Normal Forms for Dirac-Jacobi bundles and Splitting Theorems for Jacobi Structures

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Abstract
The aim of this paper is to prove a normal form Theorem for Dirac-Jacobi bundles using the recent techniques from [3]. As the most important consequence, we can prove the splitting theorems of Jacobi pairs which was proposed by Dazord, Lichnerowicz and Marle in [5]. As an application we provide a alternative proof of the splitting theorem of homogeneous Poisson structures.

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1 Introduction

Since the work of Weinstein [15], in which he proved his famous local splitting theorem for Poisson manifolds, many works appeared concerning different viewpoints on the proof and even give more general statements, namely normal form theorems. Frejlich and Marcut proved a normal form theorem around Poisson (cosymplectic) transversals of Poisson manifolds in [6]. In [7] they used the techniques of Dual Pairs to prove a similar statement for Dirac structures. And finally, there is a unified approach by Bursztyn, Lima and Meinrenken in [3] to prove normal forms for Poisson related structures.

Jacobi geometry was introduced by Kirillov in [9] as local Lie algebras. They have a deep connection to Poisson geometry, since every Poisson structure defines a Jacobi bracket. Moreover, every Jacobi structure induces a Poisson structure on a manifold of one dimension more, this is known as the symplectization or homogenezation, see [2] and its references for a detailed discussion. In Jacobi geometry there is also a local splitting theorem available, which was proven by Dazord, Lichnerowicz and Marle in [5]. Nevertheless, after this work the parallels in the work of Poisson and Jacobi geometry stopped, at least in the context of local structure. The aim of this paper is to fill these gaps, prove normal form theorems for Jacobi bundles and give a more intrinsic proof of the splitting theorems. To do so, we will chose the approach of [3] and start with so-called Dirac-Jacobi bundles which generalize the notion of Jacobi structures.

Dirac-Jacobi bundles were introduced in [12] by Vitagliano and are a slight generalization of Wade’s $\mathcal{E}^1(M)$-Dirac structures (see [14]). Moreover, these bundles are a Dirac theoretic generalizations of Jacobi bundles, as usual Dirac structures are for Poisson manifolds.

We want to stress that the methods, which are expressed in this note are also suitable for proving splittings for involutive fat anchored vector bundles $(E, L \to M, \rho)$, i.e. a vector bundle $E \to M$, a line bundle $L \to M$ and a bundle map $\rho : E \to DL$, such that $\Gamma^\infty(\rho(E))$ is closed with respect to the bracket, as well as Jacobi-algebroids (see [11]). We do not want to treat that in detail since every involutive fat anchored vector bundle is in particular, by composing the anchor $\rho$ with the anchor of $DL$, an involutive anchored vector bundle and can be treated with the methods in [3]. The same holds true for Jacobi algebroids.

This short note is organized as follows: we recall the necessary structures in order to define the setting for Dirac-Jacobi structures, the omni-Lie algebroid of a line bundle (see [4]) in Section 2. Afterwards, we introduce the notion of Euler-like derivations, which are the crucial ingredient for the proofs of the main theorems. After this we are able to provide a normal form theorem for Dirac-Jacobi bundles, which is the main part of Section 4. In the following section, we want to apply this normal form theorem to the special case of Jacobi bundles, which allows us to state and prove two normal form theorems for Jacobi bundles, which also use to give a different prove of the splitting theorems of Jacobi pairs, first provided in [3]. Moreover, we can apply this theorems to provide a splitting theorem for homogeneous Poisson structure around points where the homogeneity does not vanish, which was also done in [4]. Note that in [5] the proof works exactly the other way around: they prove a local splitting of homogeneous Poisson structures and use it to prove the splitting of Jacobi structures.

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2 Preliminaries and Notation

This introductory section is divided into two parts: first we recall the Atiyah algebroid of a vector bundle and the corresponding Der-complex with applications to contact and Jacobi geometry. Afterwards, we introduce the arena for the so-called Dirac-Jacobi bundles in odd dimensions, the omni-Lie algebroids, and give a quick reminder of Dirac-Jacobi bundles together with the properties we will need afterwards.

2.1 Notation and a brief reminder on Jacobi Geometry

The notions of Atiyah algebroid of a vector bundle and the associated Der-complex are known and are used in many other situations. This section is basically meant to fix notation. A more complete introduction to this can be found in [12] and its references. Nevertheless, the notion of Omni-Lie algebroids was first defined in [4], in order to study Lie algebroids and local Lie algebra structures on vector bundles.

For a vector bundle $E \to M$, we denote its gauge or Atiyah algebroid by $DE \to M$ and by $\sigma: DE \to TM$ its anchor. Note that $D$ is a functor from the category of vector bundles with regular, i.e. fiberwise invertible, vector bundle morphisms to Lie algebroids. Hence, we denote for a regular $\Phi: E \to E'$ by

$$D\Phi: DE \to DE'$$

the corresponding Lie algebroid morphism. We are mostly dealing with line bundles $L \to M$ for which we have the identity $DL = (J^1L)^* \otimes L$, where $J^1L$ is the first jet bundle. The gauge algebroid $DL \to M$ has a (tautological) Lie algebroid representation on $L$. The corresponding complex is denoted by

$$\left( \Omega^*_\infty(M) = \Gamma^\infty(\Lambda^*(DL)^* \otimes L), d_L \right).$$

We briefly discuss Jacobi brackets in this setting. A Jacobi bracket is a local Lie algebra structure on the smooth sections of a line bundle $L \to M$, i.e. a Lie bracket $\{-,-\}: \Gamma^\infty(L) \times \Gamma^\infty(L) \to \Gamma^\infty(L)$, such that

$$\{-,-\} \in \Gamma^\infty(DL).$$

**Remark 2.1** Let $\{-,-\}$ be a Jacobi bracket on a line bundle $L \to M$. Then there is a unique tensor, called the Jacobi tensor, $J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L)$, such that

$$\{\lambda, \mu\} = J(j^1\lambda, j^1\mu)$$

for $\lambda, \mu \in \Gamma^\infty(L)$. Conversely, every $L$-valued 2-form $J$ on $J^1L$ defines a skew-symmetric bilinear bracket $\{-,-\}$, but the latter needs not to be a Jacobi bracket. Specifically, it does not need to fulfill the Jacobi identity. However, there is the notion of a Gerstenhaber-Jacobi bracket

$$\{-,-\}: \Gamma^\infty(\Lambda^i(J^1L)^* \otimes L) \times \Gamma^\infty(\Lambda^j(J^1L)^* \otimes L) \to \Gamma^\infty(\Lambda^{i+j-1}(J^1L)^* \otimes L),$$

such that the Jacobi identity of $\{-,-\}$ is equivalent to $[J,J] = 0$ see [11, Chapter 1.3] for a detailed discussion. Finally, a Jacobi tensor defines a map $J^2: J^1L \to (J^1L)^* \otimes L = DL$.

When $L$ is the trivial line bundle, than the notion of Jacobi bracket boils down to that of Jacobi pair.
Remark 2.2 (Trivial Line bundle) Let $\mathbb{R}_M \to M$ be the trivial line bundle and let $J$ be a Jacobi tensor on it. Let us denote by $1_M \in \Gamma^\infty(\mathbb{R}_M)$ the canonical global section. Using the canonical connection

$$\nabla: TM \ni v \mapsto (f \cdot 1_M \mapsto v(f)1_M) \in D\mathbb{R}_M,$$

we can see that $DL \cong TM \oplus \mathbb{R}_M$ and hence

$$J^1\mathbb{R}_M = (D\mathbb{R}_M)^* \otimes \mathbb{R}_M = T^*M \oplus \mathbb{R}_M.$$

With this splitting, we see that

$$J = \Lambda + 1 \wedge E$$

for some $(\Lambda, E) \in \Gamma^\infty(\Lambda^2TM \oplus TM)$. The Jacobi identity is equivalent to $[\Lambda, \Lambda] + E \wedge \Lambda = 0$ and $\mathcal{L}_E \Lambda = 0$. The pair $(\Lambda, E)$ is often referred to as Jacobi pair. Moreover, if we denote by $1^* \in \Gamma^\infty(J^1\mathbb{R}_M)$ the canonical section then we can write any $\psi \in J^1\mathbb{R}_M$ as $\psi = \alpha + r1^* \in \Gamma^\infty(J^1\mathbb{R}_M)$, for some $\alpha \in T^*M$ and $r \in \mathbb{R}$. We obtain

$$J^2(\alpha + r1^*) = \Lambda^2(\alpha) + rE - \alpha(E)1.$$

A more detailed discussion about Jacobi structures on trivial line bundles can be found in [11, Chapter 2]. In a similar way, we can see that

$$\Omega^\bullet_L(M)^\bullet = \Gamma^\infty(\Lambda^\bullet(T^*M \oplus \mathbb{R}_M)) = \Gamma^\infty(\Lambda^\bullet T^*M \oplus \Lambda^\bullet \mathbb{R}_M).$$

Here $1^*$ is the canonical section of $\mathbb{R}_M$, moreover the differential $d_{\mathbb{R}_M}$ is defined by the relations

$$d_{\mathbb{R}_M}(1^*) = 0 \quad \text{and} \quad d_{\mathbb{R}_M} = d_{dR} + 1^* \wedge.$$

2.2 The Omni-Lie Algebroid of a line bundle and its automorphisms

The omni-Lie algebroid plays the same role as the generalized tangent bundle does in Dirac geometry. In fact, the parallels are evidently enormous. Moreover, since the canonical inner product of it will be line-bundle valued, one can easily drop the word local Courant algebroid. Note that the following definitions and Lemmas are obvious adaptations of the case of $H$-twisted Dirac structure, this is why we omit proofs. The non-twisted versions of the following definitions and results in Dirac-Jacobi geometry can be found in [12].

**Definition 2.3** Let $L \to M$ be a line bundle and let $H \in \Omega^3_L(M)$ be closed. The vector bundle $\mathbb{D}L := DL \oplus J^1L$ together with

i.) the (Dorfman-like, H-twisted) bracket

$$\llbracket (\Delta_1, \psi_1), (\Delta_2, \psi_2) \rrbracket_H = ([\Delta_1, \Delta_2], \mathcal{L}_\Delta_1 \psi_2 - \iota_{\Delta_2} d_{\mathbb{D}L} \psi_1 + \iota_{\Delta_1} \iota_{\Delta_2} H)$$

ii.) the non-degenerate $L$-valued pairing

$$\langle \langle \Delta_1, \psi_1), (\Delta_2, \psi_2) \rangle \rangle := \psi_1(\Delta_2) + \psi_2(\Delta_1)$$

iii.) the canonical projection $\text{pr}_D: \mathbb{D}L \to DL$

is called the $H$-twisted Omni-Lie algebroid of $L \to M$.

**Remark 2.4** If $H = 0$, we will refer to $(\mathbb{D}L, \llbracket \cdot, \cdot \rrbracket, \langle \langle \cdot, \cdot \rangle \rangle)$ as the omni-Lie algebroid.

We shall now introduce automorphisms of the omni-Lie algebroid, which mirrors the definition of automorphisms of the generalized tangent bundle.
Definition 2.5 Let $L \to M$ be line bundle and let $H \in \Omega^2_{1}(M)$ be closed. A pair $(F, \Phi) \in \text{Aut}(\mathbb{D}L) \times \text{Aut}(L)$ is called (H-twisted) Courant-Jacobi automorphism, if

i.) $D\Phi : \text{pr}_D = \text{pr}_D \circ F$

ii.) $\Phi^\ast \{ -, - \} = \{ F^\ast, F^\ast - \}$

iii.) $F^\ast [-, -]_H = [F^\ast, F^\ast -]_H$

The group of H-twisted Courant-Jacobi automorphisms is denoted by $\text{Aut}^H_{CJ}(L)$.

For a line bundle $L \to M$ and $\Phi \in \text{Aut}(L)$, we define

$$\mathbb{D}\Phi : \mathbb{D}L \ni (\Delta, \alpha) \mapsto (D\Phi(\Delta), (D\Phi^{-1})^\ast \alpha) \in \mathbb{D}L,$$

which gives canonically an automorphism $\mathbb{D}\Phi \in \text{Aut}(\mathbb{D}L)$. Moreover, the pair $(\mathbb{D}\Phi, \Phi)$ fulfills conditions i.) and ii.) in Definition 2.5, nevertheless it is not an (H-twisted) Courant-Jacobi automorphism for an arbitrary $H$. For a 2-form $B \in \Omega^2_{1}(M)$, we define

$$\exp(B) : \mathbb{D}L \ni (\Delta, \alpha) \mapsto (\Delta, \alpha + \iota_\Delta B) \in \mathbb{D}L,$$

which also fulfills conditions i.) and ii.) in Definition 2.5, seen as pair $(\exp(B), \text{id})$. We can combine this two special kinds of morphisms together with an $H$-dependent action on $\mathbb{D}L$ and find the following

Lemma 2.6 Let $L \to M$ be a line bundle and let $H \in \Omega^2_{1}(M)$ be closed. If we denote by $Z^2_{1}(M)$ the closed 2-forms, then

$$\mathcal{I}_H : Z^2_{1}(M) \times \text{Aut}(L) \ni (B, \Phi) \mapsto (\exp(B + \iota_1(H - \Phi^\ast H)) \circ \mathbb{D}\Phi, \Phi) \in \text{Aut}^H_{CJ}(L)$$

is an isomorphism of groups.

In a similar way, we can define infinitesimal automorphisms of the Omni Lie algebroid

Definition 2.7 Let $L \to M$ be line bundle and let $H \in \Omega^2_{1}(M)$ be closed. A pair $(D, \Delta) \in \Gamma^\infty(\mathbb{D}L) \times \Gamma^\infty(DL)$ is called infinitesimal (H-twisted) Courant-Jacobi automorphism, if

i.) $[\Delta, \text{pr}_D(\varepsilon)] = \text{pr}_D(D(\varepsilon))$

ii.) $\Delta \{ \varepsilon, \chi \} = \{ D(\varepsilon), \xi \} + \{ \varepsilon, D(\chi) \}$

iii.) $D([\varepsilon, \chi]_H) = [D(\varepsilon), \chi]_H + [\varepsilon, D(\chi)]_H$

for all $\varepsilon, \chi \in \Gamma^\infty(\mathbb{D}L)$. The lie algebra of infinitesimal (H-twisted) Courant-Jacobi automorphisms is denoted by $\text{aut}^H_{CJ}(L)$.

Note that it is obvious, that the flow of an infinitesimal (H-twisted) Courant-Jacobi automorphism gives a Courant-Jacobi automorphism, in this sense, we can see $\text{aut}^H_{CJ}(L)$ as the Lie algebra of $\text{Aut}^H_{CJ}(L)$. Similarly to the automorphism case, we have

Lemma 2.8 Let $L \to M$ be line bundle and let $H \in \Omega^2_{1}(M)$ be closed. Then

$$\iota_H : Z^2_{1}(M) \times \Gamma^\infty(DL) \ni (B, \Delta) \mapsto ((\Box, \beta) \mapsto ([\Delta, \Box], \mathcal{L}_\Delta \beta + \iota_\Box(B - \mathcal{L}_1H))) \in \text{aut}^H_{CJ}(L)$$

is an isomorphism of Lie algebras.
For every section \((\Delta, \alpha) \in \Gamma^\infty(DL)\) the map \([[\Delta, \alpha], -]]_H\) is an infinitesimal (H-twisted) Courant-Jacobi automorphism, in fact it is realized in \(Z^2_H(M) \rtimes \Gamma^\infty(DL)\) by

\[i_H(d_L(\iota_{\Delta} H - \alpha), \Delta) = [[\Delta, \alpha], -]_H\]

For later use, we want to talk about the flow of infinitesimal (H-twisted) Courant-Jacobi automorphisms and want to compute them as explicit as possible.

**Lemma 2.9** Let \(L \to M\) be line bundle and let \(H \in \Omega^3_L(M)\) be closed. Let additionally \((\alpha, \Delta) \in \Gamma^\infty(Z^2_L(M))\). The flow of \(i_H(B, \Delta)\) is given by

\[\mathcal{I}_H(\gamma_t, \Phi^\Delta_t) = \mathcal{I}_H\left(- \int_0^t (\Phi^\Delta_{-\tau})^* B \, d\tau, \Phi^\Delta_t\right)\]

\[\exp\left(- \int_0^t (\Phi^\Delta_{-\tau})^* (d_L \alpha + \iota\Delta H) \, d\tau\right) \circ \mathcal{D}\Phi^\Delta_t\]

**Corollary 2.10** Let \(L \to M\) be line bundle and let \(H \in \Omega^3_L(M)\) be closed. For every \((\Delta, \alpha) \in \Gamma^\infty(DL)\) the flow of \([[\Delta, \alpha], -]]_H\) is given by

\[\exp\left(\int_{-}^{t} (\Phi^\Delta_{-\tau})^* (d_L \alpha + \iota\Delta H) \, d\tau\right) \circ \mathcal{D}\Phi^\Delta_t\]

### 2.3 Dirac-Jacobi bundles

After having discussed the arena, we want to introduce the subbundles of interest: so-called Dirac-Jacobi Bundles. As the name suggest, they are the analogue of Dirac structures on the generalized tangent bundle. In fact, the definition is (up to some obvious replacements) the same.

**Definition 2.11** Let \(L \to M\) be a line bundle and \(H \in \Omega^3_L(M)\). A subbundle \(L \subseteq DL\) is called a \((H\text{-twisted})\) Dirac-Jacobi structure, if

i.) \(L\) is involutive with respect to \([[\cdot, \cdot]]_H\),

ii.) \(L\) is maximally isotropic with respect to \(\langle \cdot, \cdot \rangle\).

Moreover, if \(H = 0\), we will call \(L\) simply Dirac-Jacobi structure.

**Example 2.12** Let \(L \to M\) be a line bundle and let \(J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L)\) be a Jacobi structure, then

\[\mathcal{L}_J := \{(J^2(\psi), \psi) \in DL \mid \psi \in J^1L\}\]

is a Dirac-Jacobi structure.

**Proposition 2.13** Let \(L \to M\) be a line bundle and let \(\mathcal{L} \subseteq DL\) be a Dirac-Jacobi bundle, such that

\[DL \cap \mathcal{L} = \{0\}\]

Then there is a unique Jacobi structure \(J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L)\), such that \(\mathcal{L}_J = \mathcal{L}\)

**Proof:** The result follows the same lines as the well-known fact in Poisson geometry. \(\Box\)

Another interesting example of Dirac-Jacobi bundles, which also plops up in Jacobi geometry, is
**Definition 2.14** Let $L \to M$ be a line bundle. A Dirac-Jacobi structure $\mathcal{L} \subseteq DL$ is called of homogeneous Poisson type, if

$$\text{rank}(\mathcal{L} \cap DL) = 1.$$ 

The name of these objects is justified by the following

**Lemma 2.15** Let $L \to M$ be a line bundle and let $\mathcal{L} \subseteq DL$ a Dirac-Jacobi structure of homogeneous Poisson type, then for every point $p \in M$ there exists a local trivialization $L_U = U \times \mathbb{R}$, a flat connection $\nabla : TU \to DL_U \cong TU \oplus \mathbb{R}_U$ and a homogeneous Poisson structure $\pi \in \Gamma^\infty(\Lambda^2TU)$ with homogeneity $Z \in \Gamma^\infty(TM)$, such that

$$\mathcal{L}|_U = \{(r(1 - \nabla Z) + \nabla_{\pi^\ast(\alpha)} + \alpha(Z)\mathbb{1}^\ast) \in DL|_U \mid h \in \mathbb{R}, \alpha \in T^*U\},$$

where we use the inclusion $T^*M \to J^1L$ by $\alpha(\nabla_X) = \alpha(X)$ and $\alpha(\mathbb{1}) = 0$.

**Proof:** Let $p \in M$ and $U \subseteq M$ be an open subset containing $p$, such that $L_U \cong U \times \mathbb{R}$ with corresponding flat connection of the gauge algebroid $DL_U = TU \oplus \mathbb{R}_U$, and hence we are using the canonical flat connection $\nabla_{\text{can}} : TU \to TU \oplus \mathbb{R}_U$. In a possibly smaller neighbourhood, noted also by $U$, we find a non-vanishing section $\Delta = (-X, f) \in \Gamma^\infty(\mathcal{L} \cap DL)$. We can distinguish two cases: the first is that $f(p) \neq 0$, the we find a (possibly smaller) neighbourhood of $p$, such that $f$ is non-vanishing, hence $(-\frac{\Delta}{f}, 1) = (-Z, 1)$ spans $\mathcal{L} \cap DL$ in that neighbourhood. Exploiting the isotropy, we see that $\mathcal{L}|_U$ is of the form

$$\{(h\mathbb{1} + \nabla_{\mathbb{1}}^\ast, \alpha + \alpha(Z)\mathbb{1}^\ast) \in DL_U \mid h \in \mathbb{R}, \alpha \in T^*U\}$$

and not further specified $Y \in TU$, since the $J^1L_U$ part has to vanish at sections of the form $r(1 - \nabla_{\mathbb{1}}^\ast)$. We can write this as

$$\{(h(1 - \nabla_{\mathbb{1}}^\ast) + \nabla_{\mathbb{1}}^\ast hZ + Y, \alpha + \alpha(Z)\mathbb{1}^\ast) \in DL_U \mid h \in \mathbb{R}, \alpha \in T^*U\}.$$ 

Note that, because of the isotropy, $hZ + Y$ is completely determined by $\alpha$, hence there is a bi-vector $\pi \in \Gamma^\infty(\Lambda^2TU)$ such that $\pi^\ast(\alpha) = hZ + Y$ and we can write

$$\mathcal{L}|_U = \{(h(1 - \nabla_{\mathbb{1}}^\ast) + \nabla_{\pi^\ast(\alpha)}^\ast, \alpha + \alpha(Z)\mathbb{1}^\ast) \in DL_U \mid h \in \mathbb{R}, \alpha \in T^*U\}.$$ 

The claim follows by using the flatness of $\nabla_{\text{can}}$ and the involutivity of $\mathcal{L}$.

Now we have to treat the case $f(p) = 0$. Since $\Delta = (-X, f)$ is non-vanishing, we conclude that $X(p) \neq 0$, hence there is a closed two form $\beta \in \Gamma^\infty(T^*U)$ such that $\beta(X) = -1$ around $p$. We define the flat connection

$$\nabla : TU \ni Y \mapsto \nabla_{\text{can}}^\ast + \beta(Y)\mathbb{1} \in DL_U.$$ 

With this connection we see that $\Delta = (f - 1)\mathbb{1} - \nabla_X$ and since $f(p) = 0$, we have that $f - 1 \neq 0$ in a whole neighbourhood of $p$ and hence we choose $\Delta' = \frac{1}{f-1}\Delta$ as a generating section of $\mathcal{L} \cap DL$ around $p$. We can now repeat the same argument as for the case $f(p) \neq 0$ by using the connection $\nabla$ instead of $\nabla_{\text{can}}$, since $\Delta' = 1 - \nabla_{\mathbb{1}}$ for $Z = \frac{1}{f-1}X$. \[ \square \]

In the category of Dirac-Jacobi bundles there are not just automorphism of the omni-Lie algebroid as morphisms, one of the possibilities is to include so-called backwards transformations as in the Dirac geometry case.
Definition 2.16 Let $L_i \rightarrow M_i$ for $i = 1, 2$ be two line bundles and let $\Phi: L_1 \rightarrow L_2$ be a regular line bundle morphism covering $\phi: M_1 \rightarrow M_2$. Let $L \subseteq \mathbb{D}L_2$ be a Dirac-Jacobi bundle. The bundle

$$B_\Phi(L) := \left\{ (\Delta_p, (D\Phi)^*\alpha_{\phi(p)}) \in \mathbb{D}L_1 \mid (D\Phi(\Delta_p), \alpha_{\phi(p)}) \in L \right\}$$

is called Backwards transformation of $L$.

The backwards transform of a Dirac-Jacobi bundle need not to be Dirac-Jacobi anymore, but there are sufficient conditions on the subbundle $L$ and the line bundle morphism $\Phi$ which can be seen, i.e. in [12]:

Theorem 2.17 Let $\Phi: L_1 \rightarrow L_2$ be a regular line bundle morphism over $\phi: M_1 \rightarrow M_2$ and let $L \subseteq \mathbb{D}L_2$ be a Dirac-Jacobi bundle. If $\ker D\Phi^* \cap \phi^*L$ has constant rank, then $B_\Phi(L)$ is a Dirac-Jacobi bundle.

Proof: The proof can be found in [12, Proposition 8.4].

Remark 2.18 Note that for a line bundle automorphism $\Phi \in \text{Aut} L$, we have that $D\Phi(L) = B_{\Phi^{-1}}(L)$. But not every backwards transform needs to be of this form.

3 Submanifolds and Euler-like Vector Fields

In this subsection we want to discuss Euler-like vector fields. These vector fields, in particular, induce a homogeneity structure on the manifold, which is equivalent, under some additional conditions which are in our case always fulfilled, that the manifold is total space of a vector bundle, see e.g. [8]. This total space turns out to be the normal bundle for some submanifold, which is an input datum for an Euler-like vector field. Nevertheless, we will not go more in details with these features, since we work directly with tubular neighbourhoods. We will begin collecting facts about tubular neighbourhoods, submanifolds and corresponding mappings and describe afterwards the notion of Euler-like vector fields and extend this notion the derivations of a line bundle.

3.1 Normal Bundles and tubular neighbourhoods

For pair of manifolds $(M, N)$, i.e. a submanifold $N \hookrightarrow M$, we denote

$$\nu(M, N) = \frac{TM|_N}{TN}$$

the normal bundle. If the ambient space is clear, we will just write $\nu_N$ instead. Given a map of pairs

$$\Phi: (M, N) \rightarrow (M', N'),$$

i.e. a map $\Phi: M \rightarrow M$, such that $\Phi(N) \subseteq N'$, we denote by

$$\nu(\Phi): \nu(M, N) \rightarrow \nu(M', N')$$

the induced map on the normal bundle. For a vector field $X$ on $M$ tangent to $N$, we have that the flow $\Phi_t^X$ is a map of pairs from $(M, N)$ to itself. Hence we define

$$T\nu(X) = \left. \frac{d}{dt} \right|_{t=0} \nu(\Phi_t^X) \in \Gamma^\infty(T\nu_N).$$
Moreover, for a vector bundle $E \to M$ and $\sigma \in \Gamma^\infty(E)$, such that $\sigma|_N = 0$ for a submanifold $N \hookrightarrow M$, we denote by

$$d^N \sigma: \nu_N \to E|_N$$

the map which is $\nu(\sigma)$, for $\sigma$ seen as a map $\sigma: (M, N) \to (E, M)$, followed by the canonical identification $\nu(E, M) = E$, given by

$$C_E: E \ni v_p \to \left. \frac{d}{dt} \right|_{t=0} tv_p|_{TM} \in \nu(E, M).$$

Before we prove the next results, we want to find a useful description of $C_{E}^{-1}$. Let us therefore consider a curve $\gamma: I \to E$ for an open interval $I$ containing 0, such that $\gamma(0) = 0_p$ for $p \in M$, then one can prove in local coordinates

$$C_{E}^{-1}(\left. \frac{d}{dt} \right|_{t=0} \gamma(t)) = \lim_{t \to 0} \frac{\gamma(t)}{t}. \tag{3.1}$$

**Proposition 3.1** Let $E_i \to M_i$ vector bundles for $i = 1, 2$ and let $\Phi: E_1 \to E_2$ be a vector bundle morphism. Then, for $\Phi: (E_1, M_1) \to (E_2, M_2)$,

$$C_{E_2}^{-1} \circ \nu(\Phi) \circ C_{E_1} = \Phi$$

**PROOF:** Let $v_p \in E_1$, then

$$(C_{E_2}^{-1} \circ \nu(\Phi) \circ C_{E_1})(v_p) = (C_{E_2}^{-1} \circ \nu(\Phi))(\left. \frac{d}{dt} \right|_{t=0} tv_p|_{TM_1})$$

$$= C_{E_2}^{-1}(\left. \frac{d}{dt} \right|_{t=0} tv_p|_{TM_2})$$

$$= C_{E_2}^{-1}(\left. \frac{d}{dt} \right|_{t=0} t\Phi(v_p)|_{TM_2})$$

$$= \Phi(v_p) \quad \square$$

**Proposition 3.2** Let $E_i \to M$ be vector bundles for $i = 1, 2$ and let $\Phi: E_1 \to E_2$ be a vector bundle morphism covering the identity. Then, for every section $\sigma \in \Gamma^\infty(E_1)$, such that $\sigma|_N = 0$ for some submanifold $N \hookrightarrow M$,

$$d^N \Phi(\sigma) = \Phi(d^N \sigma)$$

holds.

**PROOF:** We consider the map $\Phi(\sigma): (M, N) \to (E_2, M)$, then we have

$$C_{E_2}^{-1} \circ \nu(\Phi(\sigma)) = C_{E_2}^{-1} \circ \nu(\Phi) \circ \nu(\sigma)$$

$$= C_{E_2}^{-1} \circ \nu(\Phi) \circ C_{E_1} \circ C_{E_1}^{-1} \circ \nu(\sigma)$$

$$= \Phi \circ C_{E_1}^{-1} \circ \nu(\sigma)$$

and the claim follows if we restrict this maps. \quad \square

**Proposition 3.3** Let $(M, N)$ be a pair of manifolds and let $X \in \Gamma^\infty(TM)$, such that $X|_N = 0$. Then

$$T\Phi^X|_N = \exp(tDX)$$

for a unique $DX \in \Gamma^\infty(End(TM|_N))$, moreover $TN \subseteq \ker(DX)$ and
Definition 3.4 Let \((M, N)\) be a pair of manifolds. A tubular neighbourhood of \(N\) is an open subset \(U \subseteq M\) containing \(N\) together with a diffeomorphism

\[
\psi: \nu_N \to U,
\]

such that \(\psi|_N: N \to N\) is the identity and for \(\psi: (\nu_N, N) \to (M, N)\) the map

\[
\nu(\psi): \nu(\nu_N, N) \to \nu_N
\]

is inverse of \(C_{\nu_N}: \nu_N \to \nu(\nu_N, N)\).

3.2 Euler-like Vector fields and Derivations

In this part, we recall basically just the notion of Euler-like vector fields from [3] and extend this notion to derivations of a line bundle.

Definition 3.5 Let \((M, N)\) be a pair of manifolds. A vector field \(X \in \Gamma^\infty(TM)\) is called Euler-like, if

i.) \(X|_N = 0\),

ii.) \(X\) has complete flow,

iii.) \(T\nu(X) = \mathcal{E}\),

where \(\mathcal{E}\) is the Euler vector field on \(\nu_N \to N\).

Proposition 3.6 Let \((M, N)\) be a pair of manifolds, then there exists an Euler-like vector field.

Proof: Let us choose a tubular neighbourhood

\[
\psi: \nu_N \to U.
\]

For the vector field \(X = \psi_\ast\mathcal{E}\) multiplied by a suitable bump function which is 1 in a neighbourhood of \(N\), we have

\[
T\nu(X) = \frac{d}{dt}\bigg|_{t=0} \nu(\Phi^X_t) = \frac{d}{dt}\bigg|_{t=0} \nu(\psi \circ \Phi^\mathcal{E}_t \circ \psi^{-1})
\]

\[
= \frac{d}{dt}\bigg|_{t=0} \nu(\psi) \circ \nu(\Phi^\mathcal{E}_t) \circ \nu(\psi^{-1})
\]

\[
= \frac{d}{dt}\bigg|_{t=0} \Phi^\mathcal{E}_t = \mathcal{E},
\]

where we used Proposition 3.1 and the fact that \(\nu(\psi) = C_{\nu_N}^{-1}\). \(\square\)

Lemma 3.7 Let \(M\) be a manifold, \(N \hookrightarrow M\) a submanifold and \(X \in \Gamma^\infty(TM)\) be a Euler-like vector field. Then there exists a tubular unique neighbourhood embedding

\[
\psi: \nu_N \to U,
\]

such that \(\psi\ast X = \mathcal{E}\).
Proof: The proof can be found in [3]. □

Proposition 3.8 Let \((M, N)\) be a pair of manifolds and let \(X \in \Gamma^\infty(TM)\) be a vector field, such that \(X|_N = 0\) and is complete. Then \(X\) is Euler-like, if and only if \(d^N X\) followed by the projection \(TM|_N \to \nu_N\) is identity.

Proof: Let \(X \in \Gamma^\infty(TM)\) be given as in the proposition. According to Proposition 3.3, there exists a unique \(D_X \in \Gamma^\infty(\text{End}(TM|_N))\), such that \(T\Phi^X_t|_N = \exp(tD_X)\). Let \([X_p] \in \nu_N\), then

\[
\nu(\Phi^X_t)([X_p]) = [T\Phi^X_t(X_p)] = [\exp(tD_X)(X_p)].
\]

This is just equal to the flow of the Euler vector field, if \(pr_{\nu_N} \circ D_X = \text{id}_{\nu_N}\). Using Proposition 3.3, we have \(d^N X = D_X\) and hence the claim. □

Note that for a pair of manifolds \((M, N)\) and an Euler-like vector field \(X \in \Gamma^\infty(TM)\), the set

\[
\{ p \in M | \lim_{t \to -\infty} \Phi^X_t(p) \text{ exists and lies in } N \}
\]

is an open subset in \(M\) containing \(N\), such that the action of \(\Phi^X_t\) shrinks to this set. Moreover, for a tubular neighbourhood \(\psi: \nu_N \to U\), such that \(\psi^* X = E\), we have that

\[
U = \{ p \in M | \lim_{t \to -\infty} \Phi^X_t(p) \text{ exists and lies in } N \}.
\]

Let us denote by \(\lambda_s = \Phi^X_{\log(s)}|_U\). We obtain, that \(\lambda_s\) is smooth for all \(s \in \mathbb{R}^+_0\). Moreover, we have that

\[
\psi \circ \lambda_s = \kappa_s \circ \psi,
\]

where we denote by \(\kappa_s: \nu_N \to \nu_N\) the map \([X_p] \mapsto [sX_p]\). Note that \(\kappa_0: \nu_N \to N\) coincides with the bundle projection, to be more precise \(k_0 = pr_\nu \circ j\), where \(pr_\nu\) is the bundle projection and \(j: N \to \nu_N\) the canonical inclusion.

Let us add now the line bundle case

Definition 3.9 Let \(L \to M\) be a line bundle and \(N \hookrightarrow M\) be a submanifold. A derivation \(\Delta \in \Gamma^\infty(DL)\) is called Euler-like, if

i.) \(\Delta|_N = 0\),

ii.) \(\sigma(\Delta)\) is an Euler-like vector field.

This definition turns out to be the correct one for our purposes, since we can prove basically all results, which are available for Euler-like vector fields. Let us start collecting them.

Proposition 3.10 Let \(L \to M\) be a line bundle and let \(\Delta \in \Gamma^\infty(DL)\) be an Euler-like derivation with respect to \(N \hookrightarrow M\), then the flow \(\Phi^\Delta_t \in \text{Aut}(L)\) of \(\Delta\) induces the map

\[
\Lambda_s = \Phi^\Delta_{\log(s)},
\]

which can be, restricted to \(U = \{ p \in M | \lim_{t \to -\infty} \Phi^\sigma_t(p) \text{ exists and lies in } N \}\), extended smoothly to \(s = 0\). Moreover, the map

\[
\Lambda_0: L \to L_N
\]

is a regular line bundle morphism.
Proof: The proof is an easy verification using a tubular neighbourhood $\psi: \nu \to U$, such that $\psi^*\sigma(X) = \mathcal{E}$. □

Definition 3.11 Let $L \to M$ be a line bundle and $N \hookrightarrow M$ be a submanifold. A fat tubular neighbourhood is a regular line bundle morphism

$$
\Psi: L_\nu \to U,
$$

where the line bundle $L_\nu$ is given by the pull-back

$$
\array{ L_\nu \ar{d} & \ar{r} \ar{d} & L_N \ar{d} \\
\nu_N & \ar{r} & N
}
$$

covering a tubular neighbourhood $\psi: \nu \to U$, such that $\Psi|_N: L_N \to L_N$ is the identity.

Lemma 3.12 Let $L \to M$ be a line bundle, let $N \hookrightarrow M$ be a submanifold and let $\psi: \nu \to U$ be a tubular neighbourhood. Then there exists a fat tubular neighbourhood covering $\psi$.

Proof: The proof can be found in [11, Chapter 3]. □

For a line bundle $L \to N$ and a vector bundle $E \to N$ there is always a canonical Derivation $\Delta_E \in \Gamma^\infty(DL_E)$, such that $\sigma(\Delta_E) = \mathcal{E}$ constructed as follows: Consider the map

$$
\array{ L_E \ar{r}^{P} & L \\
E \ar{r}^{p} & N
}
$$

and the corresponding map $DP: L_E \to L_N$. We have that canonically $\ker(DP) \cong \text{Ver}(E)$, which induces a flat (partial) connection $\nabla: \text{Ver}(E) \to DL_E$. Since the Euler vector field is canonically vertical, we can define $\Delta_E = \nabla_E$.

Proposition 3.13 Let $L \to N$ be a line bundle and let $E \to N$ be a vector bundle. Then the flow $\Phi_t$ of $\Delta_E \in \Gamma^\infty(DL_E)$ is given by

$$
\Phi_t(v_p,l_p) = (e^t \cdot v_p,l_p)
$$

for all $(v_p,l_p) \in L_E$.

Proof: This proof is an easy verification using the fact that $\Phi_t$ covers the flow of the Euler vector field. □

Note that for the flow $\Phi_t$ of the canonical Euler-like derivation $\Delta_E \in \Gamma^\infty(DL_E)$, we have that

$$
P_s = \Phi_{\log(s)}: L_E \to L_E
$$

is defined for all $s > 0$ and can be extended smoothly to $s = 0$, moreover $P_0$ coincides with the canonical projection $P: L_E \to L$ followed by the canonical inclusion $J: L \to L_E$.

Lemma 3.14 Let $L \to M$ be a line bundle, let $N \hookrightarrow M$ be a submanifold and let $\Delta \in \Gamma^\infty(DL)$ be an Euler-like derivation. Then there is a unique fat tubular neighbourhood $\Psi: L_\nu \to U$, such that $\Psi^*\Delta = \Delta_E$. 

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PROOF: First, we want to proof existence. It is clear that any such $\Psi$ has to cover the unique tubular neighbourhood $\psi: \nu_U \to U$, such that $\psi^*\sigma(\Delta) = \mathcal{E}$. So let us choose a fat tubular neighbourhood $\tilde{\Psi}: L_U \to L_U$ covering $\psi$. We consider now $\tilde{\Psi}^\ast \Delta \in \Gamma^\infty(DL_U)$. We have $\sigma(\tilde{\Psi}^\ast \Delta) = \psi^*\sigma(\Delta) = \mathcal{E}$. Hence $\sigma(\Delta_\mathcal{E}) = \sigma(\tilde{\Psi}^\ast \Delta)$. Consider now the derivation $\square = \Delta_\mathcal{E} - \tilde{\Psi}^\ast \Delta$ and

$$\square_t = -\frac{1}{t} \Phi^\ast_{\log(t)} \square,$$

where $\Phi_t$ is the flow of $\Delta_\mathcal{E}$. Let us denote the flow of $\square_t$ by $\phi_t$. Note that it is complete, since $\sigma(\square_1) = 0$, indeed there is even a explicit formula for it, which we do not use. Note however, that $\phi_t \in \text{Gau}(L_U)$ for all $t \in \mathbb{R}$. Let us compute

$$\frac{d}{dt} \phi_t^\ast (\Delta_\mathcal{E} + t \square_t) = \phi_t^\ast [([\square_t, \Delta_\mathcal{E}] + \frac{d}{dt}(\square_t)]$$

$$= \phi_t^\ast ([\square_t, \Delta_\mathcal{E}] - \frac{d}{dt} \Phi^\ast_{\log(t)} \square]\)

$$= \phi_t^\ast ([\square_t, \Delta_\mathcal{E}] - \frac{1}{t} [\Delta_\mathcal{E}, \Phi^\ast_{\log(t)} \square])$$

$$= \phi_t^\ast ([\square_t, \Delta_\mathcal{E}] + [\Delta_\mathcal{E}, \square_t])$$

$$= 0.$$

Hence we see $\Delta_\mathcal{E} = \phi_0^\ast (\Delta_\mathcal{E}) = \phi_1^\ast (\Delta_\mathcal{E} + \square_1) = \phi_1^\ast (\tilde{\Psi}^\ast \Delta)$. Therefore, we have that the map $\Psi = \tilde{\Psi} \circ \phi_1$ will do the job, since obviously $\phi_1|_N = \text{id}.$

Let us now assume that we have $\Psi_1, \Psi_2: L_\nu \to L_U$, such that $\Psi_1^\ast \Delta = \Psi_2^\ast \Delta = \Delta_\mathcal{E}$. Note that since both have to cover the unique $\psi: \nu_U \to U$, the target $L_U$ is for both the same. Let us consider $\Xi := \Psi_1^{-1} \circ \Psi_2: L_\nu \to L_\nu$, which covers the identity, which implies that there is a nowhere vanishing function $f \in \mathcal{C}^\infty(\nu_N)$, such that $\Xi(l_p) = f(p)l_p$ for all $l_p \in L_\nu$. Moreover, we have that $\Xi|_N = \text{id}_{L_\nu}|_N$, hence $f(0_n) = 1$ for all $n \in N$, and $\Xi^\ast \Delta_\mathcal{E} = \Delta_\mathcal{E}$. We consider now an arbitrary section $\lambda \in \Gamma^\infty(L_\nu)$ and compute

$$\Delta_\mathcal{E}(\lambda) = (\Xi^\ast \Delta_\mathcal{E})(\lambda)$$

$$= \Xi^\ast (\Delta_\mathcal{E}(\Xi \lambda))$$

$$= \frac{1}{f}(\Delta_\mathcal{E}(f \lambda))$$

$$= \frac{\mathcal{E}(f)}{f} \lambda + \Delta_\mathcal{E}(\lambda).$$

Hence $\mathcal{E}(f) = 0$, which means that $f = \text{pr}_n^\ast g$ for some function $g \in \mathcal{C}^\infty(N)$, but since $1 = f(0_n) = g(n)$ for all $n \in N$, we have that $\Xi = \text{id}_{L_\nu}$. \hfill \square

For a line bundle $L \to M$, a submanifold $N$ and an Euler-like derivation $\Delta \in \Gamma^\infty(DL)$, we have that

$$\Lambda_s := \Phi^\Delta_{\log(s)}: L_U \to L_U$$

is well defined for $s > 0$ and can be extended smoothly to $s = 0$, where $L_U$ is the target of the unique fat tubular neighbourhood $\Psi: L_\nu \to L_U$, such that $\Psi^\ast \Delta = \Delta_\mathcal{E}$. Moreover, we have that

$$\Lambda_s \circ \Psi = \Psi \circ P_s \quad \text{(3.3)}$$

for all $s \geq 0$. Note that if we project this equation to the manifold level, this simply gives Eq. \ref{eq:3.2}. \hfill 13
4 Normal Forms of Dirac-Jacobi bundles

Using the techniques of Euler-like derivations, we want to prove a normal form theorem for Dirac-Jacobi bundles. In fact, if the submanifold $N$ is a transversal, then we can find special Euler-like derivations which are, in some sense, controlling the behaviour of the Dirac-Jacobi bundles near $N$. The aim is now to prove the existence of this special kind of Euler-like derivations and afterwards, we are able to prove a normal form theorem, and conclude some corollaries from it.

**Definition 4.1** Let $L 	o M$ be a line bundle, let $H \in \Omega^3_M$ be closed and let $\mathcal{L} \subseteq \mathbb{D}L$ be a $H$-twisted Dirac-Jacobi bundle. A submanifold $N \hookrightarrow M$ is called transversal, if

$$DL_N + \text{pr}_D \mathcal{L}|_N = (DL)|_N.$$ 

**Proposition 4.2** Let $L 	o M$ be a line bundle, let $H \in \Omega^3_M$ be closed, let $\mathcal{L} \subseteq \mathbb{D}L$ be a $H$-twisted Dirac-Jacobi bundle and let $N \hookrightarrow M$ be a transversal. Then

$$\mathcal{B}_I(\mathcal{L}) := \{ (\Delta_p, (DI)^* \alpha_{i(p)}) \in \mathbb{D}L_1 \mid (DI(\Delta_p), \alpha_{\phi(p)}) \in \mathcal{L} \}$$

is a $I^*H$-twisted Dirac-Jacobi bundle, where $I: L_N \to L$ is the canonical inclusion.

**Proof:** This is an easy consequence of Theorem 2.17. □

**Lemma 4.3** Let $L \to M$ be a line bundle, let $H \in \Omega^3_M$ be closed, let $\mathcal{L} \subseteq \mathbb{D}L$ be a $H$-twisted Dirac-Jacobi bundle and let $\iota: N \hookrightarrow M$ be a transversal. The backwards transformation $\mathcal{B}_I(\mathcal{L})$ is canonically isomorphic (as vector bundles) to the fibered product

$$\begin{array}{ccc}
I^!\mathcal{L} & \longrightarrow & \mathcal{L} \\
\downarrow & & \downarrow \text{pr}_D \\
DL_N & \longrightarrow & DL
\end{array}$$

**Proof:** We consider the linear map

$$\Xi: I^!\mathcal{L}_p \ni (\Delta_p, (\Box_{i(p)}, \alpha_{i(p)})) \mapsto (\Delta_p, DI^* \alpha_{i(p)}) \in \mathcal{B}_I(\mathcal{L}),$$

which is well-defined since $DI(\Delta_p) = \Box_{i(p)}$. We claim now that this map is injective, let us therefore consider $(\Delta_p, (\Box_{i(p)}, \alpha_{i(p)})) \in \ker(\Xi)$. It follows immediately, that $\Delta_p = 0$ and hence $\Box_{i(p)} = 0$. If $(0, \alpha_{i(p)}) \in \mathcal{L}$ then $\alpha_{i(p)} \in \text{Ann}(\text{pr}_DL)$. Since $DI^* \alpha_{i(p)}0 = 0$, we have that $\alpha_{i(p)} \in \text{Ann}(DL_N)$, hence $\alpha_{i(p)} = 0$ and the claim follows. For dimension reasons we have that $\Xi$ is an isomorphism. □

**Proposition 4.4** Let $L \to M$ be a line bundle, let $H \in \Omega^3_M$ be closed, let $\mathcal{L} \subseteq \mathbb{D}L$ be a $H$-twisted Dirac-Jacobi bundle and let $N \hookrightarrow M$ be a transversal. Then there exists $\varepsilon \in \Gamma^\infty(\mathcal{L})$, such that $\varepsilon|_N = 0$ and $\text{pr}_D(\varepsilon)$ is Euler-like.

**Proof:** We consider the exact sequence

$$0 \to \mathcal{B}_I(\mathcal{L}) \to \mathcal{L}|_N \to \nu_N \to 0,$$

where the first arrow is defined by the identification $\mathcal{B}_I(\mathcal{L}) \cong I^!\mathcal{L}$ from Lemma 4.3 followed by the canonical map $I^!\mathcal{L} \to \mathcal{L}$. The second arrow is the projection $\text{pr}_D: \mathcal{L}|_N \to DL|_N$ followed by the symbol map $\sigma: DL|_N \to TM|_N$ and finally followed by the the projection to the normal bundle $\text{pr}_{\nu_N}: TM|_N \to \nu_N$. Let us choose a section $\varepsilon \in \Gamma^\infty(\mathcal{L})$ with $\varepsilon|_N = 0$, such that $d^N \varepsilon: \nu_N \to \mathcal{L}|_N$ defines a splitting of the sequence. We consider now
and see that if \( d^N \varepsilon \) splits the above sequence then \((\sigma \circ \text{pr}_D) d^N \varepsilon \) splits the lower sequence. Using Proposition 3.2 we see that \((\sigma \circ \text{pr}_D) d^N \varepsilon = d^N ((\sigma \circ \text{pr}_D)(\varepsilon))\) and by Proposition 3.8 we see that \( T\nu(\sigma \circ \text{pr}_D)(\varepsilon) = \mathcal{E}\). Multiplying \( \varepsilon \) by a suitable bump function we may arrange that \((\sigma \circ \text{pr}_D)(\varepsilon)\) is complete and hence an Euler-like vector field. By definition \(\text{pr}_D(\varepsilon)\) is hence an Euler-like derivation.\(\square\)

Let us fix now a \( H \)-twisted Dirac-Jacobi structure \( \mathcal{L} \subseteq \mathbb{D}L \) for a line bundle \( L \to M \). Let us also consider a transversal \( \iota: N \hookrightarrow M \) and a section \( \varepsilon = (\Delta, \alpha) \in \Gamma^\infty(\mathcal{L}) \), such that \( \varepsilon|_N = 0 \) and \( \Delta \) is an Euler-like derivation. Due to the Lemma 3.14 we find a unique fat tubular neighbourhood

\[
\begin{align*}
L_\nu & \xrightarrow{\psi} L_U \\

\nu_N & \xrightarrow{\psi} U
\end{align*}
\]

such that \( \Psi^* \Delta = \Delta_\varepsilon \). With this we have now two ways to construct a Dirac-Jacobi bundle on \( L_\nu \to \nu_N \), namely we can take the Backwards transformation \( \mathfrak{B}_\Phi(\mathcal{L}_U) \) and, if we consider

\[
\begin{align*}
L_\nu & \xrightarrow{P} L_N \xrightarrow{I} L \\

\nu_N & \xrightarrow{} N \to M
\end{align*}
\]

taking the backwards transform \( \mathfrak{B}_{I\circ P}(\mathcal{L}) = \mathfrak{B}_P(\mathfrak{B}_I(\mathcal{L})) \). The aim is now to compare these two structures. Let us therefore consider the flow of \( [(\Delta, \alpha), -]_H \), which is given by

\[
(\gamma_t, \Phi^\Delta_t) \in Z^2_{\mathfrak{I}}(M) \rtimes \text{Aut}(L),
\]

where \( \Phi^\Delta_t \) is the flow of \( \Delta \) and \( \gamma_t = \int_0^t (\Phi^\Delta_{-\tau})^*((d_L \alpha + \iota_\Delta H) \, d\tau) \). For sure we have that the action of \( (\gamma_t, \Phi^\Delta_t) \) preserves \( \mathcal{L} \), which is explicitly

\[
\exp(\gamma_t) \circ \mathbb{D}\Phi^\Delta_t(\mathcal{L}) = \mathcal{L}.
\]

This leads us directly to the following theorem.

**Theorem 4.5 (Normal form for Dirac-Jacobi bundles)** Let \( L \to M \) be a line bundle, let \( H \in \Omega^1_M \) be closed, let \( \mathcal{L} \subseteq \mathbb{D}L \) be a \( H \)-twisted Dirac-Jacobi bundle and let \( N \hookrightarrow M \) be a transversal. Then there exists an open neighbourhood \( U \subseteq M \) of \( N \) and fat tubular neighbourhood \( \Psi: L_\nu \to L_U \), such that

\[
\mathfrak{B}_\Psi(\mathcal{L}|_U) = (\mathfrak{B}_{I\circ P}(\mathcal{L}))^\omega
\]

for an \( \omega \in \Omega^2_{L_\nu}(\nu_N) \).

**Proof:** According to Proposition 4.4 we can find \( (\Delta, \alpha) \in \Gamma^\infty(\mathcal{L}) \), such that \( \Delta \) is Euler-like. Then there is a unique fat tubular neighbourhood \( \Psi: L_\nu \to L_U \), such that \( \Psi^* \Delta = \Delta_\varepsilon \), due to Lemma 3.14. Let us denote by \( (\gamma_t, \Phi^\Delta_t) \in Z^2_{\mathfrak{I}}(M) \rtimes \text{Aut}(L) \) the flow of \( [(\Delta, \alpha), -]_H \). We know that \( (\gamma_t, \Phi^\Delta_t) \) preserves \( \mathcal{L} \) for all \( t \in \mathbb{R} \) and so will \( (\gamma_{-\log(s)}, \Phi^\Delta_{-\log(s)}) \) for all \( s \) greater than 0. Let us take a closer look to

\[
\gamma_{-\log(s)} = \int_0^{-\log(s)} (\Phi^\Delta_{-\tau})^*(d_L \alpha + \iota_\Delta H) \, d\tau
\]

for an \( \omega \in \Omega^2_{L_\nu}(\nu_N) \).
we obtain that it is smoothly extendable to $s = 0$ and let us denote its limit $s \to 0$ by $\omega'$ and $\omega = \Psi^*\omega'$. We have

$$\mathcal{B}_\Psi(L|_U) = \mathcal{B}_\Psi(\exp(\gamma_{-\log(s)}) \circ J^\Delta_{-\log(s)}(L))$$

$$= \mathcal{B}_\Psi(\exp(\gamma_{-\log(s)})\mathcal{B}_{\Phi_{\log(s)}}(L))$$

$$= (\mathcal{B}_\Psi(\mathcal{B}_\Lambda_\nu(L)))^\Psi \gamma_{-\log(s)}$$

$$= (\mathcal{B}_\Lambda_\nu\mathcal{B}_\Phi(L))^\Psi \gamma_{-\log(s)}$$

$$= (\mathcal{B}_{\mathcal{B}_{\Phi,\nu}}(L))^\Psi \gamma_{-\log(s)}.$$  

which holds for all $s \geq 0$. Hence we have for $s = 0$, using that for the canonical inclusion $J : L_N \to L_\nu$ we have that $P_0 = J \circ P$ and $\Psi \circ J = I$, that

$$\mathcal{B}_\Psi(L|_U) = (\mathcal{B}_{I_0\circ P}(L))^\omega.$$  

\[\square\]

Note that this Theorem says, that up to a $B$-field, the Dirac-Jacobi structure is fully encoded in a given transversal, and hence the term ”normal form” is justified by this fact. Moreover, it is possible to distinguish two different kind of leaves in Dirac-Jacobi geometry, see [12], so it is also possible to distinguish two kinds of transversals, which are more interesting in the Jacobi setting, since in the general Dirac-Jacobi setting the normal forms will be the same. Nevertheless, we will introduce them here and use them more excessively in the next section.

**Definition 4.6 (Cosymplectic Transversal)** Let $L \to M$ be a line bundle and let $L \in \mathbb{D}L$ be a Dirac-Jacobi structure. A transversal $\iota : N \to M$ is called cosymplectic, if

$$DL_N \cap \mathcal{B}_I(L) = \{0\}.$$  

**Remark 4.7** Note that a cosymplectic transversal always inherts a Dirac-Jacobi bundle coming from a Jacobi tensor by Proposition [2,13]. So let us denote $\mathcal{L}_N = \mathcal{B}_I(L_J) \subseteq \mathbb{D}L_N$.

This transversals naturally appear as minimal transversal to locally conformal pre-symplectic leaves, see [12] for a more detailed discussion.

So a corollary of this normal form theorem using the new notion of cosymplectic transversals

**Corollary 4.8** Let $L \to M$ be a line bundle, let $L \subseteq \mathbb{D}L$ be a Dirac-Jacobi structure and let $\iota : N \to M$ be a minimal transversal to $L$ at a locally conformal pre-symplectic point $p_0$, i.e. $\mathcal{B}_{\mathcal{B}_{\Phi_D}(L)|_{p_0}} \oplus T_{p_0}N = T_{p_0}M$ and let $\nu_N = V \times N$. Then locally around $p_0$:

$$\mathcal{B}_\Psi(L|_U) = \{(\nu + J^\nu_\nu(\psi), \alpha + \psi) \in DL_\nu \mid v \in TV, \alpha \in (\text{Ann}(T^*V)) \otimes L_\nu \text{ and } \psi \in J^1L_N\}^{\omega}$$

where $J_N$ is the Jacobi structure on the transversal and the canonical identification $DL_{\nu_N} = TV \oplus DL_N$.

**Proof:** Note that it is easy to check that for a minimal transversal $N$ at a locally conformal pre-symplectic point $p_0$ the equation

$$DL_N \cap \mathcal{B}_I(L) = \{0\}$$

holds at $p_0$ and hence in a whole neighbourhood. The rest is an application of Theorem [4.5] and the usage of the splitting $DL_{\nu_N} = TV \oplus DL_N$.  

\[\square\]
The other kind of leaves of a Dirac-Jacobi structure are so-called pre-contact leaves. Their minimal transversal possess the following structure:

**Definition 4.9 (Cocontact Transversal)** Let $L \to M$ be a line bundle and let $\mathcal{L} \in \delta L$ be a Dirac-Jacobi structure. A transversal $\iota: N \hookrightarrow M$ is called cocontact, if

$$\text{rank}(DL_N \cap \mathcal{B}_I(\mathcal{L})) = 1.$$ 

**Lemma 4.10** Let $L \to M$ be a line bundle, let $\mathcal{L} \subseteq \delta L$ be a Dirac-Jacobi structure and let $\iota: N \hookrightarrow M$ be a minimal transversal to $\mathcal{L}$ at a pre-contact point $p_0$. Then

$$\text{rank}(DL_N \cap \mathcal{B}_I(\mathcal{L})) = 1$$

holds in a neighbourhood of $p_0$.

**Proof:** It is easy to see that

$$(DL_N \cap \mathcal{B}_I(\mathcal{L}))|_{p_0} = (1).$$

Now we want to argue why this holds in a whole neighbourhood. Let us therefore consider the sum $DL_N + \mathcal{B}_I(\mathcal{L}) \subseteq \delta L$ and a (local) section $\alpha \in \Omega^1_L(M)$ such that $\alpha(\mathbf{1})|_{p_0} \neq 0$. Let $(0, \beta) \in (DL_N + \mathcal{B}_I(\mathcal{L}))|_{p_0} \cap \langle \alpha \rangle|_{p_0}$, then there exists $\Delta \in D_{p_0}L$ such that $(\Delta, \beta) \in \mathcal{B}_I(\mathcal{L})$, but since $(\mathbf{1}, 0) \in \mathcal{B}_I(\mathcal{L})$, we have using the isotropy of $\mathcal{B}_I(\mathcal{L})$,

$$0 = \langle (\Delta, \beta), (\mathbf{1}, 0) \rangle = \beta(\mathbf{1}).$$

and hence, for dimensional reasons, $\delta L|_{p_0} = (DL_N + \mathcal{B}_I(\mathcal{L}))|_{p_0} \oplus \langle \alpha \rangle|_{p_0}$. Therefore this equality holds in a whole neighbourhood of $p_0$, so $\text{rank}(DL_N + \mathcal{B}_I(\mathcal{L})) = 2n + 1$ in this neighbourhood, which implies $\text{rank}(DL_N \cap \mathcal{B}_I(\mathcal{L})) = 1$ around $p_0$. \hfill $\square$

**Remark 4.11** Note that a cocontact transversal does not inhert a Jacobi structure, but nevertheless the Dirac-Jacobi structure is of homogeneous Poisson type.

**Definition 4.12** Let $L \to M$ be a line bundle and let $\mathcal{L} \in \delta L$ be a Dirac-Jacobi structure. A homogeneous cocontact transversal $\iota: N \hookrightarrow M$ is a cocontact transversal together with a flat connection $\nabla: TN \to DL_N$, such that

$$\text{im}(\nabla) \oplus (DL_N \cap \mathcal{B}_I(\mathcal{L})) = DL_N.$$ 

**Remark 4.13** The definition of a homogeneous cocontact transversal seems a bit strange, since it includes a connection. This fact can be explained quite easily using the homogenization described in [12], which turns a Dirac-Jacobi structure on a line bundle $L \to M$ into a Dirac structure on $L^* := L^* \setminus \{0_M\}$ which is homogeneous (in the sense of [10]) with respect to the shrinked Euler vector field $\mathcal{E}$ on $L^*$. The pre-symplectic leaves of this Dirac structure have the additional property that $\mathcal{E}$ is either tangential to it or transversal. If $\mathcal{E}$ is tangential, then the leaf corresponds to a pre-contact leaf on the base $M$. Hence a minimal transversal $N$ to it is transversal to the Euler vector field and defines therefore a horizontal bundle on $L^*_\text{pr}(N)$ and hence a connection.

**Proposition 4.14** Let $L \to M$ be a line bundle, let $\mathcal{L} \subseteq \delta L$ be a Dirac-Jacobi structure and let $\iota: N \hookrightarrow M$ be a minimal transversal to $\mathcal{L}$ at a pre-contact point $p_0$. Then every flat connection $\nabla$ gives $N$ locally the structure of a homogeneous cocontact transversal.
Proof: In the proof of Lemma 4.10 we have seen that
\[(DL_N \cap \mathcal{B}_I(\mathcal{L}))|_{p_0} = (1)\]
and hence for every flat connection $\nabla$, we have that $\text{im}(\nabla)|_{p_0} \oplus (DL_N \cap \mathcal{B}_I(\mathcal{L}))|_{p_0} = DL_N$ and hence this decomposition holds in a whole neighbourhood of $p_0$. \qed

An immediate consequence is:

Corollary 4.15 Let $L \to M$ be a line bundle, let $\mathcal{L} \subseteq \mathcal{B}L$ be a Dirac-Jacobi structure and let $\iota: N \hookrightarrow M$ be a homogeneous cocontact transversal with connection $\nabla$. Then there exists a local trivialization of $\nu$ such that, using the to $\nabla$ corresponding trivializations $DL_{\nu} = TV \oplus \mathbb{R}_M$ and $J^1L = T^*M \oplus \mathbb{R}_M$, $\mathcal{B}_\psi(\mathcal{L}^{|U}) = \{(v + r(1 - Z_N) + \pi^2(\psi), \alpha + \psi(Z_N)1^*) | v \in TV, \alpha \in \text{Ann}(T^*V) \text{ and } \psi \in T^*N\}^\omega$, where $(\pi_N, Z_N)$ is the homogeneous Poisson structure on the transversal from Lemma 2.14.

This last two corollaries can be seen as the Jacobi-geometric analogue of the results obtained by Blohmann in [1].

5 Normal forms and Splitting Theorems of Jacobi bundles

As explained in Example 2.12, Jacobi bundles are a special kind of Dirac-Jacobi structures. In addition, we have that Jacobi isomorphism induces an isomorphism of the corresponding Dirac structures (this holds even for morphisms if one considers forward maps of Dirac-Jacobi structures which we will not explain here, see [12]). The converse is unfortunately not true: if the Dirac-Jacobi structures of two Jacobi structures are isomorphic, it does not follow in general that the Jacobi structures are isomorphic. The parts which are not "allowed" in Jacobi geometry are the $B$-fields. Nevertheless, we can keep track of them, if we make further assumptions on the transversals, namely cosymplectic and cocontact transversals.

5.1 Cosymplectic Transversals

In this part, we are using the notion of cosymplectic transversals as explained in the previous section. The difference is now that in Jacobi geometry this transversal gives us more than on arbitrary Dirac-Jacobi manifolds. In fact, the Jacobi structure induces a line bundle valued symplectic structure on the normal bundle, to be seen in the following

Lemma 5.1 Let $L \to M$ be a line bundle, $J \in \Gamma_{\infty}(\Lambda^2(J^1L)^* \otimes L)$ be a Jacobi tensor with corresponding Dirac-Jacobi structure $\mathcal{L}_J \subseteq DL$ and let $\iota: N \hookrightarrow M$ be a cosymplectic transversal. Then
\[J^2(\text{Ann}(DL_N)) \oplus DL_N = DL|_N.\]

Proof: First we prove that $J^2|_{\text{Ann}(DL_N)}$ is injective. Let therefore $\alpha \in \text{Ann}(DL_N)$, such that $J^2(\alpha) = 0$. Hence we have for an arbitrary $\beta \in J^1L$, that
\[\alpha(J^2(\beta)) = \beta(J^2(\alpha)) = 0.\]

Hence $\alpha = \text{Ann}(DL_N) \cap \text{Ann}(\text{im}(J^2)) = \text{Ann}(DL_N + \text{im}(J^2)) = \{0\}$, and $J^2|_{\text{Ann}(DL_N)}$ is injective.

Let $\Delta \in DL_N \cap J^2(\text{Ann}(DL_N))$, then there exists an $\alpha \in \text{Ann}(DL_N)$, such that $J^2(\alpha) = \Delta$. Thus, we have that $(\Delta, \alpha) \in \mathcal{L}_J$ and moreover $(\Delta, DI^*\alpha) \in \mathcal{B}_I(\mathcal{L}_J)$, but since $\alpha \in \text{Ann}(DL_N)$, we have that $DI^*\alpha = 0$ and hence $\Delta = 0$, since $N$ is cosymplectic. Counting dimensions the claim follows. \qed
Suppose that \( \nu : N \hookrightarrow M \) is a cosymplectic transversal, then we have that

\[
\text{pr}_\nu \circ \sigma \circ J^2 : \text{Ann}(DL_N) \to \nu_N
\]
is an isomorphism. Let us chose \( \alpha \in \Gamma^\infty(J^1L) \), such that \( \alpha|_N = 0 \) and such that \( d^N\alpha : \nu_N \to \text{Ann}(DL_N) \subseteq J^1L|_N \) is a right-inverse to \( \text{pr}_\nu \circ \sigma \circ J^2 \). We have then

\[
\text{pr}_\nu(d^N\sigma(J^2(\alpha))) = \text{pr}_\nu(\sigma(J^2(d^N\alpha))) = \text{id}_{\nu_N}
\]
and hence we have that \( T\nu(\sigma(J^2(\alpha))) = \mathcal{E} \). Multiplying \( \alpha \) by a bump-function, which is 1 near \( N \), we may arrange that \( \sigma(J^2(\alpha)) \) is complete and hence \( J^2(\alpha) \) is an Euler-like derivation. By Theorem 4.5, we have that

\[
\mathfrak{B}_\Psi(\mathcal{L}_J) = \mathfrak{B}_\Psi(\mathcal{L}_{J_N})^\omega,
\]
where \( \omega = \Psi^* \int_0^1 \frac{1}{2} (\Phi^\Delta_{\log(t)})^*(d_L\alpha) \, dt \) and \( \Psi : L_\nu \to L_U \) is the unique tubular neighbourhood, such that \( \Psi^*(J^2(\alpha)) = \Delta_{\mathcal{E}} \).

**Proposition 5.2** The 2-form \( \omega \in \Omega^2_{L_\nu}(\nu_N) \) shrunk to \( N \) has kernel \( DL_N \).

**Proof:** One can show, in local coordinates, that \( d^N\alpha([\sigma(\Box)]|_N) = \mathcal{L}_\Box \alpha|_N \) for all \( \Box \in \Gamma^\infty(DL) \). Hence we have trivially \( \mathcal{L}_\Delta \alpha|_N = 0 \) for \( \Delta \in \Gamma^\infty(DL) \), such that \( \Delta|_N \in \Gamma^\infty(DL_N) \). Let now \( \Delta, \Box \in \Gamma^\infty(DL) \), such that \( \Delta|_N \in \Gamma^\infty(DL_N) \), then

\[
d_L\alpha(\Delta, \Box)|_N = -(d_L\Delta)(\Box)|_N = -\Box(\alpha(\Delta))|_N
\]

\[
= -(\mathcal{L}_\Box \alpha)(\Delta)|_N - \alpha([\Box, \Delta])|_N = -(\mathcal{L}_\Box \alpha)|_N(\Delta)
\]

\[
= d^N\alpha([\sigma(\Box)]|_N)(\Delta) = 0,
\]
where the last equality follows since \( d^N\alpha : \nu_N \to \text{Ann}(DL_N) \). Hence we have that ker\((d_L\alpha)^\Delta \supseteq DL_N \), in particular this is true for \( 4\Phi^\Delta_{\log(t)}(d_L\alpha) \), since \( \Phi^\Delta_{\log(s)}|_N \) is a gauge transformation fixing \( DL_N \). Thus it is true also for \( \omega \), since \( D\Psi|_{DL_N} = \text{id} \). \( \square \)

We want to describe the structure of \( \omega \) at \( N \). Note that for a cosymplectic transversal \( N \), the normal bundle always comes together with a canonical symplectic (i.e. non-degenerate) \( L_N \)-valued 2-form \( \Theta \in \Gamma^\infty(L^2\nu_N \otimes L_N) \) defined by

\[
\Theta(X, Y) = (\text{pr}_\nu \circ \sigma \circ J^2|_{\text{Ann}(DL_N)})^{-1}(X)(Y)
\]

**Lemma 5.3** The 2-form \( \omega \in \Omega^2_{L_\nu}(\nu_N) \) coincides, shrunk to \( \nu_N \subseteq DL_{\nu_N} \), with \( \Theta \).

**Proof:** Note that for a cosymplectic transversal, we have

\[
DL|_N = DL_N \oplus J^2(\text{Ann}(DL_N)) = DL_N \oplus \nu_N
\]

with the canonical identification

\[
J^2(\text{Ann}(DL_N)) = \frac{DL|_N}{DL_N} = \nu_N.
\]
Moreover, we have
\[ \left. DL\nu \right|_N = DL_N \oplus \nu_N, \]
where we include \( \nu_N \) by the following map:
\[ \chi: \nu_N \ni v_n \to \left( \lambda \to \frac{d}{dt} \bigg|_{t=0} P_0 \lambda (p_t(v_p)) \right) \in D_n L_{\nu}. \]

It is clear that \( D\Psi \) fixes \( DL_N \), since \( \Psi \big|_N: L_N \to L_N \) is identity. We want to show that \( D\Psi (\nu_N) \subseteq J^2(\text{Ann}(DL_N)) \). One can show that by an elementary calculation, that
\[ D\Psi (\chi(v_n)) = \lim_{t \to 0} \frac{\Delta \lambda_t (\psi(v_n))}{t} \]
using Equation [3.3] But by definition, we have that
\[ d^N \Delta (v_n) = \lim_{t \to 0} \frac{\Delta \lambda_t (\psi(v_n))}{t} \]
hence \( D\Psi \circ \chi = d^N \Delta = J^2(\text{Ann}(DL_N)) \), but \( \alpha \) was chosen in such a way that \( d^N \alpha \) takes values in \( \text{Ann}(DL_N) \). Thus \( D\Psi \big|_N \) respects the splitting. Using this and
\[ \text{ker}(\omega^\flat)|_N = DL_N \]
and the definition of \( \Theta \), we see that at \( N \) they have to coincide. \( \square \)

This leads us to the normal form theorem for Jacobi manifolds.

**Theorem 5.4 (Normal Form for Jacobi bundles I)** Let \( L \to M \) be a line bundle, let \( J \) be a Jacobi structure and let \( N \to M \) be a cosymplectic transversal. For a closed 2-form \( \omega \in \Omega^2_{L_{\nu}}(\nu_N) \), such that \( \text{ker}(\omega^\flat)|_N = DL_N \) and \( \omega \) coincides with \( \Theta \) at \( \nu_N \subseteq DL_{\nu} \). Then
\[ \mathfrak{B}_P(L_{J_N})^{\omega} \]
is the graph of a Jacobi structure near the zero section and there exists a fat tubular neighbourhood \( \Psi: L_{\nu} \to L_U \) which is a Jacobi map near the zero section.

**Proof:** We have proven this theorem for the special \( \omega \) given by
\[ \omega = \int_0^1 \frac{1}{t} (\Phi_{t(p_0)}^\Delta)^* d_L \alpha \, dt. \]
Let \( \omega' \) be a second 2-form fulfilling the requirements of the theorem, then
\[ \sigma_t := t(\omega' - \omega) \]
is a (time-dependent) 2-form such that \( \sigma_0 = 0 \) and moreover \( \sigma_t|_N = 0 \). Thus,
\[ (\mathfrak{B}_P(L_{J_N})^{\omega})^{\sigma_t} = \mathfrak{B}_P(L_{J_N})^{\omega + \sigma_t} \]
is a Jacobi structure near \( N \). Now we can apply Appendix [A] to get the result. \( \square \)

An immediate consequence of this theorem is the Splitting for Jacobi manifolds around a locally conformal symplectic leaf, proven by Dazord, Lichnerowicz and Marle in [5].
Theorem 5.5 Let \( L \rightarrow M \) be a line bundle, let \( J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L) \) be a Jacobi tensor and let \( p_0 \in M \) be a locally conformal symplectic point. Then there are a line bundle trivialization \( L_U \cong U \times \mathbb{R} \) around \( p_0 \) and a cosymplectic transversal \( N \hookrightarrow U \), such that \( U \cong U_{2i} \times N \) for an open subset \( 0 \in U_{2i} \subseteq \mathbb{R}^{2i} \) and the corresponding Jacobi pair \((\Lambda, E)\) is transformed (via this isomorphism) to
\[
(\Lambda, E) = (\pi_{\text{can}} + \Lambda_N + E_N \wedge Z_{\text{can}}, E_N),
\]
where \((\Lambda_N, E_N)\) is the induced Jacobi structure on the transversal \( N \) and the canonical structures on the fiber are given by \((\pi_{\text{can}}, Z_{\text{can}}) = (\frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}, p_i \frac{\partial}{\partial q_i})\).

Proof: We can assume from the beginning that the line bundle is trivial, since otherwise we can trivialize around \( p_0 \) and shrink the line bundle to this open neighbourhood. Let us choose an arbitrary transversal \( N \) to a leaf \( S \) at \( p_0 \) (in the sense, that \( S \times N = M \)). It is easy to see that
\[
(\mathcal{D}L_N \cap \mathfrak{B}_1(\mathcal{L}_J))|_p = \{0\},
\]
and hence we can shrink to an open neighbourhood of \( p_0 \), where this equality holds. This means every transversal to a leaf is a cosymplectic transversal near the intersection point. Let us from now on denote \( p_0 = (s_0, n_0) \), hence \( \nu_N \cong T_{s_0}S \times N \cong \mathbb{R}^{2k} \times N \). Since the line bundle is trivial, we can identify \( \nu_N \) together with \( \Theta \) as a symplectic vector bundle, hence we find a possible smaller \( N \) and a vector bundle automorphism of \( \nu_N \), such that \( \Theta \) is the constant symplectic form. We can now choose
\[
\omega = dq^i \wedge dp_i - 1^* \wedge p_i dq^i \in \Omega_{L\nu}(\nu_N)
\]
where \((q,p)\) are the symplectic coordinates on \( \nu_N \rightarrow N \). This 2-form is \( dL \)-closed and coincides with \( \Theta \) on \( N \), moreover \( \ker(\omega^\nu)|_N = \mathcal{D}L_N \). Hence the requirements of Theorem 5.4 are fulfilled and the claim follows by an easy computation. \( \square \)

5.2 Cocontact transversals

The second kind of transversals we want to discuss in the context of Jacobi geometry are cocontact transversals, which were also introduced before in Definition 4.9. In fact this notion is not enough for our purposes and we need to assume more information on the structure of the transversal, which is precisely the notion of homogeneous cocontact transversal from Definition 4.9.

Lemma 5.6 Let \( L \rightarrow M \) be a line bundle, \( J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L) \) be a Jacobi tensor with corresponding Dirac-Jacobi structure \( \mathcal{L}_j \in \mathfrak{D}L \) and let \( i : N \hookrightarrow M \) be a homogeneous cocontact transversal with connection \( \nabla : TN \rightarrow DL_N \). Then
\[
J^1(\text{Ann}(\text{im}(\nabla))) \oplus \text{im}(\nabla) = DL|_N.
\]

Proof: The proof follows the same lines as Lemma 5.4. \( \square \)

We pick now, as in the cosymplectic case, an \( \alpha \in \Gamma^\infty(J^1L) \), such that \( \alpha|_N = 0 \) and
\[
d^N\alpha : \nu_N \rightarrow \text{Ann}(\text{im}(\nabla)) \subseteq J^1L|_N
\]
defines a splitting of \( J^1\mathcal{L} \rightarrow \mathcal{L}|_N \rightarrow \nu_N \), i.e. \( \text{pr}_\nu \circ \sigma \circ J^2 \circ d^N\alpha = \text{id}_{\nu_N} \). Hence we have that \( J^2(\alpha) \), multiplied by a suitable bump function which is 1 close to \( N \), is an Euler-like derivation. By Theorem 1.5 we have that
\[
\mathfrak{B}_{\Psi}(\mathcal{L}_j) = \mathfrak{B}_P(\mathfrak{B}_1(\mathcal{L}))|_{\omega},
\]
where \( \omega = \Psi^* \int_0^1 \frac{1}{t}(\Phi_{\log(t)}^\Lambda)^*(d_L\alpha) dt \) and \( \Psi : L_\nu \rightarrow L_U \) is the unique tubular neighbourhood, such that \( \Psi^*(J^2(\alpha)) = \Delta_\varepsilon \). We can prove, as before, the following
Proposition 5.7 The 2-form $\omega \in \Omega^2_{L_\nu}(\nu_N)$ shrinked to $N$ has kernel $\text{im}(\nabla)$.

**Proof:** This proof follows the same lines as the proof of Proposition 5.2. □

As in the cosymplectic transversal case, we can define a skew symmetric 2-form

$$\Theta \in \Gamma^\infty(\Lambda^2 J^\sharp(\text{Ann}(\text{im}(\nabla))) \otimes L_N)$$

by

$$\Theta(X,Y) = (J^\sharp|_{\text{Ann}(\text{im}(\nabla))})^{-1}(X)(Y).$$

It is easy to see that $\Theta$ is non-degenerate. Moreover, we have

Lemma 5.8 The 2-form $\omega \in \Omega^2_{L_\nu}(\nu_N)$ coincides, shrinked to $\nu_N \oplus K \subseteq DL\nu$, with $\Theta$, where we denote $K := (DL_N \cap \mathcal{B}_I(L_J))$.

**Proof:** Using the ideas of the proof of Lemma 5.3 we can show that the fat tubular neighbourhood transports $J^\sharp(\text{Ann}(\text{im}(\nabla)))$ to $\nu_N \oplus K$, hence the proof is copy and paste of this Lemma. □

**Theorem 5.9 (Normal Form for Jacobi bundles II)** Let $L \to M$ be a line bundle, let $J$ be a Jacobi structure and let $N \to M$ be a cocontact transversal with connection $\nabla : TN \to DL_N$. For a closed 2-form $\omega \in \Omega^2_{L_\nu}(\nu_N)$, such that $\ker(\omega^\flat)|_N = \text{im}(\nabla)$ and $\omega$ coincides with $\Theta$ at $\nu_N \oplus (\mathcal{B}_I(L_J) \cap DL_N) \subseteq DL_\nu$. Then

$$\mathcal{B}_P(L_N)^\omega$$

is the graph of a Jacobi structure near the zero section and there exists a fat tubular neighbourhood $\Psi : L_\nu \to L_U$ which is a Jacobi map near the zero section.

**Proof:** The proof follows the lines of Theorem 5.4 with the obvious adaption. □

The next step is to prove the second splitting Theorem of Dazord and Lichnerowicz and Marle in [5], namely the splitting of Jacobi manifolds around contact leaves.

**Theorem 5.10** Let $L \to M$ be a line bundle, let $J \in \Gamma^\infty(\Lambda^2(J^1 L)^* \otimes L)$ be a Jacobi tensor and let $p_0 \in M$ be a contact point. Then there are a line bundle trivialization $L_U \cong U \times \mathbb{R}$ around $p_0$ and a homogeneous cocontact transversal $N \hookrightarrow U$, such that $U \cong U_{2q+1} \times N$ for an open subset $0 \in U_{2q+1} \subseteq \mathbb{R}^{2q+1}$ and the corresponding Jacobi pair $(\Lambda, E)$ is transformed (via this isomorphism) to

$$(\Lambda, E) = (\Lambda_{\text{can}} + \pi_N + E_{\text{can}} \wedge Z_N, E_{\text{can}}),$$

where $(\pi_N, Z_N)$ is the induced homogeneous Poisson structure on the transversal $N$ and the contact structure on the fiber is given by $(\Lambda_{\text{can}}, E_{\text{can}}) = \left((p_1 \frac{\partial}{\partial u} + p_0 \frac{\partial}{\partial q}) \wedge \frac{\partial}{\partial q}, \frac{\partial}{\partial u}\right)$.

**Proof:** Let $p_0 \in M$ be a contact point and let $N \subseteq M$ be a transversal, such that $\sigma(\text{im} J^\sharp)|_{p_0} \oplus T_{p_0}N = T_{p_0}M$. We can again assume that the line bundle $L \to M$ is trivial, since we want to prove a local statement. In a possibly smaller neighbourhood, we can assume that also the normal bundle $\nu_N = V \times N \to N$ is trivial. We want to show that there is a trivialization of $\nu_N$, such that $\Theta$ looks trivial, where we
specialize on the way through the proof what we mean by trivial. Let us therefore denote by $\lambda$ the local trivializing section of $L_N$, thus we can write

$$\Theta(\Delta, \Box) = \Omega(\Delta, \Box) \cdot \lambda$$

for $\Delta, \Box \in \nu_N \oplus K$. Since $L_N \rightarrow N$ is trivial, we identify $DL_N = TN \oplus \mathbb{R}_N$ and choose the trivial connection $\nabla$. Hence, we can find a (local) nowhere vanishing section of $K$ of the form $1 - Z$ for a unique $Z$. Let us now shrink $\Theta_{\mid_\nu_N : \nu_N \times \nu_N \rightarrow L_N}$, since $\nu_N$ is odd dimensional and $\Theta$ is a skew-symmetric pairing, we can find a local non-vanishing $X \in \Gamma^\infty(\nu_N)$, such that $\Theta(X, \cdot) = 0$, moreover, since $\Theta$ is non-degenerate, we can modify $X$ in such a way that

$$\Omega(1 - Z, X) = 1.$$ 

It is now easy to see that symplectic complement $S := \langle 1 - Z, X \rangle \perp \subseteq \nu_N$. Finally, we find a trivialization of $S$ such that $\Omega \mid_S$ is the trivial symplectic form with Darboux frame $\{e_2, e_{k+2}, \ldots\}$. Hence, by extending this trivialization to $\nu_N = V \times N$ by using the coordinate $X$ as $e_0$, we find that $\{e_0, 1 - Z, e_1, e_{k+1}, e_2, e_{k+2}, \ldots\}$ is a Darboux frame of $\Omega$ in this trivialization. with the decomposition $DL_\nu = TV \oplus TN \oplus \mathbb{R}_\nu$ we can choose

$$\omega = \sum_{i=1}^k dx^i \wedge dx^{i+k} + 1^* \wedge (dx^0 - \sum_{i=1}^k x^{i+k} dx^i)$$

which coincides with $\Theta$ on $\nu_N \oplus K$ and is $d_L$-closed. By applying Theorem 5.9 since $N$ together with $\nabla$ is a homogeneous cocontact transversal, we find a Jacobi morphism

$$\mathcal{B}_P(L_N)^\omega \cong L_J.$$

An easy computation shows that $\mathcal{B}_P(L_N)^\omega$ is the graph of the Jacobi structure of the form in the theorem. \hfill \Box

### 6 Application: Splitting theorem for homogeneous Poisson Structures

Using the homogenization scheme from [2], one can see that Jacobi bundles are nothing else but special kinds of homogeneous Poisson manifolds. Moreover, the two most important examples of Poisson manifolds are of this kind: the cotangent bundle and the dual of a Lie algebra. Using this insight, it is easy to see that proving something for Jacobi structures gives a proof for something in homogeneous Poisson Geometry. We want to apply this philosophy to give a splitting theorem for homogeneous Poisson manifolds. The first appearance of such a theorem was [5, Theorem 5.5] in order to prove the local splitting of Jacobi pairs. Here we want to attack the problem from the other side: we use the splitting of Jacobi manifolds to prove the splitting of homogeneous Poisson structures.

**Theorem 6.1** Let $(\pi, Z)$ be a homogeneous Poisson structure on a manifold $M$ and let $p_0 \in M$ be a point such that $Z_{p_0} \neq 0$. Then there exist an open neighbourhood $U$ of $p_0$, an open neighbourhood $U_{2k}$ of $0 \in \mathbb{R}^{2k}$, a manifold $N$ with a homogeneous Poisson structure $(\pi_N, Z_N)$ and a diffeomorphism $\psi: U \rightarrow U_{2k} \times N$, such that

$$\psi_{*}\pi = \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^j} + \pi_N.$$ 

Additionally,
i.) if $Z \in \text{im}(\pi^\#)$, then $\psi_* Z = p_i \frac{\partial}{\partial p_i} + \frac{\partial}{\partial q_k} + Z_N$.

ii.) if $Z \notin \text{im}(\pi^\#)$, then $\psi_* Z = p_i \frac{\partial}{\partial p_i} + Z_N$.

**Proof:** Note that since $Z_{p_0} \neq 0$, we find coordinates $\{u, x^1, \ldots, x^q\}$ with $p_0 = (1, 0, \ldots, 0)$, such that $Z = u \frac{\partial}{\partial u}$. In this chart, we have, using $\mathcal{L}_Z \pi = -\pi$,

$$\pi = \frac{1}{u} (\Lambda + u \frac{\partial}{\partial u} \wedge E)$$

for unique $\Lambda \in \Gamma^\infty(\Lambda^2 TM)$ and $E \in \Gamma^\infty(TM)$ which do not depend on $u$. It is easy to see, that we have

$$[\Lambda, \Lambda] = -E \wedge \Lambda$$

and $\mathcal{L}_E \Lambda = 0$, which means that $(\Lambda, E)$ is a Jacobi pair. This allows us to use Theorem 5.5 and Theorem 5.10 to prove the result. We will do it just for the case where $p_0$ is a contact point, which means, translated to Jacobi pairs, that $E_{p_0}$ is transversal to $\text{im}(\Lambda^\# |_{p_0}) |_{p_0}$ and thus $Z \in \text{im}(\pi^\#)$, since the other case is exactly the same. Note that, we can apply Theorem 5.10: there exists coordinates $\{x, q^i, p_i, y^j\}$ and a local non-vanishing function $a$ (which is basically the line bundle trivialization), such that

$$\Lambda = \frac{1}{a} (\Lambda_{\text{can}} + \pi_N + E_{\text{can}} \wedge Z_N)$$

and

$$E = \frac{1}{a} (E_{\text{can}} + \Lambda^2(da)),$$

where $\Lambda_{\text{can}}$ and $E_{\text{can}}$ are just depending on $\{x, q^i, p_i\}$ and $(\phi_N, Z_N)$ is a homogeneous Poisson structure just depending on $y^j$-coordinates.

If we apply the diffeomorphism $(u, x^1, \ldots, x^q) \mapsto (a \cdot u, x^1, \ldots, x^q)$, we have

$$\pi = \frac{1}{u} (\Lambda_{\text{can}} + \pi_N + E_{\text{can}} \wedge Z_N + u \frac{\partial}{\partial u} \wedge E_{\text{can}}).$$

A (quite) long and not very insightful computation shows that the diffeomorphism

$$\Phi(u, x^1, \ldots, x^q) = (u, \Phi^Z_N(u, \Phi^E_{-\log(u)}(x^1, \ldots, x^q))),$$

where $\Phi^Z_N$ (resp. $\Phi^E_{\text{can}}$) is the flow uf $Z_N$ (resp. $E_{\text{can}}$), gives us

$$\pi = \frac{1}{u} \left( \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^j} \right) + \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial x^i} + \pi_N$$

and $Z = u \frac{\partial}{\partial u} + p_i \frac{\partial}{\partial p_i} + Z_N$ and with some obvious variations and renaming coordinates of $\pi$ we get the result. \(\square\)

This Application shows us that, eventhough we can see Poisson structures as Jacobi manifolds, which suggests that they are more general objects than Poisson structures, the splitting theorems (of Jacobi pairs) are a refinement of the known splitting theorems for Poisson structures.

### 7 Generalized Contact bundles

In this last section, we want to drop a word about generalized contact bundles. They were introduced recently in [13] and they are modeled to be the odd dimensional analogue to generalized complex structures.

**Definition 7.1** Let $L \to M$ be a line bundle. A subbundle $\mathcal{L} \subseteq \mathbb{D}_\mathbb{C}L$ is called generalized contact structure on $L$, if
i.) $L$ is a (complex) Dirac-Jacobi structure

ii.) $L \cap \bar{L} = \{0\}$

A generalized contact structure can be also seen as an endomorphism of $DL$ of the form

$$
\begin{pmatrix}
\phi & J^2 \\
\alpha^b & \phi^a
\end{pmatrix},
$$

where $\phi \in \text{End}(DL)$, $J \in \Gamma^\infty((J^1L)^* \otimes L)$ and $\alpha \in \Omega^2_L(M)$ (see [13] and [10]). This endomorphism has to fulfill certain properties: it has to be almost complex, compatible with the pairing and integrable, which we do not explain what it means here and refer the reader to [13]. The $+i$-Eigenbundle produces a generalized contact structure in the sense of Definition 7.1. Moreover, we have that among many more conditions that $J$ is a Jacobi structure. Let us now pick a (cosymplectic or contact) transversal to $J$ together with an Euler-like derivation $\Delta = J^2(\alpha)$, then $(\Delta, \omega - \phi^a(\alpha)) \in \Gamma^\infty(L)$. With the techniques from Section 4 and Section 5, one can show that

$$
\mathcal{B}_\Psi(L) = \mathcal{B}_{\omega_0}(L)^{\omega + \beta},
$$

where $\omega = \int_0^1 \frac{1}{t}(\Phi^\Delta_{\text{log}(t)})^* d_L \omega_t \, dt$ and $\beta = -\int_0^1 \frac{1}{t}(\Phi^\Delta_{\text{log}(t)})^* d_L \phi^a(\alpha) \, dt$. This is nothing else but a normal form for generalized contact bundles. This can be pushed more forward to prove a local splitting of generalized bundles, but this has already be done in [10] with similar techniques.

A The Moser trick for Jacobi manifolds

Let $J \in \Gamma^\infty(\Lambda^2(J^1L)^* \otimes L)$ be a Jacobi structure on a line bundle $L \to M$. Moreover, we assume having smooth family of closed 2-forms $\sigma_t$, such that $\sigma_0 = 0$ and $L^*_J$ is a Jacobi structure for all $t$, denoted by $J_t$. For

$$
\alpha_t := -\frac{\partial}{\partial t} \sigma_t
$$

the equation

$$
\frac{\partial}{\partial t} \sigma_t = -d_L \alpha_t
$$

holds. We define the Moser-derivation by

$$
\Delta_t := -J_t^2(\alpha_t)
$$

and its flow by $\Phi_t \in \text{Aut}(L)$, where we assume it exists for on open subset containing $[0,1]$. Let us compute

$$
\frac{d}{dt} \Phi_t^* J_t = \Phi_t^* (\Delta_t, J_t) + \frac{d}{dt} J_t
$$

$$
= \Phi_t^* \left( [-J_t^2(\alpha_t), J_t] + \frac{d}{dt} J_t \right)
$$

$$
= \Phi_t^* \left( J_t (-d_L \alpha_t) + \frac{d}{dt} J_t \right).
$$

(A.1)

It is easy to see that

$$
J_t^2 = J^2 \circ (\text{id} + \alpha^b \circ J^2)^{-1}
$$
and hence we can compute
\[
\frac{d}{dt} J_t^z = \frac{d}{dt} J^z \circ (\text{id} + \sigma_t^z \circ J^z)^{-1}
\]
\[
= -J^z \circ (\text{id} + \sigma_t^z \circ J^z)^{-1} \circ \left( \frac{\partial}{\partial t} \sigma_t^z \circ J^z \right) \circ (\text{id} + \sigma_t^z \circ J^z)^{-1}
\]
\[
= -J^z \circ \left( \frac{\partial}{\partial t} \sigma_t \circ J^z \right)^z \circ J^z_t
\]
\[
= (-J^z_t \frac{\partial}{\partial t} \sigma_t)^z
\]
\[
= (J^z_t (dL_\alpha_t))^z,
\]
and hence \( \frac{d}{dt} J_t = J^z_t (dL_\alpha_t) \). If we use this equality in Equation A.1, we find
\[
\frac{d}{dt} \Phi^* J_t = 0,
\]
so we finally have \( J = \Phi_0^* J_0 = \Phi_1^* J_1 \) and hence the two Jacobi structures are isomorphic.

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