FROBENIUS DIFFERENCE EQUATIONS AND ALGEBRAIC INDEPENDENCE OF ZETA VALUES IN POSITIVE EQUAL CHARACTERISTIC

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Abstract. In analogy with the Riemann zeta function at positive integers, for each finite field \( F_p \) with fixed characteristic \( p \) we consider Carlitz zeta values \( \zeta_r(n) \) at positive integers \( n \). Our theorem asserts that among the zeta values in \( \bigcup_{r=1}^{\infty} \{ \zeta_r(1), \zeta_r(2), \zeta_r(3), \ldots \} \), all the algebraic relations are those relations within each individual family \( \{ \zeta_r(1), \zeta_r(2), \zeta_r(3), \ldots \} \). These are the algebraic relations coming from the Euler-Carlitz relations and the Frobenius relations. To prove this, a motivic method for extracting algebraic independence results from systems of Frobenius difference equations is developed.

1. Introduction

1.1. Motivic transcendence theory. Classically Grothendieck’s period conjecture for abelian varieties predicts that the dimension of the Mumford-Tate group of an abelian variety over \( \mathbb{Q} \) should be equal to the transcendence degree of the field generated by its period matrix over \( \mathbb{Q} \). Conjecturally the Mumford-Tate group is the motivic Galois group from Tannakian duality, and therefore Grothendieck’s conjecture provides an interpretation of the algebraic relations among periods in question by way of motivic Galois groups.

In this paper we are concerned with the algebraic independence of certain special values over function fields with varying finite constant fields in positive equal characteristic. In particular, we are interested in special zeta values. In the positive characteristic world, there is the concept of \( t \)-motives introduced by Anderson [1], dual to the concept of \( t \)-modules. Based on work of the third author in the 1990’s, the structure of \( t \)-modules is the key for proving many interesting linear independence results about special values in this setting (see [14]). The breakthrough in passing from linear independence to algebraic independence by way of \( t \)-motives began with Anderson, Brownawell and the second author, and in particular with the linear independence criterion of [3] (the so-called “ABP criterion”).

By introducing a Tannakian formalism for rigid analytically trivial pre-\( t \)-motives and relating it to the Galois theory of Frobenius difference equations, the second author [10] has shown that the Galois group of a rigid analytically trivial pre-\( t \)-motive is isomorphic to its difference Galois group. Furthermore, the second author has successfully used the ABP criterion to show that the transcendence degree of the field generated by the period matrix of an ABP motive (i.e. the pre-\( t \)-motive comes from a uniformizable

Date: May 4, 2009.
2000 Mathematics Subject Classification. Primary 11J93; Secondary 11M38, 11G09.
Key words and phrases. Algebraic independence, Frobenius difference equations, \( t \)-motives, zeta values.

The first author was supported by NSC and NCTS. The second author was supported by NSF Grant DMS-0600826. The third author was supported by NSC Grant No. 96-2628-M-007-006.
abelian $t$-module) is equal to the dimension of its Galois group. More generally, we say that a rigid analytically trivial pre-$t$-motive has the GP (Grothendieck period) property if the transcendence degree of the field generated by its period matrix is equal to the dimension of its Galois group (for more details of terminology, see §2).

Using a refined version of the ABP criterion proved by the first author [5], we observe that there are many pre-$t$-motives which are not ABP motives but which have the GP property. This motivates us to introduce a method of uniformizing the Frobenius twisting operators with respect to different constant fields for those pre-$t$-motives that have the GP property. The pre-$t$-motive we obtain in this way is defined over a larger constant field, but still has the GP property (cf. Corollary 2.2.4). This technique is very useful when dealing with the problem of determining all the algebraic relations among various special values of arithmetic interest in a fixed positive characteristic. It is used in this paper to study special zeta values. For another application to special arithmetic gamma values, see [6].

1.2. Carlitz zeta values. Let $p$ be a prime, and let $\mathbb{F}_{p^r}[\theta]$ be the polynomial ring in $\theta$ over the finite field with $p^r$ elements. Our aim is to determine all the algebraic relations among the following zeta values:

$$\zeta_r(n) := \sum_{a \in \mathbb{F}_{p^r}[\theta]} \frac{1}{a^n} \in \mathbb{F}_{p^r}((1/\theta)) \subseteq \mathbb{F}_p((1/\theta)),$$

where $r$ and $n$ vary over all positive integers.

The study of these zeta values was initiated in 1935 by Carlitz [4]. For a fixed positive integer $r$, Carlitz discovered that there is a constant $\tilde{\pi}_r$, algebraic over $\mathbb{F}_{p^r}(\{\frac{1}{\theta}\})$, such that $\zeta_r(n)/\tilde{\pi}_r^n$ lies in $\mathbb{F}_p(\theta)$ if $n$ is divisible by $p^r - 1$. The quantity $\tilde{\pi}_r$ arises as a fundamental period of the Carlitz $\mathbb{F}_{p^r}[t]$-module $C_r$, and Wade [12] showed that $\tilde{\pi}_r$ is transcendental over $\mathbb{F}_p(\theta)$.

We say that a positive integer $n$ is $(p, r)$-even if it is a multiple of $p^r - 1$. Thus the situation of Carlitz zeta values at $(p, r)$-even positive integers is completely analogous to that of the Riemann zeta function at even positive integers. For these $(p, r)$-even $n$, we call the $\mathbb{F}_p(\theta)$-linear relations between $\zeta_r(n)$ and $\tilde{\pi}_r^n$ the Euler-Carlitz relations. Because the characteristic is positive, there are also Frobenius $p$-th power relations among these zeta values: for positive integers $m, n$,

$$\zeta_r(p^m n) = \zeta_r(n)^{p^m}.$$

In the 1990’s, Anderson and Thakur [2] and Yu [13], [14] made several breakthroughs toward understanding Carlitz zeta values. Using the $t$-module method, the transcendence of $\zeta_r(n)$ for all positive integers $n$, in particular for “odd” $n$ (i.e., $n$ not divisible by $p^r - 1$) was proved and it was also proved that the Euler-Carlitz relations are the only $\mathbb{F}_p(\theta)$-linear relations among $\{\zeta_r(n), \tilde{\pi}_r^m; m, n \in \mathbb{N}\}$. Recently, the first and third authors in [7] used ABP motives instead of $t$-modules to show that for fixed $r$ the Euler-Carlitz relations and the Frobenius $p$-th power relations account for all the algebraic relations over $\mathbb{F}_p(\theta)$ among the Carlitz zeta values

$$\tilde{\pi}_r, \zeta_r(1), \zeta_r(2), \zeta_r(3), \ldots .$$

To complete the story of Carlitz zeta values, the next natural question is what happens if $r$ varies. In 1998, Denis [3] proved the algebraic independence of all fundamental periods $\{\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\pi}_3, \ldots \}$ as the constant field varies. Thus, in view of [7], one expects
that for the bigger set of zeta values,
\[ \bigcup_{r=1}^{\infty} \{ \zeta_r(1), \zeta_r(2), \zeta_r(3), \ldots \}, \]
the **Euler-Carlitz** relations and the **Frobenius** \( p \)-th power relations still account for all the algebraic relations. This is indeed the case as we find from the following theorem (stated subsequently as Corollary 4.5.2).

**Theorem 1.2.1.** Given any positive integers \( s \) and \( d \), the transcendence degree of the field \( \overline{\mathbb{F}_p(\theta)}(\bigcup_{r=1}^{d} \{ \tilde{\pi}_r, \zeta_r(1), \ldots, \zeta_r(s) \}) \) over \( \mathbb{F}_p(\theta) \) is
\[
\sum_{r=1}^{d} \left( s - \left\lfloor \frac{s}{p^r} \right\rfloor - \left\lfloor \frac{s}{p^r - 1} \right\rfloor + \left\lfloor \frac{s}{p(p^r - 1)} \right\rfloor + 1 \right).
\]

1.3. **Outline.** Our strategy is to construct a pre-\( t \)-motive which has the \( GP \) property and whose period matrix accounts for the Carlitz zeta values in question. In [7] an ABP motive has already been constructed for Carlitz zeta values with respect to a fixed constant field. The problem here is one concerning varying the constant fields in a fixed characteristic, and one has to uniformize Frobenius powers in order to apply the method developed in [10].

This paper is organized as follows. In §2, we review Papanikolas’ theory and investigate the pre-\( t \)-motives that have the \( GP \) property. Here we introduce the mechanism of uniformizing Frobenius twisting operators while taking direct sums. Section 3 includes discussions about rigid analytically trivial pre-\( t \)-motives of type \( SV \), i.e., their Galois groups are extensions of split tori by vector groups. The heart of this section is Theorem 3.2.2 where we determine the dimensions of Galois groups of direct sums of pre-\( t \)-motives of type \( SV \). The pre-\( t \)-motive for Theorem 1.2.1 is constructed in §4 and we prove that it satisfies the conditions of Theorem 3.2.2. Finally, we calculate its dimension explicitly in Theorem 4.5.1, which then has Theorem 1.2.1 as direct consequence.

2. **\( t \)-motivic Galois groups**

2.1. **Notation.**

2.1.1. **Table of symbols.**
- \( \mathbb{F}_p \) := the finite field of \( p \) elements, \( p \) a prime number.
- \( k := \mathbb{F}_p(\theta) \) := the rational function field in the variable \( \theta \) over \( \mathbb{F}_p \).
- \( k_{\infty} := \mathbb{F}_p((\frac{1}{\theta})) \), completion of \( k \) with respect to the infinite place.
- \( \overline{k}_{\infty} := \) a fixed algebraic closure of \( k_{\infty} \).
- \( \overline{k} := \) the algebraic closure of \( k \) in \( \overline{k}_{\infty} \).
- \( \mathbb{C}_{\infty} := \) completion of \( \overline{k}_{\infty} \) with respect to the canonical extension of the infinite place.
- \( | \cdot | := \) a fixed absolute value for the completed field \( \mathbb{C}_{\infty} \).
- \( \mathbb{T} := \{ f \in \mathbb{C}_{\infty}[t] \mid f \text{ converges on } |t|_{\infty} \leq 1 \} \). This is known as the Tate algebra.
- \( \mathbb{L} := \) the fraction field of \( \mathbb{T} \).
- \( \mathbb{G}_a := \) the additive group.
- \( \mathbb{GL}_r / F := \) for a field \( F \), the \( F \)-group scheme of invertible \( r \times r \) square matrices.
- \( \mathbb{G}_m := \mathbb{GL}_1 \), the multiplicative group.

2.1.2. **Block diagonal matrices.** Let \( A_i \in \text{Mat}_{m_i}(\mathbb{L}) \) for \( i = 1, \ldots, n \), and \( m := m_1 + \cdots + m_n \). We define \( \oplus_{i=1}^{n} A_i \in \text{Mat}_{m}(\mathbb{L}) \) to be the canonical block diagonal matrix, i.e., the matrix with \( A_1, \ldots, A_n \) down the diagonal and zeros elsewhere.
2.1.3. Twisting. For a fixed power of $p$, we let $\mathbb{F}_q$ denote the field with $q$ elements, and we define an operator $\sigma = \sigma_q$ on $\mathbb{C}_\infty$ by $x \mapsto x^{\frac{1}{q}}$. Then we extend this operator to $\mathbb{C}_\infty((t))$ as follows. For $n \in \mathbb{Z}$ and a formal Laurent series $f = \sum a_it^i \in \mathbb{C}_\infty((t))$ we define $\sigma^n(f) := f^{(-n)} := \sum a_iq^{-n}t^i$. The operation is an automorphism of the Laurent series field $\mathbb{C}_\infty((t))$ that stabilizes several subrings, e.g., $\mathbb{F}[t]$, $\mathbb{F}[t]$, and $\mathbb{T}$. More generally, for any matrix $B$ with entries in $\mathbb{C}_\infty((t))$ we define $B^{(-n)}$ by the rule $B^{(-n)}ij = B_{ij}^{(-n)}$.

2.1.4. Entire power series. A power series $f = \sum_{i=0}^{\infty} a_it^i \in \mathbb{C}_\infty[[t]]$ that satisfies

$$\lim_{i \to \infty} \sqrt[n]{|a_i|_\infty} = 0$$

and

$$[k_\infty(a_0, a_1, a_2, \ldots) : k_\infty] < \infty$$

is called an entire power series. As a function of $t$, such a power series $f$ converges on all $\mathbb{C}_\infty$ and, when restricted to $\overline{k_\infty}$, $f$ takes values in $\overline{k_\infty}$. The ring of the entire power series is denoted by $E$.

2.2. Pre-t-motives and the GP property. Let $\bar{k}(t)[\sigma, \sigma^{-1}]$ be the noncommutative ring of Laurent polynomials in $\sigma$ with coefficients in $\bar{k}(t)$, subject to the relation

$$\sigma f := f^{(-1)}\sigma$$

for all $f \in \bar{k}(t)$. A pre-t-motive $M$ over $\mathbb{F}_q$ is a left $\bar{k}(t)[\sigma, \sigma^{-1}]$-module that is finite dimensional over $\bar{k}(t)$. Letting $m \in \text{Mat}_{r \times 1}(M)$ be a $\bar{k}(t)$-basis of $M$, multiplication by $\sigma$ on $M$ is represented by

$$\sigma(m) = \Phi m$$

for some matrix $\Phi \in \text{GL}_r(\bar{k}(t))$. Furthermore, $M$ is called rigid analytically trivial if there exists $\Psi \in \text{GL}_r(\mathbb{L})$ such that

$$\sigma(\Psi) := \Psi^{(-1)} = \Phi \Psi.$$

Such a matrix $\Psi$ is called a rigid analytic trivialization of the matrix $\Phi$. We also say that $\Psi$ is a rigid analytic trivialization of $M$ (with respect to $m$). Note that if $\Psi' \in \text{GL}_r(\mathbb{L})$ is also a rigid analytic trivialization of $\Phi$, then $\Psi'^{-1} \Psi \in \text{GL}_r(\mathbb{F}_q(\bar{k}(t)))$ (cf. [10, §4.1.6]). Moreover, if we put $m' := Bm$ for any fixed $B \in \text{GL}_r(\bar{k}(t))$, then $\Phi' := B^{(-1)}\Phi B^{-1}$ represents multiplication by $\sigma$ on $M$ with respect to the $\bar{k}(t)$-basis $m'$ of $M$ and $\Psi' := B\Psi$ is a rigid analytic trivialization of $\Phi'$.

Definition 2.2.1. Suppose we are given a rigid analytically trivial pre-t-motive $M$ over $\mathbb{F}_q$ that is of dimension $r$ over $\bar{k}(t)$. If there exists a $\bar{k}(t)$-basis $m \in \text{Mat}_{r \times 1}(M)$ so that there exists $\Psi \in \text{GL}_r(\mathbb{L}) \cap \text{Mat}_r(\mathbb{Z})$ which is a rigid analytic trivialization of $M$ with respect to $m$ and satisfies

$$\text{tr. deg}_{\bar{k}(t)} \bar{k}(t)(\Psi) = \text{tr. deg}_{\bar{k}} \bar{k}(\Psi(\theta)),$$

then we say that $M$ has the GP (Grothendieck period) property, where $\bar{k}(t)(\Psi)$ (resp. $\bar{k}(\Psi(\theta))$) is the field generated by all entries of $\Psi$ (resp. $\Psi(\theta)$) over $\bar{k}(t)$ (resp. $\bar{k}$). The GP property is independent of the choices of $\Psi$ for a fixed $m$.

For any $r \in \mathbb{N}$, we let $\mathbb{F}_{q^r}$ be the finite field of $q^r$ elements. Given a rigid analytically trivial pre-t-motive $M$ over $\mathbb{F}_q$ with $(m, \Phi, \Psi)$ as above we define its $r$-th derived pre-t-motive $M^{(r)}$ over $\mathbb{F}_{q^r}$ (with respect to the operator $\sigma^r$): the underlying space of $M^{(r)}$ is the same as $M$, but it is now regarded as a left $\bar{k}(t)[\sigma^r, \sigma^{-r}]$-module. Letting

$$\Phi' := \Phi^{(-r-1)} \ldots \Phi^{(-1)} \Phi,$$
we have \( \sigma^* m = \Phi^* m \) and \( \sigma^* \Psi := \Psi^{(-r)} = \Phi^* \Psi \), and hence \( \Psi \) is also a rigid analytic trivialization of \( M^{(r)} \).

**Proposition 2.2.2.** Let \( M \) be a rigid analytically trivial pre-t-motive over \( \mathbb{F}_q \) which has the GP property. For any positive integer \( r \), the \( r \)-th derived pre-t-motive \( M^{(r)} \) over \( \mathbb{F}_{q^r} \) of \( M \) is also rigid analytically trivial and has the GP property.

Following the work of [10], one can show:

**Theorem 2.2.3.** (Chang [5] Thm. 1.2, Papanikolas [10] Thm. 5.2.2) Suppose \( \Phi \in \text{Mat}_r(k[t]) \) defines a rigid analytically trivial pre-t-motive \( M \) over \( \mathbb{F}_q \) with a rigid analytic trivialization \( \Psi \in \text{Mat}_r(\mathbb{C}) \cap \text{GL}_r(\mathbb{C}) \). If \( \det \Phi(0) \neq 0 \) and \( \det \Phi(\theta^{q^r}) \neq 0 \) for all \( i = 1, 2, 3, \ldots \), then \( M \) has the GP property.

Note that by [3] Prop. 3.1.3 the condition \( \det \Phi(0) \neq 0 \) implies \( \Psi \in \text{Mat}_r(\mathbb{C}) \). Combining Theorem [2.2.3] and Proposition [2.2.2] we have:

**Corollary 2.2.4.** Given an integer \( d \geq 2 \), we let \( \ell := \text{lcm}(1, \ldots, d) \). For each \( 1 \leq i \leq d \), let \( \ell_i := \frac{\ell}{d} \) and let \( \Phi_i \in \text{Mat}_r(k[t]) \cap \text{GL}_r(k[t]) \) define a pre-t-motive \( M_i \) over \( \mathbb{F}_q \) with a rigid analytic trivialization \( \Psi_i \in \text{Mat}_r(\mathbb{C}) \cap \text{GL}_r(\mathbb{C}) \). Suppose that each \( \Phi_i \) satisfies the hypotheses of Theorem [2.2.3] for \( i = 1, \ldots, d \). Then the direct sum

\[
M := \bigoplus_{i=1}^d M_i^{(\ell_i)}
\]

is a rigid analytically trivial pre-t-motive over \( \mathbb{F}_{q^d} \) that has the GP property.

**Proof.** For each \( 1 \leq i \leq d \), we define

\[
\Phi_i := \Phi_i^{(-\ell_i-1)} \cdots \Phi_i^{(-1)} \Phi_i.
\]

Moreover, if we define

\[
\Phi' := \bigoplus_{i=1}^d \Phi_i, \quad \Psi' := \bigoplus_{i=1}^d \Psi_i,
\]

then we have

\[
\Psi'^{(-\ell)} = \Phi' \Psi'.
\]

Note that the matrix representing multiplication by \( \sigma^\ell \) on \( M \) with respect to the evident \( k[t] \)-basis is given by \( \Phi' \).

Our task is to show that \( \Phi' \) satisfies the hypotheses of Theorem [2.2.3] (with respect to the operator \( \sigma^\ell \)), whence the result. It is obvious that \( \det \Phi'(0) \neq 0 \) since \( \det \Phi_i(0) \neq 0 \) for each \( 1 \leq i \leq d \). Suppose that \( \det \Phi' \left( \theta^{q^d} \right) = 0 \) for some \( j \in \mathbb{N} \). This implies that there exists \( 1 \leq i \leq d \) and \( 0 \leq m \leq \ell_i - 1 \) so that

\[
\det \Phi_i^{(-im)} \left( \theta^{-(\ell_j-im)} \right) = 0.
\]

However, this is equivalent to

\[
(1) \quad \det \Phi_i \left( \theta^{-(\ell_j-im)} \right) = 0.
\]

Since \( 0 \leq m \leq \ell_i - 1 \) and \( i \mid \ell_j \), we have that \( (\ell_j - im) > 0 \) and \( i \mid (\ell_j - im) \). Thus, (1) contradicts the hypothesis that \( \det \Phi_i \left( \theta^{-(ih)} \right) \neq 0 \) for all \( h = 1, 2, 3, \ldots \) \( \square \)

### 2.3. Difference Galois groups and transcendence.

In this section, we review the related theory developed in [10]. The category of pre-t-motives over \( \mathbb{F}_q \) forms a rigid abelian \( \mathbb{F}_q(t) \)-linear tensor category. Moreover, the category \( \mathcal{R} \) of rigid analytically trivial pre-t-motives over \( \mathbb{F}_q \) forms a neutral Tannakian category over \( \mathbb{F}_q(t) \). Given an object \( M \) in \( \mathcal{R} \), we let \( \mathcal{T}_M \) be the strictly full Tannakian subcategory of \( \mathcal{R} \) generated by \( M \). That is, \( \mathcal{T}_M \) consists of all objects of \( \mathcal{R} \) isomorphic to subquotients of finite direct sums of

\[
M^\otimes u \otimes (M^v)^\otimes v \text{ for various } u, v,
\]
where $M^\vee$ is the dual of $M$. By Tannakian duality, $\mathcal{T}_M$ is representable by an affine algebraic group scheme $\Gamma_M$ over $\mathbb{F}_q(t)$. The group $\Gamma_M$ is called the Galois group of $M$ and it is described explicitly as follows.

Suppose that $\Phi \in \text{GL}_r(\bar{k}(t))$ provides multiplication by $\sigma$ on $M$ with respect to a fixed basis $m \in \text{Mat}_{r 	imes 1}(M)$ over $\bar{k}(t)$. Let $\Psi \in \text{GL}_r(L)$ be a rigid analytic trivialization for $\Phi$. Let $X := (X_{ij})$ be an $r \times r$ matrix whose entries are independent variables $X_{ij}$, and define a $\bar{k}(t)$-algebra homomorphism $\nu : \bar{k}(t)[X, 1/\det X] \to L$ so that $\nu(X_{ij}) = \Psi_{ij}$. We let

$$
\Sigma_\Phi := \text{im } \nu = \bar{k}(t)[\Psi, 1/\det \Psi] \subseteq L,
$$

$$
Z_\Psi := \text{Spec } \Sigma_\Phi.
$$

Then $Z_\Psi$ is a closed $\bar{k}(t)$-subscheme of $\text{GL}_r(\bar{k}(t))$. Let $\Psi_1, \Psi_2 \in \text{GL}_r(L \otimes \bar{k}(t) L)$ be the matrices satisfying $(\Psi_1)_{ij} = \Psi_{ij} \otimes 1$ and $(\Psi_2)_{ij} = 1 \otimes \Psi_{ij}$. Let $\tilde{\Psi} := \Psi_1^{-1} \Psi_2$. We have an $\mathbb{F}_q(t)$-algebra homomorphism $\mu : \mathbb{F}_q(t)[X, 1/\det X] \to L \otimes \bar{k}(t) L$ so that $\mu(X_{ij}) = \tilde{\Psi}_{ij}$. Furthermore, we define

$$
\Delta := \text{im } \mu,
$$

$$
\Gamma_\Psi := \text{Spec } \Delta.
$$

The following theorem is proved in [10].

**Theorem 2.3.1.** (Papanikolas [10] Thm. 4.2.11, 4.3.1, 4.5.10) *The scheme $\Gamma_\Psi$ is a closed $\mathbb{F}_q(t)$-subgroup scheme of $\text{GL}_r(\mathbb{F}_q(t))$, which is isomorphic to the Galois group $\Gamma_M$ over $\mathbb{F}_q(t)$. Moreover $\Gamma_\Psi$ has the following properties:

(a) $\Gamma_\Psi$ is smooth over $\mathbb{F}_q(t)$ and is geometrically connected.
(b) $\dim \Gamma_\Psi = \text{tr.deg}_{\bar{k}(t)} \bar{k}(t)(\Psi)$.
(c) $Z_\Psi$ is a $\Gamma_\Psi$-torsor over $\bar{k}(t)$.

In particular, if $M$ has the GF property, then one has

(d) $\dim \Gamma_\Psi = \text{tr.deg}_{\bar{k}} \bar{k}(\Psi(\emptyset))$.

We call $\Gamma_\Psi$ the Galois group associated to the difference equation $\Psi^{(-t)} = \Phi \Psi$. This $\Gamma_\Psi$ is independent of the analytic trivialization $\Psi$, up to isomorphism over $\mathbb{F}_q(t)$.

Throughout this paper we always identify $\Gamma_M$ with $\Gamma_\Psi$, and regard it as a linear algebraic group over $\mathbb{F}_q(t)$ because of Theorem 2.3.1(a).

**Remark 2.3.2.** Let $r_1, r_2$ be positive integers and $0 := 0_{r_1 \times r_2}$ the zero matrix of size $r_1 \times r_2$. Suppose that the matrix

$$
\Phi := \begin{bmatrix} 
\Phi_1 & 0 \\
\Phi_3 & \Phi_2 
\end{bmatrix} \in \text{GL}_{r_1 + r_2}(\bar{k}(t))
$$

defines a rigid analytically trivial pre-$t$-motive $M$. Then one can always find its rigid analytic trivialization of the form

$$
\Psi := \begin{bmatrix} 
\Psi_1 & 0 \\
\Psi_3 & \Psi_2 
\end{bmatrix} \in \text{GL}_{r_1 + r_2}(L).
$$

By (2), we have that

$$
\Gamma_\Psi \subseteq \left\{ \begin{bmatrix} * & 0 \\ * & * \end{bmatrix} \right\} \subseteq \text{GL}_{r_1 + r_2}(\mathbb{F}_q(t)).
$$

Let $N$ be the sub-pre-$t$-motive of $M$ defined by $\Phi_1 \in \text{GL}_{r_1}(\bar{k}(t))$ with rigid analytic trivialization $\Psi_1$, then by the Tannakian theory we have a natural surjective morphism

$$
\pi : \Gamma_\Psi(\mathbb{F}_q(t)) \to \Gamma_{\Psi_1}(\mathbb{F}_q(t)).
$$

(3)
In fact, \( \pi(\gamma) \) comes from the restriction of the action of \( \gamma \) to the fiber functor of \( T_N \) (which is a full subcategory of \( T_M \)). Precisely, \( \pi(\gamma) \) is the matrix cut out from the upper left square of \( \gamma \) with size \( r \) (for detailed arguments, see [10 §6.2.2]).

### 3. A dimension criterion

3.1. **Pre-\( t \)-motives of type SV.** Let \( \{n_1, \ldots, n_h\} \) be \( h \) non-negative integers. We say that a pre-\( t \)-motive \( M \) over \( \mathbb{F}_q \) is of type \( SV \) (its Galois group being an extension of a split torus by a vector group) if it is defined by \( \Phi \) of the form

\[
\Phi := \bigoplus_{i=1}^h A_i, \quad A_i := \begin{bmatrix} a_i & 0 & \cdots & 0 \\ a_{i1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{in_i} & 0 & \cdots & 1 \end{bmatrix} \in \text{GL}_{1+n_i}(\bar{k}(t)),
\]

which has rigid analytically trivialization

\[
\Psi := \bigoplus_{i=1}^h F_i, \quad F_i := \begin{bmatrix} f_i & 0 & \cdots & 0 \\ f_{i1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_{in_i} & 0 & \cdots & 1 \end{bmatrix} \in \text{GL}_{1+n_i}(L).
\]

The entries \( f_i, i = 1, \ldots, h \), shall be called the diagonals of the rigid analytic trivialization \( \Psi \).

Let \( T \) be the Galois group associated to the difference equation

\[
\begin{bmatrix} f_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f_h \end{bmatrix} (-1) = \begin{bmatrix} a_1 & \cdots & 0 \\ a_{i1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{in_i} & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} f_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f_h \end{bmatrix}
\]

and note that by (2), \( T \) is a subtorus of the \( h \) dimensional split torus in \( \text{GL}_h(\mathbb{F}_q(t)) \).

By the same reason as (3), we have the following natural projection of Galois groups

\[
\Gamma_{\Psi} \rightarrow T
\]

given in terms of coordinates by

\[
\bigoplus_{i=1}^h \begin{bmatrix} x_i & 0 & \cdots & 0 \\ x_{i1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{in_i} & 0 & \cdots & 1 \end{bmatrix} \rightarrow [x_1] \oplus \cdots \oplus [x_h].
\]

One has also the following exact sequence of linear algebraic groups

\[
1 \rightarrow V \rightarrow \Gamma_{\Psi} \rightarrow T \rightarrow 1,
\]

where \( V \) is a vector group contained in the \( \sum_{i=1}^h n_i \)-dimensional “coordinate” vector group \( G \) which contains all elements of the form

\[
\bigoplus_{i=1}^h \begin{bmatrix} 1 & 0 & \cdots & 0 \\ x_{i1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{in_i} & 0 & \cdots & 1 \end{bmatrix}.
\]

This subgroup \( V \) is the unipotent radical of its (solvable) Galois group \( \Gamma_{\Psi} \), and \( \dim \Gamma_{\Psi} = \dim V + \dim T \).
Definition 3.1.1. Let $M$ be a rigid analytically trivial pre-$t$-motive of $SV$ type given as above. We say that its Galois group $\Gamma_\Psi$ is full if $\dim V = \sum_{i=1}^h n_i$, i.e., $V = G$.

3.2. Criterion for direct sum motives to have full Galois group. We continue with the notation of [3.1]. For each $i$, $1 \leq i \leq h$, the $1 \times 1$ matrix $[a_i]$ defines a sub-pre-$t$-motive of $M$ over $\mathbb{F}_q$, which is one-dimensional over $k(t)$. Its rigid analytic trivialization is given by the diagonal $f_i$ satisfying $[f_i]^{(-1)} = [a_i][f_i]$. We call one such sub-pre-$t$-motive a diagonal of the pre-$t$-motive $M$. Note that by Theorem 2.3.1 the Galois group of a diagonal is $\mathbb{G}_m$ if and only if the corresponding trivialization $f_i$ is transcendental over $\bar{k}(t)$. In this situation, the canonical projection $T \to \mathbb{G}_m$ on the $i$th coordinate of $T$ is surjective.

Note that there is a canonical action of $T$ on $G$, which induces an action of $T$ on $V$ compatible with the one coming from $\overline{\mathbb{Q}}$. Hence, we have that any element $\gamma \in V$ whose $x_{ij}$-coordinate is nonzero, the orbit inside $V$ given by the action of $T$ on $\gamma$ must be infinite if $f_i$ is transcendental over $\bar{k}(t)$.

Definition 3.2.1. Let $M_r$ be a rigid analytically trivial pre-$t$-motive of type $SV$ for $r = 1, \ldots, d$. We say that this set of pre-$t$-motives $\{M_r\}_{r=1}^d$ is diagonally independent if for any $1 \leq i, j \leq d$, $i \neq j$, the Galois group of $N_i \oplus N_j$ is a two-dimensional torus over $\mathbb{F}_q(t)$, where $N_i$ (resp. $N_j$) is any diagonal of $M_i$ (resp. $M_j$).

Theorem 3.2.2. Let $M_1, \ldots, M_d$ be rigid analytically trivial pre-$t$-motives over $\mathbb{F}_q$ of type $SV$ with full Galois groups. Suppose that the set $\{M_r\}_{r=1}^d$ is diagonally independent. Putting $M := \oplus_{r=1}^d M_r$, then the Galois group of $M$ is also full.

Proof. Without loss of generality, we may assume $d = 2$ since the following argument generalizes easily for arbitrary $d \geq 2$. Let the pre-$t$-motive $M_1$ be defined by a matrix $\Phi_1$, for $i = 1, 2$, with $\Psi = \Psi_1 \oplus \Psi_2$ a rigid analytic trivialization of $M$, and let $\Gamma_\Psi$, $\Gamma_{\Psi_1}$, $\Gamma_{\Psi_2}$ be the Galois groups. The unipotent radicals of these groups are denoted by $V_1$, $V_2$ respectively. Let $G$ (resp. $G_i$, $i = 1, 2$) be the coordinate vector group containing $V$ (resp. $V_i$). Suppose the matrix $\Phi_1$ has $h$ diagonal blocks, and let the coordinates of $G_1$ be denoted by $x_{ij}$, $i = 1, \ldots, h$, $j = 1, \ldots, n_i$. Similarly let $y_{ij}$, $i = 1, \ldots, \ell$, $j = 1, \ldots, m_j$ denote the coordinates of $G_2$. Any subspace $W \subseteq G$ obtained by setting some of these coordinates to 0 is called a linear coordinate subspace. The hypothesis that the Galois group of $M_i$ is full means exactly that $G_i = V_i$, for $i = 1, 2$. We are going to prove that $G = V$.

Suppose $V$ has codimension $r$ in the coordinate vector group $G$. We can find linear coordinate subspace $W \subseteq G$ of dimension $r$ such that $W \cap V$ is of dimension 0. Since $W \cap V$ is invariant under $T$, it must be $\{0\}$, because the hypothesis that $M_1$ and $M_2$ are diagonally independent implies in particular that all diagonals in the analytic trivialization $\Psi$ are transcendental over $k(t)$.

Let $W' \subseteq G$ be the linear coordinate space given by those coordinates disjoint from those of $W$. Then the natural projection from $G$ to $W'$ induces on $V$ an isomorphism of vector groups. Composing the inverse of this isomorphism with the surjective morphism $\pi_1$ in the following diagram

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & V & \longrightarrow & \Gamma_\Psi & \longrightarrow & T & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & G_1 & \longrightarrow & \Gamma_{\Psi_1} & \longrightarrow & T_1 & \longrightarrow & 1
\end{array}
\]

we obtain a morphism $\pi_1$ from $W'$ onto $G_1$ which is furthermore a $T$-morphism.

We contend that under the hypothesis that $M_1$ and $M_2$ are diagonally independent, $\pi_1$ maps $G_2 \cap W'$ to zero. This contention results from the following basic lemma by
taking any diagonal $N_1$ (resp. $N_2$) of $M_1$ (resp. $M_2$) and consider the restriction of the above morphism $\pi_1$ to a single block. Now since $G = G_1 \times G_2$, it follows that $\pi_1(G_1 \cap W') = G_1$. Thus $G_1 \subseteq W'$. Similarly we also have $G_2 \subseteq W'$, and hence $G = W'$ and $G = V$. \hfill \Box

**Lemma 3.2.3.** Let $G_1$ (resp. $G_2$) be the vector group with coordinates
\[
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
x_1 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
x_n & 0 & \cdots & 1
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & \cdots & 0 \\
y_1 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
y_m & 0 & \cdots & 1
\end{bmatrix}
\]

Let $G_m^2$ act on $G_1$ (resp. $G_2$) by
\[
G_m^2 \ni (x,y) : x_i \mapsto xx_i, 1 \leq i \leq n, \quad (\text{resp. } y_j \mapsto yy_j, 1 \leq j \leq m).
\]
If $\pi_1 : G_2 \to G_1$ is a $G_m^2$-morphism, then $\pi_1 \equiv 0$.

4. APPLICATION TO ZETA VALUES

4.1. Carlitz theory. Throughout [11] we fix $q := p$ and $\sigma = \sigma_p : \sum a_i t^i \mapsto \sum a_i^{1/p} t^i$ for $\sum a_i t^i \in C_\infty((t))$. Also, we henceforth let $r$ be a fixed positive integer. Recall the Carlitz $F_p^{[r]}[t]$-module, denoted by $C_r$, which is given by the following $F_p^{[r]}$-linear ring homomorphism:
\[
C_r = (t \mapsto (x \mapsto \theta x + x^{p^r})) : F_p^{[r]}[t] \to \text{End}_{F_p^{[r]}}(G_a).
\]
Note that when we regard $C_r$ as a Drinfeld $F_p[t]$-module, it is of rank $r$ (see [7], [11]). One has the Carlitz exponential associated to $C_r$:
\[
\exp_{C_r}(z) := \sum_{i=0}^{\infty} \frac{z^{p^{ri}}}{D_{ri}}.
\]
Here we set
\[
D_{r0} := 1, \quad D_{ri} := \prod_{j=0}^{i-1}(\theta^{p^{ri}} - \theta^{p^{rj}}), \quad i \geq 1.
\]
Now $\exp_{C_r}(z)$ is an entire power series in $z$ satisfying the functional equation
\[
\exp_{C_r}(\theta z) = \theta \exp_{C_r}(z) + \exp_{C_r}(z)^{p^r}.
\]
Moreover one has the product expansion
\[
\exp_{C_r}(z) = z \prod_{0 \neq a \in F_p^{[r]}[\theta]} \left(1 - \frac{z}{a\tilde{\alpha}_r}\right),
\]
where
\[
\tilde{\alpha}_r = \theta(-\theta)^{\frac{1}{p^{r-1}}} \prod_{i=1}^{\infty} \left(1 - \theta^{1-p^{-1}}\right)^{-1}
\]
is a fundamental period of $C_r$. Throughout this paper we will fix a choice of $(-\theta)^{\frac{1}{p^{r-1}}}$ so that $\tilde{\alpha}_r$ is well-defined element in $\overline{F_p((\frac{1}{\theta})]}$. We also choose these roots in a compatible way so that when $r | r'$ the number $(-\theta)^{\frac{1}{p^{r-1}}}$ is a power of $(-\theta)^{\frac{1}{p^{r'-1}}}$.

The formal inverse of $\exp_{C_r}(z)$ is the Carlitz logarithm $\log_{C_r}(z)$, and as a power series in $z$, one has
\[
\log_{C_r}(z) = \sum_{i=0}^{\infty} \frac{z^{p^{ri}}}{L_{ri}}.
\]
where
\[
L_{r0} := 1, \quad L_{ri} := \prod_{j=1}^{i} (\theta - \theta^{p^j}).
\]

As a function in \( z \), \( \log_{C_r}(z) \) converges for all \( z \in \mathbb{C}_\infty \) with \( |z|_\infty < |\theta|_\infty^{-\frac{1}{r}} \). It satisfies the functional equation
\[
\theta \log_{C_r}(z) = \log_{C_r}(\theta z) + \log_{C_r}(z^{\theta^r})
\]
whenever the values in question are defined.

For a positive integer \( n \), the \( n \)-th Carlitz polylogarithm associated to \( C_r \) is the series
\[
P_{\log,n}(z) := \sum_{i=0}^{\infty} \frac{z^{\theta^r}}{L_{ri}}
\]
which converges \( \infty \)-adically for all \( z \in \mathbb{C}_\infty \) with \( |z|_\infty < |\theta|_\infty^{-\frac{1}{r}} \). Its value at a particular \( z = \alpha \neq 0 \) is called the \( n \)-th polylogarithm of \( \alpha \) associated to \( C_r \). In transcendence theory we are interested in those polylogarithms of \( \alpha \in \kappa \), as analogous to classical logarithms of algebraic numbers.

4.2. Algebraic independence of special functions. For any positive integer \( r \), we let
\[
\Omega_r(t) := (-\theta)^{\frac{1}{\theta^{p^r}}} \prod_{i=1}^{\infty} \left( 1 - \frac{1}{\theta^{t+p^i}} \right) \in \kappa[[t]] \subseteq \mathbb{C}_\infty((t)).
\]
One checks that \( \Omega_r \in \mathbb{E} \). Furthermore, \( \Omega_r \) satisfies the functional equation
\[
\Omega_r^{(r)}(t) = (t - \theta) \Omega_r(t)
\]
and its specialization at \( t = \theta \) gives \( \Omega_r(\theta) = -1/\pi_r \).

By (7), the function \( \Omega_r \) provides a rigid analytic trivialization of the Carlitz motive \( C_r \) over \( \mathbb{F}_p \) that has the GP property (cf. Theorem 2.2.3). This is the pre-t-motive with underlying space \( \kappa(k(t)) \) itself and \( \sigma^r \) acts by \( \sigma^r f = f^{(-r)} \) for \( f \in C_r \).

For any \( d \in \mathbb{N} \), we let \( \ell := \text{lcm}(1, \ldots, d) \) and \( \ell_r := \frac{\ell}{r} \) for \( r = 1, \ldots, d \). Let \( C_r^{(\ell_r)} \) be the \( \ell_r \)-th derived pre-t-motive over \( \mathbb{F}_p \) of \( C_r \), we consider the direct sum \( M = M_d \) := \( \bigoplus_{r=1}^{d} C_r^{(\ell_r)} \). By Corollary 2.2.4, \( M \) also has the GP property. We note that the canonical rigid analytical trivialization of \( M \) is the diagonal matrix \( \Psi \in \text{Mat}_d(\mathbb{E}) \cap \text{GL}_d(\mathbb{L}) \) with diagonal entries \( \Omega_1, \ldots, \Omega_d \).

Lemma 4.2.1. Given any positive integer \( d \geq 2 \), let \( M = M_d \) be the rigid analytically trivial pre-t-motive with rigid analytic trivialization \( \Psi \) defined as above. Then we have \( \dim \Gamma_\Psi = d \). In particular, the functions \( \Omega_1, \ldots, \Omega_d \) are algebraically independent over \( \kappa(t) \) and the values \( \pi_1, \ldots, \pi_d \) are algebraically independent over \( \kappa \).

Proof. Suppose \( \dim \Gamma_\Psi < d \). Since \( \Psi \) is a diagonal matrix with diagonal entries \( \Omega_1, \ldots, \Omega_d \), by (2) we have that \( \Gamma_\Psi \subseteq T \), where \( T \) is the split torus of dimension \( d \) in \( \text{GL}_d(\mathbb{F}_p) \). We let \( X_1, \ldots, X_d \) be the coordinates of \( T \) and \( \chi_j \) the character of \( T \) which projects the \( j \)-th diagonal position to \( \mathbb{G}_m \). Note that \( \{\chi_j\}_{j=1}^{d} \) generates the character group of \( T \). Hence \( \Gamma_\Psi \) is the kernel of some characters of \( T \), i.e., canonical generators of the defining ideal for \( \Gamma_\Psi \) can be of the form \( X_1^m_1 \cdots X_d^m_d - 1 \) for some integers \( m_1, \ldots, m_d \), not all zero. By (2) we have that
\[
(\Omega_1^{-m_1} \cdots \Omega_d^{-m_d}) \otimes (\Omega_1^{m_1} \cdots \Omega_d^{m_d}) = 1 \in \mathbb{L} \otimes_{\kappa(t)} \mathbb{L}.
\]
and hence
\[
\beta := \Omega_1^{m_1} \cdots \Omega_d^{m_d} \in \kappa(t)^{\times}.
\]
We recall that $\Omega_r$ has zeros on $\{\theta^{p^i}\}_{i=1}^\infty$. Since $\beta \in \bar{k}(t)^\times$, it has only finitely many zeros and poles, and hence \text{ord}_{t=\theta^{p^i}}(\beta) = 0$ for $h > 0$. Choosing a prime number $p'$ sufficiently large, so that the following conditions hold:

- $p' > d$;
- the order of vanishing of $\beta$ at $t = \theta^{p^i}$ is zero.

These conditions imply that $m_1 = 0$ from \[3\]. Iterating this argument, we conclude that $m_1 = \cdots = m_d = 0$, whence a contradiction. \qed

4.3. Algebraic independence of polylogarithms. Given $n \in \mathbb{N}$ and $\alpha \in \bar{k}^\times$ with $|\alpha|_\infty < |\theta|^{\frac{1}{p^1}}$, we consider the power series

$$L_{\alpha, rn}(t) := \alpha + \sum_{i=1}^\infty \frac{\alpha^{p^i}}{(t - \theta^{p^i})^n(t - \theta^{p^i})^n},$$

which as a function on $\mathbb{C}_\infty$ converges on $|t|_\infty < |\theta|^{p^1}$. We note that $L_{\alpha, rn}(\theta)$ is exactly the $n$-th polylogarithm of $\alpha$ associated to $C_r$, i.e.,

$$L_{\alpha, rn}(\theta) = \text{Plog}_{rn}(\alpha).$$

Given a collection of such numbers $\alpha$, say $\alpha_1, \ldots, \alpha_m$, we define

$$\Phi_{rn}(\alpha_1, \ldots, \alpha_m) := \left[ \begin{array}{ccc} (t - \theta)^n & 0 & \cdots & 0 \\ \alpha_1^{(-r)}(t - \theta)^n & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_m^{(-r)}(t - \theta)^n & 0 & \cdots & 1 \end{array} \right] \in \text{GL}_{m+1}(\bar{k}(t)) \cap \text{Mat}_{m+1}(\bar{k}[t])$$

and

$$\Psi_{rn}(\alpha_1, \ldots, \alpha_m) := \left[ \begin{array}{ccc} \Omega_n \alpha_{rn} & 0 & \cdots & 0 \\ \Omega_n L_{\alpha_1, rn} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_n L_{\alpha_m, rn} & 0 & \cdots & 1 \end{array} \right] \in \text{GL}_{m+1}([L]) \cap \text{Mat}_{m+1}(E).$$

Then one has

$$\Psi_{rn}(\alpha_1, \ldots, \alpha_m)^{(-r)} = \Phi_{rn}(\alpha_1, \ldots, \alpha_m)\Psi_{rn}(\alpha_1, \ldots, \alpha_m) \quad (\text{cf.} \ [7] \ \S 3.1.2).$$

Hence, $\Phi_{rn}(\alpha_1, \ldots, \alpha_m)$ defines a rigid analytically trivial pre-$t$-motive over $\mathbb{F}_{p^1}$ that has the GP property.

In [7], we followed Papanikolas’ methods to generalize the algebraic independence of Carlitz logarithms to algebraic independence of polylogarithms. Precisely, by [7] Thm. 3.1 and Theorem 2.2.3 we have:

**Theorem 4.3.1.** (Chang-Yu, [7] Thm. 3.1) Given any positive integers $r$ and $n$, let $\alpha_1, \ldots, \alpha_m \in \bar{k}^\times$ satisfy $|\alpha_i|_\infty < |\theta|^{\frac{1}{p^i}}$ for $i = 1, \ldots, m$. Then

$$\dim_{\mathbb{F}_{p^1}(\theta)} N_{rn} = \text{tr} \deg_k \bar{k}(\pi_n, L_{\alpha_1, rn}(\theta), \ldots, L_{\alpha_m, rn}(\theta)) = \text{tr} \deg_k \bar{k}(t)(\Omega_n, L_{\alpha_1, rn}, \ldots, L_{\alpha_m, rn}),$$

where

$$N_{rn} := \mathbb{F}_{p^1}(\theta) \cdot \text{Span}\{\pi_n, L_{\alpha_1, rn}(\theta), \ldots, L_{\alpha_m, rn}(\theta)\}.$$

4.4. Formulas for zeta values.
4.4.1. Euler-Carlitz relations. In this subsection, we fix a positive integer \( r \). In \([4]\) Carlitz introduced the power sum
\[
\zeta_r(n) := \sum_{\substack{a \in \mathbb{F}_p^*[\theta] \cap k\infty \\text{a monic}}} \frac{1}{a^n} \quad (n \text{ a positive integer})
\]
which are the Carlitz zeta values associated to \( \mathbb{F}_p^*[\theta] \).

Writing down a \( p^r \)-adic expansion \( \sum_{i=0}^{\infty} n_i p^{ri} \) of \( n \), we let
\[
\Gamma_{r,n+1} := \prod_{i=0}^{\infty} D_r^{n_i}.
\]
We call \( \Gamma_{r,n+1} \) the Carlitz factorials associated to \( \mathbb{F}_p^*[\theta] \). The Bernoulli-Carlitz numbers \( B_{rn} \) in \( \mathbb{F}_p(\theta) \) are given by the following expansions from the Carlitz exponential series
\[
\frac{z}{\exp_{C_r}(z)} = \sum_{n=0}^{\infty} \frac{B_{rn}}{\Gamma_{r,n+1}} z^n.
\]
Carlitz proved the Euler-Carlitz relations:

**Theorem 4.4.2.** (Carlitz, \([3]\)) For all positive integer \( n \) divisible by \( p^r - 1 \), one has

\[
\zeta_r(n) = \frac{B_{rn}}{\Gamma_{r,n+1}} \tilde{\pi}_r^n.
\]

We call a positive integer \( n \) \((p,r)\)-even if it is divisible by \( p^r - 1 \), otherwise we call \( n \) \((p,r)\)-odd. In particular, when \( p = 2 \) and \( r = 1 \), all positive integers are even.

4.4.3. The Anderson-Thakur formula. In \([2]\), Anderson and Thakur introduced the \( n \)-th tensor power of the Carlitz \( \mathbb{F}_p^*[t] \)-module \( C_r \), and they related \( \zeta_r(n) \) to the last coordinate of the logarithm associated to the \( n \)-th tensor power of \( C_r \) for each positive integer \( n \). More precisely, they interpreted \( \zeta_r(n) \) as \( \mathbb{F}_p^*(\theta) \)-linear combinations of \( n \)-th Carlitz polylogarithms of algebraic numbers:

**Theorem 4.4.4.** (Anderson-Thakur, \([2]\)) Given any positive integers \( r \) and \( n \), one can find a sequence \( h_{rn,0}, \ldots, h_{rn,l_{rn}} \in \mathbb{F}_p^*(\theta) \), \( l_{rn} < \frac{np^r}{p^r-1} \), such that the following identity holds

\[
\zeta_r(n) = \sum_{i=0}^{l_{rn}} h_{rn,i} \text{Plog}_{rn}(\theta^i),
\]
where \( \text{Plog}_{rn}(z) \) is defined as in \([4]\). In the special case of \( n \leq p^r - 1 \),
\[
\zeta_r(n) = \text{Plog}_{rn}(1).
\]

**Definition 4.4.5.** Given any positive integer \( r \), for each \( n \in \mathbb{N} \), \((p^r - 1) \nmid n \), with \( l_{rn} \) as given by \((11)\), we fix a finite subset
\[
\{a_{0,rn}, \ldots, a_{m_{rn},rn}\} \subseteq \{1, \theta, \ldots, \theta^{l_{rn}}\}
\]
such that
\[
\{\tilde{\pi}_r^n, L_{0,rn}(\theta), \ldots, L_{m_{rn},rn}(\theta)\}
\]
and
\[
\{\tilde{\pi}_r^n, \zeta_r(n), L_{1,rn}(\theta), \ldots, L_{m_{rn},rn}(\theta)\}
\]
are \( \mathbb{F}_p^*(\theta) \)-bases for \( N_{rn} \), where \( L_j(\theta) := L_{r,\theta}(t) \) for \( j = 0, \ldots, m_{rn} \). This can be done because of \((11)\) (cf. \([7\) § 4.1]) and note that \( m_{rn} + 2 \) is the dimension of \( N_{rn} \) over \( \mathbb{F}_p^*(\theta) \). For the case of \( p = 2 \) and \( r = 1 \), we put \( m_{11} + 1 := 0 \).
Definition 4.4.6. Given any positive integers \( s \) and \( d \) with \( d \geq 2 \). For each \( 1 \leq r \leq d \) we define
\[
U_r(s) = \begin{cases} 
\{1\}, & \text{if } p = 2 \text{ and } r = 1; \\
\{1 \leq n \leq s; p \nmid n, (p^r - 1) \nmid n\}, & \text{otherwise.}
\end{cases}
\]

For each \( n \in U_r(s) \), we define that if \( p = 2 \) and \( r = 1 \),
\[
\Phi_{rn} := (t - \theta) \in \text{GL}_1(\overline{k}(t)),
\Psi_{rn} := \Omega_1 \in \text{GL}_1(\overline{L}),
\]
otherwise
\[
\Phi_{rn} := \Phi_{rn}(\alpha_{0, rn}, \ldots, \alpha_{m, rn}) \in \text{GL}_{(m + 2)}(\overline{k}(t)),
\Psi_{rn} := \Psi_{rn}(\alpha_{0, rn}, \ldots, \alpha_{m, rn}) \in \text{GL}_{(m + 2)}(\overline{L}).
\]

By Theorem 4.3.1 we have that
\[
\dim \Gamma_{\Psi_{rn}} = \text{tr. deg}_{\overline{k}(t)}(\overline{k}(t)(\Psi_{rn})) = m_{rn} + 2.
\]

Put \( \Phi_r := \bigoplus_{n \in U_r(s)} \Phi_{rn} \), then \( \Phi_r \) defines a rigid analytically trivial pre-t-motive \( M_r \)
over \( \mathbb{F}_{p^r} \) with rigid analytic trivialization \( \Psi_r := \bigoplus_{n \in U_r(s)} \Psi_{rn} \) (cf. [7]). Moreover, \( M_r \)
is of type \( SV \) and by Theorem 2.2.3 \( M_r \) has the \( GP \) property. The main theorem of
[7] is the following:

Theorem 4.4.7. (Chang-Yu, [7] Thm. 4.5) For any positive integers \( s \) and \( r \), the
Galois group \( \Gamma_{M_r} \) over \( \mathbb{F}_{p^r}(t) \) is full, i.e.,
\[
\dim \Gamma_{M_r} = 1 + \sum_{n \in U_r(s)} (m_{rn} + 1).
\]

4.5. Proof of Theorem 1.2.1. Given any integer \( d \geq 2 \), we put \( \ell := \text{lcm}(1, \ldots, d) \)
and \( \ell_r := \frac{\ell}{r} \) for \( r = 1, \ldots, d \). For each \( 1 \leq r \leq d \) let \( M_r := M_{\ell_r}^{(r)} \) be the \( \ell_r \)-th
derived pre-t-motive over \( \mathbb{F}_{p^r} \) of \( M_r \) defined as above, then it is still of type \( SV \). By
Proposition 2.2.2 each \( M_r \) has the \( GP \) property and by Theorem 4.4.7 its Galois group
\( \Gamma_{M_r} \) is full. Further, for each \( 1 \leq r \leq d \) any diagonal of \( M_r \) has canonical rigid
analytical trivialization given by \( \Omega_r^n \) for some \( n \in U_r(s) \) and hence its Galois group is
\( \Gamma_{M_r} \) because \( \Omega_r \) is transcendental over \( \overline{k}(t) \).

Since by Lemma 4.2.1 the functions \( \Omega_1, \ldots, \Omega_d \) are algebraically independent over
\( \overline{k}(t) \), particularly the Galois group of \( N_i \oplus N_j \) is a two-dimensional torus over \( \mathbb{F}_{p^r}(t) \)
for any diagonal \( N_i \) (resp. \( N_j \)) of \( M_i \) (resp. \( M_j \)) with \( i \neq j \), \( 1 \leq i, j \leq d \). Put
\( M := \bigoplus_{r=1}^d M_r \), then by Corollary 2.2.4 \( M \) has the \( GP \) property. Applying Theorem
3.2.2 to this situation, we obtain the explicit dimension of \( \Gamma_{M} \):

Theorem 4.5.1. Given any positive integers \( s \) and \( d \) with \( d \geq 2 \), let \( M \) be defined as
above. Then the Galois group \( \Gamma_{M} \) is full, i.e.,
\[
\dim \Gamma_{M} = d + \sum_{r=1}^d (m_{rn} + 1).
\]

As a consequence, we completely determine all the algebraic relations among the
families of Carlitz zeta values:

Corollary 4.5.2. Given any positive integers \( d \) and \( s \), the transcendence degree of the
field
\[
\overline{k}(\bigcup_{r=1}^d \{ \pi_r, \zeta_r(1), \ldots, \zeta_r(s) \})
\]
over \( \overline{k} \) is
\[
\sum_{r=1}^d \left( s - \left\lfloor \frac{s}{p} \right\rfloor - \left\lfloor \frac{s}{p^r - 1} \right\rfloor + \left\lfloor \frac{s}{p(p^r - 1)} \right\rfloor + 1 \right).
\]
Proof. We may assume $d \geq 2$ since the case $d = 1$ is already given in [7, Cor. 4.6]. For $1 \leq r \leq d$, let
\[ V_r(s) := \emptyset, \text{ if } p = 2; \]
\[ V_r(s) := U_r(s), \text{ otherwise.} \]
Since $M$ has the GP property, by Theorem 4.5.1 we see that the following elements
\[(12) \quad \{\Omega_1(\theta), \ldots, \Omega_d(\theta)\} \cup \left( \bigcup_{r=1}^{d} \bigcup_{n \in V_r(s)} \{L_{0, rn}(\theta), \ldots, L_{m_{rn}, rn}(\theta)\} \right) \]
are algebraically independent over $\bar{k}$. In particular, by Definition 4.4.5 we have that
\[ \{\tilde{\pi}_1, \ldots, \tilde{\pi}_d\} \cup \left( \bigcup_{r=1}^{d} \bigcup_{n \in V_r(s)} \{\zeta_r(n)\} \right) \]
is an algebraically independent set over $\bar{k}$. Counting the cardinality of $V_r(s)$ for each $1 \leq r \leq d$, we complete the proof. \hfill \Box

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