On the “scattering law” for Kasner parameters appearing in asymptotics of an exact S-brane solution

V.D. Ivashchuk\textsuperscript{1} and V.N. Melnikov\textsuperscript{2}

\textit{Center for Gravitation and Fundamental Metrology, VNIIMS, 46 Ozyornaya ul., Moscow 119361, Russia}

\textit{Institute of Gravitation and Cosmology, Peoples’ Friendship University of Russia, 6 Miklukho-Maklaya ul., Moscow 117198, Russia}

Abstract

A multidimensional cosmological model with scalar and form fields \cite{1, 2, 3, 4} is studied. An exact $S$-brane solution (either electric or magnetic) in a model with $l$ scalar fields and one antisymmetric form of rank $m \geq 2$ is considered. This solution is defined on a product manifold containing $n$ Ricci-flat factor spaces $M_1, ..., M_n$. In the case when the kinetic term for scalar fields is positive definite we singled out a special solution governed by the function $cosh$. It is shown that this special solution has Kasner-like asymptotics in the limits $\tau \to +0$ and $\tau \to +\infty$, where $\tau$ is a synchronous time variable. A relation between two sets of Kasner parameters $\alpha_\infty$ and $\alpha_0$ is found. This relation, named as “scattering law” (SL) formula, is coinciding with the “collision law” (CL) formula obtained previously in \cite{5} in a context of a billiard description of $S$-brane solutions near the singularity. A geometric sense of SL formula is clarified: it is shown that SL transformation is a map of a “shadow” part of the Kasner sphere $S^{N-2}$ ($N = n+l$) onto “illuminated” part. This map is just a (generalized) inversion with respect to a point $v$ located outside the Kasner sphere $S^{N-2}$. The shadow and illuminated parts of the Kasner sphere are defined with respect to a point-like source of light located at $v$. Explicit formulae for SL transformations corresponding to $SM2$- and $SM5$-brane solutions in 11-dimensional supergravity are presented.

\textsuperscript{1}rusg@phys.msu.ru

\textsuperscript{2}melnikov@phys.msu.su
1 Introduction

In [6] a multidimensional model describing the cosmological “evolution” of $n$ Einstein spaces in the theory with $l$ scalar fields and several antisymmetric forms was considered. When electro-magnetic composite $S$-brane ansatz was adopted, and certain restrictions on the parameters of the model were imposed, the dynamics of the model near the singularity was reduced to a billiard on the $(N-1)$-dimensional hyperbolic (Lobachevsky) space $H_{N-1}^N$, $N = n + l$.

We recall that Kasner-like solutions have the following form

\[ g = wd\tau \otimes d\tau + \sum_{i=1}^{n} A_i \tau^{2\alpha^i} g^i, \quad (1.1) \]
\[ \varphi^\beta = \alpha^\beta \ln \tau + \varphi^\beta_0, \quad (1.2) \]

where

\[ \sum_{i=1}^{n} d_i \alpha^i = 1, \quad (1.3) \]
\[ \sum_{i=1}^{n} d_i (\alpha^i)^2 + \alpha^\beta \alpha^\gamma h_{\beta\gamma} = 1, \quad (1.4) \]

$\varphi^\beta_0$ are constants $i = 1, \ldots, n$; $\beta, \gamma = 1, \ldots, l$; and $w = \pm 1$.

It was shown in [5] that the set of Kasner parameters $(\alpha^A)$ after the collision with the $s$-th wall is defined by the Kasner set before the collision $(\alpha^A)$ according to the following formula

\[ \alpha'^A = \frac{\alpha^A - 2U^s(\alpha) U^sA(U^s, U^s)^{-1} - 1 - 2U^s(\alpha)(U^s, U^A)(U^s, U^s)^{-1}}{1 - 2U^s(\alpha)(U^s, U^A)(U^s, U^s)^{-1}}. \quad (1.5) \]

Here $(\alpha^A) = (\alpha^i, \alpha^\beta) \in \mathbb{R}^N$, $N = n + l$; $U^s$ is a brane co-vector corresponding to the $s$-th wall and $U^A$ is a co-vector, corresponding to the $\Lambda$-term. These vectors and the scalar product $(.,.)$ were defined in [7, 9, 8, 5], see also Section 3 of this paper.

In the special case of one scalar field and 1-dimensional factor-spaces (i.e. $l = d_i = 1$) this formula was suggested earlier in [10]. Another special case
of collision law for multidimensional multi-scalar cosmological model with exponential potentials was considered in [11].

In this paper we consider an exact S-brane solution with one brane (either electric or magnetic) in the model with $l$ scalar fields and one antisymmetric form of rank $m \geq 2$ (see Section 2) [15, 16]. This solution is defined on a product manifold containing $n$ Ricci-flat factor spaces. In Section 3 we rewrite the solution in a so-called “minisuperspace-covariant” form that significantly simplifies forthcoming analysis.

In the case when the kinetic term for scalar fields is positive definite one we single out a special solution governed by the $\cosh$ function. In Section 4 we show that this solution has a Kasner-like asymptotics in both limits $\tau \to +0$ and $\tau \to +\infty$, where $\tau$ is the synchronous time variable. We also find a relation between two sets of Kasner parameters $\alpha_\infty$ and $\alpha_0$. This relation (“scattering law” formula) is coinciding with the CL formula from (1.5).

In Section 5 we clarify the geometric sense of the SL. We have expressed the SL transformation in terms of a function mapping a “shadow” part of the Kasner sphere $S^{N-2}$ onto “illuminated” part. We show that this function is just an inversion with respect to a point $v$ located outside the Kasner sphere $S^{N-2}$.

In Section 6 we present explicit formulae for SL transformations corresponding to $SM2$- and $SM5$-brane solutions in 11-dimensional supergravity.

2 S-brane solution

Here we deal with a model governed by the action

$$S_g = \int d^D x \sqrt{|g|} \left\{ R[g] - h_{\alpha \beta} g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta - \frac{\theta}{m!} \exp[2\lambda(\varphi)] F^2 \right\} \quad (2.1)$$

where $g = g_{MN}(x)dx^M \otimes dx^N$ is a metric, $\varphi = (\varphi^\alpha) \in \mathbb{R}^l$ is a vector of scalar fields, $(h_{\alpha \beta})$ is a constant symmetric non-degenerate $l \times l$ matrix ($l \in \mathbb{N}$), $\theta = \pm 1$, $F = dA = \frac{1}{m!} F_{M_1 \ldots M_m} dz^{M_1} \wedge \ldots \wedge dz^{M_m}$ is a $m$-form ($m \geq 1$), $\lambda$ is a 1-form on $\mathbb{R}^l$: $\lambda(\varphi) = \lambda_\alpha \varphi^\alpha$, $\alpha = 1, \ldots, l$. In (2.1) we denote $|g| = |\det(g_{MN})|$, $F^2 = F_{M_1 \ldots M_m} F_{N_1 \ldots N_m} g^{M_1 N_1} \ldots g^{M_m N_m}$. 
For pseudo-Euclidean metric of signature \((-\),\ldots,\) one should put \(\theta = 1\).

We consider \(S\)-brane solution (either electric or magnetic one) to field equations corresponding to the action (2.1) and depending upon one variable \(u\) \([15, 16]\) (see also \([14, 12, 8]\)).

This solution is defined on the manifold
\[
M = (u_-, u_+) \times M_1 \times M_2 \times \ldots \times M_n, \quad (2.2)
\]
where \((u_-, u_+)\) is an interval belonging to \(\mathbb{R}\), and has the following form
\[
g = \left[f(u) \right]^{2d(I)h/(D-2)} \left\{ \exp(2c^0 u + 2\bar{c}^0) w du \otimes du + \sum_{i=1}^{n} \left( [f(u)]^{-2h\delta^i} \right) \exp(2c^i u + 2\bar{c}^i) g^i \right\}, \quad (2.3)
\]
\[
\exp(\varphi^\alpha) = (f^h \lambda^\alpha) \exp(c^\alpha u + \bar{c}^\alpha), \quad (2.4)
\]
\[
F = Q f^{-2} du \wedge \tau(I), \quad \chi = +1, \quad (2.5)
\]
\[
= Q \tau(\bar{I}), \quad \chi = -1, \quad (2.6)
\]
\(w = \pm 1, \ \alpha = 1, \ldots, l\).

Here and in what follows
\[
\chi = +1, -1 \quad (2.7)
\]
for electric or magnetic case, respectively. \(Q \neq 0\) is a constant (charge density parameter) and \(\lambda^\alpha = h^{\alpha\beta} \lambda_\beta\) where \((h^{\alpha\beta}) = (h_{\alpha\beta})^{-1}\).

In (2.3) \(w = \pm 1\), \(g^i = g^i_{m_1n_1}(y_i) dy^{m_1}_i \otimes dy^{n_1}_i\) is a Ricci-flat metric on \(M_i\), \(i = 1, \ldots, n\),
\[
\delta^i_I = \sum_{j \in I} \delta^i_j \quad (2.8)
\]
is the indicator of \(i\) belonging to \(I\): \(\delta^i_i = 1\) for \(i \in I\) and \(\delta^i_j = 0\) otherwise.

The set \(I = \{i_1, \ldots, i_k\}\) is a subset of \(I_0 = \{1, \ldots, n\}\). It describes the location of \(S\)-brane worldvolume. Here and in what follows
\[
\bar{I} \equiv I_0 \setminus I. \quad (2.9)
\]

All manifolds \(M_i\) are assumed to be oriented and connected and the volume \(d_i\)-forms
\[
\tau_i \equiv \sqrt{|g^i(y_i)|} \ dy^{1}_i \wedge \ldots \wedge dy^{d_i}_i, \quad (2.10)
\]
and parameters
\[ \varepsilon(i) \equiv \text{sign}(\det(g_{m,n}^i)) = \pm 1 \] (2.11)
are well-defined for all \( i = 1, \ldots, n \). Here \( d_i = \dim M_i \), \( i = 1, \ldots, n \); \( D = 1 + \sum_{i=1}^n d_i \). For any set \( I = \{i_1, \ldots, i_k\} \in I_0 \), \( i_1 < \ldots < i_k \), we denote
\[ \tau(I) \equiv \tau_{i_1} \wedge \ldots \wedge \tau_{i_k}, \] (2.12)
\[ d(I) \equiv \sum_{i \in I} d_i, \] (2.13)
\[ \varepsilon(I) \equiv \varepsilon(i_1) \ldots \varepsilon(i_k). \] (2.14)

The parameters \( h \) appearing in the solution satisfy the relations
\[ h = K^{-1}, \] (2.15)
where
\[ K = d(I) + \frac{(d(I))^2}{2 - D} + \lambda_\alpha \lambda_\beta h^{\alpha\beta}. \] (2.16)

Here we assume that \( K \neq 0 \).

The moduli function reads
\[ f(u) = R \sinh(\sqrt{C}(u - u_1)), \quad C > 0, \quad K \varepsilon < 0; \] (2.17)
\[ R \sin(\sqrt{|C|}(u - u_1)), \quad C < 0, \quad K \varepsilon < 0; \] (2.18)
\[ R \cosh(\sqrt{C}(u - u_1)), \quad C > 0, \quad K \varepsilon > 0; \] (2.19)
\[ |Q| |K|^{1/2}(u - u_1), \quad C = 0, \quad K \varepsilon < 0, \] (2.20)
where \( R = |Q| |K/C|^{1/2} \), and \( C, u_1 \) are constants.

Here \( \varepsilon = \varepsilon(I) \theta \) for electric case and \( \varepsilon = -\varepsilon[g] \varepsilon(I) \theta \) for magnetic case, where \( \varepsilon[g] = \text{sign det}(g_{MN}) \).

Vectors \( c = (c^A) = (c^i, c^\alpha) \) and \( \bar{c} = (\bar{c}^A) \) obey the following constraints
\[ \sum_{i \in I} d_i c^i - \chi \lambda_\alpha c^\alpha = 0, \quad \sum_{i \in I} d_i \bar{c}^i - \chi \lambda_\alpha \bar{c}^\alpha = 0, \] (2.21)
\[ c^0 = \sum_{j=1}^n d_j c^j, \quad \bar{c}^0 = \sum_{j=1}^n d_j \bar{c}^j, \] (2.22)
\[ CK^{-1} + h_{\alpha\beta} c^\alpha \bar{c}^\beta + \sum_{i=1}^n d_i (c^i)^2 - \left( \sum_{i=1}^n d_i c^i \right)^2 = 0. \] (2.23)
Due to (2.5) and (2.6), the dimension of brane worldvolume \( d(I) \) is defined by
\[
d(I) = m - 1, \quad d(I) = D - m - 1, \tag{2.24}
\]
for electric and magnetic cases, respectively. For \( Sp \)-brane we have \( p = p(I) = d(I) - 1 \). The solution under consideration is a special one brane case of intersecting \( S \)-brane solutions from [14, 15, 16].

3 Minisuperspace-covariant notations

Our solution may be written also in the so-called “minisuperspace-covariant” form following from the sigma-model solution [12].

The metric (2.3) has the structure
\[
g = e^{2\gamma_0(u)} du \otimes du + \sum_{i=1}^{n} e^{2\phi^i(u)} g^i, \tag{3.1}
\]
where
\[
\gamma_0 = \sum_{i=1}^{n} d_i \phi^i(u). \tag{3.2}
\]
Introducing a collective variable \( x = (x^A) = (\phi^i, \varphi^{\alpha}) \) we get a minisuperspace-covariant relation (see (2.3) and (2.4)):
\[
x^A(u) = -\frac{U^A}{(U, U)} \ln |f(u)| + c^A u + \bar{c}^A, \tag{3.3}
\]
where the function \( f(u) \) was defined in (2.17)-(2.20) and \( c = (c^A) = (c^i, c^{\alpha}) \).

The linear and quadratic constraints from (2.21) and (2.23), respectively, read in a minisuperspace covariant form as:
\[
U_A c^A = 0, \quad U_A \bar{c}^A = 0, \tag{3.4}
\]
and
\[
\frac{C}{(U, U)} + \bar{G}_{ABC} c^A c^B = 0. \tag{3.5}
\]
Here
\[
(U_A) = (d_i \delta^i_I, -\chi \lambda_{\alpha}), \tag{3.6}
\]
is the so-called brane co-vector (U-vector) [7, 9]
\[
(U^A) = (\bar{G}^{AB} U_B) = (\delta^i_I - \frac{d(I)}{D - 2}, -\chi \lambda^\alpha),
\]
(3.7)

and
\[
(\bar{G}_{AB}) = \begin{pmatrix} G_{ij} & 0 \\ 0 & h_{\alpha\beta} \end{pmatrix}, \quad (\bar{G}^{AB}) = \begin{pmatrix} G^{ij} & 0 \\ 0 & h^{\alpha\beta} \end{pmatrix},
\]
(3.8)
are, correspondingly, a minisuperspace metric and inverse to it, where (see [17])
\[
G_{ij} = d_i \delta_{ij} - d_i d_j, \quad G^{ij} = \frac{\delta^{ij}}{d_i} + \frac{1}{2 - D}.
\]
(3.9)

In what follows we use a scalar product [7]
\[
(U, U') = \bar{G}^{AB} U_A U'_B, \quad \text{for} \quad U = (U_A), U' = (U'_A) \in \mathbb{R}^N, \quad N = n + l.
\]
(3.10)

In (3.5) we used the relation
\[
(U, U) = K.
\]
(3.11)

The logarithm of the lapse function (3.2) may be also written in the minisuperspace covariant form
\[
\gamma_0 = U^A x^A,
\]
(3.12)

where
\[
(U^A) = (d_i, 0)
\]
(3.13)
is U-vector, corresponding to the Λ-term [7, 9].

We will use also the relations
\[
c^0 = U^A c^A,
\]
(3.14)

and
\[
(U, U^A) = -\frac{d(I)}{D - 2},
\]
(3.15)
\[
(U^A, U^A) = -\frac{D - 1}{D - 2}.
\]
(3.16)
4 Scattering law for Kasner parameters

Here we restrict our consideration by a special solution with $K = (U, U) > 0$, $C > 0$ and $\varepsilon > 0$. We also put the matrix $(h_{\alpha\beta})$ to be positive definite.

In this case the solution is governed by the moduli function $f(u) = R \cosh(\sqrt{C}(u - u_1))$ and is defined for all $u \in (-\infty, +\infty)$.

4.1 Kasner-like behaviour

Let us consider our solution in a synchronous time:

$$\tau = \eta \int_{u_0}^{u} d\bar{u} e^{\gamma_0(u)},$$

(4.1)

where $\eta = \pm 1$, $u_0$ is constant and

$$e^{\gamma_0(u)} = |f(u)|^{d(l)h/(D-2)} \exp(c^0 u + \bar{c}^0)$$

(4.2)

is a lapse function.

Due to

$$f \sim \frac{R}{2} \exp(\pm \sqrt{C}(u - u_1)),$$

(4.3)

for $u \to \pm \infty$, we get asymptotical relations for the lapse function

$$e^{\gamma_0} \sim \text{const} \exp(b_\pm \sqrt{C} u),$$

(4.4)

as $u \to \pm \infty$, with

$$b_\pm = \pm \frac{h d(l)}{D - 2} + \frac{c^0}{\sqrt{C}}.$$  

(4.5)

Using relations (3.14), (3.15) and $h = (U, U)^{-1}$ we could rewrite the parameters $b_\pm$ in a minisuperspace-covariant form

$$b_\pm = \pm \frac{(U^{\Lambda}, U)}{(U, U)} + (s, U^{\Lambda}).$$

(4.6)

where

$$s = (s_A) = \left(\bar{G}_{AB}c^B/\sqrt{C}\right),$$

(4.7)
is a co-vector, obeying relations:

\[(s, U) = 0, \quad (4.8)\]
\[\frac{1}{(U, U)} + (s, s) = 0. \quad (4.9)\]

following just from (3.4) and (3.5). In derivation of (4.6) we used the relation

\[c^0 = (s, U^A)\sqrt{C}, \quad (4.10)\]

following from (3.14) and (4.7).

In what follows we will use an inequality

\[| (s, U^A) | > \frac{| (U^A, U) |}{(U, U)} > 0, \quad (4.11)\]

proved in Appendix. The proof uses relations (4.8), (4.9) and \((U, U) > 0\).

The parameter \(c^0\) is a non-zero one (otherwise the relation (2.23) would be incompatible with the conditions \(C > 0, \ K > 0\) and positive definiteness of the matrix \((h_{\alpha\beta})\)). It follows from inequality (4.11) that \(b_{\pm}\) are also non-zero and

\[\text{sign}(b_{\pm}) = \text{sign}((s, U^A)) = \text{sign}(c^0). \quad (4.12)\]

It may be verified that due to (4.11) the lapse function \(e^{\gamma_0(u)}\) is monotonically increasing from \(+0\) to \(+\infty\) for \(c^0 > 0\) and monotonically decreasing from \(+\infty\) to \(+0\) for \(c^0 < 0\).

We define synchronous time variable to be

\[\tau = \int_{-\infty}^{u} d\bar{u}e^{\gamma_0(\bar{u})}, \quad (4.13)\]

for \(c^0 > 0\) and

\[\tau = \int_{u}^{+\infty} d\bar{u}e^{\gamma_0(\bar{u})}, \quad (4.14)\]

for \(c^0 < 0\). Then, the function \(\tau = \tau(u)\) is monotonically increasing from \(+0\) to \(+\infty\) for \(c^0 > 0\) and monotonically decreasing from \(+\infty\) to \(+0\) for \(c^0 < 0\).

We have the following asymptotical relations for \(\tau = \tau(u)\)

\[\tau \sim \text{const} \ b_{\pm}^{-1} \exp(b_{\pm}\sqrt{C}u), \quad (4.15)\]
as \( u \to \pm \infty \).

For the collective variable \( (x^A) = (\phi^i, \varphi^\alpha) \) from (3.3) we get (see (4.3))

\[
x^A(u) \sim \mp \frac{U^A \sqrt{C} u}{(U, U)} + c^A u + \hat{c}^A,
\]

as \( u \to \pm \infty \), where \( \hat{c}^A \) are constants. Hence, due to (4.15) we are led to Kasner-like asymptotics written in a minisuperspace covariant form

\[
x^A \sim \alpha^A \pm \ln \tau \pm x^A \pm,
\]

for \( u \to \pm \infty \), where \( x^A \pm \) are constants and

\[
\alpha^A \pm = \frac{\mp U^A (U, U) + s^A}{b^A \pm}
\]

are collective Kasner-like parameters, corresponding to \( u \to \pm \infty \).

Asymptotical relations (4.17) could be also rewritten in the form assigned to proper time asymptotics, i.e.

\[
x^A \sim \alpha^A_0 \ln \tau + x^A_0, \text{ as } \tau \to 0^+,
\]

\[
x^A \sim \alpha^A_\infty \ln \tau + x^A_\infty, \text{ as } \tau \to \infty.
\]

Here

\[
\alpha^A_0 = \alpha^A_-, \quad \alpha^A_\infty = \alpha^A_+
\]

for \( c^0 > 0 \) and

\[
\alpha^A_0 = \alpha^A_+, \quad \alpha^A_\infty = \alpha^A_-
\]

for \( c^0 < 0 \) and \( x^A_0, \ x^A_\infty \) are constants.

It follows from definitions of Kasner parameters (4.18) that

\[
\bar{G}_{AB} \alpha^A_\pm \alpha^B_\pm = 0,
\]

\[
U(\alpha_\pm) = U_A \alpha^A_\pm = \mp \frac{1}{b_\pm},
\]

\[
U^A(\alpha_\pm) = 1,
\]

see (4.6), (4.8) and (4.9).
In components relations (4.23) and (4.25) read as

$$\sum_{i=1}^{n} d_i \alpha^i_\pm = \sum_{i=1}^{n} d_i (\alpha^i_\pm)^2 + \alpha^\beta_\pm \alpha^\gamma_\pm h_{\beta\gamma} = 1.$$ (4.26)

Thus, we are led to Kasner-like relations (1.3) and (1.4) for \( \alpha_\pm = (\alpha^A_\pm) \). Hence, \( \alpha_0 = (\alpha^A_0) \) and \( \alpha_\infty = (\alpha^A_\infty) \) also obey relations (1.3) and (1.4).

So, we obtain a Kasner-like asymptotical behaviour of our special solution (with \( C > 0 \) and \( \varepsilon > 0 \)) for i) \( \tau \to +0 \) and for ii) \( \tau \to +\infty \), as well. The Kasner-like behaviour in the case i) is in agreement with the general result from [6].

Using (4.12) and (4.24) we get

$$U(\alpha_0) = U_A \alpha^A_0 = \sum_{i \in I} d_i \alpha^i_0 - \chi \lambda \alpha^\beta_0 > 0, \quad (4.27)$$

$$U(\alpha_\infty) = U_A \alpha^A_\infty = \sum_{i \in I} d_i \alpha^i_\infty - \chi \lambda \alpha^\beta_\infty < 0. \quad (4.28)$$

The first inequality (4.27) is a special case of a set of inequalities derived in [6]. Any inequality of such type corresponds to a billiard wall in hyperbolic (e.g. Lobachevsky) space.

### 4.2 Scattering law

Now we derive a relation between Kasner sets \( \alpha_0 \) and \( \alpha_\infty \).

We start with the formulas:

$$b_+ \alpha_+ - b_- \alpha_- = -\frac{2U}{(U,U)} \quad (4.29)$$

and

$$b_+ - b_- = -\frac{2(U^A, U)}{(U,U)} \quad (4.30)$$

following from (4.18) and (4.6), respectively. Using these relations and (4.24) we get

$$\alpha^A_\pm = \frac{\alpha^A_\pm - 2U^A \alpha_\pm (U,U)^{-1}}{1 - 2U(U^A)(U,U)^{-1}}. \quad (4.31)$$
This formula gives a scattering law formula for Kasner parameters (see definitions (4.21) and (4.22))

$$\alpha_\infty = \alpha_0 - \frac{2UU(\alpha_0)(U,U)^{-1}}{1 - 2U(\alpha_0)(U,\Lambda)(U,U)^{-1}} = S(\alpha_0).$$

(4.32)

coinciding for $U = U^*$ with the collision law formula (1.5) derived in [5].

It should be also noted that due to (4.31) the inverse function $S^{-1}$ is given by just the same relation

$$\alpha_0 = \frac{\alpha_\infty - 2UU(\alpha_\infty)(U,U)^{-1}}{1 - 2U(\alpha_\infty)(U,\Lambda)(U,U)^{-1}} = S^{-1}(\alpha_\infty).$$

(4.33)

5 Geometric meaning of the scattering law

Let us clarify the geometric meaning of the scattering law. Since the matrix $(h_{\alpha\beta})$ is positive definite the Kasner relations (1.3) and (1.4) describe an ellipsoid isomorphic to a unit $(N - 2)$-dimensional sphere $S^{N-2}$ which is a subset of $\mathbb{R}^{N-1}$, $N = n + l$. Thus, the sets of Kasner parameters $\alpha$ may be parametrized by vectors $\vec{n} \in S^{N-2}$, i.e. $\alpha = \alpha(\vec{n})$. Let us show that the scattering law formula (1.5) in terms of $\vec{n}$-vectors reads as follows

$$\vec{n}' = \left(\vec{v}^2 - 1\right)\vec{n} + 2\left(1 - \vec{v}\vec{n}\right)\vec{v}$$

(5.1)

where $\vec{v}$ is a vector belonging to $\mathbb{R}^{N-1}$ with $|\vec{v}| > 1$.

Now we proceed with derivation of (5.1). The minisuperspace metric (3.8) has a pseudo-Euclidean signature $(-, +, \ldots, +)$, since the matrix $(G_{ij})$ has the pseudo-Euclidean signature [17] and $(h_{\alpha\beta})$ has the Euclidean one. There exists a linear transformation

$$z^a = e_A^a x^A,$$

(5.2)

diagonalizing the minisuperspace metric (3.8)

$$\bar{G}_{AB} = \eta_{ab} e_A^a e_B^a$$

(5.3)

where here and in what follows

$$(\eta_{ab}) = (\eta^{ab}) = diag(-1, +1, \ldots, +1),$$

(5.4)
\((e_A^a)\) is a non-degenerate matrix of linear transformations and \(a, b = 0, \ldots, N-1\).

The matrix \((e_A^a)\) satisfies the relation
\[
\eta^{ab} = e_A^a \bar{G}^{AB} e_B^b = (e^a, e^b),
\] (5.5)
equivalent to (5.3) where here \(e^a = (e_A^a)\).

For the inverse matrix \((e_A^a)^{-1} = (e_A^a)^{-1}\) we obtain from (5.5)
\[
e_A^a = \bar{G}^{AB} e_B^b \eta_{ba}.
\] (5.6)

Here we put as in \([6]\)
\[
e^0 = q^{-1} U^\Lambda,
\] (5.7)
where
\[
q = [-(U^\Lambda, U^\Lambda)]^{1/2} = [(D - 1)/(D - 2)]^{1/2}.
\] (5.8)

We remind that \(U^\Lambda\) is a time-like co-vector, i.e. \((U^\Lambda, U^\Lambda) < 0\). (Explicit relations for other co-vectors \(e^i, i = 1, \ldots, N - 1\), are irrelevant for our consideration. A possible choice of \(e^i\) is the following one: for \(i = 1, \ldots, n - 1\), these co-vectors could be found from relations for \(z^i\) of Ref. \([17]\), while for \(i = n, \ldots, N - 1\) these co-vectors could be readily obtained by diagonalization of the matrix \((h_{\alpha\beta})\).)

Let us define a “frame” U-vector:
\[
\hat{U}_a = e_A^a U_A = (U, e_B^b) \eta_{ba},
\] (5.9)
(see (5.6)). It follows from (3.15), (5.7) and (5.9) that
\[
\hat{U}_0 = -(U, e^0) = -q^{-1}(U, U^\Lambda) > 0.
\] (5.10)

We also define “frame” \(\alpha\)-parameters:
\[
\hat{\alpha}^a = e_A^a \alpha^A.
\] (5.11)

In terms of \(\hat{\alpha}\)-parameters the Kasner relations (written in a “minisuperspace covariant” form)
\[
\bar{G}_{AB} \alpha^A \alpha^B = 0,
\] (5.12)
\[
U^\Lambda(\alpha) = 1
\] (5.13)

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These equations imply [6]
\[ \hat{\alpha}^0 = q^{-1}, \quad \hat{\alpha}^i = q^{-1}n^i, \quad (5.16) \]
i = 1, \ldots, N - 1, where the vector \( \vec{n} = (n^i) \in \mathbb{R}^{N-1} \) has the unit length: \( \vec{n}^2 = 1 \), i.e. \( \vec{n} \in S^{N-2} \). Relations (5.11) and (5.16) define a one-to-one correspondence between the points of the Kasner sphere \( S^{N-2} \) and Kasner sets \( \alpha \).

Now we define a vector \( \vec{v} = (v_i) \in \mathbb{R}^{N-1} \) by formula
\[ v_i = -\hat{U}_i/\hat{U}_0, \quad (5.17) \]
i = 1, \ldots, N - 1 [6]. Since
\[ \eta^{ab}\hat{U}_a\hat{U}_b = -(\hat{U}_0)^2 + \sum_{i=1}^{N-1}(\hat{U}_i)^2 = (U,U) > 0, \quad (5.18) \]
we get \( |\vec{v}| > 1 \). Using relations (5.9), (5.11), (5.16) and (5.17), we obtain
\[ U(\alpha) = U_A\alpha^A = \hat{U}_a\hat{\alpha}^a = q^{-1}\hat{U}_0(1 - \vec{v}\vec{n}). \quad (5.19) \]

Due to (5.19) and \( \hat{U}_0 > 0 \), the following equivalences take place
\[ U(\alpha) > 0 \Leftrightarrow \vec{v}\vec{n} < 1, \quad (5.20) \]
\[ U(\alpha) < 0 \Leftrightarrow \vec{v}\vec{n} > 1. \quad (5.21) \]

Since the asymptotical Kasner sets \( \alpha_\infty = \alpha(\vec{n}_\infty) \) and \( \alpha_0 = \alpha(\vec{n}_0) \) obey the inequalities \( U(\alpha_\infty) < 0 \) and \( U(\alpha_0) > 0 \), we get
\[ \vec{v}\vec{n}_0 < 1, \quad \vec{v}\vec{n}_\infty > 1. \quad (5.22) \]

Geometrically, this means that the endpoint of the vector \( \vec{n}_0 \) is not illuminated by a point-like source of light located at the endpoint of the vector \( \vec{v} \).
(i.e. the endpoint of $\vec{n}_0$ belongs to the “shadow side” of the Kasner sphere), while the endpoint of the vector $\vec{n}_\infty$ is illuminated by this source.

Using the definitions (5.9) and (5.11) we rewrite the scattering law formula (1.5) (with $U^s = U$) in a (equivalent) “frame representation”

$$\hat{\alpha}'^a = \frac{\hat{\alpha}^a - 2\hat{U}(\hat{\alpha})\hat{U}^a(U,U)^{-1}}{1 - 2\hat{U}(\hat{\alpha})U^A(U,U)^{-1}}.$$  (5.23)

where $\hat{U}(\hat{\alpha}) = \hat{U}_a\hat{\alpha}^a = U(\alpha)$ is given by (5.19) and

$$\hat{U}^a = e^a_AU^A = \eta^{ab}\hat{U}_b.$$  (5.24)

Formula (5.23) is satisfied identically for $a = 0$, due to relations (5.10), (5.16) and $\hat{U}^0 = -\hat{U}_0$. It may be verified using (5.10), (5.16)-(5.19) and $\hat{U}^i = \hat{U}_i$, $i > 0$, that for $a = i > 0$ relation (5.23) coincides with (5.1). Thus, we proved the formula (5.1) with $\vec{n} = \vec{n}_0$ and $\vec{n}' = \vec{n}_\infty$.

It follows from (5.1) that

$$\vec{n}' - \vec{v} = B(\vec{n} - \vec{v}),$$  (5.25)

where $B = (\vec{v}^2 - 1)/(\vec{v} - \vec{n})^2 > 0$. Thus, the endpoints of the vectors $\vec{v}$, $\vec{n}'$ and $\vec{n}$ belong to one line. Hence, the endpoint of the vector $\vec{n}'$ may be obtained as a point of intersection of the Kasner sphere with the line connecting the endpoints of the vectors $\vec{v}$ and $\vec{n}$. Thus, the scattering law transformation (5.1) is just an inversion with respect to a point $\vec{v}$ located outside the Kasner sphere $S^{N-2}$. The point $\vec{v}$ is the endpoint of the vector $\vec{v}$. This transformation map the “shadow domain” of Kasner sphere onto “illuminated domain”.

**Remark.** We remind that according to an analysis carried out in [6] the solution under consideration is described for $\tau \to 0$ by a geodesic motion of a point-like particle in a billiard. This billiard belongs to the Lobachevsky space $H^{N-1}$ identified with the unit ball $D^{N-1} = \{\vec{y} : |\vec{y}| < 1\}$. It is described by the relation: $|\vec{y} - \vec{v}| > r$, where $r = \sqrt{\vec{v}^2 - 1}$. The billiard wall is a part (belonging to $D^{N-1}$) of a $(N-2)$-dimensional sphere with a center in the endpoint of $\vec{v}$ and radius $r$. 

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6 Example: $D = 11$ supergravity

Now we consider, as an example, $D = 11$ supergravity [18]. The bosonic sector action reads in this case as

$$S = \int d^{11}z \sqrt{|g|} \left\{ R[g] - \frac{1}{4!} F^2 \right\} + c \int_M A \wedge F \wedge F \quad (6.1)$$

where $F = dA$ is a 4-form and constant $c$ is irrelevant for our consideration. Since the second term in (6.1) (called as Chern-Simons term) does not depend upon a metric, the Hilbert-Einstein equations are not changed when it is omitted. The only modification of equations of motion is related to Maxwell-type equations

$$d \ast F = \text{const} \ F \wedge F. \quad (6.2)$$

Due to relations for $F$ in (2.5) and (2.6) we get $F \wedge F = 0$. Thus, the solution from the Section 2 in the special case $D = 11$, $w = -1$, $l = 0$ (i.e. when scalar fields are absent) and $m = 4$ gives us either $SM2$- or $SM5$-brane solution in 11-dimensional supergravity (see also [13] and references therein).

In this case $\alpha = (\alpha^i)$ and the relations on Kasner parameters (1.3) and (1.4) read

$$\sum_{i=1}^n d_i \alpha^i = \sum_{i=1}^n d_i (\alpha^i)^2 = 1. \quad (6.3)$$

For $U = (U_i)$ we get $(U, U) = 2$, $(U, U^\Lambda) = -\frac{1}{9} d(I)$ and $U^i = \delta_i^j - \frac{1}{9} d(I)$. Here $d(I) = 3$ for electric $SM2$-brane and $d(I) = 6$ for magnetic $SM5$-brane.

Thus, the scattering law (4.32) for $SM$-branes reads as follows

$$\alpha^i_\infty = \frac{\alpha^i_0 - (\delta_i^j - \frac{1}{9} d(I)) U(\alpha_0)}{1 + \frac{1}{9} U(\alpha_0) d(I)}. \quad (6.4)$$

where

$$U(\alpha_0) = \sum_{i \in I} d_i \alpha^i_0 > 0. \quad (6.5)$$

In terms of Kasner sphere parametrization of $\alpha$-parameters the scattering law relation (6.4) is given by the formula (5.1) with $\vec{n} = \vec{n}_0$, $\vec{n}' = \vec{n}_\infty$ and $N = n$. 

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Using \((U, U) = 2\) and (5.10) we get \(\vec{v}^2 = 21\) in the electric case and \(\vec{v}^2 = 6\) in the magnetic case [6]. This means that illuminated part of the Kasner sphere \(S^n\) (containing endpoints of \(\vec{n}_\infty\)) is larger in the electric case, while the shadow domain (containing endpoints of \(\vec{n}_0\)) is larger in the magnetic case.

7 Conclusions

We have considered the exact \(S\)-brane solution with one brane (either electric or magnetic) to field equations corresponding to the action (2.1) containing \(l\) scalar fields and one antisymmetric form of rank \(m \geq 2\) [15, 16]. This solution is defined on the product manifold (2.2) containing \(n\) Ricci-flat factor spaces \(M_1, ..., M_n\).

In the case when the matrix \((h_{\alpha\beta})\) is positive definite we have singled out a special solution governed by \(\cosh\) moduli function. We have shown that this solution has Kasner-like asymptotics in the limits \(u \to \pm \infty\), where \(u\) is the harmonic time variable, or, equivalently, in the limits \(\tau \to +0\) and \(\tau \to +\infty\), where \(\tau\) is the synchronous time variable.

We have found a relation between two sets of Kasner parameters \(\alpha_\infty\) and \(\alpha_0\). Remarkably, the relation between them \(\alpha_\infty = S(\alpha_0)\) is coinciding with the “collision law” formula from [5]. We have also clarified the geometrical sense of the scattering law. Namely, we have expressed the scattering law transformation in terms of a function mapping a “shadow” part of the Kasner sphere \(S^{N-2}\) onto “illuminated” one. This function is just an inversion with respect to a point \(v\) located outside the Kasner sphere \(S^{N-2}\). The shadow and illuminated parts of the Kasner sphere are defined w.r.t. a point-like source of light located at \(v\).

We have also written explicit formulae for scattering law transformations corresponding to \(SM2\)- and \(SM5\)-brane solutions in 11-dimensional supergravity.
Appendix

Let us prove the inequality (4.11)

\[ |(s, U^A)| > \frac{|(U^A, U)|}{(U, U)} > 0, \]

for a vector \( s = (s^A) \in \mathbb{R}^N \) \((N = n + l)\) obeying relations \((s, U) = 0, (s, s) = -1/(U, U)\). Here the scalar-product \((U, U') = G^{AB}U_A U'_B\) is defined by the matrix \((G^{AB})\) from (3.8) with a positive definite matrix \((h_{\alpha\beta})\) and \(G^{ij} = \delta^{ij}d_i^{-1}+(2-D)^{-1}\). We also use here the following relations \((U, U) > 0, (U^A) = (d_i, 0)\) and \((U^A, U^A) < 0\).

**Proof.** Let us define the vector

\[ U_1 = U - \frac{(U, U^A)}{(U^A, U^A)}U^A. \]  

(A.1)

It is clear that \((U_1, U^A) = 0\) and

\[ (U_1, U_1) = (U, U) - \frac{(U, U^A)^2}{(U^A, U^A)} > 0. \]  

(A.2)

since \((U, U) > 0\) and \((U^A, U^A) < 0\). Let us define vectors:

\[ s_0 = \frac{(s, U^A)}{(U^A, U^A)}U^A, \]  

(A.3)

\[ s_1 = \frac{(s, U_1)}{(U_1, U_1)}U_1, \]  

(A.4)

\[ s_2 = s - s_0 - s_1. \]  

(A.5)

\(s_0, s_1\) and \(s_2\) are mutually orthogonal and hence

\[ (s, s) = (s_0, s_0) + (s_1, s_1) + (s_2, s_2). \]  

(A.6)

For the first two terms in r.h.s. of (A.6) we get

\[ (s_0, s_0) = \frac{(s, U^A)^2}{(U^A, U^A)}, \]  

(A.7)

\[ (s_1, s_1) = \frac{(s, U_1)^2}{(U_1, U_1)} = \frac{(s, U^A)^2}{(U^A, U^A)} \frac{(U, U^A)^2}{[(U, U)(U^A, U^A) - (U, U^A)^2]} \]  

(A.8)
that implies

$$(s, s) = \frac{(s, U^A)^2(U, U)}{(U, U)(U^A, U^A) - (U, U^A)^2} + (s_2, s_2).$$  \hspace{1cm} (A.9)

For the third term in r.h.s. of (A.6) the following inequality is valid

$$(s_2, s_2) \geq 0,$$  \hspace{1cm} (A.10)

Indeed, due to $(s_2, U^A) = 0$, or, equivalently, $\sum_{i=1}^n s_i^2 d_i = 0$, and the positive definiteness of the matrix $(h_{\alpha\beta})$, we obtain

$$(s_2, s_2) = G_{AB}s_2^A s_2^B = \sum_{i=1}^n (s_i^j)^2 d_i + h_{\alpha\beta}s_2^\alpha s_2^\beta \geq 0.$$  \hspace{1cm} (A.11)

Using this inequality, (A.9), $(U^A, U^A) < 0$ and $(s, s) = -1/(U, U)$ we get

$$(s, U^A)^2 = \frac{(U, U^A)^2(U, U) - (U^A, U^A)[(U, U)^{-1} + (s_2, s_2)]}{(U, U)^2} > 0,$$  \hspace{1cm} (A.12)

that is equivalent to the inequality (4.11). Thus, (4.11) is proved.

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