Active one-way quantum computation with 2-photon 4-qubit cluster states

Giuseppe Vallone\textsuperscript{1} Enrico Pomarico\textsuperscript{1} Francesco De Martini\textsuperscript{1} and Paolo Mataloni\textsuperscript{1}\textsuperscript{*}

Dipartimento di Fisica dell’Università “La Sapienza” and Consorzio Nazionale Interuniversitario per le Scienze Fisiche della Materia, Roma, 00185 Italy

By using 2-photon 4-qubit cluster states we demonstrate deterministic one-way quantum computation in single qubit rotation algorithm. In this operation feed-forward measurements are automatically implemented by properly choosing the measurement basis of the qubits, while Pauli error corrections are realized by using two fast driven Pockels cells. We realized also a C-NOT gate for equatorial qubits and a C-Phase gate for a generic target qubit. Our results demonstrate that 2-photon cluster states can be used for rapid and efficient deterministic one-way quantum computing.

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Cluster states are the basic resource for one-way quantum computation (QC)\textsuperscript{1}. In the standard QC approach any quantum algorithm can be realized by a sequence of single qubit rotations and two qubit gates, such as C-NOT and C-Phase\textsuperscript{2}. Deterministic one-way QC is based on the initial preparation of entangled qubits in a cluster state, followed by a temporally ordered pattern of single qubit measurements and feed-forward (FF) operations depending on the outcome of the already measured qubits\textsuperscript{1}. We can consider two different types of FF operations: \textit{i)} the intermediate feed-forward measurements, i.e. the choice of the measurement basis depending on the previous measurement outcomes and \textit{ii)} the Pauli matrix feed-forward corrections on the final output state. Two qubit gates can be realized by exploiting the existing entanglement between qubits. In this way the difficulties of standard QC, related to the implementation of two qubit gates, are transferred to the preparation of the state.

One-way QC was experimentally realized by using 4-photon cluster states\textsuperscript{3} and, later, by implementing Pauli error corrections with active feed-forward\textsuperscript{4}. Very recently, a proof of principle of one-way QC (namely, the Grover’s search algorithm and a particular C-Phase gate) was given by employing 2-photons entangled in a 4-qubit cluster state, without active feed-forward\textsuperscript{3}. The possibility of encoding more qubits on the same photon determines important advantages. For instance the overall detection efficiency $\eta$, and then the detection rate, is constant. In fact, in general it scales as $\eta^N$, being $N$ the number of photons. Moreover 2-photon cluster states, realized by entangling the polarization ($\pi$) and linear momentum ($k$) degrees of freedom, outperform the earlier results obtained by 4-photon cluster states since the state fidelities can be far larger and the count rate can be higher by 3 - 4 orders of magnitude\textsuperscript{5}.\textsuperscript{6}

One may ask how feed-forward operations can be implemented in the case of 2-photon multiqubit cluster states and if the different degrees of freedom of the photons are computationally equivalent. In this Letter we demonstrate that a simple smart choice of the measurements basis can be used in this case to deterministically perform the single qubit rotation algorithm, instead of adopting intermediate active feed-forward measurements (type \textit{i)}, which are necessary in the case of QC with 4-photon cluster states\textsuperscript{5}. Moreover feed-forward corrections (type \textit{ii}) have been implemented in the same algorithm by using active fast driven electro-optics modulators. In this way, high speed deterministic one-way single qubit rotations are realized with 2-photon 4-qubit cluster states\textsuperscript{12}. In the same context, we have also verified the equivalence existing between the degrees of freedom of polarization and linear momentum by using either $k$ or $\pi$ as QC output qubit.

Besides single qubit rotations, two qubit gates, such as the C-NOT gate, are required for the realization of arbitrary quantum algorithms\textsuperscript{2}. The realization of a C-NOT gate for equatorial qubits and a universal C-Phase gate acting on arbitrary target qubits is also presented in the present Letter.

FIG. 1: (a) Top: measurement pattern for arbitrary single qubit rotations on a 4-qubit linear cluster state, carried out in three steps (I, II, III). In each measurement, indicated by a red cross, the information travels from left to right. Feed-forward measurements and corrections are respectively indicated by the red and blue arrows. Bottom: equivalent logical circuit. (b) C-NOT realization via measurement of qubits 1, 4 on the horseshoe cluster (left) and equivalent circuit (right). (c) Universal C-Phase gate realization via measurement of qubits 1 and 2 (left) and equivalent circuit (right).
were detected with fidelity
Inset: spatial mode matching on the BS. polarization, in the left (\(|\sigma\rangle\rangle\) a) is given by a 35m
corresponding degrees of freedom. Hadamard gates
acting as phase shifters and inserted before the BS. Polarization qubits \(\pi_A\) and \(\pi_B\) are measured by standard
tomographic setup, indicated by \(D_s\). BS and \(D_o\) outputs
are indicated by \(s_j = 0, 1\), where the index \(j\) refers to the
 corresponding measurement basis. This also implies
that the device can be inserted or not in the setup depending on the measurement. Inset: spatial mode matching on the BS.

In our experiment 2-photon 4-qubit cluster states were
generated by using the methods described in [8,7,5,3], to which we refer for details. Two photons belonging to
the cluster state [13]

\[
|C_4\rangle = \frac{1}{2}(|H\ell\rangle_A|Hr\rangle_B - |Hr\rangle_A|H\ell\rangle_B + |V\ell\rangle_A|Vr\rangle_B + |Vr\rangle_A|V\ell\rangle_B) \tag{1}
\]

are generated either with horizontal (\(H\)) or vertical (\(V\)) polarization, in the left (\(\ell\)) or right (\(r\)) mode of the Alice (\(A\)) or Bob (\(B\)) side (see Fig. 1 of [9]). Cluster states
were detected with fidelity \(F = 0.880 \pm 0.013\), as obtained from the measurement of the stabilizer operators of \(|C_4\rangle\)
[10].

By the correspondence \(|H\rangle \leftrightarrow |0\rangle, |V\rangle \leftrightarrow |1\rangle, |\ell\rangle \leftrightarrow |0\rangle, |r\rangle \leftrightarrow |1\rangle\), the state \(|C_4\rangle\) is equivalent to the cluster state \(|\Phi^{\text{in}}_4\rangle = \frac{1}{\sqrt{2}}(|+\rangle_{1}|0\rangle_{2}|0\rangle_{3}|+\rangle_{4} + |+\rangle_{1}|0\rangle_{2}|2\rangle_{3}|3\rangle_{4} + |\rangle_{1}|2\rangle_{2}|1\rangle_{3}|3\rangle_{4} - |\rangle_{1}|2\rangle_{2}|1\rangle_{3}|\rangle_{4}\rangle\) (with \(|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle|)| up to single qubit unitaries:

\[
|C_4\rangle = U_1 \otimes U_2 \otimes U_3 \otimes U_4|\Phi^{\text{in}}_4\rangle \equiv U|\Phi^{\text{in}}_4\rangle \tag{2}
\]

Here \(|\Phi^{\text{in}}_4\rangle\) and \(|C_4\rangle\) are respectively expressed in the
so called “computational” and “laboratory” basis, while the \(U_j\)’s (\(j = 1, \cdots, 4\)) are products of Hadamard gates
\(H = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)\) and Pauli matrices \(\sigma_z\). Their explicit expressions depend on the ordering of the four physical qubits, namely \(k_B, k_B, \pi_A, \pi_B\). In this work we use four

different ordering:

\[a) (1, 2, 3, 4) = (k_B, k_A, \pi_A, \pi_B), \quad U = \sigma_2 H \otimes \sigma_2 \otimes 1 \otimes H \]

\[b) (1, 2, 3, 4) = (\pi_B, k_A, k_B, \pi_B), \quad U = H \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \]

\[c) (1, 2, 3, 4) = (k_A, k_B, \pi_B, k_A), \quad U = \sigma_2 H \otimes \sigma_2 \otimes 1 \otimes H \]

\[d) (1, 2, 3, 4) = (\pi_A, \pi_B, k_B, k_A), \quad U = H \otimes 1 \otimes \sigma_2 \otimes \sigma_2 H. \]

In the following we refer to these expressions depending on the logical operation we consider.

The general measurement apparatus, differently used
to perform each operation, is sketched in fig. [\(2\)] (see
caption for details). The K modes corresponding to photons
\(A\) or \(B\), are respectively matched on the up and down side of a common 50:50 beam splitter (BS) (see inset).

**Single qubit rotations.** In the one-way model a
three-qubit linear cluster state is sufficient to realize
arbitrary single qubit rotations [13]. According to the
measurement basis for a generic qubit \(|j\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm e^{-i\varphi}|1\rangle\), we define \(s_j = 0, 1\) when the \(|\varphi^+\rangle_j \) outcome is obtained. With the 4-qubit cluster in
the computational basis \(|\Phi^0_4\rangle\) any arbitrary single qubit rotation expressed as
\(|\chi_{\text{out}}\rangle = R_x(\beta)R_y(\alpha)|\chi_{\text{in}}\rangle\)
[13] can be obtained by a three step procedure (cfr. [13, a]). The sequence of the measurement bases for the three qubits are the following: I) \(|\{0\rangle, |1\rangle\rangle\), which allows to obtain a three-qubit linear cluster; II) \(|\alpha_{\pm}; 2\rangle\); III) \(|\pm\rangle\rangle\) or \(|\pm\rangle\rangle\rangle\), depending on the second qubit output
\(s_2 = 0, 1\) respectively. In the previous expression
\(R_x(\alpha) = \exp \left(-\frac{i}{2}\alpha \sigma_x\right)\), \(R_y(\beta) = \exp \left(-\frac{i}{2}\beta \sigma_y\right)\) and
\(|\chi_{\text{out}}\rangle = \langle|\varphi^+\rangle_j |\rangle\rangle\rangle\) if the output of the first measurement is
\(|0\rangle \rangle\rangle\). The output state \(|\chi_{\text{out}}\rangle\) is created up to Pauli errors \((\sigma_x^2 \sigma_y^2)\), that should be corrected by proper FF
operations for deterministic QC [3].

Let’s consider ordering a), where qubit 1=\(k_B\), qubit
2=\(k_A\), qubit 3=\(\pi_A\) and qubit 4=\(\pi_B\). The output state, encoded in the polarization of photon B, can be written in the laboratory basis as

\[
|\chi_{\text{out}}\rangle_{\pi_B} = \sigma_2^x \sigma_2^z \sigma_2^x \sigma_2^y |HR_x(\beta)R_y(\alpha)|\chi_{\text{in}}\rangle, \tag{3}
\]

where the \(H\) gate derives from the change between the
computational and laboratory basis. This also implies that the actual measurement bases are \(|\pm\rangle_{k_B}\) for the
momentum of photon B (qubit 1) and \(|\alpha_{\pm}; k_B\rangle\) for the
momentum of photon A (qubit 2). The measurement basis
on the third qubit \((\pi_A)\) depends, according to the one-
way model, on the results of the measurement on the second qubit \((k_A)\). These are precisely what we call FF
measurements (type i), corresponding in this scheme to
measure \(\pi_A\) in the bases \(|\beta_{\pm}; \pi_A\rangle\) or \(|-\beta_{\pm}; \pi_A\rangle\), depending
on the BS output mode (i.e. \(s_{k_A} = 0\) or \(s_{k_A} = 1\)). Note that these deterministic FF measurements directly derive from the possibility of encoding two qubits \((k_A, \pi_A)\) in the same photon. Differently from the case of 4-photon cluster states, this avoids the need of Pockels cells to perform active FF measurements, as already said.
Pauli errors FF corrections are in any case necessary for deterministic one way QC. They were realized by using the measurement apparatus shown in fig. 2. Here two fast driven transverse LiNbO\(_3\) Pockels cells (\(\sigma_x\) and \(\sigma_z\)) with risetime \(= 1\) nsec and \(V_\frac{\lambda}{2} \sim 1\) KV are activated by the output signals of detectors \(a_i\) \((i = 1, 3, 4)\) corresponding to the different values of \(s_{\pi A}\) and \(s_{kA}\). They perform the operation \(\sigma_x^{s_{\pi A}} \sigma_x^{s_{kA}}\) on photon B, coming from the output \(s_{kB} = 0\) of BS and transmitted through a single mode optical fiber. Note that no correction is needed when photon A is detected on the output \(a_2\) \((s_{\pi A} = s_{kA} = 0)\). Temporal synchronization between the activation of the high field signal and the transmission of photon B through the Pockels cells is guaranteed by suitable choice of the delays \(D\). We used only one BS output of photon B, namely \(s_{kB} = 0\), in order to perform the algorithm with initial state \(|\chi_{in}\rangle = |+\rangle\). The other BS output corresponds to the algorithm starting with the initial state \(|\chi_{in}\rangle = |\rangle\).

In fig. 3 the values of the output state fidelity obtained with/without active FF corrections (i.e. turning on/off the Pockels cells) are compared for different values of \(\alpha\) and \(\beta\). The expected theoretical fidelities in the no-FF case are also shown. In all the cases the computational errors are corrected by the FF action, with average measured fidelity \(F = 0.867 \pm 0.018\). The overall repetition rate is about \(500\) Hz, which is more than 2 orders of magnitude larger than one-way single qubit-rotation realized with 4-photon cluster states.

We demonstrated the experimental equivalence of the two degrees of freedom of photon B by performing the same algorithm with ordering \(b\). In this case the explicit expression of the output state \(|\chi_{out}\rangle_{kB}\) in the laboratory basis is \(|\chi_{out}\rangle_{kB} = (\sigma_z^{s_1} |\sigma_x^{s_2} \sigma_z^{s_3} H R_\alpha(\beta) R_\beta(\alpha) |\chi_{in}\rangle\). By using only detectors \(a_2, a_3, b_1, b_2\) in fig. 2 we measured \(|\chi_{out}\rangle_{kB}\) choosing different values of \(\alpha\) (which correspond in the laboratory to the polarization measurement bases \(|\alpha_2\rangle_{\pi A}\) and \(\beta = 0\) (which correspond in the laboratory to the momentum bases \(|\beta_3\rangle_{kA}\)). The first qubit \((s_B)\) was always measured in the basis \(|\pm\rangle_{kB}\). By performing the \(k_B\) tomographic analysis for all the possible values of \(s_2 = s_{\pi A}\) and \(s_3 = s_{kA}\) of the input qubit (i.e. for different values of \(s_1 = s_{kB}\)) we obtained an average value of fidelity \(F > 0.9\) (see table II). In this case the realization of FF corrections could be realized by the adoption of active phase modulators. The \((\pi)-(k)\) computational equivalence and the use of active feed-forward show that the multidegree of freedom approach is feasible for deterministic one-way QC.

**C-NOT gate for equatorial qubits.** Nontrivial two-qubit operations, such as the C-NOT gate, can be realized by the four-qubit horseshoe \((180^\circ\) rotated) cluster state (see fig. II)), whose explicit expression is equal to \(|\Phi_{\text{4}}^{\text{in}}\rangle\). By simultaneously measuring qubits 1 and 4, it’s possible to implement the logical circuit shown in fig. IIb. In the computational basis the input state is \(|+\rangle_c \otimes |+\rangle_t\) \((c=\text{control}, t=\text{target})\), while the output state, encoded in qubits 2 (control) and 3 (target), is \(|\Psi_{\text{out}}\rangle = H_4 C-\text{NOT}(O) |+\rangle_c \otimes R_\pi(\alpha) |+\rangle_t\) \((s_1 = s_4 = 0)\). In the above expression we have \(O = 1\) \((O = H)\) when qubit 1 is measured in the basis \([|0\rangle_1, |1\rangle_1]\) \((|\pm\rangle_1)\). Qubit 4 is measured in the basis \(|\alpha_2\rangle_4\). It is worth noting that this circuit realizes the C-NOT gate (up to the Hadamard \(H_t\)) for arbitrary equatorial target qubit and control qubit \(|0\rangle, |1\rangle\) or \(|\pm\rangle\) depending on the measurement basis of qubit 1.

The experimental realization of this gate was performed by adopting ordering \(c\). In this case the control output qubit is encoded in the momentum \(k_B\), while the target output is encoded in the polarization \(\pi_B\). In the actual experiment we inserted \(H_1\) and \(H_t\) on photon B to compensate \(H_t\). The output state in the laboratory basis is then

\[
|\Psi_{\text{out}}\rangle = (\Sigma)^{s_4} \sigma_z^{(c)} C-\text{NOT}(O \sigma_z^{s_4} |+\rangle_c \otimes R_\pi(\alpha) |+\rangle_t),
\]
where all the possible measurement outcomes of qubits 1 and 4 are considered. The Pauli errors are \( \Sigma = \sigma_x^{(c)} \sigma_z^{(t)} \), while the matrix \( \sigma_x^{(c)} \) is due to the change between the computational and laboratory bases. Table II shows the experimental fidelities of the target qubit corresponding to the measurement of the output control qubit in the basis \( \{|0\rangle, |1\rangle\} \).

**Universal C-Phase gate.** We realized a C-Phase gate for arbitrary target qubit and fixed control \(|+\rangle^c\) (see Fig. I(c)) by measuring qubits 1 and 2 of \( |\Phi_{44}^{lin}\rangle \) in the bases \(|\alpha_k\rangle\) and \(|\pm\rangle\) respectively. By considering ordering d) we encoded the output state in the physical qubits \( k_A \) and \( k_B \). For \( s_1 = s_2 = 0 \), by using the appropriate base changing, the output state is written as

\[ \Psi_{\text{out}} = |\pm\rangle_{k_A} \otimes \sigma_x |\Phi\rangle_{k_B} + |+\rangle_{k_A} \otimes \sigma_x \sigma_z |\Phi\rangle_{k_B}. \tag{5} \]

Here \( |\Phi\rangle_{k_B} = R_x(\beta) R_z(\alpha) |+\rangle \) and the matrix \( \sigma_x \) is due to the basis changing. We measured the fidelity of target \( k_B \) corresponding to a control \(|+\rangle_{k_A} \) (\( |\pm\rangle_{k_A} \)) for different values of \( \alpha \) and \( \beta \), obtaining an average value \( F = 0.907 \pm 0.010 \) (\( F = 0.908 \pm 0.011 \)).

In this Letter 2-photon 4-qubit cluster states, realizing the full entanglement of two photons by two degrees of freedom (in our case polarization and linear momentum), have been used to perform highly efficient arbitrary single qubit rotations, either probabilistic or deterministic, and fundamental two qubit gates, such as a C-NOT gate for target qubits located in the equatorial plane of the Bloch sphere and a C-Phase gate for generic target qubits. These operations have been performed with high values of fidelity and at average repetition rates which are 2 or even 3 orders of magnitude larger than those obtained with 4-photon cluster states.

The power of computation is strongly related to the possibility of increasing the information content associated to a quantum state. For instance, 6 qubits are necessary to implement a C-NOT gate operating over the entire Bloch sphere of the target and control qubits. This could be obtained by using other degrees of freedom of the photon, such as time-energy. QC based on more complex gates and algorithms requires to work with even more qubits [11]. In our scheme this could be realized by exploiting the SPDC conical emission of a type I crystal and using a larger number of \( k \) modes. Even if this number scales exponentially with the number of qubits, up to eight qubits could be created with two photons by using only four modes per photon, besides polarization and time-bin. At the same time, complex QC operations can not be realized without the availability of a larger number of photon pairs. This number should grow contextually with the number of available degrees of freedom of the photon. For instance eight-qubit four-photon cluster states could be generated by linking together two \( |C_4\rangle \) states by a proper CP gate. This leads to conceive a hybrid approach to one-way QC based on a multiphoton multiqubit architecture, which is at the moment under investigation.

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**Table II: Experimental fidelity (F) of C-NOT gate output target qubit for different value of \( \alpha \) and \( \Omega \).**

| \( \Omega \) | \( \alpha \) | Control output | \( F(s_4 = 0) \) | \( F(s_4 = 1) \) |
|---|---|---|---|---|
| \( \pi/2 \) | \( s_1 = 0 \rightarrow |1\rangle_{c} \) | 0.965 \( \pm \) 0.004 | 0.975 \( \pm \) 0.004 |
| \( s_1 = 1 \rightarrow |0\rangle_{c} \) | 0.972 \( \pm \) 0.004 | 0.973 \( \pm \) 0.004 |
| \( \pi/4 \) | \( s_1 = 0 \rightarrow |1\rangle_{c} \) | 0.995 \( \pm \) 0.008 | 0.902 \( \pm \) 0.012 |
| \( s_1 = 1 \rightarrow |0\rangle_{c} \) | 0.946 \( \pm \) 0.010 | 0.945 \( \pm \) 0.009 |

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* URL: [http://quantumoptics.phys.uniroma1.it/](http://quantumoptics.phys.uniroma1.it/)

† Present address: Univ. de Genève GAP-Optique, Rue de l’École-de-Medicine 20, CH-1211 Genève 4, Suisse

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[12] Here “deterministic” refers to the success of the one-way algorithm and not to the overall process because of the random generation of SPDC pairs.
[13] The state \( |\Phi\rangle \) is equivalent to that generated in [3] up to single qubit transformations.
[14] The state \( |\Phi_{44}^{lin}\rangle \) is obtained by preparing a chain of qubits all prepared in the state \( |+\rangle \) and then applying the gate \( CP = |0\rangle \langle 0| \otimes |1\rangle \langle 1| \otimes \sigma_x \) for each link.
[15] Three sequential rotations are necessary to implement a generic \( SU(2) \) matrix but only two, namely \( R_x(\beta) R_z(\alpha) \), are sufficient to rotate the input state \( |\chi_{in}\rangle = |\pm\rangle \) into a generic state.