Invariant Virtual Solitary Manifold
of the Perturbed Sine-Gordon Equation

TIMUR MASHKIN
Mathematisches Institut, Universität Köln,
Weyertal 86-90, D - 50931 Köln, Germany
e-mail: tmashkin@math.uni-koeln.de

Abstract
We study the perturbed sine-Gordon equation \( \theta_{tt} - \theta_{xx} + \sin \theta = F(\varepsilon, x) \), where we assume that the perturbation \( F \) is analytic in \( \varepsilon \) and that its derivatives with respect to \( \varepsilon \) satisfy certain bounds at \( \varepsilon = 0 \). We construct implicitly an, adjusted to the perturbation \( F \), virtual solitary manifold, which is invariant in the following sense: The initial value problem for the perturbed sine-Gordon equation with an appropriate initial state on the constructed manifold has a unique solution, which follows a trajectory on the virtual solitary manifold. The trajectory is precisely described by two parameters, which satisfy a specific system of ODEs.

The approach is based on [Mas17a], where we constructed by an iteration scheme a virtual solitary manifold for the perturbed sine-Gordon equation. In [Mas17a] we proved a stability result for the perturbed sine-Gordon equation with initial data close to the virtual solitary manifold. The employed iteration scheme produces a sequence of virtual solitary manifolds such that the accuracy of the corresponding stability statements increases after each iteration step, as long as the perturbation \( F \) is sufficiently often differentiable. The invariant virtual solitary manifold constructed in this work is generated as a limit of the virtual solitary manifolds produced by the iteration scheme.

The method and the kind of result presented in this paper is to our knowledge a novelty in the field of stability of solitons.

1 Introduction
We consider the perturbed sine-Gordon equation
\[
\theta_{tt} - \theta_{xx} + \sin \theta = F(\varepsilon, x), \quad t, x \in \mathbb{R}, \quad \varepsilon \ll 1,
\] (1)
which can be written as a system in first order formulation:
\[
\partial_t \begin{pmatrix} \theta \\ \psi \end{pmatrix} = \begin{pmatrix} \psi \\ \theta_{xx} - \sin \theta + F(\varepsilon, x) \end{pmatrix}.
\] (2)

The unperturbed sine-Gordon equation (i.e., \( F(\varepsilon, x) = 0 \)) admits soliton solutions
\[
\begin{pmatrix} \theta_0(\xi(t), u(t), x) \\ \psi_0(\xi(t), u(t), x) \end{pmatrix},
\]
where
\[
\dot{\xi} = u, \quad \dot{u} = 0, \quad (\xi(0), u(0)) = (a, v) \in \mathbb{R} \times (-1, 1).
\] (3)

Here the functions \((\theta_0, \psi_0)\) are defined by
\[
\begin{pmatrix} \theta_0(\xi, u, x) \\ \psi_0(\xi, u, x) \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} \theta_K(\gamma(u)(x - \xi)) \\ -u\gamma(u)\theta'_K(\gamma(u)(x - \xi)) \end{pmatrix}, \quad u \in (-1, 1), \quad \xi, x \in \mathbb{R},
\] (4)

where
\[
\gamma(u) = \frac{1}{\sqrt{1 - u^2}}, \quad \theta_K(x) = 4 \arctan(e^x),
\]
and \(\theta_K\) satisfies \(\theta''_K(x) = \sin \theta_K(x)\) with boundary conditions \(\theta_K(x) \to \begin{pmatrix} 2\pi \\ 0 \end{pmatrix}\) as \(x \to \pm \infty\).

The states \(\begin{pmatrix} \theta_0(a, v, \cdot) \\ \psi_0(a, v, \cdot) \end{pmatrix}\) form the two dimensional classical solitary manifold
\[
\mathcal{S}_0 := \left\{ \begin{pmatrix} \theta_0(a, v, \cdot) \\ \psi_0(a, v, \cdot) \end{pmatrix} : v \in (-1, 1), \quad a \in \mathbb{R} \right\}.
\]

Let us mention some previous works before we state the main result. Orbital stability of soliton solutions under perturbations of the initial data has been proven for the unperturbed sine-Gordon equation (see [HPW82], [Stu12, Section 4]). D. M. Stuart [Stu92] considered the perturbed sine-Gordon equation
\[
\theta_{tt} - \theta_{xx} + \sin \theta + \varepsilon g = 0,
\]
for specific perturbations of the form \(g = g(\varepsilon t, \varepsilon x, \theta)\) and initial data \(\varepsilon\)-close to a kink. He proved the existence of solutions, which approximate kinks with slowly evolving in time centre and velocity, up to time \(1/\varepsilon\) and up to errors of order \(\varepsilon\). Kinks are solutions of the unperturbed equation (1), given by \(\theta(t, x) = \theta_0(\xi(t), u(t), x)\), where the centre \(\xi\) and the velocity \(u\) satisfy ODEs (3). The proof is based on an orthogonal decomposition of the solution into an oscillatory part and a one-dimensional ”zero-mode” term.

In [Mas16, Part I] we studied equation (2) for different types of perturbations. For instance, we proved for \(F(\varepsilon, x) = \varepsilon f(\varepsilon x)\) that the Cauchy problem for initial data \(\varepsilon^{1/3}\)-close
to the classical solitary manifold $S_0$ has a unique solution, which follows up to time $1/\varepsilon$ and errors of order $\varepsilon^{3/2}$ a trajectory on $S_0$, where the trajectory on $S_0$ is described precisely by ODEs for uniform linear motion. One should take into account that our perturbation $F(\varepsilon, x) = \varepsilon f(\varepsilon x)$ is not comparable to the perturbations in [Stu92] due to some specific assumptions made on $g$.

For perturbations of type $F(\varepsilon, x) = \varepsilon^2 f(\varepsilon x)$ with $f \in H^3(\mathbb{R})$, we obtained richer dynamics on the solitary manifold in [Mas17b]. We proved that the Cauchy problem for initial data $\varepsilon^{1/2}$-close to the classical solitary manifold $S_0$ has a unique solution, which follows up to time $1/\varepsilon$ and errors of order $\varepsilon^{3/2}$ a trajectory on $S_0$. The trajectory on $S_0$ is described precisely by ODEs, which contain the perturbation $\varepsilon f(\varepsilon x)$. The ODEs are obtained by considering restricted Hamilton equations and describe a fixed nontrivial perturbation of the uniform linear motion as $\varepsilon \to 0$ if $f(0) \neq 0$. The evolution of the dynamics on the solitary manifold in [Mas16, Part I]/[Mas17b] is described more accurate than the evolution of the approximated kink in [Stu92] in the following sense: In [Mas16, Part I]/[Mas17b] the parameters of the manifold satisfy exactly specific ODEs, whereas in [Stu92] the evolution of the kink parameters are determined just up to errors of order $\varepsilon$.

The proofs of [Mas16, Part I], [Mas17b], and [Stu12, Section 4] are based on a nowadays conventional method for verification of stability of solitons (for different equations), namely the decomposition of the dynamics into a part on the classical solitary manifold and a transversal part along with the application of Lyapunov-type arguments. This approach emerges, for instance, also in FGJS04 JFGS06 HZ07 HZ08 Hol11.

In [Mas17a] we extended this method by utilizing a virtual solitary manifold. There we studied the sine-Gordon equation with perturbations $\varepsilon \mapsto F(\varepsilon, \cdot)$ of class $C^m$ (mapping into a specific weighted Sobolev space on $\mathbb{R}$), whose first $k$ derivatives vanish at 0, i.e., $\partial_x^l F(0, \cdot) = 0$ for $0 \leq l \leq k$, where $k + 1 \leq n$ and $n \geq 1$. We constructed in [Mas17a] by an iteration scheme composed of $n$ steps a virtual solitary manifold, which is adjusted to the perturbation $F$. The iteration process can be thought of as a stepwise distortion of the classical solitary manifold $S_0$. Each step in the iteration scheme corresponds to solving implicitly a specific PDE. The implicit solution $\varepsilon \mapsto (\theta^\varepsilon_n(\xi, u, x), \psi^\varepsilon_n(\xi, u, x), \lambda^\varepsilon_n(\xi, u))$ obtained in the last iteration step defines the virtual solitary manifold

$$S^\varepsilon_n := \ \left\{ \begin{array}{l} \theta^\varepsilon_n(a, v, \cdot) \\
\psi^\varepsilon_n(a, v, \cdot) \end{array} : v \in (-u_s, u_s), a \in \mathbb{R} \right\}, \quad u_s \in (0, 1], \quad (5)$$

and is used to formulate the result of [Mas17a], which is as following: For $\xi_s \in \mathbb{R}$, $\varepsilon \ll 1$, the Cauchy problem

$$\partial_t \begin{pmatrix} \theta \\ \psi \end{pmatrix} = \begin{pmatrix} \psi \\ -\sin \theta + F(\varepsilon, x) \end{pmatrix}, \quad \begin{pmatrix} \theta(0, x) \\ \psi(0, x) \end{pmatrix} = \begin{pmatrix} \theta^\varepsilon_n(\xi_s, u_s, x) \\ \psi^\varepsilon_n(\xi_s, u_s, x) \end{pmatrix} + \begin{pmatrix} v(0, x) \\ w(0, x) \end{pmatrix}, \quad (6)$$

with appropriate initial data that is $\varepsilon^n$-close to $S^\varepsilon_n$, i.e., $|v(0, \cdot)|^2_{H^1(\mathbb{R})} + |w(0, \cdot)|^2_{L^2(\mathbb{R})} \leq \varepsilon^{2n}$, with initial velocity that satisfies the smallness assumption $|u_s| \leq \tilde{C} \varepsilon^{3/4}$, has a unique
solution \((\theta, \psi)\), which may be written up to time \(1/(\tilde{C}\varepsilon^{k+1})\) in the form

\[
\begin{pmatrix}
\theta(t, x) \\
\psi(t, x)
\end{pmatrix} = \begin{pmatrix}
\theta^\varepsilon_n(\bar{\xi}(t), \bar{u}(t), x) \\
\psi^\varepsilon_n(\bar{\xi}(t), \bar{u}(t), x)
\end{pmatrix} + \begin{pmatrix}
v(t, x) \\
w(t, x)
\end{pmatrix},
\]

The solution remains \(\varepsilon^n\)-close to \(S^\varepsilon_n\), i.e., \(|v(t, \cdot)|_{H^1(\mathbb{R})}^2 + |w(t, \cdot)|_{L^2(\mathbb{R})}^2 \leq \tilde{C}\varepsilon^{2n}\), and the dynamics on \(S^\varepsilon_n\) is described precisely by the parameters \((\bar{\xi}(t), \bar{u}(t))\), which satisfy exactly the ODEs

\[
\dot{\xi}(t) = \bar{u}(t), \quad \dot{\bar{u}}(t) = \lambda^\varepsilon_n(\bar{\xi}(t), \bar{u}(t)),
\]

with initial data \(\bar{\xi}(0) = \xi_s, \bar{u}(0) = u_s\). The parameters \(\bar{\xi}, \bar{u}\) describe a fixed nontrivial perturbation of the uniform linear motion as \(\varepsilon \to 0\) if the perturbation \(F\) satisfies a specific condition. The higher the differentiability class \(C^n\) of \(F\) the higher is the accuracy of the stability statement and the more first derivatives of \(F\) vanish at 0 the larger is the time scale of the result.

The sine-Gordon equation arises in various physical applications presented for instance in \[ZHQQ5, KM89, PK99, MIK78\]. In \[SKY61\] T. H. R. Skyrme proposed the equation to model elementary particles and in \[CT9\] dynamics of solitons under constant electric field were examined numerically. We focus in the present work, as also in \[MA17\], on the interaction of virtual solitons with a time independent electric field \(F(\varepsilon, x)\), which is a physically relevant problem.

**Main Result and Consequences** The iteration scheme introduced in \[MA17\] provides a sequence of implicitly given functions. In the present paper, we show that under some additional assumptions the provided sequence, denoted by \((\theta^\varepsilon_n, \psi^\varepsilon_n, \lambda^\varepsilon_n)\), converges to a limit, which we denote by \((\theta^\infty, \psi^\infty, \lambda^\infty)\). Our main result states that the virtual solitary manifold defined analogously to \([5]\) by the functions \((\theta^\infty, \psi^\infty, \lambda^\infty)\) is invariant. In greater detail, the main result is as follows. Assume that the perturbation \(\varepsilon \mapsto F(\varepsilon, \cdot)\) is analytic (mapping into a specific weighted Sobolev space on \(\mathbb{R}\)), where the derivatives with respect to \(\varepsilon\) of \(F\) satisfy specific bounds at \(\varepsilon = 0\) (stated below in \([21]\)) and \(F(0, \cdot) = 0, \partial_\varepsilon F(0, \cdot) = 0\). Let \(\xi_s \in \mathbb{R}\) and consider the Cauchy problem

\[
\partial_t \begin{pmatrix}
\theta \\
\psi
\end{pmatrix} = \begin{pmatrix}
\psi \\
\partial^2 \theta - \sin \theta + F(\varepsilon, x)
\end{pmatrix}, \quad \begin{pmatrix}
\theta(0, x) \\
\psi(0, x)
\end{pmatrix} = \begin{pmatrix}
\theta^\infty(\xi_s, u_s, x) \\
\psi^\infty(\xi_s, u_s, x)
\end{pmatrix}, \quad \varepsilon \ll 1, \quad (7)
\]

where the initial velocity satisfies the assumption \(|u_s| < u_\ast\) for a specific \(u_\ast\). Then the Cauchy problem \([7]\) has a unique solution, which may be written in the form

\[
\begin{pmatrix}
\theta(t, x) \\
\psi(t, x)
\end{pmatrix} = \begin{pmatrix}
\theta^\varepsilon_n(\bar{\xi}(t), \bar{u}(t), x) \\
\psi^\varepsilon_n(\bar{\xi}(t), \bar{u}(t), x)
\end{pmatrix},
\]

4
where the parameters \((\bar{\xi}(t), \bar{u}(t))\) satisfy the ODEs

\[
\dot{\bar{\xi}}(t) = \bar{u}(t), \quad \dot{\bar{u}}(t) = \lambda_\varepsilon \left( \bar{\xi}(t), \bar{u}(t) \right),
\]

with initial data \(\bar{\xi}(0) = \xi_s, \quad \bar{u}(0) = u_s\). The solution exists and has this form as long as the parameters stay in an appropriate parameter area, i.e., as long as \(|\bar{\xi}(t)| \leq \Xi, \quad |\bar{u}(t)| < u^*\), where \(\Xi\) depends on the initial centre \(\xi_s\). In particular, if \(|u_s| \leq \tilde{C}\varepsilon\) for a specific \(\tilde{C}\), then the unique solution exists and can be expressed in the presented form on the time scale

\[
0 \leq t \leq \frac{1}{C\varepsilon}.
\]

If additionally the perturbation \(F\) satisfies condition (21) mentioned below, then the parameters \(\xi, \bar{u}\) describe, on the nontrivial time scale \(1\), a fixed nontrivial perturbation of the uniform linear motion as \(\varepsilon \to 0\).

The result states that the solution remains on the virtual solitary manifold defined by \((\theta_\varepsilon, \psi_\varepsilon)\) and it yields a precise description of the solution \((\theta, \psi)\) to the Cauchy problem (7), since the dynamics on the manifold is exactly characterized by the ODEs (8). The maximal interval of existence (time interval) of the solution depends on the perturbation \(F\) and on the initial data, which determine the ODEs (8), whereas the ODEs determine for how long the parameters \((\bar{\xi}(t), \bar{u}(t))\) stay in the corresponding parameter area. A precise statement is found in Section 2.

The existence of the invariant virtual solitary manifold has a tremendous theoretical value. Furthermore, the invariant manifold allows us to describe the solution of (2) with appropriate initial data by far more accurate than it was done in \([\text{Mas17a}]\). Our main result can be considered as an extension of the work of \([\text{Mas17a}]\), where we corrected the classical solitary manifold of the sine-Gordon equation arbitrarily many times (finite number) and improved the accuracy of the stability statement in each correction step. In this paper the invariant virtual solitary manifold is generated by a limit process - that is, in infinitely many correction steps - in such a way that the manifold is adjusted to the perturbation term \(F\).

There exists a community, which advocates the following conjecture for specific PDEs with soliton solutions: For appropriate classes of solutions to the corresponding PDE there exists a manifold, which acts as an attractor. One expects that for appropriate initial data, not necessarily close to the manifold, the solution is going to come close to the manifold for advancing times. In case of the sine-Gordon equation the virtual solitary manifold generated in this paper is a serious candidate for such an attractive manifold, which makes our result even more interesting for further investigations.

Our approach and the fact of existence of an invariant manifold for an integrable equation with an external perturbation (invariant in the sense of our main result), is to our knowledge a novelty in the field of stability of solitons. However, singular corrections of the classical solitary manifold have been carried out in other works in different forms such as in \([\text{HL12}]\) and in \([\text{HZ08}]\) for the NLS equation, which corresponds to the first iteration in the scheme from \([\text{Mas17a}]\). The idea of modifying the classical solitary manifold of the
sine-Gordon equation by utilizing implicitly defined functions appears in [Stu12, Section 3], where the purpose was to rewrite the Hamiltonian in a neighbourhood of the manifold of virtual solitons. Neither the virtual solitary manifold (5) nor the iteration scheme introduced in [Mas17a] were considered in [Stu12].

Several long (but finite)-time results for different equations with external potentials can be found, for example, in [FGJS04, JFGS06, HZ07, Hol11]. Further results on orbital stability and long time soliton asymptotics are presented in [Wei86, Ben76, Bon75, MP12, SW90, BP92, IKV12, KMM17, CMnPS16].

Our Techniques We generate the invariant virtual solitary manifold by utilizing the iteration scheme from [Mas17a], whereby we modify the scheme in certain points. In the present paper, the scheme is implemented for an analytic function \( \varepsilon \mapsto \tilde{F}(\varepsilon) \) mapping into a specific Sobolev space on \( \mathbb{R}^2 \) such that \( \tilde{F}(\varepsilon) \) depends on \( (\xi, x) \) (for the sake of clarity, we skip the dependence on \( (\xi, x) \) in the notation). We assume that the derivatives of \( \tilde{F} \) with respect to \( \varepsilon \) satisfy specific bounds at \( \varepsilon = 0 \) (stated below in (30)) and that \( \tilde{F}(0) = 0 \), \( \partial_\varepsilon \tilde{F}(0) = 0 \). \( \tilde{F} \) will be specified later. The iteration scheme is as follows: The function \((\theta_0, \psi_0)\), given by (1), solves

\[
\begin{align*}
  u\partial_\xi \left( \frac{\theta}{\psi} \right) - \left( \psi \partial_\theta^2 - \sin \theta \right) &= 0, \\
  =: G_0(\theta, \psi)
\end{align*}
\]

which is the equation characterizing the classical solitons. In the first iteration step we amend \( G_0(\theta, \psi) = 0 \) by introducing an additional unknown variable \( \lambda \) and adding some terms involving \( (\theta_0, \psi_0) \) and \( \tilde{F} \). The amended equation is of the form

\[
\begin{align*}
  u\partial_\xi \left( \frac{\theta}{\psi} \right) - \left( \psi \theta_{xx} - \sin \theta + \tilde{F}(\varepsilon) \right) + \lambda \partial_u \left( \frac{\theta_0}{\psi_0} \right) &= 0. \\
  =: G_1^\varepsilon(\theta, \psi, \lambda)
\end{align*}
\]

Here and in the following iterations the functions \( \theta, \psi \) depend on \( (\xi, u, x) \) and \( \lambda \) depends on \( (\xi, u) \). We solve \( G_1^\varepsilon(\theta, \psi, \lambda) = 0 \) implicitly for \( (\theta, \psi, \lambda) \) in terms of \( \varepsilon \) and denote the solution by \( (\theta_1^\varepsilon, \psi_1^\varepsilon, \lambda_1^\varepsilon) \). In the next iteration step we amend \( G_1^\varepsilon(\theta, \psi, \lambda) = 0 \) by adding some terms involving \( (\theta_1^\varepsilon, \psi_1^\varepsilon) \) and solve the amended equation

\[
\begin{align*}
  u\partial_\xi \left( \frac{\theta}{\psi} \right) - \left( \psi \theta_{xx} - \sin \theta + \tilde{F}(\varepsilon) \right) + \lambda \partial_u \left( \frac{\theta_0^\varepsilon + \partial_\varepsilon \theta_0^\varepsilon}{\psi_0^\varepsilon + \partial_\varepsilon \psi_0^\varepsilon} \right) &= 0. \\
  =: G_2^\varepsilon(\theta, \psi, \lambda)
\end{align*}
\]
implicitly for \((\theta, \psi, \lambda)\) in terms of \(\varepsilon\). Continuing the iteration process we obtain in the \(n\)th step the equation

\[
 u \partial_\xi \left( \frac{\theta}{\psi} \right) - \left( \theta_{xx} - \sin \theta + \tilde{F}(\varepsilon) \right) + \lambda \partial_u \left( \sum_{i=0}^{n-1} \frac{\partial \theta_0^i}{\partial u^i} \varepsilon^i \right) = 0,
\]

where \((\theta_{n-1}^e, \psi_{n-1}^e, \lambda_{n-1}^e)\) denotes the solution of \(\mathcal{G}_{n-1}^e(\theta, \psi, \lambda) = 0\). We solve \(\mathcal{G}_{n}^e(\theta, \psi, \lambda) = 0\) implicitly for \((\theta, \psi, \lambda)\) in terms of \(\varepsilon\) and denote the solution by \((\theta_n^e, \psi_n^e, \lambda_n^e)\). Due to the assumptions on \(\mathcal{G}_{n}^e(\theta, \psi, \lambda)\) it is possible to iterate this procedure arbitrarily many times. The existence of the implicit solutions \(\varepsilon \mapsto (\theta_n^e, \psi_n^e, \lambda_n^e)\) for \(n \geq 1\) is ensured by the implicit function theorem. In the actual proof, we consider rather the transformed equations

\[
 \tilde{\mathcal{G}}_n^e(\hat{\theta}, \hat{\psi}, \lambda) := \mathcal{G}_n^e(\theta_0 + \hat{\theta}, \psi_0 + \hat{\psi}, \lambda) = 0, \quad n \geq 1,
\]

which will be solved for \((\hat{\theta}, \hat{\psi}, \lambda)\) in terms of \(\varepsilon\). This is caused by functional analytic reasons, among others, by the fact that \(\theta_0(\xi, u, x) \not\to 0\) as \(|x| \to \infty\) for fixed \(\xi\) and \(u\). We denote the solutions to the equations \(\tilde{\mathcal{G}}_n^e(\hat{\theta}, \hat{\psi}, \lambda) = 0, \quad n \geq 1\), by \((\hat{\theta}_n^e, \hat{\psi}_n^e, \lambda_n^e)\), where \((\theta_n^e, \psi_n^e, \lambda_n^e) = (\theta_0 + \hat{\theta}_n^e, \psi_0 + \hat{\psi}_n^e, \lambda)\). The application of the implicit function theorem relies on the fact that \((0, 0, 0, 0)\) solves all equations in a particular point, i.e., \(\tilde{\mathcal{G}}_n^0(0, 0, 0, 0) = 0\). As a consequence of the construction, the solution obtained in the \(n\)th iteration \(\varepsilon \mapsto (\theta_n^e, \psi_n^e, \lambda_n^e)\) solves the equation

\[
 u \partial_\xi \left( \frac{\theta}{\psi} \right) - \left( \theta_{xx} - \sin \theta + \tilde{F}(\varepsilon) \right) + \lambda \partial_u \left( \frac{\theta}{\psi} \right) = 0
\]

up to errors of order \(\varepsilon^{n+1}\) for \(n \geq 1\).

In [Mas17a], the iterative equations \(\mathcal{G}_n^e(\hat{\theta}, \hat{\psi}, \lambda) = 0\) were solved in spaces of different regularity in \(u\) such that the regularity of the spaces (which contain the corresponding iterative solutions) decreases after each iteration step by the order of 1. This technique was used for the following reason. Each iterative equation contains a derivative with respect to \(u\) of the solution of the preceding equation, as one can see in (12). This derivative leads to loss of regularity in \(u\) in the target set of the map \(\mathcal{G}_n\) after each iteration step. However, the employment of the implicit function theorem for solving the iterative equations requires that the corresponding linearizations are invertible and that the maps \(\mathcal{G}_n\) are well-defined. In [Mas17a], this is ensured by considering the maps \(\tilde{\mathcal{G}}_n\) on spaces of decreasing regularity in \(u\). Since, in the present paper, we need to execute infinitely many (and not only finitely many) iterations in order to obtain a sequence of implicit solutions, we modify the iteration scheme and proceed as follows.

Due to the analyticity assumption on \(F\) in the present paper (which was not supposed in [Mas17a]), the implicit solutions (as well as its derivatives) are analytic in \(\varepsilon\), which is a
where \( \| \cdot \| \) bounds on the derivatives of the succeeding solutions. We prove successively the same spaces as also the preceding equation \( \tilde{u} \) (derivatives with respect to \( u \in [-u_\ast, u_\ast] \) and \( \varepsilon \), evaluated at \( \varepsilon = 0 \)), which have the form

\[
\forall N \geq 2, \ 0 \leq K \leq 2 : \quad \left\| \frac{\partial^K_u \partial^N \theta_1^0}{\partial^K_u \partial^N \psi_1^0} \right\| \leq C^{2N+2K-3}(N-2)!, \quad (14)
\]

\[
\forall N \geq 2, \ K \geq 3 : \quad \left\| \frac{\partial^K_u \partial^N \theta_1^0}{\partial^K_u \partial^N \psi_1^0} \right\| \leq C^{2N+2K-3}(N-2)!(K-3)!, \quad (15)
\]

where \( \| \cdot \| \) is an appropriate norm. These bounds imply that the implicit solution \( (\theta_1^\varepsilon, \psi_1^\varepsilon, \lambda_1^\varepsilon) \) is differentiable in \( u \) in the same neighbourhood of \( \varepsilon = 0 \) where also representation \( \tilde{G}_1 \) holds. Thus the map \( \tilde{G}_2 \) is well defined on the same spaces where also \( \tilde{G}_1(\hat{\theta}, \hat{\psi}, \lambda) = 0 \) was solved initially. This eliminates the loss of regularity problem faced in [Mas17a] (in the first iteration) and we are able to solve the next iterative equation \( \tilde{G}_2^\varepsilon(\hat{\theta}, \hat{\psi}, \lambda) = 0 \) on the same spaces as also the preceding equation \( \tilde{G}_1^\varepsilon(\hat{\theta}, \hat{\psi}, \lambda) = 0 \). The process of solving the iterative equations will be continued using the same arguments, whereas we prove successively bounds on the derivatives of the succeeding solutions \( \partial^K_u \partial^N \varepsilon(\theta_n^0, \psi_n^0, \lambda_n^0) \) (derivatives with respect to \( u \in [-u_\ast, u_\ast] \) and \( \varepsilon \), evaluated at \( \varepsilon = 0 \)). The bounds are uniform in \( n \) and have the form

\[
\forall N \geq 2, \ 0 \leq K \leq 2 : \quad \left\| \frac{\partial^K_u \partial^N \theta_n^0}{\partial^K_u \partial^N \psi_n^0} \right\| \leq C^{2N+2K-3}(N-2)!, \quad (16)
\]

\[
\forall N \geq 2, \ K \geq 3 : \quad \left\| \frac{\partial^K_u \partial^N \theta_n^0}{\partial^K_u \partial^N \psi_n^0} \right\| \leq C^{2N+2K-3}(N-2)!(K-3)!, \quad (17)
\]

where \( \| \cdot \| \) is as above. Here and in \( (14)-(17) \) the higher order derivatives with respect to \( u \) are needed in order to control the first order derivative terms (derivative with respect to \( u \)) in the iterative equations (see \( (12) \)). This fact itself and the proof of bounds \( (14)-(17) \) as well rely on a recursive formula for \( \partial^K_u \partial^N \varepsilon(\theta_n^0, \psi_n^0, \lambda_n^0) \), which is proved by induction on \( N \).
and $K$. Furthermore, the assumptions on the derivatives of $\tilde{F}$ at $\varepsilon = 0$ are used in the proof of (14)-(17). Bounds (14)-(17) imply that all iterative implicit solutions are defined on the same neighbourhood, can be represented there as Taylor series around $\varepsilon = 0$ analogous to (13) and are there differentiable in $u$. Moreover, it follows from (14)-(17) that the iterative implicit solutions are all contained in the same space and that as $n \to \infty$ the sequence $(\hat{\theta}_n^\varepsilon, \hat{\psi}_n^\varepsilon, \lambda_n^\varepsilon)$ converges to the limit

$$(\hat{\theta}_\infty^\varepsilon, \hat{\psi}_\infty^\varepsilon, \lambda_\infty^\varepsilon) := \left( \sum_{i=1}^{\infty} \frac{\partial_i^0 \theta_0}{i!} \varepsilon^i, \sum_{i=1}^{\infty} \frac{\partial_i^0 \psi_0}{i!} \varepsilon^i, \sum_{i=0}^{\infty} \frac{\partial_i^0 \lambda_0}{i!} \varepsilon^i \right).$$

Using these facts and (14)-(17) we conclude that the function $\left( \theta_\infty^\varepsilon, \psi_\infty^\varepsilon, \lambda_\infty^\varepsilon \right) := (\theta_0 + \hat{\theta}_\infty^\varepsilon, \psi_0 + \hat{\psi}_\infty^\varepsilon, \lambda_\infty^\varepsilon)$ satisfies the equation

$$u \partial_x \left( \theta_\infty^\varepsilon \psi_\infty^\varepsilon \right) - \left( \theta_\infty^\varepsilon \right)_{xx} - \sin \theta_\infty^\varepsilon + \tilde{F}(\varepsilon) = 0.$$  \hspace{1cm} (18)

In order to generate the invariant virtual solitary manifold, we apply the iteration scheme to a specific $\tilde{F}$, which is a truncated version of the perturbation term $F$ from (6), given by

$\tilde{F}(\varepsilon, \xi, x) := F(\varepsilon, x) \chi(\xi),$

where $\chi \in C^\infty(\mathbb{R})$, $\chi(\xi) = 1$ for $|\xi| \leq |\xi_s| + 3$ and $\chi(\xi) = 0$ for $|\xi| \geq |\xi_s| + 4$.  \hspace{1cm} (19)

The limit of the thereby obtained sequence of iterative solutions, defines the solution of (18) with the specific $\tilde{F}$ (given by (19)), which implies our main result.

In order to simplify the computations we work in the present paper on spaces, which have lower regularity in $(\xi, x)$ than the corresponding spaces in [Mas17a].

Finally let us explain under which condition the parameters $\bar{\xi}, \bar{u}$ describe a fixed non-trivial perturbation of the uniform linear motion as $\varepsilon \to 0$. We consider the setting where the assumption $|u_s| \leq \bar{C}\varepsilon$ is satisfied and hence where the solution of (17) exists and may be expressed up to times $1/(\bar{C}\varepsilon)$ in the mentioned way. For all $n \geq 1$ the linearization of $(\theta, \psi, \lambda) \mapsto \tilde{G}_n(\hat{\theta}, \hat{\psi}, \lambda)$ carried out at $(\hat{\theta}, \hat{\psi}, \lambda) = (0, 0, 0)$, $\varepsilon = 0$ is invertible and we denote the linearization by $M_0^\alpha : (\theta, \psi, \lambda) \mapsto M_0^\alpha(\theta, \psi, \lambda)$.

Thus there exist functions $(\bar{\theta}, \bar{\psi}, \bar{\lambda})$ such that the second derivative with respect to $\varepsilon$ of a general function $\tilde{F}$ (which operates on appropriate spaces), evaluated at $\varepsilon = 0$, can be written in the form

$$\begin{pmatrix} 0 \\ \partial_\varepsilon^2 \tilde{F}(0) \end{pmatrix} = M_0^\alpha(\bar{\theta}, \bar{\psi}, \bar{\lambda}), \hspace{1cm} M_0^\alpha \text{ given by Proposition 3.2 (case } m = 0).$$  \hspace{1cm} (20)
Here the functions $\bar{\theta}, \bar{\psi}$ depend on $(\xi, u, x)$ and $\bar{\lambda}$ depends on $(\xi, u)$. ODEs (8) can be rescaled in time by introducing $s = \varepsilon t$, $\hat{\xi}(s) = \bar{\xi}(s/\varepsilon)$, and $\hat{u}(s) = \frac{1}{\varepsilon^2} \bar{u}(s/\varepsilon)$ such that the corresponding transformed ODEs have the form

$$\frac{d}{ds} \hat{\xi}(s) = \hat{u}(s), \quad \frac{d}{ds} \hat{u}(s) = \frac{1}{\varepsilon^2} \lambda_\infty^\varepsilon(\hat{\xi}(s), \varepsilon \hat{u}(s)).$$

As $\varepsilon \to 0$, the transformed ODEs converge to ODEs that describe a fixed nontrivial perturbation of the uniform linear motion if the next condition is satisfied:

$$\left\{ \begin{array}{l}
\text{There exists } \chi \text{ satisfying (19) such that for } \tilde{F} \text{ given by (19)} \\
\text{the following holds: } \tilde{\lambda}(\cdot, 0) \neq 0 \text{ in representation (20).} 
\end{array} \right. \quad (21)$$

This is for the following reason. The functions $(\theta_\infty^\varepsilon, \psi_\infty^\varepsilon, \lambda_\infty^\varepsilon)$ satisfy the relation

$$u \partial_\xi \left( \frac{\theta_\infty^\varepsilon}{\psi_\infty^\varepsilon} \right) - \left[ \theta_\infty^\varepsilon_{xx} - \sin \theta_\infty^\varepsilon + \tilde{F}(\varepsilon) \right] + \lambda_\infty^\varepsilon u \partial_u \left( \frac{\theta_\infty^\varepsilon}{\psi_\infty^\varepsilon} \right) = 0. \quad (22)$$

Due to the assumption on $F$ it holds that $\partial_\varepsilon \tilde{F}(0) = 0$ and differentiation of (22) with respect to $\varepsilon$ yields

$$\begin{pmatrix} 0 \\ \partial_\varepsilon \tilde{F}(0) \end{pmatrix} = \mathcal{M}_0^\varepsilon (\partial^l \theta_\infty^0, \partial^l \psi_\infty^0, \partial^l \lambda_\infty^0), \quad 1 \leq l \leq 2.$$

Using invertibility of $\mathcal{M}_0^\varepsilon$, condition (21) and the fact that $\lambda_0^\varepsilon = 0$ it follows that $0 \neq \lambda_\infty^0(\cdot, 0) = \mathcal{O}(\varepsilon^2)$, which implies the claim.

**Outline of the Paper** The paper is organized as follows. In Section 2, we formulate the main result. In Section 3, we modify the iteration scheme from [Mas17a], construct a sequence of iterative solutions and prove bounds on the elements of the sequence. In Section 4, we show that the sequence of iterative solutions converges and that its limit satisfies the equation of interest. Our main result, Theorem 2.2, is proved in Section 5.

**Notation and Conventions** For a Hilbert space $H$ we denote its inner product by $\langle \cdot, \cdot \rangle_H$. To simplify notation, occasionally we drop the dependence of functions on certain variables. We write $L^2_x(\mathbb{R})$, $H^k_x(\mathbb{R}^2)$ and so on for the Lebesgue and Sobolev spaces when we wish to emphasize the variables of integration. We use the notation $\theta(\xi, u, x) = \theta(u)(\xi, x)$, $\psi(\xi, u, x) = \psi(u)(\xi, x)$.
2 Main Result

To formulate our result precisely, we need some definitions.

**Definition 2.1.** Let \( \alpha, k, m \in \mathbb{N}_0 \) and \( u_\ast > 0 \). Let us denote by 

\[
I(u_\ast) := [-u_\ast, u_\ast].
\]

(a) \( H^{k,\alpha}(\mathbb{R}) \) denotes the weighted Sobolev space of functions with finite norm 

\[
|\theta|_{H^{k,\alpha}(\mathbb{R})} = |(1 + |x|^2)^{\frac{\alpha}{2}}\theta(x)|_{H^k(\mathbb{R})}.
\]

(b) \( H^{k,\alpha}(\mathbb{R}^2) \) denotes the weighted Sobolev space of functions with finite norm 

\[
|\theta|_{H^{k,\alpha}(\mathbb{R}^2)} = |(1 + |\xi|^2 + |x|^2)^{\frac{\alpha}{2}}\theta(\xi, x)|_{H^k(\mathbb{R}^2)}.
\]

(c) \( Y^\alpha \) is the space \( H^{2,\alpha}(\mathbb{R}^2) \oplus H^{1,\alpha}(\mathbb{R}^2) \oplus H^{2,\alpha}(\mathbb{R}) \) with the finite norm 

\[
|y|_{Y^\alpha} = |\theta|_{H^{2,\alpha}(\mathbb{R}^2)} + |\psi|_{H^{1,\alpha}(\mathbb{R}^2)} + |\lambda|_{H^{2,\alpha}(\mathbb{R})}.
\]

(d) \( Y_m^\alpha(u_\ast) \) is the space 

\[
\left\{ y = (\theta, \psi, \lambda) \in C^m(I(u_\ast), Y^\alpha) : |y|_{Y_m^\alpha(u_\ast)} < \infty; \quad \forall u \in I(u_\ast), \quad \forall \mu \in H^{2,\alpha}(\mathbb{R}): 
\right.
\]

\[
\left. \left\langle \left( \begin{array}{c} \theta(u)(\xi, x) \\ \psi(u)(\xi, x) \end{array} \right), \mu(\xi) \left( \begin{array}{c} \theta'(u)(\gamma(u)(x - \xi)) \\ -u\gamma(u)\theta''(u)(\gamma(u)(x - \xi)) \end{array} \right) \right\rangle_{L_x^{2,\alpha}(\mathbb{R}^2) \oplus L_x^{2,\alpha}(\mathbb{R}^2)} = 0 \right\}
\]

with the finite norm 

\[
|y|_{Y_m^\alpha(u_\ast)} = \sup_{u \in I(u_\ast)} \left( \sum_{i=0}^{m} |\partial_i^2 y(u)|_{Y^\alpha} \right).
\]

The weighted Sobolev spaces in Definition 2.1 (a), (b) are defined as in [Kop15]. We are now ready to state our main result.

**Theorem 2.2.** Let \( \xi_\ast \in \mathbb{R}, \quad \Xi := \Xi(\xi_\ast) := |\xi_\ast| + 3 \) and \( \alpha \in \mathbb{N}_0 \). Assume that \( F \in C^\infty((-1, 1), H^{0,\alpha}(\mathbb{R})) \), \( F \) is analytic and the conditions 

\[
F(0) = 0, \quad \partial_\xi F(0) = 0,
\]

\[
\forall N \geq 2: \quad |\partial_\xi^N F(0)|_{H^{0,\alpha}} \leq c^N(N - 2)!
\]

are satisfied. Then there exist \( \varepsilon^* > 0, u_\ast > 0, \hat{C} > 0 \) and a map 

\[
(-\varepsilon^*, \varepsilon^*) \to Y_0^\alpha(u_\ast), \quad \varepsilon \mapsto (\hat{\theta}_\varepsilon^\ast, \hat{\psi}_\varepsilon^\ast, \hat{\lambda}_\varepsilon^\ast)
\]
of class $C^\infty$ such that the following holds. Let $\varepsilon \in (0, \varepsilon^*)$. Consider the Cauchy problem

$$
\frac{\partial}{\partial t} \begin{pmatrix} \theta \\ \psi \end{pmatrix} = \begin{pmatrix} \psi \\ \partial_x^2 \theta - \sin \theta + F(\varepsilon, x) \end{pmatrix}, \quad \begin{pmatrix} \theta(0, x) \\ \psi(0, x) \end{pmatrix} = \begin{pmatrix} \theta^\varepsilon_\infty(\xi_s, u_s, x) \\ \psi^\varepsilon_\infty(\xi_s, u_s, x) \end{pmatrix},
$$

where $(\theta^\varepsilon_\infty, \psi^\varepsilon_\infty) = (\theta_0 + \theta^\varepsilon_\infty, \psi_0 + \psi^\varepsilon_\infty)$ with $(\theta_0, \psi_0)$ given by (4) such that the initial velocity satisfies $|u_s| < u_\ast$. Then the Cauchy problem has a unique solution, which may be written in the form

$$
\begin{pmatrix} \theta(t, x) \\ \psi(t, x) \end{pmatrix} = \begin{pmatrix} \theta^\varepsilon(\bar{\xi}(t), \bar{u}(t), x) \\ \psi^\varepsilon(\bar{\xi}(t), \bar{u}(t), x) \end{pmatrix},
$$

where $\bar{\xi}, \bar{u}$ solve the system of equations

$$
\dot{\bar{\xi}}(t) = \bar{u}(t), \quad \dot{\bar{u}}(t) = \lambda^\varepsilon \left( \bar{\xi}(t), \bar{u}(t) \right), \quad \bar{\xi}(0) = \xi_s, \quad \bar{u}(0) = u_s,
$$

and representation (27) of the solution is valid as long as $|\bar{\xi}(t)| \leq \Xi, |\bar{u}(t)| < u_\ast$.

In particular, if $|u_s| \leq \tilde{C}\varepsilon$, then the Cauchy problem (26) has a unique solution on the time interval

$$
0 \leq t \leq \frac{1}{\tilde{C}\varepsilon}
$$

and may be written in the form (27) with ODEs (28). If additionally the perturbation $F$ satisfies condition (21), then the parameters $\bar{\xi}, \bar{u}$ describe a fixed nontrivial perturbation of the uniform linear motion as $\varepsilon \to 0$.

The assumption on the first derivative of $F$ in (23) is not crucial, it is made in order to simplify the computations in the proof of the bounds on the derivatives of the iterative solutions in Section 3 (Lemma 3.6).

We work in weighted Sobolev spaces in order to ensure that symplectic decomposition (implemented by techniques of [Mas17a]) is possible in a neighbourhood of the invariant virtual solitary manifold, since this is promising to be useful in our future works. The well-definedness of a corresponding symplectic orthogonality condition formulated in analogy to [Mas17a, Theorem 2.2 (b)] is guaranteed if function (25) maps into a weighted space $Y^\alpha(\varepsilon)$ where $\alpha \geq 1$ (nevertheless symplectic decomposition is not needed in the present paper).

3 Construction of the Sequence of Iterative Solutions

In this section we modify the iteration scheme from [Mas17a] and construct a sequence of iterative solutions. By making stronger assumptions than in [Mas17a] on the function $\tilde{F}$ (utilized in the scheme below), we obtain more accurate information on the iterative solutions. We start with a definition.
Definition 3.1. Let $\alpha, m \in \mathbb{N}_0$ and $u_* > 0$.
(a) $Z^\alpha$ is the space $H^{1,\alpha}(\mathbb{R}^2) \oplus H^{0,\alpha}(\mathbb{R}^2)$ with the finite norm
$$ |z|_{Z^\alpha} = |v|_{H^{1,\alpha}(\mathbb{R}^2)} + |w|_{H^{0,\alpha}(\mathbb{R}^2)}. $$
(b) $Z_m^\alpha(u_*)$ is the space \( \{ z = (v, w) \in C^m(I(u_*), Z^\alpha) : \|z\|_{Z_m^\alpha(u_*)} < \infty \} \) with the finite norm
$$ \|z\|_{Z_m^\alpha(u_*)} = \sup_{u \in I(u_*)} \left( \sum_{i=0}^m |\partial^i_y y(u)|_{Z^\alpha} \right). $$
(c) Let us denote by $t_1(\xi, u, x) := \left( \frac{\partial_\theta \theta_0(\xi, u, x)}{\partial_\xi \psi_0(\xi, u, x)} \right)$ and by $t_2(\xi, u, x) := \left( \frac{\partial_\theta \theta_0(\xi, u, x)}{\partial_\psi \psi_0(\xi, u, x)} \right)$, where $u \in (-1, 1)$, $\xi, x \in \mathbb{R}$.

The application of the implicit function theorem in the iteration scheme is justified by the following proposition, which ensures that the corresponding linearization of $(\hat{\theta}, \hat{\psi}, \lambda) \mapsto \mathcal{G}\hat{\alpha}(\hat{\theta}, \hat{\psi}, \lambda)$, $n \geq 1$, carried out at $(\hat{\theta}, \hat{\psi}, \lambda) = (0, 0, 0)$, $\varepsilon = 0$ is invertible.

Proposition 3.2. Let $\alpha \in \mathbb{N}_0$. There exists $u^\alpha > 0$ such that for any $m \in \mathbb{N}_0$ the operator $\mathcal{M}_m^\alpha : Y_m^\alpha(u_*) \rightarrow Z_m^\alpha(u_*)$, $(\theta, \psi, \lambda) \mapsto \mathcal{M}_m^\alpha(\theta, \psi, \lambda)$, given by

$$ \mathcal{M}_m^\alpha(\theta, \psi, \lambda)(u) = \left( \begin{array}{c} u \partial_\theta \theta(u) - \psi(u) \\ -\partial^2_\theta \theta(u) + \cos(\theta(K(\gamma(u)(x - \xi))))\theta(u) + u \partial_\psi \psi(u) \end{array} \right) + \lambda(u)t_2(\xi, u, x), $$

is invertible if $0 < u_* < u^\alpha$.

Proof. The proof was given in [Mas17a, Proposition 3.2].

The modified iteration scheme is formalized in the following theorem.

Theorem 3.3. Let $\alpha \in \mathbb{N}_0$ and let $u^\alpha$ be from Proposition 3.2. Let $0 < u_* < u^\alpha$, $J = (-1, 1)$ and let $\tilde{F} : J \rightarrow H^{0,\alpha}(\mathbb{R}^2), \varepsilon \mapsto \tilde{F}(\varepsilon)$ be an analytic function such that

$$ \tilde{F}(0) = 0, \quad \partial_\varepsilon \tilde{F}(0) = 0, $$

and

$$ \forall N \geq 2 : \| \begin{pmatrix} 0 \\ \partial_\varepsilon^N \tilde{F}(0) \end{pmatrix} \|_{Z_m^\alpha(u_*)} \leq \varepsilon^N(N - 2)!. $$

Let $\mathcal{G}_1$ be given by

$$ \mathcal{G}_1 : J \times Y_0^\alpha(u_*) \rightarrow Z_0^\alpha(u_*)$, $(\varepsilon, \hat{\theta}, \hat{\psi}, \lambda) \mapsto \mathcal{G}_1^\varepsilon(\hat{\theta}, \hat{\psi}, \lambda) := G_1^\varepsilon(\hat{\theta}_0 + \hat{\theta}, \hat{\psi}_0 + \hat{\psi}, \lambda), $$

13
where $G_1$ is defined by (10). Then there exists $\varepsilon^* > 0$ and a map

$$(-\varepsilon^*, \varepsilon^*) \rightarrow Y_0^\alpha(u_\varepsilon), \varepsilon \mapsto (\hat{\theta}_1^\varepsilon, \hat{\psi}_1^\varepsilon, \lambda_1^\varepsilon),$$

of class $C^\infty$ such that $\tilde{G}_1^\varepsilon(\hat{\theta}_1^\varepsilon, \hat{\psi}_1^\varepsilon, \lambda_1^\varepsilon) = 0$. Let $\tilde{G}_2$ be given by

$$\tilde{G}_2 : J \times Y_0^\alpha(u_\varepsilon) \rightarrow Z_0^\alpha(u_\varepsilon), (\varepsilon, \hat{\theta}, \hat{\psi}, \lambda) \mapsto \tilde{G}_2^\varepsilon(\hat{\theta}, \hat{\psi}, \lambda) := G_2^\varepsilon(\theta_0 + \hat{\theta}, \psi_0 + \hat{\psi}, \lambda),$$

where $G_2$ is defined by (11) with $(\theta_1^\varepsilon, \psi_1^\varepsilon, \lambda_1^\varepsilon) = (\theta_0 + \hat{\theta}_1^\varepsilon, \psi_0 + \hat{\psi}_1^\varepsilon, \lambda_1^\varepsilon)$. Then there exists a map

$$(-\varepsilon^*, \varepsilon^*) \rightarrow Y_0^\alpha(u_\varepsilon), \varepsilon \mapsto (\hat{\theta}_2^\varepsilon, \hat{\psi}_2^\varepsilon, \lambda_2^\varepsilon),$$

of class $C^\infty$ such that $\tilde{G}_2^\varepsilon(\hat{\theta}_2^\varepsilon, \hat{\psi}_2^\varepsilon, \lambda_2^\varepsilon) = 0$. This process can be continued successively to arrive at $\tilde{G}_n$ for any $n \in \mathbb{N}$ be given by

$$\tilde{G}_n : J \times Y_0^\alpha(u_\varepsilon) \rightarrow Z_0^\alpha(u_\varepsilon), (\varepsilon, \hat{\theta}, \hat{\psi}, \lambda) \mapsto \tilde{G}_n^\varepsilon(\hat{\theta}, \hat{\psi}, \lambda) := G_n^\varepsilon(\theta_0 + \hat{\theta}, \psi_0 + \hat{\psi}, \lambda),$$

where $G_n$ is defined by (12) with $(\theta_{n-1}^\varepsilon, \psi_{n-1}^\varepsilon, \lambda_{n-1}^\varepsilon) = (\theta_0 + \hat{\theta}_{n-1}^\varepsilon, \psi_0 + \hat{\psi}_{n-1}^\varepsilon, \lambda_{n-1}^\varepsilon)$. There exists a map

$$(-\varepsilon^*, \varepsilon^*) \rightarrow Y_0^\alpha(u_\varepsilon), \varepsilon \mapsto (\hat{\theta}_n^\varepsilon, \hat{\psi}_n^\varepsilon, \lambda_n^\varepsilon),$$

of class $C^\infty$ such that $\tilde{G}_n^\varepsilon(\hat{\theta}_n^\varepsilon, \hat{\psi}_n^\varepsilon, \lambda_n^\varepsilon) = 0$. The iterative solutions may be written in the form

$$(\hat{\theta}_n^\varepsilon, \hat{\psi}_n^\varepsilon, \lambda_n^\varepsilon) = \left(\sum_{i=0}^\infty \frac{\partial^i \theta_0}{i!} \varepsilon^i, \sum_{i=0}^\infty \frac{\partial^i \psi_0}{i!} \varepsilon^i, \sum_{i=0}^\infty \frac{\partial^i \lambda_0}{i!} \varepsilon^i\right)$$

as a limit in $Y_0^\alpha(u_\varepsilon)$ for $\varepsilon \in (-\varepsilon^*, \varepsilon^*)$. We set $(\theta_n^\varepsilon, \psi_n^\varepsilon, \lambda_n^\varepsilon) := (\theta_0 + \hat{\theta}_n^\varepsilon, \psi_0 + \hat{\psi}_n^\varepsilon, \lambda_n^\varepsilon)$.

In the following we point out the relation among the derivatives of the iterative solutions from Theorem 3.3 at $\varepsilon = 0$.

**Lemma 3.4.** Let the assumptions of Theorem 3.3 hold and let $n \geq 2$. Then $(\partial^k \theta_0^{n-1}, \partial^k \psi_0^{n-1}, \partial^k \lambda_0^{n-1}) = (\partial^k \theta_1^0, \partial^k \psi_1^0, \partial^k \lambda_1^0)$ for $k = 0, \ldots, n-1$.

**Proof.** Analogous to [Mas17a, Theorem 3.4]. \qed

**Remark 3.5.** The derivatives of the iterative solutions coincide at $\varepsilon = 0$ in the following way: $(\partial^k \theta_0^0, \partial^k \psi_0^0, \partial^k \lambda_0^0) = (\partial^k \theta_1^0, \partial^k \psi_1^0, \partial^k \lambda_1^0)$ for $k = 0, 1, 2$ and so on.

Now we prove some bounds on the derivatives of the iterative solutions. These bounds will be used in the inductive proof of Theorem 3.3. Moreover, the bounds play a major key in the proof of convergence of the sequence of iterative solutions and they are also needed in order to show that the corresponding limit defines a function which satisfies the equation of interest.
Lemma 3.6. Let the assumptions of Theorem 3.3 be satisfied. There exists $C > 0$ such that the following holds. Let $n \in \mathbb{N}$ and assume that for $1 \leq j \leq n$ the iterative solutions of the equations $\tilde{G}_j^e(\hat{\theta}, \hat{\psi}, \lambda) = 0$ exist, then the following bounds are satisfied:

\begin{align}
1 \leq K \leq 2 : \quad & \left\| \left( \frac{\partial^K u}{\partial \psi_0} \right) \right\|_{Y^0_0(u^*)} \leq C, \quad (31) \\
\forall K \geq 3 : \quad & \left\| \left( \frac{\partial^K u}{\partial \psi_0} \right) \right\|_{Y^0_0(u^*)} \leq C^{2K-3}(K-3)!, \quad (32) \\
\forall N \geq 2, 0 \leq K \leq 2 : \quad & \left\| \left( \frac{\partial^K u}{\partial \psi_0} \frac{\partial^N \psi_0}{\partial \lambda_0 \lambda_n} \right) \right\|_{Y^0_0(u^*)} \leq C^{2N+2K-3}(N-2)!, \quad (33) \\
\forall N \geq 2, K \geq 3 : \quad & \left\| \left( \frac{\partial^K u}{\partial \psi_0} \frac{\partial^N \psi_0}{\partial \lambda_0 \lambda_n} \right) \right\|_{Y^0_0(u^*)} \leq C^{2N+2K-3}(N-2)!(K-3)!. \quad (34)
\end{align}

Proof. An argument for differentiability with respect to $u$ of the iterative solutions will be given in the proof of Theorem 3.3. The upper bounds in this proof are given by sums of certain types and the major key is that those sums converge. In the following we take a closer look at one of them, since the other cases can be treated similarly. It holds for $l \geq 6$ that

\begin{align}
& \sum_{k=3}^{l-3} \frac{(l - 1)(l - 2)}{(l - 1 - k)!k!} (k - 3)!(l - k - 3)! \\
= & \sum_{k=3}^{l-3} \frac{(l - 1)(l - 2)}{(l - 1 - k)(l - 2 - k)k(k - 1)(k - 2)} \\
= & \sum_{3 \leq k \leq \left\lfloor \frac{l-1}{2} \right\rfloor} \frac{(l - 1)(l - 2)}{(l - 1 - k)(l - 2 - k)k(k - 1)(k - 2)} \\
+ & \sum_{\left\lfloor \frac{l-1}{2} \right\rfloor < k \leq l-3} \frac{(l - 1)(l - 2)}{(l - 1 - k)(l - 2 - k)k(k - 1)(k - 2)}
\end{align}
In the following we use Sobolev embedding theorems. Notice that
\[ l \leq \sup_{n} = \frac{1}{k(k-1)(k-2)} \]

Taking the assumptions on \( C \) and such that \( \sup_{n} \leq 1 \)
\[ \sum_{3 \leq k \leq (l-1)/2} \frac{1}{(l-1)(l-2)} k(k-1)(k-2) \]

Thus \( u \) and such that \( \sup_{n} \leq 1 \)
\[ \sum_{l \leq j \leq (l-1)/2} \frac{1}{j(j-1)(l-1)(l-2)} \]

and thus \( R(l) < \infty \). Let us now deduce a recursive relation which will be needed later.
Taking the \( K \)-th derivative with respect to \( u \) of \( \mathcal{G}_{0}(\theta_{0}, \psi_{0}) = 0 \) yields
\[ 0 = \begin{pmatrix} u\partial_{\xi}^{K} \theta_{0} - \partial_{x}^{K} \psi_{0} \\ u\partial_{\xi} \partial_{x}^{K} \psi_{0} - \partial_{x}^{2} \partial_{x}^{K} \theta_{0} \end{pmatrix} + \begin{pmatrix} 0 \\ \sum_{m=1}^{K-1} \left( K-1 \right) \partial_{u}^{m} \cos(\theta_{0}) \partial_{x}^{K-m} \theta_{0} + \cos(\theta_{0}) \partial_{x}^{K} \theta_{0} \end{pmatrix} \]

Thus
\[ \partial_{u}^{K} \begin{pmatrix} \theta_{0} \\ 0 \end{pmatrix} = -\left[ \mathcal{M} \right]^{-1} \begin{pmatrix} 0 \\ \sum_{m=1}^{K-1} \left( K-1 \right) \partial_{x}^{m} \cos(\theta_{0}) \partial_{x}^{K-m} \theta_{0} \end{pmatrix} + K \begin{pmatrix} \partial_{\xi} \partial_{x}^{K-1} \theta_{0} \\ \partial_{\xi} \partial_{x}^{K-1} \psi_{0} \end{pmatrix}. \] (35)

We show first (31)-(32). We chose \( C > 1 \) such that the claim (31)-(32) is true for \( 0 \leq K \leq 3 \) and such that \( \sup_{u \in I(u_{0})} \partial_{u}^{m} \cos(\theta_{0}) \leq C \) for \( 0 \leq m \leq 3 \). In the following we will put some more assumptions on \( C \), where we tag each of them with an exclamation mark “!”.
We assume that the claim (31)-(32) holds for all integers up to \( K - 1 \) and prove the induction step. Let \( n \in \mathbb{N} \). Firstly, we show that for \( 3 \leq m \leq K \):
\[ \sup_{u \in I(u_{0})} \left| \partial_{u}^{m} \cos(\theta_{0}) \right|_{L_{\infty}^{2}(\mathbb{R}^{2})} \leq (m-3)!C^{2m-3+1/3}. \] (36)

We assume that (36) holds for all integers \( 3 \leq m \leq K - 1 \) and show the induction step. In the following we use Sobolev embedding theorems. Notice that
\[ \sup_{u \in I(u_{0})} \left| \sum_{k=0}^{l-1} \frac{1}{k!} \partial_{u}^{k} \cos(\theta_{0}) \partial_{\xi}^{l-k} \theta_{0} \right|_{L_{\infty}^{2}(\mathbb{R}^{2})} \]
\[ = \sup_{u \in I(u_{0})} \left| \left( \sum_{k=3}^{l-3} \frac{(l-1)!}{(l-1-k)!k!} \partial_{u}^{k} \cos(\theta_{0}) \partial_{\xi}^{l-k} \theta_{0} + \cos(\theta_{0}) \partial_{u}^{l-1} \theta_{0} + (l-1) \partial_{u} \cos(\theta_{0}) \partial_{\xi}^{l-1} \theta_{0} \right. \right. \]
\[ \left. \left. + \frac{(l-1)(l-2)}{2} \partial_{u}^{2} \cos(\theta_{0}) \partial_{u}^{l-2} \theta_{0} + \partial_{u}^{l-1} \cos(\theta_{0}) \partial_{u} \theta_{0} + (l-1) \partial_{u}^{l-2} \cos(\theta_{0}) \partial_{u}^{2} \theta_{0} \right) \right|_{L_{\infty}^{2}(\mathbb{R}^{2})} \]
\[
\begin{align*}
&\leq (l - 3)! \sum_{k=3}^{l-2} \frac{(l - 1)(l - 2)}{(l - 1 - k)!k!} (k - 3)!(l - k - 3)!C^{2k - 3 + 1/3}C^{2(l - k) - 3} \\
&+ (l - 3)!C^{2l - 3} + (l - 1)C(l - 5)!C^{2(l - 1) - 3} + 3\frac{(l - 1)(l - 2)}{2}C^{4 - 3 + 1/3}(l - 5)!C^{2(l - 2) - 3} \\
&+ (l - 4)!C^{2(l - 3)}C + (l - 1)(l - 5)!C^{2(l - 2) - 3 + 1/3}C \\
&\leq (l - 3)!C^{2l - 3 + 1/3}.
\end{align*}
\]

Using this estimate it follows for \(3 \leq m \leq K\) that

\[
\begin{align*}
&\sup_{u \in I(u_*)} \left| \left| \sum_{i=1}^{m-1} \left( \sum_{l=0}^{m-1} \left( \sum_{k=0}^{l-1} \left( \sum_{i=0}^{l-k} \partial_u^i \partial_\xi^k \partial_\theta^m \theta_0 \right) \right) \right) \right|_{L^\infty_{t,x}(\mathbb{R}^2)} \\
&= \sup_{u \in I(u_*)} \left| \left| \sum_{i=1}^{m-1} \left( \sum_{l=0}^{m-1} \left( \sum_{k=0}^{l-1} \left( \sum_{i=0}^{l-k} \partial_u^i \partial_\xi^k \partial_\theta^m \theta_0 \right) \right) \right) \right|_{L^\infty_{t,x}(\mathbb{R}^2)} \\
&= \sup_{u \in I(u_*)} \left| \left| \sum_{i=1}^{m-1} \left( \sum_{l=0}^{m-1} \left( \sum_{k=0}^{l-1} \left( \sum_{i=0}^{l-k} \partial_u^i \partial_\xi^k \partial_\theta^m \theta_0 \right) \right) \right) \right|_{L^\infty_{t,x}(\mathbb{R}^2)} \\
&= \sup_{u \in I(u_*)} \left| \left| \sum_{i=1}^{m-1} \left( \sum_{l=0}^{m-1} \left( \sum_{k=0}^{l-1} \left( \sum_{i=0}^{l-k} \partial_u^i \partial_\xi^k \partial_\theta^m \theta_0 \right) \right) \right) \right|_{L^\infty_{t,x}(\mathbb{R}^2)} \\
&+ \frac{(m - 1) \cos(\theta_0) \partial_\xi \partial_\theta + \sin \theta_0 \partial_u^m \theta_0}{2} \\
&+ \frac{(m - 1)(m - 2)}{2} \cos(\theta_0) \partial_u^2 \partial_\theta + \frac{(m - 1)(m - 2)}{2} \partial_u \cos(\theta_0) \partial_\theta \partial_\theta \right|_{L^\infty_{t,x}(\mathbb{R}^2)} \\
&\leq (m - 3)! \sum_{l=3}^{m-1} \frac{(m - 1)(m - 2)}{(m - l - 1)!!} (l - 3)!(m - l - 3)!C^{2l - 3 + 1/3}C^{2(m - l) - 3} \\
&+ (m - 1)C + (m - 2)!C^{2m - 3} + \frac{(m - 1)(m - 2)}{2} C^3 + \frac{(m - 1)(m - 2)}{2} C C \\
&\leq (m - 3)!C^{2m - 3 + 1/3}.
\end{align*}
\]
which completes the induction step for (36). In the following we denote by $\| \cdot \|$ the operator norm of $[\mathfrak{M}^e_0]^{-1}$. Now we estimate $\partial^K_u(\theta_0, \psi_0, 0)$ by using the recursive formula (35) and the bounds (36):

\[
\left\| [\mathfrak{M}^e_0]^{-1} \left[ \sum_{m=1}^{K-1} \frac{(K-1)(K-2)}{m!(m-3)!(K-m-3)!} C^{m-3+1/3} C^{2(K-m)-3} \right] \right\|_{Y^e_0(u_0)} \\
\leq \| [\mathfrak{M}^e_0]^{-1} \| \left( (K-3)! \sum_{m=3}^{K-1} \frac{(K-1)(K-2)}{m!(m-3)!(K-m-3)!} C^{m-3+1/3} C^{2(K-m)-3} \right) \\
+ (K-1)(K-4)! C^{2(K-1)-3} + \frac{(K-1)(K-2)(K-5)!}{2} C^{4-3+1/3} C^{2(K-2)-3} \\
+ (K-4)! C^{2(K-1)-3+1/3} + (K-1)(K-5)! C^{2(K-2)-3+1/3} + K(K-4)! C^{2(K-1)-3} \\
\leq (K-3)! C^{2K-3-1/3}.
\]

Assuming that $C^{2K-3-1/3} \leq C^{2K-3}$, the induction step for (31)-(32) is complete. Before proving the remaining claim, we deduce some recursive relations for further computations. Taking the $N$-th derivative with respect to $\varepsilon$ of $G^e_n(\theta^e_n, \psi^e_n, \lambda^e_n) = 0$ yields

\[
0 = \partial^N_e G^e_n(\theta^e_n, \psi^e_n, \lambda^e_n) \\
= \left( u \partial^N_e \theta^e_n - \partial^N_e \psi^e_n \right) + \left( \sum_{m=1}^{N-1} \frac{(N-1)!}{m!} \partial^m_e \cos(\theta^e_n) \partial^{N-m}_e \theta^e_n + \cos(\theta^e_n) \partial^N_e \theta^e_n \right) \\
- \left( \partial^N_e F(\varepsilon) \right) + \left( \sum_{i=0}^{n-1} \sum_{l=0}^{N} \binom{N}{l} \partial^{N-l}_e \lambda^e_n \partial^l_e \left[ \frac{\partial \theta^e_n}{\partial \varepsilon} \right] \left[ \frac{\partial \psi^e_n}{\partial \varepsilon} \right] \right) \\
- \left( \sum_{m=1}^{N-1} \frac{(N-1)!}{m!} \partial^m_e \cos(\theta^e_n) \partial^{N-m}_e \theta^e_n \right) \bigg|_{\varepsilon=0}.
\]

Thus we obtain

\[
\left( \begin{array}{c}
\partial^N_e \theta^0_n \\
\partial^N_e \psi^0_n \\
\partial^N_e \lambda^0_n
\end{array} \right) = [\mathfrak{M}^e_0]^{-1} \left[ \begin{array}{c}
0 \\
\partial^N_e F(0) \\
\sum_{1 \leq l \leq \min\{n-1,N-1\}} \binom{N}{l} \partial^{N-l}_e \lambda^0_n \partial^l_e \theta^0_n
\end{array} \right] - \left( \sum_{m=1}^{N-1} \frac{(N-1)!}{m!} \partial^m_e \cos(\theta^e_n) \partial^{N-m}_e \theta^e_n \right) \bigg|_{\varepsilon=0}.
\]

Due to assumption (29) it follows from case $N = 1$ combined with Proposition 3.2 that $(\partial^0_e \theta^0_n, \partial^0_e \psi^0_n, \partial^0_e \lambda^0_n) = (0, 0, 0)$. Taking the $K$-th derivative with respect to $u$ of (37) yields
We assume that bound (33) holds for derivatives with respect to \( \lambda \) of order 2 up to order 2. Moreover, we assume that bound (34) holds for derivatives with respect to \( \lambda \) of order 2 up to order 2. For derivatives with respect to \( \nu \) of order 0 up to order 0 such that \( c > 0 \) for \( \lambda \in H^{2, \alpha}(\mathbb{R}^2) \) and \( \theta \in H^{1, \alpha}(\mathbb{R}) \). This follows from Morrey’s inequality. Let us start the induction.

**N = 1:** The terms \( \frac{\partial^k \partial_\nu \theta_0}{\partial_\nu \partial_\nu \psi_0} \), \( \frac{\partial^k \partial_\nu \partial_\psi \theta_0}{\partial_\nu \partial_\nu \partial_\psi \psi_0} \), and \( \frac{\partial^k \partial_\nu \psi_0}{\partial_\nu \partial_\nu \psi_0} \) vanish for any \( K \) due to assumption (29).

**N = 2:** This case can be treated similarly to the following proof of the induction step. 2, \ldots, \( N - 1 \) \( \rightarrow N \): We assume that bound (33) holds for derivatives with respect to \( \nu \) of order 2 up to order 2 and for derivatives with respect to \( \nu \) of order 0 up to order 2. Moreover, we assume that bound (34) holds for derivatives with respect to \( \nu \) of order 2 up to order 2 and for all derivatives with respect to \( \nu \) from order 3. Now we show the induction step 2, \ldots, \( N - 1 \) \( \rightarrow N \). This will be done by induction on \( K \), where we use (38) and (39).

**K = 0:** We consider separately the terms of the recursive formula (38). Due to (30) we are able to estimate
We consider separately the terms of the recursive formula (39) and obtain

$$\left\| \mathcal{M}_0^\alpha \right\|_{\mathcal{L}(\mathcal{O}(u_*))} \leq (N-2)! \left( N^{-3} - 1/3 \right) \left\| \mathcal{M}_0^\alpha \right\|_{\mathcal{L}(\mathcal{O}(u_*))} \leq (N-2)! C^{2N-3} \sum_{m=3}^{N-2} \frac{(N-1)!}{(N-m-1)!m!} (m-2)! (N-m-2)! C^{2(m-3)}$$

Further we assume that $3C^{2N-3} \leq C^{2N-3}$.

$K = 1$: We consider separately the terms of the recursive formula (39) and obtain

$$\left\| \mathcal{M}_0^\alpha \right\|_{\mathcal{L}(\mathcal{O}(u_*))} \leq (N-2)! \sum_{l=2}^{N-2} \frac{N(N-1)}{(N-l)!l!} (N-l-2)! (l-2)! C^{2(N-l)-3} \leq (N-2)! C^{2N-3} - 1/3 \, .$$
We consider separately the terms of the recursive formula (39) and obtain

\[
\left\| [M_0^a]^{-1} \sum_{0 \leq k \leq 1, (l,k) \neq (0,0)} \frac{(N)}{l} \left( \begin{array}{c} N-1 \\ m \\ k \end{array} \right) \left( \begin{array}{c} \partial^{1-k} \partial^{-1} \partial_{e}^{l} \lambda_{n}^{0} \partial_{u}^{k+1} \partial_{\psi}^{0} \\ \partial^{1-k} \partial_{e}^{l} \lambda_{n}^{0} \partial_{u}^{k+1} \partial_{\psi}^{0} \\ 0 \end{array} \right) \right\|_{Y_{0}^{a}(u_{*})} \leq \left\| [M_0^a]^{-1} \right\| \left( N-2 \right) \! \! \! \sum_{0 \leq k \leq K, (m,k) \neq (0,0)} \left( \begin{array}{c} K \\ k \end{array} \right) \frac{N(N-1)}{(N-m)!m!} (m-2)! (N-m-2)! C^{2N-6} \left( N-2 \right)! C^{2N-3} + (N-2)! C^{2N-3}
\]

Further we assume that \( 2C^{2N+2-3-1/3} \leq C^{2N+2-3} \).

\( K = 2 \): We consider separately the terms of the recursive formula (39) and obtain

\[
\left\| [M_0^a]^{-1} \right\| \left( N-2 \right) \! \! \! \sum_{0 \leq k \leq 2, (m,k) \neq (0,0)} \left( \begin{array}{c} 2 \\ k \end{array} \right) \frac{N(N-1)}{(N-m)!m!} (m-2)! (N-m-2)! C^{2N-6} \left( N-2 \right)! 2C^{2N+2-1-3} + (N-2)! C^{2N-3}
\]

\( \leq (N-2)! C^{2N+2-3-1/3} \).
Further we assume that $2C^{2N+2-3-1/3} \leq C^{2N+2-3}$.

$K = 3$: This case can be proven analogously to the case $K = 2$.

$0, \ldots, K - 1 \rightarrow K$: We assume that the claim holds for all integers up to $K - 1$ and show the induction step. Recall that in the case $N = 0$ we have proven:

$$\text{0} \leq k \leq 2: \sup_{u \in I(u_+)} |\partial_u^k \cos \theta_0|_{L_{\infty}^m(\mathbb{R}^2)} \leq C, \quad \forall k \geq 3: \sup_{u \in I(u_+)} |\partial_u^k \cos \theta_0|_{L_{\infty}^m(\mathbb{R}^2)} \leq (k-3)!C^{2k-3+1/3}.$$  

To begin with, we show that for $2 \leq m \leq N - 1$:

$$\text{0} \leq k \leq 2: \sup_{u \in I(u_+)} |\partial_u^k \partial_x^m \cos \theta_n|_{|\mathbb{R}^2|} \leq (m - 2)!C^{2m+2k-3+1/3},$$  

$$\forall k \geq 3: \sup_{u \in I(u_+)} |\partial_u^k \partial_x^m \cos \theta_n|_{|\mathbb{R}^2|} \leq (k-3)!(m - 2)!C^{2k+2m-3+1/3}.$$  

(40)

(41)

The induction basis for $N = 2$ can be shown similarly to the case $N = 0$. We assume that (10) - (11) holds for all integers $2 \leq m \leq N - 2$ and show the induction step. We start with a preliminary estimate for $l \geq 4$, $i \geq 3$:

$$\sup_{u \in I(u_+)} \left|\sum_{k=0}^{l-1} \left( \begin{array}{c} l-1 \cr k \end{array} \right) \sum_{j=0}^i \left( \begin{array}{c} i \cr j \end{array} \right) \partial_u^k \partial_x^i \cos(\theta_n^2) \partial_u^i \partial_x^j \cos(\theta_n^2) \right|_{|\mathbb{R}^2|} = \sup_{u \in I(u_+)} \left|\sum_{k=2}^{l-2} \left( \begin{array}{c} l-1 \cr k \end{array} \right) \sum_{j=0}^{i-1} \left( \begin{array}{c} i \cr j \end{array} \right) \partial_u^k \partial_x^{i-1} \partial_x^j \cos(\theta_n^2) + \sum_{j=3}^{i} \left( \begin{array}{c} i \cr j \end{array} \right) \partial_u^i \cos(\theta_n^2) \partial_x^j \cos(\theta_n^2) \right|_{|\mathbb{R}^2|}$$

$$+ \sum_{k=2}^{l-2} \left( \begin{array}{c} l-1 \cr 2 \end{array} \right) i \partial_u^k \cos(\theta_n^2) \partial_x^2 \cos(\theta_n^2)$$

$$+ \sum_{k=2}^{l-2} \left( \begin{array}{c} l-1 \cr k \end{array} \right) i \partial_u^k \cos(\theta_n^2) \partial_x^{i-1} \partial_x^j \cos(\theta_n^2) + \sum_{k=2}^{l-2} \left( \begin{array}{c} l-1 \cr 2 \end{array} \right) \sum_{j=0}^{i-1} \left( \begin{array}{c} i \cr j \end{array} \right) \partial_u^i \cos(\theta_n^2) \partial_x^j \cos(\theta_n^2)$$

$$+ \sum_{k=2}^{l-2} \left( \begin{array}{c} l-1 \cr k \end{array} \right) \sum_{j=3}^{i} \left( \begin{array}{c} i \cr j \end{array} \right) \partial_u^i \cos(\theta_n^2) \partial_x^j \cos(\theta_n^2)$$

$$+ \sum_{k=2}^{l-2} \left( \begin{array}{c} l-1 \cr k \end{array} \right) \partial_u^k \partial_x^{i-1} \partial_x^j \cos(\theta_n^2) + \sum_{k=2}^{l-2} \left( \begin{array}{c} l-1 \cr 2 \end{array} \right) i \partial_u^k \cos(\theta_n^2) \partial_x^2 \cos(\theta_n^2)$$

$$+ \sum_{k=2}^{l-2} \left( \begin{array}{c} l-1 \cr k \end{array} \right) i \partial_u^k \cos(\theta_n^2) \partial_x^{i-1} \partial_x^j \cos(\theta_n^2) + \sum_{k=2}^{l-2} \left( \begin{array}{c} l-1 \cr 2 \end{array} \right) \partial_u^i \cos(\theta_n^2) \partial_x^j \cos(\theta_n^2)$$

$$+ \sum_{k=2}^{l-2} \left( \begin{array}{c} l-1 \cr k \end{array} \right) \partial_u^k \partial_x^{j-1} \partial_x^j \cos(\theta_n^2)$$

$$\left|_{\mathbb{R}^2} \right.$$
Applying the Leibniz’s formula we obtain for $2 \leq m \leq N - 1$, $r \geq 3$:

\[
\begin{align*}
\sum_{k=3}^{l-2} \sum_{j=3}^{i-3} \frac{(l-1)i(i-1)(i-2)}{(l-1-k)k(k-1)(i-j)(i-j-1)(i-j-2)j(j-1)(j-2)} \\
+ (i-3)!(l-2)!C^{2i+2l-3} + i(i-4)!(l-2)!CC^{2(i-1)+2l-3} \\
+ \frac{i(i-1)(i-5)!(l-2)!}{2} CC^{2(i-2)+2l-6} \\
+ (l-2)!(i-3)!(l-2)! \sum_{j=3}^{l-2} \frac{l-1}{k(k-1)(l-k-1)} C^{2i+2l-6} \\
+ (l-2)!(i-4)! \sum_{k=2}^{l-2} \frac{l-1}{k(k-1)(l-k-1)} C^{2i+2l-6} \\
+ (l-2)!i(i-4)! \sum_{k=2}^{l-2} \frac{(l-1)}{k(k-1)(l-k-1)} C^{2i+2l-6} \\
+ (l-2)!i(i-4)! \sum_{k=2}^{l-2} \frac{(l-1)}{k(k-1)(l-k-1)} C^{2i+2l-6} \\
+ (l-2)!(i-3)!(l-2)! \sum_{k=2}^{l-2} \frac{l-1}{k(k-1)(l-k-1)} C^{2i+2l-6} \\
\leq (l-2)!(i-3)!C^{2i+2l-3}.
\end{align*}
\]

(42)

Applying the Leibniz’s formula we obtain for $2 \leq m \leq N - 1$, $r \geq 3$:

\[
\begin{align*}
\sup_{u \in I(u_*)} \left| \partial_{\epsilon}^r \partial_{\epsilon}^m (\cos(\theta_n^\epsilon)) \right|_{\epsilon=0} \left| \right|_{L^\infty_{\xi,\epsilon}(\mathbb{R}^2)} \\
= \sup_{u \in I(u_*)} \left| \partial_{\epsilon}^r \partial_{\epsilon}^{m-1} (\sin(\theta_n^\epsilon) \partial_{\epsilon} \theta_n^\epsilon) \right|_{\epsilon=0} \left| \right|_{L^\infty_{\xi,\epsilon}(\mathbb{R}^2)}
\end{align*}
\]

23
Using bound (42) for the square brackets term we estimate the sum over indices $r,m$

\[
\sup_{u \in I(u_*)} \left| \sum_{l=0}^{m-1} \left( m - 1 \right) \sum_{i=0}^{r} \left( r \right) \partial_u^i \partial_x^j \sin(\theta_n^\varepsilon) \partial_u^{r-i} \partial_x^{m-l} \theta_n^\varepsilon \right|_{\varepsilon = 0} \leq \sup_{\xi} L^\infty_{C_\varepsilon}(\mathbb{R}^2)
\]

\[
= \sup_{u \in I(u_*)} \left| \left( \sum_{l=1}^{m-1} \left( m - 1 \right) \sum_{i=0}^{r} \left( r \right) \partial_u^i \partial_x^j \left( \cos(\theta_n^\varepsilon) \partial_x \theta_n^\varepsilon \right) \partial_u^{r-i} \partial_x^{m-l} \theta_n^\varepsilon \right) + \sum_{i=0}^{r} \left( r \right) \partial_u^i \sin(\theta_n^\varepsilon) \partial_u^{r-i} \partial_x^{m-l} \theta_n^\varepsilon \right|_{\varepsilon = 0} \leq \sup_{\xi} L^\infty_{C_\varepsilon}(\mathbb{R}^2)
\]

\[
= \sup_{u \in I(u_*)} \left| \left( \sum_{l=1}^{m-1} \left( m - 1 \right) \sum_{i=0}^{r} \left( r \right) \left( \sum_{k=0}^{l-1} \left( l - 1 \right) \sum_{j=0}^{i} \left( i \right) \partial_u^j \partial_x^k \cos(\theta_n^\varepsilon) \partial_x^{-j} \partial_u^{-k} \theta_n^\varepsilon \right) \right) \partial_u^{r-i} \partial_x^{m-l} \theta_n^\varepsilon \right|_{\varepsilon = 0}
\]

In order to control the expression

\[
\sum_{l=1}^{m-1} \sum_{i=0}^{r} \left( m - 1 \right) \left( r \right) \left( \sum_{k=0}^{l-1} \left( l - 1 \right) \sum_{j=0}^{i} \left( i \right) \partial_u^j \partial_x^k \cos(\theta_n^\varepsilon) \partial_x^{-j} \partial_u^{-k} \theta_n^\varepsilon \right)
\]

we split the sum (43) over indices $l, i$ into two sums, one over indices $I_{m,r} := \{(l, i) : 3 \leq l \leq m-2 \text{ and } 3 \leq i \leq r-2\}$ and the other over indices $\{(l, i) : 1 \leq l \leq m-1, 0 \leq i \leq r\} \setminus I_{m,r}$.

Using bound (42) for the square brackets term we estimate the sum over indices $I_{m,r}$ by

\[
\sup_{u \in I(u_*)} \left| \sum_{l=3}^{m-2} \left( m - 1 \right) \sum_{i=3}^{r-2} \left( r \right) \left( \sum_{k=0}^{l-1} \left( l - 1 \right) \sum_{j=0}^{i} \left( i \right) \partial_u^j \partial_x^k \cos(\theta_n^\varepsilon) \partial_x^{-j} \partial_u^{-k} \theta_n^\varepsilon \right) \right|_{\varepsilon = 0}
\]

\[
\leq (m - 2)!(r - 3)! C^{2r+2m-9}.
\]

\[
\sum_{l=3}^{m-2} \sum_{i=3}^{r-2} \frac{(m - 1) r (r - 1) (r - 2)}{(m - l - 1) l! (r - i)!} (l - 2)!(i - 3)!(r - i - 3)!(m - l - 2)!
\]

\[
\leq (m - 2)!(r - 3)! C^{2r+2m-9}.
\]

where the supremum over $(r, m)$ of the double sum is finite. All terms of the sum over...
indices
\[ \{ (l, i) : 1 \leq l \leq m - 1, \ 0 \leq i \leq r \} \setminus I_{m,r} \]
\[ = \{ (l, 0), \ l = 1, \ldots, m - 1 \} \cup \{ (l, 1), \ l = 1, \ldots, m - 1 \} \cup \{ (l, 2), \ l = 1, \ldots, m - 1 \} \]
\[ \cup \{ (l, r - 1), \ l = 1, \ldots, m - 1 \} \cup \{ (l, r - 2), \ l = 1, \ldots, m - 1 \} \cup \{ (1, i), \ i = 0, \ldots, r \} \]
\[ \cup \{ (2, i), \ i = 3, \ldots, r \} \cup \{ (m - 1, i), \ i = 3, \ldots, r \} \cup \{ (m - 2, i), \ i = 3, \ldots, r \} \]
can be treated in a similar way, whereby one considers separately the sums over the subsets above. For instance, for indices \( \{ (l, 0), \ l = 1, \ldots, m - 1 \} \), we obtain due to (23)

\[
\begin{align*}
\sup_{u \in I_{(u, \ast)}} & \left| \sum_{l=1}^{m-1} \left( \sum_{k=0}^{l-1} \left( \sum_{k=0}^{l-1} \right) \right) \partial_{\varepsilon}^k \cos(\theta_n) \partial_{\varepsilon}^{l-k} \theta_n \partial_{\varepsilon}^m \theta_n \right|_{\varepsilon=0} \left| L_{1, \ast}^\infty(\mathbb{R}^2) \right| \\
& \leq \sup_{u \in I_{(u, \ast)}} \left| \left( \sum_{k=0}^{l-1} \left( \sum_{k=0}^{l-1} \right) \right) \partial_{\varepsilon}^k \cos(\theta_n) \partial_{\varepsilon}^{l-k} \theta_n \partial_{\varepsilon}^m \theta_n \right|_{\varepsilon=0} \left| L_{1, \ast}^\infty(\mathbb{R}^2) \right| \\
& \quad + \sup_{u \in I_{(u, \ast)}} \left| \left( \sum_{k=0}^{l-1} \left( \sum_{k=0}^{l-1} \right) \right) \partial_{\varepsilon}^k \cos(\theta_n) \partial_{\varepsilon}^{l-k} \theta_n \partial_{\varepsilon}^m \theta_n \right|_{\varepsilon=0} \left| L_{1, \ast}^\infty(\mathbb{R}^2) \right| \\
& \quad + \sup_{u \in I_{(u, \ast)}} \left| \left( \sum_{k=0}^{l-1} \left( \sum_{k=0}^{l-1} \right) \right) \partial_{\varepsilon}^k \cos(\theta_n) \partial_{\varepsilon}^{l-k} \theta_n \partial_{\varepsilon}^m \theta_n \right|_{\varepsilon=0} \left| L_{1, \ast}^\infty(\mathbb{R}^2) \right| \\
& \leq \frac{(m-1)(m-2)}{2} \left| \cos(\theta_n) \partial_{\varepsilon}^2 \theta_n \partial_{\varepsilon}^m \theta_n \right|_{\varepsilon=0} \left| L_{1, \ast}^\infty(\mathbb{R}^2) \right| \\
& \quad + \frac{(m-1)(m-2)(m-3)}{6} \left| \cos(\theta_n) \partial_{\varepsilon}^3 \theta_n \partial_{\varepsilon}^m \theta_n \right|_{\varepsilon=0} \left| L_{1, \ast}^\infty(\mathbb{R}^2) \right| \\
& \quad + \frac{(m-1)(m-2)(m-3)(m-4)}{6} \left| \cos(\theta_n) \partial_{\varepsilon}^4 \theta_n \partial_{\varepsilon}^m \theta_n \right|_{\varepsilon=0} \left| L_{1, \ast}^\infty(\mathbb{R}^2) \right| \\
& \leq \frac{(m-1)(m-2)}{2} (m-4)! (r-3)! C^{1+2r+2(m-2)-3} \\
& \quad + \frac{(m-1)(m-2)(m-3)}{6} (m-5)! (r-3)! C^{1+2r+2(m-3)-3} \\
& \quad + (m-2)! (r-3)! C^{2r+2m-8}.
\end{align*}
\]
where the supremum over \((m, l)\) of the expression in the last line is finite.
Now we consider the sum
\[
\sum_{i=0}^{r} \binom{r}{i} \partial_u^i \sin \theta_n^\varepsilon \partial_u^{-i} \partial_\varepsilon^m \theta_n^\varepsilon \left|_{\varepsilon=0} \right. .
\]
In order to control this sum, we write it by utilizing Leibniz’s formula in the following way:
\[
\left( \sin \theta_n^\varepsilon \partial_u \partial_\varepsilon \theta_n^\varepsilon + \sum_{i=5}^{r} \binom{r}{i} \sum_{p=0}^{i-1} \left( \frac{i-1}{p} \right) \partial_u^p (\cos \theta_n^\varepsilon) \partial_u^{-p} \theta_n^\varepsilon \right.
+ \sum_{i=1}^{4} \binom{r}{i} \sum_{p=0}^{i-1} \left( \frac{i-1}{p} \right) \partial_u^p (\cos \theta_n^\varepsilon) \partial_u^{-p} \theta_n^\varepsilon
+ \sum_{i=r-2}^{r} \binom{r}{i} \sum_{p=0}^{i-1} \left( \frac{i-1}{p} \right) \partial_u^p (\cos \theta_n^\varepsilon) \partial_u^{-p} \theta_n^\varepsilon \right) \left|_{\varepsilon=0} \right. .
\]
Using the induction hypothesis we estimate the first term by
\[
\sup_{u \in I(\alpha)} | \sin \theta_n^\varepsilon \partial_u \partial_\varepsilon \theta_n^\varepsilon |_{\varepsilon=0} |_{L_\infty(\mathbb{R}^2)} \leq (m-2)! (r-3)! C^{2r+2m-3}.
\]
For the second term we obtain
\[
\sup_{u \in I(\alpha)} \left| \sum_{i=5}^{r-3} \binom{r-3}{i} \sum_{p=0}^{i-3} \left( \frac{i-1}{p} \right) \partial_u^p (\cos \theta_n^\varepsilon) \partial_u^{-p} \theta_n^\varepsilon + (\cos \theta_n^\varepsilon) \partial_u \theta_n^\varepsilon + (i-1) \partial_u \theta_n^\varepsilon \partial_u^{-1} \theta_n^\varepsilon
+ \frac{(i-1)(i-2)}{2} \partial_u^2 (\cos \theta_n^\varepsilon) \partial_u^{-2} \theta_n^\varepsilon + (i-1) \partial_u^{i-2} (\cos \theta_n^\varepsilon) \partial_u^2 \theta_n^\varepsilon
+ \partial_u^{i-1} (\cos \theta_n^\varepsilon) \partial_u \theta_n^\varepsilon \partial_u^{-i} \partial_\varepsilon^m \theta_n^\varepsilon \right|_{\varepsilon=0} |_{L_\infty(\mathbb{R}^2)} \leq (m-2)! (r-3)! C^{2r+2m-8}.
\]
\[
\sum_{i=5}^{r-3} \binom{i-3}{p=0} \frac{r(r-1)(r-2)}{(r-i)(r-i-1)(r-i-2)(r-i-3)(r-i)(r-i-1)(r-i-2)}
+ \frac{r(r-1)(r-2)}{i(i-1)(i-2)(r-i)(r-i-1)(r-i-2)}
+ \frac{r(r-1)(r-2)}{i(i-1)(r-i)(r-i-1)(r-i-2) .}
\]
\[
\begin{align*}
&+ \frac{r(r-1)(r-2)}{2i(i-3)(i-4)(r-i)(r-i-1)(r-i-2)} \\
&+ \frac{r(r-1)(r-2)}{i(i-2)(i-3)(i-4)(r-i)(r-i-1)(r-i-2)} \\
&+ \frac{r(r-1)(r-2)}{i(i-1)(i-2)(i-3)(r-i)(r-i-1)(r-i-2)}
\end{align*}
\]

where the supremum of the sum over \( r \) is finite. The summands of the sums (45)-(46) can be treated similarly. As an example we consider the case \( i = 2 \):

\[
\sup_{u \in I(u_*)} \left| \left( \begin{array}{c} r \\ 2 \end{array} \right) \sum_{p=0}^{1} \partial_{p} \left( \cos \theta_n \right) \partial_{p}^{2} \sin \theta_n \right|_{\varepsilon=0} \leq (m-2)!r(r-1)(r-5)!C^{2r+2m-8}.
\]

This completes the induction step for (11), since due to previous estimates an appropriate constant \( C \) can be found as it was done in the cases \( 0 \leq K \leq 2 \). One shows (10) similarly. Now we estimate separately the terms of the recursive formula (39). Firstly, we start for \( K \geq 5, \ N \geq 3 \) with the term

\[
\begin{align*}
\left\| [\mathcal{M}^{a}]^{-1} \left[ \sum_{\substack{0 \leq m \leq N-1, \\ 0 \leq k \leq K, \ (m,k) \neq (0,0)}} \binom{N-1}{m} \binom{K}{k} \left( \begin{array}{c} 0 \\ \partial_{p} \partial_{\varepsilon} \cos(\theta_{n}) \partial_{p} K^{k-n} \partial^{m} \theta_{n} \end{array} \right) \right] \right\|_{\varepsilon=0} \leq (N-2)!(K-3)!C^{2K+2N-5}.
\end{align*}
\]

We split the sum over indices \( m, k \), analogous to (13), into two sums, one over indices \( J_{N,K} := \{(m,k) : 3 \leq m \leq N-1 \text{ and } 3 \leq k \leq K \} \) and the other over indices \( \{(m,k) \neq 0 : 0 \leq m \leq N-1, \ 0 \leq k \leq K \} \setminus J_{N,K} \). The sum over indices \( J_{N,K} \) can be estimated by

\[
\begin{align*}
\left\| [\mathcal{M}^{a}]^{-1} \left[ \sum_{\substack{0 \leq m \leq N-1, \\ 3 \leq k \leq K, \ \delta_{m} \leq \delta_{k} \leq K}}} \binom{N-1}{m} \binom{K}{k} \left( \begin{array}{c} 0 \\ \partial_{p} \partial_{\varepsilon} \cos(\theta_{n}) \partial_{p} K^{k-n} \partial^{m} \theta_{n} \end{array} \right) \right] \right\|_{\varepsilon=0} \leq (N-2)!(K-3)!C^{2K+2N-5}.
\end{align*}
\]

\[
\sum_{\substack{3 \leq m \leq N-1, \\ 3 \leq k \leq K}} \frac{(N-1)K(K-1)(K-2)}{(N-m-1)!m!(K-k)!k!}(k-3)!(m-2)!(K-k-3)!(N-m-2)!
\]

27
Secondly, we consider for 

\[ \{ (m,k) \neq 0 : 0 \leq m \leq N-1, 0 \leq k \leq K \} \setminus J_{N,K} \]

analogously to (44) and consider the sums over the corresponding subsets. All those sums can be treated similarly. For instance, for indices \( \{(2,k), k=0, \ldots, K\} \), we obtain by using (40)-(41):

\[
\left\| \left[ \mathcal{M}_0^0 \right]^{-1} \right. \left. \frac{(N-1)(N-2)}{2} \right. \left. \sum_{k=0}^{K} \binom{K}{k} \left( \partial^k_u d^2 \cos(\theta_n^e)\partial^K_{\varepsilon} \partial_{\varepsilon}^N \delta_{n}^2 \right) \right\|_{\varepsilon=0} || Y^*_0 (u_*) \]

\[
\leq \left\| \left[ \mathcal{M}_0^0 \right]^{-1} \right. \left. \frac{(N-1)(N-2)}{2} \right. \left. \sum_{k=3}^{K} \binom{K}{k} \left( \partial^k_u d^2 \cos(\theta_n^e)\partial^K_{\varepsilon} \partial_{\varepsilon}^N \delta_{n}^2 \right) \right\|_{\varepsilon=0} || Z^*_0 (u_*) \]

\[
+ \frac{(N-1)(N-2)}{2} (K-3)! (N-4)! C^{2+2K+2(N-2)-3}
\]

\[
+ \frac{(N-1)(N-2)}{2} (K-4)! (N-4)! C^{4+2(K-1)+2(N-2)-3}
\]

\[
+ \frac{(N-1)(N-2)}{2} (K-5)! (N-4)! C^{6+2(K-2)+2(N-2)-3}
\]

\[
\leq (N-2)! (K-3)! C^{2K+2N-4}.
\]

Secondly, we consider for \( K \geq 5, N \geq 3 \) the term

\[
\left\| \left[ \mathcal{M}_0^0 \right]^{-1} \right. \left. \sum_{0 \leq l \leq \min \{ n-1, N-1 \} \atop 0 \leq k \leq K, (l,k) \neq (0,0)} \binom{N}{l} \binom{K}{k} \left( \partial^k_u d^2 \partial_{\varepsilon}^N \partial_{\varepsilon}^N \delta_{n}^2 \partial_{\varepsilon}^N \delta_{n}^2 \right) \right\|_{Y^*_0 (u_*)}
\]

from (39). We treat this term analogously to (17) and the sum over indices \( J_{N,K} \) can be estimated by

\[
\left\| \left[ \mathcal{M}_0^0 \right]^{-1} \right. \left. \sum_{3 \leq l \leq \min \{ n-1, N-1 \} \atop 3 \leq k \leq K} \binom{N}{l} \binom{K}{k} \left( \partial^k_u d^2 \partial_{\varepsilon}^N \partial_{\varepsilon}^N \delta_{n}^2 \partial_{\varepsilon}^N \delta_{n}^2 \right) \right\|_{Y^*_0 (u_*)}
\]
\[
\sum_{3 \leq m \leq N - 1, \ \ 3 \leq k \leq K} \frac{N(N - 1)K(K - 1)(K - 2)}{(N - m)!m!(K - k)!} (k - 2)!(m - 2)!(K - k - 3)!(N - m - 2)!
\]
\[
\leq \frac{\sum_{3 \leq m \leq N - 1, \ 3 \leq k \leq K} N(N - 1)K(K - 1)(K - 2)}{(N - m)(N - m - 1)m(m - 1)(K - k)(K - k - 1)(K - k - 2)k(k - 1)}
\]
\[
\leq (N - 2)!(K - 3)!C^{2K+2N-5}.
\]

We decompose the set of indices \( \{ (m,k) \neq 0 : 0 \leq m \leq N - 1, \ 0 \leq k \leq K \} \) \( \setminus J_{N,K} \) and estimate the corresponding sums as above. For instance, for indices \( \{ (0, k), \ k = 1, \ldots, K \} \), we obtain
\[
\left\| \left[ \mathfrak{M}_0^a \right]^{-1} \sum_{k=1}^{K} \binom{K}{k} \left( \frac{\partial^{K-k}_{\xi} \partial^N_{\lambda} \partial^0_{n} \partial^{k+1}_{\omega} \theta^0_{n}}{\partial^k_{\xi} \partial^N_{\lambda} \partial^0_{n} \partial^{k+1}_{\omega} \psi^0_{n}} \right) \right\|_{Y^a_0(\psi_\theta)}
\leq \left\| \left[ \mathfrak{M}_0^a \right]^{-1} \left( \sum_{k=3}^{K} \binom{K}{k} \left( \frac{\partial^{K-k}_{\xi} \partial^N_{\lambda} \partial^0_{n} \partial^{k+1}_{\omega} \theta^0_{n}}{\partial^k_{\xi} \partial^N_{\lambda} \partial^0_{n} \partial^{k+1}_{\omega} \psi^0_{n}} \right) \right) \right\|_{Z^a_0(\psi_\theta)}
\]
\[
+ K(K - 4)!(N - 2)!C^{2(K-1)+2N-5} + K(K - 1)(K - 5)!(N - 2)!C^{2(K-2)+2N-3}
\]
\[
\leq (N - 2)!(K - 3)!C^{2K+2N-3-1/3}.
\]

The last term in [32],
\[
\left\| \left[ \mathfrak{M}_0^a \right]^{-1} \left[ K \left( \frac{\partial_{\xi} \partial^{K-1}_{\xi} \partial^N_{\lambda} \partial^0_{n}}{\partial^{K-1}_{\xi} \partial^N_{\lambda} \partial^0_{n} \psi^0_{n}} \right) \right) \right\|_{Y^a_0(\psi_\theta)},
\]
can be estimated by
\[
\left\| \left[ \mathfrak{M}_0^a \right]^{-1} \right\| \left\| K \left( \frac{\partial_{\xi} \partial^{K-1}_{\xi} \partial^N_{\lambda} \partial^0_{n}}{\partial^{K-1}_{\xi} \partial^N_{\lambda} \partial^0_{n} \psi^0_{n}} \right) \right\|_{Z^a_0(\psi_\theta)}
\leq \left\| \left[ \mathfrak{M}_0^a \right]^{-1} \right\| \left\| K(K - 4)!(N - 2)!C^{2(K-1)+2N-3}
\leq (N - 2)!(K - 3)!C^{2K+2N-4},
\]
which completes the proof by the same argument as in the cases 0 \( \leq K \leq 2 \).
By using Lemma 3.6 we prove now Theorem 3.3.

**Proof** (of Theorem 3.3). In this proof, we use the notation $Y^\alpha_m = Y^\alpha_m(u_*)$, $Z^\alpha_m = Z^\alpha_m(u_*)$. We refer to [Dei85, Theorem 15.1] and check their proof of the implicit function theorem, whereas we show that $r$ and $\delta$ do not depend on $\tilde G_n$. Once $\tilde G_n : J \times Y^\alpha_0 \to Z^\alpha_0$ is defined, one obtains that its derivative with respect to $(\hat \theta, \hat \psi, \lambda)$ evaluated at $(\varepsilon, \hat \theta, \hat \psi, \lambda) = (0, 0, 0, 0)$ is given by $\mathcal{M}_{0}^\alpha$. We set

$$S_n(\varepsilon, \hat \theta, \hat \psi, \lambda) = [\mathcal{M}_{0}^\alpha]^{-1} \tilde G_n^\varepsilon(\hat \theta, \hat \psi, \lambda) - I(\hat \theta, \hat \psi, \lambda).$$

We start with $\tilde G_1$. Notice that $\tilde G_1^\varepsilon(0, 0, 0) = 0$. Let the constant $C$ be such that it satisfies the assumptions demanded in the proof of Lemma 3.6. Since $D_{(\hat \theta, \hat \psi, \lambda)} S_1(0, 0, 0, 0) = 0$ and $D_{(\hat \theta, \hat \psi, \lambda)} S_1$ is continuous, we fix $k \in (0, 1)$ and find $1 \geq \delta > 0$ such that

$$\left\| D_{(\hat \theta, \hat \psi, \lambda)} S_1(\varepsilon, \hat \theta, \hat \psi, \lambda) \right\|_{Z^0_0(u_*)} + \left\| [\mathcal{M}_{0}^\alpha]^{-1} \sum_{n=1}^{\infty} c_n \varepsilon^n \right\| \leq k$$

on $\overline{B}_{\delta}(0) \times \overline{B}_{\delta}(0)$, where $c_1 = C$, $c_n = \frac{C^{n+1}}{n(n-1)}$ for $n \geq 2$ and $\| \cdot \|$ denotes the operator norm of $[\mathcal{M}_{0}^\alpha]^{-1}$. Since $S_1(0, 0, 0, 0) = 0$ and $S_1(\cdot, 0, 0, 0)$ is continuous, there exists $r =: \varepsilon \leq \delta$ such that

$$\| S_1(\varepsilon, 0, 0, 0) \|_{Z^0_0(u_*)} < \delta(1 - k)$$

on $\overline{B}_{r}(0)$. Thus there exists by [Dei85, Theorem 15.1] a map

$$(-\varepsilon, \varepsilon) \to Y^\alpha_0, \quad \varepsilon \mapsto (\hat \theta_1^\varepsilon, \hat \psi_1^\varepsilon, \lambda_1^\varepsilon)$$

such that $\tilde G_1^\varepsilon(\hat \theta_1^\varepsilon, \hat \psi_1^\varepsilon, \lambda_1^\varepsilon) = 0$. Let $\varepsilon > 0$ be the radius of convergence of $\sum_{n=2}^{\infty} c_n \varepsilon^n$ and $\varepsilon^*: = \min\{\varepsilon, \varepsilon\}$. Since $\tilde F$ is analytic, the solution $(\hat \theta_1^\varepsilon, \hat \psi_1^\varepsilon, \lambda_1^\varepsilon)$ is also analytic and may be written in the form

$$(\hat \theta_1^\varepsilon, \hat \psi_1^\varepsilon, \lambda_1^\varepsilon) = \left( \sum_{i=0}^{\infty} \frac{\partial^i \hat \theta_0}{i!} \varepsilon^i, \sum_{i=0}^{\infty} \frac{\partial^i \hat \psi_0}{i!} \varepsilon^i, \sum_{i=0}^{\infty} \frac{\partial^i \lambda_0}{i!} \varepsilon^i \right)$$

(49) for $\varepsilon \in (-\varepsilon^*, \varepsilon^*)$ due to Lemma 3.6. Considering the map $\tilde G_{1,m}$ on spaces of higher regularity, given by

$$\tilde G_{1,m} : J \times Y^\alpha_m \to Z^\alpha_m, (\varepsilon, \hat \theta, \hat \psi, \lambda) \mapsto \tilde G_{1,m}^\varepsilon(\hat \theta, \hat \psi, \lambda) := \tilde G_1^\varepsilon(\theta_0 + \hat \theta, \psi_0 + \hat \psi, \lambda),$$

where $\tilde G_1$ is defined by (10), we obtain in the same way for any $m \in \mathbb{N}$ a constant $\varepsilon_m > 0$ and a map

$$(-\varepsilon_m, \varepsilon_m) \to Y^\alpha_m, \quad \varepsilon \mapsto (\hat \theta_{1,m}^\varepsilon, \hat \psi_{1,m}^\varepsilon, \lambda_{1,m}^\varepsilon)$$

such that $\tilde G_{1,m}^\varepsilon(\hat \theta_{1,m}^\varepsilon, \hat \psi_{1,m}^\varepsilon, \lambda_{1,m}^\varepsilon) = 0$. Since $\tilde F$ is analytic and $$(\hat \theta_1^\varepsilon, \hat \psi_1^\varepsilon, \lambda_1^\varepsilon) = (\hat \theta_{1,m}^\varepsilon, \hat \psi_{1,m}^\varepsilon, \lambda_{1,m}^\varepsilon) \in Y^\alpha_m$$ for $\varepsilon \in (-\varepsilon_m, \varepsilon_m)$, it follows from Lemma 3.6 that $$(\hat \theta_1^\varepsilon, \hat \psi_1^\varepsilon, \lambda_1^\varepsilon) \in Y^\alpha_m$$ for $\varepsilon \in (-\varepsilon^*, \varepsilon^*)$. 

30
and consequently that $\tilde{G}_2 : J \times Y_0^\alpha \rightarrow Z_0^\alpha$ is well defined. Since

$$0 = \left\| \begin{pmatrix} \partial_\varepsilon^1 \theta_0 \\ \partial_\varepsilon^1 \psi_0 \\ \partial_\varepsilon^1 \lambda_0 \\ \end{pmatrix} \right\|_{Y_0^\alpha(u_*)} \leq c_1$$

due to (23), (38) and Proposition 3.2 it follows from (48) that

$$\left\| D_{(\hat{\theta}, \hat{\psi}, \lambda)} S_2(\varepsilon, \hat{\theta}, \hat{\psi}, \lambda) \right\|_{Z_0^\alpha(u_*)} \leq k$$
on $B_\delta(0) \times B_\delta(0)$. Obviously

$$\| S_2(\varepsilon, 0, 0, 0) \|_{Z_0^\alpha(u_*)} < \delta(1 - k)$$
on $B_r(0)$. Thus there exists by the same argument as above an analytic map

$$(-\varepsilon^*, \varepsilon^*) \rightarrow Y_0^\alpha, \quad \varepsilon \mapsto (\hat{\theta}_2^\varepsilon, \hat{\psi}_2^\varepsilon, \lambda_2^\varepsilon),$$

which may be written in a form analogous to (49) for $\varepsilon \in (-\varepsilon^*, \varepsilon^*)$ such that $\tilde{G}_2(\hat{\theta}_2^\varepsilon, \hat{\psi}_2^\varepsilon, \lambda_2^\varepsilon) = 0$. We continue this process successively, whereas we use in the second and in the succeeding iteration steps the following argument. Assuming that the first $n - 1$ iterative solutions are obtained, it holds

$$\frac{1}{N!} \left\| \begin{pmatrix} \partial_\varepsilon^N \theta_0^{n-1} \\ \partial_\varepsilon^N \psi_0^{n-1} \\ \partial_\varepsilon^N \lambda_0^{n-1} \\ \end{pmatrix} \right\|_{Y_0^\alpha(u_*)} \leq c_N \quad \text{for} \quad 1 \leq N \leq n - 1,$$

due to Lemma 3.6 Thus (48) yields that

$$\left\| D_{(\hat{\theta}, \hat{\psi}, \lambda)} S_n(\varepsilon, \hat{\theta}, \hat{\psi}, \lambda) \right\|_{Z_0^\alpha(u_*)} \leq k$$
on $B_\delta(0) \times B_\delta(0)$. Since

$$\| S_n(\varepsilon, 0, 0, 0) \|_{Z_0^\alpha(u_*)} < \delta(1 - k)$$
on $B_r(0)$ there exists by the same argument as above an analytic map

$$(-\varepsilon^*, \varepsilon^*) \rightarrow Y_0^\alpha, \quad \varepsilon \mapsto (\hat{\theta}_n^\varepsilon, \hat{\psi}_n^\varepsilon, \lambda_n^\varepsilon),$$

which may be written in a form analogous to (49) for $\varepsilon \in (-\varepsilon^*, \varepsilon^*)$ such that $\tilde{G}_n^\varepsilon(\hat{\theta}_n^\varepsilon, \hat{\psi}_n^\varepsilon, \lambda_n^\varepsilon) = 0.$

$\blacksquare$

31
4 Convergence of the Sequence of Iterative Solutions

In this section, we show that the sequence of iterative solutions constructed in Section 3 converges and that its limit defines a function which solves the equation of interest.

Lemma 4.1. Let $\alpha$, $u_*$ and $\varepsilon^*$ be from Theorem 3.3. The limit

$$\left(\hat{\theta}_\infty^\varepsilon, \hat{\psi}_\infty^\varepsilon, \lambda_\infty^\varepsilon\right) := \left(\sum_{i=0}^{\infty} \frac{\partial^i \theta_0^0}{i!} \varepsilon^i, \sum_{i=1}^{\infty} \frac{\partial^i \psi_0^0}{i!} \varepsilon^i, \sum_{i=0}^{\infty} \frac{\partial^i \lambda_0^0}{i!} \varepsilon^i\right)$$

exists in $Y^\alpha_0(u_*)$ for $\varepsilon \in (-\varepsilon^*, \varepsilon^*)$. We set $(\theta_\infty^\varepsilon, \psi_\infty^\varepsilon, \lambda_\infty^\varepsilon) := (\theta_0 + \hat{\theta}_\infty^\varepsilon, \psi_0 + \hat{\psi}_\infty^\varepsilon, \lambda_\infty^\varepsilon)$ with $(\theta_0, \psi_0)$ given by (1).

Proof. The claim follows from Theorem 3.3 and Lemma 3.6 since $\varepsilon^*$ is less or equal than the radius of convergence of $\sum_{n=2}^{\infty} \frac{C^{n-2}}{n(n-1)} \varepsilon^n$ with $C$ from Lemma 3.6.

Theorem 4.2. Let $u_*$ and $\varepsilon^*$ be from Theorem 3.3. Then it holds for any $u \in I(u_*)$ and $\varepsilon \in (-\varepsilon^*, \varepsilon^*)$ that

$$u \partial_\xi \left(\frac{\theta_\infty^\varepsilon}{\psi_\infty^\varepsilon}\right) - \left(\frac{\psi_\infty^\varepsilon}{\theta_\infty^\varepsilon}\right)_{xx} - \sin \theta_\infty^\varepsilon + \tilde{F}(\varepsilon) + \lambda_\infty^\varepsilon \partial_u \left(\frac{\theta_\infty^\varepsilon}{\psi_\infty^\varepsilon}\right) = 0.$$

Proof. Let $n \in \mathbb{N}$. Notice that

$$\forall u \in I(u_*) : \ u \partial_\xi \left(\frac{\theta_n^\varepsilon}{\psi_n^\varepsilon}\right) - \left(\frac{\psi_n^\varepsilon}{\theta_n^\varepsilon}\right)_{xx} - \sin \theta_n^\varepsilon + \tilde{F}(\varepsilon) + \lambda_n^\varepsilon \partial_u \left(\sum_{i=0}^{n-1} \frac{\partial^i \theta_0^0}{i!} \varepsilon^i\right) = 0.$$

It holds due to Theorem 3.3 that

$$(\hat{\theta}_n^\varepsilon, \hat{\psi}_n^\varepsilon, \lambda_n^\varepsilon) = \left(\sum_{i=0}^{\infty} \frac{\partial^i \hat{\theta}_n^0}{i!} \varepsilon^i, \sum_{i=0}^{\infty} \frac{\partial^i \hat{\psi}_n^0}{i!} \varepsilon^i, \sum_{i=0}^{\infty} \frac{\partial^i \lambda_n^0}{i!} \varepsilon^i\right).$$

Thus using Lemma 3.4 and Lemma 3.6 we obtain for $n \geq 2$ and $\varepsilon \in (-\varepsilon^*, \varepsilon^*)$:

$$\left\|\left(\frac{\theta_n^\varepsilon}{\psi_n^\varepsilon}, \frac{\psi_n^\varepsilon}{\theta_n^\varepsilon}, \lambda_n^\varepsilon\right)\right\|_{Y^\alpha_0^0(u_*)} = \left\|\begin{pmatrix} \sum_{i=0}^{\infty} \frac{\partial^i \theta_0^0}{i!} \varepsilon^i - \sum_{i=0}^{\infty} \frac{\partial^i \theta_0^0}{i!} \varepsilon^i \\ \sum_{i=0}^{\infty} \frac{\partial^i \psi_0^0}{i!} \varepsilon^i - \sum_{i=0}^{\infty} \frac{\partial^i \psi_0^0}{i!} \varepsilon^i \\ \sum_{i=0}^{\infty} \frac{\partial^i \lambda_0^0}{i!} \varepsilon^i - \sum_{i=0}^{\infty} \frac{\partial^i \lambda_0^0}{i!} \varepsilon^i \end{pmatrix}\right\|_{Y^\alpha_0^0(u_*)}$$

32
\[
\begin{align*}
= & \left\| \left( \sum_{i=n}^{\infty} \frac{\partial_i \theta_i^0}{i!} \varepsilon^i - \sum_{i=n}^{\infty} \frac{\partial_i \theta_i^0}{i!} \varepsilon^i \right) \\
& \left( \sum_{i=n}^{\infty} \frac{\partial_i \psi_i^0}{i!} \varepsilon^i - \sum_{i=n}^{\infty} \frac{\partial_i \psi_i^0}{i!} \varepsilon^i \right) \\
& \left( \sum_{i=n}^{\infty} \frac{\partial_i \lambda_i^0}{i!} \varepsilon^i - \sum_{i=n}^{\infty} \frac{\partial_i \lambda_i^0}{i!} \varepsilon^i \right) \right\|_{Y_0^n(u_*)} \\
\leq & 2 \sum_{i=n}^{\infty} C^{2i-3} \frac{1}{i(i-1)} \varepsilon^i.
\end{align*}
\]

The claim follows since

\[
\partial_u(\theta^\varepsilon, \psi^\varepsilon, \lambda^\varepsilon) = \left( \sum_{i=0}^{\infty} \frac{\partial_u \partial_i \theta_i^0}{i!} \varepsilon^i, \sum_{i=0}^{\infty} \frac{\partial_u \partial_i \psi_i^0}{i!} \varepsilon^i, \sum_{i=0}^{\infty} \frac{\partial_u \partial_i \lambda_i^0}{i!} \varepsilon^i \right)
\]

in \(Y_0^n(u_*)\) due to Lemma 4.1 and Lemma 3.6.

\section{Proof of Theorem 2.2}

We apply Theorem 3.3 to a specific \(\tilde{F}\) which is defined below.

\begin{definition}
Let \(F, \xi, \Xi\) be from Theorem 2.2. We set \(\tilde{F}(\varepsilon, \xi, x) := F(\varepsilon, x)\chi(\xi)\), where \(\chi\) is a smooth cutoff function with \(\chi(\xi) = 1\) for \(|\xi| \leq \Xi\) and \(\chi(\xi) = 0\) for \(|\xi| \geq \Xi + 1\).
\end{definition}

The next lemma follows immediately from the assumptions on \(F\) in Theorem 2.2.

\begin{lemma}
Let \(F, \Xi\) be from Theorem 2.2 and let \(\tilde{F}\) be from Definition 5.1. Then it holds that
\begin{enumerate}
\item[(a)] \(\forall (\varepsilon, \xi, x) \in (-1, 1) \times [-\Xi, \Xi] \times \mathbb{R} : \tilde{F}(\varepsilon, \xi, x) = F(\varepsilon, x)\);
\item[(b)] \(\tilde{F}\) satisfies the assumptions of Theorem 3.3.
\end{enumerate}
\end{lemma}

We solve iteratively the equations in Theorem 3.3 with the specific \(\tilde{F}(\varepsilon, \xi, x) := F(\varepsilon, x)\chi(\xi)\) from Definition 5.1 (Theorem 3.3 is applicable due to Lemma 5.2) and obtain a sequence of solutions, which converges due to Lemma 4.1. From now on we denote its limit by \((\theta^\varepsilon, \psi^\varepsilon, \lambda^\varepsilon)\). The function \((\theta, \psi)\) given by (27) with \(\tilde{\xi}, \bar{u}\) satisfying (28), solves the Cauchy problem (26) due to Theorem 4.2 and Lemma 5.2. The claim for \(|u_\varepsilon| \leq \tilde{C}\varepsilon\) follows by using (28) and the fundamental theorem of calculus (analogous to the proof of [Mas17a, Lemma 9.2]).

\section*{References}

[Ben76] T. Brooke Benjamin. Applications of Leray-Schauder degree theory to problems of hydrodynamic stability. \textit{Math. Proc. Cambridge Philos. Soc.}, 79(2):373–392, 1976.

33
[Bon75] J. Bona. On the stability theory of solitary waves. *Proc. Roy. Soc. London Ser. A*, 344(1638):363–374, 1975.

[BP92] V. S. Buslaev and G. S. Perel’man. On nonlinear scattering of states which are close to a soliton. *Astérisque*, (210):6, 49–63, 1992. Méthodes semi-classiques, Vol. 2 (Nantes, 1991).

[CMnPS16] Raphaël Côte, Claudio Muñoz, Didier Pilod, and Gideon Simpson. Asymptotic stability of high-dimensional Zakharov-Kuznetsov solitons. *Arch. Ration. Mech. Anal.*, 220(2):639–710, 2016.

[Dei85] Klaus Deimling. *Nonlinear functional analysis*. Springer-Verlag, Berlin, 1985.

[FK39] Y. I. Frenkel, T. Kontorova. *J. Phys. Acad. Sci. USSR* 1, 137, 1939.

[FGJS04] J. Fröhlich, S. Gustafson, B. L. G. Jonsson, and I. M. Sigal. Solitary wave dynamics in an external potential. *Comm. Math. Phys.*, 250(3):613–642, 2004.

[HPW82] Daniel B. Henry, J. Fernando Perez, and Walter F. Wreszinski. Stability theory for solitary-wave solutions of scalar field equations. *Comm. Math. Phys.*, 85(3):351–361, 1982.

[HL12] Justin Holmer and Quanhui Lin. Phase-driven interaction of widely separated nonlinear Schrödinger solitons. *J. Hyperbolic Differ. Equ.*, 9(3):511–543, 2012.

[Hol11] Justin Holmer. Dynamics of KdV solitons in the presence of a slowly varying potential. *Int. Math. Res. Not. IMRN*, (23):5367–5397, 2011.

[HZ07] Justin Holmer and Maciej Zworski. Slow soliton interaction with delta impurities. *J. Mod. Dyn.*, 1(4):689–718, 2007.

[HZ08] Justin Holmer and Maciej Zworski. Soliton interaction with slowly varying potentials. *Int. Math. Res. Not. IMRN*, (10):Art. ID rnm026, 36, 2008.

[IC79] Masahiro Inoue and S. G. Chung. Bion dissociation in sine-gordon system. *Journal of the Physical Society of Japan*, 46(5):1594–1601, 1979.

[IKV12] Valery Imaykin, Alexander Komech, and Boris Vainberg. Scattering of solitons for coupled wave-particle equations. *J. Math. Anal. Appl.*, 389(2):713–740, 2012.

[JFGS06] B. Lars G. Jonsson, Jürg Fröhlich, Stephen Gustafson, and Israel Michael Sigal. Long time motion of NLS solitary waves in a confining potential. *Ann. Henri Poincaré*, 7(4):621–660, 2006.

[KM89] Yuri S. Kivshar and Boris A. Malomed. Dynamics of solitons in nearly integrable systems. *Rev. Mod. Phys.*, 61:763–915, 1989.
Michal Kowalczyk, Yvan Martel, and Claudio Muñoz. Kink dynamics in the $\phi^4$ model: Asymptotic stability for odd perturbations in the energy space. *J. Amer. Math. Soc.*, 30(3):769–798, 2017.

Elena Kopylova. Habilitationsschrift, Asymptotic stability of solitons for nonlinear hyperbolic equations. Universität Wien, 2015.

Timur Mashkin. Stability of the Solitary Manifold of the Sine-Gordon Equation. Universität zu Köln, 2016.

Timur Mashkin. Stability of the solitary manifold of the perturbed sine-gordon equation. arXiv:1705.05713, 2017.

Timur Mashkin. Solitons in the Presence of a Small, Slowly Varying Electric Field. arXiv:1712.08473, 2017.

H. J. Mikeska. Solitons in a one-dimensional magnet with an easy plane. *Journal of Physics C: Solid State Physics*, 11(1):L29, 1978.

Tetsu Mizumachi and Dmitry Pelinovsky. Bäcklund transformation and $L^2$-stability of NLS solitons. *Int. Math. Res. Not. IMRN*, (9):2034–2067, 2012.

T. H. R. Skyrme. Particle states of a quantized meson field. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 262(1309):237–245, 1961.

David M. A. Stuart. Perturbation theory for kinks. *Comm. Math. Phys.*, 149(3):433–462, 1992.

David M. A. Stuart. *Sine Gordon notes*. Unpublished notes, 2012.

A. Soffer and M. I. Weinstein. Multichannel nonlinear scattering for nonintegrable equations. *Comm. Math. Phys.*, 133(1):119–146, 1990.

Michael I. Weinstein. Lyapunov stability of ground states of nonlinear dispersive evolution equations. *Comm. Pure Appl. Math.*, 39(1):51–67, 1986.

L. Zhang, L. Huang, and X. M. Qiu. Josephson junction dynamics in the presence of microresistors and an ac drive. *Journal of Physics: Condensed Matter*, 7(2):353, 1995.