Piecewise Hereditary algebras of Dynkin and extended Dynkin type

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Abstract

We present a study on the description of incidence algebras that are piecewise hereditary, which we denominate Phia algebras.

We describe the quiver with relations of the Phia algebras of Dynkin type and introduce a new family of Phia algebras of extended Dynkin type, which we call ANS family, in reference to Assem, Nehring, and Skowroński. In this description, the important method was the one of cutting sets on trivial extensions, inspired by this we made of a computer program which shows exactly the cutting sets on the given trivial extension that result on incidence algebras.

incidence algebra, piecewise hereditary algebra, trivial extension.

1 Introduction

Throughout the paper, $K$ denotes a fixed algebraically closed field. By an algebra we will mean a finite dimensional basic associative $K$-algebra, modules are always finite dimensional right module.

Incidence algebras were introduced in the mid-1960s as a natural way of studying some combinatorial problems. In this work, we focus the study on incidence algebras $K\Delta$ associated with a finite poset $\Delta$ over $K$. 
The description of algebras is one of the most important problems in the representation of finite-dimensional algebras. In the 1980s, Assem and Happel began to characterize the piecewise hereditary algebras, see [1]. Many mathematicians contributed to the description of piecewise hereditary algebras such as Ringel [2], Keller [3], Fernandez [4], Zacharia [5], among others.

In the article [6], we consider the incidence algebras that are piecewise hereditary, which we called Phia algebras, piecewise hereditary incidence algebras. The purpose of this paper is to give a description of some classes of the Phia algebras. We describe the Phia algebras through their quivers with relations. In addition, given algebras $A$ and $B$, we use the notation $A \cong^\prime B$ to say that $A$ is derived equivalently to $B$, that is, $\mathcal{D}^b(A) \cong \mathcal{D}^b(B)$, as triangulated categories.

The paper is organized as follows: Section 2 is devoted to fixing the notation and we briefly recall some definitions and results. In section 3 we characterize the Phia algebras of type $A_n$ and the Phia algebras of type $\tilde{A}_n$. The sections 4 and 5 are dedicated to the description of Phia algebras such that $K\Delta \cong^\prime KQ$ where $\overline{Q} = \mathbb{D}_n$ and $\overline{Q} = \mathbb{E}_6$, respectively.

In sections 6 and 7 we give a presentation by quivers with relations of Phia algebras such that $K\Delta \cong^\prime KQ$ where $\overline{Q} = \mathbb{E}_7$ and $\overline{Q} = \mathbb{E}_8$, respectively. In section 8 we introduce a family Phia algebras of extended Dynkin type called the ANS family, in reference to Assem, Nehring and Skowroński. In this description of the Phia algebras, the important method was that of the cutting sets in trivial extensions, which inspired the elaboration of a computer program that produces exactly the cutting sets in the given trivial extension that result in incidence algebras. We emphasize that we are particularly interested in Phia algebras whose associated quivers admit as underlying graphs that are not tree type, otherwise, the algebra considered will be hereditary.

2 Preliminaries

In this section we will recall some definitions which we need. For more details in the subject of incidence algebras we refer to [7], [8], for the subject of piecewise hereditary algebras we refer to [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21].

There are several equivalent ways of defining incidence algebras, next we describe one way for incidence algebras of finite posets.

**Definition 2.1** (incidence algebra). Let $(\Delta, \preceq)$ be a poset with $n$ elements. The incidence algebra $K\Delta$ can be viewed as a quotient of the path algebra of the following quiver $Q$. The vertex set $Q_0$ is formed by the vertices associated
with the points of poset \( \Delta \) and the set of arrows \( Q_1 \) is described in the following way: there is exactly one arrow \( \alpha \) of a vertice \( a \) to a vertice \( b \), whenever \( a \preceq b \) and there is no \( a \preceq c \preceq b \), with \( c \neq a \) and \( c \neq b \).

Let \( I \) be the ideal generated by all commutativity relations \( \gamma - \gamma' \), with \( \gamma \) and \( \gamma' \) parallel paths. The incidence algebra \( K \Delta \) is \( KQ/I \).

Next, we begin a quick review of the definition of derived category from an abelian category \( \mathcal{A} \). First, we recall that the category \( \mathcal{K}(\mathcal{A}) \) is the quotient of the category of the complexes of \( \mathcal{A} \) with its ideal formed by the morphisms homotopic to zero.

The derived category \( \mathcal{D}(\mathcal{A}) \) is obtained from \( \mathcal{K}(\mathcal{A}) \) via localization with respect to the set of quasi-isomorphisms, that is, from the morphisms \( f: X \to Y \) of the category of the complexes on \( \mathcal{A} \), in which the morphisms induced by the cohomologies \( H^i(f): H^i(X) \to H^i(Y) \) are isomorphisms, for each integer \( i \).

Throughout this paper, when we are referring to a bounded derived category of the category \( \text{mod} \mathcal{A} \) of a \( K \)-algebra \( \mathcal{A} \), we denote by \( \mathcal{D}^{\mathbb{B}}(\mathcal{A}) \). Moreover, we assume that the \( K \)-category \( \mathcal{A} \) is abelian, skeletally small, connected, and Ext-finite (all the spaces of morphisms and the spaces of extensions \( \text{Ext}^i(M, N) \) have finite dimensions over \( K \), for every \( M \) and \( N \) objects of \( \mathcal{A} \)). An abelian category \( \mathcal{H} \) is called hereditary if the extensions groups \( \text{Ext}^n_{\mathcal{H}}(X, Y) = 0 \) for \( n \geq 2 \) for any pair of objects \( X \) and \( Y \) of \( \mathcal{H} \).

**Definition 2.2** (piecewise hereditary category). An abelian category \( \mathcal{A} \) is piecewise hereditary of type \( \mathcal{H} \) if \( \mathcal{H} \) is a hereditary abelian category with a tilting object such that \( \mathcal{D}^{\mathbb{B}}(\mathcal{A}) \) and \( \mathcal{D}^{\mathbb{B}}(\mathcal{H}) \) are equivalent as a triangulated categories.

When we say that an algebra \( A \) is piecewise hereditary, we mean that the category \( \text{mod} \mathcal{A} \) is piecewise hereditary. Moreover, there exist a tilting object \( T \) in \( \mathcal{H} \) such that \( A = (\text{End}_{\mathcal{H}} T)^{op} \). The definition of tilting object is as follows:

**Definition 2.3** (tilting object [22]). Let \( \mathcal{H} \) be a hereditary abelian Ext-finite \( K \)-category. The object \( T \in \mathcal{H} \) is called tilting if

i) \( \text{Ext}^1_{\mathcal{H}}(T, T) = 0 \), and

ii) for every \( X \in \mathcal{H} \) the condition \( \text{Ext}^1_{\mathcal{H}}(T, X) = 0 = \text{Hom}_{\mathcal{H}}(T, X) \) implies that \( X = 0 \).

The above definition was inspired by tilting module definition, which is the following.

**Definition 2.4** (tilting module). Let \( A \) be a finite dimension algebra over \( K \). An \( A \)-module \( T \) is called a tilting module if satisfies the following conditions:
i) $\text{Ext}^1(T, T) = 0$,

ii) $\text{pd} T \leq 1$, and

iii) there exists a short exact sequence $0 \to A \to T^1 \to T^2 \to 0$ with $T^1, T^2 \in \text{add } T$.

We state one of the most important theorems about piecewise hereditary algebras.

**Theorem 2.5** (Happel [12]). Let $\mathcal{H}$ be an abelian, hereditary, connected $K$-category with a tilting object. Then $\mathcal{H}$ is derived equivalent to $\text{mod } KQ$ or derived equivalent to $\text{Coh } X$ for some weighted projective line $X$.

Consequently, we call the algebras $A \cong KQ$ of piecewise hereditary algebras of quiver type or of type $Q$. Also, the algebras $A \cong \text{Coh } X$ we call piecewise hereditary algebras of sheaf type.

In order to study the $A_\alpha$ algebras of quiver type, it is enough to characterize the iterated tilted incidence algebras of type $Q$, since, according to the following theorem:

**Theorem 2.6** (Happel-Rickard-Schofield [11]). Let $A$ be a finite dimensional basic associative $K$-algebra and $Q$ be a finite quiver with no oriented cycles. Then $A$ is piecewise hereditary of type $Q$ if, and only if, $A$ is iterated tilted of type $Q$.

We recall that an algebra $A$ is called iterated tilted of type $Q$ if there exists a sequence of algebras $A = A_0, A_1, \ldots, A_n$, where $A_n$ is the path algebra $Q$, and a sequence of tilting modules $T^i_A$, for $0 \leq i < n$, such that $A_{i+1} = \text{End}(T^i_A)$, and every $A_i$-indecomposable module $M$ satisfies $\text{Hom}_{A_i}(T^i, M) = 0$ or $\text{Ext}^1_{A_i}(T^i, M) = 0$.

## 3 Type $A_n$ and type $\tilde{A}_n$

We will devote this section to the study of incidence algebras which are additionally piecewise hereditary and gentle.

**Definition 3.1** (gentle algebra). Let $A$ be an algebra with acyclic quiver $Q_A$. The algebra $A \cong KQ_A/I$ is called gentle if the bound quiver $(Q_A, I)$ has the following properties:

i) each point of $Q_A$ is the source and the target of at most two arrows;

ii) for each arrow $\alpha$ of $Q_A$, there is at most one arrow $\beta$ and one arrow $\gamma$ such that $\alpha \beta \in I$ and $\gamma \alpha \notin I$;

iii) for each arrow $\alpha$ of $Q_A$, there is at most one arrow $\delta$ and one arrow $\zeta$ such that $\alpha \delta \in I$ and $\zeta \alpha \notin I$;
iv) the ideal $I$ is generated by paths of length two.

The following proposition, characterizes the gentle incidence algebras.

**Proposition 3.2.** If the incidence algebra $K\Delta = KQ/I$ is gentle, then $K\Delta$ is hereditary.

*Proof.* We will show that the admissible ideal $I$ is zero. We assume that no, so the incidence algebra has at least one commutativity relation. Consider the following subquiver of $Q$:

$$
\begin{array}{c}
\bullet \\
\alpha_n \\
\alpha_1 \\
\bullet \\
\beta_m \\
\beta_1 \\
\bullet
\end{array}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}
\begin{array}{c}
\bullet \\
\alpha_{n-1} \\
\alpha_2 \\
\bullet \\
\beta_{m-1} \\
\beta_2 \\
\bullet
\end{array}
\begin{array}{c}
\bullet
\end{array}
\begin{array}{c}
\bullet
\end{array}
\begin{array}{c}
\bullet
\end{array}
\begin{array}{c}
\bullet
\end{array}
$$

Thus, $\alpha_1 \alpha_2 \ldots \alpha_{n-1} \beta_1 \beta_2 \ldots \beta_m \in I$, where $m, n \geq 2$. Since $K\Delta$ is a gentle algebra, the ideal $I$ is generated by paths of length two. But the commutativity relation is not a path, contradicting the assumption that $I$ is not zero.

Therefore, gentle incidence algebra is hereditary. \qed

We can state the following result:

**Corollary 3.3.** If a Phia algebra $K\Delta$ is gentle, then $K\Delta \cong KQ$, where $Q = \mathbb{A}_n$ or $Q = \mathbb{\tilde{A}}_n$.

*Proof.* From the previous proposition, we conclude that $K\Delta$ is hereditary, that is, $K\Delta$ is isomorphic to $KQ$. This implies that $KQ$ is gentle, because $K\Delta$ is a gentle algebra, by hypothesis.

We assume that there exists a subquiver of $Q$ of the form:

$$
\begin{array}{c}
\bullet
\end{array}
\begin{array}{c}
\bullet
\end{array}
\begin{array}{c}
\bullet
\end{array}
\begin{array}{c}
\bullet
\end{array}
$$

with the edges oriented respecting the first condition of the definition of gentle algebra. We will analyze a case of orientation of these edges, the other cases are similar. Let

$$
\begin{array}{c}
\bullet \\
\alpha \\
\beta
\end{array}
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\begin{array}{c}
\bullet
\end{array}
\begin{array}{c}
\bullet
\end{array}
$$

5
By the fourth condition of the definition of gentle algebra, the ideal $I$ is generated by paths of length two. In the subquiver above, we have the paths $\beta\alpha$ and $\beta\theta$ both of length two. Thus, we have $\beta\alpha \in I$ or $\beta\theta \in I$. Since the algebra $KQ$ has the ideal $I$ empty, this case does not happen.

So we only have the two options below:

We conclude that $\overline{Q}_A = \mathbb{A}_n$ or $\overline{Q}_A = \mathbb{A}_n$.

These two first results and the fact that the property of algebra is gentle is invariant by derived equivalence, see [23], we get these corollaries.

**Corollary 3.4.** If $K\Delta$ is a Phia algebra of type $\mathbb{A}_n$, then $K\Delta \cong KQ$, where $\overline{Q} = \mathbb{A}_n$.

**Corollary 3.5.** If $K\Delta$ is a Phia algebra of type $\mathbb{A}_n$, then $K\Delta \cong KQ$, where $\overline{Q} = \mathbb{A}_n$.

### 4 Type $\mathbb{D}_n$

The characterization of the piecewise hereditary algebras of type $\mathbb{D}_n$ was done by Keller in the article [3]. In her thesis [4], Elsa Fernández obtained this same characterization through an alternative tool, the concept of trivial extension. Based on this material, it is possible to state the following result:

**Theorem 4.1.** Let $K\Delta = KQ/I$ be a Phia algebra of type $\mathbb{D}_n$ such that $\overline{Q}$ is not of tree type. Then $K\Delta$ or $K\Delta^{\text{op}}$ is isomorphic to

1. 

2. 

3. 

4. 

6
We note that at the vertices $*$ of the graphs above we can attach a diagram of the form

$\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \rightarrow \bullet$

5 Type $E_6$

The first efforts to classify piecewise hereditary algebras of type $E_6$ were made by Happel, in [24]. Years later, in her thesis, Elsa Fernández obtained some updates in relation to this classification, determining a complete description for this class of algebras. This problem was also treated computationally by the Roggon's project [25], which shows the exact amount of these algebras up to isomorphisms. Particularly on piecewise hereditary algebras of type $E_6$ which are also incidence algebras, we can state the following theorem:

**Theorem 5.1.** Let $K\Delta = KQ/I$ be a Phia algebra of type $E_6$ such that $\overline{Q}$ is not of tree type. Then $K\Delta$ or $K\Delta^{op}$ is isomorphic to

1. $\bullet \leftarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftarrow \bullet$

2. $\bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftarrow \bullet$

3. $\bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet$

4. $\bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet$

5. $\bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet$

6. $\bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet$

7. $\bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet \leftrightarrow \bullet$

In the quivers above have edges are not oriented. An observation about the edges is that any of the two possible orientations can be considered.
6 Type $E_7$

The class of iterated tilted algebras of type $E_7$ has an important relation with the representation-finite trivial extensions of Cartan class $E_7$. Fernández, in her thesis [4], describes in detail these trivial extensions. We recall the definition of trivial extension of an algebra.

**Definition 6.1 (trivial extension).** Let $A$ be an $K$-algebra. The trivial extension of $A$ is the algebra $T(A) = A \rtimes D(A)$, where $D(A) = \text{Hom}_K(A,K)$, whose underlying of $K$-vector space is $A \times D(A)$ with multiplication is given by:

$$(a, f)(b, g) = (ab, ag + fb), \quad \text{for any } a, b \in A \text{ and } f, g \in D(A).$$

In this work, we will consider the trivial extensions of representation-finite type. The trivial extension is an auto-injective algebra and we can divide them into classes, called Cartan. The Cartan class of an algebra $B$ auto-injective of representation-finite type was introduced by Riedtmann in 1980 by the article “Algebren, Darstellungskocher, Ueberlagerungen und Zuruck”. We consider $(\Gamma, \tau)$ a stable Auslander-Reiten quiver of an algebra $B$. That is, in particular, the translation $\tau$ and its inverse are defined in every quiver $\Gamma$. Let $Q$ be a Dynkin diagram and $G$ be an automorphism group of $\mathbb{Z}Q$. Riedtmann related $(\Gamma, \tau)$ with $\mathbb{Z}Q/G$ for an isomorphism, such that the $G$-action in $\mathbb{Z}Q$ is admissible, that is, if each orbit of $G$ finds $\{x\} \cup x^-$ in at most one vertex and finds $\{x\} \cup x^+$ in at most one vertex for each $x$ of $(\mathbb{Z}Q)_0$. The Dynkin type of $Q$ is uniquely determined and we call the Cartan class of $B$.

With the classification of all trivial extensions of representation-finite type together with the next result, we will have an association with the iterated tilted algebras.

**Theorem 6.2 (Assem-Happel-Roldán [26]).** Let $A$ be an algebra. The following conditions are equivalent:

a) $T(A)$ is representation-finite of Cartan class $Q$;

b) $A$ is iterated tilted algebra of Dynkin type $Q$.

The focus of the study is the quiver with relations of the trivial extension $T(A)$ of a schurian algebra $A \cong KQ_A/I$. To understand the description of quiver and relations, we need the concept of maximal path. A path $p$ in $KQ_A/I$ will be called maximal if $p \neq 0$ and $\alpha p = 0 = p\alpha$ for every arrow $\alpha$ of $Q_A$. Fernández’s theorems, described the trivial extensions of schurian algebras are used. Recall that an algebras is schurian if the following condition is satisfied: $\dim_K(\text{Hom}_A(P, P')) \leq 1$ for any indecomposable projectives $P, P'$. 

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If it is not schurian, it is known that the trivial extension is representation-infinite type [4]. As our interest is in representation-finite type, the statements will be specific to schurian algebras.

The quiver of $T(A)$ is described in the following theorem.

**Theorem 6.3** (Fernández [4]). If $A = KQ_A/I$ is a schurian algebra, then the quiver of $T(A)$ is given by:

a) $(Q_{T(A)})_0 = (Q_A)_0$.

b) $(Q_{T(A)})_1 = (Q_A)_1 \cup \{\beta_{p_1}, \ldots, \beta_{p_t}\}$, where $\{p_1, \ldots, p_t\}$ is a maximal set of linearly independent maximal paths. Moreover, for each $i$, $\beta_{p_i}$ is an arrow such that $o(\beta_{p_i}) = t(p_i)$ and $t(\beta_{p_i}) = o(p_i)$.

It is necessary to describe the relations of the presentation associated with $T(A)$ and for this we will need two definitions related to the maximal paths.

**Definition 6.4** (elementary cycle). Let $M = \{p_1, \ldots, p_t\}$ be a maximal set of linearly independent maximal paths. We say that a oriented cycle $C$ of $Q_{T(A)}$ is elementary if $C = \beta_{p_i} \alpha_1 \alpha_2 \ldots \alpha_n$, where $p_i \in M$ and $\alpha_1 \ldots \alpha_n = kp_i$, for some $k \in K^*$.

Note that the length of elementary cycle $C$ is at least two. Recall that every path $q$ in $T(A)$ is contained in an elementary cycle $C$. We will say that the supplement of $q$ in $C$ is the trivial path $e_{o(q)}$, if $o(q) = t(q)$, that is the path consisting of the remaining arrows of $C$.

**Theorem 6.5** (Fernández [4]). Let $A \cong KQ_A/I_A$ be a schurian triangular algebra such that parallel paths in $Q_A$ are equal in $A$. Then $T(A) = KQ_{T(A)}/I_{T(A)}$ where the admissible ideal $I_{T(A)}$ is generated by:

a) the paths consisting of $n + 1$ arrows in an elementary cycle of length $n$,

b) the paths whose arrows do not belong to a single elementary cycle,

c) the difference $\gamma - \gamma'$ of paths $\gamma, \gamma'$ with the same origin and the same endpoint and having a common supplement in elementary cycles of $Q_{T(A)}$.

For future references in the text, we will call relations of type 1, type 2 and type 3 referring to the paths of items a), b) and c), respectively.

We will use an important procedure on a trivial extension $T(A)$. Two definitions are required to describe this method.

**Definition 6.6** (cutting set [27]). Let $T(A) = KQ_{T(A)}/I_{T(A)}$ be a trivial extension and let $\Sigma$ be a set of arrows of $Q_{T(A)}$. We say that $\Sigma$ is a cutting set if it consists of exactly one arrow in each non-zero elementary cycle of $T(A)$.
Definition 6.7 (27). Let $T(A) = KQ_{T(A)}/I_{T(A)}$ be a trivial extension given by a quiver $KQ_{T(A)}$ and an admissible ideal $I_{T(A)}$. We say that $A'$ is defined by the cutting set $\Sigma$ of $T(A)$, if $A'$ is of the form $KQ_{T(A)}/<I_{T(A)} \cup \Sigma>$.

Looking at the Theorem 6.5 and the definition of the cutting set, the algebra $A'$ defined by any cutting set does not have the relations of type 1. Therefore, in the study of these algebras, we will pay attention to relations of type 2 and type 3. We can now state the following theorem of Fernández.

Proposition 6.8 (Fernández [4]). Let $A$ be a schurian triangular algebra. Then an algebra $A'$ is defined by a cutting set of $T(A)$ if and only if $T(A) \cong T(A')$.

Let $\Gamma$ be a representation-finite trivial extension of Cartan class $E_7$. All algebras $A$ defined by a cutting set have their trivial extensions isomorphic to $\Gamma$. Therefore, the trivial extension of $A$ is representation-finite of Cartan class $E_7$, implying that the algebra $A$ is iterated tilted of type $E_7$ by Theorem 6.2.

We want to identify all algebras $A$ that are incidence algebra for each representation-finite trivial extension $\Gamma$ of Cartan class $E_7$. First, we want the algebras $A$ which have a presentations whose relations are only commutativity relations. In order to do this, we need to eliminate at least one arrow that belongs to at least two elementary cycles of the trivial extension.

Lemma 6.9. Let $\Gamma$ be a trivial extension of a schurian algebra in which the parallel paths are equal. Let $A \cong KQ_A/I_A$ be an algebra defined by a cutting set of $\Gamma$. If an arrow of the setting cut belongs to two elementary cycles of $Q_{\Gamma}$ then there is, at least, a relation of type 3 in $I_A$.

Proof. Let $B$ be a schurian algebra in which the parallel paths are equal and let $\Gamma = T(B)$ be a trivial extension. Without loss of generality, we will take a trivial extension $\Gamma$ associated with a quiver with only two elementary cycles $C_1 = \theta_1 \theta_2 \ldots \theta_i \alpha_{i+1} \ldots \alpha_n$ and $C_2 = \theta_1 \theta_2 \ldots \theta_i \alpha'_{i+1} \ldots \alpha'_m$. Let $Q_{\Gamma}$ be a quiver below:

```
\begin{verbatim}
\cdots
\cdots
\cdots
\cdots
\end{verbatim}
```
with the following relations of type 1, 2 and 3:

\[ \theta_1 \ldots \theta_l \alpha_{l+1} \ldots \alpha_n \theta_1; \ldots; \alpha_n \theta_1 \ldots \theta_l \alpha_{l+1} \ldots \alpha_n \]
\[ \theta_1 \ldots \theta_l \alpha'_l \ldots \alpha'_m \theta_1; \ldots; \alpha'_m \theta_1 \ldots \theta_l \alpha'_{l+1} \ldots \alpha'_m \]
\[ \alpha'_m \theta_1 \ldots \theta_l \alpha_{l+1}; \alpha_n \theta_1 \ldots \theta_l \alpha'_{l+1} \]
\[ \alpha_{l+1} \ldots \alpha_n - \alpha'_{l+1} \ldots \alpha'_m \]

Now, let \( \Sigma = \{ \theta_i \} \) be a cutting set for \( i \in \{1, \ldots, l\} \) such that \( \theta_i \) belonging to the two elementary cycles \( C_1 \) and \( C_2 \), we get \( I_A \) with only the relation of type 3.

Therefore the presentation of \( A \) has only the relation \( \alpha_{l+1} \alpha_{l+2} \ldots \alpha_n - \alpha'_{l+1} \alpha'_{l+2} \ldots \alpha'_m \), as desired. \( \Box \)

A question arises on the existence of a cutting set of trivial extension that defines an incidence algebra. We come across various forms of trivial extension diagrams, and we can find some particular patterns in the cutting sets that define incidence algebras. We have put these patterns together in some lemmas that we will call cutting lemmas. Before stating these lemmas, we will introduce some concepts.

For each vertex \( h \) of quiver \( Q_\Gamma \), let \( C_h \) be the set of all, not null, elementary cycles \( C \), in \( \Gamma \), with origin and terminus in \( h \).

**Lemma 6.10.** Let \( Q_\Gamma \) be a planar diagram associated with four elementary cycles. If this diagram has a vertex \( a \) such that \( C_a = \{ C_1, C_2, C_3, C_4 \} \) and has four edges in \( a \), then there are only two cutting sets defining two incidence algebras with relations.

**Proof.** We consider a trivial extension \( \Gamma \) associated with a quiver with four elementary cycles. Thus, let \( Q_\Gamma \) as below:
We will describe the generators of the admissible ideal $I_{1\Gamma}$. We denote
\[ \lambda = \lambda_1 \ldots \lambda_{m_1}, \quad \theta = \theta_1 \ldots \theta_{n_1}, \quad \mu = \mu_1 \ldots \mu_{m_2}, \quad \eta = \eta_1 \ldots \eta_{m_4}, \quad \rho = \rho_1 \ldots \rho_{n_3}, \quad \delta = \delta_1 \ldots \delta_{n_2}, \quad \gamma = \gamma_1 \ldots \gamma_{m_3} \text{ and } \sigma = \sigma_1 \ldots \sigma_{n_4}. \]

We have the relations of type 1 related to the elementary cycle $\alpha \lambda \theta \mu \beta'$:
\[ \alpha \lambda \theta \mu \beta' \alpha = \lambda \theta \mu \beta' \alpha \lambda_1 = \ldots = \lambda_{m_1} \theta \mu \beta' \alpha \lambda = \theta \mu \beta' \alpha \lambda \theta = \lambda \theta \mu \beta' \alpha \lambda \theta = \mu \beta' \alpha \lambda \theta \mu_1 = \ldots = \mu_{m_2} \beta' \alpha \lambda \theta = \beta' \alpha \lambda \theta \mu. \]

Below, we describe the relations of type 1 related to the elementary cycle $\alpha \lambda \delta \gamma \alpha'$:
\[ \alpha \lambda \delta \gamma \alpha' = \lambda \delta \gamma \alpha' \alpha \lambda_1 = \ldots = \lambda_{m_1} \delta \gamma \alpha' \alpha \lambda = \delta \gamma \alpha' \alpha \lambda \delta = \gamma \alpha' \alpha \lambda \delta \gamma_1 = \ldots = \gamma_{m_3} \alpha' \alpha \lambda \delta \gamma = \alpha' \alpha \lambda \delta \gamma \alpha'. \]

Next, we present the relations of type 1 related to the elementary cycle $\beta \eta \rho \mu \beta'$:
\[ \beta \eta \rho \mu \beta' = \eta \rho (\mu' \beta' \eta) = \ldots = \eta_{m_4} \rho \mu \beta' \eta = \rho \mu' \beta \eta \rho_1 = \ldots = \rho_{n_3} \mu' \beta \eta \rho = \mu' \beta \eta \rho \mu_1 = \ldots = \mu_{m_2} \beta' \beta \eta \rho = \beta' \beta \eta \rho \mu. \]

Finally, we describe the relations of type 1 related to the elementary cycle $\beta \eta \sigma \gamma \alpha'$:
\[ \beta \eta \sigma \gamma \alpha' = \eta \sigma \gamma \alpha' \beta \eta_1 = \ldots = \eta_{m_4} \sigma \gamma \alpha' \beta \eta = \sigma \gamma \alpha' \beta \eta \sigma_1 = \ldots = \sigma_{n_4} \gamma \alpha' \beta \eta \sigma = \gamma \alpha' \beta \eta \sigma \gamma_1 = \ldots = \gamma_{m_3} \alpha' \beta \eta \sigma = \alpha' \beta \eta \sigma \gamma \alpha'. \]
It is necessary to show the relations of type 2 and 3, which are as follows:

\[\alpha \lambda \theta - \beta \eta \rho; \theta \mu \beta' - \delta \gamma \alpha'; \alpha \lambda \delta - \beta \eta \sigma; \rho \mu \beta' - \sigma \gamma \alpha';\]
\[\theta_{n_1} \mu \beta' \beta; \rho_{n_3} \mu \beta' \alpha; \delta_{n_2} \gamma \alpha' \beta; \sigma_{n_4} \gamma \alpha' \alpha\]

We consider the cutting set with the arrows \(\alpha\) and \(\beta\). This setting cut define the quiver alongside with commutativity relations \(\theta \mu \beta' - \delta \gamma \alpha'\) and \(\rho \mu \beta' - \sigma \gamma \alpha'\).

The second incidence algebra with relations is defined by cutting set \(\{\alpha', \beta'\}\), where your quiver associated alongside with commutativity relations \(\alpha \lambda \theta - \beta \eta \rho\) and \(\alpha \lambda \delta - \beta \eta \sigma\).

Now, we observe in the relations of the trivial extension \(KQ_{\Gamma}/I_{\Gamma}\) that any other cutting set than the two setting cuts shown above will not define incidence algebras with commutativity relations. As in the case of the cutting set \(\delta_1, \beta', \sigma_{n_4}\), shown below:
with the relations $\alpha \lambda \theta - \beta \eta \rho = \delta_{n_2} \gamma \alpha' \beta$.

In the section 8, we present two more cutting lemmas 8.8 and 8.5. These lemmas inspired the search for a general algorithm of cutting set. We have been able to go one step further, in a computational way, we elaborate a program that shows exactly the cutting sets of given trivial extension that define incidence algebras. In addition, we implemented this algorithm in the site http://www.ime.usp.br/~celovg/

The first step is to name the arrows of quiver associated with the given trivial extension. We start with $\alpha_1, \alpha_2, \alpha_3, \ldots$ in arrows that make up the relations of type 2. The name of arrows that do not satisfy this condition is arbitrary. The next step is to put the necessary information from the trivial extension into the program. These initial data are:

- number of relations of type 2;
- number of elementary cycles;
- number of different arrows in these relations of type 2.

Then the program displays a table with checkboxes. In this table, the columns represent the relations of type 2 and the elementary cycles, respectively, and the lines represent the arrows involved in these elements. The user clicks on checkbox if the arrow is in the relation(s) or if it belongs to the cycle(s), and leaves unchecked otherwise. Press the “Ready!” button, and then the solution is shown. The solution is the cutting sets that define incident algebras.

Fernández’s thesis [4] showed all the representation-finite trivial extension of Cartan class $E_7$, totalling 72. Thus, for each trivial extension of the list
$\mathbb{E}_7$, we look for the cutting sets that define incidence algebras. This was a totally manual work and it was helpful to have some ideas for the program, and for some previous results. In order to make a more concise proof, we use the program for each trivial extension.

**Theorem 6.11.** Let $K\Delta = KQ/I$ be a Phia algebra of type $\mathbb{E}_7$ such that $Q$ is not of tree type. Then $K\Delta$ or $K\Delta^{op}$ is isomorphic to

\[ \begin{align*}
1. & \quad \bullet \leftrightarrow \bullet \leftrightarrow \bullet \\
2. & \quad \bullet \leftrightarrow \bullet \leftrightarrow \bullet \\
3. & \quad \bullet \leftrightarrow \bullet \leftrightarrow \bullet \\
4. & \quad \bullet \leftrightarrow \bullet \leftrightarrow \bullet \\
5. & \quad \bullet \leftrightarrow \bullet \leftrightarrow \bullet \\
6. & \quad \bullet \leftrightarrow \bullet \leftrightarrow \bullet \\
7. & \quad \bullet \leftrightarrow \bullet \leftrightarrow \bullet \\
8. & \quad \bullet \leftrightarrow \bullet \leftrightarrow \bullet \\
9. & \quad \bullet \leftrightarrow \bullet \leftrightarrow \bullet \\
10. & \quad \bullet \leftrightarrow \bullet \leftrightarrow \bullet \\
11. & \quad \bullet \leftrightarrow \bullet \leftrightarrow \bullet \\
12. & \quad \bullet \leftrightarrow \bullet \leftrightarrow \bullet \\
13. & \quad \bullet \leftrightarrow \bullet \leftrightarrow \bullet \\
14. & \quad \bullet \leftrightarrow \bullet \leftrightarrow \bullet \\
\end{align*} \]
Proof. We put the necessary information for each representation-finite trivial extension of Cartan class $E_7$ \cite{fer} in the program and we show the non-hereditary solutions. We will show this procedure to a trivial extension and the remaining work is analogous.

The relations of type 2 are: $r_1 = \alpha_1\alpha_2\alpha_3$ and $r_2 = \alpha_4\alpha_5\alpha_5$. The elementary cycles are: $C_1 = \alpha_1\alpha_2\alpha_5\alpha_6$ and $C_2 = \alpha_4\alpha_2\alpha_3\alpha_8\alpha_7$. Therefore, the program shows us the setting cut that defines the solution 1 of the theorem.

7 Type $E_8$

In the previous section, we explained the results and methods applied in the classification of all Phia algebras of type $E_7$. We will repeat the same procedure for the case $E_8$.

The representation-finite trivial extension of Cartan class $E_8$ are iterated tilted algebra of type $E_8$, and vice-versa. From the thesis of Fernández \cite{fer}, we have the list of all such trivial extensions, counting 251. From this, we will present all non-hereditary incidence algebras defined by the cutting sets of each trivial extension, as we did for the case $E_7$. We did the manual work of finding the incidence algebras defined by the cutting sets for each of the trivial extensions, except that in order to write the proof we prefer to use the program.

Theorem 7.1. Let $K\Delta = KQ/I$ be a Phia algebra of type $E_8$ such that $\overline{Q}$ is not of tree type. Then $K\Delta$ or $K\Delta^{op}$ is isomorphic to

1. $\bullet \rightarrow \bullet \leftarrow \bullet$  
2. $\bullet \rightarrow \bullet \leftarrow \bullet \leftarrow \bullet$
Proof. The systematic of the demonstration is to analyze each representation-finite trivial extension of Cartan class $E_8$, displaying its relations of type 2 and its elementary cycles. With this information, we apply in our computer program and we have the cutting sets that define the incidence algebras. Thus, we show in the list of our theorem only the non-hereditary incidence algebras. We will show this procedure to a trivial extension and the remaining work is analogous.

The relations of type 2 are:

- $r_1 = \alpha_3 \alpha_8 \alpha_1 \alpha_5 \alpha_7$, $r_2 = \alpha_4 \alpha_8 \alpha_1 \alpha_3 \alpha_{11}$,
- $r_3 = \alpha_9 \alpha_4 \alpha_8 \alpha_1$, $r_4 = \alpha_7 \alpha_4 \alpha_8 \alpha_{10}$,
- $r_5 = \alpha_3 \alpha_8 \alpha_{10}$, $r_6 = \alpha_{11} \alpha_3 \alpha_2$, $r_7 = \alpha_6 \alpha_3 \alpha_8$ and $r_8 = \alpha_4 \alpha_2$.

The elementary cycles are:

- $C_1 = \alpha_8 \alpha_1 \alpha_3 \alpha_7 \alpha_4$, $C_2 = \alpha_8 \alpha_1 \alpha_5 \alpha_{11} \alpha_3$,
- $C_3 = \alpha_8 \alpha_{10} \alpha_9 \alpha_4$ and $C_4 = \alpha_3 \alpha_2 \alpha_6$.

Therefore, the program shows us the setting cut that defines the solutions 76 and 77 of the theorem.

8 Type $\tilde{D}_n$, type $\tilde{E}_6$, type $\tilde{E}_7$, type $\tilde{E}_8$

The Phia algebras of extended Dynkin type are derived equivalent to the category of coherent sheaves of certain weighted projective line. This category is derived equivalent to the category of modules over canonical algebras, see [28], [29], [30].

We do not have a complete description of the Phia algebras of extended Dynkin type as in the Dynkin type, but we have been able to identify some families. One of them is the Phia concealed algebras of extended Dynkin type. We can identify through the works of Happel-Vossieck [31] and Bongartz [32].

The other family of Phia algebras of extended Dynkin type are the members of a new set that we call the ANS family, in reference to Assem, Nehring and Skowroński. In the Tsukuba Journal of Mathematics [33], Assem, Nehring and Skowroński published the work entitled “Domestic trivial extensions of simply connected algebras”. The main result of this article has an important role in the characterization of the ANS family.
Theorem 8.1 (Assem-Nehring-Skowroński). Let $A$ be a finite-dimensional, basic and connected algebra over an algebraically closed field $K$. If $A$ is simply connected, then the following conditions are equivalent:

1. There exists a representation-infinite tilted algebra $B$ of Euclidean type $\tilde{D}_n$ or $\tilde{E}_p$ such that $T(A) \cong T(B)$.

2. $A$ is an iterated tilted algebra of Euclidean type $\tilde{D}_n$ or $\tilde{E}_p$.

We will describe the algebras $B$ which are the members of the ANS family. In the theorem above, the algebras need to be simply connected. We will work with the concealed algebras $A$ of Euclidean type $\tilde{D}_m$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$, implying that they are tame tilted algebras with the first Hochschild cohomology group being zero. Thus, by the work [34], they are simply connected satisfying the desired hypothesis. Therefore, the difficulty is to obtain the Phia algebras $K\Delta$ such that $T(K\Delta) \cong T(A)$.

We will use the list of Happel-Vossieck [31] to work with all the concealed algebras $A$ of Euclidean type, so the algebras $B$ come from the list of Happel-Vossieck. Thus, these algebras $B$ are members of the ANS family of type HV.

The algebras which we consider are the trivial extension of schurian algebras. Thus, from the list of Happel-Vossieck [31] we will explore only the schurian algebras. In turn, we observed that the schurian concealed algebras $A$ of Euclidean type fit perfectly in the strategy used in the full description of Phia algebras of Dynkin type. The main difference is that we do not have a list of the trivial extensions of $A$. Therefore, the machinery to find Phia algebra $K\Delta$ will have one more step than that which was done in the previous sections. We will describe the quivers of the trivial extensions of $A$. An important observation is that we will not study the trivial extensions of $A^{\text{op}}$ since the trivial extension of $A^{\text{op}}$ is isomorphic to $T(A)^{\text{op}}$.

Lemma 8.2. Let $A$ be a finite dimensional algebra. We denote the trivial extension of $A$ by $T(A)$. We have $T(A)^{\text{op}} \cong T(A^{\text{op}})$.

Proof. We consider the morphism

$\Phi: (A, f) \mapsto (A, f)$

The reader can check that this is an isomorphism of algebras. □

Let a schurian concealed algebra Euclidean type $\tilde{D}_m$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$, we will do the following:
• compute the quiver of trivial extension $A$;
• describe the elementary cycles of $Q_{T(A)}$;
• describe the relations of type 2 in presentation of $T(A)$;
• identifies the arrows in these relations of type 2;
• use our computer program;
• if there is a non-hereditary solution, then we need to verify if the cutting set defines a Phia algebra $K\Delta$ that is not a concealed algebra of the Euclidean type.

This will be the script of the proof of the characterization theorems of the Phia algebras of ANS family of type HV.

The chapter XIV of the book “Elements of the Representation of Associative Algebras” [35] gives a list of all frames of the list of Happel-Vossieck [31]. We will use this list to write the series of results for the Phia algebras of type $\widetilde{D}_4$.

We will start the work of describing the Phia algebras of the ANS family of type HV-$\widetilde{D}_4$. First, we’ll start with members of type $\widetilde{D}_4$.

**Theorem 8.3.** The algebras associated with the quivers with relations below:

1. ![Diagram 1]
2. ![Diagram 2]
3. ![Diagram 3]

are Phia algebras of the ANS family of type HV-$\widetilde{D}_4$.

**Proof.** Given an algebra coming from an admissible operation on a frame, see [35], we will use the trivial extension of that algebra and extract the necessary information to use our computer program. We will do this for each permissible operation in a frame, and for each frame. Since we are only interested in type $\widetilde{D}_4$, the only frame that satisfies this condition is $Fr2$.

Observe that we are not interested in the algebras whose trivial extensions only have cutting sets that define hereditary Phia algebra. This will be examined more carefully in frame $Fr2$. The adjacent quiver $\widetilde{D}_4$ has $2^4$ possibilities of admissible operations 1, that is, we have $2^4$ Euclidean graphs.
Now, our goal is to study the trivial extension of each graph and see if there is any cutting set that defines a non-hereditary incidence algebra. For this, the trivial extension must have at least two elementary cycles that have at least one common arrow in accordance with the lemma 6.9. We will show this procedure to a trivial extension and the remaining work is analogous.

By symmetry, there are four quivers in this set. The relations of type 2 are: \( r_1 = \alpha_3 \alpha_1 \alpha_6, r_2 = \alpha_3 \alpha_1 \alpha_7, r_3 = \alpha_4 \alpha_1 \alpha_7, r_4 = \alpha_4 \alpha_1 \alpha_5, r_5 = \alpha_2 \alpha_1 \alpha_5 \) and \( r_6 = \alpha_2 \alpha_1 \alpha_6 \). The elementary cycles are: \( C_1 = \alpha_1 \alpha_5 \alpha_3, C_2 = \alpha_1 \alpha_6 \alpha_4 \) and \( C_3 = \alpha_1 \alpha_7 \alpha_2 \). With this information, the program shows us the solution 1.

In general, it is difficult to describe the Phia algebras of type \( \widetilde{D}_m \) according to the number \( m \). Thus, we begin with the type \( \widetilde{D}_4 \) and the next type is \( \widetilde{D}_5 \).

**Theorem 8.4.** The algebras associated with the quivers with relations below:

are Phia algebras of the ANS family of type HV-\( \widetilde{D}_5 \).
Proof. Following the same proof of the previous theorem, only now we are interested in the type $\tilde{\mathfrak{D}}_5$. The frame that satisfies this condition is $F_r2$, see [35]. The adjacent quiver $\tilde{\mathfrak{D}}_5$ has $2^5$ of possibilities of admissible operations 1, that is, we have $2^5$ Euclidean graphs $\tilde{\mathfrak{D}}_5$. We will show this procedure to a trivial extension and the remaining work is analogous.

$\tilde{\mathfrak{D}}_5 - F_r2.1$

The relations of type 2 are: \( r_1 = \alpha_6\alpha_4\alpha_3\alpha_2, r_2 = \alpha_4\alpha_3\alpha_1\alpha_9, r_3 = \alpha_5\alpha_3\alpha_1\alpha_6, r_4 = \alpha_9\alpha_5\alpha_3\alpha_2, r_5 = \alpha_5\alpha_3\alpha_2\alpha_8, r_6 = \alpha_7\alpha_5\alpha_3\alpha_1, r_7 = \alpha_8\alpha_4\alpha_3\alpha_1 \) and \( r_8 = \alpha_4\alpha_3\alpha_2\alpha_7 \). The elementary cycles are: \( C_1 = \alpha_1\alpha_6\alpha_4\alpha_3, C_2 = \alpha_1\alpha_9\alpha_5\alpha_3, C_3 = \alpha_2\alpha_7\alpha_5\alpha_3 \) and \( C_4 = \alpha_2\alpha_8\alpha_3\alpha_3 \). With this information, the program shows us the solutions 1, 2 and the dual of algebra 2.

After describing the Phia algebras of ANS family of type HV-$\tilde{\mathfrak{D}}_4$ and HV-$\tilde{\mathfrak{D}}_5$, we were able to generalize the demonstration and to describe the Phia algebras of ANS family of type HV-$\tilde{\mathfrak{D}}_n$ for \( n \geq 6 \).

The following lemma is used several times in proof of the theorem describing the Phia algebras of ANS family of type HV-$\tilde{\mathfrak{D}}_n$.

Lemma 8.5. Given the trivial extension $\Gamma$ of the hereditary algebra of type $A_n$, with at least two elementary cycles, we consider its diagram below, up to order of the cycles:

\[ \Gamma : \bullet \xrightarrow{\alpha_2} \bullet \xleftarrow{\beta_1} \bullet \xrightarrow{\beta_n} \bullet \xleftarrow{\gamma_n} \bullet \xrightarrow{\alpha_{n+1}} \bullet \]

We will use the dashed arrow to represent a path which may occur in the quiver. Moreover, the symbol $\bullet \bullet \bullet$ in the quiver represents the possibility of attaching subquivers of the form $\bullet \xrightarrow{\alpha} \bullet$ or dual form.

i) Let $\Sigma$ be a cutting set of $\Gamma$. Therefore, there are cutting sets $\Sigma$ of $\Gamma$ such that $KQ_{\Gamma}/ < I_\Gamma \cup \Sigma >$ are the following incidence algebras:

\[
\begin{align*}
\bullet &\xleftarrow{\cdot} \bullet \xleftarrow{\cdot} \bullet \xrightarrow{\cdot} \bullet \\
\bullet &\xrightarrow{\cdot} \bullet \xleftarrow{\cdot} \bullet \xrightarrow{\cdot} \bullet
\end{align*}
\]
ii) We also assume that there is an elementary cycle with three arrows or more in $\Gamma$.

If $\alpha_1 \in \Sigma$ or $\alpha_n \in \Sigma$, then $\mathbb{K}Q_{\Gamma} / \langle I_{\Gamma} \cup \Sigma \rangle$ is not an incidence algebra. If $\alpha_2 \in \Sigma$ or $\alpha_{n+1} \in \Sigma$, then $\mathbb{K}Q_{\Gamma} / \langle I_{\Gamma} \cup \Sigma \rangle$ is an incidence algebra with the quiver associated above.

Proof. The aim is to find a cutting set $\Sigma$ of $\Gamma$ where $\mathbb{K}Q_{\Gamma} / \langle I_{\Gamma} \cup \Sigma \rangle$ is an incidence algebra. First, let us assume that there is an elementary cycle with three arrows or more. We will show the existence of $\Sigma$ by induction in the number of elementary cycles. For clarity, we will begin the first step of the induction process with two elementary cycles:

\[
\begin{align*}
\bullet & \xrightarrow{\alpha_2} \bullet \xleftarrow{\beta_1} \bullet \xrightarrow{\alpha_4} \bullet \\
\bullet & \xrightarrow{\alpha_1} \bullet \xleftarrow{\beta_3} \bullet \xrightarrow{\alpha_3} \bullet
\end{align*}
\]

The relations of type 2 are: $r_1 = \alpha_2 \beta_3$ and $r_2 = \alpha_4 \beta_1$. We emphasize that the relation $r_2$ is defined with the arrow of the path $\beta_1$ that makes sense to have $\alpha_4 \beta_1$. If there is no path $\beta_1$, the relation will be $r_2 = \alpha_4 \alpha_1$. The elementary cycles are: $C_1 = \alpha_2 \beta_1 \alpha_1$ and $C_2 = \alpha_4 \beta_3 \alpha_3$.

Without loss of generality, we assume that exists the path $\beta_1$. Also, we start constructing the cutting set $\Sigma$ choosing the arrow of the elementary cycle $C_1$.

We consider $\alpha_2 \in \Sigma$. Then, to eliminate the relation $r_2 = \alpha_4 \beta_1$, we get $\alpha_4 \in \Sigma$. So using $\Sigma = \{\alpha_2, \alpha_4\}$ we have the incidence algebra:

\[
\mathbb{K}Q_{\Gamma} / \langle I_{\Gamma} \cup \Sigma \rangle : \bullet \xleftarrow{} \bullet \xleftarrow{} \bullet \xrightarrow{} \bullet \xrightarrow{} \bullet
\]

Let $\Gamma$ be a trivial extension of the lemma with $m+1$ elementary cycles. We call the set $\Sigma = \{\alpha_2, \alpha_4, \ldots, \alpha_{2m}\}$ for $m$ elementary cycles, where it satisfies the hypothesis of induction. We need to define which arrow of the elementary cycle $C_{m+1}$ would complete the cutting set $\Sigma$ of $\Gamma$:

\[
\begin{align*}
\bullet & \xrightarrow{\alpha_1} \bullet \xleftarrow{\beta_1} \bullet \xleftarrow{} \bullet \xleftarrow{} \bullet \xleftarrow{} \bullet \xleftarrow{} \bullet \\
\bullet & \xrightarrow{\alpha_{2m}} \bullet \xleftarrow{\beta_{2m-1}} \bullet \xrightarrow{} \bullet \xrightarrow{} \bullet \xrightarrow{} \bullet \\
\bullet & \xrightarrow{\alpha_{2m+2}} \bullet \xleftarrow{\beta_{2m+1}} \bullet \xrightarrow{} \bullet \xrightarrow{} \bullet \xrightarrow{} \bullet
\end{align*}
\]

The relations of type 2 are: $r_1 = \alpha_2 \beta_3$, $r_2 = \alpha_4 \beta_1$, $r_k = \alpha_{2m} \beta_{2m+1}$ and $r(k + 1) = \alpha_{2m+2} \beta_{2m-1}$, for some natural number $k$. The elementary cycles are: $C_1 = \beta_1 \alpha_1 \alpha_2$, $\ldots$, $C_{m+1} = \beta_{2m+1} \alpha_{2m+1} \alpha_{2m+2}$.

Therefore, by the relation $r(k + 1) = \alpha_{2m+2} \alpha_{2m-1}$, we determine the arrow $\alpha_{2m+2} \in \Sigma$. We conclude this part, and the proof is analogous to $\alpha_{n+1} \in \Sigma$.

Now, if the trivial extension $\Gamma$ have only elementary cycles with two arrows:
The relations of type 2 are: $r_1 = \alpha_2 \alpha_3$, $r_2 = \alpha_4 \alpha_1$, ..., $r_k = \alpha_{2m} \alpha_{2m+1}$ and $r(k+1) = \alpha_{2m+2} \alpha_{2m-1}$, for some natural number $k$. The elementary cycles are: $C_1 = \alpha_1 \alpha_2$, ..., $C_{m+1} = \alpha_{2m+1} \alpha_{2m+2}$.

The process is analogous to having the cutting set $\Sigma = \{\alpha_2, \alpha_4, \ldots, \alpha_{2m+2}\}$ that define the incidence algebra. And we can verify that the cutting set $\Sigma = \{\alpha_1, \alpha_3, \ldots, \alpha_{2m+1}\}$ defines the dual incidence algebra.

The last part is missing. We assume that there is an elementary cycle with three arrows or more in $\Gamma$, with $\alpha_1 \in \Sigma$ or $\alpha_n \in \Sigma$. Without loss of generality, we assume that exists the path $\beta_1$. We consider the subquiver:

The relations of type 2 are: $r_1 = \alpha_2 \beta_3$ and $r_2 = \alpha_4 \beta_1$. The elementary cycles are: $C_1 = \alpha_2 \beta_3 \alpha_1$ and $C_2 = \beta_3 \alpha_3 \alpha_4$.

By hypotheses, we have $\alpha_1 \in \Sigma$. Then we have to choose an arrow from the elementary cycle $C_2 = \beta_3 \alpha_3 \alpha_4$. Independently of the choice, we cannot rule out the relation $r_1 = \alpha_2 \beta_3$ or $r_2 = \alpha_4 \beta_1$. Therefore, $KQ_\Gamma/\langle I_\Gamma \cup \Sigma \rangle$ is not an incidence algebra. Similarly, we get the same result if $\alpha_n \in \Sigma$. \hfill $\Box$

Next, we will display the Phia algebras of ANS family of type $HV-\tilde{D}_n$.

The hardest part of the demonstration is in the case of the hereditary algebra $A$ of type $\tilde{D}_n$. The possibilities concerning the quiver of your path algebra are large, resulting in the huge amount of trivial extensions of $A$.

**Theorem 8.6.** The algebras associated with the quivers with relations below:
There are two cases in the substitution of subquiver \( \cdot \rightarrow \ldots \rightarrow \cdot \)

If we have an even number of edges then we replace by

\[
\cdot \rightarrow \cdot \leftarrow \ldots \rightarrow \cdot \leftarrow \cdot
\]

otherwise, we replace by \( \cdot \rightarrow \cdot \leftarrow \ldots \rightarrow \cdot \leftarrow \cdot \) are Phia algebras of the ANS family of type HV-\( \tilde{D}_n \) for \( n \geq 6 \).

**Proof.** We describe the script of the proof of each Phia algebra of this ANS family of type HV-\( \tilde{D}_n \) for \( n \geq 6 \):

- we consider each schurian concealed algebra \( A \) of type \( \tilde{D}_n \);
- we describe the trivial extensions of \( A \);
- we analyze the cutting sets that define an incidence algebra \( K\Delta \);
we use the theorem 8.1, proving that $K\Delta$ is a Phia algebra of type $\widehat{D}_n$.

We begin with the hereditary algebra $A$ of type $\widehat{D}_n$, with $n \geq 6$. First, we collect the configurations of the elementary cycles of the trivial extensions of $A$ of type $\widehat{D}_4$ and $\widehat{D}_5$. Then we apply in the trivial extensions of the hereditary algebra $A$ of type $\widehat{D}_n$. Thus, we can list all these trivial extensions, unless symmetry or dual algebra. We are enumerating with the characters $Fr2$ referring to the frame of the list of chapter XIV of the book “Elements of the representation theory of associative algebras” [35].

The dashed arrow symbolizes a path of length according to the number $n$, and the path may not even exist. We emphasize that in the relations of the trivial extensions that have these paths in their descriptions, we define these relations in the minimal form. If there is no path, the relation will be defined with the next arrow of the quiver that has coherence in the description.

We will show two studies on trivial extensions, the proof being analogous to the rest of the algebras.

\[ \widehat{D}_n - Fr2.1 \]

The relations of type 2 are: 
\[ r_1 = \alpha_4 \alpha_1 \beta \alpha_2 \alpha_6 \alpha_10, \] 
\[ r_2 = \alpha_3 \alpha_1 \beta \alpha_2 \alpha_6 \alpha_7, \] 
\[ r_3 = \alpha_7 \alpha_4 \alpha_1 \beta \alpha_2 \alpha_5, \] 
\[ r_4 = \alpha_9 \alpha_3 \alpha_1 \beta \alpha_2 \alpha_6, \] 
\[ r_5 = \alpha_9 \alpha_3 \alpha_1 \beta \alpha_2 \alpha_5, \] 
\[ r_6 = \alpha_8 \alpha_3 \alpha_1 \beta \alpha_2 \alpha_6, \] 
\[ r_7 = \alpha_3 \alpha_1 \beta \alpha_2 \alpha_5 \alpha_8 \] 
\[ r_8 = \alpha_4 \alpha_1 \beta \alpha_2 \alpha_5 \alpha_8. \]

The elementary cycles are: 
\[ C_1 = \alpha_9 \alpha_1 \alpha_3 \alpha_1 \beta \alpha_2, \] 
\[ C_2 = \alpha_5 \alpha_8 \alpha_3 \alpha_1 \beta \alpha_2, \] 
\[ C_3 = \alpha_5 \alpha_9 \alpha_4 \alpha_1 \beta \alpha_2 \] 
\[ C_4 = \alpha_6 \alpha_7 \alpha_4 \alpha_1 \beta \alpha_2. \]

As in the previous theorems referring to the type $\widehat{D}_4$ and $\widehat{D}_5$, there are four cutting sets define the Phia algebras 1 and 2, and their dual algebras. The Phia algebras 1 and 2 are defined from the cutting sets \{\alpha_1\} e \{\alpha_3, \alpha_4\}, respectively.

During the demonstration, we will use the dashed arrow representing a path where it is optional in the quiver. If there is no path, we will change \[ \bullet \longrightarrow \bullet \] by \[ \bullet. \] Moreover, the symbol \[ * * * \] in the quiver represents the possibility of placing subquivers of the form \[ \bullet \longrightarrow \bullet \longrightarrow \bullet \] or dual form. For example, we would replace \[ * * * \] by
In the search for non-hereditary Phia algebras, in the trivial extensions, we are using the lemma 6.9. This result requires that an arrow belongs to at least two elementary cycles. The following step shows the trivial extensions with two characteristics related to this hypothesis of the lemma 6.9:

- there is an arrow belonging to two elementary cycles;
- there are one or two pairs of elementary cycles with this property.

We will show the possible configurations for these pairs of elementary cycles:

\[ \tilde{D}_n - F_{r2.2} \]

The trivial extension has at least five elementary cycles. In this part of the proof, we will strongly use lemma 8.5.

The relations of type 2 are:

- \( r_1 = \alpha_4 \beta' \alpha_2 \alpha_8 \),
- \( r_2 = \alpha_3 \beta' \alpha_2 \alpha_7 \),
- \( r_3 = \alpha_2 \rho_2 \),
- \( r_4 = \rho_1 \alpha_7 \),
- \( r_5 = \rho_1 \alpha_8 \)

and the others is part of the subquiver below:

The elementary cycles are:

- \( C_1 = \beta' \alpha_2 \alpha_7 \alpha_4 \),
- \( C_2 = \beta' \alpha_2 \alpha_8 \alpha_3 \)

and the others are part of the previous subquiver.

Now, we will look at the configuration options pertaining to this trivial extension \( \Gamma \). First we assume that the path \( \beta \) has a length greater than or
equal to 1 or in place of the symbol \(*\ast\ast\ast\) is put some elementary cycle with three arrows or more. We will look for a cutting set \(\Sigma\) such that \(KQ_\Gamma/ \langle I_\Gamma \cup \Sigma \rangle\) is a non-hereditary incidence algebra. For this, we need to have \(\alpha_2 \in \Sigma\). Otherwise, if there is the path \(\beta'\), then any arrow in the path \(\beta'\) in \(\Sigma\) would have the relation \(\alpha_2 \rho_2\) or \(\rho_1 \alpha_7\) in \(KQ_\Gamma/ \langle I_\Gamma \cup \Sigma \rangle\). As \(\alpha_2 \in \Sigma\), implies that \(\rho_1 \in \Sigma\). Therefore, by the lemma 8.5, we obtain that \(KQ_\Gamma/ \langle I_\Gamma \cup \Sigma \rangle\) is not an incidence algebra.

It is necessary to look at the configurations where the path \(\beta\) does not exist, and in place of the symbol \(*\ast\ast\ast\) is placed only elementary cycle(s) with two arrows. As seen in the previous paragraph, we need to have \(\alpha_2 \in \Sigma\), implying \(\rho_1 \in \Sigma\). Again we use the lemma 8.5, implying a single cutting set. Consequently, we get a solution.

The solution has the quiver according to with length to the path \(\beta'\) and the number of elementary cycles in the middle of the trivial extension \(\Gamma\). For example, If \(\Gamma\) has the path \(\beta'\) with length one and only one elementary cycle in the middle of \(\Gamma\), so we get the following Phia algebra:

\[
\begin{array}{c}
\bullet & \leftrightarrow & \bullet \\
\downarrow & \downarrow & \downarrow \\
\bullet & \bullet & \bullet
\end{array}
\]

We separated in two solutions. If there is the path \(\beta'\) then we have the solution 6. Otherwise, we get Phia algebra 5.

Next we will proof two theorems related to the description of the Phia algebras of the ANS family of type HV-\(\tilde{E}_6\) and HV-\(\tilde{E}_7\).

**Theorem 8.7.** The algebras associated with the quivers with relations below:

1. \[
\begin{array}{c}
\bullet & \downarrow & \bullet \\
\downarrow & \bullet & \downarrow \\
\bullet & \bullet & \bullet
\end{array}
\]

2. \[
\begin{array}{c}
\bullet & \downarrow & \bullet \\
\downarrow & \bullet & \downarrow \\
\bullet & \bullet & \bullet
\end{array}
\]

3. \[
\begin{array}{c}
\bullet & \downarrow & \bullet \\
\downarrow & \bullet & \downarrow \\
\bullet & \bullet & \bullet
\end{array}
\]
are Phia algebras of the ANS family of type $HV\tilde{E}_6$.

Proof. The list has five frames, see [35]: $\mathcal{F}r_6$, $\mathcal{F}r_7$, $\mathcal{F}r_8$, $\mathcal{F}r_9$ and $\mathcal{F}r_{10}$. We will analyze only the schurian frames, satisfying the main hypothesis of the theory of the section 6. We will make the trivial extension of algebra originated from each frame. That is, given a frame $\mathcal{F}r$, we will apply the admissible operation and then exit the trivial extension of that algebra. Thus, we will put the necessary information for each trivial extension in the computer program and show the non-hereditary solutions.

We note that the concealed algebra of the type $\tilde{E}_6$ is always one of the solutions and we will omit it. We will show one study on the trivial extension, the proof being analogous to the rest of the algebras.

$\mathcal{F}r_6$

The adjacent Euclidean quiver $\tilde{E}_6$ has $2^6$ of possibilities of admissible operation 1, that is, we can have $2^6$ Euclidean graphs $\tilde{E}_6$. Now, our goal is to analyze the trivial extension of each graph and see if there is any cutting set that defines a non-hereditary Phia algebra. For this, in the trivial extension, we must have at least two elementary cycles that have at least one common arrow according to the lemma [6.9].
So we need to have two maximal paths that have at least length two and one arrow in common. Up to duality, there are three cases:

1. ![Diagram 1]
2. ![Diagram 2]
3. ![Diagram 3]

Up to symmetry, we will make all combinations of admissible operations 1 from the three previous cases.

The relations of type 2 are: $r_1 = \alpha_3 \alpha_2 \alpha_1 \alpha_6$ and $r_2 = \alpha_4 \alpha_2 \alpha_1 \alpha_5$.

The elementary cycles are: $C_1 = \alpha_7 \alpha_3 \alpha_2 \alpha_1 \alpha_5$ and $C_2 = \alpha_8 \alpha_4 \alpha_2 \alpha_1 \alpha_6$.

Then, the computer program shows us the cutting sets that define the algebras 1 and 2 of the theorem.

Before stating the theorem describing the Phia algebras of the ANS family of type HV-$\widehat{E}_7$, we show a lemma that we will use in the demonstration of the next theorem.

**Lemma 8.8.** Let $\Gamma$ be a trivial extension of a schurian algebra with at least three elementary cycles with the following conditions:

a) two elementary cycles have at least one arrow in common, we shall call this intersection by a path $\lambda$;

b) the third elementary cycle has a vertice in common with one of the two previous elementary cycles and has no intersection with the other;

c) the common vertice of the previous condition does not belong to the path $\lambda$;

For a better understanding of these conditions, we give the following example:
Let $\Sigma$ be a cutting set of $\Gamma$. If any arrow of $\lambda$ belongs to $\Sigma$ then $KQ_\Gamma/ < I_\Gamma \cup \Sigma >$ is not an incidence algebra.

**Proof.** Without loss of generality, we will consider $\Gamma$ as a trivial extension with three elementary cycles satisfying the conditions of the statement of the lemma:

We will denote $\theta$ as an optional path.

Thus, the relations of type 2 are: $r_1 = \gamma_1 \beta$, $r_2 = \beta' \gamma_2$, $r_3 = \alpha_m \lambda \gamma_1$ e $r_4 = \gamma_n \lambda \alpha_1$. The elementary cycles are: $C_1 = \alpha_1 \alpha_2 \ldots \alpha_{m-1} \alpha_m \lambda$, $C_2 = \lambda \gamma_1 \gamma_2 \ldots \gamma_{n-1} \gamma_n$ e $C_3 = \theta \beta' \beta$.

Assuming that some arrow in the path $\lambda$ belongs to the cutting set $\Sigma$. This hypothesis implies that it already has a representative arrow of the cycles $C_1$ and $C_2$ in the cutting set $\Sigma$. We will see that independently of the choice of the arrow of the elementary cycle $C_3$, the $KQ_\Gamma/ < I_\Gamma \cup \Sigma >$ is not an incidence algebra.

We assume that $\beta' \in \Sigma$, then the relation $r_1$ belongs to $< I_\Gamma \cup \Sigma >$. Consequently, $KQ_\Gamma/ < I_\Gamma \cup \Sigma >$ is not an incidence algebra.

Analogously, we arrive at the same result if $\beta \in \Sigma$ or any arrow of $\theta$ belongs to the cutting set. 

**Theorem 8.9.** The algebras associated with the quivers with relations below:
Proof. We consider the list of frames of concealed algebras, see \[36\]. In the frame part \(\tilde{\mathbb{E}}_7\) has 23 frames: \(\mathcal{F}r11, \ldots, \mathcal{F}r32\). As we explained at the beginning of this section, we will use only schurian frames. For each frame, we will make the admissible operation resulting in a concealed schurian algebra \(A\). Next, we will put the necessary information on the trivial extension \(T(A)\) in the computer program. Finally, the members of this ANS family of type \(HV-\tilde{\mathbb{E}}_7\) will be the algebras \(K\Delta\) defined by a cutting sets of \(T(A)\). We will show one study on the trivial extension, the proof being analogous to the rest of the algebras.

\(\mathcal{F}r11\)

We apply the admissible operation 1 in the Euclidean frame \(\tilde{\mathbb{E}}_7\), resulting in the hereditary algebra of type \(\tilde{\mathbb{E}}_7\), this algebra is a concealed algebra. Thanks to the theorem \[\ref{thm:8.1}\] we will analyze the trivial extension of the concealed algebra by noting the existence of some cutting set defines a non-hereditary Phia algebra. For this, in the trivial extension, we must have at least two elementary cycles that have at least one common arrow according to the lemma \[\ref{lem:6.9}\].

Therefore, we must have two maximal paths in hereditary algebras that are at least length 2 and one arrow in common. Up to dual graphs, the possibilities are:

\[
\begin{align*}
\mathcal{F}r11.1
\end{align*}
\]
We will investigate all the trivial extensions of the case 1. We will put aside the trivial extensions that satisfy the hypotheses of the lemma [8,8].

The symbol ** can be replaced by ● ●, or ● ● ●, or ● ● ●, or ● ● ●. One last remark, the path ● ---- ● has length one or two.

Therefore, the result of this filter, we obtain only the trivial extension:

A direct application of the lemma [6,9] we have the cutting set \{α\} that defines the solution 1.

The relations of type 2 are: \( r_1 = \alpha_3 \alpha_6 \alpha_7 \alpha_8 \alpha_5 \alpha_2, r_2 = \alpha_4 \alpha_6 \alpha_7 \alpha_8 \alpha_5 \alpha_1, r_3 = \alpha_9 \alpha_7, r_4 = \alpha_2 \alpha_4 \alpha_9 \) and \( r_5 = \alpha_10 \alpha_4 \alpha_6 \).

The elementary cycles are: \( C_1 = \alpha_1 \alpha_3 \alpha_6 \alpha_7 \alpha_8 \alpha_5, C_2 = \alpha_2 \alpha_4 \alpha_6 \alpha_7 \alpha_8 \alpha_5 \) and \( C_3 = \alpha_4 \alpha_9 \alpha_{10} \).

Then, the computer program shows us the cutting sets \{α_4, α_3\} and \{α_6, α_9\} that define the algebras 57 and 58 of the theorem.
Using these answers together with the lemma 8.8 we discard the following cases:

A work for the future is to apply the theorem 8.1 in frames $\mathcal{F}r_{33}, \ldots, \mathcal{F}r_{149}$ from the list of Happel and Vossieck [31] with an analogous demonstration of the previous theorem. This would solve the description of Phia algebras of the ANS family of type $\text{HV-}E_8$.

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