Weak type operator Lipshitz and commutator estimates for commuting tuples

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Abstract. Let \( f : \mathbb{R}^d \to \mathbb{R} \) be a Lipshitz function. If \( B \) is a bounded self-adjoint operator and if \( \{A_k\}_{k=1}^d \) are commuting bounded self-adjoint operators such that \([A_k, B] \in L_1(H)\), then
\[
\|f(A_1, \cdots, A_d), B\|_{1, \infty} \leq c(d)\|\nabla(f)\|_{\infty} \max_{1 \leq k \leq d} \|[A_k, B]\|_1,
\]
where \( c(d) \) is a constant independent of \( f, M \) and \( A, B \) and \( \| \cdot \|_{1, \infty} \) denotes the weak \( L_1 \)-norm.

If \( \{X_k\}_{k=1}^d \) (respectively, \( \{Y_k\}_{k=1}^d \)) are commuting bounded self-adjoint operators such that \( X_k - Y_k \in L_1(H) \), then
\[
\|f(X_1, \cdots, X_d) - f(Y_1, \cdots, Y_d)\|_{1, \infty} \leq c(d)\|\nabla(f)\|_{\infty} \max_{1 \leq k \leq d} \|X_k - Y_k\|_1.
\]

1. Introduction

Let \( f : \mathbb{R} \to \mathbb{R} \) be a Lipshitz function. Let \( M \) be a semi-finite von Neumann algebra and let \( M_{sa} \) be its self-adjoint part. This paper deals with differentiability properties of (multi-dimensional versions of) the mapping
\[
M_{sa} \ni A \mapsto f(A).
\]
The interest in such differentiability problems comes from very diverse directions: (i) the mapping (1.1) relates strongly to perturbations of commutators, (ii) there is a prolific series of papers devoted to differentiability and Lipshitz properties of (1.1), (iii) the map (1.1) relates to Connes’ non-commutative geometry and in particular the spectral action, see [10], [32], [35].

The roots of the results of this paper can be traced back to a problem of Krein [20] which led to a remarkable diversity of papers concerning double operator integrals and Schur multipliers. The original Krein problem asks if for a function \( f \) being Lipshitz implies that it is operator Lipshitz, meaning that (1.1) is Lipshitz for the uniform norm on \( M_{sa} \). Krein’s question is very natural but it was shown that it has a negative answer [13], unless one imposes stricter differentiability assumptions on \( f \) (like belonging to certain Besov or Sobolev spaces), see [1], [2], [28] to name just a few. Contributions to the problem were made by various people including Davies [11], Kato [18] and Kosaki [19] who found positive and negative results (under suitable conditions) for the analogue of Krein’s problem for \( L_p \)-norms.

With the development of double operator integrals (see e.g. [5], [25], [26]) significant steps forward were made on Lipshitz and differentiability properties of the

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For every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ and for every collection $A = \{A_k\}_{k=1}^d \subset B(H)$ of commuting self-adjoint operators such that $[A_k, B] \in L_1(H)$, we have

$$
\|f(A), B\|_{1, \infty} \leq c(d)\|\nabla(f)\|_{\infty} \cdot \max_{1 \leq k \leq d} \|A_k, B\|_1.
$$

For every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ and for every collections $X = \{X_k\}_{k=1}^d \subset B(H)$, $Y = \{Y_k\}_{k=1}^d \subset B(H)$ of commuting self-adjoint operators such that $X_k - Y_k \in L_1(H)$, we have

$$
\|f(X) - f(Y)\|_{1, \infty} \leq c(d)\|\nabla(f)\|_{\infty} \cdot \max_{1 \leq k \leq d} \|X_k - Y_k\|_1.
$$
As a corollary of Theorem 1.1 we extend our main result from [9] to normal operators, see Corollary 5.4 which substantially improves corresponding results in [2, 7] (see also [1]). This extension is based on a strengthened version of the transference principle from [9] as explained in Section 3. In the text we prove a somewhat stronger result than Theorem 1.1 in the terms of double operator integrals (see the next section for the definitions), of which the main Theorem 1.1 is a corollary.

**Theorem 1.2.** For every Lipschitz function \( f : \mathbb{R}^d \to \mathbb{R} \) and for every collection \( A = \{A_k\}_{k=1}^d \) of commuting self-adjoint operator in a semifinite von Neumann algebra \( \mathcal{M} \), we have

\[
\|\tau_{f_k}^A(V)\|_{1,\infty} \leq c(d) \|\nabla(f)\|_\infty \|V\|_1, \quad V \in (L_1 \cap L_2)(\mathcal{M}),
\]

for every \( 1 \leq k_0 \leq d \). Here, \( f_k \) is defined by (2.3).

Our proofs are based on weak type versions of de Leeuw theorems [22] and a delicate analysis of homogeneous Calderón–Zygmund operators.

2. Preliminaries

2.1. General notation. Throughout the paper \( d \) is an integer \( \geq 1 \). Our main result, Theorem 1.1 concerns \( d \)-tuples of commuting self-adjoint operators, whereas the proofs involve an analysis on \( \mathbb{R}^{d+1} \) and \( \mathbb{T}^{d+1} \). We use

\[
\nabla = (\partial_1, \ldots, \partial_{d+1}) = \frac{1}{d+1}(\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_{d+1}})
\]

for the gradient, which is an unbounded operator on \( L_2(\mathbb{R}^{d+1}) \). We use \( \mathcal{F} \) for the Fourier transform \( \mathcal{F}(f)(t) = (2\pi)^{-(d+1)/2} \int_{\mathbb{R}^{d+1}} f(s)e^{-i\langle s,t \rangle}ds \).

Let \( \mathcal{M} \) be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace \( \tau \). In this paper, we always presume that \( \mathcal{M} \) is represented on a separable Hilbert space.

A (closed and densely defined) operator \( x \) affiliated with \( \mathcal{M} \) is called \( \tau \)-measurable if \( \tau(E_{[x]}(s, \infty)) < \infty \) for sufficiently large \( s \). We denote the set of all \( \tau \)-measurable operators by \( S(\mathcal{M}, \tau) \). For every \( x \in S(\mathcal{M}, \tau) \), we define its singular value function \( \mu(x) \) by setting

\[
\mu(t, x) = \inf\{\|x(1 - p)\|_\infty : \tau(p) \leq t\}.
\]

Equivalently, for positive self-adjoint operators \( x \in S(\mathcal{M}, \tau) \), we have

\[
n_x(s) = \tau(E_{x}(s, \infty)), \quad \mu(t, x) = \inf\{s : n_x(s) < t\}.
\]

We have for \( x, y \in S(\mathcal{M}, \tau) \) (see e.g. [23] Corollary 2.3.16)

\[
\mu(t + s, x + y) \leq \mu(t, x) + \mu(s, y), \quad t, s > 0.
\]

Let \( S((0, \infty) \times (0, \infty)) = S(L_\infty((0, \infty) \times (0, \infty)), \int ds) \) where the integral is the Lebesgue integral. Recall that every \( x \in S(\mathcal{M}, \tau), y \in \mathcal{M} \) such \( \mu(x) \otimes \mu(y) \in S((0, \infty) \times (0, \infty)) \) we have (see [9] Eqn. (4.1) for the proof),

\[
\mu(x \otimes y) = \mu(\mu(x) \otimes \mu(y)),
\]

For a measurable function \( f \) on \( \mathbb{R}^{d+1} \) we use \( \sigma_l(f)(t) = f(t^{-1}l), l > 0 \). Note that

\[
\|\sigma_l(f)\|_1 = l^{d+1}\|f\|_1, \quad \|\sigma_l(f)\|_2 = l^{(d+1)/2}\|f\|_2,
\]

where the norms are with respect to the Lebesgue measure on \( \mathbb{R}^{d+1} \).
2.2. **Non-commutative spaces.** For $1 \leq p < \infty$ we set,

$$L_p(M) = \{ x \in S(M, \tau) : \tau(|x|^p) < \infty \}, \quad \|x\|_p = (\tau(|x|^p))^{\frac{1}{p}}.$$ 

The Banach spaces $(L_p(M), \| \cdot \|_p)$, $1 \leq p < \infty$ are separable. Define the space $L_{1,\infty}(M)$ by setting

$$L_{1,\infty}(M) = \{ x \in S(M, \tau) : \sup_{t>0} t\mu(t, x) < \infty \}.$$ 

We equip $L_{1,\infty}(M)$ with the functional $\| \cdot \|_{1,\infty}$ defined by the formula

$$\|x\|_{1,\infty} = \sup_{t>0} t\mu(t, x), \quad x \in L_{1,\infty}(M).$$

It follows from (2.1) that

$$\|x + y\|_{1,\infty} = \sup_{t>0} t\mu(t, x + y) \leq \sup_{t>0} t\mu(t, x) + \sup_{t>0} t\mu(t, y) \leq 2\|x\|_{1,\infty} + 2\|y\|_{1,\infty}.$$ 

In particular, $\| \cdot \|_{1,\infty}$ is a quasi-norm. The quasi-normed space $(L_{1,\infty}(M), \| \cdot \|_{1,\infty})$ is, in fact, quasi-Banach (see e.g. [17 Section 7] or [34]). Naturally we set $L_{1,\infty}(\mathbb{R}^{d+1}) = L_{1,\infty}(L_{\infty}(\mathbb{R}^{d+1}))$ and $L_{1,\infty}(T^{d+1}) = L_{1,\infty}(L_{\infty}(T^{d+1}))$.

2.3. **Weak type inequalities for Calderón-Zygmund operators.** Parcet [27] proved a non-commutative extension of Calderón-Zygmund theory.

Let $K$ be a tempered distribution on $\mathbb{R}^{d+1}$ which we refer to as the *convolution kernel*. We let $W_K$ be the associated Calderón-Zygmund operator, formally given by $f \mapsto K * f$. In what follows, we only consider tempered distributions having local values (that is, which can be identified with measurable functions $K : \mathbb{R}^{d+1} \to \mathbb{C}$).

Let $M$ be a semi-finite von Neumann algebra with normal, semi-finite, faithful trace $\tau$. The operator $1 \otimes W_K$ can, under suitable conditions, be defined as a non-commutative Calderón-Zygmund operator by letting it act on the second tensor leg of $L_1(M) \otimes L_1(\mathbb{R}^{d+1})$. The following theorem in particular gives a sufficient condition for such an operator to act from $L_1$ to $L_{1,\infty}$. Its proof was improved/shortened very recently by Cadilhac [6].

**Theorem 2.1 ([6], [27]).** Let $K : \mathbb{R}^{d+1} \setminus \{0\} \to \mathbb{C}$ be a kernel satisfying the conditions

$$(2.4) \quad |K|(t) \leq \frac{\text{const}}{|t|^{d+1}}, \quad |\nabla K|(t) \leq \frac{\text{const}}{|t|^{d+2}}.$$ 

Let $M$ be a semi-finite von Neumann algebra. If $W_K \in B(L_2(\mathbb{R}^{d+1}))$, then the operator $1 \otimes W_K$ defines a bounded map from $L_1(M \otimes L_{\infty}(\mathbb{R}^{d+1}))$ to $L_{1,\infty}(M \otimes L_{\infty}(\mathbb{R}^{d+1}))$.

We need a very special case of Theorem 2.1.

**Theorem 2.2.** If $g \in L_{\infty}(\mathbb{R}^{d+1})$ is a smooth homogeneous function, then $1 \otimes g(\nabla)$ defines a bounded map from $L_1(M \otimes L_{\infty}(\mathbb{R}^{d+1}))$ to $L_{1,\infty}(M \otimes L_{\infty}(\mathbb{R}^{d+1}))$.

**Proof.** Without loss of generality, the function $g$ is mean zero on the sphere $S^d$ (this can be always achieved by subtracting a constant from $g$). By Theorem 6 on p.75 in [34] and using that $g$ has mean 0, we have $g(\nabla) = W_K$, where $K = F^{-1}(g)$ is a smooth homogeneous function of degree $-d - 1$. The gradient of the function $K$ is
a smooth homogeneous function of degree \(-d-2\). These conditions guarantee that (2.4) holds for \(K\) and by Theorem 2.1, the assertion follows.

In Section 3 we prove the following compact analogue of Theorem 2.2. The transference arguments in Section 4 require such a compact form. We let \(K\) act from \(L_2(M \otimes L_{\infty}(T^{d+1}))\) to \(L_{1,\infty}(M \otimes L_{\infty}(T^{d+1}))\).

**Theorem 2.3.** If \(g\) is a smooth homogeneous function on \(\mathbb{R}^{d+1}\), then the operator \(1 \otimes g(\nabla_{T^{d+1}}) : L_2(M \otimes L_{\infty}(T^{d+1})) \to L_2(M \otimes L_{\infty}(T^{d+1}))\) admits a bounded extension acting from \(L_1(M \otimes L_{\infty}(T^{d+1}))\) to \(L_{1,\infty}(M \otimes L_{\infty}(T^{d+1}))\).

**Remark 2.4.** Theorem 2.3 should be understood as a de Leeuw theorem in the following sense. Assume for simplicity that \(M = \mathbb{C}\). \(g(\nabla)\) of Theorem 2.2 is a Fourier multiplier with symbol \(g\). \(g(\nabla_{T^{d+1}})\) is the Fourier multiplier on \(L_2(T^{d+1})\) whose symbol is the restriction of \(g\) to \(\mathbb{Z}^{d+1}\). Theorem 2.3 then shows that \(g|_{\mathbb{Z}^{d+1}}\) is the symbol of a bounded multiplier \(L_1(T^{d+1}) \to L_{1,\infty}(T^{d+1})\). This is a weak \((1,1)\) version of de Leeuw’s theorem [22].

### 2.4. Double operator integrals

Let \(A = \{A_k\}_{k=1}^d\) be a collection of commuting self-adjoint operators affiliated with \(M\). Consider projection valued measures on \(\mathbb{R}^d\) acting on the Hilbert space \(L_2(M)\) by the formulae

\[
    x \to \left( \prod_{k=1}^d E_{A_k}(B_k) \right) x, \quad x \to x \left( \prod_{k=1}^d E_{A_k}(C_k) \right), \quad x \in L_2(M).
\]

These spectral measures commute and, hence (see Theorem V.2.6 in [3]), there exists a countably additive (in the strong operator topology) projection-valued measure \(\nu\) on \(\mathbb{R}^2\) acting on the Hilbert space \(L_2(M)\) by the formula

\[
    \nu(B_1 \times \cdots \times B_d \times C_1 \times \cdots \times C_d) : x \to \left( \prod_{k=1}^d E_{A_k}(B_k) \right) x \left( \prod_{k=1}^d E_{A_k}(C_k) \right), \quad x \in L_2(M).
\]

Integrating a bounded Borel function \(\xi\) on \(\mathbb{R}^{2d}\) with respect to the measure \(\nu\) produces a bounded operator acting on the Hilbert space \(L_2(M)\). In what follows, we denote the latter operator by \(T_{\xi}^{A,A}\) (see also [20, Remark 3.1]).

In the special case when \(A_k\) are bounded and \(\text{spec}(A_k) \subset \mathbb{Z}\), we have

\[
    T_{\xi}^{A,A}(V) = \sum_{i,j \in \mathbb{Z}^d} \xi(i,j) \left( \prod_{k=1}^d E_{A_k}([i_k]) \right) V \left( \prod_{k=1}^d E_{A_k}([j_k]) \right).
\]

We are mostly interested in the case \(\xi = f_k\) for a Lipschitz function \(f\). Here, for \(1 \leq k \leq d\) and \(\lambda, \mu \in \mathbb{R}^d\),

\[
    f_k(\lambda, \mu) = \begin{cases} 
    \frac{(f(\lambda) - f(\mu))}{(\lambda - \mu)}, & \lambda \neq \mu \\
    0, & \lambda = \mu.
    \end{cases}
\]

### 3. A de Leeuw type theorem for Calderón-Zygmund operators

In this section we collect de Leeuw type results (c.f. [22]) needed in the subsequent proofs. The main result is Theorem 2.3. This theorem should be understood as a restriction theorem for (homogeneous) Fourier multipliers, see Remark 2.4.

The strategy of the proof is as follows. One finds an asymptotic embedding of \(L_1(T^{d+1})\) (resp. \(L_{1,\infty}(T^{d+1})\)) into \(L_1(\mathbb{R}^{d+1})\) (resp. \(L_{1,\infty}(\mathbb{R}^{d+1})\)) such that this
asymptotic embedding intertwines the Fourier multipliers/Calderón-Zygmund operators and their discretizations.

In what follows,
\[
G_l(t) = (l\sqrt{2\pi})^{-(d+1)} e^{-\frac{|t|^2}{2l^2}}, \quad t \in \mathbb{R}^{d+1}, \quad l > 0.
\]
We have that \(\|G_l\|_1 = 1\). Let \(\mathcal{F}\) stand for the Fourier transform. Note that
\[
(\mathcal{F}G_l)(t) = (l\sqrt{2\pi})^{-(d+1)} \int e^{-\frac{|t|^2}{2l^2}} e^{-i(t,s)} ds
\]
\[
= (l\sqrt{2\pi})^{-(d+1)} \int e^{-\frac{|s|^2}{2}} e^{-it(s)} ds = G_l(l t).
\]
We set
\[
e_k(t) := e^{i(k,t)}, \quad k, t \in \mathbb{R}^{d+1}, k \in \mathbb{Z}^{d+1}.
\]
Let \(\alpha = (\alpha_1, \ldots, \alpha_{d+1})\) where \(\alpha_k \in \mathbb{Z}_+\). The notation \(\partial^\alpha\) is used for \(h(\nabla)\), where
\[
h(t) = \prod_{k=1}^{d+1} t_{\alpha_k}, \quad t \in \mathbb{R}^{d+1}.
\]
We have
\[
M_{e_{-k}} g(\nabla) M_{e_k} = g(M_{e_{-k}} \nabla M_{e_k}) = g(\nabla + k).
\]

\textbf{Remark 3.1.} The Gaussian functions \(G_l\) are needed to normalize our asymptotic embeddings given by periodizations of functions (see Lemmas 3.8 and 3.9 for exact statements). These asymptotic embeddings are closely related to the Bohr compactification of \(\mathbb{R}^{d+1}\).

The following lemma is a \((d+1)\)-dimensional analogue of Lemma 7 in [31].

\textbf{Lemma 3.2.} For every function \(h\) on \(\mathbb{R}^{d+1}\) whose partial derivatives up to order \(d+1\) belong to \(L_2(\mathbb{R}^{d+1})\) we have
\[
\|\mathcal{F}^{-1}(h)\|_1 \leq 2^{\frac{d+1}{2}} \sum_{|\alpha| \leq d+1} \|\partial^\alpha (h)\|_2.
\]

\textbf{Proof.} For every \(\mathcal{A} \subset \{1, \ldots, d+1\}\), we define the set \(O_{\mathcal{A}} \subset \mathbb{R}^{d+1}\) by setting
\[
O_{\mathcal{A}} = \{ t \in \mathbb{R}^{d+1}: |t_k| \geq 1, \ k \in \mathcal{A}, \ |t_k| \leq 1, \ k \notin \mathcal{A}\}.
\]
We also define the function \(h_{\mathcal{A}}\) on \(\mathbb{R}^{d+1}\) by setting
\[
h_{\mathcal{A}}(t) = \prod_{k \in \mathcal{A}} t_k, \quad t \in \mathbb{R}^{d+1}.
\]
Note that the sets \(O_{\mathcal{A}}\) form a partition of \(\mathbb{R}^{d+1}\) and that for every choice of \(\mathcal{A}\) we have \(\|h_{\mathcal{A}}^{-1} \chi_{O_{\mathcal{A}}}\|_2 \leq 2^{\frac{d+1}{2}}\).

We have
\[
\|\mathcal{F}^{-1}(h)\|_1 \leq \sum_{\mathcal{A} \subset \{1, \ldots, d+1\}} \|\mathcal{F}^{-1}(h) \chi_{O_{\mathcal{A}}}\|_1.
\]
By the Hölder inequality
\[
\|\mathcal{F}^{-1}(h)\|_1 \leq \sum_{\mathcal{A} \subset \{1, \ldots, d+1\}} \|h_{\mathcal{A}} \mathcal{F}^{-1}(h) \chi_{O_{\mathcal{A}}}\|_2 \|h_{\mathcal{A}}^{-1} \chi_{O_{\mathcal{A}}}\|_2.
\]
By the previous paragraph and the Plancherel identity
\[
\|\mathcal{F}^{-1}(h)\|_1 \leq 2^{\frac{d+1}{2}} \sum_{\mathcal{A} \subset \{1, \ldots, d+1\}} \|\mathcal{F}^{-1}(h_{\mathcal{A}}(\nabla) h)\|_2 = 2^{\frac{d+1}{2}} \sum_{\mathcal{A} \subset \{1, \ldots, d+1\}} \|h_{\mathcal{A}}(\nabla) h\|_2.
\]
The proof follows as \(h_{\mathcal{A}}(\nabla) = \partial^\alpha\). \qed
For a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_{d+1}) \in \mathbb{Z}_+^{d+1} \) let \( |\alpha| = \sum_{i=1}^{d+1} \alpha_i \). We shall without further reference use the fact that \( \partial^\alpha (\sigma_l(f)) = l^{-|\alpha|} \sigma_l(\partial^\alpha(f)) \) for any smooth function \( f \) on \( \mathbb{R}^{d+1} \).

**Lemma 3.3.** Let \( g \in L_\infty(\mathbb{R}^{d+1}) \) be a smooth function with all derivatives assumed to be uniformly bounded. If \( (\partial^\alpha g)(0) = 0 \) for every multi-index \( \alpha \) with \( |\alpha| \leq d \), then

\[
\| (g(\nabla))(G_l) \|_1 \to 0, \quad l \to \infty.
\]

**Proof.** We have \( g(\nabla) = F^{-1} M_g F \), with \( M_g \) the multiplication operator with \( g \) on \( L_2(\mathbb{R}^{d+1}) \). Recall again that \( F(G_l)(t) = G_1(l t), t \in \mathbb{R}^{d+1} \). Thus, see e.g. (3.1),

\[
(g(\nabla))(G_l) = F^{-1} M_g F(G_l) = F^{-1}(g h_l),
\]

where \( h_l(t) = G_1(l t), t \in \mathbb{R}^{d+1} \). It follows from Lemma 3.2 that

\[
\| F^{-1}(g h_l) \|_1 \leq 2^{d+1} \sum_{|\alpha| \leq d+1} \| \partial^\alpha (g h_l) \|_2 \leq 2^{d+1} \sum_{|\alpha|+|\beta| \leq d+1} \| \partial^\alpha (g) \partial^\beta (h_l) \|_2.
\]

Due to the assumption that \( (\partial^\alpha g)(0) = 0 \) for every multi-index \( \alpha \) with \( |\alpha| \leq d \), all coefficients in the Taylor expansion of \( g \) around 0 of the terms of order \( \leq d \) vanish. Therefore, as all derivatives of \( g \) are assumed to be uniformly bounded functions we obtain that \( |\partial^\alpha g| \leq c(g) f^{d+1-|\alpha|}, |\alpha| \leq d+1 \), where \( f(t) = |t| \), for some constant \( c(g) \). Thus,

\[
\| F^{-1}(g h_l) \|_1 \leq 2^{d+1} c(g) \sum_{|\alpha|+|\beta| \leq d+1} \| f^{d+1-|\alpha|} \partial^\beta (h_l) \|_2.
\]

We have

\[
\partial^\beta (h_l) = l^{|eta|} \sigma_{Z_l} (\partial^\beta G_1), \quad f^{d+1-|\alpha|} = l^{|\alpha|-d-1} \sigma_{Z_l} (f^{d+1-|\alpha|}).
\]

Thus,

\[
\| f^{d+1-|\alpha|} \partial^\beta (h_l) \|_2 = l^{|\beta|+|\alpha|-d-1} \| \sigma_{Z_l} (f^{d+1-|\alpha|} \partial^\beta (G_1)) \|_2 = l^{|\beta|+|\alpha|-d-1} \| f^{d+1-|\alpha|} \partial^\beta G_1 \|_2 \to 0.
\]

This concludes the proof. \( \square \)

**Lemma 3.4.** If \( g : \mathbb{R}^{d+1} \to \mathbb{C} \) is a Schwartz function such that \( g(0) = 0 \), then

\[
\| (g(\nabla))(G_l) \|_1 \to 0, \quad l \to \infty.
\]

**Proof.** Define Schwartz functions \( g_j : \mathbb{R}^{d+1} \to \mathbb{C}, 1 \leq j \leq d+1, \) by setting

\[
g_j(t) = \frac{g(0, \cdots, 0, t_j, \cdots, t_{d+1}) - g(0, \cdots, 0, t_{j+1}, \cdots, t_{d+1})}{t_j}, \quad t \in \mathbb{R}^{d+1}.
\]

We have,

\[
g(t) = \sum_{j=1}^{d+1} t_j g_j(t).
\]

and, therefore,

\[
(3.3) \quad g(\nabla)(G_l) = \sum_{j=1}^{d+1} g_j(\nabla) \cdot (\partial_j G_l).
\]

It follows from Young inequality that

\[
\| g_j(\nabla) x \|_1 = \| F^{-1} M_{g_j} F x \|_1 = \| F^{-1} (g_j) * x \|_1 \leq \| F^{-1} (g_j) \|_1 \| x \|_1, \quad x \in L_1(\mathbb{R}^{d+1}).
\]
The proof then follows provided that for \( x = \partial_j G_l, 1 \leq j \leq d + 1 \) we have,
\[
(3.4) \quad \| \partial_j G_l \|_1 \rightarrow 0, \quad l \rightarrow \infty.
\]
Indeed, a direct computation yields,
\[
\partial_j G_l = \frac{1}{l^{d+2}} \sigma_i(h_j), \quad \text{where} \ h_j(t) := it_j G_1(t), \quad t \in \mathbb{R}^{d+1}.
\]
So appealing to (2.3), we obtain
\[
\| \partial_j(G_l) \|_1 = \frac{1}{l} \| h_j \|_1 \rightarrow 0.
\]

**Lemma 3.5.** Let \( g \in L_\infty(\mathbb{R}^{d+1}) \) be a smooth function with all its derivatives assumed to be uniformly bounded. If \( k \in \mathbb{R}^{d+1} \), then
\[
\| (g(\nabla))(G_l e_k) - g(k)G_l e_k \|_1 \rightarrow 0, \quad l \rightarrow \infty.
\]
Here \( e_k \) is given by (3.3).

**Proof.** Suppose first that \( k = 0 \) and \( g(0) = 0 \). Let \( \psi \) be a Schwartz function on \( \mathbb{R}^{d+1} \) such that \( \psi(t) = 1 \) whenever \( |t| \leq 1 \). Set
\[
\phi(t) = \sum_{|\alpha| \leq d} \frac{i^{|\alpha|}}{1!^{d+1} |\alpha_k|!} (\partial^\alpha g)(0)t^\alpha \psi(t), \quad t \in \mathbb{R}^{d+1}.
\]
Clearly, \( \phi \) is a Schwartz function, \( \phi(0) = 0 \) and \( (\partial^\alpha g)(0) = (\partial^\alpha \phi)(0) \) for \( |\alpha| \leq d \).
In other words, the function \( g - \phi \) satisfies the assumptions of Lemma 3.3. Using Lemmas 3.3 and 3.4, we obtain
\[
\| (g - \phi)(\nabla)(G_l) \|_1 \rightarrow 0, \quad \| (\phi)(\nabla)(G_l) \|_1 \rightarrow 0, \quad l \rightarrow \infty.
\]
Using triangle inequality, we obtain
\[
\| (g(\nabla))(G_l) \|_1 \rightarrow 0, \quad l \rightarrow \infty.
\]
This proves the assertion in our special case.

To prove the assertion in general, note that
\[
(3.5) \quad \| g(\nabla)(G_l e_k) - g(k)G_l e_k \|_1 = \| (M_{e_k} g(\nabla) M_{e_k} - g(k))(G_l) \|_1.
\]
Now as \( t \rightarrow g(t + k) - g(k) \) is a function satisfying the assumptions of the first paragraph, we see that (3.5) goes to 0 as \( l \rightarrow \infty \). \( \square \)

The following Lemma 3.6 is the main intertwining property as we explained in the beginning of this section.

**Lemma 3.6.** Let \( g \in L_\infty(\mathbb{R}^{d+1}) \) be a smooth (except at 0) homogeneous function of degree 0. For every \( 0 \neq k \in \mathbb{R}^{d+1} \), we have
\[
\| (g(\nabla))(G_l e_k) - g(k)G_l e_k \|_{1, \infty} \rightarrow 0, \quad l \rightarrow \infty.
\]

**Proof.** Fix \( 0 \neq k \in \mathbb{R}^{d+1} \). Fix a Schwartz function \( \phi \) supported on the ball \( \{ |t|_2 < |k|_2 \} \) such that \( \phi(t) = 1 \) whenever \( |t|_2 \leq \frac{1}{2}|k|_2 \). Clearly, both functions \( \phi \) and \( g(1 - \phi) \) satisfy the conditions of Lemma 3.3. We obtain
\[
\| (g(1 - \phi)(\nabla))(G_l e_k) - g(k)G_l e_k \|_{1, \infty} \leq \| (g(1 - \phi)(\nabla))(G_l e_k) - g(k)G_l e_k \|_{1, \infty} \rightarrow 0, \quad l \rightarrow \infty.
\]
And also
\[ \|(\phi(\nabla))(G_l\epsilon_k)\|_1 \to 0, \quad l \to \infty. \]

By Theorem 1 on p.29 in [33] (see especially Step 2 on p.30; one can also use Theorem 2.2 here), the operator \( g(\nabla) : L_1(\mathbb{R}^{d+1}) \to L_{1,\infty}(\mathbb{R}^{d+1}) \) is bounded. Thus, since \( \phi \) satisfies the assumptions of Lemma 3.3, we have
\[ \|(g(\nabla))(G_l\epsilon_k)\|_{1,\infty} \leq \|g(\nabla)\|_{L_1 \to L_{1,\infty}} \|(\phi(\nabla))(G_l\epsilon_k)\|_1 \to 0, \quad l \to \infty. \]

The assertion follows by applying triangle inequality. \( \square \)

**Lemma 3.7.** Let \( A \in L_1(\mathcal{M}_1) \) and let \( B \in L_{1,\infty}(\mathcal{M}_2) \). We have
\[ \|A \otimes B\|_{1,\infty} \leq \|A\|_1 \|B\|_{1,\infty}. \]

**Proof.** Define the function \( z \) on \((0, \infty)\) by setting \( z(t) := t^{-1}, \ t > 0 \). We have
\[ \mu(A \otimes B) \overset{\text{def}}{=} \mu(\mu(A) \otimes \mu(B)) \leq \|B\|_{1,\infty}\mu(\mu(A) \otimes z). \]

We claim that for every positive decreasing function \( x \in L_1(0, \infty) \), we have \( \mu(x \otimes z) = \|x\|_1z \). Set \( x_n = \sum_{k=0}^{n^2-1} \mu(\frac{k+1}{n}, x)\chi(\frac{k}{n}, \frac{k+1}{n}) \), \( n > 1 \). The functions \( \chi(\frac{k}{n}, \frac{k+1}{n}) \otimes z \), \( 0 \leq k < n^2 \), are disjointly supported and equimeasurable with \( \frac{1}{n}z \). Therefore,
\[ \mu(x_n \otimes z) = \mu\left( \sum_{k=0}^{n^2-1} \mu\left( \frac{k+1}{n}, x \right)\chi(\frac{k}{n}, \frac{k+1}{n}) \otimes z \right) = \mu\left( \bigoplus_{k=0}^{n^2-1} \frac{1}{n} \mu\left( \frac{k+1}{n}, x \right)z \right) = \|x_n\|_1z. \]

It is immediate that \( x_n \uparrow x \) and, therefore, \( x_n \otimes z \uparrow x \otimes z \) and \( \mu(x_n \otimes z) \uparrow \mu(x \otimes z) \). This proves the claim. \( \square \)

Let
\[ \text{per} : \mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}) \to \mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1}) \]
be the natural embedding by periodicity. Under the identification \( \mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1}) \cong L_\infty(\mathbb{R}^{d+1}, \mathcal{M}) \) (the latter being understood as weakly measurable, essentially bounded functions) and similarly for the torus, it is defined as
\[ \text{per}(f)(t) = f(t \mod 2\pi), \quad t \in \mathbb{R}^{d+1}. \]

We consider \( \mathbb{T} \) with total Haar measure \( 2\pi \). The next Lemma 3.8 provides the asymptotic embedding of \( L_1(\mathbb{T}^{d+1}) \) to \( L_1(\mathbb{R}^{d+1}) \).

**Lemma 3.8.** For every \( W \in L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1})) \), we have
\[ \lim_{l \to \infty} \|\text{per}(W) \cdot (1 \otimes G_l)\|_{L_1(M \otimes L_\infty(\mathbb{R}^{d+1}))} = \frac{1}{(2\pi)^{d+1}} \|W\|_{L_1(M \otimes L_\infty(\mathbb{T}^{d+1}))}. \]

**Proof.** For every \( m \in \mathbb{Z} \), define \( l(m), n(m) \in \mathbb{Z} \) by setting
\[ l(m) = \begin{cases} m & m \geq 0 \\ m+1 & m < 0 \end{cases}, \]
\[ n(m) = \begin{cases} m+1 & m \geq 0 \\ m & m < 0 \end{cases}. \]

Next set
\[ l(m) = (l(m_1), \ldots, l(m_{d+1})), \quad m \in \mathbb{Z}^{d+1}, \]
\[ n(m) = (n(m_1), \ldots, n(m_{d+1})), \quad m \in \mathbb{Z}^{d+1}. \]
Similarly,

\[ \|\text{per}(W) \cdot (1 \otimes G_l)\|_{L_1(M \otimes L_\infty(\mathbb{R}^{d+1}))} \]

\[ = \sum_{m \in \mathbb{Z}^{d+1}} \|\text{per}(W) \cdot (1 \otimes G_l) \cdot (1 \otimes \chi_{2\pi m + [0, 2\pi]^{d+1}})\|_{L_1(M \otimes L_\infty(\mathbb{R}^{d+1}))}. \]

By construction,

\[ G_l(2\pi n(m)) \leq G_l(t) \leq G_l(2\pi l(m)), \quad t \in 2\pi m + [0, 2\pi]^{d+1}. \]

Hence,

\[ \|\text{per}(W) \cdot (1 \otimes G_l)\|_{L_1(M \otimes L_\infty(\mathbb{R}^{d+1}))} \]

\[ \leq \sum_{m \in \mathbb{Z}^{d+1}} G_l(2\pi l(m))\|\text{per}(W) \cdot (1 \otimes \chi_{2\pi m + [0, 2\pi]^{d+1}})\|_{L_1(M \otimes L_\infty(\mathbb{R}^{d+1}))} \]

\[ = \|W\|_{L_1(M \otimes L_\infty(T^{d+1}))} \cdot \sum_{m \in \mathbb{Z}^{d+1}} G_l(2\pi l(m)). \]

Similarly,

\[ \|\text{per}(W) \cdot (1 \otimes G_l)\|_{L_1(M \otimes L_\infty(\mathbb{R}^{d+1}))} \]

\[ \geq \sum_{m \in \mathbb{Z}^{d+1}} G_l(2\pi n(m))\|\text{per}(W) \cdot (1 \otimes \chi_{2\pi m + [0, 2\pi]^{d+1}})\|_{L_1(M \otimes L_\infty(\mathbb{R}^{d+1}))} \]

\[ = \|W\|_{L_1(M \otimes L_\infty(T^{d+1}))} \cdot \sum_{m \in \mathbb{Z}^{d+1}} G_l(2\pi n(m)). \]

We have

\[ \sum_{m \in \mathbb{Z}^{d+1}} G_l(2\pi l(m)) = \left( \sum_{m \in \mathbb{Z}} G_l(2\pi l(m)) \right)^{d+1} \]

\[ = \left( \frac{1}{l\sqrt{2\pi}} + \frac{1}{l\sqrt{2\pi}} \sum_{m \in \mathbb{Z}} e^{-\frac{(2\pi m)^2}{2l^2}} \right)^{d+1} \rightarrow \frac{1}{(2\pi)^{d+1}}, \quad l \rightarrow \infty, \]

where the limit is by elementary Riemann integration. Similarly

\[ \sum_{m \in \mathbb{Z}^{d+1}} G_l(2\pi n(m)) = \left( \sum_{m \in \mathbb{Z}} G_l(2\pi n(m)) \right)^{d+1} \]

\[ = \left( -\frac{1}{l\sqrt{2\pi}} + \frac{1}{l\sqrt{2\pi}} \sum_{m \in \mathbb{Z}} e^{-\frac{(2\pi m)^2}{2l^2}} \right)^{d+1} \rightarrow \frac{1}{(2\pi)^{d+1}}, \quad l \rightarrow \infty. \]

Combining the last 4 equations completes the proof as they show that we have estimates

\[ \frac{1}{(2\pi)^{d+1}}\|W\|_{L_1(M \otimes L_\infty(T^{d+1}))} - \epsilon_l \]

\[ \leq \|\text{per}(W) \cdot (1 \otimes G_l)\|_{L_1(M \otimes L_\infty(\mathbb{R}^{d+1}))} \leq \frac{1}{(2\pi)^{d+1}}\|W\|_{L_1(M \otimes L_\infty(T^{d+1}))} + \epsilon_l, \]

for some sequences \( \epsilon_l > 0 \) that converges to 0.

The next lemma gives the asymptotic norm estimate of periodizations of elements of \( L_{1,\infty}(T^{d+1}) \) with the norms of \( L_{1,\infty}(\mathbb{R}^{d+1}) \).
Lemma 3.9. For every \( W \in L_{1, \infty}(\mathcal{M} \otimes L_{\infty}(\mathbb{T}^{d+1})) \), we have
\[
\liminf_{l \to \infty} \|\text{per}(W) \cdot (1 \otimes G_l)\|_{L_{1, \infty}(\mathcal{M} \otimes L_{\infty}(\mathbb{R}^{d+1}))} \gtrsim \|W\|_{L_{1, \infty}(\mathcal{M} \otimes L_{\infty}(\mathbb{T}^{d+1}))}.
\]
Here, \( \gtrsim \) means inequality up to some constant independent of \( W \).

Proof. We estimate crudely,
\[
G_l(t) \geq c(d)l^{-d-1}, \quad |t| \leq 4\pi l,
\]
\[
\chi_{\{|t| \leq 4\pi l\}} \geq \sum_{|m| \leq l} \chi_{2\pi m + [0, 2\pi]^d}.
\]
Hence,
\[
\|\text{per}(W) \cdot (1 \otimes G_l)\|_{L_{1, \infty}(\mathcal{M} \otimes L_{\infty}(\mathbb{R}^{d+1}))} \geq c(d)l^{-d-1}\|\text{per}(W) \cdot (1 \otimes \sum_{|m| \leq l} \chi_{2\pi m + [0, 2\pi]^d})\|_{L_{1, \infty}(\mathcal{M} \otimes L_{\infty}(\mathbb{R}^{d+1}))}.
\]

Since the elements \( \text{per}(W) \cdot (1 \otimes \chi_{2\pi m + [0, 2\pi]^d}) \) with \( |m| \leq l \) are pairwise orthogonal we have that
\[
\text{per}(W) \cdot (1 \otimes \sum_{|m| \leq l} \chi_{2\pi m + [0, 2\pi]^d}) \in L_{1, \infty}(\mathcal{M} \otimes L_{\infty}(\mathbb{R}^{d+1}))
\]
and
\[
\bigoplus_{|m| \leq l} W \in L_{1, \infty}(\mathcal{M} \otimes L_{\infty}(\mathbb{T}^{d+1}) \otimes l_{\infty})
\]
are unitarily equivalent. Then
\[
\|\text{per}(W) \cdot (1 \otimes G_l)\|_{L_{1, \infty}(\mathcal{M} \otimes L_{\infty}(\mathbb{R}^{d+1}))} \geq c(d)l^{-d-1}\|\bigoplus_{|m| \leq l} W\|_{L_{1, \infty}(\mathcal{M} \otimes L_{\infty}(\mathbb{T}^{d+1}) \otimes l_{\infty})}.
\]

Let \( n_l \) be the number of \( m \in \mathbb{Z}^{d+1} \) with \( |m|_2 \leq l \). Note that \( n_l \geq l^{d+1} \). Then
\[
\mu(t, \bigoplus_{|m| \leq l} W) = \mu(n_l^{-1}t, W)
\]
from which we may continue the estimate
\[
\|\text{per}(W) \cdot (1 \otimes G_l)\|_{L_{1, \infty}(\mathcal{M} \otimes L_{\infty}(\mathbb{R}^{d+1}))} \geq c(d)l^{-d-1}n_l\|W\|_{L_{1, \infty}(\mathcal{M} \otimes L_{\infty}(\mathbb{T}^{d+1}))} \geq c(d)\|W\|_{L_{1, \infty}(\mathcal{M} \otimes L_{\infty}(\mathbb{T}^{d+1}))}.
\]

We are now fully equipped to prove our main result.

Proof of Theorem 2.3. Let \( \mathcal{A} \subset \mathbb{Z}^{d+1} \) be a finite set. Let
\[
W = \sum_{k \in \mathcal{A}} W_k \otimes e_k, \quad W_k \in L_1(\mathcal{M}).
\]

Firstly, we prove
\[
\|(1 \otimes g(\nabla))(W)\|_{1, \infty} \lesssim \|W\|_1,
\]
for \( W \) as above. As conditional expectations are contractions on \( L_1 \) we have
\[
\|\sum_{0 \neq k \in \mathcal{A}} W_k \otimes e_k\|_1 \leq \|\sum_{k \in \mathcal{A}} W_k \otimes e_k\|_1 + \|W_0 \otimes e_0\|_1 \leq 2\|W\|_1, \quad k \in \mathcal{A}.
\]

Therefore, we may (and will) assume without loss of generality that \( 0 \notin \mathcal{A} \). By Theorem 2.1, we have
\[
\|(1 \otimes g(\nabla))(\text{per}(W) - (1 \otimes G_l))\|_{L_{1, \infty}(\mathcal{M} \otimes L_{\infty}(\mathbb{R}^{d+1}))} \leq \|\text{per}(W) - (1 \otimes G_l)\|_{L_1(\mathcal{M} \otimes L_{\infty}(\mathbb{R}^{d+1}))}.
\]
By respectively Lemma 3.7 and Lemma 3.8 we have for each $k \in \mathcal{A}$ as $l \to \infty$,
\[
\| (1 \otimes g(\nabla))(W_k \otimes G_l e_k) - g(k)(W_k \otimes G_l e_k) \|_{1, \infty} \leq \| W_k \|_1 \| g(\nabla))((G_l e_k) - g(k)G_l e_k \|_{1, \infty} \to 0.
\]
The quasi-triangle inequality gives for sums of arbitrary operators $x_\alpha$ that
\[
\| \sum_{\alpha \in \mathcal{A}} x_\alpha \|_{1, \infty} \leq 2^{\| \mathcal{A} \|} \sum_{\alpha \in \mathcal{A}} \| x_\alpha \|_{1, \infty}.
\]
So it follows that as $l \to \infty$
\[
\| \sum_{\alpha \in \mathcal{A}} (1 \otimes g(\nabla))(W_k \otimes G_l e_k) - \sum_{\alpha \in \mathcal{A}} g(k)(W_k \otimes G_l e_k) \|_{1, \infty} \to 0.
\]
In other words we have as $l \to \infty$
\[
\| (1 \otimes g(\nabla))(\text{per}(W) \cdot (1 \otimes G_l)) - \| (1 \otimes g(\nabla))((W)) \|_{1, \infty} \to 0.
\]
Thus,
\[
\text{lim inf}_{l \to \infty} \| \text{per}((1 \otimes g(\nabla))(W)) \cdot (1 \otimes G_l) \|_{L_1, \infty}(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1})) \leq \text{lim inf}_{l \to \infty} \| \text{per}(W) \cdot (1 \otimes G_l) \|_{L_1(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1}))).
\]
(3.8)

It follows now from Lemma 3.9 and Lemma 3.8 that
\[
\| (1 \otimes g(\nabla))(W) \|_{L_1, \infty}(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1})) \leq \text{lim inf}_{l \to \infty} \| \text{per}((1 \otimes g(\nabla))(W)) \cdot (1 \otimes G_l) \|_{L_1, \infty}(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1}))) \leq \| W \|_{L_1}(\mathcal{M} \otimes L_\infty(\mathbb{R}^{d+1})),
\]
(3.9)

This proves the assertion for our specific $W$.

To see the assertion in general, fix an arbitrary $W \in L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))$ and choose $W^m$ as above such that $W^m \to W$ in $L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))$ as $m \to \infty$ (see Lemma 3.2). In particular, the sequence $\{ W^m \}_{m \geq 1} \subset L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))$ is Cauchy. By (3.9), the sequence $\{ (1 \otimes g(\nabla))(W^m) \}_{m \geq 1} \subset L_1, \infty(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))$ is also Cauchy. Denote the limit by $T(W)$. If also $W \in L_2(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))$, then the sequence $\{ W^m \}_{m \geq 1} \subset L_2(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))$ (see Remark 4.1). Thus, $T(W) = (1 \otimes g(\nabla))(W)$ for $W \in (L_1 \cap L_2)(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))$. This completes the proof.

4. PROOF OF THEOREM 1.2 FOR THE CASE OF INTEGRAL SPECTRA

The next Theorem 4.1 provides the crucial connection between Calderón–Zygmund operators and commutator estimates. The equality (4.1) should be understood as a transference to Schur multipliers argument. Note that here we have an exact equality (4.1), which we did not yet obtain in [9].

**Theorem 4.1.** For every contraction $f : \mathbb{Z}^d \to \mathbb{Z}$ and for every collection of commuting self-adjoint operators $A = \{ A_k \}_{k=1}^d \subset \mathcal{M}$ with $\text{spec}(A_k) \subset \mathbb{Z}$, we have
\[
\| T^{A,A}_{f_{k_0}}(V) \|_{1, \infty} \leq c(d) \| V \|_1, \quad V \in L_1(\mathcal{M}), \quad 1 \leq k_0 \leq d.
\]
Here, $f_{k_0}$ is given by (2.7).
Proof. Fix $1 \leq k_0 \leq d$. The idea is to construct a bounded linear operator $S : L_1(M \otimes L_\infty(T^{d+1})) \to L_{1,\infty}(M \otimes L_\infty(T^{d+1}))$ (independent of $f$) and an isometric embedding $I : L_1(M) \to L_1(M \otimes L_\infty(T^{d+1}))$. By assumption, $A_k = \sum_{i_k \in Z} i_k p_{k,i_k}$, where $\{p_{k,i_k}\}_{i_k} \in Z$ are pairwise orthogonal projections such that $\sum_{i_k \in Z} p_{k,i_k} = 1$. Since $A$ is bounded, it follows that $p_{k,i_k} = 0$ for all but finitely many $i_k \in Z$. Hence, these sums are, in fact, finite. For every $i = (i_1, \ldots, i_d) \in Z^d$, let $p_i = p_{i_1,i_1} \cdots p_{i_d,i_d}$. It is immediate that $\{p_i\}_{i \in Z^d}$ are pairwise orthogonal projections and $\sum_{i \in Z^d} p_i = 1$. Consider a unitary operator $U_f = \sum_{i \in Z^d} p_i \otimes e(i,f(i))$, where $e(i,f(i))$ is given in (5.3).

We are now ready to define the operators $S$ and $I$. Set

$$S(W) = (1 \otimes g(\nabla_{T^{d+1}}))(\sum_{i,j \in Z^d, i \neq j} (p_i \otimes 1)W(p_j \otimes 1)), \quad W \in L_1(M \otimes L_\infty(R^{d+1})), \quad I(V) = U_f(V \otimes 1)U_f^*, \quad V \in L_{1,\infty}(M).$$

Since $f$ is a contraction we have that $|f(i) - f(j)| \leq |i - j|_2$ and therefore by (4.1) we obtain

$$g(i - j, f(i) - f(j)) = f_{k_0}(i,j), \quad i,j \in Z^d.$$
Recall also that \( f_k(i, i) = 0, i \in \mathbb{Z}^d \). We now prove the transference equality \((1.1)\):

\[
S(I(V)) = S\left( \sum_{i \in \mathbb{Z}^d} p_i \otimes \epsilon_{(i,f(i))} \cdot \left( \sum_{i,j \in \mathbb{Z}^d} p_i V p_j \otimes 1 \right) \right) = U_f \cdot (T^{A,A}_{f_k}(V) \otimes 1) \cdot U_f^* = I(T^{A,A}_{f_k}(V)).
\]

By Theorem \ref{thm:transference}, the mapping

\[
1 \otimes g(\nabla) : L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1})) \to L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1})).
\]

is bounded. Therefore,

\[
\|T^{A,A}_{f_k}(V)\|_{L_{1,\infty}(\mathcal{M})} = \|I(T^{A,A}_{f_k}(V))\|_{L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))} \leq \|S(I(V))\|_{L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))} \leq \|1 \otimes g(\nabla)\|_{L_1(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1}))} \leq L_{1,\infty}(\mathcal{M} \otimes L_\infty(\mathbb{T}^{d+1})) \|V\|_{L_1(\mathcal{M})}.
\]

This completes the proof. \(\square\)

5. PROOF OF THE MAIN RESULTS

In this section we collect the results announced in the abstract and its corollaries.

**Lemma 5.1.** Let \( A = \{A_k\}_{k=1}^d \subset \mathcal{M} \) be an arbitrary collection of commuting self-adjoint operators. If \( \{\xi_n\}_{n \geq 0} \) is a uniformly bounded sequence of Borel functions on \( \mathbb{R}^d \) such that \( \xi_n \to \xi \) everywhere, then

\[
T^{A,A}_{\xi_n}(V) \to T^{A,A}_{\xi}(V), \quad V \in L_2(\mathcal{M})
\]

in \( L_2(\mathcal{M}) \) as \( n \to \infty \).

**Proof.** Let \( \nu \) be a projection valued measure on \( \mathbb{R}^d \) considered in Subsection \ref{subsec:transference} (see \ref{thm:transference}). Let \( \gamma : \mathbb{R} \to \mathbb{R}^d \) be a Borel measurable bijection. Clearly, \( \nu \circ \gamma \) is a countably additive projection valued measure on \( \mathbb{R} \). Hence, there exists a self-adjoint operator \( B \) acting on the Hilbert space \( L_2(\mathcal{M}) \) such that \( E_B = \nu \circ \gamma \).

Set \( \eta_n = \xi_n \circ \gamma \) and \( \eta = \xi \circ \gamma \). We have \( \eta_n \to \eta \) everywhere on \( \mathbb{R} \). Thus,

\[
T^{A,A}_{\eta_n} = \int_{\mathbb{R}^d} \eta_n \, d\nu = \int_{\mathbb{R}} \eta_n(\lambda) \, dE_B(\lambda) = \eta_n(B) \to \eta(B)
\]

Here, the convergence is understood with respect to the strong operator topology on the space \( B(L_2(\mathcal{M})) \). In particular, \((5.1)\) follows. \(\square\)
In the next proof let \( |x| \) be the largest integer smaller than \( x \) and let \( \{x\} = x - |x| \) be the fractional part.

**Proof of Theorem 1.2**

**Step 1.** Let \( f : \mathbb{R}^d \to \mathbb{R} \) be a contraction. We claim that the mapping \( f^n : \mathbb{Z}^d \to \mathbb{Z} \) defined by the formula

\[
f^n(i) = \left\lfloor \frac{n}{2} f\left( \frac{i}{n} \right) \right\rfloor, \quad i \in \mathbb{Z}^d,
\]
is also a contraction.

Indeed, we have

\[
f^n(i) - f^n(j) = \frac{n}{2} (f\left( \frac{i}{n} \right) - f\left( \frac{j}{n} \right)) + (\{\frac{n}{2} f\left( \frac{i}{n} \right)\} - \{\frac{n}{2} f\left( \frac{j}{n} \right)\}).
\]

By assumption, we have that

\[
\frac{n}{2} |f\left( \frac{i}{n} \right) - f\left( \frac{j}{n} \right)| \leq \frac{n}{2} \left| \frac{i}{n} - \frac{j}{n} \right| \leq \frac{1}{2} |i - j|.
\]

It is immediate that

\[
\{\frac{n}{2} f\left( \frac{i}{n} \right)\} - \{\frac{n}{2} f\left( \frac{j}{n} \right)\} \in (-1, 1).
\]

Thus,

\[
|f^n(i) - f^n(j)| \leq \frac{1}{2} |i - j| + 1.
\]

If \( |i - j| \geq 2 \), then

\[
|f^n(i) - f^n(j)| \leq \frac{1}{2} |i - j| + 1 \leq |i - j|
\]

and the claim follows. If \( |i - j| < 2 \), then

\[
|f^n(i) - f^n(j)| \leq \frac{1}{2} |i - j| + 1 < 2.
\]

Since \( |f^n(i) - f^n(j)| \in \mathbb{N} \), it follows that

\[
|f^n(i) - f^n(j)| \leq 1 \leq |i - j|
\]

provided that \( i \neq j \). This proves the claim for \( |i - j| < 2 \).

**Step 2.** Let \( f : \mathbb{R}^d \to \mathbb{R} \) be a contraction. For every \( n \geq 1 \), set

\[
A_{k,n} \overset{\text{def}}{=} \sum_{i_k \in \mathbb{Z}} i_k E_A\left( \left[ \frac{i_k}{n}, \frac{i_k + 1}{n} \right) \right), \quad A_n = \{A_{k,n}\}_{k=1}^d.
\]

Fix \( 1 \leq k_0 \leq d \). Then

\[
\xi_n(t, s) = (f^n)_{k_0}(i, j), \quad t_k \in \left[ \frac{i_k}{n}, \frac{i_k + 1}{n} \right), s_k \in \left[ \frac{j_k}{n}, \frac{j_k + 1}{n} \right), \quad i, j \in \mathbb{Z}^d.
\]

It is immediate that (see e.g. Lemma 8 in [31] for a much stronger assertion)

\[
T_{\xi_n}^{A_n}(V) = T_{(f^n)_{k_0}}^{A_n}(V).
\]

It follows from Theorem 4.1 that

\[
\|T_{\xi_n}^{A_n}(V)\|_{1, \infty} \leq c(d) \|V\|_1.
\]

Note that \( \xi_n \to \frac{1}{2} f_{k_0} \) everywhere. It follows from Lemma 5.1 that

\[
T_{\xi_n}^{A_n}(V) \to T_{\frac{1}{2} f_{k_0}}^{A_n}(V), \quad V \in L_2(M)
\]
in $L_2(M)$ (and, hence, in measure — see e.g. [26]) as $n \to \infty$. Since the quasi-norm in $L_{1,\infty}(M)$ is a Fatou quasi-norm [23], it follows that
\[ \|T_{f_k}^{A,\lambda}(V)\|_{1,\infty} \leq c(d)\|V\|_1, \quad V \in (L_1 \cap L_2)(M). \]

\[ \square \]

**Corollary 5.2.** For every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ and for every collection $A = \{A_k\}_{k=1}^d$ of bounded commuting self-adjoint operators, the operator $T_{f_k}^{A,\lambda}$ extends to a bounded operator from $L_p(M)$ to $L_{p}(M)$, $1 < p < \infty$.

**Proof.** By Theorem 1.2, $T_{f_k}^{A,\lambda}$ extends to a bounded operator from $L_1(M)$ to $L_{1,\infty}(M)$ for every $1 \leq k \leq d$. Since also $T_{f_k}^{A,\lambda} : L_2(M) \to L_2(M)$, it follows from real interpolation that $T_{f_k}^{A,\lambda} : L_p(M) \to L_p(M)$, $1 < p < 2$. Thus, $(T_{f_k}^{A,\lambda})^* : L_{p'}(M) \to L_{p'}(M)$, $1 < p < 2$. Since $f_k(s,t) = f_k(t,s)$, $s,t \in \mathbb{R}^d$, it follows that $(T_{f_k}^{A,\lambda})^* = T_{f_k}^{A,\lambda}$. In particular, $T_{f_k}^{A,\lambda} : L_{p'}(M) \to L_{p'}(M)$, $1 < p < 2$. This concludes the proof.

**Lemma 5.3.** If $A_k, B \in B(H)$, $1 \leq k \leq d$, are self-adjoint operators such that $[A_k, B] \in L_2(H)$, $1 \leq k \leq d$, then, for every Lipschitz function $f$, we have
\[ \sum_{k=1}^d T_{f_k}^{A,\lambda}([A_k, B]) = [f(A), B]. \]

Here $f_k$ is given by (2.7).

**Proof.** By definition of double operator integral given in Subsection 2.4, we have for any bounded Borel function on $\mathbb{R}^d$,
\[ T_{\xi_1}^{A,\lambda}T_{\xi_2}^{A,\lambda} = T_{\xi_1 \xi_2}^{A,\lambda}. \]

Let $\xi_{1,k} = f_k$ and let $\xi_{2,k}(\lambda, \mu) = \lambda_k - \mu_k$ when $|\lambda_2|, |\mu_2| \leq \sup_{1 \leq k \leq d} \|A_k\|_{\infty}$, $\xi_{2,k}(\lambda, \mu) = 0$ when $|\lambda_2| > \sup_{1 \leq k \leq d} \|A_k\|_{\infty}$ or $|\mu_2| > \sup_{1 \leq k \leq d} \|A_k\|_{\infty}$. It is immediate that
\[ (\sum_{k=1}^d \xi_{1,k} \xi_{2,k})(\lambda, \mu) = f(\lambda) - f(\mu), \quad \lambda, \mu \in \mathbb{R}^d, \text{ s.t. } |\lambda_2|, |\mu_2| \leq \sup_{1 \leq k \leq d} \|A_k\|_{\infty}. \]

If $p$ is a finite rank projection, then $pB \in L_2(H)$ and
\[ T_{\sum_{k=1}^d \xi_{1,k} \xi_{2,k}}^{A,\lambda}(pB) = f(A)pB - pBf(A), \quad T_{\xi_{2,k}}^{A,\lambda}(pB) = A_k pB - pBA_k, \]

Applying (5.2) to the operator $pB \in L_2(H)$, we obtain
\[ \sum_{k=1}^d T_{f_k}^{A,\lambda}(A_k pB - pBA_k) = f(A)pB - pBf(A). \]

By Theorem 4.2 in [36], there exists a sequence $p_l$ of finite rank projections such that $p_l \to 1$ strongly and such that, for every $1 \leq k \leq d$, $[A_k, p_l] \to 0$ as $l \to \infty$ in $L_2(H)$ for $d > 1$ and in $L_2(H)$ if $d = 1$. In particular,
\[ A_k p_l B - p_l B A_k = p_l [A_k, B] + [A_k, p_l] B \to [A_k, B], \quad l \to \infty, \]
in $L_2(H)$. 

By the preceding paragraph and Corollary 5.2 we have
\[ T_{f_k}^{A}(A_k p_l B - p_l B A_k) \to T_{f_k}^{A}([A_k, B]), \quad l \to \infty, \]
in \( L_d(H) \). On the other hand,
\[ f(A)p_l B - p_l B f(A) \to f(A)B - B f(A), \quad l \to \infty, \]
in the strong operator topology. Substituting (5.4) and (5.5) into (5.3), we conclude the proof. \( \square \)

**Proof of Theorem 1.1.** By assumption, \([A_k, B] \in L_1(H) \subset L_2(H)\). The first assertion follows by combining Lemma 5.3 and Theorem 1.2. Applying the first assertion to the operators
\[ A_k = \begin{pmatrix} X_k & 0 \\ 0 & Y_k \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]
we obtain the second assertion. \( \square \)

**Corollary 5.4.** For every Lipschitz function \( f : \mathbb{C} \to \mathbb{R} \) and for every normal operator \( A \in B(H) \) and every \( B \in B(H) \) such that \([A, B] \in L_1(H)\), we have
\[ \|[f(A), B]\|_{1, \infty} \leq c(d)\|\nabla(f)\|_{\infty}\|[A, B]\|_1. \]

For every Lipschitz function \( f : \mathbb{C} \to \mathbb{R} \) and for every pair \( X, Y \in B(H) \) of normal operators such that \( X - Y \in L_1(H) \), we have
\[ \|f(X) - f(Y)\|_{1, \infty} \leq c(d)\|\nabla(f)\|_{\infty}\|X - Y\|_1. \]

**Proof.** An operator \( A \) is normal if and only it can be written as \( A = A_1 + iA_2 \) with \( A_1 \) and \( A_2 \) commuting self-adjoint operators. Identifying \( \mathbb{C} \simeq \mathbb{R}^2 \) we may see \( f \) as a 2 real variable Lipschitz function, say \( \tilde{f} \), and this identification is compatible with spectral calculus, i.e. \( f(A) = \tilde{f}(A_1, A_2) \). Then the corollary is a direct consequence of the statements in Theorem 1.1. \( \square \)

**Appendix A. Fejér’s lemma**

In our proof we use a von Neumann-valued Fejér’s lemma. As we could not find a reference to this type of vector valued case we prove it here for convenience of the reader.

We let \( e_l, l \in \mathbb{Z} \) denote the standard trigonometric functions on the torus. Let \( \mathcal{E} \) be the conditional expectation \( \mathcal{M} \otimes L^\infty(\mathbb{T}^{d+1}) \to \mathcal{M} \otimes 1 \). For \( k \in \mathbb{Z}^{d+1}_+ \), let
\[ S_k(x) = \sum_{l \in \mathbb{Z}^{d+1}_+ \atop -k \leq l \leq k} \mathcal{E}(x(1 \otimes e_l^*)(1 \otimes e_l)). \]
For \( n \in \mathbb{Z}_+ \), we set
\[ A_n(x) = (n + 1)^{-d-1} \sum_{k \in \mathbb{Z}^{d+1}_+ \atop k \leq (n, \ldots, n)} S_k(x). \]
Here, the order on \( \mathbb{Z}^{d+1}_+ \) is defined by \( m \leq n \) if \( m_j \leq n_j \) for all \( 1 \leq j \leq d + 1 \).

**Remark A.1.** It follows directly that for \( x \in L_2(\mathcal{M} \otimes \mathbb{T}^{d+1}) \) we have \( \|A_n x - x\|_2 \to 0 \) as \( n \to \infty \).

The assertion below is known as Fejér’s lemma.
Lemma A.2. We have \( \|A_n(x) - x\|_1 \to 0 \) for all \( x \in L_1(\mathcal{M} \otimes \mathbb{T}^{d+1}) \) as \( n \to \infty \).

Proof. We split the proof in steps.

Step 1. We claim that
\[
\|A_n x\|_1 \leq \|x\|_1, \quad x \in L_1(\mathcal{M} \otimes \mathbb{T}^{d+1}), \quad n \geq 0.
\]
To see this fact, we identify the space \( L_1(\mathcal{M} \otimes \mathbb{T}^{d+1}) \) with the space of vector-valued functions \( L_1(\mathbb{T}^{d+1}, L_1(\mathcal{M})) \). We now write a pointwise equality
\[
(A_n(x))(t) = \int_{\mathbb{T}^{d+1}} x(t+s) \Phi_n(s) ds, \quad s \in \mathbb{T}^{d+1}.
\]
Here, \( \Phi_n : \mathbb{T}^{d+1} \to \mathbb{R} \) is the Fejér kernel possessing the following properties.
\[
\Phi_n(s) \geq 0, \quad \int_{\mathbb{T}^{d+1}} \Phi_n(s) ds = 1.
\]
Thus,
\[
\|A_n x\|_1 \leq \int_{\mathbb{T}^{d+1}} \|x(\cdot + s)\|_1 \Phi_n(s) ds = \|x\|_1.
\]

Step 2. Fix \( \epsilon > 0 \) and choose a projection \( p \in \mathcal{M} \) such that \( \tau(p) < \infty \) and such that \( \|x'\|_1 < \epsilon \), where
\[
x' := x - (p \otimes 1)x(p \otimes 1).
\]
Choose \( y \in L_2(p\mathcal{M} p \otimes \mathbb{T}^{d+1}) \) such that
\[
\|y - (p \otimes 1)x(p \otimes 1)\|_1 < \epsilon.
\]
In particular, we have that \( \|y - x\|_1 < 2\epsilon \).
We clearly have \( A_n y \to y \) in \( L_2(p\mathcal{M} p \otimes \mathbb{T}^{d+1}) \). Since \( \tau(p) < \infty \), it follows that \( A_n y \to y \) in \( L_1(p\mathcal{M} p \otimes \mathbb{T}^{d+1}) \). Thus, \( A_n y \to y \) in \( L_1(\mathcal{M} \otimes \mathbb{T}^{d+1}) \). Choose \( N \) so large that \( \|A_n y - y\|_1 < \epsilon \) for \( n > N \). It follows from Step 1 that
\[
\|A_n(x - x)\|_1 \leq \|A_n(x - y)\|_1 + \|A_n y - y\|_1 + \|x - y\|_1 \leq 2\|x - y\|_1 + \|A_n y - y\|_1 \leq 4\epsilon + \|A_n y - y\|_1 < 5\epsilon, \quad n > N.
\]
Since \( \epsilon > 0 \) is arbitrarily small, the assertion follows. \( \square \)

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