Suppression of $T_c$ in superconducting amorphous wires

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The suppression of the mean field temperature of the superconducting transition, $T_c$, in homogeneous amorphous wires is studied. We develop a theory that gives $T_c$ in situations when the dynamically enhanced Coulomb repulsion competes with the contact attraction. The theory accurately describes recent experiments on $T_c$–suppression in superconducting wires, after a procedure that minimizes the role of nonuniversal mechanisms influencing $T_c$ is applied.

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Disorder suppresses the superconductivity transition in morphologically homogeneous superconductors because the diffusive character of the electron motion in dirty systems makes the Coulomb interaction more effective. As a result, the attraction between the electrons in Cooper pairs becomes weaker, and the transition temperature, $T_c$, is lowered. In two dimensions (2D) the influence of disorder on $T_c$ can be studied systematically by varying the film thickness $d$. In uniform films $T_c$, being well defined, is suppressed as the sheet resistance, $R$, increases with decreasing $d$. (For a review see Ref. [3] of the sample is such that its dimension is lowered towards the one-dimensional (1D) limit, the suppression of superconductivity should become more pronounced.

Recently, efforts have been made to extend the experiments in films to narrow wires by fabricating a series of amorphous Pb wires of different thicknesses and widths. It has been found that the $T_c$-suppression becomes stronger as the wires’ width reduces below 1000Å. The experiment of Refs. [2,4] is in the crossover region from 2D to 1D. Actually, the wires are in the 1D limit as far as superconducting fluctuations are concerned [4], but they are in the crossover region from 2D to 1D with respect to the diffusive motion of the electrons.

From the theoretical point of view, the problem of $T_c$-suppression in 1D wires is rather intriguing. As is well known, the superconductivity transition is determined by a series of logarithmically divergent terms describing the electron scattering in the two-particle Cooper channel. In 2D systems the corrections due to the electron-electron (e–e) interactions combined with disorder are logarithmically divergent as well. As the whole problem is controlled by logarithmic singularities, it can be studied by renormalization group (RG) methods. In 1D, due to the reduced dimensionality, the effect of e–e interactions is more singular. It produces corrections that diverge as the square root of the frequency. The presence of two types of singularities demands a special analysis in the calculation of $T_c$. In this paper we develop a theory that describes adequately the effect of the dynamically enhanced e–e interaction on $T_c$ in the crossover region from 2D to 1D and perform a detailed comparison with the experiment.

The mean field temperature, $T_c$, is defined as the temperature at which the electron scattering amplitude in the Cooper channel, $\Gamma_c$, becomes infinite. Fluctuations of the superconductivity order parameter lead to a broadening of the phase transition. However, its mean field temperature can be found experimentally by fitting the upper part of the resistive transition to the Aslamazov–Larkin theory [4]. The diagrammatic representation of the amplitude $\Gamma_c$ is shown in Fig. [4]. In addition to the contact BCS-interaction amplitude $\gamma$, the terms arising as a result of the interplay of the Coulomb interaction and disorder are also included in the Cooper ladder-diagram series. The impurity scattering does not influence the e–e interaction mediated by phonons because in the long wavelength limit the lattice defects oscillate together with the ions. The resulting equation for $\Gamma_c$ is:

$$\Gamma_c(\epsilon_n, \epsilon_i) = -|\gamma| + t\Lambda(\epsilon_n + \epsilon_i)$$

$$-2\pi T \sum_{m=0}^{M} \left[ -|\gamma| + t\Lambda(\epsilon_n + \epsilon_m) \right] \times \frac{1}{\epsilon_m} \Gamma_c(\epsilon_m, \epsilon_i),$$

where $\epsilon_m = 2\pi T(m + 1/2)$ is the Matsubara frequency, and the summation over $m$ is limited by $M = (2\pi T T_c)^{-1}$. In this equation $\gamma$, the bare value of the amplitude $\Gamma_c$, is rescaled in such a way that the Debye frequency as a cut off energy is substituted by $\tau^{-1}$, the inverse of the scattering time. Then, $\gamma = 1/\ln(T_{c0}\tau/1.14)$, where $T_{c0}$ is the temperature of the superconducting transition in the bulk limit. The parameter $t = (e^2/2\pi^2\hbar)R_0$ characterizes the level of disorder in a sample, where $R_0$ is the sheet resistance. The amplitude $\Lambda$ describing the combined action of the e–e interaction and disorder is given by

$$\Lambda(\omega_n) = \frac{4\pi D}{La} \sum_{q_L,q} \frac{1}{D_{q_L}^2 + D_{q'_L}^2 + \omega_n},$$

where $a$ and $L$ are the width and the length of the wire, respectively. The parameter $u$ describes the amplitude of...
the e–e interaction when the momentum $q$ transferred by
this interaction is not too small compared with the trans-
tferred frequency $\omega$, namely when $q \gtrsim q_\omega = \sqrt{\omega/D}$. (As
was explained in Refs. \cite{3,5}, the most divergent contribu-
tions from the region $q < q_\omega$ cancel each other out. In
this region of small momenta, the e–e interaction depends
only on the frequency, and therefore it can be gauged out.) Next, for amorphous Pb films the spin-orbit scat-
tering is expected to be only a few times longer than the elastic scattering time and therefore the part of the e–e interaction related to spin density fluctuations
can be neglected. In that case, we may take $u$ to be the value of the screened Coulomb interaction amplitude in
the region of momenta $q \gtrsim q_\omega$, which gives $u \approx 1/2$.

\[
\begin{align*}
\Gamma &= \gamma + t\Lambda \\
\gamma + \Gamma &= \Gamma + t\Lambda \\
t\Lambda &= \gamma + \Gamma \\
+ \Gamma &= \gamma + t\Lambda + t\Lambda
\end{align*}
\]

FIG. 2 The diagrammatic equation for the scattering ampli-
tude $\Gamma_\epsilon$ in the Cooper channel. The block $\gamma$ denotes the
BCS-interaction amplitude. The block $t\Lambda$ describes the inter-
play of the Coulomb interaction with disorder that leads to
the suppression of $T_c$. The wavy line is the screened Coulomb
interaction, dashed lines describe impurity scattering.

In 2D the summation in Eq. (2) yields $\Lambda(\omega_n) \approx
u \ln(1/\omega_n)$. Therefore, Eq. (3) combines the usual BCS
logarithms together with the ones arising due to disor-
der. Unlike the ladder diagrams in the BCS-theory, the integra-
tions in the different blocks of the diagrams in Fig. 2 cannot be factorized, because $\Lambda(\epsilon_n + \epsilon_m)$ matches
the frequency arguments of two neighboring blocks. In
order to solve this parquet-like equation with a logarith-
mic accuracy one uses the approximation
\[
\ln(z + z') \approx \ln(\max\{z, z'\}),
\]
see e.g. Ref. \cite{7}. Then, it is possible to apply the “maximum section” method. This procedure leads to the RG equation for the amplitude $\Gamma_\epsilon(z, \tau)$:
\[
\frac{d\Gamma_\epsilon}{dz} = ut - \Gamma_\epsilon^2,
\]
where $l_c = \ln(1/\epsilon\tau)$. The integration of the RG equation gives the suppression of $T_c$ by the
Coulomb interaction in 2D disordered systems:
\[
\ln\left(\frac{T_c}{T_{c0}}\right) = \frac{1}{|\gamma|} - \frac{1}{2ut} \ln \left\{ \frac{1 + \sqrt{ut}/|\gamma|}{1 - \sqrt{ut}/|\gamma|} \right\}. \tag{3}
\]
This formula accurately describes the experimental results
in MoGe films \cite{6}, with $u = 1/2$ and using only one fitting parameter, $\gamma$.

In 1D the result of the summation in Eq. (2) yields a
square root singularity in the amplitude $\Lambda(\omega_n)$. When
one deals with singularities stronger than logarithmic
ones, the approximations of the maximum section method cease to be valid, and a different method should
be invented. In this Letter we treat the problem of find-
ing $T_c$ from Eq. (2) as a sort of an eigenvalue problem,
which leads to an implicit equation for $T_c$. To see this,
we will consider $\Gamma_c(\epsilon_n, \epsilon_m)$ as the matrix elements of a
matrix $\Gamma_c$, and will write the solution of Eq. (2) for $\Gamma_c$ in
matrix notations:
\[
\hat{\Gamma}_c = \hat{\epsilon} \left( \hat{I} - |\hat{\Pi}|^{-1} \right)^{-\frac{1}{2}} \left( -|\hat{\Pi}| \hat{I} + t\hat{\Lambda} \right). \tag{4}
\]

Here $\hat{\Pi}(T) = \hat{\epsilon}^{-1/2}[1 - |\hat{\Pi}|]^{-1/2}$, $\epsilon_{nm} = \delta_{nm}(n + 1/2)$, $\Lambda_{nm} = \Lambda(\epsilon_n + \epsilon_m)$, $I_{nm} = 1$, and $\hat{I}$ is a unit matrix.
Eq. (4) is written in such a form that $\hat{\Pi}$ is a symmetric
matrix. Notice, that the dependence of $\hat{\Pi}$ on the tem-
perature $T$ is not only through the dependence of $\Lambda$ on
the Matsubara frequencies, but also through the matrix
rank $M = (2\pi T\tau)^{-1}$. The amplitude $\Gamma_c$ diverges when
the temperature is such that one of the eigenvalues of the
matrix $\Pi(T)$ is equal to $|\gamma|^{-1}$, i.e., at $T = T_c$. the
equation
\[
[|\gamma|^{-1}\hat{I} - \hat{\Pi}(T_c)]|\Psi\rangle = 0 \tag{5}
\]
holds. Thus, the equation determining $T_c$ can be ob-
tained from an eigenvalue problem. One can also obtain
an equation for $T_c$ by considering a BSC-like gap equation
with frequency dependent interaction vertex, $-|\gamma| + t\Lambda$.

The matrix elements of $\hat{\Pi} = \hat{\Pi}_0 + \hat{\Pi}_1$ are
\[
\hat{\Pi}_0^0 = [(n + 1/2)(m + 1/2)]^{-1/2}, \quad \hat{\Pi}_1^0 = -t[(n + 1/2)(m + 1/2)]^{-1/2} |\gamma|^{-1} \Lambda(\epsilon_n + \epsilon_m). \tag{6}
\]

As the matrix elements $\hat{\Pi}_0^0$ are factorized with respect to $n$ and $m$, all the eigenvalues of the matrix $\hat{\Pi}_0$, except
one, are degenerate and equal to zero. The eigen-
vector corresponding to the nonzero eigenvalue is $\Psi_0^0 =
\sqrt{c/\sqrt{n + 1/2}}$, and the equation $|\gamma|^{-1}\Psi_0^0 = \sum_m \hat{\Pi}_0^0 \Psi_m^0$
leads to the BCS relation for $T_{c0}$:
\[
|\gamma|^{-1} = l_0(T_{c0}), \quad l_0(T) = \sum_{m=0}^{M} \frac{1}{m + 1/2} = \ln \frac{1.14}{T_{c0}} \tag{7}
\]

Our strategy now will be to calculate the corrections to
this eigenvalue perturbatively in $\hat{\Pi}_1$ (notice that $\hat{\Pi}_1 \propto t$),
and in this way to get an implicit equation for $T_c$. Since
$\hat{\Pi}$ is symmetric we can perform this program using a
standard perturbation theory:
\[
|\gamma|^{-1} = l_0(T) + l_1(T) + l_2(T) + \ldots \tag{8}
\]

The first order term can be obtained straightforwardly
\[
l_1 = \langle \Psi_0^0 | \hat{\Pi}_1 | \Psi_0^0 \rangle = -\frac{t}{l_0|\gamma|} \Sigma_2(T),
\]

\[
\Sigma_2(T) = \sum_{n,m=0}^{M} \frac{\Lambda(\epsilon_n + \epsilon_m)}{(n + 1/2)(m + 1/2)}. \tag{9}
\]
The prefactor \(1/l_0\) appears in \(l_1\) because the normalization factor \(c\) of the eigenvector \(\Psi^0\) is equal to \(1/\sqrt{l_0}\). Since all the eigenvalues of the operator \(\Pi^0\) are degenerate except the one under studying, it is also possible to find the higher order corrections using only the eigenvector \(|\Psi^0\rangle\), without involving other eigenvectors. We demonstrate it here for the second order term, but a generalization to higher orders is straightforward. In the second order

\[
l_2 = \sum_{i \neq 0} \frac{\langle \Psi^0 | \hat{\Pi}^1 | \Psi^1 \rangle \langle \Psi^1 | \hat{\Pi}^1 | \Psi^0 \rangle}{l_0}.
\]

\[
= \frac{1}{l_0} \left[ \langle \Psi^0 | \hat{\Pi}^1 \hat{\Pi}^1 | \Psi^0 \rangle - (l_1)^2 \right],
\]

where

\[
\langle \Psi^0 | \hat{\Pi}^1 \hat{\Pi}^1 | \Psi^0 \rangle = \frac{t^2}{l_0 |\gamma|^2} \Sigma_3(T),
\]

\[
\Sigma_3(T) = \sum_{n,m,k=0}^M \frac{\Lambda (\epsilon_n + \epsilon_k) \Lambda (\epsilon_k + \epsilon_m)}{(n + 1/2)(m + 1/2)(k + 1/2)}.
\]  

Inverting Eq. (3) perturbatively in \(t\) and having in mind that \(|\gamma|l_0(T_{c0}) = 1\), we find

\[
\ln \frac{T_c}{T_{c0}} = -t \Sigma_2(T_{c0})
\]

\[
+ t^2 \left( \Sigma_3(T_{c0}) + T_{c0} \frac{\partial \Sigma_2(T)}{\partial T} \bigg|_{T=T_{c0}} \Sigma_2(T_{c0}) \right) + \ldots
\]

(12)

Since Eq. (12) gives an approximation for \(\ln(T_c/T_{c0})\), while the measured quantity in experiments is \(T_c/T_{c0}\), the first two terms of the perturbative series are sufficient for the description of the \(T_c\) suppression, if the parameter \(t\) is not too close to a critical value where \(T_c\) vanishes. [The parameter \(t\) should be inside the radius of convergence of the series \(\Sigma_2\).] Outside this radius the superconductivity is completely suppressed.] In the 2D case Eq. (12) reproduces the first two terms of the expansion of the right hand side of Eq. (3) in powers of \(ut/\gamma^2\):

\[
\ln \left( \frac{T_c}{T_{c0}} \right) = \sum_{n=1}^{\infty} \frac{1}{(2n+1)\gamma} \left( \frac{ut}{\gamma^2} \right)^n.
\]  

(13)

We note that expansion (13) does not contain a term \(\propto t^2/\gamma^4\). There are several diagrams that gives contributions to that order, however, finally they cancel each other. The main advantage of Eq. (13) is that it is not restricted to a logarithmic accuracy, and can be applied to the description of the crossover from 2D to 1D systems.

In the experiment of Xiong et al. [13] the mean field temperature of the superconducting transition, \(T_c\), has been measured systematically for uniform Pb wires of various widths. The effective strength of the disorder characterized by \(R_\square\) has been controlled by the wire thickness \(d\). Before going to a detailed comparison of the theory with the experiment a few remarks are in order. The theory described above deals with the universal mechanism related to large scale distances that are of the order of the thermal length \(L_T \propto \sqrt{D/\gamma}\). However, a number of other effects may also influence \(T_c\) when the thickness \(d\) is decreased. For example, the electron states quantization and the interaction of the electrons with the film’s substrate can alter the parameters of the electron liquid. These nonuniversal effects of a short range origin are not addressed by the present theory. In some systems, e.g., MoGe (see Ref. [14]), the discussed effect, originated from the interplay of the Coulomb interaction and disorder, is dominant, and the theoretical curve matches the experimental data at 2D. Unfortunately, as it is shown in Fig. 2, the theoretical curve for Pb films does not follow the experiment. This fact indicates that the effects of a short range physics are not negligible here.
width is somewhat ambiguous. For the discussed data, the extrapolation of $T_c$ to the limit $R_C \to 0$ at a fixed width yields values that are not equal to the transition temperature in the bulk limit. (Moreover, the extrapolated values behave in an irregular way as a function of the wire width.) Under this circumstance, we have normalized the theoretical curves in such a way that in the limit $R_C \to 0$ the fitting curves for each width, $a$, start from the extrapolated $T_{c0}(a) = T_c(R_C \to 0)$. After this normalization procedure and rescaling the theoretical curves by $x(R_C)$, the data for wires of different widths has been plotted together with the theoretical curves in Fig. 3. The fitting parameter $\gamma = -0.16$, determined from the initial slope of the $T_c(R_C)$ in 2D films, was the same for all wire widths. Notice that at $R_C \gtrsim 2000 \Omega$ the suppression of $T_c$ for the wire of the smallest width is about 1.5 times stronger than for the widest one. The agreement between theory, i.e. Eq. (13), and experimental data for all wires of different widths turned out to be very good.

![Graph](image)

**FIG. 3** Comparison between the theory (solid line) and the experimental data [13] for wires of different width. The short range effects are excluded assuming that they are the same as in 2D. For each wire width, $a$, the theoretical curves and the experimental data points are normalized by the extrapolated value $T_{c0}(a) = T_c(R_C \to 0)$. For all width we use $\gamma = -0.16$ as in 2D.

To summarize, we have developed a theory that describes the suppression of the mean-field temperature of the superconducting transition in amorphous systems. The theory is based on the consideration of the suppression of the contact attraction due to phonons, by the dynamically enhanced Coulomb repulsion. It is suitable for the description of the crossover region between $2D$ and 1D. By treating the problem as an eigenvalue problem, we overcame the difficulties occurring because of the coexistence of different singularities in the equation determining $T_c$. In order to compare the available experimental results with the theory, we analyzed the data in a way that minimizes the role of nonuniversal effects of a short range origin. We believe that the theory could be tested further with superconducting wires fabricated from other materials, where the initial slope of $T_c(R_C)$ is larger than in Pb films.

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