Feedback game on 3-chromatic Eulerian triangulations of surfaces

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Abstract

Recently, a new impartial game on a connected graph has been introduced, called a feedback game, which is a variant of the generalized geography. In this paper, we study the feedback game on 3-chromatic Eulerian triangulations of surfaces. We prove that the winner of the game on every 3-chromatic Eulerian triangulation of a surface all of whose vertices have degree 0 modulo 4 is always fixed. Moreover, we also study the case of 3-chromatic Eulerian triangulations of surfaces which have at least two vertices whose degrees are 2 modulo 4. In addition, as a concrete class of such graphs, we consider the octahedral path, which is obtained from an octahedron by adding octahedra in the same face, and determine the winner of the game on those graphs.

Keywords: Feedback game, Octahedron addition, Eulerian triangulation.

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1 Introduction

All graphs considered in this paper are finite simple undirected graphs. We say that a graph $G$ is Eulerian if every vertex of $G$ has even degree. We refer the reader to [3] for the basic terminologies.

Recently, a new impartial game on a connected graph has been introduced, called a feedback game.

Definition 1.1 ([8]). There are two players, Alice and Bob. Alice starts the game. For a given connected graph $G$ with a starting vertex $s$, a token is put on $s$. They alternately move the token on a vertex $u$ to another vertex $v$ which is adjacent to $u$ and the edge $uv$ will be deleted after he or she moves the token. The first player who is able to move the token back to the starting vertex $s$ or to an isolated vertex (after removing an edge used by the last move) wins the game.

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The feedback game is a variant of the generalized geography, which is a most famous two-player impartial game played on (directed) graphs. It is known that for many variants of the game to determine the winner of the game is PSPACE-complete (see [5, 7, 10] for example). Similarly, the decision problem of the feedback game is PSPACE-complete even if the maximal degree of a given connected graph is 3 or it is 4 and the graph is Eulerian [8]. Thus it is an important problem to determine the winner of the game on concrete classes of graphs.

In the recent paper [8], they focus on connected Eulerian graphs, since the token is finally moved back to the starting vertex if a given connected graph is Eulerian. It is trivial that Bob wins the game if a given connected Eulerian graph is bipartite, and hence, they deal with non-bipartite ones in that paper. In particular, they gave sufficient conditions for Bob to win the game on two concrete classes of such graphs, triangular grid graphs and toroidal grid graphs, and they also conjectured that those conditions are necessary.

One of challenging problems in this study is to determine the winner of the game on 3-chromatic graphs. In this paper, we focus on the feedback game on 3-chromatic Eulerian triangulations of surfaces and determine the winner of the game, where a triangulation is a graph of a surface with each face triangular and every two faces sharing at most one edge. (In this paper, a surface means a closed surface, i.e., a 2-dimensional manifold without boundary.) Let $c_{4,k}(G)$ be the number of vertices of a graph $G$ whose degrees are congruent to $k$ modulo $n$; note that $G$ satisfies $|V(G)| = c_{4,0}(G) + c_{4,2}(G)$ if $G$ is Eulerian.

We begin with the following general result and a remark for the game on Eulerian triangulations.

**Theorem 1.2.** Let $G$ be a 3-chromatic Eulerian triangulation of a surface. If $c_{4,2}(G) = 0$, then Bob always wins the game on $G$ with any starting vertex.

**Remark 1.3.** It is well known that every Eulerian triangulation of the sphere is 3-chromatic. Moreover, we can see by the Handshaking Lemma that we have $c_{4,2}(G) \geq 2$ for every 3-chromatic Eulerian triangulation $G$ of a surface with $c_{4,2}(G) \neq 0$ and that if $c_{4,2}(G) = 2$, then the two such vertices belong to the same partite set of $G$.

By the above theorem and remark, we mainly deal with Eulerian triangulations $G$ of surfaces with $c_{4,2}(G) \geq 2$. It is relatively easy to see that Bob wins the game on every Eulerian double wheel, which is an Eulerian triangulation on the sphere with $n \geq 6$ vertices and degree sequence $(4, 4, \ldots, 4, n-2, n-2)$, for any starting vertex (Proposition 2.5). On the other hand, we find Eulerian triangulations on which Alice wins the game, as follows.

**Theorem 1.4.** For any surface $F^2$ and any positive integer $m \geq 2$, there exists a 3-chromatic Eulerian triangulation $G$ of $F^2$ with $c_{4,2}(G) = m$ and a starting vertex such that Alice wins the game on $G$.

Thus we study one of reasonable concrete classes of Eulerian triangulations, called an octahedral path, denoted by $E_n$. (See Figure 1) This graph is obtained from the octahedron (i.e., $E_1$) by repeatedly adding a new octahedron in the same face, which way looks like a path. By the construction, the degree sequence of $E_n$ is $(4, 4, 4, 4, 4, 6, 6, \ldots, 6)$. 

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Note that every Eulerian triangulation of the sphere can be constructed from $E_1$ by two operations, one of which is the octahedron addition. (The octahedron addition is to add three vertices $a_1, a_2, a_3$ to a face $u_1, u_2, u_3$ of an Eulerian triangulation and edges so that $a_iu_ju_k$ and $a_i a_j u_k$ are faces of the resulting graph for $\{i, j, k\} = \{1, 2, 3\}$. See [2, Theorem 3] for more detail.)

![Octahedral path $E_n$](image)

The feedback game on the graph $E_n$ with its starting vertex $u_p$ or $v_p$ or $w_p$ is denoted by $E(n, p)$, where $n \geq 1$ and $0 \leq p \leq n$. Notice that the choice of $u_p, v_p, w_p$ does not matter. We give the complete characterization whether Alice or Bob wins the game on $E(n, p)$:

**Theorem 1.5.** For any integers $n$ and $p$ with $n \geq 1$ and $0 \leq p \leq n$, the winner of the game $E(n, p)$ is determined as follows:

| $n \equiv 0$ | $n \equiv 1$ | $n \equiv 2$ |
|--------------|--------------|--------------|
| $p \equiv 0$ | Alice        | Bob          |
| $p \equiv 1$ | Alice        | Bob          |
| $p \equiv 2$ | Alice        | Alice        |

Here, all "≡" mean that “congruent modulo 3”.

In the next section, we introduce an important concept for the feedback game, called an *even kernel*, and we prove Theorems 1.2 and 1.4 and Proposition 2.5. In Section 3, we shall give a proof of Theorem 1.5.

## 2 3-chromatic Eulerian triangulations

We first introduce the key notion, called an *even kernel*, which is first introduced in [5].
**Definition 2.1** (Even kernel [5]). Let $G$ be a connected graph with a starting vertex $s$. An **even kernel** for $G$ is a nonempty subset $S \subset V(G)$ such that

- $s \in S$,
- no two vertices in $S$ are adjacent, and
- every vertex not in $S$ is adjacent to an even number (possibly 0) of vertices in $S$.

**Definition 2.2** (Even kernel graph [8]). Let $S$ be an even kernel of a connected Eulerian graph $G$ with a starting vertex. An **even kernel graph** with respect to $S$ is a bipartite subgraph $H_S$ with the bipartition $V(H_S) = S \cup R$ and $E(H_S) = E_G(S, R)$, where $R$ is a superset of the set $N_G(S) = \{v \in V(G) \setminus S : v \text{ is adjacent to a vertex } u \in S\}$ and $E_G(A, B)$ denotes the set of edges between $A$ and $B$.

**Remark 2.3.** It is easy to see that if a connected graph $G$ with some starting vertex has an even kernel, then Bob wins the feedback game on $G$ since Bob can always move a token from a vertex not in $S$ back to a vertex in $S$ [5, 8]. As a corollary, Bob also wins the game on every connected Eulerian bipartite graph. It is NP-complete in general to find an even kernel of a given graph [4]. Furthermore, there exist infinitely many connected Eulerian graphs without an even kernel (for a specified starting vertex) on which Bob wins the game [8].

Now we shall prove Theorems 1.2 and 1.4.

**Proof of Theorem 1.2.** Let $G$ be a 3-chromatic Eulerian triangulation on a surface and let $(S_1, S_2, S_3)$ be the tripartite set of $G$, i.e., $V(G) = \bigcup_{i=1}^{3} S_i$ and $S_i \cap S_j = \emptyset$ for $i \neq j$. Without loss of generality, the starting vertex $s$ is in $S_1$. Since the degree of each vertex of $G$ is zero modulo 4, each vertex in $S_i$ is adjacent to an even number of vertices in $S_j$ for any $i, j \in \{1, 2, 3\}$ with $i \neq j$. (Note that the subgraph induced by the neighbors of each vertex in $G$ is an even cycle of length $4k$ for some $k \geq 1$.) Therefore, $S_1$ is clearly an even kernel.

**Proof of Theorem 1.4.** (The first step): It suffices to prove the theorem for the sphere, since we can construct the desired Eulerian triangulation on a given surface $F^2$ by “past- ing” a face of the desired one on the sphere (constructed below) and that of any 3-chromatic Eulerian triangulation on $F^2$ with only vertices of degree $4k$ with $k \geq 1$.

**Remark 2.4.** Let $G$ be a 3-chromatic Eulerian triangulation on $F^2$ and let $uvw$ and $uvw'$ be two faces of $G$. A **2-subdivision** of an edge $uv$ of $G$ is to replace $uv$ with a path $uabv$ and to add edges $aw, bw, aw', bw'$. Note that $w$ and $w'$ belong to the same partiite set of $G$ and that the remainder of their degrees modulo 4 is changed after this operation. The octahedron addition also changes the remainder modulo 4 of degrees of three vertices on the corresponding face boundary. Thus applying octahedron additions and 2-subdivisions suitably, we can construct a 3-chromatic Eulerian triangulation on $F^2$ with only vertices of degree $4k$ with $k \geq 1$ from another by Remark 1.3.
Furthermore, for any $m \equiv 0 \pmod{3}$ with $m \geq 3$, the octahedral path $E_n$ with $n \geq 2$ is the desired graph. See Theorem 1.5.

Hence, we shall construct the desired graph for any $m \geq 2$ with $m \equiv 1, 2 \pmod{3}$ in the second and third steps.

(The second step): Let $G$ be the Eulerian triangulation on the sphere shown in the left of Figure 2. Note that exactly two vertices $u_1$ and $b$ are of degree 6 and other vertices have degree $4k$ for some $k \geq 1$ (i.e., $c_{4,2}(G) = 2$). We will show that Alice can win the game on $G$ with the starting vertex $s$.

Figure 2: An Eulerian triangulation on the sphere with the starting vertex $s$ where Alice wins the game

Alice first moves the token to $u_1$. After that, unless Bob moves the token to $a$ or $u_3$, Alice moves it to a white vertex shown in the right of Figure 2 corresponding to Bob’s move. As in the argument of an even kernel (see Remark 2.3), Bob finally has to move the token to $a$ or $u_3$, and hence, Alice wins the game.

(The third step): Observe that four vertices $t_1, t_2, t_3$ and $b$ are not used for the above winning strategy of Alice in the second step. So, by repeatedly applying an octahedron addition to a face consisting three vertices of degree 4 in the interior of $t_1t_2t_3$, we can increase the number of vertices of degree 6 by $3r$ for $r \geq 1$ per one octahedron addition, keeping that Alice wins the game on the resulting graph. Thus the resulting graph is the desired graph for any $m \geq 2$ with $m \equiv 2 \pmod{3}$.

For any $m \geq 2$ with $m \equiv 1 \pmod{3}$, we construct the desired graph, as follows: First apply an octahedron addition to two faces $bu_3t_2$ and $bt_2t_3$ respectively. The resulting graph has four vertices $b, u_1, u_3$ and $t_3$ of degree $4k + 2$ for some $k \geq 1$ and Alice still wins the game on the graph. After that, similar to the case when $m \equiv 2 \pmod{3}$, we can obtain the desired graphs by repeatedly applying an octahedron addition to a face consisting of three vertices of degree 4 (in the interior of $bt_2t_3$).

We conclude this section with the following result.

Proposition 2.5. Bob wins the game on every Eulerian double wheel graph.
Proof. Let $G$ be an Eulerian double wheel graph of $n + 2$ vertices with $n \geq 4$ and $n \equiv 0 \pmod{2}$. Let $C_n = v_0v_1\ldots v_{n-1}$ be the rim of $G$ and let $x, y$ be two vertices of degree $n$. If $n \equiv 0 \pmod{4}$, then Bob wins the game by Theorem 1.2. Moreover, if $x$ or $y$ is the starting vertex of $G$, then Bob wins the game since $\{x, y\}$ is an even kernel. Thus we may assume that $n \equiv 2 \pmod{4}$ and $v_0$ is the starting vertex by symmetry.

If Alice first moves the token from $v_0$ to $v_1$ by symmetry, then Bob moves it to $v_2$. After that, since Alice can move the token to neither $x$ nor $y$, she has to move it from $v_2$ to $v_{2i+1}$ and then Bob moves it to $v_{2i+2}$, and hence, Bob can finally move the token back to $v_0$.

So Alice first moves the token from $v_0$ to $x$ by symmetry. In this case, Bob first moves the token to $v_2$, and then Alice moves it to $v_3$ and Bob moves it to $v_4$. After that, Bob can win the game as follows: If Alice moves the token to $x$, then Bob moves it to $v_3$. Since two edges $v_2v_3$ and $v_3v_4$ are already removed, Alice must move the token to $y$ and then Bob can move it back to $v_0$. Otherwise, similarly to the previous case, Bob can finally move the token back to $v_0$.

3 Octahedral Path

This section is devoted to proving Theorem 1.5. We use the same labeling of the vertices of $E_n$ as drawn in Figure 1. We can see that $E_n$ can be also drawn as Figure 3.

Lemma 3.1. Alice (resp., Bob) can win the game on $E(n, p)$ if and only if Alice (resp., Bob) can win the game on $E(n, n - p)$.

Proof. It is trivial by the symmetry of the vertices $u_p, v_p, w_p$ and $u_{n-p}, v_{n-p}, w_{n-p}$ on $E_n$, respectively. \[\square\]
For the proof of the lemmas below, we describe the neighbors of each vertex:

\[
N(u_i) = \begin{cases} 
\{v_i, w_i, u_{i+1}, v_{i+1}\} & \text{if } i = 0, \\
\{v_i, w_i, u_{i-1}, v_{i-1}\} & \text{if } i = n, \\
\{v_i, w_i, u_{i+1}, w_{i+1}, u_{i-1}, v_{i-1}\} & \text{otherwise}, 
\end{cases}
\]

\[
N(v_i) = \begin{cases} 
\{u_i, w_i, v_{i+1}, u_{i+1}\} & \text{if } i = 0, \\
\{u_i, w_i, v_{i-1}, w_{i-1}\} & \text{if } i = n, \\
\{u_i, w_i, v_{i+1}, u_{i+1}, v_{i-1}, w_{i-1}\} & \text{otherwise}, 
\end{cases}
\]

\[
N(w_i) = \begin{cases} 
\{u_i, v_i, w_{i+1}, v_{i+1}\} & \text{if } i = 0, \\
\{u_i, v_i, w_{i-1}, u_{i-1}\} & \text{if } i = n, \\
\{u_i, v_i, w_{i+1}, v_{i+1}, u_{i-1}, w_{i-1}\} & \text{otherwise}. 
\end{cases}
\]

**Lemma 3.2.** Bob can win the game on \(E(3m + 1, 3k)\) and \(E(3m + 1, 3k + 1)\) for any \(m, k \in \mathbb{Z}_{\geq 0}\).

**Proof.** Let 

\[ S = \{u_{3i}, v_{3i+1} : i = 0, 1, \ldots, m\}. \]

For proving the statement, it is enough to show that \(S\) is an even kernel. By (1), we can easily see the following:

\[
N(v_{3i}) \cap S = N(w_{3i}) \cap S = \{u_{3i}, v_{3i+1}\} \text{ for } i = 0, 1, \ldots, m;
\]

\[
N(u_{3i+1}) \cap S = N(w_{3i+1}) \cap S = \{u_{3i}, v_{3i+1}\} \text{ for } i = 0, 1, \ldots, m;
\]

\[
N(u_{3i+2}) \cap S = N(v_{3i+2}) \cap S = \{v_{3i+1}, u_{3i+3}\} \text{ for } i = 0, 1, \ldots, m - 1;
\]

\[
N(w_{3i+2}) \cap S = 0.
\]

\[ \square \]

**Lemma 3.3.** Bob can win the game on \(E(3m + 2, 3k + 1)\) for any \(m, k \in \mathbb{Z}_{\geq 0}\).

**Proof.** Let us consider \(S = S_1 \cup S_2\), where

\[
S_1 = \{u_{3i}, v_{3i+1} : i = 0, 1, \ldots, k\} \text{ and } S_2 = \{v_{3k+1+3j}, w_{3k+2+3j} : j = 0, 1, \ldots, m - k\}.
\]

Consider the bipartite graph \(H\) as depicted in Figure 4.

Note that there are two even kernel graphs \(H_1\) and \(H_2\) with respect to \(S_1\) and \(S_2\), respectively, although \(S\) is not an even kernel since \(u_{3k+1}\) and \(w_{3k+1}\) are adjacent to \(u_{3k}, v_{3k+1}\) and \(w_{3k+2}\).

If Alice moves the token to the vertex \(x\) or \(y\) (resp., \(z\) or \(y\)), then Bob may move the token to \(u_{3k}\) (resp., \(w_{3k+2}\)). After that, by following the even kernel graph \(H_1\) (resp., \(H_2\)), we see that Bob can move the token to the vertex \(u_{3k}\) (resp., \(w_{3k+2}\)). Hence, Bob can finally win the game.

\[ \square \]

**Lemma 3.4.** Alice can win the game on \(E(n, 3k + 2)\) for any \(n, k \in \mathbb{Z}_{\geq 0}\).
Proof. Let us consider the same subset as in Lemma 3.2, i.e.,
\[ S = \{u_{3i}, v_{3i+1} : i = 0, 1, \ldots, k\}, \]
and let \( s = v_{3k+2} \) be the starting vertex (see Figure 5).

First, Alice moves the token to the vertex \( v_{3k+1} \in S \). Then by following the even kernel graph with respect to \( S \), we see that Alice can move the token back to \( v_{3k+1} \) again. Finally, Bob must move the token to \( u_{3k+2} \) from \( v_{3k+1} \). Therefore, Alice can win the game. \( \square \)

Lemma 3.5. Alice can win the game on \( E(3m, 3k) \) for any \( m, k \in \mathbb{Z}_{\geq 0} \).
Proof. Let us consider the same subsets as in Lemma 3.3, i.e.,

\[ S_1 = \{ u_{3i}, v_{3i+1} : i = 0, 1, \ldots, k - 1 \} \text{ and } S_2 = \{ w_{3k+2+3j}, u_{3k+2+3j+1} : j = 0, 1, \ldots, m - k - 1 \}, \]

and let \( s = v_{3k} \) be the starting vertex (see Figure 6).

First, Alice moves the token to \( w_{3k} \). Then Bob should move the token to \( u_{3k-1} \) or \( w_{3k+1} \), otherwise Bob will lose the game immediately. Consider the case \( w_{3k-1} \) (resp., \( w_{3k+1} \)). Then Alice moves the token to \( v_{3k-2} \) (resp., \( w_{3k+2} \)). By following the even kernel graph with respect to \( S_1 \) (resp., \( S_2 \)), we see that Alice can move the token back to \( v_{3k-2} \) (resp., \( w_{3k+2} \)) again. Finally, Bob must move the token to \( v_{3k-1} \) (resp., \( u_{3k+1} \)) from \( v_{3k-2} \) (resp., \( w_{3k+2} \)). Therefore, Alice can win the game.

We can now prove Theorem 1.5.

Proof of Theorem 1.5 All of the cases directly follow from Lemmas 3.1–3.5 as follows (where each “\( \equiv \)” means that “congruent modulo 3”):

| \( p \equiv 0 \) | \( n \equiv 0 \) | \( n \equiv 1 \) | \( n \equiv 2 \) |
|---------------|---------------|---------------|---------------|
| \( \equiv 0 \) | Lemma 3.5     | Lemma 3.2     | Lemmas 3.1 & 3.4 |
| \( \equiv 1 \) | Lemmas 3.1 & 3.4 | Lemma 3.2     | Lemma 3.3     |
| \( \equiv 2 \) | Lemma 3.4     | Lemma 3.4     | Lemma 3.4     |

\[ \square \]

4 Conclusion

In this paper, focusing on 3-chromatic Eulerian triangulations of surfaces, we completely determine the winner of the feedback game on several classes of the graphs. In general, for any surface \( F^2 \) which is not the sphere, there are infinitely many non-3-colorable Eulerian triangulations of \( F^2 \). Since Theorem 1.2 strongly depends on the 3-colorability
of the graphs, it is not clear which player wins the game on a non-3-colorable Eulerian triangulation $G$ even if $c_{4,2}(G) = 0$. In fact, as shown in Figure 7, there exists a 5-chromatic Eulerian triangulation $G$ of the projective plane with $c_{4,2}(G) = 0$ such that Alice wins the game on $G$ with a starting vertex $s$; she first moves the token on $s$ to $v$, and after that, since $s$ is adjacent to all other vertices, Alice can move the token back to $s$ regardless of Bob’s move.

Figure 7: The 5-chromatic Eulerian triangulation $G$ of the projective plane with $c_{4,2}(G) = 0$

On the other hand, the proof of Theorem 1.4 does not strongly depend on the 3-colorability so much, since we can obtain a similar statement as Theorem 1.4 by showing that for a fixed surface $F^2$, there exists a non-3-colorable Eulerian triangulation $G$ of $F^2$ with $c_{4,2}(G) = 0$. Thus we guess that Theorem 1.4 also holds for non-3-colorable Eulerian triangulations.

As in the octahedral path, determining the winner of the game on a 3-chromatic Eulerian triangulation $G$ with $c_{4,2}(G) > 0$ is not easy, since a choice of the starting vertex changes the winner of the game (as Theorem 1.5). Similarly to the octahedral path, a 6-regular triangulation seems to be the next reasonable concrete class of Eulerian triangulations, since only the torus and Klein bottle admit such triangulations and they are completely classified (see [1, 6, 9]). Therefore, we conclude this paper with proposing the following problem.

**Problem 4.1.** Completely determine the winner of the feedback game on 3-chromatic 6-regular triangulations of the torus or Klein bottle.

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