A FEW COMMENTS ON MATLIS DUALITY

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Dedicated to the memory of Professor Efraim P. Armendariz

Abstract. For a Noetherian local ring \((R, m)\) with \(p \in \text{Spec}(R)\), we denote the \(R\)-injective hull of \(R/p\) by \(E_R(R/p)\). We show that it has an \(\hat{R}^p\)-module structure, and there is an isomorphism \(E_R(R/p) \cong E_{\hat{R}^p}(\hat{R}^p/p\hat{R}^p)\), where \(\hat{R}^p\) stands for the \(p\)-adic completion of \(R\). Moreover, for a complete Cohen-Macaulay ring \(R\), the module \(D(E_R(R/p))\) is isomorphic to \(\hat{R}^p\) provided that \(\dim(R/p) = 1\), where \(D(\cdot)\) denotes the Matlis dual functor \(\text{Hom}_R(\cdot, E_R(k))\). Here, \(\hat{R}^p\) denotes the completion of \(R^p\) with respect to the maximal ideal \(p^p\). These results extend those of Matlis (see [11]) shown in the case of the maximal ideal \(m\).

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1. Introduction

Throughout this paper \(R\) is a Noetherian local ring with the maximal ideal \(m\) and the residue field \(k = R/m\). We denote the (contravariant) Hom-functor \(\text{Hom}_R(\cdot, E_R(R/p))\) by \(\vee\), that is \(M^\vee := \text{Hom}_R(M, E_R(R/p))\) for an \(R\)-module \(M\) and \(E_R(R/p)\) is a fixed \(R\)-injective hull of \(R/p\) where \(p \in \text{Spec}(R)\). Moreover we denote the Matlis dual functor by \(D(\cdot) := \text{Hom}_R(\cdot, E_R(k))\). Also \(\hat{R}^p\) (resp. \(\hat{R}^p\)) stands for the completion of \(R^p\) (resp. of \(R\)) with respect to the maximal ideal \(p^p\) (resp. with respect to the prime ideal \(p\)).

Our main goal is to give the structure of \(E_R(R/p)\) as an \(\hat{R}^p\)-module. In particular we will show the following result (see Theorem 3.1):

**Theorem.** Let \((R, m)\) be a ring and \(p \in \text{Spec}(R)\). Then \(E_R(R/p)\) admits the structure of an \(\hat{R}^p\)-module and \(E_R(R/p) \cong E_{\hat{R}^p}(\hat{R}^p/p\hat{R}^p)\) as \(\hat{R}^p\)-modules.

In the case of the maximal ideal \(m\), Matlis has shown (see [11]) that the injective hull \(E_R(k)\) of the residue field \(k\) has the structure of \(R\)-module and it is isomorphic to the \(\hat{R}\)-module \(E_R(k)\). Here we will extend this result to an arbitrary prime ideal
that is $E_R(R/p)$ admits the structure of an $\hat{R}^p$-module and it is isomorphic to the $\hat{R}^p$-module $E_{\hat{R}^p}(\hat{R}^p/p\hat{R}^p)$.

Moreover, we know that $\text{Hom}_R(E_R(R/p), E_R(R/q)) = 0$ for any $p, q \in \text{Spec}(R)$ such that $p \in V(q)$ and $p \neq q$. It was proven by Enochs (see [5, p. 183]) that the module $E_R(R/q)^\vee$ has the following decomposition:

$$E_R(R/q)^\vee \cong \prod T_{p'}.$$

where $p' \in \text{Spec}(R)$ and $T_{p'}$ denotes the completion of a free $R_{p'}$-module with respect to $p' R_{p'}$-adic completion. Here we prove the following result (see Theorem 3.3):

**Theorem.** Let $(R, m)$ be a complete Cohen-Macaulay ring of dimension $n$. Suppose that $p \in \text{Spec}(R)$ with $\dim(R/p) = 1$. Then there is an isomorphism

$$D(E_R(R/p)) \cong \hat{R}_p.$$

Recently Schenzel has shown (see [17, Theorem 1.1]) that if $p$ is a one dimensional prime ideal, then $D(E_R(R/p)) \cong \hat{R}_p$ if and only if $R/p$ is complete.

Furthermore Enochs (see [5, p. 183]) has shown that the module $E_R(R/q)^\vee$ is a flat cover of some cotorsion module. In the end of the Section 3 we will show the following result (see Theorem 3.7):

**Theorem.** Let $(R, m)$ be a ring and $p, q \in \text{Spec}(R)$ such that $p \in V(q)$. Then $E_R(R/q)^\vee$ is a flat precover of $(R/q)^\vee$.

## 2. Preliminaries

In this section we will fix the notation of the paper and summarize a few preliminaries and auxiliary results. Notice that the following two propositions are known for the maximal ideal $m$ we will extend them to any prime ideal $p \in \text{Spec}(R)$.

**Proposition 2.1.** Let $(R, m)$ be a ring and $p \in \text{Spec}(R)$. Then we have:

(a) $E_R(R/p) \cong E_{R_n}(R_q/pR_q)$ for all $q \in V(p)$.

(b) Suppose that $M$ is an $R$-module and $p \in \text{Supp}_R(M)$. Then $M^\vee$ has a structure of an $R_p$-module.

**Proof.** For the proof of the statement (a) we refer to [12, Theorem 18.4]. We only prove the last claim. For this purpose note that $E_R(R/p)$ is an $R_p$-module. Now we define $(\xi \cdot f)(m) := \xi f(m)$ where $\xi \in R_p, m \in M$ and $f \in M^\vee$. Then it defines the structure of $M^\vee$ to be an $R_p$-module which completes the proof. \qed
Proposition 2.2. Let \((R, \mathfrak{m}), (S, \mathfrak{n})\) be local rings and \(R \to S\) be a surjective local homomorphism i.e. \(S = R/I\) for some ideal \(I \subseteq R\). Suppose that \(M\) is an \(R\)-module and \(p \in \text{Spec}(R)\). Then the following are true:

(a) Suppose that \(N\) is an \(S\)-module. Then for any \(p \in V(I)\)

\[ N^\vee \cong \text{Hom}_S(N, E_S(S/pS)). \]

(b) For all \(q \in V(p)\) we have

\[ M^\vee \cong \text{Hom}_{R_q}(M_q, E_{R_q}(R_q/pR_q)). \]

Proof. For the proof see [9, Example 3.6 and Exercise 13].

Lemma 2.3. For a local ring \(R\) let \(M, N\) be any \(R\)-modules. Then for all \(i \in \mathbb{Z}\) the following hold:

1. \(\text{Ext}_R^i(N, D(M)) \cong D(\text{Tor}_R^i(N, M)).\)
2. If in addition \(N\) is finitely generated, then \(D(\text{Ext}_R^i(N, M)) \cong \text{Tor}_R^i(N, D(M)).\)
3. \(\text{Hom}_R(\lim_{\leftarrow} M_n, M) \cong \lim_{\leftarrow} \text{Hom}_R(M_n, M),\) where \(\{M_n : n \in \mathbb{N}\}\) is a direct system of \(R\)-modules.

Proof. For the proof see [9, Example 3.6] and [18].

In order to prove the next results we need a few more preparations. Let \(I \subseteq R\) be an ideal and for \(i = 1, \ldots, s\) let \(q_i\) belongs to a minimal primary decomposition of the zero ideal in \(R\) where \(\text{Rad}(q_i) = p_i\). Then we denote \(u(I)\) by the intersection of all \(p_i\)-primary components \(q_i\) such that \(\dim_R(R/(p_i + I)) > 0\). Moreover we denote the functor of the global transform by \(T(\cdot) := \lim_{\rightarrow} \text{Hom}_R(m^\alpha, \cdot)\). Also note that the local cohomology functor with respect to \(I\) is denoted by \(H^i_I(\cdot), i \in \mathbb{Z}\), see [3] for its definition and applications in local algebra.

Lemma 2.4. Let \(R\) be a local ring and \(I \subseteq R\) be an ideal. Then there is an exact sequence:

\[ 0 \to \hat{R}^I/u(I\hat{R}^I) \to \lim_{\leftarrow} T(R/I^\alpha) \to \lim_{\leftarrow} H^1_m(R/I^\alpha) \to 0, \]

where \(\hat{R}^I\) denotes the \(I\)-adic completion of \(R\).

Proof. Note that for each \(\alpha \in \mathbb{N}\) there is an exact sequence

\[ 0 \to m^\alpha \to R \to R/m^\alpha \to 0. \]
For \( s \in \mathbb{N} \) apply the functor \( \text{Hom}_R(\cdot, R/I^s) \) to this sequence. Then it induces the following exact sequence

\[
0 \to \text{Hom}_R(R/m^\alpha, R/I^s) \to R/I^s \to \text{Hom}_R(m^\alpha, R/I^s) \to \text{Ext}_R^1(R/m^\alpha, R/I^s) \to 0.
\]

Now take the direct limit of this we again get an exact sequence

\[
0 \to \lim\limits_{\longleftarrow} \text{Hom}_R(R/m^\alpha, R/I^s) \to R/I^s \to \lim\limits_{\longleftarrow} \text{Hom}_R(m^\alpha, R/I^s) \to \lim\limits_{\longleftarrow} \text{Ext}_R^1(R/m^\alpha, R/I^s) \to 0.
\]

Note that the inverse systems at the left hand side of these short exact sequences satisfy the Mittag-Leffler condition. So if we take the inverse limit to them, then the resulting sequences will be exact also (see [1, Proposition 10.2]). If we combined these resulting sequences, we get the following exact sequence

\[
0 \to \lim\limits_{\longleftarrow} \text{Hom}_R(R/m^\alpha, R/I^s) \to \hat{R}^I \to \lim\limits_{\longleftarrow} \text{T}(R/I^s) \to \lim\limits_{\longleftarrow} \text{Hom}_R(m^\alpha, R/I^s) \to \lim\limits_{\longleftarrow} \text{Ext}_R^1(R/m^\alpha, R/I^s) \to 0.
\]

But by [16, Lemma 4.1] there is an isomorphism \( \lim\limits_{\longleftarrow} \text{Hom}_R(R/m^\alpha, R/I^s) \cong u(I\hat{R}^I) \). Then the exactness of the last sequence provides the required statement. \( \square \)

In the next context we need the definition of the canonical module. To do this we recall the Local Duality Theorem (see [8]). Let \((S, n)\) be a local Gorenstein ring of dimension \( t \) and \( N \) be a finitely generated \( R \)-module where \( R = S/I \) for some ideal \( I \subseteq S \). Then It was shown by Grothendieck (see [8]) that there is an isomorphism

\[
H^i_m(N) \cong \text{Hom}_R(\text{Ext}^t_S(N, S), E)
\]

for all \( i \in \mathbb{N} \) (see [8]). For a slight extension of the Local Duality to an arbitrary module \( M \) over a Cohen-macaulay local ring see [14, Lemma 3.1]. Note that a more general result was proved by Hellus (see [10, Theorem 6.4.1]). Now we are able to define the canonical module as follows:

**Definition 2.5.** With the notation of the above Local Duality Theorem we define

\[
K_N := \text{Ext}_S^{t-r}(N, S), \dim(N) = r
\]

as the canonical module of \( N \). It was introduced by Schenzel (see [15]) as the generalization of the canonical module of a Cohen-Macaulay ring (see e.g. [2]).

**Corollary 2.6.** With the notation of Lemma 2.4, suppose that \( I \) is a one dimensional ideal. Then the following are true:
(a) There is an exact sequence

\[ 0 \to \hat{R}^I / u(I\hat{R}^I) \to \bigoplus_{i=1}^s \hat{R}_{p_i} \to \lim_{\leftarrow} H^1_m(R/I^s) \to 0, \]

where \( p_i \in \text{Ass}_R(R/I) \) such that \( \dim(R/p_i) = 1 \) for \( i = 1, \ldots, s \).

(b) Suppose in addition that \( R \) is complete Cohen-Macaulay. Then there is an exact sequence

\[ 0 \to R/u(I) \to \bigoplus_{i=1}^s \hat{R}_{p_i} \to D(H^{n-1}_I(K_R)) \to 0. \]

**Proof.** Note that for the proof of the statement (a) it will be enough to prove the following isomorphism

\[ \lim_{\leftarrow} T(R/I^s) \cong \bigoplus_{i=1}^s \hat{R}_{p_i}, \]

(by Lemma 2.4). Since \( \dim(R/I) = 1 \) it implies that there exists an element \( x \in m \) such that \( x \) is a parameter of \( R/I^s \) for all \( s \in \mathbb{N} \). Then it induces the following isomorphism

\[ T(R/I^s) \cong R_x/I^sR_x \text{ for all } s \in \mathbb{N}. \]

Now suppose that \( S = \bigcap_{i=1}^s (R \setminus p_i) \). Then \( x \in S \) and by Local Global Principal for each \( s \in \mathbb{N} \) there is an isomorphism

\[ R_x/I^sR_x \cong R_S/I^sR_S. \]

Since \( R_S \) is a semi local ring, there is an isomorphism

\[ R_S/I^sR_S \cong \bigoplus_{i=1}^s R_{p_i}/I^sR_{p_i} \text{ for all } s \in \mathbb{N}. \]

(by Chinese Remainder Theorem). Since \( \dim(R/p_i) = 1 \) for \( i = 1, \ldots, s \), it follows that \( \text{Rad}(Ir_{p_i}) = p_iR_{p_i} \). By passing to the inverse limit of the last isomorphism we have

\[ \lim_{\leftarrow} T(R/I^s) \cong \bigoplus_{i=1}^s \hat{R}_{p_i}, \]

which is the required isomorphism. To prove (b), note that the canonical module of \( K_R \) of \( R \) exists (since \( R \) is complete Cohen-Macaulay). So by [14, Lemma 3.1] there is an isomorphism

\[ D(H^1_m(R/I^s)) \cong \text{Ext}^{n-i}_R(R/I^s, K_R) \]

for each \( i \in \mathbb{N} \). Since \( H^1_m(R/I^s) \) is an Artinian \( R \)-module and \( R \) is complete, by Matlis Duality, its double Matlis dual is itself. That is for each \( s \in \mathbb{N} \) there is an isomorphism

\[ H^i_m(R/I^s) \cong D(\text{Ext}^{n-i}_R(R/I^s, K_R)). \]
By passing to the inverse limit of this isomorphism induces the following isomorphism
\[ \lim_{\leftarrow} H^n_m(R/I) \cong \lim_{\leftarrow} D(\text{Ext}_{R}^{n-i}(R/I^s, K_R)). \]
Now take the Matlis dual of the isomorphism \( H^n_i(K_R) \cong \lim_{\leftarrow} \text{Ext}_{R}^{n-i}(R/I^s, K_R) \).
It induces that
\[ \lim_{\leftarrow} H^n_m(R/I^s) \cong D(H^n_i(K_R)) \]
(by Lemma 2.3 (3)). For \( i = 1 \), by the exact sequence in (a), we can get the exact sequence of (b) (since \( R \) is complete). This finishes the proof of the Corollary. \( \square \)

3. On Matlis Duality

In this section, we will prove one of the main results. Actually this result was proved by Matlis for the maximal ideal \( m \). Here we will extend this to any prime ideal \( p \in \text{Spec}(R) \).

**Theorem 3.1.** Let \((R, m)\) be a ring and \( p \in \text{Spec}(R) \). Then \( E_R(R/p) \) admits the structure of an \( \hat{R}^p \)-module and \( E_R(R/p) \cong E_{\hat{R}^p}(\hat{R}^p/p\hat{R}^p) \).

**Proof.** If \( x \in E_R(R/p) \), then by [11] \( \text{Ass}_R(Rx) = \{p\} \). It follows that some power of \( p \) annihilates \( x \).

Now let \( \hat{r} \in \hat{R}^p \). Then \( \hat{r} = (r_1 + p, \ldots, r_n + p^n, \ldots) \) such that \( r_{i+1} - r_i \in p^n \) for all \( i \geq 1 \). Since \( x \in E_R(R/p) \), then there exists a fixed \( n \in \mathbb{N} \) such that \( p^n x = 0 \). Choose \( r \in R \) such that \( \hat{r} - r \in p^n \) (by definition of the completion). Define \( \hat{r}x = rx \). Then it is clear that this gives a well-defined \( \hat{R}^p \)-module structure to \( E_R(R/p) \) which agrees with its \( R \)-module structure. Since \( E_R(R/p) \) is an essential extension of \( R/p \) as an \( R \)-module then it necessarily is also an essential extension of \( \hat{R}^p/p\hat{R}^p \) as an \( \hat{R}^p \)-module. So \( E_R(R/p) \subseteq E_{\hat{R}^p}(\hat{R}^p/p\hat{R}^p) \).

To show that \( E_R(R/p) \) is an injective hull of \( \hat{R}^p/p\hat{R}^p \) as an \( \hat{R}^p \)-module, it is enough to prove that \( E_R(R/p) = E_{\hat{R}^p}(\hat{R}^p/p\hat{R}^p) \). To do this it suffices to see that \( E_{\hat{R}^p}(\hat{R}^p/p\hat{R}^p) \) is an essential extension of \( R/p \) as an \( R \)-module since \( E_R(R/p) \) is an extension of \( R/p \) as an \( R \)-module (see [11]).

Let \( x \in E_{\hat{R}^p}(\hat{R}^p/p\hat{R}^p) \) and choose an element \( \hat{r} \in \hat{R}^p \) such that \( 0 \neq \hat{r}x \in \hat{R}^p/p\hat{R}^p \). By the argument used in the beginning of the proof applied to \( \hat{R}^p \), \( E_{\hat{R}^p}(\hat{R}^p/p\hat{R}^p) \) and \( x \) there is an \( n \in \mathbb{N} \) such that \( p^n x = 0 \). Choose \( r \in R \) such that \( \hat{r} - r \in p^n \). Then \( \hat{r}x = rx \in R/p \cong \hat{R}^p/p\hat{R}^p \) (see [1]) and \( rx \neq 0 \). Hence \( E_{\hat{R}^p}(\hat{R}^p/p\hat{R}^p) \) is an essential extension of \( R/p \) as an \( R \)-module and \( E_R(R/p) = E_{\hat{R}^p}(\hat{R}^p/p\hat{R}^p) \) which completes the proof. \( \square \)
**Remark 3.2.** (1) If \( p \in \text{Spec}(R) \), then by Theorem 3.1 for any finitely generated \( R \)-module \( M \) we have

\[
M^\vee \cong \text{Hom}_R(M, \text{Hom}_{\hat{R}}(\hat{M}, \text{Hom}_{\hat{R}}(\hat{R}/p\hat{R}, E_{\hat{R}}(\hat{R}/p\hat{R})))�).
\]

By Lemma 2.3 the later module is isomorphic to \( \text{Hom}_{\hat{R}}(\hat{M}, \text{Hom}_{\hat{R}}(\hat{R}/p\hat{R}, E_{\hat{R}}(\hat{R}/p\hat{R}, E_{\hat{R}}(\hat{R}/p\hat{R})/p\hat{R})))�. Here we use that \( \hat{M} \cong M \otimes_{\hat{R}} \hat{R} \). By Proposition 2.1 it implies that \( M^\vee \) has an \( \hat{R} \)-module structure.

(2) Since \( \text{Supp}_{\hat{R}}(E_R(\hat{R}/p)) = V(p) \) (by Proposition 2.1(a)), \( E_R(\hat{R}/p) \) is \( p \)-torsion. So from [13, Remark A.30] the natural homomorphism

\[
E_R(\hat{R}/p) \to E_R(\hat{R}/p) \otimes_R \hat{R}^p.
\]

is an isomorphism. But \( E_R(\hat{R}/p) \) is isomorphic to the module \( E_{\hat{R}}(\hat{R}/p\hat{R}^p) \) and it admits the structure of an \( \hat{R}^p \)-module (see Theorem 3.1). Moreover the above natural homomorphism is compatible with this structure. It implies that the \( p \)-adic completion of \( R \) commutes with the injective hull of \( R/p \).

It is well known that \( \text{Hom}_{\hat{R}}(E_R(\hat{R}/p), E_R(\hat{R}/q)) = 0 \) for any \( p \in V(q) \) and \( p \neq q \). Moreover Enochs has proved that

\[
E_R(\hat{R}/q)^\vee \cong \prod T_{p'}.
\]

where \( p' \in \text{Spec}(R) \) and \( T_{p'} \) denotes the completion of a free \( R_{p'} \)-module with respect to \( p'R_{p'} \)-adic completion. Here we will show the following theorem:

**Theorem 3.3.** Let \((R, m)\) be a complete Cohen-Macaulay ring and \( q \in \text{Spec}(R) \) be a one dimensional prime ideal. Then there is an isomorphism

\[
D(E_R(\hat{R}/q)) \cong \hat{R}_q.
\]

**Proof.** Since \( R \) is complete Cohen-Macaulay, \( K_R \) exists, and we have

\[
H_i^q(K_R) = 0 \text{ for all } i < \text{height}(q) = n - 1.
\]

This is true, because \( K_R \) is a maximal Cohen-Macaulay \( R \)-module of finite injective dimension and \( \text{Supp}_R(K_R) = \text{Spec}(R) \). Let \( E_R(K_R) \) be a minimal injective resolution of \( K_R \). Then by [7, Theorem 1.1], we have

\[
E_R(K_R)^i \cong \bigoplus_{\text{height}(p) = i} E_R(p).
\]

Moreover, \( \Gamma_q(E_R(\hat{R}/p)) = 0 \) for all \( p \notin V(q) \) and \( \Gamma_q(E_R(\hat{R}/p)) = E_R(\hat{R}/p) \) for all \( p \in V(q) \). Then apply \( \Gamma_q \) to \( E_R(K_R) \). It induces the following exact sequence

\[
0 \to H_i^q(K_R) \to E_R(\hat{R}/q) \to E_R(k) \to H_i^q(K_R) \to 0.
\]
Applying Matlis dual to this sequence yields the following exact sequence

\[ 0 \to D(H^n_q(K_R)) \to R \to D(E_R(R/q)) \to D(H^{n-1}_q(K_R)) \to 0. \]

(1)

Here we use that \( D(E_R(k)) \cong R \) (since \( R \) is complete). By the proof of Corollary 2.6 (b) and [16, Lemma 4.1], there are isomorphisms

\[ D(H^n_q(K_R)) \cong \lim_{\leftarrow} H^0_m(R/q^s) \cong u(q). \]

So the exact sequence (1) provides the following exact sequence

\[ 0 \to R/u(q) \to D(E_R(R/q)) \to D(H^{n-1}_q(K_R)) \to 0. \]

(2)

Now the module \( D(E_R(R/q)) \) is an \( R_q \)-module, so there is a natural homomorphism \( R_q \to D(E_R(R/q)) \). Then tensoring with \( R_q/q^s R_q \), this homomorphism induces the following homomorphism

\[ R_q/q^s R_q \to D(E_R(R/q))/q^s D(E_R(R/q)) \]

for each \( s \in \mathbb{N} \). Now take the inverse limit of it to get

\[ \hat{R}_q \to \lim_{\leftarrow} D(E_R(R/q))/q^s D(E_R(R/q)). \]

On the other side, since \( \text{Supp}_R(E_R(R/q)) = V(q) \), the module \( E_R(R/q) \) is isomorphic to \( \lim_{\to} \text{Hom}_R(R/q^s, E_R(R/q)) \). Then by Proposition 2.3 (3) there is an isomorphism

\[ D(E_R(R/q)) \cong \lim_{\leftarrow} D(\text{Hom}_R(R/q^s, E_R(R/q))). \]

But again Proposition 2.3 (2) implies that the module \( D(\text{Hom}_R(R/q^s, E_R(R/q))) \) is isomorphic to \( D(E_R(R/q))/q^s D(E_R(R/q)) \). Therefore, there is a natural homomorphism

\[ \hat{R}_q \to D(E_R(R/q)). \]

Since \( R \) is complete Cohen-Macaulay, by Corollary 2.6 (b), there is an exact sequence

\[ 0 \to R/u(q) \to \hat{R}_q \to D(H^{n-1}_q(K_R)) \to 0. \]

Then this sequence together with the sequence (2) induces the following commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \to & R/u(q) & \to & \hat{R}_q & \to & D(H^{n-1}_q(K_R)) & \to 0 \\
\| & & \downarrow & & \| & & \| & \\
0 & \to & R/u(q) & \to & D(E_R(R/q)) & \to & D(H^{n-1}_q(K_R)) & \to 0
\end{array}
\]

Then by Five Lemma, there is an isomorphism \( D(E_R(R/q)) \cong \hat{R}_q \) which is the required isomorphism. \( \square \)
Corollary 3.4. Fix the notation of Theorem 3.3. Then the following hold:

(a) There is an exact sequence

\[ 0 \to R/u(q) \to D(E_R(R/q)) \to D(H_{q-1}^n(K(R))) \to 0. \]

(b) Suppose in addition that \( R \) is a domain. Then there is an isomorphism

\[ D(H_{q-1}^n(K(R))) \cong \hat{R}_q/R. \]

Proof. Since \( R \) is complete, we apply Corollary 2.6 (b) for \( I = q \). Then it implies that there is an exact sequence

\[ 0 \to R/u(q) \to \hat{R}_q \to D(H_{q-1}^n(K(R))) \to 0. \]

Then the statement (a) is easily follows from Theorem 3.3. Note that the statement (b) follows from the above short exact sequence. Recall that \( u(q) = 0 \) since \( R \) is a domain. \( \square \)

In the next context we need the following definition of flat covers.

Definition 3.5. Let \( M \) be an \( R \)-module and \( F \) any flat \( R \)-module. Then the linear map \( \phi: F \to M \) is called a flat cover of \( M \) if the following conditions hold:

(i) For any flat \( R \)-module \( G \) the following sequence is exact

\[ \text{Hom}_R(G,F) \to \text{Hom}_R(G,M) \to 0 \]

(ii) If \( \phi = \phi \circ f \) for some \( f \in \text{Hom}_R(F,F) \), then \( f \) is an automorphism of \( F \).

Note that if condition (i) holds, then \( F \) is called a flat precover. Enochs proved in his paper (see [4, Theorem 3.1]) that if \( M \) has a flat precover, then it also admits a flat cover and it is unique up to isomorphism.

The following lemma is an easy consequence of chasing diagram (see [5, Lemma 1.1]).

Lemma 3.6. Let \( M \) be an \( R \)-module and \( F, F' \) any flat \( R \)-modules. Then we have:

(a) If \( \phi': F' \to M \) is a flat precover of \( M \) and \( \phi: F \to M \) is a flat cover of \( M \) such that \( \phi' = \phi \circ f \) for some \( f \in \text{Hom}_R(F',F) \), then \( f \) is surjective and \( \ker(f) \) is a direct summand of \( F' \).

(b) If \( \phi: F \to M \) is a flat precover of \( M \), then it is a cover if and only if \( \ker(\phi) \) contains no non-zero direct summand of \( F \).

Enochs (see [5, p. 183]) showed that the module \( \text{Hom}_R(E_R(R/q), E_R(R/p)) \) is a flat cover of some cotorsion module. Here we will show the following result:
Theorem 3.7. Let \((R, \mathfrak{m})\) be a ring and \(p, q \in \text{Spec}(R)\) such that \(p \in V(q)\). Then \(E_R(R/q)^\vee\) is a flat precover of \((R/q)^\vee\).

Proof. Note that \(E_R(R/q)\) is as essential extension of \(R/q\). Let \(F\) be any flat \(R\)-module. Then the inclusion map \(R/q \otimes_R F \hookrightarrow E_R(R/q) \otimes_R F\) induces the following exact sequence

\[
\text{Hom}_R(E_R(R/q) \otimes_R F, E_R(R/p)) \to \text{Hom}_R(R/q \otimes_R F, E_R(R/p)) \to 0.
\]

By the adjunction formula (see Lemma 2.3), we conclude that the following homomorphism is surjective for any flat \(R\)-module \(F\)

\[
\text{Hom}_R(F, E_R(R/q)^\vee) \to \text{Hom}_R(F, (R/q)^\vee),
\]

which proves that \(E_R(R/q)^\vee\) is a flat precover of \((R/q)^\vee\).

Remark 3.8. Note that if \(R\) is a complete local ring, then \(R\) is a flat cover of the residue field \(k\) (see [6, Example 5.3.19]).

Problem 3.9. (1) Let \(R\) be a complete local ring and \(p \in V(q)\) and \(p \neq q\). It would be of some interest to see whether \(E_R(R/q)^\vee\) is a flat cover of \((R/q)^\vee\) or not?

(2) Note that it was shown in [17, Theorem 1.1] that if \(\dim(R/p) = 1\), then \(R/p\) is complete if and only if \(D(E_R(R/p)) \cong \hat{R}_p\). Let \(R\) be a complete local Cohen-Macaulay ring. Is it possible to generalize Theorem 3.3 for arbitrary prime ideals \(q \subseteq p\)?

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