Since January 2020 Elsevier has created a COVID-19 resource centre with free information in English and Mandarin on the novel coronavirus COVID-19. The COVID-19 resource centre is hosted on Elsevier Connect, the company's public news and information website.

Elsevier hereby grants permission to make all its COVID-19-related research that is available on the COVID-19 resource centre - including this research content - immediately available in PubMed Central and other publicly funded repositories, such as the WHO COVID database with rights for unrestricted research re-use and analyses in any form or by any means with acknowledgement of the original source. These permissions are granted for free by Elsevier for as long as the COVID-19 resource centre remains active.
Research paper

Optimal control of a fractional order model for granular SEIR epidemic with uncertainty

Nguyen Phuong Dong\textsuperscript{a}, Hoang Viet Long\textsuperscript{b,c,*}, Alireza Kharat\textsuperscript{d}

\textsuperscript{a}Faculty of Mathematics, Hanoi Pedagogical University 2, Vietnam
\textsuperscript{b}Division of Computational Mathematics and Engineering, Institute for Computational Science, Ton Duc Thang University, Ho Chi Minh City, Vietnam
\textsuperscript{c}Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam
\textsuperscript{d}Department of Mathematics, Institute for Advanced Studies in Basic Sciences, Zanjan, Iran

ARTICLE INFO

Article history:
Received 17 January 2020
Revised 27 March 2020
Accepted 28 April 2020
Available online 30 April 2020

MSC:
34A07
28B10
58C06
54C60

Keywords:
Granular differentiability
Matrix Mittag-Leffler function
Fractional optimal control
Diseases modeling

ABSTRACT

In this study, we present a general formulation for the optimal control problem to a class of fuzzy fractional differential systems relating to SIR and SEIR epidemic models. In particular, we investigate these epidemic models in the uncertain environment of fuzzy numbers with the rate of change expressed by granular Caputo fuzzy fractional derivatives of order $\beta \in (0, 1]$. Firstly, the existence and uniqueness of solution to the abstract fractional differential systems with fuzzy parameters and initial data are proved. Next, the optimal control problem for this fractional system is proposed and a necessary condition for the optimality is obtained. Finally, some examples of the fractional SIR and SEIR models are presented and tested with real data extracted from COVID-19 pandemic in Italy and South Korea.

© 2020 Elsevier B.V. All rights reserved.

1. Introduction

Fractional differentiation and fractional integration are considered as the effective tools for characterizing the behaviors of a large category of complex dynamical systems where the integer order systems cannot be applied. The non-locality and capacity of describing hereditary properties make fractional models more practical than the usual ones, especially for systems which involve memory [38,39]. As a consequence, optimal control problems driven by fractional differential equations (DEs) have been also dramatically studied in theory with various useful applications. Especially, these models are very effective in modeling memory and hereditary properties of different materials and processes such as electrical circuits, fluid dynamics, biological models, and so forth, see [13,21,38,39]. However, the nature of phenomena being studied are vagueness, imprecision and ambiguity. Thus, the fractional calculus in fuzzy environments has gained a significant achievement in the last decade. After introducing by Agrawal et al. [4], fuzzy fractional DEs has attracted a large amount of research and achieved some noticeable results. On the theoretical side, Mazandarani and Kamyad [26] proposed a modified Euler

* Corresponding author at: Division of Computational Mathematics and Engineering, Institute for Computational Science, Ton Duc Thang University, Ho Chi Minh City, Vietnam.

E-mail addresses: nguyenphuongdong@hpu2.edu.vn (N.P. Dong), hoangvietlong@tdtu.edu.vn (H.V. Long), kharat@iasbs.ac.ir (A. Kharat).

https://doi.org/10.1016/j.cnsns.2020.105312
1007-5704/© 2020 Elsevier B.V. All rights reserved.
method to solve initial value problems for fuzzy fractional differential systems under Caputo-type fuzzy fractional derivatives. A modified fixed point approach in generalized complete semi-linear metric space, namely extended Krasnoselskii’s fixed point theorem, was discussed by Long and Dong in [25] and applied to investigate the solvability and Ulam–Hyers stability of nonlocal initial problem for implicit fractional fuzzy differential systems in [46]. In the literature [5], Ali et al. studied Riemann–Liouville fractional-order fuzzy BAM neural networks with time delay and impulsive. Based on contraction principle, Barbala’s lemma and Lyapunov function method, the authors proved the unique existence of equilibrium point of the considered model and its global asymptotic stability. For more details on theoretical results and applications, see [6,24,25,41,42]. Many effective approaches to study fuzzy fractional DEs have emerged in which the most recent approach base on the concept of granular differentiability and granular fractional differentiability introduced by Mazandarani et al. [27] and Najariyan et al. [31]. The granular differentiability is developed from the idea of horizontal membership function of Piegat [35,37]. This approach has shown its advantage for numerical methods of fuzzy fractional DEs. For more applications of granular differentiability to fuzzy differential equations and fuzzy fractional differential equations, see [28,29,43–45,49].

Optimal control theory is a branch of applied mathematics that deals with the problem to find the most suitable control input subjecting to a given dynamic system in order to minimize a performance index. In general, an optimal control is a set of differential equations describing the paths of the control variables that minimize a function of state and control variables, namely the cost function. Various optimization problems in both science and engineering associated with the integer optimal control of differential systems have studied for a long time by many authors, see for instance [10,20,28,29] and the references therein. Recently, the optimal control problem reduces to a fractional optimal control problem (FOCP) when either the performance index or the dynamic constraints or both include at least one fractional-order derivative term. It is well-known that theory of fractional differential equations has paid much attention over the past twenty years due to its capacity in describing the natural models and phenomena, that is the reason why the number of publications on fractional optimal control problem (FOCPs) have grown rapidly, see [3,8,16,22,44]. However, in the existing literature, there are a few studies investigating the optimal control problems and control problems in general for fractional dynamic systems with uncertainty. The idea of combining fractional optimal control theory with fuzzy theory opened up many exciting applied problems in various areas of science and engineering. In fact, it is an attractive topic now and have achieved some considerable results. For instance, Sharma et al. [40] proposed a scheme implemented two-layered fractional fuzzy logic controller for robotic manipulator, where the controller parameters of the proposed scheme are obtained with potential meta-heuristic technique named as cuckoo search algorithm. Next, Mazandarani et al. [28,29] gave some necessary conditions for the optimality of fuzzy Bang-bang control and sub-optimal control problems for fuzzy dynamical systems under granular differentiability. Later, Najariyan and Zhao [31] also discussed the granular fractional derivatives and fractional differential systems. Then, these concepts were applied to investigate the optimal conditions for a fuzzy fractional linear quadratic regulator problem under granular fractional derivatives. In addition, based on granular fractional differentiability, Son et al. [44] also proposed the optimal control problem for granular neutrosophic fractional differential equations with some applications in engineering. In the literature [33], Pan and Das studied an application of fuzzy fractional control problems in designing fuzzy PID controllers for hybrid power system with renewable generation using chaotic PSO. Recently, Abbasi and Jalali [1] investigated the tracking control problem for fuzzy linear dynamical systems with applications in fuzzy tracking control problems of output of a two tanks in series system and landing jet aircraft. Muñoz-Vázquez et al. [48] also studied a model-free implementation and output feedback control for robust tracking of robotic manipulators based on the combination of fuzzy logic techniques for the control of uncertain dynamic systems, as well as the intrinsic properties of fractional-order operators for modeling and control of complex engineering processes.

Epidemiology is an area concerned with the spread of disease and its effect on people. It relates to a range of science areas such as biology, sociology, philosophy, etc. and, in general, the aim is to prevent or control the spread of infection in health care facilities and the community. The modeling of infectious diseases is an effective tool which has been used to study the mechanisms by which diseases spread, to predict the future behavior of an outbreak and to evaluate strategies to control an epidemic. One of the earliest mathematical models to study infectious diseases, which is also one of the most well-known disease models, is the susceptible-infected-recovered (SIR) model proposed and studied by Kermack and McKendrick in 1927. Another early model is the Reed-Frost model developed in 1928 by Lowell Reed and Wade Hampton Frost, see [2]. Both of them describe the relationship between susceptible, infected and immune individuals in a population, however, the Kermack-McKendrick epidemic model was regarded as a successful model in predicting the behavior of outbreaks very similar to that observed in many recorded epidemics. Another famous epidemic model is the SEIR (Susceptibles, Exposed, Infectious, Recovered) model, an extension of the classical SIR. In this model, the added compartment E contains exposed persons which are infected but are not yet infectious. Moreover, individuals in all compartments of SEIR model are measured as fractions of the total population. Since the pioneering work in mathematical modeling of infectious diseases, many modified SIR & SEIR models have been developed and studied that allows us to study additional disease outbreak details, for example, the difference in disease spreads in different age groups or the difference in disease spreads in different environments. After predicting or approximating the behaviors of model’s outbreak, there is a natural necessity that we need to control the epidemic, that is the motivation for the arising of many control and optimal control problems for epidemic models. Especially, the study of optimal control problem for epidemic disease models with non-integer order differential constraints is appreciated as the most effective extension. This comes from the considerable advantage of the fractional order derivative for the best modeling of biological models whose purpose is to get a deeper understanding of the complex behavioral patterns of various communicable diseases. Moreover, mathematical epidemic models with fractional
order derivative provide a better fit to the real data instead of integer order models. Indeed, Para et al. [34] proposed a fractional order model to investigate dynamics of the outbreaks of influenza A (H1N1), and demonstrated that the fractional order model provides a better agreement with real data than integer one. In [7], Arenas et al. constructed nonstandard finite difference schemes to obtain numerical solutions of the susceptible - infected (SI) and susceptible - infected - recovered (SIR) fractional - order epidemic models. A fractional order SIR epidemic model for dengue transmission between human and mosquito was proposed by Hamdan and Kilicman [17] to show the effect of fractional order derivative in representing the index of memory and eliminating the disease. In the field of engineering, Huo and Zhao [18] applied SIR model with fractional order derivative to study the stability of the equilibrium points of heterogeneous complex networks. For more details, see [14,15,19,22] for therein. Although the issue of fractional optimal control of epidemic models has been investigated before, there still have many open questions remain. It is a fact that every real-world problem is inherently biased by uncertainty or there may be a lack of knowledge or incomplete information about variables and parameters. Hence, when we use fractional SIR & SEIR epidemic models to discuss real-world epidemic phenomena, it is a natural fact that we must accept to interpret and solve these problems in environments containing uncertainty and vagueness. An interesting fuzzy fractional problem related to SIR & SEIR epidemic models is the fuzzy fractional optimal control problem, in which we need to establish the necessary and sufficient conditions for the optimality or design the input controller for these models with incomplete initial data and uncertain parameters. However, there has been a few work done in the area of fractional optimal control problems for fuzzy fractional nonlinear differential equations, especially, fuzzy fractional optimal control problems modeled the SIR or SEIR epidemic models. Some pioneer works on fractional optimal control problem for fuzzy fractional differential systems can be mentioned such as the fuzzy time optimal problem, namely fuzzy Bang-bang control problem, for fuzzy differential equations discussed by Mazandarani and Zhao [28] or the sub-optimal control of fuzzy linear dynamical systems under granular differentiability in [29]. In a recent work, Son et al. [45] also presented an interesting result on an optimal problems governed by fractional linear differential equations in neutrosophic environment. However, it should be mentioned that most of these results are at the first stage and these considered model are just simple linear dynamical systems like linear differential systems. Although Najariyan and Zhao [31] introduced necessary conditions for the fuzzy fractional quadratic regulator problem of fuzzy fractional differential systems under granular Caputo fractional differentiability, all its applied models were in linear case. To the best of our knowledge, so far there has been no research in the area of optimal control for fuzzy fractional nonlinear differential equations governed by SIR or SEIR epidemic models. Therefore, in this paper, we propose the work on fractional optimal control problems for an abstract fuzzy fractional differential systems and apply the theoretical results to some epidemic disease model with the aim to approximate and control the behaviors of outbreak.

The main contributions of this paper can be highlighted as follows:

(i) In this paper, the horizontal membership function approach proposed by Piegat et al. [35,36] are applied to represent some concepts in the fuzzy numbers environment such as fuzzy numbers, fuzzy-valued functions and fuzzy matrices. Moreover, the paper also presents a brief on fundamental analysis properties of fuzzy-valued functions, fractional derivative and integral of fuzzy-valued functions.

(ii) We propose the concept of fuzzy matrix Mittag-Leffler function and present some basic characteristic properties. The foundation of the fuzzy matrix $\mathbb{E}_{\alpha,\beta}(A)$ plays a key role in the representation of solution formula of fuzzy fractional differential systems, i.e., it guarantees the fuzzy fractional differential systems

$$\begin{align*}
\frac{\partial^\beta}{\partial t^\beta} x(t) &= Ax(t) + f(t, x(t)), \\
x(t_0) &= x_0.
\end{align*}$$

can be transferred into the following integral equations

$$x(t) = \mathbb{E}_\beta((t - t_0)^\beta A)x_0 + \int_{t_0}^t (t - s)^{\beta - 1}\mathbb{E}_{\beta,\beta}((t - s)^\beta A)f(s, x(s))ds, \quad t \in [t_0, T],$$

and then, various analysis and numerical methods can be applied to investigate the solvability, numerical method and qualitative properties of integral solution. Here, the notion $\frac{\partial^\beta}{\partial t^\beta} (\cdot)$ is the granular Caputo fuzzy fractional derivatives introduced in [31].

(iii) Next, we investigate an abstract model of fractional differential systems with uncertainty, namely fuzzy fractional differential systems. Under the Laplace transformation, the fuzzy differential systems can be transformed equivalently into the set of deterministic fractional differential systems with RDM variable $\mu$ and the level set index $\alpha$. Especially, they still preserve the qualitative properties of the fuzzy solution. Moreover, one of the most advantages of the approach is to proceed both analytical methods and numerical methods on the deterministic fractional differential systems and then the respective fuzzy solutions can be obtained by using the transformation (1), that follows various useful applications in real-world problems with uncertainty. To illustrate the efficiency of the theoretical method, an example relating to fractional SIR epidemic model with uncertain data is presented.

(iv) At last, we contribute a great effort to introduce a general formulation for an optimal control problem to fuzzy fractional differential system (7). To find a control input $u(t)$ that minimizes the performance index (6), we establish a necessary condition for a pair $(\hat{X}, \hat{U})$ to be an optimal pair and then, applying this theoretical result to investigate an familiar epidemic disease model, namely SEIR epidemic model with fractional rate of change and fuzzy initial data.
The paper is organized as follows: In Section 2, some preliminaries are given regarding to the granular metric space of fuzzy numbers, granular fractional integrals and derivatives, fuzzy matrix Mittag-Leffler functions. Section 3 is devoted to study the Cauchy problem for a class of fuzzy fractional differential systems and prove the existence and uniqueness of fuzzy mild solutions. Section 4 presents the optimality of the fractional optimal control problem with uncertainty and proposes the numerical algorithm to implement the numerical solution of the proposed optimal problem. The fractional SIR and SEIR models are provided at the end of Sections 3 & 4 to illustrate the effectiveness of theoretical results. At last, the conclusion is given in Section 5.

2. Preliminaries

In this section, we recall some essential facts from basic concepts of fuzzy analysis [11,12,23,47].

2.1. The granular metric space of fuzzy numbers

**Definition 2.1.** In RDM (Relative Distance Measure) interval arithmetic, it is well-known that a given element \( x \) of the interval \( X = [x^-, x^+] \) can be described by using RDM variable \( \mu \in [0, 1] \) as follows
\[
x = x^- + (x^+ - x^-) \mu,
\]
and the interval \( X \) can be written as \( X = \{ x : x = x^- + \mu(x^+ - x^-), \mu \in [0, 1] \} \). Hence, this suggests an effective way to present uncertain quantities in acceptance performances for known ranges of parameter outcomes.

Now, we introduce a special type of uncertainty with various realistic applications, namely fuzzy numbers. The basic definitions of fuzzy sets and fuzzy numbers are presented clearly in [11].

**Definition 2.2** [11]. A fuzzy number \( u \) represents a mapping \( u : [a, b] \subseteq R \rightarrow [0, 1] \) satisfying 4 properties: normal, upper semi-continuous, fuzzy convex and compact supported. The space of all fuzzy numbers on \( R \) is denoted by \( R_F \).

The level sets or \( \alpha \) – cuts of a fuzzy number \( u \), denoted by \( |u|\alpha \), is defined as follows:
\[
|u|\alpha = \begin{cases} 
\{ x \in R : u(x) \geq \alpha \} & \text{if } 0 < \alpha \leq 1, \\
\text{cl}(\text{supp } u) & \text{if } \alpha = 0.
\end{cases}
\]

Notice that \( \alpha \) – cuts of a fuzzy number \( u \) is a closed and bounded interval of \( R \), that can be written in following parametric form \( |u|\alpha = [\alpha e_0, \alpha e_1] \) and the length of \( \alpha \) – cuts of \( u \) is denoted by \( \text{len}(|u|\alpha) = e_1 - e_0 \) for each \( \alpha \in [0, 1] \). Moreover, according to Bede [11], the arithmetic operations on the space \( R_F \) such as addition in Minkowski sense, scalar multiplication, \( H \)-difference, \( gH \)-difference, \ldots and other properties of fuzzy-valued mappings are defined via their \( \alpha \) – cuts.

It is well-known that in fuzzy systems, uncertain concepts are defined with membership functions \( \alpha = f(x) \) that may be called “vertical” ones. Unfortunately, there were gaps because of more complicated operations on the vertical representation of fuzzy numbers. This leads to many computational paradoxes observed in fuzzy arithmetic. Hence, based on the idea of horizontal membership functions elaborated by Piegat and Landowski [35], we introduce the concept of horizontal membership function or granular representation of fuzzy numbers

**Definition 2.3** (gr-representation [27]). Let \( u : [a, b] \subseteq R \rightarrow [0, 1] \) be a fuzzy number. Then, horizontal membership function of \( u \) is defined as follows
\[
u^{\text{gr}} : [0, 1] \times [0, 1] \rightarrow [a, b]
\]
\[
(\alpha, \mu) \mapsto u^{\text{gr}}(\alpha, \mu) = u^\alpha + \text{len}(|u|\alpha)\mu,
\]
where the notion “\( \text{gr} \)” represents for the granule of information included in \( x \in [a, b], \alpha \in [0, 1] \) and \( \mu \in [0, 1] \). The horizontal membership function of \( u \in R_F \) is also denoted by \( Q(u) \triangleq u^{\text{gr}}(\alpha, \mu) \).

**Remark 2.1.** The variable \( \mu \in [0, 1] \) is called relative-distance-measure (RDM) variable, which gives a possibility to obtain an arbitrary value between left endpoint \( u^\alpha \) and right endpoint \( u^\beta \) of fuzzy number \( u \). When \( \mu \) varies from 0 to 1, the value \( u^{\text{gr}}(\alpha, \mu) \) takes all values in the interval \( [u^\alpha, u^\beta] \). Note that by the horizontal membership function approach, we can identify each element \( x \in u(x) \) as a two-variable function \( x = f(\alpha, \mu) \) and present the number \( u \) as a granule in 3D – space.

**Example 2.1.** Let us consider an \( L \) – \( R \) fuzzy number \( u \) defined by
\[
u(x) = \begin{cases} 
x^2 & \text{if } x \in [0, 1], \\
\left(\frac{3 - x}{2}\right)^2 & \text{if } x \in [1, 3], \\
0 & \text{if } x \notin [0, 3],
\end{cases}
\]
whose \( \alpha \) – cuts are given by \( |u|\alpha = \left[ \sqrt{\alpha}, 3 - 2\alpha \right] \) for all \( \alpha \in [0, 1] \). Fig. 1 shows the fuzzy number \( u \) in granule of horizontal membership function. Here, for \( \alpha = 0.5 \) the granular representation \( |u|^{0.5} \) takes all values in the interval \( \left[ \frac{1}{\sqrt{2}}, 3 - \sqrt{2} \right] \) when
the RDM variable $\mu$ varies from 0 to 1. Moreover, we can determine uniquely an element $x \in [u]^{0.5}$ from the formula
\[
x = u^-_2 + (u^+_2 - u^-_2)\mu = \frac{1}{\sqrt{2}} + 3\left(1 - \frac{1}{\sqrt{2}}\right)\mu \text{ corresponding to a certain value } \mu \in [0, 1].
\]

**Proposition 2.1** [27], The $\alpha$–cuts of $u \in \mathbb{R}_F$ can be obtained by
\[
Q^{-1}(u^{\alpha}(\alpha, \mu)) = [u]^{\alpha} = \left[\inf_{\beta \geq \alpha} u^{\beta}(\beta, \mu), \sup_{\beta \leq \alpha} u^{\beta}(\beta, \mu)\right]. \tag{1}
\]

**Definition 2.4.** The ordered relation on the space $\mathbb{R}_F$ is defined as follows:

(i) A fuzzy number $u_1$ is said to be less than or equal to fuzzy number $u_2$, written by $u_1 \preceq u_2$ if and only if $Q(u_1) \subseteq Q(u_2)$.
(ii) A fuzzy numbers $u_1$ and $u_2$ are said to be equal and written by $u_1 = u_2$ if and only if $u_1 \preceq u_2$ and $u_2 \preceq u_1$, or equivalently, $Q(u_1) = Q(u_2)$.

**Definition 2.5** [27]. Let $u, v$ be fuzzy numbers with the horizontal membership functions $Q(u)$ and $Q(v)$, respectively. Then, the arithmetic operations on the space $\mathbb{R}_F$ are defined symbolically by
\[
Q(u \odot v) \triangleq Q(u) * Q(v),
\]
where the notations “$\odot$”, “$*$” represent for the arithmetic operations on $\mathbb{R}_F$ and $\mathbb{R}$ such addition, subtraction, multiplication or division.

**Remark 2.2.** Especially, if the notation “$\odot$” denotes for the division operator then it requires that $v^{\beta}(\alpha, \mu) \neq 0$ for all $\alpha, \mu \in [0, 1]$ and the difference in this sense is called granular difference and denoted by $v^{\beta}(\alpha, \mu)$. As a consequence, the space $\mathbb{R}_F$ equipped with these arithmetic operations becomes a linear space.

**Example 2.2.** Let $u$ be the fuzzy number in Example 2.1 and $v$ be an L–R fuzzy number given by
\[
v(x) = \begin{cases} 
(x - 2)^2 & \text{if } x \in [2, 3], \\
(4 - x)^2 & \text{if } x \in [3, 4], \\
0 & \text{if } x \notin [2, 4].
\end{cases}
\]

According to Definition 2.3, the horizontal membership functions of $u$ and $v$ are given by
\[
Q(u) = u^{\beta}(\alpha, \mu_1) = \sqrt{2} \mu_1 + 3(1 - \sqrt{2}) \mu_1,
\]
\[
Q(v) = v^{\beta}(\alpha, \mu_2) = 2 + 2\sqrt{2} + 2(1 - \sqrt{2}) \mu_2
\]
for each $\alpha, \mu_1, \mu_2 \in [0, 1]$. Hence, by Definition 2.5, the arithmetic operations between $u$ and $v$ are calculated by their respective horizontal membership functions. In particular, we have

- **Addition.** By the relation $Q(u \oplus v) = Q(u) + Q(v)$, it implies
\[
Q(u \oplus v) = 2 + 2\sqrt{2} + 3(1 - \sqrt{2}) \mu_1 + 2(1 - \sqrt{2}) \mu_2
\]
for each $\alpha, \mu_1, \mu_2 \in [0, 1]$. Next, by using the formula (1), we obtain
\[
[u \oplus v]^{0.5} = \left[2 + 2\sqrt{2}, 7 - 3\sqrt{2}\right].
\]
Moreover, the fuzzy number $u\oplus v$ is also an L - R fuzzy number

$$(u \oplus v)(x) = \begin{cases} 
\frac{x-2}{2}^2 & \text{if } x \in [2, 4], \\
\frac{7-x}{3}^2 & \text{if } x \in [4, 7], \\
0 & \text{if } x \notin [2, 7].
\end{cases}$$

- **gr-difference.** Using the relation $Q(u \ominus^\alpha v) = Q(u) - Q(v)$, we obtain

$$Q(u \ominus^\alpha v) = -2 + (1 - \sqrt{\alpha}) (3\mu_1 - 2\mu_2)$$

for each $\alpha, \mu_1, \mu_2 \in [0, 1]$. Next, the formula (1) implies that

$$[u \ominus^\alpha v]^\alpha = [-4 + 2\sqrt{\alpha}, 1 - 3\sqrt{\alpha}].$$

Moreover, the fuzzy number $u\ominus^\alpha v$ is also an L - R fuzzy number

$$(u \ominus^\alpha v)(x) = \begin{cases} 
\frac{x+4}{2}^2 & \text{if } x \in [-4, -2], \\
\frac{1-x}{3}^2 & \text{if } x \in [-2, 1], \\
0 & \text{if } x \notin [-4, 1].
\end{cases}$$

- **Scalar multiplication.** For $\lambda = 2$, since $Q(2u) = 2Q(u)$, then we obtain

$$Q(2u) = 2\sqrt{\alpha} + 6(1 - \sqrt{\alpha})\mu_1,$$

for each $\alpha, \mu_1, \mu_2 \in [0, 1]$. Next, the formula (1) implies that $[2u]^\alpha = [2\sqrt{\alpha}, 6 - 4\sqrt{\alpha}]$. Additionally, the fuzzy number $2u$ is

$$(2u)(x) = \begin{cases} 
\frac{x^2}{4} & \text{if } x \in [0, 2], \\
\frac{6-x}{4}^2 & \text{if } x \in [2, 6], \\
0 & \text{if } x \notin [0, 6].
\end{cases}$$

- **Multiplication.** By equation $Q(u \odot v) = Q(u) \times Q(v)$, we obtain

$$Q(u \odot v) = \alpha + 2\sqrt{\alpha} + (1 - \sqrt{\alpha}) (3\mu_1 (2 + \sqrt{\alpha})_2 \mu_2 \sqrt{\alpha}) + 6(1 - \sqrt{\alpha})^2 \mu_1 \mu_2,$$

for each $\alpha, \mu_1, \mu_2 \in [0, 1]$. Then, by the formula (1), it implies that

$$[u \odot v]^\alpha = [\alpha + 2\sqrt{\alpha}, 2\alpha - 11\sqrt{\alpha} + 12].$$

In addition, we can present the product fuzzy number $u \odot v$ as follows

$$(u \odot v)(x) = \begin{cases} 
\frac{\sqrt{x+1} - 1}{2}^2 & \text{if } x \in [0, 3], \\
\frac{\sqrt{8x+25} + 11}{4}^\alpha & \text{if } x \in [3, 12], \\
0 & \text{if } x \notin [0, 12].
\end{cases}$$

For an intuitive graphical representations of $u$, $v$ and their arithmetic operations, see Fig. 2.

**Definition 2.6.** A fuzzy-valued function $f$ is a mapping

$$f : [a, b] \subset \mathbb{R} \rightarrow \mathcal{R}_\mathbb{F}$$

$$t \mapsto f(t).$$

In addition, assume that $f(t)$ is represented as a combination of $n$ distinct fuzzy numbers $u_1, u_2, \ldots, u_n$. Then, the horizontal membership function of $f(t)$, denoted by $Q(f(t))$ $\triangleq f^\alpha(t, \mu, f)$, is a mapping

$$f^\alpha: [a, b] \times [0, 1] \times [0, 1] \times \ldots \times [0, 1] \rightarrow \mathbb{R},$$

for all $\alpha \in [0, 1]$ and $\mu_f = (\mu_1, \mu_2, \ldots, \mu_n)$.

**Definition 2.7 [27].** The metric on $\mathcal{R}_\mathbb{F}$, namely granular metric, is a mapping $\rho^\mathbb{F} : \mathcal{R}_\mathbb{F} \times \mathcal{R}_\mathbb{F} \rightarrow \mathbb{R}^+ \cup \{0\}$, given by

$$\rho^\mathbb{F}(u, v) = \sup_{\alpha \in [0, 1]} \max_{\mu_u, \mu_v} |u^\alpha(\alpha, \mu_u) - v^\alpha(\alpha, \mu_v)|.$$

Note that the space $\mathcal{R}_\mathbb{F}$ endowed with the granular metric $\rho^\mathbb{F}$ becomes a complete metric space (see [43]). Moreover, it is a Banach space with the norm

$$\|x\|_1 = \rho^\mathbb{F}(x, \hat{0}).$$

**Proposition 2.2 [43].** Let $u, v, w \in \mathcal{R}_\mathbb{F}$ and $\lambda \in \mathbb{R}$. Then, the following statements are fulfilled:
Fig. 2. The fuzzy number $u$, $v$ and the arithmetic operations.
Proposition 2.3. Let \( u, v \) and \( w \in \mathbb{R}_F \) be arbitrary. Then, the following statement holds
\[
\rho^{\mathcal{F}}(u \circ v, u \circ w) = \| u \|_1 \cdot \rho^{\mathcal{F}}(v, w),
\]
where the operation "\( \circ \)" represents for the multiplication on the space \( \mathbb{R}_F \).

Proof. Indeed, assume that the horizontal membership functions of the fuzzy numbers \( u, v \) and \( w \) are \( u^{\mathcal{F}}(\alpha, \mu_u), v^{\mathcal{F}}(\alpha, \mu_v) \) and \( w^{\mathcal{F}}(\alpha, \mu_w) \), respectively. Then, we have
\[
\rho^{\mathcal{F}}(u \circ v, u \circ w) = \sup_{\alpha} \max_{\mu_u, \mu_v, \mu_w} |u^{\mathcal{F}}(\alpha, \mu_u) - u^{\mathcal{F}}(\alpha, \mu_u) - w^{\mathcal{F}}(\alpha, \mu_w)|
\]
\[
\quad = \sup_{\alpha} \max_{\mu_u, \mu_v, \mu_w} |u^{\mathcal{F}}(\alpha, \mu_u) - w^{\mathcal{F}}(\alpha, \mu_w)|
\]
\[
\quad = \sup_{\alpha} \max_{\mu_u, \mu_v, \mu_w} |u^{\mathcal{F}}(\alpha, \mu_u) - w^{\mathcal{F}}(\alpha, \mu_w)|
\]
\[
\quad = \rho^{\mathcal{F}}(u, \tilde{0}) \cdot \rho^{\mathcal{F}}(v, w)
\]
\[
\quad = \| u \|_1 \cdot \rho^{\mathcal{F}}(v, w).
\]
Therefore, the proof is complete. \( \square \)

Definition 2.8. Denote \( \mathbb{R}^+_F := \mathbb{R}_F \times \mathbb{R}_F \times \cdots \times \mathbb{R}_F \). Then, the metric on the set \( \mathbb{R}_F^n \) is defined by
\[
\mathbb{D}^{\mathcal{F}}(u, v) = \sum_{i=1}^{n} \rho^{\mathcal{F}}(u_i, v_i)
\]
for all \( u = (u_1, u_2, \ldots, u_n), v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}_F^n \). By similar arguments as in Definition 2.7, we can easily prove that the space \( (\mathbb{R}_F^n, \mathbb{D}^{\mathcal{F}}) \) is a complete metric space. Moreover, it is also a Banach space with the norm
\[
\| u \| = \mathbb{D}^{\mathcal{F}}(u, \tilde{0})
\]
where \( u \in \mathbb{R}_F^n \) and \( \tilde{0} = (\tilde{0}, \ldots, \tilde{0}) \) is zero fuzzy vector.

Definition 2.9 [27]. Let \( f : (a, b) \subset \mathbb{R} \to \mathbb{R}_F \). Then, \( f \) is said to be granular differentiable (gr-differentiable for short) at a point \( t_0 \in (a, b) \) if and only if there exists a \( f'_{gr}(t_0) \in \mathbb{R}_F \) such that for all \( h > 0 \) sufficiently small, we have
\[
f(t_0 + h) \ominus^{\mathcal{F}} f(t_0) = f'_{gr}(t_0) h + O(h),
\]
where the function \( O : \mathbb{R}^+ \to \mathbb{R}_F \) satisfies \( \lim_{h \to 0} \frac{O(h)}{h} = \tilde{0} \). The value \( f'_{gr}(t_0) \) is then called granular derivative (or gr-derivative) of fuzzy-valued function \( f(t) \) at \( t_0 \).

Definition 2.10 [43]. A function \( f : (a, b) \subset \mathbb{R}^+ \to \mathbb{R}_F \) is said to be gr-differentiable on \((a, b)\) if and only if the fuzzy-valued function \( f'_{gr} : (a, b) \subset \mathbb{R} \to \mathbb{R}_F \) is well-defined for all \( t \in (a, b) \). Such fuzzy-valued function \( f'_{gr} \) is called the gr-derivative of the function \( f \) on \((a, b)\). Denote \( C^1((a, b), \mathbb{R}_F) \) by the space of all continuously granular differentiable fuzzy-valued functions on \((a, b)\).

Next, we give a necessary and sufficient condition for the gr-differentiability of \( f : (a, b) \subset \mathbb{R} \to \mathbb{R}_F \).

Proposition 2.4 [27]. The function \( f : (a, b) \subset \mathbb{R} \to \mathbb{R}_F \) is said to be gr-differentiable at a point \( t_0 \in (a, b) \) if and only if its horizontal membership function is also differentiable with respect to variable \( t \). In addition, we have
\[
Q(f'_{gr}(t_0)) = \frac{\partial f^{\mathcal{F}}(t_0, \alpha, \mu_f)}{\partial t}.
\]

Proposition 2.5 [31,45]. The function \( f : \mathbb{R}_F^n \to \mathbb{R}_F \), defined by \( t \mapsto f(x_1, x_2, \ldots, x_n) \), is said to be granular partial differential w.r.t. the ith variable \( x_i \) if and only if its horizontal membership function is differentiable w.r.t. the variable \( Q(x_i) \). Moreover,
\[
Q \left( \frac{\partial f(x_1, \ldots, x_i, \ldots, x_n)}{\partial x_i} \right) = \frac{\partial f^{\mathcal{F}}(\alpha, \mu_1, \ldots, x_i, \ldots, \mu_n)}{\partial x_i^{\mathcal{F}}(\alpha, \mu_i)}.
\]
for all \( \alpha \in [0, 1], \mu_i \in [0, 1] \).

Definition 2.11 [31,45]. Assume that:
(i) The function \( f : [a, b] \subset \mathbb{R} \to \mathbb{R}_F \) is gr-differentiable at the point \( t_0 \in [a, b] \);
(ii) The function \( g : \mathbb{R}_F \to \mathbb{R}_F \) is granular partial differentiable w.r.t. the variable \( f(t) \) at \( y_0 = f(t_0) \in \mathbb{R}_F \).
Then, the composite function \((g \circ f)(t) = g(f(t))\) is said to be granular differentiable at the point \(t_0\) if there exists an element \(\frac{dg(g(f(t_0)))}{dt} \in \mathbb{R}_f\) such that for all \(h\) sufficiently near 0, the following limit exists
\[
\lim_{h \to 0} \frac{g(f(t_0 + h)) \ominus^{gr} g(f(t_0))}{h} = \frac{dg(g(f(t_0)))}{dt}.
\]

**Proposition 2.6** [45]. Assume that:

(i) The function \(f : [a, b] \subset \mathbb{R} \to \mathbb{R}_f\) is gr-differentiable at the point \(t_0 \in [a, b]\).

(ii) The function \(g : \mathbb{R}_f \to \mathbb{R}_f\) is granular partial differentiable w.r.t the variable \(f(t)\) at \(y_0 = f(t_0) \in \mathbb{R}_f\).

Then, granular derivative of the composite function \((g \circ f)\) at the point \(t_0\) is given by
\[
\frac{d_{gr}(g(f(t_0))))}{dt} = \left(\frac{\partial_{gr} g(f(t))}{\partial f(t)} \cdot \frac{d_{gr} f(t)}{dt}\right)|_{t=t_0}.
\]

2.2. The granular fractional integral and fractional derivative of fuzzy-valued functions

For convenience to the readers, we briefly recall some notions of fractional real analysis. For more details, see [21] and the references therein.

**Definition 2.12** [21]. The fractional integral \(I^\beta_a f(t)\) of order \(\beta > 0\) of \(f \in L^1([a, b], \mathbb{R})\) is defined by
\[
I^\beta_a f(t) := \frac{1}{\Gamma(\beta)} \int_a^t (t - \tau)^{\beta-1} f(\tau) d\tau, \quad t \in [a, b].
\]
Here, \(\Gamma(\beta)\) is the gamma function \(\Gamma(\beta) = \int_0^\infty t^{\beta-1} e^{-t} dt\).

**Definition 2.13** [21]. The Caputo fractional derivative \(D^\beta_a f(t)\) of order \(\beta > 0\) of \(f \in C^n([a, b], \mathbb{R})\) is defined by
\[
D^\beta_a f(t) := \frac{1}{\Gamma(n-\beta)} \int_a^t (t - \tau)^{n-\beta-1} f^{(n)}(\tau) d\tau,
\]
where \(n - 1 < \beta < n, n \in \mathbb{N}\) and \(t \in [a, b]\).

Next, we recall the concepts of granular fractional integral and the granular Caputo fuzzy fractional derivatives of order \(p \in (0, 1]\) of fuzzy-valued functions (see [31]). Some related properties of granular integral and granular Caputo fractional derivative are also presented.

**Definition 2.14** [31]. Let \(f : [a, b] \subset \mathbb{R} \to \mathbb{R}_f\). Then, the right-sided and left-sided granular fractional integrals of order \(\beta \in (0, 1]\) of fuzzy-valued function \(f\) are defined by
\[
\ominus\mathcal{I}^\beta_a f(t) = \frac{1}{\Gamma(\beta)} \int_a^t (t - \tau)^{\beta-1} f(\tau) d\tau,
\]
\[
\ominus\mathcal{I}^\beta_b f(t) = \frac{1}{\Gamma(\beta)} \int_t^b (\tau - t)^{\beta-1} f(\tau) d\tau.
\]

**Remark 2.3.** According to **Definition 2.14**, we can define the form of horizontal membership function of granular fractional integral \(\ominus\mathcal{I}^\beta_a f(t)\) as follows
\[
\ominus\mathcal{Q}(\ominus\mathcal{I}^\beta_a f(t)) = \frac{1}{\Gamma(\beta)} \int_a^t \ominus\mathcal{Q}((t - \tau)^{\beta-1} f(\tau)) d\tau = \ominus\mathcal{I}^\beta_a \mathcal{Q}(f(t)) = \mathcal{I}^\beta_b \mathcal{Q}(f(t)).
\]
Similarly, we also have \(\ominus\mathcal{Q}(\ominus\mathcal{I}^\beta_b f(t)) = \mathcal{I}^\beta_b \mathcal{Q}(f(t))\).

**Definition 2.15** [31]. Let \(f : [a, b] \subset \mathbb{R} \to \mathbb{R}_f\) be a granular differentiable function. Then, the right-sided and left-sided granular Caputo fractional derivatives of order \(\beta \in (0, 1]\) of the function \(f\) are defined as follows
\[
\ominus\mathcal{D}^\beta_a f(t) = \frac{1}{\Gamma(1-\beta)} \int_a^t (t - \tau)^{-\beta} f^{(\beta)}(\tau) d\tau = \ominus\mathcal{I}^{1-\beta}_a (f^{(\beta)}(t)),
\]
\[
\ominus\mathcal{D}^\beta_b f(t) = -\frac{1}{\Gamma(1-\beta)} \int_t^b (\tau - t)^{-\beta} f^{(\beta)}(\tau) d\tau = -\ominus\mathcal{I}^{1-\beta}_b (f^{(\beta)}(t)).
\]
Remark 2.4. According to Definition 2.15, we can see that if \( f(t) \) is a constant function, then
\[
\frac{\partial}{\partial \alpha} D^\alpha f(t) = \frac{\partial}{\partial \beta} D^\beta f(t) = \hat{0}(t).
\]

Remark 2.5. By similar arguments as in Remark 2.3, we can also conclude that
\[
\begin{align*}
\text{(i)} & \quad Q\left( \frac{\partial}{\partial \alpha} D^\alpha f(t) \right) = \frac{\partial}{\partial \alpha} Q(f(t)). \\
\text{(ii)} & \quad Q\left( \frac{\partial}{\partial \beta} D^\beta f(t) \right) = \frac{\partial}{\partial \beta} Q(f(t)).
\end{align*}
\]

2.3. Fuzzy matrix and matrix Mittag-Leffler function

Definition 2.16. Let \( a_{ij} \) be fuzzy numbers for each \( i = 1, \ldots, m \), \( j = 1, \ldots, n \). Then, the matrix \( A = [a_{ij}]_{m \times n} \) is said to be a fuzzy matrix of order \( m \times n \). In special case, if \( m = n \), the matrix \( A \) is called a square fuzzy matrix of order \( n \). Adapting to Definition 2.3, the horizontal membership function of the matrix \( A \) is defined by
\[
A_{\alpha} (\alpha, \mu_A) = \left[ a_{ij}^\alpha (\alpha, \mu_{ij}) \right]_{n \times n}
\]
for each \( \alpha \in [0, 1] \) and \( \mu_A = \{ \mu_{ij} \in [0, 1] : i, j = 1, \ldots, n \} \).

Remark 2.6. The matrix operations on the set of fuzzy matrices, such as matrix addition - subtraction, scalar multiplication, matrix transpose and matrix inverse, can be defined based on arithmetic operations introduced in Definition 2.5 and the classical matrix operations.

Definition 2.17 [28]. Let \( A = [a_{ij}]_{n \times n} \) be a square fuzzy matrix of order \( n \). A fuzzy number \( \lambda \) is said to be an eigenvalue of the matrix \( A \) if
\[
det(\lambda^\alpha (\alpha, \mu)I_n - A_{\alpha} (\alpha, \mu_A)) = 0,
\]
where \( \det(\cdot) \) and \( I_n \) represent for the determinant and the \( n \times n \) identity matrix, respectively.

Definition 2.18 [44]. Consider a linear time-invariant (LTI) system
\[
\begin{cases}
X'_{\alpha}(t) = Ax(t), & t > 0, \\
x(0) = x_0,
\end{cases}
\]
where \( A = [a_{ij}]_{n \times n} \) is a square fuzzy matrix. Then, the exponential matrix of the LTI system (2) is defined by
\[
e^{tA} := \sum_{k=0}^{\infty} \frac{1}{k!} (tA)^k = I_n + (tA) + \frac{1}{2!} (tA)^2 + \frac{1}{3!} (tA)^3 + \ldots + \frac{1}{n!} (tA)^n + \ldots
\]

Definition 2.19. Let \( A = [a_{ij}]_{n \times n} \) be a square fuzzy matrix. The matrix Mittag-Leffler function of the matrix \( A \) with two parameters \( \alpha, \beta > 0 \) is defined by
\[
E_{\alpha, \beta}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k + \beta)},
\]
whose horizontal membership function is given by
\[
E_{\alpha, \beta}(A_{\alpha} (\alpha, \mu_A)) = \sum_{k=0}^{\infty} \frac{A_{\alpha}^k (\alpha, \mu_A)}{\Gamma(\alpha k + \beta)},
\]
where \( \alpha \in [0, 1] \) and \( \mu_A = \{ \mu_{ij} \in [0, 1] : i, j = 1, \ldots, n \} \).

Proposition 2.7. Let \( A = [a_{ij}]_{n \times n} \) be a square fuzzy matrix of order \( n \). Then, the matrix norm of \( A \) is defined by
\[
\|A\| = \max_{1 \leq j \leq n} \sum_{i=1}^{n} \rho^\alpha(a_{ij}, \hat{0}) < \infty.
\]
As a consequence, we obtain the norm of the matrix \( E_{\alpha, \beta}(A) \) as
\[
\|E_{\alpha, \beta}(A)\| \leq \sum_{k=0}^{\infty} \frac{\|A\|^k}{\Gamma(\alpha k + \beta)} < \infty.
\]

Lemma 2.1. Let \( A = [a_{ij}]_{n \times n} \) and \( B = [b_{ij}]_{n \times n} \) be square fuzzy matrices. Then, for each \( \alpha, \beta > 0 \), the following properties are fulfilled.

(i) If there exists an invertible matrix \( S \) such that \( A = SBS^{-1} \) then
\[
E_{\alpha, \beta}(A) = SE_{\alpha, \beta}(B)S^{-1}.
\]
(ii) If $A$ is a diagonal matrix, i.e., $A = \text{diag}(a_{11}, a_{22}, \ldots, a_{nn})$, then
$$E_{\alpha, \beta}(A) = \text{diag}(E_{\alpha, \beta}(a_{11}), E_{\alpha, \beta}(a_{22}), \ldots, E_{\alpha, \beta}(a_{nn})).$$

(iii) For each fuzzy matrix $A$, we have $E_{\alpha, \beta}(A^*) = \left[E_{\alpha, \beta}(A)\right]^*.$

**Example 2.3.** Let $A$ be a square fuzzy matrix defined by
$$A = \begin{pmatrix} \tilde{3} & \tilde{0} \\ \tilde{0} & \tilde{1} \end{pmatrix},$$
where $\tilde{X} = x(1)$, $\tilde{0} = x(0)$ and $\bar{3} = (2, 3, 4)$ are given fuzzy numbers. Now, our aim is to compute the matrix Mittag-Leffler function $E_{\alpha, \beta}(A)$. The procedure is divided into the following steps:

Step 1. Find the horizontal membership function of the matrix $A$ as
$$Q(A) = \begin{pmatrix} Q(\bar{3}) & 0 \\ 0 & 1 \end{pmatrix}.$$

Step 2. Find the matrix $Q(A^k) = [Q(A)]^k$. Indeed, for each $k \in \mathbb{N}$, we have
$$[Q(A)]^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, [Q(A)]^1 = \begin{pmatrix} Q(\bar{3}) & 0 \\ 0 & 1 \end{pmatrix}, \ldots$$

By induction principle, we obtain the matrix $Q(A^k) = [Q(A)]^k = \begin{pmatrix} [Q(\bar{3})]^k & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} Q(\bar{3})^k & 0 \\ 0 & 1 \end{pmatrix}$.

Step 3. Compute the matrix Mittag-Leffler function $E_{\alpha, \beta}(A)$. Firstly, we can see that the horizontal membership function of the matrix $E_{\alpha, \beta}(A)$ is given by
$$Q(E_{\alpha, \beta}(A)) = \sum_{k=0}^{\infty} Q(A^k) = \begin{pmatrix} \sum_{k=0}^{\infty} Q(\bar{3})^k / \Gamma(\alpha k + \beta) & 0 \\ 0 & \sum_{k=0}^{\infty} 1 / \Gamma(\alpha k + \beta) \end{pmatrix}.$$

Thus, by the formula (1), the matrix Mittag-Leffler function $E_{\alpha, \beta}(A)$ is
$$E_{\alpha, \beta}(A) = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{3^k}{\Gamma(\alpha k + \beta)} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)} \end{pmatrix} = \begin{pmatrix} E_{\alpha, \beta}(\bar{3}) & 0 \\ 0 & E_{\alpha, \beta}(\tilde{1}) \end{pmatrix}.$$

**Definition 2.20.** The Laplace transform of a function $f : \mathbb{R}^+ \to \mathbb{R}^n$ is defined by
$$F(\lambda) = \mathcal{L}[f(t)] = \int_0^{\infty} e^{-\lambda t} f(t) dt, \quad \lambda \in \mathbb{R}.$$

**Remark 2.7.** For each $\beta \in (0, 1]$, we have
$$\mathcal{L}[D^\beta f(t)] = \lambda^\beta \mathcal{L}[f(t)] - \lambda^{\beta-1} f(0),$$
where $f : \mathbb{R}^+ \to \mathbb{R}^n$ is a vector-valued function.

**Lemma 2.2 [38].** For each $\alpha, \beta > 0$ and $A \in \mathbb{R}^{n \times n}$, the following Laplace transform
$$\mathcal{L}[t^{\beta-1} E_{\alpha, \beta}(A t^\alpha)] = \lambda^{\alpha-\beta} (\lambda^\alpha \mathbb{I}_n - A)^{-1},$$
holds for all real number $\lambda > \|A\|^{\frac{1}{\beta}}$.

**3. The solvability of Cauchy problem to fuzzy fractional differential systems**

**3.1. State the problem**

In this work, we investigate the following Cauchy problem to fuzzy fractional differential system
$$\begin{cases} \frac{D^\beta}{D_t^\beta} x(t) = F(t, x(t)), & t \in J = [0, T], \\ x(0) = x_0. \end{cases}$$
where $\mathcal{D}_0^\beta x(t)$ is the granular Caputo fuzzy fractional derivative of order $\beta \in (0, 1]$ of state vector $x \in C([0, T], \mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$ is the initial state vector.

Here, without loss of generality, we assume that the function $F(t, x(t))$ can be rewritten as a sum of a linear term $Ax(t)$ and a nonlinear term $f(t, x(t))$, where $A$ is a square fuzzy matrix of order $n$ and $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a function that will be specified later. Then, the considered problem becomes

$$\begin{align*}
\mathcal{D}_0^\beta x(t) &= Ax(t) + f(t, x(t)), \\
x(0) &= x_0.
\end{align*} \tag{3}$$

**Remark 3.1.** Here, for simplicity in representation, denote $C_t := C([0, T], \mathbb{R}^n)$ by the space of all continuous functions from $[0, t]$ to $\mathbb{R}^n$, with the supremum metric

$$\mathcal{D}_{0,t}^\beta (x, y) = \sup_{[0, t]} \mathcal{D}_\beta (x(s), y(s)) \quad x, y \in C_t.$$ 

Moreover, the notation of norm on the space $C_t$ can be known as

$$\|x\|_t := \mathcal{D}_{0,t}^\beta (x, \bar{0}) \quad \text{for each } x \in C_t.$$ 

The solvability of the Cauchy problem (3) is obtained under the following hypotheses:

**(HA)** There exists positive constant $M_1$ such that for all $t \in [0, T]$ and $\beta \in (0, 1]$,

$$\|E_{\beta, \beta} (t^\beta A)\| \leq M_1.$$ 

**(HF)** The function $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies

**(HF1)** the fuzzy function $f(\cdot, \xi) : [0, T] \to \mathbb{R}^n$ is measurable for each $\xi \in \mathbb{R}^n$ and the fuzzy function $f(t, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is continuous for a.e. $t \in [0, T]$;

**(HF2)** $Lipschitz$ w.r.t the second variable, i.e., there exists a positive number $L_1$ such that

$$\mathcal{D}_\beta (f(t, \eta), f(t, \eta')) \leq L_1 \mathcal{D}_\beta (\eta, \eta'),$$

for all $\eta, \eta' \in \mathbb{R}^n$.

**(HF3)** bounded at the origin $\bar{0} \in \mathbb{R}^n$, i.e., there exists a positive number $L_2$ such that

$$\mathcal{D}_\beta (f(t, \bar{0}), \bar{0}) \leq L_2,$$

for all $t \in [0, T]$.

### 3.2. The existence and uniqueness of the integral solution

Firstly, we introduce a suitable concept of mild solution to Cauchy problem (3) that is associated with the matrix Mittag-Leffler function.

**Lemma 3.1.** For each $\beta \in (0, 1]$ and $t \in J$, the continuous function

$$x(t) = E_\beta (t^\beta A)x_0 + \int_0^t (t - \tau)^{\beta - 1} E_{\beta, \beta} ((t - \tau)^{\beta} A)f(\tau, x(\tau))d\tau \tag{4}$$

is said to be a mild solution to Cauchy problem (3).

**Proof.** For the proof of **Lemma 3.1**, see Appendix. $\square$

Now, we will present a fundamental result on the existence and uniqueness of mild solution of the Cauchy problem (3). For this aim, let us introduce the norm $\| \cdot \|_{\lambda}$ as follows

$$\|x\|_{\lambda} := \sup_{[0, T]} \mathcal{D}^\beta (x(t), \bar{0}) e^{-\lambda t}, \quad \lambda > 0.$$ 

According to Long et al. [24], Long and Dong [25], we can prove that the functional space $C_T$ endowed with the norm $\| \cdot \|_{\lambda}$ is a Banach space.

**Theorem 3.1.** Under the hypotheses (HA), (HF1), (HF2) and (HF3), the Cauchy problem (3) has at least one mild solution defined on $[0, T]$.

**Proof.** For the proof of **Theorem 3.1**, see Appendix. $\square$
SIR model parameters are used to model the spread of infectious diseases.

### Table 1

| Parameter                  | The world | Italy | South Korea | Data source |
|----------------------------|-----------|-------|-------------|-------------|
| Total population           | 7,773,132,415 | 60,550,000 | 51,225,000 | Link 2 |
| Infectious population ($I_0$) | 218,822 | 35,713 | 8,413 | Link 2 |
| Death population           | 8,677 | 2,978 | 84 | Link 2 |
| Recovered population ($R_0$) | 82,854 | 4,025 | 1,540 | Link 2 |
| Susceptible population ($S_0$) | "about 500,000" | "about 100,000" | "about 50,000" | Assumed |
| Infection rate $c$         | 0.65 | 0.7 | 0.35 | Assumed |
| Recovery rate $r$          | 0.37864 | 0.1127 | 0.18305 | Estimated |

### 3.3. Example

**Example 3.1.** The outbreak of the novel coronavirus disease (COVID-19) brought considerable turmoil all around the world. According to the WHO's statistical data at 24 March 2020, 7:50 GMT+7, at least 187 countries and territories around the world and 1 international conveyance (the Diamond Princess cruise ship harbored in Yokohama, Japan) are still struggling with the increasing number of new cases. Now, the problems we are interested in is what about the mechanism of the spread disease such as COVID-19 and how to control the spread of this infectious disease? The formulation of mathematical modeling for COVID-19 is considered as an effective tool for not only studying the spread of infectious diseases but also predicting the outbreak and formulate policies to control the epidemics. In this example, we use fractional order SIR epidemic model to describe the structure of how the epidemic disease COVID-19.

Next, we introduce the variables and some parameters used throughout this example. Firstly, we assume that the total population $N(t)$ is divided into three compartments: Susceptible, Infected and Recovered individuals whose sizes are denoted by $S(t)$, $I(t)$ and $R(t)$, respectively. Here, the compartment $S$ is the group of people who are vulnerable to exposure with infectious people. The group of infectious population represents the infected people. They can pass the disease to susceptible people and can be recovered in a specific period. The compartment $R$ is the group of recovered people who are assumed to get immunity so that they are not susceptible to the same illness anymore. The scheme of SIR model can be characterized in following diagram (Fig. 3) where the parameter $c$ is used to evaluate how much the disease can be transmitted through exposure and $r$ is used to express how much the disease can be recovered in a specific period. The parameters of the SIR model are given in Table 1. The initial infectious populations $I_0$ and initial recovered populations $R_0$ of the SIR model are realistic data referred from two following open data-sets

**Link 1:** [https://www.worldometers.info/coronavirus/](https://www.worldometers.info/coronavirus/)

**Link 2:** [https://worldpopulationreview.com/countries/coronavirus-by-country/](https://worldpopulationreview.com/countries/coronavirus-by-country/)

Table 4 in Appendix shows the data of infectious people and recovered people in South Korea, Italy and all the world from 18 March 2020. We set the initial time at 18 March and then, we estimate the SIR model's parameters to predict the epidemic outbreak of the COVID-19. Here is the estimated values of parameters and initial conditions.

In reality, we cannot have exact data about the number of susceptible cases and infected cases because the number of infectious individuals measured at a certain time are only confirmed cases and there have a lot of infectious people in the testing time, that have not been confirmed. Moreover, the COVID-19 outbreak are now widespread around the world with a dramatic increase in the number of infectious cases and it experienced a considerable fluctuation between the data of two compartments ($S$ and $I$). Therefore, it is a fact that we are difficult to get the exact and complete data of the epidemic and must admit that the number of susceptible cases in general is an uncertain quantity. Hence, it implies that every mathematical model modeling the outbreak of the COVID-19 should be an uncertain mathematical model. In this work, to dealing with the COVID-19's outbreak with the fuzziness in initial data, we use the mathematical SIR epidemic model with triangular fuzzy initial conditions. Furthermore, in order to express the non-local properties of the model, we propose a fractional model. Motivated by aforesaid, we introduce the following fuzzy fractional differential system for the SIR model

\[
\begin{align*}
D^\beta S(t) &= (-c)S(t)I(t) \\
D^\beta I(t) &= cS(t)I(t) - rI(t) \\
D^\beta R(t) &= rI(t),
\end{align*}
\]

with initial conditions $S(0) = S_0$, $I(0) = I_0$ and $R(0) = R_0$. 

---

**Fig. 3.** Diagram of the SIR epidemic disease model.
Next, by using the granular approach, the corresponding granular fractional differential system of the fuzzy fractional differential system (5) is

\[
\begin{aligned}
\begin{cases}
0 & D^\beta Q(S(t)) = -cQ(S(t))Q(I(t)) \\
0 & D^\beta Q(I(t)) = cQ(S(t))Q(I(t)) - rQ(I(t)) \\
0 & D^\beta Q(R(t)) = rQ(I(t)),
\end{cases}
\end{aligned}
\]

that is equivalent to following matrix form

\[
\begin{aligned}
\begin{bmatrix}
0 & 0 & 0 \\
0 & -r & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
Q(S(t)) \\
Q(I(t)) \\
Q(R(t))
\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
0 & -r & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix}
Q(S(t)) \\
Q(I(t)) \\
Q(R(t))
\end{bmatrix}
\end{bmatrix}
\end{aligned}
\]

\[
\begin{aligned}
&= -cQ(S(t))Q(I(t)) + cQ(S(t))Q(I(t)) - rQ(I(t)) + rQ(I(t)) \\
&= \begin{bmatrix}
-cQ(S(t))Q(I(t)) \\
0 \\
0
\end{bmatrix}.
\end{aligned}
\]

Here, for simplicity in representation, let us denote

- The state vector
  \[ X(t) = [S(t) \ I(t) \ R(t)]^T. \]
- \( A \) be a square matrix of the form
  \[
  A = \begin{bmatrix}
  0 & 0 & 0 \\
  0 & -r & 0 \\
  0 & 0 & 0
  \end{bmatrix}.
  \]
- The nonlinear function
  \[
  f(t, X(t)) = \begin{bmatrix}
  (-c)S(t)I(t) \\
  cS(t)I(t) \\
  r I(t)
  \end{bmatrix}.
  \]

Now, by using Theorem 3.1, we will prove that the fuzzy fractional SIR model (5) subject to the initial conditions \((S_0, I_0, R_0)\) has a unique solution. The proof is divided into following steps:

**Step 1.** According to Lemma 2.1-(ii), the matrix Mittag-Leffler function \(E_{\beta, \beta}(t^\beta A)\) is a diagonal matrix of the form

\[
E_{\beta, \beta}(t^\beta A) = \begin{bmatrix}
\frac{1}{\Gamma(\beta t^\beta)} & 0 & 0 \\
0 & \frac{1}{\Gamma(\beta I(t))} & 0 \\
0 & 0 & \frac{1}{\Gamma(\beta R(t))}
\end{bmatrix}.
\]

Moreover, since the Mittag-Leffler function \(E_{\beta, \beta}(-rt^\beta)\) is continuous on \([0, T]\), it implies that the norm of the matrix Mittag-Leffler function \(E_{\beta, \beta}(t^\beta A)\) is bounded, that means the hypothesis (HA) holds.

**Step 2.** For each \( t \in [0, T] \) and \( X, \dot{X} \in C([0, T], \mathbb{R}_+^3) \), we have

\[
D^\beta f(t, X(t), f(t, \dot{X}(t))) \leq 2c\rho^\beta(S(t)I(t), \dot{S}(t)\dot{I}(t)) + r\rho^\beta(I(t), \dot{I}(t)).
\]

Here, since \( \max \{|\dot{\rho}^\beta(I(t), \dot{I}(t))|, \rho^\beta(S(t), \dot{S}(t))\} \leq N_0 \) and by Proposition 2.3, we have

\[
\begin{aligned}
\rho^\beta(S(t)I(t), \dot{S}(t)\dot{I}(t)) &\leq \rho^\beta(S(t)I(t), \dot{S}(t)I(t)) + \rho^\beta(S(t)I(t), \dot{S}(t)\dot{I}(t)) \\
&\leq \rho^\beta(I(t), \dot{I}(t))\rho^\beta(S(t), \dot{S}(t)) + \rho^\beta(I(t), \dot{I}(t))\rho^\beta(S(t), \dot{S}(t)) \\
&\leq N_0\rho^\beta(S(t), \dot{S}(t)) + N_0\rho^\beta(I(t), \dot{I}(t)).
\end{aligned}
\]

Thus, we obtain

\[
D^\beta f(t, X(t), f(t, \dot{X}(t))) \leq 2cN_0\rho^\beta(S(t), \dot{S}(t)) + \rho^\beta(I(t), \dot{I}(t)) + r\rho^\beta(I(t), \dot{I}(t)) \\
\leq (2cN_0 + r)(\rho^\beta(S(t), \dot{S}(t)) + \rho^\beta(I(t), \dot{I}(t))) \\
\leq (2cN_0 + r)(\rho^\beta(S(t), \dot{S}(t)) + \rho^\beta(I(t), \dot{I}(t)) + \rho^\beta(R(t), \dot{R}(t))) \\
= (2cN_0 + r)\rho^\beta(X(t), \dot{X}(t)),
\]

which implies the hypothesis (HF2) holds with \( L_1 = 2cN_0 + r \).

**Step 3.** For each \( t \in [0, T] \), the boundedness of \( f(t, X(t)) \) is implied from following estimation

\[
D^\beta f(t, X(t), \dot{X}(t)) \leq 2c\rho^\beta(S(t)I(t), \dot{S}(t)) + \rho^\beta(I(t), \dot{I}(t)) \\
\leq 2c\rho^\beta(S(t), \dot{S}(t)) + \rho^\beta(I(t), \dot{I}(t)) + \rho^\beta(R(t), \dot{R}(t)) \\
\leq N_0[2c(N_0 + 1)].
\]
We can choose the constant \( L_2 = N_0[r + 2c(N_0 + 1)] \) that satisfies the hypothesis (HF3). Therefore, all assumptions of Theorem 3.1 are satisfied that guarantees the uniqueness and existence of solution of the initial value problem to the fuzzy fractional SIR model (5).

Based on the COVID-19 data in Table 4 and by using Matlab program fde12.m, we present the graphical representations of two compartments (I) and (R) of the fuzzy fractional SIR model (5). The figures shows the outbreaks of COVID-19 pandemic in three areas: all the world, Italy and Korea.

Figs. 4-6 present the number of infectious and recovered people in three populations: all the world, South Korea, Italy, that are the solutions of the fractional SIR models with parameters in Table 1. It is predicted that over a certain period of time, the rate of infections of the COVID-19 coronavirus will increase globally before reaching their peaks at a point and then, they experience a considerable decline. It is expected that the number of infectious individuals will be halted due to...
active quarantine and observatory procedures as prescribed by the WHO (2020). Moreover, there is a hope that a vaccine will be made ready in a shortest future possible. In addition, these charts also reveal that the number of recovered people will continue to increase despite the negligible increase in death rate from the COVID-19 coronavirus globally.

4. Fractional optimal control problem

4.1. The optimality of the fractional optimal control problem with uncertainty

In this section, we introduce the general formulation of a fuzzy fractional optimal control problem (FFOCP) and establish a necessary condition for the optimality of the considered FFOCP.

Here, the aim of the FFOCP is to find a control input \( u(t) \) that minimizes following performance index

\[
J(x, u) = \int_{t_0}^{t_f} G(t, x(t), u(t))dt,
\]

subject to the fuzzy fractional dynamic constraint

\[
^cD^\alpha x(t) = F(t, x(t), u(t)),
\]

with the initial condition

\[
x(t_0) = x_0.
\]

and transfers the state \( x(t, t_0, x_0) \) of the problem (7)-(8) from the state \( x(t_0) = x_0 \) to the origin at the time \( t = t_f \), where the state vector \( x : [t_0, t_f] \rightarrow \mathbb{R}^m_+ \) is granular Caputo fractional differentiable, \( u : [t_0, t_f] \rightarrow \mathbb{R}^m_+ \) is the control input vector and \( F, G : [t_0, t_f] \times \mathbb{R}^n_+ \times \mathbb{R}^m_+ \rightarrow \mathbb{R}^n_+ \) are granular differentiable functions.

**Definition 4.1.**

(i) A pair \((x, \overline{x})\) is said to be an admissible pair if it satisfies the problem (7)-(8).

(ii) A pair \((x, \overline{x})\) is said to be an optimal pair if it is an admissible pair and minimizes the performance index (6).

Next, we define the Hamiltonian function \( H(t, x, u, \lambda) \) associated with the FFOCP by

\[
H(t, x(t), u(t), \lambda(t)) = G(t, x(t), u(t)) + \lambda(t) \cdot F(t, x(t), u(t)),
\]

where the function \( \lambda(t) \in C^1([t_0, t_f], \mathbb{R}^m_+) \) is called Lagrange multiplier vector. Then, the following theorem will give a necessary condition to determine the optimality of the FFOCP.

---

**Fig. 6.** The plot of the number of COVID-19 infectious and recovered people in the world from 18 March 2020 with fractional order \( \beta = 0.9 \). where the dashed curves and dotted curves are left and right endpoints of the \( \alpha \)-cuts of the solutions, respectively and the solid curves is the exact solution corresponding with \( \alpha = 1.0 \).
Theorem 4.1. Assume that the pair $([x], [\pi])$ is an optimal pair of the FFOPCP under the fuzzy dynamic constraint (7) and initial condition (8). Then, the triplet $([x], [\pi], [\lambda])$ satisfies following conditions

\[
\begin{align*}
\frac{D}{dt} \delta([x]) &= F(t, [x](t), [\Pi](t)) \\
\frac{D}{dt} \delta([\lambda]) &= \frac{\partial}{\partial [x]} G(t, [x](t), [\Pi](t)) + \frac{\partial}{\partial [\Pi]} F(t, [x](t), [\Pi](t)) \\
\lambda(t) &= \frac{\partial}{\partial [\Pi]} G(t, [x](t), [\Pi](t)) + \frac{\partial}{\partial [x]} F(t, [x](t), [\Pi](t)) \\
\delta([x](t)) &= 0,
\end{align*}
\]

where the first and second equations are known as the state and co-state systems, the third equation is the stationary condition and the last equation is the transversality condition.

Proof. By using the formula (9) of the Hamiltonian function $H(t, x(t), u(t), \lambda(t))$, the performance index $J(x, u)$ can be rewritten as follows

\[
J(x, u) = \int_{t_0}^{t_f} \left[ H(t, x(t), u(t), \lambda(t)) \right] dt
\]

Here, for simplicity in representation, we denote $\Phi[x, u](t) := \Phi(x(t), u(t), \lambda(t), \frac{\partial}{\partial [x]} D^\beta x(t))$ whose respective horizontal membership function is denoted $\Phi^\beta[x, u](t, \mu)$ for each $\mu \in [0, 1]$. Next, for $\delta > 0$ sufficiently small, let us consider the following formation

\[
\begin{align*}
\Delta J(x, u) &= J(x, u) \circ \delta^\beta J([x], [\pi]) = \int_{t_0}^{t_f} \left[ \Phi[x + \delta x, [\pi] + \delta u](\tau) \circ \delta^\beta \Phi[x, [\pi]](\tau) \right] d\tau,
\end{align*}
\]

where $\Phi[x + \delta x, [\pi] + \delta u](t)$ is known as

\[
\Phi\left(\lambda(t) + \delta \lambda(t), [\Pi(t)] + \delta [\Pi(t)], [\lambda(t)] + \delta [\lambda(t)], \frac{\partial}{\partial [\Pi]} D^\beta [\lambda(t)] + \delta \frac{\partial}{\partial [\Pi]} D^\beta [\lambda(t)] \right).
\]

It is well-known that if the pair $([x], [\pi])$ is a minimizer of the performance index (6), then the increment $\Delta J(x, u)$ must be always non-negative, i.e., $\Delta J(x, u) \geq 0$, or equivalently,

\[
\int_{t_0}^{t_f} \left[ \Phi[x + \delta x, [\pi] + \delta u](\tau) \circ \delta^\beta \Phi[x, [\pi]](\tau) \right] d\tau \geq 0.
\]

On the other hand, by using the horizontal membership function approach, we have

\[
Q(\Delta J(x, u)) = 0 \iff Q(J(x, u)) = Q([x], [\pi]) = 0,
\]

in which

\[
\begin{align*}
Q(J(x, u)) &= \int_{t_0}^{t_f} \Phi^\beta\left([x]^\theta(\tau, \alpha, \mu_1), [\pi]^\theta(\tau, \alpha, \mu_2), \lambda^\theta(\tau, \alpha, \mu_3), \frac{\partial}{\partial \mu_1} D^\beta [x]^\theta(\tau, \alpha, \mu_1) \right) d\tau,
\end{align*}
\]

for all $\alpha, \mu_1, \mu_2, \mu_3 \in [0, 1]$. By applying Proposition 2.5, the equality (11) becomes

\[
\begin{align*}
Q(\Delta J(x, u)) &= \int_{t_0}^{t_f} \left[ \Phi^\beta[x + \delta x, [\pi] + \delta u](\tau, \alpha, \mu) - \Phi^\beta[x, [\pi]](\tau, \alpha, \mu) \right] d\tau
\end{align*}
\]
\[
\frac{\partial \Phi^g[\kappa, \mu](\tau, \alpha, \mu)}{\partial \kappa^g(\tau, \alpha, \mu_2)} \frac{\partial \lambda^g(\tau, \alpha, \mu_3)}{\partial \lambda^g(\tau, \alpha, \mu_3)} + \frac{\partial \Phi^g[\kappa, \mu](\tau, \alpha, \mu)}{\partial \mu^g(\tau, \alpha, \mu_1)} \frac{\partial \lambda^g(\tau, \alpha, \mu_3)}{\partial \mu^g(\tau, \alpha, \mu_1)} \left\{ c_{\kappa^g} D^\beta \chi^g(\tau, \alpha, \mu_1) \right\} dt.
\]

In particular, by the definition of \( \Phi(x(t), u(t), \lambda(\tau), \mu^g D^\beta x(t)) \), we have

\[
\begin{align*}
\frac{\partial \Phi^g[\kappa, \mu](\tau, \alpha, \mu)}{\partial \kappa^g(\tau, \alpha, \mu_1)} &= \frac{\partial \Phi^g(\tau, \alpha, \mu_1)}{\partial \kappa^g(\tau, \alpha, \mu_1)} \\
\frac{\partial \Phi^g[\kappa, \mu](\tau, \alpha, \mu)}{\partial \mu^g(\tau, \alpha, \mu_2)} &= \frac{\partial \Phi^g(\tau, \alpha, \mu_2)}{\partial \mu^g(\tau, \alpha, \mu_2)} \\
\frac{\partial \Phi^g[\kappa, \mu](\tau, \alpha, \mu)}{\partial \mu^g(\tau, \alpha, \mu_1)} &= \frac{\partial \Phi^g(\tau, \alpha, \mu_1)}{\partial \mu^g(\tau, \alpha, \mu_1)} \\
\frac{\partial \Phi^g[\kappa, \mu](\tau, \alpha, \mu)}{\partial \lambda^g(\tau, \alpha, \mu_1)} &= \frac{\partial \Phi^g(\tau, \alpha, \mu_1)}{\partial \lambda^g(\tau, \alpha, \mu_1)} \cdot \frac{\partial \lambda^g(\tau, \alpha, \mu_3)}{\partial \lambda^g(\tau, \alpha, \mu_3)} - c_{\mu^g} D^\beta \chi^g(\tau, \alpha, \mu_1).
\end{align*}
\]

Thus, it implies

\[
Q(\Delta J(x, u)) = \int_{t_0}^{t_f} \left\{ \left[ \frac{\partial \Phi^g(\tau, \alpha, \mu_1)}{\partial \kappa^g(\tau, \alpha, \mu_1)} \right] \delta \chi^g(\tau, \alpha, \mu_1) + \left[ \frac{\partial \Phi^g(\tau, \alpha, \mu_1)}{\partial \mu^g(\tau, \alpha, \mu_2)} \right] \delta u^g(\tau, \alpha, \mu_2) + \left[ \frac{\partial \Phi^g(\tau, \alpha, \mu_1)}{\partial \lambda^g(\tau, \alpha, \mu_3)} \right] \delta \lambda^g(\tau, \alpha, \mu_3) - \frac{\partial \Phi^g(\tau, \alpha, \mu_1)}{\partial \lambda^g(\tau, \alpha, \mu_3)} \right\} dt.
\]

By using the integral by parts formula for granular Caputo fractional derivative, it yields that

\[
\int_{t_0}^{t_f} \left[ \frac{\partial \Phi^g(\tau, \alpha, \mu_1)}{\partial \kappa^g(\tau, \alpha, \mu_1)} \right] \delta \chi^g(\tau, \alpha, \mu_1) dt = \left[ \frac{1}{\beta - 1} \frac{\partial \Phi^g(\tau, \alpha, \mu_1)}{\partial \kappa^g(\tau, \alpha, \mu_1)} \delta \chi^g(\tau, \alpha, \mu_1) \right]_{t_0}^{t_f} - \int_{t_0}^{t_f} \delta \chi^g(\tau, \alpha, \mu_1) \frac{\partial \Phi^g(\tau, \alpha, \mu_1)}{\partial \kappa^g(\tau, \alpha, \mu_1)} dt.
\]

In addition, since \( x^0(0, \alpha, \mu_1) = x_0(\alpha, \mu_1) \) is specified, then it implies that \( \delta x^g(0, \alpha, \mu_1) = 0 \) and

\[
Q(\Delta J(x, u)) = \int_{t_0}^{t_f} \left\{ \left[ \frac{\partial \Phi^g(\tau, \alpha, \mu_1)}{\partial \kappa^g(\tau, \alpha, \mu_1)} \right] \delta \chi^g(\tau, \alpha, \mu_1) + \left[ \frac{\partial \Phi^g(\tau, \alpha, \mu_1)}{\partial \lambda^g(\tau, \alpha, \mu_3)} \right] \delta \lambda^g(\tau, \alpha, \mu_3) - \frac{\partial \Phi^g(\tau, \alpha, \mu_1)}{\partial \lambda^g(\tau, \alpha, \mu_3)} \right\} dt.
\]

Hence, the necessary condition for the optimality of the FFOCP depends on the minimization of \( Q(\Delta J(x, u)) \). This requires \( Q(\Delta J(x, u)) = 0 \), or equivalently, the coefficient of \( x^g(\tau, \alpha, \mu_1) \), \( u^g(\tau, \alpha, \mu_2) \) and \( \lambda^g(\tau, \alpha, \mu_3) \) must be zero. Therefore, we
directly obtain
\[
\begin{align*}
\frac{\partial H}{\partial t} & = 0,
\frac{\partial H}{\partial \lambda} & = \frac{\partial \lambda}{\partial t} = 0,
\frac{\partial H}{\partial \alpha} & = \frac{\partial \alpha}{\partial t} = 0,
\frac{\partial H}{\partial \mu} & = \frac{\partial \mu}{\partial t} = 0.
\end{align*}
\]
Furthermore, since the function $\lambda(t)$ is continuous, the last equality of (12) implies $\lambda(t_f, \mu_3) = 0$. Hence, the system (12) is equivalent to
\[
\begin{align*}
\frac{\partial \lambda(t)}{\partial t} & = 0,
\frac{\partial \lambda(t)}{\partial \mu} & = \frac{\partial \mu(t)}{\partial \lambda(t)} = 0,
\frac{\partial \lambda(t)}{\partial \alpha} & = \frac{\partial \alpha(t)}{\partial \lambda(t)} = 0.
\end{align*}
\]
Finally, by using the transformation (1), it follows that
\[
\begin{align*}
\frac{\partial \alpha(t)}{\partial \mu} & = \frac{\partial \mu(t)}{\partial \alpha(t)} = 0,
\frac{\partial \alpha(t)}{\partial \lambda(t)} & = \frac{\partial \lambda(t)}{\partial \alpha(t)} = 0,
\frac{\partial \alpha(t)}{\partial \mu} & = \frac{\partial \mu(t)}{\partial \alpha(t)} = 0.
\end{align*}
\]
Therefore, the proof is completed. □

Remark 4.1. The system of conditions (10) represents the Euler-Lagrange equations for the FFOCP, which gives the necessary conditions for the optimality of the considered FFOCP. They are quite similar to the Euler-Lagrange equations for the classical optimal control problems excepting that the derived differential system contains both the left-sided and right-sided Caputo fractional derivatives.

4.2. Numerical method for the optimality

Nowadays, various numerical methods have been applied to numerically solve the optimality problems. In the following, we propose a modification in Adams-Bashforth-Moulton predictor-corrector algorithm to characterize the numerical solution of the fuzzy fractional optimal control problem (FFOC) with dynamic constraint
\[
\begin{align*}
\frac{d^2}{dt^2} \frac{d}{dt} x(t) & = F(t, x(t), u(t)), \\
\lambda(0) & = \lambda_0,
\end{align*}
\]
where $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a granular differentiable function and the initial condition is $x_0 \in \mathbb{R}^n$.

As a consequence of Theorem 4.1, if a pair $(\lambda, \mu)$ is an optimal pair of the FFOCP, then there exists a Lagrange multiplier vector $\lambda$ such that the triplet $(\lambda, \mu, \lambda)$ satisfies the system (10). Here, we can see that the state vector $x$ and the multiplier vector $\lambda$ can be numerically solved from the state and co-state systems.

Then, the numerical algorithm to implement the numerical solution of the FFOCP is as follows:

Step 1. Divide the interval $[t_0, t_f]$ into $N$ equal sub-intervals and consider a uniform grid
\[
\{t_j = jh : j = 0, 1, \ldots, N, h = \frac{t_f - t_0}{N}\}.
\]
Step 2. Convert the state and co-state systems of the fractional dynamic system (10) into parametric forms
\[
\begin{align*}
\frac{\partial H}{\partial t} & = 0,
\frac{\partial H}{\partial \lambda} & = \frac{\partial \lambda}{\partial t} = 0,
\frac{\partial H}{\partial \alpha} & = \frac{\partial \alpha}{\partial t} = 0,
\frac{\partial H}{\partial \mu} & = \frac{\partial \mu}{\partial t} = 0.
\end{align*}
\]
\[ \frac{\partial H^p(t, Q(\bar{x}(t)), Q(\bar{\Pi}(t)), Q(\bar{x}_h(t)))}{\partial Q(\bar{\Pi}(t))} = 0. \] (16)

**Step 3.** Choose an initial value of the control input \( Q(\bar{\Pi}) := \{ Q(\bar{\Pi}_k) \} \) for all \( k = 1, N \).

**Step 4.** Numerically solve the state Eq. (14) forward-in-time with the initial datum \( Q(\bar{x}(0)) = Q(x_0) \) and initial value of the input \( u \) by applying the following predictor-corrector formula

\[
Q(\bar{x}_{n+1}) = Q(x_0) + \frac{h^\beta}{(\beta + 2)} \left[ F^p(t_{n+1}, Q(\bar{x}_{n+1}), Q(\bar{\Pi}_{n+1})) + \sum_{j=0}^{n} p_{j,n+1} F^p(t_j, Q(\bar{x}_j), Q(\bar{\Pi}_j)) \right],
\]

where \( n = 0, 1, 2, \ldots, N - 1 \).

**Step 5.** Numerically solve the adjoint system (15) backward-in-time with the terminal conditions \( Q(\bar{x}_h(t_f)) = 0 \) to determine \( Q(\bar{x}_h(t)) \) by using the predictor-corrector formula

\[
Q(\bar{x}_k) = \frac{h^\beta}{(\beta + 2)} \left[ \frac{\partial H(t_k, Q(\bar{x}_k), Q(\bar{\Pi}_k), Q(\bar{x}_h(\tau)))}{\partial \bar{x}(\tau)} + \sum_{j=k+1}^{n} \tilde{p}_{j,k} \frac{\partial H(t_j, Q(\bar{x}_j), Q(\bar{\Pi}_j), Q(\bar{x}_h(\tau)))}{\partial \bar{x}(\tau)} \right],
\]

where \( k = N - 1, \ldots, 1, 0 \).

**Step 6.** Substitute the new numerical values of \( Q(\bar{x}(t)) \) and \( Q(\bar{x}_h(t)) \) in Steps 4 and 5 into the stationary condition (16) to solve and update the control input \( Q(\bar{\Pi}(t)) \).

**Step 7.** Checking the convergence.

**Remark 4.2.** According to the above algorithm, it should be noted that

(i) The key to obtain the predictor-corrector formula in Step 4 is transferring the state system (14) into the following equivalent integral system

\[
Q(\bar{x}(t)) = x_0 + \frac{1}{(\beta + 2)} \int_{t_0}^{t} (t - \tau)^{\beta-1} F^p(t, Q(\bar{x}(\tau)), Q(\bar{\Pi}(\tau))) d\tau.
\]

In addition, assume that we have already estimated the terms \( Q(\bar{x}_j) \approx Q(\bar{x}(t_j)) \) for all \( j \in \{0, 1, 2, \ldots, n\} \). Then, we can estimate the \((n + 1)\)th-term \( Q(\bar{x}_{n+1}) \) by using the approximation

\[
Q(\bar{x}(t_{n+1})) = Q(x_0) + \int_{t_0}^{t_{n+1}} (t_{n+1} - \tau)^{\beta-1} F^p(t, Q(\bar{x}(\tau)), Q(\bar{\Pi}(\tau))) d\tau.
\]

(17)

Here, the product trapezoidal quadrature formula can be applied to replace the integral term on the right-hand side of integral Eq. (17). Indeed, by using the standard techniques of quadrature theory, we obtain

\[
\int_{t_0}^{t_{n+1}} (t_{n+1} - \tau)^{\beta-1} F(t) d\tau \approx \int_{t_0}^{t_{n+1}} (t_{n+1} - \tau)^{\beta-1} \Phi_{n+1}(\tau) d\tau \approx \sum_{j=0}^{n+1} p_{j,n+1} \Phi(t_j).
\]

where \( \Phi_{n+1}(\cdot) \) is the piece-wise linear interpolation of \( F(\cdot) \) with nodes and knots chosen at \( t_j \) and

\[
p_{j,n+1} = \begin{cases} \frac{n^{\beta+1} - (n - \beta)(n + 1)^{\beta+1}}{(n + 1)^{\beta+1} + (n - j)^{\beta+1} - 2(n - j + 1)^{\beta+1}} & j = 0, \\
1 & 1 \leq j \leq n, \\
\frac{n^{\beta+1} - (n - \beta)(n + 1)^{\beta+1}}{(n - j + 1)^{\beta+1} + (n - j)^{\beta+1} - 2(n - j + 1)^{\beta+1}} & j = n + 1. 
\end{cases}
\]

(18)

Using product trapezoidal quadrature formula, the corrector formula is given by

\[
Q(\bar{x}_{n+1}) = Q(x_0) + \frac{h^\beta}{(\beta + 2)} \left[ F^p(t_{n+1}, Q(\bar{x}_{n+1}), Q(\bar{\Pi}_{n+1})) + \sum_{j=0}^{n} p_{j,n+1} F^p(t_j, Q(\bar{x}_j), Q(\bar{\Pi}_j)) \right],
\]

(19)

where \( p_{j,n+1} \) is defined in (18). In addition, due to the fact that the function \( F \) may be nonlinear and the unknown term \( Q(\bar{x}_{n+1}) \) appears on both sides of (19), it is difficult to solve explicitly this equation. To overcome this difficulty, we use the product rectangle rule to evaluate an approximation \( Q(\bar{x}_{n+1}^p) \), namely, predictor, which will replace the term \( Q(\bar{x}_{n+1}) \) on the right-hand side of the formula (19)

\[
Q(\bar{x}_{n+1}^p) = Q(x_0) + \frac{h^\beta}{(\beta + 1)} \sum_{j=0}^{n} q_{j,n+1} F^p(t_j, Q(\bar{x}_j), Q(\bar{\Pi}_j)),
\]

where \( q_{j,n+1} \) is given by \( q_{j,n+1} = \frac{h^\beta}{(\beta + 1)} [(n + 1 - j)^{\beta} - (n - j)^{\beta}] \).
(ii) In Step 5, by the terminal condition $Q(\tilde{x}(t_f)) = 0$, the co-state system (15) is equivalent to following fractional integral system

$$Q(\tilde{x}(t)) = \frac{1}{\Gamma(\beta)} \int_t^{t_f} (\tau - t)^{\beta-1} \frac{\partial H(\tau, Q(\tilde{x}(\tau)), Q(\tilde{u}(\tau)), Q(\tilde{\lambda}(\tau)))}{\partial Q(\tilde{x}(\tau))} d\tau.$$  \hfill (20)

Then, the predictor-corrector method numerically solves the system (20) as follows

$$Q(\tilde{x}_k) = \frac{h^\beta}{\Gamma(\beta + 2)} \left[ \frac{\partial H(t_k, Q(\tilde{x}_k), Q(\tilde{u}_k), Q(\tilde{\lambda}_k))}{\partial Q(\tilde{x}(\tau))} + \sum_{j=k+1}^{N} \tilde{p}_{jk} \frac{\partial H(t_j, Q(\tilde{x}_j), Q(\tilde{u}_j), Q(\tilde{\lambda}_j))}{\partial Q(\tilde{x}(\tau))} \right],$$

where the predictor formula is given by

$$Q(\tilde{x}_k) = \frac{h^\beta}{\Gamma(\beta + 1)} \sum_{j=k+1}^{N} \tilde{q}_{jk} \frac{\partial H(t_j, Q(\tilde{x}_j), Q(\tilde{u}_j), Q(\tilde{\lambda}_j))}{\partial Q(\tilde{x}(\tau))},$$

and the coefficients $\tilde{p}_{jk}, \tilde{q}_{jk}$ are given by

$$\tilde{p}_{jk} = \begin{cases} (N - k - 1)^{\beta+1} & j = N, \\ (j - k + 1)^{\beta+1} - (j - k - 1)^{\beta+1} - 2(j - k)^{\beta+1} & k + 1 \leq j \leq N - 1. \end{cases}$$

$$\tilde{q}_{jk} = \begin{cases} (N - k - 1)^{\beta+1} & j = N, \\ (j - k + 1)^{\beta+1} - (j - k - 1)^{\beta+1} & k + 1 \leq j \leq N - 1. \end{cases}$$

(iii) In Step 7, if the values of the variables $Q(\tilde{x}(t)), Q(\tilde{u}(t))$ and $Q(\tilde{\lambda}(t))$ in this iteration and the last iteration are negligibly close then we obtain the current values as the optimal solutions, while if they are not close enough, we return to Step 4 and continue the procedure.

4.3. Example: a fractional SEIR model

The numerical algorithm proposed above is a generalized scheme, can be applied to various types of mathematical models of the form (7). In the following, we will apply the presented scheme to solve the initial value problem and (FFOCP) for the SEIR epidemic model that describes the COVID - 19’s outbreak.

Example 4.1. The widespread infection of COVID - 19 has caused a lot of negative impacts on human life and socio-economic activities. Currently, the outbreak of acute pneumonia caused by the coronavirus has spread globally and has been raised to the highest level by WHO [see Fig. 7].

Recently, due to the global trend of the COVID - 19 pandemic, there have been several work focusing on the mathematical modeling of the epidemic outbreak caused by coronavirus such as Batista [9], Ming et al. [30] or Nesteruk [32]. Their proposed model is the mathematical SIR model whose output is used to predict the number of susceptible, infectious and recovered individuals versus time of the total population. However, some recent research pointed out that the epidemic disease caused by coronavirus has a latent phase during which the individual is infected but not yet infectious. Thus, for the better modeling of the COVID - 19’s outbreak, we propose to add a variable $E$ standing for a latent/exposed population and it allows to express the fact infected (but not yet infectious) individuals move from $S$ to $E$ and from $E$ to $I$. Hence, the mathematical SIR model is then transformed into the mathematical SEIR model. The SEIR divides the total population into four compartments:

- The compartment $S$ represents for the individuals that are susceptible to the disease;
- The compartment $E$ represents for the individuals that are infected but not able to transmit the disease;
- The compartment $I$ consists of all infected individuals capable of spreading the disease;
- The compartment $R$ are those who are immune.

In the mathematical SEIR model, assume that everyone is susceptible to the disease by birth and a susceptible individual becomes infected if he contacts with an infectious individual. Denote by $S(t), E(t), I(t), R(t)$ the number of individuals in each compartment at time $t$ and then, the total population is determined by $N(t) = S(t) + E(t) + I(t) + R(t)$. The interaction and relationship between four compartments are generalized in following diagram (see Fig. 8).

For the better modeling of reality SEIR model, we refer the data from Link 1 and Link 2. According to daily statistical data in Link 2, Italy’s COVID - 19 cases reached 63,927 on 24 March, marking the biggest coronavirus outbreak outside Asia. Italy is also the second most affected coronavirus country in the world with the cases increasing at a higher rate than any other country. Hence, this example is devoted to provide a modeling test on the effect of quarantine with the controlling of the disease. The parameters of the proposed model are estimated by a test done at 7.50 GMT+7, 24 March 2020 with the results as follows (Table 2).

Table 3 presents the estimated values of the parameters that are used in our computations. Apart from the weight parameter $\gamma$, the other parameters are estimated by using the statistical data in Italy Population Clock (Link 2, 24 March 2020, 7.50 GMT+7). The initial number of people in each compartments (I) and (R) are referred from World Population Review (Link 2, 24 March 2020, 7.50 GMT+7).
Fig. 7. The widespread infection of COVID-19 in the world (source: https://worldpopulationreview.com).

Fig. 8. The SEIR model.

| Data                        | Value   | Data source |
|-----------------------------|---------|-------------|
| The total population        | 60 485 520 | Link 2      |
| The births per day          | 1 422   | Link 2      |
| The deaths per day          | 1 762   | Link 2      |
| COVID-19 cases (active)     | 50 418  | Link 1      |
| Death cases                 | 6 077   | Link 1      |
| Recovered cases             | 7 432   | Link 1      |
| New cases in day (24 March, 2020) | 4 789 | Link 1  |
Here, is minimizing effort model triangular of the disease. Hence, when investigating an initial value problem for SEIR model, it is difficult to get the exact information about the number of individuals in the state (E) and (I) at the initial time. In practical, we must accept to consider these initial values in approximated data. For example, we say that "The initial number of exposed individuals is approximately 50 000", i.e., the number of individuals in the state (E) is 50 000 and may change a few. In computation, we often choose the value 50 000 to represent for the initial data. In this work, to dealing with the uncertainty in data and parameters and for the better modeling of the real-world SEIR model, we will use the concept of triangular fuzzy number to present the uncertain quantities. Thus, due to the fuzziness of input data, the considered SEIR model becomes a fuzzy SEIR model.

In this example, we consider the fuzzy fractional optimal control problem (FFOCP) for the fractional SEIN model whose effort to minimize the number of infected individuals in the state (I) and the cost of applying the controls. This is done by minimizing the following objective function

$$ J(x, u) = \int_0^T \left[ \gamma I(\tau) + u^2(\tau) \right] d\tau $$

subject to the dynamical constraints

$$ \begin{aligned}
\mathcal{D}^\beta S(t) &= bN(t) \otimes (dS(t) \otimes cS(t)I(t) \otimes u(t)S(t)) \\
\mathcal{D}^\beta E(t) &= cS(t)I(t) \otimes (e + d)E(t) \\
\mathcal{D}^\beta I(t) &= eE(t) \otimes (r + a + d)I(t) \\
\mathcal{D}^\beta N(t) &= (b - d)N(t) \otimes aI(t),
\end{aligned} \quad (21) $$

where the state vector $X(t) = [S(t) \ E(t) \ I(t) \ N(t)]^t$ and the control $u(t)$ represents the rate of susceptible individuals being quarantined from the infection source; $\gamma$ is the positive weight, $u^2(\tau)$ is the cost of applying control effort $u(t)$ and $T$ is the duration of the control programs.

Here, we assume that $u(t)$ satisfies the control constraint

(C1) For almost every $t \in [0, T]$, $\hat{0} \leq u(t) \leq \hat{1}$, where $\hat{0} = \chi_{[0]}$ and $\hat{1} = \chi_{[1]}$.

Remark 4.3. We can see that in the dynamical system (21), the fractional SEIR model is replaced by the fractional SEIN model. Here, the recovered population $R(t)$ is inferred from the total population $N(t) = S(t) + E(t) + I(t) + R(t)$, which gives the following fractional differential equation

$$ \mathcal{D}^\beta R(t) = rI(t) \otimes R(t) + u(t)S(t). $$

In addition, since the control variable $u(t)$ appears linearly in the dynamics, the fractional differential system (21) can be rewritten as follows

$$ \mathcal{D}^\beta X(t) = \begin{bmatrix}
bn(t) \otimes dS(t) \otimes cS(t)I(t) \\
cS(t)I(t) \otimes (e + d)E(t) \\
eE(t) \otimes (r + a + d)I(t) \\
(b - d)N(t) \otimes aI(t)
\end{bmatrix} \otimes \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} u(t). \quad (21) $$

### Table 3

Some parameters and constants.

| Parameter | Description                   | Value       | Data source |
|-----------|-------------------------------|-------------|-------------|
| $a$       | disease induced death rate    | 0.041059    | Estimated   |
| $b$       | natural birth rate            | 1.9748 x 10^{-3} | Estimated   |
| $c$       | incidence coefficient         | 1.3221 x 10^{-5} | Estimated   |
| $d$       | natural death rate            | 2.447 x 10^{-4} | Estimated   |
| $e$       | exposed to infectious rate    | 0.5         | Assumed     |
| $r$       | recovery rate                 | 0.11626     | Estimated   |
| $\gamma$  | weight parameter              | (0; 10]     |             |
| $T$       | number of days                | 20          |             |
| $N_0$     | initial population            | 60 485 520  | Link 2      |
| $S_0$     | initial susceptible population| 100 000     | Assumed     |
| $E_0$     | initial exposed population    | about “500 000” | Assumed     |
| $I_0$     | initial infected population   | 50 418      | Link 1      |
| $R_0$     | initial recovered population  | 7 432       | Link 1      |
| $N_{\max}$| the maximum total population  | 70 000 000  | Assumed     |
whose horizontal membership function is

\[ C_0, D^\beta Q(X(t)) = \begin{bmatrix} -d & 0 & 0 & 0 \\ 0 & -(e + d) & 0 & 0 \\ 0 & 0 & -(r + a + d) & 0 \\ 0 & 0 & 0 & b - d \end{bmatrix} \begin{bmatrix} \dot{Q}(S(t)) \\ \dot{Q}(E(t)) \\ \dot{Q}(I(t)) \\ \dot{Q}(N(t)) \end{bmatrix} + \begin{bmatrix} bQ(N(t)) - cQ(S(t))Q(I(t)) \\ cQ(S(t))Q(I(t)) \\ eQ(E(t)) \\ -aQ(I(t)) \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{Q}(S(t)) \\ \ddot{Q}(S(t)) \\ \dddot{Q}(S(t)) \end{bmatrix} Q(u(t)). \]

**Proposition 4.1.** Under the initial conditions \( S_0, E_0, I_0 \) and \( N_0 \), the fractional SEIN problem without control input always has a unique solution.

**Proof.** Our aim is to prove the unique existence of mild solution to the following fractional differential equation

\[ D^\beta_0 X(t) = A X(t) + f(t, X(t)), \]

subject to the initial condition \( X(0) = \begin{bmatrix} S_0 \\ E_0 \\ I_0 \\ N_0 \end{bmatrix} \), where

- The state vector is \( X(t) = \begin{bmatrix} S(t) \\ E(t) \\ I(t) \\ N(t) \end{bmatrix} \);
- The matrix \( A \) is

\[ A = \begin{bmatrix} -d & 0 & 0 & 0 \\ 0 & -(e + d) & 0 & 0 \\ 0 & 0 & -(r + a + d) & 0 \\ 0 & 0 & 0 & b - d \end{bmatrix}. \]

- The nonlinear function is \( f(t, X(t)) = \begin{bmatrix} bN(t) \odot^\beta \cS(t)I(t) \\ cS(t)I(t) \\ eE(t) \\ (1 - 1)dt(t) \end{bmatrix}. \]

According to **Lemma 3.1**, the formula of solution to (22) is given by

\[ X(t) = E_{\beta, \beta}(t^\beta A)X(0) + \int_0^t (t - \tau)^{\beta - 1} E_{\beta, \beta}(t - \tau) B f(\tau, X(\tau)) d\tau. \]

By **Lemma 2.1** (ii), since \( A \) is a diagonal matrix, the matrix Mittag-Leffler function \( E_{\beta, \beta}(t^\beta A) \), given by

\[ E_{\beta, \beta}(t^\beta A) = \begin{bmatrix} E_{\beta, \beta}(-dt^\beta) & 0 & 0 & 0 \\ 0 & E_{\beta, \beta}(-(e + d)t^\beta) & 0 & 0 \\ 0 & 0 & E_{\beta, \beta}(-(r + a + d)t^\beta) & 0 \\ 0 & 0 & 0 & E_{\beta, \beta}(-(d - b)t^\beta) \end{bmatrix}, \]

is also diagonal matrix. In addition, the norm of the matrix \( E_{\beta, \beta}(t^\beta A) \) is continuous on \([0, T]\), i.e., there exists a constant \( M > 0 \) such that \( \| E_{\beta, \beta}(t^\beta A) \| \leq 1 \) for all \( t \in [0, T] \), which means that the hypothesis (HA) holds. (See Fig. 9 for illustration).

Now, for each \( X(t), \dot{X}(t) \), we have

\[ D^\beta_f(t, X(t)), f(t, \dot{X}(t)) \leq b_{\beta}(N(t), \bar{N}(t)) + e_{\beta}(E(t), \bar{E}(t)) + a_{\beta}(I(t), \bar{I}(t)) \]

\[ + 2c_{\beta}(S(t)I(t), \bar{S}(t)\bar{I}(t)). \]

Here, note that \( \max \{ \rho_{\beta}(I(t), \dot{\bar{I}}(t)), \rho_{\beta}(S(t), \dot{\bar{S}}(t)) \leq N_{\max} \} \leq N_{\max}. \) Then, by **Proposition 2.3**, we have

\[ \rho_{\beta}(S(t)I(t), \bar{S}(t)\bar{I}(t)) \leq \rho_{\beta}(S(t)I(t), \bar{S}(t)\bar{I}(t)) + \rho_{\beta}(S(t)I(t), \bar{S}(t)\bar{I}(t)) \]

\[ \leq \rho_{\beta}(I(t), \dot{\bar{I}}(t)) + \rho_{\beta}(S(t), \bar{S}(t)) + \rho_{\beta}(S(t), \dot{\bar{S}}(t)) \rho_{\beta}(I(t), \bar{I}(t)) \]

\[ \leq N_{\max} \rho_{\beta}(S(t), \bar{S}(t)) + N_{\max} \rho_{\beta}(I(t), \bar{I}(t)). \]

Thus, we obtain

\[ D^\beta_f(t, X(t)), f(t, \dot{X}(t)) \leq 2c_{\max} \rho_{\beta}(S(t), \bar{S}(t)) + e_{\beta}(E(t), \bar{E}(t)) \]

\[ + (a + 2c_{\max}) \rho_{\beta}(I(t), \bar{I}(t)) + b_{\beta}(N(t), \bar{N}(t)) \]

\[ \leq (a + 2c_{\max}) \rho_{\beta}(X(t), \bar{X}(t)). \]
which implies the hypothesis (HF2) holds with $L_1 = a + 2cN_{\text{max}}$. In addition, since
\[
\begin{align*}
\mathcal{D}^{\beta}(f(t, X(t)), \tilde{0}) & \leq b \rho^{\beta}(N(t), \hat{0}(t)) + e \rho^{\beta}(E(t), \hat{0}(t)) + a \rho^{\beta}(I(t), \hat{0}(t)) \\
& \quad + 2c\left(\rho^{\beta}(S(t)I(t), I(t)) + \rho^{\beta}(I(t), \hat{0}(t))\right) \\
& \leq b \rho^{\beta}(N(t), \hat{0}(t)) + e \rho^{\beta}(E(t), \hat{0}(t)) + a \rho^{\beta}(I(t), \hat{0}(t)) \\
& \quad + 2c\left[\rho^{\beta}(S(t), \hat{0}(t)) \rho^{\beta}(I(t), \hat{0}(t)) + \rho^{\beta}(I(t), \hat{0}(t))\right] \\
& \leq N_{\text{max}}\left[b + e + a + 2c(N_{\text{max}} + 1)\right],
\end{align*}
\]
we can choose the constant $L_2 = N_{\text{max}}\left[b + e + a + 2c(N_{\text{max}} + 1)\right]$ such that it satisfies the hypothesis (HF3). Therefore, the proof is complete. \(\square\)

**Remark 4.4.** The numerical simulations are carried out by using Matlab program fde12.m, the parameter values given in Table 3 and different values of fractional derivative order $\beta \in (0, 1]$. The graphical representations of numerical solutions are shown in Fig. 10.

Now, we investigate the optimal level of control efforts to minimize the number of infected individuals capable of spreading the disease and the cost of applying the control $u$. Firstly, the Hamiltonian function associated with the FFOCP is defined by
\[
H(t, X(t), u(t), \lambda(t)) = \gamma I(t) + u^2(t) + \lambda_1(t)\left[bN(t) \rho^{\beta} dS(t) \rho^{\beta} cS(t)I(t) \rho^{\beta} u(t)S(t)\right] \\
+ \lambda_2(t)\left[cS(t)I(t) \rho^{\beta} (e + d)E(t)\right] + \lambda_3(t)\left[eE(t) \rho^{\beta} (r + a + d)I(t)\right] \\
+ \lambda_4(t)\left[(b - d)N(t) \rho^{\beta} al(t)\right],
\]
whose horizontal membership function is
\[
H^{\beta}\left(t, Q(X(t)), Q(u(t)), Q(\lambda(t))\right) = \gamma Q(I(t)) + [Q(u(t))]^2 + Q(\lambda_4(t))\left[(b - d)Q(N(t)) - aQ(I(t))\right] \\
+ Q(\lambda_2(t))\left[cQ(S(t))Q(I(t)) - (e + d)Q(E(t))\right] \\
+ Q(\lambda_3(t))\left[eQ(E(t)) - (r + a + d)Q(I(t))\right] \\
+ Q(\lambda_1(t))\left[bQ(N(t)) - dQ(S(t)) - cQ(S(t))Q(I(t)) - Q(u(t))Q(S(t))\right].
\]
Fig. 10. The fuzzy solution of the fractional SEIR model without control, where the outbreak (A), (B), (C) correspond to the different cases of fractional order: $\beta = 0.5$, $\beta = 0.7$, $\beta = 0.9$. 
By applying the system (13), we establish the necessary conditions for the optimality of the granular FOPC:

(i) The state system
\[
\begin{align*}
\dot{C} & = D^\beta Q(S(t)) = bQ(N(t)) - dQ(S(t)) - cQ(S(t))Q(I(t)) - Q(u(t))Q(S(t)) \\
\dot{C} & = D^\beta Q(E(t)) = cQ(S(t))Q(I(t)) - (e + d)Q(E(t)) \\
\dot{C} & = D^\beta Q(I(t)) = eQ(E(t)) - (r + a + d)Q(I(t)) \\
\dot{O} & = D^\beta Q(N(t)) = (b - d)Q(N(t)) - aQ(I(t)).
\end{align*}
\]

(ii) The co-state system
\[
\begin{align*}
\dot{C} & = D^\beta Q(\lambda_1(t)) = cQ(\lambda_2(t))Q(I(t)) - Q(\lambda_1(t))d - cQ(I(t)) - Q(u(t)) \\
\dot{C} & = D^\beta Q(\lambda_2(t)) = -(e + d)Q(\lambda_2(t)) + eQ(\lambda_3(t)) \\
\dot{C} & = D^\beta Q(\lambda_3(t)) = r - aQ(\lambda_4(t)) + cQ(\lambda_2(t))Q(S(t)) - (r + a + d)Q(\lambda_3(t)) \\
\dot{C} & = D^\beta Q(\lambda_4(t)) = (b - d)Q(\lambda_4(t)) + bQ(\lambda_1(t)).
\end{align*}
\]

(iii) The transversality conditions
\[
\begin{align*}
Q(\lambda_1(T)) & = 0 \\
Q(\lambda_2(T)) & = 0 \\
Q(\lambda_3(T)) & = 0 \\
Q(\lambda_4(T)) & = 0.
\end{align*}
\]

(iv) The stationary condition
\[
2Q(u(t)) - Q(\lambda_1(t))Q(S(t)) = 0.
\]

Furthermore, the optimal control input \(Q(\pi(t))\) is given by
\[
Q(\pi(t)) = \min \left\{ 1, \max \left\{ 0, \frac{Q(\lambda_1(t))Q(S(t))}{2} \right\} \right\}.
\]

Using the transformation (1), we obtain the optimal control \(\pi(t) = \min \left\{ \hat{1}, \max \left\{ \hat{0}, \frac{\lambda_1(t)S(t)}{2} \right\} \right\} \).

The numerical simulations are carried out by using Matlab program associated with the parameters given in Table 3, the weight parameter \(\gamma = 0.4\). In this example, we assume that the disease is in the outbreak time, where there has vague and incomplete information about the initial number of infected individuals that are unable to transmit the disease (E) and capable of spreading the disease (I). Indeed, people in the compartment (E) are someone who have had contact with COVID - 19 infectious people or whose family members have been confirmed to be infectious by coronavirus and hence, it
Fig. 12. Effect of the control input $u$ (Quarantine) on the spread of COVID-19 disease with weight parameter $\gamma = 3$ and fractional order $\beta = 0.7$, where the blue and red curves is the left and right endpoints of the state functions, respectively. The black curve is the exact solution corresponding with $\alpha = 1.0$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
is difficult to determine exactly the number of people in this compartment. For convenience in calculation and prediction, we often assume that the initial number of exposed individuals $E_0$ is "about 500000". Thus, in this work, we propose to use the fuzzy triangular fuzzy numbers (499800, 500000, 500200) to represent for this uncertain quantity with the fluctuation of 200 individuals. Furthermore, the number of susceptible and recovered individuals are estimated in Table 3.

Fig. 11 shows the fuzzy behavior of the fractional SEIR model with the fractional order $\beta = 0.7$ and the initial data $X_0 = \left[ S_0 \ E_0 \ I_0 \ R_0 \right]^T$. We can observe that the number of susceptible individuals is predicted to be dramatic increases in the next 20 days, which means that the coronavirus will be widespread in all population. The second class of people, exposed individuals (E), is going to be sharply decreasing. For people in the compartment (I), it is predicted to reach a peak at $t = 5$ and then, it will be slightly decreasing. Although the number of recovered people is going to go up, the total population in general will experience a considerable decline during the period. Therefore, it implies a natural need to carry out some treatments to reduce the number of infected individuals. In particular, the quarantining susceptible people from infected individuals is considered an effective treatment to prevent people from COVID - 19 infected.

Fig. 12 shows the effect of quarantined control $u$ on the spread of epidemic disease. Here, we consider the fractional optimal control problem for the fractional SEIR model (21) with the fractional order $\beta = 0.8$ and the initial data $X_0 = \left[ S_0 \ E_0 \ I_0 \ R_0 \right]^T$. We observe that under the effect of quarantined control $u$, the populations of four compartments (E), (I), (R) and the total population (N) have considerable change. The first important effect of the control treatment $u$ is the noticeable increase of total population during the period of 20 days. It is predicted that there are approximately $5.5 \times 10^5$ recovered people, more than the case without control. In addition, the number of exposed people is going to experience a decline. Finally, the effect of control treatment $u$ implies the sharp decrease in the number of infectious people. It is predicted to reach a peak at $t = 5$ and then, dramatically go up to nearly half of million individuals, that is less than one forth of the case without control.

5. Conclusions

We have presented a fractional-order dynamical system involving the SIR and SEIR epidemic models in the fuzzy numbers environment with granular derivatives. The fuzziness and imprecise data in the considered models are handled by using the horizontal membership function approach. The main results concentrate on the solvability and fractional optimal control problem for an abstract class of fractional differential systems with uncertainty relating to some epidemic disease models. Here, the existence and uniqueness of fuzzy solution of the abstract fractional differential systems is carried out by using the virtue of contraction mapping principle. The fractional optimal control problem corresponding to this fractional dynamic systems is investigated with the aim to minimize the infected and susceptible populations and maximize the recovered population. A scheme to numerically solve the considered problem is also presented. We use the constructive approach, thus an modifications of Adams-Bashforth-Moulton predictor-corrector algorithm to characterize the numerical solution of the fuzzy fractional optimal control problem are proposed and directly applied to fractional SEIR model. Beside epidemic models, the proposed method could also be applied on some fractional model for the dynamics of data packet delays of switching stability in wireless sensor networks. Moreover, in network security issues, the infection of malware objects such as viruses, worms or trojans in networks are based on the transmission of electric-waves and signals between wireless sensor nodes. Computer virus propagation can be modeled by some fractional models. In the future work, we will formulate some fractional mathematical model of computer virus transmission with quarantine and uncertain initial data in wireless sensor networks via Caputo Atangana - Baleanu fractional derivative.

Declaration of Competing Interest

They have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper;

They have no financial interests/personal relationships which may be considered as potential competing interests.

This statement is agreed by all the authors to indicate agreement that the above information is true and correct.

CRediT authorship contribution statement

Nguyen Phuong Dong: Software, Investigation, Visualization, Writing - original draft, Writing - review & editing. Hoang Viet Long: Conceptualization, Methodology, Investigation, Writing - review & editing, Supervision. Alireza Khasan: Writing - original draft, Writing - review & editing, Methodology.

Acknowledgments

This research was funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2018.311.
Appendix A

Proof of Lemma 3.1

Proof. Firstly, for each \( t \in J \), the fuzzy fractional differential system of the Cauchy problem (3) can be rewritten in the following parametric form

\[
\begin{align*}
\frac{\partial}{\partial t} x^\beta(t, \alpha, \mu) & = A_{gr}(\alpha, \mu) x^\beta(t, \alpha, \mu) + f(t, x^\beta(t, \alpha, \mu)) ,
\end{align*}
\]

(24)

for each \( t \in [0, T] \), \( \alpha, \mu \in [0, 1]^m \) and \( \mu_A \in [0, 1]^{n \times n} \). Then, for each \( t \in [0, T] \), by applying Laplace transform to the granular system (24) and employing Remark 2.7, we immediately obtain

\[
\lambda^\beta \mathcal{L} [x^\beta(t, \alpha, \mu)] - \lambda^\beta - 1 \mathcal{L} [f(t, x^\beta(t, \alpha, \mu))] = A_{gr}(\alpha, \mu_A) \mathcal{L} [x^\beta(t, \alpha, \mu)] + \mathcal{L} [f(t, x^\beta(t, \alpha, \mu))],
\]

that is equivalent to

\[
\mathcal{L} [x^\beta(t, \alpha, \mu)] = (\lambda^\beta \mathcal{L} - A_{gr}(\alpha, \mu_A) )^{-1} \left[ \lambda^\beta - 1 \mathcal{L} [f(t, x^\beta(t, \alpha, \mu))] \right].
\]

Next, by employing Definition 2.20 and Lemma 2.2, Eq. (25) becomes

\[
\begin{align*}
\mathcal{L} [x^\beta(t, \alpha, \mu)] & = (\lambda^\beta \mathcal{L} - A_{gr}(\alpha, \mu_A) )^{-1} \mathcal{L} [f(t, x^\beta(t, \alpha, \mu))] \\
& = \mathcal{L} \left[ \left( t^\beta \mathcal{L} A_{gr}(\alpha, \mu_A) \right) x^\beta_0(\alpha, \mu) \right] \\
& + (\lambda^\beta \mathcal{L} - A_{gr}(\alpha, \mu_A) )^{-1} \mathcal{L} [f(t, x^\beta(t, \alpha, \mu))] \\
& = \mathcal{L} \left[ t^\beta \mathcal{L} A_{gr}(\alpha, \mu_A) x^\beta_0(\alpha, \mu) \right] \\
& + \mathcal{L} \left[ t^\beta \mathcal{L} f(t, x^\beta(t, \alpha, \mu)) \right].
\end{align*}
\]

Then, the convolution theorem of Laplace transform can be applied to obtain

\[
\begin{align*}
\mathcal{L} [x^\beta(t, \alpha, \mu)] & = \mathcal{L} \left[ t^\beta A_{gr}(\alpha, \mu_A) x^\beta_0(\alpha, \mu) \right] \\
& + \mathcal{L} \left[ \int_0^t (t-s)^{\beta-1} \mathcal{L} f(s, x^\beta(s, \alpha, \mu)) ds \right].
\end{align*}
\]

Applying the inverse Laplace transform, it yields that

\[
x^\beta(t, \alpha, \mu) = t^\beta A_{gr}(\alpha, \mu_A) x^\beta_0(\alpha, \mu) \\
+ \int_0^t (t-s)^{\beta-1} \mathcal{L} f(s, x^\beta(s, \alpha, \mu)) ds.
\]

Finally, by transformation (1), we immediately obtain

\[
x(t) = t^\beta A x_0 + \int_0^t (t-s)^{\beta-1} \mathcal{L} f(s, x(s)) ds,
\]

that means the formula (4) holds for all \( t \in [0, T] \). \( \square \)

Proof of Theorem 3.1

Proof. Firstly, let us define an operator \( \mathcal{F} : C_T \to C_T \) by

\[
\mathcal{F}[x](t) = t^\beta A x_0 + \int_0^t (t-s)^{\beta-1} \mathcal{L} f(s, x(s)) ds \quad \text{for each} \ t \in [0, T].
\]

By means of Banach contraction principle, a function \( \bar{x} \in C_T \) is the unique mild solution of the Cauchy problem (3) if and only if it is a fixed point of the solution operator \( \mathcal{F} \) on a suitable closed, bounded, convex subset of the functional space \( C_T \). For this aim, let us denote

\[
\Omega_r = \{ x \in C_T : \|x\|_{\lambda} \leq r \}.
\]

where \( \lambda > 0 \) is such that \( \frac{M_1 K_{A}(\alpha, \beta)}{\lambda} < 1 \) and \( r = \frac{M_1 (\|x_0\| + L_2 T^\beta)}{\lambda} > 0 \).

Now, we proceed the proof by following steps:

Step 1. The operator \( \mathcal{F} \) is onto, i.e., \( \mathcal{F}(\Omega_r) \subseteq \Omega_r \). Indeed, for each \( x \in \Omega_r \) and \( t \in [0, T] \), we have

\[
\mathcal{D}(\mathcal{F}[x](t), \bar{y}) = \mathcal{D}\left( t^\beta A x_0 + \int_0^t (t-s)^{\beta-1} \mathcal{L} f(s, x(s)) ds, \bar{y} \right)
\]
Next, by the definition of norm $\| \cdot \|_\lambda$ and Lemma 2.3 in [24], we have

$$D^{{\alpha}}( F[x](t), \overline{0} ) \leq M_1 \left[ \|x_0\| + L_2 \frac{T^{{\alpha}}}{\beta} + \frac{e^{\lambda t}}{\beta} \gamma(\lambda, \beta) L_1 r \right].$$

(A.26)

Next, by dividing both sides of the inequality (A.26) by $e^{\lambda t}$, we obtain

$$D^{{\alpha}}( F[x](t), \overline{0} ) e^{-\lambda t} \leq M_1 \left[ \|x_0\| + \frac{M_1 L_2 T^{{\alpha}}}{\beta} + \frac{M_1 L_1 \gamma(\lambda, \beta)}{\beta} r \right] \leq \left( \frac{M_1 \lambda}{\beta} \right) r \leq r$$

that means $\| F[x] \|_\lambda \leq r$ for all $x \in \Omega_s$, or equivalently, $F$ maps $\Omega_s$ into itself.

**Step 2.** The operator $F : \Omega_s \to \Omega_s$ is continuous. Indeed, let $(x_n) \subset \Omega_s$ be such that $x_n \to x \in \Omega_s$. Then,

$$F[x_n](t) = E_{\beta}(t^{{\alpha}}A)x_0 + \int_0^t (t-s)^{\beta-1} E_{\beta}(s^{{\alpha}})(t-s)^{\beta} A f(s, x_0(s))ds, \quad t \in [0, T].$$

For each $t \in [0, T]$, by employing hypotheses (HF1), (HF2), we have

$$D^{{\alpha}}( F[x_n](t), F[x](t) ) \leq M_1 \int_0^t (t-s)^{\beta-1} D^{{\alpha}}( f(s, x_0(s)), f(s, x(s)) )ds \leq M_1 L_1 \int_0^t (t-s)^{\beta-1} D^{{\alpha}}( x_0(s), x(s) )ds \to 0$$

as $n \to \infty$, which follows that $F$ is continuous on $\Omega_s$.

**Step 3.** $F$ is a contraction mapping. Let $x, \overline{x} \in \Omega_s$ be arbitrary. Then, it suffices to show that there exists $k \in (0, 1)$ such that

$$\| F[x] \otimes^{\alpha} F[\overline{x}] \|_{\lambda} < k \| x \otimes^{\alpha} \overline{x} \|_{\lambda}.$$

Indeed, for each $t \in [0, T]$, we have

$$D^{{\alpha}}( F[x](t), F[\overline{x}](t) ) \leq \int_0^t (t-s)^{\beta-1} D^{{\alpha}}( E_{\beta}(t^{{\alpha}}A)(t-s)^{\beta} A f(s, x(s)), E_{\beta}(t^{{\alpha}}A)(t-s)^{\beta} A f(s, \overline{x}(s)) )ds \leq \int_0^t (t-s)^{\beta-1} D^{{\alpha}}( f(s, x(s)), f(s, \overline{x}(s)) )ds \leq M_1 \int_0^t (t-s)^{\beta-1} D^{{\alpha}}( f(s, x(s)), f(s, \overline{x}(s)) )ds \leq M_1 L_1 \int_0^t (t-s)^{\beta-1} e^{\lambda s} D^{{\alpha}}( x(s), \overline{x}(s) )e^{-\lambda s} ds \leq M_1 L_1 \left( \int_0^t (t-s)^{\beta-1} e^{\lambda s} ds \right) \| x \otimes^{\alpha} \overline{x} \|_{\lambda} \leq M_1 L_1 \frac{e^{\lambda t}}{\beta} \gamma(\lambda, \beta) \| x \otimes^{\alpha} \overline{x} \|_{\lambda}.$$
Then, by dividing both sides by $e^{t\lambda}$ and taking supremum for $t \in [0, T]$, we obtain

$$
\left\| \mathcal{F}[x] \right\|_\lambda < \frac{M_1 \lambda \Gamma(\lambda, \beta)}{\beta} \left\| x \right\|_\lambda.
$$

It should be noted that for a big enough $\lambda$, we have $\frac{M_1 \lambda \Gamma(\lambda, \beta)}{\beta} < 1$. Thus, it implies that the operator $\mathcal{F}$ is a contraction. Finally, by using Banach contraction principle, we can conclude that $\mathcal{F}$ has a unique fixed point $x^* \in \Omega$, that is unique mild solution of the Cauchy problem (3). □

**Realistic data about the number of infectious and recovered people**

| Date       | The world (Infectious) | The world (Recovered) | S. Korea (Infectious) | S. Korea (Recovered) | Italy (Infectious) | Italy (Recovered) |
|------------|------------------------|-----------------------|-----------------------|----------------------|---------------------|-------------------|
| 18 Mar     | 218,822                | 82,854                | 8413                  | 1540                 | 35,713              | 4025              |
| 19 Mar     | 244,933                | 86,898                | 8565                  | 1540                 | 41,035              | 4440              |
| 20 Mar     | 275,597                | 86,898                | 8565                  | 1540                 | 47,021              | 4440              |
| 21 Mar     | 305,036                | 91,137                | 8799                  | 1540                 | 53,578              | 6072              |
| 22 Mar     | 337,489                | 97,342                | 8897                  | 2909                 | 59,138              | 7024              |
| 23 Mar     | 378,860                | 97,794                | 8961                  | 2909                 | 63,927              | 7024              |
| 24 Mar     | 422,629                | 108,388               | 9037                  | 3507                 | 69176              | 8326              |

**Supplementary material**

Supplementary material associated with this article can be found in the online version, at doi:10.1016/j.cnsns.2020.105312.

**References**

[1] Abbasi SMM, Jalali A. Fuzzy tracking control of fuzzy linear dynamical systems. ISA Trans 2020;97:102–15.

[2] Abbey H. An examination of the reed-frost theory of epidemics. Hum Biol 1952;24(3):201–33.

[3] Agrawal OP. A quadratic numerical scheme for fractional optimal control problems. J Dyn Syst Meas Contr 2008;130(1). doi:10.1115/1.2814055.

[4] Agarwal RP, Lakshminantham V, Nieto JJ. On the concept of solution for fractional differential equations with uncertainty. Nonlin Anal 2010;72:2859–62.

[5] Ali MS, Narayanana G, Sevgen S, Shekher V, Arik S. Global stability analysis of fractional-order fuzzy BAM neural networks with time delay and impulsive effects. Commun Nonlin Sci Numer Simul 2019;78. doi:10.1016/j.cnsns.2019.104853.

[6] Allahviranloo T, Salahshour S, Abbasbandy S. Explicit solutions of fractional differential equations with uncertainty. Soft Comput 2012;16:297–302.

[7] Arenas AJ, González-Parrá G, Chen-Charpentier BM. Construction of nonstandard finite difference schemes for the S and SIR epidemic models of fractional order. Math Comp Simul 2016;121:48–63.

[8] Baleanu D, Defez E, Agrawal OP. A central difference numerical scheme for fractional optimal control problems. J Vib Control 2009;15(4):583–97.

[9] Batista M. Estimation of the final size of the coronavirus epidemic by the SIR model. ResearchGate; 2020.

[10] Becerra VM. Solving optimal control problems with state constraints using nonlinear programming and simulation tools. IEEE Tran Edu 2004;47(3):377–84.

[11] Bede B. Mathematics of fuzzy sets and fuzzy logic. Springer; 2013.

[12] Bede B, Stefanini L. Generalized differentiability of fuzzy-valued functions. Fuzzy Sets Syst 2013;230:119–41.

[13] Bonnet C, Partington J. Coprime factorizations and stability of fractional differential systems. Syst Cont Lett 2000;41(3):167–74.

[14] Cai L, Li X, Ghosh M, Guo B. Stability analysis of HIV/AIDS epidemic model with treatment. J Comput Appl Math 2009:229:313–23.

[15] Dibrov BF, Zhabotinsky AM, Neyfakh YA, Orlova MP, Churikova IT. Mathematical model of cancer chemotherapy. periodic schedules of phase-specific cytotoxic-agent administration increasing the selectivity of therapy. Math Biosci 1985;73:1–31.

[16] Guo TL. The necessary conditions of fractional optimal control in the sense of caputo. J Optim Theory App 2013;156(1):115–26.

[17] Hamdan N, Kilicman A. A fractional order SIR epidemic model for dengue transmission. Chaos Solitons Fractals 2018;114:55–62.

[18] Huo J, Zhao H. Dynamical analysis of a fractional SIR model with birth and death on heterogeneous complex networks. Physica A 2016;468:41–56.

[19] Hikul MM, Zahra WK. On fractional model of an HIV/AIDS with treatment and time delay. Progr Frac Differ Appl 2016;2:55–66.

[20] Hocking L. Optimal control: an introduction to the theory with applications. Oxford University Press; 1991.

[21] Kilbas AA, Srivastava HM, Trujillo J. Theory and applications of fractional differential equations, vol. 204 of North-Holland mathematics studies. Elsevier, Amsterdam, The Netherlands; 2006.

[22] Kheiri H, Jafari M. Fractional optimal control of an HIV/AIDS epidemic model with random testing and contact tracing. J Applied Math Comp 2019;60:387–411.

[23] Lakshminantham V, Mohapatra RN. Theory of fuzzy differential equations and inclusions. London, UK: Taylor & Francis, Ltd; 2003.

[24] Long HV, Son NTK, Tam HTT, Yao JC. Ulam stability for fractional partial integro-differential equation with uncertainty. Acta Math Vietnam 2017;42(4):675–700.

[25] Long HV, Dong NP. An extension of Krasnosel'skiî's fixed point theorem and its application to nonlocal problems for implicit fractional differential systems with uncertainty. J Fixed Point Theory Appl 2018;20(1):1–27.

[26] Mazandarani M, Kamyad AV. Modified fractional euler method for solving fuzzy fractional initial value problem. Commun Nonlinear Sci Numer Simul 2013;18(1):12–21.

[27] Mazandarani M, Pariz N, Kamyad AV. Granular differentiability of fuzzy-number-valued functions. IEEE Tran Fuzzy Syst 2018;26(1):310–23.

[28] Mazandarani M, Zhao Y. Fuzzy bang-bang control problem under granular differentiability. J Franklin Inst 2018;355(12):4931–51.
[29] Mazandarani M, Pariz N. Sub-optimal control of fuzzy linear dynamical systems under granular differentiability concept. ISA Trans 2018;76:1–17.

[30] Ming W.K., Huang J., Zhang C.J. Breaking down of healthcare system: Mathematical modelling for controlling the novel coronavirus. 2020. (2019-nCoV) outbreak in Wuhan, China. bioRxiv.

[31] Najarian M, Zhao Y. Fuzzy fractional quadratic regulator problem under granular fuzzy fractional derivatives. IEEE Tran Fuzzy Syst 2018;26(4):2273–88.

[32] Nesteruk I. Statistics based predictions of coronavirus 2019-nCoV spreading in mainland China. MedRxiv; 2020.

[33] Pan I, Das S. Fractional order fuzzy control of hybrid power system with renewable generation using chaotic PSO. ISA Trans 2016;62:19–29.

[34] González-Parra G, Arenas AJ, Chen-Charpentier BM. A fractional order epidemic model for the simulation of outbreaks of influenza a (h1n1). Math Methods Appl Sci 2014;37(15):2218–26.

[35] Piegat A, Landowski M. Horizontal membership function and examples of its applications. Int J Fuzzy Syst 2015;17(1):22–30.

[36] Piegat A, Landowski M. Aggregation of inconsistent expert opinions with use of horizontal intuitionistic membership functions. In: Novel developments in uncertainty representation and processing. Springer; 2016. p. 215–23.

[37] Piegat A, Landowski M. Fuzzy arithmetic type-1 with HMFs, Uncertainty modeling. Springer; 2017. 233–250

[38] Podlubny I. Fractional differential equations, vol. 198, mathematics in science and engineering. Technical University of Kosice, Slovak Republic; 1999.

[39] Samko SC, Kilbas AA, Marichev OL. Fractional integrals and derivatives, theory and applications. Switzerland, Philadelphia, Pa., USA: Gordon and Breach Science Publishers; 1993.

[40] Sharma R, Gaur P, Mittal AP. Design of two-layered fractional order fuzzy logic controllers applied to robotic manipulator with variable payload. Appl Soft Comput 2016;47:565–76.

[41] Son NTK. A foundation on semigroup of operators defined on the set of triangular fuzzy numbers and its application to fuzzy fractional evolution equations. Fuzzy Sets Syst 2018;347:1–28.

[42] Son NTK, Dong NP. Asymptotic behavior of $c^\alpha$-solutions of evolution equations with uncertainties. J Fixed Point Theory Appl 2018;20(4):1–30.

[43] Son NTK, Dong NP, Long HV. Fuzzy delay differential equations under granular differentiability with applications. Comp Appl Math 2019;38(3):107–36.

[44] Son NTK, Dong NP, Son LH, Abdel-Basset M, Manogaran G, Long HV. On the stabilizability for a class of linear time-invariant systems under uncertainty. Circuits Syst Signal Process 2020;39(2):391–60.

[45] Son NTK, Dong NP, Long HV, Son LH, Khastan A. Linear quadratic regulator problem governed by granular neutrosophic fractional differential equations. ISA Trans 2020;97:290–316.

[46] Son NTK, Dong NP. Systems of implicit fractional fuzzy differential equations with nonlocal conditions. FILOMAT 2019;33(12):3795–822.

[47] Stefanini L, Bede B. Generalized Hukuhara differentiability of interval-valued functions and interval differential equations. Nonlin Anal 2009;71(3–4):1311–28.

[48] Muñoz Vázquez AJ, Gaxiola F, Martínez-Reyes F, Manzo-Martínez A. A fuzzy fractional-order control of robotic manipulators with PID error manifolds. Appl Soft Comput 2019;83. doi:10.1016/j.asoc.2019.105646.

[49] Vu H, Hoa NV. Uncertain fractional differential equations on a time scale under granular differentiability concept. Comp Appl Math 2019;38(3). doi:10.1007/s40092-018-9373-5.