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A QUANTUM KIRWAN MAP, II: BUBBLING

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Consider a Hamiltonian action of a compact connected Lie group $G$ on an aspherical symplectic manifold $(M, \omega)$. Under suitable assumptions, counting gauge equivalence classes of (symplectic) vortices on the plane $\mathbb{R}^2$ conjecturally gives rise to a quantum deformation $Q_{\kappa_G}$ of the Kirwan map.

This is the second of a series of articles, whose goal is to define $Q_{\kappa_G}$ rigorously. The main result is that every sequence of vortices with uniformly bounded energies has a subsequence that converges to a genus 0 stable map of vortices on $\mathbb{R}^2$ and holomorphic spheres in the symplectic quotient.

Potentially, the map $Q_{\kappa_G}$ can be used to compute the quantum cohomology of many symplectic quotients. Conjecturally it also gives rise to quantum generalizations of non-abelian localization and abelianization.

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1. Main result

Let $(M, \omega)$ be a symplectic manifold and $G$ a compact connected Lie group with Lie algebra $\mathfrak{g}$. We fix a Hamiltonian action of $G$ on $M$ and an (equivariant) moment map $\mu : M \to \mathfrak{g}^*$. Throughout this article, we make the following standing assumption:

**Hypothesis (H):** $G$ acts freely on $\mu^{-1}(0)$ and the moment map $\mu$ is proper.

Then the symplectic quotient $\overline{M} := \mu^{-1}(0)/G$ is well-defined, smooth and closed (i.e., compact and without boundary). Based on ideas by D. A. Salamon, in [Zi3], I conjectured that under suitable assumptions there exists an algebra homomorphism $Q_{\kappa_G}$ from the equivariant cohomology of $M$, tensored with the equivariant Novikov ring, to the quantum cohomology of $\overline{M}$.

The idea of proof of the conjecture is to define $Q_{\kappa_G}$ by counting symplectic vortices over $\mathbb{R}^2$. Once established, this should allow to compute the quantum cohomology of many symplectic quotients (e.g. those arising from suitable linear torus actions on a symplectic vector space). Based on the map $Q_{\kappa_G}$, one can formulate quantum versions of non-abelian localization and abelianization, see [WZ].

The present article is the second of a series of papers, whose goal is to define $Q_{\kappa_G}$ rigorously. The main result is that every sequence of vortices with uniformly bounded energy has a subsequence that converges to a new kind of stable map, consisting of vortices on $\mathbb{R}^2$ and holomorphic spheres in the symplectic quotient.

To explain this, we recall the symplectic vortex equations: Let $J$ be an $\omega$-compatible $G$-invariant almost complex structure on $M$, $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ an invariant inner product on $\mathfrak{g}$, and $(\Sigma, \omega_\Sigma, j)$ a (smooth) real surface equipped with an area form and a compatible complex structure. For every principal bundle $P$ over $\Sigma$ we denote by $\mathcal{A}(P)$ the affine space of connections on $P$, and by $C_G^\infty(P, M)$ the set of smooth equivariant maps from $P$ to $M$. We denote

$$\tilde{W}(\Sigma) := \{w := (P, A, u) \mid P \text{ principal } G\text{-bundle over } \Sigma, A \in \mathcal{A}(P), u \in C_G^\infty(P, M)\}.$$
The **symplectic vortex equations** are the equations

\[
\begin{align*}
\bar{\partial}_{J,A}(u) &= 0, \\
F_A + (\mu \circ u)\omega_\Sigma &= 0
\end{align*}
\]

for a triple \((P, A, u) \in \tilde{W}(\Sigma)\). Here for a point \(x \in M\) we denote by \(L_x : g \to T_xM\) the infinitesimal action at \(x\). By \(\bar{\partial}_{J,A}(u)\) we mean the complex anti-linear part of \(d_Au := du + L_uA\), which we think of as a one-form on \(\Sigma\) with values in the complex vector bundle \((u^*TM)/G \to \Sigma\).

We view the curvature \(F_A\) of \(A\) as a two-form on \(\Sigma\) with values in the adjoint bundle \(g_P := (P \times g)/G \to \Sigma\). Finally, identifying \(g^* \) with \(g\) via \(\langle \cdot, \cdot \rangle_g\), we view \(\mu \circ u\) as a section of \(g_P\). The vortex equations (1,2) were discovered by K. Cieliebak, A. R. Gaio and D. A. Salamon [CGS], and independently by I. Mundet i Riera [Mu1, Mu2].

Two elements \(w, w' \in \tilde{W}(\Sigma)\) are called equivalent iff there exists an isomorphism \(\Phi : P' \to P\) of principal \(G\)-bundles which descends to the identity on \(\Sigma\), and satisfies

\[\Phi^*(A, u) := (A \circ d\Phi, u \circ \Phi) = (A', u').\]

In this case we write \(w \sim w'\). We define

\[W(\Sigma) := \tilde{W}(\Sigma)/\sim.\]

The equations (1,2) are invariant under equivalence. A (symplectic) vortex (on \(\Sigma\)) is by definition an equivalence class \(W \in W(\Sigma)\), such that every representative of \(W\) satisfies (1,2). We define the energy density of a class \(W \in W(\Sigma)\) to be

\[e_W := \frac{1}{2}(|d_Au|^2 + |F_A|^2 + |\mu \circ u|^2),\]

where \(w := (P, A, u)\) is any representative of \(W\). (Here the norms are induced by the Riemannian metrics \(\omega_\Sigma(\cdot, j\cdot)\) on \(\Sigma\) and \(\omega(\cdot, J\cdot)\) on \(M\), and by \(\langle \cdot, \cdot \rangle_g\). This definition does not depend on the choice of \(w\).) Vortices are absolute minimizers of the (Yang-Mills-Higgs) energy functional

\[E : W(\Sigma) \to [0, \infty], \quad E(W) := \int_{\Sigma} e_W \omega_\Sigma\]

in a given second equivariant homology class. (See [CGS]. Here we assume that \(\Sigma\) is closed, and vortices in the given class exist.) Consider \(\Sigma := \mathbb{R}^2\), equipped with the standard area form \(\omega_{\mathbb{R}^2} := \omega_0\) and complex structure \(j := i\). We define

\[\tilde{M} := \{(P, A, u) \in \tilde{W}(\mathbb{R}^2) \mid (1,2)\}, \quad M := \tilde{M}/\sim.\]
Assume that \((M,\omega)\) is (symplectically) aspherical, i.e.,

\[ \int_{S^2} u^*\omega = 0, \quad \forall u \in C^\infty(S^2, M). \]

Then heuristically, the main result of this article provides a compactification for the space of all classes in \(M\) with fixed finite energy \(E > 0\). There are three sources of non-compactness of this space: Consider a sequence \(W_\nu \in M, \nu \in \mathbb{N}\), of classes of energy \(E\). In the limit \(\nu \to \infty\), the following scenarios (and combinations) may happen:

1. The energy density of \(W_\nu\) blows up at some point in \(\mathbb{R}^2\).
2. There exists a number \(r > 0\) and a sequence of points \(z_\nu \in \mathbb{R}^2\) that converges to \(\infty\), such that the energy density of \(W_\nu\) on the ball \(B_r(z_\nu)\) is bounded above and below by some positive constants.
3. The energy densities converge to 0, i.e., the energy is spread out more and more.

In case 1, by rescaling \(W_\nu\) around the bubbling point, in the limit \(\nu \to \infty\), we obtain a non-constant \(J\)-holomorphic map from \(\mathbb{R}^2\) to \(M\). Using removal of singularity, this is excluded by the asphericity condition. In case 2, we pull \(W_\nu\) back by the translation \(z \mapsto z + z_\nu\), and in the limit \(\nu \to \infty\), obtain a vortex on \(\mathbb{R}^2\). Finally, in case 3, we “zoom out” more and more. In the limit \(\nu \to \infty\) and after removing the singularity at \(\infty\), we obtain a pseudo-holomorphic map from \(S^2\) to the symplectic quotient \(\overline{M} = \mu^{-1}(0)/G\).

Hence the limit object is a stable map, consisting of vortices on \(\mathbb{R}^2\) and pseudo-holomorphic spheres in \(\overline{M}\) (and marked points). This notion and convergence against a stable map are made precise in Section 2.

Here an important difference to Gromov-convergence for pseudo-holomorphic maps is the following: Although the vortex equations are invariant under under all orientation preserving isometries of \(\Sigma\), only translations on \(\mathbb{R}^2\) are allowed as reparametrizations used to obtain a vortex on \(\mathbb{R}^2\) in the limit. Hence we disregard some symmetries of the equations. The reasons are that otherwise the reparametrization group would not act with finite isotropy on the set of simple stable maps, and that there is no suitable evaluation map on the set of vortices which is invariant under rotation. (See Remarks 12 and 15 below.)

In order to state the main result, we also need the following. We call the quadruple \((M,\omega,\mu, J)\) (equivariantly) convex at \(\infty\) iff there exists a proper \(G\)-invariant function \(f \in C^\infty(M, [0, \infty))\) and a constant
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\[ C \in [0, \infty) \] such that
\[ \omega(\nabla_v \nabla f(x), Jv) - \omega(\nabla_{Jv} \nabla f(x), v) \geq 0, \quad df(x) J L_x \mu(x) \geq 0, \]
for every \( x \in f^{-1}([C, \infty)) \) and \( 0 \neq v \in T_x M \). Here \( \nabla \) denotes the Levi-Civita connection of the metric \( \omega(\cdot, J \cdot) \).

We define the image of a class \( W \in W(\Sigma) \) to be the set of orbits of \( u(P) \), where \( (P, A, u) \) is any representative of \( W \). This is a subset of \( M/G \). We endow \( M/G \) with the quotient topology. We are now able to formulate the main result.

**Theorem 1** (Bubbling). Assume that hypothesis (H) is satisfied, \((M, \omega)\) is aspherical, and \((M, \omega, \mu, J)\) is convex at \( \infty \). Let \( k \in \mathbb{N}_0 := \{0, 1, \ldots\} \), and for \( \nu \in \mathbb{N} \) let \( W_\nu \in \mathcal{M} \) be a vortex and \( z_1^\nu, \ldots, z_k^\nu \in \mathbb{R}^2 \) be points. Suppose that the closure of the image of each \( W_\nu \) is compact, and
\[ E(W_\nu) > 0, \quad \forall \nu \in \mathbb{N}, \quad \sup_{\nu \in \mathbb{N}} E(W_\nu) < \infty, \]
\[ \limsup_{\nu \to \infty} |z_i^\nu - z_j^\nu| > 0, \quad \text{if} \quad i \neq j. \]
Then there exists a subsequence of \((W_\nu, z_0^\nu := \infty, z_1^\nu, \ldots, z_k^\nu)\) that converges to some genus 0 stable map of vortices on \( \mathbb{R}^2 \) and pseudo-holomorphic spheres in \( \overline{M} \) with \( k + 1 \) marked points.

(6) (The reasons for the additional marked point \( z_0^\nu = \infty \) are explained in Remarks [7] and [14] below.) The relevance of Theorem 1 is the following. There is an evaluation map from the set
\[ \mathcal{M}_{< \infty} := \{ W \in \mathcal{M} \mid \text{image}(W) \text{ compact}, E(W) < \infty \}. \]
to the product of \( \overline{M} \) and the Borel construction for the action of \( G \) on \( M \). (See the forth-coming article [Zi3].) The structure constants of the quantum Kirwan map \( Q \kappa_G \) will be defined by pulling back cohomology classes via this evaluation map and integrating them over the space of vortices representing a fixed second equivariant homology class.

To make this rigorous, one has to pass to some finite-dimensional approximation of the Borel construction and show that the evaluation map is a pseudo-cycle. The proof of this will rely on Theorem 1.

Secondly, Theorem 1 will also be used to prove that \( Q \kappa_G \) is a ring homomorphism. (See the argument outlined in [Zi3].)

The proof of the theorem combines Gromov compactness for pseudo-holomorphic maps with Uhlenbeck compactness. It relies on work [CGMS, GS] by K. Cieliebak, R. Giaio, I. Mundet i Riera, and D. A. Salamon. The idea is the following. In order to capture all the energy, we “zoom out rapidly”, i.e., rescale the vortices so much that the energies
of the rescaled vortices are concentrated near the origin in $\mathbb{R}^2$. Now we “zoom back in” in such a way that we capture the first bubble, which may either be a vortex on $\mathbb{R}^2$ or a sphere in $\overline{M}$. In the first case we are done. In the second case we “zoom in” further, to obtain a finite number of vortices and spheres that are attached to the first bubble. Iterating this procedure, we construct the whole stable map.

The proof involves generalizations of results for pseudo-holomorphic maps to vortices: a bound on the energy density of a vortex, quantization of energy, compactness with bounded derivatives, and hard and soft rescaling. The proof that the bubbles connect and no energy is lost between them, uses an isoperimetric inequality for the invariant symplectic action functional, proved in [Zi2], based on a version of the inequality by R. Gaio and D. A. Salamon [GS].

Another crucial point is that when “zooming out”, no energy is lost locally in $\mathbb{R}^2$ in the limit. This relies on an upper bound of the “moment-map component” of a vortex, due to R. Gaio and D. A. Salamon.

**Related work and remarks.** Assume that $\Sigma$ is closed, (H) holds, and $M$ is symplectically aspherical and equivariantly convex at $\infty$. In this case, in [CCMS] Theorem 3.4], K. Cieliebak et al. proved compactness of the space of vortices with energy bounded above by a fixed constant. Assume that $M$ and $\Sigma$ are closed. Then in [Mu1, Theorem 4.4.2] I. Mundet i Riera compactified the space of bounded energy vortices with fixed complex structure on $\Sigma$. Assuming also that $G := S^1$, this was extended by I. Mundet i Riera and G. Tian in [MT, Theorem 1.4] to the situation of varying complex structure. This work is based on a version of Gromov-compactness for continuous almost complex structure, proved by S. Ivashkovich and V. Shevchishin in [IS].

In [Ot1, Theorem 1.8] A. Ott compactified the space of bounded energy vortices in a different way, for a general Lie group, and closed $M$ and $\Sigma$, the latter with fixed complex structure. He used the approach to Gromov-compactness by D. McDuff and D. A. Salamon in [MS]. In the case in which $\Sigma$ is an infinite cylinder, equipped with the standard area form and complex structure, the compactification was carried out by U. Frauenfelder in [Fr1, Theorem 4.12].

In [GS] R. Gaio and D. A. Salamon investigated the vortex equations with area form $C_\omega_\Sigma$ in the limit $C \to \infty$. Here $\Sigma$ is a closed surface equipped with a fixed area form $\omega_\Sigma$. They proved that three types of objects may bubble off: a holomorphic sphere in $\overline{M}$, vortices on $\mathbb{R}^2$, and holomorphic spheres in $M$. (See the proof of [GS Theorem A].)
In some earlier work (e.g. [CGS] and [Zi1]), the principal $P$ was fixed and the vortex equations were seen as equations for a pair $(A, u)$ rather than a triple $(P, A, u)$. (However, in [MT] I. Mundet i Riera and G. Tian took the viewpoint of the present article.) The motivation for making $P$ part of the data is twofold:

When formulating convergence for a sequence of vortices on $\mathbb{R}^2$ against a stable map, one has to pull back the vortices by translations of $\mathbb{R}^2$. (See Section 2.2.) If the principal bundle is fixed and vortices are defined as pairs $(A, u)$ solving (1.2), then there is no natural such pullback. However, there is a natural pullback if the principal is made part of the data for a vortex. (This is true for an arbitrary surface $\Sigma$.)

Another motivation is the following: If the area form or the complex structure on the surface $\Sigma$ vary, then in the limit we may obtain a surface $\Sigma'$ with singularities. It does not make sense to consider $P$ as a bundle over $\Sigma'$. One way of solving this problem is by decomposing $\Sigma'$ into smooth surfaces, and constructing smooth principal bundles over these surfaces. Hence the principal should be viewed as a varying object.

Once $P$ is made part of the data, it is natural to consider equivalence classes of triples $(P, A, u)$ rather than the triples themselves, since all important quantities, like energy density and energy, are invariant (or equivariant) under equivalence. Viewing the equivalence classes as the fundamental objects matches the physical viewpoint that the “gauge field”, i.e., the connection $A$, is physically relevant only “up to gauge”.

**Organization.** This article is organized as follows. In Section 2 we define the notion of a stable map of vortices on $\mathbb{R}^2$ and pseudo-holomorphic spheres in $\overline{\mathcal{M}}$ and convergence against such a stable map.

The main result of Section 3 (Proposition 18) is that given a sequence of rescaled vortices with uniformly bounded energies, there exists a subsequence that converges modulo bubbling at finitely many points. The proof is based on compactness for rescaled vortices on the punctured plane with uniformly bounded energy densities (Proposition 19). It also uses the fact that at each bubbling point at least the energy $E_{\text{min}}$ is lost, where $E_{\text{min}} > 0$ is the minimal energy of a vortex on $\mathbb{R}^2$ or pseudo-holomorphic sphere in $\overline{\mathcal{M}}$. This is the content of Proposition 20 which is proved here by a hard rescaling argument, using Proposition 19 and Hofer's lemma. We also state and prove Lemma 22, which says that the energy densities of a convergent sequence of rescaled vortices converge to the density of the limit. This is used in the proof of Proposition 18.
The main result of Section 4 is Proposition 24, which tells how to find the next bubble in the bubbling tree, at a bubbling point of a given sequence of rescaled vortices. A crucial ingredient in its proof is Proposition 25 (proven in the same section). This result states that the energy of a vortex on an annulus is concentrated near the ends, provided that it is small enough.

Based on Sections 3 and 4, the main result, Theorem 1, is proven in Section 5. In the appendix we recollect results on vortices, the invariant symplectic action, Uhlenbeck compactness, compactness for $\bar{\partial}_J$, pseudo-holomorphic maps into $\bar{M}$ and other auxiliary results, used in the proof of Theorem 1.

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2. Stable maps of vortices over the plane and holomorphic spheres in the symplectic quotient

2.1. Stable maps. Let $M, \omega, G, g, \langle \cdot, \cdot \rangle_g, \mu, J$ be as in Section 1. Our standing hypothesis (H) implies that the symplectic quotient

$$(\bar{M} = \mu^{-1}(0)/G, \bar{\omega})$$

is well-defined and closed. The structure $J$ induces a $\bar{\omega}$-compatible almost complex structure on $\bar{M}$ as follows. For every $x \in M$ we denote by $L_x : g \to T_x M$ the infinitesimal action at $x$. We define the horizontal distribution $H \subseteq T_{\mu^{-1}(0)}$ by

$$H_x := \ker d\mu(x) \cap \text{im} L_x^\perp, \quad \forall x \in \mu^{-1}(0).$$

Here $\perp$ denotes the orthogonal complement with respect to the metric $\omega(\cdot, J \cdot)$ on $M$. We denote by $\pi : \mu^{-1}(0) \to \bar{M} := \mu^{-1}(0)/G$ the canonical projection. We define $\bar{J}$ to be the unique endomorphism of $T\bar{M}$ such that

$$\bar{J} d\pi = d\pi J$$

on $H$.

We identify $\mathbb{R}^2 \cup \{\infty\}$ with $S^2$. The (Connectedness) condition in the definition of a stable map below will involve evaluation of a map.
$S^2 \to \overline{M}$ at a given point in $S^2$ and of a vortex at the point $\infty \in S^2$. In order to make sense of the latter, we need the following. We denote by $Gx$ the orbit of a point $x \in M$. Let $P$ be a smooth principal $G$-bundle over $\mathbb{R}^2$ and $u \in C^\infty_G(P,M)$ a map. We define

$$\bar{u} : \mathbb{R}^2 \to M/G, \quad \bar{u}(z) := Gu(p),$$

where $p \in P$ is an arbitrary point in the fiber over $z$. For $W \in \mathcal{W}$ we define

$$(9) \quad \bar{u}_W := \bar{u},$$

where $w = (P,A,u)$ is any representative of $W$. This is well-defined, i.e., does not depend on the choice of $w$. Recall the definition (7) of $\mathcal{M}_{<\infty}$.

**Proposition 2 (Continuity at $\infty$).** If $W \in \mathcal{M}_{<\infty}$ then the map $\bar{u}_W : \mathbb{R}^2 \to M/G$ extends continuously to a map $f : S^2 \to M/G$, such that $f(\infty) \in \overline{M} = \mu^{-1}(0)/G$.

**Proof of Proposition 2.** This follows from the estimate (53) with $R = \infty$ in Proposition 25 below. (Alternatively, one can use [GS] Proposition 11.1.)

**Definition 3.** We define the evaluation map

$$\mathcal{E}_\infty : (C^0(S^2,M/G) \times S^2) \coprod (\mathcal{M}_{<\infty} \times \{\infty\}) \to M/G$$

as follows. For $(\bar{u},z) \in C^0(S^2,M/G) \times S^2$ we define

$$(10) \quad \mathcal{E}_\infty(\bar{u},z) := \mathcal{E}(\bar{u},z) := \bar{u}(z).$$

Furthermore, for $W \in \mathcal{M}_{<\infty}$ we define

$$(11) \quad \mathcal{E}_\infty(W) := f(\infty),$$

where $f$ is as in Proposition 2.

**Definition 4.** For every $k \in \mathbb{N}_0 = \{0,1,\ldots\}$ a (genus 0) stable map of vortices on $\mathbb{R}^2$ and pseudo-holomorphic spheres in $\overline{M}$ with $k + 1$ marked points is a tuple

$$(12) \quad (W,z) := (V,\overline{T},E,(W_\alpha)_{\alpha \in V},(\bar{u}_\alpha)_{\alpha \in \overline{T}},(z_{\alpha\beta})_{\alpha E\beta},(\alpha_i,z_i)_{i=0,\ldots,k}),$$

where $V$ and $\overline{T}$ are finite sets, $E$ is a tree relation on $T := V \coprod \overline{T}$, $W_\alpha \in \mathcal{M}_{<\infty}$ (for $\alpha \in V$), $\bar{u}_\alpha : S^2 \to \overline{M} = \mu^{-1}(0)/G$ is a $\overline{J}$-holomorphic map (for $\alpha \in \overline{T}$), $z_{\alpha\beta} \in S^2$ is a point for each adjacent pair $\alpha E\beta$, $\alpha_i \in T$ is a vertex and $z_i \in S^2$ is a point, for $i = 0,\ldots,k$, such that the following conditions hold.
(i) (Special points)

- If \( \alpha_0 \in V \) then \( z_0 = \infty \).
- Fix \( \alpha \in T \). Then the points \( z_{\alpha\beta} \) with \( \beta \in T \) such that \( \alpha \text{E}\beta \) and the points \( z_i \) with \( i = 0, \ldots, k \) such that \( \alpha_i = \alpha \), are all distinct.
- If \( \alpha \in V \) and \( \beta \in T \) are such that \( \alpha \text{E}\beta \) then \( z_{\alpha\beta} = \infty \).

(ii) (Connectedness) Let \( \alpha, \beta \in T \) be such that \( \alpha \text{E}\beta \). Then
\[
\ev_{z_{\alpha\beta}}(W_\alpha) = \ev_{z_{\beta\alpha}}(W_\beta).
\]
Here \( \ev \) is defined as in (10) and (11) and we set \( W_\alpha := \bar{u}_\alpha \) if \( \alpha \in T \).

(iii) (Stability) If \( \alpha \in V \) is such that \( E(W_\alpha) = 0 \) then there exists \( i \in \{0, \ldots, k\} \) such that \( \alpha_i = \alpha \). Furthermore, if \( \alpha \in \overline{T} \) is such that \( E(\bar{u}_\alpha) = 0 \) then
\[
\# \{ \beta \in T \mid \alpha \text{E}\beta \} + \# \{ i \in \{0, \ldots, k\} \mid \alpha_i = \alpha \} \geq 3.
\]

This definition is modelled on the notion of a genus 0 stable map of pseudo-holomorphic spheres, as introduced by Kontsevich in [Ko]. (For an exhaustive exposition of those stable maps see the book by D. McDuff and D. A. Salamon [MS].)

Remarks. It follows from condition (i) that if \( \alpha \in V \) then there exists at most one \( \beta \in T \) such that \( \alpha \text{E}\beta \). This means that every vortex is a leaf of the tree \( T \). Furthermore, if \( \alpha_0 \in V \) then it follows that \( T = V \) consists only of \( \alpha_0 \). It follows that if \( T \) has at least two elements, then \( \alpha_0 \in \overline{T} \), and hence \( \overline{T} \neq \emptyset \). Furthermore, if \( \alpha \in V \) and \( \beta \in T \) are such that \( \alpha \text{E}\beta \) then \( \beta \in \overline{T} \). This means that two vortices cannot be adjacent. \( \square \)

Remark 5. If \( 1 \leq i \leq k \) is such that \( \alpha_i \in V \) then \( z_i \neq \infty \). This follows from condition (i). \( \square \)

We fix a stable map \((W, z)\) as in Definition 4 and \( \alpha \in T \). We define the set of nodal points at \( \alpha \) to be
\[
Z_\alpha := \{ z_{\alpha\beta} \mid \beta \in T, \alpha \text{E}\beta \} \subseteq S^2,
\]
the set of marked points on \( \alpha \) to be
\[
\{ z_i \mid \alpha_i = \alpha, i \in \{0, \ldots, k\} \},
\]
and the set \( Y_\alpha \) of special points to be the union of \( Z_\alpha \) and the set of marked points at \( \alpha \). The stability condition (iii) says that if \( \alpha \in V \) is such that \( E(W_\alpha) = 0 \) then \( \alpha \) carries at least one marked point on \( \mathbb{R}^2 \).
Figure 1. Stable map. The “raindrops” correspond to vortices on $\mathbb{R}^2$ and the spheres to pseudo-holomorphic spheres in $\overline{M}$. The seven dots are marked points. The dashed objects are “ghosts”, i.e., they carry no energy.

(It also carries a special point at $\infty$.) Furthermore, if $\alpha \in T$ is such that $\bar{u}_\alpha$ is a constant map, then $\alpha$ carries at least three special points.

The stability condition ensures that the action of a natural reparametrization group on the set of simple stable maps of a given type is free. (See Proposition 11 below.) This will be needed in order to show that the evaluation map on the set of non-trivial vortices (with marked points) is a pseudo-cycle.

Examples. The easiest example of a stable map consists of the tree with one vertex $T = V = \{\alpha_0\}$, a vortex $W \in \mathcal{M}_{<\infty}$, the marked point $z_0 := \infty$ and a finite number of distinct points $z_i \in \mathbb{R}^2$, $i = 1, \ldots, k$, where $k \geq 1$ if $E(W) = 0$.

As another example we set $V := \emptyset$. Then a stable map in the new sense is a genus 0 stable map of $J$-holomorphic spheres in $\overline{M}$.

Example 6. We set \( k := 0 \), choose an integer \( \ell \in \mathbb{N}_0 \), and define
\[
V := \{1, \ldots, \ell\}, \quad \mathcal{T} := \{0\}, \quad E := \{(0, 1), \ldots, (0, \ell), (1, 0), \ldots, (\ell, 0)\},
\]
\[
\alpha_0 := 0, \quad z_0 := \infty, \forall i = 1, \ldots, \ell.
\]
Let \( z_0, z_0 \in S^2 \), \( i = 1, \ldots, \ell \) be distinct points, \( W_i \in \mathcal{M}_{\leq \infty} \) be such that \( E(W_i) > 0 \), for \( i = 1, \ldots, \ell \), and \( \bar{u}_0 \) a \( \bar{J} \)-holomorphic sphere. If \( \ell \leq 1 \) then assume that \( \bar{u}_0 \) is nonconstant. Then the tuple
\[
(W, z) := (V, \mathcal{T}, E, (W_i)_{i\in\{1,\ldots,\ell\}}, \bar{u}_0, (z_{ij})_{i\in\mathcal{T}, j\in E}, (0, z_0))
\]
is a stable map. (See Figure 2) \( \square \)

Remark 7. In the previous example with \( \ell = 2 \) stability of the component \( \alpha := 0 \in \mathcal{T} \) uses the “additional” marked point \( z_0 \). This example (with \( \ell = 2 \)) will be used in the argument showing that the quantum Kirwan map is a ring homomorphism. This is one reason for having the extra marked point. (Another one is explained in Remark 14 below.) \( \square \)

Example 8. Let \((M, \omega, J, G) := (\mathbb{R}^2, \omega_0, i, S^1)\), equip \( g := \text{Lie}(S^1) = i\mathbb{R} \) with the standard inner product, and consider the action of \( S^1 \subseteq \mathbb{C} \) on \( \mathbb{R}^2 = \mathbb{C} \) by multiplication of complex numbers. We define a moment map \( \mu : \mathbb{R}^2 \to g \) for this action by \( \mu(z) := \frac{i}{2}(1 - |z|^2) \). In this setting, stable maps are classified in terms of their combinatorial structure \((V, \mathcal{T}, E)\), the location of the special points, and for each \( \alpha \in V \), a point in some symmetric product of \( \mathbb{R}^2 \). Each such point corresponds to a vortex on \( \mathbb{R}^2 \). (See the forthcoming article \([Zi5]\).) \( \square \)
For the definition of the quantum Kirwan map one needs to show that a certain natural evaluation map on the space of vortices on $\mathbb{R}^2$ (see \textcolor{red}{[Zi3]}) is a pseudo-cycle. This will rely on the fact that its omega limit set has codimension at least two. In order to show this, one needs to cut down the dimensions of the “boundary strata” by dividing by the actions of suitable “reparametrization groups”. We define these groups as follows.

We fix two finite sets $T, V$ and a tree relation $E$ on the disjoint union $T := T \coprod V$ such that every element of $V$ is a leaf. We define the reparametrization group $G_T$ as follows. We define $\text{Aut}(T) := \text{Aut}(T, V, E)$ to be the subgroup of all automorphisms $f$ of the tree $(T, E)$, satisfying $f(T) = T$ and $f(V) = V$.

We denote by $\text{PSL}(2, \mathbb{C})$ the group of Möbius transformations, i.e., biholomorphic maps on $S^2 \cong \mathbb{C}P^1$, and by $\text{TR}_2$ the group of translations of the plane $\mathbb{R}^2$. We define $\text{Aut}_\alpha := \text{PSL}(2, \mathbb{C})$ if $\alpha \in T$, and $\text{Aut}_\alpha := \text{PSL}(2, \mathbb{C})$ if $\alpha \in T$. We denote by $\text{Aut}_T$ the set of collections $(\varphi_\alpha)_{\alpha \in T}$, such that $\varphi_\alpha \in \text{Aut}_\alpha$, for every $\alpha \in T$. The group $\text{Aut}(T)$ acts on $\text{Aut}_T$ by

$$f \cdot (\varphi_\alpha)_{\alpha \in T} := (\varphi_{f^{-1}(\alpha)})_{\alpha \in T}.$$ 

**Definition 9.** We define $G_T := G_{T, V, E}$ to be the semi-direct product of $\text{Aut}(T)$ and $\text{Aut}_T$ induced by this action.

The group $\text{PSL}(2, \mathbb{C})$ acts on the set of $J$-holomorphic maps $S^2 \to \overline{M}$ by

$$\varphi^* f := f \circ \varphi.$$ 

Furthermore, the group $\text{TR}_2$ acts on the set $\mathcal{M}_{<\infty}$ by

(14) \hspace{1cm} \varphi^* [P, A, u] := [\varphi^* P, \Phi^* (A, u)],

where $\Phi : \varphi^* P \to P$ is defined by $\Phi(z, p) := p$, and $[P, A, u]$ denotes the equivalence class of $(P, A, u)$. By the combinatorial type of a stable map $(W, z)$ as in (12) we mean the tuple $T := (V, T, E)$. We denote by

$$\mathcal{M}(T) := \mathcal{M}(T, V, E)$$

the set of all stable maps of (combinatorial) type $T$. $G_T$ acts on $\mathcal{M}(T)$ as follows. For every $(f, (\varphi_\alpha)) \in G_T$ and $(W, z) \in \mathcal{M}(T)$ we define

$$W'_\alpha := \varphi_{f(\alpha)}^* W_{f(\alpha)}, \forall \alpha \in T, \quad z'_{\alpha\beta} := \varphi_{f(\alpha)}^{-1}(z_{f(\alpha)f(\beta)}), \forall \alpha E \beta,$$

$$\alpha'_i := f(\alpha_i), \quad z'_i := \varphi_{\alpha'_i}^{-1}(z_{\alpha'_i}), \quad i = 0, \ldots, k.$$
Here we set $W_{\alpha} := \bar{u}_{\alpha}$ if $\alpha \in \mathcal{T}$. Furthermore, for $\varphi \in \mathcal{T}_{\mathbb{R}^2}$ we set $\varphi(\infty) := \infty$.

**Definition 10.** We define

\[
(f, (\varphi_\alpha))^*(W, z) := (V, \bar{T}, E, (W_{\alpha})_{\alpha \in \mathcal{T}}, (z_{\alpha\beta})_{\alpha E \beta}, (\alpha_i', z_i')_{i=0, \ldots, k}).
\]

This defines an action of $G_T$ on $\mathcal{M}(T)$. Let now $(M, J)$ be an almost complex manifold. Recall that a $J$-holomorphic map $u : S^2 \to M$ is called multiply covered if there exists a holomorphic map $\varphi : S^2 \to S^2$ of degree at least two, and a $J$-holomorphic map $v : S^2 \to M$, such that $u = v \circ \varphi$. Otherwise, $u$ is called simple.

Returning to the setting of the current section, let $\bar{u} \in C^\infty(S^2, \overline{M})$ be a $\bar{J}$-holomorphic map. We call a stable map $(W, z)$ simple iff the following conditions hold: For every $\alpha \in \mathcal{T}$ the $\bar{J}$-holomorphic map $\bar{u}_\alpha$ is constant or simple. Furthermore, if $\alpha, \beta \in V$ are such that $\alpha \neq \beta$ and $E(W_{\alpha}) \neq 0$, and $\varphi \in \mathcal{T}_{\mathbb{R}^2}$, then $\varphi^*W_{\alpha} \neq W_{\beta}$. Moreover, if $\alpha, \beta \in \mathcal{T}$ are such that $\alpha \neq \beta$ and $\bar{u}_\alpha$ is nonconstant, and $\varphi \in \text{PSL}(2, \mathbb{C})$, then $\varphi^*\bar{u}_\alpha = \bar{u}_\alpha \circ \varphi \neq \bar{u}_\beta$. We denote by

\[
\mathcal{M}^*(T) := \mathcal{M}^*(\bar{T}, V, E) \subseteq \mathcal{M}(T)
\]

the subset of all simple stable maps. The action of $G_T$ on $\mathcal{M}(T)$ leaves $\mathcal{M}^*(T)$ invariant.

**Proposition 11.** The action of $G_T$ on $\mathcal{M}^*(T)$ is free.

**Proof of Proposition 11.** This follows from an elementary argument, using the stability condition (iii), the freeness of the action of $\mathcal{T}_{\mathbb{R}^2}$ on $\mathcal{M}_{<\infty}$ (see Lemma 36 in the appendix), and the fact that every simple holomorphic sphere is somewhere injective (see [MS, Proposition 2.5.1]).

Heuristically, this result implies that the quotient $\mathcal{M}^*(T)/G_T$ is canonically a smooth finite dimensional manifold. This will be important for the pseudo-cycle property of the evaluation map defined on the set of vortices on $\mathbb{R}^2$.

**Remark 12.** The action of $\mathcal{T}_{\mathbb{R}^2}$ on $\mathcal{M}_{<\infty}$ extends to an action of the group Isom$^+(\mathbb{R}^2)$ of orientation preserving isometries of $\mathbb{R}^2$. Hence one may be tempted to adjust the definition of the reparametrization group $G_T$ and its action on $\mathcal{M}^*(T)$ accordingly. However, for the purpose of defining the quantum Kirwan map, this is not possible. The reason is...
that in general there is no evaluation map on $\mathcal{M}_{<\infty}$ that is invariant under the action of $\text{Isom}^+(\mathbb{R}^2)$. This is a crucial difference between vortices and pseudo-holomorphic curves. Note also that the action of $\text{Isom}^+(\mathbb{R}^2)$ on the set of vortices of positive energy is not always free. (For an example see [Zi5].) See also the Remark 15.

2.2. Convergence against a stable map. In order to define convergence, we need the following notation. Let $\alpha \in T$ and $i = 0, \ldots, k$. We define $z_{\alpha,i} \in S^2$ as follows. If $\alpha = \alpha_i$ then we set
\begin{equation}
z_{\alpha,i} := z_i.
\end{equation}
Otherwise let $\beta \in \overline{T}$ be the unique vertex such that the chain of vertices of $T$ running from $\alpha$ to $\alpha_i$ is given by $(\alpha, \beta, \ldots, \alpha_i)$. ($\beta = \alpha_i$ is also allowed.) We define
\begin{equation}
\left(16\right)\quad z_{\alpha,i} := z_{\alpha \beta}.
\end{equation}
We define
\begin{equation}
\left(17\right)\quad M^* := \{ x \in M \mid \text{if } g \in G : gx = x \Rightarrow g = 1 \}.
\end{equation}
Note that $\mu^{-1}(0) \subseteq M^*$ by our standing hypothesis (H). Recall the definitions (9,13,14) of $\bar{u}_W, Z_\alpha$ and the action of $\mathcal{T}_{\mathbb{R}^2}$ on $\mathcal{M}_{<\infty}$. Let $k \geq 0$, for $\nu \in \mathbb{N}$ let $W_\nu \in \mathcal{M}_{<\infty}$ be a vortex and $z_1^\nu, \ldots, z_k^\nu \in \mathbb{R}^2$ be points, and let
\begin{equation}
(W, z) := (V, \mathcal{T}, E, (W_\alpha)_{\alpha \in T}, (z_{\alpha \beta})_{\alpha \beta \in \mathcal{E}}, (\alpha_i, z_i)_{i=0,\ldots,k})
\end{equation}
be a stable map. Here we use the notation $W_\alpha := \bar{u}_\alpha$ if $\alpha \in \overline{T}$. For a $J$-holomorphic map $f : S^2 \to \overline{M}$ we denote its energy by
\begin{equation}
E(f) = \int_{S^2} f^*\omega.
\end{equation}
Let $\Sigma$ be a compact smooth surface (possibly with boundary). Recall the definition (3) of $\mathcal{W}(\Sigma)$. We define the $C^\infty$-topology $\tau_\Sigma$ on this set as follows: We fix a smooth principal $G$-bundle $P$ over $\Sigma$ and a $C^\infty$-open subset $\mathcal{U} \subseteq \mathcal{A}(P) \times C^\infty_G(P, M)$. (This means that $\mathcal{U}$ is $C^k$-open for some $k \in \mathbb{N}_0$.) We define
\begin{equation}
\overline{\mathcal{U}} := \{ [P, A, u] \mid (A, u) \in \mathcal{U} \}.
\end{equation}
We define
\begin{equation}
\left(18\right)\quad \tau_\Sigma := \{ \overline{\mathcal{U}} \mid P, \mathcal{U} \text{ as above} \}.
\end{equation}
Let $\Sigma$ be a smooth surface, $W = [P, A, u] \in \mathcal{W}(\Sigma)$, and $\Omega \subseteq \Sigma$ an open subset with compact closure and smooth boundary. We define the
restriction $W|_{\Omega}$ to be the equivalence class of the pullback of $(P, A, u)$ under the inclusion map $\Omega \rightarrow \Sigma$.

**Definition 13** (Convergence). The sequence $(W_\nu, z_0^\nu := \infty, z_1^\nu, \ldots, z_k^\nu)$ is said to converge to $(W, z)$ as $\nu \rightarrow \infty$ if the limit $E := \lim_{\nu \rightarrow \infty} E(W_\nu)$ exists,

$$E = \sum_{\alpha \in T} E(W_\alpha),$$

and there exist Möbius transformations $\phi_\alpha^\nu : S^2 \rightarrow S^2$, for $\alpha \in T := V \bigsqcup T$, $\nu \in \mathbb{N}$, such that the following conditions hold.

(i) • If $\alpha \in V$ then $\phi_\alpha^\nu$ is a translation on $\mathbb{R}^2$.
   • For every $\alpha \in T$ we have $\phi_\alpha^\nu(z_{\alpha,0}) = \infty$, where $z_{\alpha,0}$ is defined as in (13), (10).
   • Let $\alpha \in T$ and $\psi_\alpha$ be a Möbius transformation such that $\psi_\alpha(\infty) = z_{\alpha,0}$. Then the derivatives $(\phi_\alpha^\nu \circ \psi_\alpha)'(z)$ converge to $\infty$, for every $z \in \mathbb{R}^2 = \mathbb{C}$.

(ii) If $\alpha, \beta \in T$ are such that $\alpha E \beta$ then $(\phi_\alpha^\nu)^{-1} \circ \phi_\beta^\nu \rightarrow z_{\alpha \beta}$, uniformly on compact subsets of $S^2 \setminus \{z_{\beta \alpha}\}$.

(iii) • Let $\alpha \in V$ and $\Omega \subseteq \mathbb{R}^2$ be an open subset with compact closure and smooth boundary. Then the restriction of $(\phi_\alpha^\nu)^*W_\nu$ to $\Omega$ converges to $W_\alpha$ with respect to the topology $\tau_{\Omega}$ (as defined in (18)).
   • Fix $\alpha \in T$. Let $Q$ be a compact subset of $S^2 \setminus (Z_\alpha \cup \{z_{\alpha,0}\})$. For $\nu$ large enough, we have

$$\bar{u}_\alpha^\nu := \bar{u}_{W_\nu} \circ \phi_\alpha^\nu(Q) \subseteq M^*/G,$$

and $\bar{u}_\alpha^\nu$ converges to $\bar{u}_\alpha$ in $C^1$ on $Q$. (Here $\bar{u}_{W_\nu}$ is defined as in (2).)

(iv) We have $(\phi_\alpha^\nu)^{-1}(z_i^\nu) \rightarrow z_i$ for every $i = 1, \ldots, k$.

(See Figure 3.) This definition is based on the notion of convergence of a sequence of pseudo-holomorphic spheres to a genus 0 stable map of pseudo-holomorphic spheres. (For that notion see for example [MS]).

**Remark.** The last part of condition (ii) and the second part of condition (iii) capture the idea of catching a pseudo-holomorphic sphere in $\overline{M}$ by “zooming out”: Fix $\alpha \in T$, and consider the case $z_{\alpha,0} = \infty$. Then there exist $\lambda_\alpha^\nu \in \mathbb{C} \setminus \{0\}$ and $z_\alpha^\nu \in \mathbb{C}$ such that $\phi_\alpha^\nu(z) = \lambda_\alpha^\nu z + z_\alpha^\nu$. 


Figure 3. Convergence of a sequence of vortices on $\mathbb{R}^2$ against a stable map.
It follows from a direct calculation that \((\varphi_\alpha^\nu)^* W_\nu\) is a vortex with respect to the area form \(\omega_\Sigma = |\lambda_\alpha^\nu|^2 \omega_0\), where \(\omega_0\) denotes the standard area form on \(\mathbb{R}^2\).

The last part of condition (i) means that \(\lambda_\alpha^\nu \to \infty\), for \(\nu \to \infty\). Hence in the limit \(\nu \to \infty\) we obtain the equations

\[
\bar{\partial}_{J,A}(u) = 0, \quad \mu \circ u = 0.
\]

These correspond to the \(\bar{J}\)-Cauchy-Riemann equations for a map from \(\mathbb{R}^2 = \mathbb{C}\) to \(\overline{M}\). (See Proposition 45.) The second part of (iii) imposes that the sequence of rescaled vortices converges (in a suitable sense) to the \(\bar{J}\)-holomorphic sphere \(\bar{u}_\alpha\). \(\square\)

**Remark.** The “energy-conservation” condition (19) has the important consequence that the stable map \((W, z)\) represents the same equivariant homology class as the vortex \(W_\nu\), for \(\nu\) large enough. (See [Zi3].) \(\square\)

**Remark 14.** One purpose of the additional marked point \((\alpha_0, z_0)\) is to be able to formulate the second part of condition (iii). (Another one is explained in Remark 7 above.) For \(\alpha \in \mathbb{T}\) and \(\nu \in \mathbb{N}\) the map \(G u_\nu \circ \varphi_\alpha^\nu\) is only defined on the subsets \((\varphi_\alpha^\nu)^{-1}(\mathbb{R}^2) \subseteq S^2\). Since by condition (i) we have \(\varphi_\alpha^\nu(z_\alpha, 0) = \infty\), the composition \(\bar{u}_W \circ \varphi_\alpha^\nu : Q \to M/G\) is well-defined for each compact subset \(Q \subseteq S^2 \setminus (Z_\alpha \cup \{z_\alpha, 0\})\). Hence the the second part of condition (iii) makes sense. \(\square\)

**Example.** Let \(M, \omega\) etc. be as in Example 8 Then a sequence \(W_\nu \in \mathcal{M}_{<\infty}\) converges to a stable map if and only if the total degree of \(W_\nu\) equals the sum of the degrees of the vortex components of the stable map, and for each \(\alpha \in V\), up to translations, the point in the symmetric product of \(\mathbb{R}^2\) corresponding to \(W_\nu\), converges to the point corresponding to the vortex \(W_\alpha\). (See [Zi5].) \(\square\)

**Remark 15.** One conceptual difficulty in defining the notion of convergence is the following. (Compare also to Remark 12.) Consider the group \(\text{Isom}^+(\Sigma)\) of orientation preserving isometries of \(\Sigma\) (with respect to the metric \(\omega_\Sigma(\cdot, j \cdot)\)). (This coincides with the group of diffeomorphisms of \(\Sigma\) that preserve the pair \((\omega_\Sigma, j)\).) This group acts on \(\mathcal{W}(\Sigma)\) (defined as in (3)), as in (14). The set \(\mathcal{M}_{<\infty}\) of finite energy vortices is invariant under this action.

Hence naively, in the definition of convergence one would allow \(\varphi_\alpha^\nu\) to be an orientation preserving isometry of \(\mathbb{R}^2\), rather than just a translation. The problem is that with this less restrictive condition, there is
no evaluation map on the set of stable maps, that is continuous with respect to convergence. (Such a map is needed for the definition of the quantum Kirwan map.)

Note here that we cannot define evaluation of a vortex $W$ at some point $z \in \Sigma$ by choosing a representative of $W$ and evaluating it at some point in the fiber over $z$, since this depends on the choices. Instead, evaluation of $W$ at $z$ yields a point in the Borel construction for the action of $G$ on $M$. (See [Zi3].) □

3. Compactness modulo bubbling for rescaled vortices

In this section we consider a sequence of rescaled vortices on $\mathbb{R}^2$ with image in a fixed compact subset of $M/G$ and uniformly bounded energies. We assume that $(M, \omega)$ is aspherical. The main result, Proposition 18 below, is that there exists a subsequence that away from finitely many bubbling points, converges to either a rescaled vortex on $\mathbb{R}^2$ or a $\bar{J}$-holomorphic sphere in $\overline{M}$. This is a crucial ingredient of the proof of Theorem 1.

In order to explain the result, let $M, \omega, G, \langle \cdot, \cdot \rangle, \mu, J, \Sigma, \omega_\Sigma, j$ be as in Section 1. Recall the definition (4) of the energy density $e_{\omega_\Sigma, j}^\Sigma$ of a class $W \in W(\Sigma)$.

Remark 16. This density has the following transformation property: Let $\Sigma'$ be another real surface, and $\varphi: \Sigma' \to \Sigma$ a smooth immersion. We define the pullback $\varphi^*W$ as in (14). Then a straight-forward calculation shows that

\begin{equation}
(e_{\varphi^*W}) = e_{\varphi^*W} \circ \varphi,
\end{equation}

and $W$ is a vortex with respect to $(\omega_\Sigma, j)$ if and only if $\varphi^*W$ is a vortex with respect to $\varphi^*(\omega_\Sigma, j)$. □

Remark 17. If $W$ is a vortex (with respect to $(\omega_\Sigma, j)$) then

\begin{equation}
e_{\omega_\Sigma, j}^\Sigma = |\partial_{J,A} u|^2 + |\mu \circ u|^2,
\end{equation}

where $\partial_{J,A} u$ is the complex linear part of $d_A u$, viewed as a one-form on $\Sigma$ with values in $(u^*TM)/G \to \Sigma$. This follows from the vortex equations [12]. □

Let $R \in [0, \infty]$ and $W \in W(\Sigma)$. Consider first the case $0 < R < \infty$. Then we define the $R$-energy density of $W$ to be

\begin{equation}
e_{W}^R := R^2 e_{W}^{R \omega_\Sigma, j}.
\end{equation}
This means that
\begin{equation}
 e^R_W = \frac{1}{2} (|d_A u|^2_{\omega_\Sigma} + R^{-2} |F_A|_{\omega_\Sigma}^2 + R^2 |\mu \circ u|^2),
\end{equation}
where the subscript “$\omega_\Sigma$” means that the norms are taken with respect to the metric $\omega_\Sigma(\cdot, j \cdot)$.

If $R = 0$ or $\infty$ then we define
\[ e^R_W := \frac{1}{2} |d_A u|^2_{\omega_\Sigma}. \]

We define the $R$-energy of $W$ on a measurable subset $X \subseteq \Sigma$ to be
\[ E^R(W, X) := \int_X e^R_W \omega_\Sigma \in [0, \infty]. \]

The density and the energy have the following rescaling property: Consider the case $(\Sigma, \omega_\Sigma, j) = (R^2, \omega_0, i)$, where $\omega_0$ denotes the standard area form on $\mathbb{R}^2$. Assume that $0 < R < \infty$, and consider the map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\varphi(z) := Rz$. Then equality (20) implies that
\[ e^R_{\varphi^* W} = R^2 e^{\omega_0, i}_{W} \circ \varphi. \]

**Remark.** The factor $R^2$ in the definition (22) is important for the subsequent analysis (bubbling, convergence with bounded energy density etc.). However, the density $e^{R^2 \omega_\Sigma}$ is more intrinsic. (Compare to (20).)

The (symplectic) $R$-vortex equations are the equations (12) with $\omega_\Sigma$ replaced by $R^2 \omega_\Sigma$, i.e., the equations
\begin{equation}
 \bar{\partial}_{J, A}(u) = 0, \quad F_A + R^2 (\mu \circ u) \omega_\Sigma = 0.
\end{equation}
In the case $R = \infty$ we interpret the second equation in (24) as
\[ \mu \circ u = 0. \]

**Remark.** Consider the case $(\Sigma, \omega_\Sigma, j) = (\mathbb{R}^2, \omega_0, i)$ and $0 < R < \infty$, and the map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\varphi(z) := Rz$. It follows from Remark 16 that a class $W \in W(\mathbb{R}^2)$ is a vortex if and only if $\varphi^* W$ is an $R$-vortex.

**Remark.** The rescaled energy density has the following important property. Let $R_\nu \in (0, \infty)$ be a sequence that converges to some $R_0 \in [0, \infty]$, and for $\nu \in \mathbb{N}_0$ let $W_\nu$ be an $R_\nu$-vortex. If $W_\nu$ converges to $W_0$ in a suitable sense then
\[ e^{R_\nu}_{W_\nu} \to e^{R_0}_{W_0}. \]
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In the proof of Theorem 1, this will be used in order to show that locally on \( \mathbb{R}^2 \) no energy is lost in the limit \( \nu \to \infty \).

We define the minimal energy \( E_{\text{min}} \) as follows. Recall the definition (5) of \( M \), and that we denote the energy of a \( \bar{J} \)-holomorphic map \( f : S^2 \to \overline{M} \) by \( E(f) = \int_{S^2} f^* \omega \). We define

\[
E_V := \inf \left\{ \left( E(W) \mid W \in M : \text{image}(W) \text{ compact} \right) \cap (0, \infty) \right\},
\]

\[
E := \inf \left\{ \{ E(f) \mid f \in C^\infty(S^2, M) : \bar{\partial}_j(f) = 0 \} \cap (0, \infty) \right\},
\]

\[
E_{\text{min}} := \min \{ E_V, E \}.
\]

Here we used the convention that \( \inf \emptyset = \infty \). Assume that \( M \) is equivariantly convex at \( \infty \). Then Corollary 30 below implies that \( E_V > 0 \).

The results of this and the next section are formulated for connections and maps of Sobolev regularity. This is a natural setup for the relevant analysis. Furthermore, we restrict our attention to the trivial bundle \( \Sigma \times G \). (Since every smooth bundle over \( \mathbb{R}^2 \) is trivial, this suffices for the proof of the main result.)

We fix \( p > 2 \) and naturally identify the affine space of connections on \( \Sigma \times G \) of local Sobolev class \( W^{1,p}_{\text{loc}} \) with the space of one-forms on \( \Sigma \) with values in \( \mathfrak{g} \), of class \( W^{1,p}_{\text{loc}} \). Furthermore, we identify the space of \( G \)-equivariant maps from \( \Sigma \times G \) to \( M \) of class \( W^{1,p}_{\text{loc}}(\Sigma, M) \). Finally, we identify the gauge group (i.e., group of gauge transformations) on \( \Sigma \times G \) of class \( W^{2,p}_{\text{loc}} \) with \( W^{2,p}_{\text{loc}}(\Sigma, G) \). We denote

\[
\tilde{W}_0^p(\Sigma) := \{ W^{1,p}_{\text{loc}} \text{-one-form on } \Sigma \text{ with values in } \mathfrak{g} \} \times C^\infty(\Sigma, M),
\]

\[
\tilde{W}_0^p(\Sigma) := \{ W^{1,p}_{\text{loc}} \text{-one-form on } \Sigma \text{ with values in } \mathfrak{g} \} \times W^{1,p}_{\text{loc}}(\Sigma, M).
\]

We call a solution \( (A, u) \in \tilde{W}_0^p(\Sigma) \) of the equations (24) an \( R \)-vortex over \( \Sigma \). (It will be clear from the notation whether the term "\( R \)-vortex" refers to such a pair \( (A, u) \) or to an equivalence class \( W \) of triples \( (P, A, u) \).) The gauge group \( W^{2,p}_{\text{loc}}(\Sigma, G) \) acts on \( \tilde{W}_0^p(\Sigma) \) by

\[
g^*(A, u) := \left( \text{ad}_{g^{-1}} A + g^{-1} dg, g^{-1} u \right),
\]

where \( \text{ad}_{g_0} : \mathfrak{g} \to \mathfrak{g} \) denotes the adjoint action of an element \( g_0 \in G \). Let \( w \in \tilde{W}_0^p(\Sigma), R \in [0, \infty] \), and \( X \subseteq \Sigma \) be a measurable subset. We
denote by \([w]\) the gauge equivalence class of \(w\), and denote
\[
e^R_w := e^R_{[w]}, \quad E^R(w, X) := E^R([w], X)
\]
For \(r > 0\) we denote by \(B_r \subseteq \mathbb{R}^2\) the open ball of radius \(r\), around 0.

**Proposition 18** (Compactness modulo bubbling). Assume that \((M, \omega)\) is aspherical. Let \(R_\nu \in (0, \infty)\) be a sequence that converges to some \(R_0 \in (0, \infty]\), \(r_\nu \in (0, \infty)\) a sequence that converges to \(\infty\), and for every \(\nu \in \mathbb{N}\) let \(w_\nu = (A_\nu, u_\nu) \in \tilde{\mathcal{W}}_0^p(B_{r_\nu})\) be an \(R_\nu\)-vortex (with respect to \((\omega_0, i)\)). Assume that there exists a compact subset \(K \subseteq M\) such that \(u_\nu(B_{r_\nu}) \subseteq K\), for every \(\nu\). Suppose also that
\[
\sup_{\nu} E^R_{R_\nu}(w_\nu, B_{r_\nu}) < \infty.
\]
Then there exist a finite subset \(Z \subseteq \mathbb{R}^2\) and an \(R_0\)-vortex \(w_0 := (A_0, u_0) \in \tilde{\mathcal{W}}_0^p(\mathbb{R}^2 \setminus Z)\), and passing to some subsequence, there exist gauge transformations \(g_\nu \in W^{2,p}_{\text{loc}}(\mathbb{R}^2 \setminus Z, G)\), such that the following conditions hold.

(i) If \(R_0 < \infty\) then \(Z = \emptyset\) and the sequence \(g_\nu^*(A_\nu, u_\nu)\) converges to \(w_0\) in \(C^\infty\) on every compact subset of \(\mathbb{R}^2\).

(ii) If \(R_0 = \infty\) then on every compact subset of \(\mathbb{R}^2 \setminus Z\), the sequence \(g_\nu^* A_\nu\) converges to \(A_0\) in \(C^0\), and the sequence \(g_\nu^{-1} u_\nu\) converges to \(u_0\) in \(C^1\).

(iii) Fix a point \(z \in Z\) and a number \(\varepsilon_0 > 0\) so small that \(B_{\varepsilon_0}(z) \cap Z = \{z\}\). Then for every \(0 < \varepsilon < \varepsilon_0\) the limit
\[
E_z(\varepsilon) := \lim_{\nu \to \infty} E^R_{R_\nu}(w_\nu, B_\varepsilon(z))
\]
exists and
\[E_z(\varepsilon) \geq E_{\min}.
\]
Furthermore, the function \((0, \varepsilon_0) \ni \varepsilon \mapsto E_z(\varepsilon) \in [E_{\min}, \infty)\) is continuous.

**Remark.** Convergence in conditions (i) should be understood as convergence of the subsequence labelled by those indices \(\nu\) for which \(B_{r_\nu}\) contains the given compact set. \(\square\)

This proposition will be proved on page 32. The strategy of the proof is the following. Assume that the energy densities \(e^{R_\nu}_w\) are uniformly bounded on every compact subset of \(\mathbb{R}^2\). Then the statement of Proposition 18 with \(Z = \emptyset\) follows from an argument involving Uhlenbeck compactness, an estimate for \(\bar{\partial}_J\), elliptic bootstrapping (for statement (i)), and a patching argument.
If the densities are not uniformly bounded then we rescale the maps $w_\nu$ by zooming in near a bubbling point $z_0$ in a “hard way”, to obtain a positive energy $\tilde{R}_0$-vortex in the limit, with $\tilde{R}_0 \in \{0, 1, \infty\}$. If $\tilde{R}_0 < \infty$ then $\tilde{R}_0 = 0$, and we obtain a $J$-holomorphic sphere in $M$. This contradicts symplectic asphericity, and thus this case is impossible.

If $\tilde{R}_0 = \infty$ then either $\tilde{R}_0 = 1$ or $\tilde{R}_0 = \infty$, and hence either a vortex on $\mathbb{R}^2$ or a pseudo-holomorphic sphere in $\mathcal{M}$ bubbles off. Therefore, at least the energy $E_{\text{min}}$ is lost at $z_0$. Our assumption that the energies of $w_\nu$ are uniformly bounded implies that there can only be finitely many bubbling points. On the complement of these points a subsequence of $w_\nu$ converges modulo gauge.

The bubbling part of this argument is captured by Proposition 20 below, whereas the convergence part is the content of the following result.

**Proposition 19 (Compactness with bounded energy densities).** Let $Z \subseteq \mathbb{R}^2$ be a finite subset, $R_\nu \geq 0$ be a sequence of numbers that converges to some $R_0 \in [0, \infty]$, $\Omega_1 \subseteq \Omega_2 \subseteq \ldots \subseteq \mathbb{R}^2 \setminus Z$ open subsets such that $\bigcup \nu \Omega_\nu = \mathbb{R}^2 \setminus Z$, and for $\nu \in \mathbb{N}$ let $w_\nu = (u_\nu, A_\nu) \in \tilde{W}_0^p(\Omega_\nu)$ be an $R_\nu$-vortex. Assume that there exists a compact subset $K \subseteq M$ such that for $\nu$ large enough

\[(27) \quad u_\nu(\Omega_\nu) \subseteq K.\]

Suppose also that for every compact subset $Q \subseteq \mathbb{R}^2 \setminus Z$, we have

\[(28) \quad \sup \left\{ \|c_{R_\nu}^{R_\nu} \|_{L^\infty(Q)} \right\} \nu \in \mathbb{N} : Q \subseteq \Omega_\nu \} < \infty.

Then there exists an $R_0$-vortex $w_0 := (A_0, u_0) \in \tilde{W}_0(\mathbb{R}^2 \setminus Z)$, and passing to some subsequence, there exist gauge transformations $g_\nu \in W^{2,p}_{\text{loc}}(\mathbb{R}^2 \setminus Z, G)$, such that the following conditions are satisfied.

(i) If $R_0 < \infty$ then $g_\nu^* w_\nu$ converges to $w_0$ in $C^\infty$ on every compact subset of $\mathbb{R}^2 \setminus Z$.

(ii) If $R_0 = \infty$ then on every compact subset of $\mathbb{R}^2 \setminus Z$, $g_\nu^* A_\nu$ converges to $A_0$ in $C^0$, and $g_\nu^{-1} u_\nu$ converges to $u_0$ in $C^1$.

The proof of this result is an adaption of the argument of Step 5 in the proof of Theorem A the paper by R. Gaio and D. A. Salamon [GS]. The proof of statement (i) is based on a compactness result in the case of a compact surface $\Sigma$ (possibly with boundary). (See Theorem 34 below. That result follows from an argument by K. Cieliebak et al. in [CGMS].) The proof also involves a patching argument for
gauge transformations, which are defined on an exhausting sequence of subsets of \( \mathbb{R}^2 \setminus Z \).

To prove statement (ii), we will show that curvatures of the connections \( A_\nu \) are uniformly bounded in \( W^{1,p} \). This uses the second rescaled vortex equations and a uniform upper bound on \( \mu \circ u_\nu \) (Lemma 31), due to R. Gaio and D. A. Salamon. The statement then follows from Uhlenbeck compactness with compact base, compactness for \( \bar{\partial}_J \), and a patching argument.

**Proof of Proposition 19.** We choose \( i_0 \in \mathbb{N} \) so big that the balls \( \bar{B}_{i_0}(z) \), \( z \in Z \), are disjoint and contained in \( B_{i_0} \). We fix \( i \in \mathbb{N}_0 \) and define

\[
X^i := \bar{B}_{i+i_0} \setminus \bigcup_{z \in Z} B_{i_0}(z) \subseteq \mathbb{R}^2.
\]

We prove statement (ii). Assume that \( R_0 < \infty \). Using the hypotheses (27)(28), it follows from Theorem 31 below that there exist an infinite subset \( I^1 \subseteq \mathbb{N} \) and gauge transformations \( g_\nu^1 \in W^{2,p}(X^1,G) \) (\( \nu \in I^1 \)), such that \( X^1 \subseteq \Omega_\nu \) and \( w_\nu^1 := (A_\nu^1, u_\nu^1) := (g_\nu^1)^*(w_\nu|X^1) \) is smooth, for every \( \nu \in I^1 \), and the sequence \( (w_\nu^1)_{\nu \in I^1} \) converges to some \( R_0 \)-vortex \( w^1 \in \mathcal{W}_0(X^1) \), in \( C^\infty \) on \( X^1 \).

Iterating this argument, for every \( i \geq 2 \) there exists an infinite subset \( I^i \subseteq I^{i-1} \) and gauge transformations \( g_\nu^i \in W^{2,p}(X^i,G) \) (\( \nu \in I^i \)), such that \( X^i \subseteq \Omega_\nu \) and \( w_\nu^i := (A_\nu^i, u_\nu^i) := (g_\nu^i)^*(w_\nu|X^i) \) is smooth, for every \( \nu \in I^i \), and the sequence \( (w_\nu^i)_{\nu \in I^i} \) converges to some \( R_0 \)-vortex \( w^i \in \mathcal{W}_0(X^1) \), in \( C^\infty \) on \( X^i \).

Let \( i \in \mathbb{N} \). For \( \nu \in I^i \) we define \( h_\nu^i := (g_\nu^{i+1}|X^i)^{-1}g^{i+1}_\nu \). We have \( (h_\nu^i)^*(A^{i+1}_{\nu}|X^i) = A^i_\nu \). Furthermore, \( (A^{i+1}_{\nu})_{\nu \in I^{i+1}} \) and \( (A^i_{\nu})_{\nu \in I^{i+1}} \) are bounded in \( W^{k,p} \) on \( X^i \), for every \( k \in \mathbb{N} \). Hence it follows from Lemma 43 below that the sequence \( (h_\nu^i)_{\nu \in I^{i+1}} \) is bounded in \( W^{k,p} \) on \( X^i \), for every \( k \in \mathbb{N} \). Hence, using the Kondrachov compactness theorem, it has a subsequence that converges to some gauge transformation \( h^i \in C^\infty(X^i,G) \), in \( C^\infty \) on \( X^i \). Note that

\[
(h^i)^*(w^{i+1}|X^i) = w^i.
\]

We choose a map \( \rho^i : X^{i+1} \rightarrow X^i \) such that \( \rho^i = \text{id} \) on \( X^{i-1} \). We define\(^1\) \( k^1 := h^1 \), and recursively,

\[
k^i := h^i(k^{i-1} \circ \rho^{i-1}) \in C^\infty(X^i,G), \forall i \geq 2.
\]

\(^1\)This patching construction follows the lines of the proofs of [Fr1] Theorem 3.6 and Theorem A.3.
Using (29) and the fact \( \rho^{-1} = \text{id} \) on \( X^{i-2} \), we have, for every \( i \geq 2 \),
\[
(k^i)^*w^{i+1} = (k^i \circ \rho^{-1})^*w^i = (k^{i-1})^*w^i,
\]
on \( X^{i-2} \).

It follows that there exists a unique \( w \in \tilde{W}_0(\mathbb{R}^2 \setminus Z) \) that restricts to \( (k^i)^*w^{i+2} \) on \( X^i \), for every \( i \in \mathbb{N} \). Let \( i \in \mathbb{N} \). We choose \( \nu_i \in I^{i+1} \) such that \( \nu_i \geq i \) and a map \( \tau^i : \mathbb{R}^2 \setminus Z \to X^i \) that is the identity on \( X^{i-1} \). We define \( g_i := (\tau^i)^*k^i \circ \tau^i \in C^\infty(\mathbb{R}^2 \setminus Z, G) \). The sequence \( g_i^*w_{\nu_i} \) converges to \( w \), in \( C^\infty \) on every compact subset of \( \mathbb{R}^2 \setminus Z \). (Here we use the \( C^\infty \)-convergence on \( X^i \) of \((u^i_\nu)_{\nu \in I^i}\) against \( w^i \) and the facts \( X_1 \subseteq X_2 \subseteq \cdots \) and \( \bigcup_{i \in \mathbb{N}} X_i = \mathbb{R}^2 \setminus Z \).) Statement (ii) follows.

We prove statement (i). Assume that \( R_0 = \infty \).

Claim 1. For every compact subset \( Q \subseteq \mathbb{R}^2 \setminus Z \) we have
\[
\sup_{\nu} \left\{ \| F_{A_\nu} \|_{L^p(Q)} \mid \nu \in \mathbb{N} : Q \subseteq \Omega_\nu \right\} < \infty.
\]

Proof of Claim 1. Let \( \Omega \subseteq \mathbb{R}^2 \) be an open subset containing \( Q \) such that \( \Omega \) is compact and contained in \( \mathbb{R}^2 \setminus Z \). Hypothesis (28) implies that
\[
\sup_{\nu} \| d_{A_\nu} u_\nu \|_{L^\infty(\Omega)} < \infty.
\]

It follows from our standing hypothesis (H) that there exists \( \delta > 0 \) such that \( G \) acts freely on
\[
K := \{ x \in M \mid |\mu(x)| \leq \delta \}.
\]

Since \( \mu \) is proper the set \( K \) is compact. It follows that
\[
\sup \left\{ \frac{|\xi|}{|L_x \xi|} \mid x \in K, 0 \neq \xi \in \mathfrak{g} \right\} < \infty.
\]

Using the second vortex equation, we have \( |\mu \circ u_\nu| \leq \sqrt{e_{R_\nu}} / R_\nu \). Hence by hypothesis (28) and the assumption \( R_\nu \to \infty \), we have \( \| \mu \circ u_\nu \|_{L^\infty(\Omega)} < \delta \), for \( \nu \) large enough. Using (32)(33), Lemma 31 implies that
\[
\sup_{\nu} R^2_{\nu} \| \mu \circ u_\nu \|_{L^p(Q)} < \infty.
\]

Estimate (31) follows from this and the second vortex equation. This proves Claim 1.

Using Claim 1, Theorem 41 (Uhlenbeck compactness) below implies that there exist an infinite subset \( I^1 \subseteq \mathbb{N} \) and gauge transformations \( g_\nu^1 \in W^{2,p}(X^1, G) \), for \( \nu \in I^1 \), such that \( X^1 \subseteq \Omega_\nu \), for every \( \nu \in I^1 \), and the sequence \( A^1_\nu := (g^1_\nu)^*(A_\nu^1 \vert X^1) \) converges to some \( W^{1,p} \)-connection \( A^1 \) over \( X^1 \), weakly in \( W^{1,p} \) on \( X^1 \). By the Kondrachov compactness
by the compact set $GK$, we may assume w.l.o.g. (without loss of generality) that $K$ is $G$-invariant. Hence passing to the subsequence $(\nu_i)_i$, we may assume w.l.o.g. that $A_{\nu}$ converges to $A$, weakly in $W^{1,p}$ and in $C^0$ on every compact subset of $\mathbb{R}^2 \setminus Z$.

**Claim 2.** The hypotheses of Proposition 42 with $k = 1$ are satisfied.

**Proof of Claim.** Let $\Omega \subset \mathbb{R}^2 \setminus Z$ be an open subset with compact closure, and $\nu_0 \in \mathbb{N}$ be such that $\Omega \subseteq \Omega_{\nu_0}$. Since the sequence $(A_{\nu})$ converges to $A$, weakly in $W^{1,p}(\Omega)$, we have

$$
\sup_{\nu \geq \nu_0} \| A_{\nu} \|_{W^{1,p}(\Omega)} < \infty.
$$

**Condition (102)** is satisfied by the assumption (27). We check **condition (103):** We denote by $|\Omega|$ the area of $\Omega$ and choose a constant $C > 0$ such that $X_\xi(x) \leq C|\xi|$, for every $x \in K$ and $\xi \in \mathfrak{g}$. For $\nu \geq \nu_0$, we have

$$
\| du_{\nu} \|_{L^p(\Omega)} \leq \| d_{A_{\nu}} u_{\nu} \|_{L^p(\Omega)} + \| LA_{\nu} \|_{L^p(\Omega)}
\leq |\Omega|^{\frac{1}{p}} \| d_{A_{\nu}} u_{\nu} \|_{L^{\infty}(\Omega)} + C \| A_{\nu} \|_{L^p(\Omega)}.
$$

Here the second inequality uses the hypothesis (27). Combining this with (28) and (34), condition (103) follows.

**Condition (104)** follows from the first vortex equation, (34), (103), and hypothesis (27). This proves Claim 2. □
By Claim 2 we may apply Proposition 42 to conclude that, passing to some subsequence, \( u_\nu \) converges to some map \( u \in W^{2,p}(\mathbb{R}^2 \setminus Z) \), weakly in \( W^{2,p} \) and in \( C^1 \) on every compact subset of \( \mathbb{R}^2 \setminus Z \). The pair \( w := (A,u) \) solves the first vortex equation. Furthermore, multiplying the second \( R_\nu \)-vortex equation with \( R^{-2}_\nu \), it follows that \( \mu \circ u = 0 \). This means that \( w \) is an \( \infty \)-vortex. By Proposition 45 below the map \( G_u : \mathbb{R}^2 \setminus Z \to \mathcal{M} \) is \( \bar{J} \)-holomorphic. Hence it is smooth. It follows that there exists a gauge transformation \( g \in W^{2,p}(\mathbb{R}^2 \setminus Z,G) \) such that \( g^* w \) is smooth. (We obtain such a \( g \) from a smooth lift of the map \( Gu \) to a map \( \mathbb{R}^2 \setminus Z \to \mu^{-1}(0) \). Such a lift exists, since by hypothesis, \( G \) is connected.) Regauging \( A_\nu \) by \( g \), statement (ii) follows. This completes the proof of Proposition 19. \( \square \)

**Remark.** One can try to circumvent the patching argument for the gauge transformations in this proof by choosing an extension \( \tilde{g}_\nu^i \) of \( g_\nu^i \) to \( \mathbb{R}^2 \setminus Z \), and defining \( g_\nu := \tilde{g}_\nu \). However, the sequence \( (g_\nu) \) does not have the required properties, since \( g^*_\nu w_\nu \) does not necessarily converge on compact subsets of \( \mathbb{R}^2 \setminus Z \). The reason is that for \( j > i \) the transformation \( g_\nu^j \) does in general not restrict to \( g_\nu^i \) on \( X^i \).

**Remark.** It is not clear if in the case \( R_0 = \infty \) the \( g_\nu^i \)'s can be chosen in such a way that \( g^*_\nu w_\nu \) converges in \( C^\infty \) on every compact subset of \( \mathbb{R}^2 \setminus Z \). To prove this, a possible approach is to fix an open subset of \( \mathbb{R}^2 \) with smooth boundary and compact closure, which is contained in \( \mathbb{R}^2 \setminus Z \). We can now try mimic the proof of [CGMS] Theorem 3.2]. In Step 3 of that proof the first and second vortex equations (and relative Coulomb gauge) are used iteratively in an alternating way. This iteration fails in our setting, because of the factor \( R^2_\nu \) in the second vortex equations, which converges to \( \infty \) by assumption. \( \square \)

The next ingredient of the proof of Proposition 18 is the following. Recall the definition (26) of \( E_{\text{min}} \). The next result shows that if the energy densities of a sequence of rescaled vortices are not uniformly bounded on some compact subset \( Q \), then at least the energy \( E_{\text{min}} \) is lost at some point in \( Q \).

**Proposition 20 (Quantization of energy loss).** Assume that \( (M,\omega) \) is aspherical. Let \( \Omega \subseteq \mathbb{R}^2 \) be an open subset, \( 0 < R_\nu < \infty \) a sequence such that \( \inf_\nu R_\nu > 0 \), and \( w_\nu \in \tilde{W}^{p}_0(\Omega) \) an \( R_\nu \)-vortex, for \( \nu \in \mathbb{N} \). Assume that there exists a compact subset \( K \subseteq M \) such that \( u_\nu(\Omega) \subseteq K \) for
every $\nu$ and that $\sup_{\nu} E_{w_\nu} < \infty$. Then the following conditions hold.

(i) For every compact subset $Q \subseteq \Omega$ we have
$$\sup_{\nu} R_{\nu}^{-2} \| e_{w_\nu} \|_{C^0(Q)} < \infty.$$

(ii) If there exists a compact subset $Q \subseteq \Omega$ such that $\sup_{\nu} \| e_{w_\nu} \|_{C^0(Q)} = \infty$ then there exists $z_0 \in Q$ with the following property. For every $\varepsilon > 0$ so small that $B_\varepsilon(z_0) \subseteq \Omega$ we have
$$\limsup_{\nu \to \infty} E_{w_\nu}(B_\varepsilon(z_0)) \geq E_{\min}.$$

The proof of Proposition 20 is built on a bubbling argument, as in step 5 in the proof of Theorem A in [GS]. The idea is that under the assumption of (i) we may construct either a $\bar{J}$-holomorphic sphere in $\overline{M}$ or a vortex over $\mathbb{R}^2$, by rescaling the sequence $w_\nu$ in a “hard way”. This means that after rescaling the energy densities are bounded. We need the following two lemmata.

**Lemma 21** (Hofer). Let $(X,d)$ be a metric space, $f : X \to [0,\infty)$ a continuous function, $x \in X$, and $\delta > 0$. Assume that the closed ball $\overline{B}_{2\delta}(x)$ is complete. Then there exists $\xi \in X$ and a number $0 < \varepsilon \leq \delta$ such that
$$d(x,\xi) < 2\delta, \quad \sup_{B_{\varepsilon}(\xi)} f \leq 2f(\xi), \quad \varepsilon f(\xi) \geq \delta f(x).$$

Proof. See [MS, Lemma 4.6.4].

The next lemma ensures that for a suitably convergent sequence of rescaled vortices in the limit $\nu \to \infty$ no energy gets lost on any compact set. Apart from Proposition 20 it will also be used in the proof of Propositions 18 and 24 and Theorem 1.

**Lemma 22** (Convergence of energy densities). Let $(\Sigma, \omega_\Sigma, j)$ be a surface without boundary, equipped with an area form and a compatible complex structure, $R_\nu \in [0,\infty)$, $\nu \in \mathbb{N}$, a sequence of numbers that converges to some $R_0 \in [0,\infty]$, and for $\nu \in \mathbb{N}_0$ let $w_\nu := (A_\nu, u_\nu) \in \tilde{W}_0^p(\Sigma)$ an $R_\nu$-vortex. Assume that on every compact subset of $\Sigma$, $A_\nu$ converges to $A_0$ in $C^0$ and $u_\mu$ converges to $u_0$ in $C^1$. Then we have
$$e_{w_\nu} \to e_{w_0}$$
in $C^0$ on every compact subset of $\Sigma$. 
Proof of Lemma 22. In the case $R_0 < \infty$ the statement of the lemma is a consequence of equality (23).

Consider the case $R_0 = \infty$. It follows from our standing hypothesis (H) that there exists a constant $\delta > 0$ such that $G$ acts freely on

$$K := \{ x \in M \mid |\mu(x)| \leq \delta \}.$$  

Properness of $\mu$ implies that $K$ is compact.

Let $Q \subseteq \Sigma$ be a compact subset. The convergence of $u_\nu$ and the fact $\mu \circ u_0 = 0$ imply that for $\nu$ large enough, we have $u_\nu(Q) \subseteq K$. Furthermore, our hypotheses about the convergence of $A_\nu$ and $u_\nu$ imply that $\sup_\nu \|d_{A_\nu}u_\nu\|_{C^0(Q)} < \infty$. Finally, since $K$ is compact and $G$ acts freely on it, we have

$$\sup_{\{ |x| \mid L_x \xi \mid \mid x \in K, \; 0 \neq \xi \in \mathfrak{g} \}} < \infty.$$  

Therefore, we may apply Lemma 31 below, to conclude that

$$\sup_Q R_\nu^{-2/p} |\mu \circ u_\nu| < \infty.$$  

Since $p > 2, R_\nu \to \infty$, and $e_\nu^\infty = \frac{1}{2} |d_{A_0}u_0|^2$, the convergence (37) follows. This completes the proof of Lemma 22. □

In the proof of Proposition 20 we will also use the following.

Remark 23. Let $(A, u) \in \widetilde{W}_0^p(\mathbb{R}^2)$ be an $\infty$-vortex, i.e., a solution of the equations $\bar{\partial}_{J,A}(u) = 0$ and $\mu \circ u = 0$. By Proposition 45 below the map $Gu : \mathbb{R}^2 \to \overline{M} = \mu^{-1}(0)/G$ is $\bar{J}$-holomorphic, and $E^\infty(A, u) = E(\bar{u})$. If this energy is finite, then by removal of singularities the map $\bar{u}$ extends to a $\bar{J}$-holomorphic map $\bar{u} : S^2 \to \overline{M}$. (See for example [MS, Theorem 4.1.2].) It follows that $E^\infty(w) \geq E_{\min}$, provided that $E^\infty(w) > 0$. □

Proof of Proposition 20. We write $(A_\nu, u_\nu) := w_\nu$. Consider the function

$$f_\nu := |d_{A_\nu}u_\nu| + R_\nu |\mu \circ u_\nu| : \Omega \to \mathbb{R}.$$  

Claim 1. Suppose that the hypotheses of Proposition 20 are satisfied and that there exists a sequence $z_\nu \in \Omega$ that converges to some $z_0 \in \Omega$, such that $f_\nu(z_\nu) \to \infty$. Then there exists

$$0 < r_0 \leq \limsup_{\nu \to \infty} \frac{R_\nu}{f_\nu(z_\nu)} \leq \infty.$$  

(38)
and an \( r_0 \)-vortex \( w_0 \in \widehat{W}_0(\mathbb{R}^2) \), such that

\[
0 < E^{r_0}(w_0) \leq \limsup_{\nu \to \infty} E^{R_\nu}(w_\nu, B_\varepsilon(z_0)),
\]

for every \( \varepsilon > 0 \) so small that \( B_\varepsilon(z_0) \subseteq \Omega \).

**Proof of Claim 7.** **Construction of** \( r_0 \): We define \( \delta_\nu := f_\nu(z_\nu)^{-\frac{1}{2}} \). For \( \nu \) large enough we have \( \hat{B}_{2\delta_\nu}(z_\nu) \subseteq \Omega \). We pass to some subsequence such that this holds for every \( \nu \). By Lemma 21 applied with \((f, x, \delta) := (f_\nu, z_\nu, \delta_\nu)\), there exist \( \zeta_\nu \in B_{2\delta_\nu}(z_0) \) and \( \varepsilon_\nu \leq \delta_\nu \), such that

\[
|\zeta_\nu - z_\nu| < 2\delta_\nu,
\]

\[
\sup_{B_{\varepsilon_\nu}(\zeta_\nu)} f_\nu \leq 2f_\nu(\zeta_\nu),
\]

\[
\varepsilon_\nu f_\nu(\zeta_\nu) \geq f_\nu(z_\nu)^{\frac{1}{2}}.
\]

Since by assumption \( f_\nu(z_\nu) \to \infty \), it follows from (40) that the sequence \( \zeta_\nu \) converges to \( z_0 \). We define

\[
c_\nu := f_\nu(\zeta_\nu), \quad \widetilde{\Omega}_\nu := \{c_\nu(z - \zeta_\nu) \mid z \in \Omega\},
\]

\[
\varphi_\nu : \widetilde{\Omega}_\nu \to \Omega, \quad \varphi_\nu(\tilde{z}) := c_\nu^{-1}\tilde{z} + \zeta_\nu,
\]

\[
\tilde{w}_\nu := \varphi_\nu^\ast w_\nu = (\varphi_\nu^\ast A_\nu, u_\nu \circ \varphi_\nu), \quad \tilde{R}_\nu := c_\nu^{-1}R_\nu.
\]

Note that \( \tilde{w}_\nu \) is an \( \tilde{R}_\nu \)-vortex. Passing to some subsequence we may assume that \( \tilde{R}_\nu \) converges to some \( r_0 \in [0, \infty] \). Since \( \varepsilon_\nu \leq \delta_\nu = f_\nu(z_\nu)^{-\frac{1}{2}} \) it follows from (42) that \( f_\nu(z_\nu) \leq f_\nu(\zeta_\nu) \). It follows that the second inequality in (38) holds for the original sequence.

**Construction of** \( w_0 \): We check the conditions of Proposition 19 with \((Z, \Omega_\nu) := (\emptyset, \bigcup_{\nu = 1, \ldots, \nu} \Omega_\nu)\) and \( R_\nu, w_\nu \) replaced by \( \tilde{R}_\nu, \tilde{w}_\nu \): **Condition (27)** is satisfied by hypothesis.

We check **condition (28)**: A direct calculation involving (41) shows that

\[
|d_{\tilde{A}_\nu} \tilde{u}_\nu| + \tilde{R}_\nu|\mu \circ \tilde{u}_\nu| = c_\nu^{-1}f_\nu \circ \varphi_\nu \leq 2, \quad \text{on } B_{\varepsilon_\nu c_\nu}(0).
\]

It follows from (42) and the fact \( f_\nu(z_\nu) \to \infty \), that \( \varepsilon_\nu c_\nu \to \infty \). Combining this with (43), condition (28) follows for every compact subset \( Q \subseteq \mathbb{R}^2 \).

Therefore, applying Proposition 19 there exists an \( r_0 \)-vortex \( w_0 = (A_0, u_0) \in \mathcal{W}_0(\mathbb{R}^2) \) and, passing to some subsequence, there exist gauge transformations \( g_\nu \in W^{2,p}(\mathbb{R}^2, G) \) with the following property. For every compact subset \( Q \subseteq \mathbb{R}^2 \), \( g_\nu^{-1}A_\nu \) converges to \( A_0 \) in \( C^0 \) on \( Q \), and \( g_\nu^{-1}u_\nu \) converges to \( u_0 \) in \( C^1 \) on \( Q \).
We prove the first inequality in (39): By Lemma 22 we have

\[ e^{\tilde{R}_v} = e^{\tilde{R}_\nu} e^{R_v} \rightarrow e^{R_0}, \]

in \( C^0(Q) \) for every compact subset \( Q \subseteq \mathbb{R}^2 \). Since \( e^{\tilde{R}_\nu}(0) = e^{-2R_\nu}(\zeta_\nu) \geq 1/2 \), it follows that \( e^{R_0}(0) \geq 1/2 \). This implies that \( E^{R_0}(w_0) > 0 \). This proves the first inequality in (39).

We prove the second inequality in (39): Let \( \varepsilon > 0 \) be so small that \( B_\varepsilon(z_0) \subseteq \Omega \), and \( \delta > 0 \). It follows from (44) that \( E^{R_\nu}(w_0) \leq \sup_{\nu} E^{R_\nu}(w_\nu) \). By hypothesis this supremum is finite. Hence there exists \( R > 0 \) such that \( E^{R_\nu}(w_0, \mathbb{R}^2 \setminus B_R) < \delta \). Since \( E^{R_\nu}(w_\nu, B_{c^{-1}_\nu R}(\zeta_\nu)) = E^{R_\nu}(\tilde{w}_\nu, B_R) \), the convergence (44) implies that

\[ \lim_{\nu \rightarrow \infty} E^{R_\nu}(w_\nu, B_{c^{-1}_\nu R}(\zeta_\nu)) = E^{R_0}(w_0, B_R) > E^{R_0}(w_0) - \delta. \]

On the other hand, since \( c_\nu \rightarrow \infty \) and \( \zeta_\nu \rightarrow z_0 \), for \( \nu \) large enough the ball \( B_{c^{-1}_\nu R}(\zeta_\nu) \) is contained in \( B_\varepsilon(z_0) \). Combining this with (45), we obtain

\[ \lim_{\nu \rightarrow \infty} \sup_{\nu} E^{R_\nu}(w_\nu, B_\varepsilon(z_0)) \geq E^{R_0}(w_0) - \delta. \]

Since this holds for every \( \delta > 0 \), the second inequality in (39) (for the original sequence) follows.

It remains to prove the first inequality in (38), i.e., that \( r_0 > 0 \). Assume by contradiction that \( r_0 = 0 \). For a map \( u \in C^\infty(\mathbb{R}^2, M) \) we denote by

\[ E(u) := \frac{1}{2} \int_{\mathbb{R}^2} |du|^2 \]

its (Dirichlet-)energy. (Here the norm is taken with respect to the metric \( \omega^*, J \) on \( M \).) By the second R-vortex equation with \( R := 0 \) we have \( F_{A_0} = 0 \). Therefore, by Proposition 14 there exists \( h \in C^\infty(\mathbb{R}^2, \mathbb{C}) \) such that \( h^* A_0 = 0 \). By the first vortex equation the map \( u'_0 := h^{-1}u_0 : \mathbb{R}^2 = \mathbb{C} \rightarrow M \) is \( J \)-holomorphic. Let \( \varepsilon > 0 \) be such that \( B_\varepsilon(z_0) \subseteq \Omega \).

Using the second inequality in (39), we have

\[ E(u'_0) = E^0(w_0) \leq \lim_{\nu \rightarrow \infty} \sup_{\nu} E^{R_\nu}(w_\nu, B_\varepsilon(z_0)). \]

Combining this with the hypothesis \( \sup_{\nu} E^{R_\nu}(w_\nu, \Omega) < \infty \), it follows that \( E(u'_0) < \infty \). Hence by removal of singularities (see e.g. [MS Theorem 4.1.2]), it follows that \( u'_0 \) extends to a smooth \( J \)-holomorphic map \( v : S^2 \rightarrow M \). By the first inequality in (39) we have \( \int_{S^2} v^* \omega = E(v) = E^0(w_0) > 0 \). This contradicts asphericity of \( (M, \omega) \). Hence \( r_0 \) must be positive. This concludes the proof of Claim 1. \( \square \)
Statement (i) of Proposition 20 follows from Claim 1, considering a sequence $z_\nu \in Q$, such that $f_\nu(z_\nu) = \|f_\nu\|_{C^0(Q)}$, and using (38).

We prove statement (ii). Assume that there exists a compact subset $Q \subseteq \Omega$ such that $\sup_\nu ||e_{R_\nu}^w||_{C^0(Q)} = \infty$. Let $z_\nu \in Q$ be such that $f_\nu(z_\nu) \to \infty$. We choose a pair $(r_0, w_0)$ as in Claim 1. Using the first inequality in (39) and Remark 23 (in the case $r_0 = \infty$), we have $E^{r_0}(w_0) \geq E_{\min}$. Combining this with the second inequality in (39), inequality (36) follows. This proves (ii) and concludes the proof of Proposition 20.

We are now ready for the proof of Proposition 18.

Proof of Proposition 18. We abbreviate $e_\nu := e_{R_\nu}^w$.

Claim 1. For every $\ell \in \mathbb{N} \cup \{0\}$ there exists a finite subset $Z_\ell \subseteq \mathbb{R}^2$ such that the following holds. If $R_0 < \infty$ then we have $Z_\ell = \emptyset$. Furthermore, if $|Z_\ell| < \ell$ then we have

$$\sup_{\nu \in \mathbb{N}} \left\{ ||e_\nu||_{C^0(Q)} \mid Q \subseteq B_{r_\nu} \right\} < \infty,$$

for every compact subset $Q \subseteq \mathbb{R}^2 \setminus Z_\ell$. Moreover, for every $z_0 \in Z_\ell$ and every $\varepsilon > 0$ the inequality (36) holds.

Proof of Claim 1. For $\ell = 0$ the assertion holds with $Z_0 := \emptyset$. We prove by induction that it holds for every $\ell \geq 1$. Fix $\ell \geq 1$. By induction hypothesis there exists a finite subset $Z_{\ell-1} \subseteq \mathbb{R}^2$ such that the assertion with $\ell$ replaced by $\ell - 1$ holds. If (46) is satisfied for every compact subset $Q \subseteq \mathbb{R}^2 \setminus Z_{\ell-1}$, then the statement for $\ell$ holds with $Z_\ell := Z_{\ell-1}$.

Hence assume that there exists a compact subset $Q \subseteq \mathbb{R}^2 \setminus Z_{\ell-1}$, such that (46) does not hold. It follows from the induction hypothesis that

$$|Z_{\ell-1}| \geq \ell - 1.$$

Applying Proposition 20 by statement (ii) of that proposition there exists a point $z_0 \in Q$ such that inequality (36) holds, for every $\varepsilon > 0$. We set $Z_\ell := Z_{\ell-1} \cup \{z_0\}$.

It follows from the fact that (46) does not hold and condition (i) of Proposition 20 that $R_0 = \lim_{\nu \to \infty} R_\nu = \infty$. Furthermore, since $z_0 \in Q \subseteq \mathbb{R}^2 \setminus Z_{\ell-1}$, (47) implies that $|Z_\ell| \geq \ell$. It follows that the statement of Claim 1 for $\ell$ is satisfied. By induction, Claim 1 follows.

We fix an integer $\ell > \sup_\nu E^{R_\nu}(w_\nu, B_{r_\nu})/E_{\min}$ and a finite subset $Z := Z_\ell \subseteq \mathbb{R}^2$ that satisfies the conditions of Claim 1. It follows from the inequality (36) that $\ell > |Z|$. Hence by the statement of Claim 1
the hypothesis (28) of Proposition 19 is satisfied with $\Omega := B_{r_1} \setminus Z$. Applying that result and passing to some subsequence, there exist an $R_0$-vortex $w_0 \in \tilde{W}_0(B_{2r_1} \setminus Z)$ and gauge transformations $g_\nu \in W^2_{\text{loc}}(\mathbb{R}^2 \setminus Z, G)$, such that the statements (iii) of Proposition 18 are satisfied. (Here we use that $Z = \emptyset$ if $R_0 < \infty$.)

We prove statement (iii). Passing to some “diagonal” subsequence, the limit $\lim_{\nu \to \infty} E^{R_\nu}(w_\nu, B_{1/i}(z))$ exists, for every $i \in \mathbb{N}$ and $z \in Z$. Let now $z \in Z$ and $\varepsilon > 0$. We choose $i \in \mathbb{N}$ bigger than $\varepsilon^{-1}$. For $0 < r < R$ we denote

$$A(z, r, R) := \overline{B_R(z)} \setminus B_r(z).$$

By Lemma 22 the limit $\lim_{\nu \to \infty} E^{R_\nu}(w_\nu, A(z, 1/i, \varepsilon))$ exists and equals $E^{R_0}(w_0, A(z, 1/i, \varepsilon))$. It follows that the limit $E_\varepsilon(\varepsilon) := \lim_{\nu \to \infty} E^{R_\nu}(w_\nu, B_\varepsilon(z))$ exists. Inequality (36) implies that $E_\varepsilon(\varepsilon) \geq E_{\text{min}}$. Since $E^{R_0}(w_0, A(z, 1/i, \varepsilon))$ depends continuously on $\varepsilon$, the same holds for $E_\varepsilon(\varepsilon)$. This proves statement (iii) and completes the proof of Proposition 18. □

**Remark.** In the above proof the set of bubbling points $Z$ is constructed by “terminating induction”. Intuitively, this is induction over the number of bubbling points. The “auxiliary index” $\ell$ in Claim 1 is needed to make this idea precise. Inequality (36) ensures that the “induction stops”.

### 4. Soft rescaling

The next proposition will be used inductively in the proof of the main result to find the next bubble in the bubbling tree, at a bubbling point of a given sequence of rescaled vortices. It is an adaption of [MS, Proposition 4.7.1.] to vortices.

**Proposition 24** (Soft rescaling). Assume that $(M, \omega)$ is aspherical. Let $r > 0$, $z_0 \in \mathbb{R}^2$, $R_\nu > 0$ a sequence that converges to $\infty$, $p > 2$, and for every $\nu \in \mathbb{N}$ let $w_\nu := (A_\nu, u_\nu) \in \tilde{W}_0^p(B_{r_1}(z_0))$ be an $R_\nu$-vortex, such that the following conditions are satisfied.

(a) There exists a compact subset $K \subseteq M$ such that $u_\nu(B_{r_1}(z_0)) \subseteq K$ for every $\nu$.

(b) For every $0 < \varepsilon \leq r$ the limit $E(\varepsilon) := \lim_{\nu \to \infty} E^{R_\nu}(w_\nu, B_\varepsilon(z_0))$ exists and $E_{\text{min}} \leq E(\varepsilon) < \infty$. Furthermore, the function

$$E(\varepsilon) \quad (0, r] \ni \varepsilon \mapsto E(\varepsilon) \in \mathbb{R}$$

is continuous.
Then there exist \( R_0 \in \{1, \infty\} \), a finite subset \( Z \subseteq \mathbb{R}^2 \), and an \( R_0 \)-vortex \( w_0 := (A_0, u_0) \in \tilde{W}_0(\mathbb{R}^2 \setminus Z) \), and passing to some subsequence, there exist sequences \( \varepsilon_\nu > 0 \), \( z_\nu \in \mathbb{R}^2 \), and \( g_\nu \in W^{2,p}_{\text{loc}}(\mathbb{R}^2 \setminus Z, G) \), such that, defining

\[
\varphi_\nu : \mathbb{R}^2 \to \mathbb{R}^2, \quad \varphi_\nu(z) := \varepsilon_\nu z + z_\nu,
\]

the following conditions hold.

(i) If \( R_0 = 1 \) then \( Z = \emptyset \) and \( E(w_0) > 0 \). If \( R_0 = \infty \) and \( E^\infty(w_0) = 0 \) then \( |Z| \geq 2 \).

(ii) The sequence \( z_\nu \) converges to \( z_0 \). Furthermore, if \( R_0 = 1 \) then \( \varepsilon_\nu = R_\nu^{-1} \) for every \( \nu \), and if \( R_0 = \infty \) then \( \varepsilon_\nu \) converges to 0 and \( \varepsilon_\nu R_\nu \) converges to \( \infty \).

(iii) If \( R_0 = 1 \) then the sequence \( g_\nu^* \varphi_\nu^* w_\nu \) converges to \( w_0 \) in \( C^\infty \) on every compact subset of \( \mathbb{R}^2 \setminus Z \). Furthermore, if \( R_0 = \infty \) then on every compact subset of \( \mathbb{R}^2 \setminus Z \), the sequence \( g_\nu^* A_\nu \) converges to \( A_0 \) in \( C^0 \), and the sequence \( g_\nu^{-1}(u_\nu \circ \varphi_\nu) \) converges to \( u_0 \) in \( C^1 \).

(iv) Fix \( z \in Z \) and a number \( \varepsilon_0 > 0 \) such that \( B_{\varepsilon_0}(z) \cap Z = \{z\} \). Then for every \( 0 < \varepsilon < \varepsilon_0 \) the limit

\[
E_z(\varepsilon) := \lim_{\nu \to \infty} E_{\varepsilon_\nu R_\nu}(\varphi_\nu^* w_\nu, B_\varepsilon(z))
\]

exists and \( E_{\min} \leq E_z(\varepsilon) < \infty \). Furthermore, the function \( (0, \varepsilon_0) \ni \varepsilon \mapsto E_z(\varepsilon) \in \mathbb{R} \) is continuous.

(v) We have

\[
\lim_{R \to \infty} \limsup_{\nu \to \infty} E_{R_\nu}(w_\nu, B_{R^{-1}(z_0)} \setminus B_R(\varepsilon_\nu)) = 0.
\]

Remarks. In the proof of Theorem 1, condition (i) will guarantee that the new bubble is stable. Condition (iv) will be used to prove that the construction of the bubbling tree terminates after finitely many steps. Finally, condition (v) will ensure that no energy is lost between the old and new bubble.

Note that in condition (iii) the pullback \( \varphi_\nu^* w_\nu \) is defined over the set \( \varphi_\nu^{-1}(B_r(z_0)) \).

The proof of Proposition 24 is given on page 39. It is based on the following result, which states that the energy of a vortex on an annulus is concentrated near the ends, provided that it is small enough. For \( 0 \leq r, R \leq \infty \) we denote the open annulus around 0 with radii \( r, R \) by

\[
A(r, R) := B_R \setminus B_r.
\]
Note that $A(r, \infty) = \mathbb{R}^2 \setminus \bar{B}_r$, and $A(r, R) = \emptyset$ in the case $r \geq R$. We define

$$d : \bigcup_M M \times M \to [0, \infty]$$

to be the distance function induced by the Riemannian metric $\omega(\cdot, J \cdot)$. (If $M$ is disconnected then $d$ attains the value $\infty$.) We define

$$(50) \quad \bar{d} : \bigcup M/G \times M/G \to [0, \infty], \quad \bar{d}(\overline{x}, \overline{y}) := \min_{x \in \overline{x}, y \in \overline{y}} d(x, y).$$

By Lemma [50] below this is a distance function on $M/G$ which induces the quotient topology.

**Proposition 25** (Energy concentration near ends). There exists a constant $r_0 > 0$ such that for every compact subset $K \subseteq M$ and every $\varepsilon > 0$ there exists a constant $E_0$, such that the following holds. Assume that $r_0 \leq r, R \leq \infty$, $p > 2$, and $w := (u, A) \in \tilde{W}_0^p (A(r, R))$ is a vortex (with respect to $(\omega_0, i)$), such that $u(A(r, R)) \subseteq K$, $E(w) = E(w, A(r, R)) \leq E_0$.

Then we have

$$E(w, A(ar, a^{-1} R)) \leq 4a^{-2+\varepsilon}E(w), \quad \forall a \geq 2,$$

$$\sup_{z, z' \in A(ar, a^{-1} R)} \bar{d}(Gu(z), Gu(z')) \leq 100a^{-1+\varepsilon} \sqrt{E(w)}, \quad \forall a \geq 4.$$

(Here $Gx \in M/G$ denotes the orbit of a point $x \in M$.)

Note that in the case $a > \sqrt{R/r}$ we have $A(ar, a^{-1} R) = \emptyset$, and hence the statement of the proposition is void. The proof of this is modelled on the proof of [Z12] Theorem 1.3], which in turn is based on the proof of [GS] Proposition 11.1]. It is based on an isoperimetric inequality for the invariant symplectic action functional (Theorem [39] in Appendix [B]). It also relies on an identity relating the energy of a vortex over a compact cylinder with the actions of its end-loops (Proposition [40] below). The proof of (53) also uses the following remark.

**Remark 26.** Let $(M, \langle \cdot, \cdot \rangle_M)$ be a Riemannian manifold, $G$ a compact Lie group that acts on $M$ by isometries, $P$ a principal $G$-bundle over $[0, 1]$, $A \in \mathcal{A}(P)$ a connection, and $u \in C^\infty_G(P, M)$ a map. We define

$$\ell(A, u) := \int_0^1 |d_A u| dt,$$
where the norm is taken with respect to the standard metric on \([0,1]\) and \(\langle \cdot, \cdot \rangle_M\). Furthermore, we define \(\bar{u} : [0,1] \to M/G\) by \(\bar{u}(t) := Gu(p)\), where \(p \in P\) is any point over \(t\). We denote by \(d\) the distance function induced by \(\langle \cdot, \cdot \rangle_M\), and define \(\bar{d}\) as in (50). Then for every pair of points \(\bar{x}_0, \bar{x}_1 \in M/G\), we have
\[
\bar{d}(\bar{x}_0, \bar{x}_1) \leq \inf \{ \ell(A,u) \mid (P,A,u) \text{ as above: } \bar{u}(i) = \bar{x}_i, i = 0,1 \}.
\]
This follows from a straightforward argument. □

**Proof of Proposition 25.** For every subset \(X \subseteq M\) we define
\[
m_X := \inf \{|L_x\xi| \mid x \in X, \xi \in g : |\xi| = 1\},
\]
where the norms are with respect to \(\omega(\cdot,J\cdot)\) and \(\langle \cdot, \cdot \rangle_g\). We set
\[
r_0 := m_{\mu^{-1}(0)}^{-1}.
\]
Let \(K \subseteq M\) be a compact subset and \(\varepsilon > 0\). Replacing \(K\) be \(GK\), we may assume w.l.o.g. that \(K\) is \(G\)-invariant. An elementary argument using our standing hypothesis (H) shows that there exists a number \(\delta_0 > 0\) such that \(G\) acts freely on \(K' := \mu^{-1}(\bar{B}_{\delta_0})\), and
\[
m_{K'} \geq \sqrt{1 - \varepsilon/2m_{\mu^{-1}(0)}}.
\]
We choose a constant \(\delta\) as in Theorem 39, corresponding to \(\langle \cdot, \cdot \rangle_M := \omega(\cdot,J\cdot),K',c := \frac{1}{2-\varepsilon}\). Shrinking \(\delta\) we may assume that it satisfies the condition of Proposition 40 (Energy action identity) for \(K'\). We choose a constant \(\tilde{E}_0 > 0\) as in Lemma 28 below (called \(E_0\) there), corresponding to \(K\). We define
\[
E_0 := \min \left\{ \tilde{E}_0, \pi r_0^2 \delta_0^2, \frac{\delta^2}{128\pi} \right\}.
\]
Assume that \(r,R,p,w\) are as in the hypothesis. Without loss of generality, we may assume that \(r < R\).

Consider first the case \(R < \infty\), and assume that \(w\) extends to a smooth vortex on the compact annulus of radii \(r\) and \(R\). We show that inequality (52) holds. We define the function
\[
E : [0,\infty), \quad E(s) := E(w,A(re^s,Re^{-s})).
\]
Claim 1. For every \(s \in [\log 2, \log(R/r)/2]\) we have
\[
\frac{d}{ds} E(s) \leq -(2-\varepsilon)E(s).
\]
Proof of Claim\footnote{4}. Using the fact \( r \geq r_0 \) and \( (51, 56) \), it follows from Lemma \footnote{28} below (with “\( r \)”:= \( |z|/2 \)) that
\begin{equation}
(59) \quad e_w(z) \leq \min \left\{ \delta_0, \frac{\delta^2}{4\pi^2|z|^2} \right\}, \quad \forall z \in A(2r, R/2).
\end{equation}

We define
\[ \Sigma_s := (s + \log r, -s + \log R) \times S^1, \forall s \in \mathbb{R}, \]
\[ \varphi : \Sigma_0 \to \mathbb{R}^2 = \mathbb{C}, \varphi(z) := e^z, \quad \bar{w} := (\bar{A}, \bar{u}) := \varphi^*w. \]
(Here we identify \( \Sigma_0 \cong \mathbb{C}/\sim \), where \( z \sim z + 2\pi in \), for every \( n \in \mathbb{Z} \).) Let \( s_0 \in \left[ \log(2r), \log(R/2) \right] \). Combining \( (59) \) with the fact \( |\mu \circ u| \leq \sqrt{e_w} \) and Remark \footnote{26} it follows that
\begin{equation}
(60) \quad \bar{u}(s_0, t) \in K' = \mu^{-1}(\bar{B}_0), \forall t \in S^1, \quad \bar{E}(G\bar{u}(s_0, \cdot)) \leq \delta.
\end{equation}

Hence the hypotheses of Theorem \footnote{39} are satisfied with \( K' \) replaced by \( K' \) and \( c := 1/(2 - \varepsilon) \). By the statement of that result the loop \((\bar{u}(s_0, \cdot))\) is admissible, and defining \( \iota_{s_0} : S^1 \to \Sigma_0 \) by \( \iota_{s_0}(t) := (s_0, t) \), we have
\begin{equation}
(61) \quad |A(\iota_{s_0}^*\bar{w})| \leq \frac{1}{2 - \varepsilon} \|\iota_{s_0}^*d_A\bar{u}\|^2 + \frac{1}{2m_K^2}\|\mu \circ \bar{u} \circ \iota_{s_0}\|^2.
\end{equation}

Here \( A \) denotes the invariant symplectic action, as defined in appendix \footnote{13}. Furthermore, the \( L^2 \)-norms are with respect to the standard metric on \( S^1 \cong \mathbb{R}/(2\pi \mathbb{Z}) \), the metric \( \omega(\cdot, J\cdot) \) on \( M \), and the operator norm \( |\cdot|_0 : g^* \to \mathbb{R} \), induced by \( \langle \cdot, \cdot \rangle_g \).

By \( (54, 55) \) and the fact \( 2r \leq e^{s_0} \), we have
\begin{equation}
(62) \quad \frac{1}{2 - \varepsilon} \|\iota_{s_0}^*d_A\bar{u}\|^2 + \frac{1}{2m_K^2}\|\mu \circ \bar{u} \circ \iota_{s_0}\|^2 \leq \frac{1}{2 - \varepsilon} e^{2s_0}e_w(e^{s_0+i}), \quad \text{on } S^1.
\end{equation}

Here the norm \( |\cdot|_0 \) is with respect to the standard metric on \( S^1 \cong \mathbb{R}/(2\pi \mathbb{Z}) \), and we used the fact \( |\varphi|_{op} \leq |\varphi| \) for \( \varphi \in g^* \), where \( |\cdot| \) denotes the norm induced by \( \langle \cdot, \cdot \rangle_g \). We fix \( s \in \left[ \log(2), \log(R/r)/2 \right] \). Recalling \( (57) \), we have \( E(s) = \int_{s_0} e^{2s_0}e_w(e^{s_0+i}) ds \). Combining this with \( (61, 62) \), it follows that
\begin{equation}
(63) \quad -A(\iota_{s_0+\log R}^*\bar{w}) + A(\iota_{s+\log R}^*\bar{w}) \leq -\frac{1}{2 - \varepsilon} \frac{d}{ds} E(s).
\end{equation}

Using \( (60) \), the hypotheses of Proposition \footnote{40} are satisfied with \( K \) replaced by \( K' \). Applying that result, we have \( E(s) = -A(\iota_{s_0+\log R}^*\bar{w}) + A(\iota_{s+\log R}^*\bar{w}) \). Combining this with \( (63) \), inequality \( (58) \) follows. This proves Claim\footnote{4}. \qed
By Claim 1 the derivative of the function \( \log 2, \log(R/r)/2 \) \( \ni s \mapsto E(s)e^{(2-\varepsilon)s} \) is non-positive, and hence this function is non-increasing. Inequality (52) follows.

We prove (53). Let \( z \in A(4r, \sqrt{rR}) \). Using (51) and the fact \( E_0 \leq \tilde{E}_0 \), it follows from Lemma 28 (with "r" := \(|z|/2\)) that

\[
(64) \quad e_w(z) \leq \frac{32}{\pi|z|^2}E(w, B_{|z|/2}(z)).
\]

We define \( a := |z|/(2r) \). Then \( a \geq 2 \) and \( B_{|z|/2}(z) \) is contained in \( A(ar, a^{-1}R) \). Therefore, by (52) we have

\[
E(w, B_{|z|/2}(z)) \leq 16r^{2-\varepsilon}|z|^{-2+\varepsilon}E(w).
\]

Combining this with (64), the fact \(|d_Au(z)| \leq \sqrt{2e_w(z)}\), and the first vortex equation, it follows that

\[
(65) \quad |d_Au(z)v| \leq C\rho^{1-\varepsilon/2}|z|^{-2+\varepsilon/2}\sqrt{E(w)}|v|, \quad \forall z \in A(4r, \sqrt{rR}), v \in \mathbb{R}^2.
\]

where \( C := 2^{9/2}\pi^{-1/2} \). A similar argument shows that

\[
(66) \quad |d_Au(z)v| \leq CR^{-1+\varepsilon/2}|z|^{-\varepsilon/2}\sqrt{E(w)}|v|, \quad \forall z \in A(\sqrt{rR}, R/4).
\]

Let now \( a \geq 4 \) and \( z,z' \in A(ar, a^{-1}R) \). Assume that \( \varepsilon \leq 1 \). (This is no real restriction.) We define \( \gamma : [0, 1] \to \mathbb{R}^2 \) to be the radial path of constant speed, such that \( \gamma(0) = z \) and \( |\gamma(1)| = |z'| \). Furthermore, we choose an angular path \( \gamma' : [0, 1] \to \mathbb{R}^2 \) of constant speed, such that \( \gamma'(0) = \gamma(1), \gamma'(1) = z' \), and \( \gamma' \) has minimal length among such paths. (See Figure 4.)
Consider the “twisted length” of $\gamma^*(A, u)$, given by $\int_0^1 |d_A u \dot{\gamma}(t)| dt$. It follows from (65,66) and the fact $\varepsilon \leq 1$, that this length is bounded above by $4C_\varepsilon \sqrt{E(w)}a^{-1+\varepsilon/2}$. Similarly, it follows that the “twisted length” of $\gamma'^*(A, u)$ is bounded above by $C_\varepsilon \sqrt{E(w)}a^{-1+\varepsilon/2}$. Therefore, using Remark 26, inequality (53) with $\varepsilon$ replaced by $\varepsilon/2$ follows.

Assume now that $w$ is not smooth. By Theorem 32 below the restriction of $w$ to any compact cylinder contained in $A(r, R)$ is gauge equivalent to a smooth vortex. Hence the inequalities (52,53) follow from what we just proved, using the $G$-invariance of $K$.

Similarly, the case $R = \infty$ can be reduced to the case $R < \infty$. This completes the proof of Proposition 25.

Proof of Proposition 24. By hypothesis (b) the function $E$ as in (48) is well-defined. Since it is increasing and bounded below by $E_{\min}$, the limit

$$m_0 := \lim_{\varepsilon \to 0} E(\varepsilon)$$

exists and is bounded below by $E_{\min}$. We fix a compact subset $K \subseteq M$ as in hypothesis (21). We choose a constant $E_0 > 0$ as in Lemma 28 depending on $K$. We may assume w.l.o.g. that $z_0 = 0$.

Claim 1. We may assume w.l.o.g. that

$$\| e_{w,\nu} \|_{C^0(B_{\hat{r}})} = e_{w,\nu}(0).$$

Proof of Claim 1. Suppose that we have already proved the proposition under this additional assumption, and let $r, z_0 = 0, R_\nu, w_\nu$ be as in the hypotheses of the proposition. We choose $0 < \hat{r} \leq r/4$ so small that

$$E(4\hat{r}) = \lim_{\nu \to \infty} E_{R_\nu}(w_\nu, B_{4\hat{r}}) < m_0 + E_0.$$

For $\nu \in \mathbb{N}$ we choose $z_\nu \in B_{2\hat{r}}$ such that

$$e_{w,\nu}(z_\nu) = \| e_{w,\nu} \|_{C^0(B_{2\hat{r}})}.$$

Claim 2. The sequence $z_\nu$ converges to 0.

Proof of Claim 2. Recall that $A(r, R)$ denotes the open annulus of radii $r$ and $R$. Let $0 < \varepsilon \leq 2\hat{r}$. Inequality (69) implies that there exists $\nu(\varepsilon) \in \mathbb{N}$ such that

$$E_{R_\nu}(w_\nu, A(\varepsilon/2, 4\hat{r})) < E_0.$$
for every $\nu \geq \nu(\varepsilon)$. Hence it follows from Lemma 28 (Bound on energy density, using $\varepsilon \leq 2\hat{r}$) that

$$(71) \quad \epsilon_{w,\nu}^R(z) < \frac{32E_0}{\pi \varepsilon^2}, \quad \forall \nu \geq \nu(\varepsilon), \forall z \in A(\varepsilon, 2\hat{r}).$$

We define $\delta_0 := \min \{2\hat{r}, \varepsilon \sqrt{m_0/(64E_0)} \}$. Increasing $\nu(\varepsilon)$, we may assume that for every $\nu \geq \nu(\varepsilon)$, we have $E_{w,\nu}^R(B_{\delta_0}) > m_0/2$, and therefore

$$\|\epsilon_{w,\nu}^R\|_{C^0(B_{\delta_0})} > \frac{32E_0}{\pi \varepsilon^2}.$$ Combining this with (70,71) and the fact $\delta \leq 2\hat{r}$, it follows that $\tilde{z}_\nu \in B_{\varepsilon}$, for every $\nu \geq \nu(\varepsilon)$. This proves Claim 2.

By Claim 2 we may pass to some subsequence such that $|\tilde{z}_\nu| < \hat{r}$ for every $\nu$. We define

$$\psi_\nu : B_{\hat{r}} \to \mathbb{R}^2, \quad \psi_\nu(z) := z + z_\nu, \quad \tilde{w}_\nu := (\tilde{A}, \tilde{u}) := \psi_\nu^* w_\nu.$$ Then (68) with $w_\nu, r$ replaced by $\tilde{w}_\nu, \hat{r}$ is satisfied. By elementary arguments the hypotheses of Proposition 24 are satisfied with $(w_\nu, r, z_0)$ replaced by $(\tilde{w}_\nu, \hat{r}, 0)$. Assuming that we have already proved the statement of the proposition for $\tilde{w}_\nu$, a straight-forward argument using Claim 2 shows that it also holds for $w_\nu$. This proves Claim 1.

So we assume w.l.o.g. that (68) holds.

**Construction of $R_0, Z,$ and $w_0$:** Recall that we have chosen $E_0 > 0$ as in Lemma 28. We choose a constants $r_0$ and $E_1$ as in Proposition 25, the latter (called $E_0$ there) corresponding to the compact set $K$ and $\varepsilon := 1$. We fix a constant

$$(72) \quad 0 < \delta < \min \{m_0, E_0/2, E_1/2 \}.$$ We pass to some subsequence such that

$$(73) \quad E_{w,\nu}^R(B_r(z_0)) > m_0 - \delta, \quad \forall \nu \in \mathbb{N}.$$ For every $\nu \in \mathbb{N}$, there exists $0 < \tilde{\varepsilon}_\nu < r$, such that

$$(74) \quad E_{w,\nu}^R(B_{\tilde{\varepsilon}_\nu}) = m_0 - \delta.$$ It follows from the definition of $m_0$ that

$$(75) \quad \tilde{\varepsilon}_\nu \to 0.$$ **Claim 3.** We have

$$(76) \quad \inf_{\nu} \tilde{\varepsilon}_\nu R_\nu > 0.$$
Proof of Claim 3. Equality (68) implies that

\[ E_{\nu}(w_{\nu}, B_{\tilde{\nu}_{\nu}}) \leq \pi \tilde{\epsilon}_{\nu}^{2} e_{w_{\nu}}^{R_{\nu}(0)}. \]

The hypotheses \( R_{\nu} \to \infty \), (ii), and (iii) imply that the hypotheses of Proposition 20 (Quantization of energy loss) are satisfied with \( \Omega := B_{\nu} \).

Thus by assertion (i) of that proposition with \( Q := \{0\} \), we have

\[ \inf_{\nu} \frac{R_{\nu}^{2}}{e_{w_{\nu}}^{R_{\nu}(0)}} > 0. \]

Combining this with (77,74) and the fact \( \delta < m_{0} \), inequality (76) follows. This proves Claim 3. \( \square \)

Passing to some subsequence, we may assume that the limit

\[ \hat{R}_{0} := \lim_{\nu \to \infty} \hat{\epsilon}_{\nu} R_{\nu} \in [0, \infty] \]

exists. By Claim 3 we have \( \hat{R}_{0} > 0 \). We define

\[ (R_{0}, \epsilon_{\nu}) := \left\{ \begin{array}{ll} (\infty, \hat{\epsilon}_{\nu}), & \text{if } \hat{R}_{0} = \infty, \\ (1, R_{\nu}^{-1}), & \text{otherwise}, \end{array} \right. \]

\( \hat{R}_{\nu} := \epsilon_{\nu} R_{\nu}, \quad \varphi_{\nu} : B_{\epsilon_{\nu}^{-1} r} \to B_{r}, \quad \varphi_{\nu}(z) := \epsilon_{\nu} z, \quad \tilde{w}_{\nu} := (\tilde{A}_{\nu}, \tilde{w}_{\nu}) := \varphi_{\nu}^{*} w_{\nu}. \)

By Proposition 18 with \( R_{\nu}, w_{\nu} \) replaced by \( \hat{R}_{\nu}, \tilde{w}_{\nu} \) and \( r_{\nu} := r/\epsilon_{\nu} \) there exist a finite subset \( Z \subseteq \mathbb{R}^{2} \) and an \( R_{0} \)-vortex \( w_{0} = (A_{0}, u_{0}) \in \mathcal{W}_{0}(\mathbb{R}^{2} \setminus Z) \), and passing to some subsequence, there exist gauge transformations \( g_{\nu} \in W^{2,p}_{0}(\mathbb{R}^{2} \setminus Z, G) \), such that the conditions of that proposition are satisfied.

We check the conditions of Proposition 24 with \( z_{\nu} := z_{0} := 0 \):

- Condition 24(iii) holds by (75,78,79).
- Condition 24(iv) follows from 18(ii).
- Condition 24(v) follows from 18(iii).

We prove condition 24(v): We define

\[ \psi_{\nu} : B_{\epsilon_{\nu}^{-1} r} \to B_{r}, \psi_{\nu}(z) := \tilde{\epsilon}_{\nu} z, \quad \tilde{w}_{\nu} := \psi_{\nu}^{*} w_{\nu}. \]

We choose \( 0 < \epsilon \leq r \) so small that \( \lim_{\nu \to \infty} E^{R_{\nu}}(w_{\nu}, B_{\epsilon}) < m_{0} + E_{1}/2. \)

Furthermore, we choose an integer \( \nu_{0} \) so large that for \( \nu \geq \nu_{0} \), we have \( E^{R_{\nu}}(w_{\nu}, B_{\epsilon}) < m_{0} + E_{1}/2. \) We fix \( \nu \geq \nu_{0} \). Using (74,72), it follows that \( E(\tilde{w}_{\nu}, A(\tilde{\epsilon}_{\nu} R_{\nu}, \epsilon R_{\nu})) < E_{1}. \) It follows that the requirements of Proposition 25 are satisfied with \( r, R, w_{\nu} \) replaced by \( \max\{r_{0}, \tilde{\epsilon}_{\nu} R_{\nu}\}, \epsilon R_{\nu}, \tilde{w}_{\nu} \). Therefore, we may apply that result (with “\( \epsilon \)” equal to 1), obtaining

\[ E^{R_{\nu}}\left(w_{\nu}, A(\max\{R_{\nu}^{-1} r_{0}, \tilde{\epsilon}_{\nu}\}, a^{-1} \epsilon)\right) \leq 4a^{-1} E_{1}, \quad \forall a \geq 2. \]
Using (79) and the fact $z_{\nu} = z_0 = 0$, the inequality (49) follows. This proves 24(v).

To see that condition 24(i) holds, assume first that $R_0 = 1$. Then $Z = \emptyset$ by statement (i) of Proposition 18. Condition 18(i) and Lemma 22 imply that $E(w_0, B_{2\tilde{R}_0}) = \lim_{\nu \to \infty} E(\tilde{w}_{\nu}, B_{2\tilde{R}_0})$. It follows from convergence $\tilde{e}_\nu R_\nu \to \tilde{R}_0 < \infty$ and (74,72) that this limit is positive. This proves condition 24(i) in the case $R_0 = 1$.

Assume now that $R_0 = \infty$ and $E^\infty(w_0) = 0$. Then condition 24(i) is a consequence of the following two claims.

Claim 4. The set $Z$ is not contained in the open ball $B_1$.

Proof of Claim 4. By 24(v) there exists $R > 0$ so that
\begin{align}
(80) \quad \limsup_{\nu \to \infty} E^{R\nu}(w_\nu, A(R\tilde{e}_\nu, R^{-1})) < \delta.
\end{align}
(Here we used that $z_0 = z_\nu = 0$.) Since $R_0 = \infty$, we have $\tilde{e}_\nu = e_\nu$.
Hence it follows from (74) and the definition (67) of $m_0$, that
\begin{align}
\lim_{\nu \to \infty} E^{R\nu}(w_\nu, A(e_\nu, R^{-1})) \geq \delta.
\end{align}
Combining this with (80), it follows that
\begin{align}
(81) \quad \liminf_{\nu \to \infty} E^{R\nu}(w_\nu, A(e_\nu, \tilde{e}_\nu R)) > 0.
\end{align}
Suppose by contradiction that $Z \subseteq B_1$. Then by 18(ii), the connection $g^*\tilde{A}_\nu$ converges to $A_0$ in $C^0$ on $\tilde{A}(1, R) := \overline{B_R \setminus B_1}$, and the map $g^{-1}_\nu\tilde{u}_\nu$ converges to $u_0$ in $C^1$ on $\tilde{A}(1, R)$. Hence Lemma 22 implies that
\begin{align}
E^\infty(w_0, A(1, R)) = \lim_{\nu \to \infty} E^{\tilde{R}\nu}(\tilde{w}_\nu, A(1, R)).
\end{align}
Combining this with (81), we arrive at a contradiction to our assumption $E^\infty(w_0) = 0$. This proves Claim 4. \hfill \Box

Claim 5. The set $Z$ contains $0$.

Proof of Claim 5. By Claim 4 the set $Z \setminus B_1$ is nonempty. We choose a point $z \in Z \setminus B_1$ and a number $\varepsilon_0 > 0$ so small that $B_{\varepsilon_0}(z) \cap Z = \{z\}$. We fix $0 < \varepsilon < \varepsilon_0$. Since $\tilde{e}_\nu \to 0$ (as $\nu \to \infty$), (68) implies that $e_{\tilde{w}_\nu}^{R\nu}(0) = \|e_{\tilde{w}_\nu}^{R\nu}\|_{C^0(B_{\varepsilon_0}(z))}$, for $\nu$ large enough. Combining this with condition 24(iv), it follows that $\liminf_{\nu \to \infty} e_{\tilde{w}_\nu}^{\tilde{R}\nu}(0) \geq E_{\min}/(\pi\varepsilon^2)$. Since $\varepsilon \in (0, \varepsilon_0)$ is arbitrary, it follows that
\begin{align}
(82) \quad e_{\tilde{w}_\nu}^{\tilde{R}\nu}(0) \to \infty, \quad \text{as } \nu \to \infty.
\end{align}
If 0 did not belong to $Z$, then by (81) and Lemma 22 we would have $e_{\tilde{\nu}}(0) \to e_{w_0}(0)$, a contradiction to (82). This proves Claim 5 and completes the proof of 24(i) and therefore of Proposition 24. □

Remark 27. Assume that $R_0, Z, w_0$ are constructed as in the proof of condition (i) of Proposition 24, and that $R_0 = \infty$ and $E_\infty(w_0) = 0$. Then $Z \subseteq B_1$ (and hence $Z \cap S^1 \neq \emptyset$ by Claim 4). This follows from the inequalities

$$\lim_{\nu \to \infty} E_{R_\nu}(\tilde{\nu}, A(1, R)) \leq \delta < E_{\min}, \quad \forall R > 1.$$ 

Here the first inequality is a consequence of condition (74). □

5. Proof of Theorem 1 (Bubbling)

Based on the results of the previous sections, we are now ready to prove the main result of this article. The proof is an adaption of the proof of [MS, Theorem 5.3.1] to the present setting. The strategy is the following: Consider first the case $k = 0$, i.e., the only marked point is $z_0 = \infty$. We rescale the sequence $W_\nu$ so rapidly that all the energy is concentrated at the origin in $\mathbb{R}^2$. Then we “zoom back in” in a soft way, to capture the bubbles (spheres in $M$ and vortices on $\mathbb{R}^2$) in an inductive way. (See Claim 1 below.)

Next we show that at each stage of this construction, the total energy of the components of the tree plus the energy loss at the unresolved bubbling points equals the limit of the energies $E(W_\nu)$. (See Claim 2.) Furthermore, we prove that the number of vertices of the tree is uniformly bounded above. (See inequality (72).) This implies that the inductive construction terminates at some point.

We also show that the components of the tree have the required properties. (See Claim 4.) Finally, we prove that the data fits together to a stable map, which is the limit of a subsequence of $W_\nu$. (See Claim 5.)

For $k \geq 1$ we then prove the statement of the theorem inductively, using the statement for $k = 0$. At each induction step we need to handle one additional marked point in the sequence of vortices and marked points. In the limit there are three possibilities for the location of this point: (I) It may lie on a vertex where it does not coincide with any special point. (II) It may coincide with the marked point $z_i$ (lying on the $\alpha_i$-th vertex), for some $i$. (III) It may lie between two already constructed bubbles.
In case (I) we can just include the new marked point into the bubble tree. In case (II) we introduce a “ghost bubble”, which carries the two marked points and is connected to \( \alpha_i \). In case (III) we introduce a “ghost bubble” between the two bubbles, which carries the new marked point.

**Proof of Theorem 1.** We consider first the case \( k = 0 \). Let \( W_\nu \) be a sequence of vortices as in the hypothesis. For each \( \nu \in \mathbb{N} \) we choose a representative \( w_\nu := (P_\nu, A_\nu, u_\nu) \) of \( W_\nu \), such that \( P_\nu = \mathbb{R}^2 \times G \).

Passing to some subsequence we may assume that \( E(w_\nu) \) converges to some constant \( E \). The hypothesis \( E(W_\nu) > 0 \) (for every \( \nu \)) implies that \( E \geq E_{\min} \). We choose a sequence \( R_\nu \geq 1 \) such that

\[
E(W_\nu, B_{R_\nu}) \to E. 
\]

We define

\[
R_\nu^\nu := \nu R_\nu, \quad \varphi_\nu : \mathbb{R}^2 \to \mathbb{R}^2, \quad w_\nu^\nu := \varphi_\nu w_\nu, \quad j_1 := 0, \quad z_1 := 0, \quad Z_0 := \{0\}, \quad z_0^\nu := 0.
\]

The next claim provides an inductive construction of the bubble tree. (Some explanations are given below. See also Figure 5.)

**Claim 1.** For every number \( \ell \in \mathbb{N} \), passing to some subsequence, there exist an integer \( N := N(\ell) \in \mathbb{N} \) and tuples

\[
(R_i, Z_i, w_i)_{i \in \{1, \ldots, N\}}, \quad (R_i^\nu, z_i^\nu)_{i \in \{1, \ldots, N\}, \nu \in \mathbb{N}}, \quad (j_i, z_i)_{i \in \{2, \ldots, N\}},
\]

where \( R_i \in \{1, \infty\} \), \( Z_i \subseteq \mathbb{R}^2 \) is a finite subset, \( w_i = (A_i, u_i) \in \tilde{W}_0(\mathbb{R}^2 \setminus Z_i) \) is an \( R_i \)-vortex, \( R_i^\nu > 0 \), \( z_i^\nu \in \mathbb{R}^2 \), \( j_i \in \{1, \ldots, i - 1\} \), and \( z_i \in \mathbb{R}^2 \), such that the following conditions hold.

(i) For every \( i = 2, \ldots, N \) we have \( z_i \in Z_{j_i} \). Moreover, if \( i, i' \in \{2, \ldots, N\} \) are such that \( i \neq i' \) and \( j_i = j_{i'} \) then \( z_i \neq z_{i'} \).

(ii) Let \( i = 1, \ldots, N \). If \( R_i = 1 \) then \( Z_i = \emptyset \) and \( E(w_i) > 0 \). If \( R_i = \infty \) and \( E(\infty)(w_i) = 0 \) then \( |Z_i| \geq 2 \).

(iii) Fix \( i = 1, \ldots, N \). If \( R_i = 1 \) then \( R_i^\nu = 1 \) for every \( \nu \), and if \( R_i = \infty \) then \( R_i^\nu \to \infty \). Furthermore,

\[
\frac{R_i^\nu}{R_{j_i}^\nu} \to 0, \quad \frac{z_i^\nu - z_{j_i}^\nu}{R_{j_i}^\nu} \to z_i.
\]

In the following we set \( \varphi_i^\nu(z) := R_i^\nu z + z_i^\nu \), for \( i = 0, \ldots, N \) and \( \nu \in \mathbb{N} \).
Figure 5. This is a “partial stable map” as in Claim 1. It is a possible step in the construction of the stable map of Figure 1. The crosses are bubbling points that have not yet been resolved. When adding marked points the components 4 and 5 will be separated by a ghost bubble which carries one marked point.

(iv) For every $i = 1, \ldots, N$ there exist gauge transformations $g_i^\nu \in W_{\text{loc}}^{2,p}(\mathbb{R}^2 \setminus Z_i, G)$ such that the following holds. If $R_i = 1$ then $(g_i^\nu)^*(\varphi_i^\nu)^* w_\nu$ converges to $w_i$ in $C^\infty$ on every compact subset of $\mathbb{R}^2$. Furthermore, if $R_i = \infty$ then on every compact subset of $\mathbb{R}^2 \setminus Z_i$ the sequence $(g_i^\nu)^*(\varphi_i^\nu)^* A_\nu$ converges to $A_i$ in $C^0$, and the sequence $(g_i^\nu)^*(\varphi_i^\nu)^* u_\nu$ converges to $u_i$ in $C^1$.

(v) Let $i = 1, \ldots, N$, $z \in Z_i$ and $\varepsilon_0 > 0$ be such that $B_{\varepsilon_0}(z) \cap Z_i = \{z\}$. Then for every $0 < \varepsilon < \varepsilon_0$ the limit

$$E_z(\varepsilon) := \lim_{\nu \to \infty} E^{R_i^w}(\varphi_i^\nu)^* w_\nu, B_{\varepsilon}(z))$$

exists, and $E_{\text{min}} \leq E_z(\varepsilon) < \infty$. Furthermore, the function $(0, \varepsilon_0) \ni \varepsilon \mapsto E_z(\varepsilon) \in [E_{\text{min}}, \infty)$ is continuous.

(vi) For every $i = 1, \ldots, N$, we have

$$\lim_{R \to \infty} \limsup_{\nu \to \infty} E(w_\nu, B_{R_{j_i}/R}(z_{j_i}^\nu + R_{j_i}^w z_i) \setminus B_{RR_i^w}(z_i^\nu)) = 0.$$

(vii) If $\ell > N$ then for every $j = 1, \ldots, N$ we have

$$Z_j = \{z_i \mid j < i \leq N, j_i = j\}.$$
To understand this claim, note that the collection \((j_i)_{i \in \{2, \ldots, N\}}\) describes a tree with vertices the numbers 1, \ldots, \(N\) and unordered edges \(\{(i, j_i), (j_i, i)\}\). Attached to the vertices of this tree are vortices and \(\infty\)-vortices. (The latter will give rise to holomorphic spheres in \(\overline{M}\).) Each pair \((R_i^\nu, z_i^\nu)\) defines a rescaling \(\varphi_i^\nu\), which is used to obtain the \(i\)-th limit vortex or \(\infty\)-vortex. (See condition (vii).)

The point \(z_i\) is the nodal point on the \(j_i\)-th vertex, at which the \(i\)-th vertex is attached. The corresponding nodal point on the \(i\)-th vertex is \(\infty\). The set \(Z_i\) consists of the nodal points except \(\infty\) (if \(i \geq 2\)) on the \(i\)-th vertex together with the bubbling points that have not yet been resolved.

Condition (i) implies that the nodal points at a given vertex are distinct. Condition (ii) guarantees that once all bubbling points have been resolved, the \(i\)-th component will be stable. (Note that in the case \(i \geq 2\) there is another nodal point at \(\infty\), and for \(i = 1\) there will be a marked point at \(\infty\), which comes from sequence \(z_0^\nu\).)

Condition (iii) implies that the rescalings \(\varphi_i^\nu\) “zoom out” less than the rescalings \(\varphi_j^\nu\). A consequence of condition (vii) is that at every nodal or unresolved bubbling point at least the energy \(E_{\text{min}}\) concentrates in the limit. Condition (vi) means that no energy is lost between each pair of adjacent bubbles. Finally, condition (vii) means that in the case \(\ell > N\) all bubbling points have been resolved.

**Proof of Claim 4.** We show that the statement holds for \(\ell := 1\). We check the conditions of Proposition 24 (Soft rescaling) with \(z_0 := 0\), \(r := 1\) and \(R_\nu, w_\nu\) replaced by \(R_0^\nu, w_0^\nu\). Condition 24(a) follows from Proposition 35 below, using the hypothesis that \(M\) is equivariantly convex at \(\infty\). Condition 24(b) follows from the facts

\[
\lim_{\nu \to \infty} E^{R_0^\nu}(w_0^\nu, B_\varepsilon) = E, \quad \forall \varepsilon > 0, \quad E \geq E_{\text{min}}.
\]

The first condition is a consequence of the facts \(R_\nu^\nu = \nu R_\nu^\nu, E(w_\nu^\nu) \to E\), and (83).

Thus by Proposition 24 there exist \(R_0^\nu \in \{1, \infty\}\), a finite subset \(Z \subseteq \mathbb{R}^2\), and an \(R_\nu\)-vortex \(w_0 \in \widetilde{W}_0^\nu(\mathbb{R}^2 \setminus Z_1)\), and passing to some subsequence, there exist sequences \(\varepsilon_\nu > 0, z_\nu, g_\nu\), such that the conclusions of Proposition 24 with \(R_\nu, w_\nu\) replaced by \(R_0^\nu, w_0^\nu\) hold. We define \(N := N(1) := 1, R_1 := R_0, Z_1 := Z, w_1 := w_0, R_1^\nu := \varepsilon_\nu R_0^\nu, \) and \(z_1^\nu := R_0^\nu z_\nu\).
We check conditions (i)-(vii) of Claim \(1 \) with \( \ell = 1 \): Conditions (i)-(vii) are void. Furthermore, conditions (ii)-(vi) follow from (24)(i)-(v). This proves the statement of the Claim for \( \ell = 1 \).

Let \( \ell \in \mathbb{N} \) and assume, by induction, that we have already proved the statement of Claim \(1 \) for \( \ell \). We show that it holds for \( \ell + 1 \). By assumption there exists a number \( N := N(\ell) \) and there exist collections \((R_i, Z_i, w_i)_{i \in \{1, \ldots, N\}}, (R'_{\nu}, z'_{\nu})_{\nu \in \mathbb{N}}, (j_i, z_i)_{i \in \{2, \ldots, N\}}\), such that conditions (i)-(vii) hold. If for every \( j = 1, \ldots, N \) we have \( Z_j = \{ z_i \mid j < i \leq N, j_i = j \} \) then conditions (i)-(vii) hold with \( N(\ell + 1) := N \), and we are done. Hence assume that there exists a \( j_0 \in \{1, \ldots, N\} \) such that

\[
Z_{j_0} \neq \{ z_i \mid j_0 < i \leq N, j_i = j_0 \}.
\]

We set \( N(\ell + 1) := N + 1 \) and choose an element

\[
z_{N+1} \in Z_{j_0} \setminus \{ z_i \mid j < i \leq N, j_i = j_0 \}.
\]

We fix a number \( r > 0 \) so small that \( B_r(z_{N+1}) \cap Z_{j_0} = \{ z_{N+1} \} \). We apply Proposition (24) with \( z_0 := z_{N+1} \) and \( R_\nu \), \( w_\nu \) replaced by \( R'_{\nu j_0} \), \( (\varphi'_{\nu j_0})^* w_\nu \). Condition (24)(a) holds by hypothesis. Furthermore, by condition (v) for \( \ell \), condition (24)(i) is satisfied. Hence passing to some subsequence, there exist \( R_0 = \{1, \infty\} \), a finite subset \( Z \subseteq \mathbb{R}^2 \), an \( R_0 \)-vortex \( w_0 \in \hat{W}_0^p (\mathbb{R}^2 \setminus Z) \), and sequences \( \varepsilon_\nu > 0 \), \( z_\nu \), such that the conclusion of Proposition (24) holds. We define \( R_{N+1} := R_0 \), \( Z_{N+1} := Z \), \( w_{N+1} := w_0 \), \( R'_{N+1} := \varepsilon_\nu R'_{\nu j_0} \), \( z'_{N+1} := R'_{\nu j_0} z_\nu + z'_{j_0} \) and \( j_{N+1} := j_0 \).

We check conditions (i)-(vii) of Claim \(1 \) with \( \ell \) replaced by \( \ell + 1 \), i.e., \( N \) replaced by \( N + 1 \). Condition (i) follows from the induction hypothesis and (87). Conditions (ii)-(vi) follow from (24)(i)-(v).

We show that (vii) holds with \( N \) replaced by \( N + 1 \): By the induction hypothesis, it holds for \( N \). Hence (86) implies that \( N \geq \ell \), i.e., \( N + 1 \geq \ell + 1 \). So there is nothing to check. This completes the induction and the proof of Claim \(1 \).

Let \( \ell \in \mathbb{N} \) be an integer and \( N := N(\ell), (R_i, Z_i, w_i), (R'_{\nu}, z'_{\nu}), (j_i, z_i) \) be as in Claim \(1 \). Recall that \( Z_0 = \{0\} \) and \( z_0' := 0 \). We fix \( i = 0, \ldots, N \). We define \( \varphi_i'(z) := R'_{\nu i} z + z_i' \), for every measurable subset \( X \subseteq \mathbb{R}^2 \) we denote

\[
E_i(X) := E_i^R(w_i, X), \quad E_i := E_i(\mathbb{R}^2 \setminus Z_i), \quad E_i'(X) := E_i^R((\varphi_i')^* w_\nu, X).
\]

Furthermore, for \( z \in Z_i \) we define

\[
m_i(z) := \lim_{\varepsilon \to 0} \lim_{\nu \to \infty} E_i'(B_\varepsilon(z)).
\]
For $i = 0$ it follows from (83) and $R_0^\nu = \nu R_\nu$ that the limit $m_0(0)$ exists and equals $E$. For $i = 1, \ldots, N$ it follows from condition (v) that the limit (88) exists and that $m_i(z) \geq E_{\min}$. For $j, k = 0, \ldots, N$ we define

\[ Z_{j,k} := Z_j \setminus \{ z_i \mid j < i \leq k, j_i = j \} \]

(This is the set of points on the $j$-th sphere that have not been resolved after the construction of the $k$-th bubble.) We define the function $f : \{1, \ldots, N\} \to [0, \infty)$ by

\[ f(i) := E_i + \sum_{z \in Z_{i,N}} m_i(z). \]

Claim 2.

\[ \sum_{i=1}^{N} f(i) = E. \]

Proof of Claim 2. We show by induction that

\[ \sum_{i=1}^{k} \left( E_i + \sum_{z \in Z_{i,k}} m_i(z) \right) = E, \]

for every $k = 1, \ldots, N$. Claim 2 is a consequence of this with $k = N$. For the proof of equality (90) we need the following.

Claim 3. For every $i = 1, \ldots, N$ we have

\[ m_{ji}(z_i) = E_i + \sum_{z \in Z_i} m_i(z). \]

Proof of Claim 3. Let $i = 1, \ldots, N$. We choose a number $\varepsilon > 0$ so small that

\[ \tilde{B}_\varepsilon(z_i) \cap Z_{ji} = \{ z_i \}, \quad Z_i \subseteq B_{\varepsilon-1-\varepsilon}, \]

and if $z \neq z'$ are points in $Z_i$ then $|z - z'| > 2\varepsilon$. By condition (v) of Claim 1, for each $z \in Z_i$ the limit $\lim_{\nu \to \infty} E_i^\nu(B_\varepsilon(z))$ exists. Lemma 22 implies that

\[ \lim_{\nu \to \infty} E_i^\nu(B_{\varepsilon-1}) = E_i \left( B_{\varepsilon-1} \setminus \bigcup_{z \in Z_i} B_\varepsilon(z) \right) + \sum_{z \in Z_i} \lim_{\nu \to \infty} E_i^\nu(B_\varepsilon(z)). \]

Combining this with condition (vi) of Claim 1, equality (91) follows from a straight-forward argument. This proves Claim 3. \[ \square \]

Since $Z_{1,1} = Z_1$, equality (90) for $k = 1$ follows from Claim 3 and the fact $m_0(0) = E$. Let now $k = 1, \ldots, N - 1$ and assume that we have proved (90) for $k$. An elementary argument using Claim 3 with
Consider the tree relation \( E \) on \( T := \{1, \ldots, N\} \) defined by \( iEi' \) iff \( i = j \) or \( i' = j \). Lemma [10] below with this pair \((T,E)\), \( f \) as in [89], \( k := 1, \alpha_1 := 1 \in T \), and \( E_0 := E_{\min} \), implies that

\[
N \leq \frac{2E}{E_{\min}} + 1.
\]

(Hypothesis (107) follows from conditions (ii,v) of Claim 1.) Assume now that we have chosen \( \ell > \frac{2E}{E_{\min}} + 1 \). By (92) we have \( \ell > N \), and therefore by condition (vii) of Claim 1, equality (85) holds, for every \( j = 1, \ldots, N \). We define

\[
T := \{1, \ldots, N\}, \quad V := \{i \in T \mid R_i = 1\}, \quad \overline{T} := T \setminus V,
\]

and the tree relation \( E \) on \( T \) by

\[
iEi' \iff i = j \text{ or } i' = j.
\]

Furthermore, for \( i,i' \in T \) such that \( iEi' \) we define the nodal points

\[
z_{ii'} := \begin{cases} \infty, & \text{if } i' = j, \\ z_i', & \text{if } i = j'. \end{cases}
\]

Moreover, we define the marked point

\[
(\alpha_0, z_0) := (1, \infty) \in T \times S^2.
\]

Claim 4. Let \( i \in T \). If \( i \in V \) then \( E(w_i) < \infty \) and \( u_i(\mathbb{R}^2 \times G) \) has compact closure. Furthermore, if \( i \in \overline{T} \) then the map \( Gu_i : \mathbb{R}^2 \setminus Z_i \to \overline{M} = \mu^{-1}(0)/G \) extends to a smooth \( \bar{J} \)-holomorphic map

\[
\bar{u}_i : S^2 \cong \mathbb{R}^2 \cup \{\infty\} \to \overline{M}.
\]

Proof. We choose gauge transformations \( g_i' \in W^{2,p}_{\text{loc}}(\mathbb{R}^2 \setminus Z_i, G) \) as in condition (iv) of Claim 1 and define \( w_i' := (g_i')^*(\varphi_i')^*w_i \).

Assume that \( i \in V \). It follows from Fatou’s lemma that \( E(w_i) \leq \liminf_{\nu \to \infty} E(w_i') = E < \infty \). Furthermore, since by hypothesis \( M \) is equivariantly convex at \( \infty \), by Proposition [35] below there exists a \( G \)-invariant compact subset \( K_0 \subseteq M \) such that \( u_i'(\mathbb{R}^2) \subseteq K_0 \), for every \( \nu \in \mathbb{N} \). Since \( w_i' \) converges to \( u_i \) pointwise, it follows that \( u_i(\mathbb{R}^2) \subseteq K_0 \). Hence \( w_i \) has the required properties.

Assume now that \( i \in \overline{T} \). By Proposition [15] below the map

\[
Gu_i : \mathbb{C} \setminus Z_i \to \overline{M} = \mu^{-1}(0)/G
\]
is \( \bar{J} \)-holomorphic, and \( e_{\bar{G}u_i} = e_{\bar{w_i}}^\infty \). It follows from Fatou’s lemma that 
\[
E^\infty(w_i, \mathbb{R}^2 \setminus Z_i) \leq \liminf_{\nu \to \infty} E^{R \nu}(w_i^\nu) = E < \infty.
\]
Therefore, by removal of singularities, it follows that \( \bar{G}u_i \) extends to a smooth \( \bar{J} \)-holomorphic map \( \bar{u}_i : S^2 \to \bar{M} \). (See e.g. [MS, Theorem 4.1.2].) This proves Claim 4. \( \square \)

Claim 5. The tuple
\[
(W, z) := (V, \bar{T}, E, ([w_i])_{i \in V}, (\bar{u}_i)_{i \in \bar{T}}, (z_{ii})_{i \in E}, (\alpha_0 := 1, z_0 := \infty))
\]
is a stable map in the sense of Definition 4, and the sequence \([w_\nu], z_\nu^\alpha := \infty\) converges to \((W, z)\) in the sense of Definition 13. (Here \([w_i]\) denotes the gauge equivalence class of \(w_i\).)

Proof of Claim 5. We check the conditions of Definition 4. Condition (i) follows from condition (i) of Claim 1 and the fact \(Z_i = \emptyset\), for \(i \in V\). (This follows from condition (ii) of Claim 1.) Condition (ii) follows from an elementary argument using Claim 1(iii,iv,vi) and Proposition 25. Condition (iii) follows from Claim 1(ii). Hence all conditions of Definition 4 are satisfied.

We check the conditions of Definition 13. Condition (19) follows from Claim 2, using condition (vii) of Claim 1. Condition 13(i) follows from a straightforward argument, using Claim 1(iii).

Condition 13(ii) follows from Claim 1(iii) by an elementary argument. Condition 13(iii) follows from Claim 1(iv). Finally, condition 13(iv) is void, since \(k = 0\). This proves Claim 5. \( \square \)

Thus we have proved Theorem 1 in the case \(k = 0\).

We prove now by induction that the Theorem holds for every \(k \geq 1\): Let \(k \geq N_0\) be an integer, \((W_\nu, z_\nu^1, \ldots, z_\nu^{k-1})\) as in the hypotheses of Theorem 1 and assume that there exists a stable map \((W, z)\) as in (12)) and a collection \((\varphi_\alpha^\nu)\) of Möbius transformations such that \((W_\nu, z_\nu^\infty := \infty, z_\nu^1, \ldots, z_\nu^{k-1})\) converges to \((W, z)\) via \((\varphi_\alpha^\nu),\)
\[
\varphi_\alpha^\nu(\infty) = \infty,
\]
and for every edge \(\alpha E \beta\) such that \(\beta\) lies in the chain of vertices from \(\alpha\) to \(\alpha_0\), we have
\[
\lim_{R \to \infty} \limsup_{\nu \to \infty} E(W_\nu, \mathbb{R}^2 \setminus \varphi_\alpha^\nu(B_\alpha)) = 0,
\]
and for every edge \(\alpha E \beta\) such that \(\beta\) lies in the chain of vertices from \(\alpha\) to \(\alpha_0\), we have
\[
\lim_{R \to \infty} \limsup_{\nu \to \infty} E(W_\nu, \varphi_\alpha^\nu(B_{R-1}(z_\beta))) \setminus \varphi_\alpha^\nu(B_\alpha) = 0.
\]
(For \(k = 0\) we proved this above. In this case condition (93) follows from (vi) of Claim 1 with \(i = 1\), and the facts \(E(W_\nu, B_{R_\nu}) \to E\) and
Furthermore, condition (94) follows from condition (vi) of Claim [I]. By hypothesis (6), passing to some subsequence, we may assume that for every $i = 1, \ldots, k - 1$, the limit
\begin{equation}
(95) \quad z_{ki} := \lim_{\nu \to \infty} (z'_{k} - z'_i) \in \mathbb{R}^2 \cup \{\infty\}
\end{equation}
exists, and $z_{ki} \neq 0$. We set $z_{k0} := \infty$.

Passing to a further subsequence, we may assume that the limit
\begin{equation}
(96) \quad z_{ak} := \lim_{\nu \to \infty} (\varphi_{\alpha} \nu)^{-1}(z'_{k}) \in S^2
\end{equation}
exists, for every $\alpha \in T$. There are three cases.

Case (I) There exists a vertex $\alpha \in T$, such that $z_{ak}$ is not a special point of $(W, z)$ at $\alpha$.

Case (II) There exists an index $0 \leq i \leq k - 1$ such that $z_{ak} = z_i$.

Case (III) There exists an edge $\alpha E \beta$ such that $z_{ak} = z_{\alpha \beta}$ and $z_{\beta k} = z_{\beta \alpha}$.

These three cases exclude each other. For the combination of the cases (II) and (III) this follows from condition (i) (distinctness of the special points) of Definition [I].

Claim 6. One of the three cases always applies.

Proof of Claim [I]. This follows from an elementary argument, using that $T$ is finite and does not contain cycles.

Assume that Case (I) holds. We fix a vertex $\alpha \in T$ such that $z_{ak}$ is not a special point. (This vertex is unique, but we do not need this.) We define $\alpha_k := \alpha$ and introduce a new marked point $z_{k}^{\text{new}} := z_{\alpha k}$ on the $\alpha_k$-sphere. Then $(W, z)$ augmented by $z_{k}^{\text{new}}$ is again a stable map and the sequence $(W_\nu, z_0', \ldots, z_k')$ converges to this new stable map via $(\varphi_{\alpha} \nu)_{\alpha \in T}$.

Assume that Case (II) holds. We fix an index $0 \leq i \leq k - 1$ such that $z_{\alpha i k} = z_i$. (It is unique.) The hypothesis (6) implies that $\alpha_i \in T$. We extend the tree $T$ by introducing an additional vertex $\alpha_i$ which is adjacent to $\alpha_i$. If $z_{ki} = \infty$ (defined as in (95)) then the new vertex
corresponds to a bubble in $\overline{M}$, otherwise it corresponds to a vortex.

We move the $i$-th marked point from the vertex $\alpha_i$ to the vertex $\gamma$ and introduce an additional marked point on $\gamma$. More precisely, we define

$$T_{new} := \begin{cases} \overline{T} \coprod \{\gamma\}, & \text{if } z_k = \infty, \\ T, & \text{otherwise}, \end{cases}$$

$$T_{new} := \overline{T} \coprod \{\gamma\}, \quad V_{new} := T_{new} \setminus T_{new},$$

$$\alpha_{i_{new}} := \alpha_k := \gamma, \quad z_{\alpha_{i_{new}}} := \infty, \quad z_{\alpha_{i\gamma}} := z_i,$$

$$z_{i_{new}} := 0, \quad z_{k_{new}} := \begin{cases} z_{ki}, & \text{if } z_k \neq \infty, \\ 1, & \text{otherwise}. \end{cases}$$

Assume first that $\gamma \in V_{new}$. We choose a point $x_0$ in the orbit $\overline{u}_{\alpha_{i_{new}}} \subseteq \mu^{-1}(0)$, and define $A_\gamma := 0 \in \Omega^1(\mathbb{R}^2, \mathfrak{g})$ and $u_\gamma : \mathbb{R}^2 \to M$ to be the map which is constantly equal to $x_0$. We identify $A_\gamma$ with a connection on $\mathbb{R}^2 \times G$ and $u_\gamma$ with a $G$-equivariant map $\mathbb{R}^2 \times G \to M$, and set

$$(96) \quad W_\gamma := [\mathbb{R}^2 \times G, A_\gamma, u_\gamma].$$

If $\gamma \in T_{new}$ then we define $\bar{u}_\gamma : S^2 \to \overline{M}$ by

$$\bar{u}_\gamma \equiv \bar{u}_{\alpha_{i_{new}}}(z_i).$$

Note that the new component $\gamma$ is a “ghost”, i.e., the map $W_\gamma$ (or $\bar{u}_\gamma$) has 0 energy. The tuple $(W_{new}, z_{new})$ obtained from $(W, z)$ in this way is again a stable map.

We define the sequence of Möbius transformations $\varphi_{\gamma}^\nu : S^2 \to S^2$ by

$$(97) \quad \varphi_{\gamma}^\nu(z) := \begin{cases} z + z_\nu^\gamma, & \text{if } \gamma \in V_{new}, \\ (z_k^\nu - z_i^\nu)z + z_i^\nu, & \text{if } \gamma \in T_{new}, i \geq 1, \\ \frac{z_k^\nu - \varphi_{\alpha_0}^\nu(w)}{z_i^\nu - \varphi_{\alpha_0}^\nu(w)} + \varphi_{\alpha_0}^\nu(w), & \text{if } \gamma \in T_{new}, i = 0, \end{cases}$$

where $w \in S^2 \setminus \{z_0 = \infty\}$ is chosen such that $\varphi_{\alpha_0}^\nu(w) \neq z_k^\nu$ for all $\nu$. Note that the last line makes sense, since $\varphi_{\alpha_0}^\nu(w) \neq \varphi_{\alpha_0}^\nu(z_0) = \infty$. Here we use the convention that $1/\infty := 0$.

Claim 7. There exists a subsequence of $(W_{\nu}, z_0^\nu, \ldots, z_k^\nu)$ that converges to $(W_{new}, z_{new})$ via the Möbius transformations $(\varphi_{\alpha}^\nu)_{\alpha \in T_{new}, \nu \in \mathbb{N}}$.

Proof of Claim 7. Condition (19) (energy conservation) holds for every subsequence, since the new component $\gamma$ carries no energy.

We denote now by $i \in \{0, \ldots, k-1\}$ the unique index such that $z_{\alpha_{i,k}} = z_i$.

Condition (ii) of Definition 13 holds (for the new collection of Möbius transformations), by an elementary argument. (To show the
third part of this condition in the case $\gamma \in T^{\text{new}}$, we set $\psi_\gamma := \text{id}$ if $i \geq 1$, and $\psi_\gamma(z) := 1/z$ if $i = 0$.

**We check condition [13][iii].** Let $\alpha E^{\text{new}} / \beta$ be an edge. Consider the case $(\alpha, \beta) := (\alpha_i, \gamma)$. (It suffices to look at this case, the case $(\alpha, \beta) := (\gamma, \alpha_i)$ can be treated analogously.) We define

\[
x := z_{\gamma \alpha_i}^{\text{new}} = \infty, \quad y := z_{\alpha_i \gamma}^{\text{new}} = z_i, \quad x_1 := z_1^{\text{new}} = 0,
\]

\[
x_2 := \begin{cases} z_k^{\prime} - z_i^{\prime}, & \text{if } \gamma \in V^{\text{new}}, \\ z_k^{\text{new}} = 1, & \text{if } \gamma \in T^{\text{new}}, \end{cases}
\]

\[
y := \begin{cases} z_{\alpha_i 0}, & \text{if } \gamma \in V^{\text{new}} \text{ or } (\gamma \in T^{\text{new}} \text{ and } i \geq 1), \\ w, & \text{if } \gamma \in T^{\text{new}} \text{ and } i = 0, \end{cases}
\]

\[
\varphi_\nu := \varphi^{\nu}_{\alpha_i \gamma} := (\varphi^{\nu}_{\alpha_i})^{-1} \circ \varphi^{\nu}_\gamma.
\]

Then the hypotheses of Lemma [18] below are satisfied, and therefore by that lemma $\varphi^{\nu}_{\alpha_i \gamma}$ converges to $y = z^{\text{new}}_{\gamma \alpha_i}$, uniformly with all derivatives on every compact subset of $S^2 \setminus \{x\} = S^2 \setminus \{z^{\text{new}}_{\gamma \alpha_i}\}$. By Remark [17] below it follows that $\varphi^{\nu}_{\gamma \alpha_i} = (\varphi^{\nu}_{\alpha_i \gamma})^{-1}$ converges to $z^{\text{new}}_{\gamma \alpha_i}$, uniformly on every compact subset of $S^2 \setminus \{z^{\text{new}}_{\gamma \alpha_i}\}$. This proves condition [13][iii].

**We check condition [13][iii] up to some subsequence.** For every $\alpha \in T$ we write

\[
W^{\nu}_\alpha := (\varphi^{\nu}_\alpha)^* W_\nu, \quad \bar{u}^{\nu}_\alpha := \bar{u}_{W^{\nu}_\alpha} : \mathbb{R}^2 \to M/G,
\]

where $\bar{u}_{W^{\nu}_\alpha}$ is defined as in (9).

**Assume that** $\gamma \in V^{\text{new}}$. This means that $z_{k_i} \neq \infty$. Since by definition $z_{k_0} = \infty$, it follows that $i \neq 0$. It follows from Proposition [18] (Compactness modulo bubbling) with $R_\nu := 1$, $r_\nu := \nu$ and $W_\nu$ replaced by $W^{\nu}_\gamma$, that there exists a vortex $\tilde{W}_\gamma$ on $\mathbb{R}^2$ such that passing to some subsequence, the sequence $W^{\nu}_\gamma$ converges to $\tilde{W}_\gamma$, with respect to $\tau_{\mathbb{R}^2}$ (defined as in [18]), and the sequence $\bar{u}^{\nu}_\alpha$ converges to $\bar{u}_{\tilde{W}_\gamma}$, uniformly on every compact subset of $\mathbb{R}^2$.

**Claim 8.** We have $\tilde{W}_\gamma = W^{\nu}_\alpha$ (defined as in [90]).

**Proof of Claim 8.** To see this, we use condition [13][iii] for the sequence $(W_\nu, z_0^{\nu}, \ldots, z_{k_i-1}^{\nu})$, recalling that $\alpha_i \in T^{\text{new}}$ and $i \neq 0$. It follows that the maps $\bar{u}^{\nu}_{\alpha_i}$ converge to $\bar{u}_{\alpha_i}$, in $C^1$ on every compact subset of $S^2 \setminus Z_{\alpha_i}$, and hence on every small enough neighbourhood of $z_i$. (To make sense of this convergence, we implicitly mean that the image of a given compact subset of $S^2 \setminus Z_{\alpha_i}$ under $\bar{u}_{W^{\nu}_{\alpha_i}}$ is contained in $M^*/G$, for $\nu$ large enough.)
Since \( \varphi_{\alpha,\gamma}^\nu := (\varphi_{\alpha,\gamma}^\nu)^{-1} \circ \varphi_{\alpha,\gamma}^\nu \) converges to \( z_{\alpha,\gamma} = z_i \), uniformly on every compact subset of \( S^2 \setminus \{ z_{\gamma,\alpha}^{\text{new}} \} = \mathbb{R}^2 \), it follows that
\[
\tilde{u}_\gamma^\nu = \tilde{u}_i^\nu \circ \varphi_{\alpha,\gamma}^\nu \to \tilde{x}_0 := \tilde{u}_{\alpha_i}(z_i),
\]
unformly on every compact subset of \( \mathbb{R}^2 \). It follows that \( \tilde{u}_{\tilde{W}_\gamma} = \tilde{x}_0 \).

We choose representatives \( (\tilde{A}_\gamma, \tilde{u}_\gamma) \in \Omega^1(\mathbb{R}^2, G) \times C^\infty(\mathbb{R}^2, G) \) of \( \tilde{W}_\gamma \) and \( x_0 \in \mu^{-1}(0) \) of \( \pi_0 \). By hypothesis (H) the action of \( G \) on \( \mu^{-1}(0) \) is free. Hence, after recharging, we may assume that \( \tilde{u}_\gamma \equiv x_0 \). (Here we use Lemma 13 below, which ensures that the gauge transformation is smooth.) It follows from the first vortex equation that \( \tilde{A}_\gamma = 0 \). This shows that \( \tilde{W}_\gamma = W_\gamma \) and hence proves Claim 8.

This proves condition 13(iii) in the case \( \gamma \in V^{\text{new}} \).

Assume now that \( \gamma \in V^{\text{new}} \). Suppose also that \( i \geq 1 \). By condition 13(iii) for the sequence \( (W_{\nu}, z^\nu_0, \ldots, z^\nu_{k-1}) \), the map \( \tilde{u}_i^\nu \) converges to \( \tilde{u}_{\alpha_i} \) in \( C^1 \) on every compact subset of \( S^2 \setminus Z_{\alpha_i} \) and hence on every small enough neighbourhood of \( z_i \). Let \( Q \subseteq \mathbb{R}^2 = S^2 \setminus Z_{\gamma} \) be a compact subset. Since \( \varphi_{\alpha,\gamma}^\nu \) converges to \( z_i \), in \( C^\infty \) on \( Q \), it follows that \( \tilde{u}_\gamma^\nu = \tilde{u}_i^\nu \circ \varphi_{\alpha,\gamma}^\nu \) converges to \( \tilde{u}_\gamma = \tilde{u}_{\alpha_i}(z_i) \) in \( C^1 \) on \( Q \), as required.

Suppose now that \( i = 0 \). An elementary argument using condition (iii) (with \( \alpha := \alpha_0 \)) in the definition of convergence, our assumption (iii), and Proposition 25 shows that for every \( \varepsilon > 0 \) there exist numbers \( R \geq r_0 \) and \( \nu_0 \in \mathbb{N} \) such that
\[
\tilde{d}(\tilde{u}_{\alpha_0}(\infty), \tilde{u}_{\alpha_0}^\nu(z)) < \varepsilon, \quad \forall \nu \geq \nu_0, \ z \in \mathbb{R}^2 \setminus B_R.
\]
Since \( \varphi_{\alpha,\gamma}^\nu \) converges to \( z_{\alpha_0,\gamma} = z_0 = \infty \), uniformly on every compact subset of \( \mathbb{R}^2 = S^2 \setminus \{ z_{\gamma,\alpha}^{\text{new}} \} \), it follows that \( \tilde{u}_\gamma^\nu \) converges to the constant map \( \tilde{u}_\gamma \equiv \tilde{u}_{\alpha_0}(\infty) \), uniformly on every compact subset of \( \mathbb{R}^2 \setminus \{0\} = S^2 \setminus (Z_{\gamma} \bigcup \{ z_0^{\text{new}} \}) \). We show that passing to some subsequence the convergence is in \( C^1 \) on every compact subset of \( \mathbb{R}^2 \setminus \{0\} \). To see this, we define \( R_\nu > 0 \) by \( \varphi_\nu \in [0, 2\pi) \), \( \tilde{\varphi}_\gamma^\nu, \tilde{w}_\gamma^\nu \) by
\[
R_\nu e^{i\varphi_\nu} := z_{k-1}^\nu - \varphi_{\alpha_0}^\nu(w), \quad \tilde{\varphi}_\gamma^\nu(\tilde{z}) := \varphi_\gamma^\nu(e^{i\varphi_\nu} / \tilde{z}) = R_\nu z + \varphi_{\alpha_0}^\nu(w),
\]
\[
\tilde{w}_\gamma := (\tilde{A}_\gamma, \tilde{u}_\gamma) := (\tilde{\varphi}_\gamma^\nu)^* w_\nu.
\]
(Recall here that we have chosen \( w \in S^2 \setminus \{ z_0 = \infty \} \) such that \( \varphi_{\alpha_0}^\nu(w) \neq z_k^\nu \) for all \( \nu \).) An elementary argument shows that \( R_\nu \) converges to \( R_0 := \infty \). Hence by Proposition 15 with \( r_\nu := \nu \) and \( w_\nu \) replaced by the \( R_\nu \)-vortex \( \tilde{w}_\gamma^\nu \) there exist a finite subset \( Z \subseteq \mathbb{R}^2 \) and an \( \infty \)-vortex \( \tilde{w}_\gamma := (\tilde{A}_\gamma, \tilde{u}_\gamma) \), and passing to some subsequence, there exist gauge
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transformations \( g^\nu \in W^{2,p}_{\text{loc}}(\mathbb{R}^2 \setminus Z, G) \), such that the assertions \([18]\) hold. By \([18]\) on every compact subset of \( \mathbb{R}^2 \setminus Z \) the sequence \( (g^\nu)\hat{A}^\nu \) converges to \( \hat{A}^\gamma \) in \( C^0 \), and the sequence \( (g^\nu)^{-1}\hat{u}^\nu \) converges to \( \hat{u}^\gamma \) in \( C^1 \).

**Claim 9.** We have \( Z \subseteq \{0\} \).

**Proof of Claim 9.** It follows from \((93)\) that there exists \( R > 0 \) such that

\[
\lim_{\nu \to \infty} E(w^\nu, \mathbb{R}^2 \setminus \varphi^\nu(\alpha_0)) < E_{\min}.
\]

Hence by an elementary argument, we have \( E^{R_0}(\hat{w}^\nu, B_\gamma(z)) < E_{\min} \), for every \( z \in \mathbb{R}^2 \setminus \{0\} \), for \( \nu \) large enough. Combining this with condition \([18](iii)\), the statement of Claim 9 follows. \( \square \)

Using Claim 9 it follows that \( G\hat{w}^\nu \) converges to \( G\hat{u}^\gamma \) in \( C^1 \) on every compact subset of \( \mathbb{R}^2 \setminus \{0\} \). We pass to some subsequence, such that \( \varphi^\nu \) converges to some number \( \varphi_0 \in [0, 2\pi] \). It follows that \( \hat{u}^\nu = Gu^\nu \) converges to the map \( \mathbb{C} \setminus \{0\} \ni z \mapsto G\hat{u}^\gamma(e^{i\varphi_0}/z) \in \overline{M} \), in \( C^1 \) on every compact subset of \( \mathbb{C} \setminus \{0\} \). Since \( \hat{u}^\nu \) also converges to \( \hat{u}^\gamma \), it follows that condition \([13](iii)\) holds in the case \( \gamma \in \mathcal{T}\) new, \( i = 0 \). Hence this condition is satisfied in all cases.

Condition \([13](iv)\) follows from the definition \((97)\) of \( \varphi^\nu \). This proves Claim 7. \( \square \)

**Assume now that Case (III) holds.** In this case we introduce a new vertex \( \gamma \) between \( \alpha \) and \( \beta \), corresponding to a bubble in \( \mathcal{M} \). Hence \( \alpha \) and \( \beta \) are no longer adjacent, but are separated by \( \gamma \). We define

\[
\mathcal{T}\text{ new} := \mathcal{T} \coprod \{\gamma\}, \quad V\text{ new} := V, \quad T\text{ new} := T \coprod \{\gamma\},
\]

\[
z_{\alpha\gamma} := z_{\alpha\beta}, \quad z_{\beta\gamma} := z_{\beta\alpha}, \quad z_{\gamma\alpha} := 0, \quad z_{\gamma\beta} := \infty, \quad \alpha_0 := \gamma, \quad z_{k\text{ new}} := 1.
\]

We define \( \hat{u}_\gamma : S^2 \to \overline{M} \) to be the constant map equal to \( \overline{\nabla}_{z_{\alpha\beta}}(W) \), where \( \overline{\nabla} \) is defined as in \((10)\). (In the case \( \alpha \in \mathcal{T} \) we denote \( W_\alpha := \hat{u}_\alpha \).) (The new component is a “ghost”, i.e., carries no energy.) The tuple \((W\text{ new}, z\text{ new})\) obtained from \((W, z)\) in this way is again a stable map. For every \( \alpha \in T\text{ new} \) we define \( z_{\alpha,0} \) as in \((15)\) and \((16)\), with \( i := 0 \) and w.r.t. to the new tree \( T\text{ new} \). By interchanging \( \alpha \) and \( \beta \) if necessary, we may assume w.l.o.g. that \( \beta \) is contained in the chain of edges from \( \alpha \) to \( \alpha_0 \). It follows that for every \( \alpha \neq \gamma \), \( z_{\alpha,0} = z_{\alpha,0} \), where \( z_{\alpha,0} \) is defined as in \((15)\) and \((16)\), with \( i := 0 \) and w.r.t. to the old tree.
$T$, and $z_{1,0}^{\text{new}} = z_{1,0}^{\text{new}} = \infty$. Furthermore, the hypotheses of Lemma 49 (Middle rescaling) are satisfied with

\begin{align*}
x := z_{\beta\gamma}^{\text{new}}, \quad x' := z_{\beta,0}, \quad x_{\nu} := z_{\nu}^{\nu} := (\varphi_{\beta}^{\nu})^{-1}(z_{k}^{\nu}), \quad y := z_{\alpha\gamma}^{\text{new}},
\end{align*}

\[\varphi_{\nu} := \varphi_{\alpha\beta}^{\nu} = (\varphi_{\alpha}^{\nu})^{-1} \circ \varphi_{\beta}^{\nu}.\]

We choose a sequence $\psi_{\nu}$ as in this lemma (satisfying $\psi_{\nu}(\infty) = z_{\beta,0}$).

We define

\[\varphi_{\gamma}^{\nu} := \varphi_{\beta}^{\nu} \circ \psi_{\nu}.\]

The sequence $(W_{\nu}, z_{\nu}^{\nu}, \ldots, z_{k}^{\nu})$ converges to $(W^{\text{new}}, z^{\text{new}})$ along the sequence of collections of M"obius transformations $(\varphi_{\alpha}^{\nu})_{\alpha \in T^{\text{new}}, \nu \in \mathbb{N}}$. This follows from elementary arguments, except for the proof of condition (i), which uses (94) and an argument as in Case (II).

This proves the induction step and hence terminates the proof of Theorem 1 in the general case. \(\square\)

**Remark.** In the above proof the stable map $(W, z)$ is constructed by “terminating induction”. Intuitively, this is induction over the integer $N$ occurring in Claim [I]. The “auxiliary index” $\ell$ in Claim [I] is needed to make this idea precise. Condition (vii) and the inequality (92) ensure that the “induction stops”. \(\square\)

### Appendix A. Vortices

Let $M, \omega, G, g, \langle \cdot, \cdot \rangle_{g}, \mu, J$ be as in Section [I]. (As always, we assume that hypothesis (H) is satisfied.) The next result is used in the proofs of Propositions [24] and [25]. For $r > 0$ and $z_{0} \in \mathbb{R}^{2}$ we denote by $B_{r}(z_{0})$ the open ball in $\mathbb{R}^{2}$, and we abbreviate $B_{r} := B_{r}(z_{0})$.

**Lemma 28** (Bound on energy density). Let $K \subseteq M$ be a compact subset. Then there exists a constant $E_{0} > 0$ such that the following holds. Let $z_{0} \in \mathbb{R}^{2}$, $r > 0$, $P$ be a smooth principal over $B_{r}(z_{0})$, $p > 2$, and $(A, u)$ a vortex on $P$ of class $W^{1,p}_{\text{loc}}$, such that

\[u(P) \subseteq K,\]

\[E(w, B_{r}(z_{0})) \leq E_{0}.\]

Then we have

\[e_{w}(z_{0}) \leq \frac{8}{\pi r^{2}} E(w, B_{r}(z_{0})).\]

For the proof of Lemma 28 we need the following lemma.
Lemma 29 (Heinz). Let $r > 0$ and $c \geq 0$. Then for every function $f \in C^2(B_r, \mathbb{R})$ satisfying the inequalities
\[ f \geq 0, \quad \Delta f \geq -cf^2, \quad \int_{B_r} f < \frac{\pi}{8c} \]
we have
\[ f(0) \leq \frac{8}{\pi r^2} \int_{B_r} f. \]

Proof of Lemma 29. This is [MS, Lemma 4.3.2]. □

Proof of Lemma 28. Since $G$ is compact, we may assume w.l.o.g. that $K$ is $G$-invariant. The result then follows from Theorem 32 below, the calculation in Step 1 of the proof of [GS, Proposition 11.1.], and Lemma 29. □

Lemma 28 has the following consequence.

Corollary 30 (Quantization of energy). If $M$ is equivariantly convex at $\infty$, then we have
\[ \inf_w E(w) > 0, \]
where $w = (P, A, u)$ ranges over all vortices on $\mathbb{R}^2$ with $P$ smooth and $(A, u)$ of class $W^{1, p}_{\text{loc}}$ for some $p > 2$, such that $E(w) > 0$ and $\bar{u}(P)$ is compact.

Proof of Corollary 30. This is an immediate consequence of Proposition 35 below and Lemma 28. □

This corollary implies that the minimal energy $E_V$ of a vortex on $\mathbb{R}^2$ (defined as in (25)) is positive, and therefore $E_{\text{min}} > 0$ (defined as in (26)).

The next lemma is used in the proofs of Proposition 19 and Lemma 22. It is a consequence of [GS, Lemma 9.1]. Let $(\Sigma, \omega_\Sigma, j)$ be a surface with an area form and a compatible complex structure. For $\xi \in \mathfrak{g}$ and $x \in M$ we denote by $L_x \xi = X_\xi(x) \in T_x M$ the (infinitesimal) action of $\xi$ at $x$.

Lemma 31 (Bounds on the moment map component). Let $c > 0$, $Q \subseteq \Sigma \setminus \partial \Sigma$ and $K \subseteq M$ be compact subsets, and $p > 2$. Then there exist positive constants $R_0$ and $C_p$ such that the following holds. Let $R \geq R_0$, $P$ a smooth principal bundle over $\Sigma$, and $(A, u)$ an $R$-vortex
on $P$ of class $W_{\text{loc}}^{1,p}$, such that

$$u(P) \subseteq K,$$
$$\|d_A u\|_{L^\infty(\Sigma)} \leq c,$$
$$|\xi| \leq c|L_{u(p)}\xi|, \forall p \in P, \forall \xi \in g.$$ 

Then

$$\int_Q |\mu \circ u|^p \omega_\Sigma \leq C_p R^{-2p}, \quad \sup_Q |\mu \circ u| \leq C_p R^{2/p - 2},$$

where we view $|\mu \circ u|$ as a function from $\Sigma$ to $\mathbb{R}$.

Proof of Lemma 31. This follows from the proof of [GS, Lemma 9.1], using Theorem 32. □

The next result is used in the proofs of Propositions 25 and 33, and Lemma 28.

**Theorem 32** (Regularity modulo gauge over compact surface). Let $k \in \mathbb{N} \cup \{\infty\}$, $P$ a smooth principal $G$-bundle over $\Sigma$, $p > 2$, and $(A, u)$ a vortex on $P$ of class $W^{1,p}$. Then there exists a gauge transformation $g \in W^{2,p}(\Sigma, G)$ such that $g^* w$ is smooth over $\Sigma \setminus \partial \Sigma$.

Proof of Theorem 32. This follows from the proof of [CGMS, Theorem 3.1], using a version of the local slice theorem allowing for boundary (see [Weh, Theorem 8.1]). □

The next result is used in the proof of Proposition 35 below.

**Proposition 33** (Regularity modulo gauge over $\mathbb{R}^2$). Let $R \geq 0$ be a number, $P$ a smooth principal $G$-bundle over $\mathbb{R}^2$, $p > 2$, and $w := (A, u)$ an $R$-vortex on $P$ of class $W_{\text{loc}}^{1,p}$. Then there exists a gauge transformation $g$ on $P$ of class $W_{\text{loc}}^{2,p}$ such that $g^* w$ is smooth.

The proof of Proposition 33 follows the lines of the proofs of [Fr1, Theorems 3.6 and Theorem A.3].

Proof of Proposition 33

Claim 1. There exists a collection $(g_j)_{j \in \mathbb{N}}$, where $g_j$ is a gauge transformation over $B_{j+1}$ of class $W^{2,p}$, such that for every $j \in \mathbb{N}$, we have

$$g_j^* w \text{ smooth over } B_{j+1},$$
$$g_{j+1} = g_j \text{ over } B_j.$$
Proof of Claim 1. By Theorem 32 there exists a gauge transformation $g_1 \in W^{2,p}(B_2, G)$ such that $g_1^*w$ is smooth. Let $\ell \in \mathbb{N}$ be an integer and assume by induction that there exist gauge transformations $g_j \in W^{2,p}(B_{j+1}, G)$, for $j = 1, \ldots, \ell$, such that (98) holds for $j = 1, \ldots, \ell$, and (99) holds for $j = 1, \ldots, \ell - 1$. We show that there exists a gauge transformation $g_{\ell+1} \in W^{2,p}(B_{\ell+2}, G)$ such that

$$g_{\ell+1}^*w \text{ smooth over } B_{\ell+2},$$

$$g_{\ell+1} = g_{\ell} \text{ over } B_\ell. \quad (101)$$

We choose a smooth function $\rho : \bar{B}_{\ell+2} \to B_{\ell+1}$ such that $\rho(z) = z$ for $z \in B_\ell$. By Theorem 32 there exists a gauge transformation $h \in W^{2,p}(B_{\ell+2}, G)$ such that $h^*w$ smooth over $\bar{B}_{\ell+2}$.

We define

$$g_{\ell+1} := h((h^{-1}g_\ell) \circ \rho).$$

Then $g_{\ell+1}$ is of class $W^{2,p}$ over $B_{\ell+2}$, and (101) is satisfied. Furthermore, $h^*w$ is of class $W^{k+1,p}$ over $B_{\ell+2}$, and

$$g_{\ell+1}^*w = (h^{-1}g_\ell)^*h^*w \text{ smooth over } B_{\ell+1}. \quad (100)$$

Therefore, Lemma 43(ii) below implies that $h^{-1}g_\ell$ is of class $W^{k+2,p}$ over $B_{\ell+1}$. It follows that $g_{\ell+1}^*w = ((h^{-1}g_\ell) \circ \rho)^*h^*w$ is smooth over $B_{\ell+2}$. Hence (100) is satisfied. This terminates the induction and concludes the proof of Claim 1. \hfill \square

We choose a collection $(g_j)$ as in Claim 1 and define $g$ to be the unique gauge transformation on $P$ that restricts to $g_j$ over $B_j$. This makes sense by (99). Furthermore, (98) implies that $g^*w$ is smooth. This proves Proposition 33. \hfill \square

The next result is used in the proof of Proposition 19.

**Theorem 34** (Compactness for vortices over compact surface). Let $\Sigma$ be a compact surface (possibly with boundary), $\omega_\Sigma$ an area form, $j$ a compatible complex structure on $\Sigma$, $P$ a principal $G$-bundle over $\Sigma$, $K \subseteq M$ a compact subset, $R_\nu \in [0, \infty)$, $p > 2$, and $(A_\nu, u_\nu)$ an $R_\nu$-vortex on $P$ of class $W^{1,p}$, for every $\nu \in \mathbb{N}$. Assume that $R_\nu$ converges to some $R_0 \in [0, \infty)$, and

$$u_\nu(P) \subseteq K, \quad \sup_\nu \|d_{A_\nu} u_\nu\|_{L^p(\Sigma)} < \infty.$$
Then there exist a smooth $R_0$-vortex $(A_0, u_0)$ on $P|\Sigma \setminus \partial \Sigma$ and gauge transformations $g_\nu$ on $P$ of class $W^{2,p}$, such that $g_\nu^*(A_\nu, u_\nu)$ converges to $(A_0, u_0)$, in $C^\infty$ on every compact subset of $\Sigma \setminus \partial \Sigma$.

Proof of Theorem 34. This follows from a modified version of the proof of [CGMS] Theorem 3.2: We use a version of Uhlenbeck compactness for a compact base with boundary, see Theorem 41 below, and a version of the local slice theorem allowing for boundary, see [Weh], Theorem 8.1. Note that the proof carries over to case in which $R_\nu = 0$ for some $\nu \in \mathbb{N}$, or $R_0 = 0$. □

The following result was used in the proofs of Theorem 1 and Corollary 30.

**Proposition 35** (Boundedness of image). Assume that $M$ is equivariantly convex at $\infty$. Then there exists a $G$-invariant compact subset $K_0 \subseteq M$ such that the following holds. Let $p > 2$, $P$ a principal $G$-bundle over $\mathbb{R}^2$, and $(A, u)$ a vortex on $P$ of class $W^{1,p}_{\text{loc}}$, such that $E(u) < \infty$ and $u(P)$ is compact. Then we have $u(P) \subseteq K_0$.

Proof of Proposition 35. Let $P$ be a principal $G$-bundle over $\mathbb{R}^2$. By an elementary argument every smooth vortex on $P$ is smoothly gauge equivalent to a smooth vortex that is in radial gauge outside $B_1$. Using Proposition 33 it follows that every vortex on $P$ of class $W^{1,p}_{\text{loc}}$ is gauge equivalent to a smooth vortex that is in radial gauge outside $B_1$. Hence the statement of Proposition 35 follows from [GS], Proposition 11.1]. □

The following lemma was used in the proof of Proposition 11. Consider the action of the group of translations of $\mathbb{R}^2$ on the set of equivalence classes of smooth vortices over $\mathbb{R}^2$ given by (14).

**Lemma 36.** The restriction of this action to the set of vortices of finite positive energy is free.

Proof of Lemma 36. Assume that $W$ is a smooth vortex over $\mathbb{R}^2$ and $v \in \mathbb{R}^2$ is a vector, such that defining $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $Tz := z + v$, we have $T^*W = W$. Then $e_W(z + nv) = e_W(z)$ for every $z \in \mathbb{R}^2$ and $n \in \mathbb{Z}$. It follows that $E(W) = \infty$, $e_W \equiv 0$, or $v = 0$. Lemma 36 follows from this. □

**Appendix B. Further auxiliary results**

The proof of Proposition 25 (Energy concentration at ends) is based on an isoperimetric inequality for the invariant action functional (Theorem 34).
Building on work by D. A. Salamon and R. Gaio, we define this functional as follows. (This is the definition from [Zi2], written in a more intrinsic way.)

We first review the usual symplectic action functional: Let \((M, \omega)\) be a symplectic manifold without boundary. We fix a Riemannian metric \(\langle \cdot, \cdot \rangle_M\) on \(M\), and denote by \(d, \exp, |v|, \iota_x > 0, \text{ and } \iota_X := \inf_{x \in X} \iota_x \geq 0\) the distance function, the exponential map, the norm of a vector \(v \in TM\), and the injectivity radii of a point \(x \in M\) and a subset \(X \subseteq M\), respectively. We define the symplectic action of a loop \(x : S^1 \rightarrow M\) of length \(\ell(x) < 2\iota_{S^1}\) to be

\[
A(x) := -\int_{\mathbb{D}} u^* \omega.
\]

Here \(\mathbb{D} \subseteq \mathbb{R}^2\) denotes the (closed) unit disk, and \(u : \mathbb{D} \rightarrow M\) is any smooth map such that

\[
u(e^{it}) = x(t), \forall t \in \mathbb{R}/(2\pi \mathbb{Z}) \cong S^1, \quad d(u(z), u(z')) < \iota_{S^1}, \forall z, z' \in \mathbb{D}.
\]

**Lemma 37.** The action \(A(x)\) is well-defined, i.e., a map \(u\) as above exists, and \(A(x)\) does not depend on the choice of \(u\).

**Proof.** The lemma follows from an elementary argument, using the exponential map \(\exp_{x(0+z)} : T_{x(0+z)}M \rightarrow M\).

Let now \(G\) be a compact connected Lie group with Lie algebra \(\mathfrak{g}\). Suppose that \(G\) acts on \(M\) in a Hamiltonian way, with (equivariant) moment map \(\mu : M \rightarrow \mathfrak{g}^*\), and that \(\langle \cdot, \cdot \rangle_M\) is \(G\)-invariant. We denote by \(\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}\) the natural contraction. Let \(P\) be a smooth principal \(G\)-bundle over \(S^1\) and \(x \in C^\infty_G(P, M)\). We call \((P, x)\) admissible iff there exists a section \(s : S^1 \rightarrow P\) such that \(\ell(s) < 2\iota_{S^1}\), and

\[
A(g(x \circ s)) - A(x \circ s) = \int_{S^1} \langle \mu \circ x \circ s, g^{-1}dg \rangle,
\]

for every \(g \in C^\infty(S^1, G)\) satisfying \(\ell(g(x \circ s)) \leq \ell(x \circ s)\).

**Definition 38.** Let \((P, x)\) be an admissible pair, and \(a\) be a connection on \(P\). We define the invariant symplectic action of \((P, a, x)\) to be

\[
A(P, a, x) := A(x \circ s) + \int_{S^1} \langle \mu \circ x \circ s, a ds \rangle,
\]

where \(s : S^1 \rightarrow P\) is a section as above.
(This is a modified version of the “local equivariant symplectic action functional” introduced by A. R. Gaio and D. A. Salamon in [GS].) To formulate the isoperimetric inequality, we need the following. If $X$ is a manifold, $P$ a principal $G$-bundle over $X$ and $u \in C_G^\infty(P, M)$, then we define $\bar{u} : X \to M$ by $\bar{u}(y) := Gu(p)$, where $p \in P$ is any point in the fiber over $y$. We define $M^\ast$ as in [17]. For a loop $\bar{x} : S^1 \to M^\ast/G$ we denote by $\bar{\ell}(\bar{x})$ its length w.r.t. the Riemannian metric on $M^\ast/G$ induced by $\langle \cdot, \cdot \rangle_M$. Furthermore, for each subset $X \subseteq M$ we define
\[
m_X := \inf \{|L_x\xi| \mid x \in X, \xi \in g : |\xi| = 1\}.
\]
The next result is Theorem 1.2 in [Z2].

**Theorem 39** (Sharp isoperimetric inequality). Assume that there exists a $G$-invariant $\omega$-compatible almost complex structure $J$ such that $\langle \cdot, \cdot \rangle_M = \omega(\cdot, J\cdot)$. Then for every compact subset $K \subseteq M^\ast$ and every constant $c > \frac{1}{2}$ there exists a constant $\delta > 0$ with the following property. Let $P$ be a principal $G$-bundle over $S^1$ and $x \in C_G^\infty(P, M)$, such that $x(P) \subseteq K$ and $\bar{\ell}(\bar{x}) \leq \delta$. Then $(P, x)$ is admissible, and for every connection $a$ on $P$ we have
\[
|A(P, a, x)| \leq c\|d_a x\|^2 + \frac{1}{2\delta^2} \|\mu \circ x\|^2.
\]

Here we view $d_a x$ as a one-form on $S^1$ with values in the bundle $(x^*TM)/G \to S^1$, and $\mu \circ x$ as a section of the co-adjoint bundle $(P \times g^\ast)/G \to S^1$. Furthermore, $S^1$ is identified with $\mathbb{R}/(2\pi \mathbb{Z})$, and the norms are taken with respect to the standard metric on $\mathbb{R}/(2\pi \mathbb{Z})$, the metric $\langle \cdot, \cdot \rangle_M$ on $M$, and the operator norm on $g^\ast$ induced by $\langle \cdot, \cdot \rangle_g$. (Note that in [Z2], Theorem 1.2] $S^1$ was identified with $\mathbb{R}/\mathbb{Z}$ instead. Note also that hypothesis (H) is not needed for Theorem 39.)

In the proof of Proposition 25 we also used the following result. For $s \in \mathbb{R}$ we denote by $t_s : S^1 \to \mathbb{R} \times S^1$ the map given by $t_s(t) := (s, t)$. Let $X, X'$ be manifolds, $f \in C^\infty(X', X)$, $P$ a principal $G$-bundle over $X$, $A$ a connection on $P$, and $u \in C_G^\infty(P, M)$. Then the pullback triple $f^*(P, A, u)$ consists of a principal $G$-bundle $P'$ over $X'$, a connection on $P'$, and an equivariant map from $P'$ to $M$.

**Proposition 40** (Energy action identity). For every compact subset $K \subseteq M^\ast$ there exists a constant $\delta > 0$ with the following property. Let $s_- \leq s_+$ be numbers, $\Sigma := [s_-, s_+] \times S^1$, $\omega_\Sigma$ an area form on $\Sigma$, $J$ a compatible complex structure, and $w := (A, u)$ a smooth vortex over $\Sigma$ (with respect to $(\omega_\Sigma, J)$, such that $u(P) \subseteq K$ and $\bar{\ell}(\bar{u} \circ t_s) < \delta$, for every
s ∈ [s_−, s_+]. Then the pairs \( \iota_{s_+}^*(P, u) \) are admissible, and
\[
E(w, \Sigma) = -A(\iota_{s_+}^*(P, A, u)) + A(\iota_{s_-}^*(P, A, u)).
\]

**Proof of Proposition 40.** This follows from [Zi2, Proposition 3.1]. □

The next result is used in the proof of Proposition 19. It is [Weh, Theorem A]. See also [Uh, Theorem 1.5].

**Theorem 41 (Uhlenbeck compactness).** Let \( n \in \mathbb{N} \), \( G \) be a compact smooth Riemannian \( n \)-manifold (possibly with boundary), \( P \) a principal \( G \)-bundle over \( X \), \( p > n/2 \) a number, and \( A_\nu \) a sequence of connections on \( P \) of class \( W^{1,p} \). Assume that
\[
\sup_{\nu \in \mathbb{N}} \| F_{A_\nu} \|_{L^p(X)} < \infty.
\]

Then passing to some subsequence there exist gauge transformations \( g_\nu \) of class \( W^{2,p} \), such that \( g_\nu^* A_\nu \) converges weakly in \( W^{1,p} \).

The next result was used in the proof of Proposition 19. Its proof goes along the lines of the proof of [MS, Proposition B.4.2].

**Proposition 42 (Compactness for \( \bar{\partial}_J \)).** Let \( M \) be a manifold without boundary, \( k \in \mathbb{N} \), \( J \) an almost complex structure on \( M \) of class \( C^k \), \( \Omega_1 \subseteq \Omega_2 \subseteq \ldots \subseteq \mathbb{C} \) open subsets, and \( u_\nu : \Omega_\nu \to M \) a sequence of functions of class \( W^{k,p}_{\text{loc}} \). Assume that \( \bar{\partial}_J u_\nu \) is of class \( W^{k+1,p}_{\text{loc}} \), for every \( \nu \), and that for every open subset \( \Omega \subseteq \bigcup_\nu \Omega_\nu \) with compact closure the following holds. If \( \nu_0 \in \mathbb{N} \) is so large that \( \Omega \subseteq \Omega_{\nu_0} \) then
\[
\exists K \subseteq M \text{ compact: } u_\nu(\Omega) \subseteq K, \quad \forall \nu \geq \nu_0.
\]
\[
\sup_{\nu \geq \nu_0} \| \partial_{\bar{\partial}_J} u_\nu \|_{L^p(\Omega)} < \infty,
\]
\[
\sup_{\nu \geq \nu_0} \| \bar{\partial}_J u_\nu \|_{W^{k+1,p}(\Omega)} < \infty.
\]

Then there exists a subsequence of \( u_\nu \) that converges weakly in \( W^{k+1,p} \) and in \( C^k \) on every compact subset of \( \bigcup_\nu \Omega_\nu \).

The next lemma is used in the proofs of Propositions 19 and 33, and Proposition I.

**Lemma 43 (Regularity of the gauge transformation).** Let \( X \) be a smooth manifold, \( G \) a compact Lie group, \( P \) a principal \( G \)-bundle over \( X \), \( k \in \mathbb{N} \cup \{0\} \), and \( p > \dim X \). Then the following assertions hold.

(i) Let \( g \) be a gauge transformation of class \( W^{1,p}_{\text{loc}} \) and \( A \) a connection on \( P \) of class \( C^k \), such that \( g^* A \) is of class \( C^k \). Then \( g \) is of class \( C^{k+1} \).
Assume that $X$ is compact (possibly with boundary). Let $U$ be a subset of the space of $W^{k,p}$-connections on $P$ that is bounded in $W^{k,p}$. Then there exists a $W^{k+1,p}$-bounded subset $V$ of the set of $W^{k+1,p}$-gauge transformations on $P$, such that the following holds. Let $A \in U$ and $g$ be a $W^{1,p}$-gauge transformation, such that $g^*A \in U$. Then $g \in V$.

**Proof of Lemma 43.** This follows from induction in $k$, using the equality $dg = g(g^*A) - Ag$ and Morrey’s inequality (for (ii)). (See [Weh] Lemma A.8.)

The next proposition is used in the proof of Proposition 20 (Quantization of energy loss).

**Proposition 44.** Let $n \in \mathbb{N}$, $G$ a compact Lie group, $P$ be a principal $G$-bundle over $\mathbb{R}^n$, and $A, A'$ smooth flat connections on $P$. Then there exists a smooth gauge transformation $g$ such that $A' = g^*A$.

**Proof of Proposition 44.** (In the case $n = 2$, see also [Fr1] Corollary 3.7.) In the case $n = 1$ such a $g$ exists, since then the condition $A' = g^*A$ can be viewed as an ordinary differential equation for $g$. Let $n \in \mathbb{N}$ and assume by induction that we have already proved the statement for $n$. Let $P$ be a principal $G$-bundle over $\mathbb{R}^{n+1}$, and $A, A'$ smooth flat connections on $P$. We define $\iota: \mathbb{R}^n \to \mathbb{R}^{n+1}$ by $\iota(x) := (x, 0)$. By the induction hypothesis there exists a smooth gauge transformation $g_0$ on $\iota^*P \to \mathbb{R}^n$, such that

$$g_0^*\iota^*A = \iota^*A'. \tag{105}$$

Since $P$ is trivializable, there exists a smooth gauge transformation $\tilde{g}_0$ on $P$ such that $\iota^*\tilde{g}_0 = g_0$.

Let $x \in \mathbb{R}^n$. We define $\iota_x: \mathbb{R} \to \mathbb{R}^{n+1}$ by $\iota_x(t) := (x, t)$. There exists a unique smooth gauge transformation $h_x$ on $\iota_x^*P \to \mathbb{R}$, such that

$$h_x^*\iota_x^*\tilde{g}_0^*A = \iota_x^*A', \quad h_x(p) = 1, \quad \forall p \in \text{fiber of } \iota_x^*P \text{ over } 0 \in \mathbb{R}. \tag{106}$$

To see this, note that these conditions can be viewed as an ordinary differential equation for $h_x$ with prescribed initial value. Since this solution depends smoothly on $x$, there exists a unique smooth gauge transformation $h$ on $P$ such that $\iota_x^*h = h_x$, for every $x \in \mathbb{R}^n$. The gauge transformation $g := \tilde{g}_0h$ on $P$ satisfies the equation $A' = g^*A$. This follows from (105,106) and flatness of $A$ and $A'$.

The next result was used in the proofs of Proposition 19, Remark 23, and Theorem 1. Let $M, \omega, G, g, \langle \cdot, \cdot \rangle, \mu, J, \Sigma, \omega_\Sigma, j$ be as in Section...
We define the almost complex structure $\bar{J}$ on $\overline{M}$ as in (8). For every open subset $\Omega \subseteq S^2 \cong \mathbb{C} \cup \{\infty\}$ the energy density of a map $f \in W^{1,p}(\Omega, \overline{M})$ is given by
\[ e_f(z) := \frac{1}{2} |df|^2, \]
where the norm is with respect to the metrics $\omega_\Sigma(\cdot, j \cdot)$ on $\Sigma$ and $\overline{\omega}(\cdot, \bar{J} \cdot)$ on $\overline{M}$. Let $P$ be a smooth principal bundle over $\Sigma$, $A$ a connection on $P$, and $u : P \to M$ an equivariant map. We define
\[ e_\infty A,u = \frac{1}{2} |d_A u|^2, \]
where the norm is taken with respect to the metrics $\omega_\Sigma(\cdot, j \cdot)$ on $\Sigma$ and $\omega(\cdot, J \cdot)$ on $M$. Furthermore, we define $\bar{u} : \Sigma \to \overline{M}, \quad \bar{u}(z) := Gu(p),$
where $p \in P$ is an arbitrary point in the fiber over $z$.

**Proposition 45** (Pseudo-holomorphic curves in symplectic quotient). Let $P$ be a smooth principal $G$-bundle over $\Sigma$, $p > 2$, $A$ a $W^{1,p}_{\text{loc}}$-connection on $P$, and $u : P \to M$ a $G$-equivariant map of class $W^{1,p}_{\text{loc}}$, such that $\mu \circ u = 0$. Then we have
\[ e_{\bar{u}} = e_\infty A,u. \]
If $(A, u)$ also solves the equation $\bar{\partial}_{J,A}(u) = 0$ then
\[ \bar{\partial}_{J,\bar{u}} = 0. \]

**Proof of Proposition 45.** This follows from an elementary argument. For the second part see also [Ga, Section 1.5].

In the proof of Theorem 1 we used the following lemma.

**Lemma 46** (Bound for tree). Let $k \in \mathbb{N} \cup \{0\}$ be a number, $(T, E)$ a finite tree, $\alpha_1, \ldots, \alpha_k \in T$ vertices, $f : T \to [0, \infty)$ a function, and $E_0 > 0$ a number. Assume that for every vertex $\alpha \in T$ we have
\[ f(\alpha) \geq E_0 \quad \text{or} \quad \#\{ \beta \in T \mid \alpha \beta \} + \#\{ i \in \{1, \ldots, k\} \mid \alpha_i = \alpha \} \geq 3. \]
Then
\[ \#T \leq \frac{2 \sum_{\alpha \in T} f(\alpha)}{E_0} + k. \]

**Proof of Lemma 46.** This follows from an elementary argument. (It is Exercise 5.1.2. in the book [MS].)
We used the following facts about sequences of Möbius transformations in the proof of the Bubbling Theorem in the case $k \geq 1$.

**Remark 47.** Let $x, y \in S^2$ be points and $\varphi_\nu$ a sequence of Möbius transformations that converges to $y$, uniformly on every compact subset of $S^2 \setminus \{x\}$. Then $\varphi_\nu^{-1}$ converges to $x$, uniformly on every compact subset of $S^2 \setminus \{y\}$. This follows from an elementary argument. (It is Exercise D.1.3 in the book [MS].) □

**Lemma 48** (Convergence for Möbius transformations). Let $\varphi_\nu$ be a sequence of Möbius transformations and $x, y \in S^2$ be points. Suppose there exist convergent sequences $x_1^\nu, x_2^\nu, y^\nu \in S^2$ such that

\[
\begin{align*}
&x \neq \lim_{\nu \to \infty} x_1^\nu \neq \lim_{\nu \to \infty} x_2^\nu \neq x, \\
&y \neq \lim_{\nu \to \infty} y^\nu, \\
&\lim_{\nu \to \infty} \varphi_\nu(x_1^\nu) = \lim_{\nu \to \infty} \varphi_\nu(x_2^\nu) = y, \\
&\lim_{\nu \to \infty} \varphi_\nu^{-1}(y^\nu) = x.
\end{align*}
\]

Then $\varphi_\nu$ converges to $y$, uniformly with all derivatives on every compact subset of $S^2 \setminus \{x\}$.

*Proof.* This follows from [MS, Lemmata D.1.4 and 4.6.6]. □

**Lemma 49** (Middle rescaling). Let $x, x_\nu, y \in S^2$ be points and $\varphi_\nu$ be a sequence of Möbius transformations that converges to $y$, uniformly on compact subsets of $S^2 \setminus \{x\}$, such that $x_\nu$ converges to $x$ and $\varphi_\nu(x_\nu)$ converges to $y$. Then there exists a sequence of Möbius transformations $\psi_\nu$ such that $\psi_\nu(1) = x_\nu$, $\psi_\nu$ converges to $x$, uniformly with all derivatives on compact subsets of $S^2 \setminus \{\infty\}$, and $\varphi_\nu \circ \psi_\nu$ converges to $y$, uniformly with all derivatives on compact subsets of $S^2 \setminus \{0\}$. Moreover, if $x' \neq x$ is any point in $S^2$ then we may choose $\psi_\nu$ such that $\psi_\nu(\infty) = x'$.

*Proof of Lemma 49.* Let $x' \neq x$ and $y'' \neq y$ be any two points in $S^2$. It follow from Remark 47 that for $\nu$ large enough the three points $x''_\nu := \varphi_\nu^{-1}(y'')$, $x_\nu$, $x'$ are all distinct. W.l.o.g. we may assume that this holds for every $\nu$. For $\nu \in \mathbb{N}$ we define $\psi_\nu$ to be the unique Möbius transformation such that

\[
\psi_\nu(0) = x''_\nu, \quad \psi_\nu(1) = x_\nu, \quad \psi_\nu(\infty) = x'.
\]

Then the hypotheses of Lemma 48 with $\varphi_\nu, x, y$ replaced by $\psi_\nu, \infty, x$ and $x''_\nu := 0$, $x'_2 := 1$ and $y''_\nu := x'$ are satisfied. Hence by that Lemma the maps $\psi_\nu$ converge to $x$, uniformly with all derivatives on compact subsets of $S^2 \setminus \{\infty\}$. Moreover, the hypotheses of the same lemma with $\varphi_\nu, x$ replaced by $\varphi_\nu \circ \psi_\nu, 0$ and $x''_1 := 1$, $x'_2 := \infty$ and $y''_\nu := y''$ are satisfied. It follows that $\varphi_\nu \circ \psi_\nu$ converges to $y$, uniformly with all derivatives on compact subsets of $S^2 \setminus \{0\}$. This proves Lemma 49. □
The next result was used in the proof of Proposition 24. Let \((X,d)\) be a metric space. \((d\) is allowed to attain the value \(\infty\).) Let \(G\) be a topological group and \(\rho : G \times X \to X\) a continuous action by isometries. By \(\pi : X \to X/G\) we denote the canonical projection. The topology on \(X\), determined by \(d\), induces a topology on the quotient \(X/G\).

**Lemma 50** (Induced metric on the quotient). Assume that \(G\) is compact. Then the map \(\bar{d} : X/G \times X/G \to [0, \infty]\) defined by

\[
\bar{d}(\bar{x}, \bar{y}) := \min_{x \in \bar{x}, y \in \bar{y}} d(x, y)
\]

is a metric on \(X/G\) that induces the quotient topology on \(X/G\).

**Proof of Lemma 50**. This follows from an elementary argument. \(\square\)

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