Abstract
Let $G$ be an acylindrically hyperbolic group. We consider a random subgroup $H$ in $G$, generated by a finite collection of independent random walks. We show that, with asymptotic probability one, such a random subgroup $H$ of $G$ is a free group, and the semidirect product of $H$ acting on $E(G)$ is hyperbolically embedded in $G$, where $E(G)$ is the unique maximal finite normal subgroup of $G$.

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1 Introduction

Acylindrically hyperbolic groups have been defined by Osin, who showed in [Osi16] that several approaches to groups that exhibit rank one behaviour [BF02, Ham08, DGO11, Sis16b] are all equivalent; see Section 2 for the precise definition. Acylindrically hyperbolic groups form a very large class of groups that vastly generalises the class of non-elementary hyperbolic groups and includes non-elementary relatively hyperbolic groups, mapping class groups [MM99, Bow06, PS16], Out($F_n$) [BF14], many groups acting on CAT(0) spaces [BHS14, CM16, Gen16, Hea16, Sis16b], and many others, see for example [GS16, MO15, Osi15].

Acylindrical hyperbolicity has strong consequences: For example, every acylindrically hyperbolic group is SQ-universal (in particular it has uncountably many pairwise non-isomorphic quotients) and its bounded cohomology is infinite dimensional in degrees 2 [HO13] and 3 [FPS15]. These results all rely on the notion of hyperbolically embedded subgroup, as defined in [DGO11] (see Section 2 for the definition), and in fact, on virtually free hyperbolically embedded subgroups. Hyperbolically embedded subgroups are hence very important for the study of acylindrically hyperbolic groups, and in fact they enjoy several nice properties such as almost malnormality [DGO11] and quasiconvexity [Sis16a].

In this paper we show that, roughly speaking, a random subgroup $H$ of an acylindrically hyperbolic group is free and virtually hyperbolically embedded. We now give a slightly simplified version of our main theorem, see Section 2 for a more refined statement. We shall write $E(G)$ for the maximal finite normal subgroup of $G$, which for $G$ acylindrically hyperbolic exists by [DGO11] Theorem 6.14], and given a subgroup $H < G$, we shall write $HE(G)$ for the subset of $G$ consisting of $\{hg \mid h \in H, g \in E(G)\}$, which in this case is a subgroup, as $E(G)$ is normal. We say that a property $P$ holds with asymptotic probability one if the the probability $P$ holds tends to one as $n$ tends to infinity.

**Theorem 1.** Let $G$ be an acylindrically hyperbolic group, with maximal finite normal subgroup $E(G)$, and let $\mu$ be a probability measure on $G$ whose support is finite and generates $G$ as a semigroup. For $k,n$ positive integers, let $H_{k,n}$ denote the subgroup of $G$ generated by $k$ independent random walks generated by $\mu$, each of length $n$, which we shall denote by $w_{i,n}$.

Then for each fixed $k$, the probability that each of the following events occurs with asymptotic probability one.

1. The subgroup $H$ is freely generated by the $\{w_{1,n_1}, \ldots, w_{k,n_k}\}$ and quasi-isometrically embedded.
2. The subgroup $HE(G)$ is a semidirect product $H \rtimes E(G)$, and is hyperbolically embedded in $G$.

The first part of Theorem 1 was previously shown by Taylor and Tiozzo [TT16], and they apply this result to study random free group and surface group extensions. The second part is definitely the main contribution of this paper. For the experts, we note that we can fix the generating set with respect to which $H \rtimes E(G)$ is hyperbolically embedded, see Theorem 8.

The study of generic properties of groups in geometric group theory goes back at least to Gromov [Gro87, Gro03], and we make no attempt to survey the substantial literature on this topic, see for example [GMO10] for a more thorough discussion, though we now briefly mention some closely related results. This model of random subgroups is used in Guivarc’h’s [Gui90] proof of the Tits alternative for linear groups, and is also developed by Rivin [Riv10] and Aoun [Aou11], who proves that a random subgroup of a non-virtually solvable linear group is free and undistorted. Gilman, Miasnikov and Osin [GMO10] consider subgroups of hyperbolic groups generated by $k$ elements arising from nearest neighbour random walks on the corresponding Cayley graph, and they show that the probability that the resulting group is a quasi-isometrically embedded free group, freely generated by the $k$ unreduced words of length $n$, tends to one exponentially quickly in $n$. The fact that the $k$ elements freely generate a free group as $n$ becomes large was shown earlier for free groups by Jitsukawa [Jit02] and Martino, Turner and Ventura [MTV], and for braid groups by Myasnikov and Ushakov [MU08]. Our argument makes use of particular group elements which we call strongly asymmetric, namely loxodromic elements $g$ contained in maximal cyclic subgroups which are equal to $\langle g \rangle \times E(G)$. We say a loxodromic element $g$ is weakly asymmetric if it is contained in a maximal cyclic subgroup which is a semidirect product $\langle g \rangle \rtimes E(G)$, see Section 2 for full details. Masai [Mas14] has previously shown that random elements of the mapping class group are strongly asymmetric, and the argument we present uses similar methods in the context of acylindrically hyperbolic groups. Mapping class groups have trivial maximal finite normal subgroups, except for a finite list of surfaces in which $E(G)$ is central, see for example [FM12, Section 3.4], so in the case of the mapping class groups there is no distinction between weakly and strongly asymmetric elements.

Theorem 1 is used in [HS16] to study the bounded cohomology of acylindrically hyperbolic groups.
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2 Background and main theorem

We say a geodesic metric space \((X,d_X)\), which need not be proper, is Gromov hyperbolic, \(\delta\)-hyperbolic or just hyperbolic, if there is a number \(\delta \geq 0\) for which every geodesic triangle in \(X\) satisfies the \(\delta\)-slim triangle condition, i.e. for any geodesic triangle, any side is contained in the \(\delta\)-neighbourhood of the other two sides.

Let \(G\) be a countable group which acts on a hyperbolic space \(X\) by isometries. We say the action of \(G\) on \(X\) is non-elementary if \(G\) contains two hyperbolic elements with disjoint pairs of fixed points at infinity. We say a group \(G\) acts acylindrically on a Gromov hyperbolic space \(X\), if there are real valued functions \(R\) and \(N\) such that for every number \(K \geq 0\), and for any pair of points \(x\) and \(y\) in \(X\) with \(d_X(x,y) \geq R(K)\), there are at most \(N(K)\) group elements \(g\) in \(G\) such that \(d_X(x, gx) \leq K\) and \(d_X(y, gy) \leq K\). We shall refer to \(R\) and \(N\) as the acylindricality functions for the action. This definition is due to Sela \([Sel97]\) for trees, and Bowditch \([Bow06]\) for general metric spaces.

We say a group \(G\) acts acylindrically hyperbolically on a space \(X\), if \(X\) is hyperbolic, and the action is non-elementary and acylindrical. A group is acylindrically hyperbolic if it admits an acylindrically hyperbolic action on some space \(X\).

A finitely generated subgroup \(H\) in \(G\) is quasi-isometrically embedded in \(X\), if for any choice of word metric \(d_H\), and any basepoint \(x_0 \in X\), there are constants \(K\) and \(c\) such that for any two elements \(h_1\) and \(h_2\) in \(H\),

\[
\frac{1}{K} d_X(h_1x_0, h_2, x_0) - c \leq d_H(h_1, h_2) \leq Kd_X(h_1x_0, h_2, x_0) + c.
\]

We say that a subgroup \(H\) of \(G\) is geometrically separated in \(X\), if for each \(x_0 \in X\) and \(R \geq 0\) there exists \(B(R)\) so that for each \(g \in G \setminus H\), we have that the diameter of \(N_R(gHx_0) \cap N_R(Hx_0)\) is bounded by \(B\), where \(N_R\) denotes the metric \(R\)-neighborhood in \(X\).

For the remainder of this paper fix an acylindrically hyperbolic group \(G\). We shall write \(E(G)\) for the maximal finite normal subgroup of \(G\),
which exists and is unique by \cite[Theorem 6.14]{DGO11}. Given an element $g \in G$, let $E(g)$ be the maximal virtually cyclic subgroup containing $g$, which is well-defined by work of Bestvina and Fujiwara \cite{BF02}. For a hyperbolic element $g$, let $\Lambda(g) = \{\lambda_+(g), \lambda(g)\}$ be the set consisting of the pair of attracting and repelling fixed points for $g$ in $\partial X$. We shall write $\text{stab}(\Lambda(g))$ for the stabilizer of this set in $G$. Dahmani, Guirardel and Osin \cite[Corollary 6.6]{DGO11} show that in fact $E(g) = \text{stab}(\Lambda(g))$.

For any hyperbolic element $g$, the group $E(g)$ is always quasi-isometrically embedded and geometrically separated.

The subgroup $E(G)$ acts trivially on the Gromov boundary $\partial X$, so in many applications it may be natural to consider $G/E(G)$, which will have a trivial maximal finite subgroup, and a reader interested in this case should feel free to assume $E(G)$ is trivial, which simplifies the arguments and statements in many places. We shall write $\langle g, \ldots, g_k \rangle$ for the subgroup of $H$ generated by $\{g_1, \ldots, g_k\}$, and in particular, $\langle g \rangle$ denotes the cyclic group generated by $g$. Recall that given a subgroup $H < G$, we will write $HE(G)$ for the subset of $G$ consisting of $\{hg \mid h \in H, g \in E(G)\}$, which is a subgroup, as $E(G)$ is normal. If $H \cap E(G) = \{1\}$, then the group $HE(G)$ is a finite extension of $H$ by $E(G)$, but in general need not be either a product $H \times E(G)$, or a semidirect product, which we shall write as $H \rtimes E(G)$. The following observation is elementary, but we record it as a proposition for future reference.

**Proposition 2.** Let $G$ be a countable group acting acylindrically hyperbolically on $X$, and let $H$ be a subgroup of $G$ with trivial intersection with $E(G)$, i.e. $H \cap E(G) = \{1\}$. Then the subgroup $HE(G)$ is a semidirect product $H \rtimes E(G)$.

**Proof.** The quotient $(HE(G))/E(G)$ corresponds to the set of cosets $hE(G)$. If $h$ is a non-trivial element of $H$ then $hE(G) \neq E(G)$, as $H \cap E(G) = \{1\}$. Therefore, $(HE(G))/E(G)$ is isomorphic to $H$, and the inclusion of $H$ into $HE(G)$ gives a section $H \to HE(G)$, i.e. a homomorphism whose composition with the quotient map is the identity on $H$. This implies that $HE(G)$ is a split extension of $(HE(G))/E(G)$ by $E(G)$, and hence a semidirect product $H \rtimes E(G)$. □

If $g$ is a hyperbolic element, then $\langle g \rangle$ is an infinite cyclic group, and the subgroup $E(g)$ always contains $\langle g \rangle E(G) = \langle g \rangle \rtimes E(G)$, but may be larger. For example, $E(g) = E(g^2)$ but $E(g^2)$ cannot be equal to $\langle g^2 \rangle E(G)$, as $E(g)$ contains $g$. Furthermore, the subgroup $\langle g^2 \rangle E(G)$ is quasi-isometrically embedded, but not geometrically separated, as $g$. 

![Image](image_url)
coarsely stabilizes this subgroup. We say that a group element \( g \in G \) is weakly asymmetric if \( E(g) \) is equal to \( \langle g \rangle \ltimes E(G) \), and strongly asymmetric when \( E(g) \) is actually the product \( \langle g \rangle \times E(G) \). Strongly asymmetric elements are sometimes called special, though we do not use this terminology in this paper. We note that strongly asymmetric elements always exist by [DGO11, Lemma 6.18].

**Example 3.** Let \( S \) be a closed genus 2 surface, and let \( \tilde{S} \to S \) be a degree 2 cover, so \( \tilde{S} \) is a closed genus 3 surface. Let \( G \) be the mapping class group of \( \tilde{S} \), which has trivial maximal finite normal subgroup, and let \( T \cong \mathbb{Z}/2\mathbb{Z} \) be the subgroup of \( G \) consisting of covering transformations. Any pseudo-Anosov map \( g: S \to S \) has a power which lifts to a map \( \tilde{g}: \tilde{S} \to \tilde{S} \), which commutes with \( T \), so \( \langle \tilde{g} \rangle \times T < E(\tilde{g}) \), and so \( \tilde{g} \) is not weakly asymmetric.

We now describe the particular model of random subgroups which we shall consider. A random subgroup of \( G \) with \( k \) generators is a subgroup whose generators are chosen to be independent random walks of length \( n_i \) on \( G \). We will require the following restrictions on the probability distributions \( \mu \) generating the random walks. We say a probability distribution \( \mu \) on \( G \) is non-elementary if the group generated by its support is non-elementary.

**Definition 4.** Let \( G \) act acylindrically hyperbolically on \( X \). We say that the probability distribution \( \mu \) on \( G \) is \((G \curvearrowright X)\)-admissible if the support of \( \mu \) generates a non-elementary subgroup of \( G \) containing a weakly asymmetric element, and furthermore, the support of \( \mu \) has bounded image in \( X \).

The set of admissible measures depends on the action of \( G \) on \( X \), though we shall suppress this from our notation and just write admissible for \((G \curvearrowright X)\)-admissible. We shall write \( \tilde{\mu} \) for the reflected probability distribution \( \tilde{\mu}(g) = \mu(g^{-1}) \), and \( \tilde{\mu} \) is admissible if and only if \( \mu \) is admissible.

We may now give a precise definition of our model for random subgroups. Let \( \mu_1, \ldots, \mu_k \) be a finite collection of admissible probability distributions on \( G \). We shall write \( H(\mu_1, \ldots, \mu_k, n_1, \ldots, n_k) \) to denote the subgroup generated by \( \langle w_{1,n_1}, \ldots, w_{k,n_k} \rangle \), where each \( w_{i,n_i} \) is a group element arising from a random walk on \( G \) of length \( n_i \) generated by \( \mu_i \). To simplify notation we shall often just write \( H(\mu_i, n_i) \), or just \( H \), for \( H(\mu_1, \ldots, \mu_k, n_1, \ldots, n_k) \). We may now state our main result.

**Theorem 5.** Let \( G \) be a countable group acting acylindrically hyperbolically on the separable space \( X \), and let \( E(G) \) be the maximal finite normal subgroup of \( G \). Let \( H(\mu_1, \ldots, \mu_k, n_1, \ldots, n_k) \) be a random subgroup of \( G \), where the \( \mu_i \) are admissible probability distributions on \( G \). Then the probability that each of the following three events occurs tends to one as \( \min n_i \).
tends to infinity.

1. All of the $w_{i,n}$ are hyperbolic and weakly asymmetric.

2. The subgroup $H$ is freely generated by the $\{w_{i,n}\}$ and quasi-isometrically embedded in $X$, and so in particular $HE(G)$ is a semidirect product $H \ltimes E(G)$.

3. The subgroup $H \ltimes E(G)$ is geometrically separated in $X$.

In general the semidirect product $H \ltimes E(G)$ need not be the product $H \times E(G)$ for random subgroups $H$, as shown below.

Example 6. Let $G$ be a group acting acylindrically hyperbolically on $X$ with trivial maximal finite normal subgroup $E(G)$, which admits a split extension

$$1 \to F \to G^+ \to G \to 1,$$

which is not a product, for some finite group $F$. Such a split extension is determined by a homomorphism $\phi: G \to \text{Aut}(F)$, where $\text{Aut}(F)$ is the automorphism group of $F$. The maximal finite normal subgroup $E(G)$ is equal to $F$. A random walk on $G^+$ pushes forward to a random walk on $G$, and then to a random walk on $\phi(G) < \text{Aut}(F)$. As $\phi(G)$ is finite, the random walk is asymptotically uniformly distributed. A hyperbolic group element $g$ has $E(g) = \langle g \rangle \times F$ if and only if the image of $g$ in $\phi(G)$ is trivial, which happens with asymptotic probability $1/|\phi(G)|$.

In the next section, Section 2.1, we recall the definition of a hyperbolically embedded subgroup, and show how Theorem 1 follows from Theorem 5.

### 2.1 Hyperbolically embedded subgroups

Osin [Osi16] showed that if a group is acylindrically hyperbolic, then there is a (not necessarily finite) generating set $Y$, such that the Cayley graph of $G$ with respect to $Y$, which we shall denote $\text{Cay}(G,Y)$, is hyperbolic, and the action of $G$ on $\text{Cay}(G,Y)$ is acylindrical and non-elementary. In general, there are many choices of $Y$ giving non-quasi-isometric acylindrically hyperbolic actions, for which different collections of subgroups will be hyperbolically embedded, but for the remainder of this section we shall assume we have chosen some fixed $Y$.

Let $H$ be a subgroup of $G$; we will write $\text{Cay}(G,Y \sqcup H)$ for the Cayley graph of $G$ with respect to the disjoint union of $Y$ and $H$ (so it might have double edges). The Cayley graph $\text{Cay}(H,H)$ is a complete subgraph of $\text{Cay}(G,Y \sqcup H)$. We say a path $p$ in $\text{Cay}(G,Y \sqcup H)$ is admissible if it does not contain edges of $\text{Cay}(H,H)$, though it may contain edges of non-trivial cosets of $H$, and may pass through vertices of $H$. We define a
restriction metric \( \hat{d}_H \) on \( H \) be setting \( \hat{d}_H(h_1, h_2) \) to be the minimal length of any admissible path in \( \text{Cay}(G, Y \sqcup H) \) connecting \( h_1 \) and \( h_2 \). If no such path exists we set \( \hat{d}_H(h_1, h_2) = \infty \).

We say a finitely generated subgroup \( H \) of a finitely generated acylindrically hyperbolic group \( G \) is hyperbolically embedded in \( G \) with respect to a generating set \( Y \subset G \) if the Cayley graph \( \text{Cay}(G, Y \sqcup H) \) is hyperbolic, and \( \hat{d}_H \) is proper. We shall denote this by \( H \hookrightarrow_{h} (G, Y) \).

We shall use the following sufficient conditions for a subgroup to be hyperbolically embedded, due to Hull [Hul13, Theorem 3.16] and Antolin, Minasyan and Sisto [AMS16, Theorem 3.9, Corollary 3.10] (both refinements of Dahmani, Guirardel and Osin [DGOT11, Theorem 4.42]), which we now describe.

**Theorem 7.** [Hul13, AMS16] Suppose that \( G \) acts acylindrically hyperbolically on \( X \). Let \( H \) be a finitely generated subgroup of \( G \), which is quasi-isometrically embedded and geometrically separated in \( X \). Then \( H \) is hyperbolically embedded in \( G \). Moreover, if \( X = \text{Cay}(G, Y) \) for some \( Y \subset G \), then \( H \hookrightarrow_{h} (G, Y) \).

Theorem 5 now follows immediately from Theorem 5 and Theorem 7, choosing \( X \) to be \( \text{Cay}(G, Y) \). In fact, we have the following refinement, which we record for future reference:

**Theorem 8.** Let the finitely generated group \( G \) act acylindrically hyperbolically on its Cayley graph \( \text{Cay}(G, Y) \), and let \( E(G) \) be the maximal finite normal subgroup of \( G \). Let \( H(\mu_1, \ldots, \mu_k, n_1, \ldots, n_k) \) be a random subgroup of \( G \), where the \( \mu_i \) are admissible probability distributions on \( G \). Then the probability that each of the following events occurs tends to one as \( \min n_i \) tends to infinity.

1. The subgroup \( H \) is freely generated by the \( \{w_{i, a_i}\} \), and in particular \( H \rtimes E(G) \) is a semidirect product \( H \rtimes E(G) \).

2. \( H \ltimes E(G) \hookrightarrow_{h} (G, Y) \).

### 2.2 Outline

We conclude this section with a brief outline of the rest of the paper, using the notation of Theorem 5. In the final part of this section, Section 2.3 we recall some basic concepts and define some notation. In Section 3 we review some estimates for the behaviour of random walks. In Section 3.1 we review some exponential decay estimates that we will use, including the fact that a random walk makes linear progress in \( X \), with the probability of a linearly large deviation tending to zero exponentially quickly, an estimate for the Gromov product of the initial and final point of a.
random walk, based at an intermediate point, and the property that the hitting measure of a shadow set in $X$ decays exponentially in its distance from the basepoint.

In Section 3.2 we review some matching estimates, which we now describe. We say two geodesics $\alpha$ and $\beta$ in $X$ have an $(A, B)$-match, if there is a subgeodesic of one of length $A$ which has a translate by some element of $G$ which $B$-fellow travels the other one; we say that a single geodesic $\gamma$ has an $(A, B)$-match if there is a subgeodesic of $\gamma$ of length $A$ which $B$-fellow travels a disjoint subgeodesic. If the constant $B$ can be chosen to only depend on $\delta$, the constant of hyperbolicity, then we may refer to an $(A, B)$-match as a match of length $A$. Let $\gamma_n$ be a geodesic in $X$ from $x_0$ to $w_nx_0$. For any geodesic $\eta$, the probability that $\eta$ has a match of length $|\eta|$ with $\gamma_n$ decays exponentially in $|\eta|$, and this can be used to show that the probability that $\gamma_n$ has a match of linear length (with itself) tends to zero as $n$ tends to infinity. We will also use the facts that if $\gamma_\omega$ is the bi-infinite geodesic determined by a bi-infinite random walk, and $\alpha_n$ is an axis for $w_n$, assuming $w_n$ is hyperbolic, then the probability that $\gamma_n, \gamma_\omega$ and $\alpha_n$ have matches of a size which is linear in $n$, tends to 1 as $n$ tends to infinity. Finally, we also use the fact that for any group element $g$ in the support of $\mu$ with axis $\gamma_g$, ergodicity implies that the bi-infinite geodesic $\gamma_\omega$ has infinitely many matches with $\gamma_g$ of arbitrarily large length.

In Section 4 we recall some standard results about free subgroups of a group $G$ acting by isometries on a Gromov hyperbolic space $X$. In particular, as shown by, e.g., Taylor and Tiozzo [TT16], if a subgroup $H$ has a symmetric generating set $A = \{a_1, a_1^{-1}, \ldots, a_k, a_k^{-1}\}$, for which the distances $d_X(x_0, ax_0)$ are large, for all $a$ in $A$, and the Gromov products $(ax_0 \cdot bx_0)x_0$ are small, for all distinct $a$ and $b$ in $A$, then $\{a_1, \ldots, a_k\}$ freely generates a free group $H$, which is quasi-isometrically embedded in $X$. We show that furthermore, if $\Gamma_H$ is a rescaled copy of the Cayley graph, in which an edge corresponding to $a \in A$ has length $d_X(x_0, ax_0)$, then $\Gamma_H$ is quasi-isometrically embedded in $H$, with quasi-isometry constants depending only on $\delta$ and the size of the largest Gromov product $(ax_0 \cdot bx_0)x_0$, and not on the lengths of the geodesics $[x_0, ax_0]$, for $a \in A$.

In Section 5 we prove a version of Theorem 5 in the case that the group has a single generator, i.e. $k = 1$, which is equivalent to showing that the probability that $w_n$ is hyperbolic and weakly asymmetric tends to one with asymptotic probability one. In Section 5.1 we define coarse analogues of the following properties of group elements: being primitive and being asymmetric, and we show that these conditions are sufficient to show that a group element is weakly asymmetric, as long as it is not conjugate to its inverse. Then in Section 5.2 we use the matching estimates to show
that the coarse analogues hold with asymptotic probability one, as does the property that \( w_n \) is not conjugate to its inverse.

A key step is to use the fact that the support of \( \mu \) contains a weakly asymmetric element, \( g \) say, with axis \( \alpha_g \). A result of Bestvina and Fujiwara \cite{BF02} says that if a group element \( h \) coarsely stabilizes a sufficiently long segment of \( \alpha_g \), then in fact \( h \) lies in \( E(g) \). Ergodicity implies that the bi-infinite geodesic \( \gamma_\omega \) fellow travels infinitely often with long segments of translates of \( \alpha_g \), and the matching estimates then imply that the axis \( \alpha_n \) for \( w_n \) also fellow travels with long segments of translates of \( \alpha_g \), with asymptotic probability one. Therefore an element \( h \in E(g) \) which coarsely fixes \( \alpha_n \) pointwise must also stabilize disjoint translates of long segments of \( \alpha_g \), say \( h_1 \alpha_g \) and \( h_2 \alpha_g \). This implies that \( h \) lies in \( (h_1(g)h_1^{-1} \ltimes (E(g)) \cap (h_2(g)h_2^{-1} \ltimes E(g)) = E(g) \), and so \( w_n \) is weakly asymmetric.

Finally, in Section 6 we extend this result to finitely generated random subgroups. Let \( w_{i,n} \) be the generators of \( H \), and let \( \gamma_{i,n} \) be a geodesic from \( x_0 \) to \( w_{i,n}x_0 \). The random walk corresponding to each generator makes linear progress, and pairs of independent random walks satisfy an exponential decay estimate for the size of their Gromov products based at the basepoint \( x_0 \), so this shows that \( H \) is asymptotically freely generated by the locations of the sample paths \( w_{i,n} \), and is quasi-isometrically embedded in \( X \). If \( H \) is not geometrically separated, then there is an arbitrarily large intersection of \( N_{R}(gH) \) and \( N_{R}(H) \), which implies that there is a pair of long geodesics \( \gamma \) and \( \gamma' \) with endpoints in \( H \) such that \( g\gamma' \) fellow travels with \( \gamma \), for \( g \in G \setminus (H \ltimes E(G)) \). The probability that \( \gamma_{i,n} \) matches any combination of shorter generators tends to zero, so for some \( i \), the group element \( g \) takes some translate \( h_1\gamma_{i,n} \) say, to another translate \( h_2\gamma_{i,n} \). This implies that \( h_2^{-1}gh_1 \) coarsely stabilizes \( \gamma_{i,n} \), and so \( g \) lies in \( h_2((w_{i,n}) \ltimes E(G))h_1 \subset H \ltimes E(G) \), by the fact that each individual random walks gives weakly asymmetric elements with asymptotic probability one. This contradicts our initial assumption that \( g \) did not lie in \( H \ltimes E(G) \).

2.3 Notation and standing assumptions

Throughout the paper we fix a group \( G \) acting acylindrically hyperbolically on a separable hyperbolic space \( X \). We will always assume that the hyperbolic space \( X \) is geodesic, but it need not be locally compact. We denote the distance in \( X \) by \( d_X \), and \( \delta \) will refer to the constant of hyperbolicity for the Gromov hyperbolic space \( X \). We will write \( O(\delta) \) to refer to a constant which only depends on \( \delta \), through not necessarily linearly. We shall write \( |\gamma| \) for the length of a path \( \gamma \). If \( \gamma \) is a geodesic, then \( |\gamma| \) is
equal to the distance between its endpoints. Geodesics will always have unit speed parameterizations, and $\gamma(t)$ will denote a point on $\gamma$ distance $t$ from its initial point, and we will write $[\gamma(t), \gamma(t')]$ for the subgeodesic of $\gamma$ from $\gamma(t)$ to $\gamma(t')$.

In all statements about a single random walk on $G$, we will assume that the random walk is generated by an admissible probability measure $\mu$ and denote the position of the walk at time $n$ by $w_n$, while the corresponding notations for multiple random walks will be $\mu_i$ for the admissible measures and $w_{i,n}$ for the locations of the random walks.

If we say that a constant $A$ depends on an admissible probability measure $\mu$, then as the set of admissible measures depends on the action of $G$ on $X$, we allow that $A$ may also depend on the action, and also on the constant of hyperbolicity $\delta$, and the acylindricity functions $R(K)$ and $N(K)$. If we say that a constant $A$ depends on the collection of probability distributions $\mu_1, \ldots, \mu_k$ corresponding to a random subgroup $H$, this includes the possibility that the constant may depend on the number $k$ of probability distributions. We will occasionally recall some of these assumptions and notations.

3 Estimates for random walks

3.1 Exponential decay

Let $\mu$ be an admissible probability distribution on $G$. We will make use of the following exponential decay estimates, as shown by Maher and Tiozzo \cite{MT14} and Mathieu and Sisto \cite{MS14}. We denote the Gromov product by $(x \cdot y)_w$, which by definition is

$$(x \cdot y)_w = \frac{1}{2}(d_X(w, x) + d_X(w, y) - d_X(x, y)).$$

Furthermore, as the distance the sample path has moved in $X$ is subadditive, the limit

$$L = \lim_{n \to \infty} \frac{1}{n} d_X(x_0, w_n x_0)$$

exists almost surely, and $L$ is the same for almost all sample paths, by ergodicity. If $\mu$ is admissible then $L$ is positive, and we say that the random walk has positive drift, or makes linear progress.

Given $x_0, x \in X$ and $R > 0$, the shadow $S_{x_0}(x, R)$ is defined to be

$$S_{x_0}(x, R) = \{ y \in X : (x \cdot y)_{x_0} \geq d_X(x_0, x) - R \}.$$

**Proposition 9.** Let $G$ be a countable group which acts acylindrically hyperbolically on a separable space $X$ with basepoint $x_0$, and let $\mu$ be an admissible probability distribution on $G$. Then the following exponential decay estimates hold:
9.1 Positive drift in $X$ with exponential decay.
There is a positive drift constant $L > 0$ such that for any $\varepsilon > 0$ there are constants $K > 0$ and $c < 1$, depending on $\mu$ and $x_0$, such that

$$\mathbb{P}((1 - \varepsilon)Ln \leq d_X(x_0, w_n x_0) \leq (1 + \varepsilon)Ln) \geq 1 - Ke^n,$$  \hspace{1cm} (1)

for all $n$.

9.2 Exponential decay for Gromov products in $X$.
There are constants $K > 0$ and $c < 1$, depending on $\mu$ and $x_0$, such that for all $i, n$ and $r$,

$$\mathbb{P}((x_0 \cdot w_n x_0)_{x_0} \geq r) \leq Ke^r$$  \hspace{1cm} (2)

9.3 Exponential decay for shadows in $X$.
There is a constant $R_0 > 0$, which only depends on the action of $G$ on $X$, and constants $K > 0$ and $c < 1$, which depend on $\mu$ and $x_0$, such that for all $g$ and $R \geq R_0$,

$$\mathbb{P}(w_n \in S_{x_0}(gx_0, R)) \leq Ke^{d_X(gx_0 x_0) - R}.$$  \hspace{1cm} (3)

3.2 Matching

A match for a pair of geodesics in $X$, is a subsegment of one geodesic, which may be translated by an element of $G$ to fellow travel with a subsegment of the other one. We now give a precise definition.

We say that two geodesics $\gamma$ and $\gamma'$ in $X$ have an $(A, B)$-match if there are disjoint subgeodesics $\alpha \subset \gamma$ and $\alpha' \subset \gamma'$ of length at least $A$, and a group element $g \in G$, such that the Hausdorff distance between $\gamma\alpha$ and $\alpha'$ is at most $B$. We may choose $\gamma$ and $\gamma'$ to be the same geodesic, or overlapping geodesics. If $\gamma$ and $\gamma'$ are the same geodesic, then we will just say that $\gamma$ has an $(A, B)$-match.

Given $w_n$, a random walk of length $n$ on $G$, we shall write $\gamma_n$ for a geodesic in $X$ from $x_0$ to $w_n x_0$. As sample paths converge to the Gromov boundary $\partial X$ almost surely [MT14], a bi-infinite sample path $\{w_n x_0\}_{n \in \mathbb{Z}}$ determines a bi-infinite geodesic in $X$ almost surely, which we shall denote $\gamma_\omega$.

In an arbitrary non-locally compact Gromov hyperbolic space, pairs of points in the boundary need not be connected by bi-infinite geodesics, however, they are always connected by $(1, O(\delta))$-quasigeodesics. With a slight abuse of language, we will call any bi-infinite $(1, O(\delta))$-quasigeodesic connecting the limit points of $g$ an axis for the hyperbolic element $g$. 
Proposition 10. Let $G$ be a countable group which acts acylindrically hyperbolically on the separable space $X$, and let $\mu$ be an admissible probability distribution on $G$. Then there is a constant $K_0$, depending only on $\delta$, such that for any $K \geq K_0$, the following matching estimates hold.

1. There are constants $B$ and $c$, depending on $\mu$ and $K$, such that for any geodesic segment $\eta$ and any constant $t$, the probability that a translate of $\eta$ is contained in a $K$-neighbourhood of $[\gamma_n(t), \gamma_n(t+|\eta|)]$ is at most $Bc|\eta|$.

2. For any $\varepsilon > 0$ the probability that $\gamma_n$ has an $(\varepsilon|\gamma_n|, K)$-match tends to zero as $n$ tends to infinity.

3. For any $\varepsilon > 0$, the probability that the $\gamma_n$ contains a subsegment of length at least $(1-\varepsilon)|\gamma_n|$ which is contained in a $K$-neighbourhood of $\gamma_\omega$ tends to one as $n$ tends to infinity. In particular, the probability that $\gamma_n$ and $\gamma_\omega$ have an $((1-\varepsilon)|\gamma_n|, K)$-match tends to one as $n$ tends to infinity.

4. Let $g$ be a hyperbolic isometry with axis $\alpha_g$ which lies in the support of $\mu$. Then for any constants $0 < \varepsilon < \frac{1}{3}$ and $L \geq 0$, the probability that $\gamma_n^-$ and $\alpha_g$ have an $(L, K)$-match tends to one as $n$ tends to infinity, where $\gamma_n^-$ is the subgeodesic of $\gamma_n$ obtained by removing $\varepsilon|\gamma_n|$-neighbourhoods of its endpoints.

5. For any $\varepsilon > 0$, the probability that $w_n$ is hyperbolic with axis $\alpha_n$, and $\gamma_n$ and $\alpha_n$ have a $((1-\varepsilon)|\gamma_n|, K)$-match tends to one as $n$ tends to infinity.

Propositions 10.1, 10.2 and 10.3 are shown by Calegari and Maher [CM15]. Proposition 10.1 is not stated explicitly, but follows directly from the proof of [CM15, Lemma 5.26]. Proposition 10.5 is shown for $\mu$ with finite support by Dahmani and Horbez [DH15, Proposition 1.5]. However, they only need finite support to ensure linear progress with exponential decay, and exponential decay for shadows, and so their argument also works for $\mu$ with bounded support in $X$.

Finally, a version of Proposition 10.4 is shown for the mapping class group acting on Teichmüller space, for $\mu$ with finite support, by Gadre and Maher [GM16], and independently by Baik, Gekhtman and Hamenstädt [BGH16]. A significantly simpler version of these arguments works in the setting of acylindrically hyperbolic groups, but we present the details below for the convenience of the reader. As a side remark, we note that it can be shown that, in fact, the largest match of $\gamma_n$ and $\alpha_g$ has logarithmic size in $n$ [ST16].
Proof (of Proposition 10.4). Let \( g \) be a hyperbolic element which lies in the support of \( \mu \), and let \( \alpha_g \) be an axis for \( g \). Let \( \gamma_\omega \) be the bi-infinite geodesic determined by a bi-infinite random walk generated by \( \mu \). We shall write \( \nu \) for the harmonic measure on \( \partial X \), and \( \check{\nu} \) for the reflected harmonic measure, i.e the harmonic measure arising from the random walk generated by the probability distribution \( \check{\mu}(g) = \mu(g^{-1}) \).

By assumption, the group element \( g \) lies in the support of \( \mu \), and so the group element \( g^{-1} \) lies in the support of \( \check{\mu} \). Given a constant \( L > 0 \), there is an \( m \) sufficiently large such that any geodesic from \( S_{x_0}(g^m x_0, R_0) \) to \( S_{x_0}(g^{-m} x_0, R_0) \) has a subsegment of length \( L \) which \( K = O(\delta) \)-fellow travels with \( \alpha_g \). The following result of Maher and Tiozzo [MT14] guarantees that the harmonic measures of these shadow sets are positive.

**Proposition 11.** [MT14 Proposition 5.4] Let \( G \) be a countable group acting acylindrically hyperbolically on a separable space \( X \), and let \( \mu \) be a non-elementary probability distribution on \( G \). Then there is a number \( R_0 \) such that for any group element \( g \) in the semigroup generated by the support of \( \mu \), the closure of the shadow \( S_{x_0}(g^m x_0, R_0) \) has positive hitting measure for the random walk determined by \( \mu \).

Therefore \( \nu(S_{x_0}(g^m x_0, R_0)) > 0 \) and \( \check{\nu}(S_{x_0}(g^{-m} x_0, R_0)) > 0 \), and so there is a positive probability \( p \) say that \( \gamma_\omega \) has a subsegment of length at least \( L \) which lies in a \( K \)-neighbourhood of \( \gamma_g \). Ergodicity now implies that the proportion of times in \( \{\lfloor n \delta \rfloor, \ldots, \lfloor 2n \delta \rfloor\} \) for which \( \gamma_\omega \) has a subsegment of length at least \( L \) which lies in a \( K \)-neighbourhood of \( w_n \gamma_g \) tends to \( p \) as \( n \) tends to infinity, for almost all sample paths \( \omega \). Proposition 10.3 then implies that the probability that \( \gamma_n \) has an \((L, K)\)-match with \( \gamma_g \) tends to one. \( \square \)

4. **Schottky groups**

In this section we collect together some standard results about free subgroups of a group \( G \) acting by isometries on a hyperbolic space \( X \), see for example Bridson and Haefliger [BH99] for a thorough discussion. For completeness, we present a mild generalization of an argument due to Taylor and Tiozzo [TT16], and show that one may rescale the Cayley graph \( \Gamma \) of a Schottky group so that the quasi-isometric embedding constants of \( \Gamma \) into \( X \) depend only on \( \delta \), the constant of hyperbolicity for \( X \), and the size of the Gromov products between the generators.

A relation \( g = g_1 g_2 \ldots g_n \) between elements of \( G \) may be thought of as a recipe for assembling a path from \( x_0 \) to \( g x_0 \) as a concatenation of translates of paths from \( x_0 \) to \( g_i x_0 \). The following proposition gives an
estimate for the distance of the endpoints of the total path in terms of the lengths of the shorter segments, and the Gromov products between adjacent segments.

Let $\eta$ be a path which is a concatenation of $k$ geodesic segments $\{\eta_i\}_{i=1}^k$, and label the endpoints of $\eta_i$ as $x_{i-1}$ and $x_i$, such that the common endpoint of $\eta_i$ and $\eta_{i+1}$ is labelled $x_i$. For $2 \leq i \leq k$, let $p_i$ be the nearest point projection of $x_{i-2}$ to $\eta_i$, and for $1 \leq i \leq k-1$, let $q_i$ be the nearest point projection of $x_{i+1}$ to $\eta_i$. We define $p_1 = x_0$ and $q_k = x_{k+1}$. We will call the subsegment $[p_i, q_i] \subset \eta$ the persistent subgeodesic of $\eta_i$. This is illustrated below in Figure 1.

![Figure 1: A concatenation of geodesic segments.](image)

The length of the persistent subgeodesic may be estimated in terms of Gromov products.

**Proposition 12.** There is a constant $C$, which only depends on $\delta$ such that if $\eta$ is a concatenation of geodesic segments $\eta_i$, with persistent subgeodesics $[p_i, q_i]$, then

\[
\begin{align*}
 d_X(p_i, q_i) &\leq d_X(x_{i-1}, x_i) - (x_{i-2} \cdot x_i)_{x_{i-1}} - (x_{i-1} \cdot x_{i+1})_{x_i} + C, \\
 d_X(p_i, q_i) &\geq d_X(x_{i-1}, x_i) - (x_{i-2} \cdot x_i)_{x_{i-1}} - (x_{i-1} \cdot x_{i+1})_{x_i} - C.
\end{align*}
\]

We omit the proof of Proposition 12 which is a straightforward application of thin triangles and the definition of the Gromov product.

We now show that if each persistent subsegment is sufficiently long, then the distance between $x_0$ and $x_k$ is equal to the sum of the lengths of the persistent subsegments, up to an additive error proportional to the number of geodesic segments.
Proposition 13. There exists a constant $C > 0$, which depends only on \( \delta \), such that if \( \eta \) is a concatenation of geodesic segments \( \eta_i \) for \( 1 \leq i \leq k \), with persistent subgeodesics \( [p_i, q_i] \) with
\[
d_X(p_i, q_i) \geq C,
\]
for all \( 1 \leq i \leq k \), then
\[
\sum_{i=1}^{k} d_X(p_i, q_i) - 2Ck \leq d_X(x_0, x_k) \leq \sum_{i=1}^{k} d_X(p_i, q_i) + 2Ck.
\]
Furthermore, any geodesic from \( x_0 \) to \( x_k \) passes within distance \( C \) of both \( p_i \) and \( q_i \).

Proof. For any three points \( x, y \) and \( z \), determining a triangle in \( X \), there is a point \( m \), known as the center of triangle, such that \( m \) is distance at most \( \delta \) from each of the three sides of the triangle. Furthermore, there is a \( C_1 \), which only depends on \( \delta \), such that if \( p \) is a closest point on \([y, z]\) to \( x \), then \( d_X(p, m) \leq C_1 \). This implies that \( d_X(q_i, p_{i+1}) \leq 2C_1 \), using the triangle with vertices \( x_{i-1}, x_i \) and \( x_{i+1} \). The upper bound
\[
d_X(x_0, x_k) \leq d_X(q_i, p_{i+1}) + 2C_1k
\]
then follows from the triangle inequality.

There are constants \( C_2 \) and \( C_3 \), which only depend on \( \delta \), such that for any point \( y \) in \( X \), whose nearest point projection to \( \eta_{i-1} \) is distance at least \( C_2 \) away from \( \eta_i \), the nearest point projection of \( y \) to \( \eta_i \) is distance at most \( C_3 \) from \( p_i \). As this also holds for \( \eta_{i-1} \), the nearest point projection of \( x_{i-3} \) to \( \eta_{i-1} \) is within distance \( C_3 \) of \( p_{i-1} \), and so the nearest point projection of \( x_{i-3} \) to \( \eta_i \) is within distance \( C_3 \) of \( p_i \). By induction, the nearest point projection of \( x_0 \) to \( \eta_i \) is within distance \( C_3 \) of \( p_i \), and similarly, the nearest point projection of \( x_k \) to \( \eta_i \) is within distance \( C_3 \) of \( q_i \).

There are constants \( C_4 \) and \( C_5 \), depending only on \( \delta \), such that if two points \( x \) and \( y \) in \( X \) have nearest point projections \( p \) and \( q \) onto a geodesic \( \alpha \), and \( d_X(p, q) \geq C_4 \), then any geodesic from \( x \) to \( y \) passes within distance \( C_5 \) of both \( p \) and \( q \).

In particular, there exists \( C_6 \) depending only on \( \delta \) so that if \( \gamma \) is a geodesic from \( x_0 \) to \( x_k \), then for each \( i \) there is a subsegment of \( \gamma \) of length at least \( d_X(p_i, q_i) - C_6 \) which is contained in a \( C_5 + 4\delta \)-neighbourhood of the persistent subsegment \( [p_i, q_i] \), and is disjoint from \( C_5 + 4\delta \)-neighbourhoods of the other persistent subsegments \([p_j, q_j] \) for \( i \neq j \). Therefore
\[
d_X(x_0, x_k) \geq \sum_{i=1}^{k} (d_X(p_i, q_i) - C_6),
\]
giving the required lower bound. \( \square \)
We now use Proposition 13 to give a lower bound on the translation length of group elements.

**Proposition 14.** There exists a constant $C > 0$, which depends only on $\delta$, such that if $g$ is an isometry of a hyperbolic space $X$ with basepoint $x_0$, which is a product of isometries $g = g_1 g_2 \ldots g_n$, where the $g_i$ satisfy the following collection of inequalities

$$
dx(x_0, g_i x_0) \geq (g_i^{-1} x_0 \cdot g_i x_0)_{x_0} + (g_i^{-1} x_0 \cdot g_{i+1} x_0)_{x_0} + C,$$

where $g_{n+1} = 1$ and $g_0 = g_n$, then the translation length of $g$ is at least

$$\tau(g) \geq \sum_{i=1}^{n} \left( \left( d_{X}(x_0, g_i x_0) - (g_i^{-1} x_0 \cdot g_i x_0)_{x_0} - (g_i^{-1} x_0 \cdot g_{i+1} x_0)_{x_0} - C \right) \right),$$

and furthermore, any geodesic from $x_0$ to $g x_0$, and any axis $\gamma$ for $g$ passes within distance $(g_i^{-1} x_0 \cdot g_i x_0)_{x_0} + (g_i^{-1} x_0 \cdot g_{i+1} x_0)_{x_0} + C$ of each $g_i \ldots g_i x_0$.

**Proof.** We first define a sequence of points $\{x_i\}_{i=0}^{n}$, and a sequence of geodesic segments $\{\eta_i\}_{i=1}^{n}$, following the index conventions of Proposition 13. Let $x_0$ be the basepoint of $X$, and for $1 \leq i \leq n$ let $x_i = g_1 \ldots g_i x_0$. For $1 \leq i \leq n$ let $\eta_i$ be a geodesic from $x_{i-1}$ to $x_i$, and let $\eta$ be the path formed from the concatenation of the geodesic segments $\eta_i$.

We may now consider the bi-infinite sequences obtained from all $g$-translates of the points $x_i$ and the geodesics $\eta_i$, labelled such that $x_{jn+i} = g^j x_i$ and $\eta_{jn+i} = g^j \eta_i$, for $j \in \mathbb{Z}$ and $1 \leq i \leq n$. The terminal point $x_n$ of $\eta_n$ is equal to $g_1 \ldots g_n x_0 = g x_0$, which is the same as the initial point of $\eta_{n+1} = g \eta_n = g x_0$, so the concatenation of the geodesics $\eta_i$ is a bi-infinite $g$-equivariant path in $X$, which we shall denote $\eta$.

If we choose $C \geq 3C_1$, where $C_1$ is the constant from Proposition 13, then the assumption 8, together with the estimate for persistent length in terms of Gromov products 4, implies that any subpath $\{\eta_i\}_{i=m}^{b}$ of $\eta$ satisfies the hypothesis of Proposition 13 and so the conclusion of Proposition 13 implies that $\tau(g) = \lim_{m \to \infty} \frac{1}{m} d_X(x_0, g^m x_0)$, is given by

$$\tau(g) = \lim_{m \to \infty} \frac{1}{m} d_X(x_0, g^{mn}) \geq C_1 n > 0.$$ 

Therefore the translation length $\tau(g)$ is positive, and so $g$ is hyperbolic, and $\eta$ is a quasi-axis for $g$. The estimate for translation length 8 then follows by combining 13 and 14, and the statements about the distance from $x_i$ to any geodesic $\gamma$ from $x_a$ to $x_b$, for $a \leq i \leq b$, and the distance from $x_i$ to any axis for $g$ follow from thin triangles and the definition of the Gromov product, for $C = 3C_1 + O(\delta)$.

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We now give conditions on the generators of a subgroup which ensure that the generators freely generate a subgroup which is quasi-isometrically embedded in $X$.

Given a symmetric generating set $A = \{a_1, a_1^{-1}, \ldots, a_k, a_k^{-1}\}$ generating a subgroup $H$ of $G$, let $\mathbb{F}_k$ be the free group generated by $A = \{a_1, \ldots, a_k\}$, and let $\Gamma_H$ be a rescaled copy of the Cayley graph for $\mathbb{F}_k$, with respect to the generating set $A$, where an edge in $\Gamma_H$ corresponding to a generator $a_i$ has length equal to $d_X(x_0, ax_0)$. We shall refer to $\Gamma_H$ as the rescaled Cayley graph for $\mathbb{F}_k$, which is quasi-isometric to the standard unscaled Cayley graph in which every edge has length one. The map from $\Gamma_H$ to $X$ which sends a vertex $h$ to $hx_0$, and an edge from $h$ to $h'$ to a geodesic from $hx_0$ to $h'x_0$ is continuous, can be made $H$-equivariant, and is an isometric embedding on each edge. The conditions we give below will in fact show that $\Gamma_H$ is quasi-isometrically embedded in $X$, with quasi-isometry constants independent of the lengths of the edges.

**Proposition 15.** There is a constant $K_0$, which only depends on $\delta$, such that for any $K \geq K_0$, if $H$ is a subgroup generated by the symmetric generating set $A = \{a_1, a_1^{-1}, \ldots, a_k, a_k^{-1}\}$, satisfying the following conditions,

\[
\begin{align*}
&d_X(x_0, ax_0) \geq 6K \quad \text{for all } a \in A \\
&(ax_0 \cdot bx_0)x_0 \leq K \quad \text{for all } a \neq b \in A
\end{align*}
\]

then $H$ is isomorphic to the free group $\mathbb{F}_k$, freely generated by the generating set $A = \{a_1, \ldots, a_k\}$, and furthermore, the subgroup $H$ is quasi-isometrically embedded in $X$, and the rescaled Cayley graph $\Gamma_H$ is $(6, O(\delta, K))$-quasi-isometrically embedded in $X$.

**Proof.** We shall choose $K_0 > 2C$, where $C$ is the constant from Proposition 14. Let $g = g_1 \ldots g_n$ be a reduced word in the generating set $A$. The $g_i$ satisfy the Gromov product inequalities from (7). Proposition 14 then implies that $\tau(g) \geq Cn$, so in particular all reduced words are non trivial, so $A$ freely generates a free group.

We now show that $\Gamma_H$ is quasi-isometrically embedded in $X$, for quasi-isometry constants that are independent of the lengths of the $g_i$. The translation length $\tau(g)$ is a lower bound for $d_X(x_0, gx_0)$, and as $g_1 \ldots g_n$ is a reduced word

\[
d_{\Gamma_H}(x_0, gx_0) = \sum_{i=1}^{n} d_X(x_0, g_i x_0).
\]

Therefore conclusion (8) of Proposition 14 implies the left hand bound below

\[
d_{\Gamma_H}(x_0, gx_0) - 5Kk \leq d_X(x_0, gx_0) \leq d_{\Gamma_H}(x_0, gx_0),
\]
where $K \geq K_0$, for the choice of $K_0$ given above. The right hand bound follows immediately from the triangle inequality. As each geodesic segment of $\Gamma_H$ has length at least $6K$, this implies
\[ \frac{1}{6}d_{\Gamma_H}(x_0, gx_0) \leq d_X(x_0, gx_0) \leq d_{\Gamma_H}(x_0, gx_0). \]
Finally, using thin triangles, we may extend this estimate to all points $x$ and $y$ in $\Gamma_H$ to obtain
\[ \frac{1}{6}d_{\Gamma_H}(x, y) - 2K + O(\delta) \leq d_{\Gamma_H}(x_0, gx_0) + 2K + O(\delta). \]
as required.

5 Special case: one generator

The probability that $w_n$ is hyperbolic tends to one, so in particular, if $X = \text{Cay}(G, Y)$, then the probability that $E(w_n)$ is hyperbolically embedded in $(G, Y)$ tends to one. In this section we show that the probability that $w_n$ is weakly asymmetric tends to one, i.e. the probability that $E(w_n) = \langle w_n \rangle \rtimes E(G)$ tends to one, and this is precisely the special case of Theorem 5 when $k = 1$.

**Proposition 16.** Let $G$ be a countable group acting acylindrically hyperbolically on the separable space $X$, and let $\mu$ be an admissible probability distribution on $G$. Then the probability that $w_n$ is hyperbolic and weakly asymmetric tends to one as $n$ tends to infinity.

We start in Section 5.1 by giving some geometric conditions which are sufficient to show that a group element is weakly asymmetric. In Section 5.2 we show that the probability that these conditions are satisfied by a random element $w_n$ tends to one as $n$ tends to infinity.

5.1 Asymmetric elements

Let $g$ be a hyperbolic isometry. Recall that a group element $g \in G$ is primitive if there is no element $h \in G$ such that $h^n = g$ for $n > 1$. We now define a notion of coarse primitivity for group elements.

**Definition 17.** Let $\gamma$ be an axis for $g$, let $p_i$ be the projection of $g^i x_0$ to $\gamma$, and set $P = \bigcup_{i \in \mathbb{Z}} p_i$. We say that $g$ is $K$-primitive if any element $h \in E(g)$ $K$-stabilizes $P$, i.e. the Hausdorff distance $d_{\text{Haus}}(P, hP) \leq K$.

If $g$ is $K$-primitive, then $g$ is primitive, for $K = O(\delta)$ sufficiently large, and if the translation distance $\tau(g)$ satisfies $\tau(g) > K + O(\delta)$.

Recall that for a hyperbolic element $g$, $\Lambda(g) = \{\lambda_+(g), \lambda_-(g)\}$ is the set consisting of the pair of attracting and repelling fixed points for $g$ in $\partial X$,
and $E(g) = \text{stab}(\Lambda(g))$. We shall write $E^+(g)$ for the subgroup of $E(g)$ which preserves $\Lambda(g)$ pointwise, i.e. $E^+(g) = \text{stab}(\Lambda_1(g)) \cap \text{stab}(\Lambda_2(g))$. This subgroup has index at most 2 in $E(g)$.

**Definition 18.** We say a hyperbolic isometry $g$ is reversible if there is an element in $E(g)$ which switches the fixed points of $g$, i.e. $E^+(g) \subseteq E(g)$. Otherwise $g$ is irreversible and $E^+(g) = E(g)$.

We say that the $K$-stabilizer of a geodesic $\gamma = [p, q]$, consists of all group elements $g$ such that if $d_X(p, gp) \leq K$ and $d_X(q, hq) \leq K$.

**Definition 19.** Let $G$ be a countable group acting acylindrically hyperbolically on a separable space $X$. We say that a group element $g$ is $K$-asymmetric if $g$ is hyperbolic with axis $\alpha_g$, and if $p$ is a closest point on $\alpha_g$ to the basepoint $x_0$, then the $K$-stabilizer for the geodesic $[p, gp]$ is equal to $E(G)$.

We first show that every non-elementary subgroup $H$ of $G$ containing a weakly asymmetric element, also contains a $K$-asymmetric element.

**Proposition 20.** Let $G$ be a countable group acting acylindrically hyperbolically on a separable space $X$, and let $H$ be a non-elementary subgroup $H$ of $G$, which contains a weakly asymmetric element. Then for any constant $K \geq 0$, the subgroup $H$ contains a $K$-asymmetric element $g$.

We will use the following lemma, which follows from work of Bestvina and Fujiwara.

**Lemma 21.** [BF02, Proposition 6] Let $g$ be a hyperbolic isometry with axis $\alpha_g$. Then for any number $K \geq 0$ there is a $D$, depending on $g, \delta$ and $K$, such that if $h$ $K$-coarsely stabilizes a segment of $\alpha_g$ of length at least $D$, then $h$ lies in $E(g)$.

**Proof (of Proposition 20):** Let $h$ be a weakly asymmetric element in $H$, with axis $\gamma_h$. As $H$ is non-elementary, and $E(h)$ is virtually cyclic, there is a hyperbolic element $f$ in $H$ which does not lie in $E(h)$. In particular, $h$ and $f$ are independent, i.e. their fixed point sets in $\partial X$ are disjoint. Consider the group element $g = h^af^b$. For all $a$ and $b$ sufficiently large, the translation lengths of $h^a$ and $f^b$ are much larger than twice any of the Gromov products between distinct elements of $\{h^{\pm a}, f^{\pm b}\}$, so we may apply Proposition 14, which in particular implies that $g$ is hyperbolic. Furthermore, for any constant $D > 0$, there is an $a$ sufficiently large such that the axis $\gamma_g$ of $g$ has a subsegment $\gamma_1$ of length at least $D$ which is contained in an $O(\delta)$-neighbourhood of $\gamma_h$, and a disjoint subsegment $\gamma_2$ of length at least $D$ which is contained in an $O(\delta)$-neighbourhood of $h^a f^b \gamma_h$. We shall choose an $a$ sufficiently large such that this holds for $D > D_h + O(K, \delta)$, where $D_h$ is the constant from Lemma 21 applied to the hyperbolic element $h$ with constant $K + O(\delta)$. Finally, we may choose
Let \( p \) be a closest point on \( \gamma_g \) to the basepoint \( x_0 \). If an element \( g' \) in \( G \) \( K \)-coarsely stabilizes \([p, gp]\), then \( g' (K + O(\delta)) \)-stabilizes \( \gamma_1 \) and \( \gamma_2 \). The segments \( \gamma_1 \) and \( \gamma_2 \) fellow travel axes of two distinct translates of \( \gamma_h \), say \( u_1 \gamma_h \) and \( u_2 \gamma_h \), and so \( g' (K + O(\delta)) \)-stabilizes segments of these axes of length at least \( D_h \). Therefore by Lemma \ref{lem:hyperbolic_and_irreversible_implies_k-irreversible} \( g' \) lies in
\[
E(u_1 h u_1^{-1}) \cap E(u_2 h u_2^{-1}),
\]
which is equal to
\[
(u_1(h) u_1^{-1} \rtimes E(G)) \cap (u_2(h) u_2^{-1} \rtimes E(G)),
\]
as \( h \) is weakly asymmetric. Hyperbolic elements in each of these subgroups have distinct fixed points in \( \partial X \), and so cannot be equal. The set of non-hyperbolic elements is equal to \( E(G) \), therefore the intersection of the two subgroups is exactly \( E(G) \), and so \( g' \in E(G), \) as required.

Finally, we show that these geometric conditions are sufficient to show that a group element \( g \) is weakly asymmetric.

**Proposition 22.** Let \( G \) be a countable group acting acylindrically hyperbolically on the separable space \( X \). Then there is a constant \( K \), depending only on \( \delta \), such that if \( g \) is an element which is hyperbolic, \( K \)-primitive, \( K \)-asymmetric and irreversible, then \( g \) is weakly asymmetric.
Proof. Let $g$ be a group element in $G$ which is hyperbolic, irreversible, $K$-primitive and $K$-asymmetric, and let $h$ be an element of $E(g)$. Let $\alpha_g$ be an axis for $g$, and let $p$ be a closest point on $\alpha_g$ to the basepoint $x_0$. As $g$ is $K$-primitive, we may multiply by a power of $g$, so that $g^n h$ is $K$-coarsely fixes $[p,gp]$. As $g$ is $K$-asymmetric, this implies that $g^n h$ lies in $E(G)$, and so $h$ lies in $(g)E(G)$. Finally, as $g$ is hyperbolic, $(g)E(G)$ is a semidirect product $(g) \rtimes E(G)$, by Proposition 2.

5.2 Random elements are asymmetric

In this section we show that the geometric properties defined in the previous section hold for random elements $w_n$ with asymptotic probability one.

We start by showing that the translation length $\tau(w_n)$ also grows linearly, using Proposition 13.

Lemma 23. Let $G$ be a countable group acting acylindrically hyperbolically on the separable space $X$, and let $\mu$ be an admissible probability distribution on $G$. For any $0 < \varepsilon < 1$ the probability that $\tau(w_n) \geq (1-\varepsilon) |\gamma_n|$ goes to 1.

Notice that, in the notation of the lemma, $|\gamma_n| \geq \tau(w_n)$ always holds.

Proof. We shall apply Proposition 13 with $g = w_n$, considered as a product of $g_1 = w_m$ and $g_2 = w_m^{-1} w_n$, where $m = \lceil n/2 \rceil$. Recall that $w_n = s_1 \ldots s_n$, where the $s_i$ are the steps of the random walk, and are independent $\mu$-distributed random variables.

By linear progress, Proposition 11 there exists $L > 0$ such that both $\mathbb{P}(d_X(x_0, w_n x_0) \geq Ln)$ and $\mathbb{P}(d_X(x_0, w_n^{-1} w_n x_0) \geq Ln)$ tend to one as $n$ tends to infinity (the $L$ here is smaller than the $L$ in Proposition 11).

By Proposition 12 the probability that the Gromov product $(w_m^{-1} x_0 \cdot w_m^{-1} w_n x_0)_{x_0} = (x_0 \cdot w_n x_0)_{w_n x_0}$ is bounded above by $\varepsilon Ln/5$ tends to one as $n$ tends to infinity. For the other Gromov product $((w_m^{-1} w_n)^{-1} x_0 \cdot w_m x_0)_{x_0}$, the two random variables $(w_m^{-1} w_n)^{-1} = s_n^{-1} \ldots s_{m+1}^{-1}$ and $w_m = s_1 \ldots s_m$ are independent, and so the distribution of

$$(\{w_m^{-1} w_n\}^{-1} x_0 \cdot w_m x_0)_{x_0} = (s_n^{-1} \ldots s_{m+1}^{-1} x_0 \cdot s_1 \ldots s_m x_0)_{x_0}$$

is the same as the distribution of

$$(s_{n-m}^{-1} x_0 \cdot s_{n-m+1} \ldots s_n x_0)_{x_0} = (x_0 \cdot w_n x_0)_{w_{n-m} x_0},$$

and so again by Proposition 12 the probability that this Gromov product is bounded above by $\varepsilon Ln/5$ tends to one as $n$ tends to infinity.

Therefore, the probability that the two inequalities (7) are satisfied tends to one as $n$ tends to infinity. Hence, by Proposition 14 for $n$
sufficiently large we have $\tau(w_n) \geq d_X(x_0, w_n x_0) + d_X(w_n x_0, w_n x_0) - \varepsilon L n \geq (1 - \varepsilon)|\gamma_n|$ with probability that tends to 1 as $n$ tends to infinity, as required.

We now show that the probability that $w_n$ is irreversible tends to one as $n$ tends to infinity.

**Proposition 24.** Let $G$ be a countable group acting acylindrically hyperbolically on the separable space $X$, and let $\mu$ be an admissible probability distribution on $G$. Then for any $K$, the probability that $w_n$ is irreversible tends to one as $n$ tends to infinity.

**Proof.** We can assume that $w_n$ is hyperbolic, with axis $\alpha_n$. Now suppose $h \in E(w_n)$ is an element which reverses the endpoints of $w_n$. Since $\alpha_n$ and $h \alpha_n$ are $O(\delta)$-fellow travelers, this gives a $\left(\frac{1}{3}\tau - O(\delta), O(\delta)\right)$-match for any subsegment of $\alpha_n$ of length $\tau(w_n)$.

Propositions 10.2 and 10.5 (in view of Lemma 23) then show that the probability that this occurs tends to zero as $n$ tends to infinity.

In fact, informally, if $\alpha_n$ had a match of size approximately $\frac{1}{3}\tau$, then by Proposition 10.5 the same would be true of $\gamma_n$, but this is ruled out by Proposition 10.2 since Lemma 23 says that $\tau(w_n)$ is approximately equal to $|\gamma_n|$.

We now show that random walks give $K$-primitive elements with asymptotic probability one.

**Proposition 25.** Let $G$ be a countable group acting acylindrically hyperbolically on the separable space $X$, and let $\mu$ be an admissible probability distribution on $G$. Then for any $K$, the probability that $w_n$ is $K$-primitive tends to one as $n$ tends to infinity.

**Proof.** Let $\alpha_g$ be an axis for a hyperbolic element with $\tau(g) > K + O(\delta)$, and suppose there is an element $h$ in $E(g)$ which does not $K$-stabilize $P$. Up to replacing $h$ with some $g^i h$, we can assume $d_X(p_0, h p_0) \leq \frac{1}{2}d_X(p_0, g p_0) + O(\delta)$. As $h$ moves $p_0$ distance at least $K$, $h$ is hyperbolic by applying Proposition 13 in the case where $n = 1$, $g = g_1 = h$ and the basepoint $x_0 = p_0$. Therefore, there is a power of $h$ such that

$$\frac{1}{3}d_X(p_0, g p_0) - O(\delta) \leq d_X(p_0, h^a p_0) \leq \frac{1}{3}d_X(p_0, g p_0) + O(\delta).$$

As $\alpha_g$ and $h^a \alpha_n$ are $O(\delta)$-fellow travelers, this gives a $\left(\frac{1}{3}\tau - O(\delta), O(\delta)\right)$-match for any subsegment of $\alpha_g$ of length $\tau(g)$. Proposition 10.5 then implies that the probability that $\gamma_n$ has a $\left(\frac{1}{3}\tau(g) - O(\delta), O(\delta)\right)$-match tends to one as $n$ tends to infinity, and the probability that this occurs tends to zero as $n$ tends to infinity, by Proposition 10.2.
We now show that the probability that $w_n$ is $K$-asymmetric tends to one as $n$ tends to infinity.

**Proposition 26.** Let $G$ be a countable group acting acylindrically hyperbolically on the separable space $X$, and let $\mu$ be an admissible probability distribution on $G$. Then for any constant $K \geq 0$ the probability that $w_n$ is $K$-asymmetric tends to one as $n$ tends to infinity.

**Proof.** By Proposition 25 the probability that $w_n$ is hyperbolic and $K$-primitive tends to one as $n$ tends to infinity. By Proposition 20 there is an element $h$ in the support of $\mu$ which is $(K + O(\delta))-\text{asymmetric}$. Let $\alpha_h$ be an axis for $h$, and let $p$ be a closest point on $\alpha_h$ to the basepoint $x_0$. Then Proposition 10.4 implies that the probability that $w_n$ is hyperbolic with axis $\alpha_n$, and $\alpha_n$ has a subsegment of length at least $2\tau(h)$ which $O(\delta)$-fellow travels with a translate of $\alpha_h$ tends to one as $n$ tends to infinity. If this happens, then if an element $g \in G$ $K$-stabilizes $[x_0, w_n x_0]$, then it also $(K + O(\delta))-\text{stabilizes}$ a translate of $[p, hp]$. As $h$ is $(K + O(\delta))-\text{asymmetric}$, this implies that $g \in E(G)$, so $w_n$ is $K$-asymmetric, as required. 

This completes the proof of Proposition 16: we have shown that all of the geometric hypotheses of Proposition 22 hold with asymptotic probability one, so Proposition 22 implies that $w_n$ is hyperbolic and weakly asymmetric with asymptotic probability one.

Although we have completed the proof of the special case of Theorem 5 in the case $k = 1$, we now conclude this section by showing a slightly stronger result, which we will need for the general case.

**Proposition 27.** Let $G$ be a countable group acting acylindrically hyperbolically on the separable space $X$, and let $\mu$ be an admissible probability distribution on $G$ with positive drift $L > 0$. Let $0 < \epsilon < \frac{1}{6}$. Then the probability that $w_n$ is $(\epsilon L n)$-asymmetric tends to 1 as $n$ tends to infinity.

**Proof.** Let $h$ be a hyperbolic element in the support of $\mu$ which is $K = O(\delta)$-asymmetric, with axis $\alpha_h$, and let $p$ be a closest point on $\alpha_h$ to the basepoint $x_0$.

The probability that $w_n$ is hyperbolic tends to one, so we may assume that $w_n$ is hyperbolic with axis $\alpha_n$. Let $q$ be a closest point on $\alpha_n$ to $x_0$, let $\gamma$ be a geodesic from $q$ to $w_n q$, and let $g$ be a group element which $(\epsilon L n)$-coarsely stabilizes $\gamma$. We have already shown the result for group elements $g$ which $K$-stabilize $\gamma$ for fixed $K$, so we may assume that $d_X(q, gq)$ and $d_X(w_n q, gw_n q)$ are both at least $K = O(\delta)$.

We now show that there is a subgeodesic $\gamma^-$ of $\gamma$ for which all points are moved a similar distance by $g$. Define $\gamma^-$ to be $\gamma \setminus (B_X(q, 2\epsilon L n) \cup B_X(w_n q, 2\epsilon L n))$. 

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Claim 28. For all $s$ and $t$ in $\gamma^-$,

$$|d_X(s, gs) - d_X(t, gt)| \leq O(\delta).$$

Proof. As $g$ is an isometry $d_X(s, t) = d_X(gs, gt)$. Let $u$ be a closest point on $g\gamma$ to $s$, and let $v$ be a closest point on $g\gamma$ to $t$, then $d_X(u, v) = d_X(s, t) + O(\delta)$. This implies that $d_X(u, gs) = d_X(v, gt) + O(\delta)$, and as $d_X(u, s) \leq 2\delta$ and $d_X(v, t) \leq 2\delta$, thus implies that $d_X(s, gs) = d_X(t, gt) + O(\delta)$, as required.

By Propositions 9.1 and 10.3 the length of $\gamma$ is at least $(1 - \varepsilon) L n$, and so the length of $\gamma^-$ is at least $(1 - 3\varepsilon) L n$. Therefore by Proposition 10.4 the probability that $\gamma^-$ has a subsegment of length at least $2\tau(h)$ which $O(\delta)$-fellow travels with $\gamma_h$ tends to 1 as $n$ tends to infinity. If $d_X(s, gs) \leq K = O(\delta)$ for $s \in \gamma^-$, then $g(K + O(\delta))$-stabilizes a translate of $[p, hp]$, and so $g \in E(G)$, which implies that $w_n$ is $K$-asymmetric, as required. Therefore the final step is to eliminate the case in which $d_X(s, gs) \geq K = O(\delta)$ for $s \in \gamma^-$, which we now consider.

Let $s$ be a point on $\gamma^-$, let $t$ be a nearest point to $gs$ on $\gamma$, and let $u$ be a nearest point on $\gamma$ to $gt$. This is illustrated below in Figure 4.

![Figure 3: Points on $\gamma^-$ are moved a similar distance.](image)

Figure 4: The image of $s$ under $g$ and $g^2$.

The distance from $gs$ to $t$ is at most $2\delta$, and the distance from $g^2 s$ to $u$ is at most $4\delta$. As $d_X(s, gs) \geq K = O(\delta)$, this gives an upper bound
on the Gromov product \((g^2 s)_{gs} = (g^{-1} s \cdot gs)_{s}\) of at most \(O(\delta)\), and so we may apply Proposition 15. Therefore, \(g\) is hyperbolic, and the axis \(\alpha_g\) for \(g\) passes within distance \(O(\delta)\) of \(t\). Furthermore, this holds for all \(t \in \gamma^-\), so the axis \(\alpha_g\) for \(g\) \(O(\delta)\)-fellow travels with \(\gamma^-\). The axis \(\alpha_g\) is \(\tau(g)\) periodic, and \(\tau(g) \leq \varepsilon Ln\), this means that \(\gamma^-\), and hence \(\gamma_n\) has an \((\varepsilon Ln + O(\delta), O(\delta))\)-match, which contradicts Proposition 10.2.

6 General case: many generators

We briefly recall the notation we use for a random subgroup \(H = H(\mu_i, n_i)\). The \(\mu_1, \ldots, \mu_k\) are admissible probability distributions on \(G\), and the \(n_1, \ldots, n_k\) are positive integers. We write \(w_{i,n_i}\) for a random walk of length \(n_i\) generated by the probability distribution \(\mu_i\), and \(\gamma_i\) for a geodesic in \(X\) from \(x_0\) to \(w_{i,n_i}x_0\). We shall write \(H\) for the subgroup generated by \(\{w_{1,n_1}, \ldots, w_{k,n_k}\}\), and set \(n = \min n_i\). Recall that the random walk generated by an admissible probability distribution \(\mu_i\) has positive drift, i.e. there is a constant \(L_i\) such that \(\frac{1}{n}d_X(x_0, w_{i,n_i}x_0) \to L_i\) as \(n \to \infty\), almost surely. We shall set \(L = \min L_i\), so in particular \(L \geq 0\), and we shall reorder the \(\mu_i\) so that \(L_1 \leq L_2 \leq \cdots \leq L_k\), as we shall need to keep track of the expected lengths of the generators in the subsequent argument. Finally, it will be convenient to have notation for paths which travel along a geodesic \(\gamma_i\) in the reverse direction, so we will extend our index set from \(I = \{1, \ldots, k\}\) to \(\pm I = \{\pm 1, \ldots, \pm k\}\), and write \(\gamma_{-i}\) for a geodesic in \(X\) from \(x_0\) to \(w^{-1}_{i,n_i}x_0\), which is a translate by \(w_{i,n_i}^{-1}\) of the reverse path along \(\gamma_i\).

In order to show that \(H\) is hyperbolically embedded in \(G\) we shall show that \(H\) is freely generated by \(\{w_{1,n_1}, \ldots, w_{k,n_k}\}\), \(H\) is quasi-isometrically embedded in \(X\), and \(H \ltimes E(G)\) is geometrically separated, with asymptotic probability one.

We start by showing some generalizations of the properties that hold for individual random walks to the case of multiple random walks. Each individual random walk makes linear progress with exponential decay. We now show that the collection of \(k\) random walks also makes linear progress with exponential decay.

**Definition 29.** Given \(0 < \varepsilon < 1\), and a random subgroup \(H\), we say that \(H\) satisfies \(\varepsilon\)-length bounds if

\[(1 - \varepsilon)L_i n_i \leq d_X(x_0, w_{i,n_i}x_0) \leq (1 + \varepsilon)L_i n_i.\]

for all \(1 \leq i \leq k\).

**Proposition 30.** Let \(H\) be a random subgroup, and let \(\varepsilon > 0\). Then there are constants \(K\) and \(c\), depending only on \(\varepsilon\), and the probability dis-
tributions \(\mu_i\), such that the probability that a random subgroup \(H\) satisfies \(\varepsilon\)-length bounds is at least \(1 - Kc^n\).

**Proof.** By Proposition 9.1, for any \(\varepsilon > 0\), for each \(\mu_i\) there are constants \(L_i, K_i, c_i\) such that

\[
P((1 - \varepsilon)L_i n_i \leq d_X(x_0, w_{i,n_i,x_0}) \leq (1 + \varepsilon)L_i n_i) \geq 1 - K_i c_i^n.
\]

If \(K' = \max K_i, c = \max c_i, n = \min n_i\), then the probability that these inequalities are satisfied simultaneously for all \(i\) is at least \(1 - kKc^n\). Therefore the required estimate holds, with \(K = kK'\), and the previous choice of \(c\). \(\square\)

We now show that the collection of \(k\) random walks satisfies the following estimates on their mutual Gromov products.

**Definition 31.** We say a random subgroup \(H\) satisfies \(K\)-Gromov product bounds if

\[
(ax_0 \cdot bx_0)_{x_0} \leq K.
\]

for all distinct \(a, b\) in the symmetric generating set \(A = \{w_{1,n_1}^\pm, \ldots, w_{k,n_k}^\pm\}\) for \(H\).

**Proposition 32.** Let \(H\) be a random subgroup. Given \(0 < \varepsilon < \frac{1}{2}\) there are constants \(K, c\), depending only on \(\varepsilon\), and the probability distributions \(\mu_i\), such that the probability that \(H\) satisfies \((\varepsilon Ln)\)-Gromov product bounds is at least \(1 - Kc^n\).

**Proof.** If \((ax_0 \cdot bx_0)_{x_0} \leq \varepsilon Ln\), then, by definition of shadows, \(ax_0 \in S_{x_0}(bx_0, d_X(x_0, bx_0) - \varepsilon Ln)\). By Proposition 9.1, the random walk determined by each \(\mu_i\) satisfies exponential decay for shadows, i.e. there are constants \(R_0, K_i, c_i\) such that for all \(R \geq R_0\), and all \(g \in G\),

\[
P(w_{i,n_i} \in S_{x_0}(gx_0, R)) \leq K_i c_i^{d_X(x_0, gx_0) - R}.
\]

We shall use (11) with \(g = b\). If

\[
d_X(x_0, bx_0) - \varepsilon Ln \geq R_0,
\]

then (11) implies that the probability that \((ax_0 \cdot bx_0)_{x_0} \leq \varepsilon Ln\) is at most \(K_i c_i^{Ln}\).

In order to apply the estimate (11), we need to check that (12) holds with asymptotic probability one. Using linear progress, Proposition 9.1,

\[
P(d_X(x_0, bx_0) \leq (1 - \varepsilon)Ln) \leq K_i' c_i'^n,
\]

for some constants \(K_i'\) and \(c_i'\) depending on \(\varepsilon\) and \(\mu_i\). Therefore

\[
P(d_X(x_0, bx_0) - \varepsilon Ln \leq (1 - 2\varepsilon)Ln) \leq K_i' c_i'^n.
\]

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As we have chosen $\varepsilon < \frac{1}{2}$, this implies that
\[ P(d_X(x_0, bx_0) - \varepsilon Ln \leq R_0) \leq K'_c c''^n. \]

for all $n \geq R_0/(L(1 - 2\varepsilon))$.

Therefore, the probability that $(ax_0 \cdot bx_0) x_0 \leq \varepsilon Ln$ is at most $K'_c c^n + K_i c_L^L n$. As there are at most $2k$ choices for each of $a$ and $b$ in $A$, the probability that any of these events occurs is at most $4k^2 K'' c^n$, where $K'' = \max\{K_i, K_i'\}$ and $c_i = \max\{c_i, c_i'\}$. The result then holds with $K = 4k^2 K''$, and the previous choice of $c$, as required.

If $H$ satisfies $\varepsilon$-length bounds and $(\varepsilon Ln)$-Gromov product bounds, then the conditions (9) are satisfied in Proposition (16) so the rescaled Cayley graph $\Gamma_H$ is $(6, O(\varepsilon Ln))$-quasi-isometrically embedded in $X$. In particular, this implies that $H$ is freely generated by $\{w_1, a_1, \ldots, w_k, a_k\}$, and $HE(G)$ is a semidirect product $H \rtimes E(G)$. As well as these properties, it will be convenient to know certain matching properties for the geodesics defined by $H$, which we now describe.

**Definition 33.** We say that a random subgroup $H$ has an $\varepsilon$-large match if a translate of $[\gamma_j(\varepsilon Ln), \gamma_j(|\gamma_j| - \varepsilon Ln)]$ is contained in a $2\delta$-neighbourhood of $\gamma_i$, for some $i < j$.

**Proposition 34.** Let $H$ be a random subgroup, and let $0 < \varepsilon < \frac{1}{2}$. Then there are constants $K$ and $c$, depending on $\varepsilon$ and the probability distributions $\mu_i$, such that the probability that $H$ has an $\varepsilon$-large match is at most $Kc^n$.

**Proof.** We may assume that $H$ satisfies $\varepsilon$-length bounds, which by Proposition (10) happens with probability at least $1 - K' c^n$, for some $K'$ and $c' < 1$, depending on the $\mu_i$ and $\varepsilon$. By $\varepsilon$-length bounds, the length of $\gamma_j$ is at least $(1 - \varepsilon)L_j n_j$, and the length of $\gamma_i$ is at most $(1 + \varepsilon)L_i n_i$.

Let $\gamma_j^-$ be the subgeodesic of $\gamma_j$ given by $[\gamma_j(\varepsilon Ln), \gamma_j(|\gamma_j| - \varepsilon Ln)]$. It will be convenient to consider a discrete set of points $\gamma_j(\ell)$ along $\gamma_j$, where $\ell \in \mathbb{N}$. If $\gamma_j^-$ is contained in a $2\delta$-neighbourhood of $[\gamma_i(t), \gamma_i(t + |\gamma_j^-|)]$, then $\gamma_j^-$ is contained in a $(2\delta + 1)$-neighbourhood of $[\gamma_i(\ell), \gamma_i(\ell + |\gamma_j^-|)]$ for some $\ell \in \mathbb{N}$.

By Proposition (10) there are constants $K_i$ and $c_i < 1$ such that the probability that a translate of $\gamma_i^-$ is contained in a $(2\delta + 1)$-neighbourhood of $\gamma_i$ starting at $\gamma_i(\ell)$ is at most
\[ K_i c_i (1 - \varepsilon)L_j n_j - 2\varepsilon Ln \leq K_c (1 - 3\varepsilon)L_j n_j, \]

where the inequality above holds with $K = \max K_i$, $c = \max c_i$, and $Ln \leq L_j n_j$. Given the length estimates for $\gamma_i$ and $\gamma_j$, the number of
possible values of $\ell$ is at most

$$(1 + \varepsilon)L_i n_i - (1 - \varepsilon)L_j n_j + 2\varepsilon Ln \leq 3\varepsilon L_j n_j,$$

where the inequality holds as $L_i n_i \leq L_j n_j$, and negative terms on the left hand side are discarded.

Therefore, the probability that a translate of $\gamma_j$ is contained in a $2\delta$-neighbourhood of $\gamma_i$ is at most

$$3\varepsilon L_j n_j K c^{(1-3\varepsilon)L_j n_j} \leq K'' c'' n,$$

for some constants $K''$ and $c''$, where the inequality holds as the function $f(x) = xc^x$ is decreasing for all $x$ sufficient large, and bounded above by a constant multiple of an exponential function. As there are at most $2k$ choices of indices for each of $i$ and $j$, the result follows.

Finally, we give an estimate for the probability that a geodesic $\gamma_j$ has an initial segment which matches a terminal segment of $\gamma_i$, concatenated with an initial segment of $\gamma_i'$. Define a collection of geodesic segments $\{\eta(i, i', K, \ell)\}_{i, i' \in \pm I, i \neq -i', \ell \in \mathbb{N}, 0 \leq \ell \leq |\gamma_i|}$ as follows. Let $i$ and $i'$ be indices in $\pm I$ with the property that $i \neq -i'$, and let $0 \leq \ell \leq |\gamma_i|$ be an integer. Let $p$ be a point on $\gamma_i$ distance $\ell$ from its endpoint, and let $q$ be a point on $w_i\gamma_i'$ distance $K$ from the initial point of $w_i\gamma_i'$. Define $\eta(i, i', K, \ell)$ to be a geodesic from $p$ to $q$.

![Figure 5: A geodesic $\eta(i, i', K, \ell)$](image)

**Definition 35.** We say that a random subgroup $H$ is $K$-unmatched if for all $i \leq j, i' \leq j$ and for $0 \leq t \leq K$, no geodesic $\eta(i, i', K, \ell)$, is contained in a $2\delta$-neighbourhood of a subgeodesic of $\gamma_j$ starting at $\gamma_j(t)$.

**Proposition 36.** Let $H$ be a random subgroup, and let $0 < \varepsilon < \frac{1}{6}$. Then there are constants $K$ and $c$, depending on $\varepsilon$ and the $\mu_i$, such that the probability that $H$ is $(3\varepsilon Ln)$-unmatched is at least $1 - K c''$.

**Proof.** We shall assume that the random subgroup $H$ satisfies the $\varepsilon$-length bounds and $(\varepsilon Ln)$-Gromov product bounds, which happens with probability at least $1 - K' c'^n$, for some constants $K'$ and $c'$, depending on $\varepsilon$ and the $\mu_i$.  

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First consider a fixed collection of indices $i, i', j$ in $\pm I$, with $i \leq j$, $i' \leq j$ and $i \neq -i'$. The Gromov product bound for $i$ and $i'$ implies that the length of $\eta = \eta(i, i', 3\varepsilon Ln, \ell)$ is at least $\ell + 3\varepsilon Ln - 2\varepsilon Ln = \ell + \varepsilon Ln$. If a translate of $\eta(i, i', 3\varepsilon Ln, \ell)$ is contained in a $2\delta$-neighbourhood of $[\gamma_j(t), \gamma_j(t + |\eta|)]$, then it is contained in a $(2\delta + 1)$-neighbourhood of $[\gamma_j(m), \gamma_j(m + |\eta|)]$, for some $m \in \mathbb{N}$.

By Proposition 10.1, the probability that a translate of $\eta(i, i', 3\varepsilon Ln, \ell)$ is contained in a $(2\delta + 1)$-neighbourhood of $[\gamma_j(m), \gamma_j(m + |\eta|)]$ is at most $Kc \ell + \varepsilon Ln$. As there are at most $3\varepsilon Ln$ choices for $m$, the probability that this occurs for some $0 \leq m \leq 3\varepsilon Ln$ is at most $3\varepsilon Ln K \varepsilon^{c + Ln}$. The sum of these probabilities over all values of $\ell$ is at most $(3\varepsilon Ln K \varepsilon^{c + Ln})/(1 - c) \leq K' \varepsilon^{c_m}$ for different constants $K'$ and $c'$.

There are at most $2^k$ possible admissible choices for each of the indices $i, i'$ and $j$, and so assuming $\varepsilon$-length bounds and Gromov product bounds, the probability that the geodesics are not $(3\varepsilon Ln)$-unmatched is at most $(2^k)^3 K' \varepsilon^{c_m}$. Therefore, the probability that $\varepsilon$-length bounds, Gromov bounds and $(3\varepsilon Ln)$-unmatching all hold simultaneously is at least $1 - Kc' \varepsilon^{c_m}$, for $K = (2^k)^3 K'$.

In order to show Theorem 5 it therefore suffices to show:

**Proposition 37.** Let $H$ be a random subgroup of $G$, and let $0 < \varepsilon < \frac{1}{6}$. If $H$ satisfies $\varepsilon$-length bounds, $(\varepsilon Ln)$-Gromov product bounds, has no $\varepsilon$-large match and is $(3\varepsilon Ln)$-unmatched, then $\Gamma_H$ is $(6, O(\varepsilon Ln))$-quasi-isometrically embedded in $X$ and $H \ltimes E(G)$ is geometrically separated in $X$.

We now prove Theorem 5 assuming Proposition 37.

**Proof (of Theorem 5).** The first property in Theorem 5, the fact that each generator $w_{i,n}$ is hyperbolic and asymmetric, follows from Proposition 16 applied to each of the random walks $w_{i,n}$.

The second property, that $\Gamma_H$ is a quasi-isometrically embedded follows (as we have already observed) if $H$ satisfies $\varepsilon$-length bounds and $(\varepsilon Ln)$-Gromov product bounds, which hold with probabilities at least $1 - K \varepsilon^{c_m}$, by Propositions 30 and 32 for constants $K$ and $c < 1$ depending only on $\varepsilon$ and the $\mu_i$. This then implies that $H$ is freely generated by its generators $w_{i,n}$, and $HE(G) = H \ltimes E(G)$.

The final property, that $H \ltimes E(G)$ is geometrically separated, holds if $H$ satisfies the four conditions, $\varepsilon$-length bounds, $(\varepsilon Ln)$-Gromov product bounds, no $\varepsilon$-large match and being $(3\varepsilon Ln)$-unmatched, and these hold with probability at least $1 - K' \varepsilon^{c_m}$, by Propositions 30, 32, 34 and 36 for some constants $K'$ and $c' < 1$, depending only on $\varepsilon$ and the $\mu_i$, as required.
The final step is to prove Proposition 37. We shall use the following properties of geodesics and quasigeodesics in a hyperbolic space $X$, see for example Bridson and Haefliger [BH90, III.H.1]. If two geodesics in $X$ are $A$-fellow travellers, then they are in fact $O(\delta)$-fellow travellers, outside balls of radius $A$ about their endpoints. Similarly, if two $(A, B)$-quasigeodesics are $C$-fellow travellers, then they are $O(\delta, A, B)$-fellow travellers outside $C$-neighbourhoods of their endpoints.

**Proof (of Proposition 37).** Recall that the (image in $X$ of the) rescaled Cayley graph $\Gamma_H$ is the union of translates of geodesic segments $\gamma_i$ from $x_0$ to $w_{i,n}x_0$ by elements of $H$. Let $\gamma$ be a geodesic in $X$ connecting two points $h_1x_0$ and $h_2x_0$ of $Hx_0$. These two points are also connected by a path $\hat{\gamma}$ in $\Gamma_H$, which is a concatenation of geodesic segments $\gamma_i$, corresponding to the reduced word determined by $h_1^{-1}h_2$ in $H$. The path $\hat{\gamma}$ is an $(6, O(\delta, \varepsilon Ln))$-quasigeodesic in $X$, which by the Morse property is contained in an $O(\delta, \varepsilon Ln)$-neighbourhood of $\gamma$.

We will show that geometric separation holds for a constant $B(R) = 4R + O(\delta, \varepsilon Ln)$. Let $\gamma$ and $\gamma'$ be geodesics in $X$ of length at least $B$, with endpoints in $H$, and an element $g \in G$, such that $g\gamma$ is an $(2R + O(\delta))$-fellow traveller with $\gamma'$. In order to show geometric separation, it suffices to show that $g$ in fact lies in $H \times E(G)$.

Let $\hat{\gamma}$ and $\hat{\gamma}'$ be the corresponding paths in $\Gamma_H$ connecting the endpoints of $\gamma$ and $\gamma'$. The quasigeodesics $\hat{\gamma}$ and $\hat{\gamma}'$ are $(2R + O(\delta, \varepsilon Ln))$-fellow travellers in $X$, and we shall denote their endpoints by $\hat{\gamma}(0)$ and $\hat{\gamma}(T)$ for $\hat{\gamma}$, and $\hat{\gamma}'(0)$ and $\hat{\gamma}'(T')$ for $\hat{\gamma}'$. Therefore, if we set $\hat{\gamma}_-$ and $\hat{\gamma}'_-$ to be the largest union of segments which are translates of the $\gamma_i$ contained in $\hat{\gamma} \setminus (B_X(\hat{\gamma}(0) \cup \hat{\gamma}(T), 2R + O(\delta, \varepsilon Ln))$ and $\hat{\gamma}' \setminus (B_X(\hat{\gamma}'(0) \cup \hat{\gamma}'(T'), 2R + O(\delta, \varepsilon Ln))$, then $\hat{\gamma}_-$ and $\hat{\gamma}'_-$ are $O(\delta, \varepsilon Ln)$-fellow travellers. By a sufficiently large choice of $B$ we may assume that the lengths of $\hat{\gamma}$ and $\hat{\gamma}'$ are at least $4R + (1 + \varepsilon)Ln + O(\delta)$, and so both $\hat{\gamma}_-$ and $\hat{\gamma}'_-$ are non-empty, as we have assumed that the $\gamma_i$ satisfy $\varepsilon$-length bounds and Gromov product bounds.

Each path $\hat{\gamma}_-$ or $\hat{\gamma}'_-$ is a concatenation of geodesic segments which are translates of the $\gamma_i$. Let $j$ be the largest index of any path segment whose translate appears in either of $\hat{\gamma}_-$ or $\hat{\gamma}'_-$. If the largest index $j$ does not appear in both paths, then up to relabelling, we may assume that $j$ occurs in $\hat{\gamma}_-$, and let $h\gamma_j$ be a corresponding geodesic segment in the path $\hat{\gamma}_-$, for some $h \in H$.

We now consider two cases. Either the nearest point projection of $h\gamma_j$ to $\hat{\gamma}_-$ is contained in the translate of a single $\gamma_i$ for $i < j$, or $h\gamma_j \subset \hat{\gamma}_-$ contains a point within distance $\varepsilon Ln$ of some point of the orbit $Hx_0$. If the first case occurs with $i < j$, then $H$ has an $\varepsilon$-large match, which we have
assumed does not happen, so $gh\gamma_j$ in fact $(\varepsilon Ln)$-fellow travels a translate of itself in $\hat{\gamma}_-$. This means that the translate $gh\gamma_j$ $(\varepsilon Ln)$-fellow travels $h'\gamma_j$ for some $h' \in H$, and so $h'^{-1}gh$ $(\varepsilon Ln + O(\delta))$-stabilizes $[p, w_{j,n,j} p]$, where $p$ is a nearest point projection of the basepoint $x_0$ to the axis $\alpha_j$ for $w_{j,n,j}$. By Proposition 24 $w_{j,n,j}$ is irreversible with asymptotic probability one, so $h'^{-1}gh$ does swap the endpoints of the geodesic $[p, w_{j,n,j} p]$, and by Proposition 27 we may assume that $w_{j,n,j}$ is $(\varepsilon Ln + O(\delta))$-asymmetric, and so this implies that $h'^{-1}gh \in \langle w_{j,n,j} \rangle \ltimes E(G) \subset H \ltimes E(G)$. As both $h$ and $h'$ lie in $H$, this implies that $g$ lies in $H \ltimes E(G)$, with asymptotic probability one, as required.

It remains to show that if the second case occurs then $H$ is $(3\varepsilon Ln)$-unmatched, as we now explain. Let $p$ be a point in $\Gamma_H$ closest to the initial point of $g'\gamma_j$, and let $q$ be the point in $\Gamma_H$ closest to the terminal point of $g'\gamma_j$. Let $hx_0$ be the first point of $HX_0$ occurring between $p$ and $q$. Let $h\gamma_i^{-1}$ be the geodesic segment of $\Gamma_H$ containing $p$, and let $h\gamma_i'$ be the next geodesic segment of $\Gamma_H$ along the geodesic in $\Gamma_H$ from $p$ to $q$. Finally, let $q'$ be a point on $h\gamma_i'$ distance $\varepsilon Ln$ from $hx_0$. This is illustrated below in Figure 6.

![Figure 6: A subsegment of the geodesic $g\hat{\gamma}_-$ fellow travels $\Gamma_H$.](image)

We now observe that the geodesic in $X$ from $p$ to $q'$ is the geodesic $\eta(i, i', 3\varepsilon Ln\delta)$ used in Definition 35 and so if the second case occurs, then $H$ is not $(\varepsilon Ln)$-unmatched, contradicting our initial assumptions on $H$.

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