Theoretical study of $d$-wave superconductivity in doped bilayer graphene

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We introduce a microscopic model on the honeycomb bilayer, which in the small-momentum limit captures the usual description of bilayer graphene. Due to assumed strong interlayer hopping it reduces to an effective honeycomb monolayer model with also third neighbor hopping. We study interaction effects in this effective model focusing on possible superconducting instabilities. We find $d_{x^2−y^2}$ superconductivity in the strong coupling limit of an effective $tJ$-model-like description that gradually transforms into $d+id$ time reversal symmetry breaking superconductivity at weak couplings. In this limit the small momentum order parameter expansion is $(k_x + ik_y)^2$ or $(k_x − ik_y)^2$ at both points (valleys) of effective, low-energy description. The relevance of our model and investigation for the physics of bilayer graphene is also discussed.

I. INTRODUCTION

Interaction effects are expected to be important for the physics of bilayer graphene and may cause a formation of correlated many-body phases.12 This needs to be contrasted to intrinsic monolayer graphene in which a vanishing density of states at the Dirac points suppresses the influence of electronic correlations.23 Recent experiments on suspended bilayer graphene12 reveal a gapped state at and around the charge neutrality point. These states may be topological3 due to the observed vanishing density of states at the Dirac points suppressing of the change in the superconducting order parameter. For experiments that point to superconducting instabilities, especially with an eye on the possibility of topological (fully gapped) superconductivity on the honeycomb bilayer. This study is motivated by a phenomenology similar to topological insulators and topological superconductors, where all transport is done via the edge states. The bilayer graphene may be also viewed as a strongly-correlated system with a possibility to support a layered antiferromagnetic state13–14 similar to the Mott physics of high $T_c$ superconductors. The existence of a layered antiferromagnetic state is supported by the most recent experiment with high quality samples1 which feature completely insulating behavior at the charge neutrality point.

There is, so far, no systematic study of superconducting instabilities in the presence of electron-electron and electron-phonon interactions on the honeycomb bilayer at finite doping (see, however, Ref. 21 for fermions in the presence of weak electron-electron interactions only at zero chemical potential). To address this question, we study in the present paper a microscopic model of a single effective honeycomb monolayer with reduced nearest neighbor hopping and third-neighbor hopping, in addition to inter-site attractive interactions. The kinetic term of the effective model is obtained by integrating out the “high-energy” degrees of freedom from the direct inter-layer hopping, and the inter-site superexchange interaction originates from the Hubbard on-site repulsion. This model is to a certain degree biased to antiferromagnetism and $d$-wave superconductivity, but preserves the usual low-energy description of the bilayer graphene and accounts moreover for the lattice symmetry of the original model (the honeycomb bilayer) that may be relevant for the symmetry of the superconducting order parameters.

Our primary interest is to find the most probable symmetry of a superconducting instability on the honeycomb bilayer. Furthermore, we aim at an understanding of the change in the superconducting order parameter and correlations as we go from a monolayer to a few-layer honeycomb lattice. This may be of relevance for experiments that point to superconducting correlations in graphite and graphite compounds (see Ref. 22 and references therein). The mean-field solution of the introduced model yields a time-reversal symmetry breaking $d+id$-wave superconducting state at weak coupling, which quickly but continuously transforms into $d_{x^2−y^2}$-wave with increasing interaction. Near 3/8 and 5/8 filling of the $\pi$-bands, i.e. near the van-Hove singularity in
the density of states, the Cooper pairing becomes much stronger. Our conclusion is that the $d + id$ superconducting instability may be present in the bilayer graphene at finite doping. However, due to the presumed smallness of coupling constant and order parameter, as well as strong quantum fluctuations in two dimensions, it may be difficult to detect this order experimentally in today’s graphene samples.

The remaining part of the paper is organized as follows. In Sec. II we define our effective two-band model on an effective honeycomb lattice with third-nearest-neighbor hopping and discuss its validity. The model is then, in Sec. III, solved by a Bogoliubov - de Gennes (BdG) transformation for a singlet bond-pairing order parameter, and we discuss the relevant symmetries. Section IV presents the phase diagram obtained from a numerical solution of the BdG equations. In Sec. V, the relevance for the physics of the bilayer graphene is discussed, and our main conclusions are presented in Sec. VI. Two Appendices summarize analytically obtained solutions in the weak-coupling BCS limit.

II. MODEL

The honeycomb bilayer lattice consists of two Bernal-stacked honeycomb lattices, each consisting of two triangular sublattices as illustrated in Fig. 1 such that the unit cell contains four lattice sites. The Hamiltonian of free electrons on such a lattice is given by

$$H_0 = -t \sum_{\vec{\imath},\vec{\jmath},\sigma} \left( a_{\vec{\imath},\vec{\jmath},\sigma}^\dagger b_{\vec{\imath},\vec{\jmath}+\vec{u},\sigma} + b_{\vec{\imath},\vec{\jmath}+\vec{v},\sigma}^\dagger a_{\vec{\imath},\vec{\jmath}+\vec{w},\sigma} + H.c. \right)$$

$$-t_\perp \sum_{\vec{\jmath},\sigma} \left( a_{\vec{\imath},\vec{\jmath},\sigma}^\dagger a_{\vec{\imath}+\vec{v},\sigma}^\dagger a_{\vec{\imath},\vec{\jmath}+\vec{w},\sigma} a_{\vec{\imath},\vec{\jmath}+\vec{u},\sigma} + H.c. \right)$$

$$-\mu \sum_{\vec{\imath},\vec{\jmath}} \left( a_{\vec{\imath},\vec{\jmath},\sigma}^\dagger a_{\vec{\imath}+\vec{v},\sigma}^\dagger a_{\vec{\imath},\vec{\jmath}+\vec{w},\sigma} a_{\vec{\imath},\vec{\jmath}+\vec{u},\sigma} + b_{\vec{\imath},\vec{\jmath},\sigma}^\dagger b_{\vec{\imath}+\vec{v},\sigma}^\dagger b_{\vec{\imath},\vec{\jmath}+\vec{w},\sigma} b_{\vec{\imath},\vec{\jmath}+\vec{u},\sigma} \right). \quad (1)$$

Here, the index $i = 1, 2$ denotes the layer and $\vec{\imath}$ enumerates primitive cells. The sum runs over $\vec{u} = \vec{u}_0, \vec{u}_1, \vec{u}_2$, where $\vec{u}_1 = \frac{3}{2}, \sqrt{3}a$ and $\vec{u}_2 = \frac{3}{2}, -\sqrt{3}a$ are the primitive vectors of the lattice, and $\vec{u}_0 = (0, 0)$ is an auxiliary vector for denoting the hopping between sites in the same primitive cell. The norm of these vectors is $|\vec{u}| = \sqrt{3}a$, in terms of the distance $a = 0.142$ nm between neighboring carbon atoms in each layer, and $t \simeq 3$ eV is the associated hopping energy, whereas $t_\perp \simeq 0.4$ eV denotes the interlayer hopping energy, between A sites in two different layers. Although the hopping energy between adjacent sites $B_1$ and $B_2$ in different layers is estimated$^{[3]}$ as $\sim 0.3$ eV and thus on the same order of magnitude as $t_\perp$, it yields a fine structure of the dispersion relation (in the form of trigonal warping and additional Dirac points with linear dispersion) that is only relevant at very low energies $\lesssim 10$ meV$^{[20]}$. In the present paper, we neglect this fine structure by setting the hopping between the $B_1$ and $B_2$ sublattices to zero. The operators $a_{i,\vec{\imath},\sigma}^\dagger, a_{i,\vec{\imath},\sigma}$ represent electron creation (annihilation) on the sublattice site $A_i$ of the layer $i$ with spin $\sigma = \uparrow, \downarrow$, and $b_{i,\vec{\imath},\sigma}^\dagger, b_{i,\vec{\imath},\sigma}$ those for electrons on the sublattice site $B_i$. $\mu$ is the chemical potential. We use units such that $\hbar = 1$.

By introducing the Fourier transforms

$$a_{i,\vec{k},\sigma} = \sum_{\vec{\imath}} a_{i,\vec{\imath},\sigma} \exp(i\vec{k} \cdot \vec{\imath}), \quad b_{i,\vec{k},\sigma} = \sum_{\vec{\imath}} b_{i,\vec{\imath},\sigma} \exp(i\vec{k} \cdot \vec{\imath}),$$

and diagonalizing the Hamiltonian, one obtains the spectrum

$$E_\alpha(\vec{k}) = \pm \left[ ( -1)^\alpha \frac{t_\perp}{2} + \sqrt{\frac{t_\parallel^2}{4} + t^2|\gamma_\vec{k}|^2} \right], \quad (2)$$

where $\alpha = 1, 2$ and $\pm$ denote 4 different branches of dispersion and

$$\gamma_\vec{k} = \sum_{\vec{\imath}} e^{i\vec{k} \cdot \vec{\imath}} = 1 + e^{i\vec{k} \cdot \vec{u}_1} + e^{i\vec{k} \cdot \vec{u}_2}. \quad (3)$$

In their orginal work$^{[20]}$, McCann and Fal’ko showed that the four-band model may be simplified to an effective two-band model if one considers energies much smaller than $t_\perp$. In momentum space, the Hamiltonian

![Image](b)
in Eq. 1 becomes
\[ H_0 = \sum_\sigma \int_{\text{BZ}} \frac{d^2 k}{(2\pi)^2} \left\{ -t \left( \gamma_{\perp 1,\sigma,\bar{k}} b_{1,\sigma,\bar{k}} \right) + \gamma_{\parallel 1,\sigma,\bar{k}} a_{1,\sigma,\bar{k}} + H.c. \right\} \]
\[ \left\{ -t_\perp a_{1,\sigma,\bar{k}} a_{2,\sigma,\bar{k}} + H.c. \right\} \]
\[ -\mu \left( a_{1,\sigma,\bar{k}}^\dagger a_{1,\sigma,\bar{k}} + a_{1,\sigma,\bar{k}}^\dagger a_{2,\sigma,\bar{k}} \right) \]
\[ + \left( b_{1,\sigma,\bar{k}}^\dagger b_{1,\sigma,\bar{k}} + b_{2,\sigma,\bar{k}}^\dagger b_{2,\sigma,\bar{k}} \right) \} . \]
\[ (5) \]
If we introduce the spinor
\[ \Psi_\sigma (\bar{k}) = (a_{1,\sigma,\bar{k}}, a_{2,\sigma,\bar{k}}, b_{1,\sigma,\bar{k}}, b_{1,\sigma,\bar{k}}) , \]
the Hamiltonian can be expressed as a 4 x 4 matrix,
\[ H_{00} (\bar{k}) = \sum_\sigma \Psi_\sigma^\dagger (\bar{k}) \left[ \begin{array}{cccc} -\mu & t_\perp & 0 & -t_\parallel \gamma_{\perp} \\ -t_\perp & -\mu & -t_\parallel \gamma_{\perp} & 0 \\ 0 & -t_\parallel \gamma_{\perp} & -\mu & 0 \\ -t_\parallel \gamma_{\perp} & 0 & 0 & -\mu \end{array} \right] \Psi_\sigma (\bar{k}) , \]
\[ (6) \]
One may further define 2 x 2 matrices \( H_{11} = -\mu I + t_\perp \sigma \), \( H_{22} = -\mu I \), \( H_{12} = -t (\Re \gamma_{\parallel} \sigma_x + \Im \gamma_{\parallel} \sigma_y ) \), \( H_{21} \), such that the eigenvalue equation can be written in the following form (\( k \) indices are implied)
\[ \left[ \begin{array}{cc} H_{11} & H_{12} \\ H_{21} & H_{22} \end{array} \right] \left[ \begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right] = E \left[ \begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right] , \]
\[ (7) \]
from which we obtain
\[ \{ H_{22} - H_{21} (H_{11} - E)^{-1} H_{12} \} \Psi_2 = E \Psi_2 . \]
\[ (8) \]
If we assume \( t_\perp \) to be the largest energy scale (i.e. \( t_\perp \gg \mu \) ) and apply the low energy limit (\( E \rightarrow 0 \) ), Eq. 6 becomes
\[ H_{\text{eff}} \Psi_2 \equiv \left[ \begin{array}{cc} -\mu & -t_\parallel \gamma_{\perp} \\ -t_\parallel \gamma_{\perp} & -\mu \end{array} \right] \Psi_2 = E \Psi_2 , \]
\[ (9) \]
with \( \Psi_2 (\bar{k}) = (b_{1,\sigma,\bar{k}}, b_{1,\sigma,\bar{k}}) \).

As mentioned above, the two-band model described by the Hamiltonian in Eq. 10, is valid in the limit where \( E \ll t_\perp \ll t \). For energies larger than \( t_\perp \), one needs to take into account the other two bands which overlap in energy with those considered in Eq. 10. If one takes into account trigonal warping at very low energies, the model is therefore valid in an energy window \( 10 \text{meV} \lesssim E \lesssim 400 \text{meV} \), where the dispersion is essentially isotropic and parabolic in the vicinity of the K-points, \( K_\pm = 4\pi \gamma_{\parallel} / 3 \sqrt{3} a \). In the following sections, however, we are interested in superconducting order parameters that account for the underlying lattice (point) symmetry beyond the parabolic approximation, and we use the simplified two-band model at even larger energies, up to the van-Hove singularity at \( \sim 3 \text{eV} \). Formally, this amounts to increasing artificially the interlayer hopping \( t_\perp \) such that it becomes the largest energy scale, \( t_\perp \gg t \).

The Hamiltonian in Eq. 10 corresponds, in real space, to a single-layer honeycomb lattice with nearest-neighbor and third-neighbor hoppings. Whereas the effective hopping amplitude of the latter is given by \( t_2^2 / t_\perp \), the effective nearest-neighbor hopping is twice as large \( \frac{2t}{t_\perp} \).

This means that due to the strong interlayer hopping, the complete low-energy physics is projected onto the B1 and B2 sublattices which themselves form a hexagonal lattice (see Fig. 1). As mentioned above, the model is naturally valid in the small-momentum limit, i.e. for \( \frac{t_2^2 / t_\perp | \bar{k} |^2}{t_\perp} \ll 1 \) and reproduces correctly the finite density of states (DOS) at \( E = 0 \) of bilayer graphene [Fig. 2].

Since we consider the effective hopping \( t_2^2 / t_\perp \) to be small and if there is a significant on-site repulsion \( U \), spin-singlet bonds between B1 and B2 sites are expected to form due to superexchange processes. Therefore, we apply the \( t - J \) model but relax the requirement of the model that double occupation of sites is excluded. As we will be working in the mean-field approximation, we just assume an effective nearest-neighbor attractive interaction between electrons on B1 and B2 sublattices, and by doing this we favor spin-singlet bond formation. If this interaction is not too strong, it can be simply added to our Hamiltonian, with the help of the term
\[ H_I = -J \sum_{j, \bar{u}} \sum_\sigma \left( a_{1,\sigma,\bar{k}} b_{1,\sigma,\bar{u}} b_{2,\sigma,\bar{u}+\bar{a}} - a_{2,\sigma,\bar{u}} b_{2,\sigma,\bar{u}+\bar{a}} - a_{1,\sigma,\bar{k}} \right) , \]
\[ (10) \]
where \( J > 0 \). Now we apply the BCS ansatz by introducing the superconducting order parameter as a 3 component complex vector
\[ \Delta \equiv (\Delta_{\tilde{n}0}, \Delta_{\tilde{n}1}, \Delta_{\tilde{n}2}) \]
apply the following transformation that diagonalizes the kinetic part of the above Hamiltonian,
\[
\begin{bmatrix}
b_{\gamma,\bar{\alpha}\sigma} \\
b_{\gamma,\alpha\sigma}
\end{bmatrix}
= \frac{1}{\sqrt{3}} \begin{bmatrix}
\delta_{\gamma,\bar{\alpha}\sigma} + c_{\gamma\sigma}
\
\delta_{\gamma,\alpha\sigma} - c_{\gamma\sigma}
\end{bmatrix},
\]
where \(\varphi_{\bar{K}} = \arg(\gamma_{\bar{K}})\).
In this basis, where \(c_{\gamma\sigma}\) and \(\delta_{\gamma\sigma}\) represent the electron states in the upper and lower band, respectively, the Hamiltonian transforms into
\[
H = \sum_{\bar{K}} \left\{ \sum_{\sigma} (\bar{\epsilon}_{\bar{K}} - \mu) c_{\bar{K}\sigma}^\dagger c_{\bar{K}\sigma} + \sum_{\sigma} (-\bar{\epsilon}_{\bar{K}} - \mu) d_{\bar{K}\sigma}^\dagger d_{\bar{K}\sigma}
+ \sqrt{2J} \left[ \sum_{\bar{u}} \Delta_{\bar{u}} \cos(\bar{K} \cdot \bar{u} - 2\varphi_{\bar{K}})(d_{\bar{K}\uparrow,\bar{u}}^\dagger c_{\bar{K}\downarrow}^\dagger - c_{\bar{K}\uparrow}^\dagger d_{\bar{K}\downarrow}^\dagger)
+ \sum_{\bar{u}} i\Delta_{\bar{u}} \sin(\bar{K} \cdot \bar{u} - 2\varphi_{\bar{K}})(c_{\bar{K}\uparrow}^\dagger d_{\bar{K}\downarrow}^\dagger - d_{\bar{K}\uparrow}^\dagger c_{\bar{K}\downarrow}^\dagger) \right] + \text{H.c.} \right\}.
\]
(16)

Here \(\bar{t} \equiv t^2/t_\perp\) and \(\epsilon_{\bar{K}} \equiv |\gamma_{\bar{K}}|^2\). The eigenvalues are given by
\[
E_{\bar{K}} = \pm \sqrt{(\bar{t}\epsilon_{\bar{K}})^2 + \mu^2 + 2J^2(|S_{\bar{K}}|^2 + |C_{\bar{K}}|^2) + 2\sqrt{A},
\]
where \(C_{\bar{K}} = \sum_{\bar{u}} \Delta_{\bar{u}} \sin(K \cdot u - 2\varphi_{\bar{K}}), \quad S_{\bar{K}} = \sum_{\bar{u}} \Delta_{\bar{u}} \cos(K \cdot u - 2\varphi_{\bar{K}})\) and
\[
A = (\mu^2 + 2J^2|S_{\bar{K}}|^2)^2 \bar{t}_{\bar{K}}^2 + 4J^4(Re C_{\bar{K}} Im S_{\bar{K}} - Im C_{\bar{K}} Re S_{\bar{K}})^2.
\]
(18)

If all \(\Delta_{\bar{u}}\) are purely real, i.e. there is no time-reversal symmetry breaking, then the second term in \(A\) is zero and the expression for the dispersion simplifies to
\[
E_{\bar{K}} = \pm \sqrt{(\bar{t}\epsilon_{\bar{K}} + \sqrt{\mu^2 + 2J^2|S_{\bar{K}}|^2})^2 + 2J^2C_{\bar{K}}^2}.
\]
(19)

In this case \(S_{\bar{K}}\) only renormalizes the chemical potential, whereas \(C_{\bar{K}}\) plays the main role in the description of the superconducting order parameter. A comparison between the Bogoliubov energy dispersion in Eq. (19) and the usual BCS expression shows that \(C_{\bar{K}}\) can be identified with the gap function. However, this name may be misleading because \(C_{\bar{K}}\) does not describe the gap, as in the example in Eq. (12) below.

The symmetry analysis of the order parameter on a honeycomb lattice yields the basis vectors which correspond to \(s, d_{x^2-y^2}\) and \(d_{xy}\) waves, respectively:
\[
\Delta = \begin{cases}
\Delta (1, 1, 1) \\
\Delta (2, -1, -1) \\
\Delta (0, 1, -1)
\end{cases}.
\]
(20)

The three symmetries are illustrated in Fig. 3 and the function \(C_{\bar{K}}\) corresponding to these symmetries is shown in Fig. 4 in comparison with the monolayer case. The

III. BOGOLIUBOV - DE Gennes ANALYSIS AND PAIRING SYMMETRIES

The complete BCS Hamiltonian in momentum space is given by
\[
H = -\frac{\hbar^2}{t_\perp} \sum_{k,\sigma} \left( \gamma_{k\sigma}^2 b_{2k,\sigma}^\dagger b_{1k,\sigma} + \text{h.c.} \right)
+ \sqrt{2J} \sum_{\bar{K}} \left[ \sum_{\bar{u}} \Delta_{\bar{u}} e^{i\bar{K} \cdot \bar{u}} (b_{2\bar{K},\uparrow}^\dagger b_{1\bar{K},\downarrow}^\dagger - b_{2\bar{K},\downarrow}^\dagger b_{1\bar{K},\uparrow}^\dagger) + \text{H.c.} \right]
- \mu \sum_{k,\sigma} \left( b_{1k,\sigma}^\dagger b_{1k,\sigma} + b_{2k,\sigma}^\dagger b_{2k,\sigma} \right).
\]
(14)

Similar to the case of the honeycomb monolayer, we can make our description much more transparent if we
last two possibilities belong to a two-dimensional subspace of irreducible representation of permutation group $S_3$. This means that any combination of these two order parameters is allowed by the symmetry.

In the case of an $s$-wave order parameter with $\Delta = (1,1,1)$, a small-wave-vector expansion ($|q|a \ll 1$) around the $K$-points yields

$$
C_{K^+ + q} \approx + \sqrt{3} q_y a \Delta, \quad S_{K^+ + q} \approx + \sqrt{3} q_x a \Delta.
$$

Thus both couplings are non-zero and no simple effective picture emerges by looking at the Hamiltonian in Eq. (16). The lower excitation energy branch can be approximated in the small-momentum limit as

$$
E_q \approx \sqrt{\mu^2 - 2\mu \epsilon_{K^+ + q} + \frac{3}{2} J^2 (|q|a)^2 \Delta^2}
$$

$$
\approx \sqrt{\mu^2 - \frac{3}{2} (3 \mu t - (J \Delta)^2)(|q|a)^2},
$$

where we have used $\epsilon_{K^+ + q} \approx 9(|q|a)^2/4$.

If the coupling strengths are such that $E_q$ has a minimum at $q = 0$, that is for $(J \Delta)^2 > 3 \mu t$, a superconducting instability may be realized. But as we pointed out in the introduction, an analysis based on the behavior in the region around $q = 0$ is not well justified for the honeycomb bilayer if triagonal warping (linearly dispersing electrons) is not taken into account. If we nonetheless proceed with analysis we will get a time-reversal invariant superconducting instability with two kinds of Cooper pairs with $p_x + ip_y$ and $p_x - ip_y$ pairings. Due to the forms of $C_{K}$ and $S_{K}$ in the above Hamiltonian in the small momentum limit, $p$-wave Cooper pairings are expected. For sufficiently large chemical potential [when we can neglect $S_{K}$ in Eq. (19)], the system may be unstable towards a $p_y$ gapless superconductor, with gap minima on the Fermi surface, i.e. on a circle.

For $\Delta = \Delta(2,-1,-1)$, the small-momentum expansion around the $K$-points yields

$$
C_{K^+ + q}(d_{x^2-y^2}) \approx -3 \frac{(q_y^2 - q_x^2)}{|q|^2} \Delta,
$$

$$
S_{K^+ + q}(d_{x^2-y^2}) \approx \mp 6 q_y q_x \frac{2}{|q|^2} \Delta \quad \text{(23)}
$$

and for $\Delta = \Delta(0,1,-1)$

$$
C_{K^+ + q}(d_{xy}) \approx 2 \sqrt{3} q_y q_x \frac{2}{|q|^2} \Delta,
$$

$$
S_{K^+ + q}(d_{xy}) \approx \mp \sqrt{3} \frac{2}{|q|^2} \Delta \quad \text{(24)}
$$

The gap function $C_{K}$ thus clearly shows the $d_{x^2-y^2}$ and the $d_{xy}$ symmetry in Eq. (23) and (24), respectively.

Notice that one may superpose two waves in the manner

$$
C_{K}(d \pm id) = C_{K}(d_{x^2-y^2}) \pm i \sqrt{3} C_{K}(d_{xy}),
$$

and

$$
S_{K}(d \pm id) = S_{K}(d_{x^2-y^2}) \pm i \sqrt{3} S_{K}(d_{xy}),
$$

which is identified with the $d + id$-wave superconducting phase in the following. In the small-wave-vector limit, the combined forms of $C_{K}$, $C_{K}(d \pm id) \approx \mp i S_{K}(d_{xy}) \approx 3 (q_x + iq_y)^2 / |q|^2 \quad \text{(27)}$

and

$$
C_{K^+}(d \pm id) \approx \pm i S_{K^+}(d_{xy}) \approx 3 (q_x - iq_y)^2 / |q|^2 \quad \text{(28)}
$$

restore the rotational symmetry – they are indeed eigenstates of rotation in two dimensions with the value of angular momentum equal to two. Only one form is allowed to exist in reality, and because it is the same irrespective of the valley $K$ or $K'$ one obtains a solution that spontaneously breaks time-reversal symmetry. Thus we can identify the solution with the broken time-reversal symmetry $d + id$ state. Something similar happens in the monolayer case, but the $d$-wave symmetry is recognized as a global dependence of the order parameter on the $K$ vector in the Brillouin zone around central $\Gamma$-point (see Ref. 23) and $p$-wave behavior around $K^\pm$ points. In the bilayer case the time-reversal symmetry breaking $d$-wave order parameter emerges as a property of the low-energy small-momentum effective description around the $K$ points, as shown above.

### IV. PHASE DIAGRAM

We have found the ground state of our model Hamiltonian for a broad range of $J$ and $\mu$ by minimizing the free
energy. At zero temperature, as a function of the order parameter, it is given by

$$F = - \sum_{\vec{k} \in \text{IBZ}} \sum_{\alpha = \pm 1} E_{\vec{k},\alpha}^f + 2N\mu \sum_{\vec{u}} |\Delta_{\vec{u}}|^2,$$  

(29)

where the first sum is over all wave vectors $\vec{k}$ in the first Brillouin zone and two Bogoliubov bands with positive energies. The ground state is defined as a global minimum of the free energy in the order parameter space. In the present study, we concentrate on superconducting order parameters in a variational approach, and thus we cannot exclude that other correlated (non-superconducting) phases may have an even lower energy.

In the mean-field approach, superconducting ground states are expected even for infinitesimal positive values of $J$.

The order parameter space is 6-dimensional, because it is defined by 3 complex numbers. However, adding the same phase to all three complex parameters does not modify the physical state, so one can always make one of the parameters purely real (we set $\Delta_{d_{x^2-y^2}}$ real) and reduce the order parameter space dimensionality to 5. We used the amoeba numerical method\cite{26} to directly minimize the free energy. Five-dimensional minimization often reveals more than one local minimum, but we were always able to identify the lowest-lying state to a satisfying level of certainty. However, for small values of $J$, the local free-energy minima are extremely shallow, with energies only slightly lower than the free energy of the normal state. Such features in the free-energy landscape are completely clouded by numerical noise due to the discretization of the first Brillouin zone. Our numerical calculations are therefore limited to higher values of $J$, which give a solution with the amplitude of the order parameter larger than $10^{-4}$. This is marked by a dashes line on Fig. 5 and a short discussion of this line in the framework of a weak-coupling analysis may be found below [Eq. (31)].

Our results are shown on Fig. 5 where the relevant quantities are represented by color in the $(\mu, J)$ plane. The amplitude of the order parameter is shown in Fig. 5(a). Upon small to moderate doping, the SC instability increases and becomes particularly favorable in Fig. 5(a). Upon small to moderate doping, the SC instability increases and becomes particularly favorable. The amplitude of the order parameter is shown in Fig. 5(a).

The contributions of different pairing symmetries are defined by the ratio $w$ of different components of $\Delta$, where

$$\Delta = \Delta_s \hat{e}_s + i \Delta_{xy} \hat{e}_{xy} + \Delta_{d_{x^2-y^2}} \hat{e}_{d_{x^2-y^2}} + i \Delta_{id_{xy}} \hat{e}_{id_{xy}},$$  

(30)

with $\hat{e}_s = (1,1,1)/\sqrt{3}$, $\hat{e}_{d_{x^2-y^2}} = (0,1,-1)/\sqrt{2}$, and $|\Delta_{id_{xy}}| = 2, -1, -1)/\sqrt{6}$. Fig. 5(c) shows the ratio $w(id_{xy}) = |\Delta_{id_{xy}}|/|\Delta_s|$, and Fig. 5(d) the ratio $w(s) = |\Delta_{d_{x^2-y^2}}|/|\Delta_s|$. The contributions of $s$ and $d_{xy}$ components are negligible in all cases, and $d_{x^2-y^2}$ is the dominant component.

The numerical results are, for clarity, also shown on Fig. 6 for three chosen values of the chemical potential, $\mu/t = 0.04, 0.55, 1$. Fig. 6(a) shows a sudden increase in the pairing amplitude with the increasing interaction $J$ (note the logarithmic scale on the $y$-axis). For small $J$, the pairing amplitude is much larger for $\mu/t = 1$.  

![FIG. 5. (Color online) (a) The order parameter amplitude, $\Delta$, in the $(\mu, J)$ parameter space, obtained by a minimization of the free energy, (b) the single-particle excitation gap, (c) the contribution of $id_{xy}$ and (d) $s$-wave component in the ground state order parameter. The green dashed line marks where $\Delta$ drops below $10^{-4}$. Below this line, our numerics is not reliable. We use $t = t^2/t_\perp$ for the unit of energy.](image)
and (b) the single-particle excitation gap as a function of $J$, for $\mu = 0.04, 0.55, 1$. (c)-(e) The contributions of 3 relevant symmetry components. $d_{x^2-y^2}$ component is the dominant one for large $J$. The contribution of $id_{xy}$ increases with decreasing $J$ until the two contributions are equal and we find a pure $d+i d$-wave symmetry. We use $t = t^*/t_\perp$ for the unit of energy.

i.e. at the van-Hove singularity, and in this case the single-particle excitation gap is also larger due the strong mixing of $d_{x^2-y^2}$ and $id_{xy}$ symmetries. Contributions of relevant components are compared in Figs. (c)-(e). At higher values of $J$ one has a pure $d_{x^2-y^2}$ symmetry, whereas a mixture of $d_{x^2-y^2}$ and $id_{xy}$ symmetries is found at lower values of $J$. The contribution of $id_{xy}$ symmetry increases with decreasing $J$ and almost pure $d+id$ symmetries are usually found at the lowest accessible values of $J$.

Our numerical calculations were performed on processors with 8GB of RAM which limited the number of $k$-points in the first Brillouin zone to $4000 \times 4000$, but we checked that results do not differ qualitatively even with a much sparser $2000 \times 2000$ $k$-grid. A much denser and probably a non-uniform discretization of the first Brillouin zone would be needed to probe the weak-coupling behavior of our model, that is for values of $J$ below the dashed lines in Fig. 5. Notice, however, that the system in the small-$J$ limit may be treated analytically within the weak-coupling limit the results of which are presented in Appendices A and B, for the cases of finite and zero chemical potential, respectively.

In this weak-coupling regime and at finite chemical potential, we find that the $d+i d$ superconducting order parameter yields the lowest mean-field energy, when compared to order parameters that respect time-reversal symmetry (Appendix A), in agreement with our numerical results for larger values of $J$. Furthermore, this behavior is also expected from a theorem discussed in Ref. 29 for the BCS description of a single band, when both, $d_{x^2-y^2}$ and $d_{xy}$, symmetries of the order parameter are allowed to exist. Although ours is a two-band (Bogoliubov) model we expect a superconducting instability at any strength of attractive interaction at finite doping since the gap opens as

$$J\Delta \propto \exp\left[-\frac{8\pi}{\sqrt{3}} \frac{1}{\rho(\mu)J}\right]$$

(31)

(see Appendix A), in terms of the DOS $\rho(E_F)$ at the Fermi level $E_F$. Notice, however, that the DOS is on the order of $1/t$ below the van-Hove singularity. Therefore, in order to resolve order parameters with a precision of $10^{-4}$, as in our numerical calculations, the minimal coupling $J/t$ should be on the order of unity, in rough agreement with the dashed line in Fig. 5. As already mentioned, the divergent DOS at the van-Hove singularity allows for a resolution of order parameters at even lower values of the coupling constant $J$. Equation (31), although derived in the weak-coupling limit, thus provides a qualitative understanding of the above-mentioned inverse-DOS behavior of the order parameter in Fig. 5.

Finally, we notice that the weak-coupling analysis yields a different picture at zero-doping (Appendix B), where a time-reversal-symmetric superconducting order parameter (with any real real combination of $d_{x^2-y^2}$ and $d_{xy}$) is energetically favored.

V. RELEVANCE FOR BILAYER GRAPHENE

In the following we discuss the relevance of our model for the physics of bilayer graphene. With an estimate for the Coulomb on-site repulsion, $U \sim 10$ eV, intralayer nearest-neighbor hopping $t_{\perp} \sim 3$ eV, and interlayer hopping $t_{\perp} \sim 0.4$ eV, bilayer graphene has a tendency to develop strongly-correlated electron phases. Notice that, although similar energy scales are found in monolayer graphene, the latter is to great accuracy described in terms of (quasi-)free electrons because of a vanishing DOS at the Fermi level, in the absence of intensive doping. On the contrary, electronic correlations are much more efficient in bilayer graphene as a consequence of the finite DOS even at the band-contact points. This finite DOS may also be invoked when considering screening. Whereas screening is highly inefficient in monolayer graphene, and one needs then to take into account the
long-range nature of the electronic interaction potential, the screening properties in bilayer graphene are similar to those in usual 2D electron systems with a parabolic band dispersion. In this sense, an approach based on the Hubbard model, as used here excluding nearest and further-neighbor interactions, is better justified in bilayer than in monolayer graphene. Though in this sense approximative our approach differs from other, mean-field and perturbative, approaches in the literature, which assume that interactions are weak despite their real magnitude. At half-filling we expect two antiferromagnetic (Heisenberg) interactions, intra, \( J \sim \frac{t^2}{U} \sim 1 \text{ eV} \), and inter, \( J_\perp \sim \frac{t_\perp^2}{U} \sim 16 \text{ meV} \). Although the origin of the antiferromagnetic behavior may be claimed due to the quadratic dispersion of juxtaposed conduction and valence band \( \rho^\perp \) (apart from triagonal warping at low energies) the magnitude of \( U \) points out to also a role of Mott physics. Upon doping, holes or electrons will move on an effective (projected monolayer) lattice because in that way less antiferromagnetic bonds will be broken. Thus low energy physics will take place on the effective (B sites) lattice. An estimate from perturbative considerations for the effective antiferromagnetic coupling of B sites is \( J_{\text{eff}} \sim \frac{t^2}{\frac{U}{2}} = \frac{t^2}{U} \sim 100 \text{ meV} \) at finite \( t_\perp \). If we rewrite the estimate for \( J_{\text{eff}} \) as \( J_{\text{eff}} \sim \frac{1}{\frac{U}{3}} \), we can identify the effective hopping between B1 and B2 sites to be \( t_{\text{eff}} = \frac{t^2}{U} = J \sim 1 \text{ eV} \). Therefore the strongly-correlated regime of the bilayer graphene may be continuously related to our model by simultaneously increasing \( t_\perp \) and decreasing \( U \) with \( t_{\text{eff}} \) becoming \( t_{\text{eff}} = \frac{t^2}{U} \), that is the hopping parameter \( \tilde{t} \) in our effective monolayer model. If the on-site interaction is very large and precludes the double occupancy we can expect a situation where doped holes or electrons are the only charge degrees of freedom and hop according \( \tilde{t} \sim \frac{t^2}{t_\perp} \) i.e. the hopping that characterize a non-interacting system (they do not pay the cost of \( U \) on their paths). We will assume that this is not the case in the graphene bilayer since \( U \) is not extremely large and keep our estimate of \( t_{\text{eff}} = \frac{t^2}{U} \sim 1 \text{ eV} \).

Therefore modeled with two effective parameters, \( J_{\text{eff}} \) and \( t_{\text{eff}} \), bilayer graphene may be compared with the effective honeycomb lattice considered in our paper and the corresponding \( t - J \) model. The main feature of bilayer graphene appears to be that \( J_{\text{eff}} \sim 0.1t_{\text{eff}} \ll t_{\text{eff}} \) and in considering the relevance of our model we should confine ourselves to weak couplings, and small or moderate dopings; because we simplified the high-momentum physics of the bilayer (by considering the large \( t_\perp \) limit) we should confine ourselves to lower dopings. First one sees from Fig. 3 that the gaps are in the meV range (2 to 5 meV for the maximal gaps) if one considers the energy scale \( t_{\text{eff}} \sim J \sim 1 \text{ eV} \). Thus our results indicate very small energy scales that are unlikely to be resolved in today’s graphene samples. Furthermore we should use \( t_{\text{eff}} \) and \( J_{\text{eff}} \) for \( t \) and \( J \) for the exponent in the weak-coupling analysis in the Appendix A. Because we estimate \( t_{\text{eff}}/J_{\text{eff}} \sim 10 \), the weak-coupling analysis yields an exponential suppression and gaps below 1 meV, in agreement with our numerical findings shown in Fig. 3.

VI. CONCLUSIONS

In conclusion, we have investigated the possible occurrence of \( d + id \) superconductivity in doped bilayer graphene. We have used an effective lattice model in which the low-energy electrons are allowed to hop on a honeycomb lattice that consists of the \( B \) sites of the two layers. This model reproduces the low-energy physical properties of bilayer graphene below \( t_\perp \sim 400 \text{ meV} \) and neglects the effect of higher-energy bands above this value by artificially increasing the value of \( t_\perp \), while maintaining the lattice point symmetry. This point symmetry happens to be important in fixing the symmetry of the superconducting order parameter when increasing the coupling constant \( J > 0 \) for a bond-singlet order within a \( t - J \) model that has been investigated in a mean-field approximation.

At low values of \( J \), a time-reversal symmetry-breaking \( d + id \) superconducting order may be stabilized, similarly to the proposed phases in monolayer\( ^{22,23,42} \) and highly doped monolayer graphene, such as to restore rotational symmetry. The time-reversal symmetry breaking in this phase is apparent in the effective (low-momentum) description \( \sim (k_x + ik_y)^2 \) at both valley points, \( K \) and \( K' \). Most saliently, the low-\( J \) \( d + id \)-wave superconducting order persists upon doping until the van-Hove singularity at 5/8 filling for electron doping (or 3/8 for hole doping). For larger values of \( J \), rotational symmetry is no longer respected because of the increased importance of lattice point symmetry. The anisotropic time-reversal symmetric \( d_{x^2−y^2} - w^2 \)-wave component then becomes increasingly important in the order parameter, such that more conventional \( d \)-wave superconductivity may be expected. One notices, however, that the superconducting energy scales, estimated both numerically and within a weak-coupling analysis, are very small (in the meV range), such that an experimental observation in presently available graphene samples is unlikely, and an important increase in sample quality is required. Similarly to monolayer graphene\( ^{59} \), one may though expect a renormalization flow to stronger coupling constants when approaching the van-Hove singularity in bilayer graphene by chemical doping.

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Appendix A: Weak-coupling analytical solution at finite chemical potential

Here, we present briefly the weak-coupling analysis of superconducting order in the effective bilayer model. In order to simplify the notation, we use the letter $t$ to denote the effective hopping $\tilde{t}$. The DOS at the Fermi level, $\rho(E_F)$, is on the order of the inverse hopping parameter $1/t$. Notice that, if only a parabolic band is taken into account it remains fixed at its $E_F = 0$ value, but corrections to the parabolic approximation immediately yield a contribution that varies linearly with the Fermi level, in agreement with the DOS plotted in Fig. 2(b).

In the case when $\Delta = \Delta(1, 1, 1)$, a weak-coupling BCS analysis that takes into account only electrons in the lower Bogoliubov band gives

$$J\Delta = \sqrt{2t}E_c \exp\left(-\frac{24\sqrt{3}\pi}{\mu \rho(E_F)}\right),$$

(A1)

with $E_c$ as an energy cut-off around the Fermi value, for the solution, and

$$\frac{\delta E_{MF}^c}{N} = -(J\Delta)^2 \frac{\mu \rho(E_F)}{t} \frac{1}{4\sqrt{3}\pi},$$

(A2)

for the gain in the mean-field energy, $\delta E_{MF}$, by the pairing instability.

The weak coupling BCS analysis in the case of electron doping ($\mu > 0$) for $d_x^2 - y^2$ and $d_{x^2 - y^2} + i\sqrt{3} d_{xy}$ gives

$$J\Delta_d = \frac{\sqrt{2}}{3} \frac{E_c}{\rho(E_F)} \exp\left(-\frac{8\pi}{\sqrt{3}} \frac{1}{\rho(E_F)} \frac{1}{2}\right),$$

(A3)

for the solution which we denoted by $\Delta = \Delta_d$, and

$$J\Delta = \sqrt{\frac{2}{3}} \frac{E_c}{\rho(E_F)} \exp\left(-\frac{8\pi}{\sqrt{3}} \frac{1}{\rho(E_F)} \frac{1}{2}\right),$$

(A4)

in the case of $d_{xy}$ wave. For the energy gain one obtains

$$\frac{\delta E_{MF}(d_x^2 - y^2)}{N} = \frac{\delta E_{MF}(d_{xy})}{N} = -(J\Delta_d)^2 \rho(E_F) \frac{3\sqrt{3}}{4\pi},$$

(A5)

and for a $d_{x^2 - y^2} + i\sqrt{3} d_{xy}$ wave, one finds

$$\frac{\delta E_{MF}^c}{N} = -(J\Delta_d)^2 \rho(E_F) \frac{3\sqrt{3}}{2\pi}.$$

(A6)

Because of its twice lower mean-field energy, the $d_x^2 - y^2 + i\sqrt{3} d_{xy}$ time-reversal symmetry breaking instability, which we call in short $d$-wave, is more likely than $d_x^2 - y^2$ and $d_{xy}$-wave order parameters. In the large-doping limit, the energy minimization is also much more efficient for $d$-wave than $p_y$ wave as seen in the small value of the ratio

$$\frac{\delta E_{MF}^c}{\delta E_{MF}^d} = \frac{\mu}{2E_c} \exp\left[-\frac{2\pi \times 8}{\sqrt{3}} \frac{1}{\rho(E_F)} \frac{9t}{2\mu} - 1\right],$$

(A7)

for $\mu < \frac{9t}{2}$. The most natural choice for $E_c$ is to be of the order of $\mu$ as a first energy scale when we start from the smallest one, i.e. $J$. The time-reversal symmetry breaking $d$-wave is the solution of our BCS mean-field Hamiltonian also due to the theorem proved in Ref. 29. In the case of weak coupling that we consider here, i.e. $J \ll \mu$, and $\mu > 0$ (electron doping), we have an effective one-band theory of upper-band electrons to which the theorem can be applied.

In the following we will look more closely into an effective, low-energy description of the $d$-wave instability in the case of high electron doping and discuss only the lower energy Bogoliubov band. Therefore our effective Hamiltonian is

$$H_e = \sum_{i\mathbf{k}\sigma}(t\epsilon_{\mathbf{k}} - \mu) c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}} \left(\Delta_{\mathbf{k}} c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{k}\downarrow}^\dagger + \text{H.c.}\right)$$

(A8)

where $\Delta_{\mathbf{k}} \sim (k_x - ik_y)^2/|k|^2$. In the weak-coupling BCS analysis it can be easily shown that the Hamiltonian is completely equivalent to the one with $\Delta_{\mathbf{k}} \sim (k_x - ik_y)^2$, because both Hamiltonians have an effective description on a Fermi circle defined by $t \epsilon_{\mathbf{k}} = \mu$. With this adjustment we have exactly the form of the BCS Hamiltonian studied in Ref. 33 on time-reversal symmetry breaking superconductors in two dimensions. In the so-called weak-pairing case for finite $\mu > 0$ that we want to study, the minimum of Bogoliubov excitations moves to finite values of $\mathbf{k}$, $t \epsilon_{\mathbf{k}} = \mu$, i.e. to the Fermi surface of free particles. The Cooper pair wave function $g(\mathbf{r})$ may be a non-universal function of $|\mathbf{r}|$ where $\mathbf{r}$ is the relative coordinate of the pair. On the other hand, the dependence of the function on the angle of vector $\mathbf{r}$ is fixed and can easily be derived in the Bogoliubov formalism to be $g(\mathbf{r}) \propto \frac{z}{\sqrt{2}} \propto (x - iy)^2$ where $z = x + iy$ is the two-dimensional complex coordinate. Thus the relative angular momentum of the Cooper pair is $l = -2$. The weak-pairing phase is topological, gapped in the bulk because $\mu > 0$, and possesses a doublet of spin $1/2$ Dirac edge modes.33 In our case, because of the fermion doublon on the honeycomb lattice and the existence of the two $\tilde{K}$ points (valleys) and because around each one we have the same effective description given by Hamiltonian in Eq. (16), we expect four Dirac modes on the edge.
Appendix B: Weak coupling analytical solution at zero chemical potential

In the weak coupling limit at $\mu = 0$, when both Bogoliubov bands are taken into account we find for $d_{x^2-y^2}$ symmetry

$$J\Delta^d = \frac{E_c}{3} \exp\left(-\frac{\delta - 11c}{2c}\right), \quad (B1)$$

with $c = \frac{1}{2\pi^2} \frac{e}{\hbar}$, for the solution, and

$$\frac{\delta E_{MF}^{d_{x^2-y^2}}}{N} = \frac{9}{2c} (J\Delta^d)^2, \quad (B2)$$

for the energy gain. On the other hand for $d + id$ symmetry we find

$$J\Delta^{d+id} = \sqrt{2E_c} \exp\left(-\frac{\delta - 5c}{2c}\right), \quad (B3)$$

and

$$\frac{\delta E_{MF}^{d+id}}{N} = -9c (J\Delta^{d+id})^2. \quad (B4)$$

Because

$$\frac{\delta E_{MF}^{d+id}}{\delta E_{MF}^{d_{x^2-y^2}}} = \frac{\delta E_{MF}^{d+id}}{\delta E_{MF}^{d_{xy}}} = 4e^{-6}, \quad (B5)$$

any real combination of $d_{x^2-y^2}$ and $d_{xy}$ waves is more likely than $d + id$ wave.

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1. A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, Rev. Mod. Phys. 81, 109 (2009).
2. V. N. Kotov, B. Uchoa, V. M. Peirera, A. H. Castro Neto, and F. Guinea, arXiv:1012.3484.
3. M. O. Goerbig, Rev. Mod. Phys. 83, 1193 (2011).
4. J. Martin, B. Feldman, T. Weitz, M. Allen, and A. Yacoby, Phys. Rev. Lett. 105, 256806 (2010).
5. R. T. Weitz, M. T. Allen, B. E. Feldman, J. Martin, and A. Yacoby, Science 330, 812 (2010).
6. F. Freitag, J. Trbovic, M. Weiss, and C. Schonenberger, arXiv:1104.3816.
7. J. Velasco Jr., L. Jing, W. Bao, Y. Lee, P. Kratz, V. Aji, M. Bockrath, C. N. Lau, C. Varma, R. Stillwell, D. Smirnov, F. Zhang, J. Jung, A.H. MacDonald, arXiv:1108.1609.
8. X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. 83, 1057 (2011).
9. H. Min, G. Borghi, M. Polini, and A. H. MacDonald, Phys. Rev. B 77, 041407(R) (2008).
10. R. Nandkishore and L. Levitov, Phys. Rev. Lett 104, 156803 (2010); Phys. Rev. B 82, 115124 (2010).
11. F. Zhang, H. Min, M. Polini, and A. H. MacDonald, Phys. Rev. B 81, 041402(R) (2010).
12. O. Vafek and K. Yang, Phys. Rev. B 81, 041401(R) (2010).
13. F. Zhang, J. Jung, G. A. Fiete, Q. Niu, and A. H. MacDonald, Phys. Rev. Lett. 106, 156801 (2011).
14. J. Jung, F. Zhang, and A. H. MacDonald, Phys. Rev. B 83, 115408 (2011).
15. Y. Lemonik, I. L. Aleiner, C. Toke, and V. I. Falko, Phys. Rev. B 82, 201408 (2010).
16. F. Zhang and A. H. MacDonald, arXiv:1107.4727.
17. R. E. Throckmorton and O. Vafek, arXiv:1111.2076.
18. M. Kharitonov, arXiv:1109.1553.
19. M. M. Scherer, S. Uebelacker, and C. Honercamp, arXiv:1112.5038.
20. E. McCann and V. I. Falco, Phys. Rev. Lett. 96, 086805 (2006).
21. O. Vafek, Phys. Rev. B 82, 205106 (2010).
22. A. M. Black-Schaffer and S. Doniach, Phys. Rev. B 75, 134512 (2007).
23. C. Bena and L. Simon, Phys. Rev. B 83, 115404 (2011).
24. D. Poletti, C. Miniatura, B. Gremaud, Euro. Phys. Lett. 93, 37008 (2011).
25. J. Linder, A. M. Black-Schaffer, T. Yokoyama, S. Doniach, and A. Sudbo, Phys. Rev. B 80, 094522 (2009).
26. D. Poletti, C. Miniatura, B. Gremaud, Euro. Phys. Lett. 93, 37008 (2011).
27. T. O. Wehling, E. Sasioglu, C. Friedrich, A. I. Lichtenstein, M. I. Katsnelson, and S. Blugel, Phys. Rev. Lett. 106, 236805 (2011).
28. K. Zou, X. Hong, and J. Zhu, Phys. Rev. B 84, 085408 (2011) and references therein.
29. M. Cheng, K. Sun, V. Galitski, and S. Das Sarma, Phys. Rev. B 81, 024504 (2010).
30. A. H. MacDonald, J. Jung, and F. Zhang, arXiv:1109.0307.
31. R. B. Laughlin, Phys. Rev. Lett. 80, 5188 (1998).
32. G. Baskaran, Phys. Rev. B 65, 212505 (2002).
33. C. Honerkamp, Phys. Rev. Lett. 100, 146404 (2008).
34. S. Pathak, V. B. Shenoy, and G. Baskaran, Phys. Rev. B 81, 085431 (2010).
35. Z.-C. Gu, H.-C. Jiang, D. N. Sheng, H. Yao, L. Balents, and X.-G. Wen, arXiv:1110.1183.
36. R. Nandkishore, L. Levitov, and A. Chubukov, Nature Physics 8, 158 (2012).
37. M. Kiesel, Ch. Platt, W. Hanke, D.A. Abanin, R. Thomale, arXiv:1109.2953.
38. N. Read and D. Green, Phys. Rev. B 61, 10267 (2000).