Localization of Gravitino Field on $f(R)$ Thick Branes

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In this paper, we consider the localization of a five-dimensional gravitino field on $f(R)$ thick branes. We get the coupled chiral equations of the Kaluza-Klein (KK) modes of gravitino by choosing the gauge condition $\Psi_z = 0$. It is found that the chiral equations of the gravitino KK modes are almost the same as the ones of the Dirac fermion. However, their chiralities are precisely opposite. The chiral KK modes of gravitino could be localized on some kinds of $f(R)$ thick branes if a coupling term is introduced. We investigate the localization of gravitino on three kinds of $f(R)$ thick branes through a Yukawa-like coupling term with background scalar fields. It has been shown that all the KK modes of gravitino can not be localized on the pure geometric $f(R)$ thick branes by adding a five-dimensional gravitino mass term. However, for the $f(R)$ thick branes generated by one or two background scalar fields, only the left- or right-handed zero mode could be localized on the branes and the massive KK resonant modes are the same for both left- and right-handed gravitinos, in spite of their opposite chiralities. All these results are consistent with that of the five-dimensional Dirac fermion except their chiralities, which may be an important sign to distinguish the gravitino field and the Dirac fermion field.
I. INTRODUCTION

The extra dimensional theory has attracted more and more attention even though the visible world is a four-dimensional spacetime \[ 4-12 \]. Some classical physical problems including the gauge hierarchy problem (the huge difference between the Planck scale and the weak scale) \[ 4, 12, 13, 15 \] and the cosmological problem \[ 5, 12, 13, 16, 17 \] could be solved via utilizing extra dimensions. In the 1920s, the Kaluza-Klein (KK) theory was proposed to unify Einstein’s gravity and electromagnetism by introducing a compact extra spacial dimension with Plank size \[ 18, 19 \]. Several decades later, Akama, Rubakov and Shaposhnikov proposed the idea of domain wall braneworld with an infinite extra dimension in five-dimensional flat spacetime \[ 1, 2 \]. In 1998, Antoniadis and Arkani-Hamed etc introduced the famous model with large extra dimension that attempts to solve the hierarchy problem \[ 3, 4 \]. One year later, Randall and Sundrum (RS) suggested that the extra dimension with warped geometry could be finite or infinite, corresponding to the RSI \[ 5 \] or RSII \[ 6 \] thin braneworld model. In both braneworld scenarios, our visible four-dimensional world is a brane without thickness along the extra dimension, and the matter fields of the Standard Model (SM) are confined on the brane while only gravity propagates in the five-dimensional bulk spacetime. Subsequently, more realistic thick branes generated dynamically by matter fields or pure gravity were introduced \[ 20, 29 \]. In these models, there exists a non-vanishing distribution of energy density along the extra dimension.

The braneworld scenario with warped infinite extra dimensions requires a natural physical mechanism to trap the matter fields on the branes in order for them to interact with the current experiments. Thus, it is significant to investigate the localization of the matter fields on various kinds of branes \[ 30, 40 \]. In order to rebuild the SM on the branes, the zero modes of these matter fields (the four-dimensional massless particles) should be localized on the branes. At the same time, the localization of massive KK modes are crucial to provide a method to explore extra dimensions. For example, we may observe some physical effects of these KK particles interacting with the SM particles in the Large Hadron Collider (LHC) \[ 17, 50 \]. In some braneworld models, there are no bounded massive KK modes for some matter fields, while there may be some resonant KK modes quasi-localized on the branes. These massive resonant KK modes may stay on the branes for a long time and interact with other particles, which could provide us with opportunities to find the massive resonant KK modes and prove the existence of extra dimensions \[ 39, 42, 45, 51, 52, 54, 50 \].

Gravitino is the gauge fermion supersymmetric partner of graviton in the supersymmetry theory. It has been suggested as a candidate for dark matter in cosmology \[ 58, 62 \]. It is a fermion of spin 3/2 and obeys the Rarita-Schwinger equation. The mass of a light gravitino is always considered around 1eV \[ 58 \], but there are still some challenges to its mass \[ 61 \]. Its mass is widely investigated in the models of hot and cold dark matters \[ 58, 60 \], and the possibility of finding light gravitinos at the LHC was discussed in Ref. \[ 63 \]. The behaviors of gravitino around a black hole also attract attentions \[ 64, 67 \]. Besides, gravitino is a kind of matter field beyond the SM with many special properties that the SM matter fields do not possess. Therefore, the localization of a five-dimensional gravitino field on a brane will be very interesting and give us new perspective to investigate the gravitino. Compared to the matter fields of the SM such as the scalar and fermion fields, the works on gravitino are few and not comprehensive \[ 37, 46, 68, 73 \]. The zero mode of a five-dimensional free gravitino can be localized on a RS-like brane only when a bulk mass term is introduced \[ 69 \]. In a \( D \)-dimensional spacetime with \( D \geq 5 \), the zero mode of the gravitino with a coupling term can be localized on the brane, and the localization property is similar to that of the Dirac fermion \[ 37, 71 \]. In addition, the behavior of the gravitino KK modes with coupling terms was investigated in Ref. \[ 72 \]. Recently, the localization and mass spectrum of the gravitino KK modes on two kinds of thin branes (the RS branes and the scalar-tensor branes) were investigated in Ref. \[ 10 \]. It should be noticed that most of these investigations focused on the RS-like thin branes.

In this paper, we pay our attention to the localization of a five-dimensional gravitino field on the \( f(R) \) thick branes. Although general relativity is a very successful theory, its non-renormalization motives the investigation of modified gravity theories, particularly the gravity theories including higher-order curvature terms \[ 74 \]. \( f(R) \) gravity is a kind of modified gravity whose Lagrangian is a function of the scalar curvature \( R \). It always contains higher-order curvature invariants, which could make the theory to be renormalizable \[ 74 \]. Furthermore, the \( f(R) \) gravity could be used to explain the dark energy or dark matter and answer the astrophysical and cosmological riddles. Therefore, it has been studied widely in cosmology and braneworld \[ 23, 29, 74, 80 \]. In Ref. \[ 29 \], the authors investigated various kinds of \( f(R) \)-branes and gave their general solutions. All of these solutions are also appropriate for the general relativity braneworlds, i.e., \( f(R) = R \).

In this paper, we would like to investigate the localization of a five-dimensional gravitino field on the \( f(R) \) thick branes, whose solutions have been given in Ref. \[ 29 \]. The conclusions of the localization of the gravitino will have some certain universality because they are also appropriate for the general relativity braneworlds. We believe it will give us some interesting results for the structure of thick branes that the thin branes do not have. Our work is organized as follows. In Sec. II, we consider the localization of a five-dimensional free massless gravitino field on a thick brane. We introduce the gauge condition \( \Psi_z = 0 \) and get the Schrödinger-like equations of the gravitino KK modes. Then we focus on the localization of a five-dimensional gravitino field with a coupling term on a thick brane in Sec. III.
Three kinds of $f(R)$ thick branes are considered and the massive KK resonances are studied. Finally, discussion and conclusion are given in Sec. IV.

II. LOCALIZATION OF FREE GRAVITINO FIELD ON THICK BRANES

Firstly, we consider the localization of a free massless gravitino field on a thick brane in a five-dimensional spacetime. Usually, it can be to assume the five-dimensional line-element as

$$ds^2 = g_{MN}dx^M dx^N = e^{2A(y)} \hat{g}_{\mu\nu}(x)dx^\mu dx^\nu + dy^2.$$  \hspace{1cm} (1)

Here, $M$ and $N$ denote the curved five-dimensional spacetime indices, $\hat{g}_{\mu\nu}$ is the metric on the brane and the warp factor $e^{2A(y)}$ is only the function of the extra dimension $y$. For convenience, the following coordinate transformation

$$dz = e^{-A(y)} dy$$  \hspace{1cm} (2)

could be performed to transform the metric (1) to be

$$ds^2 = e^{2A(z)}(\hat{g}_{\mu\nu}dx^\mu dx^\nu + dz^2).$$  \hspace{1cm} (3)

The action of a free massless gravitino field $\Psi$ in five-dimensional spacetime is given by \cite{37, 46, 71}

$$S = \int d^5 x \sqrt{-g} \bar{\Psi} M^M \Gamma^N \Gamma^R D_N \Psi_R,$$  \hspace{1cm} (4)

and the corresponding equations of motion read as

$$\Gamma^M \Gamma^N \Gamma^R D_N \Psi_R = 0.$$  \hspace{1cm} (5)

The Dirac gamma matrices $\Gamma^M$ in curved five-dimensional spacetime satisfy $\Gamma^M = e^{M_\bar{M}} \Gamma^{\bar{M}}$. $\Gamma^{\bar{M}}$ are the gamma matrices in flat five-dimensional spacetime and $(\Gamma^M, \Gamma^{\bar{N}}) = 2\eta^{MN}$, where $M$ and $N$ represent the five-dimensional local Lorentz indices. The vielbein satisfies $g_{MN} = e^{M_\bar{M}} e^{N_{\bar{N}}} \eta_{\bar{M}\bar{N}}$ and for the metric (3) it is given by

$$e^M_{\bar{M}} = \left( \begin{array}{cc} e^A \hat{e}_\mu \hat{e}_\bar{\mu} & 0 \\ 0 & e^A \end{array} \right), \quad e^M_{\bar{M}} = \left( \begin{array}{cc} e^{-A} \hat{e}_\mu \hat{e}_\bar{\mu} & 0 \\ 0 & e^{-A} \end{array} \right).$$  \hspace{1cm} (6)

From the relations $e_{M\bar{M}} = g_{MN}e^N_{\bar{N}}$ and $e^{M\bar{M}} = g^{MN}e^N_{\bar{N}}$, we can get

$$e_{M\bar{M}} = \left( \begin{array}{cc} e^A \hat{e}_\mu \hat{e}_\bar{\mu} & 0 \\ 0 & e^A \end{array} \right), \quad e^{M\bar{M}} = \left( \begin{array}{cc} e^{-A} \hat{e}_\mu \hat{e}_\bar{\mu} & 0 \\ 0 & e^{-A} \end{array} \right).$$  \hspace{1cm} (7)

Thus $\Gamma^M = e^{-A}(\gamma^\mu \gamma_\bar{\mu} \gamma^5, \gamma^5) = e^{-A}(\gamma^\mu, \gamma^5)$, where $\gamma^\mu = \hat{e}^I_{\mu} \gamma^I$, $\gamma^\bar{\mu}$ and $\gamma^5$ are the flat gamma matrices in the four-dimensional Dirac representation. In this paper, we choose the following representation for the four-dimensional flat gamma matrices:

$$\gamma^0 = \left( \begin{array}{cc} 0 & -iI \\ -iI & 0 \end{array} \right), \quad \gamma^i = \left( \begin{array}{cc} 0 & i\sigma^i \\ -i\sigma^i & 0 \end{array} \right), \quad \gamma^5 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right).$$  \hspace{1cm} (8)

Here $I$ is a two-by-two unit matrix and $\sigma^i$ are the Pauli matrices. In this paper, we only consider flat thick branes, i.e. $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$. So we have $\hat{e}^A_{\mu} = \delta^A_{\mu}$ and $\gamma^\mu = \gamma^A$. In addition, the covariant derivative of a gravitino field is defined by

$$D_N \Psi_R = \partial_N \Psi_R - \Gamma^M_{NR} \Psi_M + \omega_N \Psi_R,$$  \hspace{1cm} (9)

where the spin connection $\omega_N$ is defined by $\omega_N = \frac{1}{2} \omega^N_{\bar{M} \bar{L} \bar{R} \bar{P}} \Gamma^\bar{M}_{\bar{N}} \Gamma_{\bar{L}} \Gamma_{\bar{P}}$ and $\omega_N \bar{\omega} \bar{L}$ is given by

$$\omega^N_{\bar{M} \bar{L}} = \frac{1}{2} e^{MN} (\partial_N e_M \bar{L} - \partial_M e_N \bar{L}) - \frac{1}{2} e^{M \bar{L}} (\partial_M e_N \bar{N} - \partial_N e_M \bar{N}) - \frac{1}{2} e^{N \bar{N}} e^{P \bar{L}} (\partial_M e_{P \bar{R}} - \partial_P e_{M \bar{R}}) e_N \bar{R}.$$  \hspace{1cm} (10)

Thus we get the non-vanishing components of $\omega_N$:

$$\omega_\mu = \frac{1}{2} (\partial_z A) \gamma_\mu \gamma_5 + \hat{\omega}_\mu.$$  \hspace{1cm} (11)
Note that the four-dimensional spin connection $\omega_\mu$ on a flat brane vanishes. The non-vanishing components of $D_N \Psi_R$ are
\begin{align}
D_\mu \Psi_\nu &= \partial_\mu \Psi_\nu - \Gamma^M_{\mu \nu} \Psi_M + \omega_\mu \Psi_\nu \\
&= \hat{D}_\mu \Psi_\nu + (\partial_\mu A) \hat{g}_{\mu \nu} \Psi_z + \frac{1}{2} (\partial_\mu A) \gamma_\mu \gamma_5 \Psi_\nu, \\
D_\mu \Psi_z &= \partial_\mu \Psi_z - \Gamma^M_{\mu z} \Psi_M + \omega_\mu \Psi_z \\
&= \partial_\mu \Psi_z - (\partial_\mu A) \Psi_z + \frac{1}{2} (\partial_\mu A) \gamma_\mu \gamma_5 \Psi_z + \hat{\omega}_\mu \Psi_z, \\
D_z \Psi_\mu &= \partial_z \Psi_\mu - \Gamma^M_{z \mu} \Psi_M + \omega_z \Psi_\mu \\
&= \partial_z \Psi_\mu - (\partial_z A) \Psi_\mu, \\
D_z \Psi_z &= \partial_z \Psi_z - \Gamma^M_{z z} \Psi_M + \omega_z \Psi_z \\
&= \partial_z \Psi_z - (\partial_z A) \Psi_z.
\end{align}

Equation (5) includes five equations because $M$ runs over all five spacetime indices. There are two kinds of equations: $M = 5$ and $M = \mu$. For the first case of $M = 5$, the equation of motion reads as
\begin{align}
\Gamma^{[5 \Gamma^N \Gamma^R]} D_N \Psi_R &= \Gamma^{[5 \Gamma^\nu \Gamma^\rho]} D_\mu \Psi_\nu \\
&= ([\Gamma^\mu, \Gamma^\nu] - g^{\mu \nu}) \Gamma^5 (\hat{D}_\mu \Psi_\nu + (\partial_\mu A) \hat{g}_{\mu \nu} \Psi_z + \frac{1}{2} (\partial_\mu A) \gamma_\mu \gamma_5 \Psi_\nu) \\
&= 0.
\end{align}

In this paper, for convenience we prefer to choose the gauge condition $\Psi_z = 0$, with which we introduce the KK decomposition
\begin{equation}
\Psi_\mu = \sum \psi_\mu^{(n)}(x) \xi_n(z),
\end{equation}
where $\psi_\mu^{(n)}(x)$ is the four-dimensional gravitino field. Then Eq. (16) is reduced to
\begin{equation}
([\gamma^\mu, \gamma^\nu] - \hat{g}^{\mu \nu}) \gamma^5 \left(\hat{D}_\mu \psi_\nu^{(n)} + \frac{1}{2} (\partial_\mu A) \gamma_\mu \gamma_5 \psi_\nu^{(n)}\right) = 0.
\end{equation}

For the four-dimensional massive gravitino field $\psi_\mu$, it should satisfy the following four equations [59]
\begin{align}
\gamma^{[\lambda \gamma^\nu] \hat{D}_\mu \psi_\nu - m_{3/2}[\gamma^\lambda, \gamma^\rho] \psi_\mu &= 0, \\
\gamma^\mu \psi_\mu &= 0, \\
\hat{D}_\mu \psi_\mu &= 0, \\
(\gamma^\mu \hat{D}_\mu + m_{3/2}) \psi_\nu &= 0.
\end{align}

Here, $m_{3/2}$ is the mass of a four-dimensional gravitino field $\psi_\mu$. Thus the left-hand side of Eq. (18) is always vanished for a four-dimensional gravitino field $\psi_\mu^{(n)}$ satisfying the above equation (19). On the other hand, when we choose the gauge condition $\Psi_z = 0$, the part $\Gamma^{[5 \Gamma^N \Gamma^R]} D_N \Psi_R$ in the five-dimensional gravitino action [4] has no contribution, so Eq. (10) can be ignored. Then we will focus on the case of $M = \mu$, for which the equations of motion are
\begin{align}
\Gamma^{[5 \Gamma^\nu \Gamma^\rho]} D_\nu \Psi_L &= \Gamma^{[5 \Gamma^N \Gamma^R]} D_\nu \Psi_R + \Gamma^{[5 \Gamma^\nu \Gamma^\sigma]} D_\sigma \Psi_z + \Gamma^{[5 \Gamma^5 \Gamma^\nu]} D_\nu \Psi_\nu \\
&= e^{-3A_\gamma} [\gamma^\mu \gamma^\rho \nu] \hat{D}_\mu \psi_\nu - e^{-3A} [\gamma^\lambda, \gamma^\rho] \gamma^5 (\partial_\lambda A + \partial_\rho) \psi_\nu \\
&= 0,
\end{align}
where we have used the gauge condition $\Psi_z = 0$. When we introduce the decomposition (17) and consider the zero mode, which corresponds to the four-dimensional massless gravitino satisfying $\gamma^{[\lambda \gamma^\nu \gamma^\rho]} \hat{D}_\mu \psi_\nu^{(0)} = 0$, we get the equation of motion for the extra-dimensional configuration $\xi_0(z):
\begin{align}
\gamma^{[\lambda \gamma^\nu \gamma^\rho]} \hat{D}_\mu \psi_\nu^{(0)} (x) \xi_0(z) &= [\gamma^\lambda, \gamma^\rho] \gamma_5 \psi_\nu^{(0)}(x) (\partial_\lambda A + \partial_\rho) \xi_0(z) \\
&= - (\partial_\lambda A + \partial_\rho) \xi_0(z) = 0.
\end{align}

Obviously, the solution is
\begin{equation}
\xi_0(z) = C e^{-A(z)},
\end{equation}
where $C$ is a normalization constant. By substituting the zero mode $\xi_0(z)$ into the gravitino action (11) yields

$$S^{(0)}_\frac{3}{2} = I_0 \int d^4x \sqrt{-g} \bar{\psi}_\lambda^{(0)}(\gamma^{\mu} \gamma^{\nu}) \bar{D}_{\mu} \psi^{(0)}_\nu(x),$$

(23)

where $I_0 \equiv \frac{1}{2} \int dz \int d^2z \bar{\psi}_\lambda(z)^2 = C^2 \int dz = C^2 \int e^{-A(y)} dy$. In order to localize the spin $3/2$ gravitino on a brane, the integral $I_0$ must be finite. So if we consider a RT-type brane model, then only for a finite extra dimension the zero mode of a five-dimensional free massless gravitino can be localized on the brane.

For the massive modes, we need to introduce the following chiral decomposition:

$$\Psi_\mu(x, z) = \sum_n \left( \psi^{(n)}_{L\mu}(x) \xi_{L_n}(z) + \psi^{(n)}_{R\mu}(x) \xi_{R_n}(z) \right)$$

$$= \sum_n \left( \begin{array}{c} \bar{\gamma}^{(n)}_{L\mu} \xi_{L_n} \\ \psi^{(n)}_{R\mu} \xi_{R_n} \end{array} \right),$$

(24)

where $\bar{\gamma}^{(n)}_{L\mu}$ and $\bar{\gamma}^{(n)}_{R\mu}$ are both the two-component spinors. The effect of $P_{L,R} (P_{L,R} = \frac{1}{2}(I \mp \gamma^5))$ on the gravitino field $\Psi_M$ is to single out the left- and right-handed parts, respectively, which is equivalent to the following equations:

$$\gamma^5 \psi^{(n)}_{L\mu} = -\psi^{(n)}_{L\mu}, \quad \gamma^5 \psi^{(n)}_{R\mu} = \psi^{(n)}_{R\mu}.$$  

(25)

Thus, substituting the chiral decomposition (24) into Eq. (20), we have

$$0 = [\gamma^\lambda, \gamma^\nu] \partial_z \psi^{(n)}_{L\mu} \xi_{L_n} + \sum \left( [\gamma^\lambda, \gamma^\nu] \partial_z A \psi^{(n)}_{R\mu} \xi_{R_n} + [\gamma^\lambda, \gamma^\nu] \partial_z \psi^{(n)}_{L\mu} \xi_{R_n} - [\gamma^\lambda, \gamma^\nu] \partial_z \psi^{(n)}_{R\mu} \xi_{L_n} = 0, \right.$$  

(26)

Since the product of three gamma matrices is oblique diagonal and the product of two gamma matrices is diagonal, two equations can be obtained from above equation:

$$\gamma^\lambda [\gamma^\mu, \gamma^\nu] \partial_z \psi^{(n)}_{L\mu} \xi_{L_n} + [\gamma^\lambda, \gamma^\nu] \partial_z A \psi^{(n)}_{R\mu} \xi_{R_n} + [\gamma^\lambda, \gamma^\nu] \partial_z \psi^{(n)}_{R\mu} \xi_{L_n} - [\gamma^\lambda, \gamma^\nu] \partial_z \psi^{(n)}_{L\mu} \xi_{R_n} = 0, \right.$$  

(27a)

$$\gamma^\lambda [\gamma^\mu, \gamma^\nu] \partial_z \psi^{(n)}_{R\mu} \xi_{R_n} + [\gamma^\lambda, \gamma^\nu] \partial_z A \psi^{(n)}_{L\mu} \xi_{L_n} + [\gamma^\lambda, \gamma^\nu] \partial_z \psi^{(n)}_{L\mu} \xi_{R_n} - [\gamma^\lambda, \gamma^\nu] \partial_z \psi^{(n)}_{R\mu} \xi_{L_n} = 0. \right.$$  

(27b)

Through the method of separation of variance and defining a parameter $m_n$, we have

$$\gamma^\lambda [\gamma^\mu, \gamma^\nu] \partial_z \psi^{(n)}_{L\mu} \xi_{L_n} + [\gamma^\lambda, \gamma^\nu] \partial_z A \psi^{(n)}_{R\mu} \xi_{R_n} + [\gamma^\lambda, \gamma^\nu] \partial_z \psi^{(n)}_{R\mu} \xi_{L_n} - [\gamma^\lambda, \gamma^\nu] \partial_z \psi^{(n)}_{L\mu} \xi_{R_n} = m_n,$$  

(28a)

$$\gamma^\lambda [\gamma^\mu, \gamma^\nu] \partial_z \psi^{(n)}_{L\mu} \xi_{L_n} - [\gamma^\lambda, \gamma^\nu] \partial_z A \psi^{(n)}_{L\mu} \xi_{R_n} + [\gamma^\lambda, \gamma^\nu] \partial_z \psi^{(n)}_{R\mu} \xi_{L_n} - [\gamma^\lambda, \gamma^\nu] \partial_z \psi^{(n)}_{L\mu} \xi_{R_n} = m_n,$$  

(28b)

i.e.,

$$\gamma^\lambda [\gamma^\mu, \gamma^\nu] \partial_z \psi^{(n)}_{L\mu} \xi_{L_n} - m_n \gamma^\alpha [\gamma^\lambda, \gamma^\nu] \psi^{(n)}_{R\mu} \xi_{R_n} = m_n [\gamma^\lambda, \gamma^\nu] \psi^{(n)}_{L\mu} \xi_{R_n}, \quad \gamma^\lambda [\gamma^\mu, \gamma^\nu] \partial_z \psi^{(n)}_{R\mu} \xi_{R_n} + m_n \gamma^\nu [\gamma^\lambda, \gamma^\mu] \psi^{(n)}_{L\mu} \xi_{L_n} = -m_n [\gamma^\lambda, \gamma^\nu] \psi^{(n)}_{R\mu} \xi_{L_n}. \right.$$  

(29)

(30)

Equations (29) are the ones that four-dimensional chiral gravitino fields satisfy and Eqs. (30) are the coupled ones which KK modes $\xi_{L_n}$ and $\xi_{R_n}$ satisfy. Performing the field transformations $\xi_{R_n}(z) = \lambda_{n}^{M}(z) e^{-A}$ and $\xi_{L_n}(z) = \lambda_{n}^{L}(z) e^{-A}$, we can obtain equations for the left- and right-handed KK modes of gravitino

$$\partial^2_{z} \lambda_{n}^{L}(z) = -m_n^2 \lambda_{n}^{L}(z), \quad \partial^2_{z} \lambda_{n}^{R}(z) = -m_n^2 \lambda_{n}^{R}(z).$$  

(31a)

(31b)

When the following normalizable conditions are introduced

$$\int \lambda_{n}^{L}(z) \lambda_{n}^{R}(z) dz = \delta_{RL} \delta_{m_n},$$  

(32)

the effective action of the four-dimensional massless and massive gravitinos can be got

$$S^{m} = \sum_n \int d^4x \left( \bar{\psi}_\lambda^{(n)}(x) [\gamma^\lambda, \gamma^\nu] \partial_{\mu} \psi_{L\nu}^{(n)}(x) - m_n \bar{\psi}_\lambda^{(n)}(x) [\gamma^\lambda, \gamma^\nu] \psi_{R\mu}^{(n)}(x) \right)$$

$$+ \bar{\psi}_\lambda^{(n)}(x) [\gamma^\lambda, \gamma^\nu] \partial_{\mu} \psi_{R\nu}^{(n)}(x) - m_n \bar{\psi}_\lambda^{(n)}(x) [\gamma^\lambda, \gamma^\nu] \psi_{L\mu}^{(n)}(x) \right)$$

$$= \sum_n \int d^4x \left( \bar{\psi}_\lambda^{(n)}(x) [\gamma^\lambda, \gamma^\nu] \partial_{\mu} \psi_{L\nu}^{(n)}(x) - m_n \bar{\psi}_\lambda^{(n)}(x) [\gamma^\lambda, \gamma^\nu] \psi_{R\mu}^{(n)}(x) \right).$$  

(33)
However, obviously the solutions of Eqs. (31a) and (31b) are mediocre. Thus the four-dimensional massive gravitinos cannot be localized. This conclusion is the same as Dirac fermion.

### III. LOCALIZATION OF GRAVITINO FIELD WITH COUPLING TERM ON THICK BRANES

As we have pointed out in the previous section, the massive KK modes of a five-dimensional free massless gravitino field cannot be localized on RS-type thick branes. Therefore, it is necessary to introduce a coupling term as the case of Dirac field. In the thin brane scenario [10], one usually introduces an additional mass term which is associated with the warp factor of the thin brane. In the scenario of thick brane generated by one or multiple background scalar fields, we could introduce a coupling term between the background scalar field and gravitino field. We consider the simplest coupling, i.e., a Yukawa-like coupling, for which the action of a five-dimensional gravitino field is

\[
S = \int d^5x\sqrt{-g}\left(\bar{\Psi}_M \Gamma^{[MN} \Gamma^{R]} D_N \Psi_R - \eta F(\phi) \bar{\Psi}_M [\Gamma^M, \Gamma^N] \Psi_N \right).
\] (34)

Here, \( F(\phi) \) is a function of the background scalar field \( \phi \) and \( \eta \) is the coupling constant. The equations of motion derived from the above action are

\[
\Gamma^{[MN} \Gamma^{R]} D_N \Psi_R - \eta F(\phi)[\Gamma^M, \Gamma^N] \Psi_N = 0.
\] (35)

By using the gauge condition \( \Psi_z = 0 \) and introducing the chiral decomposition

\[
\Psi_\mu(x, z) = \sum_n e^{-\eta z} \left( \psi_L^{(n)}(x) \chi_n^L(z) + \psi_R^{(n)}(x) \chi_n^R(z) \right),
\] (36)

we can obtain the following first-order coupled equations

\[
(\partial_z - \eta e^A F(\phi)) \chi_n^L(z) = -m_n \chi_n^R(z),
\] (37a)

\[
(\partial_z + \eta e^A F(\phi)) \chi_n^R(z) = m_n \chi_n^L(z).
\] (37b)

From the above equation (37), the left- and right-handed KK modes of the gravitino field satisfy the following Schrödinger-like equations:

\[
(-\partial_z^2 + V_L(z)) \chi_n^L(z) = m_n^2 \chi_n^L(z),
\] (38a)

\[
(-\partial_z^2 + V_R(z)) \chi_n^R(z) = m_n^2 \chi_n^R(z),
\] (38b)

where the effective potentials are given by

\[
V_L(z) = (\eta e^A F(\phi))^2 + \eta \partial_z (e^A F(\phi)),
\] (39a)

\[
V_R(z) = (\eta e^A F(\phi))^2 - \eta \partial_z (e^A F(\phi)).
\] (39b)

For a five-dimensional free gravitino, we have obtained the effective action [33] of the four-dimensional left- and right-handed gravitinos [38a] and [38b] are the same as those of the KK modes of a Dirac field, while the only difference is their chiralities. For a given background solution of a thick brane, if the function \( F(\phi) \) and the coupling parameter \( \eta \) are the same, it seems that the mass spectrum of the KK gravitinos will be the same as that of the Dirac field. Here, we should note the difference of chiralities, which will give an interesting result.

Next we first review some kinds of \( f(R) \) thick branes [29, 80], and then investigate the localization of the five-dimensional gravitino on these branes and give their KK mass spectra.

In the five-dimensional spacetime, the action of a general \( f(R) \) thick brane model reads [29]

\[
S = \int d^5x\sqrt{-g} \left( \frac{1}{2\kappa_5^2} f(R) + L(\phi_i, X_i) \right),
\] (40)

where \( \kappa_5^2 = 8\pi G^5 \) is the five-dimensional gravitational constant and is set to one for convenience. \( f(R) \) is a function of the scalar curvature \( R \) and \( L(\phi_i, X_i) \) is the Lagrangian density of the background scalar fields \( \phi_i \) with the kinetic terms \( X_i = -\frac{1}{2} g^{MN} \partial_M \phi_i \partial_N \phi_i \). It is predictable that the spectra of the KK modes of the gravitino field on these \( f(R) \) thick branes will be almost the same as the ones of the Dirac field except their chiralities. These results could give us some important reference in the future experiments about extra dimension and gravitino.
A. Localization of gravitino field on the pure geometric \( f(R) \) thick branes without background scalar field

Firstly, we focus on the localization of the gravitino field on the pure geometric \( f(R) \) thick branes. In Ref. [86], the authors investigated the pure geometric \( f(R) \) thick branes, where the Lagrangian density of the background scalar fields \( L(\phi_i, X_1) \) vanishes. For the flat pure geometric \( f(R) \) thick branes, the background metric is given by [11] with \( g_{\mu\nu} = \eta_{\mu\nu} \). The solution of the warp factor \( A(y) \) is [86]

\[
A(y) = -n \ln(\cosh(ky)),
\]

where \( k \) is a positive real parameter that related to the curvature of the five-dimensional spacetime and \( n \) is a positive integer number. The solutions of the function \( f(R) \) for \( n = 1 \) and \( n = 20 \) are respectively [86]

\[
f(R) = \frac{1}{7}(6k^2 + R) \cosh(\alpha(w(R))) - \frac{2}{7}k^2 \sqrt{480 - \frac{36R}{k^2} - \frac{3R^2}{k^4} \sinh(\alpha(w(R)))}, \quad (n = 1) \tag{42}
\]

\[
f(R) = -\frac{377600}{7803}k^2 + \frac{4196}{2601}R - \frac{83}{41616k^2}R^2 + \frac{13}{39951360k^4}R^3, \quad (n = 20) \tag{43}
\]

where \( \alpha(w) = 2\sqrt{3}\arctan(\tanh(\frac{w}{R})) \) and \( w(R) = \pm\arcsch\frac{\sqrt{20n^2 + R/k^2}}{\sqrt{8n^2 + 20n^2}} \). For arbitrary \( n \), the function \( f(R) \) has no unified expression, and it is hard to get an analytical \( y(z) \) from the following relation of \( z(\mu) \) calculated from the solution [11]:

\[
z(\mu) = -\frac{\cosh^{n+1}(\mu) \sinh(\mu)}{(n + 1)k^2 \sqrt{-\sinh^2(\mu)}} \tag{44}
\]

Since there is no background scalar field in the pure geometric brane model, we may try to take \( \eta F \) as the five-dimensional mass \( M \) of the gravitino field. Then, the effective potentials \( V^L \) and \( V^R \) can be expressed in terms of the extra dimension \( y \)

\[
V^L(z(\mu)) = \text{sech}^{2n}(\mu)(M^2 - nkM \tanh(\mu)) \tag{45}
\]

\[
V^R(z(\mu)) = \text{sech}^{2n}(\mu)(M^2 + nkM \tanh(\mu)) \tag{46}
\]

It is easy to see that both potentials are asymmetric and their asymptotic behaviors are

\[
V^L(0) = M^2, \quad V^L(\pm\infty) = e^{2A(\pm\infty)}(M^2 \mp nkM) = 0, \tag{47}
\]

\[
V^R(0) = M^2, \quad V^R(\pm\infty) = e^{2A(\pm\infty)}(M^2 \pm nkM) = 0, \tag{48}
\]

which indicates that there is no bound massive KK mode. The solutions for the left- and right-handed zero modes of the gravitino field are \( \lambda_0^{L,R} \propto e^{\pm M\mu} \). It is clear that both zero modes are not normalizable, and hence cannot be localized on the pure geometric \( f(R) \) thick branes.

B. Localization of gravitino field on the \( f(R) \) thick branes with \( L = X - V(\phi) \)

Now let us consider the thick \( f(R) \) branes generated by one background scalar field. For the Lagrangian density \( L = X - V(\phi) = -\frac{1}{2} \partial^M \phi \partial_M \phi - V(\phi) \), the solution in this model with the Sine-Gordon potential is given by [20]

\[
f(\hat{R}) = \hat{R} + \alpha \left\{ \frac{24k^2 + 2\hat{R} + 2b\hat{R}}{2 + 5b} \left[ P_{K^-}^{b/2}(\Xi) - \beta Q_{K^-}^{b/2}(\Xi) \right] \\
- 4(b^2 - 2bK_+)(\Xi) \left[ P_{K_+}^{b/2}(\Xi) - \Xi P_{K_+}^{b/2}(\Xi) \right] \\
+ \beta(\Xi \left[ Q_{K_+}^{b/2}(\Xi) - Q_{K_+}^{b/2}(\Xi) \right] \right) \right) \phi^{b/2}, \tag{49a}
\]

\[
V(\phi) = \frac{3bk^2}{8} \left[ (1 - 4b) + (1 + 4b) \cos \left( \sqrt{\frac{8}{3b}} \phi \right) \right], \tag{49b}
\]

\[
\phi(y) = \sqrt{6b} \arctan \left[ \tanh \left( \frac{k y}{2} \right) \right], \tag{49c}
\]

\[
A(y) = -b \ln \left[ \cosh(\mu) \right], \tag{49d}
\]
where $b$ and $k$ are positive parameters related to the thickness of the brane, $\alpha$ is an arbitrary constant, $\hat{R} \equiv R/k^2$, $K_\pm \equiv \frac{1}{2} \sqrt{(b-14)b+1} \pm 1/2$, $\Xi \equiv \sqrt{1-\Theta^2}$, $\Theta \equiv \frac{\sqrt{b+9}+\sqrt{b-9}}{2\sqrt{b+9}}$, $P$ and $Q$ are the first and second kinds of Legendre functions, $\beta = P_{K_+}(0)/Q_{K_+}(0)$. Note that the solution $(49b)-(49d)$ is also appropriate for the case of $f(R) = R$.

Thus our following results are also appropriate for the case of general relativity thick brane. As shown in the above subsection, it is very difficult to obtain analytical $y(z)$. Therefore, in the following we will solve the equations numerically. The effective potentials $V^L$ and $V^R$ in the physical coordinate $y$ become

$$V^L(z(y)) = (\eta e^A F(\phi))^2 + \eta e^{2A} \partial_y F(\phi) + \eta (\partial_y A) e^{2A} F(\phi),$$

$$V^R(z(y)) = V^L(z(y)) |_{\eta \rightarrow -\eta}.$$  

It is obvious that for different forms of $F(\phi)$, the potentials $V^L$ and $V^R$ have different expressions, which determine the mass spectra of the KK modes. In this paper, we would like to consider one kind of Yukawa coupling, i.e., $F(\phi) = \phi^\alpha$ with positive integer $\alpha$. For a kink configuration of the scalar $\phi$, since $V^L$ and $V^R$ are demanded to be symmetrical with respect to extra dimension $y$, $\alpha$ should be odd. Next we consider two cases: the simplest case $F(\phi) = \phi$ and the case for $\alpha > 1$.

**FIG. 1.** Potentials $V^L(z)$ and $V^R(z)$ for the left- and right-handed gravitinos on the $f(R)$ thick branes with $F(\phi) = \phi$. Here, $k = 1$ and the coupling constant $\eta$ is set to 2.0 (blue thin trace), 3.0 (green thick trace) and 4.0 (red dashed trace).

1. **Case I: $F(\phi) = \phi$**

For the case of $F(\phi) = \phi$, the effective potentials $(51)$ read

$$V^L(y) = \frac{1}{2} \cosh(ky)^{-1-2b} \left[ 12bn^2 \arctan \left( \frac{\tanh \left( \frac{ky}{2} \right)}{2} \right)^2 \cosh(ky)ight.$$  

$$+ \eta \sqrt{6bk(1-2b)} \arctan \left( \frac{\tanh \left( \frac{ky}{2} \right)}{2} \right) \sinh(ky) \right],$$

$$V^R(y) = V^L(y) |_{\eta \rightarrow -\eta},$$
which are symmetrical. The values of the potentials at the original point and infinity are given by
\begin{align}
V^R(0) &= -\eta k \sqrt{\frac{3b}{2}} = -V^L(0), \\
V^R(\pm \infty) &= 0 = V^L(\pm \infty).
\end{align}

It is clear that both potentials have the same asymptotic behaviors as \( y \to \pm \infty \), while their values at \( y = 0 \) are opposite. Thus only the left- or right-handed gravitino zero mode (four-dimensional massless left- or right-handed gravitino) could be localized on the \( f(R) \) thick brane. The shapes of the potentials are shown in Fig. 1 from which it can be seen that for any positive \( b, k \) and \( \eta \), \( V^R(z(\eta)) \) is a volcano type of potential and there may exist a localized zero mode and a continuous gapless spectrum of massive KK modes. Furthermore, the depth of the potential \( V^R \) increases with values of the parameters \( \eta, b \) and \( k \). By solving Eq. (58b) with the potential, the zero mode of the right-handed gravitino becomes
\begin{align}
\chi_0^R(z) &\propto \exp \left( -\eta \int_0^z e^{A(z)} F(\phi) dz \right) \\
&= \exp \left( -\eta \int_0^y \phi(\bar{y}) dy \right) \\
&= \exp \left( -\eta \int_0^y \sqrt{6b} \arctan \left( \tanh \left( \frac{k\bar{y}}{2} \right) \right) dy \right),
\end{align}
and its normalization condition
\begin{align}
\int_{-\infty}^{\infty} (\chi_0^R(z))^2 dz &= \int_{-\infty}^{\infty} (\chi_0^R(y))^2 e^{-A(y)} dy \\
&= \int_{-\infty}^{\infty} \exp \left( -A(y) - 2\eta \int_0^y \phi(\bar{y}) dy \right) dy \\
&= \int_{-\infty}^{\infty} \exp \left( \ln(\cosh(ky)) - 2\eta \int_0^y \sqrt{6b} \arctan \left( \tanh \left( \frac{k\bar{y}}{2} \right) \right) dy \right) dy < \infty
\end{align}
is equivalent to
\begin{align}
\int_0^{\infty} \exp \left( kby - \frac{\pi \eta}{2} \sqrt{6b} y \right) dy < \infty
\end{align}
since \(-A(y) \to kby \) and \( \arctan(\tanh(\frac{ky}{2})) = \pi/4 \) as \( y \to \infty \). The above normalization condition requires
\begin{align}
\eta > \eta_0 \equiv \frac{k}{\pi} \sqrt{\frac{2b}{3}}.
\end{align}

Thus, if the coupling constant is strong enough \( (\eta > \eta_0) \), the right-handed zero mode can be localized on the brane. It is not difficult to check that the left-handed zero mode can not be localized on the brane under the condition [57].

On the other hand, the potential \( V^L(z(\eta)) \) for positive \( \eta \) is always positive and vanishes far away from the brane. This type of potential cannot trap any bound state, and hence there is no left-handed gravitino zero mode. The structure of the potential \( V^L \) is determined by the parameters \( k, b \) and \( \eta \). For given \( k \) and \( b \), the potential \( V^L \) has a barrier for a small \( \eta \). When \( \eta \) increases, there will be a quasi-potential well and the depth of the well will increase with the value of \( \eta \). However, for given \( \eta \) and \( k \) (or \( b \)), the height of the potential \( V^L \) increases with \( b \) (or \( k \)) and the quasi-potential well changes into a barrier as the growth of \( b \) (or \( k \)). The behavior of \( V^L \) around the point \( y = 0 \) is similar to that of the function \( y^4 \) and there will be three extreme points if a quasi-potential well exists around the point \( y = 0 \). Doing third-order Taylor series expansion of \( \partial_y V^L \) near the point \( y = 0 \), we will get
\begin{align}
\partial_y V^L &= \frac{1}{2} k^2 \eta [6b\eta - \sqrt{6bk}(1 + 4b)] y \\
&\quad + \frac{1}{12} k^4 \eta [\sqrt{6bk}(1 + 2b)(5 + 18b) - 24b\eta(1 + 3b)] y^3 + O(z^5).
\end{align}

For \( k = 1 \) and \( b > \frac{3}{2\sqrt{3}} \), the above function has three roots and there is a quasi-potential well when \( \eta > \frac{1}{6} \sqrt{\frac{6+48b+96b^2}{b}} \) (it equals 2.04124 when \( b = 1 \)).
For the case that there is a quasi-potential well for $V_L$, we could find resonance states of the gravitino, which are the massive four-dimensional gravitinos with finite lifetimes on the brane. To investigate the gravitino resonant modes, we give the definition of the relative probability by following Ref. [39]:

$$P_{L,R}(m^2) = \frac{\int_{-z_{\max}}^{z_{\max}} |\chi^{L,R}(z)|^2 dz}{\int_{-2z_b}^{2z_b} |\chi^{L,R}(z)|^2 dz},$$  \hspace{1cm} (59)

where $2z_b$ is approximately the width of the brane, and $z_{\max} = 10z_b$. The left- and right-handed wavefunctions $\chi^{L,R}(z)$ are the solutions of Eqs. (33). The above definition could be explained that $|\chi^{L,R}(z)|^2$ is the probability density [39, 51]. There exists a resonant mode with mass $m_n$, if the relative probability $P(m^2)$ has a peak around $m = m_n$. These peaks should have full width at half maximum and the number of these peaks is the same as the number of the resonant modes. In order to get the solutions of Eqs. (38), we always need additional two types of initial conditions

$$\begin{align*}
\chi^{L,R}_{\text{even}}(0) &= 1, \quad \partial_z \chi^{L,R}_{\text{even}}(0) = 0; \\
\chi^{L,R}_{\text{odd}}(0) &= 0, \quad \partial_z \chi^{L,R}_{\text{odd}}(0) = 1,
\end{align*}$$  \hspace{1cm} (60a-b)

where $\chi^{L,R}_{\text{even}}$ and $\chi^{L,R}_{\text{odd}}$ correspond to the even and odd parity modes of $\chi^{L,R}(z)$, respectively.

Our results are shown in Figs. 2 and Tab. I. It is obvious that the mass spectra of the left- and right-handed gravitino resonant modes are almost the same while their parities are opposite. The first resonant mode of the left-handed gravitino is even and its shape around $z = 0$ looks like a ground state. On the other hand, the first resonant mode of the right-handed gravitino is odd and it seems to be the first excited state. These results are reasonable because the effective potentials $V_L$ and $V_R$ are supersymmetric partners, which give the same spectra of the resonant modes. In fact, fermion resonances on branes have similar properties because Eqs. (58) of the KK modes of a gravitino are almost the same as the ones of a fermion. However, there is a difference between them, which will be explained as follows. For a five-dimensional Dirac fermion field with a coupling term, if we use the representation of the gamma matrices [8] and parity relation [24], the equations of motion of the left- and right-handed fermion KK modes $f^{L,R}$ are given by

$$\begin{align*}
(-\partial_z^2 + V_L(z)) f^L &= m^2 f^L, \\
(-\partial_z^2 + V_R(z)) f^R &= m^2 f^R,
\end{align*}$$  \hspace{1cm} (61a-b)

with the effective potentials

$$\begin{align*}
V_L(z) &= \eta^2 e^{2A} F^2(\phi) - \eta e^A \partial_z F(\phi) - \eta e^A(\partial_z A) F(\phi), \\
V_R(z) &= \eta^2 e^{2A} F^2(\phi) + \eta e^A \partial_z F(\phi) + \eta e^A(\partial_z A) F(\phi).
\end{align*}$$  \hspace{1cm} (62a-b)

It is obvious that the Schrödinger-like equation of the left-handed gravitino KK modes is the one of the right-handed fermion KK modes [61], and the Schrödinger-like equation for the right-handed gravitino KK modes is the one of the left-handed fermion KK modes [61]. Therefore, for a five-dimensional Dirac fermion, only the zero mode of the left-handed fermion can be localized on the $f(R)$ brane with the coupling $F(\phi) = \phi$, and the first resonant
mode of the right-handed fermion is even. This difference between the fermion and gravitino KK modes comes from the difference of their field equations. For a five-dimensional Dirac fermion field with the Yukawa coupling, the field equation reads

\[
[\gamma^\mu \partial_\mu + \gamma^5 (\partial_z + 2\partial_z A) - \eta e^A F(\phi)] \Psi = 0.
\]

(63)

It should be noticed that the sign in front of \(\gamma^5\) is plus. While for a bulk gravitino, Eq. (20) tells us that the sign in front of \(\gamma^5\) is minus, which leads to the swap of the above results. This difference is very meaningful and it could be a symbol of the distinction between Dirac fermion and gravitino fields.

In addition, the number of the resonant modes for the gravitino field increases with the coupling constant \(\eta\) but decreases with the parameter \(b\). The relative probability \(P\) decreases when the mass of the resonant mode approaches the maximum of the potentials. Furthermore, the resonant modes become closer and closer as \(m^2\) approaches the maximum of the potentials. These results are consistent with that of the Dirac fermion.

\begin{align}
(a) m^2 &= 21.9171 \\
(b) m^2 &= 38.3348 \\
(c) m^2 &= 47.9467 \\
(d) m^2 &= 21.9169 \\
(e) m^2 &= 38.3311 \\
(f) m^2 &= 47.9328
\end{align}

FIG. 3. The shapes of the massive KK resonant modes of the left-handed (upper) and right-handed (under) gravitinos for the coupling \(F(\phi) = \phi\) with different \(m^2\). Here, the parameters are set to \(k = 1, b = 1, \eta = 10,\) and \(z_{\text{max}} = 20\).

2. Case II: \(F(\phi) = \phi^\alpha\) with \(\alpha > 1\)

Next, we consider a natural generalization of the Yukawa coupling \(F(\phi) = \phi^\alpha\) with \(\alpha = 3, 5, 7, \cdots\). Note that \(\phi^\alpha\) becomes a double-kink for \(\alpha \geq 3\) since the scalar field \(\phi\) is a kink. For this case, the effective potentials (50) become

\[
V^L(y) = \frac{1}{2} 3\eta^2 k b \phi^{\alpha-1} \arctan(\tanh(ky/2)) \sech^{\alpha+1}(ky) [\alpha - 2b \arctan(\tanh(ky/2)) \sinh(ky)]
\]

\[
+ 6\eta^2 \left( \sqrt{b \arctan(\tanh(ky/2))} \right)^{2\alpha} \sech^{2\alpha}(ky),
\]

(64a)

\[
V^R(y) = V^L(y)|_{\eta \rightarrow -\eta}.
\]

(64b)

It is obvious that both the potentials are symmetry and vanish at \(y = 0\) and \(y \rightarrow \pm\infty\) and they are depicted in Fig. 11 for different values of \(b\) and \(\alpha\). There always exists a quasi-potential well for the left-handed potential \(V^L\) and a double-potential well for the right-handed one. These wells for both potentials are deeper and deeper with the increases of the parameters \(b, \eta,\) and \(\alpha,\) which means that there are more and more resonances with the increases of \(b, \eta,\) and \(\alpha.\) Since the coupling function \(\phi^\alpha\) trends to a constant as \(y \rightarrow \pm\infty,\) the zero mode of the right-handed gravitino

\[
\chi_0^R \propto \exp \left( -\eta \int_0^z e^{A(z)} \phi^\alpha \, dz \right) = \exp \left( -\eta \int_0^y \phi^\alpha \, dy \right)
\]

(65)
TABLE I. The eigenvalue $m^2$, mass $m$ and the relative probability of the left- and right-handed gravitinos with odd-parity and even-parity solutions for the coupling $F(\phi) = \phi$. In all tables of this paper, $C$ and $P$ stand for chirality and parity, $L$ and $R$ mean left- and right-handed, respectively. The parameter $k$ is set to $k = 1$.

$$\exp\left(-\eta(\pi\sqrt{6b})^\alpha |y|\right)$$

is equivalent to $\exp\left(-\eta(\pi\sqrt{6b})^\alpha |y|\right)$ since $\phi^\alpha = \pm(\pi\sqrt{6b})^\alpha$ as $y \to \pm\infty$. It is not difficult to check that the normalization condition can be satisfied for any positive coupling constant $\eta$. Thus, the right-handed zero mode can be localized on the brane for any positive coupling constant $\eta$, and at the same time the left-handed one cannot.

As for the massive modes, we consider the resonance states. As what we have done in the previous subsection, we solve the Schrödinger equations (38) numerically by using the two types of initial conditions (60). The mass spectrum of the resonances is shown in Tab. II. It is clear that in this table the masses of the resonant modes of the left- and right-handed gravitinos are still almost same, while their parities are opposite. The number of the resonances increases with the increases of the parameters $b$, $\alpha$, and $\eta$. These resonances are closer to each other as $m^2$ increasing, which is the same as the conclusion of the case of $\alpha = 1$.

TABLE II. The eigenvalue $m^2$ and mass $m$ of the left- and right-handed gravitinos with odd-parity and even-parity solutions for the coupling $F(\phi) = \phi^\alpha$. The parameters are set to $k = 1$, $\eta=1$, and $b = 1$. 

| $b$ | $\eta$ | $C$ | $P$ | $m^2$ | $m$ |
|-----|--------|-----|-----|-------|-----|
| 10  | even   | $L$ | 21.9171 | 4.68157 | 0.988172 |
|     | odd    | $L$ | 38.3348 | 6.19151 | 0.986425 |
|     | even   | $L$ | 47.9467 | 6.92436 | 0.423099 |
|     | odd    | $L$ | 21.9169 | 4.68155 | 0.999673 |
|     | even   | $L$ | 38.3311 | 6.19121 | 0.981257 |
|     | odd    | $L$ | 47.9328 | 6.92335 | 0.434111 |
|     | even   | $R$ | 34.2006 | 5.84813 | 0.999212 |
|     | odd    | $R$ | 63.1603 | 7.94735 | 0.999998 |
|     | even   | $R$ | 86.4700 | 9.29892 | 0.993796 |
|     | odd    | $R$ | 102.7540 | 10.13680 | 0.632409 |
|     | even   | $L$ | 35.3730 | 5.94752 | 0.982136 |
|     | odd    | $L$ | 54.7429 | 7.39884 | 0.287870 |
|     | even   | $R$ | 35.3712 | 5.94737 | 0.981886 |
|     | odd    | $R$ | 54.4129 | 7.37651 | 0.285428 |
|     | even   | $L$ | 56.8585 | 7.54646 | 0.999979 |
|     | odd    | $L$ | 98.8432 | 9.94199 | 0.999998 |
|     | even   | $R$ | 56.8515 | 7.53999 | 0.999923 |
|     | odd    | $R$ | 98.9253 | 9.94872 | 0.992303 |
|     | even   | $L$ | 122.4728 | 11.06670 | 0.285428 |
|     | odd    | $L$ | 122.6000 | 11.07250 | 0.277316 |
| 15  | even   | $L$ | 35.0401 | 5.96018 | 0.980016 |
|     | odd    | $L$ | 54.0392 | 7.38914 | 0.280495 |
|     | even   | $R$ | 54.0383 | 7.38814 | 0.280495 |
|     | odd    | $R$ | 54.0374 | 7.38714 | 0.280495 |
|     | even   | $L$ | 55.0465 | 7.54465 | 0.999998 |
|     | odd    | $L$ | 98.0812 | 9.94199 | 0.999998 |
|     | even   | $R$ | 55.0803 | 7.53999 | 0.999923 |
|     | odd    | $R$ | 98.0835 | 9.94872 | 0.992303 |
|     | even   | $L$ | 122.1872 | 11.06670 | 0.285428 |
|     | odd    | $L$ | 122.1904 | 11.07250 | 0.277316 |
C. Localization of the gravitino field on the \( f(R) \) thick branes with \( L = X_1 + X_2 - V(\phi_1, \phi_2) \)

In the previous subsection, the \( f(R) \)-branes are generated by a single canonical scalar field. In this subsection, we will analysis the localization of a bulk gravitino in the Bloch-\( f(R) \) brane model, where the Lagrangian density of the scalar fields is given by

\[
L = -\frac{1}{2} \partial^M \phi \partial_M \phi - \frac{1}{2} \partial^M \xi \partial_M \xi - V(\phi, \xi). \tag{66}
\]

The scalar fields \( \phi \) and \( \xi \) interact through the scalar potential \( V(\phi, \xi) \). In the following, we consider the solution given in Ref. 29:

\[
\phi(y) = v \tanh(2dv y), \tag{67a}
\]

\[
\xi(y) = \sqrt{\frac{\tilde{b} - 2d}{d}} \sech(2dv y), \tag{67b}
\]

\[
A(y) = \frac{v^2}{9d} \left[ (\tilde{b} - 3d) \tanh^2(2dv y) - 2\tilde{b} \ln \cosh(2dv y) \right], \tag{67c}
\]

where \( \tilde{b} > 2d > 0 \), and the scalar potential is

\[
V(\phi, \xi) = \frac{1}{2} \left[ \left( \tilde{b} \phi^2 - \tilde{b} \phi^2 - d\xi^2 \right)^2 + 4d^2 \phi^2 \xi^2 \right] - \frac{4}{3} \left( \tilde{b} \phi^2 - \frac{1}{3} \tilde{b} \phi^3 - d\phi \xi^2 \right)^2. \tag{68}
\]

For certain given values of the parameters \( v \) and \( \tilde{b} \), the function \( f(R) \) could have analytical expression. For example, when \( v = \sqrt{3/2} \) and \( \tilde{b} = 3d \), we have

\[
f(R) = R + \frac{2\gamma}{7} \left[ \sqrt{3(R - 48d^2)}(R + 120d^2) \sin \mathcal{Y}(R) + 2 \left( R + 36d^2 \right) \cos \mathcal{Y}(R) \right], \tag{69}
\]

where \( \gamma \) is a parameter and \( \mathcal{Y}(R) = \sqrt{3} \ln \left( \frac{\sqrt{R - 48d^2} + \sqrt{R + 120d^2}}{2\sqrt{42d}} \right) \).
Next, we investigate the localization of a bulk gravitino with the coupling function \( F(\phi) = \phi^p \xi^q \) with \( p = 1, 3, 5, \cdots \) and \( q \) any integer. Such coupling was also used to localize the Dirac fermion in Refs. 51, 53.

### 1. Case I: \( F(\phi) = \phi^p \xi^q \) with \( q > 0 \)

Firstly, we consider the case of \( F(\phi) = \phi^p \xi^q \) with \( q > 0 \). For convenience we let \( q = 1 \). The most simplest one is the Yukawa coupling between the two scalar fields and the gravitino, i.e., \( -\eta \phi \Psi_M [\Gamma^M, \Gamma^N] \Psi_N \). We also assume without loss of generality that the coupling constant \( \eta \) is positive. The asymptotic behaviors of the potentials in this case are similar to those in the last subsection. As \( z \) (or \( y \)) \( \rightarrow \infty \), both the potentials \( V^L \) and \( V^R \) vanish and their values are opposite at \( z = 0 \):

\[
V^L(0) = -V^R(0) = 2\eta v^3 \sqrt{(b - 2d)d},
\]

which shows that there is a potential well around \( z = 0 \) for \( V^R \). Thus, it seems that the left-handed zero mode of the gravitino can not be localized on the brane, while the right-handed zero mode can be localized. However, when substituting the solution of the right-handed zero mode

\[
\chi_0^R \propto \exp \left( -\eta \int_0^z d\tilde{z} e^{A(z)} \phi(z) \xi(z) \right)
\]

\[
= \exp \left( -\eta \int_0^y d\tilde{y} \phi(y) \xi(y) \right) = \exp \left( \frac{\eta v}{2d} \sqrt{\frac{b - 2d}{d} \text{sech}(2dv\tilde{y})} \right)
\]

\[
\propto \exp \left( \frac{\eta v}{2d} \sqrt{\frac{b - 2d}{d} \text{sech}(2dv\tilde{y})} \right)
\]

into the normalization condition (32), we find the integral

\[
\int_{-\infty}^{\infty} (\chi_0^R(z))^2 dz = \frac{\int_{-\infty}^{\infty} (\chi_0^R(y))^2 e^{-A(y)} dy}{\int_{-\infty}^{\infty} \exp \left( -A(y) - 2\eta \int_0^y \phi(y) \xi(y) dy \right) dy}
\]

\[
= C^2 \int_{-\infty}^{\infty} \cosh(2dv\tilde{y}) \frac{2\pi k}{2} \exp \left( \frac{2n v}{2d} \sqrt{\frac{b - 2d}{d} \text{sech}(2dv\tilde{y})} \right) dy
\]

is divergent, which means that the right-handed zero mode cannot be confined on the brane. Although the potential of the right-handed gravitino is a volcano-type one, there still does not exist the zero mode on the brane. In fact, for any \( q > 0 \) and \( p = 1, 3, 5, \cdots \), the right-handed zero mode will be a constant as \( y \rightarrow \infty \) since \( F(\phi) = \phi^p \xi^q \propto \text{tanh}^p(2dv\tilde{y}) \text{sech}^q(2dv\tilde{y}) \rightarrow 0 \). Obviously, this kind of zero mode cannot satisfy the normalization condition (32). Thus, for any \( q > 0 \), there exists no bounded zero mode of the gravitino on the brane (the left-handed zero mode can also not be localized). Since there is no localized zero mode on the brane, we turn to the case of \( q < 0 \).

### 2. Case II: \( F(\phi) = \phi^p \xi^q \) with \( q < 0 \) (or \( q = -1 \))

We let \( q = -1 \) to represent the case of \( q < 0 \) for convenience. The potentials in this case are displayed in Fig. 2. Both the potentials \( V^L \) and \( V^R \) have infinite wells. For the simplest case \( p = 1 \), both two potentials vanish as \( z \) (or \( y \)) \( \rightarrow \infty \) and their values are opposite at \( z = 0 \): \( V^L(0) = -V^R(0) = \frac{2dv}{\sqrt{(b - 2d)d}} \). The left-handed zero mode still cannot be localized on the brane since it is divergent as \( z \rightarrow \infty \). While the right-handed one

\[
\chi_0^R \propto \exp \left( -\eta \int_0^y d\tilde{y} \phi(y) \xi^{-1}(y) \right)
\]

\[
= \exp \left( -\eta \frac{1}{2} \left( \sqrt{(b - 2d)d} v \right)^{-1} \text{cosh}(2dv\tilde{y}) \right)
\]

\[
\propto \exp \left( -\eta \frac{1}{2} \left( \sqrt{(b - 2d)d} v \right)^{-1} \text{cosh}(2dv\tilde{y}) \right)
\]
FIG. 5. Potentials $V^L(z)$ and $V^R(z)$ for the left- and right-handed gravitino KK modes on the $f(R)$ thick branes with $F(\phi) = \phi^d \xi^{-1}$. Here, $v = d = 1$, $\tilde{b} = 3$, and the coupling constant $\eta$ is set to 1.0 (blue thin trace), 1.5 (green thick trace), and 2.0 (red dashed trace).

will vanish as $y \to \infty$ for any $\eta > 0$. It is not difficult to check that for any $\eta > 0$ this right-handed zero mode can be localized on the brane under the condition (57). And for any $q < 0$ and $p = 1, 3, 5 \ldots$ the right-handed zero mode will be localized. For other case of $p \geq 3$, both potentials vanish at $z = 0$: $V^L(0) = V^R(0) = 0$, and the left-handed potential $V^L$ is always non-negative while $V^R$ have a double-well. Therefore, only the right-handed zero mode could be localized on the brane.

There are infinite bounded massive KK modes in this case because both the effective potentials are infinite ones. Some of our results are shown in Tab III. It is obvious that the mass spectra of the left- and right-handed gravitino massive bounded KK modes are almost the same while their parities are opposite as shown in the previous section. When $p = 1$, the mass of the first bounded state of the left-handed gravitino (or the mass of the first excited state of the right-handed one) increases with the value of $\eta$ because the minimum of the left-handed potential $V^L$ increases with $\eta$. On the other hand, the relative width of the effective potentials decreases with the value of $\eta$ and increases with the value of $m^2$. Thus, the gaps between the bounded states will extend with the growth of $\eta$ and become narrower and narrower as $m^2$ increases. When $p \geq 3$, the mass of the first bounded state of the left-handed gravitino still increases with the growth of the $\eta$, even though the minimum of the left-handed potential $V^L$ is always zero. Other conclusions are the same with the case of $p = 1$.

IV. DISCUSSION AND CONCLUSION

In this manuscript, we investigated the localization and resonant modes of a five-dimensional gravitino field on the $f(R)$ thick branes, and gave the Schrödinger equations for the gravitino KK modes with the gauge condition $\Psi_z = 0$. As same as a five-dimensional free and massless Dirac fermion field, the zero mode of a free massless five-dimensional gravitino field could be localized on a brane only for a compactification extra dimension and its massive KK modes can not realize the localization. Therefore we introduced the coupling term $-\eta F(\phi) \bar{\Psi}_M [\Gamma^M, \Gamma^N] \Psi_N$ to investigate the localization of gravitino on three kinds of $f(R)$ thick branes. The relativity probability method has been applied to study the resonances of gravitino on these $f(R)$ thick branes. It has been shown that the localization and KK spectra of the five-dimensional gravitino field with the Yukawa coupling term $-\eta F(\phi) \bar{\Psi}_M [\Gamma^M, \Gamma^N] \Psi_N$ are very similar to
TABLE III. The eigenvalue $m^2_n$ and mass $m_n$ of the left- and right-handed gravitino bounded KK modes for the coupling $F(\phi) = \phi^p \xi^{-1}$. The parameters are set to $v = d = 1$ and $\tilde{b} = 3$.

...
Then the \( f(R) \) thick branes, which are generated by a single canonical background scalar field \( \phi \), were considered. We introduced the Yukawa coupling function, \( F(\phi) = \phi^\alpha \) with \( \alpha = 1, 3, 5, 7, \ldots \) to study the localization of the gravitino field in the \( f(R) \) thick brane model. There are two types of coupling functions \( F(\phi) \), i.e., \( \alpha = 1 \) and \( \alpha \geq 3 \). For the case of \( \alpha = 1 \), there could exist localized left- or right-handed zero mode on the brane as the coupling parameter \( \eta \) satisfies \( \eta > \frac{k}{3} \sqrt{\frac{2\varepsilon}{3}} \). Furthermore, for \( k = 1 \) and \( b > \frac{1}{2\sqrt{3}} \), we could obtain massive resonances of gravitino on the brane with the condition \( \eta > \frac{\varepsilon}{\alpha} \sqrt{\frac{b}{3}} \). The results indicate that the left- and right-handed gravitinos almost have the same resonant spectra, while their parities are opposite. With relation (25), the first resonance of the left-handed gravitino is odd and the one of the right-handed gravitino is even. There is only the right-handed zero mode of gravitino confined on the brane. These results are appropriate to other cases in this paper, while they are just opposite to the Dirac fermion. For a five-dimensional Dirac fermion field, only the left-handed zero mode of Dirac fermion could be localized on the \( f(R) \) thick branes and the first resonance of the left-handed Dirac fermion is odd. The difference results between the gravitino field and the Dirac fermion field come from the different sign in \( f \) in their dynamic equations, which may be a symbol to distinguish the Dirac fermion field and the gravitino field as they have the same coupling function \( F \) and parameter \( \eta \). In addition, the number of KK resonant modes for gravitino in this braneworld system increases with the increase of the coupling parameter \( \eta \) while decreases with the model parameter \( b \). For other case \( (\alpha \geq 3) \), there are no bounded zero modes for both left- and right-handed gravitinos, and the number of KK resonant modes increases with the growths of the parameters \( b, \alpha \), and the coupling parameter \( \eta \).

Finally, we focused on the Bloch-\( f(R) \) branes which are generated by two interacted real scalar fields. The coupling function \( F(\phi) = \phi^p \phi^q \) with \( p = 1, 3, 5, \ldots \) and \( q \) any integer was considered in this model. For the case of \( q > 0 \), there exist no bounded zero modes. For the case of \( q < 0 \), the right-handed zero mode could be localized on the brane for any \( \eta > 0 \), and there exist infinite bounded massive KK modes for both the left- and right-handed gravitinos because both the effective potentials are infinite potential wells. The gaps between the bounded states extend with the growth of \( \eta \) and become narrower and narrower as \( m^2 \) increases.

There are still some issues. As we showed in this paper, the spectra of the KK modes of a bulk gravitino are all the same as the one of a bulk Dirac fermion except for chiralitites. Thus all the results of localization of Dirac fermion on branes could be appropriate to gravitino by interchanging the chiralities. But for some kinds of branes, we found the localized KK modes of Dirac fermion by introducing a new kind of coupling term \[43, 56\]. It is not clear whether this kind of coupling term applies to gravitino and it will be our work in the future. In addition, we just consider Minkowski branes in this paper. Localization of gravitino on dS/AdS branes is also interesting.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China (Grant Nos. 11522541 and 11375075), and the Fundamental Research Funds for the Central Universities (Grant No. lzujbky-2016-k04).

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