Quantum states for perfectly secure secret sharing

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In this work, we investigate what kinds of quantum states are feasible to perform perfectly secure secret sharing, and present its necessary and sufficient conditions. We also show that the states are bipartite distillable for all bipartite splits, and hence the states could be distillable into the Greenberger-Horne-Zeilinger state. We finally exhibit a class of secret-sharing states, which have an arbitrarily small amount of bipartite distillable entanglement for a certain split.

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Introduction.— Entanglement has been considered as one of the most crucial resources for quantum communication, which has been shown to be perfectly secure against any interior/exterior eavesdropper. The perfect security seems to be due to the pure entanglement. However, in the case of the quantum key distribution, appropriately defining its perfect security as in Refs. [1,2], we can see that only pure entanglement does not guarantee the perfect security.

We here focus on another quantum communication protocol, the quantum secret sharing of classical information, originally presented by Hillery et al. [3]. Our question is what kinds of quantum states are feasible to perform the perfectly secure secret sharing (PSSS). In order to answer this question, first of all, it is required to present the conditions for the PSSS.

One of the most important problems in secret sharing of classical information is how to share random bits securely between one dealer, Alice and other players, Bob and Charlie. If each participant would securely share one of the random bit sequences as in TABLE I, then Alice could secretly make Bob and Charlie share her secret bit.

Thus, for the PSSS, the two following conditions must be satisfied: (i) Probability distributions of all participants’ secret bits should be unbiased and perfectly correlated, that is, if we let \( p_{ijk} \) be the probability that Alice, Bob, and Charlie get the random bits \( i, j, \) and \( k \), respectively, then \( p_{000} = p_{011} = p_{101} = p_{110} = 1/4 \) and \( p_{ijk} = 0 \) for other \( i, j, k \). (ii) Eavesdropper should not be able to obtain any information about participants’ secret bits.

In this work, according to the two above conditions, we show that \( \rho_{ABCA'B'C'} \) is a quantum state for the PSSS if and only if it is of the form

\[
\frac{1}{4} \sum_{i+j+k\equiv 0 \ (mod\ 2)} |ijk\rangle_{ABC} (i'j'k') \otimes U_{ijk} \rho_{A'B'C'} U_{i'j'k'}^\dagger,
\]

where \( \rho_{A'B'C'} \) is a state on subsystem \( A'B'C' \), and \( U_{ijk}'s \) are unitary operators. We call this form of states in [4] the secret-sharing states.

We also show that the states are bipartite distillable for all bipartite splits. From the results of Dür et al. [4] we can readily derive the fact that if any \( n \)-qubit state has negative partial transposition for all bipartite splits then it is distillable into the Greenberger-Horne-Zeilinger (GHZ) [3] state. Hence, the secret-sharing states could also be distillable into the GHZ state.

Furthermore, we show that our results can be generalized into multipartite cases, that is, \( \rho_n \) is an \( n \)-qubit state for the PSSS consisting of one dealer and \( n - 1 \) players if and only if it is of the form

\[
\frac{1}{2^{n-1}} \sum_{I \in \mathcal{P}_n \atop \text{even parity}} |I\rangle_{A_1A_2\ldots A_n} (J) \otimes U_{I} \rho_{A'_1A'_2\ldots A'_{n}} U_{J}^\dagger,
\]

where \( \rho_{A'_1A'_2\ldots A'_{n}} \) is a state on subsystem \( A'_1A'_2\ldots A'_{n} \), and \( U_I's \) are unitary operators, and that \( \rho_n \) is bipartite distillable for all its all bipartite splits. Hence, as in the three-party case, \( \rho_n \) could be distillable into the \( n \)-qubit GHZ state.

Secret-Sharing States.— We first provide necessary and sufficient conditions for a state to perform the PSSS. Let \( A, B, \) and \( C \) be qubit systems, and \( A', B', \) and \( C' \) be of arbitrary dimensions. Here \( AA', BB', \) and \( CC' \) are Alice’s, Bob’s, and Charlie’s systems, respectively. Then we obtain the following theorem.

Theorem 1. Any state is a quantum state for the 3-party PSSS if and only if it is a secret-sharing state of the form in (7).

Proof. We first assume that \( \rho_{ABCA'B'C'} \) is a quantum state for the PSSS, and let \( |\Phi\rangle_{ABCA'B'C'E} \) be its puri-
state when participants’ measurement result is unbiased and perfectly correlated. Thus, it suffices to show that participants get the bits that are unbiased and perfectly correlated, it is clear that where $E$ is the system of the eavesdropper, Eve. Since probability distributions of all participants’ secret bits are unbiased and perfectly correlated, it is clear that is one, that is, $\|X_{ijk,i'j'k'}\|_1 = 1$. Let the state in Eq. (11) be the purification of $\rho_{ABC'\A'B'C'}$. Then we have $X_{ijk,i'j'k'} = \mathrm{tr}_C(\rho_{ijk}|\Psi_{i,j,k}'\rangle\langle\Psi_{i,j,k}'|)$. It follows from straightforward calculations that

$$\|X_{ijk,i'j'k'}\|_1 = \mathrm{tr}_C(\rho_{ijk}\rho_{ijk}') = 1$$

where $F$ is the fidelity. Since $F(\rho_{ijk},\rho_{ijk}') = 1$ for every $i, j, k$, the proof is completed. 

We remark that, as seen in the proof of Theorem 1 in order to prove its converse, it is sufficient to use that the trace norms of three well-chosen off-diagonal blocks are one, for example, $\|X_{000,011}\|_1 = \|X_{101,110}\|_1 = 1$. Moreover, any block matrix of the form in (12) whose three well-chosen off-diagonal blocks have trace norm 1/4 forms a secret state as follows.

**Theorem 2.** $\sigma_{ABC'\A'B'C'}$ is a state which can be expressed as the following block-matrix form:

$$\sigma_{ABC'\A'B'C'} = \begin{pmatrix} * & 0 & 0 & X & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & 0 & Y & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & 0 & Z & 0 & 0 \\ * & 0 & * & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\|X\|_1 = \|Y\|_1 = \|Z\|_1 = 1/4$ if and only if $\sigma_{ABC'\A'B'C'}$ is a secret-sharing state.

**Proof.** Since any secret-sharing state is of the form (13), it suffices to show that $\sigma_{ABC'\A'B'C'}$ in (13) is a secret-sharing state. For each $ijk$, let $p_{ijk}$ be the trace of $(ijk,ijk)$ block entry of $\sigma_{ABC'\A'B'C'}$. Then

$$|\Psi\rangle = \sqrt{p_{000}p_{101}}|\Psi_{000}\rangle_{ABC'}|\Psi_{101}\rangle_{A'B'C'},$$

is its purification, and hence we obtain

$$X = \sqrt{p_{000}p_{101}}\mathrm{tr}_C(\rho_{000}|\Psi_{000}\rangle\langle\Psi_{000}|),$$

$$Y = \sqrt{p_{011}p_{101}}\mathrm{tr}_C(\rho_{011}|\Psi_{011}\rangle\langle\Psi_{011}|),$$

$$Z = \sqrt{p_{110}p_{110}}\mathrm{tr}_C(\rho_{110}|\Psi_{110}\rangle\langle\Psi_{110}|).$$

As in the proof of Theorem 1 we have

$$\|X\|_1 = \sqrt{p_{000}p_{101}}F(\rho_{000},\rho_{000}'),$$

$$\|Y\|_1 = \sqrt{p_{011}p_{101}}F(\rho_{011},\rho_{011}'),$$

$$\|Z\|_1 = \sqrt{p_{110}p_{110}}F(\rho_{110},\rho_{110}).$$
where $\rho_{ijk}^E = \text{tr}_{A'B'C'}(|\Psi_{ijk}\rangle\langle\Psi_{ijk}|)$. Since $\|X\|_1 = \|Y\|_1 = \|Z\|_1 = 1/4$, we have the following inequalities:

$$\frac{p_{000} + p_{011}}{2} \geq \sqrt{p_{000}p_{011}} \geq \frac{1}{4},$$
$$\frac{p_{011} + p_{101}}{2} \geq \sqrt{p_{011}p_{101}} \geq \frac{1}{4},$$
$$\frac{p_{011} + p_{110}}{2} \geq \sqrt{p_{011}p_{110}} \geq \frac{1}{4}. \quad (13)$$

It follows from the fact $p_{000} + p_{011} + p_{101} + p_{110} = 1$ that

$$p_{000} = p_{011} = p_{101} = p_{110} = 1/4$$
and $F(\rho_{000}^E, \rho_{011}^E) = F(\rho_{011}^E, \rho_{101}^E) = F(\rho_{101}^E, \rho_{110}^E) = F(\rho_{110}^E, \rho_{000}^E) = 1$. This implies that

$$\sigma_{ABC'}A'B'C'C''$$
is a state for the PSSS. Therefore, it is a secret-sharing state by Theorem 1.

We note that every private state is distillable [7]. By employing this note, we now show that every secret-sharing state is bipartite distillable for its all bipartite splits.

**Theorem 3.** Let $\rho$ be a secret-sharing state for the $3$-party secret sharing. Then $\rho$ is bipartite distillable for its all bipartite splits of the $3$ parties.

**Proof.** By Theorem 1, $\rho$ can be expressed as the form of $\rho_{ABC}^{AA'B'B'C'C''}$ and unitary operators $U_{ijk}$. Let CNOT$_{ij}$ be the controlled-NOT operation such that $i$ and $j$ represent its control system and target system, respectively. Then applying CNOT$_{BC}$ to $\rho$ and performing the projective measurement on system $B$ with respect to the standard basis $\{|0\rangle, |1\rangle\}$, when the measurement result is $r$, the resulting state becomes a private state, which is distillable [7].

$$\frac{1}{2} \sum_{i,j=0}^1 |ii\rangle_{AC} \langle jj| \otimes U_{iri} \rho_{ABC}^{AA'B'B'C'C''} U_{irj}^\dagger, \quad (14)$$

which is distillable [7]. Thus, a given $\rho$ is bipartite distillable for the split $AA'-BB'C'C''$. Similarly, we can show that $\rho$ is bipartite distillable for the splits $BB'-CC'A'$ and $CC'-AA'BB'$.

**Generalization into multipartite cases.** We now generalize our results into multipartite cases.

**Theorem 4.** $\rho_n$ is a quantum state for the $n$-party PSSS consisting of one dealer and $n - 1$ players if and only if it is a secret-sharing state of the form in $\{2\}$.

**Proof.** Let $|\Psi\rangle$ be a purification of $\rho_n$ as follows:

$$|\Psi\rangle = \sum_{I \in \mathbb{Z}_2^n} \sqrt{p_I} |\Psi_I\rangle |A'_1A'_2\cdots A'_nE\rangle. \quad (15)$$

As in the case of the 3-party case, it is clear that $p_I = 1/2^{n-1}$ for all $I$ with even parity and $p_I = 0$ for other $I$. For each of all participants’ measurement result $I$, Eve’s state after the measurement, $\rho_I^E$ becomes $\rho_I^E = \text{tr}_{A'_1A'_2\cdots A'_n}(|\Psi_I\rangle\langle\Psi_I|)$. Since Eve cannot obtain any information about participants’ secret bit at all, all $\rho_I^E$’s are the same. Thus, by GHJW theorem there are a state $\rho_{A'_1A'_2\cdots A'_n}$ and unitary operators $U_I$ on the system $A'_1A'_2\cdots A'_n$ such that $\rho_n$ is of the form in $\{2\}$ and $\rho_{A'_1A'_2\cdots A'_n}$ has the same spectrum as $\rho_I^E$.

Conversely, assuming that a given state $\rho_n$ is of the form $\{2\}$, it can be readily shown that $\rho_n$ is a state for a $n$-party PSSS, by the same way as the proof of Theorem 1.

We call the state in $\{2\}$ the $n$-party secret-sharing state. Remark that, as in Theorem 1 and Theorem 2 any quantum state of the form of $2^n \times 2^n$ block matrix, whose block entries vanish if they are in the rows or columns of odd parity, has well-chosen $2^{n-1} - 1$ off-diagonal block entries of the trace norm $1/2^{n-1}$ if and only if the state is an $n$-party secret-sharing state.

We now consider the bipartite distillability of the $n$-party secret-sharing states.

**Theorem 5.** Any $n$-party secret-sharing state $\rho_n$ is bipartite distillable for all bipartite splits of the $n$ parties.

**Proof.** We use the mathematical induction on $n \geq 3$. Then if $n = 3$ then this theorem is true by Theorem 3. We assume that this theorem is true for $(n - 1)$-party secret-sharing states. Let $P$ be an arbitrary bipartite split $I_0-I_1$ of the $n$ parties, $\{A_1A'_1, A_2A'_2, \ldots, A_nA'_n\}$. Then for $A_1A'_1, A_kA'_k \in I_0$ applying CNOT$_{A_k}$ to $\rho_n$ and performing the projective measurement on system $A_j$ with respect to the standard basis $\{|0\rangle, |1\rangle\}$, the resulting state becomes an $(n - 1)$-party secret-sharing state. By the induction hypothesis, the state is bipartite distillable for the split, $I_0 - \{A_1A'_1\}-I_1$, and hence $\rho_n$ is also bipartite distillable for $I_0-I_1$. This completes the proof.

**Example.** We construct a class of secret-sharing states in a similar way to one presented in [1]. Consider the following state:

$$\rho = a_0|\psi_0\rangle\langle\psi_0| \otimes \sigma_0 + a_1|\psi_1\rangle\langle\psi_1| \otimes \sigma_1 + a_2|\psi_2\rangle\langle\psi_2| \otimes \sigma_2 + a_3|\psi_3\rangle\langle\psi_3| \otimes \sigma_3, \quad (16)$$

where

$$|\psi_0\rangle = \frac{1}{2}(|000\rangle + |011\rangle + |101\rangle + |110\rangle),$$
$$|\psi_1\rangle = \frac{1}{2}(|000\rangle - |011\rangle - |101\rangle - |110\rangle),$$
$$|\psi_2\rangle = \frac{1}{2}(|000\rangle - |011\rangle + |101\rangle - |110\rangle),$$
$$|\psi_3\rangle = \frac{1}{2}(|000\rangle + |011\rangle - |101\rangle + |110\rangle), \quad (17)$$

and the states $\sigma_j$ have support on orthogonal subspaces. Then one can readily verify that $\rho$ is a $3$-party secret-sharing state, since $\|a_0\sigma_0 \pm a_1\sigma_1 \pm a_2\sigma_2 \pm a_3\sigma_3\|_1 = 1$.

As in Ref. [1], one can find a secret-sharing state which can have an arbitrarily small amount of bipartite distillable entanglement for a certain split. In order to find such
a state, take \(a_0 = a_1\) and \(a_2 = a_3\) such that \(a_1 + a_2 = 1/2\), and

\[
\begin{align*}
\sigma_0 &= \rho_s \otimes |0\rangle \langle 0|, \\
\sigma_1 &= \rho_s \otimes |0\rangle \langle 0|, \\
\sigma_2 &= \rho_s \otimes |1\rangle \langle 1|, \\
\sigma_3 &= \rho_s \otimes |1\rangle \langle 1|,
\end{align*}
\]

(18)

where \(\rho_s\) and \(\rho_a\) and two extreme \(d \otimes d\) Werner states

\[
\begin{align*}
\rho_s &= \frac{2}{d^2 + d} P_{\text{sym}} = \frac{I + \mathcal{F}}{d^2 + d}, \\
\rho_a &= \frac{2}{d^2 - d} P_{\text{as}} = \frac{I - \mathcal{F}}{d^2 - d}
\end{align*}
\]

with the identity operator \(I\) on the \(d \otimes d\) system and the flip operator \(\mathcal{F} = \sum_{i,j=0}^{d-1} |ij\rangle \langle ji|\). Then we have \(\|\rho^{AA'}\|_1 = (d + 2)/d\). Therefore, since the log-negativity is an upper bound of the distillable entanglement \([8]\), the bipartite distillable entanglement for the split \(AA'-BB'CC'\) can be arbitrarily small by increasing \(d\). Nevertheless, the state is always a secret-sharing state for any \(d\).

In conclusion, we have presented necessary and sufficient conditions for secret-sharing states, and have also shown that any secret-sharing state is bipartite distillable for its all bipartite splits, and hence the states could be distillable into the GHZ state. We have furthermore generalized our results into multipartite cases, and have exhibited a class of secret-sharing states, which have an arbitrarily small amount of bipartite distillable entanglement for a certain split.

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