A new result on the existence of periodic solutions for Liénard equations with a singularity of repulsive type

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Abstract
In this paper, the problem of the existence of a periodic solution is studied for the second order differential equation with a singularity of repulsive type

\[ x''(t) + f(x(t))x'(t) - g(x(t)) + \varphi(t)x(t) = h(t), \]

where \( g(x) \) is singular at \( x = 0 \), \( \varphi \) and \( h \) are \( T \)-periodic functions. By using the continuation theorem of Manásevich and Mawhin, a new result on the existence of positive periodic solution is obtained. It is interesting that the sign of the function \( \varphi(t) \) is allowed to change for \( t \in [0, T] \).

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1 Introduction
The aim of this paper is to search for positive \( T \)-periodic solutions for a second order differential equation with a singularity in the following form:

\[ x''(t) + f(x(t))x'(t) - g(x(t)) + \varphi(t)x(t) = h(t), \quad (1.1) \]

where \( f : [0, \infty) \to \mathbb{R} \) is an arbitrary continuous function, \( g \in C((0, +\infty), (0, +\infty)) \), and \( g(x) \) is singular of repulsive type at \( x = 0 \), i.e., \( g(x) \to +\infty \) as \( x \to 0^+ \), \( \varphi, h : \mathbb{R} \to \mathbb{R} \) are \( T \)-periodic functions with \( h \in L^2([0, T], \mathbb{R}) \) and \( \varphi \in C([0, T], \mathbb{R}) \), and the sign of the function \( \varphi \) is allowed to change for \( t \in [0, T] \).

The study of the problem of periodic solutions to scalar equations with a singularity began with work of Forbat and Huaux [1, 2], where the singular term in the equations models the restoring force caused by a compressed perfect gas (see [3–6] and the references therein). In the past years, many works used the methods, such as the approaches of critical point theory [7–12], the techniques of some fixed point theorems [13–15], and the approaches of topological degree theory, in particular, of some continuation theorems of Mawhin (see [6, 16–22]), to study the existence of positive periodic solutions for some second order ordinary differential equations with singularities. For example, in [15], by using a fixed point theorem in cones, the existence of positive periodic solutions to equation
(1.1) was investigated for the conservative case, i.e., \( f(x) \equiv 0 \). But the function \( \varphi(t) \) is required to be \( \varphi(t) \geq 0 \) for all \( t \in [0, T] \). The method of topological degree theory, together with the technique of upper and lower solutions, was first used by Lazer and Solimini in the pioneering paper [18] for considering the problem of a periodic solution to a second order differential equations with singularities. Jebelean and Mawhin in [6] considered the problem of a \( p \)-Laplacian Liénard equation of the form

\[
\left( |x'|^{p-2}x' \right)' + f(x)x' + g(x) = h(t) \tag{1.2}
\]

and

\[
\left( |x'|^{p-2}x' \right)' + f(x)x' - g(x) = h(t), \tag{1.3}
\]

where \( p > 1 \) is a constant, \( f : [0, +\infty) \to \mathbb{R} \) is an arbitrary continuous function, \( h : \mathbb{R} \to \mathbb{R} \) is a \( T \)-periodic function with \( h \in L^\infty([0, T], \mathbb{R}) \), \( g : (0, +\infty) \to (0, +\infty) \) is continuous, \( g(x) \to +\infty \) as \( x \to 0^+ \). They extended the results of Lazer and Solimini in [16] to equation (1.2) and equation (1.3). For equation (1.3), the crucial condition is that the function \( g(x) \) is bounded, which means that equation (1.3) is not singular at \( x = +\infty \).

By using a continuation theorem of Mawhin, Zhang in [18] studied the problem of periodic solutions of the Liénard equation with a singularity of repulsive type,

\[
x'' + f(x)x' + g(t, x) = 0, \tag{1.4}
\]

where \( f : \mathbb{R} \to \mathbb{R} \) is continuous, \( g : \mathbb{R} \times (0, +\infty) \to \mathbb{R} \) is an \( L^2 \)-Carathéodory function with \( T \)-periodic in the first argument, and it is singular at \( x = 0 \), i.e., \( g(t, x) \) is unbounded as \( x \to 0^+ \). Different from the equation studied in [6, 16], which is only singular at \( x = 0 \), equation (1.4) is provided with both singularities at \( x = +\infty \) and at \( x = 0 \). In [19], Wang further studied the existence of positive periodic solutions for a delay Liénard equation with a singularity of repulsive type

\[
x'' + f(x)x' + g(t, x(t - \tau)) = 0. \tag{1.5}
\]

In [18, 19], the following balance condition between the singular force at the origin and at infinity is needed.

\( (h_1) \) There exist constants \( 0 < D_1 < D_2 \) such that if \( x \) is a positive continuous \( T \)-periodic function satisfying

\[
\int_0^T g(t, x(t)) \, dt = 0,
\]

then

\[
D_1 \leq x(\tau) \leq D_2, \quad \text{for some } \tau \in [0, T]. \tag{1.6}
\]

From the proof of [18, 19], we see that the balance condition \( (h_1) \) is crucial for estimating \textit{a priori bounds} of periodic solutions. Now, the question is how to investigate the existence
of positive periodic solutions for the equations like equation (1.4) or equation (1.5) without the balance condition (h₁).

Motivated by this, in this paper, we study the existence of positive T-periodic solutions for equation (1.1) under the condition that the sign of the function ϕ is allowed to change for t ∈ [0, T]. For this case, the balance condition (h₁) may not be satisfied. By using the continuation theorem of Manásevich and Mawhin, a new result on the existence of positive periodic solutions is obtained.

2 Preliminary lemmas

Throughout this paper, let $C_T = \{x \in C(R, R) : x(t + T) = x(t) \text{ for all } t \in R\}$ with the norm defined by $|x|_\infty = \max_{t \in [0, T]} |x(t)|$. For any T-periodic solution $y(t)$ with $y \in L^1([0, T], R)$, $y_+(t)$ and $y_-(t)$ denote $\max(y(t), 0)$ and $-\min(y(t), 0)$, respectively, and $\bar{y} = \frac{1}{T} \int_0^T y(s) \, ds$. Clearly, $y(t) = y_+(t) - y_-(t)$ for all $t \in R$, and $\bar{y} = \bar{y} - \bar{y}$.

The following lemma is a consequence of Theorem 3.1 in [23].

Lemma 1 Assume that there exist positive constants $M₀, M₁,$ and $M₂$ with $0 < M₀ < M₁$, such that the following conditions hold.

1. For each $\lambda \in (0, 1)$, each possible positive T-periodic solution $x$ to the equation

$$u'' + \lambda f(u)u' - \lambda g(u) + \lambda \phi(t)u = \lambda h(t)$$

satisfies the inequalities $M₀ < x(t) < M₁$ and $|x'(t)| < M₂$ for all $t \in [0, T]$.

2. Each possible solution $c$ to the equation

$$g(c) - c\bar{\phi} + \bar{h} = 0$$

satisfies the inequality $M₀ < c < M₁$.

3. We have

$$\left( g(M₀) - \bar{\phi}M₀ + \bar{h} \right) \left( g(M₁) - \bar{\phi}M₁ + \bar{h} \right) < 0.$$

Then equation (1.1) has at least one T-periodic solution $u$ such that $M₀ < u(t) < M₁$ for all $t \in [0, T]$.

Lemma 2 ([19]) Let $x$ be a continuous T-periodic continuously differential function. Then, for any $\tau \in (0, T]$,

$$\left( \int_0^T |x(s)|^2 \, ds \right)^{1/2} \leq \frac{T}{\pi} \left( \int_0^T |x'(s)|^2 \, ds \right)^{1/2} + \sqrt{T}|x(\tau)|.$$
\[
\sigma_1 := \frac{T}{\pi} |\phi_1|_{\infty}^2 + \frac{T^2 (\int_0^T \phi_-(s)^2 \, ds)^{1/2}}{\int_0^T \phi_+(s) \, ds} \in (0, 1);
\]

[H₂] there are constants \( A > 0 \) and \( M > 0 \) such that \( g(x) \in (0, A) \) for all \( x > M \);

[H₃] \( \int_0^1 g(s) \, ds = +\infty \).

**Remark 1** If assumptions [H₁]-[H₂] hold, then there are constants \( D_1 \) and \( D_2 \) with \( 0 < D_1 < D_2 \) such that

\[
g(x) - \bar{\phi} x + \bar{\theta} > 0 \quad \text{for all} \ x \in (0, D_1)
\]

and

\[
g(x) - \bar{\phi} x + \bar{\theta} < 0 \quad \text{for all} \ x \in (D_2, \infty).
\]

Furthermore, assumption \( \sigma_1 \in (0, 1) \) in [H₁] is different from the corresponding condition \( \int_0^T \phi_+(s) \, ds < \frac{4}{\pi} \) in [20].

Now, we suppose that assumptions [H₁] and [H₂] hold, and we embed equation (1.1) into the following equation family with a parameter \( \lambda \in (0, 1] \):

\[
x'' + \lambda f(x)x' - \lambda g(x) + \lambda \varphi(t)x = \lambda h(t), \quad \lambda \in (0, 1].
\]

(2.1)

Let

\[
\Omega = \left\{ x \in C_T : x'' + \lambda f(x)x' - \lambda g(x) + \lambda \varphi(t)x = \lambda h(t), \lambda \in (0, 1]; x(t) > 0, \forall t \in [0, T] \right\},
\]

and

\[
M_0 = \left( \int_0^T \phi_-(s)^2 \, ds \right)^{1/2} \int_0^T \phi_+(s) \, ds + A \left| \bar{\theta} \right| + \left| \varphi_+ \right|_{\infty} A_0^T + A_0 T \left( \int_0^T \left| h_-(t) \right|^2 \, dt \right)^{1/2},
\]

(2.2)

where

\[
A_0 = \frac{T}{\pi (1 - \sigma_1)} \left( \int_0^T \left| h_-(t) \right|^2 \, dt \right)^{1/2} + \left( \frac{A + \left| \bar{\theta} \right|}{(1 - \sigma_1) \overline{\varphi}_+} \right)^{1/2},
\]

(2.3)

A > 0 is a constant determined by assumption [H₂]. Clearly, \( M_0 \) and \( A_0 \) are all independent of \( (\lambda, x) \in (0, 1] \times \Omega \). Let \( M > 0 \) be determined by assumption [H₂], then there is a positive integer \( k_0 \) such that

\[
k_0 M \geq M_0.
\]

**Lemma 3** Assume that assumptions [H₁]-[H₂] hold, then there is an integer \( k^* > k_0 \) such that, for each function \( u \in \Omega \), there is a point \( t_0 \in [0, T] \) satisfying

\[
u(t_0) \leq k^* M.
\]
Proof If the conclusion does not hold, then, for each \( k > k_0 \), there is a function \( u_k \in \Omega \) satisfying

\[
u_k(t) > kM \quad \text{for all } t \in [0, T].
\] (2.4)

From the definition of \( \Omega \), we see

\[
u_k'' + \lambda f(u_k)u'_k - \lambda g(u_k) + \lambda \psi(t)u_k = \lambda h(t), \quad \lambda \in (0, 1],
\] (2.5)

and by using assumption \([H_2]\),

\[
0 < g(u_k(t)) < A, \quad \text{for all } t \in [0, T].
\] (2.6)

By integrating equation (2.5) over the interval \([0, T]\), we have

\[
\int_0^T \psi(t)u_k(t) \, dt = \int_0^T g(u_k(t)) \, dt + \int_0^T h(t) \, dt,
\]
i.e.,

\[
\int_0^T \varphi_+(t)u_k(t) \, dt = \int_0^T \varphi_-(t)u_k(t) \, dt + \int_0^T g(u_k(t)) \, dt + \int_0^T h(t) \, dt.
\]

Since \( \varphi_+(t) \geq 0 \) and \( \varphi_-(t) \geq 0 \) for all \( t \in [0, T] \), it follows from the integral mean value theorem that there is a point \( \xi \in [0, T] \) such that

\[
u_k(\xi) \int_0^T \varphi_+(t) \, dt = \int_0^T \varphi_-(t)u_k(s) \, ds + \int_0^T g(u_k(t)) \, dt + Th
\]

\[
\leq \left( \int_0^T \varphi_-(s)^2 \, ds \right)^{1/2} \left( \int_0^T |u_k(s)|^2 \, ds \right)^{1/2} + \int_0^T g(u_k(t)) \, dt + Th,
\]

which together with (2.6) yields

\[
u_k(\xi) < \left( \frac{\int_0^T \varphi_-(s)^2 \, ds}{\int_0^T \varphi_+(s) \, ds} \right)^{1/2} \left( \int_0^T |u_k(s)|^2 \, ds \right)^{1/2} + \frac{A + |\tilde{h}|}{\psi_*}.
\] (2.7)

It follows from \(|u_k| \leq u_k(\xi) + T^{1/2} (\int_0^T |u'_k(s)|^2 \, ds)^{1/2} \) that

\[
|u_k| \leq \left( \frac{\int_0^T \varphi_-(s)^2 \, ds}{\int_0^T \varphi_+(s) \, ds} \right)^{1/2} \left( \int_0^T |u_k(s)|^2 \, ds \right)^{1/2} + \frac{A + \tilde{h}}{\psi_*} + T^{1/2} \left( \int_0^T |u'_k(s)|^2 \, ds \right)^{1/2}.
\] (2.8)

On the other hand, by multiplying equation (2.5) with \( u_k(t) \), and integrating it over the interval \([0, T]\), we obtain

\[
\int_0^T |u_k'(t)|^2 \, dt = \lambda \int_0^T g(u_k(t))u_k(t) \, dt + \lambda \int_0^T \varphi(t)u_k^2(t) \, dt - \lambda \int_0^T h(t)u_k(t) \, dt,
\]
which together with the fact of \( g(x) > 0 \) for all \( x > 0 \) gives

\[
\int_0^T |u_k'(t)|^2 dt < \lambda \int_0^T \varphi_+(t)u_k^2(t) dt + \lambda \int_0^T h_-(t)u_k(t) dt
\]

\[
\leq |\varphi_+|_\infty \int_0^T |u_k(t)|^2 dt + \left( \int_0^T |u_k(t)|^2 dt \right)^{\frac{1}{2}} \left( \int_0^T |h_-(t)|^2 dt \right)^{\frac{1}{2}},
\]
i.e.,

\[
\left( \int_0^T |u_k'(t)|^2 dt \right)^{\frac{1}{2}} < |\varphi_+|_\infty \left( \int_0^T |u_k(t)|^2 dt \right)^{\frac{1}{2}} + \left( \int_0^T |u_k(t)|^2 dt \right)^{\frac{1}{2}} \left( \int_0^T |h_-(t)|^2 dt \right)^{\frac{1}{2}}.
\] (2.9)

By using Lemma 2, we have

\[
\left( \int_0^T |u_k(s)|^2 ds \right)^{\frac{1}{2}} \leq \frac{T}{\pi} \left( \int_0^T |u_k'(s)|^2 ds \right)^{\frac{1}{2}} + \sqrt{T} |u_k(\xi)|.
\]

Substituting (2.7) and (2.9) into the above formula,

\[
\left( \int_0^T |u_k(t)|^2 dt \right)^{\frac{1}{2}} \leq \frac{T}{\pi} \left[ |\varphi_+|_\infty \left( \int_0^T |u_k(t)|^2 dt \right)^{\frac{1}{2}} + \left( \int_0^T |u_k(t)|^2 dt \right)^{\frac{1}{2}} \left( \int_0^T |h_-(t)|^2 dt \right)^{\frac{1}{2}} \right]
\]

\[
+ \frac{T^2}{\pi} \left( \int_0^T \varphi_+(s)^2 ds \right)^{\frac{1}{2}} \left( \int_0^T u_k(s)^2 ds \right)^{\frac{1}{2}} + \frac{(A + |\tilde{h}|)T^2}{\varphi_+}
\]

\[
= \sigma_1 \left( \int_0^T |u_k(t)|^2 dt \right)^{\frac{1}{2}} + \frac{T}{\pi} \left( \int_0^T |h_-(t)|^2 dt \right)^{\frac{1}{2}} \left( \int_0^T |u_k(t)|^2 dt \right)^{\frac{1}{2}} + \frac{(A + |\tilde{h}|)T^2}{\varphi_+},
\]

where

\[\sigma_1 = \frac{T}{\pi} |\varphi_+|_\infty^2 + \frac{T^2}{\pi} \left( \int_0^T \varphi_+(s)^2 ds \right)^{\frac{1}{2}} \left( \int_0^T u_k(s)^2 ds \right)^{\frac{1}{2}} \in (0, 1),\]

which is determined by assumption [H_1]. This gives

\[
\left( \int_0^T |u_k(t)|^2 dt \right)^{\frac{1}{2}} \leq \frac{T}{\pi (1 - \sigma_1)} \left( \int_0^T |h_-(t)|^2 dt \right)^{\frac{1}{2}} \left( \int_0^T |u_k(t)|^2 dt \right)^{\frac{1}{2}} + \frac{(A + |\tilde{h}|)T^2}{(1 - \sigma_1)\varphi_+}.
\] (2.10)

i.e.,

\[
\left( \int_0^T |u_k(t)|^2 dt \right)^{\frac{1}{2}} \leq A_0,
\] (2.11)
where
\[
A_0 = \frac{T}{\pi(1-\sigma_1)} \left( \int_0^T |h_-(t)|^2 dt \right)^{\frac{1}{2}} + \left( A + |\bar{h}| \right) \frac{T^{\frac{1}{2}}}{(1-\sigma_1)\psi_+}.
\]

It follows from (2.9) that
\[
\left( \int_0^T |u_k'(t)|^2 dt \right)^{\frac{1}{2}} < |\psi_+|\infty A_0^2 + A_0 \left( \int_0^T |h_-(t)|^2 dt \right)^{\frac{1}{2}}.
\] (2.12)

Substituting (2.11)-(2.12) into (2.8), we have
\[
|u_k|\infty < \frac{\left( \int_0^T \psi_+(s)^2 ds \right)^{\frac{1}{2}}}{\int_0^T \psi_+(s) ds} A_0^2 + \frac{A + |\bar{h}|}{\psi_+} + |\psi_+|\infty A_0^2 T^{\frac{1}{2}} + A_0 T^{\frac{1}{2}} \left( \int_0^T |h_-(t)|^2 dt \right)^{\frac{1}{2}},
\]
which together with (2.2) yields
\[
u_k(t) < M_0 \quad \text{for all } t \in [0, T]. \tag{2.13}
\]

By the definition of \(k_0\), we see from (2.3) that (2.13) contradicts (2.4). This contradiction implies that the conclusion of Lemma 3 is true. \(\square\)

3 Main results

**Theorem 1** Assume that \([H_1]-[H_3]\) hold. Then equation (1.1) has at least one positive \(T\)-periodic solution.

**Proof** Firstly, we will show that there exist \(M_1, M_2\) with \(M_1 > k^* M\) and \(M_2 > 0\) such that each positive \(T\)-periodic solution \(u(t)\) of equation (2.1) satisfies the inequalities
\[
u(t) < M_1, \quad |u'(t)| < M_2, \quad \text{for all } t \in [0, T]. \tag{3.1}
\]

In fact, if \(u\) is an arbitrary positive \(T\)-periodic solution of equation (2.1), then
\[
u'' + \lambda f(u)u' - \lambda g(u) + \lambda \psi(t)u = \lambda h(t), \quad \lambda \in (0, 1]. \tag{3.2}
\]

This implies \(u \in \Omega\). So by using Lemma 3 that there is a point \(t_0 \in [0, T]\) such that
\[
u(t_0) \leq k^* M, \tag{3.3}
\]
and then
\[
|u|\infty \leq k^* M + T^{1/2} \left( \int_0^T |u'(s)|^2 ds \right)^{1/2}. \tag{3.4}
\]

Integrating (3.2) over the interval \([0, T]\), we have
\[
- \int_0^T g(u(t)) dt + \int_0^T \varphi(t)u(t) dt = \int_0^T h(t) dt. \tag{3.5}
\]
Since \( g(x) \to +\infty \text{ as } x \to 0^+ \), we see from (3.5) that there is a point \( t_1 \in [0, T] \) such that

\[
u(t_1) \geq \gamma,
\]

(3.6)

where \( \gamma < k^*M \) is a positive constant, which is independent of \( \lambda \in (0,1] \). Similar to the proof of (2.9), we have

\[
\left( \int_0^T |u(t)|^2 \, dt \right)^{1/2} < |\varphi_1|_\infty \left( \int_0^T |u(t)|^2 \, dt \right)^{1/2} + \left( \int_0^T |u(t)|^2 \, dt \right)^{1/2} \left( \int_0^T |h_{-}(t)|^2 \, dt \right)^{1/4}.
\]

(3.7)

By using Lemma 2, we have

\[
\left( \int_0^T |u(s)|^2 \, ds \right)^{1/2} \leq \frac{T}{\pi} \left( \int_0^T |u'(s)|^2 \, ds \right)^{1/2} + \sqrt{T} |u(t_0)|,
\]

(3.8)

where \( t_0 \) is determined in (3.3). Substituting (3.7) into (3.8), we have

\[
\left( \int_0^T |u(t)|^2 \, dt \right)^{1/2} < \frac{T}{\pi} \left( \int_0^T |u(t)|^2 \, dt \right)^{1/2} + \frac{T^{1/2}}{\pi} k^*M
\]

\[
= \frac{T}{\pi} |\varphi_1|_\infty \left( \int_0^T |u(t)|^2 \, dt \right)^{1/2} + \frac{T}{\pi} \left( \int_0^T |u(t)|^2 \, dt \right)^{1/2} \left( \int_0^T |h_{-}(t)|^2 \, dt \right)^{1/4} + T^{1/2} k^*M,
\]

which results in

\[
\left( 1 - \frac{T}{\pi} |\varphi_1|_\infty \right) \left( \int_0^T |u(t)|^2 \, dt \right)^{1/2} \left( \int_0^T |h_{-}(t)|^2 \, dt \right)^{1/4} < \frac{T}{\pi} \left( \int_0^T |u(t)|^2 \, dt \right)^{1/2} + \frac{T^{1/2}}{\pi} k^*M.
\]

(3.9)

Since \( \frac{T}{\pi} |\varphi_1|_\infty < \sigma_1 \in (0,1) \), it follows from (3.9) that there is a constant \( \rho > 0 \), which is independent of \( \lambda \in (0,1] \), such that

\[
\left( \int_0^T |u(t)|^2 \, dt \right)^{1/2} < \rho,
\]

and then by (3.7), we have

\[
\left( \int_0^T |u'(t)|^2 \, dt \right)^{1/2} < |\varphi_1|_\infty \rho + \left( \int_0^T |h_{-}(t)|^2 \, dt \right)^{1/4} \rho^{1/2}.
\]
It follows from (3.4) that

$$|u|_{\infty} < k^* M + T^{1/2} \psi_{1/\psi}^{1/2} \rho + (T \rho)^{1/2} \left( \int_0^T \|h(t)\|^2 \, dt \right)^{1/2} := M_1,$$

i.e.,

$$u(t) < M_1, \quad \text{for all } t \in [0, T]. \quad (3.10)$$

Now, if $u$ attains its maximum over $[0, T]$ at $t_2 \in [0, T]$, then $u'(t_2) = 0$ and we deduce from (3.2) that

$$u'(t) = \lambda \int_{t_2}^t \left[ -f(u)u' + g(u) - \psi(t)u + h(t) \right] \, dt$$

for all $t \in [t_2, t_2 + T]$. Thus, if $F' = f$, then

$$|u'(t)| \leq \lambda |F(u(t)) - F(u(t_2))| + \lambda \int_{t_2}^{t_2 + T} g(u(t)) \, dt + \lambda \int_{t_2}^{t_2 + T} \psi(s) |u(s)| \, ds + \lambda \int_{t_2}^{t_2 + T} |h(s)| \, ds$$

$$\leq 2\lambda \max_{0 \leq u \leq M_1} |F(u)| + \lambda \int_0^T g(u(s)) \, ds + \lambda T \widehat{|\psi|} |u|_{\infty} + \lambda T \widehat{|h|}. \quad (3.11)$$

From (3.2), we see that

$$\int_0^T g(u(s)) \, ds = \int_0^T \psi(t)u(t) \, dt - T \bar{h}$$

$$\leq T \widehat{|\psi|} |u|_{\infty} + T \bar{h}.$$  

It follows from (3.10) and (3.11) that

$$|u'(t)| \leq 2\lambda \left( \max_{0 \leq u \leq M_1} |F(u)| + T \widehat{|\psi|} |u|_{\infty} + T \widehat{|h|} \right)$$

$$< 2\lambda \left( \max_{0 \leq u \leq M_1} |F(u)| + M_1 T \widehat{|\psi|} + T \widehat{|h|} \right)$$

$$:= \lambda M_2, \quad t \in [0, T], \quad (3.12)$$

and then

$$|u'(t)| < M_2, \quad \text{for all } t \in [0, T]. \quad (3.13)$$

Equations (3.10) and (3.13) imply that (3.1) holds.

Below, we will show that there exists a constant $\gamma_0 \in (0, \gamma)$, such that each positive $T$-periodic solution of equation (2.1) satisfies

$$u(t) > \gamma_0 \quad \text{for all } t \in [0, T]. \quad (3.14)$$
Suppose that \( u(t) \) is an arbitrary positive \( T \)-periodic solution of equation (2.1), then

\[
    u'' + \lambda f(u)u' - \lambda g(u) + \lambda \varphi(t)u = \lambda h(t), \quad \lambda \in (0, 1]. \tag{3.15}
\]

Let \( t_1 \) be determined in (3.6). Multiplying (3.15) by \( u'(t) \) and integrating it over the interval \([t_1, t]\) or \([t, t_1]\), we get

\[
    \frac{|u'(t)|^2}{2} + \lambda \int_{t_1}^{t} |u'(t_1)|^2 \, dt = \lambda \int_{t_1}^{t} g(u)u' \, dt - \lambda \int_{t_1}^{t} \varphi(t)uu' \, dt + \lambda \int_{t_1}^{t} h(t)u' \, dt,
\]

which yields the estimate

\[
    \lambda \left| \int_{u(t_1)}^{u(t)} g(s) \, ds \right| \leq \frac{|u'(t)|^2}{2} + \frac{|u'(t_1)|^2}{2} + \lambda \int_{0}^{T} |f(u)|^2 \, dt + \lambda \int_{0}^{T} |\varphi(t)|^2 \, dt + \lambda \int_{0}^{T} |h(t)| \, dt.
\]

From (3.10) and (3.12), we get

\[
    \lambda \left| \int_{u(t_1)}^{u(t)} g(s) \, ds \right| \leq \lambda M_2^2 + \lambda \max_{0 \leq u \leq M_1} |f(u)| TM_2^2 + \lambda M_1 M_2 T|\varphi| + \lambda M_2 T|h|,
\]

which gives

\[
    \int_{u(t_1)}^{u(t)} g(s) \, ds \leq M_3, \quad \text{for all } t \in [t_1, t_1 + T], \tag{3.16}
\]

with

\[
    M_3 = M_2^2 + \max_{0 \leq u \leq M_1} |f(u)| TM_2^2 + M_1 M_2 T|\varphi| + M_2 T|h|.
\]

From \([H_3]\) there exists \( \gamma_0 \in (0, \gamma) \) such that

\[
    \int_{\eta}^{\gamma} g(u) \, du > M_3, \quad \text{for all } \eta \in (0, \gamma]. \tag{3.17}
\]

Therefore, if there is a \( t^* \in [t_1, t_1 + T] \) such that \( u(t^*) \leq \gamma_0 \), then from (3.17) we get

\[
    \int_{u(t^*)}^{\gamma} g(s) \, ds > M_3,
\]

which contradicts (3.16). This contradiction gives that \( u(t) > \gamma_0 \) for all \( t \in [0, T] \). So (3.14) holds. Let \( m_0 = \min\{D_1, \gamma_0\} \) and \( m_1 \in (M_1 + D_2, +\infty) \) be two constants, then from (3.1) and (3.14), we see that each possible positive \( T \)-periodic solution \( u \) to equation (2.1) satisfies

\[
    m_0 < u(t) < m_1, \quad |u'(t)| < M_2.
\]

This implies that condition 1 and condition 2 of Lemma 1 are satisfied. Also, we can deduce from Remark 1 that

\[
    g(c) - \bar{\psi} c + \bar{h} > 0, \quad \text{for } c \in (0, m_0]
\]
and
\[ g(c) - \bar{c} + \bar{h} < 0, \quad \text{for } c \in [m, +\infty), \]
which results in
\[ (g(m_0) - \bar{m}_0 + \bar{h})(g(m_1) - \bar{m}_1 + \bar{h}) < 0. \]

So condition 3 of Lemma 1 holds. By using Lemma 1, we see that equation (1.1) has at least one positive T-periodic solution. The proof is complete. \(\square\)

Let us consider the equation
\[ x'' + f(x)x' - \frac{1}{x'} + \varphi(t)x = h(t), \quad (3.18) \]
where \(f : [0, +\infty) \to \mathbb{R}\) is an arbitrary continuous function, \(\varphi, h : \mathbb{R} \to \mathbb{R}\) are T-periodic functions with \(h \in L^1([0, T], \mathbb{R})\) and \(\varphi \in C([0, T], \mathbb{R})\), and the sign of the function \(\varphi\) is allowed to change for \(t \in [0, T]\), \(\gamma \geq 1\) is a constant. Corresponding to equation (1.1), \(g(x) = \frac{1}{x'}\). For this case, \(g(x) \to +\infty\) as \(x \to 0^+\), and assumptions [H2]-[H3] are satisfied.

Thus, by using Theorem 1, we have the following results.

**Corollary 1** Assume that the function \(\varphi(t)\) satisfies the following conditions:
\[ \int_0^T \varphi_+(s) \, ds > 0, \quad \sigma := \frac{\int_0^T \varphi_-(s) \, ds}{\int_0^T \varphi_+(s) \, ds} \in [0, 1) \]
and
\[ \sigma_1 := \frac{T}{\pi} \left| \varphi' \right|_\infty \frac{1}{\sqrt{2}} + \frac{T^{1/2} \left( \int_0^T \varphi_-(s)^2 \, ds \right)^{1/2}}{\int_0^T \varphi_+(s) \, ds} \in (0, 1). \]

Then, equation (3.18) possesses at least one positive T-periodic solution.

**Remark 2** Corresponding to equation (1.4) and equation (1.5), the function \(g(t, x)\) associated to equation (3.18) can be regarded as
\[ g(t, u) = -\frac{1}{u'} + \varphi(t)u - h(t), \quad (t, u) \in [0, T] \times (0, +\infty). \quad (3.19) \]

For the case of \(\varphi(t) \geq 0\) for all \(t \in [0, T]\), we see that if \(x\) is a positive T-periodic continuous function satisfying \(\int_0^T g(t, x(t)) \, dt = 0\), then
\[ \int_0^T \frac{1}{x'(t)} \, dt = \int_0^T \varphi(t)x(t) \, dt - \int_0^T h(t) \, dt. \quad (3.20) \]

By applying the integral mean value theorem to the term \(\int_0^T \varphi(t)x(t) \, dt\) in equation (3.20), one can easily verify that \(g(t, u)\) determined in (3.19) satisfies the balance condition (h1).

However, if the sign of the function \(\varphi(t)\) is changeable for \(t \in [0, T]\), then it is unclear from
whether the balance condition (h_1) is satisfied. For this case, the main results of [18, 19] cannot be applied to equation (3.18).

**Corollary 2** Assume that the function \( \varphi(t) \) satisfies \( \varphi(t) \geq 0 \) for all \( t \in [0, T] \) with \( \int_0^T \varphi(s) \, ds > 0 \), and
\[
|\varphi|_\infty < \left( \frac{\pi}{T} \right)^2.
\]
Then, equation (3.18) possesses at least one positive \( T \)-periodic solution.

**Example 1** Consider the following equation:
\[
x''(t) + f(x(t))x'(t) - \frac{1}{x^2(t)} + a(1 + 2 \sin 2t)x(t) = \cos 2t,
\]
where \( f \) is an arbitrary continuous function, \( a \in (0, +\infty) \) is a constant. Corresponding to equation (3.18), we have \( \gamma = 2 \), \( \varphi(t) = a(1 + 2 \sin 2t) \) and \( h(t) = \cos 2t \), \( T = \pi \). By simply calculating, we can verify that
\[
\int_0^T \varphi_+(t) \, dt = \left( \frac{2\pi}{3} + \frac{3}{2} \right) a, \quad \int_0^T \varphi_-(t) \, dt = \left( \frac{3}{2} - \frac{\pi}{3} \right) a,
\]
\[
\int_0^T (\varphi_-(t))^2 \, dt = \frac{3\pi a}{2},
\]
and then
\[
\sigma := \frac{\int_0^T \varphi_-(s) \, ds}{\int_0^T \varphi_+(s) \, ds} = \frac{9 - 2 \pi}{4\pi + 9} \in (0, 1)
\]
and
\[
\sigma_1 := \frac{T}{\pi} |\varphi|_\infty^{1/2} + \frac{T^{1/2} (\int_0^T \varphi_-(s)^2 \, ds)^{1/2}}{\int_0^T \varphi_+(s) \, ds} = \sqrt{3}a + \frac{3\pi \sqrt{6}}{4\pi + 9}.
\]
Thus, if \( 0 < a < \frac{1}{\sqrt{3}} \left( \frac{4\pi - 3\pi \sqrt{6}}{4\pi + 9} \right)^2 \), then \( \sigma_1 \in (0, 1) \). By using Corollary 1, we see that equation (3.21) has at least one positive \( \pi \)-periodic solution.

**Remark 3** Since the sign of \( \varphi(t) = 1 + 2 \sin t \) is changed for \( t \in [0, T] \), whether the right inequality of (1.6) in the balance condition (h_1) is satisfied remains unclear. So the conclusion of the example cannot be obtained by using the main results in [18, 19].

**Competing interests**
The author declares to have no competing interests.

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