CONJUGACY CLASSES AND CENTRALIZERS FOR
PIVOTAL FUSION CATEGORIES

SEBASTIAN BURCIU

Abstract. A criterion for M"uger centralizer of a fusion subcategory of a braided non-degenerate fusion category is given. Along the way we extend some identities on the space of class functions of a fusion category introduced by Shimizu in [Shi17b]. We also show that in a modular tensor category the product of two conjugacy class sums is a linear combination of conjugacy class sums with rational coefficients.

1. Introduction

One of the main tools in the study braided fusion categories is the notion of the centralizer of a fusion subcategory that in its full generality was introduced in [DGNO10] (see also [Mug03]). Given a braided fusion category $\mathcal{C}$ and $\mathcal{D} \subseteq \mathcal{C}$ a fusion subcategory of $\mathcal{C}$, the centralizer $\mathcal{D}'$ is defined as the full fusion subcategory $\mathcal{D}'$ of $\mathcal{C}$ generated by all simple objects $X$ of $\mathcal{C}$ that centralize any object of $\mathcal{D}$, i.e. $c_{X,Y} c_{Y,X} = \text{id}_X \otimes Y$ for all objects $Y \in O(\mathcal{D})$. The notion of the centralizer was used in important classification results for braided fusion categories, see for example papers [DGNO10, ENO05, DGNO07] and the references therein.

In general, the centralizer of a given fusion subcategory is very hard to compute. There are only few cases in the literature where concrete formulae are known. For instance, in the category of representations of a (twisted) Drinfeld double of an arbitrary finite group a formula for the centralizer of an arbitrary fusion subcategory was then given in [NNW09].

In the paper [Bur17] we have given a complete description for the M"uger centralizer of a fusion subcategory of a fusion category of the type $\mathcal{C} = \text{Rep}(H)$ for a factorizable semisimple Hopf algebra $H$. More precisely, we have shown that if $L$ is a left normal coideal subalgebra of $A$ and $\mathcal{D} = \text{Rep}(H//L)$ is a fusion subcategory of $\text{Rep}(H)$ then

$$\text{Rep}(H//L)' = \text{Rep}(H//M)$$

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where $M = f_Q((H//L)^*)$. Here $f_Q : H^* \to H$ is the Drinfeld map associated to $(H, R)$.

Denoting by $\lambda_L$ the idempotent cointegral of the quotient Hopf algebra $H//L$ this result can be written as

$$F_\lambda(f_Q(\lambda_L)) = \frac{1}{\dim_k(M)} \lambda_M$$

where $F_\lambda : H^* \to H$ is the Fourier transform associated to the Hopf algebra $H$ with $\langle \lambda, 1 \rangle = 1$.

Recall that recently Shimizu introduced in [Shi19] the notion of cointegral $\lambda_C$ of a finite tensor category $C$. For a semisimple Hopf algebra $H$, and a fusion subcategory $D = \text{Rep}(H//L)$ of $C = \text{Rep}(H)$ one has that $\lambda_D = \lambda_L$.

The main result of the paper is the following theorem generalizing Equation (1.2) to arbitrary ribbon categories.

**Theorem 1.3.** Let $D$ be a fusion subcategory of a ribbon fusion category $C$ and $\lambda_D$ be the associated idempotent cointegral of $D$. Then

$$F_\lambda(f_Q(\lambda_D)) = \frac{\dim(D')}{\dim(C)} \lambda_{D'}.$$ 

where $F_\lambda$ is the Fourier transform introduced by Shimizu in [Shi17b] and $\lambda_{D'}$ is the idempotent cointegral of $D'$.

Recently, in [Shi17b], Shimizu also extended the notion of conjugacy classes for fusion categories, similarly to the conjugacy classes introduced by Cohen and Westreich in [CW10] for semisimple Hopf algebras. They generalize the notion of conjugacy classes in finite groups. In the same paper [Shi17b] the author associated to each conjugacy class a central element called conjugacy class sum. These class sums play the role of the sum of group elements in a conjugacy class of a finite group.

In [Shi17b] the author asked if the results from [CW14, CW00] concerning conjugacy classes can be extended from semisimple Hopf algebras to fusion categories. For example, for semisimple factorizable Hopf algebras, Cohen and Westreich in [CW10] proved that a product of two conjugacy class sums is a linear combination with rational coefficients of all conjugacy class sums. In subsection 6.1 we show that this result also holds for modular tensor categories. We also extend some other results of [CW10] from semisimple factorizable Hopf algebras to modular tensor categories.

For any fusion subcategory $D$ of a spherical fusion category $C$ we exhibit an element $\ell_D \in CE(D)$ which plays the role of the integral of a
coideal subalgebra $L$ of a semisimple Hopf algebra $H$. More precisely, if $\mathcal{D} = \text{Rep}(H/L)$ is a fusion subcategory of $\mathcal{C} = \text{Rep}(H)$, then $\ell_\mathcal{D} = \Lambda_L$, the unique element of $L$ such that $l\Lambda_L = \epsilon(l)\Lambda_L$ for any $l \in L$, see [Skr07]. Using this element and a Class Equation type argument in Theorem 4.28 we show that fusion subcategories of prime index $p$ are in bijection with normal subgroups of the same prime index $p$ of the universal grading group $U_\mathcal{C}$ of $\mathcal{C}$.

This paper is organised as follows. In Section 2 we review the basic notions of fusion categories that are needed through the paper. In Section 3 we recall the notion of central Hopf comonads and adjoint algebras associated to fusion categories from [Shi17b]. This section also contains the main properties of the (co)integrals and Fourier transforms associated to fusion categories. In Section 4 we prove several identities inside the ring $\text{CF}(\mathcal{C})$ of class functions that are needed through the rest of the paper. In Section 5 we prove the main result of the paper mentioned above. In the last section we study conjugacy classes for modular tensor categories. In this section we also prove that a product of two conjugacy class sums is a linear combination with rational coefficients of all conjugacy class sums. In the Appendix we prove that a canonical natural transformation between certain coends is a morphism of Hopf comonads, result needed in the proof of the main result.

Throughout this paper we work over an algebraically closed field $k$ of arbitrary characteristic.

2. Preliminaries

In this section we review the main properties of fusion categories that are needed through the paper. For the basic theory of monoidal categories, we refer the reader to [Lan98] and [Kas95].

For a monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, 1)$ with tensor product $\otimes$ and unit object $1$, we set $\mathcal{C}^{\text{op}} = (\mathcal{C}^{\text{op}}, \otimes, 1)$ and $\mathcal{C}^{\text{rev}} = (\mathcal{C}, \otimes^{\text{rev}}, 1)$, where $\otimes^{\text{rev}}$ is the reversed tensor product given by $X \otimes^{\text{rev}} Y = Y \otimes X$. A monoidal functor [Lan98 XI.2] is a functor $F : \mathcal{C} \to \mathcal{D}$ between monoidal categories $\mathcal{C}$ and $\mathcal{D}$ endowed with a morphism $F_0 : 1 \to F(1)$ in $\mathcal{C}$ and a natural transformation $F_2(X, Y) : F(X) \otimes F(Y) \to F(X \otimes Y)$ ($X, Y \in \mathcal{O}(\mathcal{C})$) satisfying a certain coherence condition. We say that a monoidal functor $(F, F_2, F_0)$ is strong monoidal if $F_0$ and $F_2$ are invertible. We also say that $F$ is strict if the above monoidal structures are identities. A comonoidal functor from $\mathcal{C}$ to $\mathcal{D}$ is the same thing as a monoidal functor from $\mathcal{C}^{\text{op}}$ to $\mathcal{D}^{\text{op}}$.

A left dual object of $X \in \mathcal{O}(\mathcal{C})$ is a triple $(X', e, d)$ consisting of an object $X' \in \mathcal{O}(\mathcal{C})$ and morphisms $e : X' \otimes X \to 1$ and $d : 1 \to X \otimes X'$
such that
\[(e \otimes \text{id}_X') \circ (\text{id}_X' \otimes d) = \text{id}_{X'} \quad \text{and} \quad (\text{id}_X \otimes e) \circ (d \otimes \text{id}_X) = \text{id}_X.\]

We say that \(C\) is left rigid if every object of \(C\) has a left dual object. For each \(X \in \mathcal{O}(C)\), we choose a left dual object denoted by \((X^*, \text{ev}_X, \text{coev}_X)\) with the morphisms \(\text{ev}_X : X^* \otimes X \to 1\) and \(\text{coev}_X : 1 \to X \otimes X^*\) called the evaluation and the coevaluation respectively. Then the assignment \(X \mapsto X^*\) gives rise to a strong monoidal functor \((-)^* : C \to C^{\text{op,rev}},\) which we call the left duality functor. Similarly, a right dual of \(X\) is a triple \((X'', e', d')\) where \(X' \in \mathcal{O}(C)\) and the morphisms \(e' : X \otimes X' \to 1\) and \(d' : 1 \to X' \otimes X\) are defined such that the triple \((X'', e', d')\) is a left dual of \(X\) in \(C^{\text{rev}}\).

A rigid monoidal category \(C\) is a monoidal category \(C\) such that both \(C\) and \(C^{\text{rev}}\) are left rigid. If \(C\) is a rigid monoidal category, then the left duality functor \((-)^* : C \to C^{\text{op,rev}}\) is an equivalence. A quasi-inverse of \((-)^*\), denoted by \(*(-)\), is called the right duality functor.

**Remark 2.1.** As explained in [Shi15, Lemma 5.4] it may be assumed that these duality functors are strict and mutually inverse one to another. Thus one can assume through the rest of the paper that:
\[(X \otimes Y)^* = Y^* \otimes X^*, \quad 1^* = 1, \quad *(X^*) = X.\]

A finite abelian category is a \(k\)-linear category that is equivalent to the category \(A\)-mod of finite dimensional \(A\)-modules for some finite-dimensional \(k\)-module \(A\). A finite tensor category [EO04] is a rigid monoidal category \(C\) such that
\begin{enumerate}
  \item \(C\) is a finite abelian category,
  \item The tensor product \(\otimes : C \times C \to C\) is \(k\)-linear in each variable,
  \item \(\text{End}_C(1) \simeq k\) as algebras.
\end{enumerate}

A fusion category [ENO05] is a semisimple finite tensor category.

2.1. **Braided categories.** By definition, the reversed category \(C^{\text{rev}}\) of a monoidal category \(C\) is the same underlying category with the tensor product reversed.

A braiding [Kas95, XIII.1] of a monoidal category \((B, \otimes, 1)\) is a natural isomorphism \(\sigma : \otimes_B \to \otimes_B^{\text{rev}}\) satisfying the hexagon axiom. A braided monoidal category is a monoidal category endowed with a braiding. The reversed braided category \(C^{\text{rev}}\) of a braided category \(C\) is defined as the reversed monoidal category \(C\) with the braiding given by:
\[\sigma_{X,Y}^{\text{rev}} := \sigma_{Y,X}.\]
2.2. Monoidal centers. For a monoidal category $\mathcal{C}$, the monoidal center (or the Drinfeld center) of $\mathcal{C}$ is a category $Z(\mathcal{C})$ defined as follows: An object of $Z(\mathcal{C})$ is a pair $(V, \sigma_V)$ consisting of an object $V \in \mathcal{C}$ and a natural isomorphism $\sigma_{V,X} : V \otimes X \to X \otimes V$ for all $X \in \mathcal{O}(\mathcal{C})$, satisfying a part of the hexagon axiom. A morphism $f : (V, \sigma_V) \to (W, \sigma_W)$ in $Z(\mathcal{C})$ is a morphism in $\mathcal{C}$ such that $(\text{id}_X \otimes f) \circ \sigma_{V,X} = \sigma_{W,X} \circ (f \otimes \text{id}_X)$ for all $X \in \mathcal{C}$. The composition of morphisms is defined in an obvious way. The category $Z(\mathcal{C})$ is in fact a braided monoidal category, see, e.g., [Kas95, XIII.3] for details.

2.3. Pivotal tensor categories. Recall that a pivotal structure of a rigid monoidal category $\mathcal{C}$ is an isomorphism $j : \text{id}_\mathcal{C} \to (\cdot)^\ast\ast$ of monoidal functors. A pivotal monoidal category is a rigid monoidal category endowed with a pivotal structure.

Remark 2.2. As explained in [Shi15] we may assume further that in a pivotal category one has $X^{\ast\ast} = X$, i.e. the pivotal structure is the identity.

A pivotal structure a on a tensor category $\mathcal{C}$ is called spherical if $\dim(V) = \dim(V^\ast)$ for any object $V \in \mathcal{O}(\mathcal{C})$. A tensor category is spherical if it is equipped with a spherical structure.

2.4. Ribbon categories. A braided category is called pre-modular if it has a spherical structure. Equivalently, this is a ribbon fusion category, that is, a fusion category equipped with a braiding and a twist (also called a balanced structure), see [ENO05].

Assume $k = \mathbb{C}$. Then $\mathcal{C}$ is called pseudo-unitary if $\text{FPdim}(\mathcal{C}) = \dim(\mathcal{C})$. If such is the case, then by [ENO05, Proposition 8.23], $\mathcal{C}$ admits a unique spherical structure with respect to which the categorical dimensions of simple objects are all positive. It is called the canonical spherical structure. For this structure, the categorical dimension of an object coincides with its Frobenius-Perron dimension, i.e. $\text{FPdim}(X) = \dim(X)$ for any object $X \in \mathcal{O}(\mathcal{C})$. If $\mathcal{C}$ is a fusion category such that $\text{FPdim}(\mathcal{C})$ is an integer, then $\mathcal{C}$ is pseudo-unitary by [ENO05, Proposition 8.24]. Moreover, every full fusion subcategory of $\mathcal{C}$ is pseudo-unitary.

2.5. Dinatural transformations, ends and coends. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $S, T : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D}$ be two functors. A dinatural transformation $\xi : T \to S$ is a family of morphisms

$$\xi = \{\xi_X : T(X, X) \to S(X, X)\}_{X \in \mathcal{O}(\mathcal{C})}$$
such that:

\[ S(id_X, f) \circ \xi_X \circ T(f, id_X) = S(f, id_Y) \circ \xi_Y \circ T(id_Y, f) \]

for all morphisms \( f : X \to Y \) in \( C \). An end of the functor \( S \) is a pair \((E, p)\) consisting of an object \( E \in \mathcal{O}(\mathcal{D}) \) (regarded as a constant functor from \( \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D} \)) and a dinatural transformation \( p : E \Rightarrow S \) that enjoys the following universal property: For any pair \((E', p')\) consisting of an object \( E' \in \mathcal{O}(\mathcal{D}) \) and a dinatural transformation \( p' : E' \Rightarrow S \), there exists a unique morphism \( \phi : E' \to E \) in \( \mathcal{D} \) such that \( p'_{X} = p_{X} \circ \phi \) for all \( X \in \mathcal{C} \).

The end of \( S \) is expressed as \( \int_{X \in \mathcal{C}} S(X, X) \).

A coend of \( T \) is a pair \((C, i)\) consisting of an object \( C \in \mathcal{O}(\mathcal{D}) \) and a dinatural transformation \( i : T \Rightarrow C \) having a similar universal property. The coend of \( T \) is expressed as \( \int_{X \in \mathcal{C}} T(X, X) \).

See [Lan98, IX] for the basic results on (co)ends.

3. ADJUNCT ALGEBRA AND INTERNAL CHARACTERS

3.1. Hopf monads and comonads. Let \( C \) be a monoidal category. Recall, [BV07] that a bimonad \( T : C \to C \) is a monad \((T, \mu, \eta)\) with comonoidal structures

\[
T_0 : T(1) \to 1, \quad T_2(V, W) : T(V \otimes W) \to T(V) \otimes T(W)
\]

such that \( \mu \) and \( \eta \) are comonoidal natural transformations. In this case the category of \( T \)-modules \( \tau C \) is a monoidal category and the forgetful functor \( U : \tau C \to C \) is a strong monoidal functor.

If \( C \) is a rigid monoidal category then a Hopf monad on \( C \) is defined as a bimonad for which the category \( \tau C \) is also rigid. Recall that in this case \( T \) is endowed with left and right bijective antipodes \( S^l_V : T(T(V)^*) \to V^* \), \( S^r_V : T^*(T(V)) \to V^* \), see [BV07].

Also recall that a Hopf comonad is a comonad on \( C \) endowed with a monoidal structure such that \( T^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{C}^{\text{op}} \) is a Hopf monad.

3.2. The central Hopf monad \( L \). Let \( C \) be a finite tensor category and \( F : Z(C) \to C \) the forgetful functor. Then \( F \) admits a left adjoint \( L \) and \( \mathcal{L} := FL : C \to C \) is a Hopf monad defined as a coend. As in [Shi17b, Section 3.1] one has that

\[
\mathcal{L}(V) \simeq \int_{X \in \mathcal{C}} X^* \otimes V \otimes X
\]

We denote by \( \iota_{V,X} : X^* \otimes V \otimes X \to \mathcal{L}(V) \) the universal dinatural transformation associated to the coend \( \mathcal{L}(V) \).
Day and Street [DS07] showed that the functor $V \mapsto \mathcal{L}(V)$ has a structure of a monad such that the category $\mathcal{L}$ of $\mathcal{L}$-modules is canonically isomorphic to the monoidal center $\mathcal{Z}(\mathcal{C})$.

### 3.3. The central Hopf comonad $Z$

The forgetful functor $F$ also admits a right adjoint functor $R : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ and $Z := FR : \mathcal{C} \to \mathcal{C}$ is a Hopf comonad defined as an end. Indeed, following [Shi17b, Section 2.6] one has that

$$(3.2) \quad Z(V) \simeq \int_{X \in \mathcal{C}} X \otimes V \otimes X^*$$

We denote by $\pi_{V,X} : Z(V) \to X \otimes V \otimes X^*$ the universal dinatural transformation associated to the end $Z(V)$.

Recall that for a functor $T : \mathcal{B} \to \mathcal{C}$ between rigid monoidal categories, we may define $T^! : \mathcal{B} \to \mathcal{C}$ by $T^!(V) = (T^!(V))^*$, (see [Shi17b, Subsection 2.6]). Then one can take $R = L^!$ and $\pi_{V,X} = (i_{V,X}^*)^*$.

The Hopf comonad structure of $Z$ can be described in terms of the dinatural transformation $\pi$. Indeed, the comultiplication $\delta : Z \to Z^2$ is the unique natural transformation such that

$$(3.3) \quad (\text{id}_X \otimes \pi_{V,Y} \otimes \text{id}_{X^*}) \circ \pi_{Z(V),X} \circ \delta_V = \pi_{V,X \otimes Y}$$

The counit of $\epsilon : Z \to \text{id}_\mathcal{C}$ is given by $\epsilon_V := \pi_{V,1}$.

### 3.4. Adjoint algebra

It is known that $A := Z(1)$ has the structure of central commutative algebra in $\mathcal{Z}(\mathcal{C})$.

The multiplication $m : A \otimes A \to A$ and the unit $u : 1 \to A$ of the adjoint algebra $A = Z(1)$ are uniquely determined by by the universal property of the end $A$ as:

$$(3.4) \quad \pi_{1,X} \circ u = \text{coev}_X,$$

$$(3.5) \quad \pi_{1,X} \circ m = (\text{id}_X \otimes \text{ev}_X \otimes \text{id}_{X^*}) \circ (\pi_{1,X} \otimes \pi_{1,X})$$

Moreover $\epsilon_1 : A \to 1$ is a morphism of algebras, see [Shi17b]. Moreover it is well known that $A$ is a commutative algebra in the center $\mathcal{Z}(\mathcal{C})$.

### 3.5. Central elements

The vector space $\text{CE}(\mathcal{C}) := \text{Hom}_\mathcal{C}(1, A)$ is called the set of central elements. There is a canonical bijection:

$$(3.6) \quad \text{CE}(\mathcal{C}) \xrightarrow{\psi} \text{End}(\text{id}_\mathcal{C}), \quad \psi(a)_X = \rho_X(a \otimes \text{id}_X), \quad X = 1 \otimes X \xrightarrow{a \otimes \text{id}_X} X \otimes Z(1) \xrightarrow{\rho_X} X$$

The inverse of this bijection is described using ends, see [KL01, Proposition 5.2.5].
3.5.1. Multiplication in the center. For \( a, b \in \text{CE}(\mathcal{C}) \), we set \( ab := m \circ (a \otimes b) \). Then the set \( \text{CE}(\mathcal{C}) \) is a monoid with respect to this operation. Moreover, the bijection in Equation (3.6) is in fact an isomorphism of monoids.

For a natural transformation \( \alpha : S \to T \) between two functors \( S, T : \mathcal{B} \to \mathcal{C} \) we define \( \alpha^! : T^! \to S^! \) by

\[
\alpha_V : T^!(V) = T(^*V)^* \xrightarrow{(\alpha^*_V)^*} S(^*V)^* = S^!(V) \quad (V \in \mathcal{O}(\mathcal{B})).
\]

The antipodal operator \( S : \text{CE}(\mathcal{C}) \to \text{CE}(\mathcal{C}) \) on \( \text{CE}(\mathcal{C}) \) is induced by \( (-)^! : \text{End}(\text{id}_\mathcal{C}) \to \text{End}(\text{id}_\mathcal{C}), \xi \mapsto \xi^! \), via the bijection from Equation (3.6).

3.6. Internal characters of pivotal tensor categories. Recall that a pivotal structure \( j \) on a tensor category \( \mathcal{C} \) is a tensor isomorphism \( j : \text{id}_\mathcal{C} \to (^*)^* \). Using the pivotal structure one can construct a right evaluation as follows:

\[
\tilde{ev}_X : X \otimes X^* \xrightarrow{j \otimes \text{id}} X^{**} \otimes X^* \xrightarrow{\text{ev}_{X^*}} 1
\]

Then the right partial pivotal trace of \( f : A \otimes X \to B \otimes X \) is defined as follows:

\[
\text{tr}_{A,B}^X : A = A \otimes 1 \xrightarrow{\text{id} \otimes \text{coev}_X} A \otimes X \otimes X^* \xrightarrow{f \otimes \text{id}} B \otimes X \otimes X^* \xrightarrow{\text{id} \otimes \tilde{ev}_X} B
\]

The usual right pivotal trace of an endomorphism \( f : X \to X \) is obtained as a particular case for \( A = B = 1 \). In particular, the right pivotal dimension of \( X \) is defined as the right trace of the identity of \( X \).

3.6.1. Internal characters and their multiplication. Given an object \( X \in \mathcal{O}(\mathcal{C}) \) the internal character \( \text{ch}(X) \) is defined as the morphism

\[
\text{ch}(X) := \text{tr}_{A,1}^X(\rho_X) : A \to 1.
\]

Then the space \( \text{CF}(\mathcal{C}) := \text{Hom}_\mathcal{C}(A_C, 1) \) is called the space of class functions of \( \mathcal{C} \).

For two class functions \( f, g \in \text{CF}(\mathcal{C}) \) one can define a multiplication by

\[
f \ast g := f \circ Z(g) \circ \delta_1.
\]

Here \( \delta : Z \to Z^2 \) is the comultiplication structure of \( Z \) defined in the Equation (3.3). This multiplication coincides to the composition of morphisms in the co-Kleisli category of the comonad \( Z \). Thus \( \text{CF}(\mathcal{C}) \) is a monoid with respect to \( \ast \).
By [Shi17b, Theorem 3.10] one has that $\text{ch}(X \otimes Y) = \text{ch}(X)\text{ch}(Y)$ for any two objects $X$ and $Y$ of $\mathcal{C}$. For a finite tensor category $\mathcal{C}$ the space of class functions $\text{CF}(\mathcal{C})$ is a finite-dimensional algebra.

By [Shi17b, Theorem 4.1] if $\mathcal{C}$ is a finite pivotal tensor category over an algebraically closed field $k$ then the set of irreducible characters $\chi(X)$ with $X \in \text{Irr}(\mathcal{C})$ is linearly independent in $\text{CF}(\mathcal{C})$.

Recall $R : \mathcal{C} \to \mathcal{Z}^e(\mathcal{C})$ is a right adjoint to the forgetful functor $F : \mathcal{Z}^e(\mathcal{C}) \to \mathcal{C}$. As explained in [Shi17b, Theorem 3.8] this adjunction gives an isomorphism of monoids

$$
\text{CF}(\mathcal{C}) \cong \text{End}_{\mathcal{Z}(\mathcal{C})}(R(1)), \quad \chi \mapsto Z(\chi) \circ \delta_1.
$$

In the theory of semisimple Hopf algebras, it is well-known that the evaluation pairing between characters and central elements is non-degenerate. Shimizu has generalized this fact, see [Shi17b], by considering a paring $\langle \ , \ \rangle : \text{CF}(\mathcal{C}) \times \text{CE}(\mathcal{C}) \to 1$, given by $\langle f, a \rangle \text{id}_1 = f \circ a$, for all $f \in \text{CF}(\mathcal{C})$ and $a \in \text{CE}(\mathcal{C})$.

3.7. On the inclusion of class functions of a fusion subcategory. Let $\mathcal{D}$ be a fusion subcategory of a given fusion category $\mathcal{C}$. For any $V \in \mathcal{C}$ consider as in [Shi17b, Sect. 4.3] also the end $\bar{\mathcal{Z}}(V) : \mathcal{D}^{\text{op}} \times \mathcal{D} \to \mathcal{C}$, $\bar{\mathcal{Z}}(V) := \int_{X \in \mathcal{D}} X \otimes V \otimes X^*$. Let $\bar{\pi}_{V, X} : \bar{\mathcal{Z}}(V) \to X \otimes V \otimes X^*$ be the universal dinatural maps defining this end. From the universal property of $\bar{\mathcal{Z}}$ for any $V \in \mathcal{C}$ there is a unique canonical map in $\mathcal{C}$

$$
\bar{\mathcal{Z}}(V) \xrightarrow{\bar{\pi}_{V, X}} \bar{\mathcal{Z}}(V)
$$

such that $\bar{\pi}_{V, X} \circ q_V = \pi_{V, X}$ for any object $X$ of $\mathcal{D}$.

In the appendix we prove that $q$ is a map of Hopf comonads. The map $q_1 : Z(1) \to \bar{\mathcal{Z}}(1)$ induces two maps

$$
q_1^* : \text{CF}(\mathcal{D}) \to \text{CF}(\mathcal{C}), \quad \chi \mapsto \chi \circ q_1,$n

$$
q_1^* : \text{CE}(\mathcal{C}) \to \text{CE}(\mathcal{C}), \quad z \mapsto q_1 \circ z.
$$

Moreover, by Lemma 7.6 from appendix it is known that $q_1^*$ is a monomorphism and $q_1^*$ is an epimorphism and both maps are in fact $k$-algebra homomorphisms.

Remark 3.9. Since $\mathcal{C}$ is a fusion category one can give a direct description of the map $q_1^*$. Indeed, there is a canonical isomorphism $\text{ch}^C : \mathcal{G}r_k(\mathcal{C}) \cong \text{CF}(\mathcal{C}), [X] \mapsto \chi^C(X)$ where $\mathcal{G}r_k(\mathcal{C})$ is the Grothendieck group of $\mathcal{C}$, see [Shi17b, Example 4.4]. For a fusion subcategory $\mathcal{D} \subseteq \mathcal{C}$ clearly we we have $\mathcal{G}r_k(\mathcal{D}) \hookrightarrow \mathcal{G}r_k(\mathcal{C})$ as $k$-algebras. It is easy to see
that this induces an inclusion $\text{CF}(\mathcal{D}) \hookrightarrow \text{CF}(\mathcal{C})$ via the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{G}r_k(\mathcal{C}) & \xrightarrow{ch^C} & \text{CF}(\mathcal{C}) \\
\downarrow j & & \downarrow q^* \\
\mathcal{G}r_k(\mathcal{D}) & \xrightarrow{ch^D} & \text{CF}(\mathcal{D}).
\end{array}
$$

The commutativity of the diagram follows since for any $X \in \mathcal{O}(\mathcal{D})$ one has

$$
q^*(\chi^D(X)) = \chi^D(X) \circ q = e v_X \circ \pi_{1,X} \circ q = e v_X \circ \pi_{1,X} = \chi^C(X).
$$

3.8. **Unimodular tensor categories.** Let $\mathcal{C}$ be a finite tensor category and $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ be the forgetful functor. It is well known that $F$ has both left and right adjoints which are denoted by $L$ and respectively $R$.

Etingof, Nikshych and Ostrik [ENO04] introduced a distinguished invertible object $D \in \mathcal{C}$ of a finite tensor category $\mathcal{C}$ over $\mathbb{k}$. The object $D$ is a category-theoretical analogue of the modular function (also called the distinguished grouplike element) of a finite-dimensional Hopf algebra, and therefore we say that $\mathcal{C}$ is unimodular if $D \simeq 1$.

By [Shi17a] a finite tensor category is unimodular if and only if $R \simeq L$. It is well known that any fusion category is unimodular, [ENO04].

3.9. **Integrals, cointegrals and Fourier transform for finite unimodular tensor categories.** Let $\mathcal{C}$ be a unimodular finite tensor category and $A = Z(1)$ be its adjoint algebra as defined above. An integral in $\mathcal{C}$ is a morphism $\Lambda : 1 \to A$ in $\mathcal{C}$ such that

$$m \circ (\text{id}_A \otimes \Lambda) = \epsilon_1 \otimes \Lambda.$$

A cointegral in $\mathcal{C}$ is a morphism $\lambda : A \to 1$ such that

$$\bar{Z}(\lambda) \circ \delta_1 = u \otimes \lambda$$

where $u : 1 \to A$ is the unit of the algebra $A$. It is well known that the integral and cointegral of a finite unimodular tensor category are unique up to a scalar.

There is also a right action denoted by $\leftarrow$ of $\text{CE}(\mathcal{C})$ on $\text{CF}(\mathcal{C})$ given by

$$f \leftarrow b = f \circ m \circ (b \otimes \text{id}_A)$$

for all $f \in \text{CF}(\mathcal{C})$ and $b \in \text{CE}(\mathcal{C})$.

It is easy to check that $(f \leftarrow b) \leftarrow b' = f \leftarrow (bb')$, thus $\text{CF}(\mathcal{C})$ becomes naturally a right $\text{CE}(\mathcal{C})$-module. Similarly one can define a left action of $\text{CE}(\mathcal{C})$ on $\text{CF}(\mathcal{C})$. 


Let $\lambda \in \text{CF}(\mathcal{C})$ be a non-zero integral of $\mathcal{C}$. The Fourier transform of $\mathcal{C}$ associated to $\lambda$ is the linear map

$$F_\lambda : \text{CE}(\mathcal{C}) \to \text{CF}(\mathcal{C})$$

given by $a \mapsto \lambda \hookrightarrow S(a)$

where $S : \text{CE}(\mathcal{C}) \to \text{CE}(\mathcal{C})$ is the above antipodal operator. The Fourier transform is a bijective $\mathbb{k}$-linear map whose inverse is given in [Shi17b, Equation 5.17].

3.10. The case of fusion categories. For the rest of this section, suppose that $\mathcal{C}$ is a pivotal fusion category over an algebraically closed field $\mathbb{k}$. Furthermore, let $\{V_0, \ldots, V_m\}$ be a complete set of representatives of isomorphism classes of simple objects with $V_0 = 1$. For $i \in \{0, \ldots, m\}$, we define $i^* \in \{0, \ldots, m\}$ by $V_i^* \simeq V_{i^*}$. Then $i \mapsto i^*$ is an involution on $\{0, \ldots, m\}$. As an object of $\mathcal{C}$, the adjoint algebra decomposes as

$$A \simeq \bigoplus_{i=0}^{r} V_i \otimes V_i^*$$

Shimizu has defined in [Shi17b] the elements

$$E_i : 1 \xrightarrow{\text{coev}} V_i \otimes V_i^* \hookrightarrow A, \quad \chi_i : A \xrightarrow{\pi_i} V_i \otimes V_i^* \xrightarrow{\tilde{e}_i} 1$$

It is easy to see that $\{E_i\}_{i=0}^{m}$ and $\{\chi_i\}_{i=0}^{m}$ are bases for $\text{CE}(\mathcal{C})$ and $\text{CF}(\mathcal{C})$ respectively, such that

$$\langle \chi_i, E_j \rangle = \delta_{i,j}.$$ 

Moreover, $E_i E_j = \delta_{i,j}$ and $S(E_i) = E_{i^*}$, where $\tilde{S} : \text{CE}(\mathcal{C}) \to \mathcal{C}$ is the antipodal map defined above. Note that under the isomorphism Equation (3.6) $E_i$ corresponds to the natural transformation that is identity on $V_i$ and zero on the other simple objects.

By [Shi17b] Equation 6.8] one has that

$$\lambda_c = \frac{1}{\dim(\mathcal{C})} \left( \sum_{[V_i] \in \text{Irr}(\mathcal{C})} \dim(V_i^*) \chi_i \right).$$

3.11. Examples from Hopf algebras. Let $H$ be a semisimple Hopf algebra and let $\mathcal{C} = \text{Rep}_H(H)$ be the fusion category of its finite dimensional representations over $\mathbb{k}$. As in [Shi17b, Section3.7] we identify the left center of $\mathcal{C}$ with the category $H^* \text{YD}$ of left-left Yetter Drinfeld modules of $H$. Then the right adjoint $R : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ can be written as $R(V) = H \otimes V$ with $h_1(l \otimes v) = h_1 l S(h_3) \otimes h_2 v$ and
\[ \delta(h \otimes v) = h_1 \otimes (h_2 \otimes v) . \] Thus \( Z(V) = R(V) = H \otimes V \) as \( H \)-modules and the universal diantural maps are given by
\[
(3.13) \quad \pi_{M; V} : Z(V) \to M \otimes V \otimes M^*, \quad h \otimes v \mapsto \sum_i h_1 m_i \otimes h_2 v \otimes m_i^* .
\]
In particular one has that \( Z(1) = H \) whose structure as \( H \)-module is given by the left adjoint action \( h.a = h_1 a S(h_2) \). The multiplication and the unit of \( Z(1) \) are given the usual multiplication and unit of \( H \).
The universal dinatural maps are given by:
\[
\pi_{1; M} : H \to M \otimes M^*, \quad h \mapsto \sum_i h m_i \otimes m_i^* ,
\]
and the comultiplication map \( \delta_1 : Z(1) \to H \otimes Z(1) \) coincide to the usual comultiplication of \( \Delta : H \to H \otimes H \).
Moreover, in this case \( \text{CF}(\mathcal{C}) = \text{C}(H) \), the character ring of \( H \) and \( \text{CE}(\mathcal{C}) = Z(H) \), the center of \( H \). A categorical cointegral \( \lambda \in \text{CF}(\mathcal{C}) \) is the same as a Hopf algebra (two-sided) cointegral \( \lambda \in H^* \) satisfying
\[ \lambda(h_1) h_2 = \lambda(h) 1 \] for all \( h \in H \). The Fourier transform becomes as usually, \( F_\lambda : Z(H) \to \text{C}(H), z \mapsto \lambda \leftrightarrow S(z) \) for all \( z \in Z(H) \).

4. Some identities in the space of class functions

Throughout this subsection \( \mathcal{C} \) is a pivotal fusion category over an algebraically closed field \( k \) of arbitrary characteristic. Recall that by \cite[Lemma 6.2]{Shi17b} the Grothendieck ring \( \text{Gr}_k(\mathcal{C}) \) is a symmetric Frobenius algebra with the trace \( \tau : \text{Gr}_k(\mathcal{C}) \to \mathbb{C} \) given by \( [X] \mapsto \dim_k \text{Hom}_\mathcal{C}(1, X) \).

Recall also by \cite[Section 4]{Shi17b} that in the case of fusion categories the canonical map \( \text{Gr}_k(\mathcal{C}) \simeq \text{CF}(\mathcal{C}), \quad [X] \mapsto \text{ch}(X) \) is in fact an isomorphism of \( k \)-algebras.

We let \( \text{Irr}(\mathcal{C}) = \{ [V_0], \ldots, [V_m] \} \) be the set of isomorphism classes of simple objects of \( \mathcal{C} \) with \( V_0 = 1 \). In this case the characters \( \chi_i := \text{ch}(V_i) \) form a \( k \)-linear basis for \( \text{CF}(\mathcal{C}) \). We also denote by \( d_i = \dim(V_i) \) the quantum dimension of each simple object \( V_i \).

It follows by \cite[Equation (6.4)]{Shi17b} that
\[
(4.1) \quad \tau(\chi_i) = \delta_{i, 0} = \langle \chi_i, E_0 \rangle
\]
for all irreducible characters \( \chi_i \). This shows that
\[
(4.2) \quad \tau(\chi) = \langle \chi, E_0 \rangle, \quad \text{for any } \chi \in \text{CF}(\mathcal{C}),
\]
since the irreducible characters \( \chi_i \) form a basis on \( \text{CF}(\mathcal{C}) \). Also we have that \( \tau(\chi_i * \chi_j) = \delta_{i, j} \) for all irreducible characters \( \chi_i \). Note that \( E_0 \), the idempotent associated to the unit object 1 of \( \mathcal{C} \), is an integral \( \Lambda_\mathcal{C} \in \text{CE}(\mathcal{C}) \), see \cite[Lemma 6.1]{Shi17b}.
4.1. **Conjugacy classes of fusion categories.** Let \( \mathcal{C} \) be a pivotal fusion category over an algebraically closed field of arbitrary characteristic. The global dimension of \( \mathcal{C} \) is defined as

\[
\dim(\mathcal{C}) = \sum_{i=0}^{m} \dim(V_i) \dim(V_i^*) \in k
\]

Recall by [ENO05, Definition 9.1.] that a pivotal finite tensor category is called *non-degenerate* if \( \dim(\mathcal{C}) \neq 0 \) in \( k \). By [Shi17b, Theorem 6.6.] a pivotal fusion category \( \mathcal{C} \) is non-degenerate if and only if one of the following holds:

1. \( \text{Gr}_k(\mathcal{C}) \) is a semisimple algebra.
2. \( \mathcal{Z}(\mathcal{C}) \) is a semisimple abelian category.
3. \( R(1) \in \mathcal{Z}(\mathcal{C}) \) is a semisimple object.

Recall that \( R : \mathcal{C} \to \mathcal{Z}(\mathcal{C}) \) is the right adjoint of the forgetful functor \( F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C} \). Note also that in this case by [Shi17b, Theorem 6.6] the object \( R(1) \in \mathcal{O}(\mathcal{Z}(\mathcal{C})) \) is multiplicity-free since \( \text{Gr}_k(\mathcal{C}) \) is a commutative ring.

Note that in this case if \( \mathcal{D} \) is a fusion subcategory of \( \mathcal{C} \) then \( \text{Gr}_k(\mathcal{D}) \subset \text{Gr}_k(\mathcal{C}) \) is also a semisimple ring since \( \text{Gr}_k(\mathcal{C}) \) is a semisimple commutative ring. Therefore, by above, \( \mathcal{D} \) is also non-degenerate.

Let as above \( F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C} \) be the forgetful functor with its right adjoint \( R : \mathcal{C} \to \mathcal{Z}(\mathcal{C}) \). A *conjugacy class* of \( \mathcal{C} \) is defined as a simple subobject of \( R(1) \) in \( \mathcal{Z}(\mathcal{C}) \). Since the monoidal center \( \mathcal{Z}(\mathcal{C}) \) is also a fusion category we can write \( R(1) = \bigoplus_{i=0}^{m} \mathcal{C}_i \) as a direct sum of simple objects in \( \mathcal{Z}(\mathcal{C}) \). Thus \( \mathcal{C}_0, \ldots, \mathcal{C}_m \) are the conjugacy classes of \( \mathcal{C} \). Since the unit object \( 1_{\mathcal{Z}(\mathcal{C})} \) is always a subobject of \( R(1) \), we can assume \( \mathcal{C}_0 = 1_{\mathcal{Z}(\mathcal{C})} \).

Let \( \bar{F}_0, \bar{F}_1, \ldots, \bar{F}_m \in \text{End}_{\mathcal{Z}(\mathcal{C})}(R(1)) \) be the canonical projections on each of these conjugacy classes. Let also \( F_0, F_1, \ldots, F_m \) be also the corresponding primitive idempotents of \( \text{CF}(\mathcal{C}) \) under the canonical adjunction isomorphism \( \text{CF}(\mathcal{C}) \simeq \text{End}_{\mathcal{Z}(\mathcal{C})}(R(1)) \) from Equation (3.6).

We define \( \bar{\mathcal{C}}_i := \mathcal{F}_{\lambda}^{-1}(F_i) \in \text{CE}(\mathcal{C}) \) to be the *conjugacy class sums* corresponding to the conjugacy class \( \mathcal{C}_i \) where \( \lambda \in \text{CF}(\mathcal{C}) \) is a cointegral such that \( \langle \lambda, u \rangle = 1 \).

For the rest of this section we suppose that \( \mathcal{C} \) is a non-degenerate spherical fusion category with \( \text{Gr}_k(\mathcal{C}) \) a commutative ring.

**Lemma 4.4.** Let \( \mathcal{C} \) be a spherical non-degenerate fusion category. With the above notations one has that:

\[
\langle F_i, \bar{\mathcal{C}}_j \rangle = \delta_{i,j} \dim(\mathcal{C}) \tau(F_i)
\]
Proof. By [Shi17b, Equation 6.10] one has that:

\[ F^{-1}_\lambda(\chi_j) = \frac{\dim(C)}{\dim(V_j)} E_{j^*} \]  

for any irreducible character \( \chi_j \in CF(C) \), where as above \( \lambda \) is an integral with \( \langle \lambda, u \rangle = 1 \). It follows from here that for any two irreducible characters \( \chi_i, \chi_j \in CF(C) \) one has that \( \langle \chi_i, F^{-1}_\lambda(\chi_j) \rangle = \langle \chi_i, \frac{\dim(C)}{\dim(V_j)} E_{j^*} \rangle = \delta_{i,j} \frac{\dim(V_j)}{\dim(V_i)} \dim(C) \). Since \( C \) is spherical it follows that \( \dim(V_i) = \dim(V_j) \) and therefore

\[ \langle \chi_i, F^{-1}_\lambda(\chi_j) \rangle = \dim(C) \tau(\chi_i \chi_j). \]

Since \( \{ \chi_i \}_i \) form a basis for \( CF(C) \) and \( \tau \) is a bilinear form one can deduce that

\[ \langle \chi, F^{-1}_\lambda(\mu) \rangle = \dim(C) \tau(\chi \mu). \]

for any \( \chi, \mu \in CF(C) \). In particular, for \( \chi = F_i \) and \( \mu = F_j \) one has that

\[ \langle F_i, \overline{\chi_j} \rangle = \langle F_i, F^{-1}_\lambda(F_j) \rangle = \dim(C) \tau(F_i F_j) = \delta_{i,j} \dim(C) \tau(F_i). \]

We note that the size \( |\mathcal{C}| \) of \( \mathcal{C} \) is a non-zero scalar of \( k \) since it is the pivotal dimension of a simple object in a pivotal fusion category \( Z(C) \).

By the proof of [Shi17b, Lemma 6.10] one has that

\[ \langle e_1, \overline{\chi_i} \rangle = \dim(C) \langle F_i, \Lambda \rangle = \dim(C) \tau(F_i) = |\mathcal{C}| \neq 0. \]

Define

\[ n_i := \frac{1}{\langle F_i, \Lambda \rangle} = \frac{1}{\tau(F_i)} = \frac{\dim(C)}{|\mathcal{C}|}. \]

Thus in the case of a non-degenerate spherical fusion category \( C \), by Equation (4.5), the pair \( \{ F_i, \frac{m}{\dim(C)} \overline{\chi_i} \} \) constitutes another pair of dual bases for the canonical pairing \( \langle \cdot, \cdot \rangle : CF(C) \times CE(C) \rightarrow k \).

Since \( \{ \chi_i, \frac{1}{\dim(V_i)} E_i \} \) is another pair of dual bases for the same pairing it follows that in \( CF(C) \otimes CE(C) \) we have the following:

\[ \sum_{i=0}^{m} F_i \otimes \frac{n_i}{\dim(C)} \overline{\chi_i} = \sum_{i=0}^{m} \chi_i \otimes \frac{1}{\dim(V_i)} E_i. \]
Let $\lambda \in \text{CF}(\mathcal{C})$ be an idempotent integral. Applying $\text{id} \otimes \mathcal{F}_\lambda$ to the above identity one obtains

$$
\sum_{i=0}^{m} F_i \otimes \frac{n_i}{\dim(C)} F_i = \sum_{i=0}^{m} \chi_i \otimes \frac{1}{\dim(V_i)} \frac{\dim(V_i^*)}{\dim(C)} \chi_i^*
$$

which can be written as

$$(4.9) \quad \sum_{i=0}^{m} F_i \otimes n_i F_i = \sum_{i=0}^{m} \chi_i \otimes \chi_i^*.$$

since $\dim(V_i) = \dim(V_i^*)$, see [EGNO15].

Write $\chi_i = \sum_{j=0}^{m} \alpha_{ij} F_j$ with $\alpha_{ij} \in k$. Applying $\mathcal{F}_\lambda^{-1}$ to this equation one has that

$$(4.10) \quad \frac{\dim(C)}{\dim(V_i)} E_i^* = \sum_{j=0}^{m} \alpha_{ij} \overline{e}_j$$

Note that by [Shi17b, Corollary 6.11] one has

$$(4.11) \quad \alpha_{ij} = \langle \chi_i, g_j \rangle$$

where $g_j := \frac{\overline{e}_j}{\dim(C)}$.

**Lemma 4.12.** Let $\mathcal{C}$ be a non-degenerate spherical fusion category with $\text{Gr}_k(\mathcal{C})$ a commutative ring. With the above notations one has that:

$$(4.13) \quad \overline{\mathcal{E}}_i = \frac{\dim(C)}{n_i} \left( \sum_{j=0}^{m} \frac{1}{\dim(V_j)} \alpha_{ji} E_j \right)$$

**Proof.** The second orthogonality relation from [Shi17b, Theorem 6.11] can be written as

$$(4.14) \quad \sum_{i=0}^{m} \alpha_{ji} \alpha_{j'} \delta_{i,j} = \frac{\dim(C)}{\dim(\overline{\mathcal{E}}_i)}$$

Then applying Equation (4.10) one has

$$
\frac{\dim(C)}{n_i} \left( \sum_{j=0}^{m} \frac{1}{\dim(V_j)} \alpha_{ji} E_j \right) = \frac{\dim(C)}{n_i} \left( \sum_{j=0}^{m} \frac{1}{\dim(V_j)} \alpha_{ji} \left( \frac{\dim(C)}{\dim(C)} \sum_{l=0}^{m} \alpha_{j'l} \overline{e}_l \right) \right) = \\
= \frac{1}{n_i} \sum_{l=0}^{m} \sum_{j=0}^{m} \alpha_{ji} \alpha_{j'l} \overline{e}_l = \frac{\dim(C)}{\dim(\overline{\mathcal{E}}_i) n_i} \overline{\mathcal{E}}_i = \overline{\mathcal{E}}_i.
$$

□
4.2. On the central element $\ell_D$-general case. Let $D$ be a fusion subcategory of a non-degenerate spherical fusion category $\mathcal{C}$. We exhibit a central element $\ell_D := \mathcal{F}_\lambda^{-1}(\lambda_D) \in \text{CE}(\mathcal{C})$ that completely determines the fusion subcategory $D$.

**Proposition 4.15.** Suppose that $\mathcal{C}$ is a non-degenerate spherical fusion category and $\lambda$ is an integral with $\langle \lambda, 1 \rangle = 1$. With the above notations:

$$\ell_D = \frac{\text{dim}(\mathcal{C})}{\text{dim}(D)} \left( \sum_{[V_i] \in \text{Irr}(D)} E_i \right). \quad (4.16)$$

**Proof.** By Equation (3.12) one has that

$$\lambda_D = \frac{1}{\text{dim}(D)} \left( \sum_{[V_j] \in \text{Irr}(D)} \text{dim}(V_j^*) \chi_j \right).$$

Thus

$$\ell_D = \mathcal{F}_\lambda^{-1}(\lambda_D) = \frac{1}{\text{dim}(D)} \left( \sum_{[V_j] \in \text{Irr}(D)} \text{dim}(V_j^*) \mathcal{F}_\lambda^{-1}(\chi_j) \right).$$

On the other hand by Equation (4.6) one has that: $\mathcal{F}_\lambda^{-1}(\chi_j) = \frac{\text{dim}(\mathcal{C})}{\text{dim}(V_j)} E_j^*$ for any irreducible character $\chi_j$. Therefore one can write that:

$$\ell_D = \frac{\text{dim}(\mathcal{C})}{\text{dim}(D)} \left( \sum_{[V_j] \in \text{Irr}(D)} \frac{\text{dim}(V_j^*)}{\text{dim}(V_j)} E_j^* \right).$$

Since $\text{dim}(V_j^*) = \text{dim}(V_j)$ holds for spherical categories one obtains the desired formula. \qed

Suppose as above that $\mathcal{C}$ is a non-degenerate spherical fusion category and $D \subseteq \mathcal{C}$ a fusion subcategory. By Subsection 3.7 there is an inclusion of $k$-algebras $q^*_1: \text{CF}(D) \hookrightarrow \text{CF}(\mathcal{C})$. Thus there is a subset $\mathcal{L}_D \subseteq \{0, \ldots, m\}$ such that

$$\lambda_D = \sum_{j \in \mathcal{L}_D} F_j \quad (4.17)$$

since $\lambda_D$ is an idempotent element inside $\text{CF}(\mathcal{C})$. Then by applying the inverse $\mathcal{F}_\lambda^{-1}$ of the Fourier transform to the above equation it follows that

$$\ell_D = \sum_{j \in \mathcal{L}_D} \overline{\ell_j} \quad (4.18)$$

Since $\{F_i, \frac{1}{|\mathcal{J}|} \overline{\mathcal{F}}^i\}$ are dual bases for the evaluation form $\text{CF}(\mathcal{C}) \times \text{CE}(\mathcal{C}) \to k$ it follows that

$$j \in \mathcal{L}_D \iff \langle F_j, \ell_D \rangle \neq 0.$$
Proposition 4.19. Suppose that $\mathcal{C}$ is a non-degenerate fusion category and $D \subseteq \mathcal{C}$. Then

$$\epsilon_1(\ell_D) = \frac{\dim(\mathcal{C})}{\dim(D)} = \sum_{j \in \mathcal{E}_D} |\mathcal{E}_j|$$

Proof. Note that $\langle \epsilon_1, E_j \rangle = \delta_{j, 0}$ from the orthogonality relations of the characters, see Equation (4.14). Then the first equality follows from Equation (4.16). The second equality follows from Equation (4.18). □

4.3. On the lattice of fusion subcategories.

Lemma 4.21. In any spherical non-degenerate fusion category with a commutative Grothendieck ring one has

$$\ell_{D \cap E} = \frac{\dim(D) \dim(\mathcal{E})}{\dim(D \cap \mathcal{E}) \dim(\mathcal{C})} \ell_D \ell_E, \quad \lambda_{D \cap E} = \mathcal{F}_\lambda(\ell_D \ell_E),$$

Proof. Let $\mathcal{C}$ be a fusion category with commutative character ring. As above one has that

$$\ell_D = \frac{\dim(\mathcal{C})}{\dim(D)} \left( \sum_{[V_i] \in \text{Irr}(D)} E_i \right).$$

Therefore

$$\ell_D \ell_E = \frac{\dim(\mathcal{C})^2}{\dim(D) \dim(\mathcal{E})} \left( \sum_{[V_i] \in \text{Irr}(D \cap \mathcal{E})} E_i \right)$$

On the other hand one has

$$\ell_{D \cap \mathcal{E}} = \frac{\dim(\mathcal{C})}{\dim(D \cap \mathcal{E})} \left( \sum_{[V_i] \in \text{Irr}(D \cap \mathcal{E})} E_i \right) = \frac{\dim(\mathcal{C})}{\dim(D \cap \mathcal{E})} \frac{\dim(D) \dim(\mathcal{E})}{\dim(\mathcal{C})^2} \ell_D \ell_E$$

$$= \frac{\dim(D) \dim(\mathcal{E})}{\dim(D \cap \mathcal{E}) \dim(\mathcal{C})} \ell_D \ell_E.$$

□

Lemma 4.22. Let $\mathcal{C}$ be a fusion category and $D, E \subseteq \mathcal{C}$ be two fusion subcategories. Then

$$\mathcal{L}_D \subseteq \mathcal{L}_E \iff D \supseteq E.$$ 

Proof. If $\mathcal{L}_D \subseteq \mathcal{L}_E$ then by Equation (4.17) one has $\lambda_D \lambda_E = \lambda_D$. Thus $\lambda_{D \cap E} = \lambda_D$. Then Equation (3.12) implies the statement.

The converse is obvious. If $D \supseteq E$ then $\lambda_D \lambda_E = \lambda_D$ and therefore $\mathcal{L}_D \subseteq \mathcal{L}_E$. □
4.4. Fusion subcategories of prime index. In this subsection we let \( k = \mathbb{C} \) and \( \mathcal{C} \) be a pseudo-unitary fusion category with a commutative character ring \( \text{CF}(\mathcal{C}) \). We let as above \( F_i \) be the primitive central idempotents of \( \text{CF}(\mathcal{C}) \). Without loss of generality we may assume that \( F_0 = \lambda_C \), the idempotent cointegral of \( \mathcal{C} \). We denote by \( \mu_i : \text{CF}(\mathcal{C}) \to \mathbb{C} \) the characters of the semisimple commutative \( \mathbb{C} \)-algebra \( \text{CF}(\mathcal{C}) \).

By [EGNO15, Proposition 9.5.1] any pseudo-unitary fusion category admits a unique spherical structure \( a_X : X \to X^{**} \) with respect to which \( d_X = \text{FPdim}(X) \) for every simple object \( X \). Thus in this case the regular character \( r_C = \frac{1}{\dim(\mathcal{C})} (\sum_{X \in \text{Irr}(\mathcal{C})} \text{FPdim}(X) \chi(X)) \) coincides with the idempotent cointegral \( \lambda_C \).

Since \( \text{Gr}_C(\mathcal{C}) \cong \text{CF}(\mathcal{C}) \) as \( \mathbb{C} \)-algebras, it follows that the results from [Bur16] can be applied directly inside the ring \( \text{CF}(\mathcal{C}) \) instead of \( \text{Gr}_C(\mathcal{C}) \). Let \( L_{[X]} : \text{CF}(\mathcal{C}) \to \text{CF}(\mathcal{C}) \) be the operator given by the left multiplication by \( \chi(X) \) on \( \text{CF}(\mathcal{C}) \). Recall that \( L_{[X]} \) has all non-negative entries with respect to the the basis of \( \text{CF}(\mathcal{C}) \) given by the characters \( \chi(S) \) of all simple objects \( S \in \text{Irr}(\mathcal{C}) \). Moreover the largest eigenvalue (in absolute value) of \( L_{[X]} \) is positive and it is called \( \text{FPdim}(X) \). Any eigenvector corresponding to this eigenvalue is called a principal eigenvector.

In [Bur16] for any object \( X \in \mathcal{O}(\mathcal{C}) \) the kernel of \( X \) is defined as:

\[
\ker_C(X) = \{ \mu_i | \chi(X) F_i = \text{FPdim}(X) F_i \},
\]

the set of characters \( \mu_i \) corresponding to the principal eigenvectors \( F_i \) of the operator \( L_{[X]} \). Note that \( \mu_0 = \text{FPdim}() \in \ker_C(X) \) since \( F_0 = \lambda_C \).

Lemma 4.24. Let \( \mathcal{C} \) be a pseudo-unitary fusion category \( \mathcal{C} \) commutative Grothendieck ring. For any two fusion subcategories \( \mathcal{D}, \mathcal{E} \subseteq \mathcal{C} \) with the above notations one has:

\[
\lambda_{\mathcal{D} \vee \mathcal{E}} = \lambda_{\mathcal{D}} \lambda_{\mathcal{E}}, \quad \ell_{\mathcal{D} \vee \mathcal{E}} = \mathcal{F}_\chi^{-1}(\lambda_{\mathcal{D} \vee \mathcal{E}}) = \sum_{j \in \mathcal{L}_{\mathcal{D} \cap \mathcal{E}}} \bar{t}_j, \quad \mathcal{L}_{\mathcal{D} \vee \mathcal{E}} = \mathcal{L}_{\mathcal{D}} \cap \mathcal{L}_{\mathcal{E}}.
\]

Proof. By [Shi17b, Subsection 6.1], in a fusion category \( \mathcal{C} \) the cointegral element \( \lambda_C \) is the unique element \( t \in \text{CF}(\mathcal{C}) \) with the property that

\[
\chi t = \langle \chi, 1 \rangle t, \quad \text{for any } \chi \in \text{CF}(\mathcal{C}).
\]

Since \( \text{CF}(\mathcal{C}) \) is commutative one has \( \lambda_{\mathcal{D}} \lambda_{\mathcal{E}} = \lambda_{\mathcal{E}} \lambda_{\mathcal{D}} \) and \( \lambda_{\mathcal{D}} \lambda_{\mathcal{E}} = \lambda_{\mathcal{D} \vee \mathcal{E}} \in \text{CF}(\mathcal{D} \vee \mathcal{E}) \). Moreover, since \( \mathcal{C} \) is pseudo-unitary note that for any object \( X \in \mathcal{C} \) one has that \( [X] \in \mathcal{O}(\mathcal{D}) \) if and only if \( \ker_C(X) \subseteq \mathcal{L}_D \). Therefore any \( F_i \) with \( i \in \mathcal{L}_D \) is a principal eigenvector for any \( L_{[X]} \) with \( X \in \mathcal{D} \) or \( X \in \mathcal{E} \). By [Bur16, Proposition 3.3] it follows that any \( F_i \) with \( i \in \mathcal{L}_D \) is a principal eigenvector for any \( L_{[X]} \) with \( X \in \mathcal{D} \vee \mathcal{E} \). Thus \( \chi(X)(\lambda_{\mathcal{D}} \lambda_{\mathcal{E}}) = \langle \chi(X), 1 \rangle \lambda_{\mathcal{D}} \lambda_{\mathcal{E}} \) for any \( X \in \text{Irr}(\mathcal{D} \vee \mathcal{E}) \). Therefore \( \lambda_{\mathcal{D} \vee \mathcal{E}} = \lambda_{\mathcal{D}} \lambda_{\mathcal{E}} \).
On the other hand note that:
\[ \lambda_{D \vee E} = \lambda_D \lambda_E = \left( \sum_{j \in \mathcal{L}_D} F_j \right) \left( \sum_{j \in \mathcal{L}_E} F_j \right) = \sum_{j \in \mathcal{L}_D \cap \mathcal{L}_E} F_j \]
This implies that \( \mathcal{L}_{D \vee E} = \mathcal{L}_D \cap \mathcal{L}_E. \)

**Lemma 4.25.** Let \( \mathcal{C} \) be a pseudo-unitary fusion category with a commutative character ring \( \text{CF}(\mathcal{C}) \) over \( k = \mathbb{C} \). Then
\[ (4.26) \quad \mathcal{L}_{\mathcal{C}_\text{ad}} = \{ i \mid |\mathcal{C}_i| = 1 \}. \]

**Proof.** Let \( \chi_{\text{ad}} = \sum_{i=0}^{m} \chi_i \chi_i^* \) be the character of the adjoint subcategory. From Equation (4.9) it follows that \( \chi_{\text{ad}} = \sum_{i=0}^{m} n_i F_i \), where by Equation (4.8) one has \( n_i = \dim(\mathcal{C}) |\mathcal{C}_i| \). Thus
\[ \ker_C(\chi_{\text{ad}}) = \{ \mu_i \mid n_i = \dim(\mathcal{C}) \}. \]

By [Bur16, Proposition 3.12] it follows that the regular character \( r_{\mathcal{C}_\text{ad}} = \frac{1}{\dim(\mathcal{C})} (\sum_{X \in \text{Irr}(\mathcal{C}_\text{ad})} \text{FPdim}([X])[X]) \) of \( \mathcal{C}_\text{ad} \) with respect to \( \text{FPdim} \) can be written in \( \text{CF}(\mathcal{C}) \) as follows
\[ (4.27) \quad r_{\mathcal{C}_\text{ad}} = \sum_{\{ \mu_i \mid |\mathcal{C}_i| = 1 \}} F_i. \]

On the other hand since \( \mathcal{C} \) is pseudo-unitary it follows that \( r_{\mathcal{C}_\text{ad}} = \lambda_{\mathcal{C}_\text{ad}} \) and therefore \( \mathcal{L}_{\mathcal{C}_\text{ad}} = \{ i \mid |\mathcal{C}_i| = 1 \}. \) □

**Theorem 4.28.** Let \( \mathcal{C} \) be an integral fusion category with a commutative character ring \( \text{CF}(\mathcal{C}) \) over \( \mathbb{C} \). Then fusion subcategories \( \mathcal{D} \) of the smallest prime index dividing \( \text{FPdim}(\mathcal{C}) \) are in bijection with normal subgroups of the same prime index \( p \) of the universal grading group \( U_{\mathcal{C}} \) of \( \mathcal{C} \). Moreover, \( \mathcal{D} \supseteq \mathcal{C}_\text{ad} \) for any such fusion subcategory.

**Proof.** Clearly the center \( Z(\mathcal{C}) \) is also an integral fusion category. By Equation (4.20) one has that
\[ \epsilon_1(\ell_{\mathcal{D}}) = p = 1 + \sum_{j \in \mathcal{L}_{\mathcal{D}} \setminus \{0\}} |\mathcal{C}'| \]
and \( |\mathcal{C}'| |\dim(\mathcal{C}) \). Thus \( |\mathcal{C}'| = 1 \) and therefore \( \mathcal{L}_{\mathcal{D}} \subseteq \{ j \mid \chi_j \in \text{Irr}(\mathcal{C}_{\text{pt}}) \} = \mathcal{L}_{\mathcal{C}_\text{ad}} \). This implies that \( \mathcal{D} \supseteq \mathcal{C}_\text{ad} \).

Moreover, by [DGNO07, Lemma 2.5] fusion subcategories containing \( \mathcal{C}_\text{ad} \) are in bijection with subgroups of the universal grading group \( U_{\mathcal{C}} \). Thus \( \mathcal{D} = \mathcal{C}(H) \) where \( H \leq U_{\mathcal{C}} \) is a subgroup of \( U_{\mathcal{C}} \) and \( \mathcal{C}(H) = \oplus_{h \in H} \mathcal{C}_h \). Since \( \text{FPdim}(\mathcal{C}(H)) = |H| \text{FPdim}(\mathcal{C}_\text{ad}) \) it follows that \( H \) is a subgroup of index \( p \) of \( G \). Since \( p \) is also the least divisor of \( G \) it follows that \( H \) is a normal subgroup of \( G \). □
5. Proof of the main centralizer result

The goal of this section is to prove Theorem 1.3. In order to do that we need to recall the following preliminaries.

Recall that two objects $X, Y$ of a braided fusion category $\mathcal{C}$ centralize each other if and only if the monodromy $c_{X,Y}c_{Y,X} : X \otimes Y \to X \otimes Y$ is the identity map. Given two simple objects $V_i$ and $V_j$ of a ribbon fusion category $\mathcal{C}$ one can define

$$s_{ij} := \text{tr}_q(c_{V_j, V_i^*}c_{V_i^*, V_j}),$$

where $\text{tr}_q$ denotes the canonical quantum trace in $\mathcal{C}$.

Throughout this section we consider a pre-modular fusion category $\mathcal{C}$, i.e. a braided fusion category with a spherical structure. Recall that by [ENO05, Prop 8.23] any braided weakly integral fusion category is pre-modular. As in the previous section we use the same notation $V_0, \ldots, V_m$ for the set of simple objects of $\mathcal{C}$ up to isomorphism. We let also $E_0, \ldots, E_m$ be their associated primitive idempotents inside $\text{CE}(\mathcal{C})$. Without loss of generality we assume that $V_0 = 1$. Then by [Shi17b, Section 6] $E_0$ is the idempotent integral of $\mathcal{C}$ and $\epsilon_1 = \chi_0$. It is also well known that with this notations one has $d_i = s_{0i} = s_{i0}$, the quantum dimension of the simple object $V_i$.

**Remark 5.1.** Let $\mathcal{C}$ be a pre-modular fusion category and $\mathcal{D}, \mathcal{E}$ be two fusion subcategories. It is well known that in a pre-modular fusion category two simple objects $V_i$ and $V_j$ centralize each other if and only if $s_{ij} = d_id_j$.

Extending linearly the $S$-matrix by $S(\sum_i \alpha_i \chi_i, \sum_j \beta_j \chi_j) = \sum_{ij} \alpha_i \beta_j s_{ij}$ we see that if $\mathcal{D}' \subseteq \mathcal{E}$ then $S(\chi, \mu) = d(\chi)d(\mu)$ for any $\chi \in \text{CF}(\mathcal{D})$ and $\mu \in \text{CF}(\mathcal{E})$. Here $d : \text{CF}(\mathcal{C}) \to \mathbb{K}$ is the quantum dimension homomorphism defined on the basis by $d(\chi_i) = d_i$.

5.1. Definition of the Drinfeld map. In this subsection we recall the construction of the Drinfeld map in a braided fusion category. Consider as in Section 3 also the central Hopf monad of $\mathcal{C}$ given by:

$$\mathcal{L}(V) = \int^{X \in \mathcal{C}} X^* \otimes V \otimes X$$

and $\iota_{V,X} : X^* \otimes V \otimes X \to \mathcal{L}(V)$ the associated universal dinatural maps. As already mentioned is well known that $\mathcal{L}$ is a Hopf monad and $\mathcal{L}$ is a left adjoint of the forgetful functor $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$.

Following [FGR19, Subsection 4.4] one can define the map $T_{X,Y} : X \otimes X^* \to Y^* \otimes Y$ as the classical Hopf link map. Since this is dinatural with respect to $X$ and $Y$, by the universal properties of ends and coends
there exists a unique $Q : \mathcal{L}(1) \to \mathbb{Z}(1)$ such that $T_{X,Y}$ factors as

$$T_{X,Y} = \pi_1, X \circ Q \circ \iota_1, Y$$

The map $Q$ is called the Drinfeld map associated to $\mathcal{C}$.

Since $\mathcal{L}(1) \simeq \mathbb{Z}(1)^*$ this map induces a map on the ring of class functions $f_Q : \text{CF}(\mathcal{C}) \to \text{CE}(\mathcal{C})$. In the case of a ribbon category $\mathcal{C}$ the map $f_Q$ has the following formula (see [Shi17b, Example 6.14])

$$f_Q(\chi_i) = \sum_{j=0}^{m} s_{ij} E_j$$

(5.2)

where as above $s_{ij} := \text{tr}_q(c_{V_j}, V_i^*, c_{V_i^*}, V_j)$, Here $E_i \in \text{CE}(\mathcal{C})$ represents the primitive idempotent of $\text{CE}(\mathcal{C})$ associated to the simple object $V_i$.

Recall that for a pivotal fusion category we denoted by $\lambda_C \in \text{CF}(\mathcal{C})$ the idempotent integral of $\mathcal{C}$.

**Lemma 5.3.** Let $\mathcal{C}$ be a ribbon fusion category and $f_Q : \text{CF}(\mathcal{C}) \to \text{CE}(\mathcal{C})$ defined as above. Then

$$\epsilon_1(f_Q(\chi)) = \langle \chi, u \rangle,$$

for any character $\chi \in \text{CF}(\mathcal{C})$.

**Proof.** As mentioned above $\epsilon_1 = \chi_0$ and $\langle \chi_i, E_j \rangle = \delta_{i,j}d_i$. Then the statement follows from Equation (5.2). \qed

### 5.2. Proof of the main result.

As explained in [Shi17b, Section 6] for a braided fusion category the spaces $\text{CF}(\mathcal{C})$ and $\text{CE}(\mathcal{C})$ are $k$-semisimple commutative algebras. Thus, since $f_Q$ is an algebra map, it follows that $f_Q(q_1^*(\lambda_\mathcal{D}))$ can be written as sum of the central primitive idempotents $E_j \in \text{CE}(\mathcal{C})$. By abuse of notations we will write $f_Q(\lambda_\mathcal{D})$ instead of $f_Q(q_1^*(\lambda_\mathcal{D}))$ in the next theorem.

Now we are ready to prove the main result Theorem 1.3 concerning the centralizer in pre-modular fusion categories.

**Theorem 5.4.** Let $\mathcal{D}$ be a fusion subcategory of a ribbon fusion category $\mathcal{C}$ and $\lambda_\mathcal{D}$ be the associated idempotent cointegral of $\mathcal{D}$. Then

$$\mathcal{F}_\lambda(f_Q(\lambda_\mathcal{D})) = \frac{\dim(\mathcal{D}')}{\dim(\mathcal{C})} \lambda_{\mathcal{D}'}.$$

where $\mathcal{F}_\lambda$ is the Fourier transform introduced by Shimizu in [Shi17b] and $\lambda_{\mathcal{D}'}$ is the idempotent cointegral of $\mathcal{D}'$. 

Proof. Let \( \mathcal{D} \) be a fusion subcategory of a ribbon tensor category \( \mathcal{C} \). Suppose that for the idempotent integral \( \lambda_\mathcal{D} \) of \( \mathcal{D} \) has the following writing in \( \text{CE}(\mathcal{C}) \):

\[
(5.5) \quad f_Q(\lambda_\mathcal{D}) = \sum_{j \in \mathcal{A}_0} E_j.
\]

First we will prove that

\[
(5.6) \quad \text{Irr}(\mathcal{D}') = \{ V_j | j \in \mathcal{A}_0 \}.
\]

For any irreducible character \( \chi_i \in \text{CF}(\mathcal{D}) \) one has by Equation (6.1)

\[
f_Q(\chi_i) = \sum_{j=0}^{m} s_{ij} d_j E_j.
\]

Moreover, by the definition of the categorical cointegral one has \( \chi_i \lambda_\mathcal{D} = d_i \lambda_\mathcal{D} \) for any \( \chi_i \in \text{Irr}(\mathcal{D}) \). This implies that

\[
d_i f_Q(\lambda_\mathcal{D}) = f_Q(\chi_i \lambda_\mathcal{D}) = f_Q(\chi_i) f_Q(\lambda_\mathcal{D}).
\]

Using Equation (5.5) this gives that \( s_{ij} = d_i d_j \) for any \( j \in \mathcal{A}_0 \) and \( \chi_i \in \mathcal{D} \). Thus \( \{ \chi_j | j \in \mathcal{A}_0 \} \subseteq \mathcal{D}' \).

Conversely if \( \chi_j \in \text{CF}(\mathcal{D}') \) then \( S(\chi_j, \lambda_\mathcal{D}) = d_j \) by Remark 5.1. Since \( r_\mathcal{D} = \text{dim}(\mathcal{D}) \lambda_\mathcal{D} \) it follows that

\[
(5.7) \quad \langle \chi_j, f_Q(r_\mathcal{D}) \rangle = \text{dim}(V_j) \text{dim}(\mathcal{D})
\]

From formula (5.5) this happens if and only if \( j \in \mathcal{A}_0 \).

As above one has that \( \mathcal{F}_\lambda(E_i) = \frac{d_i}{\text{dim}(\mathcal{C})} \chi_i^{**} \). Thus

\[
\mathcal{F}_\lambda(f_Q(\lambda_\mathcal{D})) = \sum_{j \in \mathcal{A}_0} \mathcal{F}_\lambda(E_j) = \sum_{j \in \mathcal{A}_0} \frac{d_j}{\text{dim}(\mathcal{C})} \chi_j^{**} = \frac{\text{dim}(\mathcal{D}')}{\text{dim}(\mathcal{C})} \lambda_{\mathcal{D}'}.
\]

Corollary 5.8. Let \( \mathcal{C} \) be a ribbon category. Consider the particular case \( \mathcal{D} = \mathcal{C}' \). Then \( f_Q(\lambda_\mathcal{D}) = 1_{\text{CE}(\mathcal{C})} \), the sum of all primitive central the idempotents.

Proof. The Corollary follows since \( \mathcal{D}' = (\mathcal{C}')' = \mathcal{C} \).

In the next example we show how to recover the centralizer formula [Bur17] from our main result Theorem 1.3.

Example 5.9. Suppose that \((H, R)\) is a semisimple quasi-triangular Hopf algebra and let \( L \) be a left normal coideal subalgebra of \( H \). Let \( \mathcal{D} := \text{Rep}(H//L) \). Then \( \lambda_\mathcal{D} = \lambda_L \) is the integral of the Hopf algebra \((H//L)^*\). Recall that \( H \) is a ribbon Hopf algebra with ribbon element given by \( v = u^{-1} \), the inverse of the Drinfeld element.
It was shown in [Bur17, Lemma 4.1] that \( f_Q(\lambda_L) = \Lambda_M \) where \( M := \phi_R((H//L)^*) \) is a normal left coideal subalgebra of \( H \). Then in this case \( F(\Lambda_M) = \lambda_M \) by [CW17, Lemma 1.1]. Note that the result \( \text{Rep}(H//L) = \text{Rep}(H//M) \) from [Bur17, Theorem 1.1] follows now from the fact \( M = M \hookrightarrow H^* \), see [CW17, Lemma 1.1].

6. On the conjugacy classes of fusion categories

A pre-modular category \( C \) is called modular if its \( S \)-matrix is non-degenerate. By [DGNO10, Proposition 3.7] the \( S \)-matrix is non-degenerate if and only if \( C' = \text{Vec} \).

In the case of a modular tensor category \( C \), the map \( f_Q \) is a bijective map and after a relabelling of indices one may suppose that \( f_Q(F_j) = E_j \) for any primitive idempotent \( E_j \in \text{CE}(C) \) associated to the character \( \chi_j \in \text{CF}(C) \). In this case as in [Shi17b, Example 6.14] one can also write:

\[
\chi_i = \sum_{j=0}^{m} \frac{s_{ij}}{d_j} F_j.
\]

Lemma 6.2. Let \( C \) be a modular category. With the above notations it follows that \( F_0 = \lambda_C \) and \( f_Q(F_0) = \Lambda_C \) is the categorical integral of \( C \).

Proof. Note that from [Shi17b, Lemma 6.9] for \( r = 0 \) it follows that \( F_0 = \lambda_C \) is the idempotent integral of \( C \). One has that \( f_Q(f) f_Q(F_0) = f_Q(f F_0) = \langle f, u \rangle f_Q(F_0) \). Since \( \langle f, u \rangle = \langle \epsilon_1, f_Q(f) \rangle \) it follows that \( f_Q(f) f_Q(F_0) = \langle \epsilon_1, f_Q(f) \rangle \) for all \( f \in \text{CF}(C) \). On the other hand since \( C \) is modular it follows that \( f_Q \) is surjective and therefore \( f_Q(F_0) \) is an integral of \( C \).

6.1. Product of conjugacy class sums in the modular case.

Assume that \( C \) is a modular tensor category and let as above \( F_j \) be the primitive idempotents of \( \text{CF}(C) \). As in Section 4 one can write \( \chi_i = \sum_{j=0}^{m} \alpha_{ij} F_j \) with \( \alpha_{ij} \in \mathbb{k} \).

Lemma 6.3. Assume that \( C \) is a modular tensor category. With the above notations one has that:

\[
\alpha_{ij} = \frac{d_i}{d_j} \alpha_{ji}.
\]
Proof. In the modular case one has that $h_i : \text{CF}(\mathcal{C}) \to \mathbb{k}$, $[X_i] \mapsto \frac{s_{ij}}{d_j}$ are all the characters of the character ring $\text{CF}(\mathcal{C})$. Then as in [Shi17b] it follows directly that $\alpha_{ij} = \frac{s_{ij}}{d_j}$. Thus the equality $\alpha_{ij} = \frac{d_i d_j}{|\mathcal{C}|} \alpha_{ji}$ follows from the symmetry of the $S$-matrix, i.e. $s_{ij} = s_{ji}$. □

**Theorem 6.5.** Let $\mathcal{C}$ be a modular fusion category. Then

\[
(6.6) \quad f_Q(\chi_i) = \frac{d_i n_i}{\text{dim}(\mathcal{C})} \overline{\mathcal{C}}_i = \frac{d_i}{|\mathcal{C}|} \overline{\mathcal{C}}_i.
\]

**Proof.** Using Lemma (4.12) one has that

\[
(6.7) \quad f_Q(\chi_i) = \sum_{j=0}^{m} \alpha_{ij} E_j = \sum_{j=0}^{m} \frac{d_i}{d_j} \alpha_{ji} E_i = d_i \sum_{j=0}^{m} \frac{1}{d_j} \alpha_{ji} E_i = \frac{d_i n_i}{\text{dim}(\mathcal{C})} \overline{\mathcal{C}}_i.
\]

The second equality of the theorem follows from Equation (4.8). □

6.2. $\ell_D$ in the pre-modular case. Note that in the case of a ribbon category $\mathcal{C}$, the main Theorem 1.3 can be written as

\[
(6.8) \quad \ell_D' = \frac{\text{dim}(\mathcal{D}')}{\text{dim}(\mathcal{C})} f_Q(\lambda_{\mathcal{D}})
\]

**Theorem 6.9.** Let $\mathcal{C}$ be a modular tensor category and $\mathcal{D}$ be a fusion subcategory of $\mathcal{C}$. With the above notations one has

1. $d_j^2 = |\mathcal{C}^j|$ for all $j$.
2. For the centralizer $\mathcal{D}'$ one has that:

\[
\ell_{\mathcal{D}'} = \sum_{\{j \mid \chi_j \in \text{Irr}(\mathcal{D})\}} \overline{\mathcal{C}}_j,
\]

i.e. $\mathcal{L}_{\mathcal{D}'} = \{j \mid \chi_j \in \text{Irr}(\mathcal{D})\}$.

**Proof.** Note that $\text{dim}(\mathcal{D}) \cdot \text{dim}(\mathcal{D}') = \text{dim}(\mathcal{C})$ since $\mathcal{C}$ is a modular tensor category. By Equation (6.8) one has

\[
\ell_{\mathcal{D}'} = \frac{\text{dim}(\mathcal{D}')}{\text{dim}(\mathcal{C})} f_Q(\lambda_{\mathcal{D}}) = \frac{\text{dim}(\mathcal{D}')}{\text{dim}(\mathcal{C})} \left( \frac{1}{\text{dim}(\mathcal{D})} \sum_{j \in \text{Irr}(\mathcal{D})} d_j \cdot f_Q(\chi_j) \right) = \sum_{j \in \text{Irr}(\mathcal{D})} d_j \cdot \frac{d_j}{|\mathcal{C}^j|} \overline{\mathcal{C}}_j
\]

On the other hand by Equation (4.17) one has that

\[
\ell_{\mathcal{D}'} = \sum_{j \in \mathcal{L}_{\mathcal{D}'}^j} \overline{\mathcal{C}}_j
\]

This implies both statements of the theorem. □

**Corollary 6.10.** Let $\mathcal{C}$ be a modular tensor category. Then

1. $|\mathcal{C}^j| = 1 \iff \chi_j \in \text{Irr}(\mathcal{C}_{pt}) \iff F(\mathcal{C}^j) = 1$. 


(2) \( \ell_{\text{ad}} = \sum_{j \mid \chi_j \in \text{Irr}(\mathcal{C}_{\text{pt}})^*} \mathcal{C}_j \), \( \mathcal{L}_{\text{ad}} = \{ j \mid \chi_j \in \text{Irr}(\mathcal{C}_{\text{pt}})^* \} \), and
\( \lambda_{\text{ad}} = \sum_{j \mid \chi_j \in \text{Irr}(\mathcal{C}_{\text{ad}})} F_j \).
\( \ell_{\text{pt}} = \sum_{j \mid \chi_j \in \text{Irr}(\mathcal{C}_{\text{ad}})^*} \mathcal{C}_j \), and \( \lambda_{\text{pt}} = \sum_{j \mid \chi_j \in \text{Irr}(\mathcal{C}_{\text{ad}})} F_j \).

Proof. The first item follows from the fact that \( |\mathcal{C}_j| = d_j^2 \) and under the forgetful functor \( F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C} \) the object \( F(\mathcal{C}_j) \) always contains \( 1_{\mathcal{C}} \). The last two items follow from the fact that \( \mathcal{C}_{\text{ad}}' = \mathcal{C}_{\text{pt}} \) and \( \mathcal{C}_{\text{pt}}' = \mathcal{C}_{\text{ad}} \) in a modular tensor category. \( \square \)

6.3. On the structure constants and product of conjugacy classes. In [Shi17b] the author asked if results from [CW14, CW00] concerning conjugacy classes can be extended from semisimple Hopf algebras to fusion categories.

Since the class sums \( \mathcal{C}_i \) form a basis for the center \( \text{CF}(\mathcal{C}) \) one has that
\[
(6.11) \quad \mathcal{C}_i \mathcal{C}_j = \sum_{l=0}^{r} c^l_{ij} \mathcal{T}_l
\]
for some scalars \( c^l_{ij} \in \mathbb{k} \). These scalars were called the structure constants of conjugacy classes in [CW10]. For factorizable Hopf algebras, by [CW10, Theorem 4.3], one has that
\[
(6.12) \quad c^l_{ij} = \frac{d_i d_j}{d_l} N^l_{ij}
\]
where \( \chi_i \chi_j = \sum_{l=0}^{r} N^l_{ij} \chi_l \).

The following result follows from Equation (6.14) and extends the above mentioned result to arbitrary modular tensor categories.

**Proposition 6.13.** Let \( \mathcal{C} \) be a modular tensor category. With the above notations we have that the conjugacy class sums form a \( \mathbb{Q} \)-subalgebra of the algebra \( \text{CE}(\mathcal{C}) \) of central elements. More precisely,
\[
\mathcal{C}_i \mathcal{C}_j = \sum_{l=0}^{m} \frac{d_i d_j}{d_l} N^l_{ij} \mathcal{T}^l
\]
where \( N^l_{ij} \) are the fusion coefficients of \( \mathcal{C} \), i.e. \( \chi_i \chi_j = \sum_{l=0}^{m} N^l_{ij} \chi_l \).

Proof. Since \( |\mathcal{C}_i| = d_i^2 \), Equation (6.7) implies that \( f_Q(\chi_i) = \frac{1}{d_i} \mathcal{T}_i \). For any two irreducible characters write \( \chi_i \chi_j = \sum_{l=0}^{m} N^l_{ij} \chi_l \) for some non-negative integers \( N^l_{ij} \). Then one can write
\[
(6.14) \quad \frac{1}{d_i d_j} \mathcal{C}_i \mathcal{C}_j = f_Q(\chi_i) f_Q(\chi_j) = f_Q(\chi_i \chi_j) = \sum_{l=0}^{m} N^l_{ij} f_Q(\chi_l) = \sum_{l=0}^{m} \frac{1}{d_l} N^l_{ij} \mathcal{T}^l
\]
which proves the statement. \qed

7. Appendix

Let $\mathcal{C}$ be any finite tensor category and $\mathcal{D}$ a full subcategory of $\mathcal{C}$. In this appendix we prove the main properties of the ends

\[ Z_{\mathcal{C}}(M) = \int_{X \in \mathcal{C}} X \otimes M \otimes X^* \]

and

\[ Z_{\mathcal{D}}(M) = \int_{X \in \mathcal{D}} X \otimes M \otimes X^* \]

that were used in the previous sections of the paper. For simplicity we write $Z := Z_{\mathcal{C}}$ and $\bar{Z} := Z_{\mathcal{D}}$. We will also denote by $\pi_{M;X} : Z(M) \to X \otimes M \otimes X^*$ and respectively $\bar{\pi}_{M;X} : \bar{Z}(M) \to X \otimes M \otimes X^*$ their universal dinatural transformations.

By Fubini theorem for ends one has that

\[ Z^2(M) = \int_{(X,Y) \in \mathcal{C} \times \mathcal{C}} (X \otimes Y) \otimes M \otimes (X \otimes Y)^* \]

and

\[ \bar{Z}^2(M) = \int_{(X,Y) \in \mathcal{D} \times \mathcal{D}} (X \otimes Y) \otimes M \otimes (X \otimes Y)^* \]

with universal dinatural transformations:

\[ \bar{\pi}^2_{M,(X,Y)} = Z^2(M) \xrightarrow{\pi_{Z(M);X}} X \otimes Z(M) \otimes X^* \xrightarrow{id \otimes \bar{\pi}^1_{M,Y} \otimes id} (X \otimes Y) \otimes M \otimes (X \otimes Y)^* \]

respectively

\[ \bar{\pi}^2_{M,(X,Y)} = \bar{Z}^2(M) \xrightarrow{\bar{\pi}_{Z(M);X}} X \otimes \bar{Z}(M) \otimes X^* \xrightarrow{id \otimes \bar{\pi}^1_{M,Y} \otimes id} (X \otimes Y) \otimes M \otimes (X \otimes Y)^* \]

Using the universal property of the ends there are unique maps

\[ \delta_M : Z(M) \to Z^2(M), \quad \epsilon_M : Z(M) \to M \]

making $Z : \mathcal{C} \to \mathcal{C}$ a comonad on $\mathcal{C}$. Similarly, $\bar{Z} : \mathcal{C} \to \mathcal{C}$ is a comonad with structure maps

\[ \bar{\delta}_M : \bar{Z}(M) \to \bar{Z}^2(M), \quad \bar{\epsilon}_M : \bar{Z}(M) \to M \]

As in subsection 3.7 there is a unique natural transformation $q : Z \to \bar{Z}$ such that

\[ \bar{\pi}_{M;X} \circ q_M = \pi_{M;X} \]

for any object $X$ of $\mathcal{D}$. 
By the universal property of the coend $Z(M \otimes N)$, the canonical morphisms

$$Z(M) \otimes Z(N) \xrightarrow{\pi_{M,X} \otimes \pi_{N,X}} X \otimes M \otimes X^* \otimes X \otimes N \otimes X^* \xrightarrow{id \otimes ev_X \otimes id_X} X \otimes (M \otimes N) \otimes X$$

induces a canonical map $Z_{M,N}^2 : Z(M) \otimes Z(N) \to Z(M \otimes N)$. Together with $Z^0 := \pi_{1,1} : Z(1) \to 1$ one can verify that $(Z, Z^2, Z^0)$ is a monoidal functor.

Similarly one can define a monoidal structure $(\bar{Z}, \bar{Z}^2, \bar{Z}^0)$ on the relative coend $\bar{Z}$.

**7.1. Morphisms of monoidal functors.** A monoidal morphism between two monoidal functors on a monoidal category $C$ is a morphism $q : T \to S$ such that the diagrams

$$
\begin{array}{c}
T(M) \otimes T(N) \\
\downarrow \phi_{M,N} \quad \downarrow \phi_{M,N} \\
S(M) \otimes S(N)
\end{array}
\quad \text{and}
\begin{array}{c}
T(M \otimes N) \\
\downarrow \phi_{M,N} \\
S(M \otimes N)
\end{array}
$$

are commutative.

**Lemma 7.1.** The natural transformation $q : Z \to \bar{Z}$ from above is a morphism of monoidal functors.

**Proof.** The compatibility with the unit follows directly from the definition of $q_1$ since $\epsilon_1 = \pi_{1,1}$ and $\bar{\epsilon}_1 = \bar{\pi}_{1,1}$.

In order to show that $q$ is a morphism of monoidal functors one needs to verify:

$$q_{M \otimes N} \circ Z_{M,N}^2 = \bar{Z}_{M,N}^2 \circ (q_M \otimes q_N) : Z(M) \otimes Z(N) \to \bar{Z}(M \otimes N)$$

i.e. that the upper rectangle of the following diagram is commutative:

Note that the left and right rectangles of the above diagram are commutative by the definition of $q$. The bottom bent rectangle commutes by the definition of $\bar{Z}^2$ while the whole bent rectangle by the definition of $Z^2$. It follows that the two maps from above are the universal maps corresponding to the same dinatural map $Z(M) \otimes Z(N) \to X \otimes (M \otimes N) \otimes X^*$. By the unicity of the universal map it follows that they are equal. □
7.1.1. **Definition of** $q^{(2)}_M$. By the universal property of the end $\bar{Z}^2(M) = \int_{(X,Y) \in D \times D} X \otimes Y \otimes M \otimes (X \otimes Y)^*$ there is a unique natural transformation $q^{(2)} : Z^2 \rightarrow \bar{Z}^2$ such that for any two objects $X, Y \in O(D)$ the following diagram

\[
\begin{array}{c}
\begin{array}{ccc}
Z^2(M) & \xrightarrow{\pi_Z^{(M), X}} & X \otimes Z(M) \otimes X^* \\
\downarrow{\pi^{(2)}_M (X,Y)} & & \downarrow{\pi^{(2)}_M (X,Y)} \\
(X \otimes Y) \otimes M \otimes (X \otimes Y)^* & \xrightarrow{id_X \otimes q_M \otimes id_X^*} & X \otimes \bar{Z}(M) \otimes X^*
\end{array}
\end{array}
\]

is commutative.

7.1.2. **Definition of** $r^{(2)}_M$. There is a dinatural map in $X \in O(D)$ given by

\[
Z^2(M) \xrightarrow{\pi_Z^{(M), X}} X \otimes Z(M) \otimes X^* \xrightarrow{id_X \otimes q_M \otimes id_X^*} X \otimes \bar{Z}(M) \otimes X^*.
\]

Since $\bar{Z}^2(M) = \int_{X \in C} X \otimes \bar{Z}(V) \otimes X^*$ it follows that there exists a unique map

\[
r^{(2)}_M : Z^2(M) \rightarrow \bar{Z}^2(M)
\]

such that:

\[
\begin{array}{c}
\begin{array}{ccc}
Z^2(M) & \xrightarrow{r^{(2)}_M} & \bar{Z}^2(M) \\
\downarrow{\pi_Z^{(M), X}} & & \downarrow{\pi^{(2)}_Z (M), X} \\
X \otimes Z(M) \otimes X^* & \xrightarrow{id_X \otimes q_M \otimes id_X^*} & X \otimes \bar{Z}(M) \otimes X^*
\end{array}
\end{array}
\]

7.1.3. $r^{(2)}_M = q^{(2)}_M$.

**Lemma 7.2.** With the above notations $r^{(2)}_M = q^{(2)}_M$.

**Proof.** By the definition of $q^{(2)}_M$ this we have to show that

\[
\begin{array}{c}
\begin{array}{ccc}
Z^2(M) & \xrightarrow{\pi_Z^{(M), X}} & X \otimes Z(M) \otimes X^* \\
\downarrow{\pi^{(2)}_M (X,Y)} & & \downarrow{\pi^{(2)}_M (X,Y)} \\
(X \otimes Y) \otimes M \otimes (X \otimes Y)^* & \xrightarrow{id_X \otimes q_M \otimes id_X^*} & X \otimes \bar{Z}(M) \otimes X^*
\end{array}
\end{array}
\]

Expanding $\pi^2$ and $\bar{\pi}^2$ we have to show that:

\[
\begin{array}{c}
\begin{array}{ccc}
Z^2(M) & \xrightarrow{r^{(2)}_M} & \bar{Z}^2(M) \\
\downarrow{\pi^{(2)}_Z (M), X} & & \downarrow{\pi^{(2)}_Z (M), X} \\
X \otimes Z(M) \otimes X^* & \xrightarrow{id_X \otimes q_M \otimes id_X^*} & X \otimes \bar{Z}(M) \otimes X^*
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
\xrightarrow{id_X \otimes q_M \otimes id_X^*} & \xrightarrow{id_X \otimes q_M \otimes id_X^*} \\
\xrightarrow{id_X \otimes \bar{\pi}^2_{M,Y} \otimes id_X^*} & \xrightarrow{id_X \otimes \bar{\pi}^2_{M,Y} \otimes id_X^*} \\
(X \otimes Y) \otimes M \otimes (X \otimes Y)^* & \xrightarrow{id_X \otimes q_M \otimes id_X^*} & X \otimes \bar{Z}(V) \otimes X^*
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
\xrightarrow{id_X \otimes \bar{\pi}^2_{M,Y} \otimes id_X^*} & \xrightarrow{id_X \otimes \bar{\pi}^2_{M,Y} \otimes id_X^*} \\
\xrightarrow{id_X \otimes q_M \otimes id_X^*} & \xrightarrow{id_X \otimes q_M \otimes id_X^*} \\
(X \otimes Y) \otimes M \otimes (X \otimes Y)^* & \xrightarrow{id_X \otimes q_M \otimes id_X^*} & X \otimes \bar{Z}(M) \otimes X^*
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
\xrightarrow{id_X \otimes \bar{\pi}^2_{M,Y} \otimes id_X^*} & \xrightarrow{id_X \otimes \bar{\pi}^2_{M,Y} \otimes id_X^*} \\
\xrightarrow{id_X \otimes q_M \otimes id_X^*} & \xrightarrow{id_X \otimes q_M \otimes id_X^*} \\
(X \otimes Y) \otimes M \otimes (X \otimes Y)^* & \xrightarrow{id_X \otimes q_M \otimes id_X^*} & X \otimes \bar{Z}(M) \otimes X^*
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
\xrightarrow{id_X \otimes \bar{\pi}^2_{M,Y} \otimes id_X^*} & \xrightarrow{id_X \otimes \bar{\pi}^2_{M,Y} \otimes id_X^*} \\
\xrightarrow{id_X \otimes q_M \otimes id_X^*} & \xrightarrow{id_X \otimes q_M \otimes id_X^*} \\
(X \otimes Y) \otimes M \otimes (X \otimes Y)^* & \xrightarrow{id_X \otimes q_M \otimes id_X^*} & X \otimes \bar{Z}(M) \otimes X^*
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
\xrightarrow{id_X \otimes \bar{\pi}^2_{M,Y} \otimes id_X^*} & \xrightarrow{id_X \otimes \bar{\pi}^2_{M,Y} \otimes id_X^*} \\
\xrightarrow{id_X \otimes q_M \otimes id_X^*} & \xrightarrow{id_X \otimes q_M \otimes id_X^*} \\
(X \otimes Y) \otimes M \otimes (X \otimes Y)^* & \xrightarrow{id_X \otimes q_M \otimes id_X^*} & X \otimes \bar{Z}(M) \otimes X^*
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
\xrightarrow{id_X \otimes \bar{\pi}^2_{M,Y} \otimes id_X^*} & \xrightarrow{id_X \otimes \bar{\pi}^2_{M,Y} \otimes id_X^*} \\
\xrightarrow{id_X \otimes q_M \otimes id_X^*} & \xrightarrow{id_X \otimes q_M \otimes id_X^*} \\
(X \otimes Y) \otimes M \otimes (X \otimes Y)^* & \xrightarrow{id_X \otimes q_M \otimes id_X^*} & X \otimes \bar{Z}(M) \otimes X^*
\end{array}
\end{array}
\]
The bottom triangle is commutative by the definition of $q_M$. The upper rectangle is commutative by the definition of $i^{(2)}_M$. □

**Lemma 7.3.** One has that

$$q^{(2)}_M = \tilde{Z}(q_M) \circ q_{Z(M)} = q_{Z(M)} \circ Z(q_M).$$

**Proof.** For the first equality one has to show that the following diagram is commutative:

$$\begin{array}{ccc}
Z^2(M) & \xrightarrow{q_{Z(M)}} & \tilde{Z}(M) \\
\downarrow{\pi_{Z(M)}; X} & & \downarrow{\tilde{Z}(M); X} \\
X \otimes Z M \otimes X^* & \xrightarrow{id_X \otimes q_M \otimes id_X^*} & X \otimes \tilde{Z}(M) \otimes X^*
\end{array}$$

Note that the left triangle is commutative by the definition of $q_{Z(M)}$. The next square commutes by the definition of $\tilde{Z}(q_M)$.

For the second equality one has to show that the following diagram is commutative:

$$\begin{array}{ccc}
Z^2(M) & \xrightarrow{Z(q_M)} & Z\tilde{Z}(M) \\
\downarrow{\pi_{Z(M)}; X} & & \downarrow{\tilde{Z}(M); X} \\
X \otimes Z M \otimes X^* & \xrightarrow{id_X \otimes q_M \otimes id_X^*} & X \otimes \tilde{Z}(M) \otimes X^*
\end{array}$$

Note that the left square commutes by the definition of $Z(q_M)$ while the right triangle commutes by the definition of $q_{Z(M)}$. □

7.2. **On the map** $v_M$.

**Lemma 7.4.** There is a unique natural transformation $v_M : Z(M) \rightarrow \tilde{Z}^2(M)$ such that

$$\tilde{\pi}_M^{(2)}; (X,Y) \circ v_M = \tilde{\pi}_M; X \otimes Y \circ q_M$$

i.e. such that the following diagram is commutative:

$$\begin{array}{ccc}
Z(M) & \xrightarrow{v_M} & \tilde{Z}^2(M) \\
\downarrow{q_M} & & \downarrow{\tilde{\pi}_M; (X,Y)} \\
\tilde{Z}(M) & \xrightarrow{\pi_{M,X \otimes Y}} & (X \otimes Y) \otimes M \otimes (X \otimes Y)^*.
\end{array}$$

One has that

$$v_M = \tilde{\delta}_M \circ q_M.$$  

**Proof.** One has as above

$$\tilde{Z}^2(M) = \int_{(X,Y) \in \mathcal{D} \times \mathcal{D}} (X \otimes Y) \otimes M \otimes (X \otimes Y)^*. $$
The maps
\[ Z(M) \xrightarrow{q_M} \bar{Z}(M) \xrightarrow{\bar{\pi}_{M,Z}} \bar{Z}((X \otimes Y) \otimes M \otimes (X \otimes Y)^*) \]
are dinatural with respect to \((X, Y)\). Thus by the universal property of the coend there is a unique map \(v_M \) making the above diagram commutative. It remains to show that \(v_M = \bar{\delta}_M \circ q_M\). Thus we have to show that the following diagram is commutative:
\[ \begin{array}{ccc}
Z(M) & \xrightarrow{q_M} & \bar{Z}(M) \\
\downarrow{\bar{\pi}_{M,X \otimes Y}} & & \downarrow{\bar{\pi}_{M,X \otimes Y}} \\
\bar{Z}(M) & \xrightarrow{q_M} & \bar{Z}((X \otimes Y) \otimes M \otimes (X \otimes Y)^*) \\
\end{array} \]

This follows since all three inside triangles are commutative. 

\[ \square \]

7.3. Morphism of comonads. Let \((T, \delta, \epsilon), (S, \delta', \epsilon')\) be two comonads on a monoidal category \(\mathcal{C}\). Recall that a morphism \(q : T \to S\) is a morphism of comonads if
\[ \epsilon' \circ q = \epsilon, \quad \delta' \circ q = q_{S(1)} \circ T(q) \circ \delta, \quad \delta' \circ q = S(q) \circ q_{T(1)} \circ \delta \]
For any \(M \in \mathcal{O}(\mathcal{C})\) these relations are equivalent to the commutativity of the following diagrams:
\[ \begin{array}{ccc}
T(M) & \xrightarrow{T(q_M)} & S(M) \\
\downarrow{\lambda_M} & & \downarrow{\lambda_M} \\
T^2(M) & \xrightarrow{T(S(q_M))} & S^2(M) \\
\end{array} \quad \text{and} \quad \begin{array}{ccc}
T(M) & \xrightarrow{S(q_M)} & S(M) \\
\downarrow{\lambda_M} & & \downarrow{\lambda_M} \\
T^2(M) & \xrightarrow{S(T(q_M))} & S^2(M) \\
\end{array} \]

\[ \text{Theorem 7.5. Let } \mathcal{D} \subseteq \mathcal{C} \text{ be a tensor subcategory of a finite tensor category. With the above notations } q : Z \to \bar{Z} \text{ is a morphism of Hopf comonads.} \]

\[ \text{Proof. We have already shown that } q \text{ is a morphism of monoidal functors. It remains to show that } q \text{ is a morphism of comonads.} \]

In order to show that \(q\) is a natural transformation of comonads we need to show that for any object \(M \in \mathcal{O}(\mathcal{D})\) one has
\[ \bar{\epsilon}_M \circ q_M = \epsilon_M \]
and the following diagrams are commutative:
\[ \begin{array}{ccc}
Z(M) & \xrightarrow{\delta_M} & Z^2(M) \\
\downarrow{\lambda_M} & & \downarrow{\lambda_M} \\
\bar{Z}(M) & \xrightarrow{\bar{Z}(q_M)} & \bar{Z}(M) \\
\end{array} \quad \begin{array}{ccc}
Z(M) & \xrightarrow{\delta_M} & Z^2(M) \\
\downarrow{\lambda_M} & & \downarrow{\lambda_M} \\
\bar{Z}(M) & \xrightarrow{\bar{Z}(q_M)} & \bar{Z}(M) \\
\end{array} \quad \begin{array}{ccc}
Z(M) & \xrightarrow{\delta_M} & Z^2(M) \\
\downarrow{\lambda_M} & & \downarrow{\lambda_M} \\
\bar{Z}(M) & \xrightarrow{\bar{Z}(q_M)} & \bar{Z}(M) \\
\end{array} \]
Since $\epsilon_M = \pi_{M;1}$ and $\bar{\epsilon}_M = \bar{\pi}_{M;1}$ the first formula follows from the definition of $q_M$.

We first show that the second diagram is commutative. By Lemma 7.3, it's enough to show that

\[ v_M = q_{Z(M)} \circ Z(q_M) \circ \delta_M \]

i.e. the following diagram is commutative:

\[
\begin{array}{cccccccc}
Z(M) & \xrightarrow{\delta_M} & Z^2(M) & \xrightarrow{Z(q_M)} & Z\bar{Z}(M) & \xrightarrow{q_{Z(M)}} & \bar{Z}^2(M) \\
\downarrow{q_M} & & \downarrow{\bar{\pi}_{M;X,Y}} & & \downarrow{\bar{\pi}_{M;X,Y}} & & \downarrow{\bar{\pi}^2_{M;X,Y}} \\
\bar{Z}(M) & & \xrightarrow{\bar{\pi}_{M;X,Y}} & & (X \otimes Y) \otimes M \otimes (X \otimes Y)^* \\
\end{array}
\]

The most left bottom triangle is commutative by the definition of $q_M$. The middle triangle is commutative by the definition of $\delta_M$. Thus it remains to show that the diagram:

\[
\begin{array}{cccccccc}
Z^2(M) & \xrightarrow{Z(q_M)} & Z\bar{Z}(M) & \xrightarrow{q_{Z(M)}} & \bar{Z}^2(M) \\
\downarrow{\bar{\pi}^2_{M;X,Y}} & & \downarrow{\bar{\pi}^2_{M;X,Y}} & & \downarrow{\bar{\pi}^2_{M;X,Y}} \\
(X \otimes Y) \otimes M \otimes (X \otimes Y)^* & & (X \otimes Y) \otimes M \otimes (X \otimes Y)^* \\
\end{array}
\]

is commutative. Writing down the maps $\bar{\pi}^2_{M;X,Y}$ and $\bar{\pi}^2_{M;X,Y}$ one needs to show that the following diagram is commutative:

\[
\begin{array}{cccccccc}
Z^2(M) & \xrightarrow{Z(q_M)} & Z\bar{Z}(M) & \xrightarrow{q_{Z(M)}} & \bar{Z}^2(M) \\
\downarrow{\bar{\pi}^2_{Z(M);X,Y}} & & \downarrow{\bar{\pi}^2_{Z(M);X,Y}} & & \downarrow{\bar{\pi}^2_{Z(M);X,Y}} \\
X \otimes Z(M) \otimes X^* & \xrightarrow{id_X \otimes id_M \otimes id_{X^*}} & X \otimes Z\bar{Z}(M) \otimes X^* & \xrightarrow{id_X \otimes \bar{\pi}_{M,Y} \otimes id_{X^*}} & X \otimes \bar{Z}(M) \otimes X^* \\
\downarrow{id_X \otimes \pi_{M,Y} \otimes id_{X^*}} & & \downarrow{id_X \otimes \pi_{M,Y} \otimes id_{X^*}} & & \downarrow{id_X \otimes \pi_{M,Y} \otimes id_{X^*}} \\
(X \otimes Y) \otimes M \otimes (X \otimes Y)^* & & (X \otimes Y) \otimes M \otimes (X \otimes Y)^* \\
\end{array}
\]

Note that the upper-left rectangle is commutative by the definition of $Z(q_M)$. The upper right rectangle is commutative by the definition of $q_{Z(M)}$. The bottom rectangle is commutative by the definition $q_M$.

Similarly for the proof of the first diagram we have to show that

\[ v_M = Z(q_M) \circ q_{Z(M)} \circ \delta_M \]

i.e. the following diagram is commutative:

\[
\begin{array}{cccccccc}
Z(M) & \xrightarrow{\delta_M} & Z^2(M) & \xrightarrow{Z(q_M)} & Z\bar{Z}(M) & \xrightarrow{Z(q_M)} & \bar{Z}^2(M) \\
\downarrow{\pi_M} & & \downarrow{\bar{\pi}_{M;X,Y}} & & \downarrow{\bar{\pi}_{M;X,Y}} & & \downarrow{\bar{\pi}^2_{M;X,Y}} \\
(X \otimes Y) \otimes M \otimes (X \otimes Y)^* & & (X \otimes Y) \otimes M \otimes (X \otimes Y)^* \\
\end{array}
\]
Writing down again the map \( \tilde{\pi}^2 \) one needs to show that the following diagram is commutative:

\[
\begin{array}{c}
\begin{array}{c}
Z(M) \xrightarrow{\delta_M} Z^2(M) \xrightarrow{\pi(M)} Z^2(M) \\
\end{array}
\end{array}
\]

Note that the most left down triangle commutes by the definition of \( q_M \). The middle inside rectangle commutes by definition of \( \delta_M \). The upper right corner rectangle commutes by the definition of \( Z(q_M) \). The most right bottom triangle commutes by the definition of \( q_M \). \( \square \)

**Remark 7.6.** Using Lemma 7.3 note that the commutativity of both diagrams is equivalent to the equality

\[
v_M = \delta_M \circ q_M^{(2)}.
\]

**7.4. Compatibility with characters and central elements.** The next lemma shows that the two maps induced by \( q_1 : Z(1) \to \bar{Z}(1) \) are \( k \)-algebra homomorphisms.

**Lemma 7.7.** With the above notations:

1. The epimorphism \( q \) induces a surjective algebra map

\[
q^* : CE(C) \to CE(D), \quad z \mapsto q \circ z
\]

2. The epimorphism \( q \) induces an injective algebra map

\[
q_{*1} : CF(D) \to CF(C), \quad \mu \mapsto \mu \circ q
\]

**Proof.** By [Shi17b, Section 4.3] one has that \( q_1 \) is an epimorphism of algebras. It follows that \( q^*_1 \) is surjective and \( q_{*1} \) is injective.

1. Let \( a, b \in CF(C) \). Recall that \( a \cdot b = m \circ (a \otimes b) \). Since \( q^*(a) = a \otimes q \) this is clearly a surjective map. In order to see that \( q^*(a \cdot b) = q^*(a) \cdot q^*(b) \) one has to consider the following diagram:

\[
\begin{array}{c}
\begin{array}{c}
1 \simeq 1 \otimes 1 \xrightarrow{a \otimes b} Z(1) \otimes Z(1) \xrightarrow{m} Z(1) \\
\end{array}
\end{array}
\]

The left rectangle is commutative by the naturality of the tensor product and the second one since \( q_1 \) is an algebra map.
(2) Let \( f, g \in \text{CF}(\mathcal{D}) \). In order to show that \( q_\ast 1(f \ast g) = q_\ast 1(f) \ast q_\ast 1(g) \) one has to consider the following diagram.

\[
\begin{array}{cccccccc}
Z(1) & \overset{\delta_1}{\longrightarrow} & Z(Z(1)) & \overset{Z(q_1)}{\longrightarrow} & Z(\tilde{Z}(1)) & \overset{Z(g)}{\longrightarrow} & Z(1) & \overset{q_1}{\longrightarrow} & \tilde{Z}(1) & \overset{f}{\longrightarrow} & 1 \\
\downarrow{q_1} & & \downarrow{q_4} & & \downarrow{\delta_4} & & \downarrow{\delta_1} & & \downarrow{q_1} & & \\
\tilde{Z}(1) & \overset{\delta_1}{\longrightarrow} & Z(\tilde{Z}(1)) & \overset{\tilde{Z}(g)}{\longrightarrow} & Z(1) & \overset{f}{\longrightarrow} & \tilde{Z}(1) & \end{array}
\]

Note that the left rectangle is commutative since \( q \) is a natural transformations of comonads. The second rectangle is commutative by the naturality of \( q \). \( \square \)

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Inst. of Math. “Simion Stoilow” of the Romanian Academy
P.O. Box 1-764, RO-014700, Bucharest, Romania
E-mail address: sebastian.burciu@imar.ro