APPLICATION OF SURVIVAL THEORY IN TAXATION

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Abstract. The paper deals with the application of the survival theory in economic systems. Theory and methodology of survival is used to evaluate fiscal policy. The survival of the system reduces to a problem of maximizing a radius of a cube inscribed into a polyhedral set so-called the target-oriented purpose [1-5]. We show that the survival theory can be applied to the government fiscal policy optimizing a taxation system. Numerical simulations were conducted using Mongolian statistical data for 2015.

1. Introduction. The problem of survival is a new branch of optimization theory. Survival is the ability of any system to cope with unfavorable situations and effects. Formalization of this theory was first given in [2]. Further development of the survival theory for a system described by optimal control formulation has been done in [2]. Applications of survival theory related to ecology issues and economics have been considered in [1]. In general mathematical model of improving survival of any system is defined by control parameter \( u \in U \), perturbation parameter \( v \in V \) and state parameter \( x \in X \). Implementation of survival theory can be considered in several steps.

First, the target-oriented purpose of a system is defined as
\[
F(x, u, v) \in Q,
\]
where \( F : X \times U \times V \rightarrow \mathbb{R}^* \), \( Q \) is a set of admissible solutions. Then a set of safe system perturbation can be defined in the following.
\[
K(x, u) = \left\{ v \in V | F(x, u, v) \in Q \right\}
\]

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for any $x \in X$ and $u \in U$. Finally, the survival of the system is defined by

$$J(x, u) = \frac{\mu(K(x, u))}{\mu(V)},$$

(2)

where $J : X \times U \rightarrow [0, 1]$, $\mu(V)$ is a measure of the set $V$.

The higher the value $J(x, u)$ is the higher survival of the system. This means that in order to improve the survival of a system one has to solve the following optimization problem

$$\max J(x, u)$$

$$(x, u) \in X \times V.$$

This paper is devoted to application of the survival theory to the government fiscal theory [8] which must run a balanced budget and maintain an optimal taxation level. The paper is organized as follows. In section 2, a problem of survival theory has been discussed. Section 3 is devoted to a mathematical model of the government taxation system. In Section 4, numerical experiments of the survival theory are provided.

2. **Survival model specification.** Let $D$ be polyhedral set in $\mathbb{R}^n$ defined as

$$D = \{ x \in \mathbb{R}^n | Ax \leq b \},$$

where $A = \{ a_{ij} \}_{m \times n}$, $b \in \mathbb{R}^m$.

Introduce the norm $\| \cdot \|$ as

$$\| x \| = \max_{1 \leq j \leq n} | x_j |.$$

Define a cube with a center $x \in \mathbb{R}^n$ and radius $r \geq 0$ in the following:

$$B(x, r) = \prod_{i=1}^{n} [x_i - r, x_i + r], r \geq 0$$

Denote by the vector $N$:

$$N = (N_1, N_2, ..., N_m), N_i = \sum_{j=1}^{n} | a_{ij} |, i = 1, 2, ..., m.$$ 

Now we formulate the following assertion.

**Lemma 2.1.** $B(x, r) \subset D$ if and only if

$$Ax + rN \leq b, \ r \geq 0$$

**Proof.** We follow up a proof given in [1].

Necessity. Let $B(x, r) \subset D$ and $r \geq 0$.

For $\alpha \geq 0$, construct points $z$ in the following,

$$z(\alpha) = x + r \sum_{j=1}^{n} \alpha_j e_j, |\alpha_j| = 1, j = 1, 2, ..., n.$$ 

$$e_j = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^n, j = 1, 2, ..., n.$$
It is clear that \( z(\alpha) \in B \). Indeed,

\[
\|z(\alpha) - x\| = r \| \sum_{j=1}^{n} \alpha_j e_j \| = r \ \max_{1 \leq j \leq n} \{ \alpha_1, \alpha_2, \ldots, \alpha_n \} = r
\]

and the points \( z(\alpha) \) are boundary points of \( B(x, r) \). Then we have

\[
\sum_{j=1}^{n} a_{ij} x_j + r \sum_{j=1}^{n} a_{ij} \alpha_j \leq b_i, \ i = 1, 2, \ldots, m
\]

\[
Ax \leq b.
\]

**Sufficiency.** For any \( y \in B(x, r) \) and \( r \geq 0 \), we get

\[
Ay - Ax = A(y - x) \leq \|y - x\| N \leq rN \leq b - Ax
\]

Hence, we conclude that \( Ay \leq b \) and consequently, \( y \in D \). \qed

Based on Lemma 2.1, we can define the maximum radius cube inscribed in \( D \) by solving the following linear programming.

\[
\max \ r,

Ax + rN \leq b \tag{3}
\]

A solution to problem (3) helps to formulate the following theorem.

**Theorem 2.1.** [1] Let \((x^*, r^*)\) be a solution to problem (3). Then

a) If \( r^* < 0 \), then \( D = \emptyset \).

b) If \( r^* = 0 \), then there is no \( x \in \mathbb{R}^n \) such that \( Ax < b \).

c) If \( r^* > 0 \), then \( B(x^*, r^*) \) is the cube with the maximum radius \( r^* \).

d) If \( N > 0 \) and problem (3) has no solution then \( D \) is unbounded with int \( D \neq \emptyset \).

Note that we can inscribe three nonoverlapping disks with the maximum area in \( D \) in order to increase the survival of the system. In this case the problem reduces to Malfatti’s problem [9] which was examined from a view of point of global optimization in [6, 7]. The Malfatti’s problem is formulated as:

\[
\max \pi[r_1^2 + r_2^2 + r_3^2]
\]

\[
< a^i, w^j > + r_j \| a^i \| \leq b_i, \ i, j = 1, 2, 3
\]

\[
\| u^i - w^j \| \geq r_i + r_j, \ i, j = 1, 2, 3
\]

\[
r_j \geq 0, \ i, j = 1, 2, 3
\]

where \( u^i, \ j = 1, 2, 3 \) are centers of disks, \( \| \cdot \| \) is Euclidean norm, \( r_j \) are radii of disks.

Unlike the problem (3), Malfatti’s problem is nonconvex, that’s why it is preferable to consider one disk instead 3 disks. In this case, we solve the following linear programming:

\[
\max \ r
\]

\[
< a^i, u > + r \| a^i \| \leq b_i, \ i = 1, 2, 3
\]
3. Application of survival theory to taxation. It is known that in every country the governments and central banks implement various policies in order to maintain the level of development, economic performance and monetary stability. The government fiscal policy takes actions to influence the economy through the change in revenues, taxes and expenditures of state budget. General government budget consists mainly of tax revenues. Denote by $T_i$ i-th type of taxable income, $i = 1, 2, ..., n,$ and $C_j$ j-th type of expenditure, $j = 1, 2, ..., m.$ $C_j$ and $\overline{C}_j$ are lower and upper values of $C_j.$ Let $M$ be a total nontax income, $\alpha_i$ are tax rates for income $T_i,$ $i = 1, 2, ..., n$ with corresponding lower and upper values $\underline{\alpha}_i$ and $\overline{\alpha}_i.$

Then the government budget revenue $T$ is as follows:

$$T = \sum_{i=1}^{n} \alpha_i T_i + M$$  \hspace{1cm} (4)

Let $G$ be the government budget deficit and $\gamma$ its growth $0 \leq \gamma \leq 1.$ Then the target-oriented purpose of the government budget revenue system is defined as follows:

$$\sum_{j=1}^{n} C_j - T \leq G - G\gamma$$  \hspace{1cm} (5)

According to survival theory [2] introduce control parameters $u_i$ and perturbation parameters $v_j$ in the following.

$$C_j = \overline{C}_j + v_j (\overline{C}_j - \underline{C}_j),$$  \hspace{1cm} (6)

$$0 \leq v_j \leq 1, \ j = 1, 2, ..., m,$$  \hspace{1cm} (7)

$$\alpha_i = \alpha_i + u_i (\overline{\alpha}_i - \underline{\alpha}_i),$$  \hspace{1cm} (8)

$$0 \leq u_i \leq 1, \ i = 1, 2, ..., n.$$  \hspace{1cm} (9)

Substituting (6)-(9) and (4) into (5), we get

$$\sum_{j=1}^{m} (C_j + v_j (\overline{C}_j - \underline{C}_j)) - \sum_{j=1}^{m} \alpha_i + u_i (\overline{\alpha}_i - \underline{\alpha}_i) \sum_{i=1}^{n} T_i - M \leq G - G\gamma, \ 0 \leq \gamma \leq 1$$  \hspace{1cm} (10)

$$\sum_{j=1}^{m} C_j + \sum_{j=1}^{m} v_j (\overline{C}_j - \underline{C}_j) - \sum_{i=1}^{n} \alpha_i T_i - \sum_{i=1}^{n} u_i (\overline{\alpha}_i - \underline{\alpha}_i) T_i - M \leq G - G\gamma$$  \hspace{1cm} (11)

Denote by

$$a_j = \overline{C}_j - \underline{C}_j, \ j = 1, 2, ..., m, \ b_i = (\overline{\alpha}_i - \underline{\alpha}_i) T_i, \ i = 1, 2, ..., n,$$

$$d = (1 - \gamma) G + \sum_{i=1}^{n} \alpha_i T_i - \sum_{j=1}^{m} C_j + M.$$

Then (11) has the form

$$\sum_{j=1}^{m} a_j v_j \leq d + \sum_{i=1}^{n} b_i u_i$$  \hspace{1cm} (12)

Now we define the set of safe system perturbation as:

$$K = \left\{ u \in V = [0, 1]^m \ \mid \ \sum_{j=1}^{m} a_j v_j \leq d + \sum_{i=1}^{n} b_i u_i, \right\}$$

$$0 \leq v_j \leq 1, \ 0 \leq u_i \leq 1, \ i = 1, 2, ..., m; j = 1, 2, ..., n \}$$

Assume that $K \neq \emptyset.$
The wider this set is, the higher survival of the system. In order to improve the survival of the system, due to Lemma 2.1, we need to solve the linear programming problem.

\[
\begin{align*}
\text{max} & \quad r
\sum_{j=1}^{m} a_j v_j + r \sum_{j=1}^{m} a_j - \sum_{j=1}^{m} b_i u_i \leq d, \\
- v_j + r & \leq 0, j = 1, 2, \ldots, m; \\
v_j + r & \leq 1, j = 1, 2, \ldots, m; \\
0 & \leq u_i \leq 1, i = 1, 2, \ldots, n
\end{align*}
\]

(13)

4. Numerical experiments. The survival model was tested on the statistical data of Mongolian economy for 2015. We considered 5 types of taxable incomes and 2 types of government expenditure. The taxable income and government expenditures are the following.

- \( T_1 \): Corporations and other enterprises annual income not exceeding 3 billion tugrig (Mongolian currency).
- \( T_2 \): Corporations and other enterprises annual income greater than 3 billion tugrig.
- \( T_3 \): Individuals income,
- \( T_4 \): Progressive income,
- \( T_5 \): Customs income,
- \( M \): other nontax income.

Parameters of the model were:

\[
\begin{align*}
C_1 &= 3749, \quad C_1^* = 5749.8 \\
C_2 &= 1000, \quad C_2^* = 1387, \quad T_1 = 558.8, \quad T_2 = 140, \\
T_3 &= 551.8, \quad T_4 = 1037.1, \quad T_5 = 924.8, \quad M = 3964.1, \quad \alpha_1 = 0.1, \\
\bar{\alpha}_1 &= 0.15, \quad \alpha_2 = 0.15, \quad \bar{\alpha}_2 = 0.25, \\
\alpha_3 &= 0.06, \quad \bar{\alpha}_3 = 0.1, \quad \alpha_4 = 0.15, \\
\alpha_5 &= 0.20, \quad \bar{\alpha}_5 = 0.15, \quad \bar{\alpha}_5 = 0.20, \quad \gamma = 0.2
\end{align*}
\]

Optimal solutions to problems (5) and (13) are: \( \alpha_1^* = 0.15, \alpha_2^* = 0.25, \alpha_3^* = 0.1, \alpha_4^* = 0.2, \alpha_5^* = 0.2, r^* = 0.1464, C_1 = 4043, C_2 = 1057 \). It means that with the above solutions then government deficit will be 20%.

5. Conclusion. We have applied theory and methodology of survival to Mongolian government tax revenue. The problem reduces to finding a cube or a disk with the maximum radius inscribed into a set defined by a set of safe perturbation of the system. Numerical experiment shows that under certain changes of current expenditure, the government could have decreased the budget deficit by 20%. The proposed methodology can be applied to other fields of economics such as management, accounting and banking.

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