Double $B$-tensors and quasi-double $B$-tensors

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Abstract

In this paper, we propose two new classes of tensors: double $B$-tensors and quasi-double $B$-tensors, give some properties of double $B$-tensors and quasi-double $B$-tensors, discuss their relationships with $B$-tensors and positive definite tensors and show that even order symmetric double $B$-tensors and even order symmetric quasi-double $B$-tensors are positive definite. These give some checkable sufficient conditions for positive definiteness of tensors.

Keywords: $B$-tensors, Double $B$-tensors, Quasi-double $B$-tensors, Positive definite.

2010 MSC: 47H15, 47H12, 34B10, 47A52, 47J10, 47H09, 15A48, 47H07.

1. Introduction

A real order $m$ dimension $n$ tensor $A = (a_{i_1\ldots i_m})$ consists of $n^m$ real entries:

$$a_{i_1\ldots i_m} \in \mathbb{R},$$

where $i_j \in \mathbb{N} = \{1, 2, \ldots, n\}$ for $j = 1, \ldots, m$ [4, 6, 8, 16, 20]. It is obvious that a matrix is an order 2 tensor. Moreover, a tensor $A = (a_{i_1\ldots i_m})$ is called
symmetric \[17, 20\] if

\[ a_{i_1 \ldots i_m} = a_{\pi(i_1 \ldots i_m)}, \forall \pi \in \Pi_m, \]

where \( \Pi_m \) is the permutation group of \( m \) indices. And an order \( m \) dimension \( n \) tensor is called the unit tensor denoted by \( I \) \[4, 32\], if its entries are \( \delta_{i_1 \ldots i_m} \) for \( i_1, \ldots, i_m \in N \), where

\[
\delta_{i_1 \ldots i_m} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_m, \\ 0, & \text{otherwise}. \end{cases}
\]

For a tensor \( \mathcal{A} \) of order \( m \) dimension \( n \), if there is a nonzero vector \( x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \) and a number \( \lambda \in \mathbb{R} \) such that

\[ \mathcal{A}x^{m-1} = \lambda x^{[m-1]}, \]

where

\[
(Ax^{m-1})_i = \sum_{i_2, \ldots, i_m \in N} a_{i_1 \ldots i_m} x_{i_2} \cdots x_{i_m}
\]

and \( x^{[m-1]} = (x_1^{m-1}, \ldots, x_n^{m-1})^T \), then \( \lambda \) is called an H-eigenvalue of \( \mathcal{A} \) and \( x \) is called an H-eigenvector of \( \mathcal{A} \) \[20\].

As a natural extension of \( B \)-matrices \[18, 19\], \( B \)-tensors is introduced by Song and Qi \[28\].

**Definition 1.** \[28\] Let \( \mathcal{B} = (b_{i_1 \ldots i_m}) \) be a real tensor of order \( m \) dimension \( n \). \( \mathcal{B} \) is called a \( B \)-tensor if for all \( i \in N \)

\[
\sum_{i_2, \ldots, i_m \in N} b_{i_1 \ldots i_m} > 0
\]

and

\[
\frac{1}{n^{m-1}} \left( \sum_{i_2, \ldots, i_m \in N} b_{i_1 \ldots i_m} \right) > b_{i_{j_2} \ldots i_{j_m}}, \text{ for } j_2, \ldots, j_m \in N, \delta_{i_{j_2} \ldots i_{j_m}} = 0.
\]

By Definition 1 Song and Qi \[28\] gave the following property of \( B \)-tensors.

**Proposition 1.** \[28, Proposition 3\] Let \( \mathcal{B} = (b_{i_1 \ldots i_m}) \) be a real tensor of order \( m \) dimension \( n \). Then \( \mathcal{B} \) is a \( B \)-tensor if and only if for each \( i \in N \),

\[
\sum_{i_2, \ldots, i_m \in N} b_{i_1 \ldots i_m} > n^{m-1} \beta_i(\mathcal{B}),
\]

(1)
where

$$\beta_i(B) = \max_{j_2, \ldots, j_m \in N, \delta_{i_2 \ldots j_m} = 0} \{0, b_{i_2 \ldots j_m}\}.$$  

It is easy to see that Inequality (1) is equivalent to

$$b_{i_2 \ldots i} - \beta_i(B) > \Delta_i(B),$$  

(2)

where

$$\Delta_i(B) = \sum_{i_2 \ldots i_m \in N, \delta_{i_2 \ldots i_m} = 0} (\beta_i(B) - b_{i_2 \ldots i_m}).$$  

(3)

Hence, we by Inequality (2) obtain another property for $B$-tensors.

**Proposition 2.** Let $B = (b_{i_1 \ldots i_m})$ be a real tensor of order $m$ dimension $n$. Then $B$ is a $B$-tensor if and only if for each $i \in N$, Inequality (2) holds.

$B$-tensors are linked with positive definite tensors and $M$-tensors, which are useful in automatical control, magnetic resonance imaging and spectral hypergraph theory [2, 3, 6, 7, 8, 9, 10, 11, 12, 13, 23, 24, 25, 26, 28, 30, 31, 33].

**Definition 2.** [27, 28] Let $A = (a_{i_1 \ldots i_m})$ be a real tensor of order $m$ dimension $n$. $A$ is called positive definite if for any nonzero vector $x$ in $\mathbb{R}^n$,

$$Ax^m > 0,$$

and positive semi-definite if for any vector $x$ in $\mathbb{R}^n$,

$$Ax^m \geq 0,$$

where $Ax^m = \sum_{i_1, i_2, \ldots, i_m \in N} a_{i_1 i_2 \ldots i_m} x_{i_1} \cdots x_{i_m}$.

One of the most important properties of $B$-tensors is listed as follows.

**Theorem 1.** [27] Let $B = (a_{i_1 \ldots i_m})$ be a real tensor of order $m$ dimension $n$. If $B$ is an even order symmetric $B$-tensor, then $B$ is positive definite.
The definition of DB-matrix is a generalization of the B-matrix [10]. Here we call a matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ a DB-matrix if for any $i, j \in N$, $i \neq j$,

$$(a_{ii} - \beta_i(A)) (a_{jj} - \beta_j(A)) > \sum_{k \neq i} (\beta_i(A) - a_{ik}) \sum_{k \neq j} (\beta_j(A) - a_{jk}).$$

A natural question is that can we extend the class of DB-matrices to tensors with order $m \geq 3$ such that it has the property like that in Theorem 1, that is, whether or not an even order symmetric tensor $A = (a_{i_1...i_m})$ satisfying

$$(a_{i...i} - \beta_i(A)) (a_{j...j} - \beta_j(A)) > \Delta_i(A) \Delta_j(A),$$

is positive definite? We see an example firstly for discussing the question.

Consider the symmetric tensor $A = (a_{ijkl})$ of order 4 dimension 2 defined as follows:

$$a_{1111} = a_{2222} = 2, a_{1222} = a_{2122} = a_{2212} = a_{2221} = -1,$$

and other $a_{ijkl} = 0$. By calculation, we have $\beta_1(A) = \beta_2(A) = 0$, and

$$a_{1111}a_{2222} = 4 > 3 = \sum_{\delta_{1ijkl}=0} (-a_{1ijkl}) \sum_{\delta_{2ijkl}=0} (-a_{2ijkl}),$$

which satisfies Inequality (1). However, $A$ is not positive definite. In fact, for any entrywise positive vector $x = (x_1, x_2)^T$. If $Ax^4 > 0$, that is,

$$\begin{cases}
a_{1111}x_1^4 + \sum_{j,k,l \in \{1,2\}, \delta_{1ijkl}=0} a_{1ijkl}x_1x_jx_kx_l > 0, \\
a_{2222}x_2^4 + \sum_{j,k,l \in \{1,2\}, \delta_{2ijkl}=0} a_{2ijkl}x_2x_jx_kx_l > 0,
\end{cases}$$

equivalently,

$$\begin{cases}
2x_1^4 - x_1x_2^3 > 0, \\
2x_1^4 - 3x_1^3x_2 > 0,
\end{cases}$$

then

$$\begin{cases}
2x_1 > x_2, \\
x_2 > \frac{3}{2} x_1,
\end{cases}$$

which implies

$$2x_1^3 > x_2^3 > \frac{27}{8} x_1^3.$$
This is a contradiction. Hence, for any vector \( x = (x_1, x_2)^T > 0 \), \( Ax^4 > 0 \) doesn’t hold. Hence \( A \) is not positive definite by Definition 2.

The example shows that Inequality (11) doesn’t guarantee the positive definiteness of tensor \( A \). In this paper, we introduce two new classes of tensors by adding other conditions: double \( B \)-tensors and quasi-double \( B \)-tensors as generalizations of \( B \)-tensors, and prove that an even order symmetric (quasi-)double \( B \)-tensor is positive definite.

2. Double \( B \)-tensor and quasi-double \( B \)-tensor

Now, we present the definitions of double \( B \)-tensors and quasi-double \( B \)-tensors.

**Definition 3.** Let \( B = (b_{i_1 \ldots i_m}) \) be a real tensor of order \( m \) dimension \( n \) with \( b_{i \ldots i} > \beta_i(B) \) for all \( i \in N \). \( B \) is called a double \( B \)-tensor if \( B \) satisfies

(I) for any \( i \in N \),

\[
    b_{i \ldots i} - \beta_i(B) \geq \Delta_i(B),
\]

where \( \Delta_i(B) \) is defined as (3);

(II) for all \( i, j \in N, i \neq j \), Inequality (II) holds.

**Definition 4.** Let \( B = (b_{i_1 \ldots i_m}) \) be a real tensor of order \( m \) dimension \( n \) with \( b_{i \ldots i} > \beta_i(B) \) for all \( i \in N \). \( B \) is called a quasi-double \( B \)-tensor if for all \( i, j \in N, i \neq j \),

\[
    (b_{i \ldots i} - \beta_i(B)) (b_{j \ldots j} - \beta_j(B) - \Delta_j^i(B)) > (\beta_j(B) - b_{ji \ldots i}) \Delta_i(B),
\]

where

\[
    \Delta_j^i(B) = \Delta_j(B) - (\beta_j(B) - b_{ji \ldots i}) = \sum_{\delta_{jj_2 \ldots j_m} = 0, \delta_{jj_2 \ldots j_m} = 0} (\beta_j(B) - b_{jj_2 \ldots j_m}).
\]

We now give some properties of double \( B \)-tensors and quasi-double \( B \)-tensors.

**Proposition 3.** Let \( B = (b_{i_1 \ldots i_m}) \) be a real tensor of order \( m \) dimension \( n \). (I) If \( B \) is a double \( B \)-tensor, then there is at most one \( i \in N \) such that

\[
    b_{i \ldots i} - \beta_i(B) = \Delta_i(B).
\]

(II) If \( B \) is a quasi-double double \( B \)-tensor, then there is at most one \( i \in N \) such that

\[
    b_{i \ldots i} - \beta_i(B) \leq \Delta_i(B).
\]
Proof. We only prove that (II) holds, and (I) is proved similarly. Suppose that there are $i_0$ and $j_0$ such that
\[ b_{i_0} - \beta_{i_0}(\mathcal{B}) \leq \Delta_{i_0}(\mathcal{B}), \]
and
\[ b_{j_0} - \beta_{j_0}(\mathcal{B}) \leq \Delta_{j_0}(\mathcal{B}), \]
equivalently
\[ b_{j_0} - \beta_{j_0}(\mathcal{B}) - \Delta_{i_0}(\mathcal{B}) \leq \beta_{j_0}(\mathcal{B}) - b_{i_0}. \]
If $b_{j_0} - \beta_{j_0}(\mathcal{B}) - \Delta_{i_0}(\mathcal{B}) < 0$, then
\[ (b_{i_0} - \beta_{i_0}(\mathcal{B})) (b_{j_0} - \beta_{j_0}(\mathcal{B}) - \Delta_{i_0}(\mathcal{B})) \leq (\beta_{j_0}(\mathcal{B}) - b_{i_0}) \Delta_{i_0}(\mathcal{B}), \]
otherwise, $b_{j_0} - \beta_{j_0}(\mathcal{B}) - \Delta_{i_0}(\mathcal{B}) \geq 0$, which also leads to Inequality (6). This contradicts to the definition of quasi-double double $\mathcal{B}$-tensors. The conclusion follows. \[ \square \]

The relationships of $\mathcal{B}$-tensors, double $\mathcal{B}$-tensors and quasi-double $\mathcal{B}$-tensors are given as follows.

**Proposition 4.** Let $\mathcal{B} = (b_{i_1...i_m})$ be a tensor of order $m$ dimension $n$. If $\mathcal{B}$ is a $\mathcal{B}$-tensor, then $\mathcal{B}$ is a double $\mathcal{B}$-tensor and a quasi-double $\mathcal{B}$-tensor. Furthermore, if $\mathcal{B}$ is a double $\mathcal{B}$-tensor, then $\mathcal{B}$ is a quasi-double $\mathcal{B}$-tensor.

Proof. If $\mathcal{B}$ is a $\mathcal{B}$-tensor, then by Proposition 2 for any $i \in \mathbb{N}$,
\[ b_{i_1...i} - \beta_i(\mathcal{B}) > \Delta_i(\mathcal{B}), \]
that is,
\[ b_{i_1...i} - \beta_i(\mathcal{B}) - \Delta_i(\mathcal{B}) > \beta_i(\mathcal{B}) - b_{i_1...i}, \]
for $k \neq i$. Obviously, Inequality (0) holds for any $i \neq j$. This implies that $\mathcal{B}$ is a double $\mathcal{B}$-tensor. On the other hand, note that for $i, j \in \mathbb{N}$, $j \neq i$,
\[ b_{i_1...i} - \beta_i(\mathcal{B}) > \Delta_i(\mathcal{B}), \]
and
\[ b_{i_1...i} - \beta_i(\mathcal{B}) - \Delta_i(\mathcal{B}) > \beta_i(\mathcal{B}) - b_{i_1...i}. \]
It is easy to see that Inequality (5) holds, i.e., $\mathcal{B}$ is a quasi-double $B$-tensor by Definition 4.

Furthermore, if $\mathcal{B}$ is a double $B$-tensor, then there is at most one $k \in N$ such that $b_{i\ldots i} - \beta_i(\mathcal{B}) = \Delta_i(\mathcal{B})$. If there is not $k \in N$ such that $b_{k\ldots k} - \beta_k(\mathcal{B}) = \Delta_k(\mathcal{B})$, then $\mathcal{B}$ is a $B$-tensor, consequently, $\mathcal{B}$ is a quasi-double $B$-tensor. If there is only one $k \in N$ such that $b_{k\ldots k} - \beta_k(\mathcal{B}) = \Delta_k(\mathcal{B})$, then we have for any $j \neq k$:

$$b_{j\ldots j} - \beta_j(\mathcal{B}) \geq \Delta_j(\mathcal{B}).$$

Note that for any $j \in N$, $j \neq k$,

$$b_{k\ldots k} - \beta_k(\mathcal{B}) = \Delta_k(\mathcal{B}), \quad b_{k\ldots k} - \Delta_{i}^k(B) = \beta_k(B) - b_{k\ldots k},$$

and

$$b_{j\ldots j} - \beta_j(\mathcal{B}) \geq \Delta_j(\mathcal{B}), \quad b_{j\ldots j} - \Delta_j^k(B) = \beta_j(B) - b_{j\ldots j}.$$

This implies that Inequality (5) holds, i.e., $\mathcal{B}$ is a quasi-double $B$-tensor. □

**Remark 1.** (I) It is not difficult to see that the class of $B$-tensors is a proper subclass of double $B$-tensors and quasi-double $B$-tensors, that is,

$$\{B - tensors\} \subset \{double\ B - tensors\}$$

and

$$\{B - tensors\} \subset \{quasi - double\ B - tensors\}.$$

(II) The class of double $B$-tensors is a proper subclass of quasi-double $B$-tensors. Consider the tensor $\mathcal{A} = (a_{ijk})$ of order 3 dimension 2 defined as follows:

$$\mathcal{A} = [A(1,\cdot,\cdot), A(2,\cdot,\cdot)],$$

where

$$A(1,\cdot,\cdot) = \begin{pmatrix} 2 & 0 \\ 0 & -0.3 \end{pmatrix}, \quad A(2,\cdot,\cdot) = \begin{pmatrix} -1 & -0.3 \\ -1.5 & 2 \end{pmatrix}.$$ 

By calculation, $\beta_1(\mathcal{A}) = \beta_2(\mathcal{A}) = 0$, and $a_{222} = 2 < 2.8 = \sum_{\delta_{2jk}=0} (-a_{2jk})$. Hence $\mathcal{A}$ is not a double $B$-tensor. Since

$$a_{111} \left(a_{222} - \Delta_2^1(\mathcal{A})\right) = 0.4 > 0.3 = (-b_{211}) \Delta_1(\mathcal{B})$$

...
and
\[ a_{222} (a_{111} - \Delta_2(A)) = 4 > 0.84 = (-a_{122})\Delta_2(A), \]
then \( A \) is a quasi-double \( B \)-tensor. Hence, the class of double \( B \)-tensors is a proper subclass of quasi-double \( B \)-tensors. By (I), we have
\[
\{B - tensors\} \subset \{double B - tensors\} \subset \{quasi - double B - tensors\}.
\]

As is well known, a \( B \)-matrix is a \( P \)-matrix [19, 19]. This is not true for higher order tensors, that is, a \( B \)-tensor may not be a \( P \)-tensor. In [28], Song and Qi proved that a symmetric tensor is a \( P \)-tensor if and only if it is positive definite.

**Definition 5.** [28] A real tensor \( A = (a_{i_1 \cdots i_m}) \) of order \( m \) dimension \( n \) is called a \( P \)-tensor if for any nonzero \( x \) in \( \mathbb{R}^n \),
\[
\max_{i \in \mathbb{N}} x_i (Ax^{m-1})_i > 0.
\]

It is pointed out in [27, 28] that an odd order \( B \)-tensor may not be a \( P \)-tensor. Furthermore, Yuna and You [33] gave an example to show that an even order nonsymmetric \( B \)-tensor may not be a \( P \)-tensor. Hence, by Proposition 4, we conclude that an odd order (quasi-)double \( B \)-tensor may not be a \( P \)-tensor, and an even order nonsymmetric (quasi-)double \( B \)-tensor may not be a \( P \)-tensor. On the other hand, it is pointed out in [27] that an even order symmetric \( B \)-tensor is a \( P \)-tensor. A natural question is that whether or not an even order symmetric (quasi-)double \( B \)-tensor is a \( P \)-tensor? In the following section, we will answer this question by discussing the positive definiteness of (quasi-)double \( B \)-tensors.

### 3. Positive definiteness

Now, we discuss the positive definiteness of (quasi-)double \( B \)-tensors. Before that some definitions are given.

**Definition 6.** [6, 8, 34] Let \( A = (a_{i_1 \cdots i_m}) \) be a real tensor of order \( m \) dimension \( n \). \( A \) is called a \( Z \)-tensor if all of the off-diagonal entries of \( A \) are non-positive;
Definition 7. Let $\mathbf{A} = (a_{i_1 \ldots i_m})$ be a tensor of order $m$ dimension $n \geq 2$. $\mathbf{A}$ is called a doubly strictly diagonally dominant tensor (DSDD) if

(I) when $m = 2$, $\mathbf{A}$ satisfies
\[
|a_{i \ldots i}| |a_{j \ldots j}| > r_i(\mathbf{A}) r_j(\mathbf{A}), \text{ for any } i, j \in N, \ i \neq j, \tag{7}
\]

(II) when $m > 2$, $\mathbf{A}$ satisfies $|a_{i \ldots i}| \geq r_i(\mathbf{A})$ for any $i \in N$ and Inequality (7) holds.

Note here that when $m > 2$, the similar condition that $|a_{i \ldots i}| \geq r_i(\mathbf{A})$ for any $i \in N$, is necessary for DSDD tensors to have the properties of doubly strictly diagonally dominant matrices; for details, see [16, 17].

Definition 8. Let $\mathbf{A} = (a_{i_1 \ldots i_m})$ be a tensor of order $m$ dimension $n \geq 2$. $\mathbf{A}$ is called a quasi-doubly strictly diagonally dominant tensor (Q-DSDD) if for $i, j \in N, j \neq i$,
\[
|a_{i \ldots i}| (|a_{j \ldots j}| - r_j(\mathbf{A})) > r_i(\mathbf{A}) |a_{ji \ldots i}|, \tag{8}
\]

where
\[
r_j^i(\mathbf{A}) = \sum_{j_2 \ldots j_m \in N, \delta_{j_2 \ldots j_m} = 0} |a_{jj_2 \ldots j_m}| = \sum_{j_2 \ldots j_m \in N, \delta_{j_2 \ldots j_m} = 0} |a_{jj_2 \ldots j_m}| - |a_{ji \ldots i}| = r_j(\mathbf{A}) - |a_{ji \ldots i}|.
\]

The relationships between (Q-)DSDD tensors and (quasi-)double $B$-tensors are established as follows.

Proposition 5. Let $\mathbf{B} = (b_{i_1 \ldots i_m})$ be a $Z$-tensor of order $m$ dimension $n$. Then

(I) $\mathbf{B}$ is a double $B$-tensor if and only if $\mathbf{B}$ is a DSDD tensor.

(II) $\mathbf{B}$ is a quasi-double $B$-tensor if and only if $\mathbf{B}$ is a Q-DSDD tensor.

Proof. We only prove that (II) holds, (I) can be obtained similarly. Since $\mathbf{B}$ be a $Z$-tensor, all of its off-diagonal entries are non-positive. Thus, we have that for any $i \in N$, $\beta_i(\mathbf{B}) = 0$,
\[
|b_{ii_2 \ldots i_m}| = -b_{ii_2 \ldots i_m}, \text{ for all } i_2, \ldots, i_m \in N, \delta_{ii_2 \ldots i_m} = 0,
\]

\[
r_i(\mathbf{B}) = \Delta_i(\mathbf{B}) = \sum_{i_2 \ldots i_m \in N, \delta_{ii_2 \ldots i_m} = 0} (\beta_i(\mathbf{B}) - b_{ii_2 \ldots i_m}),
\]
and

\[ r_j^i(B) = \Delta_j^i(B) = \sum_{\delta_{jj_2\cdots j_m} = 0, \delta_{ij_2\cdots j_m} = 0} (\beta_j^i(B) - b_{jj_2\cdots j_m}), \text{ for } j \neq i. \]

which implies that Inequality (5) is equivalent to Inequality (8). The conclusion follows.

In [16], Li et al. gave some sufficient conditions for positive definiteness of tensors.

**Lemma 2.** [16, Theorem 11] Let \( A = (a_{i_1\cdots i_m}) \) be an even order real symmetric tensor of order \( m \) dimension \( n > 2 \) with \( a_{k\cdots k} > 0 \) for all \( k \in \mathbb{N} \). If \( A \) satisfies the condition (II) in Definition 7, then \( A \) is positive definite.

**Lemma 3.** [16, Theorem 13] Let \( A = (a_{i_1\cdots i_m}) \) be an even order real symmetric tensor of order \( m \) dimension \( n > 2 \) with \( a_{k\cdots k} > 0 \) for all \( k \in \mathbb{N} \). If there is an index \( i \in \mathbb{N} \) such that for all \( j \in \mathbb{N}, j \neq i \), such that Inequality (8) holds and \( |a_{i\cdots i}| \geq r_i(A) \), then \( A \) is positive definite.

By Lemmas 2 and 3, we can easily obtain the following result.

**Theorem 4.** An even order real symmetric (Q-)DSDD tensor is positive definite.

Now according to Theorem 4 we research the positive definiteness of symmetric (quasi-)double \( B \)-tensors. Before that we give the definition of partially all one tensors, which proposed by Qi and Song [27]. Suppose that \( A \) is a symmetric tensor of order \( m \) dimension \( n \), and has a principal sub-tensor \( A'_J \) with \( J \in \mathbb{N} \) and \( |J| = r_1 \leq r \leq n \) such that all the entries of \( A'_J \) are one, and all the other entries of \( A \) are zero, then \( A \) is called a partially all one tensor, and denoted by \( \varepsilon'_J \). If \( J = \mathbb{N} \), then we denote \( \varepsilon'_J \) simply by \( \varepsilon \) and call it an all one tensor. And an even order partially all one tensor is positive semi-definite; for details, see [27].

**Theorem 5.** Let \( B = (b_{i_1\cdots i_m}) \) be a symmetric quasi-double \( B \)-tensor of order \( m \) dimension \( n \). Then either \( B \) is a Q-DSDD symmetric \( Z \)-tensor itself, or we have

\[ B = M + \sum_{k=1}^{s} h_k \varepsilon_{\hat{J}_k}, \tag{9} \]

where \( M \) is a Q-DSDD symmetric \( Z \)-tensor, \( s \) is a positive integer, \( h_k > 0 \) and \( \hat{J}_k \subseteq \mathbb{N} \), for \( k = 1, 2, \cdots, s \). Furthermore, If \( m \) is even, then \( B \) is positive definite, consequently, \( B \) is a \( P \)-tensor.
Proof. Let $\hat{J}(\mathcal{B}) = \{i \in N : \text{there is at least one positive off-diagonal entry in the } i\text{th row of } \mathcal{B}\}$. Obviously, $\hat{J}(\mathcal{B}) \subseteq N$. If $\hat{J}(\mathcal{B}) = \emptyset$, then $\mathcal{B}$ is a Z-tensor. The conclusion follows in the case.

Now we suppose that $\hat{J}(\mathcal{B}) \neq \emptyset$, let $\mathcal{B}_1 = \mathcal{B} = (b^{(1)}_{i_1 \cdots i_m})$, and let $d^{(1)}_i$ be be the value of the largest off-diagonal entry in the $i$th row of $\mathcal{B}_1$, that is,

$$d^{(1)}_i = \max_{b^{(1)}_{i_2 \cdots i_m} \neq 0} b^{(1)}_{i_1 \cdots i_m}.$$ 

Furthermore, let $\hat{J}_1 = \hat{J}(\mathcal{B}_1)$, $h_1 = \min_{i \in \hat{J}_1} d^{(1)}_i$ and

$$J_1 = \{i \in \hat{J}_1 : d^{(1)}_i = h_1\}.$$

Then $J_1 \subseteq \hat{J}_1$ and $h_1 > 0$.

Consider $\mathcal{B}_2 = \mathcal{B}_1 - h_1 \varepsilon^{\hat{J}_1} = (b^{(2)}_{i_1 \cdots i_m})$. Obviously, $\mathcal{B}_2$ is also symmetric by the definition of $\varepsilon^{\hat{J}_1}$. Note that

$$b^{(2)}_{i_1 \cdots i_m} = \begin{cases} b^{(1)}_{i_1 \cdots i_m} - h_1, & i_1, i_2, \ldots, i_m \in \hat{J}_1 \\ b^{(1)}_{i_1 \cdots i_m}, & \text{otherwise}, \end{cases}$$

for $i \in J_1$,

$$\beta_i(\mathcal{B}_2) = \beta_i(\mathcal{B}_1) - h_1 = 0,$$  \quad (11)

and that for $i \in \hat{J}_1 \setminus J_1$,

$$\beta_i(\mathcal{B}_2) = \beta_i(\mathcal{B}_1) - h_1 > 0.$$

(12)

Combining (10), (11), (12) with the fact that for each $j \notin \hat{J}_1$, $\beta_i(\mathcal{B}_2) = \beta_i(\mathcal{B}_1)$, we easily obtain by Definition 1 that $\mathcal{B}_2$ is still a symmetric quasi-double $B$-tensor.

Now replace $\mathcal{B}_1$ by $\mathcal{B}_2$, and repeat this process. Let $\hat{J}(\mathcal{B}_2) = \{i \in N : \text{there is at least one positive off-diagonal entry in the } i\text{th row of } \mathcal{B}_2\}$. Then $\hat{J}(\mathcal{B}_2) = \hat{J}_1 \setminus J_1$. Repeat this process until $\hat{J}(\mathcal{B}_{s+1}) = \emptyset$. Let $\mathcal{M} = \mathcal{B}_{s+1}$.

Then (9) holds.

Furthermore, if $m$ is even, then $\mathcal{B}$ a symmetric quasi-double $B$-tensor of even order. If $\mathcal{B}$ itself is a Q-DSDD symmetric Z-tensor, then it is positive definite by Lemma 3. Otherwise, (9) holds with $s > 0$. Let $x \in \mathbb{R}^n$. Then by (9) and that fact that $\mathcal{M}$ is positive definite, we have

$$\mathcal{B} x^m = \mathcal{M} x^m + \sum_{k=1}^{s} h_k \varepsilon^{\hat{J}_k} x^m = \mathcal{M} x^m + \sum_{k=1}^{s} h_k ||x|_{\hat{J}_k}|_m^m \geq \mathcal{M} x^m > 0.$$
This implies that \( B \) is positive definite. Note that a symmetric tensor is a \( P \)-tensor if and only it is positive definite \([28]\), therefore \( B \) is a \( P \)-tensor. The proof is completed. \( \Box \)

Similar to the proof of Theorem 5, by Lemma 2 we easily have that an even order symmetric double \( B \)-tensor is positive definite and a \( P \)-tensor.

**Theorem 6.** Let \( B = (b_{i_1 \cdots i_m}) \) be a symmetric double \( B \)-tensor of order \( m \) dimension \( n \). Then either \( B \) is a DSDD symmetric \( Z \)-tensor itself, or we have

\[
B = \mathcal{M} + \sum_{k=1}^{s} h_k \mathcal{J}_k,
\]

(13)

where \( \mathcal{M} \) is a DSDD symmetric \( Z \)-tensor, \( s \) is a positive integer, \( h_k > 0 \) and \( \mathcal{J}_k \subseteq N \), for \( k = 1, 2, \cdots, s \). Furthermore, If \( m \) is even, then \( B \) is positive definite, consequently, \( B \) is a \( P \)-tensor.

Since an even order real symmetric tensor is positive definite if and only if all of its H-eigenvalues are positive \([20]\), by Theorems 5 and 6 we have the following results.

**Corollary 1.** All the H-eigenvalues of an even order symmetric double \( B \)-tensor are positive.

**Corollary 2.** All the H-eigenvalues of an even order symmetric quasi-double \( B \)-tensor are positive.

4. Conclusions

In this paper, we give two generalizations of \( B \)-tensors: double \( B \)-tensors and quasi-double \( B \)-tensors, and prove that an even order symmetric (quasi-)double \( B \)-tensor is positive definite.

On the other hand, we could consider the problem that whether an even order symmetric tensor is positive semi-definite by weakening the condition of Definition 4 as follows.

**Definition 9.** Let \( B = (b_{i_1 \cdots i_m}) \) be a tensor of order \( m \) dimension \( n \). \( B \) is a quasi-double \( B_0 \)-tensor if and only if for all \( i, j \in N \) \( i \neq j \),

\[
(b_{i_1 \cdots i} - \beta_i(B)) (b_{j_1 \cdots j} - \beta_j(B) - \Delta_j^i(B)) \geq (\beta_j(B) - b_{j_1 \cdots j}) \Delta_i(B).
\]

(14)
However, it can’t be proved by using the technique in this paper that an even order symmetric quasi-double $B_0$-tensor is positive semi-definite. We here only give the following conjecture.

**Conjecture 1.** An even order symmetric quasi-double $B_0$-tensor is positive semi-definite.

**Acknowledgements**

The authors would like to thank Professor L. Qi for his many valuable comments and suggestions.

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