Maximum-likelihood Soft-decision Decoding for Binary Linear Block Codes Based on Their Supercodes

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Abstract—Based on the notion of supercodes, we propose a two-phase maximum-likelihood soft-decision decoding (tpMLSD) algorithm for binary linear block codes in this work. The first phase applies the Viterbi algorithm backwardly to a trellis derived from the parity-check matrix of the supercode of the linear block code. Using the information retained from the first phase, the second phase employs the priority-first search algorithm to the trellis corresponding to the linear block code itself, which guarantees finding the ML decision. Simulations on Reed-Muller codes show that the proposed two-phase scheme is an order of magnitude more efficient in average decoding complexity than the recursive maximum-likelihood decoding (RMLD) [1] when the signal-to-noise ratio per information bit is 4.5 dB.

I. INTRODUCTION

Linear block codes have been deployed for error control in communication systems for many years, while algebraic structures of such codes are generally used for their decoding [2]. Since the inputs of algebraic decoders are commonly required to be quantized into two levels, they are classified as hard-decision decoding technique. In comparison with soft-decision decoding technique, a loss of information is induced due to quantization and hence the decoding performance is restricted. By contrast, the soft-decision decoding is developed to eliminate the performance loss due to quantization. The input of soft-decision decoding is thus unquantized (or practically quantized into more than two levels). In the literature, many maximum-likelihood (ML) soft-decision decoding algorithms for linear block codes have been proposed [1], [3]–[11], and the priority-first search algorithm (PFSA) is one of them [6]. It has been shown in [6] that the PFSA can provide the optimal ML decoding performance within practically acceptable decoding complexity.

In this paper, a novel two-phase maximum-likelihood soft-decision decoding (tpMLSD) scheme based on supercodes of linear block codes is proposed. Specifically, in the first phase, the Viterbi algorithm (VA) is applied to a trellis, derived from the parity-check matrix of the supercode of the linear block code to be decoded, in a backward fashion (i.e., operated from the last trellis level to the first trellis level). Upon the completion of the first phase, each state will retain a path metric that is used later in the second phase. Because the trellis derived from the parity-check matrix of the supercode of the linear block code has fewer states than that derived from the parity-check matrix of the linear block code itself, the computational complexity is considerably reduced.

In the second phase, the priority-first search algorithm is applied to the trellis corresponding to the parity-check matrix of the linear block code. With a properly designed evaluation function, the optimal ML decision is guaranteed to be located. Notably, the path metric information obtained from the first phase is incorporated into the evaluation function for priority-first search, by which the decoding procedure can be significantly sped up. Simulations on Reed-Muller codes are then performed to confirm the efficiency of the proposed two-phase ML soft-decision decoding scheme.

It should be pointed out that the idea of decoding linear block codes based on their super codes is not new in the literature. It has been used in the hard-decision decoding in [12], where super codes are designed based upon covering sets and split syndromes. In addition, a suboptimal hard-decision list decoding of linear block codes based on trellises of supercodes was presented in [13]. Further generalization of [13] to ML soft-decision list decoding and to soft-output decoding can be found in [14] and [15], respectively.

The rest of this paper is organized as follows. Notions of supercodes and ML soft-decision decoding of linear block codes are introduced in Section II. The proposed two-phase ML soft-decision decoding algorithm for binary linear block codes based on their supercodes is presented in Section III. The optimality of the proposed algorithm is proved in Section IV. Section V evaluates the complexity of the proposed algorithm for practical linear block codes, and Section VI concludes the paper.

II. NOTIONS OF SUPERCODES AND ML SOFT-DECISION DECODING OF LINEAR BLOCK CODES

Let \( \mathcal{C} \) be an \((n, k)\) binary linear block code with parity-check matrix \( \mathbb{H} \) of size \((n-k) \times n\). Denote by \( \mathcal{S} \) an \((n, k)\) supercode of \( \mathcal{C} \) with parity-check matrix \( \mathbb{H} \) of size \((n-k) \times n\),
where completes the construction of trellis the hard-decision sequence matrix
\( (\text{and its parity-check matrix} \ r) \)
probability density function (pdf) for continuous (unquantized) for some matrix \( a \) of proper size. Obviously, 

The trellis \( T \) derived from \( H \) is then formed by picking up all paths corresponding to codewords of \( C \).

By convention, state \( s_\ell \) identifies a node on trellis \( T \) at level \( \ell \). In particular, \( s_{\ell-1} \) and \( s_{\ell+1} \) identify the initial and the final node on trellis \( T \) at levels \( -1 \) and \( n \), respectively. In addition, the branch connecting state \( s_{\ell-1} \) and state \( s_\ell \) is labeled with code bit \( v_\ell \). As such, the one-to-one mapping between codewords of \( C \) and paths over \( T \) is built. This completes the construction of trellis \( T \) based on parity-check matrix \( H \). The super-trellis \( T \) corresponding to supercode \( \overline{C} \) and its parity-check matrix \( \overline{H} \) can be similarly constructed, of which its state at level \( \ell \) is denoted by \( \overline{s}_\ell \).

We next introduce the ML soft-decision decoding for codes with trellis representation. Denote again by \( v \triangleq (v_0, v_1, \ldots, v_{n-1}) \) a binary zero-one codeword of \( C \). Define the hard-decision sequence \( y \triangleq (y_0, y_1, \ldots, y_{n-1}) \) corresponding to the received vector \( r \triangleq (r_0, r_1, \ldots, r_{n-1}) \) as

\[
y_j = \begin{cases} 
1, & \text{if } \phi_j < 0; \\
0, & \text{otherwise,}
\end{cases}
\]

where

\[
\phi_j \triangleq \log \frac{\Pr(r_j|0)}{\Pr(r_j|1)}
\]
is the log-likelihood ratio, and \( \Pr(r_j|0) \) and \( \Pr(r_j|1) \) are the conditional probabilities of receiving \( r_j \) given 0 and 1 were transmitted, respectively. Here, \( \Pr(r_j|0) \) can be either a probability density function (pdf) for continuous (unquantized) \( r_j \) or a probability mass function (pmf) for discrete (softly quantized) \( r_j \).

The syndrome of \( y \) is given by \( yH^T \), where superscript “\(^T\)” denotes the matrix transpose operation. Let \( E(\alpha) \) be the collection of all error patterns whose syndrome is \( \alpha \). Then, the maximum-likelihood (ML) decoding output \( \hat{v} \) for received vector \( r \) satisfies:

\[
\hat{v} = y \oplus e^*;
\]

where \( e^* = (e_0^*, e_1^*, \ldots, e_{n-1}^*) \in E(yH^T) \) is the error pattern satisfying

\[
\sum_{j=0}^{n-1} e_j^*|\phi_j| \leq \sum_{j=0}^{n-1} e_j|\phi_j|
\]

for all \( e = (e_0, e_1, \ldots, e_{n-1}) \in E(yH^T) \), and “\( \oplus \)” denotes component-wise modulo-two addition. We thereby define a new metric for paths in a trellis as follows.

Definition 1 (ML path metric): For a path with labels \( x_{(\ell)} = (x_0, x_1, \ldots, x_\ell) \), which ends at level \( \ell \) on trellis \( T \), define the metric associated with it as

\[
M(x_{(\ell)}) \triangleq \sum_{j=0}^{\ell} M(x_j),
\]

where \( M(x_j) \triangleq (y_j \oplus x_j)|\phi_j| \) is the bit metric. Similarly, for a backward path with labels \( \bar{x}_{(\ell)} = (\bar{x}_\ell, \bar{x}_{\ell+1}, \ldots, \bar{x}_{n-1}) \) on super-trellis \( \overline{T} \), define the metric associated with it as

\[
M(\bar{x}_{(\ell)}) \triangleq \sum_{j=\ell}^{n-1} M(\bar{x}_j).
\]

After giving the notions of supercode and super-trellis as well as path metrics, we proceed to present the proposed two-phase decoding scheme in the next section.

III. TWO-PHASE ML SOFT-DECISION DECODING ALGORITHM FOR BINARY LINEAR BLOCK CODES

As mentioned in the introduction section, the proposed decoding algorithm has two phases.

The first phase applies the Viterbi algorithm backwardly to the super-trellis derived from the parity-check matrix \( \overline{H} \) of supercode \( \overline{C} \) using the path metric defined in (2), during which the path metric of the backward survivor starting from the final node at level \( n \) and ending at a node corresponding to state \( \bar{s}_\ell \) at level \( \ell \) is retained for use in the second phase. For convenience of referring it, we denote this path metric by \( c(\bar{s}_\ell) \). At the end of the first phase, a backward survivor path ending at the initial node at level \( 1 \) is selected. The backward Viterbi algorithm in the first phase is summarized below.

Phase 1: The backward Viterbi Algorithm

Step 1. Associate zero initial metric with the backward path containing only the final state \( \bar{s}_{n-1} \) on super-trellis \( \overline{T} \), and let \( c(\bar{s}_{n-1}) = 0 \). Set \( \ell = n-1 \).

Step 2. Decrease \( \ell \) by one. Compute the metrics for all backward paths extending from the backward survivors ending at level \( \ell + 1 \) (and hence entering a state at level \( \ell \)). For each state \( \bar{s}_\ell \) at level \( \ell \), keep the entering path with the least metric as its survivor,

\[1\]It is clear that a path on a trellis can not only be identified by its labels, but also be determined by the states it traverses. Accordingly, path \( x_{(\ell)} \) can be equivalently designated by \( (\bar{s}_0, \bar{s}_{\ell+1}, \ldots, \bar{s}_{n-1}) \).
and delete the remaining. Let $c(s_t)$ be this least metric.

Step 3. If $\ell = 0$, stop the algorithm; otherwise, go to Step 5.

In the second phase, the priority-first search algorithm is operated on trellis $\mathcal{T}$ in the usual forward fashion (i.e., from level 0 to level $n-1$); hence, the second phase always outputs a codeword in $\mathcal{C}$.

Now for each path with labels $x_{(\ell)}$ on trellis $\mathcal{T}$, an evaluation function $f$ associated with it is defined as:

$$ f (x_{(\ell)}) = g (x_{(\ell)}) + h (x_{(\ell)}) ,$$

where the value of $g$-function is assigned according to:

$$ g (x_{(\ell)}) = \begin{cases} 0, & \ell = -1; \\ g (x_{(\ell-1)}) + M (x_{(\ell)}), & \ell = 0, 1, \ldots, n-1 \end{cases}$$

(3)

and the value of $h$-function is given by:

$$ h (x_{(\ell)}) = c (\beta (s_{(\ell)})) .$$

(4)

In $\mathcal{A}$, $s_{\ell}$ is the ending state of the path with label $x_{(\ell)}$, and $\beta (s_{\ell})$ is the state $s_{\ell}$ on super-trellis $\mathcal{T}$ that has the same first $(n-k)$ components as $s_{\ell}$. Note that $\beta (s_{\ell})$ exists and is well-defined for every $s_{\ell}$ on trellis $\mathcal{T}$ since the parity-check matrices of $\mathcal{C}$ and $\mathcal{C}$ satisfy (1).

It can be verified that $f (x_{(n-1)}) = g (x_{(n-1)})$ since $\beta (s_{n-1}) = 0$ and $h (x_{(n-1)}) = c (\beta (s_{n-1})) = 0$. This implies that the path with the minimum $f$-function value on trellis $\mathcal{T}$ is exactly the one with the minimum ML path metric.

Two storage spaces are necessary for the priority-first search over trellis $\mathcal{T}$. The Open Stack records the paths visited thus far by the priority-first search, while the Close Table keeps the starting and ending states and ending levels of the paths that have ever been on top of the Open Stack. They are so named because the paths in the Open Stack can be further extended and hence remain open, but the paths with information in the Closed Table are closed for further extension.

We summarize the priority-first search algorithm over trellis $\mathcal{T}$ in the following.

(Phase 2: The Priority-First Search Algorithm)

Step 1. Let $\rho = \infty$, and assign $x = \emptyset$.

Step 2. Load into the Open Stack the path containing only the initial state $s_{-1}$ at level $-1$.

Step 3. If the Open Stack is empty, output $x$ as the final ML decision, and stop the algorithm.

Step 4. If the starting and ending states and ending level of the top path in the Open Stack have been recorded in the Close Table, discard the top path from the Open Stack, and go to Step 5; otherwise, record the starting and ending states and ending level of this top path in the Close Table.

Step 5. Compute the $f$-function values of the successors of the top path in the Open Stack, and delete the top path from the Open Stack. If the $f$-function value of any successor is equal to or greater than $\rho$, just delete it.

Step 6. For all remaining successor paths that reach level $n-1$, set $\rho$ to be the least path metric among them, and update $x$ as the successor path corresponding to this least path metric and discard all the others.

Step 7. Insert the remaining successor paths (from Steps 5 and 6) into the Open Stack, and re-order the paths in the Open Stack according to ascending $f$-function values. Go to Step 3.

IV. Optimality of the Proposed Algorithm

This section proves the optimality of the proposed two-phase decoding algorithm. We begin with two essential lemmas required for the optimality proof.

Lemma 1: Let path $x_{(\ell+1)}$ be an immediate successor of path $x_{(\ell)}$ on trellis $\mathcal{T}$. Denote the ending states of $x_{(\ell+1)}$ and $x_{(\ell)}$ by $s_{\ell+1}$ and $s_{\ell}$, respectively. Then,

$$ \beta (s_{\ell+1}) = \beta (s_{\ell}) + x_{(\ell+1)} h_{\ell+1},$$

where $h_{\ell+1}$ is the $(\ell + 2)$th column of parity-check matrix $\mathcal{H}$. In other words, there exists a branch between $\beta (s_{\ell})$ and $\beta (s_{\ell+1})$ with label $x_{(\ell+1)}$ over super-trellis $\mathcal{T}$.

Proof: Recall that

$$ s_{\ell+1} = s_{\ell} + x_{(\ell+1)} h_{\ell+1}$$

and

$$ h_{\ell+1} = \begin{bmatrix} h_{\ell+1} \\ p_{\ell+1} \end{bmatrix}$$

for some $p_{\ell+1}$ according to (1). It is thus obvious that $\beta (s_{\ell+1}) = \beta (s_{\ell}) + x_{(\ell+1)} h_{\ell+1}$ since $h_{\ell+1}$ contains the first $(n-k)$ components of $h_{\ell+1}$.

Lemma 2: $f$ is a non-decreasing function along any path on trellis $\mathcal{T}$, i.e.,

$$ f (x_{(\ell)}) \leq f (x_{(\ell+1)}) ,$$

where path $x_{(\ell+1)}$ is an immediate successor of path $x_{(\ell)}$ over trellis $\mathcal{T}$.

Proof: The fundamental attribute of the backward Viterbi algorithm in the first phase gives that $c (\beta (s_{\ell}))$ is the minimum metric among all backward paths that end at state $\beta (s_{\ell})$ at level $\ell$. By Lemma 1 we have:

$$ (c (\beta (s_{\ell})) \leq c (\beta (s_{\ell+1})) + M (x_{(\ell+1)}) ,$$

where $s_{\ell+1}$ and $s_{\ell}$ are respectively the states that paths $x_{(\ell+1)}$ and $x_{(\ell)}$ end at. Hence, we derive:

$$ f (x_{(\ell+1)}) = g (x_{(\ell+1)}) + h (x_{(\ell+1)}) = g (x_{(\ell)}) + M (x_{(\ell+1)} + c (\beta (s_{\ell+1}))) \geq g (x_{(\ell)}) + c (\beta (s_{\ell})) = f (x_{(\ell)}) .$$

Based on these two lemmas, the next theorem proves the optimality of the proposed two-phase algorithm.
Theorem 1: In the second phase, the priority-first search algorithm always output an ML path.

Proof: It suffices to prove that if the Open Stack is empty, the algorithm will output an ML path as claimed in Step 5. This can be confirmed by showing that Steps 4 and 5 never delete any ML path.

Suppose that in Step 4 the starting and ending states and ending level of the new top path \( x_\ell \) have been recorded in the Close Table at some previous time due to path \( \hat{x}_\ell \). Since path \( x_\ell \) must be an offspring of some path \( x_{j\ell} \) that once coexisted with path \( \hat{x}_\ell \) in the Open Stack at the time path \( \hat{x}_\ell \) was on top of the Open Stack, where \( j < \ell \), we have

\[
f(x_\ell) \geq f(x_{j\ell}) \geq f(\hat{x}_\ell). \tag{5}
\]

Notably, the first inequality in \( f \) follows from Lemma \( 2 \) and the second inequality in \( f \) is valid because the top path in the Open Stack always carries the minimum \( f \)-function value among all coexisting paths. As a result, the offsprings of path \( x_\ell \) ending at level \( n - 1 \) cannot yield smaller metrics than those length-\( n \) offsprings of path \( \hat{x}_\ell \), and hence deletion of path \( x_\ell \) will not compromise the optimality of the decoding algorithm.

For Step 5, we argue that \( \rho \) is either a trivial upper bound of the final ML path (cf. Step \( 1 \)) or the metric of a valid path that reaches level \( n - 1 \) (cf. Step \( 5 \)), so deletion of any successor paths whose \( f \)-function values are no less than \( \rho \) will never eliminate any ML path. This completes the proof of optimality of the proposed algorithm.

V. Evaluation of Computational Efforts

In this section, we investigate by simulations the computational effort of the proposed decoding algorithm over the additive white Gaussian noise (AWGN) channels. We assume that the codeword is antipodally modulated, and hence the received vector is given by

\[ r_j = (-1)^{v_j} \sqrt{E} + \lambda_j, \]

for \( 0 \leq j \leq n - 1 \), where \( E \) is the signal energy per channel bit, and \( \{\lambda_j\}_{j=0}^{n-1} \) are independent noise samples of a white Gaussian process with single-sided noise power per hertz \( N_0 \). The signal-to-noise ratio (SNR) for the channel is therefore given by \( \text{SNR} = \frac{E}{N_0} \). In order to account for the code redundancy for different code rates, we will use the SNR per information bit in the following discussion, which is defined as

\[ \text{SNR}_b = \frac{nE/k}{N_0} = \frac{E}{k} \left( \frac{E}{N_0} \right). \]

It can be easily verified that for antipodal-input AWGN channels, the log-likelihood ratio \( \phi_j \) is a fixed multiple of the received scalar \( r_j \); thus, the metric associated with a path \( x_{\ell} \) can be equivalently simplified to

\[ M(x_{\ell}) \triangleq \sum_{j=0}^{\ell} (y_j \oplus x_j)|r_j|, \]

where \( y_j \triangleq \begin{cases} 1, & \text{if } r_j < 0; \\ 0, & \text{otherwise.} \end{cases} \)

The decoding complexity in the first phase is clearly determined by the number of bit metric computations performed. We emphasize that the decoding complexity in the second phase can also be regulated by the number of \( f \)-function evaluations (equivalently, the number of bit metric computations as indicated in \( f \)) during the priority-first search. This is due to that the cost of searching and re-ordering of stack elements can be made a constant multiple of the computational complexity by adopting a priority-queue data structure in stack implementation. One can even employ a hardware-based stack structure and attain constant complexity in stack maintenance. Therefore, to use the number of overall metric computations as the key determinant of algorithmic complexity for our proposed two-phase decoding algorithm is justified.

We now turn to empirical examination of the average decoding complexity of the proposed tpMLSD algorithm. The linear block code considered is the \( r \)th order binary Reed-Muller code, \( \text{RM}(r, m) \), which is an \( (n, k) \) linear block code with \( n = 2^m \) and \( k = 1 + \sum_{i=1}^{m} \binom{m}{i} \). It is known that \( \text{RM}(r + i, m) \) is a supercode of \( \text{RM}(r, m) \) for \( i \geq 1 \). In our simulations, \( \text{RM}(2, 6) \) and \( \text{RM}(4, 6) \); hence, \( n = 64, k = 22 \) and \( k = 57 \). Under the same optimal ML performance, we compare the proposed two-phase ML soft-decision decoding (tpMLSD) algorithm with the recursive ML decoding (RMLD) algorithm \( 1 \) and the list ML decoding (LMLD) algorithm \( 14 \) in average decoding complexity, and summarize the results in Table \( 1 \).

Note that instead of listing the decoding complexity of the LMLD, lower bounds obtained from decoding its supercode counterpart using the marking algorithm are given \( 14 \). Apparently, the real decoding complexity of the LMLD is higher than this lower bound. The table then shows that the LMLD is much more complex than the other two algorithms, and our two-phase decoding algorithm consumes only 1/13 of the computational effort of the RMLD at \( \text{SNR}_b = 4.5 \) dB, in which circumstance the bit error rate (BER) is around \( 10^{-5} \). Further, when \( \text{SNR}_b \) is reduced to 3 dB, the average computational complexity of the proposed two-phase decoding scheme can still reach 1/8 of that of the RMLD.
VI. CONCLUSION

In this work, we proposed a two-phase scheme for ML soft-decision decoding of linear block codes. This novel decoding algorithm has two phases, where the backward Viterbi algorithm is employed on a supercode of the linear block code in the first phase, while the priority-first search algorithm is performed on the trellis of the linear block code in the second phase. Simulations showed that the computational complexity of the proposed two-phase scheme is one order of magnitude better than that of the RMLD when SNR_b = 4.5 dB. Since such a new approach can be extended to decoding any linear block codes when their supercodes are obtainable, a possible future work is to extend this two-phase decoding scheme to codes like Reed-Solomon, for which maximum-likelihood soft-decision decoding is generally considered a challenging task.

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