On the application of homotopy perturbation method to differential equations

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Abstract

We show that a recent application of homotopy perturbation method to a class of ordinary differential equations yields either useless or wrong results.

There has recently been great interest in the application of several approximate procedures, like the homotopy perturbation method (HPM), the Adomian decomposition method (ADM), and the variation iteration method (VIM), to a variety of linear and nonlinear problems of interest in theoretical physics [1–15]. For brevity I will call VAPA all those variational and perturbation approaches. In a series of papers I have shown that most of the results produced by those methods are useless, nonsensical, and worthless [16–19].

In a recent paper Rafiq et al [13] applied the HPM to some ordinary second–order differential equations, and the purpose of this comment is to discuss their results.

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Rafiq et al [13] solved second order differential equations of the form

\[\begin{align*}
\begin{cases}
y''(x) + p(x)y'(x) + f(x, y) = 0 \\
y(0) = A, \ y'(0) = B
\end{cases}
\end{align*}\]

(1)

where \( f(x, y) = F(x, y) - g(x) \) in their notation. In order to apply HPM they wrote \( y''(x) + p(x)y'(x) + \theta f(x, y) = 0 \) and expanded the solution in a \( \theta \)-power series

\[y(x) = y_0(x) + \theta y_1(x) + \ldots\]

(2)

Finally, they set \( \theta = 1 \) in order to obtain an approximate solution to equation (1).

In the particular HPM implementation proposed by Rafiq et al [13] the perturbation corrections \( y_j(x) \) result to be polynomials so that the partial sums of the HPM series (2) are just \( x \)-power series of the form

\[y(x) = A + Bx + a_2x^2 + a_3x^3 + \ldots\]

(3)

In what follows we analyze the examples chosen by Rafiq et al [13]. The first one is a textbook exercise for an introductory course on differential equations:

\[\begin{align*}
\begin{cases}
y''(x) + \frac{8}{x} y'(x) + xy = x^5 + 44x^2 - 30x \\
A = B = 0
\end{cases}
\end{align*}\]

(4)

We appreciate that \( y(x) \sim x^4 \) as \( x \to \infty \). Therefore, if we substitute the polynomial \( y(x) = a_2x^2 + a_3x^3 + a_4x^4 \) into equation (4) we easily obtain the exact solution \( y(x) = x^4 - x^3 \). However, Rafiq et al [13] apply a cumbersome perturbation method and obtain an infinite number of nonzero perturbation corrections \( y_j(x) \) that cancel out to (hopefully) produce the exact solution.
The authors do not prove that their HPM already yields the exact result, and any partial sum gives the exact result plus terms that are cancelled by corrections of higher order. In this case the HPM partial sum of any order is an inexact approach to an extremely simple solution of a trivial differential equation that one easily derives by the straightforward method just indicated.

The second example is as trivial as the first one:

\[
\begin{align*}
  y''(x) + \frac{2}{x} y'(x) + y &= 6 + 12x + x^2 + x^3 \\
  A &= B = 0
\end{align*}
\]  

(5)

Any undergraduate student will try \( y(x) = a_2 x^2 + a_3 x^3 \) and obtain the exact solution \( y(x) = x^2 + x^3 \) without effort. Again, the HPM yields an infinite series and the authors do not prove their convergence. Although it seems that the spurious terms cancel out, any partial sum yields a wrong result.

A simple inspection of the third example

\[
\begin{align*}
  y''(x) + \frac{2}{x} y'(x) + y^3 &= 6 + x^6 \\
  A &= B = 0
\end{align*}
\]  

(6)

suggests that \( y(x) = a_2 x^2 \) and thus one obtains the exact result \( y(x) = x^2 \). By means of the HPM Rafiq et al [13] obtain the wrong result \( y(x) = x^2 + x^8/72!!! \). Chowdhury and Hashim [4] also obtain this wrong result but then they choose a different starting point of the perturbation approach in order to derive the expected exact solution. In the first two examples Rafiq et al [13] mention the appearance of noise terms. It seems to me that it was that noise that already affected the calculation in this example.
The fourth example is much more interesting (at least it is not trivial):

\[
\begin{cases}
  y''(x) + \frac{2}{x} y'(x) + e^{xy^2} = x + 1 \\
  A = B = 0
\end{cases}
\]  

(7)

After a long and tedious perturbation calculation the authors derive the first correct terms of the Taylor series expansion about \( x = 0 \) \[13\]

\[
y(x) = \frac{x^3}{12} - \frac{x^9}{12960} + \ldots
\]

(8)

It is unnecessary to say that one can easily obtain the same approximate result more easily by means of the power-series method. We expect that the power series will not reveal the most interesting features of the solution to this equation but a mere indication of what happens in a neighbourhood of \( x = 0 \).

At first sight one guesses that the solution to equation (7) should behave approximately as \( y(x) \sim \sqrt{\ln(x)/x} \) as \( x \to \infty \); however, its actual behaviour is richer. Fig. 1 shows that \( y(x) \) (calculated numerically with sufficient accuracy) oscillates about \( \sqrt{\ln(x)/x} \) and approaches this asymptotic function as \( x \to \infty \). The \( x \)-power series (8) only accounts for the behaviour of \( y(x) \) up to about the first maximum. If equation (7) represented an actual physical problem we would be missing its most interesting features when using the HPM to obtain its solution.

Summarizing: the HPM proposed by Rafiq et al [13] gives cumbersome approximations to simple solutions of trivial differential equations, yields a wrong result in one of the cases, and fails to provide the most interesting features of the solution to the only nontrivial example. Unfortunately the authors do not indicate any physical application of the examples chosen. They seem to be just toy problems for fiddling around with the HPM.

The conclusion above is not surprising because all our previous analysis of
several applications of VAPA have shown that they produce useless or trivial results [16–19]. See, for example, the earlier papers by Chowdhury and Hashim [4] and Yıldırım and Öziş [3] who applied similar cumbersome perturbation equations and obtained the power series for the solutions of exactly solvable differential equations.

It seems that one of the greatest feats of many VAPA applications is to produce power–series expansions to simple and trivial problems in a cumbersome and laborious way [3,4,13]; the reader may find the analysis of other such examples in our earlier communications [16–19]. In fact, VAPA have produced the worst research papers ever written.

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Fig. 1. Exact (accurate numerical) solution, power series \( [\ln(x)/x]^{1/2} \) and asymptotic expansion for example (7)