HILBERT CUBES IN PROGRESSION-FREE SETS AND IN THE SET OF SQUARES II

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Abstract. Let $S_2$ be the set of integer squares. We show that the dimension $d$ of a Hilbert cube $a_0 + \{0,a_1\} + \cdots + \{0,a_d\} \subset S_2$ is bounded by $d = O(\log \log N)$. 

Let

$$H(a_0; a_1, \ldots, a_d) := a_0 + \{0, a_1\} + \cdots + \{0, a_d\}$$

$$= \left\{ a_0 + \sum_{i=1}^d \epsilon_i a_i : \epsilon_i \in \{0,1\} \right\}$$

denote a Hilbert cube of dimension $d$. Brown, Erdős and Freedman \cite{BEF} asked whether the maximal dimension of a Hilbert cube in the set of squares is absolutely bounded or not. Experimentally, one finds only very small cubes such as

$$1 + \{0,840\} + \{0,840\} + \{0,528\} = \{1^2, 23^2, 29^2, 37^2, 37^2, 41^2, 47^2\}.$$ 

Observe that in this example $29^2$ and $37^2$ occur as sums in two different ways. Cilleruelo and Granville \cite{CG} and Solymosi \cite{S} and Alon, Angel, Benjamin and Lubetzky \cite{HAL} explain that the Bombieri-Lang conjecture implies that $d$ is absolutely bounded. Hegyvári and Sárközy \cite{HS}, Theorem 1) proved that for the set of integer squares $S_2 \cap [1,N]$ the maximal dimension is bounded by $d = O((\log N)^{1/3})$. Dietmann and Elsholtz \cite{DE}, Theorem 3) improved this to $d = O((\log \log N)^2)$. Here we further reduce that bound.

Theorem 1 (Main theorem). Let $S_2$ denote the set of integer squares. Let $N$ be sufficiently large, let $a_0$ be a non-negative integer and let $A = \{a_1, \ldots, a_d\}$ be a set of distinct positive integers such that $H(a_0; a_1, \ldots, a_d) \subset S_2 \cap [1,N]$. Then

$$d \le 7 \log \log N.$$ 

A comparable bound was proved in the author’s earlier paper \cite{DE}, Theorem 1) for higher powers instead of squares.

Remark. The special case of subsetsums, i.e. Hilbert cubes with $a_0 = 0$, was previously studied by Csikvári \cite{C}, Corollary 2.5), who proved in this case the same bound $d = O(\log \log N)$. His method of proof would not extend to the general case of $a_0 \neq 0$.

Corollary 1. Let $f(x) = ax^2 + bx + c$ be a quadratic polynomial where $a, b, c \in \mathbb{Z}$ such that $a > 0$, and let $S = \{f(x) : x \in \mathbb{N}\}$. Let $a_0$ be a non-negative integer and
\[ A = \{a_1, \ldots, a_d\} \text{ be a set of distinct positive integers such that } H(a_0; a_1, \ldots, a_d) \subset S \cap [1, N]. \text{ Then for sufficiently large } N, \text{ we have} \]
\[ d \leq 7 \log \log N. \]

**Theorem 2.** Let \( k \geq 3 \) be a positive integer, and let \( S \) denote a set of integers without an arithmetic progression of length \( k \). Moreover, let \( c \) be a real number such that
\[ 1 < c < \frac{k}{k - 1}. \]
Then for sufficiently large \( N \), the following holds true: If \( a_0 \) is a non-negative integer and \( A = \{a_1, \ldots, a_d\} \) is a set of distinct positive integers such that \( H(a_0; a_1, \ldots, a_d) \subset S \cap [1, N] \), then
\[ d \leq \frac{2(k - 2)}{(k - 1) \log c \log N}. \]

This last theorem can be easily deduced from a recent paper of Schoen [13], but we state it for completeness. The two following lemmas are also implicitly contained in Schoen [13], Lemma 2.1.

**Lemma 1.** Let \( k \geq 3 \) be a positive integer, and let \( S \) denote a set of integers without an arithmetic progression of length \( k \). Moreover, let \( c \) be a real number such that
\[ 1 < c < \frac{k}{k - 1}. \]
Let \( H = a_0 + \{0, a_1\} + \cdots + \{0, a_d\} \subset S \). Then \( |H| \geq 2c^{d-1} \).

It is well known that for sets without progressions of length \( k = 3 \) one even has that \( |H| = 2^d \). This is an exercise in Solymosi [15], see also Lemma 3 in [7].

**Lemma 2.** Let \( 0 < \alpha < 1 \), let \( h \) be an integer and let \( B \) be a non-empty set of distinct integers. If \( |B \cap (B + h)| > (1 - \alpha)|B| \), then \( B \) contains an arithmetic progression of length \( \left\lfloor \frac{1}{\alpha} \right\rfloor + 1 \) and difference \( h \).

**Proof of Lemma 1.** Consider the shift operator \( f : \mathbb{Z} \to \mathbb{Z} \) defined by \( f(b) = b + h \) and its iterates. For given \( b \in \mathbb{Z} \), let \( r(b) \) denote the least non-negative integer \( r \) with
\[ \{b, f(b), \ldots, f^{r-1}(b)\} \subset B, \text{ but } f^r(b) = b + rh \notin B. \]

The assumption \(|\{b \in B : b + h \in B\}| > (1 - \alpha)|B| \) implies that for each fixed non-negative integer \( r \), there are less than \( \alpha|B| \) elements \( b \in B \) with this given value \( r = r(b) \). Hence, for \( k \in \mathbb{N} \) the number of elements \( b \in B \) with \( r(b) \leq k \) is less than \( k\alpha|B| \). If \( k\alpha|B| < |B| \), then there exists a \( b \in B \) with \( r(b) \geq k + 1 \geq \left\lfloor \frac{1}{\alpha} \right\rfloor + 1 \). Therefore \( B \) contains the arithmetic progression \( \{b, b+h, \ldots, b+\left(\lfloor\frac{1}{\alpha}\rfloor+1\right)h\} \) of length \( r \geq \left\lfloor \frac{1}{\alpha} \right\rfloor + 1 \). \( \square \)

**Proof of Lemma 2.** Let
\[ H_i = a_0 + \{0, a_1\} + \cdots + \{0, a_i\}. \]
Suppose that \( |H| < 2c^{d-1} \), then there is some \( i \in \{1, \ldots, d - 1\} \) such that
\[ \frac{|H_{i+1}|}{|H_i|} < c. \]
For this \( i \) we have
\[ |(H_i + a_{i+1}) \cap H_i| > (2 - c)|H_i|. \]
Then by Lemma 2 and our assumption on \( c \) the set \( H_i \) contains an arithmetic progression of length 
\[
\left\lfloor \frac{1}{c-1} \right\rfloor + 1 \geq k
\]
which is a contradiction, as \( S \) does not contain a progression of length \( k \). \( \square \)

As in our previous work [7] we make use of the following two results on squares:

**Lemma 3** (Theorem 9 of Gyarmati [10]). Let \( S_2 \) denote the set of integer squares. For sufficiently large \( N \) the following holds true: If \( C, D \subseteq \{1, \ldots, N\} \) such that \( C + D \subseteq S_2 \), then
\[
\min(|C|, |D|) \leq 8 \log N.
\]

**Lemma 4** (Fermat, Euler (see Volume II, page 440 of [6])). There are no four integer squares in arithmetic progression.

**Corollary 2.** Let \( f(x) = ax^2 + bx + c \) be a quadratic polynomial where \( a, b, c \in \mathbb{Z} \) such that \( a > 0 \), and let \( S = \{f(x) : x \in \mathbb{N}\} \). Then the set \( S \) does not contain four integer squares in arithmetic progression.

Let us remark that Setzer [14] proved this corollary using elliptic curves. Here we show that it is a simple consequence of Lemma 4, which can be proved with elliptic curves, but also allows for an elementary proof.

**Proof of Corollary 2.** From \( 4af(x) + b^2 - 4ac = (2ax + b)^2 \), it follows that if the arithmetic four-progression \( P = \{n, n + m, n + 2m, n + 3m\} \) is contained in \( S \), then the shifted four-progression \( \{4an + b^2 - 4ac, 4an + b^2 - 4ac + 4am, 4an + b^2 - 4ac + 8am, 4an + b^2 - 4ac + 12am\} \) will be in the set of squares, contradicting Lemma 4. \( \square \)

**Proof of Theorem 1.** By Lemma 4 Lemma 1 is applicable for \( S = S_2 \) with any \( c < \frac{4}{3} \). Let
\[
C = H(a_0; a_1, \ldots, a_{\lfloor d/2 \rfloor})
\]
and
\[
D = H(0; a_\lfloor (d+1)/2 \rfloor, \ldots, a_d),
\]
then by Lemma 1 and Lemma 2 we obtain
\[
c^{\lfloor d/2 \rfloor} \leq \min(|C|, |D|) \leq 8 \log N.
\]
Hence
\[
d \leq \frac{2}{\log c} \log \log N + O(1) \leq 6.96 \log \log N,
\]
for sufficiently large \( N \). \( \square \)

**Proof of Theorem 2.** The proof is as the one above, except that instead of Lemma 2 we use the bound
\[
\min(|C|, |D|) \leq 3N^{1-\frac{1}{k-1}},
\]
by Croot, Ruzsa and Schoen [4]. \( \square \)
Proof of Corollary. From $4af(x) = (2ax + b)^2 - b^2 + 4ac$, it follows that if
\[ H(a_0; a_1, \ldots, a_d) \subset S \cap [1, N], \]
then
\[ 4aH(a_0; a_1, \ldots, a_d) + b^2 - 4ac \subset S_2 \cap [4a + b^2 - 4ac, 4aN + b^2 - 4ac] \]
\[ \subset S_2 \cap [1, 4aN + b^2 - 4ac]. \]
Moreover,
\[ 4aH(a_0; a_1, \ldots, a_d) + b^2 - 4ac = H(4aa_0 + b^2 - 4ac; 4aa_1, \ldots, 4aa_d). \]
The Corollary now follows immediately from the proof of Theorem and observing that
\[ d < 6.96 \log \log(4aN + b^2 - 4ac) + O(1) \leq 7 \log \log N \]
for sufficiently large $N$.

Remark. It is easy to rephrase our results for multisets where multiplicity of elements $a_i$ is allowed: If there are no $k$-progressions in $S$, then it is immediately seen that the maximum multiplicity can be $k - 1$, so after allowing for an extra factor $k - 1$ our bounds established above still hold true. In particular, Theorem holds with
\[ d \leq 21 \log \log N. \]

Remark. While the proof follows the general strategy outlined in [7], the paper by Schoen [13] (which independently studied related questions) inspired us to rephrase his Lemma 2.1 in the form of our Lemma. This allows us to simplify and improve some of the steps described in [7] considerably.

With regard to the earlier results in [7], Noga Alon kindly pointed out to us that Lemma 5 of [7] is actually a version of a result of Erdős and Rado [9] on $\Delta$-systems (or sunflowers). Small quantitative improvements here are due to Kostochka [12], which together with the previous argument, for the set $S_2$ of squares, would lead to the tiny improvement
\[ d = O((\log N)^2 \log \log N), \]
the $\log_i N$ denoting the $i$-fold iterated logarithm. Moreover, the Erdős-Rado conjecture on these $\Delta$-systems, for which Erdős [8] offered a prize of $1000, would have implied $d = O(\log \log N)$. Fortunately, Lemma allowed us to bypass the realm of $\Delta$-systems.

Acknowledgements: We would like to thank Noga Alon for pointing out to us the connection to the work of Erdős and Rado [9]. In an attempt to avoid their conjecture we got the idea of the present approach.

During the preparation of this paper, R. Dietmann was supported by EPSRC Grant EP/I018824/1, C. Elsholtz by FWF-DK Discrete Mathematics Project W1230-N13.

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