Transmission stabilization and destabilization involving Kerr and Raman effects in broadband soliton-based fiber optics systems

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Abstract

We study stabilization and destabilization of propagating soliton sequences in broadband fiber optics systems with \( N \) frequency channels, taking into account second-order dispersion, Kerr nonlinearity, delayed Raman response, and linear gain-loss. We employ a propagation model consisting of a system of \( N \) coupled nonlinear Schrödinger (NLS) equations and a reduced \( N \)-dimensional predator-prey model for amplitude dynamics. Numerical simulations with the coupled-NLS model with \( 2 \leq N \leq 4 \) show stable oscillatory dynamics of soliton amplitudes at short-to-intermediate distances, in agreement with predictions of the predator-prey model. Furthermore, the main destabilizing mechanism at long distances is due to generation of radiative sidebands, where the sidebands for a given channel form at the frequencies of solitons in the neighboring channels. This destabilizing process can be partially mitigated by employing frequency dependent linear gain-loss. Moreover, significant enhancement of transmission stability is achieved in a nonlinear \( N \)-waveguide coupler with frequency shifting of the linear gain-loss profile.

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I. INTRODUCTION

Transmission of information in broadband optical waveguide links can be significantly enhanced by launching many pulse sequences through the same waveguide [1–5]. Each pulse sequence propagating through the waveguide is characterized by the central frequency of its pulses, and is therefore called a frequency channel. Applications of these multichannel systems, which are also known as wavelength-division-multiplexed (WDM) systems, include fiber optics transmission lines [2–5], data transfer between computer processors through silicon waveguides [6–8], and multiwavelength lasers [9–12]. Since pulses from different frequency channels propagate with different group velocities, interchannel pulse collisions are very frequent, and can therefore lead to error generation and cause severe transmission degradation [1–5, 13, 14].

In the current paper, we study pulse propagation in broadband multichannel fiber optics transmission lines, considering optical solitons as an example for the pulses. In this case, the two main processes affecting interchannel collisions are due to the fiber’s instantaneous nonlinear response (Kerr nonlinearity) and delayed Raman response. The only effects of Kerr nonlinearity on a single interchannel collision between two isolated solitons in a long optical fiber are a phase shift and a position shift, which scale as $1/\Delta \beta$ and $1/\Delta \beta^2$, respectively, where $\Delta \beta$ is the difference between the frequencies of the colliding solitons [14–16]. Thus, in this long fiber setup, the amplitude, frequency, and shape of the solitons do not change due to the collision. However, the situation is different when the colliding solitons propagate in a closed fiber loop. In this case, the collision leads to emission of small amplitude waves (continuous radiation) with peak power that is also inversely proportional to $\Delta \beta$. The emission of continuous radiation in many collisions in an $N$-channel transmission system can eventually lead to pulse-shape distortion and as a result, to transmission destabilization. The main effect of delayed Raman response on single-soliton propagation is an $O(\epsilon_R)$ frequency downshift, where $\epsilon_R$ is the Raman coefficient [17–19]. This Raman-induced self frequency shift is a result of energy transfer from high frequency components of the pulse to lower frequency components. The main effect of delayed Raman response on an interchannel two-soliton collision is an $O(\epsilon_R)$ amplitude shift, which is called Raman-induced crosstalk [16, 20–27]. It is a result of energy transfer from the high frequency pulse to the low frequency one. The amplitude shift is accompanied by an $O(\epsilon_R/\Delta \beta)$ collision-induced frequency downshift.
(Raman cross frequency shift) and by emission of continuous radiation [16, 22, 24–28]. Note that the Raman-induced amplitude shift in a single collision is independent of the magnitude of the frequency difference between the colliding solitons. Consequently, the cumulative amplitude shift experienced by a given pulse in an \(N\)-channel transmission line is proportional to \(N^2\), a result that is valid for linear transmission [20, 21, 29, 30], conventional soliton transmission [22–24, 26], and dispersion-managed soliton transmission [25]. Thus, in a 100-channel system, for example, Raman crosstalk effects are larger by a factor of \(2.5 \times 10^3\) compared with a two-channel system operating at the same bit rate per channel. For this reason, Raman-induced crosstalk is considered to be one of the most important processes affecting the dynamics of optical pulse amplitudes in broadband fiber optics transmission lines [1, 2, 20, 21, 30–34].

The first studies of Raman crosstalk in multichannel fiber optics transmission focused on the dependence of the energy shifts on the total number of channels [20], as well as on the impact of energy depletion and group velocity dispersion on amplitude dynamics [29, 35]. Later studies turned their attention to the interplay between bit-pattern randomness and Raman crosstalk in on-off-keyed (OOK) transmission, and showed that this interplay leads to lognormal statistics of pulse amplitudes [16, 21, 26, 30, 31, 36]. This finding means that the \(n\)th normalized moments of the probability density function (PDF) of pulse amplitudes grow exponentially with both propagation distance and \(n^2\). Furthermore, in studies of soliton-based multichannel transmission, it was found that due to the strong coupling between frequency dynamics and amplitude dynamics, the \(n\)th normalized moments of the PDFs of the Raman self and cross frequency shifts also grow exponentially with propagation distance and \(n^2\) [32, 33]. The exponential growth of the normalized moments of pulse parameter PDFs can be interpreted as intermittent dynamics, in the sense that the statistics of the amplitude and frequency is very sensitive to bit-pattern randomness. Moreover, it was shown in Refs. [32–34] that this intermittent dynamics has important practical consequences in massive multichannel transmission, by leading to relatively high bit-error-rate values at intermediate and large propagation distances. Additionally, the different scalings and statistics of Raman-induced and Kerr-induced effects lead to loss of scalability in these systems. As a result, analysis of pulse dynamics in a few central frequency channels does not provide an accurate evaluation of system performance in massive multichannel amplitude-keyed transmission [34].
One of the ways to overcome the detrimental effects of Raman crosstalk on massive OOK multichannel transmission is by employing encoding schemes, which are less susceptible to these effects. The differential phase shift keying (DPSK) scheme, in which the information is encoded in the phase difference between adjacent pulses, is among the most promising encoding methods, and has thus become the focus of intensive research [37, 38]. Since in DPSK transmission the information is encoded in the phase, the amplitude patterns are deterministic, and as a result, the Raman-induced amplitude dynamics is also approximately deterministic. A key question about this deterministic dynamics concerns the possibility to achieve stable steady-state transmission with nonzero predetermined amplitude values in all channels. In Ref. [29], it was demonstrated that this is not possible in unamplified optical fiber lines. However, the experiments in Refs. [39, 40] showed that the situation is very different in amplified multichannel transmission. More specifically, it was found that the introduction of amplification enables transmission stabilization and reduction of the cumulative Raman crosstalk effects. In Ref. [41], we provided a dynamical explanation for the stabilization of DPSK soliton-based multichannel transmission, by demonstrating that the Raman-induced amplitude shifts can be balanced by an appropriate choice of amplifier gain in different channels. Our approach was based on showing that the collision-induced amplitude dynamics in an $N$-channel system can be described by an $N$-dimensional predator-prey model. We obtained the Lyapunov function for the model and used it to show that stable transmission with nonzero amplitudes in all channels can be realized by overamplification of high frequency channels and underamplification of low frequency channels. We also generalized the treatment to incorporate the perturbative effects of the Raman self and cross frequency shifts and of the Kerr-induced position shift on amplitude dynamics [41].

All the results in Ref. [41] were obtained with the $N$-dimensional predator-prey model, which is based on several simplifying assumptions, whose validity might break down with increasing number of channels and at large propagation distances. In particular, the predator-prey model neglects high-order effects due to radiation emission and modulation instability, intrasequence interaction, and temporal inhomogeneities. These effects can lead to pulse shape distortion and eventually to transmission destabilization. The distortion of the solitons shapes can also lead to the breakdown of the predator-prey model description at large distances (see, for example Refs. [42–44], for the case of crosstalk induced by nonlinear gain or loss). In contrast, the complete propagation model, which consists of a system of $N$
perturbed coupled nonlinear Schrödinger (NLS) equations, fully incorporates the effects of radiation emission, intrachannel interaction, and temporal inhomogeneities. Thus, in order to check whether stable long-distance multichannel transmission can indeed be realized by a proper choice of amplifier gain, it is important to carry out numerical simulations with the full coupled-NLS model.

In the current paper, we take on this important task. For this purpose, we employ a perturbed coupled-NLS model, which takes into account the effects of second-order dispersion, Kerr nonlinearity, delayed Raman response, and frequency dependent linear gain-loss. We perform numerical simulations with the coupled-NLS model with two, three, and four channels. We then analyze the simulations results in comparison with the predator-prey model predictions, looking for processes leading to transmission stabilization and destabilization. The numerical simulations with the coupled-NLS propagation model show that at short-to-intermediate distances soliton amplitudes exhibit stable oscillatory dynamics, in agreement with the predictions of the predator-prey model of Ref. [41]. The frequency differences between the soliton sequences exhibit similar oscillatory dynamics. These results mean that radiation emission and intrachannel interaction effects can indeed be neglected at short-to-intermediate distances. However, at larger distances, transmission destabilization due to formation of radiative sidebands is observed, where the $j$th channel’s sidebands form at the frequencies $\beta_k(z)$ of the solitons in the neighboring frequency channels. We also find that generation of radiative sidebands is partially mitigated by the presence of frequency dependent linear gain-loss and that this leads to a significant increase in the distance along which stable transmission is observed. However, this mitigation of radiative instability is limited due to the fact that in a single fiber it is not possible to employ strong linear loss at the frequencies of the propagating solitons.

The limitation of transmission stabilization in a single-fiber system can be overcome by employing a nonlinear waveguide coupler, consisting of $N$ nearby waveguides. We assume that the linear gain-loss of the $j$th waveguide is equal to the value required to balance Raman crosstalk inside a distance-dependent frequency interval centered about the soliton frequency $\beta_j(z)$, and is equal to a relatively large negative value outside of this interval. Numerical simulations with the corresponding coupled-NLS model show that the distances along which stable transmission is observed in the $N$-waveguide coupler are significantly larger compared with the distances in the single-fiber system. Furthermore, the simulations confirm that the
enhanced transmission stability in the $N$-waveguide coupler is due to suppression of radiative sideband generation at all frequencies outside of the central amplification bandwidth for each of the $N$ waveguides in the waveguide coupler.

We consider optical solitons as an example for the pulses carrying the information for the following reasons. First, due to the integrability of the unperturbed NLS equation and the shape-preserving property of NLS solitons, derivation of the predator-prey model for Raman-induced amplitude dynamics is done in a rigorous manner [41]. Second, the soliton stability and shape-preserving property make soliton-based transmission in broadband fiber optics links advantageous compared with other transmission methods [1, 3, 13, 45]. Third, as mentioned above, the Raman-induced energy exchange in pulse collisions is similar in linear transmission, conventional soliton transmission, and dispersion-managed soliton transmission. Thus, even though pulse dynamics in these different transmission systems is different, analysis of soliton-based transmission stabilization and destabilization might give a rough idea about the processes leading to stabilization and destabilization of the optical pulse sequences in other transmission setups.

The remainder of the paper is organized as follows. In Sec. II, we present the coupled-NLS model for pulse propagation in a single-fiber $N$-channel transmission line together with the $N$-dimensional predator-prey model for collision-induced amplitude dynamics. We then review the results of Ref. [41] for stability analysis of the equilibrium states of the predator-prey model. In Sec. III, we present the results of numerical simulations with the coupled-NLS model for single-fiber multichannel transmission and analyze these results in comparison with the predictions of the predator-prey model. In Sec. IV, we present the coupled-NLS model for pulse propagation in a nonlinear $N$-waveguide coupler. We then analyze the results of numerical simulations with this model and compare the results with the predator-prey model’s predictions. Our conclusions are presented in Sec. V. In Appendix A, we discuss the perturbed predator-prey model with Raman self and cross frequency shifts.

II. THE PROPAGATION MODEL FOR SINGLE-FIBER TRANSMISSION AND THE PREDATOR-PREY MODEL FOR AMPLITUDE DYNAMICS

We consider propagation of pulses of light in a single-fiber $N$-channel transmission link, taking into account second-order dispersion, Kerr nonlinearity, delayed Raman response, and
frequency-dependent linear loss or gain. The net linear gain-loss is the difference between amplifier gain and fiber loss, where we assume that the gain is provided by distributed Raman amplification [48, 49]. In addition, we assume that the frequency difference $\Delta \beta$ between adjacent channels is much larger than the spectral width of the pulses, which is the typical situation in many soliton-based WDM systems [13, 14]. Under these assumptions, the propagation is described by the following system of $N$ perturbed coupled-NLS equations [46]:

$$
i \partial_z \psi_j + \partial_t^2 \psi_j + 2|\psi_j|^2 \psi_j + 4 \sum_{k=1}^{N} (1 - \delta_{jk})|\psi_k|^2 \psi_j = i \mathcal{F}^{-1}(g(\omega) \hat{\psi}_j)/2 - \epsilon_R \psi_j \partial_t |\psi_j|^2$$

$$- \epsilon_R \sum_{k=1}^{N} (1 - \delta_{jk}) \left[ \psi_j \partial_t |\psi_k|^2 + \psi_k \partial_t (\psi_j \psi_k^*) \right],$$

(1)

where $\psi_j$ is proportional to the envelope of the electric field of the $j$th sequence, $1 \leq j \leq N$, $z$ is propagation distance, and $t$ is time [47]. In Eq. (1), $\epsilon_R$ is the Raman coefficient, $g(\omega)$ is the net frequency dependent linear gain-loss function, $\hat{\psi}$ is the Fourier transform of $\psi$ with respect to time, $\mathcal{F}^{-1}$ stands for the inverse Fourier transform, and $\delta_{jk}$ is the Kronecker delta function. The second term on the left hand side of Eq. (1) describes second-order dispersion effects, while the third and fourth terms represent intrachannel and interchannel interaction due to Kerr nonlinearity. The first term on the right hand side of Eq. (1) describes the effects of frequency dependent linear gain or loss, the second corresponds to Raman-induced intrachannel interaction, while the third and fourth terms describe Raman-induced interchannel interaction.

The form of the net frequency dependent linear gain-loss function $g(\omega)$ is chosen so that Raman crosstalk and radiation emission effects are suppressed. More specifically, $g(\omega)$ is equal to a value $g_j$, required to balance Raman-induced amplitude shifts, inside a frequency interval of width $W$ centered about the initial frequency of the $j$th-channel solitons $\beta_j(0)$, and is equal to a negative value $g_L$ elsewhere. Thus, $g(\omega)$ is given by:

$$g(\omega) = \begin{cases} 
g_j & \text{if } \beta_j(0) - W/2 < \omega \leq \beta_j(0) + W/2, 
g_L & \text{if } \beta_j(0) + W/2 < \omega \leq \beta_{j+1}(0) - W/2, \text{ or } \omega \leq \omega_L, \text{ or } \omega > \omega_R,
\end{cases}$$

(2)

where $1 \leq j \leq N - 1$, $g_L < 0$, $\omega_L = \beta_1(0) - W/2$, and $\omega_R = \beta_N(0) + W/2$. The width $W$ in Eq. (2) satisfies $0 < W \leq \Delta \beta$, where $\Delta \beta = \beta_{j+1}(0) - \beta_j(0)$ for $1 \leq j \leq N - 1$. Note that the actual values of the $g_j$ coefficients are determined by the predator-prey model for
collision-induced amplitude dynamics. In addition, the values of $g_L$ and $W$ are determined such that instability due to radiation emission is mitigated. In practice, we determine the latter values by carrying out numerical simulations with the coupled-NLS model (1), while looking for the set of values, which yields the longest distance for stable propagation. As an example, Fig. 1 illustrates a typical linear gain-loss function $g(\omega)$ for a two-channel system with $g_1 = -0.02133$, $g_2 = 0.02133$, $g_L = -0.5$, $\beta_1(0) = -20$, $\beta_2(0) = 20$, and $W = 20$. Note that the description of the frequency dependent linear gain-loss in terms of a single function $g(\omega)$ is the only self-consistent characterization of the linear gain-loss for a single fiber.

In the current paper we study soliton-based transmission systems, and therefore the optical pulses in the $j$th frequency channel are fundamental solitons of the unperturbed NLS equation $i \partial_z \psi_j + \partial^2_t \psi_j + 2|\psi_j|^2 \psi_j = 0$. The envelopes of these solitons are given by $\psi_{sj}(t, z) = \eta_j \exp(i \chi_j) \text{sech}(x_j)$, where $x_j = \eta_j (t - y_j - 2 \beta_j z)$, $\chi_j = \alpha_j + \beta_j(t - y_j) + (\eta_j^2 - \beta_j^2) z$, and the four parameters $\eta_j$, $\beta_j$, $y_j$, and $\alpha_j$ are related to the soliton amplitude, frequency (and group velocity), position, and phase, respectively. The assumption of a large frequency (and group velocity) difference between adjacent channels, means that $|\beta_j - \beta_k| \gg 1$ for $1 \leq j \leq N$, $1 \leq k \leq N$, and $j \neq k$. As a result of the large group velocity difference, the solitons undergo a large number of intersequence collisions. The Raman-induced crosstalk during these collisions can lead to significant amplitude and frequency shifts, which can in turn lead to severe transmission degradation.

In Ref. [41], we showed that the dynamics of soliton amplitudes in an $N$-channel system can be approximately described by an $N$-dimensional predator-prey model. The derivation of
the predator-prey model was based on the following simplifying assumptions. (1) The soliton sequences are deterministic in the sense that all time slots are occupied and each soliton is located at the center of a time slot of width $T$, where $T \gg 1$. In addition, the amplitudes are equal for all solitons from the same sequence, but are not necessarily equal for solitons from different sequences. This setup corresponds, for example, to return-to-zero phase-shift-keyed transmission. (2) The sequences are either (a) infinitely long, or (b) subject to periodic temporal boundary conditions. Setup (a) is an approximation for long-haul transmission systems, while setup (b) is an approximation for closed fiber-loop experiments. (3) The linear gain-loss coefficients $g_j$ in the frequency intervals $(\beta_j(0) - W/2 < \omega \leq \beta_j(0) + W/2]$, defined in Eq. (2), are determined by the difference between distributed amplifier gain and fiber loss. In particular, for some channels this difference can be slightly positive, resulting in small net gain, while for other channels this difference can be slightly negative, resulting in small net loss. (4) Since $T \gg 1$, the solitons in each sequence are temporally well-separated. As a result, intrachannel interaction is exponentially small and is neglected. (5) The Raman coefficient and the reciprocal of the frequency spacing satisfy $\epsilon_R \ll 1/\Delta \beta \ll 1$. Consequently, high-order effects due to radiation emission are neglected, in accordance with the analysis of the single-collision problem [16, 22–27].

By assumptions (1)-(5), the propagating soliton sequences are periodic, and as a result, the amplitudes of all pulses in a given sequence undergo the same dynamics. Taking into account collision-induced amplitude shifts due to delayed Raman response, and single-pulse amplitude changes due to linear gain-loss, we obtain the following equation for amplitude dynamics of $j$th-channel solitons [41]:

$$\frac{d\eta_j}{dz} = \eta_j \left[ g_j + C \sum_{k=1}^{N} (k - j) f(|j - k|) \eta_k \right],$$

(3)

where $C = 4\epsilon R \Delta \beta / T$, and $1 \leq j \leq N$. The coefficients $f(|j - k|)$ on the right hand side of Eq. (3) are determined by the frequency dependence of the Raman gain. In particular, for the commonly used triangular approximation for the Raman gain curve [1, 20], in which the gain is a piecewise linear function of the frequency, $f(|j - k|) = 1$ for $1 \leq j \leq N$ and $1 \leq k \leq N$.

In WDM systems it is often desired to achieve steady state transmission, in which pulse amplitudes in all channels are equal and constant (independent of $z$) [1]. We therefore look for a steady state of the system (3) in the form $\eta_j^{eq} = \eta > 0$ for $1 \leq j \leq N$, where $\eta$ is the
desired equilibrium amplitude value. This yields the following expression for the \( g_j \):

\[
g_j = -C\eta \sum_{k=1}^{N} (k - j)f(|j - k|). \tag{4}
\]

Thus, in order to maintain steady-state transmission with equal amplitudes in all channels, high-frequency channels should be overamplified and low-frequency channels should be underamplified, compared with central frequency channels. Substituting Eq. (4) into Eq. (3), we obtain the unperturbed model for amplitude dynamics [41]:

\[
\frac{d\eta_j}{dz} = C\eta_j \sum_{k=1}^{N} (k - j)f(|j - k|)(\eta_k - \eta), \tag{5}
\]

which has the form of a predator-prey model for \( N \) species [50].

The steady states of the predator-prey model (5) with nonzero amplitudes in all channels are determined by solving the following system of linear equations:

\[
\sum_{k=1}^{N} (k - j)f(|j - k|)(\eta_k^{(eq)} - \eta) = 0, \quad 1 \leq j \leq N. \tag{6}
\]

The trivial solution of Eq. (6), i.e., the solution with \( \eta_k^{(eq)} = \eta > 0 \) for \( 1 \leq k \leq N \), corresponds to steady state transmission with equal nonzero amplitudes. Note that the coefficients \( (k - j)f(|j - k|) \) in Eq. (6) are antisymmetric with respect to the interchange of \( j \) and \( k \). As a result, for WDM systems with an odd number of channels, Eq. (6) has infinitely many nontrivial solutions, which correspond to steady states of the predator-prey model (5) with unequal nonzero amplitudes. This is also true for WDM systems with an even number of channels, provided that the Raman gain is described by the triangular approximation [41].

The stability of all the steady states with nonzero amplitudes, \( \eta_j = \eta_j^{(eq)} > 0, 1 \leq j \leq N \), was established in Ref. [41], by showing that the function

\[
V_L(\eta) = \sum_{j=1}^{N} \left[ \eta_j - \eta_j^{(eq)} + \eta_j^{(eq)} \ln \left( \frac{\eta_j^{(eq)}}{\eta_j} \right) \right], \tag{7}
\]

where \( \eta = (\eta_1, \ldots, \eta_j, \ldots, \eta_N) \), is a Lyapunov function for the predator-prey model (5). This stability was found to be independent of the \( f(|j - k|) \) values, i.e., of the specific details of the approximation to the Raman gain curve. Furthermore, since \( dV_L/dz = 0 \) along trajectories of (5), rather than \( dV_L/dz < 0 \), typical dynamics of the amplitudes \( \eta_j(z) \) for
input amplitudes that are off the steady state value is oscillatory \[41\]. This behavior also means that the steady states with nonzero amplitudes in all channels are nonlinear centers of Eq. \[5\], i.e., their stability properties might change due to incorporation of perturbation terms into the model. In particular, high-order effects, such as changes in collision rates due to Raman self and cross frequency shifts, can make these steady states unstable. This issue is discussed in appendix \[A\] where we introduce the perturbed predator-prey model that takes into account the Raman self and cross frequency shifts.

III. NUMERICAL SIMULATIONS FOR SINGLE-FIBER TRANSMISSION

The predator-prey model, described in section \[II\] is based on several simplifying assumptions, whose validity might break down with increasing number of channels or at large propagation distances. In particular, the predator-prey model neglects radiation emission and modulation instability, intrasequence interaction, and temporal inhomogeneities. These effects can lead to instabilities and pulse-pattern corruption, and also to the breakdown of the predator-prey model description (see, for example Refs. \[42–44\], for the case of crosstalk induced by nonlinear gain or loss). In contrast, the coupled-NLS model \[1\] provides a fuller description of the propagation, which includes all these effects. Thus, in order to check whether stable long-distance transmission can indeed be realized, it is important to carry out numerical simulations with the full coupled-NLS model.

In the current section, we first present numerical simulations with the system \(1\) without the Raman term. We then present a comparison between simulations with the full coupled-NLS model \(1\) with the linear gain-loss profile \(2\) and predictions of the predator-prey model \(5\) for collision-induced amplitude dynamics. A comparison with coupled-NLS simulations with the model \(1\) and a simpler linear gain-loss profile is also presented. We proceed with an analysis of frequency difference dynamics. We then analyze amplitude dynamics in the important case of soliton transmission at 10 Gb/s per channel. We conclude the section by analyzing pulse-pattern deterioration at large distances, as observed in the simulations.

The coupled-NLS system \(1\) is numerically solved using the split-step method with periodic boundary conditions \[1\]. The use of periodic boundary conditions means that the numerical simulations describe pulse dynamics in a closed fiber loop. The initial condition is in the form of \(N\) periodic sequences of \(2J+1\) solitons with initial amplitudes \(\eta_j(0)\), initial
TABLE I: The dimensionless parameters

| № | N  | $\epsilon R$ | $T$ | $\Delta \beta$ | $\beta_j(0)$ | W | $g_L$ | $z_f$ | $dz$ | $dt$ | Figures | Eqs. |
|---|----|-------------|-----|--------------|-------------|---|------|------|------|------|---------|-----|
| 1 | 2  | 0.0024      | 18  | 40           | $-20, 20$   | 20| -0.5 | 2500 | 0.001 | 0.0588 | [3(a), 5(a)-(b), 6] | [1], [2] |
| 2 | 2  | 0           | 18  | 40           | $-20, 20$   | 40| -0.5 | 1000 | 0.001 | 0.0588 | [4(a), 5(c)-(d)] | [1], [10] |
| 3 | 2  | 0.0024      | 18  | 40           | $-20, 20$   | 14| -0.5 | 1100 | 0.0005 | 0.0588 | [3(b)] | [1], [2] |
| 4 | 3  | 0.0006      | 10  | 20           | $-20, 20$   | 14| -0.1 | 400  | 0.0001 | 0.03 | [7(a)] | [1], [2] |
| 5 | 3  | 0.0024      | 18  | 20           | $-20, 0, 20$| 10| -0.5 | 5000 | 0.001 | 0.0588 | [9(a), 10(a)-(b)] | [11] & [12] |
| 6 | 4  | 0           | 18  | 15           | $-15, 0, 15, 30$| 0 | 0  | 400  | 0.001 | 0.0588 | [2(c)-(d)] | [9] |
| 8 | 4  | 0.0024      | 18  | 15           | $-15, 0, 15, 30$| 12| -0.5 | 700  | 0.0005 | 0.0588 | [3(c)] | [1], [2] |
| 9 | 4  | 0.0024      | 18  | 15           | $-15, 0, 15, 30$| 12| -0.5 | 740  | 0.0005 | 0.0588 | [4(a)-(b)] | [1], [2] |
| 10| 4  | 0.0024      | 18  | 15           | $-15, 0, 15, 30$| 15| -0.5 | 700  | 0.0005 | 0.0588 | [4(b)] | [1], [10] |
| 11| 4  | 0.0006      | 10  | 15           | $-15, 0, 15, 30$| 10| -0.1 | 270  | 0.0001 | 0.03 | [7(b)] | [1], [2] |
| 12| 4  | 0.0024      | 18  | 15           | $-15, 0, 15, 30$| 10| -0.5 | 2000 | 0.001 | 0.0588 | [9(b), 10(c)-(d)] | [11] & [12] |
| 13| 4  | 0.0024      | 18  | 15           | $-15, 0, 15, 30$| 14| -0.5 | 1400 | 0.001 | 0.0588 | [11] | [11] & [13] |

frequencies $\beta_j(0)$, and initial zero phases:

$$\psi_j(t, 0) = \sum_{k=\pm J} \eta_j(0) \exp\left[ i \beta_j(0)(t - kT - \delta_j) \right] \cosh\left[ \eta_j(0)(t - kT - \delta_j) \right],$$

where $1 \leq j \leq N$. The coefficients $\delta_j$ in Eq. (5) correspond to the initial position shift of the pulses in the $j$th sequence relative to pulses located at $kT$ for $-J \leq k \leq J$. To maximize propagation distance in the presence of delayed Raman response, we take $\delta_j = (j - 1)T/N$ for $1 \leq j \leq N$.

We simulate pulse propagation in $N$-channel transmission systems with $2 \leq N \leq 4$. To illustrate collision-induced amplitude dynamics in broadband soliton-based WDM systems, we consider two setups, one corresponding to transmission at bit-rate $B = 22.22$ Gb/s per channel (setup I), and the other corresponding to transmission at bit-rate $B = 10$ Gb/s per channel (setup II). The pulse width and time slot width in setup I are $\tau = 2.5$ ps and $\tilde{T} = 45$ ps, and the frequency spacing is taken as $\Delta \nu = 2.55, 1.27, 0.95$ THz for $N = 2, 3, 4$ channels. Thus, the total bandwidth of the system is smaller than 13.2
THz, and all channels lie within the main body of the Raman gain curve. The values of the dimensionless parameters for this system are $\epsilon_R = 0.0024$, $T = 18$, and $\Delta\beta = 40, 20, 15$ for $N = 2, 3, 4$, respectively. Assuming $\tilde{\beta}_2 = -2 \text{ ps}^2\text{km}^{-1}$ and $\gamma = 2 \text{ W}^{-1}\text{km}^{-1}$ for the second-order dispersion and Kerr nonlinearity coefficients, the soliton peak power is $P_0 = 160 \text{ mW}$. The pulse width and time slot width in setup II ($B = 10 \text{ Gb/s}$) are $\tau = 10 \text{ ps}$ and $\tilde{T} = 100 \text{ ps}$, and the frequency spacing values are $\Delta\nu = 0.32 \text{ THz}$ for $N = 3$, and $\Delta\nu = 0.24 \text{ THz}$ for $N = 4$. The values of the dimensionless parameters in setup II are $\epsilon_R = 0.0006$, $T = 10$, $\Delta\beta = 20$ for $N = 3$, and $\Delta\beta = 15$ for $N = 4$. In addition, for $\tilde{\beta}_2 = -2 \text{ ps}^2\text{km}^{-1}$ and $\gamma = 2 \text{ W}^{-1}\text{km}^{-1}$, the soliton peak power is $10 \text{ mW}$. Tables I and II summarize the values of the dimensionless and dimensional physical parameters used in the simulations. In these tables, $W$ and $\tilde{W}$ stand for the dimensionless and dimensional width of the linear gain-loss function $g(\omega)$ in Eq. (2), while $z_f$ and $X_f$ correspond to the dimensionless and dimensional final propagation distance. In addition, $dz$ is the dimensionless spatial simulation step and $dt$ is the dimensionless temporal simulation grid spacing.

Note that the Kerr nonlinearity terms appearing in Eq. (1) are nonperturbative. Even though these terms are not expected to affect the shape, amplitude, and frequency of a single soliton, propagating in an ultralong optical fiber, the situation can be very different for multiple soliton sequences, circulating in a fiber loop. In the latter case, Kerr-induced effects might lead to radiation emission, modulation instability, and eventually to pulse-pattern corruption. It is therefore important to first analyze the effects of Kerr nonlinearity alone on the propagation. For this purpose, we carry out numerical simulations with the following coupled-NLS model, which incorporates second-order dispersion and Kerr nonlinearity, but neglects delayed Raman response and linear gain-loss:

$$i \partial_z \psi_j + \partial_t^2 \psi_j + 2|\psi_j|^2\psi_j + 4 \sum_{k=1}^{N} (1 - \delta_{jk})|\psi_k|^2\psi_j = 0,$$

where $1 \leq j \leq N$. The simulations are carried out for two and four channels with the physical parameter values of setup I. The initial soliton amplitudes are $\eta_1(0) = 0.9$, $\eta_2(0) = 1.2$ for $N = 2$, and $\eta_1(0) = 0.9$, $\eta_2(0) = 0.95$, $\eta_3(0) = 1.05$, $\eta_4(0) = 1.15$ for $N = 4$. The initial frequencies are specified in Table I. The final propagation distances are $z_{f_1} = 4000$ for $N = 2$ and $z_{f_2} = 400$ for $N = 4$. Figure 2 shows the final pulse patterns $|\psi_j(t, z_f)|$ along with their Fourier transforms $|\hat{\psi}_j(\omega, z_f)|$, as obtained by the numerical solution of Eq. (9). Also shown, is the theoretical prediction for the final pulse patterns for $N = 2$. This prediction
for $|\psi_j(t, z_f)|$ is obtained by summation over fundamental NLS solitons with amplitudes $\eta_j(0)$, frequencies $\beta_j(0)$, and positions $y_jk(z_f)$ for $-J \leq k \leq J$, which are measured from the simulations. The theoretical prediction for $|\hat{\psi}_j(\omega, z_f)|$ is the Fourier transform of the latter sum. As can be seen, the numerically obtained results for the final pulse patterns are strikingly different for the two-channel and the four-channel systems. In the two-channel case, the theoretical prediction is in excellent agreement with the numerical result, i.e., the soliton sequences retain their shape throughout the propagation [see Fig. 2 (a) and (b)]. Furthermore, the amplitudes and frequencies of the solitons do not change and no significant radiation emission is observed. This means that the behavior of the two circulating soliton sequences is similar to the behavior of a single soliton propagating through a long fiber line. In contrast, in the four-channel case, significant pulse distortion due to radiation emission is observed for the final pulse patterns in the $j = 3$ and $j = 4$ channels, while the final pulse patterns in the $j = 1$ and $j = 2$ channels are still intact [see Fig. 2 (c)]. Moreover, as can be

| №  | N  | $B$ (Gb/s) | $\tau_0$ (ps) | $T_0$ (ps) | $P_0$ (mW) | $\Delta \nu$ (THz) | $\tilde{W}$ (THz) | $X_f$ (km) | Figures | Eqs. |
|----|----|------------|--------------|----------|-----------|----------------|--------------|----------|--------|------|
| 1  | 2  | 22.22      | 2.5          | 45       | 160       | 2.55          | 1.27         | 15625    |        | 2(a) | (1), (2) |
| 2  | 2  | 22.22      | 2.5          | 45       | 160       | 2.55          | 0            | 25000    |        | 2(a) | (1)     |
| 3  | 2  | 22.22      | 2.5          | 45       | 160       | 2.55          | 2.55         | 6250     |        | 2(a), 2(a)-(b) | (1) |
| 4  | 3  | 22.22      | 2.5          | 45       | 160       | 1.27          | 0.89         | 6875     |        | 2(b) | (1), (2)   |
| 5  | 3  | 10         | 10           | 100      | 10        | 0.32          | 0.22         | 40000    |        | 2(a) | (1), (2)   |
| 6  | 3  | 22.22      | 2.5          | 45       | 160       | 1.27          | 0.64         | 31250    |        | 2(a), 2(a)-(b) | (11)&(12) |
| 7  | 4  | 22.22      | 2.5          | 45       | 160       | 0.95          | 0            | 2500     |        | 2(c)-(d) | (9) |
| 8  | 4  | 22.22      | 2.5          | 45       | 160       | 0.95          | 0.76         | 4375     |        | 2(b) | (1), (2)   |
| 9  | 4  | 22.22      | 2.5          | 45       | 160       | 0.95          | 0.76         | 4625     |        | 2(b) | (1), (2)   |
| 10 | 4  | 22.22      | 2.5          | 45       | 160       | 0.95          | 0.95         | 4375     |        | 2(b) | (1), (2)   |
| 11 | 4  | 10         | 10           | 100      | 10        | 0.24          | 0.16         | 27000    |        | 2(b) | (1), (2)   |
| 12 | 4  | 22.22      | 2.5          | 45       | 160       | 0.95          | 0.64         | 12500    |        | 2(b), 2(a)-(b) | (11)&(12) |
| 13 | 4  | 22.22      | 2.5          | 45       | 160       | 0.95          | 0.89         | 8750     |        | 2(b), 2(a)-(b) | (11)&(12) |
seen from Fig. 2 (d), the formation of temporal radiative tails is related to the emergence of radiative sidebands for the $j = 3$ and $j = 4$ soliton sequences. The $j = 3$ sideband forms at the frequency of the $j = 4$ sequence $\beta_4(z_{f2}) = 29.05$, while the $j = 4$ sideband forms at the frequency of the $j = 3$ sequence $\beta_3(z_{f2}) = 16.06$. The fact that each sideband forms at the other sequence’s frequency, strongly suggests that cross phase modulation [i.e. terms of the form $|\psi_k|^2\psi_j$ in Eq. (9)] is involved in sideband generation. It is also worth noting that the $j = 3$ and $j = 4$ sequences experience significant frequency shifts, whereas the final frequencies of the $j = 1$ and $j = 2$ sequences [$\beta_1(z_{f2}) = -15.02$ and $\beta_2(z_{f2}) = -0.008$] are very close to their original values. This finding coincides with the significant pulse pattern distortion experienced by the $j = 3$ and $j = 4$ sequences, and the absence of such distortion for the $j = 1$ and $j = 2$ sequences.

We now take into account the effects of delayed Raman response and frequency dependent linear gain-loss on the propagation. Our first objective is to check the validity of the predator-prey model’s predictions for stable transmission in all channels. For this purpose, we carry out numerical simulations with the full coupled-NLS model (1) with the linear gain-loss function (2) for two, three, and four channels with the physical parameter values of setup I. The initial soliton amplitudes are $\eta_1(0) = 0.9$, $\eta_2(0) = 1.2$ for $N = 2$, $\eta_1(0) = 0.8$, $\eta_2(0) = 0.9$, $\eta_3(0) = 1.1$ for $N = 3$, and $\eta_1(0) = 0.9$, $\eta_2(0) = 0.95$, $\eta_3(0) = 1.05$, $\eta_4(0) = 1.15$ for $N = 4$. The initial frequencies are specified in Table I. The final propagation distances are $z_{f2} = 2500$ for $N = 2$, $z_{f4} = 1100$ for $N = 3$, and $z_{f5} = 700$ for $N = 4$. The $z$ dependence of soliton amplitudes obtained by numerical solution of the coupled-NLS model (1) is shown in Fig. 3 along with the prediction of the predator-prey model (5). In all three cases the soliton amplitudes oscillate about their equilibrium value $\eta = 1$, i.e., soliton transmission is stable. Furthermore, the agreement between the coupled-NLS simulations and the predator-prey model predictions is excellent. Based on this comparison we conclude that stable long-distance transmission can be realized by employing the frequency dependent linear gain-loss function (2), in accordance with stability analysis for the predator-prey model (5). Additionally, the effects of radiation emission, modulation instability, intrachannel interaction, and other high-order perturbations can indeed be neglected for the propagation distances considered in the coupled-NLS simulations.

It is interesting to compare the final propagation distances for the coupled-NLS simulations with and without Raman and linear gain-loss effects. In the absence of delayed Raman
FIG. 2: (Color online) The final pulse patterns $|\psi_j(t, z_f)|$ and their Fourier transforms $|\hat{\psi}_j(\omega, z_f)|$ for a two-channel system (a)-(b) and a four-channel system (c)-(d) in setup I. The final distances are $z_{f_1} = 4000$ for $N = 2$ and $z_{f_2} = 400$ for $N = 4$. (a) The red circles and green squares represent $|\psi_1(t, z_f)|$ and $|\psi_2(t, z_f)|$ as obtained by numerical solution of Eq. (9), while the brown solid and gray dashed curves correspond to the theoretical predictions for $|\psi_1(t, z_f)|$ and $|\psi_2(t, z_f)|$. (b) The red circles and green squares represent $|\hat{\psi}_1(\omega, z_f)|$ and $|\hat{\psi}_2(\omega, z_f)|$ as obtained by numerical solution of Eq. (9), while the brown diamonds and gray left-pointing triangles correspond to the theoretical predictions. (c) The red solid, green dashed-dotted, blue dotted, and magenta dashed curves represent $|\psi_j(t, z_{f_2})|$ with $j = 1, 2, 3, 4$, obtained by numerical solution of Eq. (9). (d) The red circles, green squares, blue up-pointing triangles, and magenta down-pointing triangles represent $|\hat{\psi}_j(\omega, z_{f_2})|$ with $j = 1, 2, 3, 4$, obtained by numerical solution of Eq. (9).

response and linear gain-loss, the final transmission distances are $z_f = 4000$ for $N = 2$, $z_f = 1300$ for $N = 3$, and $z_f = 400$ for $N = 4$. When delayed Raman response and linear gain-loss are present, we find $z_f = 2500$ for $N = 2$, $z_f = 1100$ for $N = 3$, and $z_f = 700$ for $N = 4$. Thus, for two- and three-channel systems with frequency differences $\Delta \beta = 40$
and $\Delta \beta = 20$, respectively, the inclusion of Raman terms in the propagation model leads to transmission destabilization at shorter distances, and the effect is stronger in the two-channel system compared with the three-channel one. However, for the four-channel system with $\Delta \beta = 15$, the presence of frequency dependent linear gain-loss leads to a significant increase in the distance along which stable propagation is observed. This strongly indicates that frequency dependent linear gain-loss effectively mitigates radiation emission induced by both Kerr and Raman effects and that the impact of this mitigation increases with increasing number of channels.

From the discussion in the previous paragraph it follows that the frequency dependent linear gain-loss function plays a key role in mitigation of radiative instability. It is therefore important to study the impact of the form of this function on transmission stability. For this purpose, we consider the following simpler from of $g(\omega)$:

$$
\begin{align*}
g(\omega) &= \begin{cases} 
g_j & \text{if } \beta_j(0) - \Delta \beta/2 < \omega \leq \beta_j(0) + \Delta \beta/2, \\
g_L & \text{if } \omega \leq \omega_L, \text{ or } \omega > \omega_R,
\end{cases}
\end{align*}
$$

(10)

where $1 \leq j \leq N - 1$, $g_L < 0$, $\omega_L = \beta_1(0) - \Delta \beta/2$, $\omega_R = \beta_N(0) + \Delta \beta/2$, and $g_j$ is defined by Eq. (4). We expect the from (10) to be more effective in suppression of radiation instability compared with (2), due to the presence of significant linear loss $g_L$ inbetween the central frequencies $\beta_j(0)$. To check the influence of $g(\omega)$ on transmission stability, we numerically solve the coupled-NLS model (1) with the linear gain-loss function (10) for two and four channels with the parameter values of setup I. The initial amplitude and frequency values are the same as the ones used in the simulations described in the previous paragraphs. The final propagation distances are $z_{f_6} = 1000$ for $N = 2$, and $z_{f_7} = 700$ for $N = 4$.

Figure 4 shows the $z$ dependence of soliton amplitudes as obtained by numerical solution of Eqs. (1) and (10) along with the prediction of the predator-prey model (5). We observe stable oscillatory dynamics of soliton amplitudes and good agreement between coupled-NLS simulations and predator-prey model predictions for both two- and four-channel systems.

Note that for a two-channel system, the final propagation distance obtained with the linear gain-loss (10), $z_{f_6} = 1000$, is significantly smaller than the final distance obtained with the linear gain-loss (2), $z_{f_4} = 2500$. In contrast, there is no difference between the final distances for the four-channel system ($z_{f_6} = z_{f_4} = 700$). The latter finding can be explained by noting that in a four-channel system with frequency spacing $\Delta \beta = 15$, radiation emission due to Kerr nonlinearity is strong, and the main radiative sidebands are generated at the central
FIG. 3: (Color online) The $z$ dependence of soliton amplitudes $\eta_j$ for two-channel (a), three-channel (b), and four-channel (c) systems in setup I with the linear gain-loss (2). The red circles, green squares, blue up-pointing triangles, and magenta down-pointing triangles represent $\eta_1(z)$, $\eta_2(z)$, $\eta_3(z)$, and $\eta_4(z)$ as obtained by numerical solution of Eqs. (1) and (2). The solid brown, dashed gray, dashed-dotted black, and dotted orange curves correspond to $\eta_1(z)$, $\eta_2(z)$, $\eta_3(z)$, and $\eta_4(z)$ as obtained by the predator-prey model (5).

frequencies $\beta_j(z)$ of the propagating solitons, as seen in Fig. 2 (d). As a result, the detailed form of the linear gain-loss function is not very important in this case.

To explain the difference in final propagation distances for the two-channel system, we compare the final pulse patterns obtained with the two different gain-loss functions. Figure 5 shows the final pulse patterns $|\psi_j(t, z_f)|$ and their Fourier transforms $|\hat{\psi}_j(\omega, z_f)|$ as obtained by numerical solution of Eq. (1) with the linear gain-loss functions (2) and (10). Also shown are the theoretical predictions, which are obtained in the same manner as in Fig. 2 (a)-(b).
FIG. 4: (Color online) The $z$ dependence of soliton amplitudes $\eta_j$ for a two-channel system (a) and a four-channel system (b) in setup I with the linear gain-loss function $\text{(10)}$. The initial amplitudes are the same as in Fig. 2. The red circles, green squares, blue up-pointing triangles, and magenta down-pointing triangles represent $\eta_1(z)$, $\eta_2(z)$, $\eta_3(z)$, and $\eta_4(z)$ obtained by numerical solution of Eqs. $\text{(1)}$ and $\text{(10)}$. The solid brown, dashed gray, dashed-dotted black, and dotted orange curves correspond to $\eta_1(z)$, $\eta_2(z)$, $\eta_3(z)$, and $\eta_4(z)$ obtained by the predator-prey model $\text{(5)}$.

It is seen that the pulse patterns $|\hat{\psi}_1(\omega,z_f)|$ and $|\hat{\psi}_2(\omega,z_f)|$ obtained with the gain-loss $\text{(2)}$ at $z_{f_3} = 2500$ develop two small radiation sidebands at frequencies $\beta_2(z_f)$ and $\beta_1(z_f)$, respectively. These radiative sidebands cannot be suppressed by either gain-loss function $\text{(2)}$ or $\text{(10)}$, since they form at the central frequencies of the propagating solitons. The situation is different for the pulse patterns obtained with the gain-loss $\text{(10)}$ at $z_{f_6} = 1000$, where only $|\hat{\psi}_2(\omega,z_f)|$ develops a sideband at $\beta_{sb} = 33.85 \simeq 2/\beta_2(z_f)$. The formation of this sideband can be understood by noting that for gain-loss $\text{(10)}$, radiation with frequencies in the interval $(20, 40]$ propagates in the presence of net linear gain $g(\omega) = 0.02133$. This enables the generation of sidebands within this interval at intermediate distances. In contrast, for gain-
FIG. 5: (Color online) The final pulse patterns $|\psi_j(t, z_f)|$ and their Fourier transforms $|\hat{\psi}_j(\omega, z_f)|$ for a two-channel system in setup I with linear gain-loss functions (2) (a)-(b) and (10) (c)-(d). The final distances are $z_{f_3} = 2500$ in (a)-(b) and $z_{f_6} = 1000$ in (c)-(d). (a) The red circles and green squares represent $|\psi_1(t, z_{f_3})|$ and $|\psi_2(t, z_{f_3})|$ obtained by numerical solution of Eqs. (1) and (2), while the brown solid and gray dashed curves correspond to the theoretical predictions. (b) The red circles and green squares represent $|\hat{\psi}_1(\omega, z_{f_3})|$ and $|\hat{\psi}_2(\omega, z_{f_3})|$ as obtained by Eqs. (1) and (2), while the brown diamonds and gray left-pointing triangles correspond to the theoretical predictions. (c) The red circles and green squares represent $|\psi_1(t, z_{f_6})|$ and $|\psi_2(t, z_{f_6})|$ obtained by numerical solution of Eqs. (1) and (10), while the brown solid and gray dashed curves correspond to the theoretical predictions. (d) The red circles and green squares represent $|\hat{\psi}_1(\omega, z_{f_6})|$ and $|\hat{\psi}_2(\omega, z_{f_6})|$ as obtained by Eqs. (1) and (10), while the brown diamonds and the gray left-pointing triangles correspond to the theoretical predictions. The insets in (b) and (d) are magnified versions of the same curves for small $|\hat{\psi}_j(\omega, z_f)|$ values.

loss (2), radiation with frequencies in the interval $(30, 40]$ propagates in the presence of strong net linear loss $g(\omega) = g_L = -0.5$, and as a result, generation of sidebands in this interval is efficiently suppressed.

The collision-induced amplitude shift is only one of three main effects of delayed Raman
response on soliton propagation in broadband optical fiber systems. The other two effects are the single-pulse and collision-induced frequency shifts that are also known as the Raman self and cross frequency shifts, respectively. Since both frequency shifts depend on soliton amplitudes, the dynamics of soliton frequency in the presence of Raman crosstalk becomes coupled to amplitude dynamics. Furthermore, as shown in Ref. [41], one can construct a perturbed predator-prey model, which captures the coupling between frequency and amplitude dynamics. This perturbed predator-prey model is given by Eqs. (A2) and (A3) in Appendix A. Numerical solution of the perturbed predator-prey model for a two-channel system shows that the equilibrium state \((\eta, \eta, \Delta \beta)\) is stable for \(\Delta \beta > \Delta \beta_{bif} = (16T\eta^3/15 + 8\eta^2/3)^{1/2}\), and that in this case, both the frequency difference \(\beta_{21}(z) = \beta_2(z) - \beta_1(z)\) and the soliton amplitudes exhibit stable oscillatory dynamics. In order to test these predictions, we carry out numerical simulations with the coupled-NLS model (1) and the linear gain-loss (2) for two channels with the parameter values of setup I. The initial amplitudes and frequencies are the same as in Fig. (2) (a). The final propagation distance is \(z_{f_3} = 2500\). The \(z\) dependence of the frequency difference \(\beta_{21}\) obtained by the simulations is shown in Fig. (6). Also shown are the predictions of the perturbed predator-prey model (A2)-(A3) with both Raman self and cross frequency shifts, and with Raman self frequency shift only. We observe stable oscillations of \(\beta_{21}(z)\) throughout the propagation in accordance with the predictions of both versions of the perturbed predator-prey model [51]. Based on the comparison in Fig. (6) and similar results obtained with other sets of the physical parameters, we conclude that the perturbed predator-prey model (A2)-(A3) correctly captures the coupling between frequency and amplitude dynamics.

We now turn to analyze collision-induced amplitude dynamics in setup II, which corresponds to soliton-based WDM transmission at bit rate \(B = 10\) Gb/s per channel. This example is of special importance due to the large number of experimental and theoretical works on such systems; see for example, Ref. [14] and references therein. To analyze soliton dynamics, we perform numerical simulations with Eqs. (1) and (2) for three and four channels. The initial pulse amplitudes are \(\eta_1(0) = 0.85, \eta_2(0) = 0.9, \eta_3(0) = 1.15\) for \(N = 3\), and \(\eta_1(0) = 0.9, \eta_2(0) = 0.95, \eta_3(0) = 1.1, \eta_4(0) = 1.2\) for \(N = 4\). The initial frequencies are specified in Table I. The coupled-NLS simulations are carried out up to final distances \(z_{f_a} = 400\) for \(N = 3\) and \(z_{f_b} = 270\) for \(N = 4\), corresponding to \(X_{f_a} = 40000\) km and \(X_{f_b} = 27000\) km, respectively. Figure 7 shows the \(z\) dependence of soliton amplitudes ob-
FIG. 6: (Color online) The $z$ dependence of the frequency difference $\beta_{21}$ for a two-channel system in setup I. The initial amplitudes and frequencies are the same as in Fig. 2(a). The blue circles represent $\beta_{21}(z)$ obtained by numerical solution of Eqs. (1) and (2). The red curve corresponds to the prediction of the perturbed predator-prey model (A2) and (A3) with both Raman self and cross frequency shifts. The green dashed-dotted curve is the prediction of the perturbed predator-prey model (A2) and (A3) with Raman self frequency shift only.

Obtained by the coupled-NLS simulations along with the prediction of the predator-prey model (5). We observe stable periodic oscillations of soliton amplitudes with periods $z_{p_1} = 527.6$ for $N = 3$ and $z_{p_2} = 389.7$ for $N = 4$, corresponding to $X_{p_1} = 52760$ km and $X_{p_2} = 38970$ km, respectively. Additionally, the agreement between the coupled-NLS simulations and the predator-prey’s model predictions is good up to the final distances $z_{f_a}$ and $z_{f_b}$. Note that these final distances are significantly smaller than the final distances obtained in setup I (see $z_f$ column in Table I) despite of the fact that the $\epsilon_R$ value in setup II is smaller by a factor of 4 compared with the value in setup I. We attribute the relative decrease in transmission stability in setup II to intrachannel interaction. Indeed, the major difference between the two setups besides the $\epsilon_R$ values, is in time-slot widths. The $T$ values in setups I and II are $T = 18$ and $T = 10$, respectively, and as a result, intrachannel interaction is significantly stronger in setup II.

We now discuss the main processes leading to pulse-pattern degradation at large distances. Of particular interest is the dependence of these processes on the number of channels $N$ and the frequency spacing $\Delta \beta$. For two-channel systems with frequency difference $\Delta \beta = 40$, we saw that the presence of delayed Raman response and linear gain-loss leads to a significant decrease in the distance along which the transmission is stable; compare Figs. 2(a)-(b) and
FIG. 7: The $z$ dependence of soliton amplitudes $\eta_j$ for three-channel (a) and four-channel (b) transmission lines operating at 10 Gb/s per channel. The physical parameter values correspond to transmission setup II. The red circles, green squares, blue up-pointing triangles, and magenta down-pointing triangles represent $\eta_1(z)$, $\eta_2(z)$, $\eta_3(z)$, and $\eta_4(z)$ as obtained by numerical solution of the coupled-NLS model (1) with linear gain-loss (2). The solid brown, dashed gray, dashed-dotted black, and dotted orange curves correspond to $\eta_1(z)$, $\eta_2(z)$, $\eta_3(z)$, and $\eta_4(z)$ obtained by the predator-prey model (5). The final distances for the coupled-NLS simulations are $z_{f_a} = 400$ for $N = 3$ and $z_{f_b} = 270$ for $N = 4$.

Furthermore, as observed from Fig. 5 (b), the main mechanism leading to pulse pattern degradation is due to the generation of radiation sidebands, where the $j$th sequence sideband forms at the frequency $\beta_k(z)$ of the other soliton sequence. For four-channel systems with frequency difference $\Delta \beta = 15$, pulse pattern degradation in the absence of delayed Raman response is caused by the same processes of radiative sideband generation; see Fig. 2 (c)-(d). On the other hand, in contrast to the situation in two-channel systems, the introduction of frequency dependent linear gain-loss leads to a significant increase in the distance...
along which the transmission is stable despite of the presence of delayed Raman response; see Fig. 3 (c). It is therefore interesting to study the impact of delayed Raman response and linear gain-loss on pulse pattern degradation in the four-channel system. Figure 8 shows the final pulse patterns $|\psi_j(t, z_{fc})|$ and their Fourier transforms $|\hat{\psi}_j(\omega, z_{fc})|$ at $z_{fc} = 740$ as obtained by numerical solution of Eqs. (1) and (2) in setup I. The initial amplitudes and frequencies are the same as those used in Fig. 3 (c). It is seen that the pulse patterns at $z_{fc} = 740$ are highly distorted, and that the distortion is caused by the formation of radiative sidebands. Similar to the two-channel case, the $j$th sequence sidebands are formed at frequencies $\beta_k(z)$ of the neighboring soliton sequences. More specifically, the $j = 1$ sideband is formed at frequency $\beta_2(z)$, the $j = 2$ and $j = 3$ sidebands are formed at frequencies $\beta_{j-1}(z)$ and $\beta_{j+1}(z)$, and the $j = 4$ sideband is formed at frequency $\beta_3(z)$. Furthermore, as seen from Fig. 8 (a), the radiation emitted by each soliton sequence eventually develops into pulses, which do not possess the soliton sech form. This in turns leads to further degradation of the soliton sequences.

**IV. NONLINEAR WAVEGUIDE COUPLER TRANSMISSION**

The results of the numerical simulations in Sec. III show that in a single fiber, radiative instabilities can be partially mitigated by employing the frequency dependent linear gain-loss (2). However, as described in the last paragraph, suppression of radiation emission in this setup is still limited, and generation of radiative sidebands leads to pulse pattern degradation at large distances. It is therefore interesting to look for waveguide setups that can significantly enhance transmission stability. A very promising approach for enhancing transmission stability is by employing a nonlinear waveguide coupler, consisting of $N$ very close waveguides. In this case each pulse sequence propagates through its own waveguide and each waveguide is characterized by its own frequency dependent linear gain-loss function $\tilde{g}_j(\omega, z)$. This enables better suppression of radiation emission, since the linear gain-loss of each waveguide can be set equal to the required $g_j$ value within a certain $z$-dependent bandwidth $(\beta_j(z) - W/2, \beta_j(z) + W/2]$ around the central frequency $\beta_j(z)$ of the solitons in that waveguide, and equal to a negative value $g_L$ outside of that bandwidth. This leads to enhancement of transmission stability compared with the single fiber setup, since generation of all radiative sidebands outside of the interval $(\beta_j(z) - W/2, \beta_j(z) + W/2]$ is suppressed.
FIG. 8: (Color online) The final pulse patterns $|\psi_j(t, z_f)|$ (a) and their Fourier transforms $|\hat{\psi}_j(\omega, z_f)|$ (b) at $z_f = 740$ as obtained by numerical solution of Eqs. (1) and (2) for a four-channel system in setup I. The initial amplitudes and frequencies are the same as in Fig. 3 (c).

(a) The red solid, green dashed-dotted, blue dotted, and magenta dashed curves represent the numerically obtained $|\psi_j(t, z_f)|$ with $j = 1, 2, 3, 4$, respectively. (b) The red circles, green squares, blue up-pointing triangles, and magenta down-pointing triangles correspond to the numerically obtained $|\hat{\psi}_j(\omega, z_f)|$ with $j = 1, 2, 3, 4$, respectively.

by the linear loss $g_L$.

To enable comparison with results of Sec. II for single-fiber transmission, we assume that the other terms in the propagation model are the same as in Eq. (1). Thus, the propagation of the pulse sequences through the waveguide coupler is described by the following coupled-
NLS model:

\[ i \partial_z \psi_j + \partial_t^2 \psi_j + 2|\psi_j|^2 \psi_j + 4 \sum_{k=1}^{N} (1 - \delta_{jk})|\psi_k|^2 \psi_j = iF^{-1}(\tilde{g}_j(\omega,z)\hat{\psi}_j)/2 - \epsilon_R \psi_j \partial_t |\psi_j|^2 \]

\[ -\epsilon_R \sum_{k=1}^{N} (1 - \delta_{jk}) \left[ \psi_j \partial_t |\psi_k|^2 + \psi_k \partial_t (\psi_j \psi_k^*) \right], \tag{11} \]

where \(1 \leq j \leq N\). The linear gain-loss function of the \(j\)th waveguide \(\tilde{g}_j(\omega,z)\), appearing on the right-hand side of Eq. \((11)\), is defined by:

\[ \tilde{g}_j(\omega,z) = \begin{cases} 
  g_j & \text{if } \beta_j(z) - W/2 < \omega \leq \beta_j(z) + W/2, \\
  g_L & \text{if } \omega \leq \beta_j(z) - W/2, \text{ or } \omega > \beta_j(z) + W/2, 
\end{cases} \tag{12} \]

where the \(g_j\) coefficients are determined by Eq. \((4)\), the \(z\) dependence of the frequencies \(\beta_j(z)\) is determined from the numerical solution of the coupled-NLS model \((11)\), and \(g_L < 0\). Notice the following important properties of the gain-loss \((12)\). First, the gain-loss is equal to \(g_j\) inside a single central frequency interval, and is equal to \(g_L\) everywhere else. This property is expected to enable suppression of radiative sidebands for any frequency outside of the central frequency interval. Second, the end points of the central frequency interval are shifting with \(z\), such that the interval is centered around \(\beta_j(z)\) throughout the propagation. This shifting of the central amplification interval is introduced to compensate for the significant Raman-induced frequency shifts experienced by the solitons during the propagation \([52]\). The combination of the two properties of \(\tilde{g}_j(\omega,z)\) is expected to lead to a significant enhancement of transmission stability in the nonlinear \(N\)-waveguide coupler compared with the single-fiber system considered in Sec. \(\textbf{III}\).

In order to check whether the \(N\)-waveguide coupler setup leads to enhancement of transmission stability, we numerically solve Eq. \((11)\) with the gain-loss \((12)\) and \(W = 10\) for three and four channels. The values of the other physical parameters are the same as in Figs. \(\textbf{3}\) (b) and (c). Figure \(\textbf{9}\) shows the \(z\) dependence of soliton amplitudes as obtained by the coupled-NLS simulations along with the prediction of the predator-prey model \((5)\). It is seen that the amplitudes exhibit stable oscillations about the equilibrium value \(\eta = 1\) for both three and four channels, in good agreement with the predator-prey model predictions. Furthermore, stable transmission is observed over distances \(z_{fs} = 5000\) for \(N = 3\) and \(z_{fs} = 2000\) for \(N = 4\), which are larger by factors of 4.5 and 2.9, respectively, compared with the distances obtained with the single-fiber WDM system in Sec. \(\textbf{III}\). Further insight into
FIG. 9: (Color online) The z dependence of soliton amplitudes $\eta_j$ in a three-channel (a) and a four-channel (b) nonlinear waveguide coupler with linear gain-loss $W$ and $W = 10$. The values of the other physical parameters are the same as the ones used in Figs. (b) and (c). The red circles, green squares, blue up-pointing triangles, and magenta down-pointing triangles represent $\eta_1(z)$, $\eta_2(z)$, $\eta_3(z)$, and $\eta_4(z)$ obtained by numerical solution of the coupled-NLS model (11) with the gain-loss (12). The solid brown, dashed gray, dashed-dotted black, and dotted orange curves correspond to $\eta_1(z)$, $\eta_2(z)$, $\eta_3(z)$, and $\eta_4(z)$ obtained with the predator-prey model (5).

The enhanced transmission stability is gained by analyzing the final pulse patterns $|\psi_j(t, z_f)|$ and their Fourier transforms $|\hat{\psi}_j(\omega, z_f)|$. Figure 10 shows the results obtained by numerical solution of Eqs. (11) and (12) together with the theoretical prediction for both three and four channels. The agreement between theory and simulations is very good. Furthermore, no radiative sidebands are observed at the final distances $z_{fs} = 5000$ and $z_{fs} = 2000$, i.e., the solitons retain their shape throughout the propagation apart from small distortions appearing in their tails. Based on these observations we conclude that transmission stability is significantly enhanced in the nonlinear waveguide coupler system with frequency dependent gain-loss (12) compared with the single-fiber system of Sec. III.

We note that the frequency shifts experienced by the propagating solitons at large prop-
FIG. 10: (Color online) The final pulse patterns $|\psi_j(t, z_f)|$ and their Fourier transforms $|\hat{\psi}_j(\omega, z_f)|$ for the three-channel waveguide coupler [(a)-(b)] and the four-channel waveguide coupler [(c)-(d)] of Fig. 9. The final distances are $z_{f_8} = 5000$ in (a)-(b) and $z_{f_9} = 2000$ in (c)-(d). The red circles, green squares, blue up-pointing triangles, and magenta down-pointing triangles represent $|\psi_j(t, z_f)|$ or $|\hat{\psi}_j(t, z_f)|$ with $j = 1, 2, 3, 4$ as obtained by numerical solution of Eqs. (11) and (12). The solid brown, dashed gray, dashed-dotted black, and dotted orange curves correspond to the theoretical prediction for $|\psi_j(t, z_f)|$ with $j = 1, 2, 3, 4$. The brown diamonds, gray left-pointing triangles, black right-pointing triangles, and orange stars represent the theoretical prediction for $|\hat{\psi}_j(\omega, z_f)|$ with $j = 1, 2, 3, 4$, respectively.

Agitation distances are quite large for both the single-fiber systems of Sec. III and the $N$-waveguide coupler systems of the current section. For example, the total frequency shifts measured at $z_{f_9} = 2000$ from the coupled-NLS simulations for the four-channel waveguide coupler of Fig. 9 (b) are $\Delta \beta_1(z_{f_9}) = -4.86$, $\Delta \beta_2(z_{f_9}) = -4.55$, $\Delta \beta_3(z_{f_9}) = -4.66$, and $\Delta \beta_4(z_{f_9}) = -4.72$. These Raman-induced frequency shifts make the gain-loss functions with fixed frequency intervals [such as the function in Eq. (2)] less effective in stabilizing
soliton amplitude dynamics. In order to compensate for the effects of these frequency shifts, a shifting of the central amplification interval was introduced into the gain-loss function (12). We now complete the analysis of transmission stabilization in the waveguide coupler, by evaluating the impact of shifting of the amplification interval in the gain-loss function (12). For this purpose, we consider the following alternative gain-loss functions with fixed amplification intervals:

$$\tilde{g}_j(\omega) = \begin{cases} 
g_j & \text{if } \beta_j(0) - W/2 < \omega \leq \beta_j(0) + W/2, 
g_L & \text{if } \omega \leq \beta_j(0) - W/2, \text{ or } \omega > \beta_j(0) + W/2, 
\end{cases}$$

(13)

where \(1 \leq j \leq N\). We carry out numerical simulations with Eq. (11) and the gain-loss functions (13) with \(W = 14\) for a four-channel waveguide coupler. The final propagation distance is \(z_{f10} = 1400\) and the values of the other physical parameters are the same as the ones used in Fig. 9 (b). Figure 11 shows the \(z\) dependence of soliton amplitudes obtained by these coupled-NLS simulations along with the prediction of the predator-prey model (5). We observe stable oscillatory dynamics and good agreement with the predator-prey model’s prediction throughout the propagation. We note that the final distance \(z_{f10} = 1400\) is larger by a factor of 2 compared with the final distance for the corresponding single-fiber system [see Fig. 4 (c)], but smaller by a factor of 0.7 compared with the distance obtained with the waveguide coupler and the gain-loss (12). Based on this comparison we conclude that the introduction of shifting of the central amplification interval does lead to a significant enhancement of transmission stability. On the other hand, we also observe that even in the absence of shifting of the amplification interval, the waveguide coupler setup enables stable propagation along significantly larger distances compared with the single-fiber systems considered in Sec. III.

V. CONCLUSIONS

We investigated the physical mechanisms leading to stabilization and destabilization of propagating soliton sequences in broadband WDM optical fiber transmission systems. For this purpose, we carried out numerical simulations with coupled-NLS models, which take into account second-order dispersion, Kerr nonlinearity, delayed Raman response, and frequency dependent linear gain-loss. The simulations were carried out for two, three, and four frequency channels in both single-fiber and waveguide coupler setups. The results of
FIG. 11: (Color online) The $z$ dependence of soliton amplitudes $\eta_j$ in a four-channel nonlinear waveguide coupler with linear gain-loss [13] and $W = 14$. The values of the other physical parameters are the same as the ones used in Fig. 9(b). The red circles, green squares, blue down-pointing triangles, and magenta up-pointing triangles represent $\eta_1(z)$, $\eta_2(z)$, $\eta_3(z)$, and $\eta_4(z)$ obtained by numerical solution of the coupled-NLS model [11] with the gain-loss [13]. The solid brown, dashed gray, dashed-dotted black, and dotted orange curves correspond to $\eta_1(z)$, $\eta_2(z)$, $\eta_3(z)$, and $\eta_4(z)$ obtained with the predator-prey model [5].

The coupled-NLS simulations were compared with the predictions of a reduced predator-prey model for amplitude dynamics [41], which incorporates amplitude shifts due to linear gain-loss and delayed Raman response, but neglects radiation emission and intrachannel interaction.

We found that, in the absence of delayed Raman response and linear gain-loss, the main mechanism leading to transmission destabilization is due to emission of continuous waves and generation of radiative sidebands. Furthermore, we observed that the radiative sidebands for the $j$th soliton sequence form at frequencies $\beta_k(z)$ of the solitons in the neighboring channels. This observation strongly indicates that Kerr-induced cross-phase modulation plays a key role in sideband generation. Additionally, we found that the strength of sideband formation increases with increasing number of channels $N$ and decreases with increasing frequency spacing $\Delta \beta$.

We then incorporated the effects of delayed Raman response and frequency dependent linear gain-loss in the coupled-NLS simulations. We assumed that the gain-loss function $g(\omega)$ for single-fiber transmission is given by Eq. [2]. That is, $g(\omega)$ is equal to the constants $g_j$, determined by the predator-prey model, in frequency intervals $(\beta_j(0) - W/2, \beta_j(0) + W/2]$ of constant width $W$ centered about the initial soliton frequencies $\beta_j(0)$, and is equal to a
negative value $g_L$ outside of these intervals. Numerical simulations with the full coupled-NLS model showed that at short-to-intermediate distances soliton amplitudes exhibit stable oscillatory dynamics, in agreement with the predictions of the predator-prey model developed in Ref. [41]. The frequency differences between the soliton sequences exhibit similar oscillatory dynamics. These findings mean that the effects of radiation emission and intra-channel interaction can indeed be neglected at intermediate distances. However, at larger distances, transmission destabilization due to formation of radiative sidebands is observed, where the $j$th channel’s sidebands form at frequencies $\beta_k(z)$ of the solitons in the neighboring frequency channels. The formation of radiative sidebands leads to distortion of the soliton shapes, and eventually to generation of new pulses, which do not possess the soliton sech form. Another important finding of our simulations was that for a four-channel system, the inclusion of frequency dependent linear gain-loss leads to a significant increase in the distance along which stable transmission is observed compared with the result obtained in the absence of linear gain-loss and delayed Raman response. In addition, for a two-channel system with large frequency spacing ($\Delta \beta = 40$), the presence of linear loss $g_L$ at intermediate frequency intervals is essential for further transmission stabilization. In contrast, for a four-channel system with smaller frequency spacing ($\Delta \beta = 15$), the presence of linear loss $g_L$ at intermediate frequency intervals does not make a noticeable difference in transmission stability, since in this case the radiative sidebands form at frequencies $\beta_k(z)$ of the propagating soliton sequences.

Mitigation of transmission instabilities in single-fiber WDM soliton transmission is limited, since the main radiative sidebands are generated at the frequencies $\beta_k(z)$ of the propagating solitons, and it is therefore not possible to employ strong linear loss $g_L$ at these frequencies. A promising approach for overcoming this limitation is by employing a non-linear waveguide coupler, consisting of $N$ very close waveguides. In this case, each soliton sequence propagates through its own waveguide, and each waveguide is characterized by its own frequency dependent linear gain-loss function. We assumed that the linear gain-loss for the $j$th waveguide $\tilde{g}_j(\omega, z)$ is equal to the required value $g_j$ inside a $z$-dependent frequency interval centered about the soliton frequency $\beta_j(z)$, and is equal to a negative value $g_L$ outside of this interval. The waveguide coupler setup is expected to lead to enhanced transmission stability compared with the single-fiber setup, since generation of all radiative sidebands outside of the central amplification interval is suppressed for each of the waveguides in the
waveguide coupler. To enable comparison with the results obtained for single-fiber transmission, we assumed that second-order dispersion, Kerr nonlinearity, and delayed Raman response in the waveguide coupler system can be described in the same manner as in the single-fiber system. Accordingly, the only difference between the coupled-NLS models for the single-fiber and the waveguide coupler is in the form of the linear gain-loss terms. Numerical simulations with the new coupled-NLS model showed that the distances along which stable transmission is observed increase by factors of 4.5 for three channels and of 2.9 for four channels compared with the distances in the single-fiber system. Furthermore, the solitons retain their shape and no radiative sidebands appear throughout the propagation. Thus, transmission stability is indeed significantly enhanced in the nonlinear waveguide coupler system compared with the single-fiber system.

To complete the analysis of transmission stabilization in the waveguide coupler, we evaluated the impact of the shifting of the central amplification interval of the gain-loss functions $\tilde{g}_j(\omega, z)$. For this purpose, we replaced each $\tilde{g}_j(\omega, z)$ by a $z$-independent gain-loss function $\tilde{g}_j(\omega)$ that is equal to $g_j$ inside a fixed frequency interval centered about the initial soliton frequency $\beta_j(0)$, and is equal to a negative value $g_L$ outside of this interval. Numerical simulations with the coupled-NLS model for a four-channel system showed that the distance along which stable transmission is observed is larger by a factor of 2 compared with the distance for the single-fiber system, but smaller by a factor of 0.7 compared with the distance obtained with the waveguide coupler with shifting of the central amplification interval of the linear gain-loss. Based on these findings we conclude that the introduction of shifting of the central amplification interval does lead to a significant enhancement of transmission stability. On the other hand, even in the absence of shifting of the amplification interval, the waveguide coupler setup enables stable propagation along significantly larger distances compared with the single-fiber setup.

**Appendix A: The perturbed predator-prey model with Raman self and cross frequency shifts**

In this appendix, we present the perturbed predator-prey model for amplitude dynamics in an $N$-channel WDM transmission line, which takes into account the effects of Raman self and cross frequency shifts. We also provide a brief discussion of stability of the steady
states of this model. As explained in Sec. III, the Raman-induced frequency shifts depend on soliton amplitudes. As a result, in the presence of Raman crosstalk, frequency dynamics becomes coupled to amplitude dynamics. This leads to a $z$ dependence of the frequency differences between the different soliton sequences: $\beta_{kj}(z) = \beta_k(z) - \beta_j(z)$, and to a $z$ dependence of the corresponding collision rates, which results in the coupling of amplitude dynamics to frequency dynamics. Since the frequency shifts affect amplitude dynamics indirectly via high-order changes in the interchannel collision rates, we refer to the new predator-prey model as a perturbed model.

The perturbed predator-prey model with Raman self and cross frequency shifts is derived by introducing $z$-dependent frequency differences $\beta_{kj}(z) = \beta_k(z) - \beta_j(z)$, and by replacing the constant intercollision distance $\Delta z_c^{(1)} = T/(2\Delta \beta)$ with a $z$-dependent one: $\Delta z_c^{(1)}(z) = (k - j)T/(2\beta_{kj}(z))$ [41]. Employing these changes, we obtain a new equation for amplitude dynamics of the $j$th-channel solitons:

$$\frac{d\eta_j}{dz} = \eta_j \left[ g_j + \frac{4\epsilon_R}{T} \sum_{k=1}^{N} f(|j-k|)\beta_{kj}\eta_k \right].$$  \hfill (A1)

In addition, taking into account the dependence of the Raman self and cross frequency shifts on the amplitudes, we obtain the following equation for the rate of change of the frequency difference $\beta_{kj}$ [41]:

$$\frac{d\beta_{kj}}{dz} = -\frac{8}{15}\epsilon_R \left( \eta_k^4 - \eta_j^4 \right) - \frac{16\epsilon_R}{3T} \left[ \eta_k\eta_j(\eta_k - \eta_j) + \sum_{m=1}^{N} \eta_m(\eta_k^2 - \eta_j^2)(1 - \delta_{mk})(1 - \delta_{mj}) \right].$$  \hfill (A2)

Note that the first term on the right-hand side of Eq. (A2) is due to Raman self frequency shift, while the second and third terms are due to Raman cross frequency shift. Requiring steady-state transmission with equal amplitudes in all channels and with frequency differences $\beta_{kj}^{(eq)} = (k - j)\Delta \beta$ for $1 \leq j \leq N$ and $1 \leq k \leq N$, we find that the linear gain-loss coefficients are still given by Eq. (4). Substituting this relation into Eq. (A1), we obtain [41]:

$$\frac{d\eta_j}{dz} = \frac{4\epsilon_R}{T} \eta_j \sum_{k=1}^{N} f(|j-k|) [\beta_{kj}\eta_k - (k - j)\Delta \beta \eta].$$  \hfill (A3)

Equations (A2) and (A3) are the perturbed predator-prey model for collision-induced amplitude and frequency dynamics in an $N$-channel WDM transmission line.
Let us briefly discuss the stability of the steady states of the perturbed predator-prey model (A2)-(A3) for a two-channel system \((N = 2)\). In this case, the model has infinitely many steady states with nonzero amplitude values in both channels, which all lie on the half line \((b, b, \eta \Delta \beta / b)\), where \(b > 0\) is a free parameter. We focus attention on the desired steady state \((\eta, \eta, \Delta \beta)\), where \(\eta\) is the desired equilibrium amplitude, and \(\Delta \beta\) is the desired equilibrium frequency difference. The Jacobian matrix of the linearized system at the equilibrium point \((\eta, \eta, \Delta \beta)\) has the eigenvalues

\[\lambda_{1,2} = \pm \frac{4\epsilon R \eta T}{\gamma} \left[16T \eta^3 / 15 + 8\eta^2 / 3 - (\Delta \beta)^2 \right]^{1/2}, \lambda_3 = 0. \tag{A4}\]

Based on this, one would expect a bifurcation at \(\Delta \beta_{bif} = (16T \eta^3 / 15 + 8\eta^2 / 3)^{1/2}\), such that \((\eta, \eta, \Delta \beta)\) is unstable if \(\Delta \beta < \Delta \beta_{bif}\) and stable if \(\Delta \beta > \Delta \beta_{bif}\). However, since \((\eta, \eta, \Delta \beta)\) is a nonhyperbolic steady state, linear stability analysis might fail \[53\], and one has to determine stability by numerical simulations with Eqs. (A2)-(A3). The numerical simulations confirm the predictions of linear stability analysis, and show that \((\eta, \eta, \Delta \beta)\) is indeed unstable for \(\Delta \beta < \Delta \beta_{bif}\) and stable for \(\Delta \beta > \Delta \beta_{bif}\). Additionally, when \(\Delta \beta > \Delta \beta_{bif}\), the amplitudes and frequency difference exhibit oscillations about their equilibrium values, i.e., the steady state \((\eta, \eta, \Delta \beta)\) is a center for the predator-prey model (A2)-(A3) in this case.

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Note that similar perturbed coupled-NLS models with second-order dispersion, Kerr nonlinearity, and delayed Raman response, but without frequency dependent linear gain-loss, were used in Refs. [24, 25].

The dimensionless $z$ in Eq. (11) is $z = (|\tilde{\beta}_2|X)/(2\tau_0^2)$, where $X$ is the actual position, $\tau_0$ is the soliton width, and $\tilde{\beta}_2$ is the second order dispersion coefficient. The dimensionless retarded time is $t = \tau/\tau_0$, where $\tau$ is the retarded time. The spectral width is $\nu_0 = 1/(\pi^2\tau_0)$ and the frequency difference is $\Delta\nu = (\pi\Delta\beta\nu_0)/2$. $\psi = E/\sqrt{P_0}$, where $E$ is proportional to the electric field and $P_0$ is the peak power. The dimensionless second order dispersion coefficient is $d = -1 = \tilde{\beta}_2/(\gamma P_0\tau_0^2)$, where $\gamma$ is the Kerr nonlinearity coefficient. The coefficient $\epsilon_R$ is given by $\epsilon_R = 0.006/\tau_0$, where $\tau_0$ is in picoseconds.

Note that the shifting of the central amplification bandwidth of the linear gain-loss [12] is somewhat similar to the shifting of the central frequency in sliding frequency filters, which
were widely studied in the context of soliton-based transmission. See Ref. [13] and references therein.

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