TRANSITORY QUEUEING NETWORKS

BY HARSHA HONNAPPA, RAHUL JAIN

Purdue University* and University of Southern California†

Queueing networks are notoriously difficult to analyze sans both Markovian and stationarity assumptions. Much of the theoretical contribution towards performance analysis of time-inhomogeneous single class queueing networks has focused on Markovian networks, with the recent exception of work in Liu and Whitt (2011) and Mandelbaum and Ramanan (2010). In this paper, we introduce transitory queueing networks as a model of inhomogeneous queueing networks, where a large, but finite, number of jobs arrive at queues in the network over a fixed time horizon. The queues offer FIFO service, and we assume that the service rate can be time-varying. The non-Markovian dynamics of this model complicate the analysis of network performance metrics, necessitating approximations. In this paper we develop fluid and diffusion approximations to the number-in-system performance metric by scaling up the number of external arrivals to each queue, following Honnappa et al. (2014). We also discuss the implications for bottleneck detection in tandem queueing networks.

1. Introduction. Single class queueing networks (henceforth ‘queueing networks’) have been studied extensively in the literature, with much effort focused on understanding the steady-state joint distribution of the state of the network (typically defined as the number of jobs in each queue). In this paper, we consider a variation of the generalized Jackson queueing network model ((Chen and Yao 2001, Chapter 7)) where a finite, but large, number of jobs arrive at some of the nodes in the network from an extraneous source. We characterize fluid and diffusion approximations to the queue length state process, as the population size scales to infinity.

To motivate this model, consider a manufacturing facility that produces engines. Each part of the engine is produced and assembled in a separate machine that requires some human supervision, with final assembly occurring at the end of the job-shop. Typically, there is a finite, but large number of jobs that need to be completed in a shift spanning a few hours. Furthermore, jobs cannot be carried over to the next shift. It is typically the case that the shift horizon is not long enough for the system to reach a steady state. Furthermore, the bottleneck machine can change over the shift horizon, as

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a consequence of the variation in the arrival of jobs to the job-shop, or due to stochastic variation in the assembly time in each machine. This is also known as the ‘shifting bottleneck’ phenomenon in production engineering; see Roser et al. (2002), Lawrence and Buss (1994, 1995). In the latter two papers, Jackson networks are used to analyze the shifting bottleneck network. More generally, there is less reason to believe that the arrival and service processes are stationary and ergodic, and that they are Poisson processes. We introduce *transitory queueing networks* as a broadly applicable model of such systems.

Transitory queueing networks consist of a number of infinite buffer, FIFO, single server queues (a.k.a. ‘nodes’) interconnected by customer routes. We assume that the routing matrix satisfies a so-called *Harrison-Reiman* (H-R) condition that the matrix has a spectral radius of less than one. This implies that, on completion of service at a particular node, a customer is routed to another node or exits the network altogether. Jobs enter a given node at random time epochs modeled as the ordered statistics of independent and identically distributed (i.i.d.) random variables. The arrival epoch of the jobs at different nodes can be correlated (for instance, in the manufacturing context, jobs can be submitted to multiple nodes simultaneously). We assume that the service processes at different nodes are independent with time-inhomogeneous service rates, and modeled as a time change of a unit rate renewal counting process, generalizing the construction of a time-inhomogeneous Poisson process. The transient analysis of generalized Jackson networks is non-trivial, as noted before. The conventional heavy-traffic diffusion approximation that relied on long-run average rates has been used to approximate the evolution of the state process. However, these rates do not exist in transitory queueing networks. Here, we develop a ‘population acceleration’ approximation, by increasing the number of jobs arriving at the network in the interval of interest to infinity, and suitably scaling (or ‘accelerating’) the service process in each queue by the population size.

Our results complement the existing literature on the analysis of single-class queueing networks by establishing the following results:

(i) we develop a large population approximation framework for studying single class queueing networks in a transitory setting, complementing and extending Markovian network analyses to non-Markovian queueing networks,

(ii) Our diffusion approximations use the recently developed directional derivative oblique reflection map in Mandelbaum and Ramanan (2010) to establish a diffusion scale approximation; this is substantially different from the conventional heavy-traffic approximations used to study
single-class queueing networks, and
(iii) we study the evolution of the bottleneck process over the time horizon, identifying the bottleneck station as time progresses. This analysis extends the standard bottleneck analyses, where bottlenecks are identified in terms of the long-term average arrival and service rates.

1.1. Analytical Results. We consider a sequence of queueing networks wherein \( n \) jobs arrive at each node that receives external traffic in the \( n \)th network. We first establish a functional strong law of large numbers (FSLLN) to the arrival process, as the population size scales to infinity, by generalizing the Glivenko-Cantelli Theorem to multiple dimensions. We also assume the service processes satisfy a FSLLN and functional central limit theorem (FCLT) in the population acceleration scale. The queue length fluid limit is shown to be equal to the oblique reflection of the difference of the fluid arrival and service processes (or the ‘fluid netput’ process). On the other hand, for the FCLT we introduce the notion of a multidimensional Brownian bridge process, as a generalization of a one-dimensional Brownian bridge process, and show that the diffusion scaled arrival process converges to a multidimensional Brownian bridge in the large population limit. The diffusion limit turns out to be complicated, and it is shown to be a reflection of a multidimensional Gaussian bridge process - however, the reflection is through a directional derivative of the oblique reflection of the netput in the direction of the diffusion limit of the fluid netput process. This is a highly non-standard result. Indeed, it is only in the recent past that Mandelbaum and Ramanan (2010) have investigated the existence of a directional derivative to the oblique reflection map.

Leveraging the results of Mandelbaum and Ramanan (2010), we can only establish a pointwise diffusion limit for an arbitrary transitory queueing network. This is due to the fact that the directional derivative limit can have sample paths with discontinuities that are both right- and left-discontinuous. Thus, establishing convergence in a sample path space under a suitably weak topology such as the (‘strong’ or ‘weak’) \( M_1 \), for instance, is not straightforward. Instead, we focus on the case of tandem queueing networks, with uniform and unimodal arrival time distribution functions. In this case, we show that the discontinuities in the limit are either right or left continuous, and hence we can establish \( M_1 \) convergence. Using these approximations, we next address the question of bottleneck prediction. We generalize the standard definition of a bottleneck in a single class network, defined as the queues whose fluid arrival rate exceeds the fluid service capacity to the transitory setting.
1.2. *Implications for Bottleneck Detection.* Bottleneck detection and prediction is likely the most important question that a system operator faces. Heavy traffic theory has been immensely successful at characterizing steady state bottlenecks in general queueing networks, under minimal network data assumptions. However, there are many circumstances, ranging from manufacturing, to healthcare, transportation and computing, where transient bottleneck detection and analysis is critical. In the purely transient, or ‘transitory’ setting, it is common for the bottleneck node to change over the shift horizon. As a consequence the plant manager moves workers around trying to ease bottlenecks, increasing costs and increasing the likelihood of job overages. Another example is in healthcare where patient diagnosis relies on a number of tests that must be done with different machines. Furthermore, in many time critical settings, the horizon within which tests must be conducted is fixed. An important question in these situations is whether transitory bottlenecks can be accurately predicted, given network data.

Note that the standard definition of a bottleneck is a ‘capacity level’ one, defined in terms of long-term averages. This definition, of course, is not satisfactory in the transitory setting where steady states might not be reached. Second, standard heavy-traffic analysis completely ignores non-stationarities in the network data, which is generally prevalent in the examples presented above, making it an inappropriate analytical tool to use. In particular, it is possible that the number of bottleneck nodes in the network changes over time, thus necessitating a more nuanced definition and analysis of bottlenecks. We introduce a natural definition of a ‘transitory bottleneck’ in terms of the diffusion approximations developed in this paper, and then use these approximations to analyze the evolution of the number of bottlenecks in tandem transitory networks. This is particularly relevant to job-shop analysis of production systems.

1.3. *Related Literature.* There has been significant interest in the analysis of single class queueing networks. Under the assumption of Poisson arrival and service processes, Jackson (1957) showed that the steady state distribution of the state of the network (the number of jobs waiting in each node) is equal to the product of the distribution of the state of each node in the network. This desirable property implies that, in steady state, the network exhibits a nice independence property. This property does not extend to networks with general arrival and service processes; these are also known as generalized Jackson networks.

Reiman first established the heavy-traffic diffusion approximation to open generalized Jackson networks in Reiman (1984). In particular, the diffu-
ation approximation is shown to be a multi-dimensional reflected Brownian motion in the non-negative orthant, reflected through the oblique reflection mapping. Such reflection maps have come to be called as Harrison-Reiman maps following the seminal work in Harrison and Reiman (1981).

Chen and Mandelbaum (1991b,a) characterize a homogeneous fluid network, and establish fluid and diffusion approximations. The analysis of non-stationary and time inhomogeneous queueing systems is non-trivial in general. For single server queues, see Keller (1982), Massey (1985), Mandelbaum and Massey (1995) among others. In Honnappa et al. (2014, 2017) we develop fluid and diffusion approximations to the $\Delta_{(i)}/GI/1$ transitory single server queue in the population acceleration scaling regime. Recent work in Bet, G. and Hofstad, R. v. d. and Leeuwarden, J. s. h. v. (2015), Bet, G. and Hofstad, R. v. d. and Leeuwarden, J. s. h. v. (2016) considers the $\Delta_{(I)}/GI/1$ queue under a ‘uniform’ acceleration regime, where the initial work in the network is assumed to satisfy a heavy-traffic like condition. For networks of queues, Mandelbaum et al. (1998) develops strong approximations to queueing networks with nonhomogeneous Poisson arrival and service processes. In Duffield et al. (2001), the authors study the offered load process in a bandwidth sharing network, with nonstationary traffic and general bandwidth requirements. The closest work in the literature to the current paper is Liu and Whitt (2011) who study a network of non-Markovian fluid queues with time-varying traffic and customer abandonments. To be precise, they consider a $(G_t/M_t/s_t + GI_t)^m/M_t$ network with $m$ nodes, time-varying arrivals, staffing and abandonments, and inhomogeneous Poisson service and routing, and characterize the performance of the network as a direct extension of the single-server queue case. However, their scaling regime and the limit processes are completely different from the our results.

The rest of the paper is organized as follows. We collect relevant notation in Section 2. We start our analysis with a description of the transitory generalized Jackson network model in Section 3, and we develop fluid and diffusion approximations to the network primitives. In Section 4, we develop functional strong law of large numbers approximations to the queueing equations, and identify the fluid model corresponding to the transitory network. We identify the diffusion network model in Section 5, and establish a weak convergence result for a tandem network with unimodal arrival time distribution. We end with conclusions and future research directions in Section 7.

2. Notation. Following standard notation, $\mathcal{C}^K$ represents the space of continuous $\mathbb{R}^K$-valued functions, and $\mathcal{D}^K$ the space of functions that are
right continuous with left limits (RCLL) and are $\mathbb{R}^K$-valued. The space $D^K_{l,r}$ consists of $\mathbb{R}^K$-valued functions that are either right- or left-continuous at each point in time, while $D^K_{\text{lim}}$ is the space of $\mathbb{R}^K$-valued functions that have right and left limits at all points in time. $D^K_{usc}$ is the space of RCLL functions that are upper semi-continuous as well. We represent the space of $L \times L$ matrices by $M^L$. The space and mode of convergence of a sequence of stochastic elements is represented by $(X, Y)$, where $X$ is the space in which the stochastic elements take values and $Y$ the mode of convergence. In this paper our results will be proved under the uniform mode of convergence and occasionally in the “strong” $M_1$ ($SM_1$) topology (see (Whitt 2001a, Chapter 11)). Weak convergence of measures will be represented by $\Rightarrow$. Finally, $\text{diag}(x_1, \ldots, x_K)$ represents a $K \times K$ diagonal matrix with entries $x_1, \ldots, x_K$.

3. Transitory Queueing Network. Let $(\Omega, F, \mathbb{P})$ be an appropriate probability space on which we define the requisite random elements. Let $K := \{1, \ldots, K\}$ be the set of nodes in the network, and $\mathcal{E} \subset K$ the set of nodes where exogenous traffic enters the network. Each node in $\mathcal{E}$ receives $n$ jobs that arrive exogenously to the node. We assume a very general model of the traffic: let $T_{e,m} := (T_{e,1,m}, \ldots, T_{e,J,m})$, $m \leq n$, represent the tuple of arrival epoch random variables where $T_{e,j,m}$ is the arrival epoch of the $m$th job to node $e_j \in \mathcal{E}$ (here $J := |\mathcal{E}|$). By assumption $T_{e,j,m} \in [0, T]$ for all $e_j \in \mathcal{E}$ and $1 \leq m \leq n$. We also assume that $\{T_m; m = 1, \ldots, n\}$ forms a sequence of independent random vectors. Let $F_{e_j}$ be the distribution function of the arrival epochs to node $e_j \in \mathcal{E}$; that is $\mathbb{E}[1\{T_{e,j,m} \leq t\}] = F_{e_j}(t)$ with support $[0, T]$. Users sample a time epoch to arrive at the node and enter the queue in order of the sampled arrival epochs; thus the arrival process to each node is a function of the ordered statistics of the arrival epoch random variables. In many situations, it is plausible that there is correlation between the arrival processes to the nodes in $\mathcal{E}$. To model such phenomena, we assume that the joint distribution of the arrival epochs are fully specified. To be precise, we assume that $\mathbb{P}(T_{e_1,m} \leq t, \ldots, T_{e_J,m} \leq t)$ for all $m \in \{1, \ldots, n\}$ is well defined. Let $a_m(t) := (1\{T_{e_1,m} \leq t\}, \ldots, 1\{T_{e_J,m} \leq t\}) \in D^J[0, \infty)$ and

$$A_{n,e_j} := \sum_{m=1}^n 1\{T_{e_j,m} \leq t\} \text{ for } 1 \leq j \leq J,$$

then $A_n(t) := \sum_{m=1}^n a_m(t) = (A_{n,e_1}(t), \ldots, A_{n,e_J}(t)) \in D^J[0, \infty)$ is the vector of cumulative arrival processes to the nodes in $\mathcal{E}$. Then,

$$\mathbb{E}[A_n(t)] = nF(t) := n(F_{e_1}(t), \ldots, F_{e_J}(t))$$
and $\mathbb{E}[A_n(t)A_n(t)^T] = [nF_{e_i}(t) + n(n-1)F_{e_j}(t)F_{e_j}(t)]$, where $F_{e_i,e_j}(t) := P(T_{e_i,m} \leq t, T_{e_j,m} \leq t)$. This ‘multivariate empirical process’ representation for the traffic affords a very natural model of correlated traffic in networks, and stands in contrast with generalized Jackson networks where external traffic to each node in $\mathcal{E}$ is independent.

Recall from Donsker’s Theorem (for empirical sums) that $n^{-1/2}(A_{n,e_i} - nF_{e_i}) \Rightarrow W_{e_i}^0 \circ F_{e_i}$, where $W_{e_i}^0$ is a standard Brownian bridge process. The Brownian bridge process $W^0$ is also well defined as a ‘tied-down’ Brownian motion process equal in distribution to $(W(t) - tW(1), t \in [0,1])$, for all $t \in [0,1]$, where $W$ is a standard Brownian motion process.

**Definition 1** Let $W = (W_1, \ldots, W_J)$ be a $J$-dimensional standard Brownian motion process with identity covariance matrix. If $R$ is a $J \times J$ positive-definite matrix with lower-triangular Cholesky factor $L$, then $W_R = LW$ is a $J$-dimensional Brownian motion with covariance matrix $R$. By directly extending the definition of a one-dimensional Brownian bridge process,

$$(W^0(t) = W_R(t) - tW_R(1), \ t \in [0,1])$$

is a $J$-dimensional Brownian bridge process with covariance matrix $R$.

It is straightforward to see that $\mathbb{E}[W^0(t)] = 0$ for all $t \in [0,1]$ and $\mathbb{E}[W^0(t)W^0(s)^T] = t(1-s)R = [t(1-s)r_{i,j}]$ when $t \leq s$. More generally, we define a Brownian bridge process with time-dependent covariance matrix as follows. Recall that a stochastic process is defined as a Gaussian process provided its finite dimensional distributions are jointly Gaussian.

**Definition 2** Let $R : [0,T] \times [0,T] \rightarrow \mathcal{M}_J$ be a right continuous, symmetric function such that $R(t,s)$ is positive-definite for each $t, s \in [0,T]$, with the restriction that $R(0,0) = 0$ and $r_{i,j}(T,T) = r_i(T)r_j(T)$, where $r_{i,j}$ is the $(i,j)$th entry of $R$ and $r_i$ is the $i$th diagonal element of $R$. The Gaussian process $W^0$ is a $J$-dimensional Brownian bridge process if it has mean zero and covariance function $\mathbb{E}[W^0(t)W^0(s)^T] = R(t,s) - \text{diag}(R(t,t))\text{diag}(R(s,s)) = [r_{i,j}(t,s) - r_i(t)r_j(s)]$, where $\text{diag} : \mathcal{M}_J \rightarrow \mathbb{R}^J$ is a function that maps a matrix to a vector of the diagonal elements.

Note that this definition is a natural generalization of the bridge process in Definition 1. The terminal condition $r_{i,j}(T,T) = r_i(T)r_j(T)$ ensures that $W^0(T) = 0$ a.s. While we do not argue the existence of this object rigorously, the right continuity of the covariance matrix and the assumed Gaussianity of the marginals imply that it can be inferred from the Daniell-Kolmogorov theorem; see Rogers and Williams (2000). Now, observe that the covariance function of the pre-limit traffic process satisfies $R_n(t,s) =$
\[ \mathbb{E}[(A_n(t) - \mathbb{E}[A_n(t)]) (A_n(s) - \mathbb{E}[A_n(s)])^T] = n[F_{e_i,e_j}(t,s) - F_{e_i}(t)F_{e_j}(s)] \in C^{J \times J}, \]

where \( F_{e_i,e_j}(t,s) := \mathbb{P}(T_{e_i,m} < t, T_{e_j,m} < s) \) and \( t \leq s \). Now, following Definition 2 let \( W^0 \circ F \) represent a multidimensional Brownian bridge with covariance function

\[ R(t,s) = [F_{e_i,e_j}(t,s) - F_{e_i}(t)F_{e_j}(s)], \]

when \( t \leq s \). Note that we are “overloading” the composition operator \( \circ \) in this notation, but the usage should be clear from the context. Theorem 1 below establishes multivariate generalizations of the classical Glivenko-Cantelli and Donsker’s theorems.

**Theorem 1** We have,

(i) \( n^{-1} A_n \to F \) in \((C^J, U)\) a.s. as \( n \to \infty \), and

(ii) \( \hat{A}_n := n^{-1/2} \left( \sum_{m=1}^n a_m - nF \right) \Rightarrow W^0 \circ F \) in \((C^J, U)\) as \( n \to \infty \), where \( W^0 \circ F \in C^J[0, \infty) \) is the \( J \)-dimensional Brownian bridge process with covariance matrix defined in (2).

The proof of the theorem is available in the appendix. We refer to the \( k \)th component process by \( W_k \circ F_k \).

Next, we consider a sequence of service processes indexed by the population size \( n \geq 1 \), \( S_{n,k} : \Omega \times [0, \infty) \to \mathbb{N} \) for \( k \in \mathcal{K} \). We assume that for each \( k \in \mathcal{K} \) the function \( \mu_{n,k} : [0, \infty) \to [0, \infty) \) is Lebesgue-integrable and that \( M_{n,k}(t) := \int_0^t \mu_{n,k}(s)ds \) satisfies \( M_{n,k} \to M_k \) in \((C, U)\) as \( n \to \infty \), where \( M_k : [0, \infty) \to [0, \infty) \) is non-decreasing and continuous. We also assume that

\[ M_n := (M_{1,n}, \ldots, M_{n,k}) \to M := (M_1, \ldots, M_K) \text{ in } (C^K, U) \text{ as } n \to \infty, \]

where \( K = |\mathcal{K}| \). Let \( S_n := (S_{1,n}, \ldots, S_{n,k}) \) represent the ‘network’ service process, where the component service processes are independent of each other. We assume that \( S_n \) satisfies the following fluid and diffusion limits.

**Assumption 1** The service processes \( \{S_n, \ n \geq 1\} \) satisfies

(i) \( n^{-1} S_n - M_n \to 0 \) in \((C^K, U)\) a.s. as \( n \to \infty \), and

(ii) \( \hat{S}_n(t) := n^{-1/2} (S_n - nM) \Rightarrow W \circ M \) in \((C^K, U)\) as \( n \to \infty \), where \( W := (W_1, \ldots, W_K) \) is a \( K \)-dimensional Brownian motion process with identity covariance matrix.

Note that the covariance function of the process \( W \circ M \) is the diagonal matrix with entries \( (M_1, \ldots, M_K) \). This service process is analogous to the time-dependent ‘general’ traffic process \( G_t \) proposed in Liu and Whitt (2014). It’s possible to anticipate a proof of this result when the centered service process \( S_n - M_n \) is a martingale. This would be the case when
$S_n$ is a $K$-dimensional stochastic process where the marginal processes are nonhomogeneous Poisson processes and $M_{n,k} = \mathbb{E}[S_{n,k}]$. Here, we leave the development of a general result to a separate paper and, instead, assume that such a sequence of service processes exist.

On completion of service at node $i$, a job will join node $j$ with probability $p_{i,j} \geq 0$, $i, j \in \{1, \ldots, K\}$, or exit the network with probability $1 - \sum_j p_{i,j}$. Thus, the routing matrix $P := [p_{i,j}]$ is sub-stochastic. Note that, we also allow feedback of jobs to the same node; i.e., $p_{i,i} \geq 0$. Let $\phi_i^l : \Omega \to \{1, \ldots, K\}$, $\forall i \in K$ and $\forall l \in \mathbb{N}$, be a measurable function such that $\phi_i^l = j$ implies that the $l$th job at node $i$ will be routed to node $j$ and $\mathbb{E}[\{\phi_i^l = j\}] = p_{i,j}$. Define the random vector $R_l(m) := \sum_{i=1}^{m} e_{\phi_i^l}$, where $e_i$ is the $i$th $K$-dimensional unit vector and the $k$th component of $R_l(m)$, denoted $R_{l,k}(m)$, represents the number of departures from node $l$ to node $k$ out of $m$ departures from that node. Then, $R(m) := (R_1(m), \ldots, R_K(m))$ is a $K \times K$ matrix whose columns are the routing vectors from the nodes in the network.

**Assumption 2** The stationary routing process $\{R(m), \ m \geq 1\}$ satisfies the following functional limits:

(i) $n^{-1}R(ne) \to Pe$ in $(C^{K \times K}, U)$ a.s. as $n \to \infty$, where $e : [0, \infty) \to [0, \infty)$ is the identity function, and

(ii) $\hat{R}_n := n^{-1/2}(R(ne) - nPe) \Rightarrow \tilde{R}$, in $(C^{K \times K}, U)$ as $n \to \infty$, where $\tilde{R} = [W_{i,j}]$ and $W_{i,j}$ are independent Brownian motion processes with mean zero and diffusion coefficient $p_{i,j}(1 - p_{i,j})$.

As a direct consequence of Assumption 2 we have the following corollary, which will prove useful in our analysis of the network state process in the next section.

**Corollary 1** The routing process $R$ also satisfies the following fCLT:

$$\hat{R}_n^T1 \Rightarrow \tilde{R}^T1 = \tilde{W} \text{ in } (C^K, U) \text{ as } n \to \infty,$$

where $1 = (1, \ldots, 1)$ is a $K$-dimensional vector of one’s and

$$\tilde{W} = \left(\sum_{k=1}^{K} W_{1,k}, \ldots, \sum_{k=1}^{K} W_{K,k}\right)$$

is a $K$-dimensional Brownian motion with mean zero and diagonal covariance matrix with entries

$$\left(\sum_{k=1}^{K} p_{1,k}(1 - p_{1,k}), \ldots, \sum_{k=1}^{K} p_{K,k}(1 - p_{K,k})\right).$$
Finally, we claim the following joint convergence result that summarizes and generalizes the convergence results in the afore-mentioned theorems.

**Proposition 1** Assume that for each \( n \geq 1 \), \( A_n, S_n, R(n) \) are mutually independent. Then,

(i) \( n^{-1}(A_n, S_n, R(ne)) \to (F, M, P') \) in \((C^J \times C^K \times C^K, U)\) a.s. as \( n \to \infty \), and

(ii) \((\hat{A}_n, \hat{S}_n, \hat{R}_n) \Rightarrow (W^0 \circ F, W \circ M, \hat{R}) \) in \((C^J \times C^K \times C^K, U)\) as \( n \to \infty \).

The joint convergence follows from the assumed independence of the pre-limit random variables, and is straightforward to establish under the uniform convergence criterion.

**4. Functional Strong Law of Large Numbers.** Let \( Q_{n,k}(t) = E_{n,k}(t) - D_{n,k}(t) \) be the queue length sample path at node \( k \), where \( E_{n,k}(t) \) is the total number of jobs arriving at node \( k \) in the interval \([0, t]\) and \( D_{n,k} \) is the cumulative departure process. We assume that the server is non-idling implying that \( D_{n,k}(t) = S_{n,k}(B_{n,k}(t)) \), where \( B_{n,k}(t) := \int_0^t \mathbf{1}_{\{Q_{n,k}(s) > 0\}} ds \) is the total busy time of the server. Therefore, the queue length process is

\[
Q_{n,k}(t) := A_{n,k}(t) + \sum_{l=1}^K R_{l}^k(S_{n,l}(B_{n,l}(t))) - S_{n,k}(B_{n,k}(t)).
\]

The \( K \)-dimensional multivariate stochastic process \( Q_n := (Q_{n,1}, \ldots, Q_{n,K}) \in \mathcal{D}^K \) represents the network state. Our first result establishes a fluid limit approximation to a rescaled version of \( Q_n \) by establishing a functional strong law of large number result as the exogenous arrival population size \( n \) scales to infinity. Consider the queue length in the \( k \)th node, \( Q_k \). Centering each term on the right hand side by the corresponding fluid limits (and subtracting
those terms), and introducing the term \( \int_0^t \mu_{n,k}(s) \, ds \), we obtain \( n^{-1} Q_{n,k}(t) \)

\[
\begin{align*}
&= \left( \frac{1}{n} A_{n,k}(t) - F_k(t) \right) + \left( \frac{1}{n} \sum_{l=1}^K \left[ R_{l}^k (S_{n,l}(B_{n,l}(t))) - p_{l,k} S_{n,l}(B_{n,l}(t)) \right] \right) \\
&\quad - \left( \frac{S_{n,k}(B_{n,k}(t))}{n} - \int_0^{B_{n,k}(t)} \mu_{n,k}(s) \, ds \right) \\
&\quad + \left( F_k(t) - \int_0^{B_{n,k}(t)} \mu_{n,k}(s) \, ds + \frac{1}{n} \sum_{l=1}^K p_{l,k} S_{n,l}(B_{n,l}(t)) \right) \\
&\quad = \left( \frac{1}{n} A_{n,k}(t) - F_k(t) \right) + \left( \frac{1}{n} \sum_{l=1}^K \left[ R_{l}^k (S_{n,l}(B_{n,l}(t))) - p_{l,k} S_{n,l}(B_{n,l}(t)) \right] \right) \\
&\quad - \left( \frac{S_{n,k}(B_{n,k}(t))}{n} - \int_0^{B_{n,k}(t)} \mu_{n,k}(s) \, ds \right) \\
&\quad + \left( F_k(t) - \int_0^t \mu_{n,k}(s) \, ds \right) + (1 - p_{k,k}) \int_{B_{n,k}(t)}^t \mu_{n,k}(s) \, ds \\
&\quad + \left( \frac{1}{n} \sum_{l=1}^K p_{l,k} \left[ S_{n,l}(B_{n,l}(t)) - n \int_0^{B_{n,l}(t)} \mu_{n,l}(s) \, ds \right] \right) \\
&\quad + \sum_{l=1}^K p_{l,k} \left( \int_0^t \mu_{n,l}(s) \, ds \right) - \sum_{l \neq k} p_{l,k} \int_{B_{n,l}(t)}^t \mu_{n,l}(s) \, ds.
\end{align*}
\]

(6)
the sum of two processes, \( \bar{X}_{n,k} \) and \( \bar{Y}_{n,k} \), where

\[
\bar{X}_{n,k}(t) = \left( \frac{1}{n} A_{n,k}(t) - F_k(t) \right)
+ \left( \frac{1}{n} \sum_{l=1}^{K} \left[ R_l^k(S_{n,l}(B_{n,l}(t))) - p_{l,k} S_{n,l}(B_{n,l}(t)) \right] \right)
- \frac{S_{n,k}(B_{n,k}(t))}{n} - \int_0^{B_{n,k}(t)} \mu_{n,k}(s)ds
\]

(7)

and

\[
\bar{Y}_{n,k}(t) = (1 - p_{k,k}) \int_{B_{n,k}(t)}^{t} \mu_{n,k}(s)ds - \sum_{l \neq k} p_{l,k} \int_{B_{n,l}(t)}^{t} \mu_{n,l}(s)ds.
\]

(8)

While this expression appears formidable, the analysis is simplified significantly by the fact that \( \bar{Q}_n := n^{-1}(Q_{n,1}, \ldots, Q_{n,K}) \) and \( \bar{Y}_n := (\bar{Y}_{n,1}, \ldots, \bar{Y}_{n,K}) \) are solutions to the \( K \)-dimensional Skorokhod/oblique reflection problem. First, we recall the definition of the oblique reflection problem.

**Theorem 2** [Oblique Reflection Problem] Let \( \mathbf{V} \) be a \( K \times K \) \( M \)-matrix\(^1\), also known as the reflection matrix. Then, for every \( x \in \mathcal{D}_0^K := \{ x \in \mathcal{D}^K : x(0) \geq 0 \} \), there exists a unique tuple of functions \((y, z)\) in \( \mathcal{D}^K \times \mathcal{D}^K \) satisfying

\[
\begin{align*}
    z &= x + \mathbf{V}y \geq 0, \\
    dy &\geq 0 \text{ and } y(0) = 0, \\
    z_jdy_j &= 0, \quad j = 1, \ldots, K.
\end{align*}
\]

The process \((z, y) := (\Phi(x), \Psi(x))\) is the so-called oblique reflection map, where \( \Phi(x) = x + \mathbf{V}\Psi(x) \).

Note that, in general, if \( \mathbf{G} \) is a nonnegative \( M \)-matrix then so is \( \mathbf{V} = \mathbf{I} - \mathbf{G}^T \) (Lemma 7.1 of Chen and Yao (2001)). The following lemma shows that the queue length satisfies the Oblique Reflection Mapping.

**Lemma 1** Consider \( \mathbf{X}_n(t) = (\mathbf{X}_{n,1}(t), \ldots, \mathbf{X}_{n,K}(t)) \in \mathcal{D}_0^K \), where \( \mathbf{X}_{n,k}(t) \) \( k \in \{1, \ldots, K\} \) is defined in (7), \( \mathbf{Q}_n \in \mathcal{D}^K \) and \( \mathbf{Y}_n \in \mathcal{D}_0^K \). Then, \((\mathbf{Q}_n, \mathbf{Y}_n) = (\Phi(\mathbf{X}_n), \Psi(\mathbf{X}_n))\), with reflection matrix \( \mathbf{V} = \mathbf{I} - \mathbf{P}^T \).

\(^1\)An \( M \)-matrix is a square matrix with spectral radius less than one.
Next, we establish a functional strong law of large numbers result for (7), which will subsequently be used in Theorem 3 for the queue length approximation.

**Lemma 2** The fluid-scaled netput process $\bar{X}_n$ converges to a deterministic limit as $n \to \infty$: 
$$\bar{X}_n(t) \to \bar{X}(t) := (\bar{X}_1(t), \ldots, \bar{X}_K(t)) \text{ in } (C^K, U) \text{ a.s.,}$$

where,

$$\bar{X}_k(t) = F_k(t) - \int_0^t \mu_k(s) ds + \sum_{l=1}^{K} p_{l,k} \int_0^t \mu_l(s) ds. \quad (10)$$

We can now establish the functional strong law of large numbers limit for the queue length process. The proof essentially follows from the continuity of the oblique reflection map $(\Phi(\cdot), \Psi(\cdot))$.

**Theorem 3** Let $\bar{X}_n(t)$ and $\bar{X}(t)$ be as defined in (7) and (10) respectively. Then, $(Q_n(t), Y_n(t))$ satisfy Theorem 2 and, as $n \to \infty$,

$$(Q_n(t), Y_n(t)) \to (\Phi(\bar{X}(t)), \Psi(\bar{X}(t))) \text{ in } (C^K \times C^K) \text{ a.s. } \forall t \in [0, \infty).$$

**Proof:** It follows by Lemma 1 that $(Q_n(t), Y_n(t))$ satisfy the oblique reflection mapping theorem. Therefore, $(Q_n(t), Y_n(t)) \equiv (\Phi(\bar{X}_n(t)), \Psi(\bar{X}_n(t)))$. Now, the reflection regulator map, $\Psi(\cdot)$, is Lipschitz continuous under the uniform metric (Theorem 7.2, Chen and Yao (2001)). By the Continuous Mapping Theorem and Lemma 2 it follows that,

$$(\Phi(\bar{X}_n(t)), \Psi(\bar{X}_n(t))) \to (\Phi(\bar{X}(t)), \Psi(\bar{X}(t))) \text{ u.o.c. a.s. as } n \to \infty, \forall t \in [0, \infty).$$

**Proof:** [Proof of Lemma 1] First, by definition we have $Q_n = \bar{X}_n + (I - P^T)Y_n$. Note that $P$ is a non-negative (sub-stochastic) matrix with spectral radius less than unity and, therefore, an $M$-matrix, implying that $I - P^T$ is also an $M$-matrix. Once again by definition $Q_{n,k}$ and $Y_{n,k}$ satisfy the conditions in (9) for all $k \in K$. Thus, Theorem 2 is satisfied and the lemma is proved.

**Proof:** [Proof of Lemma 2] The result follows by an application of part (i) of Proposition 1 to (7). Noting that $B_{n,k}(t) \leq t$, the random time change
Theorem (Theorem 5.5, Chen and Yao (2001)) and Assumption 1(i) together imply that,
\[
\frac{1}{n} S_{n,k}(B_{n,k}(t)) - \int_0^{B_{n,k}(t)} \mu_{n,k}(s)ds \to 0 \text{ in } (C, U) \text{ a.s. as } n \to \infty \forall t \in [0, \infty).
\]
Similarly, applying the random time change theorem along with Assumption 2(i) and Assumption 1(i) we obtain
\[
\frac{1}{n} \left( R^k_t(S_{n,k}(B_{n,k}(t))) - p_{t,k}S_{n,k}(B_{n,k}(t)) \right) \to 0 \text{ in } (C, U)
\]
a.s. as \( n \to \infty \forall t \in [0, \infty) \). Applying these results to (7) it follows that \( \bar{X}_{n,k}(t) \to \bar{X}_k(t) \) in \( (C, U) \) a.s. as \( n \to \infty \). The joint convergence follows automatically from these results and Proposition 1.

Note that neither Theorem 2 nor Theorem 3 provide an explicit functional form for the reflection regulator \( \Psi(\cdot) \). It can be shown (see (Chen and Yao 2001, Chapter 7)) that the regulator map is the unique fixed point, \( y^* \in D^K \), of the map \( \pi(x, y)(t) := \sup_{0 \leq s \leq t} [-x(s) + Gy(s)]^+ \forall t \in [0, \infty) \), where \( G \) is an \( M \)-matrix. Extracting a closed form expression for \( y^* \) is not straightforward, barring a few special cases. The following corollary shows that the reflection map and fluid limit of the queue length process for a parallel node queueing network is particularly simple and an obvious generalization of that of a single queue.

**Corollary 2** Consider a \( K \)-node parallel queueing network. The fluid limit to the queue length and cumulative idleness processes are
\[
(Q, Y) = (\Phi(\bar{X}, \Psi(\bar{X}))) \in C^K \times C^K,
\]
where \( \bar{X} = (X_1, \ldots, X_K) \), \( \Psi(\bar{X}(t)) = \sup_{0 \leq s \leq t} [-\bar{X}(s)]^+ \) and \( \Phi(\bar{X}) = \bar{X} + \Psi(\bar{X}) \).

**Proof:** Note that for a parallel queueing network \( P = 0 \). Therefore, the fixed point of the map \( \pi(\cdot, \cdot) \) is simply \( \sup_{0 \leq s \leq t} [-\bar{X}(s)]^+ \). It follows that the regulator map of the fluid scaled queue length process is \( \Psi(\bar{X}_n(t)) = \sup_{0 \leq s \leq t} [-\bar{X}_n(s)]^+ \). It follows by Theorem 3 that
\[
\Psi(\bar{X}_n(t)) \to \sup_{0 \leq s \leq t} [-\bar{X}(s)]^+
\]
and
\[
\Phi(\bar{X}_n(t)) \to \bar{X}(t) + \Psi(\bar{X}(t)) \text{ in } (C^K, U) \text{ a.s.}
\]
as $n \to \infty$.

A slightly more complicated example would be a series queueing network. Corollary 3 establishes the fluid limit to the network state of a two queue tandem network, when a large, but finite, number $n$ of users arrive at queue 1 over a finite time horizon $[-T_0, T]$. This result can be rather straightforwardly extended to a network of more than two queues. Let $P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, be the matrix of Markov routing probabilities.

**Corollary 3** Consider a tandem queueing network and recall that $V = I - PT$. Let $F = F_1$ be the arrival epoch distribution with support $[-T_0, T]$ where $T_0, T > 0$, and assume that $\mu_1$ and $\mu_2$ are the fixed service rates. Then, the (joint) fluid limit to the queue length and cumulative idleness processes is $(\bar{Q}, \bar{Y}) = \Phi(\bar{X}), \Psi(\bar{X})) \in \mathcal{C}^K \times \mathcal{C}^K$, where

\[
\bar{X} := (X_1, X_2) = ((F_1 - \mu_1 e), (\mu_1 - \mu_2)e),
\]

and

\[
\Psi(\bar{X}) = (Y_1, Y_2)
\]

with $Y_1(t) = \sup_{0 \leq s \leq t} (-X_1(s))_+$ and $Y_2(t) = \sup_{0 \leq s \leq t} (-X_2(s) + Y_1(s))_+ = \sup_{0 \leq s \leq t} [-X_2(s) + \sup_{0 \leq r \leq s} (-X_1(r))_+]_+$, and $\Phi(\bar{X})(t) = \bar{X} + V \Psi(\bar{X}) = (X_1 + Y_1, X_2 + Y_2 - Y_1)$.

The proof is straightforward by substitution and we omit it. Note that the queue length fluid limit to the downstream queue appears quite complicated: $\bar{Q}_2 = X_2 + Y_2 - Y_1$ where $Y_2(t) = \sup_{0 \leq s \leq t} (-X_2(s) + Y_1(s))_+$. By substituting in the expression for $X_2$ we have

\[
\bar{Q}_2 = (\mu_1 - \mu_2)e + F_1 - F_1 - Y_1 + Y_2 = (F_1 - \bar{Q}_1 - \mu_2 e) + Y_2.
\]

Note that $F_1 - \bar{Q}_1$ is just the cumulative fluid departure function from the upstream queue, which is precisely the input to the downstream queue.

Next, suppose the service process is stationary such that $\mu_k(t) = \mu_k$ for all $t \geq 0$ and $k \in K$. Then, the busy time process satisfies the following theorem.

**Theorem 4** Let $\bar{B}_n(t) = (B_{n,1}(t), \ldots, B_{n,k}(t))$. Then, as $n \to \infty$,

\[
\bar{B}_n \to e1 - M \Psi(\bar{X}) \text{ in } (\mathcal{C}^K, U) \text{ a.s.,}
\]

where, $M = \text{diag}(1/\mu_1, \ldots, 1/\mu_K)$. 

Proof: By definition $\bar{B}_n(t) = t1 - \bar{I}_n(t)$, where $\bar{I}_n(t) = (I_{n,1}(t), \ldots, I_{n,k}(t))^T$. Recalling the definition of the process $\bar{Y}_n(t)$ it is straightforward to see that $\bar{I}_n(t) = (I - PT)^{-1}\bar{Y}_n(t)$ for all $t \geq 0$. Therefore, $\bar{B}_n(t) = 1 - (I - PT)^{-1}\bar{Y}_n(t)$. Theorem 3 implies that, as $n \to \infty$, $\bar{B}_n(t) \to 1 - \Psi(\bar{X}(t))$ in $(C^K, U)$ a.s. $\forall t \in [0, \infty)$.

The following corollary establishes the fluid busy time process for the parallel queue case. The proof follows that of Corollary 2 and we omit it.

**Corollary 4** Consider a $K$-node parallel queueing network. Then,

$$\bar{B}_n \to e1 - (I - PT)^{-1} \sup_{0 \leq s \leq t} [-\bar{X}(s)]^+ \text{ in } (C^K, U) \text{ a.s. } n \to \infty.$$  

In the stationary case we considered here, the busyness time-scale is effectively fixed by the service rate through the matrix $M$. On the other hand, if the service processes are non-stationary this time-scale itself is time-varying. Thus, computing the busy time (or equivalently the idle time) process when the service process is non-stationary is complicated. Note that the function $\bar{Y}$ represents the number of “blanks” or the amount of unused capacity in the network at each point in time, providing an indication of whether a particular queue in the network is busy or not.

Note that the population acceleration scale we use in the current analysis ensures that (in the limit) the amount of time each user spends in service is infinitesimally small. The ‘behavior’ of the queue state under population acceleration scaling is akin to the conventional heavy-traffic scaling introduced in Reiman (1984) for stationary single class queueing networks. The corresponding diffusion heavy-traffic scaling identifies the critical time-scale of the stationary queueing network. The population acceleration scaling differs from the conventional heavy-traffic scaling by the fact that the fluid limit process (in general) is non-linear in nature. This implies that queues in the network can enter idle and busy periods, and arriving jobs will only face delays in the latter time intervals. We should anticipate that the critical time-scale of the queue state in the diffusion scale should itself change depending on whether the queue is busy or idle, leading to a non-stationary diffusion approximation. Indeed, this is precisely what is implied by the results in the next section.

5. **Functional Central Limit Theorems.** We now consider the second order refinement to the fluid limit by establishing a functional central limit theorem (FCLT) satisfied by the queue length state process. We show, in particular, that the FCLT is a reflected diffusion, where the diffusion
process \( \hat{X} \) is a function of the multi-dimensional Brownian bridge process as in Definition 1. Unlike the heavy traffic limits for generalized Jackson networks (see (Chen and Yao 2001, Chapter 7) Reiman (1984)), the diffusion is not reflected through the oblique reflection map (see (Chen and Yao 2001, Definition 7.1)). The non-homogeneous traffic and non-stationary service processes induce a time-varying critical time-scale under the population acceleration scaling. We show that this time-varying critical time-scale manifests as a time-varying reflection boundary in transitory queueing networks. To be precise, the reflection regulator for the queue length diffusion is the directional derivative of the oblique reflection of \( \bar{X} \) (from Lemma 2) in the direction of the diffusion limit \( \bar{X} \) to the netput process. A similar result was observed in the case of a single \( \Delta_\text{G}/\text{GI}/\text{1} \) transitory queue in Honnappa et al. (2014).

Recall that \( V \) is a \( K \times K \) matrix and \( P^T = I - V \). Let \( x \in C_0^K \), then, under the hypothesis of Theorem 2, there exists a unique oblique reflection map \((z, y) := (\Phi(x), \Psi(x)) \in C \times C\) such that \( z = x + Vy \), \( y_j \) is non-decreasing and \( y_j \) grows only when \( z_j \) is zero (for all \( j = 1, \ldots, K \)). The directional derivative of the oblique reflection of \( x \) in the direction of the process \( \chi \in C \) is defined as follows (see Mandelbaum and Ramanan (2010) as well):

**Definition 3 (Directional Derivative Reflection Map)** Given \((x, \chi) \in C_0^K \times C^K \) and matrix \( V \), the directional derivative of the oblique reflection map \( \Phi(x) = x + V\Psi(x) \) in the direction of \( \chi \) is the pointwise limit of

\[
\Delta^n_\chi(x) := \sqrt{n} \left( \Phi \left( \frac{\chi}{\sqrt{n}} + x \right) - \Phi(x) \right) \in C \quad n \geq 1
\]
as \( n \to \infty \).

Theorem 1.1 (ii) of Mandelbaum and Ramanan (2010) identifies the limit process, which we state as a lemma for completeness. Here,

**Lemma 3** If \((x, \chi) \in C_0^K \times C^K \) then the directional derivative limit \( \Delta_\chi(x) \) exists and convergence in Definition 3 is uniformly on compact subsets of continuity points of the limit \( \Delta_\chi(x) \). Further, if \((z, y) \) solve the oblique reflection problem for \( x \) then

\[
\Delta_\chi(x) = \chi + V\gamma(x, \chi),
\]

where \( V = I - P^T \) \( \gamma := \gamma(x, \chi) \) lies in \( D^K_{usc} \) and is the unique solution to the system of equations

\[
\gamma^i(t) = \begin{cases} 
  \sup_{s \in \tau^i} [-\chi^i(s) + [P\gamma]^i(s)]_+ & t \in [0, t_u], \\
  \sup_{s \in \tau^i} [-\chi^i(s) + [P\gamma]^i(s)] & t > t_u,
\end{cases}
\]

\( \tau^i \) is the time of the \( i \)-th arrival at the \( i \)-th server, and \( [u]_+ := \max(0, u) \).
for $i = 1, \ldots, K$, where $\nabla_i := \{ s \in [0, t] | z^i(s) = 0 \text{ and } y^i(s) = y^i(t) \}$, and $t^i_u := \inf\{ t \geq 0 : y^i(t) > 0 \}$.

Consider the second order refinement to the netput process,

$$\hat{X}_n := n^{-1/2} (X_n - n\bar{X}) \in \mathcal{D}^K.$$  

Using Proposition 1, and the fact that the limit processes have sample paths in $\mathcal{C}^K$, the following Lemma is straightforward to establish.

**Lemma 4** The diffusion-scaled netput process satisfies,

$$\hat{X}_n \Rightarrow \bar{X} in (\mathcal{C}^K, U) as n \rightarrow \infty,$$

where $\hat{X}_k := W_k^0 \circ F_k - W_k \circ \int_0^t \mu_k(s)ds + 1^T (\hat{R}_k \circ M)$, $\hat{R}_k$ is the $k$th row of the matrix valued process $\hat{R}$ defined in part (ii) of Assumption 2, and $M$ is defined in (3).

The proof of the lemma follows from an application of part (ii) of Proposition 1 and using the fact that the addition operator is a continuous map under the uniform metric. We omit it for brevity.

Now, the diffusion scale queue length process is

$$\hat{Q}_n := n^{-1/2} (Q_n - n\bar{Q}) \in \mathcal{D}^K.$$  

Recall, from Lemma 1, that $\bar{Q}_n = \bar{X}_n + V\Psi(\bar{X}_n)$ and, from Theorem 3, that $\bar{Q} = \bar{X} + V\Psi(\bar{X})$. It follows that

$$\hat{Q}_n = \sqrt{n} (\bar{X}_n + V\Psi(\bar{X}_n) - \bar{X} - V\Psi(\bar{X})).$$

$$= \hat{X}_n + V\sqrt{n} \left( \Psi \left( \frac{\bar{X}_n}{\sqrt{n}} + \bar{X} \right) - \Psi (\bar{X}) \right)$$

$$+ \Psi(\bar{X}_n) - \Psi \left( \frac{\bar{X}_n}{\sqrt{n}} + \bar{X} \right).$$

Our next result shows that $\Delta^n_{\bar{X}_n}(\bar{X})$ is asymptotically equal to $\Delta^n_{\bar{X}}(\bar{X})$.

**Lemma 5** Let $\Delta^n_{\bar{X}_n}(\bar{X})$ and $\Delta^n_{\bar{X}}(\bar{X})$ be defined as in Definition 3. Then,

$$\Delta^n_{\bar{X}_n}(\bar{X}) - \Delta^n_{\bar{X}}(\bar{X}) \rightarrow 0 in (\mathcal{C}^K, U) a.s. as n \rightarrow \infty.$$
Lemma 5 implies it suffices to consider the process

\[
\hat{Q}_n \equiv \Delta \hat{X}(\hat{X}) + V \sqrt{n} \left( \Psi(\hat{X}_n) - \Psi\left( \frac{\hat{X}}{\sqrt{n}} + \hat{X} \right) \right)
\]

(where, by an abuse of notation, we call this process \( \hat{Q}_n \) as well). Now, if we show that

\[
\sqrt{n} \left( \Psi(\hat{X}_n) - \Psi\left( \frac{\hat{X}}{\sqrt{n}} + \hat{X} \right) \right) \to 0 \text{ a.s. as } n \to \infty
\]

then Lemma 3 implies that \( \hat{Q}_n \) converges to the process \( \Delta \hat{X}(\hat{X}) \) pointwise in the large population limit. The following lemma establishes the required result under general conditions.

**Lemma 6** Let \( x_n, x \in D^K \) be stochastic processes that satisfy

\[
\|\sqrt{n}(x_n - x)\| \to \chi \text{ a.s. as } n \to \infty.
\]

Then,

\[
\left\| \sqrt{n} \left( \Psi(x_n) - \Psi\left( \frac{\chi}{\sqrt{n}} + x \right) \right) \right\| \to 0 \text{ a.s. as } n \to \infty,
\]

where \( \chi \in C^K \).

The main result of this section follows as a consequence of these lemma’s.

**Theorem 5** For any fixed \( t \in [0, \infty) \), as \( n \to \infty 
\]

\[
\hat{Q}_n(t) \Rightarrow \hat{Q}(t) = \Delta \hat{X}(\hat{X})(t),
\]

where \( \Delta \hat{X}(\hat{X})(t) = \hat{X}(t) + V \gamma(\hat{X}, \hat{X})(t) \).

**Proof:** First, using the Skorokhod representation theorem Billingsley (1968), it follows from Lemma 4 that there exist versions of the stochastic processes \( \{\hat{X}_n\} \) and \( \hat{X} \), referred to using the same notation, such that

\[
\hat{X}_n \to \hat{X} \text{ in } (C^K, U) \text{ a.s. as } n \to \infty.
\]

It follows that \( \hat{X}_n = \hat{X} + (\sqrt{n})^{-1} \hat{X} + o(1) \) a.s. Lemma 6 implies that

\[
\left\| \sqrt{n} \left( \Psi(\hat{X}_n) - \Psi\left( \frac{\hat{X}}{\sqrt{n}} + \hat{X} \right) \right) \right\| \to 0
\]

a.s. as \( n \to \infty \). Next, using Lemma 5 and Lemma 3, it follows that \( \hat{Q}_n(t) \to \Delta \hat{X}(\hat{X})(t) \) a.s. as \( n \to \infty \) for any fixed \( t \in [0, \infty) \), which in turn implies weak
convergence of the stochastic processes thus proving the desired result.

Proof: [Proof of Lemma 5] First, recall that

$$
\Delta_{X_n}^n (\hat{X}) = \hat{X}_n + V \sqrt{n} \left( \Psi \left( \frac{\hat{X}_n}{\sqrt{n}} + \bar{X} \right) - \Psi(\bar{X}) \right).
$$

By Lemma 4 and the Skorokhod representation theorem Billingsley (1968), it follows that

(15) $\hat{X}_n \to \hat{X}$ in $(C^K, U)$ a.s. as $n \to \infty$. The lemma is proved once we show that

$$
\sqrt{n} \left( \Psi \left( \frac{\hat{X}_n}{\sqrt{n}} + \bar{X} \right) - \Psi \left( \frac{\hat{X}}{\sqrt{n}} + \bar{X} \right) \right) \to 0 \text{ in } (C^K, U) \text{ a.s. as } n \to \infty.
$$

Chen and Whitt Chen and Whitt (1993) show that the oblique reflection map and the reflection regulator are Lipschitz continuous with respect to the uniform metric topology. Recall that $\| \cdot \|$ represents the uniform metric on $C^K$ over the interval $[0, T]$. Then,

$$
\left\| \sqrt{n} \left( \Psi \left( \frac{\hat{X}_n}{\sqrt{n}} + \bar{X} \right) - \Psi \left( \frac{\hat{X}}{\sqrt{n}} + \bar{X} \right) \right) \right\| \leq K \left\| \hat{X}_n - \hat{X} \right\|,
$$

where $K$ is the Lipshitz constant associated with the oblique reflection map. The conclusion follows as a consequence of (15).

Proof: [Proof of Lemma 6] The condition on $x_n, x$ implies that $x_n \overset{a.s.}{=} x + (\sqrt{n})^{-1} \chi + o(\sqrt{n})$. Therefore, it follows that

$$
\left\| \sqrt{n} \left( \Psi(x_n) - \Psi \left( \frac{\chi}{\sqrt{n}} + x \right) \right) \right\| \overset{a.s.}{=} \left\| \sqrt{n} \left( \Psi \left( \frac{\chi}{\sqrt{n}} + x + o(1) \right) - \Psi \left( \frac{\chi}{\sqrt{n}} + x \right) \right) \right\|
\leq K \sqrt{n} \| o(1) \|,
$$

where the last inequality follows from the Lipshitz continuity of the oblique reflection map. The final conclusion follows from the fact that the indeterminate form on the right hand side converges to 0 as $n \to \infty$.  ■
Remarks: We include a short summary of the relevant results in Mandelbaum and Ramanan (2010) that imply that process-level convergence might be near impossible to prove (in general) in a transitory queueing network. Lemma 2 in Honnappa et al. (2014) (an extension of Theorem 3.2 in Mandelbaum and Massey (1995)) proves the process-level diffusion limit result in the $M_1$ topology for a single queue. The fact that the limit process has right- or left-discontinuity points that are ‘unmatched’ by the pre-limit process necessitates that convergence be proved in the $M_1$ topology as opposed to the more natural $J_1$ topology. On the other hand, Mandelbaum and Ramanan (2010) show that it is not possible to prove a process-level convergence result even in the $WM_1$ topology (‘weak’ $M_1$ topology (see Whitt (2001a)), due to the fact that the multidimensional limit process can have discontinuity points that are both right- and left-discontinuous. For completeness, we state the relevant portion of Theorem 1.2 of Mandelbaum and Ramanan (2010) that encapsulates the various necessary conditions for discontinuities in the sample paths of the directional derivative limit process, $\Delta \hat{X}(\bar{X})$. First, given $(z, y)$ as the solution to the oblique reflection problem for $x \in C_0$ define, for each $t \in [0, \infty),\hfill
\begin{align*}
O(t) &:= \{i \in \{1, \ldots, K\} : z^i(t) > 0\}, \\
U(t) &:= \{i \in \{1, \ldots, K\} : z^i(t) = 0, \ \Delta y^i(t+) \neq 0, \ \Delta y^i(t-) \neq 0\}, \\
C(t) &:= \{1, \ldots, K\} \setminus (O(t) \cup U(t)), \\
EO(t) &:= \{i \in C(t) : \exists \delta > 0 \text{ such that } z^i(s) > 0 \ \forall s \in (t - \delta, t)\}, \\
SU(t) &:= \{i \in C(t) : \Delta z^i(t-) = 0, \ \Delta z^i(t+) \neq 0\}.
\end{align*}
\]When $x = \bar{X}$, $O(t)$ is the set of nodes in the network that are overloaded at time $t$, $U(t)$ is the set of underloaded nodes, $C(t)$ the set of critically loaded nodes, $EO(t)$ is the set of critically loaded queues that are at the end of overloading and $SU(t)$ is the set of critically loaded nodes that are at the start of under-loading. Note that the definitions of overloading, under-loading and critical loading conform to the standard notions for $G/G/1$ queues, as noted in Honnappa et al. (2014). Next, we also require the notion of critical and sub-critical chains, as in Definition 1.5 of Mandelbaum and Ramanan (2010):

**Definition 4 (Def. 1.5 Mandelbaum and Ramanan (2010))** Given a $K \times K$ routing matrix $P$ and the oblique reflection map $\Psi$ and $x \in \mathcal{O}^K$ so that $y = \Psi(x)$. Then a sequence $j_0, j_1, \ldots, j_m$ with $j_k \in \{1, \ldots, K\}$ for $k = 0, 1, \ldots, m$ that satisfies $P_{j_{k-1}, j_k} > 0$ for $k = 0, 1, \ldots, m$ is said to be a chain. The chain is said to be a cycle if there exist distinct $k_1, k_2 \in \{0, \ldots, m\}$ such that $j_{k_1} = j_{k_2}$, the chain is said to precede $i$ if $j_0 = i$ and is said to be empty.
at \( t \) if \( y_{jk}(t) = 0 \) for every \( k = 1, \ldots, m \). For \( i = 1, \ldots, K \) and \( t \in [0, \infty) \), we consider the following two types of chains:

1. An empty chain preceding \( i \) is said to be critical at time \( t \) if it is either cyclic or \( j_m \) is at the end of overloading at \( t \).
2. An empty chain preceding \( i \) is said to be sub-critical at time \( t \) if it is either cyclic or \( j_m \) is at the start of overloading at \( t \).

Theorem 1.2 of Mandelbaum and Ramanan (2010) gives necessary conditions so that, in general, the sample paths of the directional derivative can have both a right and left discontinuity at \( t \in [0, \infty) \). Simply put, the structure of the routing matrix \( P \) determines whether we see such a point.

**Proposition 2 (Thm. 1.2 Mandelbaum and Ramanan (2010))** Under the conditions of Definition 4 and given a process \( \chi \in \mathcal{C}^k \), if the directional derivative \( \Delta \chi(x) \) has both a right and a left discontinuity at \( t \in [0, \infty) \) then one of the following conditions must hold at time \( t \):

a) \( i \) is at the end of overloading, and a sub-critical chain precedes \( i \), in which case

\[
\Delta \chi(x)^i(t^-) < \Delta \chi(x)^i(t)^i = 0 < \Delta \chi(x)^i(t^+),
\]

b) \( i \) is at the start of under-loading and a critical chain precedes \( i \), in which case

\[
\Delta \chi(x)^i(t^-) > \Delta \chi(x)^i(t) > \Delta \chi(x)^i(t^+) = 0,
\]

c) \( i \) is not underloaded and there exist both critical and sub-critical chains preceding \( i \); if, in addition, \( i \) is overloaded then the discontinuity is a separated discontinuity of the form

\[
\Delta \chi(x)^i(t) < \min\{\Delta \chi(x)^i(t^-), \Delta \chi(x)^i(t^+)}\}.
\]

Note that the sample paths of \( \Delta \chi(\mathbf{X}) \) lie in \( D_{\text{lim}}^K \) and establishing \( M_1 \) convergence in this space is non-trivial. Recall that the standard description of \( M_1 \) convergence is through the graphs of the functions - which can be described via linear interpolations in \( D \) and \( D_{\text{lim}}^K \). However, in \( D_{\text{lim}}^K \) no such simple description exists (see Chapter 12 of Whitt (2001a) and Chapter 6, 8 of Whitt (2001b) for further details on these issues).

Given the inherent difficulty in establishing a general process-level result, we first focus on a two queue tandem network, where the arrival time distribution is uniform on the interval \([-T_0, T]\) and \(T_0, T > 0\) where the difficulties will become apparent.
Theorem 6 Consider a tandem queueing network with $P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and
\[
R = I - PT = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.
\]
Assume that $F = F_1$ is uniform over $[-T_0, T]$, and service rate at node 1 is $\mu_1$ and at node 2 $\mu_2$. Then, $\hat{Q}_n \Rightarrow \hat{Q} := \Delta_X(\bar{X})$ in $(\mathcal{D}_{t,r}^2, SM_1)$ as $n \to \infty$, where $\bar{X} = (\hat{X}_1, \hat{X}_2)$ with $\hat{X}_1 = W_1^0 \circ F_1 - W_1 \circ M_k$, $\hat{X}_2 = W_1 \circ M_k - W_2 \circ M_2$ and $M_k(\cdot) = \int_0^t \mu_k(s)ds$ for $k \in \{1, 2\}$, $\bar{X} = ((F_1 - \mu_1), (\mu_1 - \mu_2)e)^T$ and $e : \mathbb{R} \to \mathbb{R}$ is the identity map.

Proof: Recall that $F(t) = \frac{t+T_0}{T_0}$ for all $t \in [-T_0, T]$. We consider three subcases and establish the weak convergence result for each of them separately.

(i) Let $\mu_1 < \mu_2$. Then,
\[
\hat{Q}_1(t) = \begin{cases} (F(t) - \mu_1 t 1_{t \geq 0}) & \forall t \in [-T_0, \tau_1], \\
0 & \forall t \in [\tau_1, \infty), \end{cases}
\]
and $\hat{Q}_2(t) = 0 \ \forall t \geq 0$, where $\tau_1 := \inf\{t > 0 | F(t) = \mu_1 t\}$. These follow as a consequence of Corollary 3, and noting that $\bar{X} = (F(t) - \mu_1, (\mu_1 - \mu_2)e)$. Thus, we have

\[
\nabla_t^1 := \begin{cases} \{-T_0\} & \forall t \in [0, \tau_1), \\
\{-T_0, \tau_1\} & t = \tau_1, \\
\{t\} & \forall t > \tau_1, \end{cases}
\]

\[
\nabla_t^2 := \{t\} \ \forall t \in [0, \infty).
\]

Thus, node 1 is in $\mathcal{O}(t)$ for all $t \in [-T_0, \tau_1)$, $\mathcal{C}(t)$ for $t = \tau_1$ and in $\mathcal{U}(t)$ for $t > \tau_1$, and node 2 is in $\mathcal{U}(t)$ for all $t$.

The limit process $\hat{Q}$ has a discontinuity only in the first component at $\hat{Q}_1(\tau_1) = \bar{X}^1(\tau_1) + \max\{0, -\bar{X}^1(\tau_1)\}$. Note that $\hat{Q}_1(\tau_1-1) = \bar{X}^1(\tau_1)$ and $\hat{Q}_1(\tau_1+1) = 0$, implying that $\hat{Q}_1$ has either a right or left discontinuity at $\tau_1$. If $\bar{X}^1(\tau_1) \geq 0$ then $\hat{Q}_1(\tau_1) = \bar{X}^1(\tau_1) = \hat{Q}_1(\tau_1-) = 0$ and has a right discontinuity. Else, if $\bar{X}^1(\tau_1) < 0$ then $\hat{Q}_1(\tau_1) = 0 = \hat{Q}_1(\tau_1+) > \hat{Q}_1(\tau_1-)$ and has a left discontinuity. Thus, the limit process $\hat{Q}$ has sample paths in $\mathcal{D}_{t,r}^2$. The proof of convergence for $\hat{Q}_n = (\hat{Q}_{n,1}, \hat{Q}_{n,2})$ in this case is simple. First, Theorem 2 of Honnappa et al. (2014) shows that $\hat{Q}_{n,1} \Rightarrow \hat{Q}_1 := \bar{X}^1 + \sup_{s \in \mathcal{V}} (-\hat{X}(s))$ in $(\mathcal{D}_{t,r}, M_1)$ as $n \to \infty$, and $\hat{Q}_{n,2} \Rightarrow 0$ in $(\mathcal{D}_{t,r}, M_1)$. Recall that $\text{Disc}(\hat{Q}_1)$ and $\text{Disc}(\hat{Q}_2)$ are the (respective) sets of discontinuity point, and it is obvious that $\text{Disc}(\hat{Q}_1) \cap \text{Disc}(\hat{Q}_2) = \phi$. Therefore, by (Whitt 2001b, Corollary 6.7.), $\hat{Q}_{n,1} + \hat{Q}_{n,2} \Rightarrow \hat{Q}_1$ in $(\mathcal{D}_{t,r}(\mathbb{R}), M_1)$ as $n \to \infty$. Consequently, by (Whitt 2001b, Theorem 6.7.2), it follows that $\hat{Q}_n \Rightarrow \hat{Q} := (\hat{Q}_1, 0)^T$ in $(\mathcal{D}_{t,r}^2, SM_1)$ as $n \to \infty$. 

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(ii) Let $\mu_1 > \mu_2$. Then, $\tilde{Q}_1$ and $\nabla^1_t$ follow (16) and (17) (resp.). $\tilde{Q}_2$ on the other hand, is more complex now:

$$\tilde{Q}_2(t) = \begin{cases} (\mu_1 - m_2)t & \forall t \in [0, \tau_1], \\ (F_1(t) - \mu_2) & \forall t \in [\tau_1, \tau_2], \\ 0 & \forall t > \tau_2, \end{cases}$$

where $\tau_2 := \inf\{t > \tau_1 : F_1(t) = \mu_2 t\}$ (note that $\tau_2 > \tau_2$ since $\mu_1 > \mu_2$). It follows that

$$\nabla^2_t = \begin{cases} \{0\} & \forall t \in [0, \tau_2), \\ \{0, \tau_2\} & t = \tau_2, \\ \{t\} & \forall t > \tau_2. \end{cases}$$

It follows that node 2 is in $O(t)$ for all $t \in [0, \tau_2)$, $C(t)$ at $t = \tau_2$ and $U(t)$ for $t > \tau_2$.

The diffusion limit $\hat{Q} := (\hat{Q}_1, \hat{Q}_2)$ has discontinuities in both components. For node 1, if $\hat{X}^1(\tau_1) > 0$ then $\hat{Q}_1(\tau_1)$ has a right discontinuity, while $\hat{X}^1(\tau_1) < 0$ then $\hat{Q}_1(\tau_1)$ has a left discontinuity. Similarly, if $\hat{X}^2(\tau_2) \geq 0$ then $\hat{Q}_2(\tau_2)$ has a right discontinuity, and if $\hat{X}^2(\tau_2) < 0$ it has a left discontinuity. It follows that $\hat{Q}$ has sample paths in $D^2_{l,r}$. Furthermore, it is clear that $Disc(\hat{Q}_1) \cap Disc(\hat{Q}_2) = \emptyset$. Therefore, the weak convergence result follows by the same reasoning as in part (i).

(iii) Assume $\mu_1 = \mu_2$. Once again, $\hat{Q}_1$ and $\nabla^1_t$ follow (16) and (17) (resp.). On the other hand, for node 2 $\hat{Q}_2 = 0$, but unlike case (i), the queue is empty but the server operates at full capacity till $\tau_1$, and then enters underload. Thus,

$$\nabla^2_t = \begin{cases} [0, \tau_1] & \forall t \in [0, \tau_1], \\ \{t\} & \forall t > \tau_1. \end{cases}$$

It is clear that node 2 switches from $C(t)$ in $[0, \tau_1]$ to $U(t)$ for $t > \tau_1$. Furthermore, at $\tau_1$ itself, the node is in $SU(t)$ (the regulator is flat to the left of $\tau_1$ and increasing to the right).

The diffusion limit, once again, has discontinuities in both components. However, it is clear that $Disc(\hat{Q}_1) = Disc(\hat{Q}_2) = \{\tau_1\}$. For any $T > -T_0$, it is straightforward to see that $(\hat{Q}_1(t) - \hat{Q}_1(t^-))(\hat{Q}_2(t) - \hat{Q}_2(t^-)) \geq 0$ for all $-T_0 \leq t \leq T$: clearly, for any $t < \tau_1$, $\hat{Q}_i, i = 1, 2$ are both continuous. On the other hand, at $\tau_1$, $\hat{Q}_1(\tau_1) \geq \hat{Q}_1(\tau_1^-)$ and $\hat{Q}_2(\tau_1) = \hat{Q}_2(\tau_1^-)$. Finally, for any $t > \tau_1$, $\hat{Q}_1(\tau_1) = \hat{Q}_1(\tau_1^-)$ and $\hat{Q}_2(\tau_1) = \hat{Q}_2(\tau_1^-)$. Now, by Theorem 6.7.3 of Whitt (2001b), it follows that $\hat{Q}_{n,1} + \hat{Q}_{n,2} \Rightarrow \hat{Q}_1 + \hat{Q}_2$ in $(D_{l,r}(\mathbb{R}), M_1)$ as $n \to \infty$. Then, by Theorem 6.7.2 of Whitt (2001b), $\hat{Q}_n \Rightarrow \hat{Q}$ in $(D^2_{l,r}, SM_1)$
as $n \to \infty$. This concludes the proof.

Theorem 6 shows that in the case of a tandem network, with uniform arrival time distribution, the weak convergence result can be established in the space $D^2_{l,r}$ and in the $SM_1$ topology. In fact this result is true, if $F_1$ is unimodal such that node 1 is overloaded in the initial phase (i.e., in the interval $[-T_0, \tau_1]$, with $T_0 \geq 0$ now). We capture this fact in the following corollary. Without loss of generality, we will assume that $T_0 = 0$.

**Corollary 5** Let $F_1$ be a unimodal distribution function with finite support $[0, T]$, and consider a tandem queue as defined in Theorem 6. Then, $\hat{Q}_n \Rightarrow \hat{Q} := \Delta_{\bar{X}}(\bar{X})$ in $(D^2_{l,r}, SM_1)$ as $n \to \infty$, where

$$
\bar{X} = \left( (F_1 - \mu_1 e, (\mu_1 - \mu_2)e) \right)^T, \\
\hat{X} := \left( W_1^0 \circ F_1 - \sigma_1 \mu_1^{3/2} W_1, (\sigma_1 \mu_1^{3/2} W_1 - \sigma_2 \mu_2^{3/2} W_2) \right)^T
$$

The proof follows that of Theorem 6 and is omitted. Note that the compact support assumption is required, due to the fact that we prove weak convergence over compact intervals of time (see Section 7.2 of Honnappa et al. (2014) for a discussion on this point).

**6. High-intensity Analysis of Tandem Networks.** We illustrate the utility of the afore-developed approximations in bottleneck analysis of transitory tandem networks. Almost all of the analysis in the literature has focused on the characterization and detection of bottlenecks in stationary queueing networks. Of particular relevance to our results in this paper is the heavy-traffic bottleneck phenomenon identified in Suresh and Whitt (1990), Whitt (2001a). To recall, the heavy-traffic bottleneck phenomenon corresponds to the state space collapse that is observed when the traffic intensity at a single queue approaches 1, while the traffic intensity at other queues remains below 1. In this case, the well known heavy-traffic approximations in Iglehart and Whitt (1970), Reiman (1984), Chen and Mandelbaum (1991a) imply that the network workload process will collapse to a single dimensional process determined by the bottleneck node. In other words, the non-bottleneck nodes behave like ‘switches’ where the service time is effectively zero. In general, exact bottleneck analysis is very difficult (if not impossible), and several approximations have been proposed in the literature, particularly the parametric-decomposition approach Whitt (1983), Buzacott and Shanthikumar (1992), the stationary-interval method Whitt (1984), and Reiman’s individual (IBD) and sequential bottleneck decomposition (SBD) algorithms Reiman (1990). Nonetheless, the natural metric to
use to study bottlenecks would be the waiting time at each node. The fluid and diffusion workload approximations can be established as a corollary to Theorem 3 and Theorem 5, assuming that the service process is stationary:

**Corollary 6 (Workload Approximation)** Recall that \( M \) is a diagonal matrix defined as

\[
M := \text{diag}(1/\mu_1, \ldots, 1/\mu_1, 1/\mu_K).
\]

Then the fluid workload process \( \bar{Z} = M\bar{Q} \), and for each \( t \in [0, \infty) \) the diffusion workload process is \( \hat{Z}(t) = M\hat{Q}(t) \).

The proof of this corollary follows by analogous arguments to (Honnappa et al. 2014, Proposition 4), and we omit it.

Bottleneck analysis, however, has largely been ignored in transitory networks in particular. The key difference (and difficulty) in the transitory setting is that, for general arrival epoch distributions \( F \) it is possible that the number of bottleneck queues can change with time. The situation is considerably simpler when \( F \) is uniform, however, and we focus on this case first to illustrate the main ideas. We commence with a definition of a bottleneck queue in a transitory network, in the large population limit.

**Definition 5 (Transitory Bottleneck Queue)** Queue \( k \in K \) in the transitory queueing network is a bottleneck at time \( t \) if and only if the diffusion workload process satisfies

\[
|\hat{Z}_k(t)| > 0.
\]

Note that we choose to use a sample path definition of the bottleneck node owing to the fact that the temporal stochastic variations can produce differing numbers of bottlenecks, even compared with the average/fluid variation. This definition is natural to consider in job-shop type production systems and complements the definitions in (Lawrence and Buss 1994, P. 23) that classifies bottlenecks in terms of short, intermediate and long time horizons.

**Example 1 (Tandem network with uniform traffic)** Consider a series network of \( K \) queues. Let the service rate at queues 1 through \( K-1 \) be \( \mu_1 \) and \( \mu_K \) at queue \( K \). Without loss of generality we assume that \( \mu_K < 1 \leq \mu_1 \). Assume that the traffic arrival epochs are randomly scattered per a uniform distribution function, over the interval \([0,1]\). Then, in the fluid population acceleration limit as observed in Theorem 3, it can be observed that each of the queues 1, \ldots, \( K-1 \) behave like instantaneous switches and \( O(n) \) fluid accumulates at the final queue. Extending the analysis in Corollary 3 to a \( K \)-node tandem network it is straightforward to compute that

\[
\bar{X} = (\bar{X}_1, \ldots, \bar{X}_{K-1}, \bar{X}_K), \quad \text{where } \bar{X}_1(t) = F_1(t) - \mu_1 t \leq 0 \text{ and } \bar{X}_k(t) = 0 \text{ for } k \geq 2.
\]
all $k = 2, \ldots, K - 1$, and $\bar{X}_K(t) = (\mu_1 - \mu_K)t > 0$. Since the routing matrix is
\[
P = \begin{pmatrix}
0 & 0 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots \\
0 & 0 & \ldots & 1 & 0
\end{pmatrix}
\]
a simple (if tedious) calculation shows that
\[
\bar{Q}(t) = \begin{cases}
(0, \ldots, 0, (\mu_1 - \mu_K)t) & t \in [0, 1/\mu_K], \\
(0, \ldots, 0) & t > 1/\mu_K.
\end{cases}
\]
Now, it follows that, in the case of the tandem queueing network under consideration
\[
\bar{Z}(t) = \begin{cases}
(0, \ldots, 0, (\mu_1 \mu_K^{-1} - 1)t) & t \in [0, 1/\mu_K], \\
(0, \ldots, 0) & t > 1/\mu_K.
\end{cases}
\]
Thus, in the fluid limit, we find that the tandem queueing network “collapses” to a single queue in the fluid limit (this is an example of a state space collapse as defined in Reiman (1984)), and the sojourn time through the network, in the fluid scale and large population limit, is determined entirely by the delay at node $K$.

On the other hand, as the diffusion limit in Theorem 6 shows, there is non-zero variability in the queue length at each node in the network. Indeed, Theorem 6 and Corollary 6 imply that the diffusion limit of the workload vector in a tandem network is
\[
\hat{Z} = \mathbf{M}\Delta X(\bar{X}),
\]
where
\[
\hat{X}(t) = \left( (W_0^0(t) - \sigma_1^{3/2} W_1(t)), (\sigma_1 \mu_1^{3/2} W_1(t) - \sigma_1 \mu_1^{3/2} W_2(t)), \ldots, \\
\sigma_1 \mu_1^{3/2} W_{K-1}(t) - \sigma_K \mu_K^{3/2} W_K(t) \right).
\]
Now, if $\mu_1 > 1$, then $\bar{Z}_k \overset{D}{=} 0$ for $k = 1, \ldots, K - 1$ and $\bar{Z}_K(t) \overset{D}{=} \mu_K^{-1}(\dot{X}_K(t) + \sup_{0 \leq s \leq t}(-\dot{X}_K(s)))$ with $\dot{X}_K = \sigma_1 \mu_1^{3/2} W_{K-1} - \sigma_K \mu_K^{3/2} W_K$. That is, in the population acceleration scaling, the distribution of the sojourn time through the network is asymptotically equal to the delay distribution of the last queue.

On the other hand, if $\mu_1 = 1$, then $\bar{Z}_1 = \mu_1^{-1}(\dot{X}_1(t) + \sup_{0 \leq s \leq t}(-\dot{X}_1(s)))$
with \( \hat{X}_1 = W_1^0 - \sigma \mu_1^{3/2}W_1 \), \( Z_k = D_k \) for \( k = 2, \ldots, K - 1 \) and

\[
\hat{Z}_k = \begin{cases} 
\mu_K^{-1}(\sigma_1 \mu_1^{3/2}W_{K-1} - \sigma_K \mu_K^{3/2}W_K) & \forall t \in [0, 1] \\
\mu_K^{-1}(-\sigma_K \mu_K^{3/2}W_K) & \forall t \in (1, 1/\mu_K] \\
0 & \forall t > 1/\mu_K.
\end{cases}
\]

This indicates that there are two bottlenecks at queues 1 and K. Thus, there is a state space collapse to a two-dimensional vector \( \hat{Z} = (Z_1, Z_K) \), and the sojourn time through the network is asymptotically equal in distribution to the sum of the delays in these two queues.

**Example 2 (Tandem network with unimodally traffic)** Now, suppose \( F_1 \) is not uniform, but unimodal with support on \([0, 1]\). For simplicity, we assume that the distribution function is symmetric around \( \tau := \arg\max \{ F_1'(t) : t \in [0, 1] \} \), where \( F_1' \) represent the first derivative of the arrival epoch distribution (assuming it is well defined), and that the service rates are the same as in Example 1. The uni-modality of the arrival epoch distribution implies that up to time \( \tau \) the distribution function is convex increasing, while after \( \tau \) it is concave decreasing. A simple example of such a distribution function would be,

\[
F_1'(t) = \begin{cases} 
4t & t \in [0, 1/2] \\
4(1-t) & t \in (1/2, 1].
\end{cases}
\]

In this case, \( \tau = 1/2 \) and \( F_1'(\tau) = 2 \).

We first focus on the case where the service rates satisfy \( \mu_K < \mu_1 < F_1'(\tau) \). Observe that

\[
\check{X}_1(t) = F_1(t) - \mu_1 \begin{cases} 
\leq 0 & \forall t \in [0, \tau_1) \\
> 0 & \forall t \in [\tau_1, \tau_2), \\
\leq 0 & \forall t \in [\tau_2, 1],
\end{cases}
\]

where \( \tau_1 := \inf\{ t > 0 : F'(t) = \mu_1 \} \) and \( \tau_2 := \inf\{ t > \tau_1 : F'(t) = \mu_1 \} \); that is, these are the two points in time where the derivative of the arrival epoch distribution equals the service rate in queue 1. We also have \( \check{X}_k(t) = 0 \) for all \( k = 2, \ldots, K - 1 \) and \( \check{X}_K(t) = (\mu_1 - \mu_K)t > 0 \), for all \( t \in [0, 1] \). Consider the fluid queue length at node 1 \( \check{Q}_1(t) = \check{X}_1(t) + \sup_{0 \leq s \leq t}[-\check{X}_1(s)]^+ \), and observe that

\[
\check{Q}_1(t) = \begin{cases} 
0 & t \in [0, \tau_1) \\
\check{X}_1(t) - \check{X}_1(\tau_1) & t \in [\tau_1, \tau_2) \\
0 & t \geq \tau_2.
\end{cases}
\]
Following the arguments in Example 1 it can be shown that the fluid queue length in the downstream nodes satisfies $Q_k(t) = 0$ for all $k = 2, \ldots, K - 1$ for all $t > 0$. Similarly, in the terminal node

$$Q_K(t) = \begin{cases} 
0 & t \in [0, \tau_1') \\
(F_1(t) - F_1(\tau_1')) - \mu_K(t - \tau_1') & t \in [\tau_1', \tau_1) \\
(F_1(\tau_1) - F_1(\tau_1')) - \mu_1(\tau_1 - \tau_1') + (\mu_1 - \mu_K)(t - \tau_1') & t \in [\tau_1, \tau_2) \\
(F_1(t) - F_1(\tau_1')) - \mu_K(t - \tau_1') - (F_1(\tau_1) - \mu_1\tau_1) & t \in [\tau_2, \tau_2') \\
0 & t \geq \tau_2,
\end{cases}$$

where $\tau_1' := \inf\{t > 0 : F_1'(t) \geq \mu_K \}$ and $\tau_2' := \sup\{t > \tau_1' : F_1'(t) \geq \mu_K \}$. This follows from the fact that

$$F_1'(t) - \mu_K \begin{cases} 
\leq 0 & \forall t \in [0, \tau_1') \\
> 0 & \forall t \in [\tau_1', \tau_2'), \\
\leq 0 & \forall t \in [\tau_2, 1],
\end{cases}$$

In contrast to Example 1 the state-space collapse is not straightforward here. The fluid tandem queueing network switches between collapsing to a single queue network in time intervals $[\tau_1', \tau_1)$ and $[\tau_2, \tau_2')$ and a two queue network in the interval $[\tau_1, \tau_2)$. Thus, the state-space collapse itself exhibits non-stationary behavior.

Next, considering the diffusion limit, extending Corollary 5 to a $K$-node network we have

$$\hat{X} = \left( (W_1^0 \circ F_1 - \sigma_1\mu_1^{3/2}W_1)\sigma_1\mu_1^{3/2}(W_1 - W_2), \right.$$

$$\left. \ldots, (\sigma_1\mu_1^{3/2}W_{K-1} - \sigma_K\mu_K^{3/2}W_K) \right),$$

and from Corollary 6 the workload diffusion limit process is $\hat{Z} = M\Delta_X(\hat{X})$. Note that $\hat{X}_k := \sigma_1\mu_1^{3/2}(W_{k-1} - W_k) \overset{D}{=} \sigma_1\mu_1^{3/2}W_k^*$ for $k = 2, \ldots, K - 1$ where $W_k^*$ are independent but identically distributed Brownian motion processes.

Following the fluid limit discussion above, the diffusion limit workload process at queue 1 satisfies

$$\hat{Z}_1(t) = \begin{cases} 
0 & t \in [0, \tau_1) \\
\mu_1^{-1}(\hat{X}_1(t) - \hat{X}_1(\tau_1)) & t \in [\tau_1, \tau_2) \\
0 & t \geq \tau_2.
\end{cases}$$
On the other hand, following Corollary 6 and using the description of the directional derivative regulator map in Lemma 3, the processes \( \hat{Z}_k \) for \( k = 2, \ldots, K - 1 \) can be shown to satisfy

\[
\hat{Z}_k(t) = \begin{cases} 
0 & t \in [0, \tau_1) \\
\mu_1^{-1} \left( \hat{X}_k(t) + \sup_{\tau_1 \leq s \leq t} [\hat{X}(s)]^+ \right) & t \in [\tau_1, \tau_2) \\
0 & t \geq \tau_2.
\end{cases}
\]

Observe that jobs flowing through queues \( 2, \ldots, K - 1 \) will experience non-zero delays in the interval \( [\tau_1, \tau_2) \), determined by the reflected Brownian motion process. The reason why this happens is manifest: the departure rate from queue 1 reaches its maximum value (\( \mu_1 \)) in this interval, so that the downstream queues \( 2, \ldots, K - 1 \) become critically loaded in this interval. Note that the jobs experience (effectively) zero service delay in the latter queues, and they are instantaneously switched through to downstream nodes. Thus, the “surge” period \( [\tau_1, \tau_2) \) is the same in all of these nodes. Finally, in queue \( K \) the diffusion limit workload process satisfies

\[
\hat{Z}_K(t) = \begin{cases} 
0 & t \in [0, \tau'_1) \\
\mu_K^{-1} \hat{X}_K(t) & t \in [\tau'_1, \tau'_2) \\
0 & t \geq \tau'_2.
\end{cases}
\]

Unlike Example 1, jobs experience delays in the last queue only in the interval \( [\tau'_1, \tau'_2) \). Note that this interval is includes the interval \( [\tau_1, \tau_2] \), due to the assumption that \( F_1 \) is unimodal.

Now, consider the alternative case where the serve rates satisfy \( \mu_K < F'_1(\tau) \leq \mu_1 \). In the fluid approximation, nodes \( k = 1, \ldots, K - 1 \) are always 'underloaded' and thus, \( \bar{Q}_k = 0 \). On the other hand, the queue length at node \( K \) satisfies

\[
\bar{Q}_K(t) = \begin{cases} 
0 & t \in [0, \tau'_1) \\
\bar{X}_1(t) - \bar{X}_1(\tau'_1) & t \in [\tau'_1, \tau'_2) \\
0 & t \geq \tau'_2,
\end{cases}
\]

where, as before, \( \tau'_1 := \inf \{ t > 0 : F'_1(t) \geq \mu_K \} \) and \( \tau'_2 := \sup \{ t > \tau'_1 : F'_1(t) \geq \mu_K \} \). Consequently, it follows that the diffusion approximation satisfies \( \hat{Z}_k \overset{D}{=} 0 \) for \( k = 1, \ldots, K - 1 \) and \( \hat{Z}_K(t) \) follows \( (20) \). That is, the only bottleneck node in the network manifests at node \( K \) in the interval \( [\tau'_1, \tau'_2) \).

As a final observation, note that the diffusion limit processes all exhibit discontinuities - at time \( \tau_2 \) for queues \( k = 1, \ldots, K - 1 \) and at \( \tau'_2 \) for queue \( K \). This parallels similar observations for the single server transitory queue.
in Honnappa et al. (2014). Again unlike Example 1, the bottlenecks in the network change over time: there are no bottlenecks in $[0, \tau_1')$, one bottleneck (queue $K$) in $[\tau_1', \tau_1)$, all the queues are bottlenecks in $[\tau_1, \tau_2)$, one bottleneck (queue $K$) in $[\tau_2, \tau_2')$ and zero bottlenecks in $[\tau_2', \infty)$. Thus, the state space “collapse” of the diffusion approximation is much more complicated.

7. Concluding Statements. In this paper we developed asymptotic ‘population acceleration’ approximations of the queue length and (implicitly) the workload processes in a network of transitory queues. These results complement and add to the body of research studying single class generalized Jackson networks. In particular, our fluid limit results accommodate rather general traffic and service models. On the other hand, we can only establish point-wise diffusion approximations in the most general case, owing to the difficulties in the existence of the so-called directional derivative oblique reflection map. Nonetheless, we establish functional central limit theorems in the special case of a tandem network and we also present direct consequences of these developments on bottleneck analysis.

There are several directions in which this research will be expanded in the future. The extension of these results to general polling queueing networks will be interesting, exploiting some recently observed connections between acceleration scalings and polling networks in Rawal et al. (2014). Second, the arrival counts in non-overlapping intervals under the $\Delta(i)$ traffic model have strong negative association. How soon will this correlation be ‘forgotten’ as traffic passes through multiple stages of service? This requires a study of the possible sample paths of the workload process. We believe this question has deep connections with directed percolation models; this is not a novel observation: Glynn and Whitt (1991) identify this connection when there are no traffic dynamics. In on-going work we are working towards extending their analysis to transitory networks. A further interesting question is how the last passage percolation time scales with the population size in a non-stationary setting (as opposed to the classical setting where the percolation model is only studied in the stationary setting). The connection between percolation time and the sojourn time through the network affords yet another bottleneck/performance analysis measure in networks of transitory queues that will be highly relevant in the context of manufacturing lines. We will consider these questions in future papers.

7.1. Appendix subsection. Proof of Theorem 1 We prove lemma’s for each of the claims in the theorem. The first lemma establishes the FSLLN.

**Lemma 7 (FSLLN)** The multivariate traffic process $A_n = (A_1, \ldots, A_J)$ :=
\[ \sum_{n=1}^{\infty} a_n \text{ satisfies } \lim_{n \to \infty} n^{-1} A_n \to F \text{ in } (C^J, U) \text{ a.s.} \]
as \( n \to \infty \), where \( F = (F_1, \ldots, F_J) \) and \( F_j(t) = E[1_{\{T_j \leq t\}}] \) for all \( t \in [0, T] \).

**Proof:** First, for each \( j \in \mathcal{E} \), the classical Glivenko-Cantelli theorem implies that
\[ \lim_{n \to \infty} n^{-1} A_j \to F_j \text{ in } (C, U) \text{ a.s.} \]
as \( n \to \infty \). By the multivariate strong law of large numbers it follows that \( n^{-1} A_j \to F_j \text{ a.s.} \) as \( n \to \infty \). The functional limit follows as a consequence of (21).

This proves part (i) of Theorem 1. The next lemma establishes part (ii).

**Lemma 8** The multivariate traffic process \( A_n \) satisfies a functional central limit theorem where
\[ \sqrt{n} (n^{-1} A_n - F) \Rightarrow W^0 \circ F \text{ in } (C^J, U), \]
where \( W^0 \circ F \) is a \( J \)-dimensional Brownian bridge process as defined in Definition 2, with covariance function \( (R(t), t \geq 0) = ([F_{i,j}(t) - F_i(t)F_j(t), t \geq 0) \).

**Proof:** Once again, Donsker’s theorem for empirical processes implies that
\[ \hat{A}_j := \sqrt{n} (n^{-1} A_j - F_j) \Rightarrow W^0_j \circ F_j \text{ in } (C, U) \]
as \( n \to \infty \) for every \( j \in \mathcal{K} \). This implies that the marginal arrival processes are tight. (Whitt 2001a, Theorem 11.6.7) implies that the multivariate process \( A_n \) is also tight. The multivariate central limit theorem (Whitt 2001a, Theorem 4.3.4) implies that the scaled process \( \hat{A}_n(t) = (\hat{A}_1(t), \ldots, \hat{A}_J(t)) \) (for fixed \( t \in [0, T] \)) satisfies
\[ \hat{A}_n(t) = \sqrt{n} \left( \frac{A_n(t)}{n} - F(t) \right) \Rightarrow \mathcal{N}(0, R(t)), \]
where \( \mathcal{N}(0, R(t)) \) is a mean zero \( J \)-dimensional Gaussian random vector with covariance matrix \( R(t) = [F_{i,j}(t) - F_i(t)F_j(t)] \). The Cramér-Wold device together with this result implies that the finite-dimensional distributions of \( A_n \) converge weakly to a tuple of Gaussian random vectors. The tightness of the processes \( \{A_n\} \), the continuity of the limit process and Prokhorov’s theorem implies that \( A_n \) converges weakly to the multivariate Gaussian stochastic process \( W^0 \circ F \) with mean zero and covariance function \( (R(t), t \geq 0) \) in \((C^J, U)\).
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School of Industrial Engineering, Purdue University, West Lafayette IN 47906. E-mail: honnappa@purdue.edu

EE, CS & ISE Departments, University of Southern California, Los Angeles, CA 90089. E-mail: rahul.jain@usc.edu