted as EΛΛ. The other is BΛΛ which is the binding energy of two Λ particles with respect to the ground-state nuclear core α + x.

The calculated values of BΛΛ can be compared with some experimental data, though they are quite limited at present. The most recent and clear data of the NAGARA event is used as a basic input of our model so that our ΛΛ interaction is adjusted to reproduce the experimental value BΛΛ(exp) (6ΛΛ He) = 7.25 ± 0.19 ± 0.18 MeV [1]. It is of particular interest to compare the present result with another data which is not used in the fitting procedure. There is an event found in the E373 experiment, named as Demachi-Yanagi event [30, 31]; the most probable interpretation of this event is a production of a bound state of 10ΛΛ Be having BΛΛ(exp) = 12.33 ± 0.35 MeV which is obtained by assuming BΞ(exp) = 0.15 ± 0.1 MeV. In the emulsion analysis there is no direct evidence for the production of 10ΛΛ Be in an excited state. However, if the produced 10ΛΛ Be is interpreted to be in the ground state, the resultant ΛΛ bond energy becomes repulsive, contradictorily to the NAGARA event. From the viewpoint of the present study, the Demachi-Yanagi event can be interpreted most probably as observation of the 2+ excited state in 10ΛΛ Be; our calculated value of BΛΛ 10ΛΛ Be(2+) ) = 12.28 MeV, which
agrees with the above experimental value. This good agreement suggests that our level structures calculated systematically are predictive and useful for coming events expected to be found in the further analysis for the E373 data. Now it should be stressed that the above experimental data of $^{10}_{\Lambda\Lambda}\text{Be}(2^+)$ leads to no information for the ground-state value of $B_{\Lambda\Lambda}$ unless the theoretical value (2.86 MeV in our case) of the excitation energy of $^{10}_{\Lambda\Lambda}\text{Be}(2^+)$ is utilized.

On the other hand, the old experiment by Danysz et al. \cite{2} on the pionic decay of $^{10}_{\Lambda\Lambda}\text{Be}(0^+) \rightarrow ^9_{\Lambda}\text{Be}(1/2^+) + p + \pi^-$ gave $B^\text{exp}_{\Lambda\Lambda}(^{10}_{\Lambda\Lambda}\text{Be}(0^+)) = 17.7 \pm 0.4$ MeV. This value has been used for a long time, which means the strongly attractive $\Lambda\Lambda$ interaction. However, it should be noted that the authors also suggested the possibility of another decay $^{10}_{\Lambda\Lambda}\text{Be}(0^+) \rightarrow ^9_{\Lambda}\text{Be}(3/2^+, 5/2^+) + p + \pi^-$ (Table 5 of Ref.\cite{2}); the same was pointed out in Ref. \cite{12}, too. In this case, the value of $B^\text{exp}_{\Lambda\Lambda}(^{10}_{\Lambda\Lambda}\text{Be}(0^+))$ is modified to $14.6 \pm 0.4$ MeV, which is obtained by using the excitation energy of $^9_{\Lambda}\text{Be}(3/2^+, 5/2^+)$ to be $3.05$ MeV \cite{32}. This modified value turns out to be not contradictory with our calculated value, 15.14 MeV. A similar re-interpretation, with the hypernuclear excited states taken into account, may be needed also for the E176 event which was identified as $^{13}_{\Lambda\Lambda}\text{B}(^{10}_{\Lambda\Lambda}\text{Be})$ with the strongly attractive (repulsive) $\Lambda\Lambda$ interaction.

Thus, we have understood the consistency between the experimental data and our theoretical results of $^{10}_{\Lambda\Lambda}\text{Be}$. We, therefore, discuss on the level structures of double-$\Lambda$ hypernuclei in more detail. As seen in Figs. 2 – 7 and Table V, the $\Lambda$ particle plays a glue-like role so that a whole system becomes of stronger binding. This effect in a double-$\Lambda$ nucleus is more enhanced than that in the corresponding single-$\Lambda$ nucleus. One can see a typical example in the case of $^7_{\Lambda\Lambda}\text{Li}$ in Fig. 3. For the unbound nuclear system of $^5\text{Li}$, a single $\Lambda$ cannot make a bound system of $^6_{\Lambda}\text{Li}$, but, the addition of one more $\Lambda$ particle leads to a bound system of $^7_{\Lambda\Lambda}\text{Li}$ whose ground state is of weakly-binding with respect to the $^6_{\Lambda\Lambda}\text{He} + p$ threshold.

The bound excited states of double-$\Lambda$ hypernuclei predicted in the present cluster model are summarized as follows: In $^7_{\Lambda\Lambda}\text{He}$ and $^7_{\Lambda\Lambda}\text{Li}$, the ground states are both bound but no excited states are predicted. It is needless to say, there are no bound excited states in double-$\Lambda$ hypernuclei with $A \leq 6$ since there is no bound excited state in their core nuclei. The lightest double-$\Lambda$ hypernucleus that has at least one excited state is $^8_{\Lambda\Lambda}\text{Li}$. In $^8_{\Lambda\Lambda}\text{Li}$ we predict two $T = 0$ excited states in the bound-state region. It is expected to have a $T = 1, 0^+$ bound excited state in $^8_{\Lambda\Lambda}\text{Li}$ which corresponds to the $T = 1, 0^+$ state in $^6\text{Li}$.
nucleus at $E_x = 3.56$ MeV, but the state is not shown in Fig. 4 because the $T = 1$ state may have five-body structure and is out of scope of the present cluster model. We predict three bound excited states in $^{9}_{\Lambda\Lambda}\text{Li}$ ($^{9}_{\Lambda\Lambda}\text{Be}$). There is one bound excited state in $^{10}_{\Lambda\Lambda}\text{Be}$ as mentioned before. It will be challenging to discover these excited states one by one as well as the ground states.

B. Dynamical change of the core nucleus

It is interesting to look at the dynamical change of the $\alpha + x$ nuclear cores which occurs due to the successive participation of two $\Lambda$ particles. The possibility that a nuclear core shrinks due to an attached $\Lambda$ particle has been theoretically pointed out using the $\alpha + x + \Lambda$ cluster model of light $p$-shell $\Lambda$ hypernuclei [33]. An updated prediction [34] was made specifically on a shrinkage in size by 21% to be seen in $^{7}_{\Lambda}\text{Li}$. The recent measurement of $\gamma$-ray transition rate in $^{7}_{\Lambda}\text{Li}$ [35] has confirmed quantitatively the shrinkage effect predicted in both the old calculation and the updated one. It is quite reasonable, therefore, that in a double-$\Lambda$ hypernucleus the participation of one more $\Lambda$ particle can induce further shrinkage of the nuclear core. Such an effect has been also investigated systematically using the molecular orbital model for $^{8+n}_{n\Lambda}\text{Be}$ ($n = 1 - 4 = \alpha + \alpha + n\Lambda$ [36].

In order to see such shrinkage effect, we show three physical quantities: first in Table VI we list the r.m.s. distance between $\alpha$ and $x$, $\bar{r}_{\alpha x}$. As the number of the $\Lambda$ particles increases, $\bar{r}_{\alpha x}$ turns out to shrink significantly due to the glue-like role of the attached $\Lambda$ particles. For example, one sees $\bar{r}_{\alpha x}$ changes as $4.11 \rightarrow 3.44 \rightarrow 3.16$ fm for $^{6}_{\Lambda}\text{Li} \rightarrow ^{7}_{\Lambda}\text{Li} \rightarrow ^{8}_{\Lambda\Lambda}\text{Li}$. Participation of the second $\Lambda$ gives rise to about 8% reduction of $\bar{r}_{\alpha x}$ except $x = n$. Secondly, in more detail, it is worthwhile to demonstrate in Fig. 8 the change of the $\alpha - n$ two-body density (correlation function) $\rho(r_{\alpha n})$ in the $^5\text{He}$, $^6\text{He}$ and $^{7}_{\Lambda}\text{He}$ when $\Lambda$ particles participate successively, which again manifests the shrinkage effect. Thirdly, this shrinkage effect is seen in the large change of the expectation value of the relative kinetic energy, $<T_{\alpha x}>$, and that of the potential energy, $<V_{\alpha x}>$, in the $\alpha - x$ subsystems. When the $\alpha$ and $x$ clusters approach to each other, the increase of $<T_{\alpha x}>$ overcomes the gain of $<V_{\alpha x}>$, and the sum $<T_{\alpha x} + V_{\alpha x}>$ increases appreciably. In spite of this energy loss in the $\alpha - x$ core system, the core shrinkage is realized by the stronger energy gain of the $\Lambda - \alpha$ and $\Lambda - x$ parts.
C. \( \Lambda \Lambda \) bond energy

In Fig. 9 we reveal the contributions of the \( \Lambda \Lambda \) interaction to the total binding energies of double-\( \Lambda \) hypernuclei \( ^A_{\Lambda\Lambda}Z \). Here the calculated values of \( B_{\Lambda\Lambda}(^A_{\Lambda\Lambda}Z) \) in the ground states are shown by closed circles. In order to extract the contribution of the \( \Lambda \Lambda \) interaction, we perform the same calculations by putting \( V_{\Lambda\Lambda} = 0 \). The obtained values are denoted as \( B_{\Lambda\Lambda}(^A_{\Lambda\Lambda}Z; V_{\Lambda\Lambda} = 0) \) and shown by open circles in the figure. It should be noted that the effect of the dynamical change of the \( \alpha + x \) core due to the \( \Lambda N \) interactions is included in the four-body estimate of \( B_{\Lambda\Lambda} \) and \( B_{\Lambda\Lambda}(V_{\Lambda\Lambda} = 0) \). Since the \( \Lambda \Lambda \) interaction is not so strong compared with the \( \Lambda N \) interaction, the core-rearrangement effects included in \( B_{\Lambda\Lambda} \) and \( B_{\Lambda\Lambda}(V_{\Lambda\Lambda} = 0) \) are similar to each other. Then, naturally the pure effect of the \( \Lambda \Lambda \) interaction is given by the difference

\[
\gamma_{\Lambda\Lambda}^{\text{bond}}(^A_{\Lambda\Lambda}Z) \equiv B_{\Lambda\Lambda}(^A_{\Lambda\Lambda}Z) - B_{\Lambda\Lambda}(^A_{\Lambda\Lambda}Z; V_{\Lambda\Lambda} = 0). \tag{4.1}
\]

We consider \( \gamma_{\Lambda\Lambda}^{\text{bond}} \) as the \( \Lambda \Lambda \) bond energy which should be determined essentially by the strength of the \( \Lambda \Lambda \) interaction. Now in Fig. 9, we find that the magnitude of \( \gamma_{\Lambda\Lambda}^{\text{bond}} \), the energy difference between the closed and open circles, is almost constant at \( \sim 1 \text{ MeV} \) for all the double-\( \Lambda \) hypernuclei with \( A = 6 - 10 \). The detailed values of \( \gamma_{\Lambda\Lambda}^{\text{bond}} \) are listed in Table V.

So far the following intuitive formula has been often used to estimate the \( \Lambda \Lambda \) interaction strength:

\[
\Delta B_{\Lambda\Lambda}(^A_{\Lambda\Lambda}Z) \equiv B_{\Lambda\Lambda}(^A_{\Lambda\Lambda}Z) - 2B_{\Lambda}(^{A-1}_\Lambda Z). \tag{4.2}
\]

It is worthwhile to point out the problems underlying in this formula: This expression includes three problems which come from i) the mass-polarization term of the three-body kinetic-energy operator, ii) the \( \Lambda N \) spin-spin interaction and iii) the dynamical change of the core nuclear structure.

The problem i) is stated as follows: In the \( \alpha + \Lambda + \Lambda \) three-body model for \( ^6_{\Lambda\Lambda}\text{He} \) (generally, ",\( \alpha \)" may be replaced by "spinless frozen-core nucleus"), if one takes the \textit{non}-Jacobian coordinate set \( r_{\alpha\Lambda_1} \) and \( r_{\alpha\Lambda_2} \), the Shrödinger equation may be written, in a self-explanatory notation, as

\[
\left[ - \frac{\hbar^2}{2\mu_{\alpha\Lambda_1}} \nabla^2_{\alpha\Lambda_1} - \frac{\hbar^2}{2\mu_{\alpha\Lambda_2}} \nabla^2_{\alpha\Lambda_2} - \frac{\hbar^2}{m_\alpha} \nabla_{\alpha\Lambda_1} \cdot \nabla_{\alpha\Lambda_1} + V_{\alpha\Lambda_1} + V_{\alpha\Lambda_2} + V_{\Lambda_1\Lambda_2} - E \right] \Psi_{JM}(^6_{\Lambda\Lambda}\text{He}) = 0. \tag{4.3}
\]
If the third term of the kinetic energy, the so-called mass-polarization term, and $V_{\Lambda_1\Lambda_2}$ are neglected, we have the trivial solution $-E(=B_{\Lambda\Lambda}) = 2B_\Lambda$. Therefore, the quantity $\Delta B_{\Lambda\Lambda} = B_{\Lambda\Lambda} - 2B_\Lambda$ stands for the contribution from the neglected two terms. In $^6\Lambda\Lambda$He, the contribution to $B_{\Lambda\Lambda}$ from the mass-polarization term is +0.13 MeV which explains the difference between $\Delta B_{\Lambda\Lambda} = 1.01$ MeV and the $\Lambda\Lambda$ bond energy $V_{\Lambda\Lambda}^{\text{bond}} = 0.88$ MeV in Table V. This contribution decreases rapidly as the core-nuclear mass increases (+0.01 MeV in $^{10}\Lambda\Lambda$Be).

Next, we discuss about the second problem, an effect of the $\Lambda N$ spin-spin interaction on $\Delta B_{\Lambda\Lambda}$ of Eq.(4.2). In Fig. 10, the calculated values of $\Delta B_{\Lambda\Lambda}$ are illustrated by the dashed bars. One notices clearly that $\Delta B_{\Lambda\Lambda}$ has peculiar mass dependence in which some interesting mechanism is included. It should be remarked here, however, that, as was already pointed out by Danysz et al. [2], the traditional definition of Eq.(4.2) is of simple meaning only when the nuclear core is spinless. On the other hand, in the case of nuclear core with spin, the single-$\Lambda$ binding energy $B_\Lambda$ to be subtracted from $B_{\Lambda\Lambda}$ is distributed over the ground-state doublet of the corresponding single-$\Lambda$ hypernucleus.

Here, we remark the fact that the $\Lambda N$ spin-spin interaction is not effective (cancelled out) in the double-$\Lambda$ hypernuclei having the $\Lambda\Lambda$ spin-singlet pairs. In the parent single-$\Lambda$ hypernuclei, however, the spin-spin interaction plays an important role in giving rise to the energy splitting of the ground-state doublet. The typical and unique example known experimentally is the spin-doublet in $^7\Lambda\Lambda$Li with $J = \frac{1}{2}^+$(ground; $B_\Lambda = 5.58$ MeV) and $J = \frac{3}{2}^+(E_x = 0.69$ MeV; $B_\Lambda = 4.49$ MeV). Considering this effect, one should use the spin-averaged value $\bar{B}_\Lambda(\Lambda\Lambda\text{Li}) = \frac{1}{3}B_\Lambda(\frac{1}{2}_{g.s.}^+) + \frac{2}{3}B_\Lambda(\frac{3}{2}^+)$ instead of $B_\Lambda(\frac{1}{2}_{g.s.}^+)$ when one likes to deduce $\Delta B_{\Lambda\Lambda}$ from the $^8\Lambda\Lambda\text{Li}(1^+)$ ground state data, if any. If we adopt this prescription also for the adjacent systems, we may use

\[
\begin{align*}
\bar{B}_\Lambda(^6\Lambda\Lambda\text{He}) &= \frac{1}{4}B_\Lambda(1^-_{g.s.}) + \frac{3}{4}B_\Lambda(2^-) = 4.02 \text{ MeV}, \\
\bar{B}_\Lambda(^6\Lambda\Lambda\text{Li}) &= \frac{1}{4}B_\Lambda(1^-_{g.s.}) + \frac{3}{4}B_\Lambda(2^-) = 4.31 \text{ MeV}, \\
\bar{B}_\Lambda(^7\Lambda\Lambda\text{Li}) &= \frac{1}{3}B_\Lambda(\frac{1}{2}_{g.s.}^+) + \frac{2}{3}B_\Lambda(\frac{3}{2}^+) = 5.12 \text{ MeV}, \\
\bar{B}_\Lambda(^7\Lambda\Lambda\text{Be}) &= \frac{1}{4}B_\Lambda(1^-_{g.s.}) + \frac{3}{4}B_\Lambda(2^-) = 6.48 \text{ MeV}, \\
\bar{B}_\Lambda(^8\Lambda\Lambda\text{Be}) &= \frac{1}{4}B_\Lambda(1^-_{g.s.}) + \frac{3}{4}B_\Lambda(2^-) = 6.58 \text{ MeV}. 
\end{align*}
\]
Here, $B_{\Lambda}$ of the excited states are taken from our calculation. In general, we have

$$
\bar{B}_{\Lambda}(A^{-1}Z) = \frac{J_0}{2J_0 + 1}B_{\Lambda}(A^{-1}Z; J_1 = J_0 - \frac{1}{2}) + \frac{J_0 + 1}{2J_0 + 1}B_{\Lambda}(A^{-1}Z; J_1 = J_0 + \frac{1}{2}),
$$

where $J_1 = J_0 \pm \frac{1}{2}$ denote the doublet spins of the $\alpha + x + \Lambda$ system, $J_0$ being the ground-state spin of the $\alpha + x$ nuclear core. For the two spinless cases ($x = 0$ and $\alpha$), needless to say, $\bar{B}_{\Lambda}^{5\Lambda}{\rm He}) = B_{\Lambda}(5\Lambda{\rm He; \frac{1}{2}_{g.s.}})$ and $\bar{B}_{\Lambda}^{9\Lambda}{\rm Be}) = B_{\Lambda}(9\Lambda{\rm Be; \frac{1}{2}_{g.s.}})$.

Thus, replacing $B_{\Lambda}$ with $\bar{B}_{\Lambda}$ in Eq.(4.2), we modify $\Delta B_{\Lambda\Lambda}$ by $\Delta \bar{B}_{\Lambda\Lambda}$ as

$$
\Delta \bar{B}_{\Lambda\Lambda}(A\Lambda\Lambda Z) \equiv B_{\Lambda\Lambda}(A\Lambda\Lambda Z) - 2\bar{B}_{\Lambda}(A^{-1}Z).
$$

In Fig. 11, the solid bars illustrate $\Delta \bar{B}_{\Lambda\Lambda}$. Though $\Delta \bar{B}_{\Lambda\Lambda}$ is free from the effect of the $\Lambda N$ spin-spin interaction, its magnitude for $A = 7 - 10$ deviates significantly from $\Delta \bar{B}_{\Lambda\Lambda} (\Lambda\Lambda 6\Lambda{\rm He}) = 1.01$ MeV. The deviation comes from the effect of the dynamical change in the core nucleus structure (shrinkage in the $\alpha - x$ distance) due to the partition of the $\Lambda$ hyperons, and turns out to be maximum in the case of $\Lambda\Lambda 10\Lambda{\rm Be}$. We emphasize that, even if one employs $\Delta \bar{B}_{\Lambda\Lambda}$, it is impossible to extract any consistent value of the $\Lambda\Lambda$ bond energy from Fig. 11 in which $\Delta \bar{B}_{\Lambda\Lambda}$ scatters in a range of a factor of two.

As mentioned above, a consistent estimation of the $\Lambda\Lambda$ bond energy ($0.9 - 1.0$ MeV, nearly independent of the mass number, as seen in Table V) can be obtained by taking $\mathcal{V}_{\Lambda\Lambda}^{\text{bond}}$ of Eq. (4.1) as the definition of that energy, though help of the theoretical calculation with $V_{\Lambda\Lambda} = 0$ is necessary.

V. SUMMARY

We have carried out structure calculations of $\Lambda\Lambda 6\Lambda{\rm He}, \Lambda\Lambda 7\Lambda{\rm He}, \Lambda\Lambda 7\Lambda{\rm Li}, \Lambda\Lambda 8\Lambda{\rm Li}, \Lambda\Lambda 9\Lambda{\rm Li,}, \Lambda\Lambda 9\Lambda{\rm Be}$ and $\Lambda\Lambda 10\Lambda{\rm Be}$ taking the framework of $\alpha + x + \Lambda + \Lambda$ model with $x = 0, n, p, d, t, 3\Lambda{\rm He}$ and $\alpha$, respectively. We determined the interactions between constituent particles so as to reproduce reasonably the observed low-energy properties of the $\alpha + x$ nuclei and the existing data of $\Lambda$-binding energies of the $x + \Lambda$ and $\alpha + x + \Lambda$ systems. The $\Lambda\Lambda$ interaction was constructed so as to reproduce the $B_{\Lambda\Lambda}(6\Lambda{\rm He})$ given by the NAGARA event within our $\alpha + \Lambda + \Lambda$ model, where the long-range part of our interaction was adjusted to simulate the behavior of the appropriate OBE model (NF). With no adjustable parameters, the four-body calculations of
the $\alpha + x + \Lambda + \Lambda$ systems were performed accurately using the Jacobian-coordinate Gaussian-basis coupled-rearrangement-channel method. Obtained energy spectra of the double-$\Lambda$ hypernuclei with $A = 6 − 10$ are summarized in Fig. 12.

Major results to be emphasized here are as follows:

(1) It is striking that the calculated $B_{\Lambda\Lambda}$ of the $2^+$ excited state in $^{10}_{\Lambda\Lambda}$Be, 12.28 MeV, agrees with the experimental value $B_{\Lambda\Lambda}^{\exp}(^{10}_{\Lambda\Lambda}$Be) = 12.33$^{+0.35}_{-0.21}$ MeV in the Demachi-Yanagi event [30, 31]. We therefore interpret this event as observation of the $2^+$ excited state of $^{10}_{\Lambda\Lambda}$Be. The agreement suggests that our systematic calculations are predictive for coming events expected to be found in the further analysis of the E373 data, etc.

(2) Together with the energy spectrum of each double-$\Lambda$ hypernucleus, those of the corresponding core nucleus and single-$\Lambda$ hypernucleus are exhibited side by side in Figs. 2 − 7 so as to see clearly that injection of one and two $\Lambda$ particles leads to stronger binding of the whole system and prediction of more bound states. In the bound states of any double-$\Lambda$ hypernucleus, two $\Lambda$ particles are dominantly coupled to $S = 0$ and hence the spin and parity become the same as those of its nuclear core, but the theoretical $B_{\Lambda\Lambda}$ values are of importance to guide the analysis of the emulsion experiments.

(3) Dynamical change of the $\alpha + x$ nuclear core by the participation of the $\Lambda$ particles is substantially seen in double-$\Lambda$ hypernuclei; there occurs, averagely speaking, about 8% shrinkage of the $\alpha − x$ distance compared with the distance in the single-$\Lambda$ hypernucleus. This shrinkage is realized by the large energy gain in the $\Lambda − \alpha$ and $\Lambda − x$ parts which overcomes the energy loss in the $\alpha − x$ relative motion.

(4) We estimated the $\Lambda\Lambda$ bond energy using the faithful definition $\nu_{\Lambda\Lambda}^{\text{bond}} = B_{\Lambda\Lambda} - B_{\Lambda\Lambda}(V_{\Lambda\Lambda} = 0)$ and found it to be 0.88 MeV for $^{6}_{\Lambda\Lambda}$He and 0.93 − 0.98 MeV for the other double-$\Lambda$ hypernuclei. We demonstrated that the quantity $\Delta B_{\Lambda\Lambda} = B_{\Lambda\Lambda} - 2\bar{B}_{\Lambda}$ is not a good measure of the $\Lambda\Lambda$ bond energy since $\Delta B_{\Lambda\Lambda}$ is free from neither the contribution from the splitting of the ground-state doublet in the single-$\Lambda$ hypernucleus nor that of the structure change of the core nucleus. In fact, the value of $\Delta B_{\Lambda\Lambda}$ scatters from 0.28 to 1.68 MeV for the double-$\Lambda$ hypernuclei with $A = 6 − 10$. We then modified $\Delta B_{\Lambda\Lambda}$ by $\Delta \bar{B}_{\Lambda\Lambda} = B_{\Lambda\Lambda} - 2\bar{B}_{\Lambda}$ with $\bar{B}_{\Lambda}$ being the spin-average of $B_{\Lambda}$'s for the ground-state spin-doublet. We found, however, that $\Delta \bar{B}_{\Lambda\Lambda}$ still ranges from 0.83 to 1.68 MeV due to the structure change of the core nucleus. Direct use of $B_{\Lambda\Lambda}$ itself rather than the use of $\Delta B_{\Lambda\Lambda}$ or $\Delta \bar{B}_{\Lambda\Lambda}$ is recommended when the experimental result and calculational result are compared to each other.

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In conclusion, the present precise and extensive four-body cluster-model calculation can be an opening of the spectroscopic study of double-$\Lambda$ hypernuclei.

Acknowledgments

The authors would like to thank Professor K. Nakazawa and Dr. H. Takahashi for valuable discussions and information on the experimental project KEK-E373. They are also thankful to Professor Y. Akaishi, Professor K. Ikeda and Professor A. Gal for helpful discussions and encouragement. One of the authors (T.M.) thanks S. Kahana, L. McLerran, D.J. Millener and Physics Department of Brookhaven National Laboratory for their hospitality and support. He is also grateful to the Institute for Nuclear Theory at the University of Washington for its hospitality. This work was supported by the Grant-in-Aid for Scientific Research of Monbukagakushou of Japan.

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TABLE I. Parameters of (a) $\alpha\alpha$ interaction, (b) $\alpha t$ ($^3\text{He}$) interaction, (c) $\alpha d$ interaction and (d) $\alpha N$ interaction defined in Eq.(3.2). Size parameters are in fm$^{-2}$ and strengths are in MeV. The $^1S_0$ scattering length is -0.575 fm and the effective range is 6.45 fm.

|        | (a) $\alpha\alpha$ interaction |       |       |       |
|--------|---------------------------------|-------|-------|-------|
|        | $i$                             | 1     | 2     | 3     |
| $\beta_i$ |                               | 0.1111| 0.2777| 0.3309|
| $V_i$   |                                | -1.742| -395.9| 299.4 |
| $V_i^p$ |                                | 0.0   | 0.0   | 0.0   |

|        | (b) $\alpha t$ ($^3\text{He}$) interaction |       |       |       |
|--------|----------------------------------------------|-------|-------|-------|
|        | $i$                                          | 1     | 2     | 3     |
| $\beta_i$ |                                         | 0.0913| 0.1644| 0.2009|
| $V_i$   |                                         | 6.9   | -43.35| -51.7 |
| $\beta_i^p$ |                                    | 0.0913| 0.1644| 0.2009|
| $V_i^p$ |                                    | 6.9   | 43.35 | -51.7 |
| $\gamma_i$ |                                | 0.28  |       |       |
| $V_i^{ls}$ |                              | -1.2  |       |       |
| $\gamma_i^p$ |                            | 0.28  |       |       |
| $V_i^{ls,p}$ |                         | 1.2   |       |       |

|        | (c) $\alpha d$ interaction |       |       |       |
|--------|----------------------------|-------|-------|-------|
|        | $i$                        | 1     |       |       |
| $\beta_i$ |                        | 0.2   |       |       |
| $V_i$   |                       | -64.21|       |       |
| $\beta_i^p$ |                   | 0.2   |       |       |
| $V_i^p$ |                   | -10.21|       |       |
| $\gamma_i$ |               | 0.3   |       |       |
| $V_i^{ls}$ |            | -4.0  |       |       |
| $\gamma_i^p$ |           | 0.3   |       |       |
| $V_i^{ls,p}$ |        | -4.0  |       |       |

|        | (d) $\alpha N$ interaction |       |       |       |
|--------|-----------------------------|-------|-------|-------|
|        | $i$                        | 1     | 2     | 3     |
| $\beta_i$ |                     | 0.36  | 0.9   |       |
| $V_i$   |                   | -96.3 | 77.0  |       |
| $\beta_i^p$ |               | 0.2   | 0.53  | 2.5   |
| $V_i^p$ |               | 34.0  | -85.0 | 51.0  |
| $\gamma_i$ |           | 0.396 | 0.52  | 2.2   |
| $V_i^{ls}$ |       | -20.0 | -16.8 | 20.0  |
| $\gamma_i^p$ |     | 0.396 | 2.2   |       |
| $V_i^{ls,p}$ |   | 6.0   | -6.0  |       |
Table II. Parameters of (a) Λ α interaction, (b) Λ t (Λ³He) interaction and (c) Λ d interaction defined in Eq.(3.4). Size parameters are in fm⁻² and strengths are in MeV.

|   | (a) Λ α interaction |   |   |   | (b) Λ t (Λ³He) interaction |   |   |   | (c) Λ d interaction |   |   |   |
|---|---------------------|---|---|---|----------------------------|---|---|---|---------------------|---|---|---|
| i | 1                   | 2 | 3 |   | i                           | 1 | 2 | 3 | i                   | 1 | 2 | 3 |
| β_i | 0.2752              | 0.4559 | 0.6123 |   | β_i                          | 0.2874 | 0.4903 | 0.6759 | β_i                   | 0.3153 | 0.5773 | 0.8532 |
| V_i | −17.49              | −127.0 | 497.8   |   | V_i                          | −14.16 | −108.0 | 425.9   | V_i                   | −10.84 | −88.36 | 167.2   |
| V_is | 0.0                 | 0.0    | 0.0     |   | V_is                         | 2.379  | 10.91  | −126.9  | V_is                 | 2.734  | 14.35  | −179.9  |
| γ_i | 0.1808              | 0.1808 | 0.1808  |   | γ_i                          | 0.2033 | 0.2033 | 0.2033  | γ_i                   | 0.2710 | 0.2710 | 0.2710  |
| δ_i | 0.4013              | 0.9633 | 2.930   |   | δ_i                          | 0.3383 | 0.8234 | 2.521   | δ_i                   | 0.2470 | 0.4870 | 1.924   |
| U_i | −0.3706             | −12.94 | −331.2  |   | U_i                          | −0.2701 | −9.553 | −231.6  | U_i                  | −0.1862 | −5.844 | −3.065  |
| U_is | 0.0                 | 0.0    | 0.0     |   | U_is                         | −0.2615 | 1.433  | 97.05   | U_is                | −0.2705 | 1.566  | 100.4   |

TABLE III. Parameters of the Λ N interaction defined in Eq.(3.5) which is used only in the α + N + Λ and α + N + Λ + Λ systems (x = N). Size parameters are in fm⁻² and strengths are in MeV.

|   | Λ N interaction when x = N |   |   |   |   |   |   |   |   |
|---|---------------------------|---|---|---|---|---|---|---|---|
| i | 1 | 2 | 3 |   |   |   |   |   |
| μ_i | 0.5487 | 1.384 | 6.250 |   |   |   |   |   |
| v_i^{even} | −10.40 | −87.05 | 1031 |   |   |   |   |   |
| v_i^{even,σ} | 0.2574 | 17.09 | −256.3 |   |   |   |   |   |
| v_i^{odd} | −5.816 | −18.29 | 4029 |   |   |   |   |   |
| v_i^{odd,σ} | −0.959 | −9.184 | −573.8 |   |   |   |   |   |
TABLE IV. Parameters of the Λ Λ interaction defined in Eq.(3.6). Size parameters are in fm$^{-2}$ and strengths are in MeV. The $^1S_0$ scattering length is -0.575 fm and the effective range is 6.45 fm.

| $i$ | 1   | 2   | 3   |
|-----|-----|-----|-----|
| $\mu_i$ | 0.555 | 1.656 | 8.163 |
| $v_i$ | -10.67 | -93.51 | 4884 |
| $v_i^\sigma$ | 0.0966 | 16.08 | 915.8 |

Table V. Calculated energies of the ground states of $A = 6 - 10$ double-Λ hypernuclei based on the $\alpha + x + \Lambda + \Lambda$ four-body model ($x = 0, n, p, d, t, ^3$He, and $\alpha$). $E_{\Lambda\Lambda}$ are measured from the $\alpha + x + \Lambda + \Lambda$ threshold. The Λ Λ bond energy $V_{\Lambda\Lambda}^{\text{bond}}$ is defined by Eq.(4.1). Information on the $2^+$ excited state of $^{10}_{\Lambda\Lambda}$Be is specially added so as to demonstrate the agreement with the experimental result.

| $J^\pi$ | $E_{\Lambda\Lambda}$ (MeV) | $B_{\Lambda\Lambda}$ (MeV) | $B_{\Lambda\Lambda}^{\text{exp}}$ (MeV) | $V_{\Lambda\Lambda}^{\text{bond}}$ (MeV) |
|--------|----------------|----------------|----------------|----------------|
| $^{6}_{\Lambda\Lambda}$He | $0^+$ | $-7.25$ | $7.25$ | $7.25 \pm 0.19^a$ | 0.88 |
| $^{7}_{\Lambda\Lambda}$He | $\frac{3}{2}^-$ | $-8.47$ | $9.36$ | - | 0.96 |
| $^{7}_{\Lambda\Lambda}$Li | $\frac{3}{2}^-$ | $-7.48$ | $9.45$ | - | 0.95 |
| $^{8}_{\Lambda\Lambda}$Li | $1^+$ | $-12.10$ | $11.44$ | - | 0.98 |
| $^{9}_{\Lambda\Lambda}$Li | $\frac{3}{2}^-$ | $-17.05$ | $14.55$ | - | 0.98 |
| $^{9}_{\Lambda\Lambda}$Be | $\frac{3}{2}^-$ | $-16.00$ | $14.40$ | - | 0.97 |
| $^{10}_{\Lambda\Lambda}$Be | $0^+$ | $-15.05$ | $15.14$ | $17.7 \pm 0.4^b$ | 0.93 |
| $^{10}_{\Lambda\Lambda}$Be | $2^+$ | $-12.19$ | $12.28$ | $12.33 \pm 0.35^c$ | 0.93 |

$^a$ Ref.[1].
$^b$ Ref.[2]. Also see text for the second value.
$^c$ Ref.[30, 41].

Table VI. Calculated r.m.s. distances between $\alpha$ and $x$, $\bar{r}_{\alpha x}$, in core nuclei, single Λ hypernuclei and double-Λ hypernuclei ($x = n, d, t, \alpha$). The expectation values of kinetic energy
and potential energy between \( \alpha \) and \( x \), \( <T_{\alpha x}> \), \( <V_{\alpha x}> \) and \( <T_{\alpha x} + V_{\alpha x}> \) are also listed. For \(^{5}\text{He}\) and \(^{8}\text{Be}\), \( \bar{r}_{\alpha-x} \) are not calculated since they are resonant states.

| \( \bar{r}_{\alpha x} \) | \( <T_{\alpha x}> \) | \( <V_{\alpha x}> \) | \( <T_{\alpha x} + V_{\alpha x}> \) |
|-----------------|----------------|----------------|----------------|
| \(^{5}\text{He}\) | 7.86 | -6.97 | 0.89 |
| \(^{6}\Lambda\text{He}\) | 5.79 | 11.38 | -9.92 | 1.46 |
| \(^{7}\Lambda\text{He}\) | 3.92 | 15.19 | -11.95 | 2.24 |
| \(^{6}\text{Li}\) | 4.10 | 11.59 | -13.06 | -1.47 |
| \(^{7}\Lambda\text{Li}\) | 3.44 | 15.59 | -16.70 | -1.11 |
| \(^{8}\Lambda\text{Li}\) | 3.16 | 18.86 | -19.54 | -0.68 |
| \(^{7}\text{Li}\) | 3.69 | 17.45 | -19.95 | -2.50 |
| \(^{8}\Lambda\text{Li}\) | 3.30 | 21.85 | -24.00 | -2.15 |
| \(^{9}\Lambda\text{Li}\) | 3.05 | 26.74 | -28.33 | -1.59 |
| \(^{8}\text{Be}\) | 7.21 | -7.12 | 0.09 |
| \(^{9}\Lambda\text{Be}\) | 3.78 | 14.90 | -14.14 | 0.76 |
| \(^{10}\Lambda\Lambda\text{Be}\) | 3.44 | 19.49 | -17.96 | 1.53 |

**Figure captions**

**FIG. 1.** Jacobian coordinates for all the rearrangement channels (\( c = 1 \) – 9) of the \( \alpha + x + \Lambda + \Lambda \) four-body system. Two \( \Lambda \) particles are to be antisymmetrized, and \( \alpha \) and \( x \) are to be symmetrized when \( x = \alpha \).

**FIG. 2.** Calculated energy levels of \(^{5}\text{He}\), \(^{6}\Lambda\text{He}\) and \(^{7}\Lambda\Lambda\text{He}\) on the basis of the \( \alpha + n \), \( \alpha + n + \Lambda \) and \( \alpha + n + \Lambda + \Lambda \) models, respectively. The level energies are measured from the particle break-up thresholds.

**FIG. 3.** Calculated energy levels of \(^{5}\text{Li}\), \(^{6}\Lambda\text{Li}\) and \(^{7}\Lambda\Lambda\text{Li}\) on the basis of the \( \alpha + p \), \( \alpha + p + \Lambda \) and \( \alpha + p + \Lambda + \Lambda \) models, respectively. The level energies are measured from the particle break-up thresholds.
FIG. 4. Calculated energy levels of $^6$Li, $^7$Li and $^8$Li on the basis of the $\alpha + d$, $\alpha + d + \Lambda$ and $\alpha + d + \Lambda + \Lambda$ models, respectively. The level energies are measured from the particle break-up thresholds or are given by excitation energies $E_x$.

FIG. 5. Calculated energy levels of $^7$Li, $^8$Li and $^9$Li on the basis of the $\alpha + t$, $\alpha + t + \Lambda$ and $\alpha + t + \Lambda + \Lambda$ models, respectively. The level energies are measured from the particle break-up thresholds or are given by excitation energies $E_x$.

FIG. 6. Calculated energy levels of $^7$Be, $^8$Be and $^9$Be on the basis of the $\alpha + ^3$He, $\alpha + ^3$He + $\Lambda$ and $\alpha + ^3$He + $\Lambda + \Lambda$ models, respectively. The level energies are measured from the particle break-up thresholds or are given by excitation energies $E_x$.

FIG. 7. Calculated energy levels of $^8$Be, $^9$Be and $^{10}$Be on the basis of the $\alpha + \alpha$, $\alpha + \alpha + \Lambda$ and $\alpha + \alpha + \Lambda + \Lambda$ models, respectively. The level energies are measured from the particle break-up thresholds or are given by excitation energies $E_x$.

FIG. 8. The $\alpha - n$ two-body densities (correlation functions), $\rho(r_{\alpha n})$, of $^5$He$(3/2^{-})$, $^6$He$(1^{-})$ and $^7$He$(3/2^{-})$. Here, it is multiplied by $r_{\alpha n}^2$.

FIG. 9. Calculated values of $B_{\Lambda\Lambda}(^A_{\Lambda\Lambda}Z)$ in the ground states are given by closed circles. The same quantities but calculated by putting $V_{\Lambda\Lambda} = 0$, namely $B_{\Lambda\Lambda}(^A_{\Lambda\Lambda}Z; V_{\Lambda\Lambda} = 0)$, are shown by open circles.

FIG. 10. Calculated values of $\Delta B_{\Lambda\Lambda}(^A_{\Lambda\Lambda}Z)$ defined in Eq. (4.2).

FIG. 11. Calculated values of $\Delta \bar{B}_{\Lambda\Lambda}(^A_{\Lambda\Lambda}Z)$ defined in Eq. (4.3).

FIG. 12. Summary of the energy levels of the double-$\Lambda$ hypernuclei $^6_{\Lambda\Lambda}$He, $^7_{\Lambda\Lambda}$He, $^7_{\Lambda\Lambda}$Li, $^8_{\Lambda\Lambda}$Li, $^9_{\Lambda\Lambda}$Li, $^8_{\Lambda\Lambda}$Be and $^{10}_{\Lambda\Lambda}$Be calculated using the $\alpha + x + \Lambda + \Lambda$ model with $x = 0, n, p, d, t, ^3$He and $\alpha$, respectively.
\[ \begin{align*}
\Lambda \Lambda \Lambda \Lambda & = 9.36 \\
\Lambda \Lambda \Lambda \Lambda \Lambda & = 4.18 \text{ (exp. 4.18)}
\end{align*} \]

\[
\begin{aligned}
\alpha + n & \rightarrow {}^5\text{He} \\
\alpha + n + \Lambda & \rightarrow {}^6\Lambda \text{He} \\
\alpha + n + \Lambda + \Lambda & \rightarrow {}^7\Lambda \Lambda \text{He}
\end{aligned}
\]
\[ \begin{align*}
\Sigma_\text{Li}^{5} & = 9.45 \\
\Sigma_\text{Li}^{6} & = 4.52 \text{ (exp. 4.50)} \\
\Sigma_\text{Li}^{7} & = 1.97 \\
\Sigma_\text{He}^{5} + p & = 0.0 \\
\Sigma_\text{He}^{6} + p & = 2.0 \\
\Sigma_\text{He}^{7} + p & = -2.55 \\
\Sigma_\text{He}^{\Lambda + \Lambda} + p & = -7.48 \\
\end{align*} \]
\[
\begin{align*}
2^+ &: 3.99 \\
3^+ &: 0.83 \\
\alpha + d &: -1.47 \\
\end{align*}
\]
\[ ^8\text{Be} \quad ^9\text{Be} \quad ^{10}\Lambda\Lambda\text{Be} \]

\( \Lambda \Lambda \)

\( B_{\Lambda} = 6.73 \text{ (exp. 6.71)} \)

\( 5\text{He} + \alpha \quad ^{10}\Lambda\Lambda\text{He} + \alpha \)

\( B_{\Lambda\Lambda}(2^+) = 12.28 \text{ (exp. 12.33)} \)

\( 0^+ -15.05 \quad 0.00 \text{ (0.00)} \)

\( E_x \quad B_{\Lambda\Lambda}(0^+) = 15.14 \)

\( 2^+ -3.00 \quad 3.00 \)

\( 1/2^+ -6.64 \quad 0.00 \text{ (0.00)} \)
The graph shows the relationship between the binding energy $B_{\Lambda\Lambda}$ (MeV) and the number of neutrons in the nucleus. The x-axis represents the number of neutrons, ranging from $^6\Lambda\Lambda\text{He}$ to $^{10}\Lambda\Lambda\text{Be}$, while the y-axis represents the binding energy in MeV. The data points are marked with either black circles or white circles, indicating different sets of measurements.
\[ r^2 \cdot \rho(r_{\alpha-p})(\text{fm}^{-1}) \]

- \( ^5 \text{He} (3/2^-) \)
- \( ^6 \Lambda \text{He} (1^-) \)
- \( ^7 \Lambda \Lambda \text{He} (3/2^-) \)
