INTEGRAL LATTICES OF THE SU(2)-TQFT-MODULES

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Abstract. We find explicit bases for naturally defined lattices over certain rings of integers in the SU(2) theory modules of surfaces. We consider the TQFT where the Kauffman’s A variable is a root of unity of order four times an odd prime. As an application, we show that the Frohman Kania-Bartoszynska ideal invariant for 3-manifolds with boundary using the SU(2) theory is equal to the product of the ideals using the 2-theory and the SO(3) theory under a certain change of coefficients.

Introduction

We let \( p \) denote an odd prime or twice an odd prime. Also, we let \( \Sigma \) denote a connected surface of genus \( g \) with a (possibly empty) collection of colored banded points, where the color is an integer \( i \in \{0, 1, \ldots, p - 2\} \).

Based on the integrality results of the Witten-Reshetikhin-Turaev quantum invariants of closed 3-manifolds [MR, M], an integral functor \( \mathcal{S}_p \) is defined in [G1]. This is a functor that associates to a surface \( \Sigma \), a lattice \( \mathcal{S}_p(\Sigma) \) over the cyclotomic ring of integers

\[
\mathcal{O}_p = \begin{cases} 
\mathbb{Z}[A_p], & \text{if } p \equiv -1 \pmod{4}, \\
\mathbb{Z}[\alpha_p], & \text{if } p \equiv 1 \text{ or } 2 \pmod{4},
\end{cases}
\]

here and elsewhere \( A_p, \alpha_p \) are \( \zeta_{2p} \) and \( \zeta_{4p} \) respectively for \( p \geq 3 \).

Gilmer in [G1] showed that these lattices are free in the case of \( p \) is an odd prime and projective in the case \( p \) is twice an odd prime. In [GMW], the authors gave explicit bases for these lattices in genus one and two at roots of unity of odd prime order. Recently, Gilmer and Masbaum announced a basis for \( \mathcal{S}_p(\Sigma) \) and hence they gave an independent proof of freeness in the case \( p \) is an odd prime.

In the 2-theory, the author showed that the lattice \( \mathcal{S}_2(\Sigma) \) is free by constructing an explicit basis. Also he showed that the lattice \( \mathcal{S}_p(S^1 \times S^1) \) is free by constructing two explicit bases in the case \( p \) is twice an odd prime. The surfaces considered in [Q] have no colored points on the boundary. In this paper, we consider the other version of the 2-theory and give an explicit basis for the lattice \( \mathcal{S}_2(\Sigma) \). The goal to consider this other version of the 2-theory is to provide an explicit basis for the lattice \( \mathcal{S}_p(\Sigma) \) based on the results of [GM].

Frohman and Kania-Bartoszynska in [EK] defined an ideal invariant of 3-manifolds with boundary using the SU(2) theory that is not finitely generated by definition. In fact, they make use of another ideal that they defined to give an estimate for this ideal. Later on, Gilmer and Masbaum in [GM] defined an analogous ideal invariant using the SO(3) theory for 3-manifolds with boundary. Also, they showed that this

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ideal is finitely generated by giving a finite set of generators. In this paper, we give a similar result concerning the ideal using the \( SU(2) \) theory. Moreover, we show that this ideal is equal to the product of the ideals using the \( 2 \)- and the \( SO(3) \) theories.

In the first section, we review the integral TQFT-functor that was first introduced in \( [G1] \). The quantization functor for \( p = 2 \) is discussed in the next section, following \( [BHMV3] \). Also in this section, we give a basis for \( S_p^2(\Sigma) \). In the third section, we list some of the results concerning the surface \( S^1 \times S^1 \) stated in \( [GM] [GMW] [Q] \). We reformulate some of the results given in \( [BHMV3] \) concerning the relation between the \( SO(3)- \) and the \( SU(2)-\TQFTs \) in the fourth section to serve our need. The main result will be given in the fifth section. In the next section, we give an explicit basis for the module \( S_p^q(\Sigma) \). In the last section, we give an application of our main result concerning the Frohman Kania-Bartoszynska ideal.

1. The Integral TQFT-functor

We consider the \((2+1)\)-dimensional TQFT constructed as the main example of \( [BHMV3] \) Page. 456 with some modifications. In particular, we use the cobordism category \( \mathcal{C} \) discussed in \( [G1] [GQ] \) where the 3-manifolds have banded links and surfaces possibly have colored points. Hence the objects are oriented surfaces with extra structure (Lagrangian subspaces of their first real homology). The cobordisms are equivalence classes of compact oriented 3-manifolds with extra structure (an integer weight) with banded links sitting inside of them. Two cobordisms with the same weight are said to be equivalent if there is an orientation preserving diffeomorphism that fixes the boundary.

Now, we consider the TQFT-functor \((V_p, Z_p)\) from \( \mathcal{C} \) to the category of finitely generated free \( k_p \)-modules, where

\[
k_p = \begin{cases} 
\mathbb{Z}[A_p, \frac{1}{A_p}], & \text{if } p \equiv -1 \pmod{4}; \\
\mathbb{Z}[\alpha_p, \frac{1}{\alpha_p}], & \text{if } p \equiv 1 \text{ or } 2 \pmod{4}.
\end{cases}
\]

The functor \((V_p, Z_p)\) is defined as follows. \( V_p(\Sigma) \) is a quotient of the \( k_p \)-module generated by all cobordisms with boundary \( \Sigma \), and \( Z_p(M) \) is the \( k_p \)-linear map from \( V_p(\Sigma) \) to \( V_p(\Sigma') \) (where \( \partial M = -\Sigma \sqcup \Sigma' \)) induced by gluing representatives of elements of \( V_p(\Sigma) \) to \( M \) along \( \Sigma \) via the identification map.

If \( M \) is a closed cobordism, then \( Z_p[M] \) is the multiplication by the scalar \( \langle M \rangle_p \) defined in \( [BHMV3] \) Section. 2. This invariant is normalized in two other ways. The first normalization of this invariant is \( I_p(M) = D_p(M) \). Here and elsewhere \( M \) is the 3-manifold \( M \) with a reassigned weight zero, and \( D_p = \langle S^3 \rangle_p^{-1} \).

The second normalization is \( \theta_p(M) = D_p^{\beta_1(M)+1}(M) \), i.e. \( \theta_p(M) = D_p^{\beta_1(M)} I_p(M) \).

If \( \partial M = \Sigma \) and \( M \) is considered as a cobordism from \( \emptyset \) to \( \Sigma \), then \( Z_p(M)(1) \in V_p(\Sigma) \) is denoted by \( [M]_p \) and called a vacuum state and it is connected if \( M \) is connected. Finally, note that \( V_p \) is generated over \( k_p \) by all vacuum states.

The modules \( V_p(\Sigma) \) are free modules over \( k_p \), and carry a nonsingular Hermitian sesquilinear form

\[
(\ ,\ )_\Sigma : V_p(\Sigma) \times V_p(\Sigma) \longrightarrow k_p,
\]
given by

\[
([M_1], [M_2])_\Sigma = \langle M_1 \cup \Sigma - M_2 \rangle_p.
\]
Here $-M$ is the cobordism $M$ with the orientation reversed and multiplying the integer weight by $-1$, and leaving the Lagrangian subspace on the boundary the same.

A standard basis $\{u_\sigma\}$ for $V_p(\Sigma)$ is given (see [BHMV3]) in terms of $p$-admissible colorings $\sigma$ of a banded uni-trivalent graph in a handlebody of genus $g$ whose boundary is $\Sigma$.

Let $\mathcal{C}'$ be the subcategory of $\mathcal{C}$ consisting of the nonempty connected surfaces and connected cobordisms between them.

**Definition 1.1.** For the surface $\Sigma$, we define $S_p(\Sigma)$ to be the $O_p$-submodule of $V_p(\Sigma)$ generated by all connected vacuum states.

If $M : \Sigma \rightarrow \Sigma'$ is a cobordism of $\mathcal{C}'$, then $Z_p(M)([N]_p) = [M \cup \Sigma - N]_p \in S_p(\Sigma')$. Hence we obtain a functor from $\mathcal{C}'$ to the category of $O_p$-modules. These modules are projective as they are finitely generated and torsion-free over Dedekind domains [G1, Theorem. 2.5]. Also, these modules carry an $O_p$-Hermitian sesquilinear form $(\ , \ )_\Sigma : S_p(\Sigma) \times S_p(\Sigma) \rightarrow O_p,$

given by

\[
(\{M_1\}, [M_2])_\Sigma = D_p([M_1], [M_2]) = D_p(M_1 \cup \Sigma - M_2)_p,
\]

The value of this form always lies in $O_p$ by the integrality results for closed 3-manifolds in [MR, M].

If $R \subseteq S_p(\Sigma)$ is an $O_p$-submodule define

\[
R^\sharp = \{v \in V_p(\Sigma) | (r, v)_\Sigma \in O_p, \forall r \in R\},
\]

then we can conclude

\[
R \subseteq S_p(\Sigma) \subseteq S_p^\sharp(\Sigma) \subseteq R^\sharp.
\]

**Definition 1.2.** A Hermitian sesquilinear form on a projective module over a Dedekind domain is called non-degenerate if the adjoint map is injective, and unimodular if the adjoint map is an isomorphism.

For our use, if the matrix of the form has a nonzero (unit) determinant, then the form will be non-degenerate (unimodular) respectively.

All of the above elements $u_\sigma$ lie in $S_p(\Sigma)$ when $p$ is twice an odd prime. This follows from the fact that the quantum integers (denominators of the Jones-Wenzl idempotents) are units in $O_p$ (see [Q, Corollary. 6.4]). An admissible colored uni-trivalent graph [BHMV3] is to be interpreted, here and elsewhere, as an $O_p$-linear combination of links.

We can describe the modules $S_p(\Sigma)$ in terms of ‘mixed graph’ notation in a fixed connected 3-manifold $M$ whose boundary is $\Sigma$. By a mixed graph, we mean a $p$-admissibly uni-trivalent graph whose simple closed curves may be colored $\omega_p$ or an integer from the set $\{0, 1, \ldots, p - 2\}$ where

\[
\omega_p = D_p^{-1} \sum_{i=0}^{d_p-1} (-1)^i [i + 1]_p e_i.
\]

Using the surgery axiom (S2) in [BHMV3], we can choose this fixed 3-manifold to be a handlebody whose boundary is $\Sigma$. Thus we have
Proposition 1.3. A mixed graph in a connected 3-manifold with boundary \( \Sigma \) represents an element in \( S_p(\Sigma) \). Moreover, \( S_p(\Sigma) \) is generated over \( O_p \) by all the elements given by a mixed graph in a fixed handlebody whose boundary is \( \Sigma \) with the same genus.

Proof. The first statement follows from that fact that \( V_p \) satisfies the second surgery axiom. The second statement follows from the fact that every 3-manifold with boundary \( \Sigma \) is obtained by a sequence of 2-surgeries to a handlebody of the same boundary and the definition of \( S_p(\Sigma) \). \( \square \)

2. The TQFT theory for \( p = 2 \)

To relate between the \( SU(2) \) and the \( SO(3) \) theories, we need to discuss the 2-theory.

We start by recalling the ring used in this theory and its ring of integers:

\[ k_2 = \mathbb{Z}[\alpha, \frac{1}{2}] \quad \text{and} \quad O_2 = \mathbb{Z}[\alpha]. \]

The surgery element for this theory is \( \omega_2 = \sqrt{2} \Omega_2 \) where \( \Omega_2 = 1 + z \). One has \( D_2 = \sqrt{2} \), and \( \kappa_2 = \zeta_8 \). Therefore, the invariant of a closed connected 3-manifold \( M \) that is obtained by doing surgery on \( S^3 \) along the link \( L \) is given by

\[
\langle M \rangle_2 = (-2)^{-n} \frac{1}{\sqrt{2}} \kappa_2^{\sigma(L)} < L(\omega_2) >,
\]

where \( < L(\omega_2) > \) denotes the Kauffman bracket of the link \( L \) colored \( \Omega_2 \) and \( n \) is the number of components of the banded link in \( M \). From this formula, we can easily verify that

\[
\langle M_1 \sharp M_2 \rangle_2 = \sqrt{2} \langle M_1 \rangle_2 \langle M_2 \rangle_2.
\]

Now this invariant \( \langle M \rangle_2 \) defined in [BHMV3, Section. 2] is involutive and extended to be multiplicative, hence (by [BHMV3, Proposition. 1.1]) there exits a unique cobordism generated quantization functor that extends \( \langle \rangle_2 \) which is denoted by \( (V_2, Z_2) \). The modules \( V'_2(\Sigma) \) carry a Hermitian sesquilinear form defined as follows.

\[
( , )_\Sigma : V'_2(\Sigma) \times V'_2(\Sigma) \rightarrow k_2,
\]

given by

\[
\langle [M_1], [M_2] \rangle_\Sigma = \langle M_1 \cup_{\Sigma} -M_2 \rangle_2.
\]

By [BHMV3] 1.5 and 6.3, \( V'_2(S^1 \times S^1) \) is generated by two elements, named 1 and \( z \), each of which is a solid torus where the core is colored either 0 or 1. If we restrict this theory to the category of nonempty connected objects and connected cobordisms between them, then we have an integral cobordism theory as before. This follows from the fact \( \langle \rangle_2 \) is integral as stated in the proof of [MR, Theorem. 1.1].

Definition 2.1. We define \( S'_2(\Sigma) \) to be the \( O_2 \)-submodule of \( V'_2(\Sigma) \) generated by all connected vacuum states, and we define an \( O_2 \)-Hermitian sesquilinear form on \( S'_2(\Sigma) \) given by \( ( , )_\Sigma = \sqrt{2} \langle , \rangle_\Sigma \).

The above basis for \( V'_2(S^1 \times S^1) \) does not generate \( S'_2(S^1 \times S^1) \). The following theorem gives a basis for \( S'_2(S^1 \times S^1) \).
Theorem 2.2. We have that $\mathcal{B} = \{\omega_2, t(\omega_2)\}$ is a basis for $\mathcal{S}_2^1(S^1 \times S^1)$, and the form is unimodular on $\mathcal{S}_2^1(S^1 \times S^1)$. Moreover, the matrix of the form defined in the previous definition in terms of $\mathcal{B}$ is given by

$$
\begin{pmatrix}
\sqrt{2} & 1 \\
\frac{1+i}{\sqrt{2}} & \frac{1-i}{\sqrt{2}}
\end{pmatrix}
$$

Proof. Let $\omega_2$ and $t(\omega_2)$ stand for the elements in the Kauffman skein module of the solid torus where the core is colored $\omega_2$ and $t(\omega_2)$ respectively. From the definition we know that these two elements lie in $\mathcal{S}_2^1(S^1 \times S^1)$, hence $\mathcal{W} = \text{Span}_{\mathbb{C}} \mathcal{B} \subseteq \mathcal{S}_2^1(S^1 \times S^1)$. The matrix of the form $(\ , \ )_{S^1 \times S^1}$ in terms of $\mathcal{B}$ is given by

$$
\begin{pmatrix}
\sqrt{2} & 1 \\
\frac{1+i}{\sqrt{2}} & \frac{1-i}{\sqrt{2}}
\end{pmatrix}
$$

So the form restricted on $\mathcal{W}$ has a unit determinant. Hence $\mathcal{W} = \mathcal{W}^\perp$. Using equation (2.2), we get that $\mathcal{W}$ is all of $\mathcal{S}_2^1(S^1 \times S^1)$. In conclusion, $\{\omega_2, t(\omega_2)\}$ is a basis for $\mathcal{S}_2^1(S^1 \times S^1)$.

Theorem 2.3. Let $\Sigma_{g,2n}$ be a surface of genus $g$ and $2n$ points colored 1. Then the basis elements of $\mathcal{S}_2^1(\Sigma_{g,2n})$ is given by an admissible colorings of a banded uni-trivalent graph by 0 or 1 that meets the colored points nicely and whose loops are colored by either $\omega_2$ or $t(\omega_2)$.

Proof. Let $(S^1 \times S^2)_{ij}$ denote $S^1 \times S^2$ formed by gluing two solid tori whose cores are colored $t_i^j(\omega_2)$, and $t_i^j(\omega_2)$ where $i, j \in \{0, 1\}$. Also, let $H_1, H_2$ be any two elements. Let us look at the pairing

$$(H_1, H_2)_\Sigma = \sqrt{2}(H_1, H_2)_\Sigma
$$

$$
= \sqrt{2}\sum_{k=1}^g ((S^1 \times S^2)_{ijk})_2
$$

$$
= \sqrt{2} \prod_{k=1}^g ((S^1 \times S^2)_{ijk})_2
$$

$$
= \prod_{k=1}^g ((S^1 \times S^2, t_i^j(w_2) \cup t_j^k\omega_2))_2.
$$

With a natural order, the matrix of the form in terms of this set is given by $\bigotimes^g \mathcal{B}$ ($\mathcal{B}$ is defined in the proof of the previous theorem). This implies that the determinant of this form is a unit in $\mathcal{O}_2$. By a similar argument as in the proof of Theorem (2.2), the module generated by this set is all of $\mathcal{S}_2^1(\Sigma_{g,2n})$.

Corollary 2.4. The same set as above forms a basis for $\mathcal{S}_2^1(\Sigma_{g,2n})$.

Proof. The result follows since the determinant of the form $(\ , \ )_{\Sigma}$ is a unit in $\mathcal{O}_2$.

3. Basis for $\mathcal{S}_p(S^1 \times S^1)$

In this section, we consider the surface $S^1 \times S^1$ with no colored points. Moreover, we let $d_p = [(p - 1)/2]$.

In [GMW], the authors gave two explicit bases for $\mathcal{S}_p(S^1 \times S^1)$ where $p$ is an odd prime. To serve our need, we state the first basis.
Theorem 3.1. The set \( \{1, v, \ldots, v^{d_p-1}\} \) forms a basis for \( S_p(S^1 \times S^1) \), where 
\[ v = \frac{z+2}{1+A_p}. \]

Later on, more bases for the same lattice were given in \( \[Q\] \) that were needed in the construction of the main result.

Theorem 3.2. The set \( \{t^{2i}(\omega_p) \mid 0 \leq i \leq d_p - 1\} \) and \( \{t^{2i+1}(\omega_p) \mid 0 \leq i \leq d_p - 1\} \) form bases for \( S_p(S^1 \times S^1) \).

Also, an explicit basis for the lattice in the case where \( p \) is twice an odd prime is given in \( \[Q\] \).

Theorem 3.3. The set \( \{t^{i+\delta_p}(\omega_p) \mid 0 \leq i \leq d_p - 1\} \) forms a basis for \( S_p(S^1 \times S^1) \), where

\[
\delta_i = \begin{cases} 
0, & \text{if } i + p \equiv 2 \text{ or } 3 \pmod{4}; \\
1, & \text{if } i + p \equiv 0 \text{ or } 1 \pmod{4}.
\end{cases}
\]

From now on, we let \( p = 2r \) where \( r \) is an odd prime. As a corollary of the proof of the above theorem, we obtain:

Corollary 3.4. The isomorphism \( F \) in theorem (4.4) induces an isomorphism between \( S_p(S^1 \times S^1) \) and \( S'_2(S^1 \times S^1) \otimes_{k_p} S_r(S^1 \times S^1) \).

Therefore, we obtain elements \( u^{ij} \) whose images are equal to \( t^i(\omega_2) \otimes (z+2)^j \) for any \( 0 \leq j \leq d_r - 1 \) and \( i = 0, 1 \). It is easy to see that

\[
u^{0j} = \sum_{i=0}^{d_p-1} k_i t^{i+\delta_p}(\omega_p), \quad i+\delta_p \equiv 0 \pmod{4}
\]

and

\[
u^{1j} = \sum_{i=0}^{d_p-1} k'_i t^{i+\delta_p}(\omega_p), \quad i+\delta_p \equiv 1 \pmod{4}
\]

for \( k_i, k'_i \in \mathcal{O}_p \) such that \( F(u^{ij}) = t^i(\omega_2) \otimes (z+2)^j \).

As \( t^i(\omega_2) \otimes (z+2)^j \) is divisible by \( (1+\zeta_p)^j \) in \( S'_2(S^1 \times S^1) \otimes_{k_p} S_r(S^1 \times S^1) \), then \( u^{ij} \) is divisible by \( (1+\zeta_p)^j \) in \( S_p(S^1 \times S^1) \). Now we define new basis for \( S_p(S^1 \times S^1) \) that will be used in our main result.

Theorem 3.5. The set \( \{u^{ij}/(1+\zeta_p)^j \mid 0 \leq j \leq d_r - 1, i = 0, 1\} \) forms a basis for \( S_p(S^1 \times S^1) \).

Also, there is a special element \( u \) in \( S_p(S^1 \times S^1) \) that maps to \( 1 \otimes v \in S'_2(S^1 \times S^1) \otimes_{k_p} S_r(S^1 \times S^1) \). The following lemma gives the explicit formula for \( u \).

Lemma 3.6. \( u = \frac{v^2}{1+\zeta_p} \).
Proof. From the above isomorphism, there is only one element that maps to \(1 \otimes v\). Now it is enough to prove that
\[
F\left(\frac{e_{r-3} + 2}{1 + \zeta_p}\right) = 1 \otimes v.
\]
\[
F\left(\frac{e_{r-3} + 2}{1 + \zeta_p}\right) = \frac{1}{1 + \zeta_p} (F(e_{r-3}) + F(2))
= \frac{1}{1 + \zeta_p} (1 \otimes z + 1 \otimes 2)
= 1 \otimes \frac{z + 2}{1 + \zeta_p}
= 1 \otimes v.
\]
where the second equality \((F(e_{r-3}) = 1 \otimes z)\) follows from the fact \(z^{2k} = 1\) in the 2-theory and the following figure.

\[\begin{array}{c}
r-3 = 1 \\
< 1 > \\
< r-2, r-3, 1 > \\
= 1
\end{array}\]

**Figure 1.** Changing the color on a loop from \(r-3\) to 1 in the \(SO(3)\) theory

4. Relating the \(SU(2)\) and the \(SO(3)\) Theories

The results of this section are slight variations of results of [BHMV3 Section. 6] and [BHMV2 Section. 2]. The ring \(k_p\) is not exactly the same as the ring denoted this way in [BHMV3].

We consider the ring \(k_p\) as a \(k_2\) (or a \(k_r\))-module via the homomorphisms defined below that are slight variation of the maps defined in [BHMV2 Section. 2]. We list the results that we use later (see [Q Section. 6] for proofs and details).

**Lemma 4.1.** There are well-defined ring homomorphisms \(i_r : k_2 \rightarrow k_p\), \(j_r : k_r \rightarrow k_p\) given by
\[
i_r(\alpha_2) = \alpha_p^{-r^2}, \quad j_r(\alpha_r) = \alpha_p^{1+r^2} \text{ for } r \equiv 1 \pmod{4}, \text{ and }
\]
\[
j_r(A_r) = A_p^{1+r^2} \text{ for } r \equiv -1 \pmod{4}.
\]
The above ring homomorphisms provide a \( k_p \)-module structure on any \( k_2 \)-module or \( k_r \)-module. We let \( V_2(\Sigma) \) or \( V_r(\Sigma) \) be the \( k_p \)-module obtained in this way. We give a relation between \( V_2' \), \( V_r \), and \( V_2r \) for any surface \( \Sigma \), but before that we need the following slight reformulation of [BHMV2, Theorem. 2.1].

**Theorem 4.2.** For any closed 3-manifold \( M \) with possibly a banded link sitting inside of it we have,

\[
I_{2r}(M) = i_r(I_2'(M))j_r(I_r(M)).
\]

We let \( \kappa_n \) to be an element that plays the role of \( \kappa_3 \) in [BHMV3]. We define this element as follows:

\[
\kappa_n = \begin{cases} 
\alpha_{4n}^{-6 \frac{n(n+1)}{2}}, & \text{if } n \text{ is an odd prime;} \\
-\alpha_{4n}^{-6 \frac{n(n+1)}{2}}, & \text{if } n \text{ is twice an odd prime.}
\end{cases}
\]

Changing the weight by one multiplies the invariant \( \langle \rangle_n \) by \( \kappa_n \).

**Lemma 4.3.** For the above ring homomorphisms. We have

\[
\kappa_p = i_r(\kappa_2)j_r(\kappa_r)
\]

**Theorem 4.4.** There is a natural \( k_p \)-isomorphism \( F : V_p(\Sigma) \to V_2(\Sigma) \otimes_{k_p} V_r(\Sigma) \) such that

\[
F([M]_p) = [M_2'] \otimes_{k_p} [M]_r,
\]

where \( M \) is a 3-manifold with banded link (but not linear combination of links) sitting inside of it.

**Lemma 4.5.** Let \( V, W \) be free modules over an integral domain \( R \) (with involution) equipped with Hermitian sesquilinear forms \( \langle , \rangle_V, \langle , \rangle_W \), and let \( F : V \to W \) be a form-preserving linear map. Let \( (V, \langle , \rangle_V) \) be the quotient of \( V \) by the radical of \( \langle , \rangle_V \). Suppose that \( \langle , \rangle_W \) is non-degenerate. Suppose either that \( F \) is surjective, or \( V \) and \( W \) are free of finite rank and \( \langle , \rangle_V \) is unimodular and furthermore that \( \text{rank}(W) \leq \text{rank}(V) \). Then \( F \) induces an isometry \( \langle , \rangle_V \to \langle , \rangle_W \).

5. Basis for \( S_p(\Sigma) \)

We are ready now to state our main result concerning basis of \( S_p(\Sigma) \). The basis is given in terms of a \( p \)-admissible colorings of a fixed lollipop tree that was first introduced in [GM, Section. 3].

Let \( H_g \) be a fixed handlebody whose boundary is \( \Sigma \) with a fixed collection of colored banded points in \( \Sigma \) denoted by \( l(\Sigma) \). Also, let \( G \) be any uni-trivalent banded graph having the same homotopy type as the handlebody \( H_g \) which meets \( \Sigma \) at \( l(\Sigma) \). We review the definition of a lollipop tree.

**Definition 5.1.** Let \( G \) be as above, and \( s \) be the number of colored points in \( l(\Sigma) \). Then \( G \) is a lollipop tree if it satisfies:

1. \( G \) has exactly \( g \) loop edges, and the complement of the loop edges in \( G \) is a tree and denoted by \( T \).
2. If \( s > 0 \), then there exist a single edge that plays the role of trunk for a tree and this edge is called a trunk edge.
We describe a special coloring of the above lollipop graph $G$. The set of permissible colors is \( \{0, 2, \ldots, d_p - 2\} \). We call an edge that is incident to a loop edge a stick edge and ordinary otherwise. The stick edges have always even colors denoted by \( 2a_1, 2a_2, \ldots, 2a_y \), and by \( c_1, c_2, \ldots \) for the ordinary edges. Here

\[
0 \leq a_i \leq d_r - 1.
\]

The case \( g = 2, s = 0 \) is a special case as we count the stick edge twice with \( a_1 = a_2 \). The color of the trunk edge is even denoted by \( 2e \), where \( e \) satisfies Eq. (5.1).

Using \[GMW\] Lemma (8.2), we can use small colors for the loop edges that are denoted by \( a_i + b_i \) for the loop edge incident to the stick edge colored \( 2a_i \). Moreover, we have

\[
0 \leq b_i \leq d_r - 1 - a_i.
\]

Coloring the graph \( G \) by \((a, b, c)\), where \( a = (a_1, a_2, \ldots, a_y) \), \( b = (b_1, b_2, \ldots, b_y) \), and \( c = (c_1, c_2, \ldots) \) that satisfy Eqs. (5.1) and (5.2) and such that the pair \((2a, c)\) is an \( r \)-admissible coloring of \( T \) extending the coloring of \( l(\Sigma) \), gives a vector \( g(a, b, c) \) in \( V_p(\Sigma) \). Each such vector represents a linear combination over \( \mathcal{O}_p \) of skein elements in the handlebody \( H_g \) by replacing the edges of \( G \) with the appropriate Jones-Wenzl idempotents. We define a new vector \( \hat{g}(a, b, c) \) by replacing \([i]\) by \((-1)^{i+1}[i] \) and changing the color on each loop colored \( 1 \) by a loop colored \( r - 3 \) in the above linear combination.

The basis for \( \mathcal{S}_p(\Sigma) \) will be denoted by \( \mathcal{B} \). It consists of elements indexed by \((a, b, c)\) denoted by \( b(a, b, c) \). They are defined as follows:

\[
b(a, b, c) = h^{-\sum_{i=1}^{n} b_i - \left\lceil \frac{1}{2}(-e + \sum_{i} a_i) \right\rceil} u_1^{m_1 b_1} \ldots u_y^{m_y b_y} \hat{g}(a, b, c),
\]

for \( m_i = 0, 1 \) and \( h = 1 + \zeta_p \).

The above multiplication is defined since the Kauffman bracket skein module is a module over the absolute skein module by stacking elements on top of each other.

**Theorem 5.2.** \( \mathcal{B} \) forms a basis for \( \mathcal{S}_p(\Sigma) \).

The rest of this section will be devoted to prove this result. The proof goes in two steps: by showing that the elements of \( \mathcal{B} \) lie in \( \mathcal{S}_p(\Sigma) \) and then by showing that they generate it. Hence we conclude that \( \mathcal{S}_p(\Sigma) \) is a free lattice.

We proceed in the proof by quoting the 3-ball lemma whose proof consists of a series of lemmas that can be found in \[GM\]. We let \( K_{\mathcal{O}_p}(D^3, l_{n,2}) \) denote the Kauffman bracket skein module of \( D^3 \) relative \( n \) points in \( S^2 = \partial D^3 \) colored 2. This result is needed to prove that \( b(a, b, c) \) lie in \( \mathcal{S}_p(\Sigma) \).

**Theorem 5.3.** (\[GM\],3-ball lemma). If \( n \) is even, \( K_{\mathcal{O}_p}(D^3, l_{n,2}) \) is generated by skein elements which can be represented by a collection of \( n/2 \) disjoint arcs colored 2. If \( n \) is odd, \( K_{\mathcal{O}_p}(D^3, l_{n,2}) \) is generated by skein elements which can be represented by a collection of \((n - 3)/2 \) disjoint arcs colored 2 union one \( Y \) shaped component also colored 2.

**Remark 5.4.** The proof given in \[GM\] will go through since the conditions of \[GM\] Remark (5.6) hold for the \( SU(2) \) theory.

We quote some notation and terminology from \[GM\] for the next lemma. Let \( \Sigma_g \) denote the boundary of the handlebody \( H_g \) of genus \( g \) with no colored points. The
vector \( \hat{g}((1, 1), (0, 0)) \in \mathcal{S}_p(\Sigma_2) \) is called an eyeglass, and the vector \( \hat{g}((1, 1), (0, 0, 0)) \in \mathcal{S}_p(\Sigma_3) \) is called a tripod.

**Lemma 5.5.** The eyeglass and the tripod are divisible by \( h \) in \( \mathcal{S}_p \).

**Proof.** We let \( e_{r-3}^i \) denotes \( e_{r-3} \) that encloses the \( i \)th hole. Also, we let \( e_{r-3}^{ij} \) denotes \( e_{r-3} \) that encloses only the \( i \)th and \( j \)th holes. If we want these curves colored \( u \), we just change the \( e_{r-3} \) to a \( u \). Now we expand \( \hat{g}((1, 1), (0, 0)) \) linearily and we obtain:

\[
\hat{g}((1, 1), (0, 0)) = e_{r-3}^{12} - \frac{1}{2} e_{r-3}^{12} e_{r-3}^{12} = (hu^{12} - 2) - \frac{1}{2} (hu^1 - 2)(hu^2 - 2) = hu^{12} - [2]^{-1}(h^2 u^1 u^2 - 2hu^1 - 2hu^2 + 4 + 2[2]) = hu^{12} - [2]^{-1}(h^2 u^1 u^2 - 2hu^1 - 2hu^2 + 2\zeta h^2)
\]

The result follows as \([2]\) is a unit in \( \mathcal{O}_p \) and the \( u \)-graphs lie in \( \mathcal{S}_p \). The divisibility of the tripod is proved in a similar way. \( \square \)

**Proposition 5.6.** The vectors \( b(a, b, c) \) lie in \( \mathcal{S}_p(\Sigma) \).

**Proof.** It is enough to consider the case where \( b = 0 \) since \( b(a, b, c) \) is obtained from \( b(a, 0, c) \) by doing surgery along simple loops. So we need to show that \( b(a, 0, c) \) is divisible by \( h^{-\left(\frac{1}{2}(e+\Sigma, a)\right]} \).

We apply the same procedure given in the [GM Proposition 6.2] to the colored graph representing \( g(a, 0, c) \). Thus \( \hat{g}(a, 0, c) \) is represented as a linear combination over \( \mathcal{O}_p \) of diagrams with \( h^{-\left(\frac{1}{2}(e+\Sigma, a)\right]} \) eyeglasses and tripods as above. Using the previous lemma, we obtain the required result. \( \square \)

**Proposition 5.7.** For the map \( F \) defined in Theorem (4.4), we have

\[
F(b(a, b, c)) = H_{i_1...i_g} \otimes k_p b(a, b, c)
\]

where \( H_{i_1...i_g} \) is a basis element for \( \mathcal{S}_p(\Sigma) \) described in Theorem (4.4) and \( b(a, b, c) \) is a basis element for \( \mathcal{S}_p(\Sigma) \) defined in [GM Section 4].

To obtain a proof for the above proposition, we apply the map \( F \) to each term of the linear combination that represents \( b(a, b, c) \). Hence we conclude that \( F(B) \) generates \( \mathcal{S}_p(\Sigma) \otimes k_p \mathcal{S}_p(\Sigma) \), but it is clear that \( F(\mathcal{S}_p(\Sigma)) \subseteq \mathcal{S}_p(\Sigma) \otimes k_p \mathcal{S}_p(\Sigma) \). Therefore, we conclude that \( B \) is a basis for \( \mathcal{S}_p(\Sigma) \)

**Corollary 5.8.** The isomorphism \( F \) in theorem (4.4) induces an isomorphism between \( \mathcal{S}_p(\Sigma) \) and \( \mathcal{S}_p(\Sigma) \otimes k_p \mathcal{S}_p(\Sigma) \).

6. **Basis for \( \mathcal{S}_p(\Sigma) \)**

The basis for \( \mathcal{S}_p(\Sigma) \) will be denoted by \( B^\Sigma \). The basis elements are indexed by the same set as before and denoted by \( B^\Sigma(a, b, c) \). Using the algebra/module structure described above, we define these elements as follows:

\[
(6.1) \quad B^\Sigma(a, b, c) = h^{-\sum_{i=1}^g b_i - \left[\frac{1}{2}(e+\Sigma, a)\right]} u_1^{m_1 b_1} \ldots u_g^{m_g b_g} \hat{g}(a, b, c).
\]

**Theorem 6.1.** \( B^\Sigma \) forms a basis for \( \mathcal{S}_p^\Sigma(\Sigma) \).
Proof. The isomorphism $F$ induces a module isomorphism

$$F^\sharp : (S'_2(\Sigma) \otimes_{k_p} S_r(\Sigma))^\sharp \longrightarrow S'_p(\Sigma),$$

where $F^\sharp(H_{i_1\ldots i_g} \otimes_{k_p} b(a, b, c)) = b^\sharp(a, b, c)$.

Finally, we need to show that $H_{i_1\ldots i_g} \otimes_{k_p} b^\sharp(a, b, c)$ is a basis for the lattice $(S'_2(\Sigma) \otimes_{k_p} S_r(\Sigma))^\sharp$.

Let $\mathbb{G}$ denote the free $\mathcal{O}_p$-lattice spanned by $H_{i_1\ldots i_g} \otimes_{k_p} b(a, b, c)$. Also, let $\mathbb{W}$ denote the free $\mathcal{O}_p$-lattice spanned by $H_{i_1\ldots i_g} \otimes_{k_p} b^\sharp(a, b, c)$.

We define the index $[\mathbb{L}^\sharp : \mathbb{L}]$ of an inclusion of free lattices $\mathbb{L} \subseteq \mathbb{L}'$ of the same rank over $\mathcal{O}_p$, to be the determinant up to units in $\mathcal{O}_p$ of a matrix representing a basis for $\mathbb{L}$ in terms of a basis for $\mathbb{L}'$.

Proposition 6.2. $[\mathbb{G}^\sharp : \mathbb{G}] = [\mathbb{W} : \mathbb{G}]$.

Proof. With a natural order, the matrix representing the basis for $\mathbb{G}$ in terms of the basis for $\mathbb{W}$ is just the tensor product of two matrices: the first is the identity matrix and the second is the matrix representing the elements $b(a, b, c)$ in terms of the elements $b^\sharp(a, b, c)$ say $A$. Therefore, $[\mathbb{W} : \mathbb{G}] = \det(A)$. In a similar manner, we show that $[\mathbb{G}^\sharp : \mathbb{G}] = \det(A)$. \qed

Proposition 6.3. There is an isomorphism between $\mathbb{W}$ and $(S'_2(\Sigma) \otimes_{k_p} S_r(\Sigma))^\sharp$.

Proof. The result follows since $\mathbb{W} \subseteq (S'_2(\Sigma) \otimes_{k_p} S_r(\Sigma))^\sharp$ and the fact $[\mathbb{G}^\sharp : \mathbb{G}] = [\mathbb{W} : \mathbb{G}]$. \qed

7. The Frohman Kania-Bartoszynska ideal

This ideal was first introduced by Frohman and Kania-Bartoszynska in \cite{FK}.

Definition 7.1. Let $N$ be a 3-manifold with boundary, we define $\mathcal{J}_p(N)$ to be the ideal generated over $\mathcal{O}_p$ by

$$\{I_p(M)\} \text{ where } M \text{ is a closed connected 3-manifold containing } N\}.$$

The importance of this ideal is in being an invariant of 3-manifolds with boundary and an obstruction to embedding as stated below (see \cite{FK} for more details).

Proposition 7.2. The ideal $\mathcal{J}_p$ is an invariant of oriented 3-manifolds with boundary.

Proposition 7.3. If $N_1$, $N_2$ are an oriented compact 3-manifolds, and $N_1$ embeds in $N_2$, then $\mathcal{J}_p(N_2) \subseteq \mathcal{J}_p(N_1)$.

Remark 7.4. Frohman and Kania-Bartoszynska defined this ideal using the $SU(2)$ theory. Afterward, Gilmer defined this ideal using the $SO(3)$ theory and the 2-theory.

In general, it is not easy to compute this ideal because we have infinitely many closed connected 3-manifolds that contains $N$. Following his work with Masbaum in the case $p$ an odd prime, Gilmer observed that $\mathcal{J}_p(N)$ is finitely generated based on his result that $S_p(\Sigma)$ is finitely generated in the case $p$ twice an odd prime as well. We give a finite set of generators for this ideal as an application of our main theorem in the case $p$ is twice an odd prime.
**Theorem 7.5.** Let $N$ be an oriented 3-manifold with boundary $\Sigma$, then $J_p(N)$ is finitely generated by all scalars $([N], b)_{\Sigma}$ as $b$ varies over a basis for $S_p(\Sigma)$.

**Remark 7.6.** This result is analogous to [GM, Theorem 16.5].

Finally, a good question would be: “Is there a relation between the Frohman Kania-Bartoszynska ideals using the SU(2)- and the SO(3) theories?” The answer is given by the following theorem.

**Theorem 7.7.** Let $N$ be an oriented compact 3-manifold with boundary. Then we have

$$J_p(N) = i_\tau(J_2(N))j_\tau(J_r(N)),$$

where $i_\tau$ and $j_\tau$ are defined as before.

**Proof.** The result follows as a consequence of corollary [5.8]. □

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