DISCRIMINANTS AND AUTOMORPHISM GROUPS
OF VERONESE SUBRINGS OF SKEW POLYNOMIAL RINGS

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Abstract. We study important invariants and properties of the Veronese sub-algebras of \( q \)-skew polynomial rings, including their discriminant, center and automorphism group, as well as cancellation property and the Tits alternative.

Introduction

The determination of the full automorphism group of an algebra is a fundamental problem in mathematics. This is generally extremely difficult, for example, even for the polynomial ring in three variables its automorphism group is not well understood. Aside from a remarkable result of Shestakov-Umirbaev [SU] which shows that the Nagata automorphism is a wild automorphism, the general structure of this automorphism group eludes our grasp.

Since the 1990s, researchers have successfully computed the full automorphism group of several interesting families of noncommutative algebras of finite Gelfand-Kirillov dimension, including certain quantum groups, generalized quantum Weyl algebras, skew polynomial rings – see [AlC, AlD, AnD, BJ, GTK, GY, LL, SAV]. A few years ago, by using a rigidity theorem for quantum tori, Yakimov proved the Andruskiewitsch-Dumas conjecture [Y1] and the Launois-Lenagan conjecture [Y2], each of which determines the full automorphism group of an important class of quantized algebras. Recently, Ceken-Palmieri-Wang and the third-named author introduced a discriminant method to control the automorphism group of certain classes of algebras [CPWZ1, CPWZ2] and then were able to compute the automorphism group of several more families of Artin-Schelter regular algebras that satisfy a polynomial identity. The wisdom behind much of this progress is that noncommutative algebras are more rigid, so that more techniques are available for detecting their symmetries.

In most of the results mentioned above, the algebras are deformations (in some weak sense) of the commutative polynomial rings. In this paper we apply the discriminant method to certain noncommutative algebras that are not deformations of polynomial rings. We are mainly interested in the automorphism problem, but will briefly touch upon the cancellation problem and the Tits alternative.

Throughout the introduction let \( k \) denote our base field and \( k^\times \) be the its group of units. For a \( k \)-algebra \( A \), let \( \text{Aut}(A) \) denote the group of \( k \)-algebra automorphisms of \( A \).

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Fix a $q \in \mathbb{k}^\times$. Let $\mathbb{k}_q[x_1, \cdots, x_n]$ denote the skew polynomial ring generated by $x_1, \cdots, x_n$ subject to the relations
\begin{equation}
(x_jx_i = q x_ix_j, \quad \forall 1 \leq i < j \leq n).
\end{equation}
(E0.0.1)

In this paper we assume that $q$ is a nontrivial root of unity. First we consider the case when $q = -1(\neq 1)$. By [CPWZ1, Theorem 4.10(1)], if $n$ is even, then
\begin{equation}
\text{Aut}(\mathbb{k}_q[x_1, \cdots, x_n]) = S_n \ltimes (\mathbb{k}^\times)^n
\end{equation}
(E0.0.2)
which is virtually abelian [Definition 0.6(1)]. On the other hand, if $n$ is odd, then [CPWZ3, Theorem 2] says that Aut($\mathbb{k}_q[x_1, \cdots, x_n]$) contains a free group on two generators. In the case of $q = -1$, these two results present a dichotomy depending on the parity of $n$. This is a version of the Tits alternative [Definition 0.6(3)].

For any $\mathbb{Z}$-graded algebra $A = \bigoplus_{i \in \mathbb{Z}} A_i$ and for any positive integer $v$, the $v$th Veronese subring of $A$ is defined to be
\[A^{(v)} := \bigoplus_{i \in \mathbb{Z}} A_{vi}.
\]
The following theorem generalizes the result [CPWZ1, Theorem 4.10(1)].

**Theorem 0.1.** Suppose that $\text{char } \mathbb{k} \neq 2$. Let $A$ be $\mathbb{k}_q[x_1, \cdots, x_n]^{(v)}$ where $v$ is a positive integer. If $n$ and $v$ have different parity, then $\text{Aut}(A) \cong S_n \ltimes (\mathbb{k}^\times)^n$.

Considering elements in $(\mathbb{k}^\times)^n$ as $(a_1, \cdots, a_n)$, the $S_n$-action on $(\mathbb{k}^\times)^n$ in Theorem 0.1 does not follow the standard rule
\begin{equation}
\sigma : (a_1, \cdots, a_n) \mapsto (a_{\sigma^{-1}(1)}, \cdots, a_{\sigma^{-1}(n)})
\end{equation}
for all $\sigma \in S_n$, due to asymmetry of the automorphisms corresponding to $(\mathbb{k}^\times)^n$, see Lemma 0.2 for some details. On the other hand, the $S_n$-action appearing in (E0.0.2) does follow the standard rule (E0.1.1).

If $n$ and $v$ have the same parity, we are unable to determine the automorphism group of $A$, but we conjecture that it contains a free subgroup of rank 2. Also in this case we are unable to decide whether or not $A$ is cancellative [Definition 0.3], see Theorem 0.4 and Question 0.5 below for related results and questions.

We can generalize the above theorem to the case when $q$ is arbitrary of finite order. Let $m$ be the order of $q$ and assume that $m$ is bigger than 2 (or equivalently, $q \neq \pm 1$). We have two different hypotheses dependent on the parity of $n$ in the following theorem.

**Theorem 0.2.** Let $A$ be $\mathbb{k}_q[x_1, \cdots, x_n]^{(v)}$ where $v$ is a positive integer and $m > 2$. Suppose that one of the following is true.

(a) $n$ is even and $m$ does not divide $v$.
(b) $n$ is odd and $\gcd(m, v) \neq 1$.

Then the following statements hold.

1. If $q^v$ is either 1 or $-1$, then $\text{Aut}(A) \cong \mathbb{Z}/(n) \ltimes (\mathbb{k}^\times)^n$.
2. If $q^v \neq \pm 1$, then $\text{Aut}(A) \cong (\mathbb{k}^\times)^n$.

It is very difficult to describe the group $\text{Aut}(A)$ if $n \geq 3$ and $(n, m, v)$ does not satisfy Theorem 0.2(a,b). Hypotheses (a) and (b) have other significant consequences. Furthermore, for any tensor product of algebras in above two theorems, the automorphism group is also computable, see Remark 7.8(1).
The proofs of the first two theorems are based on calculations of the discriminant of the algebra $A$ over its center. Further, the discriminant method can also be used to answer the cancellation problem which is closely related to the automorphism problem. We recall a definition.

**Definition 0.3.** An algebra $A$ is called **cancellative** if $A[t] \cong B[t]$ for any algebra $B$ implies that $A \cong B$.

One famous open problem in affine algebraic geometry is the Zariski Cancellation Problem which asks if the polynomial ring $k[x_1, \ldots, x_n]$, for $n \geq 3$, is cancellative. It is well-known that $k[x]$ and $k[x_1, x_2]$ are cancellative for any field $k$. In 2013, Gupta [Gu1, Gu2] settled the Zariski Cancellation Problem negatively in positive characteristic for $n \geq 3$. The Zariski Cancellation Problem in characteristic zero remains open for $n \geq 3$, see [BZ, Gu3] for more details and relevant references.

Our methods of using the discriminant can be applied to show that certain Veronese subalgebras of the skew polynomial rings are cancellative.

**Theorem 0.4.** Let $A$ be $k_q[x_1, \ldots, x_n](v)$ where $v$ is a positive integer and let $m$ be the order of $q$. Suppose that one of the following is true.

(a) $n$ is even and $m$ does not divide $v$.
(b) $n$ is odd and $\gcd(m, v) \neq 1$.

Then $A$ is cancellative.

This says that all the algebras appearing in the first two theorems are cancellative, see Remark [7,8(2)] for a more general result. As mentioned above, we can not decide whether or not $k_q[x_1, \ldots, x_n](v)$ is cancellative if it does not fit into Theorem [0.4]. We formally ask

**Question 0.5.** Let $A$ be $k_q[x_1, \ldots, x_n](v)$ where $v$ is a positive integer and let $2 \leq m < \infty$ be the order of $q$. Suppose that one of the following is true.

(a) $n$ is even and $m$ divides $v$.
(b) $n$ is odd and $\gcd(m, v) = 1$.

Is then $A$ cancellative?

The Zariski Cancellation Problem is connected to several other open problems in affine algebraic geometry – see [BZ, Gu3]. In the noncommutative setting, it is also related to certain properties of the Nakayama automorphism [LMZ] and the Makar-Limanov invariant [BZ].

The last result in this paper concerns the Tits alternative for automorphism groups of the Veronese subalgebras of skew polynomial rings. In 1972, Tits proved a remarkable and surprising dichotomy [14]: for any subgroup $G$ of the general linear group $\text{GL}(\mathbb{C}^n)$, either $G$ is virtually solvable, or $G$ contains a free group of rank $2$. Since then, similar dichotomy results have generally been referred as the *Tits alternative*. The original Tits alternative and its variations have many applications in dynamical systems, geometric group theory, Diophantine geometry, topology and so on. There is a version of the Tits alternative for the class of the automorphism groups of skew polynomial rings following [CPWZ3, Theorem 2]. In general it would be very interesting to prove that some classes of algebraic objects must satisfy certain non-obvious dichotomy such as the Tits alternative.

To state our result we recall some definitions.

**Definition 0.6.** Let $G$ be a group.
(1) $G$ is called **virtually abelian** if there is a normal abelian subgroup $N \subseteq G$ such that $G/N$ is finite.

(2) $G$ is called **virtually solvable** if there is a normal solvable subgroup $N \subseteq G$ such that $G/N$ is finite.

(3) Let $\mathcal{C}$ be a class of groups. We say $\mathcal{C}$ satisfies the **Tits Alternative** if the following dichotomy holds: any $G \in \mathcal{C}$ is either virtually solvable or it contains a free subgroup of rank 2.

For any fixed $n \geq 2$, let $\mathcal{C}_n$ consist of groups $\text{Aut}(A)$ where $A = k_q[x_1, \cdots, x_n]^v$ for all $q \in k^\times$ being a root of unity and all $v \in \mathbb{N}$.

**Theorem 0.7.** Retain the above notation.

1. If $n$ is odd, the Tits alternative holds for $\mathcal{C}_n$.
2. The Tits alternative holds for $\mathcal{C}_2$.

This theorem leaves the following question.

**Question 0.8.** Does the Tits alternative hold for $\mathcal{C}_n$ for even integer $n \geq 4$?

In principle, the discriminant method introduced in [CPWZ1, CPWZ2] can be applied to any algebras, though in applications (and examples) given there most algebras are Artin-Schelter regular. In this paper we consider a class of algebras that are not Artin-Schelter regular and show that the discriminant method is still very effective in solving several classical problems.

The paper is organized as follows. We provide background material and recall the definition of the discriminant in the noncommutative setting in Section 1. In Section 2, we study some basic properties of the discriminant. In Section 3, we provide some information about the center and Veronese subrings of the $q$-skew polynomial rings. Detailed discriminant computations are given in Section 4 (when $n$ is odd) and Section 5 (when $n$ is even). Main theorems (Theorems 0.1 and 0.2) are proved in Section 6. In Section 7 we deal with the cancellation problem and prove Theorem 0.4. The Tits alternative is discussed in Section 8 where Theorem 0.7 is proved.

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1. **Definitions**

Throughout the paper let $k$ be a commutative domain, and sometimes we further assume that $k$ is a field. Modules, vector spaces, algebras, tensor products, and morphisms are over $k$. All algebras are associative with unit.

We will recall some definitions given in [CPWZ1, CPWZ2] and introduce some new definitions. In particular, we will introduce a new variant of the discriminant in this section.

Let $B = M_w(R)$ be the $w \times w$-matrix algebra over a commutative domain $R$. We have the internal trace

$$\text{tr}_{\text{int}} : B \to R, \quad (b_{ij})_{w \times w} \mapsto \sum_{i=1}^w b_{ii}$$
which is the usual matrix trace. Now let $B$ be a general $R$-algebra and $F$ be a localization of $R$ such that that $B_F := B \otimes_R F$ is finitely generated and free over $F$. Then the left multiplication defines a natural embedding of $R$-algebras

$$\text{im} : B \to B_F \to \text{End}_F(B_F) \cong M_w(F)$$

where $w$ is the rank $\text{rk}_F(B_F)$. We define the regular trace map by composing

$$\text{tr}_{\text{reg}} : B \xrightarrow{\text{im}} M_w(F) \xrightarrow{\text{tr}_{\text{reg}}} F \subseteq Q(R)$$

where $Q(R)$ is the field of fractions of $R$. Note that $\text{tr}_{\text{reg}}$ is independent of the choices of $F$. In this paper, a trace (function) means the regular trace unless otherwise stated. In computation, we also need to assume that the image of $\text{tr}_{\text{reg}}$ is in $R$.

Let $R^\times$ denote the set of invertible elements in $R$. If $f, g \in R$ and $f = cg$ for some $c \in R^\times$, then we write $f \sim_R g$.

Let $A$ be a domain. We say a normal element $x \in A$ divides $y \in A$ if $y = xz$ for some $z \in A$. If $D$ is a set of elements in $A$, a normal element $x \in A$ is called a common divisor of $D$ if $x$ divides $d$ for all $d \in D$. We say a normal element $x \in A$ is the greatest common divisor or gcd of $D$, denoted by $\gcd_D(A)$, if

1. $x$ is a common divisor of $D$, and
2. any common divisor $y$ of $D$ divides $x$.

It follows from part (2) that the gcd of any subset $D \subseteq A$ (if it exists) is unique up to a scalar in $A^\times$.

In practice, we often choose a domain $A$ such that $R \subseteq A \subseteq B$. Note that given $D \subseteq R$, the elements $\gcd_D(A), \gcd_A(D), \gcd_B(D)$ may not all exist. Even when they exist, they may not be equal.

**Definition 1.1.** Let $R$ be a commutative domain and $B$ be an $R$-algebra. Suppose that the image of $\text{tr} := \text{tr}_{\text{reg}}$ in (E1.0.2) is in $R$. Let $(r, p)$ be a pair of positive integers. Let $A$ be a fixed domain between $R$ and $B$ in part (3).

1. ([CPWZ2] Definition 1.2(1)] Let $Z = \{z_i\}_{i=1}^r$ and $Z' = \{z'_i\}_{i=1}^r$ be two $r$-element subsets of $B$. The discriminant of the pair $(Z, Z')$ is defined to be

$$d_r(Z, Z') = \det(\text{tr}(ziz'_j))_{r \times r} \in R.$$  

2. The $p$-power discriminant ideal of rank $r$, denoted by $D_r^p(B/R)$, is the ideal of $R$ generated by the set of elements of the form

$$d_r(Z_1, Z_2) \cdots d_r(Z_{2p-1}, Z_{2p})$$

for all possible $r$-element subsets $Z_1, Z_2, \ldots, Z_{2p} \subseteq B$.

3. The $p$-power discriminant of rank $r$, denoted by $d_r^p(B/R)$, is defined to be the gcd in $A$ of the elements of the form (E1.1.1). Equivalently, the $p$-power discriminant $d_r^p(B/R)$ of rank $r$ is the gcd in $A$ of the elements in $D_r^p(B/R)$.

The notation $d_r^p(B/R)$ suppresses the dependence of the $p$-power discriminant of rank $r$ on the choice of an intermediate domain $A$. In applications, this choice of $A$ will be clearly specified. Allowing different choices of $A$, as well as different $p$ and $r$, increases the probability for the existence of the gcd of elements in (E1.1.1). When $p = 1$, the above definition agrees with [CPWZ2] Definition 1.2. If $d_r^p(B/R)$
Lemma 1.2. Suppose $B$ is finitely generated and free over $R$ of rank $r$. Then
\[ d_r^p(B/R) = A^\times (d_r(B/R))^p \]
and $D_r^p(B/R)$ is the principal ideal of $R$ generated by $(d_r(B/R))^p$.

Proof. Let $X = \{x_1, \cdots, x_r\}$ be a basis of $B$ over $R$. By \cite{CPWZ1} Definition 1.3(3)], $d_r(B/R) = R^\times d_r(X, X)$. For any $r$-element subset $Z := \{z_1, \cdots, z_r\} \subset B$, we can write $z_i = \sum_j r_{ij}x_j$ for an $r \times r$-matrix $(r_{ij})$. Similarly for $Z'$. Then
\[ d_r(Z, Z') = d_r(X, X) \det(r_{ij}) \det(r_{ij}') = d_r(B/R) \det(r_{ij}) \det(r_{ij}'). \]
Then the assertion follows from the definition.

Lemma 1.3. Let $\Psi$ be a subset of $B$ that generates $B$ as an $R$-module.

1. $D_r^p(B/R)$ is the ideal of $R$ generated by the set
\[(E_{1.3.1}) \quad \{d_r(X_1, X_2) \cdots d_r(X_{2p-1}, X_{2p}) \mid X_i \subseteq \Psi, \forall i\}.\]

2. $d_r^p(B/R)$ is the gcd in $A$ of elements in set (E_{1.3.1}).

Proof. Every element $z \in B$ is an $R$-linear combination of $\phi_i \in \Psi$. By bilinearity of $tr(zz')$ and multi-linearity of $\det$, every $d_r(Z, Z')$ is an $R$-linear combination of $d_r(X, X')$ where $X, X'$ are $r$-element subsets of $\Psi$. Therefore every element of the form (E_{1.1}) is an $R$-linear combination of elements in (E_{1.3.1}). The assertions follow.

In this paper we will see that some discriminants satisfy the following.

Definition 1.4. Retain the notation as in Definition 1.1. The $p$-power $r$-rank discriminant $d_r^p(B/R)$ is called stable if
\[ d_r^p(B/R) = A^\times (d_r^p(B/R))^i \]
for all positive integers $i$.

Under the hypotheses of Lemma 1.2, $d_r^p(B/R)$ is always stable for every $p$.

2. Properties of the discriminant

In this section we list of elementary properties of $d_r^p(B/R)$. The following lemma is similar to \cite{CPWZ2} Lemma 1.4].

Lemma 2.1. Suppose that the image of the regular trace $tr$ is in $R$. Let $g$ be an automorphism of $B$ such that $g$ and $g^{-1}$ preserve $R$.

1. The $p$-power $r$-rank discriminant ideal $D_r^p(B/R)$ is $g$-invariant.

2. The $p$-power $r$-rank discriminant $d_r^p(B/R)$ (if exists) is $g$-invariant up to a unit in $A$. 
(3) Suppose \( r_1 \leq r_2 \) and \( p_1 \leq p_2 \) are positive integers. Then
\[
D^{[p_2]}_{r_2}(B/R) \subseteq D^{[p_1]}_{r_1}(B/R).
\]

If both \( d^{[p_2]}_{r_2}(B/R) \) and \( d^{[p_1]}_{r_1}(B/R) \) exist, then
\[
d^{[p_1]}_{r_1}(B/R) \mid d^{[p_2]}_{r_2}(B/R).
\]

As a consequence, the quotient \( d^{[p_2]}_{r_2}(B/R)/d^{[p_1]}_{r_1}(B/R) \) is \( g \)-invariant up to
a unit in \( A \).

Proof. (1) By \([\text{CPWZ1}] \) Lemma 1.8(3), \( g(d_r(Z, Z')) = d_r(g(Z), g(Z')) \). Then \( g \)
maps an element of the form \([E.1.4]\) to another element of the same form. Similarly,
this holds for \( g^{-1} \). Hence, \( g \) (and \( g^{-1} \)) preserves \( D^{[p]}(B/R) \).

(2) This follows from part (1) and the fact that the gcd is well-defined up to
a unit.

(3) When \( p_1 = p_2 = 1 \), this is \([\text{CPWZ2}] \) Lemma 1.4(5)]. For general \( p_1 \leq p_2 \),
the proof is similar to the proof of \([\text{CPWZ2}] \) Lemma 1.4(5)], so it is omitted. □

We recall some definitions from \([\text{CPWZ2}] \) p.766. Let \( C \) be a domain such that \( \mathbb{k} \subseteq C \) and that \( C/\mathbb{k} \) is \( \mathbb{k} \)-flat. We say that \( A \odot C \) is \( A \)-closed if, for every \( 0 \neq f \in A \)
and \( x, y \in A \odot C \), the equation \( xy = f \) implies that \( x, y \in A \) up to units of \( A \odot C \).
For example, if \( C \) is connected graded and \( A \odot C \) is a domain, then \( A \odot C \) is
\( A \)-closed. The next lemma is similar to \([\text{CPWZ2}] \) Lemma 1.12.

**Lemma 2.2.** Retain the hypotheses as above. Assume that \( B \odot C \) is a domain.

1. \( D^{[p]}(B \odot C/R \odot C) = D^{[p]}(B/R) \odot C. \)

2. Suppose \( A \odot C \) is \( A \)-closed. If \( d^{[p]}_r(B/R) \) exists, then \( d^{[p]}_r(B \odot C/R \odot C) \)
exists and equals \( d^{[p]}_r(B/R). \)

Proof. (1) First of all, the regular trace \( \text{tr} \) of \( B \odot C \) over \( R \odot C \) is equal to
the regular trace \( \text{tr} \) of \( B \) over \( R \) when restricted to elements in \( B \).

Let \( \Psi \) be a subset of \( B \) such that \( B \) is generated by \( \Psi \) as an \( R \)-module. Then
\( B \odot C \) is generated by \( \Psi \) as an \( R \odot C \)-module. By Lemma 1.3(1), \( D^{[p]}_r(B \odot C/R \odot C) \)
is the ideal of \( R \odot C \) generated by the set \([E.1.3.1] \), which is just \( D^{[p]}_r(B/R) \odot C \) by
Lemma 1.3(1).

(2) Suppose \( d := d^{[p]}_r(B/R) \) exists. Then it is the gcd in \( A \) of the set \( D^{[p]}_r(B/R) \)
by definition. By part (1), \( d^{[p]}(B \odot C/R \odot C) \) (if exists) is the gcd in \( A \odot C \) of the
set \( D^{[p]}_r(B/R) \odot C \), which is the gcd in \( A \odot C \) of the set \( D^{[p]}_r(B/R) \).

Let \( d' \in A \odot C \) be a common divisor in \( A \odot C \) of the set \( D^{[p]}_r(B/R) \). By \( A \)-
closedness of \( A \odot C \), we may assume that \( d' \) is in \( A \) (up to a unit). This implies
that \( d' \) divides \( d \). It is clear that \( d \) is a common divisor in \( A \odot C \) of the set \( D^{[p]}_r(B/R) \).
Therefore \( d \) is the gcd in \( A \odot C \) of the set \( D^{[p]}_r(B/R) \). The assertion follows. □

Let \( F \) be a localization of a commutative domain \( R \) (and \( R \) is the center \( Z(B) \)
in most of applications) and \( F \) may not be the fraction field of \( R \). We assume that
\( BF := B \odot R \) is finitely generated and free over \( F \). We recall a definition from
\([\text{CPWZ2}] \).

**Definition 2.3.** \([\text{CPWZ2}] \) Definition 1.10] Retain the above notations.
A subset \( b = \{b_1, \cdots, b_w\} \subseteq B \) is called a semi-basis of \( B \) if it is an \( F \)-basis of \( B_F \), where \( b_i \) is viewed as \( b_i \otimes_R 1 \in B_F \). In this case \( w \) is the rank of \( B \) over \( R \).

(2) Let \( b \) be a semi-basis of \( B \) and \( T \) be a subset of \( B \) containing \( b \) which generates \( B \) as an \( R \)-module. We call such a set \( T \) an \( R \)-generating set of \( B \). Then \( b \) is called a quasi-basis (with respect to \( T \)) of \( B \) if every \( t \in T \) can be written as \( t = cb \) for some \( b \in b \). We denote \( c \) by \( (t : b) \).

We continue to introduce some notation. Again let \( w \) be the rank of \( B \) over \( R \). Let \( Z := \{z_1, \cdots, z_w\} \) be a subset of \( B \). If \( b \) is a semi-basis, then for each \( i \),

\[
z_i = \sum_{j=1}^{w} a_{ij} b_j
\]

for some \( a_{ij} \in F \). In this case, the \( w \times w \)-matrix \((a_{ij})_{w \times w}\) is denoted by \((Z : b)\).

Let \( T \) be as in Definition (2.3) (2). Let \( T/b \) denote the subset of \( F \) consisting of nonzero scalars of the form \( \det(Z : b) \) for all \( Z \subseteq T \) with \( |Z| = w \). Let

\[
\mathcal{D}(T/b) = \{d_w(b, b)f f' \mid f, f' \in T/b\},
\]

and

\[
\mathcal{D}_w(T) = \{d_w(Z, Z') \mid Z, Z' \subseteq T, |Z| = |Z'| = w\}.
\]

Note that if \( Z \) and \( Z' \) are \( w \)-element subsets of \( T \), then \( d_w(Z, Z') \in \mathcal{D}(T/b) \) by [CPWZ2 (1.10.1)]. In fact, we have \( \mathcal{D}(T/b) = \mathcal{D}_w(T) \).

If \( b = \{b_1, \cdots, b_w\} \) is a quasi-basis with respect to an \( R \)-generating set \( T \). Then for each \( i \), let

\[
(C_i) = \{(t : b_i) \mid t \in T\}\backslash\{0\}.
\]

It is easy to see that every element in \( T/b \) is of the form \( c_1 c_2 \cdots c_w \), where \( c_i \in C_i \) for each \( i \). Let

\[
\mathcal{D}'(T/b) = \left\{ d_w(b, b) \prod_{i=1}^{w} (c_i c'_i) \mid c_i, c'_i \in C_i \right\}.
\]

If \( b \) is a quasi-basis with respect to \( T \), then \( \mathcal{D}(T/b) = \mathcal{D}'(T/b) \).

Let \( S \) be a subset of an algebra \( R \). Let \( S^p \) denote the subset of \( R \) consisting of \( s_1 s_2 \cdots s_p \) for all \( s_i \in S \). The following lemma is similar to [CPWZ2 Lemma 1.11].

**Lemma 2.4.** Let \( T \) be a set of generators of \( B \) as an \( R \)-module and \( w = \text{rank}(B/R) \). Let \( p \) be a positive integer.

1. \( D_w^{[p]}(B/R) \) is generated by the set \( \mathcal{D}_w(T)^p \).
2. \( d_w^{[p]}(B/R) = \text{gcd} \mathcal{D}_w(T)^p \).
3. If \( b \) is a semi-basis of \( B \), then \( d_w^{[p]}(B/R) = \text{gcd} \mathcal{D}(T/b)^p \).
4. If \( b \) is a quasi-basis of \( B \), then \( d_w^{[p]}(B/R) = \text{gcd} \mathcal{D}'(T/b)^p \).

**Proof.** (1) This follows from Lemma [2.3 (1)].

(2), (3) and (4) follow from the definition, the above discussion and part (1). □

In the rest of the section we assume that \( B^1 \) and \( B^2 \) are two algebras that are \( A \)-flat. If \( X^1 \subseteq B^1 \) and \( X^2 \subseteq B^2 \) are two subsets, then \( X^1 \otimes X^2 \) denotes the set \( \{x \otimes y \mid x \in X^1, y \in X^2\} \). We say that the pair \((X^1, X^2)\) is hereditary if for \( x \in X^1 \) and \( y \in X^2 \), every divisor of \( x \otimes y \) is of the form \( x' \otimes y' \) (up to a unit in \( B^1 \otimes B^2 \)). The following lemma is easy.
Lemma 2.5. Let $X^1 \subseteq B^1$ and $X^2 \subseteq B^2$ be two subsets such that

(a) $\gcd X^1$ and $\gcd X^2$ exist.
(b) $(X^1, X^2)$ is hereditary.

Then $\gcd(X^1 \otimes X^2) = (B^1 \otimes B^2) \times \gcd X^1 \otimes \gcd X^2$. \hfill \Box

Lemma 2.6. Let $B^1$ and $B^2$ be two $k$-algebras containing central subalgebra domains $R^1$ and $R^2$ respectively. Let $w_i = \text{rank}_{R^i}(B^i)$ for $i = 1, 2$. Assume that

(a) $R := R^1 \otimes R^2$ is a domain.
(b) $b^i$ is a quasi-basis of $B^i$ over $R^i$ with corresponding $R^i$-module generating set $T^i$ (and $b^i \subseteq T^i$).

Then the following hold.

(1) $b := b^1 \otimes b^2$ is a quasi-basis of $B := B^1 \otimes B^2$ over $R$ with corresponding $R$-generating set being $T := T^1 \otimes T^2$.

(2) Let $w := w_1 w_2$. Then $D^r(T/b) = D^r((T^1/b^1)^{w_2} \otimes D^r(T^2/b^2)^{w_1})$.

(3) Suppose that $(D^r(T^1/b^1)^{w_2}, D^r(T^2/b^2)^{w_1})$ is hereditary. Let $p$ be an integer. If $d[p]_w(B^1/R^1)$ and $d[p]_w(B^2/R^2)$ are stable discriminants, then so is $d[p]_w(B/R)$. Further,

\[ d[p]_w(B/R) = B \times d[p]_w(B^1/R^1)^{w_2} \otimes d[p]_w(B^2/R^2)^{w_1}. \]

Proof. (1) Let $i$ be either 1 or 2. Let $b^i = \{b^i_1, b^i_2, \ldots, b^i_{w_i}\}$ and $T^i = \{t^i_j\}_{j \in J_i}$. By definition, it is routine to check that $b^1 \otimes b^2$ is a quasi-basis of $B^1 \otimes B^2$ over $R$ with corresponding $R$-generating set being $T = T^1 \otimes T^2$.

(2) Let $1 \leq j \leq w_1$ and $1 \leq k \leq w_2$. Following (2.3.3), define $C^1_j$ to be the set of elements $c$ in $B^1$ such that $cb^1_j \in T^1$. Similarly, we define $C^2_k$. Let $C_{j,k}$ consist of elements of the form $c_j \otimes d_k$ where $c_j \in C^1_j$ and $d_k \in C^2_k$. Then $C_{j,k}$ consist of elements of the form $t(b^1_j \otimes b^2_k)^{-1}$ for all $t \in T$.

By linear algebra, $d[w]_w(b, b) = d[w_1](b^1, b^1)^{w_2} \otimes d[w_2](b^2, b^2)^{w_1}$. And we have the following computation, for all $(c_j \otimes d_k), (c'_j \otimes d'_k) \in C_{j,k}$ for different $(j, k)$,

\[ \prod_{j,k} [(c_j \otimes d_k)(c'_j \otimes d'_k)] \in X^{w_2} \otimes Y^{w_1} \]

where $X = \{ (\prod_{j} (c_j c'_j)) \mid c_j, c'_j \in C^1_j \}$ and $Y = \{ (\prod_{k} (d_k d'_k)) \mid d_k, d'_k \in C^2_k \}$. Now assertion follows from (2.3.3).

(3) This follows from the definition, Lemma (2.3.4), part (2) and Lemma (2.5). \hfill \Box

3. CENTER AND VERONESE SUBRINGS OF $q$-POLYNOMIAL RINGS

From now on we fix two integers $m, n \geq 2$ and a primitive $m$th root of unity, say $q$, in $k$. The $q$-skew polynomial ring is generated by $x_1, \ldots, x_n$ and subject to the relations

\[ (E3.0.1) \quad x_j x_i = q x_i x_j, \quad \forall 1 \leq i < j \leq n. \]

and is denoted by $k_q[x_1, \ldots, x_n]$, or simply by $k_q[x]$.

We will adopt the following notation for monomials $x^s := x_1^{s_1} \cdots x_n^{s_n}$ where $s = (s_1, \ldots, s_n) \in \mathbb{N}^n$ is its degree vector. We will also denote by $e_i$ the standard basis vector, with 1 in its $i$th component and 0 elsewhere. For any $0 \leq k \leq m$, define

\[ (E3.0.2) \quad y_k := q^{-|n/2|k(k+1)/2} \cdot x^{(k, m-k, m-k, \ldots)}, \]
in particular,
\[
\begin{align*}
y_0 &= x_1^m x_2^m \cdots x_n^m, \\
y_m &= (-1)^{\lfloor n/2 \rfloor} x_1^m x_2^m \cdots x_n^m.
\end{align*}
\]
Note that both \(y_0\) and \(y_m\) are in the central subalgebra generated by \(\{x_n^m, \ldots, x_1^m\}\). One can easily check that the \(y_i\)s satisfy the following relations
\[
y_i y_j = q^{-\lfloor n/2 \rfloor (i+j)/(i+j+1)/2} x_1^{i+j+1} y_j / x_1^i y_j, \quad \forall i, j \leq m.
\]
As a consequence,
\[
y_i y_j = y_k y_{i+k}, \quad \forall i + j = k + \ell.
\]
Equations (E3.0.3)-(E3.0.4) also imply that
\[
\begin{align*}
y_{i} y_{j} &= q^{-\lfloor n/2 \rfloor (i+j)/(i+j+1)/2} x_1^{i+j+1} y_j / x_1^i y_j, \quad \forall i, j \leq m.
\end{align*}
\]

The following is a consequence of [CYZ] Lemma 4.1. Let \(Z(A)\) denote the center of an algebra \(A\).

Lemma 3.1.  
1. If \(n\) is even, then \(Z(k_q[x])\) is a polynomial ring generated by 
\(x_1^m, \ldots, x_n^m\).
2. If \(n\) is odd, then \(Z(k_q[x])\) is generated by \(x_1^m, \ldots, x_n^m, y_1, \ldots, y_{m-1}\).

Proof. (1) This is [CPWZ2, Example 2.4(2)].
(2) One can check it directly or use [CYZ] Lemma 4.1. We use some of the notation in [CYZ] Section 4. Let \(Y\) be the skew symmetric \(n \times n\)-matrix with \(1/m\) in all entries above the diagonal.

Let \(t \in \mathbb{N}^n\). By [CYZ] Lemma 4.1, the monomial \(x^t\) is in the center \(Z(k_q[x])\) if and only if \(Y t \in \mathbb{Z}^n\). Let \(S = m Y\). Then \(x^t \in Z(k_q[x])\) if and only if \(S t \in m \mathbb{Z}^n\). Let \(\tilde{S}\) be the endomorphism of \((\mathbb{Z}/m \mathbb{Z})^n\) represented by the matrix \(S\). Then \(S t \in m \mathbb{Z}^n\) if and only if \(t\) is a lift of an element in \(\ker(\tilde{S})\).

Since \(n\) is odd, \(\text{rank}(\tilde{S} \otimes \mathbb{F}_p) = n - 1\) for all primes \(p\). It is easy to check that \(\ker(\tilde{S})\) is generated by \((i, -i, i, \ldots, -i, i) \in (\mathbb{Z}/m \mathbb{Z})^n\) for \(i = 0, \ldots, m\). Lifting these to \(\mathbb{Z}^n\) gives \((i, m-i, i, \ldots, m-i, i)\) for \(i = 1, \ldots, m-1\) and \(m\) for \(i = 1, \ldots, n\). \(\square\)

When \(n\) is even, the center \(Z(k_q[x])\) is easy to understand, namely
\[
Z(k_q[x]) = \mathbb{k}[x_1^m, \ldots, x_n^m].
\]
If \(n\) is odd, every element of \(Z(k_q[x])\) can be expressed as a linear combination of terms of the form \(x^{a b m}\) or \(x^{a b m} y_b\), with \(a \in \mathbb{N}^n\) and \(0 < b < m\). Each such term can be rewritten as follows,
\[
x^{a b m} y_b = x_1^{a_1 m} \cdots x_n^{a_n m} y_b = q^{-\lfloor n/2 \rfloor b(b+1)/2} x_1^{a_1 m} x_2^{a_2 m} \cdots x_n^{a_n m} y_b.
\]
Since the above polynomials form a \(\mathbb{k}\)-linear basis of \(Z(k_q[x])\), we have
\[
Z(k_q[x]) \cong \mathbb{k}[x_1^m, \ldots, x_n^m, y_1, \ldots, y_{m-1}].
\]
For example, if \( n = 3 \),
\[
Z(\kappa_q[\mathbf{x}]) \cong \frac{\kappa[x_1^n, x_2^m, x_3^m, y_0, \ldots, y_m]}{y_0 - x_2^m, \; \; y_m - (-1)^{m+1}x_1^m x_3^m, \; \; y_i y_j - y_k y_i, \; \; \forall i + j = k + \ell}.
\]
and if \( m = 2 \),
\[
Z(\kappa_q[\mathbf{x}]) \cong \frac{\kappa[x_1^2, x_2^2, y_1]}{y_1^2 - (\pm\ell)^{(n-1)/2} x_1^2 x_2^2 \cdots x_n^2}.
\]
Hopefully this gives some idea on what the center should be.

For any \( v \in \mathbb{N} \), the \( v \)th Veronese subalgebra of \( \kappa_q[\mathbf{x}] \), denoted by \( \kappa_q[\mathbf{x}]^{(v)} \), is the subalgebra generated by elements of total degree \( v \).

As before we fix positive integers \( m, n, v \). Let
\[
g := \gcd(v, m).
\]
Let \( \kappa_q[\mathbf{x}]^{\pm 1} \) be the localization of \( \kappa_q[\mathbf{x}] \) by inverting all \( x_i \)'s. We extend the notation \( \mathbf{x}^s \) for all \( s \in \mathbb{Z}^n \) in a natural way.

Let \( F \) be the center of \( \kappa_q[\mathbf{x}]^{\pm 1}^{(v)} \) which is a localization of \( Z := Z(\kappa_q[\mathbf{x}]^{(v)}) \), and let \( \kappa_q[\mathbf{x}]^{(v)} = \kappa_q[\mathbf{x}]^{(v)} \otimes_Z F \). Since \( F \) is a \( \mathbb{Z}^n \)-graded field, we have

1. \( \kappa_q[\mathbf{x}]^{(v)} = \kappa_q[\mathbf{x}]^{\pm 1}^{(v)} \) which is a \( \mathbb{Z}^n \)-graded skew field.
2. \( \kappa_q[\mathbf{x}]^{(v)} \) is free over \( F \).

Since each \( x_i^{mv} \in F \), we have that \( \kappa_q[\mathbf{x}]^{(v)} \) is finite dimensional over \( F \), and we denote
\[
w := \dim_F \kappa_q[\mathbf{x}]^{(v)}.
\]
Let \( H_v = \{ s \in \mathbb{Z}^n \mid \sum_{i=1}^n s_i \leq v \mathbb{N} \} \), and let \( H_v^+ = H_v \cap \mathbb{N}^n \), so that \( \kappa_q[\mathbf{x}]^{(v)} \) (respectively, \( \kappa_q[\mathbf{x}]^{\pm 1}^{(v)} \)) is the span of \( \mathbf{x}^{H_v^+} \) (respectively, \( \mathbf{x}^{H_v} \)).

**Lemma 3.2.** Retain the above notation. Suppose that \( n \) is odd.

1. \( Z(\kappa_q[\mathbf{x}]^{(v)}) = Z(\kappa_q[\mathbf{x}]) \cap \kappa_q[\mathbf{x}]^{(v)} = \kappa\langle x_i^m, y_j \rangle \cap \kappa_q[\mathbf{x}]^{(v)} \).
2. The center \( Z(\kappa_q[\mathbf{x}]^{\pm 1}^{(v)}) \) is spanned by \( \mathbf{x}^M \) where
\[
M = \left( m\mathbb{Z}^n + g\mathbb{Z} \sum_{i=1}^n (-1)^{i-1} e_i \right) \cap H_v.
\]
As a consequence,
\[
Z(\kappa_q[\mathbf{x}]^{(v)}) = \kappa\langle x_i^m, y_j \rangle \cap \kappa_q[\mathbf{x}]^{(v)}.
\]

**Proof.** Let \( \mathbf{x}^s \in Z(\kappa_q[\mathbf{x}]^{\pm 1}^{(v)}) \) for some \( s \in \mathbb{Z}^n \). Since \( x_i x_i^{m+1} - 1 \in \kappa_q[\mathbf{x}]^{(v)} \),
\[
\mathbf{x}^s x_i x_i^{m+1} = x_i x_i^{m+1} \mathbf{x}^s = q^{-(s_i + s_i + 1)} \mathbf{x}^s x_i x_i^{m+1}.
\]
Hence, \( s_i + s_{i+1} \in m\mathbb{Z} \) for all \( i \). Then, for each \( i \),
\[
\text{(E3.2.1) } s_i = \begin{cases} a_i m + b & \text{if } i \text{ is odd}, \\ a_i m + (m - b) & \text{if } i \text{ is even}. \end{cases}
\]
for some \( a_1, \ldots, a_n \in \mathbb{Z} \) and \( 0 \leq b \leq m - 1 \). This part of the proof works for both even and odd \( n \).
(1) When $x^s \in Z(k_q[x^{(v)}])$ for $s \in \mathbb{N}^n$. We obtain that, if $b > 0$, then $a_i \geq 0$ for all $a_i$ in (E4.2.1) and if $b = 0$, $a_i \geq 0$ for odd $i$ and $a_i \geq -1$ for even $i$. This is equivalent to

$$x^s \equiv_{k_q} x_1^{a_1} \cdots x_n^{a_n} y_b \quad b = 0,$$

for some $a_i \geq 0$. The assertion follows.

(2) Recall that $n$ is odd. Note that, if $x_1^{a_1} \cdots x_n^{a_n} y_b \in Z(k_q[x^{\pm 1}])$, then

$$b + m \left( \frac{n - 1}{2} + \sum_{i=1}^{n} a_i \right) \in v\mathbb{Z},$$

and hence, $b \in g\mathbb{Z}$. This means that if $x^s \in Z$, then $s \in M$. Conversely, it is straightforward to check that if $s \in M$, then $x^s \in Z$. □

We are interested the discriminant of $k_q[x^{(v)}]$ over its center. We examine separately the case when $n$ is odd, and the case when $n$ is even.

We conclude this section with the hereditary property (as mentioned before Lemma 2.3) for monomials in $k_q[x^{(v)}].$

**Lemma 3.3.** Let $A^1, \ldots, A^s$ be algebras of type $k_q[x^{(v)}]$. For each $i$, let $X^i \subseteq A^i$ be a set of monomials. Then, for any $f^i \in X^i$, every divisor of $f^1 \otimes f^2 \otimes \cdots \otimes f^s$ is of the form $g_1 \otimes g_2 \otimes \cdots \otimes g^s$ where each $g^i$ is a divisor of $f^i$.

**Proof.** Consider $A^i = k_q[x^{(v)}]$ as an $\mathbb{N}^n$-graded algebra for all $i$. Let $n = \sum_{i=1}^{s} n_i$. Then $A^1 \otimes \cdots \otimes A^s$ is an $\mathbb{N}^n$-graded algebra. Since each $f^i$ is $\mathbb{N}^n$-homogeneous, $F := f^1 \otimes \cdots \otimes f^s$ is $\mathbb{N}^n$-homogeneous. Note that $\mathbb{N}^n$ is an ordered semigroup. Then any divisor $G$ of $F$ is $\mathbb{N}^n$-homogeneous. Equivalently, $G = g_1 \otimes \cdots \otimes g^s$ where each $g^i$ is a divisor of $f^i$. □

4. **Discriminant computation: when $n$ is odd**

We will freely use the notation introduced in the last section, and further assume that $n$ is odd in a large part of this section.

Recall from Lemma 3.2 that, if $n$ is odd, then

(E4.0.1) \[ M = \left( m\mathbb{Z}^n + g\mathbb{Z} \left( \sum_{i=1}^{n} (-1)^{i-1} e_i \right) \right) \cap H_v. \]

Then $M$ is a subgroup of $H_v$. We can partition $H_v$ into cosets mod $M$. It is easy to see the total number of these cosets is equal to $w$ (E3.1.2).

**Lemma 4.1.** Assume $n$ is odd.

(1) For each coset of $M$ in $H_v$, there is a unique representative $p := (p_1, \ldots, p_n)$ such that
   (a) $0 \leq p_1 < g$,
   (b) for each $1 \leq i < n$, we have $0 \leq p_i < m$, and
   (c) $0 \leq p_n < vm/g$.
   Moreover, the above remains true with indices $(1, n)$ replaced by any $(\mu, \nu)$ with $\mu \neq \nu$.

(2) $w = m^{n-1}$.

(3) $w \neq 0$ in $k$. 

Proof. (1) Pick an arbitrary coset $M'$ of $M$, and let $p = (p_1, \ldots, p_n) \in M'$. Since $g = \gcd(m, v)$, there exists $c \in \mathbb{Z}$ such that $cm \equiv g \mod v$. Hence $(g, -g, g, -g, \ldots, -g, g) = \mathbb{Z}$. and we can translate $p$ by some multiple of this vector to obtain $0 \leq p_1 < g$. Furthermore, if $t \in M$ then $t_1 \in g\mathbb{Z}$, so there is no vector in $M'$ whose first component is any other $0 \leq r' < g$.

For each $1 < i < n$, we have $m(e_i - e_n) \in M$, so we can apply the translation trick above and assume that $0 \leq p_i < m$. Furthermore, if $t \in M$ and $t_1 = 0$, then each other $t_i \in m\mathbb{Z}$. This implies that there is no other set of possible values of $p_1, \ldots, p_{n-1}$ subject to the conditions $0 \leq p_1 < g$ and $0 \leq p_i < m$ for every $1 < i < n$.

Finally, $(vm/g)e_n \in M$, so there exists a representative $p \in M'$ subject to constraints (a)-(c) of the lemma. If $ce_n \in M$, then $c \in m\mathbb{Z} \cap v\mathbb{Z} = (vm/g)\mathbb{Z}$, so this representative is unique.

The last statement is clear since the above calculations do not depend on the ordering of the indices $1, \ldots, n$. This finishes the proof of part (1).

(2) The value $w$ can be determined by counting the cosets by their representatives. For every sequence of integers $p_1, \ldots, p_{n-1}$ such that $0 \leq p_1 < g$ and $0 \leq p_i < m$ for all $1 < i < n$, there are $m/g$ possible values of $p_n$ such that $0 \leq p_n < vm/g$ and $(p_1, \ldots, p_n) \in H_c$. Therefore, $w = g \cdot m^{n-2} \cdot m/g = m^{n-1}$.

(3) Since $g \in k$ and $o(g) = m$, the characteristic of $k$ cannot divide $m$. Or $m \neq 0$ and $w \neq 0$ in $k$. □

In this paper we mainly consider the case when $B = k[x]^{(v)}$ for both even and odd $n$. Using [CPWZ1] Definition 1.10 and notation in Section 2, if $B := \{b_1, \ldots, b_m\} \subseteq H^+_{\mathbb{Z}}$ is a set of representatives of each coset of $M$, then $b := x^w$ is a quasi-basis of $k[x]^{(v)}$ with respect to $T := x^{H^+_{\mathbb{Z}}}$, and

(E4.1.1) $\mathcal{D}^c(T/b) = \mathbb{Z}[d_w(b, b)] \left\{ \prod_{i=1}^{w} x^{s_i - b_i} x^{s_i'} - b_i \mid s_i, s_i' \in \mathbb{N} \cap (M + b_i) \right\}$.

The following lemmas hold for both even and odd $n$.

**Lemma 4.2.** Retain the above notation. Let $s \in H^+_{\mathbb{Z}}$ such that $x^s$ is not central. Then $\text{tr}(x^s) = 0$. As a consequence, the trace map $\text{tr}$ sends $k_q[x]^{(v)}$ to $Z(k_q[x]^{(v)})$.

**Proof.** Since $A := k_q[x]^{(v)}$ is $\mathbb{Z}^n$-graded, so is the center $Z := Z(A)$. Let $F$ be the graded field of fractions of $Z$. Then $A$ is a free module over $F$ with $F$-basis $B$. Then, for all $i, j$, there is a unique $k$ such that $b_i b_j = c_{ij}^k b_k$ for some $0 \neq c_{ij}^k \in F$. If $b_i$ is not in the center, then $j \neq k$. Therefore $\text{tr}(b_i) = \sum_{j=k} c_{ij}^k = 0$. Every element $x^s$ is of the form $c b_i$ for some $i$ and $c \in F$. The assertion follows. □

**Lemma 4.3.** Retain the above notation. Suppose that $w$ is invertible. Then

$$\mathcal{D}^c(T/b) =_{k[x]} \left\{ \left( \prod_{i=1}^{w} x^{s_i} \right) \left( \prod_{i=1}^{w} x^{s_i'} \right) \mid s_i, s_i' \in \mathbb{N} \cap (M + b_i) \right\}.$$

**Proof.** For each $b_i \in B$, let $b_i^* \in B$ be such that $b_i + b_i^* \in M$. For any $s \in H^+_{\mathbb{Z}}$, if $s \notin M$, then $x^s$ is not central, and $\text{tr}(x^s) = 0$ by Lemma 4.2. If $s \in M$, then $x^s$ is central, and $\text{tr}(x^s) = w x^s =_{k[x]} x^s$, where the last equation follows from the hypothesis that $w$ is invertible. Therefore, in the matrix $(\text{tr}(x^s x^{s'}))_{w \times w}$, the only
nonzero terms appear where $b_i = b_i^*$, and
\[ d_w(b, b) = k^x \det(\text{tr}(x^b \cdot x^{b^*})))_{w \times w} = k^x \prod_{i=1}^{w} x^{b_i}x^{b_i^*} = k^x \left( \prod_{i=1}^{w} x^{b_i} \right)^2. \]

The assertion follows by the above formula and equation (E4.1.1). \qed

Recall that $m$ is the order of $q$, the rank of $k_q[x]^{(v)}$ over its center is $w = m^{n-1}$ and $g = \gcd(v, m)$.

**Theorem 4.4.** Let $B = k_q[x]^{(v)}$ when $n$ is odd. Suppose that $m$ is invertible in $k$. Let $R$ be the center of $B$. Assume that $v$ divides $wp(g - 1)$. Then
\[ d_w^{[p]}(B/R) = k^x (x_1x_2 \cdots x_n)^{wp(g-1)} = k^x (x_1^w x_2^w \cdots x_n^w)^{wp(g-1)/v}. \]

As a consequence, $d_w^{[p]}(B/R)$ is stable.

**Proof.** By Lemmas 2.4(4) and 4.3, we have $d_w^{[p]}(B/R) = \gcd \Lambda^{2p}$ where
\[ \Lambda := \left\{ \prod_{i=1}^{m} x^{s_i} \mid s_i \in \mathbb{N}^n \cap (M + b_i) \right\}. \]

For each $1 \leq s \leq n$, let $f_s \in \mathbb{N}$ be maximal such that $x_i^{f_s}$ divides all elements of $\Lambda$. This gives $x^{2pf}$ as the gcd of $\Lambda^{2p}$ in the over-algebra $k_q[x] \supseteq B$ where $f = (f_1, \ldots, f_n)$. If $2pf \in H^+_v$, then it is the gcd of $\Lambda^{2p}$ in $k_q[x]^{(v)}$ as well, but, otherwise, this is not true.

We first calculate $f_1$ by summing the lowest powers of $x_1$ in each coset of $M$ (or more precisely, in each $\mathbb{N}^n \cap (M + b_i)$ for different $i$). These lowest powers can be found by using the representatives outlined in Lemma 4.1(1), which also shows that this power cannot exceed $g - 1$. For each $0 \leq k \leq g - 1$, there are $m^{n-1}/g$ cosets with lowest power $x_1^k$. Therefore, the sum is
\[ f_1 = \frac{m^{n-1}}{g} \frac{g(g-1)}{2} = \frac{w(g-1)}{2}. \]

For $f_i$ with $i \neq 1$, we can use the last assertion of Lemma 4.1(1) to relabel indices, so the above calculation remains valid for $i \neq 1$ and we conclude that $f_1 = f_2 = \cdots = f_n$.

Now $2pf = wp(g-1)(1, 1, \cdots, 1) \in H^+_v$ as $v$ divides $wp(g - 1)$. The assertion follows from the last paragraph, and stability of $d_w^{[p]}(B/R)$ follows from the main assertion. \qed

5. Discriminant computation: when $n$ is even

In this section we assume that $n$ is even. The following is similar to Lemma 4.22

**Lemma 5.1.** Suppose that $n$ is even.

1. \[ Z(k_q[x]^{(v)}) = k(x_1^m, y_{jm/g}) \cap k_q[x]^{(v)}. \]

2. The center $Z(k_q[x^{\pm 1}]^{(v)})$ is spanned by $x^M$ where
\[ M := \left( m\mathbb{Z}^n + \left( \frac{m}{g} \right) \mathbb{Z} \sum_{i=1}^{n} (-1)^{i-1} e_i \right) \cap H_v. \]
Proof. (1) We copy the first part of the proof of Lemma 3.2.

Let \( x^s \in Z(k_q[x_1^\pm 1](v)) \) for some \( s \in \mathbb{Z}^n \). Since \( x_i x_i^{-1} \in k_q[x_1](v) \), we have

\[
x^s x_i x_i^{-1} = x_i x_i^{-1} x^s = q^{-(s_i + s_i)} x^s x_i x_i^{-1}.
\]

Hence, \( s_i + s_i \in m\mathbb{Z} \) for all \( i \). Then, for each \( i \),

\[
(E5.1.1) \quad s_i = \begin{cases} a_i m + b & \text{if } i \text{ is odd}, \\ a_i m + (m - b) & \text{if } i \text{ is even}. \end{cases}
\]

for some \( a_i, b \in \mathbb{Z} \) and \( 0 \leq b \leq m - 1 \). Considering \( x^s \in Z(k_q[x]^\pm(v)) \) for \( s \in \mathbb{N}^n \). We obtain that, if \( b > 0 \), then \( a_i \geq 0 \) for all \( a_i \) in \( (E5.1.1) \) and if \( b = 0, a_i \geq 0 \) for odd \( i \) and \( a_i \geq -1 \) for even \( i \). This is equivalent to

\[
x^s =_{k^x} \begin{cases} x_1^{ma_1} \cdots x_n^{ma_n} & b = 0, \\ x_1^{ma_1} \cdots x_n^{ma_n} y_b & b \neq 0, \end{cases}
\]

for some \( a_i \geq 0 \). Next we need to determine the values of \( b \) such that \( y_b \in Z(k_q[x]^\pm(v)) \). Note that \( x_i y_b = q^{vb} y_b x_i \). Hence \( vb \in m\mathbb{Z} \), or equivalently, \( b \) is a multiple of \( m/g \). The assertion follows.

(2) By the proof of part (1), every monomial \( x^s \in Z(k_q[x]^\pm(v)) \) is in \( x^M \).

Conversely, it is straightforward to check that every element in \( x^M \) is also in \( Z(k_q[x]^\pm(v)) \)

\( \square \)

Much of the work of last section can be reapplied. When \( n \) is even we define \( M \) as in Lemma 5.1(2):

\[
(E5.1.2) \quad M = \left( m\mathbb{Z}^n + \left( \frac{m}{g} \right) \mathbb{Z} \left( \sum_{i=1}^n (-1)^{i-1} e_i \right) \right) \cap H_v.
\]

Lemma 5.2. Suppose that \( n \) is even.

(1) For each coset of \( M \) in \( H_v \), there is a unique representative \( p := (p_1, \cdots, p_n) \) such that

(a) \( 0 \leq p_1 < m/g \),

(b) for each \( 1 < i < n \), \( 0 \leq p_i < m \), and

(c) \( 0 \leq p_n < vm/g \).

(2) \( w = m^2 / g^2 \).

(3) \( w \neq 0 \) in \( k \).

Proof. The following proof is similar to the proof of Lemma 4.1.

(1) Pick an arbitrary coset \( M' \) of \( M \), and let \( p = (p_1, \cdots, p_n) \in M' \). Since

\[
(m/g, -m/g, m/g, -m/g, \cdots, m/g, -m/g) \in M,
\]

we can replace \( p_1 \) by \( r \) where \( 0 \leq r \leq g \) and \( r \equiv p_1 \mod m/g \) within the coset \( M' \). Therefore we can assume, without loss of generality, that \( 0 \leq p_1 < m/g \). Furthermore, if \( t \in M \) then \( t_1 \in (m/g)\mathbb{Z} \), so there is no vector in \( M' \) whose first component is any other \( 0 \leq r' < m/g \).

For each \( 1 < i < n \), \( m(e_i - e_n) \in M \), so we can assume, without loss of generality, that \( 0 \leq p_i < m \). Furthermore, if \( t \in M \) and \( t_1 = 0 \), then each other \( t_i \in m\mathbb{Z} \). This implies that there is no other set of possible values of \( p_1, \cdots, p_{n-1} \) subject to the conditions \( 0 \leq p_i < m/g \) and \( 0 \leq p_i < m \) for every \( 1 < i < n \).
Finally, \((vm/g)e_n \in M\), so there exists a representative \(p \in M'\) subject to constraints (a)-(c) of the lemma. If \(ce_n \in M\), then \(c \in m\mathbb{Z} \cap v\mathbb{Z} = (vm/g)\mathbb{Z}\), so this representative is unique. This finishes the proof of part (1).

(2) The value \(w\) can be determined by counting the cosets by their representatives. For every sequence of integers \(p_1, \ldots, p_{n-1}\) such that \(0 \leq p_i < m/g\) and \(0 \leq p_i < m\) for all \(1 < i < n\), there are \(m/g\) possible values of \(p_n\) such that \(0 \leq p_n < vm/g\) and \((p_1, \ldots, p_n) \in H_w\). Therefore, \(w = (m/g) \cdot m^{n-2} \cdot (m/g) = m^{n-1}/g^2\).

(3) Since \(q \in k\) and \(o(q) = m\), then the characteristic of \(k\) cannot divide \(m\). Consequently \(m \neq 0\) and \(w \neq 0\) in \(k\).

\[\Box\]

**Theorem 5.3.** Let \(B = k_q[x]^{(v)}\) when \(n\) is even and let \(R\) be the center of \(B\). Suppose that \(m\) is invertible in \(k\) and that \(v\) divides \(wp(m/g - 1)\). Then

\[d_w^p(B/R) = k^\times \cdot (x_1 x_2 \cdots x_n)^{wp(m/g - 1)} = k^\times \cdot (x_1^{x_1} x_2^{x_2} \cdots x_n^{x_n})^{wp(m/g)}\]

As a consequence, \(d_w^p(B/R)\) is stable.

**Proof.** This proof is similar to the proof of Theorem 4.4.

By Lemmas 2.4(4) and 4.3, \(d_w^p(B/R) = \gcd \Lambda^{2p}\) where

\[\Lambda := \left\{ \prod_{i=1}^n x_i^{s_i} \bigg| s_i \in \mathbb{N}^n \cap (M + b_i) \right\}.

For each \(1 \leq s \leq n\), let \(f_s \in \mathbb{N}\) be maximal such that \(x_1^{s_1}\) divides all elements of \(\Lambda\). This gives \(x^{2p}\) as the gcd of \(\Lambda^{2p}\) in the over-algebra \(k_q[x]\) \(\supseteq B\) where \(f = (f_1, \ldots, f_n)\). If \(2p\mathbf{f} \in H_w^\times\), then it is the gcd of \(\Lambda^{2p}\) in \(k_q[x]^{(v)}\) as well.

By symmetry (see the proof of Theorem 4.4 for a similar argument), \(f_1 = f_2 = \cdots = f_n\), and we will only work out \(f_1\). We calculate \(f_1\) by summing the lowest powers of \(x_1\) in each coset of \(M\) (or more precisely, in each \(\mathbb{N}^n \cap (M + b_i)\) for different \(i\)). These lowest powers can be found by using the representatives outlined in Lemma 5.2 which also shows that this power cannot exceed \(m/g - 1\). For each \(0 \leq k \leq m/g - 1\), there are \(m^{n-1}/g\) cosets with lowest power \(x_1^k\). Therefore, the sum is

\[f_1 = \frac{m^{n-1}}{g} \cdot \frac{(m/g)(m/g - 1)}{2} = \frac{w}{2} \cdot \frac{m}{g} - 1\]

Now \(2p\mathbf{f} = wp(m/g - 1)(1, 1, \ldots, 1) \in H_w^\times\) as \(v\) divides \(wp(m/g - 1)\). The assertion follows from the last paragraph, and stability of \(d_w^p(B/R)\) follows clearly from the main assertion.

\[\Box\]

6. **APPLICATION I: AUTOMORPHISM GROUP**

For any algebra \(A\), let \(\text{Aut}(A)\) denote the group of all algebra automorphisms of \(A\). When \(A\) is \(\mathbb{N}\)-graded, let \(\text{Aut}_{gr}(A)\) denote the group of all graded algebra automorphisms of \(A\).

In this section we only consider the algebra \(A := k_q[x]^{(v)}\) and use \(g\) for an algebra automorphism of \(A\). First we consider an algebra automorphism \(g\) satisfying

\[(E6.0.1) \quad g((x_1^a \cdots x_n^a)^a) = k^\times \cdot (x_1^a \cdots x_n^a)^a, \quad \text{for some positive integer} \ a.

The first few lemmas discuss some easy properties of \(g\) satisfying \((E6.0.1)\).

There is a natural \(\mathbb{N}\)-grading on the skew polynomial ring \(k_q[x]\) with \(\deg x_i = e_i\) for \(i = 1, \ldots, n\). We consider \(k_q[x]^{(v)}\) as an \(\mathbb{N}\)-graded subalgebra of \(k_q[x]\). Both
\(k_q[x]\) and \(k_q[x]^{(v)}\) are also \(\mathbb{N}\)-graded by considering the total degree. We will use both gradings in this section.

For any permutation \(\pi\) of \(\{1, \ldots, n\}\), we denote the linear function \(\pi : \mathbb{Z}^n \to \mathbb{Z}^n\) determined by \(\pi : e_i \mapsto e_{\pi(i)}\). For a permutation \(\pi \in S_n\), we have
\[
(E6.0.2) \quad x_1^{s_{\pi(1)}} \cdots x_n^{s_{\pi(n)}}
\]
and denote
\[
(E6.0.3) \quad x_{\pi} := x_{\pi(1)}^{s_1} \cdots x_{\pi(n)}^{s_n}.
\]

It is clear that \(x^\pi = x_{\pi}^s\).

**Lemma 6.1.** Let \(g \in \text{Aut}(k_q[x]^{(v)})\) satisfying (E6.0.1) in parts (1)-(4).

1. The image of every monomial through \(g\) is a \(k^\times\)-multiple of a monomial.
2. \(\deg f(g) = \deg f\) for any monomial \(f\). As a consequence, \(g\) is a graded algebra automorphism.
3. The image of each \(x_i^v\) is a \(k^\times\)-multiple of some \(x_i^v\).
4. There exists a permutation \(\pi_g\) of \(\{1, \ldots, n\}\) such that each monomial \(x^s\) is mapped to a \(k^\times\)-multiple of \(x^{\pi(s)}\).
5. \(g\) satisfies (E6.0.1) if and only if, for each \(i\), there is a \(j\) such that
\[
g(x_i^v) =_{k^\times} x_j^v.
\]

**Proof.**

1. By (E6.0.1), \(g((x_1 x_2 \cdots x_n)^v) =_{k^\times} (x_1 x_2 \cdots x_n)^v\) for all \(N > 0\).
2. **Proof of (1)** Let \(f\) be any monomial in \(k_q[x]^{(v)}\). Then \(f\) is a factor of \((x_1 x_2 \cdots x_n)^v\) for some \(N > 0\). Let \(f'\) be a monomial such that \(ff' =_{k^\times} (x_1 x_2 \cdots x_n)^v\). Then
\[
g(f)g(f') = g((x_1 x_2 \cdots x_n)^v) =_{k^\times} (x_1 x_2 \cdots x_n)^v.
\]

Since \(\mathbb{N}^n\) is an ordered semigroup and \(k_q[x]^{(v)}\) is an \(\mathbb{N}^n\)-graded domain, both \(g(f)\) and \(g(f')\) are \(\mathbb{N}^n\)-homogeneous. Every \(\mathbb{N}^n\)-homogeneous element is a \(k^\times\)-multiple of a monomial. The assertion follows.

2. (2) Note that the lowest total degree of a non-scalar element in \(k_q[x]^{(v)}\) is \(v\). Applying \(g\) to the monomials \(f\) of (total) degree \(v\), we have that \(\deg g(f) \geq v = \deg f\).

Since every monomial in \(k_q[x]^{(v)}\) is a product of monomials of degree \(v\), \(\deg f\) for all monomials. By symmetry, \(\deg g^{-1}(f) \geq \deg f\). The assertion follows.

3. If \(f\) is a degree \(v\) monomial of \(k_q[x]^{(v)}\), \(f^2\) can be decomposed as
\[
f^2 =_{k^\times} f_1 f_2
\]
where \(f_1, f_2\) are degree \(v\) monomials. The decomposition is unique if and only if \(f = x_i^v\) for some \(i\). This property is invariant under \(g\).

4. We choose \(\pi\) so that, for each \(i\), we have \(g(x_i^v) =_{k^\times} x_{\pi(i)}^v\) by part (3). For any monomial \(x^s\), we have
\[
g(x^s)^v = g(x^{s\pi}) =_{k^\times} x_{\pi(1)}^{s_1} \cdots x_{\pi(n)}^{s_n} =_{k^\times} (x^{s_{\pi}(s)})^v,
\]
which implies that \(g(x^s) =_{k^\times} x^{s_{\pi}(s)}\).

5. (5) One implication is part (3) and the other implication is clear. \(\square\)

Next we wish to understand the coefficients of the image of \(g\). The next lemma deals with the case when \(\pi_g\) is the identity. For any automorphism \(g\) of \(k_q[x]^{(v)}\), we say \(\pi_g = 1\) if
\[
g(x_i^v) =_{k^\times} x_i^v.
\]
for all \(i = 1, \ldots, n\). Let \(\text{Aut}_1(k_q[x]^{(v)})\) be the subgroup of \(\text{Aut}(k_q[x]^{(v)})\) consisting of automorphisms \(g\) with \(\pi_g = 1\). It is clear that \(\text{Aut}_1(k_q[x]^{(v)}) \subseteq \text{Aut}_{g_1}(k_q[x]^{(v)})\).

**Lemma 6.2.** Retain the above notation.

1. Let \(g \in \text{Aut}_1(k_q[x]^{(v)})\). Then there exist \((c, k_2, \ldots, k_n) \in (k^\times)^n\) such that for each monomial \(x^a\) of degree \(Nv\),

\[
(E6.2.1) \quad g(x^a) = c N k_2^{s_2} \cdots k_n^{s_n} x^a.
\]

2. Conversely, given \((c, k_2, \ldots, k_n) \in (k^\times)^n\), then \((E6.2.1)\) defines a unique algebra automorphism \(g \in \text{Aut}_1(k_q[x]^{(v)})\).

As a consequence, \(\text{Aut}_1(k_q[x]^{(v)}) \cong (k^\times)^n\).

**Proof.** (1) Let \(c\) be such that \(g(x_1^1) = cx_1^1\), and let \(k_1 = 1\). For each \(i \neq 1\), let \(k_i\) be such that \(g(x_i^{v-1} x_i) = ck_i x_i^{v-1} x_i\). For any monomial \(x^a\) of degree \(Nv\) (which means that it is in \(k_q[x]^{(v)}\)), there exists a scalar \(r \in k^\times\) such that

\[
x_1^{N(v-1)} x^a = r(x_1^{v-1} x_1)^s_1 \cdots (x_1^{v-1} x_n)^s_n.
\]

Therefore

\[
c^{N(v-1)} x_1^{N(v-1)} g(x^a) = g(c N x_1^{N(v-1)} x^a) = r (ck_1 x_1^{v-1} x_1)^s_1 \cdots (ck_n x_1^{v-1} x_n)^s_n = c^{N(s_1 \cdots s_n)} k_1^{s_1} \cdots k_n^{s_n} x_1^{N(v-1)} x^a,
\]

which implies that

\[
g(x^a) = c N k_2^{s_2} \cdots k_n^{s_n} x^a.
\]

(2) This is easy and the proof is omitted. \(\square\)

For any \(g_1, g_2 \in \text{Aut}_1(k_q[x]^{(v)})\) such that \(\pi_{g_1} = \pi_{g_2}\), we have \(\pi_{g_1 \circ g_2} = 1\), and there exist \(c, k_1 = 1, k_2, \ldots, k_n \in k^\times\) such that for any monomial \(x^a\) of degree \(Nv\),

\[
(E6.2.2) \quad g_2(x^a) = c N k_1^{s_1} \cdots k_n^{s_n} g_1(x^a).
\]

The automorphism group can therefore be fully determined by determining the possible values of \(\pi_g\) and producing an example automorphism for each. We discuss possible \(\pi_g\) in the next lemma.

**Lemma 6.3.** Let \(g\) denote an automorphism of \(k_q[x]^{(v)}\) satisfying \((E6.0.1)\).

1. If \(q = \pm 1\), for every permutation \(\pi\) of \(\{1, \ldots, n\}\), there exists \(g\) such that \(\pi_g = \pi\), and for each \(x^a\), \(g(x^a) = x_\pi^a\).
2. If \(q^v \neq \pm 1\), then, for any \(g\), we have \(\pi_g = 1\).
3. If \(q^v = \pm 1\), then, for each \(m \in \mathbb{Z}\), there exists \(g\) such that \(\pi_g\) is addition by \(m\ modulo\ n\), and \(g(x^a) = x_{\pi_g}^a\).
4. If \(q^v = \pm 1\) and \(q \neq \pm 1\), then, for any \(g\), there exists \(m \in \mathbb{Z}\) such that \(\pi_g\) is addition by \(m\ modulo\ n\).

**Proof.** (1) The relations of \(k_q[x]\) are simply \(x_i x_j = qx_j x_i\) for all \(i \neq j\). Therefore any permutation \(\pi\) of the generators \(x_1, \ldots, x_n\) extends to an automorphism \(g\) of \(k_q[x]\), and \(g\) restricts to an automorphism of \(k_q[x]^{(v)}\).

(2) For any distinct \(i, j\), we have

\[
(E6.3.1) \quad (x_i^{v-1} x_j)x_i^v = r_{i,j} x_i^v (x_i^{v-1} x_j)
\]
where
\[ r_{i,j} = \begin{cases} 
q^{-v} & j < i, \\
q^v & i < j.
\end{cases} \]

We apply \( g \) to both sides of (6.3.1). Then Lemma 6.1(5) shows that \( r_{i,j} = r_{\pi(i), \pi(j)} \), where \( \pi = \pi_g \). Since \( q^v \neq q^{-w} \), we have that \( i < j \) implies \( \pi(i) < \pi(j) \). Therefore \( \pi \) is the identity.

(3) It suffices to prove the assertion in the case \( m = 1 \). Let \( s = (s_1, \cdots, s_n) \) and \( \pi(i) \equiv i + 1 \mod n \). Then
\[ x_s^\pi = x_2^{s_1} \cdots x_n^{s_{n-1}} x_1^{s_n} = q^{\sum_{i=1}^{n-1} s_is_i} x_2^{s_1} \cdots x_n^{s_{n-1}} = q^{\sum_{i=1}^{n-1} s_is_i} x_{\pi(s)}. \]

For all \( s \) and \( t \),
\[ x^s x^t = q^{\sum_{i<j} s_i t_j} x^{s+t}. \]

Let \( \alpha(s) = s_2^2 \) and define
\[ g : x^s \mapsto q^{\alpha(s)} x^s \]
for all monomials \( x^s \) in \( k_q[x] \). Note that \( g \) cannot extend to an automorphism of \( k_q[x] \). But we show next that \( g \) extends to an automorphism of \( k_q[x]^{(v)} \).

To show this, it suffices to show that
\[ g(x^s)g(x^t) = g(x^s x^t) = q^{\sum_{i<j} s_i t_j} g(x^{s+t}) \]
for all \( x^s \) and \( x^t \) in \( k_q[x]^{(v)} \). Using the above computation, we have
\[ g(x^s x^t) = q^{\sum_{i<j} s_i t_j} g(x^{s+t}) = q^{\alpha(s+t)} q^{\sum_{i<j} s_i t_j} q^{\sum_{i=1}^{n-1} (s_n+t_n)(s_i+t_i)} x_{\pi(s+t)}, \]
and
\[ g(x^s)g(x^t) = q^{\alpha(s)+\alpha(t)} q^{\sum_{i=1}^{n-1} s_i t_i} q^{\sum_{i=1}^{n-1} t_i s_i} x_{\pi(s)} x_{\pi(t)} \]
\[ = q^{\alpha(s)+\alpha(t)} q^{\sum_{i=1}^{n-1} s_i t_i} q^{\sum_{i=1}^{n-1} t_i s_i} q^{\sum_{i<j} \pi(s) \pi(t)} x_{\pi(s+t)}. \]

By direct calculation, the difference between the \( q \)-powers in the expressions of \( g(x^s x^t) \) and \( g(x^s)g(x^t) \) is \( 2 s_n \sum_{i=1}^{n-1} t_i \). Since \( v \) divides \( \sum_{i=1}^{n-1} t_i \) and \( q^v = \pm 1 \), we have \( q^{2 s_n \sum_{i=1}^{n-1} t_i} = (\pm 1)^2 = 1 \). Therefore \( g(x^s x^t) = g(x^s)g(x^t) \) so \( g \) is an algebra automorphism.

(4) For distinct \( i, j \), let \( y_{i,j} = x_i^{v-1} x_j \). For any distinct \( i, j, k \), we have
\[ y_{i,j} y_{i,k} = r y_{i,k} y_{i,j} , \]
where
\[ r = \begin{cases} 
q^{-1} & \text{if } i < j < k \text{ or } j < k < i \text{ or } k < i < j, \\
q & \text{if } i < k < j \text{ or } k < j < i \text{ or } j < i < k. 
\end{cases} \]

Recall \( q \neq q^{-1} \). For any \( i, j \), the number of values of \( k \) that yield \( r = q \) is equal to \( j - i - 1 \mod n \). Since this is true for all \( i \neq j \), we have \( \pi_g(j) - \pi_g(i) - 1 \equiv j - i - 1 \mod n \). Therefore \( \pi_g(j) - \pi_g(i) \equiv j - i \mod n \), and the assertion follows by letting \( m = \pi_g(n) \).

We are now ready to prove Theorems 0.1 and 0.2.

Proof of Theorem 0.1. For each \( \sigma \in S_n \), let \( F_\sigma \) be the algebra automorphism of \( k_{-1}[x] \) induced by sending \( x_i \) to \( x_{\sigma(i)} \) for all \( i \). This automorphism restricts to an algebra automorphism of \( k_{-1}[x]^{(v)} \), which is still denoted by \( F_\sigma \) — see Lemma 6.3(1). Then the subgroup generated by all \( \{ F_\sigma \mid \sigma \in S_n \} \) is isomorphic to \( S_n \).
Now assume that \( n \) and \( v \) have different parity and that \( g \) is an algebra automorphism of \( k[-1][x]^{(v)} \). Recall that \( m = 2 \). If \( n \) is odd, \( \gcd(m, v) = 2 \) and we can apply Theorem 1.4. If \( n \) is even, \( \gcd(m, v) = 1 \), so we can apply Theorem 5.3. In both cases, by Theorem 1.4 or 5.3, the \( v \)-power discriminant \( d^{[v]}_{m}(k[-1][x]^{(v)}/R) \) is of the form \( (x_1^{n} \cdots x_n^{n})^N \) for some \( N > 0 \). By Lemma 2.1(1), this discriminant is \( g \)-invariant. This means that \( g \) satisfies (E6.0.1). Let \( \pi_g \) be the permutation defined in Lemma 6.1(4). It is easy to see that the map \( \phi : g \to F_{\pi_g} \) is a surjective group homomorphism from \( \text{Aut}(k[-1][x]^{(v)}) \) to \( S_n \) with kernel being \( \text{Aut}_1(k[-1][x]^{(v)}) \). By Lemma 6.2 we have \( \text{Aut}_1(k[-1][x]^{(v)}) \cong (k^x)^n \). Therefore
\[
\text{Aut}(k[-1][x]^{(v)}) \cong S_n \rtimes \text{Aut}_1(k[-1][x]^{(v)}) \cong S_n \rtimes (k^x)^n.
\]
\( \square \)

The proof of Theorem 0.2 is similar.

**Proof of Theorem 0.2.** The proofs of (1) and (2) are similar, we only provide the proof of (2) here.

(2) Under hypotheses (a) or (b), we use Theorem 1.4 or 5.3 to conclude that the \( v \)-power discriminant \( d^{[v]}_{m}(k_q[x]^{(v)}/R) \) is of the form \( (x_1^{n} \cdots x_n^{n})^N \) for some \( N > 0 \). By Lemma 2.1(1), this discriminant is \( g \)-invariant. This means that \( g \) satisfies (F6.0.1). By Lemma 6.3(2), we have \( \pi_g = 1 \), or equivalently, \( g \in \text{Aut}_1(k_q[x]^{(v)}) \). Therefore
\[
\text{Aut}(k_q[x]^{(v)}) = \text{Aut}_1(k_q[x]^{(v)}) \cong (k^x)^n.
\]
\( \square \)

7. **Application II: cancellation problem**

The second application of the discriminant method is the cancellation problem. We need to recall some definitions and results from [BZ].

**Definition 7.1.** [BZ] Definition 1.1 Let \( A \) be an algebra.

1. We call \( A \) cancellative if \( A[t] \cong B[t] \) for some algebra \( B \) implies that \( A \cong B \).
2. We call \( A \) strongly cancellative if, for any \( d \geq 1 \), \( A[t_1, \cdots, t_d] \cong B[t_1, \cdots, t_d] \) for some algebra \( B \) implies that \( A \cong B \).
3. We call \( A \) universally cancellative if, for any \( k \)-flat finitely generated commutative domain \( R \) such that \( R/I = k \) for some ideal \( I \subset R \) and any \( k \)-algebra \( B \), any algebra isomorphism \( A \otimes R \cong B \otimes R \) implies that \( A \cong B \).

The first result is

**Lemma 7.2.** [BZ] Proposition 1.3 Let \( k \) be a field and \( A \) be an algebra with center \( C(A) = k \). Then \( A \) is universally cancellative, hence, strongly cancellative.

We only need the following definition for connected graded domains.

**Definition 7.3.** [CPWZ1] Definition 2.1(2) Let \( A \) be a connected graded domain generated by \( A_1 = \bigoplus_{i=1}^{f} k x_i \). An element \( f \in A \) is called dominating if, for every testing \( \mathbb{N} \)-filtered PI algebra \( T \) with \( \text{gr}_F T \) being a connected graded domain, and for every testing subset \( \{y_1, \cdots, y_r\} \subseteq T \) that is linearly independent in the quotient \( k \)-module \( T/F_0T \), there is a presentation of \( f \) of the form \( f(x_1, \cdots, x_r) \) in the free algebra \( k(x_1, \cdots, x_r) \) such that the following hold: either \( f(y_1, \cdots, y_r) = 0 \), or

1. \( \deg f(y_1, \cdots, y_r) \geq \deg f \), and
(b) \( \deg f(y_1, \cdots, y_r) > \deg f \), further, \( \deg y_{i_0} > 1 \) for some \( i_0 \).

**Lemma 7.4.** [BZ, Theorem 4.6] Let \( A \) be a connected graded PI domain generated in degree 1, of finite Gelfand-Kirillov dimension. Suppose that the discriminant power \( (d_w^p(A/C))^a \) is dominating for some \( p, w \) and \( a \). Then \( A \) is strongly cancellative.

**Proof.** The original [BZ, Theorem 4.6] was proved for discriminant \( d_w(A/C) \). But the proof works for this more general setting when [BZ, Lemma 4.5(2)] is replaced by Lemma 2.2. So we are not going to repeat the rest of the proof. \( \square \)

The following lemma is easy.

**Lemma 7.5.** Let \( A \) be the algebra \( k_q[x]^{(v)} \) for some \( n, q, v \). Let \( f \) be an element of the form \( (x_1 \cdots x_n)^N \) for some \( N > 0 \). Then there is an integer \( a > 0 \) such that \( f^{ab} \) is dominating for all integer \( b > 0 \).

**Proof.** Let \( \Phi := \{ x_1^d_1 \cdots x_n^{d_n} \mid d_s \geq 0, \sum_{s=1}^n d_s = v \} \) be the set of monomials of degree \( v \), which is a \( k \)-basis of the degree 1 component of \( A \) after regrading. Let \( P \) be the product of elements in \( \Phi \). Then \( P = k_x (x_1 \cdots x_n)^a \) for some \( a > 0 \). Then \( f^{ab} = k_x P^{bN} \). It suffices to show that \( P^{bN} \) is dominating. But this is [CPWZ1, Lemma 2.2(1)]. \( \square \)

Now we are ready to prove Theorem 0.4. In fact we prove that the algebras are strongly cancellative.

**Theorem 7.6.** Let \( A \) be the algebra \( k_q[x_1, \cdots, x_n]^{(v)} \) where \( v \) is a positive integer and let \( m \geq 2 \) be the order of \( q \). Suppose that one of the following is true.

(a) \( n \) is even and \( m \) does not divide \( v \).
(b) \( n \) is odd and \( \gcd(m, v) \neq 1 \).

Then \( A \) is strongly cancellative.

**Proof.** Under the hypotheses (a) or (b), by Theorems 4.4 and 5.3, there is some \( p \) and \( w \) such that \( d_w^p(A/C) \) is of the form \( (x_1 \cdots x_n)^N \) for some \( N > 0 \). By Lemma 7.5, the element \( (d_w^p(A/C))^{ab} \) is dominating for some \( a > 0 \) and all \( b > 0 \). The assertion follows from Lemma 7.4. \( \square \)

We make some comments and remarks for the rest of this section.

**Lemma 7.7.** Let \( \{ A^1, \cdots, A^s \} \) be a set of algebras as in Theorem 7.6(a,b) with possible repetition. Let \( A \) be the tensor product \( A^1 \otimes \cdots \otimes A^s \). Then some \( p \)-power discriminant of \( A \) over its center is dominating.

**Proof.** Each algebra \( A' \) has some \( p \)-power discriminant (over its center) that is dominating by Theorems 4.4 and 5.3. The assertion follows from Lemma 2.6(3) together with induction. Some of the hypotheses in Lemma 2.6 can be verified by using Lemma 5.3. \( \square \)

**Remark 7.8.** Let \( A \) be as in Lemma 7.7.

(1) By using the discriminant method [CPWZ1, CPWZ2], we obtain that every automorphism of \( A \) is graded. Therefore it is a linear algebra problem to determine the full automorphism group of \( A \). In many case (when the Gelfand-Kirillov dimension of \( A \) is small), one can explicitly work out the full automorphism group of \( A \).

(2) By Lemma 7.4 and 7.7, \( A \) is strongly cancellative.
8. Tits alternative

Recall that, in the last few sections, we are only considering the case when $q \neq 1$, which implies that

\[(E8.0.1) \quad k \neq \mathbb{Z}/(2).\]

In this section, if $q = 1$, we will further assume that $k \neq \mathbb{Z}/(2)$. Note that \[(E8.0.1)\] is one of the hypotheses in [CPWZ3, Proposition 2.5].

Firstly we consider the case when $n = 2s + 1$ is odd and $g := \gcd(m, v) = 1$ where $m \geq 2$ is the order of $q$. Since $\gcd(m, v) = 1$, there are two positive integers $\alpha$ and $\beta$ such that

\[(E8.0.2) \quad (\alpha + s)m - \beta v = 1.\]

**Lemma 8.1.** Retain the above hypotheses.

1. The following are locally nilpotent derivations of $k_q[x]$ of degree $\beta v$.

   (a) \[\partial_1 : x_i \rightarrow \begin{cases} \frac{x_2^m (x_2 x_3 \cdots x_5 \cdots x_2 x_{2s+1})}{x_i} & i = 1, \\ 0 & i \neq 1. \end{cases}\]

   (b) \[\partial_3 : x_i \rightarrow \begin{cases} \frac{x_2^m (x_2 x_3 \cdots x_5 \cdots x_2 x_{2s+1})}{x_i} & i = 3, \\ 0 & i \neq 3. \end{cases}\]

2. Let $g_1 = \exp(\partial_1)$ and $g_3 = \exp(\partial_3)$. Then $g_1$ and $g_3$ are two automorphisms of $k_q[x]$ that generate a free subgroup of $\text{Aut}(k_q[x])$.

3. Both $g_1$ and $g_2$ send a homogeneous element $f$ of degree $h$ to a linear combination of homogeneous elements of degrees $h + \beta v n$. As a consequence, both $g_1$ and $g_2$ restrict to algebra automorphisms of $k_q[x]^{(v)}$.

**Proof.** (1) (a) The degree of $\partial_1$ is $\alpha m + sm - 1 = \beta v$. To check that $\partial_1$ is a derivation we just verify that $\partial_1(x_i x_j - qx_j x_i) = 0$ for all $1 \leq i < j \leq n$, which is straightforward by the choice of $\partial_1(x_i)$. It is clear that $\partial_1^2(x_i) = 0$. Then $\partial_1^2$ is locally nilpotent. The proof of (1)(b) is similar.

(2) By definition, for any power $d$, we have

\[g_1^d(x_i) = \begin{cases} x_1 + d x_2^{\alpha m} (x_2 x_3 \cdots x_5 \cdots x_2 x_{2s+1}) & i = 1, \\ x_i & i \neq 1, \end{cases}\]

and

\[g_3^d(x_i) = \begin{cases} x_1 + d x_2^{\alpha m} (x_2 x_3 \cdots x_5 \cdots x_2 x_{2s+1}) & i = 1, \\ x_i & i \neq 1, \end{cases}\]

Let $R$ be the subalgebra of $k_q[x]$ generated by $x_2, x_4, x_5, \cdots, x_n$. Then $g_1$ satisfies [CPWZ3, (E2.1.1)] with $a_0 = 0$ and $a_1 = x_2^{\alpha m} (x_2 x_3 \cdots x_5 \cdots x_2 x_{2s+1})$ and $g_3$ satisfies [CPWZ3, (E2.1.2)] (when $x_2$ is replaced by $x_3$) with $b_0 = 0$ and $b_1 =$.
\[x_2^{m}(x_1^{m-2}x_2x_4x_5^{m-1} \cdots x_2x_5^{m-1}).\] It is clear that \(a_1b_1\) is transcendental over \(k\). Also \(R + Rx_1 + Rx_3\) is a free \(R\)-module of rank 3. Thus we have checked all hypotheses of \([\text{CPWZ3}, \text{Proposition 2.5}].\) By \([\text{CPWZ3, Proposition 2.5(2)}]\) and its proof, the subgroup generated by \(g_1\) and \(g_3\) is free.

(3) Since \(\partial_1\) has degree \(\beta v\), the first assertion follows because \(g_1 = \exp(\partial_1)\). In particular, \(g_1\) maps a homogeneous element of degree \(v\) to a linear combination of homogeneous elements of degrees in \(v + \beta vN\). Thus \(g_1\) restricts to an automorphism of \(k_q[x]^{(v)}\). The same statement holds for \(g_3\). \(\square\)

**Lemma 8.2.** Let \(A\) be a connected graded domain generated in degree one. Let \(g\) be an automorphism of \(k_q[x]\) such that

(a) \(g(x) = x + \) higher degree terms for all \(x\) of degree 1, and

(b) \(g\) and \(g^{-1}\) send a homogeneous element of degree \(v\) to a linear combination of homogeneous elements of degrees in \(vN\).

Then \(g\) restricts to an automorphism \(g'\) of \(A^{(v)}\). Further \(g\) is the identity if and only if \(g'\) is.

**Proof.** The first assertion is easy to show. Now we assume that \(g'\) is the identity. Then \(g'(x^v) = x^v\) for all \(x \in A\) of degree 1. This implies that

\[v \deg g(x) = \deg g(x^v) = \deg g'(x^v) = \deg x^v = v.\]

Hence \(\deg g(x) = 1\) and \(g(x) = x\) by hypothesis (a). \(\square\)

Now we are ready to prove the first Tits alternative theorem.

**Theorem 8.3.** Suppose that \(n\) is odd.

(1) If \(\gcd(m, v) > 1\), then \(\text{Aut}(k_q[x]^{(v)})\) is virtually abelian.

(2) If \(\gcd(m, v) = 1\), then \(\text{Aut}(k_q[x]^{(v)})\) contains a free subgroup of rank 2.

**Proof.** (1) This follows from Theorems 0.1 and 0.2

(2) Let \(g_1\) and \(g_3\) be the automorphisms of \(k_q[x]\) given in Lemma 8.1. By Lemma 8.2, the elements \(g_1\) and \(g_3\) generate a free subgroup of rank 2, by Lemma 8.1(3), they restrict to automorphisms \(g'_1\) and \(g'_3\) of \(k_q[x]^{(v)}\). We claim that \(g'_1\) and \(g'_3\) generates a free subgroup of \(\text{Aut}(k_q[x]^{(v)})\). Let \(\text{Id} \neq g \in \langle g_1, g_3 \rangle \subseteq \text{Aut}(k_q[x])\) and let \(g'\) be the corresponding element in \(\langle g'_1, g'_3 \rangle \subseteq \text{Aut}(k_q[x]^{(v)})\). By Lemma 8.2 the element \(g'\) is not the identity. Therefore \(\langle g'_1, g'_3 \rangle\) is a free group of rank 2. \(\square\)

Secondly we consider the case when \(n\) is even and write \(n = 2s\). As before let \(m\) be the order of \(q\). We consider the case where \(m\) divides \(v\) and write \(v = m\gamma\).

**Lemma 8.4.** Let \(n = 2\) and \(v = m\gamma\) and \(A = k_q[x]^{(v)}\). Then \(\text{Aut}(A)\) contains a free subgroup of rank 2.

**Proof.** By direct computation or equations similar to (E3.0.1), if \(n = 2\), \(A\) is isomorphic to the commutative ring \(k[x_1, x_2]^{(v)}\). So we identify these two algebras.

Consider two derivations

\[\partial_1 : x_1 \to x_2^{v+1}, \quad x_2 \to 0\]

and

\[\partial_2 : x_1 \to 0, \quad x_2 \to x_1^{v+1}.\]
Let $g_1 = \exp(\partial_1)$ and $g_2 = \exp(\partial_2)$. Then, by [CPWZ3, Proposition 2.5], $g_1$ and $g_2$ generate a free subgroup of rank 2. Since the degree of $\partial_i$ is $v$, we see that $g_1$ and $g_2$ restrict to automorphisms $g_1'$ and $g_2'$ of $\mathbb{k}[x_1, x_2]^{(v)}$. By Lemma 8.2, the subgroup of $\text{Aut}(A)$ generated by $g_1'$ and $g_2'$ is free of rank 2. □

**Proof of Theorem 0.7.** When $n$ is odd, this follows from Theorem 8.3. When $n = 2$, this follows from Theorem 0.2 and Lemma 8.4. □

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