THE MOVE FROM FUJITA TO KATO TYPE EXPONENT FOR A CLASS OF SEMILINEAR EVOLUTION EQUATIONS WITH TIME-DEPENDENT DAMPING

MARCELO REMPEL EBERT, JORGE MARQUES, WANDERLEY NUNES DO NASCIMENTO

Abstract. In this paper, we derive suitable optimal $L^p - L^q$ decay estimates, $1 \leq p \leq q \leq \infty$, for the solutions to the $\sigma$-evolution equation, $\sigma > 1$, with scale-invariant time-dependent damping and power nonlinearity $|u|^p$,

$$u_{tt} + (-\Delta)^\sigma u + \frac{\mu}{t} u_t = |u|^p, \quad t \geq 0, \quad x \in \mathbb{R}^n,$$

where $\mu > 0$, $p > 1$. The critical exponent $p = p_c$ for the global (in time) existence of small data solutions to the Cauchy problem is related to the long time behavior of solutions, which changes accordingly when $\mu > 1$. Under the assumption of small initial data in $L^1 \cap L^2$, we find the critical exponent

$$p_c = 1 + \max \left\{ \frac{2\sigma}{|n - \sigma + \sigma\mu|}, \frac{2\sigma}{\sigma} \right\} = \begin{cases} \frac{1 + 2\sigma}{n - \sigma + \sigma\mu}, & \mu \in (0, 1) \\ \frac{1 + 2\sigma}{\sigma}, & \mu > 1. \end{cases}$$

For $\mu > 1$ it is well known as Fujita type exponent, whereas for $\mu \in (0, 1)$ one can read it as a shift of Kato exponent.

1. Introduction

In this paper we study the global (in time) existence of small data solutions to the Cauchy problem for the semilinear damped $\sigma$-evolution equations with scale-invariant time-dependent damping

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + \frac{\mu}{t} u_t = f(u), & t \geq 0, \quad x \in \mathbb{R}^n, \\ u(0, x) = 0, & x \in \mathbb{R}^n, \\ u_t(0, x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.1)$$

where $\mu > 0$, $\sigma > 1$ and $f(u) = |u|^p$ for some $p > 1$. The nonlinearity may have several shapes, for instance, the derived results in this paper also hold if $f(u) = |u|^{p-1}u$ or if $f$ is locally Lipschitz-continuous satisfying

$$f(0) = 0, \quad |f(u) - f(v)| \leq C|u - v|(|f(u)|^{p-1} + |f(v)|^{p-1}),$$

for some $p > 1$. The important information is that the nonlinearity is a perturbation which may create blow-up in finite time, well known in the literature as source nonlinearity. If the initial condition $u(0, x)$ is small, then $f(u)$ becomes small for large $p$. For this reason one is often able to prove such a global (in time) existence result only for some $p > p_c$.

Let us introduce some previous results to the Cauchy problem for the semilinear free $\sigma$-evolution equations

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u = |u|^p, & \quad u(0, x) = 0, \quad u_t(0, x) = u_1(x). \end{cases} \quad (1.2)$$

We begin with results for $\sigma = 1$. If $1 < p < p_K(n) = \frac{n+1}{n-1}$, Kato [19] proved the nonexistence of global generalized solutions to (1.2), for small initial data with compact support. On the other hand, John [18] showed that $p = 1 + \sqrt{2}$ is the critical exponent for the global existence of classical solutions with small initial data in space dimension $n = 3$. A bit later, Strauss [32] conjectured that the critical exponent $p_S(n)$, $n \geq 2$, is the positive root of

$$(n - 1)p^2 - (n + 1)p - 2 = 0.$$
existence result up to $n \leq 8$ and for all $n$ in the case of radial initial data (see also [20] for the case of odd space dimension). In [14], the authors removed the assumption of spherical symmetry. Then, for $\sigma > 1$ and for space dimensions $1 \leq n \leq 2\sigma$, in [9] it was obtained the critical exponent to [12], $p_k(n) = \frac{n + 2}{n - 2}$, which is of Kato type.

In [36], the authors proved global existence of small data solutions for the semilinear damped wave equation

$$u_{tt} - \Delta u + u_t = |u|^p, \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x),$$

in the supercritical range $p > 1 + 2/n$, by assuming initial data with compact support from the energy space. A previous existence result in space dimensions $n = 1$ and $n = 2$ was proved in [22]. The compact support assumption on the initial data can be weakened. By only assuming initial data in Sobolev spaces, the existence result was proved in space dimensions $n = 1$ and $n = 2$ in [17], by using energy methods, and in space dimensions $n \leq 5$ in [24], by using $L^r - L^q$ estimates, $1 \leq r \leq q \leq \infty$. Nonexistence of the global small data solution is proved in [36] for $1 < p < 1 + 2/n$ and in [41] for $p = 1 + 2/n$. The critical case for more general nonlinearities has been recently discussed in [8]. The exponent $p_F(n) := 1 + 2/n$ is well known as Fujita exponent and it is the critical index for the semilinear parabolic problem [11]:

$$v_t - \Delta v = v^p, \quad v(0, x) = v_0(x) \geq 0.$$

The diffusion phenomenon between linear heat and linear classical damped wave models (see [15, 23, 24] and [25]) explains the parabolic nature of classical damped wave models with power nonlinearities from the point of view of decay estimates of solutions.

In [49] the author considered a more general model

$$u_{tt} - \Delta u + b(t)u_t = 0, \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x),$$

with a class of time dependent damping $b(t)u_t$ for which the critical exponent is still Fujita exponent $1 + 2/n$ for the associate semilinear Cauchy problem with power nonlinearity $|u|^p$ (see [5] and [7]).

We state now well known results for the semilinear wave equation with scale-invariant time-dependent damping

$$\begin{cases}
    u_{tt} - \Delta u + \frac{\mu}{1 + \tau}u_t = |u|^p, \\
    u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x).
\end{cases} \quad (1.3)$$

This model is critical, in the sense that it is relevant the size of the parameter $\mu$ to describe the asymptotic behavior of solutions. If $\mu \geq \frac{q}{q - 1}$ for $n = 1$ or $\mu \geq 3$ for $n = 2$, by assuming initial data in the energy spaces with additional regularity $L^1(R^n)$, a global (in time) existence result for [24] was proved in [5] for $p > p_F(n) := 1 + \frac{2}{n}$. This result was extended by same author for higher space dimensions $n \geq 3$ by assuming initial data in spaces with weighted norms for $\mu \geq n + 2$. The exponent $p_F$ is critical for this model, that is, for $1 < p \leq p_F$ and suitable, arbitrarily small initial data, there exists no global weak solution [5]. In [6] the authors studied the special case $\mu = 2$ and showed that the critical exponent for [13] is given by $p_c = \max\{p_S(n + 2), p_F(n)\}$. In the same paper the authors also conjectured that $p_c \geq \max\{p_S(n + \mu), p_F(n)\}$ for \(\mu \in (2, n + 2)\). The threshold value $\mu_*$ is the solution to $p_S(n + \mu_*) = p_F(n)$ and it is given by

$$\mu_* = \frac{n^2 + n + 2}{n + 2}.$$

In [16], for suitable initial data, the authors obtained blow-up in finite time and gave the upper bound for the lifespan of solutions to [13] if $1 < p \leq p_S(n + \mu)$ with $\mu \in (0, \mu_*]$. It is worth noticing that if $\mu \in (0, \mu_*]$, then $p_F(n) < p_S(n + \mu)$.

As far as we know, it is still an open problem to prove global existence of small initial data solutions for $p > p_F(n)$ in the cases $\frac{2}{3} < \mu < \frac{3}{2}$ for $n = 1, 2 < \mu < 3$ for $n = 2$, or $\mu_* < \mu < n + 2$ for $n \geq 3$.

A related model to [13] is the semilinear wave equation with scale-invariant mass and dissipation

$$\begin{cases}
    u_{tt} - \Delta u + \frac{\mu}{1 + \tau}u_t + \frac{\mu^2}{(1 + \tau)^2}u = |u|^p, \\
    u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x).
\end{cases}$$

For results about existence and non-existence of global (in time) small initial data solutions, we address the reader to [23, 28, 29, 30] and the references therein.

The main goals in this paper are to derive $L^p - L^q$ estimates and energy estimates for solutions to the linear Cauchy problem associated to [11] and to obtain the critical exponent for the global (in time) existence of small initial data solutions to [11]. We conclude that $\mu = 1$ is the threshold for the asymptotic behavior of
solution to \(1.1\), it means that the critical exponent is a shift of Kato type exponent \(p_K(n + \sigma \mu) := \frac{n + \sigma + \sigma \mu}{n - \sigma + \sigma \mu}\) for \(0 < \mu < 1\) and of Fujita type \(p_F(n, \sigma) := 1 + \frac{2 \sigma}{n}\) for \(\mu > 1\).

The plan of the paper is the following:

- in Section 2 we collect and discuss our main results;
- in Section 3 we derive the \(L^p - L^q\) estimates for solutions to the associate linear Cauchy problem;
- in Section 4 we apply the decay estimates previously derived to prove Theorems 2.1 and 2.2 for the nonlinear problems \(1.1\);
- in Section 5 we apply the test function method to prove Proposition 2.1;
- in Appendix, we include some notations, well known estimates for multipliers and properties of special functions used to prove our results throughout the paper.

2. Main results

Our first result is for small \(\mu\) and \(\sigma > 1\), it shows that the critical exponent is a shift of the Kato exponent, unlike other case \(\sigma = 1\) where it appears a shift of Strauss exponent \([6, 16]\). In the next theorem we are going to use the following notation

\[
\mu_\sharp = \begin{cases} 
\infty, & \text{if } \mu \leq 2 - \frac{2n}{\sigma}, \\
\frac{2\sigma}{n - \sigma + \sigma \mu} (\sigma - n + \sqrt{9\sigma^2 - 10n\sigma + n^2}), & \text{if } \mu > 2 - \frac{2n}{\sigma}.
\end{cases}
\]  

(2.1)

**Theorem 2.1.** Let \(\sigma > 1\), \(1 \leq n < 1\), \(1 - \frac{2}{\sigma} < \mu < \min\{\mu_\sharp; 1\}\), with \(\mu_\sharp\) as in (2.1) and \(\mu \neq 2 - \frac{2n}{\sigma}\). If

\[
1 + \frac{2\sigma}{n - \sigma + \sigma \mu} := p_K(n + \sigma \mu) < p \leq 1 + \frac{2\sigma - \sigma \mu}{2n - 2\sigma + \sigma \mu} := q_1,
\]  

(2.2)

then there exists \(\epsilon > 0\) such that for any initial data

\[
u_1 \in \mathcal{A} = L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n), \quad \|\nu_1\|_{\mathcal{A}} \leq \epsilon,
\]

there exists a unique energy solution \(u \in C([0, \infty), H^p(\mathbb{R}^n)) \cap C^1([0, \infty), L^2(\mathbb{R}^n)) \cap L^\infty([0, \infty) \times \mathbb{R}^n)\) to \(1.1\). Moreover, for \(2 \leq q \leq q_1\) the solution satisfies the following estimates

\[
\|u(t, \cdot)\|_{L^p} \lesssim (1 + t)^{-\frac{1}{2} + \frac{1}{p} + \frac{1}{p} - \mu} \|\nu_1\|_{\mathcal{A}},
\]

\[
\|u(t, \cdot)\|_{L^\infty} \lesssim (1 + t)^{-\min\{\frac{1}{2} + \frac{1}{p} - \frac{1}{q}, \frac{1}{2} + \frac{1}{p} - \frac{1}{q_1}\}} \|\nu_1\|_{\mathcal{A}},
\]

and

\[
\|u(t, \cdot)\|_{H^s} + \|\partial_t u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{s}{2}} \|\nu_1\|_{\mathcal{A}}, \quad \forall t \geq 0.
\]

**Remark 2.1.** The condition \(\mu < \min\{\mu_\sharp; 1\}\) implies that the range for \(p\) in (2.2) is not empty, i.e., if \(\mu > 2 - \frac{2n}{\sigma}\), \(\mu_\sharp\) is the positive root of \(\sigma \mu^2 + (n - \sigma) \mu + 2(n - \sigma) = 0\). In particular, \(\mu_\sharp \geq 1\) if \(3n \leq 2\sigma\). Moreover, \(\mu = 2 - \frac{2n}{\sigma} \left(1 - \frac{1}{q_1}\right)\) and \(\mu < 2 - \frac{2n}{\sigma} \left(1 - \frac{1}{q}\right)\) for all \(q < q_1\). If \(\mu \leq 2 - \frac{2n}{\sigma}\), then \(q_1 = \infty\) in (2.2).

**Remark 2.2.** If \(\mu = 2 - \frac{2n}{\sigma}\), under the assumptions of Theorem 2.1 it is possible to obtain global (in time) unique energy solutions to \(1.1\), however a logarithm term appears on the estimates for the solutions.

**Remark 2.3.** In the limit value of \(\mu, \mu = 0\), we get \(p_K(n) = \frac{n + \sigma}{n - \sigma}\), so we recover the critical index obtained in \([9]\).

**Remark 2.4.** Using the properties of the Hankel functions and the representation \([6, 34]\) we conclude that

\[
g_{j, k}(t) = \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\partial_t^j \hat{u}(t, \xi)|^2 d\xi, \quad j + k \leq 1,
\]

are continuous functions on \([0, \infty)\). The validity of the Fourier inversion formula implies \(u \in C([0, \infty), H^p(\mathbb{R}^n)) \cap C^1([0, \infty), L^2(\mathbb{R}^n))\). But the same argument cannot be used to conclude the continuity of \(h(t) = \|u(t, \cdot)\|_{L^\infty}\).

**Example 2.1.** For the plate equation \(\sigma = 2\) in one space dimension \(n = 1\), the conclusions of Theorem 2.1 hold for all \(p > 1 + \frac{1}{2\sigma} = \frac{3}{2}\) and \(\frac{1}{2} < \mu < 1\).

The next result is an extension of Theorem 2 in \([3]\) done for the case \(\sigma = 1\).
Theorem 2.2. Let $\sigma > 1$, $n < 2\sigma$ and $\mu > \max\left\{\frac{n}{\sigma} + \frac{2\sigma}{n + 2\sigma}, 1\right\}$, $\mu \neq \frac{n}{\sigma}$ and $\mu \neq \frac{n}{\sigma} + 2$. If

$$1 + \frac{2\sigma}{n} < p \leq \frac{n}{2n - \sigma \mu} := q_0, \quad (2.3)$$

then there exists $\epsilon > 0$ such that for any initial data

$$u_1 \in A = L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n), \quad ||u_1||_A \leq \epsilon,$$

there exists a unique energy solution $u \in C([0, \infty), H^{\sigma}(\mathbb{R}^n)) \cap C^1([0, \infty), L^2(\mathbb{R}^n)) \cap L^\infty([0, \infty) \times \mathbb{R}^n)$ to $(1.2)$. Moreover, for $2 \leq q \leq q_0$ the solution satisfies the following estimates

$$||u(t, \cdot)||_{L^q} \lesssim (1 + t)^{-\frac{n}{2n - \sigma \mu}} ||u_1||_A, \quad \forall t \geq 0, \quad (2.4)$$

$$||u(t, \cdot)||_{L^n} \lesssim (1 + t)^{-\min\left\{\frac{n - \sigma}{n}, \frac{1}{2}\right\}} ||u_1||_A, \quad \forall t \geq 0, \quad (2.5)$$

$$||u(t, \cdot)||_{H^{\sigma}} + ||\partial_t u(t, \cdot)||_{L^2} \lesssim (1 + t)^{-\min\left\{\frac{n - \sigma}{n}, \frac{1}{2}\right\}] ||u_1||_A, \quad \forall t \geq 0. \quad (2.6)$$

Remark 2.5. The condition $\mu > \frac{n}{\sigma} + \frac{2\sigma}{n + 2\sigma}$ implies that the range for $p$ in $(2.3)$ is not empty. Moreover, $q \leq q_0$, with $q_0$ defined by $(2.3)$, is equivalent to $\mu \geq \frac{2n}{\sigma} \left(1 - \frac{1}{q}\right)$. If $\mu \geq \frac{2n}{\sigma}$ then $q_0 = \infty$ in $(2.3)$.

Remark 2.6. The cases $\mu = 1$, $\mu = \frac{2n}{\sigma}$ and $\mu = \frac{n}{\sigma} + 2$ can also be included in Theorem 2.2, but it appears an additional logarithm loss on the derived estimates for the solutions. Moreover, to obtain the result for higher space dimension $n \geq 2\sigma$, one also have to derive $L^p - L^q$ estimates, with $p \in [1, 2]$, for solutions to the linear problem at low frequencies and combine it with the already obtained estimates at high frequencies.

Example 2.2. For the plate equation $\sigma = 2$, Theorem 2.2 applies for $\mu > 1$ and $\mu \neq \frac{n}{2}$ if $n = 1$, for $\mu > \frac{n}{2}$ and $\mu \notin \{2; 3\}$ if $n = 2$ and for $\mu > \frac{2n}{\sigma}$ and $\mu \notin \{3; \frac{7}{2}\}$ if $n = 3$.

For the sake of simplicity, in the next two results we restrict our analysis for integer $\sigma$. However, the test function method was recently applied in $[3]$ for a class of $\sigma -$ evolution operators with non-integer $\sigma$.

First let us discuss into details the non-existence result for the non-effective damping cases $0 < \mu \leq 1$. The proof of the next result can be obtained with a slightly change in the proof of Theorem 1.5 in $[3]$. 

Proposition 2.1. Let $\sigma \in \mathbb{N}$, $0 < \mu \leq 1$ and

$$1 < p \leq p_K(n + \sigma \mu) := \frac{n + \sigma + \sigma \mu}{n - \sigma + \sigma \mu}. \quad (2.7)$$

If $u_1 \in L^1(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} u_1(x) \, dx > 0,$$

then there exists no global (in time) weak solution $u \in L^p_{loc}([0, \infty) \times \mathbb{R}^n)$ to $(1.1)$. 

Remark 2.7. The proof of Proposition 2.1 also holds for $\mu > 1$, however it is not optimal (see Proposition 2.2).

Since

$$1 + \frac{2\sigma}{n} = \frac{n + \sigma + \sigma \mu}{n - \sigma + \sigma \mu}$$

is equivalent to $\mu = 1$ and $p_K(n + \sigma \mu) < 1 + \frac{2\sigma}{n}$ for $\mu > 1$, so Proposition 2.1 is not the counterpart of Theorem 2.2 for $\mu > 1$. Applying Theorem 2.2 in $[3]$, one may have the following improvement of Proposition 2.1 and the counterpart of Theorem 2.2 is obtained.

Proposition 2.2. Let $\sigma \in \mathbb{N}$, $\mu > 1$ and

$$1 < p \leq 1 + \frac{2\sigma}{n},$$

If $u_1 \in L^1(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} u_1(x) \, dx > 0,$$

then there exists no global (in time) weak solution $u \in L^p_{loc}([0, \infty) \times \mathbb{R}^n)$ to $(1.1)$. 

Remark 2.8. From Theorem 2.2 and Proposition 2.2 we conclude that for $\mu > 1$ the Fujita type index

$$ p = 1 + \frac{2\sigma}{n} $$

is the critical exponent to (1.1), whereas for $0 < \mu \leq 1$, Theorem 2.1 and Proposition 2.1 implies that the Kato type index

$$ p_K(n + \sigma \mu) = \frac{n + \sigma + \sigma \mu}{|n - \sigma + \sigma \mu|_+} $$

is the critical exponent to (1.1).

The next remark was suggested by Prof. M. D’Abbicco and says that Proposition 2.1 could also be obtained by applying Theorem 2.2 in [5].

Remark 2.9. Let us consider

$$ u_{tt} + (-\Delta)^\sigma u + \frac{\mu}{1 + t} u_t = (1 + t)^{2\gamma} |u|^p, $$

with $\nu > 0$. Applying Theorem 2.2 in [5], one may derive a nonexistence result for

$$ 1 + \gamma < p \leq p_c = 1 + \frac{2(1 + \gamma)\sigma}{n}. $$

If $\mu \in [0, 1)$, applying the change of variable $v = (1 + t)^{-\mu} u$, so that

$$ v_{tt} + (-\Delta)^\sigma v + \frac{2 - \mu}{1 + t} v_t = (1 + t)^{(p - 1)(1 - \mu)} |v|^p. $$

Setting $\nu = 2 - \mu$ and $\gamma = (p - 1)(1 - \mu)/2$, Theorem 2.2 in [5] implies the nonexistence of solutions if

$$ 1 + (p - 1)\frac{1 - \mu}{2} < p \leq 1 + \frac{(2 + (p - 1)(1 - \mu))\sigma}{n}. $$

The left-hand side is clearly true, due to $1 - \mu < 2$, and the right-hand side gives the condition for the desired critical exponent:

$$ p \leq 1 + \frac{(2 + (p - 1)(1 - \mu))\sigma}{n}, \quad \text{i.e.,} \quad p \leq \frac{n + \sigma + \sigma \mu}{|n - \sigma + \sigma \mu|_+}. $$

3. $L^p - L^q$ estimates for solutions

Let us consider the Cauchy problem for the linear $\sigma$-evolution equation with scale-invariant time-dependent damping

$$ u_{tt} + (-\Delta)^\sigma u + \frac{\mu}{1 + t} u_t = 0, \quad u(s, x) = 0, \quad u_t(s, x) = u_1(x) \quad (3.1) $$

in $[0, \infty) \times \mathbb{R}^n$, with $s \leq t$, $\mu > 0$ and $\sigma > 1$.

Taking the partial Fourier transform with respect to the $x$ variable in (3.1) we obtain

$$ \hat{u}_{tt} + |\xi|^{2\sigma} \hat{u} + \frac{\mu}{1 + t} \hat{u}_t = 0, \quad \hat{u}(s, \xi) = 0, \quad \hat{u}_t(s, \xi) = \hat{u}_1(\xi). \quad (3.2) $$

According to [27] and [59], we have the following representation for the solution to (3.2) in terms of the Hankel functions $H_{\rho}^{\pm}$:

**Proposition 3.1.** Assume that $u$ solves the Cauchy problem (3.1) for data $u_1 \in S(\mathbb{R}^n)$. Then the Fourier transform $\hat{u}(t, s, \xi)$ can be represented as

$$ \hat{u}(t, s, \xi) = \psi(t, s, \xi) \hat{u}_1(\xi), $$

where the multiplier $\psi$ satisfies

$$ i|\xi|^\sigma \partial^k_x \psi(t, s, \xi) = \frac{\pi}{4 (1 + s)^{\rho - 1} |\xi|^{(k + j)\sigma}} \begin{bmatrix} H_{\rho}^{-} ((1 + s)|\xi|^\sigma) & H_{\rho}^{-}((1 + t)|\xi|^\sigma) & H_{\rho-}^{-}((1 + t)|\xi|^\sigma) \\ H_{\rho}^{+} ((1 + s)|\xi|^\sigma) & H_{\rho}^{+}((1 + t)|\xi|^\sigma) & H_{\rho-}^{+}((1 + t)|\xi|^\sigma) \end{bmatrix} $$

with $k + j = 0, 1$ and

$$ \rho = \frac{1 - \mu}{2}. $$
In order to derive estimates for $\hat{u}$ and its derivatives, we divide the extended phase space into zones to analyse the behavior of the Hankel functions $H^j_n$ (see Lemma 5.2 in Appendix):

$$Z_{\text{high}} = \{ \xi; |\xi| \geq 1 \} \text{ and } Z_{\text{low}} = Z_1 \cup Z_2 \cup Z_3$$

where

$$Z_1 = \{ \xi; (1 + s)^{-1} \leq |\xi|^r \leq 1 \}; \quad Z_2 = \{ \xi; (1 + s)|\xi|^r \leq 1 \leq (1 + t)|\xi|^r \}; \quad Z_3 = \{ \xi; (1 + t)|\xi|^r \leq 1 \}.$$ 

We consider the cut-off function $\chi \in C^\infty(\mathbb{R}^n)$ with $\chi(r) = 1$ for $r \leq \frac{1}{2}$ and $\chi(r) = 0$ for $r \geq 1$ and define

$$\chi_1(s, \xi) = 1 - \chi((1 + s)|\xi|^r),$$

$$\chi_2(t, s, \xi) = \chi((1 + s)|\xi|^r)(1 - \chi((1 + t)|\xi|^r)),$$

$$\chi_3(t, s, \xi) = \chi((1 + s)|\xi|^r)\chi((1 + t)|\xi|^r),$$

such that $\chi_1 + \chi_2 + \chi_3 = 1$. In the following we decompose the multiplier

$$m(t, s, \xi) = |\xi|^{(k+j)s} \begin{vmatrix} H^j_n^- ((1 + s)|\xi|^r) & H^j_n^- ((1 + t)|\xi|^r) \\ H^j_n^+ ((1 + s)|\xi|^r) & H^j_n^+ ((1 + t)|\xi|^r) \end{vmatrix}$$

as $m = (1 - \chi)m + \chi m$ and $\chi m = m \sum \chi_i$ and estimate each of the summands $(1 - \chi)m$ and $m_i := m \chi_i, i = 1, 2, 3$.

**Considerations in $Z_1$:** In $Z_1$ we may estimate

$$|\chi(|\xi|)\chi_1(s, \xi)m(t, s, \xi)| \lesssim (1 + s)^{-1/2}(1 + t)^{-1} |\xi|^{(k+j-1)}$$

so that

$$|\chi(|\xi|)\chi_1(s, \xi)| |\xi|^{(j|\sigma| + k)} \hat{u}(t, s, \xi)| \lesssim (1 + s)^{\frac{j}{p}}(1 + t)^{\frac{k}{q}} |\xi|^{(k+j-1)}.$$ 

By using Hausdorff-Young inequality and Hölder inequality, setting

$$\frac{1}{r} = \frac{1}{q'} = \frac{1}{p'} - \frac{1}{q'},$$

for $1 \leq p \leq 2 \leq q \leq \infty$ and $k + j = 0, 1$ one may estimate

$$\|\mc{S}^{-1}(\chi(|\xi|)\chi_1(s, \xi)| |\xi|^{(j|\sigma| + k)} \hat{u}(t, s, \xi)) \ast u_1\|_{L^p} \lesssim \|\chi(|\xi|)\chi_1(s, \xi)| |\xi|^{(j|\sigma| + k)} \hat{u}(t, s, \xi)\|_{L^p} \lesssim \|\chi(|\xi|)\chi_1(s, \xi)| |\xi|^{(j|\sigma| + k)} \hat{u}(t, s, \xi)\|_{L^p} \lesssim (1 + s)^{\frac{j}{p}}(1 + t)^{\frac{k}{q}} \|u_1\|_{L^p}$$

thanks to

$$\| |\xi|^{\sigma(k+j-1)}\|_{L^r(Z_1)} = \int_{|\xi|^{1+s}}^{1} d\xi = \int_{(1 + s)^{-1}}^{1} d\xi \lesssim (1 + s)^{-\frac{j}{p} + r(1 - k - j)}, \quad r\sigma(k + j - 1) + n < 0$$

$$\ln(e + s), \quad r\sigma(k + j - 1) + n = 0$$

$$1, \quad r\sigma(k + j - 1) + n > 0.$$ 

**Considerations in $Z_2$:** In $Z_2$ we may estimate

$$|\chi_2(t, s, \xi)m(t, s, \xi)| \lesssim \begin{cases} (1 + s)^{-\frac{|\rho|}{\rho}}(1 + t)^{-\frac{1}{p'}}|\xi|^{\sigma(k+j-|\rho| - \frac{1}{p'})} & \text{if } \mu \neq 1 \\ (1 + t)^{-\frac{1}{p'}}|\xi|^{\sigma(k+j-\frac{1}{p'})} \ln \left(\frac{1}{1 + s|\xi|^r}\right) & \text{if } \mu = 1 \end{cases}$$

so that

$$|\chi_2(t, s, \xi)| |\xi|^{(j|\sigma| + k)} \hat{u}(t, s, \xi)| \lesssim \begin{cases} (1 + s)^{1 - \frac{|\rho|}{\rho}}(1 + t)^{\frac{-1}{p'}}|\xi|^{\sigma(k+j-|\rho| - \frac{1}{p'})} & \text{if } \mu \neq 1 \\ (1 + s)(1 + t)^{-\frac{1}{p'}}|\xi|^{\sigma(k+j-\frac{1}{p'})} \ln \left(\frac{1}{1 + s|\xi|^r}\right) & \text{if } \mu = 1 \end{cases}$$

If $\mu \neq 1$ and $j + k \leq 1$ then, by using Hausdorff-Young inequality and Hölder inequality, setting

$$\frac{1}{r} = \frac{1}{q'} = \frac{1}{p'} - \frac{1}{q'},$$
for $1 \leq p < 2 \leq q < \infty$, one may estimate

$$\|\hat{\mathcal{S}}^{-1}(\chi_2(t, s, \xi)\xi^\sigma \partial_t^k \psi(t, s, \xi)) * u_1\|_{L^p} \lesssim \|\chi_2(t, s, \xi)\xi^\sigma \partial_t^k \psi(t, s, \xi)\|_{L^q} \lesssim (1 + s)^{1 - \sigma - |\rho|} (1 + t)^{\rho - \frac{k}{2}} \|u_1\|_{L^p}$$

$$\lesssim (1 + s)^{1 - \sigma - |\rho|} (1 + t)^{\rho - \frac{k}{2}} \lesssim \begin{cases} (1 + s)^{1 - \sigma - |\rho|} (1 + t)^{\rho - \frac{k}{2}} & r\sigma(k + j - |\rho| - \frac{1}{2}) + n > 0 \\
\ln \left(\frac{e^t}{e^s + s}\right), & r\sigma(k + j - |\rho| - \frac{1}{2}) + n = 0 \\
(1 + t)^{-\frac{k}{2} + r(|\rho| + \frac{1}{2} - k)}, & r\sigma(k + j - |\rho| - \frac{1}{2}) + n < 0 \end{cases},$$

thanks to

$$\|\chi_2(t, s, \xi)\xi^\sigma (k + j - |\rho| - \frac{1}{2}) \|_{L^p} = \int_{(1 + t)^{-\frac{k}{2}}}^{(1 + s)^{-\frac{k}{2}}} \|\xi\|^{r\sigma(k + j - |\rho| - \frac{1}{2})} d\xi \lesssim (1 + s)^{-\frac{k}{2}}$$

and obtain

$$\|\hat{\mathcal{S}}^{-1}(\chi_2(t, s, \xi)\xi^\sigma \partial_t^k \psi(t, s, \xi)) * u_1\|_{L^p} \lesssim (1 + s)^{-\frac{k}{2}} (1 + t)^{-\frac{k}{2}} \|u_1\|_{L^p}$$

whereas if $\mu = 1$ and $j = k = 0$ we may estimate

$$\|\hat{\mathcal{S}}^{-1}(\chi_2(t, s, \xi)\psi(t, s, \xi)) * u_1\|_{L^p} \lesssim (1 + s)(1 + t)^{-\frac{k}{2} - \frac{k}{2}} \|u_1\|_{L^p}$$

and obtain

$$\|\hat{\mathcal{S}}^{-1}(\chi_2(t, s, \xi)\psi(t, s, \xi)) * u_1\|_{L^p} \lesssim (1 + s)^{-\frac{k}{2} + \frac{k}{2}} (1 + t)^{-\frac{k}{2}} \|u_1\|_{L^p}$$

and obtain

$$\|\hat{\mathcal{S}}^{-1}(\chi_2(t, s, \xi)\psi(t, s, \xi)) * u_1\|_{L^p} \lesssim \|\chi_2(t, s, \xi)\psi(t, s, \xi)\|_{L^q} \lesssim \|\chi_2(t, s, \xi)\psi(t, s, \xi)\|_{L^q} \lesssim \|\chi_2(t, s, \xi)\psi(t, s, \xi)\|_{L^q}$$

$$\lesssim \|u_1\|_{L^p} (1 + s)(1 + t)^{-\frac{k}{2}} \ln \left(\frac{e^t}{e^s + s}\right), 2n > r\sigma$$

$$\ln \left(\frac{e^t}{e^s + s}\right), 2n = r\sigma$$

$$\ln \left(\frac{e^t}{e^s + s}\right), 2n < r\sigma$$

thanks to

$$\|\chi_2(t, s, \xi)\xi^{-\frac{k}{2}}\|_{L^p} = \int_{(1 + t)^{-\frac{k}{2}}}^{(1 + s)^{-\frac{k}{2}}} \|\xi\|^{-\frac{k}{2}} d\xi \lesssim \begin{cases} (1 + s)^{-\frac{k}{2}} & 2n > r\sigma \\
\ln \left(\frac{e^t}{e^s + s}\right), & 2n = r\sigma \\
(1 + t)^{-\frac{k}{2}} & 2n < r\sigma \end{cases}.$$
so that
\[ |\chi_3(t, s, \xi)|^{j \sigma} |D_t^k \psi(t, s, \xi)| \lesssim (1 + s)^{1 - 2\rho}(1 + t)^{2\rho - k}\|\xi\|^{(k + j)\sigma}.\]
By using Hausdorff-Young inequality and Hölder inequality, setting
\[ \frac{1}{r} = \frac{1}{q'} - \frac{1}{p'} = \frac{1}{p} - \frac{1}{q}, \]
for \(1 \leq p \leq 2 \leq q \leq \infty\) one may estimate for \(k + j = 0, 1,\)
\[ \|\mathcal{S}^{-1}(\chi_3(t, s, \xi))|^{j \sigma} |D_t^k \psi(t, s, \xi)| * u_1\|_{L^r} \lesssim \|\xi|^{j \sigma} \chi_3(t, s, \xi)|^{j \sigma} \|D_t^k \psi(t, s, \xi)\| \|u_1\|_{L^{r'}} \]
\[ \lesssim \|\chi_3(t, s, \xi)|^{j \sigma} \|D_t^k \psi(t, s, \xi)\| \|u_1\|_{L^{r'}} \]
\[ \lesssim \|(1 + t)^{-\frac{3}{2}(\frac{1}{p'} - \frac{1}{q'}) + \rho + |\rho| - k - j} (1 + s)^{1 - \rho - |\rho|} \|u_1\|_{L^p} \]
thanks to
\[ \|\chi_3(t, s, \xi)|^{j \sigma} \|_{L^r}^{a} = \int_{|\xi| \leq (1+t)^{-\frac{1}{r}}} |\xi|^{ra} d\xi \lesssim (1 + t)^{-\frac{3}{2}a}, \]
with \(a \geq 0\). In the case, \(\rho, \rho - k \in \mathbb{Z}\), we obtain
\[ |\chi_3(t, s, \xi)m(t, s, \xi)| \lesssim (1 + s)^{-\rho}(1 + t)^{\rho - k}\|\xi\|^{j \sigma} + (1 + s)^\rho(1 + t)^{-\rho + k}\|\xi\|^{(k + j)\sigma} \]
+ \( (1 + s)^\rho(1 + t)^{-\rho + k}\|\xi\|^{(k + j)\sigma} \ln \left( \frac{e + t}{e + s} \right) \)
if \(\rho - k \geq 0\) or
\[ |\chi_3(t, s, \xi)m(t, s, \xi)| \lesssim (1 + s)^{-\rho}(1 + t)^{\rho - k}\|\xi\|^{j \sigma} + (1 + s)^\rho(1 + t)^{-\rho + k}\|\xi\|^{(k + j)\sigma} \]
+ \( (1 + s)^{-\rho}(1 + t)^{\rho - k}\|\xi\|^{j \sigma} \ln \left( \frac{e + t}{e + s} \right) \)
if \(\rho - k < 0\). In fact, we use the relation \(J_{\rho - k}((1 + t)|\xi|^{\sigma}) = (-1)^{k - \rho}J_{k - \rho}((1 + t)|\xi|^{\sigma})\) if \(\rho - k \geq 0\) and \(J_{\rho}((1 + s)|\xi|^{\sigma}) = (-1)^{\rho}J_{-\rho}((1 + s)|\xi|^{\sigma})\) if \(\rho - k < 0\). By using Hausdorff-Young inequality and Hölder inequality, setting
\[ \frac{1}{r} = \frac{1}{q'} - \frac{1}{p'} = \frac{1}{p} - \frac{1}{q}, \]
for \(1 \leq p \leq 2 \leq q \leq \infty\) one may estimate for \(k + j = 0, 1,\)
\[ \|\mathcal{S}^{-1}(\chi_3(t, s, \xi))|^{j \sigma} |D_t^k \psi(t, s, \xi)| * u_1\|_{L^r} \lesssim \|\xi|^{j \sigma} \chi_3(t, s, \xi)|^{j \sigma} \|D_t^k \psi(t, s, \xi)\| \|u_1\|_{L^{r'}} \]
\[ \lesssim \|\chi_3(t, s, \xi)|^{j \sigma} \|D_t^k \psi(t, s, \xi)\| \|u_1\|_{L^{r'}} \]
\[ \lesssim \|u_1\|_{L^p} \left\{ (1 + t)^{\rho + |\rho| - k - j} \left( \frac{1}{2} - \frac{3}{2} \right) (1 + s)^{1 - \rho - |\rho|} \right\} \]
if \(\rho \neq 0\)
\[ \left\{ (1 + s)(1 + t)^{-k - j} \left( \frac{1}{2} - \frac{3}{2} \right) (1 + \ln \left( \frac{e + t}{e + s} \right) \right\} \]
if \(\rho = 0\).
Hence, if \(\mu \neq 1\), then we have the same estimate for both cases.
In particular, if \(\mu > 1\) we conclude
\[ \|\mathcal{S}^{-1}(\chi_3(t, s, \xi))|^{j \sigma} |D_t^k \psi(t, s, \xi)| * u_1\|_{L^p} \lesssim (1 + s)(1 + t)^{1 - \mu - \frac{3}{2}(\frac{1}{p'} - \frac{1}{q'}) - k - j}\|u_1\|_{L^p}, \]
and, if \(\mu < 1\) we conclude
\[ \|\mathcal{S}^{-1}(\chi_3(t, s, \xi))|^{j \sigma} |D_t^k \psi(t, s, \xi)| * u_1\|_{L^p} \lesssim (1 + s)^{\mu + 1 - \frac{3}{2}(\frac{1}{p'} - \frac{1}{q'}) - k - j}\|u_1\|_{L^p}. \]

Considerations in \(Z_{\text{high}}\): Thanks to Lemma 5.22 (see Appendix), we may decompose \(m_0 := (1 - \chi)m\) as the sum of two multipliers
\[ e^{\pm i(x + z)|\xi|(k+j)\sigma} a((1 + s)|\xi|^{\sigma}) b((1 + t)|\xi|^{\sigma}), \]
where \(a, b\) are symbols of order \(\frac{1}{2} - \frac{3}{2} \).
If one tries to follow the analysis of the previous zones, in \(Z_{\text{high}}\) it appears the additional restriction \(\frac{a}{n} \left( \frac{1}{p'} - \frac{1}{q'} \right) < 1\) on the \(L^p - L^n\) estimates, for \(1 \leq p \leq 2 \leq q \leq \infty\). To relax this range, in this zone of the extended phase space we may employ the strategy used in [32] to study the damping-free problem. By using duality argument, it is enough to prove the estimates for \(\frac{1}{p'} + \frac{1}{q'} \geq 1\).
Let $\phi \in C^\infty_c(\mathbb{R}^n)$ be a non-negative function supported in $\{\xi : \frac{1}{3} \leq |\xi| \leq 2\}$ and $\phi_\ell(\xi) := \phi(2^{-\ell}|\xi|)$, with $\ell$ an integer satisfying

$$\sum_{\ell \in \mathbb{Z}} \phi_\ell(\xi) = 1, \quad \forall \xi \neq 0.$$ 

In particular, $(1 - \chi)\phi_\ell = 0$ if $\ell < -1$ and $(1 - \chi)\phi_\ell = \phi_\ell$ if $\ell \geq 1$, hence one may write

$$\phi_\ell(\xi)m_0(t, s, \xi) = \sum_{\ell = -1}^{\infty} \phi_\ell(\xi)m_0(t, s, \xi).$$

By using Plancherel’s theorem and putting $\eta := 2^{-\ell} \xi$ we have

$$\|\phi_\ell \cdot (1 - \chi)m(t, s, \cdot)\|_{M^2} = \sup_{\eta \in \text{supp} \phi} |\phi(\eta)m_0(t, x, 2^\ell|\eta|)| \leq C 2^\ell (1 + t)^{-\frac{\ell}{\sigma}} (1 + s)^{-\frac{\ell}{q}}.$$ (3.6)

Now, by using Littman’s lemma (see Appendix) we conclude

$$\left\| \delta_\xi^{-1}(c^\ell \phi_\ell(\xi)\xi^{(k+j)\sigma} a((1 + s)|\xi|^\sigma)b((1 + t)|\xi|^\sigma)) \right\|_{L^\infty}$$

$$= \left\| \delta_\xi^{-1}(c^\ell \phi_\ell(\xi)\xi^{(k+j)\sigma} a((1 + s)|\xi|^\sigma)b((1 + t)|\xi|^\sigma)) \right\|_{L^\infty}$$

$$\leq C 2^\ell (1 + (t - s)2^{\ell\sigma})^{-\frac{1}{\sigma}} \sum_{|\alpha| \leq L} \|D_\eta^\alpha \phi(\eta)|\eta|^{(k+j)\sigma} a((1 + s)|\eta|^\sigma)b((1 + t)|\eta|^\sigma)\|_{L^\infty}$$

$$\leq C 2^\ell (1 + (t - s)2^{\ell\sigma})^{-\frac{1}{\sigma}} (1 + t)^{-\frac{1}{\sigma}} (1 + s)^{-\frac{1}{q}}.$$ 

We remark that for $\sigma \neq 1$ the rank of the Hessian $H_{|\eta|^\sigma}$ is equal to $n$.

Hence, Young’s Inequality implies

$$\left\| \delta_\xi^{-1}(m_0(t, s, \cdot)\phi_\ell(\xi)\eta(f)) \right\|_{L^\infty(\mathbb{R}^n)} \leq C 2^\ell (1 + (t - s)2^{\ell\sigma})^{-\frac{1}{\sigma}} (1 + t)^{-\frac{1}{\sigma}} (1 + s)^{-\frac{1}{q}} \|f\|_{L^1},$$ (3.7)

for all integer $\ell$, or equivalent,

$$\|\phi_\ell \cdot m_0(t, s, \cdot)\|_{M^\infty} \leq C 2^\ell (1 + (t - s)2^{\ell\sigma})^{-\frac{1}{\sigma}} (1 + t)^{-\frac{1}{\sigma}} (1 + s)^{-\frac{1}{q}}.$$ (3.8)

As a consequence of (3.6) and the Riesz-Thorin interpolation theorem we get

$$\|\phi_\ell \cdot m_0(t, s, \cdot)\|_{M^p_0} \leq C 2^\ell (1 + (t - s)2^{\ell\sigma})^{-\frac{1}{\sigma}} (1 + t)^{-\frac{1}{\sigma}} (1 + s)^{-\frac{1}{q}}$$ (3.9)

for $\frac{1}{p_0} + \frac{1}{q_0} = 1$.

In order to derive an estimate for $\|\phi_\ell \cdot m_0\|_{M^\infty}$, we consider the following estimates

$$\|\partial_\xi^\ell \phi m_0(t, s, \cdot)\|_{L^2} \leq C(t - s)^{-\frac{1}{\sigma}} (1 + s)^{-\frac{1}{q}} \left( \int_{2^{-\ell} \leq |\xi| \leq 2^{\ell+1}} |\xi|^{2(k+j)\sigma + 2\sigma - 2(\sigma - 1)|\xi|} d\xi \right)^{\frac{1}{2}}$$

$$\leq C \gamma \cdot 2^\ell (1 + (t - s)^{-\frac{1}{\sigma}} (1 + t)^{-\frac{1}{\sigma}} (1 + s)^{-\frac{1}{q}}$$

and applying the Bernstein’s inequality (see Proposition 5.1 in Appendix) for $N > \frac{\tau}{\sigma}$ we get

$$\|\phi m_0(t, s, \cdot)\|_{M^\infty} \leq C 2^\ell (1 + (t - s)^{-\frac{1}{\sigma}} (1 + t)^{-\frac{1}{\sigma}} (1 + s)^{-\frac{1}{q}}.$$ (3.10)

Using (3.9), (3.10) and Riesz-Thorin interpolation theorem we conclude that

$$\|\phi \cdot m_0(t, s, \cdot)\|_{M^p_0} \leq C 2^\ell \left(1 + \frac{q_0}{p_0} - q \right)^{-\frac{1}{q}} \left( 1 + \frac{q_0}{p_0} - q \right) \left( 1 + \frac{q_0}{p_0} - q \right) \left( 1 + \frac{q_0}{p_0} - q \right)$$

where $0 < \theta < 1$, with $\frac{1}{2} = \frac{1}{p_0} + \theta$ and $\frac{1}{q} = \frac{1}{q_0} + \theta$.

Therefore, for large frequencies, using the Littlewood-Paley dyadic decomposition we conclude the estimate

$$\|m_0(t, s, \cdot)\|_{M^p_0} \leq C (t - s)^{-\frac{1}{\sigma}} (1 + t)^{-\frac{1}{\sigma}} (1 + s)^{-\frac{1}{q}},$$ (3.11)

which is convergent if

$$\frac{1}{p} + \frac{\sigma - 1}{q} < \sigma \left( \frac{1}{2} - \frac{k + j - 1}{n} \right), \quad \text{for} \quad \frac{1}{p} + \frac{1}{q} \geq 1.$$ (3.12)
By duality arguments, the analogous estimate is true if
\[
\frac{1 - \sigma}{p} - \frac{1}{q} < \sigma\left(\frac{k + j - 1}{n} - \frac{1}{2}\right), \quad \text{for } \frac{1}{p} + \frac{1}{q} \leq 1.
\]
However, in the special case \(1 < p \leq 2 \leq q < \infty\), the latter estimates may be refined by using the embeddings for Besov spaces (see, for instance, [11]): \(L^p \hookrightarrow B^0_{p,2}\) for \(p \in (1,2]\) and \(B^0_{q,2} \hookrightarrow L^q\) for \(q \in [2,\infty)\). Indeed, since the sum in (3.5) is finite for any given \(\xi\) (see Appendix), in particular, \(\#\{\ell : \phi_\ell(\xi) \neq 0\} \leq 3\), we obtain the chain of inequalities (see also [11])
\[
\|\hat{\mathcal{S}}^{-1}(m_0f)\|_{B^0_{p,2}} \leq C_1 \sup_\ell \|\hat{\mathcal{S}}^{-1}(m_0\phi_\ell f)\|_{L^1} \leq C_2 \|f\|_{L^p} \leq C_3 \|f\|_{B^0_{p,2}}.
\]

Summing up we have:

Proposition 3.2. Let \(n \in \mathbb{N}\) and \(\sigma \neq 1\). Assume \(1 \leq p < q \leq \infty\) and \(j + k \leq 1\) such that
\[
\frac{n}{\sigma}\left(\frac{1}{p} - \frac{1}{q}\right) + n \max\left\{\left(\frac{1}{2} - \frac{1}{p}\right), \left(\frac{1}{q} - \frac{1}{2}\right)\right\} + j + k < 1.
\]
Then \(u^{\text{high}}(t,s,\cdot) := \mathcal{S}^{-1}\left((1 - \chi(|\xi|))\psi(t,s,\xi)\right) + u_1(x)\) satisfies
\[
\|\partial_t^k(-\Delta)^{\frac{p-1}{2}}u^{\text{high}}(t,s,\cdot)\|_{L^p} \leq C(t - s)^{n\Gamma(p,q)}(1 + t)^{-\frac{1}{2}}(1 + s)^{\frac{q}{2}}\|u_1\|_{L^p},
\]
with
\[
\Gamma(p,q) = \left\{\frac{1}{q} - \frac{1}{2}, \frac{1}{2} - \frac{1}{p}, \frac{n}{\sigma} + \frac{1}{2}\right\}.
\]
Moreover, if equality holds in (3.13), estimate (3.14) remains valid for \(1 < p \leq 2 \leq q < \infty\).

Remark 3.1. If \(\frac{1}{p} + \frac{1}{q} \geq 1\) with \(1 \leq p \leq 2\), (3.12) is true for all \(\hat{q} < q \leq \infty\) (\(q \geq \hat{q} \geq 2\)), with \(\hat{q}\) given by
\[
\frac{1}{\hat{q}} := \frac{1}{\sigma - 1}\left(\frac{1 - j - k}{n} + \frac{1}{p}\right).
\]
In particular, if \(j = k = 0\), then for \(2\sigma = \left(\frac{2}{p} - 1\right)n\) we get \(\hat{q} = 2\), whereas \(\hat{q} < 2 \leq 2\sigma > \left(\frac{2}{p} - 1\right)n\) and \(\hat{q} > 2\) for \(2\sigma < \left(\frac{2}{p} - 1\right)n\) if \(j + k = 1\), then \(p = 2 = \hat{q}\) and \(1 \leq p < 2 < \hat{q}\).

Remark 3.2. For \(p = 1\) the term \((t - s)^{n\Gamma(p,q)}\) may be singular but \(n\Gamma(p,q) > -1\) for \(q < \frac{2n}{n - 2\sigma}\).

Remark 3.3. For \(n < 2\sigma(1 - k - j)\) we may also have \(L^1 - L^2\) estimate for \(u^{\text{high}}\). Indeed,
\[
\|\partial_t^k(-\Delta)^{\frac{p-1}{2}}u^{\text{high}}(t,s,\cdot)\|_{L^2} \lesssim \|\partial_t^k(-\Delta)^{\frac{p-1}{2}}u^{\text{high}}(t,s,\cdot)\|_{L^2} \|u_1\|_{L^\infty} \lesssim \|\chi^{(k+j-1)\sigma}\|_{L^\infty} \|u_1\|_{L^1} \lesssim (1 + s)^{\frac{q}{2}}\|u_1\|_{L^1}.
\]

Corollary 3.1. Let \(u_1 \in L^2\), then \(u^{\text{high}}(t,s,\cdot) := \mathcal{S}^{-1}\left((1 - \chi(|\xi|))\psi(t,s,\xi)\right) + u_1(x)\) satisfies
\[
\|u^{\text{high}}(t,s,\cdot)\|_{L^2} \leq C(1 + t)^{-\frac{1}{2}}(1 + s)^{\frac{q}{2}}\|u_1\|_{L^2}, \quad 2 \leq q \leq \frac{2n}{n - 2\sigma} + \frac{1}{2}.
\]

and for \(j + k = 1\)
\[
\|\partial_t^k(-\Delta)^{\frac{p-1}{2}}u^{\text{high}}(t,s,\cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{1}{2}}(1 + s)^{\frac{q}{2}}\|u_1\|_{L^2}.
\]

Remark 3.4. It is worth to mention that different from \(Z_{\text{high}}\) and \(Z_1\), in zones \(Z_2\) and \(Z_3\) additional derivatives produce additional decay.

In the following, we state the estimates for solutions to the linear problem for \(s = 0\) and \(s \neq 0\) that will be used in Section [H].

Theorem 3.1. Let \(\sigma > 1\), \(1 \leq n < 2\sigma\) and \(q \geq 2\).

(i): If \(\mu > \max\left\{\frac{2n}{n - 2\sigma}, 1\right\}\), then the solution to (5.1) satisfies
\[
\|u(t,\cdot)\|_{L^p} \lesssim (1 + t)^{-\frac{p}{2}}(1 + s)\left(\|u_1\|_{L^1} + (1 + s)^{\frac{q}{2}}\|u_1\|_{L^2}\right)
\]
or
\[
\|u(t,\cdot)\|_{L^p} \lesssim (1 + t)^{-\frac{p}{2}}\left((1 + s)^{\max\left\{1,\frac{p(1 - \frac{1}{p})}{2}\right\}}\|u_1\|_{L^1} + (1 + s)^{\frac{q}{2}}\|u_1\|_{L^2}\right), \quad \frac{n}{\sigma} - \frac{1}{q} \neq 1;
\]
(ii): If \( \max\left\{ 2 - \frac{2n}{\sigma} \left( 1 - \frac{1}{q} \right); 0 \right\} < \mu < 1 \) or \( 1 < \mu < 2 - \frac{2n}{\sigma} \left( 1 - \frac{1}{q} \right) \), then the solution to (3.1) satisfies
\[
\|u(t, \cdot)\|_{L^q} \lesssim (1 + t)^{-\frac{\mu}{q}} (1 + s)^{1 + \frac{n}{\sigma} \cdot \frac{1}{q} (1 - \frac{1}{q})} \|u_1\|_{L^1} + (1 + s)^{\frac{n}{\sigma} (1 - \frac{1}{q})} \|u_1\|_{L^2}
\]
or
\[
\|u(t, \cdot)\|_{L^q} \lesssim (1 + t)^{-\frac{\mu}{q}} (1 + s)^{\frac{n}{\sigma} \max\{0, 1 - \frac{1}{q}, 1 - \frac{1}{q}\}} \|u_1\|_{L^1} + \|u_1\|_{L^2}, \quad \frac{n}{\sigma} \left( 1 - \frac{1}{q} \right) \neq 1.
\]
Moreover, if \( \mu = 2 - \frac{2n}{\sigma} \left( 1 - \frac{1}{q} \right) > 1 \), then
\[
\|u(t, \cdot)\|_{L^q} \lesssim (1 + t)^{-\frac{\mu}{q}} (1 + s)^{-\frac{n}{\sigma} (1 - \frac{1}{q})} \left( \ln \left( \frac{e + t}{e + s} \right) \right)^{1 - \frac{1}{q}} \|u_1\|_{L^1} + (1 + s)^{\frac{n}{\sigma} (1 - \frac{1}{q})} \|u_1\|_{L^2}
\]
whereas if \( \mu = 2 - \frac{2n}{\sigma} \left( 1 - \frac{1}{q} \right) < 1 \), then
\[
\|u(t, \cdot)\|_{L^q} \lesssim (1 + t)^{-\frac{\mu}{q}} (1 + s)^{\frac{n}{\sigma} (1 - \frac{1}{q})^{-1} (1 + s)^{\mu}} \left( \ln \left( \frac{e + t}{e + s} \right) \right)^{1 - \frac{1}{q}} \|u_1\|_{L^1} + (1 + s)^{\frac{n}{\sigma} (1 - \frac{1}{q})^{-1}} \|u_1\|_{L^2};
\]
(iii): If \( 0 < \mu < \min\{2 - \frac{2n}{\sigma} \left( 1 - \frac{1}{q} \right); 1\} \), then the solution to (3.1) satisfies
\[
\|u(t, \cdot)\|_{L^q} \lesssim (1 + t)^{1 - \mu} \left( 1 + \frac{n}{\sigma} \cdot \frac{1}{q} (1 - \frac{1}{q}) (1 + s)^{\mu} \|u_1\|_{L^1} + (1 + s)^{\mu - 1} \left( \frac{1}{q} + \frac{1}{q} \right) } \|u_1\|_{L^2}
\]
or
\[
\|u(t, \cdot)\|_{L^q} \lesssim (1 + t)^{1 - \mu} \left( 1 + \frac{n}{\sigma} \cdot \frac{1}{q} (1 - \frac{1}{q}) (1 + s)^{\mu} \|u_1\|_{L^1} + (1 + s)^{\mu - 1} \left( \frac{1}{q} + \frac{1}{q} \right) } \|u_1\|_{L^2}, \quad \frac{n}{\sigma} \left( 1 - \frac{1}{q} \right) \neq 1.
\]
(iv): If \( \mu = 1 \), then the solution to (3.1) satisfies
\[
\|u(t, \cdot)\|_{L^q} \lesssim (1 + t)^{-\frac{\mu}{q}} (1 + s)^{\frac{n}{\sigma} \cdot \frac{1}{q} (1 - \frac{1}{q})} \|u_1\|_{L^1}
\]
\[
+ \|u_1\|_{L^2} (1 + t)^{-\min\left\{ \frac{\mu}{q} \left( 1 - \frac{1}{q} \right), \frac{\mu}{q} \right\}} \left\{ \frac{(1 + s)^{\frac{n}{\sigma} \cdot \frac{1}{q} (1 - \frac{1}{q})}}{(1 + s)^{\frac{n}{\sigma} \cdot \frac{1}{q} (1 - \frac{1}{q})}} \ln \left( \frac{e + t}{e + s} \right), \quad q > \frac{2n}{[2n - n_{\sigma}]_+}, \right.
\]
\[
\left. \left( 1 + s \right)^{\frac{n}{\sigma} \cdot \frac{1}{q} (1 - \frac{1}{q})} \ln \left( \frac{e + t}{e + s} \right), \quad 1 \leq q < \frac{2n}{[2n - n_{\sigma}]_+}. \right.
\]

**Remark 3.5.** We point out that
\[
\max\left\{ \frac{2n}{\sigma} \left( 1 - \frac{1}{q} \right); 1 \right\} = \left\{ \begin{array}{ll}
\frac{2n}{\sigma} \left( 1 - \frac{1}{q} \right) & \text{if} \quad 1 \leq q \leq \frac{2n}{[2n - n_{\sigma}]_+}, \\
1 & \text{if} \quad q > \frac{2n}{[2n - n_{\sigma}]_+}.
\end{array} \right.
\]
is equivalent to
\[
\min\left\{ 2 - \frac{2n}{\sigma} \left( 1 - \frac{1}{q} \right); 1 \right\} = \left\{ \begin{array}{ll}
2 - \frac{2n}{\sigma} \left( 1 - \frac{1}{q} \right) & \text{if} \quad 1 \leq q \leq \frac{2n}{[2n - n_{\sigma}]_+}, \\
1 & \text{if} \quad q > \frac{2n}{[2n - n_{\sigma}]_+}.
\end{array} \right.
\]

**Proof.** For \( n < 2\sigma \), from (6.10) we get
\[
\|u_{\text{high}}^n(t, s, \cdot)\|_{L^q} \leq C(1 + t)^{-\frac{\mu}{q}} (1 + s)^{\frac{n}{\sigma} \cdot \frac{1}{q} (1 - \frac{1}{q}) } \|u_1\|_{L^2}, \quad q \geq 2.
\]
The proof of (i): Suppose that \( \mu > \max\left\{ \frac{2n}{\sigma} \left( 1 - \frac{1}{q} \right); 1 \right\}, \) \( \mu > \frac{2n}{\sigma} \left( 1 - \frac{1}{q} \right) \) then
\[
(1 + t)^{-\frac{\mu}{q}} (1 + s)^{\frac{n}{\sigma} \cdot \frac{1}{q} (1 - \frac{1}{q})} \leq (1 + t)^{-\frac{\mu}{q} \left( 1 - \frac{1}{q} \right)} (1 + s)^{\frac{n}{\sigma} \cdot \frac{1}{q} (1 - \frac{1}{q})}.
\]
For \( n < 2\sigma \) we get
\[
\|u_{\text{high}}^n(t, s, \cdot)\|_{L^q} \leq C(1 + t)^{-\frac{\mu}{q} \left( 1 - \frac{1}{q} \right)} (1 + s)^{\frac{n}{\sigma} \cdot \frac{1}{q} (1 - \frac{1}{q}) } \|u_1\|_{L^2}, \quad q \geq 2.
\]
Applying the derived estimates at zone \( Z_{1} \) with \( p = 2 \) or \( p = 1 \), respectively, we may estimate
\[
\|\mathcal{H}_{-1}(\chi(\xi)) \chi_1(s, \xi) \psi(t, s, \xi) \ast u_1\|_{L^2} \lesssim \| (1 + t)^{-\frac{\mu}{q} \left( 1 - \frac{1}{q} \right)} (1 + s)^{\frac{n}{\sigma} \cdot \frac{1}{q} (1 - \frac{1}{q}) } \|u_1\|_{L^2} \]
\[
\lesssim (1 + t)^{-\frac{\mu}{q} \left( 1 - \frac{1}{q} \right)} (1 + s)^{\frac{n}{\sigma} \cdot \frac{1}{q} (1 - \frac{1}{q}) + \frac{1}{q} (1 - \frac{1}{q}) } \|u_1\|_{L^2} \]
\[
= (1 + t)^{-\frac{\mu}{q} \left( 1 - \frac{1}{q} \right)} (1 + s)^{1 + \frac{n}{\sigma} \cdot \frac{1}{q} (1 - \frac{1}{q}) } \|u_1\|_{L^2}.
\]
or
\[
\|\tilde{\mathcal{F}}^{-1}(\chi(|\xi|)\chi_1(s,\xi)\psi(t, s, \xi))\|_{L^2} \lesssim (1 + t)^{-\frac{\sigma}{\sigma + 1}}(1 + s)^{\frac{\sigma}{\sigma + 1} + \max\{0, 1 - \frac{\sigma}{\sigma + 1}\}}\|u_1\|_{L^1}.
\]

In \( Z_2 \) and \( Z_3 \) we have the following estimate:
\[
\|u(t, \cdot)\|_{L^\infty} \lesssim (1 + t)^{-\frac{\sigma}{\sigma + 1}}(1 + s)\|u_1\|_{L^1}.
\]
Thus (i) is concluded.
The proof of (ii): In \( Z_1 \) we have
\[
\|u(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{\sigma}{\sigma + 1}}(1 + s)^{1 + \frac{\sigma}{\sigma + 1} + \frac{\sigma}{\sigma + 1} - \frac{\sigma}{\sigma + 1}}\|u_1\|_{L^2}
\]
or
\[
\|\tilde{\mathcal{F}}^{-1}(\chi(|\xi|)\chi_1(s,\xi)\psi(t, s, \xi))\|_{L^2} \lesssim (1 + t)^{-\frac{\sigma}{\sigma + 1}}(1 + s)^{\frac{\sigma}{\sigma + 1} + \max\{0, 1 - \frac{\sigma}{\sigma + 1}\}}\|u_1\|_{L^2}.
\]
Suppose that \( 1 < \mu \leq \frac{2a}{\sigma} \left( 1 - \frac{1}{q} \right) \). We have in \( Z_2 \)
\[
\|u(t, \cdot)\|_{L^\infty} \lesssim (1 + t)^{-\frac{\sigma}{\sigma + 1}}(1 + s)^{1 + \frac{\sigma}{\sigma + 1} + \frac{\sigma}{\sigma + 1} - \frac{\sigma}{\sigma + 1}}\|u_1\|_{L^1} \quad \text{for } \mu = \frac{2a}{\sigma} \left( 1 - \frac{1}{q} \right).
\]
For \( \mu \leq \frac{2a}{\sigma} \left( 1 - \frac{1}{q} \right) \) we have
\[
(1 + t)^{-\frac{\sigma}{\sigma + 1}}(1 + s) \leq (1 + t)^{-\frac{\sigma}{\sigma + 1} + \frac{\sigma}{\sigma + 1} - \frac{\sigma}{\sigma + 1}}(1 + s) \leq (1 + t)^{-\frac{\sigma}{\sigma + 1}}(1 + s)^{1 + \frac{\sigma}{\sigma + 1} + \frac{\sigma}{\sigma + 1} - \frac{\sigma}{\sigma + 1}},
\]
\[
\|u(t, \cdot)\|_{L^\infty} \lesssim (1 + t)^{-\frac{\sigma}{\sigma + 1}}(1 + s)^{1 + \frac{\sigma}{\sigma + 1} + \frac{\sigma}{\sigma + 1} - \frac{\sigma}{\sigma + 1}}\|u_1\|_{L^1} \quad \text{for } 1 < \mu < \frac{2a}{\sigma} \left( 1 - \frac{1}{q} \right).
\]
Suppose that \( \max\{2 - \frac{2a}{\sigma} \left( 1 - \frac{1}{q} \right) : 0\} < \mu < 1 \) or \( \mu = 2 - \frac{2a}{\sigma} \left( 1 - \frac{1}{q} \right) \). We have in \( Z_2 \)
\[
\|u(t, \cdot)\|_{L^\infty} \lesssim (1 + t)^{-\frac{\sigma}{\sigma + 1}}(1 + s)^{1 + \frac{\sigma}{\sigma + 1} + \frac{\sigma}{\sigma + 1} - \frac{\sigma}{\sigma + 1}}\|u_1\|_{L^1} \quad \text{for } \mu = 2 - \frac{2a}{\sigma} \left( 1 - \frac{1}{q} \right).
\]
If \( \mu \geq 2 - \frac{2a}{\sigma} \left( 1 - \frac{1}{q} \right) \), then
\[
(1 + t)^{1 - \mu - \frac{\sigma}{\sigma + 1}}(1 + s)^{\mu} \leq (1 + t)^{-\frac{\sigma}{\sigma + 1}}(1 + t)^{-\frac{\sigma}{\sigma + 1} + \frac{\sigma}{\sigma + 1} - \frac{\sigma}{\sigma + 1}}(1 + s)^{\mu} \leq (1 + t)^{-\frac{\sigma}{\sigma + 1}}(1 + s)^{\frac{\sigma}{\sigma + 1} + \frac{\sigma}{\sigma + 1} - \frac{\sigma}{\sigma + 1}}.
\]
Hence, we have in \( Z_3 \)
\[
\|u(t, \cdot)\|_{L^\infty} \lesssim (1 + t)^{1 - \mu - \frac{\sigma}{\sigma + 1}}(1 + s)^{\mu}\|u_1\|_{L^1} \quad \text{for } \mu \leq 2 - \frac{2a}{\sigma} \left( 1 - \frac{1}{q} \right).
\]
\[
\|u(t, \cdot)\|_{L^\infty} \lesssim (1 + t)^{1 - \mu - \frac{\sigma}{\sigma + 1}}(1 + s)^{\mu}(1 + s)^{\frac{\sigma}{\sigma + 1} + \frac{\sigma}{\sigma + 1} - \frac{\sigma}{\sigma + 1}}\|u_1\|_{L^1}.
\]
The proof of (iii): Suppose that \( 0 < \mu < \min\{2 - \frac{2a}{\sigma} \left( 1 - \frac{1}{q} \right) : 1\} \). If \( \mu < 2 - \frac{2a}{\sigma} \left( 1 - \frac{1}{q} \right) \), then
\[
(1 + t)^{-\frac{\sigma}{\sigma + 1}}(1 + s)^{\frac{\sigma}{\sigma + 1} + \frac{\sigma}{\sigma + 1} - \frac{\sigma}{\sigma + 1}} \leq (1 + t)^{1 - \mu - \frac{\sigma}{\sigma + 1}}(1 + s)^{\mu - \frac{\sigma}{\sigma + 1} + \frac{\sigma}{\sigma + 1} - \frac{\sigma}{\sigma + 1}}.
\]
For \( n < 2\sigma \) we get
\[
\|u^{high}(t, s, \cdot)\|_{L^\infty} \leq C(1 + t)^{1 - \mu - \frac{\sigma}{\sigma + 1}}(1 + s)^{\mu - \frac{\sigma}{\sigma + 1} + \frac{\sigma}{\sigma + 1} - \frac{\sigma}{\sigma + 1}}\|u_1\|_{L^2}, \quad q \geq 2.
\]
We obtain the following estimates: in \( Z_2 \cup Z_3 \)
\[
\|u(t, \cdot)\|_{L^\infty} \lesssim (1 + t)^{1 - \mu - \frac{\sigma}{\sigma + 1}}(1 + s)^{\mu}\|u_1\|_{L^1}.
\]
and in $Z_1$

$$\|u(t, \cdot)\|_{L^s} \lesssim (1 + t)^{-\frac{\mu}{2}} (1 + s)^{1 + \frac{\sigma}{2} - \frac{n}{s} \left(\frac{1}{2} - \frac{1}{q}\right)} \|u_1\|_{L^2}$$

and in $Z_2$

$$\|u(t, \cdot)\|_{L^s} \lesssim \|u_1\|_{L^1}$$

with

$$\begin{cases}
(1 + t)^{-\frac{\mu}{2}} (1 + s)^{-\frac{n}{2} \left(\frac{1}{2} - \frac{1}{q}\right)} \ln \left(\frac{e + t}{e + s}\right), & q > \frac{2n}{2n - \sigma} + 1
\end{cases}$$

or

$$\begin{cases}
(1 + t)^{-\frac{\mu}{2}} (1 + s) \ln^{\frac{n}{2}} \left(\frac{e + t}{e + s}\right), & q = \frac{2n}{2n - \sigma} + 1
\end{cases}$$

and in $Z_1$

$$\|u(t, \cdot)\|_{L^s} \lesssim (1 + t)^{-\frac{\mu}{2}} (1 + s)^{\frac{1}{2} - \frac{n}{2} \left(\frac{1}{2} - \frac{1}{q}\right)} \|u_1\|_{L^2}.$$
(i): If $\mu > \max\{\frac{n+2\gamma}{\sigma}; 1\}$, then the solution to (3.1) satisfies
\[
\|u(t, \cdot)\|_{H^1} \leq (1 + t)^{-\frac{\mu}{2}} \|u_1\|_{L^1} + (1 + s)^{\frac{\mu+1}{2\alpha}} \|u_1\|_{L^2};
\]

(ii): If $2 - \frac{\mu}{2} - \frac{n+2\gamma}{\sigma} < 0$, then the solution to (3.1) satisfies
\[
\|u(t, \cdot)\|_{H^1} \leq (1 + t)^{-\frac{\mu}{2}} \|u_1\|_{L^1} + (1 + s)^{\frac{\mu+1}{2\alpha}} \|u_1\|_{L^2},
\]

Moreover, if $\mu = \frac{n+2\gamma}{\sigma} > 1$, then
\[
\|u(t, \cdot)\|_{H^1} \leq (1 + t)^{-\frac{\mu}{2}} \|u_1\|_{L^1} + (1 + s)^{\frac{\mu+1}{2\alpha}} \|u_1\|_{L^2},
\]

whereas if $\mu = 2 - \frac{\mu}{2} - \frac{n+2\gamma}{\sigma} < 1$, then
\[
\|u(t, \cdot)\|_{H^1} \leq (1 + t)^{-\frac{\mu}{2}} \|u_1\|_{L^1} + (1 + s)^{\frac{\mu+1}{2\alpha}} \|u_1\|_{L^2};
\]

(iii): If $0 < \mu < \min\{2 - \frac{\mu}{2} - \frac{n+2\gamma}{\sigma}; 1\}$, then the solution to (3.1) satisfies
\[
\|u(t, \cdot)\|_{H^1} \leq (1 + t)^{1-\mu} \|u_1\|_{L^1} + (1 + s)^{\frac{\mu+1}{2\alpha}} \|u_1\|_{L^2}.
\]

Moreover, the $\partial_t u(t, \cdot)$ satisfies the same decay estimates of $\|(-\Delta) u(t, \cdot)\|_{L^2}$.

Proof. For $n < 2\sigma$, from Corollary 3.1 we get
\[
\|\xi\|_{H^{\alpha}} \leq C(1 + t)^{-\frac{\mu}{2}} \|u_1\|_{L^1}, \quad \gamma \in [0, \sigma].
\]

Putting $q = 2, \sigma = \gamma$, $k = 0$ and $\sigma \leq 2 - \frac{\mu}{2} - \frac{n+2\gamma}{\sigma}$, following the calculations on Section 3 we get:

The proof of (i): Suppose $\mu > \max\{\frac{n+2\gamma}{\sigma}; 1\}$. In $Z_3$ we have for $p = 1$, the estimate
\[
\|\mathbf{S}^{-1}(\chi_3(t, s, \xi))\|_{H^1} \leq (1 + t)^{-\frac{\mu}{2}} \|u_1\|_{L^1}.
\]

In $Z_2$ we have for $p = 1$, the estimate
\[
\|\mathbf{S}^{-1}(\chi_2(t, s, \xi))\|_{H^1} \leq (1 + t)^{-\frac{\mu}{2}} \|u_1\|_{L^1}.
\]

In $Z_1$ we have for $p = 2$, the estimate
\[
\|\mathbf{S}^{-1}(\chi_1(t, s, \xi))\|_{H^1} \leq (1 + t)^{-\frac{\mu}{2}} \|u_1\|_{L^1}.
\]

The proof of (ii): Suppose that $1 < \mu \leq \frac{n+2\gamma}{\sigma}$. In $Z_1$ we have for $p = 2$, the estimate
\[
\|\mathbf{S}^{-1}(\chi_1(t, s, \xi))\|_{H^1} \leq (1 + t)^{-\frac{\mu}{2}} \|u_1\|_{L^1}.
\]

In $Z_2$ we have
\[
\|\mathbf{S}^{-1}(\chi_2(t, s, \xi))\|_{H^1} \leq (1 + t)^{-\frac{\mu}{2}} \|u_1\|_{L^1}.
\]

If $\mu \leq \frac{n+2\gamma}{\sigma}$, then we have in $Z_3$
\[
\|\mathbf{S}^{-1}(\chi_3(t, s, \xi))\|_{H^1} \leq (1 + t)^{-\frac{\mu}{2}} \|u_1\|_{L^1}.
\]

Suppose that $\max\{2 - \frac{\mu}{2} - \frac{n+2\gamma}{\sigma}; 0\} < \mu < 1$ or $\mu = 2 - \frac{\mu}{2} - \frac{n+2\gamma}{\sigma}$. We have in $Z_2$
\[
\|\mathbf{S}^{-1}(\chi_2(t, s, \xi))\|_{H^1} \leq (1 + t)^{-\frac{\mu}{2}} \|u_1\|_{L^1}.
\]

If $\mu \geq 2 - \frac{\mu}{2} - \frac{n+2\gamma}{\sigma}$, then we have in $Z_3$
\[
\|\mathbf{S}^{-1}(\chi_3(t, s, \xi))\|_{H^1} \leq (1 + t)^{-\frac{\mu}{2}} \|u_1\|_{L^1}.
\]
The proof of (iii): Suppose that $0 < \mu < \min\{2 - \frac{p}{n}, 2\sigma - \frac{2p}{n}\}$. If $\mu < 2 - \frac{p}{n} - \frac{2p}{n}$, then for $n < 2\sigma$ we get
\[ \|\xi\|^n u^\text{high}(t, s, \cdot)\|_{L^2} \leq C(1 + t)^{-\frac{p}{n}} (1 + s)^{\frac{p}{n}} u_1\|_{L^2} \leq C(1 + t)^{1-\mu - \frac{n}{2\sigma}} (1 + s)^{\mu - \frac{n}{2\sigma}} \|u_1\|_{L^2}, \quad q \geq 2.\]
We obtain the following estimates: in $Z_2 \cup Z_3$
\[ \|\delta^{-1}(\chi_2(t, s, \xi)\chi_3(t, s, \xi)\|_{L^2} \leq (1 + t)^{1-\mu} (1 + s)^{\mu} \|u_1\|_{L^2} \]
and in $Z_1$
\[ \|\delta^{-1}(\chi(t, s, \xi)\chi_3(t, s, \xi)\|_{L^2} \leq (1 + t)^{1-\mu} (1 + s)^{\mu} \|u_1\|_{L^2} \]
\[ \lesssim (1 + t)^{\mu} (1 + s)^{\mu} \|u_1\|_{L^2}. \]

4. Proof of the Global existence results

By Duhamel’s principle, a function $u \in Z$, where $Z$ is a suitable space, is a solution to (1.1) if, and only if, it satisfies the equality
\[ u(t, x) = u^\text{lin}(t, x) + \int_0^t K_1(t, s, x) * |u(s, x)|^p ds, \quad \text{in } Z, \quad (4.1) \]
where $K_1(t, s, x) = \delta^{-1}(\psi)(t, s, x)$ and
\[ u^\text{lin}(t, x) := K_1(t, 0, x) * u_1(x), \]
is the solution to the linear Cauchy problem (3.1) with $s = 0$. The proof of our global existence results is based on the following scheme. We define an appropriate data function space
\[ A := L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n), \quad (4.2) \]
and an evolution space for solutions
\[ Z(T) := C([0, T], H^s(\mathbb{R}^n)) \cap C^1([0, T], L^2(\mathbb{R}^n)) \cap L^\infty([0, \infty) \times \mathbb{R}^n), \quad (4.3) \]
equipped with a norm relate to the estimates of solutions to the linear problem (3.1) with $s = 0$ such that
\[ \|u^\text{lin}(t, \cdot)\|_Z \leq C \|u_1\|_A. \quad (4.4) \]

We define the operator $F$ such that, for any $u \in Z$,
\[ F(u, t, x) := \int_0^t K_1(t, s, x) * |u(s, x)|^p ds, \]
then we prove the estimates
\[ \|Fu\|_Z \leq C \|u\|_Z^p, \quad (4.5) \]
\[ \|Fu - Fv\|_Z \leq C \|u - v\|_Z \|u\|_{L^\infty}^{p-1} \|v\|_{L^\infty}^{-1}. \quad (4.6) \]
By standard arguments, since $u^\text{lin}$ satisfies (4.4) and $p > 1$, from (4.5), it follows that $e^{tF}$ maps balls of $Z$ into balls of $Z$, and for small data in $A$, from (4.6) $F$ is a contraction. So, the estimates (4.5), (4.6) lead to the existence of a unique solution to (4.1), that is, $u = u^\text{lin} + Fu$, satisfying (4.4). We simultaneously gain a locally in time for large data and globally in time for small data existence result [10].

Proof. (Theorem 2.2) We have to prove (4.4), (4.5), and (4.6), with $A$ as in (4.2) and $Z(T)$ as in (4.3) equipped with the norm
\[ \|u\|_{Z(T)} := \sup_{t \in [0, T]} \left\{ (1 + t)^{\frac{p}{n}} \|u(t, \cdot)\|_{L^2} + (1 + t)^{\frac{p}{n}} (1 + s)^{\frac{p}{2\sigma}} \|u(t, \cdot)\|_{L^\infty} + (1 + t)^{\min\{\frac{p}{n}, \frac{p}{2\sigma}\}} \|u(t, \cdot)\|_{L^\infty} \right\}, \]
where $q_0$ is defined as in (2.3).

Thanks to Corollary 3.2 and Theorem 3.2, $u^\text{lin} \in Z(T)$ and it satisfies (4.4).

Let us prove (4.5). We omit the proof of (4.6), since it is analogous to the proof of (4.5).
Let $u \in Z(T)$. If $\mu > \max\left\{\frac{2n}{q_0}; 1\right\}$, by Theorem 3.1 for $q \geq 2$ we have

\[
\|Fu(t, \cdot)\|_{L^q} \lesssim \int_0^t (1 + t)^{-\frac{q}{p}} (1 + s)^{1 + \frac{1}{q} - \frac{1}{p}} \left(\|u(s, \cdot)\|_{L^1} + (1 + s)^{\frac{1}{2}} \|u(s, \cdot)\|^p_{L^2}\right) ds
\]

\[
\lesssim (1 + t)^{-\frac{q}{p}} \int_0^t (1 + s)^{1 + \frac{1}{q} - \frac{1}{p}} \left(\|u(s, \cdot)\|^p_{L^2}\right) ds \|u\|^p_{Z(T)}
\]

for all $p > 1 + \frac{2n}{n+2\sigma}$, that is, $\frac{n}{\sigma} (p - 1) - 1 > 1$ and

\[
\frac{n}{\sigma} \left(p - \frac{1}{2}\right) - 1 - \frac{n}{2\sigma} > 1.
\]

If $\max\left\{\frac{n}{\sigma} + \frac{2n}{n+2\sigma} - 1\right\} < \mu < \frac{2n}{\sigma}$ and $p \leq \frac{n}{2\sigma + 2\sigma}$, then $\mu \geq \frac{2n}{\sigma} \left(1 - \frac{1}{2p}\right)$, hence $L^q$ norm of $u$, with $2 \leq q \leq q_0$ may be estimate as in the previous case, whereas

\[
\|Fu(t, \cdot)\|_{L^q} \lesssim \int_0^t (1 + t)^{-\frac{q}{p}} (1 + s)^{1 + \frac{1}{q} - \frac{1}{p}} \left(\|u(s, \cdot)\|^p_{L^2}\right) ds \|u\|^p_{Z(T)}
\]

for all $p > 1 + \frac{2n}{n+2\sigma} > \frac{2n}{\sigma} + \frac{2n}{\sigma}$.

Finally, if $\mu > \max\left\{\frac{n}{\sigma} + \frac{2n}{n+2\sigma} - 1\right\}$, by Theorem 3.2 we have

\[
\|Fu(t, \cdot)\|_{L^q} \lesssim (1 + t)^{-\min\left\{\frac{n}{\sigma} + 1, \frac{2n}{n+2\sigma}\right\}} \int_0^t (1 + s)^{1 - \frac{2n}{q_0} - \frac{1}{q_0}} \left(\|u(s, \cdot)\|^p_{L^2}\right) ds \|u\|^p_{Z(T)}
\]

and for all $p > 1 + \frac{2n}{n+2\sigma}$.

**Proof.** (Theorem 2.1) We have to prove 3.4, 3.5 and 3.6, with $A$ as in 2.2 and $Z(T)$ as in 2.3 equipped with the norm

\[
\|u\|_{Z(T)} := \sup_{t \in [0, T]} \left\{(1 + t)^{\frac{n}{2\sigma} + \mu - 1} \|u(t, \cdot)\|_{L^2} + (1 + t)^{\frac{1}{2}} (1 + s)^{\frac{1}{2}} \|u(t, \cdot)\|_{L^{q_1}} + (1 + t)^{\min\left\{\frac{n}{\sigma} + 1, \frac{2n}{n+2\sigma}\right\}} \|u(t, \cdot)\|_{L^\infty}
\]

\[
+ (1 + t)^{\frac{1}{2}} \|u(t, \cdot)\|_{L^2} + \|(-\Delta)^{\frac{1}{2}} u(t, \cdot)\|_{L^2}\right\}
\]

Thanks to Corollary 3.2 and Theorem 3.2 $u^{\text{lin}} \in Z(T)$ and it satisfies 3.3.

Let us prove 3.5. We omit the proof of 3.4, since it is analogous to the proof of 3.5.

Let $u \in Z(T)$. If $1 - \frac{2n}{q_0} < \mu < \min\{2 - \frac{2n}{q_0}; 1\}$ by Theorem 3.1 for $q \geq 2$ we have

\[
\|Fu(t, \cdot)\|_{L^q} \lesssim \int_0^t (1 + t)^{-\frac{1}{p} + 1 - \mu} (1 + s)^{\mu} \left(\|u(s, \cdot)\|^p_{L^2}\right) ds \|u\|^p_{Z(T)}
\]

\[
\lesssim (1 + t)^{-\frac{1}{p} + 1 - \mu} \int_0^t (1 + s)^{\mu} \left(\|u(s, \cdot)\|^p_{L^2}\right) ds \|u\|^p_{Z(T)}
\]

for all $p > \frac{n+\sigma+1}{n-\sigma+1}$.
If $2 - \frac{2n}{p} < \mu < \min \{\mu; 1\}$ and $2p \leq q_1$, then $L^q$ norm of $u$, with $2 \leq q \leq q_1$ may be estimate as in the previous case, whereas

$$\|Fu(t, \cdot)\|_{L^q} \lesssim \int_0^t (1 + t)^{-\frac{\mu}{2} + (1 - \frac{\mu}{2})} \|u(s, \cdot)\|_{L^1} \|u(s, \cdot)\|_{W^{1, q}} \|u(s, \cdot)\|_{L^2} ds$$

for all $p > \frac{n + \sigma + \mu}{n - \sigma - \mu} > \frac{2\sigma + \mu}{n - \sigma - \mu}$.

Now, if $p \leq q_1 < 2p$ we use the interpolation

$$\|u\|_{L^{2p}} \leq \|u\|_{L^{\theta_1}} \|u\|_{L^{\theta_2}}, \quad \theta = q_1 / 2p. \quad (4.7)$$

By Theorem 3.1(iii) for $2 \leq q \leq q_1$ we have

$$\|Fu(t, \cdot)\|_{L^q} \lesssim \int_0^t (1 + t)^{1 - \mu - \frac{\mu}{2} + (1 - \frac{\mu}{2})} \|u(s, \cdot)\|_{L^1} \|u(s, \cdot)\|_{W^{1, q}} \|u(s, \cdot)\|_{L^2} ds$$

$$\lesssim (1 + t)^{1 - \mu - \frac{\mu}{2} + (1 - \frac{\mu}{2})} \times \int_0^t \|u(s, \cdot)\|_{L^1} \|u(s, \cdot)\|_{W^{1, q}} \|u(s, \cdot)\|_{L^2} ds$$

for all $p > p_K(n + \sigma \mu) > 1 + \frac{2n}{p}$, thanks to

$$- \frac{n(q_1 - 1)}{2\sigma} + \frac{(1 - \mu)q_1}{2} + \frac{q_1 \mu}{4} = 0$$

and

$$\mu - 1 + \frac{n}{\sigma} \left(1 - \frac{1}{q}\right) - \frac{\mu}{2} \leq \mu - 1 + \frac{2 - \mu}{2} - \frac{\mu}{2} = \frac{\mu(1 - p)}{2} < -1.$$

To estimate the $\|Fu(t, \cdot)\|_{L^\infty}$ for $\mu > 2 - \frac{2n}{\sigma}$ and $q_1 < 2p$, one may use Theorem 3.1(ii) and apply again (4.7), namely

$$\|Fu(t, \cdot)\|_{L^\infty} \lesssim \int_0^t (1 + t)^{-\frac{\mu}{2} + (1 + s)^{1 - \frac{\mu}{2}} \|u(s, \cdot)\|_{L^1} + \|u(s, \cdot)\|_{L^2} ds$$

for all $p > p_K(n + \sigma \mu) > 1 + \frac{2n}{p}$, thanks to

$$\frac{\mu}{2} + 1 - \frac{n}{\sigma} < \mu.$$

Finally, if $1 - \frac{n}{\sigma} < \mu < \min \{\mu; 1\}$, by Theorem 3.2 we have

$$\|Fu(t, \cdot)\|_{H^\sigma} \lesssim (1 + t)^{-\frac{\mu}{2} + (1 + s)^{\frac{\mu}{2}} \|u(s, \cdot)\|_{L^1} + \|u(s, \cdot)\|_{L^2} ds$$

and

$$\|\partial_t Fu(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{\mu}{2}} \|u\|_{Z(T)}^p,$$

for all $p > p_K(n + \sigma \mu)$.  

\[\square\]
5. Proof of the Non existence result via test function method

Proof. (Proposition 2.1) Let us multiply (1.1) by the function \( g(t) = g(0)(1 + t)^{\mu} \), with \( g(0) > 0 \), so that

\[
(gu)_t + (-\Delta)^{\sigma} (gu) - (g' u)_x = g(t) |u|^p.
\] (5.1)

We fix a nonnegative, non-increasing, test function \( \varphi \in \mathcal{C}^\infty_c([0, \infty)) \) with \( \varphi = 1 \) in \([0, 1/2]\) and \( \text{supp} \varphi \subset [0, 1] \), and a nonnegative, radial, test function \( \psi \in \mathcal{C}^\infty_c(\mathbb{R}^n) \), such that \( \psi = 1 \) in the ball \( B_{1/2} \), and \( \text{supp} \psi \subset B_1 \). We also assume \( \psi(x) \leq \psi(y) \) when \( |x| \geq |y| \). Here \( B_r \) denotes the ball of radius \( r \), centered at the origin. We may now apply Young inequality to estimate:

\[
\int_0^\infty \int_{\mathbb{R}^n} u (g\varphi''_R \psi_R + g' \varphi' \psi_R + g\varphi_R (-\Delta)^{\sigma} \psi_R) \, dx \, dt - g(0) \int_{\mathbb{R}^n} u_1(x) \psi_R(x) \, dx = I_R,
\] (5.4)

where:

\[
I_R = \int_0^\infty \int_{\mathbb{R}^n} g(t) |u|^p \varphi_R \psi_R \, dx \, dt.
\]

We may now apply Young inequality to estimate:

\[
\int_0^\infty \int_{\mathbb{R}^n} |u| \left( |g\varphi''_R | \psi_R + g' |\varphi'| \psi_R | + g\varphi_R |(-\Delta)^{\sigma} \psi_R| \right) \, dx \, dt \leq \frac{1}{p} I_R
\]

\[
+ \frac{1}{p'} \int_0^\infty \int_{\mathbb{R}^n} \left( \varphi_R \psi_R \right)^{-\frac{p'}{p}} \left( |\varphi''_R | \psi_R | + |\varphi_R |(-\Delta)^{\sigma} \psi_R | \right) \, dx \, dt
\]

Due to

\[
\varphi'_R(t) = R^{-\sigma} (\varphi')(R^{-\sigma} t), \quad \varphi''_R(t) = R^{-2\sigma} (\varphi'')(R^{-\sigma} t),
\]

\[-(\Delta)^{\sigma} \psi_R(x) = R^{-2\sigma} (-\Delta)^{\sigma} \psi(R^{-1} x),
\]

recalling (5.2), we may estimate

\[
\int_0^\infty \int_{\mathbb{R}^n} g(t) (\varphi_R \psi_R)^{-\frac{p'}{p}} \left| \varphi''_R \psi_R \right| \, dx \, dt \leq C R^{-2\sigma p' + n + (1 + \mu)\sigma},
\]

\[
\int_0^\infty \int_{\mathbb{R}^n} (g\varphi_R \psi_R)^{-\frac{p'}{p}} |g' \varphi' \psi_R|^\sigma \, dx \, dt \leq C R^{-2\sigma p' + n + (1 + \mu)\sigma},
\]

\[
\int_0^\infty \int_{\mathbb{R}^n} g(t) (\varphi_R \psi_R)^{-\frac{p'}{p}} \left| \varphi_R |(-\Delta)^{\sigma} \psi_R \right|^\sigma \, dx \, dt \leq C R^{-2\sigma p' + n + (1 + \mu)\sigma}.
\]

Summarizing, we proved that

\[
\frac{1}{p'} I_R \leq C R^{-2\sigma p' + n + (1 + \mu)\sigma} - g(0) \int_{\mathbb{R}^n} u_1(x) \psi_R(x) \, dx.
\]

Assume, by contradiction, that the solution \( u \) is global (in time). Recalling assumption (5.7), in the subcritical case \( p < \frac{n+\sigma+\mu}{n-\sigma+\mu} \), it follows that \( I_R < 0 \), for any sufficiently large \( R \), and this contradicts the fact that \( I_R \geq 0 \). The critical case \( p = \frac{n+\sigma+\mu}{n-\sigma+\mu} \) is treated in standard way, but we omit the details for the sake of brevity. Therefore, \( u \) cannot be a global solution (in time). \( \square \)

APPENDIX

In this section we include notations, well known results of Harmonic Analysis and properties of special functions used throughout the paper.

Notation 1. By \([x]_+\) we denote the non-negative part of \( x \in \mathbb{R} \), i.e. \([x]_+ = \max\{x, 0\}\).

Notation 2. We write \( f \lesssim g \) if there exists a constant \( C > 0 \) such that \( f \leq C g \), and \( f \approx g \) if \( g \lesssim f \lesssim g \).
Notation 4. By $L^p = L^p(\mathbb{R}^n)$, $p \in [1, \infty]$, we denote the space of measurable functions $f$ such that $|f|^p$ has finite integral over $\mathbb{R}^n$, if $p \in [1, \infty)$, or has finite essential supremum over $\mathbb{R}^n$ if $p = \infty$. We denote by $W^{m,p}$, $m \in \mathbb{N}$, the space of $L^p$ functions with weak derivatives up to the $m$-th order in $L^p$. We denote by $H^s(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$, $s \geq 0$, the spaces of tempered distributions $S'(\mathbb{R}^n)$ with $(1 + |\xi|^2)^s \hat{u} \in L^2$ and $|\xi|^s \hat{u} \in L^2$, respectively.

Notation 5. By $L^q_\mu = L^q_\mu(\mathbb{R}^n)$ we denote the space of tempered distributions $T \in S'(\mathbb{R}^n)$ such that $T \ast f \in L^q$ for any $f \in \mathcal{S}$, and

$$
\|T \ast f\|_{L^q} \leq C\|f\|_{L^p}
$$

for all $f \in \mathcal{S}$ with a constant $C$, which is independent of $f$. In this case, the operator $T \ast$ is extended by density from $\mathcal{S}$ to $L^p$.

By $M^p_\mu = M^p_\mu(\mathbb{R}^n)$, $p \leq q$, we denote the set of Fourier transforms $\hat{T}$ of distributions $T \in L^q_\mu$, equipped with the norm

$$
\|m\|_{M^p_\mu} := \sup \left\{ \|\hat{\mathcal{F}}^{-1}(m\hat{f})\|_{L^q} : f \in \mathcal{S}, \|f\|_{L^p} = 1 \right\},
$$

and we set $M^p_\mu = M^p_\mu$. A function $m$ in $M^p_\mu$ is called a multiplier of type $(p,q)$.

Now, let us introduce the Besov spaces (see [3]).

Notation 6. We fix a nonnegative function $\psi \in C^\infty$, having compact support in $\{ \xi \in \mathbb{R}^n : 2^{-1} \leq |\xi| \leq 2 \}$, such that:

$$
\sum_{k=-\infty}^{+\infty} \psi_k(\xi) = 1, \quad \text{where} \quad \psi_k(\xi) := \psi(2^{-k}\xi). \quad (5.5)
$$

(This property is easily obtained if $\psi(\xi) = \varphi(\xi/2) - \varphi(\xi)$, for some $\varphi \in C^\infty$, with $\varphi(\xi) = 1$ for $|\xi| \leq 1/2$ and $\varphi(\xi) = 0$ if $|\xi| \geq 1$). For any $p \in [1,\infty]$, we define the Besov space

$$
B^p_{p,2} = \{ f \in \mathcal{S} : \forall k \in \mathbb{Z}, \hat{\mathcal{F}}^{-1}(\psi_k \hat{f}) \in L^p, \|f\|_{B^p_{p,2}} < \infty \},
$$

where

$$
\|f\|_{B^p_{p,2}} = \|\hat{\mathcal{F}}^{-1}(\psi_k \hat{f})\|_{L^p} = \left( \sum_{k=-\infty}^{+\infty} \|\hat{\mathcal{F}}^{-1}(\psi_k \hat{f})\|_{L^p}^2 \right)^{\frac{1}{2}}.
$$

We are interested in obtain $L^p - L^q$ estimates to the solutions of the Cauchy problem (3.1). For this purpose it is used the following results about multipliers and special functions:

**Lemma 5.1** (Litman’s Lemma). Suppose that the function $v = v(\eta) \in C^\infty_0$ with support in $\{ \eta \in \mathbb{R}^n : 1/2 \leq |\eta| \leq 2 \}$ and the function $\omega = \omega(\eta) \in C^\infty$ in a neighborhood of the support of $v$. Assume $\tau_0$ a large positive number and the rank of the Hessian $H_{\omega}(\eta)$ satisfies rank $H_{\omega}(\eta) \geq k$ on the support of $v$. Then there exists an integer number $L$, such that for all $\tau \geq \tau_0$ holds

$$
\left\| \hat{\mathcal{F}}_{\eta \rightarrow x}^{-1} \left( e^{-i\tau\omega(\eta)} v(\eta) \right) \right\|_{L^\infty(\mathbb{R}^n)} \lesssim (1 + \tau)^{-\frac{k}{2}} \sum_{|\alpha| \leq L} \|D^\alpha v(\eta)\|_{L^\infty(\mathbb{R}^n)}.
$$

In Proposition 2.5 of [33], one can find a simple proof of Lemma 5.1 from which it is easy to check that the statement remains valid whenever $\omega$ and $v$ depend on some parameter $t$, provided that $|\det H_{\omega}(t,\eta)| \geq c > 0$, with $c$ uniform with respect to $t$.

In [33] one can find the following result:

**Proposition 5.1** (Berstein’s inequality). Let $n \geq 1$ and $N > \frac{n}{2}$. If $f \in H^N$, then $\hat{\mathcal{F}}^{-1} m \in L^1$ and there exists a constant $C > 0$ such that

$$
\|\hat{\mathcal{F}}^{-1} m\|_{L^1} \leq C\|f\|_{L^2}^{\frac{n}{2}}\|D^N f\|_{L^2}^{\frac{n}{2}}.
$$

In [2] one can find the following properties for Bessel and Hankel functions:

**Lemma 5.2.** The function

$$
\Gamma_\gamma(\tau) = \tau^{-\gamma} J_\gamma(\tau),
$$

where $J_\gamma(\tau)$ is the Bessel function, is entire in $\gamma$ and $\tau$, in particular,

$$
|J_\gamma(\tau)| \lesssim \tau^{-\gamma}, \quad 0 < \tau < 1.
$$

$$
(5.6)
$$
The Weber’s function $Y_n(\tau)$ satisfies for every integer $n$

$$Y_n(\tau) = \frac{2}{\pi} J_n(\tau) \ln \tau + A_n(\tau),$$

where $\tau^n A_n(\tau)$ is entire, non-null for $\tau = 0$ and

$$|A_n(\tau)| \lesssim \tau^{-n}, \quad 0 < \tau < 1. \quad (5.7)$$

The Hankel functions $H_\gamma^\pm = J_\gamma \pm iY_\gamma$ satisfy

$$2(H_\gamma^\pm)'(\tau) = H_{\gamma-1}^\pm(\tau) - H_{\gamma+1}^\pm(\tau), \quad \text{and} \quad \tau(H_\gamma^\pm)'(\tau) = \tau H_{\gamma-1}^\pm(\tau) - \gamma H_{\gamma+1}^\pm(\tau).$$

Moreover, $H_\gamma^\pm(\tau), \tau \geq K$ can be written as

$$H_\gamma^\pm(\tau) = e^{\pm i\gamma} a_\gamma^\pm(\tau), \quad (5.8)$$

where $a_\gamma^\pm(\tau) \in S^{\frac{-1}{2}}(K, \infty)$ is a classical symbol of order $-\frac{1}{2}$.

For small arguments $0 < \tau < K < 1$ we have

$$|H_\gamma^\pm(\tau)| \lesssim \begin{cases} |\tau^{-\gamma'}|, & \text{if } \gamma \neq 0 \\ -\ln(\tau), & \text{if } \gamma = 0. \end{cases} \quad (5.9)$$

REFERENCES

[1] P. Brenner, On $L_p - L_{p'}$ estimates for the wave equation. Math. Z. 145 (1975), 251–254.
[2] H. Bateman, A. Erdélyi, Higher Transcendental Functions, Vol. II, MacGraw-Hill Book Company, Inc., 1953.
[3] D’Abbicco M. The threshold of effective damping for semilinear wave equation, Math. Methods Appl. Sci., 38 (2015), 1032–1045.
[4] M. D’Abbicco, K. Fujiwara, A test function method for evolution equations with fractional powers of the Laplace operator, arXiv:2005.12056.
[5] M. D’Abbicco, Lucente S. A modified test function method for damped wave equations. Advanced Nonlinear Studies 13(2013), 867–892.
[6] M. D’Abbicco, S. Lucente, M. Reissig, A shift in the Strauss exponent for semilinear wave equations with a not effective damping, J. Differential Equations 259 (2015) 5040–5073.
[7] M. D’Abbicco, S. Lucente, M. Reissig, Semi-linear wave equations with effective damping, Chin. Ann. Math. Ser. B 34 (2013), 345–380.
[8] M. R. Ebert, G. Girardi, M. Reissig, Critical regularity of nonlinearities in semilinear classical damped wave equations, J. Math. Soc. Japan, 71 (2019), 153–177.
[9] M. R. Ebert, M. Reissig, Methods for partial differential equations, Qualitative Properties of Solutions, Phase Space Analysis, Semilinear Models (Birkhäuser/Springer, Cham, 2018).
[10] H. Fujita, On the blowing up of solutions of the Cauchy Problem for $u_t = \Delta u + u^1+\alpha$, J. Fac. Sci. Univ. Tokyo, 13 (1966), 109–124.
[11] R. T. Glassey, Finite-time blow-up for solutions of nonlinear wave equations. Math. Z. 177, no. 3, 323–340 (1981).
[12] V. Georgiev, H. Lindblad, C.D. Sogge, Weighted Strichartz estimates and global existence for semilinear wave equations, Amer. J. Math. 119 (1997) 1291–1319.
[13] R. T. Glassey, Existence in the large for $u_t = F(u)$ in two space dimensions. Math. Z. 178, no. 2, 233–261 (1981).
[14] R. Kubo, Slowly decaying solutions for semilinear wave equations in odd space dimensions, Nonlinear Anal., 28 (1997), 327–357.
[15] R. Kubo, Long-time existence for small amplitude semilinear wave equations. Amer. J. Math. 118, no. 5, 1047–1135 (1995).
[16] I. Kukavica, I. Lasiecka, Uniform decay rates for the solution of the linear thermoelastic plate equation in unbounded domains, J. Differential Equations 137 (1997), 103–154.
[17] T. Nara, Blow-up of solutions for nonlinear wave equations with damping, J. Differential Equations 129 (1996), 56–81.
[18] T. Nara, Blow-up of solutions for nonlinear wave equations with damping, J. Differential Equations 129 (1996), 56–81.
[19] T. Nara, Blow-up of solutions for nonlinear wave equations with damping, J. Differential Equations 129 (1996), 56–81.
[20] T. Nara, Blow-up of solutions for nonlinear wave equations with damping, J. Differential Equations 129 (1996), 56–81.
[26] K. Nishihara, $L^p-L^q$ estimates for solutions to the damped wave equations in 3-dimensional space and their applications, Math. Z. 244, 631–649 (2003).

[27] A. Palmieri, Linear and non-linear sigma-evolution equations. Master thesis, University of Bari (2015), 117pp.

[28] A. Palmieri, Global existence of solutions for semi-linear wave equation with scale-invariant damping and mass in exponentially weighted spaces, J Math Anal Appl. 461 (2018), 1215–1240.

[29] A. Palmieri, M. Reissig, Semi-linear wave models with power non-linearity and scale-invariant time-dependent mass and dissipation, II, Math. Nachr. 291 (2018), 1859–1892.

[30] A. Palmieri, M. Reissig, A competition between Fujita and Strauss type exponents for blow-up of semi-linear wave equations with scale-invariant damping and mass, J. Differential Eq. 266 (2019), 1176–1220.

[31] W. Sickel, H. Triebel, Hölder inequalities and sharp embeddings in function spaces of $B^{s}_{p,q}(\mathbb{R}^n)$ and $F^{s}_{p,q}(\mathbb{R}^n)$ type. Z. Anal. Anwendungen 14 (1995), 105–140.

[32] T. C. Sideris, Nonexistence of global solutions to semilinear wave equations in high dimensions. J. Differential Eq. 52, no. 3, 378–406 (1984).

[33] S. Sjöstrand. On the Riesz means of the solutions of the Schrödinger equation. Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, 24 (1970), 331–348.

[34] E. M. Stein and Rami Shakarchi, Functional analysis, Introduction to further topics in analysis, volume 4 of Princeton Lectures in Analysis. Princeton University Press, Princeton, NJ, 2011.

[35] W. A. Strauss, Nonlinear scattering theory at low energy. J. Funct. Anal. 41, no. 1, 110–133 (1981).

[36] G. Todorova, B. Yordanov, Critical Exponent for a Nonlinear Wave Equation with Damping, J. Differential Equations 174, 464–489 (2001).

[37] H. Triebel. Theory of function spaces. Basel, Birkhäuser, 1983.

[38] Y. Wakasugi, Critical exponent for the semilinear wave equation with scale invariant damping, in: M. Ruzhansky, V. Turunen (Eds.), Fourier Analysis, in: Trends Math., Springer, Basel, 2014, pp. 375–390.

[39] J. Wirth, Solution representations for a wave equation with weak dissipation. Math. Meth. Appl. Sci. 27 (2004)

[40] J. Wirth, Wave equations with time-dependent dissipation II. Effective dissipation, J. Differ. Equ. 232 (2007), 74–103.

[41] Q. S. Zhang, A blow-up result for a nonlinear wave equation with damping: the critical case, C. R. Acad. Sci. Paris Sér. I Math. 333, 109–114 (2001).