Problems associated with use of the logarithmic equivalent strain in high pressure torsion

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Abstract. The logarithmic “equivalent” strain is frequently recommended for description of the experimental flow curves determined in high pressure torsion (HPT) tests. Some experimental results determined at -196 and 190 °C on a 2024 aluminum alloy are plotted using both the von Mises and logarithmic equivalent strains. Three types of problems associated with use of the latter are described. The first involves the lack of work conjugacy between the logarithmic and shear stress/shear strain curves, a topic that has been discussed earlier. The second concerns the problems associated with testing at constant logarithmic strain rate, a feature of particular importance when the material is rate sensitive. The third type of problem involves the “history dependence” of this measure in that the incremental logarithmic strain depends on whether the prior strain accumulated in the sample is known or not. This is a difficulty that does not affect use of the von Mises equivalent strain. For these reasons, it is concluded that the qualifier “equivalent” should not be used when the logarithmic strain is employed to describe HPT results.

Key Words: High pressure torsion, logarithmic equivalent strain, von Mises strain, work conjugacy, rate sensitivity

1. Introduction
In an earlier paper [1], some of the problems associated with use of the logarithmic “equivalent” strain in the description of the stress/strain curves derived from high pressure torsion testing were described. These concerned two principal inconsistencies, which were described in detail. One is that when the logarithmic strain is used in conjunction with the stress derived from the Fields and Backofen (F & B) analysis for torsion [2], the work done/unit volume derived from the flow curve does not agree with the actual torque/twist work done/unit volume during testing. The former is then much smaller than the experimental torque/twist work done, so that such “equivalent stress/equivalent strain” curves are not in fact work conjugate with the actual work done.

Conversely, if an energy balance approach is employed, according to which the flow stress is work conjugate with the logarithmic strain, then the derived flow stresses adopt absurd values that do not have any possible physical interpretation. This problem has already been pointed out by Stüwe [3], as well as by Pippan and co-workers [4], in earlier publications. Thus it is evident that, with regard to the derivation of flow curves from torque/twist data, there is no “equivalent stress” available that is compatible with the logarithmic “equivalent strain”. In this way, the description of this strain as “equivalent” is inappropriate and this qualifier is therefore a candidate for elimination from common usage. This point has also been made with respect to the derivation of equivalent stress/equivalent strain curves more generally from standard torsion testing data [5].

Here, some further problems associated with the use of the logarithmic strain are introduced and described. The two issues considered here involve the additivity of such “equivalent”
strains and the issues associated with testing at constant “equivalent strain rate”. The latter arises because the test results reported in Ref. 6 and analyzed below were carried out at the following temperatures: -196, room temperature, 100, 190, 250, 300 and 350 °C. At the four highest temperatures, the rate sensitivity is no longer negligible, as it is at room temperature and below. Accordingly, questions arise regarding control of the testing machine when it is required to carry out tests at constant strain rate. This is one of the issues that will be examined in detail below.

2. Torque/Twist vs. Stress/Strain Curves
We begin with a description of the equipment employed by Vafaei et al. [6] and Pippan et al. [4] in their experiments on high pressure torsion. This is illustrated in Fig. 1 [4]. In the experiments described below, the anvil was rotated at 0.2 rpm and a normal pressure of 2.5 GPa was employed [6]. It is important to note that the Fields and Backofen analysis only provides the shear stress τ at the maximum radius and cannot directly specify the value at any other location [5]. Thus, it is essential in the present type of work to relate the F & B stress to the strain at the same radius. Once the shear stress τ at the external radius is known, it can readily be converted into equivalent stress $\sigma_{VM}$ using the von Mises relation, $\sigma_{VM} = \sqrt{3}\tau$. In the experiments described below, the external radius was 7 mm, the diameter of the sample 14 mm, and the sample thickness was 1.5 mm [6].

The torque/twist curves determined at -196 and 190 °C in the experiments of Ref. 6 are reproduced here as Fig. 2a. These concerned tests on aluminum alloy 2024, whose composition is listed in Table 1. By integration of the torque/twist data cited here and taking into account the geometry of the specimens, the work done/unit volume in the two cases turns out to be 55,437 and 40,528 MJ/m$^3$, respectively. These quantities will be compared below to those obtained from the derived stress/strain curves and it should be noted that no account is taken of the work done by the compressive stress.

Figure 1. Schematic illustration of the HPT equipment employed in the experiments described in Fig. 2 below [4, 6]. a) tooling; b) specimen geometry.
Table 1. Composition of the 2024 Al as received in the T3 condition

| Element | Cu | Mg | Mn | Si | Fe | Zn | Cr | Al  |
|---------|----|----|----|----|----|----|----|-----|
|         | 4.2| 1.2| 0.7| 0.5| 0.5| 0.25| 0.1| Bal.|

Figure 2. a) Torque/twist curves determined at two different temperatures on the aluminum 2024 alloy specimens of reference 6. Here the twist angle has been derived from the shear strain at a radius of 6 mm employed in Ref. 6. b) von Mises equivalent stress/equivalent strain curves corresponding to a radius of 7 mm derived from the data in a) using Fields and Backofen for the stress and eq. (1) for the strain.

The shear stress/shear strain curves that correspond to the torque/twist curves of Fig. 2a can be readily derived using F & B for the flow stress and the usual formula, \( \gamma = r \theta / t \), for the shear strain, where \( \gamma \) is the shear strain, the thickness \( t \) is the equivalent here of the length in conventional torsion testing, and \( \theta \) is the angle of twist in radians. As mentioned above, F & B provides the shear stress at the outer radius, \( r = 7 \) mm. To be consistent, the shear strain should also be evaluated at the same radius, something that was not done in the original paper [6], but is the procedure that is followed here.

The shear stress/shear strain curves derived in this way were converted into von Mises equivalent stress/equivalent strain curves using the usual relations:

\[
\sigma_{\text{VM}} = \sqrt{3} \tau \quad \text{and} \quad \varepsilon_{\text{VM}} = \frac{\gamma}{\sqrt{3}}
\]  

The curves obtained in this way are illustrated in Fig. 2b and can be readily converted into shear stress/shear strain form using the conversion factors shown in eq. (1). By integrating the stress/strain curves, the work done/unit volume values that correspond to these curves were determined to be 83,155 and 60,792 MJ/m\(^3\), respectively. These quantities apply, of course, to both the shear stress/shear strain as well as the von Mises stress/strain curves.

The work done/unit volume determined from a stress/strain curve is a local quantity, as it increases (in a non-linear manner that depends on the current work hardening rate and rate sensitivity [5]) with the radius. By contrast, the overall work done/unit volume obtained from the torque/twist curve represents an integrated quantity and so cannot be compared directly with the stress/strain value associated with the external radius. It corresponds instead to the work done/unit volume at a specific radius somewhere between the axis and
the outside radius. As shown in more detail elsewhere [5], for room temperature tests at
which the rate sensitivity is close to zero, and when a steady state of flow is achieved, the
values determined from external radius stress/strain data will always be exactly 1.5 times
greater than the overall values. Examination of the numbers quoted above confirms that this
ratio is respected in the present calculations, indicating that both the shear stress/shear
strain as well as the von Mises curves are work conjugate with the torque/twist curves.

3. Logarithmic Stress/Strain Curves
We turn now to deriving the curves obtained by employing the logarithmic strain, as
recommended in References 7 and 8 and employed by many authors [9-11]. The relation
used here is reproduced below as eq. 2.

\[
\varepsilon_{ln} = \frac{2}{\sqrt{3}} \ln \left[ \sqrt{\left(1 + \frac{\gamma^2}{4}\right) + \frac{\gamma}{2}} \right]
\]

The curves obtained in this way are illustrated in Fig. 3, where it can be seen that the F & B-
based flow stresses have not changed, but the strains have been reduced by a factor of about
16 (which depends on the maximum angle of twist employed). The work done/unit volume
quantities that correspond to the test results represented in Fig. 3 were calculated, leading to
values of 4490 and 3709 MJ/m³, respectively. These are too low to be work conjugate by
factors of 4490/83,155 = 0.054 and 3709/60,792 = 0.061, respectively. That is, when the
logarithmic strain is employed, the work done estimate is in error by factors in the range 16 –
19, i.e. by about the extent to which the strain is reduced by choosing this formalism. Thus
this notation leads to errors of one to two orders of magnitude, depending on the number of
turns employed in the tests.

Figure 3. von Mises equivalent stress vs. logarithmic “equivalent” strain curves applicable to a radius
of 7 mm derived using equation (2) from the data of Fig. 2a.
4. Problems Associated with the Logarithmic “Equivalent” Strain Rate

At elevated temperatures, the rate sensitivity rises well above zero, so that it is imperative to carry out tests at constant strain rate. This can be the shear strain rate, as in torsion, or the von Mises equivalent strain rate more generally, an approach that applies to all strain paths. Under torsion testing conditions, there is no difference between testing at constant shear strain rate or von Mises strain rate because of the linearities defined in eq. (1). Thus, when a state of steady state flow or constant flow stress is achieved during testing in compression or tension, steady state flow is also observed during torsion testing.

However, this simple state of affairs does not apply if testing is carried out at constant logarithmic “equivalent” strain rate. Just how this comes about is now considered in more detail. We first define the logarithmic strain rate $\dot{\varepsilon}_{in}$ by deriving the definition of the logarithmic strain provided in eq. (2) with respect to time:

$$\frac{d\varepsilon_{\text{ln}}}{dt} = \frac{2}{\sqrt{3}} \left[ \frac{1}{4 + \gamma^2 + \frac{\gamma}{2}} \right] \left[ \frac{1}{2} \sqrt{\frac{4}{4 + \gamma^2}} \right] (\frac{1}{4}) \gamma + (\frac{1}{2}) \right] \cdot \frac{d\gamma}{dt}$$

(3)

$$\dot{\varepsilon}_{in} = \frac{2}{\sqrt{3} \left[ \sqrt{4 + \gamma^2} + \gamma \right]} \cdot \dot{\gamma}$$

(4)

$$\dot{\varepsilon}_{\text{ln}} = \frac{2}{(\sqrt{3}) \sqrt{4 + \gamma^2}} \cdot \dot{\gamma}$$

(5)

This expression provides the multiplying factor required to convert the shear strain rate (as well as the von Mises strain rate) into their logarithmic equivalents. For convenience, this conversion factor is plotted in the form of the two ratios $\varepsilon_{\text{ln}}/\varepsilon_{vM}$ and $\dot{\varepsilon}_{\text{ln}}/\varepsilon_{vM}$ in Fig. 4. Here it can be seen that a test carried out at constant logarithmic strain rate up to a shear strain of 80 (as in Fig. 2a) requires the twist rate in the testing machine to be gradually increased to more than 1500 times the initial value. For a material with a rate sensitivity of 0.2, the flow stress would then increase by a factor of 4.3 over the steady state value determined in compression or tension. Thus, testing at a constant logarithmic “equivalent” strain rate clearly does not lead to results that are equivalent to tests carried out along other strain paths.

Similar remarks apply to tests carried out at a constant twist or shear strain rate. In this case, as also shown in Fig. 4, if the logarithmic “equivalent” strain rate at the end of a test carried out under these conditions must be specified, it will have decreased by more than three orders of magnitude below the initial value. Thus specifying this type of “equivalent” strain rate cannot lead to results that are equivalent to the test results obtained using other testing methods.
Figure 4. Dependence of the ratios $\dot{\varepsilon}_{\text{ln}}/\dot{\varepsilon}_{\text{vM}}$ and $\dot{\varepsilon}_{\text{vM}}/\dot{\varepsilon}_{\text{ln}}$ on applied shear strain. Both quantities vary by more than three orders of magnitude over the experimental range.

5. Non-additivity of Incremental Logarithmic Equivalent Strains

Measures such as the shear strain and von Mises equivalent strain are clearly additive in the same sense that angles of twist are additive. This useful property arises because they are linearly related to the latter measure. However, because of the non-linear nature of the relation between the logarithmic strain and the shear strain (or twist angle) expressed in eq. (2), this simple property does not apply to the former quantity. The ambiguities entailed by the use of eq. (2) to describe test results are depicted graphically in Fig. 5.

Figure 5. Three alternative representations of the data of Fig. 3. The leftmost curve is a copy of the -196 °C curve of Fig. 3; the two segmented curves represent the results of the -196 °C torque/twist test of Fig. 2a carried out in: i) 5 and ii) 14 incremental steps.
Here the data of Fig. 2a relating to -196 °C are shown plotted in three different ways. The most leftward and highest curve in Fig. 5 is the logarithmic stress/strain curve of Fig. 3, foreshortened here because of the change in scale of the horizontal axis. The two other curves represent the -196 °C test carried out in: i) 5 separate segments of 6.8 radians each; and ii) 14 separate segments of 2.43 radians each. For the construction of each individual segment curve, the angle of twist begins at zero and increases to 6.8 or 2.43 radians, as required. That is, all the segment curves are of equal (strain) length, as each of the 5 (and 14) technicians carrying out the tests is unaware of the previous history of the sample being tested. These two representations of the material properties are thus examples of what is known as the “history effect”.

Although this is a somewhat unrealistic testing scenario, it nevertheless reveals the problems associated with a strain measure that is not additive. This is a problem that has already been discussed, although somewhat more briefly, in References 3 and 4. It should also be noted that, if the segments are infinitesimally small, the resulting logarithmic flow curve is identical to the von Mises curve of Fig. 2b.

5. Conclusions
1. When the logarithmic strain is employed in conjunction with the Fields and Backofen flow stress, the resulting stress/strain curve is not work conjugate with the shear stress/shear strain curve (nor with the external work done). Here a second example is provided of a problem that has already been pointed out in an earlier publication.

2. When tests are carried out at a constant logarithmic “equivalent” strain rate, the twist rate of the testing equipment must increase over several orders of magnitude, by an amount that increases with the total twist or shear being applied.

3. When tests are conducted in increments, the resulting flow curves depend on whether and how much of the accumulated strain is known or taken into account, i.e. on knowledge of the prior history of the sample. In this way, the logarithmic strain is “history dependent”, a shortcoming that does not affect the shear strain or the von Mises equivalent strain.

4. For the above reasons, researchers who want to describe or control experiments using the logarithmic strain should not refer to it as an “equivalent” strain. This usage should be avoided because the results obtained up to an HPT equivalent strain expressed in this way are in no way equivalent to those obtained by deforming to the same equivalent strain along other strain paths.

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7. References
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