Are Damage Spreading Transitions Generically in the Universality Class of Directed Percolation?

Peter Grassberger

Physics Department, University of Wuppertal, D-42097 Wuppertal, FRG

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Abstract

We present numerical evidence for the fact that the damage spreading transition in the Domany-Kinzel automaton found by Martins et al. is in the same universality class as directed percolation. We conjecture that also other damage spreading transitions should be in this universality class, unless they coincide with other transitions (as in the Ising model with Glauber dynamics) and provided the probability for a locally damaged state to become healed is not zero.

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Among all critical phenomena, directed percolation (DP) is maybe that which has been associated with the most wide variety of phenomena.

First there are interpretations where the preferred direction is a spatial direction. This was of course proposed to apply to material and charge transport in disordered media under the influences of external forces. Also, it should model the propagation of epidemics and forest fires under some directional bias, e.g. strong wind.

More interesting are interpretations where the preferred direction is time. Here, the primary interpretation is as an epidemic without immunization, the so called “contact process” \( \mathbb{1} \) or the “simple epidemic” \( \mathbb{2} \).

But these are by no means all possible applications. A very early application (even if it took rather long until it was understood as such \( \mathbb{3}, \mathbb{4} \)) was to “reggeon field theory”, a theory for ultrarelativistic particle collisions popular in the 70’s \( \mathbb{5} \). Here, the preferred direction is that of “rapidity”, while the directions transverse to it are provided by the impact parameter plane. This connection is interesting since it was through it that first precise estimates of critical exponents and amplitudes were obtained for DP \( \mathbb{3} \).

Another realization of the DP transition occurs in simple models of heterogeneous catalysis. The first such model was proposed by Ziff et al. \( \mathbb{3} \) (ZGB). The simulations of these and subsequent authors indicated that this model was in a different universality class, and it is only after some controversies that it is now generally accepted to be in the DP universality class \( \mathbb{7} \). Similar models are invented again and again \( \mathbb{8}, \mathbb{9} \). Repeatedly they are claimed to be in different universality classes, and repeatedly these claims are refuted \( \mathbb{10}, \mathbb{11}, \mathbb{12} \).

In \( \mathbb{13}, \mathbb{14} \) it was proposed that the universality class of DP contains all continuous transitions from a “dead” or “absorbing” state to an “active” one with a single scalar order parameter, provided the dead state is not degenerate (and provided some technical points are fulfilled: short range interactions both in space and time, nonvanishing probability for any active state to die locally, translational invariance [absence of ‘frozen’ randomness], and absence of multicritical points). It seems fair to say that there is now ample evidence for this proposal. It predicts, e.g. immediately that the ZGB model is in this universality class. A rather subtle question is whether also chaotic systems where the random noise is replaced by deterministic chaos are in the same class \( \mathbb{15}, \mathbb{16} \).

As far as I am aware of, no model with non-fluctuating absorbing state and a multi-component order parameter has ever been studied in the literature. Notice that this has to be distinguished from models for which some mean field approximation has a multi-component order parameter. Such models are quite common (e.g. the Bethe-Peierls approximation of the Ising model, or the mean field approximation of ZGB), and it was just the study of such a model which had led to the conjecture...
in \[14\]. A supposed generalization \[17\] of the above conjecture is thus already fully contained in the original conjecture of \[14\].

A more interesting question is what happens if the dead state is degenerate. Counter examples with twofold degeneracy were studied in \[18, 19, 20\]. They involve conservation laws which prevent some active states from dying, making it thus immediately clear that any transition — if it occurs at all — has to be in a different universality class.

But the main open problem is whether models can be generically in the DP class if they have an absorbing state with positive entropy (for obvious reason, we prefer not to call it “dead” in this case). One might conjecture that such a state is essentially unique on a coarse scale, provided its evolution is ergodic and mixing — and provided it does not involve long range correlations (long correlations should be entirely due to patches of “active” states). Since only coarse-grained properties should influence critical behavior, this would suggest that such transitions are in the DP class. This seems contradicted by simulations of some catalysis models \[21, 22\]. But as we pointed out already, systematic errors are often underestimated, and recent simulations of these and similar models support the conjecture \[23, 24, 25, 26\] (as explained below, we believe that violations of universality observed in some of the latter papers for ‘dynamic’ properties should be disregarded).

In this note we propose that there is a rather large and well studied class of transitions which are exactly of the latter type, and which are thus all in the DP class. These are so called “damaging” transitions. In these models one considers two replicas of a stochastic spin system, and lets them evolve with identical realizations of the stochastic noise. The initial conditions can be either completely independent, or one can start with two states which are identical except for a single spin. This single flip is considered as a “damage”, and the question is whether this damage will finally heal, so that both replicas converge towards identical states — or whether it will spread. If the two states are uncorrelated initially, the transition is between a situation where their rescaled Hamming distance (= density of damaged sites) stays finite and one where this distance goes to zero.

More precisely, we propose that such damaging transitions are in the DP class if they do not coincide with another transition (since then there would be long range correlations in the absorbing state), and if there is no frozen randomness. The former applies to the 2-d Ising model with Glauber dynamics, since there the damaging transition coincides with the ordinary critical point \[27\] (the situation is less clear in 3 dimensions \[28, 29\]). Frozen randomness is involved in damaging in spin glasses \[30, 31, 32\] and in the extensive studies of damaging in Kauffman models \[33, 34, 35\]. This should be in the same universality class as DP with frozen randomness \[36\], but for the Kauffman models there is a further complication: there a damage typically does not heal completely, whence the ‘dead’ state is not absorbing in our sense. We might mention that it was already pointed out that damaging in the annealed Kauffman model is in the DP class \[34\], but this is
much more trivial than our present claim. The annealed model can be mapped exactly onto DP, which is not the case in general.

We support our claim with simulations of damaging in the Domany-Kinzel cellular automaton (CA) \[37\]. This is a CA with one space and one time dimension, and with two states per site: \( s_i = 0, 1 \). Dynamics is defined by the following rule involving two real parameters \( p_1 \) and \( p_2 \) (we make a trivial modification which slightly simplifies the simulation):

(i) if \( s_i = 0 \) and \( s_{i+1} = 0 \) then \( s'_i = 0 \)
(ii) if \( s_i \) XOR \( s_{i+1} = 1 \) then \( s'_i = 1 \) with probability \( p_1 \) and \( s'_i = 0 \) with probability \( 1 - p_1 \)
(iii) if \( s_i \) AND \( s_{i+1} = 1 \) then \( s'_i = 1 \) with probability \( p_2 \) and \( s'_i = 0 \) with probability \( 1 - p_2 \).

For \( p_1 < 1/2 \) and any \( p_2 < 1 \), it is obvious that any state will converge towards the dead state \( \ldots 000 \ldots \). Actually, this state is an attractor for all values of \( p_1 \) below a critical curve \( p_1^c(p_2) \). This curve is indicated as curve \( C \) in fig.1. To the right of \( C \), one has an active state (the dead state still is stationary, but it no longer attracts all initial states) with \( \rho \equiv \langle s_i \rangle > 0 \).

The above conjecture suggests that the transition all along \( C \) is in the DP class, except at its upper limit point \((p_1, p_2) = (1/2, 1)\) where the model is a discrete time variant of the exactly solvable voter model \[1\] ("compact directed percolation" \[38\]). This is supported by all numerical evidence \[39, 40\] (except for a renormalization group analysis and Monte Carlo simulations presented in \[41\]; in high precision Monte Carlo simulations \[12\] we could not confirm these claims). In particular, bond and site DP correspond to \( p_2 = (2 - p_1)p_1 \) and \( p_1 = p_2 \), respectively.

It was found recently in \[42\] that the active phase can be further subdivided into a phase in which damage does not spread ("healing active phase") and one where it does ("chaotic"). The transition between these two phases is indicated by curve \( D \) in fig.1. It corresponds to \( p_1 = p_1^d(p_2) \) where \( p_1^d > p_1^c \) for all \( p_2 > 0 \), while \( p_1^d(0) = p_1^c(0) \) \[43\]. Indeed, one can consider two different variants of the damaging process: in the first one uses different random numbers when applying rules (ii) and (iii) above, in the second one uses the same. Curve \( D \) is computed with the second variant. The first variant would give a different curve slightly to the left of \( D \).

As pointed out in \[10\], one can describe a pair of replicas by an extended phase space with 4 states per site: \((00)\), \((01)\), \((10)\) and \((11)\). Damage spreading corresponds to the (directed) percolation of states \((10)\) and \((01)\), while any state with \((00)\) and \((11)\) only is healed. Since \( p_1^d > p_1^c \) for all \( p_2 > 0 \), the healing state has positive entropy at the damage spreading transition, and it does not immediately

\[\footnote{The very existence of these two variants shows that it is misleading to speak of different phases in the Domany-Kinzel CA, as done in \[12\]. Instead these are different phases for very specific algorithms for simulating pairs of such automata.} \]
follow from the conjecture of [13, 14] that this transition is in the DP universality class.

To check our conjecture that it is in this class nevertheless, we performed extensive simulations at $p_1 = 1$ where both variants coincide. Less extensive runs were made at several other values of $p_1$, where we studied both variants.

We worked on lattices of length $L$ with periodic boundary conditions. To speed up our simulations, we simulated 64 lattices simultaneously (we worked on machines with 64 bit words) by assigning the $k$-th bit of the $i$-th word in an integer array of length $L$ to the spin $s_i^{(k)}$ in the $k$-th lattice. The dynamics is then easily implemented by standard bit operations.

To measure the degree of damage in simulations which start with independent random initial configurations (thus with half of the sites damaged initially), we count the number of set bits in each word. If this number is $n_i$ for the $i$-th word, then the number of pairs of lattices which are damaged at site $i$ is $(64 - n_i)n_i$. The sum of Hamming distances between all $64 \times 63/2 = 2016$ pairs of lattices is thus

$$d = \sum_{i=1}^{L} (64 - n_i)n_i.$$  \hspace{1cm} (1)

For simulations with initial single site damage, this is not possible since we cannot build an initial state in which each pair is damaged at only one site. Instead, we introduced single site damage only between successive bits, i.e. we initially damaged the $(2k+1)$-st bit ($k = 0, 1, 2, \ldots 31$) in 32 different words, and counted how often the $(2k+1)$-st bit differed from the $(2k)$-th one.

Results from runs with random and independent initial states on lattices of size $L = 2^{22}$ are presented in fig.2. There we show the total damage as function of time, for $p_1 = 1$ and several values of $p_2$. At the critical point we expect an algebraic decay, corresponding to a straight line in fig.2. If the transition is in the DP class, this decay is governed by an exponent $\delta = 0.1596 \pm 0.0001$ \cite{44, 45, 46}. We see indeed a nearly perfect straight line for $p_2 \approx 0.3122$. Together with the data described below this gives our estimate

$$p_2^d = 0.31215 \pm 0.00004 \quad \text{for} \quad p_1 = 1,$$  \hspace{1cm} (2)

and the exponent extracted from it ($0.157 \pm 0.002$) is in good agreement with DP. Similar results (although somewhat less precise) were obtained for both variants of the damage spreading at several values of $p_1$. For $p_1 = 0.85$, e.g., they gave $p_2^d = 0.1957 \pm 0.0002$ (variant 1) resp. $p_2^d = 0.1400 \pm 0.0002$ (variant 2). In all cases $\delta$ was compatible with the DP value.

It is well known from studies of DP that the exponent $\beta$ defined by

$$d \sim (p_2^d - p_2)^\beta$$  \hspace{1cm} (3)
is not easily measured precisely due to the very long transients close to the critical point (i.e., due to the smallness of $\delta$) and due to finite size effects. The latter are absent in our simulations due to the very large lattice size. Nevertheless, extrapolating the data from fig.2 to $t \to \infty$ gave only a crude estimate $\beta = 0.272 \pm 0.006$ which is however in perfect agreement with DP where $\beta = 0.2766 \pm 0.0003$.

In order to measure an independent critical exponent, we made in addition runs with initial single site damage on much smaller lattices ($L \leq 7000$) and for shorter times ($t \leq 40000$). Apart from the damaged sites, the initial configurations were randomly chosen active states (they were set to the final configuration of the first lattice in the preceding run by setting the $i$-th word to 0 if $s_i^{(1)} = 0$, and to -1 if $s_i^{(1)} = 1$). From universality with DP we expect that at the critical point

$$d \sim t^\eta, \quad \eta = 0.314 \pm 0.001,$$

which is nicely fulfilled. Off the critical point we should have

$$\langle d \rangle \propto t^{-\delta}(p_2^d - p_2)t^{1/\nu|\rangle} \quad \text{(full initial damage)}$$

and

$$\langle d \rangle \propto t^{\eta}(p_2^d - p_2)t^{1/\nu|\rangle} \quad \text{(single site initial damage)}$$

with universal scaling functions $\phi(z)$ and $\psi(z)$ which are regular at $z = 0$, and with $\nu|\rangle = 1.7336 \pm 0.0005$. To see that our data are fully consistent with this, we plotted in fig.4 $d/t^\eta$ against $(p_2^d - p_2)t^{1/\nu|\rangle}$ for both types of initial conditions. We see indeed a perfect data collapse as predicted by the above ansatz. We just mention that similar results (again with somewhat smaller statistics and with significantly larger corrections to scaling) were obtained for several other values of $p_1$, and allowed us to locate curve $\mathcal{D}$ in fig.1 with high precision.

In conclusion we have given numerical evidence that the damage spreading transition in the Domany-Kinzel CA is in the DP universality class, although the undamaged state has positive entropy. We expect this to be true in general, not only for the Domany-Kinzel CA.

Of course, we have to set initial conditions such that we are not confined to atypical states carrying zero measure. In the present case, such atypical behavior would e.g. result if we would start with one of the configurations being dead (all $s_i = 0$) or nearly dead ($s_i \neq 0$ only in a finite region). In the latter case, we would then have a linear increase of $d$ instead of eq.(4). We believe that not taking into account this caveat is the reason why only partial universality with DP was observed in [24, 25]. In these papers, ‘dynamical’ simulations were done where the active region was bounded and expanding with time. Outside this region the configurations were not allowed to evolve, but were (artificially) kept in atypical states. It seems trivial that this modification of the model can lead to violations of universality.
Unfortunately, our conjecture does not immediately apply to the case of Kauffman automata [33] where damage spreading had been studied quite intensively [34, 35]. First of all, these models involve frozen randomness and should thus — if at all — be compared to DP in disordered media. Secondly, healing is not perfect in Kauffman models even in the phase in which damage does not spread. In this phase a finite damage has a non-zero probability to persist forever, and the healed state is not absorbing in our sense. It would be most interesting to study modified Kauffman models where such healing takes place (e.g. stochastic versions — apart from the randomness in the attribution of local rules, Kauffman models are strictly deterministic), and to compare them with DP in disordered media.

We have added one more item to the already long list of possible physical realizations of the DP transition. It is vexing that in spite of this ubiquity in models, and in spite of its conceptual simplicity (DP is by far the simplest critical phenomenon to study on a computer and to explain to a high school student), there have not yet been reported any experiments where the critical behavior of DP was observed even crudely! Maybe the present realization can lead to such an observation.

On the more practical side, we have introduced a new and very efficient method for simulating damage spreading which might find applications also in similar problems.

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Figure Captions:

**Fig.1:** Part of the phase diagram for the Domany-Kinzel CA. Curve $\mathcal{C}$ separates dead (left) from active (right) phases. Curve $\mathcal{D}$ (which joins $\mathcal{C}$ at its lower end point, but runs otherwise entirely in the active phase) separates a healing phase (left) from a chaotic phase (right) where any damage has non-zero probability not to heal. Here the damaging is implemented according to the first variant described in the text. With the second variant, $\mathcal{D}$ would be somewhat further to the left. The transition curves were determined by runs with single active/damaged initial sites, and demanding that the exponent $\eta$ (see eq.(4)) has the value of DP. The precision of the curves is everywhere better than the thickness of the lines. Quantitatively, the phase diagram agrees with data from [43], but not with the diagram given in [42]. It also deviates significantly from that in [47].

**Fig.2:** Log-log plot of the total number of damaged sites in 2016 pairs of lattices, with $2^{22}$ sites each. For all curves $p_1 = 1$, while $p_2$ ranges from 0.3086 to 0.3155 (from top to bottom). Initial configurations were random.

**Fig.3:** Log-log plot of the total number of damaged sites in runs where each pair of lattices was initially damaged at a single site. Again $p_1 = 1$. The four central values of $p_2$ are the same as in fig.2.

**Fig.4:** The same data as in figs.2 (panel a) and 3 (panel b), but plotted such that all data should collapse onto a single curves if eqs.(5,6) are correct. Only data for $t > 40$ are plotted in panel a, and for $t > 10$ in panel b, in order to reduced finite-time corrections.