2-Reconstructibility of Weakly Distance-Regular Graphs

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Abstract

A graph is $\ell$-reconstructible if it is determined by its multiset of induced subgraphs obtained by deleting $\ell$ vertices. We prove that strongly regular graphs with at least six vertices are 2-reconstructible.

The $k$-deck of an $n$-vertex graph is the multiset of its $\binom{n}{k}$ induced subgraphs with $k$ vertices. The famous Reconstruction Conjecture of Ulam [4, 12] asserts that when $n \geq 3$, every $n$-vertex graph is determined by its $(n-1)$-deck. One can consider more generally whether an $n$-vertex graph is determined by its $(n-\ell)$-deck. A graph or graph property is $\ell$-reconstructible if it is determined by the deck obtained by deleting $\ell$ vertices. In light of the following observation, we seek the maximum $\ell$ such that a graph is $\ell$-reconstructible.

The observation holds because each card in the $k'$-deck appears as an induced subgraph in the same number of cards in the $k$-deck.

Observation 1. For $k' < k$, the $k$-deck of a graph determines the $k'$-deck.

In light of this observation, Manvel [8, 9] posed a more general version of the Reconstruction Conjecture, and he called this more general version “Kelly’s Conjecture”.

Conjecture 2 ([8, 9]). For each natural number $\ell$, there is a threshold $M_\ell$ such that every graph with at least $M_\ell$ vertices is $\ell$-reconstructible.

The original Reconstruction Construction is $M_1 = 3$. Since the graph $C_4 + K_1$ and the tree $K_{1,3}'$ obtained by subdividing one edge of $K_{1,3}$ have the same 3-deck, $M_2 \geq 6$. Since $P_{2\ell}$ and $C_{\ell+1} + P_{\ell-1}$ have the same $\ell$-deck ([11]), in general $M_\ell \geq 2\ell + 1$, and a difficult result of Nýdl [10] implies that $M_\ell$, if it exists, must grow superlinearly. (We use $C_n$, $P_n$, $K_n$ for the cycle, path, and complete graph with $n$ vertices, $K_{r,s}$ for the complete bipartite graph with

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parts of sizes \( r \) and \( s \), and \( G + H \) for the disjoint union of graphs \( G \) and \( H \). Our graphs have no loops or multi-edges.)

Kostochka and West [7] surveyed results on \( \ell \)-reconstructibility of graphs, so we do not provide an exhaustive review here. One theme is to prove that graphs in a particular family are \( \ell \)-reconstructible. We consider 2-reconstructibility of a special family of regular graphs, where a graph is regular if all vertices have the same degree. One of the first results about reconstruction is that regular graphs having at least three vertices are 1-reconstructible (Kelly [5]). By Observation 1, the \((n-1)\)-deck of a graph \( G \) determines the 2-deck and hence the number of edges, so in each card of the \((n-1)\)-deck we know the degree of the missing vertex. We then know we have a \( k \)-regular graph, and in any card the neighbors of the missing vertex are those with degree \( k-1 \) in the card.

Bojan Mohar asked whether regular graphs are 2-reconstructible. Although Chernyak [1] proved that the degree list is 2-reconstructible for graphs with at least six vertices (\( C_4 + K_1 \) and \( K_{1,3} \) show that this is sharp), knowing that the graph is \( k \)-regular does not generally provide enough information to decide which of the deficient vertices in a card adjacent to which of the two vertices missing from the card (those of degree \( k-2 \) in the card are adjacent to both missing vertices). Nevertheless, Kostochka, Nahvi, West, and Zirlin [6] proved that 3-regular graphs are 2-reconstructible.

In this note, we consider regular graphs of higher degree but restrict the structure of common neighbors. A graph is strongly regular with parameters \((k, \lambda, \mu)\) if it is \( k \)-regular, every two adjacent vertices have exactly \( \lambda \) common neighbors, and every two nonadjacent vertices have exactly \( \mu \) common neighbors. Discussion of strongly regular graphs and their properties can be found for example in the book by van Lint and Wilson [13].

Most of our argument applies to graphs in a more general family. A graph is distance-regular if for any two vertices \( u \) and \( v \), the number of vertices at distance \( i \) from \( u \) and distance \( j \) from \( v \) depends only on \( i, j \), and the distance between \( u \) and \( v \). For graphs with diameter \( d \), an equivalent condition is the existence of parameters \((b_0, \ldots, b_d; c_0, \ldots, c_d)\) (called the intersection array of \( G \)) such that for all \( u, v \in V(G) \) separated by distance \( j \), the numbers of neighbors of \( u \) having distance \( j+1 \) or \( j-1 \) from \( v \) are \( b_j \) and \( c_j \), respectively (Brouwer et al.[3]). A strongly regular graph with parameters \((k, \lambda, \mu)\) is distance-regular with intersection array \((k; k-\lambda-1, 0; 0, 1, \mu)\). In fact, a non-complete distance-regular graph is strongly regular if and only if it has diameter 2 (Biggs [2]).

A disjoint union of complete graphs with at least six vertices is 2-reconstructible, because we know the degree list and we know that no three vertices induce \( P_3 \). Also, connectedness of an \( n \)-vertex graph is determined by the \((n-2)\)-deck when \( n \geq 6 \) (Manvel [9]). Hence in our discussion we may assume that we are given the deck of an \( n \)-vertex connected graph. We will prove 2-reconstructibility for all strongly regular graphs and all graphs in a family that includes all distance-regular graphs where vertices at distance 2 have at least two common neighbors. The latter family includes all distance-regular graphs.
We define a regular graph to be *weakly distance-regular* if any two adjacent vertices have \( \lambda \) common neighbors and any two vertices separated by distance 2 have \( \mu' \) common neighbors. In particular, we are requiring the existence of only one of the parameters of distance-regular graphs for nonadjacent vertices. We prove that strongly regular graphs (even with \( \mu = 1 \)) and weakly distance-regular graphs with \( \mu' \geq 2 \) are 2-reconstructible.

For strongly regular graphs, the method is analogous to the proof of 1-reconstructibility of regular graphs. We use all the cards in the \((n - 2)\)-deck to recognize that any graph having this deck is strongly regular and to determine the parameters \((k, \lambda, \mu)\). We then use a single card to reconstruct the graph.

**Theorem 3.** *Strongly regular graphs with at least six vertices are 2-reconstructible.*

*Proof.* Let \( G \) be an \( n \)-vertex graph, where \( n \geq 6 \), and let \( D \) be the \((n - 2)\)-deck of \( G \). By the result of Chernyak [1], \( D \) determines the degree list of \( G \) and hence whether \( G \) is \( k \)-regular. If so, then any card \( C \) in \( D \) is missing \( 2k - 1 \) or \( 2k \) of the \( kn/2 \) edges in \( G \), depending on whether the two omitted vertices are adjacent or not. Hence we also see whether the vertices omitted by \( C \) are adjacent. Their number of common neighbors is the number of vertices with degree \( k - 2 \) in \( C \). The graph \( G \) is strongly regular with parameters \((k, \lambda, \mu)\) if and only if that number is \( \lambda \) in each card missing \( 2k - 1 \) edges and \( \mu \) in each card missing \( 2k \) edges.

Having recognized that \( G \) is strongly regular with parameters \((k, \lambda, \mu)\), consider one card \( C \), and let \( u \) and \( v \) be the two omitted vertices. We know whether \( u \) and \( v \) are adjacent. If \( G \) is not \( K_n \), which we can determine, then we may choose \( C \) so that \( u \) and \( v \) are not adjacent. We know the \( \mu \) common neighbors of \( u \) and \( v \), and we know the set \( S \) of \( 2k - 2\mu \) vertices that are adjacent to exactly one of \( \{u, v\} \).

For \( x, y \in S \), each of \( x \) and \( y \) has one neighbor in \( \{u, v\} \); the neighbors may be the same or distinct. The vertices \( x \) and \( y \) have \( \lambda \) or \( \mu \) common neighbors in \( G \), depending on whether they are adjacent. We see in \( C \) whether they are adjacent, so we know their number of common neighbors in \( G \); call it \( \rho \). If \( x \) and \( y \) have \( \rho \) common neighbors in \( C \), then they have different neighbors in \( \{u, v\} \); if they have \( \rho - 1 \) common neighbors in \( C \), then they have the same neighbor in \( \{u, v\} \).

This labels each pair of vertices in \( S \) as “same” or “different”. Also, the relation defined by “same” is an equivalence relation. Hence it partitions \( S \) into two sets. We assign one of those sets to the neighborhood of \( u \) and the other to the neighborhood of \( v \). It does not matter which set we assign to which neighborhood, because in both cases we obtain the same graph, and it is \( G \). \( \square \)

The proof of Theorem 3 applies to all connected strongly regular graphs. In particular, we allow the possibility \( \mu = 1 \). For the more general class of weakly distance-regular graphs, we need to work harder, and the proof does not apply to the case \( \mu' = 1 \).
Theorem 4. Weakly distance-regular graphs with at least six vertices and parameters \((k, \lambda, \mu')\) with \(\mu' \geq 2\) are 2-reconstructible.

Proof. As in the proof of Theorem 3, we know the degree list and thus can recognize both that \(G\) is \(k\)-regular and whether the missing vertices in any card are adjacent in \(G\). The number of common neighbors of the two missing vertices in a card is the number of vertices having degree \(k-2\) in the card. To recognize that \(G\) is in the specified class, we check that these numbers all equal \(\lambda\) when the missing vertices are adjacent and equal \(\mu'\) when the missing vertices are nonadjacent and the number is positive. The number is 0 when the distance between the missing vertices in \(G\) exceeds 2. Hence we can recognize that \(G\) is weakly distance-regular with parameters \((k, \lambda, \mu')\) (including when \(\mu' = 1\)).

Given a card \(C\), again let \(S\) be the set of vertices adjacent to exactly one of the two vertices \(u\) and \(v\) missing from \(C\); these are the vertices having degree \(k-1\) in \(C\). Let \(x\) and \(y\) be two vertices in \(S\). We see in \(C\) whether \(x\) and \(y\) are adjacent. If so, then they have \(\lambda\) common neighbors in \(G\). Their number of common neighbors in \(C\) is then \(\lambda\) or \(\lambda - 1\), which tells us whether they have the same neighbor in \(\{u, v\}\).

Since \(\mu' \geq 2\), when \(x\) and \(y\) are nonadjacent in \(G\) we see a common neighbor of \(x\) and \(y\) in \(C\) if and only if the distance between \(x\) and \(y\) in \(G\) is 2. Hence for the pairs of vertices in \(S\) separated by distance 2, we can again tell whether their neighbors in \(\{u, v\}\) are the same or different. The pairs of vertices in \(S\) that are separated by distance more than 2 in \(G\) are those having no common neighbor in \(C\). They must have distinct neighbors in \(\{u, v\}\), and the distance between them is 3.

With these arguments, we know for all pairs of vertices in \(S\) whether their neighbors in \(\{u, v\}\) are the same or different. Hence again we have two equivalence classes and assign one class to the neighborhood of each of these vertices to complete the reconstruction of \(G\). \(\Box\)

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