A SIMPLE HOMOTOPY-THEORETICAL PROOF OF THE SULLIVAN CONJECTURE

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Abstract. We give a new proof, using comparatively simple tech-
niques, of the Sullivan conjecture: map_*(B\mathbb{Z}/p, K) \sim * for every finite-
dimensional CW complex K.

Introduction

Haynes Miller proved the Sullivan conjecture in [15]. Another proof can
be deduced from the extension due to Lannes [10], whose proof depends on
the insights into unstable modules and algebras afforded by the T functor.
There are three main problems with these proofs: they are very complicated;
the content is almost entirely encoded in pure algebra; and it is difficult to
tease out the fundamental properties of B\mathbb{Z}/p that make the proofs work.

Our purpose in this paper is to offer a new proof avoiding these com-
plaints. The Sullivan conjecture is an easy consequence of the following
main theorem.

Theorem 1. Let X be a CW complex of finite type, and assume that
\bar{H}^*(X; \mathbb{Z}[\frac{1}{p}]) = 0. Then each of the following conditions implies the next.

(1) \bar{H}^*(X; \mathbb{F}_p) is a reduced unstable A_p-module and \bar{H}^*(X; \mathbb{F}_p) \otimes J(n) is
an injective unstable A_p-module for all n \geq 0.
(2) Ext^t_*(\Sigma^{2m+1}\mathbb{F}_p, \bar{H}^*(\Sigma^{s+t}X)) = 0 for every s, t \geq 0 and all m \geq 0.
(3) map_*(X, S^{2m+1}) \sim * for all sufficiently large m.
(4) map_*(X, K) \sim * for all simply-connected finite complexes K.
(5) map_*(X, \bigvee_{i=1}^{\infty} S^{n_i}) \sim * for any countable set \{n_i\} with each n_i > 1.
(6) map_*(X, K) \sim * for all simply-connected finite-dimensional CW complexes K.
(7) map_*(X, K) \sim * for all simply-connected CW complexes K with
ch_W(K) < \infty.

Furthermore, if \pi_1(X) has no perfect quotient groups (that is, if \pi_1(X) is
hypabelian), then the simply-connected hypotheses on K are not needed.

The notation ch_W(K) in part (7) denotes the cone length of K with respect
to the collection W of all wedges of spheres; see [17] §1 for a brief overview

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of the main properties of cone length. Since (7) implies (3), the last five statements are actually equivalent.

Every finite-dimensional CW complex $K$ of course has finite cone length with respect to $W$. The extension from part (6) to part (7) is a geometric parallel to the passage from finite-dimensional spaces in [15] to spaces with locally finite cohomology in [10].

Since it is known that the reduced cohomology $\tilde{H}^*(B\mathbb{Z}/p; \mathbb{F}_p)$ is reduced \cite[Lem. 2.6.5]{16} and that $\tilde{H}^*(B\mathbb{Z}/p; \mathbb{F}_p) \otimes J(n)$ is an injective unstable $A_p$-module for all $n \geq 0$ \cite[Thm. 3.1.1]{16}, we obtain the Sullivan conjecture as an immediate consequence.

**Corollary 2** (Miller). If $\text{cl}_W(K) < \infty$ (and, in particular, if $K$ is finite-dimensional), then $\text{map}_*(B\mathbb{Z}/p, K) \sim *$.

The claim that our proof is simple should be justified: we make no use whatsoever of spectral sequences, except implicitly in making use of the well-known cohomology of Eilenberg-Mac Lane spaces; the existence of unstable algebras over the Steenrod algebra is mentioned only to make sense of Massey-Peterson towers; finally, the only homological algebra in this paper is the usual abelian kind. It has seemed appropriate at points to emphasize the simplicity of the present approach by laying out some results that might just as well have been cited from their sources.

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## 1. Preliminaries

We begin by reviewing some preliminary material on the category $U$ of unstable $A_p$-algebras, Massey-Peterson towers and phantom maps.

1.1. **Unstable Modules over the Steenrod Algebra.** The cohomology functor $H^*(?; \mathbb{F}_p)$ takes its values in the category $U$ of unstable modules and their homomorphisms. An unstable module over the Steenrod algebra $A_p$ is a graded $A_p$-module $M$ satisfying $P^I(x) = 0$ if $e(I) > |x|$, where $e(I)$ is the excess of $I$ and $|x|$ is the degree of $x \in M$. We begin with some basic algebra of unstable modules, all of which is (at least implicitly) in [16].

**Suspension of Modules.** An unstable module $M \in U$ has a suspension $\Sigma M \in U$ given by $(\Sigma M)^n = M^{n-1}$. The functor $\Sigma : U \to U$ has a left adjoint $\Omega$ and a right adjoint $\tilde{\Sigma}$. A module $M$ is called reduced if $\tilde{\Sigma}M = 0$.

**Projective and Injective Unstable Modules.** In the category $U$, there are free modules $F(n) = A_p/E(n)$, where $E(n)$ is the smallest left ideal containing all Steenrod powers $P^I$ with excess $e(I) > n$. It is easy to see that the assignment $f \mapsto f([1])$ defines natural isomorphisms

$$\text{Hom}_U(F(n), M) \xrightarrow{\cong} M^n.$$
This property defines $F(n)$ up to natural isomorphism, and shows that $F(n)$ deserves to be called a free module on a single generator of dimension $n$. More generally, the free module on a set $X = \{x_\alpha\}$ with $|x_\alpha| = n_\alpha$ is (up to isomorphism) the sum $\bigoplus F(n_\alpha)$ (see [16 §1.6] for details).

A graded $\mathbb{F}_p$-vector space $M$ is of finite type if $\dim_{\mathbb{F}_p}(M_k) < \infty$ for each $k$. Since $A_p$ is of finite type, so is $F(n)$.

The functor which takes $M \in \mathcal{U}$ and returns the dual $\mathbb{F}_p$-vector space $(M^n)^*$ is representable: there is a module $J(n) \in \mathcal{U}$ and a natural isomorphism

\[ \text{Hom}_\mathcal{U}(M, J(n)) \cong \text{Hom}_{\mathbb{F}_p}(M^n, \mathbb{F}_p). \]

Since finite sums of vector spaces are also finite products, these functors are exact, so the module $J(n)$ is an injective object in $\mathcal{U}$.

**The Functor $\bar{\tau}$.** In [16 Thm 3.2.1] it is shown that for any module $H \in \mathcal{U}$, the functor $H \otimes_{A_p} ?$ has a left adjoint, denoted $(? : H)_\mathcal{U}$. Fix a module $H$ (to stand in for $\tilde{H}^*(X; \mathbb{F}_p)$) and write $\bar{\tau}$ for the functor $(? : H)_\mathcal{U}$; this is intended to evoke the standard notation $\bar{T}$ for the special case $H = \tilde{H}^*(B\mathbb{Z}/p; \mathbb{F}_p)$.

**Lemma 3.** Let $H \in \mathcal{U}$ be a reduced unstable module of finite type and suppose that $H \otimes J(n)$ is injective in $\mathcal{U}$ for every $n \geq 0$. Then

(a) $\bar{\tau}$ exact,
(b) $\bar{\tau}$ commutes with suspension,
(c) if $M$ is free and of finite type, then so is $\bar{\tau}(M)$, and
(d) if $H^0 = 0$, then $\bar{\tau}(M) = 0$ for any finite module $M \in \mathcal{U}$.

**Proof.** These results are covered in Sections 3.2 and 3.3 of [16]. Specifically, parts (a) and (b) are proved as in [16 Thm. 3.2.2 & Prop. 3.3.4], Parts (c) and (d) may be proved following [16 Lem. 3.3.1 & Prop. 3.3.6], but since there are some changes needed, we prove those parts here.

Write $d_k = \dim_{\mathbb{F}_p}(H^k)$; then there are natural isomorphisms

\[ \text{Hom}_\mathcal{U}(\bar{\tau}(F(n)), M) \cong \text{Hom}_\mathcal{U}(F(n), H \otimes M) \]

\[ \cong \text{Hom}_\mathcal{U} \left( \bigoplus_{i+j=n} F(i) \otimes d_j, M \right), \]

proving (c) in the case of a free module on one generator. Since $\bar{\tau}$ is a left adjoint, it commutes with colimits (and sums in particular), we derive the full statement of (c).

If $H^0 = 0$, then $d_0 = 0$ and $\bar{\tau}(F(n))$ is a sum of free modules $F(k)$ with $k < n$. Since $F(0) = \mathbb{F}_p$, we see that $\bar{\tau}(\mathbb{F}_p) = 0$; then (a), together with the fact that $\bar{\tau}$ commutes with colimits, implies that $\bar{\tau}(M) = 0$ for all trivial modules $M$. Finally, any finite module $M$ has filtration all of whose subquotients are trivial, and (d) follows.

**1.2. Massey-Peterson Towers.** Cohomology of spaces has more structure than just that of an unstable $A_p$-module. It has a cup product which makes $H^*(X; \mathbb{F}_p)$ into an unstable algebra over $A_p$. The category of unstable algebras is denoted $\mathcal{K}$. 

The forgetful functor $K \to U$ has a left adjoint $U : U \to K$. A space $X$ is said to have very nice cohomology if $H^*(X) \cong U(M)$ for some unstable module $M$ of finite type.

Since $U(F(n)) \cong H^*(K(\mathbb{Z}/p, n))$, there is a contravariant functor $K$ which carries a free module $F$ to a generalized Eilenberg-Mac Lane space (usually abbreviated GEM) $K(F)$ such that $H^*(K(F)) \cong U(F)$. If $F$ is free, then so is $\Omega F$, and $K(\Omega F) \cong \Omega K(F)$.

**Lemma 4.** For any $X$, $[X, K(F)] \cong \text{Hom}_U(F, \tilde{H}^*(X))$.

It is shown in [8,11,14] that if $H^*(Y) \cong U(M)$ and $P_\ast \to M \to 0$ is a free resolution in $U$, then $Y$ has a Massey-Peterson tower

$$
\cdots \to Y_s \to Y_{s-1} \to \cdots \to Y_1 \to Y_0
$$

in which

1. $Y_0 = K(P_0)$,
2. each homotopy group $\pi_k(Y_s)$ is a finite $p$-group,
3. the limit of the tower is the completion $Y_p^\wedge$,
4. each sequence $Y_s \to Y_{s-1} \to K(\Omega^{s-1} P_s)$ is a fiber sequence, and
5. the compositions $\Omega K(\Omega^{s-1} P_s) \to Y_s \to K(\Omega^s P_{s+1})$ can be naturally identified with $K(\Omega^s d_{s+1})$, where $d_{s+1} : P_{s+1} \to P_s$ is the differential in the given free resolution.

1.3. **Phantom Maps.** A **phantom map** is a map $f : X \to Y$ from a CW complex $X$ such that the restriction $f|_{X_n}$ of $f$ to the $n$-skeleton is trivial for each $n$. We write $\text{Ph}(X,Y) \subseteq [X,Y]$ for the set of pointed homotopy classes of phantom maps from $X$ to $Y$. See [12] for an excellent survey on phantom maps.

If $X$ is the homotopy colimit of a telescope diagram $\cdots \to X(n) \to X(n+1) \to \cdots$, then there is a short exact sequence of pointed sets

$$
* \to \lim^1 [X(n), Y] \to [X, Y] \to \lim [X(n), Y] \to *,
$$

and dually, if $Y$ is the homotopy limit of a tower $\cdots \leftarrow Y(n) \leftarrow Y(n+1) \leftarrow \cdots$, then there is a short exact sequence

$$
* \to \lim^1 [X, \Omega Y(n)] \to [X, Y] \to \lim [X, Y(n)] \to *.
$$

In the particular case of the expression of a CW complex $X$ as the homotopy colimit of its skeleta or of a space $Y$ as the homotopy limit of its Postnikov system, the kernels are the phantom sets.

We will be interested in showing that all phantom maps are trivial. One useful criterion is that if $G$ is a tower of compact Hausdorff topological spaces, then $\lim^1 G = *$ (see [12, Prop. 4.3]). This is used to prove the following lemma.
Lemma 5. Let $\cdots \leftarrow Y_s \leftarrow Y_{s+1} \leftarrow \cdots$ be a tower of spaces such that each homotopy group $\pi_k(Y_s)$ is finite. If $Z$ is of finite type, then $\lim^1[Z, \Omega Y_s] = \ast$.

**Proof.** The homotopy sets $[Z_n, \Omega Y_s]$ are finite, and we give them the discrete topology, resulting in towers of compact groups and continuous homomorphisms. Fixing $s$ and letting $n$ vary, we find that $\lim^1[Z_n, \Omega Y_s] = \ast$, and hence the exact sequence

$$0 \rightarrow \lim^1_n[Z_n, \Omega Y_s] \rightarrow [Z, \Omega Y_s] \rightarrow \lim_n[Z_n, \Omega Y_s] \rightarrow 1$$

(of groups) reduces to an isomorphism $[Z, \Omega Y_s] \cong \lim_n[Z_n, \Omega Y_s]$. Since $[Z, \Omega Y_s]$ is an inverse limit of finite discrete spaces, it is compact and Hausdorff; and since the structure maps $Y_s \rightarrow Y_{s-1}$ induce maps of the towers that define the topology, the induced maps $[Z, \Omega Y_s] \rightarrow [Z, \Omega Y_{s-1}]$ are continuous. Thus $\lim^1[Z, \Omega Y_s] = \ast$. \qed

The Mittag-Leffler condition is another useful criterion for the vanishing of $\lim^1$. A tower of groups $\cdots \leftarrow G_n \leftarrow G_{n+1} \leftarrow \cdots$ is **Mittag-Leffler** if there is a function $\kappa : \mathbb{N} \rightarrow \mathbb{N}$ such that for each $n$ $\operatorname{Im}(G_{n+k} \rightarrow G_n) = \operatorname{Im}(G_{n+\kappa(n)} \rightarrow G_n) \subseteq G_n$ for every $k \geq \kappa(n)$.

Proposition 6. Let $\cdots \leftarrow G_n \leftarrow G_{n+1} \leftarrow \cdots$ be a tower of groups.

(a) If the tower is Mittag-Leffler, then $\lim^1 G_n = \ast$.

(b) If each $G_n$ is a countable group, then the converse holds: if $\lim^1 G_n = \ast$, then the tower is Mittag-Leffler [12, Thm. 4.4].

Importantly, the Mittag-Leffler condition does not refer to the algebraic structure of the groups $G_n$. This observation plays a key role in the following result (cf. [13, §3]).

Proposition 7. Let $X$ be a CW complex of finite type, and let $Y_1$ and $Y_2$ be countable CW complexes with $\Omega Y_1 \simeq \Omega Y_2$. Then $\text{Ph}(X, Y_1) = \ast$ if and only if $\text{Ph}(X, Y_2) = \ast$.

**Proof.** The homotopy equivalence $\Omega Y_1 \simeq \Omega Y_2$ gives levelwise bijections $\{[\Sigma X_n, Y_1]\} \cong \{[X_n, \Omega Y_1]\} \cong \{[X_n, \Omega Y_2]\} \cong \{[\Sigma X_n, Y_2]\}$ of towers of sets. Since $X$ is of finite type and $Y_1, Y_2$ are countable CW complexes, these towers are towers of countable groups. Now the triviality of the first phantom set implies that the first tower is Mittag-Leffler; but then all four towers must be Mittag-Leffler, and the result follows. \qed

2. (1) IMPLIES (2)

Write $H = \tilde{H}^*(X; \mathbb{F}_p)$; thus $H \in \mathcal{U}$ is a reduced module of finite type and $H \otimes J(n)$ is injective for all $n$. If $P_* \rightarrow M \rightarrow 0$ is a free resolution of $M$ in
U, then Lemma 3 implies that \( \tau(P_s) \to 0 \to 0 \) is a free resolution of 0, so
\[
\text{Ext}_U^s (M, \Sigma^{s+t}H) = \text{Ext}_U^s \left( M, H \otimes \Sigma^{s+t}F_p \right) = H^s \left( \text{Hom} (P_s, H \otimes \Sigma^{s+t}F_p) \right) = H^s \left( \text{Hom} \left( \tau(P_s), \Sigma^{s+t}F_p \right) \right) = \text{Ext}_U^s \left( 0, \Sigma^{s+t}F_p \right) = 0.
\]

3. (2) IMPLIES (3)

**Theorem 8.** Suppose \( H^*(Y) = U(M) \) for some finite \( M \in U \), and \( Z \) is a CW complex of finite type with \( \tilde{H}^*(Z; \mathbb{Z}[\frac{1}{p}]) = 0 \). If \( \text{Ext}_U^s(M, \Sigma^s \tilde{H}^*(Z)) = 0 \) for all \( s \geq 0 \), then \([Z, Y] = *\).

Condition (2) allows us to apply Theorem 8 to \( Z = \Sigma^tX \) and \( Y = S^{2m+1} \) to deduce condition (3): \( \pi_t(\text{map}_s(X, S^{2m+1})) \cong [\Sigma^tX, S^{2m+1}] = * \).

**Proof of Theorem 8.** According to [14, Thm. 4.2], the natural map \( Y \to Y_p^\wedge \) induces a bijection \([Z, Y] \cong [Z, Y_p^\wedge] \), so it suffices to show \([Z, Y_p^\wedge] = *\). Since \( H^*(Y) = U(M) \), \( Y \) has a Massey-Peterson tower, whose homotopy limit is \( Y_p^\wedge \). Let \( f_s \) be the composite \( Z \to Y \to Y_p^\wedge \); we will show by induction that \( f_s \cong * \) for all \( s \).

Since \( Y_0 \) is a GEM, \( f_0 \) is determined by its effect on cohomology; and since \( \text{Hom}_U(M, \tilde{H}^*(Z)) = \text{Ext}_U^0(M, \Sigma^0 \tilde{H}^*(Z)) = 0 \), \( f_0 \) is trivial on cohomology, and hence trivial. Inductively; suppose \( f_{s-1} \) is trivial. We have the following situation

\[
\begin{align*}
K(\Omega^s P_{s-1}) & \xrightarrow{K(\Omega^s d_s)} \Omega Y_{s-1} & \xrightarrow{K(\Omega^s d_{s+1})} K(\Omega^s P_s) & \xrightarrow{Y_s} K(\Omega^s P_{s+1}) \\
& \downarrow \uparrow & \downarrow & \downarrow \\
& \downarrow & \downarrow & \downarrow \\
& Y_{s-1}. & & \end{align*}
\]

Now apply \([Z, ?] \) to this diagram and observe that Lemma 4 together with the isomorphism \( \text{Hom}_U(\Omega^s P, H) \cong \text{Hom}_U(P, \Sigma^s H) \) (with \( H = \tilde{H}^*(Z) \)), gives
\[
\begin{align*}
\text{Hom}_U(P_{s-1}, \Sigma^s H) & \xrightarrow{d^*_s} \text{Hom}_U(P_s, \Sigma^s H) & \xrightarrow{\alpha} [Z, Y_s] & \xrightarrow{\beta} \text{Hom}_U(P_{s+1}, \Sigma^s H) \\
& \downarrow & \downarrow & \downarrow \\
& [Z, Y_{s-1}]. & & \end{align*}
\]

Exactness at \([Z, Y_s] \) implies that the homotopy class \([f_s] \) is equal to \( \alpha(g_s) \) for some \( g_s \in \text{Hom}_U(P_s, \Sigma^s H) \). Since \([f_s] \) is in the image of the vertical
map from \([Z, Y_{s+1}]\), it is in the kernel of \(\beta\), so \(d^s_{s+1}(g_s) = \beta([f_s]) = 0\); in other words, \(g_s\) is a cycle representing an element \([g_s] \in \text{Ext}^s_U(M, \Sigma^sH)\). Since \(\text{Ext}^s_U(M, \Sigma^sH) = 0\), we conclude that \(g_s = d^s_{s}(g_{s-1})\) and so \([f_s] = \alpha(d^s_{s}(g_{s-1})) = [\ast]\).

Since every map \(f : Z \to Y\) is trivial on composition to \(Y_s\) for each \(s\), the exact sequence \(* \to \lim^1[Z, \Omega Y_s] \to \lim[Z, Y^\wedge_s] \to \lim[Z, Y_s] \to *\) reduces to an isomorphism \([Z, Y^\wedge_s] \cong \lim^1[Z, \Omega Y_s]\), and Lemma 5 finishes the proof. □

4. (3) IMPLIES (4)

The statement that (3) implies (4) is very similar to Corollary 11 from [17], which is proved using only classical results: the splitting of a product after suspension; the James construction [9]; the Hilton-Milnor theorem in the form proved by Gray [6]; a result of Ganea on the homotopy type of the suspension of a homotopy fiber [5, Prop. 3.3] (see also Gray’s paper [7]); and a Blakers-Massey-type theorem for \(n\)-ads due to Barratt and Whitehead [1] and Toda [19].

The difference is that here we restrict our attention to odd spheres. To get this stronger statement, write \(\mathcal{R} = \{K \mid \text{map}_a(X, K) \sim *\}\), and suppose that \(\mathcal{R}\) contains all odd-dimensional spheres of sufficiently high dimension; to make the argument of [17] work, it suffices to show that \(\mathcal{R}\) contains all finite-type wedges of odd-dimensional spheres of sufficiently high dimension.

If \(W = V_1 \vee V_2\) where both \(V_1\) and \(V_2\) are wedges of odd-dimensional spheres, then the homotopy fiber of the quotient map \(q : W \to V_2\) is

\[V_1 \vee \Omega V_2 \simeq V_1 \wedge U,\]

where \(U\) is a wedge of even-dimensional spheres. Now an examination of the proof of [17] Prop. 7 reveals that if the initial wedge \(V_0\) is a wedge of odd-dimensional spheres, then so are all of the later wedges \(V_n\), and so the argument carries through unchanged.

5. (4) IMPLIES (5)

Write \(W = \bigvee_{i=1}^{\infty} S^{n_i}\), and let \(f : \Sigma^i X \to W\). Since \(X\) has finite type, \((\Sigma^i X)_k\) is compact, and so \(f((\Sigma^i X)_k)\) is contained in a finite subwedge \(V \subseteq W\). The homotopy commutative diagram

\[
\begin{array}{ccc}
(S^i X)_k & \xrightarrow{f|_{(S^i X)_k}} & (S^i X)_k \\
\downarrow f & & \downarrow f|_{(S^i X)_k} \\
\Sigma^i X & \xrightarrow{i} & W \\
\downarrow q & & \downarrow i \\
W & \xrightarrow{\ast} & V \\
\end{array}
\]

in which \(q\) is the collapse map to \(V\) and \(i\) is the inclusion, shows that \(f|_{(S^i X)_k} \simeq \ast\), and hence that \(f\) is a phantom map.
The conclusion \( f \simeq * \) follows from taking \( Z = \Sigma^t X \) in Proposition 9.

**Proposition 9.** If \( Z \) is rationally trivial and of finite type, then
\[
\text{Ph}(Z, \bigvee_{i=1}^{\infty} S^{m_i}) = *.
\]

**Proof.** The Hilton-Milnor theorem implies that there is a weak product of spheres \( P = \prod_\alpha S^{m_\alpha} \) such that \( \Omega (\bigvee_{i=1}^{\infty} S^{m_i}) \simeq \Omega P \) (that is, \( P \) is the (homotopy) colimit of the diagram of finite subproducts of the categorical product). By Proposition 7 it suffices to show that \( \text{Ph}(Z, P) = * \).

Since the skeleta of \( Z \) are compact, every map \( \Sigma Z_k \to P \) factors through a finite subproduct of \( P \), so \( [\Sigma Z_k, P] \) is a weak product \( \prod_\alpha [\Sigma Z_k, S^{m_\alpha}] \).

Since \( Z \) is rationally trivial, we have \( \lim^1 [\Sigma Z_k, S^m] = * \) for each \( m \), and since these are towers of countable groups, they are all Mittag-Leffler. Write \( \lambda(n, m) \) for the first \( k \) for which the images stabilize. Since \( \lambda(n, m) = 0 \) for \( m > n + 1 \), the set \( \{ \lambda(n, m) \mid m \geq 0 \} \) is finite, and we define \( \kappa(n) \) to be its maximum. Now it is clear that the images
\[
\text{Im}(\prod_\alpha [\Sigma Z_{n+k}, S^m] \to \prod_\alpha [\Sigma Z_n, S^m])
\]
stabilize. Since \( \lambda(n, m) = 0 \) for \( m > n + 1 \), the set \( \{ \lambda(n, m) \mid m \geq 0 \} \) is finite, and we define \( \kappa(n) \) to be its maximum. Now it is clear that the images
\[
\text{Im}(\prod_\alpha [\Sigma Z_{n+k}, S^{m_\alpha}] \to \prod_\alpha [\Sigma Z_n, S^{m_\alpha}])
\]
are independent of \( k \) for \( k \geq \kappa(n) \). Thus the tower \( \{ \prod_\alpha [\Sigma Z_k, S^{m_\alpha}] \} \) is Mittag-Leffler, \( \text{Ph}(Z, P) = * \), and the proof is complete. \( \Box \)

6. (5) implies (6)

Let \( f : \Sigma^t X \to K \), where \( K \) is a simply-connected finite-dimensional CW complex, which we may assume has trivial 1-skeleton. Since \( X \) is of finite type, the image \( f(\Sigma^t X_k) \) of each skeleton is contained in a finite subcomplex of \( K \), and hence \( f \) factors through the inclusion of a countable subcomplex of \( K \), which is necessarily simply-connected and finite-dimensional. Thus to show \( f \simeq * \), it suffices to prove that \( \text{map}_{\ast}(X, L) \sim * \) for all finite-dimensional countable simply-connected CW complexes \( L \).

Write \( \mathcal{A} = \{ \bigvee_{i=1}^{\infty} S^{n_i} \} \) (with each \( n_i > 1 \)). Since \( \mathcal{A} \) is closed under suspension and smash product, and \( \mathcal{A} \subseteq R = \{ K \mid \text{map}_{\ast}(X, K) \sim * \} \), Theorem 8 of [17] implies that every simply-connected space which has finite cone length with respect to \( \mathcal{A} = \mathcal{A}^v \) is also in \( R \). But this includes all simply-connected countable finite-dimensional CW complexes.

7. (6) implies (7)

We continue to write \( R = \{ K \mid \text{map}_{\ast}(X, K) \sim * \} \). We know that \( R \) contains all simply-connected finite-dimensional wedges of spheres, a collection that is closed under suspension and smash product. Since generic wedges \( \bigvee S^{m_\alpha} \) are finite-type wedges of such wedges, Theorem 8 of [17] implies that \( R \) contains \( W \). The same theorem now implies that every simply-connected space \( X \) with \( \text{cl}_{\ast}(X) < \infty \) is also in \( R \).
8. NON-SIMPLY-CONNECTED TARGETS

Suppose, finally, that $K$ is not simply-connected and that $\pi_1(X)$ has no perfect quotients. If $f : \Sigma^t X \to K$ is trivial on fundamental groups, then there is a lift in the diagram

\[
\begin{array}{ccc}
\Sigma^t X & \xrightarrow{f} & K \\
p & \downarrow & \downarrow \\
\tilde{K} & \xrightarrow{\pi} & K
\end{array}
\]

where $p$ is the universal cover of $K$. Since $K$ is finite-dimensional, so is $\tilde{K}$, and so $f \simeq \ast$. On the other hand, if $f$ is nontrivial on fundamental groups, then we write $G = \text{Im}(f_*)$ and consider the covering space $q : L \to K$ corresponding to the subgroup $G \subseteq \pi_1(K)$. Again, we have a lift in the diagram

\[
\begin{array}{ccc}
\Sigma^t X & \xrightarrow{f} & L \\
\phi & \downarrow & \downarrow q \\
\Sigma^t X & \xrightarrow{f} & K
\end{array}
\]

and $\phi$ induces a surjection on fundamental groups. But since $G$ is not perfect, there is a nontrivial map $u : L \to K(A, 1)$ (which must be nonzero on fundamental groups) for some abelian group $A$. Thus $\phi$ is nonzero on cohomology and so $\Sigma \phi : \Sigma^{t+1} X \to \Sigma L$ is nontrivial.

To finish the proof, we observe that if $L \to K$ is a covering with $L$ path-connected, then forming the pullback squares

\[
\begin{array}{cccccc}
L_0 & \to & L_1 & \to & \cdots & \to & L_n-1 & \to & L_n & \to & L \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_0 & \to & K_1 & \to & \cdots & \to & K_{n-1} & \to & K_n & \to & K
\end{array}
\]

do over a $W$-cone decomposition of $K$ yields a $W$-cone decomposition of $L$. Thus the nontriviality of $\Sigma \phi$ contradicts Theorem 7 because $\Sigma L$ is simply-connected and $\text{ch}_W(\Sigma L) \leq \text{ch}_W(L) < \infty$.

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