Directed Intersection Representations and the Information Content of Digraphs

Alexandr V. Kostochka and Xujun Liu
Department of Mathematics
University of Illinois, Urbana-Champaign
Urbana, Illinois 61801
Email: kostochk@math.uiuc.edu, xliu150@illinois.edu

Roberto Machado and Olgica Milenkovic
Department of Electrical and Computer Engineering
University of Illinois, Urbana-Champaign
Urbana, Illinois 61808
Email: robertomachado@ime.unicamp.br, milenkov@illinois.edu

Abstract—Consider a directed graph (digraph) in which two user vertices are connected if and only if they share at least one unit of common information content and the head vertex has a strictly smaller content than the tail. We seek to estimate the smallest possible global information content that can explain the observed digraph topology. To address this problem, we introduce the new notion of a directed intersection representation of a digraph, and show that it is well-defined for all directed acyclic graphs (DAGs). We then proceed to describe the directed intersection number (DIN), the smallest number of information units needed to represent the DAG. Our main result is a nontrivial upper bound on the DIN number of DAGs based on the longest terminal path decomposition of the vertex set. In addition, we compute the exact values of the DIN number for several simple yet relevant families of connected DAGs and construct digraphs that have near-optimal DIN values.

I. INTRODUCTION

Consider the following problem, illustrated by a small-scale directed graph depicted in Figure 1. Assume that the vertices \( A, B, C, D \) correspond to web-pages that contain different collections of topics or files, represented by color-coded rectangles. Two web-pages are linked to each other if they have at least one topic in common. This is a natural generative assumption, which has been exploited in a number of data mining contexts [1, 2]. For a directed graph, in addition to a shared content assumption one needs to explain the direction of the links, i.e., which vertex in the arc represents the head and which vertex in the arc represents the tail. In the context of web-page linkages, it is reasonable to assume that a webpage links to another terminal webpage if the latter covers more topics, i.e., contains additional information compared to the source page. In Figure 1, the link between web-pages \( A \) and \( B \) is directed from \( A \) to \( B \), since \( B \) lists three topics, while \( A \) lists only two. Hence, we assume two generative constraints for the existence of a directed edge: shared information content and content size dominance.

In many practical settings, one is only presented with the directed graph topology of a directed graphs and asked to determine the latent vertex content leading to the observed topology. A problem of particular interest is to determine the smallest topic/information content that explains the observed digraph. This question may be formally described as follows. Let \( D = (V, A) \) be a directed graph with vertex set \( V \) and arc set \( A \), and assume that each vertex \( v \in V \) is associated with a subset \( \varphi(v) \) of a finite ground set \( C \), termed the color set, such that \( (u, v) \in A \) if and only if \( |\varphi(u) \cap \varphi(v)| \geq 1 \) and \( |\varphi(u)| < |\varphi(v)| \) (i.e., two vertices share an arc if their color sets intersect and the color set of the head is strictly smaller than the color set of the tail). If such a representation is possible, we refer to it as a directed intersection representation. The question of interest is to determine the smallest cardinality of the ground set \( C \) which allows for a directed intersection representation of a digraph \( D \) with \( |V| = n \) vertices, henceforth termed the directed intersection number of \( D \). Clearly, not all digraphs allow for such a representation. For example, a directed triangle \( D(V, A) \) with \( V = \{1, 2, 3\} \) and \( A = \{(1, 2), (2, 3), (3, 1)\} \) does not admit a directed intersecting representation, as such a representation would require \( |\varphi(1)| < |\varphi(2)| < |\varphi(3)| < |\varphi(1)| \), which is impossible. The same is true of every digraph that contains cycles, but as we subsequently show, every acyclic directed graph (DAG) admits a directed intersection representation. We focus on connected DAGs, although our results apply to unconnected graphs with either no or some small modifications.

The problem of finding directed intersection representations of digraphs is closely associated with the intersection representation problem for undirected graphs. Intersection representations are of interest in many applications such as keyword
II. REPRESENTATIONS OF DIRECTED ACYCLIC GRAPHS

We use the notation and terminology described below. Whenever clear from the context, we omit the argument \( n \).

The in-degree of a vertex \( v \) is the number of arcs for which \( v \) is a tail, while the out-degree is the number of arcs for which \( v \) is the head. The set of in-neighbors of \( v \) is the set of vertices sharing an arc with \( v \) as the tail, and is denoted by \( N^{-}(v) \). The set of out-neighbors \( N^{+}(v) \) is defined similarly.

For a given acyclic digraph \( D(V,A) \), let \( \Gamma : V \rightarrow \mathbb{N} \) be a mapping that assigns to each vertex \( v \in V \) the length of the longest directed path that terminates in \( v \). The map \( \Gamma \) induces a partition of the vertex set \( V \) into levels \((V_{0}, \ldots, V_{\ell})\), such that \( V_{i} = \{ v \in V : \Gamma(v) = i \} \). Clearly, there is no arc between any pair of vertices \( u \) and \( v \) at the same level \( V_{i} \), \( i = 1, \ldots, \ell \), as this would violate the longest path partitioning assumption. Note that although the longest path problem is NP-hard for general graphs, it is linear time for DAGs, and based on topological sorting [8].

**Lemma II.1.** Every DAG \( D(V,A) \) on \( n \) vertices admits a directed intersection representation. Moreover, one has \( \text{DIN}(n) \leq \left[ \frac{n^{2}}{2} \right] \).

**Proof.** Let \( V = V_{0} \cup V_{1} \cup \ldots \cup V_{\ell} \) describe the longest path partition of the vertex set of a DAG \( D \), with \( \ell \leq n \). Assume that the vertices in \( V_{0}, V_{1}, \ldots, V_{\ell} \) are labeled in an arbitrary fashion, so that \( V_{i} = \{ v_{i}^{1}, \ldots, v_{i}^{n_{i}} \} \) and \( n_{i} = |V_{i}| \). The initial color sets of vertices are empty and we add colors as follows. We start from \( v_{1}^{1} \in V_{1} \) and examine its in-neighbors \( N^{-}(v_{1}^{1}) \). We assign a new color, say \( c \), to each vertex \( v \in N^{-}(v_{1}^{1}) \) to represent the edge \((v, v_{1}^{1})\) and subsequently add the same color \( c \) to the list of colors assigned to \( v_{1}^{1} \). This process is repeated for the remaining vertices in \( V_{1} \), and subsequently, for all vertices in \( V_{i} \), \( 2 \leq i \leq \ell \). During the assignment, we used up at most \( n(n-1) \) colors since each vertex can have at most \( n-1 \) in-neighbors and there are at most \( n \) vertices in \( V_{1} \cup \ldots \cup V_{\ell} \). In addition, the size of the color set \( \varphi(v) \) of any vertex \( v \) at this stage is at most \( n-1 \). To satisfy the size constraint in the definition of a directed intersection, we add at most \( n \) colors to the lists of colors of vertices at level \( V_{i} \), for each \( 1 \leq i \leq \ell \), so that all vertices at the same level have color sets of the same cardinality \( n \). Note that these colors may be reused at each level \( 1 \leq i \leq \ell \). We then add \( i \) new colors to the color sets \( \varphi(v_{i}^{j}) \) of all vertices \( v_{i}^{j} \in V_{i} \), with the color sets being the same for each vertex in \( V_{i} \), but disjoint for different labels \( i \) and \( j \). As a result, all vertices in \( V_{i} \) have color sets of size \( n+i \), and consequently, the size condition is satisfied. During the second stage of the color set augmentation procedure, we added at most \( 1+2+\ldots+\ell-1 = \frac{\ell(\ell-1)}{2} \) new colors. Hence, the total number of different colors used is upper bounded by \( n(n-1) + \ell(n-1) + \frac{\ell(\ell-1)}{2} + n \leq \frac{3}{2} n^{2} \). This proves the claimed result. \( \square \)

Clearly, the above procedure, although simple, is clearly suboptimal. On the example of the directed rooted tree shown in Figure 3 we see that more careful book-keeping and reuse of the colors used at the different levels allows one to
Theorem III.1. Let $D = (V, A)$ be an acyclic digraph on $n$ vertices. If $n$ is even, then

$$\text{DIN}(D) \leq \frac{5n^2}{8} - \frac{n}{4}.$$  

Proof. We will prove a stronger statement that asserts that $D$ has a representation $\varphi$ such that

(a) $|\varphi(w)| \leq \frac{3}{2}n - 1$ for every vertex $w \in V$.
(b) $|\bigcup_{v \in V} \varphi(v)| \leq \frac{5n^2}{8} - \frac{n}{4}$.

The claim is easy to verify for $n = 2$ and $n = 4$. When $n = 2$, we need at most $2 \leq 2 = \frac{2}{3}2 - 1$ colors to represent a DAG and each vertex has a color set of maximum size $2 \leq 2$. When $n = 4$, we need at most $8 \leq 9 = \frac{5n^2}{8} - \frac{3}{2}$ colors to represent any DAG, and there exists a representation using 8 colors in which all vertices have a color set of size bounded by $5 \leq 5 = \frac{5}{2}4 - 1$.

We next assume that a representation of the form described above exists for all DAGs with fewer than $n$ vertices and prove the existence of the same type of representation for DAGs with $n$ vertices. We separately analyze two cases, depending on the number of sink vertices in the DAG. Note that every DAG has at least one sink.

Case 1: There are at least two sinks in $D$. Pick two sinks, say $u$ and $v$. By induction hypothesis, the digraph $D - \{u, v\}$ has a representation $\varphi$ such that

(a) $|\varphi(w)| \leq \frac{3}{2}(n - 2) - 1 = \frac{3}{2}n - 4$ for each vertex $w \in V - \{u, v\}$ and

(b) the number of colors in the representation $\varphi$ is at most

$$\frac{5(n - 2)^2}{8} - \frac{n - 2}{4} = \frac{5n^2}{8} - \frac{n}{4} - \left(\frac{5}{2}n - 3\right).$$

Next, we classify the vertices $w \in V - \{u, v\}$ in terms of $|\varphi(w)|$ and extend $\varphi$ to a representation $\varphi'$ of $D$ using at most $\frac{3}{2}n - 3$ new colors. The procedure goes through the following steps.

1) Initialize the color assignments by setting $\varphi'(u) = \varphi'(v) = \emptyset$.

2) To each vertex $w \in V - \{u, v\}$ such that $(w, u) \in A$ and $(w, v) \notin A$, assign a distinct new color (i.e., a color not used in $\varphi$), say $c_{uw}$, to represent the arc, i.e.,

$$\varphi'(w) = \varphi(w) \cup \{c_{uw}\} \quad \text{and} \quad \varphi'(u) = \varphi'(u) \cup \{c_{uw}\}.$$

3) To each vertex $w \in V - \{u, v\}$ such that $(w, v) \in A$ and $(w, u) \notin A$, assign a distinct new color $c_{uw}$ to represent the arc, i.e.,

$$\varphi'(w) = \varphi(w) \cup \{c_{uw}\} \quad \text{and} \quad \varphi'(v) = \varphi'(v) \cup \{c_{uw}\}.$$

4) To each vertex $w \in V - \{u, v\}$ such that both $(w, u) \in A$ and $(w, v) \in A$, assign a distinct new color $c_{uw}$ to represent both arcs, i.e.,

$$\varphi'(w) = \varphi(w) \cup \{c_{uw}\}, \quad \varphi'(u) = \varphi'(u) \cup \{c_{uw}\}, \quad \text{and} \quad \varphi'(v) = \varphi'(v) \cup \{c_{uw}\}.$$

5) To each vertex $w \in V - \{u, v\}$ such that $(w, v) \notin A$ and $(w, u) \notin A$, assign a distinct new color $c_w$ to satisfy the size condition, i.e.,

$$\varphi'(w) = \varphi(w) \cup \{c_w\}.$$
Denote the number of colors introduced in 2) by $m_2$, the number of colors introduced in 3) by $m_3$, in 4) by $m_4$, and in 5) by $m_5$. Without loss of generality, assume that $m_2 \leq m_3$. Since
\[ m_2 + m_3 + m_4 \leq n - 2, \]
we need to add colors to $\varphi'(u)$ and $\varphi'(v)$ to obtain two color sets of size $\frac{3}{2}n - 1$. Let $C$ be a set of $\frac{3}{2}n - 1 - m_2 - m_4$ distinct new colors and let $C' \subseteq C$ be a set of colors of size $\frac{3}{2}n - 1 - m_3 - m_4$. Furthermore, let $\varphi'(u) = \varphi'(u) \cup C$ and $\varphi'(v) = \varphi'(v) \cup C'$.

We hence introduced a total of
\[ m_5 + m_2 + m_3 + m_4 + \frac{3}{2}n - 1 - m_2 - m_4 \]
\[ = \frac{3}{2}n - 1 + m_5 + m_3 \leq \frac{3}{2}n - 1 + n - 2 = \frac{5}{2}n - 3 \]
new colors.

Claim III.2. The sets $\varphi'$ provide a valid directed intersection representation.

Proof. We validate the claim through the following observations:

i) For vertices $w_1, w_2 \in V - \{u, v\}$, we added a distinct new color to $w_1$ and another distinct new color to $w_2$. Hence, if $\varphi$ is valid, then $\varphi'$ is valid as well.

ii) For vertices $w_1, w_2$ with $w_1 \in V - \{u, v\}$ and $w_2 \in \{u, v\}$, we have two cases. If $(w_1, w_2) \notin A$, then since $\varphi'(w_1) \cap \varphi'(w_2) = \emptyset$, $\varphi'$ is valid. If $(w_1, w_2) \in A$, we added a distinct new color to $w_1$ and $w_2$ to represent the arc $(w_1, w_2)$ so that the intersection condition holds. The size condition holds as well, since
\[ \varphi'(w_1) \leq \frac{3}{2}n - 4 + 1 < \frac{3}{2}n - 1 = \varphi'(w_2). \]

(iii) For the sink vertices $u$ and $v$, $\varphi'$ is valid since $|\varphi'(u)| = |\varphi'(v)| = \frac{3}{2}n - 1$ and there is no arc between $u$ and $v$.

(iv) Clearly, $|\varphi'(u)| = |\varphi'(v)| = \frac{3}{2}n - 1$, and $\varphi'(w) \leq \frac{3}{2}n - 4 + 1 \leq \frac{3}{2}n - 1$ for $w \in V - \{u, v\}$. \qed

Case 2: There is exactly one sink in $D$, say $v$. Then, if we delete $v$, we create at least one sink, say $u$, in $D - \{v\}$. This claim follows since the digraph is acyclic and $(u, v) \in A$, since otherwise $u$ would be a sink in $D$, which contradicts the assumption that there is only one sink in $D$.

Recall that by induction hypothesis, $D - \{u, v\}$ has a representation $\varphi$ such that $|\varphi(w)| \leq \frac{3}{2}n - 4$ for each vertex $w \in V - \{u, v\}$ and $\varphi$ uses at most
\[ \frac{5(n - 2)^2}{8} - \frac{n - 2}{4} = \frac{5n^2}{8} - \frac{n}{4} - \left(\frac{5}{2}n - 3\right) \]
colors. Once again, we classify each vertex $w$ in $D - \{u, v\}$ based on $|\varphi(w)|$ and extend $\varphi$ to a representation $\varphi'$ of $D$ by using at most $\frac{3}{2}n - 3$ new colors. The assignments used are the same as those used for Case 1, and hence omitted.

Claim III.3. The sets $\varphi'$ provide a valid directed intersection representation.

Proof. The cases $w_1, w_2 \in V - \{u, v\}$ and $w_1, w_2$ with $w_1 \in V - \{u, v\}$ and $w_2 \in \{u, v\}$ are analyzed in the same manner as for Case 1. The only difference arises for the following two settings:

(i) For $u$ and $v$, $\varphi'$ is valid since
\[ 1) \frac{\varphi'(u)}{\varphi'(v)} = \frac{3}{2}n - 2 < \frac{\varphi'(v)}{\varphi'(u)} = \frac{3}{2}n - 1; \]
\[ 2) \min\left\{ \frac{3}{2}n - 2 - m_2 - m_4, \frac{3}{2}n - 1 - m_3 - m_4 \right\} > 0, \]
and thus $\varphi'(u) \cap \varphi'(v) \neq \emptyset$.

(ii) We have $|\varphi'(w)| = \frac{3}{2}n - 2$, $|\varphi'(v)| = \frac{3}{2}n - 1$, and $\varphi'(w) \leq \frac{3}{2}n - 4 + 1 \leq \frac{3}{2}n - 1$ for $w \in V - \{u, v\}$.

IV. EXTREMAL DIN DIGRAPHS

Using significantly more elaborate techniques, the bound in Theorem III.1 can be further improved both in a nonconstructive and constructive manner. Our best nonconstructive upper bound asserts that for any DAG $D$ on $n$ vertices, one has
\[ DIN(D) \leq \frac{5n^2}{8} - \frac{3n}{4} + 1. \tag{1} \]

Fig. 4: Examples of DIN-extremal graphs for $n \leq 7$.

In comparison, the intersection number of any graph on $n$ vertices is upper bounded by $\frac{n^2}{4}$ [6]. Furthermore, there exist undirected graphs that meet the intersection number bound $\frac{n^2}{4}$, the existence of which can be established by observing that the intersection number of a graph is equivalent to its edge-clique cover number and by invoking Mantel’s theorem [9] that asserts that any triangle-free graph on $n$ vertices can have at most $\frac{n^2}{4}$ edges. The extremal graphs with respect to the intersection number are the well-known Turan $T(n, 2)$ graphs [10].

Consequently, the following question is of interest in the context of directed intersection representations: do there exist DAGs that meet the upper bound in (1) and which DIN values are achievable? To address this issue, we introduce the notion of DIN-extremal DAGs: a DAG on $n$ vertices is said to be DIN-extremal if it has the largest DIN among all DAGs with the same number of vertices.

Figure IV provides examples of DIN-extremal DAGs for $n \leq 7$ vertices. These graphs were obtained by combining computer simulations and proof techniques used in establishing the upper bound of (1). Direct verification for large $n$
through exhaustive search only is prohibitively complex, as the number of connected/unconnected DAGs with \( n \) vertices follows a “fast growing” recurrence \( [11] \). For example, even for \( n = 6 \), there exist 5984 different unlabeled DAGs. All the listed extremal DAGs are Hamiltonian, e.g., they contain a directed path visiting each of the \( n \) vertices exactly once.

As such, the digraphs have a unique topological order induced by the directed path, and \( |V_i| = 1 \), for \( i \in [n] \). Note that the bound in \( [11] \) for \( n = 2, 3, 4, 5, 6, 7 \) equals 2, 4, 8, 12, 19, 26, respectively. Hence, our best upper bound is loose for \( n \geq 6 \).

The DIN-extremal graphs for \( n \leq 7 \) are what we refer to as source arc-paths, illustrated in Figure 5 a),b). A source arc-path on \( n \) vertices has the following arc set
\[
A = \{ (v_1, v_{2k}) : k \in \lfloor n/2 \rfloor \} \cup \{ (v_k, v_{k+1}) : k \in [n-1] \}.
\]

It is straightforward to prove the following result.

**Proposition IV.1.** A source arc-path on \( n \) vertices has a DIN equal to \( \lfloor n^2/2 \rfloor = \lfloor 4n^2/8 \rfloor \). Hence, the DIN of source arc-paths is by \( n^2/2 \) smaller than the leading term of the upper bound \( [11] \).

**Proof.** A directed triangle in a digraph \( D = (V, A) \) is a collection of three vertices \( \{v_1, v_2, v_3\} \) such that \( (v_1, v_2) \in A, (v_2, v_3) \in A \), and \( (v_1, v_3) \in A \). Since a source arc-path avoids directed triangles and since every vertex has a color set of different size than another due to the presence of the directed Hamiltonian path, every color may be used at most twice. We need \( 2 \) colors for \( \varphi(v_1) \) to represent the arcs \( v_1v_{2i} \), where \( 1 \leq i \leq \frac{n}{2} \). Since the size of the color sets \( \varphi \) increases along the directed path, vertex \( v_j \) in the natural ordering has \( \varphi(v_j) \geq \frac{n}{2} + j - 1 \). Furthermore, \( (v_{2i}, v_{2j}) \notin A \) for a source arc-path, for all \( 1 \leq i < j \leq \frac{n}{2} \). Thus, \( \varphi(v_{2i}) \cap \varphi(v_{2j}) = \emptyset \), \( 1 \leq i < j \leq \frac{n}{2} \). This implies the number of colors needed is
\[
\geq \frac{n}{2} + \frac{n}{2} + 3 + \ldots + n - 1 = \frac{n}{2} \cdot \frac{1 + n - 1}{2} = \frac{n^2}{2} - \frac{n}{2}.
\]

To show that the above lower bound is met, we exhibit the following representation \( \varphi \) with \( \frac{n^2}{2} \) colors.

1. \( \varphi(v_1) = \{c_1, \ldots, c_{\frac{n}{2}}\} \), \( \varphi(v_2) = \{c_{\frac{n}{2}+1}, g_1, \ldots, g_{\frac{n}{2}+1}\} \).

2. For \( 2 \leq i \leq \frac{n}{2} - 1 \),
   \[
   \varphi(v_{2i}) = \{c_i, d_i, f_i, g_{1,i}, \ldots, g_{\frac{n}{2}+2i-1}, i\}.
   \]

3. For \( 2 \leq i \leq \frac{n}{2} - 1 \),
   \[
   \varphi(v_{2i-1}) = \{d_i, f_{i-1}, g_{1, i}, \ldots, g_{\frac{n}{2}+2i-4}, i\}.
   \]

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