Explicit and sharp two-sided estimates for the killed Langevin process

Mouad Ramil∗

1Research Institute of Mathematics, Seoul National University, Seoul, Republic of Korea

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Abstract

We prove explicit and sharp two-sided estimates for the transition density of the Langevin process with quadratic potential, killed outside of the position interval (0, 1). The long-time asymptotics of this transition density are also obtained. In particular, this allows us to show that the killed semigroup is uniformly conditionally ergodic.

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1 Introduction

Let \((q_t, p_t)_{t \geq 0}\) be the Langevin process in \(\mathbb{R} \times \mathbb{R}\) satisfying the following stochastic differential equation:

\[
\begin{align*}
\frac{dq_t}{dt} &= p_t, \\
\frac{dp_t}{dt} &= -V'(q_t)dt - \gamma p_t dt + \sigma dB_t,
\end{align*}
\]

where \(\gamma > 0\), \(\sigma = \sqrt{2\gamma k_{\text{B}}T}\), \(k_{\text{B}} > 0\) is the Boltzmann constant, \(T\) is the fixed medium temperature, \(V\) is the potential function and \((B_t)_{t \geq 0}\) is a one-dimensional Brownian process. The Langevin process is used in statistical physics to model the evolution of a thermostated molecular system. In the above two-dimensional case, the particle is described at all time \(t \geq 0\) by its position \(q_t \in \mathbb{R}\) and velocity \(p_t \in \mathbb{R}\). It is also subjected to a force \(F(q) = -V'(q)\) and to collisions modeled by a friction coefficient \(\gamma > 0\) along with a random inflow of energy carried by the Brownian component.

We are interested here in sharp two-sided estimates for the transition density of the process (1) killed when leaving the domain \(D := (0, 1) \times \mathbb{R}\). Namely, let

\[
\tau_D := \inf\{t > 0 : q_t \notin (0, 1)\}.
\]

We define the transition density of the process (1) killed outside of \(D\) as the function \(p_D\) defined in \(\mathbb{R}_{+} \times D \times D\) such that for all \(t > 0\), for all \((q, p) \in D\), for all borelian \(A \subset D\),

\[
\mathbb{P}(q_t, p_t) ((q, p) \in A, \tau_D > t) = \int_A p^D_t(q, p, q', p')dq'dp'.
\]

We shall prove here the existence of time-dependent constants \(c_t, c'_t > 0\) (which may also depend on the coefficients of (1)) and explicit functions \(H, H^*\) defined on \(D\) such that for all \(t > 0\),

\[
\forall (q, p), (q', p') \in D, \quad c_t H(q, p) H^*(q', p') \leq p^D_t(q, p, q', p') \leq c_t H(q, p) H^*(q', p').
\]

These estimates have been proved in the literature for multidimensional elliptic processes such as

\[
\frac{d\eta_t}{dt} = F(\eta_t)dt + \sigma(\eta_t)dB_t,
\]

∗E-mail: ramil.mouad@gmail.com
on a smooth bounded domain \( O \) when \( F, \sigma \) are smooth and \( \sigma \) is uniformly elliptic, see for instance [7, 16, 15]. The proofs often rely on sharp estimates on the associated Green function which have been obtained for uniformly elliptic diffusions in [10]. In our case, given the absence of noise in the \( q \)-direction in (1), the associated infinitesimal generator of (1) is not elliptic anywhere in the domain and therefore such methods do not apply in this case.

Sharp two-sided estimates are important as they can help us understand how the killed semigroup of (1) and associated eigenvectors decay at the boundary and therefore such methods do not apply in this case.

In the elliptic case, its boundary behaviour shall depend on the distance to the boundary of the domain. In the case of (1), given the kinetic nature of the process the boundary behaviour is in fact not valid anywhere here. From the results we obtain we see that there is in fact a difference of nature between the boundary behaviour of an elliptic diffusion process and a kinetic process like (1). The approach used here shall be different and will rely on a careful study of the first exit time probability.

Let us now take a look at the behaviour of the first exit time probability \( \tau \) of the process (1) in the domain \( D \). Let us mention that convergence results of the conditioned distribution towards the quasi-stationary distribution were already obtained recently for the Langevin process in [21, 11]. However, in these results the prefactor \( C = C(\theta) \) strongly depends on the initial distribution \( \theta \) and blows up at the boundary \( \partial D \).

Let us mention that this is the first work to provide a two-sided estimates and/or a uniform convergence result towards the QSD for a Langevin process killed outside of a domain. We shall also provide here explicit expressions of the functions \( H, H^* \) involved in the two-sided estimates (2), thus showing explicitly how the first exit time probability behaves close to the boundary of the domain. As said previously, the main difficulty of such proofs is that the tools used in the literature for the study of two-sided estimates of diffusion processes require an ellipticity condition on the generator which is not valid anywhere here. From the results we obtain we see that there is in fact a difference of nature between the boundary behaviour of an elliptic diffusion process and a kinetic process like (1). The approach used here shall be different and will rely on a careful study of the first exit time probability.

In the elliptic case, its boundary behaviour shall depend on the distance to the boundary of the domain. In the case of (1), given the kinetic nature of the process the boundary behaviour is in fact expected to be more complex and shall depend on the velocity and its sign. Examples of different behaviours are for instance provided in the next paragraph.

If we integrate the two-sided estimates (2) over \( (q', p') \in D \), we obtain constants \( c_1, c'_1 > 0 \) such that for all \( t > 0 \),

\[
\forall (q, p) \in D, \quad c'_1 H(q, p) \leq \mathbb{P}_{(q,p)}(\tau_\theta > t) \leq c_1 H(q, p).
\]

Let us now take a look at the behaviour of the first exit time probability \( \mathbb{P}_{(q,p)}(\tau_\theta > t) \) at a fixed time \( t > 0 \), depending on the vector \( (q, p) \in D \), in the following two-dimensional example:

![Figure 1: Exit event from \( D \).](image)

Consider the point \( A \in \partial D \) with velocity directed towards the interior of \( D \) in Figure 1. In this case, even though \( A \in \partial D \), the probability \( \mathbb{P}_{(q,p)}(\tau_\theta > t) \) is positive because the process (1) re-enters the domain immediately almost-surely. On the contrary, it shall vanish if the velocity is directed towards the exterior of the domain \( D \). Consider now the point \( B \in D \) which is away from the boundary \( \partial D \) with however a very large velocity. In this case the probability \( \mathbb{P}_{(q,p)}(\tau_\theta > t) \) should
be very small and is expected to vanish when the velocity goes to infinity, even though the distance of $B$ to the boundary is bounded from below. In fact, since the velocity is very high the process is likely to exit $D$ "quickly". However, this velocity should be high compared to the distance to the boundary in some sense which shall be clarified throughout the computations. Consider now the last case, the point $C \in D$ is close to the boundary with tangential velocity. In this case the velocity component does not play any role but given the random oscillations in the position coordinates, which are propagated by the velocity coordinates, the probability $P_{(q,p)}(\tau_B > t)$ should be small as well but in this case its boundary behaviour shall mostly depend on the distance to the boundary. Therefore, the kinetic nature of the process exhibits multiple boundary behaviours depending on the distance to the boundary as well as the velocity.

The main objective of this work is to prove the two-sided estimates (2) for the process (1) when the potential is quadratic, i.e. $V(q) = \alpha q^2/2 + \beta q + \delta$ for some $\alpha \geq 0$ and $\beta, \delta \in \mathbb{R}$. We shall also assume that $\sigma > 0$ can be independent of $\gamma$ and $\xi \in \mathbb{R}$ can be zero or negative. The strategy of this work is to first prove estimates on the first exit time probability as in (5) and then show that these estimates actually ensure the aimed two-sided estimates (2) using estimates obtained on the transition density $p_B^2$ in previous works [20, 21].

Nonetheless, let us mention that some recent works have focused on the analytic counter-part of (1). Namely in [12], the authors have shown that weak solutions to the parabolic equation $\partial_t u = \Delta_p u$ vanishing at the exit boundary admit a Hölderian behaviour $(\alpha, 3\alpha)$ in (position,velocity) on the boundary set with tangential velocities (called singular set), for any $\alpha < 1/6$. Actually, it shall follow from the two-sided estimates (2) and with the expression of $H, H^*$ that the Hölderian behaviour on the singular set is attained for $\alpha = 1/6$ for the transition density $p_B^2$.

Some attention has also been drawn in the literature towards the long-time behavior of the following process when killed outside of the position half-line $(0, \infty)$:

$$
\begin{align*}
\frac{d\hat{q}_t}{dt} &= \hat{p}_t dt, \\
\frac{d\hat{p}_t}{dt} &= dB_t.
\end{align*}
$$

The solution $(\hat{q}_t)_{t \geq 0}$ of (6), corresponds just to a time integrated Brownian process, up to the initial conditions $(\hat{q}_0, \hat{p}_0)$:

$$
\forall t \geq 0, \quad \hat{q}_t = \hat{q}_0 + t\hat{p}_0 + \int_0^t B_s ds.
$$

Namely, previous works in [14, 6, 8, 19, 17] have led to an explicit expression of the law of $(\hat{q}_0, B_{\tau_{\hat{q}}})$ starting from any couple $(\hat{q}_0, \hat{p}_0) \in \mathbb{R}_+ \times \mathbb{R}$, where $\tau_{\hat{q}} = \inf\{t > 0 : \hat{q}_t = 0\}$.

The long-time behaviour of the probability $P_{(q,p)}(\tau_B > t)$ was also studied in [24] where the author showed that this probability behaved as $h(q,p)/t^{1/4}$ for $q > 0, p \in \mathbb{R}$ when $t \to \infty$. Later, an explicit expression of the prefactor $h(q,p)$ was provided in the following works [13, 9]. This function $h$ is crucial in this work as it will appear in the definition of the explicit two-sided estimates (2).

The literature regarding the study of the first exit event is less extensive for bounded position-domains. For instance, there is no explicit description of the law of the first exit event $(\tau_B, B_{\tau_B})$ from $D := (0, 1) \times \mathbb{R}$. However, the long-time behavior of the first exit-time probability was studied in [21, Theorem 2.17] where it is shown that $P_{(q,p)}(\tau_B > t) \sim \phi(q,p) e^{-\lambda_0 t}$ for some $\lambda_0 > 0$. The function $\phi$ is the eigenvalue, up to a multiplicative constant, of the infinitesimal generator of (1). However no estimates on $\phi$ at the boundary $\partial D$ was obtained but shall follow in this work for (1) in the one-dimensional case, using (2).

This work is divided as follows: we shall first prove sharp estimates for the integrated Brownian process on the probability $P_{(q,p)}(\tau_B > t)$ where $\tau_B = \inf\{t > 0 : \hat{q}_t \notin (0, 1]\}$. These estimates are then used to deduce two-sided estimates on its killed transition density. Then, using a Girsanov argument we will be able to extend these two-sided estimates to the process (1) when $V(q) = \alpha q + \beta$. Finally, combining the long-time asymptotics of the killed semigroup obtained in [21] and the two-sided estimates (2) we shall obtain the uniform conditional ergodicity property stated in (4).

We provide here a few notations used throughout this work before detailing in the next section the main results of this work.

**Notations:** For any subset $A$ of $\mathbb{R}^n$ $(n \geq 1)$, we denote by:

(i) $\mathcal{B}(A)$ the Borel $\sigma$-algebra on $A$,

(ii) for $1 \leq p \leq \infty$, $L^p(A)$ the set of $L^p$ scalar-valued functions on $A$ and $\| \cdot \|_p$ the associated norm,
(iii) $\mathcal{C}(A)$ (resp. $\mathcal{C}^b(A)$) the set of scalar-valued continuous (resp. continuous and bounded) functions on $A$.
(iv) for $1 \leq k \leq \infty$, $\mathcal{C}^k(A)$ the set of scalar-valued $\mathcal{C}^k$ functions on $A$.
(v) for $a, b \in \mathbb{R}$, $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$

Finally for families of functions $(f_t)_{t \geq 0}$, $(g_t)_{t \geq 0}$ defined on $D \subset \mathbb{R}^n$, we shall say that for all $t > 0$, $f_t(x) \propto g_t(x)$ if for all $t > 0$, there exist constants $c_t, c'_t > 0$ such that for all $x \in D$,

$$c'_t g_t(x) \leq f_t(x) \leq c_t g_t(x).$$

## 2 Main results

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be the probability space under consideration. Consider the Langevin process $(q_t, p_t)_{t \geq 0}$ solution to the following equation:

\[
\begin{aligned}
&dq_t = p_t dt, \\
&dp_t = (-\alpha q_t + \beta) dt - \gamma p_t dt + \sigma dB_t,
\end{aligned}
\]  

where $\alpha \geq 0$, $\beta, \gamma \in \mathbb{R}$, $\sigma > 0$ and $(B_t)_{t \geq 0}$ is a one-dimensional $(\mathcal{F}_t)_{t \geq 0}$-Brownian process. Its infinitesimal generator, also called kinetic Fokker-Planck operator is defined for $(q, p) \in \mathbb{R} \times \mathbb{R}$ by:

\[
\mathcal{L} = p \partial_q - (\alpha q + \beta) \partial_p - \gamma p \partial_p + \frac{\sigma^2}{2} \Delta_p,
\]

with formal adjoint $\mathcal{L}^*$ in $L^2(dq dp)$ given by:

\[
\mathcal{L}^* = -p \partial_q + (\alpha q + \beta) \partial_p + \gamma p \partial_p + \frac{\sigma^2}{2} \Delta_p.
\]

Let $\mathcal{O} := (0, 1)$ and $D := \mathcal{O} \times \mathbb{R}$. Let us introduce the following partition of $\partial D$:

\[\Gamma^+ = \{(0, p) : p < 0\} \cup \{(1, p) : p > 0\}, \]

\[\Gamma^- = \{(0, p) : p > 0\} \cup \{(1, p) : p < 0\}, \]

\[\Gamma^0 := \{(0, 0), (1, 0)\}.\]

The sets $\Gamma^\pm$ correspond to the boundary points with exiting or entering velocity in $D$ and $\Gamma^0$ is the boundary set with zero velocity also called singular set.

Let

\[\tau_0 := \inf\{t > 0 : q_t \notin \mathcal{O}\}.\]

It was shown in [20, Theorem 2.20] that the transition kernel of the Langevin process (8) killed outside of $D$ admits a smooth transition density, i.e. there exists a function

\[(t, (q, (q', p')) \in \mathbb{R}_+ \times D \times D \mapsto p_t^D(q, p, q', p') \in C^\infty(\mathbb{R}_+^n \times D \times D) \cap C^b_{\partial}(\mathbb{R}_+^n \times \overline{D} \times \overline{D})\]

such that for all $t > 0$, $(q, p) \in \overline{D}$ and $A \in \mathcal{B}(D)$,

\[\mathbb{P}_{(q,p)}(\{(q_t, p_t) \in A, \tau_0 > t\} = \int_A p_t^D(q, p, q', p')dq'dp', \]

where we denote by $\mathbb{P}_{(q,p)}$ the probability measure under which $(q_0, p_0) = (q, p)$ almost surely.

The main result of this work is the two-sided estimates satisfied by the transition density $p_t^D$. In order to state this result, we will first define the eigenvector $\phi$ (resp. $\psi$) of the generator $\mathcal{L}$ (resp. $\mathcal{L}^*$) obtained in [21, Theorem 2.13].

Let $(\lambda_0, \phi)$ be the unique solution, up to a multiplicative constant on $\phi$, such that $\phi \in C^2(D) \cap C^b(D \cup \Gamma^+)$ is a non-zero, non-negative classical solution to the following problem

\[
\begin{aligned}
&\mathcal{L}\phi(q, p) = -\lambda_0 \phi(q, p) \quad (q, p) \in D, \\
&\phi(q, p) = 0 \quad (q, p) \in \Gamma^+.
\end{aligned}
\]
Let also \((\lambda_0, \psi)\) be the unique solution, up to a multiplicative constant on \(\psi\), such that \(\psi \in C^2(D) \cap C^b(\Gamma \cup \Gamma^-)\) is a non-zero, non-negative classical solution to:

\[
\begin{aligned}
\mathcal{L}^* \psi(q, p) &= -\lambda_0 \psi(q, p) \quad (q, p) \in D, \\
\psi(q, p) &= 0 \quad (q, p) \in \Gamma^-.
\end{aligned}
\]

(13)

In particular, if we choose the specific \(\psi\) which satisfies \(\int_D \psi = 1\) we can define the following probability measure \(\mu\) on \(D\):

\[
\mu(dqdp) = \psi(q, p) dq dp
\]

(14)

which is called the quasi-stationary distribution (QSD) on \(D\) of the process \((8)\) [21, Theorem 2.14]. Let us now state the two-sided estimates obtained for \(p_t^D\).

**Theorem 2.1** (Two-sided estimates). For all \(t > 0\),

\[
p_t^D(q, p, q', p') \propto \phi(q, p) \psi(q', p').
\]

(15)

Besides, there exists \(\alpha > 0\) such that for all \(t_0 > 0\) there exists \(C > 0\) such that for all \(t \geq t_0\) and for all \((q, p), (q', p') \in D\),

\[
|e^{\lambda t} p_t^D(q, p, q', p') - \frac{\phi(q, p) \psi(q', p')}{\int_D \phi \psi}| \leq C \phi(q, p) \psi(q', p') e^{-\alpha t}.
\]

(16)

Furthermore, we can exhibit sharp and explicit estimates satisfied by the eigenvectors \(\phi, \psi\). In order to do this, let us define a few functions. Let

\[
g : z \in \mathbb{R} \mapsto \begin{cases} 
\left(\frac{2}{3}\right)^{1/6} z U \left(\frac{1}{5}, \frac{4}{3}, \frac{2}{3} z^3\right) & z > 0, \\
\left(-\frac{2}{3}\right)^{1/6} \frac{1}{6} z V \left(\frac{1}{5}, \frac{4}{3}, \frac{2}{3} z^3\right) & z < 0, \\
\left(\frac{2}{3}\right)^{1/6} \Gamma(1/3)/\Gamma(1/6) & z = 0,
\end{cases}
\]

where \(U, V\) are the confluent hypergeometric functions, see [23, p. 256]. The function \(g\) also satisfies the following asymptotics see [9, Lemma 2.1, Sections 3 and 5]:

**Remark 2.2** (Asymptotics of \(g\)). The function \(g\) is a positive, analytic, non-decreasing function satisfying

\[
g(z) \sim \begin{cases} 
\sqrt{z} & z \to +\infty, \\
\left(\frac{2}{3}\right)^{-5/6} \frac{1}{6} e^{-2|z|^{3/9}} |z|^{-5/2} & z \to -\infty.
\end{cases}
\]

Let us define

\[
h : (q, p) \in D \mapsto q^{1/6} g(p/q^{1/3}).
\]

(17)

The function \(h\) is used in [13] and [9, Lemma 2.1] to characterize the long-time behaviour of \(\mathbb{P}_{(q,p)}(\hat{\tau}_0 > t)\) for the integrated Brownian process \((6)\) through the asymptotics:

\[
\forall (q, p) \in \mathbb{R}^+ \times \mathbb{R}, \quad \mathbb{P}_{(q,p)}(\hat{\tau}_0 > t) \sim_{t \to \infty} 3\Gamma(1/4) \frac{h(q, p)}{4^{1/4} \pi^{1/2}}.
\]

(18)

Symmetrically, the function \(h(1-q, -p)\) can be used to describe the long-time asymptotics of \(\mathbb{P}_{(q,p)}(\hat{\tau}_1 > t)\) where \(\hat{\tau}_1 = \inf \{t > 0 : \hat{\tau}_t = 1\}\). This allows us to think that the function \(H\) defined as follows:

\[
\forall (q, p) \in D, \quad H(q, p) = h(q, p) \wedge h(1-q, -p)
\]

(19)

captures the behaviour in \((0,1)\) of the probability \(\mathbb{P}_{(q,p)}(\hat{\tau}_0 > t)\) for a fixed \(t > 0\). We shall in fact prove this result and later extend it to the process \((8)\), using a Girsanov argument, obtaining the following function:

\[
H_{\alpha, \beta, \gamma, \sigma}(q, p) = T_{\alpha, \beta, \gamma, \sigma}(q, p) G_{\sqrt{\alpha + \gamma^2/2}, \sqrt{\pi \sigma}}(q, p),
\]

(20)

where for \((q, p) \in D,\) and \(\lambda \geq 0, \sigma > 0,\)

\[
G_{\lambda, \sigma}(q, p) = h \left(\frac{q, (p+3\lambda q)/\sigma^{2/3}}{\sigma^{2/3}}\right) \wedge \left(e^{-3\lambda p/\sigma^2} h \left(1-q, -(p+3\lambda q)/\sigma^{2/3}\right)\right),
\]

(21)

\[
T_{\alpha, \beta, \gamma, \sigma}(q, p) = \exp \left(-\frac{p^2}{\sigma^2} \left(\frac{\gamma}{2} - 2 \sqrt{\frac{\alpha + \gamma^2/2}{11}}\right) - \frac{q p}{\sigma^2} \left(\frac{8\alpha}{11} \frac{3\gamma^2}{22} - \frac{\beta}{\sigma^2}\right)\right).
\]
Corollary 2.3 (Eigenvectors estimates).

\[ \phi(q,p) \propto H_{\alpha,\beta,\gamma,\sigma}(q,p), \quad \psi(q,p) \propto H_{\alpha,\beta,-\gamma,\sigma}(q,-p). \]

Remark 2.4 (Comparability). In particular, in contrary to the elliptic case, see for instance [7, Proposition 3], we do not have that \( \phi(q,p) \propto \psi(q,p) \). However, one has that \( \phi(q,-p) \propto \psi(q,p) \) if \( \gamma = 0 \).

Remark 2.5 (Integrability of \( H_{\alpha,\beta,\gamma,\sigma} \)). Notice that \( H_{\alpha,\beta,\gamma,\sigma} \in L^r(D) \) for any \( r \in (0,\infty] \). In fact, since \( g \) is non-decreasing, for all \((q,p) \in D\),

\[ G_{\lambda,\sigma}(q,p) \leq e^{\alpha q^2/2} \int_{\|q-p\| \geq \lambda} h(1-q, -(p+3\lambda q)/\sigma^2) + 1_{p+3\lambda q \leq 0} h(q, (p+3\lambda q)/\sigma^2)^2/3) \]

\[ \leq e^{\alpha x^2/2} \left( 1_{p+3\lambda q \geq 0} (1-q)^{1/6} g(-p+3\lambda q)/\sigma^2/3\right) + 1_{p+3\lambda q \leq 0} (1-q)^{1/6} g(p+3\lambda q)/\sigma^2/3\)

\[ \leq e^{\alpha x^2/2} g(-p+3\lambda q)/\sigma^2/3). \]

Therefore, the asymptotics of \( g \) in Remark 2.2 and the expression of \( H_{\alpha,\beta,\gamma,\sigma} \) in (20) ensure that \( H_{\alpha,\beta,\gamma,\sigma} \in L^r(D) \) for any \( r \in (0,\infty] \).

Furthermore, Theorem 2.1 allows us to define the semigroup associated to the killed transition kernel (11) on the Banach space \( L_{H_{\alpha,\beta,\gamma,\sigma}} \) given by:

\[ L_{H_{\alpha,\beta,\gamma,\sigma}} := \{ f \text{ measurable } : (q',p') \in D \mapsto |f(q',p')|H_{\alpha,\beta,-\gamma,\sigma}(q',-p') \in L^1(D) \}, \]

endowed with the norm \( \|f\|_{H_{\alpha,\beta,\gamma,\sigma}} := \int_D |f(q',p')|H_{\alpha,\beta,\gamma,\sigma}(q',-p') dq'dp'. \)

Remark 2.6 (Set \( L_{H_{\alpha,\beta,\gamma,\sigma}} \)). Using the inequality on \( H_{\alpha,\beta,\gamma,\sigma} \) provided in Remark 2.5 along with the asymptotics of \( g \) in Remark 2.2, one deduces that \( L_{H_{\alpha,\beta,\gamma,\sigma}} \) contains the set of functions:

\[ (q,p) \in D \mapsto \eta(q)e^{\|p\|/\sigma^2}, \]

where \( \eta \in L^1(\mathcal{O}) \) and \( c \in [0,2/9) \).

The semigroup \( (P^D_t)_{t \geq 0} \) associated to the transition kernel (11) in \( L_{H_{\alpha,\beta,\gamma,\sigma}} \) is defined as follows:

\[ P^D_0 f = f \text{ for } f \in L_{H_{\alpha,\beta,\gamma,\sigma}} \text{ and for } t > 0, \]

\[ \forall f \in L_{H_{\alpha,\beta,\gamma,\sigma}}, \forall (q,p) \in D, \quad P^D_t f(q,p) = \mathbb{E}_{(q,p)} [f(q_t,p_t)|\mathcal{F}_{\tau_0 > \tau}] = \int_D f(q',p') P^D_t(q,p,q',p')dq'dp'. \]

As a result, Theorem 2.1 and Corollary 2.3 ensure the following immediate corollary.

Corollary 2.7 (Killed semigroup estimates). Let \( f \) be a non-negative function in \( L_{H_{\alpha,\beta,\gamma,\sigma}} \), then for all \( t > 0 \), there exist \( c_t > 0 \), \( c_t^* > 0 \) such that for all \( f \in L_{H_{\alpha,\beta,\gamma,\sigma}} \):

\[ \forall (q,p) \in D, \quad c_t||f||_{H_{\alpha,\beta,\gamma,\sigma}} H_{\alpha,\beta,\gamma,\sigma}(q,p) \leq P^D_t f(q,p) \leq c_t||f||_{H_{\alpha,\beta,\gamma,\sigma}} H_{\alpha,\beta,\gamma,\sigma}(q,p). \]

Remark 2.8 (Hölder property). Given the asymptotics of \( g \) when \( p \rightarrow +\infty \) in Remark 2.2, there exists \( C > 0 \) such that

\[ \forall q > 0, p \in \mathbb{R}, \quad h(q,p) \leq 1_{p < 0} q^{1/6} g(0) + 1_{p \geq 0} C(q^{1/6} + \sqrt{p}) \]

\[ \leq C'(q^{1/6} + \sqrt{|p|}). \]

for some constant \( C' > 0 \) and \( p_+ = p \vee 0 \). As a result, in the case \( \alpha = \beta = \gamma = 0 \), by Corollary 2.7 for any \( t > 0 \), there exists \( c_t > 0 \) such that for all \( f \in L_{H_{\alpha,0,0,\sigma}} \), for all \( (q,p) \in D \),

\[ |P^D_t f(q,p)| \leq c_t||f||_{H_{\alpha,0,0,\sigma}} \left((q \wedge (1-q))^{1/6} + \sqrt{|p|}\right). \]

This result can be related to the work by Hwang, Jang and Velazquez in [12] where the authors showed that when \( \gamma = \alpha = \beta = 0 \) weak solutions to \( \partial_t u = Lu \) with zero boundary condition on \( \Gamma^+ \) and initial condition \( f \in L^1(D) \cap L^\infty(D) \) are \((\alpha,3\alpha)\)-Hölderian at the boundary \( \Gamma_0 \) for any \( \alpha \in (0,1/6) \). Here, we are able to show that this Hölder regularity is actually attained for \( \alpha = 1/6 \).
The strategy of the proof of Theorem 2.1 consists in showing first sharp estimates on the first exit time probability as in (5). Second, these estimates are shown to ensure the two-sided estimates and their long-time asymptotics. Namely, we prove Proposition 2.9 in Section 4.1 which yields the two-sided estimates from the results of Section 3. Corollary 2.3 is also proven in Section 4.1. Finally, we prove the long-time asymptotics of Theorem 2.1 along with Theorem 2.10 in Section 4.2.

Proposition 2.9 (Equivalence of two-sided estimates). Let $d \geq 1$. Let $\mathcal{O}$ be a $C^2$ bounded connected open set of $\mathbb{R}^d$. Let $D = \mathcal{O} \times \mathbb{R}^d$ and let $(q_t^{F,\gamma,\sigma}, p_t^{F,\gamma,\sigma})_{t \geq 0}$ be the process in $\mathbb{R}^d \times \mathbb{R}^d$ solution to

\[
\begin{align*}
\frac{dt}{dt} F^{\gamma,\sigma} &= p_t^{F,\gamma,\sigma} dt, \\
(\partial_t F^{\gamma,\sigma}) &= F(q_t^{F,\gamma,\sigma}) dt - \gamma p_t^{F,\gamma,\sigma} dt + \sigma dB_t,
\end{align*}
\]

where $F \in C^\infty(\mathbb{R}^d)$, $\gamma \in \mathbb{R}$ and $\sigma > 0$. Let $\tau_0^{F,\gamma,\sigma} = \inf\{t > 0 : q_t^{F,\gamma,\sigma} \notin \mathcal{O}\}$. Assume that there exists a function $H_{\alpha,\beta,\gamma,\sigma}$ in $D$ such that for all $t > 0$,

\[
\mathbb{P}_{(q,p)}(\tau_0^{F,\gamma,\sigma} > t) \propto H_{\alpha,\beta,\gamma,\sigma}(q,p) \quad \text{and} \quad \mathbb{P}_{(q,p)}(\tau_0^{F,\gamma,\sigma} < t) \propto H_{\alpha,\beta,\gamma,\sigma}(q,p).
\]

Then, for all $t > 0$,

\[
p_t^{D}(q,p,q',p') \propto \phi(q,p)\psi(q',p'),
\]

where the eigenvectors $\phi$ and $\psi$ are defined analogously to (12) and (13) using the infinitesimal generator of (23).

As a result, in the general multidimensional Langevin case it is sufficient to obtain sharp estimates on the first exit time probability to obtain the two-sided estimates. This is namely the object of future research.

Last but not least, these two-sided estimates allow us in particular to sharpen a convergence result, stated in [21, Theorem 2.22] regarding the convergence of the law of the process (8) conditioned on not being killed. Unlike the results found in the literature in [21, 11, 1], the convergence result below is stated with a prefactor independent of the initial distribution $\theta$.

Theorem 2.10 (Long-time convergence). There exists $\alpha > 0$ such that for all $t_0 > 0$ there exists $C_{t_0} > 0$ such that for all $t > t_0$, for all $f \in L_{H_{\alpha,\beta,\gamma,\sigma}}$, for any probability measure $\theta$ on $D$,

\[
\left| \mathbb{E}_{\theta} \left[ f(q_t, p_t) | \tau_0 > t \right] - \int_D f d\mu \right| \leq C_{t_0} \| f \|_{H_{\alpha,\beta,\gamma,\sigma}} e^{-\alpha t},
\]

where $\mu$ is the quasi-stationary distribution defined in (14).

Remark 2.11 (Total-variation convergence). In particular, taking the supremum in (24) for $f \in L^\infty(D) \subset L_{H_{\alpha,\beta,\gamma,\sigma}}(D)$ such that $\| f \|_{L^\infty(D)} \leq 1$, one has the existence of $\alpha, C > 0$ such that for all $t \geq 0$, for all probability measure $\theta$ on $D$,

\[
\| \mathbb{P}_{\theta} \left( (q_t, p_t) \in | \tau_0 > t \right) - \mu(\cdot) \|_{TV} \leq Ce^{-\alpha t}.
\]

Following a criterion established in [3, Proposition 3.8], the convergence in [21] can be extended directly to the uniform convergence (25) if there exists $t_0 > 0$ and a compact set $K_0 \subset D$ such that the following estimate is satisfied,

\[
\inf_{q,p \in D} \frac{\mathbb{P}_{(q,p)}(\tau_{\theta} > t_0)}{\mathbb{P}_{(q,p)}(\tau_{\theta} > t_0)} > 0,
\]

which follows directly from the two-sided estimates in Theorem 2.1. The purpose of the estimates in Theorem 2.10 is therefore mainly to extend this convergence for the largest set of functions $L_{H_{\alpha,\beta,\gamma,\sigma}}(D)$.
3 First exit time probability estimates

In this section we shall prove sharp and explicit estimates for the first exit time probability of the Langevin process (8).

Proposition 3.1 (First exit time probability estimates). For all $t > 0$,
\[ \mathbb{P}_{(q,p)}(\tau_0 > t) \propto H_{\alpha,\beta,\gamma,\sigma}(q,p), \]
where $H_{\alpha,\beta,\gamma,\sigma}$ is defined in (20).

The proof is first completed in Section 3.1 for the integrated Brownian process (6) and extended in Section 3.2 to the Langevin process (8) using a Girsanov argument.

3.1 Integrated Brownian process

Let us consider the following process for $\tau_0 > t$:
\[
\begin{aligned}
dq^2 = \tilde{p}_t^2 dt, \\
d\tilde{p}_t = \sigma dB_t.
\end{aligned}
\tag{27}
\]
We denote by $\tau_0^\sigma$ its first exit time from $D$. The main result of this section is the following.

Proposition 3.2 (First exit time probability estimates). For all $t > 0$,
\[ \mathbb{P}_{(q,p)}(\tau_0^\sigma > t) \propto H(q,p/\sigma^{2/3}), \]
where $H$ is defined in (19).

In addition, recall that for $c > 0$, the integrated Brownian process satisfies the following equality-in-law:
\[ \left( \int_0^t B_s ds \right)_{t \geq 0} \overset{\mathcal{L}}{=} \left( c^{3/2} \int_0^t B_s ds \right)_{t \geq 0}. \tag{28} \]
Therefore, for any $t > 0$, $(q,p) \in D$,
\[ \mathbb{P}_{(q,p)}(\tau_0^\sigma > t) = \mathbb{P}_{(q,p/\sigma^{2/3})}(\tau_0 > \sigma^{2/3} t), \tag{29} \]
where $\tau_0$ corresponds to the first exit time from $D$ of (27) for $\sigma = 1$. As a result, it is sufficient to complete the proof in the case $\sigma = 1$. We shall from now on denote simply by $(\tilde{q}_t, \tilde{p}_t)_{t \geq 0}$ the process (27) defined with $\sigma = 1$ and by $\tau_0$ its first exit time from $D$.

Therefore we shall prove the following proposition which directly implies Proposition 3.2.

Proposition 3.3 (First exit time probability estimates). For all $t > 0$,
\[ \mathbb{P}_{(q,p)}(\tau_0 > t) \propto H(q,p). \]

The proof is divided as follows: Section 3.1.1 is devoted to the proof of the upper-bound and Section 3.1.2 focuses on the proof of the lower-bound. In order to achieve this proof we will use mostly the two following properties satisfied by the integrated Brownian process along with the long-time asymptotics (18) from [13, 9].

Let us define $\tilde{\tau}_0, \tilde{\tau}_1$ as the following hitting times
\[ \tilde{\tau}_0 := \inf\{ t > 0 : \tilde{q}_t = 0 \}, \quad \tilde{\tau}_1 := \inf\{ t > 0 : \tilde{q}_t = 1 \}. \]
We shall make use of the following properties.

Remark 3.4 (Timescale change). Given (28), it follows that for all $q > 0$, $p \in \mathbb{R}$ and $\lambda > 0$,
\[ \mathbb{P}_{(q,p)}(\tilde{\tau}_0 > t) = \mathbb{P}_{(\lambda^3 q, \lambda p)}(\tilde{\tau}_0 > \lambda^2 t). \]

Remark 3.5 (Invariant transformation). Let us notice that the equality-in-law $(B_t)_{t \geq 0} \overset{\mathcal{L}}{=} (-B_t)_{t \geq 0}$ ensures that
\[ \forall (q,p) \in D, \quad \mathbb{P}_{(q,p)}(\tau_0 > t) = \mathbb{P}_{(1-q,-p)}(\tau_0 > t), \quad \text{and} \quad \mathbb{P}_{(1-q,-p)}(\tau_0 > t) = \mathbb{P}_{(q,p)}(\tau_1 > t). \]
3.1.1 Upper-bound on the first exit time probability

The goal of this subsection is to provide an upper-bound on the probability $P_{(q,p)}(\hat{\tau}_0 > t)$. Namely, we shall prove the following proposition.

**Proposition 3.6 (Upper-bound).** For all $t > 0$ there exists $c_t > 0$ such that

$$\forall(q,p) \in D, \quad P_{(q,p)}(\hat{\tau}_0 > t) \leq c_t H(q,p). \quad (30)$$

We resort in particular to the following lemma.

**Lemma 3.7 (Martingales).** For all $(q,p) \in D$, the processes $(h(\hat{\tau}_t \wedge \hat{\tau}_0, \hat{\tau}_t \wedge \hat{\tau}_0))_{t \geq 0}$ and $(h(1-\hat{\tau}_t \wedge \hat{\tau}_0, -\hat{\tau}_t \wedge \hat{\tau}_0))_{t \geq 0}$ are $(\mathcal{F}_t)_{t \geq 0}$-martingales under $P_{(q,p)}$, where $(\mathcal{F}_t)_{t \geq 0}$ is the natural filtration of the Brownian process.

**Proof.** Let $(q,p) \in D$. Since $h \in C^\infty(D)$ and $Lh = 0$ on $D$, see [9, Lemma 2.1], the Itô formula ensures that both processes are $(\mathcal{F}_t)_{t \geq 0}$-local martingales under $P_{(q,p)}$. They are martingales if one can prove that for all $t > 0$,

$$E_{(q,p)} \left[ \sup_{s \in [0,t]} h(\hat{q}_s, \hat{p}_s) \right] < \infty \quad \text{and} \quad E_{(q,p)} \left[ \sup_{s \in [0,t]} h(1-\hat{q}_s, -\hat{p}_s) \right] < \infty.$$

Using the definition of $h$ in (17) and the asymptotics of $g$ at $+\infty$ in Remark 2.2, one can see that both inequalities follow if one can show that

$$E_{(q,p)} \left[ \sup_{s \in [0,t]} \sqrt{|p_s|} \right] < \infty.$$

The Cauchy-Schwarz inequality ensures that it is sufficient to prove that

$$E \left[ \sup_{s \in [0,t]} |B_s| \right] < \infty.$$

Moreover,

$$E \left[ \sup_{s \in [0,t]} |B_s| \right] = \int_0^\infty P \left( \sup_{s \in [0,t]} |B_s| > x \right) dx \leq \int_0^\infty 4P(B_t > x) dx.$$

Using the inequality $P(B_t > x) \leq 1 \wedge \frac{e^{-x^2/2t} \sqrt{x}}{\sqrt{2\pi}}$ concludes the proof. $\square$

Let us now prove Proposition 3.6.

**Proof of Proposition 3.6.** Let $t > 0$, $(q,p) \in D$. Assume that $\frac{|p|}{q^{1/3}} \leq 3/t$, one has using Remark 3.4 that

$$P_{(q,p)}(\hat{\tau}_0 > t) \leq P_{(q,p)}(\hat{\tau}_0 > t)$$

$$= P_{(1,p/q^{1/3})}(\hat{\tau}_0 > t/q^{2/3})$$

$$\leq P_{(1,\lambda/3)}(\hat{\tau}_0 > t/q^{2/3}),$$

since $P_{(q,p)}(\hat{\tau}_0 > t)$ is a non-decreasing function of $p$. Therefore,

$$P_{(q,p)}(\hat{\tau}_0 > t) \leq \frac{P_{(1,\lambda/3)}(\hat{\tau}_0 > t/q^{2/3})}{q^{1/6} g(p/q^{1/3})} \leq \frac{1}{g(-3/\lambda)} \frac{P_{(1,\lambda/3)}(\hat{\tau}_0 > t/q^{2/3})}{q^{1/6}},$$

since $g$ is a non-decreasing function. Besides, the term in the right-hand side of the inequality above is bounded when $q \to 0$. In fact, using the long-time asymptotics (18) from [13, 9] one has the existence of a constant $\alpha_t > 0$ such that

$$\frac{P_{(1,\lambda/3)}(\hat{\tau}_0 > t/q^{2/3})}{q^{1/6}} \sim \frac{\alpha_t q^{1/6}}{q^{1/6}} \sim 0.$$
This ensures that for any $t > 0$, there exists $\beta_t > 0$ such that for all $(q, p) \in D$ satisfying $|p|/q^{1/3} \leq 3/t$,
\[
P_{(q, p)}(\hat{\tau}_\partial > t) \leq \beta_t h(q, p).
\] (31)

Assume now that $|p|/q^{1/3} > 3/t$. Let

\[
\hat{\tau}_t := \inf\{ s \geq 0 : |\hat{p}_s|/q_s^{1/3} \leq 3/t \}.
\]

Necessarily, $\hat{\tau}_\partial \wedge \hat{\tau}_t \leq t/2$ almost surely. In fact if $\hat{\tau}_\partial \wedge \hat{\tau}_t > t/2$, then, depending on the sign of $\hat{p}_0$, one has by continuity of the trajectory $(\hat{p}_s)_{s \geq 0}$ that, almost-surely, for all $s \in [0, t/2]$,

\[
\frac{\hat{p}_s}{q_s^{1/3}} = \frac{d\hat{q}_s}{q_s^{1/3}} \leq -3/t \quad \text{or} \quad \frac{\hat{p}_s}{q_s^{1/3}} = \frac{d\hat{q}_s}{q_s^{1/3}} \geq 3/t.
\]

As a result, integrating $\hat{p}_s/q_s^{1/3}$ over $s \in [0, t/2]$, one obtains that

\[
\frac{q_s^{2/3}}{t} \leq q^{2/3} - 1 < 0, \quad \text{or} \quad \frac{q_s^{2/3}}{t} \geq q^{2/3} + 1 > 1,
\]

which contradicts the fact that $\hat{\tau}_\partial > t/2$. As a result, one has by the strong Markov property and (31)

\[
P_{(q, p)}(\hat{\tau}_\partial > t) = \mathbb{E}_{(q, p)} \left[ 1_{\hat{\tau}_\partial > \hat{\tau}_t} P_{(\hat{q}_\hat{\tau}_t, \hat{p}_\hat{\tau}_t)} (\hat{\tau}_\partial > t - s) | s = \hat{\tau}_t \right]
\]

\[
\leq \mathbb{E}_{(q, p)} \left[ 1_{\hat{\tau}_\partial > \hat{\tau}_t} P_{(\hat{q}_\hat{\tau}_t, \hat{p}_\hat{\tau}_t)} (\hat{\tau}_\partial > t/2) \right]
\]

\[
\leq \beta_{t/2} \mathbb{E}_{(q, p)} \left[ 1_{\hat{\tau}_\partial > \hat{\tau}_t} h(\hat{q}_\hat{\tau}_t, \hat{p}_\hat{\tau}_t) \right]
\]

\[
\leq \beta_{t/2} \mathbb{E}_{(q, p)} \left[ h(\hat{q}_{\hat{\tau}_t}, \hat{p}_{\hat{\tau}_t}, \hat{q}_{\hat{\tau}_0}, \hat{p}_{\hat{\tau}_0}) \right] = \beta_{t/2} h(q, p),
\]

by Lemma 3.7 and Doob’s optional sampling theorem since $\hat{\tau}_t \wedge \hat{\tau}_\partial \leq t/2$ almost-surely. Therefore, for all $t > 0$, there exists $c_t > 0$ such that for all $(q, p) \in D$,

\[
P_{(q, p)}(\hat{\tau}_\partial > t) \leq c_t h(q, p).
\]

Using Remark 3.5, we also have that

\[
P_{(q, p)}(\hat{\tau}_\partial > t) = P_{(1-q, -p)}(\hat{\tau}_\partial > t) \leq c_t h(1-q, -p),
\]

hence (30), which concludes the proof.

\[\square\]

### 3.1.2 Lower-bound on the first exit time probability

We shall prove in this section the lower-bound in Proposition 3.3. This proof is more complex as it requires a careful study depending on the location in the phase space. We represent below the domain $D$ with its boundary $\Gamma^+$, $\Gamma^-$ and $\Gamma^0$ along with a partition 1, 2 and 3 of $D$ used in the lower-bound proof.

![Figure 2: Domain decomposition](image-url)
The idea is that the minimum defining $H$ will be lead by $h(1 - q, -p)$ for large positive velocities occurring in 1 and by $h(q, p)$ for large negative velocities occurring in 3. As said in the introduction large velocities have to considered compared to the distance boundary. Given the expression of $h$, it seems that the proper scaling to be considered are $p/q^{1/3}$ or $-p/(1 - q)^{1/3}$. The domain 2 is therefore delimited by the curves of equations $p = -3q^{1/3}/t$ and $p = 3(1 - q)^{1/3}/t$. In 2, we shall prove directly the lower bound $H$ where we shall use the fact that 2 is a compact in $(q, p)$.

We first prove the lower-bound until some time $t_0 > 0$ and then generalize it in Proposition 3.12 for any time $t > 0$. Therefore, we prove here the existence of a constant $c_t' > 0$ depending on $t > 0$ such that for all $t \in (0, t_0]$,

1. $\forall (q, p) \in 1$, $\mathbb{P}_{(q, p)}(\tilde{\tau}_0 > t) \geq c_t'h(1 - q, -p)$.
2. $\forall (q, p) \in 2$, $\mathbb{P}_{(q, p)}(\tilde{\tau}_0 > t) \geq c_t'(h(1 - q, -p) \wedge h(q, p))$.
3. $\forall (q, p) \in 3$, $\mathbb{P}_{(q, p)}(\tilde{\tau}_0 > t) \geq c_t'h(q, p)$,

thus leading to the lower-bound proof. The generalization for any $t > 0$ is done later in this section. Besides, since 1 and 3 are symmetric through the transformation $(q, p) \mapsto (1 - q, -p)$ it is sufficient to prove the first two inequalities.

**Proposition 3.8** (Lower-bound). There exists $t_0 > 0$ such that for all $t \in (0, t_0]$ there exists a constant $c'_t > 0$ such that

\[ \forall (q, p) \in D, \quad \mathbb{P}_{(q, p)}(\tilde{\tau}_0 > t) \geq c'_t H(q, p). \]

This proposition will later be extended for all time $t > 0$ in Proposition 3.12. The proof relies on the two following lemmas.

**Lemma 3.9.** For any $t > 0$ there exists $c'_t > 0$ such that

\[ \forall q \in (0, 1), \forall p \in [-3q^{1/3}/t, 3(1 - q)^{1/3}/t], \quad \mathbb{P}_{(q, p)}(\tilde{\tau}_0 > t) \geq c'_t h(q, p). \]

**Lemma 3.10.** There exists $t_0 > 0$ such that for all $t \in (0, t_0]$ there exists a constant $\alpha_t > 0$ such that

\[ \forall q \in (0, 1), \forall p \in [-3q^{1/3}/t, 3(1 - q)^{1/3}/t], \quad \mathbb{P}_{(q, p)}(\tilde{\tau}_0 > t) \geq \alpha_t\mathbb{P}_{(q, p)}(\tilde{\tau}_0 > t) \wedge \mathbb{P}_{(q, p)}(\tilde{\tau}_1 > t). \]

Assuming Lemmas 3.9 and 3.10 are satisfied, let us prove Proposition 3.8.

**Proof of Proposition 3.8.** Let $t_0 > 0$ be as defined in Lemma 3.10 and let $t \in (0, t_0)$. Let $q \in (0, 1)$ and $p \in \mathbb{R}$ such that $p \in [-3q^{1/3}/t, 3(1 - q)^{1/3}/t]$. Notice that for such $(q, p)$, the couple $(1 - q, -p)$ also satisfies the hypothesis of Lemma 3.9. As a result, applying Lemma 3.9 along with Remark 3.5 one obtains that

\[ \mathbb{P}_{(q, p)}(\tilde{\tau}_1 > t) \geq c'_t h(1 - q, -p). \]

Therefore, it follows from Lemma 3.10 that for all $q \in (0, 1)$ and $p \in [-3q^{1/3}/t, 3(1 - q)^{1/3}/t],

\[ \mathbb{P}_{(q, p)}(\tilde{\tau}_0 > t) \geq \alpha_t c'_t (h(q, p) \wedge h(1 - q, -p)) = \alpha_t c'_t H(q, p). \]

Thus, it remains to prove (32) for $q \in (0, 1)$ and $p/q^{1/3} < -3/t$ or $p/(1 - q)^{1/3} > 3/t$. We start with the case $p/q^{1/3} < -3/t$.

Let $q \in (0, 1)$ and $p/q^{1/3} < -3/t$. Let

\[ \hat{\pi}_t := \inf\{s > 0 : \hat{\pi}_s/q_s^{1/3} \geq -3/t\}. \]

Necessarily, $\hat{\tau}_0 \wedge \hat{\pi}_t \leq t/2$ almost-surely. Otherwise, integrating $\hat{\pi}_s/q_s^{1/3} = (d\hat{q}_s/ds)/q_s^{1/3}$ for $s \in [0, t/2]$, one obtains that $\hat{q}_s/t \leq 0$ contradicting that $\hat{\tau}_0 > t/2$. Besides, the continuity of the trajectories ensures that $\hat{\tau}_1 > \hat{\pi}_t$ almost-surely. Therefore, applying the strong Markov property at the stopping time $\hat{\pi}_t$,

\[ \mathbb{P}_{(q, p)}(\tilde{\tau}_0 > t) = \mathbb{E}_{(q, p)}\mathbb{I}_{\{\tau_0 > \hat{\pi}_t\}} \mathbb{P}_{(\hat{q}_{\hat{\pi}_t}, \hat{\pi}_t)}(\tilde{\tau}_0 > t - s) | s = \hat{\pi}_t \]

\[ \geq \mathbb{E}_{(q, p)}\mathbb{I}_{\{\tau_0 > \hat{\pi}_t\}} \mathbb{P}_{(\hat{q}_{\hat{\pi}_t}, \hat{\pi}_t)}(\tilde{\tau}_0 > t) \]

(33)
On the event $\{\hat{\tau}_0 > \hat{\tau}_1\}$, $\hat{q}_{\hat{\tau}_1} \in (0,1)$ and $\hat{p}_{\hat{\tau}_1}/\hat{q}_{\hat{\tau}_1}^{1/3} = -3/t$. Therefore, applying (32) to $\mathbb{P}(\hat{q}_{\hat{\tau}_1}, \hat{p}_{\hat{\tau}_1})(\hat{\tau}_0 > t)$ in the expectation above, one has
\[
\mathbb{P}(\hat{q}_{\hat{\tau}_1}, \hat{p}_{\hat{\tau}_1})(\hat{\tau}_0 > t) \geq \alpha_t c'_{t}(h(\hat{q}_{\hat{\tau}_1}, \hat{p}_{\hat{\tau}_1}) \land h(1- \hat{q}_{\hat{\tau}_1}, -\hat{p}_{\hat{\tau}_1})).
\]
(34)

By definition of $h$ and $\hat{\tau}_1$,
\[
h(\hat{q}_{\hat{\tau}_1}, \hat{p}_{\hat{\tau}_1}) = \hat{q}_{\hat{\tau}_1}^{1/6} g \left( \frac{\hat{p}_{\hat{\tau}_1}}{\hat{q}_{\hat{\tau}_1}^{1/3}} \right) = \hat{q}_{\hat{\tau}_1}^{1/6} g \left( \frac{3}{t} \right).
\]
(35)

Besides, since $q$ is positive and satisfies the asymptotics in Remark 2.2, there exists $\beta > 0$ such that for all $z \geq 0$, $g(z) \geq \beta \sqrt{z}$. As a result,
\[
h(1- \hat{q}_{\hat{\tau}_1}, -\hat{p}_{\hat{\tau}_1}) = (1- \hat{q}_{\hat{\tau}_1})^{1/6} g \left( \frac{\hat{p}_{\hat{\tau}_1}}{(1- \hat{q}_{\hat{\tau}_1})^{1/3}} \right)
= (1- \hat{q}_{\hat{\tau}_1})^{1/6} g \left( \frac{3}{t} \right) \left( \frac{\hat{q}_{\hat{\tau}_1}}{1- \hat{q}_{\hat{\tau}_1}} \right)^{1/3}
\geq \beta \sqrt[3]{ \frac{3}{t} } \hat{q}_{\hat{\tau}_1}^{1/6} = \frac{\beta}{g(-3/t)} \sqrt[3]{ \frac{3}{t} } h(\hat{q}_{\hat{\tau}_1}, \hat{p}_{\hat{\tau}_1}).
\]
(36)

Hence, reinserting (35) and (36) into (34) one has the existence of a constant $\gamma_t > 0$ such that on the event $\{\hat{\tau}_0 > \hat{\tau}_1\}$,
\[
\mathbb{P}(\hat{q}_{\hat{\tau}_1}, \hat{p}_{\hat{\tau}_1})(\hat{\tau}_0 > t) \geq \gamma_t h(\hat{q}_{\hat{\tau}_1}, \hat{p}_{\hat{\tau}_1}).
\]
Therefore, reinserting into (33) it follows that
\[
\mathbb{P}(\tau_0 > t) \geq \gamma_t E_{(q,p)}[1_{\{\hat{\tau}_0 > \hat{\tau}_1\}} h(\hat{q}_{\hat{\tau}_1}, \hat{p}_{\hat{\tau}_1})]
= \gamma_t E_{(q,p)}[h(\hat{q}_{\hat{\tau}_1}, \hat{p}_{\hat{\tau}_1})] = \gamma_t h(q,p)
\]
using Lemma 3.7 and Doob’s optional sampling theorem since $\hat{\tau}_1 \land \hat{\tau}_0 \leq t/2$ almost-surely and since $h$ vanishes continuously on the set $\{(0,p) : p \leq 0\}$, which follows from the asymptotics in Remark 2.2.

Now let us take $q \in (0,1)$ and $p/(1-q)^{1/3} > 3/t$. In this case, since $(1-q, -p)$ satisfy the hypothesis of the previous case, we immediately have that $\mathbb{P}(1- q, -p)(\hat{\tau}_0 > t) \geq \gamma_t h(1-q, -p)$. Using Remark 3.5, we deduce that for $q \in (0,1)$ and $p/(1-q)^{1/3} > 3/t$
\[
\mathbb{P}(\tau_0 > t) \geq \gamma_t h(1-q, -p).
\]
Hence the proof for all $q \in (0,1)$ and $p \in \mathbb{R}$.

Let us now prove Lemma 3.9.

Proof of Lemma 3.9. Let $t > 0$. Let us take sequences $(q_n)_{n \geq 0}$, $(p_n)_{n \geq 0}$ such that for all $n \geq 0$, $q_n \in (0,1)$ and $p_n \in [-3q_n^{1/3}/t, 3(1-q_n)^{1/3}/t]$ and prove that $\liminf_{n \to \infty} \mathbb{P}(q_n, p_n)(\hat{\tau}_0 > t) / h(q_n, p_n) > 0$.

Both sequences $(q_n)_{n \geq 0}$ and $(p_n)_{n \geq 0}$ are bounded. Therefore, up to extracting appropriate subsequences, we can assume that they both converge. For $n \geq 0$, using Remark 3.4,
\[
\mathbb{P}(q_n, p_n)(\hat{\tau}_0 > t) = \mathbb{P}(1, p_n/q_n^{1/3})(\hat{\tau}_0 > t/q_n^{2/3}).
\]
(37)

First assume that $\limsup_{n \to \infty} p_n/q_n^{1/3} < \infty$, then the sequence $(p_n/q_n^{1/3})_{n \geq 0}$ is bounded from above by a constant $M > 0$. Therefore, since for all $n \geq 0$, $-3/t \leq p_n/q_n^{1/3} \leq M$,
\[
\frac{\mathbb{P}(q_n, p_n)(\hat{\tau}_0 > t)}{h(q_n, p_n)} \geq \frac{\mathbb{P}(1, -3/t)(\hat{\tau}_0 > t/q_n^{2/3})}{q_n^{1/6} g(M)}.
\]

If $q_n \to q_\infty > 0$, then the term in the right-hand side of the inequality above is clearly bounded from below by a positive time-dependent constant. If $q_n \to 0$, using the long-time asymptotics in (18) from [13, 9] one has the existence of a constant $\alpha_t > 0$ such that
\[
\mathbb{P}(1, -3/t)(\hat{\tau}_0 > t/q_n^{2/3}) \sim \frac{\alpha_t q_n^{1/6}}{q_n^{1/6}} = \alpha_t > 0.
\]
Assume now that $\limsup_{n \to \infty} p_n / q_n^{1/3} = \infty$, up to taking a subsequence we can assume that for all $n \geq 0$, $p_n > 0$ and that $p_n / q_n^{1/3} \to \infty$.

First, consider the case $p_n \to p_\infty > 0$. Then up to taking a subsequence we can assume that $p_n \geq p_\infty / 2$ for all $n \geq 0$. Besides, $\hat{P}_{(q,p)}$ almost-surely for all $s \geq 0$,

$$\hat{q}_s = q + \int_0^s (B_r + p) \, dr,$$

one has that

$$\mathbb{P}_{(q_n,p_n)}(\hat{\tau}_0 > t) \geq \mathbb{P}\left( \inf_{s \in [0,t]} B_s \geq -p_n \right) \geq \mathbb{P}\left( \inf_{s \in [0,t]} B_s \geq -p_\infty / 2 \right).$$

Therefore, by the asymptotics of $g$ in Remark 2.2,

$$\mathbb{P}_{(q_n,p_n)}(\hat{\tau}_0 > t) \sim \mathbb{P}\left( \inf_{s \in [0,t]} B_s \geq -p_\infty / 2 \right) / q_n^{1/6} g(p_n / q_n^{1/3}) > 0.$$

Second, consider the case $\limsup_{n \to \infty} p_n / q_n^{1/3} = \infty$ and $p_n \to 0$. Again we consider a subsequence such that $p_n > 0$ for all $n$. Let $\tilde{\tau}_0 := \inf \{ s > 0 : \hat{\tau}_0 = 0 \}$. Then, by the strong Markov property at $\tilde{\tau}_0$ and using the first step of the proof, there exists a constant $c_1 > 0$ such that

$$\mathbb{P}_{(q_n,p_n)}(\tilde{\tau}_0 > t) \geq \mathbb{P}_{(q_n,p_n)}(\tilde{\tau}_0 > t, \tilde{\tau}_0 \leq t \wedge \tilde{\tau}_1)$$

$$= \mathbb{E}_{(q_n,p_n)} \left[ 1_{\tilde{\tau}_0 \leq t \wedge \tilde{\tau}_1} \mathbb{P}_{(\hat{q}_n, \hat{p}_n)}(\hat{\tau}_0 > t - s) | s = \tilde{\tau}_0 \right]$$

$$\geq c_1 \mathbb{E}_{(q_n,p_n)} \left[ 1_{\tilde{\tau}_0 \leq t \wedge \tilde{\tau}_1} \mathbb{P}_{(\hat{q}_n, \hat{p}_n)}(\hat{\tau}_0 > t) \right]$$

$$= c_1 h(q_n, p_n) - c_1 \mathbb{E}_{(q_n,p_n)} \left[ 1_{\tilde{\tau}_0 > t \wedge \tilde{\tau}_1} h(\hat{q}_n, \hat{p}_n) \right].$$

(38)

by Lemma 3.7. Furthermore, by the Hölder inequality with coefficients 4/3 and 4 and considering the asymptotics of $h$ in Remark 2.2 one has that

$$\mathbb{E}_{(q_n,p_n)} \left[ 1_{\tilde{\tau}_0 > t \wedge \tilde{\tau}_1} h(\hat{q}_n, \hat{p}_n) \right] \leq \mathbb{P}_{(q_n,p_n)}(\tilde{\tau}_0 > t \wedge \tilde{\tau}_1)^{3/4} \mathbb{E}_{(q_n,p_n)} \left[ h(\hat{q}_n, \hat{p}_n) \right]^{1/4}$$

$$\leq \mathbb{P}_{(q_n,p_n)}(\tilde{\tau}_0 > t \wedge \tilde{\tau}_1)^{3/4} \mathbb{E}_{(q_n,p_n)} \left[ \sup_{s \in [0,t]} C(1 + |\hat{p}_s|^2) \right]^{1/4}$$

$$\leq C^{1/4} \mathbb{P}_{(q_n,p_n)}(\tilde{\tau}_0 > t \wedge \tilde{\tau}_1)^{3/4} \mathbb{E}_{(q_n,p_n)} \left[ 1 + p_n^2 + 2p_n \sup_{s \in [0,t]} |B_s| + \sup_{s \in [0,t]} |B_s|^2 \right]^{1/4}$$

$$\leq C^{1/4} \mathbb{P}_{(q_n,p_n)}(\tilde{\tau}_0 > t \wedge \tilde{\tau}_1)^{3/4} \left( 1 + p_n^2 + \frac{2\sqrt{2}t}{\sqrt{\pi}} p_n + 2t \frac{\sqrt{2}}{\sqrt{\pi}} \right)^{1/4},$$

(39)

Furthermore,

$$\mathbb{P}_{(q_n,p_n)}(\tilde{\tau}_0 > t \wedge \tilde{\tau}_1) \leq \mathbb{P}_{(q_n,p_n)}(\tilde{\tau}_0 > t) + \mathbb{P}_{(q_n,p_n)}(\tilde{\tau}_0 > \tilde{\tau}_1).$$

Since $p_n > 0$ for all $n \geq 0$,

$$\mathbb{P}_{(q_n,p_n)}(\tilde{\tau}_0 > t) = \mathbb{P}\left( \inf_{s \in [0,t]} B_s > -p_n \right)$$

$$= \mathbb{P}( |B_t| < p_n )$$

$$= \mathbb{P}( |B_t| < p_n / \sqrt{t} )$$

$$\leq \frac{\sqrt{2}}{\sqrt{\pi} p_n}.$$
Now notice that
\[ P_{(q_n,p_n)}(\hat{\tau}_0 > \hat{\tau}_1) = P_{(q_n,p_n)}(\hat{\tau}_0 > 1) \]
In addition, the explicit expression of the law of $\hat{\tau}_0$, is given in Lachal’s work in [17, Equation 6]. Therefore, if we take $N \geq 1$ large enough such that for all $n \geq N$, $\bar{q}_n \in (0, 1/2)$ then
\[ P_{(q_n,p_n)}(\hat{\tau}_0 > \hat{\tau}_1) = P_{(q_n,p_n)}(\hat{\tau}_0 > 1) = \int_1^\infty \frac{\Gamma(2/3)}{\pi^{2/3}3^{1/6}} \frac{p_n}{|z - q_n|^4} e^{-2p_n^3/9|z - q_n|^3} dz \]
\[ = \frac{\Gamma(2/3)}{\pi^{2/3}3^{1/6}} \int_0^{1/(1-q_n)} p_n u^{-2/3} e^{-2up_n^2/9} du \leq \frac{\Gamma(2/3)}{\pi^{2/3}3^{1/6}} p_n \int_0^2 u^{-2/3} du. \]

Consequently, there exists a constant $C'_1 > 0$ such that for all $n \geq N$,
\[ P_{(q_n,p_n)}(\hat{\tau}_0 > t \wedge \hat{\tau}_1) \leq C'_1 p_n. \]

It follows then from (38), (39) and (40) that
\[ \frac{P_{(q_n,p_n)}(\hat{\tau}_0 > t)}{h(q_n,p_n)} \geq c_4 - c_4 C^{1/4}(C'_1)^{3/4} \left(1 + p_n^2 + 2\sqrt{\pi} p_n + 2t \sqrt{\pi} \right)^{1/4} \frac{p_n^{3/4}}{h(q_n,p_n)}. \]

Moreover,
\[ \left(1 + p_n^2 + 2\sqrt{\pi} p_n + 2t \sqrt{\pi} \right)^{1/4} \frac{p_n^{3/4}}{h(q_n,p_n)} \to 0 \text{ as } n \to \infty \]
Taking the limit in (41) concludes the proof of Lemma 3.9.

Let us now prove Lemma 3.10.

Proof of Lemma 3.10. Let $t > 0$. Since $P_{(q,p)}(\hat{\tau}_0 > t)$ continuously vanishes on the set $\{(0,p) : p \leq 0\}$ and $P_{(q,p)}(\hat{\tau}_1 > t)$ continuously vanishes on the set $\{(1,p) : p \geq 0\}$, it is enough to prove that for any sequences $(q_n)_{n \geq 0}$, $(p_n)_{n \geq 0}$ such that $q_n \to 0$ and $\limsup_{n \to \infty} p_n \leq 0$ which satisfy
\[ \forall n \geq 0, \quad q_n \in (0,1), \quad p_n \in [-3q_n^{1/3}/t, 3(1-q_n)^{1/3}/t], \]
we have
\[ \liminf_{n \to \infty} \frac{P_{(q_n,p_n)}(\hat{\tau}_0 > t)}{P_{(q_n,p_n)}(\hat{\tau}_0 > t)} > 0 \quad \text{and} \quad \liminf_{n \to \infty} \frac{P_{(1-q_n,p_n)}(\hat{\tau}_0 > t)}{P_{(1-q_n,p_n)}(\hat{\tau}_1 > t)} > 0. \]

Using the invariance in Remark 3.5 this is equivalent to the proof of the first liminf. Moreover, for $n \geq 0$,
\[ P_{(q_n,p_n)}(\hat{\tau}_0 > t) = \frac{P_{(q_n,p_n)}(\hat{\tau}_0 > t)}{P_{(q_n,p_n)}(\hat{\tau}_0 > t)} P_{(q_n,p_n)}(\hat{\tau}_0 > t) \]
\[ = \left(1 - \frac{P_{(q_n,p_n)}(\hat{\tau}_0 > t, \hat{\tau}_1 \leq t)}{P_{(q_n,p_n)}(\hat{\tau}_1 > t)}\right) P_{(q_n,p_n)}(\hat{\tau}_0 > t). \]

Our aim here is to obtain appropriate upper-bounds of $P_{(q_n,p_n)}(\hat{\tau}_0 > t, \hat{\tau}_1 \leq t)$ in order to obtain a positive lower-bound for $P_{(q_n,p_n)}(\hat{\tau}_0 > t)/P_{(q_n,p_n)}(\hat{\tau}_0 > t)$.

Let $\bar{\tau}_0 := \inf\{s \geq 0 : \bar{p}_s = 0\}$. If $q \in (0,1)$ and $p \leq 0$, the continuity of the velocity ensures that $\bar{\tau}_0 < \hat{\tau}_1 P_{(q,p)}$ almost-surely. Therefore, if $p_n \leq 0$, applying the strong Markov property at $\bar{\tau}_0$ one obtains that
\[ P_{(q_n,p_n)}(\bar{\tau}_0 > t, \hat{\tau}_1 \leq t) \leq E_{(q_n,p_n)} \left[\mathbf{1}_{\bar{\tau}_0 > \bar{\tau}_0} P_{(q_n,p_n)}(\bar{\tau}_0 > t - \bar{\tau}_0, \hat{\tau}_1 \leq t - \bar{\tau}_0)_{\bar{\tau}_0 = \bar{\tau}_0} \right] \]
\[ \leq E_{(q_n,p_n)} \left[\mathbf{1}_{\bar{\tau}_0 > \bar{\tau}_0} P_{(q_n,p_n)}(\bar{\tau}_0 > \hat{\tau}_1) \right]. \]
Besides, thanks to the work of Lachal in [18, Equation 4] one has an exact expression of $P_{(q,0)}(\hat{\tau}_0 > \hat{\tau}_1)$ for $q \in (0, 1)$ where $F, \beta$ are respectively the hypergeometric and the beta function,

$$P_{(q,0)}(\hat{\tau}_0 > \hat{\tau}_1) = \frac{\Gamma(1/3)}{\Gamma(1/6)} \frac{\Gamma(1/6)}{\Gamma(1/2)} \int_{0}^{1} x^{-1/6}(1-x)^{-2/3}(1-qx)^{-1/6} dx \beta(5/6, 1/3)$$

using the fact that $\beta(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$ along with $\Gamma(a + 1) = a\Gamma(a)$. As a result, reinserting into (42), one has

$$P_{(q_n, p_n)}(\hat{\tau}_0 > t, \hat{\tau}_1 \leq t) \leq \mathbb{E}_{(q_n, p_n)} \left[ 1_{\hat{\tau}_0 > \hat{\tau}_1} g(0) \right]$$

where

$$g(0) = h(q_n, p_n) = \frac{\Gamma(1/3)}{\Gamma(1/6)} \frac{\Gamma(1/6)}{\Gamma(1/2)} \int_{0}^{1} x^{-1/6}(1-x)^{-2/3}(1-qx)^{-1/6} dx \beta(5/6, 1/3)$$

$$= q^{1/6}$$

Therefore, using Lemma 3.7 it follows from Doob’s optional sampling theorem that $\mathbb{E}_{(q_n, p_n)} \left[ h(q_n, p_n) \right] = h(q_n, p_n)$. As a result,

$$P_{(q_n, p_n)}(\hat{\tau}_0 > t, \hat{\tau}_1 \leq t) \leq \frac{h(q_n, p_n)}{g(0)} \leq \frac{1}{g(0)} \int_{0}^{1} x^{-1/6}(1-x)^{-2/3}(1-qx)^{-1/6} dx \beta(5/6, 1/3)$$

However, in the case $p_n > 0$, one cannot ensure that $\hat{\pi}_0 < \hat{\tau}_1$ almost-surely. Therefore,

$$P_{(q_n, p_n)}(\hat{\tau}_0 > t, \hat{\tau}_1 \leq t) = \mathbb{E}_{(q_n, p_n)} \left[ 1_{\hat{\tau}_0 > \hat{\tau}_1} \right] \leq \mathbb{E}_{(q_n, p_n)} \left[ 1_{\hat{\tau}_0 > \hat{\tau}_1} \right] = \frac{1}{g(0)} h(q_n, p_n) \leq \frac{1}{g(0)} \int_{0}^{1} x^{-1/6}(1-x)^{-2/3}(1-qx)^{-1/6} dx \beta(5/6, 1/3)$$

Further more, one has that

$$P_{(q_n, p_n)}(\hat{\tau}_0 \geq \hat{\tau}_1) \leq \frac{1}{g(0)} \int_{0}^{1} x^{-1/6}(1-x)^{-2/3}(1-qx)^{-1/6} dx \beta(5/6, 1/3)$$

Since $q_n \to 0$, up to taking a subsequence, we can assume that $q_n \in (0, 1/2)$ for $n \geq 1$. In addition, reinserting the explicit expression of the law of $\tilde{q}_{\pi_0}$, which follows from Lachal’s work in [17, Equation 6], one has, as in the proof of Lemma 3.9, that

$$P_{(q_n, p_n)}(\tilde{q}_{\pi_0} > 0) \leq \frac{\Gamma(1/3)}{\Gamma(1/6)} \frac{\Gamma(1/6)}{\Gamma(1/2)} \int_{0}^{1} u^{-2/3} du$$

Let us now compute an upper-bound for $\limsup_{n \to \infty} \frac{P_{(q_n, p_n)}(\hat{\tau}_0 > t, \hat{\tau}_1 \leq t)}{P_{(q_n, p_n)}(\hat{\tau}_0 > t)}$.

Assuming there is some $N \geq 0$ such that for all $n \geq N$, $P_{q_n, p_n} \leq 0$. Then using (43) it follows that for $n \geq N$,

$$\frac{P_{(q_n, p_n)}(\hat{\tau}_0 > t, \hat{\tau}_1 \leq t)}{P_{(q_n, p_n)}(\hat{\tau}_0 > t)} \leq \frac{1}{g(0)} \frac{P_{(q, p_n)}(\hat{\tau}_0 > t)}{P_{(q, p_n)}(\hat{\tau}_0 > t)} \leq \frac{1}{g(0)} \frac{\Gamma(1/3)}{\Gamma(1/6)} \frac{\Gamma(1/6)}{\Gamma(1/2)} \int_{0}^{1} u^{-2/3} du$$
Since $p_n/q_n^{1/3} \in [-3/t, 0]$ for all $n \geq N$, up to taking an appropriate subsequence, one can assume that the sequence $(p_n/q_n^{1/3})_{n \geq N}$ converges to a limit $l \leq 0$.

Let $\epsilon > 0$. Let $N' \geq 1$ such that for all $n \geq N'$, $p_n/q_n^{1/3} \geq l - \epsilon$. Then for all $n \geq N' \lor N$,

$$\frac{\mathbb{P}_{(q_n,p_n)}(\hat{\tau}_0 > t, \hat{\tau}_1 \leq t)}{\mathbb{P}_{(q_n,p_n)}(\hat{\tau}_0 > t)} \leq \frac{1}{g(0)} \frac{q_n^{1/6}g(p_n/q_n^{1/3})}{\mathbb{P}_{(1, l-\epsilon)}(\hat{\tau}_0 > t/q_n^{2/3})}.$$  

It follows from (18) (see [9, Lemma 2.1]) that $\mathbb{P}_{(q,p)}(\hat{\tau}_0 > t) \sim C(h(q,p)t^{-1/4}$ where $C$ is a universal constant. Therefore,

$$\limsup_{n \to \infty} \frac{\mathbb{P}_{(q_n,p_n)}(\hat{\tau}_0 > t, \hat{\tau}_1 \leq t)}{\mathbb{P}_{(q_n,p_n)}(\hat{\tau}_0 > t)} \leq \frac{t^{1/4}}{Cg(0)} \limsup_{n \to \infty} \frac{g(p_n/q_n^{1/3})}{g(l - \epsilon)}.$$  

Since the term in the left hand-side of the inequality above does not depend on $\epsilon$, one can take the limit $\epsilon \to 0$ thus obtaining

$$\limsup_{n \to \infty} \frac{\mathbb{P}_{(q_n,p_n)}(\hat{\tau}_0 > t, \hat{\tau}_1 \leq t)}{\mathbb{P}_{(q_n,p_n)}(\hat{\tau}_0 > t)} \leq \frac{t^{1/4}}{Cg(0)},$$  

which can be made as small as possible for $t \in (0, t_0)$ with $t_0 > 0$ small enough since $Cg(0)$ is a universal constant.

Assume now that for all $n \geq 0$, there exists a $n' \geq n$ such that $p_{n'} > 0$. Then, since $\limsup_{n \to \infty} p_n \leq 0$ necessarily $p_n \to 0$. Besides, up to taking an appropriate subsequence, one can assume that for all $n \geq 0$, $p_n > 0$. It follows from (44), (45) and (46) that there exist a universal constant $C' > 0$ such that

$$\frac{\mathbb{P}_{(q_n,p_n)}(\hat{\tau}_0 > t, \hat{\tau}_1 \leq t)}{\mathbb{P}_{(q_n,p_n)}(\hat{\tau}_0 > t)} \leq \frac{1}{g(0)} \frac{h(q_n,p_n)}{\mathbb{P}_{(q_n,p_n)}(\hat{\tau}_0 > t)} + C' \limsup_{n \to \infty} \frac{p_n}{\mathbb{P}_{(q_n,p_n)}(\hat{\tau}_0 > t)}.$$  

It follows from (47) that

$$\limsup_{n \to \infty} \frac{\mathbb{P}_{(q_n,p_n)}(\hat{\tau}_0 > t, \hat{\tau}_1 \leq t)}{\mathbb{P}_{(q_n,p_n)}(\hat{\tau}_0 > t)} \leq \frac{t^{1/4}}{Cg(0)} + C' \limsup_{n \to \infty} \frac{p_n}{\mathbb{P}_{(q_n,p_n)}(\hat{\tau}_0 > t)}.$$  

First, consider the case $\limsup_{n \to \infty} p_n/q_n^{1/3} < \infty$. Using the fact that

$$\mathbb{P}_{(q_n,p_n)}(\hat{\tau}_0 > t) \geq \mathbb{P}_{(q_n,0)}(\hat{\tau}_0 > t) = \mathbb{P}_{(1,0)}(\hat{\tau}_0 > t/q_n^{2/3}),$$

we have

$$\limsup_{n \to \infty} \frac{\mathbb{P}_{(q_n,p_n)}(\hat{\tau}_0 > t, \hat{\tau}_1 \leq t)}{\mathbb{P}_{(q_n,p_n)}(\hat{\tau}_0 > t)} \leq \frac{t^{1/4}}{Cg(0)} + C' \limsup_{n \to \infty} \frac{p_n}{\mathbb{P}_{(1,0)}(\hat{\tau}_0 > t/q_n^{2/3})}.$$  

In addition,

$$\frac{q_n^{1/3}}{\mathbb{P}_{(1,0)}(\hat{\tau}_0 > t/q_n^{2/3})} \sim \frac{t^{1/4}}{Cg(0)q_n^{1/6}} \longrightarrow 0,$$

since $q_n \to 0$. As a result, the inequality (47) is still satisfied.

Consider now the case $\limsup_{n \to \infty} p_n/q_n^{1/3} = \infty$. Up to taking a subsequence we can assume that $p_n/q_n^{1/3} \sim \infty$. It follows from Lemma 3.9 that there exists a constant $\alpha_t > 0$ such that for all $n \geq 0$, $\mathbb{P}_{(q_n,p_n)}(\hat{\tau}_0 > t) \geq \alpha_t h(q_n,p_n)$. Therefore,

$$\frac{p_n}{\mathbb{P}_{(q_n,p_n)}(\hat{\tau}_0 > t)} \leq \alpha_t \frac{1}{h(q_n,p_n)} \frac{p_n}{\mathbb{P}_{(1,0)}(\hat{\tau}_0 > t/q_n^{2/3})} \sim \frac{p_n}{\sqrt{p_n}} \longrightarrow 0,$$

since $p_n \rightarrow 0$. Again, the inequality (47) is still satisfied. Hence the proof of Lemma 3.10.

In order to extend the lower-bound obtained in Proposition 3.8 for any time $t > 0$, let us first prove the following lemma which is stated for a general Langevin process.
Lemma 3.11 (Control in a compact set). Let $d \geq 1$. Let $O$ be a $C^2$ bounded connected open set of $\mathbb{R}^d$. Let $D = O \times \mathbb{R}^d$ and let $(q_t, p_t)_{t \geq 0}$ be the process in $\mathbb{R}^d \times \mathbb{R}^d$ solution to
\[
\begin{aligned}
    dq_t &= p_t dt, \\
    dp_t &= F(q_t)dt - \gamma p_t dt + \sigma dB_t,
\end{aligned}
\]
where $F \in C^\infty(\mathbb{R}^d)$, $\gamma \in \mathbb{R}$ and $\sigma > 0$. Let $\tau_0 = \inf\{t > 0 : q_t \notin O\}$. Assume that there exist $t_0 > 0$, a function $H_{\alpha,\beta,\gamma,\sigma}$ in $D$ such that for all $t \in (0, t_0)$,
\[
    \mathbb{P}_{(q,p)}(\tau_0 > t) \propto H_{\alpha,\beta,\gamma,\sigma}(q,p).
\]
Then, there exists a compact set $K_0 \subset D$ such that
\[
    \inf_{(q,p) \in D} \frac{\mathbb{P}_{(q,p)}((q_0, p_0) \in K_0, \tau_0 > t_0)}{\mathbb{P}_{(q,p)}(\tau_0 > t_0)} > 0.
\]

Proof. Let $t_0 > 0$ be defined as in the assumption. Let $K_0 \subset D$ be any given compact set then for any $(q, p) \in D$,
\[
    \mathbb{P}_{(q,p)}((q_0, p_0) \in K_0, \tau_0 > t_0) = \mathbb{P}_{(q,p)}(\tau_0 > t_0) - \mathbb{P}_{(q,p)}((q_0, p_0) \notin K_0, \tau_0 > t_0).
\]

Besides, using the existence of a transition density $p^D$ for the killed kernel, see [20, Theorem 2.20], along with the Chapman-Kolmogorov relation, one has that
\[
    \mathbb{P}_{(q,p)}((q_0, p_0) \notin K_0, \tau_0 > t_0) = \iint_{D \times D \times K_0} p^D_{t_0/3}(q, y)p^D_{t_0/3}(y, z)p^D_{t_0/3}(z, w)dydw.
\]

Furthermore, $p^D$ satisfies a Gaussian upper-bound, see [20, Theorem 2.19] which ensures in particular that $p^D_{t_0/3}$ is uniformly bounded on $D \times D$ by a constant $\alpha_0 > 0$. As a result,
\[
    \mathbb{P}_{(q,p)}((q_0, p_0) \notin K_0, \tau_0 > t_0) \leq \alpha_0 \int_D p^D_{t_0/3}(q, y)dy \iint_{D \times K_0} p^D_{t_0/3}(z, w)dzdw
\]
\[
    = \alpha_0 \mathbb{P}_{(q,p)}(\tau_0 > t_0/3) \iint_{D \times K_0} p^D_{t_0/3}(z, w)dzdw.
\]

Consequently,
\[
    \mathbb{P}_{(q,p)}((q_0, p_0) \notin K_0, \tau_0 > t_0) \leq \alpha_0 \mathbb{P}_{(q,p)}(\tau_0 > t_0/3) \iint_{D \times K_0} p^D_{t_0/3}(z, w)dzdw.
\]

By assumption there exists a constant $c_0 > 0$ independent of $(q, p) \in D$ such that
\[
    \frac{\mathbb{P}_{(q,p)}(\tau_0 > t_0/3)}{\mathbb{P}_{(q,p)}(\tau_0 > t_0)} \leq c_0.
\]

Furthermore, it was shown in [21, Lemma 3.1] that for any $t > 0$, $p^D_t \in L^1(D \times D)$. Consequently, there exists a compact set $K_0 \subset D$ large enough such that
\[
    \iint_{D \times K_0} p^D_{t_0/3}(z, w)dzdw \leq \frac{1}{2\alpha_0 c_0}.
\]

The two inequalities above ensure that for $K_0$ defined as such, for all $(q, p) \in D$,
\[
    \frac{\mathbb{P}_{(q,p)}((q_0, p_0) \in K_0, \tau_0 > t_0)}{\mathbb{P}_{(q,p)}(\tau_0 > t_0)} \geq \frac{1}{2}.
\]

We are now able to prove Proposition 3.12.
Proposition 3.12 (Lower-bound). For all \( t > 0 \), there exists a constant \( c'_t > 0 \) such that
\[
\forall (q, p) \in D, \quad \mathbb{P}_{(q, p)}(\tilde{\tau}_\theta > t) \geq c'_t H(q, p).
\]  

Proof of Proposition 3.12. Let \( t_0 > 0 \) be as defined in Proposition 3.8. Using Propositions 3.6 and 3.8 we can apply Lemma 3.11 to obtain
\[
c_0 := \inf_{(q, p) \in D} \frac{\mathbb{P}_{(q, p)}((\tilde{q}_{t_0}, \tilde{\mu}_t) \in K_0, \tilde{\tau}_\theta > t_0)}{\mathbb{P}_{(q, p)}(\tilde{\tau}_\theta > t_0)} > 0.
\]  

Let us now prove (50). In order to do that, we start by considering, for any integer \( k \geq 1 \), the probability \( \mathbb{P}_{(q, p)}(\tilde{\tau}_\theta > (k + 1)t_0) \). Using the Markov property one has
\[
\mathbb{P}_{(q, p)}(\tilde{\tau}_\theta > (k + 1)t_0) = \mathbb{E}_{(q, p)} \left[ \mathbb{1}_{\tilde{\tau}_\theta > k t_0} \mathbb{P}_{(q, p)}(\tilde{\tau}_\theta > t_0) \right]
\[
\quad \geq \mathbb{E}_{(q, p)} \left[ \mathbb{1}_{\tilde{\tau}_\theta > k t_0} \mathbb{P}_{(q, p)}(\tilde{\tau}_\theta > t_0) \right].
\]

Let \( \delta_0 := \inf_{z \in K_0} \mathbb{P}_z(\tilde{\tau}_\theta > t_0) \) then \( \delta_0 > 0 \) since \( K_0 \) is a compact subset of \( D \) and such probability is continuous and positive on \( D \) by [20, Theorem 2.20]. Therefore,
\[
\mathbb{P}_{(q, p)}(\tilde{\tau}_\theta > (k + 1)t_0) \geq \delta_0 \mathbb{P}_{(q, p)}((\tilde{q}_{k t_0}, \tilde{\mu}_{k t_0}) \in K_0, \tilde{\tau}_\theta > k t_0)
\[
\quad = \delta_0 \mathbb{E}_{(q, p)} \left[ \mathbb{1}_{\tilde{\tau}_\theta > (k-1)t_0} \mathbb{P}_{(q, p)}((\tilde{q}_{(k-1)t_0}, \tilde{\mu}_{(k-1)t_0}) \in K_0, \tilde{\tau}_\theta > t_0) \right].
\]

As a result, by (51),
\[
\mathbb{P}_{(q, p)}(\tilde{\tau}_\theta > (k + 1)t_0) \geq \delta_0 c_0 \mathbb{E}_{(q, p)} \left[ \mathbb{1}_{\tilde{\tau}_\theta > (k-1)t_0} \mathbb{P}_{(q, p)}((\tilde{q}_{(k-1)t_0}, \tilde{\mu}_{(k-1)t_0}) \in K_0, \tilde{\tau}_\theta > t_0) \right]
\[
\quad = \delta_0 c_0 \mathbb{P}_{(q, p)}(\tilde{\tau}_\theta > k t_0).
\]

As a result, for all \( k \geq 1 \),
\[
\mathbb{P}_{(q, p)}(\tilde{\tau}_\theta > (k + 1)t_0) \geq (\delta_0 c_0)^k \mathbb{P}_{(q, p)}(\tilde{\tau}_\theta > t_0) \geq (\delta_0 c_0)^k c'_t H(q, p)
\]
by Proposition 3.8.

Now let us take any \( t > t_0 \). Then there exists an integer \( k \geq 1 \) and \( s \in [0, t_0] \) such that \( t = k t_0 + s \). Since \( \mathbb{P}_{(q, p)}(\tilde{\tau}_\theta > t) \geq \mathbb{P}_{(q, p)}(\tilde{\tau}_\theta > (k + 1)t_0) \) and \( k = \lfloor t/t_0 \rfloor \), one has from the inequality above that
\[
\mathbb{P}_{(q, p)}(\tilde{\tau}_\theta > t) \geq (\delta_0 c_0)^{\lfloor t/t_0 \rfloor} c'_t H(q, p)
\]
which concludes the proof of Proposition 3.12. \( \square \)

Finally, Propositions 3.6 and 3.12 conclude the proof of Proposition 3.3 which yields Proposition 3.2. Let us mention that applying Proposition 2.9 to the process (27) now ensures two-sided estimates for the transition density of the killed process (27) which we shall use in the next section. Let us also recall that the proof of Proposition 2.9 is completed in Section 4.

3.2 Langevin process

In this section we shall extend the estimates obtained in Proposition 3.2 to the first exit time probability of the Langevin process (8). Namely we shall here prove Proposition 3.1. Let \( (q_t, p_t)_{t \geq 0} \) satisfying (8) and let \( \tau_\theta \) be its first exit time from \( D \).

As mentioned previously, the estimates obtained in Proposition 3.2 in the previous section yield the two-sided estimates (15) for the process (27) using Proposition 2.9 which is proven in Section 4.

For a fixed \( \sigma > 0 \) and any \( \eta \geq 0 \) let us define the following process \( (q^\eta_t, p^\eta_t)_{t \geq 0} \):
\[
\begin{cases}
 dq^\eta_t = p^\eta_t dt, \\
p^\eta_t = -4np^\eta_t dt - 3q^2 p^\eta_t dt + \sigma dB_t.
\end{cases}
\]

Let
\[
\tau^\eta_\theta := \inf \{ t > 0 : q^\eta_t \notin (0, 1) \}.
\]

We shall first prove sharp estimates on the first exit time probability \( \mathbb{P}_{(q, p)}(\tau^\eta_\theta > t) \) for the process (52) in Lemma 3.13. Such estimates are then used in Lemma 3.14 to provide an estimate on an expectation which shall appear later in the proof of Proposition 3.1. We will then conclude with the proof of Proposition 3.1.

We start this section with the proof of the following two lemmas.
Lemma 3.13. For all $t > 0$, 

$$
P_{(q,p)}(\tau^q_0 > t) \propto G_{q,\sigma}(q,p),$$

where $G_{q,\sigma}$ is defined in (21).

Lemma 3.14. Let $\lambda \geq 0$ and let $f : D \mapsto \mathbb{R}_+$ such that there exists $a,b > 0$ satisfying for all $(q,p) \in D$,

$$
|f(q,p)| \leq e^{a|q|^2 + b}.
$$

Then, for all $t > 0$,

$$
\mathbb{E}_{(q,p)} \left[ 1_{\tau^q_\ell > t} f(q,\tilde{\nu}_t, \tilde{\tau}_t) \exp \left( -\frac{\lambda^2}{\sigma^2} \int_0^t (\tilde{\nu}_s^q)^2 ds \right) \right] \propto \exp \left( \frac{2\lambda p^2}{\sigma^2 \sqrt{11}} + \frac{3qp\lambda^3}{11\sigma^2} \right) G_{q,\sigma}(q,p),
$$

where $(\tilde{q}_t^q, \tilde{\nu}_t, \tilde{\tau}_t)$ is defined on (27) and $\tilde{\tau}_0^q$ is its first exit time from $D$.

Proof of Lemma 3.13. Let $(q,p) \in D$. It can be seen that the solution $\tilde{q}_t^q$ of (52) at any time $t \geq 0$ is given by, under $\mathbb{P}_{(q,p)}$ almost-surely,

$$
\tilde{q}_t^q = \frac{q}{2} (3e^{-nt} - e^{-3nt}) + \frac{p}{2\eta} (e^{-nt} - e^{-3nt}) + \frac{\sigma}{2\eta} \int_0^t (e^{-n(t-s)} - e^{-3n(t-s)}) dB_s
$$

$$
= \frac{q}{2} e^{-3nt} (3e^{2nt} - 1) + \frac{p}{2\eta} e^{-3nt} (e^{2nt} - 1) + \frac{\sigma}{2\eta} \int_0^t (e^{-n(t-s)} - e^{-3n(t-s)}) dB_s.
$$

In addition, one can check the following equality in law for the below Gaussian processes:

$$
\frac{1}{2\eta} \int_0^t (e^{-n(t-s)} - e^{-3n(t-s)}) dB_s \overset{d}{=} e^{-3nt} \frac{(e^{2nt} - 1)}{2\eta} B_s ds.
$$

As a result, the process $(\tilde{q}_t^q)_{t \geq 0}$ shares the same law as the following process given at any time $t \geq 0$ by:

$$
qe^{-3nt} + (p + 3\eta q)e^{-3nt} \frac{(e^{2nt} - 1)}{2\eta} + \sigma e^{-3nt} \int_0^{(e^{2nt} - 1)/2\eta} B_s ds.
$$

It follows from the change of time $s = (e^{2nt} - 1)/2\eta$ that

$$
P_{(q,p)}(\tau^q_0 > t)
$$

$$
= P \left( \forall 0 \leq s \leq (e^{2nt} - 1)/2\eta, \quad 0 < q + (p + 3\eta q)s + \sigma \int_0^s B_r dr < (1 + 2\eta s)^{3/2} \right).
$$

Let us start by showing the upper-bound, i.e. there exists $C_t > 0$ such that for all $(q,p) \in D$,

$$
P_{(q,p)}(\tau^q_0 > t) \leq C_t G_{q,\sigma}(q,p).
$$

By (54) one has

$$
P_{(q,p)}(\tau^q_0 > t)
$$

$$
\leq P \left( \forall 0 \leq s \leq (e^{2nt} - 1)/2\eta, \quad 0 < q e^{-3nt} + (p + 3\eta q)e^{-3nt} s + \sigma e^{-3nt} \int_0^s B_r dr < 1 \right).
$$

Therefore, it follows from Proposition 3.2 that for all $t > 0$, there exists $c_t > 0$ such that for all $(q,p) \in D$,

$$
P_{(q,p)}(\tau^q_0 > t) \leq c_t \left( q e^{-3nt}, (p + 3\eta q)e^{-nt}/\sigma^{2/3} \right)
$$

$$
= c_t e^{-nt/2} \left( q, (p + 3\eta q)/\sigma^{2/3} \right).
$$

Furthermore, by (54),

$$
P_{(q,p)}(\tau^q_0 > t)
$$

$$
= P \left( \forall 0 \leq s \leq (e^{2nt} - 1)/2\eta, \quad 1 - (1 + 2\eta s)^{3/2} < q + (p + 3\eta q)s + \sigma \int_0^s \left( B_r - \frac{3\eta}{\sigma} \sqrt{1 + 2\eta r} \right) dr < 1 \right)
$$

$$
= P \left( \forall 0 \leq s \leq (e^{2nt} - 1)/2\eta, \quad 0 < 1 - q - (p + 3\eta q)s + \sigma \int_0^s \left( B_r + \frac{3\eta}{\sigma} \sqrt{1 + 2\eta r} \right) dr < (1 + 2\eta s)^{3/2} \right).
$$
Now let 
\[ \tau^\sigma_\eta = \inf\{t > 0 : \tilde{q}^\sigma_\eta \notin (0, (1 + 2\eta t)^{3/2})\}. \]

Then, by Girsanov Lemma, under the probability \( \mathbb{Q}_t \) defined on \( \mathcal{F}_t \) for \( t \geq 0 \) where \( (\mathcal{F}_t)_{t \geq 0} \) is the natural filtration of the Brownian motion,
\[
\frac{d\mathbb{Q}_t}{d\mathbb{P}} = \exp \left( -\frac{3\eta}{\sigma} \int_0^t \sqrt{1 + 2\eta s} dB_s - \frac{9\eta^2}{2\sigma^2} \int_0^t (1 + 2\eta s) ds \right),
\]
the process \( (B_t + \frac{3\eta s}{\sigma} \sqrt{1 + 2\eta t})_{t \geq 0} \) is a Brownian motion. Therefore,
\[
P_{(q,p)}(\tau^\sigma_\eta > t) = E_{(1 - q, -(p + 3\eta q))} \left[ \mathbb{1}_{\tau^\sigma_\eta > (e^{2\eta t} - 1)/2\eta} Z^\sigma_t \right],
\]
where 
\[ Z^\sigma_t = \exp \left( \int_0^{(e^{2\eta t} - 1)/2\eta} -\frac{3\eta}{\sigma^2} \sqrt{1 + 2\eta s} dB_s - \frac{9\eta^2}{2\sigma^2} \int_0^{(e^{2\eta t} - 1)/2\eta} (1 + 2\eta s) ds \right). \]

By integration-by-parts, one has \( P_{(1 - q, -(p + 3\eta q))} \) almost-surely,
\[
\int_0^{(e^{2\eta t} - 1)/2\eta} \frac{3\eta}{\sigma^2} \sqrt{1 + 2\eta s} dB_s = \frac{3\eta^2}{\sigma^2} e^{-\eta t} \tilde{q}^\sigma (e^{2\eta t} - 1)/2\eta - \frac{3\eta^2}{\sigma^2} (p + 3\eta q) + \int_0^{(e^{2\eta t} - 1)/2\eta} \frac{3\eta^2}{\sigma^2} \hat{\eta}^\sigma s (1 + 2\eta s)^{3/2} ds.
\]

Furthermore, additional integration-by-parts yields the following equality
\[
\int_0^{(e^{2\eta t} - 1)/2\eta} \frac{3\eta^2}{\sigma^2} \sqrt{1 + 2\eta s} dB_s = \frac{3\eta^2}{\sigma^2} e^{-\eta t} \tilde{q}^\sigma (e^{2\eta t} - 1)/2\eta - \frac{3\eta^2}{\sigma^2} (1 - q) + \int_0^{(e^{2\eta t} - 1)/2\eta} \frac{3\eta^2}{\sigma^2} \hat{\eta}^\sigma s (1 + 2\eta s)^{3/2} ds.
\]

As a result, since \( \tilde{q}^\sigma (e^{2\eta t} - 1)/2\eta \) is bounded under the event \( \hat{\eta}^\sigma > (e^{2\eta t} - 1)/2\eta \), it follows from (58) that for all \( t > 0 \), there exists \( c_t > 0 \) such that
\[
P_{(q,p)}(\tau^\sigma_\eta > t) \leq c_t e^{-3\eta p/\sigma^2} E_{(1 - q, -(p + 3\eta q))} \left[ \mathbb{1}_{\tau^\sigma_\eta > (e^{2\eta t} - 1)/2\eta} \exp \left( -\frac{3\eta}{\sigma^2} e^{2\eta t} \tilde{q}^\sigma (e^{2\eta t} - 1)/2\eta \right) \right].
\]

For any \( (q', p') \in D \) and \( t > 0 \), the process \( (\tilde{q}^\sigma_\eta, \tilde{p}^\sigma_\eta)_{s \geq 0} \) starting from \( (q', p') \) shares the same law as the process \( (e^{q't} \tilde{q}^\sigma_\eta e^{-3\eta t}, e^{p't} \tilde{p}^\sigma_\eta e^{-3\eta t}) \) \( s \geq 0 \) whenever the process \( (\tilde{q}^\sigma_\eta e^{-3\eta t}, \tilde{p}^\sigma_\eta e^{-3\eta t}) \) \( s \geq 0 \) starts from \( (q' e^{-3\eta t}, p' e^{-3\eta t}) \). In addition, if for all \( s \in (0, (e^{2\eta t} - 1)/2\eta), e^{3\eta t} \tilde{q}^\sigma_\eta e^{-3\eta t} \in (0, (1 + 2\eta s)^{3/2}) \) implies that for all \( s \in (0, (e^{2\eta t} - 1)/2\eta), q^\sigma_\eta e^{-3\eta t} \in (0, 1) \). Consequently, reinserting into the above expectation,
\[
P_{(q,p)}(\tau^\sigma_\eta > t) \leq c_t e^{-3\eta p/\sigma^2} E_{(1 - q, -(p + 3\eta q))} \left[ \mathbb{1}_{\tau^\sigma_\eta > (e^{2\eta t} - 1)/2\eta} \exp \left( -\frac{3\eta}{\sigma^2} e^{2\eta t} \tilde{q}^\sigma (e^{2\eta t} - 1)/2\eta \right) \right].
\]

Using the two-sided estimates on the transition density of the killed process (27) ensured by Proposition 2.9, there exists \( c'_t > 0 \) such that
\[
E_{(1 - q) e^{-3\eta t}, -(p + 3\eta q) e^{-3\eta t}} \left[ \mathbb{1}_{\tau^\sigma_\eta > (e^{2\eta t} - 1)/2\eta} \exp \left( -\frac{3\eta}{\sigma^2} e^{2\eta t} \tilde{q}^\sigma (e^{2\eta t} - 1)/2\eta \right) \right]
\]
\[
\leq c'_t (1 - q) e^{-3\eta t} \left( p + 3\eta q \right) e^{-\eta t} / \sigma^{2/3} \left( 1 - q, -(p + 3\eta q) / \sigma^{2/3} \right),
\]
for some \( c'_t > 0 \), hence the existence of a constant \( \tilde{c}_t > 0 \) such that
\[
P_{(q,p)}(\tau^\sigma_\eta > t) \leq \tilde{c}_t e^{-3\eta p/\sigma^2} h \left( 1 - q, -(p + 3\eta q) / \sigma^{2/3} \right),
\]
where
which ensures (55) taking the minimum with (56). It remains now to prove the analogous lower-bound, i.e. for all $t > 0$, there exists $C'_t > 0$ such that

$$
P_{(q,p)}(\tau^q_0 > t) \geq C'_t \sigma(q,p).$$

(61)

We proceed similarly to the phase space decomposition used in Section 3.1.2. Let $t > 0$, we first consider the case $(p + 3\eta q)/\sigma^2/3 \in [-3q^{1/3}/t, 3(1 - q)^{1/3}/t]$. By (54) and Proposition 3.2, there exists $c_t > 0$ such that

$$\mathbb{P}_{(q,p)}(\tau^q_0 > t) \geq \mathbb{P}_{(q,p+3\eta q)}(\tau^q_0 > (e^{2\eta q} - 1)/2\eta)$$

$$\geq c_t H\left(q, (p + 3\eta q)/\sigma^2/3\right)$$

$$\geq c'_t \sigma(q,p),$$

(62)

for some $c'_t > 0$ since $e^{-3\eta p/\sigma^2}$ is bounded from below and above by assumption.

Assume now that $(p + 3\eta q)/\sigma^2/3 < -3q^{1/3}/t$. Then one has that $h(q, (p + 3\eta q)/\sigma^2/3) \leq q^{1/3}g(-3/t)$. Additionally, since there exist $\lambda > 0$ such that $g(z) \geq \lambda\sqrt{z}$ for all $z \geq 0$ by Remark 2.2, one has

$$h \left(1 - q, -(p + 3\eta q)/\sigma^2/3\right) \geq (1 - q)^{1/6}g\left(\frac{3}{t} \left(\frac{q^{1/3}}{1 - q}\right)^{1/3}\right)$$

$$\geq \lambda \sqrt{\frac{3}{t}} q^{1/6} \geq \frac{\lambda}{g(-3/t)} \sqrt{\frac{3}{t}} h\left(q, (p + 3\eta q)/\sigma^2/3\right).$$

(63)

As a result, using (62) and Proposition 3.2 there exists $c''_t > 0$ such that

$$\mathbb{P}_{(q,p)}(\tau^q_0 > t) \geq c''_t H\left(q, (p + 3\eta q)/\sigma^2/3\right).$$

Consider now the case $(p + 3\eta q)/\sigma^2/3 > 3(1 - q)^{1/3}/t$. By (57) and Girsanov lemma, one has that

$$\mathbb{E}_{(q,p)}(\tau^q_0 > t) \geq \mathbb{E}_{(1 - q, -(p + 3\eta q)/\sigma^2/3)}\left[\mathbb{1}_{\tau^q_0 > (e^{2\eta q} - 1)/2\eta} Z^q_t\right].$$

Moreover, by the previous computation of $Z^q_t$ in (59), (60) there exists $c_t > 0$ such that

$$\mathbb{E}_{(1 - q, -(p + 3\eta q)/\sigma^2/3)}\left[\mathbb{1}_{\tau^q_0 > (e^{2\eta q} - 1)/2\eta} Z^q_t\right] \geq e^{-3\eta p/\sigma^2} \mathbb{E}_{(1 - q, -(p + 3\eta q)/\sigma^2/3)}\left[\mathbb{1}_{\tau^q_0 > (e^{2\eta q} - 1)/2\eta} \exp\left(-\frac{3\eta}{\sigma^2} (\eta^q\sigma^q_{(e^{2\eta q} - 1)/2\eta})\right)\right].$$

Using the two-sided estimates for the killed process (27) ensured by Proposition 2.9, there exists $c'_t > 0$ such that

$$\mathbb{E}_{(1 - q, -(p + 3\eta q)/\sigma^2/3)}\left[\mathbb{1}_{\tau^q_0 > (e^{2\eta q} - 1)/2\eta} \exp\left(-\frac{3\eta}{\sigma^2} (\eta^q\sigma^q_{(e^{2\eta q} - 1)/2\eta})\right)\right]$$

$$\geq c'_t \mathbb{P}_{(1 - q, -(p + 3\eta q)/\sigma^2/3)}(\tau^q_0 > (e^{2\eta q} - 1)/2\eta)$$

$$\geq c'_t H\left(1 - q, -(p + 3\eta q)/\sigma^2/3\right).$$

(64)

Besides, similarly to the computation in (63) one can see that $H\left(1 - q, -(p + 3\eta q)/\sigma^2/3\right) \geq \tilde{c}_t h\left(1 - q, -(p + 3\eta q)/\sigma^2/3\right)$ for some $\tilde{c}_t > 0$. Therefore, this ensures that for some $c'_t > 0$

$$\mathbb{P}_{(q,p)}(\tau^q_0 > t) \geq c'_t e^{-3\eta p/\sigma^2} h\left(1 - q, -(p + 3\eta q)/\sigma^2/3\right),$$

which concludes the proof of the lower bound.

\[\Box\]

Proof of Lemma 3.14. Let $f : D \mapsto \mathbb{R}_+$ satisfying (53). For $t > 0$, $(q,p) \in D$ and $\eta \geq 0$, let us consider the following expectation:

$$u_t(q,p) := \mathbb{E}_{(q,p)}\left[\mathbb{1}_{\tau^q_0 > t} f(q^q_t, p^q_t) \exp\left(-\frac{2\eta}{\sigma^2}(p^q_t)^2 - \frac{3\eta^2}{\sigma^2} \eta^q p^q_t\right)\right].$$

By Girsanov’s lemma, one has that

$$u_t(q,p) = \mathbb{E}_{(q,p)}\left[\mathbb{1}_{\tau^q_0 > t} f(q^q_t, p^q_t) \exp\left(-\frac{2\eta}{\sigma^2}(p^q_t)^2 - \frac{3\eta^2}{\sigma^2} \eta^q p^q_t\right) Z^q_t\right].$$

(65)
where
\[ Z_t^n = \exp \left( \int_0^t \left( \frac{4n}{\sigma^2} \rho_s^n + \frac{3n^2}{\sigma^2} \dot{\rho}_s^n \right) \, ds - \frac{1}{2} \int_0^t \left( \frac{4n}{\sigma^2} \rho_s^n + \frac{3n^2}{\sigma^2} \dot{\rho}_s^n \right)^2 \, ds \right). \]

By integration by parts, one has \( \mathbb{P}_{(q,p)} \)-almost surely,
\[ \int_0^t \left( \frac{4n}{\sigma^2} \rho_s^n + \frac{3n^2}{\sigma^2} \dot{\rho}_s^n \right) \, ds = \frac{2n}{\sigma^2} (\rho_t^n)^2 - \frac{2n}{\sigma^2} \rho_0^n - \frac{3n^2}{\sigma^2} \int_0^t (\dot{\rho}_s^n)^2 \, ds. \]

In addition,
\[ \frac{1}{2} \int_0^t \left( \frac{4n}{\sigma^2} \rho_s^n + \frac{3n^2}{\sigma^2} \dot{\rho}_s^n \right)^2 \, ds = \frac{8n^2}{\sigma^2} \int_0^t (\dot{\rho}_s^n)^2 \, ds + \frac{12n^3}{\sigma^2} \int_0^t \rho_s^n (\dot{\rho}_s^n)^2 \, ds + \frac{9n^4}{2\sigma^2} \int_0^t (\dot{\rho}_s^n)^4 \, ds. \]

As a result, for all \( t > 0 \),
\[ u_t(q,p) \propto \exp \left( -\frac{2n}{\sigma^2} p^2 - \frac{3n^2}{\sigma^2} qp \right) \mathcal{E}_{(q,p)} \left[ 1_{\tau_0^n > t} f(\rho_t^n, \dot{\rho}_t^n) \exp \left( -\frac{11n^2}{2\sigma^2} \int_0^t (\dot{\rho}_s^n)^2 \, ds \right) \right]. \]

In order to conclude the proof, we need to show that \( u_t(q,p) \propto \mathbb{P}_{(q,p)} (\tau_0^n > t) \) and use Lemma 3.13 to obtain the result with \( \lambda = \sqrt{\Pi \eta} \). As a first step, notice that there exists \( C > 0 \) such that for all \( (q,p) \in D \),
\[ u_t(q,p) \leq C \mathcal{E}_{(q,p)} \left[ 1_{\tau_0^n > t} f(\rho_t^n, \dot{\rho}_t^n) \right]. \]

In addition, by (53) and Remark 2.6, the function \( f \) is integrable against \( H(q, -p/\alpha^{2/3}) \). Therefore, by the two-sided estimates from Proposition 2.9 there exists a constant \( C_t > 0 \) such that for all \( (q,p) \in D \),
\[ u_t(q,p) \leq C_t H(q,p/\alpha^{2/3}). \]

Let us start by proving the upper-bound in the statement \( u_t(q,p) \propto \mathbb{P}_{(q,p)} (\tau_0^n > t) \). Calling \( p_t^{\eta,D} \) the transition density of \( (q_t^n, p_t^n)_{t \geq 0} \) killed outside of \( D \), we have using Chapman-Kolmogorov relation that for all \( (q,p), (q', p') \in D \),
\[ p_t^{\eta,D}(q,p, q', p') = \int_D \int_D p_t^{\eta,D}(y,z) p_t^{\eta,D}(z, (q', p')) \, dy \, dz. \]

Using the uniform Gaussian upper-bound \( \alpha_t \) of \( p_t^{\eta,D} \) one has that
\[ p_t^{\eta,D}(q,p, q', p') \leq \alpha_t \mathbb{P}_{(q,p)} (\tau_0^n > t/3) \int_D p_t^{\eta,D}(z, (q', p')) \, dz. \]

Therefore, multiplying the above inequality by \( f(q', p') \exp \left( -\frac{2n}{\sigma^2} p^2 - \frac{3n^2}{\sigma^2} q' p' \right) \) and integrating over \( (q', p') \in D \),
\[ u_t(q,p) \leq \alpha_t \mathbb{P}_{(q,p)} (\tau_0^n > t/3) \int_D u_t(z) \, dz, \]
where the integral in the right-hand side of the inequality above is finite given (64). This gives us the wanted upper-bound.

Let us now prove the lower-bound. In order to do that let us apply Lemma 3.11 to the process \( (q_t^n, p_t^n)_{t \geq 0} \) in order to obtain the existence of a compact set \( K_t \subset D \) such that for all \( t > 0 \),
\[ c_t := \inf_{(q,p) \in D} \frac{\mathbb{P}_{(q,p)} ((q_t^n, p_t^n) \in K_t, \tau_0^n > t)}{\mathbb{P}_{(q,p)} (\tau_0^n > t)} > 0. \]

Applying Chapman-Kolmogorov and using the positivity and continuity of the killed transition density [20, Theorem 2.20] with \( c'_t := \inf_{y,z \in K_t} p_{t/3}^{\eta,D} (y, z) \), one has
\[ p_t^{\eta,D}(q,p, q', p') \geq \int_{K_t^{1/3} \times K_t^{1/3}} p_t^{\eta,D}(y,z) p_t^{\eta,D}(z, (q', p')) \, dy \, dz \geq c'_t \mathbb{P}_{(q,p)} ((q_t^n, p_t^n) \in K_t^{1/3}, \tau_0^n > t/3) \int_{K_t^{1/3}} p_t^{\eta,D}(z, (q', p')) \, dz \geq c_t c'_t \mathbb{P}_{(q,p)} (\tau_0^n > t/3) \int_{K_t^{1/3}} p_t^{\eta,D}(z, (q', p')) \, dz. \]
Again, multiplying the above inequality by \(f(q', p')\exp\left(-\frac{2\eta}{\sigma^2}(p')^2 - \frac{3\eta^2}{4\sigma^2}q'p'\right)\) and integrating over \((q', p') \in D\),

\[ u_t(q, p) \geq c_t c'_t \mathbb{P}(q, p) (r_{\alpha}^2 > t/3) \int_{K_{t/3}} u_{t/3}(z) dz, \]

which concludes the proof that \(u_t(q, p) \propto \mathbb{P}(q, p) (r_{\alpha}^2 > t)\).

We are now ready to prove Proposition 3.1.

**Proof of Proposition 3.1.** Let \(t > 0\) and \((q, p) \in D\). By Girsanov’s lemma,

\[ \mathbb{P}(q, p)(\tau_0 > t) = \mathbb{E}_{(q, p)} \left[ 1_{\tau_0^q > t} Z_t^q \right], \]

where

\[ Z_t^q = \exp \left( \int_0^t \left( \frac{\alpha \tilde{q}_s^2}{\sigma^2} + \beta \tilde{p}_s^2 + \frac{\gamma \tilde{p}_s^2}{\sigma^2} \right) ds - \frac{1}{2} \int_0^t \left( \frac{\alpha \tilde{q}_s^2}{\sigma^2} + \beta \tilde{p}_s^2 + \frac{\gamma \tilde{p}_s^2}{\sigma^2} \right)^2 ds \right). \]

In addition, by integration by parts,

\[
\mathbb{P}(q, p)(\tau_0 > t) = \mathbb{E}_{(q, p)} \left[ 1_{\tau_0^q > t} Z_t^q \right] \frac{\mathbb{P}(q, p) M_t^q}{\mathbb{P}(q, p) Z_t^q} = \mathbb{E}_{(q, p)} \left[ 1_{\tau_0^q > t} \left( \frac{\mathbb{P}(q, p) M_t^q}{\mathbb{P}(q, p) Z_t^q} \right) \right].
\]

Moreover,

\[
\mathbb{P}(q, p)(\tau_0 > t) = \mathbb{E}_{(q, p)} \left[ 1_{\tau_0^q > t} \left( \frac{\mathbb{P}(q, p) M_t^q}{\mathbb{P}(q, p) Z_t^q} \right) \right] = \mathbb{E}_{(q, p)} \left[ 1_{\tau_0^q > t} \left( \frac{\mathbb{P}(q, p) M_t^q}{\mathbb{P}(q, p) Z_t^q} \right) \right]
\]

As a result, since \((\tilde{q}_s^q)_{s \in [0, t]}\) is bounded on the event \(\{\tau_0^q > t\}\), for all \(t > 0\),

\[ \mathbb{P}(q, p)(\tau_0 > t) \propto \mathbb{E}_{(q, p)} \left[ 1_{\tau_0^q > t} f(\tilde{q}_t^q, \tilde{p}_t^q) \exp \left( -\frac{\alpha + \gamma^2/2}{\sigma^2} \int_0^t (\tilde{p}_s^q)^2 ds \right) \right], \]

where \(f(q, p) = \exp \left( \frac{\alpha}{\sigma^2} q p - \frac{\beta}{\sigma^2} p + \frac{\gamma}{2\sigma^2} p^2 \right)\) which satisfies (53) by the expression of \(H(19)\) and Remark 2.2. Therefore, Lemma 3.14 concludes the proof.

4 Two-sided estimates and long-time asymptotics

The goal of this section is to prove the two-sided estimates given the first exit time probability estimates provided in the previous sections. Namely in Section 4.1, we shall prove the more general result in multi-dimension given in Proposition 2.9 ensuring two-sided estimates whenever estimates on the first exit-time probability are given. Additionally we shall investigate the long-time asymptotics of the two-sided estimates appearing in Theorem 2.1, in Section 4.2. We shall also provide the proof of the long-time asymptotics of the semigroup conditioned on not being killed stated in Theorem 2.10.

4.1 Proof of Proposition 2.9

Let \(d \geq 1\). Let \(O\) be a \(C^2\) bounded connected open set of \(\mathbb{R}^d\) and let \(D = O \times \mathbb{R}^d\). Let \((q_t, p_t)_{t \geq 0}\) be the process in \(\mathbb{R}^d\) satisfying

\[
\begin{align*}
\frac{dq_t}{dt} &= p_t dt, \\
\frac{dp_t}{dt} &= F(q_t) dt - \gamma p_t dt + \sigma dB_t,
\end{align*}
\]

where \(F \in C^\infty(\mathbb{R}^d), \gamma \in \mathbb{R}, \sigma > 0\). For \((x, y) \in D\), we denote by \(p_t^D(x, y)\) the transition density of \((q_t, p_t)_{t \geq 0}\) killed outside of \(D\) and \(\tau_0\) its first exit time from \(D\). Let us also define the process \((\tilde{q}_t, \tilde{p}_t)_{t \geq 0}\) in \(\mathbb{R}^d\) satisfying

\[
\begin{align*}
\frac{d\tilde{q}_t}{dt} &= -\tilde{p}_t dt, \\
\frac{d\tilde{p}_t}{dt} &= -F(\tilde{q}_t) dt + \gamma \tilde{p}_t dt + \sigma dB_t.
\end{align*}
\]
We also denote by \( \tilde{p}^D_t(x,y) \) the transition density of \((\tilde{q}_t, \tilde{p}_t)_{t \geq 0} \) killed outside of \( D \) and \( \tau_0 \) its first exit time from \( D \). It was shown in [20, Theorem 6.2] that for all \( t > 0, x,y \in D \),

\[
p^D_t(x, y) = e^{d_0 t} \tilde{p}^D_t(y, x).
\]

(67)

Additionally, it was shown in [21, Theorem 2.13] that both killed semigroups admit smooth positive and bounded eigenvectors \( \phi, \psi \) for their respective infinitesimal generators. In particular, there exists \( \lambda_0 > 0 \) such that for all \( t \geq 0, (q,p) \in D \),

\[
E_{(q,p)}[1_{\tau_0 > t}\phi(q, p)] = e^{-\lambda_0 t} \phi(q, p), \quad E_{(q,p)}[1_{\tau_0 > t}\psi(\tilde{q}_t, \tilde{p}_t)] = e^{-(\lambda_0 + d_\gamma) t} \psi(q, p).
\]

(68)

We are now ready to prove Proposition 2.9.

Proof of Proposition 2.9. By Lemma 3.11 and Proposition 3.1, one has for all \( t > 0 \), the existence of a compact set \( K_t \subset D \) such that

\[
\beta_t := \inf_{(q,p) \in D} \frac{P_{(q,p)}((q_t, p_t) \in K_t, \tau_0 > t)}{P_{(q,p)}(\tau_0 > t)} > 0.
\]

In particular, we can choose the compact set \( K_t \) above large enough such that it satisfies the following property:

\[
(q, p) \in K_t \iff (q, -p) \in K_t.
\]

(69)

In addition, notice that the process \((\tilde{q}_t, -\tilde{p}_t)_{t \geq 0} \) satisfies the same equation as (65) where \(-\gamma \) is replaced by \( \gamma \). Since the assumptions in Lemma 3.11 and Proposition 3.1 do not depend on the sign of \( \gamma \), we obtain the same control for the process \((\tilde{q}_t, -\tilde{p}_t)_{t \geq 0} \)

\[
\tilde{\beta}_t := \inf_{(q,p) \in D} \frac{P_{(q,p)}((\tilde{q}_t, \tilde{p}_t) \in K_t, \tilde{\tau}_0 > t)}{P_{(q,p)}(\tilde{\tau}_0 > t)} > 0.
\]

We used here the fact that \( P_{(q,p)}((\tilde{q}_t, -\tilde{p}_t) \in K_t, \tilde{\tau}_0 > t) = P_{(q,p)}((\tilde{q}_t, \tilde{p}_t) \in K_t, \tilde{\tau}_0 > t) \) by the symmetry of \( K_t \). We denote now by \( \alpha_t \) the uniform bound of \( \tilde{p}^D_t \) on \( D \times D \) which follows from [20, Theorem 2.19].

Let \((q,p), (q',p') \in D \). We start by proving the upper-bound. Using the Chapman-Kolmogorov relation, one has

\[
p^D_t(q, p, q', p') = \int_{D \times D} p^D_{t/3}(q, p, y)p^D_{t/3}(y, z)p^D_{t/3}(z, (q', p'))dydz
\]

\[
\leq \alpha_{t/3} P_{(q,p)}(\tau_0 > t/3) \int_D p^D_{t/3}(z, (q', p'))dz
\]

\[
= \alpha_{t/3} e^{d_{t/3}} P_{(q,p)}(\tau_0 > t/3) \int_D \tilde{p}^D_{t/3}((q', p'), z)dz.
\]

(70)

by (67). In addition,

\[
\int_D \tilde{p}^D_{t/3}((q', p'), z)dz = P_{(q', p')}(\tilde{\tau}_0 > t/3).
\]

(71)

Consider now the probability \( P_{(q,p)}(\tau_0 > t/3) \). By definition of \( \beta_{t/3} \),

\[
P_{(q,p)}(\tau_0 > t/3) \leq \frac{1}{\beta_{t/3}} P_{(q,p)}((q_{t/3}, p_{t/3}) \in K_{t/3}, \tau_0 > t/3).
\]

Let \( m_{t/3} = \inf_{(q,p) \in K_{t/3}} \phi(q, p) > 0 \) since \( \phi \) is smooth and positive on \( D \). It follows that

\[
P_{(q,p)}((q_{t/3}, p_{t/3}) \in K_{t/3}, \tau_0 > t/3) \leq \frac{1}{m_{t/3}} E_{(q, p)}[1_{\tau_0 > t/3} \phi(q_{t/3}, p_{t/3})] = \frac{e^{-\lambda_0 t/3}}{m_{t/3}} \phi(q, p).
\]

by (68). As a result,

\[
P_{(q,p)}(\tau_0 > t/3) \leq \frac{e^{-\lambda_0 t/3}}{\beta_{t/3} m_{t/3}} \phi(q, p).
\]

(72)
Similarly,
\[
P_{(q',p')}((\tau_0 > t/3) \leq \frac{e^{-(\lambda_0 + d\gamma) t/3}}{C_{t/3}} \psi(q',p'),
\]
where \(\tilde{m}_t\) is the \(\inf_{(q,p) \in K_{t/3}} \psi(q,p)\). Reinjecting into (70) we obtain the existence of a constant \(\gamma_t > 0\) such that
\[
p_{t}^D (q, p, q', p') \leq \gamma_t \phi(q, p) \psi(q', p').
\]

Let us now prove the lower-bound for \(p_{t}^D\). Let
\[
M_{t/3} = \inf_{(q,p),(q',p') \in K_{t/3}} p_{t/3}^D(q, p, q', p') > 0,
\]
by continuity and positivity of \(p_{t/3}^D\), see [20, Theorem 2.20]. Again, by the Chapman-Kolmogorov relation, one has for all \(t > 0\), and for all \((q, p), (q', p') \in D\),
\[
p_{t}^D (q, p, q', p') \geq \int_{K_{t/3} \times K_{t/3}} p_{t/3}^D(q, p, y) p_{t/3}^D(y, z) p_{t/3}^D(z, (q', p')) d\gamma d\gamma d\gamma
\]
\[
\geq M_{t/3} e^{d^2 t/3} P((q_{t/3}, p_{t/3}) \in K_{t/3}, \tau_0 > t/3) P((\tilde{q}_{t/3}, \tilde{p}_{t/3}) \in K_{t/3}, \tilde{\tau}_0 > t/3)
\]
using (67). Furthermore, for all \((q, p) \in D\),
\[
P((q_{t/3}, p_{t/3}) \in K_{t/3}, \tau_0 > t/3) \geq \beta_{t/3} E(q, p) \left[ 1_{\tau_0 > t/3} \phi(q_{t/3}, p_{t/3}) \right]
\]
\[
= \beta_{t/3} e^{-\lambda_0 t/3} \phi(q, p).
\]
Similarly, for all \((q', p') \in D\),
\[
P((\tilde{q}_{t/3}, \tilde{p}_{t/3}) \in K_{t/3}, \tilde{\tau}_0 > t/3) \geq \beta_{t/3} e^{-\lambda_0 t/3} \psi(q', p').
\]
Reinjecting into (73) we obtain the existence of a constant \(\gamma'_t > 0\) such that for all \((q, p), (q', p') \in D\),
\[
p_{t}^D (q, p, q', p') \geq \gamma'_t \phi(q, p) \psi(q', p').
\]
Hence, for all \(t > 0\), \((q, p), (q', p') \in D\),
\[
\gamma'_t \phi(q, p) \psi(q', p') \leq p_{t}^D (q, p, q', p') \leq \gamma_t \phi(q, p) \psi(q', p'),
\]
which concludes the proof of Proposition 2.9.

The proof of the two-sided estimates in Theorem 2.1 follows now immediately from Proposition 2.9 and Proposition 3.1. We conclude this section with the proof of Corollary 2.3 which provides now explicit control in the two-sided estimates.

**Proof of Corollary 2.3.** Integrating over \(y \in D\) in Theorem 2.1 yields that for all \(t > 0\),
\[
P((q, p) \tau_0 > t) \propto \phi(q, p).
\]
The estimate on \(\phi\) then follows from Proposition 3.1. Regarding the estimate on \(\psi\), we use the equality (67) in the two-sided estimates and integrate this time over \(x \in D\), we obtain then that for all \(t > 0\),
\[
P((q, p) \tau_0 > t) \propto \psi(q, p).
\]
In addition, since the process \((\tilde{q}_t, -\tilde{p}_t)_{t \geq 0}\) satisfies the same equation as (65) with \(\gamma\) instead of \(-\gamma\). Therefore,
\[
P((q, p) \tau_0 > t) \propto H_{\alpha, \beta, -\gamma, \sigma}(q, -p),
\]
which ensures the desired estimate for \(\psi\). 
\[\square\]
4.2 Long-time asymptotics

The objective of this final section is to prove the long-time asymptotics (16) in Theorem 2.1. Additionally, we shall prove the uniform conditional ergodicity on the killed semigroup of (8) stated in Theorem 2.10. The main ingredient of the proofs are the long-time asymptotics of the killed semigroup obtained in [21, Theorem 2.19] combined with the two-sided estimates from Theorem 2.1.

Let us start this subsection with the proof of the following proposition where we extend the long-time asymptotics of the killed semigroup obtained in [21, Theorem 2.12] to the set of functions \( f \in L_{H_{\alpha,\beta,\gamma,\sigma}}(D) \).

**Proposition 4.1 (Long time asymptotics).** There exists a spectral gap \( \alpha > 0 \) such that for all \( t_0 > 0 \), there exists \( C_{t_0} > 0 \) such that for all \( t \geq t_0 \), for all \( f \in L_{H_{\alpha,\beta,\gamma,\sigma}}(D) \),

\[
\left| E_{(q,p)} \left[ f(q_t, p_t) \mathbb{1}_{\tau > t} \right] - e^{-\lambda_0(t-t_0/2)} \int_D \frac{\partial}{\partial \psi} \phi(q,p) \right| \leq C_{t_0} \left\| f \right\|_{H_{\alpha,\beta,\gamma,\sigma}} e^{-(\lambda_0+\alpha)t}
\]

**Proof of Proposition 4.1.** We first consider the case \( f \in L^\infty(D) \). Let us fix \( t_0 > 0 \) and let \( t > t_0 \). By the Markov property at time \( t_{0}/2 \), one has

\[
E_{(q,p)} \left[ f(q_{t-t_0/2}, p_{t-t_0/2}) \mathbb{1}_{\tau > t-t_0/2} \right] = E_{(q,p)} \left[ \mathbb{1}_{\tau > t_0/2} \int_0^{t-t_0/2} \int_D \frac{\partial}{\partial \psi} \phi(q,p) \right]
\]

since \( E_{(q,p)} \left[ \mathbb{1}_{\tau > t_0/2} \phi(q_{t_0/2}, p_{t_0/2}) \right] = e^{-\lambda_0 t_0/2} \phi(q,p) \). Besides, it follows from the long-time asymptotics in [21, Theorem 2.19] that there exists \( \alpha > 0 \) and \( C > 0 \) such that almost-surely, for all \( t > t_0 \),

\[
\left| E_{(q_{t_0/2}, p_{t_0/2})} \left[ f(q_{t-t_0/2}, p_{t-t_0/2}) \mathbb{1}_{\tau > t-t_0/2} \right] - e^{-\lambda_0(t-t_0/2)} \int_D \frac{\partial}{\partial \psi} \phi(q_{t_0/2}, p_{t_0/2}) \right| \leq C \left\| f \right\|_{\infty} e^{-(\lambda_0+\alpha)(t-t_0/2)}.
\]

As a result, reinserting into (74) we deduce that

\[
\left| E_{(q,p)} \left[ f(q_{t-t_0/2}, p_{t-t_0/2}) \mathbb{1}_{\tau > t-t_0/2} \right] - e^{-\lambda_0(t-t_0/2)} \int_D \frac{\partial}{\partial \psi} \phi(q,p) \right| \leq C P_{(q,p)}(\tau_{t_0/2} > t_0/2) \left\| f \right\|_{\infty} e^{-(\lambda_0+\alpha)(t-t_0)} = C t_{0/2} \left\| f \right\|_{\infty} e^{-(\lambda_0+\alpha)(t-t_0)}
\]

where the constant \( C_{t_{0/2}} > 0 \) follows from Theorem 2.1.

Assume now that \( f \in L_{H_{\alpha,\beta,\gamma,\sigma}}(D) \), then since \( H_{\alpha,\beta,\gamma,\sigma} \) is bounded it follows from the upper-bound in Theorem 2.1 that for all \( s > 0 \), \( P^D_s f \in L^\infty(D) \) where the semigroup \( (P^D_t)_{t \geq 0} \) is defined in (22). Applying the inequality above for \( P^D_{t_{0/2}} f \in L^\infty(D) \) instead of \( f \) and using the Markov property in the left-hand side of the inequality, we obtain the existence of a constant \( C_{t_{0/2}} > 0 \) such that for all \( t > t_0 \),

\[
\left| E_{(q,p)} \left[ f(q_{t-t_0/2}, p_{t-t_0/2}) \mathbb{1}_{\tau > t-t_0/2} \right] - e^{-\lambda_0(t-t_0/2)} \int_D \frac{\partial}{\partial \psi} \phi(q_{t_0/2}, p_{t_0/2}) \right| \leq C_{t_{0/2}} \int_D f \left\| P^D_{t_{0/2}} \right\|_{\infty} e^{-(\lambda_0+\alpha)_t}.
\]

Moreover, by (67) and using the Fubini permutation,

\[
\int_D \psi(q,p) P^D_{t_{0/2}} f(q,p) dq dp = e^{\gamma_{t_{0/2}}} \int_D \left( \int_D \psi(q,p) P^D_{t_{0/2}}(q',p',q,p) dq dp \right) f(q',p') dq' dp' = e^{\gamma_{t_{0/2}}} \int_D \left( \int_D \psi(q',p') P^D_{t_{0/2}}(q_{t_{0/2}}, p_{t_{0/2}}) \mathbb{1}_{\tau_{t_{0/2}} > t_0/2} \right) f(q',p') dq' dp' = e^{-\lambda_0 t_{0/2}} \int_D \psi(q',p') f(q',p') dq' dp',
\]

by (68). In addition, using the upper-bound in Theorem 2.1 along with Corollary 2.3, there exists a constant \( \alpha_{t_{0/2}} > 0 \) for all \( (q,p) \in D \),

\[
\left| P^D_{t_{0/2}} f(q,p) \right| \leq \alpha_{t_{0/2}} H_{\alpha,\beta,\gamma,\sigma}(q,p) \int_D \left| f(q',p') \right| H_{\alpha,\beta,\gamma,\sigma}(q', -p') dq' dp' \leq \alpha_{t_{0/2}} \left\| H_{\alpha,\beta,\gamma,\sigma} \right\|_{\infty} \left\| f \right\|_{H_{\alpha,\beta,\gamma,\sigma}}.
\]
Reinjecting (76) and (77) into (75) we obtain that for all \( f \in L_{H_{\alpha,\beta,\gamma,\sigma}}(D) \) and \( t > t_0 \),
\[
\left| E_{(q,p)}[f(q_t,p_t)1_{\tau_0 > t}] - e^{-\lambda_0 t} \frac{\int_D \psi f}{\int_D e^{\psi}} \phi(q,p) \right| \leq C_{t_0/2\alpha t_0/2} \| H_{\alpha,\beta,\gamma,\sigma} \|_L^\infty(D) \| f \| H_{\alpha,\beta,\gamma,\sigma} \phi(q,p) e^{-(\lambda_0 + \alpha)t}.
\]
which concludes the proof. \( \square \)

Let us now prove Theorem 2.10. The idea of the proof is similar to the proof of [21, Theorem 2.22] but uses the sharper estimate provided by Proposition 4.1.

**Proof of Theorem 2.10.** Let \( \theta \) be a probability measure on \( D \). First notice that for \( t > 0 \) and \( (q,p) \in D \),
\[
P_{(q,p)}(\tau_0 > t) \geq E_{(q,p)} \left[ \frac{\| \phi(q_t,p_t) \|}{\| \phi \|} \right] = e^{-\lambda_0 t} \phi(q,p).
\]
Consequently,
\[
P_\theta(\tau_0 > t) = \int_D P_{(q,p)}(\tau_0 > t) \theta(dqdp) \geq \frac{\int_D \phi \theta(dqdp)}{\| \phi \|} e^{-\lambda_0 t}.
\]
(78)

Let us now take \( t_0 > 0 \) and \( t > t_0 \). By Proposition 4.1 there exists \( C_{t_0} > 0 \) and \( \alpha > 0 \) such that for all \( f \in L_{H_{\alpha,\beta,\gamma,\sigma}}(D) \),
\[
\left| \frac{E_{(q,p)}[f(q_t,p_t)1_{\tau_0 > t}]}{P_\theta(\tau_0 > t)} - \int_D f \psi \right| \leq \int_D \left( \frac{E_{(q,p)}[f(q_t,p_t)1_{\tau_0 > t}] - \int_D f \psi}{P_\theta(\tau_0 > t)} \right) \theta(dqdp)
\]
\[
+ \int_D \psi f \int_D \left( \frac{e^{-\lambda_0 t} \frac{\phi(q,p)}{\int_D e^{\psi}} - P_{(q,p)}(\tau_0 > t)}{P_\theta(\tau_0 > t)} \right) \theta(dqdp)
\]
\[
\leq C_{t_0} e^{-(\lambda_0 + \alpha)t} \left( \left( \int_D \phi \theta \right) \| f \|_{H_{\alpha,\beta,\gamma,\sigma}} + \left( \int_D \psi f \right) \left( \int_D \psi \phi \theta \right) \| 1 \|_{H_{\alpha,\beta,\gamma,\sigma}} \right)
\]
\[
\leq C_{t_0} \| \phi \|_{\infty} e^{-\alpha t} \left[ \| f \|_{H_{\alpha,\beta,\gamma,\sigma}} + \left( \int_D \psi f \right) \| 1 \|_{H_{\alpha,\beta,\gamma,\sigma}} \right]
\]
(79)

using (78). Moreover, by Corollary 2.3 there exists a constant \( c > 0 \) independent of \( f \in L_{H_{\alpha,\beta,\gamma,\sigma}}(D) \) such that
\[
\int_D \psi f \leq c \| f \|_{H_{\alpha,\beta,\gamma,\sigma}}.
\]
Reinjecting this inequality above in (79) concludes the proof. \( \square \)

Let us conclude this work with the proof of the long-time asymptotics (16) in Theorem 2.1.

**Proof of (16) in Theorem 2.1.** This proof follows from applying Proposition 4.1 to the killed transition density \( (q,p) \in D \mapsto p^D_{\tau_0}(q,p,q',p') \) for \( s > 0 \) and \( (q',p') \in D \). The fact that such function is in \( L_{H_{\alpha,\beta,\gamma,\sigma}}(D) \) for all \( (q',p') \in D \) follows from the two-sided estimates (15) and the fact that \( H_{\alpha,\beta,\gamma,\sigma} \) is bounded and integrable, see Remark 2.5. As a result, the Chapman-Kolmogorov relation ensures that for all \( (q,p), (q',p') \in D, t, s > 0 \),
\[
\left| p^{D+\tau_s}_{\tau_0}(q,p,q',p') - e^{-\lambda_0 t} \frac{\int_D \psi P^{D+\tau_s}_{\tau_0}(q',p') \phi(q)}{\int_D e^{\psi}} \phi(q,p) \right| \leq C_s \phi(q,p) \| p^D_{\tau_s}(\cdot,\cdot,q',p') \|_{H_{\alpha,\beta,\gamma,\sigma}} e^{-(\lambda_0 + \alpha)t}.
\]
(80)

Furthermore, by (67) and (68),
\[
\int_D \psi(q,p)P^D_s(q,p,q',p') dqdp = e^{\gamma_s} \int_D \psi(q,p) \tilde{p}^D_s(q',p',q,p) dqdp
\]
\[
= e^{\gamma_s} E_{(q',p')} [\psi(q_t,\tilde{p}_s) 1_{\tau_0 > s}]
\]
\[
= \psi(q',p') e^{-\lambda_0 s}.
\]
In addition, one has from Corollary 2.3 the existence of a constant $c > 0$ such that for all $(q', p') \in D$,
\[
\|p_{D}^{D}(\cdot, \cdot, q', p')\|_{H_{\alpha, \beta, \gamma, \sigma}} = \int_{D} p_{s}^{D}(q, p, q', p') H_{\alpha, \beta, \gamma, \sigma}(q, -p) dq dp \nonumber \\
= c e^{\gamma s} \int_{D} p_{s}^{D}(q', p', q, p) \psi(q, p) dq dp 
\]

As a result, reinjecting into (80) one has
\[
\left| p_{t+s}^{D}(q, p, q', p') - e^{-\lambda_{0}(t+s)} \frac{\phi(q, p) \psi(q', p')}{\int_{D} \phi} \right| \leq C_{s} c e^{\alpha s} \phi(q, p) \psi(q', p') e^{-(\lambda_{0} + \alpha)(t+s)}. 
\]

Therefore, taking $t+s \to \infty$ for fixed $s > 0$ ensures (16), which concludes the proof of Theorem 2.1.

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