Cotton flow

Ali Ulaş Özgür Kışisel\textsuperscript{1}, Özgür Sarıoğlu\textsuperscript{2} and Bayram Tekin\textsuperscript{2}

\textsuperscript{1} Department of Mathematics, Faculty of Arts and Sciences, Middle East Technical University, 06531, Ankara, Turkey
\textsuperscript{2} Department of Physics, Faculty of Arts and Sciences, Middle East Technical University, 06531, Ankara, Turkey

E-mail: akisisel@metu.edu.tr, sarioglu@metu.edu.tr and btekin@metu.edu.tr

Received 17 March 2008, in final form 17 June 2008
Published 5 August 2008
Online at stacks.iop.org/CQG/25/165019

Abstract

Using the conformally-invariant Cotton tensor, we define a geometric flow, the Cotton flow, which is exclusive to three dimensions. This flow tends to evolve the initial metrics into conformally flat ones, and is somewhat orthogonal to the Yamabe flow, the latter being a flow within a conformal class. We define an entropy functional, and study the flow of nine homogeneous spaces both numerically and analytically. In particular, we show that the arbitrarily deformed homogeneous 3-sphere flows into the round 3-sphere. Two of the nine homogeneous geometries, which are degenerated by the Ricci flow, are left intact by the Cotton flow.

PACS numbers: 02.40.Ky, 02.40.Ma, 02.40.Vh, 04.60.Kz

1. Introduction

In this paper, inspired by Hamilton’s work on the Ricci flow\cite{1} that culminated in Perelman’s proof of Thurston’s geometrization conjecture and the longstanding Poincaré conjecture\cite{2,3}, we introduce a new flow, based on the conformally-invariant Cotton tensor. This flow is exclusive to three dimensions, and its fixed points are the conformally flat metrics.

Historically, the Ricci flow equations seem to have first arisen in a work by Friedan\cite{4} that deals with the renormalization group (RG) flow in a two-dimensional nonlinear sigma model. Just like any other coupling in quantum field theories, the Riemannian metric, considered as a coupling ‘constant’ between the sigma model fields defined on a manifold $M$, runs as the energy scale changes:

$$\beta_g = \partial_t g_{ij} = k \alpha' R_{ij},$$

where $k$ is a constant uniquely fixed by the 1-loop result, and $\alpha'$ is a dimensionful parameter (such as the tension). Of course, in general, the beta function for the RG flow of the metric will have infinitely many terms, including the Ricci tensor. In fact, in\cite{4}, the 2-loop result
with a Riemann-square correction is given. In string theory, which is again a two-dimensional sigma model with additional fields, gravitation (including general relativity as the lowest order approximation) that governs the dynamics of the target space, is derived by the requirement that the beta function vanishes as a result of Weyl invariance on the worldsheet. Since a non-anomalous Weyl symmetry is crucial for string theory, in the physics literature, it is common practice to set $\beta_g = 0$ and consequently, end up not with a flow, but with a usual gravity theory (containing perhaps some additional fields and higher curvature terms). On the other hand, Hamilton [1] introduced the Ricci flow as a means to deform a given metric on a manifold and developed a program aiming to prove the Poincaré conjecture. In particular, he classified closed 3-manifolds of positive Ricci curvature.

Obviously, one could introduce many different flows, or depending on one’s point of view, derive flows that come from the beta functions of two-dimensional nonlinear sigma models. As examples of the former, see [5–7]. For an example of the latter, see [8] where a string inspired flow with scalar, Maxwell and antisymmetric tensor fields is studied.

Our motivation was to find a flow whose critical points are the conformally flat spaces, and this is achieved in three dimensions with a unique 2-tensor called the Cotton tensor. We study the behaviour of the nine homogeneous geometries under this flow in detail. Four of these are fixed points of our flow. (We remark that the eight geometries appearing in Thurston’s geometrization conjecture are maximally symmetric; viewed in this way, two of the nine $(\text{Isom}(\mathbb{R}^2))$ and $\mathbb{R}^3$) collapse to the same maximally symmetric geometry.)

Interesting physics could come out of the Ricci flow or other flows such as that we introduce here, especially in Euclidean quantum gravity defined by the path integral. This is somewhat an unexplored territory. Yet, we can point out several works in this direction: In [9], the Ricci flow is applied to four-dimensional black hole physics as a means to understand the phase structure of this theory in Euclidean space. The Ricci flow is also used [10] to find inequalities in general relativity regarding the evolution of the area of a surface and the enclosed Hawking mass. See [11] for a compilation of possible applications of the Ricci flow in physics.

The outline of the paper is as follows. In section 2, we introduce the Cotton flow and show how some geometric quantities evolve accordingly. In section 3, we give an entropy functional which is non-decreasing under the Cotton flow. This also gives us a gradient formulation of the Cotton flow. In section 4, we show that the Cotton flow is in a certain sense orthogonal to the Yamabe flow. Section 5 is the bulk of our paper where we study the evolution of homogeneous spaces under the Cotton flow both numerically and analytically. In the conclusion, we summarize our results and mention some topics for further study. Details of the computations regarding the homogeneous geometries are given in the appendices.

2. The Cotton flow

In dimensions greater than or equal to four, the Weyl tensor determines whether or not a given manifold is locally conformally flat. In three dimensions, the Weyl tensor vanishes identically, and the role of the Weyl tensor is played by the Cotton tensor [12–16]

$$ C^{ij} = \epsilon^{imn} \nabla_m \left( R^j_n - \frac{1}{4} \delta^j_n R \right), $$

where $\epsilon$ is a tensor density, in an orthonormal frame $\epsilon^{123} = 1$, $R_{ij}$ is the Ricci tensor, and $R$ is the curvature scalar. (Note that the Cotton tensor is also referred to as the ’Cotton–York tensor’ since York reintroduced [15] and used it extensively in the initial value formulation of general relativity.) Here and throughout, the signature of the space will be $(+, +, +)$, the
indices will range from \(\{1, 2, 3\}\), and the sign convention for the Riemann tensor will follow from \([\nabla_i \nabla_j]v_k = R_{ijk}^l v_l\). The tensor \(C^{ij}\) is covariantly conserved, symmetric and traceless (see [17], where the Cotton tensor was used to introduce the ‘topologically massive gravity’). The symmetry can be directly seen from the following representation of \(C^{ij}\):

\[
C^{ij} = \frac{1}{2\sqrt{g}}(\epsilon^{imn} \nabla_m R^i_n + \epsilon^{imn} \nabla_n R^i_m).
\]  

(3)

The tensor \(C^i_j\) is conformally invariant with weight 0.

Let \(M\) be a smooth 3-manifold with a positive definite metric \(g_{ij}\). We will consider the flow defined by the equation

\[
\partial_t g_{ij} = \kappa C_{ij},
\]

(4)

where \(\kappa\) is a positive constant (note that \(t\) is not one of the coordinates of the manifold). Without loss of generality, we may scale \(t\) to set \(\kappa = 1\). The reason for taking \(\kappa > 0\) is to make homogeneous metrics on the 3-sphere converge to the round sphere rather than diverge from it. Since the right-hand side is traceless, \(\sqrt{g}\) is independent of \(t\) although it could depend on the coordinates. In contrast, the Ricci tensor is not traceless in general, so a normalization factor is necessary for the Ricci flow (1) in order for the volume density or volume to be preserved.

Assuming that \(g_{ij}\) flows under the evolution equation (4), one can compute how some other geometric quantities evolve:

\[
\partial_t C_{ij} = \epsilon^{m(k} (\partial_m C_{k}^{|i|} + \partial_m C_{|i|}^k - \frac{1}{2} \nabla^2 C_{ij}),
\]

(5)

\[
\partial_t R_{ij} = \frac{3}{2} R_{(i}R_{j)} - R_{i}^{(m}C_{j)m} - \frac{1}{2} \nabla^2 R_{ij},
\]

\[
\partial_t R = -C_{ij} R_{ij},
\]

\[
\partial_t (R_{jk} R_{jk}) = 4 R_{j}C_{i} R_{ik} - 3 R R_{jk} C_{jk} - R_{jk} \nabla^2 C_{jk},
\]

(5)

where the usual shorthand notation for symmetrization is used: \(A_{(i|j|k)} = (A_{ijk} + A_{kji})/2\).

The Cotton flow equations form a system of third-order, nonlinear partial differential equations. About a flat background, the linearization of the system, setting \(g_{ij} = \delta_{ij} + h_{ij}\), gives

\[
\partial_t h^{ij} = -\frac{1}{4} \partial^2 \partial_k (\epsilon^{ikl} h^j_l + \epsilon^{jkl} h^i_l) + \partial^i \xi^j + \partial^j \xi^i,
\]

(6)

where

\[
\xi^i = -\frac{1}{4} \epsilon^{ikl} \partial_k \partial^a h_{al},
\]

and \(\partial^2\) denotes the Laplacian in flat space. The \(\xi\) terms can be viewed as coming from a diffeomorphism of the manifold. Unlike the Ricci flow case, whose linearized form is of heat equation type in the highest order after DeTurck’s modification by a diffeomorphism [18], our equation is of third order. In this respect, it seems impossible to apply results from elliptic operator theory directly for the short time existence problem. Understanding the conditions for the initial metric under which the equations are locally well posed seems to be a delicate problem, which will not be studied in this paper.

### 3. Cotton entropy

In order to show that the Ricci flow can be regarded as a gradient flow, Perelman [2] defined an entropy functional \(\mathcal{F}(g, f) = \int_M (R + |\nabla f|^2) e^{-f} \sqrt{g} \ d^3x\), where \(g\) denotes the metric, and \(f\) is a scalar field. He proved that if \(\partial_t f = \partial_t \ln \sqrt{g}\) and \(g\) obeys the Ricci flow equations,
then $\mathcal{F}$ is a (non-strictly) increasing functional. For the Cotton flow, an analogous functional is the well-known gravitational Chern–Simons action [17]:

$$\mathcal{F}(g) = -\frac{1}{2} \int_M \epsilon^{ijk} \Gamma^l_{im} (\partial_j \Gamma^m_{kl} + \frac{2}{3} \Gamma^m_{ja} \Gamma^n_{kl}) \, d^3x.$$  \hspace{1cm} (8)

Note the absence of an extra scalar field, in contrast to the case of the Ricci flow. The functional $\mathcal{F}$ is conformally invariant, and up to a boundary term, diffeomorphism invariant. Defining $\partial_t g_{ij} = v_{ij}$, after a straightforward but lengthy computation, one obtains

$$\frac{d\mathcal{F}}{dt} = \int_M \sqrt{g} v_{ij} C^{ij} \, d^3x.$$  \hspace{1cm} (9)

The choice $v_{ij} = C_{ij}$ gives the steepest descent, and leads to the Cotton flow $\partial_t g_{ij} = C_{ij}$. The functional $\mathcal{F}(g)$ is increasing if $g$ evolves under the Cotton flow. The Cotton entropy $\mathcal{F}(g)$ is constant if and only if $C_{ij}$ is identically zero, which means that $M$ is locally conformally flat at all points.

### 4. Cotton flow and the Yamabe flow

One can define various flows in the space of metrics. One of these is the Yamabe flow defined by the equation [6]

$$\partial_\tau g_{ij} = -\mathcal{R} g_{ij}. $$  \hspace{1cm} (10)

Under this flow, the conformal class of a metric does not change. The Yamabe flow was introduced in order to solve Yamabe’s conjecture [6], which states that any metric is conformally equivalent to a metric with constant scalar curvature. The Cotton flow has a somewhat complementary behaviour since by definition it changes the conformal class unless the metric is conformally flat. We will now make this behaviour precise in a sense, by computing the commutator of the vector fields for these two flows.

Let $t$ and $\tau$ denote, respectively, the parameters for the Cotton and the Yamabe flows. Then

$$[\partial_t, \partial_\tau] g_{ij} = \partial_t \partial_\tau g_{ij} - \partial_\tau \partial_t g_{ij} = -\partial_t (\mathcal{R} g_{ij}) - \partial_\tau C_{ij}. $$  \hspace{1cm} (11)

Since $C_{ij}$ is conformally invariant, $\partial_\tau C_{ij} = 0$. From here, it follows that $\partial_t C_{ij} = -\mathcal{R} C_{ij}$. Using the third equality of (5), one gets $\partial_t (\mathcal{R} g_{ij}) = \mathcal{R} C_{ij} - R^{kl} C_{kl} g_{ij}$. Therefore,

$$[\partial_t, \partial_\tau] g_{ij} = R^{kl} C_{kl} g_{ij}. $$  \hspace{1cm} (12)

Since the result is proportional to the metric, the commutator of the two flows gives another flow preserving the conformal class, just like the Yamabe flow.

### 5. Cotton flow on homogeneous 3-manifolds

In this section, we will be interested in studying the behaviour of the Cotton flow on homogeneous 3-manifolds. There are nine such homogeneous geometries: $SU(2), SL(2, \mathbb{R}), Isom(\mathbb{R}^2), Solv, Nil, \mathbb{R}^3, H^3, S^2 \times \mathbb{R}, H^2 \times \mathbb{R}$. As explained in the introduction, they give rise to the eight maximally symmetric geometries which are the basic building blocks that take the stage in Thurston’s geometrization conjecture for 3-manifolds [19]. This conjecture, which includes the Poincaré conjecture as a special case, was settled by Perelman’s work [2, 3].
The first six of these nine geometries can be obtained by the following construction: Suppose that \( G \) is a three-dimensional, unimodular, simply connected Lie group, and that \( g \) is a left-invariant metric on \( G \). Let \( M \) be a compact 3-manifold obtained as the quotient of \( G \) with respect to a discrete subgroup. The analysis of the Ricci flow on these manifolds was carried out in detail in [20, 21] (see also [22]). (A similar analysis for cross curvature flow was carried out in [23].) As in the case of the Ricci flow, the Cotton flow equations on a homogeneous 3-manifold reduce to a set of three coupled ordinary, nonlinear, autonomous differential equations.

We now proceed to find this system of differential equations. In [24], it is shown that there exists an orthogonal left-invariant frame \( \{ F_1, F_2, F_3 \} \) on \( G \) such that \( \{ F_1, F_2, F_3 \} = 2\nu F_1, [F_2, F_3] = 2\lambda F_1, [F_3, F_1] = 2\mu F_2 \) where \( \mu, \nu, \lambda \in \{-1, 0, 1\} \) (see also [22]). Let \( \{ \omega_1, \omega_2, \omega_3 \} \) be the dual basis of 1-forms to \( \{ F_1, F_2, F_3 \} \). In this basis, the metric \( g \) takes the form

\[
g = A(t)\omega_1 \otimes \omega_1 + B(t)\omega_2 \otimes \omega_2 + C(t)\omega_3 \otimes \omega_3. \tag{13}\]

Defining \( e^1 = \sqrt{A} \omega^1, e^2 = \sqrt{B} \omega^2, e^3 = \sqrt{C} \omega^3 \) we obtain an orthonormal coframe, with respect to which the metric takes the form \( g = e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 \). One has

\[
d\omega^1 = 2\lambda \omega^2 \wedge \omega^3, \quad d\omega^2 = 2\nu \omega^3 \wedge \omega^1, \quad d\omega^3 = 2\nu \omega^1 \wedge \omega^2. \tag{14}\]

The Cotton flow equations can be written as \( \partial_t e^a = \ast C^a \), where \( C^a \) is the Cotton 2-form and \( \ast \) denotes the Hodge dual. Since the Cotton flow preserves the volume density, from now on we assume that \( ABC = 1 \). Referring the reader to the appendix for the details of the computation, we state the equations which take the form

\[
\begin{align*}
\frac{dA}{dt} &= 4A[-\lambda^2 \lambda^2 (\mu B + \nu C) - 2\lambda A - (\mu B + \nu C)(\mu B - \nu C)], \\
\frac{dB}{dt} &= 4B[-\mu^2 \mu^2 (\nu C + \lambda A) - 2\mu B - (\nu C + \lambda A)(\nu C - \lambda A)], \\
\frac{dC}{dt} &= 4C[-\nu^2 \nu^2 (\lambda A + \mu B) - 2\nu C - (\lambda A + \mu B)(\lambda A - \mu B)].
\end{align*} \tag{15}\]

The entropy functional \( \mathcal{F} \) defined in section 3 can alternatively be computed using

\[
\mathcal{F} = -\frac{1}{4} \int \left( \omega^a_{\ b} \wedge d\omega^b_{\ a} + \frac{2}{3} \omega^a_{\ b} \wedge \omega^b_{\ c} \wedge \omega^c_{\ a} \right),
\]

which for our case yields

\[
\mathcal{F} = \int [4\nu \mu + 2(\lambda A + \mu B - \nu C)(\lambda A - \mu B + \nu C)(\nu C + \lambda A)(\nu C - \lambda A)] \ast 1, \tag{16}\]

where \( \ast 1 = e^1 \wedge e^2 \wedge e^3 \). Recall that by the result in section 3, \( \mathcal{F} \) is a non-decreasing functional. (The name ‘entropy’ may be misleading since \( \mathcal{F} \) is not necessarily nonnegative. In fact, the maximum value of \( \mathcal{F} \) in the SU(2) case below is \(-2\). All that matters is that \( \mathcal{F} \) is non-decreasing.) The following simpler functional, which was obtained by manipulating (15), also acts like an entropy in certain cases, and will also be useful in what follows:

\[
f(t) = \frac{1}{8} \left( \frac{\mu^2 \nu^2}{A^2} + \frac{\lambda^2 \nu^2}{B^2} + \frac{\mu^2 \lambda^2}{C^2} \right). \tag{17}\]

Note also that, using (15)

\[
\frac{df}{dt} = \frac{\mu^2 \nu^2}{A^2} (\mu B - \nu C)(\mu B + \nu C) + \frac{\lambda^2 \nu^2}{B^2} (\nu C + \lambda A)(\nu C - \lambda A) + \frac{\lambda^2 \mu^2}{C^2} (\lambda A + \mu B)(\lambda A - \mu B). \tag{18}\]
5.1. $SU(2)$

When we take $\lambda = \mu = v = -1$, the Lie group $G$ is isomorphic to $SU(2)$, which is homeomorphic to $S^1$ as a topological manifold. The proof that the flow equations (15) take any initial homogeneous metric to the round metric $A = B = C = 1$ on $S^3$ is remarkably simple. Since $\lambda = \mu = v = -1$, one has $f(t) \geq 3/8$, $df/dt \leq 0$, and in both cases equality occurs if and only if $A = B = C = 1$. This proves the claim.

It seems impossible to find an exact analytical solution to this system. Therefore we carry out an estimate analysis similar to that in [20]. From (15), one gets

$$\frac{d(A - B)}{dt} = 8(B - A)[2(B^3 + B^2A + BA^2 + A^3) - C(A + B)^2 - C^3].$$

(19)

Since the right-hand side vanishes when $A = B$, by the uniqueness of solutions, the ordering between $A$ and $B$, and likewise the ordering of the three variables cannot change during the flow. Therefore, without loss of generality, we may assume that $A \geq B \geq C$ due to the symmetry of the equations. From the last equation of (15), it is clear that $C$ is not decreasing, so $A \geq B \geq C \geq C_0$ where $C_0$ is the initial value of $C$. Moreover,

$$\frac{d(A - C)}{dr} = -8(A - C)[2C^3 + 2CA^2 + 2A^3 - B(A + C)^2 - B^3]
\leq -8(A - C)[2C^3 + 2CA^2 + 2A^3 - A(A + C)^2 - A^3]
\leq -24C_0^3(A - C).$$

(20)

We deduce that $A - C \leq (A_0 - C_0) \exp(-24C_0^3t)$. Since the difference between the greatest and smallest of $A, B, C$ decays at least at this rate to zero, $(A, B, C)$ also converges at least at this rate to $(1, 1, 1)$.

Linearizing the system around the fixed point $(A, B) = (1, 1)$ after setting $C = 1/AB$ gives

$$\frac{dA}{dt} = -24A,
\frac{dB}{dt} = -24B,$$

(21)

which is in accordance with the above estimate since $\max(C_0) = 1$.

One can alternatively solve the equations numerically. Starting from various initial values for $(A_0, B_0, C_0)$, a sketch of the solution curves can be seen in figure 1.

5.2. $SL(2, \mathbb{R})$

When we take $\lambda = \mu = -1, v = 1$, the group $G$ is isomorphic to the universal cover of $SL(2, \mathbb{R})$. From (15) one gets

$$\frac{d(A - B)}{dt} = 4(B - A)[2(B^3 + B^2A + BA^2 + A^3) + C(B^2 + A^2) + 2 + C^3].$$

(22)

Since the right-hand side is zero when $A = B$, we deduce that the ordering between $A$ and $B$ does not change throughout the flow. Without loss of generality, assume $A \geq B$ (note that the equations for $A$ and $B$ are symmetric). The equation for $A$ reads

$$\frac{dA}{dt} = 4A[A^2B - A^2C - 2A^3 + B^3 + B^2C - BC^2 - C^3]
\leq -4(ABC^2 + AC^3) = -4(C + AC^3).$$

(23)

This shows that $A$ is strictly decreasing (However $B$ is not necessarily decreasing). The equation for $C$ gives a lower bound

$$\frac{dC}{dt} = 4C[C^2(A + B + 2C) + (A + B)(A - B)^2] \geq 8C^4.$$

(24)
Therefore, $C \to \infty$ in a finite time $t_f$. We want to show that $A(t_f) = B(t_f) = 0$. Since there exists a point $t_1$ of time after which $C \geq A + B$, we have $dC/dt \leq 16C^4$. For $t \geq t_1$

$$\frac{d(A^5C)}{dt} \leq -4A^5C^4 \leq 0,$$  

(25)

which implies that $A(t_f) = 0$. Since $B(t_f) \leq A(t_f)$, we also get $B(t_f) = 0$. In the terminology of [20], the geometry always tends to a ‘cigar degeneracy’ in finite time. Some numerical solutions of the system are given in figure 2.

5.3. Isom($\mathbb{R}^2$)

When we take $\lambda = \mu = -1, v = 0, G$ is isomorphic to the universal cover of the group of Euclidean motions of $\mathbb{R}^2$. From (15) one obtains

$$\frac{d(A - B)}{dt} = 8(B^4 - A^4), \quad \frac{dC}{dt} = 4C(A^2 - B^2)(A - B).$$  

(26)

From these two equations, one gets

$$C(A - B) \frac{d(A - B)}{dt} + 2 \left( (B - A)^2 + \frac{2}{C} \right) \frac{dC}{dt} = 0.$$  

(27)

The function $C^3$ is an integrating factor for this equation. So, there is a second conserved quantity besides $ABC = 1$:

$$C^4(A - B)^2 + \frac{k}{8}C^3 = k,$$  

(28)

where $k$ is a constant. Since there are two conserved quantities and three variables, the phase space foliates into one-dimensional leaves. From (26), we see that the ordering between
the symmetric variables $A$ and $B$ does not change during the flow. Assume without loss of
generality that $A \geq B$. Then,

$$\frac{dA}{dt} = 4A(A^2B - 2A^3 + B^3) \leq 0, \quad (29)$$

$$\frac{dB}{dt} = 4B(A^3 + AB^2 - 2B^3) \geq 0. \quad (30)$$

Thus $A$ is decreasing, $B$ is increasing, and they are both bounded away from zero. This implies
that the flow exists for all $t$. If $B_0$ is the initial value of $B$, then since $B \geq B_0$, from (26) one
gets

$$\frac{d(A - B)}{dt} = -8(A - B)(B^3 + B^2A + BA^2 + A^3) \leq -32B_0^3(A - B). \quad (31)$$

So, $A - B$ decays to zero. In fact, using the conserved quantities, the equations can be
integrated. From (28), it follows that $(A - B)^2 = \frac{k}{C^3} - \frac{8}{3C}$. Using $AB = 1/C$, one obtains
$(A + B)^2 = \frac{k}{C^3} + \frac{4}{3C}$. Then, the second equation in (26) yields

$$\frac{dC}{dt} = 4 \left( \frac{k}{C^3} - \frac{8}{3} \right) \sqrt{\frac{k}{C^4} + \frac{4}{3C}}.$$

Defining

$$C = \left( \frac{3}{4}(a^2 - k) \right)^{1/3}.$$

---

Figure 2. The solid lines indicate flows, for various initial metrics on a homogeneous geometry modelled on $SL(2, \mathbb{R})$: $(A_0, B_0) = (0.5, 1), (2, 0.7), (1.5, 1.8), (0.2, 2)$. 
one finds
\[ \frac{u^2 - k}{3k - 2u^2} \frac{du}{dt} = \frac{32}{3}, \]
whose integration leads to an implicit definition of \( u(t) \) as
\[ - \frac{u}{2} + \frac{\sqrt{k}}{2\sqrt{6}} \tanh^{-1}(u\sqrt{2/3k}) = \frac{32}{3} t + l, \]
and \( l \) is another constant. The remaining metric functions follow easily as
\[ A = \frac{1}{2} \left( \frac{3}{4} (u^2 - k) \right)^{2/3} (u + \sqrt{3k - 2u^2}), \]
\[ B = \frac{1}{2} \left( \frac{3}{4} (u^2 - k) \right)^{2/3} (u - \sqrt{3k - 2u^2}). \]

Since \( A - B \to 0 \), by (28), the limiting value of \( C \) as \( t \to \infty \) is \((3k/8)^{1/3}\). In the limit, \( A = B \) and the geometry becomes flat. Even though we have analytically solved the equations, we also present some numerical solutions of the system for various initial conditions in figure 3.

The conserved quantity (28) can be, more invariantly, written in terms of the Ricci tensor and the scalar curvature as
\[ -\frac{164}{6} \frac{R^3 Ric^2}{(3R^2 - Ric^2)^2} = \text{const.} \tag{33} \]
To get a unitless quantity, one should divide this by \( \text{Vol}(M)^2 \), if the manifold is of finite volume.
Figure 4. The solid lines indicate flows, for various initial metrics on a $\text{Solv}$ geometry: $(A_0, B_0) = (2, 0.1), (1.5, 0.3), (1.8, 1)$.

5.4. Solv geometry

When we take $\lambda = -1$, $\mu = 1$, $\nu = 0$, the group $G$ is isomorphic to the group of isometries of the Minkowski plane. Equations (15) take the form

\begin{align*}
\frac{dA}{dt} &= -4A(A^2(B + 2A) + B^3), \\
\frac{dB}{dt} &= 4B(B^2(A + 2B) + A^3), \\
\frac{dC}{dt} &= -4C(B^2 - A^2)(A + B). 
\end{align*} 

(34)

The first two equations imply that $A$ is decreasing and $B$ is increasing. In fact, $\frac{dB}{dt} \geq 8B^4$, which shows that $B \to \infty$ in finite time. As in section 5.3, one can find a second conserved quantity which now reads

\begin{equation}
(A + B)^2C^4 - \frac{k}{4}C^3 = k, 
\end{equation}

(35)

where $k$ is a constant. Then, since $A$ is decreasing, $C$ must decay to zero at a rate like $1/\sqrt{B}$. Recalling $ABC = 1$, we see that $A$ also decays to zero like $1/\sqrt{B}$. All geometries in this class thus tend to a cigar degeneracy. Again, the equations can be completely integrated to give

\begin{align*}
A &= \frac{1}{2} \left( \frac{3}{4}(k - u^2) \right)^{-2/3} (\sqrt{3k - 2u^2} - u), \\
B &= \frac{1}{2} \left( \frac{3}{4}(k - u^2) \right)^{-2/3} (\sqrt{3k - 2u^2} + u), \\
C &= \left( \frac{3}{4}(k - u^2) \right)^{1/3},
\end{align*}

where

\begin{equation}
\frac{u}{2} + \frac{\sqrt{k}}{2\sqrt{6}} \tanh^{-1}(u\sqrt{3k}) = \frac{32}{3} t + l,
\end{equation}

and $l$ is another constant. We present some numerical solutions of the system for various initial conditions in figure 4.
5.5. Nil geometry

When we take $\lambda = -1, \mu = \nu = 0$, $G$ is isomorphic to the Heisenberg group. The flow equations

\[
\frac{dA}{dt} = -8A^4, \quad \frac{dB}{dt} = 4BA^3, \quad \frac{dC}{dt} = 4CA^3,
\]

(36)
can be easily integrated to yield

\[
A = (24t + A_0^{-3})^{-1/3}, \quad B = B_0\sqrt{A_0}A^{-1/2}, \quad C = A^{-1/2}/(B_0\sqrt{A_0}).
\]

(37)

Therefore, $A \to 0$ whereas $B$ and $C$ diverge. In the terminology of [20], this degeneracy is called a ‘pancake degeneracy’.

5.6. $\mathbb{R}^3$

In this case, $\lambda = \mu = \nu = 0$, the metric is flat and the flow equations are trivial.

5.7. $H^3$, $S^2 \times \mathbb{R}$, and $H^2 \times \mathbb{R}$

These cases do not arise from the construction that uses a three-dimensional Lie group as described, however the metrics can be written explicitly for each case. The metrics for the three geometries are of the form $g = \Lambda(t)g_{H^3}$, $g = D(t)g_{S^2}$ and $g = D(t)g_{H^2}$, where $g_{H^3}$, $g_{S^2}$, $g_{S^2}$ and $g_{H^2}$ are the standard metrics on the hyperbolic 3-space, the unit 2-sphere, the real line and the hyperbolic 2-space, respectively. Each of the resulting metrics is already conformally flat, therefore they are fixed points of the Cotton flow. Note, however that the Ricci flow degenerates $H^2 \times \mathbb{R}$ and $S^2 \times \mathbb{R}$ [20].

6. Conclusions

We have introduced a new flow in three dimensions, whose fixed points are conformally flat metrics. We have presented a gradient formulation of this flow by finding an entropy functional. We have also shown that the commutator of this flow with the Yamabe flow, which preserves the conformal class of a metric, is another flow preserving conformal class. We have studied the evolution of the nine homogenous geometries in detail. Four of these, $\mathbb{R}^3$, $H^3$, $H^2 \times \mathbb{R}$, $S^2 \times \mathbb{R}$, are fixed under the flow. Note that the last two of these are degenerated by the Ricci flow. Every homogenous metric in the $SU(2)$ class converges to the round $S^3$ metric. For the Isom($\mathbb{R}^2$), Solv and Nil cases, the equations can be solved analytically. Every metric in the Isom($\mathbb{R}^2$) class evolves to a flat metric. The metrics in the Nil class tend to a pancake degeneracy. The metrics in the $SL(2, \mathbb{R})$ and Solv classes develop cigar degeneracies.

There are several points which require further study. As in the Ricci flow case, one can define solitons

\[
C_{ij} + \nabla_i \xi_j + \nabla_j \xi_i + \lambda g_{ij} = 0.
\]

(38)

A metric satisfying this equality, which we call a ‘Cotton soliton’, can be thought of as evolving under the Cotton flow by just a diffeomorphism and/or a scaling of the underlying manifold. In the case of a compact manifold, the conservation of total volume implies that $\lambda$ vanishes; however, to allow for non-compact manifolds, we keep it. We have not been able to find a non-trivial solution of this equation, or of the simpler gradient Cotton soliton equation where $\xi_j = \nabla_j \varphi$. However, it may be useful to consider the techniques introduced in [25, 26] in this
regard. Another open question is to determine the conditions on the initial metric for which the short time existence problem can be solved, but this seems to be much more difficult than in the second-order case. One obviously interesting avenue, for geometric purposes, is the study of singularities formed under the Cotton flow, or under other possible flows obtained by combining the Ricci and Cotton flows. It would also be interesting to find a nonlinear sigma model whose beta function is the Cotton tensor.

Acknowledgments

This work is partially supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK). BT is also partially supported by the ‘Young Investigator Grant’ of the Turkish Academy of Sciences (TUBA), and both AUÖK and BT are supported by the TÜBİTAK Kariyer grant no 104T177. BT would also like to thank M Headrick and A Waldron for a discussion and the UC Davis Math Department where early stages of this work took place. We would also like to thank a very conscientious referee for useful remarks.

Appendix A. Computation of the flow equations

Here we show the details of the calculation of the Cotton tensor and how (15) is derived using the differential forms. The connection 1-forms $\omega^a{}_{b}$ can be computed using the vanishing of torsion, $de^a + \omega^a{}_{b} \wedge e^b = 0$. For the metric (13) they specifically take the following form:

$$\omega^1{}_{2} = \frac{1}{\sqrt{ABC}}(\lambda A + \mu B - vC)e^3,$$
$$\omega^1{}_{3} = \frac{1}{\sqrt{ABC}}(-\lambda A + \mu B - vC)e^2,$$
$$\omega^2{}_{3} = \frac{1}{\sqrt{ABC}}(-\lambda A + \mu B + vC)e^1.$$  \hfill (A.1)

Using the formula $R^a{}_{b} = d\omega^a{}_{b} + \omega^a{}_{c} \wedge \omega^c{}_{b}$, one computes the curvature 2-forms as

$$R^1{}_{2} = \frac{1}{ABC}(\lambda^2 A^2 + \mu^2 B^2 - 3\mu^2 C^2 - 2\lambda \mu AB + 2\lambda vAC + 2\mu vBC)e^1 \wedge e^2,$$
$$R^1{}_{3} = \frac{1}{ABC}(\lambda^2 A^2 + \mu^2 B^2 - 3\mu^2 C^2 - 2\lambda \mu AB + 2\lambda vAC + 2\mu vBC)e^1 \wedge e^3,$$
$$R^2{}_{3} = \frac{1}{ABC}(\mu^2 B^2 + v^2 C^2 - 3\mu^2 A^2 - 2\mu vBC + 2\lambda \mu AB + 2\lambda vAC)e^2 \wedge e^3.$$ \hfill (A.2)

Next, we compute the Ricci 1-forms and the curvature scalar using the formulæ $(Ric)_a = \iota_b R^b{}_{a}$ and $\mathcal{R} = \iota_a (Ric)^a$, respectively, as

$$(Ric)_1 = \frac{2}{ABC}(\lambda^2 A^2 - \mu^2 B^2 - v^2 C^2 + 2\mu v BC)e^1,$$
$$(Ric)_2 = \frac{2}{ABC}(\mu^2 B^2 - v^2 C^2 - \lambda^2 A^2 - 2\mu vAC)e^2,$$
$$(Ric)_3 = \frac{2}{ABC}(v^2 C^2 - \lambda^2 A^2 - \mu^2 B^2 + 2\lambda \mu AB)e^3,$$
$$\mathcal{R} = \frac{2}{ABC}(-\lambda^2 A^2 - \mu^2 B^2 - v^2 C^2 + 2\lambda \mu AB + 2\lambda vAC + 2\mu v BC).$$ \hfill (A.3)
The Cotton 2-form is given by $C^a = dY^a + \omega^a_b \wedge Y^b$, where $Y^a = (Ric)^a - \frac{1}{4}Re^{a}$, which leads to

$$C^1 = \frac{4}{(ABC)^{3/2}}[-\mu^2 A^2 (\mu B + \nu C - 2\lambda A) - (\mu B + \nu C) (\mu B - \nu C)^2] e^2 \wedge e^3,$$

$$C^2 = \frac{4}{(ABC)^{3/2}}[-\mu^2 B^2 (\nu C + \lambda A - 2\mu B) - (\nu C + \lambda A) (\nu C - \lambda A)^2] e^3 \wedge e^1,$$

$$C^3 = \frac{4}{(ABC)^{3/2}}[-\nu^2 C^2 (\lambda A + \mu B - 2\nu C) - (\lambda A + \mu B) (\lambda A - \mu B)^2] e^1 \wedge e^2.$$  \hspace{1cm} (A.4)

Finally, using $\partial_i e^a = sF^a$, one obtains (15).

**Appendix B. Coordinate-invariant computation of the Cotton tensor**

It is sometimes useful to consider the coordinate free version of the computation above. For this purpose, we follow [27] and [22]. Define the $(0, 2)$ tensor $s$ by the formula

$$s = Ric - \frac{\mathcal{R}}{4}g.$$  \hspace{1cm} (B.1)

A Cotton 3-form on a generic $D$-dimensional manifold $\mathcal{M}$ can be defined by

$$C(X, Y)(Z) = (\nabla X s)(Y, Z) - (\nabla Y s)(X, Z),$$  \hspace{1cm} (B.2)

where $X, Y, Z$ are vector fields on $\mathcal{M}$. Let us now return to the case of a homogeneous three-dimensional manifold. Recall $\{F_1, F_2, F_3\}$ from section 5. First, we compute $s$ using the formulae for $Ric(F_i, F_j)$ and $\mathcal{R}$ directly from [22]:

$$s(F_1, F_i) = \frac{2}{BC} ((\mu^2 A^2 - (\mu B + \nu C)^2) - \frac{1}{4} (2\mu \nu BC + 2\nu \lambda AC + 2\lambda \mu AB$$  

$$- \lambda^2 A^2 - \mu^2 B^2 - \nu^2 C^2),$$

$$s(F_2, F_i) = \frac{2}{AC} ((\mu^2 B^2 - (\lambda A - \nu C)^2) - \frac{1}{4} (2\mu \nu BC + 2\nu \lambda AC + 2\lambda \mu AB$$  

$$- \lambda^2 A^2 - \mu^2 B^2 - \nu^2 C^2),$$

$$s(F_3, F_i) = \frac{2}{AB} ((\mu^2 C^2 - (\lambda A - \mu B)^2) - \frac{1}{4} (2\mu \nu BC + 2\nu \lambda AC + 2\lambda \mu AB$$  

$$- \lambda^2 A^2 - \mu^2 B^2 - \nu^2 C^2)),$$

and clearly, $s(F_i, F_j) = 0$ if $i \neq j$. One has

$$(\nabla X s)(Y, Z) = \partial_X (s(Y, Z)) - s(\nabla X Y, Z) - s(Y, \nabla X Z).$$  \hspace{1cm} (B.4)

When $\{X, Y, Z\} \subset \{F_1, F_2, F_3\}$, the first term on the right is zero by homogeneity. $\nabla X F_i = 0$ for any $i$, and

$$\nabla F_1 F_2 = \frac{1}{2} (F_1, F_2) - (ad F_1){^*} F_2 - (ad F_2){^*} F_1 = -\frac{\lambda A + \mu B + \nu C}{C} F_3,$$

$$\nabla F_1 F_3 = -\frac{\mu B + \nu C + \lambda A}{A} F_1, \quad \nabla F_1 F_1 = -\frac{\nu C + \lambda A + \nu B}{B} F_2, \quad \nabla F_1 F_1 = \frac{\mu B - \lambda A - \nu C}{C} F_3,$$

$$\nabla F_2 F_1 = \frac{\lambda A - \mu B}{B} F_2.$$  \hspace{1cm} (B.5)

From the definition, $C(X, Y)(Z)$ is antisymmetric in $(X, Y)$, we have $C(F_i, F_j)(F_k) = 0$ for every $i, j$. Since the other antisymmetry is not apparent, we prove
Lemma. For any \( i, j \), \( C(F_i, F_j)(F_j) = 0 \).

Proof. We may assume \( i \neq j \) since the equality case is taken care of above. Let \( k \) be the third index, different from \( i \) and \( j \). Recall that \( s \) is diagonal. Now,

\[
C(F_i, F_j)(F_j) = (\nabla_{F_i} s)(F_j, F_j) - (\nabla_{F_j} s)(F_i, F_j)
\]

\[
= -s(\nabla_{F_i} F_j, F_j) - s(F_j, \nabla_{F_i} F_j) + s(F_i, \nabla_{F_j} F_j)
\]

\[
= s(F_j, aF_k) = 0,
\]

(B.6)

where \( a \) is some constant. Note that we have again used homogeneity to drop terms that involve \( \partial F_i \). This finishes the proof. \( \square \)

Therefore, the only interesting components of the tensor are those involving all three indices:

\[
C(F_1, F_2, F_3)(F_3) = -s(\nabla_{F_1} F_2, F_3) - s(F_2, \nabla_{F_1} F_3) + s(F_1, \nabla_{F_2} F_3)
\]

\[
= -s\left(\frac{-\lambda A + \mu B + vC}{C} F_3, F_3\right) - s\left(\frac{\mu B - \lambda A - vC}{C} F_3, F_3\right) + s\left(\frac{-\mu B + vC + \lambda A}{A} F_3, F_3\right)
\]

\[
= -2\nu s(F_3, F_3) - \frac{\lambda A - vC - \mu B}{B} s(F_2, F_2) + \frac{-\mu B + vC + \lambda A}{A} s(F_1, F_1),
\]

and the other \( C(F_i, F_j)(F_k) \) can be computed in a similar way. Defining \( C_{ijk} = C(F_i, F_j)(F_k) \), the flow equations (15) can be written as

\[
\partial_t g_{ij} = \frac{1}{2\sqrt{g}} \epsilon^{kmn} C_{kmn} g_{jm}.
\]

(B.7)

In fact, in a general coordinate frame, to relate this formulation to that presented in the main text, we define

\[
C_{ijk} = C(\partial_i, \partial_j)(\partial_k)
\]

\[
= (\nabla_{\partial_i} s)(\partial_j, \partial_k) - (\nabla_{\partial_j} s)(\partial_i, \partial_k),
\]

\[
= \partial_i s(\partial_j, \partial_k) - s(\nabla_{\partial_i} \partial_j, \partial_k) - s(\partial_j, \nabla_{\partial_i} \partial_k) - \partial_j s(\partial_i, \partial_k) + s(\nabla_{\partial_j} \partial_i, \partial_k) + s(\partial_i, \nabla_{\partial_j} \partial_k).
\]

Using \( s(\nabla_{\partial_j} \partial_i, \partial_k) = s(\Gamma^\ell_{ij} \partial_\ell, \partial_k) = \Gamma^\ell_{ij} s(\partial_\ell, \partial_k) \), this simplifies to

\[
C_{ijk} = \nabla_i s_{jk} - \nabla_j s_{ik},
\]

where \( s_{jk} = s(\partial_j, \partial_k) \). In the case of three dimensions, one can get a symmetric, traceless and covariantly conserved 2-tensor out of this by contracting it with the completely antisymmetric Levi-Civita tensor:

\[
C_{ij} = \frac{1}{2\sqrt{g}} \epsilon^{kmn} C_{kmn} g_{jm}.
\]

References

[1] Hamilton R S 1982 Three-manifolds with positive Ricci curvature J. Differ. Geom. 17 255
[2] Perelman G 2002 The entropy formula for the Ricci flow and its geometric applications Preprint math/0211159
[3] Perelman G 2003 Ricci flow with surgery on three-manifolds Preprint math/0303109
[4] Friedan D 1980 Nonlinear models in two-epsilon dimensions Phys. Rev. Lett. 45 1057
[5] Eells J Jr and Sampson J H 1964 Harmonic mappings of Riemannian manifolds Am. J. Math. 86 109
[6] Hamilton R S 1988 The Ricci flow on surfaces Mathematics and General Relativity (Santa Cruz, CA, 1986) Contemp. Math. vol 71 (Providence, RI: American Mathematical Society) pp 237–62
[7] Fischer A E 2004 An introduction to conformal Ricci flow Class. Quantum Grav. 21 S171
[8] Gegenberg J and Kunstatter G 2004 Using 3D string-inspired gravity to understand the Thurston conjecture Class. Quantum Grav. 21 1197
[9] Headrick M and Wiseman T 2006 Ricci flow and black holes Class. Quantum Grav. 23 6683 (Preprint hep-th/0606086)
[10] Samuel J and Chowdhury S R 2008 Energy, entropy and the Ricci flow Class. Quantum Grav. 25 035012 (Preprint arXiv:0711.0430)
[11] Woolgar E 2007 Some applications of Ricci flow in physics Preprint arXiv:0708.2144
[12] Cotton E 1899 Sur les variétés à trois dimensions Ann. Fac. Sci. Toulouse (II) 1 385
[13] Eisenhart L P 1997 Riemannian Geometry 8th printing (Princeton, NJ: Princeton University Press) p 91
[14] Arnowitt R, Deser S and Misner C W 1960 Phys. Rev. 117 1595
    Arnowitt R, Deser S and Misner C W 1962 Gravitation: An Introduction to Current Research ed I. Witten (New York: Wiley)
    Arnowitt R, Deser S and Misner C W 2004 The dynamics of general relativity Preprint gr-qc/0405109
[15] York J W 1972 Role of conformal three geometry in the dynamics of gravitation Phys. Rev. Lett. 28 1082
[16] Garcia A, Hehl F W, Heinicke C and Macias A 2004 The Cotton tensor in Riemannian spacetimes Class. Quantum Grav. 21 1099 (Preprint gr-qc/0309008)
[17] Deser S, Jackiw R and Templeton S 1982 Topologically massive gauge theories Ann. Phys. 140 372
[18] DeTurck D M 1983 Deforming metrics in the direction of their Ricci tensors J. Differ. Geom. 18 157
[19] Thurston W 1982 Three-dimensional manifolds, Kleinian groups, and hyperbolic geometry Bull. Am. Math. Soc. (N. S.) 6 357
[20] Isenberg J and Jackson M 1992 Ricci flow on locally homogeneous geometries on closed manifolds J. Differ. Geom. 35 723
[21] Knopf D and McLeod K 2001 Quasi-convergence of model geometries under the Ricci flow Anal. Geom. 9 879
[22] Chow B and Knopf D 2004 The Ricci Flow: An Introduction (Math. Surveys and Monographs vol 110) (Providence, RI: American Mathematical Society)
[23] Cao X, Ni Y and Saloff-Coste L 2007 Cross curvature flow on locally homogenous three-manifolds (I) Preprint arXiv:0708.1922 (to appear in Pacific J. Math.)
[24] Milnor J 1976 Curvatures on left invariant metrics on Lie groups Adv. Math. 21 293
[25] Lott J 2007 On the long-time behavior of type-III Ricci flow solutions Math. Ann. 339 627
[26] Glickenstein D 2008 Riemannian groupoids and solitons for three-dimensional homogeneous Ricci and cross-curvature flows Int. Math. Res. Not. 2008 34–49
[27] Belgun F A 2001 Null-geodesics in complex conformal manifolds and the LeBrun correspondence J. Reine Angew. Math. 536 43