Linear differential equations to solve nonlinear mechanical problems: A novel approach

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Abstract

Often a non-linear mechanical problem is formulated as a non-linear differential equation. A new method is introduced to find out new solutions of non-linear differential equations if one of the solutions of a given non-linear differential equation is known. Using the known solution of the non-linear differential equation, linear differential equations are set up. The solutions of these linear differential equations are found using standard techniques. Then the solutions of the linear differential equations are put into non-linear differential equations and checked whether these solutions are also solutions of the original non-linear differential equation. It is found that many solutions of the linear differential equations are also solutions of the original non-linear differential equation.

1 Introduction

Nonlinear mechanics is often said [1] to be the last and third major development of physics in the 20th century. The other two are quantum mechanics and relativity. One of the major tools for studying nonlinear mechanics is nonlinear differential equations [2, 3]. Nonlinear differential equations appear prominently in the study of fluid dynamics, cooperative and nondissipative phenomena [1, 5, 6], General theory of relativity [7] etc. Probably one of the best ways to obtain maximum information about a nonlinear mechanical system is to set up a nonlinear differential equation and to find out the maximum number of explicit solutions of the nonlinear differential equation. The larger the number of the explicit solutions we know, the better our understanding of the mechanical system, represented by the nonlinear differential equations. Thus to find out the maximum number of solutions of a given nonlinear differential equation is of utmost importance. But to find out new explicit solutions of nonlinear differential equations is often extremely difficult. For the last three hundred years, mathematicians, physicists and engineers were mostly concerned with the solutions of linear differential equations. As a result we now know far more about solving linear differential equations than solving nonlinear differential equations. Here we show how a knowledge of solving a linear differential equation can help us to locate new solutions of a given nonlinear differential equation if one of the solutions of the nonlinear differential equation is already known.

In the next paragraph, we shall discuss the basic properties of the differential equations and their solutions that led to the present work.

It is common knowledge that many differential equations have some common solutions. Examples are in plenty. The harmonic oscillator equation

\[ \frac{d^2 x}{dt^2} + x = 0 \]  

has two linearly independent solutions

\[ x = \cos t \quad \text{and} \quad x = \sin t \]  

The sixth order linear differential equation

\[ \frac{d^6 x}{dt^6} + x = 0 \]  

was also solved by the above method.
has also \( x = \cos t \) and \( x = \sin t \) as solutions. Again the second degree nonlinear differential equation

\[
\left( \frac{dx}{dt} \right)^2 + x^2 - 1 = 0
\]  

(5)

has \( x = \cos t \) and \( x = \sin t \) as solutions. Therefore, it seems reasonable to conclude that there is something common for the three systems represented by the equations (1), (4) and (5). What is this something that is common?

An explicit solution of a differential equation is a function. A function has a set of symmetries associated with it. For example, \( \sin x \) is an odd function with a set of symmetries associated with it. Thus when a linear differential equation and a nonlinear differential equation have one common solution, the two different differential equations represent two different systems having one common set of symmetries. This idea of common set of symmetries can be extended to any number of common solutions for linear differential equations or for any set of nonlinear differential equations. Warning! The symmetries associated with differential equations can be altered by initial and boundary conditions of the problem.

What we have just discussed about the ordinary differential equations and their solutions is also true of partial differential equations. The well known K dV equation

\[
u_t - 6uvu_x + u_{xxx} = 0
\]  

(6)

has

\[
u(x, t) = -\frac{c}{2} \text{sech}^2 \left\{ \frac{c^{1/2}}{2} (x - ct - x_0) \right\}
\]

(7)

as one of its solutions [8]. Equation (7) is also the solution of the well known second order wave equation

\[
u_{xx} = \frac{1}{c^2} \nu_{tt}
\]  

(8)

Equation (7) is also the solution of many other wave equations. To cite one more example, the nonlinear Burgers’ equation

\[
u_t + uu_x = 0
\]  

(9)

has \( u = \frac{x}{t} \) as one of its explicit solutions. The linear partial differential equations

\[
x u_x + tu_t = 0
\]

(10)

\[
2xu_x + t^2 u_t = 0
\]

(11)

have also \( u = \frac{x}{t} \) as their solution.

We have earlier remarked that common solutions for different differential equations imply that the corresponding differential equations have common symmetry properties. This aspect is of fundamental importance from the point of view of the theory of differential equations. But here we show how the commonness of some solutions of a linear differential equation and a nonlinear differential equation can be exploited to find new solutions of nonlinear differential equations.

## 2 Illustration

We shall illustrate our method of finding new solutions for both ordinary nonlinear differential equations and nonlinear partial differential equations.

We shall start with the ordinary nonlinear differential equation

\[
\left( \frac{dx}{dt} \right)^4 - x^4 + 2x^2 - 1 = 0
\]  

(12)

We shall take \( x = \cos t \) as the already known solutions of (12). We shall call this solution as the seed solution. By differentiating the seed solution four times, we get the fourth order linear differential equation

\[
\frac{d^4 x}{dt^4} - x = 0
\]  

(13)
By using the standard techniques of solving linear differential equations, we can obtain three other solutions of (13). They are

\[ x = \sin t \quad (3) \]
\[ x = \cosh t \quad (14) \]
\[ x = \sinh t \quad (15) \]

The solutions (3), (14) and (15) are solutions of a linear differential equation generated from one of the solutions of the nonlinear differential equation. Therefore, since the seed is the same for both linear and nonlinear differential equations, there is a possibility that some or all the solutions of the linear equation (13) will also be the solutions of the nonlinear differential equation (12).

Substitution shows that \( x = \sin t \) and \( x = \cosh t \) are also solutions of the nonlinear equation (12). In a nutshell, by setting up the fourth order linear differential equation

\[ \frac{d^4}{dt^4} x - x = 0 \]

from the known solution \( x = \cos t \) of the nonlinear differential equation

\[ (\frac{dx}{dt})^4 - x^4 + 2x^2 - 1 = 0 \]

we have succeeded in obtaining two new solutions, namely \( x = \sin t \) and \( x = \cosh t \) of the nonlinear differential equation.

From the known solution \( x = \cos t \) of the nonlinear differential equation (12), we can also set up the linear differential equation,

\[ \frac{d^2}{dt^2} x + x = 0 \quad (1) \]

As we have stated earlier, this equation has two linearly independent solutions \( x = \cos t \) and \( x = \sin t \). Therefore, if we had set up the second order linear differential equation (1), instead of the fourth order equation (13), we would have got only one new solution \( x = \sin t \) for the nonlinear differential equation \( (\frac{dx}{dt})^4 - x^4 + 2x^2 - 1 = 0 \).

In general, the higher the order of the linear differential equation set up, the greater the probability for finding new solutions to the original nonlinear differential equation.

Note that the linear superposition \( x = \cos t + \sin t \) of the linearly independent solutions \( x = \cos t \) and \( x = \sin t \), is a solution of the linear differential equation (13), but \( x = \cos t + \sin t \) is not a solution of the nonlinear differential equation (12).

Now we shall move to the realm of partial differential equations. Let us consider the first order, second degree nonlinear wave equation

\[ u_x^2 = \frac{k^2}{\omega^2} u_t^2 \quad (16) \]

where
\[
\begin{align*}
  u &= u(x, t) \text{ is the dependent variable} \\
  x, t &= \text{independent variables} \\
  k, \omega &= \text{parameters}
\end{align*}
\]

is taken as the known seed solution from (16). By differentiating (17) twice with respect to \( x \) and \( t \), we can obtain the second order linear partial differential equation

\[ u_{xx} = \frac{1}{c^2} u_{tt} \quad (18) \]

It can be easily found by standard methods that

\[
\begin{align*}
  u &= \sin(kx - \omega t) \quad (19) \\
  u &= \cos h(kx - \omega t) \quad (20) \\
  u &= \sin h(kx - \omega t) \quad (21) \\
  u &= \sin(kx + \omega t) \quad (22) \\
  u &= \cos(kx + \omega t) \quad (23) \\
  u &= \sin h(kx + \omega t) \quad (24) \\
  u &= \cos h(kx + \omega t) \quad (25)
\end{align*}
\]

are solutions of the second order wave equation \( u_{xx} = \frac{1}{c^2} u_{tt} \). Now we can verify whether any of these solutions is also a solution of the original nonlinear differential equation (16). In fact all the solutions (19) to (25) are also solutions of the nonlinear partial differential equation (16).
It is interesting to note that the linear superpositions
\[ u = \sin(kx + \omega t) + \cos(kx + \omega t) \] and
\[ u = \sin(kx + \omega t) + \cos(kx - \omega t) \]
are solutions of the second order linear wave equation \( u_{xx} = k^2 \omega^2 u_{tt} \) where as only \( u = \sin(kx + \omega t) + \cos k(x + \omega t) \) is a solution of the nonlinear differential equation
\[ u^2_x = \frac{k^2}{\omega^2} u^2_t \]
\( u = \sin(kx + \omega t) + \cos(kx - \omega t) \) is not a solution of the above nonlinear differential equation. This implies that the nonlinear wave equation \( u^2_x = \frac{k^2}{\omega^2} u^2_t \) can represent only superposition of waves moving in the same direction. This simple result is of immense importance in the theory of wave equations and the application of wave theory to fluid dynamics.

New solutions to well known nonlinear partial differential equations, such as K dV equation and Burgers’ equation will be reported somewhere else.

3 Limitation

The above method suffers from a serious limitation. Starting from the solution of a nonlinear differential equation, a higher order differential equation can be set up only if the derivatives higher than the first derivative do exist. For example, if for the solution of a given first order nonlinear differential equation, if the derivatives higher than the first derivative do not exist, then we can not set up linear differential equations of second or higher order, starting from that particular seed solution.

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