Putting the Agents Back in the Domain: A Two-Sorted Term-Modal Logic

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June 1st, 2017

Abstract

In the present paper, syntax and semantics will be presented for an expansion of ordinary n-agent QML with constant domain, non-rigid constants, rigid variables and including both functions, relations, and equality. Further, the number of agents will be specified axiomatically thus ensuring maximal flexibility wrt. the cardinality of the set of agents. Domain, variables, and constants will be partitioned in an agent-part and an object-part and the syntax will be expanded to include strings of the type $\forall xK_x \phi$ as wff’s of the language. This will enhance expressiveness regarding the epistemic status of agents. Such a term-modal version of the logic $K$ is shown to be sound and complete wrt. the class of (appropriate) frames, and a term-version of $S4$ is shown to be sound and complete wrt. the class of (appropriate) frames in which the relations are transitive. It should be noted that completeness is shown via the framework of canonical models and thus allows for non-complicated generalizations to other logics than the term-versions of $K$ and $S4$. 
"Remember... All I'm offering is the truth. Nothing more."

— Morpheus
1 Introduction

In a passage of [2] Descartes reflects on what can be known and famously concludes that “I think, therefore I am”. As the insight presented in “the cogito” has a central place in classic epistemology it is of natural interest for the student of formal epistemology to answer the question of how such a “quote” might be formalized. What is noticeable is the reflexive nature of the cogito, i.e. the fact that it deals with an agent’s reflections on itself. One might go about formalizing the cogito in some epistemic modal logic: Propositional modal logic, represented by the Backus-Naur form, is given as

\[ P ::= \neg P \mid P \rightarrow Q \mid K_i P \]

and to express the sentiment of the cogito we would have to devise propositional formulas \( P_1, ..., P_n \) and interpret \( P_i \) as “agent \( i \) is”. Under this ascription, \( K_i P_i \) reads “agent \( i \) knows that she is”, and the question then becomes how to formalize “I think”. A natural choice would be \( K_i \phi \) for some tautology \( \phi \), since for instance any thinking being ought to “know” e.g. \( p \lor \neg p \), and anyone knowing such a thing is certainly a thinking being. The cogito then, would amount to

\[ K_i (p \lor \neg p) \rightarrow K_i P_i, \]

the \( K_i \)'s being modal operators, and the index set in one-to-one correspondence with some fitting set of agents. Three things are noteworthy; first, in order to express the cogito for a set of agents one would have to do so rather crudely by conjunction and secondly, any relation between “agent \( i \)” and the subscript of the modal operator has to be stated in meta-language, i.e. the agents are nowhere to be seen in the semantics and thirdly, dealing with propositional logic there is no intra-logical connection between the entities of which the propositions speak and the semantics. The situation is slightly better in standard first-order modal logic due to the enhanced expressivity that comes with the introduction of quantifiers and relations; represented by its Backus-Naur form standard first-order modal logic yields the following syntax:

\[ \phi ::= R(t_1, ..., t_n) \mid \neg \phi \mid \phi \rightarrow \psi \mid \forall x \phi \mid K_i \phi \]

where \( R \) is an \( n \)-ary relation symbol, and all the \( t_i \)'s are terms. In this language, the sentence “agent \( i \) knows that she is” is formalized along the lines of \( \exists x K_i(x = \alpha) \), assuming that a constants \( \alpha \) for each agent, and the cogito would amount to something along the lines of \( K_i(\phi \lor \neg \phi) \rightarrow \exists x K_i(x = \alpha) \).

What is important to notice is that in ordinary first-order \( n \)-agent modal logic we have to state the relation between the subscript \( i \) of the modal operator and the referent of \( \alpha \) in meta-language, i.e. it is not built-in in the semantics when “quantifying” over agents by means of the conjunction. What we really want is the ability to state \( \forall y (K_y(\phi \lor \neg \phi) \rightarrow \exists x K_y(x = y)) \) for some agent-designating variable \( y \), i.e. we want modal operators indexed by terms, we want the ability to distinguish between those terms that are appropriate for modal indexation and those that are not, and we want the agents back in the domain!

\footnote{Will be defined later in the context of two-sorted term-modal logic.}
In Hintikka’s landmark 1962 book “Knowledge and Belief: An Introduction to the Logic of the Two Notions” he lays the foundation of what is today called epistemic logic. In this context he discusses syntax for formulas of the form $K_\alpha P$ for some individual constant $\alpha$, and writes in a footnote:

“Strictly speaking, we ought to distinguish those free individual symbols which can only take names of persons as their substitution-values and which can therefore serve as subscripts of epistemic operators from those which cannot do so.” [3, pp. 11]

Nevertheless, Hintikka does not present either syntax or semantics for such a logic, which is what will be accomplished here.

The purpose of the present paper is thus to first state syntax and semantics for a two-sorted term-modal logic with constant domain, non-rigid constants, and equality and secondly to state and prove soundness and completeness. As such the aim is simply for me to hone my technical skills by treating a logic somewhat more complex than what is normally encountered on undergraduate level. The following is a merge between the exposition of a term-modal logic in [5] and the exposition of a many-sorted modal logic in [6]. To remain true to the epistemic interpretation and for notational convenience I remain two-sorted even though the generalization to general many-sorted logic is straightforward. The interested reader is encouraged to do the proper adjustments in order to obtain the many-sorted version of the logic presented here.

First, language and syntax for a language for a two-sorted term-modal logic is stated in section 2. Then, in section 3 we present semantics including frames, interpretations, models and valuations leading up to the notions of truth and validity in 3.1 and 3.2. In section 4 we turn to an axiomatic system for the logic, and in 5 a couple of useful results are proved as preparation for a proof of soundness in section 6. Sections 7 and 8 contains the bulk of the work in this paper, namely completeness results for the two-sorted term-modal versions of $K$ and $S4$, and section 9 relates the results to Hintikka’s thoughts on the axiom 4 as presented in [3]. As the completeness proof is lengthy and technical we present a quick overview here meant to ease the acquisition.

### 1.1 A Brief Survey of the Proof of Completeness

The key insight regarding the overall proof strategy is contained in Proposition 2 which states the following: Showing completeness essentially amounts to, given a consistent set of formulas, constructing an appropriate model satisfying this set of formulas. The next part of the completeness proof thus consists of constructing this model starting with the worlds of the model, which we choose to be a fitting maximally consistent set of formulas. Lindenbaum’s Lemma (Lemma 7) and the Saturation Lemma (Lemma 8) ensures that it is possible to extend a given set of formulas to a maximally consistent set, and Lemma 6 then states some crucial properties of such maximally consistent sets. Definition 25 then presents the canonical model, and Propositions 4, 5 and 6 are technical
results ensuring that the canonical model is well-defined and stands in appropriate relations to the semantics of the logical connectives. Having defined the worlds of the canonical model as sets of formulas the idea is that a world should satisfy all and only those formulas which enjoy membership in this particular world, and that the construction actually has this feature is the content of the Truth Lemma (Lemma 9). With the Truth Lemma in hand the canonical class Theorem (Theorem 2) is stated and proved, the consequence of which is that any appropriate logic is complete wrt. the class of canonical models. As an application of the above, completeness of the “term-modal” version of $\mathbf{S4}$ is proven in Theorem 3.

2 The Language $L_{TM}$ and Syntax

Fix an agent-set $A = \{\alpha_1...\alpha_n\}$ at the outset. In this section we add these agents to the domain of quantification resulting in a two-sorted term-modal logic. The two sorts will correspond to agents and objects respectively, and “term” refers to the ability to quantify over indexes in modal operators thus enhancing the ability to express agents reflecting reflexively about themselves and other agents. In this section language and syntax for a two-sorted term-modal logic with constant domain, non-rigid constants, and equality is defined. We start with the syntax.

It is worth noting that the language is not parametrized by the agent-set – rather the number of agents is specified axiomatically. Throughout, if $v = (x_1,...,x_n)$ is a vector, denote by $v_i$ the $i$'th element i.e $v_i = x_i$.

Definition 1. (Language). A two-sorted term-modal language $L_{TM}^{\sigma}$ consists of

1. A set $\sigma := \{\text{agt}, \text{obj}\}$ of two sorts.
2. A countable, infinite set $VAR$ of variables, each assigned a sort $\sigma \in \sigma$. Let $VAR{\sigma}$ denote the set of $\sigma$ – variables and assume that both $VAR{\text{agt}}$ and $VAR{\text{obj}}$ are infinite.
3. A countable (possibly empty) set $CON$ of constants with each constant assigned a sort. Denote the set of $\sigma$ – constants $CON{\sigma}$.
4. A countable (possibly empty) set $FUN$ of function symbols, each assigned an arity $\alpha \in \sigma^{k+1}$ for some $k \geq 0$. Denote the set of function symbols with $\alpha$ – arity $FUN_{\alpha}$.
5. A countable (possibly empty) set $REL$ of relation symbols, each assigned an arity $\beta \in \sigma^k$ for some $k \geq 1$. Denote the set of relations with $\beta$ – arity $REL_{\beta}$.
6. The equality symbol $=$.
7. A set of modal operators $K_t$ indexed by agent-referring terms (to be defined shortly).

\footnote{See axiom $N$ in section 4.}
8. The logical connectives \( \neg \) and \( \rightarrow \).

9. The universal quantifier \( \forall \).

A couple of comments are in order before we proceed to define terms. First, the set \( \mathcal{S} \) of sorts is meant to model the distinction between agents and objects. For readability the sorts are simply denoted “agt” and “obj” but readers interested in generalizing to n-sorted logic might wish to think of these in more generic terms\(^3\). For the special notion of arity employed in 4. notice that the arity is now a vector which the \( i \)’th entry is the sort of the \( i \)’th input term except from the last entry describing the sort of the resulting term. The case with relations are completely analogous so we turn towards the notion of a term.

**Definition 2.** [Terms] The set \( \text{Term}(\mathcal{L}_{TM}^\mathcal{S}) \) of terms is defined inductively by

1. \( \text{VAR} \cup \text{CON} \subseteq \text{Term}(\mathcal{L}_{TM}^\mathcal{S}) \), and
2. If \( f \in \text{FUN}_\alpha \) with \( \alpha \in \mathcal{S}^{k+1} \) and \( t_1, ..., t_k \in \text{Term}(\mathcal{L}_{TM}^\mathcal{S}) \) where \( t_i \) is of sort \( \alpha_i \) then \( f(t_1, ..., t_k) \in \text{Term}(\mathcal{L}_{TM}^\mathcal{S}) \).

The set \( \text{Term}_\sigma \) of \( \sigma \)– terms is defined as the smallest subset of \( \text{Term}(\mathcal{L}_{TM}^\mathcal{S}) \) such that

1. \( \text{VAR}_\sigma \cup \text{CON}_\sigma \subseteq \text{Term}_\sigma \), and
2. If \( f \in \text{FUN}_\alpha \) with \( \alpha \in \mathcal{S}^{k+1} \) such that \( \alpha_{k+1} = \sigma \) and \( t_1, ..., t_k \in \text{Term}(\mathcal{L}_{TM}^\mathcal{S}) \) where \( t_i \) is of sort \( \alpha_i \) then \( f(t_1, ..., t_k) \in \text{Term}_\sigma \).

Specifically, the set \( \text{Term}_{\text{agt}} \) is what we will call the set of agent-referring terms.

We can now qualify point 7. in Definition\(^1\) by defining the set \( K \) of modal operators by

\[ K := \{ K_i \mid t \in \text{Term}_{\text{agt}} \} \]

The set of \( \mathcal{L}_{TM}^\mathcal{S} \) well-formed formulas is now definable:

**Definition 3.** [Well-formed formulas] The set of \( \mathcal{L}_{TM}^\mathcal{S} \) well-formed formulas is defined inductively by:

1. If \( t_1, t_2 \in \text{Term}(\mathcal{L}_{TM}^\mathcal{S}) \) then the expression \( t_1 = t_2 \) is an atomic wff of the language, and
2. If \( P \in \text{REL}_\beta \) with \( \beta \in \mathcal{S}^k \) and \( t_1, ..., t_n \in \text{Term}(\mathcal{L}_{TM}^\mathcal{S}) \) where \( t_i \) is of sort \( \beta_i \) then \( P(t_1, ..., t_k) \) is an atomic wff of the language, and
3. If \( \phi \) and \( \psi \) are atomic wffs, \( t \in \text{Term}_{\text{agt}} \), and \( x \in \text{VAR} \) then the remaining wffs of the language is defined by the Backus-Naur form:

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\(^3\)One option would be to simply define the set of sorts as \( \mathcal{S} = \{ \sigma_1, \sigma_2 \} \) thus making the generalization completely straightforward.
\[ \phi ::= \neg \phi \mid \phi \to \psi \mid \forall x \phi \mid K_t \phi \]

Please note that the remaining logical connectives are defined in the ordinary way while the existential quantifier \( \exists \) and the modal operators \( P_t \) are defined as duals of \( \forall \) and \( K_t \) respectively – this is also completely standard.

Before moving on we define the notion of a free variable. For the most part it is completely standard but for the set \( K \) of modal operators something new is going on.

**Definition 4.** [Free variable] We define a free variable inductively by

1. For any term \( t \) the free variables are just the variables occurring in \( t \), and
2. For any atomic formula \( \phi \) the free variables are the free variables of the terms of \( \phi \), and
3. For formulas \( \phi, \psi \) the free variables of \( \neg \phi \) and \( \phi \to \psi \) are just the free variables of \( \phi \) and \( \psi \), and
4. For each formula of the form \( K_t \phi \) the free variables are just the variables of \( t \) joined with the free variables of \( \phi \), and
5. The free variables of \( \forall x \phi \) are any free variable of \( \phi \) excluding \( x \).

Any occurring variable not free is referred to as bound. Furthermore, any formula without any free variables is called a sentence. Where \( \phi \) is a formula, \( t \) is a term, and \( x \) a variable we denote by \( \phi(t/x) \) the result of substituting any free occurrence of \( x \) for \( t \), demanding that no free variable of \( t \) becomes bound by doing so.\[^1\]

This concludes the syntactic definitions, which do not differ much from the syntax of standard many-sorted modal logic as presented in e.g. in [6]. The new part is that we can quantify over indexes for modal operators, that is expressions of the type \( \forall x K_x \phi \) are now wffs of the language. Of course such syntactic additions are in need of a semantic counterpart; thus turning to semantics we put the agents \( \mathcal{A} \) back in the domain!

### 3 Semantics

Before moving on one bit of notation is needed. Where \( A \) and \( B \) are sets, take \( A \cup B \) to mean the disjoint union of \( A \) and \( B \).

**Definition 5.** [\( TM^\mathcal{A} \) – frame] A \( TM^\mathcal{A} \) – frame \( \mathcal{F} \) for the language \( L^\mathcal{T_M} \) is a triple \( \mathcal{F} := (W, R, DOM) \) where

1. \( W \) is a non-empty set of worlds, and

\[^1\]What then do we take \( \phi(t/x) \) to mean, if in fact some free variable in \( t \) does become bound by the substitution? We simply substitute \( \phi \) for an appropriate alphabetic variant which is always possible. See [4] pp. 241.
2. $\mathcal{R}$ is a map associating binary accessibility relations on $\mathcal{W}$ to each agent, that is $\mathcal{R}: \mathcal{A} \rightarrow \mathcal{P}(\mathcal{W} \times \mathcal{W})$, and

3. $\text{DOM} = \text{DOM}_{\text{agt}} \cup \text{DOM}_{\text{obj}}$ is the domain of quantification, where neither of the cojoints are empty and $\text{DOM}_{\text{agt}} = \mathcal{A}$.

Take $R_i$ to mean $\mathcal{R}(\alpha_i)$ and write $wR_iw'$ whenever $w, w' \in \mathcal{W}$ are such that $(w, w') \in \mathcal{R}(\alpha_i)$.

Next up is the notion of an interpretation but before bringing the technical definition a few comments are in order. We are currently defining a logic with non-rigid functions, relations, and constants which is naturally reflected in the definition above. Also, as we now have to take care what sort of term goes into each place of functions and relations the notion of arity is slightly more complex than the ordinary natural number cf. Definition 1.

**Definition 6.** [Interpretation] An interpretation $\mathcal{I}$ is a map such that simultaneously

1. For each $\beta \in \mathcal{P}^k$ let $\mathcal{I}: \text{REL}_\beta \times \mathcal{W} \rightarrow \mathcal{P}(\prod_{i \in \{1, \ldots, k\}} \text{DOM}_{\beta_i})$, and

2. For each $\alpha \in \mathcal{P}^{k+1}$ let $\mathcal{I}: \text{FUN}_\alpha \times \mathcal{W} \rightarrow \mathcal{P}(\prod_{i \in \{1, \ldots, k+1\}} \text{DOM}_{\alpha_i})$ and

3. $\mathcal{I}(=, w) = \{(d, d) | d \in \text{DOM}\}$ for every $w \in \mathcal{W}$, and

4. For each $\sigma \in \mathcal{P}$ let $\mathcal{I}: \text{CON}_\sigma \times \mathcal{W} \rightarrow \text{DOM}_\sigma$.

**Definition 7.** [Model] The two-tuple $\langle \mathcal{F}, \mathcal{I} \rangle$ consisting of a frame and an interpretation is called a model and we write $\mathcal{M} = \langle \mathcal{F}, \mathcal{I} \rangle$. In this case we say that $\mathcal{M}$ is based on $\mathcal{F}$.

Having dealt with relation symbols, function symbols, equality and constants the only remaining task is to interpret free variables. For this we need a valuation:

**Definition 8.** [Valuation] A valuation $v$ is a (surjective) map such that for each $\sigma \in \mathcal{P}$ we have $v: \text{VAR}_\sigma \rightarrow \text{DOM}_\sigma$.

For technical reasons we need one more definition before we turn to truth. Namely, we need the notion of an $x$-variant to deal with formulas involving $\forall$.

**Definition 9.** [$x$-variant] An $x$-variant $v'$ of a valuation $v$ is a valuation such that $v'$ and $v$ agrees on all variables except possibly for $x$. We note that any valuation is an $x$–variant of itself.

For brevity we shall write $t^{w,v}$ for the extension of the term $t$ at world $w$ under valuation $v$, that is:

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5In reality, $\mathcal{I}$ maps into the power set of a subset of $\prod_{i \in \{1, \ldots, k+1\}} \text{DOM}_{\alpha_i}$: Namely the subset defined by the rule that if $\{\text{DOM}_{\alpha_i} \}_{i \in \{1, \ldots, k+1\}}$ and $\{\text{DOM}_{\gamma_i} \}_{i \in \{1, \ldots, k+1\}}$ are such that $\text{DOM}_{\alpha_i} = \text{DOM}_{\gamma_i}$ for $i \in \{1, \ldots, k\}$ it hold that $\text{DOM}_{\gamma_{k+1}} = \text{DOM}_{\alpha_{k+1}}$. 

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It is useful to note that for any \( \text{agent-referring term } t \) the composite \( R(t^{w,v}) \) denotes an accessibility relation in \( \mathcal{R} \).

### 3.1 Truth

**Definition 10.** [Truth for non-modal formulas] Let \( \mathcal{M} = \langle W, \mathcal{R}, \text{DOM}, \mathcal{I} \rangle \) be a model, \( w \in W \), and \( v \) a valuation. We define truth in the model \( \mathcal{M} \) at \( w \) for the formula \( \phi \in L_{TM} \) inductively by:

1. \( \mathcal{M},w \models_v t_1 \ldots t_k \iff (t_1^{w,v} \ldots t_k^{w,v}) \in \mathcal{I}(P,w) \), and
2. \( \mathcal{M},w \models_v t_1 = t_2 \iff t_1^{w,v} = t_2^{w,v} \), and
3. \( \mathcal{M},w \models_v \neg \phi \iff \) it is not the case that \( \mathcal{M},w \models_v \phi \), and
4. \( \mathcal{M},w \models_v \phi \rightarrow \psi \iff \) either \( \mathcal{M},w \models_v \psi \) or \( \mathcal{M},w \models_v \neg \phi \), and
5. \( \mathcal{M},w \models_v \forall x \phi \iff \mathcal{M},w \models_v \phi \) for every \( x \)-variant \( v' \) of \( v \).

And we notice that whenever \( \phi \) is a sentence we can omit explicit mentioning of the valuation and simply write \( \mathcal{M},w \models \phi \).

**Definition 11.** [Truth for modal formulas] Let \( \mathcal{M} = \langle \mathcal{M}, \mathcal{R}, \text{DOM}, \mathcal{I} \rangle \) be a model, \( w \in W \), \( v \) a valuation, \( \phi \in L_{TM}^{\mathcal{R}} \), and \( t \) an agent-referring term. We define truth in the model \( \mathcal{M} \) at \( w \) for modal formulas inductively by:

1. \( \mathcal{M},w \models_v K_t \phi \iff \mathcal{M},w' \models_v \phi \) for every \( w' \) such that \( (w,w') \in \mathcal{R}(t^{w,v}) \), and
2. \( \mathcal{M},w \models_v P_t \phi \iff \mathcal{M},w' \models_v \phi \) for at least one \( w' \) such that \( (w,w') \in \mathcal{R}(t^{w,v}) \).

### 3.2 Validity

Above we defined the most basic notion of truth namely that of truth in a world of a model under a given valuation. As the aim of the project is to state and prove completeness, it shall be of paramount importance for us to be able to distinguish truth at different levels. As we shall see later the notion of satisfiability will play a key role in proving completeness.

**Definition 12.** [Validity and Satisfiability] A formula \( \phi \) is satisfiable if there exists a model \( \mathcal{M} \), a world \( w \) and a valuation \( v \) such that \( \mathcal{M},w \models_v \phi \). Further, if \( \phi \) is such that \( \mathcal{M},w \models_v \phi \) for all valuations we say that \( \phi \) is valid at world \( w \), and we write \( \mathcal{M},w \models \phi \). We say that a formula \( \phi \) is valid in the model \( \mathcal{M} \) if \( \mathcal{M},w \models \phi \) for all \( w \in W \) and we write \( \mathcal{M} \models \phi \). If a formula \( \phi \) is valid in all
models based on a frame $F$ we say that $\phi$ is valid in $F$ and we write $F \models \phi$. We say that a formula $\phi$ is valid on the class $F$ of frames if $F \models \phi$ for all $F \in F$ and we write this $F \models \phi$. If $\phi$ is valid on the class of all frames we simply say that $\phi$ is valid and write $\models \phi$.

Please recall that we require formulas to be finite. Yet, sometimes we wish to speak about infinite strings of symbols so it is advantageous for us to introduce a bit of notation to deal with this. So, whenever $\Gamma$ is an arbitrarily large set of formulas we write $M, w \models_v \varphi \Gamma$, $M, w \models \Gamma$ and so forth to mean the obvious. We ask the reader to also note that whenever $S$ is a class of models a model from $S$ simply means some model $M$ for which $M \in S$, while if $F$ is a class of frames a model based on some frame $F \in F$, that is $M = (F, I)$ for some interpretation $I$. I shall write $M \in F$ for a model based on some frame $F \in F$, and write that $M$ is a model from $F$. We are now in position to define the semantic consequence relation:

**Definition 13.** [Semantic Consequence] Let $\phi$ be a formula, $\Gamma$ be a set of formulas, and $S$ a class of structures (either models or frames). We say that $\phi$ is a semantic consequence of $\Gamma$ and write $\Gamma \models_S \phi$ if it holds in all models $M$ from $S$, for all valuations $v$ and all worlds $w$ of $M$ that if $M, w \models_v \Gamma$ then $M, w \models_v \phi$.

Having defined the semantic consequence relation we are in position to clarify the notion of a semantically specified logic:

**Definition 14.** [Semantically Specified Logic] Given a language $L$, and a class $S$ of structures (either models or frames) formulated in $L$, we define the logic $L_S := \{ \phi \mid \models_S \phi \}$.

We proceed directly to the first Proposition of the project.

**Proposition 1.** [Principle of Replacement] Let $\phi$ be a formula, $x, y$ variables, $M$ a model, $w$ a world, and $v$ a valuation. Then it holds that if $v'$ is an $x$-variant of $v$ with $v(x) = v'(y)$ then

$$M, w \models_v \phi \text{ iff } M, w \models_{v'} \phi(y/x)$$

**Proof.** Seeing that the only leeway after fixing model and world is the interpretation of free variables via the valuation, and $\phi(y/x)$ differs only from $\phi$ in that $\phi(y/x)$ has a free occurrence of $y$ everywhere that $\phi$ has a free occurrence of $x$, and $v$ and $v'$ agrees everywhere except for $x$ where $v(x) = v'(y)$ the Proposition follows.

Having established a thorough semantic description of the two-sorted term-modal logic we now aim to provide the proof-theoretical counterpart; indeed the overall purpose of the project is to show that these two descriptions amount to the same thing. As such we now turn to the axioms.
4 An Axiomatic System of $K_{TM}^{A,\sigma}$

In this section we define the two-sorted term-modal logic version of the concept of normal logics. We will denote the resulting logic for $n$ agents over the language $L_{TM}$ by $K_{TM}^{A,\sigma}$ and begin by stating what axioms are to hold. Then we proceed to the definition of a $K_{TM}^{A,\sigma}$ proof before we are ready to define normality in the new setting. The section concludes with three Propositions that shall be of help to us when proving completeness later on.

4.1 Axiom Schemas of $K_{TM}^{A,\sigma}$

First of all, we let all substitution instances of valid formulas from Propositional modal logics be axioms of $K_{TM}^{A,\sigma}$. Furthermore, we add the following axiom schemas:

Let $\phi \in L_{TM}$ be any formula and $y$ any variable free in $\phi$. Then every instance of

$$\forall x \varphi \to \varphi(y/x)$$

is an axiom. Further, for any term $t$

$$t = t$$

are axioms. Also, in order to accommodate the partition of terms in sorts we include for each $x \in VAR_{agt}$ and $y \in VAR_{obj}$ the axiom

$$x \neq y$$

as an axiom. We note that as variables are rigid the stated version of PS should be of no concern. We also add Existence of Identicals, such that for any constant $c$

$$(c = c) \to \exists x (x = c)$$

is an axiom. For technical reasons we also need to make sure that the agent-part of the domain always contain the appropriate number of element. So, for any $x_1, ..., x_{|A|}, y \in VAR_{agt}$ we have as an axiom:

$$\exists x_1, ..., x_{|A|} \left( x_1 \neq x_2 \land ... \land x_{|A|} \neq x_{|A|} \land \forall y (y = x_1 \lor ... \lor y = x_{|A|}) \right)$$

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6 That is, any formula obtained from a Propositional axiom by substituting Propositional variables for formulas from $L_{TM}$.

7 Where $\phi$ involves the modal operator $K_x$ we obviously requires $x$ and $y$ to be agent-referring.
We further include a version of the axiom $K$, such that for any agent-referring term $t$ and any formulas $\phi$ and $\psi$ we have

$$K_t(\phi \rightarrow \psi) \rightarrow (K_t\phi \rightarrow K_t\psi) \quad (K)$$

as an axiom of the system $K_{TM,A,\sigma}$ too. We also choose to add the Barcan Formula for interplay between quantifiers and modal operators such that for any agent-referring term $t$ and any variable $x \neq t$ we have the axiom

$$\forall xK_t\phi \rightarrow K_t\forall x\phi \quad (BF)$$

and lastly; for any variables $x$ and $y$, and any agent-referring term $t$ we add the axiom Knowledge of Non-identity:

$$(x \neq y) \rightarrow K_t(x \neq y) \quad (KNI)$$

And we note that the axiom Dual is unnecessary since we have simply defined the operator $P_t$ such that $P_t := \neg K_t\neg$.

### 4.2 Inference Rules of $K_{TM,A,\sigma}$

The strategy when defining a logic in a syntactical matter is to first state a set $\Lambda$ of axioms and then to close it off under rules of inference such that the resulting logic can be defined as $\Lambda^\overline{8}$ by a slight abuse of notation. First, we choose to include *modus ponens*. That is for any formulas $\phi$ and $\psi$

$$\frac{\phi, \phi \rightarrow \psi}{\psi} \quad (MP)$$

is a valid inference. Also, we include *Knowledge Generalization* among our rules of inference such that for every agent-referring term $t$ we have

$$\frac{\phi}{K_t\phi} \quad (KG)$$

as a valid inference. Finally, if $\phi$ is without free occurrences of $x$ we let

$$\frac{\phi \rightarrow \psi}{\phi \rightarrow \forall x\psi}. \quad (Gen)$$

Having established axioms and rules of inference we are ready to define the notion of a $K_{TM,A,\sigma}$ proof.

**Definition 15.** $[K_{TM,A,\sigma} – \text{proof}] \Lambda K_{TM,A,\sigma} – \text{proof}$ is a finite sequence of formulas from $\mathcal{L}_{TM,\sigma}$, each of which is either a $K_{TM,A,\sigma}$ – axiom or the result of using some rule of inference on one or more earlier formulas of the sequence. If $\phi$ is a formula we say that $\phi$ is $K_{TM,A,\sigma}$ – provable if it is the last element in such a sequence and we write $\vdash K_{TM,A,\sigma} \phi$. 

8The overline denoting closure.
Note that at this point we know nothing about the relation between truth as defined in definitions 10 and 11 on the one side and provability as just defined on the other. It will be the aim of this project to establish the appropriate equivalence! Three more concepts shall be defined here before turning to a couple of handy results – which will also yield an opportunity to see $K_{TM}^{A,σ}$ – proofs in action.

**Definition 16.** [Normality] A normal $n$-agent two-sorted term-modal logic $Λ$ is any set of $L_{TM}$ – formulas containing all of the above axioms closed under all of the above rules of inference.

Whenever $φ ∈ Λ$ we say that $φ$ is a Theorem of $Λ$, and write $⊢_Λ φ$. If $Λ_1$ and $Λ_2$ are two logics such that $Λ_1 ⊆ Λ_2$ we say that $Λ_2$ is an extension of $Λ_1$.

**Definition 17.** [$Λ$ – deducibility] Let $Γ$ be any set of wffs of the language, $φ$ be any formula, and $Λ$ a logic. Then $φ$ is $Γ$ – deducible from $Λ$ if there exist a finite subset $Γ_0 ⊆ Γ$ such that $⊢_Λ Γ_0 → φ$.

Whenever this obtains, write $Γ ⊢_Λ φ$. Write $Γ ⊬_Λ φ$ if $φ$ is not $Λ$ – deducible from $Γ$.

**Definition 18.** [$Λ$ – consistency] Let $Γ$ be any set of wffs of the language. We say that $Γ$ is $Λ$ – consistent if for all formulas $φ$ we have

$$Γ ⊬_Λ φ ∧ ¬φ$$

and $Λ$ – inconsistent otherwise. Please note that if some arbitrary set of wffs $Γ$ is inconsistent then there has to exist $Γ_0 ⊆ Γ$ finite which is incomplete since proofs are finite by definition.

Having defined the framework we end this section by the proof-theoretical counterpart to definition 14:

**Definition 19.** [Proof-Theoretically Specified Logic] If $L$ is a language, and $K$ denote a set of axioms together with some rules of inference formulated in $L$, we define the logic $L_K := \{ φ \mid ⊢_K φ \}$.

Now, we can express completeness as the inclusion $L_S ⊆ L_K$ and soundness as $L_K ⊆ L_S$. We proceed to state and prove some Lemmas that shall be of great use during the proof of completeness.

## 5 A Couple of Handy Lemmas

**Lemma 1.** Let $x, y ∈ VAR$ and $t ∈ Term_{agt}$. Then

$$⊢_{K_{TM}^{A,σ}} (x = y) → K_t(x = y).$$
Proof. Let $\phi(z) = K_t(x = z)$ for any $t \in \text{Term}_{agt}$. Then we have by Propositional calculus that $\vdash_{K_{TM}} \phi(t(x) \rightarrow (K_t(x = x) \rightarrow K_t(x = y)))$. Using the Theorem from Propositional calculus that $(\phi \rightarrow (\psi \rightarrow \phi)) \rightarrow (\phi \rightarrow (\psi \rightarrow \phi))$ yields $\vdash_{K_{TM}} (x = y) \rightarrow K_t(x = x)$. It follows directly from KG and Id that $\vdash_{K_{TM}} K_t(x = x)$, and so an application of MP yields the desired result.

In the following the formula

$$(x = y) \rightarrow K_t(x = y) \quad (KI)$$

will be called Knowledge of Identity.

Lemma 2. Let $t \in \text{Term}_{agt}$, and $\phi_1, \ldots, \phi_n$ be wffs of the language. Then the following holds

$$\vdash_{K_{TM}} K_t(\phi_1 \land \ldots \land K_t) \rightarrow (K_t \phi_1 \land \ldots \land K_t \phi_n)$$

In the following the formula

$$K_t(\phi_1 \land \ldots \land \phi_n) \rightarrow (K_t \phi_1 \land \ldots \land K_t \phi_n) \quad (KD)$$

will be called K-Distribution.

Proof. We omit the proof as it is fairly standard. See [4] pp. 28.

Lemma 3. Let $\phi$ and $\psi$ be wffs of the language, and $t \in \text{Term}_{agt}$. Then it holds that

$$\text{if } \vdash_{K_{TM}} \phi \rightarrow \psi \text{ then } \vdash_{K_{TM}} K_t \phi \rightarrow K_t \psi$$

Proof. Assume $\vdash_{K_{TM}} \phi \rightarrow \psi$. By KG we obtain $\vdash_{K_{TM}} K_t(\phi \rightarrow \psi)$ and thus K yields the desired result.

The last Lemma to be proven here is a small result that is simply practical to have at hand.

Lemma 4. [Consistency Lemma] Let $\Gamma$ be any $\Lambda$-consistent set of formulas, and $\phi$ any formula. Then either is $\Gamma \cup \{\phi\}$ consistent or $\Gamma \cup \{\neg \phi\}$ is.

Proof. Assume $\vdash_{K_{TM}} \phi \rightarrow \psi$. By KG we obtain $\vdash_{K_{TM}} K_t(\phi \rightarrow \psi)$ and thus $K$ yields the desired result.

We are now ready to undertake the work of showing soundness, i.e that any Theorem of our logic is valid wrt. the class of all $TM^\Lambda$-frames, or equivalently containment of the logic specified via syntax in the logic we have specified semantically.
6 Soundness

We start by the central definition:

**Definition 20.** [Soundness] Let $S$ be a class of structures (either models or frames), and let $\Lambda$ be a logic. We say that $\Lambda$ is **sound wrt.** $S$ if for every formula $\phi$ of the language we have

\[
\text{if } \vdash_\Lambda \phi \text{ then } \models_S \phi.
\]

And we go about showing soundness for $K_{TM}^{\mathbf{A},\mathbf{F}}$ wrt. the class $F$ of all $TM_{\mathbf{A}}$-frames in two steps: First we show that all axioms are valid, and then we show that validity is preserved by each of the rules of inference.

### 6.1 Validity of Axioms

We have to show that all of the axioms $\forall$, $\mathbf{Id}$, $\mathbf{PS}, \exists \mathbf{Id}$, $\mathbf{N}$, $\mathbf{K}$, $\mathbf{BF}$, and $\mathbf{KNI}$ are valid. In the following let $\mathcal{F} = (W, \text{DOM}, R)$.

**$\forall$:** We have to show that $\models_F \forall x \phi \rightarrow \phi(y/x)$ where $x, y \in \text{VAR}$ and $y$ has only free occurrences in $\phi$ so assume $\mathcal{M}, w \models_v \forall x \phi$ for arbitrary $\mathcal{M} \in F$, $w \in W$, and $v$ a valuation. Let $v'$ be an $x$-variant of $v$ such that $v(x) = v'(y)$ such that we have $\mathcal{M}, w \models_v \phi$ by the semantics for $\forall$. By Proposition we obtain $\mathcal{M}, w \models_{v'} \phi$ which in turn implies $\mathcal{M}, w \models \phi(y/x)$, and as $\mathcal{M}, w$ and $v$ was arbitrary this yields the validity of the axiom $\forall$ with respect to $F$.

**$\mathbf{Id}$:** We have to show that where $t$ is any term we have $\models_F (t = t)$, but this is immediately clear from the semantics of $=$.

**$\mathbf{MSC}$:** By the semantics of $=$ and the definition of the valuation $\mathbf{MSC}$ is trivially valid.

**$\mathbf{PS}$:** We have to show that where $x$ and $y$ are variables and $\phi$ any wff we have $\models_F (x = y) \rightarrow (\phi(x) \rightarrow \phi(y))$, which will be accomplished by induction on the complexity of $\phi$. Assume $\mathcal{M}, w \models_v (x = y)$ and $\mathcal{M}, w \models_v \phi(x)$ for arbitrary $\mathcal{M} \in F$, $w \in W$ and valuation $v$. By the semantics for $=$ we immediately get $v(x) = v(y)$ and thus if $\phi$ is atomic $\models_v \phi(y)$ follows readily. This establishes the induction base. Assume now that we have the result for $\psi$ and wish to show $\mathbf{PS}$ for $\neg \psi$. If $\mathcal{M}, w \models_v (x = y)$ we get by the induction hypothesis that $\models_v (\neg \psi(x) \leftrightarrow \neg \psi(y))$ and as we by assumption have that $\models_v \neg \psi(x)$ it yields a contradiction if $\models_v \psi(y)$. As $\mathcal{M}, w$ and $v$ was arbitrary we conclude that $\mathbf{PS}$ holds for $\neg \psi$.

If we know $\mathbf{PS}$ for formulas $\phi$ and $\psi$ we show that it holds for $\phi \rightarrow \psi$, so assume $\mathcal{M}, w \models_v (x = y)$ and $\mathcal{M}, w \models_v (\phi(x) \rightarrow \psi(x))$. Now, either $\mathcal{M}, w \models_v \neg \phi(x)$ or $\mathcal{M}, w \models_v \psi(x)$ by the semantics for $\rightarrow$, but since $\mathbf{PS}$ holds for $\neg \phi$ and $\psi$ by the previous and the induction hypothesis this means that either $\mathcal{M}, w \models_v \neg \phi(y)$ or $\mathcal{M}, w \models_v \psi(y)$ which in turn implies $\mathcal{M}, w \models_v \phi(y) \rightarrow \psi(y)$. As $\mathcal{M}, w$ and $v$ was arbitrary we conclude validity of $\mathbf{PS}$ for $\phi \rightarrow \psi$. 

13
If we assume PS for φ it vacuously follows that PS holds for ∀xφ so we turn to the modal case.

Assume PS for φ and let t ∈ Term_{agt}. We show PS for Ktφ. To this end, assume that M, w ⊩v (x = y) for variables x, y ∈ VAR_{agt} and M, w ⊩ v Ktφ(x). But by the semantics of Kt this means that M, w′ ⊩ v φ(x) for any w′ such that (w, w′) ∈ R(x^w,v) but then the induction hypothesis yields that M, w′ ⊩ v φ(y) and as by assumption x^w,v = y^w,v we get M, w′ ⊩ v φ(y) for any w′ such that (w, w′) ∈ R(y^w,v). This is exactly the definition of M, w ⊩ v Ktφ(y) as wanted.

∀Id: We need to show that for any constant c, any M ∈ F, w ∈ W and valuation v we have M, w ⊩ v (c = c) → ∃x(x = c) so assume M, w ⊩ v (c = c). By the semantics of ∃ and the definition of ⊩v we need to produce a valuation v′ such that v′(x) = c^w,v but since all valuations are surjective by definition this is indeed possible for all constants c and all w ∈ W.

N: This is simply true by construction. The axiom says that there are exactly |A| = n elements which are quantified over using variables from VAR_{agt}, but this is precisely the content of definition 5.3.

K: We have to show that for any M ∈ F, w ∈ W, valuation v, agent-denoting term t, and wff φ and ψ we have M, w ⊩ v Kt(φ → ψ) → (Ktφ → Ktψ), so assume M, w ⊩ v Kt(φ → ψ). By the semantics of Kt this means that M, w′ ⊩ v φ → ψ for every w′ ∈ W with (w, w′) ∈ R(t^w,v) but this means that for all such w′ we have either M, w′ ⊩ v ¬φ or M, w′ ⊩ v ψ. All in all, we either have M, w′ ⊩ v ψ for all w′ ∈ W with the property that (w, w′) ∈ R(t^w,v) in which case it holds that M, w ⊩ v Ktφ, or we have for some such w′ that M, w′ ⊩ v ψ in which case M, w ⊩ v ¬Ktφ. By the semantics of → we conclude M, w ⊩ v Ktφ → Ktψ which was what we wanted.

BF: We need to show that for any M ∈ F, w ∈ W, valuation v, agent-denoting term t, variable x = t, and wff φ we have M, w ⊩ v ∀xKtφ → Kt∀xφ. Assume for contradiction that M, w ⊩ v ∀xKtφ yet M, w ⊩ v ¬Kt∀xφ. By the latter there is some world w′ for which (w, w′) ∈ R(t^w,v) such that M, w′ ⊩ v ¬∀xφ which in turn means that there is v′ an x-variant of v such that M, w′ ⊩ v′ ¬φ. However, by the former we get that M, w ⊩ v Ktφ and thus M, w′ ⊩ v Ktφ, yielding a contradiction.

KNI: We have to prove that for any M ∈ F, w ∈ W, valuation v, agent-denoting term t, variables x and y, we have M, w ⊩ v (x ≠ y) → Kt(x ≠ y). Assume M, w ⊩ v (x ≠ y). As variables are rigid this will hold in any world, thus especially those worlds w′ ∈ W such that (w, w′) ∈ R(t^w,v).

Having dealt with all the axioms we turn to show that the rules of inference MP, KG, and Gen preserves validity.
6.2 Rules of Inference Preserves Validity

**MP:** We have to show that if \( \phi \rightarrow \psi \) and \( \phi \) are valid on \( F \) then \( \psi \) is valid on \( F \), so assume that \( \models_F \phi \rightarrow \psi \) and \( \models_F \phi \). By the latter we see that \( \phi \) is satisfied in every model based on \( F \), in every world, under every valuation. But then the semantics for \( \rightarrow \) yields that so must \( \psi \) be which is what we wanted.

**KG:** We have to show that if \( \phi \) is valid on \( F \) then so is \( K_t \phi \) for any agent-referring term \( t \), but this is immediately clear from the semantics for \( K_t \).

**Gen:** We have to show that if \( \phi \rightarrow \psi \) is valid on \( F \) for wffs \( \phi \) and \( \psi \) with the variable \( x \) not having any free occurrences in \( \phi \) then \( \phi \rightarrow \forall x \psi \) is also valid on \( F \). Since \( \phi \rightarrow \psi \) is valid on \( F \) we know from the semantics for \( \rightarrow \) that in every model \( M \in F \), in every world \( w \in W \) for every valuation \( v \) we have either \( M, w \models_v \neg \phi \) or \( M, w \models_v \psi \) so the only way for Gen fail is if in some such model, world and valuation we have \( M, w \models_v \phi \), and \( M, w \models_v \psi \) yet somehow \( M, w \models_{v'} \neg \forall x \psi \). As \( x \) does not occur free in \( \phi \) \( M, w \models_v \phi \) implies \( M, w \models_{v'} \phi \), contradicting the validity of \( \phi \rightarrow \psi \).

Having established validity of the axioms and the validity preservation of the rules of inference soundness of \( K_{TM}^{\Lambda_\pi} \) wrt. \( F \) is a sitting duck:

**Theorem 1.** [Soundness] As all \( K_{TM}^{\Lambda_\pi} \) axioms are valid on the class \( F \) of all \( TM \) \( \Lambda_\pi \) frames, and all rules of inference preserves validity we conclude that \( K_{TM}^{\Lambda_\pi} \) is sound wrt. \( F \), i.e if \( \vdash_{K_{TM}^{\Lambda_\pi}} \phi \) then \( \models_F \phi \).

**Proof.** See above.

We now know, that the logic defined by the syntax presented is included in the logic that follows from the semantic definitions, that is \( L_{K_{TM}^{\Lambda_\pi}} \subseteq L_F \) in the language of definitions [13] and [19]. What remains is the opposite inclusion which will be the topic of the next section.

7 Completeness

We start with the most central definition.

**Definition 21.** [Completeness] When \( F \) is a class of structures (models or frames) and \( \Lambda \) is a logic, we say that \( \Lambda \) is strongly complete wrt. \( F \) if for any set \( \Gamma \) of wffs, and \( \phi \) a single wff, if \( \Lambda \models_F \phi \) then \( \Gamma \vdash_{\Lambda} \phi \). That is, if \( \phi \) is a semantic consequence of \( \Gamma \) on \( F \) then \( \Gamma \) proves \( \phi \) in \( \Lambda \).

Based on the proof of soundness one might infer that we should prove completeness by manually show for each valid formula that a proof existed. However, this will not be the strategy. Consider the following Proposition.

**Proposition 2.** [IFF] The logic \( \Lambda \) is strongly complete wrt. the class of structures \( F \) (models or frames) iff any \( \Lambda \) – consistent set of formulas \( \Gamma \) is satisfiable on some structure from \( F \).
Proof. We first prove that if $\Lambda$ is complete wrt. $F$, then every $\Lambda$– consistent set of formulas $\Gamma$ is satisfiable on some structure from $F$, so assume completeness and pick a $\Lambda$– consistent set of formulas $\Gamma \cup \{ \phi \}$. If $\Gamma \cup \{ \phi \}$ is not satisfiable on any structure from $F$ we have $\Gamma \models_F \neg \phi$ but then by completeness we get $\Gamma \vdash \Lambda \neg \phi$ meaning that $\Gamma \cup \{ \phi \}$ was $\Lambda$– inconsistent after all, a contradiction.

Now, we prove that if any $\Lambda$– consistent set of formulas is satisfiable on $F$, then $\Lambda$ is complete wrt. $F$. Assume for contradiction that any $\Lambda$– consistent set of formulas is satisfiable on $F$, yet $\Lambda$ is not complete wrt. $F$. This means for some set of wffs $\Gamma \cup \{ \phi \}$ that $\Gamma \models_F \phi$ yet $\Gamma \not\models \Lambda \phi$. By the latter it follows that the set $\Gamma \cup \{ \neg \phi \}$ is consistent and so by assumption satisfiable on some structure from $F$, but this contradicts that $\Gamma \models_F \phi$.

What is the significance of Proposition 2? It means essentially that proving completeness is a question of model-hunting, for all we need to do, given a consistent set of formulas, is to produce a model (and a valuation) satisfying that set of formulas. This insight has given rise to a more or less standardized construction of what is called canonical models and this will be the focus of the present inquiry in the following. First, we need to produce a set of worlds.

### 7.1 Worlds of the Canonical Model

Seeing that the overall aim is to produce, given a consistent set of formulas, a model and a valuation satisfying that set of formulas it is natural to let the worlds of the canonical model be sets of consistent formulas subject to appropriate extra conditions. Truth will then be defined as membership. Imagine in the following that a $\Lambda$– consistent set $\Omega$ of formulas is given, and that our task is to produce a model $M^\Lambda_\Omega$ from $F$ satisfying $\Omega$. First we bring a couple of important definitions.

**Definition 22.** [Maximal $\Lambda$– Consistent] Let $\Lambda$ be a logic and $\Gamma$ a set of wffs.
We say that $\Gamma$ is maximally $\Lambda$– consistent if $\Gamma$ is $\Lambda$– consistent and no proper extension of $\Gamma$ is $\Lambda$– consistent.

We shall abbreviate such that we write that $\Gamma$ is a $\Lambda$– MSC whenever $\Gamma$ is a maximally $\Lambda$– consistent set. In order to make the machinery work we need to make sure that whenever a formula of the form $\forall x \phi$ is not included in some world of the canonical model there must be some “witness” of the falsity. This motivates the next definition:

**Definition 23.** [$\forall$ – property] If $\Gamma$ is a set of formulas, we say that it has the $\forall$ – property if for every wff $\phi$, for every variable $x$, there is some variable $y$ such that $(\phi(y/x) \to \forall x \phi) \in \Gamma$. Note, that if some set $\Gamma$ of wffs has the $\forall$ – property then so will every set of wffs of which $\Gamma$ is a subset.

What follows from definition 23 is that if for some $\Lambda$– MSC $\Gamma \phi$ we have $\forall x \phi \not\in \Gamma$, then there is some variable $y$ such that $\phi(y/x) \not\in \Gamma$. This is the reason why the $\forall$ – property is sometimes referred to as “bearing witness”.

16
For technical reasons we need to enlarge our language in order to make sure our construction will function, and so we add to $\mathcal{L}_{TM}^{A\sigma}$ infinitely many new variables equally divided between $VAR_{agr}$ and $VAR_{obj}$. The resulting language is called $\mathcal{L}^+$, and note that trivially any wff of $\mathcal{L}_{TM}^{A\sigma}$ is also a wff of $\mathcal{L}^+$. Now we need to be better acquainted with $\Lambda - MSC$’s, and for that we need a Lemma:

**Lemma 5.** [Completeness Lemma] Let $\Gamma$ be a consistent set of formulas such that for all formulas $\phi$ either $\phi \in \Gamma$ or $\neg \phi \in \Gamma$. Then $\Gamma$ is deductively closed, that is whenever $\Gamma \vdash \Lambda \psi$ we have $\psi \in \Gamma$.

*Proof.* Assume that $\Gamma$ has the mentioned property, and that for some formula $\phi$ we have $\Gamma \vdash \Lambda \phi$. By assumption either $\phi \in \Gamma$ or $\neg \phi \in \Gamma$ but the latter contradicts the consistency of $\Gamma$. $\square$

**Lemma 6.** [Properties of $\Lambda - MCS$’s] If $\Lambda$ is a logic and $\Gamma$ is a $\Lambda - MCS$, then

1. $\Lambda \subseteq \Gamma$
2. for all formulas $\phi$, either $\phi \in \Gamma$ or $\neg \phi \in \Gamma$
3. $\Gamma$ is deductively closed
4. for all formulas $\phi, \psi : (\phi \rightarrow \psi) \in \Gamma$ iff $\neg \phi \in \Gamma$ or $\psi \in \Gamma$

*Proof.* i) If $\phi \in \Lambda \setminus \Gamma$ then the sets $\Gamma \cup \{\phi\}$ and $\Gamma \cup \{\neg \phi\}$ are both proper extensions of $\Gamma$ and by Lemma 4 one of them is consistent contradicting the maximality of $\Gamma$.

ii) Assume for contradiction that for some formula $\phi$ neither $\phi \in \Gamma$ nor $\neg \phi \in \Gamma$, meaning that both $\Gamma \cup \{\phi\}$ and $\Gamma \cup \{\neg \phi\}$ are proper extensions of $\Gamma$. By Lemma 4 one of them is consistent contradicting the maximality of $\Gamma$.

iii) By assumption $\Gamma$ is consistent, and so it follows by ii) and Lemma 5 that $\Gamma$ is deductively closed.

iv) Assume for contradiction that for some formulas $\phi$ and $\psi$ we have $\phi \rightarrow \psi \in \Gamma$ yet $\phi \in \Gamma$ and $\neg \psi \in \Gamma$ yielding $\{(\phi \rightarrow \psi), \phi, \neg \psi\} \subseteq \Gamma$. By iii) this implies that $\psi \in \Gamma$ by an application of MP, contradicting the consistency of $\Gamma$. $\square$

The next needed result is the well-known Lindenbaum’s Lemma. As the proof is completely standard it will be omitted.

**Lemma 7.** [Lindenbaum’s Lemma] Let $\Gamma$ be a $\Lambda - consistent set of formulas. Then there exists some $\Lambda - MCS \Gamma^+$ such that $\Gamma \subseteq \Gamma^+$.

Given a consistent set of formulas $\Gamma$ we, in accordance with Proposition 2, wish to produce a model and a valuation satisfying $\Gamma$. We know by Lemma 7 that we can extend $\Gamma$ to a maximally consistent set, but $\Gamma$ is formulated over $\mathcal{L}_{TM}^{A\sigma}$ and the model we are about to build will be formulated over $\mathcal{L}^+$ so we need to ascertain ourselves that we can find an appropriate set of formulas over $\mathcal{L}^+$extending $\Gamma$. This is the content of the next Lemma:

---

9For a proof see [1] or [5].
Lemma 8. [Saturation] Let \( \Gamma \) be a \( \Lambda \)– consistent set of formulas over \( \mathcal{L}_{TM}^{\Lambda} \). Then there exist a \( \Lambda \)– consistent set \( \Gamma_+ \) over \( \mathcal{L}^+ \) with the \( \forall \) – property such that \( \Gamma \subseteq \Gamma_+ \).

Proof. Let \( \Gamma \) be a \( \Lambda \)– consistent set of formulas over \( \mathcal{L}_{TM}^{\Lambda} \), and observe that \( \mathcal{L}^+ \) is countable and as any formula is a finite string of symbols from \( \mathcal{L}^+ \) the set of wffs is also countable, meaning that we can enumerate all formulas of the type \( \forall x \phi \) over \( \mathcal{L}^+ \). Define now a sequence of sets \( \{\Delta_i\}_{i \in \mathbb{N}} \) by

\[
\begin{align*}
\Delta_0 & := \Gamma \\
\Delta_{n+1} & := \Delta_n \cup \{ \phi(y/x) \rightarrow \forall x \phi \}
\end{align*}
\]

where we take \( \phi \) to be the \( n+1 \)'st formula wrt to the enumeration, and \( y \) to be the first variable (again, relative to the enumeration) not to occur in \( \Delta_n \) nor \( \phi \) (and note that this a fortiori means that \( y \) does not occur free anywhere in neither \( \Delta_n \) nor \( \phi \)). The reason why we introduced the enlarged language \( \mathcal{L}^+ \) was exactly to ensure that this construction is possible; as \( \Delta_0 = \Gamma \subseteq \mathcal{L}_{TM}^{\Lambda} \) and only finitely many new variables are introduced in each step we can always pick the variable \( y \in \mathcal{L}^+ \).

From here the strategy is to show that each \( \Delta_i \) is consistent and then choose our appropriate set of \( \mathcal{L}^+ \)– formulas as \( \Gamma_+ = \bigcup_{i \in \mathbb{N}} \Delta_i \). We proceed by induction. To establish the induction base simply note that \( \Gamma \) is \( \Lambda \)– consistent by assumption. Now, assume that \( \Delta_n \) is \( \Lambda \)– consistent while \( \Delta_{n+1} \) is not. This means that for some \( \phi_1, \ldots, \phi_k \in \Delta_n \) we have

\[
\vdash_{\Lambda} (\phi_1 \ldots \phi_k) \rightarrow \phi(y/x) \quad \text{and} \quad \vdash_{\Lambda} (\phi_1 \ldots \phi_k) \rightarrow \neg \forall x \phi
\]

As \( y \) does not occur free in any of the \( \phi_i \)'s we get from the first above and \textbf{Gen} that \( \vdash_{\Lambda} (\phi_1 \ldots \phi_k) \rightarrow \forall y \phi(y/x) \) and since \( y \) does not occur free in \( \phi \) this is equivalent to

\[
\vdash_{\Lambda} (\phi_1 \ldots \phi_k) \rightarrow \forall x \phi(x), \quad \text{but this means that}
\]

\[
\vdash_{\Lambda} (\phi_1 \ldots \phi_k) \rightarrow \forall x \phi \quad \text{and} \quad \vdash_{\Lambda} (\phi_1 \ldots \phi_k) \rightarrow \neg \forall x \phi
\]

contradicting the consistency of \( \Delta_n \). We conclude that \( \Delta_i \) is \( \Lambda \)– consistent for every \( i \in \mathbb{N} \). Put \( \Gamma_+ = \bigcup_{i \in \mathbb{N}} \Delta_i \) and observe that \( \Gamma_+ \) trivially has the \( \forall \) – property by construction so it remains to show that consistency is “preserved in the limit”.

For \( \Gamma_+ \) to be inconsistent there would have to exist some inconsistent, finite subset \( \Gamma_0 \subseteq \Gamma_+ \) by definition \[18\] but any such \( \Gamma_0 \) would be contained in some \( \Delta_i \), a contradiction. We conclude that \( \Gamma_+ \) is \( \Lambda \)– consistent and has the \( \forall \) – property. \(\square\)

\[10\]See \[4\] pp. 242, 258.
Note that the Saturation Lemma does not explicitly ensure preservation of the \( \forall - \) property; this is not necessary as a set of formulas has the \( \forall - \) property if any subset has it. We are almost ready to actually define the canonical model but one more technicality is still pressing. Since variables are rigid and we have included equality in our language we need to make sure that the same equality statements are true in each world in the canonical model. To ensure that this is the case we shall restrict attention to a cohesive subset of the set of worlds which motivates the following definition:

**Definition 24.** [\( R - \) path connected] Given sets \( \mathcal{X} \) and \( \mathcal{Y} \) and a map \( R : \mathcal{Y} \rightarrow \mathcal{P}(\mathcal{X} \times \mathcal{X}) \) associating binary relations on \( \mathcal{X} \) to each member of \( \mathcal{Y} \), we say that the subset \( B \subseteq \mathcal{X} \) is \( R - \) path connected if for every \( b, b' \in B \) there is a sequence \( (y_1, \ldots, y_k) \subseteq \mathcal{Y} \), and a sequence \( (b_1, \ldots, b_{k-1}) \subseteq B \) such that \( (b, b_1) \in R(y_1), (b_1, b_2) \in R(y_2), \ldots, (b_{k-1}, b') \in R(y_k) \).

Note that if \( M \) is a model, then so is any \( R - \) path connected subset \( M' \subseteq M \) since by definition truth in \( M' \) cannot depend on worlds in \( M \setminus M' \). We are now ready to define the canonical model for a \( \Lambda - \) consistent set \( \Omega \).

### 7.2 Canonical Models

**Definition 25.** [Canonical Model] Given a \( \Lambda - \) consistent set \( \Omega \) from an \( n - \) agent, two-sorted term-modal logic \( \Lambda \) in the language \( \mathcal{L}_T \) with extension \( \mathcal{L}^\Lambda \), we define the canonical model \( M_\Omega \) to be the four-tuple \( \langle \mathcal{W}_\Omega^\Lambda, Dom^\Lambda, \mathcal{R}^\Lambda, \mathcal{I}^\Lambda \rangle \) where

1. \( \mathcal{W}_\Omega^\Lambda \) is an \( \mathcal{R}^\Lambda - \) path connected subset of all \( \Lambda - \) MCS’s with the \( \forall - \) property in \( \mathcal{L}^\Lambda \) containing some \( \Lambda - \) MCS extending \( \Omega \).

2. \( Dom^\Lambda = \{ [x] \mid x \in VAR_{agt} \} \cup \{ [y] \mid y \in VAR_{obj} \} \) = \( \mathcal{A}^\Lambda \cup \{ [y] \mid y \in VAR_{obj} \} \) with \( [z] = \{ v \in VAR^+ \mid z \sim v \} \) where \( z \sim v \) iff \( (z \in v) \in w \) for any one \( w \in \mathcal{W}_\Omega^\Lambda \). Note that we indeed have a partition of the domain by MSD.

3. \( \mathcal{R}^\Lambda : \mathcal{A}^\Lambda \rightarrow \mathcal{P}(\mathcal{W}_\Omega^\Lambda \times \mathcal{W}_\Omega^\Lambda) \) is a map associating binary accessibility relations to agents such that \( (w, w') \in \mathcal{R}^\Lambda(\alpha^\Lambda) \) iff for every formula \( K_x \psi \in \mathcal{L}^\Lambda \) with \( x \in \alpha^\Lambda \in \mathcal{A}^\Lambda \) if \( K_x \psi \in w \), then \( \psi \in w' \).

4. For each \( \beta \in \tau^\Lambda, w \in \mathcal{W}_\Omega^\Lambda \) and \( P \) some relation symbol with arity \( \beta \) let \( \mathcal{I}^\Lambda(P,w) = \{ \{ [x_1], \ldots, [x_k] \} \in \prod_{i \in \{1, \ldots, k\}} DOM^\Lambda_{\beta_i} \mid P(x_1, \ldots, x_k) \in w \} \).

For each \( \alpha \in \tau_{\mathcal{A}}^{\Lambda+1}, f \in \mathcal{W}_\Omega^\Lambda \) and \( \eta \) some function symbol with arity \( \alpha \) let \( \mathcal{I}^\Lambda(f,w) = \{ ([x_1], \ldots, [x_k], [x_{k+1}]) \in \prod_{i \in \{1, \ldots, k+1\}} DOM^\Lambda_{\alpha_i} \mid (f(x_1, \ldots, x_k) = x_{k+1}) \in w \} \). For any world \( w \in \mathcal{W}_\Omega^\Lambda \) take \( \mathcal{I}^\Lambda(\cdot, w) = \{ (d, d) \mid d \in Dom^\Lambda \} \).

For any constant \( c \in CON \), for any world \( w \in \mathcal{W}_\Omega^\Lambda \), \( \mathcal{I}^\Lambda(c, w) = [x] \in Dom^\Lambda \) such that \( (x = c) \in w \).

Before defining the canonical valuation we need to make sure that the canonical model is well-defined. This motivates the following Proposition:
**Proposition 3.** [Canonical Interpretation of Constants] The canonical interpretation does indeed ascribe extensions to all constants in $\mathcal{L}^+$, i.e. for all worlds $w \in \mathcal{W}_\mathcal{M}^\Lambda$ for every constant $c \in \text{CON}$ there is $y \in \text{VAR}^+$ such that $(y = c) \in w$.

*Proof.* By the $\forall$–property we have for any wff $\phi$ and every variable $x$ that there exists a variable $y$ such that $(\phi(y/x) \to \forall x \phi) \in w$, and we recall that $w$ is deductively closed by Lemma[8]. By contraposition this yields $(\neg \forall x \phi \to \neg \phi(y/x)) \in w$ and thus by substituting $\psi := \neg \phi$ that $(\neg \forall x \neg \psi \to \neg \neg \psi(y/x)) \in w$. Canceling double-negation and abbreviating using $\exists$ gets us $(\exists x \psi \to \psi(y/x)) \in w$. By $\exists \text{Id}$ and $\text{MP}$ we get $(\exists x \psi) \in w$ and thus by another application of $\text{MP}$ that $\psi(y/x) \in w$. Putting $\psi := (x = c)$ now yields $(y = c) \in w$ as desired[11]. $\square$

The only thing left to specify is the canonical valuation:

**Definition 26.** [Canonical Valuation] The Canonical Valuation $v^\Lambda$ is defined by $v^\Lambda(x) = [x]$ for all $x \in \text{VAR}^+$.

By construction the domain of the Canonical Model is depending on what equalities between variables are to be found in the $\Lambda$ – $\text{MCS}$’s that constitutes the worlds. As such one might be a little queasy that maybe the occurrence of different such equalities in different worlds might pose incommensurable claims on the cardinality of the domain. That this is not a problem needs to be proven:

**Proposition 4.** [Well – Defined Domain] Let $w$ and $w'$ be any two worlds from $\mathcal{W}_\mathcal{M}^\Lambda$ and $x$ and $y$ any two variables from $\mathcal{L}^+$ we have $(x = y) \in w$ iff $(x = y) \in w'$.

*Proof.* This is where we need that $\mathcal{W}_\mathcal{M}^\Lambda$ is $\mathcal{R}^\Lambda$– path connected. Assume for $x, y \in \text{VAR}^+$ and world $w \in \mathcal{W}_\mathcal{M}^\Lambda$ that $(x = y) \in w$. As $\mathcal{W}_\mathcal{M}^\Lambda$ is $\mathcal{R}^\Lambda$– path connected we can find $(\alpha_1^\Lambda, \alpha_2^\Lambda) \subseteq \mathcal{A}^\Lambda$ and $(w_1, \ldots, w_{k-1}) \subseteq \mathcal{W}_\mathcal{M}^\Lambda$ such that $(w, w_1) \in \mathcal{R}(\alpha_1^\Lambda)$, $(w_1, w_2) \in \mathcal{R}(\alpha_2^\Lambda)$, ..., and $(w_{k-1}, w') \in \mathcal{R}(\alpha_k^\Lambda)$. Furthermore, let the terms $t_1, \ldots, t_k \in \text{Term}^\Lambda_{\text{agg}}$ be such that $t_i$ designates $\alpha_i^\Lambda$ for $i = 1, \ldots, k$.

As worlds are deductively closed cf. Lemma[8] we get by an application of $\text{KG}$ that $K_{t_i}(x = y) \in w$ and thus by the definition of $\mathcal{R}^\Lambda$ we have $(x = y) \in w'$. Repeating this procedure $k$ times yields the desired result. $\square$

In the definition of the canonical model we made sure that whenever a formula of the form $K_i \phi$ where contained in some world $w$, we had $(w, w') \in \mathcal{R}^\Lambda(\ell^{w,w'})$ only when $\phi \in w'$. However, we also need to make sure that the interplay between $\mathcal{R}^\Lambda$ and formulas of the type $P_i \phi$ is appropriate, i.e. if $P_i \phi \in w$ then there must be some $w'$ such that $(w, w') \in \mathcal{R}^\Lambda(\ell^{w,w'})$ with $\phi \in w'$. This is the content of the next Proposition:

**Proposition 5.** [Existence] If $w \in \mathcal{W}_\mathcal{M}^\Lambda$ such that $P_i \phi \in w$, then there is $w' \in \mathcal{W}_\mathcal{M}^\Lambda$ such that $(w, w') \in \mathcal{R}(\alpha_i^\Lambda)$ and $\phi \in w'$.

[11]Strictly speaking the proof uses that there is a one-to-one correspondence between formulas $\phi$ and $\neg \phi$ but this is trivial. Note also that contraposition and tertium non datur are substitution instances of valid formulas from propositional modal logic, and thus at our disposal.
Proof. The proof is constructive so assume \( P_x \phi \in w \) for some world \( w \in \mathcal{W}_1^\Lambda \) and some variable \( x \) with \( v^\Lambda(x) = \alpha^\Lambda_1 \). The aim is then to produce a world \( w' \) such that \( (w, w') \in \mathcal{R}(\alpha^\Lambda_1) \) and \( \phi \in w' \). Consider the set \( \Gamma = \{ \psi \} \cup \{ \phi | K_x \phi \in w \} \). We first show that \( \Gamma \) is indeed \( \Lambda \) consistent, so assume otherwise. As \( \{ \phi | K_x \phi \in w \} \) is consistent this means that there are \( \phi_1, ..., \phi_n \in \{ \phi | K_x \phi \in w \} \) such that

\[
\vdash_{\Lambda} (\phi_1 \land ... \land \phi_n) \rightarrow \neg \psi
\]

by KG and two applications of K this yields

\[
\vdash_{\Lambda} (K_x \phi_1 \land ... \land K_x \phi_n) \rightarrow K_x \neg \psi.
\]

Since \( w \) is an \( \Lambda - \text{MCS} \) containing \( K_x \phi_i \) for \( i = 1, ..., n \) we can conclude that \( (K_x \phi_1 \land ... \land K_x \phi_n) \in w \) and thus by MP also \( K_x \neg \psi \in w \), but by the definition of \( P_x \) this means that \( \neg P_x \psi \in w \) contradicting the consistency of \( w \). We conclude that the set \( \Gamma \) is indeed \( \Lambda \) consistent. The idea from here is to construct a sequence of formulas \( \{ \psi_i \}_{i \in \mathbb{N}} \) in a fitting manner such that \( \{ \bigcup_{i \in \mathbb{N}} \psi_i \} \cup \{ \phi | K_x \phi \in w \} \) is a \( \Lambda \) consistent set with the \( \forall \) property that can be extended by Lindenbaum’s Lemma to an \( \Lambda - \text{MCS} \) with the required properties.

To define the sequence \( \{ \psi_i \}_{i \in \mathbb{N}} \) we first define a couple of enumerations; first we enumerate all formulas of the type \( \forall y \lambda \), and then letting \( \forall y \lambda \) be the \( n + 1 \)th such formula we define the sequence \( \{ \psi_i \}_{i \in \mathbb{N}} \) inductively by \( \psi_0 = \psi \) (from \( \Gamma \)) and \( \psi_{n+1} = \psi_n \land \{ \lambda(z/y) \rightarrow \forall y \lambda \} \) where \( z \neq x \) is a variable such that \( \{ \phi | K_x \phi \in w \} \cup \{ \psi_n \land \{ \lambda(z/y) \rightarrow \forall y \lambda \} \} \) is consistent. That it is always possible to choose such variable \( z \) is now shown:

Assume that \( \{ \psi_n \} \cup \{ \phi | K_x \phi \in w \} \) is consistent and assume for contradiction that \( \{ \phi | K_x \phi \in w \} \cup \{ \psi_n \land \{ \lambda(z/y) \rightarrow \forall y \lambda \} \} \) is inconsistent for every variable \( z \) of \( \mathcal{L}_+^\Lambda \). This means that for every variable \( z \) we could find \( \{ \chi_1, ..., \chi_n \} \subseteq \{ \phi | K_x \phi \in w \} \) such that

\[
\vdash_{\Lambda} (\chi_n \land ... \land \chi_n) \rightarrow \neg (\{ \psi_n \land \{ \lambda(z/y) \rightarrow \forall y \lambda \} \})
\]

which by standard logic is equivalent to

\[
\vdash_{\Lambda} (\chi_n \land ... \land \chi_n) \rightarrow (\psi_n \rightarrow \{ \lambda(z/y) \land \neg \forall y \lambda \})
\]

and by Lemma 3 we get

\[
\vdash_{\Lambda} K_x (\chi_n \land ... \land \chi_n) \rightarrow K_x (\psi_n \rightarrow \{ \lambda(z/y) \land \neg \forall y \lambda \})
\]

further, by Lemma 2 we get

\[
\vdash_{\Lambda} (K_x \chi_n \land ... \land K_x \chi_n) \rightarrow K_x (\psi_n \rightarrow \{ \lambda(z/y) \land \neg \forall y \lambda \}).
\]

Seeing that \( \{ \chi_1, ..., \chi_n \} \subseteq w \) by definition we get by Lemma 4 that \( (K_x \chi_n \land ... \land K_x \chi_n) \subseteq w \) and thus by an application of MP that \( K_x (\psi_n \rightarrow \{ \lambda(z/y) \land \neg \forall y \lambda \}) \subseteq w \) for every variable \( z \) of the language. Now, if \( z \neq x \) is some variable occurring in neither \( \lambda \) nor \( \psi_n \) we consider the formula
∀zK_x(ψ_n → ¬(λ(z/y) → ∀yλ))

and see that by the ∀ - property the formula there is some variable z’ such that

K_x(ψ_n → ¬(λ(z’/y) → ∀yλ)) → (∀zK_x(ψ_n → ¬(λ(z/y) → ∀yλ))) ∈ w.

But we just showed that K_x(ψ_n → {λ(z/y) ∧ ¬∀yλ}) ⊆ w for every z ∈ VAR⁺

and so by an application of MP we obtain

∀zK_x(ψ_n → ¬(λ(z/y) → ∀yλ)) ∈ w.

Furthermore, as x ≠ z we see that

∀zK_x(ψ_n → ¬(λ(z/y) → ∀yλ)) → K_x∀z(ψ_n → ¬(λ(z/y) → ∀yλ))

is an instance of BF and so included in w. An application of MP yields

K_x∀z(ψ_n → ¬(λ(z/y) → ∀yλ)) ∈ w

and utilizing that whenever x does not occur free in φ, ∀x(φ → ψ) → (φ → ∀xψ)

is a Theorem of first-order logic we obtain that

K_x(ψ_n → ∀z¬(λ(z/y) → ∀yλ)) ∈ w

since by assumption z does not occur (free) in ψ_n. Recall that whenever y is not free in ∀xφ, ∃x(φ(y/x) → ∀xφ) is a Theorem of first-order logic, from which we get

∃z(λ(z/y) → ∀yλ).

(2)

Seeing that (1) can be re-written as K_x(ψ_n → ¬∃z(λ(z/y) → ∀yλ)) ∈ w we conclude that K_x¬ψ_n ∈ w contradicting the consistency of {ψ_n} ∪ {φ | K_xφ ∈ w}.

We conclude that it is indeed possible to choose z ≠ x such that

{φ | K_xφ ∈ w} ∪ (ψ_n ∧ {λ(z/y) → ∀yλ}) = {φ | K_xφ ∈ w} ∪ {ψ_n+1}

(3)

is Λ - consistent. Now, consider the set {φ | K_xφ ∈ w} ∪ {∪_{n∈ℕ} ψ_n}. First, for all n, {φ | K_xφ ∈ w} ∪ {ψ_n} is consistent by the above. Second, we see that ⊢_Λ ψ_n → ψ_m for all n ≥ m such that {φ | K_xφ ∈ w} ∪ {∪_{n∈ℕ} ψ_n} is consistent too. Further, by construction the set has the ∀ - property, and so can be extended to an Λ - MCS by Lindenbaum - call this set w’. This is the world we set out to produce; by construction φ ∈ w’ and so we have by definition that (w, w’) ∈ Λ(x,w,φ⁺).

Having secured the needed-in-a-second existence-result another worry presents itself; for if K_xφ ∈ w for some world w, some wff φ, and some agent-referring variable x yet for some variable y designating the same agent as x we had

K_yφ ∉ w we have a problem. To see this note that by Lemma (6) that ¬K_yφ ∈ w or equivalently P_y¬φ ∈ w, but by the Proposition just proved we have a world w’ such that (w, w’) ∈ Λ(y,w,φ⁺) and ¬φ ∈ w’ contradicting the assumption that K_xφ ∈ w. That this does not obtain is the content of the next Proposition:

**Proposition 6.** [Uniformity] Let w ∈ Λ such that K_xφ ∈ w for wff φ and variable x with v⁺(x) = α⁺ ∈ A⁺. Then for all variables y with v⁺(x) = v⁺(y)

we have K_yφ ∈ w.

---

¹²See pp. 242.
Proof. Let \( w \in W^A_\Omega \) such that \( K_x \phi \in w \) for \( \text{wff} \) and variable \( x \) with \( v^A(x) = \alpha^A_i \in A^A \).

Further, let \( y \) be such that \( v^A(x) = v^A(y) \), but this means that \( [x] = [y] \) so by the definition of the canonical model and Proposition \( \text{4} \) we have \( (x = y) \in w' \), for every \( w' \in W^A_\Omega \), hence a fortiori \((x = y) \in w \). By \( \text{PS} \) we get

\[
(x = y) \rightarrow (K_x \phi \rightarrow K_y \phi) \in w
\]

and thus by \( \text{MP} \) that \((K_x \phi \rightarrow K_y \phi) \in w \). By assumption we have \( K_x \phi \in w \) and so one more application of \( \text{MP} \) yields \( K_y \phi \in w \) as desired. \( \Box \)

Obviously, the idea is that “truth is membership” in the canonical model, and we are now in position to prove that this is exactly the case.

**Lemma 9.** [Truth] For every \( w \in W^A_\Omega \), for every \( \phi \in L^+ \) we have \( \mathcal{M}^A_\Omega, w \models_{v^A} \phi \) iff \( \phi \in w \).

As the proof of the truth Lemma will be by induction of the complexity of \( \phi \) we need to make this notion precise first.

**Definition 27.** [Complexity] For any \( \text{wff} \) \( \phi \) of the language, for any variable \( x \) and any term-referring term \( t \), the complexity \( c(\phi) \) of \( \phi \) is given by

\[
\begin{align*}
c(\phi) &= 0, & \text{for all atomic formulas } \phi \\
c(\phi \rightarrow \psi) &= \max\{c(\phi), c(\psi)\} + 1 \\
c(\neg \phi) &= c(\forall x \phi) = c(K_1 \phi) = c(\phi) + 1
\end{align*}
\]

**Proof.** Proof of the Truth Lemma by induction on the complexity of \( \phi \).

**Equality:** If \( t_1, t_2 \) are terms and \( w \in W^A_\Omega \) then we have \( \mathcal{M}^A_\Omega, w \models_{v^A} (t_1 = t_2) \) iff \( (t_1^w, v^A, t_2^w, v^A) \in \mathcal{I}^A(=, w) \) which in turn holds iff \( (t_1^w, v^A) = (t_2^w, v^A) \). Now, by the definition of extensions in the canonical model this is the case iff for some \( x \in VAR^+ \) we have \( t_1^w, v^A, t_2^w, v^A \in [x] \) iff \((t_1 = x) \in w \) and \((t_2 = x) \in w \) and as \( w \) is deductively closed \((t_1 = t_2) \in w \) as desired.

**Atomic Formulas:** Let \( P \) be a predicate symbol of arity \( \beta \in \mathcal{P}^A \) and \( t_1, \ldots, t_k \in \text{Term}^+ \) such that \( t_i \) is of sort \( \beta_i \). Then, \( \mathcal{M}^A_\Omega, w \models_{v^A} P(t_1, \ldots, t_k) \) iff \( (t_1^w, v^A, \ldots, t_k^w, v^A) = ([x_1], \ldots, [x_k]) \in \mathcal{I}^A(P, w) \) which in turn holds iff \( P(x_1, \ldots, x_k) \in w \) and as by assumption \((t_i = x_i) \in w \) for \( i = 1, \ldots, k \) this is equivalent to \( P(t_1, \ldots, t_k) \in w \) by deductive closedness.

**Negation:** Assume the truth Lemma for a \( \text{wff} \) \( \phi \), then we have \( \mathcal{M}^A_\Omega, w \models_{v^A} \neg \phi \) iff not \( \mathcal{M}^A_\Omega, w \models_{v^A} \phi \) iff \( \neg \phi \notin w \) by the induction hypothesis. By \( \Box \) we get \( \neg \phi \in w \) as desired.

**Implication:** Assume the truth Lemma for \( \text{wffs} \) \( \phi, \psi \). Then we have \( \mathcal{M}^A_\Omega, w \models_{v^A} \phi \rightarrow \psi \) iff either \( \mathcal{M}^A_\Omega, w \models_{v^A} \neg \phi \) or \( \mathcal{M}^A_\Omega, w \models_{v^A} \psi \) which in turn holds iff either \( \neg \phi \in w \) or \( \psi \in w \) by the induction hypothesis. Then Lemma \( \Box \) yields \( \phi \rightarrow \psi \in w \) as desired.
Universal: We first show that if $M^\Lambda_\Omega, w \models_{\omega} \forall x \phi$ then $\forall x \phi \in w$, so assume truth Lemma for wffs of complexity lower than $\forall x \phi$, and that $M^\Lambda_\Omega, w \models_{\omega} \forall x \phi$. Thus, for all $x$ – variants $v^\Lambda$, $M^\Lambda_\Omega, w \models_{v^\Lambda} \phi$. Now, by the $\forall$ – property there exists variable $y_0$ such that $(\phi(y_0/x) \rightarrow \forall x \phi) \in w$, and as $M^\Lambda_\Omega, w \models_{v^\Lambda} \phi$ holds for all $x$ – variants we can in particular choose $v^\Lambda$ such that $v^\Lambda(x) = v^\Lambda(y_0)$ but then we get by Proposition 1 that $M^\Lambda_\Omega, w \models_{v^\Lambda} \phi(y_0/x)$, and by the induction hypothesis $\phi(y_0/x) \in w$. An application of MP yields $\forall x \phi \in w$ as desired.

For the converse we proceed by contraposition, so assume $\forall x \phi \notin w$. By Lemma 9 we get $\neg \forall x \phi \in w$, and by a contrapositive application of the $\forall$ – property we get $\neg \phi(y_0/x) \in w$ for some variable $y_0$ and thus by the induction hypothesis $M^\Lambda_\Omega, w \models_{v^\Lambda} \neg \phi(y_0/x)$. This means for the specific $x$ – variant $v^\Lambda$ such that $v^\Lambda(x) = v^\Lambda(y_0)$ we get by Proposition 1 that $M^\Lambda_\Omega, w \models_{\alpha^\Lambda} \neg \phi(x)$ and thus $M^\Lambda_\Omega, w \models_{\alpha^\Lambda} \exists x \neg \phi(x)$ or equivalently $M^\Lambda_\Omega, w \models_{\alpha^\Lambda} \neg \forall x \phi(x)$ as desired.

Modal: We first show that if $K_x \phi \in w$ then $M^\Lambda_\Omega, w \models_{v^\Lambda} K_x \phi$, so assume the truth Lemma for wffs of lower complexity than $K_x \phi$, and let $K_x \phi \in w$ for some variable $x$ such that $v^\Lambda(x) = \alpha^\Lambda_x$. If $w' \in \mathcal{W}^\Lambda_\Omega$ is any world such that $(w, w') \in \mathcal{R}^\Lambda(\alpha^\Lambda_x)$ we have by definition that $\phi \in w'$ and thus by the induction hypothesis that $M^\Lambda_\Omega, w' \models_{v^\Lambda} \phi$. As this holds for every such $w'$ we conclude that $M^\Lambda_\Omega, w \models_{\alpha^\Lambda} K_x \phi$ as desired.

We prove the converse by contraposition so assume $K_x \phi \notin w$ for some variable $x$ such that $v^\Lambda(x) = \alpha^\Lambda_x$. By Lemma 9 we thus have $\neg K_x \phi \in w$, or equivalently $P_x \neg \phi \in w$. Now, Proposition 1 yields the existence of some world $w' \in \mathcal{W}^\Lambda_\Omega$ such that $\neg \phi \in w'$ such that $(w, w') \in \mathcal{R}^\Lambda(\alpha^\Lambda_x)$. By the induction hypothesis $M^\Lambda_\Omega, w' \models_{v^\Lambda} \neg \phi$ and thus $M^\Lambda_\Omega, w \models_{\alpha^\Lambda} P_x \neg \phi$ or equivalently $M^\Lambda_\Omega, w \models_{\alpha^\Lambda} \neg K_x \phi$ as desired.

Before moving on, let us ascertain what we have accomplished so far. Given a $\Lambda$ – consistent set $\Omega$ formulated in language $L_{TM}^{\Lambda, \omega, \omega}$ we have produced a model formulated in language $L^+$ satisfying $\Omega$, but in order to be able to argue for completeness via Proposition 2 we have rather to produce a model formulated in $L_{TM}^{\Lambda, \omega, \omega}$. Luckily, this is not really a problem since all we have to do is to restrict the canonical valuation $v^\Lambda_{\lambda, TM}$ and throw away the excess variables from the language. To convince oneself that this is legitimate simply consider why we introduced new variables in the first place; they were never to be used as part of the language but rather as part of the domain. Having acquainted ourselves with the canonical model we shall zoom out; instead of considering, for each $\Lambda$ – consistent set $\Omega$ a model, we shall rather consider the class of such models. This motivates the following definition:

Definition 28. [Canonical Class] Let $\Lambda$ be a normal two-sorted term-modal logic. Then we define the class $M^\Lambda_\Omega$ of canonical models as the set of models $M^\Lambda_\Omega$ for each $\Lambda$ – consistent set $\Omega$.  

24
Which brings us to the main result:

**Theorem 2.** [Canonical Class Theorem] Let $\Lambda$ be a normal two-sorted term-modal logic. Then $\Lambda$ is complete wrt. the class $M^\Lambda$ of canonical models for $\Lambda$.

*Proof.* By Proposition 2 proving completeness of $\Lambda$ wrt. $M^\Lambda$ is simply a question of, given some $\Lambda$-consistent set $\Omega$, producing some element in $M^\Lambda$ on which $\Omega$ is satisfied. By Lemmas 7 and 8 we can extend $\Omega$ to an $\Lambda-MCS$ $w$ with the $\forall$-property, and we can find a model $M^\Lambda_\Omega \in M^\Lambda$ such that $\Omega \subseteq w \in M^\Lambda_\Omega$ which by Lemma 9 gives that $M^\Lambda_\Omega, w |\models v_{\Lambda} \Omega$. As this holds for every $\Lambda$-consistent set $\Omega$ we conclude that $\Lambda$ is complete wrt. the class $M^\Lambda$ of canonical models. \(\square\)

A corollary of this is that $K_{TM}^A, \sigma$ is complete wrt. the class of all $TM^\sigma_A$-frames.

**Corollary 1.** The logic $K_{TM}^A, \sigma$ is complete wrt. the class of all $TM^\sigma_A$-frames.

*Proof.* By Proposition 2 we have to produce, for each $K_{TM}^A, \sigma$-consistent set $\Omega$ a world $w$ in a model $M$ based on some $TM^\sigma_A$-frame and a valuation $v$ such that $M, w |\models v_{K_{TM}^A, \sigma} \Omega$. Now, choose the model to be $M_{\Omega, \sigma}$, the world $w$ to be some $\Omega$-extending $K_{TM}^A, \sigma$-MCS, and the valuation to be $v_{K_{TM}^A, \sigma}$. Now we have $M_{\Omega, \sigma}, w |\models v_{K_{TM}^A, \sigma} \Omega$ by Lemma 9 and we conclude that $K_{TM}^A, \sigma$ is complete wrt. the class of all $TM^\sigma_A$-frames by Theorem 2. \(\square\)

Conclusively, $K_{TM}^A, \sigma$ is sound and complete wrt. the class of all $TM^\sigma_A$-frames.

8 Applications of the Canonical Class Theorem

In this section applications of the Canonical Class Theorem will be explored, and soundness and completeness for the term-modal version of $S4$ will be proved. It is a fact from standard modal logic that the axioms $T$ (i.e. $\forall x (K_x \phi \rightarrow \phi)$), $5$ (i.e. $\forall x (P_x \phi \rightarrow K_x P_x \phi)$) and $4$ (i.e. $\forall x (K_x \phi \rightarrow K_x K_x \phi)$) characterizes the class of reflexive, euclidian, and transitive frames respectively (cf. [1, pp. 128]) and the appropriate term-modal version of these results will now be stated and proved.

**Lemma 10.** [Axiom $T$] Let $x$ be any agent-referring variable, and $\phi$ any wff of $L_{TM}^{A, \sigma}$. The axiom $\forall x (K_x \phi \rightarrow \phi)$ characterizes the class of frames in which $R(\alpha)$ is reflexive for all $\alpha \in A$.

*Proof.* Recall that a relation $R$ on a set $W$ is reflexive iff for all $w \in W$ we have that $wRw$. We show first that if for some frame $F = \langle W, R, DOM \rangle, R(\alpha)$ is reflexive for all $\alpha \in A$ then the axiom $\forall x (K_x \phi \rightarrow \phi)$ is valid on $F$ for any agent-referring variable $x$. Fix some agent-referring variable $x$, world $w \in W$, and wff
Assume further that $\mathcal{R}(x^{w,v})$ is reflexive. It suffices to show that $\mathcal{M}, w \models K_x \phi \rightarrow \phi$ by surjectivity of valuations, so assume $\mathcal{M}, w \models K_x \phi$. By reflexivity we have $(w, w) \in \mathcal{R}(x^{w,v})$ and so by the semantics for $K_x$ that $\mathcal{M}, w \models \phi$ as desired.

The converse is shown by contraposition: Assume $\mathcal{R}(\alpha)$ is non-reflexive for some agent $\alpha \in \mathcal{A}$, i.e. for some world $w \in \mathcal{W}$ we have $(w, w) \notin \mathcal{R}(\alpha)$. Now we can pick the interpretation such that for the resulting model $\mathcal{M}$, we have $\mathcal{M}, w \not\models \phi$ while for all $w' \in \mathcal{W} \setminus \{w\}$ we have $\mathcal{M}, w' \models \phi$. By the semantics of $K_x$ we get that $\mathcal{M}, w \models K_x \phi$ yet $\mathcal{M}, w \not\models \phi$.

Lemma 11. [Axiom 5] Let $x$ be any agent-referring variable, and $\phi$ any wff of $\mathcal{L}_{TM}$. The axiom $\forall x (P_x \phi \rightarrow K_x P_x \phi)$ characterizes the class of frames in which $\mathcal{R}(\alpha)$ is euclidian for all $\alpha \in \mathcal{A}$.

Proof. Recall that a relation $R$ on a set $W$ is euclidian iff. for all $v, u, w \in W$ if $uRv$ and $uRw$ then $vRw$. We show first that if for some frame $\mathcal{F} = (\mathcal{W}, \mathcal{R}, \text{DOM}), \mathcal{R}(\alpha)$ is euclidian for all $\alpha \in \mathcal{A}$ then the axiom $\forall x (P_x \phi \rightarrow K_x P_x \phi)$ is valid on $\mathcal{F}$ for any agent-referring variable $x$, so fix some agent-referring variable $x$, world $w \in \mathcal{W}$, and any wff $\phi$. Assume further that $\mathcal{R}(x^{w,v})$ is euclidian and see that it suffices to show that $\mathcal{M}, w \models P_x \phi \rightarrow K_x P_x \phi$, so assume $\mathcal{M}, w \models P_x \phi$. By assumption there is $w' \in \mathcal{W}$ such that $\mathcal{M}, w' \models \phi$. If $w'$ is the only world accessible from $w$ we have by definition that $\mathcal{M}, w \models K_x P_x \phi$ so assume otherwise, i.e. there is some $u \neq w' \in \mathcal{W}$ s.t. $(w, u) \in \mathcal{R}(x^{w,v})$. As $\mathcal{R}(x^{w,v})$ is euclidian we get that $(u, w') \in \mathcal{R}(x^{w,v})$ and thus $\mathcal{M}, u \models P_x \phi$ and then $\mathcal{M}, w \models K_x P_x \phi$ as desired.

The converse is shown by contraposition, so let $\mathcal{R}(\alpha)$ is non-euclidian for some agent $\alpha \in \mathcal{A}$, i.e. for worlds $w, v, u \in \mathcal{W}$ we have $(w, v) \in \mathcal{R}(\alpha)$ and $(w, u) \in \mathcal{R}(\alpha)$ while $(v, u) \notin \mathcal{R}(\alpha)$. We can now choose our interpretation such that in the resulting model $\mathcal{M}$, we have $\mathcal{M}, v \models \phi$ yet $\mathcal{M}, w \not\models \phi$ for every $w \in \mathcal{W} \setminus \{v\}$. Now, $\mathcal{M}, w \models P_x \phi$ yet $\mathcal{M}, w \not\models K_x P_x \phi$ as desired.

Lemma 12. [Axiom 4] Let $x$ be any agent-referring variable, and $\phi$ any wff of $\mathcal{L}_{TM}$. The axiom $\forall x (K_x \phi \rightarrow K_x K_x \phi)$ characterizes the class of frames in which $\mathcal{R}(\alpha)$ is transitive for all $\alpha \in \mathcal{A}$.

Proof. Recall that a relation $R$ on a set $W$ is transitive iff. for all $v, u, w \in W$ if $vRw$ and $uRw$ then $vRw$. We show first that if for some frame $\mathcal{F} = (\mathcal{W}, \mathcal{R}, \text{DOM}), \mathcal{R}(\alpha)$ is transitive for all $\alpha \in \mathcal{A}$ then the axiom $\forall x (K_x \phi \rightarrow K_x K_x \phi)$ is valid on $\mathcal{F}$ for any agent-referring variable $x$, so fix some agent-referring variable $x$, world $w \in \mathcal{W}$, and any wff $\phi$, and see that it suffices to show that $\mathcal{M}, w \models K_x \phi \rightarrow K_x K_x \phi$, so assume $\mathcal{M}, w \models P_x \phi$. If $v, u \in \mathcal{W}$ is such that $(w, v) \in \mathcal{R}(\alpha)$ and $(v, u) \in \mathcal{R}(\alpha)$ we have by transitivity that $(w, u) \in \mathcal{R}(\alpha)$ and thus by assumption that $\mathcal{M}, u \models \phi$, such that $\mathcal{M}, v \models K_x \phi$ meaning that $\mathcal{M}, w \models K_x K_x \phi$ as desired.

The converse is shown by contraposition, so let $\mathcal{R}(\alpha)$ be non-transitive for some agent $\alpha \in \mathcal{A}$, i.e. for worlds $w, v, u \in \mathcal{W}$ we have $(w, v) \in \mathcal{R}(\alpha)$
and \((v,u) \in R(\alpha)\) while \((w,u) \notin R(\alpha)\). We can now choose an interpretation such that in the resulting model, we have \(\mathcal{M}, u \models \phi\) while \(\mathcal{M}, v \models \phi\) for all \(v \in W \setminus \{u\}\). By construction we have \(\mathcal{M}, w \models K_x \phi\) yet \(\mathcal{M}, w \not\models K_x K_x \phi\) as desired.

**Definition 29.** \([\text{TMS}_\sigma, K4], \text{TMS}_\sigma, K5, \text{and TMS}_\sigma, KT]\) Denote by \(\text{TMS}_\sigma, K4\), \(\text{TMS}_\sigma, K5\), and \(\text{TMS}_\sigma, KT\) the logics resulting from adding to \(K_{\text{TMS}\sigma}^\sigma\) the axiom \(\forall x (K_x \phi \rightarrow K_x K_x \phi)\), \(\forall x (P_x \phi \rightarrow K_x P_x \phi)\), and \(\forall x (K_x \phi \rightarrow \phi)\) respectively for any agent-referring variable \(x\) and wff \(\phi\), and closing the resulting collection of formulas under MP, KG, and Gen.

**Corollary 2.** [Soundness \(\text{TMS}_\sigma, K4\), \(\text{TMS}_\sigma, K5\), \(\) and \(\text{TMS}_\sigma, KT\)] The logic \(\text{TMS}_\sigma, K4\) is sound wrt. the class of transitive \(\text{TMS}_\sigma^\sigma\) – frames, the logic \(\text{TMS}_\sigma, K5\) is sound wrt. the class of euclidian \(\text{TMS}_\sigma^\sigma\) – frames, and the logic \(\text{TMS}_\sigma, KT\) is sound wrt. the class of reflexive \(\text{TMS}_\sigma^\sigma\) – frames.

**Proof.** This follows from Theorem 1 and Lemmas 12, 11 and 10.

**Theorem 3.** [Completeness \(\text{TMS}_\sigma, K4\)] The logic \(\text{TMS}_\sigma, K4\) is complete wrt. the class of transitive \(\text{TMS}_\sigma^\sigma\) – frames.

**Proof.** By Proposition 4, it suffices to produce, given any \(\text{TMS}_\sigma, K4\) – consistent set \(\Omega\), a model based on some frame from the class of all transitive \(\text{TMS}_\sigma^\sigma\) – frames, a world and a valuation satisfying \(\Omega\). Obviously, we are going to choose the model to be \(\mathcal{M}_{\Omega}^{\text{TMS}_\sigma K4}\) the world to be some \(\Omega\) – extending \(\text{TMS}_\sigma, K4\) – MCS \(w\), and the valuation to be the canonical valuation \(v_{\text{TMS}_\sigma, K4}\), such that it follows from Lemma 9 that \(\mathcal{M}_{\Omega}^{\text{TMS}_\sigma K4}, w \models \Omega\). It remains to show that \(\mathcal{M}_{\Omega}^{\text{TMS}_\sigma K4}\) is indeed based on a transitive frame, but cf. Lemma 12, this amounts to showing that the axiom \(\forall x (K_x \phi \rightarrow K_x K_x \phi)\) is satisfied on \(\mathcal{M}_{\Omega}^{\text{TMS}_\sigma K4}\), which is fulfilled by Lemma 12. We conclude that \(\mathcal{M}_{\Omega}^{\text{TMS}_\sigma, K4}\) is based on a transitive \(\text{TMS}_\sigma^\sigma\) – frame and the desired conclusion follows.

The interested reader may note that compactness is a sitting duck at this point - I will however, not make this point explicit since I am already in excess of keystrokes. Recall our brief discussion of Descartes cogito from the introduction; we are now equipped with a logic in which \(\forall y \exists x K_y(x = y)\) is a wff of the language, and we know that it exhibits appropriate behavior as far as the relation between syntax and semantics go. Before turning the key a few words on Hintikka and the logic \(\text{TMS}_\sigma, K4\) are in order.

## 9 Hintikka Revisited

As an answer to Hintikka’s 1962 footnote we can now present the logic \(\text{TMS}_\sigma, K4\). We have, as queried, partitioned terms in those that form well-formed sentences of the kind \(K_i \phi\) and those which do not, and we know that the logic is sound and complete wrt. the class of transitive \(\text{TMS}_\sigma^\sigma\) – frames. Yet, a comment is in order. We have included the 4 – axiom \(\forall x (K_x \phi \rightarrow K_x K_x \phi)\), but the equivalent
axiom in [3] reads \( K_a p \rightarrow K_a K_a p \). Further, Hintikka insists that the agent whom is referred to by \( a \) must know who he is for the axiom to make sense, and formalizes this demand as \( \exists x K_a (x = a) \). As it turns out, if interpreted in the framework presented here, there is a rather nice motivation for taking the axiom \( \exists x K_a (x = a) \) as necessary for \( K_a p \rightarrow K_a K_a p \) for assume we evaluate the latter over the class of transitive frames; it turns out that the axiom is not even valid.

**Proposition 7.** \( [K_a p \rightarrow K_a K_a p] \) is not valid on the class of transitive \( T M^A_\sigma \) – frames] Let \( a \) be any agent-referring constant. Then, the formula \( K_a p \rightarrow K_a K_a p \) is not valid over the class of transitive \( T M^A_\sigma \) – frames.

**Proof.** We show the Proposition by constructing a counter-example. Choose some model \( M \), some world \( w \) in the class of transitive \( T M^A_\sigma \) – frames, some wff \( \phi \), a valuation \( v \), and some agent-referring constant \( a \) such that \( M, w \models_v K_a \phi \). It is certainly possible to choose the above such that for some world \( w' \) we have \((w, w') \in R(a^{w,v}). Now, since constants are non-rigid we can even choose the interpretation such that \( I(a, w) \neq I(a, w') \), and thus if \( t \) is yet another world such that \((w, t) \in R(a^{w,v}) \) we do not have by transitivity that \((w, t) \in R(a^{w,v}) \). We can even choose our model such that \( M, t \not\models_v \phi \), yielding \( M, w' \not\models_v K_a \phi \) and subsequently \( M, w \not\models_v K_a K_a \phi \).

Comparing the above Proposition with the proof of Lemma 12 we see that, unsurprisingly, what goes wrong is an effect of the non-rigidity of constants. In plain English we do not know, that the referent of \( a \) is the same in all worlds and that enables us to construct a counter example. What would be the effect of adding Hintikka’s “Knowing who” – axiom \( \exists x K_a (x = a) \)? Take world \( w \), a valuation \( v \), and model \( M \) from the proof of Proposition 7 and assume \( \exists x K_a (x = a) \) as an axiom. Now, as \( M, w \models_v \exists x K_a (x = a) \) we get by the definition of \( \exists \) that \( M, w \models_v \neg \forall x \neg K_a (x = a) \) and thus by the semantics of \( \forall \) that \( M, w \models_v \neg K_a (x = a) \) for some \( x \) – variant \( v' \) of \( v \). By the semantics of \( K_a \) we get for any world \( w' \) with \((w, w') \in R(a^{w,v'}) \) that \( M, w' \models_{v'} (x = a) \). Denote by \( d \) the referent of \( x \) and note that the consequence of the above is that \( I(a, w') = d \) for any world \( w' \) with \((w, w') \in R(a^{w,v'}) \). This makes way for the last result of this paper:

**Proposition 8.** [Knowing Who] Assuming \( \exists x K_a (x = a) \) as an axiom yields validity of \( K_a p \rightarrow K_a K_a p \) for any agent-referring constant \( a \) on the class of transitive \( T M^A_\sigma \) – frames.

**Proof.** Take world \( w \), valuation \( v \), and model \( M \) based on the class of transitive \( T M^A_\sigma \) – frames and assume \( M, w \models_v K_a p \) for agent-referring constant \( a \) and wff \( p \). If \( w' \) is a world such that \((w, w') \in R(a^{w,v}) \) we get \( M, w' \models_v p \) and if \( t \) is yet another world such that \((w', t) \in R(a^{w',v}) \) we make the observation that

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\(^{13}\)I will not go into Hintikka’s semantics which differs from the kripkean semantics employed throughout the present work.

\(^{14}\)Again, this is formally a translation from Hintikka’s framework into the framework developed here. However, it seems rather harmless.
\( R(a^{w,v}) = R(a^{w',v}) \). By transitivity this yields \((w, t) \in R(a^{w,v})\) and then by assumption \( M, t \models v p \) which in turn implies \( M, w' \models K_a p \) and \( M, w \models K_a K_a p \) as desired.

Hintikka’s motivation for demanding the “Knowing Who” – axiom as prerequisite for his version of the axiom 4 is quite different than the above sketched, but internal to the Kripkean framework employed here Propositions 7 and 8 offers separate justification.

10 Conclusion

I have stated language and syntax for a two-sorted term-modal logic. I have presented an axiomatic system for the logic \( K_{TM^A, \sigma} \), and shown it to be both sound and complete wrt. the class of all \( TM^A_{\sigma} \) – frames. Then I have added the “term-modal” version of axiom 4 and shown the logic \( TM^A_{\sigma}, K4 \) to be both sound and complete wrt. the class of transitive \( TM^A_{\sigma} \) – frames. Lastly, I have discussed the version of axiom 4 that Hintikka puts forward in [3], and suggested a motivation for presupposing the “Knowing Who” – axiom in a Kripkean framework.
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