Spanier–Whitehead duality for topological coHochschild homology

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1 INTRODUCTION

Background

Topological Hochschild homology (THH) for ring spectra extends the notion of Hochschild homology for algebras. The topological perspective provides new insight on rings. For instance, the Hochschild homology of the finite field \( \mathbb{F}_p \) is a divided power algebra, but Bökstedt shows that the THH of its Eilenberg–Mac Lane spectrum \( \mathbb{H} \mathbb{F}_p \) is a polynomial ring. More recent examples can be found in the work of [7]. THH plays also an important role as an invariant for algebras in spectra thanks to its connections with \( K \)-theory via the Dennis trace map. Furthermore, THH is of interest for string topology. For a connected topological space \( X \), there is an equivalence

\[
\text{THH}(\Sigma_+^\infty \Omega X) \simeq \Sigma_+^\infty \mathcal{L}X,
\]

where \( \mathcal{L}X \) denotes the free loop space on \( X \). This goes back to [27, Theorem 7.3.11]; see [32, IV.3.2, IV.3.3] for a modern reference.
Topological coHochschild homology (coTHH) is a homology theory for coalgebras in spectra introduced by Hess–Shipley in [20], analogous to THH for ring spectra. Furthermore, coTHH extends the notion of coHochschild homology (coHH) introduced in [11] and [18] to coalgebras in a symmetric monoidal model category [4, 2.3]. One reason why this invariant of coalgebras is important is because coalgebraic structures are shown to contain valuable information regarding topological spaces. For instance, [38] establish that homotopy type of a topological space is entirely captured by the coalgebraic structure of its singular chain complex. Furthermore, Hess and Shipley prove that Waldhausen $K$-theory of a topological space $X$ can be obtained using the category of comodules over the coalgebra $\Sigma_+^\infty X$ [19].

Just like THH, coTHH also provides a new model for free loop spaces. Namely, Hess and Shipley show that there is an equivalence
\[ \text{coTHH}(\Sigma_+^\infty X) \simeq \Sigma_+^\infty \mathcal{L}X \]
for every ‘EMSS-good space’ $X$; in particular, for every simply connected space $X$ [20]. Connecting with the result above, for every EMSS-good space $X$, one obtains an instance of Koszul duality between THH and coTHH:
\[ \text{THH}(\Sigma_+^\infty \Omega X) \simeq \text{coTHH}(\Sigma_+^\infty X). \]

In [4, 2.3], the authors construct a coBökstedt spectral sequence for coTHH and show that this spectral sequence has a ‘□-Hopf algebra structure’, that is, it is endowed with a comultiplicative structure that is compatible with its multiplicative structure. The $E_2$-page of this spectral sequence is computed for various interesting coalgebra spectra in [23]. In [6], Bohmann, Gerhardt and Shipley use the □-Hopf algebra structure on the coBökstedt spectral sequence and the equivalence above to obtain homology computations for various free loop spaces generalizing earlier computations of [25]. Recently, Ayala and Francis defined factorization cohomology which generalizes coTHH for $\mathbb{E}_n$-coalgebras to obtain a Poincaré duality result for factorization homology [1]. From work in progress, the second author and Klanderman provide a shadow structure in the sense of [35] for coHH, resulting in new interesting bicategorical traces for coHH (forthcoming paper by Klanderman and Péroux).

Computations

In this work, we compute the coTHH of new coalgebras not considered before. For this, we carry the definition of coTHH to the $\infty$-categorical setting and we prove a duality relationship between coTHH and THH which provides further insight to coTHH, and also to THH.

Our main computation is the coHH groups of the Steenrod algebra spectrum $[HF_p, HF_p]_S$. Here $[HF_p, HF_p]_S$ denotes the spectrum of $S$-module endomorphisms of $HF_p$. First, we show that $[HF_p, HF_p]_S$ inherits the structure of an $\mathbb{E}_\infty$-$HF_p$-coalgebra, that is, an $\mathbb{E}_\infty$-coalgebra in $HF_p$-modules, as it is the linear dual of the $\mathbb{E}_\infty$-$HF_p$-algebra $HF_p \wedge HF_p$. Note that this does not follow by the correspondence between $\mathbb{E}_\infty$-algebras and $\mathbb{E}_\infty$-coalgebras on dualizable objects (Proposition 3.11), given by the dualization functor. This is because $[HF_p, HF_p]_S$ is not dualizable in $HF_p$-modules as the Steenrod algebra
\[ A \cong \pi_*([HF_p, HF_p]_S) \]
is infinite-dimensional, see Example 3.9. On the other hand, our Theorem 5.8, which generalizes
the correspondence on dualizable objects, does equip \([HF_p, HF_p]_S\) with the structure of an \(E_\infty\)-
\(HF_p\)-coalgebra coming from the \(E_\infty\)-\(HF_p\)-algebra structure of \(HF_p \wedge HF_p\). Using our new duality
result for co\(THH\) (Theorem 1.3), together with Bökstedt periodicity, we compute the co\(HH\) groups
of \([HF_p, HF_p]_S\).

**Theorem 1.1** (Theorem 7.7). There is an equivalence of graded \(F_p\)-modules:

\[
\pi_* \left( \text{co\(THH\)}^{HF_p} \left( [HF_p, HF_p]_S \right) \right) \cong A \otimes F_p [x_{-2}],
\]

where \(|x_{-2}| = -2\).

Moreover, using Theorem 1.3, we do the following computations.

- **Theorem 7.2**: there is an \(E_\infty\)-coalgebra structure on \(HF_p \wedge_{HZ} HF_p\) as an \(HF_p\)-module, and its
coh\(H\) is given by:

\[
\pi_* \left( \text{co\(THH\)}^{HF_p} \left( HF_p \wedge_{HZ} HF_p \right) \right) \cong \Lambda F_p (x_1) \otimes F_p [[t]],
\]

where \(|t| = 0\) and \(|x_1| = 1\). This completes the result of [23] in which the \(E_2\)-page of the relative
cob\(B\)ökstedt spectral sequence from [4] computing the homotopy groups above was calculated.

- **Theorem 7.8**: there is an \(A_\infty\)-coalgebra structure on \(\Omega HF_p\) as an \(HZ\)-module and its co\(THH\) is
given by:

\[
\pi_* \left( \text{co\(THH\)}^{HZ} \left( \Omega HF_p \right) \right) \cong \Omega F_p [x_{-2}],
\]

where \(|x_{-2}| = -2\) and \(\Omega\) on the right-hand side denotes the functor that decreases the grading
by 1.

**From model categories to higher categories**

We mentioned earlier the importance of coalgebraic structures in spectra. However, there is no
model category of spectra that accurately captures the homotopy theory of its coalgebras, see [36]
and [34, 5.6]. An \(A_\infty\)-coalgebra in spectra is a spectrum endowed with a comultiplication that is
coassociative and counital up to higher homotopy. Given an \(A_\infty\)-coalgebra, there is not, in gen-
eral, a choice of monoidal model category of spectra in which the \(A_\infty\)-coalgebra structure can be
rigidified to a strictly coassociative counital coalgebra structure. The major disadvantage of model
categories is that the Spanier–Whitehead duality, that relates \(A_\infty\)-algebras with \(A_\infty\)-coalgebras in
finite spectra (see [30, 3.2.5]), cannot be realized at the level of strict algebras and coalgebras.
Essentially, model categories currently can only capture coalgebras of the form \(\Sigma X\) induced by
the diagonal on the space \(X\). It is for this very reason that the work of [4, 20], and [23] is limited
to the computation of co\(THH\)(\(\Sigma X\)).

If a symmetric monoidal \(\infty\)-category \(C\) is endowed with an internal hom \([\cdot, \cdot]\), we can define
the notion of linear duality that generalizes the classical dual of a vector space over a field, or the
Spanier–Whitehead dual of a spectrum. For \(X\) an object in \(C\) we denote \(X^\vee = [X, \mathbb{1}]\) the linear dual
of \(X\), where \(\mathbb{1}\) denotes the monoidal unit of \(C\). If \(C\) is a coalgebra in \(C\), then \(C^\vee\) is always an algebra
in $C$. It is well-known that algebras and coalgebras in finite-dimensional vector spaces are anti-equivalent, see [40, I.1.1]. Similarly, by [30, 3.2.5] (see also Corollary 3.13), the linear dual provides an anti-equivalence between $\mathbb{A}_\infty$-algebras and $\mathbb{A}_\infty$-coalgebras in dualizable objects in $C$.

To be able to incorporate Spanier–Whitehead duality into our computational framework, we extend the definition of coTHH to any symmetric monoidal $\infty$-category $C$, following the THH approach of [32]. Indeed, our computations provide an instance demonstrating the strength of $\infty$-categories for computational purposes.

**A duality relationship between coTHH and THH**

Using the anti-equivalence between $\mathbb{A}_\infty$-algebras and $\mathbb{A}_\infty$-coalgebras in dualizable objects mentioned above, and our $\infty$-categorical definition of coTHH, we prove the following theorem that relates THH and coTHH of algebras and coalgebras in dualizable objects via the linear dual.

**Theorem 1.2** (Theorem 4.1). Let $R$ be an $E_\infty$-ring spectrum. If $C$ is an $\mathbb{A}_\infty$-coalgebra over $R$, whose underlying $R$-module is dualizable, then there is an equivalence of $R$-module spectra

$$\text{coTHH}^R(C) \simeq \left(\text{THH}^R(C^\vee)\right)^\vee.$$ 

We in fact generalize the anti-equivalence on dualizable objects to larger subclasses of algebras and coalgebras where the dualization functor is strong monoidal, see Theorem 3.31. We also extend the duality relationship between THH and coTHH to these new subclasses. The main ingredient for these generalizations is our notions of quasi-dualizability (Definition 3.20) and quasi-proper coalgebras (Definition 3.27) which are weaker conditions than dualizability, see Remark 3.22. We show in Theorem 4.1 that the equivalence in Theorem 1.2 also holds for quasi-proper $\mathbb{A}_\infty$-coalgebras.

With specific examples in mind, such as the Steenrod algebra spectrum mentioned earlier, we are led to consider a more general class of $HA$-coalgebras than those whose underlying $HA$-modules are dualizable. Here $A$ denotes a discrete commutative ring. Through careful Tor and Ext spectral sequence considerations, we are able to show that these $HA$-coalgebras are indeed quasi-proper, see Section 5. This provides Theorem 1.3, which extends Theorem 1.2 for $HA$-coalgebras. Indeed, this generalization turns out to be very useful for computations. For instance, our computation of the coTHH of the Steenrod algebra spectrum relies on this generalization.

We say an $HA$-module $X$ is of finite type if $\pi_i X$ is a finitely generated $A$-module for each $i$. Recall that a ring $A$ is said to have finite global dimension if there is an integer $d$ such that every $A$-module $M$ admits a projective resolution of length at most $d$.

**Theorem 1.3** (Theorem 5.1). Let $A$ be a discrete commutative Noetherian ring with finite global dimension and let $C$ be a connective or coconnective $\mathbb{A}_\infty$-coalgebra over $HA$ of finite type. Then there is an equivalence of $HA$-module spectra

$$\text{coTHH}^{HA}(C) \simeq \left(\text{THH}^{HA}(C^\vee)\right)^\vee.$$ 

We obtain the following new insight on THH and coTHH.
• Example 6.4: The suspension spectrum of a finite CW-complex $X$ is a dualizable object in spectra. When $X$ is simply connected, one has

$$(\text{THH}((\Sigma^\infty_+ X)^\vee))^\vee \simeq \Sigma^\infty_+ \mathcal{L}X,$$

due to [24] and [31]. We obtain a new proof of this equality using Theorem 1.2 and the results of [20]. Indeed, this generalizes the equality above to EMSS-good finite CW-complexes. Furthermore, this provides the following Koszul duality relationship for THH:

$$(\text{THH}((\Sigma^\infty_+ X)^\vee))^\vee \simeq \text{THH}(\Sigma^\infty_+ \Omega X).$$

• Example 6.6: if $G$ is a compact Lie group, then the Thom spectrum $G^{-\tau}$ of its stable normal bundle is endowed with an $A_\infty$-coalgebra structure in spectra, induced by the group multiplication of $G$, and its coTHH is given by:

$$\text{coTHH}(G^{-\tau}) \simeq (\Sigma^\infty_+ \mathcal{L}BG)^\vee,$$

where $\mathcal{L}BG$ is the free loop space of the classifying space of $G$. By [34], such a computation could not have been considered in the framework of [20].

**Outline**

In Section 2, we define coTHH in general symmetric monoidal $\infty$-categories. After this, we discuss duality between coalgebras and algebras and define our notions of quasi-dualizability and quasi-properness in Section 3. Using this, we prove general duality results between coTHH and THH in Section 4. In Section 5, we apply these duality results to $HA$-module spectra and prove Theorems 1.3 and 5.8. Section 6 is devoted to a discussion of examples of $A_\infty$-coalgebras in spectra; we show that many examples of spectra such as $ku$, $MU$ and $ko$, and so on, are not $A_\infty$-coalgebras in spectra. In Section 7, we do explicit coHH computations using Theorem 1.3. The arguments in this section only make use of Theorems 1.3 and 5.8; therefore, the reader interested in these computations may jump to Section 7.

**Notation 1.4.** We begin by setting notation that we use throughout and recalling some elementary notions of the theory of $\infty$-categories, following [28, 29]. The notions of symmetric monoidal $\infty$-categories and $\infty$-operads are defined, respectively, in [29, 2.0.0.7, 2.1.1.10].

1. We denote by $\text{Assoc}^\otimes$ the associative $\infty$-operad as in [29, 4.1.1.3]. If $C$ is a monoidal $\infty$-category, we denote the $\infty$-category of $A_\infty$-algebras in $C$, as described in [29, 4.1.1.6], by $\text{Alg}_{A_\infty}(C)$. Recall there is an approximation of $\infty$-operads $N(\Delta^{op}) \rightarrow \text{Assoc}^\otimes$, see [29, 4.1.2.11]. Here $\Delta$ denotes the usual simplex category and $N$ is the nerve of a category.

2. We denote by $\text{Com}^\otimes = N(\text{Fin}_\ast)$ the commutative $\infty$-operad as in [29, 2.1.1.18]. Here $\text{Fin}_\ast$ is the category of finite pointed sets. If $C$ is a symmetric monoidal $\infty$-category, we denote the $\infty$-category of $E_\infty$-algebras in $C$, as described in [29, 2.1.3.1], by $\text{Alg}_{E_\infty}(C)$.

3. Let $\Theta$ be an $\infty$-operad. Let $C$ and $D$ be $\Theta$-monoidal $\infty$-categories, as defined in [29, 2.1.2.13]. We say a functor $F : C \rightarrow D$ is lax $\Theta$-monoidal if it is a map of $\infty$-operads in the sense of [29, 2.1.2.13].
2.1.2.13]. We denote the $\infty$-category of lax $\mathcal{O}$-monoidal functors from $C$ to $D$ by $\text{Fun}^{\text{lax}}_{\mathcal{O}}(C, D)$. We say $F$ is strong $\mathcal{O}$-monoidal if it is $\mathcal{O}$-monoidal in the sense of [29, 2.1.3.7]. When $\mathcal{O}^\otimes = \text{Com}^\otimes$, we prefer to say symmetric monoidal instead of $\mathcal{O}$-monoidal.

(4) Given a symmetric monoidal $\infty$-category $C$ and an $E_\infty$-algebra $A$ in $C$, we denote by $\text{Mod}_A(C)$ the $\infty$-category of left modules over $A$ in $C$ as in [29, 4.2.1.13]. As $A$ is an $E_\infty$-algebra, the $\infty$-category is also equivalent to the $\infty$-category of right modules over $A$ in $C$ by [29, break 4.5.1.6].

(5) We denote the $\infty$-category of spectra by $\text{Sp}$ as in [29, 1.4.3.1]. By a commutative ring spectrum we mean an $E_\infty$-algebra in $\text{Sp}$. We denote by $S$ the sphere spectrum. If $A$ is a discrete commutative ring, we denote by $HA$ its associated Eilenberg–Mac Lane spectrum.

(6) Given a commutative ring spectrum $R$, by an $R$-algebra we mean an $A_\infty$-algebra in $\text{Mod}_R(\text{Sp})$. By a commutative $R$-algebra, we mean an $E_\infty$-algebra in $\text{Mod}_R(\text{Sp})$. See also Definition 2.3 for the dual case of coalgebras.

(7) For a discrete commutative ring $A$, an $A$-differential graded algebra is a monoid object in the symmetric monoidal category of chain complexes in $A$-modules. We call these $A$-DGAs.

2 | TOPOLOGICAL COHOCHSCHILD HOMOLOGY

We provide here, in Definition 2.10, the notion of coTHH for an $A_\infty$-coalgebra in a general symmetric monoidal $\infty$-category. We show that it is equivalent to the model categorical approach of [20] and [4] in Proposition 2.13.

Cocommutative coalgebras in $\infty$-categories were introduced in [30], and a more general definition of coalgebras was given in [33]. Essentially, coalgebras are algebras in the opposite category. If $\mathcal{O}$ is an $\infty$-operad, and $C$ is an $\mathcal{O}$-monoidal $\infty$-category, as in [29, 2.1.1.18], then we need to describe the $\mathcal{O}$-monoidal structure on its opposite category $C^{\text{op}}$. The case $\mathcal{O}^\otimes = \text{Com}^\otimes$ is done in [29, 2.4.2.7], and it can be generalized using [5].

**Definition 2.1** [33, 2.1]. Let $\mathcal{O}$ be an $\infty$-operad. Let $C$ be an $\mathcal{O}$-monoidal $\infty$-category. The $\infty$-category of $\mathcal{O}$-coalgebras in $C$ is defined as:

$$\text{CoAlg}_{\mathcal{O}}(C) = \left( \text{Alg}_{\mathcal{O}}(C^{\text{op}}) \right)^{\text{op}}.$$ 

We shall mostly be interested in the case where $\mathcal{O}^\otimes = \text{Assoc}^\otimes$ or $\mathcal{O}^\otimes = \text{Com}^\otimes$.

**Remark 2.2.** By [33, 2.3], if $C$ is the nerve of a symmetric monoidal (ordinary) category $M$, then the $\infty$-category $\text{CoAlg}_{A_\infty}(C)$ corresponds precisely to the nerve of the category of (strictly) coassociative counital coalgebras in $M$. Similarly, the $\infty$-category $\text{CoAlg}_{E_\infty}(C)$ corresponds precisely to the nerve of the category of (strictly) cocommutative counital coalgebras in $M$.

**Definition 2.3.** Let $R$ be a commutative ring spectrum. An $R$-coalgebra is an $A_\infty$-coalgebra in $\text{Mod}_R(\text{Sp})$. A cocommutative $R$-coalgebra is an $E_\infty$-coalgebra in $\text{Mod}_R(\text{Sp})$.

Recall that an $\mathcal{O}$-algebra in $C$ is simply a lax $\mathcal{O}$-monoidal functor $\mathcal{O} \to C$. In other words, we have the equivalence $\text{Alg}_{\mathcal{O}}(C) \simeq \text{Fun}_{\mathcal{O}}^{\text{lax}}(\mathcal{O}, C)$. We have a dual characterization for coalgebras.
Definition 2.4. Let $\mathcal{O}$ be an $\infty$-operad. Let $C$ and $D$ be $\mathcal{O}$-monoidal $\infty$-categories. We say that $F : C \to D$ is colax $\mathcal{O}$-monoidal (sometimes also called op lax $\mathcal{O}$-monoidal) if its opposite $F^{\text{op}} : C^{\text{op}} \to D^{\text{op}}$ is lax $\mathcal{O}$-monoidal. We denote the category of colax $\mathcal{O}$-monoidal functor by $\text{Fun}_{\mathcal{O}}^{\text{colax}}(C, D) = (\text{Fun}_{\mathcal{O}}^{\text{lax}}(C^{\text{op}}, D^{\text{op}}))^{\text{op}}.$

Proposition 2.5. Let $\mathcal{O}$ be an $\infty$-operad. Let $C$ be an $\mathcal{O}$-monoidal category. Then we have an equivalence:

$$\text{CoAlg}_{\mathcal{O}}(C) \simeq \text{Fun}_{\mathcal{O}}^{\text{colax}}(\mathcal{O}, C).$$

Proof. Recall from [29, 2.4.2.7] and [5] that $C^{\text{op}}$ is $\mathcal{O}$-monoidal as follows. Given a coCartesian fibration $p : C^{\otimes} \to \mathcal{O}^{\otimes}$ that provides $C$ its $\mathcal{O}$-monoidal structure, its straightening is a functor $F : \mathcal{O}^{\otimes} \to \mathcal{C}^\infty,$ with value in the $\infty$-category of not necessarily small $\infty$-categories, as in [29, 3.0.0.5]. The functor $F$ also classifies a Cartesian fibration $\hat{p} : \hat{C}^{\otimes} \to (\mathcal{O}^{\otimes})^{\text{op}}.$ The fiber of $X \in \mathcal{O}$ over $\hat{p}$ is equivalent to $(C_X) –$ the fiber of $X$ over $p.$ Taking the opposite yields a coCartesian fibration $\hat{p}^{\text{op}} : (\hat{C}^{\otimes})^{\text{op}} \to \mathcal{O}^{\otimes}.$ This coCartesian fibration provides an $\mathcal{O}$-monoidal structure on $C^{\text{op}}.$ Note moreover that if we view $\mathcal{O}$ as a $\mathcal{O}$-monoidal category, then its $\mathcal{O}$-monoidal structure is given by the coCartesian fibration $\mathcal{O}^{\otimes} \to \mathcal{O}^{\otimes}.$ Thus, the associated Cartesian fibration $\check{\mathcal{O}}^{\otimes} \to (\mathcal{O}^{\otimes})^{\text{op}}$ is simply $(\mathcal{O}^{\otimes})^{\text{op}} \to (\mathcal{O}^{\otimes})^{\text{op}}.$ Hence, taking opposites again, the opposite $\mathcal{O}$-monoidal structure of $\mathcal{O}$ is given by the coCartesian fibration $\mathcal{O}^{\otimes} \to \mathcal{O}^{\otimes},$ that is, is $\mathcal{O}$ again. Hence, we obtain:

$$\text{Fun}_{\mathcal{O}}^{\text{colax}}(\mathcal{O}, C) = (\text{Fun}_{\mathcal{O}}^{\text{lax}}(\mathcal{O}, C^{\text{op}}))^{\text{op}} \simeq (\text{Alg}_{\mathcal{O}}(C^{\text{op}}))^{\text{op}} = \text{CoAlg}_{\mathcal{O}}(C).$$

Remark 2.6. A functor $F : C \to D$ is a strong $\mathcal{O}$-monoidal functor if and only if its opposite $F : C^{\text{op}} \to D^{\text{op}}$ is a strong $\mathcal{O}$-monoidal functor. This follows from [5, 1.3]. In particular, we recover the well-known result that strong $\mathcal{O}$-monoidal functors are colax $\mathcal{O}$-monoidal.

To define the cyclic bar and cobar construction in the $\infty$-category setting, we use the approach of [32]. For any symmetric monoidal $\infty$-category $C^{\otimes}$ (or more generally any $\infty$-operad), we can define its active part $(C^{\otimes})_{\text{act}}$ as the pullback (see [29, 2.2.4.1])

$$\begin{array}{ccc}
(C^{\otimes})_{\text{act}} & \longrightarrow & N(\text{Fin}) \\
\downarrow & & \downarrow \\
C^{\otimes} & \longrightarrow & N(\text{Fin}_n).
\end{array}$$

Here $\text{Fin} \subseteq \text{Fin}_n$ is the subcategory consisting of the active morphisms. The fiber of $(C^{\otimes})_{\text{act}}$ over a finite set $I$ in $\text{Fin}$ is given by $C^I.$ An $\mathbb{A}_\infty$-algebra $A$ in $C$ is a lax monoidal functor $A^{\otimes} : \text{Assoc}^{\otimes} \to C^{\otimes}.$ It extends to $A^{\otimes} : (\text{Assoc}^{\otimes})_{\text{act}} \to (C^{\otimes})_{\text{act}}.$ The adjoint of the identity functor $C \to C$ is the functor

$$\otimes : (C^{\otimes})_{\text{act}} \to C,$$

defined by $(X_1, \ldots, X_n) \mapsto X_1 \otimes \cdots \otimes X_n,$ as in [32, III.3.2]. From [32, B.1, B.2], we obtain a functor:

$$N(\Delta^{\text{op}}) \longrightarrow (\text{Assoc}^{\otimes})_{\text{act}}.$$
We therefore obtain a simplicial object $\mathcal{B}(A)$ in $\mathcal{C}$, that is, an object in the $\infty$-category $\text{Fun}(N(\Delta^{\text{op}}), \mathcal{C})$, as the composite of functors:

$$N(\Delta^{\text{op}}) \to (\text{Assoc})_{\text{act}}^{\otimes} \to (C^{\otimes})_{\text{act}}^{\otimes} \to \mathcal{C}.$$ 

This is making precise the cyclic bar construction in $\mathcal{C}$:

$$\cdots \to A \otimes A \otimes A \to A \otimes A \to A.$$

**Remark 2.7.** If $\mathcal{C}$ is the nerve of a symmetric monoidal ordinary category $\mathcal{M}$ and $A$ is a (strictly) associative algebra in $\mathcal{M}$, then $\mathcal{B}(A)$ corresponds to the usual cyclic bar complex of $A$ in $\mathcal{M}$.

**Definition 2.8.** Let $\mathcal{C}$ be a symmetric monoidal $\infty$-category that admits geometric realizations. For $A$ an $\mathbb{A}_\infty$-algebra in $\mathcal{C}$, we define its *topological Hochschild homology* $\text{THH}(A)$ to be the geometric realization in $\mathcal{C}$ of the simplicial object $\mathcal{B}(A)$.

**Remark 2.9.** By [32, B.5], our definition of $\text{THH}(A)$ is equivalent to the one of [32, III.2.3] where we ignore the cyclotomic structures.

Since our description of the cyclic bar construction is completely natural in the $\mathbb{A}_\infty$-algebra $A$, it defines a functor:

$$\mathcal{B}_*(\mathcal{C})(-): \text{Alg}_{\mathbb{A}_\infty}(\mathcal{C}) \to \text{Fun}(N(\Delta^{\text{op}}), \mathcal{C}),$$

for a symmetric monoidal $\infty$-category $\mathcal{C}$. In particular, if we apply this to the opposite symmetric monoidal $\infty$-category $\mathcal{C}^{\text{op}}$, we obtain the functor:

$$\text{Alg}_{\mathbb{A}_\infty}(\mathcal{C}^{\text{op}}) \to \text{Fun}(N(\Delta^{\text{op}}), \mathcal{C}^{\text{op}}).$$

Taking opposite on each side of this functor, we obtain:

$$\left(\text{Alg}_{\mathbb{A}_\infty}(\mathcal{C}^{\text{op}})\right)^{\text{op}} \to \left(\text{Fun}(N(\Delta^{\text{op}}), \mathcal{C}^{\text{op}})\right)^{\text{op}} = \text{Fun}(N(\Delta), (\mathcal{C}^{\text{op}})^{\text{op}}),$$

that is, we constructed a functor

$$\text{coBar}_*(\mathcal{C})(-): \text{CoAlg}_{\mathbb{A}_\infty}(\mathcal{C}) \to \text{Fun}(N(\Delta), \mathcal{C}).$$

This defines a *cobar* cyclic complex in $\mathcal{C}$:

$$\cdots \to C \otimes C \otimes C \to C \otimes C \to C.$$

**Definition 2.10.** Let $\mathcal{C}$ be a symmetric monoidal $\infty$-category that admits totalizations. Let $C$ be an $\mathbb{A}_\infty$-coalgebra in $\mathcal{C}$. Its *topological co Hochschild homology* $\text{coTHH}(C)$ is defined to be the totalization in $\mathcal{C}$ of the cosimplicial object $\text{coBar}_*(\mathcal{C})(C)$. 

$$\cdots \to C \otimes C \otimes C \leftarrow C \otimes C \leftarrow C \leftarrow C.$$
Notation 2.11. We shall sometimes write $\text{coTHH}^C(C)$ to emphasize the symmetric monoidal $\infty$-category $C$ that is being considered. If $R$ is an $E_\infty$-algebra in $C$, we may write $\text{coTHH}^{\text{Mod}_R(C)}(C)$ simply by $\text{coTHH}^R(C)$, where $C$ is an $A_\infty$-algebra in $\text{Mod}_R(C)$. For $C = \text{Sp}$ we follow the notation of the algebraic case and denote by $\text{coTHH}_n^R(C)$ the homotopy groups $\pi_n(\text{coTHH}^R(C))$, for any $R$-coalgebra $C$. Furthermore if $R = \mathbb{S}$, we omit $\mathbb{S}$ and write $\text{coTHH}(C)$ instead of $\text{coTHH}^\mathbb{S}(C)$.

Let $\mathcal{M}$ be a symmetric monoidal model category (see [17, 4.2.6]), and denote by $W$ its class of weak equivalences. Let $\mathcal{M}_c$ denote the full subcategory spanned by the cofibrant objects. Let us denote $N(M_c)[W^{-1}]$ its underlying symmetric monoidal $\infty$-category as defined in [29, 4.1.7.6]. This is also sometimes called the (symmetric monoidal) Dwyer–Kan localization of the (symmetric monoidal) model category $\mathcal{M}$.

In [4, 2.3], the authors defined the topological coHochschild homology $\text{coTHH}^\mathcal{M}(C)$ of a (strictly) coassociative counital coalgebra $C$ in $\mathcal{M}$ to be the homotopy limit in $\mathcal{M}$ of the cosimplicial object given by the cyclic cobar complex. We show that our definition agrees with the one in model categories in Proposition 2.13.

Lemma 2.12. Let $F : C \to D$ be a strong symmetric monoidal functor between symmetric monoidal $\infty$-categories. Then:

$$F \circ \text{Bar}^C_{\mathcal{M}}(-) \simeq \text{Bar}^D_{\mathcal{M}}(-) \circ F.$$  

Dually, we also obtain:

$$F \circ \text{coBar}^C_{\mathcal{M}}(-) \simeq \text{coBar}^D_{\mathcal{M}}(-) \circ F.$$  

Proof. By [22, 4.4], we obtain the commutative diagram of $\infty$-categories:

$$
\begin{array}{ccc}
(C \otimes)^\otimes_{\mathcal{M}} & \to & C \\
F_{\text{act}} \downarrow & & \downarrow F \\
(D \otimes)^\otimes_{\mathcal{M}} & \to & D.
\end{array}
$$

Let $A$ be an $A_\infty$-algebra in $C$. By the above result, we obtain that the composite

$$
(\text{Assoc}^\otimes)^\otimes_{\mathcal{M}} \xrightarrow{A^\otimes} (C \otimes)^\otimes_{\mathcal{M}} \xrightarrow{C} D
$$

is equivalent to the composite

$$
(\text{Assoc}^\otimes)^\otimes_{\mathcal{M}} \xrightarrow{F(A)^\otimes} (D \otimes)^\otimes_{\mathcal{M}} \xrightarrow{D} D.
$$

Therefore, we obtain $F(\text{Bar}^C_{\mathcal{M}}(A)) \simeq \text{Bar}^D_{\mathcal{M}}(F(A))$. We obtain the dual statement for cobar by applying our above argument to the strong symmetric monoidal functor $F^{\text{op}} : C^{\text{op}} \to D^{\text{op}}$.  

Proposition 2.13. Let $\mathcal{M}$ be a combinatorial symmetric monoidal model category. Let $C$ be a coassociative coalgebra in $\mathcal{M}$ that is cofibrant as an object in $\mathcal{M}$. Then $\text{coTHH}^\mathcal{M}(C)$ is weakly equivalent to $\text{coTHH}^{N(\mathcal{M}_c)[W^{-1}]}(C)$.  


Proof. In [34, 3.1], there is a functor of \( \infty \)-categories:

\[ \iota : N(\coAlg(M_c))[W^{-1}] \to \CoAlg_{\infty}(N(M_c)[W^{-1}]) \]

which is the identity on objects in \( N(M_c)[W^{-1}] \). Here \( W' \) denotes the weak equivalences in \( M \) that are also maps in \( \coAlg(M_c) \). The functor \( \iota \) allows one to consider the (strictly) coassociative counital coalgebra \( C \) in \( M \) as an \( \mathbb{A}_\infty \)-coalgebra \( \iota(C) \) in the Dwyer–Kan localization of \( M \). Since the localization \( N(M_c) \to N(M_c)[W^{-1}] \) is a strong symmetric monoidal functor, then by Remark 2.7, [29, 1.3.4.25], and Lemma 2.12, the two definitions of the cyclic cobar complexes agree. The desired result follows by [29, 1.3.4.23], as the homotopy limit of the cyclic cobar construction of \( C \) in \( M \) is equivalent to the totalization of the cyclic cobar construction of \( \iota(C) \) in the Dwyer–Kan localization of \( M \).

\[ \square \]

Remark 2.14. In the proof of Proposition 2.13, we are not claiming that the \( \infty \)-category \( N(\coAlg(M_c))[W^{-1}] \) is the Dwyer–Kan localization of a model structure on \( \coAlg(M) \) as such a model structure may not exist, see Remark 2.15. Here we are instead referring to the Dwyer–Kan localization of the \( \infty \)-category \( N(\coAlg(M_c)) \) with respect to the class of edges induced by the weak equivalences \( W' \), as in [29, 1.3.4.1].

Remark 2.15. The model category approach faces several challenges. First of all, given a combinatorial symmetric monoidal model category \( M \), there are no conditions in general to lift the model structure on \( M \) to a model structure on the category of (strictly) coassociative counital coalgebras \( \coAlg(M) \). In the cases where a model structure exists, the model structure of \( M \) is generally replaced by a Quillen equivalent one that is not a monoidal model category, see [13, 16]. Second, given a model structure in \( \coAlg(M) \) induced by one on \( M \), there is no reason to expect that strictly coassociative coalgebras in \( M \) are equivalent to \( \mathbb{A}_\infty \)-coalgebras in the Dwyer–Kan localization of \( M \), unlike the case for algebras (as in [29, 4.1.8.4]). As a matter of fact, there is indication that these do not correspond. In particular, for \( M \) any symmetric monoidal model category of spectra, such as symmetric spectra or orthogonal spectra, it is shown that strictly coassociative coalgebras do not correspond to \( \mathbb{A}_\infty \)-coalgebras in the \( \infty \)-category of spectra \( \Sp \), see [36] and [34, 5.6].

3 | DUALITY BETWEEN ALGEBRAS AND COALGEBRAS

From a coalgebra, one can always obtain an associated algebra by taking the dual. The converse is in general not true. We make this fact precise in this section. We also show here equivalences between subclasses of algebras and coalgebras that we introduce.

3.1 | The linear dual functor

We introduce the notion of dualization for closed symmetric monoidal \( \infty \)-categories. We recall the well-known result that the dual of a coalgebra is always an algebra.

Definition 3.1. Let \( C \) be a closed symmetric monoidal \( \infty \)-category. Denote

\[ [-, -] : C^{op} \times C \to C, \]

\[ \text{Definition 3.1} \]
its internal hom defined by the universal property \( C(X \otimes Y, Z) \simeq C(X, [Y, Z]) \). It is lax symmetric monoidal, see [15, A.5.3] or [14, I.3]. Let \( I \) be the unit of the symmetric monoidal structure of \( C \). For any object \( X \) in \( C \), define \( X^\vee \) to be \([X, I]\). This defines a lax symmetric monoidal functor:

\[
(\dashv)^\vee : \mathcal{C}^{\text{op}} \to \mathcal{C},
\]
called the linear dual functor. We denote its opposite \(((\dashv)^\vee)^{\text{op}} : \mathcal{C} \to \mathcal{C}^{\text{op}}\) again by:

\[
(\dashv)^\vee : \mathcal{C} \to \mathcal{C}^{\text{op}},
\]
and also refer to it as the linear dual functor.

**Example 3.2.** If \( C \) is the nerve of the category of vector spaces over a field \( k \), then the linear dual functor \((\dashv)^\vee\) is precisely the dual of a vector space over \( k \).

**Example 3.3.** If \( C \) is the \( \infty \)-category of spectra \( \mathcal{S} \), then the linear dual is precisely the Spanier–Whitehead dual.

The linear dual is what is called a self-adjoint (contravariant) functor: a property that we describe in the following result.

**Proposition 3.4.** Let \( C \) be a closed symmetric monoidal \( \infty \)-category. Then we obtain an adjunction:

\[
\begin{array}{ccc}
\mathcal{C} & \xleftarrow{\dashv} & \mathcal{C}^{\text{op}} \\
\downarrow & & \downarrow \\
(\dashv)^\vee & \xrightarrow{\dashv} & (\dashv)^{\text{op}} \end{array}
\]

**Proof.** This is a classical result in ordinary categories. It follows from the string of equivalences:

\[
\mathcal{C}^{\text{op}}(X^\vee, Y) \simeq C(Y, X^\vee) \\
\simeq C(Y \otimes X, I) \\
\simeq C(X \otimes Y, I) \\
\simeq C(X, Y^\vee),
\]
for every \( X \) and \( Y \) in \( C \). Here we used the natural symmetry equivalence \( X \otimes Y \simeq Y \otimes X \). If we choose \( Y = X^\vee \), then the identity morphism \( X^\vee \to X^\vee \) in \( \mathcal{C}^{\text{op}} \) defines a natural morphism \( X \to X^{\vee\vee} \) via the equivalence

\[
C(X, X^{\vee\vee}) \simeq \mathcal{C}^{\text{op}}(X^\vee, X^\vee),
\]
defined above. This provides the unit of the adjunction (see [28, 5.2.2.8]).

The right adjoint functor \((\dashv)^\vee : \mathcal{C}^{\text{op}} \to \mathcal{C}\), being lax symmetric monoidal, lifts to the category of algebras

\[
(\dashv)^\vee : \text{CoAlg}_\mathcal{C}(\mathcal{C})^{\text{op}} \simeq \text{Alg}_\mathcal{C}(\mathcal{C}^{\text{op}}) \longrightarrow \text{Alg}_\mathcal{C}(\mathcal{C}),
\]
for any $\infty$-operad $\mathcal{O}$. This functor provides a description of the well-known fact that the dual of a coalgebra is always an algebra. Since limits in algebras are computed in their underlying category, the linear dual $(-)^\vee : \text{CoAlg}_\mathcal{O}(C)^{\text{op}} \to \text{Alg}_\mathcal{O}(C)$ remains a limit-preserving functor.

However, as the linear dual is not strong monoidal in general, it does not lift to a functor $(-)^\vee : \text{Alg}_\mathcal{O}(C) \to \text{CoAlg}_\mathcal{O}(C)^{\text{op}}$. In other words, if $A$ is an $\mathcal{O}$-algebra in $C$, then we cannot claim $A^\vee$ is an $\mathcal{O}$-coalgebra in $C$ in general. We are interested in the cases where an $\mathcal{O}$-algebra $A$ induces an $\mathcal{O}$-coalgebra structure on its linear dual via a comultiplication induced by:

$$A^\vee \longrightarrow (A \otimes A)^\vee$$

where the vertical map is an equivalence. We are also interesting in knowing conditions for which the linear dual functor $(-)^\vee : \text{CoAlg}_\mathcal{O}(C)^{\text{op}} \to \text{Alg}_\mathcal{O}(C)$ is an equivalence of $\infty$-categories.

### 3.2 Proper algebras and coalgebras

It is a well-known result that algebras and coalgebras over finite-dimensional vector spaces are anti-equivalent. We remind here the result of [30] that shows this in greater generality.

**Definition 3.5** [29, 4.6.1]. An object $X$ is said to be dualizable in a symmetric monoidal $\infty$-category $C$ if there exists another object $Y$ in $C$, together with maps:

$$e : Y \otimes X \to \mathbb{1}, \quad c : \mathbb{1} \to X \otimes Y,$$

for which the composite maps:

$$X \simeq \mathbb{1} \otimes X \xrightarrow{c \otimes \text{id}_X} X \otimes Y \otimes X \xrightarrow{\text{id} \otimes e} X \otimes \mathbb{1} \simeq X,$$

$$Y \simeq Y \otimes \mathbb{1} \xrightarrow{\text{id} \otimes c} Y \otimes X \otimes Y \xrightarrow{e \otimes \text{id}_Y} \mathbb{1} \otimes Y \simeq Y,$$

are homotopic to the identity. In other words, an object $X$ is dualizable if and only if it is dualizable in the homotopy category $hC$, see [29, 4.6.1.6]. Let $C_{fd}$ denote the full subcategory of $C$ spanned by the dualizable objects. By [30, 3.2.4], the tensor product of $C$ is closed in $C_{fd}$ and thus $C_{fd}$ is also symmetric monoidal.

**Example 3.6** [30, 3.2.1]. If $C$ is the nerve of the category of vector spaces over a field $k$, then the dualizable objects are precisely the finite-dimensional vector spaces over $k$.

**Example 3.7.** The dualizable objects in the $\infty$-category of spectra $\text{Sp}$ are precisely the finite spectra.

**Example 3.8** [30, 3.2.3]. Let $R$ be a commutative ring spectrum. Let $C$ be the $\infty$-category of $R$-modules $\text{Mod}_R(\text{Sp})$. Then the dualizable objects in $C$ are precisely the compact objects in $C$, that
is, the perfect $R$-modules. An $R$-module is said to be perfect if it lies in the smallest subcategory of $\text{Ho} (\text{Mod}_R(\text{Sp}))$ that contains $R$ and is closed under triangles and retracts.

**Example 3.9.** Let $k$ be a field. An $Hk$-module $X$ in $\text{Sp}$ is dualizable in $\text{Mod}_{Hk}(\text{Sp})$ if and only if $\pi_* X$ is finitely generated as a graded $k$-module.

**Lemma 3.10.** Let $C$ be a closed symmetric monoidal $\infty$-category. If $X$ is dualizable in $C$, then its dual is equivalent to the linear dual $X^\vee$.

**Proof.** Let $X$ be a dualizable object in $C$, with dual $Y$. Then $Y \otimes - : C \to C$ is a right adjoint of $X \otimes - : C \to C$. The unit and counit of the adjunction are induced by the maps $e : Y \otimes X \to \mathbb{I}$ and $c : \mathbb{I} \to X \otimes Y$. The universal property of tensor product with its internal hom gives the equivalence of functors on $C$:

$$[X, -] \simeq Y \otimes -,$$

as they are both the right adjoint of the functor $X \otimes - : C \to C$. Then in particular we get the equivalence $X^\vee \simeq Y$. □

**Proposition 3.11** [30, 3.2.4]. Let $C$ be a closed symmetric monoidal $\infty$-category. Then the linear dual determines an equivalence of symmetric monoidal $\infty$-categories:

$$(-)^\vee : (C_{\text{fd}})^{\text{op}} \xrightarrow{\simeq} C_{\text{fd}},$$

with the inverse equivalence given by itself.

**Definition 3.12** [29, 4.6.4.2]. Let $C$ be a symmetric monoidal $\infty$-category. Let $\mathcal{O}$ be an $\infty$-operad. An $\mathcal{O}$-algebra $A$ in $C$ is said to be proper if $A$ is dualizable as an object in $C$. Similarly, we say that an $\mathcal{O}$-coalgebra $C$ in $C$ is proper if it is dualizable as an object in $C$.

There is an anti-equivalence between proper algebras and proper coalgebras, over any $\infty$-operad, given by the linear dual functor. This follows directly from Proposition 3.11.

**Corollary 3.13** [30, 3.2.5]. Let $C$ be a closed symmetric monoidal $\infty$-category. Let $\mathcal{O}$ be an $\infty$-operad. Then the linear dual induces an equivalence:

$$(-)^\vee : \text{CoAlg}_{\mathcal{O}}(C_{\text{fd}})^{\text{op}} \xrightarrow{\simeq} \text{Alg}_{\mathcal{O}}(C_{\text{fd}}),$$

where the inverse equivalence is given by itself.

### 3.3 Quasi-proper algebras and coalgebras

We want to generalize the anti-equivalence between proper algebras and coalgebras of Corollary 3.13, by considering a larger class of algebras and coalgebras. We do so in Theorem 3.31. We introduce the notions of quasi-proper algebras and coalgebras. This allows us to extend the equivalence between $\text{THH}$ and $\text{coTHH}$ (Theorem 1.2) in the next section, see Theorem 4.1. Given $A$ an algebra, we are also seeking conditions for its linear dual $A^\vee$ to be a coalgebra.
Condition 1. Let $C$ be a closed symmetric monoidal $\infty$-category. We say an object $X$ in $C$ satisfies this condition if the lax symmetric monoidal structure map of the linear dual functor $(-)^\vee : C^{\text{op}} \to C$:

$$(X^\vee)^{\otimes n} \to (X^{\otimes n})^\vee$$

is an equivalence in $C$ for all $n \geq 0$.

Condition 2. Let $C$ be a closed symmetric monoidal $\infty$-category. We say an object $X$ in $C$ satisfies this condition if the natural unit map from the self-adjoint property of the linear dual

$$X \to X^{\vee \vee}$$

is an equivalence in $C$.

Remark 3.14. In Condition 1, the case $n = 0$ is stating that $\mathbb{1}^\vee \simeq \mathbb{1}$ which is always satisfied, where $\mathbb{1}$ is the unit of the closed symmetric monoidal $\infty$-category $C$. The case $n = 1$ is also always satisfied for any $X$ in $C$.

We provide examples of spectra and examples of chain complexes that do not satisfy these conditions. These examples provide a justification for the hypothesis of Theorem 1.3.

Example 3.15. The Eilenberg–Mac Lane spectrum $HF_p$ satisfies Condition 1 but does not satisfy Condition 2 in the $\infty$-category of spectra $\text{Sp}$. It is known that the Spanier–Whitehead dual of $HF_p$ is trivial [26]. This shows that

$$HF_p^{\vee \vee} \simeq 0$$

and that $HF_p$ does not satisfy Condition 2 in $\text{Sp}$. Furthermore, we have the equivalences

$$[HF_p^{\wedge n}, S] \simeq [HF_p^{\wedge (n-1)}, [HF_p, S]] \simeq 0$$

and therefore both sides of the map in Condition 1 are trivial. Therefore, $HF_p$ satisfies Condition 1 in $\text{Sp}$.

Example 3.16. We provide here an example of an object that satisfies Condition 2 but not Condition 1 in the $\infty$-category of $Hk$-modules $\text{Mod}_{Hk}(\text{Sp})$, where $k$ is a field. The homotopy category of $Hk$-modules is equivalent to the category of graded $k$-modules as a symmetric monoidal category. This follows from the Küneth and Ext spectral sequences in [12, IV.4.1]. Let $X$ denote the $Hk$-module corresponding to the chain complex $k[x_1, x_1^{-1}]$ with trivial differentials, where $|x_1| = 1$. Note that the homology of this chain complex and therefore the homotopy of the corresponding $Hk$-module is $k[x_1, x_1^{-1}]$. Since dualization in $Hk$-modules results in graded $k$-module dualization at the level of homotopy groups, we have:

$$\pi_* (X^\vee) \cong k[x_1, x_1^{-1}]^\vee \cong k[x_1, x_1^{-1}]$$

By inspection, it is clear that $X$ satisfies Condition 2 as an $Hk$-module. Note that $X \wedge_{Hk} X$ has countably infinite-dimensional homotopy groups at each degree. For instance, the basis for degree
0 homotopy group is given by $x_i^1 \otimes x_i^{-1}$ over all $i \in \mathbb{Z}$. Therefore, $(X \wedge_{Hk} X)^\vee$ has \textit{uncountably} infinite-dimensional homotopy at each degree. On the other hand, $X^\vee \wedge_{Hk} X^\vee$ has countably infinite-dimensional homotopy at each degree. Therefore, the map

$$X^\vee \wedge_{Hk} X^\vee \to (X \wedge_{Hk} X)^\vee$$

cannot be surjective in homotopy showing that $X$ does not satisfy Condition 1. This example justifies the (co)connectivity hypothesis in Theorem 1.3.

\textbf{Example 3.17.} Let $X$ be a countably infinite-dimensional $k$-vector space, where $k$ is a field. Let $HX$ denote the corresponding $Hk$-module. Neil Strickland explains that $HX$ does not satisfy Condition 1 in $Hk$-modules because the map mentioned in Condition 1 is not surjective in homotopy for $HX$. To see this, let $\{x_i\}_{i \in \mathbb{Z}}$ be a basis for $X$. We consider the map in Condition 1 for the case $n = 2$. At the level of homotopy, this map

$$\varphi : X^\vee \otimes_k X^\vee \to (X \otimes_k X)^\vee$$

is given by:

$$\varphi(f \otimes_k g)(x \otimes_k y) = f(x)g(y).$$

The $k$-linear map $h : X \otimes_k X \to k$ given by $h(x_i \otimes_k x_j) = 0$ if $i \neq j$ and $h(x_i \otimes_k x_i) = 1$ for all $i$ is not in the image of $\varphi$. This is due to the fact that the matrix $(h(x_i \otimes_k x_j))_{i,j}$ has infinite rank where the matrix $(\varphi(\sum_{r = 1}^r f_r \otimes g_r)(x_i \otimes_k x_j))_{i,j}$ has rank at most $r$. Therefore, $\varphi$ is not surjective and $X$ does not satisfy Condition 1 in $Hk$-modules. This example justifies the finiteness hypothesis in Theorem 1.3. Furthermore, dualization increases cardinality of an infinite-dimensional vector space. In particular, $\pi_0((HX)^{\vee \vee}) = X^{\vee \vee}$ has a larger cardinality than $\pi_0 HX = X$ and therefore $X$ does not satisfy Condition 2 either.

We now record some results that capture if the linear dual preserves the conditions above.

\textbf{Lemma 3.18.} Let $C$ be a closed symmetric monoidal $\infty$-category. Let $X$ be an object in $C$. If $X$ satisfies Condition 2, then $X^\vee$ satisfies Condition 2.

\textbf{Proof.} We have the equivalence:

$$X^\vee \to (X^\vee)^{\vee \vee} \simeq (X^{\vee \vee})^\vee \simeq X^\vee,$$

using Condition 2 on $X$. \hfill $\square$

\textbf{Lemma 3.19.} Let $C$ be a closed symmetric monoidal $\infty$-category. Suppose $X$ satisfies Conditions 1 and 2. Then $X^\vee$ satisfies Condition 1 if and only if $X^{\otimes n}$ satisfies Condition 2 for all $n \geq 0$.

\footnote{https://mathoverflow.net/questions/56255/duals-and-tensor-products}
Proof. We have the following commutative diagram in $C$:

\[
\begin{array}{ccc}
X^\otimes_n & \longrightarrow & (X^\otimes_n)^{\vee}\vee \\
\downarrow \cong & & \downarrow \cong \\
(X^{\vee\vee})^\otimes_n & \longrightarrow & ((X^{\vee})^\otimes_n)^{\vee}.
\end{array}
\]

The left vertical map is an equivalence as $X$ satisfies Condition 2. The right vertical map is an equivalence as $X$ satisfies Condition 1. The result follows. □

Definition 3.20. Let $C$ be a closed symmetric monoidal $\infty$-category. An object $X$ in $C$ is said to be quasi-dualizable if $X$ satisfies both Conditions 1 and 2. It is called weak quasi-dualizable if $X$ satisfies Condition 1.

Corollary 3.21. Let $C$ be a closed symmetric monoidal $\infty$-category. If an object in $C$ is dualizable, then it is quasi-dualizable in $C$.

Proof. Apply Lemma 3.10 and Proposition 3.11. □

Remark 3.22. Corollary 3.21 shows that we have the implications:

\[
dualizable \Rightarrow \text{quasi-dualizable} \Rightarrow \text{weak quasi-dualizable}.
\]

Example 3.15 provides an instance of an object that is weak quasi-dualizable but not quasi-dualizable; therefore, weak quasi-dualizability is a strictly weaker condition than quasi-dualizability. For a Noetherian ring $A$ with finite global dimension, we show in Section 5 that a (co)connective finite type $HA$-module is quasi-dualizable. If such an $HA$-module has unbounded homotopy, then it is not dualizable due to Proposition 5.3. This shows that quasi-dualizability is a strictly weaker condition than dualizability. For instance, $H^F_p \wedge H^F_p$ is quasi-dualizable but not dualizable in $H^F_p$-modules.

Remark 3.23. The distinction made in the previous remark is not seen in the classical case. Indeed, let $C$ be the nerve the category of vector spaces over a field $k$. We saw in Example 3.6 that a vector space is dualizable if and only if it is finite-dimensional. In fact, the discussion in Example 3.17 shows that infinite-dimensional vector spaces cannot satisfy Condition 1 nor Condition 2. Therefore, given a vector space $V$ over $k$, the following are equivalent:

1. $V$ is finite-dimensional,
2. $V$ is dualizable,
3. $V$ is quasi-dualizable,
4. $V$ is weak quasi-dualizable, that is, satisfies Condition 1,
5. $V$ satisfies Condition 2.

Definition 3.24. Let $C$ be a closed symmetric monoidal $\infty$-category. Let $\mathcal{O}$ be an $\infty$-operad. We say an $\mathcal{O}$-algebra $A$ in $C$ is weak quasi-proper if $A$ is weak quasi-dualizable as an object in $C$. We denote by $(\text{Alg}_{\mathcal{O}}(C))_w$ the full subcategory in $\text{Alg}_{\mathcal{O}}(C)$ spanned by the weak quasi-proper algebras.
We first show that a weak quasi-dualizable algebra $A$ has the property that its linear dual $A^\vee$ has a natural induced coalgebra structure.

**Proposition 3.25.** Let $C$ be a closed symmetric monoidal $\infty$-category. Let $\mathcal{O}$ be an $\infty$-operad. Let $A$ be a weak quasi-proper $\mathcal{O}$-algebra in $C$, then $A^\vee$ is an $\mathcal{O}$-coalgebra in $C$. More generally, the linear dual functor induces a functor:

$$(-)^\vee : \text{Alg}_{\mathcal{O}}(C) \to \text{CoAlg}_{\mathcal{O}}(C)^{\text{op}}.$$ 

We prove the above proposition after showing the following lemma.

**Lemma 3.26.** Let $C$ and $D$ be symmetric monoidal $\infty$-categories. Let $\mathbb{1}_C$ and $\mathbb{1}_D$ be the unit objects of the monoidal structures of $C$ and $D$, respectively. Let $F : C \to D$ be a lax symmetric monoidal functor. If for all objects $X$ and $Y$ in $C$, the natural maps $F(X) \otimes F(Y) \to F(X \otimes Y)$ are equivalences, and the natural map $\mathbb{1}_D \to F(\mathbb{1}_C)$ is an equivalence, then $F$ is a strong symmetric monoidal functor.

**Proof.** Let $p : C^\otimes \to N(\text{Fin}_n)$ and $q : D^\otimes \to N(\text{Fin}_n)$ be the coCartesian fibrations that endow $C$ and $D$ of a symmetric monoidal structure. We need to show that $F$ sends $p$-coCartesian lifts to $q$-coCartesian lifts.

We first recall how we obtain the natural maps $F(X) \otimes F(Y) \to F(X \otimes Y)$. Let $m : \{2\} \to \{1\}$ be the map in $\text{Fin}_n$ that sends both 1 and 2 to 1. Let $X$ and $Y$ be objects in $C$. On the one hand, we have the $p$-coCartesian lift of $m$ with respect to $(X, Y)$ given by the edge $(X, Y) \to X \otimes Y$ in $C^\otimes$ that represents the tensor product:

$$C \times C \xleftarrow{\sim} C^\otimes_{\{2\}} \xrightarrow{m} C.$$

Similarly, we obtain the $q$-coCartesian lift $(F(X), F(Y)) \to F(X) \otimes F(Y)$ in $D^\otimes$. On the other hand, since $m$ is not an inert morphism in $\text{Fin}_n$, then the induced map $F(X, Y) \to F(X \otimes Y)$ is not a priori a $q$-coCartesian lift. However, since the Segal condition $C^\otimes_{\{n\}} \simeq C^{\text{inert}}$ is induced by the inert morphisms $\langle n \rangle \to \langle 1 \rangle$, and $F$ is lax symmetric monoidal, we obtain an equivalence $F(X, Y) \xrightarrow{\simeq} (F(X), F(Y))$. By the universal property of coCartesian lifts, we obtain a unique map up to contractible choice $F(X) \otimes F(Y) \to F(X \otimes Y)$ such that we have the commutative diagram:

$$\begin{align*}
(F(X), F(Y)) &\longrightarrow F(X) \otimes F(Y) \\
&\downarrow \\
&F(X \otimes Y).
\end{align*}$$

If we assume that the dashed map is an equivalence, then by unicity of the coCartesian lifts, we see that $F$ sends $p$-coCartesian lifts of $m$ to $q$-coCartesian lifts.

Similarly, the unique map $\iota : \langle 0 \rangle \to \langle 1 \rangle$ in $\text{Fin}_n$ induces the map $\mathbb{1}_D \to F(\mathbb{1}_C)$, and if we require it to be an equivalence, $F$ sends $p$-coCartesian lifts of $\iota$ to a $q$-coCartesian lift.

Note that morphisms in $\text{Fin}_n$ are generated under wedge sum, composition, inert morphisms and the maps $\iota$ and $m$. If $\phi : \langle n \rangle \to \langle k \rangle$ and $\phi' : \langle n' \rangle \to \langle k' \rangle$ are maps in $\text{Fin}_n$, then by the Segal...
condition and the universality of coCartesian lifts, we obtain an equivalence:

\[
\begin{array}{ccc}
C_{(n+n')} & \xrightarrow{\phi \cdot \phi'} & C_{(k+k')}\\
\downarrow \cong & & \downarrow \cong \\
C_{(n)} \times C_{(n')} & \xrightarrow{\phi \times \phi'} & C_{(k)} \times C_{(k')}.
\end{array}
\]

Therefore, \( F \) sends \( p \)-coCartesian lifts to \( q \)-coCartesian lifts. \( \square \)

**Proof of Proposition 3.25.** For any object \( X \) in \( C \), let us denote \( C_{X} \) the full subcategory of \( C \) spanned by objects equivalent to \( X^{\otimes n} \) for any \( n \geq 0 \). By [29, 2.2.1.2], since \( C_{X} \) is closed under the tensor product and contains the unit, the \( \infty \)-category \( C_{X} \) inherits a symmetric monoidal structure from \( C \).

Suppose \( X \) is weak quasi-dualizable. Then the composition

\[
C_{X} \xrightarrow{\cong} C \xrightarrow{(-)^{\vee}} C^{\text{op}}
\]

is strong symmetric monoidal, by applying the dual of Lemma 3.26 to the colax symmetric monoidal functor \((-)^{\vee} : C \to C^{\text{op}}\).

If \( X \) is an \( \mathcal{O} \)-algebra in \( C \), then \( X \) is also an \( \mathcal{O} \)-algebra in \( C_{X} \) as the inclusion \( C_{X} \subseteq C \) is strong symmetric monoidal.

Therefore, if we let \( A \) be a weak quasi-proper \( \mathcal{O} \)-algebra in \( C \), we apply the discussion above to \( C_{A} \) and we obtain that \( A^{\vee} \) is an \( \mathcal{O} \)-coalgebra in \( C \). The coalgebra structure is completely natural from the choice of algebra structure on \( A \) and thus we obtain the desired functor. \( \square \)

Now that we have obtained conditions that ensure the linear dual of an algebra is a coalgebra, we restrict further the conditions to show that the assignment is an equivalence.

**Definition 3.27.** Let \( C \) be a closed symmetric monoidal \( \infty \)-category. Let \( \mathcal{O} \) be an \( \infty \)-operad.

1. We say an \( \mathcal{O} \)-algebra in \( C \) is **quasi-proper** if \( A \) is quasi-dualizable as an object in \( C \). We denote by \((\Alg_{\mathcal{O}}(C))_{\ast} \) the full subcategory in \( \Alg_{\mathcal{O}}(C) \) spanned by the quasi-proper algebras.
2. We say an \( \mathcal{O} \)-coalgebra \( C \) is **quasi-proper** if \( C^{\vee} \) satisfies Condition 1 as an object in \( C \), and \( C \) satisfies Condition 2 as an object in \( C \). We denote by \((\CoAlg_{\mathcal{O}}(C))_{\ast} \) the full subcategory in \( \CoAlg_{\mathcal{O}}(C) \) spanned by the quasi-proper coalgebras.

**Remark 3.28.** From Remark 3.22, we obtain the following sequence of subcategories in \( \Alg_{\mathcal{O}}(C) \):

\[
\Alg_{\mathcal{O}}(C_{id}) \subseteq (\Alg_{\mathcal{O}}(C))_{\ast} \subseteq (\Alg_{\mathcal{O}}(C)).
\]

**Remark 3.29.** We do not define the notion of weak quasi-proper coalgebra as the linear dual of a coalgebra is always an algebra since the linear dual functor \((-)^{\vee} : C \to C^{\text{op}}\) is colax symmetric monoidal. We have defined the notion of quasi-proper coalgebra so that it is exactly the essential image of the linear dual functor defined in Proposition 3.25 restricted to quasi-proper algebras, as we show in Theorem 3.31.
Proposition 3.30. Let $C$ be a closed symmetric monoidal $\infty$-category. Let $\mathcal{O}$ be an $\infty$-operad. Let $C$ be an $\mathcal{O}$-coalgebra in $C$. If $C$ satisfies Conditions 1 and 2 as an object in $C$, and $C^\otimes n$ satisfies Condition 2 as an object in $C$ for all $n \geq 0$, then $C$ is a quasi-proper coalgebra in $C$.

Proof. We only need to check that $C^\vee$ satisfies Condition 1, but this follows from Lemma 3.19. $\square$

Theorem 3.31. Let $C$ be a closed symmetric monoidal $\infty$-category. Let $\mathcal{O}$ be an $\infty$-operad. Then the linear dual functor induces an equivalence of $\infty$-categories

$$(\cdot)^\vee : ((\text{CoAlg}_\mathcal{O}(C))_\ast)^{op} \simeqto (\text{Alg}_\mathcal{O}(C))_\ast,$$

where the inverse equivalence is also given by the linear dual functor $(\cdot)^\vee$.

Proof. We first begin to show that the linear dual functor

$$(\cdot)^\vee : \text{CoAlg}_\mathcal{O}(C)^{op} \to \text{Alg}_\mathcal{O}(C),$$

restricts and corestricts to the functor:

$$(\cdot)^\vee : ((\text{CoAlg}_\mathcal{O}(C))_\ast)^{op} \longrightarrow (\text{Alg}_\mathcal{O}(C))_\ast.$$

Let $C$ be a quasi-proper $\mathcal{O}$-coalgebra in $C$. Then by assumption, its linear dual $C^\vee$ satisfies Condition 1 as an object in $C$. Moreover, it also satisfies Condition 2 by Lemma 3.18. Thus, $C^\vee$ is a quasi-proper $\mathcal{O}$-algebra in $C$.

We now provide its inverse. In Proposition 3.25, we have shown that the linear dual functor lifts to a functor

$$(\cdot)^\vee : (\text{Alg}_\mathcal{O}(C))_\circ \longrightarrow (\text{CoAlg}_\mathcal{O}(C))^{op}.$$

We now show it restricts and corestricts to:

$$(\cdot)^\vee : (\text{Alg}_\mathcal{O}(C))_\ast \longrightarrow ((\text{CoAlg}_\mathcal{O}(C))_\ast)^{op}. $$

We need to verify that for $A$ a quasi-proper $\mathcal{O}$-algebra in $C$, then $A^\vee$ is a quasi-proper $\mathcal{O}$-coalgebra in $C$. Since $A$ satisfies Condition 2 as an object in $C$, then so does $A^\vee$ by Lemma 3.18. Since $A$ satisfies Condition 1 and that $A^\vee \simeq A$, then $A^\vee$ satisfies Condition 1 as an object of $C$. Thus, $A^\vee$ is indeed a quasi-proper $\mathcal{O}$-coalgebra.

We now argue that given a quasi-proper $\mathcal{O}$-algebra $A$, the unit map $A \to A^\vee$ in $C$ of the adjunction of Proposition 3.4 is a map in $\text{Alg}_\mathcal{O}(C)$. If we let $X$ be a quasi-dualizable object in $C$, then as in the proof of Proposition 3.25, the linear dual functor restricts to a strong monoidal functor $C_X \to C^{op}$. Note that its image is precisely $(C_X^\vee)^{op}$. Restricting the linear dual functor $C^{op} \to C$ to $(C_X^\vee)^{op}$, we obtain a lax symmetric monoidal functor $(C_X^\vee)^{op} \to (C_X)^{op}$ as we have the equivalence of symmetric monoidal $\infty$-categories $C_X \simeq C_X^{\vee \vee}$ by [29, 2.1.3.8] as $X$ satisfies Condition 2. Therefore, the adjunction of Proposition 3.4 restricts and corestricts to the full subcategories:
The left adjoint is strong symmetric monoidal and thus lifts to an adjunction:

\[
\text{Alg}_\sigma(C_X) \overset{(-)^\vee}{\underset{(-)^\wedge}{\leftrightarrow}} \text{Alg}_\sigma((C_X)^{op}).
\]

If we apply the discussion above to a quasi-proper algebra \( X = A \), we obtain in particular that the unit of the above adjunction \( A \to A^{\vee\vee} \) is a map of \( \mathcal{O} \)-algebras in \( C \). We can argue similarly to show that \( C \to C^{\vee\vee} \) is a map of \( \mathcal{O} \)-coalgebras for \( C \) a quasi-proper \( \mathcal{O} \)-coalgebra.

We can therefore lift the adjunction of Proposition 3.4 to an adjunction:

\[
(\text{Alg}_\sigma(C))^\ast \overset{(-)^\vee}{\underset{(-)^\wedge}{\leftrightarrow}} ((\text{CoAlg}_\sigma(C))^\ast)^{op}.
\]

By [29, 3.2.2.6], if \( A \) is a quasi-proper \( \mathcal{O} \)-algebra in \( C \), then unit map of the adjunction \( A \cong A^{\vee\vee} \) is an equivalence of \( \mathcal{O} \)-algebras in \( C \). Similarly if \( C \) is a quasi-proper \( \mathcal{O} \)-coalgebra in \( C \), then the map \( C \cong C^{\vee\vee} \) is an equivalence of \( \mathcal{O} \)-coalgebras in \( C \). Therefore, the unit and counit of the adjunction are equivalences. Thus, the adjunction is an equivalence of \( \infty \)-categories.

\[\square\]

**Remark 3.32.** In [33, 3.9], it was shown that for \( C \) a presentably symmetric monoidal \( \infty \)-category, the linear dual functor \((-)^\vee : \text{CoAlg}_\mathcal{O}(C)^{op} \to \text{Alg}_\mathcal{O}(C) \) is a right adjoint, for any essentially small \( \infty \)-operad \( \mathcal{O} \). Its left adjoint:

\[
(-)^\circ : \text{Alg}_\mathcal{O}(C) \to \text{CoAlg}_\mathcal{O}(C)^{op},
\]

is called the **finite dual**. Our arguments above show that \( A^\circ \cong A^\vee \), as \( \mathcal{O} \)-coalgebras in \( C \), for any weak quasi-dualizable \( \mathcal{O} \)-algebra \( A \) in \( C \). This follows by unicity of left adjoint functors.

## 4 DUALITY BETWEEN THH AND coTHH

We show here one of our main results: there is a duality between THH and coTHH. Recall that we have introduced the notion of weak quasi-proper algebra in Definition 3.24 and quasi-proper coalgebra in Definition 3.27.

**Theorem 4.1.** Let \( C \) be a closed symmetric monoidal \( \infty \)-category that admits geometric realizations and totalizations.

(i) Let \( A \) be a weak quasi-proper \( A_\infty \)-algebra in \( C \). Then:

\[
(\text{THH}^C(A))^\vee \cong \text{coTHH}^C(A^\vee).
\]

(ii) Let \( C \) be a quasi-proper \( A_\infty \)-coalgebra in \( C \). Then:

\[
\text{coTHH}^C(C) \cong (\text{THH}^C(C^\vee))^\vee.
\]

We shall prove the above theorem at the end of the section. We first make some observations.
Remark 4.2. Even for $C$ a proper $A_\infty$-coalgebra, we have examples where:

$$(\text{coTHH}^C(C))^\vee \not\cong \text{THH}^C(C^\vee),$$

as shown in Remark 7.6.

Remark 4.3. Let $A$ be a weak quasi-proper $A_\infty$-algebra in $C$. Combining (i) of Theorem 4.1 above with the unit of the adjunction of Proposition 3.4, we obtain a natural map in $A$, dashed in the diagram below:

![Diagram](image)

Similarly, let $C$ be a quasi-proper $A_\infty$-coalgebra in $C$. Combining (ii) of Theorem 4.1 above with the unit of the adjunction of Proposition 3.4, we obtain a natural map in $C$, dashed in the diagram below:

![Diagram](image)

As mentioned in the previous remark, these maps are *not* equivalences in general.

Remark 4.4. Let $C$ be a quasi-proper $S$-coalgebra. Combining the natural map in the above remark with the trace map in $K$-theory, we obtain:

$$K(C^\vee) \longrightarrow \text{THH}(C^\vee) \longrightarrow (\text{coTHH}(C))^\vee.$$

We therefore obtain a map by applying the Spanier–Whitehead dual:

$$\text{coTHH}(C) \longrightarrow K(C^\vee)^\vee,$$

as the composite $\text{coTHH}(C) \longrightarrow (\text{THH}(C^\vee))^\vee \longrightarrow K(C^\vee)^\vee$. By adjointness, this provides a map

$$K(C^\vee) \wedge \text{coTHH}(C) \longrightarrow S.$$

We now prove Theorem 4.1. It essentially follows from the linear dual assigning the cyclic bar construction of a weak quasi-proper algebra $A$:

$$\cdots \longrightarrow A \otimes A \otimes A \longrightarrow A \otimes A \longrightarrow A.$$
to the cyclic cobar construction of the linear dual coalgebra $A^\vee$:

\[
\begin{array}{c}
\cdots \\
\longleftarrow & (A \otimes A \otimes A)^\vee & \longleftarrow & (A \otimes A)^\vee & \longleftarrow & A^\vee \\
\上升 & \cong & \上升 & \cong & \上升 \\
\longleftarrow & A^\vee \otimes A^\vee \otimes A^\vee & \longleftarrow & A^\vee \otimes A^\vee & \longleftarrow & A^\vee.
\end{array}
\]

We make precise the above idea in the following lemma.

**Lemma 4.5.** Let $C$ be a closed symmetric monoidal $\infty$-category. Let $A$ be a weak quasi-proper $\mathbb{A}_\infty$-algebra in $C$. Then we obtain an equivalence of cosimplicial objects in $C$:

\[
(\text{Bar}_C^\ast(A))^\vee \simeq \text{coBar}_C^\ast(A^\vee).
\]

**Proof.** For $A$ a weak quasi-proper $\mathbb{A}_\infty$-algebra in $C$, we consider the subcategory $C_A$ of $C$ just as in the proof of Proposition 3.25. Note that $A$ is also an $\mathbb{A}_\infty$-algebra in $C_A$. Since the composition

\[
C_A \hookrightarrow C \xrightarrow{(-)^\vee} C^{\text{op}}
\]

is strong symmetric monoidal, by Lemma 2.12, we obtain an equivalence:

\[
\left(\text{Bar}_A^\ast(A)\right)^\vee \simeq \text{Bar}_A^{\text{coop}}(A^\vee).
\]

By applying again Lemma 2.12 on the strong symmetric monoidal functor $C_A \subseteq C$, we have:

\[
\left(\text{Bar}_A^\ast(A)\right)^\vee \simeq \left(\text{Bar}_A^\ast(A)\right)^\vee.
\]

By definition, we also obtain:

\[
\text{Bar}_A^{\text{coop}}(A^\vee) \simeq \text{coBar}_A^\ast(A^\vee).
\]

Combining the above equivalences, we obtain the desired result. \hfill \Box

**Proof of Theorem 4.1.** We first prove (i). By Lemma 4.5, we have:

\[
\left(\text{Bar}_C^\ast(A)\right)^\vee \simeq \text{coBar}_C^\ast(A^\vee).
\]

The functor $(-)^\vee : C^{\text{op}} \rightarrow C$, as a right adjoint (see Proposition 3.4), preserves limits (or sends colimits to limits when regarded as a contravariant functor). Therefore, if we take the totalization of the cosimplicial objects on each side of the equivalence above we obtain the desired equivalence:

\[
\left(\text{THH}_C^\ast(A)\right)^\vee \simeq \text{coTHH}_C^\ast(A^\vee).
\]

We now prove (ii). Let $C$ be a quasi-proper $\mathbb{A}_\infty$-algebra in $C$. Then $C^\vee$ is a quasi-proper $\mathbb{A}_\infty$-algebra and in particular weak quasi-proper, by Theorem 3.31. Therefore, we can apply (i) to $A = C^\vee$, and we obtain:

\[
\left(\text{THH}_C^\ast(C^\vee)\right)^\vee \simeq \text{coTHH}_C^\ast(C^{\vee\vee}).
\]
By Theorem 3.31, we have \( C^\vee \simeq C \) as \( \Lambda_\infty \)-coalgebras in \( C \). Thus, we obtain:

\[
\text{coTHH}^C(C^\vee) \simeq \text{coTHH}^C(C).
\]

Combining the two equivalences, we have obtained the desired result. □

5 APPLICATIONS TO THE \( \infty \)-CATEGORY OF \( H\mathbb{Z} \)-MODULES

In this section, we start with the proof Theorem 1.3. This is an application of Theorem 4.1 to \( HA \)-modules, where \( A \) denotes a discrete commutative ring. After that, we prove Theorem 5.8 which provides a contravariant equivalence of \( \infty \)-categories between algebras and coalgebras in (co)connective \( HA \)-modules with finite and free homology. We apply Theorem 3.31 to prove Theorem 5.8.

5.1 Duality between THH and coTHH in \( HA \)-modules

Let \( A \) denote a discrete commutative Noetherian ring for the rest of this section. Furthermore, we assume that \( A \) has finite global dimension, that is, there is an \( d \) such that every \( A \)-module has a projective resolution of length at most \( d \). Recall that we say an \( HA \)-module (or an \( HA \)-(co)algebra) \( M \) is of finite type if \( \pi_i M \) is a finitely generated \( A \)-module for every \( i \). We restate Theorem 1.3.

\[ \text{Theorem 5.1.} \text{ Let } C \text{ be a connective or coconnective } HA \text{-coalgebra of finite type where } A \text{ is as above. Then there is an equivalence of } HA \text{-module spectra:} \]

\[
\text{coTHH}^{HA}(C) \simeq (\text{THH}^{HA}(C^\vee))^\vee.
\]

\[ \text{Proof.} \text{ We apply Theorem 4.1. We need to show that } C \text{ is a quasi-proper } HA \text{-coalgebra. It follows by Proposition 3.30 that it is sufficient to show that } C \text{ satisfies Conditions 1 and 2 and } C^{\wedge HA}_n \text{ satisfies Condition 2 in } HA \text{-modules for every } n. \text{ This follows by Lemmas 5.6 and 5.7.} \]

Remark 5.2. In [20, Remark 2.6], Hess and Shipley claim a duality relation between coHH and Hochschild homology in \( k \)-chain complexes, where \( k \) denotes a field. Their claim is an equivalence:

\[
(\text{coHH}^k_*(C))^\vee \cong \text{HH}^k_*(C^\vee),
\]

for a general coalgebra \( C \) in \( k \)-chain complexes. However, we believe that this is not true in this generality and that one needs finiteness and (co)connectivity conditions as in Theorem 1.3. This is due to our Examples 3.16 and 3.17 showing that the linear dual functor is not strong monoidal without these assumptions.

The rest of this subsection is devoted to the proof of Lemmas 5.6 and 5.7. For this, we begin with an identification of dualizable objects in \( HA \)-modules, see Proposition 5.3. From Proposition 5.3, we deduce that for a connective \( C \) as in Theorem 1.3, every Postnikov section of \( C \) is dualizable and therefore satisfies the relevant conditions. In the proofs of Lemmas 5.6 and 5.7, we consider the Postnikov section maps \( C \to \tau_{\leq n} C \) for various \( n \) to prove that \( C \) also satisfies Conditions 1 and
2 and $C^{|HA}|^n$ satisfies Condition 2 as desired. For coconnective $C$, we use the connective cover functors instead of the Postnikov section functors and argue similarly.

Recall from Example 3.8 that dualizable objects in $HA$-modules are precisely what are called the perfect $HA$-modules, see [12, III.7.9] and [30, 3.2.3]. An $HA$-module is said to be perfect if it lies in the smallest subcategory of the homotopy category of $HA$-modules that contains $HA$ and is closed under triangles and retracts. For instance, finite coproducts of shifted copies of $HA$ are examples of perfect $HA$-modules.

Since $A$ is Noetherian of finite projective dimension, every finitely generated $A$-module $M$ admits a finite length resolution of finitely generated projective $A$-modules. This shows that for every finitely generated $A$-module $M$, $HM$ is perfect and therefore dualizable in $HA$-modules. Furthermore, we have the following proposition that characterize dualizable objects in $HA$-modules.

**Proposition 5.3.** Let $A$ be a discrete Noetherian ring of finite global dimension. An $HA$-module $N$ is dualizable if and only if $\pi_*N$ is finitely generated as a graded $A$-module, that is, $\pi_iN$ is a finitely generated $A$-module for every $i$ and $\pi_iN \neq 0$ for only finitely many $i$.

**Proof.** Let $N$ be a dualizable $HA$-module. This implies that $N$ can be obtained from $HA$ via taking finitely many triangles and retracts. The conclusion is immediate for the case $N = HA$ as $\pi_*HA = A$ is a finitely generated $A$-module. Therefore, it is sufficient to show that the property of having finitely generated homotopy is closed under retracts and triangles. The closeness under retracts follows form the fact that the codomain of a surjective map is finitely generated when the domain is finitely generated. For triangles, assume that there is a triangle

$$M \longrightarrow N \longrightarrow L,$$

in $HA$-modules, where $M$ and $L$ have finitely generated homotopy groups. We need to show that $N$ also has finitely generated homotopy groups. From the induced long exact sequence, we get that $\pi_iN \neq 0$ for only finitely many $i$. Therefore, it is sufficient to show that $\pi_iN$ is finitely generated for each $i$. The induced long exact sequence in homotopy can be broken into short exact sequences of the form:

$$0 \longrightarrow Y \longrightarrow \pi_iN \longrightarrow Z \longrightarrow 0,$$

where $Z$ is a submodule of the finitely generated $A$-module $\pi_iL$. Since $A$ is Noetherian, $Z$ is also finitely generated. The $A$-module $Y$ is the image of the finitely generated $A$-module $\pi_iM$ under an $A$-module homomorphism and therefore $Y$ is also finitely generated. This proves that $\pi_iN$ is finitely generated. We conclude that $\pi_*N$ is a finitely generated graded $A$-module as desired.

Now we prove the other direction. Since dualizable $HA$-modules are closed under triangles, we get that an $HA$-module is dualizable if and only if one of its (de)suspensions is dualizable. Therefore, we can assume without loss of generality that $N$ is connective. For a given $HA$-module $N$ with finitely generated homotopy, let $h(N)$ be the largest $i$ for which $\pi_iN \neq 0$. We argue inductively on $h(N)$.

For the base case of our induction, we have $N = HM$, for some finitely generated $A$-module $M$. We already argued that $N$ is a dualizable object in this situation.

Assume that every connective $A$-module $M$ with $h \geq h(M)$ and finitely generated homotopy is dualizable. Let $N$ be an $HA$-module with finitely generated $\pi_*N$ and $h(N) = h + 1$. We need to
show that $N$ is dualizable. Since $\pi_0 N$ is finitely generated, we have a map:

$$\bigvee_{j=1}^{m} HA \rightarrow N,$$

that induces a surjective map on degree 0 homotopy where the domain denotes a finite coproduct. Let $N'$ be defined by the cofiber sequence:

$$\bigvee_{j=1}^{m} HA \rightarrow N \rightarrow N'.$$

Using the long exact sequence induced by this cofiber sequence, one observes that $\pi_0 N' = 0$ and:

$$\pi_i N' = \pi_i N = 0,$$

for every $i > h(N) = h + 1$ and $i < 0$. Since $A$ is Noetherian, every submodule of a finitely generated $A$-module is also finitely generated. This fact, together with the aforementioned long exact sequence implies that $\pi_i N'$ is finitely generated for every $i$. It follows by our induction hypothesis that $\Sigma^{-1} N'$ is dualizable because $h(\Sigma^{-1} N') \leq h$. Therefore, $N'$ is also dualizable and it is generated by $HA$ under triangles and retracts. The triangle above proves that $N$ is also generated by $HA$ under triangles and retracts. In other words, $N$ is also dualizable. □

For the rest of this section, let $d$ denote the global dimension of $A$. We say a spectrum $E$ is $n$-connective if $\pi_i E = 0$ for every $i < n$. Similarly, we say $E$ is $n$-coconnective if $\pi_i E = 0$ for every $i > n$. Before we start the proof of Lemmas 5.6 and 5.7, we need to prove the following technical lemmas.

**Lemma 5.4.** If $N$ is an $n$-connective $HA$-module, then $N^\vee$ is an $-n$-coconnective $HA$-module. If $N$ is an $n$-coconnective $HA$-module, then $N^\vee$ is an $-(n+d)$-connective $HA$-module.

**Proof.** For this, we use the Ext spectral sequence of [12, IV.4.1] to compute the mapping space defining $N^\vee$. The $E_2$-page of this spectral sequence is given by:

$$E_2^{p,q} = \text{Ext}_A^{p,-q}(\pi_* N, A),$$

and it abuts to:

$$\pi_{-(p+q)}([N, A]),$$

where $[-,-]$ denotes the internal mapping spectrum in $HA$-modules and the Ext groups $\text{Ext}_A^{*, -q}(-, -)$ are computed by considering the maps that decrease the internal degree by $q$. For the first statement, note that $\pi_* N$ has a projective resolution consisting of graded modules that are trivial below degree $n$ and this ensures that $E_2^{p,q} = 0$ for $q < n$. Since $p \geq 0$, we have $q < n$ whenever $-(p+q) > -n$ and therefore $E_2^{p,q} = 0$ whenever $-(p+q) > -n$. This proves the first statement.

Since $A$ has global dimension $d$, $\pi_* N$ has a projective resolution of length $d$ in graded $A$-modules. This shows that $E_2^{p,*} = 0$ for every $p > d$. Since $N$ is $n$-coconnective, $E_2^{*,q} \neq 0$ implies $q \leq n$. Therefore, the possible non-trivial entries contributing to the homotopy group in degree
\(-(p + q) < -(n + d)\) should also satisfy \(q \leq n\). But in this situation, we have \(p > d\) and therefore \(E^2_{p,q} = 0\). In other words, \(E^2_{p,q} = 0\) whenever \(-(p + q) < -(n + d)\). □

**Lemma 5.5.** Let \(M\) and \(N\) be \(A\)-modules. If \(M\) is \(m\)-connective and \(N\) is \(n\)-connective, then \(M \wedge_{HA} N\) is \((m + n)\)-connective. If \(M\) and \(N\) are \(m\)-coconnective and \(n\)-coconnective, respectively, then \(M \wedge_{HA} N\) is \((m + n + d)\)-coconnective.

**Proof.** In this case, we use the Künneth spectral sequence in [12, IV.4.1] to compute \(M \wedge_{HA} N\). The \(E^2\)-page is given by:

\[
E^2_{p,q} = \text{Tor}^A_{p,q}(\pi_* M, \pi_* N) \Rightarrow \pi_{p+q}(M \wedge_{HA} N).
\]

In the first situation, there is a flat resolution of \(\pi_* M\) given by graded \(A\)-modules that are trivial below degree \(m\). Tensoring this resolution with \(\pi_* N\) gives a resolution that is trivial in degrees below \(m + n\). Therefore, \(E^2_{p,q} = 0\) for \(q < m + n\). This gives the desired result.

For the second statement, note that \(\pi_* M\) admits a projective and therefore a flat resolution of length at most \(d\). Therefore, \(E^2_{p,*} = 0\) for every \(p > d\). Due to the coconnectivity assumptions, we also have \(E^2_{*,q} = 0\) whenever \(q > m + n\). This provides the second statement in the lemma. □

**Lemma 5.6.** If \(C\) is a (co)connective finite type \(HA\)-module, then \(C^{\wedge_{HA} n}\) satisfies Condition 2 for every \(n \geq 0\). In particular, \(C\) satisfies Condition 2.

**Proof.** We start with the proof of the case \(n = 1\). Let \(C\) be a connective and finite type \(HA\)-module. To prove that \(C\) satisfies Condition 2, we need to show that the natural map:

\[
\eta_C : C \longrightarrow C^{\vee\vee},
\]

is a weak equivalence. We are going to show that this map induces an isomorphism on each homotopy group. Let \(i \in \mathbb{Z}\), to show that \(\pi_i \eta_C\) is an isomorphism, we use the cofiber sequence:

\[
\tau_{\geq n} C \longrightarrow C \longrightarrow \tau_{\leq n-1} C.
\]  

(5.1)

Considering the natural transformation \(\eta\) on this cofiber sequence, we obtain the following diagram where the vertical sequences are the canonical cofiber sequences:

\[
\begin{array}{ccc}
\tau_{\geq n} C & \longrightarrow & (\tau_{\geq n} C)^{\vee\vee} \\
\downarrow & & \downarrow \\
C & \longrightarrow & C^{\vee\vee} \\
\downarrow & & \downarrow \\
\tau_{\leq n-1} C & \longrightarrow & (\tau_{\leq n-1} C)^{\vee\vee}.
\end{array}
\]  

(5.2)

Let \(n = i + d + 1\). Due to Proposition 5.3, \(\tau_{\leq i+d+1} C\) is dualizable and therefore Corollary 3.21 implies that the bottom horizontal arrow is an equivalence. Since \(\tau_{\geq i+d+1} C\) is \((i + d + 1)\)-connective, \((\tau_{\geq i+d+1} C)^{\vee\vee}\) is \(-(i + d + 1)\)-coconnective due to Lemma 5.4. This, together with Lemma 5.4 shows that \((\tau_{\geq i+d+1} C)^{\vee\vee}\) is \((i + 1)\)-connective. In particular, both sides of the top horizontal arrow are \((i + 1)\)-connective. The map of long exact sequences induced by the diagram
of cofiber sequences above shows that \( C \to C^{\vee \vee} \) induces an isomorphism in degree \( i \) homotopy groups. This proves that \( C \) satisfies Condition 2.

For the case where \( C \) is coconnective, we use the cofiber sequence in (5.1) for \( n = i - d \). In this case, \( \tau_{\geq i-d} C \) is dualizable due to Proposition 5.3 and therefore the top horizontal arrow in Diagram (5.2) is an equivalence. Also, \( \tau_{\leq i-d-1} C^{\vee \vee} \) is \((i-1)\)-coconnective due to Lemma 5.4. In particular, both sides of the bottom horizontal arrow are \((i-1)\)-coconnective. It follows by considering the map of long exact sequences induced by Diagram (5.2) that \( \eta_C \) induces an isomorphism on degree \( i \) homotopy.

Now we prove that for every \( C \) as in Theorem 1.3, \( C^n \wedge HA \) also satisfies Condition 2 for every \( n \geq 0 \). For simplicity, we write \( C^n \) for the \( n \)-fold smash product \( C \wedge HA^n \). Note that \( n = 0 \) case is satisfied trivially because \( HA \) is dualizable. The \( n = 1 \) case is what we prove above. We proceed by induction. Assume that this is true for some \( (n-1) \geq 1 \), we need to show that the natural transformation:

\[
\eta_{C^n} : C^n \to (C^n)^{\vee \vee},
\]

is a homotopy isomorphism, that is, a weak equivalence. Let \( C \) be connective and finite type. Given \( i \in \mathbb{Z} \), let \( \ell' = i + d \), there is a cofiber sequence:

\[
(\tau_{\geq \ell' + 1} C) \wedge HA C^{n-1} \to C \wedge HA C^{n-1} \to (\tau_{\leq \ell'} C) \wedge HA C^{n-1}.
\]

Due to Lemma 5.5, \( (\tau_{\geq \ell' + 1} C) \wedge HA C^{n-1} \) is \((i + d + 1)\)-connective and it follows from Lemma 5.4 that \( ((\tau_{\geq \ell' + 1} C) \wedge HA (C^{n-1}))^{\vee \vee} \) is \((i + 1)\)-connective. We apply the natural transformation \( \eta \) to this cofiber sequence and consider the induced map of long exact sequences. This shows that in order to prove \( \pi \eta_{C^n} \) is an isomorphism, it is sufficient to show that:

\[
\eta_{(\tau_{\leq \ell'} C) \wedge HA C^{n-1}} : (\tau_{\leq \ell'} C) \wedge HA C^{n-1} \to ((\tau_{\leq \ell'} C) \wedge HA C^{n-1})^{\vee \vee}
\]

is an equivalence. Note that this map is an equivalence when \( \tau_{\leq \ell'} C \) is replaced by \( HA \) by our induction hypothesis. Furthermore, both the domain and the codomain of this natural transformation preserves triangles and retracts in the first factor of the smash product. Due to Proposition 5.3, \( \tau_{\leq \ell'} C \) is dualizable and therefore it is generated by \( HA \) under triangles and retracts. This shows that \( \eta_{(\tau_{\leq \ell'} C) \wedge HA C^{n-1}} \) is also an equivalence.

We also need to show that \( C^n \) satisfies Condition 2 under the induction hypothesis whenever \( C \) is coconnective and finite type. Fix an integer \( i \) and let \( \ell' \) be a negative integer such that:

\[
\ell' + nd < i - 1 \quad \text{and} \quad \ell' + (n - 1)d < i - 1.
\]

We need to show that \( \pi \eta_{C^n} \) is an isomorphism. We consider the similar cofiber sequence:

\[
(\tau_{\geq \ell' + 1} C) \wedge HA C^{n-1} \to C \wedge HA C^{n-1} \to (\tau_{\leq \ell'} C) \wedge HA C^{n-1}.
\]

The right-hand side is \((\ell' + (n - 1)d)\)-coconnective and therefore its double dual is \((\ell + nd)\)-coconnective. Therefore, both the right-hand side and its double dual are \((i-1)\)-coconnective. Applying the natural transformation \( \eta \) to this cofiber sequence, one sees that it is sufficient to show that \( \eta_{(\tau_{\geq \ell' + 1} C) \wedge HA C^{n-1}} \) is an equivalence. This follows as before due to the fact that \( \tau_{\geq \ell' + 1} C \) is dualizable. □
Lemma 5.7. If $C$ is a (co)connective finite type $HA$-module, then $C$ is weak quasi-dualizable, that is, $C$ satisfies Condition 1.

Proof. As before, we write $C^n$ for the $n$-fold smash product $C^\wedge_{HA^n}$. By induction, it is sufficient to show that the lax symmetric monoidal structure map:

$$\varphi: C^\vee \wedge_{HA} (C^{n-1})^\vee \to (C^n)^\vee,$$

is an equivalence for every $n$.

We first prove this for the case where $C$ is a connective and finite type $HA$-module. For a given integer $i$, we need to show that the map above induces an isomorphism in degree $i$ homotopy. Let $\ell$ be an integer such that $-\ell + d < i - 1$ and $-\ell < i - 1$. There is a cofiber sequence:

$$\tau \geq \ell C \to C \to \tau \leq \ell - 1 C.$$

Recall that dualization functor is a contravariant functor. From this cofiber sequence and the natural transformation $\varphi$, we obtain the following diagram:

$$\begin{align*}
\xymatrix{
(	au \leq \ell - 1 C)^\vee \wedge_{HA} (C^{n-1})^\vee 
& ((\tau \leq \ell - 1 C) \wedge_{HA} C^{n-1})^\vee \\
C^\vee \wedge_{HA} (C^{n-1})^\vee 
& (C \wedge_{HA} C^{n-1})^\vee. \\
}\end{align*}$$

(5.3)

The vertical lines above are cofiber sequences and the horizontal maps are given by lax monoidal structure map of the dualization functor. Since $\tau \geq \ell C$ is $\ell$-connective and $C$ is $0$-connective, $(\tau \geq \ell C)^\vee$ is $-\ell$-coconnective and $(C^{n-1})^\vee$ is $0$-coconnective due to Lemmas 5.4 and 5.5. This, together with Lemma 5.5 shows that the bottom left corner is $(-\ell + d)$-coconnective; in particular, it is $(i - 1)$-coconnective. Similarly, the bottom right corner is $-\ell$-coconnective and therefore $(i - 1)$-coconnective.

The map of long exact sequences induced by the diagram above shows that it is sufficient to prove that the top horizontal arrow is an equivalence. Note that $\tau \leq \ell - 1 C$ is dualizable in $HA$-modules due to Proposition 5.3. Therefore, $\tau \leq \ell - 1 C$ can be obtained from $HA$ via finitely many triangles and retracts. Note that the arrow above is an equivalence if we replace $\tau \leq \ell - 1 C$ by $HA$ and that both the domain and the codomain of the top horizontal arrow above preserve triangles and retracts in the first variable. Since $\tau \leq \ell - 1 C$ can be built from $HA$ via triangles and retracts, this shows that the top arrow in the diagram above is an equivalence.

Finally, we need to show that an $HA$-module $C$ satisfies Condition 1 whenever it is coconnective and finite type. Fix $i$ and let $\ell$ be an integer such that:

$$-\ell + 1 - nd > i + 1.$$

We need to show that $\pi_i \varphi$ is an isomorphism. Again, we consider Diagram (5.3). In this case, $\tau \leq \ell - 1 C$ is $(\ell - 1)$-coconnective and $C$ is $0$-coconnective. Therefore, $(\tau \leq \ell - 1 C)^\vee$ is $(-\ell + 1 - d)$-connective and $C^{n-1}$ is $(n - 2)d$-coconnective. Also, $(C^{n-1})^\vee$ is $-(n - 1)d$-connective. Therefore, the domain of the top horizontal map in Diagram (5.2) is and $(-\ell + 1 - dn)$-connective. Similarly, the codomain of the top horizontal map is also $(-\ell + 1 - dn)$-connective. In particular,
both the domain and the codomain of the top horizontal map are \((i + 1)\)-connective. Therefore, it is sufficient to show that the bottom horizontal map is an equivalence.

This follows in a way similar to the connective case. Note that \(\tau_{\geq t} C\) is dualizable due to Proposition 5.3. The bottom horizontal map is an equivalence if \(\tau_{\geq t} C\) is replaced by \(HA\). The domain and the codomain of this map preserves triangles and retracts in the first variable and \(\tau_{\geq t} C\) can be built from \(HA\) via retracts and triangles. Therefore, the bottom horizontal map is also an equivalence as desired. □

5.2  A duality between algebras and coalgebras in \(H\mathbb{Z}\)-modules

In this section, we prove a duality result between algebras and coalgebras in the \(\infty\)-category of chain complexes over a general discrete commutative ring \(B\). For the theorem below, let

\[
\text{Alg}_\mathcal{O}(\text{Mod}_{HB})_{\text{ft}} \quad \text{and} \quad \text{CoAlg}_\mathcal{O}(\text{Mod}_{HB})_{\text{ft}}
\]

denote the full subcategory of finite type \(\mathcal{O}\)-algebras in \(HB\)-modules whose homotopy groups are free \(B\)-modules and the full subcategory of finite type \(\mathcal{O}\)-coalgebras in \(HB\)-modules whose homotopy groups are free \(B\)-modules, respectively. Furthermore, \(\geq 0(\cdot)\) and \(\leq 0(\cdot)\) denotes the restriction to the full subcategories of connective and coconnective and objects, respectively.

Theorem 5.8. There are equivalences of \(\infty\)-categories:

\[
\geq 0 \text{CoAlg}_\mathcal{O}(\text{Mod}_{HB})_{\text{ft}}^{\text{op}} \simeq \leq 0 \text{Alg}_\mathcal{O}(\text{Mod}_{HB})_{\text{ft}},
\]

and

\[
\leq 0 \text{CoAlg}_\mathcal{O}(\text{Mod}_{HB})_{\text{ft}}^{\text{op}} \simeq \geq 0 \text{Alg}_\mathcal{O}(\text{Mod}_{HB})_{\text{ft}},
\]

given by the dualization functor \((-)^\vee\) in all directions.

To prove this theorem, we use Theorem 3.31 for the case \(C = \text{Mod}_{HB}\). This shows that there is an equivalence of \(\infty\)-categories:

\[
((\text{CoAlg}_\mathcal{O}(\text{Mod}_{HB}))^\ast)^{\text{op}} \simeq (\text{Alg}_\mathcal{O}(\text{Mod}_{HB}))^\ast,
\]

given in both directions by the dualization functor in \(HB\)-modules. We show that the equivalences of \(\infty\)-categories given in Theorem 5.8 are given by restrictions of the equivalence above to the mentioned subcategories. The lower script \(\ast\) in (5.4) denotes the restriction to the quasi-proper objects, see Definition 3.27. Recall that an \(\mathcal{O}\)-coalgebra \(C\) in \(HB\)-modules is quasi-proper if \(C\) satisfies Condition 2 and \(C^\vee\) satisfies Condition 1. On the other hand, an \(\mathcal{O}\)-algebra \(X\) in \(HB\)-modules is said to be quasi-proper if it satisfies Conditions 1 and 2 in \(HB\)-modules.

Lemma 5.9. Let \(M\) be a (co)connective \(HB\)-module whose homotopy groups are finite-dimensional free \(B\)-modules. In this situation, \(M\) satisfies Conditions 1 and 2. Furthermore, \(M^\vee\) also satisfies Condition 1. In other words, \(M\) is quasi-dualizable and \(M^\vee\) is weak-quasi dualizable in \(HB\)-modules.

Proof. We first prove this for the case where \(M\) is connective. Note that the homotopy classes of maps of \(HB\)-modules with free homotopy groups are given by graded \(B\)-module maps of the
corresponding homotopy groups. This follows by the Ext spectral sequence computing mapping spectra [12, IV.4.1]. The Ext spectral sequence also shows that $M$ is a wedge of suspensions of $HB$. One could also see this by choosing maps $\Sigma^k HB \to M$ for every basis element in $\pi_k M$ and taking a coproduct over these maps.

In particular, the Postnikov section $\tau_{\leq n} M$ is a wedge of finitely many suspensions of $HB$. Since dualizable $HB$-modules are those that can be obtained from $HB$ via finitely many triangles and retracts [30, 3.2.3], this shows that $\tau_{\leq n} M$ is dualizable for every $n$. In particular, $\tau_{\leq n} M$ and $(\tau_{\leq n} M)^\vee$ satisfies Conditions 1 and 2, see Corollary 3.21.

At this point, we could argue as in the proof of Lemmas 5.6 and 5.7 but this situation is simpler. For instance, to show that the map

$$M \to M^\vee$$

is an isomorphism in degree $i$ homotopy, we use the map $M \to \tau_{\leq i+1} M$ to compare the map above with the map:

$$\tau_{\leq i+1} M \to \tau_{\leq i+1} M^\vee.$$

Since dualization in these cases results in dualization of graded $B$-modules at the level of homotopy, these two maps agree on degree $i$ homotopy and the second map is an equivalence because $\tau_{\leq i+1} M$ is dualizable. This shows that $M$ satisfies Condition 2.

To see that $M$ satisfies Condition 1, note that smash products of $HB$-modules with free homotopy are given by tensor products over $B$ at the level of homotopy groups. This follows by the Künneth spectral sequence [12, IV.4.1]. Again, to show that the relevant map

$$(M^\vee)^{\wedge HB} \to (M^{\wedge HB})^\vee$$

gives an equivalence in degree $i$ homotopy, we use the map $M \to \tau_{\leq i+1} M$ to compare this with the map

$$((\tau_{\leq i+1} M)^\vee)^{\wedge HB} \to ((\tau_{\leq i+1} M)^{\wedge HB})^\vee$$

which is an equivalence because $\tau_{\leq i+1} M$ is dualizable. Observing that these maps agree on degree $i$ homotopy, we deduce that the first map is also an equivalence and that $M$ satisfies Condition 1.

Using the Ext spectral sequence one more time, one observes that $M^\vee$ is a coconnective $HB$-module where each homotopy group is a finite-dimensional free $B$-module. By using the connective cover functor $\tau_{\geq i} -$ instead of the Postnikov section functor $\tau_{\leq i} -$ one argues as before to see that $M^\vee$ also satisfies Condition 1. The coconnective case of the lemma also follows similarly. □

**Proof of Theorem 5.8.** We start with the proof of the first equivalence in the theorem. We show that this equivalence is a restriction of the equivalence in (5.4) to a full subcategory. For this, it is sufficient to show that that the $\infty$-category

$$\gtrless 0 \CoAlg_{\mathcal{O}}(\text{Mod}_{HB})^\text{op}_{\text{ft}}$$

is a full subcategory of $((\CoAlg_{\mathcal{O}}(\text{Mod}_{HB}))^\text{op}_{\text{ft}}$ and the essential image of this full subcategory is given by

$$\lesssim 0 \Alg_{\mathcal{O}}(\text{Mod}_{HB})_{\text{ft}}$$
under the dualization functor. For the first part, we need to show that every \( C \) in \( \geq 0 \text{CoAlg}_{\mathcal{O}}(\text{Mod}_{HB})_{\text{ft}}^{\text{op}} \) is quasi-proper, that is, \( C \) satisfies Condition 2 and \( C^\vee \) satisfies Condition 1. This follows by Lemma 5.9.

What is left is to show that the essential image of \( \geq 0 \text{CoAlg}_{\mathcal{O}}(\text{Mod}_{HB})_{\text{ft}}^{\text{op}} \) under the dualization functor is \( \leq 0 \text{Alg}_{\mathcal{O}}(\text{Mod}_{HB})_{\text{ft}} \). First, we show that for every \( C \) in \( \geq 0 \text{CoAlg}_{\mathcal{O}}(\text{Mod}_{HB})_{\text{ft}}^{\text{op}} \), \( C^\vee \) lies in \( \leq 0 \text{Alg}_{\mathcal{O}}(\text{Mod}_{HB})_{\text{ft}} \). This follows by the Ext spectral sequence showing that on an \( HB \)-module whose homotopy groups are free \( B \)-modules, the dualization functor results in graded \( B \)-module dualization at the level of homotopy groups. Given \( X \) in \( \leq 0 \text{Alg}_{\mathcal{O}}(\text{Mod}_{HB})_{\text{ft}} \), we need to show that \( X = C^\vee \) for some \( C \) in \( \geq 0 \text{CoAlg}_{\mathcal{O}}(\text{Mod}_{HB})_{\text{ft}}^{\text{op}} \). For this, note that \( X \) is quasi-dualizable due to Lemma 5.9, therefore \( X \) lies in:

\[
(\text{Alg}_{\mathcal{O}}(\text{Mod}_{HB}))^*. 
\]

The equivalence in (5.4) shows that \( (X^\vee)^\vee \cong X \) and that \( X^\vee \) is an \( \mathcal{O} \)-coalgebra in \( HB \)-modules. It follows by inspection on homotopy groups that \( X^\vee \) lies in \( \geq 0 \text{CoAlg}_{\mathcal{O}}(\text{Mod}_{HB})_{\text{ft}}^{\text{op}} \) as desired. This finishes the proof of the first equivalence in the theorem. The proof of the second equivalence follows in the same way.

\[\Box\]

6 | COALGEBRAS IN SPECTRA

We have introduced the definition of coTHH associated to an \( \mathbb{A}_\infty \)-coalgebra in any symmetric monoidal \( \infty \)-category. We are interested here in \( \mathbb{A}_\infty \)-coalgebras in the \( \infty \)-category of spectra \( \text{Sp} \) or more generally in \( \mathbb{A}_\infty \)-coalgebras in \( R \)-modules, where \( R \) is a commutative ring spectrum.

In symmetric monoidal model categories of spectra, such as symmetric spectra, an \( S \)-coalgebra structure is quite a restrictive feature on a spectrum. However, unlike for algebras, the category of strictly coassociative and counital \( S \)-coalgebras do not represent the \( \mathbb{A}_\infty \)-coalgebras in \( \text{Sp} \). See [36] and [34]. The main point is that the model categories only capture coalgebras that behave as the cocommutative \( S \)-coalgebra \( \Sigma^\infty_+X \) with comultiplication induced by the diagonal:

\[ X_+ \longrightarrow (X \times X)_+ \cong X_+ \wedge X_+, \]

where \( X \) is a space and \( X_+ \) is the free pointed space on \( X \). In particular, strictly coassociative and counital \( S \)-coalgebras do not capture the Spanier–Whitehead duality between \( \mathbb{A}_\infty \)-algebras and \( \mathbb{A}_\infty \)-coalgebras in finite spectra of Corollary 3.13.

Nonetheless, we show in this section that an \( \mathbb{A}_\infty \)-coalgebra structure on a spectrum \( C \) is also a restrictive feature. If \( R \) is a commutative ring spectrum, then an \( \mathbb{A}_\infty \)-algebra in \( R \)-modules in \( \text{Sp} \) is also an \( \mathbb{A}_\infty \)-algebra in \( \text{Sp} \). In other words, we have that spectra in \( \text{Alg}_{\mathbb{A}_\infty}(\text{Sp}) \) require less structure than spectra in \( \text{Alg}_{\mathbb{A}_\infty}(\text{Mod}_R(\text{Sp})) \). This is not true for coalgebras. In fact, we can say that spectra in \( \text{CoAlg}_{\mathbb{A}_\infty}(\text{Sp}) \) have a more restrictive structure than spectra in \( \text{CoAlg}_{\mathbb{A}_\infty}(\text{Mod}_R(\text{Sp})) \). For instance, if \( C \) is an \( R \)-module endowed with an \( \mathbb{A}_\infty \)-coalgebra structure in \( \text{Sp} \), then \( C \) is an \( \mathbb{A}_\infty \)-coalgebra in \( R \)-modules in \( \text{Sp} \).

One of the main restrictive feature for an \( \mathbb{A}_\infty \)-algebra \( C \) in \( \text{Sp} \) is the requirement of a map \( C \rightarrow S \) that is counital (up to higher homotopy).
Theorem 6.1. Let $C$ be a connective $\mathbb{A}_\infty$-coalgebra in the $\infty$-category of spectra. If $\pi_0 C \neq 0$, then the $p$-localization of $C$ contains the $p$-local sphere spectrum $S(p)$ as a retract in the stable homotopy category for every $p \gg 0$.

Proof. Since $C$ is an $\mathbb{A}_\infty$-coalgebra in $\text{Sp}$, it is in particular a coassociative counital coalgebra up to homotopy: that is, it is a strictly coassociative counital coalgebra in the homotopy category of $\text{Sp}$. Let $\Delta$ denote the comultiplication map and $\varepsilon$ denote the counit map of $C$ in the homotopy category of $\text{Sp}$, respectively. The following composite is homotopic to the identity map on $C$:

$$C \xrightarrow{\Delta} C \wedge C \xrightarrow{\varepsilon \wedge \text{id}_C} S \wedge C \simeq C,$$

where the smash product denotes the derived smash product. Since $C$ is connective, by the Künneth spectral sequence [12, IV.4.1] we have:

$$\pi_0(C \wedge C) \cong \pi_0 C \otimes \pi_0 C \text{ and } \pi_0(S \wedge C) \cong \pi_0 S \otimes \pi_0 C.$$

We apply $\pi_0(\cdot)$ to (6.1) and due to the functoriality of the Künneth spectral sequence, we obtain the following composite:

$$\pi_0 C \rightarrow \pi_0 C \otimes \pi_0 C \xrightarrow{\pi_0 \varepsilon \otimes \pi_0 \text{id}_C} \mathbb{Z} \otimes \pi_0 C \cong \pi_0 C.$$

If $\pi_0 \varepsilon$ is the trivial map, it follows that the composite above is trivial but this contradicts the fact that this composite is the identity map and that $\pi_0 C \neq 0$. Therefore, $\pi_0 \varepsilon$ is non-trivial. Let $m$ be the smallest positive integer in the image of $\pi_0 \varepsilon$ and let

$$c : S \rightarrow C$$

denote a map whose image under $\pi_0 \varepsilon$ is $m$. If $m = 1$, then this makes sure that $C$ contains the sphere spectrum $S$ as a retract in the stable homotopy category through the composite:

$$S \xrightarrow{\varepsilon} C \xrightarrow{\varepsilon} S.$$

For $m > 1$, let $p$ be a prime greater than $m$. In this situation, $m$ is a unit in $\mathbb{Z}(p)$ and therefore a unit in $\pi_0(S(p))$. Therefore, there is a map $m^{-1} : S(p) \rightarrow S(p)$ representing $m^{-1}$ in $\pi_0(S(p))$. We obtain the following composite:

$$S(p) \xrightarrow{\varepsilon} C(p) \xrightarrow{\varepsilon} S(p) \xrightarrow{m^{-1}} S(p),$$

which is the identity map of $S(p)$ (in the homotopy category) as desired. In particular, this implies that $\pi_p C(p)$ contains the $p$-local stable homotopy groups of spheres for every $p > m$. 

By inspection on the homotopy groups of various spectra, we obtain the following.

Corollary 6.2. The spectra $k_0, k_u, MU$, connective covers of the Morava E-theory $E_n$ and $K(n)$ are not $\mathbb{A}_\infty$-coalgebras in the $\infty$-category of spectra $\text{Sp}$. For a discrete commutative ring $A$, the Eilenberg–Mac Lane spectrum $HA$ is not an $\mathbb{A}_\infty$-coalgebra in $\text{Sp}$. 
Example 6.3. Let $R$ be a commutative ring spectrum. Let $S$ be the $\infty$-category of spaces. Then the functor

$$R \wedge - : S \rightarrow \text{Sp}$$

is strong symmetric monoidal. Here $S$ is endowed with its Cartesian symmetric monoidal structure. In particular, for any space $X$, we get that $R \wedge X_+$ is a cocommutative $R$-coalgebra. For instance, for $R = \mathbb{S}$, we have that $\Sigma_+^\infty X$ is a cocommutative $\mathbb{S}$-coalgebra in $\text{Sp}$.

Example 6.4. From [20, 3.6], we say a space is EMSS-good if $X$ is connected and $\pi_i X$ acts nilpotently on $H_i(\Omega X; \mathbb{Z})$ for all $i$. In particular, every simply connected space is EMSS-good. If $X$ is EMSS-good, then it was shown in [20, 3.7] that

$$\text{coTHH}(\Sigma_+^\infty X) \simeq \Sigma_+^\infty \mathcal{L}X,$$

where $\mathcal{L}X$ is the free loop space on $X$. If $X$ is a finite CW-complex, then $\Sigma_+^\infty X$ is dualizable in $\text{Sp}$, and by Theorem 4.1, we get:

$$\text{coTHH}(\Sigma_+^\infty X) \simeq \left(\text{THH}(\Sigma_+^\infty X^\vee)\right)^\vee.$$

Thus, if $X$ is a finite CW-complex and EMSS-good, we obtain:

$$\text{coTHH}(\Sigma_+^\infty X) \simeq \left(\text{THH}(\Sigma_+^\infty X^\vee)\right)^\vee \simeq \Sigma_+^\infty \mathcal{L}X.$$

The last equivalence was proved for $X$ a finite CW-complex and simply connected in [24] and [31]. Thus, our result generalizes the last equivalence above to EMSS-good finite CW-complexes.

Example 6.5. Let $M$ be a compact smooth manifold. It is homotopic to a finite CW-complex and thus $\Sigma_+^\infty M$ is dualizable in $\text{Sp}$. Then by Atiyah duality (see [2, 3.3]), the Spanier–Whitehead dual of the spectrum $\Sigma_+^\infty M$ is the Thom spectrum $M^{-\tau}$ of its stable normal bundle. The geometric construction of $M^{-\tau}$ equips it with the structure of a commutative ring spectrum and, by [10], this structure agrees with the induced commutative ring structure on the Spanier–Whitehead dual of $\Sigma_+^\infty M$ from Corollary 3.13. Moreover, by Theorem 4.1, we obtain

$$\text{coTHH}(\Sigma_+^\infty M) \simeq \left(\text{THH}(M^{-\tau})\right)^\vee.$$

If we further assume that $M$ is EMSS-good, then we obtain by Example 6.4:

$$\text{coTHH}(\Sigma_+^\infty M) \simeq \left(\text{THH}(M^{-\tau})\right)^\vee \simeq \Sigma_+^\infty \mathcal{L}M.$$

This recovers the cohomological version described in [9, Comment 3, section I.1.5].

Example 6.6. Let $G$ be a compact Lie group. Then $\Sigma_+^\infty G$ is a ring spectrum. Its Spanier–Whitehead is the Thom spectrum $G^{-\tau}$ of its stable normal bundle, just as in Example 6.5. By Corollary 3.13, we obtain that $G^{-\tau}$ is an $\mathbb{S}$-coalgebra, induced by the group structure on $G$. By Theorem 4.1, we obtain:

$$\text{coTHH}(G^{-\tau}) \simeq \left(\text{THH}(\Sigma_+^\infty G)\right)^\vee.$$
We also have that $\text{THH}(\Sigma_+^\infty G) \simeq \Sigma_+^\infty \mathcal{L}BG$, and thus we obtain:

$$\text{coTHH}(G^{-\tau}) \simeq (\Sigma_+^\infty \mathcal{L}BG)^\vee.$$

If $G$ is connected, then $BG$ is simply connected and thus by our discussion in Example 6.5, we obtain:

$$\text{coTHH}(G^{-\tau}) \simeq (\text{coTHH}(\Sigma_+^\infty BG))^\vee.$$  

In [4, 5.7], the mod $p$-homology of $\text{coTHH}(\Sigma_+^\infty BG)$ was computed for $G$ equals one of the Lie groups $U(n), SU(n), Sp(n), SO(2k), G_2, F_4, E_6, E_7$ and $E_8$.

Remark 6.7. Let $A \to B$ be a map of commutative $\mathbb{E}_\infty$-algebras in a symmetric monoidal $\infty$-category $C$. One might expect $B \otimes_A B$ to be an $A_\infty$-coalgebra in $B$-modules in $C$ due to the comultiplication map

$$B \otimes_A B \cong B \otimes_A A \otimes_A B \to B \otimes_A B \otimes_A B$$

$$\cong B \otimes_A (B \otimes_B B) \otimes_A B \cong (B \otimes_A B) \otimes_B (B \otimes_A B),$$

and the counit map

$$B \otimes_A B \to B$$

given by the multiplication on $B$. However, this does not work. This is due to the fact that the comultiplication map above is not a map of $B$-modules in general but of $B$-bimodules. The $B$-module structure on $B \otimes_A B$ is given by the map

$$B \cong B \otimes_A A \to B \otimes_A B,$$

but the $B$-module structure of the right-hand side is given by the one induced by the tensor product in $B$-modules. We believe that there is no general way of turning this comultiplication map into a map of $B$-modules.

7 | COMPUTATIONS

In this section, we apply Theorem 5.8 to construct interesting examples of coalgebras in module spectra and then we use Theorem 1.3 to compute the coTHH groups of these coalgebras. Furthermore, all our coTHH computations are algebraic. In other words, in each case, we compute $\text{coTHH}^{HR}(C)$ for some discrete ring $R$ and $HR$-coalgebra $C$.

Each subsection below is devoted to the construction of a particular coalgebra and its coHH computation. In Subsection 7.1, we show that $HF_p \wedge_{HZ} HF_p$ is an $HF_p$-coalgebra in a unique way and compute its coHH groups. Subsection 7.2 is devoted to the study of the Steenrod algebra spectrum as an $HF_p$-coalgebra and in Subsection 7.3, we construct an $HF_p$-coalgebra structure on $\Omega HF_p$; we compute the relevant coHH groups in each of these cases.
7.1 CoHochschild homology of the dual Steenrod algebra in $H\mathbb{Z}$-modules

Here, we study the dual Steenrod algebra spectrum $HF_p \wedge_{H\mathbb{Z}} HF_p$ in $H\mathbb{Z}$-modules and its coHH. We claim in Remark 6.7 that such objects, that is, objects of the form $B \otimes_A B$ for some map of $E_\infty$-ring spectra $A \to B$, do not carry a canonical $B$-coalgebra structure in general. In contrast, we show below that $HF_p \wedge_{H\mathbb{Z}} HF_p$ carries a unique $HF_p$-coalgebra structure. Subsequently, we compute the coHH of $HF_p \wedge_{H\mathbb{Z}} HF_p$ with this $HF_p$-coalgebra structure using Theorem 1.3.

**Theorem 7.1.** There is a unique $HF_p$-coalgebra structure on $HF_p \wedge_{H\mathbb{Z}} HF_p$. Furthermore, this $HF_p$-coalgebra structure lifts to a cocommutative $HF_p$-coalgebra structure.

We postpone the proof of this theorem to the end of the subsection and start the computation of the coHH groups of $HF_p \wedge_{H\mathbb{Z}} HF_p$. Note the theorem above allows us to consider $HF_p \wedge_{H\mathbb{Z}} HF_p$ as an $HF_p$-coalgebra.

**Theorem 7.2.** The coHH groups of $HF_p \wedge_{H\mathbb{Z}} HF_p$ are given by:

$$\operatorname{coTHH}_{i}^{HF_p}(HF_p \wedge_{H\mathbb{Z}} HF_p) \cong \begin{cases} \Pi_{n \in \mathbb{N}} F_p & i = 0 \\ \Pi_{n \in \mathbb{N}} F_p & i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

In other words, there is an equivalence of graded $F_p$-modules:

$$\operatorname{coTHH}_{*}^{HF_p}(HF_p \wedge_{H\mathbb{Z}} HF_p) \cong \Lambda_{F_p}(x_1) \otimes F_p[[t]],$$

(7.1)

where $|t| = 0$ and $|x_1| = 1$.

By a Tor calculation, one obtains that:

$$\pi_* (HF_p \wedge_{H\mathbb{Z}} HF_p) \cong \Lambda_{F_p}(y_1),$$

where $|y_1| = 1$, see [12, Theorem 4.1 in IV.4]. In particular, $HF_p \wedge_{H\mathbb{Z}} HF_p$ satisfies the hypothesis of Theorem 1.3. Indeed, $HF_p \wedge_{H\mathbb{Z}} HF_p$ is a dualizable $HF_p$-module due to Example 3.9. We obtain the following equivalence of $HF_p$-modules:

$$\operatorname{coTHH}_{*}^{HF_p}(HF_p \wedge_{H\mathbb{Z}} HF_p) \cong \left( \operatorname{THH}_{*}^{HF_p}((HF_p \wedge_{H\mathbb{Z}} HF_p)^\vee) \right)^\vee.$$

Therefore, our question reduces to the calculation of the $F_p$-Hochschild homology groups of $(HF_p \wedge_{H\mathbb{Z}} HF_p)^\vee$. Recall that dualization in $HF_p$-modules results in dualization of graded $F_p$-modules at the level of homotopy groups which carries coproducts to products. Therefore, Proposition 7.3 together with the equivalence above provide Theorem 7.2. Let $X$ denote the $HF_p$-algebra $(HF_p \wedge_{H\mathbb{Z}} HF_p)^\vee$ for the rest of this subsection.

**Proposition 7.3.** There is an isomorphism of graded $F_p$-modules:

$$\operatorname{THH}_{*}^{HF_p}((HF_p \wedge_{H\mathbb{Z}} HF_p)^\vee) \cong F_p[y] \otimes \Lambda_{F_p}(z_{-1}),$$

where $|y| = 0$ and $|z_{-1}| = -1$. 
Proof. By the Ext spectral sequence calculating the homotopy groups of mapping spectra [12, IV.4.4.1], we have an equivalence of $\mathbb{F}_p$-modules

$$\pi_* X \cong \Lambda_{\mathbb{F}_p}(z^{-1}),$$

where $|z^{-1}| = -1$. This is also an isomorphism of rings since there is a unique ring structure on the right-hand side.

We show below in Proposition 7.4 that the $\mathbb{F}_p$-DGA $X$ is formal, that is, it is quasi-isomorphic to an $\mathbb{F}_p$-DGA with trivial differentials. Note that in the spectral sequence corresponding to a double complex with trivial vertical differentials, all the differentials are trivial on the $E^2$ page and after. This shows that the differentials in the standard spectral sequence calculating the Hochschild homology groups of $X$ are trivial after the first page.

For this spectral sequence, we have:

$$E^2_{s,t} = \text{HH}^p_{s,t}(\Lambda_{\mathbb{F}_p}(z^{-1}), \Lambda_{\mathbb{F}_p}(z^{-1})) \cong \text{Tor}_{s,t}^{\Lambda_{\mathbb{F}_p}(z^{-1})}(\Lambda_{\mathbb{F}_p}(z^{-1}), \Lambda_{\mathbb{F}_p}(z^{-1}))$$

$$\implies \text{THH}^H_{s,t}(X).$$

To calculate the $E^2$ page of this spectral sequence, we use the automorphism of $\Lambda_{\mathbb{F}_p}(z^{-1}) \otimes \Lambda_{\mathbb{F}_p}(z^{-1})$ given by:

$$z^{-1} \otimes 1 \to z^{-1} \otimes 1 \quad \text{and} \quad 1 \otimes z^{-1} \to z^{-1} \otimes 1 - 1 \otimes z^{-1}.$$}

Precomposing with this automorphism, the action of the second factor of $\Lambda_{\mathbb{F}_p}(z^{-1}) \otimes \Lambda_{\mathbb{F}_p}(z^{-1})$ on $\Lambda_{\mathbb{F}_p}(z^{-1})$ becomes trivial, that is, this action is the one obtained by the augmentation map of $\Lambda_{\mathbb{F}_p}(z^{-1})$. Using the K"{u}nneth formula [3, eq. (2)], we obtain the following:

$$E^2 \cong \text{Tor}^{\Lambda_{\mathbb{F}_p}(z^{-1})}(\Lambda_{\mathbb{F}_p}(z^{-1}), \Lambda_{\mathbb{F}_p}(z^{-1}))$$

$$\cong \text{Tor}^{\Lambda_{\mathbb{F}_p}(z^{-1})}(\Lambda_{\mathbb{F}_p}(z^{-1}), \Lambda_{\mathbb{F}_p}(z^{-1})) \otimes \text{Tor}^{\Lambda_{\mathbb{F}_p}(z^{-1})}(F_p, F_p)$$

$$\cong \Lambda_{\mathbb{F}_p}(z^{-1}) \otimes \Gamma_{\mathbb{F}_p}(\sigma z^{-1}).$$

Here: $\deg(\sigma z^{-1}) = (1, -1)$ and $\Gamma_{\mathbb{F}_p}(\sigma z^{-1})$ denotes the divided power algebra on a single generator. There is an equivalence of $\mathbb{F}_p$-modules $\Gamma_{\mathbb{F}_p}(\sigma z^{-1}) \cong \mathbb{F}_p[\sigma z^{-1}]$.

As mentioned before, all the differentials in this spectral sequence are trivial on the $E^2$ page and after. Therefore, there is an isomorphism of graded $\mathbb{F}_p$-modules

$$\text{THH}^H_{s,t}(X) \cong \mathbb{F}_p[y] \otimes \Lambda_{\mathbb{F}_p}(z^{-1}),$$

where $|y| = 0$.

To finish the proof of Theorem 7.2, we need to prove the following proposition. Recall that an $\mathbb{F}_p$-DGA is said to be formal if it is quasi-isomorphic to an $\mathbb{F}_p$-DGA with trivial differentials.

**Proposition 7.4.** Every $\mathbb{F}_p$-DGA $Z$ with homology

$$H_* Z = \Lambda_{\mathbb{F}_p}(z^{-1})$$

is formal as an $\mathbb{F}_p$-DGA. In particular, $X = (HF_p \wedge HZ HF_p)^\vee$ is a formal $\mathbb{F}_p$-DGA.
Remark 7.5. In [3, proof of Theorem 1.6], the first author shows that every $\mathbb{F}_p$-DGA with homology $\Lambda_{\mathbb{F}_p}(z_{-1})$ is formal as a DGA. However, we need a slightly stronger result. We need to show that such $\mathbb{F}_p$-DGAs are formal as $\mathbb{F}_p$-DGAs.

We work on this problem in $H\mathbb{F}_p$-algebras and use the obstruction theory of Hopkins and Miller [37]. This obstruction theory provides obstructions to lifting a map of monoids in the homotopy category of $H\mathbb{F}_p$-modules to a map of $H\mathbb{F}_p$-algebras [21, 4.5].

Proof of Proposition 7.4. Let $Z$ also denote an $H\mathbb{F}_p$-algebra corresponding to the $Z$ in Proposition 7.4 and let $T$ denote the $H\mathbb{F}_p$-algebra corresponding to the formal DGA with homology $\Lambda_{\mathbb{F}_p}(z_{-1})$.

Since the homotopy category of $H\mathbb{F}_p$-modules is the category of graded $\mathbb{F}_p$-modules as a monoidal category via the functor $\pi^*$, there is an isomorphism of monoids from $Z$ to $T$ in the homotopy category of $H\mathbb{F}_p$-modules. By [21, Theorem 4.5], the obstructions to lifting this map to a map of $H\mathbb{F}_p$-algebras lie in the following André–Quillen cohomology groups for graded associative $\mathbb{F}_p$-algebras [21, section 5.2.1]:

$$\text{Der}^{s+1}(\Lambda_{\mathbb{F}_p}(z_{-1}), \Omega^s \Lambda_{\mathbb{F}_p}(z_{-1})) \text{ for } s \geq 1.$$ 

Here, $\Omega^n$ shifts down a graded module $n$ times. By the identification of André–Quillen cohomology groups with Hochschild cohomology groups, one obtains that the obstructions lie in the following groups [21, section 5.2.1]:

$$\text{Ext}^{s+2}_{\Lambda_{\mathbb{F}_p}(z_{-1}) \otimes \Lambda_{\mathbb{F}_p}(z_{-1})^{op}}(\Lambda_{\mathbb{F}_p}(z_{-1}), \Omega^s \Lambda_{\mathbb{F}_p}(z_{-1})) \text{ for } s \geq 1. \quad (7.2)$$

By the discussion above, it is sufficient to show that the Ext groups in $(7.2)$ are trivial. Note that all the rings in sight are graded commutative and therefore we actually have $\Lambda_{\mathbb{F}_p}(z_{-1})^{op} = \Lambda_{\mathbb{F}_p}(z_{-1})$. There is an automorphism of

$$\Lambda_{\mathbb{F}_p}(z_{-1}) \otimes \Lambda_{\mathbb{F}_p}(z_{-1}),$$

given by:

$$z_{-1} \otimes 1 \rightarrow z_{-1} \otimes 1 \text{ and } 1 \otimes z_{-1} \rightarrow z_{-1} \otimes 1 - 1 \otimes z_{-1}.$$ 

We consider the obstruction groups in $(7.2)$ after precomposing with this automorphism. This ensures that the action of the first factor in $\Lambda_{\mathbb{F}_p}(z_{-1}) \otimes \Lambda_{\mathbb{F}_p}(z_{-1})$ on $\Lambda_{\mathbb{F}_p}(z_{-1})$ and $\Omega^s \Lambda_{\mathbb{F}_p}(z_{-1})$ are the canonical non-trivial actions and the action of the second factor on $\Lambda_{\mathbb{F}_p}(z_{-1})$ and $\Omega^s \Lambda_{\mathbb{F}_p}(z_{-1})$ are the trivial actions given by the augmentation map of $\Lambda_{\mathbb{F}_p}(z_{-1})$.

Let

$$\varphi : \Lambda_{\mathbb{F}_p}(z_{-1}) \longrightarrow \Lambda_{\mathbb{F}_p}(z_{-1}) \otimes \Lambda_{\mathbb{F}_p}(z_{-1})$$

denote the map of rings given by the inclusion of the second factor. By the derived extension of scalars functor induced by $\varphi$, we obtain the first isomorphism below. The second isomorphism is obtained using the fact that $\Lambda_{\mathbb{F}_p}(z_{-1}) \otimes \Lambda_{\mathbb{F}_p}(z_{-1})$ is a flat $\Lambda_{\mathbb{F}_p}(z_{-1})$-module through the action
induced by \( \varphi \):

\[
\text{Ext}_{\Lambda_{F_p}(z)}^{s+2}(F_p, \Omega^s(F_p \oplus \Omega F_p)) \\
\cong \text{Ext}_{\Lambda_{F_p}(z) \otimes \Lambda_{F_p}(z)}^{s+2}((\Lambda_{F_p}(z) \otimes \Lambda_{F_p}(z)) \otimes \Lambda_{F_p}(z), \Omega^s \Lambda_{F_p}(z)) \\
\cong \text{Ext}_{\Lambda_{F_p}(z) \otimes \Lambda_{F_p}(z) \otimes \Lambda_{F_p}(z)}^{s+2}(\Lambda_{F_p}(z), \Omega^s \Lambda_{F_p}(z)).
\]

Therefore, it is sufficient to show that:

\[
\text{Ext}_{\Lambda_{F_p}(z)}^{s+2}(F_p, \Omega^s(F_p \oplus \Omega F_p)) = 0 \text{ for } s \geq 1.
\]

We have the following free resolution of \( F_p \) in \( \Lambda_{F_p}(z) \)-modules:

\[
\cdots \rightarrow \Omega^2 \Lambda_{F_p}(z) \rightarrow \Omega \Lambda_{F_p}(z) \rightarrow \Lambda_{F_p}(z) \rightarrow F_p.
\]

Applying \( \text{Hom}_{\Lambda_{F_p}(z)}(-, \Omega^s(F_p \oplus \Omega F_p)) \) to this resolution gives a resolution which is given by the following in resolution degree \( s + 2 \):

\[
\text{Hom}_{\Lambda_{F_p}(z)}(\Omega^{s+2} \Lambda_{F_p}(z), \Omega^s(F_p \oplus \Omega F_p)) = 0.
\]

Therefore, the groups in (7.2) containing the obstructions to the formality of \( Z \) are trivial.

\[\square\]

Remark 7.6. Our computation above shows that:

\[
(\text{coTHH}^{H_{F_p}}(C))^\vee \cong \text{THH}^{H_{F_p}}(C)^\vee.
\]

Indeed we have that \( \text{coTHH}^{H_{F_p}}(H_{F_p} \wedge_{HZ} H_{F_p}) \) is an uncountably infinite-dimensional \( F_p \)-vector space, see (7.1). We also know that \( \text{THH}^{H_{F_p}}\left((H_{F_p} \wedge_{HZ} H_{F_p})^\vee\right) \) is a countably infinite-dimensional \( F_p \)-vector space, see Proposition 7.3. Note that dualization in \( H_{F_p} \)-modules results in \( F_p \)-dualization at the level of homotopy groups [12, IV.4.1]. In particular, the dual of \( \text{coTHH}^{H_{F_p}}(H_{F_p} \wedge_{HZ} H_{F_p}) \) is again an uncountable-dimensional vector space and therefore it is not isomorphic to \( \text{THH}^{H_{F_p}}\left((H_{F_p} \wedge_{HZ} H_{F_p})^\vee\right) \).

Proof of Theorem 7.1. To prove this theorem, we apply Theorem 5.8 for the case \( A = F_p \) and \( O = \text{Assoc} \). Recall that \( H_{F_p} \)-modules are uniquely determined by their homotopy groups; this follows by the Ext spectral sequence calculating mapping spectra [12, IV.4.4.1]. Therefore, our goal is to show that there is a unique \( H_{F_p} \)-coalgebra with homotopy groups given by:

\[
\Lambda_{F_p}(y_1) \cong \pi_*((H_{F_p} \wedge_{HZ} H_{F_p})^\vee).
\]

It follows by the Ext spectral sequence that the dualization functor in \( H_{F_p} \)-modules results in graded \( F_p \)-module dualization at the level of homotopy groups. In particular, we have:

\[
\pi_*((H_{F_p} \wedge_{HZ} H_{F_p})^\vee) \cong \Lambda_{F_p}(z).\]
where $|z_{-1}| = -1$. Similarly, the dual of an $HF_p$-algebra with homotopy $\Lambda_{HF_p}(z_{-1})$ has homotopy groups given by $\Lambda_{HF_p}(y_{1})$. Using this together with Theorem 5.8, we obtain that the set of equivalence classes of $HF_p$-coalgebras with homotopy groups $\Lambda_{HF_p}(y_{1})$ is in bijective correspondence with the set of equivalence classes of $HF_p$-algebras with homotopy groups $\Lambda_{HF_p}(z_{-1})$. Furthermore, observe that the $F_p$-module $\Lambda_{HF_p}(z_{-1})$ has a unique ring structure on it.

Therefore, to prove the first part of Theorem 7.1, it is sufficient to show that there is a unique $HF_p$-algebra with homotopy ring $\Lambda_{HF_p}(z_{-1})$. This is provided by Proposition 7.4. Indeed, this shows that $HF_p \wedge_{HZ} HF_p$ is the dual of the formal $F_p$-DGA with homology $\Lambda_{HF_p}(z_{-1})$. Since this formal $F_p$-DGA is commutative, we deduce that $HF_p \wedge_{HZ} HF_p$ is the dual of a commutative $HF_p$-algebra. Together with Theorem 5.8, this shows that $HF_p \wedge_{HZ} HF_p$ is a cocommutative $HF_p$-coalgebra.

\[\square\]

### 7.2 CoHochschild homology of the Steenrod algebra spectrum

In this section, we apply Theorem 1.3 to compute the coHH of the Steenrod algebra spectrum. Recall that

$$\pi_* (HF_p \wedge HF_p) \cong A_*$$

is the dual Steenrod algebra. Furthermore, we have:

$$[HF_p \wedge HF_p, HF_p]_{HF_p} \cong [HF_p, [HF_p, HF_p]_S]_{HF_p} \cong [HF_p, HF_p]_S,$$

where $[-,-]_R$ denotes the internal hom in $R$-modules for a given commutative ring spectrum $R$. Recall that $[HF_p, HF_p]_S$ is the spectrum of cohomology operations on the cohomology theory defined by $HF_p$. Indeed:

$$A = \pi_* ([HF_p, HF_p]_S)$$

is the Steenrod algebra. There is an $HF_p$-coalgebra structure on $[HF_p, HF_p]_S$ given by Theorem 5.8 and the fact that it is the dual of the $HF_p$-algebra $HF_p \wedge HF_p$. This induces the usual comultiplication on the Steenrod algebra [39, section II.10.2]. We use Theorem 1.3 to compute the coHH of $[HF_p, HF_p]_S$ with this $HF_p$-coalgebra structure. Let $(-)^\vee$ denote the linear dual functor in $HF_p$-modules. There are equivalences of $HF_p$-algebras

$$\left([HF_p, HF_p]_S\right)^\vee \cong (HF_p \wedge HF_p)^\vee \cong HF_p \wedge HF_p$$

where the first equivalence follows by our definition of the $HF_p$-coalgebra structure on $[HF_p, HF_p]_S$ and the second equality follows by Theorem 5.8. Using Theorem 1.3, we obtain:

$$\text{coTHH}^{HF_p}_*([HF_p, HF_p]_S) \cong \left(\text{THH}^{HF_p}_*(HF_p \wedge HF_p)\right)^\vee,$$

where $(-)^\vee$ here denotes the linear dual functor in graded $F_p$-modules. To compute the right-hand side of this equality, we use the following equalities

$$\text{THH}^{HF_p}_*(HF_p \wedge HF_p) \cong HF_p \wedge \text{THH}(HF_p) \cong (HF_p \wedge HF_p) \wedge_{HF_p} \text{THH}(HF_p)$$
and the Bökstedt periodicity [8]

$$\text{THH}_s(HF_p) = F_p[x_2].$$

This shows that:

$$\text{coTHH}_s^{HF_p}(\{HF_p, HF_p\}_S) \cong (A_\ast \otimes F_p[x_2])^\vee,$$

where \((-)^\vee\) on the right-hand side denotes the dualization functor in graded $F_p$-modules. Note that $(M^\vee)_i = (M_i)^\vee$ for every graded $F_p$-module $M$. The right-hand side of the isomorphism above is a finite-dimensional $F_p$-module at each degree and $F_p$-dualization is symmetric monoidal on finite-dimensional $F_p$-vector spaces, see Remark 3.23. We therefore have the following isomorphisms of graded $F_p$-modules:

$$(A_\ast \otimes F_p[x_2])^\vee \cong (A_\ast)^\vee \otimes F_p[x_2]^\vee \cong A \otimes F_p[x_{-2}],$$

where $|x_{-2}| = -2$. We obtain the following result.

**Theorem 7.7.** There is an equivalence of graded $F_p$-modules:

$$\text{coTHH}_s^{HF_p}(\{HF_p, HF_p\}_S) \cong A \otimes F_p[x_{-2}],$$

where $A$ denotes the Steenrod algebra and $|x_{-2}| = -2$.

### 7.3 An interesting coalgebra in $HZ$-modules

Recall that dualizable $HZ$-modules are precisely those that can be obtained from $HZ$ via finitely many cofiber sequences and retracts [30, 3.2.3]. Therefore, the $HZ$-algebra $HF_p$ is dualizable as an $HZ$-module. This is due to the cofiber sequence

$$HZ \xrightarrow{p} HZ \to HF_p.$$ 

Thus, $HF_p^\vee$, the dual of $HF_p$ in $HZ$-modules, is an $HZ$-coalgebra, see Corollary 3.13. Applying the dualization functor to the sequence above, one obtains a fiber sequence

$$\Sigma(HF_p^\vee) \leftarrow HZ \xrightarrow{p} HZ \leftarrow HF_p^\vee,$$

which shows that $\Sigma(HF_p^\vee) = HF_p$. Therefore, we have an equivalence

$$HF_p^\vee \simeq \Omega HF_p,$$

of $HZ$-modules. We consider $\Omega HF_p$ as an $HZ$-coalgebra through this equivalence.

**Theorem 7.8.** With the $HZ$-coalgebra structure on $\Omega HF_p$ described above, the coHH of $\Omega HF_p$ is given by:

$$\text{coTHH}_s^{HZ}(\Omega HF_p) \cong \Omega F_p[x_{-2}],$$
as a graded $\mathbb{F}_p$-module, where $|x_{-2}| = -2$ and $\Omega$ on the right-hand side denotes the functor that decreases the grading by 1.

**Proof.** We compute the coHH of $\Omega H \mathbb{F}_p$ using Theorem 1.3. Since $H \mathbb{F}_p$ is dualizable in $Hz$-modules, we have the following equivalences of $Hz$-algebras:

$$(\Omega H \mathbb{F}_p)^\vee \simeq H \mathbb{F}_p^{\vee},$$

where the first equivalence follows by our definition of $\Omega H \mathbb{F}_p$ as an $Hz$-coalgebra and the second equivalence follows by Corollary 3.13. Using Theorem 1.3, we obtain:

$$\text{coTHH}^{Hz}_e(\Omega H \mathbb{F}_p) = \pi_*(\text{THH}^{Hz}(H \mathbb{F}_p))^\vee.$$ 

There is an equivalence of graded $\mathbb{F}_p$-modules

$$\pi_*(\text{THH}^{Hz}(H \mathbb{F}_p)) \cong \Gamma_{\mathbb{F}_p}(x_2) \cong \mathbb{F}_p[x_2].$$

Here: $\Gamma_{\mathbb{F}_p}(x_2)$ denotes the divided power algebra over $\mathbb{F}_p$ on a single generator, where $|x_2| = 2$. Hochschild homology groups above are obtained via the Künneth spectral sequence calculating

$$\pi_*(H \mathbb{F}_p \wedge H \mathbb{F}_p \wedge Hz H \mathbb{F}_p \mathbb{F}_p),$$

whose $E^2$ page, as a bigraded $\mathbb{F}_p$-module, is given by:

$$\text{Tor}^e_{*,*}(\mathbb{F}_p, \mathbb{F}_p) \cong \Gamma_{\mathbb{F}_p}(x_{1,1}) \cong \mathbb{F}_p[x_{1,1}].$$

Since equivalence type of Eilenberg–Mac Lane spectra are uniquely determined by their homotopy groups, we have an equivalence of $Hz$-modules:

$$\text{THH}^{Hz}(H \mathbb{F}_p) \simeq \bigvee_{i=0}^\infty \Sigma^{2i} H \mathbb{F}_p.$$ 

Therefore,

$$\text{THH}^{Hz}(H \mathbb{F}_p)^\vee \simeq [\bigvee_{i=0}^\infty \Sigma^{2i} H \mathbb{F}_p, Hz] \simeq \prod_{i=0}^\infty \Omega^{2i}[H \mathbb{F}_p, Hz]$$

$$\simeq \prod_{i=0}^\infty \Omega^{2i} H \mathbb{F}_p^\vee \simeq \prod_{i=0}^\infty \Omega^{2i} \Omega H \mathbb{F}_p.$$ 

We obtain:

$$\text{coTHH}^{Hz}_e(\Omega H \mathbb{F}_p) \cong \Omega_{\mathbb{F}_p}[x_{-2}],$$

where $|x_{-2}| = -2$ and $\Omega$ denotes the functor that decreases the grading by 1.

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