Inequalities on Complex $L_p$ Centroid Bodies

CHENG Manli, ZHOU Yanping†

College of Science & Three Gorges Mathematical Research Center, China Three Gorges University, Yichang 443002, Hubei, China

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Abstract: Based on the notion of the complex $L_p$ centroid body, we establish Brunn-Minkowski type inequalities and monotonicity inequalities for complex $L_p$ centroid bodies in this article. Moreover, we obtain the affirmative form of Shephard type problem for the complex $L_p$ centroid bodies and its negative form.

Key words: complex $L_p$ centroid body; Brunn-Minkowski type inequalities; Shephard type problem

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Biography: CHENG Manli, female, Master candidate, research direction: convex geometric analysis. E-mail: 1806857113@qq.com
† To whom correspondence should be addressed. E-mail: zhyp5208@163.com

0 Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex sets with non-empty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors, the set of origin-symmetric convex bodies, we write $\mathcal{K}^n_o$ and $\mathcal{K}^n_{os}$, respectively. Let $V(K)$ denote the volume of $K$ and $S^{n-1}$ the unit sphere.

Centroid bodies are a classical notion from geometry which have attracted increasing attention in recent years [1-11]. In 1997, Lutwak and Zhang [12] introduced the concept of $L_p$ centroid body as follows: For each compact star-shaped about the origin $K \in \mathbb{R}^n$ and $1 < p \leq \infty$, the $L_p$ centroid body, $\Gamma_p K$, of $K$ is the origin-symmetric convex body whose support function is defined by

$$h(\Gamma_p K, u) = \frac{1}{(n + p)c_{n,p}V(K)} \int_{S^{n-1}} |u \cdot v|^p \rho(K, v)^{n+p} dS(v)$$

for any $u \in S^{n-1}$. Refs. [13-19] had conducted a series of studies on the $L_p$ centroid body, and many scholars were attracted. The $L_p$ centroid body has got many results from these articles. Particularly, Refs. [20, 21] gave the Brunn-Minkowski inequality and monotonicity inequality for the $L_p$ centroid body. Grinberg and Zhang [22] gave the Shephard problems for the $L_p$ centroid body.

However, complex convex geometry has been studied in many works [23-28]. In this paper, we mainly study the complex centroid body. First, we introduce some notations in complex vector space $\mathbb{C}^n$. Let $C(\mathbb{C}^n)$ denote
the set of compact convex subsets of complex vector space \( \mathbb{C}^n \). Let \( \mathcal{K}(\mathbb{C}^n) \), \( \mathcal{K}_o(\mathbb{C}^n) \), and \( \mathcal{K}_{os}(\mathbb{C}^n) \) denote the set of complex convex bodies, the set of complex convex bodies containing the origin in their interiors, and the set of origin symmetric complex convex bodies, respectively. Let \( \mathcal{S}(\mathbb{C}^n) \), \( \mathcal{S}_o(\mathbb{C}^n) \), and \( \mathcal{S}_{os}(\mathbb{C}^n) \) denote the set of complex star bodies, the set of complex star bodies containing the origin in their interiors, and the set of origin symmetric complex star bodies, respectively. \( S^{2n} \) stands for the complex unit sphere.

Harber\[29\] firstly proposed the complex centroid body of \( K \) and established the Busemann-Petty centroid inequality. In 2021, Wu\[30\] introduced the concept of the \( L_p \) complex centroid body \( \Gamma_{p,C}K \) as follows: If \( p > 1, K \in \mathcal{K}_a(\mathbb{C}^n) \) and \( C \in \mathcal{K}(\mathbb{C}) \), the complex \( L_p \) centroid body \( \Gamma_{p,C}K \) is the convex body with support function

\[
h(\Gamma_{p,C}K,u)^p = \frac{1}{(2n + p)V(K)} \int_{S^{2n}} h(Cu,v)^p \rho(K,v)^{2n+p} dS(v)
\]

where the integration is with respect to the push forward of the Lebesgue measure under the canonical isomorphism \( \eta \) and as for \( \eta \), it is the canonical isomorphism between \( \mathbb{C}^{n} \) and \( \mathbb{R}^{2n} \), i.e.,

\[
\eta(c)=(\Re[c_1], \ldots, \Re[c_n], \Im[c_1], \ldots, \Im[c_n]), c \in \mathbb{C}^{n}
\]

where \( \Re, \Im \) are the real part and imaginary part, respectively. It is obvious to get that if \( p \geq 1, K \in \mathcal{S}_a(\mathbb{C}^n) \), then

\[
\Gamma_{p,C}(\lambda K) = \lambda \Gamma_{p,C}K
\]

In this article, associated with the definition of complex \( L_p \) centroid body, we continuously study the complex \( L_p \) centroid body. Let \( \Gamma^*_p K \) denote the polar of \( \Gamma_{p,C}K \) and \( \Gamma^*_p K \) denote the polar for complex conjugate of \( \Gamma_{p,C}K \). First, we establish the Brunn-Minkowski type inequalities for complex \( L_p \) centroid bodies.

**Theorem 1** If \( p \geq 1, K,L \in \mathcal{S}_{os}(\mathbb{C}^n) \) and \( C \in \mathcal{K}(\mathbb{C}) \), then

\[
V\left( \Gamma_{p,C}(K^p + L^p) \right)^{\frac{p}{2n}} \geq V\left( \Gamma_{p,C}K \right)^{\frac{p}{2n}} + V\left( \Gamma_{p,C}L \right)^{\frac{p}{2n}}
\]

with equality if and only if \( K \) and \( L \) are real dilation.

**Theorem 2** If \( p \geq 1, K,L \in \mathcal{S}_{os}(\mathbb{C}^n) \) and \( C \in \mathcal{K}(\mathbb{C}) \), then

\[
V\left( \Gamma_{p,C}(K^p + L^p) \right)^{\frac{p}{2n}} \geq V\left( \Gamma_{p,C}K \right)^{\frac{p}{2n}} + V\left( \Gamma_{p,C}L \right)^{\frac{p}{2n}}
\]

with equality if and only if \( K \) and \( L \) are real dilation.

Then we obtain monotonicity inequalities for complex \( L_p \) centroid bodies.

**Theorem 3** For \( \rho \geq 1, K,L \in \mathcal{S}_o(\mathbb{C}^n) \), \( C \in \mathcal{K}(\mathbb{C}) \), if \( \tilde{V}_\rho(K,Q) \leq \tilde{V}_\rho(L,Q) \) for any \( Q \in \mathcal{S}_o(\mathbb{C}^n) \), then

\[
\frac{V(\Gamma_{p,C}K)^{\frac{p}{2n}}}{(K)^{\frac{p}{2n}}} \leq \frac{V(\Gamma_{p,C}L)^{\frac{p}{2n}}}{(L)^{\frac{p}{2n}}}
\]

with equality if and only if \( K=L \).

**Theorem 4** For \( \rho \geq 1, K,L \in \mathcal{S}_o(\mathbb{C}^n) \), \( C \in \mathcal{K}(\mathbb{C}) \), if \( \tilde{V}_\rho(K,Q) \leq \tilde{V}_\rho(L,Q) \) for any \( Q \in \mathcal{S}_o(\mathbb{C}^n) \), then

\[
\frac{V(\Gamma^*_pK)^{\frac{p}{2n}}}{(K)^{\frac{p}{2n}}} \geq \frac{V(\Gamma^*_pL)^{\frac{p}{2n}}}{(L)^{\frac{p}{2n}}}
\]

with equality if and only if \( K=L \).

Finally, we study the \( L_p \) Shephard type problem of complex \( L_p \) centroid bodies and give the negative form.

**Theorem 5** Let \( \mathcal{Z}^*_p \) denote the set of polar for complex conjugate of \( \Gamma_{p,C}K \). For \( K \in \mathcal{S}_o(\mathbb{C}^n) \), \( L \in \mathcal{Z}^*_p, p \geq 1 \), if \( \Gamma_{p,C}K \subset \Gamma_{p,C}L \), then \( V(K) \leq V(L) \) with equality if and only if \( K=L \).

**Theorem 6** For \( p \geq 1, L \in \mathcal{S}_o(\mathbb{C}^n) \), if \( L \) is not origin symmetric star body, then there exists \( K \in \mathcal{S}_{os}(\mathbb{C}^n) \) such that \( \Gamma_{p,C}K \subset \Gamma_{p,C}L \), but \( V(K) > V(L) \).

Throughout this paper, we assume that \( \dim C > 0 \).

## 1 Preliminaries

In this section, we collect complex reformulations of well-known results from convex geometry. These complex versions can be directly deduced from their real counterparts by an appropriate application of \( \eta \). For standard reference, the readers may consult the books of Gardner\[31\] and Schneider\[32\].

### 1.1 Complex Support Functions and Radial Functions

For a complex number \( c \in \mathbb{C}^n \), we write \( \tau \) for its conjugate and \( \|c\| \) for its norm. If \( \phi \in \mathbb{C}^{\text{conj}} \), then \( \phi^* \) denotes the conjugate transpose of \( \phi \) and if \( \phi \) is in-
verible, $\phi^{-1}$ denotes the inverse of $\phi$. A complex convex body $K \in \mathcal{K}(\mathbb{C}^n)$ is uniquely determined by its support function $h(K,x): \mathbb{C}^n \to \mathbb{R}$,

$$h(K,x) = \max \{ \Re[x \cdot y] : y \in K \}$$

where “$\cdot$” means the standard Hermitian inner production in $\mathbb{C}^n$ and $\Re[x \cdot y]$ is the real part of $x \cdot y$. It is easy to see that $h_{\lambda K} = \lambda h_K$ for all $\lambda > 0$ and $h_{\phi K} = \phi^* \circ h_K$ for all $\phi \in \text{GL}(n, \mathbb{C})$. The complex radial function $\rho_K(x) = \rho(K,x) : \mathbb{C}^n \setminus \{ 0 \} \to [0, \infty)$ of a compact star-shaped (about the origin) $K$ is defined, for $x \in \mathbb{C}^n \setminus \{ 0 \}$, by

$$\rho(K,x) = \max \{ \lambda \geq 0 : \lambda x \in K \}$$

It is easy to see that $\rho_{\lambda K} = \lambda \rho_K$ for all $\lambda > 0$ and $\rho_{\phi K} = \phi^{-1} \circ \rho_K$ for all $\phi \in \text{GL}(n, \mathbb{C})$. If $\rho_K$ is positive and continuous, $K$ will be called a star body. Moreover, if $K \in \mathcal{K}_o(\mathbb{C}^n)$, it is easy to certify that

$$h_K = \frac{1}{\rho_K} \rho_K$$

An application of polar coordinates to the volume of a complex star body $K \in \mathcal{S}_o(\mathbb{C}^n)$ gives that

$$V(K) = \frac{1}{2^n} \int_{S^{2n-1}} \rho(K,u)^{2n} \, d\mathcal{S}(K,u)$$

1.2 Complex $L_p$ Mixed Volume and Dual $L_p$ Mixed Volume

For $p \geq 1$, $K, L \in \mathcal{K}_o(\mathbb{C}^n)$ and $\alpha, \beta \geq 0$ (not both zero), the complex $L_p$ Minkowski combination $\alpha \cdot K +_p \beta \cdot L$ is defined by

$$h(\alpha \cdot K +_p \beta \cdot L,u)^p = \alpha h(K,u)^p + \beta h(L,u)^p$$

The complex $L_p$ mixed volume, $V_p(K,L)$ of $K, L \in \mathcal{K}_o(\mathbb{C}^n)$ is defined by (see Ref.[33])

$$\frac{2n}{p} V_p(K,L) = \lim_{\varepsilon \to 0^+} \frac{V(K + \varepsilon \cdot L) - V(K)}{\varepsilon}$$

By (9) we have $V_p(K,L) = V_p(\eta K, \eta L)$ and for $\phi \in \text{GL}(n, \mathbb{C})$,

$$V_p(\phi K, \phi L) = |\phi|^p V_p(K,L)$$

For every Borel set $\sigma \subseteq S^{2n-1}$, the complex surface area measure $S_K$ of $K \in \mathcal{K}(\mathbb{C}^n)$ is defined by

$$S_K(\sigma) = \mathcal{H}^{2n-1}(\{ x \in K, \exists u \in \sigma, \Re[x \cdot y] = h_K(u) \})$$

where $\mathcal{H}^{2n-1}$ stands for $(2n-1)$-dimensional Hausdorff measure on $\mathbb{R}^{2n}$.

In addition, the complex surface area measures are translation invariant and $S_{K + \sigma} = S_K(\sigma)$ for all $c \in S^{n-1}$ and each Borel set $\sigma \subseteq S^{2n-1}$. If $p \geq 1$, we define the complex $L_p$ surface area measure $S_{p,K}$ of $K \in \mathcal{K}(\mathbb{C}^n)$ as

$$S_{p,K}(\sigma) = \int_{\sigma} h(K,v)^{-p} \, d\mathcal{S}(K,v)$$

For $K, L \in \mathcal{K}_o(\mathbb{C}^n)$, there is the $L_p$ surface area measure $S_{p,K}$ of $K$ on $S^{2n-1}$ such that

$$V_p(K,L)^{2n} \geq V(K)^{2n-p} V(L)^p$$

with equality if and only if $K$ and $L$ are real dilation. The real $L_p$ Minkowski inequality and its proof are shown in Ref.[32].

For $p \geq 1$, $K, L \in \mathcal{S}_o(\mathbb{C}^n)$ and $\alpha, \beta \geq 0$ (not both zero), the complex $L_p$ harmonic radial combination $\alpha \cdot K +_p \beta \cdot L$ is defined by

$$\rho(\alpha \cdot K +_p \beta \cdot L,u)^p = \alpha \rho(K,u)^p + \beta \rho(L,u)^p$$

Then the dual complex $L_p$ mixed volume $\tilde{V}_{-p}(K,L)$ is defined by (see Ref.[33])

$$\frac{2n}{p} \tilde{V}_{-p}(K,L) = \lim_{\varepsilon \to 0^+} \frac{\tilde{V}(K + \varepsilon \cdot L) - \tilde{V}(K)}{\varepsilon}$$

The polar coordinate formula for volume yields

$$\tilde{V}_{-p}(K,L) = \frac{1}{2^n} \int_{S^{2n-1}} \rho(K,u)^{2n+p} \rho(L,u)^{-p} \, d\mathcal{S}(u)$$

Particularly, $\tilde{V}_{-p}(K,K) = V(K)$.

The integral representation (12), together with the Hölder inequality[34] immediately gives that

$$\tilde{V}_{-p}(K,L)^{2n} \geq V(K)^{2n+p} V(L)^{-p}$$

with equality if and only if $K$ and $L$ are real dilation. For the real $L_p$ harmonic radial combination and real $L_p$ dual Minkowski inequality, we refer to Ref. [35].
1.3 The Complex $L_p$ Harmonic Blaschke Combination

The notion of real $L_p$ harmonic Blaschke combination was given by Lu and Leng\[36\]. Then, we extend real $L_p$ harmonic Blaschke combination to the complex case.

For $p \geq 1$, $K, L \in \mathcal{S}_p(\mathbb{C}^n)$ and $\lambda, \mu \geq 0$ (not both zero), the complex $L_p$ harmonic Blaschke combination $\lambda \ast K \hat{+}_p \mu \ast L$ of $K$ and $L$ is defined by

$$\rho \left( \lambda \ast K \hat{+}_p \mu \ast L, r \right) \frac{2n+p}{2n} = \frac{\lambda \rho(K, r) + \mu \rho(L, r)}{V(K) + \mu V(L)}$$

where $\lambda \ast K$ is $L_p$ harmonic Blaschke scalar multiplication and $\lambda \ast K = \lambda \hat{K}$ if $\lambda \hat{=} \mu \geq 0$. Taking $\lambda = \mu = \frac{1}{2}$, $K = L$ in $\lambda \ast K \hat{+}_p \mu \ast L$, then the complex $L_p$ harmonic Blaschke body $\hat{V}_{p,C} K$ is introduced by

$$\hat{V}_{p,C} K = \frac{1}{2} \ast K \hat{+}_p \frac{1}{2} \ast (-K)$$

Obviously, $\hat{V}_{p,C} K$ is origin symmetric.

2 Proofs of Theorems

In this section, we will prove Theorem 1-Theorem 6.

**Proof of Theorem 1** For $p \geq 1$ and $C \in \mathcal{K}(\mathbb{C})$, the $L_p$ harmonic Blaschke combination (14) together with (2) yields

$$h(\Gamma_{p,C}(\lambda \ast K \hat{+}_p \mu \ast L), u)^p = \lambda h(\Gamma_{p,C,K}, u)^p + \mu h(\Gamma_{p,C,L}, u)^p$$

(16)

From (10) and for any $Q \in \mathcal{S}_p(\mathbb{C}^n)$, we obtain

$$V_p (Q, \Gamma_{p,C}(K \hat{+}_p L)) = \frac{1}{2n} \int_{S^{n-1}} h(\Gamma_{p,C}(K \hat{+}_p L), u)^p dS_p(Q, u)$$

$$= \frac{1}{2n} \int_{S^{n-1}} (h(\Gamma_{p,C,K}, u)^p + h(\Gamma_{p,C,L}, u)^p) dS_p(Q, u)$$

$$= V_p (Q, \Gamma_{p,C,K}) + V_p (Q, \Gamma_{p,C,L})$$

Therefore, by (11), we get

$$V_p (Q, \Gamma_{p,C}(K \hat{+}_p L)) \geq V(Q) \frac{2n+p}{2n} \left( V(\Gamma_{p,C,K}) \frac{p}{2n} + V(\Gamma_{p,C,L}) \frac{p}{2n} \right)$$

(17)

with equality if and only if $Q, \Gamma_{p,C,K}$ and $\Gamma_{p,C,L}$ are real dilation. Taking $Q = \Gamma_{p,C}(K \hat{+}_p L)$ in (17), one has

$$V(\Gamma_{p,C}(K \hat{+}_p L)) \frac{p}{2n} \geq V(\Gamma_{p,C,K}) \frac{p}{2n} + V(\Gamma_{p,C,L}) \frac{p}{2n}$$

Together (16) with the equality condition of (17), we know that the equality holds if and only if $K$ and $L$ are real dilation.

**Proof of Theorem 2** From (8) and (16), one has

$$\rho \left( \Gamma_{p,C}(\lambda \ast K \hat{+}_p \mu \ast L), u \right)^p = \lambda \rho \left( \Gamma_{p,C,K}, u \right)^p + \mu \rho \left( \Gamma_{p,C,L}, u \right)^p$$

(18)

Then by (12) and the inverse Minkowski’s integral inequality\[34\], we obtain

$$V_p(Q, \Gamma_{p,C}(K \hat{+}_p L)) \geq \frac{1}{2n} \int \left( \rho \left( \Gamma_{p,C}(K \hat{+}_p L), u \right)^p \right)^{\frac{2n}{p}} dS_p(Q, u)$$

$$= \frac{1}{2n} \int \left( \rho \left( \Gamma_{p,C,K}, u \right)^p + \rho \left( \Gamma_{p,C,L}, u \right)^p \right)^{\frac{2n}{p}} dS_p(Q, u)$$

$$\geq \frac{1}{2n} \int \rho \left( \Gamma_{p,C,K}, u \right)^{2n} dS_p(Q, u) + \frac{1}{2n} \int \rho \left( \Gamma_{p,C,L}, u \right)^{2n} dS_p(Q, u)$$

(19)

Taking $Q = \Gamma_{p,C}(K \hat{+}_p L)$ in (19) and by (13), one yields the inequality (6). According to the equality conditions of Minkowski’s integral inequalities, we see that equality holds in (19) if and only if $K$ and $L$ are real dilation.

Next, we turn to prove Theorem 3 and Theorem 4. Lemma 1 provides a connection of $\Gamma_{p,C,K}$ and $\Pi_{p,C} K$ in terms of mixed volumes and their dual.

**Lemma 1** If $C \in \mathcal{K}(\mathbb{C})$ and $K, L \in \mathcal{S}_p(\mathbb{C}^n)$, then,

$$V_p \left( K, \Gamma_{p,C,L} \right) = \frac{1}{(2n+p)V(L)} \tilde{V}_{-p} \left( L, \Pi_{p,C} K \right)$$

(20)
Proof From (3), (8), (10) and definition of $L_p$ projection body\[^{29}\] , we have

$$V_p (K, \Gamma_{p,C} L) = \frac{1}{2n} \int_{S^{2n+1}} h (\Gamma_{p,C} L, u)^p dS_p(K, u)$$

$$= \frac{1}{2n(2n+p)} V(L) \int \int \rho(L, u)^{2n+p} h(Cu, v)^p dS(v)dS_p(K, u)$$

$$= \frac{1}{2n(2n+p)} V(L) \int \int \rho(L, u)^{2n+p} h \left( \Pi_{p,C} K, u \right)^p dS(v)$$

$$= \frac{1}{2n(2n+p)} V(L) \int \rho(L, u)^{2n+p} \left( \Pi_{p,C} K, u \right)^p dS(v)$$

$$= \frac{1}{2n(2n+p)} V(L) \int \rho(L, u)^{2n+p} \left( \Pi_{p,C} K, u \right)^p dS(v)$$

which ends the proof of Lemma 1.

Lemma 2 If $p \geq 1, K, L \in S^c_o(C^n)$, then

$$\bar{\nu}_p (K, \Gamma_{p,C} L) = V_p (K, \Gamma_{p,C} L)$$

Proof From (3), (8) and (12), it easily gets

$$\bar{\nu}_p (K, \Gamma_{p,C} L) = \frac{1}{2n} \int_{S^{2n+1}} \rho(L, u)^{2n+p} \left( \Pi_{p,C} K, u \right)^p dS(u)$$

$$= \frac{1}{2n} \int_{S^{2n+1}} \rho(L, u)^{2n+p} h \left( \Pi_{p,C} K, u \right)^p dS(u)$$

$$= \frac{1}{2n} \int_{S^{2n+1}} \rho(L, u)^{2n+p} h \left( \Pi_{p,C} K, u \right)^p dS(u)$$

$$= \frac{1}{2n} \int_{S^{2n+1}} \rho(L, u)^{2n+p} h \left( \Pi_{p,C} K, u \right)^p dS(u)$$

That is to say,

$$\frac{\bar{\nu}_p (K, \Gamma_{p,C} L)}{V(L)} = \frac{\bar{\nu}_p (K, \Gamma_{p,C} L)}{V(K)}$$

which yields (21).

Remark 1 If $p \geq 1, K \in S^c_o(C^n)$, then $\bar{\nu}_p (K, \Gamma_{p,C} K) \geq V(K)$. If $K$ is a central ellipsoid or an Hermitian ellipsoid, then the equality holds.

Now we are in a position to prove Theorem 3 and Theorem 4.

Proof of Theorem 3 Since $K, L \in S^c_o(C^n)$ and $\bar{\nu}_p (K, Q) \leq \bar{\nu}_p (L, Q)$ for any $Q \in K^o_o(C^n)$, then taking $Q=\Gamma_{p,C} M$ for any $M \in S^c_o(C^n)$, we have

$$\bar{\nu}_p (K, \Gamma_{p,C} M) \leq \bar{\nu}_p (L, \Gamma_{p,C} M)$$

with equality if and only if $K=L$. By Lemma 1, we obtain

$$V(K)\bar{\nu}_p (M, \Gamma_{p,C} K) \leq V(L)\bar{\nu}_p (M, \Gamma_{p,C} L)$$

Taking $M=\Gamma_{p,C} L$ in (23) and by (11), one has

$$V(L)\bar{\nu}_p (M, \Gamma_{p,C} L) \leq V(L)\bar{\nu}_p (M, \Gamma_{p,C} L)$$

i.e.,

$$V(\Gamma_{p,C} K) \leq V(\Gamma_{p,C} L)$$

From Lemma 1, we see that inequalities (22) and (23) are equivalent. Thus, equality holds in (25) if and only if $K=L$.

Proof of Theorem 4 Since $\bar{\nu}_p (K, Q) \leq \bar{\nu}_p (L, Q)$ for any $Q \in K^o_o(C^n)$, then, taking $Q=\Gamma_{p,C} M$ for any $M \in S^c_o(C^n)$, we get

$$\bar{\nu}_p (K, \Gamma_{p,C} M) \leq \bar{\nu}_p (L, \Gamma_{p,C} M)$$

with equality if and only if $K=L$. Combining (21) and (26), we obtain

$$V(K)\bar{\nu}_p (M, \Gamma_{p,C} K) \leq V(L)\bar{\nu}_p (M, \Gamma_{p,C} L)$$

Taking $M=\Gamma_{p,C} L$ and by (13), it yields

$$V(L)\bar{\nu}_p (\Gamma_{p,C} L)$$
\[ \geq V(K) \tilde{V}_p(\Gamma_{p,C}^* L, \Gamma_{p,C}^* K) \]
\[ \geq V(K) \tilde{V} \left( \Gamma_{p,C}^* 2^{n+p} V \left( \Gamma_{p,C}^* K \right) ^{ \frac{p}{2n} } \right) \tag{28} \]
with equality in the second inequality of (30) if and only if \( \Gamma_{p,C}^* L \) and \( \Gamma_{p,C}^* K \) are real dilation. Thus, it follows from (28) that we have
\[ \frac{V \left( \Gamma_{p,C}^* K \right) ^{ \frac{p}{2n} } }{V(L)} \geq \frac{V \left( \Gamma_{p,C}^* L \right) ^{ \frac{p}{2n} } }{V(L)} \tag{29} \]
with equality if and only if \( K = L \).

Now, we are dedicated to proving Theorem 5 and Theorem 6.

**Proof of Theorem 5** For \( p \geq 1 \) and \( M \in S'_a(C^n) \), it follows from the Lemma 2,
\[ \tilde{V}_p \left( K, \Gamma_{p,C}^* M \right) = \tilde{V}_p \left( M, \Gamma_{p,C}^* K \right) \]
\[ \tilde{V}_p \left( L, \Gamma_{p,C}^* M \right) = \tilde{V}_p \left( M, \Gamma_{p,C}^* L \right) \tag{30} \]
Since \( \Gamma_{p,C}^* K \subseteq \Gamma_{p,C}^* L \), then \( \Gamma_{p,C}^* L \subseteq \Gamma_{p,C}^* K \), hence for all \( u \in S_{2n-1} \), we have
\[ \rho \left( \Gamma_{p,C}^* K \right) ^p \geq \rho \left( \Gamma_{p,C}^* L \right) ^p \tag{31} \]
Combining (30) and (31), we get
\[ \tilde{V}_p \left( K, \Gamma_{p,C}^* M \right) \leq \tilde{V}_p \left( L, \Gamma_{p,C}^* M \right) \frac{V(L)}{V(J)} \tag{32} \]
For \( L \in S_{p,C}^* \) and taking \( \Gamma_{p,C}^* M \) for \( L \) in (32), then from (13), we get \( V(K) \leq V(L) \) with equality if and only if \( K = L \).

**Proof of Theorem 6** By (3), (15) and (16), we have
\[ h \left( \Gamma_{p,C} \left( \tilde{V}_p(K), u \right) \right) ^p \]
\[ = h \left( \Gamma_{p,C} \left( \frac{1}{2} K + \frac{1}{2} (-K) \right), u \right) ^p \]
\[ = \frac{1}{2} h \left( \Gamma_{p,C} K, u \right) ^p + \frac{1}{2} h \left( \Gamma_{p,C} (-K), u \right) ^p \]
\[ = h \left( \Gamma_{p,C} K, u \right) ^p \tag{33} \]
Meanwhile, according to (12), (13) and (14), it yields
\[ \tilde{V}_p \left( \lambda * K + \mu * L, Q \right) \]
\[ = \lambda \tilde{V}_p \left( K, Q \right) + \mu \tilde{V}_p \left( L, Q \right) \]
\[ \geq V(K) \tilde{V}_p \left( \Gamma_{p,C}^* L, \Gamma_{p,C}^* K \right) \]
\[ \geq V(K) \tilde{V} \left( \Gamma_{p,C}^* 2^{n+p} V \left( \Gamma_{p,C}^* K \right) ^{ \frac{p}{2n} } \right) \tag{28} \]
with equality in the second inequality of (30) if and only if \( \Gamma_{p,C}^* L \) and \( \Gamma_{p,C}^* K \) are real dilation. Thus, it follows from (28) that we have
\[ \frac{V \left( \Gamma_{p,C}^* K \right) ^{ \frac{p}{2n} } }{V(L)} \geq \frac{V \left( \Gamma_{p,C}^* L \right) ^{ \frac{p}{2n} } }{V(L)} \tag{29} \]
with equality if and only if \( K = L \).

Taking \( Q = \lambda * K + \mu * L \) and \( \lambda = \mu = \frac{1}{2} \), then \( V(K) \geq V(L) \) with equality if and only if \( L \) is an origin symmetric body.

Since \( L \) is not an origin symmetric, we get \( V(K) \geq V(L) \). Choose \( \epsilon > 0 \) such that
\[ V \left( (1-\epsilon) \tilde{V}_p^* L \right) > V(L) \]
Let \( K = (1-\epsilon) \tilde{V}_p^* L \), then \( V(L) < V(K) \). According to (3), we see that
\[ \Gamma_{p,C}^* K = \Gamma_{p,C} \left( (1-\epsilon) \tilde{V}_p^* L \right) \]
\[ = (1-\epsilon) \Gamma_{p,C} \left( \tilde{V}_p^* L \right) \]
\[ = (1-\epsilon) \Gamma_{p,C}^* \left( \tilde{V}_p^* L \right) \subset \Gamma_{p,C}^* L \]
which ends the proof of Theorem 6.

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