Octonions and Super Lie Algebra

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Abstract

We discuss how to represent the non-associative octonionic structure in terms of the associative matrix algebra using the left and right octonionic operators. As an example we construct explicitly some Lie and Super Lie algebra. Then we discuss the notion of octonionic Grassmann numbers and explain its possible application for giving a superspace formulation of the minimal supersymmetric Yang-Mills models.

Usually we define an almost complex manifold as a real manifold equipped with a complex structure \( \mathcal{I} \) such that \( \mathcal{I}^2 = -1 \) which may be a matrix like

\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]

and the same holds equally well for a quaternionic manifold but we would have \( \mathcal{I}, \mathcal{J}, \mathcal{K} \) respecting an \( su(2) \) algebra. Generalizing this notion to octonions we meet a puzzle, how to

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represent the non-associative structure of octonions. Can it be only in terms of some defined
operators or can we find an easy way to do it with matrices? the answer is indeed yes, we
can again do it with matrices but we need a trick.

We use the symbols $e_i$ to denote the imaginary octonionic units where $i, j, k = 1..7$ and $e_i.e_j = -\delta_{ij} + \epsilon_{ijk}e_k$ or $[e_i, e_j] = \epsilon_{ijk}e_k$ such that $\epsilon_{ijk}$ equals 1 for one of the following seven combinations $\{(123),(145),(176),(246),(257),(347),(365)\}$.

We know that, from a topological point of view, any $\mathbb{R}^8$ is a trivial octonionic manifold. So we can represent an octonion as 8 dimensions column matrix but as octonions are non-commutative and non-associative different action from right or left and taking into account their peculiar non-associativity property may give rise to 106 left/right operators which may be constructed completely from the following 14 operators $\{E_1, ..., E_7, 1|E_1, ..., 1|E_7\}$, we mean by $e_i$ an octonionic number whereas $E_i$ are their corresponding matrices and $1|E_i$ represent action from right, i.e they are the corresponding matrix form of $1|e_i$ given by\textsuperscript{1}

$$1|e_i \ g = g \ e_i \quad g \in \mathcal{O}. \quad (1)$$

We have given a matrix as well as a tensorial representation of these fundamental 14 operators in a separate appendix.

One can check explicitly that any of these matrices square to -1 but they don’t obey the octonionic multiplication table.

\begin{align*}
(E_i)^2 &= -\mathbb{1}, \\
(1|E_i)^2 &= -\mathbb{1}, \\
E_i \ 1|E_i &= 1|E_i \ E_i, \quad (4) \\
\{E_i, E_j\} &= -2\delta_{ij}\mathbb{1}, \\
\{1|E_i, 1|E_j\} &= -2\delta_{ij}\mathbb{1}, \quad (6) \\
[E_i, E_j] &= 2\epsilon_{ijk}E_k - 2[E_i, 1|E_j], \quad (7)
\end{align*}

\textsuperscript{1}We use the elegant notations of [3,4].
Moreover, any of the left or right set alone doesn’t close an algebra but only when we allow their mixing, we can get something useful. To this moment we should recall how an octonionic structure is usually constructed. In any work about octonions, one extracts the octonionic structure by geometric meaning from spaces with $SO(8)$ or $SO(7)$ holonomy. The argument is simple any unit octonion is isomorphic to the Reimannian $S_7 \sim SO(8)/SO(7)$ or to any of its homeomorphic squashed versions $S'_7 \sim SO(7)/G_2$ or $S''_7 \sim SU(4)/SU(3)$ and lastly $S'''_7 \sim Sp(2)/Sp(1)$ so having a manifold with holonomy group $SO(8), SO(7), SU(4) \sim SO(6)$, or even $G_2$, the octonionic automorphism group, one can in principle extract the octonionic structure. But any of these groups $(SO(8), SO(7), SO(6), SO(5), G_2)$ admits an explicit construction using the left and right $E_i, 1|E_i$. So they represent the “associativizing” form of the non-associative octonionic imaginary units $e_i, 1|e_i$. Simply

$$\{1/2 E_i, 1/4 [E_i, E_j]\} \quad i, j = 1..7 \quad \text{close Spin(8)}$$

$$\{-1/4 [E_i, E_j]\} \quad i, j = 1..7 \quad \text{close Spin(7)}$$

$$\{-1/4 [E_i, E_j]\} \quad i, j = 1..6 \quad \text{close Spin(6)}$$

$$\{-1/4 [E_i, E_j]\} \quad i, j = 1..5 \quad \text{close Spin(5)}$$

$$\{-1/4 [E_i, E_j]\} \quad i, j = 1..4 \quad \text{close Spin(4)}$$

$$\{-1/4 [E_i, E_j]\} \quad i, j = 1..3 \quad \text{close Spin(3)}$$

and the same construction can be done using the $\{1|E_i\}$ set. For further study of $G_2$ look at [3]. Actually the logic behind this construction is very easy, upon the use of (2, 3, 5, 6), it is easy to see that the two sets $\{E_1, ..., E_7\}$ and $\{1|E_1, ..., 1|E_7\}$ generate Clifford Algebra $\text{Cliff}(0,7)$ then all the above given construction follows except for $Spin(8)$ which follows from $SO(8) \sim S_7 \times SO(7)$.

In summary, our philosophy is: these matrices can be used to investigate/detect octonions easily using matrices. The non-associativity will be represented by the non closure of
the algebra. Our left/right 14 operators, \( \{ E_1, \ldots, E_7, 1|E_1, \ldots, 1|E_7 \} \) satisfy the Jacobi identity but they don’t close an algebra. To close an algebra we should allow their mixing as it is clear from (9)–(14) whereas octonions close an algebra but they don’t satisfy the Jacobi identity.

One may even go further and check if these matrices admit a Super Lie algebra (SLA). We know that any Super Lie Algebra is defined as a vector space which is the union of an even (bosonic) part \( B \) and another odd (fermionic) part \( F \) such that

\[
\{ F, F \} \in B, \quad [B, B] \in B, \quad [F, B] \in F, \quad (15)
\]

and we have the following four Super Jacobi Identities (SJI)

\[
[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0, \quad a, b, c \in B, \quad (18)
\]

\[
[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0, \quad a, b \in B \text{ and } c \in F, \quad (19)
\]

\[
[a, \{ b, c \}] + \{ b, [c, a] \} - \{ c, [a, b] \} = 0, \quad a \in B \text{ and } b, c \in F, \quad (20)
\]

\[
[a, \{ b, c \}] + [b, \{ c, a \}] + [c, \{ a, b \}] = 0, \quad a, b, c \in F, \quad (21)
\]

Amazingly enough three of these SJI are satisfied by the octonionic elements \( e_i \) under the following decomposition

\[
B = \{ e_1, e_2, e_3 \} \quad \text{and} \quad F = \{ e_0, e_4, e_5, e_6, e_7 \}, \quad (22)
\]

problems arise because of (19) which is the true reflection of the non-associativity, but (18)–(21) are satisfied by the matrices \( E_i, 1|E_i \) for

\[
B_1 = \{ E_1, E_2, E_3 \} \quad \text{and} \quad F_1 = \{ 1, E_4, E_5, E_6, E_7 \}, \quad (23)
\]

and the same for the right combination

\[
B_2 = \{ 1|E_1, 1|E_2, 1|E_3 \} \quad \text{and} \quad F_2 = \{ 1, 1|E_4, 1|E_5, 1|E_6, 1|E_7 \}. \quad (24)
\]
Using (5) and (6), one can check easily that $F_1$ or $F_2$ close a fermionic algebra, But our bosonic algebra should be modified to

$$B'_1 = \{[E_1, E_2], [E_1, E_3], [E_2, E_3]\},$$  \hspace{1cm} (25)$$

$$B'_2 = \{[1|E_1, 1|E_2], [1|E_1, 1|E_3], [1|E_2, 1|E_3]\},$$  \hspace{1cm} (26)$$

using (7) and (8), it is obvious that $B'_1$ and $B'_2$ close properly under the commutation relation and generate an $so(3) \sim su(2)$ algebra. It remains to check, for example, $[F_1, B'_1] \in F$, to appreciate how difficult to work analytically, let’s try to use (7) and (8), for (we will work with the $\{E_i\}$ set but everything holds well for the $\{1|E_i\}$)

$$[E_4, [E_1, E_2]] = E_4(2E_3 - 2[E_1, (1|E_2)]) - (2E_3 - 2[E_1, (1|E_2)])E_4$$

$$= 2[E_4, E_3] - 2[E_4, [E_1, (1|E_2)]]$$

$$= 4E_7 - 4[E_4, (1|E_3)] - 2[E_4, [E_1, (1|E_2)]]],$$  \hspace{1cm} (27)$$

we should prove that the last equation belongs to $F$, one may even try to work with the tensorial notation given in the appendix and invoke some octonionic identities to find the answer. But, we have an easy way, simply, we used the matrix representation given in the appendix and find \(^1\)

$$[F_1, B'_1] = [F_2, B'_2] = 0.$$  \hspace{1cm} (28)$$

Now using (2–6) and (28) the four SJI follow directly. So $\{B'_1, F_1\}$ or $\{B'_2, F_2\}$ is an SLA composed of a bosonic $su(2)$ and a four dimensional fermionic part. Even, we can generalize the previous construction to include all the bosonic algebra $so(3) \sim su(2)$ to $so(6) \sim su(4)$

$$B_6 = \left\{\frac{1}{4}[E_i, E_j]\right\} \hspace{0.5cm} i, j = 1..6 \hspace{0.5cm} \text{close Spin}(6)$$  \hspace{1cm} (29)$$

$$F_6 = \{\Pi, E_7\}. $$  \hspace{1cm} (30)$$

\(^2\)One can use any computer system and after entering the matrices given in the appendix, all what he has to do is to define the commutator and then he can check the next equation.
Whatever we believe that the correct niche, for octonions, is local supersymmetry, it seems to be a good idea to try, first, something easier like global higher dimensional supersymmetry. But we should first solve the following puzzle. In trying to write supersymmetry over quaternions or octonions, one faces the following difficulty. What is the correct definition of a quaternionic or an octonionic Grassmann variables? Usually, a Grassmann Algebra (GA) is defined by the following relation: \( \{ \theta_i, \theta_j \} = 0 \). We want to show that this relation holds for quaternionic or octonionic Grassmann variables without any modification. Actually, this should be anticipated from the start as Grassmann variables are nothing but fermions. Quaternionizing or octonionizing fermions is nothing but writing a quaternionic or an octonionic representation of the corresponding Clifford algebra with a reduction of the number of components of the spinor.

Grassmann numbers are defined as the set of anticommuting numbers \( \{ \theta_1, \theta_2, ..., \theta_n \} \) such that \( \forall i, j = 1...n \)

\[
\{ \theta_i, \theta_j \} = 0, \tag{31}
\]
\[
\theta_i^2 = 0. \tag{32}
\]

Whereas Cliff(p,q) is defined as the set of \( \{ \gamma_1, ..., \gamma_p, \gamma_{p+1}, ..., \gamma_{p+q} \} \) satisfying the following anticommutation relations (\( \forall n, m = 1...p+q \)) and \( \eta_{nm} \equiv (1, ..., 1, -1, ..., -1) \)

\[
\{ \gamma_n, \gamma_m \} = 2\eta_{nm}, \tag{33}
\]

For simplicity, consider \( p = q \), because of the signature, we will have

\[
\gamma_1^2 = \ldots = \gamma_p^2 = 1 \text{ and } \gamma_{p+1}^2 = \ldots = \gamma_{2p}^2 = -1. \tag{34}
\]

then by coupling two elements of different signature, we have

\[
(\gamma_1 + \gamma_{p+1})^2 = 0,
\]

\[
(\gamma_p + \gamma_{2p})^2 = 0,
\]

\( (35) \)
i.e p Grassmann variables. It is evident that we can not construct more than $p$-Grassmann variables. To see this explicitly, consider $p=2$, we have

$$\gamma_0, \gamma_1, i\gamma_2, \gamma_3$$

where $(\gamma_i)$ are the standard Cliff(2,2) for example in the Dirac representation

$$\gamma_0^2 = (i\gamma_2)^2 = 1 \text{ whereas } \gamma_1^2 = \gamma_3^2 = -1.$$ 

Then our Grassmann variables are nothing but

$$\theta_1 = \gamma_0 + \gamma_1 \text{ and } \theta_2 = i\gamma_2 + \gamma_3.$$ 

If one tries to introduce a third Grassmann variables as

$$\theta_3 = \gamma_0 + \gamma_3$$

we have the following situation

$$\{\theta_1, \theta_2\} = 0, \{\theta_2, \theta_3\} \neq 0 \text{ and } \{\theta_1, \theta_3\} \neq 0.$$ 

And generally for any $2p$-dimensional Cliff. of signature $(p,p)$, one can construct $p$-Grassmann variables. It seems really that Clifford algebra is too fundamental. It would be fantastic if the above construction can be extended somehow to give the exact number of Grassmann variables needed for the construction of supersymmetric theories.

Actually, the relation between supersymmetry and ring division algebra is very clear in the construction of the minimal supersymmetric Yang-Mills theory [7–11]. For example representing the $D=6, 10$ Lorentz group as $sl(2, \mathcal{Q})$ and $sl(2, \mathcal{O})$ respectively [12], then they admit the natural $D=4, N=1$ superspace construction as a solution. A more general solution may be generated because the Taylor expansion (used for example to find the solution of the chiral field constraint) is not well defined at the level of a quaternionic or an octonionic formulation. In principle left and right as well as their mixing is allowed, explicitly (consult [13] for notations)
\[ D_\alpha \Phi = 0 \].

This defines the chiral superfield, \( \Phi \). We can solve the constraint (37) by writing \( \Phi \) as a function of \( y \) and \( \theta \), where

\[ y^m = x^m + i\theta \sigma^m \bar{\theta} \].

(38)

Since \( \bar{D} \theta = D y = 0 \), the field \( \Phi(y, \theta) \) automatically satisfies the constraint (37).

To find the component fields, we expand \( \Phi(y, \theta) \) in terms of \( \theta \),

\[
\Phi(y, \theta) = A(y) + \sqrt{2} \theta \chi(y) + \theta \bar{\theta} F(y)
\]

\[ = A(x) + i\theta \sigma^m \bar{\theta} \partial_m A(x) + \frac{1}{4} \theta \bar{\theta} \bar{\theta} \partial \, A(x) \]

\[ + \sqrt{2} \partial \chi(x) - \frac{i}{\sqrt{2}} \theta \bar{\theta} \partial_m \chi(x) \sigma^m \bar{\theta} + \theta \bar{\theta} F(x) \].

(39)

The problem is simply the following: quaternionic or octonionic \( A(x) \) doesn’t commute anymore with the quaternionic or octonionic \( \sigma^m \) i.e

\[ i\theta \sigma^m \bar{\theta} \partial_m A(x) \neq i \partial^m A(x) \theta \sigma^m \bar{\theta} \],

(40)

also, it is better to use the momentum operator \( P^m \) instead of \( i\partial^m \), this is a technical problem related to the quantization process of a quaternionic or octonionic fields, look in [14] for more details. We think that this problem may be related to the construction of the off-shell formulation of the ten dimensional super Yang-Mills.

We think that supergravitational (gauged supersymmetric) theories can be a good candidate for an octonionic gauge theory since they are torsionfull version of the general relativity with specific conditions imposed on the torsion tensor by the action of the Bianchi identities and it is also well known that any octonionic manifold is a full torsion space. When

\footnote{Simply, adopting a complex scalar product, the quantization process is the same. Adler [15] goes further and proposes that a complex scalar quantum mechanics has the same Hilbert space as the standard quantum theory. From our point of view, this argument is not clear since, even after the use of complex scalar, the theory still carries a quaternionic or an octonionic structure.}
we compactify the simple N=1 D=10 super Yang-Mills to 4 dimensional, we get an N=4 SU(4) super Yang-Mills; where the $SU(4) \sim SO(6)$ represents the remnant of the higher dimensional Lorentz group. By the same token, When we compactify on $S_7$ the D=11 N=1 supergravity to 4 dimensions, we should look to the resultant theory, namely the N=8 D=4 supergravity, as a full gauged $S_7$ theory. To prove such conjectures, we should construct explicitly the octonionic version of the D=4 N=8 or D=11 N=1 supergravity.

Lastly, octonion is a consistent wonderful part of mathematics and finding their correct physical application, from our point of view, is highly needed rather than just being a challenge or a conjecture. In [16], it has been proposed to use the fact that octonions are “almost Lie algebra” i.e locally, they close a Lie algebra with the structure constant being a function of the coordinate. Unfortunately, one can not have a topological support of this notion, by applying different Hopf fibrations

$$S_7 \rightarrow S_4 \times S_3 \rightarrow S_4 \times S_2 \times S_1,$$

also, this localization may miss some important global features. As

$$\pi_7(S_7) \neq \pi_7(S_4 \times S_2 \times S_1).$$

Until finding the correct way, one may try every possible physical/mathematical formulation keeping in mind that our job as physicists is to try our best to describe nature not to choose it.

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APPENDIX:

We introduce the following notation:

\[
\{ a, b, c, d \}_{(1)} \equiv \begin{pmatrix}
    a & 0 & 0 & 0 \\
    0 & b & 0 & 0 \\
    0 & 0 & c & 0 \\
    0 & 0 & 0 & d \\
\end{pmatrix},
\]

(A1)

\[
\{ a, b, c, d \}_{(2)} \equiv \begin{pmatrix}
    0 & a & 0 & 0 \\
    b & 0 & 0 & 0 \\
    0 & 0 & c & 0 \\
    0 & 0 & 0 & d \\
\end{pmatrix},
\]

(A2)

\[
\{ a, b, c, d \}_{(3)} \equiv \begin{pmatrix}
    0 & 0 & a & 0 \\
    0 & 0 & 0 & b \\
    c & 0 & 0 & 0 \\
    0 & d & 0 & 0 \\
\end{pmatrix},
\]

(A3)

\[
\{ a, b, c, d \}_{(4)} \equiv \begin{pmatrix}
    0 & 0 & 0 & a \\
    0 & 0 & b & 0 \\
    0 & c & 0 & 0 \\
    d & 0 & 0 & 0 \\
\end{pmatrix},
\]

(A4)

where \( a, b, c, d \) and 0 represent \( 2 \times 2 \) real matrices.

In the following \( \sigma_1, \sigma_2, \sigma_3 \) represent the standard Pauli matrices.
\[ e_1 \leftrightarrow \{-i\sigma_2, -i\sigma_2, -i\sigma_2, i\sigma_2\}_{(1)}, \quad 1 \mid e_1 \leftrightarrow \{-i\sigma_2, i\sigma_2, i\sigma_2, -i\sigma_2\}_{(1)} \right] \\
\[ e_2 \leftrightarrow \{-\sigma_3, \sigma_3, -1, 1\}_{(2)}, \quad 1 \mid e_2 \leftrightarrow \{-1, 1, 1, -1\}_{(2)} \right] \\
\[ e_3 \leftrightarrow \{-\sigma_1, \sigma_1, -i\sigma_2, -i\sigma_2\}_{(2)}, \quad 1 \mid e_3 \leftrightarrow \{-i\sigma_2, -i\sigma_2, i\sigma_2, i\sigma_2\}_{(2)} \right] \\
\[ e_4 \leftrightarrow \{-\sigma_3, 1, \sigma_3, -1\}_{(3)}, \quad 1 \mid e_4 \leftrightarrow \{-1, -1, 1, 1\}_{(3)} \right] \quad (A5) \\
\[ e_5 \leftrightarrow \{-\sigma_1, i\sigma_2, \sigma_1, i\sigma_2\}_{(3)}, \quad 1 \mid e_5 \leftrightarrow \{-i\sigma_2, -i\sigma_2, -i\sigma_2, -i\sigma_2\}_{(3)} \right] \\
\[ e_6 \leftrightarrow \{-1, -\sigma_3, \sigma_3, 1\}_{(4)}, \quad 1 \mid e_6 \leftrightarrow \{-\sigma_3, \sigma_3, -\sigma_3, \sigma_3\}_{(4)} \right] \\
\[ e_7 \leftrightarrow \{-i\sigma_2, -\sigma_1, \sigma_1, -i\sigma_2\}_{(4)}, \quad 1 \mid e_7 \leftrightarrow \{-\sigma_1, \sigma_1, -\sigma_1, \sigma_1\}_{(4)} \right] \\
\]

Following [17], it is easy to realize that our matrices, in tensorial notation, are anti-hermitian

\[ < l | E_i | k > = - < k | E_i | l >, \quad (A6) \]

moreover,

\[ < 0 | E_i | k > = - < k | E_i | 0 > = -\delta_{ik}, \quad (A7) \]

and finally

\[ < l | E_i | k > = \epsilon_{ikl}. \quad (A8) \]

Whereas, for right operators, we have

\[ < l | (1 | E_i) | k > = - < k | (1 | E_i) | l >, \quad (A9) \]
\[ < 0 | (1 | E_i) | k > = - < k | (1 | E_i) | 0 > = -\delta_{ik}, \quad (A10) \]
\[ < l | (1 | E_i) | k > = -\epsilon_{ikl}, \quad (A11) \]

We think, it is clear, that these fundamental matrices are the direct generalizations of 't Hooft matrices [18] (the quaternionic case \( i, j, k = 1, 2, 3 \)). So, they can play a dual role, for a SLA, as we see in this article, and a solitonic construction as any 7 or 8 dimensions instanton is a direct generalization from quaternions to octonions. Problems arise only for finding the suitable embedding of \( S_7 \) in a Lie algebra which can be solved directly by using (9–14) and that is all for the time being.