A locally calculable $P^3$-pressure in a decoupled method for incompressible Stokes equations

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Abstract

This paper will suggest a new finite element method to find a $P^4$-velocity and a $P^3$-pressure solving incompressible Stokes equations at low cost. The method solves first the decoupled equation for a $P^4$-velocity. Then, using the calculated velocity, a locally calculable $P^3$-pressure will be defined component-wisely. The resulting $P^3$-pressure is analyzed to have the optimal order of convergence.

Since the pressure is calculated by local computation only, the chief time cost of the new method is on solving the decoupled equation for the $P^4$-velocity. Besides, the method overcomes the problem of singular vertices or corners.

1 Introduction

High order finite element methods for incompressible Stokes equations have been developed well in 2 dimensional domain and analyzed along with the inf-sup condition [1, 2, 6, 8, 11]. They, however, endure their large degrees of freedom and have to avoid singular vertices or corners.

In the Scott-Vogelius finite element method, the inf-sup condition fails if the mesh has an exact singular vertex. Even on nearly singular vertices, the pressure solution is easy to be spoiled. Recently, to fix the problem, we have found a cause of singular vertex and devised a new error analysis based on a so called sting function. As a result, the ruined pressure can be restored by simple post-process [9].

In this paper, employing the previous new error analysis, we will suggest a new finite element method to find a $P^4$-velocity and a $P^3$-pressure solving incompressible Stokes equations at low cost.

The method will solve first the decoupled equation for a divergence-free $P^4$-velocity which is almost same as the one from the Falk-Neilan finite element method except corners [6]. Then, utilizing the calculated velocity and orthogonal decomposition of $P^3$, the 5 locally calculable components of a $P^3$-pressure will be defined by exploring locally calculable components in the Falk-Neilan and Scott-Vogelius finite element spaces [6, 8, 11]. The resulting $P^3$-pressure is analyzed to have the optimal order of convergence.

Since the $P^3$-pressure is calculated by local computation only, the chief time cost of the new method is on solving the decoupled equation for the $P^4$-velocity. If the pressure has a region of interest in $\Omega$, the regional computation is enough for it. Besides, the method overcomes the problem arising from the singular vertices or corners by using the jump of the a priori calculated pressure components.

In the overall paper, the characteristics of sting functions depicted in Figure 1-(a) play key roles as in the previous work in [9]. Since the sting function exists in $P^k$ for every integer $k \geq 0$, the results for $P^4 - P^3$ in this paper are easily extended for $P^{k+1} - P^k$, $k \geq 4$ [10].

The paper is organized as follows. In the next section, the detail on finding a $P^4$-velocity will be offered. We will introduce an orthogonal decomposition of the space of $P^3$-pressures, based on the orthogonality of sting and non-sting functions in Section 3. Then, the most sections will be devoted to defining the non-sting component for each triangle in Section 4 and the sting component for each

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vertex classified by regular vertices, nearly singular ordinary vertices and dead corners in Section 5.

After the piecewise constant component is done in Section 5, the final \( P^3 \)-pressure will be defined by summing up all the components in Section 6. In the last two sections, a summary of the method and numerical tests will be given.

Throughout the paper, for a set \( S \subset \mathbb{R}^2 \), standard notations for Sobolev spaces are employed and \( L^2(S) \) is the space of all \( f \in L^2(S) \) whose integrals over \( S \) vanish. We will use \( \| \cdot \|_{m,S}, | \cdot |_{m,S} \) and \( (\cdot,\cdot)_S \) for the norm, seminorm for \( H^m(S) \) and \( L^2(S) \) inner product, respectively. If \( S = \Omega \), it may be omitted in the subscript. Denoting by \( P^k \), the space of all polynomials of degree less than or equals \( k \), \( f|_S \in P^k \) will mean that \( f \in L^2(\Omega) \) coincides with a polynomial in \( P^k \) on \( S \).

2 Velocity from the decoupled equation

Let \( \Omega \) be a simply connected polygonal domain in \( \mathbb{R}^2 \). In this paper, we will approximate a pair of velocity and pressure \( (u,p) \in [H^1_0(\Omega)]^2 \times L^2_0(\Omega) \) which satisfies an incompressible Stokes equation:

\[
(\nabla u, \nabla v) + (p, \text{div} v) + (q, \text{div} u) = (f, v) \quad \text{for all } (v, q) \in [H^1_0(\Omega)]^2 \times L^2_0(\Omega), \tag{1}
\]

for a body force \( f \in [L^2(\Omega)]^2 \).

Given a family of shape-regular triangulations \( \{T_h\}_{h>0} \) of \( \Omega \), define \( P_h^k(\Omega) \) as the following space of piecewise polynomials:

\[
P_h^k(\Omega) = \{v_h \in L^2(\Omega) : v_h|_K \in P^k \text{ for all triangles } K \in T_h\}, \quad k \geq 0.
\]

Let \( \Sigma_{h,0} \) be a space of \( C^1 \)-Argyris \( P^5 \) triangle elements \([4, 5]\) such that

\[
\Sigma_{h,0} = P_h^5(\Omega) \cap H^2_0(\Omega), \tag{2}
\]

where

\[
H^2_0(\Omega) = \{\phi \in H^2(\Omega) : \phi, \phi_x, \phi_y \in H^1_0(\Omega)\}.
\]

The degrees of freedom of \( \phi \in \Sigma_{h,0} \) are \( \phi_{xx}, \phi_{xy}, \phi_{yy}, \phi_x, \phi_y, \phi \) at interior vertices, \( \phi_{nn} \) at midpoints of interior edges and \( \phi_{mn} \) at non-corner boundary vertices, where \( \nu, n \) are unit vectors normal to edges, \( \partial \Omega \), respectively.

Define a divergence-free space \( Z_{h,0} \) as

\[
Z_{h,0} = \{(\phi_{hx}, -\phi_{hx}) : \phi_h \in \Sigma_{h,0}\} \subset [P_h^1(\Omega) \cap H^1_0(\Omega)]^2. \tag{3}
\]

We note that

\[
\dim \Sigma_{h,0} = \dim Z_{h,0} = 6\nu_{\text{in}} + E_{\text{in}} + \nu_{\text{bdy}} - \nu_{\text{cnr}},
\]

where \( \nu_{\text{in}}, E_{\text{in}}, \nu_{\text{bdy}}, \) and \( \nu_{\text{cnr}} \) are the numbers of interior vertices, interior edges, boundary vertices and corners, respectively \([6]\).

Then, we can solve \( u_h \in Z_{h,0} \) satisfying the following decoupled equation:

\[
(\nabla u_h, \nabla v_h) = (f, v_h) \quad \text{for all } v_h \in Z_{h,0}. \tag{4}
\]

**Theorem 2.1.** Let \( (u, p) \in [H^1_0(\Omega)]^2 \times L^2_0(\Omega) \) and \( u_h \in Z_{h,0} \) satisfy \([1], [4]\), respectively. Then we estimate

\[
|u - u_h|_1 \leq Ch^4|u|_5, \tag{5}
\]

if \( u \in [H^5(\Omega)]^2 \), where \( C \) is a constant independent of \( h \).

**Proof.** Since \( (u, p) \in [H^1_0(\Omega)]^2 \times L^2_0(\Omega) \) satisfies \([1]\), we have \( \text{div} u = 0 \). Thus, there exists a stream function \( \phi \in H^2_0(\Omega) \) such that

\[
\phi = (\phi_y, -\phi_x). \tag{6}
\]
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Let \( \Pi_h \phi \in \Sigma_{h,0} \) be a projection of \( \phi \) such that the Hessians, gradients and values of \( \phi - \Pi_h \phi \) at vertices and its normal derivatives at midpoints of edges vanish. Then, if \( u \in [H^5(\Omega)]^2 \), by Bramble-Hilbert lemma, we have

\[
|\phi - \Pi_h \phi|_2 \leq Ch^4|\phi|_6. \tag{7}
\]

If we denote \( \Pi_h u = ((\Pi_h \phi)_y) - ((\Pi_h \phi)_x) \in Z_{h,0} \), then from (6) and (7), we estimate

\[
|u - \Pi_h u|_1 \leq Ch^4|u|_5. \tag{8}
\]

We note that \((p, \text{div} \, v_h) = 0\) for all \(v_h \in Z_{h,0}\). Thus, from (1) and (4), we deduce

\[
(\nabla u - \nabla u_h, \nabla v_h) = 0 \quad \text{for all } v_h \in Z_{h,0}. \tag{9}
\]

Then we can establish (5) from (8) and (9) with \( v_h = \Pi_h u - u_h \in Z_{h,0} \).

3 Orthogonal decomposition of \( P^3\)-pressures

For a triangle \( K \in \mathcal{T}_h \) and an integer \( k \geq 0 \), define

\[
P^k(K) = \{ q \in L^2(\Omega) : q|_K \in P^k, \; q = 0 \; \text{on} \; \Omega \setminus K \}.
\]

In the remaining of the paper, we will use the following notations:

- \( C \): a generic constant which does not depend on \( h \) of \( \mathcal{T}_h \),
- \( K(V) \): the union of all triangles in \( \mathcal{T}_h \) sharing a vertex \( V \),
- \( \xi^\perp \): the counterclockwise 90° rotation of a vector \( \xi \),
- \(|S|\): the area or length of a set \( S \),
- \( m(f)\): the average of a function \( f \) over \( \Omega \).

We assume the following on \( \mathcal{T}_h \) to exclude pathological meshes.

**Assumption 3.1.** No triangle in \( \mathcal{T}_h \) has two corner points of \( \partial \Omega \).

3.1 sting function

Let \( V \) be a vertex of a triangle \( K \). Then there exists a unique function \( s_{V,K} \in P^3(K) \) satisfying the following quadrature rule:

\[
\int_K s_{V,K} \, q \, dxdy = \frac{|K|}{100} q(V) \quad \text{for all } q \in P^3, \tag{10}
\]

since the both sides of (10) are linear functionals on \( P^3 \). If \( \hat{K} \) is a reference triangle with vertices \((0,0),(1,0),(0,1)\) and \( \hat{V} = (0,1) \), we have

\[
s_{V,K}(x,y) = \frac{28}{5} y^4 - \frac{63}{10} y^2 + \frac{9}{5} y - \frac{1}{10}, \tag{11}
\]

as depicted in Figure 1(a). Given a vertex \( V \) of \( K \), we note that

\[
s_{V,K} = s_{\hat{V},\hat{K}} \circ F^{-1} \quad \text{for an affine transformation } F : \hat{K} \rightarrow K. \tag{12}
\]
Thus, the values of $s_{\mathbf{V}K}$ are inherited from those of $s_{\mathbf{V}h}$ as

$$s_{\mathbf{V}K}(\mathbf{V}) = 1, \quad s_{\mathbf{V}K} \big|_E = -\frac{1}{10} \text{ on the opposite edge } E \text{ of } \mathbf{V}. \quad (13)$$

If $q = \alpha s_{\mathbf{V}K}$ for a scalar $\alpha$, we call it a sting function of $\mathbf{V}$ on $K$, named after the shape of its graph as in Figure 1(a).

For a triangle $K$, define a subspace of $P^3(K)$ as

$$S(K) = \langle s_{\mathbf{V}1K}, s_{\mathbf{V}2K}, s_{\mathbf{V}3K} \rangle, \quad (14)$$

where $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3$ are 3 vertices of $K$. From (13), it is easy to prove that

$$\dim S(K) = 3. \quad (15)$$

![Figure 1: examples of sting and non-sting functions](image)

(a) a sting function $s_{\mathbf{V}K}$ of $\mathbf{V}$ on $K$

(b) a non-sting function $\mathbf{n}$ on $K$

### 3.2 non-sting function

For a triangle $K$, let

$$B(K) = \{ \mathbf{v} \in [P^4(K)]^2 : \mathbf{v} = \mathbf{0} \text{ on } \partial K \}, \quad (16)$$

and define a subspace of $P^3(K)$ as

$$N(K) = \{ \text{div } \mathbf{v} : \mathbf{v} \in B(K) \}. \quad (17)$$

If $q \in N(K)$, we will call it a non-sting function on $K$. By definition in (16), (17), every non-sting function $q \in N(K)$ has the following properties:

$$q(\mathbf{V}) = 0 \text{ for every vertex } \mathbf{V} \text{ of } K, \quad \int_K q \, dxdy = 0. \quad (18)$$

An example of its graph is depicted in Figure 1(b).

Then, the following orthogonality is clear from (14), (18) and the quadrature rule in (10),

$$N(K) \perp S(K). \quad (19)$$

The fact $\dim B(K) = 6$ induces the following, with an aid of Lemma 3.1 below,

$$\dim N(K) = 6. \quad (20)$$
Lemma 3.1. If \( v \in B(K) \) and \( \text{div} \ v = 0 \), then \( v = 0 \).

Proof. Since \( \text{div} \ v = 0 \) on \( K \) and \( v = 0 \) on \( \partial K \), there exists \( \phi \in P^5 \) such that
\[
(\phi_y, -\phi_x) = v \quad \text{on} \quad K, \quad \phi = 0 \quad \text{on} \quad \partial K.
\]

Let \( \ell_1, \ell_2, \ell_3 \) be 3 infinite lines containing 3 line segments of \( \partial K \), respectively. Then \( \phi, \nabla \phi \) vanish on \( \ell_1, \ell_2, \ell_3 \). It implies that \( \phi \) vanishes on any line which passes 3 points in \( \ell_1 \cup \ell_2 \cup \ell_3 \). Thus we have \( \phi = 0 \) and \( v = 0 \) on \( K \).

The above lemma tells that \( \| \text{div} \ v \|_{0,K} \) is a norm of \( v \in B(K) \). Furthermore, we can show that \( |v|_{1,K} \leq C \| \text{div} \ v \|_{0,K} \) for all \( v \in B(K) \). \( \Box \)

3.3 orthogonal decomposition

Let \( 1^K \in P^3(K) \) be a constant function of value 1 on \( K \). Then, we can decompose \( P^3(K) \) as in the following lemma. We will notate \( \bigoplus A \) for \( A \perp B \).

Lemma 3.2.

\[
P^3(K) = \mathcal{N}(K) \bigoplus \left( \mathcal{S}(K) \bigoplus <1^K> \right).
\] \hspace{1cm} (22)

Proof. From (21), we have \( <1^K> \perp \mathcal{N}(K) \). Thus, by (15), (19), (20), it is enough to prove (22) that \( 1^K \notin \mathcal{S}(K) \).

For the vertices \( V_1, V_2, V_3 \) of \( K \), let
\[
s_1^K = \frac{5}{4}(s_{V_1,K} + s_{V_2,K} + s_{V_3,K}) \in \mathcal{S}(K).
\]

Then, by (13), \( 1^K - s_1^K \) vanishes at all vertices of \( K \). It means, from (10), that
\[
\int_K q(1^K - s_1^K) \; dx\,dy = 0 \quad \text{for all} \quad q \in \mathcal{S}(K). \hspace{1cm} (23)
\]

Assume \( 1^K \in \mathcal{S}(K) \). Then, \( 1^K - s_1^K \in \mathcal{S}(K) \) and \( 1^K - s_1^K = 0 \) from (23). It contradicts to
\[
\int_K s_1^K \; dx\,dy = \frac{5}{4} \int_K s_{V_1,K} + s_{V_2,K} + s_{V_3,K} \; dx\,dy = \frac{3}{80} |K| \neq |K| = \int_K 1^K \; dx\,dy. \hspace{1cm} \Box
\]

Let’s define the following subspaces of \( P_h^3(\Omega) = \bigoplus_{K \in T_h} P^3(K) \):
\[
\mathcal{N}_h = \bigoplus_{K \in T_h} \mathcal{N}(K), \quad \mathcal{S}_h = \bigoplus_{K \in T_h} \mathcal{S}(K), \quad \mathcal{C}_h = \bigoplus_{K \in T_h} <1^K>. \hspace{1cm} (24)
\]

Then by Lemma 3.2 we have
\[
P_h^3(\Omega) = \mathcal{N}_h \bigoplus \left( \mathcal{S}_h \bigoplus \mathcal{C}_h \right). \hspace{1cm} (25)
\]
3.4 decomposition of $\Pi_h p$

For $(u, p)$ satisfying (1), let $\Pi_h p \in \mathcal{P}_h^3(\Omega)$ be a Hermite interpolation of $p$ such that

$$\nabla \Pi_h p(V) = \nabla p(V), \quad \Pi_h p(V) = p(V), \quad \Pi_h p(G) = p(G),$$

(26)

at all vertices $V$ and gravity centers $G$ of triangles in $\mathcal{T}_h$. Then, if $p \in H^4(\Omega)$, we have

$$\|p - \Pi_h p\|_0 \leq C h^4 |p|_4.$$

(27)

By [25], we can decompose $\Pi_h p$ into

$$\Pi_h p = \Pi_h p^N + \Pi_h p^S + \Pi_h p^C,$$

(28)

for

$$\Pi_h p^N \in \mathcal{N}_h, \quad \Pi_h p^S \in \mathcal{S}_h, \quad \Pi_h p^C \in \mathcal{C}_h,$$

(29)

called the non-sting, sting and piecewise constant components of $\Pi_h p$, respectively. We will approximate them component-wisely, exploiting the following equation for $\Pi_h p$ from [1]:

$$(\Pi_h p, \nabla v) = (f, v) - (\nabla u, \nabla v) - (p - \Pi_h p, \nabla v) \quad \text{for all } v \in [H^1_0(\Omega)]^2.$$

(30)

4 Non-sting component for a triangle

Fix a triangle $K \in \mathcal{T}_h$ and define an operator $L : \mathcal{N}(K) \rightarrow \mathcal{B}(K)'$ so that, if $q \in \mathcal{N}(K),$

$$Lq(v_h) = (q, \nabla v_h) \text{ for all } v_h \in \mathcal{B}(K).$$

Then, $L$ is an isomorphism from the definition of $\mathcal{N}(K)$ in [17] and

$$\dim \mathcal{N}(K) = \dim \mathcal{B}(K).$$

Thus, for each triangle $K \in \mathcal{T}_h$, there exists a unique $p^K_h \in \mathcal{N}(K)$ such that

$$(p^K_h, \nabla v_h) = (f, v_h) - (\nabla u, \nabla v_h) \quad \text{for all } v_h \in \mathcal{B}(K).$$

(31)

Lemma 4.1. Define

$$p^K_h = \sum_{K \in \mathcal{T}_h} p^K_h.$$

Then for $\Pi_h p^N$ in [28], we estimate

$$\|\Pi_h p^N - p^K_h\|_{0,K} \leq C(|u - u_h|_{1,K} + \|p - \Pi_h p\|_{0,K}) \quad \text{for each } K \in \mathcal{T}_h.$$

(32)

Proof. For each triangle $K \in \mathcal{T}_h$, we note that $\Pi_h p^N$ in [28] satisfies that

$$(\Pi_h p^N, \nabla v_h) = (f, v_h) - (\nabla u, \nabla v_h) - (p - \Pi_h p, \nabla v_h) \quad \text{for all } v_h \in \mathcal{B}(K),$$

(33)

from (30) and orthogonality in Lemma 3.2.

Denote $e^K_h = \Pi_h p^N|_K - p^K_h|_K$. Then from (31) and (33), we have

$$(e^K_h, \nabla v_h) = -(\nabla u - \nabla u_h, \nabla v_h) - (p - \Pi_h p, \nabla v_h) \quad \text{for all } v_h \in \mathcal{B}(K).$$

(34)

Since $e^K_h \in \mathcal{N}(K)$, the estimation (32) comes from (21), (34) and the definition of $\mathcal{N}(K)$ in (17).
5 Clustering sting functions by vertex

5.1 regular and nearly singular vertices

A vertex $V$ is called exactly singular if the union of all edges sharing $V$ is contained in the union of two infinite lines. To be precise, let $K_1, K_2, \cdots, K_J$ be all triangles sharing $V$ and denote by $\theta(K_j)$, the angle of $K_j$ at $V$, $j = 1, 2, \cdots, J$. Define

$$\Upsilon(V) = \{\theta(K_i) + \theta(K_j) : K_i \cap K_j \text{ is an edge, } i, j = 1, 2, \cdots, J\}.$$  

Then $V$ is called exactly singular if and only if $\Upsilon(V) = \{\pi\}$ or $\emptyset$.

For the quantitative definition of nearly singularity, fix $\vartheta$ such that $0 < \vartheta \leq \inf\{\theta : \theta \text{ is an angle of a triangle } K \in T_h, h > 0\}$. Then call a vertex $V$ to be nearly singular if

$$|\Theta - \pi| < \vartheta \text{ for all } \Theta \in \Upsilon(V),$$

otherwise regular. We note that nearly singular vertices are isolated from each others in the sense of the following lemma [9].

**Lemma 5.1.** There is no interior edge connecting two nearly singular vertices.

5.2 clustering sting functions by vertex

For each vertex $V$, let $S(V)$ be the space of all sting functions of $V$, that is,

$$S(V) = \langle s_{V K_1}, s_{V K_2}, \cdots, s_{V K_J} \rangle,$$

where $K_1, K_2, \cdots, K_J$ are all triangles in $T_h$ sharing $V$. An example of a function in $S(V)$ is represented in Figure 2. The support of a function in $S(V)$ belongs to $K(V)$.

![Figure 2: the support of $q^V_h = \alpha_1 s_{V K_1} + \alpha_2 s_{V K_2} + \cdots + \alpha_5 s_{V K_5} \in S(V)$](image)

Then we note that

$$S_h = \bigoplus_{K \in T_h} S(K) = \bigoplus_{V \text{ vertex}} S(V).$$

Thus the sting functions forming $\Pi_h p^S$ of $\Pi_h p$ in (28), (29) can be clustered by vertex so that

$$\Pi_h p^S = \sum_{V \text{ vertex}} \Pi_h p^V \text{ for } \Pi_h p^V \in S(V).$$

(37)
If a vertex $V$ does not meet any interior edge as in Figure 6, $V$ is called dead, otherwise, ordinary. Then, all vertices are classified into 3 classes: regular vertices, nearly singular ordinary vertices and dead corners as in Figure 4, 5, 6, respectively.

In the next 3 sections, we will define the sting component $p_h^V \in S(V)$ for each vertex $V$ in order of those 3 classes to approximate $\Pi_h p^V$ in (37). The local functions in the following subsection will play roles of test functions on defining $p_h^V$.

### 5.3 test functions on two adjacent triangles

Let $K_1, K_2$ be two adjacent triangles sharing an edge and a vertex $V$ as in Figure 4. Denote other 3 vertices and a unit tangent vector by $W_0, W_1, W_2, \tau$ so that

$$\nabla W_0 = K_1 \cap K_2, \quad \tau = \frac{\nabla W_0}{|\nabla W_0|}, \quad W_j \in K_j \setminus \{V, W_0\}, j = 1, 2.$$ 

![Figure 3: The union of two adjacent triangles sharing V](image)

Then, there exists a function $w \in P_h^0(\Omega) \cap H^1_0(\Omega)$ such that \[9\]

$$\frac{\partial w}{\partial \tau}(V) = 1, \quad \frac{\partial w}{\partial \tau}(W_0) = 0, \quad \int_{\nabla W_0} w \, d\ell = 0, \quad \text{the support of } w \text{ is } K_1 \cup K_2. \quad (38)$$

Assuming $K_1, K_2$ are counterclockwisely numbered with respect to $V$, by simple calculation, we have

$$\nabla w|_{K_1}(V) = \frac{|\nabla W_0|}{2|K_1|} \nabla W_1 \cdot \xi, \quad \nabla w|_{K_2}(V) = -\frac{|\nabla W_0|}{2|K_2|} \nabla W_2 \cdot \xi. \quad (39)$$

For a vector $\xi = (\xi_1, \xi_2)$, denote $w_h^\xi = w_1 = (\xi_1 w, \xi_2 w)$. Then from (38) and (39), $\nabla w_h^\xi$ vanishes at all vertices in $T_h$ except

$$\text{div } w_h^\xi|_{K_1}(V) = \frac{|\nabla W_0|}{2|K_1|} \nabla W_1 \cdot \xi, \quad \text{div } w_h^\xi|_{K_2}(V) = -\frac{|\nabla W_0|}{2|K_2|} \nabla W_2 \cdot \xi. \quad (40)$$

Let $q_h \in S(K_1) \bigoplus S(K_2)$ be a sting function on $K_1 \cup K_2$. It is represented with some constants $a, b, j = 1, 2, 3$ as

$$q_h = \alpha_1 \delta V_{K_1} + \alpha_2 \delta W_{0,K_1} + \alpha_3 \delta W_{1,K_1} + \beta_1 \delta V_{K_2} + \beta_2 \delta W_{0,K_2} + \beta_3 \delta W_{1,K_2}.$$ 

Then, by (40) and quadrature rule of sting functions in (10), we have that

$$(q_h, \text{div } w_h^\tau) = \alpha_1 \text{div } w_h^\tau|_{K_1}(V) \left|\frac{K_1}{100}\right| + \beta_1 \text{div } w_h^\tau|_{K_2}(V) \left|\frac{K_2}{100}\right| = \frac{1}{200} \left(\alpha_1 \nabla W_1 \cdot \xi - \beta_1 \nabla W_2 \cdot \xi\right) \cdot \nabla W_0,$$

$$(q_h, \text{div } w_h^{\tau^+}) = \alpha_1 \text{div } w_h^{\tau^+}|_{K_1}(V) \left|\frac{K_1}{100}\right| + \beta_1 \text{div } w_h^{\tau^+}|_{K_2}(V) \left|\frac{K_2}{100}\right| = \frac{1}{200} \left(\alpha_1 \nabla W_1 \cdot \xi - \beta_1 \nabla W_2 \cdot \xi\right) \cdot \nabla W_0.$$
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It can be written in simpler form:

\[
(q_h, \text{div } \mathbf{w}_h^\tau) = \ell_1 \ell_0 \sin \theta_1 \alpha_1 + \frac{\ell_0 \ell_2 \sin \theta_2}{200} \beta_1 = \frac{1}{100} (|K_1| \alpha_1 + |K_2| \beta_1),
\]

\[
(q_h, \text{div } \mathbf{w}_h^{\tau^\perp}) = \ell_1 \ell_0 \cos \theta_1 \alpha_1 - \frac{\ell_0 \ell_2 \cos \theta_2}{200} \beta_1,
\]

where \(\theta_j\) is the angle of \(K_j\) at \(V\), \(j = 1, 2\) and \(\ell_j = |VV_j|, j = 0, 1, 2\) as in Figure 3.

6 \(p_h^V\) for a regular vertex \(V\)

Let’s fix a vertex \(V\) and \(K_1, K_2, \cdots, K_J\) be all triangles in \(T_h\) sharing \(V\), counterclockwisely numbered as in Figure 4, 5. Denote by \(J\), the number of interior edges which meet \(V\), that is,

\[
J = \begin{cases} J, & \text{if } V \text{ is an interior vertex}, \\ J - 1, & \text{if } V \text{ is a boundary vertex}. \end{cases}
\]

We will use the indices modulo \(J\), if \(V\) is an interior vertex. Then, for \(j = 1, 2, \cdots, J\), let \(V_j\) and \(\tau_j\) be a vertex and a unit vector, respectively, such that

\[
\nabla V_j = K_j \cap K_{j+1}, \quad \tau_j = \frac{\nabla V_j}{|\nabla V_j|}.
\]

In case of boundary vertex \(V\) as in Figure 4-(b) and 5-(a),(b), denote by \(V_0, V_J\), the vertices such that

\[
V_0 \in K_1 \setminus \{V, V_1\}, \quad V_J \in K_J \setminus \{V, V_{J-1}\}.
\]

Figure 4: examples of regular vertices \(V\) (dashed lines belong to \(\partial \Omega\))

6.1 least square solution of a system by test functions

For each \(j = 1, 2, \cdots, J\), similarly to (38), there exists a function \(w_j \in P_h^4(\Omega) \cap H_0^1(\Omega)\) such that

\[
\frac{\partial w_j}{\partial \tau_j}(V) = 1, \quad \frac{\partial w_j}{\partial \tau_j}(V_j) = 0, \quad \int_{\nabla V_j} w_j \, d\ell = 0, \quad \text{the support of } w_j \text{ is } K_j \cup K_{j+1}.
\]
For the uniqueness of $w_j$, we add the following conditions:

$$w_j(G) = 0, \nabla w_j(G) = 0$$

at each gravity center $G$ of $K_j, K_{j+1}$.

Then we have $2J$ test functions in $[P^1_h(\Omega) \cap H_0^1(\Omega)]^2$ such that

$$w_h^+ = w_j \tau_j, \quad w_h^- = w_j \tau_j^+, \quad j = 1, 2, \ldots, J.$$

Consider the following system of $2J$ equations for unknown $q_h^V \in S(V)$:

$$(q_h^V, \text{div } w_h^+ ) = a_j, \quad (q_h^V, \text{div } w_h^- ) = b_j,$$

for given scalars $a_j, b_j, j = 1, 2, \ldots, J$. Since $q_h^V \in S(V)$ can be represented for $J$ unknown constants $\alpha_1, \alpha_2, \ldots, \alpha_J$ as

$$q_h^V = \alpha_1 s_{VK_1} + \alpha_2 s_{VK_2} + \cdots + \alpha_J s_{VK_J},$$

the system (48) is of $2J$ equations for $J$ unknowns $\alpha_1, \alpha_2, \ldots, \alpha_J$ in (49).

**Lemma 6.1.** Let $q_h^V \in S(V)$ be the least square solution of the system (48). Then, if $V$ is a regular vertex, we have

$$||q_h^V||_0 \leq C|K(V)|^{-1/2} \sum_{j=1}^{J} (|a_j| + |b_j|).$$

*Proof.* Let $\theta_j$ be the angle of $K_j$ at $V$, $j = 1, 2, \ldots, J$ and $\ell_j = |VV_j|, j = 0, 1, \ldots, J$ as in Figure 4 Then from (41), (45), (47) and (49), we can rewrite (48) for $j = 1, 2, \ldots, J$ as

$$\frac{\ell_{j-1}\ell_j \sin \theta_j}{200} \alpha_j + \frac{\ell_j \ell_{j+1} \sin \theta_{j+1}}{200} \alpha_{j+1} = a_j, \quad \frac{\ell_{j-1}\ell_j \cos \theta_j}{200} \alpha_j - \frac{\ell_j \ell_{j+1} \cos \theta_{j+1}}{200} \alpha_{j+1} = b_j.$$

Denote

$$\beta_j = \frac{\ell_{j-1}\ell_j \alpha_j}{200} \quad \text{for } j = 1, 2, \ldots, J.$$  

Then for $j = 1, 2, \ldots, J$, we simplify (50) into

$$\sin \theta_j \beta_j + \sin \theta_{j+1} \beta_{j+1} = a_j, \quad (52a) \quad \cos \theta_j \beta_j - \cos \theta_{j+1} \beta_{j+1} = b_j. \quad (52b)$$

If $V$ is a regular vertex, then by (35), we can assume without loss of generality,

$$0 < C \leq |\sin(\theta_1 + \theta_2)|,$$

which tells the bound of the determinant of two equations (52a), (52b) for $j = 1$.

Consider the following subsystem of (52) consisting of $J$ equations:

$$\sin \theta_1 x_1 + \sin \theta_2 x_2 = a_1, \quad (54a) \quad \cos \theta_1 x_1 - \cos \theta_2 x_2 = b_1, \quad (54b)$$

$$\sin \theta_j x_j + \sin \theta_{j+1} x_{j+1} = a_j, \quad j = 2, 3, \ldots, J - 1. \quad (54c)$$

We can solve first (54a), (54b) to get $x_1, x_2$. Then, solve (54c) consecutively to obtain

$$x_{j+1} = (\sin \theta_{j+1})^{-1}(a_j - \sin \theta_j x_j), \quad j = 2, 3, \ldots, J - 1.$$
From (53) and $0 < C \leq \sin \theta_j, j = 1, 2, \cdots, J$ by shape-regularity, we have
\[ \|(x_1, x_2, \cdots, x_J)\|_2 \leq C\|(b_1, a_1, a_2, \cdots, a_{J-1})\|_2. \]  
(55)

Let $A \in \mathbb{R}^{J \times J}$ be the matrix for the system (54). Then from (55), we deduce that
\[ \|A^{-1}y\|_2 \leq C\|y\|_2 \quad \text{for all } y \in \mathbb{R}^J. \]  
(56)

Thus, for the smallest singular value $s$ of $A$, we can estimate the following from (56):
\[ s = \min_{x \in \mathbb{R}^J \setminus \{0\}} \frac{\|Ax\|_2}{\|x\|_2} = \min_{y \in \mathbb{R}^J \setminus \{0\}} \frac{\|y\|_2}{\|A^{-1}y\|_2} \geq \frac{1}{C}. \]  
(57)

Since the smallest singular value of a matrix is greater than that of its submatrix, we can estimate
\[ \|((\beta_1, \beta_2, \cdots, \beta_J))\|_2 \leq \frac{1}{s} \|(a_1, a_2, \cdots, a_J, b_1, b_2, \cdots, b_J)\|_2, \]  
(58)

for the least square solution of (62). From (11) and (12), we note that
\[ \|s_{VJ}\|_0 \leq C|K_j|^{1/2}, \quad j = 1, 2, \cdots, J. \]  
(59)

Thus, combining (49), (51), (57)-(59), the proof is completed. \qed

### 6.2 Definition of $p_h^V$ for a regular vertex $V$

We test (30) with $w_{h}^{T_j}$, $w_{h}^{T_j}$ in (47). Then for $\Pi_h p^V$ in (37), we have
\[ (\Pi_h p^V, \text{div } w_{h}^{T_j}) = (f, w_{h}^{T_j}) - (\nabla u, \nabla w_{h}^{T_j}) - (\Pi_h p^N, \text{div } w_{h}^{T_j}) - (p - \Pi_h p, \text{div } w_{h}^{T_j}), \]  
(60)

for $j = 1, 2, \cdots, J$, since $\text{div } w_{h}^{T_j} \text{ div } w_{h}^{T_j}$ vanish at all vertices except $V$ and
\[ (1, \text{div } w_{h}^{T_j})_K = (1, \text{div } w_{h}^{T_j})_K = 0 \quad \text{for all } K \in T_h. \]

Reflecting (60), create the following system of $2J$ equations for unknown $p_h^V \in \mathcal{S}(V)$:
\[ (p_h^V, \text{div } w_{h}^{T_j}) = (f, w_{h}^{T_j}) - (\nabla u, \nabla w_{h}^{T_j}) - (p_h^N, \text{div } w_{h}^{T_j}), \]  
(61a)

\[ (p_h^V, \text{div } w_{h}^{T_j}) = (f, w_{h}^{T_j}) - (\nabla u, \nabla w_{h}^{T_j}) - (p_h^N, \text{div } w_{h}^{T_j}), \]  
(61b)

for $j = 1, 2, \cdots, J$, where $p_h^N$ is the non-sting component, already defined in Lemma 4.1.

#### Lemma 6.2. If $V$ is a regular vertex as in Figure 3, define $p_h^V \in \mathcal{S}(V)$ as the least square solution of (61). Then, for $\Pi_h p^V$ in (37), we estimate
\[ \|\Pi_h p^V - p_h^V\|_0 \leq C \left( \|u - u_h|_{1,K(V)} + \|p - \Pi_h p|_{0,K(V)} \right). \]  
(62)

**Proof.** If we denote the error by $e_h^V = \Pi_h p^V - p_h^V$, then from (60) and (61), $e_h^V \in \mathcal{S}(V)$ is the least square solution of the following system of $2J$ equations:
\[ (e_h^V, \text{div } w_{h}^{T_j}) = - (\nabla (u - u_h), \nabla w_{h}^{T_j}) - (\Pi_h p^N - p_h^N, \text{div } w_{h}^{T_j}) - (p - \Pi_h p, \text{div } w_{h}^{T_j}), \]  
(63)

for $j = 1, 2, \cdots, J$, since the least square solution of a system is the solution of its normal equation and $\Pi_h p^V$ can be regarded as the solution of the normal equation of (60).

By definition of $w_{h}^{T_j}$, $w_{h}^{T_j}$ in (45)-(47), we note that
\[ |w_{h}^{T_j}|_1, |w_{h}^{T_j}|_1 \leq C(|K_j| + |K_{j+1}|)^{1/2}, \quad j = 1, 2, \cdots, J. \]  
(64)

Thus (62) comes from (63), (64) and Lemma 4.1. \qed
7 $p_h^V$ for a nearly singular vertex $V$, not a dead corner

When a vertex $V$ is exactly singular, the system (61) is underdetermined, since the determinant of $(52a)$, $(52b)$ for each $j = 1, 2, \cdots, J$, makes

\[-\sin(\theta_j + \theta_{j+1}) = -\sin \pi = 0.\]

Although $V$ is not exactly singular, if it is nearly singular, the error $\Pi_h p^V - p^V$ in Lemma 6.2 might be large from the tiny smallest singular value of (63).

To overcome the problem on nearly singular vertices, we will replace the equations in (61b) with new ones utilizing jumps of $p_N^V$ and $p^V$ for regular vertices $V$ defined in Lemma 4.1 and 6.2, respectively.

7.1 jump of tangential derivative

For simple motivation, let's fix a boundary vertex $V$ which is not a corner point of $\partial \Omega$. If $V$ is nearly singular, it is exact singular and has only two triangles $K_1, K_2$ which share $V$ as in Figure 5-(a).

Then, since the system (61) is singular, in order to define $p^V \in S(V)$ approximating $\Pi_h p^V$, we have to create a new equation for $p^V$, reflecting some condition for $\Pi_h p^V$.

Define a jump of a function $q_h$ at $V$ across $K_1 \cap K_2$ as

\[ J_{12}(q_h) = |K_1 \cap K_2|^3 \left( \frac{\partial}{\partial \tau_1} (q_h |_{K_1}) (V) - \frac{\partial}{\partial \tau_1} (q_h |_{K_2}) (V) \right). \]  

(65)

Then, since $\Pi_h p$ is continuous on $K_1 \cap K_2$, we have $J_{12}(\Pi_h p) = 0$. It is written in

\[ J_{12}(\Pi_h p) = J_{12} (\Pi_h p^N + \Pi_h p^V + \Pi_h p^{V0} + \Pi_h p^{V1} + \Pi_h p^{V2}) = 0. \]  

(66)

We note that $s_{V_1 K_1}, s_{V_2 K_2}$ are constant on $K_1 \cap K_2$ from [12], [13]. It results in

\[ J_{12} (\Pi_h p^{V0}) = J_{12} (\Pi_h p^{V1}) = 0. \]  

(67)

Thus, from (66) and (67), $\Pi_h p^V$ satisfies

\[ J_{12} (\Pi_h p^V) = - J_{12} (\Pi_h p^N + \Pi_h p^{V1}). \]  

(68)

We note $V_1$ is a regular vertex by Lemma 5.1, since $V_1$ and $V$ are connected by an interior edge. It means that we have $p_h^{V1}$, already defined in Lemma 6.2.

Thus we can impose a following new condition for unknown $p_h^V$, which is similar to (68):
$$J_{12}(p^V_h) = -J_{12}(p^N_h + p^V_i).$$  

(69)

Then, replacing (61b) with (69), consider the following system for unknown $p^V_h \in \mathcal{S}(V)$:

$$(p^V_h, \text{div } w^{\tau_1} - (\nabla u_h, \nabla w^{\tau_1}) - (p^N_h, \text{div } w^{\tau_1}),$$

(70)

$$J_{12}(p^V_h) = -J_{12}(p^N_h + p^V_i).$$



**Lemma 7.1.** If $V$ is a nearly singular vertex on $\partial \Omega$, not a corner as in Figure 5-(a), define $p^V_h \in \mathcal{S}(V)$ as the solution of (70). Then, for $\Pi_h p^V$ in (37), we estimate

$$\|\Pi_h p^V - p^V_h\|_0 \leq C \left( |u - u_h|_{1,K(V_i)} + \|p - P_h p\|_{0,K(V_i)} \right),$$

(71)

where $V_1$ is a regular vertex which shares an interior edge with $V$.

**Proof.** Denote $e^V_h = \Pi_h p^V - p^V_h$. Then from (68), (70) and the same argument inducing (63), we have

$$e^V_h = (e^V_h, \text{div } w^{\tau_1}) = (\nabla(u - u_h), \nabla w^{\tau_1} - (p^N_h, \text{div } w^{\tau_1})), -J_{12}(e^V_h) = -J_{12}(p^N_h - p^V_i) - J_{12}\left(\Pi_h p^V - p^V_i\right).$$

(72)

We can represent $e^V_h \in \mathcal{S}(V)$ with 2 unknown constants $e_1, e_2$ as

$$e^V_h = e_1 \mathbf{v}_K + e_2 \mathbf{v}_{K_2}.$$  

(73)

Let $\theta_j$ be the angle of $K_j$ at $V$, $j = 1, 2$ and $\ell_j = |\nabla V_j|$, $j = 0, 1, 2$ as in Figure 5-(a). Then, from (11), (12), we calculate

$$J_{12}(e^V_h) = -6\ell_1^2 e_1 + 6\ell_2^2 e_2.$$  

(74)

Abbreviating the right hand sides in (72) by $a, b$, respectively, we can rewrite (72) by (41), (74) into

$$\frac{\ell_0 \ell_1 \sin \theta_1}{200} e_1 + \frac{\ell_1 \ell_2 \sin \theta_2}{200} e_2 = a,$$

$$-6\ell_1^2 e_1 + 6\ell_2^2 e_2 = b.$$  

(75)

From Lemma 4.1 and (63), we have

$$|a| \leq C \left( |K_1| + |K_2| \right)^{1/2} \left( |u - u_h|_{1,K_1 \cup K_2} + \|p - \Pi_h p\|_{0,K_1 \cup K_2} \right).$$  

(76)

For $q_h \in P_0^3(\Omega)$ in (65), we can estimate

$$\left| J_{12}(q_h) \right| \leq \ell_1^3 \left( \left\| \nabla q_h |_{K_1}(V) \right\|_2 + \left\| \nabla q_h |_{K_2}(V) \right\|_2 \right) \leq C \ell_1^3 \left( |K_1| + |K_2| \right)^{-1/2} |q_h|_{1,K_1 \cup K_2} \right)^{1/2}$$

(77)

$$\leq C \ell_1^2 |q_h|_{1,K_1 \cup K_2} \leq C \ell_1 \|q_h\|_{0,K_1 \cup K_2} \leq C \left( |K_1| + |K_2| \right)^{1/2} \|q_h\|_{0,K_1 \cup K_2}.$$  

(77)

Then, since $V_1$ is regular, Lemma 4.1, (6.2) and (77) deduce

$$|b| \leq C \left( |K_1| + |K_2| \right)^{1/2} \left( |u - u_h|_{1,K(V_i)} + \|p - \Pi_h p\|_{0,K(V_i)} \right).$$  

(78)

We note that the system (75) is far from singular, since $0 < C \leq \sin \theta_j$, $j = 1, 2$ by shape-regularity. Thus from (75), (76), (78), we have

$$|e_1| + |e_2| \leq C \left( |K_1| + |K_2| \right)^{-1/2} \left( |u - u_h|_{1,K(V_i)} + \|p - \Pi_h p\|_{0,K(V_i)} \right).$$  

(79)

Then (71) comes from (59), (73), (79).
7.2 definition of $p_h^V$ for a nearly singular ordinary vertex $V$

Let $V$ be a nearly singular vertex meeting an interior edge and $K_1, K_2, \ldots, K_J$, all triangles in $T_h$ sharing $V$ as in Figure 5. Using the same notations in (42)-(44), define a jump of a function $q_h$ at $V$ across an interior edge $K_j \cap K_{j+1}$ as

$$\mathcal{J}_{j,j+1}(q_h) = |K_j \cap K_{j+1}|^3 \left( \frac{\partial}{\partial \tau_j} \left( q_h|_{K_j} \right)(V) - \frac{\partial}{\partial \tau_{j+1}} \left( q_h|_{K_{j+1}} \right)(V) \right),$$

for $j = 1, 2, \ldots, J$, similarly to (65).

We note that the adjacent vertices $V_1, V_2, \ldots, V_J$ are all regular from Lemma 5.1. That is, $p_{h,V_1}, p_{h,V_2}, \ldots, p_{h,V_J}$ are already defined in Lemma 6.2.

Thus, replacing (61b) with new equations using the jumps in (80), we can consider the following system of $2J$ equations for unknown $p_{h,V} \in S(V)$:

$$\begin{align*}
(p_{h,V}, \text{div } w^{T_j}) &= (f, w^{T_j}) - (\nabla u_h, \nabla w^{T_j}) - (p_{h,V}, \text{div } w^{T_j}), \\
\mathcal{J}_{j,j+1}(p_{h,V}) &= -\mathcal{J}_{j,j+1}(p_{h,V} + p_{h,V_j}),
\end{align*}$$

for $j = 1, 2, \ldots, J$.

Then, we can repeat the arguments for Lemma 6.1, 6.2, 7.1 to establish the following lemma.

Lemma 7.2. If $V$ is a nearly singular vertex meeting an interior edge as in Figure 5, define $p_{h,V} \in S(V)$ as the least square solution of (81). Then, for $\Pi_h p_{h,V}$ in (37), we estimate

$$\|\Pi_h p_{h,V} - p_{h,V}\|_0 \leq C \sum_{j=1}^J (|u - u_h|_{1,K(V_j)} + \|p - \Pi_h p\|_{0,K(V_j)}),$$

where $V_1, V_2, \ldots, V_J$ are all vertices sharing interior edges with $V$.

8 $p_h^V$ for a dead corner $V$

Let $V$ be a vertex meeting no interior edge. Then $V$ is a dead corner and has only one triangle $K_1$ as in Figure 6. There exists a triangle $K$ in $T_h$ sharing two vertices $W_1, W_2$ with $K_1$. Denote by $W_3$, the third vertex of $K$ not shared with $K_1$.

Define a jump of a function $q_h$ at $W_1$ across $K_1 \cap K$ as

$$\mathcal{J}(q_h) = \ell^3 \left( \frac{\partial}{\partial n} \left( q_h|_{K_1} \right)(W_1) - \frac{\partial}{\partial n} \left( q_h|_K \right)(W_1) \right),$$

where $n$ is a unit outward normal vector on $K_1 \cap K$ of $K_1$ and $\ell$ is the distance between $V$ and $K_1 \cap K$.
We note that \( p_{W_1} \), \( p_{W_2} \), \( p_{W_3} \) are already defined in Lemma 6.2 and 7.2 since \( W_1, W_2, W_3 \) are not corners by Assumption 3.1 on \( T_h \). Thus we can consider the following equation for unknown \( p^V \in S(V) \),

\[
\mathcal{J}(p^V) = -\mathcal{J}(p^N + p_{W_1} + p_{W_2} + p_{W_3}).
\]  

(83)

For a vertex \( W \), let \( V(W) \) be a set of all vertices sharing interior edges with \( W \), then denote \( \mathcal{K}(W) = \bigcup_{U \in V(W)} \mathcal{K}(U) \).

Lemma 8.1. If \( V \) is a vertex meeting no interior edge as in Figure 6, define \( p^V \in S(V) \) as the solution of (83). Then, for \( \Pi_h p^V \) in (37), we estimate

\[
\|\Pi_h p^V - p^V\|_0 \leq C \sum_{j=1}^3 (\|u - u_h|_{1,\mathcal{K}(W_j)}\| + \|p - \Pi_h p\|_{0,\mathcal{K}(W_j)}),
\]  

(84)

where \( W_1, W_2, W_3 \) are all vertices of the triangle whose intersection with \( \mathcal{K}(V) \) is an edge.

Proof. We remind that \( \nabla \Pi_h p \) is continuous at \( W_1 \) by (26). It can be written in

\[
\mathcal{J}(\Pi_h p^V) = -\mathcal{J}(\Pi_h p^N + \Pi_h p_{W_1} + \Pi_h p_{W_2} + \Pi_h p_{W_3}).
\]  

(85)

Set the error \( e^V = \Pi_h p^V - p^V = e_{S_{W_1}} \) for some constant \( e \), then (83) and (85) make

\[
\mathcal{J}(e_{S_{W_1}}) = -\mathcal{J}\left(\Pi_h p^N - p^N + \sum_{j=1}^3 \Pi_h p_{W_j} - p_{W_j}\right).
\]  

(86)

From (11), (12), we have

\[
\mathcal{J}(e_{S_{W_1}}) = -\frac{9}{5} \ell^2.
\]  

(87)

For the right hand side in (86), we can estimate the following for \( q_h \in P^3_h(\Omega) \),

\[
|\mathcal{J}(q_h)| \leq C (|K| + |K|^{1/2}) \|q_h\|_{0,\mathcal{K}(V)},
\]  

(88)

same as in (77) by similarity of (65) and (82).

Then, we can deduce (84) from (59), (86)-(88) and Lemma 4.1, 6.2, 7.2.

We have defined the sting components \( p^V \in S(V) \) for all vertices \( V \) in Lemma 6.2, 7.2 and 8.1. All the results are summarized in the following lemma.

Lemma 8.2. Define a sting component \( p^S_h \in S_h \) as

\[
p^S_h = \sum_{V \text{vertex}} p^V.
\]

Then, for the sting component \( \Pi_h p^S \) of \( \Pi_h p \) in (28), we estimate

\[
\|\Pi_h p^S - p^S_h\|_0 \leq C (|u - u_h|_1 + \|p - \Pi_h p\|_0).
\]
9 Piecewise constant component

9.1 inf-sup condition

Define the following spaces:

\[ V_{h,0} = \{ v_h \in [P_h^0(\Omega) \cap C^0(\Omega)]^2 : \nabla v_h \text{ is continuous at all vertices in } T_h \}, \]
\[ V_{h,00} = \{ v_h \in V_{h,0} : \nabla v_h \text{ vanishes at all vertices in } T_h \}. \]  \hspace{1cm} (89)

Then we have the following inf-sup condition for \( V_{h,00} \times P_h^0(\Omega) \cap L_0^2(\Omega) \).

**Lemma 9.1.** For each \( c_h \in P_h^0(\Omega) \cap L_0^2(\Omega) \), there exists a nontrivial \( v_h \in V_{h,00} \) such that

\[ \beta \| c_h \|_0 \| v_h \|_1 \leq (c_h, \text{div } v_h), \]

for a constant \( \beta > 0 \) regardless of \( h \).

**Proof.** Given \( c_h \in P_h^0(\Omega) \cap L_0^2(\Omega) \), there exists a nontrivial \( w \in [P_h^2(\Omega) \cap H_0^1(\Omega)]^2 \) such that

\[ \beta \| c_h \|_0 \| w \|_1 \leq (c_h, \text{div } w), \]  \hspace{1cm} (90)

for a constant \( \beta > 0 \) regardless of \( h \).

For each triangle \( K \) in \( T_h \), define \( z_K \in [P^4]^2 \) so that

\[ \nabla z_K = \nabla w \text{ at all vertices of } K, \]
\[ \int_E z_K \, d\ell = 0 \text{ for each edge } E \text{ of } K, \]
\[ z_K = 0 \text{ at all } 3 \text{ vertices and midpoints of } 3 \text{ medians of } K. \]  \hspace{1cm} (91)

Then, for a reference triangle \( \hat{K} \) and an affine map \( F : \hat{K} \rightarrow K \), we have

\[ |z_K|_{1, K} \leq C |z_K \circ F|_{1, \hat{K}} \leq C |w \circ F|_{1, \hat{K}} \leq C |w|_{1, K}. \]  \hspace{1cm} (92)

If we define \( z \in [P_h^4(\Omega)]^2 \) by \( z|_K = z_K \) for all \( K \in T_h \), then \( z \) belongs to \([H_0^1(\Omega)]^2\), since derivatives of \( w \) along to edges are continuous. We note that \( w \neq z \). If so, \( w = 0 \) from the second and third conditions in \((91)\).

Thus, from \((91)\) and \((92)\), \( z \) satisfies

\[ w - z \in V_{h,00} \setminus \{ 0 \}, \quad |z|_1 \leq C |w|_1, \quad (1, \text{div } z)_K = 0 \text{ for all } K. \]  \hspace{1cm} (93)

Then, the following comes from \((90)\) and \((93)\), which completes the proof:

\[ \beta / (1 + C) \| c_h \|_0 \| w - z \|_1 \leq \beta \| c_h \|_0 \| w \|_1 \leq (c_h, \text{div } (w - z)). \]

\( \square \)

9.2 definition of piecewise constant component

**Lemma 9.2.** There exists a unique \( p_h^c \in P_h^0(\Omega) \cap L_0^2(\Omega) \) satisfying

\[ (p_h^c, \text{div } v_h) = (f, v_h) - (\nabla u_h, \nabla v_h) - (p_h^N, \text{div } v_h) \text{ for all } v_h \in V_{h,00}. \]  \hspace{1cm} (94)

**Proof.** The uniqueness comes from the inf-sup condition in Lemma 9.1. For the existence, let

\[ Q_{h,0} = \{ q_h \in P_h^0(\Omega) \cap L_0^2(\Omega) : q_h \text{ is continuous at all vertices in } T_h \}, \]
\[ Q_{h,00} = \{ q_h \in Q_{h,0} : q_h \text{ vanishes at all corners of } \partial \Omega \}. \]  \hspace{1cm} (95)
Then, there exists a unique \((\bar{u}_h, r_h)\) \(\in V_{h,0} \times Q_{h,00}\) satisfying the following discrete Stokes equation:

\[
(\nabla \bar{u}_h, \nabla v_h) + (r_h, \nabla v_h) + (q_h, \nabla \bar{u}_h) = (f, v_h) \quad \text{for all } (v_h, q_h) \in V_{h,0} \times Q_{h,00},
\]

(96)

since the following Stokes complex is exact for \(\Sigma_{h,0}, V_{h,0}, Q_{h,00}\) in [2, (89), (95)], respectively [5]:

\[
0 \rightarrow \Sigma_{h,0} \xrightarrow{\text{curl}} V_{h,0} \xrightarrow{\text{div}} Q_{h,00} \rightarrow 0.
\]

From (3), we note that \(\bar{u}_h \in V_{h,0}\) in (96) coincides with the previous \(u_h \in Z_{h,0}\) in [4]. Thus \(r_h \in Q_{h,00}\) in (96) satisfies that

\[
(r_h, \text{div} v_h) = (f, v_h) - (\nabla \bar{u}_h, \nabla v_h) \quad \text{for all } v_h \in V_{h,0}.
\]

(97)

Similarly to (28), decompose \(r_h\) into

\[
r_h = r_h^N + r_h^S + r_h^C \quad \text{for } r_h^N \in N_h, \ r_h^S \in S_h, \ r_h^C \in C_h.
\]

(98)

From the quadrature rule in [10] and definition of \(V_{h,00}\) in [89], we have

\[
(r_h^S, \text{div} v_h) = 0 \quad \text{for all } v_h \in V_{h,00}.
\]

(99)

Then, by (24) and (97)-(99), \(r_h^C - m(r_h^C) \in P_0^0(\Omega) \cap L_0^2(\Omega)\) satisfies

\[
(r_h^C - m(r_h^C), \text{div} v_h) = (f, v_h) - (\nabla \bar{u}_h, \nabla v_h) - (r_h^N, \text{div} v_h) \quad \text{for all } v_h \in V_{h,00}.
\]

(100)

For each triangle \(K\), we note \(B(K) \subset V_{h,00}\). Thus, from (17), [18], (100), we have

\[
(r_h^N|_K, \text{div} v_h) = (f, v_h) - (\nabla \bar{u}_h, \nabla v_h) \quad \text{for all } v_h \in B(K).
\]

(101)

Since \(p_h^K \in N(K)\) satisfying (31) is unique, (101) implies that \(r_h^N\) in (100) coincides with \(p_h^N\) in (94).

Lemma 9.3. Define the piecewise constant component as \(p_h^C\) in Lemma 9.2. Then for \(\Pi_h p^C\) in (25), we estimate

\[
\|\Pi_h p^C - m(\Pi_h p^C) - p_h^C\|_0 \leq C(|u - u_h|_1 + \|p - \Pi_h p\|_0).
\]

(102)

Proof. By (30) and quadrature rule in (10), \(\Pi_h p^C\) satisfies for all \(v_h \in V_{h,00},\)

\[
(\Pi_h p^C, \text{div} v_h) = (f, v_h) - (\nabla u, \nabla v_h) - (\Pi_h p^N, \text{div} v_h) - (p - \Pi_h p, \text{div} v_h).
\]

(103)

Set \(e_h = \Pi_h p^C - m(\Pi_h p^C) - p_h^C \in P_0^0(\Omega) \cap L_0^2(\Omega)\). Then, from (94) and (103), it satisfies

\[
(e_h, \text{div} v_h) = - (\nabla u - \nabla u_h, \nabla v_h) - (\Pi_h p^N - p_h^N, \text{div} v_h) - (p - \Pi_h p, \text{div} v_h),
\]

(104)

for all \(v_h \in V_{h,00}\).

From Lemma 9.1, there exists a nontrivial \(v_h \in V_{h,00}\) such that

\[
\beta\|e_h\|_0|v_h|_1 \leq (e_h, \text{div} v_h) \quad \text{for } \beta > 0 \text{ regardless of } h.
\]

(105)

Then (102) comes from (104), (105) and Lemma 4.1. \(\square\)
9.3 local calculation of piecewise constant component

For each triangle \( K \in T_h \), let \( C_K \) be a constant such that

\[
C_K = p_h^K|_K.
\]

(106)

If \( K_1, K_2 \) are two adjacent triangles sharing an edge, there exists a test function \( \mathbf{v}_h \in V_{h,00} \) such that

the support of \( \mathbf{v}_h \) is in \( K_1 \cup K_2 \),

\[
\int_{K_j} \text{div}\, \mathbf{v}_h \, dx\,dy = (-1)^{j+1}, \quad j = 1, 2.
\]

(107)

Then from (94), (106), (107), we can calculate the adjacent difference by

\[
C_{K_1} - C_{K_2} = (f, \mathbf{v}_h) - (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (p_h^N, \text{div}\, \mathbf{v}_h).
\]

(108)

Fix a triangle \( K_0 \) and denote \( C_0 = p_h^E|_{K_0} \). Then we can calculate \( C_K - C_0 \) for all triangles \( K \in T_h \) by the following iterative algorithm.

(i) Set \( \tilde{\Omega} = K_0 \) and \( \mathcal{V} = \emptyset \).

(ii) Choose a vertex \( \mathbf{V} \notin \mathcal{V} \) on the boundary of \( \tilde{\Omega} \).

(iii) Calculate \( C_K - C_0 \) for all triangles \( K \subset K(\mathbf{V}) \) by adding adjacent differences in (108).

(iv) Update \( \tilde{\Omega} = \tilde{\Omega} \cup K(\mathbf{V}) \), \( \mathcal{V} = \mathcal{V} \cup \{ \mathbf{V} \} \) and go to (ii), if \( \tilde{\Omega} \neq \Omega \).

The unknown \( C_0 \) is calculated from the knowledges of \( C_K - C_0 \) for all \( K \in T_h \), since

\[
0 = \int_{\Omega} p_h^E \, dx\,dy = \sum_{K \in T_h} C_K|_K = \sum_{K \in T_h} (C_K - C_0)|_K + C_0|\Omega|.
\]

Remark 9.4. For an efficient choice of \( \mathbf{V} \) in (ii), we could use a structure of the mesh such as a hierarchy. Dividing \( \Omega \) into subdomains, the above algorithm would be easily parallelized.

10 A locally calculable \( P^3 \)-pressure

We have prepared the locally calculable components \( p_h^N, p_h^S, p_h^E \) in Lemma 4.1, 8.2, 9.3 respectively. At last, we arrive at the following final definition of a pressure \( p_h \in \mathcal{P}_h^4(\Omega) \cap L^2_0(\Omega) \):

\[
p_h = p_h^N + p_h^S + p_h^E - m(p_h^S).
\]

(109)

**Theorem 10.1.** Let \( (\mathbf{u}, p) \in [H^1(\Omega)]^2 \times L^2(\Omega) \) satisfy (1). Then, for \( p_h \) defined in (109), we estimate

\[
\|p - p_h\|_0 \leq Ch^4(|\mathbf{u}|_5 + |p|_4),
\]

(110)

if \( (\mathbf{u}, p) \in [H^5(\Omega)]^2 \times H^4(\Omega) \).

**Proof.** By (28) and (109), we expand

\[
\Pi_h p - p_h = (\Pi_h p_h^N - p_h^N) + (\Pi_h p_h^S - p_h^S) + (\Pi_h p_h^C - p_h^C + m(p_h^S))
\]

(111)

Since \( m(\Pi_h p) = m(\Pi_h p_h^C) + m(\Pi_h p_h^S) \), the last term in (111) makes

\[
\Pi_h p_h^C - p_h^C + m(p_h^S) = (\Pi_h p_h^C - m(\Pi_h p_h^C) - p_h^S) + m(p_h^S - \Pi_h p_h^S) + m(\Pi_h p).
\]

(112)

We note that

\[
|m(\Pi_h p)| = |m(\Pi_h p - p)| \leq |\Omega|^{-1/2}\|\Pi_h p - p\|_0.
\]

(113)

Therefore, (110) is established from (27), (111)-(113) and Lemma 4.1, 8.2, 9.3, and Theorem 2.1. \( \square \)
11 A summary of the method

Step 1. Calculate \( u_h \in Z_{h, 0} \) in (3) such that
\[
(\nabla u_h, \nabla v_h) = (f, v_h) \quad \text{for all } v_h \in Z_{h, 0}.
\]

Step 2. For each triangle \( K \in T_h \), calculate \( p^K_h \in N(K) \) in (17) such that
\[
(p^K_h, \text{div } v_h) = (f, v_h) - (\nabla u_h, \nabla v_h) \quad \text{for all } v_h \in B(K) \text{ in (16)}.
\]

Then, denote
\[
p^N_h = \sum_{K \in T_h} p^K_h.
\]

Step 3. For each regular vertex \( V \) as in Figure 4, calculate the least square solution \( p^V_h \in S(V) \) in (36) for a system of 2\( J \) equations:
\[
(p^V_h, \text{div } w^V_{h j}) = (f, w^V_{h j}) - (\nabla u_h, \nabla w^V_{h j}) - (p^N_h, \text{div } w^V_{h j}),
\]
\[
(p^V_h, \text{div } w^V_{h+ j}) = (f, w^V_{h+ j}) - (\nabla u_h, \nabla w^V_{h+ j}) - (p^N_h, \text{div } w^V_{h+ j}),
\]
for \( j = 1, 2, \ldots, J \), where \( J \) is the number of interior edges of \( V \) and \( w^V_{h j}, w^V_{h+ j} \) are test functions defined in (46)–(47).

Step 4. For each nearly singular vertex \( V \) meeting an interior edge as in Figure 5, calculate the least square solution \( p^V_h \in S(V) \) for a system of 2\( J \) equations:
\[
(p^V_h, \text{div } w^V_{h j}) = (f, w^V_{h j}) - (\nabla u_h, \nabla w^V_{h j}) - (p^N_h, \text{div } w^V_{h j}),
\]
\[
3_j j+1(p^V_h) = -3_j j+1(p^V_h + p^V_h'),
\]
for \( j = 1, 2, \ldots, J \), where \( 3_{j j+1} \) is the jump defined in (80) and \( V_1, V_2, \ldots, V_J \) are all regular vertices sharing interior edges with \( V \).

Step 5. For each nearly singular vertex \( V \) meeting no interior edge as in Figure 6, calculate \( p^V_h \in S(V) \) such that
\[
3(p^V_h) = -3(p^N_h + p^W_1 + p^W_2 + p^W_3),
\]
where \( 3 \) is the jump defined in (82) and \( W_1, W_2, W_3 \) are all vertices of the triangle whose intersection with \( K(V) \) is an edge.

Step 6. Calculate \( p^e_h \in P^3_h(\Omega) \cap L^2(\Omega) \) such that, for every two triangles \( K_1, K_2 \) sharing an edge,
\[
p^e_h|_{K_1} - p^e_h|_{K_2} = (f, v_h) - (\nabla u_h, \nabla v_h) - (p^N_h, \text{div } v_h),
\]
where \( v_h \in V_{h, 0} \) in (80) is a test function satisfying (107).

Step 7. Denote
\[
p^S_h = \sum_{V: \text{vertex}} p^V_h.
\]

Then, define \( p_h \in P^3_h(\Omega) \cap L^2(\Omega) \) as
\[
p_h = p^N_h + p^S_h + p^e_h - \int_\Omega p^S_h \, dx dy.
\]
12 Numerical tests

All of the numerical tests were done with the velocity $u$ and pressure $p$ on $\Omega = [0,1]^2$ such that

$$u = (s(x)s'(y), -s'(x)s(y)), \quad p = \sin(4\pi x)e^{\pi y},$$

where $s(t) = (t^2 - t)\sin(2\pi t)$.

12.1 suggested method over singular meshes

We tested the suggested method over singular meshes as in Figure 7. For triangulations, we formed first the meshes of uniform squares over $\Omega$, then added one exactly singular vertex in every non-corner square. For the corners, we made them singular as in Figure 7, an example of $8 \times 8 \times 4$ mesh.

We calculated locally the components $p_N^h, p_C^h$ as well as $p_V^h$ for all vertices $V$ in order: regular vertices, interior singular vertices, dead corners. In Figure 11, their superposition on making $p_h$ in (109) are depicted for the mesh in Figure 7.

The errors in Table 1 show the optimal order of convergence, expected in Theorem 2.1 and 10.1. We used a direct linear solver in LAPACK on solving the problems (4) for $u_h$ in double precision.

![Figure 7: 8 × 8 × 4 singular mesh, each unit square has a singular vertex](image)

12.2 comparison with mixed FEM

We calculated the discrete solutions $(u_{FN}^h, p_{FN}^h), (u_{SV}^h, p_{SV}^h)$ and $(u_h, p_h)$ by the Falk-Neilan [6], Scott-Vogelius [8, 11] and suggested methods, respectively, over regular meshes as in Figure 8 and nearly singular meshes as in Figure 9. For easy comparison, a common direct linear solver was used for all involving linear systems.

As shown in Table 2 and 3, the suggested and Falk-Neilan methods were almost same for the velocity, since the divergence-free subspace of Falk-Neilan for $u_{FN}^h$ is slightly different to $Z_{h,0}$ for $u_h$ in (4) by merely a few elements for the corners. In those tables, a little advantage was lying on the Scott-Vogelius method as expected from its larger divergence-free subspace.

For the pressure, the suggested and Falk-Neilan methods offered more favorable errors as in Table 4 and 5. It is acceptable since they reflect some continuity of pressure.
The results in Table 5 implied that the pressures by Scott-Vogelius were ruined over nearly singular meshes. It is also confirmed in Figure 10, where the pressures calculated over the $8 \times 8 \times 4$ nearly singular mesh are depicted. This unstable phenomena is well known and recently turns out due to the characteristic of the sting function on the singular vertex. Based on that, we could recover a stable pressure from the ruined one by simple postprocess utilizing vertex continuity of pressure [9].

It is interesting that the results over nearly singular meshes outperformed those over regular meshes, except the pressures by Scott-Vogelius. The reason why is that the largest triangles in Figure 9 are smaller than those in Figure 8.

In all the tests, the suggested method was comparable with other two mixed finite element methods, while it would cost less, since it used only local computations for the pressures.

![Figure 8: regular mesh, the vertex $V$ divides the diagonal of positive slope in the ratio 2 : 3](image)

![Figure 9: nearly singular mesh, the vertex $V$ divides the diagonal of positive slope in the ratio 99 : 100](image)

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Locally calculable pressure

Table 2: the errors in velocity over regular meshes as in Figure 8

| mesh         | $|\mathbf{u} - \mathbf{u}_h|_1$ | order | $|\mathbf{u} - \mathbf{u}^{\text{FN}}_h|_1$ | order | $|\mathbf{u} - \mathbf{u}^{\text{SV}}_h|_1$ | order |
|--------------|-------------------------------|-------|------------------------------------------|-------|------------------------------------------|-------|
| 4 x 4 x 4    | 1.4450E-2                     |       | 1.4450E-2                                |       | 1.1706E-2                                |       |
| 8 x 8 x 4    | 8.5476E-4                     | 4.08  | 8.5476E-4                                | 4.08  | 7.5823E-4                                | 3.95  |
| 16 x 16 x 4  | 5.1606E-5                     | 4.05  | 5.1606E-5                                | 4.05  | 4.7135E-5                                | 4.01  |
| 32 x 32 x 4  | 3.1882E-6                     | 4.02  | 3.1882E-6                                | 4.02  | 2.9271E-6                                | 4.01  |

Table 3: the errors in velocity over nearly singular meshes as in Figure 9

| mesh         | $|\mathbf{u} - \mathbf{u}_h|_1$ | order | $|\mathbf{u} - \mathbf{u}^{\text{FN}}_h|_1$ | order | $|\mathbf{u} - \mathbf{u}^{\text{SV}}_h|_1$ | order |
|--------------|-------------------------------|-------|------------------------------------------|-------|------------------------------------------|-------|
| 4 x 4 x 4    | 1.1266E-2                     |       | 1.1266E-2                                |       | 8.5523E-3                                |       |
| 8 x 8 x 4    | 6.1513E-4                     | 4.19  | 6.1513E-4                                | 4.19  | 5.4485E-4                                | 3.97  |
| 16 x 16 x 4  | 3.5952E-5                     | 4.10  | 3.5952E-5                                | 4.10  | 3.3934E-5                                | 4.01  |
| 32 x 32 x 4  | 2.2009E-6                     | 4.03  | 2.2009E-6                                | 4.03  | 2.1180E-6                                | 4.00  |

Table 4: the errors in pressure over regular meshes as in Figure 8

| mesh         | $\|p - p_h\|_0$ | order | $\|p - p^{\text{FN}}_h\|_0$ | order | $\|p - p^{\text{SV}}_h\|_0$ | order |
|--------------|-----------------|-------|-----------------------------|-------|----------------------------|-------|
| 4 x 4 x 4    | 6.1948E-2       |       | 7.5405E-2                   |       | 9.0916E-2                  |       |
| 8 x 8 x 4    | 3.1862E-3       | 4.28  | 3.5251E-3                   | 4.42  | 5.3241E-3                  | 4.09  |
| 16 x 16 x 4  | 1.9879E-4       | 4.00  | 2.1695E-4                   | 4.02  | 3.2844E-4                  | 4.02  |
| 32 x 32 x 4  | 1.2413E-5       | 4.00  | 1.3499E-5                   | 4.01  | 2.0319E-5                  | 4.01  |

Table 5: the errors in pressure over nearly singular meshes as in Figure 9

| mesh         | $\|p - p_h\|_0$ | order | $\|p - p^{\text{FN}}_h\|_0$ | order | $\|p - p^{\text{SV}}_h\|_0$ | order |
|--------------|-----------------|-------|-----------------------------|-------|----------------------------|-------|
| 4 x 4 x 4    | 5.7969E-2       |       | 7.6090E-2                   |       | 1.1022E+0                  |       |
| 8 x 8 x 4    | 2.7017E-3       | 4.42  | 3.3691E-3                   | 4.50  | 4.1561E-2                  | 4.73  |
| 16 x 16 x 4  | 1.6761E-4       | 4.01  | 2.0466E-4                   | 4.04  | 1.3696E-3                  | 4.92  |
| 32 x 32 x 4  | 1.0455E-5       | 4.00  | 1.2636E-5                   | 4.02  | 4.6032E-5                  | 4.89  |
Locally calculable pressure

\[ p = \sin(4\pi x)e^{\pi y} \]

\( p_h \) by local computation

\( p_{h}^{\text{FN}} \) by Falk-Neilan

\( p_{h}^{\text{SV}} \) by Scott-Vogelius

Figure 10: the pressures calculated over the \( 8 \times 8 \times 4 \) nearly singular mesh in Figure 9.
Locally calculable pressure

(a) non-sting component \( p_h^N \)

(b) adding piecewise constant component \( p_h^F \) to (a)

(c) adding \( p_h^V \) for regular vertices \( V \) to (b)

(d) adding \( p_h^V \) for interior singular vertices \( V \) to (c)

(e) \( p_h = (d) + (p_h^V \text{ for dead corners } V) - m(p_h^S) \)

Figure 11: superposition of the 5 components on making \( p_h \) over the mesh in Figure 7
Data availability statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

References

[1] M. Ainsworth and C. Parker, Mass Conserving Mixed $hp$-FEM Approximations to Stokes Flow. Part I: Uniform Stability, SIAM Journal on Numerical Analysis, 59 (2021), 1218-1244, DOI: 10.1137/20M1359109

[2] M. Ainsworth and C. Parker, Mass Conserving Mixed $hp$-FEM Approximations to Stokes Flow. Part II: Optimal Convergence, SIAM Journal on Numerical Analysis, 59 (2021), 1245-1272, DOI: 10.1137/20M1359110

[3] C. Bernardi and G. Raugel, Analysis of some finite elements for the Stokes problem, Mathematics of Computation, 44 (1985), 71-79, DOI: 10.2307/2007793

[4] S. C. Brenner and L. R. Scott, The mathematical theory of finite element methods, Springer-Verlag, New York, 2nd edition, 2002

[5] P. G. Ciarlet, The finite element method for elliptic equations, North-Holland, Amsterdam, 1978

[6] R. S. Falk and M. Neilan, Stokes complexes and the construction of stable finite elements with pointwise mass conservation, SIAM journal of Numerical Analysis, 51 (2013), 1308-1326, DOI: 10.1137/120888132

[7] V. Girault and P. A. Raviart, Finite element methods for the Navier-Stokes equations: Theory and Algorithms, Springer-Verlag, New York, 1986

[8] J. Guzman and L. R. Scott, The Scott-Vogelius finite elements revisited, Mathematics of Computation, electronically published (2018), DOI: 10.1090/mcom/3346

[9] C. Park, Spurious pressure in Scott–Vogelius elements, Journal of Computational and Applied Mathematics, 363 (2020), 370-391, DOI: 10.1016/j.cam.2019.06.007

[10] C. Park, Local computation of pressure in decoupled methods of high order for incompressible Stokes equations, in preparation

[11] L. R. Scott and M. Vogelius, Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials, ESIAM: M2AN, 19 (1985), 111-143, DOI: 10.1051/m2an/1985190101111