1. Introduction. It is well-known that the classical variational principle is a powerful tool in various disciplines such as physics, engineering and mathematics. However, the classical variational principle cannot describe many important physical processes. In 1930 Herglotz [22] proposed a generalized variational principle with one independent variable, which generalizes the classical variational principle. As reported in [14, 15, 16], the principle of Herglotz gives a variational description of nonconservative processes, even when the Lagrangian is autonomous, something that cannot be done with the classical approach. For the importance to include nonconservativism in the calculus of variations, we refer the reader to [26].

The generalized variational calculus proposed by Herglotz deals with the following problem: determine the trajectories $x \in C^2([a, b], \mathbb{R})$ satisfying given boundary conditions $x(a) = \alpha, x(b) = \beta$, for fixed real numbers $\alpha, \beta$, that extremize (minimize or maximize) the value
\[
z(b) \rightarrow \text{extr},
\]
where $z$ satisfies the differential equation
\[
\dot{z}(t) = L(t, x(t), \dot{x}(t), z(t)), \quad t \in [a, b],
\]
subject to the initial condition
\[
z(a) = \gamma,
\]
where $\gamma$ is a fixed real number. The Lagrangian $L$ is assumed to satisfy the following hypotheses:

1. $L$ is a $C^1([a, b] \times \mathbb{R}^3, \mathbb{R})$ function;
functions \( t \mapsto \frac{\partial L}{\partial x}(t, x(t), \dot{x}(t), z(t)), \) \( t \mapsto \frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t), z(t)) \) and \( t \mapsto \frac{\partial L}{\partial z}(t, x(t), \dot{x}(t), z(t)) \), are differentiable for any admissible trajectory \( x \).

Note that \((1)\) represents a family of differential equations: for each function \( x \) a different differential equation arises. Therefore, \( z \) depends on \( x \), a fact that can be made explicit by writing \( z\{x; t\} \) (or \( z(t, x(t), \dot{x}(t)) \)), but for brevity and convenience of notation it is usual to write simply \( z(t) \). Observe that Herglotz’s variational problem reduces to the classical fundamental problem of the calculus of variations (see, e.g., \([12]\)) if the Lagrangian \( L \) does not depend on the variable \( z \): if

\[
\dot{z}(t) = L(t, x(t), \dot{x}(t)), \quad t \in [a, b],
\]

\[
z(a) = \gamma, \quad \gamma \in \mathbb{R},
\]

then we obtain the classical variational problem

\[
z(b) = \int_a^b \dot{L}(t, x(t), \dot{x}(t))\,dt \rightarrow \text{extr},
\]

where

\[
\dot{L}(t, x, \dot{x}) = L(t, x, \dot{x}) + \frac{\gamma}{b - a}.
\]

Herglotz proved that a necessary condition for a trajectory \( x \) to be an extremizer of the generalized variational problem \( z(b) \rightarrow \text{extr} \) subject to \((1)-(2)\) is given by

\[
\frac{\partial L}{\partial x}(t, x(t), \dot{x}(t), z(t)) - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t), z(t))
+ \frac{\partial L}{\partial z}(t, x(t), \dot{x}(t), z(t)) \cdot \frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t), z(t)) = 0, \quad (3)
\]

t \in [a, b]. Herglotz called \((3)\) the generalized Euler–Lagrange equation \([20, 21, 29]\).

Observe that for the classical problem of the calculus of variations one has \( \frac{\partial L}{\partial z} = 0 \), and the differential equation \((3)\) reduces to the classical Euler–Lagrange equation:

\[
\frac{\partial L}{\partial x}(t, x(t), \dot{x}(t)) - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t)) = 0.
\]

A generalized Euler–Lagrange differential equation for Herglotz-type higher-order variational problems was recently proved in \([32]\).

It is well-known that the notions of symmetry and conservation law play an important role in physics, engineering and mathematics \([34]\). The interrelation between symmetry and conservation laws in the context of the calculus of variations is given by the first Noether theorem \([28]\). The first Noether theorem, usually known as Noether’s theorem, guarantees that the invariance of a variational integral under a group of transformations depending smoothly on a parameter \( \epsilon \) implies the existence of a conserved quantity along the Euler–Lagrange extremals. Such transformations are global transformations. Noether’s theorem explains all conservation laws of mechanics, e.g., conservation of energy comes from invariance of the system under time translations; conservation of linear momentum comes from invariance of the system under spatial translations; while conservation of angular momentum reflects invariance with respect to spatial rotations. The first Noether theorem is nowadays a well-known tool in modern theoretical physics, engineering and the calculus of variations \([36]\). Inexplicably, it is still not well-known that the famous paper of Emmy Noether \([28]\) includes another important result: the second Noether theorem...
subject to the initial condition

\[ z = \gamma \]

where \( \gamma \) is a fixed real number. The Lagrangian \( L \) is assumed to satisfy the following hypotheses:

\[ (H1): L \text{ is a } C^1([a, b] \times \mathbb{R}^5, \mathbb{R}) \text{ function}; \]

\[ (H2): \text{ functions } t \mapsto \partial_i L (t, x(t), \dot{x}(t), x(t - \tau), \dot{x}(t - \tau), z(t)) \text{ for } i = 2, \ldots, 6 \text{ are differentiable for any admissible trajectory } x. \]

Observe that problem \( z(b) \to \text{extr} \) subject to (4)-(5) reduces to the classical fundamental problem of the calculus of variations with time delay if the Lagrangian
$L$ does not depend on $z$. Also note that problem $(P)$ reduces to the generalized variational problem of Herglotz when $\tau = 0$.

The structure of the paper is as follows. In Section 2 we review some preliminaries about the generalized variational calculus (without time delay). In particular, we recall the notion of invariance and the first Noether theorem for variational problems of Herglotz type. Our main results are given in Section 3: a generalized Euler–Lagrange necessary optimality condition (Theorem 3.3), a DuBois–Reymond necessary optimality condition (Theorem 3.5), and Noether’s first theorem for variational problems of Herglotz type with time delay (Theorem 3.9), are proved. We end with an illustrative example of our results in Section 4. The results of the paper are trivially generalized for the case of vector functions $x : [a - \tau, b] \to \mathbb{R}^n$, $n \in \mathbb{N}$, but for simplicity of presentation we restrict ourselves to the scalar case.

2. Review of Noether’s first theorem for variational problems of Herglotz type. For the convenience of the reader, we present here the definition of generalized extremals, the definition of invariance of functional $z$, defined by $\dot{z} = L(t, x, \dot{x}, z)$ and $z(a) = \gamma$, and we recall Noether’s first theorem for the generalized variational problem of Herglotz type. For simplicity of notation, we introduce the operator $\langle \cdot, \cdot \rangle$ defined by

$$\langle x, z \rangle(t) := (t, x(t), \dot{x}(t), z(t)).$$

**Theorem 2.1** (Generalized Euler–Lagrange equation [22]). If $x \in C^2([a, b], \mathbb{R})$ is a solution to problem $z(b) \rightarrow \text{extr}$ subject to (1)–(2) and the boundary conditions $x(a) = \alpha$ and $x(b) = \beta$, for some fixed real numbers $\alpha, \beta$, then $x$ satisfies the generalized Euler–Lagrange equation

$$\partial_2 L(x, z)(t) - \frac{d}{dt} \partial_3 L(x, z)(t) + \partial_4 L(x, z)(t) \partial_1 L(x, z)(t) = 0, \quad t \in [a, b]. \quad (6)$$

**Definition 2.2** (Generalized extremals—cf. [15]). The solutions $x \in C^2([a, b], \mathbb{R})$ of the generalized Euler–Lagrange equation (6) are called generalized extremals.

Consider a one-parameter group of infinitesimal transformations on $\mathbb{R}^2$,

$$\bar{t} = \phi(t, x, \epsilon), \quad \bar{x} = \psi(t, x, \epsilon), \quad (7)$$

in which $\epsilon$ is the parameter and $\phi$ and $\psi$ are invertible $C^1$ functions such that $\phi(t, x, 0) = t$ and $\psi(t, x, 0) = x$. The infinitesimal representation of transformations (7) is given by

$$\bar{t} = t + \sigma(t, x) \epsilon + o(\epsilon), \quad \bar{x} = x + \xi(t, x) \epsilon + o(\epsilon),$$

where $\sigma$ and $\xi$ denote the first degree coefficients of $\epsilon$. Explicitly,

$$\sigma(t, x) = \frac{d\phi}{d\epsilon}(t, x, \epsilon) \bigg|_{\epsilon=0}, \quad \xi(t, x) = \frac{d\psi}{d\epsilon}(t, x, \epsilon) \bigg|_{\epsilon=0}.$$

**Definition 2.3** (Invariance—cf. Proposition 3.1 of [15]). The one-parameter group of transformations

$$\begin{cases} \bar{t} = \phi(t, x, \epsilon) \\ \bar{x} = \psi(t, x, \epsilon) \end{cases}$$

leave the functional $z$, defined by $\dot{z} = L(t, x, \dot{x}, z)$ and $z(a) = \gamma$ for some fixed real number $\gamma$, invariant, if

$$\frac{d}{d\epsilon} \left[ L \left( \bar{t}(\bar{t}), \bar{x}(\bar{t}), \frac{d\bar{x}}{d\bar{t}}(\bar{t}), \bar{z}(\bar{t}) \right), \frac{d\bar{t}}{d\bar{t}}(\bar{t}) \right] \bigg|_{\epsilon=0} = 0.$$
We now prove the following useful result.

Lemma 2.4 (Necessary condition for invariance). If the functional \( z = z\{x; t\} \) defined by \( \dot{z}(t) = L(t, x(t), \dot{x}(t), z(t)) \) and \( z(a) = \gamma \), for some fixed real number \( \gamma \), is invariant under the one-parameter group of transformations (7), then

\[
\frac{d\dot{z}}{d\epsilon}(t) \bigg|_{\epsilon=0} = 0
\]

for each \( t \in [a, b] \).

Proof. Note that

\[
\frac{d\dot{z}}{dt}(t) = L \left( \dot{t}, \dot{x}(t), \frac{d\ddot{x}}{dt}(t), \ddot{z}(t) \right)
\]

and by multiplying both sides of the equality by \( \frac{d\ddot{x}}{dt} \) we have, by the chain rule, that

\[
\frac{d\dot{z}}{dt}(t) = \frac{d\dot{z}}{d\epsilon}(t) \frac{d\ddot{x}}{dt}(t) = L \left( \dot{t}, \dot{x}(t), \frac{d\ddot{x}}{dt}(t), \ddot{z}(t) \right) \frac{d\ddot{x}}{dt}(t).
\]

Now, differentiating with respect to \( \epsilon \) and setting \( \epsilon = 0 \), we find, by definition of invariance, that

\[
\frac{d}{dt} \left( \frac{d\dot{z}}{d\epsilon} \right) \bigg|_{\epsilon=0} = \frac{d}{d\epsilon} \left( \frac{d\dot{z}}{d\epsilon} \right) \bigg|_{\epsilon=0} = \frac{d}{d\epsilon} \left[ L \left( \dot{t}, \dot{x}(t), \frac{d\ddot{x}}{dt}(t), \ddot{z}(t) \right) \frac{d\ddot{x}}{dt}(t) \right]_{\epsilon=0} = 0.
\]

Defining \( h(t) := \frac{d\dot{z}}{d\epsilon}(t) \bigg|_{\epsilon=0} \), we get that \( \frac{dh}{dt}(t) = 0 \) for all \( t \in [a, b] \). Since we are supposing the initial condition \( z(a) \) to be fixed \( (z(a) = \gamma) \), then \( \ddot{z}(\bar{a}) \) is also fixed \( (\ddot{z}(\bar{a}) = \ddot{\gamma}) \) and hence \( \frac{d}{d\epsilon}(\ddot{z}(\bar{a})) \bigg|_{\epsilon=0} = 0 \). Observe that if \( \bar{a} = a \), then \( \frac{d\dot{z}}{d\epsilon}(a) \bigg|_{\epsilon=0} = 0; \) if \( \bar{a} \neq a \), then

\[
0 = \frac{\ddot{z}}{d\epsilon}(\ddot{z}(\bar{a})) \bigg|_{\epsilon=0} = \frac{d\ddot{z}}{d\epsilon}(\ddot{a}) \bigg|_{\epsilon=0} \frac{d\ddot{a}}{d\epsilon} \bigg|_{\epsilon=0} = \frac{d\ddot{z}}{d\epsilon}(a) \bigg|_{\epsilon=0} \sigma(a, x)
\]

and because \( \sigma(a, x) \neq 0 \), we can write that \( \frac{d\ddot{z}}{d\epsilon}(a) \bigg|_{\epsilon=0} = 0 \). By definition of \( h \), this means that \( h(a) = 0 \). Since \( h \) is constant on \( [a, b] \), we conclude that

\[
h(t) := \frac{d\dot{z}}{d\epsilon}(t) \bigg|_{\epsilon=0} = 0
\]

for all \( t \in [a, b] \).

\[\square\]

Theorem 2.5 (Noether’s first theorem for variational problems of Herglotz type [15]). If functional \( z = z\{x; t\} \) defined by \( \dot{z} = L(t, x(t), \dot{x}(t), z(t)) \) and \( z(a) = \gamma \), for some fixed real number \( \gamma \), is invariant under the one-parameter group of transformations (7), then

\[
\lambda(t) \cdot \left( [L(x, z)(t) - \dot{x} \partial_3 L(x, z)(t)] \sigma(t, x) + \partial_3 L(x, z)(t) \xi(t, x) \right)
\]

is conserved along the generalized extremals, where \( \lambda(t) := e^{-\int_a^t \partial_4 L(x, z)(\theta) d\theta} \).
3. Main results. We prove some important results for variational problems of Herglotz type with time delay: a generalized Euler–Lagrange necessary optimality condition (Theorem 3.3), a DuBois–Reymond necessary optimality condition (Theorem 3.5) and a Noether’s first theorem for variational problems of Herglotz type with time delay (Theorem 3.9). To simplify the presentation, we suppress most of the arguments and the following notation is used throughout:

\[ [x, z]_\tau(t) := (t, x(t), \dot{x}(t), x(t - \tau), \dot{x}(t - \tau), z(t)). \]

**Definition 3.1** (Admissible function). A function \( x \in C^2([a - \tau, b], \mathbb{R}) \) is said to be admissible for problem \((P)\) if it satisfies the endpoint condition \( x(b) = \beta \) and \( x(t) = \delta(t) \) for all \( t \in [a, \beta]\).

**Definition 3.2** (Admissible variation). We say that \( \eta \in C^2([a - \tau, b], \mathbb{R}) \) is an admissible variation for problem \((P)\) if \( \eta(t) = 0 \) for \( t \in [a - \tau, a] \) and \( \eta(b) = 0 \).

The following result gives a necessary condition of Euler–Lagrange type for an admissible function \( x \) to be an extremizer of the functional \( z(x; b)_\tau \), where \( z \) is defined by (4)–(5).

**Theorem 3.3** (Generalized Euler–Lagrange equations for variational problems of Herglotz type with time delay). If \( x \in C^2([a - \tau, b], \mathbb{R}) \) is a solution to problem \((P)\), then \( x \) satisfies the following generalized Euler–Lagrange equations with time delay:

\[
\lambda(t + \tau) \left[ \partial_4 L[x, z]_\tau(t + \tau) - \frac{d}{dt} \partial_5 L[x, z]_\tau(t + \tau) \right] + \partial_3 L[x, z]_\tau(t + \tau) \partial_6 L[x, z]_\tau(t + \tau) + \lambda(t) \left[ \partial_2 L[x, z]_\tau(t) - \frac{d}{dt} \partial_3 L[x, z]_\tau(t) + \partial_3 L[x, z]_\tau(t) \partial_6 L[x, z]_\tau(t) \right] = 0, \quad (8)
\]

where \( \lambda(t) := e^{-\int_a^t \partial_5 L[x, z]_\tau} \), and

\[
\partial_2 L[x, z]_\tau(t) - \frac{d}{dt} \partial_3 L[x, z]_\tau(t) + \partial_3 L[x, z]_\tau(t) \partial_6 L[x, z]_\tau(t) = 0, \quad (9)
\]

\( a \leq t \leq b - \tau \), where \( \eta(t) := e^{-\int_a^t \partial_5 L[x, z]_\tau} \), and

\[
\dot{\zeta}(t) := \frac{d}{de} z\{x + \epsilon \eta; t\}_\tau \bigg|_{\epsilon=0}.
\]

Obviously, \( \zeta(a) = 0 \) and, since \( x \) is an extremizer, we conclude that \( \zeta(b) = 0 \). Observe that

\[
\dot{\zeta}(t) = \frac{d}{dt} \frac{d}{de} z\{x + \epsilon \eta; t\}_\tau \bigg|_{\epsilon=0} = \frac{d}{dt} \frac{d}{de} z\{x + \epsilon \eta; t\}_\tau \bigg|_{\epsilon=0} = \frac{d}{dt} \partial_2 L[x, \epsilon \eta, z]_\tau(t) \bigg|_{\epsilon=0},
\]

which means that

\[
\dot{\zeta}(t) = \partial_2 L[\eta(t)] + \partial_3 L[\eta(t)] \dot{\eta}(t) + \partial_4 L[\eta(t)] \dot{\eta}(t - \tau) + \partial_5 L[\eta(t)] \dot{\eta}(t - \tau) + \partial_6 L[\eta(t)] \dot{\eta}(t - \tau) + \partial_6 L[\zeta(t)].
\]

Consequently, \( \zeta \) is solution of the first order linear differential equation

\[
\dot{\zeta} = \partial_2 L \eta(t) + \partial_3 L \dot{\eta}(t) + \partial_4 L \eta(t - \tau) + \partial_5 L \dot{\eta}(t - \tau) + \partial_6 L \zeta.
\]
Hence, $\zeta$ satisfies the equation

$$\lambda(t) \zeta(t) - \zeta(a) = \int_a^t \lambda(s) \left[ \partial_2 L[x, z, \tau](s) \eta(s) + \partial_3 L[x, z, \tau](s) \dot{\eta}(s) + \partial_4 L[x, z, \tau](s) \eta(s - \tau) + \partial_5 L[x, z, \tau](s) \dot{\eta}(s - \tau) \right] ds,$$

where $\lambda(t) := e^{-\int_a^t \partial_6 L[x, z, \tau](\theta)d\theta}$. The previous equation is valid for all $t \in [a, b]$, in particular for $t = b$. Because $\zeta(a) = \zeta(b) = 0$, we have

$$\int_a^b \lambda(s) [\partial_2 L[x, z, \tau](s) \eta(s) + \partial_3 L[x, z, \tau](s) \dot{\eta}(s)] ds + \int_a^b \lambda(s) [\partial_4 L[x, z, \tau](s + \tau) \eta(s) + \partial_5 L[x, z, \tau](s + \tau) \dot{\eta}(s)] ds = 0.$$

Applying the change of variable $s = t + \tau$ in the second integral and recalling that $\eta$ is null in $[a - \tau, a]$, we obtain that

$$\int_a^b \lambda(s) [\partial_2 L[x, z, \tau](s) \eta(s) + \partial_3 L[x, z, \tau](s) \dot{\eta}(s)] ds + \int_a^{b-\tau} \lambda(s + \tau) [\partial_4 L[x, z, \tau](s + \tau) \eta(s) + \partial_5 L[x, z, \tau](s + \tau) \dot{\eta}(s)] ds = 0,$$

that is,

$$\int_a^{b-\tau} \left[ \lambda(s) \partial_2 L[x, z, \tau](s) + \lambda(s + \tau) \partial_4 L[x, z, \tau](s + \tau) \right] \eta(s) ds + \int_a^{b-\tau} \left[ \lambda(s) \partial_3 L[x, z, \tau](s) + \lambda(s + \tau) \partial_5 L[x, z, \tau](s + \tau) \right] \dot{\eta}(s) ds = 0.$$

Integration by parts gives

$$\int_a^{b-\tau} \left\{ \lambda(s) \partial_2 L[x, z, \tau](s) + \lambda(s + \tau) \partial_4 L[x, z, \tau](s + \tau) - \frac{d}{ds} [\lambda(s) \partial_3 L[x, z, \tau](s) + \lambda(s + \tau) \partial_5 L[x, z, \tau](s + \tau)] \right\} \eta(s) ds + \int_a^{b-\tau} \left[ (\lambda(s) \partial_3 L[x, z, \tau](s) + \lambda(s + \tau) \partial_5 L[x, z, \tau](s + \tau)) \eta(s) \right]_{a}^{b-\tau} + \int_a^{b} \left[ \lambda(s) \partial_2 L[x, z, \tau](s) - \frac{d}{ds} (\lambda(s) \partial_3 L[x, z, \tau](s)) \right] \eta(s) ds + \int_{b-\tau}^{b} \left[ \lambda(s) \partial_3 L[x, z, \tau](s) \eta(s) \right]_{b-\tau}^{b} = 0.$$

Since previous equation holds for all admissible variations, it holds also for those admissible variations $\eta$ such that $\eta(t) = 0$ for all $t \in [b - \tau, b]$ and, therefore, we get

$$\int_a^{b-\tau} \left\{ \lambda(s) \partial_2 L[x, z, \tau](s) + \lambda(s + \tau) \partial_4 L[x, z, \tau](s + \tau) - \frac{d}{ds} [\lambda(s) \partial_3 L[x, z, \tau](s) + \lambda(s + \tau) \partial_5 L[x, z, \tau](s + \tau)] \right\} \eta(s) ds = 0.$$
From the fundamental lemma of the calculus of variations (see, e.g., [12]), we conclude that
\[
\lambda(t + \tau)\partial_4 L[x, z]_\tau(t + \tau) + \lambda(t)\partial_2 L[x, z]_\tau(t) - \frac{d}{dt} \left[ \lambda(t + \tau)\partial_3 L[x, z]_\tau(t + \tau) + \lambda(t)\partial_3 L[x, z]_\tau(t) \right] = 0
\]
for \( a \leq t \leq b - \tau \), proving equation (8). Now, if we restrict ourselves to those admissible variations \( \eta \) such that \( \eta(t) = 0 \) for all \( t \in [a, b - \tau] \) we get
\[
\int_{b-\tau}^{b} \left[ \lambda(s)\partial_2 L[x, z]_\tau(s) - \frac{d}{ds} (\lambda(s)\partial_3 L[x, z]_\tau(s)) \right] \eta(s) ds = 0
\]
and from the fundamental lemma of the calculus of variations we conclude that
\[
\lambda(t)\partial_2 L[x, z]_\tau(t) - \frac{d}{dt} (\lambda(t)\partial_3 L[x, z]_\tau(t)) = 0
\]
for \( b - \tau \leq t \leq b \), proving equation (9).

**Remark 1.** Note that if there is no time delay, that is, if \( \tau = 0 \), then problem \((P)\) reduces to the classical variational problem of Herglotz and Theorem 2.1 is a corollary of our Theorem 3.3.

In order to simplify expressions, and in agreement with Theorem 3.3, from now on we use the notation
\[
\lambda(t) := e^{-\int_{s}^{t} \partial_4 L[x, z]_\tau(\theta) d\theta}.
\]
The following theorem gives a generalization of the DuBois–Reymond condition for classical variational problems [4] and generalizes the Dubois–Reymond condition for variational problems with time delay of [11].

**Theorem 3.5** (DuBois–Reymond conditions for variational problems of Herglotz type with time delay). If \( x \) is a generalized extremal with time delay such that
\[
\partial_4 L[x, z]_\tau(t + \tau) \cdot \dot{x}(t) + \partial_3 L[x, z]_\tau(t + \tau) \cdot \ddot{x}(t) = 0
\]
for all \( t \in [a - \tau, b - \tau] \), then \( x \) satisfies the following equations:
\[
\frac{d}{dt} \{ \lambda(t)L[x, z]_\tau(t) - \dot{x}(t) [\lambda(t)\partial_3 L[x, z]_\tau(t) + \lambda(t + \tau)\partial_3 L[x, z]_\tau(t + \tau)] \}
\]
\[
= \lambda(t)\partial_1 L[x, z]_\tau(t)
\]
(11)
for \( a \leq t \leq b - \tau \), and
\[
\frac{d}{dt} \{ \lambda(t) [L[x, z]_\tau(t) - \dot{x}(t)\partial_3 L[x, z]_\tau(t)] \} = \lambda(t)\partial_1 L[x, z]_\tau(t)
\]
(12)
for \( b - \tau \leq t \leq b \).
Proof. In order to prove equation (11), let \( t \in [a, b - \tau] \) be arbitrary. Note that

\[
\int_{a}^{t} \frac{d}{ds} \left\{ \lambda(s)L[x, z],(s) - \dot{x}(s) [\lambda(s)\partial_{3}L[x, z],(s) + \lambda(s + \tau)\partial_{5}L[x, z],(s + \tau)] \right\} ds
\]

\[
= \int_{a}^{t} \left\{ - \partial_{6}L[x, z],(s)\lambda(s)L[x, z],(s) + \lambda(s) \left[ \partial_{1}L[x, z],(s) + \partial_{2}L[x, z],(s)\dot{x}(s) + \partial_{3}L[x, z],(s)\ddot{x}(s) + \partial_{4}L[x, z],(s)\dot{x}(s - \tau) + \partial_{5}L[x, z],(s)\ddot{x}(s - \tau) \right] \\
+ \partial_{6}L[x, z],(s)L[x, z],(s) - \dot{x}(s) \left[ \lambda(s)\partial_{3}L[x, z],(s) + \lambda(s + \tau)\partial_{5}L[x, z],(s + \tau) \right]
\]

\[
- \dot{x}(s) \frac{d}{ds} \left[ \lambda(s)\partial_{3}L[x, z],(s) + \lambda(s + \tau)\partial_{5}L[x, z],(s + \tau) \right] \right\} ds.
\]

Canceling symmetrical terms, we get

\[
\int_{a}^{t} \frac{d}{ds} \left\{ \lambda(s)L[x, z],(s) - \dot{x}(s) [\lambda(s)\partial_{3}L[x, z],(s) + \lambda(s + \tau)\partial_{5}L[x, z],(s + \tau)] \right\} ds
\]

\[
= \int_{a}^{t} \left( \lambda(s)\partial_{1}L[x, z],(s) + \lambda(s)\partial_{2}L[x, z],(s)\dot{x}(s) - \dot{x}(s)\lambda(s + \tau)\partial_{5}L[x, z],(s + \tau) \\
- \dot{x}(s) \frac{d}{ds} \left[ \lambda(s)\partial_{3}L[x, z],(s) + \lambda(s + \tau)\partial_{5}L[x, z],(s + \tau) \right] \right) ds
\]

\[
+ \int_{a}^{t} \left( \lambda(s)\partial_{4}L[x, z],(s)\ddot{x}(s - \tau) + \lambda(s)\partial_{5}L[x, z],(s)\ddot{x}(s - \tau) \right) ds.
\]

Observe that, by hypothesis (10), the last integral is null and by substitution of the Euler–Lagrange equation (8) one gets

\[
\int_{a}^{t} \frac{d}{ds} \left\{ \lambda(s)L[x, z],(s) - \dot{x}(s) [\lambda(s)\partial_{3}L[x, z],(s) + \lambda(s + \tau)\partial_{5}L[x, z],(s + \tau)] \right\} ds
\]

\[
= \int_{a}^{t} \left( \lambda(s)\partial_{1}L[x, z],(s) - \lambda(s + \tau) \left[ \partial_{1}L[x, z],(s + \tau)\dot{x}(s) + \ddot{x}(s)\partial_{5}L[x, z],(s + \tau) \right] \right) ds.
\]

Using hypothesis (10) in the right hand side of the last equation, we conclude that

\[
\int_{a}^{t} \frac{d}{ds} \left\{ \lambda(s)L[x, z],(s) - \dot{x}(s) [\lambda(s)\partial_{3}L[x, z],(s) + \lambda(s + \tau)\partial_{5}L[x, z],(s + \tau)] \right\} ds
\]

\[
= \int_{a}^{t} \lambda(s)\partial_{1}L[x, z],(s) ds.
\]

Condition (11) follows from the arbitrariness of \( t \in [a, b - \tau] \). In order to prove equation (12), let \( t \in [b - \tau, b] \) be arbitrary. Note that

\[
\int_{t}^{b} \frac{d}{ds} \left\{ \lambda(s)L[x, z],(s) - \lambda(s)\dot{x}(s)\partial_{5}L[x, z],(s) \right\} ds
\]

\[
= \int_{t}^{b} \left\{ - \partial_{6}L[x, z],(s)\lambda(s)L[x, z],(s) + \lambda(s) \left[ \partial_{1}L[x, z],(s) + \partial_{2}L[x, z],(s)\dot{x}(s) + \partial_{3}L[x, z],(s)\ddot{x}(s) + \partial_{4}L[x, z],(s)\dot{x}(s - \tau) + \partial_{5}L[x, z],(s)\ddot{x}(s - \tau) \right] \\
+ \partial_{6}L[x, z],(s)L[x, z],(s) - \dot{x}(s)\lambda(s)\partial_{3}L[x, z],(s) - \dot{x}(s) \left[ \lambda(s)\partial_{3}L[x, z],(s) \right] \right\} ds.
\]
Cancelling symmetrical terms, the previous equation becomes
\[
\int_t^b \frac{d}{ds} \{ \lambda(s)L[x,z]_\tau(s) - \lambda(s)\dot{x}(s)\partial_3 L[x,z]_\tau(s) \} \, ds \\
= \int_t^b \left\{ \lambda(s)(\partial_1 L[x,z]_\tau(s) + \partial_2 L[x,z]_\tau(s)\dot{x}(s)) - \dot{x}(s)\frac{d}{ds} [\lambda(s)\partial_3 L[x,z]_\tau(s)] \right\} ds \\
+ \int_t^b \left\{ \lambda(s)(\partial_4 L[x,z]_\tau(s)\dot{x}(s - \tau) + \partial_5 L[x,z]_\tau(s)\ddot{x}(s - \tau)) \right\} ds.
\]
Substituting the Euler–Lagrange equation (9) and using the hypothesis (10) in the last integral, we conclude that
\[
\int_t^b \frac{d}{ds} \{ \lambda(s)L[x,z]_\tau(s) - \lambda(s)\dot{x}(s)\partial_3 L[x,z]_\tau(s) \} \, ds = \int_t^b \lambda(s)\partial_1 L[x,z]_\tau(s) \, ds.
\]
Condition (12) follows from the arbitrariness of \( t \in [b - \tau, b] \). \( \square \)

Remark 2. For the classical variational problem and for the variational problem of Herglotz (without delayed arguments), the hypothesis (10) is trivially satisfied. There is an inconsistency in the proof of the DuBois–Reymond equations for the classical variational problem with time delay recently obtained in [11]: the proof is correct if we suppose that
\[
\partial_4 L[x,z]_\tau(t + \tau) \cdot \dot{x}(t) + \partial_5 L[x,z]_\tau(t + \tau) \cdot \ddot{x}(t) = 0
\]
along any extremal with time delay for all \( t \in [a - \tau, b - \tau] \). Such condition is trivially satisfied for the examples presented in [7, 11].

Before presenting the extension of the famous Noether’s first theorem to variational problems of Herglotz type with time delay, we introduce the definition of invariance and give two useful necessary conditions for invariance.

Definition 3.6 (Invariance with time delay). The one-parameter group of invertible \( C^1 \) transformations

\[
\begin{aligned}
\bar{t} &= \phi(t,x,\epsilon) = t + \sigma(t,x)\epsilon + o(\epsilon) \\
\bar{x} &= \psi(t,x,\epsilon) = x + \xi(t,x)\epsilon + o(\epsilon)
\end{aligned}
\]

(13)
leave the functional \( z \) defined by (4)–(5) invariant if
\[
\left. \frac{d}{de} \left[ L \left( \bar{t}, \bar{x}(\bar{t}), \frac{d\bar{x}}{d\bar{t}}(\bar{t}), \frac{d\bar{x}}{d\bar{t}}(\bar{t} - \tau), \bar{z}(\bar{t}) \right) \cdot \frac{d\bar{t}}{d\bar{t}} \right] \right|_{\epsilon=0} = 0.
\]

Lemma 3.7 (Necessary condition for invariance with time delay I). If the functional \( z \) defined by (4)–(5) is invariant under the one-parameter group of transformations (13), then
\[
\left. \frac{d\bar{z}}{d\epsilon}(t) \right|_{\epsilon=0} = 0
\]
for each \( t \in [a,b] \).

Proof. The proof is similar to the one of Lemma 2.4. \( \square \)

The next result is a consequence of Lemma 3.7 and is useful in the proof of our Noether’s first theorem for variational problems of Herglotz with time delay.
Lemma 3.8 (Necessary condition for invariance with time delay II). If the functional $z$ defined by (4)–(5) is invariant under the one-parameter group of transformations (13), then

$$\int_a^t \lambda(s) \left[ \partial_1 L[x, z, \tau](s) \sigma(s) + \partial_2 L[x, z, \tau](s) x(s) + \partial_3 L[x, z, \tau](s) (\dot{\xi}(s) - \dot{x}(s) \sigma(s)) \\
+ \partial_4 L[x, z, \tau](s) (\dot{\xi}(s) - \dot{x}(s) \sigma(s)) + \partial_5 L[x, z, \tau](s) (\dot{\xi}(s) - \dot{x}(s) \sigma(s)) \right] ds = 0$$

(14)

for each $t \in [a, b]$.

Proof. Since

$$\frac{d\bar{z}}{dt}(t) = L\left(\bar{t}, \bar{x}(\bar{t}), \frac{d\bar{x}}{d\bar{t}}(\bar{t}), \bar{x}(\bar{t} - \tau), \frac{d\bar{x}}{d\bar{t}}(\bar{t} - \tau), \bar{x}(\bar{t})\right) \frac{d\bar{t}}{dt}(t)$$

and

$$\left. \frac{d\bar{t}}{dt}(t) \right|_{\epsilon = 0} = 1, \quad \left. \frac{d\bar{x}}{d\epsilon}(t) \right|_{\epsilon = 0} = \frac{d\bar{t}}{dt}(t) \sigma(t, x),$$

we get

$$\left. \frac{d}{d\epsilon} \left( \frac{d\bar{z}}{dt}(t) \right) \right|_{\epsilon = 0} = \left. \frac{dL}{d\epsilon}(t) \right|_{\epsilon = 0} \cdot \left. \frac{d\bar{x}}{dt}(t) \right|_{\epsilon = 0} + L \cdot \left. \frac{d\bar{t}}{dt}(t) \right|_{\epsilon = 0} = \left. \frac{dL}{d\epsilon}(t) \right|_{\epsilon = 0} + L \cdot \frac{d\bar{t}}{dt}(t) \sigma(t, x).$$

Defining $h(t) := \left. \frac{d\bar{z}}{d\epsilon}(t) \right|_{\epsilon = 0}$,

$$\dot{h}(t) = \partial_1 L \left. \frac{d\bar{x}}{d\epsilon}(t) \right|_{\epsilon = 0} + \partial_2 L \left. \frac{d\bar{x}}{d\epsilon}(t) \right|_{\epsilon = 0} + \partial_3 L \left. \frac{d\bar{x}}{d\epsilon}(t) \right|_{\epsilon = 0} + \partial_4 L \left. \frac{d\bar{x}}{d\epsilon}(t - \tau) \right|_{\epsilon = 0} + \partial_5 L \left. \frac{d\bar{x}}{d\epsilon}(t - \tau) \right|_{\epsilon = 0} + \partial_6 L \left. \frac{d\bar{x}}{d\epsilon}(t) \right|_{\epsilon = 0} + L \dot{\bar{\sigma}}.$$  (15)

Next we prove that

$$\left. \frac{d\bar{t}}{d\epsilon} \right|_{\epsilon = 0} = \dot{\bar{t}} - \dot{\bar{x}} \dot{\sigma}.$$  (16)

Because

$$\frac{d\bar{x}}{dt} = \frac{d\bar{x}}{d\epsilon} \frac{d\epsilon}{dt} + \frac{d\bar{x}}{d\epsilon} \left( \frac{d\bar{t}}{dt} \frac{\partial\bar{t}}{\partial\bar{x}} + \frac{d\bar{t}}{d\epsilon} \frac{\partial\bar{t}}{\partial\bar{x}} \right),$$

one has

$$\left. \frac{d\bar{t}}{d\epsilon} \right|_{\epsilon = 0} = \left. \frac{d\bar{t}}{d\epsilon} \left( \frac{d\bar{x}}{d\epsilon} \left( \frac{\partial\bar{t}}{\partial\bar{x}} + \frac{\partial\bar{t}}{\partial\bar{x}} \right) \right) \right|_{\epsilon = 0} \left. \frac{d\bar{x}}{d\epsilon} \right|_{\epsilon = 0}$$

(16)

On the other hand, since

$$\left. \frac{d\bar{t}}{d\epsilon} \right|_{\epsilon = 0} = \left. \frac{d\bar{t}}{d\epsilon} \left( \frac{\partial\bar{t}}{\partial\bar{x}} + \frac{\partial\bar{t}}{\partial\bar{x}} \right) \right|_{\epsilon = 0},$$

we get from equality (16) that

$$\left. \frac{\partial\bar{t}}{\partial t} \frac{d\bar{x}}{d\epsilon} + \dot{\bar{t}} \frac{\partial\bar{t}}{\partial\bar{x}} \right|_{\epsilon = 0} = \left. \frac{d\bar{t}}{d\epsilon} \right|_{\epsilon = 0} + \dot{\bar{t}} \left( \frac{\partial\bar{t}}{\partial\bar{x}} + \frac{\partial\bar{t}}{\partial\bar{x}} \right).$$
and therefore
\[ \frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x} = \left. \frac{d}{dc} \frac{d \bar{x}}{dt} \right|_{c=0} + \dot{x} \dot{\sigma}, \]
which is equivalent to
\[ \left. \frac{d}{dc} \frac{d \bar{x}}{dt} \right|_{c=0} = \dot{\xi} - \dot{x} \dot{\sigma}. \]
Substituting (17) into (15), we get
\[ \dot{h} = \partial_1 L \dot{\sigma} + \partial_2 L \dot{\xi} + \partial_2 L (\dot{\xi} - \dot{x} \dot{\sigma}) + \partial_4 L \dot{\xi}(s - \tau) \]
\[ + \partial_5 L (\dot{\xi}(t - \tau) - \dot{x}(t - \tau) \dot{\sigma}(t - \tau)) + \partial_6 L \dot{\sigma}. \]
Therefore, \( h \) satisfies a first order differential equation whose solution is
\[ \lambda(t)h(t) - h(a) = \int_a^t \lambda(s) \left[ \partial_1 L \dot{\sigma} + \partial_2 L \dot{\xi} + \partial_2 L (\dot{\xi} - \dot{x} \dot{\sigma}) + \partial_4 L \dot{\xi}(s - \tau) \right. \]
\[ + \partial_5 L \left( \dot{\xi}(s - \tau) - \dot{x}(s - \tau) \dot{\sigma}(s - \tau) \right) + \partial_6 L \dot{\sigma} \] \( ds. \]
Finally, since functional \( z \) defined by (4)–(5) is invariant under the one-parameter group of transformations (13), \( h \equiv 0 \) by Lemma 3.7 and we obtain (14).

The next result establishes an extension of the celebrated Noether first theorem to variational problems of Herglotz type with time delay.

**Theorem 3.9** (Noether’s first theorem for variational problems of Herglotz type with time delay). If functional \( z \) defined by (4)–(5) is invariant under the one-parameter group of transformations (13), then the quantities defined by
\[ [\lambda(t) \partial_1 L[x, z]_{\tau}(t) + \lambda(t + \tau) \partial_2 L[x, z]_{\tau}(t + \tau)](t) \]
\[ + [\lambda(t) L[x, z]_{\tau}(t) - \dot{x}(t) (\lambda(t) \partial_2 L[x, z]_{\tau}(t) + \lambda(t + \tau) \partial_5 L[x, z]_{\tau}(t + \tau))] \sigma(t) \]
for \( a \leq t \leq b - \tau \), and
\[ \lambda(t) \left[ \partial_1 L[x, z]_{\tau}(t) \xi(t) + (L[x, z]_{\tau}(t) - \dot{x}(t) \partial_3 L[x, z]_{\tau}(t)) \sigma(t) \right] \]
for \( b - \tau \leq t \leq b \), are conserved along the generalized extremals with time delay that satisfy
\[ \partial_4 L[x, z]_{\tau}(t + \tau) \dot{x}(t) + \partial_5 L[x, z]_{\tau}(t + \tau) \dot{\bar{x}}(t) = 0 \]
for all \( t \in [a - \tau, b - \tau] \), and
\[ \partial_1 L[x, z]_{\tau}(t + \tau) \xi(t) + \partial_5 L[x, z]_{\tau}(t + \tau) \left( \dot{\xi}(t) - \dot{x}(t) \dot{\sigma}(t) \right) = 0 \]
for all \( t \in [a, b - \tau] \).

**Proof.** Suppose that the functional \( z \) defined by (4)–(5) is invariant under the one-parameter group of transformations (13) and that \( x \) is a solution of the Euler–Lagrange equations (8)–(9). From the necessary condition for invariance with time delay II (Lemma 3.8), we get that
\[ \int_a^t \lambda(s) \left[ \partial_1 L \dot{\sigma} + \partial_2 L \dot{\xi} + \partial_2 L (\dot{\xi} - \dot{x} \dot{\sigma}) + \partial_4 L \dot{\xi}(s - \tau) \right. \]
\[ + \partial_5 L \left( \dot{\xi}(s - \tau) - \dot{x}(s - \tau) \dot{\sigma}(s - \tau) \right) + \partial_6 L \dot{\sigma} \] \( ds = 0 \)
for each $t \in [a, b]$. Proceeding with a linear change of variable and noticing that we can assume $\xi$ and $\sigma$ to be null outside $[a, b]$, the last equation is equivalent to

$$
\mathcal{J}(\alpha, \beta) = \int_a^b \lambda(s) \left[ \partial_1 L \sigma + \partial_2 L \xi + \partial_3 L (\xi - \dot{x} \sigma) + L \dot{\xi} \right] dt
+ \lambda(s + \tau) \left[ \partial_1 L (s + \tau) \xi + \partial_2 L (s + \tau) \left( \dot{\xi}(s) - \dot{x} \sigma(s) \right) \right] ds
+ \int_{t - \tau}^{t} \lambda(s) \left[ \partial_1 L \sigma + \partial_2 L \xi + \partial_3 L (\xi - \dot{x} \sigma) + L \dot{\xi} \right] ds = 0. \quad (22)
$$

Using hypothesis (21), equation (22) implies that

$$
\int_a^b \lambda(s) \left[ \partial_1 L \sigma + \partial_2 L \xi + \partial_3 L (\xi - \dot{x} \sigma) + L \dot{\xi} \right] ds = 0.
$$

From the arbitrariness of $t \in [a, b]$ we conclude that

$$
\partial_1 L \sigma + \partial_2 L \xi + \partial_3 L (\xi - \dot{x} \sigma) + L \dot{\xi} = 0 \quad (23)
$$

for all $t \in [a, b]$. Then, equation (22) becomes

$$
\mathcal{J}(\alpha, \beta) = \int_a^b \left( \lambda(s) \partial_1 L \sigma + \lambda(s) \partial_2 L + \lambda(s + \tau) \partial_4 L (s + \tau) \right) \xi
+ \left[ \lambda(s) \partial_3 L + \lambda(s + \tau) \partial_5 L (s + \tau) \right] \dot{\xi}
+ \left[ \lambda(s) L - \dot{x} \left( \lambda(s) \partial_3 L + \lambda(s + \tau) \partial_5 L (s + \tau) \right) \right] \dot{\sigma} ds = 0
$$

for $t \in [a + \tau, b]$. Using integration by parts, one has

$$
\mathcal{J}(\alpha, \beta) = \int_a^b \left( \lambda(s) \partial_1 L \sigma + \lambda(s) \partial_2 L + \lambda(s + \tau) \partial_4 L (s + \tau) \right) \xi
- \frac{d}{ds} \left[ \lambda(s) \partial_3 L + \lambda(s + \tau) \partial_5 L (s + \tau) \right] \xi
- \frac{d}{ds} \left[ \lambda(s) L - \dot{x} \left( \lambda(s) \partial_3 L + \lambda(s + \tau) \partial_5 L (s + \tau) \right) \right] \dot{\sigma} ds
+ \left[ \lambda(s) \partial_3 L + \lambda(s + \tau) \partial_5 L (s + \tau) \right] \dot{\xi}
+ \left[ \lambda(s) L - \dot{x} \left( \lambda(s) \partial_3 L + \lambda(s + \tau) \partial_5 L (s + \tau) \right) \right] \dot{\sigma} \bigg|_a^{a + \tau} = 0.
$$

Observe that the terms in $\xi$ inside the integral are null because $x$ satisfies the Euler–Lagrange equation on $[a, b - \tau]$ and that, from the DuBois–Reymond equation (11), the sum of the remaining terms of the integral is zero. This leads to

$$
\left[ \lambda(s) \partial_3 L + \lambda(s + \tau) \partial_5 L (s + \tau) \right] \dot{\xi}
+ \left[ \lambda(s) L - \dot{x} \left( \lambda(s) \partial_3 L + \lambda(s + \tau) \partial_5 L (s + \tau) \right) \right] \dot{\sigma} \bigg|_a^{a + \tau} = 0
$$

for every $t \in [a + \tau, b]$, which means that

$$
\left( \lambda(s) \partial_3 L + \lambda(t + \tau) \partial_5 L (t + \tau) \right) \dot{\xi}
+ \left( \lambda(s) L - \dot{x} \left( \lambda(s) \partial_3 L + \lambda(t + \tau) \partial_5 L (t + \tau) \right) \right) \dot{\sigma}
$$
is constant for \( t \in [a, b - \tau] \). Consider \([t_1, t_2] \subseteq [b - \tau, b]\). From equation (23) one has
\[
\int_{t_1}^{t_2} (\lambda(s)\partial_1 L \sigma + \lambda(s)\partial_2 L \xi + \lambda(s)\partial_3 L \dot{\sigma} + \lambda(s)(L - \dot{x}\partial_3 L) \sigma) \, ds = 0.
\]
Using integration by parts, we get
\[
\int_{t_1}^{t_2} (\lambda(s)\partial_1 L \sigma + \lambda(s)\partial_2 L \xi - \frac{d}{ds} (\lambda(s)\partial_3 L) \xi - \frac{d}{ds} [\lambda(s)(L - \dot{x}\partial_3 L)] \sigma) \, ds
\]
\[
+ [\lambda(s)\partial_3 L \xi + \lambda(s)(L - \dot{x}\partial_3 L) \sigma]_{t_1}^{t_2} = 0.
\]
Observe that the terms in \( \xi \) inside the integral are null because \( x \) satisfies the Euler–Lagrange equation (9) and that, from the DuBois–Reymond equation (12), the sum of the remaining terms of the integral is zero. This leads to
\[
[\lambda(s)\partial_3 L \xi + \lambda(s)(L - \dot{x}\partial_3 L) \sigma]_{t_1}^{t_2} = 0.
\]
From the arbitrariness of \( t_1, t_2 \in [b - \tau, b] \), we conclude that
\[
\lambda(s)\partial_3 L \xi + \lambda(s)(L - \dot{x}\partial_3 L) \sigma
\]
is constant in \([b - \tau, b]\). This ends the proof of our main result. \( \square \)

**Remark 3.** In the classical variational problem and in the variational problem of Herglotz, the hypotheses (20) and (21) are trivially satisfied.

**Remark 4.** Our first Noether-type theorem is a generalization of Noether’s first theorem for the classical variational problem of Herglotz type presented in [15], that is, Theorem 2.5 is a corollary of Theorem 3.9.

Our results provide generalizations of the variational results with time delay presented in [11]. If the Lagrangian \( L \) in the definition of \( z \), (4), does not depend on \( z \), then \( \partial_3 L \equiv 0 \) and \( \lambda(t) \equiv 1 \). In that case, problem \( (P) \) reduces to the classical variational problem with time delay. The Euler–Lagrange equations, the DuBois–Reymond conditions and Noether’s first theorem with time delay obtained by Frederico and Torres in [11] are particular cases of Theorem 3.3, Theorem 3.5 and Theorem 3.9, respectively. In what follows we use the notation
\[
[x]_\tau := (t, x(t), \dot{x}(t), x(t - \tau), \dot{x}(t - \tau)).
\]

**Corollary 1** (See [11]). If \( x \) is an extremizer to the functional
\[
\int_a^b L(t, x(t), \dot{x}(t), x(t - \tau), \dot{x}(t - \tau)) \, dt,
\]
then \( x \) satisfies the Euler–Lagrange equations
\[
\partial_4 L[x]_\tau(t + \tau) - \frac{d}{dt} \partial_3 L[x]_\tau(t + \tau) + \partial_2 L[x]_\tau(t) - \frac{d}{dt} \partial_3 L[x]_\tau(t) = 0,
\]
a \leq t \leq b - \tau, and
\[
\partial_2 L[x]_\tau(t) - \frac{d}{dt} \partial_3 L[x]_\tau(t) = 0,
\]
b - \tau \leq t \leq b.
Corollary 2 (Cf. [11]). If \( x \) is an extremizer to the functional \((24)\) and
\[
\partial_t L[x, \tau(t + \tau)] \cdot \dot{x}(t) + \partial_{\tau} L[x, \tau(t + \tau)] \cdot \dot{\tau}(t) = 0,
\]
t \in [a - \tau, b - \tau], then \( x \) satisfies the DuBois–Reymond equations
\[
\frac{d}{dt} \left[ L[x, \tau(t)] - \dot{x}(t) \left( \partial_t L[x, \tau(t)] + \partial_{\tau} L[x, \tau(t + \tau)] \right) \right] = \partial_t L[x, \tau(t)],
\]
a \leq t \leq b - \tau, and
\[
\frac{d}{dt} \left[ L[x, \tau(t)] - \dot{x}(t) \partial_{\tau} L[x, \tau(t)] \right] = \partial_t L[x, \tau(t)],
\]
b - \tau \leq t \leq b.  

Corollary 3 (Cf. [11]). If functional \((24)\) is invariant in the sense of Definition 2.3, then the quantities
\[
[\partial_3 L[x, \tau(t)] + \partial_{3\tau} L[x, \tau(t + \tau)]] \xi(t)
+ [L[x, \tau(t)] - \dot{x}(t) \left( \partial_3 L[x, \tau(t)] + \partial_{3\tau} L[x, \tau(t + \tau)] \right)] \sigma(t),
\]
a \leq t \leq b - \tau, and
\[
\partial_t L[x, \tau(t)] \dot{x}(t) + [L[x, \tau(t)] - \dot{x}(t) \partial_3 L[x, \tau(t)] \sigma(t),
\]
b - \tau \leq t \leq b, are conserved along the solutions of the Euler–Lagrange equations \((25)-(26)\) that satisfy
\[
\partial_t L[x, \tau(t)] \cdot \dot{x}(t) + \partial_3 L[x, \tau(t + \tau)] \cdot \dot{\tau}(t) = 0,
\]
t \in [a - \tau, b - \tau], and
\[
\partial_3 L[x, \tau(t + \tau)] \dot{x}(t) + \partial_{3\tau} L[x, \tau(t + \tau)] \left( \dot{\tau}(t) - \dot{x}(t) \sigma(t) \right) = 0,
\]
t \in [a, b - \tau].  

4. Illustrative example. We present an example that shows the usefulness of our results. Consider the following Herglotz’s variational problem with time delay \( \tau = 1 \):
\[
\begin{align*}
z(2) & \rightarrow \text{extr} \\
\dot{z}(t) & = L[x, z]_1(t) := (\dot{x}(t - 1))^2 + z(t), \quad t \in [0, 2], \\
x(t) & = -t, \quad t \in [-1, 0], \\
x(2) & = 1, \quad z(0) = 0.
\end{align*}
\]
For this problem, Euler–Lagrange optimality conditions \((8)-(9)\) given by Theorem 3.3 assert that
\[
\begin{cases}
\dot{x}(t) - \ddot{x}(t) = 0, & t \in [0, 1], \\
0 = 0, & t \in [1, 2].
\end{cases}
\]
Solving the equation of previous system with the initial condition \( x(0) = 0 \), we obtain
\[
x(t) = -k + ke^t, \quad t \in [0, 1],
\]
for some constant \( k \in \mathbb{R} \). Since in \([0, 1]\) \( z \) is defined by \( \dot{z}(t) = 1 + z(t) \), with \( z(0) = 0 \), we obtain
\[
z(t) = e^t - 1, \quad t \in [0, 1].
\]
In order to illustrate our remaining results (Theorems 3.5 and 3.9), we look for trajectories \( x \) that satisfy hypothesis \((10)\): \( 2\dot{x}(t) \ddot{x}(t) = 0, \quad t \in [-1, 1] \). This condition is trivially satisfied in the interval \([-1, 0]\), but leads to \( k = 0 \) and, consequently,
$x(t) = 0$ in $[0,1]$. Hence, we get a family $x_\phi$ of generalized extremals with time delay given by

$$x_\phi(t) = \begin{cases} 
-t, & t \in [-1,0], \\
0, & t \in [0,1], \\
\phi(t), & t \in [1,2], \\
1, & t = 2,
\end{cases} \tag{28}$$

where the continuous function $\phi$ is chosen to guarantee that $x_\phi$ is a $C^2$ function. With $x$ defined by (28) for some $\phi$, and $z$ defined in $[1,2]$ as $\dot{z}(t) = z(t)$ with $z(1) = e - 1$, it follows that $z(t) = e^{t-1}(e - 1)$ for $t \in [1,2]$ and, consequently,

$$z(t) = \begin{cases} 
  e^t - 1, & t \in [0,1], \\
  e^{t-1}(e - 1), & t \in [1,2],
\end{cases} \tag{29}$$

for which $z(2) = e^2 - e$. Next we show that DuBois–Reymond conditions (11)–(12) given by Theorem 3.5 are valid for $x$ and $z$ given by (28)–(29). In this case, (11) reduces to

$$\frac{d}{dt} \left[ \lambda(t) \left( \dot{x}^2(t - 1) + z(t) \right) - \dot{x}(t) (2\lambda(t + 1)\dot{x}(t)) \right] = 0, \quad t \in [0,1],$$

which is equivalent to

$$\frac{d}{dt} \left[ \lambda(t)e^t \right] = 0, \quad t \in [0,1].$$

Since $\lambda(t) = e^{-t}$, condition (11) holds for $t \in [0,1]$. Similarly, it can be proved that condition (12) holds for $t \in [1,2]$. Finally, we show the relevance of our main result (Theorem 3.9). First we define a one-parameter group of transformations on $t$ and $x$ with generators $\sigma(t) \equiv 1$ and $\xi(t) \equiv 0$, respectively. Since the Lagrangian defined in (27) is autonomous, i.e., does not depend explicitly on $t$, then it is invariant in the sense of Definition 3.6. Observe that in this case hypothesis (21) is trivially satisfied. Theorem 3.9 asserts that (18) and (19) are constant in $t$, in intervals $[0,1]$ and $[1,2]$, respectively, along generalized extremals with time delay that satisfy hypotheses (20) and (21). Observe that (18) is equal to

$$\left[ \lambda(t)L[x,z]_1(t) - 2 (\dot{x}(t))^2 \lambda(t + 1) \right] \sigma(t) = e^{-t} \left[ \dot{x}^2(t - 1) + z(t) \right]$$

$$= e^{-t} \left[ 1 + e^t - 1 \right], \quad t \in [0,1],$$

which is equal to one. Similarly, it can be easily proved that quantity (19) is also constant in $t$ and equal to $1 - e^{-1}$.

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