We study the Lasry-Lions approximation using the kernel determined by the fundamental solution with respect to a time-dependent Tonelli Lagrangian. This approximation process is also applied to the viscosity solutions of the discounted Hamilton-Jacobi equations.

1. Introduction

The method of Lasry-Lions regularization is a kind of variational approximation which is a generalization of the Moreau-Yosida approximation in convex analysis, see, for instance [19] and [1]. Beyond the analytic aspect of such regularization using the standard kernel \(|x - y|/2t\) (\(x, y \in \mathbb{R}^n\) and \(t > 0\)), more dynamical aspect of this method has already been studied widely in the past decade, especially with an emphasis on the weak KAM theory and Mather theory:

- An explanation of such a method using the fundamental solution of the associated Hamilton-Jacobi equations instead of the quadratic kernels was first given by Bernard ([2]). In the context of weak KAM theory, this method is closely connected to the Lax-Oleinik operators \(T_{s,t}^\pm\) ([4], [5] and [18]).
- Ilmanen’s lemma on insertion of \(C^{1,1}\) functions between a semiconvex function less than a semiconcave function ([3] and [17]).
- In [14], the authors also obtained the limiting behavior of the derivatives of the approximating sequence and the relation between the regular and singular dynamics of the associated Hamiltonian dynamical systems.
- There also exists a connection to the theory of generalized characteristics by the recent work on global propagation of singularities of the viscosity solutions of Hamilton-Jacobi equations ([9]), and [6, 7, 10] for more about the singularities propagation of weak KAM solutions.
- An application of standard Lasry-Lions approximation to the minimal homoclinic orbits ([8]).

Let \(H: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\) be a Tonelli Hamiltonian (i.e., \(H = H(x,p)\) is of \(C^2\) class and it is strictly convex in \(p\) and uniformly superlinear in \(p\)), and let \(L: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\) be the associated Tonelli Lagrangian. In this paper, we extend the Lasry-Lions regularization procedure to the viscosity solution of the discounted Hamilton-Jacobi equation

\[
\lambda u^\lambda(x) + H(x, Du^\lambda(x)) = 0, \quad x \in \mathbb{R}^n
\]
with a discount factor $\lambda > 0$. The associated dynamical system is dissipative system and it is a very special kind of contact type Hamiltonian systems (see, for instance, [21], [22], [11] and [23]). In fact, by defining a new Hamiltonian $H^\lambda(t, x, p) = e^{\lambda t}H(x, e^{-\lambda t}p)$, this equation can be reduced to a time-dependent evolutionary Hamilton-Jacobi equation
\[
D_t v + H^\lambda(t, x, D_xv) = 0.
\]
Therefore, one can define a kind of intrinsic Lasry-Lions regularization with respect to $u^\lambda$ as
\[
\hat{T}_t u^\lambda(x) = \sup_{y \in \mathbb{R}^n} \{u^\lambda(y) - A^\lambda_{s,t}(x, y)\},
\]
where $A^\lambda_{s,t}(x, y)$ is the fundamental solution with respect to the time-dependent Lagrangian $L^\lambda(t, x, v) = e^{\lambda t}L(x, v)$.

The main result of this paper clarifies the approximation property of this kind of Lasry-Lions regularization. We obtain not only the uniform convergence of $\hat{T}_t u^\lambda$ to $u^\lambda$ but also the limit of $DT_t u^\lambda$ as $t \to 0^+$. It is worth noting that the latter is closely connected to the intrinsic explanation of the propagation of singularities along generalized characteristics of the associated viscosity solutions ([9]).

To study the aforementioned intrinsic approximation, we need the regularity properties of the fundamental solutions $A^\lambda_{s,t}(x, y)$, which is the least action of the absolutely continuous curve connecting $x$ to $y$ from time $s$ to $t$. The required regularity result is a generalization of the relevant result in [9].

The paper is organized as follows: In section 2, we briefly review some fundamental facts of semiconcave functions and Tonelli’s theory in the calculus of variation. In section 3, we prove a Lasry-Lions approximation result for the time-dependent Lagrangians, then discuss this approximation method in a model of discounted Hamilton-Jacobi equations and its connection to the propagation of singularities of associated viscosity solutions.

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2. Viscosity Solutions and Semiconcave Functions

In this section, we briefly review some basic properties of semiconcave functions and the viscosity solutions of Hamilton-Jacobi equations.

Let $\Omega \subset \mathbb{R}^n$ be a convex open set. A function $u : \Omega \to \mathbb{R}^n$ is semiconcave (with linear modulus) if there exists a constant $C > 0$ such that
\[
\lambda u(x) + (1 - \lambda)u(y) - u(\lambda x + (1 - \lambda)y) \leq \frac{C}{2} \lambda(1 - \lambda)|x - y|^2
\]
for any $x, y \in \Omega$ and $\lambda \in [0, 1]$. The constant $C$ that satisfies the above inequality is called a semiconcavity constant of $u$ in $\Omega$. A function $u$ is said to be locally semiconcave if for each $x \in \Omega$ there exists an open ball $B(x, r) \subset \Omega$ such that $u$ is a semiconcave function on $B(x, r)$. 
If $D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function, $x \in \Omega$, the following closed convex sets

\[ D^+ u(x) = \{ p \in \mathbb{R}^n : \limsup_{y \rightarrow x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \leq 0 \} \]

\[ D^- u(x) = \{ p \in \mathbb{R}^n : \liminf_{y \rightarrow x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \geq 0 \} \]

are called the \textit{superdifferential} and \textit{subdifferential} of $u$ at $x$ respectively.

**Definition 2.2.** Let $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ be locally Lipschitz. We call a vector $p \in \mathbb{R}^n$ a \textit{limiting differential} of $u$ at $x$ if there exists a sequence $\{ x_k \} \subset \Omega \setminus \{ x \}$ such that $u$ is differentiable at $x_k$ for all $k \in \mathbb{N}$ and

\[ \lim_{k \rightarrow \infty} x_k = x, \quad \text{and} \quad \lim_{k \rightarrow \infty} Du(x_k) = p. \]

The set of all limiting differentials of $u$ at $x$ is denoted by $D^+ u(x)$.

**Proposition 2.3** ([12]). Let $u : \Omega \rightarrow \mathbb{R}$ be a semiconcave function and $x \in \Omega$. Then the following properties hold:

1. $D^+ u(x)$ is a nonempty closed convex set in $\mathbb{R}^n$ and $D^+ u(x) \subset \partial D^+ u(x)$, where $\partial D^+ u(x)$ denotes the topological boundary of $D^+ u(x)$.
2. The set-value function $x \mapsto D^+ u(x)$ is upper semi-continuous.
3. $D_+ u(x) = \text{co} D^+ u(x)$.
4. If $D^+ u(x)$ is a singleton, then $u$ is differentiable at $x$. Moreover, if $D^+ u(x)$ is a singleton for every point in $\Omega$, then $u \in C^1(\Omega)$.

Recall that a continuous real-valued function $u$ on $(0, +\infty) \times \mathbb{R}^n$ is called a \textit{viscosity supersolution} (resp. \textit{viscosity subsolution}) of the Hamilton-Jacobi equation

\[ D_t u + H(t, x, D_x u) = 0 \]

if for any $(t, x) \in (0, +\infty) \times \mathbb{R}^n$

\[ p_t + H(t, x, p_x) \geq 0 \quad (\text{resp.} \leq 0), \quad \forall (p_t, p_x) \in D^- u(t, x) \quad (\text{resp.} D^+ u(t, x)). \]

A continuous function $u$ is called a \textit{viscosity solution} of the equation if it is both a viscosity subsolution and a viscosity supersolution.

In this paper, we concentrate on Lagrangians on Euclidean configuration space $\mathbb{R}^n$. We say that a function $\theta : [0, +\infty) \rightarrow [0, +\infty)$ is \textit{superlinear} if $\theta(r)/r \rightarrow +\infty$ as $r \rightarrow +\infty$.

**Definition 2.4.** A function $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is called a (time-dependent) \textit{Tonelli Lagrangian} if $L$ is a function of class $C^2$ satisfying the following conditions:

1. $L_{vv}(t, x, v)$ is positive definite for all $(t, x, v) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$.
2. There exist two superlinear functions $\theta, \bar{\theta} : [0, +\infty) \rightarrow [0, +\infty)$ and a constant $c_0 \geq 0$ such that

\[ \bar{\theta}(|v|) \geq L(t, x, v) \geq \theta(|v|) - c_0, \quad (t, x, v) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n. \]
3. There exists a constant $c > 0$ such that

\[ |L_t(t, x, v)| \leq c(1 + L(t, x, v)), \quad (t, x, v) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n. \]

Let $L$ be a Tonelli Lagrangian and let $H$ be the associated Hamiltonian. Given $x \in \mathbb{R}^n$, $y \in B(x, R)$ with $R > 0$, and $s < t$, we define

\[ \Gamma_{x,y}^{s,t} = \{ \xi \in W^{1,k}([s,t], \mathbb{R}^n) : \xi(s) = x, \xi(t) = y \}, \]
and

\begin{equation}
A_{s,t}(x, y) = \inf_{\xi \in \Gamma_{s,t}^{x,y}} \int_s^t L(\tau, \xi(\tau); \dot{\xi}(\tau)) d\tau.
\end{equation}

The existence the minimizers in (2.4) is a well known result in Tonelli’s theory, (see, for instance, [12]). We call $\xi \in \Gamma_{s,t}^{x,y}$ a minimizer for $A_{s,t}(x, y)$ if

\[ A_{s,t}(x, y) = \int_s^t L(\tau, \xi(\tau); \dot{\xi}(\tau)) d\tau. \]

It is well known that such a minimizer $\xi$ must be of class $C^2$.

It is known that, for any $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$, the function $u(t, x) = A_{t_0,t}(x_0, x)$ is called a fundamental solution of the Hamilton-Jacobi equation

\begin{equation}
D_t u(t, x) + H(t, x, D_x u(t, x)) = 0 \quad x \in \mathbb{R}^n, t > t_0.
\end{equation}

When considering the Cauchy problem with initial condition $u(t_0, x) = u_0(x)$ with $u_0 \in \text{Lip}(\mathbb{R}^n)$, the associated unique viscosity solution has the following representation:

\begin{equation}
A_{s,t}(x, y) = \sup_{\xi \in \Gamma_{s,t}^{x,y}} \left\{ u_0(y) - A_{s,t}(x, y) \right\}, \quad x \in \mathbb{R}^n, t > t_0.
\end{equation}

Let us recall the Lax-Oleinik operators for time-dependent Lagrangians. For any $s < t$, we define

\begin{align}
T^+_n t u_0(x) &:= \sup_{y \in \mathbb{R}^n} \{ u_0(y) - A_{s,t}(x, y) \}, \\
T^-_n t u_0(x) &:= \inf_{y \in \mathbb{R}^n} \{ u_0(y) + A_{s,t}(x, y) \}.
\end{align}

Therefore, $u(t, x) = T_{t_0,t} u_0(x)$ is the unique viscosity solution of (2.2) with the initial condition $u(t_0, x) = u_0(x)$. For any $t_1 > t_0$, it is well known that $u_1(x) = u(t_1, x) = T_{t_0,t_1} u_0(x)$ is a locally semiconcave function (see [12]).

3. LASRY-LIONS APPROXIMATION FOR DISCOUNTED EQUATIONS

3.1. Positive type Lax-Oleinik Operators in time-dependent case. In [6], the authors studied the intrinsic relation between propagation of singularities and the procedure of sup-convolution. In this section, we concentrate on the case of sup-convolution $T^+_n t u$ with $u$ a locally semiconcave function.

Let $u : \mathbb{R}^n \to \mathbb{R}$ be a locally semiconcave function. Fixed $x \in \mathbb{R}^n$, $t_0 > 0$, $\kappa > 0$ and $0 < T < 1$. For any $t \in [t_0, t_0 + T]$, we define the local barrier function $\psi^{x,t}_{t_0,t} : B(x, \kappa(t-t_0)) \to \mathbb{R}$ as

\[ \psi^{x,t}_{t_0,t}(y) := u(y) - A_{t_0,t}(x, y). \]

Now we need the following condition:

\[ (M) \quad \psi^{x,t}_{t_0,t} \text{ attains a unique maximum point in } B(x, \kappa(t-t_0)). \]

The following result shows condition (M) is satisfied if $u \in \text{Lip}(\mathbb{R}^n, \mathbb{R})$. It is a slight generalization of Lemma 3.1 in [9].

**Lemma 3.1.** Suppose $L$ is a Tonelli Lagrangian and let $u_0 \in \text{Lip}(\mathbb{R}^n, \mathbb{R})$. Then, the supremum in (2.4) is attained for every $(t, x) \in (t_0, +\infty) \times \mathbb{R}^n$. Moreover,
there exists a constant $\kappa_0 > 0$, depending only on $\text{Lip}(u_0)$, such that, for any $(t, x) \in (t_0, +\infty) \times \mathbb{R}^n$ and any maximum point $y_{t,x}$ of $\psi_{t,x}^\tau(y)$, we have

$$|y_{t,x} - x| \leq \kappa_0 (t - t_0).$$

If $\xi : [t_0, t] \to \mathbb{R}^n$ is the unique minimizer for $A_{t_0,t}(x, y)$, we define the associated dual arc $p_t$ as

$$p_t(s) = L_v(s, \xi_t(s), \dot{\xi}_t(s)), \quad s \in [t_0, t].$$

**Theorem 3.2.** Suppose $u : \mathbb{R}^n \to \mathbb{R}$ is a locally semiconcave function and $L$ is a Tonelli Lagrangian. If condition $[\mathcal{M}]$ is satisfied, then $T_{t_0,t}^+ u$ is of class $C^{1,1}_{\text{loc}}$ for all $t \in [t_0, t_0 + T]$. Moreover, $\lim_{t \to t_0^+} DT_{t_0,t}^+ u(x) = q_x$, where $q_x$ is the unique element of $D^+ u(x)$ such that

$$H(t_0, x, q_x) = \min_{p \in D^+ u(x)} H(t_0, x, p)$$

**Proof.** Fix $(t_0, x) \in \mathbb{R} \times \mathbb{R}^n$, we have that $\tilde{\psi}_{t_0,t}^\tau$ attains the maximum at $y_t \in B(x, \kappa(t - t_0))$ for each $t \in [t_0, t_0 + T]$ by condition $[\mathcal{M}]$. Let $\xi_t \in \Gamma_{t_0,t}^+$ be the minimizer for $A_{t_0,t}(x, y_t)$, by $[\text{A.2}]$, we have

$$L_v(t, \xi_t(t), \dot{\xi}_t(t)) = D_y A_{t_0,t}(x, y_t) \in D^+ u(y_t),$$

since $y_t$ is a maximizer of $\psi_{t_0,t}^\tau$. Moreover, the family $\{\xi_t(\cdot)\}_{t \in [t_0, t_0 + T]}$ is equi-Lipschitz by Lemma 3.1 and Proposition A.2. Let $v_t := (\xi_t(t) - x)/(t - t_0)$, we obtain

$$\left|\frac{\xi_t(t) - x}{t - t_0} - \dot{\xi}_t(t_0)\right| \leq \frac{1}{t - t_0} \int_{t_0}^{t} |\xi_t(s) - \dot{\xi}_t(t_0)| ds \leq \frac{C_1}{t - t_0} \int_{t_0}^{t} (s - t_0) ds = \frac{C_1}{2} (t - t_0).$$

Thus, we have

$$v_0 := \lim_{t \to t_0^+} v_t = \lim_{t \to t_0^+} \dot{\xi}_t(t_0).$$

Since $u$ is a locally semiconcave function, for any $y \in B(x, \kappa(t - t_0))$, $p_x \in D^+ u(x)$ and $p_y \in D^+ u(y)$, we have $([12] \text{ Proposition 3.3.10})$

$$\langle p_y - p_x, y - x \rangle \leq C_2 |y - x|^2.$$

Taking any $t_k \to t_0$, we have

$$p_{y_k} = L_v(t_k, \xi_{t_k}(t_k), \dot{\xi}_{t_k}(t_k)) \in D^+ u(y_k).$$

Then, for any $p_x \in D^+ u(x)$

$$\langle p_x - L_v(t_k, \xi_{t_k}(t_k), \dot{\xi}_{t_k}(t_k)), v_{t_k} \rangle + C_2 (t_k - t_0)|v_{t_k}|^2 \geq 0.$$

Taking the limit in the above inequality as $k \to \infty$ we obtain

$$\langle p_x, v_0 \rangle \geq \langle L_v(t_0, x, v_0), v_0 \rangle = \langle q_x, v_0 \rangle, \quad \forall p_x \in D^+ u(x),$$

where $q_x := L_v(t_0, x, v_0) \in D^+ u(x)$ by the upper semicontinuity of $x \mapsto D^+ u(x)$. Thus, for all $p_x \in D^+ u(x)$,

$$H(t_0, x, p_x) \geq \langle L_v(t_0, x, v_0), v_0 \rangle - L(t_0, x, v_0) = H(t_0, x, q_x),$$

$^1$\text{Lip}(u_0) stands for the least Lipschitz constant of $u_0$.
and \( q_x \) is the unique minimum point of \( H(t_0, x, \cdot) \) on \( D^+ u(x) \). The uniqueness of \( p_x \) implies the uniqueness of \( v_0 \) since \( L_v(t_0, x, \cdot) \) is injective, and we have
\[
\lim_{t \to t_0^+} DT_{t_0,t}^+ u(x) = \lim_{t \to t_0^+} L_v(t_0, \xi(t_0), \xi(t_0)) = q_x.
\]
This completes the proof of the theorem. \( \square \)

### 3.2. Lasry-Lions regularization on discounted equations

For \( \lambda > 0 \), we consider the Hamilton-Jacobi equations with discount factors,
\[
(HJ_d) \quad \lambda u^\lambda(x) + H(x, Du^\lambda(x)) = 0, \quad x \in \mathbb{R}^n.
\]
Multiplying \( e^{\lambda t} \) in \((HJ_d)\) and defining \( v^\lambda(t, x) = e^{\lambda t} u^\lambda(x) \), one can check that \( v = v^\lambda \) is a viscosity solution of
\[
(HJ_e) \quad D_t v + H^\lambda(t, x, D_x v) = 0
\]
with \( H^\lambda(t, x, p) = e^{\lambda t} H(x, e^{-\lambda t} p) \), if \( u^\lambda \) is a viscosity solution of \((HJ_d)\). The associated Lagrangian \( L^\lambda \) with respect to \( H^\lambda \) has the form
\[
L^\lambda(t, x, v) = e^{\lambda t} L(x, v).
\]

**Proposition 3.3.** \( u^\lambda(x) \) is a viscosity solution of \((HJ_d)\) if and only if \( v^\lambda(t, x) \) is a viscosity solution of \((HJ_e)\).

**Proof.** It is not hard to check that \( u^\lambda \) is a locally semiconcave function if and only if so is \( v^\lambda \) when restricted to any compact time interval. The local semiconcavity properties of viscosity solutions of \((HJ_d)\) and \((HJ_e)\) are well known results, see, for instance, [12]. Thus, our conclusion is a direct consequence of Proposition 5.3.1 in [12]. \( \square \)

Now, one can define a kind of **intrinsic Lasry-Lions regularization** with respect to \( u^\lambda \) as follows: let \( u^\lambda \) be a viscosity solution of the discounted Hamilton-Jacobi equation \((HJ_d)\), define
\[
\hat{T}_t u^\lambda(x) = \sup_{y \in \mathbb{R}^n} \{ u^\lambda(y) - A_{0,t}^\lambda(x, y) \} = T_{0,t}^+ v^\lambda(0, x),
\]
where \( A_{0,t}^\lambda(x, y) \) is the fundamental solution with respect to the Lagrangian \( L^\lambda \).

**Theorem 3.4.** If \( u^\lambda \) is a viscosity solution of the discounted Hamilton-Jacobi equation \((HJ_d)\), and \( \hat{T}_t u^\lambda \) is the associated intrinsic Lasry-Lions regularization, then we have \( \hat{T}_t u^\lambda \) is of class \( C^{1,1}_{loc} \) and \( \hat{T}_t u^\lambda \) tends to \( u^\lambda \) uniformly as \( t \to 0^+ \). Moreover, there exists an unique \( q_x^\lambda \in D^+ u^\lambda(x) \) such that
\[
H(x, q_x^\lambda) = \min_{p \in D^+ u^\lambda(x)} H(x, p)
\]
and \( \lim_{t \to 0^+} D \hat{T}_t u^\lambda(x) = q_x^\lambda \).

**Proof.** Notice that \( L^\lambda \) satisfies conditions (L1)-(L3) for any fixed \( \lambda > 0 \). The \( C^{1,1} \) regularity of \( \hat{T}_t u^\lambda \) is a direct consequence of the \( C^{1,1} \) regularity of \( A_{0,t}(x, \cdot) \), (A.6) and condition (M) which holds by a slight generalization of Lemma 3.1 in [9], since \( v^\lambda(0, \cdot) = u^\lambda \) is semiconcave and Lipschitz. Now, fix \( T > 0 \) as in Theorem 3.2, then for any \( t \in (0, T] \), there exists a unique maximizer \( y_{t,x} \) of \( u^\lambda(\cdot) - A_{0,t}^\lambda(x, \cdot) \), and \( \lim_{t \to 0^+} y_{t,x} = x \), therefore \( \hat{T}_t u^\lambda \) tends to \( u^\lambda \) uniformly as \( t \to 0^+ \).
Applying Theorem 3.2 to the solution \( v^\lambda \) of (HJ\( \lambda \)), there exists a unique \( q^\lambda_x \in D^+ v^\lambda(0, x) = D^+ u^\lambda(x) \) such that
\[
\lim_{t \to t_0^+} DT_{0,t}^+ v^\lambda(0, x) = q^\lambda_x \in D^+ v^\lambda(0, x) = D^+ u^\lambda(x)
\]
and
\[
H^\lambda(0, x, q^\lambda_x) = \min_{p \in D^+_0 v^\lambda(0, x)} H^\lambda(0, x, p).
\]
which is equivalent to (3.4). \( \square \)

Let \( \lambda > 0 \), a recent work by Davini, et al. (15) shows the unique solution \( u^\lambda \) of (HJ\( \lambda \)) converges uniformly, as \( \lambda \to 0^+ \), to a certain viscosity solution of the stationary Hamilton-Jacobi equation
\[
(HJ_\lambda) \quad H(x, Du(x)) = 0,
\]
when 0 is Mańe’s critical value. Comparing to the results in [14], there exists a unique \( q_x \in D^+ u(x) \) such that
\[
(3.5) \quad H(x, q_x) = \min_{p \in D^+ u(x)} H(x, p),
\]
\[
\lim_{\lambda \to 0^+} DT_t^+ u(x) = q_x. \quad \text{Therefore, one can raise the following problem:}
\]
**Problem:** For \( q^\lambda_x \) and \( q_x \) defined in (3.4) and (3.5), does \( \lim_{\lambda \to 0^+} q^\lambda_x = q_x \)?

**Remark 3.5.** To answer Problem above, a possible systematic approach will be based on the recent works [21] and [22]. Moreover, one can understand such a problem as follows (11 and 23): We suppose \( L \) is a function of \( C^2 \) class and it satisfies the following conditions:

(L1) \( L_{vv}(x, r, v) > 0 \) for all \( (x, r, v) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \);

(L2) For each \( r \in \mathbb{R} \), there exist two superlinear and nondecreasing function \( \overline{\theta}_r, \theta_r : [0, +\infty) \to [0, +\infty) \), \( \theta_r(0) = 0 \) and \( c_r > 0 \), such that
\[
\overline{\theta}_r(|p|) \geq L(x, r, v) \geq \theta_r(|p|) - c_r, \quad (x, v) \in \mathbb{R}^n \times \mathbb{R}^n.
\]

(L3) There exists \( K > 0 \) such that
\[
|L_r(x, r, v)| \leq K, \quad (x, r, v) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n.
\]
Fix \( x, y \in \mathbb{R}^n \), \( u \in \mathbb{R} \) and \( t > 0 \). Define \( \Gamma^t_{x,y} = \{ \xi \in AC([0, t], \mathbb{R}^n) : \xi(0) = x, \xi(t) = y \} \). Let \( \xi \in \Gamma^t_{x,y} \), we consider the Carathéodory equation
\[
(3.6) \quad \dot{u}_\xi(s) = L(\xi(s), u_\xi(s), \dot{\xi}(s)), \quad \text{a.e.} \ s \in [0, t]
\]
with initial conditions \( u_\xi(0) = u \). We define
\[
(3.7) \quad A(t, x, y, u) = u + \inf_{\xi} \int_0^t L(\xi(s), u_\xi(s), \dot{\xi}(s)) \, ds,
\]
where the infimum is taken over of \( \xi \in \Gamma^t_{x,y} \) and \( u_\xi : [0, t] \to \mathbb{R}^n \) is a absolutely continuous curve determined by (3.6). In the case of discounted equations, \( L(x, u, v) = -\lambda u + L(x, v) \). One can define the negative type Lax-Oleinik operator \( T^-_t : C(\mathbb{R}^n, \mathbb{R}) \to C(\mathbb{R}^n, \mathbb{R}) \) for any \( t > 0 \):
\[
(3.8) \quad (T^-_t \phi)(x) = \inf_{y \in \mathbb{R}^n} A(t, y, x, \phi(y)).
\]
It is not very difficult to show the fundamental solution \( A(t, x, y, u) \) is locally semiconcave with constants depending on \(|x - y|/t \) and \( \lambda \). Thus, the key point of the
uniform semiconcavity of $u^\lambda$ is to show that there exists $\kappa_0 > 0$ independent of $x$ such that the minimum of $A(t, \cdot, x, \phi(\cdot))$ on $\mathbb{R}^n$ is attained and all the minimum points is contained in $B(x, \kappa_0 t)$. If $\phi = u^\lambda$ is a Lipschitz weak KAM solution of the equation $H(x, u(x), Du(x)) = 0$ with respect to $H(x, u, p) = \lambda u + H(x, p)$, then this gives a uniform bound of the velocity all of backward calibrated curves which leads to the uniform constants in the associated semiconcavity estimate. We will answer Problem above in a much more general context in the future.

3.3. Connection to singularities. The intrinsic Lasry-Lions regularization is closely connected to the propagation of singularities of the solution $u^\lambda$ of (HJ). It is obvious that $u^\lambda$ shares the singularities of $v^\lambda$ in (HJ). In this section, we suppose that 0 is Mañe’s critical value.

Recall that a point $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ is called a singular point of a semiconcave function $u(t, x)$ if $D^+ u(t, x)$ is not a singleton. The set of all singular points of $u$ is denoted by $\text{Sing}(u)$. It is obvious that $(t, x) \in \text{Sing}(v^\lambda)$ if and only if $x \in \text{Sing}(u^\lambda)$.

Using the representation formula of $v^\lambda$ (see, for instance, [15 Proposition 3.5]), for any $\tau, x \in \mathbb{R}$, we obtain that

$$v^\lambda(\tau, x) = v^\lambda(\tau - t, \gamma_x(\tau - t)) + \int_{\tau - t}^\tau L^\lambda(s, \gamma_x(s), \dot{\gamma}_x(s)) \, ds,$$

where the infimum is taken over all absolutely continuous curves $\gamma : (-\infty, \tau) \to \mathbb{R}^n$, with $\gamma(\tau) = x$. Moreover, there exists a Lipschitz and $C^2$ curve $\gamma_x : (-\infty, \tau) \to \mathbb{R}^n$, with $\gamma_x(\tau) = x$, such that, for any $t > \tau$,

$$v^\lambda(\tau, x) = v^\lambda(\tau - t, \gamma_x(\tau - t)) + \int_{\tau - t}^\tau L^\lambda(s, \gamma_x(s), \dot{\gamma}_x(s)) \, ds.$$  

(3.10)

It is clear that $v^\lambda$ is differentiable at $(\tau - t, \gamma_x(\tau - t))$ for all $t > \tau$, and $\gamma_x$ is an extremal of the associated Euler-Lagrange equation with respect to $L^\lambda$. As in the classical weak KAM theory, it is not difficult to prove that $(\tau, x)$ is a differentiable point of $v^\lambda$ if and only if there exists a unique $\gamma_x$ satisfying (3.10) (see also [12 Theorem 6.4.9]).

**Theorem 3.6.** Let $x_0 \in \text{Sing}(u^\lambda)$, then any maximizer $y_{t_1, x_0}$ of $v^\lambda(t_0, \cdot) - A^\lambda_{0, t_1}(x_0, \cdot)$ is contained in $\text{Sing}(u^\lambda)$ for all $t > 0$ and there exists $t_1 > 0$ such that the map $t \mapsto y_{t, x_0}$, the maximizers with respect to $v^\lambda(t_0, \cdot) - A^\lambda_{0, t}(x_0, \cdot)$ for $0 < t < t_1$, is continuous. Moreover, the right derivative $\frac{d}{dt} y_{t_0, x_0} |_{t=0^+}$ exists and it is equal to $q^\lambda_{x_0}$ as in Theorem 3.4, i.e., $q^\lambda_{x_0}$ is the unique element in $D^+ v^\lambda(t_0, x_0)$ such that

$$H(x_0, q^\lambda_{x_0}) = \min_{p \in D^+ v^\lambda(t_0, x_0)} H(x_0, p).$$

(3.11)

**Proof.** Fix $t_0 \in \mathbb{R}$, then $(t_0, x_0) \in \text{Sing}(v^\lambda)$ since $x_0 \in \text{Sing}(u^\lambda)$. For any $t > 0$ and $y_{t, x_0} \in \text{arg max} \{v^\lambda(t_0, \cdot) - A^\lambda_{0, t}(x_0, \cdot)\}$ (which is nonempty since $v^\lambda(t_0, \cdot)$ is Lipschitz and Lemma 3.1), suppose $y_{t, x_0}$ is a differentiable point of $v^\lambda(t_0, \cdot)$. Thus

$$0 \in D^+ \{v^\lambda(t_0, \cdot) - A^\lambda_{0, t}(x_0, \cdot)\}(y_{t, x_0}) = D_y v^\lambda(t_0, y_{t, x_0}) - D^- \{A^\lambda_{0, t}(x_0, \cdot)\}(y_{t, x_0}),$$

equivalently, $D_y v^\lambda(t_0, y_{t, x_0}) \in D^- \{A^\lambda_{0, t}(x_0, \cdot)\}(y_{t, x_0})$. It follows that $A^\lambda_{0, t}(x_0, \cdot)$ is differentiable at $y_{t, x_0}$ and

$$p_{t, x_0} = D_y v^\lambda(t_0, y_{t, x_0}) = D_y A^\lambda_{0, t}(x_0, y_{t, x_0}).$$
since $A^\lambda_0(x_0,\cdot)$ is locally semiconcave (see, for instance, \[12\]). Therefore, there exists two $C^2$ curves $\xi_{t,x_0} : [0, t] \to \mathbb{R}^n$ and $\gamma_{x_0} : (-\infty, t] \to \mathbb{R}^n$ such that $\xi_{t,x_0}(0) = x_0$, $\gamma_{x_0}(t) = \gamma_{t,x_0}(t) = y_{t,x_0}$ and $p_{t,x_0} = L_v(\gamma_{x_0}(t), \dot{\gamma}_{x_0}(t)) = L_v(\xi_{t,x_0}(t), \dot{\xi}_{t,x_0}(t))$.

Since $\xi_{t,x_0}$ and $\gamma_{x_0}$ has the same endpoint condition at $t$, then they coincide on $[0, t]$. Thus, $x_0 = \gamma_{x_0}(0)$ and $(0, x_0)$ is a differentiable point of $v^\lambda$ since $\gamma_{x_0}$ is a backward calibrated curve by \[10\]. On the other hand, $(0, x_0)$ is contained in $\text{Sing}(v^\lambda)$ since $(0, x_0) \in \text{Sing}(v^\lambda)$. This leads to a contradiction.

To prove \[11\], we need a slight modification of Theorem 3.2. Notice that $v^\lambda(t, \cdot)$ is equi-Lipschitz and equi-semiconcave for $t \in [0, t_1]$. By the regularity properties of the fundamental solutions, $v^\lambda(t_0, \cdot) - A^\lambda_0(t_0, \cdot)$ is strictly concave for $t \in (0, t_2]$, where $t_2 \leq t_1$ is determined by Proposition A.4 and the semiconcavity of $u^\lambda$. Therefore, $t \mapsto y_{t,x_0}$ is a continuous selection since the function $v^\lambda(t_0, \cdot) - A^\lambda_0(t_0, \cdot)$ is continuous. By the same argument as in the proof of Theorem 3.2 we obtain that

$$\frac{d}{dt} y_{t,x_0} |_{t=0^+} = q^\lambda_{x_0}$$

with $q^\lambda_{x_0}$ satisfying \[11\].

\[APPENDIX A. REGULARITY PROPERTIES OF FUNDAMENTAL SOLUTIONS\]

Here we collect some relevant regularity results with respect to the fundamental solutions of \[2.2\] on $\mathbb{R}^n$. The proofs of these regularity results are similar to those in \[9\] in the time-independent case. The difference is that we need an extra condition (L3) to ensure the uniform Lipschitz estimate of the minimizers in the relevant Tonelli-like variational problem. We omit the proof.

**Proposition A.1.** Let $a \leq s < t \leq b$, $R > 0$ and suppose $L$ satisfies condition (L1)-(L3). Given any $x \in \mathbb{R}^n$ and $y \in B(x, R)$, let $\xi \in \Gamma^s_{\infty} x, y$ be a minimizer for $A_{s,t}(x, y)$ and let $p(\cdot)$ be the dual arc. Then we have that

$$\sup_{\tau \in [s, t]} |\dot{\xi}(\tau)| \leq \kappa_T(R/(t-s)), \quad \sup_{\tau \in [s, t]} |p(\tau)| \leq \kappa_T(R/(t-s))$$

and

$$\sup_{\tau \in [s, t]} |\xi(\tau) - x| \leq \kappa_T(R/(t-s)),$$

where $\kappa_T : (0, \infty) \to (0, \infty)$ is nondecreasing and $T = b - a$.

Fix $x \in \mathbb{R}^n$ and suppose $R > 0$ and $L$ is a Tonelli Lagrangian. For any $a \leq s < t \leq b$, $T = b - a$ and $y \in B(x, R)$, let $\xi \in \Gamma^s_{\infty} x, y$ be a minimizer for $A_{s,t}(x, y)$ and let $p$ be its dual arc. Then there exists a nondecreasing function $\kappa_T : (0, \infty) \to (0, \infty)$ such that

$$\sup_{\tau \in [s, t]} |\dot{\xi}(\tau)| \leq \kappa_T(R/(t-s)), \quad \sup_{\tau \in [s, t]} |p(\tau)| \leq \kappa_T(R/(t-s)),$$

by Proposition A.1. Now, $a < b$, $x \in \mathbb{R}^n$ and $\lambda > 0$ define compact sets

$$K_{a,b,x,\lambda} := [a, b] \times B(x, \kappa(4\lambda)) \times B(0, \kappa(4\lambda)) \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n,$$

$$K^*_{a,b,x,\lambda} := [a, b] \times B(x, \kappa(4\lambda)) \times \overline{B}(0, \kappa(4\lambda)) \subset \mathbb{R} \times \mathbb{R}^n \times (\mathbb{R}^n)^*.$$
Suppose there exists a constant $C_T = \text{Proposition A.4.}$

Then any minimizer $\xi \in \Gamma_{x,y+z}^{s,t+h}$ for $A_{s,t+h}(x,y+z)$ and corresponding dual arc $p$ satisfy the following inclusions

$$
\{(\tau, \xi(\tau), \dot{\xi}(\tau)) : \tau \in [s, t + h]\} \subset K_{s,s+1,x,\lambda},
$$

$$
\{(\tau, \xi(\tau), p(\tau)) : \tau \in [s, t + h]\} \subset K_{s,s+1,x,\lambda}^*.
$$

**Proposition A.3.** Suppose $L$ is a Tonelli Lagrangian. Then for any $\lambda > 0$ there exists a constant $C_\lambda > 0$ such that for any $x \in \mathbb{R}^n$, $s < t$ with $T = t - s < 2/3$, $y \in B(x, \lambda T)$, and $(h, z) \in \mathbb{R} \times \mathbb{R}^n$ satisfying $|h| < T/2$ and $|z| < \lambda T$ we have

$$
A_{s,t+h}(x, y + z) + A_{s,t-h}(x, y - z) - 2A_{s,t}(x, y) \leq \frac{C_\lambda}{T} (|h|^2 + |z|^2).
$$

Consequently, $(t, y) \mapsto A_{s,t}(x, y)$ is locally semiconcave in $(0,1) \times \mathbb{R}^n$, uniformly with respect to $x$ and $s$.

**Proposition A.4.** Suppose $L$ is a Tonelli Lagrangian and, for any $\lambda > 0$, there exists $T_\lambda > 0$ such that for any $x \in \mathbb{R}^n$, $s < t$, the function $(t, y) \mapsto A_{s,t}(x, y)$ is semiconvex on the cone

$$
S_\lambda(x, T_\lambda) := \{(t, y) \in \mathbb{R} \times \mathbb{R}^n : T = t - s < T_\lambda, |y - x| < \lambda T\},
$$

and there exists a constant $C''_{\lambda} > 0$ such that for all $(t, y) \in S_\lambda(x, T_\lambda)$, all $h \in [0, T/2)$, and all $z \in B(0, \lambda T)$ we have that

$$
A_{s,t+h}(x, y + z) + A_{s,t-h}(x, y - z) - 2A_{s,t}(x, y) \geq -\frac{C''_{\lambda}}{T} (h^2 + |z|^2).
$$

Moreover, there exists $T''_{\lambda} \in (0, T'_\lambda]$ and $C'''_{\lambda} > 0$ such that for all $T \in (0, T''_{\lambda})$ the function $A_{s,t}(x, \cdot)$ is uniformly convex on $B(x, \lambda T)$ and for all $y \in B(x, \lambda T)$ and $z \in B(0, \lambda T)$ we have that

$$
A_{s,t}(x, y + z) + A_{s,t}(x, y - z) - 2A_{s,t}(x, y) \geq \frac{C'''_{\lambda}}{T} |z|^2.
$$

**Proposition A.5.** Suppose $L$ is a Tonelli Lagrangian and, for any $\lambda > 0$, there exists $T'_{\lambda} > 0$ such that for any $x \in \mathbb{R}^n$ the functions $(t, y) \mapsto A_{s,t}(x, y)$ and $(t, y) \mapsto A_{s,t}(y, x)$ are of class $C^{1,1}_{\text{loc}}$ on the cone $S_\lambda(x, T'_{\lambda})$ defined in (A.2). Moreover, for all $(t, y) \in S(x, T'_{\lambda})$

$$
D_y A_{s,t}(x, y) = L_v(t, \xi(t), \dot{\xi}(t)),
$$

$$
D_x A_{s,t}(x, y) = -L_v(s, \xi(s), \dot{\xi}(s)),
$$

where $\xi \in \Gamma_{x,y}^{s,t}$ is the unique minimizer for $A_{s,t}(x, y)$.

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