Statistics of the Number of Zero Crossings : from Random Polynomials to Diffusion Equation.

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(Dated: February 1, 2008)

We consider a class of real random polynomials, indexed by an integer $d$, of large degree $n$ and focus on the number of real roots of such random polynomials. The probability that such polynomials have no real root in the interval $[0, 1]$ decays as a power law $n^{-\theta(d)}$ where $\theta(d) > 0$ is the exponent associated to the decay of the persistence probability for the diffusion equation with random initial conditions in space dimension $d$. For $n$ even, the probability that such polynomials have no root on the full real axis decays as $n^{-2(\theta(d)+\Theta(2))}$. For $d = 1$, this connection allows for a physical realization of real random polynomials. We further show that the probability that such polynomials have exactly $k$ real roots in $[0, 1]$ has an unusual scaling form given by $n^{-\tilde{\varphi}(k/\log n)}$ where $\tilde{\varphi}(x)$ is a universal large deviation function.

PACS numbers: 02.50.-r, 05.40.-a,05.70.Ln, 82.40.Bj

Persistence properties and related first passage problems have been the subject of intense activities, both theoretically \cite{1} and experimentally \cite{2,3} these last few years. The persistence probability $p(t)$ for a time dependent stochastic process of zero mean, is defined as the probability that it has not changed sign up to time $t$. In many physical situations, $p(t)$ was found to decay at large time as a power law $p(t) \propto t^{-\theta}$. Surprisingly, the persistence exponent $\theta$ was found to be highly non trivial even for simple systems. One example is the diffusion equation in space dimension $d$, the solution $p_0(t, L)$ is simply the probability that $\phi(x, t)$ does not change sign up to time $t$. It was found \cite{4} that for $t \gg 1$, $p_0(t, L)$ has a finite size scaling form

$$p_0(t, L) \propto L^{-\theta(d)} h(L^z/t)$$

with $h(u) \sim c^u$, a constant independent of $L$ and $t$, for $u \ll 1$ and $h(u) \propto u^{\theta(d)}$ for $u \gg 1$, where $\theta(d)$ is a non trivial exponent, e.g. $\theta(1) = 0.1207$, $\theta(2) = 0.1875$. The scaling form of $p_0(t, L)$ indicates that $p_0(t, L) \propto t^{-\theta(d)}$ for large $t$ in a infinite system. Alternatively, $\theta(d)$ can be extracted by measuring $p_0(t, L) \propto L^{-\theta(d)}$ for very large time $t \gg L^z$. Remarkably, the persistence for $d = 1$ was observed in experiments on magnetization of spin polarized Xe gas and the exponent $\theta_{exp}(1) \approx 0.12$ was measured \cite{5}, in good agreement with analytical approximations and numerical simulations \cite{4,5}.

Another apparently unrelated problem concerns the roots of random polynomials (i.e. polynomials with random coefficients), which have attracted renewed interest over the last few years \cite{6,7} in the context of probability and number theory. A recent work \cite{8} proposed a physical realization of the complex roots of Weyl complex polynomials in a system of a rotating quasi-ideal atomic Bose gas. Here we focus instead on the real roots of a class of real random polynomials indexed by an integer $d$

$$f_n(x) = a_0 + \sum_{i=1}^{n-1} a_i x^{(d-2)/4} x^i$$

Here $a_i$'s are real Gaussian random variables of zero mean and with correlations $(a_i a_j) = \delta_{ij}$. We will see below that, for $d = 2$, where $f_n(x)$ reduce to the famous Kac polynomials \cite{9}, the statistics of the real roots of $f_n(x)$ is identical in the 4 subintervals $(-\infty, -1]$, $[-1, 0)$, $[0, 1]$ and $[1, +\infty]$. However, for $d \neq 2$, the statistics of real roots of $f_n(x)$ depend on $d$ in the two inner intervals $[-1, 0]$ and $[0, 1]$, while it is identical to the case $d = 2$ in the two outer ones. In this letter we will focus primarily on the interval $[0, 1]$ and ask : what is the probability $P_0(1, n)$ that $f_n(x)$ has no real root in $[0, 1]$ ? Recently, it was found, for $d = 2$, that $P_0(1, n) \propto n^{-\zeta(2)}$ for large $n$ where the exponent $\zeta(2) \approx 0.19(1)$ was computed numerically \cite{10}. In addition, for the special case $d = 2$, the authors of Ref. \cite{10} showed that $P_0(1, n)$ is related to the probability of no zero crossing of a Gaussian stationary process (GSP) with correlator $\text{sech}(|T|/2)$.

The purpose of this Letter is to provide a link between the persistence of the diffusion equation and the probability $P_0(1, n)$ that $f_n(x)$ has no real root in $[0, 1]$. For arbitrary dimension $d$, we show that $P_0(1, n) \propto n^{-\zeta(d)}$ with $\zeta(d) = \theta(d)$. Given that $\theta(1)$ was measured experimentally \cite{4}, this demonstrates an experimental realization of real random polynomials. The connection between these two problems in arbitrary $d$ is achieved by showing that both problems can be mapped to the same GSP. In addition, we compute the probability that a "smooth" GSP, such as the one that appears in the context of dif-
fusion equation, crosses zero exactly k times up to time $T$. Translated into the language of random polynomials, our analysis shows that the probability $P_0(t, L)$ that $f_n(x)$ has exactly k real roots in the interval $[0, 1]$ has a rather unusual scaling form (for large k, large n, but keeping the ratio $k/L$ fixed)

$$P_k(1, n) \propto n^{-\tilde{\varphi}\left(\frac{k}{\log n}\right)} \tag{3}$$

where $\tilde{\varphi}(x)$ is a large deviation function, with $\tilde{\varphi}(0) = \zeta(d)$. Besides, our numerical analysis suggests that $\tilde{\varphi}(x)$ is universal in the sense that it is independent of the distribution of $a_i$ provided $(a_i^2)$ is finite.

To study the persistence probability $p_0(t, L)$ of the diffusion equation, it is customary to study the normalized process $X(t) = \phi(x(t))/\phi(x(t), t)^{1/2}$. Its autocorrelation function $\rho(t, t') = \langle X(t)X(t') \rangle$ is given, in the limit $t, t' \ll L^2$ by $\rho(t, t') = |4t/(t+t')|^{2d/4}$. In terms of logarithmic time variable $T = \log t$, $X(T)$ is a GSP with correlator $\rho(T, T') \equiv \langle X(T)X(T') \rangle = \text{sech}(T-T'/2)^{d/2}$, which decays exponentially for large $|T-T'|$. Thus the persistence probability $p_0(t, L)$, for $t \ll L^2$, reduces to the computation of the probability $P_0(T)$ of no zero crossing of $X(T)$ in the interval $[0, T]$. It is well known that if $a(T) < 1/T$ at large $T$ then $P_0(T) \sim \exp[-\beta T]$ decays exponentially for large $T$ where the decay constant $\beta$ depends on the full stationary correlator $a(T)$.

Reverting back to the original time $t = e^T$, one finds $p_0(t, L) \sim t^{-\beta(d)}$, for $t \ll L^2$. In the opposite limit $t \gg L^2$, one has $p_0(t, L) \rightarrow A_L$, a constant which depends on $L$. These two limiting behaviors of $p_0(t, L)$ can be combined into a single finite size scaling form in Eq. (1) where $\beta(d)$ is the decay constant associated with the no zero crossing probability of the GSP with correlator $a(T) = \text{sech}(T/2)^{d/2}$. \[\text{Eq. (1)}\]

The mapping of real random polynomials $f_n(x)$ in Eq. (2) to a GSP is more subtle. We first observe that for large $n$ the real roots of $f_n(x)$ are concentrated around $x = \pm 1$. To show this, we have generalized the Kac’s method to compute the mean density of real roots $\rho_n(x)$ of $f_n(x)$ (2). We do not give the details of the computation and simply quote the results here. We find that $\rho_n(\pm 1) \sim 2n^{1/2}/(d+1)/\pi(d+2)$ and away from these singularities $\rho_\infty(x \neq \pm 1)$ is given, for $|x| < 1$, by

$$\rho_\infty(x) = \frac{(\text{Li}_{-1-d/2}(x^2)(1+\text{Li}_{-1-d/2}(x^2)) - \text{Li}_{-d/2}(x^2))}{\pi |x|(1+\text{Li}_{-1-d/2}(x^2))} \tag{4}$$

where $\text{Li}_n(z) = \sum_{i=1}^{\infty} z^i/i^n$ is the polylogarithm function, yielding back $\rho_\infty(x \neq \pm 1) = (\pi/|x^2-1|)^{-1}$ for $d = 2$. In particular, one has $\rho_\infty(0) = 1/\pi$ for all $d$, and $\rho_\infty(x) \sim (d/2)^{1/2} (2\pi (1-x^2))^{-1}$ for $x \rightarrow 1^-$. For $|x| > 1$, $\rho_\infty(x) = 1/|\pi(x^2-1)|$ for all $d$. In Fig. (1a), we plot $\rho_\infty(x)$ for $d = 1$ where the divergence at $x = \pm 1$ indicates that the real roots concentrate around $x = \pm 1$ for large $n$.

![FIG. 1: a) $\rho_\infty(x)$ given analytically in Eq. (4) and below as a function of $x$ for $d = 1$. The divergences for $x = \pm 1$ indicate that the real roots concentrate around $x = \pm 1$ for large $n$. b) Plot of $P_0(n)$ as a function of $n$ for different values of $d = 1, 2, 3, 4, 10$. The lines are guide to the eyes. The measured exponents are given by 0.12, 0.18, 0.23, 0.27, 0.46 for $d = 1, 2, 3, 4, 10$ respectively, in full agreement with the numerical values for $\eta(d)$ obtained in Ref. 4, 6.](image)

The random polynomial $f_n(x)$ being a Gaussian process, it is completely determined by its two-point correlator $C_n(x, x') = \langle f_n(x) f_n(x') \rangle$. Using the fact that $a_i$’s are uncorrelated, it is easy to see that

$$C_n(x, x') = 1 + \sum_{i=1}^{n-1} i^{(d-2)/2} (xx')^i. \tag{5}$$

It is useful to introduce the normalized correlator $C_n(x, x') = C_n(x, x')/C_n(0, x)^{1/2}C_n(x', x')^{1/2}$. From the analysis of $C_n(x, x')$ in the large $n$ limit, one can show that, asymptotically, $C_n(x)$ takes independent values in the 4 subintervals $]-\infty, -1[, [-1, 0[, [0, 1[, [1, +\infty[$. For $d = 2$, $C_n(x, x')$ has a special symmetry, namely $C_n(-x, x') = C_n(x, -x') = C_n(x/1, x')$, meaning that the normalized Gaussian processes in these 4 subintervals are independent and isomorphic. Denoting $P_0(1, n)$ as the probability of no zero crossing in $[0, 1]$, it follows that the probability $q_0(n)$ that there are no real roots for $d = 2$ is precisely equal to $P_0(1, n)^4$ for large $n$. For $d \neq 2$, one can show that the behavior of $C_n(x, x')$ depends on $d$ in the two inner intervals (see below), while it behaves like for $d = 2$ in the two outer ones.

We now focus on the interval $[0, 1]$ and to make a precise connection with the persistence probability $p_0(t, L)$ defined in the context of the diffusion equation we define $P_0(x, n)$, for $x \leq 1$, as the probability that $f_n(x)$ has no real root in the interval $[0, x]$. We next reparametrize the polynomial with a change of variable, $x = 1 - 1/t$. One finds that the relevant scaling limit of $C_n(t, t')$ is obtained for $t, t' \rightarrow \infty$ keeping $\tilde{t} = t/n$ and $\tilde{t}' = t'/n$ fixed (see also Ref. 17). In that scaling limit one finds that $C_n(t, t') \rightarrow C(\tilde{t}, \tilde{t}')$ with the asymptotic behaviors

$$C(\tilde{t}, \tilde{t}') \sim \begin{cases} \frac{\Gamma(\tilde{t}^{2/3})}{\Gamma(\tilde{t}'^{2/3})} \frac{d}{4}, & \tilde{t}, \tilde{t}' \ll 1 \\ 1, & \tilde{t}, \tilde{t}' \gg 1 \end{cases} \tag{6}$$

For $\tilde{t}, \tilde{t}' \ll 1$, this correlator is exactly the same as the one found for diffusion, $C(\tilde{t}, \tilde{t}') = a(t, t')$ in this regime.
Since a Gaussian process is completely characterized by its two point correlator, we conclude that the diffusion process and the random polynomial are essentially the same Gaussian process and hence have the same zero crossing properties. In the opposite limit, $\hat{t}, \hat{t}' \gg 1$ the fact that $C(\hat{t}, \hat{t}') \rightarrow 1$ suggests that $P_0(x, n)$ goes to a constant when $x \rightarrow 1$. Therefore, in complete analogy with Eq. (10) we propose the scaling form for random polynomials

$$P_0(x, n) \propto n^{-\theta(d)} \tilde{h}(n(1 - x))$$

with $\tilde{h}(u) \sim e^{\mu u}$ for $u \ll 1$ and $\tilde{h}(u) \sim u^{\theta(d)}$ for $u \gg 1$, where $\theta(d)$ is the persistence exponent associated to the diffusion equation in dimension $d$. Note that $n$ here plays the role of $L^2$ in diffusion problem while the variable $1 - x$ is the analogue of the inverse time $1/t$. This scaling form (7), which we verify numerically, thus establishes a direct link between the real roots of real random polynomials and the diffusion equation with random initial conditions. It also follows from our earlier discussion on the 4 subintervals that the probability that $f_{n}(x)$ has no real root decays like $n^{-b(d)}$ with $b(d) = 2(\theta(d) + \theta(2))$ where $2\theta(d)$ is contributed by the 2 inner intervals and $\theta(2)$ by the two outer intervals.

We have verified this scaling form (7) numerically by computing the number of real roots in the interval $[0, x]$ of random polynomials such as in Eq. (2) for different degrees $n$. In each case, the probability distribution $P_0(x, n)$ is obtained by averaging over $10^4$ realizations of the random variables $\theta_i$’s, drawn independently from a Gaussian distribution of unit variance. The plot shown on Fig. 2 for $d = 2$, shows a good agreement with the scaling in Eq. (7) with $\theta(2) = 0.1875$, in agreement with the numerical value reported in Ref. [10]. We have also calculated the exponent $\theta(d)$ in Eq. (1) for various dimensions, by measuring $P_0(1, n)$ for different values of $d$.

The results are shown in Fig. (1) b) where, according to our prediction in Eq. (2) $P_0(1, n) \sim n^{-\theta(d)}$ for large $n$. The value of $\theta(d)$ obtained this way is fully compatible with previous numerical estimates [4, 5] of $\theta(d)$ for the diffusion equation.

We now generalize our analysis and consider the probability $p_k(t, L)$ that the diffusing field $\phi(x, t)$ crosses zero exactly $k$ times up to time $t$ (similarly, one considers the probability $P_k(x, n)$, $x < 1$, that such polynomials have exactly $k$ real roots [10] in the interval $[0, x]$, see below). Let us first consider the regime $1 \ll t \ll L^2$. In this regime, $p_k(t, L)$ is given by the probability $P_k(T)$ that $X(T)$ crosses zero exactly $k$ times where $X(T)$ is a GSP with correlations $a(|T - T'|) = |\text{sech}(|T - T'|)|^{d/2}$, where $T = \log t$. Since, $a(T) = 1 - d(T^2 + o(T^2))$ for small $T$, our GSP is a smooth process with a finite density of zero crossings given by the Rice’s formula $\mu = (-a''(0))^1/2/\pi$. We propose the following scaling form for large $T$ and large $k$

$$\log P_k(T) = -T \varphi \left( \frac{k}{\mu T} \right).$$

To understand the origin of this scaling form, let us consider the generating function $\hat{P}(p, T) = \sum_{k=0}^{\infty} p^k P_k(T)$ [14]. It turns out that, $\hat{P}(p, T) \sim \exp(-\tilde{\theta}(p)T)$, where for a smooth GSP $\tilde{\theta}(p)$ depends continuously on $p$ : this was shown exactly for the random acceleration process and approximately using the independent interval approximation for arbitrary smooth GSP – and checked numerically for the diffusion equation with random initial conditions [11]. If the scaling in Eq. (8) holds, one gets by steepest descent method valid for large $T$, $\tilde{\theta}(p) = \min_{x > 0} [\mu x \log p - \varphi(x)]$. Inverting the Legendre transform we get

$$\varphi(x) = \max_{0 \leq x \leq 1} [\mu x \log p + \varphi(x)]$$

Notice that although $\tilde{\theta}(p)$ is a priori defined on the interval $[0, 1]$, the computation of $\varphi(x)$ involves an analytical continuation of $\tilde{\theta}(p)$ on $[0, 2]$. Going back to real time $t$, Eq. (9) then yields a rather unusual scaling form valid in the limit $1 \ll t \ll L^2$

$$p_k(t, L) \sim t^{-\varphi \left( \frac{1}{\mu \log t} \right)}.$$ 

For $k$ close to $\mu \log t$, one expects $p_k(t, L)$ to behave locally as a Gaussian and $\varphi(x)$ is thus quadratic around $x = 1$. Away from the minimum, we have not been able to obtain $\varphi(x)$ analytically. We have thus tested the scaling form (10) numerically. We used a space time discretized diffusion equation

$$\phi_i(t + 1) = \phi_i(t) + \sum_j [\phi_j(t) - \phi_i(t)]$$

where $j$ runs over the nearest neighbours of $i$ on a $d$-dimensional square lattice. A stability analysis shows
that the solution is unstable for \( a > a_c = 1/2d \) and we chose \( a = a_c/2 \) that provided the quickest onset of the asymptotic behavior. The initial values \( \phi_i(0) \)'s were chosen independently from a Gaussian distribution of zero mean and unit variance. Fig. 3 shows the results for \( d = 2 \), i.e. the case of Kac’s polynomials [2]. We computed \( p_k(t, L) \) for different times \( t \) for a system of linear size \( L = 256 \) and averaging over 100 different realizations of the random initial condition. In the inset of Fig. 3 we plot \( -\log p_k(t, L)/\log t \) as a function of the rescaled variable \( t, L \). In Fig. 3 these curves for different time \( t \) fall on the same master curve when plotted as a function of the rescaled variable \( k/\mu \log t \) with \( \mu = 1/2\pi \) in this case, confirming the validity of the scaling form in Eq. (10). We also checked that a different choice of the distribution of \( \phi_i(0) \), such as \( \phi_i(0) = \pm 1 \) or rectangular, gave, within the error bars, the same function \( \varphi(x) \) thus indicating the universality of this large deviation function.

In the opposite limit \( t \gg L^2 \), one simply replaces \( t \) in Eq. (10) by \( L^2 \). Translating into random polynomials, this regime corresponds to \( (1 - x) \ll n^{-1} \) since one just replaces \( t \) by \( 1/(1 - x) \) and \( L^2 \) by the degree \( n \) as discussed before. Thus, in this regime, we arrive at the announced scaling form for \( q_k(n) \) in Eq. (9). For the special case of Kac’s polynomials \( (d = 2) \), this scaling form, in the neighbourhood of \( k = \log n/2\pi \), is consistent with the rigorous result [13] that in this neighbourhood \( q_k(n) \) is a Gaussian with mean \( \log n/2\pi \) and variance \( V_n \sim \frac{2}{\pi} \log n \) in the large \( n \) limit.

In fact, the scaling in Eq. (10) holds more generally for any smooth GSP. To illustrate this, we considered the random acceleration process \( d^2x(t)/dt^2 = \eta(t) \) where \( \eta(t) \) is a white noise, and for which \( \mu = \sqrt{3}/(2\pi) \). For this particular smooth GSP \( \theta(p) \) has been computed exactly [18], yielding \( \theta(p) = \frac{1}{\pi}(1 - \frac{6}{\pi}\sin^{-1}(\frac{p}{2})) \). By performing the Legendre transform \([9]\) one obtains the asymptotic behaviors as

\[
\varphi(x) \sim \begin{cases} 
\frac{1}{4} + \frac{x}{\pi} \log x, & x \to 0 \\
\frac{3\pi^2}{16} (x - 1)^2, & x \to 1 \\
\frac{\sqrt{2}x}{\pi} \log 2, & x \to \infty 
\end{cases}
\]

which gives back the exact result \( \varphi(0) = 1/4 \) [19].

To conclude, we have established in this Letter a connection between the persistence probability \( p_0(t, L) \) for the diffusion equation in dimension \( d \) and the probability \( P_0(n, x) \) that generalized Kac’s real random polynomials, indexed by \( d \) as in Eq. (2), have no real roots in the interval \([0, x]\), with \( x < 1 \). This connection is useful in predicting new results for random polynomials, such as the unusual scaling form (3) for the probability of having \( k \) real roots in \([0, 1]\). Besides, we hope that this connection may also shed some light in calculating the exponents \( \theta(d) \) exactly which still remains a challenge.

S.N.M thanks J. Unterberger for pointing out Ref. [10].

FIG. 3: \((-\log p_k(t, L))/\log t\) for the diffusion equation with random initial conditions as a function of \( k/\mu \log t \) with \( \mu = 1/2\pi \) for different times \( t = 256, 512, 1024, 2048 \). Inset: \((-\log p_k(t, L))/\log t\) as a function of \( k \) for different times.

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