IKEDA TYPE CONSTRUCTION OF CUSP FORMS

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Abstract. This is a survey of results on the construction of holomorphic cusp forms on tube domains originally initiated by Ikeda [9]. Besides a survey it includes conjectures and possible applications of our work [19].

1. Introduction

There are five simple tube domains (cf. [6]). They are of the form $D = \{ Z = X + iY | X \in \mathbb{R}^n, Y \in C \}$, where $C$ is a self-adjoint homogeneous cone in $\mathbb{R}^n$. Let $G$ be (the real points of) the simply connected, simple real algebraic group which acts transitively on $D$. We list the group $G$ and the cone $C$:

1. $Sp_{2n}$ (rank $n$); $n \times n$ positive definite matrices over $\mathbb{R}$;
2. $SU(n,n)$; $n \times n$ positive definite hermitian matrices over $\mathbb{C}$;
3. $SU(2n,H) = Spin^*(4n)$; $n \times n$ positive definite hermitian matrices over $H$ (quaternions);
4. $SO(2n)^0$; the cone in $\mathbb{R}^{n+1}$ of $(x_0, \ldots, x_n)$ with $x_0 > (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$;
5. $E_7,3$; $3 \times 3$ positive definite hermitian matrices over $\mathfrak{e}$ (Cayley numbers).

It is an important problem to explicitly construct holomorphic cusp forms on $D$ with respect to $G(\mathbb{Z})$ (we will call such a modular form on $D$ “a level one form”). In particular, we focus on the lifting from normalized Hecke cusp eigenforms on the complex upper half-plane $\mathbb{H}$ with respect to $SL_2(\mathbb{Z})$ to holomorphic cusp forms on $D$.

Ikeda [9] (see also [8]) gave a (functorial) construction of Siegel cusp forms of weight $n + k$, $n \equiv k \mod 2$ (so that $n + k$ is even) for $Sp_{4n}$ from normalized Hecke eigenforms in $S_{2k}(SL_2(\mathbb{Z}))$ which has been conjectured by Duke and Imamoglu (Independently Ibukiyama formulated a conjecture in terms of Koecher-Maass series). He made use of the uniform property of the Fourier coefficients of Siegel Eisenstein series for $Sp_{4n}$ and together with various deep facts established in [9] to prove Duke-Imamoglu conjecture. When $n = 1$, it is nothing but a Saito-Kurokawa lift. Since then, his construction was generalized to unitary

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groups $U(n,n)(K/Q)$ or $SU(n,n)$ for an imaginary quadratic field $K/Q$ ([10]), quaternion unitary groups $SU(2n,H)$ for a definite quaternion algebra $H$ over $Q$ ([25]), symplectic groups $Sp_{2n}$ over totally real fields ([11],[12] including some levels), and the exceptional group of type $E_{7,3}$ with $Q$-rank 3 [19].

In this note we explain main ideas of Ikeda and how they generalize to above cases. We do not discuss a further development by Ikeda [11] though it is important because his new ideas will work beyond “level one” case. We can give a uniform treatment except the case (4), which we will omit since it has been studied thoroughly by Oda [21] and Sugano [22].

Let $G$ be $Sp_{4n}$, $SU_{2n+1} := SU(2n+1,2n+1)(K/Q)$ (to ease the notation, we restrict ourselves to this case), $SU(2n,H)$, or $E_{7,3}$, and $P = MN$ the Siegel parabolic subgroup of $G$ with the Levi subgroup $M$ and the abelian unipotent radical $N$. For any ring $R$, let $Tr_G : N(R) \rightarrow R$ be the trace on $N$, which is defined as:

$$Tr_G(n(B)) := \begin{cases} Tr(B) & \text{if } G = Sp_{4n}, N = \left\{ n(B) = \begin{pmatrix} 1_{2n} & B \\ 0_{2n} & 1_{2n} \end{pmatrix} \mid ^tB = B \right\} \\
\frac{1}{2}Tr(B + \overline{B}) & \text{if } G = SU_{2n+1}, N = \left\{ n(B) := \begin{pmatrix} 1_{2n+1} & B \\ 0_{2n+1} & 1_{2n+1} \end{pmatrix} \mid ^t\overline{B} = B \right\} \\
\frac{1}{2}Tr(B + \tau(B)) & \text{if } G = SU(2n,H), N = \left\{ n(B) := \begin{pmatrix} 1_n & B \\ 0_n & 1_n \end{pmatrix} \mid ^t(\overline{B}) = B \right\} 
\end{cases}$$

where $^tx = x_0 - ix_1 - jx_2 - kx_3$ for $x = x_0 + ix_1 + jx_2 + kx_3 \in H$, and $\tau(x) = x + ^tx$.

For $E_{7,3}$, see [19].

Set $K = Q$ if $G = Sp_{4n}$ or $E_{7,3}$, and $K = \mathbb{H}$ if $G = SU(2n,H)$. Let $O$ be the ring of integers of $K$ if $G \neq SU(2n,H)$, and a maximal order of $H$ if $G = SU(2n,H)$. An element $T$ of $N(K)$ is semi-integral if $Tr_G(TX) \in Z$ for any $N(O)$. We denote by $L$ the set of all semi-integral elements in $N(K)$ and denote by $L^+$ the subset of $L$ consisting of positive definite elements. Here the positivity has the usual meaning as matrices for $G \neq E_{7,3}$, and see [19] for $E_{7,3}$. For instance, if $G = Sp_{4n}$, $L$ consists of matrices $(x_{ij})_{1 \leq i, j \leq 2n}$ so that $x_{ii} \in Z$ and $x_{ij} = x_{ji} \in \frac{1}{2}Z$ for $i \neq j$.

For the integers $k$ and $d$, we denote by $\mathfrak{d}_d$ the discriminant of $Q \left( \sqrt{(-1)^k d} \right) / Q$ and $\chi_d$ the Dirichlet character associated to $Q \left( \sqrt{(-1)^k d} \right) / Q$. Let $f_d$ be the positive rational number so that $d = \mathfrak{d}_d f_d^2$. Let $L(s, \chi_d)$ be the Dirichlet L-function of $\chi_d$. For $T \in L^+$, put $D_T = \det(2T)$ (resp. $\gamma(T) = (-D_K)^{1/2} \det(T)$ where $-D_K$ stands for the fundamental discriminant of $K/Q$) if $G = Sp_{4n}$ (resp. if $G = SU_{2n+1}$). For $G = SU(2n,H)$, put $D_T = (D_H)^{1/2} \text{Paf}(T)$ where $D_H$ is the product of rational primes $p$ so that $H \otimes Q_p$ is a skew field and $\text{Paf}$ is defined in
Section 1 of [25]. When \( G = E_{7,3} \), \( \det(T) \) is as in [19]. Set

\[
\ell(k) := \begin{cases} 
  k + n, & \text{if } G = Sp_{4n}, \\
  2k + 2n, & \text{if } G = SU_{2n+1}, \\
  2k + 2n - 2, & \text{if } G = SU(2n, H), \\
  2k + 8, & \text{if } G = E_{7,3}.
\end{cases}
\]

For each \( \gamma \in G(\mathbb{R}) \) and \( Z \in \mathcal{D} \), one can associate the automorphic factor \( j(\gamma, Z) \in \mathbb{C} \) so that \( j(\gamma, Z)^k \) is used to define modular forms on \( \mathcal{D} \) of weight \( k \) for any integer \( k \geq 0 \). For example, if \( \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2n}(\mathbb{R}) \), then \( j(\gamma, Z) = \det(CZ + D) \). Put \( \Gamma := G(\mathbb{Z}) \) and \( \Gamma_\infty = \Gamma \cap N(\mathbb{Q}) \). Let us consider the Siegel Eisenstein series of weight \( \ell(k) \):

\[
E_{\ell(k)}(Z) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} j(\gamma, Z)^{-\ell(k)}.
\]

Then we have the Fourier expansion

\[
\mathcal{E}_{\ell(k)}(Z) = \frac{1}{C(\ell(k))} E_{\ell(k)}(Z) = \sum_{T \in \mathcal{L}} A(T) \exp(2\pi \sqrt{-1} \cdot \text{Tr}_G(TZ)),
\]

for a constant \( C(\ell(k)) \), and for \( T \in L^+ \), \( A(T) \) is given as follows:

\[
A(T) = \begin{cases} 
  L(1 - k, \chi_{D_T})^{k^2 - \frac{1}{2}} \prod_{p \mid \delta_T} \tilde{F}_p(T; p^{k - \frac{1}{2}}) & \text{if } G = Sp_{4n} \\
  |\gamma(T)|^{k^2 - \frac{1}{2}} \prod_{p \mid \gamma(T)} \tilde{F}_p(T; p^{k - \frac{1}{2}}) & \text{if } G = SU_{2n+1} \\
  D_T^{k^2 - \frac{1}{2}} \prod_{p \mid D_T} \tilde{f}_{p,T}(X_p) & \text{if } G = SU(2n, H) \\
  \det(T)^{k^2 - \frac{1}{2}} \prod_{p \mid \det(T)} \tilde{f}^p_I(X_p) & \text{if } G = E_{7,3},
\end{cases}
\]

where \( \tilde{F}_p(T; X), \tilde{f}_{p,T}(X) \) and \( \tilde{f}^p_I(X) \) are Laurent polynomials over \( \mathbb{Q} \) with one variable \( X \) which are depending only on \( T, p \) and both are identically 1 for all but finitely many \( p \).

Introducing multi-variables \( \{ X_p \}_p \) indexed by rational primes \( p \), we may consider

\[
A(\{ X_p \}_p) := \begin{cases} 
  L(1 - k, \chi_{D_T})^{k^2 - \frac{1}{2}} \prod_{p \mid \delta_T} \tilde{F}_p(T; X_p) & \text{if } G = Sp_{4n} \\
  |\gamma(T)|^{k^2 - \frac{1}{2}} \prod_{p \mid \gamma(T)} \tilde{F}_p(T; X_p) & \text{if } G = SU_{2n+1} \\
  D_T^{k^2 - \frac{1}{2}} \prod_{p \mid D_T} \tilde{f}_{p,T}(X_p) & \text{if } G = SU(2n, H) \\
  \det(T)^{k^2 - \frac{1}{2}} \prod_{p \mid \det(T)} \tilde{f}^p_I(X_p) & \text{if } G = E_{7,3}.
\end{cases}
\]
Then $A(\{X_p\}_p)$ can be regarded as an element of $\otimes_p \mathbb{C}[X_p, X_p^{-1}]$. For each normalized Hecke eigenform $f = \sum_{n=1}^{\infty} a(n)q^n$, $q = \exp(2\pi \sqrt{-1}\tau)$, $\tau \in \mathbb{H}$ in $S_{2k}(SL_2(\mathbb{Z}))$ and each rational prime $p$, we define the Satake $p$-parameter $\alpha_p$ by $a(p) = p^{k-\frac{1}{2}}(\alpha_p + \alpha_p^{-1})$. For such $f$, consider the following formal series on $\mathcal{D}$:

$$F_f(Z) := \sum_{T \in L^+} A_{F_f}(T) \exp(2\pi \sqrt{-1}\text{Tr}_G(TZ)), \ Z \in \mathcal{D}, \ A_{F_f}(T) = A(\{\alpha_p\}_p).$$

Then

**Theorem 1.1.** Assume that $H$ is the Hurwitz quaternion when $G = SU(2n, H)$. Then $F_f$ is a non-zero Hecke eigen cusp form on $\mathcal{D}$ of weight $\ell(k)$ with respect to $G(\mathbb{Z})$.

Of course, we have to specify what kind of Hecke theory we use for each case. At any late, the issue is only on the normalization factor of a Hecke action and it does not matter as long as we deal with the adelic form attached to $F_f$ on $G(\mathbb{A}_\mathbb{Q})$ because since $G$ is semi-simple, it does not contain the central torus. By virtue of Theorem 1.1, $F_f$ gives rise to a cuspidal automorphic representation $\pi_F = \pi_\infty \otimes \otimes_p \pi_p$ of $G(\mathbb{A}_\mathbb{Q})$. Here $\pi_\infty$ is a holomorphic discrete series of $G(\mathbb{R})$ of the lowest weight $\ell(k)$, and for each prime $p$, $\pi_p$ is unramified at every finite place (but a few exception when $G = SU(2n, H)$), since $F_f$ is of “level one”. In fact, $\pi_p$ turns out to be a degenerate principal series $\pi_p \simeq I(s_p)$, where $s_p \in \mathbb{C}$ so that $p^{s_p} = \alpha_p$ and

$$I(s) = \begin{cases} \text{Ind}^{G(\mathbb{Q}_p)}_{P(\mathbb{Q}_p)} |\nu(g)|^s_p & \text{if } G = Sp_{4n} \\ \text{Ind}^{G(\mathbb{Q}_p)}_{P(\mathbb{Q}_p)} |\nu(g)|^s_p & \text{if } G = SU_{2n+1}, \\ \text{Ind}^{G(\mathbb{Q}_p)}_{P(\mathbb{Q}_p)} |\nu(g)|^{2s}_p & \text{if } G = SU(2n, H) \text{ and } p \nmid D_H, \\ \text{Ind}^{G(\mathbb{Q}_p)}_{P(\mathbb{Q}_p)} |\nu(g)|^{2s}_p & \text{if } G = E_{7,3}, \end{cases}$$

where $\nu : P(\mathbb{Q}_p) \rightarrow \mathbb{Q}_p^\times$ is defined as follows:

$$\nu(g) := \begin{cases} \det(A) & \text{if } G = Sp_{4n}, P = \left\{ g = \begin{pmatrix} A & B \\ 0_{2n} & t A^{-1} \end{pmatrix} \right\}^t B = B \\ \left|\det(A)\right|^2 & \text{if } G = SU_{2n+1}, P = \left\{ g = \begin{pmatrix} A & B \\ 0_{2n+1} & t A^{-1} \end{pmatrix} \right\}^t B = B \end{cases}$$

For $SU(2n, H)$ and $E_{7,3}$, see [25] and [19], resp. The relationship between $I(s)$ and the Eisenstein series is explained in [18]: Let $\Phi(g, s) = \Phi_\infty(g, s) \otimes \otimes_p \Phi_p(g, s)$ be a standard section in $I(s)$ such that $\Phi_\infty(k, s) = \nu(k)^{\ell(k)}$, and $\Phi_p(g, s) = \Phi_p^0(g, s)$ is the normalized spherical section for all $p$. Then one can define the adelic and classical Eisenstein series

$$E(g, s, \Phi) = \sum_{\gamma \in \mathbb{P}(\mathbb{Q}) \setminus G(\mathbb{Q})} \Phi(\gamma g, s), \ E_{\ell(k), s}(Z) = \det(Y)^{\frac{s}{2}} \sum_{\gamma \in \mathbb{P}(\mathbb{Q}) \setminus \Gamma} j(\gamma, Z)^{-\ell(k)} |j(\gamma, Z)|^{-s}.$$
Hence the degenerate principal series $I(k - \frac{1}{2})$ corresponds to $E_{\ell(k)}(Z)$ if $G \neq E_{7,3}$, and $I(2k - 1)$ corresponds to $E_{\ell(k)}(Z)$ if $G = E_{7,3}$.

In terms of representation theory, Theorem 1.1 can be reformulated as follows: Let $\pi_{\infty}$ be the holomorphic discrete series of $G(\mathbb{R})$ of the lowest weight $\ell(k)$, and let $\pi_p$ be the above degenerate principal series which is irreducible. Then we can form an irreducible admissible $\pi$-packet, which is a cuspidal automorphic representation of $G(\mathbb{A})$.

Arthur’s trace formula [1] may give a more general result as follows: By Adams-Johnson’s result on $A$-packets, $\pi_{\infty}$ belongs to a packet with the local character $(-1)^n$. Since $\pi$ is unramified at every finite place, by the multiplicity formula, $\pi$ is a cuspidal automorphic representation if and only if the global character $(-1)^n$ is equal to the root number of $L(s, f)$ which is $(-1)^k$. Hence we have the parity condition $k \equiv n \pmod{2}$. We have similar results for $SU_{2n+1}$ and $SU(2n, H)$. However, the advantage of Theorem 1.1 is that one can write down the modular form explicitly. Let $L(s, \pi_f) = \prod_p ((1 - \alpha_p p^{-s})(1 - \alpha_p^{-1} p^{-s}))^{-1}$ be the (normalized) automorphic $L$-function of the cuspidal representation $\pi_f$ attached to $f$. In the case of $SU_{2n+1}$, let $\chi(p) = (-\frac{D\mathfrak{a}}{p})$ be the quadratic character attached to $K/\mathbb{Q}$, and $L(s, f, \chi) = \prod_p ((1 - \alpha_p \chi(p)p^{-s})(1 - \alpha_p^{-1} \chi(p)p^{-s}))^{-1}$. For each local component $\pi_p$, one can associate the local $L$-factor $L(s, \pi_p, St)$ of the standard $L$-function of $\pi_F$. Set

$$L(s, \pi_F, St) = \prod_p L(s, \pi_p, St)$$

**Theorem 1.2.**

$$L(s, \pi_F, St) = \begin{cases} 
\zeta(s) \prod_{i=1}^{2n} L(s + n + \frac{1}{2} - i, f) & \text{if } G = Sp_{4n}, \\
\prod_{i=1}^{2n+1} L(s + n + 1 - i, f)L(s + n + 1 - i, f, \chi) & \text{if } G = SU_{2n+1} \\
\prod_{i=1}^{2n} L(s + n + \frac{1}{2} - i, f) & \text{if } G = SU(2n, H) \\
L(s, \text{Sym}^3 \pi_f)L(s, f)^2 \prod_{i=1}^{4} L(s \pm i, f)^2 \prod_{i=5}^{8} L(s \pm i, f) & \text{if } G = E_{7,3}, 
\end{cases}$$

where $L(s, \text{Sym}^3 \pi_f)$ is the symmetric cube $L$-function.
Notice that $\pi_p$ for $G = E_{7,3}$ is slightly different from other cases (Note $2s_p$ rather than $s_p$) and the third symmetric power $L$-function appears in the standard $L$-function. Note also that in the case $G = SU_{2n+1}$, $L(s, f)L(s, f, \chi) = L(s, \pi_K)$, the $L$-function of the base change $\pi_K$ of $\pi_f$ to $K$.

In Section 2, we review the tube domains. In Section 3, we review the Jacobi group, Jacobi forms, and a key property of the Fourier-Jacobi expansion of Siegel Eisenstein series, namely, the Fourier-Jacobi coefficients of Eisenstein series are a sum of products of theta functions and Eisenstein series. In Section 4, we will give a sketch of proof of the main theorem. Except for $G = Sp_{4n}$, the situations are similar, in that we do not need to consider half-integral modular forms. Finally in Section 5, we discuss conjectures and problems related to the results in [19].

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2. Description of tube domains

2.1. $Sp_{2n}$. The tube domain is given by

$$\mathbb{H}_n := \{Z \in M_n(\mathbb{C}) \mid ^tZ = Z, \text{Im}(Z) > 0\} \subset \mathbb{C}^{n(n+1)/2}$$

and $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2n}(\mathbb{R})$ acts on $\mathbb{H}_n$ as $\gamma(Z) = (AZ + B)(CZ + D)^{-1}$. Put $j(\gamma, Z) = \det(CZ + D)$.

2.2. $SU_{2n+1}$. The tube domain is given by

$$\mathcal{H}_{2n+1} := \{Z \in M_{2n+1}(\mathbb{C}) \mid \frac{1}{2\sqrt{-1}}(Z - ^t\bar{Z}) > 0\} \subset \mathbb{C}^{(2n+1)^2}$$

and $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU_{2n+1}(\mathbb{R})$ acts on $\mathcal{H}_{2n+1}$ as $\gamma(Z) = (AZ + B)(CZ + D)^{-1}$. Put $j(\gamma, Z) = \det(CZ + D)$.

2.3. $SU(2n, H)$. Let $H$ be a definite quaternion algebra with basis $1, i, j, k = ij$ over $\mathbb{Q}$. By Lemma 1.1 of [25], there exists a unique polynomial map (with $4n$ variables) $P : M_n(H) \longrightarrow \mathbb{Q}$ such that $\nu(X) = P(X)^2$ and $P(I_n) = 1$. Put $\text{Paf}(X) = P(X)$ for any $X \in M_n(H)$. The tube domain is given by

$$\mathcal{S}_n := \{Z \in M_n(H \otimes \mathbb{Q} \mathbb{C}) \mid ^*Z = Z, \text{Im}(Z) > 0\} \subset \mathbb{C}^{2n(n+1)}$$
and $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(2n, \mathbb{H})(\mathbb{R})$ acts on $\mathfrak{h}_n$ as $\gamma(Z) = (AZ + B)(CZ + D)^{-1}$. Put $j(\gamma, Z) = \nu(CZ + D)^{\frac{1}{2}}$.

2.4. $E_{7,3}$. This group is defined by using Cayley numbers and the structure is rather complicated than previous cases. We refer [2], [4], [15], and Section 2 of [19]. For any field $K$ whose characteristic is different from 2 and 3, the Cayley numbers $\mathcal{C}_K$ over $K$ is an eight-dimensional vector space over $K$ with basis $\{e_0 = 1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ satisfying certain rules of multiplication. Let $J_K$ be the exceptional Jordan algebra consisting of the element:

$$X = (x_{ij})_{1 \leq i,j \leq 3} = \begin{pmatrix} a & x & y \\ \bar{x} & b & z \\ \bar{y} & \bar{z} & c \end{pmatrix},$$

where $a, b, c \in K e_0 = K$ and $x, y, z \in \mathcal{C}_K$. We also define

$$\mathfrak{J}_2 = \left\{ \begin{pmatrix} a & x \\ \bar{x} & b \end{pmatrix} \; | \; a, b \in K, \; x \in \mathcal{C}_K \right\}.$$

Then the exceptional domain is

$$\mathfrak{D} := \{ Z = X + Y \sqrt{-1} \in \mathfrak{J}_\mathbb{C} \; | \; X, Y \in \mathfrak{J}_\mathbb{R}, \; Y > 0 \}$$

which is a complex analytic subspace of $\mathbb{C}^{27}$. We also define

$$\mathfrak{D}_2 := \{ X + Y \sqrt{-1} \in \mathfrak{J}_2(\mathbb{C}) \; | \; X, Y \in \mathfrak{J}_2(\mathbb{R}), \; Y > 0 \}$$

which is the tube domain of $Spin(2, 10)$, i.e., $Spin(2, 10)$ acts on $Z \in \mathfrak{D}_2$.

3. JACOBI GROUPS AND JACOBI FORMS

In this section we review Jacobi groups and Jacobi forms with a matrix index.

3.1. Jacobi groups. We are concerned with the Jacobi group $J$ realized in $G$, which is a semi-direct product $J \simeq V \rtimes H$ of a semisimple group $H$ and a Heisenberg group $V$ with a 2 step unipotency which has a form $V = X \cdot Y \cdot Z$, where each factor is an additive group (scheme), $\dim(X) = \dim(Y)$, and the center of $V$ is $Z$. We further require that the action of $H$ on $Z$ is trivial.

In our case, $H = SL_2$ if $G \neq SU_{2n+1}$, and $H = SU_1$ if $G = SU_{2n+1}$. If we write an element as $v = v(x, y, z)$, then by definition, an alternating form $\langle *, * \rangle$ is furnished on $X \oplus Y$ such that the multiplication of two elements in $V$ is given by

$$v(x, y, z) \cdot v(x', y', z') = v(x + x', y + y', z + z' + \frac{1}{2} \langle (x, y), (x', y') \rangle)$$
and further $SL_2$ or $U_1$ acts on $V$ as

$$\gamma \cdot v(x, y, z) = v(ax + cy, bx + dy, z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2 \text{ or } U_1.$$  

We shall give a table of the dimension $\dim(X)$ of $X$ as a vector scheme over $\mathbb{Z}$ which will be related to the difference of the weights between the original form and the lift.

| $G$       | $Sp_{4n}$ | $SU_{2n+1}$ | $SU(2n, H)$ | $E_7, 3$ |
|-----------|-----------|-------------|-------------|---------|
| $\dim(X)$| $2n - 1$  | $4n$        | $4(n - 1)$  | 16      |

Table 1.

The difference between $\ell(k)$ and $2k$ is given by $\frac{1}{2} \dim(X)$ except for $Sp_{4n}$. For $Sp_{4n}$, we first obtain a cusp form of the half-integral weight $k + \frac{1}{2}$ via Shimura correspondence $S_{2k}(SL_2(\mathbb{Z})) \simeq S_{k + \frac{1}{2}}(\Gamma_0(4))^{+}$ from the cusp form $f \in S_{2k}(SL_2(\mathbb{Z}))$. Then the difference should be understood as $\ell(k) - (k + \frac{1}{2}) = n - \frac{1}{2}$, which is nothing but $\frac{1}{2} \dim(X)$ for $Sp_{4n}$.

For $Sp_{4n}$,

$$V = \left\{ v(x, y, z) = \begin{pmatrix} 1_{2n-1} & x & z \\ 0 & 1 & ty \\ 1_{2n-1} & 0 & -t^x \end{pmatrix} \in Sp_{4n} \right\} \quad \left\{ t_z - y(t^x) = z - x(t^y) \right\} = X \cdot Y \cdot Z,$$

where $X = \{v(x, 0, 0) \in V\}$, $Y = \{v(x, 0, 0) \in V\}$, and $Z = \{v(0, 0, z) \in V\}$, and

$$SL_2 \simeq H := \left\{ \begin{pmatrix} 1_{2n-1} & 0 & 0_{2n-1} \\ 0 & a & 0 \\ 0_{2n-1} & 0 & 1_{2n-1} \end{pmatrix} \in Sp_{4n} \right\}.$$  

For $SU_{2n+1}$,

$$V = \left\{ v(x, y, z) = \begin{pmatrix} 1_{2n} & x & z \\ 0 & 1 & ty \\ 1_{2n} & 0 & -t^x \end{pmatrix} \in SU_{2n+1} \right\} \quad \left\{ t_z - y(t^x) = z - x(t^y) \right\} = X \cdot Y \cdot Z,$$
where \( X = \{ v(x, 0, 0) \in V \}, \ Y = \{ v(x, 0, 0) \in V \}, \) and \( Z = \{ v(0, 0, z) \in V \}, \) and

\[
U_1 \simeq H := \left\{ \begin{pmatrix} 1_{2n} & 0 & 0 \\ 0 & a & 0 \\ 0_{2n} & 0 & 1_{2n} \\ 0 & c & 0 \\ \end{pmatrix} \in SU_{2n+1} \right\}.
\]

We omit details for \( SU(2n, H) \) or \( E_{7,3}. \) Instead we refer Section 5 of [25], and Section 3 and 4 of [19].

Recall \( L \) in the introduction. This is the parameter space of Fourier expansion of a modular form on \( \mathcal{O}. \) Let \( Z' \) be a subgroup of the unipotent radical \( N \) of the Siegel parabolic subgroup consisting of matrices whose last low and last column are zero. Then \( Z' \) is naturally identified with \( Z. \) We denote by \( L' \) (resp. \( L'_+ \)) the subset of \( Z'(\mathbb{Q}) \) consisting of semi-positive (resp. positive), semi-integral matrices. For any \( T \in L^+, \) there exists \( S \in L'_+ \) such that \( T = (S \alpha \beta \ x) \) with \( x \in \mathbb{Z}_+ \) and

\[
\beta = \begin{cases} 1_{\alpha}, & \text{if } G = Sp_{2n}, \\ 1_{\tilde{\alpha}}, & \text{if } G = SU_{2n+1} \text{ or } SU(2n, H) \\ 1_{t\tilde{\alpha}}, & \text{if } G = E_{7,3}. \\ \end{cases}
\]

Henceforth we fix \( S \in L'_+ . \) We define the map \( \lambda_S \) on \( Z \) by \( z \mapsto \frac{1}{2} \text{Tr}_G(Sz) \) if \( G \neq E_{7,3} \) and \( z \mapsto \frac{1}{4}(S, z) \) for \( E_{7,3}. \) Then for any domain ring \( R \) with characteristic zero, the map \( V(R) \rightarrow X \oplus Y \oplus R, \ v(x, y, z) \mapsto (x, y, \lambda_S(z)) \) gives rise to the Heisenberg structure on \( X \oplus Y \oplus R. \) Hence for any two elements \( (x, y, a), (x', y', b) \in X \oplus Y \oplus R, \) the multiplication is given by

\[
(x, y, a) * (x', y', b) = (x + x', y + y', a + b + \frac{1}{2}((x, y), (x', y'))) \in \mathbb{S}
\]

where \( \langle (x, y), (x', y') \rangle = \sigma_S(x, y') - \sigma_S(x', y). \) Here \( \sigma_S(*, *) \) on \( X \oplus Y \) is given by

\[
\sigma_S(x, y) = \begin{cases} t_{\tilde{x}y}, & \text{if } G = Sp_{4n}, \\ t_{\tilde{x}y}, & \text{if } G = SU_{2n+1} \text{ or } SU(2n, H) \\ (S, x^t\overline{y} + y^t\overline{x}) & \text{if } G = E_{7,3}. \\ \end{cases}
\]

Put \( \mathcal{X} := \mathcal{X}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \) and \( \mathcal{D}_1 := \mathbb{H} \times \mathcal{X}. \) The group \( J(\mathbb{R}) \) acts on \( \mathcal{D}_1 \) by

\[
\beta(\tau, u) := (\gamma \tau, \frac{u}{c\tau + d} + x(\gamma \tau) + y), \ \beta = v(x, y, z)h, \ v(x, y, z) \in V(\mathbb{R}), \ h = h(\gamma) \in H(\mathbb{R})
\]

where \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}). \) Here \( \gamma \tau = \frac{a\tau + b}{c\tau + d} \) and put \( j(\gamma, \tau) := c\tau + d \) for simplicity.

For each positive half integer \( k, \) the automorphy factor on \( J(\mathbb{R}) \times \mathcal{D}_1 \) is defined by

\[
j_{k, S}(\beta, (\tau, u)) := j(\gamma, \tau)^k e^{-2\lambda_S(z)} + \frac{c}{j(\gamma, \tau)} \sigma_S(u, u) - \frac{2\sigma_S(x, u)}{j(\gamma, u)} - \sigma_S(x, x)(\gamma \tau) - \sigma_S(x, y)) \text{,}
\]
where \( e(*) = \exp(2\pi \sqrt{-1} \ast) \). When \( k \) is not an integer, \( j(\gamma, z)^k = (j(\gamma, z) \frac{\ast}{2})^{2k} \) is defined by the automorphy factor \( j(\gamma, z) \frac{\ast}{2} \) of the metaplectic covering \( \widetilde{SL}_2(\mathbb{R}) \) of \( SL_2(\mathbb{R}) \).

For each function \( f : \mathcal{D}_1 \rightarrow \mathbb{C} \) and \( \beta \in V(\mathbb{R}) \), we define the “slash” operator \( f|_{k,S}[\beta] : \mathcal{D}_1 \rightarrow \mathbb{C} \) by
\[
f|_{k,S}[\beta](\tau, u) := j_{k,S}(\beta, (\tau, u))^{-1} f(\beta, u).
\]

### 3.2. Jacobi forms with a matrix index.

We define and study Jacobi forms of matrix index on \( \mathcal{D}_1 = \mathbb{H} \times \mathbb{X} \) in the classical setting. Set
\[
\Gamma_J := J(\mathbb{Q}) \cap G(\mathbb{Z}).
\]

**Definition 3.1.** Let \( k \) be a positive even integer if \( G \neq Sp_{4n} \), a positive half-integral integer if \( G = Sp_{4n} \), and \( S \) be an element of \( L' \). We say a holomorphic function \( \phi : \mathcal{D}_1 \rightarrow \mathbb{C} \) is a Jacobi form (resp. Jacobi cusp form) of weight \( k \) and index \( S \) if \( \phi \) satisfies the following conditions:

1. \( \phi|_{k,S}[\beta] = \phi \) for any \( \beta \in \Gamma_J \)
2. \( \phi \) has a Fourier expansion of the form
\[
\phi(\tau, u) = \sum_{\xi \in X(\mathbb{Q}), N \in \mathbb{Z}} c(N, \xi) e(N \tau + \sigma_S(\xi, u)),
\]
where \( c(N, \xi) = 0 \) unless \( S_{\ast}(\xi, N) : \left( \begin{array}{cc} S & S\xi \\ \ast\xi & N \end{array} \right) \) belongs to \( L' \) (resp. \( L'_+ \)) where \( \ast \xi \) stands for \( t\xi \) if \( G = Sp_{4n} \), \( t\xi \) if \( G = SU_{2n+1} \) or \( SU(2n, H) \), and \( t\xi \) if \( G = E_{7,3} \).

We denote by \( J_{k,S}(\Gamma_J) \) (resp. \( J_{k,S}^{\text{cusp}}(\Gamma_J) \)) the space of Jacobi forms (resp. Jacobi cusp forms) of weight \( k \) and index \( S \).

Let us extend the quadratic form \( \sigma_S \) linearly to that on \( \mathbb{X} \). We denote by \( S(X(\mathbb{A}_f)) \) the space of Schwartz functions on \( X(\mathbb{A}_f) \). For each \( \varphi \in S(X(\mathbb{A}_f)) \), the classical theta function on \( \mathcal{D}_1 := \mathbb{H} \times \mathbb{X} \) is given by
\[
\theta_S^\mathbb{X}(\tau, u) := \sum_{\xi \in X(\mathbb{Q})} \varphi(\xi) e(\sigma_S(\xi, \xi) \tau + 2\sigma_S(\xi, u)).
\]

Define the dual of the lattice \( \Lambda := X(\mathbb{Z}) \) with respect to the quadratic form \( \sigma_S \) by
\[
\tilde{\Lambda}(S) = \{ x \in X(\mathbb{Q}) \mid \sigma_S(x, y) \in \mathbb{Z} \text{ for all } y \in \Lambda \}.
\]
If \( S \in L'_+ \), then the quotient \( \tilde{\Lambda}(S)/\Lambda \) is a finite group. Fix a complete representative \( \Xi(S) \) of \( \tilde{\Lambda}(S)/\Lambda \) and denote by \( \varphi_\xi \) the characteristic function \( \xi + \prod_{p < \infty} X(\mathbb{Z}_p) \in S(X(\mathbb{A}_f)) \). Any
Jacobi form turns to be the linear combination of products of elliptic modular forms and theta functions by the following lemma.

**Lemma 3.2.** Assume \( S \in L'_+ \). Let \( \Xi(S) \) be a complete representative of \( \tilde{\Lambda}(S)/\Lambda \). Then any Jacobi form \( \phi \in J_{k, S}(\Gamma_J) \) can be written as

\[
\phi(\tau, u) = \sum_{\xi \in \Xi(S)} \phi_{S, \xi}(\tau) \theta^S_{\psi}(\tau, u), \quad \phi_{S, \xi}(\tau) = \sum_{N \in \mathbb{Z}} c(N, \xi) e((N - \sigma_S(\xi, \xi))\tau).
\]

Furthermore, for each \( \xi \in \Xi(S) \), \( \phi_{S, \xi}(\tau) \) is an elliptic modular form of weight \( k - \frac{1}{2} \text{dim}(X) \).

**Proof.** See example (iv) at Section 2 of [17] and also the argument at p.656 of [9]. \( \square \)

Let \( k \) be a positive integer and \( F \) be a modular form of weight \( k \) on \( \mathfrak{D} \). We rewrite the variable \( Z \) on \( \mathfrak{D} \) as \( \begin{pmatrix} W & u \\ v & \tau \end{pmatrix} \) where \( \tau \in \mathbb{H}, u \in X(\mathbb{R}) \otimes \mathbb{R} \mathbb{C} \), and \( W \in \mathbb{H}_{2n-1}, \mathcal{H}_{2n}, \mathcal{H}_{n-1}, \text{or } \mathfrak{D}_2 \). Note that \( v \) is determined by \( u \). Then we have the Fourier-Jacobi expansion

\[
F \left( \begin{pmatrix} W & u \\ v & \tau \end{pmatrix} \right) = \sum_{S \in L'} F_S(\tau, u) e((S, W)).
\]

**Lemma 3.3.** Keep the notation as above. Assume \( S \in L'_+ \). Then \( F_S(\tau, u) \in J_{k, S}(\Gamma_J) \).

**Remark 3.4.** Consider any holomorphic function \( F(Z) \) with \( Z = \begin{pmatrix} W & u \\ v & \tau \end{pmatrix} \) on \( \mathfrak{D} \) which is invariant under \( P(\mathbb{Z}) \). Then one has the Fourier and the Fourier-Jacobi expansion

\[
F(Z) = \sum_{T \in L} A_F(T) e((T, Z)) = \sum_{S \in L'} F_S(\tau, u) e((S, W)),
\]

as in (3.3). By Lemma 3.2,

\[
F_S(\tau, u) = \sum_{\xi \in \Xi(S)} F_{S, \xi}(\tau) \theta^S_{\psi}(\tau, u), \quad F_{S, \xi}(\tau) = \sum_{N \in \mathbb{Z}} A_S(S_N, N) e((N - \sigma_S(\xi, \xi))\tau),
\]

where \( S_N = \begin{pmatrix} S & S \\ * & S \end{pmatrix} \). The function \( F_{S, \xi} \) will be called \( (S, \xi) \)-component of \( F \).

**Definition 3.5.** For a sufficiently large \( k_0 \), a compatible family of Eisenstein series is a family of elliptic modular forms, for even integer \( k' \geq k_0 \),

\[
g_{k'}(\tau) = b_{k'}(0) + \sum_{N \in \mathbb{Q}_{>0}} N^{k'-\frac{1}{2}} b_{k'}(N) q^N, \quad q = e(\tau),
\]

satisfying the following three conditions:
(1) \( g_{k'} \in \mathcal{V}(E_{k'}) \) for all \( k' \geq k_0 \)

(2) for each \( N \in \mathbb{Q}_+ \), there exists \( \Phi_N \in \mathcal{R} \) such that \( b_{k'}(N) = \Phi_N(\{p^{k'-1}\}_p) \).

(3) there exists a congruence subgroup \( \Gamma \subset SL_2(\mathbb{Z}) \) such that \( g_{k'} \in M_{k'}(\Gamma) \) for all \( k' \geq k_0 \).

Here \( M_{k'}(\Gamma) \) stands for the space of elliptic modular forms of weight \( k \) with respect to \( \Gamma \).

The following theorem plays a key role in the proof of Theorem 1.1:

**Theorem 3.6.** Keep the notations above. Let \( E_{\ell(k)} \) be the Siegel Eisenstein series in Section 1. Assume \( S \in L'_+ \). Then any \( (S, \xi) \)-component of \( E_{\ell(k)} \) is an Eisenstein series of weight \( k - \frac{1}{2}\dim(X) \).

This theorem was first proved by Böcherer [3] for \( G = Sp_{4n} \) in the classical language. However the proof there involves many complicated terms and seems difficult to read off what we need. More sophisticated proof was given by Ikeda [7]. He made a good use of Weil representation and worked over the adelic language. In [25], [19], the authors followed his method. However in case \( E_{7,3} \), the group structure is much more complicated than others. So the proof is not a routine at all.

The following Lemma 10.2 of [10] is a crucial ingredient.

**Theorem 3.7.** Let \( f(\tau) = \sum_{n=1}^{\infty} c(n)q^n \) be a Hecke eigenform of weight \( k \) with respect to \( SL_2(\mathbb{Z}) \) with \( c(p) = p^{k-1}(\alpha_p + \alpha_p^{-1}) \). Assume that there is a finite dimensional representation \( (u, \mathcal{C}^d) \) of \( SL_2(\mathbb{Z}) \) and

\[
\bar{\Phi}_N := t(\Phi_{1,N}, \ldots, \Phi_{d,N}) \in \mathcal{R}^d, \quad N \in \mathbb{Q}_{>0}
\]

satisfying the following two conditions:

(1) there exists a vector valued modular form \( \bar{g}_{k'} = t(g_{1,k'}, \ldots, g_{d,k'}) \) which has

\[
\bar{g}_{k'}(\tau) = \bar{b}_{k'}(0) + \sum_{N \in \mathbb{Q}_{>0}} N^{k'-1} b_{k'}(N)q^n, \quad (\bar{b}_{k'}(N) = t(b_{1,k'}(N), \ldots, b_{d,k'}(N)), \quad N \in \mathbb{Q}_{>0})
\]

of weight \( k' \) with type \( u \) for each sufficiently large even integers \( k' \), hence this means that

\[
\bar{g}_{k'}(\tau)|_{k'\gamma} := t(g_{1,k'}|_{k'\gamma}, \ldots, g_{d,k'}|_{k'\gamma}) = u(\gamma)\bar{g}_k(\tau) \text{ for any } \gamma \in SL_2(\mathbb{Z}),
\]

(2) each component \( g_{i,k'} \), \( 1 \leq i \leq d \) of \( \bar{g}_{k'}(\tau) \) is a compatible family of Eisenstein series such that

\[
b_{i,k'}(N) = \Phi_{i,N}(\{p^{k'-1}\}_p).
\]
Then $\tilde{h}(\tau) := \sum_{N \in \mathbb{Q}_{>0}} N^{k+1} \Phi_N(\{\alpha_p\}_p) q^N$ is a vector valued modular form of weight $k$ with type $u$, hence it satisfies

$$\tilde{h}(\tau)|_{k}[\gamma] = u(\gamma)\tilde{h}$$

for any $\gamma \in SL_2(\mathbb{Z})$.

4. Proof of Theorem 1.1 and 1.2

Recall that for each normalized Hecke eigenform $f = \sum_{n=1}^{\infty} a(n)q^n \in S_{2k}(SL_2(\mathbb{Z}))$, we have considered the following formal series on $D$:

$$F_f(Z) := \sum_{T \in L^+} A_f(T) \exp(2\pi \sqrt{-1} \text{Tr}_G(TZ)), \quad Z \in D, \quad A_f(T) = A(\{\alpha_p\}_p).$$

The first task is to check the absolute convergence for $F_f$: This is done by using explicit formula of Fourier coefficients of Siegel Eisenstein series and Ramanujan bound $|a(p)| \leq 2p^{k-\frac{1}{2}}$.

Next, we use the fact that $\Gamma = G(\mathbb{Z})$ is generated by $P(\mathbb{Z})$ and $H(\mathbb{Z})$, where $H$ is in (3.1) or (3.2). We can easily check, by property of Fourier coefficients of Siegel Eisenstein series, that $F_f$ is invariant under the action of $P(\mathbb{Z})$. Therefore to prove the automorphy of $F_f$, we have to check only the invariance of $F_f$ under the action of $H(\mathbb{Z})$. For this, we need to use the Fourier-Jacobi expansion.

To unify notation we write the Fourier coefficient of $F_f$ as $A_f(T) = C_1(T)C_2(T)^{k-\frac{1}{2}} \prod_p \widetilde{F}_p(T; \alpha_p)$ for $T \in L^+$ where $C_1(T) = L(1-k, \chi_T)$ if $G = Sp_{4n}$, $C_1(T) = 1$ otherwise, and other terms should be clear from the definition as in the introduction. Since $F(Z) := F_f(Z)$ is invariant under $P(\mathbb{Z})$, by Remark 3.4 one has the Fourier-Jacobi expansion:

$$F \left( \begin{array}{ccc} W & u \\ v & \tau \end{array} \right) = \sum_{S \in L_+^*} F_S(\tau, u) e(\text{Tr}_G(SW)), \quad F_S(\tau, u) = \sum_{\xi \in \Xi(S)} F_{S,\xi}(\tau) \varphi^S_{\xi}(\tau, u),$$

and

$$F_{S,\xi}(\tau) = \sum_{N \in \mathbb{Z}} A_f(S_{\xi, N}) e((N - \sigma_S(\xi, \xi))\tau), \quad S_{\xi, N} := \left( \begin{array}{cc} S & S \xi \\ \eta S & N \end{array} \right)$$

$$= \sum_{N \in \mathbb{Z}} C_1(S_{\xi, N})C_2(S_{\xi, N})^{k-\frac{1}{2}} \prod_p \widetilde{F}_p(S_{\xi, N}; \alpha_p) e((N - \sigma_S(\xi, \xi))\tau)$$

$$= D(S)^{k-\frac{1}{2}} \sum_{N \in \mathbb{Z}} C_1(S_{\xi, N})(N - \sigma_S(\xi, \xi))^{k-\frac{1}{2}} \prod_p \widetilde{F}_p(S_{\xi, N}; \alpha_p) e((N - \sigma_S(\xi, \xi))\tau)$$

where there exists the constant $D(S)$ depending only on $S$ such that $C_2(S_{\xi, N}) = D(S)(N - \sigma_S(\xi, \xi))$. The invariance under $H(\mathbb{Z})$ is equivalent to claiming that $F_S(\tau, u) \in J_{k, S}(\Gamma_f)$ for any $S \in L_+^*$. 
By (2.1), p.124 of [23], for each \( \gamma \in SL_2(\mathbb{Z}) \), there exists a unitary matrix \( u_{S}(\gamma) = (u_{S}(\gamma)\xi,\eta)_{\xi,\eta \in \Xi(S)} \) such that

\[
\theta_{\varphi_{\xi}}^{S} |_{k,S} [\gamma](\tau,u) = \sum_{\eta \in \Xi(S)} u_{S}(\gamma)\theta_{\varphi_{\eta}}^{S}(\tau,u).
\]

Further there exists a positive integer \( \Delta_{S} \) depending on \( S \) such that \( u_{S} \) is trivial on \( \Gamma(\Delta_{S}) \subset SL_2(\mathbb{Z}) \). Since \( \{ \theta_{\varphi_{\xi}}^{S} | \xi \in \Xi(S) \} \) are linearly independent over \( \mathbb{C} \), it suffices to prove that \( \{ F_{S,\xi} \}_{\xi \in \Xi(S)} \) is a vector valued modular form with type \( u_{S} \).

For a sufficiently large positive integer \( k' \), we now turn to consider \((S,\xi)\)-component \((E_{\ell(k')})_{S,\xi}\) of the classical Eisenstein series

\[
E_{\ell(k')}(Z) = \sum_{T \in L} A(T) \exp(2\pi \sqrt{-1} \cdot \text{Tr}_{G}(TZ)), \quad A(T) = C_{1}(T)C_{2}(T)^{k'-\frac{1}{2}} \prod_{p} \tilde{F}_{p}(T;p^{k'-\frac{1}{2}}),
\]

Then one has

\[
D(S)^{-k'+\frac{1}{2}}(E_{\ell(k')})_{S,\xi}(\tau) = \sum_{N \in \mathbb{Z}} C_{1}(S_{\xi,N})(N - \sigma_{S}(\xi,\xi))^{-k'+\frac{1}{2}} \prod_{p | \det(S_{\xi,N})} \tilde{F}_{p}(S_{\xi,N};p^{k'-\frac{1}{2}})\mathrm{e}((N - \sigma_{S}(\xi,\xi))\tau)
\]

Then by Theorem 3.6 \( \{ D(S)^{-k'+\frac{1}{2}}(E_{\ell(k')})_{S,\xi} \}_{k' \geq 0} \) makes up a compatible family of Eisenstein series in the sense of Ikeda (see Section 10 of [9] for \( G = Sp_{4n} \) and Section 7 of [10] for other cases). Applying Lemma 3.7 one can conclude that

\[
F_{S,\xi} = D(S)^{k'-\frac{1}{2}} \sum_{n \in \mathbb{Z}, n \geq 0} C_{1}(S_{\xi,N})n^{k'-\frac{1}{2}} \prod_{p | \det(S_{\xi,N})} \tilde{F}_{p}(S_{\xi,N};p^{k'-\frac{1}{2}})q^{n},
\]

is a vector valued modular form with type \( u_{S} \). The non-vanishing is easy to check except for \( Sp_{4n} \). In this case, a bit of careful study was needed (see p.651 of [9]). At any late one can prove the non-vanishing of \( F_{f} \).

Since we know Satake parameters of \( \pi_{F} \), it is easy to compute \( L(s,\pi_{F},St) \). For \( G = E_{7,3} \), we can use the Langlands-Shahidi method for the case \( GE_{7} \subset E_{8} \) (cf. [16], section 2.7.8).

5. SOME CONJECTURES AND PROBLEMS

In this section we are concerning with some conjectures and problems related to the results in [19].
5.1. **Conjectural Arthur parameter.** It is worth considering the compatibility with Arthur conjecture in the case $E_{7,3}$: We write the degree 56 standard $L$-function of $F := F_f$ as

$$L(s, \pi_F, St) = L(s, \text{Sym}^3 \pi_f) \prod_{i=-4}^{4} L(s+i, \pi_f) \prod_{i=-8}^{8} L(s+i, \pi_f).$$

This suggests the following parametrization of $\pi_F$:

Let $\mathcal{L}$ be the (hypothetical) Langlands group over $\mathbb{Q}$, and let $\rho_f : \mathcal{L} \longrightarrow SL_2(\mathbb{C})$ be the 2-dimensional irreducible representation of $\mathcal{L}$ corresponding to $\pi_f$. Let $\text{Sym}^n$ be the irreducible $(n+1)$-dimensional representation of $SL_2(\mathbb{C})$. Note that if $n = 2m - 1$, $\text{Im} (\text{Sym}^n) \subset Sp_{2m}(\mathbb{C})$, and if $n = 2m$, $\text{Im} (\text{Sym}^n) \subset SO_{2m+1}(\mathbb{C})$. We have the tensor product maps $SL_2(\mathbb{C}) \times Sp_{2m}(\mathbb{C}) \longrightarrow Sp_{4m}(\mathbb{C})$ and $SL_2(\mathbb{C}) \times SO_{2m+1}(\mathbb{C}) \longrightarrow Sp_{4m+2}(\mathbb{C})$. Hence

$$\rho_f \otimes \text{Sym}^6 : \mathcal{L} \times SL_2(\mathbb{C}) \longrightarrow Sp_{34}(\mathbb{C}),$$

and $\rho_f \otimes \text{Sym}^8 : \mathcal{L} \times SL_2(\mathbb{C}) \longrightarrow Sp_{18}(\mathbb{C}).$

Let $\text{Sym}^3 \rho_f : \mathcal{L} \times SL_2(\mathbb{C}) \longrightarrow Sp_4(\mathbb{C})$ be the parameter of $\text{Sym}^3 \pi_f$, where it is trivial on $SL_2(\mathbb{C})$. Consider the parameter

$$\rho = \text{Sym}^3 \rho_f \oplus (\rho_f \otimes \text{Sym}^6) \oplus (\rho_f \otimes \text{Sym}^8) : \mathcal{L} \times SL_2(\mathbb{C}) \longrightarrow Sp_4(\mathbb{C}) \times Sp_{34}(\mathbb{C}) \times Sp_{18}(\mathbb{C}) \subset Sp_{56}(\mathbb{C}).$$

Note that $E_7(\mathbb{C}) \subset Sp_{56}(\mathbb{C})$. We expect that $\rho$ will factor through $E_7(\mathbb{C})$, and give rise to a parameter $\rho : \mathcal{L} \times SL_2(\mathbb{C}) \longrightarrow E_7(\mathbb{C})$, which parametrizes $\pi_F$.

5.2. **Ikeda lift as CAP form.** If $G = Sp_{4n}$, the Ikeda lift $F_f$ is a CAP form. Namely, $\pi_F$ is nearly equivalent to the quotient of the induced representation

$$\text{Ind}_{P_{2,\ldots,2}}^{Sp_{4n}} \pi_f |\text{det}|^{n-\frac{1}{2}} \otimes \pi_f |\text{det}|^{n-\frac{3}{2}} \otimes \cdots \otimes \pi_f |\text{det}|^{\frac{1}{2}},$$

where $P_{2,\ldots,2}$ is the standard parabolic subgroup of $Sp_{4n}$ with the Levi subgroup $GL_2 \times \cdots \times GL_2$ ($n$ factors) (see also p.114 of [S]).

If $G = E_{7,3}$, $\pi_F$ cannot be a CAP form in a usual sense since there are not many $\mathbb{Q}$-parabolic subgroups of $E_{7,3}$. We expect that $\pi_F$ will be a CAP form in a more general sense: Namely, there exists a parabolic subgroup $Q = M'N'$ of the split $E_7$, and a cuspidal representation $\tau = \otimes_p' \tau_p$ of $M'$, and a parameter $\Lambda_0$ such that for all finite prime $p$, $\pi_p$ is a quotient of $\text{Ind}_{Q(\mathbb{Q}_p)}^{E_7(\mathbb{Q}_p)} \tau_p \otimes \exp(\Lambda_0, H_Q( ))$.

5.3. **Miyawaki type lift to $GSpin(2,10)$.** This work is in progress [20]. For $Z \in \mathcal{O}_2$, let

$$\begin{pmatrix} Z & 0 \\ 0 & \tau \end{pmatrix} \in \mathcal{O}.$$ For $f \in S_2k(SL_2(\mathbb{Z}))$, let $F$ be the Ikeda lift of $f$, which is a cusp form of weight $2k+8$ on $\mathcal{O}$. For $h \in S_{2k+8}(SL_2(\mathbb{Z}))$, consider the integral
When $F_{f,h}$ is not zero, it is a cusp form of weight $2k + 8$ on $\mathcal{D}_2$. It is expected that $F_{f,h}$ is a Hecke eigen form, and it would give rise to a cuspidal representation $\pi_{F_{f,h}}$ on $\text{GSpin}(2,10)$: Let $\pi_{F_{f,h}} = \pi_\infty \otimes \pi_p$. Let $\{\alpha_p, \alpha_p^{-1}\}$ and $\{\beta_p, \beta_p^{-1}\}$ be the Satake parameter of $f$ and $h$ at the prime $p$, resp. Then for each prime $p$, it is expected that the Satake parameter of $\pi_p$ is
\[
\{(\beta_p \alpha_p)^{\pm 1}, (\beta_p \alpha_p^{-1})^{\pm 1}, 1, 1, p^{\pm 1}, p^{\pm 2}, p^{\pm 3}\}.
\]

Then the standard $L$-function of $\pi_{F_{f,h}}$ is
\[
L(s, \pi_{F_{f,h}}, St) = L(s, h \times f)\zeta(s)\zeta(s + 1)\zeta(s + 2)\zeta(s + 3),
\]
where the first factor is the Rankin-Selberg $L$-function. This can be explained by Arthur parameter as follows: Let $\phi_f, \phi_h: \mathcal{L} \rightarrow SL_2(\mathbb{C})$ be the hypothetical Langlands parameter. Then due to the tensor product map $SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \rightarrow SO_4(\mathbb{C})$, we have $\phi_f \otimes \phi_h: \mathcal{L} \rightarrow SO_4(\mathbb{C})$. The distinguished unipotent orbit $(7,1)$ of $SO_8(\mathbb{C})$ gives rise to the map $SL_2(\mathbb{C}) \rightarrow SO_8(\mathbb{C})$. It defines the map $\phi_u: \mathcal{L} \times SL_2(\mathbb{C}) \rightarrow SO(8,\mathbb{C})$. Then consider
\[
\phi = (\phi_h \otimes \phi_f) \oplus \phi_u: \mathcal{L} \times SL_2(\mathbb{C}) \rightarrow SO_4(\mathbb{C}) \times SO_8(\mathbb{C}) \subset GSO_{12}(\mathbb{C}).
\]

We expect that $\phi$ parametrizes $\pi_{F_{f,h}}$.

5.4. Petersson formula and its possible application. In case $E_{7,3}$, it may be interesting to give an explicit formula of the Petersson inner product formula for $F_f$. (See [5] for its importance.) Since $L(s, \pi_F, St)$ involves the third symmetric power $L$-function $L(s, \text{Sym}^3 \pi_f)$, we expect to somehow figure out an “algebraic part” of $L(s, \text{Sym}^3 \pi_f)$.

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