Stochastic modelling of entropy-growth and dissipation in mixed-chaotic system that are driven quasi-statically

Yehoshua Winsten and Doron Cohen
Department of Physics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel

We analyze entropy-growth for a system that features mixed chaotic phase-space, whose control parameters (or slow degrees of freedom) vary quasi-statically. For demonstration purpose we consider the restricted 3 body problem, where the distance between the two central stars is modulated due to their Kepler motion. If the system featured hard-chaos, one would expect diffusive spreading with coefficient that can be estimated using linear-response (Kubo) theory. But for mixed phasespace the chaotic sea is multi-layered. Consequently, it becomes a challenge to find a robust procedure that translates the sticky dynamics into a stochastic model. We propose a Poincaré-sequencing method that reduces the multi-dimensional motion into a one-dimensional random-walk in impact-space. We test the implied relation between stickiness and the rate of entropy-growth.

I. INTRODUCTION

Considering a closed Hamiltonian driven system, such as a particle in a box with moving wall (the piston paradigm), the textbook assumption is that quasi-static processes are adiabatic, and therefore reversible. This claim can be established for an integrable system by recognizing that the action-variables are adiabatic invariants [1]. At the opposite extreme, analysis of slowly driven completely chaotic systems [2–4] has led to a mesoscopic version of Kubo linear-response theory and its associated fluctuation-dissipation phenomenology [5–8]. However, generic systems are neither integrable nor completely chaotic. Rather their phasespace is mixed, resulting in the failure of the adiabatic picture [9–13], and of linear-response theory. Namely, the phasespace structure varies with the control parameter: tori are destroyed; chaotic corridors are opened allowing migration between different regions in phasespace [14, 15]; stochastic regions merge into chaos; sticky regions are formed [16–19]; sets of tori re-appear or emerge. Some of those issues can be regarded as a higher-dimensional version of non-linear scenarios that are relate to bifurcations of fixed points, notably swallow-tail loops [20–24], or as a higher-dimensional version of the well-studied separatrix crossing [25–36], where the Kruskal-Neishtadt-Henrard theorem is followed.

Motivation.– Irreversibility is observed in hysteresis experiments with ultracold atoms [37]. For ring geometry see [38, 39]. Signatures of mixed phasespace naturally arise in the analysis of the Bose-Hubbard Hamiltonian [40–42]. In this context, the effect of chaos has been taken into account while studying the efficiency of a nonlinear stimulated Raman adiabatic passage (STIRAP) [12]; the efficiency of quasistatic transfer protocols [15]; and the Hamiltonian hysteresis that follows the reversal of the driving [13]. All the latter studies concern the dynamics of systems that feature an underlying mixed-chaotic phase-space.

Objective and choice of model system.– One would like to provide a comprehensive set of tools for the design and the optimization of quasi-static protocols, e.g. in the context of Bose-Hubbard circuits. The feasibility and the efficiency of such protocols is related to phasespace spreading (‘entropy-growth’) as demonstrated in the above cited works [12, 13, 15]. In a broader perspective, the analysis of driven systems that feature an underlying mixed-chaotic phase-space is a rather universal theme, that has relevance to many fields in physics, notably celestial mechanics. Inspired by the restricted 3 body problem, we focus below on the dynamics that is generated by a generalized Hill’s Hamiltonian [43–45]. This describes the mixed-chaotic motion of a test-particle in the field of force of a binary system. More interestingly, we can have in mind a binary system immersed in a cloud of dust. The dust is driven quasi-statically by the Kepler motion of the stars.

Outline.– In Sec. (II) we introduce the generalized Hill’s Hamiltonian. This Hamiltonian will be used as a test case for the application of our approach. It features a mixed chaotic phasespace whose parametric evolution can be visualized using a Poincaré landscape plot. In Sec. (III) we use a Poincaré-sequencing method in order to characterize the time dependent dynamics. This inspires the introduction of an effective stochastic model in Sec. (IV) and Sec. (V). This simplified model is used in Sec. (VI) to provide an explicit relation between stickiness and the rate of entropy growth. In Sec. (VII) we explain that the dependence on the directionality of a cycle is linked to asymmetry that can be detected in the Poincaré-sequencing analysis. For completeness we present in Sec. (VIII) the theoretical reasoning that relates the rate of dissipation to the stochastic characterisation of the dynamics.

II. THE GENERALIZED HILL PROBLEM

The Hamiltonian under consideration concerns the motion of a dust particle (satellite) in the vicinity of massive bodies (stars). The stars are performing a cycle \((X(t), Y(t))\) that has frequency \(\Omega\) and constant angular momentum \(\ell\). It might be, but not have to be, the Kepler motion of Appendix A. We define the character-
istic radius \( c \) such that the scaled angular momentum is \( \ell = c^2 \Omega \). In polar coordinates the cycle is parameter-
ized by \( R(t) = cR(\theta(t)) \). By definition of \( c \), one observes that the \( d\theta/(2\pi) \) integral over \( |R(\theta)|^2 \) is unity. Regarding \( \theta \) as the time variable, one obtains, after a sequence of transformations (see Appendix B), the generalized Hill’s Hamiltonian:

\[
\mathcal{H} = \frac{1}{2}(p - r_\perp)^2 + gR(\theta)u(r) - \frac{1}{2}K(\theta) r^2
\]  

(1)

where (prime indicates theta derivative):

\[
K(\theta) = 1 + \left( \frac{1}{R(\theta)} \right)'' R(\theta)
\]  

(2)

and the scaled version of the attractive potential is

\[
u(r) = -\frac{\mu_2}{\sqrt{(x - \mu_1)^2 + y^2}} - \frac{\mu_1}{\sqrt{(x + \mu_2)^2 + y^2}}
\]  

(3)

with \( \mu_1 + \mu_2 = 1 \). The parameter \( g \) is the scaled attraction constant for the force between the satellite and the stars. It can be due to gravitation, or (in different context) it can be of Coulomb origin.

For an arbitrary quasi-Kepler motion (as defined above) the Hamiltonian is controlled by two parameters \((R,K)\). So in general the satellite experiences a cycle. But for a proper Kepler motion \( K(\theta) = g_e/R(\theta) = 1/[1 + \varepsilon \cos(\theta)] \) with \( g_e = (1 - \varepsilon^2)^{-3/4} \). Consequently, see Appendix C, we get a Hamiltonian that depends on a single parameter,

\[
\mathcal{H}(r,p) = \frac{1}{2}(p - r_\perp)^2 + R(\theta) \left( gu(r) - \frac{1}{2}g_e r^2 \right)
\]  

(4)

Thus, a proper Kepler motion should be regarded as a modulation and not as a cycle. As explained at the last paragraph of Appendix C, the dimensionless constant that controls the quasi-static aspect of the Kepler driving is \( \varepsilon g_e/g \). For simulations we used \( g=25 \), and \( \varepsilon=0.2 \), and \( \mu_1=\mu_2=1/2 \).

**Poincaré landscape.**— Fig.1 displays the Poincaré section for the Hill’s Hamiltonian: Two (blue) regions contain quasi-regular trajectories around each of the two stars; There are additional quasi-regular regions; and there is a large (red) chaotic sea. In order to demonstrate the variation of phasespace with respect to the energy \( E \), or with respect to a control parameter (here it is \( \theta \) that parameterize the Kepler motion), we propose to look on the Poincaré landscape that is displayed in the additional panels of Fig.1. Each row in those additional panels encodes the information regarding the phase-space structure for a different value of \( E \) or \( \theta \), respectively. In a later Section we display on top of this landscape, the evolving cloud of the time-depended Hamiltonian \( \mathcal{H}(r,p,\theta(t)) \).

The time-dependence of \( \theta \) induces spreading in energy and migration of trajectories between different regions.

**FIG. 1.** Poincaré landscape. The upper panel is a standard Poincaré section for Hill’s Hamiltonian Eq.(1) with \( \mu_1=\mu_2=1/2 \), and \( \varepsilon=0.2 \), and \( g=25 \), at energy \( E = -22.2 \). The control parameter \( R \) is frozen at the value \( \theta = 0 \). In the middle panel each row is a \( p_\perp = 0 \) stripe of the Poincaré section at different \( E \), while \( R \) is frozen at the value \( \theta = 0 \). In the lower panel the Poincaré stripes are plotted for frozen \( R \) at different values of \( \theta \). Initially \( E = -22.2 \), and later we follow \( E \) adiabatically. The color code is such that chaotic trajectories are red, while quasi-regular chaotic trajectories are blue.
III. POINCARÉ SEQUENCING

The spreading of energy is determined by the fluctuations of the generalized force $\mathcal{F}$ that is associated with the control variable $\theta$. Note that we assume periodic driving, and that the scaled Hamiltonian is defined such that $\theta$ is the time variable. For typical model systems, e.g. Billiards with moving piston, and also for the Hill’s Hamiltonian, we can factorize $\mathcal{F}$ as follows:

$$\mathcal{F} = -\frac{\partial \mathcal{H}}{\partial \theta} = g(\theta) f(r)$$

(5)

The variation of the energy is an integral over $-\dot{\theta}\mathcal{F}(t)$, but for the analysis it is more convenient to consider

$$Q = \int_0^t f(r(t))dt \equiv \sum_j F_j$$

(6)

This allows to get rid of the intra-cycle modulation that is observed in the variation of the energy, see Fig.2, while the features that are essential for the analysis are preserved. The last equality express the integral as a sum over pulses, whose area is defined in the illustration of Fig.3.

**FIG. 3. Poincaré pulses.** Representative piece of the signal $f(r(t))$. The vertical red lines indicate the moments $t_j$ that are selected by the Poincaré section. The signal is regarded as a sequence of rectangular pulses, displayed in green color. Each pulse has duration $T_j$ and average height $\bar{F}_j = F_j/T_j$. In the figure $\bar{F}$ is scaled vertically ($\times10$) to improve resolution.

Entropy growth.— In the simulations we consider the following scenario. Initially we launch a narrow cloud in the middle of the chaotic sea. After a short transient this cloud fills most of the chaotic sea. Some extra time might be required in order to penetrate peripheral regions where the dynamics is sticky. We shall come back to this stickiness issue later on. For a time-dependent Hamiltonian we also have to consider: (a) Spreading away from the initial energy surface; (b) Migration between regions due to the time-dependence of the Hamiltonian. The dissipation aspect (growth of the average-energy) is directly related to #a and indirectly related to #b. In order to quantify the growth of entropy that is associated with the spreading in the energy, we define $S_Q$ as the range of energy (around the median) where 50% of the probability is located. There are traditional measures for entropy growth in terms of phasespace volume, but we prefer a definition that has a direct practical meaning.

A very long chaotic trajectory that explores the whole chaotic sea is regarded as a Poincaré sequence of $F_j$ pulses. The spreading $S_Q$ as a function of time is presented in Fig.4. In order to get the spreading in, the chaotic trajectory is divided into sub-sequences of length $T$ (upper panel) or of length $10T$ (lower panel), where $T$ is the period of a cycle. Equivalently, we start the simulation with a cloud of initial points at the middle of the chaotic sea, evolve them, and care to exclude the initial transient.

In order to figure out whether temporal correlations are important we randomize the original $F_j$ sequence, and then divide it again into sub-sequences. In Fig.4, the spreading of the randomized-trajectories is displayed too, for sake of comparison. The ratio between the actual rate of spreading, and that of the randomized-trajectories, is a robust measure for correlations.
Identifying correlations. – We would like to “see” the correlation by looking on the “signal”. For this purpose we plot images of the non-randomized subsequences in Fig. 5. The sub-sequences are ordered according to their average. If the sub-sequences originate from a randomized-trajectory, this average is close to zero, and the ordering has no significance (and no visual effect). But sequences of the non-randomized-trajectory are correlated. The correlations can be identified by inspection of the figure. Namely, the sequences of the time-independent \( \theta=0 \) Hamiltonian exhibit long red stretches, and the Kepler-driven sequences exhibit also blue stretches. This should be contrasted with the sequences of the \( \theta=\pi \) Hamiltonian, that look rather uncorrelated.

Phasespace exploration. – Having identified correlations in the ‘signal’, we would like to trace out their phaspace origin. For this purpose we point out that the value \( F \) of the pulse provides information about the location of the phaspace region that supports the pulse, as demonstrated in Fig. 6. Roughly speaking, we can re-
FIG. 6. **Poincaré mapping of pulses.** The upper panel displays the Poincaré section for the $\theta = 0$ Hamiltonian. Those are the same chaotic trajectories as in Fig. 1 (the quasi-integrable regions are left empty), but the points are color-coded by the values of $\tilde{F}_j$. The black line indicates the border of the energy surface (the forbidden region is outside). The lower panel shows a chaotic trajectory $(x(t), y(t))$. The color encodes the time. The black stretch indicates motion in the "red" sticky region of the upper panel. It is characterized by an "∞" shaped loops.

FIG. 7. **Driven system dynamics.** In the upper panel a trajectory of the Kepler-driven Hamiltonian is presented using the same section as that of Fig. 6. Gray color is used for all the points of the trajectory, while red and blue colors are used for the correlated stretches of Fig. 5. Parameters are the same as in Fig. 1. The Hamiltonian is time dependent. Accordingly, the cloud of trajectories spreads in energy, and can migrate between different regions. This is demonstrated in the lower panel, where we use the same presentation method as in Fig. 1. The native chaotic region, that corresponds to the red areas in Fig. 1, is colored in black. The evolving cloud of the Kepler-driven system is displayed using blue, red and gray dots. Red color indicates sticking in periphery regions, while blue indicates sticking in the native chaotic sea. The non-sticking gray points expand into "swamp" regions that are located outside of the native chaotic sea. See text for further details.

The stickiness to peripheral regions is expected. It has been studied in past literature. What we find rather surprising is the extra stickiness that we find in the dynamics that is generated by the Kepler driven system: the additional blue stretches indicate stickiness in phase-space regions that have distinct typical non-zero value of $F$.

A more careful inspection, see Fig. 7, reveals that the stickiness in the central region of the chaotic sea is in the region that was chaotic also in the absence of driving. So roughly we have the following regions: (a) Native chaotic sea region; (b) Swamp chaotic region; (c) Peripheral chaotic regions; (d) Quasi regular regions. The swamp regions appear due to the driving. They form in some sense a barrier between the native chaotic sea and its periphery. As for the quasi regular regions: they are excluded from our simulation, and not penetrated by the chaotic trajectories.
IV. STOCHASTIC MODELLING

Hard-chaos dynamics can be described as a random hopping between cells in phasespace. We have mixed-chaotic phasespace, with tendency for stickiness in e.g. peripheral regions, and therefore an effective stochastic description becomes a challenge. We would like to introduce a robust procedure for this purpose. First of all we recall that: (i) chaotic motion is ergodic. (ii) the pulse strength $F$ is like a radial coordinate. It is therefore rather natural to divide phasespace into $F$-cells. The size of the $F$-bins is determined such that all the (binned) values have the same rate of occurrence in the $F_j$ sequence. In particular we distinguish in Fig.6 the blue and the red regions, that corresponds to the bins that contain the smallest and the largest pulses respectively.

Stochastic Kernel.– Having done the $F$-binning of phasespace regions, it becomes possible to define a matrix $P$ whose element $P_{n,m}$ provide the probability to make a transition form bin $m$ to bin $n$. Note that the calculation of $P$ is a straightforward ‘signal analysis’ procedure that is based solely on the inspection of the $F_j$ sequence.

An image of the $P_{n,m}$ matrix is provided in Fig.8. Qualitatively, we see that the images reflects our expectation for enhanced probability to stay in red and blue regions whenever stickiness is observed in Fig.5. But this is misleading. In fact $P$ is not capable of providing an explanation for the stickiness. We explain this point in the subsequent paragraph.

We can generate artificial $F_j$ sequences using $P$ as the propagator (kernel) for a memory-less Markov process. Naively one might have the hope to get sequences that have the same statistical properties as the original Poincaré sequences. But this is not the case: the Markov process does not reproduce red/blue stretches that we see in Fig.5. On the quantitative side we define the probability $P(\tau)$ for survival in (say) the “red” region after $\tau$ steps. It is defined as the relative number of “red” pulses that have at least $\tau$ consecutive “red” pulses. (In other words, it is the inverse-accumulative distribution of the dwell time in the red region). The one-step survival probability is $P \equiv P(\tau=1)$. Accordingly, for a Markov process with $P_{n,m}$ we get

$$P(\tau) \bigg|_{\text{Naive}} = P^\tau$$

(7)

The actual $P(\tau)$ clearly does not agree with exponential decay, as shown in Fig.9. The naive expectation grossly underestimates the stickiness.

V. MINIMAL STOCHASTIC MODEL

The failure of $P_{n,m}$ to reproduce $P(\tau)$ is easily understood by inspection of phasespace. For presentation purpose we focus on the stickiness in the “red” region(s) of Fig.6. Regarding this region as composed of tiny phasespace cells, we can determine what is the ‘survival time’ in the red region for each cell. Then we realize that only fraction of red cells have large survivals time. Accordingly, $P_{n,m}$ should be regarded as the coarse-graining of a finer kernel $P_{\nu,\mu}$. Note that the type of index (Roman vs Greek) is used in order to distinguish the coarse-grained version from the “microscopic” version.

We would like to construct a minimal version for $P_{\nu,\mu}$, that corresponds to $P_{n,m}$, and that reproduces $P(\tau)$ correctly. Using this model we would like to illuminate semi-quantitatively how the rate of entropy growth depends on the stickiness. The calculation of $P(\tau)$ given a Markov kernel $P$ is done as follows:

$$P(\tau) \bigg|_{\text{Markov}} = \left( \frac{1}{p^\dagger p} \right) p^\dagger (Q P)^\tau p$$

(8)

In this expression $p$ is a vector that contains the initial distribution within the bins. Specifically, we assume uniform distribution within the “red” bins. Note that $(p^\dagger p)^{-1}$ is the number of participating red bins. The
Specifically, we assume that the transition probabilities between the 3 regions would be transitions of probabilities between the 3 regions. We could have added also direct transitions with probability \( p_2 \) between the \( n_0 \) and the \( n_2 \) cells, but it turns out that this would be a redundancy for our purpose. Also the total number of cells is insignificant for the analysis.

All the probabilities in the \( P \) matrix must be less than 1. This impose some constraints over the valid range of the model parameters. In particular one realizes that if \( R > 1 \) then \( P > (1/2) \). Therefore, in order to describe a model that exhibit stickiness (small \( S \)) we have to assume \( R < 1 \), and then the same constraints imply that \( P > [R/(R + 1)] \).

In practice the parameters \( P_0 \) and \( P \) are determined from the statistics of the signal, while \( R \) and \( Q \) are effective parameters that are determined by fitting to \( \mathcal{P}(\tau) \). These parameters control the stretch and the area of this function. The latter, using Eq.(9), is given by the following expression

\[
S = \frac{1}{1-p} + \left[ \frac{R}{R+1} \right] \frac{1}{Q}
\]

(16)

For \( R = 0 \) we get the naive result \( S = 1/(1-P) \). In the limit \( Q \to 0 \) there is no decay from the \( n_2 \) cells, and then the survival probability approach \( \mathcal{P}(\infty) = R/(1+R) \), and \( S \) diverges. The minimal value \( S = S_0 = 1/(1-P_0) \) is obtained for a fully connected chaos.

On the basis of the simulations we have determined \( \mathcal{P}(\tau) \) for the blue and for the red regions, for both the \( \theta=0 \) Hamiltonian and for the Kepler-driven Hamiltonian.

The results are displayed in Fig.9. The effective model parameters have been extracted via fitting, are are listed in Appendix D.
VI. STICKINESS AND ENTROPY GROWTH

Using the stochastic model we can try to figure out a relation between the stickiness and the spreading. The latter is determined by the time dependent diffusion coefficient

\[ D(t) = \frac{1}{2} \sum_{\tau=-t}^{t} C(\tau) \]

with the correlation function

\[ C(\tau) = \langle F_{\tau} F_0 \rangle = \frac{1}{N} \sum_{\mu,\nu} f_\mu [P^\tau]_{\mu,\nu} f_\nu \]

where \( N \) is the number of cells, and \( f_\mu \) is the \( F \) value that is associated with the phase space cell that is indexed as \( \mu \). Note that a finite result for \( P(\tau) \) is obtained provided \( \sum f_\mu = 0 \), reflecting that the correlation function is defined for the signal after subtraction of the average.

Without the sticky red region the correlation function is of the form \( C(\tau) = g(0) \). With the red region included we get correlations due to the sticking. To evaluate the contribution of the latter we use the reduced matrix of Eq.(11), where all the cells are grouped into 3 regions. In such case the sum over \( \mu \) in Eq.(18) is replaced by a sum over regions, and the number of cells in each region should be introduced as a weight factor. Then we obtain

\[ C(0) = C_0 + P_0 S_0 f_{\text{red}}^2 \]

where \( f_{\text{red}} \) is the average \( F \) value of the red region. Note that the implied average \( F \) value of the non-red region is \( -P_0 S_0 f_{\text{red}} \). The summation over \( \tau \) leads to Eq.(18) with \( P^\tau \) replaced by \( 1/(1-P) \). The zero mode has to be excluded from the inversion. Including \( C_0 \) we get

\[ \sum_{\tau=0}^{\infty} C(\tau) = C_0 + P_0 S_0 f_{\text{red}}^2 \]

We define the correlation factor as follows:

\[ c = \frac{\sum C(\tau)}{C(0)} = 2 \left( \frac{\sum_{\tau=0}^{\infty} C(\tau)}{C(0)} \right) - 1 \]

It is the correlation “time” in terms of iteration with the Poincaré map. For the minimal model of Eq.(11) we obtain

\[ c = 2 \left( \frac{S}{S_0} \right) - 1, \quad \text{for } C_0=0 \]

Note that for fully connected chaos we get \( c = 1 \) as expected.

For the Kepler-driven system we observed an additional “blue” sticky region. Therefore we have to generalize the minimal model, such as to have two sticky regions, “red” and “blue”. The total number of cells is \( N = N_0 + N_{\text{red}} + N_{\text{blue}} \). We define \( F_{0}^{\text{red}} = N_{\text{red}}/N \) and \( F_{0}^{\text{blue}} = N_{\text{blue}}/N \). Each of the regions has its own \( P(\tau) \), with effective parameters that have been determined in Appendix D. Note that the implied average \( F \) value of the red region is \( f_0 = -(1/N_0)[N_{\text{red}} f_{\text{red}} + N_{\text{blue}} f_{\text{blue}}] \). Then one obtains

\[ C(0) = C_0 + \frac{N_0}{N} f_0^2 + P_0^{\text{red}} f_{\text{red}}^2 + P_0^{\text{blue}} f_{\text{blue}}^2 \]

and

\[ \sum_{\tau=0}^{\infty} C(\tau) = C_0 + P_0^{\text{red}} S_{\text{red}}^2 f_{\text{red}}^2 + P_0^{\text{blue}} S_{\text{blue}}^2 f_{\text{blue}}^2 \]

Note that for \( N_{\text{blue}} = 0 \) these equations reduce back to Eq.(22).

The correlation factor can be extract numerically from the \( S_Q \) plots of Fig.4. Namely, it is the ratio between the slope of \( S_Q^2 \) for the true pulse sequence, and that of the randomized sequence (of the same pulses). By inspection of Fig.10 we see that the true \( S_Q^2 \) exhibits a super-diffusive transient that indicates long-time correlations that are not captured by our simplified model. The agreement with the minimal model is qualitative rather than quantitative. Some extra details about the quantitative aspect are provided in Appendix D.

We see that a model that faithfully reproduces \( P(\tau) \) is not enough for the determination of \( C(\tau) \). In principle we could have introduced a more elaborated stochastic model, that features a hierarchy of “red” and “blue” regions, but such an approach has no practical value, and does not allow the derivation of analytical results.
VII. CYCLE VS MODULATION

It is important to distinguish between Cycle and Modulation. Consider an Hamiltonian \( H_R \) where \( R \) is a set of control parameters. In a time dependent scenario we can say that the Hamiltonian varies along a curve in a parametric manifold. A modulation can be parametrized by a single non-cyclic parameter, say \( R(t) = A \cos(\Omega t) \), while a cycle requires an angle parameter, say \( \theta(t) = \Omega t \), where \( \theta \) is defined modulo \( 2\pi \). A prototype example for cyclic driving is presented in Fig.11.

It is sometimes difficult to determine whether the time dependence in the Hamiltonian should be regarded as Constant or as Modulated or as Cyclic driving. For example: the time dependence for particle in a rotating box can be removed by transforming into a rotating frame. Similarly, the time dependence for a particle in an expanding box can be removed via a dilation transformation. In the case of a cycle the outcome depends in general on the sense of the cycle, and furthermore for a mixed phasespace we expect difference in the rate of entropy growth.

At first glance one may naively think that a Kepler-driven system qualifies as Cyclic driving. The two parameters might be \((X(t), Y(t))\) or equivalently \((\theta(t), K(\theta))\) as in Eq.(1). But it turns out that for a proper Kepler driving the cycle degenerates into a modulation. In order to avoid such ‘degeneracy’ we have to assume an asymmetric \( R(\theta) \), for example \( R(\theta) \propto [1 + \epsilon \cos(\theta) + \epsilon' \sin(2\theta)]^{-1} \), that is illustrated in Fig.12.

If we have a non-degenerate cycle, we can ask whether the rate of entropy-growth depends on its sense (cycle vs reversed cycle). For a system with mixed-chaotic phasespace indeed we can have such dependence, as discussed for e.g. the mushroom billiard in [11]. A different illustration of the same idea is provided in Fig.11. During the cycle the space is divided by a barrier (that serves as a “valve”) into two regions. This is done periodically, and out-of-phase with respect to the piston movement. Specifically, in the plotted illustration, the splitting ratio of the cloud is roughly 1:2 for forward cycle, and roughly 1:1 for reversed cycle. Changes of energy due to changes in the volume obey a simple “ideal gas” multiplicative law \( E \mapsto \alpha E \), where \( \alpha \) is given by Eq.(E7). The value of \( \alpha \) depends on the sense of the cycle due to the different splitting ratio, and we get \( \alpha = 10/9 \) and \( \alpha = 9/8 \) respectively.

However, the billiard examples are rather artificial. They are based on construction that allows a sharp distinction between regions in (phase)space. Generic systems, such as the Hill’s Hamiltonian, do not feature dramatic splitting and merging of well defined (phase)space regions. Consequently, dependence on the sense of the cycle is not a prominent effect, and careful numerical procedure is required to detect it. This motivates the following discussion of directionality dependence.

Directionality.– The dependence on the sense of the cycle is related to the directionality dependence of a modulation. The argument is as follows: A modulation can be encoded by a sequence \( AAAAA \cdots AA \cdots \). The inverse modulation is clearly the same sequence. A cycle can be encoded as \( ABA \cdots ABA \cdots \). The reversed cycle \( BAB \cdots B \cdots \) is distinct if the cycle is not degenerated \( A \neq B \), and provided \( A \) and \( B \) are not characterized by the same spreading rate. It is therefore enough to establish dependence on directionality.

Regarding the sequence \( F_j \) as a ‘signal’, we ask whether it looks statistically the same during the ‘forward’ half period when \( R(\theta) \) changes from \( R(0) \) to \( R(\pi) \), and the ‘backward’ half period when it changes from \( R(\pi) \) to \( R(0) \). In the standard paradigm of quasi-static processes the directionality has no significance. In Fig.13 we plot the distribution of the \( F \) values for the two groups of pulses. For the full signal the distribution of the \( F_j \) over the bin is uniform by definition. But if we look only on the pulses that belong to the ‘backward’ half-periods we see that the small values (blue pulses) become slightly more frequent, as opposed to the large values (red pulses) that become slightly less frequent. The difference is very small. Still, it indicates that the steady state is not the same for “forward” and “backward” driving.

VIII. DISSIPATION

Dissipation is associated with energy spreading. The standard theory [2–4] assumes a globally chaotic energy surface that instantly ergodizes at any moment. It follows that the phase-space volume \( \mathcal{N}(E; \theta) \) is an adiabatic invariant, where \( \theta(t) \) is a slowly varying control parameter. For a closed cycle, the conservative work
is zero. Still, beyond the zero-order adiabatic result, there is diffusion in energy with coefficient \( D_E = \nu \theta^2 \), where \( \nu \) is the intensity of the fluctuations, i.e. the algebraic area of \( C(t) \). From the Fokker-Plank description of the spreading process, one deduces rate of absorption \( \dot{E} = \mu \theta^2 \), aka the Kubo formula, with dissipation coefficient \( \mu \) that is related to \( \nu \) via a fluctuation-dissipation relation [5–8], namely \( \mu = (1/2)\beta \nu \), where \( \beta(E) \) is some version of microcanonical inverse temperature, as defined in Appendix E. Consequently, for periodic driving with frequency \( \Omega \equiv \dot{\theta} \) one expects an amount \( Q = 2\pi \mu \Omega \) of dissipated energy per cycle, which vanishes in the quasi-static limit (\( \Omega \to 0 \)).

For a driven mixed chaotic system, we expect parametric dissipation, meaning that the dissipated energy per cycle (\( Q \)) approaches a finite non-zero constant in the limit (\( \Omega \to 0 \)), and depends on the directionality of the driving as discussed in the previous paragraph. Billiard example that have been discussed in the past, as well as that of Fig.11 are illuminating, but do not fully reflect some complications that are encountered once we deal with a generic system, such as Hill’s. In what follows we highlight those zero-order subtitles, and also generalize the first-order formulation.

**Zero order Dissipation.**— In systems with mixed-chaotic dynamics we can get irreversibility (entropy growth), as well as dissipation (growth of the average energy) even in the quasi-static limit. The derivation of this claim requires phase-space generalization of [11]. This generalization is presented in Appendix E. We write the phasespace area as \( A = \sum_\mu A_\mu \), where \( \mu \) distinguishes different regions. Each region might have a different “inverse temperature” \( \beta_\mu \). Then we obtain the following result:

\[
\langle Q \rangle_0 \approx -\sum_{steps} \sum_\mu \delta A_\mu / \beta_\mu A + \mathcal{O}(\delta A^2) \quad (25)
\]

This expression is obtained from Eq.(E6) after expansion with respect to \( \delta A = A_\mu - A_\mu^{(0)} \). Based on Eq.(25) our observation is that we can get an \( \mathcal{O}(\delta A) \) non-zero result provided the \( \beta_\mu \) are non-identical. In such case \( \langle Q \rangle_0 \) can switch sign for a reversed cycle. This should be contrasted with a Billiard system for which the \( \beta_\mu = 1/E \) are identical, and \( \langle Q \rangle_0 \) is always positive.

**First order Dissipation.**— We define \( F = -\partial H / \partial \theta \). For a periodically driven Hamiltonian with \( \dot{\theta} = \Omega \) we have \( \dot{E} = -\Omega F(t) \). Integrating over a cycle, squaring, and averaging over an ensemble, we get \( \text{Var}(Q) = 2\pi \nu \Omega \), where \( \nu \) is the intensity of the fluctuations (the area of the \( F \) autocorrelation function). This assumes a globally chaotic energy surface. If we have a fragmented phase-space (as in the previous Billiard example) we get \( \text{Var}(Q) = \text{Var}_0(Q) + 2\pi \nu \Omega \), where \( \text{Var}_0(Q) \) is the variance that is associated with \( \langle Q \rangle_0 \). Using the above explained fluctuation-dissipation reasoning we deduce that the energy increase per cycle is

\[
\langle Q \rangle = \langle Q \rangle_0 + \frac{1}{2} \beta_{\text{eff}} [\text{Var}_0(Q) + 2\pi \nu \Omega] \quad (26)
\]

This expression goes beyond Kubo, because the zero order spreading is taken into account. Note that if correlations persist over a time duration that is longer than a cycle, the result is a long super-diffusive transient as in Fig.4. In any case the appropriate correlation factor \( c \) has to be incorporated in the calculation of \( \nu \), as discussed in Sec. (VI).
IX. SUMMARY AND OUTLOOK

We have introduced an effective stochastic theory for quasi-static entropy growth in systems with mixed chaotic phasespace. The main objective was to provide tools for the analysis energy spreading, which is useful for the calculation of the average energy growth (dissipation), and possibly for estimating the rate of evaporation (dust that escapes the binary system).

The agenda was, on the one hand, to characterize the multi-dimensional phasespace dynamics using signal-analysis of a single chaotic trajectory. On the other hand we wanted to reproduce the essential statistical features of the ‘signal’ using a minimal Markovian model.

For the characterization of the chaotic motion we represent the chaotic trajectory as a Poincaré sequence of pulses ($F_i$). The value of $F$ is regarded as a radial phasespace coordinate that is used in order to divide phasespace into regions (indexed by $n$). We realize that this coarse-graining is too rough: we cannot build on it a Markov process that reproduces the observed stickiness. We therefore have to define a refined version of the Markov process that reflects the hierarchic structure of phasespace. Consequently we constructed a minimal model that allows to reproduce the observed stickiness. This model suggests a relation between the stickiness and the enhancement that is observed is the rate of entropy growth. Unfortunately, in the present model, the quantitative agreement is poor due to long range correlations that were neglected.

Specifically, for the frozen dynamics we have identified stickiness in peripheral regions of the chaotic sea, and the minimal stochastic model required 3 regions (central chaotic region; non-sticky peripheral chaotic region; sticky peripheral chaotic region). Surprisingly in the Kepler-driven system extra stickiness manifests in the native chaotic sea. This extra stickiness is related to the appearance of an additional “swamp chaotic region” where chaos penetrates due to the time-dependence of the Hamiltonian.

We also looked for directionality dependence, implying that the rate of entropy growth is not the same if a cycle is reversed. We have clarified that also this effect can be identified from the ‘signal analysis” of the Poincaré sequence. Specifically for Kepler-driving this is a very weak effect (a few percent difference). Finally, for sake of generality, we have explained how the Kubo theory of dissipation can be generalized in order to incorporate both the zero-order and first-order irreversibility. This picture implies exponential energy growth if $\langle Q \rangle$ of Eq. (26) is proportion to $E$. This is indeed the case for Billiard systems if $\langle Q \rangle_0 \neq 0$ as discussed originally by [9–11]. More generally we can get from Eq. (26) different energy dependence, say $\dot{E} = \lambda E^\alpha$. Note that for $\alpha > 1$ one obtains hyperbolic-like growth that leads to escape $E \sim 1/(t - t_e)^{1/(\alpha-1)}$ within a finite time $t_e$. The exploration of such scenario requires further study of possibly different model systems.

Appendix A: Basic formulas for Kepler motion

The constant of motion in Kepler problem is the angular momentum. In terms of polar coordinates $(\theta,R)$ we define $\ell = R^2 \dot{\theta}$. Kepler’s area law is the statement

$$\frac{d}{dt} \text{Area} = \frac{1}{2} \ell$$

(A1)

The Kepler motion is along an ellipse with major axes $a$ and $b = \sqrt{1 - \varepsilon^2}a$. We also define $c = \sqrt{ab}$. From the area law it follows that $T = 2\pi ab/\ell$. Accordingly the frequency is $\Omega = \ell/(ab)$. So we have the relation

$$\ell = ab\Omega = c^2 \Omega$$

(A2)

For a circular motion of radius $R = a = b = c$, the frequency the motion is determined by the equation

$$\Omega^2 a^3 = G(M_1 + M_2) \equiv GM$$

(A3)

This result applies also if the motion is along an ellipse. The equation of the ellipse is

$$R(\theta) = \frac{(1 - \varepsilon^2)a}{1 + \varepsilon \cos(\theta)} \equiv cR(\theta)$$

(A4)

Note that with this definition $R(\theta)$ is square-normalized to unity. The equation of motion for the radial motion is

$$\ddot{R} = \frac{\ell^2}{R^3} - \frac{GM}{R^2}$$

(A5)

which implies conservation of energy (here we are in the non-rotating ”lab” frame):

$$E = \frac{1}{2} R^2 + \frac{\ell^2}{2 R^2} - \frac{GM}{R} = -\frac{1}{2}a^2 \Omega^2$$

(A6)

Given $(GM,E,\ell)$, the orbit, up to orientation, is described by $(\Omega,a,c)$. The $\Omega$ and the $a$ are determined by Eq. (A3) and Eq. (A6), while $c$ is determined by Eq. (A2), and we have the ratio $c/a = (1 - \varepsilon^2)^{1/4}$.

Appendix B: The generalized Hill Hamiltonian

We use the notations $r = (x,y)$ and $p = (p_x, p_y)$. We consider time dependent $R(t)$ and $\theta(t)$. Without loss of generality we set $m = 1$ for the mass of the satellite. The Hamiltonian is:

$$\mathcal{H} = \frac{1}{2} p^2 + U(r; R(t)\theta(t))$$

(B1)

In order to transform the Hamiltonian with $T = \exp[-i\alpha(t)G]$ we use the formula

$$\mathcal{H} = T^\dagger \mathcal{H} T - i T^\dagger \frac{\partial T}{\partial \alpha}$$

(B2)

$$= T^\dagger \mathcal{H} T - \dot{\alpha} G$$

(B3)

First stage go to a rotated reference frame with $T = \exp[-i\theta(t)L]$ where $L = r \wedge p = xp_y - xp_y$

$$\mathcal{H} = \frac{1}{2} p^2 - \dot{\theta} L + U(r; R(t),0)$$

(B4)
Second stage we apply time dependent dilation with $T = \exp[-i\ln R/\Lambda]$ where $K = r \cdot p = xp_x + yp_y$. Note that $T^\dagger x T = R x$ and $T^\dagger p T = (1/R)p$. Using the notation $U(r) = U(r; 1, 0)$ we get

\begin{align*}
\mathcal{H} &= \frac{1}{2R^2}(p + R\dot{R}r)^2 - \frac{\dot{R}}{R} \cdot (p + R\dot{R}r) - \dot{\theta} L \\
&+ \frac{1}{R} U(r) + \frac{1}{2} \left[ R^2 + R\dot{R} r^2 \right] r^2 \\
&= \frac{1}{2R^2} p^2 - \dot{\theta} L + \frac{1}{R} U(r) + \frac{1}{2} R\dot{R} r^2 \\
&= \frac{1}{2R^2} (p - A)^2 + \frac{1}{R} U(r) - \frac{1}{2} \left[ R^2 \dot{\theta}^2 - R\dot{R} \right] r^2 
\end{align*}

where $A = R^2 \dot{\theta} r_{\perp}$ with $r_{\perp} = (-y, x)$. Given $R(t)$ and $\dot{\theta}(t)$, the above Hamiltonian can be written schematically as

\[ \mathcal{H}(r, p, \theta(t), R(t)) = \frac{1}{R(t)^2} \left\{ \frac{1}{2}(p - \ell(t)r_{\perp})^2 + R(t)U(r) - \frac{1}{2} Q(t)r^2 \right\} \]

where $\ell(t) \equiv R^2 \dot{\theta}$, and $Q(t) \equiv (R^2 \dot{\theta})^2 - R^3 \ddot{R}$. If we assume Kepler motion we get $Q(t) = \Omega^2 \alpha^2 R(t)$ from the radial equation of motion.

Appendix C: Hamiltonian for a Kepler system

Due to the dilation transformation $r$ is dimensionless, and the distance between the stars is unity, while $p$ has the same units as $\ell$. We now assume that $\ell$ is constant for the cycles of interest. Consequently we can re-scale the momentum $p := \ell p$. It is convenient to define the characteristic radius $c$ of the orbit through $\ell \equiv \Omega c^2$, where $\Omega$ is the frequency of the cycle. We also define the notation $R(t) = cR(\theta(t))$. By definition of $c$ and from $\dot{\theta} = \ell/R(t)^2$ it follows that

\[ \int |R(\theta)|^2 \, d\theta = 1 \]  

(C1)

Given $R(\theta)$ we have the identity

\[ \frac{\dot{R}}{\Omega^2} R^3 = - \left( \frac{1}{R} \right)^{''} R \]  

(C2)

where dot (') is for time derivative and prime ('') is for theta derivative.

We write the attraction constant between the satellite and the stars as $G_0 M$, such that $U(r) = G_0 M u(r)$. The Hamiltonian takes the form

\[ \mathcal{H} = \frac{\Omega}{R^2} \left\{ \frac{1}{2}(p - r_{\perp})^2 + gR\dot{u}(r) - \frac{1}{2}Kr^2 \right\} \]  

(C3)

where

\[ K = 1 + \left( \frac{1}{R} \right)^{''} R \]  

(C4)

and

\[ g = \frac{G_0 M}{c^3\Omega^2} = \frac{\Omega_0^2}{\Omega^2} \]  

(C5)

For a Kepler driven system we use the notation

\[ g_c = \frac{GM}{c^3\Omega^2} = (1 - \varepsilon^2)^{-3/4} \]  

(C6)

and get the simpler Hamiltonian

\[ \mathcal{H} = \frac{\Omega}{R^2} \left\{ \frac{1}{2}(p - r_{\perp})^2 + R \left( g\dot{u}(r) - \frac{1}{2}g_r r^2 \right) \right\} \]  

(C7)

Given $R(\theta)$ and $\Omega$ and $c$ we have $\dot{\theta} = \Omega/R^2$. Consequently, if we use $\theta$ as time variable, we get the Hamiltonian Eq.(C7) without the $\Omega/R^2$ term.

The simple minded slowness condition is $\Omega \ll \Omega_0$ which can be written as $1 \ll g$. In analogy with the piston paradigm we have to assure that $\dot{R} \ll \dot{r}$ where the typical velocity of the dust particles is $\dot{r} \sim c\Omega_0$. For Kepler motion the maximum velocity of the “piston” is $\dot{R} \sim c g_\varepsilon \Omega$. Consequently, the slowness condition takes the form $\varepsilon g_\varepsilon \ll g$, which always breaks down if $\varepsilon$ is too close to unity.

Appendix D: Determination of effective parameters

The $F_j$ values have been grouped into 10 bins. The pulses that belong to a given bin define a region $n$ in phasespace. It is implied that the same number of pulses is associated with each region. In our jargon $n = 1$ is the “blue” region and $n = 10$ is the “red” region. The matrix $P_{n,m}$ of Fig.8 characterizes the statistics of the transitions between regions. Independently of that we can determine numerically the probability $P(\tau)$ to stay, say, in the blue region, and the associated staying probability $P = P_{1,1} = P(\tau = 1)$. We have obtained $P$ in $[\theta=0, \text{red}]$, in $[\theta=0, \text{blue}]$, in $[\text{Kepler,red}]$, and in $[\text{Kepler,blue}]$. Respectively we obtained $\tilde{P}(\tau)$ in Fig.9, extracted $S$, and determined $R$ and $Q$ via fitting. The results were:

| $P$ | $S$ | $R$ | $Q$ |
|-----|-----|-----|-----|
| 0.27, 0.06, 0.24, 0.23 | 1.57, 1.06, 1.52, 1.53 | 0.15, 0.0003, 0.092, 0.067 | 0.65, 0.83, 0.42, 0.26 |

(D1) (D2) (D3) (D4)
The main difference between the Kepler-driven Hamiltonian and the frozen Hamiltonian is related to the stickiness in the blue region.

The "digitized" signal is obtained as follows. We define $f_n$ as the average value that characterizes the $n$-th bin. Then we set digitized $[F_j] = f_n$, if $F_j$ belongs to the $n$-th bin. In order to analyze the stickiness-related correlations, we have regarded all the intermediate bins ($n = 2 \cdots 9$) as one region that is characterized by an average value $f_0$, while bins $n = 1, 10$ are characterized by $f_{\text{blue}}$ and $f_{\text{red}}$ respectively. Due to this digitization the noise is reduced by factor $\sim 2.5$. We are left with a signal that contains information that is related to the stickiness, and we can set $C_0 = 0$ in Eq.(19). Consequently, this digitization procedure allows a meaningful comparison between the numerical results and the minimal model in Fig.10.

The correlation factor $c$ can be extract numerically by inspection of Fig.10. For the Kepler-driven system we get $c = 18$, while for the $\theta = 0$ Hamiltonian we get $c = 9$. This is consistent with what we observed in Fig.4. The minimal model does not take into account the observed long-time correlations and therefore predicts much smaller values, namely, $c = 2.0$ and $c = 1.7$ respectively.

**Appendix E: Quasistatic energy spreading**

The energy landscape of phase-space is described by the function $E = \mathcal{H}(r,p;\theta)$. The $d\mathbf{r}d\mathbf{p}/(2\pi)^{\text{dof}}$ volume of an energy surface is denoted $\mathcal{N}(E;\theta)$, and corresponds to the number of phasespace cells in semiclassical mechanics. The area of the energy surface is defined as $A(E;\theta) = \partial_E \mathcal{N}$, and corresponds to the density of states. The microcanonical-like inverse temperature is $\beta = \partial_E \ln \mathcal{N} = A/\mathcal{N}$. For a particle in a billiard of area $A$, setting appropriate units for the mass, we get $\mathcal{N} = AE$, and $\beta = 1/E$. For a mixed phasespace the total area is written as

$$A(E;\theta) \equiv \partial_E \mathcal{N} = \sum_{\mu} A_{\mu}(\theta)$$

This assumes that there is a way to identify distinct regions as in the billiard example of Fig.11 where $\mu = L, R$ distinguishes the left and right regions, and $A_{\mu}(\theta)$ is the respective geometric area of the $\mu$-th region, while $\theta$ is a parameter that is used to specify the position of the piston. Without any approximation we always have

$$\dot{E} = \left\langle \frac{\partial \mathcal{H}}{\partial \theta} \right\rangle_t \dot{\theta} \equiv -\Omega F(\theta(t))$$

In the Ott-Wilkinson-Kubo formulation of linear response theory [2–8], it is assumed that for a quasi-static process the instantaneous average can be replaced by an evolving microcanonical average due to quasi-ergodicity. Accordingly the variation of the energy becomes parametric:

$$dE = \left\langle \frac{\partial \mathcal{H}}{\partial \theta} \right\rangle_{E,\theta} d\theta = \left( \frac{\partial \mathcal{N}}{\partial E \mathcal{N}} \right) d\theta$$

(E3)

From the last relation it is implied that $d\mathcal{N} = 0$, meaning that $\mathcal{N}(E;\theta)$ is an adiabatic invariant. With the definition of phasespace area this can be written as

$$dE = -\frac{1}{\beta} [\partial_\theta \ln \mathcal{N}] d\theta \equiv -\frac{1}{\beta_{\text{eff}}} [\partial_\theta \ln A] d\theta$$

(E4)

where the latter equality defines $\beta_{\text{eff}}$. Adjusting notations to mixed phasespace we write the change of the energy per-cycle as

$$dE = -\sum_{\mu} P_{\mu}(\theta) \frac{1}{\beta_{\mu}} [\partial_\theta \ln A_{\mu}] d\theta$$

(E5)

where $P_{\mu}(\theta)$ is the probability at region $\mu$ of the energy surface, and it is assumed that the regions are well defined. Ref [11] consider a more complicated case where the borders between regions is affected by $\theta$. But such complication does not affect the big picture.

For a Billiard system that undergoes a multi-step process of the type that is illustrated in Fig.11, the dissipated energy per cycle is

$$\langle Q \rangle_0 = -\frac{1}{\beta} \sum_{\text{steps}} \sum_{\mu} \frac{A_{\mu}^{(0)}}{A_{\mu}^{(0)}} \ln \left[ \frac{A_{\mu}}{A_{\mu}^{(0)}} \right]$$

(E6)

where $\beta = 1/E$ assumes a narrow distribution around $E$. Here the outer summation is over steps of the cycle. We assume global chaos at transitions between steps. The superscript "0" indicates the area at the beginning of a step. Without "0" it is the area at the end of the step.

Billiard systems are simple enough to allow an improved (exact) version of Eq.(E6) that does not assume a narrow distribution around a fixed energy. Changes of energy due to changes in the volume obey the simple "ideal gas" multiplicative law $E \rightarrow \alpha E$, with

$$\alpha = \sum_{\text{steps}} \sum_{\mu} \left( \frac{A_{\mu}^{(0)}}{A_{\mu}} \right)^2 \frac{A_{\mu}^{(0)}}{A_{\mu}}$$

(E7)

One can easily verify that $\langle Q \rangle_0$ of Eq.(E6) is consistent with $(\alpha - 1)E$. Note that we always have $\alpha > 1$.

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