$F$-factors in hypergraphs via absorption

Allan Lo and Klas Markström

Department of Mathematics and Mathematical Statistics,
Umeå University, S-901 87 Umeå, Sweden
allan.lo@math.umu.se, klas.markstrom@math.umu.se

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Abstract

For integers $n \geq k > l \geq 1$ and $k$-graphs $F$, define $t_{k,l}^F(n,F)$ to be the smallest integer $d$ such that every $k$-graph $H$ of order $n$ with minimum $l$-degree $\delta_l(H) \geq d$ contains an $F$-factor. A classical theorem of Hajnal and Szemerédi [9] implies that $t_{2,1}^F(n,K_t^t) = (1 - 1/t)n$ for integers $t$. For $k \geq 3$, $t_{k-1,k}^F(n,K_k^k)$ (the $\delta_{k-1}(H)$ threshold for perfect matchings) has been determined by Kühn and Osthus [17] (asymptotically) and Rödl, Ruciński and Szemerédi [22] (exactly).

In this paper, we generalise the absorption technique of Rödl, Ruciński and Szemerédi [22] to $F$-factors. We determine the asymptotic values of $t_{k,1}^F(n,K_k^k(m))$ for $k = 3, 4$ and $m \geq 1$. In addition, we show that $t_{k-1,k}^F(n,K_k^k) \leq \left(1 - \frac{1 + \gamma}{\binom{k}{k-1}}\right)n$ for some $\gamma > 0$ as well as constructing a lower bound on $t_{3,2}^F(n,K_3^4)$. In particular, we deduce that $t_{3,2}^F(n,K_3^4) = (3/4 + o(1))n$ answering a question of Pikhurko [20].

1 Introduction

Given graphs $G$ and $F$, an $F$-factor (or perfect $F$-tiling) in $G$ is a spanning subgraph consisting of vertex disjoint copies of $F$. Clearly, $G$ contains an $F$-factor only if $|F|$ divides $|G|$. Given an integer $t$ and a graph $G$ of order $n$ with $t|n$, we would like to know the minimum degree threshold that guarantees a $K_t$-factor in $G$. Notice that the minimum degree must be at least $(t-1)n/t$ by considering a complete $t$-partite graph with partition classes $V_1, \ldots, V_t$ with $|V_i| = n/t - 1$, $|V_i| = n/t + 1$ and $|V_t| = n/t$ for $1 < i < t$. In fact, $\delta(G) \geq (t-1)n/t$ suffices. For $t = 2$, this can be easily verified using Dirac’s Theorem [4]. Corrádi and Hajnal [5] proved the case for $t = 3$. All remaining cases $t \geq 4$ can be verified by a classical theorem of Hajnal and Szemerédi [9]. In this paper, we ask the same question for hypergraphs.

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We denote by $\binom{V}{k}$ the set of all $k$-sets of $U$. A $k$-uniform hypergraph, $k$-graph for short, is a pair $H = (V(H), E(H))$, where $V(H)$ is a finite set of vertices and $E(H) \subseteq \binom{V(H)}{k}$. Often we write $V$ instead of $V(H)$ when it is clear from the context. For a $k$-graph $H$ and an $l$-set $T \in \binom{V}{l}$, let $\deg(T)$ be the number of $(k-l)$-sets $S \in \binom{V}{k-l}$ such that $S \cup T$ is an edge in $H$, and let $\delta_l(H)$ be the minimum $l$-degree of $H$, that is, $\min \deg(T)$ over all $T \in \binom{V}{l}$. Note that $\delta_l(H)$ (or $\delta_{k-l}(H)$) are the minimum vertex degree (or codegree respectively) of $H$.

Analogously, given a hypergraph $H$ and a family $\mathcal{F}$ of hypergraphs, an $\mathcal{F}$-factor is a spanning subgraph consisting of vertex disjoint copies of members of $\mathcal{F}$. For a family $\mathcal{F}$ of $k$-graphs, define $t^F_k(n, \mathcal{F})$ to be the smallest integer $d$ such that every $k$-graph $H$ of order $n$ satisfying $\delta_l(H) \geq d$ contains an $\mathcal{F}$-factor. Throughout this paper, $\mathcal{F}$ is assumed to be $\{F\}$, so we simply write $F$-factor and $t^F_k(n, F)$. Moreover, we always assume that $|F|$ divides $n$ whenever we talk about $t^F_k(n, F)$. If $l = k-1$, we simply write $t^F(n, F)$. Let $t^F_k(n, t)$ denote $t^F_k(n, K^F_t)$, where $K^F_t$ is a complete $k$-graph on $t$ vertices. Thus, $t^F_k(n, t) = (t-1)n/t$. For graphs (that is, 2-graphs) $F$, there is a large body of research on $t^F(n, F)$, for surveys see [18, 23].

However, only a few values of $t^F_k(n, F)$ are determined for $k$-graphs $F$, $k \geq 3$ and $1 \leq t < k$. Note that $K^F_3$ is a single edge, so a $K^F_3$-factor is equivalent to a perfect matching. Kühn and Osthus [17] showed that $t^3(n, k) = n/2 + O(\sqrt{n \log n})$. Later, Rödl, Ruciński and Szemerédi [22] evaluated the exact value of $t^3(n, k)$ using an absorption technique. Hán, Person and Schacht [19] conjectured that

$$t^3_k(n, k) \approx \max \left\{ \frac{1}{2}, 1 - \left(1 - \frac{1}{k}\right)^{k-l} \right\} \binom{n}{k}.$$

We recommend [21] for a survey of the recent developments in $t^3_k(n, k)$. Pikhurko [20] showed that $3n/4 - 2 \leq t^3(n, 4) \leq 0.860n$ and asked whether $t^3(n, 4) = 3n/4 - 2$. For the unique 3-graph $F$ of order 4 and size 2, Kühn and Osthus [18] showed that $t^3(n, F) = (1/4 + o(1))n$. Recently, Kierstead and Mubayi [14] proved a generalisation of Hajnal-Szemerédi theorem for 3-graphs and vertex degree, which implies that

$$t^3_k(n, t) \leq \left(1 - \frac{c}{t^2 \log^2(n/t)}\right) \binom{n}{2}$$

for some constant $c > 0$.

One of the key techniques in finding perfect matchings (evaluating $t^F_k(n, \mathcal{F})$) is the absorption technique, which was first introduced by Rödl, Ruciński and Szemerédi [22]. Roughly speaking, the absorption technique reduce the task of finding a perfect matching in $H$ to finding an almost perfect matching, that is, a matching covering all but at most $\varepsilon|H|$ vertices for some small $\varepsilon > 0$. Here, we generalise the absorption technique to $F$-factors.

For a $k$-graph $H$ and a vertex set $U \subset V(H)$, $H[U]$ is the subgraph of $H$ induced by the vertices of $U$. We write $v$ to mean the set $\{v\}$ when it is clear from the context. Given a $k$-graph $F$ with $|F| = t$, an integer $i \geq 1$ and a constant $\eta > 0$, we say that a vertex $x$ is $(F, i, \eta)$-close to a vertex $y$ if there exist $\eta^{t-i-1}$ sets $S \in \binom{V}{t-i}$ of size $it - 1$ such that $S \cap \{x, y\} = \emptyset$ and both $H[S \cup x]$ and $H[S \cup y]$ contain $F$-factors. Moreover, $H$ is said to be $(F, i, \eta)$-closed if every two vertices are $(F, i, \eta)$-close to each other.
We say that $H$ has an almost $F$-factor $T$ to mean that $T$ is a set of vertex disjoint copies of $F$ in $H$ such that $|V(H)\setminus V(T)| < \epsilon|H|$ for small $\epsilon > 0$.

We now state the absorption lemma for $F$-factors.

**Lemma 1.1** (Absorption lemma for $F$-factors). Let $t$ and $i$ be positive integers and let $\eta > 0$. Let $F$ be a hypergraph of order $t$. Then, there is an integer $n_0$ satisfying the following: Suppose that $H$ is a hypergraph of order $n \geq n_0$ and $H$ is $(F,i,\eta)$-closed, then there exists a vertex subset $U \subset V(H)$ of size $|U| \leq (\eta/2)^2n/4$ such that there exists an $F$-factor in $H[\{U\cup W]\setminus W]$ for every vertex set $W \subset V(U)$ of size $|W| \leq (\eta/2)^2tn/32$ with $|W| \leq t\mathbb{Z}$.

Note that in the above lemma $H$ and $F$ are not necessarily $k$-graphs, but we only consider $k$-graphs here. Equipped with the absorption lemma, we can break down the task of finding an $F$-factor in large hypergraphs $H$ into the following algorithm.

**Algorithm for finding $F$-factors.**

1. Remove a small set $T_1$ of vertex disjoint copies of $F$ from $H$ such that the resultant graph $H_1 = H[V\setminus V(T_1)]$ is $(F,i,\eta)$-closed for some integer $i$ and constant $\eta > 0$.
2. Find a vertex set $U$ in $V(H_1)$ satisfying the conditions of the absorption lemma. Set $H_2 = H_1[V(H_1)\setminus U]$.
3. Show that $H_2$ contains an almost $F$-factor, i.e. a set $T_2$ of vertex disjoint copies of $F$ such that $|V(H_2)\setminus V(T_2)| < \epsilon|H_2|$ for small $\epsilon > 0$.
4. Set $W = V(H_2)\setminus V(T_2)$. Since $H_2[U \cup W]$ contains an $F$-factor $T_3$ by the choice of $U$, $T_1 \cup T_2 \cup T_3$ is an $F$-factor in $H$.

We now apply the algorithm to various $k$-graphs $F$. We would like to point out that Steps 1 and 3 of the algorithm require most of the work.

A $k$-graph $H$ is $k$-partite with partition $V_1,\ldots,V_k$, if $V = V_1 \cup \cdots \cup V_k$ and every edge intersects every $V_i$ in exactly one vertex. We denote by $K_k^n(m_1,\ldots,m_k)$ the complete $k$-partite $k$-graph with parts of sizes $m_1,\ldots,m_k$. If $m = m_l$ for $1 \leq i \leq m_k$, we simply write $K_k^k(m)$. Clearly,

$$t^i_k(n,K_k^k(m)) \geq t^i_k(n,K_k^k) = t^i_k(n,k).$$

As a simple application of the absorption lemma, we show that $t^i_k(n,K_k^k(m))$ equals $t^i_k(n,k)$ asymptotically for integers $k = 3, 4$ and $m \geq 1$.

**Theorem 1.2.** For integers $m \geq 1$ and $k = 3, 4$

$$t^i_k(n,K_k^k(m)) = t^i_k(n,k) + o(n^{k-1}) = \begin{cases} \left(\frac{k}{2} + o(1)\right) \binom{n}{2} & \text{if } k = 3 \\ \left(\frac{k}{2^k} + o(1)\right) \binom{n}{k} & \text{if } k = 4. \end{cases}$$

For $m = 1$, Hán, Person, Schacht [10] evaluated $t^i_1(n,3)$ asymptotically. Köhn, Osthus and Treglown [19] and independently Khan [12] determined the exact value of $t^i_1(n,3)$. Also, Khan [13] evaluated $t^i_1(n,4)$ exactly.

Recall that $t^k(n,t) = t^k_{t-1}(n,K_k^t)$. For $t = k + 1$, we give a lower bound on $t^k(n,k+1)$ for $k \geq 4$ as well as bounding $t^k(n,k+1)$ from below for $k \geq 4$ even.

**Theorem 1.3.** Given an integer $k \geq 3$ and a constant $\gamma > 0$, there exists an integer $n_0 = n_0(k,\gamma)$ such that for $n \geq n_0$ and $k+1|n$

$$t^k(n,k+1) \leq \left(1 - \frac{k+1_{k,\text{odd}}}{2k^2} + \gamma\right)n,$$

where $1_{k,\text{odd}} = 1$ if $k$ is odd and $1_{k,\text{odd}} = 0$ otherwise. Moreover, $t^k(n,k+1) \geq 2n/3$ for integers $k \geq 4$ even.
Furthermore, we bound from above \( t^k(n, t) \) from above for \( t > k \geq 3 \).

**Theorem 1.4.** For integers \( n \geq t > k \geq 3 \) with \( t|n \),

\[
t^k(n, t) \leq \left( 1 - \frac{1 + \gamma}{\binom{k-1}{t-1}} \right) n,
\]

where \( \gamma \) is a strictly positive function tending to zero as \( t \) and \( n \) tends to infinity. In particular, \( t^3(n, 4) \leq (3/4 + o(1))n \).

Pikhurko [20] showed that \( t^3(n, 4) \geq 3n/4 - 2 \), so \( t^3(n, 4) = (3/4 + o(1))n \). In fact, Theorem 1.4 is derived from the following stronger statement.

**Theorem 1.4.** For integers \( t > k \geq 3 \) and constants \( \gamma > 0 \), there exists an integer \( n_0 = n_0(k, t, \gamma) \) such that for \( n \geq n_0 \) and \( t|n \)

\[
t^k(n, t) \leq \left( 1 - \min \left\{ \beta(k, t), \frac{t - 1}{\binom{k-1}{t-1}} \right\} + \gamma \right) n,
\]

where the function \( \beta(k, t) \) is explicitly defined in Section 2.

We would like to point out that the terms \( \beta(k, t) \) and \( \frac{t - 1}{\binom{k-1}{t-1}} \) come from Step 1 and Step 3 of the algorithm respectively. We believe that the bound in Theorem 1.4 is sharp in the sense that \( \delta_k(H) > \left( 1 - \frac{t - 1}{\binom{k-1}{t-1}} \right) n \) is needed to guarantee that every \( K^k_{t-1} \) in \( H \) can be extended to a \( K^k_t \). For \( k = 3 \), we give the following lower bound on \( t^3(n, t) \).

**Proposition 1.5.** For integers \( n > t \geq 4 \) with \( n \) sufficiently large and \( t|n \),

\[
t^3(n, t) \geq (1 - C \log t/t^2)n,
\]

where \( C > 0 \) is an absolute constant independent of \( n \) and \( t \).

However, for \( t > k \geq 4 \), we know that \( t^k(n, t) \geq (1 - (k - 1)/t) n - k + 2 \). Indeed, this is true by considering the following \( k \)-graph \( H \) of order \( n \) such that there exists a vertex subset \( W \) of size \( |W| = (t - k + 1)n/t - 1 \) and every edge in \( H \) meets \( W \). Thus, we ask the following question.

**Question 1.6.** How does \( t^k(n, t) \) behave for \( t \geq k \geq 3 \) as \( n \to \infty \)? Is \( t^k(n, t) \sim (1 - C \log(t/t^2))n \) or \( t^k(n, t) \sim (1 - C/t^2)n \) for \( n \) large and some constant \( C \)?

As an auxiliary result, we also prove the following.

**Theorem 1.7.** Let \( c, \epsilon > 0 \) be constants. Let \( t \geq 3 \) and \( \lambda \geq 1 \) be integers such that \( (\lambda t^2)\log(\lambda t) \leq c t^2 \). Then, there exist absolute constants \( 0 < c_1 \leq c_2 \) independent of \( c, \epsilon, t \) and \( \lambda \) such that

\[
\frac{c_1 \lambda t^2}{\log(\lambda t)} \leq R(B_\lambda, K^3_t) \leq \frac{c_2 \lambda t^2}{\log(\lambda t)},
\]

where \( B_\lambda = K^3_3(1, 1, \lambda + 1) \).
2 Layout of the paper and preliminaries

The paper is structured as follows. First, we set up some basic notations in Section 2.1 as well as studying some properties of being \((F, i, \eta)\)-close. Next, we look at the Ramsey number \(R(B_\lambda, K_\lambda^t)\), Theorem 1.2, in Section 2.2. In Section 5 we prove the absorption lemma, Lemma 1.1. Then, we apply the absorption lemma to prove Theorem 1.2. In Section 2.3, we prove that a vertex set \(U\) is \((F, i, \eta)\)-closed in \(H\) if for distinct vertices \(x, y \in U\), \(x, y\) is \((F, i, \eta)\)-close to \(y\).

The remainder of the paper is focused on bounding \(t^*(n, t)\). In Section 2.3 we bound \(t^*(n, t)\) below by constructions. We give a lower bound on \(t^*(n, k + 1)\) for \(k \geq 4\) even, followed by a lower bound on \(t^*(n, t)\), which proves Proposition 1.3. In Section 5, we bound \(t^*(n, k + 1)\) from above, which proves Theorem 1.3. With further works, we bound \(t^*(n, t)\) from above, which proves Theorem 1.4 in Sections 7.

2.1 Notations

For \(a \in \mathbb{N}\), we refer to the set \(\{1, \ldots, a\}\) as \([a]\).

Throughout this paper, \(H\) is assumed to be a \(k\)-graph \(H\) of order \(n\). The maximum \(l\)-degree \(\Delta_l(H)\) is simply the maximal \(\deg(T)\) over all \(T \in \binom{V}{l}\). Given an \(l\)-set \(T \in \binom{V}{l}\), the neighbourhood \(N(T)\) of \(T\) is the set of \((k - l)\)-sets \(S \in \binom{V}{k-l}\) such that \(T \cup S\) is an edge in \(H\). Clearly, \(\deg(T) = |N(T)|\). For \(s\)-sets \(S \in \binom{V}{s}\), define

\[
L(S) = \begin{cases} 
V \setminus S & \text{if } s < k - 1 \\
\bigcap_{T \in \binom{V}{s}} N(T) & \text{if } s \geq k - 1.
\end{cases}
\]

For \(k\)-graphs \(F\), integers \(i\) and constants \(\eta > 0\), denote by \(\tilde{N}_{F, i, \eta}(x)\) the set of vertices \(y \in V(H)\) that are \((F, i, \eta)\)-close to vertex \(x\). We say that a vertex set \(U\) is \((F, i, \eta)\)-closed in \(H\) if for distinct vertices \(x, y \in U\), \(x\) is \((F, i, \eta)\)-close to \(y\).

In the next two propositions, we show that the property of being \((F, i, \eta)\)-close is ‘additive’.

Proposition 2.1. Let \(i_1, i_2\) and \(t\) be positive integers and let \(\eta_1, \eta_2, \epsilon > 0\) be constants. Let \(F\) and \(H\) be \(k\)-graphs of order \(t\) and \(n \geq n_0(i_1, i_2, t, \eta_1, \eta_2, \epsilon)\) respectively. If

\[
|\tilde{N}_{F, i_1, \eta_1}(x) \cap \tilde{N}_{F, i_2, \eta_2}(y)| \geq \epsilon n
\]

for two distinct vertices \(x, y \in V(H)\), then \(x\) is \((F, i_1 + i_2, \eta)\)-close to \(y\), where \(\eta = \eta(i_1, i_2, t, \eta_1, \eta_2, \epsilon) > 0\).

Proof. Let \(m_j = j + t - 1\) for \(j = 1, 2\) and \(m = m_1 + m_2 + 2\). Let \(n_0 = \max\{2(n_0 - 1), 2t_0 n_0 - 1\}\) and \(\epsilon = \epsilon n_0 (4m)!\). Fix \(z \in \tilde{N}_{F, i_1, \eta_1}(x) \cap \tilde{N}_{F, i_2, \eta_2}(y)\). Since \(z \in \tilde{N}_{F, i_1, \eta_1}(x)\), there are at least

\[
\eta n^{m_1} - n^{m_1 - 1} \geq \eta n^{m_1}/2
\]

\(m_1\)-sets \(S_x\) such that \(S_x \cap \{x, y, z\} = \emptyset\) and both \(H[S_x \cup x]\) and \(H[S_x \cup y]\) contain \(F\)-factors, and fix one such \(S_x\). Similarly, there are at least

\[
\eta_2 n^{m_2} - (i_1 + 1)n^{m_2 - 1} \geq \eta n^{m_2}/2
\]

\(m_2\)-sets \(S_y\) such that \(S_y \cap \{x, y, z\} = \emptyset\) and both \(H[S_y \cup y]\) and \(H[S_y \cup z]\) contain \(F\)-factors, and fix one such \(S_y\). Set \(S = S_x \cup S_y \cup z\).
Note that both \( H[S \cup x] \) and \( H[S \cup y] \) contain \( F \)-factors and \( |S| = n \). Furthermore, note that there are at least
\[
\frac{cn \times n!}{m!} \geq \frac{cn \times n^{m+1}/2 \times n^{m+2}/2}{m!} = \eta n^m
\]
such \( S \), so \( x \) is \((F, i_1 + i_2, \eta)\)-close to \( y \). \( \square \)

By a similar argument, we have the following proposition of which the proof is omitted.

**Proposition 2.2.** Let \( i_1, i_2 \) and \( t \) be positive integers and let \( \eta_1, \eta_2, \epsilon > 0 \) be constants. Let \( F \) and \( H \) be \( k \)-graphs of order \( t \) and \( n \geq n_0(i_1, i_2, t, \eta_1, \eta_2, \epsilon) \) respectively. If \( y \) is \((F, i_1, \eta_1)\)-close to \( x \) and \(|N_{F,i_2,\eta_2}(x)| \geq cn\), then \( x \) is \((F, i_1 + i_2, \eta)\)-close to \( y \), where \( \eta = \eta(i_1, i_2, t, \eta_1, \eta_2, \epsilon) > 0 \).

Since \( \tilde{N}_{F,i_0,\eta_0}(x) \subset \tilde{N}_{F,i_0,\eta_0}(x) \) by a suitable choice of \( \eta_1 \), by the above proposition the following two statements are equivalent:

- For every pair of vertices \( x, y \in V(H) \), \( x \) is \((F, i_{x,y}, \eta_{x,y})\)-close to \( y \) for \( i_{x,y} \leq i_0 \) and \( \eta_{x,y} > 0 \).

- \( H \) is \((F, i_{00}, \eta_0)\)-close for some \( \eta_0 > 0 \).

### 2.2 Ramsey number of 3-graphs

Recall that the Ramsey number \( R(S, T) \) of \( k \)-graphs \( S \) and \( T \) is the minimum integer \( N \) such that if we edge colour \( K_N^k \) with colours red and blue then there exists a red monochromatic \( S \) or blue monochromatic \( T \). Given an integer \( \lambda \geq 0 \), let \( B_\lambda \) be the 3-graph on vertex set \( \{x, y, z_1, \ldots, z_{\lambda+1}\} \) with edges \( xyz_i \) for \( 1 \leq i \leq \lambda + 1 \). In other words, \( B_\lambda = K_3^3(1, 1, \lambda+1) \).

First we bound \( R(B_\lambda, K_3^3) \) from below. A partial \( t - (n, k, \lambda) \) design is a family \( J \) of \( k \)-sets in \( [n] \) such that every \( t \)-set \( T \) is contained in at most \( \lambda + 1 \) \( k \)-sets in \( J \). Note that a partial \( 2 - (n, 3, \lambda) \) design does not contain a \( B_\lambda \). We are going to construct a partial \( t - (n, k, \lambda) \) design with small independence number by modifying a construction of Kostochka, Mubayi, Rödl and Tetali [15]. It should be noted that Grable, Phelps and Rödl [8] constructed \( 2 - (n, k, \lambda) \) design with small independence number, but \( n \) is a even power of a sufficient large prime.

**Proposition 2.3.** For integers \( t, k, \lambda, x > 0 \), there exists a partial \( t - (n, k, \lambda) \) design \( H \) with
\[
n = \left\lfloor \frac{8(k-1)}{t} \frac{\left( \frac{k-1}{t} \right) \left( \frac{k}{t} \right) \left( \frac{\lambda x^{k-1}}{\log \lambda x} \right)^{1/(k-1)} }{4} \right\rfloor
\]
and the independence number of \( H \) is less than \( x \).

**Proof.** We consider the following constrained random process. First we order all \( k \)-sets of \( [n] \) at random: \( E_1, \ldots, E_{\binom{n}{k}} \). Let \( H_0 \) be the empty graph on vertex set \( [n] \). For \( 1 \leq j \leq \binom{n}{k} \), set \( H_j = H_{j-1} + E_j \) if \( H_{j-1} + E_j \) is a partial \( t - (n, k, \lambda) \) design, otherwise \( H_j = H_{j-1} \). Let \( H = H_{\binom{n}{k}} \). Our aim is to show that with positive probability that the independence number of \( H \) is less than \( x \).

Fix an \( x \)-set \( X \). Let \( B_X \) be the event that \( X \) is an independent set in \( H(n, k) \). Observe that \( B_X \) implies that for every \( k \)-set \( T \subset X \) there exist an \( S \in \binom{\binom{n}{k}}{j} \) and edges \( E_1, \ldots, E_{\lambda-1} \) not in \( X \) preceded \( T \) (in the ordering) with \( S \subset E_i \) for \( 1 \leq j \leq \lambda \). This implies \( t \leq |E_j \cap X| \leq k-1 \).
for $1 \leq j \leq \lambda$. Call an edge $E$ in $H$ a witness for $T \in {X \choose k}$ not to be included in $H$ if $E$ precedes $T$ in the ordering and $|E \cap T| \geq t$. Thus, in order for $B_X$ to happen, each $T \in {X \choose k}$ must have at least $\lambda$ witnesses. Each edge $E$ in $H$ can be a witness for at most $\binom{k-1}{t} \binom{\lambda - 1}{k-t}$ $k$-sets $T \subset X$. Therefore, if $B_X$ happens, there are at least
\[
m = \frac{\lambda \binom{\lambda}{t}}{\binom{k-1}{t} \binom{\lambda - 1}{k-t}}\]
witnesses.

For $j \geq 1$, let $A_j = A_{X,j}$ denote the event that the first $j$ edges $E_{i_1}, E_{i_2}, \ldots, E_{i_j}$ in $H$ satisfy $t \leq |E_{i_t} \cap X| < k$ for $1 \leq i \leq j$. Note that $B_X$ implies $A_m$. Our task is to bound the probability of $A_m$ from above by $\binom{\lambda}{t}^{-1}$. We further assume that $E_{i_1}$ is the witness that appears first in the ordering, and that for each $1 \leq j \leq m$, $E_{i_j}$ is the first witness which comes after $E_{i_{j-1}}$. Let $H' = H_{i_{j-1}}$ be the family of all $k$-sets included in $H(n,k)$ before $j$th witness $E_{i_j}$ is chosen. For $1 \leq j \leq \lceil m/2 \rceil$, let $S_j$ be the collection of all $k$-sets such that $|X \cap S| \geq t$ and $|E \cap S| < t$ for all but at most $\lambda$ edges $E \in H'$. The k-graph $H'$ contains precisely $j - 1$ edges $E$ with $t < |E \cap X| < k - 1$. Each of these is a witness for at most $\binom{|E \cap X|}{t} \binom{x-t}{k-t} \binom{x-t}{k-t}$ $k$-sets $T \in {X \choose k}$. Consequently, the number of $k$-sets $T$ in $X$ with less than $\lambda$ witnesses at this stage is
\[
\left| S_j \cap {X \choose k} \right| \geq \binom{x}{k} - \frac{j - 1}{\lambda} \binom{x-t}{k-t} \binom{x-t}{k-t} \geq \binom{x}{k} - \frac{\lfloor m/2 \rfloor - 1}{\lambda} \binom{x-t}{k-t} \binom{x-t}{k-t} \geq \frac{1}{2} \binom{x}{k}.
\]
Since $A_m \subset A_{m-1} \subset \cdots \subset A_1$, we have
\[
P(A_m) = P(A_1)P(A_2|A_1) \cdots P(A_m|A_{m-1}).
\]

Note that each of the events $A_1$ and $A_{j+1}|A_j$, $j = 1, \ldots, m-1$ corresponds to a random choice from the set $S_j$ with the result that the chosen set belongs to $S_j \setminus {X \choose k}$. Since $|S_1| \leq \binom{\lambda}{t} \binom{x}{k-t}$ we have
\[
P(A_1) = \frac{|S_1| - \binom{\lambda}{t}}{|S_1|} \leq 1 - \frac{\binom{\lambda}{t}}{\binom{\lambda}{t} \binom{x}{k-t}}.
\]

Furthermore, for $1 < j \leq \lceil m/2 \rceil$,
\[
P(A_j|A_{j-1}) = \frac{|S_j \setminus {X \choose k}|}{|S_j|} = 1 - \frac{|S_j \cap {X \choose k}|}{|S_j|} \leq 1 - \frac{\binom{\lambda}{t}}{2 \binom{\lambda}{t} \binom{x}{k-t}}.
\]

This yields
\[
P(A_m) \leq P(A_1) \prod_{1< j \leq \lceil m/2 \rceil} P(A_j|A_{j-1}) \leq \left(1 - \frac{\binom{\lambda}{t}}{2 \binom{\lambda}{t} \binom{x}{k-t}}\right)^{\lceil m/2 \rceil} \leq \exp \left(-\frac{\lambda(x)_k}{4(n)_{k-\lambda}(k)_{t-1} \binom{\lambda}{t}}\right).
\]
Note that
\[
\left( \frac{n}{x} \right) P(A_m) \leq \exp \left\{ x \left( \log \frac{en}{x} - \frac{\lambda(x-1)_{k-1}}{4(n)_k (\frac{1}{\sqrt{t}})^2} \right) \right\} \\
\leq \exp \left\{ x \left( \log \frac{en}{x} - \frac{\lambda x^{k-1}}{8n^{k-1} (\frac{1}{\sqrt{t}})^2} \right) \right\} < 1,
\]
where the last inequality holds due to our choice of \(n\). Thus, by the union bound with positive probability that the independence number of \(H\) is less than \(x\).

\[\square\]

Proof of Theorem 1.7. The lower bound is proved by Proposition 2.1 so it is enough to prove the upper bound. Let \(n = c_2 \lambda t^2 / \log(\lambda t)\) and \(\tau = \sqrt{\lambda n}\), where \(c_2\) is a constant to be chosen later. Let \(H\) be a 3-graph of order \(n\) with \(\Delta_2(H) \leq \lambda\), so \(H\) does not contain a \(B_3\). It is enough to show that the independence number of \(H\) \(\alpha(H) \geq t\). Note that \(\Delta_1(H) \leq \lambda n = \tau^2\). The number of 2-cycles, that is the number of \(B_1\) in \(H\), is at most \((\lambda n) / (\sqrt{t})^2 \leq n \tau^{2-t}\) by our choices of \(\epsilon, c, \lambda\) and \(t\). Then, by a theorem of Duke, Lefmann and Rödl [5], there exists a constant \(c''\) such that
\[
\alpha(H) \geq \epsilon n \frac{\sqrt{\log \tau}}{t} = c'' \sqrt{n \log(\lambda n) / 2\lambda}
\]
which is greater than \(t\) by choosing \(c_2\) large.

\[\square\]

3 Proof of the absorption lemma

Here we prove the absorption lemma, Lemma 1.1, of which the proof is based on Hán, Person and Schacht [10].

Proof of Lemma 1.1. Let \(H\) be a hypergraph of order \(n \geq n_0\) such that \(H\) is \((F, i, \eta)\)-closed. Throughout the proof we may assume that \(n_0\) is chosen to be sufficiently large. Set \(m_1 = it - 1\) and \(m = (t-1)(m_1 + 1)\). Furthermore, call a \(m\)-set \(A \in \binom{[n]}{m}\) an absorbing \(m\)-set for a \(t\)-set \(T \in \binom{[n]}{t}\) if \(A \cap T = \emptyset\) and both \(H[A]\) and \(H[A \cup T]\) contain \(F\)-factors. Denote by \(\mathcal{L}(T)\) the set of all absorbing \(m\)-sets for \(T\). Next, we show that for every \(t\)-set \(T\), there are many absorbing \(m\)-sets for \(T\).

Claim 3.1. For every \(t\)-set \(T \in \binom{[n]}{t}\), \(|\mathcal{L}(T)| \geq (\eta/2)^l \binom{n}{m}\).

Proof. Let \(T = \{v_1, \ldots, v_l\}\) be fixed. Since \(v_1\) and \(u\) are \((F, i, \eta)\)-connected for \(u \notin T\), there are at least \(\eta n^{m-1}\) \(m_1\)-set \(S\) such that \(H[S \cup v_1]\) contains an \(F\)-factor. Hence, by an averaging argument there are at least \(\eta n^{m-1}\) copies of \(F\) containing \(v_1\). Since \(n_0\) was chosen large enough, there are at most \((t-1)n^{l-2} \leq \eta n^{m-1}/2\) copies of \(F\) containing \(v_1\) and \(v_j\) for some \(2 \leq j \leq t\). Thus, there are at least \(\eta n^{m-1}/2\) copies of \(F\) containing \(v_1\) but none of \(v_2, \ldots, v_l\). We fix one such copy of \(F\) with \(V(F) = \{v_1, u_2, \ldots, u_l\}\). Set \(U_1 = \{u_2, \ldots, u_l\}\) and \(W_0 = T\).

For each \(2 \leq j \leq t\) and each pair \(u_{j-1}, v_j\) suppose we succeed to choose an \(m_1\)-set \(U_j\) such that \(U_j\) is disjoint from \(W_{j-2} = U_{j-1} \cup W_{j-2}\) and both \(H[S_j \cup u_j]\) and \(H[S_j \cup v_j]\) contain \(F\)-factors. Then for a fixed \(2 \leq j \leq t\) we call such a choice \(U_j\) good, motivated by \(A = \bigcup_{1 \leq j \leq t} S_j\) being an absorbing \(m\)-set for \(T\).

In each step \(2 \leq j \leq t\), recall that \(u_j\) is \((F, i, \eta)\)-closed to \(v_j\), so there are at least \(\eta n^{m-1}\) \(m_1\)-sets \(S\) such that \(H[S \cup u_j]\) and \(H[S \cup v_j]\) contain
For each $2 \leq j \leq t$ there are at least $\eta n^{m_j}/2$ choices for $S_j$ and in total we obtain $(\eta/2)^t n^m$ absorbing $m$-sets for $T$ with multiplicity at most $m!$, so the claim holds.

Now, choose a family $\mathcal{F}$ of $m$-sets by selecting each of the $\binom{n}{m}$ possible $m$-sets independently with probability $p = (\eta/2)^t n/(8\binom{n}{m})$. Then, by Chernoff’s bound (see e.g. [1]) with probability $1 - o(1)$ as $n \to \infty$, the family $\mathcal{F}$ satisfies the following properties:

$$|\mathcal{F}| \leq (\eta/2)^t n/4$$

and

$$|\mathcal{L}(T) \cap \mathcal{F}| \geq (\eta/2)^{2t} n/16$$

for all $t$-sets $T$. Furthermore, we can bound the expected number of intersecting $m$-sets by

$$\binom{n}{m} \times m \times \binom{n}{m-1} \times p^2 \leq \left(\frac{\eta}{2}\right)^{2t} \frac{n}{32}.$$

Thus, using Markov’s inequality, we derive that with probability at least $1/2$

$$\mathcal{F} \text{ contains at most } \left(\frac{\eta}{2}\right)^{2t} \frac{n}{32} \text{ intersecting pairs.}$$

Hence, with positive probability the family $\mathcal{F}$ has all properties stated in [1], [2] and [3]. By deleting all the intersecting $m$-sets and non-absorbing $m$-sets in such a family $\mathcal{F}$, we get a subfamily $\mathcal{F}'$ consisting of pairwise disjoint balanced $m$-sets, which satisfies

$$|\mathcal{L}(T) \cap \mathcal{F}'| \geq (\eta/2)^{2t} n/16 - (\eta/2)^{2t} n/32 = (\eta/2)^{2t} n/32$$

for all $t$-sets $T$. Set $U = V(\mathcal{F}')$. Since $\mathcal{F}'$ consists only of absorbing $m$-sets, $H[U]$ has an $F$-factor. For a set $W \subseteq V\setminus U$ of size $|W| \leq (\eta/2)^{2t} m/32$ and $|W| \in \mathbb{Z}$, $W$ can be partition in to at most $(\eta/2)^{2t} n/32$ $t$-sets. Each $t$-set can be successively absorbed using a different absorbing $m$-set, so $H[U \cup W]$ contains an $F$-factor.

4 $K^k_k(m)$-factors

Our aim of this section is to prove Theorem [12] which determines the asymptotic values of $t_2^1(n, K^2_3(m))$ and $t_2^1(n, K^2_2(m))$. Without loss of generality, we may assume that $m \geq 2$ for the remainder of this section.

For $k$-graphs $H$ and constants $\alpha > 0$, a $(k-1)$-set $S \subseteq \binom{V}{k-1}$ is said to be $\alpha$-good for vertices $x$ and $y$ if $\deg(S) \geq \alpha n$ and $x, y \in N(S)$. Otherwise, $S$ is $\alpha$-bad. A pair of vertices $(x, y)$ is $\alpha$-good if there are at least $\alpha \binom{n}{k-1}$ $\alpha$-good $(k-1)$-sets $S \subseteq \binom{V}{k-1}$ for $x$ and $y$. First, we need the following result of Erdős [8].

Theorem 4.1 (Erdős [8]). For integers $k, m \geq 2$, there exists an integer $n_0 = n_0(k, m) > 0$ such that every $k$-graphs $H$ on $n \geq n_0$ vertices with at least $n^{k-1}k^{-k}$ edges contains a $K^k_k(m)$.
Corollary 4.2. For integers $k, m \geq 2$ and constant $\beta > 0$, there exist a constant $c = c(\beta, k, m) > 0$ and an integer $n_0 = n_0(\beta, k, m) > 0$ such that for every $k$-graphs $H$ on $n \geq n_0$ vertices with at least $\beta n^{k}$ edges, there are at least $cn^{km}$ copies of $K_{k}^{m}(m)$ in $H$.

Now, we show that if $(x, y)$ is $\alpha$-good, then $x$ and $y$ are $(K_{k}^{m}(m), 1, \eta)$-close to each other.

Lemma 4.3. Let $k, m \geq 2$ be integers and $\alpha > 0$. There exist a constant $\eta = \eta(\alpha, k, m)$ and an integer $n_0 = n_0(\alpha, k, m)$ such that for every $k$-graph $H$ of order $n \geq n_0$, if $(x, y)$ are $\alpha$-good in $H$ for $x, y \in V(H)$ then $x$ and $y$ are $(K_{k}^{m}(m), 1, \eta)$-close to each other.

Proof. Let $H$ be a $k$-graph of order $n$ sufficiently large. Suppose that $(x, y)$ are $\alpha$-good for $x, y \in V(H)$. By the definition of $(K_{k}^{m}(m), 1, \eta)$-close, it is enough to show that there exist at least $cn^{(k-1)m}$ copies of $K_{k}^{m}(m, \ldots, m, m + 1)$ (not necessarily vertex disjoint), which contains $x$ and $y$ in the partition of size $m + 1$ for some constant $c = c(\alpha, k, m) > 0$. Since $(x, y)$ is $\alpha$-good, by an averaging argument there exist at least $\alpha n/2$ vertices $z$ satisfying

$$|N(x) \cap N(y) \cap N(z)| \geq \alpha \binom{n}{k-1}.$$

Thus, there are at least $c'n^{(k-1)m}$ copies of $K_{k-1}^{m}(m)$ in $N(x) \cap N(y) \cap N(z)$ by Corollary 4.2. For each $(k-1)m$-set $T \subset V$, denote by $\deg'(T)$ the number of vertices $z \in V \setminus \{x, y\}$ such that $T \cup \{x, y, z\}$ forms a $K_{k}^{m}(m, \ldots, m, 3)$ of which the partition of size 3 is exactly $\{x, y, z\}$. Hence,

$$\sum_{T \in (k-1)m} \deg'(T) \geq c'n^{(k-1)m} \times \alpha n/2.$$

Therefore, the number of copies of $K_{k}^{m}(m, \ldots, m, m + 1)$, of which the partition of size $m + 1$ contains both $x$ and $y$, is equal to

$$\sum_{T \in (k-1)m} \frac{\left(\sum \deg'(T)/\binom{n}{k-1}\right)}{m - 1} \geq \left(\frac{n}{k-1}\right) \left(\frac{\alpha' n^{(k-1)m+1}/2}{m - 1}\right) \geq cn^{km-1},$$

for some suitable chosen constant $c = c'(\gamma, k, m) = c(\alpha, k, m)$ and $n$ sufficiently large, where the first inequality is due to Jensen.

By a similar argument, we also have the following result.

Proposition 4.4. Let $k, m \geq 2$ be integers and $\gamma > 0$. There exist a constant $c = c(\gamma, k, m)$ and an integer $n_0 = n_0(\gamma, k, m)$ such that every $k$-graph $H$ of order $n$ with $\delta_1(H) \geq (1/2 + \gamma)\binom{n}{k-1}$ and every vertex $v \in V$, there exists at least $cn^{km-1}$ copies of $K_{k}^{m}(m)$ containing $v$.

Also, we would need the following result from Khan [12, 13], which shows the existence of an almost $K_{k}^{m}(m)$-factors for $k = 3, 4$. The result was not stated explicitly, but it is easily seen from his proofs (for the non extremal cases).
Lemma 4.5 (Khan). Let $k = 3, 4$ and $m \geq 2$ be integers. Let $\gamma$ be a strictly positive constant. Then there exists an integer $n_0 = n_0(\gamma, k, m)$ such that every $k$-graph $H$ of order $n \geq n_0$ with

$$
\delta_1(H) \geq \begin{cases} 
\left(\frac{\alpha}{12} + \gamma\right) \binom{n}{2} & \text{if } k = 3 \\
\left(\frac{\alpha}{12} + \gamma\right) \binom{n}{4} & \text{if } k = 4.
\end{cases}
$$

contains a set $T$ of vertex disjoint copies of $K^k_n(m)$ in $H$ covering all but at most $\gamma n$ vertices.

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $k = 3, 4$ and $m \geq 2$. Recall that

$$t^k_1(n, K^k_n(m)) \geq t^k_1(n, K^k_n),$$

so it is enough to show the upper bound. Let $\gamma$ and $\alpha$ be strictly positively small constants with $\alpha \ll \gamma$. Let $H$ be a $k$-graph of order $n$ sufficiently large with

$$\delta_1(H) \geq \begin{cases} 
\left(\frac{\alpha}{12} + \gamma\right) \binom{n}{2} & \text{if } k = 3 \\
\left(\frac{\alpha}{12} + \gamma\right) \binom{n}{4} & \text{if } k = 4.
\end{cases}
$$

We are going to show that $H$ contains a $K^k_n(m)$-factor. We follow the algorithm for finding $F$-factors as stated in Section 1.

**Step 1** Since $\delta_1(H) \geq (1/2 + \gamma') \binom{n}{k-1}$, if $(x, y)$ is $\alpha$-bad then there are at least $2(k - 1)$-sets $S \subseteq N(x) \cap N(y)$. We claim that there are at most $\frac{1}{2}(\frac{\alpha}{12})(k - 1)$-sets $S \subseteq \binom{n}{k-1}$ with $d(S) < (1 + \gamma)n$. Otherwise,

$$n \left(\frac{1}{2} + \gamma\right) \binom{n}{k-1} \leq \sum_{x \in V} d(x) = \sum_{s \in \binom{n}{k-1}} d(S) < \frac{(1 + \gamma)n}{2} \binom{n}{k-1},$$

a contradiction. Therefore, there are at most

$$\frac{(\frac{\alpha}{12})}{(2(k - 1))} \binom{n}{k-1} \leq \frac{\alpha^2}{\gamma} \binom{n}{2}$$

$\alpha$-bad pairs in $H$. Hence, we have at most $4\alpha^2 n/\gamma$ vertices that are in at least $n/4$ $\alpha$-bad pairs. By Proposition 1.1 and greedy algorithm, there exists a collection $T_1$ of vertex disjoint copies of $K^k_n(m)$ covering these vertices. Clearly, $|V(T_1)| \leq 4\alpha^2 kmn/\gamma$. Let $H_1 = H[V \setminus V(T_1)]$. Note that for $x, y \in V(H_1)$ if $(x, y)$ is $\alpha$-good in $H$, then $(x, y)$ is $(\alpha/2)$-good in $H_1$. Moreover, each vertex in $H_1$ is in at least $2n/3$ $(\alpha/2)$-good pairs. By Lemma 1.5

$$|N^H_{K^k_n(m), 2}(x)| \geq 2n/3 > 2|H'|/3$$

for $x \in V(H_1)$. Therefore $H_1$ is $(K^k_n(m), 2, \eta)$-closed by Proposition 1.1 and Proposition 2.2.

**Step 2** Since $H_1$ is $(K^k_n(m), 2, \eta)$-closed, there exists a vertex set $U$ in $H_1$ satisfying the absorption lemma, Lemma 1.1. Set $H_2 = H[V(H_1) \setminus U]$.

**Step 3** By taking $\eta$ sufficiently small, $|V(T_1)| + |U| \leq \gamma n/2$. Thus,

$$\delta_1(H_2) \geq \delta_1(H_2) - \gamma \binom{n}{k-1}/2 \geq \begin{cases} 
\left(\frac{\alpha}{12} + \gamma\right) \binom{n}{2} & \text{if } k = 3 \\
\left(\frac{\alpha}{12} + \gamma\right) \binom{n}{4} & \text{if } k = 4.
\end{cases}$$
where $\gamma'' < (\eta/2)^{2kmn/32}$. By Lemma 5.3, there exists a set $T_2$ of vertex disjoint copies of $K^i_k(m)$ such that $|V(H_2)\setminus V(T_2)| < \gamma''|H_2|$.  

**Step 4** Set $W = V(H_2)\setminus V(T_2)$. Note that $H_3[U \cup W]$ contains a $K^j_k(m)$-factor $T_3$ by the choice of $U$ and $W$. Thus, $T_1 \cup T_2 \cup T_3$ is a $K^j_k(m)$-factor in $H$. □

**Remark 4.6.** With more work, we believe that one can determine the exact values of $t^j_k(n, K^i_k(m))$ for $k = 3, 4$ and $m \ge 2$.

## 5 Some lower bounds on $t^k(n,t)$

In this section, we give some lower bounds on $t^k(n,t)$ by constructions. First, we show that $t^k(n,k+1) \ge 2n/3$ for $k \ge 4$ even.

**Proposition 5.1.** For integers $k \ge 4$ even, $t^k(n,k+1) \ge 2n/3$.

**Proof.** We define a $k$-graph $H$ on $n$ vertices as follows. Partition $V(H)$ into three sets $V_1$, $V_2$ and $V_3$ of roughly the same size such that $|V_1| \neq |V_2|$ (mod 2). A $k$-set $S$ is an edge of $H$ if $|S \cap V_i|$ is odd for some $1 \le i \le 3$. Observe that $\delta_{k-1}(H) \ge 2n/3 - 1$. We now claim that $H$ does not contain a $K^j_{k+1}$-factor. Let $T$ be a $K^j_{k+1}$ in $H$. Observe that $|T \cap V_i|$ is odd for $1 \le i \le 3$. Otherwise we can deduce that for some $1 \le i \le 3$ $|T \cap V_i|$ is odd and $|T \cap V_j|$ is even for $j \neq i$, but $T \cap v$ is not an edge for $v \in T \cap V_i$. Recall that $|V_1| \neq |V_2|$ (mod 2), so $H$ does not contain a $K^j_{k+1}$-factor. □

Our next task is to prove Proposition 1.3 that is to show that $t^k(n,t) \ge (1 - C'\log t/t^2)$ for some constant $C'$, where we generalise a construction given in Proposition 1 of Pikhurko [20].

**Proof of Proposition 1.13.** Let $\lambda$ be an integer and let $c, \epsilon > 0$ be constants such that $(\lambda t)^{\epsilon (\log \lambda)t^{-\epsilon}} \le ct^2$. Let $l = R(B_{\lambda}, K^3_{k-1}) - 1$. By Theorem 1.17 $l = c_0\lambda^{c/2} / \log(\lambda t)$ for some constant $c_0$ independent of $t$.

Let $H_0$ be the 3-graph on vertex set $[l]$ with

$$\Delta_2(H_0) \le \lambda$$

which exists by the choice of $l$. Partition $[n]$ into $A_0$, $A_1$, $\ldots$, $A_l$ of size $a_0$, $a_1$, $\ldots$, $a_l$ such that $a_0 + a_1 + \cdots + a_l = n$, $a_0$ is odd and $a_0/\lambda$, $a_1$, $\ldots$, $a_l$ are nearly equal, that is, $|a_0/\lambda - a_i|, |a_i - a_j| \le 1$ for $1 \le i, j \le l$. Let $H$ be a 3-graph on vertex set $[n]$ with edges satisfying one of following (mutually exclusive) properties:

(a) lie inside $A_0$,

(b) have two vertices in inside $A_i$ and one in $A_0$ for $1 \le i \le l$,

(c) have one vertex in each of $A_{i_1}$, $A_{i_2}$ and $A_{i_3}$ with $i_1i_2i_3 \in E(H_0)$.

We claim that $\overline{H}$, the complement of $H$, does not contain a $K^3_k$-factor. Let $T$ be a $K^3_k$ in $\overline{H}$, so $T$ is an independent set in $H$. By (a), $|T \cap A_0| < 3$. If $|T \cap A_0| = 1$, then without loss of generality $|T \cap A_1| = 1$ for $1 \le i \le t$ by (b), which implies $[l]$ is an independent set in $H_0$ contradicting the fact that $\alpha(H_0) < t - 1$. Thus, every $K^3_k$ in $\overline{H}$ has even number of vertices in $A_0$ and so there is no $K^3_k$-factor in $\overline{H}$ as $|A_0| = a_0$ is odd. Also, it is easy to see that

$$\max_{j \in \{0,1\}} \sum_{i \in \mathbb{Z}} a_i \approx \Delta_2(H) = \frac{\lambda n}{\lambda + 1} + o(n) = \frac{n\log(\lambda t)}{\log(\lambda t) + c_0t^2} + o(n) \le \frac{C'\log t}{t^2}n$$

for some constant $C' > 0$. Hence, $\delta_2(\overline{H}) \ge (1 - C'\log t/t^2)n$ and so the proposition follows. □
6 $K^k_{k+1}$-factors

Here, we prove Theorem 1.23 which bounds $t^k(n, k + 1)$ from above for $k \geq 4$. We are going to show that if $H$ is a $k$-graph with $\delta_{k-1}(H) \geq \left(1 - \frac{k-1}{2k} + \gamma\right)n$, then $H$ contains a $K^k_{k+1}$-factor. The proof can be split into the two main steps:

(a) Showing $H$ is $(K^k_{k+1}, 1, \eta)$-closed.

(b) Finding an almost $K^k_{k+1}$-factor.

We are going to tackle (b) first. The almost $K^k_{k+1}$-factor that we are seeking will covering all but at most $t^k$ vertices of $H$. By adopting the proof of Theorem 2.1 of Fischer [7], we prove the following lemma.

Lemma 6.1. Let $3 \leq k < t$ be integers. Let $H$ be a $k$-graph of order $n$ with $\delta_{k-1}(H) \geq \left(1 - \frac{k-1}{\ell_{k-1}}\right)n$ and $|t|n$. There, there exists a set $T$ of vertex disjoint copies of $K^k_{k+1}$ in $H$ covering all but at most $t^k(t-1)$ vertices.

Proof. Consider a partition $P$ of the vertex set $V(H)$ into $t$-sets $V_1, \ldots, V_t$. Let $G_i$ be the largest complete graph in $V_i$, where we allow independent sets of size $k - 1$ to be a complete graph. Denote by $w : \{0, \ldots, t\} \to \mathbb{R}$ the function defined by $w(0) = w(1) = 0$ and $w(j + 1) - w(j) = 1 - \frac{t-i}{(t+1)!}$. We define the weighting $w(P)$ of $P$ to be $\sum_{1 \leq i \leq t} w(|G_i|)$. Assume that $P$ is chosen such that $w(P)$ is maximal. We are going to show that for each $0 \leq i \leq t-1$ there are at most $t-i$ vertices in $P$.

Thus, there are at least $(t-i)n$ vertices $v$ such that $(v, j)$ is a connection. Thus, there are at least $(t-i)n$ connections $(v, j)$ with $v \notin V_1 \cup \cdots \cup V_t$ and $1 \leq j \leq t$. There exists $t < j' \leq n/(t)$ such that there are more than $t(t-i)$ connections $(v', j)$ for $v' \in V_{j'}$ and $1 \leq j' \leq t$. Without loss of generality, we may assume by the König-Egerváry Theorem (see [2] Theorem 8.32) that $(v', j)$ is a connection for distinct $v' \in V_{j'}$ and $1 \leq j' \leq t - i + 1$. By moving $v$ to $V_{j'}$, we may assume that $v'$ is in $V_{j'}$ for $1 \leq j \leq t - i + 1$ and $\{v_1, \ldots, v_t\} \to V_{j'}$, $w(P)$ increases by

$$(t-i+1)(w(i + 1) - w(i)) - (w(|G_i|) - w(\max\{k-1, |G_i'| - t + 1 + i\}))$$

$$\geq (t-i+1)\left(1 - \frac{(t-i+1)!}{(t+1)!}\right) - \left(t + i - \frac{t-i+1}{t}\right)$$

$$= \frac{(t-i+1)!}{(t+1)!} > 0,$$

a contradiction. \(\square\)

Next, we are going to verify (a), that is, show that $H$ is $(K^k_{k+1}, 1, \eta)$-closed.
Lemma 6.2. Let \( k \geq 3 \) be an integer and \( \gamma > 0 \). Let \( H \) be a \( k \)-graph of order \( n \geq n_0 \) with
\[
\delta_{k-1}(H) \geq \left( 1 - \frac{k + 1_{k, \text{odd}}}{2k^2} + \gamma \right)n.
\]
where \( 1_{k, \text{odd}} = 1 \) if \( k \) is odd and \( 1_{k, \text{odd}} = 0 \) otherwise. Then, \( H \) is \((K_{k+1}, 1, \eta)\)-closed for some \( \eta > 0 \).

Proof. Let \( x \) and \( y \) be distinct vertices of \( H \). Let \( G \) be the \((k-1)\)-graph on vertex set \( V(H) \setminus \{x, y\} \) and edge set \( N(x) \cap N(y) \), so
\[
\delta_{k-2}(G) \geq \left( 1 - \frac{k + 1_{k, \text{odd}}}{k^2} + 2\gamma \right)n > \left( 1 - \frac{1}{k-1} + \gamma \right)n. \tag{5}
\]
Hence, we can find a \( k \)-set \( T = \{v_1, \ldots, v_k\} \) in \( V(G) \) that forms a \( K_{k-1}^h \) in \( G \). In fact, there are at least \( C\gamma n^k \) choices for some constant \( C > 0 \) independent of \( \gamma \) and \( n \). For \( u \in V(G) \setminus T \), we claim that if \( u \) is in more than \( \lfloor k(k-2)/2 \rfloor \) neighbourhoods \( N^G(S) \) for \( S \in \binom{T}{k-2} \), then \( u \cup T \setminus v_i \) forms a \( K_{k-1}^h \) for some \( 1 \leq i \leq k \). Indeed, this is true by considering the 2-graph \( G' \) on \( T \) such that \( v_i, v_j \in E(G'_u) \) if and only if \( u \in N^G(T \setminus \{v_i, v_j\}) \). Note that \( v_i \) has degree \( k-1 \) in \( G'_u \) and if only if \( u \cup T \setminus v_i \) forms a \( K_{k-1}^h \) in \( G \). Therefore, if \( u \) is in more than \( \lfloor k(k-2)/2 \rfloor \) neighbourhoods \( N^G(S) \) for \( S \in \binom{T}{k-2} \), then \( e(G'_u) > k(k-2)/2 \) and so \( G'_u \) contains a vertex \( v_i \) of degree at least \( k-1 \) in \( G'_u \), i.e. \( u \cup T \setminus v_i \) forms a \( K_{k-1}^h \) for some \( 1 \leq i \leq k \) as claimed.

Thus, there exists a vertex \( v_i \in T \) such that \( u \cup T \setminus v_i \) forms a \( K_{k-1}^h \) in \( G \) for at least
\[
\frac{1}{k} \left( \begin{array}{c} k \\ k-2 \end{array} \right) \delta_{k-2}(G) - \frac{k(k-2) - 1_{k, \text{odd}}}{2} \leq \left( \frac{k + 1_{k, \text{odd}}}{2k^2} + (k-1)\gamma \right)n
\]
vertices \( u \in V(G) \setminus T \) by (5). Fix such \( i \) and let \( U \) be the set of such corresponding \( u \). Note that \( |N^H(T \setminus v_i) \cap U| \geq k\gamma n \). Moreover, for each \( z \in N^H(T \setminus v_i) \cap U \), \( \{x, z\} \cup T \setminus v_i \) and \( \{y, z\} \cup T \setminus v_i \) form \( K_{k+1}^h \) in \( H \). Thus, \( x \) is \((K_{k+1}, 1, \eta)\)-closed to \( y \) for \( \eta = C\gamma n^k/k > 0 \). Since \( x \) and \( y \) are arbitrary, \( H \) is \((K_{k+1}, 1, \eta)\)-closed as required.

We are now ready to prove Theorem \[.4\] Note that the second assertion of the theorem is implied by Proposition \[.1\] so it is enough to prove the first assertion.

Proof of Theorem \[.4\]. Let \( k \geq 4 \) be an integer and \( \gamma > 0 \). Let \( H \) be a \( k \)-graph of order \( n \) with
\[
\delta_{k-1}(H) \geq \left( 1 - \frac{k + 1_{k, \text{odd}}}{2k^2} + \gamma \right)n
\]
and \( k + 1|n \). Throughout this proof, we may assume that \( \gamma \) and \( n \) are sufficiently small and large respectively. Note that \( H \) is \((K_{k+1}, 1, \eta)\)-closed by Lemma \[.2\]. Let \( U \) be the vertex set given by Lemma \[.3\] and so \( |U| \leq (\eta/2)^{k+1}n/4 \). Let \( H' = H[V \setminus U] \). Note that
\[
\delta_{k-1}(H') \geq \left( 1 - \frac{k + 1_{k, \text{odd}}}{2k^2} + \frac{\gamma}{2} \right)n' \geq \left( 1 - \frac{1}{k+1} \right)n'
\]
where \( n' = n - |U| \). There exists a family \( \mathcal{T} \) of vertex disjoint copies of \( K_{k+1}^h \) in \( H' \) covering all but at most \((k+1)^2\) vertices by Lemma \[.4\] taking
Let \( W = V(H^+) \setminus V(T) \), so \(|W| < (k + 1)^2 < (\eta/2)^{2k+2} n/32\). By Lemma 7.1 there exists a \( K^k_{k+1} \)-factor \( T' \) in \( H[U \cup W] \). Thus, \( T \cup T' \) is a \( K^k_{k+1} \)-factor in \( H \).

### 7 \( K^k_t \)-factors

For integers \( k, t, l \), define the function \( \beta(k, t, l) \) as

\[
\beta(k, t, l) = \frac{2}{(k-1) + (t-1) + (t+1) \choose k-1}.
\]

Set \( l_0(3, t) = 1 \) and for \( k \geq 4 \), let \( l_0(k, t) \) be the largest integer \( l \) such that

\[
2(t + 1) \leq t, \beta(k, t, l) \leq \frac{1}{(k-1) + (t-1) + (t+1) \choose k-1} \quad \text{and} \quad \beta(k, t, l) \leq \frac{1}{2} \frac{2(t+1)}{k-1}
\]

if \( \binom{2(t+1)}{k-1} > 0 \). Note that \( l_0(k, t) \geq \lceil \min\{k, t - 2\}/2 \rceil \geq 1 \) for \( k \geq 4 \). Set

\[
\beta(k, t) = \beta(k, t, l_0(k, t)) \quad \text{and} \quad d(k, t) = 1 - \left( \frac{t - 1}{k - 1} \right) \beta(k, t).
\]

The aim of this section is to show that if \( \delta_{k-1}(H) \geq (1 - \beta(k, t) + \gamma)n \) with \( \gamma > 0 \) and \( n \) is sufficiently large, then \( H \) is \( (K^k_t, i, \eta) \)-closed for some \( i, \eta > 0 \) independent of \( n \).

**Lemma 7.1.** Let \( 3 \leq k < t \) be integers and \( 0 < \gamma < \beta(k, t) \). Let \( H \) be a \( k \)-graph of order \( n \geq n_0 \) with

\[
\delta_{k-1}(H) \geq (1 - \beta(k, t) + \gamma)n.
\]

Then, \( H \) is \( (K^k_t, i, \eta) \)-closed for some \( i \) and \( \eta > 0 \).

Observe that this lemma implies Theorem 1.4 by modifying the proof of Theorem 1.3. Thus, it is enough to prove Lemma 7.1. We advise the reader to consider the special case when \( k = 3 \) and \( t = 4 \) which illustrate all the ideas in the proof. We now give an outline of its proof. Let \( H \) be a \( k \)-graph satisfying the hypothesis of the above lemma. Lemma 7.3 shows that we can partition the vertex set \( V(H) \) into \( W_1, \ldots, W_p \) such that \( W_j \) is \((K^k_t, i', \eta')\)-closed in \( H \) for \( 1 \leq j \leq p \) with \( i', \eta', p \geq 0 \) independent of \( n \). Then, we ‘glue’ \( W_1, \ldots, W_p \) together using Lemma 7.3.

First, we show that \(|N_{K^k_{k+1}, \eta}(v)|\) is large for every \( v \in V(H) \).

**Proposition 7.2.** Let \( 3 \leq k < t \) be integers and \( 0 < \gamma < \beta(k, t) \). Let \( H \) be a \( k \)-graph of order \( n \) with \( \delta_{k-1}(H) \geq (1 - \beta(k, t) + \gamma)n \). Then, \(|N_{K^k_{k+1}, \eta}(v)| \geq (d(k, t) + 2\gamma)n \) for \( v \in V(H) \), where

\[
\eta = \frac{\gamma}{2(t - 1)!} \prod_{k-1 \leq s \leq t-1} \left( 1 - \frac{s}{k - 1} \right) \beta.
\]

**Proof.** Fix \( k \) and \( t \). Write \( \delta, \beta \) and \( d \) for \( \delta_{k-1}(H) \), \( \beta(k, t) \) and \( d(k, t) \) respectively. Recall that \( L(S) = \bigcap_{U \in \binom{V}{k-1}} N(U) \) for \( s \)-sets \( S \) and \( s \geq k-1 \). Thus,

\[
|L(S)| \geq n - \left( \frac{s}{k-1} \right) (n - \delta) \geq 1 - \left( \frac{s}{k-1} \right) \beta + \left( \frac{s}{k-1} \right) \gamma n > 0
\]

for
for $s$-sets $S$ and $k - 1 \leq s \leq t - 1$. Hence, each vertex $v$ is contained in at least

$$\frac{n^{k-1}}{2(t-1)!} \prod_{k-1 \leq s \leq t-1} \left(1 - \frac{s}{k-1}\right)$$

copies of $K_t^k$. Fix $x \in V(H)$. Let $G_x$ be a bipartite graph with the following properties. The vertex classes of $G_x$ are $V(H)$ and $W_x$, where $W_x$ is the set of $(t-1)$-sets $T$ in $V(H)$ such that $T \cup x$ forms a $K_t^k$. For $y \in V(H)$ and $T \in W_x$, $(y,T)$ is an edge in $G_x$ if and only if $T \cup y$ forms a $K_t^k$. Note that for $T \in W_x$

$$d^G(T) = |L(T)| \geq (d + 3\gamma)n.$$

We claim that there are greater than $(d + 2\gamma)n$ vertices $y \in V(H)$ with $d^G(y) \geq \gamma|W_x|$. Indeed, it is true or else

$$(d + 3\gamma)n|W_x| \leq e(G) \leq \gamma|W_x|(1 - d - 2\gamma)n + |W_x|(d + 2\gamma)n,$$

a contradiction. The proposition follows as $y$ is $(K_t^k, 1, d^G(y)/n^{t-1})$-close to $x$.

Now, we show that we can partition the vertex set $V(H)$ into $W_1, \ldots, W_p$ such that each $W_j$ is $(K_t^k, \tilde{i}, \eta)$-closed in $H$.

**Lemma 7.3.** Let $3 \leq k < t$ be integers and $0 < \gamma < \beta(k,t)$. Let $H$ be a $k$-graph of order $n \geq n_0$ with $\delta_{k-1}(H) \geq (1 - \beta(k,t) + \gamma)n$. Then, there exists a vertex partition of $V(H)$ into $W_1, \ldots, W_p$ with

$$|W_j| \geq (d(k,t) + \gamma)n$$

and each $W_j$ is $(K_t^k, 2^{i+1}/(2^{i+2}d^2(k,t)\gamma^{i+1}), \eta)$-closed in $H$ for some $p \leq (d(k,t) + \gamma)^{-1}$, $i \leq 2 - \log_2 d(k,t)$ and $\eta > 0$.

**Proof.** Fixed $k$ and $t$. Write $\delta$, $\beta$ and $d$ for $\delta_{k-1}(H)$, $\beta(k,t)$ and $d(k,t)$ respectively. Let $\eta_0 = \eta$ that defined in Proposition [22]. We further assume that $\gamma < d^{2^{i+1}/(2^{i+2}d\log_2 d)}$. For $1 \leq i \leq 2 - \log_2 d$, let

$$\eta_i = \frac{\eta_0^2 - 1}{(2^{i+1})!} \geq \frac{\eta_0^2 - 1}{(2^{i+1})!} \text{ and } n_0 = \left[\eta_0^{2^{i+1}/(2^{i+2}d\log_2 d)}\right].$$

We say $i$-close to mean $(K_t^k, 2^i, \eta_i)$-close and write $\tilde{N}_i(v)$ for $\tilde{N}_{K_t^k, 2^i, \eta_i}(v)$, so $\tilde{N}_i(v) \geq (d + 2\gamma)n$ by Proposition [22]. Moreover, $\tilde{N}_{i+1}(v) \subset \tilde{N}_i(v)$. Set $V^0 = V$. Iteratively for $i \geq 0$ and $V^i \neq \emptyset$, we partition $V^i$ into $W_1^i, \ldots, W_{n_0}^i, V^{i+1}$ such that:

(i) $|W_j^i| \geq (2d + \gamma)n$ for $1 \leq j \leq r_i$,

(ii) $W_j^i$ is $(i + 1/(2^{i-2}d\gamma)) + 1)$-closed in $H$ for $1 \leq j \leq r_i$,

(iii) $|\tilde{N}_i(v) \cap V^{i+1}| \geq (2^{i+1}d + 2\gamma)n$ for $v \in V^{i+1}$.

First, assume that the iteration is valid. Since $\sum |W_j^i| \leq n$, $r_i < 1/(2d)$ by properties (i). In addition, by (iii) the iteration must terminate after $i$ steps, i.e. $V^{i+1} = \emptyset$, for $i < - \log_2 d - 1$. Therefore, it is enough to show the algorithm is valid. Suppose that we are given $V^i$ for $i \geq 0$. We are going to partition $V^i$ into $W_1^i, \ldots, W_{n_0}^i, V^{i+1}$ satisfying properties (i)–(iii) and the following additional property:

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Recall that \(|\tilde{N}_i(v) \cap V_j^i| \geq (2d + (2 - 2^{-1}d)\gamma)n\) for \(v \in V_j^i\), where \(V_j^i = V_i^j \cup_{i' \leq j+1} W_j^i\).

Further suppose that we have already constructed \(W_1^i, \ldots, W_j^i\). Recall that \(r_i \leq 1/(2d)\), so by property (iv)

\[|\tilde{N}_i(u) \cap V_j^i| \geq (2d + 3\gamma/2)n\]

for \(v \in V_j^i\). If \(|\tilde{N}_{i+1}(v) \cap V_j^i| \geq (2d + 2\gamma)n\) for all \(v \in V_j^i\), then set \(V_j^i = V_j^{i+1}\) and \(r_i = j\). Thus, we may assume that there exists \(v \in V_j^i\) such that \(|\tilde{N}_{i+1}(v) \cap V_j^i| < (2d + 2\gamma)n\).

Let \(U\) be the set of vertices \(u \in V_j^i \cap \tilde{N}_i(v)\) such that

\[|\tilde{N}_i(u) \cap \tilde{N}_{i+1}(v) \cap V_j^i| \geq (2d + 3\gamma/3)n.\]

Claim 7.4. The size of \(U\) is at least \((2d + \gamma)n\) and \(U\) is 2-closed in \(H\).

Proof of claim. Note that if \(|\tilde{N}_i(u) \cap \tilde{N}_i(v)| \geq \gamma^2/8\), then \(w \in \tilde{N}_{i+1}(v)\) by Proposition 2.1. Thus, for each \(w \in V \setminus \tilde{N}_{i+1}(v)\), \(|\tilde{N}_i(v) \cap \tilde{N}_i(w)| < \gamma^2/18\). Therefore, by summing over all \(w \in V \setminus \tilde{N}_{i+1}(v)\)

\[
\sum_{w \in \tilde{N}_i(v)} |\tilde{N}_i(u) \cap \tilde{N}_i(v)| = \sum_{w \in \tilde{N}_i(v) \cap \tilde{N}_i(w)} |\tilde{N}_i(v) \cap \tilde{N}_i(w)|
\]

\[\leq \gamma^2|V \setminus \tilde{N}_{i+1}(v)|/\gamma^2 = \gamma^2/2.\] (6)

Recall that \(|\tilde{N}_i(u') \cap V_j^i| \geq (2d + 3\gamma/2)n\) for \(u' \in V_j^i\). For \(u' \in V_j^i \cap \tilde{N}_i(v)/U\)

\[|\tilde{N}_i(u') \cap \tilde{N}_{i+1}(v)| \geq |(\tilde{N}_i(u') \cap \tilde{N}_{i+1}(v)) \cap V_j^i| = |\tilde{N}_i(u') \cap V_j^i| - |\tilde{N}_i(u') \cap \tilde{N}_{i+1}(v)| \cap V_j^i| > \frac{2n}{6}.
\]

Therefore, by summing over \(u' \in V_j^i \cap \tilde{N}_i(v)/U\) and (6)

\[
\frac{7n}{3} |V_j^i \cap \tilde{N}_i(v)/U| \leq \sum_{u' \in V_j^i \cap \tilde{N}_i(v)/U} |\tilde{N}_i(u') \cap \tilde{N}_{i+1}(v)|
\]

\[\leq \sum_{u \in \tilde{N}_i(v)} |\tilde{N}_i(u) \cap \tilde{N}_{i+1}(v)| \leq \gamma^2/18.
\]

Again recall that \(|\tilde{N}_i(u) \cap \tilde{N}_i(v)| \geq (2d + 3\gamma/2)n\), so \(|U| \geq (2d + \gamma)n\) as desired. Furthermore, for \(u, u' \in U\), \(\tilde{N}_i(u) \cap \tilde{N}_i(u')\) is of size at least

\[|\tilde{N}_i(u) \cap \tilde{N}_{i+1}(v) \cap V_j^i| + |\tilde{N}_i(u') \cap \tilde{N}_{i+1}(v) \cap V_j^i| - |\tilde{N}_{i+1}(v) \cap V_j^i| \geq 2\gamma n/3
\]

and so \(u\) and \(u'\) are \((i + 1)\)-close to each other by Proposition 2.1. □

Set \(U_0 = U\). For integers \(i' > 0\), we define \(U_{i'}\) to be the set of vertices \(u' \in V_j^i \cup_{p=0}^{i-1} U_p\) such that \(|\tilde{N}(u') \cap \bigcup_{p=0}^{i-1} U_p| \geq 2^{i-2}d\gamma n\).

By induction on \(i',\) together with Proposition 2.1 and Proposition 2.2, we deduce that \(\bigcup_{p=0}^{i'} U_p\) is \((i + i' + 1)\)-closed in \(H\). Let \(i_0 \geq 1\) be the minimal integer such that \(|U_{i'}| < 2^{-2}d\gamma n\). Clearly \(i_0 \leq \lfloor 1/(2^{i-2}d\gamma)\rfloor\) as \(U_0, U_1, \ldots\) are pairwise disjoint.

We set \(W_{j+1}^i = \bigcup_{p=0}^{i_0} U_p\). Note that \(|W_{j+1}^i| \geq |U_0| \geq (2d + \gamma)n\) and \(W_{j+1}^i\) is \((i + i_0 + 1)\)-closed in \(H\). Hence, \(W_{j+1}^i\) satisfies properties (i)
and (ii). If $W_{j+1}^i = V_j^i$, then we are done. Thus, we may assume that $W_{j+1}^i \neq V_j^i$. For $w \in V_{j+1}^i \setminus V_j^i$, we have by (iv)

$$\left| \tilde{N}(w) \cap V_{j+1}^i \right| \geq \left| \tilde{N}(w) \cap V_j^i \right| - \left| \tilde{N}(w) \cap \bigcup_{p=0}^{i-1} U_p \right| - |U_{n|}$$

$$(2d \left| \frac{2-2^{-1}d}{} \right| n - 2^{-2}d \gamma n - 2^{-2}d \gamma n$$

and so $V_{j+1}^i$ satisfies (iv). Thus, the algorithm is valid and the proof of the lemma is completed.

The next lemma helps us to join two of $W_i$’s together.

**Lemma 7.5.** Let $3 \leq k < t$ be integers and $0 < \gamma < \beta(k, t)$. Then, there exist an integer $n_0 = n_0(k, t, \gamma)$ and $\varepsilon_0 = \varepsilon_0(k, t, \gamma) > 0$ such that the following holds. Let $H$ be a $k$-graph of order $n \geq n_0$ with $\delta_{k-1}(H) \geq (1 - \beta(k, t) + \gamma)n$. Let $X$ and $Y$ be a partition of $V(H)$ with $|X|, |Y| \geq (\ell(k, t) + \gamma)n$, where $\ell(k, t)$ is defined in Lemma 7.3. Then, there exist at least $\varepsilon_0 n^{t+1}$ triples $(x, y, T)$ with $x \in X$, $y \in Y$, $T \in V_{k-1}$ such that both $x \cup T$ and $y \cup T$ form $K^k_T$ in $H$.

**Proof.** Fix $k$, $t$ and $\gamma$. Write $\delta$, $\beta$, $d_l$ and $l_0$ for $\delta_{k-1}(H)$, $\beta(k, t)$, $d(k, t)$ and $l_0(k, t)$ respectively. Set $\varepsilon = \gamma(\frac{t}{k-1})/3\ell$ and

$$\varepsilon_0 = \min \left\{ C\ell \cdot \frac{\varepsilon d^2}{2(t-1)} \right\},$$

where $C$ to be a small constant specified later. Without loss of generality, we may assume that $|Y| \geq |X| \geq (d + \gamma)n$. We say that a triple $(x, y, T)$ is good if $x \in X$, $y \in Y$, $T \in V_{k-1}$ such that both $x \cup T$ and $y \cup T$ form $K^k_T$ in $H$. Thus, we are going to show that there are at least $\varepsilon_0 n^{t+1}$ good triples. We now consider various cases.

**Case 1:** There exist $Cn^{t-1}$ copies of $K^k_{t-1} T$ such that $|L(T) \cap X| \geq \varepsilon n$ and $|L(T) \cap Y| \geq \varepsilon n$.

For each such $T$, $(x, y, T)$ is good with $x \in L(T) \cap X$, $y \in L(T) \cap Y$. Therefore, there are at least $C^2 n^{t+1}$ good triples.

**Case 2:** For some $l_0 \leq s \leq t/2$, there exist $2Cn^{t-1}$ copies of $K^k_{t-1} T$ such that $|T \cap Y| = s$ and $|L(T) \cap Y| > \varepsilon n$. By Case 1, we may assume that there exist $Cn^{t-1}$ of such $K^k_{t-1} T$ with $|L(T) \cap X| \leq \varepsilon n$. Fixed one such $T = \{v_1, \ldots , v_{r-1}\}$ with $T_X = \{v_1, \ldots , v_r\} \subset X$ and $T_Y = \{v_{r+1}, \ldots , v_{r+s}\} \subset Y$, where $r + s + 1 = t$. Fix $z \in L(T) \cap Y$. Recall that $L(T) \geq (d + 3\gamma)n$. Let $L = L(T) \cap Y \setminus z$, so

$$|L| = |L(T)| - |L(T) \cap X| - 1 \geq (d + \gamma)n - \varepsilon n \geq (d + \varepsilon)n.$$

**Claim 7.6.** One of the followings holds:

(a) $|X \cap L(T \cup z \setminus v_i)| \geq \varepsilon n$ for some $1 \leq i \leq r$, or

(b) $|X \cap L(T \cup z \setminus v_{r+j})| \geq \varepsilon n$ for some $1 \leq j \leq s$.  

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Proof of claim. Let \( M \) be the set of vertices \( v \in V \setminus (T \cup z) \) such that \( v \) is in the neighbourhoods of exactly \( \binom{k-2}{i} \) \((k-1)\)-sets \( S \in \binom{T}{k-1} \). For \( 1 \leq i \leq \binom{k-2}{i} \), let \( n_i \) be the number of vertices \( v \) such that \( v \) is in the neighbourhoods of exactly \( i \) \((k-1)\)-sets \( S \in \binom{T}{k-1} \). Note that \( \sum n_i = n \), \( \sum in_i \geq \binom{k-2}{i} \delta \), \(|M| = n \binom{k-2}{i} \) and \( n \binom{k-2}{i} \leq |L| + \varepsilon n + 3 \). Therefore,

\[
2|L| + |M| + 3\varepsilon \geq \sum in_i - \left( \binom{t-1}{k-1} - 2 \right) \sum n_i \\
\geq 2n - \left( \frac{t-1}{k-1} \right)(n - \delta). \tag{7}
\]

Denote by \( R \) and \( S \) be the sets \( M \cap X \cap L(T_X) \) and \( M \cap Y \cap L(T_Y) \) respectively. Thus, we have

\[
|R| + |S| \geq |M| - \left( \binom{r}{k-1} + \binom{s}{k-1} \right)(n - \delta). \tag{8}
\]

Set \( S_1 = S \cap L(T_{v_1}) \) and for \( 2 \leq i \leq r \), set \( S_i = S \cap L(T_{v_i}) \setminus \bigcup_{j<i} S_j \). Observe that \( S_1, \ldots, S_r \) form a partition of \( S \). Note that

\[
|S_1 \cap \bigcap_{u \in \binom{T_{v_1}}{k-2}} N(z \cup U)| \geq \sum_{u \in \binom{T_{v_1}}{k-2}} |S_1 \cap N(z \cup U)| - \left( \frac{t-2}{k-2} - 1 \right)|S_1|. \tag{10}
\]

The left hand side is at most \(|Y \cap L(T \cup z \setminus v_1)|\), so we may assume that it is less than \( \varepsilon n \) or else (a) holds for \( i = 1 \). Similarly, we may assume that for \( 1 \leq i \leq r \)

\[
\sum_{u \in \binom{T_{v_1}}{k-2}} |S_i \cap N(z \cup U)| - \left( \frac{t-2}{k-2} - 1 \right)|S_i| \leq \varepsilon n. \tag{9}
\]

Next, we apply similar arguments on \( R \). Set \( R_1 = R \cap L(T_{v_{r+1}}) \) and for \( 2 \leq j \leq s \), \( R_j = R \cap L(T_{v_{r+1}}) \setminus \bigcup_{i<j} R_i \). Again, \( R_1, \ldots, R_s \) form a partition of \( R \). Similarly, we may assume that for \( 1 \leq j \leq s \),

\[
\sum_{u \in \binom{T_{v_{r+1}}}{k-2}} |R_j \cap N(z \cup U)| - \left( \frac{t-2}{k-2} - 1 \right)|R_j| \leq \varepsilon n. \tag{10}
\]

or else (b) holds. Let \( L' \) to be the set \( L \cap L(T_Y \cup z) \), so

\[
|L'| = |L \cap \bigcap_{u \in \binom{T_{v_1}}{k-2}} N(z \cup U)| \geq |L| - \sum_{u \in \binom{T_{v_1}}{k-2}} |L \setminus N(z \cup U)| \geq |L| - \left( \frac{s}{k-2} \right)(n - \delta) \\
\geq |L| - \sum_{u \in \binom{T_{v_1}}{k-2}} \left( \sum_{1 \leq i < r} |S_i \setminus N(z \cup U)| + \sum_{1 \leq j \leq s} |R_j \setminus N(z \cup U)| \right). \tag{11}
\]

In addition, for \( U \in \binom{T_{v_1}}{k-2} \), we have

\[
|L' \cap N(z \cup U)| \geq |L'| + \delta - n + \sum_{1 \leq i < r} |S_i \setminus N(z \cup U)| + \sum_{1 \leq j \leq s} |R_j \setminus N(z \cup U)|. \tag{12}
\]
We claim that $2|L \cap \bigcup_{1 \leq i \leq r} L(T \cup z \setminus v_i)|$ is at least

$$\sum_{U \in (T_{k-1}) \setminus (T_{k-2})} |L' \cap N(z \cup U)| - \left(\frac{t-1}{k-2} + \frac{s}{k-2}\right)(n-\delta).$$

(13)

Note that this is twice the number of vertices $y \in L'$ such that $y$ is in the neighbourhood of $N(z \cup U)$ for all but at most one $U \in (T_{k-2}) \setminus (T_{k-2})$. Let $y$ be such a vertex. In order to show that (13) holds, it is enough to show that $y \in L \cap L(T \cup z \setminus v_i)$ for some $1 \leq i \leq r$. Let $U$ be the set of $U \in (T_{k-2})$ such that $y \in N(z \cup U)$. Note that $(T_{k-2}) \subset U$ as $y \in L'$ and so $|U| \geq (\binom{k-1}{2} - 1)$. Define $A$ to be the family of sets in $T_X$ such that if $A \in A$ then $A \cup B \in U$ for every $B \in (T_{k-2} \setminus A)$. Since $U$ misses at most one set in $(T_{k-2})$, $A$ contains all but at most one subsets $A_0$ in $T_X$, say $v_1 \in A_0$ if exists. Therefore, $A$ contains all subsets of $\{v_2, \ldots, v_r\}$. Moreover, the set $\{v_2, \ldots, v_{r-1}, z, y\}$ forms a $K^*_n$ and so holds.

Finally, by (13), (12), (11), (9), (10), (8) and (7), we can deduce that

$$2|L \cap \bigcup_{1 \leq i \leq r} L(T \cup z \setminus v_i)| \geq 2|L| - \left(\frac{t-1}{k-2} + \frac{s}{k-2}\right)(n-\delta)$$

$$+ \sum_{U \in (T_{k-2}) \setminus (T_{k-3})} \left(\sum_{1 \leq i \leq r} |S_i \setminus N(z \cup U)| + \sum_{1 \leq j \leq s} |R_j \setminus N(z \cup U)|\right)$$

$$\geq 2|L| + |S| + |R| - \left(\frac{t-1}{k-2} + \frac{s}{k-2}\right)(n-\delta) - (t-1)\epsilon n$$

$$\geq (2 - (t + 2)\epsilon) n - \left(\frac{t}{k-1} + \frac{s+1}{k-1}\right)(n-\delta)$$

$$\geq (2 - (t + 2)\epsilon) n - \left(\frac{t}{k-1} + \frac{1}{k-1}\right)(n-\delta)$$

$$\geq 2(t - 1)\epsilon n,$$

and so (a) holds for some $1 \leq i \leq r$. Hence, the proof of Claim 7.6 is completed.

Suppose that Claim 7.6(a) holds for $i = 1$ say, then $|Y \cap L(T \cup z \setminus v_1)| \geq \epsilon n$. The triple $(v_1, y, T')$ is a good triple for $y \in Y \cap L(T \cup z \setminus v_1)$. Similarly, if Claim 7.6(b) holds for $j = 1$ say, then $(x, n_{r+1}, T')$ is a good triple for $x \in X \cap L(T \cup z \setminus v_{r+1})$. Thus, for each such $T$ and $z$, we can find at least $\epsilon n$ good triples $(x, y, T')$ (counting with multiplicities) such that $T \cup z \subset T' \cup \{x, y\}$. Therefore, there are at least

$$\frac{Cn^{t-1} \times d n \times c n}{2(t-1)} \geq \epsilon_0 n^{t+1}$$

good triples in $H'$.

Case 3: For some $l_0 \leq r < t/2$, there exist $2Cn^{t-1}$ copies of $K^*_n$ in $T$ such that $|T \cap X| = r$ and $|L(T) \cap X| > \epsilon n$. Notice that this is the symmetrical case of Case 2. Since we do not require the fact $|X| < |Y|$ in the arguments in Case 2, we can deduced that there are at least $\epsilon_0 n^{t+1}$ good triples in $H$. 

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Case 4: We have \( k = 3 \) and there exist \( 2C^{t-1} \) copies of \( K^3 \) T such that \( T \subseteq X \) and \( |L(T) \cap Y| > cn \). By Case 1, we may assume that there exist \( Cn^{t-1} \) copies of \( T \) with \( |L(T) \cap X| \leq cn \). Fixed one such \( T = \{ v_1, \ldots, v_{t-1} \} \) and \( z \in L(T) \cap Y \). Recall that \( |L(T)| \geq (d+3 \gamma)n \). Let \( L = L(T) \cap Y \cap z \), so
\[
|L| \geq |L(T)| - cn - 1 \geq (d + \epsilon)n.
\]
Similarly to Case 2, the following claim would imply that there are at least \( r_0n^{t+1} \) good triples in \( H \).

Claim 7.7. For some \( 1 \leq i \leq t-1 \), \( |Y \cap L(T \cup z \setminus v_i)| \geq cn \).

Proof of claim. Let \( M \) be the set of vertices \( v \in V \setminus (T \cup \{ z \}) \) such that \( v \) is in the neighbourhoods of exactly \( \binom{t+1}{2} - 1 \) 2-sets \( S \in \binom{X}{2} \). For \( 1 \leq i \leq \binom{t+1}{2} - 1 \), let \( n_i \) be the number of vertices \( v \) such that \( v \) is in the neighbourhoods of exactly \( i \) 2-sets of \( S \in \binom{T}{2} \). Note that \( \sum n_i = n \), \( \sum n_i \geq \binom{t+1}{2} \delta \), \( |M| = n_{(t+1) - 1} \) and \( n_{(t+1) - 1} \leq |L| + cn + 3 \). Therefore,
\[
2|L| + |M| + 3 \epsilon \geq 2n - \left( \frac{t-1}{2} \right)(n - \delta). \tag{14}
\]
Let \( S = M \cap Y \), so \( |S| \geq |M| - n/2 \). Set \( S_1 = S \cap L(T \setminus v_1) \) and for \( 2 \leq i \leq t-1 \), \( S_i = S \cap L(T \setminus v_i) \setminus \bigcup_{j<i} S_j \). Observe that \( S_1, \ldots, S_{t-1} \) form a partition of \( S \). Note that
\[
|S_i \cap \bigcap_{u \in T \setminus v_1} N(zu)| \geq \sum_{u \in T \setminus v_1} |S_i \cap N(zu)| - (t - 3) |S_i|.
\]
Observe that \( S_i \cap \bigcap_{u \in T \setminus v_1} N(zu) \subset L(T \cup z \setminus v_1) \). Therefore, we may assume that the left hand side of the above inequality is less than \( cn \) or else the claim holds for \( i = 1 \). Similarly, we may assume that for \( 1 \leq j \leq t-1 \)
\[
\sum_{u \in T \setminus v_j} |S_j \cap N(zu)| - (t - 3) |S_j| \leq cn. \tag{15}
\]
It is easy to see that for \( u \in T \),
\[
|L \cap N(zu)| \geq |L| + \delta - n + \sum_{1 \leq i \leq t-1} |S_i \setminus N(zu)| \tag{16}
\]
Note that
\[
2|L \cap \bigcup_{1 \leq j \leq t-1} L(T \cup z \setminus v_j)| \geq \sum_{u \in T \setminus v_1} |L \cap N(zu)| - (t - 3) |L|.
\]
Thus, by (15), (16) and (14), we have
\[
\geq 2|L| - (t - 1)(n - \delta) + \sum_{1 \leq i \leq t-1} \sum_{u \in T \setminus v_1} |S_i \setminus N(zu)|
\geq 2|L| - (t - 1)(n - \delta) + \delta - (t - 1)cn
\geq 2|L| + |M| - (t - 1)(n - \delta) - n/2 - (t - 1)cn
\geq (3/2 - (t + 2) \epsilon)n - \left( \frac{t}{2} \right)(n - \delta) \geq 2(t - 1)cn.
\]
Hence, \( |Y \cap L(T \cup z \setminus v_i)| \geq |L \cap L(T \cup z \setminus v_i)| \geq cn \) for some \( 1 \leq i \leq t-1 \).

The proof of Claim 7.7 is completed.
Case 5: All of Cases 1-4 do not hold. We first consider the case when $k = 3$. Since $\delta > n/2$, there exist at least $d^2(\delta - 1/2)n^2/2$ edges with 2 vertices in $X$ and one in $Y$. Furthermore, each edge can be extended to a $K^3_{t-1}$, so there are at least
\[
\frac{n^{t-1}}{(t-1)!}d^2(\delta - 1/2) \prod_{3 \leq s \leq t-2} \left(1 - \left(\frac{s}{2}\right)\beta\right) = C' n^{t-1}
\]
copies of $K^3_{t-1}$ with at least two vertices in $X$ and one in $Y$. Hence, we may assume that there exist at least $C'n^{t-1}/t$ copies $T$ of $K^3_{t-1}$ such that $|T \cap X| = r$ and $|T \cap Y| = s$ for some $r \geq 2$ and $s \geq 1$. Assume that $1 \leq s \leq t/2$. Since Case 1 and Case 2 do not hold, for all but at most $2C'n^{t-1}$ such $T$ we have $|L(T) \cap Y| < \epsilon n$ and so $|L(T) \cap X| \geq (1 - (t-1)/2) \beta + \epsilon)n$. Therefore, there are at least
\[
(d + 2\epsilon) \frac{(C' - 2C)n^t}{t^2}
\]
copies $U$ of $K^3_1$ with $|U \cap X| = r + 1$ and $|U \cap Y| = s$. Suppose that $C$ is sufficiently small such that $d(C' - 2C)/t^2 \geq 2C$. By an averaging argument, there exist at least
\[
d(C' - 2C)n^{t-1}/t^2 \geq 2Cn^{t-1}
\]
copies $T'$ of $K^3_{t-1}$ such that $|T' \cap Y| = s - 1$ and $|L(T') \cap Y| \geq \epsilon n$ contradicting either Case 2 or Case 4. A similar argument holds for $r < t/2$. Hence, Lemma 7.6 holds for $k = 3$.

For $k \geq 4$, recall the definition of $l_0, |X| \geq (d + \gamma)n$ and $|Y| \geq n/2 - 1$. Hence, by greedy algorithm we can first find a $K^3_{t+1}$ in $X$, extend it to a $K^3_{2(t+1)}$ with $l + 1$ vertices in each of $X$ and $Y$, and further extend it to a $K^3_{l-1}$. Thus, there are at least $C'n^{t-1}$ copies of $K^3_{l-1}$ with at least $l + 1$ vertices in each of $X$ and $Y$. By a similar argument as the case $k = 3$ we obtain a contradiction. The proof of Lemma 7.1 is completed.

We are now ready to prove Lemma 7.1.

Proof of Lemma 7.1: Fixed $k$ and $t$. Write $\beta$ and $d$ to be $\beta(k, t)$ and $d(k, t)$ respectively. Let $H$ be a $k$-graph of order $n$ sufficiently large with $\delta_{k-1}(H) \geq (1 - \beta + \gamma)n$. By Lemma 7.6, we can partition $V(H)$ into $W_1, \ldots, W_p$ such that $|W_j| \geq dn$ and $W_j$ is $(K^k_t, \delta_0, \eta_1)$-closed in $H$ for $1 \leq j \leq p$ and some $\delta_0, \epsilon$ and $\eta_1$ independent of $n$.

For $1 \leq j \leq p$, we say that $(j, \eta_j)$-close to mean $(K^k_t, j(i_0 + 1) - 1, \eta_j)$-close, where $\eta_j > 0$ is a sufficiently small constant independent of $n$ and its value will be become clear. Therefore, by Proposition 2.2 in order to prove the lemma, it is enough to show that for vertices $x_0$ and $y_0$, they are $(j, \eta_j)$-closed for some $1 \leq j \leq p$. There is nothing to prove if $x_0$ and $y_0$ are in the same $W_j$. Thus, we may assume that $x_0 \in W_1$ and $y_0 \in W_p$ with $p \geq 2$.

Let $X = W_0$ and $Y = \bigcup_{i=2}^p W_i$. By Lemma 7.5, there are at least $\epsilon_0 n^{t+1}$ triples $(x, y, T)$ with $x \in X, y \in Y, T \in (t-1)$ such that both $x \cup T$ and $y \cup T$ form $K^3_t$ in $H$. Thus, there is an integer $2 \leq j \leq p$ such that there are at least $\epsilon_0 n^{t+1}/(p - 1)$ such triples with $x \in W_1$ and $y \in W_j$. Arbitrarily pick $y_j \in W_j$. Therefore, there are at least $\epsilon_0 n^{t+1}/2(p - 1)$ such triples with $x \in W_1 \cup x_0$ and $y \in W_j \cup y_j$. Fix one such triple $(x, y, T)$. Let $m = i_0 t - 1$. Since $x \in W_1$ and so $x$ is $(1, \eta_1)$-close to $x_0$, there exist at least
\[
\eta_1 n^m - (t + 1)n^{m-1} \geq \eta_1 n^m/2
\]
m-sets $S_X$ such that $S_X \cap (T \cup \{x_0, y_1, x, y\}) = \emptyset$ and both $H[S_X \cup x_0]$ and $H[S_X \cup x]$ contain $K_t^k$-factors, and fix one such $S_X$. Similarly, as $y \in W_j$, there are at least
\[ \eta_1n^m - (m + t + 1)n^{m-1} \geq \eta_1n^m/2 \]
m-sets $S_Y$ such that $S_Y \cap (S_X \cup T \cup \{x_0, y_1, x, y\}) = \emptyset$ and both $H[S_Y \cup y_1]$ and $H[S_Y \cup y]$ contain $K_t^k$-factors, and fix one such $S_Y$. Set $S = S_X \cup S_Y \cup T \cup \{x, y\}$. Note that both $H[S \cup x_0]$ and $H[S \cup y_1]$ contain $K_t^k$-factors and $|S| = (2t + 1)t - 1$. Therefore, there at least
\[
\frac{1}{((2t + 1)t - 1)!} \times \frac{\epsilon_0 n^{t+1}}{2(p - 1)} \times \frac{\eta_1 n^m}{2} \times \frac{\eta_1 n^m}{2} = \frac{\epsilon_0 \eta_1^2 n^{2t + 1}t - 1}{((2t + 1)t - 1)!}8(p - 1)
\]
distinct $S$, so $x_0$ and $y_1$ are $(2, \eta_2)$-close to each other with $\eta_2 = \epsilon_0 \eta_1^2/((2t + 1)t - 1)!8(p - 1))$. Recall that $y_1 \in W_j$ is chosen arbitrarily, so $x_0$ is $(2, \eta_2)$-close to every vertex in $W_j$. If $j = p$, then we are done. Without loss of generality, we may assume that $j = 2$. By repeating the argument for the vertex partition $W_1 \cup W_2$, $W_3, \ldots, W_p$, we conclude that $x_0$ is $(p, \eta_p)$-close to $y_0$. Thus, $H$ is $(p, \eta_p)$-closed. \[ \square \]

8 Remarks on Theorem [1.4]

Note that $\lambda_0(k, k + 1) = 1$ for $t = 3, 4$ and $\lambda_0(k, k + 1) \geq 2$ for $t \geq 5$. Thus, we have the following corollary by combining Theorem [1.3] and Theorem [1.4] for $t = k + 1$.

Corollary 8.1. For integers $n \geq k \geq 3$ with $k + 1|n$,
\[
t^k(n, k + 1) \leq \begin{cases} \left(\frac{t}{2} + o(1)\right) n & \text{if } k = 3 \\
\left(\frac{t}{2} + o(1)\right) n & \text{if } k = 4 \\
\left(\frac{t}{2} + o(1)\right) n & \text{if } k = 5 \\
\left(\frac{t}{2} + o(1)\right) n & \text{if } k = 6 \\
\left(1 - \frac{k + 1}{k + 2} + o(1)\right) n & \text{if } k \geq 7. \end{cases}
\]

Moreover, equality holds for $k = 3$.

Note that the bounds in Theorem [1.4] are due to Lemma [6.1] and Lemma [7.1]. If $t$ is large compared to $k$, then the bound from Lemma [6.1] dominates. It should be noted that by a simple modification of the proof of Lemma [6.1] we can show that if
\[
\delta_{k-1}(H) \geq \left(1 - \frac{t - 1}{t(t - 1)} - \gamma\right) n
\]
with $n$ sufficiently large then there exist vertex disjoint copies of $K_t^k$ covering all but at most $\gamma' n$, where $\gamma, \gamma' > 0$. Thus, we can deduced that
\[
t^k(n, t) \leq \max \left\{1 - \beta(k, t) + \gamma, 1 - \frac{t - 1}{t(t - 1)} - \gamma\right\} n,
\]
where $\gamma$ are strictly positively functions tending to zero as $n \to \infty$. As we have mentioned in the introduction, we would like to know the asymptotic values of $t^k(n, t)$.
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During the submission of this paper, we learned that Keevash and Mycroft \cite{11} determined the exact value of $t^3(n, 4)$ for $n$ sufficiently large using a different method.

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