Stochastic Modeling of Star Formation Histories. III. Constraints from Physically Motivated Gaussian Processes

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Abstract

Galaxy formation and evolution involve a variety of effectively stochastic processes that operate over different timescales. The extended regulator model provides an analytic framework for the resulting variability (or “burstiness”) in galaxy-wide star formation due to these processes. It does this by relating the variability in Fourier space to the effective timescales of stochastic gas inflow, equilibrium, and dynamical processes influencing giant molecular clouds’ creation and destruction using the power spectral density (PSD) formalism. We use the connection between the PSD and autocovariance function for general stochastic processes to reformulate this model as an autocovariance function, which we use to model variability in galaxy star formation histories (SFHs) using physically motivated Gaussian processes in log star formation rate (SFR) space. The extended regulator synthesis models, we then explore how changes in model stochasticity can affect spectral signatures across galaxy populations with properties similar to the Milky Way and present-day dwarfs, as well as at higher redshifts. We find that, even at fixed scatter, perturbations to the stochasticity model (changing timescales vs. overall variability) leave unique spectral signatures across both idealized and more realistic galaxy populations. Distributions of spectral features including Hα and UV-based SFR indicators, Hβ and Ca H and K absorption-line strengths, $D_n(4000)$, and broadband colors provide testable predictions for galaxy populations from present and upcoming surveys with the Hubble Space Telescope, James Webb Space Telescope, and Nancy Grace Roman Space Telescope. The Gaussian process SFH framework provides a fast, flexible implementation of physical covariance models for the next generation of spectral energy distribution modeling tools. Code to reproduce our results can be found at https://github.com/kartheikiyer/GP-SFH.

Unified Astronomy Thesaurus concepts: Galaxy evolution (594); Galaxy processes (614); Spectral energy distribution (2129); Computational methods (1965); Astrostatistics techniques (1886)

1. Introduction

Large galaxy surveys like the Sloan Digital Sky Survey (SDSS; York et al. 2000; Strauss et al. 2002; Abazajian et al. 2009), GAMA (Driver et al. 2009), and COSMOS (Lilly et al. 2007; Scoville et al. 2007; Weaver et al. 2022) reveal an enormous diversity in galaxy demographics across different epochs, and many studies in modern galaxy evolution have devoted considerable effort in trying to explain this diversity from a physical standpoint, usually through analytical models and cosmological simulations (see review by Somerville & Davé 2015).

A part of this picture involves the stochasticity in star formation, which is regulated by a variety of physical processes acting over many orders of magnitude in spatial and temporal scales. This stochasticity or “burstiness” can be physically observed through short-timescale star formation rate (SFR) indicators such as Hα or UV-based SFRs, which probe recent star formation averaged over the most recent $\sim 4–10$ Myr and $\sim 10–100$ Myr, respectively (Madau & Dickinson 2014; Flores Velázquez et al. 2021; Tacchella et al. 2022b). It can also be observed by studying resolved star formation in galaxies, where most star formation appears to occur in discrete clumps traced by the rest-UV that are spatially offset from the bulges of galaxies where most older stellar populations live (Guo et al. 2016; Huertas-Company et al. 2020), with the creation and destruction of these clumps due to interstellar medium (ISM)
physics and feedback leading to stochasticity on the timescales of the clump lifetimes (Semenov et al. 2018, 2021). On a population level, this could leave signatures in the scatter of scaling relations like the SFR–$M_\star$ correlation (Daddi et al. 2007; Elbaz et al. 2007; Noeske et al. 2007) and the mass–metallicity relation (Tremonti et al. 2004), which show overall coherent behavior for the “average” galaxy that is tied to the growth of their parent dark matter halos, but significant variation (at the $\sim$0.3–0.5 dex level) in the SFRs of individual galaxies that seem to fluctuate around these average relations (Kauffmann et al. 2006; Tacchella et al. 2016; Matthee & Schaye 2019).

These fluctuations in SFR are regulated by a variety of physical processes ranging from the local creation and destruction of stars in giant molecular clouds (GMCs), to dynamical processes like disk formation and bulge growth, to galaxy-wide processes that include mergers, galactic winds from stellar and active galactic nucleus (AGN) feedback, and baryon cycling that couples a galaxy to its surrounding circumgalactic medium (Iyer et al. 2020; Tacchella et al. 2020). On the largest scales, however, a galaxy’s growth is tied to the reservoir of fuel available to it to form stars, which are in turn tied to the accretion rates of their parent halos, galactic depletion times and outflows, and the large-scale structure of the environment they live in (see review by Wechsler & Tinker 2018).

Analyzing the stochasticity of star formation across a range of timescales\(^{18}\) therefore provides us with a way to constrain the relative strengths of these physical processes. A particularly effective way to quantify and assess this is by quantifying the relative strengths of these physical processes. A particularly effective way to quantify and assess this is by quantifying the growth of pure dark matter halos, but significant variation (at the $\sim$0.3–0.5 dex level) in the SFRs of individual galaxies that seem to fluctuate around these average relations (Kauffmann et al. 2006; Tacchella et al. 2016; Matthee & Schaye 2019).

These studies hit on a fundamental aspect of galaxy evolution, that the growth of pure dark matter halos is essentially a scale-free process that leads to a power-law PSD (Guszejnov et al. 2018; Kelson et al. 2020). This is the reason why (for first order) the halo mass is such a good predictor of galaxy properties and methods like subhalo abundance matching have met with such remarkable success. Other aspects, including stellar feedback, baryon cycling, and multiphase ISM astrophysics, decouple star formation from this hierarchical buildup and add additional stochasticity to this on a range of (often interwoven) timescales, leading to an overall complex power spectrum that can be studied and understood with careful analysis. Caplar & Tacchella (2019, Paper I in this series) define the PSD of a galaxy’s star formation history (SFH) and, assuming the shape of a simple broken power law, linked it to SFR distributions averaged over different timescales. Tacchella et al. (2020, Paper II in this series, hereafter TFC20) build on this, using the widely successful\(^{19}\) gas regulator model (Lilly et al. 2013) coupled with the stochastic inflow of gas (Kelson et al. 2020, TFC20) to derive a more general form for the PSD of galaxy SFHs.

The PSD of the extended regulator model depends only on an overall level of stochasticity for gas inflow and GMCs and characteristic timescales for effective gas inflow, equilibrium, and GMC lifetimes. TFC20 formulates this PSD and proposes the timescales for a few galaxy populations (Milky Way analogs, dwarf galaxies, galaxies at high redshift, and massive galaxies at cosmic noon) given our current knowledge of burstiness in these galaxy populations. However, to verify these models and observationally probe these timescales, we need a framework in which the PSDs can be tested observationally.

To observationally measure the PSD (i.e., the variability in SFR over different timescales), we can leverage the fact that a range of spectral features measure changes in the SFR averaged over different timescales. Some of the commonly measured spectral features include the nebular H$\alpha$ line, rest-UV+IR spectral energy distribution (SED) that traces emission from young massive stars and reemitted radiation from dust, H$\delta$ absorption from the photospheres of smaller (mostly A-type) stars, and the 4000 Å break from the accumulation of ionized metal absorption lines for older stellar populations (Gonçalves et al. 2012; Flores Velázquez et al. 2021). Taken together, these indicators probe fluctuations in SFRs on timescales ranging from $\sim$5 Myr to $\sim$10 Gyr, and their ratios have previously been used to obtain estimates of the burstiness of galaxy populations (Guo et al. 2016; Broussard et al. 2019; Emami et al. 2019; Faisst et al. 2019; Wang & Lilly 2020a).

SED fitting methods go a step further and estimate the full SFHs of individual galaxies using the full range of available multiwavelength spectral information, whether it is photometry, spectroscopy, or a combination of the two (Heavens et al. 2000; Tojeiro et al. 2007; Dye 2008; Pacifici et al. 2012; Smith & Hayward 2015; Pacifici et al. 2016; Iyer & Gawiser 2017; Leja et al. 2017; Carnall et al. 2018; Leja et al. 2019a; Iyer et al. 2019). A few of these methods that implement nonparametric SFHs (Pacifici et al. 2012; Leja et al. 2017; Iyer et al. 2019) also allow users to implement priors on SFR burstiness on one or more timescales. For example, the nonparametric dense basis method (Iyer & Gawiser 2017; Iyer et al. 2019) for SFH reconstruction allows us to incorporate physical priors on SFR stochasticity through a Gaussian process (GP) covariance function (also called the kernel). These kernels are related to the PSDs through the Weiner–Khinchin theorem\(^{20}\) (Wiener 1930; Khinchin 1938), which allows us to relate the frequency domain PSDs to the time domain autocovariance functions (ACFs).

For the first time, this opens up a way to explicitly incorporate an analytical framework for correlated SFRs over a range of timescales into an SED modeling framework. This is a crucial development for multiple reasons:

1. It provides a physical explanation for how SFRs vary over time through the three timescales—the timescale

\(^{17}\) However, building falsifiable tests to test these signatures can be challenging (Kelson 2014; Abramson et al. 2016).

\(^{18}\) We use population-level statistics in this work, since we currently lack the constraining power in our observations to perform this analysis for individual galaxies.

\(^{19}\) A review by Tacconi et al. (2020) finds that the model can reproduce the combined evolution of molecular gas fractions, SFRs, and gas-phase metallicities.

\(^{20}\) Assuming that galaxy SFHs, or their oscillation around a fiducial “main sequence,” are stationary processes; see Section 5.2 for more discussion about this.
over which stochastic gas inflow is correlated, the mass-loss or equilibrium timescale on which gas is consumed/removed from the reservoir, and the average GMC lifetime timescales. While a range of different physical processes are responsible for regulating star formation in galaxies, these three timescales have been shown to capture a significant portion of the effective dynamics of galaxy populations21 (TFC20; Tacconi et al. 2020).

2. It offers a framework to forward-model galaxy observations based on a stochasticity model to compare against existing models and to determine sensitive spectral features for future observations (e.g., Whitaker et al. 2014; Parul & Bailin 2021).

3. It allows us to explicitly compare against existing priors, e.g., uncorrelated SFRs in adjacent time steps, or the Dirichlet and continuity type priors (Leja et al. 2017; Iyer et al. 2019) that assume an arbitrary amount of stochasticity and/or correlation between adjacent SFR bins in a model, to test their efficacy and their relation to the effective timescales in the regulator model.

4. It provides an intuitive framework for modeling SFH priors, or incorporating SFH priors from cosmological simulations by estimating the extended regulator model parameters directly from their SFHs.

In this paper, we incorporate the physically motivated SFR stochasticity model proposed in TFC20 within the framework of a GP (Rasmussen & Williams 2006; Iyer et al. 2019). Using this, we then use the flexibility of the extended regulator model of TFC20 to define GPs corresponding to various regimes of stochasticity that we might find in galaxy populations—ranging from the bursty behavior of galaxies at high redshifts to the long-timescale correlated behavior of Milky Way analogs at lower redshifts. We then use these GPs to generate mock SFHs in a computationally inexpensive manner, which is crucial if these are to be used in SED fitting. By running these SFHs through a flexible stellar population synthesis (FSPS) framework (Conroy et al. 2009; Conroy & Gunn 2010; Johnson et al. 2021), we then model spectra corresponding to these SFHs and use them to identify observables that can be used to tell the models apart, laying the foundations of future work where this can be incorporated into SED fitting packages. A schematic representation of our main approach is shown in Figure 1, and the code used to implement the GPs is publicly available on GitHub at https://github.com/kartheikiyer/GP-SFH. A copy of Version 1 has been deposited to Zenodo (Iyer 2023).

This paper is structured as follows. In Section 2, we go through the formalism and provide a description of the extended regulator model first presented in TFC20, derive the associated ACF, and highlight specific cases likely to correspond to various galaxy populations of interest. In Section 3, we describe the implementation of the derived ACF as a “physical kernel” in a GP and how it can be used as a prior in SED fitting codes that have flexible models for galaxy SFHs. In Section 4, we investigate how differences in the underlying ACFs can translate over to spectral signatures using stellar population synthesis (SPS) models and investigate how these differences manifest in populations of simulated galaxies. We discuss our findings in Section 5 and conclude in Section 6.

Note that while it is most natural to have a process using the base-e logarithm $\ln\text{SFR}(t)$, we convert to the base-10 logarithm $\log\text{SFR}(t)$ in most plots to facilitate comparisons with other quantities and measurements in the literature. Throughout this paper magnitudes are in the AB system; we use a standard $\Lambda$CDM cosmology with $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$, and $H_0 = 70$ km s$^{-1}$ Mpc$^{-1}$.

2. Modeling Stochasticity in Star Formation Rates

We start by building physical intuition of how different physical processes related to galaxies can affect stochasticity and correlations in the SFRs of individual galaxies across cosmic time, summarized through their PSDs and associated ACFs. In Section 2.1, we provide a brief set of definitions for the PSD and ACF and their relationship with each other. In Section 2.2, we derive results for the extended regulator model presented in TFC20. Finally, in Section 2.3 we highlight the four special cases from TFC20—Milky Way analogs, dwarfs, galaxies at cosmic noon, and galaxies at high redshift—along with a description of their general expected behavior. A more detailed set of derivations are presented in Appendix A.

Following TFC20, we will assume that the stochastic processes described by the extended regulator model correspond to variability in the natural log of the SFR, $\ln\text{SFR}$, around a time-dependent mean $\mu(t)$ and are captured entirely by the corresponding PSD/ACF. This will allow us to model SFHs as log GPs, which we will return to in Section 3.

2.1. Terms and Definitions

We start by informally defining a stochastic process as something that can generate infinite realizations of a time series $\{x_1, x_2, ..., x_n\} \equiv \{x_i\}_n \equiv x_n$ at any times $t = 1, ..., n$ (i.e., the $x_i$ values change every time we simulate from the process). The collection of $x_n$ values will then follow some joint probability distribution $P(x_n)$ that is defined by the stochastic process.

The simplest way to explore the correlation structure in a given stochastic process is to compute the ACF,22

$$C(t, t') = \sum_{-\infty}^{+\infty} \{x_i - \mu(t)][x_{i'} - \mu(t')] P(x_t, x_{t'}) \, dx_t \, dx_{t'} \tag{1}$$

between $x_t$ and $x_{t'}$ at two times $t$ and $t'$, where $\mu(t)$ is the time-dependent mean and $P(x_t, x_{t'})$ is the joint distribution of $x_t$ and $x_{t'}$ defined by the process. Assuming that our process is stationary so that the ACF only depends on the separation (i.e., time lag) between any two given times $\tau \equiv t - t'$ rather than the individual times $t$ and $t'$ themselves, we can instead write the ACF as

$$C(t, t') = C(t - t') \equiv C(\tau). \tag{2}$$

In addition to defining correlation structure as a function of time $t$, we can also do the same as a function of frequency $f$. We first define a “windowed” version of $x(t)$,

$$x_{\tau}(t) \equiv x_t w(\tau) = \begin{cases} x_i , & t - \frac{T}{2} \leq t < t + \frac{T}{2}, \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

21 While we currently consider the TFC20 model in this work, other models for galaxy growth (Hirashita & Kamaya 2000; Davé et al. 2012; Alarcon et al. 2023) that can be reformulated as an autocorrelation function for a GP could also be tested in future studies.

22 The prefix “auto-” is often used to emphasize that the calculation is done at two different times for the same process, rather than between two different processes.
for a window function $w_T(t)$ with some width (duration) $T$ centered around $t$. Taking its Fourier transform

$$\hat{x}_T(t) = \int_{-\infty}^{+\infty} x_T(t) e^{-2\pi i f t} dt,$$

the PSD is then

$$S(f) = \lim_{T \to \infty} \frac{1}{T} |\hat{x}_T(f)|^2,$$

where the limit $T \to \infty$ assumes that the stochastic process is not localized in time. We can interpret the PSD as the relative amount of variance as a function of frequency, where larger values indicate stronger correlations at particular frequencies.

While the ACF and PSD can be computed directly from a given stochastic process, they can also be directly computed from each other. Based on the Wiener–Khinchin theorem, in the continuous-time limit the PSD $S(f)$ and ACF $\mathcal{C}(\tau)$ are Fourier pairs, and we can convert between the two using

$$S(f) = \int_{-\infty}^{+\infty} \mathcal{C}(\tau) e^{-2\pi i f \tau} d\tau \quad \iff \quad \mathcal{C}(\tau) = \int_{-\infty}^{+\infty} S(f) e^{+2\pi i f \tau} df.$$

This property is extremely useful, as many stochastic processes (such as the extended regulator model in Section 2.2) can be much easier to describe in frequency rather than in time (and vice versa).

### 2.2. Extended Regulator Model

TFC20 introduced the extended regulator model as a way to characterize how stochastic processes that drive (1) gas inflow rates, (2) gas cycling (between atomic and molecular gas) in equilibrium, and (3) the formation and disruption of GMCs relate to $\lnSFR(t)$. Assuming that the behavior of each component follows a damped random walk with some decorrelation timescale $\tau_{\text{dec}}$ and variability $\sigma$, each PSD can
be shown to have a broken power-law PSD of the form

\[ S_{\text{DRW}}(f) = \frac{s^2}{1 + (2\pi \tau_{\text{dec}})^2 f^2}, \]

where \( s^2 \) is the absolute normalization (scatter squared) for \( f = 0 \). Making the well-justified assumptions that (1) the process describing the behavior of GMCs is largely independent of those describing gas inflow and cycling in equilibrium, and (2) the processes describing gas inflow and equilibrium gas cycling are coupled, the full PSD of the extended regulator model can be written as

\[ S_{\text{ExReg}}(f) = S_{\text{gas}}(f) + S_{\text{dyn}}(f) + S_{\text{eq}}(f) \]

\[ = \frac{s^2_{\text{gas}}}{1 + (2\pi \tau_{\text{eq}})^2 f^2} + \frac{s^2_{\text{dyn}}}{1 + (2\pi \tau_{\text{dyn}})^2 f^2} + \frac{s^2_{\text{eq}}}{1 + (2\pi \tau_{\text{eq}})^2 f^2}, \]

(8)

where \( s^2_{\text{gas}} = s^2_{\text{in}} s^2_{\text{eq}} \) is the total variability in gas inflows and equilibrium cycling, \( s^2_{\text{dyn}} \) is the variability in dynamical processes including the creation and destruction of GMCs, and \( \tau_{\text{in}}, \tau_{\text{eq}} \) and \( \tau_{\text{dyn}} \) are the decorrelation timescales associated with gas inflows, cycling in equilibrium, and GMC formation/disruption, respectively, and we use the values of \( \beta_1 = 0 \) and \( \beta_2 = 2 \) for the power-law slopes of the gas inflow term \( s^2_{\text{in}} = s^2_{\text{in}}/(\tau_{\text{in}} f)^{1/2} + (\tau_{\text{in}} f)^{1/2}) \) as defined in Equation (23) and Table 2 of TFC20.

Using the Wiener–Khintchine theorem, the corresponding ACF is then

\[ C_{\text{ExReg}}(\tau) = C_{\text{gas}}(\tau) + C_{\text{dyn}}(\tau) = \sigma^2_{\text{gas}} \times e^{-\tau/\tau_{\text{in}}} - \sigma^2_{\text{eq}} \times e^{-\tau/\tau_{\text{eq}}} + \sigma^2_{\text{dyn}} \times e^{-\tau/\tau_{\text{dyn}}}, \]

(9)

for all \( \tau_{\text{in}} \neq \tau_{\text{eq}} \) and where we have replaced \( \tau \to \sigma \) to emphasize that \( S(f = 0) = s^2 = \sigma^2 = C(\tau = 0) \). See Appendix A for further details and more general results.

Compared to the PSD, the ACF offers different insights into the correlation structure. In particular, it shows that a damped random walk leads to correlations that decay exponentially with time \( (\propto e^{-\tau/\tau_{\text{dec}}} \)\). Note also that the variance of the extended regulator model now becomes

\[ \sigma^2_{\text{ExReg}} = C_{\text{ExReg}}(\tau = 0) = \sigma^2_{\text{gas}} + \sigma^2_{\text{eq}} + \sigma^2_{\text{dyn}} \]

(10)

since there are now multiple independent stochastic processes involved.

We can quantify the extent and strength of the correlations using the autocorrelation time

\[ \tau_e \equiv \frac{1}{\sigma^2} \int_{-\infty}^{+\infty} C(\tau) \, d\tau, \]

(11)

which is a measure of the relative correlation contributed by all possible time lags \( \tau \). For the extended regulator model, evaluating this expression gives

\[ \tau_{\text{ExReg}} = 2 \times \left( \frac{\sigma^2_{\text{gas}}}{\sigma_{\text{ExReg}}} + \frac{\sigma^2_{\text{dyn}}}{\sigma_{\text{ExReg}}} \right) \]

\[ \equiv 2(\sigma_{\text{gas}} \tau_{\text{gas}} + (1 - \sigma_{\text{gas}}) \tau_{\text{dyn}}), \]

(12)

where we have defined \( \sigma_{\text{gas}} \) as the fractional contribution to the variance from the gas component. In the limit where \( \sigma_{\text{gas}} \ll \sigma_{\text{dyn}} \) so GMC formation/disruption is the dominant source of variability, this gives

\[ \tau_{\text{ExReg}} = 2 \tau_{\text{dyn}}, \]

(13)

which is just twice the GMC formation/disruption time (since \( \tau \) can range from \( -\infty \) to \( +\infty \)). In the limit where \( \sigma_{\text{gas}} \gg \sigma_{\text{dyn}} \), we instead have

\[ \tau_{\text{ExReg}} = 2 \tau_{\text{gas}} = 2(\tau_{\text{in}} + \tau_{\text{eq}}), \]

(14)

which is related instead to the combined timescales involved in gas inflows \( \tau_{\text{in}} \) and cycling \( \tau_{\text{eq}} \). This final case, where the GMC contribution to the variability is assumed to be negligible, is what TFC20 refer to as the regulator model.

The PSD and ACF for the extended regulator model are shown in Figure 1, while comparisons between the regulator model (two damped random walks) and a single damped random walk are shown in Figure 2.

2.3. Special Cases

Following TFC20, we consider the following special illustrative cases to highlight the behavior of the extended regulator model in four different regimes:

1. Milky Way Analog. \( (\tau_{\text{in}}, \tau_{\text{eq}}, \tau_{\text{dyn}}) = (0.15, 2.5, 0.025) \) Gyr. Based on the long-term secular evolutionary trends seen in the Milky Way, this includes an extremely large \( \tau_{\text{eq}} (2.5 \text{ Gyr}) \) along with approximate order-of-magnitude \( (>5) \) differences between various timescales, with \( \tau_{\text{eq}} \gg \tau_{\text{in}} \gg \tau_{\text{dyn}} \). SFHs will be dominated by the long-running equilibrium timescale, with small perturbations from changes to the inflow rate along with small amounts of additional white noise from GMCs on much shorter timescales.

2. Dwarf. \( (\tau_{\text{in}}, \tau_{\text{eq}}, \tau_{\text{dyn}}) = (0.15, 0.03, 0.01) \) Gyr. Although it has the same \( \tau_{\text{in}} \) as the Milky Way, \( \tau_{\text{eq}} \) has substantially decreased to account for the much more rapid gas cycling (and \( \tau_{\text{dyn}} \) to a lesser extent for similar reasons) expected in these low-mass galaxies. This has the effect of making the SFHs substantially burrier on short timescales \( (\lesssim 0.1 \text{ Gyr}) \). It also includes smaller changes in scale \( (\sim 5) \), leading to somewhat larger impacts from \( \tau_{\text{in}} \gg \tau_{\text{eq}} \gg \tau_{\text{dyn}} \). SFHs will be dominated by the variability timescales associated with gas inflows, but with larger perturbations from equilibrium and white noise from GMCs compared to our Milky Way analog.

3. Cosmic Noon. \( (\tau_{\text{in}}, \tau_{\text{eq}}, \tau_{\text{dyn}}) = (0.1, 0.2, 0.05) \) Gyr. This case is designed to simulate a typical \( 10^9 M_\odot \) galaxy around \( z \sim 2 \). The equilibrium time is larger by an order of magnitude relative to the dwarf case owing to the larger overall mass, with smaller changes in scale \( (\sim 2) \).
and longer-lived GMCs. As $\tau_{\text{eq}} \lesssim \tau_{\text{d}} \lesssim \tau_{\text{dyn}}$, all timescales remain quite relevant, leading to larger and more correlated fluctuations.

4. High-$z$ ($\tau_{\text{inn}}, \tau_{\text{eq}}, \tau_{\text{dyn}} = (0.016, 0.015, 0.006)$ Gyr). Our last case is designed to simulate the conditions for a galaxy at $z \sim 4-6$. Here $\tau_{\text{eq}} \approx \tau_{\text{inn}}$ with both only $\sim 2\tau_{\text{dyn}}$, with extremely short timescales due to the lower masses involved along with the more disruptive environments that many galaxies find themselves in. Since the gas-related timescales are almost identical, we expect this case to have behavior most similar to a Matern32 kernel (see Section A.4.1) but with additional perturbations caused by GMCs on somewhat similar timescales.

We consider two classes of models when deciding on the values of the scatter $\sigma_{\text{gas}}$ and $\sigma_{\text{dyn}}$.

1. Fixed (0.4 dex). We normalize our results such that $\sigma_{\text{gas}} = 0.39$ dex and $\sigma_{\text{dyn}} = 0.07$ dex, so that the relative contribution from gas inflows/cycling versus GMC formation/disruption is always fixed. This allows us to fix the scatter in logSFR at 0.4 dex and isolate the impact that relative changes in timescales may have on SFHs and associated observables. We choose 0.4 dex since it is close to the commonly measured value for the scatter in the SFR$-M_*$ correlation (Kurczynski et al. 2016; Iyer et al. 2018; Leja et al. 2022).

2. Variable (TFC20). We normalize our results to the values reported in TFC20 (see their Figure 9) of $\sigma_{\text{MWV}} = 0.17$ dex, $\sigma_{\text{Dwarf}} = 0.53$ dex, $\sigma_{\text{Neutral}} = 0.24$ dex, and $\sigma_{\text{High-z}} = 0.27$ dex. This involves relative changes in both the overall scatter and the relative contributions from gas inflows/cycling versus GMC formation/disruption.

The general behavior of each case for fixed and variable scatter is highlighted in Figures 3 and 4, respectively.

3. Gaussian Process Implementation

We now describe how we use them to generate realizations of synthetic galaxy spectra. Our implementation is publicly available at https://github.com/kartheikiyer/GP-SFH and summarized in Figure 1.

3.1. Brief Overview of Gaussian Processes

A GP is a generalization of the Gaussian distribution to the space of functions (Rasmussen & Williams 2006). Similar to our definition of a stochastic process, this means that our GP can generate an infinite set of values $y_t = \{y_t\}_{t=1}^n$ at any time $t = 1, \ldots, n$ whose joint probability distribution will always follow a multidimensional Gaussian distribution

$$P(y_n) = \mathcal{G}(y_n | \mu_n, C_{n,n}),$$

where $\mu_n$ is the $n$-vector of mean values generated from some mean function $\mu(t)$ at times $t = 1, \ldots, n$ and $C_{n,n}$ is the $n \times n$ covariance matrix evaluated at each pair of times $t = 1, \ldots, n$ and $t' = 1, \ldots, n$. If we have some values $y_m$ that are known and want to predict a set of new possible values at our given times $t = 1, \ldots, n$, this can be done by exploiting the fact that the joint distribution is

$$P(y_n, y_m) = \mathcal{G}\left(\begin{bmatrix} y_n \\ y_m \end{bmatrix} \mid \begin{bmatrix} \mu_n \\ \mu_m \end{bmatrix}, \begin{bmatrix} C_{n,n} & C_{n,m} \\ C_{m,n} & C_{m,m} \end{bmatrix}\right),$$

which gives a conditional distribution of

$$P(y_n | y_m) = \mathcal{G}(\mu_n + C_{n,m} C_{m,m}^{-1} (y_m - \mu_m), C_{n,n} - C_{n,m} C_{m,m}^{-1} C_{m,n}).$$

Together, these properties, along with the functional nature of GPs, are often summarized using the following notation:

$$y(t) \sim \mathcal{G}(\mu(t), C(t, t')),$$

where the $\sim$ indicates "is a realization of" rather than the "is of the same order of magnitude as" definition usually used in the astrophysics literature.

Taken together, the above results imply that we can use GPs to quickly and easily generate realizations of our data $y_t$, either completely from scratch or conditioning on some known values $y_m$ based on some input mean $\mu(t)$ and covariance $C(t, t')$.
functions that we can easily replace with any of the ACFs derived in Section 2. In particular, switching over to our variable of interest \( \ln SFR \) and the model of interest (the extended regulator model discussed in Section 2.2), the model we explore in our paper takes the form

\[
\ln SFR(t) \sim \mathcal{GP}(\ln SFR_{\text{base}}(t), C_{\text{ExReg}}(\tau)),
\]

(19)

where \( \ln SFR_{\text{base}}(t) \) is some baseline SFH we are interested in studying and again \( \tau = t - t' \). In other words, any given realization of the SFH will depend on both the “baseline” (mean) SFH, \( \ln SFR_{\text{base}}(t) \), and the particular ACF \( C_{\text{ExReg}}(\tau) \) defined by the extended regulator model.

In practice, our GP is implemented following a similar procedure to Iyer et al. (2019) using a multidimensional Gaussian prior initialized at every point of an input time array. This is done through an instance of the \GP_{\text{SFH}}() class, which is initialized with a user-determined kernel at a particular redshift, along with an astropy.cosmology() object and a \fsps.stellarpopulation() object for generating spectra and other observables. Upon initialization, the instance computes the covariance matrix specified by the ACF at a range of time values ranging from 0 to the \( t_{\text{univ}} \) at the specified redshift. Once this matrix is computed, realizations of SFHs can be generated simply by sampling a multivariate normal distribution at each time in the array with the covariance
structure determined by the kernel. This can then be conditioned on observable constraints using Equation (17).

### 3.2. Implementation with Stellar Population Synthesis Models

To generate spectra corresponding to draws from the GP, we pass the SFHs though the FSPS code (Conroy et al. 2009; Conroy & Gunn 2010). To highlight the differences in galaxy spectra that arise solely from differences in the ACF, we choose a simple set of modeling assumptions (listed in Table 1) when generating spectra while keeping everything else fixed to their default FSPS values. We also demonstrate the effects of varying some of these parameters on our observables of interest in Section 5.1.

In practice, changes to the stellar population parameters can be made simply by reassigning the input fspss.stellar-population object linked to the GP-SFH class instance. This modular implementation allows for a precomputed covariance matrix to be rapidly associated with many different stellar population parameters while modeling and fitting SEDs, since that is the rate-limiting step to generating SFH realizations.

For spectral features, we choose to consider features sensitive to star formation on a range of timescales (Kauffmann et al. 2003). Hα emission from O and B stars probes star formation on 4–10 Myr timescales (Flores Velázquez et al. 2021; Tacchella et al. 2022b). Hδ absorption traces star formation over the past 0.1–1 Gyr (Worthey & Ottaviani 1997; Wang & Lilly 2020a). Finally, the 4000 Å break strength \( (D_n(4000)) \) provides a reliable tracer of the median age of the stellar populations that make up a galaxy, as can be seen in Figure 2 of Kauffmann et al. (2003), who point out that these indices are largely insensitive to dust attenuation effects, and the distribution of galaxies in this space is sensitive to stochasticity in star formation over the most recent \( \sim 2 \) Gyr in a galaxy’s past. This happens because galaxies that form stars at a steady rate tend to occupy a very narrow locus in HδEW–\( D_n(4000) \) space. In addition to these, we also consider the equivalent width of the Ca H and K lines at \( \lambda \approx 3933.6 \) and 3968.5 Å. The Ca II K line traces older stellar populations, while the Ca H absorption line is blended with Hr and [Ne III] and effectively traces intermediate-age populations (Mayya et al. 2004; Zhu & Ménard 2013). The resulting equivalent widths probe a range of timescales as seen in Figure 5 and Appendix B.

For our spectra, we get the Hα line luminosity directly from the FSPS outputs, and following a procedure similar to Kauffmann et al. (2003), we use the 3850–3950 Å and 4000–4100 Å continuum bands introduced by Balogh et al. (1999) to compute the strength of the \( D_n(4000) \) break, and we compute the HδEW using the bandpasses of 4083.50–4122.25 Å, and 4128.50–4161.00 Å for the index and blue/red continuum bands, respectively, defined in Table 1 of Worthey & Ottaviani (1997). For the Ca KEW, we use 3929.51–3941.22 Å, 3907.01–3929.51 Å, and 3941.22–3961.02 Å, and for Ca HEW, we use 3961.02–3980.83 Å, 3941.22–3961.02 Å, and 3980.82–3997.03 Å for the index, blue bandpass, and red bandpass, respectively.

For better comparison across the different stellar masses that could be produced owing to bursts and troughs in individual realizations of SFHs, we normalize these quantities by the stellar masses of each realization, effectively reporting, e.g., the distribution of Hα luminosity (in L\( \odot \)) per solar mass for the different cases discussed in Section 4. The HδEW, \( D_n(4000) \), and broadband galaxy colors remain unaffected by this normalization.

### 4. Spectrophotometric Signals of Changing Stochastic Behavior

Following the procedure highlighted in Figure 1, in this section we identify the particular spectrophotometric signals that can help to distinguish different types of stochastic behavior (i.e., varying correlation timescales \( \tau_{\text{eq}} \), \( \tau_{\text{dyn}} \), and \( \delta_{\text{gas}} \) and fluctuation strengths \( \sigma_{\text{gas}} \) and \( \sigma_{\text{dyn}} \)). We generate 10,000 realizations of various SFHs (logSFR(\( t \))) for each of the cases outlined in Section 2.3 with the scatter fixed (Figure 3) and the scatter matched to TFC20 (Figure 4). We then feed these into FSPS to generate a set of UV-to-IR galaxy SEDs as described in Section 3.2. To compare the effects of the stochasticity models on the same footing, we generate SEDs corresponding to the four toy models at the same epoch (\( z = 0.1 \)). In this context, the toy models are meant to illustrate the effects of a certain amount and timescale of stochasticity in mock galaxy populations, rather than provide an estimate of the properties of these galaxies, i.e., the SFHs are not the SFHs of high-z galaxies, but rather of galaxies observed at \( z = 0.1 \) that have experienced “high-z”-like stochasticity in their lifetimes.

To highlight the behavior of our model, we first investigate overall effects that the parameters in the extended regulator model may have on a few key observables. Our results are highlighted in Figure 5, where we vary single parameters in the extended regulator model while holding the rest fixed at fiducial values (\( \sigma_{\text{gas}} = 1.0 \), \( \tau_{\text{eq}} = 0.5 \) Gyr, \( \tau_{\text{dyn}} = 0.15 \) Gyr, \( \delta_{\text{gas}} = 0.1 \), \( \delta_{\text{dyn}} = 0.01 \) Gyr). We find that the exact mechanism for adjusting the “burstiness” of star formation—whether through the overall level of variability in the gas inflows/cycling (\( \sigma_{\text{gas}} \)) or GMC formation/disruption (\( \sigma_{\text{dyn}} \)) or through the duration of the (gas equilibrium) correlation time \( \tau_{\text{eq}} \)—leaves different imprints on various observables even at fixed SFR scatter. In particular, while varying \( \sigma_{\text{gas}} \) affects both the stellar mass and (s)SFR distributions, varying \( \tau_{\text{eq}} \) has a much bigger effect on long-term SFR as measured using HδEW. This gives at least one way to distinguish populations with differing amounts of variability about the same base set of SFHs.

Since changing correlation timescales and the level of scatter can lead to large differences in the final stellar mass formed, we choose to normalize all SEDs based on their final stellar mass before investigating possible (relative) differences. This helps to highlight trends as a function of specific SFR (sSFR) rather than just SFR, and it helps to account for the increasing (expected) variance in the total stellar mass formed for more bursty SFHs.

| Table 1: Modeling Assumptions for Generating Spectra Corresponding to Different ACF Cases |
|----------------------------------|----------------------------------|
| Input/Parameter                  | Fixed Option/Value               |
| Ischrones, stellar tracks        | MILES+MIST                       |
| Redshift                         | 0.1                              |
| SFH \( t_{\text{max}} \)         | Constant (1.0 \( M_\odot \) yr\(^{-1} \)) |
| IMF                              | Chabrier                         |
| Dust attenuation                 | Calzetti (\( A_V = 0.2 \))       |
| \( \log Z/Z_\odot \)             | 0.0 (solar)                      |

For spectral features, we choose to consider features sensitive to star formation on a range of timescales (Kauffmann et al. 2003). Hα emission from O and B stars probes star formation on 4–10 Myr timescales (Flores Velázquez et al. 2021; Tacchella et al. 2022b). Hδ absorption traces star formation over the past 0.1–1 Gyr (Worthey & Ottaviani 1997; Wang & Lilly 2020a). Finally, the 4000 Å break strength \( (D_n(4000)) \) provides a reliable tracer of the median age of the stellar populations that make up a galaxy, as can be seen in Figure 2 of Kauffmann et al. (2003), who point out that these indices are largely insensitive to dust attenuation effects, and the distribution of galaxies in this space is sensitive to stochasticity in star formation over the most recent \( \sim 2 \) Gyr in a galaxy’s past. This happens because galaxies that form stars at a steady rate tend to occupy a very narrow locus in HδEW–\( D_n(4000) \) space. In addition to these, we also consider the equivalent width of the Ca H and K lines at \( \lambda \approx 3933.6 \) and 3968.5 Å. The Ca II K line traces older stellar populations, while the Ca H absorption line is blended with Hr and [Ne III] and effectively traces intermediate-age populations (Mayya et al. 2004; Zhu & Ménard 2013). The resulting equivalent widths probe a range of timescales as seen in Figure 5 and Appendix B.

For our spectra, we get the Hα line luminosity directly from the FSPS outputs, and following a procedure similar to Kauffmann et al. (2003), we use the 3850–3950 Å and 4000–4100 Å continuum bands introduced by Balogh et al. (1999) to compute the strength of the \( D_n(4000) \) break, and we compute the HδEW using the bandpasses of 4083.50–4122.25 Å, and 4128.50–4161.00 Å for the index and blue/red continuum bands, respectively, defined in Table 1 of Worthey & Ottaviani (1997). For the Ca KEW, we use 3929.51–3941.22 Å, 3907.01–3929.51 Å, and 3941.22–3961.02 Å, and for Ca HEW, we use 3961.02–3980.83 Å, 3941.22–3961.02 Å, and 3980.82–3997.03 Å for the index, blue bandpass, and red bandpass, respectively.

For better comparison across the different stellar masses that could be produced owing to bursts and troughs in individual realizations of SFHs, we normalize these quantities by the stellar masses of each realization, effectively reporting, e.g., the distribution of Hα luminosity (in L\( \odot \)) per solar mass for the different cases discussed in Section 4. The HδEW, \( D_n(4000) \), and broadband galaxy colors remain unaffected by this normalization.
4.1. Fixed (0.4 dex) Scatter

The results of this procedure assuming a fixed (0.4 dex) scatter are that the SFR is shown in Figures 6 and 7 for all four of the models we consider. Figure 6 shows the median spectra for each model compared to a reference spectrum corresponding to the base SFH, and Figure 7 shows a corner plot comparing the full distributions of the spectral features described in Section 3.2. As expected based on our results in Figure 5, since the gas inflow/cycling physics dominates the main behavior of the model, decreasing the associated gas timescale (i.e., $\tau_{\text{gas}} = \tau_{\text{in}} + \tau_{\text{eq}}$) to
make SFHs more bursty leads to both larger stellar masses and a tighter overall distribution for a given scatter. While the mass-normalized spectra have very similar medians across the four archetypes, they diverge in the optical and ultraviolet. This, combined with subtle differences in the Hα EW (rising with the decreasing τgas as SFHs form more of their stars in recent bursts) and the slope of the Hα–Dn(4000) correlation (becoming shallower and more dispersed as SFHs become less correlated), indicates that, even in the case where the only thing varying are the timescales, constraining them should be possible with a large enough sample size.

Since obtaining spectroscopic data for large ensembles of galaxies is expensive and may not always be feasible, we also consider distributions of broadband colors corresponding to the different archetypes. We consider three different color–color spaces: (i) the commonly used UVJ diagram (Wuyts et al. 2007; Williams et al. 2009; Muzzin et al. 2013), where the different archetypes can be differentiated based on their sSFR distributions; (ii) the NUV−r−K diagram (Arnouts et al. 2013; Moutard et al. 2016), which traces the differences in stellar mass distributions since the Ks band probes the rest-frame 1.6 μm feature and the NUV−r probes the effects of dust and SFR; and (iii) the recently introduced wide-baseline FUV–V–(Wise)W3 diagram following Leja et al. (2019b), which is more sensitive to lower sSFRs than the UVJ diagram, where galaxies with low sSFRs tend to populate the upper left portion of the space. As shown in Figure 8, all three color–color spaces provide a means to differentiate between the different regimes of stochasticity typified by the four archetypes with sufficient sample sizes and signal-to-noise ratio (S/N). While summary statistics from the distribution of colors might not be sufficient to distinguish between the models, using metrics (e.g., the KL divergence or mutual information) between models in the color–color space that encodes more information might allow us to do so. Since defining this distance can be tricky, we use implicit-likelihood-based methods such as simulation-based inference (SBI) to quantify the amount of information given the observations. We stress that while individual color–color spaces might not contain enough information by themselves to distinguish such populations, a combination of observables sensitive to a range of timescales would be able to do so, as shown in Appendix E. In addition to this, upcoming observations with JWST will help push rest-frame colors out to higher redshifts and provide extremely high S/N probes of differences in the stochasticity across the four archetypes considered here.

4.2. Variable (TCF20) Scatter

In the more realistic case from TCF20 where the scatter among each archetype also varies, we see larger differences in the corresponding observables (Figures 9 and 10). To first order, this is associated with the large variations in σgas, leading to shifts and/or broadening of the stellar mass and SFR distributions that propagate down to Hα, HβEQW, Dn(4000), and other spectral features. These can be seen most clearly by looking at the distribution of values for the dwarf model (which
has the largest $\sigma_{\text{gas}}$ and the shortest $\tau_{\text{gas}}$ and the Milky Way analog (which has the shortest $\sigma_{\text{gas}}$ and the longest $\tau_{\text{gas}}$). The dramatic changes in the distributions due to changing $\sigma_{\text{gas}}$ make it much more tractable to distinguish between the models. However, they can also make it more difficult to disentangle the subtler changes due to changing correlation timescales. Perhaps the most informative change in terms of the spectral features is the H$\delta$$_{\text{EW}}$−$D_{\lambda}(4000)$ or H$\delta$$_{\text{EW}}$−log$F_{\nu,\text{NUV}}$ distributions, which show opposing trends to changes in $\sigma_{\text{gas}}$ versus $\tau_{\text{gas}}$ as seen in Figure 5. This allows us to see changes in the slope of this relation as a function of changing $\tau_{\text{gas}}$ independent of the changes in scatter induced by changes in $\sigma_{\text{gas}}$. Another noticeable change due to the lower $\sigma_{\text{gas}}$ for all (except the dwarf) models is a reduced scatter, which provides less opportunity for the bursty episodes of star formation to form large amounts of stellar mass and thereby increase the sSFRs on average. In terms of the broadband colors, changing the scatter has the effect of scaling the corresponding distribution of colors by a proportional amount, while maintaining the differences in shape due to varying stochasticity.

Figure 7. Observational signatures caused by only varying correlation timescales $\tau_{\text{in}}$, $\tau_{\text{eq}}$, and $\tau_{\text{dyn}}$ based on the four cases highlighted in Figure 3 with fixed scatter. Using 10,000 SFH realizations for each case, we highlight differences in the distributions of spectral features sensitive to star formation across a range of timescales, including H$\alpha$, H$\delta$, and $D_{\lambda}(4000)$. While the differences in the median value of observables are subtle, the joint distributions do show some differences that arise from changes in the underlying sSFR distribution.
5. Discussion

Having established the GP-SFH formalism and demonstrated sensitivity to the extended regulator model parameters in Section 2, we consider the implications and predictions we can make with JWST observations using this formalism in Section 5.1. We consider the effects of other stellar population variables such as dust and metallicity in Section 5.2. Section 5.3 demonstrates how the GP-SFH can be used as a stochasticity prior for binned SFHs, and Section 5.4 considers how populations of galaxies with specific spectral features can provide novel constraints on stochasticity. Section 5.5 considers relaxing the assumption of a fixed base SFH, Section 5.6 considers the assumption of stationarity and shows an example of relaxing it with a time-varying kernel, and Section 5.7 considers some caveats and challenges of the GP-SFH formalism.

5.1. Constraints with Existing Data and Implications for JWST

The large amount of high-quality spectrophotometric data from existing surveys across a range of redshifts, including SDSS-IV MaNGA (Bundy et al. 2015; Abdurro'uf et al. 2022), SAMI (Allen et al. 2015; Bryant et al. 2015),
CALIFA (Sánchez et al. 2012; Husemann et al. 2013), MOSFIRE (Kriek et al. 2015), MUSE-Wide (Herenz et al. 2017), VANDELS (Pentericci et al. 2018), LEGA-C (van der Wel et al. 2016) and more, contain spectral features at an S/N and resolution where a large portion of the methods developed in this work can already be applied to real data, either through careful forward modeling or through integration into state-of-the-art SED codes. These constraints can also be contrasted against complementary approaches based on semianalytic or physically motivated empirical methods that connect galaxies across different epochs (Behroozi et al. 2019; Chaves-Montero & Hearin 2020, 2021; Kipper et al. 2021; Zhou et al. 2022).

The release of JWST data from the early release observations (ERO, aka “Webb’s First Deep Field”; Pontoppidan et al. 2022), GLASS (Merlin et al. 2022), and CEERS (Finkelstein et al. 2022) have demonstrated the incredible potential for probing star formation in galaxies across an incredible range of redshifts, environments, and stellar masses. This also enables studies of star formation stochasticity using the colors and spectral features in this work to constrain the ExReg model parameters, i.e., stochasticity amplitudes and timescales.

Figure 10. As in Figure 7, but now for the four cases highlighted in Figure 4 with variable (TFC20) scatter. The variation in scatter dramatically expands the set of sSFR distributions owing to the changing distribution of total stellar mass formed after a fixed time given a constant SFH. These differences lead to greater distinguishing power using the distributions of the spectral features we consider.
At redshifts around cosmic noon, we will see a major improvement in being able to directly measure rest-frame colors. In contrast to Hubble Space Telescope (HST) ACS +WFC3, which can measure rest-frame UVJ colors at $0.21 \lesssim z \lesssim 0.29$, JWST’s NIRCam filters will extend this to $1.51 \lesssim z \lesssim 2.55$, and a combination of HST+JWST will span the entire $0.21 \lesssim z \lesssim 2.55$ range. This is also similar to the redshift range in which slitless spectroscopy using NIRISS will be able to measure the spectral features discussed in this work for large populations of galaxies (Willott et al. 2022).

At higher redshifts, as we attempt the challenging task of measuring the SFRs and histories of these galaxies, care must be taken to account for the dependence of the results on the assumed priors (Tacchella et al. 2022a; Whitler et al. 2023), and we use a combination of spectroscopic and photometric data where available (Tacchella et al. 2023). The GP-SFH formalism described here can help motivate SFH priors for the next generation of SED-fitting-based studies based on estimates of the stochasticity from lower-redshift analogs or simulations.

Additionally, although we did not consider its effects in this work, the evolution of gas-phase and stellar metallicity in galaxies is also tied to their star formation, and further studies of their correlated properties (as in Camps-Fariña et al. 2022; Zhou et al. 2022) could provide further observable tests of the ExReg model timescales and priors for SED fitting codes that explicitly allow for evolution in chemical enrichment over time (Pacifici et al. 2013; Thorne et al. 2021).

5.2. Effects of Varying Other SED Parameters

SED modeling depends on a host of assumptions about the stellar populations that make up a galaxy, in addition to dust attenuation and emission from dust heating, nebular regions, and AGNs. In our current analysis we have held most of these constant in order to isolate and study the effects of perturbing the SFH stochasticity model, but it is informative to consider the extent to which varying these additional parameters will broaden the distributions we expect to observe.

Figure 11 shows the effect of varying the stellar metallicity and the dust attenuation (assuming a Calzetti dust law; Calzetti et al. 2000) on the spectral indices we consider. This analysis is done for a galaxy with a fixed SFH of $1 \: M_\odot \: yr^{-1}$ and other parameters corresponding to Table 1. The effects of a distribution of values in either of these stellar population parameters would correspond to a broadening in the distribution of spectral indices by an amount proportional to the mean and width of the dust/metallicity distribution. For example, a distribution of $\sim 1$ dex in metallicity centered around solar metallicity would correspond to a spread of $\sim 0.04$ in $D_n(4000)$ and $\sim 0.3$ dex in log $H_\alpha$ luminosity. While convolving the distributions in Figure 10 does make the different models harder to discriminate between, it is still distinct enough to be possible with a large enough sample size. This is additionally helped by the fact that the broadening of distributions in the spectral indices is not homogeneous and in fact displays quite different signatures across the three indices for dust and metallicity—notably that dust attenuation does not affect the $H_\delta$ EW. We have not shown the effects of varying SPS models or the initial mass function (IMF), since that would correspond to an overall shift in the indices rather than a broadening of the distribution. In the rest-optical part of the SED that we study in this work, we are also not significantly affected by AGNs, dust reemission, and other factors that manifest in the mid-to-far-IR portions of the SED. These effects have also been studied in relation to SFR stochasticity in the literature (Broussard et al. 2019; Emami et al. 2019; Wang & Lilly 2020a).
One additional factor to note is that the broadening predicted by Figure 11 assumes that variations in dust and metallicity are independent of SFR stochasticity and SFH. However, given that stochasticity determines the frequency of sharp bursts of star formation, it is likely that it will be correlated with the chemical enrichment of the galaxy. Although this is outside the scope of this work, cosmological simulations of galaxy evolution could shed light on the link between these parameters and help develop correlated priors for use with future observations.

5.3. A Prior for Binned SFHs

The GP implementation described in this paper can also be adapted as a prior for binned SFHs that are used by spectrophotometric fitting codes like Prospector (Leja et al. 2017; Johnson et al. 2021), STARLIGHT (Cid Fernandes et al. 2005), MOPED (Heavens et al. 2000), VESPA (Tojeiro et al. 2007), STECMAP (Ocvirk et al. 2006), Beagle (Chevallard & Charlot 2016), MCSED (Bowman et al. 2020), or color–magnitude diagram (CMD) based methods like Match (Dolphin 2002; Weisz et al. 2014). Prospector in particular uses either a Dirichlet prior or a continuity prior that parameterizes the logSFR ratios between bins using a Student’s t distribution. To incorporate the covariance models described in this work, it suffices to replace these priors with the covariance values $c_{i,j}(t_i, t_j)$ where $t_i$, $t_j$ correspond to the centers of the individual bins. In Appendix D, we verify that this procedure yields SFHs identical to sampling from the high-resolution GP-SFH and degrading the resolution to match the logarithmically spaced time bins used in these codes.

For other SED fitting codes, CIGALE (Boquien et al. 2019) allows custom SFHs to be input using the sfhfromfile module. PiXedit (Abdurrouf et al. 2021) is a similar modular package that allows custom SFHs to be added through modifications to pixedit_model. Bagpipes (Carnall et al. 2018) already contains an implementation of the Iyer et al. (2019) GP-SFH module, which can be augmented to add the kernel developed in this work. DiFFstar (Alarcon et al. 2023) implements a physically motivated parametric form for the SFHs that can be augmented to include fluctuations described by a GP. Additionally, methods like ProSpect (Robotham et al. 2020; Thorne et al. 2021) that include models for chemical enrichment based on gas consumption can be used to couple the SFHs and their variability to metallicity histories as well.

Repeating our analysis to determine the distribution of observational metrics in Figures 6–9 using this formalism, we find that they are very similar due to the effective sampling of variability on different timescales due to the logarithmic binning and the lack of information encoded in galaxy SEDs about short-term variability at large look-back times for $N_{\text{log bin}} > 5$. For fewer bins, the discretization of the ACF due to the effective “smoothing” of the SFHs results in a loss of ACF information, which can bias estimates of timescales. Figure 12 shows samples of binned SFHs corresponding to the four stochasticity regimes used in this work.

5.4. How Do We Choose a “Population of Galaxies” to Study Observationally?

An underlying assumption in comparing distributions of spectral features or colors is that galaxies in the sample belong to the same underlying population and thus can be described using the same model for star formation stochasticity. Caution should therefore be exercised when creating a galaxy sample to minimize contamination by other populations. Methods like clustering in colors can be used to select galaxies that are likely to belong to the same population, e.g., to select all the star-forming galaxies at a given epoch. Additional methods like selecting specific galaxy subpopulations using self-organizing maps (Davidzon et al. 2022; Holwerda et al. 2022; Teimoorinia et al. 2022) or in the latent space of variational autoencoders (Portillo et al. 2020) can also be used for this purpose. For small $(O(10^2))$ populations, nearest neighbor or density-based methods applied directly in observational feature spaces or galaxy latent spaces from self-organizing maps or variational autoencoders can be used to select galaxies with similar properties.

It is currently not well understood whether properties of star formation stochasticity (i.e., properties like $\sigma_{\text{gas}}$ or $\tau_{\text{eq}}$ in the extended regulator model) correlate strongly with other physical properties like galaxy size, chemical enrichment history, environment, morphology, or dynamics. Since the variability in these quantities can span a wide range of timescales, that may or may not relate to the timescales on which SFR fluctuations are correlated. It would be interesting to study this further, since the stochasticity model can also influence the presence of certain galaxy populations.

For example, in Figure 8 we notice that the dwarf and cosmic noon populations show a slight excess of galaxies with $(r - K < 0.75)$ and $(\text{NUV} - r \sim 3)$. This region is highlighted in the left panel of Figure 13. Since spectral sensitivity falls off as a function of age at different rates depending on the wavelengths under consideration, an assumed model for SFH stochasticity can produce unique spectral signatures depending on a combination of broadband filters. In the NUV–r–K color–color space, for example, the r–K color has a mild linear dependence with age, except for a short period between ~5 and 60 Myr during which the color sharply decreases. In complement to this, the NUV–r color is relatively flat until ~20 Myr, after which it shows a linear dependence with age. Because of this combination of sensitivities, a portion of NUV–r–K color space (e.g., $(\text{NUV} - r > 2)$ and $(r - K < 0.5)$) is uniquely sensitive to galaxies that recently experienced a sudden recent rise and fall in their SFHs.25

Since the probability of such an event is directly proportional to the amount of burstiness and effective timescales over which SFR is correlated, the four stochasticity models considered above make differing predictions for the probability that a galaxy can have such an event (and therefore on the number of galaxies in a given sample). Therefore, if a given stochasticity model has the ability to produce post-starburst galaxies with a certain width for the starburst episode (which is proportional to the correlation timescale), it will have a greater number of galaxies in the NUV–r–K diagram in that region. Since the dwarf and cosmic noon cases have SFRs that are correlated over the intermediate timescales sensitive to these colors, they should produce more galaxies with post-starburst colors in the NUV–r–K space. We examine this portion of the NUV–r–K color space better in Figure 13, finding that the SFHs of galaxies with these colors indeed tend to show a strong post-starburst feature in their SFHs, with the dwarf and cosmic noon ACFs resulting in a higher number of these galaxies compared

25 A similar region exists in the UVJ diagram as well (Suess et al. 2020; Wild et al. 2020; Akins et al. 2022).
to the Milky Way or high-z populations. Additional quantities, like the fraction of PSB galaxies and the timescales of the recent burst, could therefore be useful tracers of star formation stochasticity in future studies.

We also stress that the primary focus of observationally driven analysis using the current ExReg kernel should be on populations of star-forming galaxies for which we can measure emission-line EWs. This is because while a small number of...
quiescent galaxies arise directly out of the extended regulator model PSDs for cases with long correlation timescales such as the Milky Way analog case, the model does not by default include mechanisms that quench galaxies. In addition, star-forming galaxies are much more likely to have measurable SFR timescales, and it can be extremely challenging to measure SFRs for quiescent galaxies (often allowing us to set upper limits on the integrated SFRs even with deep IR imaging; e.g., Fumagalli et al. 2014). Additionally, since the GP-SFH method treats the different spectral features as “response curves” that are essentially convolved with the SFH over different timescales, we believe that the strongest constraints on the parameters come from relatively recent SFRs (out to a few Gyr) and are thus less affected by spectral degeneracies for old stellar populations (Ocvirk et al. 2006; Ferreras et al. 2023).

The current paper proposes a population-level method to constrain the ExReg model parameters using the spectral features or SEDs for a sample of galaxies. In Appendix E, we show that a sample of $N_{gal} \sim 30$ is sufficient to constrain the model parameters using SBI. In practice, observational noise due to instrumental effects and measurement uncertainties might necessitate larger samples. While a detailed modeling of the noise considering wavelength coverage, resolution, extent of correlated noise across wave bands, and overall S/N is outside the scope of this work, including an accurate noise model in the forward modeling “simulator” in the SBI while training can significantly improve its robustness, especially with newer methods (e.g., SBI++; Wang et al. 2023) being developed.

5.5. Varying the Base SFHs

The analysis in Sections 4.1 and 4.2 assumes that the individual SFHs can be described as perturbations around a base SFH, which we assume to be a constant SFH with SFR $= 1 \, M_\odot \, yr^{-1}$. While it is possible to relax this assumption in our current framework, caution needs to be exercised when the variability of the base SFHs is comparable to that of the timescales in the extended regulator model, since this will modify the ACF by adding power on the longer timescales when the autocorrelation time of the base SFH is of the same order as that for the extended regulator model $\tau_{A,\text{baseSFH}} \approx \tau_{A,\text{ExReg}}$. Appendix C presents a detailed discussion of the effects on spectral features with an implementation where the base SFHs themselves vary across the sample and are drawn from a distribution. While this leads to a broadening of the distributions of individual spectral features, we find that it is still possible to differentiate between the models by comparing distributions of spectral features.

5.6. The Assumption of Stationarity and Ergodicity

For simplicity, the derivation in Section 2 assumes that the SFHs of a population of galaxies are stationary and ergodic. The assumption of stationarity requires that the PSD or ACF of a galaxy SFH does not have an explicit time dependence. However, it is not necessary that SFHs in the real universe follow this, with either the stochasticity or timescales of the PSD model evolving with time. However, (i) for most science cases that discriminate between different models of stochasticity, the evolution is slow enough that this assumption is expected to hold (see the discussion in Section 3.2 of Wang & Lilly 2020b); and (ii) if/when we decide to relax this assumption of stationarity, the kernel in our GP formalism can be updated to account for that. Indeed, nonstationary kernels are an open topic of research in GPs (Rhode 2020), and models for the time evolution of the ACF are an important part of the future work enabled by this formalism.

In fact, some of our own toy models for stochasticity in this paper (Milky Way, cosmic noon, and high-$z$) are likely to be causally linked over different epochs as galaxy populations evolve over time, and though we assume a fixed stochasticity for these models, it is seemingly in contrast with the expectation that some high-$z$ galaxies, for example, will evolve to galaxies analogous to the present-day Milky Way. We stress here that our assumption of stationarity, in practice, is weaker than the theoretical expectation, since the SFHs we probe are only sensitive to the SFHs convolved with the response functions for the various spectral features (as seen in Appendix B). Thus, the various indicators at $z \sim 0$ only probe the average SFR out to $z \sim 0.2–0.5$, over which range we expect the stochasticity to remain fairly uniform from CMD studied and cosmological simulations. Thus, even if the stochasticity is different at higher redshifts, this does not affect the model definitions and the relevant timescales. This might not, however, hold at higher redshifts, where the changes in stochasticity are rapid enough that they need to be explicitly modeled.

As observational data with future telescopes unlock new timescales and large populations of galaxies across different cosmic epochs, our models can be updated to include variations in the extended regulator model parameters as a function of time. In this case, it is possible to relax the assumption of stationarity (i.e., Equation (2)) and implement a more general GP for galaxy (log) SFRs, as shown in Figure 14. A simple extension in this direction would be to make one or more of the parameters in the extended regulator model time dependent. As an example of this case, we consider a simple time dependence to both $\sigma_{\text{gas}}$ and $\tau_{eq}$ given by

$$\sigma_{\text{gas}} = 0.3(-0.03t_{\text{univ}}^2 + 0.4t_{\text{univ}}) \quad & \quad \tau_{eq} = 0.01(t_{\text{univ}}^2 + 1).$$

(20)

This form is chosen for the $\sigma$ to allow for increased variability at higher redshifts, while the increasing $\tau_{eq}$ is linked to the increasing dynamical times of galaxies with decreasing redshifts.

Ergodicity is a more subtle issue and concerns the fact that we perform our analysis using populations of galaxies. For a dynamical system, ergodicity implies that the variability of a single galaxy’s SFH over time is equivalent to the variability of an ensemble of SFHs observed at a given epoch, after accounting for selection functions and completeness. This is considered in Section 3.2 of Wang & Lilly (2020b), which also considers the need for ergodicity while working with galaxy PSDs. While it is unlikely that galaxy SFHs are fully ergodic, due to the changing conditions in which galaxies form stars at different epochs, the extent to which this is violated is expected to be minor and can be tested in the future using cosmological simulations.

5.7. Challenges and Caveats

Techniques studying the stochasticity of star formation across timescales rely on a host of modeling assumptions, and these should be kept in mind while using inference to
The median behavior of the galaxy sample varying component to starbursts. An additional factor is the correlation between the gas compression by spiral arms or compaction-induced GMCs, which can be due to not only stellar feedback but also accounts for dynamical processes that create and destroy SFH stochasticity model can be folded into existing SED developed here is modular and runs off FSPS. As a result, the distributions of spectral features we consider in this paper show significant overlaps for the different toy models, which makes it difficult to distinguish between them using only simple summary statistics or marginal distributions for individual observables. This is complicated by additional systematics, either astrophysical (e.g., variations in dust and metallicity) or observational (e.g., measurement uncertainties or selection effects), that can potentially broaden the distributions and make it more difficult to distinguish between them. However, this belongs to a class of noisy inference problems that can be solved by considering the full distributions and their covariances and tackled with methods like SBI. In Appendix E we demonstrate a proof of concept where we infer the regulator model parameters using this technique to distinguish between the toy models studied in this paper.

A minor concern is the scalability of the GP formalism in the limit of high-resolution SFHs and/or computationally intensive forward modeling comparisons to large data sets of observations. However, fast GP-based methods that include automatic differentiation (e.g., tinyGP; Foreman-Mackey et al. 2022) can provide work-arounds to these issues as they arise.

6. Conclusion

Modeling the stochasticity of star formation over a range of timescales provides a way to connect observations of galaxy SFHs to the underlying physical processes driving galaxy growth and quenching.

In this paper, we propose a fast, modular, GP-based formalism implementing the extended regulator model based on TFC20 as an ACF or kernel. This model provides a parametric form for SFR stochasticity as a combination of different physical processes and is completely characterized by three effective timescales corresponding to stochastic gas inflows, equilibrium, and dynamical processes influencing GMC creation and destruction.

Figure 14. Implementing a nonstationary kernel in the GP-SFH formalism. While following most of the behavior of the Milky Way analogs, this kernel has a time-varying component to $\sigma_{\text{gas}}$, which rises to its maximum value at $z \sim 1$, after which it falls again; and $\tau_{\text{dyn}}$, which keeps rising with time, leading to smoother, less "bursty" SFHs at lower redshifts. Left: the time-dependent “kernel” for the GP, with the dotted lines showing a deviation of 1 Gyr to the future and past; right: the median behavior of the galaxy sample (black line and shaded region) and individual realizations (red lines) of galaxy SFHs.
Implementing a GP with this kernel allows us to make extremely fast draws of galaxy SFHs with a particular SFH autocovariance structure and to use them to forward-model galaxy spectra and dependent observables. Studying these observables as a function of the kernel parameters allows us to quantify differences as a function of the extended regulator model’s timescales and thus differentiate between different regimes of stochasticity. This is illustrated by considering four toy models for galaxy populations: Milky Way analogs, dwarf galaxies, massive galaxies at cosmic noon, and galaxies at high redshifts. We model the spectra of these galaxies using FSPS and study distributions of spectral features including Hα and UV-based SFR indicators, Hδ and Ca H and K absorption-line equivalent widths, the $D_n(4000)$ spectral break, and broadband galaxy colors, finding that these distributions are sensitive to the extended regulator model parameters and that their distributions and covariances can be used to discriminate between the models.

Since increasing the amount of stochasticity leads to greater stellar masses formed in intense bursts of star formation, the sSFR distribution, and therefore the flux in SFR tracers like Hα and rest-UV, is sensitive to the overall level of stochasticity. Complementary to this, the Hβ versus $D_n(4000)$ space traces star formation over longer timescales and is extremely sensitive to both the overall level of variability and the timescales on which SFR is correlated. We also find that the rest-frame broadband colors reveal populations of galaxies such as post-starburst galaxies that are preferentially found in models that allow SFR correlations on the timescales that the colors are sensitive to and can thus be used as additional constraints on the ExReg model parameters.

The GP-SFH formalism can also be easily incorporated into existing SED fitting codes to provide realistic priors for SFH covariance or infer them from future high-S/N spectro-photometric observations using JWST, used to study the effective timescales in cosmological simulations and further expanded to include factors like nonstationarity. Code to reproduce our results can be found at https://github.com/kartheikiyer/GP-SFH.

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Appendix A

Connecting the PSD and ACF in the Extended Regulator Model

We start by building physical intuition of how different physical processes related to galaxies can affect stochasticity and correlations in the SFRs of individual galaxies across cosmic time, summarized through their PSDs and associated ACFs. We focus in particular on derivations of the ACF, which can in some cases be easier to interpret than the PSD. In Section A.1, we provide a brief set of definitions for the PSD and ACF, their relationship with each other, and useful associated quantities. In Section A.2, we outline the expected behavior for completely uncorrelated SFRs. In Section A.3, we outline the relationship between a damped random walk process, the PSD, and the ACF. In Sections A.4 and A.5, we derive results for the regulator and extended regulator models presented in TFC20.

A.1. Overview of Formalism

We start by informally defining a stochastic process as something that can generate infinite realizations of a time series \( \{ x_t \} \equiv \{ x_t \}_{t=1}^{\infty} \) at any times \( t = 1, \ldots, n \) (i.e., the \( x_t \) values change every time we simulate from the process). The collection of \( x_t \) values will then follow some joint probability distribution \( P(x_t) \) that is defined by the stochastic process.

We can use the time-dependent mean

\[
\mu(t) = \int_{-\infty}^{\infty} x_t \ P(x_t) \ dx_t
\]  
(A1)

as a simple summary statistic to describe how this process evolves over time, given the marginal distribution \( P(x_t) \) of \( x_t \) defined by our process. Many stochastic processes are defined with \( \mu(t) = 0 \), so modifying them to follow some nonzero mean is as simple as adding in a chosen mean function to the generated data \( x_t \).

The simplest way to explore the correlation structure in a given stochastic process is to compute the ACF,\(^{28}\)

\[
\mathcal{C}(t, t') = \int_{-\infty}^{\infty} [x_t - \mu(t)][x_{t'} - \mu(t')] \ P(x_t, x_{t'}) \ dx_t \ dx_{t'}
\]  
(A2)

between \( x_t \) and \( x_{t'} \) at two different times \( t \) and \( t' \). As with the mean, \( P(x_t, x_{t'}) \) is the joint distribution of \( x_t \) and \( x_{t'} \) defined by the process.

Assuming that our process is stationary such that the ACF only depends on the separation (i.e., time lag) between any two given times \( \tau \equiv t - t' \) rather than the individual times \( t \) and \( t' \) themselves, we can instead write the ACF as

\[
\mathcal{C}(\tau) = \int_{-\infty}^{\infty} [x_t - \mu(t)][x_{t+\tau} - \mu(t + \tau)] \ P(x_t, x_{t+\tau}) \ dx_t \ dx_{t+\tau}
\]  
(A3)

\(^{28}\) The prefix “auto-” is often used to emphasize that the calculation is done at two different times for the same process, rather than between two different processes.
We can use the ACF $\mathcal{C}(\tau)$ to also define the autocorrelation function as

$$\rho(\tau) \equiv \mathcal{C}(\tau)/\mathcal{C}(0) \equiv \mathcal{C}(\tau)/\sigma^2,$$  \hspace{1cm} (A4)

which is normalized to be between 1 and $-1$. Note that at $\tau = 0$ the autocorrelation function is always 1 since it is normalized by the variance

$$\sigma^2 \equiv \mathcal{C}(\tau = 0).$$  \hspace{1cm} (A5)

It can also be useful to define a timescale over which a stochastic process is correlated. One definition is the autocorrelation time $\tau_A$, which tries to account for contributions from correlations across all possible time lags $\tau$. This can be computed via

$$\tau_A \equiv \int_{-\infty}^{+\infty} \rho(\tau) \, d\tau.$$  \hspace{1cm} (A6)

In addition to defining and investigating correlation structure as a function of time $t$, we can also do the same as a function of frequency $f$. Defining a “windowed” version of $x(t)$,

$$x_T(t) \equiv x_t w_T(t) = \begin{cases} x_t & t - \frac{T}{2} < t < t + \frac{T}{2}, \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (A7)

for a window function $w_T(t)$ with some width (duration) $T$ centered around $t$, the average power of a signal can be computed via

$$\mathcal{P} = \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{+\infty} |x_T(t)|^2 \, dt,$$  \hspace{1cm} (A8)

where we take the limit $T \to \infty$ assuming that the stochastic process is not localized in time. Using Parseval’s theorem, which states that power is preserved if we move from describing our process in the time domain to the frequency domain, we can rewrite this expression in terms of the frequency $f$ as

$$\mathcal{P} = \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{+\infty} |\hat{x}_T(f)|^2 \, df,$$  \hspace{1cm} (A9)

where

$$\hat{x}_T(f) = \int_{-\infty}^{+\infty} x_T(t) e^{-2\pi i ft} \, dt$$  \hspace{1cm} (A10)

is the Fourier transform of $x_T(t)$. We now define the PSD as the integrand of the above expression, i.e.,

$$S(f) \equiv \lim_{T \to \infty} \frac{1}{T} |\hat{x}_T(f)|^2.$$  \hspace{1cm} (A11)

We can interpret the PSD as the relative amount of power as a function of frequency, where larger values indicate stronger correlations across particular frequencies.

While the ACF and PSD can be computed directly from a given stochastic process, they can also be directly computed from each other. Based on the Wiener–Khinchin theorem, which states that in the continuous-time limit the PSD $S(f)$ and ACF $\mathcal{C}(\tau)$ are Fourier pairs, we can convert between the two using the following relations:

$$S(f) = \int_{-\infty}^{+\infty} \mathcal{C}(\tau) e^{-2\pi i ft} \, d\tau \iff \mathcal{C}(\tau) = \int_{-\infty}^{+\infty} S(f) e^{+2\pi i ft} \, df.$$  \hspace{1cm} (A12)

This property is extremely useful, as many stochastic processes (such as the ones discussed below) can be much easier to describe in frequency than in time (and vice versa).

Note that, due to the nature of the Fourier transform, there will be offsets in the overall normalization depending on how the PSD and ACF are parameterized. In other words, a maximum value of $\sigma^2$ in the PSD may not correspond to a maximum value of $\sigma^2$ in the ACF. Since these normalizations are often arbitrary, we will use the variables $s$ and $\sigma^2$ to refer to the variance/overall amplitude in the ACF/PSD, respectively.

In the following subsections, we will include expressions for $S(f)$, $\mathcal{C}(\tau)$, $\sigma^2$, $\rho(\tau)$, and $\tau_A$ for all cases under consideration.

### A.2. White Noise

The simplest stochastic process is white noise. This process has equal power at all frequencies and is defined by the PSD

$$S_{\text{WN}}(f) = s^2,$$  \hspace{1cm} (A13)

where $s^2$ is a constant that defines the intrinsic stochasticity. While the direct Fourier transform is ill-defined, taking the expanding window limit gives an ACF of

$$\mathcal{C}_{\text{WN}}(\tau) = \sigma^2 \times \delta_1(\tau) = \begin{cases} \sigma^2 & \tau = 0, \\ 0 & \text{otherwise}, \end{cases}$$  \hspace{1cm} (A14)

where $\delta_1(\tau)$ is the Kronecker delta function that gives 1 when $\tau = 0$ and 0 otherwise. This corresponds to a variance, autocorrelation function, and autocorrelation time of

$$\sigma^2_{\text{WN}} = \sigma^2$$  \hspace{1cm} (A15)

$$\rho_{\text{WN}}(\tau) = \delta_1(\tau)$$  \hspace{1cm} (A16)

$$\tau_{A,\text{WN}} = 0.$$  \hspace{1cm} (A17)

This makes sense for a white-noise process with no intrinsic correlations—the autocorrelation time is 0, and the autocorrelation function is only nonzero for $\tau = 0$.

### A.3. Damped Random Walk

Many natural processes have some characteristic timescale $\tau_{\text{decor}}$ where for $\tau < \tau_{\text{decor}}$ it is strongly correlated and for $\tau_{\text{decor}}$ it becomes uncorrelated (i.e., it loses its “memory” of the previous values and behaves like the white-noise process described in Section A.2). One way to describe such a process is by defining a damped random walk with a broken power-law PSD of

$$S_{\text{DRW}}(f) = \frac{s^2}{1 + (2\pi \tau_{\text{dec}})^2 f^2}.$$  \hspace{1cm} (A18)

This is damped as a function of $f^2$ with a characteristic damping scale of $2\pi \tau_{\text{decor}}$.

As shown in TFC20, assuming that the gas mass is directly related to the SFR of the galaxy, that the conversion from gas mass to SFR follows a stochastic process with an equilibrium timescale $\tau_{\text{eq}}$, and that the gas inflow rate is a white-noise process, the galaxy’s SFR will follow a damped random walk with $\tau_{\text{dec}} = \tau_{\text{eq}}$. The normalization $\sigma$ is the “long-term”
variability and is directly related to the stochasticity of the inflow rate. This gives (see also Paper I and TFC20)

\[ C_{\text{DRW}}(\tau) = \sigma^2 \times e^{-|\tau|/\tau_{\text{eq}}}. \]  

(A19)

The corresponding variance, autocorrelation function, and autocorrelation time are

\[ \sigma^2_{\text{DRW}} = \sigma^2 \]  
(A20)

\[ \rho_{\text{DRW}}(\tau) = e^{-|\tau|/\tau_{\text{eq}}} \]  
(A21)

\[ \tau_{A,\text{DRW}} = 2\tau_{\text{eq}}. \]  
(A22)

A.4. Regulator Model

In TFC20, the “regulator model” is defined as the case where the gas inflow rate onto the galaxy is also a stochastic process (i.e., it is not just white noise). Assuming that this process also follows a damped random walk with an inflow timescale \( \tau_{\text{in}} \), the combined PSD will be the product of the two PSDs:

\[ S_{\text{Reg}}(f) = S_{\text{eq}}(f) \times S_{\text{in}}(f) \]

\[ = \frac{s_{\text{eq}}^2}{1 + (2\pi\tau_{\text{eq}})^2 f^2} \times \frac{s_{\text{in}}^2}{1 + (2\pi\tau_{\text{in}})^2 f^2} \]

\[ \approx \frac{1 + (2\pi\tau_{\text{eq}})^2 f^2 + (2\pi\tau_{\text{in}})^2 f^2}{1 + (2\pi\tau_{\text{eq}})^2 f^2 + (2\pi\tau_{\text{in}})^2 f^2}. \]  

(A23)

This now includes two damping terms: one that scales with \( f^2 \), and one that scales with \( f^4 \). Since they suppress at large \( f \) (short timescales), they lead to even longer correlations.

Using the Wiener–Khinchin theorem, this PSD corresponds to an ACF of

\[ C_{\text{Reg}}(\tau) = \sigma_{\text{gas}}^2 \times \frac{\tau_{\text{in}} e^{-|\tau|/\tau_{\text{in}}} - \tau_{\text{eq}} e^{-|\tau|/\tau_{\text{eq}}}}{\tau_{\text{in}} - \tau_{\text{eq}}}. \]  

(A24)

The corresponding variance, autocorrelation function, and autocorrelation time are

\[ \sigma_{\text{Reg}}^2 = \sigma_{\text{gas}}^2 \]  

(A25)

\[ \rho_{\text{Reg}}(\tau) = \frac{\tau_{\text{in}} e^{-|\tau|/\tau_{\text{in}}} - \tau_{\text{eq}} e^{-|\tau|/\tau_{\text{eq}}}}{\tau_{\text{in}} - \tau_{\text{eq}}} \]  

(A26)

\[ \tau_{A,\text{Reg}} = 2(\tau_{\text{in}} + \tau_{\text{eq}}). \]  

(A27)

A.4.1. \( \tau_{\text{in}} = \tau_{\text{eq}} \) and the Matern32 Connection

In the limit where \( \tau_{\text{in}} = \tau_{\text{eq}} \), the ACF from Section A.4 becomes undefined even though the PSD is simply

\[ S_{\text{Reg}}(f) = \frac{s_{\text{gas}}^2}{1 + (2\pi\tau_{\text{eq}})^2 f^2 + (2\pi\tau_{\text{in}})^2 f^4}. \]  

(A28)

However, simply recomputing the ACF from the above PSD (or taking the limit as \( \tau_{\text{in}} \rightarrow \tau_{\text{eq}} \)) gives the well-defined expression

\[ C_{\text{Reg}}(\tau) = \sigma_{\text{gas}}^2 \times \left(1 + \frac{|\tau|}{\tau_{\text{eq}}}ight) e^{-|\tau|\tau_{\text{eq}}}. \]  

(A29)

which corresponds to

\[ \sigma_{0,\text{Reg}}^2 = \frac{\sigma_{\text{gas}}^2}{2\tau_{\text{eq}}} \]  

(A30)

\[ \rho_{\text{Reg}}(\tau) = \left(1 + \frac{|\tau|}{\tau_{\text{eq}}}ight) e^{-|\tau|\tau_{\text{eq}}} \]  

(A31)

\[ \tau_{A,\text{Reg}} = 4\tau_{\text{eq}}. \]  

(A32)

This parallels the original damped random walk case in Section A.3 closely, except the prefactor has changed from \( 1 \rightarrow 1 + \tau_{\text{eq}} \), which doubles the autocorrelation time.

The ACF for this special case can be shown to reduce exactly to that of the Matern32 kernel, a common choice of ACF when modeling a range of stochastic processes. In particular, Iyer et al. (2019) found the Matern32 kernel to best reproduce observed SFR correlation structure from simulations compared to several alternatives. As a result, we should interpret the regulator model with \( \tau_{\text{in}} = \tau_{\text{eq}} \) to be a direct generalization of that work.

A.5. Extended Regulator Model

TFC20 introduced the extended regulator model, which—in addition to gas inflow physics—also includes a prescription for star formation within GMCs. The formation and disruption of GMCs introduce additional stochasticity and a new correlation timescale in the system \( \tau_{\text{dyn}} \). This arises because the SFR of the galaxy is correlated over the timescale of the star formation processes. While this is originally linked to the lifetime of GMCs, we expand the definition to include the effects of dynamical processes (such as spiral arms and bars) that affect local star formation in galaxies (Krumholz & Kruijssen 2015; Forbes et al. 2019; Semenov et al. 2021) and thus call it \( \tau_{\text{dyn}} \) as opposed to \( \tau_{\text{fl}} \) as in TFC20. TFC20 show that the formation and disruption of GMCs follow a damped random walk such that the PSD and ACF are

\[ S_{\text{dyn}}(f) = \frac{s_{\text{dyn}}^2}{1 + (2\pi\tau_{\text{dyn}})^2 f^2} \]  

(A33)

\[ C_{\text{dyn}}(\tau) = \sigma_{\text{dyn}}^2 \times e^{-|\tau|/\tau_{\text{dyn}}}, \]  

(A34)

with timescale \( \tau_{\text{dyn}} \) and scatter \( \sigma_{\text{dyn}} \). Following TFC20 and assuming that this star formation process is decoupled from the processes related to gas cycling (converting gas into stars; \( \tau_{\text{eq}} \)) and inflows (bringing in gas; \( \tau_{\text{in}} \)), the PSD of the extended regulator model is the sum of the two PSDs:

\[ S_{\text{ExReg}}(f) = S_{\text{Reg}}(f) + S_{\text{gas}}(f) \]

\[ = \frac{s_{\text{gas}}^2}{1 + (2\pi\tau_{\text{eq}})^2 f^2 + (2\pi\tau_{\text{in}})^2 f^4} + \frac{s_{\text{dyn}}^2}{1 + (2\pi\tau_{\text{dyn}})^2 f^2}. \]  

(A35)

The corresponding ACF is then likewise the sum of the two ACFs:

\[ C_{\text{ExReg}}(\tau) = \sigma_{\text{gas}}^2 \times \frac{\tau_{\text{in}} e^{-|\tau|/\tau_{\text{in}}} - \tau_{\text{eq}} e^{-|\tau|/\tau_{\text{eq}}}}{\tau_{\text{in}} - \tau_{\text{eq}}} + \sigma_{\text{dyn}}^2 \times e^{-|\tau|/\tau_{\text{dyn}}}. \]  

(A36)

The corresponding variance, autocorrelation function, and autocorrelation time are

\[ \sigma_{\text{ExReg}}^2 = \sigma_{\text{gas}}^2 + \sigma_{\text{dyn}}^2 \]  

(A37)
Appendix B
Response Curves for Individual Spectral Features

To better understand the timescales over which the spectral features we consider in this work are sensitive to star formation, we evaluate the relative strength of each spectral feature used in Section 4 using a simple stellar population at various ages using the same assumptions as Table 1, and we plot the normalized results in Figure 15.

Figure 15. Response curves for spectral features we consider, ranging from Hα and rest-UV flux, which traces star formation on short timescales, to the \(D_n(4000)\) break, which traces the median age of the stellar population.
Appendix C
Varying the Base SFHs

The results in Section 4 mainly deal with the case where each underlying population is given identical base SFHs. Here we additionally consider the case where each individual base SFH is itself drawn around some mean SFH, lnSFH_{pop}(t). This type of doubly stochastic process is generally known as a (log-)Cox process. We construct our SFHs based on the framework in Iyer et al. (2019), which models the SFR as a smooth interpolation (using a GP) over points in time where galaxies formed evenly spaced quantiles of their total mass (e.g., t_{25}, t_{50}, and t_{75} being the times when the galaxy formed 25%, 50%, and 75% of its total stellar mass, respectively) in addition to the present-day SFR and a particular formation time t_{0}. Following Iyer et al. (2019), the times t_{25}, t_{50}, and t_{75} are drawn from a Dirichlet distribution with α = 5.0, which has been shown to agree well with SFHs from cosmological simulations. For simplicity, the stellar masses of the base SFHs are fixed at $10^9 M_\odot$, and SFRs at the time of observation are drawn from a normal distribution designed to mimic a portion of observed $\sim 0.3$ dex scatter in the observed SFR–$M^*$ correlation. Based on this model, we then construct a base population SFH lnSFH_{pop}(t) that is relatively constant across several gigayears, as seen in the top left panel of Figure 16.

The SFH realizations generated using this sample of varying SFHs are shown in Figure 16, with the four panels on the top right showing perturbations to the base SFH realizations in the top left. Noticeably, even in this case where the underlying population itself possesses some intrinsic variability in their base SFHs around some population mean, we still observe some differences in Figure 17 in the joint distribution of observables, noticeably Hα and Hδ. The former difference comes from the varying sSFR distributions due to the perturbations of the different stochasticity archetypes; the latter, due to the varying amount of SFH burstiness in the past $\sim 1–2$ Gyr, with the median of the distribution rising with decreasing $\tau_{\text{gas}}$.

Figure 16. Implementing the ExReg kernel with the realistic SFHs described in Iyer et al. (2019). The top panel shows draws from a Dirichlet distribution, with perturbations from the various toy models added on to the base SFHs. The bottom panels follow the same format as Figure 6 and show that, despite the increased scatter due to SFH variations, the differences between the models are still distinguishable.

Following Iyer et al. (2019), $t_0$ here is set by the age of the universe at a given redshift.
Appendix D
Verifying the Covariance for Binned SFHs

In practice, many non-parametric SFH codes use SFHs binned in roughly logarithmic bins in look-back time with varying priors on stochasticity. In this appendix, we verify that directly sampling from the covariance function using bin centers is equivalent to sampling high-resolution SFHs and degrading them to the same coarse time bins.

We start with 1000 samples of high-resolution SFHs from a GP corresponding to each of the four galaxy regimes described in Section 2. We then degrade them to binned SFHs with 10 equally spaced bins in log look-back time such that the SFR in a given bin is the average of the SFR in that interval, as shown in the middle panels of Figure 18. As expected, the covariance matrix computed from the binned SFHs matches the analytical estimate, with small differences due to spectral leakage from the finite length of the time series. Repeating this analysis, now drawing from the SFHs with the coarse time array corresponding to the bin centers with the same kernels, confirms that the coarse SFHs are consistent with the distribution obtained by coarsening the high-resolution SFHs. This also results in a significantly faster GP that may be more suitable for forward modeling large ensembles of observations that require repeated computation of the covariance matrix.

Figure 17. Implementing the ExReg kernel with the realistic SFHs described in Iyer et al. (2019). The top panel shows draws from a Dirichlet distribution, with perturbations from the various toy models added on to the base SFHs. The bottom panels follow the same format as Figure 7 and show that, despite the increased scatter due to SFH variations, the differences between the models are still distinguishable.
While the primary purpose of the paper is to provide a proof of concept for observational signatures of SFR stochasticity in galaxies in a physically motivated manner, it is also illustrative to consider the problem of inferring these physical timescales from a distribution of observations. Since the distributions in Figure 10 overlap significantly, it is difficult in practice to distinguish between the models using summary statistics like the mean and dispersion alone. This is also complicated by the added effects of broadening the distributions due to dispersion introduced by metallicity and dust variations, as well as observational systematics.

Thus, we need to consider methods that can utilize the full distributions and covariances for all the observables simultaneously and use the forward modeling technique described in this work to create a mapping between the extended regulator model parameters and the observed distributions $p(\theta_{\text{obs}} | \theta_{\text{ExReg}}, \theta_{\text{SPS}}, \theta_{\text{meas}})$. Doing so will allow us to account for potential systematics (both modeling and observational) while constraining the ExReg model parameters.

While a full treatment of this inference problem is outside the scope of this work, here we show a proof of concept using SBI (also sometimes called likelihood-free inference or implicit likelihood inference) to constrain $\sigma_{\text{reg}}, \tau_{\text{inv}}$, and $\tau_{\text{eq}}$ from observations. We condition a neural flow using simulated observables for various values of the input parameters, to infer the joint distribution $P(\theta_{\text{obs}} | \theta_{\text{Reg}})$, where $\theta_{\text{obs}}$ are the spectral features used in this paper and $\theta_{\text{ExReg}}$ are the regulator model parameters. We then estimate the posterior distribution of model parameters conditioned on an ensemble of observations, with results shown in Figure 19. Using a procedure similar to Hahn & Melchior (2022) and related papers (Khullar et al. 2022; Legin et al. 2023; Lemos et al. 2023; Wang et al. 2023), we implement SBI combined with amortized neural posterior estimation to determine that ensemble observations of about 30 galaxies allow us to constrain the timescales and scatter enough to differentiate between the toy models in this paper. Adding additional systematics like variable metallicity or observational noise broadens the posteriors, but nevertheless allows us to distinguish between the toy models and obtain constraints on the timescales, as shown in Figure 20. In most cases, the increased uncertainties can be mitigated using larger ensembles of galaxy observations.
Figure 19. SBI to estimate the regulator model parameters from a sample of galaxy observations.

Figure 20. Similar to Figure 19, but showing the derived posteriors for a distribution of galaxies with varying metallicity.

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