Transient of the kinetic spherical model between two temperatures *

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Abstract

We solve the dynamic equation for the kinetic spherical model that initially is in an arbitrary equilibrium state and then is left to evolve in a heat-bath with another temperature. Flows of the Renormalization group are determined.

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1. Introduction

In recent years, the universal scaling in non-equilibrium states have attracted much attention\cite{1,2,3,4,5}. The phase-ordering process (POP) \cite{6} and the short-time critical dynamics (SCD) \cite{7} are two fruitful examples. In the POP, the system initially at a very high temperature then is quenched to a heat-bath of very low temperature. In the SCD, the heat-bath has the critical temperature of the system.

In both POP and SCD, the initial correlation length is assumed to be zero. For finite initial correlation, the scaling invariance of the initial state is broken. One would expect a crossover phenomenon that is usually quite difficult to study either by theoretic methods or by numerical simulation. In order to gain an insight into this phenomenon, in the present paper we investigate the kinetic spherical model (KSM) with initial correlation.

The short-time dynamics of the kinetic spherical model with zero initial correlation length has been studied in \cite{8,9,10,11,12,13,14,15,16}. In the latest publication \cite{9} the effect of non-zero initial order was emphasized.

The system to be considered in the present paper is initially at an arbitrary equilibrium state. Then it is suddenly put into a heat-bath of another temperature. The external field is also removed instantly. The system is assumed to evolve following the Langevin equation. We are interested in the non-equilibrium transient state in the following time. We will concentrate on the time-dependent order parameter $m(t)$. To include the non-zero correlation, one must integrate the non-trivial properties of the initial equilibrium state. Though most properties of the spherical model can be found in literatures, e.g., the famous book by Baxter \cite{17}, it is still quite tricky to connect the equilibrium initial state with the non-equilibrium dynamics. We find that besides the subtraction of mass that is known in the zero initial correlation, one need an extra renormalization of the dynamic equation.

2. The Model

The Hamiltonian of the spherical model is

$$H = \frac{\alpha}{2} \sum_i S_i^2 - \beta J \sum_{ij} <ij> S_i S_j - \beta \sum_i h_i S_i$$  \hspace{1cm} (1)
with the constraint
\[ \sum_i S_i^2 = N \] (2)
where \(<ij>\) are bonds of a 3-dimensional regular lattice, \(N\) is the total number of spins; \(\beta = \frac{1}{k_B T}\). In the dynamic process, \(\alpha\) is a time-dependent Lagrange multiplier corresponding to the constraint.

The fluctuations of spins are defined to be \(\tilde{S}_i = S_i - <S_i>\). In the momentum space, one has
\[ \tilde{S}(p, t) = \frac{1}{\sqrt{N}} \sum_i \tilde{S}_i e^{i p \cdot r_i} \] (3)
with \(r_i\) the position vector of site \(i\). Define
\[ \Omega(p) = 1 - \frac{1}{3} (\cos(p_1) + \cos(p_2) + \cos(p_3)) \] (4)
and a function
\[ w(x) = \frac{1}{N} \sum_p \frac{1}{x + \Omega(p)} \] (5)
The correlation length is inversely proportional to the square root of \(z_0\) that is the solution of the equation
\[ w(2z_0) = \beta_0 J - \frac{\beta_0 h^2}{4 J z_0^3} \] (6)
for the initial temperature \(T_0\) (\(\beta_0 = 1/k_B T_0\)) and the homogenous field \(h\). The initial magnetization is given by
\[ m_0 = \frac{h}{2 J z_0} \] (7)
The equation of equilibrium state turns out to be
\[ w\left(\frac{h}{J m_0}\right) = \beta_0 J (1 - m_0^2) \] (8)
The initial correlation function is
\[ C_0(p, p') = <\tilde{S}(p, 0)\tilde{S}(p', 0)> = \frac{1}{\beta_0 J (2z_0 + \Omega(p))} \delta_{p,p'} \] (9)
It is easy to see that
\[ w(2\tau_0) = \frac{\beta_0 J}{N} \sum_p C_0(p, p) \] (10)

The Langevin equation in the momentum space for this constrained spin
system is \[ \frac{\partial \tilde{S}(p, t)}{\partial t} = -\lambda (\tau(t) + \beta J \Omega(p)) \tilde{S}(p, t) + \eta(p, t) \] (11)
where \( T (\beta = 1/k_B T) \) is the temperature of the heat-bath. The consistency
condition gives
\[ \tau(t) = \tau_{sub} + \beta J \left[ m^2(t) - 1 \right] + \frac{\beta J}{N} \sum_p (1 - \Omega(p)) \langle \tilde{S}(-p, t) \tilde{S}(p, t) \rangle > + \frac{1}{\lambda N} \sum_p \langle \tilde{S}(-p, t) \eta(p, t) \rangle > \] (12)
where the first term comes from the mass subtraction which guarantees \( \tau_c(\infty) = 0 \) at the critical point. Recalling that the equilibrium correlation has a zero pole at the critical temperature, one has
\[ \langle \tilde{S}(-p, \infty) \tilde{S}(p, \infty) \rangle > \big|_{\beta_c} \sim \frac{1}{\Omega(p)} \]
It defines the critical temperature \( \beta_c J = w(0) \) [17]. Substituting it into (12) at the critical temperature, one can find \( \tau_{sub} = 1 \) for the Ito prescription with which the last term of (12) is zero due to causality. If the Stratonovich prescription is used, the last term of (12) is 1 and \( \tau_{sub} = 0 \). Through the paper, Ito prescription will be used.

By solving (11), it is not difficult to obtain the response propagator
\[ G_p(t, t') = \frac{1}{2\lambda} < \tilde{S}(-p, t) \eta(p, t') >= \Theta(t - t') e^{-\lambda \beta J \Omega(p)(t - t') - \lambda \int_{t'}^t dt'' \tau(t'')} \] (13)
with the Heaviside step function \( \Theta(t - t') = 1 \) for \( t > t' \), otherwise \( \Theta(t - t') = 0 \); and the full correlation function (correlation function including the initial correlation)
\[ \tilde{C}_p(t, t') = < \tilde{S}(p, t) \tilde{S}(-p, t') > = C_0(p, p) G_p(t, 0) G_{-p}(t', 0) + C_p(t, t') \] (14)
with the correlation function

$$C_p(t, t') = 2\lambda \int_0^\infty dt''G_p(t, t'')G_p(t', t'')$$  \hspace{1cm} (15)$$

3. Laplace transformation

Introducing $f(t) = m^{-2}(t)$ with the time-dependent magnetization $m(t) = \frac{1}{N} < \sum_i S_i >$, one can convert the dynamic equation into a linear integro-differential equation for $f(t)$

$$\frac{\partial f(t)}{\partial t} = 2\lambda\beta J - 2\lambda(\beta J - 1)f(t) + \frac{2\lambda\beta J}{N} \sum_p (1 - \Omega(p))\bar{C}_p(t, t)f(t)$$  \hspace{1cm} (16)$$

For convenient, define $\gamma = (2\lambda\beta J)^{-1}$. By Laplace transformation

$$F(q) = \int_0^\infty dt f(t)e^{-qt}$$

Equation (16) is transformed to

$$F(q) = \frac{1}{q + \frac{2\lambda}{m_0} \frac{1}{N} \sum_p C_0(p, p) \frac{1}{\gamma q + \Omega(p)}}$$  \hspace{1cm} (17)$$

Substituting $C_0(p, p)$ by equation (9), one has

$$F(q) = \frac{1}{\frac{1}{q} + \frac{2\lambda}{m_0} \frac{1}{N} \sum_p C_0(p, p) \frac{1}{\gamma q + \Omega(p)}} \frac{1}{1 - \frac{1}{\beta J} w(\gamma q)} \frac{1}{1 - \frac{1}{\beta J} w(2z_0)}$$  \hspace{1cm} (18)$$

The last square bracket in the numerator can be written as

$$w(\gamma q) - w(2z_0) = [\beta J - w(2z_0)] - \Lambda [\beta J - w(\gamma q)]$$  \hspace{1cm} (19)$$

Where the constant $\Lambda = 1$. In fact, we will see that $\Lambda$ plays as a renormalization multiplier that should be determined self-consistently. The reason
will be clear soon. For the infinite system the sum in (5) is replaced by an integral. In the continuum-limit one has

\[ w(\gamma q) = w(0) - D(\gamma q)^{1/2} \]  

(20)

where the constant \( D = \left( \frac{9}{2\pi^2} \right)^{3/2} \). In this expansion of \( w(x) \), the spatial and temporal microscopic detail is lost. However, this microscopic detail would have macroscopic effects through equation (19) since it associates with the factor

\[ \frac{1}{2z_0 - \gamma q} \]

which in Laplace reversion is a factor increasing versus time exponentially. Therefore, a renormalization multiplier \( \Lambda \) is needed to rescue the error introduced by equation (19).

Recalling \( q \) has the inverse unit of time, one easily finds two characteristic time-scales

\[ t_h = \frac{\gamma}{2z_0} = \frac{m_0}{2\lambda \beta h} \]

\[ t_\beta = \left[ \frac{2\lambda \gamma^{3/2}D}{\mu} \right]^{1/2} \]

(21)

where \( \mu = 1 - T/T_c \). By use of (19), \( F(q) \) is written as

\[ F(q) = \frac{1}{\mu q(1 + \sqrt{t_\beta q})} - \frac{\Lambda \beta t_h}{\beta_0 m_0^2} \frac{1}{1 - t_h q} + \frac{A}{\mu} \frac{1}{1 - t_h q} \frac{1}{1 + \sqrt{t_\beta q}} \]

(22)

where

\[ A = \frac{\beta(1 - \beta w(2z_0))}{\beta_0 m_0^2} = \frac{\beta J - w\left( \frac{h}{Jm_0^2} \right)}{\left( \frac{h}{Jm_0} \right)} 1 - m_0^2 \]

(23)

When the heat-bath is at the critical temperature, \( F(q) \) is

\[ F(q) = Bq^{-3/2} - \frac{\Lambda \beta t_h}{\beta_0 m_0^2} \frac{1}{1 - t_h q} + \frac{AB}{(1 - t_h q)q^{1/2}} \]

(24)

where

\[ B = \frac{(2\lambda\beta J)^{3/2}}{2\lambda D} \].

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4. Laplace reversion

Let us first consider the case of $T < T_c$, i.e., $\mu > 0$. A direct Laplace reversion to (22) gives

$$m(t) = \sqrt{\mu} \left\{ 1 + \frac{At_h}{t_{\beta} - t_h} \sqrt{\frac{t_{\beta}}{t_h}} e^{t/t_h} \text{erfc}\left(\sqrt{\frac{t}{t_h}}\right) \right\}$$

$$- \left[ 1 + \frac{At_h}{t_{\beta} - t_h} \right] e^{t/t_{\beta}} \text{erfc}\left(\sqrt{\frac{t}{t_{\beta}}}\right)$$

$$+ \left[ \frac{\beta \mu}{\beta_0 m_0^2} \Lambda - \frac{A\sqrt{t_h}}{\sqrt{t_{\beta}} - \sqrt{t_h}} \right] e^{t/t_h} \right\}^{-1/2} \right\}^{-1/2}$$

(25)

where

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\tau^2} d\tau$$

(26)

is the complementary error function. For large $x$, it has the asymptotic expansion

$$\text{erfc}(x) = \frac{e^{-x}}{\sqrt{\pi}} \left( x^{-1/2} - \frac{1}{2} x^{-3/2} + \frac{3}{4} x^{-5/2} + \cdots \right)$$

(27)

In equation (25), the last term is not welcome since it leads to a fault exponential decay of magnetization. It can be killed by choosing the renormalization multiplier as

$$\Lambda = \frac{A\sqrt{t_h}}{\sqrt{t_{\beta}} - \sqrt{t_h}} \frac{\beta_0 m_0^2}{\beta \mu}$$

(28)

One can show that $\Lambda = 1$ in the limit $z_0 \to 0$. The final result is

$$m(t) = \sqrt{\mu} \left\{ 1 + \frac{At_h}{t_{\beta} - t_h} \sqrt{\frac{t_{\beta}}{t_h}} e^{t/t_h} \text{erfc}\left(\sqrt{\frac{t}{t_h}}\right) \right\}$$

$$- \left[ 1 + \frac{At_h}{t_{\beta} - t_h} \right] e^{t/t_{\beta}} \text{erfc}\left(\sqrt{\frac{t}{t_{\beta}}}\right)$$

$$+ \left[ \frac{\beta \mu}{\beta_0 m_0^2} \Lambda - \frac{A\sqrt{t_h}}{\sqrt{t_{\beta}} - \sqrt{t_h}} \right] e^{t/t_h} \right\}^{-1/2} \right\}^{-1/2}$$

(29)

The case of $T > T_c$ can be attained by similar method

$$m(t) = \sqrt{|\mu|} \left\{ -1 + \frac{At_h}{t_{\beta} - t_h} \sqrt{\frac{t_{\beta}}{t_h}} e^{t/t_h} \text{erfc}\left(\sqrt{\frac{t}{t_h}}\right) \right\}$$

$$- \left[ 1 + \frac{At_h}{t_{\beta} - t_h} \right] e^{t/t_{\beta}} \left( 2 - \text{erfc}\left(\sqrt{\frac{t}{t_{\beta}}}\right) \right) \right\}^{-1/2} \right\}^{-1/2}$$

(30)
One can attain the critical evolution of magnetization from the Laplace reversion of (24),

\[ m(t) = \left[ \frac{B^2 t_h}{\pi} \right]^{-1/4} \left[ 2 \left( \frac{t}{t_h} \right)^{1/2} + A \left( \frac{t}{t_h} \right)^{1/2} g \left( \frac{t}{t_h} \right) \right]^{-1/2} \]  

(31)

where

\[ g(x) = \int_{0}^{\infty} dy \frac{e^{-y}}{(1 + xy)^{1/2}} \]  

(32)

It is clear that \( g(0) = 1 \). The critical behavior also can be attained by taking the limitation \( T \to T_c \) in equation (29).

To recover the results of zero initial correlation of \([9]\), one only need to take the limitation of \( \beta_0 \to \infty \), that is \( z_0 \to \infty \) (or \( t_h \to 0 \)), in \((29), (30)\) and \((31)\), respectively. In this limit, \( A = \frac{2\pi}{t_h} \) with \( t_i \) defined in \([9]\) and \( \beta_\beta = \pi t_\mu \).

5. Discussions and Conclusions

In summary, we have studied the transient behavior of KSM that is quenched from an arbitrary temperature into another. The formula can describe ordering/disordering phenomena and the critical dynamics. We find that the correct long-time behavior can be only recovered after the subtraction of mass as well as the renormalization of the dynamic equation.

From the exact magnetization obtained in the present paper, one can find out the flows of renormalizational group of the bare parameters \( \beta, z_0 \) and \( m_0 (or \beta_0) \) under the scale transformation. Changing the time scale by a factor \( b^z \), in order to keep the macroscopic quantity \( m(t) \) unchange up to a scaling factor, \( t_h \) and \( t_\beta \) must be transformed in the same way,

\[ t_h(b) = b^z t_h(1), t_\beta(b) = b^z t_\beta(1) \]  

(33)

while \( A \) must be an invariant,

\[ A(b) = A(1) \]  

(34)

The relations of bare parameters and the scaling factor \( b \) are implicitly defined in the above equations. These are the so-called characteristic functions for the dynamic crossover phenomena \([8, 9, 16, 19]\).
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