On sum edge-coloring of regular, bipartite and split graphs

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An edge-coloring of a graph \(G\) with natural numbers is called a sum edge-coloring if the colors of edges incident to any vertex of \(G\) are distinct and the sum of the colors of the edges of \(G\) is minimum. The edge-chromatic sum of a graph \(G\) is the sum of the colors of edges in a sum edge-coloring of \(G\). It is known that the problem of finding the edge-chromatic sum of an \(r\)-regular (\(r \geq 3\)) graph is \(NP\)-complete. In this paper we give a polynomial time \((1 + \frac{2r}{(r+1)^2})\)-approximation algorithm for the edge-chromatic sum problem on \(r\)-regular graphs for \(r \geq 3\). Also, it is known that the problem of finding the edge-chromatic sum of bipartite graphs with maximum degree 3 is \(NP\)-complete. We show that the problem remains \(NP\)-complete even for some restricted class of bipartite graphs with maximum degree 3. Finally, we give upper bounds for the edge-chromatic sum of some split graphs.

Keywords: edge-coloring, sum edge-coloring, regular graph, bipartite graph, split graph

1. Introduction

We consider finite undirected graphs that do not contain loops or multiple edges. Let \(V(G)\) and \(E(G)\) denote sets of vertices and edges of \(G\), respectively. For \(S \subseteq V(G)\), let \(G[S]\) denote the subgraph of \(G\) induced by \(S\), that is, \(V(G[S]) = S\) and \(E(G[S])\) consists of those edges of \(E(G)\) for which both ends are in \(S\). The degree of a vertex \(v \in V(G)\) is denoted by \(d_G(v)\), the maximum degree of \(G\) by \(\Delta(G)\), the chromatic number of \(G\) by \(\chi(G)\), and the chromatic index of \(G\) by \(\chi'(G)\). The terms and concepts that we do not define can be found in [4, 26].

A proper vertex-coloring of a graph \(G\) is a mapping \(\alpha : V(G) \rightarrow N\) such that \(\alpha(u) \neq \alpha(v)\) for every \(uv \in E(G)\). If \(\alpha\) is a proper vertex-coloring of a graph \(G\), then \(\Sigma(G, \alpha)\)
denotes the sum of the colors of the vertices of \( G \). For a graph \( G \), define the vertex-chromatic sum \( \Sigma(G) \) as follows: \( \Sigma(G) = \min_\alpha \Sigma(G, \alpha) \), where minimum is taken among all possible proper vertex-colorings of \( G \). If \( \alpha \) is a proper vertex-coloring of a graph \( G \) and \( \Sigma(G) = \Sigma(G, \alpha) \), then \( \alpha \) is called a sum vertex-coloring. The strength of a graph \( G \) \((s(G))\) is the minimum number of colors needed for a sum vertex-coloring of \( G \). The concept of sum vertex-coloring and vertex-chromatic sum was introduced by Kubicka [16] and Supowit [22]. In [17], Kubicka and Schwenk showed that the problem of finding the vertex-chromatic sum is \( NP \)-complete in general and polynomial time solvable for trees. Jansen [12] gave a dynamic programming algorithm for partial \( k \)-trees. In papers [8, 10, 13, 18], some approximation algorithms were given for various classes of graphs. For the strength of graphs, Brook’s-type theorem was proved in [11]. On the other hand, there are graphs with \( s(G) > \chi(G) \) [8]. Some bounds for the vertex-chromatic sum of a graph were given in [23].

Similar to the sum vertex-coloring and vertex-chromatic sum of graphs, in [5, 9, 11], sum edge-coloring and edge-chromatic sum of graphs was introduced. A proper edge-coloring of a graph \( G \) is a mapping \( \alpha : E(G) \rightarrow \mathbb{N} \) such that \( \alpha(e) \neq \alpha(e') \) for every pair of adjacent edges \( e, e' \in E(G) \). If \( \alpha \) is a proper edge-coloring of a graph \( G \), then \( \Sigma'(G, \alpha) \) denotes the sum of the colors of the edges of \( G \). For a graph \( G \), define the edge-chromatic sum \( \Sigma'(G) \) as follows: \( \Sigma'(G) = \min_\alpha \Sigma'(G, \alpha) \), where minimum is taken among all possible proper edge-colorings of \( G \). If \( \alpha \) is a proper edge-coloring of a graph \( G \) and \( \Sigma'(G) = \Sigma'(G, \alpha) \), then \( \alpha \) is called a sum edge-coloring. The edge-strength of a graph \( G \) \((s'(G))\) is the minimum number of colors needed for a sum edge-coloring of \( G \). For the edge-strength of graphs, Vizing’s-type theorem was proved in [11]. In [9], Bar-Noy et al. proved that the problem of finding the edge-chromatic sum is \( NP \)-hard for multigraphs. Later, in [9], it was shown that the problem is \( NP \)-complete for bipartite graphs with maximum degree 3. Also, in [9], the authors proved that the problem can be solved in polynomial time for trees and that \( s'(G) = \chi'(G) \) for bipartite graphs. In [20], Salavatipour proved that determining the edge-chromatic sum and the edge-strength are \( NP \)-complete for \( r \)-regular graphs with \( r \geq 3 \). Also he proved that \( s'(G) = \chi'(G) \) for regular graphs. On the other hand, there are graphs with \( \chi'(G) = \Delta(G) \) and \( s'(G) = \Delta(G) + 1 \) [11]. Recently, Cardinal et al. [7] determined the edge-strength of the multicycles.

In the present paper we give a polynomial time \( \frac{11}{8} \)-approximation algorithm for the edge-chromatic sum problem of \( r \)-regular graphs for \( r \geq 3 \). Next, we show that the problem of finding the edge-chromatic sum remains \( NP \)-complete even for some restricted class of bipartite graphs with maximum degree 3. Finally, we give upper bounds for the edge-chromatic sum of some split graphs.

### 2. Definitions and necessary results

A proper \( t \)-coloring is a proper edge-coloring which makes use of \( t \) different colors. If \( \alpha \) is a proper \( t \)-coloring of \( G \) and \( v \in V(G) \), then \( S(v, \alpha) \) denotes set of colors appearing on edges incident to \( v \). Let \( G \) be a graph and \( R \subseteq V(G) \). A proper \( t \)-coloring of a graph \( G \) is called an \( R \)-sequential \( t \)-coloring [11, 2] if the edges incident to each vertex \( v \in R \) are
colored by the colors $1, \ldots, d_G(v)$. For positive integers $a$ and $b$, we denote by $[a, b]$, the set of all positive integers $c$ with $a \leq c \leq b$. For a positive integer $n$, let $K_n$ denote the complete graph on $n$ vertices.

We will use the following four results.

**Theorem 1** [15]. If $G$ is a bipartite graph, then $\chi'(G) = \Delta(G)$.

**Theorem 2** [24]. For every graph $G$,

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$ 

**Theorem 3** [25]. For the complete graph $K_n$ with $n \geq 2$,

$$\chi'(K_n) = \begin{cases} n - 1, & \text{if } n \text{ is even}, \\ n, & \text{if } n \text{ is odd}. \end{cases}$$ 

**Theorem 4** [9, 11]. If $G$ is a bipartite or a regular graph, then $s'(G) = \chi'(G)$.

### 3. Edge-chromatic sums of regular graphs

In this section we consider the problem of finding the edge-chromatic sum of regular graphs. It is easy to show that the edge-chromatic sum problem of graphs $G$ with $\Delta(G) \leq 2$ can be solved in polynomial time. On the other hand, in [19], it was proved that the problem of finding the edge-chromatic sum of an $r$-regular ($r \geq 3$) graph is $NP$-complete. Clearly, $\Sigma'(G) \geq \frac{nr(r+1)}{4}$ for any $r$-regular graph $G$ with $n$ vertices, since the sum of colors appearing on the edges incident to any vertex is at least $\frac{r(r+1)}{2}$. Moreover, it is easy to see that $\Sigma'(G) = \frac{nr(r+1)}{4}$ if and only if $\chi'(G) = r$ for any $r$-regular graph $G$ with $n$ vertices.

First we give a result on $R$-sequential colorings of regular graphs and then we use this result for constructing an approximation algorithm.

**Theorem 5** If $G$ is an $r$-regular graph with $n$ vertices, then $G$ has an $R$-sequential $(r+1)$-coloring with $|R| \geq \left\lceil \frac{n}{r+1} \right\rceil$.

**Proof.** By Theorem 2 there exists a proper $(r+1)$-coloring $\alpha$ of the graph $G$. For $i = 1, 2, \ldots, r + 1$, define the set $V_\alpha(i)$ as follows:

$$V_\alpha(i) = \{ v \in V(G) : i \notin S(v, \alpha) \}.$$ 

Clearly, for any $i', i'', 1 \leq i' < i'' \leq r + 1$, we have

$$V_\alpha(i') \cap V_\alpha(i'') = \emptyset \quad \text{and} \quad \bigcup_{i=1}^{r+1} V_\alpha(i) = V(G).$$
Hence,

\[ n = |V(G)| = \left| \bigcup_{i=1}^{r+1} V_\alpha(i) \right| = \sum_{i=1}^{r+1} |V_\alpha(i)|. \]

This implies that there exists \( i_0, 1 \leq i_0 \leq r + 1 \), for which \(|V_\alpha(i_0)| \geq \left\lceil \frac{n}{r+1} \right\rceil \). Let \( R = V_\alpha(i_0) \).

If \( i_0 = r + 1 \), then \( \alpha \) is an \( R \)-sequential \( (r + 1) \)-coloring of \( G \); otherwise define an edge-coloring \( \beta \) as follows: for any \( e \in E(G) \), let

\[ \beta(e) = \begin{cases} 
\alpha(e), & \text{if } \alpha(e) \neq i_0, r + 1, \\
i_0, & \text{if } \alpha(e) = r + 1, \\
r + 1, & \text{if } \alpha(e) = i_0.
\end{cases} \]

It is easy to see that \( \beta \) is an \( R \)-sequential \( (r + 1) \)-coloring of \( G \) with \(|R| \geq \left\lceil \frac{n}{r+1} \right\rceil \).

**Corollary 6** If \( G \) is a cubic graph with \( n \) vertices, then \( G \) has an \( R \)-sequential \( 4 \)-coloring with \(|R| \geq \left\lceil \frac{n}{4} \right\rceil \).

Note that if \( n \) is odd, then the lower bound in Theorem 5 cannot be improved, since the complete graph \( K_n \) has an \( R \)-sequential \( n \)-coloring with \(|R| = 1 \).

The theorem we are going to prove will be used in section 5.

**Theorem 7** For any \( n \in \mathbb{N} \), we have

\[ \Sigma'(K_n) = \begin{cases} 
\frac{n(n^2-1)}{4}, & \text{if } n \text{ is odd,} \\
\frac{(n-1)n^2}{4}, & \text{if } n \text{ is even.}
\end{cases} \]

**Proof.** Since for any \( r \)-regular graph \( G \) with \( n \) vertices, \( \Sigma'(G) = \frac{nr(r+1)}{4} \) if and only if \( \chi'(G) = r \) and, by Theorems 3 and 4, we obtain \( \Sigma'(K_n) = \frac{(n-1)n^2}{4} \) if \( n \) is even.

Now let \( n \) be an odd number and \( n \geq 3 \). In this case by Theorems 3 and 4 we have \( s'(K_n) = \chi'(K_n) = n \). It is easy to see that in any proper \( n \)-coloring of \( K_n \) the missing colors at \( n \) vertices are all distinct. Hence,

\[ \Sigma'(K_n) = \frac{n^2(n+1)}{2} - \frac{n(n+1)}{2} = \frac{n(n^2 - 1)}{4}. \]

□

In [5], it was shown that there exists a 2-approximation algorithm for the edge-chromatic sum problem on general graphs. Now we show that there exists a \( \left( 1 + \frac{2r}{(r+1)^2} \right) \)-approximation algorithm for the edge-chromatic sum problem on \( r \)-regular graphs for \( r \geq 3 \). Note that \( 1 + \frac{2r}{(r+1)^2} \) decreases for increasing \( r \) and \( \frac{11}{8} \) is its maximum value achieved for \( r = 3 \). Thus, we show that there is a \( \frac{11}{8} \)-approximation algorithm for the edge-chromatic sum problem on regular graphs.
For any \( r \geq 3 \), there is a polynomial time \( \left(1 + \frac{2r}{(r+1)^2}\right) \)-approximation algorithm for the edge-chromatic sum problem on \( r \)-regular graphs.

**Proof.** Let \( G \) be an \( r \)-regular graph with \( n \) vertices and \( m \) edges. Now we describe a polynomial time algorithm \( A \) for constructing a special proper \((r+1)\)-coloring of \( G \). First we construct a proper \((r+1)\)-coloring \( \alpha \) of \( G \) in \( O(mn) \) time \([21]\). Next we recolor some edges as it is described in the proof of Theorem 5 to obtain an \( R \)-sequential \((r+1)\)-coloring \( \beta \) of \( G \) with \( \Delta(G) \geq \left\lceil \frac{n}{r+1} \right\rceil \). Clearly, we can do it in \( O(m) \) time. Now, taking into account that the sum of colors appearing on the edges incident to any vertex is at most \( \frac{r(r+3)}{2} \), we have

\[
\Sigma'_A(G) = \Sigma'(G, \beta) \leq \frac{r(r+1)}{2} \left\lfloor \frac{n}{r+1} \right\rfloor + \left( n - \left\lceil \frac{n}{r+1} \right\rceil \right) \frac{r+3}{2} \leq \frac{r(r+1)}{2} \frac{n}{r+1} + \left( n - \frac{n}{r+1} \right) \frac{r+3}{2}
\]

\[
= \frac{r(r+1)}{2} \frac{n}{r+1} + \frac{nr}{r+1} + \frac{nr}{r+1} = \frac{nr^2 + 4r + 1}{4(r+1)}.
\]

On the other hand, since \( \Sigma'(G) \geq \frac{nr(r+1)}{4} \), we get

\[
\frac{\Sigma'_A(G)}{\Sigma'(G)} \leq \frac{nr^2 + 4r + 1}{4(r+1)} \cdot \frac{4}{nr(r+1)} = \frac{r^2 + 4r + 1}{(r+1)^2} = 1 + \frac{2r}{(r+1)^2}.
\]

This shows that there exists a \( \left(1 + \frac{2r}{(r+1)^2}\right) \)-approximation algorithm for the edge-chromatic sum problem on \( r \)-regular graphs. Moreover, we can construct the aforementioned coloring \( \beta \) for a regular graph in \( O(mn) \) time. \( \square \)

4. Edge-chromatic sums of bipartite graphs

In this section we consider the problem of finding the edge-chromatic sum of bipartite graphs. Let \( G = (U \cup W, E) \) be a bipartite graph with a bipartition \((U, W)\). By \( U_i \subseteq U \) and \( W_i \subseteq W \), we denote sets of vertices of degree \( i \) in \( U \) and \( W \), respectively. Define sets \( V_{\geq i} \subseteq V(G) \) and \( U_{\geq i} \subseteq U \) as follows: \( V_{\geq i} = \{v : v \in V(G) \land d_G(v) \geq i\} \) and \( U_{\geq i} = \{u \in V(G) : u \in U \land d_G(u) \geq i\} \). It was proved the following:

**Theorem 9** \([4, 7, 8, 4]\) If \( G = (U \cup W, E) \) is a bipartite graph with \( d_G(u) \geq d_G(w) \) for every \( uw \in E(G) \), where \( u \in U \) and \( w \in W \), then \( G \) has a \( U \)-sequential \( \Delta(G) \)-coloring.

By this theorem, we obtain the following corollary:

**Corollary 10** If \( G = (U \cup W, E) \) is a bipartite graph with \( d_G(u) \geq d_G(w) \) for every \( uw \in E(G) \), where \( u \in U \) and \( w \in W \), then a \( U \)-sequential \( \Delta(G) \)-coloring of \( G \) is a sum edge-coloring of \( G \) and \( \Sigma'(G) = \sum_{u \in U} \frac{d_G(u)(d_G(u)+1)}{2} \).
In [9], it was shown that the problem of finding the edge-chromatic sum of bipartite graphs $G$ with $\Delta(G) = 3$ is $NP$-complete. Now we give a short proof of this fact. First we need the following

**Problem 1.** \[2, 4, 14\]

Instance: A bipartite graph $G = (U \cup W, E)$ with $\Delta(G) = 3$.

Question: Is there a $U$-sequential 3-coloring of $G$?

It was proved the following:

**Theorem 11** \[2, 14\] Problem 1 is $NP$-complete.

Now let us consider the following

**Problem 2.**

Instance: A bipartite graph $G = (U \cup W, E)$ with $\Delta(G) = 3$.

Question: Is $\Sigma'(G) = \sum_{i=1}^{3} i \cdot |U_i| \geq |U|\geq i$?

**Theorem 12** Problem 2 is $NP$-complete.

**Proof.** Clearly, Problem 2 belongs to $NP$. For the proof of the $NP$-completeness, we show a reduction from Problem 1 to Problem 2. We prove that a bipartite graph $G = (U \cup W, E)$ with $\Delta(G) = 3$ admits a $U$-sequential 3-coloring if and only if $\Sigma'(G) = \sum_{i=1}^{3} i \cdot |U_i|$. Let $G = (U \cup W, E)$ be a bipartite graph with $\Delta(G) = 3$ and $\alpha$ be a $U$-sequential 3-coloring of $G$. In this case the colors 1, 2, 3 appear on the edges incident to each vertex $u \in U_3$, the colors 1, 2 appear on the edges incident to each vertex $u \in U_2$ and the color 1 appears on the pendant edges incident to each vertex $u \in U_1$. Hence, $\Sigma'(G, \alpha) = \sum_{i=1}^{3} i \cdot |U_i|$. On the other hand, clearly, $\Sigma'(G) \geq \sum_{i=1}^{3} i \cdot |U_i|$, thus $\Sigma'(G) = \sum_{i=1}^{3} i \cdot |U_i|$. Now suppose that $\Sigma'(G) = \sum_{i=1}^{3} i \cdot |U_i|$. By Theorems 11 and 12, there exists a proper $3$-coloring $\beta$ of a bipartite graph $G$ with $\Delta(G) = 3$. This implies that the colors 1, 2, 3 appear on the edges incident to each vertex $u \in U_3$. If the color 3 appears on the edges incident to some vertices $u \in U_2$ or the colors 2 or 3 appear on the pendant edges incident to some vertices $u \in U_1$, then it is easy to see that $\Sigma'(G, \beta) > \sum_{i=1}^{3} i \cdot |U_i|$. Hence, $\beta$ is a $U$-sequential 3-coloring of $G$. □

Now we prove that the problem of finding the edge-chromatic sum of bipartite graphs $G$ with $\Delta(G) = 3$ and with additional conditions is $NP$-complete, too. We need the following

**Problem 3.** \[2, 14\]

Instance: A bipartite graph $G = (U \cup W, E)$ with $\Delta(G) = 3$ and $|U_i| = |W_i|$ for $i = 1, 2, 3$.

Question: Is there a $V(G)$-sequential 3-coloring of $G$?

It was proved the following:
Theorem 13 [2, 14] Problem 3 is NP-complete.

Now let us consider the following
Problem 4.
Instance: A bipartite graph \( G = (U \cup W, E) \) with \( \Delta(G) = 3 \) and \( |U_i| = |W_i| \) for \( i = 1, 2, 3 \).

Question: Is \( \Sigma'(G) = \frac{1}{2} \sum_{i=1}^{3} i \cdot |V_{\geq i}| \)?

Theorem 14 Problem 4 is NP-complete.

Proof. Clearly, Problem 4 belongs to \( NP \). For the proof of the \( NP \)-completeness, we show a reduction from Problem 3 to Problem 4. We prove that a bipartite graph \( G = (U \cup W, E) \) with \( \Delta(G) = 3 \) and \( |U_i| = |W_i| \) for \( i = 1, 2, 3 \), admits a \( V(G) \)-sequential 3-coloring if and only if \( \Sigma'(G) = \frac{1}{2} \sum_{i=1}^{3} i \cdot |V_{\geq i}| \). Let \( \alpha \) be a \( V(G) \)-sequential 3-coloring of \( G \). In this case the colors 1, 2, 3 appear on the edges incident to each vertex \( v \in V(G) \) with \( d_G(v) = 3 \), the colors 1, 2 appear on the edges incident to each vertex \( v \in V(G) \) with \( d_G(v) = 2 \) and the color 1 appears on the pendant edges incident to each vertex \( v \in V(G) \) with \( d_G(v) = 1 \). Hence, \( \Sigma'(G, \alpha) = \frac{1}{2} \sum_{i=1}^{3} i \cdot |V_{\geq i}| \). On the other hand, clearly,
\[
\Sigma'(G) \geq \frac{1}{2} \sum_{i=1}^{3} i \cdot |V_{\geq i}|, \text{ thus } \Sigma'(G) = \frac{1}{2} \sum_{i=1}^{3} i \cdot |V_{\geq i}|.
\]

Now suppose that \( \Sigma'(G) = \frac{1}{2} \sum_{i=1}^{3} i \cdot |V_{\geq i}| \). By Theorems 11 and 13 there exists a proper 3-coloring \( \beta \) of a bipartite graph \( G \) with \( \Delta(G) = 3 \) and \( |U_i| = |W_i| \) for \( i = 1, 2, 3 \). This implies that the colors 1, 2, 3 appear on the edges incident to each vertex \( v \in V(G) \) with \( d_G(v) = 3 \). If the color 3 appears on the edges incident to some vertices \( v \in V(G) \) with \( d_G(v) = 2 \) or the colors 2 or 3 appear on the pendant edges incident to some vertices \( v \in V(G) \) with \( d_G(v) = 1 \), then it is easy to see that \( \Sigma'(G, \beta) > \frac{1}{2} \sum_{i=1}^{3} i \cdot |V_{\geq i}| \). Hence, \( \beta \) is a \( V(G) \)-sequential 3-coloring of \( G \). \( \square \)

In [19], it was proved that the problem of finding the edge-chromatic sum of bipartite graphs \( G \) with \( \Delta(G) = 3 \) remains \( NP \)-hard even for planar bipartite graphs.

5. Edge-chromatic sums of split graphs

In this section we consider the problem of finding the edge-chromatic sum of split graphs. A split graph is a graph whose vertices can be partitioned into a clique \( C \) and an independent set \( I \). Let \( G = (C \cup I, E) \) be a split graph, where \( C = \{u_1, u_2, \ldots, u_n\} \) is clique and \( I = \{v_1, v_2, \ldots, v_m\} \) is independent set. Define a number \( \Delta_I \) as follows:
\[
\Delta_I = \max_{1 \leq j \leq m} d_G(v_j).
\]
Define subgraphs \( H \) and \( H' \) of a graph \( G \) as follows:
\[
H = (C \cup I, E(G) \setminus E(G[C])) \text{ and } H' = G[C].
\]
Clearly, $H$ is a bipartite graph with a bipartition $(C, I)$, and $d_H(u_i) = d_G(u_i) - n + 1$ for $i = 1, 2, \ldots, n$, $d_H(v_j) = d_G(v_j)$ for $j = 1, 2, \ldots, m$.

**Theorem 15** Let $G = (C \cup I, E)$ be a split graph, where $C = \{u_1, u_2, \ldots, u_n\}$ is clique and $I = \{v_1, v_2, \ldots, v_m\}$ is independent set. If $d_G(u_i) - d_G(v_j) \geq n - 1$ for every $u_iv_j \in E(G)$, then:

1. if $n$ is even, then

   $\min \left\{ \sum_{i=1}^{n} \left( \frac{d_G(u_i) - n + 1}{2} \right)^2, \frac{\sum' \left( G \right)}{4}, \sum'(K_n) + \sum_{i=1}^{n} \left( \frac{d_G(u_i) - n + 1}{2} \right)^2 \right\}$;

2. if $n$ is odd, then

   $\min \left\{ \sum_{i=1}^{n} \left( \frac{d_G(u_i) - n + 1}{2} \right)^2, \frac{\sum' \left( G \right)}{4}, \sum'(K_n) + \sum_{i=1}^{n} \left( \frac{d_G(u_i) - n + 1}{2} \right)^2 \right\}$.

**Proof.** For the proof, we are going to construct edge-colorings that satisfy the specified conditions.

Since $d_G(u_i) - d_G(v_j) \geq n - 1$ for every $u_iv_j \in E(G)$, we have $d_H(u_i) \geq d_H(v_j)$ for each $u_iv_j \in E(H)$. By Theorem [9] there exists a $C$-sequential $\Delta(H)$-coloring $\alpha$ of the graph $H$ and, by Corollary [10] we obtain

$\Sigma'(H) = \Sigma'(H, \alpha) = \sum_{i=1}^{n} \frac{d_H(u_i)(d_H(u_i) + 1)}{2}$.

Now we consider two cases.

Case 1: $n$ is even.

In this case, by Theorem [3] we have $\chi'(H') = n - 1$. Let $\beta$ be a proper edge-coloring of a graph $H'$ with colors $\Delta(G) - n + 2, \ldots, \Delta(G)$. Clearly, for each vertex $u_i$, $i = 1, 2, \ldots, n$, the set of colors appearing on edges incident to $u_i$ in $H'$ is $[\Delta(G) - n + 2, \Delta(G)]$. Thus, we obtain

$\Sigma'(G) \leq \Sigma'(H) + \frac{(\Delta(G) - n + 2)n(n-1)}{4}$.

On the other hand, let $\beta'$ be a proper edge-coloring of a graph $H'$ with colors $1, 2, \ldots, n - 1$. Clearly, for each vertex $u_i$, $i = 1, 2, \ldots, n$, the set of colors appearing on edges incident to $u_i$ in $H'$ is $[1, n - 1]$. Next, we define an edge-coloring $\gamma$ of the graph $H$ as follows: for every $e \in E(H)$, let $\gamma(e) = \alpha(e) + n - 1$. Thus, we obtain

$\Sigma'(G) \leq \Sigma'(K_n) + \sum_{i=1}^{n} \left( \frac{d_G(u_i) - n + 1}{2} \right)^2$.

Case 2: $n$ is odd.

In this case, by Theorem [3] we have $\chi'(H') = n$. Let $\beta$ be a proper edge-coloring of a graph $H'$ with colors $\Delta(G) - n + 2, \ldots, \Delta(G) + 1$. Without loss of generality, we may assume that for each vertex $u_i$, $i = 1, 2, \ldots, n$, the set of colors appearing on edges incident to $u_i$ in $H'$ is $[\Delta(G) - n + 2, \Delta(G) + 1] \setminus \{\Delta(G) - n + 1 + i\}$. Thus, we obtain
\[ \Sigma'(G) \leq \Sigma'(H) + \frac{(2\Delta(G) - n + 3)n(n-1)}{4}. \]

On the other hand, let \( \beta' \) be a proper edge-coloring of a graph \( H' \) with colors 1, 2, \ldots, \( n \). Without loss of generality, we may assume that for each vertex \( u_i \), \( i = 1, 2, \ldots, n \), the set of colors appearing on edges incident to \( u_i \) in \( H' \) is \([1, n] \setminus \{i\}\). Next, we define an edge-coloring \( \gamma \) of the graph \( H \) as follows: for every \( e \in E(H) \), let \( \gamma(e) = \alpha(e) + n \). Thus, we obtain

\[ \Sigma'(G) \leq \Sigma'(K_n) + \sum_{i=1}^{n} \frac{(d_G(u_i) - n + 1)(d_G(u_i) + n + 2)}{2}. \]

\( \square \)

**Theorem 16** Let \( G = (C \cup I, E) \) be a split graph, where \( C = \{u_1, u_2, \ldots, u_n\} \) is clique and \( I = \{v_1, v_2, \ldots, v_m\} \) is independent set. If \( d_G(u_i) - d_G(v_j) \leq n - 1 \) for every \( u_i, v_j \in E(G) \), then:

1. if \( n \) is even, then

\[ \min \left\{ \sum_{j=1}^{m} \frac{d_G(v_j)(d_G(v_j) + 1)}{2} + \frac{(2\Delta_I + n)(n-1)}{4}, \Sigma'(K_n) + \sum_{j=1}^{m} \frac{d_G(v_j)(d_G(v_j) + 2n - 1)}{2} \right\}; \]

2. if \( n \) is odd, then

\[ \min \left\{ \sum_{j=1}^{m} \frac{d_G(v_j)(d_G(v_j) + 1)}{2} + \frac{(2\Delta_I + n + 1)(n-1)}{4}, \Sigma'(K_n) + \sum_{j=1}^{m} \frac{d_G(v_j)(d_G(v_j) + 2n + 1)}{2} \right\}. \]

**Proof.** For the proof, we are going to construct edge-colorings that satisfies the specified conditions.

Since \( d_G(u_i) - d_G(v_j) \leq n - 1 \) for every \( u_i, v_j \in E(G) \), we have \( d_H(u_i) \leq d_H(v_j) \) for each \( u_i, v_j \in E(H) \). By Theorem 13 there exists an \( I \)-sequential \( \Delta_I \)-coloring \( \alpha \) of the graph \( H \) and, by Corollary 10 we obtain

\[ \Sigma'(H) = \Sigma'(H, \alpha) = \sum_{j=1}^{m} \frac{d_H(v_j)(d_H(v_j) + 1)}{2} = \sum_{j=1}^{m} \frac{d_G(v_j)(d_G(v_j) + 1)}{2}. \]

Now we consider two cases.

Case 1: \( n \) is even.

In this case, by Theorem 31 we have \( \chi'(H') = n - 1 \). Let \( \beta \) be a proper edge-coloring of a graph \( H' \) with colors \( \Delta_I + 1, \ldots, \Delta_I + n - 1 \). Clearly, for each vertex \( u_i \), \( i = 1, 2, \ldots, n \), the set of colors appearing on edges incident to \( u_i \) in \( H' \) is \([\Delta_I + 1, \Delta_I + n - 1]\). Thus, we obtain

\[ \Sigma'(G) \leq \Sigma'(H) + \frac{(2\Delta_I + n)(n-1)}{4}. \]

On the other hand, let \( \beta' \) be a proper edge-coloring of a graph \( H' \) with colors 1, 2, \ldots, \( n - 1 \). Clearly, for each vertex \( u_i \), \( i = 1, 2, \ldots, n \), the set of colors appearing on edges incident to \( u_i \) in \( H' \) is \([1, n - 1]\). Next, we define an edge-coloring \( \gamma \) of the graph \( H \) as follows: for every \( e \in E(H) \), let \( \gamma(e) = \alpha(e) + n - 1 \). Thus, we obtain
\[
\Sigma'(G) \leq \Sigma'(K_n) + \sum_{j=1}^{m} \frac{d_G(v_j)(d_G(v_j)+2n-1)}{2}.
\]

Case 2: \(n\) is odd.

In this case, by Theorem 3, we have \(\chi'(H') = n\). Let \(\beta\) be a proper edge-coloring of a graph \(H'\) with colors \(\Delta_{\bar{I}} + 1, \ldots, \Delta_{\bar{I}} + n\). Without loss of generality, we may assume that for each vertex \(u_i, i = 1, 2, \ldots, n\), the set of colors appearing on edges incident to \(u_i\) in \(H'\) is \([\Delta_{\bar{I}} + 1, \Delta_{\bar{I}} + n]\) \(\backslash\) \(\{\Delta_{\bar{I}} + i\}\). Thus, we obtain

\[
\Sigma'(G) \leq \Sigma'(H) + \frac{(2\Delta_{\bar{I}}+n+1)n(n-1)}{4}.
\]

On the other hand, let \(\beta'\) be a proper edge-coloring of a graph \(H'\) with colors \(1, 2, \ldots, n\). Without loss of generality, we may assume that for each vertex \(u_i, i = 1, 2, \ldots, n\), the set of colors appearing on edges incident to \(u_i\) in \(H'\) is \([1, n]\) \(\backslash\) \(\{i\}\). Next, we define an edge-coloring \(\gamma\) of the graph \(H\) as follows: for every \(e \in E(H)\), let \(\gamma(e) = \alpha(e) + n\). Thus, we obtain

\[
\Sigma'(G) \leq \Sigma'(K_n) + \sum_{j=1}^{m} \frac{d_G(v_j)(d_G(v_j)+2n+1)}{2}.
\]

\(\square\)

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