GIBB’S MINIMIZATION PRINCIPLE FOR APPROXIMATE SOLUTIONS OF SCALAR CONSERVATION LAWS

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ABSTRACT. In this work we study variational properties of approximate solutions of scalar conservation laws. Solutions of this type are described by a kinetic equation which is similar to the kinetic representation of admissible weak solutions due to Lions-Perthame-Tadmor[12], but also retain small scale non-equilibrium behavior. We show that approximate solutions can be obtained from a BGK-type equation with equilibrium densities satisfying Gibb’s entropy minimization principle.

1. INTRODUCTION

1.1. Motivation. We consider a Cauchy problem for a scalar conservation law

\[ \begin{align*}
\partial_t \rho + \text{div}_x A(\rho) &= 0, \quad (x,t) \in \mathbb{R}^{d+1}_+, \\
\rho(x,0) &= \rho_0(x), \quad x \in \mathbb{R}^d,
\end{align*} \tag{1} \]

where \( A : \mathbb{R} \to \mathbb{R}^d \) is a Lipschitz continuous function. For initial data

\( \rho_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \),

the problem is uniquely solvable in the class of admissible (entropy) solutions, as was established in [10]. When an admissible solution \( \rho(x,t) \) is represented by a kinetic density as

\[ \rho(x,t) = \int f(x,t,v) \, dv, \]

with

\[ f(x,t,v) = \begin{cases} 
\mathbb{1}_{[0,\rho(x,t)]}, & \rho(x,t) \geq 0 \\
-\mathbb{1}_{[\rho(x,t),0]}, & \rho(x,t) < 0 
\end{cases}, \tag{2} \]

then \( f \) is a weak solution of a kinetic equation

\[ \partial_t f + A'(v) \cdot \nabla_x f = -\partial_v m, \quad \mathcal{D}'(\mathbb{R}^{2d+1}_+), \tag{3} \]

where \( m \) is non-negative Radon measure on \( \mathbb{R}^{2d+1}_+ \). Conversely, any solution of (3) constrained by condition (2) for some \( \rho(x,t) \) defines an admissible weak solution of conservation law in (1), see [12]. Kinetic methods for obtaining admissible solutions originate in works [5,9]. References [1,2,3,4,11,13,16] is an short list of some representative results of the kinetic approach to solving systems of quasilinear PDEs.

Given a kinetic density \( f \), with \( \rho = \int f \, dv \), we will denote an equilibrium density in (2) by \( \Pi_f^{eq} \).
A class of approximate weak solutions of scalar conservation laws and equations of gas dynamics was introduced in [14, 15]. An approximate solution \( \rho(x, t) \) of (1) is characterized by the following properties. For any \( \varepsilon > 0 \) there is \( \rho \) such that

P1: \( \rho \) is a weak solution of the equation

\[
\partial_t \rho + \text{div}_x \rho \left( A(\rho)/\rho + O(\varepsilon) \right) = 0,
\]

where \( O(\varepsilon) \) is function of \((x, t)\) such that

\[
|O(\varepsilon)| \leq C\varepsilon, \quad \text{a.e. } (x, t) \in \mathbb{R}_+^{d+1};
\]

P2: \( \rho \) has a kinetic representation

\[
\rho(x, t) = \int f(x, t, v) \, dv,
\]

with \( f \) solving a kinetic equation

\[
\partial_t f + A'(v) \cdot \nabla_x f = -\partial_v m,
\]

where both \( m \) and \( \partial_v m \) are Radon measures \( \mathbb{R}_+^{2d+1} \), \( m \) is being non-negative;

P3: the kinetic density \( f \) deviates slightly from the equilibrium density:

\[
D(f) = \frac{\int v(f - \Pi_{eq}^f) \, dv}{\int vf \, dv} \leq \varepsilon, \quad \text{a.e. } (x, t) \in \mathbb{R}_+^{d+1};
\]

P4: there is a parametrized, unit mass, measure \( \mu_{x,t} \) on \( \mathbb{R}_y^d \), such that \( \mu_{x,t} \) is a measure-valued solution of the equation in (1):

\[
\partial_t \langle \rho, \mu_{x,t} \rangle + \text{div}_x \langle A(\rho), \mu_{x,t} \rangle = 0,
\]

and \( \mu_{x,t} \) is close to a delta mass concentrated at \( \rho(x, t) \):

\[
\mu_{x,t} = \delta(v - \rho(x, t)) + \mu_{x,t}^\varepsilon,
\]

with

\[
\text{mass} |\mu_{x,t}^\varepsilon| \leq C\varepsilon, \quad \text{a.e. } (x, t) \in \mathbb{R}_+^{d+1}.
\]

An example of an approximate solution corresponding to stationary shock data for Burger’s equation was constructed in [14]. The solution has a sharp interface of discontinuity (shock) which is \( \varepsilon \)-close to a classical shock, but it also contains \( \varepsilon \)-small rarefaction waves that interact with the shock and travel through it. It is unlikely that conditions P1–P4 determine approximate solutions in a unique fashion: there is large “amount of indeterminacy” in condition P1. However, the method that is used to construct them in [14] (described below), in dimension 1, results in approximate solutions that coincide with smooth solutions of (1), and for some initial data, coincide with shocks of (1) as well. In fact, it is possible to show that a sequence of approximate solutions \( \{\rho^\varepsilon\} \) with \( \varepsilon \to 0 \), accumulates on an admissible solution of (1).

In [14] approximate solutions are constructed by taking zero relaxation limit of a family of solutions of a BGK model

\[
\partial_t f + A'(v) \cdot \nabla_x f = \frac{\Pi_{eq}^f - f}{h} \mathbb{1}_{\{(x,t) : D(f(x,t,:)) > \varepsilon\}},
\]
where the deviation $D(f)$ is defined in (4).

Note that at the points $(x,t)$ where $D(f) \leq \epsilon$, equation (5) is a linear transport equation, which results in small (but non-vanishing) regularization of $f$ due to dispersion (mixing). This regularization is expressed in property P2 above, by the condition that $\partial_v m$ is a Radon measure.

A limiting point $f = \lim f^h$ is located near the set of equilibrium densities, as expressed by the condition $D(f) \leq \epsilon$, a.e. $(x,t)$. Thus, approximate solutions retain some small scale non-equilibrium features of kinetic equation (5).

This framework applies equally well to systems of conservation laws that have a kinetic representation, see [15] for an example of equations of isentropic gas dynamics.

In this paper we show that approximate solutions can obtained in a zero relaxation limit of BGK model

$$\partial_t f + A'(v) \cdot \nabla_x f = \frac{\Pi^\epsilon_f - f}{h},$$

where $\Pi^\epsilon_f$ is a solution of minimization problem

$$(7) \quad \min \left\{ \int \eta^\epsilon(v) f dv \right\},$$

constrained by conditions

$$\int f dv = \text{const.}, \quad f(v) \in [0, 1],$$

where $\eta^\epsilon(v)$ is a piece-wise constant approximation of entropy $\eta(v) = v$ which defines Gibb’s entropy $S(f) = \int vf dv$. The minimizer of the later is the equilibrium kinetic densities $\Pi^\epsilon_{eq}$, as in (2). The restriction of $f$ to have non-negative values can be made by considering only non-negative solutions $\rho(x,t)$, which can be assumed without the loss of generality. This approach formally puts the kinetic equation for approximate solutions (6) into a classical framework of kinetic equations in gas dynamics, in which the equilibrium density is a minimizer of an entropy, subject to prescribed moments. The important difference is that minimizers of (7) are not unique. In fact, we use this non-uniqueness to select a minimizer that is regularized by dispersion at $\epsilon$ scales, see lemma 2.

In our variational approach $\epsilon$ has different interpretation. Whereas in (5) it was a non-dimensional quantity measuring relative deviation of the entropy, here, we measure the deviation of $f$ from the equilibrium by

$$D(f) = \frac{\int v(f - \Pi^\epsilon_{eq}) dv}{\int f dv}.$$ 

Thus, $\epsilon$ has the dimension of the kinetic variable $v$.

Our main result established an approximate solution $\rho$ that verifies properties P1–P4, with the above $D(f)$. In addition, we improve condition P4, by showing that a measure-valued representation of $\rho(x,t) = (\rho, \mu_{x,t})$ with the measure $\mu_{x,t}$ is supported near $v = \rho(x,t)$:

$$\text{diam}(\text{supp } \mu_{x,t}) \leq C\epsilon, \quad \text{a.e. } (x,t).$$
1.2. Main results. In the rest of the paper we always assume that $A \in C^2(\mathbb{R}^d)$ and verifies a non-degeneracy condition:

\begin{equation}
\forall \sigma \in \mathbb{S}^{d-1}, \forall \xi \in \mathbb{R}, \quad \left| \left\{ v \in (-\|\rho_0\|_{L^\infty},\|\rho_0\|_{L^\infty}) : A'(v) \cdot \sigma = \xi \right\} \right| = 0,
\end{equation}

where $\mathbb{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$.

**Theorem 1.** Let $\rho_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. There are functions $\rho = \rho(x,t)$ and $f = f(x,t,v)$, with

$$
\rho = \int f \, dv, \quad \text{a.e.} \ (x,t),
$$

and $C > 0$, that verify the following properties.

1. $\rho \in L^\infty(\mathbb{R}^d_{+1}) \cap C([0, +\infty); W_{loc}^{1,p}(\mathbb{R}^d_{+1}))$, for any $p \in (1, +\infty)$, and verifies in the weak sense the initial condition in (1), $\rho$ is a weak solution of the equation

$$
\partial_t \rho + \text{div}_x \rho \left( A(\rho) / \rho + \tilde{A}(x,t) \right) = 0,
$$

for some functions $\tilde{A}(x,t)$ with

$$
\| \tilde{A} \|_{L^\infty(\mathbb{R}^d_{+1})} \leq C\varepsilon;
$$

2. for every convex function $\eta$ on $[0,M]$,

$$
\partial_t \int \eta' f \, dv + \text{div}_x \int \eta' A' f \, dv \leq 0, \quad \mathcal{D}'(\mathbb{R}^d_{+1});
$$

3. there is $m$ – a non-negative Radon measure on $\mathbb{R}^d_{+1}$ such that $\partial_x m$ is signed Radon measure and $f$ is a distributional solution of the equation

$$
\partial_t f + A' \cdot \nabla_x f = -\partial_x m.
$$

4. for a.e. $(x,t)$,

$$
\int v(f - \Pi_{f}^{eq}) \leq 4\varepsilon \int v f \, dv;
$$

5. there is a parametrized, unit mass, measure $\mu_{x,t}$ on $\mathbb{R}^d_v$, such that $\mu_{x,t}$ is a measure-valued solution of the equation in (1):

$$
\partial_t \langle \rho, \mu_{x,t} \rangle + \text{div}_x \langle A(\rho), \mu_{x,t} \rangle = 0,
$$

and $\mu_{x,t}$ is close to a delta mass concentrated at $\rho(x,t)$:

$$
\mu_{x,t} = \delta(v - \rho(x,t)) + \mu_{x,t}^\varepsilon,
$$

with

$$
\text{mass} | \mu_{x,t}^\varepsilon |, \text{diam} (\text{supp} \mu_{x,t}^\varepsilon) \leq C\varepsilon, \quad \text{a.e.} \ (x,t) \in \mathbb{R}^d_{+1}.
$$
1.3. **Proof of theorem**

**Proof.** Assume that $\rho_0$ is non-negative and denote by $M = 1 + \text{ess sup } \rho_0$. Define a piecewise constant function $\eta_\varepsilon$ as
\[
\eta_\varepsilon(v) = k, \quad v \in [(k-1)\varepsilon, k\varepsilon), \quad k = 1, \ldots, \lceil M/\varepsilon \rceil.
\]
For a non-negative constant $\rho \in [0, \text{ess sup } \rho_0]$ consider a minimization problem
\[
\min \left\{ \int \eta_\varepsilon(v)f(v) \, dv : f(v) \in [0, 1], \, \int f \, dv = \rho \right\}.
\]
Here and below $\int f \, dv = \int_M f \, dv$.

**Lemma 1.** Let $N = \lfloor \rho / \varepsilon \rfloor$. The minimum of the above problem equals
\[
\begin{cases}
\varepsilon \sum_{k=0}^{N-1} k + \varepsilon N (\rho - N \varepsilon), & N \geq 1, \\
0, & N = 0.
\end{cases}
\]
It is achieved on the minimizers
\[
f_{\min}(v) = 1_{[0,N\varepsilon]}(v) + \tilde{f}(v),
\]
where $\tilde{f}$ is an arbitrary function verifying conditions:
- $\tilde{f}(v) \in [0,1], \quad \forall v \in [0,M]$;
- $\text{supp } \tilde{f} \subset [N\varepsilon, (N+1)\varepsilon]$;
- $\int \tilde{f} \, dv = \rho - N \varepsilon$.

**Proof.** $\eta_\varepsilon(v)$ is a non-decreasing function. To minimize the functional $\int \eta_\varepsilon f \, dv$ we need to pick $f$ that has all its mass as close to $v = 0$ as possible, and is less than or equal 1. This leads to the statement of the lemma.

In the next lemma we show that the decrease of the entropy controls $L^1$ distance between function $f$ and a certain minimizer $f_{\min}$.

**Lemma 2.** Let $f$ be any function with values in $[0,1]$, with mass equal to $\rho$. If $f_{\min}$ is a minimizer from the last lemma, and
\[
f_{\min}(v) \geq f(v), \quad v \in [N\varepsilon, (N+1)\varepsilon],
\]
Then, for $\varepsilon \leq 1$,
\[
\int |f - f_{\min}| \, dv \leq \frac{3}{\varepsilon} \int \eta_\varepsilon(f - f_{\min}) \, dv.
\]

**Proof.** Consider first the case $N = 0$, or $N = \rho / \varepsilon = 1$. Under these conditions
\[
\int \eta_\varepsilon f_{\min} \, dv = 0.
\]
Then,
\[
\int |f - f_{\min}| \, dv = \int_0^\varepsilon f_{\min} - f \, dv + \int_{\varepsilon}^M f \, dv = 2 \int_{\varepsilon}^M f \, dv
\leq \frac{2}{\varepsilon} \int_{\varepsilon}^M \eta v \, dv = \frac{2}{\varepsilon} \int_0^M \eta v \, dv = \frac{2}{\varepsilon} \int_0^M \eta (f - f_{\min}).
\]

Suppose now \( N > 1 \).
\[
\int |f - f_{\min}| \, dv = \int_0^{N\varepsilon} (1 - f) \, dv + \int_{N\varepsilon}^{(N+1)\varepsilon} (f_{\min} - f) \, dv + \int_{(N+1)\varepsilon}^M f \, dv
\leq \int_{N\varepsilon}^{(N+1)\varepsilon} (f_{\min} - f) \, dv + 2 \int_{(N+1)\varepsilon}^M f \, dv
\leq \frac{1}{\varepsilon} \left( \int_{N\varepsilon}^{(N+1)\varepsilon} \eta (f_{\min} - f) \, dv + 2 \int_{(N+1)\varepsilon}^M \eta f \, dv \right) \leq \frac{3}{\varepsilon} \int \eta (f - f_{\min}) \, dv.
\]

A particular minimizer that verifies the conditions of the last lemma will be denoted by
\( \Pi_f^\varepsilon (v) = \mathbb{1}_{[0,N\varepsilon + v_0]}(v) + \mathbb{1}_{(N\varepsilon + v_0,(N+1)\varepsilon)}(v) f(v), \)
where number \( v_0 \in [0,\varepsilon] \) equals
\[
v_0 = \max \left\{ 0, \int_0^{N\varepsilon} + \int_{(N+1)\varepsilon}^M f \, dv - N\varepsilon \right\}.
\]

**Lemma 3.** Let \( \eta \) be a convex function on \([0,M]\). Then,
\[
\int \eta(v)(f(v) - \Pi_f^\varepsilon(v)) \, dv \geq 0.
\]

**Proof.** Restricted to the compliment of \([v_0,(N+1)\varepsilon]\), function \( \Pi_f^\varepsilon \) coincides with the equilibrium density of \( f \), restricted to the same set. For an equilibrium density \( \Pi_f^{eq} \) the inequality is a well-know fact, shown for example in [5]. Since \( f \) and \( \Pi_f^\varepsilon \) coincide on \([v_0,(N+1)\varepsilon]\), the inequality follows. \( \square \)

We consider the Cauchy problem
\[
\partial_t f + A'(v) \cdot \nabla_x f = \frac{\Pi_f^\varepsilon - f}{h}, \quad (x,t,v) \in \mathbb{R}^d \times \mathbb{R}_+ \times [0,M],
\]
\[
f(x,0,v) = f_0(x,v), \quad (x,v) \in \mathbb{R}^d \times [0,M].
\]

The proof of the next theorem can deduced by repeating the arguments of a result of [6], or theorem 4.7 of [14], that apply to the same problem with \( \Pi_f^{eq} \), instead of \( \Pi_f^\varepsilon \) on the right-hand side of the equation (10). We omit the proof.

**Theorem 2.** Let \( f_0 \in L^\infty(\mathbb{R}^d \times [0,M]) \), with values in \([0,1]\) for a.e. \((x,v)\), with the support
\[
\text{supp } f_0(x,\cdot) \subset [0,M], \quad a.e. \ x,
\]
and finite moments
\[ \iint (1 + v)f_0(x, v) \, dx \, dv < +\infty. \]

For any \( h > 0 \) there is a weak solution of the problem (10), (11): for any \( p \in [1, +\infty) \),
\[ f \in L^\infty(\mathbb{R}^{d+1} \times [0, M]) \cap L^\infty(0, +\infty; L^1(\mathbb{R}^d \times [0, M])) \cap C([0, +\infty); L^p_{loc,weak}(\mathbb{R}^d \times [0, M])), \]
with the following properties: for all \( t > 0 \) and a.e. \( (x, v), f(x, t, v) \in [0, 1] \);
\[ \text{supp } f(x, t, \cdot) \subset [0, M], \quad \text{a.e. } (x, t); \]
for all \( t > 0 \),
\[ \iint (1 + v)f(x, t, v) \, dx \, dv \leq \iint (1 + v)f_0(x, v) \, dx \, dt. \]

Solutions of a BGK model verify the following estimates.

**Lemma 4** (Entropy estimates). Let \( f \) be a solution of (10), (11) with properties listed in the above theorem. There exists \( C > 0 \), independent of \( h \), such that \( \forall T > 0 \),
\[
\begin{align*}
(12) & \quad \sup_{[0, T]} \int \eta_f(x, t, v) \, dv \, dx + \frac{1}{h} \int_0^T \int \eta_f(f - \Pi_f^\varepsilon) \, dv \, dx \, dt \leq C, \\
(13) & \quad \frac{\varepsilon}{h} \int_0^T \int |f - \Pi_f^\varepsilon| \, dv \, dx \, dt \leq C, \\
(14) & \quad \frac{\varepsilon}{h} \int_0^T \int f \mathbb{1}_{[\rho(x, t) + \varepsilon, M]}(v) \, dv \, dx \, dt \leq C, \\
(15) & \quad \frac{\varepsilon}{h} \int_0^T \int (1 - f) \mathbb{1}_{[0, \max(0, \rho(x, t) - \varepsilon)]}(v) \, dv \, dx \, dt \leq C.
\end{align*}
\]

Let \( f_0(x, v) \) be the equilibrium density corresponding to initial data \( \rho_0(x) \). Let \( f^h \) be a sequence of solutions of (10), (11) with such \( f_0(x, t) \), and consider the compactness properties of \{\( f^h \)\} as \( t \to 0 \). Since \( f^h \) are bounded in \( L^\infty \), and the right-hand sides of (10) are bounded \( L^1(\mathbb{R}^{d+1} \times [0, M]) \), due to estimate (13), the compactness theorem of Gérard, see [8], implies that for any test function \( \psi(v) \) the moments
\[
\left\{ \int \psi(v) f^h \, dv \right\} \quad \text{pre-compact in } L^p_{loc}(\mathbb{R}^{d+1}),
\]
for any \( p \in [1, +\infty) \). Thus, we can select a subsequence (still labeled by \( h \)) such that for some \( f \in L^\infty(\mathbb{R}^{d+1} \times [0, M]) \), with values in \([0, 1]\), for which
\[
\begin{align*}
\int f^h \, dv & \quad \text{a.e. } (x, t) \quad \text{and in } L^p_{loc}(\mathbb{R}^{d+1}); \\
\int v f^h \, dv & \quad \text{a.e. } (x, t) \quad \text{and in } L^p_{loc}(\mathbb{R}^{d+1}); \\
\int \eta_f f^h \, dv & \quad \text{a.e. } (x, t) \quad \text{and in } L^p_{loc}(\mathbb{R}^{d+1}); \\
\int \eta f^h \, dv & \quad \text{a.e. } (x, t) \quad \text{and in } L^p_{loc}(\mathbb{R}^{d+1}); \\
\int \eta f \, dv & \quad \text{a.e. } (x, t) \quad \text{and in } L^p_{loc}(\mathbb{R}^{d+1}); \\
\eta f^h & \quad \text{a.e. } (x, t) \quad \text{and in } L^p_{loc}(\mathbb{R}^{d+1}).
\end{align*}
\]
Estimates (14), (15) imply that
\[ f(x, t, v) = 0, \quad (x, t, v) \in \mathbb{R}^{d+1}_+ \times (\rho(x, t) + \varepsilon, M), \]
\[ f(x, t, v) = 1, \quad (x, t, v) \in \mathbb{R}^{d+1}_+ \times (0, \max\{0, \rho(x, t) - \varepsilon\}). \]

This implies that a.e. \((x, t)\),
\[ \int v(f - \Pi_{f}^{eq}) \, dv \leq \int_{\max\{0, \rho - \varepsilon\}}^{\rho + \varepsilon} v \, dv \leq 4\varepsilon \int f \, dv, \]
which establishes part 2 of the theorem.

Similarly, for any \(i\),
\[ (16) \quad \left| \int A_i(v)(f - \Pi_{f}^{eq}) \, dv \right| \leq C\varepsilon\rho, \]
for some \(C\) determined by \(A_i\). This establishes the equation (9).

It remains to show that there is a measure \(\mu_{x,t}\) with the properties stated in the theorem. We follow the approach from [14], where a similar fact is established.

Let \(a_i(v)\) be a continuously differentiable extension of \(A_i'(v)\) restricted to the interval \((\max\{0, \rho - \varepsilon\}, \rho + \varepsilon)\):
\[ a_i(v) = A_i'(v), \quad v \in (\max\{0, \rho - \varepsilon\}, \rho + \varepsilon) \]
\[ a_i(v) = 0, \quad v \in (\max\{0, \rho - 2\varepsilon\}, \rho + 2\varepsilon). \]
Functions \(a_i\) depend on \((x, t)\) through \(\rho = \rho(x, t)\), which we implicitly assume.

Condition (8) implies that set \(\{a_i\}\) is linearly independent on \([0, M]\). Let \(f_0\) be the projection of \(f - \Pi_{f}^{eq}\) to \(\text{Span}\{a_1, ..., a_d\} \subset L^2([0, M])\). Thus,
\[ f_0(v) = \sum_{i=1}^{d} \alpha_i a_i(v), \]
and due to estimate (16), all \(|\alpha_i| \leq C\varepsilon\), for some \(C > 0\), independently of \((x, t)\). Note, that also \(\text{diam}(\text{supp} f_0) \leq 4\varepsilon\).

The measure \(\mu_{x,t}\) can be defined as
\[ \mu_{x,t} = f_0'(v) \, dv + \delta(v - \rho(x, t)). \]

\[\square\]

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