REIDEMEISTER THEORY OF ITERATIONS OF ENDMORPHISMS AND POLY-BIEBERBACH GROUPS

ALEXANDER FEL’SHHTYN AND JONG BUM LEE

ABSTRACT. We develop the Reidemeister theory of iterations of a group endomorphism \( \varphi \) and study the asymptotic behavior of the sequence of the Reidemeister numbers \( \{ R(\varphi^k) \} \), the essential periodic [\( \varphi \)]-orbits and the heights of \( \varphi \) on poly-Bieberbach groups.

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1. Introduction

Let \( f : X \to X \) be a map on a connected compact polyhedron \( X \) and \( p : \tilde{X} \to X \) be the universal covering projection onto \( X \) and \( \tilde{f} : \tilde{X} \to \tilde{X} \) a fixed lift of \( f \). Let \( \Pi \) be the group of covering transformations of the projection \( p : \tilde{X} \to X \). Then \( f \) induces an endomorphism \( \varphi = \varphi_f : \Pi \to \Pi \) by the following identity \( \varphi(\alpha) \tilde{f} = \tilde{f} \alpha, \ \forall \alpha \in \Pi \). The subsets \( p(\text{Fix}(\alpha \tilde{f})) \subset \text{Fix}(f) \), \( \alpha \in \Pi \), are called fixed point classes of \( f \). A fixed point class is called essential if its index is nonzero. The number of essential fixed point classes is called the Nielsen number of \( f \), denoted by \( N(f) \) [20].

The Nielsen number is always finite and is a homotopy invariant lower bound for the number of fixed points of \( f \). In the category of compact, connected polyhedra the Nielsen number of a map is, apart from in certain exceptional cases, equal to the least number of fixed points of maps with the same homotopy type as \( f \).

Let \( \varphi : \Pi \to \Pi \) be an endomorphism on an arbitrary group \( \Pi \). Consider the Reidemeister action of \( \Pi \) on \( \Pi \) determined by the endomorphism \( \varphi \) as follows:

\[
\Pi \times \Pi \to \Pi, \quad (\gamma, \alpha) \mapsto \gamma \alpha \varphi(\gamma)^{-1}.
\]

The Reidemeister (twisted conjugacy) class containing \( \alpha \) is an orbit of this action and will be denoted by \( [\alpha] \), and the set of Reidemeister classes of \( \Pi \) determined by \( \varphi \) will be denoted by \( \mathcal{R}[\varphi] \). Write \( R(\varphi) = \# \mathcal{R}[\varphi] \), called the Reidemeister number of \( \varphi \). When the endomorphism \( \varphi : \Pi \to \Pi \) is induced from a self-map \( f : X \to X \), i.e., when \( \varphi = \varphi_f \), we also refer to \( \mathcal{R}[\varphi] \) as the set \( \mathcal{R}[f] \) of Reidemeister classes of \( f \), and \( R(\varphi) \) as...
the Reidemeister number $R(f)$ of $f$. In favorable situations the Reidemeister number $R(f)$ calculates the Nielsen number $N(f)$.

It is easy to observe that if $\psi$ is an automorphism on $\Pi$, then $\psi$ sends the Reidemeister class $[\alpha]$ of $\varphi$ to the Reidemeister class $[\psi(\alpha)]$ of $\psi \varphi^{-1}$. Hence the Reidemeister number is an automorphism invariant. For any $\beta \in \Pi$, let $\tau_\beta$ denote the inner automorphism determined by $\beta$. We will compare $R[\varphi]$ with $R[\tau_\beta \varphi]$. Observe that the right multiplication $r_{\beta^{-1}}$ on $\Pi$ induces a bijection $R[\varphi] \to R[\tau_\beta \varphi]$, $[\alpha] \mapsto [\alpha \beta^{-1}]$. Indeed,

$$r_{\beta^{-1}} : \gamma \cdot \alpha \cdot \varphi(\gamma)^{-1} \longmapsto (\gamma \cdot \alpha \cdot \varphi(\gamma)^{-1}) \beta^{-1} = \gamma \cdot (\alpha \beta^{-1}) \cdot \beta \varphi(\gamma)^{-1} \beta^{-1} = \gamma \cdot (\alpha \beta^{-1}) \cdot (\tau_\beta \varphi)(\gamma)^{-1}.$$  

Similarly, we can show that $r_{\beta \varphi(\beta) \cdots \varphi^{n-1}(\beta)^{-1}}$ induces a bijection $R[\varphi^n] \to R[(\tau_\beta \varphi)^n]$, $[\alpha]^n \mapsto [\alpha(\beta \varphi(\beta) \cdots \varphi^{n-1}(\beta)^{-1})^{-1}]^n$. Hence the Reidemeister number is a conjugate invariant. This is well-expected because of the fact that if $f$ and $g$ are homotopic, then their induced endomorphisms differ by an inner automorphism $\tau_\beta$.

The set $\text{Fix}(f^n)$ of periodic points of $f$ splits into a disjoint union of periodic point classes $p(\text{Fix}(\alpha f^n))$ of $f$, and these sets are indexed by the Reidemeister classes $[\alpha]^n \in R[\varphi^n]$ of the endomorphism $\varphi^n$ where $\varphi = \varphi_f$. Namely,

$$(D) \quad \text{Fix}(f^n) = \coprod_{[\alpha]^n \in R[\varphi^n]} p\left(\text{Fix}(\alpha f^n)\right).$$

From the dynamical point of view, it is natural to consider the Nielsen numbers $N(f^k)$ and the Reidemeister numbers $R(f^k)$ and $R(\varphi^k)$ of all iterations of $f$ and $\varphi$ simultaneously. For example, N. Ivanov [18] introduced the notion of the asymptotic Nielsen number, measuring the growth of the sequence $N(f^k)$ and found the basic relation between the topological entropy of $f$ and the asymptotic Nielsen number. Later on, it was suggested in [7, 30, 8, 9, 10] to arrange the Nielsen numbers $N(f^k)$, the Reidemeister numbers $R(f^k)$ and $R(\varphi^k)$ of all iterations of $f$ and $\varphi$ into the Nielsen and the Reidemeister zeta functions

$$N_f(z) = \exp\left(\sum_{k=1}^{\infty} \frac{N(f^k)}{k} z^k\right),$$

$$R_f(z) = \exp\left(\sum_{k=1}^{\infty} \frac{R(f^k)}{k} z^k\right), \quad R_\varphi(z) = \exp\left(\sum_{k=1}^{\infty} \frac{R(\varphi^k)}{k} z^k\right).$$

The Nielsen and Reidemeister zeta functions are nonabelian analogues of the Lefschetz zeta function

$$L_f(z) = \exp\left(\sum_{k=1}^{\infty} \frac{L(f^k)}{k} z^k\right),$$

where

$$L(f^n) := \sum_{k=0}^{\dim X} (-1)^k \text{tr} \left[f_{*k}^n : H_k(X; \mathbb{Q}) \to H_k(X; \mathbb{Q})\right]$$

is the Lefschetz number of the iterate $f^n$ of $f$.

Nice analytical properties of $N_f(z)$, $R_f(z)$ and $R_\varphi(z)$ [30, 11, 10, 4, 4] indicate that the numbers $N(f^k)$, $R(f^k)$ and $R(\varphi^k)$ are closely interconnected. Another manifestations of this are Gauss congruences. Whenever all $R(f^k)$ are finite, we have

$$\sum_{d \mid k} \mu\left(\frac{k}{d}\right) \frac{k}{d} R(f^d) = \sum_{d \mid k} \mu\left(\frac{k}{d}\right) N(f^d) \equiv 0 \mod k.$$
for any $k > 0$, where $f$ is a map on an infra-solvmanifold of type $(R)$ \[13\]. It is known that the Reidemeister numbers of the iterates of an automorphism $\varphi$ of an almost polycyclic group also satisfy Gauss congruences \[15\], \[16\].

The fundamental invariants of $f$ used in the study of periodic points are the Lefschetz numbers $L(f^k)$, and their algebraic combinations, the Nielsen numbers $N(f^k)$ and the Nielsen-Jiang periodic numbers $NP_n(f)$ and $N\Phi_n(f)$, and the Reidemeister numbers $R(f^k)$ and $R(\phi^k)$.

The study of periodic points by using the Lefschetz theory has been done extensively by many authors in the literatures such as \[21\], \[2\], \[5\], \[29\]. Utilizing the arguments employed mainly in \[2\] and \[19\], Chap. III for the Lefschetz theory for the iterations of any endomorphism $f$ using the Nielsen theory of maps $\Pi$. In this paper, we will study the asymptotic behavior of the sequence $\{R(f^k)\}$, the essential periodic $[\varphi]$-orbits and the heights of $\varphi$ on poly-Bieberbach groups. We refer to \[13\], \[14\] for background of our present work.

Acknowledgments. The first author is indebted to the Max-Planck-Institute for Mathematics(Bonn) and Sogang University(Seoul) for the support and hospitality and the possibility of the present research during his visits there. The second author is partially supported by Basic Science Researcher Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (No.2013R1A1A2058693) and by the Sogang University Research Grant of 2010 (10022).

2. Preliminaries

Recall that the periodic point set $\text{Fix}(f^n)$ splits into a disjoint union of periodic point classes

$$\text{Fix}(f^n) = \bigcup_{[\alpha] \in R(\varphi^n)} p\left(\text{Fix}(\alpha f^n)\right).$$

Consequently, there is an 1-1 correspondence $\eta$ from the set of periodic point classes $p(\text{Fix}(\alpha f^n))$ to the set of Reidemeister classes $[\alpha]^n$ of $\varphi^n$. When $m \mid n$, $\text{Fix}(f^m) \subset \text{Fix}(f^n)$. Let $x \in \text{Fix}(f^m)$ and $\tilde{x} \in p^{-1}(x)$. Then there exist unique $\alpha, \beta \in \Pi$ such that $\alpha f^m(\tilde{x}) = \tilde{x}$ and $\beta f^n(\tilde{x}) = \tilde{x}$. It can be easily derived that

$$\beta = \alpha \varphi^m(\alpha) \varphi^{2m}(\alpha) \cdots \varphi^{n-m}(\alpha).$$

This defines two natural functions, called boosting functions,

$$\gamma_{m,n} : p\left(\text{Fix}(\alpha f^m)\right) \mapsto p\left(\text{Fix}(\alpha \varphi^m(\alpha) \varphi^{2m}(\alpha) \cdots \varphi^{n-m}(\alpha) f^n)\right),$$

$$\iota_{m,n} = \iota_{m,n}(\varphi) : [\alpha]^m \mapsto [\alpha \varphi^m(\alpha) \varphi^{2m}(\alpha) \cdots \varphi^{n-m}(\alpha)]^n$$

so that the following diagram is commutative

$$
\begin{array}{ccc}
\text{Fix}(f^m) & \xrightarrow{\gamma_{m,n}} & \text{Fix}(\alpha \varphi^m(\alpha) \varphi^{2m}(\alpha) \cdots \varphi^{n-m}(\alpha) f^n) \\
\text{Fix}(f^m) & \xrightarrow{\iota_{m,n}} & [\alpha]^m \\
\end{array}
$$

Moreover, it is straightforward to check the commutativity of the diagram
Similarly, \( f \) induces a well-defined function on the Reidemeister classes of \( \varphi^n \), which we will denote by \([\varphi]\), given by \([\varphi] : [\alpha]^n \mapsto [\varphi(\alpha)]^n\). Then the following diagram is commuting:

\[
\begin{array}{ccc}
[\alpha]^n & \xrightarrow{[\varphi]} & [\varphi(\alpha)]^n \\
\downarrow{\varphi} & & \downarrow{[\varphi]} \\
[\alpha]^n & \xrightarrow{[\varphi]} & [\varphi(\alpha)]^n
\end{array}
\]

By [20] Theorem III.1.12, \([f]\) is an index-preserving bijection on the periodic point classes of \( f^n \). We say that \([\alpha]^n\) is **essential** if the corresponding class \( p(\Fix(\alpha \tilde{f}^n)) \) is essential. Evidently,

\[
\Fix(\alpha \tilde{f}^n) \xrightarrow{\tilde{f}} \Fix(\varphi(\alpha) \tilde{f}^n) \xrightarrow{\alpha \tilde{f}^{-1}} \Fix(\alpha \tilde{f}^n)
\]

This implies that for each \( \alpha \in \Pi \), the restrictions of \( f \)

\[
f| : p\left(\Fix(\alpha \tilde{f}^n)\right) \longrightarrow p\left(\Fix(\varphi(\alpha) \tilde{f}^n)\right)
\]

are homeomorphisms such that \([f]^n\) is the identity. In particular,

\[
p\left(\Fix(\alpha \tilde{f}^n)\right) = \emptyset \iff p\left(\Fix(\varphi(\alpha) \tilde{f}^n)\right) = \emptyset.
\]

Moreover, \([\varphi]^n\) is the identity, \( t_{\Pi, n} \circ [\varphi] = [\varphi] \circ t_{\Pi, n} \) and \( \gamma_{\Pi, n} \circ [f] = [f] \circ \gamma_{\Pi, n} \).

The **length** of the element \([\alpha]^n \in \mathcal{R}[\varphi^n]\), denoted by \( \ell([\alpha]^n) \), is the smallest positive integer \( \ell \) such that \([\varphi]^\ell([\alpha]^n) = [\alpha]^n\). The **\( [\varphi] \)-orbit** of \([\alpha]^n\) is the set

\[
([\alpha]^n) = \{ [\alpha]^n, [\varphi([\alpha]^n)], \ldots, [\varphi]^{\ell-1}([\alpha]^n) \},
\]

where \( \ell = \ell([\alpha]^n) \). We must have that \( \ell \mid n \). The element \([\alpha]^n \in \mathcal{R}[\varphi^n]\) is **reducible** to \( m \) if there exists \([\beta]^m \in \mathcal{R}[\varphi^m]\) such that \( t_{\Pi, n}([\beta]^m) = [\alpha]^n \). Note that if \([\alpha]^n\) is reducible to \( m \), then \( m \mid n \). If \([\alpha]^n\) is not reducible to any \( m < n \), we say that \([\alpha]^n\) is **irreducible**. The **depth** of \([\alpha]^n\), denoted by \( d([\alpha]^n) \), is the smallest integer \( m \) to which \([\alpha]^n\) is reducible. Since clearly \( d([\alpha]^n) = d([\varphi([\alpha]^n)]) \), we can define the **depth** of the orbit \( ([\alpha]^n) \): \( d(([\alpha]^n)) = d([\alpha]^n) \). If \( n = d([\alpha]^n) \), the element \([\alpha]^n\) or the orbit \( ([\alpha]^n) \) is called **irreducible**.

Clearly, as a set \( p(\Fix(\alpha \tilde{f}^m)) \subset p(\Fix(\alpha \varphi^m(\alpha) \varphi^{2m}(\alpha) \cdots \varphi^{n-m}(\alpha) \tilde{f}^n)) \). This implies that if \( p(\Fix(\alpha \tilde{f}^m)) \) is the periodic point class of \( f^m \) determined by \( x \), then

\[
p\left(\Fix(\alpha \varphi^m(\alpha) \varphi^{2m}(\alpha) \cdots \varphi^{n-m}(\alpha) \tilde{f}^n)\right)
\]

is the periodic point class of \( f^n \) determined by \( x \).
Note that if \([\alpha]^n\) is irreducible, then every element of the fixed point class \(p(\text{Fix}(\alpha \tilde{f}^n))\) is a periodic point of \(f\) with minimal period \(n\). Let \([\alpha]^n\) be an essential class with depth \(m\) and let \(\iota_{m,n}(\beta^m) = [\alpha]^n\). Then there is a periodic point \(x\) of \(f\) with minimal period \(m\). Consequently, the irreducibility of a periodic Reidemeister class of \(\varphi\) is an algebraic counterpart of the minimal period of a periodic point of \(f\). We say that a periodic Reidemeister class \([\alpha]^n\) of \(\varphi\) has height \(n\) if it is irreducible. The set \(\mathcal{I}(\varphi^n)\) of all classes in \(\mathcal{R}[\varphi^n]\) with height \(n\) is an algebraic analogue of the set \(P_n(\varphi)\) of periodic points of \(\varphi\) with minimal period \(n\). Let \(\mathcal{I}(\varphi)\) be the set of all irreducible classes of \(\varphi\). That is,

\[
\mathcal{I}(\varphi) = \{[\alpha]^k \in \mathcal{R}[\varphi^k] \mid \alpha \in \Pi, k > 0, [\alpha]^k\text{ is irreducible}\}.
\]

We define the set \(\mathcal{H}(\varphi)\) of all heights of \(\varphi\) to be

\[
\mathcal{H}(\varphi) = \{k \in \mathbb{N} \mid \text{some } [\alpha]^k \text{ has height } k\}.
\]

Then \(\mathcal{H}(\varphi)\) is an algebraic analogue of the set \(\text{Per}(\varphi)\) of all minimal periods of \(\varphi\). Motivated from homotopy minimal periods of \(\varphi\), we may define the set of all homotopy heights of \(\varphi\) as follows:

\[
\mathcal{HI}(\varphi) = \bigcap_{\beta \in \Pi} \{n \in \mathbb{N} \mid \mathcal{I}(\tau(\beta^n\varphi^k)) \neq \emptyset\}\).
\]

However, as we have observed before, since the boosting functions \(\iota_{m,n}\) commutes with “right multiplications”, i.e.,

\[
\iota_{m,n}(\tau(\beta^m\varphi^k)) \circ r_{(\beta^m\varphi^k)^{-1}} = r_{(\beta^m\varphi^k)^{-1}} \circ \iota_{m,n}(\varphi),
\]

it follows that the height is a conjugate invariant. Consequently, we have \(\mathcal{HI}(\varphi) = \mathcal{H}(\varphi)\).

### 3. Poly-Bieberbach groups

The fundamental group of an infra-solvmanifold is called a poly-Bieberbach group, which is a torsion free poly-crystallographic group. It is known (see for example [3] Theorem 2.12]) that every poly-Bieberbach group is a torsion-free virtually poly-\(\mathbb{Z}\) group. We refer to [32] Theorem 3] for a characterization of poly-crystallographic groups. Recall also from [32] Corollary 4] that for any poly-Bieberbach group \(\Pi\) there exist a connected simply connected supersolvable Lie group \(S\), a compact subgroup \(K\) of \(\text{Aut}(S)\) and an isomorphism \(\iota\) of \(\Pi\) onto a discrete cocompact subgroup of \(S \times K\) such that \(\iota(\Pi) \cdot S\) is dense in \(S \times K\). By [11] Lemma 2.1], the supersolvable Lie groups are the Lie groups of type (R), i.e., if \(\text{ad } X : \mathfrak{G} \to \mathfrak{G}\) has only real eigenvalues for all \(X\) in the Lie algebra \(\mathfrak{G}\) of \(S\). Assuming \(\iota\) to be an inclusion or identifying \(\Pi\) with \(\iota(\Pi)\), we have the following commutative diagram

\[
\begin{array}{ccc}
1 & \longrightarrow & S \\
\uparrow & & \uparrow \\
1 & \longrightarrow & \Pi \cap S \\
\end{array}
\begin{array}{ccc}
& \longrightarrow & S \times K \\
& \uparrow & \uparrow \\
& \Pi & \longrightarrow \pi(\Pi) \\
\end{array}
\begin{array}{ccc}
& \longrightarrow & K \\
& \uparrow & \uparrow \\
& & 1 \\
\end{array}
\]

However, we cannot longer assume that \(\pi(\Pi)\) is a finite group and that the translations \(\Pi \cap S\) form a lattice of the solvable Lie group \(S\) of type (R).

In this paper, we will assume the following: Let \(\Pi\) be a poly-Bieberbach group which is the fundamental group of an infra-solvmanifold of type (R), i.e., \(\Pi\) is a discrete cocompact subgroup of \(\text{Aff}(S) := S \times \text{Aut}(S)\), where \(S\) is a connected, simply connected solvable Lie group of type (R) and \(\Pi \cap S\) is of finite index in \(\Pi\) and a lattice of \(S\). The finite group \(\Phi := \Pi/\Pi \cap S\) is called the holonomy group of the poly-Bieberbach group \(\Pi\) or the infra-solvmanifold \(\Pi \cap S\) of type (R). Naturally \(\Phi\) sits in \(\text{Aut}(S)\). Let \(\varphi : \Pi \to \Pi\) be an endomorphism. Then by [24] Theorem 2.2], \(\varphi\) is semi-conjugate by an “affine map”. Namely, there exist \(d \in S\) and a Lie group endomorphism \(D : S \to S\) such that \(\varphi(\alpha)(d, D) = (d, D)\alpha\) for all \(\alpha \in \Pi \subset \text{Aff}(S)\). From this identity condition, the
affine map \( \hat{f} := (d, D) : S \to S \) restricts to a map \( f : \Pi \backslash S \to \Pi \backslash S \) for which it induces the endomorphism \( \varphi \). Conversely, if \( f \) is a self-map on an infra-solvmanifold \( \Pi \backslash S \) of type \((R)\), \( f \) induces an endomorphism \( \varphi = \varphi_f \), see Section \[1\]. As remarked above, \( f \) is homotopic to a map induced by an affine map on \( S \). Since the Lefschetz, Nielsen and Reidemeister numbers of \( f \) are homotopy invariants, we may assume that our \( f \) has an affine lift \((d, D)\) on \( S \).

**Theorem 3.1** ([3] Corollary 7.6]). Let \( \varphi : \Pi \to \Pi \) be an endomorphism on a poly-Bieberbach group \( \Pi \) of \( S \) with holonomy group \( \Phi \). If \( \varphi \) is the semi-conjugate by an affine map \((d, D)\) on \( S \), then we have

\[
R(\varphi^k) = \frac{1}{\# \Phi} \sum_{A \in \Phi} \sigma \left( \det(I - A_* D^k) \right)
\]

where \( \sigma : \mathbb{R} \to \mathbb{R} \cup \{\infty\} \) is given by \( \sigma(0) = \infty \) and \( \sigma(x) = |x| \) for all \( x \neq 0 \). Furthermore, if \( R(\varphi^k) < \infty \) then \( R(\varphi^k) = N(f^k) \) where \( f \) is a map on \( \Pi \backslash S \) which induces \( \varphi \).

When all \( R(\varphi^k) \) are finite, Theorem 3.1 says that the Reidemeister theory for poly-Bieberbach groups follows directly from the Nielsen theory for infra-solvmanifolds of type \((R)\). In this paper, whenever possible, we will state our results in the language of Reidemeister theory.

**Proposition 3.2** ([3] Proposition 9.3]). Let \( f \) be a map on an infra-solvmanifold \( \Pi \backslash S \) of type \((R)\) induced by an affine map. Then every essential fixed point class of \( f \) consists of a single element.

Let \( \varphi : \Pi \to \Pi \) be an endomorphism on a poly-Bieberbach group \( \Pi \). We assume as before that \((d, D)\) be an affine map on \( S \) and \( f : \Pi \backslash S \to \Pi \backslash S \) be the map induced by \((d, D)\) and inducing \( \varphi \). We assume further that all \( R(\varphi^n) < \infty \). Hence by Theorem 3.1, \( R(\varphi^n) = N(f^n) \) for all \( n > 0 \). This implies that for every \( n > 0 \) all fixed point classes of \( f^n \) are essential and hence consist of a single element by Proposition 3.2. Consequently, we can refer to essential fixed point classes of \( f^n \) as essential periodic points of \( f \) with period \( n \). Moreover, for every \( n > 0 \) all Reidemeister classes of \( \varphi^n \) are essential.

For \( m \mid n \) and for \( \beta \in \Pi \), let \( \alpha = \beta \varphi^m(\beta) \cdots \varphi^n-m(\beta) \) and consider the commuting diagram

\[
\begin{array}{ccc}
[\alpha]^n & \xrightarrow{\eta} & p\left(\text{Fix}(\alpha \hat{f}^n)\right) = \{x\} \\
\downarrow \iota_{m,n} & & \downarrow \gamma_{m,n} \\
[\beta]^m & \xrightarrow{\eta} & p\left(\text{Fix}(\beta \hat{f}^m)\right) = \{x\}
\end{array}
\]

This shows that the observation in Section 2 can be refined as follows: \([\alpha]^n\) is irreducible if and only \([\alpha]^n\) has height \( n \) if and only if the corresponding essential periodic point \( x \) of \( f \) has minimal period \( n \). Moreover, \([\alpha]^n\) has depth \( d \) if and only if the corresponding essential periodic point \( x \) of \( f \) has minimal period \( d \). Let \( \ell \) be the length of \([\alpha]^n\). That is, \([\varphi^\ell(\alpha)]^n = [\alpha]^n\). Equivalently, we have \( \hat{f}^\ell(x) = x \). This implies that \([\alpha]^n\) is reducible to \( \ell \). Further, \( d = \ell \). In particular, if \([\alpha]^n\) is irreducible, then its length is the height, \( \ell = n \), and so \( \#([\alpha]^n) = n \).

We denote by \( O([\varphi], k) \) the set of all (essential) periodic orbits of \([\varphi] \) with length \( \leq k \). Then we have

\[
O([\varphi], k) = \{ ([\alpha]^m) \mid \alpha \in \Pi, m \leq k \} = \{ (x) \mid x \text{ is an essential periodic point of } f \text{ with length } \leq k \} = O(f, k).
\]
Recall that the set of essential periodic points of $f$ with minimal period $k$ is

$$\text{EP}_k(f) = \text{Fix}_e(f^k) - \bigcup_{d|k, d<k} \text{Fix}_e(f^d).$$

Then we have the algebraic counterpart. Namely,

$$\text{EP}_k(\varphi) = \{[\alpha]^k \in R[\varphi^k] \mid [\alpha]^k \text{ is irreducible (and essential)} \}$$

$$= \{[\alpha]^k \in R[\varphi^k] \mid [\alpha]^k \text{ has height } k \}$$

$$= \mathcal{IR}(\varphi^k),$$

and $\#\mathcal{IR}(\varphi^k) = \#\text{EP}_k(f)$. Hence the set of (essential) Reidemeister classes of $\varphi^k$ can be identified with a disjoint union of irreducible classes, that is, $\mathcal{R}[\varphi^k]$ is decomposed by heights:

$$\mathcal{R}[\varphi^k] = \coprod_{d|k} \mathcal{IR}(\varphi^d),$$

and hence its cardinality satisfies

$$R(\varphi^k) = \#\mathcal{R}[\varphi^k] = \sum_{d|k} \#\mathcal{IR}(\varphi^d).$$

Recall that if we denote by $O_k(\varphi)$ the number of essential and irreducible periodic orbits of $\mathcal{R}[\varphi^k]$, i.e., if $O_k(\varphi) = \#\{([\alpha]^k) \mid [\alpha]^k \in \text{EP}(\varphi)\}$ then by definition, the prime Nielsen-Jiang periodic number of period $k$ is

$$\text{NP}_k(\varphi) = k \times O_k(\varphi).$$

As observed earlier, each such orbit $([\alpha]^k)$ has length $k$. Therefore, $\text{NP}_k(\varphi) = \#\mathcal{IR}(\varphi^k)$ and topologically $\text{NP}_k(f) = \#\text{EP}_k(f)$, the number of essential periodic points of $f$ with minimal period $k$.

**Theorem 3.3.** Let $\varphi : \Pi \to \Pi$ be an endomorphism on a poly-Bieberbach group such that all $R(\varphi^k)$ are finite. Then

$$\text{NP}_k(\varphi) = \#\mathcal{IR}(\varphi^k).$$

Let $f : \Pi \setminus S \to \Pi \setminus S$ be a map on an infra-solvmanifold $\Pi \setminus S$ of type (R). Then

$$\text{NP}_k(f) = \#\text{EP}_k(f).$$

Consider all periodic orbits $([\alpha]^m)$ of $\varphi$ with $m \mid k$. A set $\mathcal{G}$ of periodic orbits $([\alpha]^m)$ of $\varphi$ with $m \mid k$ is said to be a set of $k$-representatives if every essential orbit $([\beta]^m)$ with $m \mid k$ is reducible to some element of $\mathcal{G}$. Then the full Nielsen-Jiang periodic number of period $k$ is defined to be

$$\text{NF}_k(\varphi) = \min \left\{ \sum_{A \in \mathcal{G}} d(A) \mid \mathcal{G} \text{ is a set of } k\text{-representatives} \right\}.$$ 

Recall that all periodic orbits are essential, and every periodic orbit $([\beta]^m)$ is boosted to a periodic orbit $([\alpha]^k)$. Hence to compute $\text{NF}_k(\varphi)$, we need first to consider only $\mathcal{R}[\varphi^k]$ and a set of representatives $[\alpha_i]^k$ of the orbits in $\mathcal{R}[\varphi^k]$. Then $\text{NF}_k(\varphi)$ is the sum of depths of all $[\alpha_i]^n$. Topologically, $\text{NF}_k(f)$ is the sum of minimal periods of all essential periodic point classes $(x)$ in $\text{Fix}_e(f^k)$. 
4. Reidemeister numbers of iterations $R(\varphi^k)$

Concerning the Reidemeister numbers $R(\varphi^k)$ of all iterates of $\varphi$, we shall assume that all $R(\varphi^k)$ are finite. Whenever all $R(\varphi^0)$ are finite, we can consider following \cite{3} \cite{10} the Reidemeister zeta function of $\varphi$

$$R_\varphi(z) = \exp \left( \sum_{k=1}^{\infty} \frac{R(\varphi^k)}{k} z^k \right).$$

Let $\varphi : \Pi \to \Pi$ be an endomorphism on a poly-Bieberbach group $\Pi$ with $\Pi \subset \text{Aff}(S) = S \rtimes \text{Aut}(S)$, where $S$ is a connected, simply connected solvable Lie group of type (R). By Section \cite{3}, $\varphi$ is a homomorphism induced by a self-map $f$ on the infra-solvmanifold $\Pi \setminus S$ of type (R). First we recall the following results:

**Theorem 4.1** (\cite{13} Theorem 11.4). Let $\varphi : \Pi \to \Pi$ be an endomorphism on a poly-Bieberbach group $\Pi$ such that all $R(\varphi^k)$ are finite. Then

\begin{equation}
(DN) \quad \sum_{d|k} \mu \left( \frac{k}{d} \right) R(\varphi^d) \equiv \sum_{d|k} \mu \left( \frac{k}{d} \right) N(f^d) \equiv 0 \mod k
\end{equation}

for all $k > 0$.

Consider the sequences of algebraic multiplicities $\{A_k(f)\}$ and Dold multiplicities $\{I_k(f)\}$ associated to the sequence $\{N(f^k)\}$:

$$A_k(f) = \frac{1}{k} \sum_{d|k} \mu \left( \frac{k}{d} \right) N(f^d), \quad I_k(f) = \sum_{d|k} \mu \left( \frac{k}{d} \right) N(f^d).$$

Then $I_k(f) = k A_k(f)$ and all $A_k(f)$ are integers by (DN). From the Möbius inversion formula, we immediately have

$$N(f^k) = \sum_{d|k} d A_d(f).$$

Because we are assuming that all $R(\varphi^k)$ are finite, by Theorem 4.1 $R(\varphi^k) = N(f^k)$. Consequently, we obtain the sequences of algebraic multiplicities $\{A_k(\varphi)\}$ and Dold multiplicities $\{I_k(\varphi)\}$ associated to the sequence $\{R(\varphi^k)\}$. Thus $I_k(\varphi) = k A_k(\varphi)$ and all $A_k(\varphi)$ are integers. Furthermore, we immediately have

$$R(\varphi^k) = \sum_{d|k} d A_d(\varphi).$$

**Theorem 4.2** (\cite{13}). Let $\varphi : \Pi \to \Pi$ be an endomorphism on a poly-Bieberbach group $\Pi$ such that all $R(\varphi^k)$ are finite. Then the Reidemeister zeta function of $\varphi$

$$R_\varphi(z) = \exp \left( \sum_{k=1}^{\infty} \frac{R(\varphi^k)}{k} z^k \right)$$

is a rational function.

Since $R_\varphi(0) = 1$ by definition, $z = 0$ is not a zero nor a pole of the rational function $R_\varphi(z)$. Thus we can write

$$R_\varphi(z) = \frac{u(z)}{v(z)} = \frac{\prod_i (1 - \beta_i z)}{\prod_j (1 - \gamma_j z)} = \prod_{i=1}^{r} (1 - \lambda_i z)^{-\rho_i}$$

with all $\lambda_i$ distinct nonzero algebraic integers and $\rho_i$ nonzero integers. This induces

\begin{equation}
(R1) \quad R(\varphi^k) = \sum_{i=1}^{r(\varphi)} \rho_i \lambda_i^k.
\end{equation}

Note that $r(\varphi)$ is the number of zeros and poles of $R_\varphi(z)$. Since $R_\varphi(z)$ is a homotopy invariant, so is $r(\varphi)$. 

Consider another generating function associated to the sequence \( \{R(\varphi^k)\} \):

\[
S_\varphi(z) = \sum_{k=1}^{\infty} R(\varphi^k) z^{k-1}.
\]

Then it is easy to see that

\[
S_\varphi(z) = \frac{d}{dz} \log R_\varphi(z).
\]

Moreover,

\[
S_\varphi(z) = \sum_{k=1}^{\infty} \sum_{i=1}^{r(\varphi)} \rho_i \lambda_i^k z^{k-1} = \sum_{i=1}^{r(\varphi)} \rho_i \lambda_i \frac{1}{1 - \lambda_i z}
\]

is a rational function with simple poles and integral residues, and 0 at infinity. The rational function \( S_\varphi(z) \) can be written as \( S_\varphi(z) = u(z)/v(z) \) where the polynomials \( u(z) \) and \( v(z) \) are of the form

\[
u(z) = R(\varphi) + \sum_{i=1}^{s} a_i z^i, \quad v(z) = 1 + \sum_{j=1}^{t} b_j z^j
\]

with \( a_i \) and \( b_j \) integers, see (3) \( \Rightarrow \) (5), Theorem 2.1 in [2] or [19, Lemma 3.1.31]. Let \( \tilde{v}(z) \) be the conjugate polynomial of \( v(z) \), i.e., \( \tilde{v}(z) = z^{\tilde{t}} v(1/z) \). Then the numbers \( \{\lambda_i\} \) are the roots of \( \tilde{v}(z) \), and \( r(\varphi) = \tilde{t} \).

The following can be found in the proof of (3) \( \Rightarrow \) (5), Theorem 2.1 in [2], see also [14, Lemma 2.4].

**Lemma 4.3.** If \( \lambda_i \) and \( \lambda_j \) are roots of the rational polynomial \( \tilde{v}(z) \) which are algebraically conjugate (i.e., \( \lambda_i \) and \( \lambda_j \) are roots of the same irreducible polynomial), then \( \rho_i = \rho_j \).

Let \( \tilde{v}(z) = \prod_{\alpha=1}^{s} \tilde{v}_\alpha(z) \) be the decomposition of the monic integral polynomial \( \tilde{v}(z) \) into irreducible polynomials \( \tilde{v}_\alpha(z) \) of degree \( r_\alpha \). Of course, \( r = r(\varphi) = \sum_{\alpha=1}^{s} r_\alpha \) and

\[
\tilde{v}(z) = z^r + b_1 z^{r-1} + b_2 z^{r-2} + \cdots + b_{r-1} z + b_r
\]

\[
= \prod_{\alpha=1}^{s} (z^{r_\alpha} + b_1^{\alpha} z^{r_\alpha-1} + b_2^{\alpha} z^{r_\alpha-2} + \cdots + b_{r_\alpha-1}^{\alpha} z + b_r^{\alpha}) = \prod_{\alpha=1}^{s} \tilde{v}_\alpha(z).
\]

If \( \{\lambda_\alpha^{(\alpha)}\} \) are the roots of \( \tilde{v}_\alpha(z) \), then the associated \( \rho \)'s are the same \( \rho_\alpha \). Consequently, we can rewrite (1) as

\[
R(\varphi^k) = \sum_{\alpha=1}^{s} \rho_\alpha \left( \sum_{i=1}^{r_\alpha} (\lambda_\alpha^{(\alpha)})^i \right)
\]

\[
= \sum_{\rho_\alpha > 0} \rho_\alpha^+ \left( \sum_{i=1}^{r_\alpha} (\lambda_\alpha^{(\alpha)})^i \right) - \sum_{\rho_\alpha < 0} \rho_\alpha^- \left( \sum_{i=1}^{r_\alpha} (\lambda_\alpha^{(\alpha)})^i \right).
\]

Consider the \( r_\alpha \times r_\alpha \)-integral square matrices

\[
M_\alpha = \begin{bmatrix}
0 & 0 & \cdots & 0 & -b_{r_\alpha}^\alpha \\
1 & 0 & \cdots & 0 & -b_{r_\alpha-1}^\alpha \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & -b_2^\alpha \\
0 & 0 & \cdots & 1 & -b_1^\alpha
\end{bmatrix}.
\]

The characteristic polynomial is \( \det(z I - M_\alpha) = \tilde{v}_\alpha(z) \) and therefore \( \{\lambda_\alpha^{(\alpha)}\} \) are the eigenvalues of \( M_\alpha \). This implies that \( R(\varphi^k) = \sum_{\alpha=1}^{s} \rho_\alpha \text{ tr } M_\alpha^k \). Set

\[
M_+ = \bigoplus_{\rho_\alpha > 0} \rho_\alpha^+ M_\alpha, \quad M_- = \bigoplus_{\rho_\alpha < 0} \rho_\alpha^- M_\alpha.
\]
Then
\[
R(\varphi^k) = \text{tr } M_+^k - \text{tr } M_-^k = \text{tr } (M_+ \bigoplus M_-)^k.
\]

We will show in Proposition \[\text{(2)}\] that if \(A_k(f) \neq 0\) then \(N(f^k) \neq 0\) and hence \(f\) has an essential periodic point of period \(k\). In the following we investigate some other necessary conditions under which \(N(f^k) \neq 0\). Recall that

\[
N(f^k) = \text{the number of essential fixed point classes of } f^k.
\]

If \(\mathbb{F}\) is a fixed point class of \(f^k\), then \(f^k(\mathbb{F}) = \mathbb{F}\) and the length of \(\mathbb{F}\) is the smallest number \(p\) for which \(f^p(\mathbb{F}) = \mathbb{F}\), written \(p(\mathbb{F})\). We denote by \(\langle \mathbb{F} \rangle\) the \(f\)-orbit of \(\mathbb{F}\), i.e.,

\[
\langle \mathbb{F} \rangle = \{F, f(\mathbb{F}), \ldots , f^{p-1}(\mathbb{F})\}
\]

where \(p = p(\mathbb{F})\). If \(\mathbb{F}\) is essential, so is every \(f^i(\mathbb{F})\) and \(\langle \mathbb{F} \rangle\) is an essential periodic orbit of \(f\) with length \(p(\mathbb{F})\) and \(p(\mathbb{F}) \mid k\). These are variations of Corollaries 2.3, 2.4 and 2.5 of \([2]\).

Assuming that all \(R(\varphi^k)\) are finite, we have

**Corollary 4.4.** If \(r(\varphi) \neq 0\), then \(R(\varphi^i) \neq 0\) for some \(1 \leq i \leq r(\varphi)\). In particular, \(\varphi\) has an essential periodic orbit with the length \(p \mid i, i \leq r(\varphi)\).

Recalling the identity \(R(\varphi^k) = \sum_{i=1}^{r(\varphi)} \rho_i \lambda_i^k\), we define

\[
\rho(\varphi) = \sum_{i=1}^{r(\varphi)} \rho_i, \quad M(\varphi) = \max \left\{ \sum_{\rho_i \geq 0} \rho_i, -\sum_{\rho_i < 0} \rho_j \right\}.
\]

**Corollary 4.5.** If \(\rho(\varphi) = 0\) and \(r(\varphi) \geq 1\), then \(r(\varphi) \geq 2\) and \(R(\varphi^i) \neq 0\) for some \(1 \leq i < r(\varphi)\). In particular, \(\varphi\) an essential periodic orbit with the length \(p \mid i, i \leq r(\varphi) - 1\).

**Corollary 4.6.** If \(r(\varphi) > 0\), then \(R(\varphi^i) \neq 0\) for some \(1 \leq i \leq M(\varphi)\). In particular, \(\varphi\) has an essential periodic orbit with the length \(p \mid i, i \leq M(\varphi)\).

5. **Radius of convergence of the Reidemeister zeta function** \(R_\varphi(z)\)

From the Cauchy-Hadamard formula, we can see that the radii \(R\) of convergence of the infinite series \(R_\varphi(z)\) and \(S_\varphi(z)\) are the same and given by

\[
\frac{1}{R} = \limsup_{k \rightarrow \infty} \left( \frac{R(\varphi^k)}{k} \right)^{1/k} = \limsup_{k \rightarrow \infty} R(\varphi^k)^{1/k}.
\]

We will understand the radius \(R\) of convergence from the identity \(R(\varphi^k) = \sum_{i=1}^{r(\varphi)} \rho_i \lambda_i^k\). Recall that the \(\lambda_i^{-1}\) are the poles or the zeros of the rational function \(R_\varphi(z)\). We define

\[
\lambda(\varphi) = \max\{ |\lambda_i| \mid i = 1, \ldots , r(\varphi) \}.
\]

If \(r(\varphi) = 0\), i.e., if \(R(\varphi^k) = 0\) for all \(k > 0\), then \(R_\varphi(z) \equiv 1\) and \(1/R = 0\). In this case, we define customarily \(\lambda(\varphi) = 0\). We shall assume now that \(r(\varphi) \neq 0\). In what follows, when \(\lambda(\varphi) > 0\), we consider

\[
n(\varphi) = \# \{ i \mid |\lambda_i| = \lambda(\varphi) \}.
\]

Remark that if \(\lambda(\varphi) < 1\) then \(R(\varphi^k) = \sum_{i=1}^{r(\varphi)} \rho_i \lambda_i^k \rightarrow 0\) and so the sequence of integers are eventually zero, i.e., \(R(\varphi^k) = 0\) for all \(k\) sufficiently large. This shows that \(1/R = 0\) and furthermore, \(R_\varphi(z)\) is the exponential of a polynomial. Hence the rational function \(R_\varphi(z)\) has no poles and zeros. This forces \(R_\varphi(z) \equiv 1\); hence \(\lambda(\varphi) = 0 = 1/R\).

Assume \(|\lambda_j| \neq \lambda(\varphi)\) for some \(j\); then we have

\[
\frac{R(\varphi^k)}{\lambda_j^k} = \sum_{i \neq j} \rho_i \left( \frac{\lambda_i}{\lambda_j} \right)^k + \rho_j, \quad \lim \sum_{i \neq j} \rho_i \left( \frac{\lambda_i}{\lambda_j} \right)^k = \infty.
\]
Theorem 5.1. Let $\varphi : \Pi \to \Pi$ be an endomorphism on a poly-Bieberbach group $\Pi$ such that all $R(\varphi^k)$ are finite. Let $R$ denote the radius of convergence of the Reidemeister zeta function $R_\varphi(z)$ of $\varphi$. Then $\lambda(\varphi) = 0$ or $\lambda(\varphi) \geq 1$, and

$$\frac{1}{R} = \lambda(\varphi).$$

In particular, $R > 0$. 

Recall that

$$S_\varphi(z) = \sum_{i=1}^{r(\varphi)} \frac{\rho_i \lambda_i}{1 - \lambda_i z},$$

$$R_\varphi(z) = \prod_{i=1}^{\rho(\varphi)} (1 - \lambda_i z)^{-\rho_i} = \prod_{\rho_i < 0} (1 - \lambda_i z)^{-\rho_i} \prod_{\rho_i > 0} (1 - \lambda_i z)^{\rho_i}.$$ 

These show that all of the $1/\lambda_i$ are the poles of $S_\varphi(z)$, whereas the $1/\lambda_i$ with corresponding $\rho_i > 0$ are the poles of $R_\varphi(z)$. The radius of convergence of a power series centered at a point $a$ is equal to the distance from $a$ to the nearest point where the power series cannot be defined in a way that makes it holomorphic. Hence the radius of convergence of $S_\varphi(z)$ is $1/\lambda(\varphi)$ and the radius of convergence of $R_\varphi(z)$ is $1/\max\{1/|\lambda_i| : \rho_i > 0\}$. In particular, we have shown that

$$\lambda(\varphi) = \max\{|\lambda_i| : i = 1, \ldots, r(\varphi)\} = \max\{|\lambda_i| : \rho_i > 0\}.$$

Theorem 5.2. Let $\varphi : \Pi \to \Pi$ be an endomorphism on a poly-Bieberbach group $\Pi$ of $S$ such that all $R(\varphi^k)$ are finite. Let $R$ denote the radius of convergence of the Reidemeister zeta function of $\varphi$. If $\varphi$ is the semi-conjugate by an affine map $(d, D)$ on $S$ and if $D_s$ has no eigenvalue 1, then

$$\frac{1}{R} = \text{sp} (\bigwedge D_s) = \lambda(\varphi).$$

Proof. Recall that $R_\varphi(z) = R_f(z)$ and the radius $R$ of convergence of $R_f(z)$ satisfies $1/R = \text{sp} (\bigwedge D_s)$ by [14, Theorem 3.4]. With Theorem 5.1, we obtain the required assertion. 

We recall that the asymptotic Reidemeister number of $\varphi$ is defined to be

$$R^\infty(\varphi) := \max \left\{ 1, \limsup_{k \to \infty} R(\varphi^k) \right\}.$$
We also recall that the most widely used measure for the complexity of a dynamical system is the topological entropy $h(f)$. A basic relation between these two numbers is $h(f) \geq \log N^\infty(f)$, which was found by Ivanov in [18]. There is a conjectural inequality $h(f) \geq \log(\text{sp}(f))$ raised by Shub [31]. This conjecture was proven for all maps on infra-solvmanifolds of type (R), see [27, 28] and [13]. Consider a continuous map $f$ on a compact connected manifold $M$ of the group $\Pi$ of covering transformations on the universal cover of $M$. Since $M$ is compact, $\Pi$ is finitely generated. Let $T = \{\tau_1, \cdots, \tau_n\}$ be a set of generators for $\Pi$. For any $\gamma \in \Pi$, let $L(\gamma, T)$ be the length of the shortest word in the letters $T \cup T^{-1}$ which represents $\gamma$. For each $k > 0$, we put

$$L_k(\varphi, T) = \max \left\{ L(\varphi^k(\tau_i), T) \mid i = 1, \cdots, n \right\}. $$

Then the algebraic entropy $h_\text{alg}(f) = h_\text{alg}(\varphi)$ of $f$ or $\varphi$ is defined as follows:

$$h_\text{alg}(f) = \lim_{k \to \infty} \frac{1}{k} \log L_k(\varphi, T).$$

The algebraic entropy of $f$ is well-defined, i.e., independent of the choices of a set $T$ of generators for $\Pi$ and a homomorphism $\varphi$ induced by $f$ ([22, p. 114]). We refer to [22] for backgrounds. Among others, we recall that R. Bowen in [3] and A. Katok in [21] have proved that the topological entropy $h(f)$ of $f$ is at least as large as the algebraic entropy $h_\text{alg}(\varphi)$ of $\varphi$. Furthermore, for any inner automorphism $\tau_\gamma$ by $\gamma$, we have $h_\text{alg}(\tau_\gamma \varphi) = h_\text{alg}(\varphi)$ ([22, Proposition 3.1.10]). The problem of determining an algebraic entropy and the growth rate of a group endomorphism, initiated by R. Bowen in [3], is now an area of active research (see [12] and references therein).

Now we can state relations between $R^\infty(\varphi)$, $\lambda(\varphi)$, $h(f)$ and $h_\text{alg}(\varphi)$.

**Corollary 5.3.** Let $\varphi : \Pi \to \Pi$ be a homomorphism on a poly-Bieberbach group $\Pi$ of $S$ and all $R(\varphi^k)$ are finite. Let $(d, D)$ be an affine map on $S$ such that $\varphi(\alpha) \circ (d, D) = (d, D) \circ \alpha$ for all $\alpha \in \Pi$. Let $f$ be the map on $\Pi \setminus S$ induced by $(d, D)$ and let $f$ be any map on $\Pi \setminus S$ which is homotopic to $f$. Then

$$R^\infty(\varphi) = \text{sp} \left( \bigwedge D_\ast \right) = \lambda(\varphi),$$

$$h_\text{alg}(\varphi) = h_\text{alg}(\tilde{f}) = h_\text{alg}(f) \leq h(\tilde{f}) = \log R^\infty(\varphi) \leq h(f),$$

provided that 1 is not an eigenvalue of $D_\ast$.

**Proof.** From [13, Theorem 4.3] and Theorem 5.2 we obtain the first assertion, $R^\infty(\varphi) = \text{sp} \left( \bigwedge D_\ast \right) = \lambda(\varphi)$. By [13, Theorem 5.2], $h(f) \geq h(\tilde{f}) = \log \lambda(\varphi)$ and by the remark mentioned just above, we have that $h(\tilde{f}) \geq h_\text{alg}(\tilde{f}) = h_\text{alg}(f) = h_\text{alg}(\varphi)$.

**Remark 5.4.** The inequality

$$\log R^\infty(\varphi) \geq h_\text{alg}(\varphi)$$

in Corollary 5.3 can be regarded as an algebraic analogue of the Ivanov inequality $h(f) \geq \log N^\infty(f)$.

### 6. Asymptotic behavior of the sequence $\{R(\varphi^k)\}$

In this section, we study the asymptotic behavior of the Reidemeister numbers of iterates of maps on poly-Bieberbach groups. We can restate our Theorem 4.1 in [13] as follows:

**Theorem 6.1.** Let $\varphi : \Pi \to \Pi$ be an endomorphism on a poly-Bieberbach group such that all $R(\varphi^k)$ are finite. Then one of the following two possibilities holds:

1. $\lambda(\varphi) = 0$, which occurs if and only if $R_\varphi(z) \equiv 1$. 


(2) The sequence \( \{R(\varphi^k)/\lambda(\varphi)^k\} \) has the same limit points as a periodic sequence
\( \{\sum_j \alpha_j \psi_j^k\} \) where \( \alpha_j \in \mathbb{Z}, \psi_j \in \mathbb{C} \) and \( \psi_j \neq 1 \) for some \( q > 0 \).

In Theorem 5.2 we showed that if \( D_\ast \) has no eigenvalue 1 then \( \lambda(\varphi) = \text{sp}(\bigvee D_\ast) \). In fact, we have the following:

**Lemma 6.2.** Let \( \varphi \) be a homomorphism on a poly-Bieberbach group \( \Pi \) and let \( \varphi \) be the semi-conjugate by an affine map \((d, D)\) on \( S \). If \( \lambda(\varphi) \geq 1 \), then \( \lambda(\varphi) = \text{sp}(\bigvee D_\ast) \).

It is important to know not only the rate of growth of the sequence \( \{R(\varphi^k)\} \) but also the frequency with which the largest Reidemeister number is encountered. The following theorem shows that this sequence grows relatively dense. The following are variations of Theorem 2.7, Proposition 2.8 and Corollary 2.9 of [2].

**Theorem 6.3.** Let \( \varphi : \Pi \rightarrow \Pi \) be an endomorphism on a poly-Bieberbach group \( \Pi \) such that all \( R(\varphi^k) \) are finite. If \( \lambda(\varphi) \geq 1 \), then there exist \( N \), \( \gamma > 0 \) and a natural number \( N \) such that for any \( m > N \) there is an \( \ell \in \{0, 1, \cdots, n(\varphi) - 1\} \) such that \( R(\varphi^{m+\ell})/\lambda(\varphi)^{m+\ell} \geq \gamma \).

**Proposition 6.4.** Let \( \varphi : \Pi \rightarrow \Pi \) be an endomorphism on a poly-Bieberbach group such that all \( R(\varphi^k) \) are finite and such that \( \lambda(\varphi) \geq 1 \). Then for any \( \epsilon > 0 \), there exists \( N \) such that if \( R(\varphi^m)/\lambda(\varphi)^m \geq \epsilon \) for \( m > N \), then the Dold multiplicity \( I_m(\varphi) \) satisfies
\[
|I_m(\varphi)| \geq \frac{\epsilon}{2}\lambda(\varphi)^m.
\]

Theorem 6.3 and Proposition 6.4 imply immediately the following:

**Corollary 6.5.** Let \( \varphi : \Pi \rightarrow \Pi \) be an endomorphism on a poly-Bieberbach group such that all \( R(\varphi^k) \) are finite and such that \( \lambda(\varphi) > 1 \). Then there exist \( N \), \( \gamma > 0 \) and a natural number \( N \) such that if \( m \geq N \) then there exists \( \ell \) with \( 0 \leq \ell \leq n(\varphi) - 1 \) such that \( |I_m+\ell(\varphi)|/\lambda(\varphi)^{m+\ell} \geq \gamma/2 \). In particular \( I_{m+\ell}(\varphi) \neq 0 \) and so \( A_{m+\ell}(\varphi) \neq 0 \).

**Remark 6.6.** We can state a little bit more about the density of the set of algebraic periods \( A(\varphi) = \{m \in \mathbb{N} \mid A_m(\varphi) \neq 0\} \). We consider the notion of the lower density \( DA(\varphi) \) of the set \( A(\varphi) \subseteq \mathbb{N} \):
\[
DA(\varphi) = \liminf_{k \to \infty} \frac{\#(A(\varphi) \cap [1, k])}{k}.
\]

By Corollary 6.5 when \( \lambda(\varphi) > 1 \), we have \( DA(\varphi) \geq 1/n(\varphi) \). On the other hand, when \( \lambda(\varphi) \geq 1 \) by Theorem 6.1 the sequence \( \{R(\varphi^k)/\lambda(\varphi)^k\} \) has the same limit points as the periodic sequence \( \{\sum_{j=1}^{n(\varphi)} \rho_j e^{2\pi i (k q_j)}\} \) of period \( q = \text{LCM}(q_1, \cdots, q_{n(\varphi)}) \). Hence by Theorem 6.3 we have \( DA(\varphi) \geq 1/q \).

7. Periodic \([\varphi]\)-orbits

In this section, we shall give an estimate from below the number of periodic \([\varphi]\)-orbits of an endomorphism \( \varphi \) on a poly-Bieberbach group based on facts discussed in Section 3. We keep in mind that all periodic classes are essential, see Proposition 3.2.

We denote by \( O([\varphi], k) \) the set of all (essential) periodic orbits of \([\varphi]\) with length \( \leq k \). Thus
\[
O([\varphi], k) = \{\langle \alpha \rangle^m \mid \alpha \in \Pi, m \leq k\}.
\]

Recalling from Section 3 that \( O([\varphi], k) = O(f, k) \), we can restate our Theorem 5.3 in [14] as follows:

**Theorem 7.1.** Let \( \varphi : \Pi \rightarrow \Pi \) be an endomorphism on a poly-Bieberbach group such that all \( R(\varphi^k) \) are finite. Suppose that the sequence \( R(\varphi^k) \) is unbounded. Then there exists a natural number \( N_0 \) such that
\[
k \geq N_0 \implies |O([\varphi], k)| \geq \frac{k - N_0}{r(\varphi)}.
\]
Proposition 7.2. Let \( \varphi : \Pi \to \Pi \) be an endomorphism on a poly-Bieberbach group such that all \( R(\varphi^k) \) are finite. For every \( k > 0 \), we have

\[
\# \mathcal{IR}(\varphi^k) = \sum_{d|k} \mu\left(\frac{k}{d}\right) R(\varphi^d) = I_k(\varphi).
\]

Proof. We apply the Möbius inversion formula to the identity

\[
R(\varphi^k) = \sum_{d|k} \# \mathcal{IR}(\varphi^d)
\]

in Section 3 to obtain \( \# \mathcal{IR}(\varphi^k) = \sum_{d|k} \mu\left(\frac{k}{d}\right) R(\varphi^d) \), which is exactly the Dold multiplicity \( I_k(\varphi) \).

Definition 7.3. When all \( R(\varphi^k) \) are finite, we consider the mod 2 reduction of the Reidemeister number \( R(\varphi^k) \) of \( f^k \), written \( R^{(2)}(\varphi^k) \). A positive integer \( k \) is a \( R^{(2)} \)-period of \( \varphi \) if \( R^{(2)}(\varphi^{k+i}) = R^{(2)}(\varphi^i) \) for all \( i \geq 1 \). We denote the minimal \( R^{(2)} \)-period of \( \varphi \) by \( \alpha^{(2)}(\varphi) \).

Proposition 7.4 ([29, Proposition 1]). Let \( p \) be a prime number and let \( A \) be a square matrix with entries in the field \( \mathbb{F}_p \). Then there exists \( k \) with \( (p,k) = 1 \) such that

\[
\text{tr } A^{k+i} = \text{tr } A^i
\]

for all \( i \geq 1 \).

Recalling [R2]: \( R(\varphi^k) = N(f^k) = \text{tr } M_+^k - \text{tr } M_-^k = \text{tr } (M_+ \oplus -M_-)^k \), we can see easily that the minimal \( R^{(2)} \)-period \( \alpha^{(2)}(\varphi) \) always exists and must be an odd number.

Now we obtain a result which resembles [29, Theorem 2].

Theorem 7.5. Let \( \varphi : \Pi \to \Pi \) be an endomorphism on a poly-Bieberbach group such that all \( R(\varphi^k) \) are finite. Let \( k > 0 \) be an odd number. Suppose that \( \alpha^{(2)}(\varphi)^2 \mid k \) or \( p \mid k \) where \( p \) is a prime such that \( p \equiv 2^i \mod \alpha^{(2)}(\varphi) \) for some \( i \geq 0 \). Then

\[
\frac{NP_k(\varphi)}{k} = \frac{\# \mathcal{IR}(\varphi^k)}{k}
\]

is even.

8. Heights of \( \varphi \)

In this section, we study (homotopy) heights \( \mathcal{HI}(\varphi) = \mathcal{H}(\varphi) \) of Reidemeister classes of endomorphisms \( \varphi \) on poly-Bieberbach groups. We like to determine the set \( \mathcal{H}(\varphi) \) of all heights only from the knowledge of the sequence \( \{R(\varphi^k)\} \). Recalling that when all \( R(\varphi^k) \) are finite, \( R(\varphi^k) = \sum_{i=1}^{r(\varphi)} \rho_i \lambda_i^k \) and \( \lambda(\varphi) = \max\{|\lambda_i| \mid i = 1, \ldots, r(\varphi)\} \), we define

\[
R^{[\lambda]}(\varphi^k) = \sum_{|\lambda_i| = |\lambda|} \rho_i \lambda_i^k, \quad \tilde{R}^{[\lambda]}(\varphi^k) = \frac{1}{|\lambda|^k} R^{[\lambda]}(\varphi^k).
\]

Lemma 8.1. When all \( R(\varphi^k) \) are finite, if \( \lambda(\varphi) \geq 1 \), then we have

\[
\limsup_{k \to \infty} \frac{R(\varphi^k)}{\lambda(\varphi)^k} = \limsup_{k \to \infty} |\tilde{R}^{[\lambda]}(\varphi^k)|.
\]

Proof. We have

\[
R(\varphi^k) = \frac{1}{\lambda(\varphi)^k} \sum_{|\lambda_i| < \lambda(\varphi)} \rho_i \lambda_i^k.
\]

Since for \( |\lambda_i| < \lambda(\varphi) \), \( \lim \lambda_i^k / \lambda(\varphi)^k = 0 \), it follows that the proof is completed.

\[\square\]
Lemma 8.5 and 8.6 are essential.

From Proposition 6.4 it follows that the Dold multiplicity is periodic and nonzero, because \( \limsup_{k \to \infty} |\tilde{R}^{\lambda}(\varphi^k)| > 0 \) by Lemma 8.1. Consequently, there exists \( m \) with \( 1 \leq m \leq q \) such that \( \tilde{R}^{\lambda}(\varphi^m) \neq 0 \).

Let \( \psi = \varphi^m. \) Then \( \lambda(\psi) = \lambda(\varphi^m) = \lambda(\varphi)^m \geq 1. \) The periodicity \( R^{\lambda}(\varphi^{m+\ell q}) = \tilde{R}^{\lambda}(\varphi^m)(\psi^{1+\ell q}) = \tilde{R}^{\lambda}(\varphi^m)(\psi) \) for all \( \ell > 0. \) By Lemma 8.1 or Theorem 6.1, we can see that there exists \( \gamma > 0 \) such that \( R(\psi^{1+\ell q}) \geq \gamma \lambda(\psi)^{1+\ell q} \) for all \( \ell \) sufficiently large. From Proposition 6.4 it follows that the Dold multiplicity \( I_{1+\ell q}(\psi) \) satisfies \( |I_{1+\ell q}(\psi)| \geq (\gamma/2)\lambda(\psi)^{1+\ell q} \) when \( \ell \) is sufficiently large.

According to Dirichlet prime number theorem, since \((1, q) = 1, \) there are infinitely many primes \( p \) of the form \( 1 + \ell q. \) Consider all primes \( p_i \) satisfying \( |I_{p_i}(\psi)| \geq (\gamma/2)\lambda(\psi)^{p_i}. \)

By Proposition 6.2, \( \#IR(\psi^{p_i}) = I_{1+\ell q}(\psi) > 0, \) each \( p_i \) is the height of some (essential) Reidemeister class \( [\alpha]^{p_i} \in \mathcal{R}[\varphi^{p_i}]. \) That is, \( [\alpha]^{p_i} \) is an irreducible Reidemeister class of \( \psi^{p_i}. \) Consider the Reidemeister class \( [\alpha]^{mp_i} \) determined by \( \alpha \) of \( \varphi^{mp_i}. \) Let \( d_i \) be the depth of the Reidemeister class \( [\alpha]^{mp_i} \in \mathcal{R}[\varphi^{mp_i}]. \) Then \( d_i = m p_i \) for some \( m_i \) \( m \) and so there is an irreducible Reidemeister class \( [\beta]^{d_i} \in \mathcal{R}[\varphi^{d_i}] \) which is boosted to \( [\alpha]^{mp_i}. \) This means that \( d_i \) is the height of \( [\beta]^{d_i}. \) Choose a subsequence \( \{m_{i_k}\} \) of the sequence \( \{m_i\} \) bounded by \( m \) which is constant, say \( m_0. \) Consequently, the infinite sequence \( \{m_0 p_{i_k}\} \) consists of heights of \( \varphi, \) or \( \{m_0 p_{i_k}\} \subset \mathcal{H}(\varphi). \)

In the proof of Theorem 8.2, we have shown the following, which proves that the algebraic period is a (homotopy) height when it is a prime number.

Corollary 8.3. Let \( \varphi : \Pi \to \Pi \) be an endomorphism on a poly-Bieberbach group such that all \( R(\varphi^k) \) are finite. For all primes \( p, \) if \( A_p(\varphi) \neq 0 \) then \( p \in \mathcal{H}(\varphi). \)

Corollary 8.4. Let \( \varphi : \Pi \to \Pi \) be an endomorphism on a poly-Bieberbach group such that all \( R(\varphi^k) \) are finite. If the sequence \( \{R(\varphi^k)\} \) is strictly monotone increasing, then there exists \( N \) such that the set \( \mathcal{H}(\varphi) \) contains all primes larger than \( N. \)

Proof. By the assumption, we have \( \lambda(\varphi) > 1. \) Thus by Theorem 6.3, there exist \( \gamma > 0 \) and \( N \) such that if \( k > N \) then there exists \( \ell = \ell(k) < r(\varphi) \) such that \( R(\varphi^{k-\ell})/\lambda(\varphi)^{k-\ell} > \gamma. \) Then for all \( k > N, \) the monotonicity induces

\[
\frac{R(\varphi^k)}{\lambda(\varphi)^k} \geq \frac{R(\varphi^{k-\ell})}{\lambda(\varphi)^{k-\ell} \lambda(\varphi)^\ell} \geq \frac{\gamma}{\lambda(\varphi)^\ell} \geq \frac{\gamma}{\lambda(\varphi)^r(\varphi)}.
\]

Applying Proposition 6.4 with \( \epsilon = \gamma/\lambda(\varphi)^r(\varphi), \) we see that \( I_k(\varphi) \neq 0 \) and so \( A_k(\varphi) \neq 0 \) for all \( k \) sufficiently large. Now our assertion follows from Corollary 8.3.

A endomorphism \( \varphi : \Pi \to \Pi \) is essentially reducible if any Reidemeister class of \( \varphi^k \) being boosted to an essential Reidemeister of \( \varphi^{kn} \) is essential, for any positive integers \( k \) and \( n. \) The group \( \Pi \) is essentially reducible if every endomorphism on \( \Pi \) is essentially reducible.

Lemma 8.5 (II, Lemma 6.7). Every poly-Bieberbach group is essentially reducible.

This means that for any \( n, \) if \( [\alpha]^n \) is essential and if \( i_{m,n}(\beta)^m) = [\alpha]^n \) then \( [\beta]^m \) is essential.

Lemma 8.6 (II, Proposition 2.2). Let \( \varphi : \Pi \to \Pi \) be an endomorphism such that all \( R(\varphi^k) \) is finite. If

\[
\sum_{\text{prime}} R(\varphi^k) < R(\varphi^m),
\]
then \( \varphi \) has a periodic Reidemeister class with height \( m \), i.e., \( m \in \mathcal{H}(\varphi) \).

**Proof.** Let \( r = R(\varphi^m) \) and let \([\alpha_1]^m, \ldots, [\alpha_r]^m\) be the Reidemeister classes of \( \varphi^m \). If some \([\alpha_j]^m\) is irreducible, then we are done. So assume no \([\alpha_j]^m\) is irreducible. Then, for each \( j \), there is a \( k_j \) so that \( m/k_j \) is prime and \([\alpha_j]^m\) is reducible to \([\beta_j]^{k_j} \in R(\varphi^{k_j})\). But this shows that \( R(\varphi^m) \leq \sum_{\text{prime}} R(\varphi^k) \), a contradiction. \( \square \)

We can not only extend but also strengthen Corollary 8.4 as follows:

**Proposition 8.7.** Let \( \varphi : \Pi \rightarrow \Pi \) be an endomorphism on a poly-Bieberbach group such that all \( R(\varphi^k) \) are finite. Suppose that the sequence \( \{R(\varphi^k)\} \) is strictly monotone increasing. Then:

1. All primes belong to \( \mathcal{H}(\varphi) \).
2. There exists \( N \) such that if \( p \) is a prime \( N \) then \( \{p^n \mid n \in \mathbb{N}\} \subset \mathcal{H}(\varphi) \).

**Proof.** Observe that for any prime \( p \)

\[
R(\varphi^p) - \sum_{p \text{ prime}} R(\varphi^k) = R(\varphi^p) - R(\varphi) = I_p(\varphi).
\]

The strict monotonicity implies \( A_p(\varphi) = pI_p(\varphi) > 0 \), and hence \( p \in \mathcal{H}(\varphi) \), which proves (1).

Under the same assumption, we have shown in the proof of Corollary 8.4 that there exists \( N \) such that \( k > N \Rightarrow I_k(\varphi) > 0 \). Let \( p \) be a prime \( N \) and \( n \in \mathbb{N} \). Then

\[
R(\varphi^{pn}) - \sum_{p \text{ prime}} R(\varphi^k) = \sum_{i=0}^n I_{pn}(\varphi) - R(\varphi^{pn-1}) = I_{pn}(\varphi) > 0.
\]

By Lemma 8.6 we have \( p^n \in \mathcal{H}(\varphi) \), which proves (2). \( \square \)

In Remark 6.6 we observed under the lower density \( DA(\varphi) \) of the set of algebraic periods \( A(\varphi) = \{m \in \mathbb{N} \mid A_m(\varphi) \neq 0\} \). We can consider as well the lower density of the set \( \mathcal{H}(\varphi) \) of heights, see also [24], [17] and [14]:

\[
\text{DH}(\varphi) = \liminf_{k \to \infty} \frac{\#(\mathcal{H}(\varphi) \cap [1,k])}{k}.
\]

Since \( I_k(\varphi) = \#IR(\varphi^k) \) by Proposition 7.2 it follows that \( A(\varphi) \subset \mathcal{H}(\varphi) \). Hence we have \( DA(\varphi) \leq \text{DH}(\varphi) \).

**Corollary 8.8.** Let \( \varphi : \Pi \rightarrow \Pi \) be an endomorphism on a poly-Bieberbach group such that all \( R(\varphi^k) \) are finite. Suppose that the sequence \( \{R(\varphi^k)\} \) is strictly monotone increasing. Then \( \mathcal{H}(\varphi) \) is cofinite and \( DA(\varphi) = \text{DH}(\varphi) = 1 \).

**Proof.** Under the same assumption, we have shown in the proof of Corollary 8.4 that there exists \( N \) such that if \( k > N \) then \( I_k(\varphi) > 0 \). This means \( IR(\varphi^k) \) is nonempty by Proposition 7.2 and hence \( k \in \mathcal{H}(\varphi) \). \( \square \)

Let \( \varphi : \Pi \rightarrow \Pi \) be an endomorphism on a poly-Bieberbach group \( \Pi \) of \( S \) such that all \( R(\varphi^k) \) are finite. When \( \varphi \) is the semi-conjugate by an affine map \((d,D)\) on \( S \), we say that \( \varphi \) is **expanding** if all the eigenvalues of \( D_\ast \) have modulus \( > 1 \).

Now we can prove the main result of [25].

**Corollary 8.9 ([23 Theorem 4.6], [25 Theorem 3.2]).** Let \( \varphi \) be an expanding endomorphism on an almost Bieberbach group. Then \( H\text{Per}(\varphi) \) is cofinite.

**Proof.** Since \( \varphi \) is expanding, we have that \( \lambda(\varphi) = \text{sp}(\bigwedge D_\ast) > 1 \). For any \( k > 0 \), we can write \( R(\varphi^k) = \Gamma_k + \Omega_k \), where

\[
\Gamma_k = \lambda(\varphi)^k \left( \sum_{j=1}^{n(\varphi)} \rho_j e^{2\pi i (k \theta_j)} \right), \quad \Omega_k = \sum_{i=n(\varphi)+1}^r \lambda_i^k \quad \text{with } |\lambda_i| < \lambda(\varphi).
\]
Here $\Omega_k \to 0$ and $\Gamma_k \to \infty$ as $k \to \infty$. This implies that $R(\varphi^k)$ is eventually strictly monotone increasing. We can use Corollary 8.3 and then Corollary 8.8 to conclude the assertion.

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Instytut Matematyki, Uniwersytet Szczeciński, ul. Wielkopolska 15, 70-451 Szczecin, Poland
E-mail address: fels@wmf.univ.szczecin.pl

Department of mathematics, Sogang University, Seoul 121-742, KOREA
E-mail address: jlee@sogang.ac.kr