A simple and low redundancy method of image compressed sampling

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Abstract

A problem is addressed of minimization of the number of measurements needed for image acquisition and reconstruction with a given accuracy. In last several years, the compressed sensing approach to solving this problem was advanced, which promises reducing the number of required measurements by means of obtaining sparse approximations of images. However, the number of measurements required by compressive sensing substantially exceeds the theoretical minimum defined by sparsity of the image sparse approximation. In the paper, a sampling theory based method of image sampling is suggested that represents a practical and substantially more economical alternative to the compressed sensing approach. Presented and discussed are also results of experimental verification of the method, its possible applicability extensions and some its limitations.

OCIS codes: (100.0100) Image processing; (100.2000) Digital image processing; (100.3020) Image reconstruction-reconstruction; (110.6980) Transforms

1. Introduction

Digital display devices and image processing software assume by default that sampling over regular square sampling grids is used for discrete representation of images. As it is well known, digital images acquired in this way are, as a rule, highly compressible. Hence images are compressed for storage and transmission and then, for displaying or processing, are reconstructed to the standard sampled representation. For instance, image of 1024x1024 pixels shown in Figure 1 can be compressed by JPEG to a file of 155 Kbytes, i.e. 6 times lesser than the initial 1Mb file required for the non-compressed image.

Figure 1. A 1024x1024 pixel image that is JPEG compressible to 155Kb file with root mean square error 4 gray levels (PSRN 36 Db)

The phenomenon of ubiquitous compressibility raises very natural questions: why go to so much effort to acquire all the data when most of what we get will be thrown away? Can we not just directly measure the part that will not end up being thrown away ([1])?

Compressed images are, of course, not identical to non-compressed ones and reproduce them with a certain admissible error. In particular, root mean square (RMS) of JPEG reconstruction error for the image in Figure 1 is 4 of 256 gray levels. One can reconstruct this image with exactly the same reconstruction error from just its 290 455 largest DCT transform coefficients from 1024x1024 coefficients, i.e. from 3.6 times lesser amount of data. Theoretically, this means that, only 290 455 samples will be sufficient for acquiring this 1024x1024 image: one can compute from these samples those required 290 455 image DCT coefficients, and then reconstruct the 1024x1024 image by inverse DCT setting the rest 758121 coefficients to zero. The problem is that positions of the required DCT coefficients in spectral domain are not known.

For solving this problem of image economic sampling, the “Compressed sensing” approach was advanced ([1] - [4]). It was mathematically proven ([5]) that if an image of N samples is known to have, in domain of a certain transform, only K non-zero transform coefficients out of N, the image can be precisely reconstructed from M measurements by means of L1 norm minimization in the transform domain, provided the following relationship between the required number of samples N, the required number of measurements M and the number K of signal nonzero spectral coefficients holds:

\[ M > 2 \cdot K \cdot \log(N/K). \]  

(1)
However the “Compressed sensing” approach has several drawbacks:

- As it follows from Eq. 1, \( M > K \), i.e. the number \( M \) of required image samples for images with \( K \) nonzero spectral coefficients is considerably redundant with respect to the minimal required number, which is \( K \). For the range of sparsity \( K / N \) of real images from \( 10^{-1} \) to \( 10^{-3} \), the required redundancy \( M/K \) reaches 5 to 10 times ([7]).

- While “Compressive sampling” approach guarantees precise reconstruction of images with sparse spectrum, it does not specify, what specific kind of sparse approximation it provides for real images with spectrum, which is not precisely sparse, and what is expected error of the image approximation it provides. It is especially true if noise is present in image signal.

In what follows a sampling theory based alternative approach to solving the problem of image economical sampling is outlined.

2. Sampling theory based economical image sampling and reconstruction

The Sampling Theorem states that the minimal number of image samples per unit of image area required for image reconstruction with a given mean square error is equal to the area of image Fourier spectrum, which contains \((1-\varepsilon^2)\) fraction of image energy, where \( \varepsilon^2 \) is the acceptable mean square error normalized to the image energy ([8]). The corresponding Discrete Sampling Theorem ([6], [8]) that assumes image numerical reconstruction states that given \( K \) samples of an image of \( N \) samples, one can approximate this image by an image with \( K \) non-zero spectral coefficients in a domain of a certain transform with mean square approximation error equal to the energy of the rest \( N-K \) transform coefficients. The approximation error is minimal if selected are \( K \) the largest transform coefficients. In case of Discrete Fourier and Cosine transforms, \( K \) samples can be taken in arbitrary positions.

As it was mentioned, for image reconstruction from these \( K \) samples, one needs to precisely specify, which particular spectral coefficients of the approximating image should be nonzero. The “Compressed sensing” approach does not require any specification of signal non-zero spectral components and is capable of “blind”, in this sense, reconstruction. It has its price, the above-mentioned considerable redundancy in the required number of measurements.

In practical tasks of image acquisition, the total uncertainty regarding signal nonzero spectral components assumed by “Compressed sensing” is a too pessimistic scenario. There are at least the following four reasons in favor of this assertion:

- The relevant energy compacting transforms, such as DCT, DFT and wavelets, have a feature to compact image energy into few transform coefficients that form in transform domain more or less compact groups rather than are chaotically spread over it.

- Practical experience shows that although these groups do not have sharp borders, the density of non-zero spectral components quite gradually decays toward the periphery of the groups. This means that the groups can be, with a reasonably good accuracy in terms of preservation of the group total energy, circumscribed by some sufficiently simple standard shapes, such as oval, rectangle or similar ones with few shape parameters, and no fine tuning of shape parameters is required. This property of sparse spectra is illustrated on a test image in Figure 2. One can see in this image a certain prevalence of horizontal edges. This prevalence causes anisotropy of image spectrum seen in boxes b) –d), where shown are marked as white dots non-zero DCT coefficients that reconstruct this image with RMS error 3.85 gray levels of 255 levels (36.4 dB), the same as the reconstruction error of this image after its standard JPG compression by the Matlab tools.

Figure 2. Test image (a), its nonzero spectral components for the reconstruction error 3.85 of image gray levels (white dots) and approximative oval and rectangular shapes (b) –d))
Additionally in boxes b) – d) shown are borders of oval and rectangular shapes that have the same area (0.275 of the total area) and different aspect ratios (0.3, 0.45 and 0.35, correspondingly) that when used as band limitation shapes, reconstruct the test image with practically the same RMS reconstruction errors (38, 3.8 and 4.1 of image gray levels correspondingly).

As soon as one believes that an image can be replaced by its copy with sparse spectrum, one usually knows, at least roughly, the region in spectral domain and its shape, where non-zero spectral coefficients are concentrated; otherwise this belief has no substance. In particular, for overwhelming number of real images it is well known that transforms with good energy compaction capability, such as DCT, compact image energy into the lower frequency part of spectral domain. After all, it is this property that is put in the base of transform coefficient zonal quantization tables in JPEG image coding such.

How can one use this intuitive a priori knowledge for compressed sampling, i.e. sampling with number of samples $M$ lesser than the number $N$ required for image display? First of all, a trivial solution comes to mind: assume that relevant $M$ image spectral components can be approximately encompassed by a square shape and sample the image in a regular way over a square sampling grid to obtain an $M$-pixel image that then can be, using one or another interpolation method, up-sampled to the required $N$-pixel image. The perfect interpolation is secured by the discrete sinc interpolation ([8]). There are various algorithms of discrete sinc-interpolation, of which one of the most simple is DCT zero-padding.

This simplest solution is, however, not sufficiently economical because square shapes do not sufficiently tightly approximate shapes of image spectra. Shapes of spectra of majority of images are better approximated by circular shapes, if they are isotropic, or by oval or rectangular shapes of different orientation, if they are anisotropic. The degree of spectra isotropy or anisotropy can be roughly evaluated on the base of physical nature of images.

Therefore, one can, in addition to specifying the number $N$ of desired images samples and the number $M$ of samples to be taken, choose an appropriate standard shape say, oval or rectangular one, that encompasses image $M$ most important for image reconstruction spectral components where they are expected to be concentrated. As it was mentioned, the area of concentration is, most frequently, that of lowest spatial frequencies. Then one can take $M$ image samples in arbitrary, in the case of sparsity of DCT or DFT spectra, positions.

The choice of the number $M$ of samples taken at image sampling determines image restoration quality and image resolving power. The choice of the number $N$ of desired image samples is governed by the required accuracy of rounding up positions of image samples to integer indices of nodes of the sampling grid (for instance, in case of a square $\sqrt{N}$-pixel sampling grid, $(m,n)$-th sample has to be placed at $(\text{round}(m/\sqrt{N}), \text{round}(n/\sqrt{M}))$-th node of sampling grid), by parameters of image display or storage or other similar considerations.

For image reconstruction, there are two options:

- Direct inversion of the $MxN$ transform matrix for computing $M$ transform non-zero coefficients specified by the selected spectral shape from $M$ samples with given indices, setting the rest $N-M$ transform coefficients to zero and applying inverse transform to the found zero-pad spectrum. Generally, matrix inversion is a very hard computational problem and no fast matrix inversion algorithms are known. In our specific case, a pruned fast transform matrix should be inverted. There exist pruned versions of fast transforms for computing subsets of transform coefficients of signals with all but some subset of samples being zeros ([9] [10]), which is inverse to what is required in the given case. The question, whether these pruned algorithms can be adapted for computing a subset of transform non-zero coefficients from a subset of signal samples is open.

- An iterative Papoulis-Gerchberg type algorithm, in which direct and inverse transforms are performed alternatively at each iteration: in transform domain, transform coefficients that are supposed to be zero are zeroed and then, in image domain, image samples in positions, where they were actually taken at sampling, are replaced by the corresponding available samples.

For the sake of brevity we’ll call the described method of image compressed sampling and reconstruction SCS (“Sighted compressed sampling”)-method because it provides, on the base of intuitive insight into the structure of image spectra, compressed sampled representations of images.

3. Experimental verification and practical considerations

The suggested SCS-method of image compressed sampling and reconstruction has been experimentally verified on a considerable database of test images. In the experiments, the above-described iterative Gerchberg-Papoulis type algorithm was used and three types of sampling grids were tested: (i) uniform sampling grid, in which $M$ image samples are uniformly distributed, with appropriate rounding up of their positions to nearest nodes of a dense sampling grid of $N$ samples; (ii) uniform sampling grid with jitter, in which horizontal and vertical positions of each of $M$ samples are randomly chosen, independently in each of two image coordinates, within primary uniform sampling intervals; and (iii) random sampling grid, in which positions of samples are totally uniformly and randomly distributed over the dense sampling grid of $N$ samples. As the image transform that compacts image spectrum, Discrete Cosine Transform was used. Two types of spectrum shapes of lower frequency part of image DCT spectra were tested in the experiments: rectangular and oval, with aspect ratio manually set on the base of visual evaluation of possible presence or absence of image spectrum anisotropy. For instance, if vertical edges prevailed in a particular image, the aspect ratio was set larger than one and when horizontal edges prevailed, it was set lower than one.

As a zero-order approximation, from which iterative reconstruction starts, not available image samples were interpolated from the nearest 3 available samples taken with weights inversely proportional to their distance from the interpolated sample. It was found in experiments that when the number $M$ of samples is equal to its theoretical minimum, i.e. to the area of the chosen spectral shape, the RMS of reconstruction error decays at first hundreds iterations quite rapidly but after it reaches value of 3-5 quantization intervals, the error decay slows down. The error decay can be considerably accelerated if images are 25-50% oversampled.

Figures 3 and 4 illustrate outcomes of experiments with two images of the tested set: test images “Man” of 1024x1024 pixels and “Blood Vessels” of 512x512 pixels. The former is the image, which was used for demonstrating potentials of the “Compressed Sensing” approach in Ref. [11]. In order to make results of experiments comparable with those reported in [11], spectrum of the image was artificially sparsified by zeroing its all but 25000 largest DCT coefficients.

Shown in figures are: (a) initial test image band limited according to the chosen image spectrum shape, (b) reconstructed image, (c) sparsely sampled test image, (d) reconstruction error (difference between the initial test image band limited according to the chosen spectrum shape and the reconstructed image), (e) image sparse DCT spectrum, and (f) two plots of RMS of reconstruction error vs iteration number, one for overall error and another for 90% of smallest errors.
Figure 3. Results of experiments on sampling and reconstruction of test image “Man”. Root mean square (RMS) of reconstruction error is given in units of image gray levels in the range 0-255.
Figure 4. Results of experiments on sampling and reconstruction of test image “Blood Vessels”. Root mean square (RMS) of reconstruction error is given in units of image gray levels in the range 0-255.
RMS error was evaluated in units of image gray levels from range (0-255). Image sparse spectra were evaluated by selecting image largest spectral components that reconstruct image with the same RMS error as that for standard JPEG compression by the Matlab tools. Test image “Man” was sampled over a uniform sampling grid with jitter; test image “Blood Vessels” was sampled over a random sampling grid. In both cases iterative reconstruction was stopped, when RMS of overall reconstruction error became lower than RMS of error of image quantization to 256 gray levels (0.289).

As one can see in the figures, test image “Man” is reconstructed from 0.0377x1024x1024=39531 samples with RMS error 0.18 (PSNR 63 Db) achieved after 2000 iterations, i.e. sampling redundancy with respect to image spectrum sparsity (25000 non-zero coefficients) is 158 (coefficient 1.26 of approximation of image sparse spectrum by the chosen spectrum shape times the chosen oversampling factor 12.5). Note that according to ([11]) the “Compressive sensing” approach required 96000 samples for the same image, i.e. 2.4 times more.

Band-limited copy of test image “Blood Vessels” of 512x512 pixels was reconstructed from 0.395x512x512=103547 samples with RMS error 0.27 (PSNR 59.5 Db) achieved after 1500 iterations. Sampling redundancy in this case is 2.4 (coefficient 1.6 of approximation of image sparse spectrum by the chosen spectrum shape times the chosen oversampling factor 15). RMS of band limitation error is 4.2 (35.7 Db) i.e. of the order of JPG compression error for this image (3.85, PSNR 36.4).

Plots of RMS of overall reconstruction errors and of 90% smallest errors (Figure 3, f and Figure 4, f) and images of reconstruction errors (Figure 3, d and Figure 4, d) evidence the presence in reconstructed images of isolated outliers. Apparently it is these outliers that retard decay of the reconstruction error with iterations.

Several practical considerations should be added to conclude the discussion:
- In experiments with the image data set, the number of iterations sufficient for image reconstruction with RMS error lesser than image RMS quantization error was of the order of couple of thousands. The necessity of a certain oversampling for accelerating the reconstruction error decay in the iterative reconstruction process does not seem to be of fundamental nature.
- The method is by necessity somewhat redundant with respect to the potential sparsity of image spectra because, in order to preserve the quality of image reconstruction, chosen image band limitation shapes that approximate image sparse spectra should contain more spectral coefficients than image sparse spectra, which, by definition, contain the image most intensive spectral components. For instance, for above shown test images this redundancy of spectra shape turned to be 1.26 and 1.6, correspondingly. The ratio M/K of the number M of samples needed for image reconstruction by the SCS-method, to the number K of image sparse spectra non-zero coefficients determines sampling redundancy required by the method. In experiments with other images from the tested data set, the typical level of the SCS-method redundancy was less than 2. This is substantially less than the redundancy required by the “Compressive sensing” approach ([5],[7]) even with an account for the above-mentioned oversampling.
- There are some texture images, which contain very few periodic components such as an image shown in Figure 5, a), for which the assumption that image spectra most intensive components are concentrated in low frequency part of spectral domain, will require substantially excessive number of image samples. For example, the area of image low frequency components bounded by image spectral components shown by white points in Figure 5, b) contains substantially larger number of spectral components than the number of non-zero spectral components of the image. In such cases, for efficient application of the SCS-method, more smart evaluation of signal spectrum shape using appropriate simple measurements of spatial parameters of the image texture is required.

![Figure 5. A texture image of extremely low sparsity (a) and its DCT spectrum (b, white dots)](image)

### 4. Other possible applications

The described method can find applications in solving other computational imaging tasks as well. At least these three potential applications can be listed: (i) in-painting of occlusions in images; (ii) image reconstruction from projections and (iii) image reconstruction from magnitude of its Fourier spectrum.

In-painting of image occlusions can be implemented with exactly the same algorithm as the above described iterative algorithm of image reconstruction from sparse samples, in this case those samples that are not occluded.

In quite frequent cases, when body slice occupies only a fraction of the area of entire image, the SCS-method can be used for implementing the direct Fourier method of image reconstruction from projections. For this, samples of Fourier spectra of projections should be placed in their corresponding nodes of a regular square sampling grid and then the iterative reconstruction should be run, applying, at each iteration, zeroing of the area of image domain, which is known to be empty, and replacing in Fourier domain spectra samples with corresponding initial samples of spectra of projections.

For image reconstruction from magnitude of its Fourier spectrum, object has to be illuminated through a randomized binary (opaque-transparent) mask with the fraction of transparent area equal to the required by SCS-method sampling rate. The module of Fourier spectrum of illuminated in such a way object has to be recorded and then used for reconstruction of the masked image using the Gerchberg-Papoulis iterative procedure, in which, at each iteration, a known image support defined by the mask is applied in image domain and module of image spectrum is restored in spectral domain. The convergence of the iteration procedure is enabled by the randomized shape of the illumination mask. Once masked image is reconstructed from its spectrum, reconstruction of the entire correspondingly band-limited image can be accomplished in the same way as in above mentioned image in-painting.

### 5. Conclusion

The described SCS-method of image sampling and reconstruction represents a simple, practical and substantially more economical alternative to “Compressed Sensing” for solving the problem of
minimization of the number of measurements required for image reconstruction with a given accuracy. For a given transform, in which the image is supposed to have a sparse spectrum, and for a given number of image samples $M$ taken on an arbitrary sampling grid, the method guarantees approximation of images of $N>M$ samples with RMS error equal to the energy of $N/M/R$ transform coefficients outside of the chosen spectrum shape set to zero, where $R$ is the sampling redundancy factor of the order of 2. Contrary to Compressive Sensing methods, the efficiency of the suggested SCS method does not depend on noise that might be present in image signals. Practical application of the method does not require any special prior knowledge on images for setting sampling and image reconstruction parameters except that, which is usually required in conventional image sampling.

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