Non-stationarity in financial time series: Generic features and tail behavior

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Abstract - Financial markets are prominent examples for highly non-stationary systems. Sample averaged observables such as variances and correlation coefficients strongly depend on the time window in which they are evaluated. This implies severe limitations for approaches in the spirit of standard equilibrium statistical mechanics and thermodynamics. Nevertheless, we show that there are similar generic features which we uncover in the empirical multivariate return distributions for whole markets. We explain our findings by setting up a random matrix model.

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The great success of statistical mechanics and thermodynamics is borne out by their ability to characterize, in the equilibrium, large systems with many degrees of freedom in terms of a few state variables, for example temperature and pressure. Ergodicity (or quasi-ergodicity) is the prerequisite needed to introduce statistical ensembles. Systems out of equilibrium or, more generally, non-stationary systems still pose fundamental challenges [1–4].

Complex systems — the term “complex” is used in a broad sense — show a wealth of different aspects which can be traced back to non-stationarity [5,6]. Financial markets are presently in the focus, because they demonstrated their non-stationarity in a rather drastic way during the recent years. To assess a financial market as a whole, the correlations between the prices of the individual stocks are of crucial importance [7–10]. They fluctuate considerably in time, e.g., because the market expectations of the traders change, the business relations between the companies change, particularly in a state of crisis, and so on.

The motion of the stock prices is in this respect reminiscent of that of particles in many-body systems such as heavier atomic nuclei. Depending on the excitation energy, the motion of the individual particles can be incoherent, i.e., uncorrelated in the above terminology, or coherent (collective), i.e., correlated, or even somewhere in-between [11–13]. This non-stationarity on the energy scale leads to very different spectral properties, see [12,13]. Such an analogy can be helpful, but we do not want to overstretch it.

Here, we want to show that the non-stationarity, namely the fluctuation of the correlations, induces generic features in financial time series. These become visible when looking at quantities which measure the multivariate stock price changes for the entire market. We have four goals. First, we carry out a detailed data analysis revealing the generic features. Second, we set up a random matrix model to explain them. Third, we demonstrate that the non-stationarity of the correlations leads to heavy tails in the multivariate return distribution. Fourth, we argue that our approach maps a non-invariant situation to an effectively invariant one. For an economic audience we discuss the consequences for portfolio management elsewhere [14].

Consider K companies with stock prices \( S_k(t), k = 1, \ldots, K \) as functions of time \( t \). The relative price changes over a fixed time interval \( \Delta t \), i.e., the returns

\[
r_k(t) = \frac{S_k(t + \Delta t) - S_k(t)}{S_k(t)}
\]

are well known to have distributions with heavy tails, the smaller \( \Delta t \), the heavier. This is linked to the also well-established fact that the sample standard deviations \( \sigma_k \), referred to as volatilities, strongly fluctuate for different time windows of the same length \( T \) [15,16], as shown in fig. 1. This non-stationarity is not grasped by the conventional description of stock prices using Brownian motion [17], because the latter assumes a constant volatility. Other schematic processes such as the
The dataset is given by $M$, the time series (generalized) autoregressive conditional heteroscedasticity from 1992 to 2012. The return interval is $\Delta t = 1$ trading day and the time window has length $T = 60$ trading days.

Surprising, as recent studies\cite{20,21} show that the heavy parameters lack a clear economic interpretation. This is not too surprising, as recent studies\cite{20,21} show that the heavy tails result, among other reasons, from the very procedure of how stock market trading takes place.

Another equally fundamental non-stationarity of stock market data is the time-dependent fluctuation of the sample Pearson correlation coefficients

$$C_{kl} = \langle M_k(t)M_l(t) \rangle_T,$$

$$M_k(t) = \frac{r_k(t) - \langle r_k(t) \rangle_T}{\sigma_k}, \quad (2)$$

between the two companies $k$ and $l$ in the time window of length $T$, where $\sigma_k$ is evaluated in the same time window. The time series $M_k(t)$ are normalized to zero mean and unit variance. To illustrate how strongly the $K \times K$ correlation matrix $C$ as a whole changes in time, we show it for subsequent time windows in fig. 2. The dataset used here and in all analyses in the sequel consists of $K = 306$ continuously traded companies in the SP500 index between 1992 and 2012 [22]. For later discussion, we emphasize that the stripes in these correlation matrices indicate the structuring of the market in industrial sectors, see, e.g., ref. [9]. We will also use the covariance matrix $\Sigma = \sigma C \sigma$ where the diagonal matrix $\sigma$ contains the volatilities $\sigma_k$, $k = 1, \ldots, K$.

We now show that the returns are to a good approximation multivariate Gaussian distributed, if the covariance matrix $\Sigma$ is fixed. Hence, we assume that the distribution of the $K$ dimensional vectors $r(t) = (r_1(t), \ldots, r_K(t))$ for a fixed return interval $\Delta t$ while $t$ is running through the dataset is given by

$$g(r|\Sigma) = \frac{1}{\sqrt{\text{det}(2\pi \Sigma)}} \exp \left( -\frac{1}{2} r^t \Sigma^{-1} r \right), \quad (3)$$

where we suppress the argument $t$ of $r$ in our notation. We test this assumption with the SP500 dataset. We divide the time series in windows of length $T$ (not to be confused with the return intervals of length $\Delta t$) which are so short that the sampled covariances can be viewed as constant within these windows. However, the corresponding covariance matrices $\Sigma$ are non-invertible because their rank is lower than $K$. Mathematically, this is not a problem, because the distribution (3) is still well defined in terms of proper $\delta$ functions. To carry out the data analysis, we take all pairs $r_k, r_l$ of returns which, according to our assumption, should be bivariate Gaussian distributed with a $2 \times 2$ covariance matrix $\Sigma^{(k,l)}$, which always is invertible. We rotate the two-component vectors $(r_k, r_l)$ into the eigenbasis of $\Sigma^{(k,l)}$, and normalize the axes with the eigenvalues. Then the components become comparable and can be aggregated into a single univariate distribution.

Figure 3 shows the results for daily returns and a window length of $T = 25$ trading days. We observe a good agreement with a Gaussian. Again, we emphasize that we require this agreement on small time horizons only. This is the case in the dataset.

However, as argued above, the covariances change in time, i.e., the covariance matrix is not constant. Thus there is no contradiction to the heavy-tailed distributions observed for individual stocks over longer time horizons. Here, we are not aiming at making statements about individual stocks. Rather, we focus on the multivariate return
distribution for the whole correlated market. Our idea is now to take the non-stationarity of the covariance matrix into account by replacing the fixed covariance matrix with a random matrix,

\[ \Sigma \rightarrow \frac{1}{N} A A^\dagger, \quad (4) \]

with \( A \) being a rectangular \( K \times N \) real random matrix without any symmetries and with the dagger indicating the transpose. The product form \( A A^\dagger/N \) is essential to model a proper covariance matrix. This is seen from the definition (2) implying that \( C = M M^\dagger/T \) where the rectangular \( K \times T \) data matrix \( M \) contains the normalized time series \( M_k(t) \) as rows. Hence, when viewing the rows of \( A \) as model time series, their length is \( N \), not \( T \). This important ingredient of our model and the meaning of \( N \), which is arbitrary at the moment, will be discussed later on. What is the probability distribution of the random matrices \( A \)? Following Wishart \([23,24]\), we assume the Gaussian

\[ w(A|\Sigma) = \frac{1}{\text{det}^{N/2}(2\pi \Sigma)} \exp \left(-\frac{1}{2} \text{tr} A^\dagger \Sigma^{-1} A \right), \quad (5) \]

where \( \Sigma \) is now the empirical covariance matrix evaluated over the entire time interval, i.e., it is fixed. The Wishart covariance matrices \( A A^\dagger/N \) fluctuate around the sampled empirical one. By definition, one has \( \langle A A^\dagger \rangle/N = \Sigma \), where the angular brackets without an index \( T \) always indicate the average over the ensemble defined by the distribution (5). The parameter \( N \) acquires the meaning of an inverse variance characterizing the fluctuations around \( \Sigma \). The larger \( N \), the more terms contribute to the individual matrix elements of \( A A^\dagger/N \), eventually making them sharp for \( N \to \infty \).

In our model, the fluctuating covariances alter the multivariate Gaussian (3), implying the introduction of an ensemble averaged return distribution

\[ \langle g(r|\Sigma, N) \rangle = \int g \left(r \bigg| \frac{1}{N} A A^\dagger \right) w(A|\Sigma) d[A], \quad (6) \]

which parametrically depends on the fixed empirical covariance matrix \( \Sigma \) as well as on \( N \). The construction (6) states the most important conceptual point of our study. The measure \( d[A] \) is simply the product of all differentials. To calculate this ensemble averaged return distribution, we write the Gaussian (3) as a Fourier transform in terms of a \( K \) component vector \( \omega \). The ensemble average is then Gaussian, leading to

\[ \langle g(r|\Sigma, N) \rangle = \frac{1}{\text{det}^{N/2}(2\pi \Sigma)} \int \text{det}^{K/2}/(2\pi)^K \exp(-i\omega \cdot r) \]

\[ \times \int d[A] \exp \left(-\frac{1}{2} \text{tr} A^\dagger \Sigma^{-1} A \right) \]

\[ \times \exp \left(-\frac{1}{2N} A^\dagger \Sigma A \right), \]

\[ = \frac{1}{(2\pi)^K} \int \text{det}^{N/2}(1_K + \Sigma \omega^\dagger/N) \]  

\[ \langle g(r|\Sigma, N) \rangle = \frac{1}{2^{N/2}/\sqrt{\Gamma(N/2)\Gamma(K)} \sqrt{\text{det}(2\pi \Sigma)/N}} \]

\[ \times \frac{\text{exp}(-i\omega \cdot r)\text{det}(2\pi \Sigma)/N)}{\sqrt{N^2/\sqrt{\text{det}(2\pi \Sigma)/N}}}, \quad (7) \]

The matrix adding to the \( K \times K \) unit matrix \( 1_K \) in the determinant has rank unity, implying that the whole determinant is equal to the positive definite quantity \( 1 + \omega^\dagger \Sigma \omega/N \). Furthermore, expressing the use the expression

\[ \frac{1}{a^\eta} = \frac{1}{2^{\eta/2} \Gamma(\eta)} \int_0^\infty z^{\eta-1} \exp \left(-\frac{z}{2} \right) dz \]

for real and positive variables \( a \) and \( \eta \), we find

\[ \langle g(r|\Sigma, N) \rangle \]

\[ \times \int \text{det}^{K/2}/\sqrt{\text{det}(2\pi \Sigma)/N} \]

\[ \times \exp(-i\omega \cdot r - z/N^2/\sqrt{\sqrt{\text{det}(2\pi \Sigma)/N}}) \]. \quad (9) \]

The \( \omega \) integral yields the multivariate Gaussian of the form (3), but now with the covariance matrix \( z \Sigma/N \). Hence, we arrive at the remarkable expression

\[ \langle g(r|\Sigma, N) \rangle = \int_0^\infty \chi^2_N(z) \exp \left(-\frac{z}{2} \right) dz \]

\[ = \frac{1}{2^{N/2}/\sqrt{\Gamma(N/2)\Gamma(K)} \sqrt{\text{det}(2\pi \Sigma)/N}} \]

\[ \times \frac{\text{det}(2\pi \Sigma)/N)}{\sqrt{N^2/\sqrt{\text{det}(2\pi \Sigma)/N}}}, \quad (10) \]

which maps the whole random matrix average to a one-dimensional average involving the \( \chi^2 \) distribution of \( N \) degrees of freedom,

\[ \chi^2_N(z) = \frac{1}{2^{N/2}/\sqrt{\Gamma(N/2)\Gamma(K)} \sqrt{\text{det}(2\pi \Sigma)/N}} \]

\[ \times \exp(-i\omega \cdot r - z/N^2/\sqrt{\sqrt{\text{det}(2\pi \Sigma)/N}}) \]. \quad (11) \]

for \( z \geq 0 \) and zero otherwise. The parameter \( N \) can now be interpreted as an effective number of degrees of freedom characterizing the ensemble induced by the fluctuations of the covariance matrices. In fact, the Wishart distribution can be viewed as a matrix generalized \( \chi^2 \) distribution. The integral (10) can be done in closed form,

\[ \langle g(r|\Sigma, N) \rangle = \frac{1}{2^{N/2}/\sqrt{\Gamma(N/2)\Gamma(K)} \sqrt{\text{det}(2\pi \Sigma)/N}} \]

\[ \times \frac{\text{det}(2\pi \Sigma)/N)}{\sqrt{N^2/\sqrt{\text{det}(2\pi \Sigma)/N}}}, \quad (12) \]

where \( K_\nu \) is the modified Bessel function of the second kind of order \( \nu \). In the data analysis below, we will find \( K > N \). Since the empirical covariance matrix \( \Sigma \) is fixed, \( N \) is the only free parameter in the distribution (12). For large \( N \) it approaches a multivariate Gaussian. The smaller \( N \), the heavier the tails, for \( N = 2 \) the distribution is exponential. Importantly, the returns enter \( \langle g(r|\Sigma, N) \rangle \) only via the bilinear form \( r^\dagger \Sigma r \). This high degree of invariance is due to the invariance of the Wishart distribution (5). Such features are common to all random matrix models. The derivation and the result (12) extend and generalize our previous study [25] by allowing arbitrary correlation structures. There we aimed at rough estimates for another purpose concerning credit risk, and we did not carry out a data analysis. Hence, the decisive question is, if our new result describes the data.
To test this, we rotate the vector $r$ into the eigenbasis of the covariance matrix $\Sigma$, and normalize its elements with the eigenvalues. Integrating out all but one of the components of the rotated vector, which we denote $\tilde{r}$, we find

$$\langle \tilde{r} \rangle = \frac{\sqrt{2}^{-N} \sqrt{N} N^{(N-1)/2} \Gamma(N/2)}{\sqrt{N} \Gamma(N/2) \Gamma(K/2)}.$$

This formula is compared in fig. 4 with the aggregated distributions evaluated for the entire SP500 dataset from 1992 to 2012, i.e. $T = 5275$ days. We determine the parameter $N$ with a Cramer-von Mises test [26] considering only integer values and find $N = 5$ for daily returns and $N = 14$ for $\Delta t = 20$ trading days. We find a good agreement between model and data. Importantly, the distributions have heavy tails which result from the fluctuations of the covariances, the smaller $N$, the heavier. For small $N$ there are deviations between theory and data in the tails. However, our theory describes the center of the empirical distribution much better than a Gaussian. Figure 5 shows that $N$ increases monotonically with the return interval $\Delta t$. This *ad hoc* Gaussian choice (5) is justified *a posteriori* as it leads to a good description of the data.

As already mentioned, our random matrix model has invariances. On the other hand, fig. 2 clearly shows that the empirical ensemble of correlation matrices has inner structures and is thus not at all invariant. Interestingly, our *invariant* random matrix model can handle this. The effective number of degrees of freedom $N$ also contains information about these inner structures.

The determination of the parameter $N$ by fitting is strongly corroborated by another, independent check: the expectation value for the bilinear form $\langle r^\dagger \Sigma^{-1} r \rangle$ reads

$$\langle r^\dagger \Sigma^{-1} r \rangle = \frac{2 \Gamma((N + 1)/2) \Gamma((K + 1)/2)}{\sqrt{N} \Gamma(N/2) \Gamma(K/2)}.$$

Comparing this result to the average value of the empirical bilinear form yields a value of $N$ which is consistent with the fitted value.

Our formula (10) can be viewed as an average of the Gaussian with argument $\sqrt{N} r^\dagger \Sigma^{-1} r$ over the variance $z$ which is $\chi^2$ distributed. It is thus a certain non–uniform average of Gaussians. Procedures of the type (10), i.e., construction of a new distribution by averaging a parameter of a distribution, are known as *compounding* [27] or *mixture* [28,29] in the statistics literature. Starting from a compounding ansatz, the distribution (12) was found recently in ref. [30] when studying scattering of microwaves in random potentials. In this case, however, the argument was simply the squared intensity and not a bilinear form. To the best of our knowledge, we have given here, in the context of finance, the first *explanation* of such a compounding ansatz by deriving it from fluctuating covariances.

Random matrix models are widely used in quantum chaos, many-body and mesoscopic physics and in related fields, see ref. [13] for a review. The systems in question are quantum dynamical systems defined by a Hamiltonian. Thus, there is a fundamental difference to be underlined when comparing to systems such as financial markets. Of course, the latter are also dynamical systems, but of a very different nature. In particular, there is nothing like the mean level spacing in quantum chaotic or mesoscopic systems which sets the scale when investigating spectral fluctuations. Thus, there cannot be the type of universality as discussed so much in the context of random matrix models based on a Hamiltonian. In view of this, it is even more encouraging that our present random matrix model,
which is just based on the Gaussian Wishart distribution, leads to a quantitative description of the heavy-tailed multivariate return distributions. Another important caveat is in order. When measuring correlations, the effect of noise-dressing leads to an eigenvalue density for individual correlation matrices which is consistent with Wishart random matrices [31,32]. This is not related to our study. We model fluctuating correlations by an ensemble of covariance or correlation matrices. The only correlation matrix which we directly extract from the data is for the entire observation period. Measurement noise is thus negligible.

The deviations in the tails between the data and our model, as seen in fig. 4, seem to have two possible explanations: they could be due to a non-Gaussian shape of the multivariate return distribution or of the covariance matrix distribution. This will be studied in future work.

In summary, we have shown that the fluctuations of correlations and covariances induce generic properties for the multivariate return distribution of the corresponding market. We uncovered them by constructing a random matrix model which reduces the high complexity of a whole correlated market to one single parameter that characterizes the fluctuations. The model describes the SP500 dataset from 1992 to 2012 well. Although this finding is reminiscent of equilibrium statistical mechanics, we emphasize once more that we treated a substantially non-stationary system. Our model is capable of describing the empirical non-invariant ensembles of covariance matrices in terms of an invariant random matrix ensemble. It also quantitatively explains how the fluctuations yield heavy tails.

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