The Superalgebraic Approach to Supergravity

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Abstract

We formulate classical actions for \(N=1\) supergravity in \(D=(1,3)\) as a gauge theory of \(OSp(1|4)\). One may choose the action such that it does not include a cosmological term.

1 Introduction

The theory of supergravity \([1, 2]\) was discovered more than 20 years ago, but in spite of determined efforts we do not understand some very basic properties of this theory. There is, as of now, no manifestly supersymmetric action for the most interesting cases, namely in \(D=10\) and \(D=11\). It seems that ordinary superspace becomes exceedingly difficult to handle beyond \(N=1\) in \(D=4\). The introduction of harmonic variables \([3, 4]\) helps significantly, but they are introduced more as a (very clever) trick than from first principles. A further mystery arises from nonperturbative phenomena in string theory: the supersymmetry algebra includes tensor charges which point to \(OSp(1|32)\), but the relation of this symmetry group to \(D=11\) supergravity \([5]\), though suspected from the beginning, remains unclear.

It seems that we are missing some essential ingredients in the formulation of supergravity, which probably are not strictly necessary in the case \(N=1, D=4\), but which
become indispensable in higher dimensions or for higher $N$. In this paper we take a first step in our search for those ingredients by (partially) disentangling gauge symmetries and reparametrizations.

This was first done by MacDowell and Mansouri for gravity and supergravity [6]. The gauged superalgebra approach to supergravities they used is developed and carefully explained in [10]. A formulation in terms of compensating fields was given by Chamseddine [8] for gravity coupled to Spins 3/2, 1 and 1/2. The gravity case was examined in detail by Stelle and West [9]. We present supergravity as a partially compensated gauge theory of $OSp(1|4)$.

2 Gravity as a gauge theory

In order to describe gravity as a gauge theory of $SO(2, 3) = Sp(4, R)$ (in the anti-de Sitter case) or $SO(1, 4)$ (in the de Sitter case), one introduces a tangent space metric $\eta^{MN} = (-1, 1, 1, 1, \mp 1)$ and a connection 1-form $\omega^{MN}$, with $N = 0, 1, 2, 3, 5$. The connection should decompose as the usual fourdimensional Lorentz connection $\omega^{mn}$ and the vierbein $e^m = \rho^a e^m_5$. We may perform this split in a gauge-covariant way at the cost of introducing a compensator field $U^M$ that satisfies $U^M U^M = \mp \rho^2$. Then the role of the vierbein is played by the frame field

$$E^M = DU^M,$$

which has the property $E^M U_M = 0$. The covariant Lorentz-connection

$$\omega^M_L = \omega^{MN} \pm \frac{1}{\rho^2} \left(U^M E^N - E^M U^N\right),$$

is defined by $D_L U^M = dU^M + \omega^M_L U^N = 0$. $U^M$ describes locally a fourdimensional subspace of the fivedimensional tangent space we started with. The de Sitter curvature

$$\frac{1}{2} R^{MN} = \frac{1}{2} d\omega^m d\omega^n R_{mn}^{MN} = d\omega^{MN} + \omega^M_K \omega^K_N$$

(3)

decomposes as follows:

$$\frac{1}{2} R^{MN} = \frac{1}{2} R_{L}^{MN} \pm \frac{1}{\rho^2} \left(U^M T^N - T^M U^N\right) \pm \frac{1}{\rho^2} E^M E^N,$$

(4)

with $\frac{1}{2} R_{L}^{MN} = d\omega^M_L + \omega^M_K \omega^K_L$ and $T^M = DE^M = \frac{1}{2} R^{MN} U_N$. The action

$$S = \mp \frac{1}{64\pi \rho} \int_M \epsilon N_1...N_5 U^{N_1} R^{N_2 N_3 N_4 N_5}$$

$$= \mp \frac{1}{64\pi \rho} \int_M \epsilon N_1...N_5 U^{N_1} \left(R_{L}^{N_2 N_3 N_4 N_5} \pm \frac{4}{\rho^2} E^{N_2} E^{N_3} E^{N_4} E^{N_5} + \frac{4}{\rho^4} E^{N_2} E^{N_3} E^{N_4} E^{N_5}\right)$$

(5)
$(\epsilon^{01235} = 1)$ is manifestly reparametrization invariant and de Sitter gauge invariant. Upon gauge fixing $U^M = \mp \rho \delta^M_5$ we obtain

$$S = \mp 2\pi \chi(M_4) - \frac{1}{2\kappa^2} \int_{M_4} |e| R(e, \omega) \pm \frac{6\pi}{\kappa^4} \int_{M_4} |e|$$

with $R = e_m^m e_n^m R_{mn}^{nm}$ and $\kappa^2 = 2\pi \rho^2$, i.e. the usual Einstein action with a cosmological constant and a topological term. The presence of such terms is due to the simple choice of $S$ above, and has nothing to do with the fact that we formulated a gauge theory of the (anti-) de Sitter group. We may write a slightly more complex action that reproduces precisely Einstein gravity:

$$S_E = -\frac{1}{8\kappa^2 \rho} \int_{M_4} \epsilon_{N_1...N_5} U^{N_1} E^{N_1} E^{N_2} R^{N_1N_5}_{L} = -\frac{1}{2\kappa^2} \int_{M_4} |e| R(e, \omega) ,$$

which is again reparametrization and gauge covariant. We learn that the vacuum algebra, i.e. the symmetry algebra of (anti-) de Sitter or Minkowski space has little to do with the gauge algebra: we may choose the Poincare or the (anti-) de Sitter algebra for either one, and independently. The main advantage of choosing (5) or (7) is that the vacuum solutions $R^{MN} = 0$ resp. $R_{Lmn}^{np} = 0$ can be read off the action immediately.

### 3 $Sp(4, R)$

In order to formulate supergravity as a gauge theory it will prove useful to convert the vector notation of the previous section to a spinor one. We define the fundamental representation of $Sp(4, R) \sim O(2, 3)$ in terms of an $Sl(2, C)$-spinor as

$$\Psi^a = \left( \Psi^\alpha, \Psi^{\dot{\alpha}} \right) ,$$

with $(\Psi^\alpha)^* = \overline{\Psi}^{\dot{\alpha}}$. The invariant tensor $C^{ab} = (\epsilon_{\alpha\beta}, \epsilon_{\dot{\alpha}\dot{\beta}})$ and its inverse $\tilde{C}^{ab} = (\tilde{\epsilon}^{\alpha\beta}, \tilde{\epsilon}^{\dot{\alpha}\dot{\beta}})$ may be used to raise and lower spinor indices. The $Sp(4, R)$-generators $J_{ab}$ are symmetric bispinors. The covariant derivative $D\Psi^a = d\Psi^a + \omega^a_b \Psi^b$ leads to the curvatures $\frac{1}{2} R^{ab} = d\omega^{ab} + \omega^a_c \omega^{cb}$, which decompose as follows:

$$R^{\alpha\beta} = \frac{1}{4} \sigma^{\alpha\beta}_{mn} R^{mn} ; \quad R^{\dot{\alpha}\dot{\beta}} = \frac{1}{4} \tilde{\sigma}^{\dot{\alpha}\dot{\beta}}_{mn} R^{mn}$$

$$R^{\alpha}_{\beta} = \frac{1}{2} \sigma^{\alpha}_{\beta} R^{m5} ; \quad R^{\dot{\alpha}}_{\dot{\beta}} = \frac{1}{2} \tilde{\sigma}^{\dot{\alpha}}_{\dot{\beta}} R^{m5} .$$

The vector field $U^M$ now appears as an antisymmetric traceless bispinor:

$$U^{[ab]} = i \begin{pmatrix} U^5 \epsilon_{\alpha\beta} & U_m \sigma^{m\alpha\dot{\beta}} \\ -U_m \sigma^{m\dot{\alpha}\beta} & -U^5 \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} .$$

After gauge fixing we obtain the action (6) in the form given by MacDowell and Mansouri (3):

$$S = \frac{i \rho^2}{8\kappa^2} \int_{M_4} \left( R^{\alpha}_{\beta} R^{\beta}_{\alpha} - R^{\dot{\alpha}}_{\dot{\beta}} R^{\dot{\beta}}_{\dot{\alpha}} \right) .$$
4 \textit{OSp}(1|4) and Supergravity

We now upgrade the gauge algebra to \textit{OSp}(1|4), with fundamental representation $\Psi^A = (\Psi^a, \Psi^j) = (\Psi^A)^*$, grading

\[ (-)^A = \begin{cases} +1 & \text{for } A \in \{\alpha, \dot{\alpha}\} ; \quad \alpha, \dot{\alpha} \in \{1, 2\} \\ -1 & \text{for } A = j ; \quad j \in \{1\} \end{cases} \tag{12} \]

and invariant tensors

\[ C_{[AB]} = -(-)^A C_{BA} = (C_{ab}, i\delta_{ij}) ; \quad \tilde{C}^{[AB]} = -(-)^A \tilde{C}^{BA} = (\tilde{C}^{ab}, -i\delta_{ij}) \]

\[ C_{AB} C^{BC} = \tilde{C}^{CB} C_{BA} = \delta^B_A = (\delta^c_a, \delta^k_i) , \tag{13} \]

which raise and lower indices as follows:

\[ \Psi_A = C_{AB} \Psi^B ; \quad \Psi^A = \tilde{C}^{AB} \Psi_B . \tag{14} \]

The standard index contraction is

\[ \Psi_A \Phi^A = -(-)^A \Psi^A \Phi_A ; \quad (\Psi_A \Phi^A)^* = \Phi^A \Psi_A , \tag{15} \]

and upon introducing the graded symmetric gauge connection $\omega^{[AB]}$ we arrive at curvatures $R^{AB}$, which now contain fermionic (gravitino) curvatures $R^{aj}$ in addition to those listed in (9). The compensator field $U^{[AB]}$ is graded antisymmetric and traceless, and we require it to satisfy the conditions

\[ U^A_A = 0 ; \quad U_{AB} U^{BA} = 4 \rho^2 ; \quad U_{AB} U^B_C U^{CA} = 0 . \tag{16} \]

The action

\[ S = \frac{\rho}{4 \kappa^2} \int_{M_4} U^A_B R^B_C \left( \delta^C_E + \frac{1}{2\rho^2} U^C_D U^D_E \right) R^E_A (-)^A \tag{17} \]

is then manifestly reparametrization invariant (since it is a 4-form), and \textit{OSp}(1|4)-gauge invariant. In addition, it possesses a local supersymmetry which stays hidden because we are not working in superspace. We gauge fix $U^{AB}$ by the local \textit{OSp}(1|4) as follows:

\[ U^A_B = \begin{pmatrix} i\rho \delta^\alpha_\beta & 0 & 0 \\ 0 & -i\rho \delta^{\dot{\alpha}}_{\dot{\beta}} & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{18} \]

and (17) takes the form

\[ S = \frac{i \rho^2}{8 \kappa^2} \int_{M_4} \left( R^\alpha_\beta R^\beta_\alpha + 2 R^\alpha_j R^j_\alpha - R^\alpha_{\dot{\beta}} R^{\dot{\beta}}_\alpha - 2 R^{\dot{\alpha}}_j R^{\dot{j}}_\dot{\alpha} \right) , \tag{19} \]
which may be rewritten as

\[
S = -\frac{\rho^2}{32\kappa^2} \int_{M_4} d\left[ \epsilon_{mnpq} (\omega^{mn} d\omega^{pq} + \frac{2}{3} \omega^{mn} \omega^{pl} \omega_{pq}^l) + \frac{16\kappa^2}{\rho} \psi_\alpha D^\alpha \psi_\alpha - \frac{16\kappa^2}{\rho} \overline{\psi} \cdot D\overline{\psi} \right] \\
- \frac{1}{2\kappa^2} \int_{M_4} |e| R(e, \omega) \\
+ \frac{1}{2} \int_{M_4} |e| \epsilon_{mnpq} \left( \overline{\psi}_m \stackrel{\dot{\alpha}}{\sigma} \alpha \beta \gamma D^\alpha \psi_{n\alpha} - \psi_m \sigma_{\alpha \beta} \overline{\psi}_q \right) \\
+ \int_{M_4} |e| \left( \frac{3}{\kappa^2 \rho^2} - \frac{i}{2\rho} \left[ \psi_{m\alpha} \sigma^{mn\alpha \beta} \psi_n \beta + \overline{\psi}_{m\dot{\alpha}} \sigma^{mn\dot{\alpha} \dot{\beta}} \overline{\psi}_n \dot{\beta} \right] \right),
\]

(20)

where \( D^\alpha_m \) contains only the Lorentz connection. It is well known that (20) is supersymmetric.

### 5 Supersymmetry

In order to make this symmetry transparent, we compute the variation of the above action under infinitesimal variations of the fermionic compensators:

\[
\delta S \propto \int_{M_4} R^\alpha_\beta \left( \delta U^\dot{\beta} \dot{j} R^j_\alpha + R^\dot{\beta} \dot{j} \delta U^j_\alpha \right)
\]

(21)

and compare it with the effect of arbitrary infinitesimal variations of the Lorentz connection:

\[
\delta S \propto \int_{M_4} R^\alpha_\beta \omega^\gamma \gamma R^\alpha \beta \delta \omega \dot{\beta} + R^\alpha_\beta \omega^\gamma \gamma R^\alpha \dot{\beta} \delta \omega \dot{\beta} .
\]

(22)

we may either determine \( \delta \omega^\alpha_\beta, \delta \omega^\dot{\alpha}_\dot{\beta} \) such that the sum of both variations vanishes (1st order formalism) or we may solve the algebraic equations of motion for \( \omega^\alpha_\beta, \omega^\dot{\alpha}_\dot{\beta} \), i.e. set \( R^\alpha_\dot{\beta} = 0 \) (1.5 order formalism). In either case we have shown that there is a hidden fermionic symmetry, which turns out to be supersymmetry:

\[
\delta \omega^\alpha_\beta = \omega^\alpha_\beta \lambda^\beta \beta - \lambda^\alpha_\beta \omega\dot{\beta} \dot{\beta} \\
\delta \omega^\dot{\alpha}_\dot{\beta} = D^\alpha \lambda^\alpha_\beta + \omega^\alpha_\beta \lambda^\beta \dot{\beta} .
\]

(23)

In this picture supersymmetry looks like a gauge symmetry, because it is inherited from a bona fide gauge theory of a graded Lie algebra. The hidden aspect of this symmetry is, that in the sense expressed by (21) and (22), the action is in fact independent of the fermionic fields \( U^{\alpha\dot{\beta}} \), and hence the fermionic gauge symmetry is true and uncompensated.

### 6 Projectors

The compensators \( U^{A B} \) allow us to distinguish an \( Sl(2, C) \) subgroup within \( OSp(1|4) \) in a gauge covariant way. This is most clearly seen by the construction of projection operators:

\[
\Pi^{(a)}_{(b)} = -\frac{1}{\rho^2} U^A \epsilon^B = \left( \delta^\alpha_\beta, \delta^\dot{\alpha}_\dot{\beta}, 0 \right)
\]
\[ \Pi^{(\alpha)}_{(\beta)} = \frac{1}{2} \left( -\frac{i}{\rho} U^{A} B - \frac{1}{\rho^2} U^{A} C U^{C} B \right) = (\delta^\alpha_{\beta}, 0, 0) \]

\[ \Pi^{(\bar{\alpha})}_{(\bar{\beta})} = \frac{1}{2} \left( \frac{i}{\rho} U^{A} B - \frac{1}{\rho^2} U^{A} C U^{C} B \right) = (0, \delta^{\bar{\alpha}}_{\bar{\beta}}, 0) \]

\[ \Pi^{(i)}_{(j)} = \delta^{A}_{B} + \frac{1}{\rho^2} U^{A} C U^{C} B = (0, 0, \delta^{i}_{j}) . \]  

(24)

The last equality in each line holds of course only after fixing the gauge \([18]\). Proving the projection property requires some computation, but relies only on the properties \([16]\) of the compensator field, which imply among other things \(U^{a}_{j} = \frac{1}{\rho^2} U^{a}_{j} U^{b}_{j} U^{j}_{a} \) and as a consequence \(U^{A} B U^{B} C U^{C} D = -\rho^2 U^{A} D \). We use the notation \(\Psi^{(\alpha)} = \Pi^{(\alpha)}_{(\beta)} \Psi^{B} \), and may then formulate \([17]\) as

\[ S = \frac{i \rho^2}{8 \kappa^2} \int_{M_4} \left( R^{(\alpha)}_{(\beta)} R^{(\bar{\beta})}_{(\alpha)} + 2 R^{(\alpha)}_{(j)} R^{(\bar{\beta})}_{(\alpha)} - R^{(\alpha)}_{(\beta)} R^{(\bar{\beta})}_{(\alpha)} - 2 R^{(\bar{\alpha})}_{(j)} R^{(\bar{\beta})}_{(\alpha)} \right) , \]  

(25)

which is the \(OSp(1|4)\)-invariant form of the MacDowell-Mansouri action. Once one knows the projectors it is also possible to perform the \(SL(2,C)\)-decomposition covariantly. We set for simplicity \(\rho = 1\) and start with

\[ \omega^{A}_{B} = \omega^{A}_{C} B + U^{A} C E^{C} D (1 + U^2)^{D}_{B} - (1 + U^2)^{A} C E^{C} D U^{D} B \]

\[ -\frac{1}{4} \left( U^{A} C E^{C} D (U^2)^{D}_{B} - (U^2)^{A} C E^{C} D U^{D} B \right) , \]  

(26)

where the Lorentz connection \(\omega^{A}_{C} B\) is defined by \(D^{C} U^{A} B = 0\), and \(E^{A}_{B} = D U^{A}_{B}\). Then \([17]\) is decomposed as follows:

\[ S = \frac{1}{8 \kappa^2} \int_{M_4} \left( -A U^{A}_{B} \left[ R_{C} R_{C} \right]^{B}_{A} + 8 d \left( -A U^{A}_{B} E B_{C} D^{C} E^{C} \right) \right) \]

\[ + 4(-A U^{A}_{B} \left[ E (1 + U^2) R_{C} E - E (1 + U^2) E U^{2} E (1 + U^2) E \right]^{B}_{A} \]

\[ -(-A U^{A}_{B} \left[ E U^{2} E R_{C} - 8 E U E D^{C} E \right]^{B}_{A} \]

\[ + (-A U^{A}_{B} \left[ \frac{1}{4} E U E U E U E + 4 E U E E (1 + U^2) E \right]^{B}_{A} . \]  

(27)

The first line of \([27]\) is topological, the second line vanishes upon gauge fixing, since the index \(j\) can take only one value, the third line yields the kinetic terms for gravitons and gravitinos, and the last line describes the supersymmetric cosmological constant.

### 7 The Cosmological Term

The decomposition of the action \([17]\) in the form \([20]\) shows that we have obtained a supersymmetric cosmological term. One may be tempted to regard this as an inherent feature of any \(OSp(1|4)\) gauge theory of gravity. \([27]\) implies that this is not so. We
easily extract pure supergravity without cosmological constant:

\[
S_{\varepsilon+3/2} = \frac{1}{8\rho k^2} \int_{M_4} (-)^A U^A_B \left( 4DU_B^{(c)} DU^{(c)}_D R^D_A - 3DU^{(d)}_B DU^{(d)}_A \right) \\
- \frac{2}{\rho^2} DU^B_C DU^C_D DU^D_E DU^E_A + \frac{3}{2\rho^2} DU^{(c)}_B DU^{(d)}_A DU^{(d)}_A DU^{(c)}_A \\
= - \frac{1}{8k^2\rho^3} \int_{M_4} (-)^A U^A_B \left[ EU^2_1 E R_{L} - 8 EU ED^c E \right] B_A \\
= - \frac{1}{2k^2} \int_{M_4} |e| R(e, \omega) + \frac{1}{2} \int_{M_4} |e| \epsilon^{mnpq} \left( \bar{\psi}_m^{\dot{\alpha}} \sigma_n^{\dot{\alpha} \dot{\beta}} D^c \psi_{p\beta} - \psi_m^{\alpha} \sigma_n^{\alpha \beta} D^c \bar{\psi}_{p\dot{\beta}} \right).
\]

(28)

Admittedly the first line of (28) does not look particularly elegant. We understand it only by rewriting it in the form of line two. Without mentioning compensators Chamseddine and West wrote this action quite early in the development of supergravity [7]. It proves our point that the gauge algebra is independent of the vacuum algebra, i.e. the algebra of isometries of the vacuum state.

8 Conclusions

We have presented both gravity and supergravity as partially compensated gauge theories. The results of section 6 make it easy to formulate any Lorentz-covariant theory in terms of OSp(1|4), without the need for a group contraction. The significance of this gauge group lies then in the simplicity of (17) and the hidden supersymmetry.

We believe that one can extend this analysis to include additional fermionic as well as bosonic coordinates. This should lead to a natural and simple form of supergravity in superspace.

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