Bi-directional half-duplex protocols with multiple relays

Sang Joon Kim, Natasha Devroye, and Vahid Tarokh

Abstract

In a bi-directional relay channel, two nodes wish to exchange independent messages over a shared wireless half-duplex channel with the help of relays. Recent work has considered information theoretic limits of the bi-directional relay channel with a single relay. In this work we consider bi-directional relaying with multiple relays. We derive achievable rate regions and outer bounds for half-duplex protocols with multiple decode and forward relays and compare these to the same protocols with amplify and forward relays in an additive white Gaussian noise channel. We consider three novel classes of half-duplex protocols: the \((m, 2)\) 2 phase protocol with \(m\) relays, the \((m, 3)\) 3 phase protocol with \(m\) relays, and general \((m, t)\) Multiple Hops and Multiple Relays (MHMR) protocols, where \(m\) is the total number of relays and \(3 < t \leq m + 2\) is the number of temporal phases in the protocol. The \((m, 2)\) and \((m, 3)\) protocols extend previous bi-directional relaying protocols for a single \(m = 1\) relay, while the \((m, t)\) protocol efficiently combines multi-hop routing with network coding. Finally, we provide a comprehensive treatment of the MHMR protocols with decode and forward relaying and amplify and forward relaying in the Gaussian noise, obtaining their respective achievable rate regions, outer bounds and relative performance under different SNRs and relay geometries.

Index Terms

bi-directional communication, achievable rate regions, decode and forward, amplify and forward, multiple relays

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I. INTRODUCTION

In bi-directional channels, two terminal nodes (a and b) wish to exchange independent messages. In wireless channels or mesh networks, this communication may take place with the help of \( m \) other nodes \( r_i, i \in \{1, 2, \cdots m\} \) termed relays. This two-way channel [2] was first considered in [9], where full-duplex operation where nodes could transmit and receive simultaneously, was assumed. Since full-duplex operation is, with current technology, of limited practical significance, in this work we assume that the nodes are half-duplex, i.e. at each point in time, a node can either transmit or receive symbols, but not both.

Our main goal is to determine the limits of bi-directional communication with multiple relays. To do so, we propose and determine the achievable rate regions, as well as outer bounds obtained using several protocols. The protocols we propose for the multiple-relay bi-directional channel may be described in terms of two parameters: the number of relays, \( m \), and the number of temporal phases \( t \), called hops. Throughout this work, phases and hops are used interchangeably. We also define an intermediate hop as a hop in which only relays transmit (and not the terminal nodes). Note that our protocols are all composed of a number of temporal phases/hops due to the half-duplex nature of the channel. We denote our proposed protocols as \((m, t)\) MHMR (Multiple Hops and Multiple Relays) protocols, for general positive integers \( m \geq 2 \) and \( t \geq 2 \). For the special case of two hops \((t = 2)\), the terminal nodes may simultaneously transmit in phase 1 as in the MABC (Multiple Access Broadcast Channel) protocol of [5], while the relays transmit the decoded messages to the terminal nodes in phase 2. For the special case of three hops \((t = 3)\) the terminal nodes may sequentially transmit in the first two phases as in the TDBC (Time Division Broadcast Channel) protocol of [5], after which the relays transmit in phase 3.

While a protocol in this work defines the temporal aspect (phases) of bi-directional communication, it does not specify the type of relaying a node may perform, or relaying scheme. That is, for each of the MHMR protocols, the relays may process and forward the received signals differently. Standard forwarding techniques include decode-and-forward, amplify-and-forward, compress-and-forward, and de-noise and forward. We consider only the first two relaying schemes. In the Decode and Forward (DF) scheme, the relays decode messages from the other nodes before re-encoding them for transmission. The DF scheme requires the full codebooks of all nodes and a large amount of computation at the relays \( \{r_i\} \). In the Amplify and Forward (AF) scheme, the relays \( \{r_i\} \) construct their symbol by symbol replications of the received symbols. The AF scheme does not require any computation for relaying, and carries the noise incurred in the first stage(s) forward during the latter relaying stage(s). The relative benefits
and merits of the temporal protocols and relaying schemes are summarized in Tables I and II where we compare the amount of knowledge/computation at the relays as well as the amount of interference and side information present. By side information we mean information obtained from the wireless channel in a particular phase which may be combined with information obtained in different stages to potentially improve decoding or increase transmission rates.

Some of the protocols and relaying schemes have been previously considered. In [6], the DF TDBC protocol with a single relay is considered. There, network coding in $\mathbb{Z}_2^k$ is used to encode the message of relay $r$ from the estimated messages $\tilde{w}_a$ and $\tilde{w}_b$. The works [7] and [8] consider the MABC protocol with multiple hops, where an amplification and denoising relaying scheme are introduced. In [5], achievable rate regions and outer bounds of the MABC protocol and the TDBC protocol for a single DF relay are derived. In [1], a comprehensive analysis of the AF scheme in large networks is provided.

The main contributions of this work are: (1) the extension of previously defined single relay MABC and TDBC protocols to multiple DF relays, (2) a novel class of general $(m, t)$ MHMR bi-directional DF relaying protocols, (3) all the associated achievable rate regions and outer bounds, and (4) a comprehensive comparison of these schemes with their AF analogs in Gaussian noise. Some of the main conclusions drawn are that, in multiple-relay bi-directional relay channels, it may be beneficial to have information flowing in both directions along a series of hops, where the information is carefully combined in a network-coding-like fashion. When the number of hops is large or when the SNR is low, DF outperforms AF as noise is not carried forward. Simulations show that the careful choice of the number of hops and which relays participate in each hop can lead to significant gains in terms of the achievable rates.

This paper is structured as follows: in Section II we introduce our notation and review previously determined achievable rate regions. In Section III we introduce novel $(m, t)$ MHMR protocols. In Section IV we derive achievable rate regions for the $(m, t)$ MHMR protocols with DF relaying. In Section V we derive outer bounds for the MHMR protocols. In Section VI we obtain explicit expressions for achievable rate regions and outer bounds and their corresponding AF analogs in Gaussian noise. In Section VII we numerically compute these bounds in the Gaussian noise channel and compare the results for different powers and channel conditions.

II. PRELIMINARIES

A. Definitions

Nodes $a$ and $b$ are the two terminal nodes and $\mathcal{R} := \{r_1, r_2, \cdots, r_m\}$ is the set of relays which aid the communication between nodes $a$ and $b$. For convenience of analysis we define $r_0 := a$, $r_{m+1} := b$.
and use these notations interchangeably in the following sections. Also define \( \mathcal{R}^* := \mathcal{R} \cup \{a, b\} = \{r_0, r_1, \ldots, r_{m+1}\} \). We use \( R_{i,j} \) for the transmitted data rate from node \( i \) to node \( j \), i.e. the rate of the message between node \( i \) and node \( j \), \( W_{i,j} \), lies in the set \( \mathcal{S}_{i,j} := \{0, \ldots, \lfloor 2^n R_{i,j} \rfloor - 1\} \). In our case, two terminal nodes denoted \( a \) and \( b \) exchange their messages. The message to be transmitted from node \( a \) to node \( b \) (\( b \) to \( a \)) are denoted by \( W_a := W_{a,b} \) (\( W_b := W_{b,a} \)) at the rate \( R_a := R_{a,b} \) (\( R_b := R_{b,a} \)). The two distinct messages \( W_a \) and \( W_b \) are taken to be independent and uniformly distributed in the set of \( \{0, \ldots, \lfloor 2^n R_a \rfloor - 1\} := \mathcal{S}_a \) and \( \{0, \ldots, \lfloor 2^n R_b \rfloor - 1\} := \mathcal{S}_b \), respectively.

Each node \( i \) has channel input alphabet \( \mathcal{X}_i^* = \mathcal{X}_i \cup \{\emptyset\} \) and channel output alphabet \( \mathcal{Y}_i^* = \mathcal{Y}_i \cup \{\emptyset\} \).

Because of the half-duplex constraint, not all nodes transmit/receive during all phases and we use the dummy symbol \( \emptyset \) to denote that there is no input or no output at a particular node during a particular phase. The half-duplex constraint forces either \( X_i^{(\ell)} = \emptyset \) or \( Y_i^{(\ell)} = \emptyset \) for all \( \ell \) phases. The channel is assumed discrete memoryless. For convenience, we drop the notation \( \emptyset \) from entropy and mutual information terms when a node is not transmitting or receiving. Communication takes place over \( n \) of channel uses and rates are achieved in the classical asymptotic sense as \( n \to \infty \). At channel use \( k \), we use \( X_i^k \) to denote the input distribution and \( Y_i^k \) to denote the distribution of the received signal of node \( i \). Similarly, during phase \( \ell \) we use \( X_i^{(\ell)} \) to denote the input distribution and \( Y_i^{(\ell)} \) to denote the distribution of the received signal of node \( i \). \( \Delta_{i,n} \) is the phase duration of phase \( i \) with block size \( n \) and \( \Delta_i \) is the phase duration of phase \( i \) when \( n \to \infty \). It is also convenient to define \( X_S^k := \{X_i^k | i \in S\} \),
the set of input distributions by all nodes in the set $S$ at time $k$ and similarly $X_S^{(\ell)} := \{X_i^{(\ell)} | i \in S\}$, a set of input distributions during phase $\ell$. Lower case letters $x_i$ denote instances of the upper case $X_i$ which lie in the calligraphic alphabets $X_i^*$. Boldface $x_i$ represents a vector indexed by time at node $i$. Finally, it is convenient to denote by $x_S := \{x_i | i \in S\}$, a set of vectors indexed by time. We define $W_{S,T} := \{W_{i,j} | i \in S, j \in T, S,T \in \mathbb{R}^*\}$. We use the notation $x_{S}(w_{S,T})$ to denote the dependence of $x_{S}$ on the message set $w_{S,T}$.

For the $(m, 2)$, $(m, 3)$ DF MHMR protocols we define $A$ (resp. $B$) as the set of relays which are able to decode $w_{a}$ (resp. $w_{b}$). We define $I_{S}^{\min}(X_i^{(\ell)};Y_s^{(\ell)}) := \min_{s \in S} I(X_i^{(\ell)};Y_s^{(\ell)})$. For example, $I_{A}^{\min}(X_a^{(1)};Y_r^{(1)}) = \min_{r \in A} I(X_a^{(1)};Y_r^{(1)})$, i.e. the minimum mutual information between node $a$ and a relay in the set of relays which can decode $w_{a}$.

For a block length $n$, encoders and decoders are functions $X_k^{\ell}(W_{\{i\},R^*}, Y_1^{\ell}, \cdots, Y_{k-1}^{\ell})$ producing an encoded message, and $\tilde{W}_{j,i}(Y_1^{\ell}, \cdots, Y_{k-1}^{\ell}, W_{\{i\},R^*})$ producing a decoded message or error, for sending a message from node $j$ to node $i$ for time $k = 1, 2, \cdots, n$. We define error events $E_{S,T} := \{W_{i,j} \neq \tilde{W}_{i,j(.)} | i \in S, j \in T\}$ for decoding the messages $W_{S,\{j\}}$ at nodes $j \in T$ at the end of the block of length $n$, and $E_{S,T}^{(\ell)}$ as the error event at nodes $j \in T$ in which nodes $j \in T$ independently attempt to decode $W_{S,T}$ at the end of phase $\ell$ using a joint typicality decoder.

Let $A_{S,T}^{(\ell)}$ represent the set of $\epsilon$-typical $(x_S^{\ell},y_T^{\ell})$ sequences of length $n \cdot \Delta_{\ell,n}$ according to the input distributions employed in phase $\ell$. Define the events $D_{S,T}^{(\ell)}(w_{S,T}) := \{(x_S^{(\ell)}(w_{S,T}),y_T^{(\ell)}) \in A_{S,T}^{(\ell)}\}$ and $L_{S,T}^{(\ell)}$ as the messages $w_{S,T}$ whose codewords $x_S^{(\ell)}(w_{S,T})$ are jointly typical with $y_T^{(\ell)}$ in phase $\ell$. 

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Fig. 1. The two phase MABC and three phase TDBC protocols of [5].
B. Previous results

We next outline the relevant previously derived [5] achievable rate regions of bi-directional decode and forward protocols with a single relay, \( r \), and terminal nodes \( a \) and \( b \), as shown in Fig. 1. These regions will be used in the discussions in Sections III and IV. The three phase protocol is called the \textit{Time Division Broadcast} (TDBC) protocol, while the two phase protocol is called the \textit{Multiple Access Broadcast} (MABC) protocol. One of the main conceptual differences between these two protocols is the possibility of \textit{side-information} in the TDBC protocol but not in the MABC protocol. The two previously considered protocols may be described as:

1) TDBC protocol: this consists of the three phases \( a \rightarrow r \), \( b \rightarrow r \) and \( a \leftarrow r \rightarrow b \). In this protocol, only a single node is transmitting at any given point in time. Therefore, by the broadcast nature of the wireless channel, the non-transmitting nodes may listen in and obtain “side information” about the other nodes’ transmissions. This may be used to improve the rates of transmission.

2) MABC protocol: this protocol combines the first two phases of the TDBC protocol and consists of the two phases \( a \rightarrow r \leftarrow b \) and \( a \leftarrow r \rightarrow b \). Due to the half-duplex assumption, during phase 1 both source nodes are transmitting and thus cannot obtain any “side information” regarding the other nodes’ transmission. It may nonetheless be spectrally efficient since it has less phases than the TDBC protocol and may take advantage of the multiple-access channel in phase 1.

We now state the results of [5] for completeness.

\textit{Theorem 1:} An achievable rate region of the half-duplex bi-directional relay channel with the decode and forward MABC protocol is the closure of the set of all points \((R_a, R_b)\) satisfying

\[
R_a < \min \left\{ \Delta_1 I(X_a^{(1)}; Y_r^{(1)}|X_b^{(1)}, Q), \Delta_2 I(X_r^{(2)}; Y_b^{(2)}|Q) \right\}
\]

\[
R_b < \min \left\{ \Delta_1 I(X_b^{(1)}; Y_r^{(1)}|X_a^{(1)}, Q), \Delta_2 I(X_r^{(2)}; Y_a^{(2)}|Q) \right\}
\]

\[
R_a + R_b < \Delta_1 I(X_a^{(1)}, X_b^{(1)}; Y_r^{(1)}|Q)
\]

over all joint distributions \( p(q)p^{(1)}(x_a|q)p^{(1)}(x_b|q)p^{(2)}(x_r|q) \) with \( |Q| \leq 5 \) over the restricted alphabet \( X_a \times X_b \times X_r \).

\textit{Theorem 2:} An achievable region of the half-duplex bi-directional relay channel with the decode and forward TDBC protocol is the closure of the set of all points \((R_a, R_b)\) satisfying

\[
R_a < \min \left\{ \Delta_1 I(X_a^{(1)}; Y_r^{(1)}|Q), \Delta_1 I(X_a^{(1)}; Y_b^{(1)}|Q) + \Delta_3 I(X_r^{(3)}; Y_b^{(3)}|Q) \right\}
\]

\[
R_b < \min \left\{ \Delta_2 I(X_b^{(2)}; Y_r^{(2)}|Q), \Delta_2 I(X_b^{(2)}; Y_a^{(2)}|Q) + \Delta_3 I(X_r^{(3)}; Y_a^{(3)}|Q) \right\}
\]
over all joint distributions \( p(q)p^{(1)}(x_a|q)p^{(2)}(x_b|q)p^{(3)}(x_r|q) \) with \( |Q| \leq 4 \) over the restricted alphabet \( \mathcal{X}_a \times \mathcal{X}_b \times \mathcal{X}_r \).

III. PROTOCOLS

We next describe a class of bi-directional multiple-relay protocols which we term \((m, t)\) DF MHMR (Decode and Forward, Multiple Hop Multiple Relay) protocols, where \( m \) is the number of relays and \( t \) is the number of hops. A protocol is a series of temporal phases through which bi-directional communication between nodes \( a \) and \( b \) is enabled. A single protocol may employ different types of relaying schemes, which specify how relays process and forward the received signals. In Section VI and VII we consider Amplify and Forward relaying in the Gaussian channel and use the term \((m, t)\) AF MHMR protocol to denote the protocols described next with Amplify and Forward relaying rather than Decode and Forward relaying.

When the number of hops is 2, i.e. \( t = 2 \) we re-name the \((m, 2)\) MHMR protocol the MABC MHMR protocol. When the number of hops is 3, we re-name the \((m, 3)\) MHMR protocol the TDBC MHMR protocol. These names reflect the similarity of the protocols to the previously defined MABC and TDBC protocols [5].

For \( t > 2 \), we define the \((m, t)\) regular MHMR protocol for \( m \mod (t−2) = 0 \) as the MHMR protocol which has the same number of relays in each intermediate hop, equal to \( m/(t−2) \). For example, the \((8, 6)\) regular MHMR protocol consists of two relays in each of the four hops.

A. \((m, 2)\) MABC and \((m, 3)\) TDBC DF MHMR protocols

If multiple relays are permitted to transmit in a single temporal phase, or hop, the protocols match those of when only a single relay is present [5]. The added complication lies in which subset of relays will transmit in that phase. We thus extend the MABC and TDBC two protocols previously proposed for the single relay bi-directional channel [5] to allow for multiple relays. During the relay transmission phase, each relay lies in one of the four following sets, which partition \( \mathcal{R} \):

1) \((A \cup B)^c\) : cannot decode \( w_a \) or \( w_b \)
2) \(A \setminus B\) : decode \( w_a\) only
3) \(B \setminus A\) : decode \( w_b\) only
4) \(A \cap B\) : decode both \( w_a\) and \( w_b\)

Note that the relays in case 1) do not re-transmit any messages as they were not able to decode any and the single relay protocol is contained in case 4) (where all relays in \( A \cap B \) may be viewed as a single
relay with multiple antennas). With the MHMR protocols, relays in the sets $A \setminus B$ and $B \setminus A$ can be used for the relay transmission. The detailed protocol is as follows:

(a) Two terminal nodes transmit their own messages.

- with the MABC protocol, terminal nodes transmit simultaneously in a single multiple access phase
- with the TDBC protocol, terminal nodes transmit in two sequential phases

(b) Relays in $A \cap B$ generate $x_{A \cap B}(w_a \oplus w_b)$, relays in $A \setminus B$ generate $x_{A \setminus B}(w_a)$ and relays in $B \setminus A$ generates $x_{B \setminus A}(w_b)$ which they simultaneously transmit during the relay transmit phase.

(c) Node $a$ receives $y_a$ and decodes $\tilde{w}_b$ from the jointly typical sequences $(x_{A \cap B}, x_{A \setminus B}, x_{B \setminus A}, y_a)$. Since $a$ knows $w_a$, we can remove $x_{A \setminus B}$ and the total cardinality is bounded by $\lceil 2^{nR_b} \rceil$. Node $b$ similarly decodes $\tilde{w}_a$.

Fig. 2 illustrates the impact of having different subsets $A, B \subseteq R$ in the DF MABC MHMR protocol with two relays, consider the labeled rate regions, corresponding to different sets $A$ and $B$ specified in Table III. We see that if smaller subsets of $R$ decode messages $w_a$ and $w_b$ then larger rates $R_a$ and $R_b$ may be possible (as in for example region (4)).

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Fig. 2. Achievable regions of $r_1$ and $r_2$ for the first phase and the corresponding achievable region for the MABC protocol with two relays.
TABLE III

\(A\) AND \(B\) FOR EACH ACHIEVABLE REGION WITH \(R = \{r_1, r_2\}\)

| Region | \(A\) | \(B\) |
|--------|-------|-------|
| (1)    | \(\{r_1, r_2\}\) | \(\{r_1, r_2\}\) |
| (2)    | \(\{r_1, r_2\}\) | \(\{r_1\}\) |
| (3)    | \(\{r_2\}\) | \(\{r_1, r_2\}\) |
| (4)    | \(\{r_2\}\) | \(\{r_1\}\) |

B. \((m, t)\) DF MHMR protocol

In this section, we consider a relay network with \(m\) relays and \(3 < t \leq m + 2\). For simplicity, we first describe the \((m, m+2)\) MHMR protocol: our general protocol with the maximal number of phases. From the \((m, m+2)\) MHMR protocol the \((m, t)\) for \(3 < t < m + 2\) protocol and corresponding achievable rate regions readily follow. The multi-hop network may be represented graphically: each node is represented as a vertex and a directed edge \((s, t)\) exists if node \(t\) can decode \(w_a\) or \(w_b\) at the end of the transmission of node \(s\). For example,

\[
\begin{align*}
  r_0 (= a) &\rightarrow r_1 \rightarrow r_2 \rightarrow \cdots \rightarrow r_m \rightarrow r_{m+1} (= b)
\end{align*}
\]

is one possible graphical representation of our multi-hop network with \(m\) relays. A simple naïve protocol for the above example network is: \(r_0 \rightarrow r_1 \rightarrow \cdots \rightarrow r_m \rightarrow r_{m+1}\) and then \(r_{m+1} \rightarrow r_m \rightarrow \cdots \rightarrow r_1 \rightarrow r_0\). This is one possible \((m, 2m + 2)\) MHMR protocol, which may be spectrally inefficient as the number of phases is large. Intuitively, spectral efficiency may be improved by combining phases through the use of network coding. In the following, we reduce the number of phases needed from \(2m + 2\) to \(m + 2\).

In the \((m, m+2)\) protocol only a single relay transmits during each phase. This is extended to allow for multiple relays transmitting in each phase in Theorem 7 of the next section. The protocol may be described in the following algorithm:

\[\text{[ALGORITHM] - } (m, m+2) \text{ DF MHMR protocol}\]

0) Preparation  
\(a\) and \(b\) divide their respective messages into \(B\) sub-messages, one for each block. Thus, \(a\) has the message set \(\{w_{a,(0)}, w_{a,(1)}, \cdots, w_{a,(B-1)}\}\). Likewise \(b\) has \(\{w_{b,(0)}, w_{b,(1)}, \cdots, w_{b,(B-1)}\}\).

1) Initialization  
For \(i = 0\) to \(m - 1\)
For $j = 0$ to $i$

- $r_{i-j}$ transmits $x_{r_{i-j}}(w_{a,(j)})$
- $r_{i-j+1}$ decodes $w_{a,(j)}$

end

end

2) Main routine

For $i = 0$ to $B - m - 1$

- $r_{m+1}$ transmits $x_{r_{m+1}}(w_{b,(i)})$
- $r_m$ decodes $w_{b,(i)}$ and generates $x_{r_{m}}(w_{a,(i)} \oplus w_{b,(i)})$

For $j = 0$ to $m - 1$

- $r_{m-j}$ transmits $x_{r_{m-j}}(w_{a,(i+j)} \oplus w_{b,(i)})$
- $r_{m-j-1}$ decodes $w_{b,(i)}$ and generates $x_{r_{m-j-1}}(w_{a,(i+j+1)} \oplus w_{b,(i)})$
- $r_{m-j+1}$ decodes $w_{a,(i+j)}$

end

- $r_0$ transmits $x_{r_0}(w_{a,(m+i)})$
- $r_1$ decodes $w_{a,(m+i)}$

end

3) Termination

For $i = B - m$ to $B - 1$

- $r_{m+1}$ transmits $x_{r_{m+1}}(w_{b,(i)})$
- $r_m$ decodes $w_{b,(i)}$ and generates $x_{r_{m}}(w_{a,(i)} \oplus w_{b,(i)})$

For $j = 0$ to $m - 1$

- $r_{m-j}$ transmits $x_{r_{m-j}}$
- $r_{m-j-1}$ decodes $w_{b,(i)}$ and generates
  \[
  \begin{cases} 
  x_{r_{m-j-1}}(w_{a,(i+j+1)} \oplus w_{b,(i)}), & \text{if } i+j \leq B-2 \\
  x_{r_{m-j-1}}(w_{b,(i)}), & \text{otherwise}
  \end{cases}
  \]
- $r_{m-j+1}$ decodes $w_{a,(i+j)}$ if $i+j \leq B-1$

end

end

After initialization, relay $r_i$ has the following messages from node $a$: $\{w_{a,(0)}, w_{a,(1)}, \cdots, w_{a,(m-i)}\}$ ($1 \leq i \leq m$). In other words message $w_{a,(i)}$ has reached $r_{m-i}$ at the end of the initialization. In the
main routine, which, when the number of blocks $B \to \infty$ makes up the majority of this protocol, $w_{b,(i)}$ travels along the path $r_{m+1} \to r_m \to \cdots \to r_1 \to r_0$ in the $i^{th}$ loop. During the same loop, as the single sub-message from node $b$ travels to node $a$, the stream of messages from node $a$ sitting in the each of the relays are all shifted to the right by one through the use of network coding. Overall then, we require 2 transmissions from the terminal nodes, and $m$ relay transmissions to transfer two individual sub-messages. When node $a$ finishes sending its all sub-messages to $r_1$, the termination step starts. The remaining $w_{a,(i)}$s in the relays and $w_{b,(i)}$s in node $b$ are processed in this step. The number of transmissions in the main routine depends only on the number of blocks $B$ while the number of transmissions in the initialization and termination steps are a function of the hop size $m$. In the following theorem we formally prove that by increasing the block size $B$, our algorithm asymptotically results in $m + 2$ phases.

**Theorem 3:** The number of phases achieved by the $(m, m + 2)$ algorithm with $B$ blocks approaches $m + 2$ as the number of blocks $B \to \infty$.

**Proof:** In the $(m, m + 2)$ DF MHMR protocol, the total number of transmissions $N_T(m)$ for $B$ blocks is:

$$N_T(m) = \frac{m(m+1)}{2} + (B - m)(m + 2) + m(m + 1)$$

$$= B(m + 2) + \frac{m(m - 1)}{2}$$

Therefore, the number of phases per block is given by

$$N_T(m) / B = (m + 2) + \frac{m(m - 1)}{2B}$$

As $B \to \infty$, $m + 2$ phases result.

### IV. Achievable Rate Regions

**A. $(m, 2)$ DF MABC protocol**

We now derive an achievable rate region for the multi-hop bi-directional relay channel with $m$ relays and two phases, an extension of the MABC protocol of [5] to multiple relays. We note that relays which share the same message set may cooperate in transmitting their messages, as seen in the joint distributions of phase 2.

**Theorem 4:** An achievable region of the half-duplex bi-directional channel under the $(m, 2)$ DF MABC
protocol is the closure of the set of all points \((R_a, R_b)\) satisfying

\[
R_a < \min \left\{ \Delta_1 I_{A \cap B}^{\text{min}}(X_a^{(1)}; Y_r^{(1)} | X_b^{(1)}, Q), \Delta_1 I_{A \cap B}^{\text{min}}(X_a^{(1)}; Y_r^{(1)} | Q), \Delta_2 I(X_a^{(2)}; Y_b^{(2)} | Q) \right\}
\]

\[
R_b < \min \left\{ \Delta_1 I_{A \cap B}^{\text{min}}(X_b^{(1)}; Y_r^{(1)} | X_a^{(1)}, Q), \Delta_1 I_{A \cap B \setminus A}^{\text{min}}(X_b^{(1)}; Y_r^{(1)} | Q), \Delta_2 I(X_b^{(2)}; Y_a^{(2)} | Q) \right\}
\]

\[
R_a + R_b < \Delta_1 I_{A \cap B}^{\text{min}}(X_a^{(1)}, X_b^{(1)}; Y_r^{(1)} | Q)
\]

over all joint distributions \(p(q)p^{(1)}(x_a | q)p^{(1)}(x_b | q)p^{(2)}(x_A | B | q)p^{(2)}(x_B | A | q)\) with \(|Q| \leq 3m + 2\) over the restricted alphabet \(\prod_{i=0}^{m+1} \mathcal{X}_r\) for all possible \(A, B \subseteq \mathcal{R}\).

**Proof:** Random code generation: For simplicity of exposition, we take \(|Q| = 1\) and therefore consider distributions \(p^{(1)}(x_a), p^{(1)}(x_b), p^{(2)}(x_A | B), p^{(2)}(x_A | B)\) and \(p^{(2)}(x_B | A)\). First we generate random \((n \cdot \Delta_{1,n})\)-length sequences \(x_a^{(1)}(w_a)\) with \(w_a \in \mathcal{S}_a\) and \(x_b^{(1)}(w_b)\) with \(w_b \in \mathcal{S}_b\) according to \(p^{(1)}(x_a), p^{(1)}(x_b)\) to be used in phase 1. For phase 2, we generate random \((n \cdot \Delta_{2,n})\)-length sequences \(x_A^{(2)}(w_A)\) with \(w_A \in \mathcal{Z}_L (L = \max([2^n R_a], [2^n R_b]))\), \(x_A^{(2)}(w_a)\) and \(x_B^{(2)}(w_b)\), according to \(p^{(2)}(x_A | B)\), \(p^{(2)}(x_A \setminus B)\) and \(p^{(2)}(x_B \setminus A)\) respectively.

**Encoding:** During phase 1, encoders of node a and b send the codewords \(x_a^{(1)}(w_a)\) and \(x_b^{(1)}(w_b)\) respectively. Relays in \(A \cap B\) estimate \(\hat{w}_a\) and \(\hat{w}_b\) after phase 1 using jointly typical decoding, then construct \(w_r = \hat{w}_a \oplus \hat{w}_b\) in \(\mathcal{Z}_L\) and send \(x_A^{(2)}(w_r)\) during phase 2. Likewise relays in \(A \setminus B\) (resp. \(B \setminus A\)) estimate \(\hat{w}_a\) (resp. \(\hat{w}_b\)) after phase 1 and send \(x_A^{(2)}(\hat{w}_a)\) (resp. \(x_A^{(2)}(\hat{w}_b)\)).

**Decoding:** a and b estimate \(\hat{w}_b\) and \(\hat{w}_a\) after phase 2 using jointly typical decoding. Since \(w_r = w_a \oplus w_b\) and a knows \(w_a\), node a can reduce the cardinality of \(w_r\) to \([2^n R_b]\). b similarly decodes \(\hat{w}_a\).

**Error analysis:**

\[
P[E_{\{a\}, \{b\}}] \leq P[E_{\{a\}, A \cap B} \cup E_{\{b\}, A \cap B} \cup E_{\{a\}, A \setminus B} \cup E_{\{b\}, A \setminus B}]
\]

\[
\leq P[E_{\{a\}, A \cap B} \cup E_{\{b\}, A \cap B} \cup P[E_{\{a\}, A \setminus B} + \sum P[D_{\{a\}, \{b\}, \{r\}}(w_a, w_b)]
\]

\[
+ 2^n R_a 2^{-n \Delta_{1,n}(I(X_a^{(1)}; Y_r^{(1)} | X_b^{(1)}) - 3\epsilon)} + 2^n R_b 2^{-n \Delta_{1,n}(I(X_a^{(1)}; Y_r^{(1)} | X_b^{(1)}) - 3\epsilon)} + 2^n (R_a + R_b) 2^{-n \Delta_{1,n}(I(X_a^{(1)}; Y_r^{(1)} | X_b^{(1)}) - 4\epsilon)}
\]

Following the well-known MAC error analysis from (15.72) in [2]:

\[
P[E_{\{a\}, A \cap B} \cup E_{\{b\}, A \cap B}] \leq \sum_{r \in \mathcal{A} \cap B} P[D_{\{a\}, \{b\}, \{r\}}(w_a, w_b)]
\]

\[
+ 2^n R_a 2^{-n \Delta_{1,n}(I(X_a^{(1)}; Y_r^{(1)} | X_b^{(1)} - 3\epsilon)} + 2^n R_b 2^{-n \Delta_{1,n}(I(X_a^{(1)}; Y_r^{(1)} | X_b^{(1)}) - 3\epsilon)} + 2^n (R_a + R_b) 2^{-n \Delta_{1,n}(I(X_a^{(1)}; Y_r^{(1)} | X_b^{(1)}) - 4\epsilon)}
\]
Also,

\[
P[E_{\{a\},A\setminus B}^{(1)}] \leq \sum_{r \in A \setminus B} P[E_{\{a\},\{r\}}^{(1)}] \leq \sum_{r \in A \setminus B} P[\bar{D}_{\{a\},\{r\}}(w_a)] + 2^nR_a2^{-n\Delta_1,n}(I(X^{(1)}_a;Y^{(1)}_r) - 3\epsilon),
\]

and

\[
P[E_{\{a\},A\setminus B}^{(2)} | E_{\{a\},A \cap B}^{(1)} \cap \bar{E}_{\{b\},A \cap B}^{(1)} \cap \bar{E}_{\{a\},A \setminus B}^{(1)}] \leq P[\bar{D}_{\{a\},\{b\}}(w_r, w_a)] + 2^nR_a2^{-n\Delta_2,n}(I(X^{(2)}_a;Y^{(2)}_b) - 3\epsilon).
\]

Since \(\epsilon > 0\) is arbitrary, the conditions of Theorem 4 and the AEP property will guarantee that the right hand sides of (11), (12) and (13) vanish as \(n \to \infty\). Similarly, \(P[E_{b,a}] \to 0\) as \(n \to \infty\). By Fenchel-Bunt’s extension of Carathéodory theorem in [3], it is sufficient to restrict \(|Q| \leq 3m + 2\).

### B. \((m, 3)\) DF TDBC protocol

We now derive an achievable rate region for the multi-hop bi-directional relay channel with \(m\) relays and 3 phases, an extension of the DF TDBC protocol of [5]. We note that relays which share the same message set may cooperate in transmitting their messages, as seen in the joint distributions of phase 3.

**Theorem 5:** An achievable region of the half-duplex bi-directional channel under the \((m, 3)\) DF TDBC protocol is the closure of the set of all points \((R_a, R_b)\) satisfying

\[
R_a < \min \{\Delta_1I_{\mathcal{A}}^\text{min}(X^{(1)}_a;Y^{(1)}_r|Q), \Delta_1I(X^{(1)}_a;Y^{(1)}_b|Q) + \Delta_3I(X^{(3)}_a;Y^{(3)}_b|Q)\}
\]

\[
R_b < \min \{\Delta_2I_{\mathcal{B}}^\text{min}(X^{(2)}_b;Y^{(2)}_r|Q), \Delta_2I(X^{(2)}_b;Y^{(2)}_a|Q) + \Delta_3I(X^{(3)}_b;Y^{(3)}_a|Q)\}
\]

over all joint distributions \(p(q)p^{(1)}(x_a|q)p^{(2)}(x_b|q)p^{(3)}(x_{\mathcal{A}\setminus B}|q)p^{(3)}(x_{\mathcal{A}\cap B}|q)p^{(3)}(x_{\mathcal{B}\setminus A}|q)\) with \(|Q| \leq 2m + 2\) over the restricted alphabet \(\prod_{i=0}^{m+1} X_r\) for all possible \(\mathcal{A}, \mathcal{B} \subseteq \mathcal{R}\).

Theorem 5 is proven in a similar manner to Theorem 4 in Appendix II.

### C. \((m, t)\) DF MHMR protocol

The \((m, 2)\) and \((m, 3)\) protocols were extensions of previously derived MABC and TDBC protocols to multiple relays. In this section we derive the rates achieved by the novel \((m, m + 2)\) protocol which does not resemble the MABC and TDBC protocols. We recall that in the \((m, m + 2)\) MHMR protocol a single relay transmits in each hop. We then extend the ideas of the \((m, m + 2)\) MHMR protocol to derive achievable rate regions for general \((m, t)\) protocols with \(3 < t < m + 2\). Recalling that \(a\) and \(b\) are denoted as \(r_0\) and \(r_{m+1}\) respectively, our main result lies in the following Theorem.
Theorem 6: An achievable rate region of the half-duplex bi-directional multi-hop relay channel under the \((m, m + 2)\) DF MHMR protocol \((m > 1)\) is the closure of the set of all points \((R_a, R_b)\) satisfying

\[
R_a < \min_{1 \leq k \leq m + 1} \left\{ \sum_{i=1}^{k} \Delta_{m+3-i} I(X_{r_{t-i}}^{(m+3-i)}, Y_{r_k}^{(m+3-i)} | Q) \right\}
\]

(16)

\[
R_b < \min_{1 \leq k \leq m + 1} \left\{ \sum_{i=1}^{k} \Delta_i I(X_{r_{t+1-i}}^{(t+1-i)}, Y_{r_{t+1-i}}^{(t+1-i)} | Q) \right\}
\]

(17)

over all joint distributions \(p(q) \prod_{i=1}^{n+2} p^{(i)}(x_{r_{m+1-i}}, q)\) with \(|Q| \leq 2m + 2\) over the restricted alphabet \(\prod_{i=0}^{n+1} X_r\).

The proof is provided in Appendix III.

The minimization is over the number of hops, and results from the need for a series of relays to decode each message. The summation for a given \(k\) represents the accumulated amount of information the node \(k\) may use to decode message \(w_a\) or \(w_b\).

We can extend Theorem 6 to allow for multiple relays in each hop. In order to use network coding, we make the assumption that each relay is able to decode both \(w_a\) and \(w_b\). In each hop or phase then, a subset of the nodes will be able to decode both messages \(w_a\) and \(w_b\) and may cooperate in re-transmitting the obtained messages. We denote this subset of relays in the \(i\)-th hop as \(R_i\). The following theorem takes into account all possible subsets \(R_i \subset R\) and does not consider how these subsets are chosen.

Corollary 7: An achievable rate region of the half-duplex bi-directional channel in the \((m, t)\) DF MHMR protocol for \(3 < t < m + 2\) is the closure of the set of all points \((R_a, R_b)\) satisfying

\[
R_a < \min_{1 \leq k \leq t-1} \min_{r_i \in R_a} \left\{ \sum_{i=1}^{k} \Delta_{t+1-i} I(X_{r_{t-i}}^{(t+1-i)}, Y_{r_k}^{(t+1-i)} | Q) \right\}
\]

(18)

\[
R_b < \min_{1 \leq k \leq t-1} \min_{r_i \in R_{t-k}} \left\{ \sum_{i=1}^{k} \Delta_i I(X_{r_{t-i}}^{(i)}, Y_{r_{t-i}}^{(i)} | Q) \right\}
\]

(19)

over all joint distributions \(p(q) \prod_{i=1}^{t} p^{(i)}(x_{r_{t-i}}, q)\) with \(|Q| \leq 2m + 2\) over the restricted alphabet \(\prod_{i=0}^{n+1} X_r\), for all possible \(R_i \subset R\) such that \(R_i \cap R_j = \emptyset\) for all \(i, j \in [0, t-1]\), where \(R_0 = \{a\}\) and \(R_{t-1} = \{b\}\).

The proof of Corollary 7 follows the same argument as the proof of Theorem 6.

V. OUTER BOUNDS

In this section we derive outer bounds for each MHMR protocol using the following cut-set bound lemma [5]. Again, given subsets \(S, T \subseteq \mathcal{M} = \{1, 2, \ldots, m\}\), and \(\bar{S} := \mathcal{M}\setminus S\), we define \(W_{S,T} := \{W_{i,j}|i \in S, j \in T\}\) and \(R_{S,T} = \lim_{n \to \infty} \frac{1}{n} H(W_{S,T})\).
Lemma 8: If in some network the information rates \( \{R_{i,j}\} \) are achievable for a protocol \( P \) with relative durations \( \{\Delta_{\epsilon}\} \), then for every \( \epsilon > 0 \) and all \( S \subset M \)

\[
R_{S,S} \leq \sum_{\ell} \Delta_{\epsilon} I(X_{S}^{(\ell)}; Y_{S}^{(\ell)} | X_{S}^{(\ell)}, Q) + \epsilon,
\]

for a family of conditional distributions \( p^{(\ell)}(x_{1}, x_{2}, \ldots, x_{m}|q) \) and a discrete time-sharing random variable \( Q \) with distribution \( p(q) \). Furthermore, each \( p^{(\ell)}(x_{1}, x_{2}, \ldots, x_{m}|q)p(q) \) must satisfy the constraints of phase \( \ell \) of protocol \( P \).

We next state the outer bounds, which will be numerically evaluated and discussed in the following sections.

A. \((m, 2)\) MABC protocol

Theorem 9: (Outer bound) The capacity region of the half-duplex bi-directional relay channel with the \((m, 2)\) MABC protocol is outer bounded by the set of rate pairs \( (R_{a}, R_{b}) \) satisfying

\[
R_{a} \leq \min_{S_{R}} \left\{ \Delta_{1} I(X_{a}^{(1)}; Y_{S_{R}}^{(1)} | X_{b}^{(1)}, Q) + \Delta_{2} I(X_{S_{R}}^{(2)}; Y_{b}^{(2)} | X_{S_{R}}^{(2)}, Q) \right\}
\]

\[
R_{b} \leq \min_{S_{R}} \left\{ \Delta_{1} I(X_{b}^{(1)}; Y_{S_{R}}^{(1)} | X_{a}^{(1)}, Q) + \Delta_{2} I(X_{S_{R}}^{(2)}; Y_{a}^{(2)} | X_{S_{R}}^{(2)}, Q) \right\}
\]

for all choices of the joint distribution \( p(q)p^{(1)}(x_{a}|q)p^{(1)}(x_{b}|q)p^{(2)}(x_{S_{R}}|q) \) with \( |Q| \leq 2^{m+1} \) over the restricted alphabet \( \prod_{i=0}^{m+1} X_{r_{i}} \) for all possible \( S_{R} \subseteq R \).

Proof: We use Lemma 8 to prove the Theorem 9. For every \( S_{R} \subseteq R \), there exist 4 types of cut-sets such that \( S_{1} = \{a\} \cup S_{R} \), \( S_{2} = \{b\} \cup S_{R} \), \( S_{3} = \{a, b\} \cup S_{R} \) and \( S_{4} = S_{R} \), as well as two rates \( R_{a} \) and \( R_{b} \). Also, in the MABC protocol,

\[
Y_{a}^{(1)} = Y_{b}^{(1)} = X_{R}^{(1)} = \emptyset
\]

\[
X_{a}^{(2)} = X_{b}^{(2)} = Y_{R}^{(2)} = \emptyset.
\]

Thus, the corresponding outer bounds for a given subset \( S_{R} \) are:

\[
S_{1}: R_{a} \leq \Delta_{1} I(X_{a}^{(1)}; Y_{S_{R}}^{(1)} | X_{b}^{(1)}, Q) + \Delta_{2} I(X_{S_{R}}^{(2)}; Y_{b}^{(2)} | X_{S_{R}}^{(2)}, Q) + \epsilon,
\]

\[
S_{2}: R_{b} \leq \Delta_{1} I(X_{b}^{(1)}; Y_{S_{R}}^{(1)} | X_{a}^{(1)}, Q) + \Delta_{2} I(X_{S_{R}}^{(2)}; Y_{a}^{(2)} | X_{S_{R}}^{(2)}, Q) + \epsilon,
\]

where the cut sets \( S_{3} \) and \( S_{4} \) yield no constraints. Since \( \epsilon > 0 \) is arbitrary, \( \Delta_{2} \), \( \Delta_{1} \) and the fact that the half-duplex nature of the channel constrains \( X_{b}^{(1)} \) to be conditionally independent of \( X_{b}^{(1)} \) given \( Q \) yields Theorem 9. By Fenchel-Bunt’s extension of the Carathéodory theorem in [3], it is sufficient to restrict \( |Q| \leq 2^{m+1} \).
B. \((m, 3)\) TDBC protocol

**Theorem 10:** (Outer bound) The capacity region of the half-duplex bi-directional relay channel with the \((m, 3)\) TDBC protocol is outer bounded by the set of rate pairs \((R_a, R_b)\) satisfying

\[
R_a \leq \min_{S_R} \left\{ \sum_{r_i \in S_R \cup \{a\}} \Delta_{m+2-i} I(X^{(m+2-i)}; Y^{(m+2-i)}|Q) \right\}
\]

\[
R_b \leq \min_{S_R} \left\{ \sum_{r_i \in S_R \cup \{b\}} \Delta_{m+2-i} I(X^{(m+2-i)}; Y^{(m+2-i)}|Q) \right\}
\]

for all choices of the joint distribution \(p(q) p^{(1)}(x_{a}|q) p^{(2)}(x_{b}|q) p^{(3)}(x_{S_R}|q)\) with \(|Q| \leq 2^{m+1}\) over the restricted alphabet \(\prod_{i=0}^{m+1} \mathcal{X}_{r_i}\) for all possible \(S_R \subseteq \mathcal{R}\).

**Theorem 10** is proven in a similar manner to Theorem 9 in Appendix III.

C. \((m, t)\) MHMR protocol

**Theorem 11:** (Outer bound) The capacity region of the half-duplex bi-directional multi-hop relay channel under the \((m, m + 2)\) MHMR protocol \((m > 1)\) is outer bounded by the set of rate pairs \((R_a, R_b)\) satisfying

\[
R_a \leq \min_{S_R} \left\{ \sum_{r_i \in S_R \cup \{a\}} \Delta_{t-i} I(X^{(t-i)}; Y^{(t-i)}|Q) \right\}
\]

\[
R_b \leq \min_{S_R} \left\{ \sum_{r_i \in S_R \cup \{b\}} \Delta_{t-i} I(X^{(t-i)}; Y^{(t-i)}|Q) \right\}
\]

for all choices of the joint distribution \(p(q) \prod_{i=0}^{m+2} p^{(i)}(x_{r_{m+2-i}}|q)\) with \(|Q| \leq 2^{m+1}\) over the restricted alphabet \(\prod_{i=0}^{m+1} \mathcal{X}_{r_i}\) for all possible \(S_R \subseteq \mathcal{R}\).

The proof of Theorem 11 follows the same argument as the proofs of Theorem 9 and Theorem 10.

**Corollary 12:** (Outer bound) The capacity region of the half-duplex bi-directional channel in the \((m, t)\) MHMR protocol for \(3 < t < m + 2\) is outer bounded by the set of rate pairs \((R_a, R_b)\) satisfying

\[
R_a \leq \min_{S_R} \left\{ \sum_{i=0}^{t-1} \Delta_{t-i} I(X^{(t-i)}; Y^{(t-i)}|Q) \right\}
\]

\[
R_b \leq \min_{S_R} \left\{ \sum_{i=0}^{t-1} \Delta_{t-i} I(X^{(t-i)}; Y^{(t-i)}|Q) \right\}
\]

for all choices of the joint distribution \(p(q) \prod_{i=1}^{t-1} p^{(i)}(x_{r_{t-i}}|q)\) with \(|Q| \leq 2^{m+1}\) over the restricted alphabet \(\prod_{i=0}^{m+1} \mathcal{X}_{r_i}\), for all possible \(\mathcal{R}_i \subseteq \mathcal{R}\) such that \(\mathcal{R}_i \cap \mathcal{R}_j = \emptyset\) for all \(i, j \in [0, t-1]\), where \(\mathcal{R}_0 = \{a\}\) and \(\mathcal{R}_{t-1} = \{b\}\) for all possible \(S_R \subseteq \mathcal{R}\).

The proof of Corollary 12 follows the same argument as the proofs of Theorem 9 and Theorem 10.
VI. THE GAUSSIAN RELAY NETWORK

In this section, we apply the bounds obtained in the previous section to a Gaussian relay network. We assume that there are two terminal nodes a and b, and m relays \( r_1, r_2, \ldots, r_m \). Also, for convenience of analysis, we denote a as \( r_0 \) and b as \( r_{m+1} \). The corresponding mathematical channel model is, for each channel use \( k \):

\[
Y[k] = HX[k] + Z[k]
\]

where,

\[
Y[k] = \begin{bmatrix}
Y_{r_0}[k] \\
Y_{r_1}[k] \\
\vdots \\
Y_{r_{m+1}}[k]
\end{bmatrix},\quad X[k] = \begin{bmatrix}
X_{r_0}[k] \\
X_{r_1}[k] \\
\vdots \\
X_{r_{m+1}}[k]
\end{bmatrix},\quad Z[k] = \begin{bmatrix}
Z_{r_0}[k] \\
Z_{r_1}[k] \\
\vdots \\
Z_{r_{m+1}}[k]
\end{bmatrix}
\]

and

\[
H = \begin{bmatrix}
0 & h_{r_1,r_0} & \cdots & h_{r_{m+1},r_0} \\
h_{r_0,r_1} & 0 & \cdots & h_{r_{m+1},r_1} \\
\vdots & \ddots & \ddots & \vdots \\
h_{r_0,r_{m+1}} & h_{r_1,r_{m+1}} & \cdots & 0
\end{bmatrix}
\]

where \( Y[k], X[k] \) and \( Z[k] \) are in \( \mathbb{C}^{(m+2)\times 1} = (\mathbb{C} \cup \{\emptyset\})^{(m+2)\times 1} \), and \( H \in \mathbb{C}^{(m+2)\times (m+2)} \). In phase \( \ell \), if node \( r_i \) is in transmission mode \( X_{r_i}[k] \) follows the input distribution \( X_{r_i}(\ell) \sim \mathcal{N}(0, P_{r_i}) \). Otherwise, \( X_{r_i}[k] = \emptyset \), which means that the input symbol does not exist in the above mathematical channel model.

In each phase, the total transmit power is bounded by \( P \), i.e. \( \sum_{r \in \mathcal{R}_\ell} E[X_r^2] \leq P \) for all \( \ell \), where \( \mathcal{R}_\ell \) is the set of nodes which transmit during phase \( \ell \). While ideally the per-phase power of \( P \) could be distributed amongst the nodes in \( \mathcal{R}_\ell \) arbitrarily, as a first step, we allocate equal power \( P/|\mathcal{R}_\ell| \) for each relay in \( \mathcal{R}_\ell \). Equal power allocation between participating nodes may also be simpler to implement. We will later investigate the gain achieved by allowing for arbitrary power allocations.

In each phase, we also allow for cooperation between relays which have the same messages. For example, in the \((m, 2)\) DF MABC protocol, we have three different subsets of relays in phase 2: \( A \cap B \), \( A \setminus B \) and \( B \setminus A \). We first allocate equal power \( P/|A \cup B| \) to each relay in \( A \cup B \) and then allow cooperation in each subset which has the same messages. For convenience of analysis we denote \( P_{\mathcal{R}_\ell} \) as the total power of relays in \( \mathcal{R}_\ell \). \( h_{i,j} \) is the effective channel gain between transmitter \( i \) and receiver \( j \).

We assume the channel is reciprocal \((h_{i,j} = h_{j,i})\) and that each node is fully aware of the channel gains,
i.e., full CSI. The noise at all receivers is independent, of unit power, additive, white Gaussian, complex and circularly symmetric. For convenience of analysis, we also define the function $C(x) := \log_2(1 + x)$.

### A. Amplify and Forward

As a comparison point for the DF MHMR protocols, we derive an achievable region of the same temporal protocols in which the relays use a simple amplify and forward relaying scheme rather than a decode and forward scheme. “Simple” means that there is no power optimization in each phase, i.e. each node during phase $\ell$ has equal transmit power $P / |R_\ell|$. Also, in the amplify and forward scheme, all phase durations are equal since relaying is performed on a symbol by symbol basis. Thus, $\Delta_\ell = \frac{1}{t}$, where $t$ is the number of phases and $\ell \in [1, t]$. Furthermore, relay $r$ scales the received symbol by $\sqrt{P_r}$ to meet the transmit power constraint. We now state the achievable rate regions for the analogous MHMR protocols with AF relaying.

- **$(m, 2)$ AF MABC Protocol**

\[
R_a < \frac{1}{2} C \left( \frac{\frac{P}{2} \left( \sum_{i=1}^{m} |h_{b,r_i}|^2 |h_{a,r_i}|^2 \tilde{P}_r \right)^2}{\sum_{i=1}^{m} |h_{b,r_i}|^2 \tilde{P}_r + 1} \right) \\
R_b < \frac{1}{2} C \left( \frac{\frac{P}{2} \left( \sum_{i=1}^{m} |h_{b,r_i}|^2 |h_{a,r_i}|^2 \tilde{P}_r \right)^2}{\sum_{i=1}^{m} |h_{a,r_i}|^2 \tilde{P}_r + 1} \right)
\]

where $\tilde{P}_r = \frac{\frac{P}{2}}{P_r (|h_{r,a}|^2 + |h_{r,b}|^2) + 1}$.

- **$(m, 3)$ AF TDBC Protocol**

\[
R_a < \frac{1}{3} C \left( |h_{a,b}|^2 P + \frac{P \left( \sum_{i=1}^{m} |h_{b,r_i}|^2 |h_{a,r_i}|^2 \tilde{P}_r \right)^2}{2 \sum_{i=1}^{m} |h_{b,r_i}|^2 \tilde{P}_r + 1} \right)
\]

\[
R_b < \frac{1}{3} C \left( |h_{a,b}|^2 P + \frac{P \left( \sum_{i=1}^{m} |h_{b,r_i}|^2 |h_{a,r_i}|^2 \tilde{P}_r \right)^2}{2 \sum_{i=1}^{m} |h_{a,r_i}|^2 \tilde{P}_r + 1} \right)
\]

where $\tilde{P}_r = \frac{\frac{P}{2}}{P_r (|h_{r,a}|^2 + |h_{r,b}|^2) + 2}$.
• \((m, m + 2)\) AF MHMR Protocol

We now consider the \((m, m + 2)\) MHMR protocol with Amplify and Forward relaying. The temporal protocol consists of the same phases: Initialization, Main routine, and Termination, but we replace Decode and Forward relaying with Amplify and Forward relaying. In the main routine, we schedule the transmissions as \(r_{m+1} \rightarrow r_m \rightarrow \cdots \rightarrow r_0\). To simplify the analysis, we assume node \(r_i\) which receives \(Y_{r_i}^{(\ell)}\) during phase \(\ell\) constructs its transmission \(X_{r_i}^{(m-i+2)}\) as a function of \(Y_{r_i}^{(m-i+1)}\) and \(Y_{r_i}^{(m-i+3)}\). That is, it considers only the received symbols from its neighboring nodes \(r_{i+1}\) and \(r_{i-1}\), where \(i \in [1, m]\) rather than all previously heard transmissions providing a simpler but smaller achievable rate region. The corresponding channel model is then

\[
Y_{r_i}^{(m-i+1)} = h_{r_{i+1}, r_i}X_{r_{i+1}}^{(m-i+1)} + Z_{r_i}^{(m-i+1)} \tag{40}
\]

\[
Y_{r_i}^{(m-i+3)} = h_{r_{i-1}, r_i}X_{r_{i-1}}^{(m-i+3)} + Z_{r_i}^{(m-i+3)}. \tag{41}
\]

We construct the channel input symbol \(X_{r_i}^{(m-i+2)}\) as

\[
X_{r_i}^{(m-i+2)} = \sqrt{P_{r_i}} \left( \tilde{h}_{a,r_i}X_{a}^{(m+2)} + \tilde{h}_{b,r_i}X_{b}^{(1)} + \tilde{Z}_{a,r_i} + \tilde{Z}_{b,r_i} \right), \tag{42}
\]

where \(\tilde{h}_{a,r_i}\) is the effective channel gain from a to \(r_i\) that captures the channel \(a \rightarrow (r_1, r_2, \cdots, r_{i-1}) \rightarrow r_i\) and \(\tilde{h}_{b,r_i}\) is the effective channel gain from b to \(r_i\) that captures \(b \rightarrow (r_m, r_{m-1}, \cdots, r_{i+1}) \rightarrow r_i\) where \(\tilde{Z}_{a,r_i}\) and \(\tilde{Z}_{b,r_i} \sim \mathcal{N}(0, 1)\) and \(\tilde{P}_{r_i} = \frac{P}{p(\|h_{a,r_i}\|^2 + \|h_{b,r_i}\|^2) + 2}\). We apply (42) to \(X_{r_i}^{(m-i+1)}\) and \(X_{r_i}^{(m-i+3)}\) in (40) and (41) to obtain:

\[
Y_{r_i}^{(m-i+1)} = h_{r_{i+1}, r_i} \sqrt{P_{r_{i+1}}} \left( \tilde{h}_{a,r_{i+1}}X_{a}^{(m+2)} + \tilde{h}_{b,r_{i+1}}X_{b}^{(1)} + \tilde{Z}_{a,r_{i+1}} + \tilde{Z}_{b,r_{i+1}} \right) + Z_{r_i}^{(m-i+1)} \tag{43}
\]

\[
Y_{r_i}^{(m-i+3)} = h_{r_{i-1}, r_i} \sqrt{P_{r_{i-1}}} \left( \tilde{h}_{a,r_{i-1}}X_{a}^{(m+2)} + \tilde{h}_{b,r_{i-1}}X_{b}^{(1)} + \tilde{Z}_{a,r_{i-1}} + \tilde{Z}_{b,r_{i-1}} \right) + Z_{r_i}^{(m-i+3)}. \tag{44}
\]

In (43), since \(X_{a}\) flows from \(r_i\) to \(r_{i+1}\), \(r_i\) knows \(X_{a}^{(m+2)}\) when it receives \(Y_{r_i}^{(m-i+1)}\). Thus, \(r_i\) can eliminate \(X_{a}^{(m+2)}\) from \(Y_{r_i}^{(m-i+1)}\). By the same reasoning, \(r_i\) can eliminate \(X_{b}^{(1)}\) from \(Y_{r_i}^{(m-i+2)}\).

After elimination and normalization, the modified \(\hat{Y}_{r_i}^{(m-i+1)}\) and \(\hat{Y}_{r_i}^{(m-i+3)}\) are given by:

\[
\hat{Y}_{r_i}^{(m-i+1)} = \sqrt{\left( \frac{|h_{r_{i+1}, r_i}|^2|\tilde{h}_{b,r_{i+1}}|^2\tilde{P}_{r_{i+1}}P}{2|h_{r_{i+1}, r_i}|^2|\tilde{P}_{r_{i+1}} + 1} \right)} \cdot X_{b}^{(1)} + \tilde{Z}_{b,r_i} \tag{45}
\]

\[
= \tilde{h}_{b,r_i}X_{b}^{(1)} + \tilde{Z}_{b,r_i}, \tag{46}
\]

\[
\hat{Y}_{r_i}^{(m-i+3)} = \sqrt{\left( \frac{|h_{r_{i-1}, r_i}|^2|\tilde{h}_{a,r_{i-1}}|^2\tilde{P}_{r_{i-1}}P}{2|h_{r_{i-1}, r_i}|^2|\tilde{P}_{r_{i-1}} + 1} \right)} \cdot X_{a}^{(m+2)} + \tilde{Z}_{a,r_i}, \tag{47}
\]

\[
= \tilde{h}_{a,r_i}X_{a}^{(m+2)} + \tilde{Z}_{a,r_i}. \tag{48}
\]
We obtain $X_r^{(m-i+2)}$ by adding the two terms $\tilde{Y}_r^{(m-i+1)}$ and $\tilde{Y}_r^{(m-i+3)}$ and scaling it all by $\sqrt{P_r}$, as
\begin{equation}
X_r^{(m-i+2)} = \sqrt{P_r}(\tilde{Y}_r^{(m-i+1)} + \tilde{Y}_r^{(m-i+3)}).
\end{equation}

From (43) - (48), we derive the recurrence relations for $\{|\tilde{h}_{b,r_i}|^2\}$ and $\{|\tilde{h}_{a,r_i}|^2\}$ for $2 \leq i \leq m - 1$ as
\begin{align}
|\tilde{h}_{b,r_i}|^2 &= \frac{P^2|h_{r_{i+1},r_i}|^2|\tilde{h}_{b,r_{i+1}}|^2}{2P|h_{b,r_{i+1}}|^2 + P(|h_{a,r_{i+1}}|^2 + |\tilde{h}_{b,r_{i+1}}|^2) + 2}, \\
|\tilde{h}_{a,r_i}|^2 &= \frac{P^2|h_{r_{i-1},r_i}|^2|\tilde{h}_{a,r_{i-1}}|^2}{2P|h_{r_{i-1},r_i}|^2 + P(|h_{b,r_{i-1}}|^2 + |\tilde{h}_{a,r_{i-1}}|^2) + 2},
\end{align}
where $|\tilde{h}_{b,r_m}|^2 = |h_{b,r_m}|^2$ and $|\tilde{h}_{a,r_1}|^2 = |h_{a,r_1}|^2$. Then an achievable rate region is given by:
\begin{align}
R_a &< \frac{1}{m + 2} C \left( \frac{1}{2} |h_{a,b}|^2 P + \sum_{i=1}^m \frac{|h_{r_{i,b}}|^2 |\tilde{h}_{a,r_i}|^2 \tilde{P}_r P}{2|h_{r_{i,b}}|^2 \tilde{P}_r + 1} \right) \\
R_b &< \frac{1}{m + 2} C \left( \frac{1}{2} |h_{a,b}|^2 P + \sum_{i=1}^m \frac{|h_{r_{i,a}}|^2 |\tilde{h}_{b,r_i}|^2 \tilde{P}_r P}{2|h_{r_{i,a}}|^2 \tilde{P}_r + 1} \right).
\end{align}

B. Decode and Forward

We can likewise obtain the achievable rate regions from Theorems 45 56 and Corollary 7 in Gaussian noise, under the same power allocation assumptions as above as:

- (m,2) DF MABC Protocol
The achievable rate region of the (m,2) DF MABC Protocol is the union over all $\Delta_1 + \Delta_2 = 1$, $\Delta_1, \Delta_2 \geq 0$ of
\begin{align}
R_a &< \min \left\{ \min_{r_i \in A \cap B} \Delta_1 C \left( \frac{P}{2} |h_{a,r_i}|^2 \right), \min_{r_j \in A \setminus B} \Delta_1 C \left( \frac{P|h_{a,r_i}|^2}{P|h_{b,r_i}|^2 + 2} \right) \right\} \\
R_a &< \Delta_2 C \left( \sum_{r_i \in A \cap B} |h_{r_{i,b}}|^2 P_{A \cap B} + \sum_{r_j \in A \setminus B} |h_{r_{j,b}}|^2 P_{A \setminus B} \right) \\
R_b &< \min \left\{ \min_{r_i \in A \cap B} \Delta_1 C \left( \frac{P}{2} |h_{b,r_i}|^2 \right), \min_{r_j \in B \setminus A} \Delta_1 C \left( \frac{P|h_{b,r_i}|^2}{P|h_{a,r_i}|^2 + 2} \right) \right\} \\
R_b &< \Delta_2 C \left( \sum_{r_i \in A \cap B} |h_{r_{i,a}}|^2 P_{A \cap B} + \sum_{r_j \in B \setminus A} |h_{r_{j,a}}|^2 P_{B \setminus A} \right) \\
R_a + R_b &< \min_{r_i \in A \cap B} \Delta_1 C \left( \frac{P}{2} (|h_{a,r_i}|^2 + |h_{b,r_i}|^2) \right)
\end{align}
where $P_{A \cap B} = \frac{|A \cap B|}{|A|} P$, $P_{A \setminus B} = \frac{|A \setminus B|}{|A|} P$, and $P_{B \setminus A} = \frac{|B \setminus A|}{|B|} P$ over all $A, B \subseteq \mathcal{R}$. 
• \((m, 3)\) DF TDBC Protocol

The achievable rate region of the \((m, 3)\) DF TDBC Protocol is the union over all \(\Delta_1 + \Delta_2 + \Delta_3 = 1, \Delta_1, \Delta_2, \Delta_3 \geq 0\) of

\[
R_a < \min_{r_i \in A} \Delta_1 C(P|h_{a,r_i}|^2) \tag{59}
\]

\[
R_a < \Delta_1 C(P|h_{a,b}|^2) + \Delta_3 C \left( \sum_{r_i \in A \cap B} |h_{r_i,b}|^2 P_{A \cap B} + \sum_{r_j \in A \setminus B} |h_{r_j,b}|^2 P_{A \setminus B} \right) \tag{60}
\]

\[
R_b < \min_{r_i \in B} \Delta_2 C(P|h_{b,r_i}|^2) \tag{61}
\]

\[
R_b < \Delta_2 C(P|h_{a,b}|^2) + \Delta_3 C \left( \sum_{r_i \in A \cap B} |h_{r_i,a}|^2 P_{A \cap B} + \sum_{r_j \in B \setminus A} |h_{r_j,a}|^2 P_{B \setminus A} \right) \tag{62}
\]

where \(P_{A \cap B} = \frac{|A \cap B|}{|A| \cup |B|} P\), \(P_{A \setminus B} = \frac{|A \setminus B|}{|A|} P\), and \(P_{B \setminus A} = \frac{|B \setminus A|}{|B|} P\) over all \(A, B \subseteq \mathcal{R}\).

• \((m, m + 2)\) DF MHMR Protocol

The achievable rate region of the \((m, m + 2)\) DF MHMR Protocol is the union over all \(\sum_{j=1}^{m+2} \Delta_j = 1, \Delta_j \geq 0\) of the rate pairs \((R_a, R_b)\) satisfying

\[
R_a < \min_{1 \leq k \leq m+1} \left\{ \sum_{i=1}^{k} \Delta_{m+3-i} C(P|h_{r_{i-1},r_i}|^2) \right\} \tag{63}
\]

\[
R_b < \min_{1 \leq k \leq m+1} \left\{ \sum_{i=1}^{k} \Delta_i C(P|h_{r_{m+2-i},r_{m+1-k}}|^2) \right\} \tag{64}
\]

• \((m, t)\) DF MHMR Protocol

For \(3 < t < m + 2\), the achievable rate region will be the union over all \(\sum_{j=1}^{t} \Delta_j = 1, \Delta_j \geq 0\) and all \(\mathcal{R}_{j-1} \subseteq \mathcal{R}\) for \(j \in [1, t]\) of the rate pairs \((R_a, R_b)\) satisfying

\[
R_a < \min_{1 \leq k \leq t-1} \min_{r_k \in \mathcal{R}_k} \left\{ \sum_{i=1}^{k} \Delta_{t+1-i} C \left( \sum_{r_{i-1} \in \mathcal{R}_{i-1}} |h_{r_{i-1},r_i}|^2 P \right) \right\} \tag{65}
\]

\[
R_b < \min_{1 \leq k \leq t-1} \min_{r_{t-1-k} \in \mathcal{R}_{t-1-k}} \left\{ \sum_{i=1}^{k} \Delta_i C \left( \sum_{r_{i-1} \in \mathcal{R}_{i-1}} |h_{r_{i-1},r_{i-1-k}}|^2 P \right) \right\} . \tag{66}
\]

C. Outer Bounds

We derive outer bounds from Theorems 9, 10 and 11 in Gaussian channel.

• \((m, 2)\) MABC Protocol
The capacity region of the \((m, 2)\) MABC Protocol is outer bounded by the set of rate pairs \((R_a, R_b)\) satisfying

\[
R_a \leq \min_{S_R} \left\{ \Delta_1 C \left( \sum_{t_i \in S_R} \frac{P}{2} |h_{a,t_i}|^2 \right) + \Delta_2 C \left( \sum_{t_i \in S_R} P|h_{r_i,b}|^2 \right) \right\} \tag{67}
\]

\[
R_b \leq \min_{S_R} \left\{ \Delta_1 C \left( \sum_{t_i \in S_R} \frac{P}{2} |h_{b,t_i}|^2 \right) + \Delta_2 C \left( \sum_{t_i \in S_R} P|h_{r_i,a}|^2 \right) \right\}, \tag{68}
\]

over all \(\Delta_1 + \Delta_2 = 1, \Delta_1, \Delta_2 \geq 0\) and all \(S_R \subseteq \mathcal{R}\).

- \((m, 3)\) TDBC Protocol

The capacity region of the \((m, 3)\) TDBC Protocol is outer bounded by the set of rate pairs \((R_a, R_b)\) satisfying

\[
R_a \leq \min_{S_R} \left\{ \Delta_1 C \left( \sum_{t_i \in S_R} P|h_{a,t_i}|^2 + P|h_{a,b}|^2 \right) + \Delta_3 C \left( \sum_{t_i \in S_R} P|h_{r_i,b}|^2 \right) \right\} \tag{69}
\]

\[
R_b \leq \min_{S_R} \left\{ \Delta_2 C \left( \sum_{t_i \in S_R} P|h_{b,t_i}|^2 + P|h_{b,a}|^2 \right) + \Delta_3 C \left( \sum_{t_i \in S_R} P|h_{r_i,a}|^2 \right) \right\}, \tag{70}
\]

over all \(\Delta_1 + \Delta_2 + \Delta_3 = 1, \Delta_1, \Delta_2, \Delta_3 \geq 0\) and all \(S_R \subseteq \mathcal{R}\).

- \((m, m + 2)\) MHMR Protocol

The capacity region of the \((m, m + 2)\) MHMR Protocol is outer bounded by the set of rate pairs \((R_a, R_b)\) satisfying

\[
R_a \leq \min_{S_R} \left\{ \sum_{t_i \in S_R \cup \{a\}} \Delta_{m+2-i} C \left( \sum_{t_j \in S_R \cup \{b\}} P|h_{r_i,t_j}|^2 \right) \right\} \tag{71}
\]

\[
R_b \leq \min_{S_R} \left\{ \sum_{t_i \in S_R \cup \{b\}} \Delta_{m+2-i} C \left( \sum_{t_j \in S_R \cup \{a\}} P|h_{r_i,t_j}|^2 \right) \right\}, \tag{72}
\]

over all \(\sum_{j=1}^{m+2} \Delta_j = 1, \Delta_j \geq 0\) and all \(S_R \subseteq \mathcal{R}\).

VII. NUMERICAL ANALYSIS

A. Rate region comparisons with one to two relays

In this section we numerically evaluate the rate regions obtained in the previous section for a variety of parameters, which include the number of relays, the type of relaying (DF or AF), as well as the number of hops \(t\) and whether these hops are regular. Specifically, we look at:
- **One relay versus two relays under with DF relaying:** We compare the achievable regions of two single relay protocols (MABC and TDBC) and three two-relay MHMR protocols with DF schemes at low (Fig. 3) and high (Fig. 4) SNRs.

- **DF versus AF relaying:** We compare the regions of DF and AF relaying in the MHMR protocols at low (Fig. 5) and high (Fig. 6) SNRs.

We use the following channel gain matrix $H$:

$$
H = \begin{bmatrix}
0 & 1.2 & 0.8 & 0.2 \\
1.2 & 0 & 2 & 0.8 \\
0.8 & 2 & 0 & 1.2 \\
0.2 & 0.8 & 1.2 & 0
\end{bmatrix}
$$

(73)

In the DF relaying protocols, the (2,4) MHMR protocol outperforms the other protocols at both low and high SNR. This improved performance may be attributed to this protocol’s effective use of side information. During each phase, every node which is not transmitting can receive the current transmission which it may employ as side information to aid decoding during later stages. It may also subtract off the part of the transmission corresponding to the message(s) it already knows. There is naturally a tradeoff between the number of phases and the amount of information broadcasted in each phase. However, as seen by our simulations in this particular channel, the effect of reducing the number of phases to 2 or 3 does not outweigh the effect of broadcasting information.

It is interesting to note that the (1,2) DF MABC and (1,3) DF TDBC protocols may outperform the (2,2) DF MABC and (2,3) DF TDBC protocols in some scenarios. This reveals, as suspected, that using one relay and allocating all transmit power to that single node is sometimes better than using multiple relays with equal power allocated to each of them. However, if we allow power optimization between different subsets of relays then multiple relaying protocols outperform the single relay protocols (dotted lines in Fig. 3 and 4). This reveals that we achieve larger gain if power allocation between the relays participating in the transmission of messages is permitted.

The inner and outer bounds differ for a number of reasons, with the prevailing one being that our inner bounds use a DF scheme. For the MABC scheme using DF relaying, in equations (67)–(70), every relay contributes to enlarging the outer bound regions, while only the subset of relays $\mathcal{A} \cup \mathcal{B}$ are used in determining the achievable regions. At low SNR, when $\mathcal{A} \cup \mathcal{B}$ is relatively small, the gaps, shown in Fig. 5 are larger than the gaps at high SNRs shown in Fig. 6 where the number of relays in $\mathcal{A} \cup \mathcal{B}$

\[1\] If other channel gains are chosen, the numerical results may change.
are relatively larger. In addition to simply having more relays contribute to the outer bound regions, their effect is summed up outside of the logarithm for the outer bound, and inside of it for the inner bounds. For example, in the MABC protocol, $R_a \leq \Delta_1 C(\cdot) + \Delta_2 C(\cdot)$ for the outer bound, as opposed to $R_a \leq C(\sum \cdot)$ for the inner bound. Lastly, the achievable rate regions for DF relaying are significantly reduced by the necessity of having all relays decode the message(s) $w_a$ or $w_b$ individually, resulting in the min function which significantly diminishes the region. This requirement to decode all messages is not present in the outer bounds. The inner bounds for the AF relaying schemes are relatively small as (a) noise is carried forward, (b) no power optimization is performed and (c) no phase-length optimization is performed. The inner bounds may be improved through the use of compress and forward relaying [4] or de-noising, which may be able to capture the optimal tradeoff between eliminating the noise while not requiring the messages to be decoded. The exploration of different relaying schemes as well as the analytical impact of different channel gain matrices is left for future work.

In the proposed protocols, the (2,4) DF MHMR protocol achieves the largest rate region in most scenarios. In the high SNR regime, the (2,2) AF MABC protocol may achieve rates slightly better than the (2,4) DF MHMR protocol, as noise amplification is less of an issue. Furthermore, the (2,2) AF MABC protocol outperforms the (2,3) AF TDBC protocol since it employs less phases and the interference is perfectly canceled at each terminal node. However as a general rule, multiple hops with DF relaying is the optimum protocol in this bi-directional half-duplex channel.
B. Rate region comparison with 8 relays on a line.

In this subsection 8 relays are placed on the line between a and b. The distance from a to \(r_i\) is \(d_{ar} = \frac{i}{9}d_{ab}\) (1 \(\leq i \leq 8\)). Thus, \(d_{br} = (1 - \frac{i}{9})d_{ab}\). We let \(h_{ab} = 0.2\) and \(|h_{ij}|^2 = k/d_{ij}^{3.8}\) for \(k\) constant and a path-loss exponent of 3.8.

In Fig. 7 and 8 the (8,10) DF MHMR protocol dominates the other protocols both in the low SNR and high SNR regime. As we explained in the previous subsection, this may be attributed to the broadcasted side information. While increasing the number of phases means that less information may be transmitted during each time phase, the accumulated side information and improved channel gains (shorter distances) for each hop outweighs these detrimental effects, yielding higher overall rates.

In contrast to the DF scheme, the achievable rate region for the AF schemes decreases as the number of hops increases, as the noise is increasingly amplified and carried forward. Similarly, in the low SNR regime (Fig. 7) when the noise is very large to begin with, as expected, the AF schemes performs very poorly.

C. Sum data rate with an increasing number of relays

We consider the same geometric location of the relays as in the previous subsection but increase their number. We compare the optimized sum rates \((R_a + R_b)\) of the different relaying schemes.

In Fig. 9 and 10 the \((m, m + 2)\) DF MHMR protocol outperforms the other protocols. Also, with more relays, the \((m, m + 2)\) DF MHMR protocol improves its performance, while the \((m, m + 2)\) AF MHMR protocol’s performance deteriorates. The tendency can be seen more significantly in the high
In this paper, we proposed protocols for the half-duplex bi-directional channel with multiple relays: the $(m, 2)$ MABC protocol, the $(m, 3)$ TDBC protocol and the general $(m, t)$ protocol for $m$ relays and $3 < t \leq m + 2$ phases. We derived achievable rate regions as well as outer bounds for 3 half-duplex bi-directional multiple relay protocols with decode and forward relays. We compared these regions to those achieved by the same protocols with amplify and forward relays in the Gaussian noise channel. Numerical evaluations suggest that the $(m, m + 2)$ DF MHMR protocol achieves the largest rate.
region under simulated channel conditions. As expected, for a low number of hops or at high SNR AF relaying protocols perform well, but rapidly degrade when the number of hops is increased or the SNR is decreased.

APPENDIX I

PROOF OF THEOREM 5

Random code generation: For simplicity of exposition, we take \(|Q| = 1\) and therefore consider distributions \(p^{(1)}(x_a), p^{(2)}(x_b), p^{(3)}(x_{A \cap B}), p^{(3)}(x_{A \setminus B})\) and \(p^{(3)}(x_{B \setminus A})\). First we generate random \((n \cdot \Delta_{1,n})\)-length sequences \(x_a^{(1)}(w_a)\) and \((n \cdot \Delta_{2,n})\)-length sequences \(x_b^{(2)}(w_b)\) i.i.d. according to \(p^{(1)}(x_a)\) and \(p^{(2)}(x_b)\) respectively. We also generate random \((n \cdot \Delta_{3,n})\)-length sequences \(x_{A \cap B}^{(3)}(w_r)\) with \(w_r \in \mathbb{Z}_L\) \((L = \max(\lfloor 2^n R_a \rfloor, \lfloor 2^n R_b \rfloor))\), \(x_{A \setminus B}^{(3)}(w_a)\) and \(x_{B \setminus A}^{(3)}(w_b)\), according to \(p^{(3)}(x_{A \cap B}), p^{(3)}(x_{A \setminus B})\) and \(p^{(3)}(x_{B \setminus A})\) respectively.

Encoding: During phase 1, the encoder of node a sends the codeword \(x_a^{(1)}(w_a)\). Node b similarly sends the codeword \(x_b^{(1)}(w_b)\) in phase 2. Relays in \(A \cap B\) estimate \(\hat{w}_a\) and \(\hat{w}_b\) after phase 1 and 2 using jointly typical decoding, then construct \(w_r = \hat{w}_a \oplus \hat{w}_b\) in \(\mathbb{Z}_L\) and send \(x_{A \cap B}^{(3)}(w_r)\) during phase 3. Likewise relays in \(A \setminus B\) (resp. \(B \setminus A\)) estimate \(\hat{w}_a\) (resp. \(\hat{w}_b\)) after phase 1 (resp. phase 2) and send \(x_{A \setminus B}^{(2)}(\hat{w}_a)\) (resp. \(x_{B \setminus A}^{(2)}(\hat{w}_b)\)).

Decoding: a estimates \(\tilde{w}_b\) after phase 3 using two independent message lists \(L_{\{b,\{a\}}^{(2)}\) and \(L_{\{B,\{a\}}^{(3)}\). In \(B\), there are two different messages \(w_r(= w_a \oplus w_b) \in A \cap B\) and \(w_b \in B \setminus A\). However, since a knows \(w_a\) as side information \(w_r\) is equivalent to \(w_b\) for a. Therefore, \(L_{\{b,\{a\}}^{(2)}\) and \(L_{\{B,\{a\}}^{(3)}\) are both subsets of \(S_b\). If there is a unique \(\tilde{w}_b\) in \(L_{\{b,\{a\}}^{(2)} \cap L_{\{B,\{a\}}^{(3)}\), a declares it as the decoded message. Otherwise an error is declared. Similarly, b decodes \(\tilde{w}_a\) after phase 3.

Error analysis:

\[
P[E_{\{b,\{a\}}^{(2)}] \leq P[E_{\{b,\{a\}}^{(2)} \cup E_{\{b\} \cup B,\{a\}}^{(3)}]
\leq P[E_{\{b,\{a\}}^{(2)}] + P[E_{\{b\} \cup B,\{a\}] E_{\{b,\{a\}}^{(2)}]
\]

(74)

(75)

\[
P[E_{\{b,\{a\}}^{(2)}] \leq \sum_{r \in B} P[E_{\{b,\{a\}}^{(2)}]
\]

(76)

\[
\leq \sum_{r \in B} P[D_{\{b,\{a\}}^{(2)}(w_b)] + 2^{nR_b} 2^{-n \cdot \Delta_{2,n}(I(X_a^{(2)};Y_{r}^{(2)}) - \epsilon)}
\]

(77)

\[
P[E_{\{b\} \cup B,\{a\}] E_{\{b,\{a\}}^{(2)}] \leq P[D_{\{b,\{a\}}^{(2)}(w_b)] + P[D_{B,\{a\}}^{(3)}(w_a \oplus w_b, w_b)]
\]
\[ P[\bigcup \tilde{u}_b \neq u_a D^{(2)}_{\{b\},\{a\}}(\tilde{u}_b)] \cdot P[\bigcup \tilde{u}_b \neq u_a D^{(3)}_{B\{a\}}(\tilde{u}_b \oplus w_a, w_a)] \leq 2\epsilon + 2^n R_0 2^{-n(\Delta_2,n I(X^{(2)}_b; Y^{(2)}_a) + \Delta_3,n I(X^{(3)}_b; Y^{(3)}_a) - 6\epsilon)} \]

Since \( \epsilon > 0 \) is arbitrary, the conditions of Theorem 5 and the AEP property will guarantee that the right hand sides of (77) and (79) vanish as \( n \to \infty \). Similarly, \( P[E_{a,b}] \to 0 \) as \( n \to \infty \). By Fenchel-Bunt’s extension of Carathéodory theorem in [3], it is sufficient to restrict \( |Q| \leq 2m + 2 \).

**APPENDIX II**

**PROOF OF THEOREM 6**

**Random code generation:** For simplicity of exposition, we take \(|Q| = 1\). \( a(= r_0) \) and \( b(= r_{m+1}) \) divide \( w_a \) and \( w_b \) into \( B \) blocks respectively. Then \( a \) has a message set \( \{w_{a(0)}, w_{a(1)}, \ldots, w_{a(B-1)}\} \) and \( b \) has a message set \( \{w_{b(0)}, w_{b(1)}, \ldots, w_{b(B-1)}\} \). We generate random \((n - \Delta_{i}, m+2-i,n)\)-length sequences \( x_i(m+2-i-w(r_i)) \) with \( w_i \in \mathbb{Z}_L \) \( L = \max(\{2n R_a, 2n R_b\}) \), according to \( p(m+2-i)(x_i) \) for \( i \in [0, m+1] \).

**Encoding:** We divide the total time period into \( B \) time slots. Each time slot consists of \( m+2 \) phases. Node \( a \) transmits \( x_i(m+2-w_{a(j-1)}) \) during slot \( j \) and phase \( m+2 \), where \( j \in [1, B] \) and node \( b \) transmits \( x_i(1-w_{b(j-m-1)}) \) during slot \( j \) and phase 1, where \( j \in [m+1, B+m] \). Intermediate node \( r_i (i \in [1, m]) \) transmits \( x_i(m+2-w_{r_i(j)}) \) during slot \( j \) and phase \( m+2-i \), where \( j \in [1, B+m] \) and \( i \in [1, m] \). \( w_{r_i(j)} \) is defined as follows:

\[
 w_{r_i(j)} := \begin{cases} 
 \tilde{w}_{a, (j-i-1)} \oplus \tilde{w}_{b, (j-m-1)}, & (1 \leq j - i \leq B, m + 1 \leq j \leq B + m) \\
 \tilde{w}_{a, (j-i-1)}, & (1 \leq j - i \leq B, 1 \leq j \leq m) \\
 \tilde{w}_{b, (j-m-1)}, & (j - i > B, m + 1 \leq j \leq B + m) \\
 \emptyset, & (1 > j - i, 1 \leq j \leq m) 
\end{cases} \tag{80}
\]

**Decoding:** After slot \( j \) and phase \( (m+2-i) \), \( r_{i+1} \) decodes \( \tilde{w}_{a, (j-i-1)} \) using \( i \) independent message lists \( L_{\{r_{i}\},\{r_{i+1}\}}, \ldots, L_{\{r_{i}\},\{r_{i+1}\}}, \) when \( j \in [i, B+i-1] \). If there is a unique \( \tilde{w}_{a, (j-i-1)} \) in \( \bigcap_{k=0}^{i} L_{\{r_{k}\},\{r_{i+1}\}, \{r_{i+1}\}} \), \( r_{i+1} \) declares it as the decoded message. Otherwise an error is declared. Similarly, \( r_{i-1} \) decodes \( \tilde{w}_{b, (j-m-1)} \) using \( m+2-i \) independent message lists \( L_{\{r_{i}\},\{r_{i-1}\}}, \ldots, L_{\{r_{m+1}\},\{r_{i-1}\}}, \) when \( j \in [m+1, B+m] \). If there is a unique \( \tilde{w}_{b, (j-m-1)} \) in \( \bigcap_{k=i}^{m+1} L_{\{r_{k}\},\{r_{i-1}\}, \{r_{i-1}\}} \), \( r_{i-1} \) declares it as the decoded message. However, after phase 1 (resp. phase \( m+2 \)), \( r_m \) (resp. \( r_1 \)) only decodes \( \tilde{w}_{b, j} \) (resp. \( \tilde{w}_{a, j} \)).

**Error analysis:**

\[
P[E_{\{a\},\{b\}}] \leq P[\bigcup_{i=1}^{m+1} E_{\{r_{i}\}, \{r_{i+1}\}, \{r_{i+1}\}, \{r_{i+1}\}}] \leq \sum_{i=1}^{m+1} P[E_{\{r_{i}\}, \ldots, \{r_{i+1}\}, \{r_{i+1}\}] \bigcap_{j=1}^{i-1} E_{\{r_{0}, \ldots, r_{j-1}\}, \{r_{j}\}}] \tag{82}
\]
for \( i \in [1, m + 1] \), we have
\[
P[E_{\{\tau_0, \cdots, \tau_{i-1}\},\{\tau_i\}}] \cap \bigcup_{j=0}^{i-1} P[D_{\{\tau_j\},\{\tau_i\}}] \leq \sum_{j=0}^{i-1} P[D_{\{\tau_j\},\{\tau_i\}}] (w_{\tau_j}) + \prod_{j=0}^{i-1} P[\bigcup_{\tilde{w}_{\tau_j} \neq w} D_{\{\tau_j\},\{\tau_i\}} (w_{\tau_j} (\tilde{w}_{\tau_j}, w))] \]
\[
\leq i \cdot \epsilon + 2^n R_a 2^{-n \sum_{j=0}^{i-1} \Delta (m+2-j), n I(X_{\{\tau_j\}}^{(m+2-j)}; Y_{\{\tau_j\}}^{(m+2-j)}) - \epsilon')
\] (83)

Since \( \epsilon > 0 \) is arbitrary, the conditions of Theorem 6 and the AEP property will guarantee that the right hand side of (84) vanishes as \( n \to \infty \). Similarly, \( P[E_{b,a}] \to 0 \) as \( n \to \infty \). By Fenchel-Bunt’s extension of Carathéodory theorem in [3], it is sufficient to restrict \( |Q| \leq 2m + 2 \).

### APPENDIX III

**PROOF OF THEOREM 10**

We use Lemma 8 to prove the Theorem 10. For every \( S_R \subseteq \mathcal{R} \), we have 4 kinds of cut-sets, \( S_1 = \{a\} \cup S_R \), \( S_2 = \{b\} \cup S_R \), \( S_3 = \{a,b\} \cup S_R \) and \( S_4 = S_R \), as well as two rates \( R_a \) and \( R_b \). Also, in the TDBC protocol,
\[
Y_a^{(1)} = X_b^{(1)} = X_R^{(1)} = \emptyset \quad (85)
\]
\[
X_a^{(2)} = Y_b^{(2)} = X_R^{(2)} = \emptyset \quad (86)
\]
\[
X_a^{(3)} = X_b^{(3)} = Y_R^{(3)} = \emptyset \quad (87)
\]

The corresponding outer bounds for a given subset \( S_R \) are :
\[
S_1: R_a \leq \Delta_1 I(X_a^{(1)}; Y_b^{(1)}_{\tilde{S}_R}, Y_b^{(1)}_{S_R} | Q) + \Delta_3 I(X_b^{(3)}; Y_b^{(3)}_{\tilde{S}_R}, Y_a^{(3)}_{\tilde{S}_R} | Q) + \epsilon, \quad (88)
\]
\[
S_2: R_b \leq \Delta_2 I(X_b^{(2)}; Y_b^{(2)}_{\tilde{S}_R}, Y_a^{(2)}_{S_R} | Q) + \Delta_3 I(X_b^{(3)}; Y_a^{(3)}_{\tilde{S}_R}, Y_a^{(3)}_{S_R} | Q) + \epsilon, \quad (89)
\]

The cut-sets \( S_3 \) and \( S_4 \) yield no constraints. Since \( \epsilon > 0 \) is arbitrary, (88) and (89) yields the Theorem 10. By Fenchel-Bunt’s extension of the Carathéodory theorem in [3], it is sufficient to restrict \( |Q| \leq 2m + 1 \).

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