Dilaton Gravity in $2 + \epsilon$ Dimensions

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Abstract

Quantum theory of dilaton gravity is studied in $2 + \epsilon$ dimensions. Divergences are computed and renormalized at one-loop order. The mixing between the Liouville field and the dilaton field eliminates $1/\epsilon$ singularity in the Liouville-dilaton propagator. This smooth behavior of the dilaton gravity theory in the $\epsilon \to 0$ limit solves the oversubtraction problem which afflicted the higher orders of the Einstein gravity in $2 + \epsilon$ dimensions. As a nontrivial fixed point, we find a dilaton gravity action which can be transformed to a CGHS type action.

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1. Introduction

Ultraviolet divergences and the apparent nonrenormalizability have been a major source of difficulties to formulate a consistent quantum theory of gravity. Simple dimensional analysis shows that gravity is power counting renormalizable in two dimensions. Therefore it is useful to consider the quantum theory of gravity near two dimensions and to make an analytic continuation to higher dimensions by means of the $\epsilon$ expansion \[1\]–\[3\]. Recently there has been a progress in this $(2+\epsilon)$-dimensional approach for quantum gravity \[4\]–\[7\]. The dynamics of the Liouville mode has been better understood and an ultraviolet fixed point has been found.

Initiated by the work on the black hole evaporation \[8\], much efforts have been devoted to study gravity interacting with a dilaton field and matter fields, especially in two dimensions \[9\]–\[13\]. As a toy model imitating the spherically symmetric situation, two-dimensional models are considered in most of the works. The semiclassical approximation has often been used to study the Hawking radiation and the black hole evaporation in the model \[8\]–\[10\], which is often blamed to be the possible origin of diseases in this problem. Therefore it is very desirable to have a quantum theory of the dilaton gravity aiming at the higher dimensional situation.

The purpose of the present work is to study the quantum theory of the dilaton gravity in $2 + \epsilon$ dimensions, which can be used as a starting point for the higher dimensional quantum theory of gravity. As a bonus from the study of the dilaton gravity, we find that the dilaton gravity can solve the oversubtraction problem \[4\], which has been a basic obstacle to renormalization at higher orders in the usual $(2+\epsilon)$-dimensional approach to quantum gravity. We obtain divergences and beta functions to one-loop which exhibit a nontrivial fixed point. The fixed point is ultraviolet stable for the gravitational coupling constant $G$, if $\epsilon > 0$ and $N < 24$. However, it is not ultraviolet stable for the strength of the dilaton coupling function. We find that the fixed point theory can be transformed to an action of the usual CGHS type \[8\].

It has been known for some time that there is a subtlety in the $(2+\epsilon)$-dimensional approach to quantum gravity because of the following intrinsic problem of gravity in two spacetime dimensions. In the limit of two dimensions, the usual Einstein action becomes a topological invariant, which is dynamically meaningless. This peculiarity of the Einstein action at two dimensions implies that the Liouville field $\rho$ corresponding to the conformal degree of freedom does not appear in the Einstein
action. However, quantum theory possesses a conformal anomaly, which resuscitates the Liouville field to play a dynamical role. One way to see this nontrivial dynamics of the Liouville field is to consider the quantum gravity in $2 + \epsilon$ dimensions. The Liouville field has a kinetic term of order $\epsilon$ in $2 + \epsilon$ dimensions, since the Einstein action becomes a topological invariant only at two dimensions. Therefore the Liouville field propagator has a $1/\epsilon$ singularity, which is the origin of a number of subtleties in the $(2 + \epsilon)$-dimensional approach to quantum gravity. One finds divergences associated to the traceless mode $h_{\mu\nu}$ (graviton), but no divergences corresponding to the Liouville field kinetic term at one-loop order. On the other hand, general covariance dictates that the $1/\epsilon$ counter term for the graviton $h_{\mu\nu}$ inevitably accompanies a finite amount of Liouville field kinetic term. Since the Liouville field kinetic term is of order $\epsilon$ at the tree level, this finite counter term is an oversubtraction.

At one-loop order, we can perform this oversubtraction as is dictated by the general coordinate invariance in the $2 + \epsilon$ dimensions. However, it has been noted that the multiple insertion of this finite counter term produces extra singularities in $1/\epsilon$, because of the $1/\epsilon$ singularity of the Liouville field propagator. These extra singularities cannot be removed by any conventional renormalization procedure. An unconventional procedure has been proposed to resum infinitely many diagrams in defining a bare coupling before the renormalization, but a concrete procedure including higher orders is still to be worked out [5].

Since this difficulty is associated with an intrinsic problem of the topological nature of the Einstein action at two dimensions, it seems difficult to cure the problem. In fact, this difficulty has even led to a proposal to abandon the general coordinate invariance of quantum gravity in $2 + \epsilon$ dimensions. Namely the action should be only approximately invariant under general coordinate transformations at large distances, but is not invariant at short distances [6]. We admit that this ambitious possibility is not ruled out by any experimental facts. However, it seems to us more natural and satisfactory if we can maintain general coordinate invariance while keeping the idea of expanding the quantum theory around two dimensions. As we shall see, the dilaton gravity in $2 + \epsilon$ dimensions offers an alternative possibility to define the quantum theory of Einstein gravity in higher dimensions by overcoming this oversubtraction problem.

In sect. 2, we show that a general model of the dilaton gravity in $2 + \epsilon$ dimensions can be reduced to a standard form containing an arbitrary function of dilaton coupling to matter fields. We also explain that the quantum theory of gravity in higher
dimensions can be defined by means of the dilaton gravity instead of the usual Einstein gravity in $2+\epsilon$ dimensions. In sect. 3, the standard form of the dilaton gravity in $2+\epsilon$ dimensions is quantized and our solution to the oversubtraction problem is pointed out. In sect. 4, one-loop divergences are computed in a general model for the dilaton gravity. In sect. 5, beta functions are obtained. We find a nontrivial fixed point and examine its stability. Sect. 6 is devoted to a discussion.

2. Dilaton gravity in $2+\epsilon$ dimensions

The Einstein action for the metric $g_{\mu\nu}$ has been considered in $d = 2+\epsilon$ dimensions together with a number of free matter fields $X^i$ ($i = 1, \cdots, N$)

$$
S = \int d^d x \sqrt{-g} \left[ \frac{\mu^\epsilon}{16 \pi G} R^{(d)} - \frac{1}{2} g^\mu_\nu \partial_\mu \partial_\nu X^i X^j \delta_{ij} \right],
$$

(2.1)

where $G$ and $\mu$ are the renormalized gravitational constant and the renormalization scale respectively. String theory and other models have inspired the notion of the spacetime-dependent gravitational constant which is described by a field $\phi$ called dilaton. In order to allow an arbitrary interaction of the dilaton and to be able to renormalize the theory around two spacetime dimensions, we shall start from the following general action containing only parameters that become dimensionless in the limit of two dimensions

$$
S = \int d^d x \sqrt{-g} \left[ \frac{\mu^\epsilon}{16 \pi G} R^{(d)} L(\phi, X) - \frac{1}{2} g^\mu_\nu \partial_\mu \phi \partial_\nu \phi G^{(d)} \delta_{ij} \Psi(\phi, X) \right],
$$

(2.2)

which contains four arbitrary functions $L, G^{(d)}, G_{i}, G_{ij}$ of $\phi$ and $X^i$. Since we consider matter fields $X^i$ to be free apart from possible gravitational interactions with metric and dilaton, we shall demand the action to be invariant under the $N$-dimensional Euclidean transformations (translations and rotations) among matter fields $X^i$ (target space). This requirement distinguishes the matter fields $X^i$ from the dilaton field $\phi$. Then the action is restricted to

$$
S = \int d^d x \sqrt{-g} \left[ \frac{\mu^\epsilon}{16 \pi G} R^{(d)} L(\phi) - \frac{1}{2} g^\mu_\nu \partial_\mu \phi \partial_\nu \phi \Psi(\phi) - \frac{1}{2} g^\mu_\nu \partial_\mu X^i \partial_\nu X^j \delta_{ij} e^{-2\phi(\phi)} \right].
$$

(2.3)
Within the class of models of the form (2.3), we still have a freedom of field redefinitions. There are three kinds of possible transformations. The first one is the rigid rescaling of matter fields

\[ X^i \to aX^i, \quad (2.4) \]

where \( a \) is a constant. The second one is the local Weyl rescaling of the metric

\[ g_{\mu\nu} \to e^{-2\Lambda(\phi)} g_{\mu\nu}, \quad (2.5) \]

where \( \Lambda \) is a function of the dilaton \( \phi \). The third one is an arbitrary field redefinition of the dilaton with a function \( f \) of the dilaton \( \phi \)

\[ \phi \to f(\phi). \quad (2.6) \]

Using the freedom corresponding to the rigid rescaling (2.4), we can always make \( \Phi(0) = 0 \). With this choice, the first freedom is fixed completely. The local Weyl rescaling (2.5) can be used to fix one of the functions \( L, \Psi, \) or \( \Phi \). The function \( \Psi \) changes by terms of order \( \epsilon^0 \) by the local Weyl rescaling, whereas \( L, \Phi \) change only by terms of order \( \epsilon \). Therefore we shall choose to fix the function \( \Psi \) by means of the local Weyl rescaling. We can find the local Weyl rescaling \( \Lambda \) which transforms a generic model to the model with \( \Psi(\phi) = 0 \), and is finite as we let \( \epsilon \to 0 \). Let us note that this choice fixes only \( \Lambda'(\phi) \), namely the nonzero modes of \( \Lambda(\phi) \). Finally the arbitrary redefinition (2.6) of the dilaton field \( \phi \) can be used to fix the form of the function \( L(\phi) \). We shall choose \( L(\phi) = \exp(-2\phi) \). This fixes only nonzero modes of \( f(\phi) \). The zero modes of \( \Lambda(\phi) \) and \( f(\phi) \) are used to fix coefficients of two reference operators, which will be introduced into the action later. We shall discuss reference operators in sect. 6. With these choices of \( a, \Lambda(\phi), f(\phi) \), we can fix \( L(\phi), \Psi(\phi) \) in the action (2.3) and obtain the following standard form of dilaton gravity

\[ S = \int d^dx \sqrt{-g} \left[ \frac{\mu^6}{16\pi G} R^{(d)} e^{-2\phi} - \frac{1}{2} g^{\mu\nu} \partial_\mu X^i \partial_\nu X^j \delta_{ij} e^{-2\Phi(\phi)} \right], \quad (2.7) \]

where \( \Phi(0) = 0 \).

Let us note that the field redefinition generally produces Jacobian factors in the path integral measure in the usual regularization schemes. In \( 2 + \epsilon \) dimensions,
however, these Jacobian factors do not contribute to the effective action since they are proportional to $\delta^d(0) = 0$ which are discarded in the analytic regularization such as the $(2 + \epsilon)$-dimensional approach [15].

It is important to realize that the dilaton gravity is equivalent to the Einstein gravity with one extra scalar field. In order to demonstrate this point, let us take the dilaton gravity without matter fields in $2 + \epsilon$ dimensions

$$S = \frac{\mu^{\epsilon}}{16\pi G} \int d^d x \sqrt{-g} R^{(d)} e^{-2\phi}.$$  \hspace{1cm} (2.8)

If we make a field redefinition involving a local Weyl rescaling $g_{\mu\nu} \to e^{4\phi/\epsilon} g_{\mu\nu}$, we find

$$S = \frac{\mu^{\epsilon}}{16\pi G} \int d^d x \sqrt{-g} \left[ R^{(d)} - \frac{4(1+\epsilon)}{\epsilon} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right].$$ \hspace{1cm} (2.9)

This clearly shows that the dilaton gravity is equivalent to the Einstein gravity with one scalar field. However, the coefficient of the $\phi$ kinetic term is singular for $\epsilon \to 0$ in the action (2.9). If we are precisely at two dimensions, we cannot transform the Einstein action, which is a topological invariant given by the first term of eq. (2.9), back into the dilaton gravity action (2.8). Therefore the dilaton gravity is different from the Einstein gravity precisely at two dimensions, even though they are equivalent in dimensions other than two. Since we are going to use an expansion in $\epsilon = d - 2$, we should only consider field redefinitions which keep the action regular for $d \to 2$. The dilaton gravity and the Einstein gravity in $2 + \epsilon$ dimensions give different divergences around two dimensions. Therefore the Einstein gravity and the dilaton gravity in $(2 + \epsilon)$-dimensional approach give inequivalent quantum gravity theories in higher dimensions, even though they are classically equivalent in dimensions other than two.

Since the dilaton gravity is equivalent to the Einstein gravity in higher dimensions, the dilaton gravity in $2 + \epsilon$ dimensions is as legitimate as the Einstein gravity in $2 + \epsilon$ dimensions, in order to define the quantum gravity in higher dimensions. The dynamical content of the Einstein action does not have a smooth limit as $\epsilon \to 0$. This is precisely the origin of a number of problems in the usual $(2 + \epsilon)$-dimensional approach, such as the oversubtraction problem [5]. On the other hand, the dilaton gravity action still has a dynamical content even at two dimensions and has a smooth limit as $\epsilon \to 0$. This smooth behavior of dilaton gravity in $\epsilon \to 0$ suggests that the
dilaton gravity is more natural than the Einstein gravity to define the quantum gravity theories in higher dimensions in the $(2 + \epsilon)$-dimensional approach.

It is customary to incorporate matter fields as free fields. One should note that the notion of free fields is invariant under the local Weyl rescaling of the form (2.5) only at two dimensions. Since we need to consider the Weyl rescaling in order to relate the dilaton gravity action and the Einstein action, we are led to consider the free matter fields $X^i$ to interact with dilaton in addition to the metric as in eq. (2.7). This is quite natural, since the dilaton is better regarded as a part of gravity.

A number of different forms of dilaton gravity has been proposed so far. A popular form of the action for the dilaton gravity is the CGHS type action [8]. It can be obtained from our standard action (2.7) by a local Weyl rescaling $g_{\mu\nu} = e^{-2\phi} g'_{\mu\nu}$.

The transformed action becomes in $2 + \epsilon$ dimensions

$$S_{\text{CGHS}} = \int d^d x \sqrt{-\tilde{g}} \left[ \frac{\mu^\epsilon}{16\pi G} \left( R^{(d)} + (\epsilon + 1)(\epsilon + 4) g'^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right) e^{-(2+\epsilon)\phi} 
- \frac{1}{2} g'^{\mu\nu} \partial_\mu X^i \partial_\nu X^j \delta_{ij} e^{-2\Phi(\phi) - \epsilon\phi} \right].$$

(2.10)

3. Gauge fixing and quantization

We shall use the background field method [16], [17] to compute one-loop divergences. We define an expansion parameter $\kappa$ as

$$\kappa^2 = \frac{16\pi G}{\mu^\epsilon}.$$  

(3.1)

Let us decompose the metric $g_{\mu\nu}$ into the traceless field $h_{\mu\nu}$ and the Liouville field $\rho$ and introduce the background metric $\tilde{g}_{\mu\nu}$

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} e^{-2\rho} = \tilde{g}_{\mu\lambda} (e^{\kappa h})^{\lambda\nu} e^{-2\rho}.$$  

(3.2)

Our standard action (2.7) for the dilaton gravity becomes

$$S = \frac{\mu^\epsilon}{16\pi G} \int d^d x \sqrt{-\tilde{g}} e^{-\epsilon\rho} \left[ \tilde{R}^{(d)} e^{-2\phi} + \epsilon (\epsilon + 1) \tilde{g}^{\mu\nu} \partial_\mu \rho \partial_\nu \rho e^{-2\phi} 
+ 4(\epsilon + 1) \tilde{g}^{\mu\nu} \partial_\mu \rho \partial_\nu \phi e^{-2\phi} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu X^i \partial_\nu X^j \delta_{ij} e^{-2\Phi(\phi)} \right],$$

(3.3)
where \(i, j = 1, \ldots, N\). We also introduce the background fields of the Liouville field, the dilaton field and the matter fields which are denoted by putting a hat \(\hat{\ }\) on:

\[
\rho = \hat{\rho} + \kappa \rho_q, \quad \phi = \hat{\phi} + \kappa \phi_q, \quad X^i = \hat{X}^i + X_q^i. \tag{3.4}
\]

In the following we will omit the suffix \(q\) of the quantum fields \(\rho_q, \phi_q, X_q^i\) for simplicity. We use the background vielbein \(\hat{e}_\nu^\alpha\) to convert world indices \(\mu, \nu, \cdots\) to local Lorentz indices \(\alpha, \beta, \cdots\).

To fix the gauge we introduce the Faddeev-Popov ghosts \(b_\alpha, c_\alpha\) and the Nakanishi-Lautrup auxiliary field \(B_\alpha\). The BRST transformations are given by

\[
\delta_B h_{\alpha\beta} = \hat{D}_\alpha c_\beta + \hat{D}_\beta c_\alpha - \frac{2}{d} \eta_{\alpha\beta} \hat{D}_\gamma c^\gamma + \cdots, \quad \delta_B \rho = -\frac{1}{d} \hat{D}_\alpha c^\alpha + \cdots,
\]

\[
\delta_B \phi = c^\alpha \partial_\alpha \hat{\phi} + \cdots, \quad \delta_B b_\alpha = i B_\alpha, \quad \delta_B B_\alpha = 0, \tag{3.5}
\]

where the dots represent terms quadratic and of higher powers in the quantum fields.

The gauge fixing term and the ghost action are given by

\[
S_{\text{GF+FP}} = \int d^d x \delta_B (-i b^\alpha F_\alpha)
\]

for a gauge function \(F_\alpha\) [13]. We use the following gauge function to eliminate the mixing between \(h_{\mu\nu}\) and other fields

\[
F_\alpha = \sqrt{-\hat{g}} e^{-2\hat{\phi}} \left( \hat{D}^3 h_{\beta\alpha} + \hat{D}_\alpha (\epsilon \rho + 2\phi) + \frac{1}{2} B_\alpha \right). \tag{3.6}
\]

The total action quadratic in the quantum fields is given by

\[
S^{(2)}_{\text{tot}} = S^{(2)} + S^{(2)}_{\text{GF+FP}}
\]

\[
= \int d^d x \sqrt{-\hat{g}} e^{-2\hat{\phi}} \left[ \frac{1}{2} B'^\alpha B'_\alpha - \frac{1}{4} \hat{D}_\mu h_{\alpha\beta} \hat{D}^\mu h^{\alpha\beta} + \frac{1}{2} \epsilon (\epsilon + 2) \hat{g}^{\mu\nu} \partial_\mu \rho \partial_\nu \rho 
\right.
\]

\[
+2(\epsilon + 2) \hat{g}^{\mu\nu} \partial_\mu \rho \partial_\nu \phi - 2 \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \hat{R}^{(d)}_{\mu\rho\sigma\nu} h^{\mu\nu} h^{\rho\sigma} + \frac{1}{2} (\epsilon \rho + 2\phi)^2 \hat{R}^{(d)}
\]

\[
+ (\epsilon \rho + 2\phi) h^{\mu\nu} \hat{R}^{(d)}_{\mu\nu} - 2 h^{\mu\lambda} \hat{D}_\nu h^{\nu\lambda} \partial_\mu \hat{\phi} - 2 (\epsilon \rho + 2\phi) \hat{D}_\nu h^{\mu\nu} \partial_\mu \hat{\phi}
\]

\[
- 4(\epsilon + 1)(\epsilon \rho + 2\phi) \hat{g}^{\mu\nu} \partial_\mu \rho \partial_\nu \hat{\phi} - 4(\epsilon + 1) h^{\mu\nu} \partial_\mu \rho \partial_\nu \hat{\phi} + \text{ghost terms}
\]

\[
+ S^{(2)}_{\text{matter}}, \tag{3.7}
\]

where \(B'_\alpha = B_\alpha + \hat{D}^3 h_{\beta\alpha} + \partial_\alpha (\epsilon \rho + 2\phi)\) is a shifted auxiliary field. One can easily see that the terms quadratic in the Liouville field \(\rho\) are of order \(\epsilon\) similarly to the
case of the Einstein action. However, it is important to note that the mixing of the Liouville field $\rho$ with the dilaton field $\phi$ is of order $\epsilon^0$. Therefore the kinetic term of the Liouville field has to be considered together with the dilaton field.

By inverting the kinetic term of various fields, we obtain propagators. The nonvanishing propagators are given by

\begin{align}
\langle h_{\alpha\beta}(x)h_{\gamma\delta}(y) \rangle &= \left( \eta_{\alpha\gamma} \eta_{\beta\delta} + \eta_{\alpha\delta} \eta_{\beta\gamma} - \frac{2}{d} \eta_{\alpha\delta} \eta_{\beta\gamma} \right) \Delta_F(x - y), \\
\langle \rho(x)\rho(y) \rangle &= -\frac{1}{2(\epsilon + 1)(\epsilon + 2)} \Delta_F(x - y), \\
\langle \phi(x)\phi(y) \rangle &= \frac{\epsilon}{8(\epsilon + 1)} \Delta_F(x - y), \\
\langle c^\alpha(x)b_\beta(y) \rangle &= -i \delta^\alpha_\beta \Delta_F(x - y), \\
\langle X^i(x)X^j(y) \rangle &= \delta^{ij} \Delta_F(x - y), \\
\Delta_F(x - y) &= -i \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2} e^{ip(x-y)}. \quad (3.8)
\end{align}

All the propagators including that for the Liouville field have well-defined limits as $\epsilon \to 0$. This nonsingular behavior is a direct consequence of the mixing between the Liouville field $\rho$ and the dilaton field $\phi$. Although the Liouville field kinetic term alone vanishes in the $\epsilon \to 0$ limit, the dilaton field mixes with the Liouville field nontrivially even in the two-dimensional limit. Therefore the Liouville field $\rho$ and the dilaton field $\phi$ should be considered as inseparable parts of a single entity in the dilaton gravity.

In the case of the Einstein action in the $(2 + \epsilon)$-dimensional approach, the propagator of the Liouville field is singular. Since there are divergences proportional to the scalar curvature, the general coordinate invariance with respect to the physical background metric $\bar{g}_{\mu\nu} = \hat{g}_{\mu\nu} e^{-2\hat{\rho}}$ dictates that the counter term contains the Liouville field kinetic term of order $\epsilon^0$

\begin{align}
S_{\text{counter}} \propto \int d^d x \frac{1}{\epsilon} \sqrt{-\bar{g}} \hat{R}^{(d)} \\
= \int d^d x \sqrt{-\bar{g}} e^{-\epsilon \rho} \left[ \frac{1}{\epsilon} \hat{R}^{(d)} + (1 + \epsilon) \hat{g}^{\mu\nu} \partial_\mu \rho \partial_\nu \rho \right]. \quad (3.9)
\end{align}

This oversubtraction is the origin of the nontrivial dynamics of the Liouville field even in the limit of two dimensions. At the same time, this oversubtraction causes the following problem at higher orders. If one inserts this counter term into a Liouville field propagator of any diagram, one finds extra $1/\epsilon$ singularities for each insertion because of the singular behavior of the Liouville field propagator. The multiple insertions of the counter term provide more and more singular diagrams.
Moreover, these diagrams are nonlocal and cannot be renormalized in any conventional way. This is the oversubtraction problem \[5\].

If we consider the dilaton gravity instead of the Einstein gravity in the \((2+\epsilon)\)-dimensional approach, we find that the Liouville field should be considered with the dilaton field and two by two matrix of their propagators \(3,8\) are nonsingular. Even though we still need to subtract the finite counter term for the Liouville kinetic term, we no longer obtain additional divergences from the multiple insertion of the finite counter terms. Therefore the oversubtraction problem that has afflicted the Einstein gravity in \(2+\epsilon\) dimensions can be overcome in the dilaton gravity. Since the dilaton gravity has a smooth limit at two dimensions unlike the Einstein gravity, the dilaton gravity is similar to all the other field theoretical models in the \(\epsilon\) expansion approach. Therefore we expect that the dilaton gravity does not have any subtleties in higher orders of the \((2+\epsilon)\)-dimensional approach contrary to the Einstein gravity.

4. One-loop divergences in a general model

We shall compute divergences at one-loop order for a general action instead of the standard one \(2,7\) in order to allow a more flexible treatment. Combining all the scalar fields and the Liouville field into \(Y^I = (\rho, \phi, X^i)\), where the index runs \(I = (\rho, \phi, i)\), we obtain a kind of nonlinear sigma model \(14\) using \(\tilde{g}_{\mu\nu} = \hat{g}_{\lambda\lambda}(e^{\kappa h})^\lambda_\nu\)

\[
S = \frac{\mu^\epsilon}{16\pi G} \int d^d x \sqrt{-\tilde{g}} \left[ \tilde{R}^{(d)} L(Y) - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu Y^I \partial_\nu Y^J G_{IJ}(Y) \right]. \tag{4.1}
\]

Although we do not require the invariance under the Weyl rescaling of the physical metric \(\delta g_{\mu\nu}(x) = -2\sigma(x)g_{\mu\nu}(x)\), we demand the invariance under the Weyl rescaling of the background metric \(\tilde{g}_{\mu\nu}\) accompanied by a shift of \(Y^I\)

\[
\delta \tilde{g}_{\mu\nu} = -2\sigma \tilde{g}_{\mu\nu}, \quad \delta Y^I = -2(\epsilon + 1)\sigma G_{IJ} \partial_J L. \tag{4.2}
\]

In the case of our standard model for the dilaton gravity, this transformation of \(Y^I\) reduces to \(\delta \rho = -\sigma, \, \delta \phi, \, \delta X^i = 0\). The invariance under the transformation \(4.2\) implies that the action depends on \(\tilde{g}_{\mu\nu}\) and \(\rho\) only through a combination \(g_{\mu\nu}\) in eq. \(3.2\), which assures the general coordinate invariance with respect to the physical
metric $g_{\mu\nu}$. This requirement constrains the function $L$ and the target space metric $G_{IJ}$ as

$$
-\epsilon G_{IJ} - 4(\epsilon + 1) D_I \partial_J L = 0,
$$
$$
-\epsilon L - 2(\epsilon + 1) G^{IJ} \partial_I \partial_J L = 0.
$$

(4.3)

We will discuss the general solution of these conditions in the appendix. We can define the following general coordinate transformation

$$
\delta G \tilde{g}_{\mu\nu} = \tilde{D}_\mu \tilde{v}_\nu + \tilde{D}_\nu \tilde{v}_\mu - \frac{2}{\epsilon + 2} \tilde{g}_{\mu\nu} \tilde{D}_\lambda \tilde{v}^\lambda,
$$
$$
\delta G \tilde{Y}^I = v^\mu \partial_\mu \tilde{Y}^I - \frac{2(\epsilon + 1)}{\epsilon + 2} G^{IJ} \partial_J \tilde{L} \partial_I \tilde{L},
$$

(4.4)

where $\tilde{v}_\mu = \tilde{g}_{\mu\nu} v^\nu$. The action (4.1) is invariant under eq. (4.4) when the functions $L$ and $G_{IJ}$ satisfy eq. (4.3).

We decompose the fields into background fields and quantum fields as

$$
\tilde{g}_{\mu\nu} = \hat{g}_{\mu\nu}(e^{\kappa h})_{\lambda} \nu, \quad \tilde{Y}^I = \hat{Y}^I + \kappa \xi^I + O(\kappa^2 \xi^2),
$$

(4.5)

where $\kappa$ is defined in eq. (3.1) and $\xi^I$ are normal coordinates with respect to the target space metric $G_{IJ}$ [19]. Terms quadratic in the quantum fields in the action (4.1) are given by

$$
S^{(2)} = \int d^d x \sqrt{-\hat{g}} \left[ \hat{L} \left\{ -\frac{1}{4} \hat{D}_\mu h_{\rho\sigma} \hat{D}_\nu h^{\rho\sigma} + \frac{1}{2} \hat{D}_\mu h^{\mu\lambda} \hat{D}_\nu h_{\nu}^\lambda - \frac{1}{2} \hat{R}^{(d)}_{\rho\sigma\nu\sigma} h^{\rho\sigma} - \hat{D}_\mu (h^{\mu\lambda} \hat{D}_\nu h_{\nu}^\lambda) \right\} - \frac{\epsilon}{8(\epsilon + 1)} \xi^I \hat{G}_{IJ} \hat{R}^{(d)} + \left( \hat{D}_\mu \hat{D}_\nu h^{\mu\nu} - \hat{R}^{(d)}_{\mu\nu} h^{\mu\nu} \right) \xi^I \partial_I \hat{L}
$$
$$
- \frac{1}{2} h^{\mu\lambda} h^{\nu\lambda} \partial_\mu \hat{Y}^I \partial_\nu \hat{Y}^J \hat{G}_{IJ} + h^{\mu\nu} \hat{D}_\mu \xi^I \partial_\nu \hat{Y}^J \hat{G}_{IJ} - \frac{1}{2} \hat{D}_\mu \xi^I \hat{D}_\nu \xi^J \hat{G}_{IJ}
$$
$$
+ \frac{1}{2} \partial_\mu \hat{Y}^I \partial_\nu \hat{Y}^J \hat{R}_{IKJL} \xi^K \xi^L \right],
$$

(4.6)

where $\hat{G}_{IJ} = G_{IJ}(\hat{Y})$, $\hat{L} = L(\hat{Y})$ and we have used the first condition in eq. (4.3).

To fix the gauge we introduce the Faddeev-Popov ghosts $b_\alpha$, $c^\alpha$ and the Nakanishi-Lautrup auxiliary field $B_\alpha$. The BRST transformations are

$$
\delta_B h_{\alpha\beta} = \hat{D}_\alpha c_\beta + \hat{D}_\beta c_\alpha - \frac{2}{\epsilon + 2} \eta_{\alpha\beta} \hat{D}_\gamma c^\gamma + \cdots,
$$
$$
\delta_B \xi^I = c^\alpha \partial_\alpha \hat{Y}^I - \frac{2(\epsilon + 1)}{\epsilon + 2} \hat{D}_\alpha c^\alpha \hat{G}^{IJ} \partial_J \hat{L} + \cdots,
$$
$$
\delta_B b_\alpha = i B_\alpha, \quad \delta_B B_\alpha = 0,
$$

(4.7)
where the dots represent terms quadratic and of higher powers in the quantum fields.
To eliminate the mixing of $h_{\mu\nu}$ and other fields, we use the following gauge function

$$F_{\alpha} = \sqrt{-g} \hat{L} \left[ \hat{D}^\beta h_{\beta\alpha} - \frac{1}{\kappa} \partial_\alpha \left( \frac{L(Y)}{\hat{L}} \right) + \frac{1}{2} B_\alpha \right].$$  

(4.8)

Then the gauge fixing term and the ghost action are given by

$$S_{\text{GF+FP}} = \int d^d x \delta_B (-i b^\alpha F_{\alpha})$$

$$= \int d^d x \sqrt{-g} \hat{L} \left[ \frac{1}{2} B^\alpha B'^\alpha - \frac{1}{2} \left( \hat{D}^\beta h_{\alpha\beta} - \partial_\alpha \left( \xi^I \partial_I \ln \hat{L} \right) \right)^2 
+ i b^\alpha \hat{D}^\beta \hat{D}_\beta c_\alpha + i b^\alpha \hat{D}_\alpha \left( c^\beta \partial_\beta \ln \hat{L} \right) + \cdots \right],$$  

(4.9)

where $B'_\alpha$ is a shifted auxiliary field and we have used the second condition in eq. (4.3). The total action is

$$S^{(2)}_{\text{tot}} = S^{(2)} + S^{(2)}_{\text{GF+FP}}$$

$$= \int d^d x \sqrt{-g} \hat{L} \left[ -\frac{1}{4} \hat{D}_\mu h_{\mu\rho} \hat{D}^\rho h_{\rho\sigma} - \frac{1}{2} \hat{R}^{(d)}_{\mu\rho\nu\sigma} h^{\mu\nu} h^{\rho\sigma} 
- \hat{D}_\mu \left( h^{\mu\lambda} \hat{D}_\lambda h^{\nu\nu} \right) - \hat{R}^{(d)}_{\mu\nu\lambda} h^{\mu\nu} \xi^I \partial_I \ln \hat{L} - \hat{D}_\mu h^{\mu\nu} \xi^I \partial_I \ln \hat{L} \partial_\nu \ln \hat{L} 
- \frac{1}{2} \hat{D}_\alpha \left( \xi^I \partial_I \ln \hat{L} \right) \hat{D}_\beta \left( \xi^J \partial_J \ln \hat{L} \right) + \frac{1}{2} B^\alpha B'_\alpha + \text{ghost terms} \right]$$

$$- \frac{\epsilon}{8(\epsilon + 1)} \xi^I \xi^J \hat{G}^{(d)}_{IJ} \hat{R}^{(d)}_{IJ} - \frac{1}{4} h^{\mu\lambda} \hat{D}_\mu \hat{D}_\nu \chi \partial_\mu \hat{Y}^I \partial_\nu \hat{Y}^J \hat{G}_{IJ} + h^{\mu\nu} \hat{D}_\mu \xi^I \partial_\nu \hat{Y}^J \hat{G}_{IJ} 
- \frac{1}{2} \hat{D}_\mu \xi^I \hat{D}^\mu \xi^J \hat{G}_{IJ} + \frac{1}{2} \hat{D}_\mu \hat{Y}^I \partial_\mu \hat{Y}^J \hat{R}_{IKJL} \xi^K \xi^L \right].$$  

(4.10)

It is better to remove interaction terms with two derivatives. We rescale

$$h_{\alpha\beta} \rightarrow \hat{L}^{-\frac{1}{2}} h_{\alpha\beta}, \quad B'_\alpha \rightarrow \hat{L}^{-\frac{1}{2}} B'_\alpha, \quad b_\alpha \rightarrow \hat{L}^{-1} b_\alpha.$$  

(4.11)

To reassemble the terms quadratic in $\xi^A$, we define the effective target space metric $G'_{IJ}$ as

$$G'_{IJ} = G_{IJ} + \frac{\partial_I L \partial_J L}{L}, \quad G'^{IJ} = G'^{IJ} = \frac{2(\epsilon + 1) \partial_I L \partial_J L}{\epsilon + 2}.$$  

(4.12)
where \( \partial^IL = G^{IJ} \partial_JL \) and the second condition in eq. (4.3) is used to obtain the inverse \( G'^IJ \). The vielbein can be chosen as

\[
G_{IJ} = E^A_I E^B_J \eta_{AB}, \quad G'_{IJ} = E'^A_I E'^B_J \eta_{AB},
\]

and

\[
E'^A_I = E^A_I + a \frac{\partial_I L \partial^A L}{L}, \quad E'_A^I = E_A^I + b \frac{\partial_A L \partial^I L}{L}, \tag{4.13}
\]

where \( \partial^A L = E^A_I \partial_I L \) and

\[
a = \frac{2(\epsilon + 1) - \sqrt{2(\epsilon + 1)(\epsilon + 2)}}{\epsilon} = \frac{1}{2} + O(\epsilon),
\]

\[
b = -\frac{2(\epsilon + 1)a}{2(\epsilon + 1) - \epsilon a} = -\frac{1}{2} + O(\epsilon). \tag{4.14}
\]

We define the quantum fields with the index \( A \) and the covariant derivative on them as

\[
\xi^A = \xi^I E^A_I, \tag{4.15}
\]

\[
\hat{D}_\mu \xi^A = \partial_\mu \xi^A + \partial_\mu \hat{Y}^I \hat{G}_{A}^I B \xi^B,
\]

\[
\Omega^{A} = E'^J_B \Gamma^K_{IJ} E^I_A - \partial_I L \partial^A L \partial_B LL^{-2}. \tag{4.16}
\]

Here \( \Gamma^K_{IJ} \) is the Christoffel connection of the metric \( G_{IJ} \) (not \( G'_{IJ} \)). Using eqs. (4.13), (4.14) and (4.3) it can be shown that the the effective spin connection \( \Omega_{IAB} \) is not antisymmetric in \( A, B \)

\[
\Omega_{IAB} + \Omega_{IBA} = -\partial_I L \partial_A L \partial_B LL^{-2} + O(\epsilon). \tag{4.17}
\]

In terms of the new fields (4.11), (4.15) the total action (4.10) becomes

\[
S^{(2)}_{\text{tot}} = \int d^d x \sqrt{-\hat{g}} \left[ -\frac{1}{4} \hat{D}_\mu h_{\alpha\beta} \hat{D}_\mu h^{\alpha\beta} - \frac{1}{2} \hat{R}^{(d)}_{\alpha\gamma\delta\beta} h^{\alpha\beta} h^{\gamma\delta} 
\right. 
\left. - \frac{1}{16} h_{\alpha\beta} h^{\alpha\beta} \left( 2 \partial^\mu \partial_\mu \ln \hat{L} + \partial^\mu \ln \hat{L} \partial_\mu \ln \hat{L} \right) 
\right. 
\left. - \frac{1}{4} h^{\alpha \beta} h^{\gamma} \left( 2 \partial_\alpha \ln \hat{L} \partial_\gamma \ln \hat{L} + \partial_\alpha \hat{Y}^I \partial_\gamma \hat{G}_{IJ} \hat{L}^{-1} \right) + \partial_\alpha h^{\alpha \beta} h^{\gamma} \partial_\gamma \ln \hat{L} 
\right. 
\left. - i \hat{D}_\beta b^\alpha \hat{c}_\alpha + i \hat{R}^{(d)}_{\alpha\beta} b^\alpha c^\beta + i \hat{D}_\alpha b^\alpha c^\beta \partial_\beta \ln \hat{L} + \frac{1}{2} B^\alpha B^\alpha' 
\right. 
\left. - \frac{1}{2} \hat{D}_\mu \xi^A \hat{D}_\mu \xi^B \eta_{AB} + \frac{1}{2} \xi^A \xi^B \hat{Y}^I \partial_\mu \hat{Y}^J \left( \hat{R}_{IAB} - \partial_I \hat{L} \partial_J \hat{L} \partial_A \hat{L} \partial_B \hat{L} \hat{L}^{-3} \right) 
\right. 
\left. + h^{\alpha \beta} \partial_\alpha \xi^A \hat{E}_{IA} \partial_\beta \hat{Y}^I \hat{L}^{-\frac{1}{2}} + \ldots \right]. \tag{4.18}
\]
where we have used eq. (4.3) and the dots represent terms which do not contribute to one-loop divergences. They are either terms of order $\epsilon$ or terms proportional to $\xi^A h_{\mu\nu}$ (without derivatives). Non-vanishing propagators are the same as those in eq. (3.8), except those for $\rho, \phi, X^i$ should be replaced by those for the fields $\xi^A$

$$\langle \xi^A(x)\xi^B(y) \rangle = \eta^{AB} \Delta_F(x-y). \quad (4.19)$$

Divergences of the effective action at one-loop order should take the form

$$\Gamma_{\text{div}} = \int d^d x \sqrt{-\hat{g}} \left[ \hat{R}^{(d)} A(\hat{Y}) + \hat{g}^{\mu\nu} \partial_\mu \hat{Y}^I \partial_\nu \hat{Y}^J B_{IJ}(\hat{Y}) \right]. \quad (4.20)$$

To obtain the coefficient $A(\hat{Y})$ in the first term we only need to consider the case $\hat{Y}^I = \text{constant}$. Then the action becomes almost the same as the action of the Einstein gravity in refs. [5], [7] except for the terms which do not contribute to the divergence. Therefore, the coefficient of the first term is given by

$$A = \frac{24 - N}{24\pi \epsilon}. \quad (4.21)$$

We note that there is no factor $L(\hat{Y})$ in this divergence. To obtain $B_{IJ}(\hat{Y})$ in the second term of the divergence (4.20) we only need to consider the case $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$. There are two kinds of divergent diagrams as shown in Fig. 1. The internal line of the diagram (a) can be

(a1) : $\langle hh \rangle$,

(a2) : $\langle \xi \xi \rangle$. \quad (4.22)
Their divergences are
\[ B_{IJ}^{(a1)} = \frac{1}{4\pi\epsilon} \left( \hat{G}_{IJ} \hat{L}^{-1} + \frac{5}{2} \partial_I \ln \hat{L} \partial_J \ln \hat{L} \right), \]
\[ B_{IJ}^{(a2)} = -\frac{1}{4\pi\epsilon} \left( \hat{R}_{IJ} - \hat{\Omega}_{IAB} \hat{\Omega}_{J}^{AB} \right). \] (4.23)

The internal lines of the diagram (b) can be
\[ (b1) : \langle hh \rangle, \langle hh \rangle, \]
\[ (b2) : \langle cb \rangle, \langle cb \rangle, \]
\[ (b3) : \langle hh \rangle, \langle \xi \xi \rangle, \]
\[ (b4) : \langle \xi \xi \rangle, \langle \xi \xi \rangle. \] (4.24)

Their divergences are
\[ B_{IJ}^{(b1)} = -\frac{1}{2\pi\epsilon} \partial_I \ln \hat{L} \partial_J \ln \hat{L}, \]
\[ B_{IJ}^{(b2)} = -\frac{1}{8\pi\epsilon} \partial_I \ln \hat{L} \partial_J \ln \hat{L}, \]
\[ B_{IJ}^{(b3)} = -\frac{1}{4\pi\epsilon} \left( \hat{G}_{IJ} \hat{L}^{-1} + \partial_I \ln \hat{L} \partial_J \ln \hat{L} \right), \]
\[ B_{IJ}^{(b4)} = -\frac{1}{8\pi\epsilon} \hat{\Omega}_{IAB} \left( \hat{\Omega}_{J}^{AB} \right. \left. - \hat{\Omega}_{J}^{BA} \right). \] (4.25)

The divergences of the $\Omega^2$ terms in (a2) and (b4) cancel each other. A sum of the diagrams with $h_{\alpha\beta}$ propagators and the diagrams with ghost propagators gives non-zero contributions to both of $A$ and $B_{IJ}$ in the divergence (4.20). From eqs. (4.21), (4.23), (4.25) we obtain the total one-loop divergence
\[ \Gamma_{\text{div}} = \int d^d x \sqrt{-\hat{g}} \left[ \frac{24 - N}{24\pi\epsilon} \hat{R}^{(d)} - \frac{1}{4\pi\epsilon} \hat{g}^{\mu\nu} \partial_\mu \hat{Y}^I \partial_\nu \hat{Y}^J \left( \hat{R}_{IJ} + \partial_I \ln \hat{L} \partial_J \ln \hat{L} \right) \right]. \] (4.26)

The third term came from $h_{\mu\nu}$ and ghost diagrams. It vanishes for the Einstein gravity, in which $L = \text{constant}$.

Using eqs. (A.1), (A.2), (A.3) in the appendix, the first two terms of eq. (4.26) are shown to be invariant under $\delta \hat{Y}^I$ in eq. (4.2). Therefore, they are independent of the Liouville field $\hat{\rho}$ since $\delta \hat{Y}^I$ in eq. (4.2) is a shift of the Liouville field. The third term changes under $\delta \hat{Y}^I$ in eq. (4.2) as
\[ \delta_{\hat{Y}} \left[ \int d^d x \sqrt{-\hat{g}} \hat{g}^{\mu\nu} \partial_\mu \hat{Y}^I \partial_\nu \hat{Y}^J \partial_I \ln \hat{L} \partial_J \ln \hat{L} \right] = 2\epsilon \int d^d x \sqrt{-\hat{g}} \hat{g}^{\mu\nu} \partial_\mu \sigma \partial_\nu \ln \hat{L}. \] (4.27)
The $\hat{\rho}$-dependence is of order $\epsilon$. Therefore the divergent part of (4.26) is independent of the Liouville field $\hat{\rho}$. Since the finite part of the counter terms can be chosen at will, we shall choose a counter term which depends on the background Liouville field $\hat{\rho}$ only through $\hat{g}_{\mu\nu} e^{-2\hat{\rho}}$. Then we should just replace the background metric $\hat{g}_{\mu\nu}$ by the background physical metric $\hat{g}_{\mu\nu} e^{-2\hat{\rho}}$ which we shall denote $g_{\mu\nu}$ without a hat $\hat{\cdot}$. This assures the general coordinate invariance with respect to the physical metric $g_{\mu\nu} = \hat{g}_{\mu\nu} e^{-2\hat{\rho}}$. The invariance under the background Weyl transformation (4.2) also becomes manifest.

Let us apply the result of one-loop divergence (4.26) to our standard model (2.7), which is rewritten in terms of the Liouville field in eq. (3.3). By computing the target space metric and curvature, we obtain one-loop divergences of the dilaton gravity as

$$\Gamma_{\text{div}} = \int d^d x \sqrt{-g} \left[ \frac{24 - N}{24\pi\epsilon} R^{(d)} - \frac{1}{4\pi\epsilon} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \left( N \left( \Phi'' + 2\Phi' - (\Phi')^2 \right) + 4 \right) \right],$$

(4.28)

where we have dropped to write a hat $\hat{\cdot}$ for the background field.

5. Beta functions and fixed points

The counter terms can be summarized with three types of coefficients $A, B, C$

$$S_{\text{counter}} = -\mu^\epsilon \int d^d x \sqrt{-g} \left[ R^{(d)} A(\phi) + g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi B(\phi) \right] - \frac{1}{2} g^{\mu\nu} \partial_\mu X^i \partial_\nu X^i C(\phi).$$

(5.1)

At one-loop level these coefficients are given by

$$A(\phi) = \frac{24 - N}{24\pi\epsilon}, \quad C(\phi) = 0,$$

$$B(\phi) = -\frac{1}{\pi\epsilon} + \frac{N}{4\pi\epsilon} \left[ (\Phi'(\phi))^2 - \Phi''(\phi) - 2\Phi'(\phi) \right].$$

(5.2)

The action including counter terms is expressed in terms of bare quantities as

$$S_0 = S + S_{\text{counter}}$$

$$= \int d^d x \sqrt{-g_0} \left[ \frac{1}{16\pi G_0} R^{(d)}_0 e^{-2\phi_0} - \frac{1}{2} g_0^{\mu\nu} \partial_\mu X^i_0 \partial_\nu X^i_0 e^{-2\Phi_0(\phi_0)} \right].$$

(5.3)
If we define

\[ \Phi_0(\phi_0) = \Phi(\phi) + F(\phi) \quad (F(0) = 0), \]
\[ g_{\mu\nu} = g_{\mu\nu} e^{-2\Lambda(\phi)} \quad (\Lambda(0) = 0), \]
\[ \phi_0 = \phi + f(\phi) \quad (f(0) = 0), \]
\[ X_0^i = \sqrt{Z} X^i, \quad (5.4) \]

the action becomes

\[
S_0 = \int d^d x \sqrt{-g} \left[ \frac{1}{16\pi G_0} R^{(d)} e^{-2\phi - \epsilon \Lambda - 2f} 
+ \frac{\epsilon + 1}{16\pi G_0} \left( 4\Lambda' + \epsilon (\Lambda')^2 + 4f'\Lambda' \right) e^{-2\phi - \epsilon \Lambda - 2f} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi 
- \frac{1}{2} Z e^{-2\phi - \epsilon \Lambda - 2F} g^{\mu\nu} \partial_\mu X^i \partial_\nu X^i \right]. \quad (5.5)
\]

To reproduce the counter terms (5.1) we require

\[
\frac{1}{16\pi G_0} e^{-2\phi - \epsilon \Lambda(\phi) - 2f(\phi)} = \mu^\epsilon \left[ \frac{1}{16\pi G} e^{-2\phi} - A(\phi) \right],
\]
\[
\frac{\epsilon + 1}{16\pi G_0} \left( 4\Lambda' + \epsilon (\Lambda')^2 + 4f'\Lambda' \right)(\phi) e^{-2\phi - \epsilon \Lambda(\phi) - 2f(\phi)} = -\mu^\epsilon B(\phi),
\]
\[
Z e^{-2\phi - \epsilon \Lambda(\phi) - 2F(\phi)} = e^{-2\phi} - C(\phi). \quad (5.6)
\]

At one-loop level we can simplify these equations. Using the fact that \( A(\phi) \) is a constant, \( C(\phi) = 0 \) and \( \Lambda, f, F = O(G) \), and neglecting higher order terms in \( G \), we find that the above equations become

\[
\frac{1}{16\pi G_0} e^{-2\phi - \epsilon \Lambda(\phi) - 2f(\phi)} = \mu^\epsilon \left( \frac{1}{16\pi G} e^{-2\phi} - A \right),
\]
\[
\frac{\epsilon + 1}{4\pi G_0} \Lambda'(\phi) e^{-2\phi} = -\mu^\epsilon B(\phi),
\]
\[
\epsilon \Lambda(\phi) + 2F(\phi) = 0, \quad Z = 1. \quad (5.7)
\]

Substituting the solution of these equations into eq. (5.4) we obtain the relation between the bare and renormalized quantities as

\[
\frac{1}{G_0} = \mu^\epsilon \left( \frac{1}{G} - 16\pi A \right),
\]
\[
\rho_0 = \rho - \frac{4\pi G}{\epsilon + 1} \int_0^\phi d\phi' e^{2\phi'} B(\phi'),
\]

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\[ \phi_0 = \phi + 8\pi AG \left( e^{2\phi} - 1 \right) + \frac{2\pi\epsilon G}{\epsilon + 1} \int_0^\phi d\phi' e^{2\phi'} B(\phi'), \]

\[ \Phi_0(\phi_0) = \Phi(\phi) + \frac{2\pi\epsilon G}{\epsilon + 1} \int_0^\phi d\phi' e^{2\phi'} B(\phi'), \]

\[ = \Phi(\phi_0) - 8\pi AG \left( e^{2\phi_0} - 1 \right) \Phi'(\phi_0) - \frac{2\pi\epsilon G}{\epsilon + 1} \left( \Phi'(\phi_0) - 1 \right) \int_0^\phi d\phi' e^{2\phi'} B(\phi'). \] (5.8)

We find that the beta function \( \beta \) and the anomalous dimension \( \gamma \) are functions of \( \phi \) in general

\[ \beta_G \equiv \frac{\partial G}{\partial \mu} = \epsilon G - 16\pi\epsilon AG^2, \]

\[ \beta_\Phi(\phi_0) \equiv \frac{\partial \Phi(\phi_0)}{\partial \mu} \]

\[ = 8\pi\epsilon AG \left( e^{2\phi_0} - 1 \right) \Phi'(\phi_0) + \frac{2\pi\epsilon G}{\epsilon + 1} \left( \Phi'(\phi_0) - 1 \right) \int_0^\phi d\phi' e^{2\phi'} B(\phi'), \]

\[ \gamma_\rho \equiv \frac{\partial \rho}{\partial \mu} = \frac{4\pi\epsilon G}{\epsilon + 1} \int_0^\phi d\phi' e^{2\phi'} B(\phi'), \]

\[ \gamma_\phi \equiv \frac{\partial \phi}{\partial \mu} = -8\pi\epsilon AG \left( e^{2\phi} - 1 \right) - \frac{2\pi\epsilon G}{\epsilon + 1} \int_0^\phi d\phi' e^{2\phi'} B(\phi'). \] (5.9)

The beta function \( \beta_G \) for \( G \) is similar to that of the Einstein gravity. For \( N < 24 \), \( G = 0 \) is an infrared stable fixed point and \( G = G^* \) is an ultraviolet stable fixed point, where

\[ G^* = \frac{3\epsilon}{2(24 - N)}, \quad \beta_G(G^*) = 0, \quad \beta'_G(G^*) < 0. \] (5.10)

In order to find fixed points for the beta function \( \beta_\Phi = 0 \), we consider an ansatz

\[ \Phi(\phi) = \lambda \phi \quad (\lambda = \text{constant}). \] (5.11)

Then the beta function becomes

\[ \beta_\Phi(\phi) = \left( e^{2\phi} - 1 \right) \left[ \frac{24 - N}{3} G\lambda + \frac{\epsilon G}{\epsilon + 1} \left( \frac{N}{4} (\lambda^2 - 2\lambda) - 1 \right) (\lambda - 1) \right]. \] (5.12)
The above beta function shows that fixed points can be obtained as a solution of the cubic equation. We find that there is only one fixed point for the real value of the parameter $\lambda$

$$\Phi(\phi) = \lambda^* \phi, \quad \lambda^* = -\frac{3\epsilon}{24 - N} + O(\epsilon^2). \quad (5.13)$$

The other two solutions are purely imaginary and of order $\epsilon^{-1/2}$ at the fixed point for the gravitational coupling constant $G^*$

$$\lambda^* = \pm 2i \sqrt{\frac{24 - N}{3N\epsilon}} + O(\epsilon^0). \quad (5.14)$$

Since the imaginary solution corresponds to a wildly oscillating dilaton coupling, we consider the real solution (5.13) to be the only physical solution.

Let us study stability of this fixed point (5.10), (5.13). We expand the beta functions near the fixed point

$$G = G^* + \delta G, \quad \Phi = \lambda^* \phi + \delta \Phi, \quad (5.15)$$

assuming the fluctuations $\delta G$ and $\delta \Phi$ to be small. By examining the beta functions near the fixed point to first order in the fluctuation we find

$$\beta_G = -\epsilon \delta G,$$

$$\beta_\Phi = \epsilon(1 - e^{2\Phi}) \delta G - \frac{1}{2} \epsilon(1 - e^{2\Phi}) \frac{d}{d\phi} \delta \Phi + O(\epsilon^2). \quad (5.16)$$

We can diagonalize these equation by considering the beta function for

$$\tilde{\Phi}(\phi) \equiv \Phi(\phi) + \left(1 + \frac{2\phi e^{2\phi}}{1 - e^{2\phi}}\right) G. \quad (5.17)$$

Then we obtain

$$\beta_G = -\epsilon \delta G,$$

$$\beta_{\tilde{\Phi}} = -\frac{1}{2} \epsilon(1 - e^{2\phi}) \frac{d}{d\phi} \delta \tilde{\Phi} + O(\epsilon^2). \quad (5.18)$$
Since $e^\phi$ plays a role of the loop expansion parameter, we shall only consider the region $-\infty < \phi \leq 0$, in which $e^\phi \leq 1$. If we define a variable $\psi$

$$\psi = \frac{1}{2} \ln(e^{-2\phi} - 1), \quad \begin{cases} \psi \to +\infty \iff \phi \to -\infty, \\ \psi \to -\infty \iff \phi \to 0, \end{cases} \quad (5.19)$$

the second equation in eq. (5.18) becomes

$$\beta_{\Phi} = \frac{1}{2} \epsilon \frac{d}{d\psi} \delta \Phi. \quad (5.20)$$

Eigenfunctions of the differential operator on the right hand side is

$$\delta \Phi = e^{\Lambda \psi} = (e^{-2\phi} - 1)^{\frac{1}{2} \Lambda} \quad (5.21)$$

with eigenvalues

$$\beta_{\Phi} = \frac{1}{2} \epsilon \Lambda \delta \Phi. \quad (5.22)$$

The condition $\delta \Phi(\phi = 0) = 0$ requires $\Lambda > 0$. Therefore, the fixed point (5.10), (5.13) is not ultraviolet stable in the direction $\delta \Phi$.

We have also studied more general solutions without using the linear ansatz (5.11). Since we are interested in the fixed point for the $\beta_{\Phi}$, we take the gravitational coupling at the fixed point $G = G^*$. We shall change variables from $\Phi$ to $\delta \Phi$ by eq. (5.13), but we no longer assume $\delta \Phi$ to be small. We also use the variable $\psi$ defined in eq. (5.19) instead of $\phi$. By neglecting terms which are small for small $\epsilon$, we obtain

$$\beta_{\delta \Phi} = \frac{24 - N}{3} G^* \frac{d \delta \Phi}{d \psi} + \frac{\epsilon G^* N}{2(\epsilon + 1)} \left[ 1 + (1 + e^{-2\psi}) \frac{d \delta \Phi}{d \psi} \right]$$

$$\times \int_{-\infty}^{\psi} d\psi' \left[ e^{-2\psi'} \left( \frac{d \delta \Phi}{d \psi'} \right)^2 - \frac{d}{d \psi'} \left( e^{-2\psi'} \frac{d \delta \Phi}{d \psi'} \right) \right]. \quad (5.23)$$

We can explicitly solve the fixed point condition $\beta_{\delta \Phi} = 0$ and find that there are only three solutions (5.13) and (5.14) that has been obtained using the linear ansatz (5.11).

The fixed point is found at (5.10) for the beta function $\beta_G = 0$ of the gravitational coupling and at (5.13) for the beta function $\beta_{\Phi} = 0$ of the dilaton weight. This fixed
point is ultraviolet stable in the direction of $G$, but is unstable in the direction of $\tilde{\Phi}$. Therefore we can consider a renormalized theory with the gravitational coupling constant near the fixed point $G^*$, as long as we fine tune the dilaton weight $\Phi$ to be precisely at the fixed point $\delta \Phi = 0$

$$\Phi(\phi) = \lambda^* \phi - \left(1 + \frac{2\phi e^{2\phi}}{1 + e^{2\phi}}\right)(G - G^*). \tag{5.24}$$

It is interesting to observe that the dilaton gravity theory at the fixed point can be recast into a form without matter coupling to the dilaton by means of a nonsingular local Weyl rescaling, since the dilaton weight $\Phi$ is of order $\epsilon$. After the Weyl rescaling $g_{\mu\nu} \to g_{\mu\nu} \exp \left(\frac{-2\lambda^*}{\epsilon} \phi\right)$, we obtain an action

$$S = \int d^d x \sqrt{-g} \left[\frac{\mu^d}{16\pi G^*} e^{-2(1-\lambda^*)\phi} \left(\mathcal{R}^{(d)} - \frac{4(1 + \epsilon)}{\epsilon} \lambda^*(2 - \lambda^*) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi\right) - \frac{1}{2} g^{\mu\nu} \partial_\mu X^i \partial_\nu X^j \delta_{ij}\right], \tag{5.25}$$

This action is similar to the CGHS action (2.10) since the coefficient in front of the dilaton kinetic term is finite ($\frac{24}{24-N}$) in the limit $\epsilon \to 0$ and is of the same sign as the CGHS action.

6. Discussion

It has been observed in the Einstein gravity theory in $2+\epsilon$ dimensions that one has to fix the scale of the metric before one can meaningfully discuss the renormalization of the gravitational coupling constant [4]. One can fix the scale of the metric by introducing a reference operator $\lambda \int d^d x \ O(x)$ with a coupling constant $\lambda$ into the action. The most convenient choice of the reference operator is a gravitational dressing of an operator $\Psi$ which has a dimensionless coupling constant $\lambda$ in the limit of $\epsilon \to 0$, such as the Thirring interaction. The gravitational dressing of such an operator takes the form of

$$O = e^{\alpha x^\rho} \Psi, \tag{6.1}$$

where the exponent $\alpha$ is of order $\epsilon$. The merit of the decomposition (3.2) of the traceless mode $h_{\mu\nu}$ and the Liouville field $\rho$ is the fact that no one-loop divergence
arises for gravitational dressing of operators of the above form

\[\langle e^{\alpha \kappa \rho} \rangle = 1 + \frac{1}{2} (\kappa \alpha)^2 \langle \rho(0) \rho(0) \rangle + \cdots = \text{finite}, \quad (6.2)\]

since the Liouville field propagator is of order $1/\epsilon$ and the propagator at the coincidence point produces an additional $1/\epsilon$ singularity both of which are canceled by $\alpha^2 = O(\epsilon^2)$. Since the reference operator is automatically finite, one can perform the renormalization of the gravitational coupling without considering the reference operator explicitly.

Now let us discuss renormalization of gravitationally dressed operators in the dilaton gravity theory. We need to introduce two reference operators in the action and use them to fix the freedom of field redefinitions (zero modes of $\Lambda(\phi)$ and $f(\phi)$) as we discussed in sect. 2. We would like to choose reference operators $O_i$ ($i = 1, 2$) to be gravitational dressing of operators $\Psi_i$ which have dimensionless coupling constants $\lambda_i$ in the limit of $\epsilon \to 0$, such as the Thirring interaction. In the case of the dilaton gravity, however, the Liouville field should be considered together with the dilaton because of the mixing. Therefore we have a freedom in the gravitational dressing to choose the amount of the dilaton dressing $\beta$ in addition to the Liouville dressing $\alpha$.

\[O_i = e^{\alpha_i \kappa \rho + \beta_i \kappa \phi} \Psi_i. \quad (6.3)\]

If we take as reference operators those that become dimensionless in the $\epsilon \to 0$ limit, we expect that the gravitational dressing exponents $\alpha_i$ for the Liouville field to be $O(\epsilon)$. On the other hand, we have no particular reason to specify the gravitational dressing exponent $\beta_i$ for the dilaton field. Therefore $\beta_i$ is naturally expected to be $O(\epsilon^0)$. These two reference operators serve to fix the scale of the metric $g_{\mu \nu}$ (origin of $\rho$) and the origin of $\phi$.

By using our propagators (3.8), we obtain divergences of the expectation value as

\[\langle e^{\alpha \kappa \rho + \beta \kappa \phi} \rangle = 1 + \frac{1}{2} \kappa^2 \langle (\alpha \rho + \beta \phi)^2 \rangle + \cdots = 1 + \frac{G}{\epsilon \mu^*} (\alpha^2 + 2\alpha\beta) + O(\epsilon^0) + \cdots. \quad (6.4)\]

When the parameters satisfy $\alpha = O(\epsilon), \beta = O(\epsilon^0)$, there is no one-loop divergence. Since these reference operators are automatically one-loop finite, the gravitational
coupling $G$ and the dilaton weight $\Phi(\phi)$ in our theory can be identified as those after the coupling constants $\lambda_i$ for these reference operators are fixed.

After writing this paper, we have received a preprint which has worked out more details on the Einstein gravity in $2 + \epsilon$ dimensions [20]. They showed that one can choose a renormalization group trajectory which respects the general covariance, although their theory is invariant only under volume preserving diffeomorphisms.

Two of the authors (N.S. and S.K.) thank Yoshihisa Kitazawa for interesting comments on our work and giving ref. [20] prior to publication. One of the authors (N.S.) also thanks Hikaru Kawai for a useful discussion on the $(2 + \epsilon)$-dimensional approach. One of the authors (Y.T.) would like to thank the Theoretical Physics Group of Imperial College for hospitality, and the Japan Society for the Promotion of Science and the Royal Society for a grant. This work is supported in part by Grant-in-Aid for Scientific Research (S.K.) and (No.05640334) (N.S.), and Grant-in-Aid for Scientific Research for Priority Areas (No.05230019) (N.S.) from the Ministry of Education, Science and Culture.

Appendix A. Conditions of background Weyl invariance

In this appendix we discuss the conditions (4.3), which have been obtained by requiring the background Weyl invariance of the action (4.1). First, let us derive several identities which are used in the text. From the trace of the first condition in eq. (4.3) we obtain

$$D^I \partial_I L = \text{constant}. \quad (A.1)$$

By applying $D_K$ on the first condition in eq. (4.3) and antisymmetrizing the indices $K$ and $I$ we obtain

$$\partial^M LR_{MIK} = 0, \quad \partial^J LR_{IJ} = 0. \quad (A.2)$$

By applying $D^K$ on the first equation in eq. (A.2) and using the Bianchi identity and the first of eq. (4.3) we obtain

$$\partial^K LD_K R_{IJ} + D_I \partial^K LR_{KJ} + D_J \partial^K LR_{IK} = 0. \quad (A.3)$$
Next, we shall obtain the general couplings $G_{IJ}$, $L$ which satisfy the conditions (4.3). We split the target space coordinates as $Y^I = (\rho, \phi, X^i)$ ($i = 1, \ldots, N$). By appropriately choosing the coordinates $Y^I$ we can put $G_{\rho I}$ to arbitrary functions in the target space. This is most easily understood by noting that $G_{\rho\rho}$ and $G_{\rho I}$ ($I \neq \rho$) are analogous to the lapse and the shift functions in the canonical formulation of general relativity. We choose them as

$$G_{I\rho} = 2(\epsilon + 1)\partial_I L. \quad (A.4)$$

This choice corresponds to a coordinate system in which the background Weyl transformation of $Y^I$ in eq. (4.2) becomes $\delta \rho = -\sigma$, $\delta Y^I = 0$ ($I \neq \rho$). Using eq. (A.4) the second condition in eq. (4.3) becomes

$$-\epsilon L - \partial_\rho L = 0. \quad (A.5)$$

The solution is

$$L(Y) = e^{-\epsilon \rho} \bar{L}(\phi, X), \quad (A.6)$$

where $\bar{L}(\phi, X)$ is an arbitrary function of $\phi$ and $X^i$. Substituting this into eq. (A.4) we obtain

$$G_{\rho\rho}(Y) = -2\epsilon(\epsilon + 1) e^{-\epsilon \rho} \bar{L}(\phi, X),$$

$$G_{IJ}(Y) = 2(\epsilon + 1) e^{-\epsilon \rho} \partial_0 \bar{L}(\phi, X) \quad (I \neq \rho). \quad (A.7)$$

Then the first condition in eq. (4.3) becomes

$$-\epsilon G_{IJ} - \partial_\rho G_{IJ} = 0. \quad (A.8)$$

This condition is automatically satisfied for $I = \rho$ or $J = \rho$ by eq. (A.7). The remaining cases are solved by

$$G_{IJ}(Y) = e^{-\epsilon \rho} \bar{G}_{IJ}(\phi, X) \quad (I \neq \rho \text{ and } J \neq \rho), \quad (A.9)$$

where $\bar{G}_{ij}(\phi, X)$ is an arbitrary function of $\phi$ and $X^i$. Therefore, the general solution of eq. (4.3) is given by eqs. (A.6), (A.7) and (A.9) up to coordinate transformations.
in the target space. Substituting this solution into eq. (4.1) the action becomes

\[
S = \frac{\mu^k}{16\pi G} \int d^d x \sqrt{-g} \left[ R^{(d)} \bar{L}(\phi, X) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \bar{G}_{\phi\phi}(\phi, X) 
- g^{\mu\nu} \partial_\mu \phi \partial_\nu X^j \bar{G}_{\phi j}(\phi, X) - \frac{1}{2} g^{\mu\nu} \partial_\mu X^i \partial_\nu X^j \bar{G}_{ij}(\phi, X) \right],
\]

(A.10)

where \( g_{\mu\nu} = \tilde{g}_{\mu\nu} e^{-2\rho} \). This sigma model type action is the same as the general action (2.2) after a trivial rescaling of the target space metric \( \bar{G}_{IJ} \). Therefore the \( \rho \)-dependence is completely absorbed into the physical metric \( g_{\mu\nu} \).

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