From Supermembrane to Matrix String

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Abstract

We develop a systematic method of directly embedding supermembrane wrapped around a circle into matrix string theory. Our purpose is to study connection between matrix string and membrane from an entirely 11 dimensional point of view. The method does neither rely upon the DLCQ limit nor upon string dualities. In principle, this enables us to construct matrix string theory with arbitrary backgrounds from the corresponding supermembrane theory. As a simplest application of the formalism, the matrix-string action with a 7 brane background (Kaluza-Klein Melvin solution) with nontrivial RR vector field is given.

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1. Introduction

Supermembrane is expected to play a pivotal role in the quest for the fundamental degrees of freedom of the conjectured M-theory. For example, in Matrix theory as the first concrete proposal along such direction, the matrix-regularized form of light-cone supermembrane action is reinterpreted as the effective theory of D-particles in the infinite-momentum frame boosted along the compactified 11th direction. In particular, the diagonal elements of matrix coordinates are identified with the transverse coordinates of D-particles. This means that the membrane itself is boosted along the compactified direction as a whole. In this picture, it is not possible to see fundamental strings directly. Although we assume that the off-diagonal elements would correspond to fundamental open strings attached to D-particles, it is difficult to exhibit such properties from the viewpoint of the regularization of membrane world volume. In the context of the so-called DLCQ limit, this is not unreasonable, since in this limit the length of open strings connecting D-particles must be regarded as being far too shorter than the typical string scale $\ell_s$, and therefore the stringy behavior of the open strings does not manifest itself. In spite of some nontrivial confirmations to two-loop order on the behaviors of graviton in the DLCQ limit, it is not at all clear whether this theory gives the complete description of gravity in 11 (and even in 10) dimensional space-time.

On the other hand, in matrix string theory which followed the proposal of Matrix theory, fundamental strings regain the role of the basic degrees of freedom. This was originally explained, on the basis of the Matrix theory conjecture, by combining T and S-dualities with the flipping of the compactified direction from 11th to 9th, which is now transverse to the light-like directions. Namely, the T-duality converts D-particles into D-strings. Then, D-strings are turned into fundamental (F) strings after using S-duality and making an inverse T-duality transformation. Also the D-particle quantum number is understood as electric flux. Direct visibility of fundamental strings in this way ensures that gravity is consistently described in the sense of 10 dimensional space-time, to the extent that matrix string is reducible to ordinary light-cone string in the weak string coupling limit $g_s \to 0$. However, as in the case of Matrix theory, the situation in 11 dimensions is quite obscure.
Now if we recall that fundamental string can also be understood, at least classically, as double-dimensionally reduced membrane from 11 dimensions, it is natural to ask whether and how precisely matrix string is related to such reduced membrane. We expect that the wrapped membrane should be directly mapped to matrix string without invoking duality arguments. This is not a trivial question, and to the best of our knowledge, such a possibility has never been studied in the published literature. We think that clarification of this correspondence will be important and useful for at least three reasons: (1) First of all, it is of intrinsic interest of its own to check overall consistency between the web of string dualities and the membrane-string connection from a purely 11 dimensional viewpoint. (2) It would provide a new hint on the treatment of the dynamics both of membrane and of matrix string, especially, with respect to the nature of the large $N$ limit of matrix string theory, and also on the nature of its decompactification limit $g_s \to \infty$, by clarifying the meaning and role of the off-diagonal elements of matrix string variables. (3) From a more practical point of view, it would help us to formulate the matrix string theory on general backgrounds, given the corresponding formulation of supermembrane, going beyond linearized approximation. All of these can be a first step towards the more crucial task of deriving 11 dimensional gravity from membrane-matrix-string theories.

With this motivation, we develop a systematic method of directly relating the wrapped (super)membrane to matrix string. Basically, we show that the off-diagonal matrix variables of matrix string theory are directly identified, in the large $N$ limit, with the higher Kaluza-Klein modes with respect to the world-volume momentum along the wrapped direction. This provides us a general prescription of embedding arbitrary membrane action into matrix-string action in the large $N$ limit. We also briefly study the nature of the double dimensional reduction of supermembrane quantum-mechanically. Our discussion shows that the double-dimensional reduction is a subtle problem which is common to the infra-red reduction from matrix string to the ordinary light-cone string theory in the weak coupling limit. As a simplest application of the general correspondence for the extension of the theory to nontrivial backgrounds, we discuss the matrix-string action in the background, so-called Kaluza-Klein Melvin solution, representing a 7-brane with a Ramond-Ramond (magnetic) vector field of arbitrary strength.

In the next section, we start from a brief review of light-cone supermembrane theory
and then discuss its compactification on a circle and quantum mechanical reduction. Some preliminary discussions on the quantum-mechanical double dimensional reduction, which is based on a strong-coupling expansion, is given in Appendix A. In section 3, we discuss the correspondence of membrane with the matrix string and present a general formula which enables us to map arbitrary trace-integrals of the matrix string variables into the corresponding integrals over the membrane volume. In section 4, we derive the matrix-string action for a 7-brane background with magnetic RR vector field. The relation of our result with previous works in the linearized approximation is briefly summarized in Appendix B. In the final section, we conclude the paper with discussions on remaining problems and future possibilities.

2. Light-cone supermembrane and its compactification

It is well known that the light-cone dynamics of supermembrane, in the sector of fixed total longitudinal momentum $P^+$, is summarized by the following effective action.

$$A = \frac{1}{\ell_M^3} \int d\tau \int_0^{2\pi L} d\sigma \int_0^{2\pi L} d\rho \mathcal{L},$$

$$w^{-1}\mathcal{L} = \frac{1}{2}(D_0 X^a)^2 + \bar{\psi} \gamma_- D_0 \psi - \frac{1}{4}(\{X^a, X^b\})^2 + i \bar{\psi} \gamma_- \Gamma_a \{X^a, \psi\}, \quad (2.1)$$

where the covariant time derivative $D_0$ is with respect to gauge field $A_0 : D_0 X^a = \partial_\tau X^a - \{A_0, X^a\}$ and the spatial index $a$ runs through $1, 2, \ldots, 9$. The density function $w$ is introduced in the gauge fixing process such that the longitudinal momentum $P^+(\sigma, \rho)$ satisfies $P^+(\sigma, \rho) = P^+ w(\sigma, \rho)/L^2$. The bracket notation is defined by

$$\{X^a, X^b\} = w^{-1}(\partial_\sigma X^a \partial_\rho X^b - \partial_\rho X^a \partial_\sigma X^b).$$

The time coordinate $\tau$ is related to the light-cone time by

$$\ell_M^3 P^+ \tau/(2\pi L)^2 = X^+, \quad (2.2)$$

such that the total center of mass (transverse) momentum $P^a$ contributes to the Hamiltonian in the standard form,

$$P^- dX^+ = \left(\frac{(P^a)^2}{2P^+} + \cdots\right) dX^+.$$
The membrane tension $1/\ell_M^3$ is assumed to be the fundamental M-theory scale, namely $\ell_M = g_s^{1/3}\ell_s$ in terms of the standard string theory parameters up to some numerical constant. We assumed that the space-time dimension is 11. This is justified if we can establish the validity of the double-dimensional reduction quantum-mechanically in the limit of small compactification radius, as we discuss later. We use the purely real Majorana representation for the $\Gamma$ matrices $(\gamma_{\pm} = (\Gamma^{10} \pm \Gamma^0)/\sqrt{2}, (\Gamma^0)^T = -\Gamma^0, (\Gamma^{10})^T = \Gamma^{10}, (\Gamma^a)^T = \Gamma^a)$. In ref. [7], everything is dimensionless. We recovered dimensions by normalizing the spatial world-volume parameters $(\sigma, \rho)$ as

$$0 \leq \sigma \leq 2\pi L, \quad 0 \leq \rho \leq 2\pi L, \quad \int d\sigma d\rho \ w = L^2$$

with $L$ being some arbitrary length parameter, which will later be chosen to be the radius of the compactification circle. Note that the arbitrariness of $L$ is manifest in the action: $L$ can be eliminated from the action by performing the rescaling $\tau \rightarrow L^2 \tau, \sigma \rightarrow L\sigma, \rho \rightarrow L\rho, \psi \rightarrow \psi/L$. For simplicity, we choose the density function $w(\sigma, \rho)$ to be constant so that $w = (2\pi)^{-2}$, by assuming that the topology of membrane is simply torus. In this convention, the Lagrangian density is dimensionless. The supersymmetry transformation law is given by

$$\delta_\epsilon X^a = -2i\tau\Gamma^a\psi,$$

$$\delta_\epsilon \psi = \frac{1}{2} \gamma_+ (D_0 X^a \Gamma_a + \gamma_-) \epsilon + \frac{1}{4} \{X^a, X^b\} \gamma_+ \Gamma_{ab} \epsilon,$$

$$\delta_\epsilon A = -i2\tau \psi. \quad (2.3)$$

The Gauss-law constraint derived from this action by the variation with respect to the gauge field $A_0$ gives the constraint corresponding to the area-preserving diffeomorphism (APD) which is the residual reparametrization symmetry $\delta X^a = \{\Lambda, X\}, \delta A_0 = \partial_0 \Lambda + \{\Lambda, A_0\}, etc$, after fixing to the light-cone gauge. More precisely, the Gauss-law constraint

$$\{D_0 X^a, X^a\} + i\{\bar{\psi}, \gamma_- \psi\} = 0$$

is the integrability condition for the equation determining the longitudinal coordinate $X^-$. The latter is

$$\frac{\ell_M^3}{(2\pi L)^2} P^+ \partial_j X^- + \partial_+ X^a \partial_j X^a + i\bar{\psi} \gamma_- \partial_j \psi = 0 \quad (2.4)$$
which is locally equivalent with the condition (2.4). When there exist topologically non-trivial cycles on membrane spatial world-volume, we have to further impose the global constraint of the form
\[ \int d\sigma \, (\partial_\tau X^a \partial_j X^a + i \bar{\psi} \gamma_\tau \partial_j \psi) = 0 \] (2.6)
to ensure that \( X^- \) is periodic along the cycles. Of course, if the \( X^- \) itself is compactified as in the DLCQ treatment which is not adopted in the present paper, the right-hand side should be proportional to integers times the corresponding compactification radius.

Now we study the situation where one, 9th, of the transverse directions is compactified along a circle by assuming this direction to be the ‘eleventh’ direction of M-theory. We set the radius of the circle to be \( L = g_s \ell_s \), by identifying it with the arbitrary parameter of the membrane action. Then, of course, \( L \) becomes a physical parameter. Denoting the 9-th coordinate by \( Y = X^9(\sigma, \rho, \tau) \) and choosing the world-volume coordinate \( \rho \) along the \( Y \)-direction, the compactification amounts to the condition
\[ Y(\sigma, \rho + 2\pi L, \tau) = 2\pi L + Y(\sigma, \rho, \tau). \] (2.7)
We denote the remaining eight transverse directions \((1, 2, \ldots, 8)\) by the indices \( i, j, k, \ldots \).

Classically, the double dimensional reduction assumes that everything is \( \rho \)-independent,
\[ \partial_\rho X^i = 0, \quad \partial_\rho A = 0, \quad \partial_\rho \psi = 0, \quad \partial_\rho Y = 1. \]
The action then reduces to
\[ A = \frac{2\pi L}{\ell^3_M} \int d\tau \int_0^{2\pi L} d\sigma \left[ \frac{1}{2} (\partial_\rho X^i)^2 - \frac{1}{2} (2\pi)^4 (\partial_\sigma X^i)^2 + i \bar{\psi} \gamma_\tau \partial_\rho \psi - i (2\pi)^2 \bar{\psi} \gamma_\tau \Gamma_9 \partial_\sigma \psi \right]. \] (2.8)
By the identification \((2\pi)^3 L/\ell^3_M = 1/2\pi\alpha'\) which is kept finite in the limit \( L \to 0 \), and by rescaling \( \sigma \to L\sigma, \tau \to L\tau/(2\pi)^2 \) and also by a change of normalization of the fermionic coordinate \( \psi \), this is nothing but the standard world-sheet action of the light-cone superstring in the Green-Schwarz formalism.

The naive double-dimensional reduction completely ignores the nonzero momenta along the compactified direction \( \rho \). It is, however, important to note that quantum

\[ ^\dagger \text{In ref. } \mathbb{8}, \text{it was emphasized that we should supplement the term } D_0 X^- \text{ to the lagrangian in the presence of winding. We drop this term, since it is a total derivative, as long as we assume periodicity for } X^-, \text{and hence does not play any substantial role below.} \]
mechanically the suppression of higher momentum (Kaluza-Klein) modes along the $\rho$ direction can not be so straightforward. One might naively expect that, just as the usual Kaluza-Klein compactification in ordinary field theory, the modes with nonzero momenta along the compactified direction would be suppressed by the large masses of order $1/L$ viewed from lower dimensional space. This is not the case in the double compactification of membrane. It is simply not possible to apply this standard argument to the membrane, since there arises no mass term in the usual sense. Instead of the ordinary mass term, we have the leading quadratic terms in the potential part of the action as

$$\frac{1}{4}\{X^i, X^j\}^2 \rightarrow \frac{1}{4}\left( (\partial_{\sigma}x^i + \partial_{\sigma}X^i)\partial_{\rho}X^j - (\partial_{\sigma}x^i + \partial_{\sigma}X^i)\partial_{\rho}X^j \right)^2$$

$$\rightarrow \frac{1}{2}(\partial_{\sigma}x^i)^2(\partial_{\rho}X^j)^2 - \frac{1}{2}(\partial_{\sigma}x^i \partial_{\rho}X^j)^2 + \cdots,$$ (2.9)

where we have separated the zero mode part $x^i(\sigma)$ by making the shift $X^i(\sigma) \rightarrow x^i(\sigma) + X^i(\sigma, \rho)$. Naively, this form behaves as $\sigma$-dependent mass terms for nonzero modes $X^i$. However, whether ignoring higher modes on the basis of this is justified seems a subtle question. A possible approach is to show explicitly that the effective theory after integrating over the infinite set of the higher modes is given by the superstring action in the limit $L \rightarrow 0$. Since $i\partial_{\rho} \sim O(1/L)$, we see that the strength of the fluctuations of nonzero Kaluza-Klein modes is in fact proportional $O(L)$. However, the Gaussian fluctuation generically gives rise to an order $O(L^0)$ contribution to the action, owing to the dependence of the coefficients on the ($\sigma$-dependent) zero modes. This cannot be neglected since the reduced action itself is of the same order $O(L^0)$ as long as $\alpha'$ is fixed in the limit $L \rightarrow 0$.

Note that the relevant expansion here is essentially the strong-coupling expansion with respect to the (gauge) coupling $g \sim 1/L = 1/g_s \ell_s \rightarrow \infty$ as $g_s \rightarrow 0$. The weak coupling perturbation theory (or semi-classical argument) is not suitable. As discussed in Appendix A, it is easy to check that, in the leading order in strong-coupling $1/g \sim L$ expansion, integrations over the higher modes precisely cancel between the bosonic and fermionic modes, leaving the double-dimensionally reduced action, except possibly at the

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$^5$ The subtlety here is of the same nature as has been discussed in ref. [10] for infrared reduction of $D = 4, N = 4$ Yang-Mills theory to a conformal field theory. The problem is common to the reduction to orbifold string from matrix string.
point where $(\partial_\sigma x^i)^2$ vanishes. Such points would just correspond to where interactions are occurring by the joining or splitting of strings. In the membrane picture with one more dimension, this kind of topology change would correspond to a smooth dynamical process at the vicinity of the interaction point.

However, strong coupling perturbative expansions in continuum field theory such as we encounter here are necessarily singular due to its ultra-local character, and hence it is not easy to rigorously justify the double-dimensional reduction in this way. We give a preliminary discussion on the higher-order effects in the strong coupling expansion in Appendix A. In any case, our argument implies that supersymmetry is very crucial for justifying the double-dimensional reduction quantum-mechanically: In the limit of vanishing string coupling $L \to 0$, the susy transformation law for the zero modes reduces to the ordinary linear transformation law, since the nonlinear term $\{X^a, X^b\} \Gamma_{ab} \epsilon$ which is the only possible term correcting the susy transformation law is of order $L^2/L = L$, and hence the zero mode action must take the ordinary form without any correction. This also shows that the critical dimension of supermembrane theory is $D = 11$ in the limit of vanishing compactification radius. In the case of bosonic membrane, there is no cancellation as above and the integration gives complicated (singular) contribution for the zero mode $x^i$, and hence it does not seem possible to justify the double-dimensional reduction quantum-mechanically.

Justification of the double-dimesional reduction is not the main concern of the present work. Rather, we would like to establish some direct relation between the compactified membrane and the large $N$ limit of matrix string theory. To prepare for our discussion about this direction in the next section, let us make more explicit how the compactified direction $Y$ behaves in the action. Using the same scaled variables as the naive double-dimension, the full action is

$$A = \frac{(2\pi)^2}{\ell_M^4} \int d\tau \int_0^{2\pi} d\sigma \int_0^{2\pi L} d\rho \left\{ \frac{1}{2} (D_0 X^i)^2 + \frac{1}{2} (D_0 Y)^2 - \frac{1}{4} \{X^i, X^j\}^2 - \frac{1}{2} \{X^i, Y\}^2 + i\psi^T D_0 \psi + i\psi^T \Gamma_1 \{X^i, \psi\} + i\psi^T \Gamma_9 \{Y, \psi\} \right\},$$

(2.10)

where the Poisson bracket is now defined as $\{A, B\} = \partial_\sigma A \partial_\rho B - \partial_\rho A \partial_\sigma B$ because of

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*In preparing the present manuscript, the work [9] appeared, discussing the critical dimension of bosonic membrane on the basis of the ordinary perturbative treatment of (bosonic) membrane.*
the rescaling. Note that in this form the action is invariant with respect to a global rescaling of two-dimensional space \((\tau, \sigma) \rightarrow (\lambda \tau, \lambda \sigma)\) with \(\psi \rightarrow \sqrt{\lambda}^{-1} \psi\). The light-cone gauge condition \(\gamma^+ \psi = 0\) implies \(\overline{\psi} \gamma^- = \sqrt{2} \psi^T\). We have changed the normalization of the fermion field to eliminate the factor \(\sqrt{2}\).

To take the condition of compactification into account, we redefine the \(Y\) coordinate by making a shift

\[ Y \rightarrow \rho + Y. \]

After this shift, all world-volume fields are assumed to be periodic with respect to \(\rho\). The terms affected by this substitution are

\[
\frac{1}{2} (D_0Y)^2 \rightarrow \frac{1}{2} (\partial_0 Y - \partial_\sigma A - \{A, Y\})^2,
\]

\[
\frac{1}{2} \{X^i, Y\}^2 \rightarrow \frac{1}{2} (\partial_\sigma X^i - \{Y, X^i\})^2,
\]

\[
\psi^T \Gamma_9 \{Y, \psi\} \rightarrow \psi^T \Gamma_9 (\partial_\sigma \psi - \{Y, \psi\}).
\]

Thus the final form of the action is, after rescaling \(\rho\) by \(\rho \rightarrow L \rho\),

\[
A = (2\pi)^2 L/\ell_M^3 \int d\tau \int_0^{2\pi} d\sigma \int_0^{2\pi} d\rho \\
\left[ \frac{1}{2} F_{0,\sigma}^2 + \frac{1}{2} (D_0X^i)^2 - \frac{1}{2} (D_\sigma X^i)^2 - \frac{1}{4L^2} \{X^i, X^j\}^2 \\
+ i\psi^T D_0 \psi - i\psi^T \Gamma_9 D_\sigma \psi + i\frac{1}{L} \psi^T \Gamma_i \{X^i, \psi\} \right]
\]

where now the covariant derivatives and the field strength are defined as

\[
F_{0,\sigma} = \partial_0 Y - \partial_\sigma A - \frac{1}{L} \{A, Y\},
\]

\[
D_0 X^i = \partial_0 X^i - \frac{1}{L} \{A, X^i\},
\]

\[
D_\sigma X^i = \partial_\sigma X^i - \frac{1}{L} \{Y, X^i\}
\]

and similarly for \(\psi\)'s. This is nothing but the two-dimensional gauge theory of APD, where \((A_0 = A, A_1 = A_\sigma = Y)\) plays the role of gauge field and the inverse compactification radius \(1/L\) is the gauge coupling constant. The (infinitesimal) gauge transformation is

\[
\delta A_r = L \partial_r \Lambda + \{\Lambda, A_r\}, \quad \delta X^i = \{\Lambda, X^i\}, \quad \delta \psi = \{\Lambda, \psi\}.
\]
At this point, the reader must recognize that the structure exhibited here is very close to that of matrix string theory. Indeed, if Poisson bracket is replaced by matrix commutator, it seems that the above action formally reduces to the matrix string action. However, the usual correspondence between $U(N)$ matrices and the two-dimensional phase space $(\sigma, \rho)$ does not work here. If it worked, the theory would have been reduced to $0+1$ dimensional matrix theory, but matrix string theory is a two($=1+1$)-dimensional gauge theory. A resolution of this small puzzle will be given in the next section by establishing a new direct correspondence between the two, which is an extension of the familiar correspondence introduced in [7].

3. Correspondence of wrapped membrane with matrix string

In order to motivate our method, let us first start from considering the Poisson bracket between a zero ($x(\sigma)$) and a nonzero mode part $X(\sigma, \rho)$ which is Fourier-decomposed as

$$X^j(\sigma, \rho) = \sum_n X^j_n(\sigma)e^{in\rho},$$

$$\{x^i, X^j\}(\sigma, \rho) = \partial_\sigma x^i \partial_\rho X^j(\sigma, \rho) = \sum_n \partial_\sigma x^i(\sigma)X^j_n(\sigma)ine^{in\rho}. \quad (3.1)$$

We compare this expression with the commutator in matrix string theory:

$$[x^i(\theta), X^j(\theta)]_{nm} = (x^i_n(\theta) - x^i_m(\theta))X^j_{nm}(\theta) \quad (3.2)$$

between $x^i$ with only diagonal matrix elements $x^i_{nm} \equiv x^i_n$ and a generic matrix with nonzero off-diagonal elements.

Suppose we consider a long string which satisfies the orbifold condition

$$x^i_k(\theta + 2\pi) = x^i_{k+1}(\theta), \quad (k = 1, 2, \ldots, N - 1), \quad x^i_N(\theta + 2\pi) = x^i_1(\theta) \quad (3.3)$$

for $N \gg 1$. Then it is natural to identify the diagonal components $x^i_n(\theta)$ with the membrane zero mode $x^i(\sigma)$ by

$$x^i(\sigma) = \begin{cases} x^i_1(\theta), & 0 \leq \sigma \leq 2\pi/N \\ x^i_2(\theta), & 2\pi/N \leq \sigma \leq 4\pi/N \\ \vdots \\ x^i_N(\theta), & 2(N-1)\pi/N \leq \sigma \leq 2\pi. \end{cases} \quad (3.4)$$
For sufficiently large $N$ and for generic $k, \ell$ such that $|k - \ell| \ll N$, this leads to

$$x^i_k(\theta) - x^i_\ell(\theta) = x^i(\sigma_{k\ell} + \frac{1}{2}(\sigma_k - \sigma_\ell)) - x^i(\sigma_{k\ell} - \frac{1}{2}(\sigma_k - \sigma_\ell)) = (\sigma_k - \sigma_\ell)\partial_\sigma x^i(\sigma_{k\ell}),$$

where we set as

$$\sigma_k = \frac{2(k - 1)}{N}\pi + \frac{\theta}{N}, \quad \sigma_\ell = \frac{2(\ell - 1)}{N}\pi + \frac{\theta}{N}$$

and

$$\sigma_{k\ell} = (\sigma_k + \sigma_\ell)/2.$$  

This shows that the commutator (3.2) of matrix-string theory and the Poisson bracket of doubly compactified membrane is identical in the large $N$ limit under the correspondence

$$\{x^i, X^j\}_n(\sigma) \equiv \int \frac{dp}{2\pi} e^{-inp}\{x^i, X^j\}(\theta, \rho) \leftrightarrow i\left(\frac{2\pi}{N}\right)^{-1}[x^i, X^j]_{k\ell}(\theta)$$

with

$$n = k - \ell, \quad \sigma = \sigma_{k\ell} = \frac{k + \ell - 2}{N}\pi + \frac{\theta}{N},$$

by identifying the matrix element $X_{k\ell}$ with the $n(= k - \ell)$-th Fourier mode

$$X^i_k(\sigma) = X^i_{k\ell}(\theta),$$

which obeys the condition

$$X^i_{k\ell}(\theta + 2\pi) = X^i_{k+1\ell+1}(\theta)$$

corresponding to the decomposition (3.4). This provides us a first hint for a direct mapping between membrane and matrix string. Namely, we start from the the diagonal elements (zero modes) and include the off-diagonal elements (higher Kaluza-Klein (KK) Fourier modes), gradually from near to far off-diagonals.

Our next task is to check whether this correspondence can be generalized to brackets between arbitrary functions of $\rho$. Let us first rewrite the general commutator between two matrices with nonzero off-diagonal components as

$$[X^i, X^j]_{k\ell} = X^i_{km}X^j_{m\ell} - X^j_{km}X^i_{m\ell} = X^i_{k-m}(\sigma_{km})X^j_{m-\ell}(\sigma_{m\ell}) - X^j_{k-m}(\sigma_{km})X^i_{m-\ell}(\sigma_{m\ell}).$$
Then, by using
\[ \sigma_{km} = \sigma_{k\ell} \frac{m - \ell}{N}, \quad \sigma_{m\ell} = \sigma_{k\ell} \frac{m - k}{N}, \]
we find (\(\sigma = \sigma_{k\ell}\))

\[
[X^i, X^j]_{k\ell}(\theta) = X^i_{km}(\theta) X^j_{m\ell}(\theta) - X^j_{km}(\theta) X^i_{m\ell}(\theta)
= X^i_{k-m}(\sigma) X^j_{m-\ell}(\sigma) - X^j_{k-m}(\sigma) X^i_{m-\ell}(\sigma)
+ X^i_{k-m}(\sigma) \frac{(m - k)\pi}{N} \partial_\sigma X^j_{m-\ell}(\sigma) + \frac{(m - \ell)\pi}{N} \partial_\rho X^i_{k-m}(\sigma) X^j_{m-\ell}(\sigma)
- X^j_{k-m} \frac{(m - k)\pi}{N} \partial_\sigma X^i_{m-\ell}(\sigma) - \frac{(m - \ell)\pi}{N} \partial_\rho X^j_{k-m}(\sigma) X^i_{m-\ell}(\sigma)
+ O(N^{-2}).
\]

After the summation over \(m\) (making the shift \(m \rightarrow -m + \ell + k\)), the first line vanishes, while the second and the third terms give the \(n(= k - \ell)\)-th Fourier mode of the corresponding Poisson bracket,

\[i(\frac{2\pi}{N})^{-1}[X^i, X^j]_{k\ell} \Leftrightarrow \int \frac{d\rho}{2\pi} e^{-i\rho} \{X^i, X^j\}\]

with the same identification (3.9) between the mode numbers in the large \(N\) limit and the world-volume coordinates.

The dictionary of the correspondence is thus summarized as

| Long string of matrix string theory | Doubly compactified membrane |
|------------------------------------|-----------------------------|
| \(\text{Tr} \frac{1}{N} \int_0^{2\pi} d\theta\) | \(\int_0^{2\pi} d\sigma \frac{1}{2\pi} \int_0^{2\pi} d\rho\) |
| \(i \frac{N}{2\pi} [A, B]\) | \(\{A, B\}\) |
| \(A_{k\ell}(\theta)\) | \(\int \frac{d\rho}{2\pi} e^{-i\rho} A(\sigma, \rho)\) |
| \(k - \ell\) | \(n\) |
| \(\theta\) | \(\sigma = \frac{(k + \ell - 2)\pi}{N} + \frac{\theta}{N}\) |
Essentially, the off-diagonal matrix elements of matrix string are nothing but the higher Kaluza-Klein momentum modes, and the average value of row and column indices indicates the position with respect to the $\sigma$ world-volume coordinate. The $\sigma$ space of periodicity $2\pi$ is decomposed into $N$ segments of length $2\pi/N$. Because of this, the integral over $\theta$ together with the trace operation can cover the spatial world volume of the membrane. Note that the segments corresponding to the off-diagonal elements $(k, \ell)$ with even $k + \ell$ and odd $k + \ell$, respectively, are shifted by half unit $\pi/N$ from each other.

In the above table, we assumed that the indices $k$ and $\ell$ take generic values such that $|k - \ell| \ll N$. When $N$ is finite, we have to specify the appropriate boundary condition with respect to the period of $\theta$ including the boundary region of the indices. Only natural choice, which is of the form of gauge transformation and is consistent with the above condition (3.11), is

$$A(\theta + 2\pi) = SA(\theta)S^\dagger, \quad S_{k\ell} = \delta^{(N)}_{k+1 \ell}, \quad (SS^\dagger = 1, \quad SN = 1) \quad (3.15)$$

where the Kronecker symbol $\delta^{(N)}_{k\ell}$ should be understood as being valid modulo $N$ with respect to the matrix indices $k, \ell$. For generic off-diagonal matrix elements, this coincides with (3.11) and also with the original orbifold condition for diagonal elements. To preserve the modulo $N$ property exactly including the off-diagonal elements, however, it becomes necessary to modify the assignment of $\sigma$-coordinates such that the off-diagonal elements can also be regarded as fields on the base $\sigma$-space. For example, the above boundary condition (3.15) indicates that the matrix element $A_{1N}(\theta)$ with shift $\theta \to \theta + 2\pi$ is continued to $A_{2N+1}(\theta) = A_{21}(\theta)$, the first Kaluza-Klein mode $\{A_{k+1}(\theta)\}$ in terms of the membrane picture. Similarly, $A_{2N}(\theta)$ is continued to $A_{3N+1} = A_{31}$ which is the second Kaluza-Klein mode. In general, this shows that the Kaluza-Klein modes are cut off such that the KK mode number does not exceed $N/2$.

This modulo $N$ property ($SN = 1$) requires the following modifications of the naive correspondence explained in the table. First, we require that all the fields be periodic under the shift $\sigma \to \sigma + \pi$, since the shift of the matrix indices $k \to k + N$ or $\ell \to \ell + N$ is equivalent to the shift $\sigma \to \sigma + \pi$. In the usual treatments of matrix string theory, off-

\[ \text{\[\text{Actually, there is a phase degree of freedom of the form } \delta^{(N)}_{k\ell} \exp[i(\beta_k - \beta_\ell)\theta]. \text{ The phase can however is absorbed by redefining the field } A(\theta). \]}

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diagonal matrix elements are completely neglected except at the interaction points, and then the orbifold boundary condition would not be important for off-diagonal elements. However, when the off-diagonal elements are kept on equal footing with the diagonal ones as in the present work, the half periodicity is a natural requirement in order to satisfy the orbifold condition (3.15) if we interpret the matrix variables as fields on the $\sigma$ space. Note that the half period can equivalently be formulated as the truncation of the Fourier modes with respect to $\sigma$ to only even modes. This is a consistent reduction in quantum theory, since it is closed under arbitrary algebraic manipulation. Also the quantum states of light-cone strings represented by the diagonal elements are not reduced at all, provided that one makes appropriate redefinition of operators both for zero and nonzero modes, because of global scale invariance remaining in light-cone string theory. Similarly, the change of periodicity can also be trivially done on the side of membrane, without changing the string-theory parameters, by using the global scale invariance of the action with respect to two-dimensional base space $(\tau, \sigma)$.

The second modification is that the $\rho$ space must be discretized so that we can restrict the KK momentum $|n|$ up to $|n| \leq N/2$. Thus, instead of a circle, the $\rho$ space must be assumed to be a $\mathbb{Z}_N$ 'clock' space, $\rho \in [2\pi n/N] \ (n = n + N)$. In the large $N$ limit, the clock space would smoothly be approximated by a continuous circle. To summarize these two modifications, the right hand column in the first line of the above table should be understood as

$$\int_0^{2\pi} d\sigma \frac{1}{2\pi} \int_0^{2\pi} d\rho \to 2 \int_0^{\pi} d\sigma \frac{1}{N} \sum_{\rho \in \mathbb{Z}_N} . \quad (3.16)$$

Correspondingly, the right hand column in the third line should also be modified to

$$\int \frac{d\rho}{2\pi} e^{-i\rho A(\sigma, \rho)} \to \frac{1}{N} \sum_{\rho \in \mathbb{Z}_N} e^{-i\rho A(\sigma, \rho)} . \quad (3.17)$$

With these modifications, the appropriate assignment of the $\sigma$-coordinate, corresponding to the treatment of the off-diagonal elements as Kaluza-Klein fields on the base $\sigma$-space, can be given as follows:

When $N=$odd,

$$\sigma = \begin{cases} \frac{(k+\ell-2)\pi}{N} + \frac{\theta}{N}, & |k - \ell| \leq \frac{N-1}{2} \\ \frac{(k+\ell-N-2)\pi}{N} + \frac{\theta}{N}, & |k - \ell| > \frac{N-1}{2} \end{cases} \quad (3.18)$$
When $N=\text{even}$,

$$\sigma = \begin{cases} 
\frac{(k+\ell-2)\pi}{N} + \theta, & |k-\ell| \leq \frac{N}{2} \\
\frac{(k+\ell-2)\pi}{N} + \theta, & k-\ell = \frac{N}{2} \\
\frac{(k+\ell-N-2)\pi}{N} + \theta, & k-\ell = -\frac{N}{2} \\
\frac{(k+\ell-N-2)\pi}{N} + \theta, & |k-\ell| > \frac{N}{2} 
\end{cases} \tag{3.19}$$

We have used the modulo $N$ property in translating the $\sigma$ coordinate by $\pi$ for the off-diagonal matrix elements with $|k-\ell| \geq N/2$. In the following, in order to avoid unnecessary complications, we always use the notation of the naive correspondence. But for finite $N$, all the above modifications must be tacitly assumed.

Under these caveats, we can now establish the following general formula between the integral of traces of matrix string variables and the corresponding membrane variables.

$$\frac{1}{N} \int d\theta \text{Tr}(M^{(1)}(\theta)M^{(2)}(\theta) \cdots M^{(\ell)}(\theta)) = \frac{1}{2\pi} \int d\rho \int d\tilde{\rho} \exp \left[ -\frac{i}{N} \sum_{\ell \geq \ell \geq 1} (\partial_\sigma, \partial_\rho, - \partial_{\tilde{\rho}}) \right] M^{(1)}(\sigma_1, \rho_1) \cdots M^{(\ell)}(\sigma_\ell, \rho_\ell) \bigg|_{\sigma_i=\sigma, \rho_i=\rho}.$$

In order to prove this, let us first consider a product of two arbitrary matrix-string fields $A(\theta), B(\theta)$.

$$(AB)_{k\ell}(\theta) = \sum_m \int d\rho_1 \int d\rho_2 \frac{2\pi}{\pi} e^{-i(k-m)\rho_1} e^{-i(m-\ell)\rho_2}$$

$$\times A(\sigma_{k\ell}(\theta), \rho_1) \exp \left( \frac{m-\ell}{N} \frac{\pi}{\partial_\sigma} \right) \exp \left( \frac{m-k}{N} \frac{\pi}{\partial_\sigma} \right) B(\sigma_{k\ell}(\theta), \rho_2).$$

Here the exponential differential operators appeared to take account of the difference of the $\sigma$ coordinates of the two fields, and the $\sigma_{k\ell}(\theta)$ is the $\sigma$-coordinate determined by the indices $k, \ell$ as above. By making a change of integration variables $(\rho_1, \rho_2) \to (\rho = \rho_1, \tilde{\rho} = \rho_2 - \rho_1)$, this can be rewritten as

$$(AB)_{k\ell}(\theta) = \sum_m \int d\rho \int d\tilde{\rho} \frac{2\pi}{\pi} e^{-i(k-\ell)\rho} e^{-i(m-\ell)\tilde{\rho}}$$

$$\times A(\sigma_{k\ell}(\theta), \rho) \exp \left( \frac{m-\ell}{N} \frac{\pi}{\partial_\sigma} \right) \exp \left( \frac{m-\ell}{N} \frac{\pi}{\partial_\sigma} \right).$$
Performing the $\tilde{\rho}$ integral (sum) which leads to Kronecker $\delta$ constraint $m - \ell = -i\partial_\rho$, we can eliminate the summation over $m$ and further make the partial integral (sum) over $\rho$, we arrive at the formula

$$(AB)_{k\ell}(\theta) = \int \frac{d\rho}{2\pi} e^{-i(k-\ell)\rho} \left( A(\sigma_{k\ell}(\theta), \rho) \exp \left[ -i\frac{\pi}{N} (\partial_\sigma \partial_\rho - \partial_\rho \partial_\sigma) \right] B(\sigma_{k\ell}(\theta), \rho) \right)$$

where the derivatives on the exponential inside the big bracket act only within the bracket.

Note that this general expression is valid for finite $N$ too under the replacements explained before. In particular, the exponential operator $\exp \left[ -i\frac{\pi}{N} (\partial_\sigma \partial_\rho - \partial_\rho \partial_\sigma) \right]$ and those appeared in the above manipulation are all well defined in the discrete clock space $\mathbb{Z}_N$ since the eigenvalues of $i\partial_\sigma$ are even integers. Thus the naive correspondence, including the correspondence between commutator and Poisson bracket which motivated our discussion, is naturally extended to finite $N$ theory with modulo $N$ property. For example, expansion in $1/N$ trivially gives the correspondence (3.14).

Now, applying the above product formula to a general multiple product, we obtain

$$\sum_{i_2,i_3,\ldots,i_{\ell}} M_{i_1i_2}^{(1)}(\theta) M_{i_2i_3}^{(2)}(\theta) \cdots M_{i_{\ell}i_{\ell+1}}^{(\ell)}(\theta) =$$

$$\int \frac{d\rho}{2\pi} e^{-i(i_1-i_{\ell+1})\rho} \exp \left[ -i\frac{\pi}{N} \sum_{\ell_1,\ell_1+1} (\partial_{\sigma_{\ell_1}} \partial_{\rho_{\ell_1}} - \partial_{\rho_{\ell_1}} \partial_{\sigma_{\ell_1}}) \right] M_{i_1i_2}^{(1)}(\sigma_1, \rho_1) \cdots M_{i_{\ell}i_{\ell+1}}^{(\ell)}(\sigma_{\ell}, \rho_{\ell}) \bigg|_{\sigma_{i_1}=\sigma, \rho_{i_1}=\rho},$$

where $\sigma = \sigma_{i_1+i_{\ell+1}}$. Taking trace of this expression implies $i_1 = i_{\ell+1}$ and $\sum_{i_1=i_{\ell+1}} \int d\theta/N = \int d\sigma$, so that we arrive at the promised formula (3.20). Although the formula does not look manifestly cyclically symmetric, we can easily prove cyclic symmetry using partial integrations $\sum_{i=1}^\ell \partial_{\rho_i} = \sum_{i=1}^\ell \partial_{\sigma_i} = 0$: Indeed, the exponentiated differential operator in the formula can be replaced by

$$\exp \left[ -i\frac{\pi}{N} \sum_{\ell_1,\ell_1+2} (\partial_{\sigma_{\ell_1}} \partial_{\rho_{\ell_1}} - \partial_{\rho_{\ell_1}} \partial_{\sigma_{\ell_1}}) \right] = \exp \left[ -i\frac{\pi}{N} \sum_{\ell-1,\ell+1} (\partial_{\sigma_{\ell}} \partial_{\rho_{\ell}} - \partial_{\rho_{\ell}} \partial_{\sigma_{\ell}}) \right]$$

which allows us to cyclically change the positions of the matrices located at ends inside the trace.

We note that the above formulas are very similar to the well known Moyal product but are not identical: The similarity comes from the resemblance of our identification of KK
modes and \( \sigma \)-coordinate with matrix indices to the Wigner representation of matrices. The difference comes from our asymmetrical treatment of the base space coordinates \( \sigma \) and \( \rho \), in that the former is combined with the base-space coordinate \( \theta \) of matrix fields while the latter is not (and is discretized for finite \( N \)). Note that the combination of \( \sigma \) and \( \theta \) is directly responsible to the orbifold condition on the matrix-string side.

Now, using the above formula, we can derive, for instance, the correspondence

\[
\frac{1}{N} \int d\theta \text{STr} \left( [M^{(1)}(\theta), M^{(2)}(\theta)] [M^{(3)}(\theta), M^{(4)}(\theta)] M^{(3)}(\theta) \cdots M^{(l)}(\theta) \right)
\]

\[
= -\frac{1}{2\pi} \left( \frac{2\pi}{N} \right)^2 \int d\sigma d\rho \left\{ M^{(1)}(\sigma, \rho), M^{(2)}(\sigma, \rho) \right\} \left\{ M^{(3)}(\sigma, \rho), M^{(4)}(\sigma, \rho) \right\}
\times M^{(3)}(\sigma, \rho) \cdots M^{(l)}(\sigma, \rho) (1 + O(1/N^2))
\] (3.24)

in the large \( N \) limit, where the symmetrized trace (STr) means to treat the commutators in the left hand side as single matrices. The fact that the correction is of order \( O(1/N^2) \) is owing to the symmetrized trace. Using this and similar formulas, we can convert arbitrary terms of the membrane action into the corresponding ones of matrix string theory in the large \( N \) limit. Thus it is clear that most symmetry properties of the supermembrane theory are also valid in the matrix-string representation interpreted in our way up to the order \( O(1/N^2) \) corrections, provided that the symmetry transformation can consistently be expressed using the matrix-string degrees of freedom. Furthermore, when the theory is extended to various nontrivial backgrounds, the corresponding symmetries should also be ensured in similar ways.

It is now straightforward to map the supermembrane action into matrix representation by using the established correspondence. Using (3.20) (in particular (3.24)), the membrane action (2.12) is rewritten as, up to \( O(1/N^2) \) corrections,

\[
A = \frac{(2\pi)^2 L}{\ell_M^3} \int d\tau \frac{2\pi}{N} \int_0^{2\pi} d\theta \text{Tr} \left( \frac{1}{2} F_{0,\theta}^2 + \frac{1}{2} (D_{\theta} X^i)^2 - \frac{1}{2} N^2 (D_{\theta} X^i)^2 + \frac{1}{4 L^2} \left( \frac{N}{2\pi} \right)^2 [X^i, X^j]^2 
+ i \psi^T D_{\theta} \psi - N i \psi^T \Gamma_9 D_{\theta} \psi - \frac{1}{L} \frac{N}{2\pi} \psi^T \Gamma_i [X^i, \psi] \right),
\] (3.25)

where

\[
D_{\theta} X^i = \partial_{\theta} X^i - i \frac{1}{2\pi L} [Y, X^i],
\] (3.26)

\[
D_0 X^i = \partial_0 X^i - i \frac{N}{2\pi L} [A, X^i],
\] (3.27)
\[ F_{0,\theta} = \partial_{\tau} Y - N \partial_{\theta} A - i \frac{N}{2\pi L} [A, Y], \quad (3.28) \]

and similarly for fermion variables. By performing the redefinition

\[ \tau \to \tau/N, \quad L \to L/2\pi, \quad \psi \to \sqrt{N} \psi, \]

the \( N \) dependence is eliminated and the action is reduced to the standard matrix-string theory action. Assuming that the physical light-cone time \( X^+ \) must be independent on \( N \), this rescaling of the time coordinate requires, by the relation (2.2), that the total longitudinal momentum \( P^+ \) scales with \( N \) as

\[ P^+ \to NP^+ \quad (3.29) \]

which coincides with the correct scaling for the matrix-string theory interpretation. Namely, the diagonal matrix elements of \( X^i \) represent a fundamental string bit in the large \( N \) limit and consequently the total longitudinal momentum is proportional to the number \( N \) of the string bits.\(^{**}\)

In this way, we have succeeded to derive the matrix-string theory in the large \( N \) limit directly from the supermembrane action. One might wonder what is the relation of this method with that based on the standard method [11] of compactifying general matrix models of D-branes. The above result shows that each segment, parametrized by \( \theta \), of the \( \sigma \)-space corresponds to the collection of an infinite number of image spaces of a D0-brane. The \( \theta \) parameter is nothing but the conjugate coordinate to the winding number of the image spaces. At this point, we would like to remind the reader of the fact that usual matrix models of D-branes should however be regarded as the effective low-energy description of D-branes keeping only the lowest modes of open strings. Then, results obtained by the standard prescription should also be regarded as being rigorously valid only in some special situation such as in the DLCQ limit, where the low-energy approximation can be trusted. In contrast to this, our method is basically independent of any such assumptions, and hence it seems reasonable to expect more general applicability of our method than the standard approach at least in the context of establishing connection between matrix string and membrane.

\(^{**}\) As discussed in the Appendix of the first reference in [4], the normalization of the matrix-string action in the convention where explicit \( N \) dependence is completely eliminated gives \( P^+ \propto N \) in agreement with (3.29).
We note also that our identification of the off-diagonal components of matrix strings with the Kaluza-Klein modes of membrane along the compactified direction is consistent with the usual approach based on T- and S-dualities, which suggests that the off-diagonal components would correspond to the fluctuating fields associated with bound states of D0 and F1 string bits. Indeed, D0 charge is carried by the Kaluza-Klein momentum along the compactified M-theory. If we assume further that each string bit can only carry the smallest unit of Kaluza-Klein momentum, the mass of the fluctuating field should be proportional to $|\partial_\sigma x|/g_s$, which is indeed the case as exhibited in (3.1).

In the language of matrix string theory, the interaction of strings in the limit of small $g_s$ has been shown to be understood as resulting from the world-sheet instanton effect \[12\] where the coincident diagonal matrix elements are permuted. By our mapping between the membrane picture and matrix-string picture, the singular topology change of the strings corresponding to the vanishing of $\partial_\sigma x^{\tau}$ in the membrane picture is now mapped into this instanton effect in the matrix-string picture. However, we have to keep in mind that the same difficulty as we have discussed in connection with quantum-mechanical double-dimensional reduction of membrane still remains in reducing the matrix string action to the light-cone string action by integrating over the off-diagonal matrix variables. Also, the fact that membrane should actually be interpreted as the second quantized theory by the matrix representation is equally true as in the ordinary matrix regularization. As in the latter case, taking the large $N$ limit of the matrix string theory should really amount to providing a proper way of defining the supermembrane theory.

Finally, we have to recall that, for the existence of longitudinal coordinate $X^{-}$, the condition (2.6) must be imposed along the $\rho$-direction. This will be important for discussing the dynamics. Keep in mind however that, in a finite $N$ approximation, there is no obvious counterpart to this condition, since at least apparently the Gauss-law constraint for finite $N$ cannot be interpreted as the integrability condition.

\[11\] To our knowledge, there has been no literature discussing explicitly the physical meaning of the off-diagonal components of matrix string theory. Only reference which is related to this question seems to be \[12\], where it is shown that the one-loop quantum fluctuation of the off-diagonal components leads to D0-D0 creation.
4. Matrix-string theory action in nontrivial background: An example

We have emphasized that one of the possible merits of establishing direct correspondence between membrane and matrix string is that it enables us to write down the matrix-string theory in nontrivial backgrounds, since on the side of membrane we can in principle know the form of the action on arbitrary background which satisfies the field equation of 11 dimensional supergravity \[13\]. Indeed, one of the natural methods of studying string theory in the presence of nontrivial RR backgrounds has been to start from supermembrane in 11 dimensions and to perform classical double-dimensional reduction. In this section, as a simplest nontrivial application of our method to this direction, we construct the matrix-string action in a RR background which is called Kaluza-Klein Melvin (flux 7-brane) background. Recently, the backgrounds of this type have become a focus of some interests in connection to the possible duality relation between type 0 and II theories. See, e.g., \[16, 17, 18\].

The Kaluza-Klein Melvin background in 10 dimensions is obtained from the flat space-time in 11 dimensions by the compactification with non-trivial topology. We pick up two, say, 7th and 8th, of the transverse coordinates and make the following identification mixing them with the compactified 9th coordinate:

\[
(r, y, \varphi) \simeq (r, y + 2\pi L m, \varphi + 2\pi q L m + 2\pi n) \tag{4.1}
\]

where \(y \equiv x^9\) and \(x^7 + ix^8 = re^{i\varphi}\). That is, we combine the \(2\pi L\) shift in the \(y\)-direction with the \(2\pi q L\) rotation in the \(x^7\)-\(x^8\) plane. For our later purpose, it is more convenient to define the coordinates which are single-valued in the \(y\)-direction:

\[
x^7_{\text{flat}} + ix^8_{\text{flat}} = e^{iy}(x^7 + ix^8), \quad \psi_{\text{flat}} = e^{-\frac{2}{3} F_{78} y} \psi \tag{4.2}
\]

where the subscript ‘flat’ denotes the original coordinates of flat 11-dimensions, which after imposition of the periodicity condition \((4.1)\) are no more single-valued when \(qL \neq \text{integers}\) (in the case of fermion when \(qL \neq \text{even integers}\)). Because of this, the system after this transformation can describe a nontrivial curved background. The 11-dimensional metric after this transformation is given as

\[
ds_{11}^2 = -dt^2 + dx_1^2 + \cdots + dx_6^2 + dr^2 + r^2(d\varphi + q dy)^2 + dy^2 + dx_{10}^2. \tag{4.3}
\]
Following the standard relation between the metric in 11 dimensions and the string-frame metric, dilaton and RR 1-form in 10 dimensions

\[ ds^{2}_{11} = e^{-2\phi/3} ds^{2}_{10} + e^{4\phi/3} (dx_{11} + A_\mu dx^\mu)^2, \]  

(4.4)

we observe that the 10-dimensional string-theory background corresponding to the above 11D metric is given as

\[ ds^{2}_{10} = f(r)[-dt^2 + dx_1^2 + \cdots + dx_6^2 + dr^2 + r^2 f^{-2}(r) d\varphi^2 + dx_{10}^2] \]

\[ e^\phi = f^{3/2}(r), \quad A_\varphi = qr^2 f^{-2}(r) \]

\[ f(r) = (1 + q^2 r^2)^{1/2} \]  

(4.5)

The RR vector field is magnetic, and both 10 D metric and the dilaton are nonpolynomial. Since the fermion can acquire \(-1\) (\(\sim\) anti-periodic boundary condition along the \(y\)-direction) under the shift of \(q\), the meaningful range of the magnetic charge \(q\) is \(-1/L < q \leq 1/L\). Since our purpose here is only to demonstrate a simple application of our formalism for nontrivial backgrounds, we assume the ordinary periodic boundary condition for spinor field \(\psi\). To treat the case of antiperiodic boundary condition, it is necessary to modify our prescription appropriately. We will touch upon this only very briefly in the end of this section.

The matrix-string action in this background is obtained from the supermembrane action in the flat background, simply by rewriting it in terms of the new rotated (single-valued) fields via (4.2) and by using the correspondence established in the last section.

We start from the flat space action

\[ A = \int d\tau d\sigma dp \left[ \frac{1}{2} (D_0 X^a)^2 + i \psi^T D_0 \psi - \frac{1}{4} \{X^a, X^b\}^2 + i \psi^T \Gamma_a \{X^a, \psi\} \right] \]  

(4.6)

where we have already redefined the normalization of \(\psi\) from (2.1). Also note that in this section, we set the length scales \(L = 1, 2\pi a' = 1\) for simplicity of formulas. Now by applying the transformation (4.2), the action in terms of the rotated fields is given

\[ \text{‡‡ In preparing the present manuscript, we became aware of the work [14] which discusses a matrix-string version of the Kaluza-Klein Melvin background from the viewpoint of the standard DLCQ approach to the compactification of Matrix models. We hope that our treatment provides a complementary account to this important problem.} \]
by replacing the world volume derivatives $\partial_\alpha$ ($\alpha = \tau, \sigma, \rho$) with the following ‘covariant
derivatives’ $\nabla_\alpha$

$$
\nabla_\alpha X^m = \partial_\alpha X^m + q \partial_\alpha Y \epsilon^{mn} X^n, \\
\nabla_\alpha X^i = \partial_\alpha X^i, \ \nabla_\alpha Y = \partial_\alpha Y, \\
\nabla_\alpha \psi = \partial_\alpha \psi - \frac{q}{2} \partial_\alpha Y \Gamma_{78} \psi,
$$

(4.7)

where the indices $i$ denote the ‘trivial’ transverse directions ($i = 1, \ldots, 6$) and the directions where rotation is performed are indicated by $m, n, p = 7, 8$. The action reads

$$
A = \int d\tau d\sigma d\rho \left[ \frac{1}{2} (D_0 Y)^2 + \frac{1}{2} (D_0 X^i)^2 + \frac{1}{2} (\nabla_0 X^m - \{A, X^m\}_\nabla)^2 - \frac{1}{2} (Y, X^i)^2 \\
- \frac{1}{2} \{Y, X^m\}_\nabla^2 - \frac{1}{4} \{X^i, X^j\}_\nabla^2 - \frac{1}{2} \{X^i, X^m\}_\nabla^2 - \frac{1}{4} \{X^m, X^n\}_\nabla^2 + i \psi^T \nabla_0 \psi \\
- i \psi^T \{A, \psi\}_\nabla + i \psi^T \Gamma_i \{X^i, \psi\}_\nabla + i \psi^T \Gamma_8 \{Y, \psi\}_\nabla + i \psi^T \Gamma_m \{X^m, \psi\}_\nabla \right],
$$

(4.8)

where subscript $\nabla$ means that the Poisson bracket is evaluated using the above covariant
derivatives. The action has the order $O(q^0)$-, $O(q^1)$- and $O(q^2)$- parts. The $q^0$-part is

course the same as the original action except for the fact that the fields are now the
redefined ones. The $q^1$- and $q^2$- parts are given as

$$
A^{q^1} = q \int d\tau d\sigma d\rho \epsilon^{mn} \left[ - D_0 Y D_0 X^m X^n + \{X^i, Y\}_\nabla \{X^i, X^m\}_\nabla X^n \\
+ \{X^p, Y\}_\nabla \{X^p, X^m\}_\nabla X^n - i \psi^T \Gamma_m X^n \{Y, \psi\} \\
- \frac{i}{4} \psi^T \Gamma_{mn} \psi D_0 Y - \frac{i}{4} \psi^T \Gamma_i \Gamma_{mn} \psi \{X^i, Y\} \right],
$$

(4.9)

$$
A^{q^2} = q^2 \int d\tau d\sigma d\rho \left[ \frac{1}{2} (D_0 Y)^2 (X^m)^2 - \frac{1}{2} \{X^i, Y\}_\nabla^2 (X^m)^2 + \frac{1}{2} (X^m \{X^m, Y\})^2 \right]
$$

(4.10)

Here we have directly applied the coordinate transformation to the light-cone action.

However, it is easy to check that we obtain the same result if we first make the coordinate
transformation and afterwards go to the light-cone gauge. The covariant action resulting
from the latter procedure is in fact given in [16]. We have explicitly checked that the
Hamiltonian obtained from our procedure (i.e. using the action (4.9), (4.10)) in the
$A_0 = 0$ gauge agrees with the light-cone-gauge Hamiltonian obtained from the covariant
action of [16]. This is as expected since the coordinate rotation is performed in the
transverse directions, so the light-cone gauge fixing and the rotation should commute.
We also note that when the gauge field $A_0$ is integrated over, the dependence on the background charge $q$ becomes nonpolynomial.

To study the compactified membrane, we perform the shift $Y \rightarrow \rho + Y$ in the above action. As we have seen in section 2, after the substitution, $Y$ plays the role of the $\sigma$-component of 2-dimensional gauge field. The resulting order $O(q)$ and $O(q^2)$ actions are

$$A^q = q \int d\tau d\sigma d\rho \epsilon^{mn} \left[ -F_{0,\sigma} D_0 X^m X^n + D^Y_\sigma X^i \{X^i, X^m\} X^n 
+ D^Y_\sigma X^p \{X^p, X^m\} X^n + i\psi^T \Gamma_m X^n D^Y_\sigma \psi 
- \frac{i}{4} \psi^T \Gamma_{mn} \psi F_{0,\sigma} - \frac{i}{4} \psi^T \Gamma_i \Gamma_{mn} \psi D^Y_\sigma X^i \right],$$

(4.11)

$$A^{q^2} = q^2 \int d\tau d\sigma d\rho \left[ \frac{1}{2} (F_{0,\sigma})^2 (X^m)^2 - \frac{1}{2} (D^Y_\sigma X^i)^2 (X^m)^2 - \frac{1}{2} (X^m D^Y_\sigma X^m)^2 \right],$$

(4.12)

where $D^Y_\sigma$ and $F_{0,\sigma}$ are those defined in section 2 (with $L = 1$).

Now we follow the correspondence between compactified membrane and Matrix string and obtain the Matrix-string representation of the action. The dictionary of the correspondence is given in the table in section 3. Especially, the Poisson brackets of membrane fields correspond to the commutators of matrices in the leading order in the large $N$ limit. Also, the orbifold boundary condition (3.15) must be kept in mind. $q$-independent part of the action is quadratic in the fields when we regard Poisson brackets as a single unit, and it was shown in section 3 that this part is mapped to the Matrix-string action in the flat background. The order $q^1$-part and $q^2$-part are cubic and quartic in the fields respectively, treating the Poisson bracket as a single unit. Using the general formula (3.20), we can easily see that to the leading order in the large $N$ limit up to $O(1/N^2)$ correction, they are expressed using the symmetrized trace as follows.

$$A^q = q \int d\tau d\theta \epsilon^{mn} \text{STr} \left[ -F_{0,\theta} D_0 X^m X^n + i D_\theta X^i \{X^i, X^m\} X^n 
+ i D_\theta X^p \{X^p, X^m\} X^n + i\psi^T \Gamma_m X^n D_\theta \psi 
- \frac{i}{4} \psi^T \Gamma_{mn} \psi F_{0,\theta} - \frac{i}{4} \psi^T \Gamma_i \Gamma_{mn} \psi D_\theta X^i \right],$$

(4.13)

$$A^{q^2} = q^2 \int d\theta \text{STr} \left[ \frac{1}{2} (F_{0,\theta})^2 (X^m)^2 - \frac{1}{2} (D_\theta X^i)^2 (X^m)^2 - \frac{1}{2} (X^m D_\theta X^m)^2 \right].$$

(4.14)
where we have performed the rescaling as in the flat case. The definitions of covariant derivatives and field strength are as follows.

\[ D_\theta X^i = \partial_\theta X^i - i[Y, X^i], \quad D_0 X^i = \partial_\tau X^i - i[A, X^i], \]

\[ F_{0,\theta} = \partial_\tau Y - \partial_\theta A - i[A, Y]. \]

In the linearized approximation where we ignore the \( O(q^2) \) terms, we can derive equivalent results as ours using the membrane vertex operator or matrix-string vertex operator as studied in refs. [19] or [20], respectively, provided we perform a field redefinition of the space-time spinor field corresponding to a change of local Lorentz frame. In contrast to the linearized approximation, however, our result should be valid to all orders in \( q \). For the sake of future reference and also as a consistency check, we briefly describe the correspondence of our result with the linearized approximation in Appendix B.

Finally, let us briefly touch upon the case where the transformed spinor \( \psi \) is antiperiodic along the \( \rho \) direction. In this case, we need to introduce half-integer Kaluza-Klein modes for spinors. The Fourier decomposition

\[ \psi(\sigma, \rho) = \sum_n \psi_{n+1/2}(\sigma) e^{i(n+1/2)\rho} \] (4.15)

and the Poisson bracket between the zero mode coordinate

\[ \{ x(\sigma), \psi(\sigma, \rho) \} = i \sum \partial_\sigma x(\sigma) (n + \frac{1}{2}) \psi_{n+1/2}(\sigma) e^{i(n+1/2)\rho} \] (4.16)

suggest that a natural extension of our procedure discussed in section 3 is to double the range of matrix indices \( N \to 2N \) and to assume that the bosonic matrix variables have only even-even and odd-odd elements while the spinor matrix variables have odd-even (or even-odd) elements. Thus the number of (real) matrix components for bosons is \( 2N^2 \). As \( 2N \times 2N \) matrices, the off-diagonal matrix elements of bosonic matrix-string fields are nonzero only for ‘even’ off diagonal lines with the differences between row and column being restricted to even integers. Correspondingly, the fermion matrices are now assumed to be complex \( N \times N \). Let us denote even integers by \( m, n, \ldots \) and odd ones by \( p, q, \ldots \). We associate the integer KK mode numbers with differences among \( (m, n, \ldots) \) and among \( (p, q, \ldots) \), while the half-integer KK modes numbers with differences between \( (m, n, \ldots) \)
and \((p, q, \ldots)\). Then the correspondence of matrix-string field and membrane field for spinors is

\[
\psi_{p,m}(\theta) \leftrightarrow \psi_{(p-m)/2}(\sigma)
\]

with the hermiticity condition \(\psi_{p,m}^\dagger = \psi_{m,p}\). As before the correspondence of the \(\sigma\) and \(\theta\) coordinates is \(\sigma = (p + m - 2)\pi/2N + \theta/2N\) \((p, m \text{ modulo } 2N)\). The bracket relation (4.16) can then be naturally interpreted as a commutator between bosonic and spinor matrices, for the derivative \((n + 1/2)\partial_n x\) can be interpreted as the difference between the odd-odd and even-even diagonal matrices of bosons.

The gauge symmetry is now \(U(N) \times U(N)\) under which the bosonic matrix variables transform as adjoint representation, while the spinors transform as the bifundamental \((\overline{N}, N)\) representation. Note also that the boundary condition with respect to \(\theta \rightarrow \theta + 2\pi\) connects the even-even and odd-odd matrices in the bosonic case, while in the fermionic case it connects even-odd to odd-even. The appearance of similar gauge structure for type 0A or Melvin background has previously been pointed out in \([15, 14]\) within the standard approach to matrix string. In our approach, the theory is completely local on the world volume because of our transformation (4.2) to the single-valued (double-valued for antiperiodic fermions) world-volume fields at the level of the membrane theory. Although we do not elaborate along this direction further in the present paper, it would be an interesting problem to investigate membrane and matrix strings in the Kaluza-Klein Melvin background using our approach from the viewpoint of duality of type 0 and II theories.

5. Concluding remarks

In the present work, we have developed a method of directly mapping the theory of supermembrane wrapped along a circle to matrix string theory and discussed its implications and applications. We hope that our observations may provide new impetus toward further exploration of the dynamics of membranes and matrix strings. Here we mention some remaining problems and future possibilities.

An interesting possibility is to apply our method to the covariant treatments of (super) membrane. For a relatively recent attempt of covariant quantization of membrane in 10
dimensional sense, see e.g. ref. [21]. Unfortunately, the method of the latter reference does not seem convenient for double compactification, because of their gauge choice for fermion degrees of freedom. If this were successfully done, we would have a covariant version of matrix string theory. It is also desirable to prove Lorentz invariance directly using the matrix-string language.

In connection to covariantization, another intriguing question might be whether our procedure of making correspondence between Poisson bracket and matrix commutator can be extended usefully to higher bracket structures such as the Nambu bracket, which seems to be relevant [22] for fully covariantized formulation of membranes.

Besides these and other possibilities of extending our formalism, one of the most crucial problems related to our work is to study the possibility of generating 11 dimensional gravity dynamically from supermembrane or matrix string. Since our correspondence provides a clear physical picture for matrix string variables including the off-diagonal elements from the viewpoint of 11 dimensional membrane theory, we may proceed to explore the behavior of D-particles using membrane-matrix picture in the decompactification limit $L \to \infty$. For this direction, it seems important to formulate the detailed dynamical properties of D-particles in our membrane-matrix-string approach.

We hope to return to some of these issues in the near future.

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Appendix A

In this Appendix, we report preliminary results of a quantum mechanical study on the double dimensional reduction. It is believed that the supermembrane in 11 dimensions wrapped around a circle becomes the type IIA superstring in 10 dimensions by a simultaneous dimensional reduction of the world-volume and space-time. In [23], this was
shown classically, i.e. by simply discarding the dependence of the fields on the compact
direction. Let us try to give a justification to this picture by studying the quantum effective action. Contrary to the classical argument, we keep the Kaluza-Klein modes on the
membrane world-volume and integrate them out. We analyze the behavior in the limit of
small compactification radius by a strong coupling expansion and see whether quantum
corrections affect the string action.

Start from light-cone supermembrane action

\[ A = \frac{(2\pi)^2}{\ell_M^3} \int d\tau \int_0^{2\pi} d\sigma \int_0^{2\pi L} d\rho \left( \frac{1}{2} (D_0 X^i)^2 + \frac{1}{2} (D_0 Y)^2 - \frac{1}{4} \{X^i, X^j\}^2 - \frac{1}{2} \{X^i, Y\}^2 + i\psi^T D_0 \psi + i\psi^T \Gamma_i \{X^i, \psi\} + i\psi^T \Gamma_9 \{Y, \psi\} \right). \]  

(A.1)

To study the compactified membrane, we set the background of \( Y \) equal to the world volume spatial direction \( \rho \).

\[ Y(\sigma, \rho, \tau) \rightarrow \rho + Y(\sigma, \rho, \tau), \]  

(A.2)

\[ Y(\sigma, \rho, \tau) = \sum_{n \neq 0} e^{in\rho/L} Y_n(\sigma, \tau). \]  

(A.3)

Other fields are decomposed into \( \rho \)-independent and \( \rho \)-dependent parts.

\[ X^i(\sigma, \rho, \tau) \rightarrow x^i(\sigma, \tau) + X^i(\sigma, \rho, \tau), \quad \psi(\sigma, \rho, \tau) \rightarrow \psi(\sigma, \tau) + \Psi(\sigma, \rho, \tau), \]  

(A.4)

\[ A(\sigma, \rho, \tau) \rightarrow a(\sigma, \tau) + A(\sigma, \rho, \tau), \]  

where \( X, A \) and \( \Psi \) are periodic in \( \rho \):

\[ X^i(\sigma, \rho, \tau) = \sum_{n \neq 0} e^{in\rho/L} X_n^i(\sigma, \tau), \quad \Psi(\sigma, \rho, \tau) = \sum_{n \neq 0} e^{in\rho/L} \Psi_n(\sigma, \tau), \]  

(A.5)

\[ A(\sigma, \rho, \tau) = \sum_{n \neq 0} e^{in\rho/L} A_n(\sigma, \tau). \]

We can treat the \( \rho \)-independent part as background and \( \rho \)-dependent part as fluctuations, since the \( \rho \)-dependent part starts from quadratic order. Classical action for the backgrounds is nothing but the type IIA string action in 10 dimensions.

The most convenient gauge choice for the fluctuations \( (A, X^i, Y) \) is the standard background field gauge condition:

\[ \partial_0 A - \{a, A\} + \{x^i, X^i\} + \{\rho, Y\} = 0. \]  

(A.6)
This condition remains invariant if the gauge parameter is independent of $\rho$, for which
the background and fluctuations transform as
\[
\delta a = \partial_0 \lambda, \quad \delta x^i = 0, \quad \delta \rho = \partial_\sigma \lambda, \\
\delta A = \{\lambda, A\}, \quad \delta X^i = \{\lambda, X^i\}, \quad \delta Y = \{\lambda, Y\}.
\] (A.7)
\[
\delta A = \{\lambda, A\}, \quad \delta X^i = \{\lambda, X^i\}, \quad \delta Y = \{\lambda, Y\}.
\] (A.8)

Note that this residual gauge freedom is used to bring $Y$ to the form (A.2). The gauge-fixing term and the ghost action for the background field gauge are as follows.
\[
\mathcal{L}_{gf} = -\frac{1}{2} (\partial_0 A - \{a, A\} + \{x^i, X^i\} - \partial_\sigma Y)^2, \\
\mathcal{L}_{gh} = -\partial_0 \overline{C}(\partial_0 C - \{a, C\} - \{A, C\}) + \{a, \overline{C}\}(\partial_0 C - \{a, C\} - \{A, C\}) + \partial_\sigma \overline{C}(\partial_\sigma C - \{Y, C\}) + \{x^i, \overline{C}\}(\{x^i, C\} + \{X^i, C\}),
\] (A.9)

where $C$ and $\overline{C}$ are also periodic in $\rho$

\[
C(\sigma, \rho, \tau) = \sum_{n \neq 0} e^{in\rho/L} C_n(\sigma, \tau), \quad \overline{C}(\sigma, \rho, \tau) = \sum_{n \neq 0} e^{in\rho/L} \overline{C}(\sigma, \tau).
\] (A.10)

Now to study the small compactification limit $L \to 0$, we make a rescaling
\[
\rho \to L \rho.
\] (A.11)

As mentioned in section 2, $1/L$ plays the role of the gauge coupling. We compute the effective action in expansion of $L$, which is essentially the strong coupling expansion. We use the Euclidean formulation
\[
\tau \to -i\tau, a \to ia, A \to iA.
\] (A.12)

The fluctuations are also rescaled as
\[
A \to LA, \quad X^i \to LX^i, \quad Y \to LY, \quad \Psi \to \sqrt{L} \Psi, \quad C \to LC.
\] (A.13)

The action at each order of $L$ is given as follows. The subscript ‘0, 1, 2’ indicate the order with respect to $L$ and the subscript ‘bg’ means the term containing only the background field. First, the parts which contain no spinor fields ($\psi, \Psi$) are
\[
\mathcal{L}_B = \mathcal{L}_{B,0} + \mathcal{L}_{B,1} + L^2 \mathcal{L}_{B,2},
\] (A.14)
The parts containing spinor fields are

\[ \mathcal{L}_{B,bg} = \frac{1}{2}(\partial_0 x^i)^2 + \frac{1}{2}(\partial_\sigma x^i)^2 + \frac{1}{2}(\partial_\sigma a)^2, \quad (A.16) \]

\[ \mathcal{L}_{B,0} = \frac{1}{2}((\partial_\sigma a)^2 + (\partial_\sigma x^i)^2)(\partial_\rho X^j)^2 + \frac{1}{2}((\partial_\sigma a)^2 + (\partial_\sigma x^i)^2)(\partial_\rho Y)^2 + \frac{1}{2}((\partial_\sigma a)^2 + (\partial_\sigma x^i)^2)(\partial_\rho A)^2 - i((\partial_\sigma a)^2 + (\partial_\sigma x^i)^2)\partial_\rho C \partial_\rho C, \quad (A.17) \]

\[ \mathcal{L}_{B,1} = -2\partial_0 x^i \{ A, X^i \} - 2\partial_\sigma a \{ Y, A \} - 2\partial_\sigma x^i \{ Y, X^i \} - \partial_\sigma a \partial_0 A \partial_\rho A - \partial_\sigma a \partial_0 Y \partial_\rho Y - \partial_\sigma a \partial_0 X^i \partial_\rho X^i - \partial_\sigma a \partial_0 Y \{ Y, A \} - \partial_\sigma a \partial_0 X^i \{ X^i, A \} - \partial_\sigma x^i \partial_\rho A \{ A, X^i \} - \partial_\sigma x^i \partial_\rho Y \{ Y, X^i \} - \partial_\sigma x^i \partial_\rho X^j \{ X^i, X^j \} + i\partial_\sigma a \partial_0 C \partial_\rho C + i\partial_\sigma a \partial_\rho C \partial_\rho C + i\partial_\sigma x^i \partial_\rho C \{ C, X^i \}, \quad (A.18) \]

\[ \mathcal{L}_{B,2} = \frac{1}{2}(\partial_0 A)^2 + \frac{1}{2}(\partial_0 X^i)^2 + \frac{1}{2}(\partial_0 Y)^2 + \frac{1}{2}(\partial_\sigma A)^2 + \frac{1}{2}(\partial_\sigma X^i)^2 + \frac{1}{2}(\partial_\sigma Y)^2 + \partial_\sigma A \{ A, Y \} + \partial_\sigma X^i \{ X^i, Y \} + \partial_0 X^i \{ X^i, A \} + \partial_0 Y \{ Y, A \} + \frac{1}{2} \{ A, X^i \}^2 + \frac{1}{2} \{ A, Y \}^2 + \frac{1}{2} \{ X^i, Y \}^2 + \frac{1}{4} \{ X^i, X^j \}^2 - i\partial_0 \overline{C} \partial_0 C - i\partial_\sigma \overline{C} \partial_\sigma C + i\partial_0 \overline{C} \{ A, C \} + i\partial_\sigma \overline{C} \{ Y, C \}. \quad (A.19) \]

The parts containing spinor fields are

\[ \mathcal{L}_F = \mathcal{L}_{F,bg} + \mathcal{L}_{F,0} + L^{1/2} \mathcal{L}_{F,1/2} + L^1 \mathcal{L}_{F,1}, \quad (A.20) \]

\[ \mathcal{L}_{F,bg} = \psi^T \partial_0 \psi + i\Psi^T \Gamma_9 \partial_\sigma \psi, \quad (A.21) \]

\[ \mathcal{L}_{F,0} = -\Psi^T \partial_\sigma a \partial_\rho \Psi - i\Psi^T \Gamma_i \partial_\sigma x^i \partial_\rho \Psi, \quad (A.22) \]

\[ \mathcal{L}_{F,1/2} = 2\Psi^T \partial_\rho A \partial_\sigma \psi + 2i\Psi^T \Gamma_9 \partial_\rho Y \partial_\sigma \psi + 2i\Psi^T \Gamma_i \partial_\rho X^i \partial_\sigma \psi, \quad (A.23) \]

\[ \mathcal{L}_{F,1} = \Psi^T \partial_0 \Psi + \Psi^T \Gamma_9 \partial_\sigma \Psi - \Psi^T \{ A, \Psi \} - i\Psi^T \Gamma_9 \{ Y, \Psi \} - i\Psi^T \Gamma_i \{ X^i, \Psi \}. \quad (A.24) \]
We regard the order $L^0$-parts as the free action. Substituting the Fourier expansions \((A.3)\), \((A.5)\) and \((A.11)\) into the action, $L^0$-part in terms of the Kaluza-Klein modes reads

\[
A_0 = \frac{(2\pi)^3 L}{2\ell_3^3 M} \left[ \int d\tau d\sigma \left[ \frac{n^2}{2} ((\partial_\sigma a)^2 + (\partial_\sigma x^i)^2) A_n A_{-n} + n^2 ((\partial_\sigma a)^2 + (\partial_\sigma x^i)^2) X_n^i X_{-n}^i + n^2 ((\partial_\sigma a)^2 + (\partial_\sigma x^i)^2) Y_n Y_{-n} - 2n^2 ((\partial_\sigma a)^2 + (\partial_\sigma x^i)^2) \overline{C}_n C_{-n} - 2n \Psi_n (\partial_\sigma a + i \Gamma_i \partial_\sigma x^i) \Psi_{-n} \right] \right].
\]

The propagators for $X_n^i$, $\Psi_n$ are thus given as

\[
\langle X_{-n}^i(\xi) X_n^j(\xi') \rangle = \frac{\delta^{ij}}{n^2} G(\xi, \xi'),
\]

\[
\langle \Psi_{-n}(\xi) \Psi_n(\xi') \rangle = -\frac{i}{2n} (\partial_\sigma a - i \Gamma_i \partial_\sigma x^i) G(\xi, \xi'),
\]

\[
G(\xi, \xi') = \frac{1}{(\partial_\sigma a)^2 + (\partial_\sigma x^i)^2} \delta^{(2)}(\xi - \xi'),
\]

where $\xi = \tau, \sigma$ and we set $(2\pi)^3 L/\ell_3^3 = 1$ for brevity. Propagators for other fields are given similarly. We restrict the background configurations where the U(1) gauge field which have no dynamics is set to zero ($a = 0$).

In our strong coupling expansion, the free part contains no derivatives of the fluctuations with respect to the world-sheet coordinates $\tau, \sigma$. As emphasized in the text, this ultra-local action necessarily leads to the propagators which are proportional to the delta function and lead to the UV divergences $\delta(0)$ upon loop calculations, thus for a rigorous treatment, we need a regularization. Since, as already alluded to in the discussion in section 2, it seems difficult to find a suitable regularization (which respects supersymmetry etc.), we only give a formal and partial argument for the vanishing quantum correction at low orders in $L$, by demonstrating that the coefficients of the divergence vanish after appropriately arranging these singular terms.

First of all, we see from \((A.25)\) that the lowest order correction, coming from logarithm of the one-loop determinant actually vanishes due to the matching of the bosonic and fermionic degrees of freedom ($8 (=10-2)$ bosonic d.o.f: $X^i, Y, A, C, \overline{C}$; $8 (=16/2)$ fermionic d.o.f: $\Psi$). We shall see the vanishing of a few low order terms in similar way.

The contribution to the effective action obviously does not have terms of half-integer order in $L$, for they are associated with odd number of fermionic fluctuations. The effective
Figure 1: The order $L^2$ contributions to the effective action. (a1),(a2): $\langle \mathcal{L}_{B,1}\mathcal{L}_{B,1} \rangle$, (b1),(b2): $\langle \mathcal{L}_{B,2} \rangle$, and (c1),(c2): $\langle \mathcal{L}_{F,1}\mathcal{L}_{F,1} \rangle$, respectively. Solid line denotes the propagators of bosonic fields or ghosts and dotted line denotes the propagators of spinor fields.

action at order $L$ would come from $\langle \mathcal{L}_{B,1} \rangle$, $\langle \mathcal{L}_{F,1} \rangle$ and $\langle \mathcal{L}_{F,1/2}\mathcal{L}_{F,1/2} \rangle$. However, these three contributions vanish separately for the following reasons. First, $\langle \mathcal{L}_{B,1} \rangle = 0$ for there is no way to self-contract $\mathcal{L}_{B,1}$ as we can see from (A.18). Also, $\langle \mathcal{L}_{F,1} \rangle = 0$ as we can see from (A.24) by noting that $\langle \Psi^T \partial_0 \Psi \rangle = \langle \Psi^T \Gamma_9 \partial_\sigma \Psi \rangle = 0$ due to $\text{Tr}\Gamma_i = \text{Tr}\Gamma_i\Gamma_9 = 0$. Finally, $\langle \mathcal{L}_{F,1/2}\mathcal{L}_{F,1/2} \rangle = 0$, for it is proportional to $\sum_{n \neq 0} \frac{1}{n}$ which can be set to zero by symmetry.

At order $L^2$, possible contributions which do not contain background spinor $\psi$ come from $\langle \mathcal{L}_{B,1}\mathcal{L}_{B,1} \rangle$, $\langle \mathcal{L}_{B,2} \rangle$ and $\langle \mathcal{L}_{F,1}\mathcal{L}_{F,1} \rangle$. Each one has one-loop and two-loop terms as shown in the Figure. There are ambiguities in the evaluation of the effective action at this order for we are dealing with the divergent quantities by formally treating the divergence as $\delta(0)$. (One-loop terms are proportional to $\delta(0)$ and two-loop terms are proportional to $(\delta(0))^2$, and hence they have different dimensions.)

We shall show that the one-loop contributions cancel. Explicit form of the one-loop contributions are as follows. First, from the diagram of Figure (a1),

$$\langle \mathcal{L}_{B,1}\mathcal{L}_{B,1} \rangle^{\text{(one-loop)}} = -4 \int d^2\xi \int d^2\xi' \sum_{n \neq 0} \frac{1}{n^2} \left[ \partial_0 x^i \partial_0' \partial_\sigma x^i' \partial_\sigma G(\xi, \xi') G(\xi, \xi') + \partial_\sigma x^i \partial_\sigma' x^i' \partial_\sigma G(\xi, \xi') G(\xi, \xi') \right],$$

(A.27)

where $\partial_\sigma x^i$, $\partial_\sigma' x^i$ means the argument of $x^i$ is $\xi$, $\xi'$ respectively and similarly for $\partial_0 x^i$, $\partial_0' x^i$. From Figure (b1),

$$\langle \mathcal{L}_{B,2} \rangle^{\text{(one-loop)}} = -4 \int d^2\xi \sum_{n \neq 0} \frac{1}{n^2} \lim_{\xi \to \xi'} \left[ \partial_0 \partial_0' G(\xi, \xi') + \partial_\sigma \partial_\sigma' G(\xi, \xi') \right].$$

(A.28)
We rewrite this term by inserting the delta function
\[
\lim_{\xi \to \xi'} 1 = \int d^2 \xi' \delta(\xi - \xi') = \int d^2 \xi' \partial_\sigma x^i \partial'_\sigma x^i G(\xi, \xi')
\] (A.29)
as
\[
\langle L_B, 1 \rangle_{(\text{one-loop})}^{(B2)} = -4 \int d^2 \xi \int d^2 \xi' \sum_{n\neq 0} \frac{1}{n^2} \partial_\sigma x^i \partial'_\sigma x^i [\partial_0 \partial' G(\xi, \xi') G(\xi, \xi') + \partial_\sigma \partial'_\sigma G(\xi, \xi') G(\xi, \xi')].
\] (A.30)

From figure (c1),
\[
\langle L_F, 1 \rangle_{(\text{one-loop})}^{(F1)} = 4 \int d^2 \xi \int d^2 \xi' \sum_{n\neq 0} \frac{1}{n^2} \partial_\sigma x^i \partial'_\sigma x^i [\partial_0 \partial' G(\xi, \xi') G(\xi, \xi') + \partial_\sigma \partial'_\sigma G(\xi, \xi') G(\xi, \xi')]
\] (A.31)

Here, we have to note an ambiguity due to a formal treatment of divergent quantities. In writing (A.31), we have assumed that the propagators for spinors are related to those of bosons by \(\langle \Psi_n(\xi) \Psi_n(\xi') \rangle = \frac{1}{2n} \Gamma_i \partial_\sigma x^i G(\xi, \xi')\), but if we have assigned the argument of \(x^i\) in a different way, (e.g. \((\partial_\sigma x^i(\xi) + \partial'_\sigma x^i(\xi'))/2\)), we would have a different answer. The sum of the one-loop contributions to the effective action (A.27), (A.30) and (A.31) vanish
\[
\langle L_{B,1} L_{B,1} \rangle_{(\text{one-loop})}^{(B1)} + \langle L_{B,2} \rangle_{(\text{one-loop})}^{(B2)} + \langle L_{F,1} L_{F,1} \rangle_{(\text{one-loop})}^{(F1)} = 0.
\] (A.32)

To prove this, we have used \(\partial_\sigma G(\xi, \xi') G(\xi, \xi') = \frac{1}{2} \partial_\sigma (G(\xi, \xi') G(\xi, \xi'))\) and performed partial integration.

We have also checked in a similar way that the order \(L^2\)-contributions containing background spinor \(\psi\)'s (two \(\psi\)'s: \(\langle L_{F,1/2} L_{F,1/2} L_{B,1} \rangle\); four \(\psi\)'s: \(\langle L_{F,1/2} L_{F,1/2} L_{F,1/2} L_{F,1/2} \rangle\)) vanish essentially due to the antisymmetry of the \(\psi\)'s.

Two-loop contributions are more ambiguous and moreover we need an appropriate regularization scheme for the infinite sum over the Kaluza-Klein level \(n\). This is a difficult question for which we do not have a definite answer.

Almost the same computations can be performed for the matrix-string case as well. Therefore it is very difficult to really justify the reduction to 10 dimensions in the infrared limit too. However, vanishing of all these corrections seems essential for justifying the
reduction to diagonal elements and for establishing the equivalence of matrix string theory with the perturbative superstring theory, such that the only remaining effect of the string coupling is the usual string interaction which is described by the instanton effect corresponding to the exchange of coincident eigenvalues of the diagonal matrices.

Appendix B

In section 4, we studied the correspondence between supermembrane and matrix string on the Kaluza-Klein Melvin background. We started with a light-cone supermembrane action in flat space-time in 11 dimensions and rewrote it in terms of the new coordinates which are single-valued along the compactified direction. As a result of the coordinate transformation, the action in terms of the new coordinates describes the supermembrane in a curved background. In this appendix, as a simple consistency check, we confirm that the leading part of the expansion of the curvature \((q^1\text{-}part)\) of the resulting membrane action agrees with the linearized interaction between supermembrane and backgrounds derived in [19]. Actually, this problem might sound trivial. However, it is not necessarily so in the presence of fermions and we decided to include it here for the sake of avoiding possible confusion, since to relate these two we have to perform a local Lorentz transformation for spinors appropriately. In the usual treatment of linearized approximation such as [19], it is not clear which local Lorentz frame is used for spinor fields.

The 11-dimensional background which we consider is

\[
ds_{11}^2 = -dt^2 + dx_1^2 + \cdots + dx_6^2 + dr^2 + r^2(d\varphi + q dy)^2 + dy^2 + dx_{10}^2. \tag{B.1}
\]

This in the linearized approximation \(q \ll 1\) corresponds to the flat 10 dimensional space with constant dilaton and a nontrivial magnetic vector field \(A_\varphi = qr^2\). The action at the linear order in \(q\) is given as

\[
A_{q^1} = q \int d\tau d\sigma d\rho \epsilon^{mn}[-D_0 Y D_0 X^m X^n + \{X^i, Y\} \{X^i, X^m\} X^n \\
+\{X^p, Y\} \{X^p, X^m\} X^n - i\psi^T \Gamma_m X^n \{Y, \psi\} \\
- i \frac{4}{\psi^T \Gamma_{mn} \psi D_0 Y - i \frac{4}{\psi^T \Gamma_i \Gamma_{mn} \psi} \{X^i, Y\}} \tag{B.2}
\]

Now, in ref [19], ‘vertex operators for supermembrane’ are derived from the consistency with supersymmetry, gauge symmetry and with vertex operators for string or particle
upon reduction. For example, the vertex operator for graviton with transverse polarization is given as

\[ V_h = h_{ab}[D_0 X^a D_0 X^b - \{ X^a, X^c \} \{ X^b, X^c \} + i \psi \Gamma^a \{ X^b, \psi \} + \frac{1}{2} D_0 X^a \psi^T \Gamma^{bc} \psi_k c + \frac{1}{2} \{ X^a, X^c \} \psi^T \Gamma^{bcd} \psi_k d + \frac{1}{8} \psi^T \Gamma^{ac} \psi \psi^T \Gamma^{bd} \psi_k c_k d \} e^{-ik^d X^d}, \]  

(B.3)

where we have flipped the signs of the coefficients of the fermion bilinears from the ones in [13] because of the difference of our convention.

Using (B.3) and the linearization of the the background (B.1)

\[ h_{m9} = -q \epsilon^{mn} x^n = -i q \epsilon^{mn} \frac{\partial}{\partial k n}, \]

the coupling to the background at \( O(q^1) \) is obtained as

\[ A^{(q^1)}_{int} = q \int d\tau d\sigma d\rho \epsilon^{mn} [-D_0 Y D_0 X^m X^n + \{ X^i, Y \} \{ X^i, X^m \} X^n + \{ X^p, Y \} \{ X^p, X^m \} X^n - i \frac{1}{2} \psi^T \Gamma^n \{ Y, \psi \} X^n - i \frac{1}{2} \psi^T \Gamma^n \{ X^m, \psi \} X^n - i \frac{1}{4} \psi^T \Gamma^n \{ X^m, X^i \} \psi - i \frac{1}{4} \psi^T \Gamma^n \{ X^m, X^p \} \psi - i \frac{1}{4} \psi^T \Gamma^n \{ Y, X^i \} \psi]. \]  

(B.4)

The purely bosonic part agrees with our result as it should. It turns out that the fermion part (B.2) is related to that of (B.4) by a redefinition of the fermionic fields by

\[ \psi \rightarrow e^{\frac{i}{4} \epsilon^{mn} X^m \Gamma_{n9}} \psi = \psi + \frac{q}{4} \epsilon^{mn} X^m \Gamma_{n9} \psi + O(q^2). \]  

(B.5)

This is an allowed transformation, which does not change the fermion boundary condition, and corresponds to a transformation of local Lorentz frame for spinors. In curved space-time, there is always such an ambiguity. In fact, curved space action must be formulated such that it is invariant under the change of local Lorentz frame. However, the discussion of vertex operators in ref. [13] tacitly assumed a special gauge choice for this gauge degrees of freedom to ensure manifest world-volume supersymmetry. After the rotation (B.5), the action acquires additional contribution \( \tilde{A}^{q^1} \) to the \( q^1 \) part

\[ \tilde{A}^{q^1} = q \int d\tau d\sigma d\rho \epsilon^{mn} [-i \frac{1}{4} D_0 X^m \psi^T \Gamma^{9n} \psi - i \frac{1}{4} \psi^T \Gamma^{9n} \{ X^m, X^i \} \psi - i \frac{1}{4} \psi^T \Gamma^{9n} \{ X^m, X^p \} \psi - i \frac{1}{4} \psi^T \Gamma^{9n} \{ Y, \psi \}]. \]  

(B.6)
The sum $A^v + \tilde{A}^v$ is precisely the linearized coupling obtained from the vertex operator for membranes (B.4).

By making conversion to the matrix-string theory following our prescription, this should be equivalent with the result we obtain by using the approach of ref. [20], which is based on the standard duality arguments and only analyzed, though, the zero-th moments of external fields.

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