ALMOST SURE RATES OF MIXING FOR RANDOM INTERMITTENT MAPS

MARKS RUZIBOEV

Dedicated to Abdulla Azamov and Leonid Bunimovich on the occasion of their 70th birthday

ABSTRACT. We consider a family $F$ of maps with two branches and a common neutral fixed point $0$ such that the order of tangency at $0$ belongs to the interval $[\alpha_0, \alpha_1] \subset (0, 1)$. Maps in $F$ do not necessarily share common Markov partition. At each step a member of $F$ is chosen independently with respect to uniform distribution on $[\alpha_0, \alpha_1]$. We obtain upper bounds for the quenched decay of correlations of form $n^{-1/\alpha_0 + \delta}$ for any $\delta > 0$.

1. INTRODUCTION

In recent years many papers have been devoted to the study statistical properties of random dynamical systems induced by random compositions of different maps (see for example [1]-[6], [10], [13], [14], [17] and references therein). In [4] i.i.d. random compositions of two Liverani-Saussol-Vaienti (LSV)\(^1\) maps were considered and it was shown that the rate of decay of the annealed (averaged over all realizations) correlations is given by the fast dynamics. Recently the general results on quenched decay rates (i.e. decay rates for almost every realization) for the random compositions of non-uniformly expanding maps were obtained in [5]. As an illustration it was shown in [5] that the general results are applicable to the random map induced by compositions of LSV maps with parameters in $[\alpha_0, \alpha_1] \subset (0, 1)$ chosen with respect to a suitable distribution $\nu$ on $[\alpha_0, \alpha_1]$. In the current note we fix uniform distribution on $[\alpha_0, \alpha_1]$ and consider a family of maps with common neutral fixed point. Our maps do not share a common Markov partition. We show that the construction of the random tower of [5] with general return time can be carried out for the random compositions of such maps. Hence general result

\(^1\)These are subclass of the so called intermittent or Pomeau-Manneville maps, which popularised in [15]. Such systems have attracted the attention of both mathematicians and physicists. Pomeau-Manneville was first introduced in [18] to model an intermittent transition to turbulence, (see [14] for a recent work in this area).
of \([5]\) is applicable. We obtain upper bounds for the quenched decay of correlations of form \(n^{-1/\alpha_0+\delta}\) for any \(\delta > 0\).

The rest of the papers is organized as follows. In Section \([2]\) we give formal definition of the family \(\mathcal{F}\) and state the main result of the paper (Theorem \([1]\)). In Section \([3]\) we construct uniformly expanding induced random map and show that the assumptions required in \([5]\) are satisfied i.e. we check uniform expansion, bounded distortion, decay rates for the tail of the return time and aperiodicity.

\[\text{2. THE SET UP AND THE MAIN RESULTS}\]

In this section we define the main object of the current note: the random maps. Fix two real numbers \(0 < \alpha_0 < \alpha_1 < 1\). Let \(I = [0, 1]\) and let \(\mathcal{F}\) be a parametrized family of maps \(T_\alpha : I \to I, \alpha \in [\alpha_0, \alpha_1]\) with the following properties.

(A1) There exists a \(C^1\) function \(x : [\alpha_0, \alpha_1] \to (0, 1), \alpha \mapsto x_\alpha\) such that \(T_\alpha : [0, x_\alpha) \to [0, 1)\) and \(T_\alpha : [x_\alpha, 1] \to [0, 1]\) are increasing diffeomorphisms.

(A2) \(T_\alpha'(x) > 1\) for any \(x > 0\).

(A3) There exists \(\varepsilon_0 > 0\) and a continuous functions \(\alpha \mapsto c_\alpha, (x, \alpha) \mapsto f_\alpha(x)\) such that \(f_\alpha(0) = 0\) and \(T_\alpha(x) = x + c_\alpha x^{1+\varepsilon}(1 + f_\alpha(x))\) for any \(x \in [0, \varepsilon_0]\).

(A4) Every \(T_\alpha\) is \(C^3\) on \([0, x(\alpha)]\) with negative Schwarzian derivative.

(A5) \((x, \alpha) \mapsto T_\alpha'(x)\) and \((x, \alpha) \mapsto T_\alpha''(x)\) are continuous on \(I \times [\alpha_0, \alpha_1]\).

Notice that the elements of \(\mathcal{F}\) are parametrized according to the tangency near \(0\). Now, we describe the randomizing dynamics. Let \(\eta\) be the normalized Lebesgue measure on \([\alpha_0, \alpha_1]\). Let \(\Omega = [\alpha_0, \alpha_1]^\mathbb{Z}\) and \(\mathbb{P} = \eta^\mathbb{P}\). Then the shift map \(\sigma : \Omega \to \Omega\) preserves \(\mathbb{P}\), i.e. \(\sigma_* \mathbb{P} = \mathbb{P}\). For \(\omega \in \Omega, \omega = \ldots \omega_{-1}, \omega_0, \omega_1, \ldots\) let \(\alpha(\omega) = \omega_0 \in [\alpha_0, \alpha_1]\). The random map is formed by random compositions of maps \(T_{\alpha(\omega)} : I \to I\) from \(\mathcal{F}\), where the compositions are defined as \(T_{\alpha(\omega)}^n(x) = T_{\alpha(\sigma^{n-1}(\omega))} \circ \cdots \circ T_{\alpha(\omega)}(x)\). Below we use more shortcut notation \(T_{\alpha(\omega)}^n := T_{\alpha(\omega)}^n(\omega)\). We will denote the random map by \(T_{\mathcal{F}}\) to indicate the underlying family. We are interested in the statistical properties of equivariant families of measures i.e. families of measures \(\{\mu_\omega\}_{\omega \in \Omega}\) such that \((T_{\alpha(\omega)})_* \mu_\omega = \mu_{\sigma^{n\alpha(\omega)}}\). Let \(\mu\) be a probability measure on \(I \times \Omega\) such that \(\mu(A) = \int_\Omega \mu_\omega(A) d\mathbb{P}(\omega)\) for \(A \subset I \times \Omega\). We say that the system \(\{f_\omega, \mu_\omega\}_{\omega \in \Omega}\) (or simply \(\{\mu_\omega\}_\omega\)) is mixing if for all \(\varphi, \psi \in L^2(\mu)\),

\[
\lim_{n \to \infty} \left| \int_\Omega \int_0^1 \varphi_{\sigma^{n\omega}} \circ f_\omega \cdot \psi_\omega d\mu_\omega dP - \int_\Omega \int_0^1 \varphi_\omega d\mu_\omega dP \int_\Omega \int_0^1 \psi_\omega d\mu_\omega dP \right| = 0.
\]

Further, future and past correlations are defined as follows. Let \(\varphi, \psi : I \to \mathbb{R}\) be two observables on \(I\). Then we define future correlations as

\[
\text{Cor}_{\mathcal{F}}(\varphi, \psi) := \left| \int \varphi \circ T_{\alpha(\omega)}^n \psi d\mu_{\sigma^{n\alpha(\omega)}} - \int \varphi d\mu_{\sigma^{n\alpha(\omega)}} \int \psi d\mu_\omega \right|
\]
and past correlations as

\[ \text{Cor}^p_\mu(\varphi, \psi) := \left| \int (\varphi \circ T^n_{\sigma^{-n}\omega}) \psi d\mu_\omega - \int \varphi d\mu_\omega \int \psi d\mu_{\sigma^{-n}\omega} \right|. \]

**Theorem 1.** Let \( T_F \) be the random map described as above. Then for almost every \( \omega \in \Omega \) there exists a family of absolutely continuous equivariant measures \( \{ \mu_\omega \}_\omega \) on \( I \) which is mixing. Moreover, for every \( \delta > 0 \) there exists a full measure subset \( \Omega_0 \subset \Omega \) and a random variable \( C_\omega : \Omega \to \mathbb{R}_+ \) which is finite on \( \Omega_0 \) such that for any \( \varphi, \psi \in L^\infty(I) \), \( \psi \in C^\alpha(I) \) there exists a constant \( C_{\varphi, \psi} > 0 \) so that

\[ \text{Cor}^p_\mu(\varphi, \psi) \leq C_\omega C_{\varphi, \psi} n^{1-\frac{\alpha_0}{\alpha}+\delta} \quad \text{and} \quad \text{Cor}^p_\mu(\varphi, \psi) \leq C_\omega C_{\varphi, \psi} n^{1-\frac{\alpha_0}{\alpha}+\delta}. \]

Furthermore, there exist constants \( C > 0, u' > 0 \) and \( 0 < \nu' < 1 \) such that

\[ P\{C_\omega > n\} \leq C e^{-u'n^{\nu'}}. \]

**Remark 1.** Notice that in the deterministic setting every mapping in the family \( F \) admits an absolutely continuous invariant probability measure, which is polynomially mixing at the rate \( n^{1-\alpha} \) if \( T_\alpha(x) = x + c_\alpha x^{1+\alpha}(1 + f_\alpha(x)) \) (see [20], [8]). In the random setting the upper bounds we give are arbitrarily close to the sharp decay rates of the fastest mixing system in the family. Since the result holds for almost every \( \omega \in \Omega \), and in principle there can be arbitrarily long compositions of systems in \( T_\alpha^n \) whose mixing rates are slower than that of \( T_\alpha \), it is not expected that the mixing rate of the random system will be the same as the mixing rate of the fastest mixing system in the family \( F \) and \( C_\omega \) integrable at the same time.

**Remark 2.** We also remark that we are choosing the family \( F \) so that all the maps in it share the common neutral fixed point 0. If we choose the family by allowing different maps having distinct neutral fixed points i.e. \( T_\alpha(p(\alpha)) = p(\alpha), T'_\alpha(p(\alpha)) = 1 \) and \( p(\alpha) \neq 0 \) for a positive (with respect to \( \nu \)) measure set of parameters \( \alpha \in [\alpha_0, \alpha_1] \) and expanding elsewhere, then the resulting random map is expanding on average. Whence one can apply spectral techniques as in [7] on the Banach space of quasi-Hölder functions from [12] or [19] and obtain exponential decay rates. Such systems are out of context in our setting since we are after systems with only polynomial decay of correlations.

To prove the theorem we construct a random induced map (or Random Young Tower) for \( T_F \) with the properties described in [5]. Below we briefly recall the definition of induced map.

Let \( m \) denote Lebesgue measure on \( I \) and \( \Lambda \subset I \) be a measurable subset. We say \( T_F \) admits a Random Young Tower with the base if for almost every \( \omega \in \Omega \) there exists a countable partition \( \{ \Lambda_j(\omega) \}_j \) of \( \Lambda \) and a return time function \( R_\omega : \Lambda \to \mathbb{N} \) that is constant on each \( \Lambda_j(\omega) \) such that

(P1) for each \( \Lambda_j(\omega) \) the induced map \( T^{R_\omega}_\omega \mid_{\Lambda_j(\omega)} \) is a diffeomorphism and there exists a constant \( \beta > 1 \) such that \( (T^{R_\omega}_\omega)' > \beta. \)
(P2) There exists $D > 0$ such that for all $\Lambda_j(\omega)$ and $x, y \in \Lambda_j(\omega)$

$$\left| \frac{(T^R_\omega)^t x}{(T^R_\omega)^t y} - 1 \right| \leq D\beta^{-s(x,y)},$$

where $s(x,y)$ is the smallest $n$ such that $(T^R_\omega)^n x$ and $(T^R_\omega)^n y$ lie in distinct elements.

(P3) There exists $M > 0$ such that $\sum_n m \{ x \in \Lambda \mid R_\omega(x) > n \} \leq M$ for all $\omega \in \Omega$;

There exist $C, u, v > 0, a > 1, b \geq 0$, a random variable $n_1 : \Omega_1 \to \mathbb{N}$ so that

$$\begin{cases} m \{ x \in \Lambda \mid R_\omega(x) > n \} \leq C \frac{(\log n)^{b}}{n^a}, & \text{whenever } n \geq n_1(\omega), \\ \Pr \{ n_1(\omega) > n \} \leq Ce^{-un^v}; \end{cases}$$

(2.1)

(2.2)

$$\int m \{ x \in \Lambda \mid R_\omega = n \} d\Pr(\omega) \leq C \frac{(\log n)^{b}}{n^{a+1}}.$$ 

(P4) There are $N \in \mathbb{N}$ and $\{ t_i \in \mathbb{Z}_+ \mid i = 1, 2, ..., N \}$ such that g.c.d. $\{ t_i \} = 1$ and $\epsilon_i > 0$ so that for almost every $\omega \in \Omega$ and $i = 1, 2, \ldots, N$ we have

$$m \{ x \in \Lambda \mid R_\omega(x) = t_i \} > \epsilon_i.$$ 

Under the above assumptions it is proven in [5] that there exists a family of absolutely continuous equivariant measures (Theorem 4.1, [5]), which is mixing and the mixing rates have upper bound of form $n^{1+\delta-a}$ for any $\delta > 0$ (Theorem 4.2, [5]). Therefore to prove the Theorem it is sufficient to construct an induced map $T^R_\omega$ with the properties (P1)-(P4), which is carried out in the next section.

3. INDUCING SCHEME

Here we will construct a uniformly expanding full branch induced random map on $\Lambda = [0, 1]$ for almost every $\omega \in \Omega$. Let $X_0(\omega) = 1$, $X_1(\omega) = x(\omega_0) = x_{\alpha}(\omega)$ and

$$X_n(\omega) = (T^R_\omega |_{[0,x(\omega_0)]})^{-1} X_{n-1}(\sigma\omega) \text{ for } n \geq 2.$$ 

Let $I_n(\omega) = (X_n(\omega), X_{n-1}(\omega))$. Then by definition $T_\omega(I_n(\omega)) = I_{n-1}(\sigma\omega)$. By induction we have

$$I_n(\omega) \xrightarrow{T_\omega} I_{n-1}(\sigma\omega) \xrightarrow{T_\omega} \cdots \xrightarrow{T_\omega} I_1(\sigma^{n-1}\omega) \xrightarrow{T_\omega^{n-1}} I.$$ 

Hence, every interval $I_n(\omega)$ first is mapped onto $I_1(\omega)$ and then is mapped onto $\Lambda$ by the next iterate of $T_\omega$. Define a return time $R_\omega : [0, 1] \to \mathbb{N}$ by setting $R_\omega |_{(X_n(\omega), X_{n-1}(\omega))} = n$. Then the induced map $T^R_\omega : [0, 1] \to [0, 1]$ defined as $T^R_\omega |_{I_n(\omega)} = T^R_\omega^n$, for $n \geq 1$ is full branch. By assumptions (A1) and (A2) there exists $\beta > 1$ such that $T^R_\omega > \beta$ for all $\omega \in \Omega$. In fact, we can choose

$$\beta = \min_{\omega \in [0,\alpha]} \min_{x \in [x(\omega_0), 1]} |T^R_\omega(x)|.$$ 

This proves (P1). By (A1) all the maps in $\mathcal{F}$ have two full branches with $x_\alpha < 1$. Hence, the interval where $R_\omega = 1$ has strictly positive length and thus (P4) is obviously satisfied.
To prove the remaining properties we need the following.

**Proposition 3.1.** 1) For every $\omega \in \Omega$ the sequence $\{X_n(\omega)\}_n$ is decreasing and $\lim_{n \to \infty} X_n(\omega) = 0$. Moreover, there exists a constant $C_0 > 0$ such that for all $\omega \in \Omega$

\[
(3.2) \quad \frac{1}{C_0 n^{1/\alpha_0}} \leq X_n(\omega) \leq \frac{C_0}{n^{1/\alpha_0}}.
\]

2) There exists $C, u > 0, v \in (0, 1)$ and a random variable $n_1 : \Omega \to \mathbb{N}$ which is finite for $\mathbb{P}$-almost every $\omega \in \Omega$ such that

\[
(3.3) \quad X_n(\omega) \leq C n^{-1/\alpha_0} (\log n)^{1/\alpha_0}, \quad \forall n \geq n_1,
\]

\[
(3.4) \quad \mathbb{P}\{\omega \mid n_1(\omega) > n\} \leq C e^{-v n^u},
\]

\[
(3.5) \quad \int (X_n(\omega) - X_{n-1}(\omega)) d\mathbb{P}(\omega) \leq C n^{-1-1/\alpha_0} (\log n)^{1/\alpha_0}.
\]

The proof of the proposition proceeds as in [5], but needs extra care due to the general form of the maps under the consideration. First we prove an auxiliary lemma.

**Lemma 3.2.** For any $k \in \mathbb{N}$, $c \geq 1$ and $t > 0$ we have

\[
E_\mathbb{P}[e^{-(\alpha_0 k^h \omega - \alpha_0)t}] = \frac{1}{\alpha_1 - \alpha_0} \frac{e^{\alpha_0 t(1-c)}}{ct} (1 - e^{-ct(\alpha_1 - \alpha_0)}).
\]

**Proof.** Since $\sigma$ preserves $\mathbb{P}$ we have

\[
E_\mathbb{P}[e^{-(\alpha_0 k^h \omega - \alpha_0)t}] = E_\mathbb{P}[e^{-(\alpha_0 \omega - \alpha_0)t}]
\]

\[
= \frac{1}{\alpha_1 - \alpha_0} \int_{\alpha_0}^{\alpha_1} e^{-(\alpha_0 - \alpha_0)t} dx = \frac{1}{\alpha_1 - \alpha_0} \frac{e^{\alpha_0 t(1-c)}}{ct} (1 - e^{-ct(\alpha_1 - \alpha_0)}).
\]

Proof of Proposition 3.1. The first two assertions in item 1) are obvious, since $T'(x) > 1$ for $x > 0$ and $x = 0$ is the unique fixed point in $[0, 1/2]$. Since all the maps in $\mathcal{F}$ are uniformly expanding except 0, there exists $n_0 \in \mathbb{N}$ independent of $\omega$ such that $X_n(\omega) \in (0, \varepsilon_0)$ for all $n \geq n_0$. Thus, it is sufficient to prove inequality (3.2) for any $n \geq n_0$. We now define a sequence $\{Z_n\}_n$ which bounds $X_n(\omega)$ from below and has desired asymptotic. Let $K_0 = [0, \varepsilon_0] \times [0, \alpha_1]$ and $C_1 = \max_{(x, \omega) \in K_0} c_\alpha (1 + k(\alpha))$.

Set $G(x) = x(1 + C_1 x^{\alpha_0})$. Define $\{Z_n\}_{n \geq n_0}$ as follows: $Z_{n_0} = \min_{\omega \in \Omega} X_{n_0}(\omega)$ and let $Z_n = (G|_{[0, \varepsilon_0]})^{-1}(Z_{n-1})$ for $n > n_0$. Since $G(x) \geq T_{\alpha}(\omega)(x)$ for any $x \in [0, \varepsilon_0]$ and for any $\omega \in \Omega$, one can easily verify by induction that $Z_n \leq X_n(\omega)$ for $n \geq n_0$. Finally note that $Z_n \sim n^{-1/\alpha_0}$ [8]. Defining $C_1' = \min_{(x, \alpha) \in K_0} c_\alpha (1 + k(\alpha))$, $G'(x) = x(1 + C_1' x^{\alpha_1})$, $Z_{n_0}' = \max_{\omega \in \Omega} X_{n_0}(\omega)$ and $Z_n' = (G'|_{[0, \varepsilon_0]})^{-1}(Z_{n-1}')$ for $n > n_0$ we obtain a sequence $\{Z_n'\}$ such that $X_n(\omega) \leq Z_n'$ and $Z_n' \sim n^{-1/\alpha_1}$. This finishes the proof of item 1).

To prove item 2) first note that by the choice of $n_0$ for any $n \geq n_0$ we have

\[
(3.6) \quad X_n(\sigma \omega) = X_{n+1}(\omega)[1 + c_\alpha(\omega) X_{n+1}(\omega)^{\alpha}(1 + f_{\alpha}(\omega) \circ X_{n+1}(\omega))].
\]
The latter equality together with the standard estimate \((1 + x)^{-a} \leq 1 - ax + \frac{a(a+1)}{2}x^2\) for \(x, a > 0\) implies that
\[
\frac{1}{X_{n+1}(\omega)^{\alpha_0}} - \frac{1}{X_n(\sigma \omega)^{\alpha_0}} \geq C_1 \alpha_0 X_{n+1}(\omega)^{\alpha(\omega) - \alpha_0} - C_2 X_{n+1}(\omega)^{2\alpha(\omega) - \alpha_0},
\]
where \(C_2 = \frac{\alpha_0(\alpha_0+1)}{2} \min_{\alpha(x) \in K_0} \{ c_\alpha (1 + f_\alpha(x)) \}^2 \). Hence,
\[
\frac{1}{X_n(\omega)^{\alpha_0}} \geq \frac{1}{x^{\alpha_0(\omega)}} + C_1 \alpha_0 \sum_{k=2}^{n} X_k(\omega)^{\alpha(\sigma^{-k}\omega) - \alpha_0} - C_2 \sum_{k=2}^{n} X_k(\omega)^{2\alpha(\sigma^{-k}\omega) - \alpha_0},
\]
Notice that take \(C_1\) and \(C_2\) are independent of \(\omega\). Therefore, by inequality (3.2) we have
\[
(3.7) \quad \frac{1}{X_n(\omega)^{\alpha_0}} \geq 1 + C_3 \sum_{k=2}^{n} \left(k^{1/\alpha_0}\right)^{\alpha_0 - \alpha(\sigma^{-k}\omega)} - C_2 \sum_{k=2}^{n} \left(k^{1/\alpha_1}\right)^{2\alpha(\sigma^{-k}\omega) + \alpha_0},
\]
To obtain estimates for the right hand side of the latter inequality we set
\[
a_k := \left(k^{1/\alpha_0}\right)^{\alpha_0 - \alpha(\sigma^{-k}\omega)}, \quad b_k := \left(k^{1/\alpha_1}\right)^{2\alpha(\sigma^{-k}\omega) + \alpha_0}
\]
and
\[
S_n = \sum_{k=2}^{n} C_3 a_k - C_2 b_k.
\]
Applying the above lemma to \(E_P(e^{\log a_k})\) with \(c = 1\) and \(u = \log k^{1/\alpha_1}\) and using the fact \(\sum_{k \leq n} \frac{1}{\log k} \sim \frac{n}{\log n}\) we obtain
\[
\sum_{k=2}^{n} E_P(a_k) = \frac{\alpha_0}{\alpha_1 - \alpha_0} \sum_{k=2}^{n} \frac{1}{\log k} (1 - k^{-\frac{\alpha_1 - \alpha_0}{\alpha_0}}) = \frac{\alpha_0}{\alpha_1 - \alpha_0} \frac{n}{\log n} + O\left(n^{-\frac{\alpha_1 - \alpha_0}{\alpha_0}} (\log n)^{-1}\right)
\]
and hence,
\[
(3.8) \quad \frac{\log n}{n} \sum_{k=2}^{n} E_P(a_k) = \frac{\alpha_0}{\alpha_1 - \alpha_0} + O\left(n^{-\frac{\alpha_1 - \alpha_0}{\alpha_0}} \right).
\]
Similarly, applying Lemma 3.2 to \(E_P(b_k)\) with \(c = 2\) and \(t = \log k^{1/\alpha_1}\), we obtain
\[
\sum_{k=2}^{n} E_P(b_k) := \frac{\alpha_1}{2(\alpha_1 - \alpha_0)} \sum_{k=2}^{n} \frac{1}{\log k} \left(k^{-\frac{\alpha_0}{\alpha_1}} - k^{-\frac{\alpha_0}{\alpha_1} + 1}\right) = \frac{\alpha_1}{2(\alpha_1 - \alpha_0)} \frac{n^{1-\alpha_0/\alpha_1}}{\log n} + o(n)
\]
and hence,
\[
(3.9) \quad \lim_{n \to \infty} \frac{\log n}{n} \sum_{k=2}^{n} E_P(b_k) = \lim_{n \to \infty} n^{-\alpha_0/\alpha_1} = 0.
\]
Combining (3.8) and (3.9) implies
\[
\lim_{n \to \infty} \frac{\log n}{n} E_P(S_n) = \lim_{n \to \infty} \frac{\log n}{n} \sum_{k=2}^{n} E_P(C_3 a_k - C_2 b_k) = C_4,
\]

where $C_4 = C_3 \alpha_0 / \alpha_1 - \alpha_0$. Thus, there exists $N$ independent of $\omega$ such that

\begin{equation}
\frac{C_4}{2} \leq \frac{\log n}{n} E_P(S_n) \leq \frac{3C_4}{2}
\end{equation}

for all $n \geq N$. On the other hand, by Theorem 1 of Hoeffding \[11\], there exists $C > 0$ such that for every $t > 0$ and $n \in \mathbb{N}$ we have

$$
P \left\{ \frac{\log n}{n} \left| S_{n+1} - E_P S_{n+1} \right| < t \right\} \leq e^{-\frac{Ct^2}{(\log n)^2}}.
$$

Thus, by letting $C_5 = CC_4^2 / 16$ we obtain

$$
P \left\{ \frac{\log n}{n} S_{n+1} < \frac{C_4}{4} \right\} \leq P \left\{ \frac{\log n}{n} (S_{n+1} - E_P S_{n+1}) < -\frac{C_4}{4} \right\} \leq e^{-\frac{C_5 n}{(\log n)^2}}.
$$

Define $n_1(\omega) = \inf \{ n \geq N \mid \forall k \geq n, \frac{\log k}{k} S_k \geq \frac{C_4}{4} \}$. Then, the above inequality implies that

\begin{equation}
P \{ n_1(\omega) > n \} \leq \sum_{k=n}^{\infty} e^{-\frac{C_5 k}{(\log k)^2}} \leq C_6 \sum_{k=n}^{\infty} e^{-uk} v \leq Ce^{-un}
\end{equation}

for some $C > 0$, $u > 0$ and $v \in (0, 1)$.

For any $n \geq n_1$ by (3.7) we have

$$X_n(\omega)^{\alpha_0} \leq \frac{\log n}{n} \frac{4}{C_4}.
$$

Hence, for some positive $C > 0$ we have

$$X_n(\omega) \leq C \left( \frac{\log n}{n} \right)^{1/\alpha_0}.
$$

This together with inequality (3.11) implies (3.3) and (3.4). It remains to prove (3.5). Recall that there exists $n_0$ which depends only on $\varepsilon_0$ in (A3) such that (3.6) holds for all $n \geq n_0$. Thus, recalling that $\sigma$ preserves $P$ we have

$$
\int m \{ R_{\omega} = n \} dP(\omega) \leq \frac{1}{\beta} \int (X_{n-1}(\sigma \omega) - X_n(\sigma \omega)) dP(\omega) = \frac{1}{\beta} \int (X_{n-1}(\sigma \omega) - X_n(\omega)) dP(\omega) = \int_{\{n_1(\omega) > n\}} (X_{n-1}(\sigma \omega) - X_n(\omega)) dP(\omega) + \int_{\{n_1(\omega) \leq n\}} (X_{n-1}(\sigma \omega) - X_n(\omega)) dP(\omega)
\leq Ce^{-u n} + \int_{\{n_1(\omega) \leq n\}} c_{\alpha(\omega)} X_n(\omega)^{\alpha(\omega) + 1} (1 + f_{\alpha(\omega)} \circ X_n(\omega)) dP(\omega)
\leq Ce^{-u n} + C \int \left( \frac{\log n}{n} \right)^{(\alpha(\omega) + 1) / \alpha_0} dP(\omega) \leq C \left( \frac{\log n}{n} \right)^{(\alpha_0 + 1) / \alpha_0}.
$$
This finishes the proof for all $n \geq n_0$. For $n < n_0$ the assertion follows by increasing the constant $C$ if necessary. □

Now we will prove (P3). By definition of $R_\omega$ and inequality (3.2) we have

$$m\{R_\omega > n\} = X_n(\omega) - x(\omega_0) \leq \frac{1}{\beta} X_n(\sigma\omega) \leq C_0 n^{-1/\alpha_1}.$$  

Since $\alpha_1 < 1$ we have $\sum n^{-1/\alpha_1} < +\infty$ and hence, there exists $M > 0$ such that

$$\sum_{n \geq 1} m\{R_\omega > n\} \leq M.$$  

Inequalities (3.3) and (3.4) in Proposition 3.1 directly imply the second assumption in (P3). Similarly, (3.5) implies inequality (2.2) in (P3). It remains to show distortion estimates (P2) for the induced map. Our proof is based on Koeb e principle. Recall that the Schwarzian derivative of a $C^3$ diffeomorphism $g$ is defined as

$$Sg(x) = \frac{g'''(x)}{g'(x)} - \frac{3}{2} \left( \frac{g''(x)}{g'(x)} \right)^2.$$  

It can be easily checked that if $f$ and $g$ are two maps with negative Schwarzian derivative then the composition $f \circ g$ has negative Schwarzian derivative.

Let $J \subset J'$ be two intervals. $J'$ is called $\tau > 0$ scaled neighbourhood of $J$ if both components of $J' \setminus J$ has length at least $\tau |J|$, where $|J|$ denotes the length of $J$. The Koebe principle (Chapter IV, Theorem 1.2, [16]) states that, if $g$ is a diffeomorphism onto its image with $Sg < 0$ and $J \subset J'$ are two intervals such that $g(J')$ contains $\tau$ scaled neighbourhood of $g(J)$ then there exists $\tilde{K}(\tau)$ such that for any $x, y \in J$

$$\left| \frac{g'(x)}{g'(y)} - 1 \right| \leq \tilde{K}(\tau) \frac{|x - y|}{|J|}.$$  

Thus there exists a constant $\tilde{K}(\tau)$ such that

(3.12)$$\left| \frac{g'(x)}{g'(y)} - 1 \right| \leq K(\tau) \frac{|g(x) - g(y)|}{|g'(y)|}. $$  

Recall that by (A4) the left branch of $T_\omega$ has negative Schwarzian derivative for all $\omega \in \Omega$. This fact will we used in the proof of the following lemma.

**Lemma 3.3.** There exists $K > 0$ such that for all $\omega \in \Omega$, $n \in \mathbb{N}$ and for $x, y \in I_n(\omega)$

$$\left| \frac{(T_\omega^n)'(x)}{(T_\omega^n)'(y)} - 1 \right| \leq K|T_\omega^n(x) - T_\omega^n(y)|.$$  

**Proof.** Notice, that $M = \max_{\omega \in [\omega_0, \omega_1]} \max_{x \in I_1(\omega)} |T_\omega''(x)| < +\infty$ by (A5). Also, recall that $T_\alpha |_{I_\omega} > \beta > 1$ for any $T_\alpha \in \mathcal{F}$. Thus for $n = 1$, we have

$$\left| \frac{(T_\omega)'(x)}{(T_\omega)'(y)} - 1 \right| \leq \frac{1}{\beta} \left| (T_\omega)'(x) - (T_\omega)'(y) \right| \leq \frac{M}{\beta^2} |T_\omega(x) - T_\omega(y)|.$$
For \( n \geq 2 \) we use Koebe principle mentioned above. Set \( J = \{X_n(\omega), X_{n-1}(\omega)\} \) and \( J' = \{X_{n+1}(\omega), 2\} \). Let \( g \) denote compositions of analytical extensions of the left branches of \( T_{\omega_{n-2}}, \ldots, T_{\omega_0} \) to \((0, +\infty)\). We will show that \( g(J') \) contains \( \tau \) scaled neighbourhood of \( g(J) \) for some \( \tau > 0 \), which is independent of \( \omega \). Since \( g(X_n(\omega)) = X_1(\sigma^{n-1}\omega) \) and \( g(X_{n+1}(\omega)) = X_2(\sigma^{n-1}\omega) \). It is sufficient to show that \( X_1(\omega) - X_2(\omega) \) is bounded below by a constant independent of \( \omega \). By definition of \( X_n \) we have

\[
|X_1(\omega) - X_2(\omega)| = |T^{-1}_\omega(1) - T^{-1}_\omega \circ T_{\sigma^{n-1}}^{-1}(1)| \geq \frac{1}{\beta'}|1 - T^{-1}_{\sigma^{n}}(1)| \geq \kappa > 0,
\]

where \( \beta' = \min\{T'_\omega(x) \mid (x, \omega_0) \in [\bar{X}, 1] \times [\alpha_0, \alpha_1]\} > 1 \) with \( \bar{X} = \min_\omega X_2(\omega) \) and \( \kappa = \beta'(1 - \min_\alpha x(\alpha)) > 0 \) by (A1). Thus, using the fact \( |g(J)| > 1 - \max_\alpha x(\alpha) > 0 \) from (3.12) we obtain

\[
\left| \frac{g'(x)}{g'(y)} - 1 \right| \leq K|g(x) - g(y)|,
\]

with \( K = K(\tau)/1 - \max_\alpha x(\alpha) \) which finished the proof. \( \Box \)

**Lemma 3.4.** There exists a constant \( C > 0 \) independent of \( \omega \) such that for all \( \omega \in \Omega \) and for any \( x, y \in I_n(\omega) \)

\[
\left| \log \left( \frac{T^{R}_{\omega}(x)}{T^{R}_{\omega}(y)} \right) \right| \leq C|T^{R}_{\omega}(x) - T^{R}_{\omega}(y)|.
\]

**Proof.** Note that \( T^{R}_{\omega}(x) \) is the composition of the right branch of \( T_{\sigma^{n-1}} \) and \( g \) i.e. \( T^{R}_{\omega}(x) = T_{\sigma^{n-1}} \circ g(x) \). Therefore, by definition of \( \beta \) in (3.1) by Lemma 3.3 we have

\[
\log \left| \frac{(T^{R}_{\omega})'(x)}{(T^{R}_{\omega})'(y)} \right| \leq K|T^{R}(x) - T^{R}(y)| + K|g(x) - g(y)| \leq K(1 + \frac{1}{\beta})|T^{R}(x) - T^{R}(y)|.
\]

\( \Box \)

Now, we will prove (P2). Together with an elementary inequality \( |x - 1| \leq C|\log(x)| \) (for some \( C > 0 \), whenever \( |\log x| \) is bounded above) Lemma 3.4 implies that for any \( x, y \in I_n(\omega) \) we have

\[
\left| \frac{(T^{R}_{\omega})'(x)}{(T^{R}_{\omega})'(y)} - 1 \right| \leq D(K, \beta)|T^{R}(x) - T^{R}(y)| \leq D\beta^{-s(x,y)},
\]

where \( D = D(K, \beta) \) is a constant that depends only on \( K \) and \( \beta \) and the last inequality follows from the observation: if \( x, y \in (0, 1) \) are such that \( s(x, y) = n \) then \( |x - y| \leq \beta^{-n} \). Indeed, by definition \((T^{R}_{\omega})^{i}(x)\) and \((T^{R}_{\omega})^{i}(y)\) belong to the same element of the partition \{\( I_k(\omega) \)\} for all \( i = 0, \ldots, n - 1 \). Thus by the mean value theorem

\[
|x - y| = |[(T^{R}_{\omega})^{n}](\xi)|^{-1}|(T^{R}_{\omega})^{n}(x) - (T^{R}_{\omega})^{n}(y)| \leq \beta^{-n}.
\]
REFERENCES

1. R. Aimino, H. Hu, M. Nicol, A. Török, S. Vaienti, *Polynomial loss of memory for maps of the interval with a neutral fixed point*. Discrete Contin. Dyn. Syst. 35 (2015), no. 3, 793–806.

2. A. Ayyer, C. Liverani, M. Stenlund, *Quenched CLT for random toral automorphisms*. Discrete Contin. Dyn. Syst. 24, no. 2 (2009), 331–348.

3. W. Bahsoun, C. Bose, *Mixing rates and limit theorems for random intermittent maps*. Nonlinearity. Vol. 29 (2016), no. 4, 1417–1433.

4. W. Bahsoun, C. Bose, Y. Duan, *Decay of correlation for random intermittent maps*. Nonlinearity. Vol. 27 (2014), no. 7, 1543–1554.

5. W. Bahsoun, C. Bose, M. Ruziboev, *Quenched decay of correlations for slowly mixing systems*, Available at https://arxiv.org/abs/1706.04158.

6. V. Baladi, M. Benedicks, V. Maume-Deschamps, *Almost sure rates of mixing for i.i.d. unimodal maps*. (English, French summary) Ann. Sci. cole Norm. Sup. (4) 35 (2002), no. 1, 77–126.

7. J. Buzzi, *Exponential decay of correlations for random Lasota-Yorke maps*. Comm. Math. Phys. 208 (1999), no. 1, 25–54.

8. S. Gouëzel, *Sharp polynomial estimates for the decay of correlations*. Israel J. Math. 139 (2004), 29–65.

9. N. Haydn, M. Nicol, A. Török, S. Vaienti, *Almost sure invariance principle for sequential and non-stationary dynamical systems*. Trans. Amer. Math. Soc. 369 (2017), no. 4, 1481–1518.

10. W. de Melo, S. van Strien, *One-dimensional dynamics*. Springer-Verlag, Berlin, 1993.

11. M. Nicol, A. Török, S. Vaienti, *Central limit theorems for sequential and random intermittent dynamical systems*. To appear in Ergodic Theory Dynam. Systems. DOI: https://doi.org/10.1017/etds.2016.69.

12. Y. Pomeau, P. Manneville, *Intermittent transition to turbulence in dissipative dynamical systems*. Comm. Math. Phys. 74 (1980), 189–197.

13. B. Saussol, *Absolutely continuous invariant measures for multidimensional expanding maps*. Israel J. Math. 116 (2000), 223–248.

14. L-S. Young, *Recurrence times and rates of mixing*. Israel J. Math. 110 (1999), 153–188.