On monoidal functors between (braided) Gr-categories

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Abstract. In this paper, we state and prove precise theorems on the classification of the category of (braided) categorical groups and their (braided) monoidal functors, and some applications obtained from the basic studies on monoidal functors between categorical groups.

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1 Introduction and Preliminaries

Monoidal categories (symmetric monoidal categories) can be “refined” to become categories with (abelian) group structure if the notion of invertible objects is added (see [7], [13]). Then, if the underlying category is a groupoid, we have the notion of (symmetric) categorical groups, or Gr-categories (Picard categories) (see [14]). The structure of Gr-categories and Picard categories was deeply dealt with by H. X. Sinh in [14]. Braided categorical groups were originally considered by A. Joyal and R. Street in [5] as extensions of Picard categories. The category $\mathcal{BCG}$ of braided categorical groups and braided monoidal functors was classified by the category $\mathcal{Quad}$ of quadratic functions. These classification theorems were applied and extended in works of (braided) graded categorical groups (see [1], [4]), and they led to many interesting results. It is interesting to revisit even the most basic theory of monoidal categories for further improvements.

In this paper, we state quite adequate studies on Gr-functors and use them as a common technique to state classification theorems of the category of categorical groups and the category of braided categorical groups. Applications motivate these basic studies.

The plan of this paper is, briefly, as follows. In the first section we recall the construction of a Gr-category of the type $(\Pi, A, h)$, a reduction of an arbitrary Gr-category.

In the second section, we prove that each Gr-functor between reduced Gr-categories is one of the type $(\varphi, f)$. Then, we introduce the notion of the obstruction of a functor of type $(\varphi, f)$, and cohomological classify these functors.

The next section is devoted to showing the precise theorem of the category of Gr-categories and Gr-functors, a more complete version of the Classification
Theorem of H. X. Sinh.

As a consequence of Section 3, it is appropriate to have a different version of
the Classification Theorem of A. Joyal and R. Street for the category of braided
Gr-categories and braided Gr-functors, and this is the goal of Section 4.

The following section is delicated to the first application of the obstruc-
tion theory of Gr-functors. We construct the Gr-category of an abstract kernel
as an example for general theory. This leads to an interesting consequence: a Gr-
category can be transformed into a strict one (H. X. Sinh proved this result in
a completely different way [15]).

In the last section, we focus on using the Gr-category of an abstract kernel to
classify group extensions by means of Gr-functors. Then we obtain well-known
results on the group extension problem.

We would stress that Theorem 5 is used in the method of factor sets to
introduce a new proof of the Classification Theorem for graded Gr-
categories (see [11]), and a version of Theorem 5 for Ann-functors (see [9]) is u-
sed to classify Ann-functors thanks to the Mac Lane cohomology.

Let us start by some elementary concepts of monoidal categories.

A monoidal category $(G, \otimes, I, a, l, r)$ is a category $G$ together with a
tensor product $\otimes: G \times G \to G$; and an object $I$, called the unit object
and the natural isomorphisms

$$a_{X,Y,Z}: X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z,$$

$$l_X : I \otimes X \to X, \quad r_X : X \otimes I \to X,$$

which are, respectively, called associativity, left and right unit constraints. These
constraints satisfy the pentagon axiom

$$(a_{X,Y,Z} \otimes id_T) \circ a_{X,Y\otimes Z,T} \circ (id_X \otimes a_{Y,Z,T}) = a_{X\otimes Y,Z,T} \circ (a_{X,Y} \otimes FZ).$$

and the triangle axiom

$$(r_X \otimes id_Y) \circ a_{X,I,Y} = id_X \otimes l_Y.$$
and
\[ \alpha_{X \otimes Y} \circ \tilde{F}_{X,Y} = \tilde{G}_{X,Y} \circ (\alpha_X \otimes \alpha_Y). \]

A monoidal equivalence between monoidal categories is a monoidal functor \( F : \mathcal{G} \to \mathcal{G}' \), such that there exists a monoidal functor \( G : \mathcal{G}' \to \mathcal{G} \) and homotopies \( \alpha : G.F \to id_{\mathcal{G}} \) and \( \beta : F.G \to id_{\mathcal{G}'} \). \((F, \tilde{F}, F_{\alpha})\) is a monoidal equivalence if and only if \( F \) is an equivalence.

A Gr-category is a monoidal category, where every object is invertible and every morphism is isomorphic. If \((F, \tilde{F}, F_{\alpha})\) is a monoidal functor between Gr-categories, it is called a Gr-functor. Then the isomorphism \( F_* : I' \to FI \) can be deduced from \( F \) and \( \tilde{F} \).

Let us recall some well-known results (see [14]). Each Gr-category \( \mathcal{G} \) is equivalent to a Gr-category of the type \((\Pi, A)\), which can be described as follows. The set \( \pi_0\mathcal{G} \) of iso-classes of objects of \( \mathcal{G} \) is a group with the operation induced by the tensor product in \( \mathcal{G} \), and the set \( \pi_1\mathcal{G} \) of automorphisms of the unit object \( I \) is an abelian group with the operation, denoted by +, induced by the composition of morphisms. Moreover, \( \pi_1\mathcal{G} \) is a \( \pi_0\mathcal{G} \)-module with the action
\[ su = \gamma_X^{-1}\delta_X(u), \quad X \in s, \quad s \in \pi_0\mathcal{G}, \quad u \in \pi_1\mathcal{G}, \]
where \( \delta_X, \gamma_X \) are defined by the following commutative diagrams

\[
\begin{array}{ccc}
X & \xrightarrow{\gamma_X(u)} & X \\
1_X & \downarrow & 1_X \\
I \otimes X & \xrightarrow{u \otimes id} & I \otimes X
\end{array}
\quad \quad \quad
\begin{array}{ccc}
X & \xrightarrow{\delta_X(u)} & X \\
r_X & \downarrow & r_X \\
X \otimes I & \xrightarrow{id \otimes u} & X \otimes I
\end{array}
\]

The reduced Gr-category \( S_\mathcal{G} \) of a Gr-category \( \mathcal{G} \) is a category whose objects are the elements of \( \pi_0\mathcal{G} \) and whose morphisms are automorphisms \((s, u) : s \to s\), where \( s \in \pi_0\mathcal{G}, \quad u \in \pi_1\mathcal{G} \). The composition of two morphisms is induced by the addition in \( \pi_1\mathcal{G} \)
\[ (s, u).(s, v) = (s, u + v). \]
The category \( S_\mathcal{G} \) is equivalent to \( \mathcal{G} \) by canonical equivalences constructed as follows. For each \( s = [X] \in \pi_0\mathcal{G} \), choose a representative \( X_s \in \mathcal{G} \); and for each \( X \in s \), choose an isomorphism \( i_X : X_s \to X \) such that \( i_{X_s} = id_{X_s} \). For the set of representatives, we obtain two functors
\[
\begin{align*}
G : \mathcal{G} & \to S_\mathcal{G}, \\
G(X) & = [X] = s, \\
G(X \xrightarrow{f} Y) & = (s, \gamma^{-1}_{X_s}(i_Y f i_X)),
\end{align*}
\]
\[
\begin{align*}
H : S_\mathcal{G} & \to \mathcal{G}, \\
H(s) & = X_s, \\
H(s, u) & = \gamma_X(u).
\end{align*}
\]

Two functors \( G \) and \( H \) are categorical equivalences by natural transformations
\[ \alpha = (i_X) : HG \cong id_{S_\mathcal{G}}; \quad \beta = id : GH \cong id_{S_\mathcal{G}}. \]

They are called canonical equivalences.

With the structure transport (see [13], [14]) by the quadruple \((G, H, \alpha, \beta)\), \( S_\mathcal{G} \) becomes a Gr-category together with the following operation:
\[ s \otimes t = s.t, \quad s, t \in \pi_0\mathcal{G}, \]
\[ (s, u) \otimes (t, v) = (st, u + sv), \quad u, v \in \pi_1\mathcal{G}. \]
The set of representatives \((X_s, i_X)\) is called a stick of the Gr-category \(G\) for \(X_1 = I, i_I \otimes X_s = 1_{X_s}, i_{X_s \otimes I} = r_{X_s}\).

The unit constraints of the Gr-category \(S_G\) are therefore strict, and its associativity constraint is a normalized 3-cocycle \(h \in Z^3(\pi_0 G, \pi_1 G)\). Moreover, the equivalences \(G, H\) become Gr-equivalences together with natural isomorphisms

\[
\tilde{G}_{A,B} = G(i_A \otimes i_B), \quad \tilde{H}_{s,t} = i_{X_s \otimes X_t}^{-1}.
\]

The Gr-category \(S_G\) is called a reduction of the Gr-category \(G\). \(S_G\) is said to be of the type \((\Pi, A, h)\) or simply the type \((\Pi, A)\) if \(\pi_0 G, \pi_1 G\) are, respectively, replaced with the group \(\Pi\) and the \(\Pi\)-module \(A\).

2 Classification of Gr-functors of the type \((\varphi, f)\)

In this section, we show that each Gr-functor \((F, \tilde{F}) : G \to G'\) induces a Gr-functor \(S_F\) between their reduced Gr-categories. This allows us to study the problem of the existence of Gr-functors and classify them on Gr-categories of the type \((\Pi, A)\).

The following proposition is mentioned in many works related to categorical groups.

**Proposition 2.1.** [14] Let \((F, \tilde{F}) : G \to G'\) be a Gr-functor. Then, \((F, \tilde{F})\) induces a pair of group homomorphisms

\[
F_0 : \pi_0 G \to \pi_0 G', \quad [X] \mapsto [FX],
F_1 : \pi_1 G \to \pi_1 G', \quad u \mapsto \gamma_{F_1}^{-1}(Fu)
\]

satisfying \(F_1(su) = F_0(s)F_1(u)\).

Our first result is to strengthen Proposition 2.1 by Proposition 2.4 as asserting that each Gr-functor \((F, \tilde{F}) : G \to G'\) induces a Gr-functor \(S_F\) between their reduced Gr-categories. This allows us to study the problem of the existence of Gr-functors and classify them on Gr-categories of the type \((\Pi, A)\).

**Lemma 2.2.** Let \(G, G'\) be two \(\otimes\)-categories with, respectively, unit constraints \((I, l, r)\), \((I', l', r')\), and \((F, \tilde{F}, F_*) : G \to G'\) be an \(\otimes\)-functor which is compatible with the unit constraints. Then, the following diagram commutes:

\[
\begin{array}{ccc}
FI & \xrightarrow{\gamma_{F_1}(u)} & FI \\
F_* & \downarrow & F_* \\
I' & \xrightarrow{u} & I'
\end{array}
\]

It follows that \(\gamma_{F_1}^{-1}(Fu) = F_*^{-1}F(u)F_*\), i.e., the notions of the map \(F_1\) in [1] and in Proposition 1 coincide.

**Proof.** Clearly, \(\gamma_{F}(u) = u\). Moreover, the family \(\gamma_{X'}(u)\), \(X' \in G'\), is an endomorphism of the identity functor \(id_{G'}\). So the above diagram commutes.

The final conclusion is deduced from the above commutative diagram, when \(u\) is replaced by \(\gamma_{F_1}^{-1}(Fu)\). \(\square\)
Lemma 2.3. If the hypothesis of Lemma 2.2 holds, we have

\[ F\gamma_X(u) = \gamma_{FX}(\gamma_{I^I}^{-1}Fu). \]

Proof. Consider the following diagram

\[
\begin{array}{cccc}
I' \otimes FX & \xrightarrow{F \otimes \text{id}} & FI \otimes FX & \xrightarrow{F(I \otimes X)} FX \\
\text{(1)} & \downarrow F(u \otimes \text{id}) & \text{(2)} & \downarrow F(u \otimes \text{id}) \\
I' \otimes FX & \xrightarrow{F \otimes \text{id}} & FI \otimes FX & \xrightarrow{F(I \otimes X)} FX \\
\text{(4)} & \downarrow F \gamma_X(u) & \text{(3)} & \downarrow F \gamma_X(u) \\
I' & \xrightarrow{\gamma_{I^I}^{-1}Fu \otimes \text{id}} & I' & \xrightarrow{(5)} \\
\end{array}
\]

In this diagram, the regions (4) and (5) commute thanks to the compatibility of the functor \((F, \tilde{F})\) with the unit constraints. The region (3) commutes due to the definition of \(\gamma_X\) (with image through \(F\)), the region (1) commutes by Lemma 2.2. The commutativity of the region (2) follows from the naturality of the isomorphism \(\tilde{F}\). Therefore, the outer perimeter commutes, i.e., \(F\gamma_X(u) = \gamma_{FX}(\gamma_{I^I}^{-1}Fu)\).

Remark on notations: Hereafter, if there is no explanation, \(S, S'\) refer to Gr-categories \((\Pi, A, h), (\Pi', A', h')\).

A functor \(F : S \rightarrow S'\) is called a functor of the type \((\varphi, f)\) if

\[ F(x) = \varphi(x), \ F(x, u) = (\varphi(x), f(u)), \]

where \(\varphi : \Pi \rightarrow \Pi', f : A \rightarrow A'\) is a pair of group homomorphisms satisfying \(f(xa) = \varphi(x)f(a)\) for \(x \in \Pi, a \in A\).

Proposition 2.4. Each Gr-functor \((F, \tilde{F}) : \mathbb{G} \rightarrow \mathbb{G}'\) induces a Gr-functor \(S_F : S_\mathbb{G} \rightarrow S_{\mathbb{G}'}\) of the type \((\varphi, f)\), with \(\varphi = F_0, f = F_1\). Moreover, \(S_F = G'FH\), where \(H, G'\) are canonical equivalences.

Proof. Let \(K = G'FH\) be the composition functor. One can verify that \(K(s) = F_0(s)\), for \(s \in \pi_0G\). We now prove that \(K(s, u) = (F_0s, F_1u)\) for each morphism \(u : I \rightarrow I\). We have

\[ K(s, u) = G'FH(s, u) = G'(F\gamma_X(u)). \]

Since \(H'G' \simeq \text{id}_{\mathbb{G}'}\), by the natural equivalence \(\beta = (\iota_X')\), the following diagram commutes (note that \(X' = H'G'FX_s\)):

\[
\begin{array}{ccc}
X'_s & \xrightarrow{\iota'} & FX_s \\
H'G'F\gamma_X(u) \downarrow & & \downarrow F\gamma_X(u) \\
X'_s & \xrightarrow{\iota'} & FX_s
\end{array}
\]

According to Lemma 2.3, we have

\[ F\gamma_X(u) = \gamma_{FX}(\gamma_{I^I}^{-1}Fu). \]
Besides, since the family \((\gamma_X)\) is a natural equivalence of the identity functor \(id_{\mathcal{C}}\), the following diagram commutes:

\[
\begin{array}{ccc}
X'_{\phi} & \xrightarrow{\gamma} & FX_{\phi} \\
\downarrow & & \downarrow \\
\gamma_{X,\phi}(\gamma_{\phi}^{-1}F_{\phi}) & = & \gamma_{F_{\phi},\phi}(\gamma_{\phi}^{-1}F_{\phi}) \\
\end{array}
\]

Hence, \(H'G'F\gamma_{X,\phi}(u) = \gamma_{X,\phi}(\gamma_{F,\phi}^{-1}F_{\phi}).\) By the definition of \(H'\), we have

\[G'F\gamma_{X,\phi}(u) = (F_0s, \gamma_{F,\phi}^{-1}F_{\phi}) = (F_0s, F_1(u)).\]

This means \(K = SF.\)

We now describe \(\mathsf{Gr}\)-functors on \(\mathsf{Gr}\)-categories of the type \((\Pi, A)\).

**Theorem 2.5.** Any \((F, \tilde{F}) : \mathcal{S} \to \mathcal{S}'\) is a \(\mathsf{Gr}\)-functor of the type \((\varphi, f)\).

**Proof.** For \(x, y \in \Pi\), \(\tilde{F}_{x,y} : Fx \otimes Fy \to F(x \otimes y)\) is a morphism in \(\mathcal{S}'\). It follows that \(Fx.Fy = F(xy)\). So if one sets \(\varphi(x) = Fx, \varphi : \Pi \to \Pi'\) is a group homomorphism.

We write \(F(x, a) = (\varphi(x), f_x(a)).\) Since \(F\) is a functor, we have

\[F((x, a), (x, b)) = F(x, a).F(x, b).\]

It follows that

\[f_x(a + b) = f_x(a) + f_x(b).\]

Therefore, \(f_x : A \to A'\) is a group homomorphism for each \(x \in \Pi\). Besides, since \((F, \tilde{F})\) is an \(\otimes\)-functor, the following diagram commutes

\[
\begin{array}{ccc}
Fx.Fy & \xrightarrow{\tilde{F}} & F(xy) \\
\downarrow_{F_{u\otimes Fv}} & & \downarrow_{F(u \otimes v)} \\
Fx.Fy & \xrightarrow{\tilde{F}} & F(xy) \\
\end{array}
\]

for all \(u = (x, a), v = (y, b)\). Hence, we have

\[F(u \otimes v) = Fu \otimes Fv \iff f_{xy}(a + xb) = f_x(a) + \varphi(x).f_y(b) \iff f_{xy}(a) + f_{xy}(xb) = f_x(a) + \varphi(x).f_y(b).\]  
(3)

In the relation (3), let \(x = 1\), we obtain \(f_y(a) = f_1(a)\). Thus, \(f_y = f_1\) for all \(y \in \Pi\). Write \(f_y = f\) and use (3), we obtain \(f(xb) = \varphi(x).f(b).\)

Note that if \(\Pi'\)-module \(A'\) is regarded as a \(\Pi\)-module by the action \(xa' = \varphi(x).a'\), then \(f : A \to A'\) is a homomorphism of \(\Pi\)-modules. Since

\[\tilde{F}_{x,y} = (F(xy), g_F(x, y)) : Fx.Fy \to F(xy),\]

where \(g_F : \Pi^2 \to A'\) is a function, it is said that \(g_F\) is associated with \(\tilde{F}\). The compatibility of \((F, \tilde{F})\) with the associativity constraint leads to the relation:

\[\varphi^*h' - f_1h = \partial(g_F),\]
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where

\[(f, h)(x, y, z) = f(h(x, y, z)),\]
\[(\varphi^*h')(x, y, z) = h'(\varphi x, \varphi y, \varphi z).\]

One can see that two Gr-functors \((F, \tilde{F})\), \((F', \tilde{F}') : S \to S'\) are homotopic if and only if \(F' = F\), i.e., they are of the same type \((\varphi, f)\), and there is a function \(t : \Pi \to A'\) such that \(g_F' = g_F + \partial t\).

We refer to \(\text{Hom}_{(\varphi, f)}[S, S']\) as the set of homotopy classes of Gr-functors of the type \((\varphi, f)\).

In order to find the sufficient condition to make a functor of the type \((\varphi, f)\) become a Gr-functor, we introduce the notion of the obstruction like in the case of Ann-functors (see [10]). If \(h, h'\) are, respectively, associativity constraints of Gr-categories \(S, S'\) and \(F : S \to S'\) is a functor of the type \((\varphi, f)\), then the function

\[k = \varphi^*h' - f_*h\]  \hspace{1cm} (4)

is called an obstruction of \(F\).

Keeping in mind that \(S = (\Pi, A, h), S' = (\Pi', A', h')\), we move on the following theorem

**Theorem 2.6.** The functor \(F : S \to S'\) of the type \((\varphi, f)\) induces a Gr-functor if and only if its obstruction \([k] = 0\) in \(H^3(\Pi, A')\). Then, there exist bijections:

i) \(\text{Hom}_{(\varphi, f)}[S, S'] \to H^2(\Pi, A')\),

ii) \(\text{Aut}(F) \to Z^1(\Pi, A')\).

**Proof.** If \((F, \tilde{F}) : S \to S'\) is a Gr-functor, then \((F, \tilde{F}) = (\varphi, f, g_F)\), where

\[\varphi^*h' - f_*h = \partial(g_F) \in B^3(\Pi, A').\]

Therefore, \([\varphi^*h'] - [f_*h] = 0\) in \(H^3(\Pi, A')\).

Conversely, since \([\varphi^*h'] - [f_*h] = 0\) there exists a 2-cochain \(g \in Z^2(\Pi, A')\) such that \(\varphi^*h' - f_*h = \partial g\). Take \(\tilde{F}\) be associated with \(g\), one can see that \((F, \tilde{F})\) is a Gr-functor.

i) If \((F, \tilde{F}) : S \to S'\) is a Gr-functor, then \(F = (\varphi, f, g_F)\). Let \(g_F\) be fixed. Now if

\[(K, \tilde{K}) : S \to S'\]

is a Gr-functor of the type \((\varphi, f)\), then \(\partial(g_F) = \varphi^*h' - f_*h = \partial(g_K)\). It follows that \(g_F - g_K\) is a 2-cocycle. Consider the correspondence

\[\Phi : [(K, \tilde{K})] \to [g_F - g_K]\]

between the set of congruence classes of Gr-functors of the type \((\varphi, f)\) from \(S\) to \(S'\) and the group \(H^2(\Pi, A')\).

First, we show that the above correspondence is a map. Indeed, let

\[(K', \tilde{K}') : S \to S'\]
be a Gr-functor and \( u : K \to K' \) be a homotopy. Then \( K, K' \) are of the same type \((\varphi, f)\) and \( g_{K'} = g_K + \partial t \) where \( g_K, g_{K'} \) are, respectively, associated with \( K, K' \), i.e., \([g_{K'} - g_K] = [g_F - g_K] \in H^2(\Pi, A')\). Furthermore, \( \Phi \) is an injection.

Finally, we show that the correspondence \( \Phi \) is a surjection. Indeed, assume that \( g \) is an arbitrary 2-cocycle. We have

\[
\partial(g_F - g) = \partial g_F - \partial g = \varphi^* h' - f_* h.
\]

Then, there exists a Gr-functor

\[(K, \tilde{K}) : S \to S'\]

of the type \((\varphi, f)\), with a functor isomorphism \( \tilde{K} = (\bullet, g_F - g) \). So \( \Phi \) is a surjection.

ii) Let \( F = (F, F') : S \to S' \) be a Gr-functor and \( t \in \text{Aut}(F) \). Then, the equality \( g_F = g_F + \partial t \) implies that \( \partial t = 0 \), i.e., \( t \in Z^1(\Pi, A') \).

\[\square\]

3 The general case

Let \( CG \) be a category whose objects are Gr-categories, and whose morphisms are monoidal functors between them. We determine the category \( H^3_{Gr} \) whose objects are triplets \((\Pi, A, [h])\) where \([h] \in H^3(\Pi, A)\). A morphism \((\varphi, f) : (\Pi, A, [h]) \to (\Pi', A', [h'])\) in \( H^3_{Gr} \) is a pair \((\varphi, f)\) such that there is a function \( g : \Pi^2 \to A'\) so that \((\varphi, f, g) : (\Pi, A, h) \to (\Pi', A', h')\) is a Gr-functor, i.e., \([\varphi^* h'] = [f_* h] \in H^3(\Pi, A')\). The composition in \( H^3_{Gr} \) is given by

\[(\varphi', f') \circ (\varphi, f) = (\varphi' \varphi, f' f).
\]

One can observe that two Gr-functors \( F, F' : G \to G' \) are homotopic if and only if \( F_i = F'_i, i = 0, 1 \) and \([g_F] = [g_{F'}]\). Denote the set of homotopy classes of Gr-functors \( G \to G' \) which induce the same pair the map \((\varphi, f)\) by

\[\text{Hom}_{(\varphi, f)}(G, G').\]

We now state the main result of this section

**Theorem 3.1.** (The Classification Theorem) There is a classifying functor

\[
d : CG \to H^3_{Gr},
\]

\[
G \mapsto (\pi_0 G, \pi_1 G, [h_G]),
\]

\[
(F, F') \mapsto (F_0, F_1)
\]

which has the following properties:

i) \( d^F \) is an isomorphism if and only if \( F \) is an equivalence.

ii) \( d \) is a surjection on objects.

iii) \( d \) is full, but not faithful. For \((\varphi, f) : dG \to dG'\), there is a bijection

\[
d : \text{Hom}_{(\varphi, f)}(G, G') \to H^2(\pi_0 G, \pi_1 G').
\]

**Proof.** In the Gr-category \( G \), for each stick \((X_s, i_X)\) we can construct a reduced Gr-category \((\pi_0 G, \pi_1 G, h)\). If the choice of the stick is modified, then the 3-cocycle \( h \) changes to a cohomologous 3-cocycle \( h' \). Therefore, \( G \) determines a unique element \([h] \in H^2(\pi_0 G, \pi_1 G)\). This shows that \( d \) is a map on objects.
For Gr-functors

\[ \mathcal{G} \xrightarrow{F} \mathcal{G}' \xrightarrow{F'} \mathcal{G}'' \]

one can see that \((F'F)_a = F'_{aF_0}.\) Since \((F'F)^*_a\) is the composition

\[ IF'F'F'F_0 \xrightarrow{\text{Comp}} F'F'F_0 \]

then for \(u \in \text{Aut}(I)\) we have

\[(F'F)_1u = (F'F)^*_a(F'F)u(F'F)^*_a \]

That is

\[ d(F' \circ F) = (dF') \circ (dF). \]

Clearly, \(d(id_{\mathcal{G}}) = id_{d\mathcal{G}}.\) Therefore, \(d\) is a functor.

i) According to Proposition 1.1.

ii) If \((\Pi, A, [h])\) is an object of \(H^2_{\text{Gr}}\), \(S = (\Pi, A, h)\) is a Gr-category of the type \((\Pi, A)\) and obviously \(dS = (\Pi, A, [h])\).

iii) Let \((\varphi, f)\) be a morphism in \(\text{Hom}_{H^2_{\text{Gr}}}(d\mathcal{G}, d\mathcal{G}').\) Then, there exists a function \(g: (\pi_0\mathcal{G})^2 \rightarrow \pi_1\mathcal{G}'\) such that

\[ \varphi^*h_{\mathcal{G}'} = f, h_{\mathcal{G}} + \partial g. \]

Hence, by Theorem 2.6,

\[ K = (\varphi, f, g): (\pi_0\mathcal{G}, \pi_1\mathcal{G}, h_{\mathcal{G}}) \rightarrow (\pi_0\mathcal{G}', \pi_1\mathcal{G}', h_{\mathcal{G}'}) \]

is a Gr-functor. Then, the composition Gr-functor \(F = H'K: \mathcal{G} \rightarrow \mathcal{G}'\) induces \(dF = (\varphi, f).\) This shows that the functor \(d\) is full.

To prove that \(\varpi\) is a bijection, we prove the correspondence

\[ \varpi: \text{Hom}_{\Pi, A, h}([\mathcal{G}, \mathcal{G}']) \rightarrow \text{Hom}_{\Pi, A, h}([S_\mathcal{G}, S_\mathcal{G}']), \]

is a bijection.

Clearly, if \(F, F' : \mathcal{G} \rightarrow \mathcal{G}'\) are homotopic, then the induced Gr-functors \(S_F, S_{F'} : S_\mathcal{G} \rightarrow S_\mathcal{G}'\) are homotopic. Conversely, if \(F, F'\) are Gr-functors such that \(S_F, S_{F'}\) are homotopic, then the compositions \(E = H'(S_F)G\) and \(E' = H'(S_{F'})G\) are homotopic, where \(H', G\) are canonical Gr-equivalences. The Gr-functors \(E, E'\) are respectively homotopic to \(F, F'.\) Hence, \(F\) and \(F'\) are homotopic. This shows that \(\varpi\) is an injection.

Now, if \(K = (\varphi, f, g): S_\mathcal{G} \rightarrow S_\mathcal{G}'\) is a Gr-functor, then the composition

\[ F = H'K: \mathcal{G} \rightarrow \mathcal{G}' \]

is a Gr-functor satisfying \(S_F = K,\) i.e., \(\varpi\) is a surjection. Finally, the bijection \(\varpi\) is the composition of (5) and (7). \(\Box\)

By Theorem 3.1, one can simplify the problem of equivalence classification of Gr-categories by the one of Gr-categories with the same (up to an isomorphism) two first invariants.
Let $\Pi$ be a group and $A$ be a $\Pi$-module. It is said that the Gr-category $G$ has a pre-stick of the type $(\Pi, A)$ if there exists a pair of group isomorphisms
\[ p: \Pi \to \pi_0 G, \quad q: A \to \pi_1 G \]
which is compatible with the action of modules
\[ q(su) = p(s)q(u), \]
where $s \in \Pi, u \in A$. The pair $\epsilon = (p, q)$ is called a pre-stick of the type $(\Pi, A)$ to the Gr-category $G$.

A morphism between the two Gr-categories $G, G'$ whose pre-sticks are of the type $(\Pi, A)$ (with, respectively, the pre-sticks $\epsilon = (p, q), \epsilon' = (p', q')$) is a Gr-functor $(F, \tilde{F}): G \to G'$ such that the following diagrams commute
\[ \pi_0 G \xrightarrow{F_0} \pi_0 G', \quad \pi_1 G \xrightarrow{F_1} \pi_1 G'. \]

where $F_0, F_1$ are two homomorphisms induced from $(F, \tilde{F})$.

Clearly, it follows from the definition that $F_0, F_1$ are isomorphisms and therefore $F$ is an equivalence. Let
\[ C\mathcal{G}[\Pi, A] \]
denote the set of equivalence classes of Gr-categories whose pre-sticks are of the type $(\Pi, A)$. We can prove the Classification Theorem of H. X. Sinh [15] based on these results as follows.

**Theorem 3.2.** (H. X. Sinh) There exists a bijection
\[ \Gamma: C\mathcal{G}[\Pi, A] \to H^3(\Pi, A), \]
\[ [G] \mapsto q_1^{-1}p^*\epsilon[h_G]. \]

**Proof.** By Theorem 3.1, each Gr-category $G$ determines uniquely an element $[h_G] \in H^3(\pi_0 G, \pi_1 G)$, and then determines an element
\[ \epsilon[h_G] = q_1^{-1}p^*\epsilon[h_G] \in H^3(\Pi, A). \]

Now, if $F: G \to G'$ is a morphism between two Gr-categories whose pre-sticks of the type $(\Pi, A)$, then the induced Gr-functor $S_F = (\varphi, f, g_F)$ satisfies
\[ \varphi^*[h_{G'}] = f_*[h_G]. \]

It follows that
\[ \epsilon'[h_{G'}] = \epsilon[h_G]. \]

This means $\Gamma$ is a map. Moreover, it is an injection. Indeed, suppose that $\Gamma[G] = \Gamma[G']$, we have
\[ \epsilon'(h_{G'}) - \epsilon(h_G) = \partial g. \]

Therefore, there exists a Gr-functor $J$ of the type $(id, id)$ from $J = (\Pi, A, \epsilon(h_G))$ to $J' = (\Pi, A, \epsilon'(h_{G'}))$. The composition
\[ G \xrightarrow{S_G} G' \xrightarrow{\epsilon^{-1}} J \xrightarrow{J'} J' \xrightarrow{\epsilon'} G' \xrightarrow{\tilde{F}'} G' \]
implies $[G] = [G']$, and $\Gamma$ is an injection. Obviously, $\Gamma$ is a surjection. \qed
The case of braided Gr-categories

A Gr-category $\mathcal{B}$ is called a braided Gr-category if there is a braiding $\mathcal{c}$, i.e., a natural isomorphism $\mathcal{c} = c_{X,Y} : X \otimes Y \to Y \otimes X$, which is compatible with $a, l, r$ in the sense of satisfying the following coherence conditions:

\[
(id_Y \otimes c_{X,Z})a_{Y,Z,X}(c_{X,Y} \otimes id_Z) = a_{Y,Z,X}c_{X,Y\otimes Z}a_{X,Y,Z}, \quad (8)
\]

\[
(c_{X,Z} \otimes id_Y)a_{X,Y,Z}^{-1}(id_X \otimes c_{Y,Z}) = a_{X,Y,Z}^{-1}c_{X\otimes Y,Z}a_{X,Y,Z}^{-1}. \quad (9)
\]

If the braiding $\mathcal{c}$ satisfies $c_{X,Y} \cdot c_{Y,X} = id$ then braided Gr-categories are called symmetric categorical groups, or Picard categories. Then the relations (9) and (8) coincide.

If $(\mathcal{B}, \mathcal{c})$ and $(\mathcal{B}', c')$ are braided Gr-categories, then a braided Gr-functor $(F, F') : \mathcal{B} \to \mathcal{B}'$ is a Gr-functor which is compatible with the braidings $\mathcal{c}, c'$ in the sense that the following diagram commutes

\[
\begin{array}{ccc}
F(X \otimes Y) & \overset{F(c)}{\longrightarrow} & F(Y \otimes X) \\
F_X \downarrow & & \downarrow F_Y \\
FX \otimes FY & \overset{c'}{\longrightarrow} & FY \otimes FX
\end{array}
\]

First, let us briefly recall the result on classification of A. Joyal and R. Street [5].

An abelian 3-cocycle for $M$ with coefficients in $N$ is a pair $(h, \eta)$, where $h : M^3 \to N$ is a “normalized 3-cocycle”, satisfying

\[
h(y, z, t) - h(x + y, z, t) + h(x, y + z, t) - h(x, y, z + t) + h(x, y, z) = 0,
\]

\[
h(x, y, z) - h(y, x, z) + h(y, z, x) + \eta(x, y + z) - \eta(x, y) - \eta(x, z) = 0,
\]

\[
h(x, y, z) - h(x, y, z) + h(z, x, y) + h(x, y, z) - h(x, y + z) + h(y, z) + h(y, z) = 0.
\]

For any function $g : M^2 \to N$ satisfying $g(x, 0) = g(0, y) = 0$, the coboundary of $g$ is the abelian 3-cocycle $\partial_{ab}(g) = (h, \eta)$ defined by the equations

\[
h(x, y, z) = g(y, z) - g(x + y, z) + g(x, y + z) - g(x, y),
\]

\[
\eta(x, y) = g(x, y) - g(y, x).
\]

A function $\nu : M \to N$ between abelian groups is called a quadratic map when the function $M \times M \to N$, $(x, y) \mapsto \nu(x) + \nu(y) - \nu(x + y)$ is bilinear and $\nu(-x) = -\nu(x)$.

The trace of an abelian 3-cocycle $(h, \eta) \in Z^3_{ab}(M, N)$ is a function

\[
t_\eta : M \to N, \quad t_\eta(x) = \eta(x, x).
\]

A simple calculation shows that traces are quadratic maps, and Eilenberg - MacLane [2, 3, 8] proved that the traces determine an isomorphism

\[H^3_{ab}(M, N) \cong Quad(M, N), \quad [(h, \eta)] \mapsto t_\eta.\]
where \( \text{Quad}(M, N) \) is the abelian group of quadratic maps from \( M \) to \( N \). This result plays a fundamental role in the proof of the Classification Theorem (Theorem 3.3 [5]).

A. Joyal and R. Street proved that each braided Gr-category \( \mathcal{B} \) determines a quadratic function \( \mathcal{B} \rightarrow \text{Quad} \) and let \( \text{Quad} \) be the category whose objects \( (M, N, t) \) are quadratic maps \( t : M \rightarrow N \) between abelian groups \( M, N \) and whose morphisms \( (\varphi, f) : (M, N, t) \rightarrow (M', N', t') \) consist of homomorphisms \( \varphi, f \) such that we have a commutative square

\[
\begin{array}{ccc}
M & \xrightarrow{\varphi} & M' \\
t & \downarrow & t' \\
N & \xrightarrow{f} & N'
\end{array}
\]

Let \( \mathcal{BCG} \) denote the category whose objects are braided categorical groups and whose morphisms are braided monoidal functors.

**Theorem 4.1.** (Theorem 3.3 [5]) The functor

\[
T : \mathcal{BCG} \rightarrow \text{Quad},
\]

\[
\mathcal{B} \mapsto (\pi_0 \mathcal{B}, \pi_1 \mathcal{B}, \mathcal{Q} \text{quad})
\]

has the following properties:

i) For each object \( \mathcal{Q} \) of \( \text{Quad} \), there exists an object \( \mathcal{B} \) of \( \mathcal{BCG} \) with an isomorphism \( T(\mathcal{B}) \cong \mathcal{Q} \);

ii) For any isomorphism \( \rho : T(\mathcal{B}) \cong T(\mathcal{B}') \), there is an equivalence \( F : \mathcal{B} \cong \mathcal{B}' \) such that \( T(F) = \rho \); and

iii) \( T(F) \) is an isomorphism if and only if \( F \) is an equivalence.

Now, we state the solution to the classification problem of braided Gr-categories by the same technique performed for Gr-categories.

If \( \mathcal{B} \) is a braided Gr-category with the braiding \( c \), then \( \pi_0 \mathcal{B} \) is an abelian group and acts trivially on \( \pi_1 \mathcal{B} \). Then the reduced Gr-category \( \mathcal{S} \mathcal{B} \) becomes a braided Gr-category, with the induced braiding \( c^* = (\bullet, \eta) \) given by the following commutative diagram:

\[
\begin{array}{ccc}
X_r \otimes X_s & i_{X_r \otimes X_s} & X_{rs} \\
\downarrow^c & \gamma_{X_{rs}}(\eta(r,s)) & \downarrow \\
X_s \otimes X_r & i_{X_s \otimes X_r} & X_{sr}
\end{array}
\]

Moreover, \( (H, \tilde{H}) \) and \( (G, \tilde{G}) \) defined by (1) and (2) are then braided Gr-equivalences.

Therefore, each pair \( (h, \eta) \) of associativity and braiding constraints of \( \mathcal{S} \mathcal{B} \) is an abelian 3-cocycle, and \( \mathcal{B} \) determines uniquely an element \( [(h, \eta)] \in H^3_{ab}(\pi_0 \mathcal{B}, \pi_1 \mathcal{B}) \).

It follows from Theorem 2.5 that

**Corollary 4.2.** Each braided Gr-functor \( (F, \tilde{F}) : \mathcal{S} \rightarrow \mathcal{S}' \) is a triplet \( (\varphi, f, g) \), where

\[
\varphi^* (h', \eta') - f_* (h, \eta) = \partial_{ab}(g).
\]

Let \( H^3_{\text{BGr}} \) denote the category whose objects are triplets \( (M, N, [(h, \eta)]) \), where \( [(h, \eta)] \in H^3_{ab}(M, N) \). A morphism \( (\varphi, f) : (M, N, [(h, \eta)]) \rightarrow (M', N', [(h', \eta')]) \)
On monoidal functors between (braided) Gr-categories

in \( H^3_{\text{Gr}} \) is a pair \((\varphi, f)\) such that there is a function \( g : M^2 \to N' \) making \((\varphi, f, g) : (M, N, (h, \eta)) \to (M', N', (h', \eta'))\) become a braided monoidal functor, i.e., \([\varphi^* (h', \eta')] = [f_*(h, \eta)] \in H^3_{\text{Gr}}(M, N')\).

We write
\[ \text{Hom}_{[\varphi, f]}[\mathbb{B}, \mathbb{B}'] \]
for the set of homotopy classes of braided Gr-functors \( \mathbb{B} \to \mathbb{B}' \) inducing the same pair \((\varphi, f)\).

Now, Corollary 4.2 and the proofs of Theorem 3.1, Theorem 3.2 with some appropriate modifications lead to the following theorem

**Theorem 4.3.** (The Classification Theorem) There is a classifying functor
\[
d : \mathcal{BCG} \to H^3_{\text{Gr}},
\]
which has the following properties:
1) \( dF \) is an isomorphism if and only if \( F \) is an equivalence,
2) \( d \) is a surjection on objects,
3) \( d \) is full, but not faithful. For \((\varphi, f) : d\mathbb{B} \to d\mathbb{B}'\), we have
\[
\text{Hom}_{[\varphi, f]}[\mathbb{B}, \mathbb{B}'] \cong H^2(\mathbb{B}, \mathbb{B}').
\]

We write
\[ \mathcal{BCG}[M, N] \]
for the set of equivalence classes of braided Gr-categories whose pre-sticks are of the type \((M, N)\).

**Theorem 4.4.** There exists a bijection
\[
\Gamma : \mathcal{BCG}[M, N] \to H^3_{\text{ab}}(M, N),
\]
\[ [\mathbb{B}] \mapsto q^{-1}\varphi'^*[\mathbb{A}].\]

5 Gr-category of an abstract kernel

The notion of *abstract kernel* was introduced in [9]. It is a triplet \((\Pi, G, \psi)\), where \( \psi : \Pi \to \text{Aut}G/\text{In}G \) is a group homomorphism. In this section, we describe the Gr-category structure of an abstract kernel and apply it to make the constraints of a Gr-category be strict. The operation of \( G \) is denoted by +. The *center* of \( G \), denoted by \( ZG \), consists of elements \( c \in G \) such that \( c + a = a + c \) for all \( a \in G \).

Let us recall that the obstruction of \((\Pi, G, \psi)\) is an element \([k] \in H^3(\Pi, ZG)\), defined as follows. For each \( x \in \Pi \), choose \( \varphi(x) \in \psi(x) \) such that \( \varphi(1) = id_G \).

Then, there is a function \( f : \Pi^2 \to G \) satisfying
\[
\varphi(x)\varphi(y) = f_{x,y}(x, y).
\]
(10)
The pair \((\varphi, f)\) therefore induces an element \( k \in Z^3(\Pi, ZG) \) by the relation
\[
\varphi(x)[f(y, z)] + f(x, yz) = k(x, y, z) + f(x, y) + f(xy, z).
\]
(11)
For each group $G$, we can construct a category, denoted by $\text{Aut}_G$, whose objects are elements of the group of automorphisms $\text{Aut}G$. For two elements $\alpha, \beta$ of $\text{Aut}G$, we write

$$\text{Hom}(\alpha, \beta) = \{c \in G | \alpha = \mu_c \circ \beta\},$$

where $\mu_c$ is an inner-automorphism of $G$, induced by $c \in G$. For two morphisms $c : \alpha \rightarrow \beta; \ d : \beta \rightarrow \gamma$ in $\text{Aut}_G$, the composition is defined by $d \circ c = c + d$ (the addition in $G$).

The category $\text{Aut}_G$ is a strict Gr-category with the tensor product defined by $\alpha \otimes \beta = \alpha \circ \beta$, and

$$\alpha \otimes \alpha' \circ (\beta \otimes \beta') = \alpha \otimes (\beta + \gamma') \circ \beta'.$$

(12)

The following proposition describes the reduced Gr-category of the Gr-category of an abstract kernel.

**Proposition 5.1.** Let $(\Pi, G, \psi)$ be an abstract kernel with $[k] \in H^3(\Pi, ZG)$ be its obstruction. Let the reduced Gr-category of the strict one $\text{Aut}_G$ be $S_{\text{Aut}_G} = (\Pi', C, h)$. Then

i) $\Pi' = \pi_0(\text{Aut}_G) = \text{Aut}G / \text{In}G, C = \pi_1(\text{Aut}_G) = ZG$,

ii) $\psi^*h$ belongs to the cohomology class of $k$.

**Proof.** i) It follows from the definition of the category $\text{Aut}_G$ and the reduced Gr-category.

ii) Let $(H, \tilde{H})$ be a canonical Gr-equivalence from $S$ to $\text{Aut}_G$. Then, the following diagram

$$
\begin{align*}
Hr(HsHt) & \xrightarrow{id \otimes \tilde{H}} HrH(st) \xrightarrow{\tilde{H}} H(r(st)) \\
H(r(s))Ht & \xrightarrow{H(r, s) \otimes id} H(rs)Ht \xrightarrow{\tilde{H}} H((rs)t)
\end{align*}
$$

(13)

commutes for all $r, s, t \in \Pi'$. Since $\text{Aut}_G$ is a strict Gr-category, we have

$$\gamma_a(u) = u, \forall a \in \text{Aut}_G, \forall u \in ZG = C.$$

Associating with the definition of $H$, we obtain $H(\bullet, c) = c, \forall c \in C$. From the commutativity of the diagram (13) and the relation (12), we have

$$Hr[g(s, t)] + g(r, st) = g(r, s) + g(rs, t) - h(r, s, t)$$

(14)

where $g = g_{\Pi'} : \Pi' \times \Pi' \rightarrow G$ is associated with $\tilde{F}$.

For the abstract kernel $(\Pi, G, \psi)$, choose a function $\varphi = H.\psi : \Pi \rightarrow \text{Aut}(G)$. Clearly, $\varphi(1) = id_G$. Moreover, since

$$\tilde{H}_\psi(x, \psi(y)) : H\psi(x)H\psi(y) \rightarrow H\psi(xy)$$

is a morphism in $\text{Aut}_G$, for all $x, y \in \Pi$ we have

$$\varphi(x)\varphi(y) = H\psi(x)H\psi(y) = \mu_{f(x, y)}H\psi(xy) = \mu_{f(x, y)}\varphi(xy),$$

where $f(x, y) = \tilde{H}_\psi(x, \psi(y))$. Thus, the pair $(\varphi, f)$ is a factor set of the abstract kernel $(\Pi, G, \psi)$. It induces an obstruction $k(x, y, z) \in Z^3(\Pi, ZG)$ satisfying [11].

Now, for $r = \psi(x), \ s = \psi(y), \ t = \psi(z)$, the equation (14) becomes

$$\varphi(x)[f(y, z)] + f(x, yz) = +f(x, y) + f(xy, z) - (\psi^*h)(x, y, z).$$

In comparison with [11], $[\psi^*h] = [k]$. 

\[\square\]
We now use Proposition 5.1 and the Theorem on the realization of the obstruction in the group extension problem to prove Theorem 5.3. First, we need the following lemma

**Lemma 5.2.** Let $\mathbb{H}$ be a strict Gr-category and $S_{\mathbb{H}} = (\Pi, C, h)$ be its reduced Gr-category. Then, for each group homomorphism $\psi : \Pi' \to \Pi$, there exists a strict Gr-category $\mathcal{G}$ which is Gr-equivalent to the Gr-category $J = (\Pi', C, h')$, where $C$ is regarded as a $\Pi'$-module with an operator $xc = \psi(x)c$, and $h'$ belongs to the same cohomology class as $\psi^*h$.

**Proof.** We construct a strict Gr-category $\mathcal{G}$ as follows

- $\text{Ob}(\mathcal{G}) = \{(x, X) \mid x \in \Pi', X \in \psi(x)\}$,
- $\text{Hom}((x, X), (x, Y)) = \{x\} \times \text{Hom}_{\mathbb{H}}(X, Y)$.

The tensor products on objects and morphisms of $\mathcal{G}$ are defined by

- $(x, X) \otimes (y, Y) = (xy, X \otimes Y)$,
- $(x, u) \otimes (y, v) = (xy, u \otimes v)$.

The unit object of $\mathcal{G}$ is $(1, I)$ where $I$ is the unit object of $\mathbb{H}$. One can verify that $\mathcal{G}$ is a strict Gr-category. Moreover, we have isomorphisms

- $\lambda : \pi_0 \mathcal{G} \to \Pi'$, $f : \pi_1 \mathcal{G} \to \pi_1 \mathbb{H} = C$,
- $[(x, X)] \mapsto x$, $(1, c) \mapsto c$,

and a Gr-functor $(F, \tilde{F}) : \mathcal{G} \to \mathbb{H}$ given by

- $F(x, X) = X$, $F(x, u) = u$, $\tilde{F} = \text{id}$.

Let $S_F = (\phi, \tilde{\phi}) : S_{\mathcal{G}} \to S_{\mathbb{H}}$ be a Gr-functor induced by $(F, \tilde{F})$ between the reduced categories. Then, we have

- $\phi(x, X) = F_0(x, X) = [F(x, X)] = [X] = \psi(x)$,
- $\phi(1, u) = F_1(1, u) = \gamma_{F(1, 1)}F(1, u) = \gamma_1(u) = u$,

where $u$ is a morphism in $\mathbb{G}$. This means $F_0 = \psi \lambda$ and $F_1 = f$, or $S_F$ is a functor of the type $(\psi \lambda, f)$.

Now if $h_G$ is the associativity constraint of $S_{\mathcal{G}}$. By Theorem 2.6, the obstruction of the pair $(\psi \lambda, f)$ must vanish in $H^3(\pi_0 \mathcal{G}, \pi_1 \mathbb{H}) = H^3(\pi_0 \mathcal{G}, C)$, i.e.,

- $(\psi \lambda)^*h = f_! h_{G} + \delta \tilde{\phi}$.

Now, if we denote $h' = f_* h_{G}$, the pair $J = (\lambda, f)$, $\tilde{J} = \text{id}$ is a Gr-functor from $S_{\mathcal{G}}$ to $J = (\Pi', C, h')$. Then, the composition

- $G \xrightarrow{(G, \tilde{G})} S_{\mathcal{G}} \xrightarrow{(J, \tilde{J})} J$

is a Gr-equivalence from $\mathcal{G}$ to $J = (\Pi', C, h')$.

Finally, we prove that $h'$ belongs to the same cohomology class as $\psi^*h$.

Let $K = (\lambda^{-1}, f^{-1}) : (\Pi', C, h') \to S_{\mathcal{G}}$. Then $K$ together with $\tilde{K} = \text{id}$ is a Gr-functor, and the composition

- $(\phi, \tilde{\phi}) \circ (K, \tilde{K}) : (\Pi', C, h') \to S_{\mathbb{H}}$
is a Gr-functor making the following diagram commute

\[
\begin{array}{ccc}
S_G & \xrightarrow{\phi} & S_H \\
\downarrow K & & \downarrow (\Psi, C, h') \\
J = (\Pi', C, h') & \xrightarrow{\phi \circ K} & \end{array}
\]

Clearly, \(\phi \circ K\) is a Gr-functor of the type \((\psi, id)\) and therefore its obstruction vanishes. By (4), we have \(\psi^* h - h' = \partial g\), i.e., \([h'] = [\psi^* h]\).

**Theorem 5.3.** Each Gr-category is Gr-equivalent to a strict one.

**Proof.** Let \(C\) be a Gr-category whose reduced Gr-category is \(S_C = (\Pi', C, k)\). By the theorem on the realization of the obstruction (Theorem 9.2 Section IV [9]), the realization of 3-cocycle \(k \in H^3(\Pi', C)\) is the group \(G\) with the center \(ZG = C\) and group homomorphism \(\psi : \Pi' \to \text{Aut}G/\text{In}G\) such that \(\psi\) induces a \(\Pi'\)-module structure on \(C\) and the obstruction of the abstract kernel \((\Pi', G, \psi)\) is \(k\). By Proposition 5.1, \(S_{\text{Aut}_G} = (\text{Aut}G/\text{In}G, C, h)\) is the reduced Gr-category of the strict Gr-category \(\text{Aut}_G\), where \([\psi^* h] = [k]\).

Using Lemma 5.2 for \(H = \text{Aut}_G\), the homomorphism \(\psi : \Pi' \to \text{Aut}G/\text{In}G\) defines a strict Gr-category \(G\), which is Gr-equivalent to the strict Gr-category \(\Pi'\)-module structures of \(C\) on \(S_C\) and on \(J\) coincide. Moreover, \([\psi^* h] = [h']\). It follows that \([h'] = [k]\). So there is a function \(g : \Pi' \times \Pi' \to C\) such that \(h' - k = \partial g\). Then, by Theorem 2.6,

\[(K, \tilde{K}) = (id_{\Pi'}, id_{\Pi'}, g) : S_C \to J\]

is a Gr-equivalence. Therefore, \(C\) is equivalent to the strict Gr-category \(G\).

Readers can see a different proof of Theorem 5.3 in [15].

**6 Gr-functors and the group extension problem**

In this section, we apply Theorem 3.1 to obtain Schreier classical Theorem on group extensions.

**Theorem 6.1.** Let \(G\) and \(\Pi\) be groups. Then

i) There is a canonical partition

\[\text{Ext}(\Pi, G) = \bigsqcup_{\psi} \text{Ext}(\Pi, G, \psi),\]

where, for each morphism \(\psi : \Pi \to \text{Aut}G/\text{In}G\), \(\text{Ext}(\Pi, G, \psi)\) is the set of equivalence classes of group extensions \(E : G \to B \to \Pi\) of \(G\) by \(\Pi\) which induce \(\psi\).

ii) Each abstract kernel \((\Pi, G, \psi)\) determines a (normalized) third-dimensional cohomology class \(\text{Obs}(\Pi, G, \psi) \in H^3(\Pi, ZG)\) (with respect to the \(\Pi\)-module structure on \(ZG\) obtained via \(\psi\)), called the obstruction of \((\Pi, G, \psi)\). The abstract kernel has extensions if and only if its obstruction vanishes. Then, there is a bijection

\[\text{Ext}(\Pi, G) \leftrightarrow H^3_\text{G}(\Pi, ZG).\]
As below, each factor set \((\varphi, f)\) of a group extension can be lifted to a Gr-functor \(F : \text{Dis}\Pi \to \text{Aut}G\), when \(\text{Dis}\Pi\) is regarded as a Gr-category of the type \((\Pi, 0, 0)\), and therefore we can classify group extensions by means of Gr-functors.

We write \(\text{Hom}_\psi[\text{Dis}\Pi, \text{Aut}G]\) for the set of homotopy class of Gr-functors from \(\text{Dis}\Pi\) to \(\text{Aut}G\), inducing the pair of homomorphisms \((\psi, 0)\) and \(\text{Ext}_\psi(\Pi, G)\) for the set of equivalence classes of group extensions of \(G\) by \(\Pi\), we have

**Theorem 6.2.** There exists a bijection

\[ \Delta : \text{Hom}_\psi[\text{Dis}\Pi, \text{Aut}G] \to \text{Ext}_\psi(\Pi, G). \]

**Proof.** Step 1: The construction of the group extension \(E_F\) of \(G\) by \(\Pi\), induced by Gr-functor \(F\).

Let \((F, \tilde{F}) : \text{Dis}\Pi \to \text{Aut}G\) be a Gr-functor. Then, \(\tilde{F}_{x,y} = f(x, y)\) is a function from \(\Pi^2\) to \(G\) such that

\[ Fx \circ Fy = \mu_{f(x,y)} \circ Fxy. \]  

(15)

The compatibility of \((F, \tilde{F})\) with the unit and associativity constraints, respectively, implies

\[ Fx[f(y, z)] + f(x, yz) = f(x, y) + f(xy, z), \]  

(16)

\[ f(x, 1) = f(1, y) = 0. \]  

(17)

Set \(B_F = \{(a, x) | a \in G, x \in \Pi\}\) and the operation

\[ (a, x) + (b, y) = (a + Fx(b) + f(x, y), xy). \]

Then, \(B_F\) is an extension of \(G\) by \(\Pi\),

\[ E_F : 0 \to G \xrightarrow{i} B_F \xrightarrow{p} \Pi \to 1, \]

where \(i(a) = (a, 1), p(a, x) = x\). The relations (15), (16) imply the associativity of the operation in \(B_F\). Indeed, the unit of the addition in \(B_F\) is \((0, 1)\), the opposite element \((a, x) \in B_F\) is \((b, x^{-1}) \in B_F\), where \(b\) is an element such that \(Fx(b) = -a + f(x, x^{-1})\).

The conjugation homomorphism \(\psi : \Pi \to \text{Aut}G/\text{In}G\) is determined by \(\psi(x) = [\mu_{(0,x)}]\). By a simple calculation, we have \(\mu_{(0,x)}(a, 1) = (Fx(a), 1)\). Let \(G\) and its image \(iG\) be identical, we obtain \(\psi(x) = [Fx]\).

Step 2: \(F\) and \(F'\) are homotopic if and only if \(E_F\) and \(E_{F'}\) are congruent.

Let \(F, F' : \text{Dis}\Pi \to \text{Aut}G\) be Gr-functors and \(\alpha : F \to F'\) be a homotopy. Then, by the definition of Gr-morphisms, the following diagram commutes

\[ \begin{array}{ccc}
Fx \otimes Fy & \xrightarrow{F} & F(xy) \\
\alpha_x \otimes \alpha_y & & \alpha_{xy} \\
F'x \otimes F'y & \xrightarrow{F'} & F'(xy)
\end{array} \]

That is,

\[ \tilde{F}_{x,y} + \alpha_{xy} = \alpha_x \otimes \alpha_y + \tilde{F}'_{x,y}, \]
or

\[ f(x, y) + \alpha_{xy} = \alpha_x + F'x(\alpha_y) + f'(x, y). \]  \hspace{1cm} (18)

Now, we write

\[ \beta : BF \rightarrow BF', \]

\[ (a, x) \mapsto (a + \alpha_x, x). \]

Note that \( Fx = \mu_{\alpha_x} \circ F'x \), and by (15) one can see that \( \beta \) is a homomorphism. Moreover, it is an isomorphism making the following diagram commute, i.e., \( E_F \) and \( E_{F'} \) are congruent.

\[
\begin{array}{ccc}
E_F : 0 & \xrightarrow{i} & G \\
\downarrow{id} & & \downarrow{\beta} \\
E_{F'} : 0 & \xrightarrow{i'} & G \\
\end{array}
\]

The conversion of the proposition can be obtained as we see by retracing our steps.

**Step 3: \( \Delta \) is a surjection.**

Suppose that the group extension

\[ E : 0 \rightarrow G \rightarrow B \rightarrow \Pi \rightarrow 1, \]

associates with the homomorphism \( \psi : \Pi \rightarrow \text{Aut}G/\text{In}G \). Select a “representative” \( u_x, x \in \Pi \), in \( B \), that is \( p(u_x) = x \). In particular, choose \( u_1 = 0 \). Then, the elements of \( B \) can be written uniquely as \( a + u_x \), for \( a \in G, x \in \Pi \), and

\[ u_x + a = \mu_{u_x}(a) + u_x. \]

The sum \( u_x + u_y \) must be in the same coset as \( u_{xy} \), so there are unique elements \( f(x, y) \in G \) such that

\[ u_x + u_y = f(x, y) + u_{xy}. \]

The function \( f \) is called a factor set of the extension \( E \). Thus, it satisfies the relations

\[ \mu_{u_x}[f(y, z)] + f(x, yz) = f(x, y) + f(xy, z), \quad x, y, z \in \Pi. \]  \hspace{1cm} (19)

\[ f(x, 1) = f(1, y) = 0. \]  \hspace{1cm} (20)

We construct a Gr-functor \( F = (F, \tilde{F}) : \text{Dis}\Pi \rightarrow \text{Aut}_G \) as follows: \( Fx = \mu_{u_x}, \tilde{F}_{x,y} = f(x, y) \).

Clearly, the relations (19), (20) show that \( (F, \tilde{F}) \) is a monoidal functor between Gr-categories.

We now prove Theorem 6.1.

Let \( (\Pi, G, \psi) \) be an abstract kernel. For each \( x \in \Pi \), choose \( \varphi(x) \in \psi(x) \) such that \( \varphi(1) = id_G \). The family of \( \varphi(x) \) induces a function \( f : \Pi^2 \rightarrow G \) satisfying the relation (10). The pair \( (\varphi, f) \) induces an obstruction \( k \in Z^3(\Pi, ZG) \) by the relation (11). Write \( F(x) = \varphi(x) \), we obtain a functor \( \text{Dis}\Pi \rightarrow \text{Aut}_G \).
Let $S = (\text{Aut}G/\text{In}G, ZG, h)$ be the reduced Gr-category of $\text{Aut}_G$. Then $F$ induces the pair of group homomorphisms $(\psi, 0) : (II, 0) \rightarrow (\text{Aut}G/\text{In}G, ZG)$ and by the relation \( \{ k \} \) an obstruction of the functor $F$ is $\psi^* h$. By Proposition 5.1, $[\psi^* h] = [k]$, i.e., the obstruction of the abstract kernel $(II, G, \psi)$ and the obstruction of the functor $F$ coincide. Then, by Theorem 2.6, $(II, G, \psi)$ has extensions if and only if its obstruction vanishes.

According to Theorem 3.1, there is a bijection
\[
\text{Hom}_{(\psi, 0)}[\text{Dis}II, \text{Aut}_G] \leftrightarrow H^2(II, ZG),
\]
since $\pi_0(\text{Dis}II) = II, \pi_1(\text{Aut}_G) = ZG$. Combination with Theorem 6.1 yields:
\[
\text{Ext}(II, G) \leftrightarrow H^2(II, ZG).
\]
This completes the proof.

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