Lax representations with non-removable parameters and integrable hierarchies of PDEs via exotic cohomology of symmetry algebras. II

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Abstract. We derive new Lax representations for the hyper-CR equation of Einstein–Weyl structures and for the associated integrable hierarchy.
1. Introduction

In this paper we continue to study the relations between structure of symmetry algebras and Lax representations of integrable partial differential equations \[10, 11, 12\]. We consider the hyper-CR equation for Einstein–Weyl structures \[7, 9, 13, 2\]

\[ u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}, \]

(1)

and generalize the Lax representation

\[
\begin{align*}
  v_t &= (\lambda^2 - \lambda u_x - u_y) v_x, \\
  v_y &= (\lambda - u_x) v_x,
\end{align*}
\]

(2)

thereof. In \[12\] we show that (2) can be inferred from nontriviality of the second exotic cohomology group of Sym\(_0\)(\(E\)). A nontrivial cocycle provides an extension of the symmetry algebra Sym\(_0\)(\(E\)). The linear combination of the Maurer–Cartan (mc) form \(\sigma\) of the extension and certain horizontal mc form of Sym\(_0\)(\(E\)) gives the Wahlquist–Estabrook form that generates the Lax representation (2). In the section 2 of the present paper we find new Lax representation for equation (1). We consider the linear combination of \(\sigma\) with arbitrary basic horizontal mc form of Sym\(_0\)(\(E\)) and find conditions under which this combination produces a Lax representation for equation (1). As a result we generalize the the Lax representation (2).

Expansion of (2) into the Taylor series w.r.t. \(\lambda\) yields an infinite-dimensional Lax representation for (1). This infinite-dimensional Lax representation can be used to study nonlocal symmetries \[1\] and nonlocal conservation laws \[8\] of equation (1). In section 3 we expand the new Lax representation into the Taylor series and find explicit expressions for the coefficients of the obtained infinite-dimensional Lax representation. This Lax representation can be considered as a pair of hydrodynamic chains (see \[13\] and references therein) whose compatibility conditions coincide with (1).

In section 4 we derive new Lax representation (14) for the integrable hierarchy associated with equation (1).

We follow definitions and notation of \[3, 4, 12\], see also \[5, 6, 14\].

2. The generalized Lax representation

As it was shown in \[12\], the structure equations for the symmetry algebra Sym\(_0\)(\(E\)) of (1) have the form

\[
\begin{align*}
  d\alpha_0 &= 0, \\
  d\alpha_1 &= \alpha_0 \wedge \alpha_1, \\
  d\Theta &= \nabla_1(\Theta) \wedge \Theta + (h_0 \alpha_0 + h_0^2 \alpha_1) \wedge \nabla_0(\Theta),
\end{align*}
\]

(3)

where

\[
\Theta = \sum_{k=0}^{3} \sum_{m=0}^{\infty} \frac{h_k h_1^m}{m!} \theta_{k,m},
\]

(4)
where \( h_0 \) and \( h_1 \) are formal parameters such that \( dh_i = 0 \) and \( h_0^k = 0 \) when \( k > 3 \), \( \nabla_0 \) is the derivative with respect to \( h_0 \) in \( \mathbb{R}[h_0] = \mathbb{R}[h_0]/(h_0^3 = 0) \), \( \nabla_1 = \partial_{h_1} \), while \( \alpha_0, \alpha_1 \), and \( \theta_{k,m} \) are MC forms of \( \text{Sym}_0(\mathcal{E}) \). The second exotic cohomology group \( H^{2,0}_c(\text{Sym}(\mathcal{E})) \) is nontrivial when \( c = 1 \), the nontrivial 2-cocycle \( \alpha_0 \land \alpha_1 \) of the differential \( d_{\alpha_0} \) defines a non-central extension of the Lie algebra \( \text{Sym}(\mathcal{E}) \). The additional Maurer–Cartan form \( \sigma \) for the extended Lie algebra is a solution to \( d_{\alpha_0} \sigma = \alpha_0 \land \alpha_1 \), that is, to equation

\[
d\sigma = \alpha_0 \land \sigma + \alpha_0 \land \alpha_1.
\]  

This equation is compatible with the structure equations (3) of the Lie algebra \( \text{Sym}_0(\mathcal{E}_1) \).

We find the following expressions for the MC forms: \( \alpha_0 = dq \), \( \alpha_1 = -e^q ds \), \( \sigma = e^q (dv - q ds) \), \( \theta_{0,0} = r dt \), \( \theta_{1,0} = r e^q (dy - (u_x - 2 s) dt) \), \( \theta_{2,0} = r e^{2q} (dx - (u_x - s) dy - (u_y + s u_x - s^2) dt) \), \( \theta_{3,0} = r e^{3q} (du - u_t dt - u_x dx - u_y dy) \), where \( q, s, v, \) and \( r \) are free parameters. As it was shown in [12], the form \( \sigma - \theta_{2,0} \) is the Wahlquist–Estabrook form of the Lax representation (2).

Consider the linear combination

\[
\sigma - \sum_{k=0}^{2} c_k \theta_{k,0} = e^q (dv - q ds - c_2 r e^q dx - r (c_1 + c_2 e^q(s - u_x)) dy
\]

\[
- r (c_0 e^{-q} + c_1 (2 s - u_x) + c_2 e^q(s^2 - s u_x - u_y)) dt
\]

of the form \( \sigma \) and the basic horizontal forms \( \theta_{0,0}, \theta_{1,0}, \theta_{2,0} \). Put without loss of generality \( c_2 = 1 \) and rename \( q = v_s, r = v_x \exp(-v_s) \). Then the form

\[
\sigma - c_0 \theta_{0,0} - c_1 \theta_{1,0} - \theta_{2,0} = e^q (dv - v_s ds - v_x (dx + (s - u_x + c_1 e^{-v_s}) dy
\]

\[
+ (s - u_x - u_y + c_1 e^{-v_s} (2 s - u_x) + c_0 e^{-2v_s}) dt)
\]

is equal to zero whenever there hold

\[
\begin{align*}
  v_t &= (s^2 - s u_x - u_y + c_1 e^{-v_s} (2 s - u_x) + c_0 e^{-2v_s}) v_x, \\
  v_y &= (s - u_x + c_1 e^{-v_s}) v_x.
\end{align*}
\]  

The compatibility condition \( (v_t)_y = (v_y)_t \) reads

\[
(c_1^2 - c_0) v_x e^{-2v_s} (2 v_x + u_{xx}) - v_x (u_{yy} - u_{tx} - u_y u_{xx} + u_x u_{xy}) = 0.
\]  

In Appendix we prove that system (3) does not define a Lax representation for equation (1) when \( c_0 \neq c_1^2 \). Consider case \( c_0 = c_1^2 \). When \( c_1 = 0 \), we get the Lax representation (2) with \( \lambda = s \). When \( c_1 \neq 0 \) we put \( c_1 = 1 \) without loss of generality. This yields the new Lax representation

\[
\begin{align*}
  v_t &= (s^2 - s u_x - u_y + e^{-v_s} (2 s - u_x) + e^{-2v_s}) v_x, \\
  v_y &= (s - u_x + e^{-v_s}) v_x
\end{align*}
\]  

of equation (1).
3. Infinite-dimensional Lax representation

For the infinite sequence \( b = (a_1, a_2, \ldots, a_n, \ldots) \) put \( R_0(b) = 1 \) and

\[
R_k(b) = S_k \left( -\frac{a_1}{1}, -\frac{a_m}{m!}, \ldots, -\frac{a_k}{k!} \right)
\]
for \( k \geq 1 \), where the elementary Schur polynomials \( S_k \) are defined by the generating function

\[
\exp \left( \sum_{m=1}^{\infty} a_m z^m \right) = 1 + \sum_{k=1}^{\infty} S_k(a_1, \ldots, a_k) z^k.
\]

For expansion

\[
v = \sum_{k=0}^{\infty} v_k s_k \frac{s^k}{k!}, \quad v_k = v_k(t, x, y),
\]

of function \( v \) into the Taylor series w.r.t. \( s \) denote \( v = (v_2, v_3, \ldots, v_m, \ldots) \). Then we have

\[
v_s = \sum_{k=0}^{\infty} v_{k+1} s_k \frac{s^k}{k!},
\]

and

\[
e^{-v_s} = e^{-v_1} \left( \sum_{k=0}^{\infty} R_k(v) s^k \right).
\]

Substituting for (9) into (8) yields the infinite-dimensional Lax representation

\[
\begin{align*}
v_{0,t} & = - (u_y + u_x e^{-v_1} + e^{-2v_1}) v_{0,x}, \\
v_{1,t} & = (v_2 e^{-v_1} (u_x - 2e^{-v_1}) + 2e^{-v_1} - u_x) v_{0,x} + (e^{-2v_1} - e^{-v_1} u_x - u_y) v_{1,x}, \\
v_{k,t} & = k (k-1) v_{k-2,x} - k u_x v_{k-1,x} - u_y v_{k,x} + 2 e^{-v_1} \sum_{m=0}^{k-1} R_{k-m-1}(v) \frac{k!}{m!} v_{m,x} \\
& + e^{-v_1} \sum_{m=0}^{k} (e^{-v_1} R_{k-m}(2v) - u_x R_{k-m}(v)) \frac{k!}{m!} v_{m,x}, \\
v_{0,y} & = (-u_x + e^{-v_1}) v_{0,x}, \\
v_{1,y} & = (1 - v_2 e^{-v_1}) v_{0,x} + (-u_x + e^{-v_1}) v_{1,x}, \\
v_{k,y} & = k v_{k-1,x} - u_x v_{k,x} + e^{-v_1} \sum_{m=0}^{k} R_{k-m}(v) \frac{k!}{m!} v_{m,x}.
\end{align*}
\]

with \( k \geq 2 \). This Lax representation is a generalization of the ‘positive’ covering of (1) from [II § 4.1] and can be considered as a pair of hydrodynamic chains whose compatibility conditions coincide with (1).
4. Lax representation for the associated hierarchy

The symmetry algebra $\text{Sym}_0(E)$ of equation (1) admits an increasing sequence of natural extensions $\text{Sym}_0(E) = p_4 \subset p_5 \subset \ldots \subset p_n \subset p_{n+1} \subset \ldots$, where the Lie algebra $p_{n+1}$ has the same structure equations (3) with (4) replaced by

$$\Theta = \sum_{k=0}^{n} \sum_{m=0}^{\infty} \frac{h_k^0 h_1^m}{m!} \theta_{k,m},$$

and $h_0^k = 0$ for $k \geq n + 1$. The nontrivial 2-cocycle $\alpha_0 \wedge \alpha_1$ of $d_{\alpha_0}$ defines a non-central extension $\hat{p}_{n+1}$ of the Lie algebra $p_{n+1}$. The structure equations for $\hat{p}_{n+1}$ are given by (3), (10), and (5).

As we show in [12], the basic horizontal forms $\theta_{0,0}, \ldots, \theta_{n-1,0}$ for the Lie algebra $\hat{p}_{n+1}$ with fixed $n > 4$ can be expressed as follows. Put $p_0 = 1$ and for $i \geq 0, j \in \{0, \ldots, i\}$ define polynomials $P_{ij} = P_{ij}(s)$ of variable $s$ by the formula

$$P_{ij} = \sum_{k=0}^{j} (-1)^k \binom{i-j+k-1}{k} p_{j-k} s^k.$$  

(11)

Coefficients of $P_{ij}$ depend on parameters $p_1, \ldots, p_j$. We have

$$\theta_{k,0} = e^{kq} r \sum_{j=0}^{k} P_{kj} dx_{n-k+j-1}$$

(12)

for $k \in \{0, \ldots, n\}$. Then we put $x_{-1} = u$ and enforce $\theta_{n,0}$ to be the contact form:

$$\theta_n = e^{nq} r \left( du - \sum_{i=0}^{n-1} u_{x_i} dx_i \right).$$

This requirement yields the linear triangular system

$$P_{k,k-i} = -u_{x_{k-i-1}}, \quad i \in \{0, \ldots, k-1\}, \quad k \in \{1, \ldots, n\}$$

(13)

with unknows $p_1, p_2, \ldots, p_n$. The final expressions for forms $\theta_{k,0}$ are obtained by substituting for the solution of (13) into (12).

As it was shown in [12], form $\sigma - \theta_{n-1}$ is the Wahlquist–Estabrook form for the Lax representation of the $n$-th element of the integrable hierarchy associated to equation (1). To generalize this result we consider the linear combination

$$\sigma - \sum_{k=0}^{n-1} c_k \theta_{k,0}.$$  

where $c_{n-1} = 1$ without loss of generality. This form is equal to zero whenever an overdetermined system of PDEs for function $v = v(x_0, \ldots, x_{n-1}, s)$ holds. The compatibility conditions of this system coincide with equation (1) when either $c_i = 0$ for $i \in \{0, \ldots, n-2\}$, that is, we have the Wahlquist–Estabrook form $\sigma - \theta_{n-1}$ from.
Lax representations and integrable hierarchies via exotic cohomology. II

\[ c_k = c_{n-2}^{n-k-1} \text{ for } k \in \{0, \ldots, n-3\}. \]

In the last case we put \( c_{n-2} = 1 \) without loss of generality and obtain

\[
\begin{align*}
v_{x_1} &= (s - u_{x_0}) v_{x_0}, \\
v_{x_2} &= v_{x_0} \mathcal{D}_2 (s^2 - s u_{x_0} - u_{x_1}), \\
&\quad \vdots \\
v_{x_i} &= v_{x_0} \mathcal{D}_i \left( s^i - \sum_{j=0}^{i-1} s^{i-j-1} u_{x_j} \right), \\
&\quad \vdots \\
v_{x_{n-1}} &= v_{x_0} \mathcal{D}_{n-1} (s^{n-1} - s^{n-2} u_{x_0} - s^{n-3} u_{x_1} - \ldots - s u_{x_{n-3}} - u_{x_{n-2}})
\end{align*}
\]

with differential operators

\[
\mathcal{D}_m = \sum_{k=0}^{m} \frac{e^{-k s}}{k!} \frac{\partial^k}{\partial s^k}.
\]

The compatibility conditions of this system give the \( n \)-th element of the integrable hierarchy

\[
u_{x_m x_k} = u_{x_{m-1} x_{k+1}} + u_{x_k} u_{x_0 x_{m-1}} - u_{x_{m-1}} u_{x_0 x_k},
\]

\( m \in \{1, \ldots, k\}, \quad k \in \{1, \ldots, n-2\}, \) associated with equation (1). When \( n = 3 \), the change of notation \( x_0 \mapsto x, \ x_1 \mapsto y, \ x_2 \mapsto t \) in (14) and (15) gives (8) and (1).

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Appendix

When $c_1^2 \neq c_0$, equation (17) yields either $v_x = 0$, as so $v_t = v_y = 0$, or $v_x = -\frac{1}{2} u_{xx}$. In the last case we have $v_{xys} = 0$, and therefore

$$0 = v_{xys} = -v_{xx} e^{-v_s} \left( c_1 v_{ss} - e^{-v_s} \right).$$

If there holds $c_1 v_{ss} - e^{-v_s} \neq 0$, then $v_{xx} = 0$, and

$$0 = v_{txs} = -\frac{1}{2} v_x^2 e^{-v_s} \left( c_1 v_{ss} - e^{-v_s} \right)$$

gives $v_x = 0$ again, while $c_1 v_{ss} - e^{-v_s} = 0$ entails

$$0 = v_{txs} = \frac{2}{c_1} (c_1^2 - c_0) v_{xx} e^{-v_s}$$

and so $v_{xx} = -\frac{1}{2} u_{xxx} = 0$, or

$$u = W_2(t, y) x^2 + W_1(t, y) x + W_0(t, y).$$

Then from the second equation of (8) we have $v_{xy} = -v_x u_{xx}$, or

$$u_{xxy} = -u_{xx}^2.$$

(17)

Substituting for (16) into (11) and (17) gives a family

$$u = (A_2 y + A_1) x + \frac{1}{6} (A_2' - A_2^2) y^3 + \frac{1}{2} (A_1' - A_1 A_2) y^2 + A_3 y + A_4,$$

(18)

of solutions to equation (11) that include arbitrary functions $A_1 = A_1(t), \ldots, A_4 = A_4(t)$. For a solution from (18) we have $u_{xxx},$ so $v_x = v_t = v_y = 0$ again.

Note that (18) is a subset of the family of solutions to (11) that are defined by restriction $u_{xxx} = 0$ and have the form

$$u = A_0(t) x^3 + W_2(t, y) x^2 + W_1(t, y) x + W_0(t, y),$$

(19)
where there holds

\[
\begin{align*}
W_{2,yy} &= -2 W_2 W_{2,y} + 3 (A_0 W_{1,y} + A_0'), \\
W_{1,yy} &= 2 (W_{2,t} - W_1 W_{2,y} + 3 A_0 W_{0,y}), \\
W_{0,yy} &= 2 W_2 W_{0,y} - W_1 W_{1,y} + W_{1,t}. 
\end{align*}
\]