THE SOLAR PROTON BURNING PROCESS REVISITED IN CHIRAL PERTURBATION THEORY

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ABSTRACT

The proton burning process \( p + p \rightarrow d + e^+ + \nu_e \), important for the stellar evolution of main-sequence stars of mass equal to or less than that of the Sun, is computed in effective field theory by means of chiral perturbation expansion to the next to next to leading chiral order. This represents a model-independent calculation consistent with low-energy effective theory of QCD comparable in accuracy to the radiative np capture at thermal energy previously calculated by first using very accurate two-nucleon wave functions backed up by an effective field theory technique with a finite cutoff. The result obtained thereby is found to support within theoretical uncertainties the previous calculation of the same process by Bahcall and his coworkers.

Subject headings: nuclear reactions, nucleosynthesis, abundances — stars: evolution — stars: fundamental parameters — Sun: evolution — Sun: fundamental parameters

1. INTRODUCTION

The proton fusion reaction

\[
p + p \rightarrow d + e^+ + \nu_e,
\]

which plays an important role for stellar evolution and—as the dominant neutrino source—for the solar neutrino problem, has quite a long history of investigation. Indeed the reaction rate of this process (hereafter called the pp rate) was first calculated by Bethe & Critchfield (1938). Salpeter (1952) recalculated the pp rate using the effective range approximation and argued that the relevant nuclear matrix element squared could be estimated with an accuracy of the ~5% level. (The pp rate itself was subject to much larger uncertainty, ~20%, because of the limited precision with which the Fermi coupling constant was known at that time.) Bahcall & May (1969) examined the dependence of the pp rate on explicit forms of the two-nucleon wave functions generated by two-parameter nuclear potentials of various forms adjusted so as to reproduce the scattering length and effective range (for the pp channel) and the low-energy properties of the deuteron (for the np channel). The pp rate was found to vary by ~1.5% corresponding to the changes in the deuteron wave function, and by ~1.2% due to the change in the pp wave function. The most updated work along this line was done by Kamionkowski & Bahcall (1994) employing deuteron wave functions obtained from much more accurate potentials such as the Argonne \( v_{14} \), \( v_{18} \), Urbana \( v_{14} \), super—soft-core (SSC) and Reid soft-core potentials. Changes in the pp rates arising from the different potentials were found to be ~1%. Thus it seems that the presently available calculated pp rate is robust and needs no further scrutiny, the famous solar neutrino problem remaining unresolved from this angle and hence persisting as one of the outstanding unsolved problems in astrophysics (Bahcall 1997).

There are however two reasons for revisiting this issue. One is that although the calculated pp rate seems to have converged to a “canonical” value given in Kamionkowski & Bahcall (1994, hereafter KB), there lingers the unsettling feeling that the strong interaction involved in nuclear physics of the two-nucleon systems is infested with uncontrollable uncertainties associated with model dependence in the treatment, making it difficult to assess the accuracy achieved. Thus it is not unexpected that this canonical value will be—as it has been in the past—challenged. Indeed it has recently been argued by Ivanov et al. (1997) and Oberhummer et al. (1998) that the nonrelativistic potential models used in the previous works could be seriously in error. They show that in their version of a relativistic field theory model, the pp rate comes out to be as big as 2.9 times the previous estimates. Should their new result turn out to be correct, it would have profound consequences on theories of stellar evolution in general and on the solar neutrino problem in particular. In a nutshell, the issue comes down to whether or not a more general framework such as relativistic field theory would invalidate the calculation made in the traditional nonrelativistic potential models. The claim of the authors in Ivanov et al. (1997) and Oberhummer et al. (1998) is that it indeed does. Our aim is to address this issue using a low-energy effective field theory of QCD that has found a quantitative success in other nuclear processes.

The second reason is really more theoretical, independent of the above important astrophysical issue. Along with the thermal np capture, the proton fusion process is the simplest nuclear process amenable to an accurate calculation—something rare in hadronic physics—and it is of interest on its own to test how well a calculation faithful to a “first-principle approach” can tackle this problem. In particular, we are interested in checking how accurately the effective field theory approach, found to be stunningly successful for the np capture \( n + p \rightarrow d + \gamma \), low-energy nucleon-nucleon (NN) scattering and static properties of the deuteron (Park, Min & Rho 1995, 1996; Park et al. 1997b), fares with the

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They use a procedure that seems to disagree with other physical properties of low-energy pp systems, as was pointed out by Bahcall & Kamionkowski (1997). It has also been pointed out by Degl’Innocenti, Fiorentini, & Ricci (1998) that such a large deviation from the value used by KB would be inconsistent with helioseismology in the Sun.
proton fusion problem, the weak-interaction sector of the standard model.

The strategy we shall adopt here is quite close to that used for the \( np \) capture process (Park et al. 1995, 1996). We shall use chiral perturbation theory to the next to next to leading order (NNLO) in chiral counting; as defined precisely below, this corresponds to \( O(Q^3) \) relative to the leading-order term. This is roughly the same order as considered for the \( np \) capture. However the relative importance of terms of various chiral orders is somewhat different here. As we explain later, in the present case, the corrections to the leading order are not suppressed by what is called the “chiral filter” \(^7\) and so the accuracy with which these can be calculated is not as good as in the \( np \) capture case. Even so, using the argument developed in Park et al. (1997b), we shall suggest that the procedure used here provides a model-independent result in the same sense as in Park et al. (1995, 1996, 1997b).

To streamline the presentation, we first give our result and then discuss (as briefly as possible) how we arrive at it in the rest of the paper. Apart from the meson-exchange contributions that are \( O(Q^3) \) and that, for the reason mentioned above and further stressed in our concluding section, are the main uncertainty to the order considered, our chiral perturbation theory result in terms of the reduced matrix element \( \Lambda \) defined in Bahcall & May (1969) is

\[
\Lambda_{\text{PT}}^2 = (1 \pm 0.003) \times 6.93 ,
\]

(2)

where the uncertainty is due to experimental errors.\(^8\) The above result is to be compared with the value obtained by KB

\[
\Lambda_{\text{KB}}^2 = (1 \pm 0.002^{+0.014}_{-0.009}) \times 6.92 .
\]

(3)

As we shall explain, there are some differences in details between our calculational framework and that of KB, but our final numerical result is in good agreement with that of KB and disagrees with that of Ivanov et al. (1997) and Oberhummer et al. (1998).

The paper is organized as follows. In § 2, our strategy for carrying out a chiral perturbation calculation for two-nucleon systems is outlined. Our approach here is similar to the one used in the previous calculation of the \( np \) capture process. We shall sketch a justification of this approach from the standpoint of low-energy effective field theory of QCD (with concrete supporting evidence summarized in § 5.2). Section 3 describes chiral counting of the terms appearing in the relevant weak current. In § 4, the wave functions for the initial \( pp \) state and the final \( d \) state are specified. Our numerical results are given in § 5, and a brief discussion including a comment on the main uncertainty in the calculation is given in § 6.

2. EFFECTIVE FIELD THEORY FOR NUCLEI

Before presenting our calculation, we sketch the chain of reasoning that leads us to believe that the result we obtained is model independent. The argument is essentially identical to that used for thermal \( np \) capture, and therefore we review briefly the key ingredients of the calculations in Park et al. (1995, 1996, 1997b) for the process

\[
n + p \rightarrow d + \gamma .
\]

(4)

The basic premise of our approach is an application of the “Weinberg theorem” (see, e.g., Weinberg 1996, 1997), which for the case in hand implies that at very low energies at which the process (4) is probed, QCD can be described by chiral perturbation theory. In setting up chiral expansion for two-nucleon systems we are interested in, we follow Weinberg (1990, 1991, 1992) and separate the relevant transition amplitude into the “reducible” and “irreducible” terms. Bound states and resonances involve reducible graphs that are kinematically enhanced. To incorporate this feature, we need to sum up a given set of reducible terms to all orders, a procedure equivalent to solving the Schrödinger or Lippmann-Schwinger equation. The irreducible graphs that appear as kernels in the latter are calculated to a specified order in chiral expansion. How this can be done in a systematic way consistent with chiral counting has been extensively discussed in the literature (Ordonez, Ray, & van Kolck 1994, 1996; Kaplan, Savage, & Wise 1996, 1998; Luke & Manohar 1997; Beane, Cohen, & Phillips 1998, and references therein). In Park et al. (1995, 1996), we have written the relevant M1 matrix element for the process (4) with the EM current given by the irreducible graphs calculated to NNLO and accurate wave functions for the initial and final states, namely, the Argonne \( v_{18} \) wave function. The Argonne \( v_{18} \) potential (Wiringa, Stoks, & Schiavilla 1995) provides an impressive description of all two-nucleon data at low energies up to \( \sim 350 \text{ MeV} \). So the wave functions obtained with it correspond to summing both the irreducible and reducible graphs to all orders. It seems then that the procedure used in Park et al. (1995, 1996) is not quite consistent since the current is computed only to NNLO. We argued however in Park et al. (1995, 1996) that this is consistent with the chiral counting to the chiral order considered, and the error made in the terms of higher order than that taken in the current cannot be important. We have subsequently demonstrated (Park et al. 1997b) the correctness of this conjecture by showing in a cutoff field theory that a systematic summation of the reducible diagrams with the irreducible graphs appearing as a kernel—which accurately describes all static properties of the deuteron and NN scattering lengths—gives precisely the same result for the leading chiral-order M1 matrix element as that obtained with the Argonne \( v_{18} \) wave function. Now, as we know, the ratios of subleading chiral terms to the leading term are almost model independent and insensitive to the renormalization schemes used, so these ratios can be calculated with reasonable accuracy. Thus the result obtained in Park et al. (1995, 1996),

\[
\sigma_{\text{PT}} = 334 \pm 3 \text{ m barn} \tag{5}
\]

(where the error represents our ignorance regarding the short-distance part of the interaction in chiral perturbation theory), which agrees well with the experiment,

\[
\sigma_{\text{exp}} = 334.2 \pm 0.5 \text{ m barn} \tag{6}
\]

can be taken as a first-principle calculation as argued in Park et al. (1997b).
For the weak process (1) we will follow the same procedure and, to supply for it, carry out an effective field theory calculation up to NNLO with the use of the same cutoff scheme as in Park et al. (1997b), where the leading-order M1 matrix element was evaluated.

3. CHIRAL COUNTING AND THE WEAK CURRENT

As in KB, we shall include the vacuum-polarization effect pertinent to the initial $pp$ channel as well as the effect of meson-exchange currents. In addition, we include the effect of the two-photon–exchange Coulomb interaction for the $pp$ channel.

3.1. Chiral Counting

The core temperature of the Sun is believed to be about $T_C = 15.5 \times 10^6$ K, for which the corresponding nucleon momentum is $p \sim (2m_p T_C)^{1/2} \sim 1.6$ MeV. We are therefore interested in the process (1) at very low energies, a situation similar to the thermal $np$ capture process. This is an arena where the low-energy effective field theory can be powerful.

At these low energies, the reaction is dominated by the $^1S_0 \rightarrow d$ transition, involving only the transition operators that carry orbital angular momentum, $L = (0, 2)$, spin $S = 1$, and total angular momentum $J = 1$. The momentum carried by the leptons, $k = p_e + p_n$, is also very small, $0 \leq |k| \leq [4m_p m_n - 2m_p(4m_p m_n + m_p^2 - m_n^2)]^{1/2} \approx 0.78$ MeV, where $m_p$, $m_n$, and $m$ are the mass of proton, deuteron, and electron, respectively. We can therefore safely ignore terms of $O(|k|^4)$ and work only to $O(|k|)$. These selection rules will be implicit throughout our subsequent arguments.

To carry out the procedure explained in § 2, we need to specify a counting rule for irreducible terms for the electro-weak hadronic current $J^{u,a} = \equiv J^{u,1} - i J^{u,2}$, where

$$J^{u,a} = V^{u,a} - A^{u,a}. \quad (7)$$

This rule has been extensively discussed in Park et al. (1995, 1996, 1993, 1994) and Park, Jung, & Min (1997a), so we shall simply summarize it here.

Let $Q$ denote the typical momentum scale probed in the process and $\lambda_x$ be the chiral scale $\sim m_N$. The chiral expansion is made in powers of $Q/\Lambda_x$. A Feynman diagram for an $A$-nucleon process (Weinberg 1990, 1991, 1992) goes as $Q^3$. Now for a Feynman graph with $L$ loops, $N_k$ external fields, and $C$ separately connected pieces, the exponent $v$ is given by

$$v = 4 - A - 2C + 2L - N_k + \sum_i v_i,$$

with

$$v_i = d_i + e_i + \frac{n_i}{2} - 2, \quad (8)$$

where $n_i$ is the number of nucleon lines, $d_i$ the number of derivatives or powers of $m_N$, and $e_i$ the number of external fields at the $i$th vertex. Since $v_i$ is defined so that chiral symmetry guarantees $v_i \geq 0$ (Weinberg 1990, 1991, 1992) even in the presence of external fields (Rho 1991), the effective Lagrangian can be ordered according to $v_i$:

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \cdots, \quad (9)$$

where $\mathcal{L}_n$ has vertices with $v_i = n$.

3.2. Current

Let us now apply this counting rule to our case. The leading contribution arises from a one-body diagram with $v = v_0 = -3$ (i.e., $L = 0$, $C = A = 2$, $N_k = 1$, and $v_i = 0$), and the next to leading order ($v = v_0 + 1$) also comes from a one-body diagram involving a $v_i = 1$ vertex. These one-body operators are given by

$$V_1^{u,a} = \sum_i \frac{e_i}{2} \left(1, \frac{p_i + p_i'}{2m_N} + \frac{i\mu \nu}{2m_N} k \times \sigma_i\right), \quad (10)$$

$$A_1^{u,a} = g_A \sum_i \frac{e_i}{2} \left(1, \frac{p_i + p_i'}{2m_N} \sigma_i\right), \quad (11)$$

with $p_i (p_i')$ the incoming (outgoing) momentum of the $i$th nucleon, $g_A \approx 1.2601(25)$ (Particle Data Group 1996), the axial-vector coupling constant, $\mu \nu \equiv \mu_p - \mu_n \approx 4.7059$ the nucleon isovector magnetic moment, and $m_N$ the nucleon mass. Of these one-body operators only the Gamow-Teller term contributes at the leading order, whereas the weak magnetism (WM) term that comes at the next order is further suppressed kinematically for the small lepton momentum $|k| \ll Q$. The others do not contribute by the selection rule. We will keep the WM term but neglect all other higher chiral-order $k$-dependent terms. Thus the one-body current we calculate is

$$J_{1B} = J_{GT} + J_{WM} + \cdots, \quad (12)$$

with

$$J_{GT} = - g_A \sum_i \tau_i \sigma_i, \quad (13)$$

$$J_{WM} = \frac{1}{2m_N} k \times \sum_i \tau_i \sigma_i. \quad (14)$$

To proceed to higher orders, it is convenient to do chiral counting relative to the leading order. Expressed in this way, the leading order one-body current equation (12) is taken to be $\sim O(1)$, and the next to leading order term is $\sim O(Q)$. The two-body exchange current can enter at the next to next to leading order (NNLO), that is at $O(Q^3)$. Because of the selection rule, however, there is no term of this order contributing to the process (1). Hence the two-body exchange current at tree order enters at $O(Q^4)$. Note the difference in the role of the exchange currents between the present case and the $np$ capture process. In the latter, the exchange-current (two-body) M1 operator is $\sim O(Q)$ relative to the leading one-body M1 operator. This is because the one-body M1 operator is formally suppressed by one order of chiral counting, so that the one–soft-pion exchange two-body current is enhanced relative to the one-body operator. This is just the chiral filter enhancement referred to above. In the Gamow-Teller case, the chiral filter mechanism is not operative, making the leading two-body correction come at $O(Q^3)$. This renders evaluation of the two-body terms less precise, although their overall importance relative to the one-body term should be considered diminished here.

We will return to this matter later.

Calculation of the meson-exchange Gamow-Teller operator is discussed in Park et al. (1997a). Here we skip the details and simply quote the result. The leading two-body contributions of $O(Q^3)$ come from the one-pion exchange and four-Fermi contact graphs with $L = 0$, $C = A - 1 = 1$, and one $v_i = 1$ interaction ($v = v_0 + 3$). The relevant part of
the Lagrangian with the $v_i = 1$ interaction is of the form
(Bernard, Kaiser, & Meissner 1997; Cohen et al. 1996)
\[
\mathcal{L}_1 = B \left( \frac{v^{\mu\nu} - g^{\mu\nu}}{2m_N} D_{\mu} D_{\nu} + 4c_3 iA \cdot i\Delta \right)
+ \left( 2c_4 + \frac{1}{2m_N} \right) \left[ S^\mu, S^\nu \right] [iA_{\mu}, iA_{\nu}] B
- 4id_iBS \cdot \Delta B B + 2id_2 e^{abc} \epsilon_{\mu\nu\lambda} v^\nu A^\lambda \cdot a
\times BS^4 v^5 BS^4 v^5 B + \cdots ,
\]
where $\epsilon_{0123} = +1$, and the definitions of the “covariant derivatives” $D_{\mu}$, $A_{\mu}$, and the spin operator $S^\mu$ can be found in Bernard et al. (1997) and Cohen et al. (1996). There are no vector-current contributions, and the surviving two-body Gamow-Teller operator in momentum space is
\[
A^2_B = - \frac{g_A}{2m_N f_\pi^2 m_\pi^2 + q^2} \left[ - i \frac{q^2}{2} p^\sigma \sigma \cdot q + \hat{c}_3 q \right]
\times q (r_{1 \sigma}^\sigma + r_{2 \sigma}^\sigma) + \left( \frac{\hat{c}_4 + 1}{4} \right) q \times (\sigma \times q)
- \frac{g_A}{2m_N f_\pi^2} \left[ \hat{d}_1 (r_{1 \sigma}^\sigma + r_{2 \sigma}^\sigma) - 2\hat{d}_2 r_{1 \sigma}^\sigma \right].
\]
Among the corrections, terms proportional to $1/m_N$ have been dropped. The $p$-dependent terms, which come from $(2m_N)BD^*B$ in equation (15), shall be referred to as the “kinetic term.” The low-energy constants $c_3$ and $c_4$ have been determined from pion-nucleon experiments by Bernard et al. (1997):
\[
c_3 = -5.29 \pm 0.25 \text{ GeV}^{-1},
c_4 = 3.63 \pm 0.10 \text{ GeV}^{-1}.
\]
They are almost completely saturated by the resonance-exchange contributions
\[
\begin{align*}
\epsilon_3^{\text{res}} &= c_3^\Delta + c_3^\rho + c_3^S = (-3.83 - 1.40 - 0.06) \text{ GeV}^{-1} = -5.29 \text{ GeV}^{-1},
\epsilon_4^{\text{res}} &= c_4^\Delta + c_4^\rho + c_4^S = (1.92 + 1.63 + 0.12) \text{ GeV}^{-1} = 3.67 \text{ GeV}^{-1},
\end{align*}
\]
where the superscripts $\Delta$, $\rho$, $S$, and $R$ denote the contributions from the exchange of $\Delta(1232)$, $\rho$ mesons, scalar mesons, and the Roper resonance, respectively. In terms of the dimensionless constants $\hat{c}_3$ and $\hat{c}_4$, equation (18) gives us
\[
\hat{c}_3 \approx -4.97 \pm 0.23 ,
\hat{c}_4 \approx 3.41 \pm 0.09.
\]
There are no low-energy data to determine the constants $\hat{d}_1$ and $\hat{d}_2$. However, it was argued in Park et al. (1995, 1996) that the zero-range operators corresponding to the $\hat{d}_1$ and $\hat{d}_2$ terms should be killed by the short-range correlation. We shall justify this argument here by showing that for “natural” coefficients in the Lagrangian (as required by chiral symmetry) their matrix elements are indeed negligible.

Finally, when Fourier-transformed, the two-body current in coordinate space reads
\[
\begin{align*}
\bar{J}^B_2(r) &= - \bar{A}^B_2(r) + \cdots \\
&= - \frac{g_A m_n^2}{2m_N f_\pi^2} \left( \frac{y_1(m_r n) r}{2m_N^2 r} \right) \bar{r}^\sigma \sigma \cdot \bar{r}^\sigma \\
&\times \left( \bar{r}_{1 \sigma}^\sigma + \bar{r}_{2 \sigma}^\sigma \right) + \left( \hat{c}_4 + \frac{1}{4} \right) \bar{r}^\sigma \\
&\times \left[ - \bar{r}^\sigma y_2(m_r n_r) \frac{2\delta_{ij}}{3} y_0(m_r n_r) \right] \sigma^\sigma \\
&\times \left( \bar{r}_{1 \sigma}^\sigma + \bar{r}_{2 \sigma}^\sigma \right) + \left( -2\hat{d}_2 + \frac{2}{3} \hat{c}_3 \right) \bar{r}^\sigma \sigma \cdot \bar{r}^\sigma \\
&\times \left( \frac{1}{6} \bar{r}_{1 \sigma}^\sigma \sigma^\sigma + \bar{r}_{1 \sigma}^\sigma \sigma^\sigma \right) \delta^{(3)}(r) + \cdots ,
\end{align*}
\]
where the ellipses denote irrelevant terms and the Yukawa functions, $y_i$, are defined as
\[
y_i(x) = e^{-x} \frac{1}{4\pi x} ,
y_i(x) = (1 + x) e^{-x} \frac{1}{4\pi x} = -x \frac{d}{dx} y_0(x) ,
y_i(x) = \left( 1 + \frac{3}{x} + \frac{3}{x^2} \right) e^{-x} \frac{1}{4\pi x} = x \left( x - \frac{1}{d} \frac{d}{dx} \frac{1}{dx} x + \frac{1}{x^2} x \right).
\]

4. WAVE FUNCTIONS
In the scheme we are using, it is essential to have accurate wave functions for the initial and final states so that the leading single-particle Gamow-Teller matrix element is accurately given. In Park et al. (1995, 1996), we have shown that the single-particle M1 matrix element for the $np$ capture process (4) is accurately given by the wave functions obtained with the Argonne $v_{18}$ potential. The single-particle Gamow-Teller operator is identical in structure to the isovector M1 operator, so provided that electromagnetic effects are suitably taken into account, we expect that the Gamow-Teller matrix element for the proton fusion process (1) will also be given very accurately by the wave functions of the Argonne $v_{18}$ potential. Once this argument is justified, then we can conclude immediately that the answer was essentially given by Bahcall and collaborators since they used, among others, the Argonne $v_{18}$ potential in calculating the Gamow-Teller matrix element. One difference in their calculation from ours is that in their treatment, the initial and final wave functions were calculated asymmetrically with the initial scattering wave function treated approximately, whereas the final deuteron state treated exactly. Our strategy in the chiral perturbation approach requires that they be treated with the same “exact” poten-
tial. Numerically, however, their approximation for the initial state should be quite good, so it is not unreasonable to expect that not much will be gained by the symmetric treatment. We confirm this.

We now turn to discussion of the wave functions used in our calculation.

4.1. The Deuteron Wave Function

The deuteron wave function obtained with the Argonne $v_{18}$ potential (referred to as the “Argonne potential” in what follows) is well known, so without going into any details, we will merely define the notations for later use.

The deuteron wave function will be written in the conventional form

$$\psi_d(r) = \frac{A_s}{\sqrt{4\pi}} \left[ u_d(r) + \frac{S_{12}(r)}{\sqrt{8}} w_d(r) \right] (3S_1),$$

(24)

where $S_{12}(r) = 3\sigma_1 \cdot r \sigma_2 \cdot r - \sigma_1 \cdot \sigma_2$ and $A_s$ is the normalization factor defined below. Asymptotically, the radial functions take the form

$$\lim_{r \to \infty} u_d(r) = \phi_d(r) \equiv e^{-\gamma r},$$

$$\lim_{r \to \infty} w_d(r) = \eta_d \left[ 1 + \frac{3}{\gamma r} + \frac{3}{(\gamma r)^2} \right] \phi_d(r),$$

(25)

where $\gamma$ is the wavenumber, $\gamma = (m_dB_d^{1/2})$, with $B_d$ the binding energy; $\eta_d$ is the deuteron $D/S$ ratio, and the normalization factor $A_s$ is defined by

$$A_s^2 = \int_0^\infty dr [u_d^2(r) + w_d^2(r)] = 1.$$

(26)

The experimental values (van der Leun & Alderliesten 1982; Ericson & Rosa-Clot 1983; Rodning 1990) are

$$B_d^{exp} = 2.224575(9) \text{ MeV}, \quad A_s^{exp} = 0.88468(6) \text{ fm}^{1/2}, \quad \eta_d^{exp} = 0.02564(4).$$

(27)

It is also interesting to note that $A_s$ can be expressed in terms of the effective range defined at the pole position,

$$\rho_d = 2 \int_0^\infty dr \left[ \phi_d^2(r) - \frac{u_d^2(r) + w_d^2(r)}{1 + \eta_d^2} \right]$$

$$= \frac{1}{\gamma} - \frac{2}{(1 + \eta_d^2)A_s^2}$$

(28)

or

$$A_s^2 = \frac{2\gamma}{(1 + \eta_d^2)(1 - \gamma \rho_d^2)}.$$  

(29)

With equation (28), the experimental data (eq. [27]) lead to

$$\rho_d^{exp} = 1.7635(46) \text{ fm}.$$  

(30)

One can get an idea as to how accurately the Argonne wave function describes the properties of the deuteron by looking at the predictions: $B_d = 2.22460 \text{ MeV}, \ A_s = 0.88506 \text{ fm}^{1/2}, \ \eta_d = 0.02504,$ and $\rho_d^{18} = 1.7660 \text{ fm}$. The close agreement between the Argonne results and the experimental values invites us to use the two interchangeably in referring to “experiments.” Thus for instance in Park et al. (1997b), we called the single-particle M1 matrix element given by the Argonne wave function “experimental” although strictly speaking it cannot be so called. In what follows, we will use equations (27) and (30) in calculating the Gamow-Teller matrix element with the Argonne wave function.

4.2. The pp $^1S_0$ Wave Function

An accurate description of the pp wave function involves certain subtlety and is interesting on its own right. KB made an extensive analysis of the pp wave function. Since our result is a bit different from theirs, we discuss here our approach to this problem in some detail.

4.2.1. Potential

We begin by decomposing the full potential $V_{full}$ into four different components: the nuclear potential $V_N$, the pure Coulomb potential $V_C$, the two-photon–exchange Coulomb potential $V_{C2}$, and the vacuum-polarization potential $V_{VP}$:

$$V_{full} = V_N + V_C + V_{C2} + V_{VP},$$  

(31)

where (Carlson et al. 1991)

$$V_C(r) = \alpha x, \quad \alpha \equiv \frac{1 + 2p^2/m_p^2}{1 + p^2/m_p^2},$$  

(32)

$$V_{C2}(r) = -\frac{1}{2m_p^2} \left( (V^2 + p^2)^2 + \frac{2}{r} (V^2 + p^2) \right)$$

$$= -\frac{\alpha}{m_p^2} V_{full}(r),$$  

(33)

$$V_{VP}(r) = \frac{2\alpha x}{3\pi} \int_0^\infty dx e^{-2mx} \left( 1 + \frac{1}{2x^2} \right) \frac{\sqrt{x^2 - 1}}{x^2},$$  

(34)

where, in arriving at the second line of equation (33), we have removed the nonlocal piece $(V^2 + p^2)$ using the equation of motion

$$(V^2 + p^2)\psi_{pp}(r) = m_p V_{full}(r)\psi_{pp}(r).$$  

(35)

The nuclear potential $V_N$ contains not only strong interactions but also other electromagnetic interactions such that the Argonne potential for the $pp$ channel is of the same form as equation (31). In what follows, when we refer to the “Argonne potential” for this channel, we mean equation (31).

4.2.2. Solution

Our treatment of the potential $V_N \equiv V_{C2} + V_{VP}$, which is somewhat different from that in KB, goes as follows. First we solve exactly the Schrödinger equation with the Coulomb plus nuclear potential

$$V_{C+N}(r) \equiv V_C(r) + V_N(r) = \frac{\alpha}{r} + V_N(r),$$  

(36)

where we have set $\alpha' = \alpha$ since we are confining ourselves to the case $p \approx 0$. Then $V_{C+N}$, which is $O(x^2)$, is treated perturbatively.

Focusing on the $S$ wave, let $u_{C+N}$ be a solution of the radial Schrödinger equation for the potential equation (36)
satisfying the boundary condition \( u_{C+N}(0) = 0 \). Thus
\[
\left[ \frac{d^2}{dr^2} + \frac{p^2 - m_p V_{C+N}(r)}{r} \right] u_{C+N}(r) = 0 .
\] (37)

Let \( r_1 \sim (10-20) \text{ fm} \) be the range of the nuclear potential so that
\[ V_r(r) = 0, \ r > r_1 . \] (38)

For \( r > r_1 \), the solution \( u_{C+N} \) is given by a linear combination of the regular and irregular Coulomb wave functions, \( F_0 \) and \( G_0 \):
\[
u_{C+N}(r) = \phi_c(r), \ r > r_1 ,
\]
\[
\phi_c(r) = C_0(\eta)[G_0(r) + \cot \delta^C F_0(r)]
\] (39)
with \( C_0(\eta) = 2\pi\eta/(e^{2\pi\eta} - 1) \), \( \eta = m_p a_p/2p \). Equation (39) defines the normalization of the wave function and the phase shift \( \delta^C \). The properties of the Coulomb functions imply
\[
\phi_c(0) = 1 ,
\]
\[
\lim_{r \to \infty} \phi_c(r) = \frac{C_0(\eta)}{\sin \delta^C} \sin \left[ p r - \eta \ln(2p r) + \sigma_c + \delta^C \right] ,
\] (40)
where \( \sigma_c \equiv \text{Arg} \Gamma(1 + i\eta) \) is the Coulomb phase. The scattering length \( a^C \) is defined by
\[
\lim_{p \to 0} C_0^2(\eta) p \cot \delta^C = -\frac{1}{a^C} ,
\] (41)
with which the asymptotic wave function at threshold reads
\[
\lim_{p \to 0} \phi_c(r) = \phi_1(r) - \frac{R}{a^C} \phi_2(r)
\] (42)
with
\[
\phi_1(r) \equiv \lim_{p \to 0} C_0(\eta) G_0(r) = 2 \sqrt{\frac{R}{r}} K_1 \left( 2 \sqrt{\frac{R}{r}} \right),
\]
\[
\phi_2(r) \equiv \lim_{p \to 0} \frac{F_0(r)}{C_0(\eta) P R} = \sqrt{\frac{R}{r}} I_1 \left( 2 \sqrt{\frac{R}{r}} \right),
\] (43)
where \( R \equiv (m_p a_p)^{-1} = 28.8198 \text{ fm} \) is the so-called proton Bohr radius and \( I_v(z) \) and \( K_v(z) \) are the modified Bessel functions of order \( v \). The scattering length \( a^C \) can be determined by matching the logarithmic derivative at \( r = r_1 \),
\[
\frac{R}{a^C} \equiv \left[ u_{C+N}(r) \phi_1(r) - u_{C+N}(r) \phi_1'(r) \right]_{r=r_1},
\] (44)
and the effective range \( r^C \) is given by the standard effective range formula
\[
r^C = 2 \int_0^\infty dr \left[ \phi_2^2(r) - u_{C+N}^2(r) \right] .
\] (45)

For the Argonne nuclear potential, the scattering length and the effective range come out to be
\[
a^C = -7.8202 \text{ fm} , \ r^C = 2.782 \text{ fm} ,
\] (46)
in very good agreement with the “experimental values,”
\[
a_{C+N}^C = -7.8196(26) \text{ fm} , \ r_{C+N}^C = 2.790(14) \text{ fm} ,
\] (47)
given by the Nijmegen multienergy analysis (Bergervoet et al. 1988).

In taking into account the effects of the residual long-range potential, \( V_r = V_{C2} + V_{P} \), some subtleties appear. First, although the magnitudes of \( V_{C2} \) and \( V_{P} \) are small \([\sim O(\alpha^2)]\), they are both extremely long ranged, making it difficult to solve the Schrödinger equation directly. We bypass this difficulty by treating them perturbatively. Second, depending on how the potential \( V_r \) is treated, there exist in the literature (Bergervoet et al. 1988) a number of different definitions of phase shifts and low-energy scattering parameters extracted therefrom. The \( \delta^C \) in equation (39) represents the phase shift due to \( V_{C+N} \) relative to the phase corresponding to the pure Coulomb potential.\(^{10}\) Third, we need to keep track of the changes in the large-\(r \) behavior, and in the wave function normalization factor that may cause difference in the flux density. They should be taken into account self-consistently in the definition of the cross section, as pointed out by KB. We resolve this problem in such a way that the \( V_r \) potential affects the cross section only through the reduced matrix element, i.e., by requiring that \( u_{pp}(r) \) the radial wave function from the full potential—have, modulo change in the phase shift, exactly the same large-\(r \) behavior as that of \( u_{C+N}(r) \):
\[
\lim_{r \to \infty} u_{pp}(r) = \frac{C_0(\eta)}{\sin \delta^C} \sin \left[ p r - \eta \ln(2p r) + \sigma_c + \delta^C + \Delta_{0} \right] ,
\] (48)
where \( \Delta_{0} \) is the improved Coulomb-Foldy correction (Bergervoet et al. 1988). This means that in order to have the correct flux density, equation (48) requires the full \( pp \) wave function to be of the form,
\[
\psi_{pp}(r) = \frac{N_{pp}}{r} u_{pp}(r)|S_0\rangle ,
\]
\[
N_{pp} = \frac{\sin \delta^C}{p C_0(\eta)} - C_0(\eta)a^C .
\] (49)

The \( u_{pp}(r) \) that satisfies the boundary condition (eq. [48]) is
\[
u_{pp}(r) = \cos \Delta_{0} u_{C+N}(r) - \frac{\sin \delta^C}{p C_0(\eta)} \times \left[ u_{C+N}(r) P_0(r) + v_{C+N}(r) P_{1}(r) \right] ,
\]
\[
P_0(r) = m_p \int_0^\infty dr' v_{C+N}^2(r') V_r(r') \delta_{pp}(r') ,
\]
\[
P_{1}(r) = m_p \int_0^\infty dr' v_{C+N}(r') V_r(r') \delta_{pp}(r') ,
\] (50)
where \( v_{C+N}(r) \) is a second solution of equation (37) that satisfies the boundary condition
\[
v_{C+N}(r) = C_0(\eta)[\cos \delta^C G_0(r) - \sin \delta^C F_0(r)] , \ r > r_1 .
\] (51)

The first-order correction in \( V_r \) can be obtained by replacing \( u_{pp} \) by \( u_{C+N} \) in \( P_0 \) and \( P_{1} \). The threshold limit is readily
obtained by noticing that \( \Delta_0 \to 0 \) and \( \nu_{C+N}(r; r > r_0) \to \phi_1(r) \) when \( p \to 0 \).

A remark is in order here about our treatment of the contribution from the C2 potential given in equation (33). Up to first order in \( V_{C2} \) and \( V_{VP} \), we have

\[
V_{C2}(r) \approx -\frac{\alpha}{m_p r} \left[ \frac{\alpha}{r} + V_0(r) \right] = V_{C2}^0(r) + V_{C2}^N(r). \tag{52}
\]

Since the first term, \( V_{C2}^0(r) \), is attractive, its contribution to the phase shift adds destructively to the effect of the repulsive VP potential. An approximation frequently used in calculating the phase shifts is to take only this term, ignoring the second term, \( V_{C2}^N(r) \). This is reasonable for scattering amplitudes since, as one can see from equation (50), \( V_{C2}^N(r) \) cannot affect significantly the large-r behavior of \( u_{pp}(r) \). For the calculation in question, however, we need to keep the full potential equation (52). The reason is that we are concerned here with not only the phase shifts but also the wave function in short and intermediate ranges. In fact, we find that the effect of \( V_{C2}^N(r) \) is opposite in sign to, and greater than, that of \( V_{C2}^0(r) \) for \( r \lesssim 49 \text{ fm} \) (except for very short distances, say, \( r \lesssim 0.05 \text{ fm} \)). Consequently, the C2 contribution to the pp amplitude adds constructively to the VP contribution. These features are exhibited in Figure 1, where we plot the ratio \( (u_{pp} - u_{C+N})/u_{C+N} \) as a function of \( r \).

5. NUMERICAL RESULTS

It is convenient to define the reduced matrix element of the weak current as

\[
\mathcal{M} = \mathcal{M}_{1B} + \mathcal{M}_{2B} \tag{53}
\]

such that

\[
\frac{\langle \psi_4 | J | \psi_{pp} \rangle}{\sqrt{8\pi g_A} A_S N_{pp}} = \left( \hat{s}_d - i \frac{\mu \nu}{2g_A m_N} \mathbf{k} \times \hat{s}_d \right) \mathcal{M}_{1B} + \hat{s}_d \mathcal{M}_{2B} = \left( \hat{s}_d - i \frac{\mu \nu}{2g_A m_N} \mathbf{k} \times \hat{s}_d \right) \mathcal{M} + O(Q^4, Q^2 | \mathbf{k} |, | \mathbf{k} |^2), \tag{54}
\]

where \( \hat{s}_d \) is the spin polarization vector of the deuteron and the normalization factors \( A_S \) and \( N_{pp} \) have been defined in equations (26) and (49). Although the reshuffling of terms in the last line introduces a term of \( O(Q^3 | \mathbf{k} |) \) (i.e., the product of the WM term and two-body term), the error made thereby should be negligible. We give our numerical results in the increasing orders in chiral counting.

5.1. \( O(1) \) and \( O(Q) \) Contributions: The Single-Particle Matrix Element

It follows from equations (12) and (54) that the single-particle (reduced) matrix element simplifies to

\[
\mathcal{M}_{1B} = \int_0^\infty dr u_1(r)u_{pp}(r), \tag{55}
\]

Following the discussion given in the preceding section, we formally separate the contributions into the “Coulomb plus nuclear part” and the “large-r part”

\[
\mathcal{M}_{1B}^{C+N} = \mathcal{ER}_{1B} + \delta \mathcal{M}_{1B}^{C+N}, \tag{56}
\]

\[
\mathcal{ER}_{1B} = \int_0^\infty dr \phi_1(r)u_{pp}(r) - u_{C+N}(r), \tag{57}
\]

\[
\delta \mathcal{M}_{1B}^{C+N} = \int_0^\infty dr u_1(r)[u_{pp}(r) - u_{C+N}(r)]. \tag{58}
\]

Clearly the most important term is the “Coulomb plus nuclear term” (eq. [57]), so it needs to be calculated as accurately as possible. This term can in turn be decomposed into a term that can be extracted from experiments using the effective range formula and the remainder that requires theoretical input from the Argonne potential:

\[
\mathcal{M}_{C+N}^{C+N} = \mathcal{ER}_{1B} + \delta \mathcal{M}_{1B}^{C+N}. \tag{59}
\]

If the next to leading order approximation is made to the effective range formula, the resulting matrix element is

\[
\mathcal{M}_{1B}^{ER} = \int_0^\infty dr \phi_1(r)u_{pp}(r) + \frac{1}{2} \int_0^\infty dr \left[ \frac{u_1^2(r) + w_1^2(r)}{1 + \eta_1^2} - \phi_1^2(r) \right]
\]

\[
+ \left[ u_{C+N}(r) - \phi_1^2(r) \right], \tag{60}
\]

\[
= -\frac{e^2}{\alpha^2 \gamma^2} \left[ 1 + \frac{\alpha^2}{R} \left( E_1(\zeta) - \frac{e^{-\zeta}}{\zeta} \right) \right] - \rho_1 + \frac{\alpha^2}{4}, \tag{61}
\]

where \( \zeta \equiv (\gamma R)^{-1} \) and

\[
E_1(\zeta) = \int_1^\infty dt e^{-\zeta t} = \int_1^\infty dt \frac{e^{-t}}{t}. \tag{62}
\]

Now we can evaluate \( \mathcal{M}_{1B}^{ER} \) with the experimental values for the low-energy constants given in equations (27), (30), and (47). The result is

\[
\mathcal{M}_{1B}^{ER, \text{exp}} = 5.986(1) - 1.138(5) \text{ fm}, \tag{63}
\]

where we have given the individual contributions of the two terms in equation (61). If one calculates with the Argonne potential instead of using the experimental data, one obtains

\[
\mathcal{M}_{1B}^{ER, \text{Argonne}} = 5.986 - 1.137 \text{ fm}, \tag{64}
\]

which shows once more that it is reasonable to identify the results given by the Argonne potential with “experiments.” We can now estimate accurately the remaining small contribution coming from the difference between equations (57) and (60) using the Argonne wave functions:

\[
\delta \mathcal{M}_{1B}^{C+N} = 0.011 \text{ fm}. \tag{65}
\]
Since this quantity is very small and the Argonne wave function is accurate enough, we have not included the error associated with this correction. Combining equations (63) and (65), we obtain

\[ \mathcal{M}_{1B}^{C+N} = (1 \pm 0.02\% \mp 0.07\% \mp 0.02\%) \times 4.859 \text{ fm}, \tag{66} \]

where the first error comes from the uncertainty in \( \alpha_C \), the second from \( r_C \), and the third from \( \rho_{Q} \). (The use of "\( \mp \)" instead of "\( \pm \)" indicates that \( \mathcal{M}_{1B}^{C+N} \) decreases as each of \( \left| \alpha_C \right|, r_C \), and \( \rho_{Q} \) increases.)

Finally, the effects of the residual C2 and VP potentials are calculated from equation (58):

\[ \mathcal{M}_{1B}^{C+N} = \mathcal{M}_{1B}^{VP} + \mathcal{M}_{1B}^{C2} = -0.031 \text{ fm} \tag{67} \]

with \( \mathcal{M}_{1B}^{VP} = -0.022 \text{ fm} \) and \( \mathcal{M}_{1B}^{C2} = -0.009 \text{ fm} \). For later purpose, we evaluate the ratio

\[ \delta_{1B} = \frac{\mathcal{M}_{1B}^{C+N}}{\mathcal{M}_{1B}^{C+N}} = \delta_{1B}^{VP} + \delta_{1B}^{C2} = -0.63\% \tag{68} \]

with

\[ \delta_{1B}^{VP} = -0.45\% \]
\[ \delta_{1B}^{C2} = \delta_{1B}^{C2,C} + \delta_{1B}^{C2,N} = (0.03\%) + (-0.21\%) = -0.18\% \tag{69} \]

where we have presented each contribution from \( V_{C2} \) and \( V_{N}^{C2} \). It is worth noting that the contribution from the C2 potential is about 40\% of the VP contribution, and has the same sign. This is contrary to what one would expect if the \( V_{N}^{C2} \) were ignored.

We follow KB and express the result obtained up to this point in terms of \( \Lambda \) defined by

\[ \Lambda = \frac{(\alpha_C)^{2,3/2}}{2} A_S \mathcal{M}_{1B}. \tag{70} \]

In terms of \( \Lambda \), our result can be summarized as [see eq. (66)]

\[ \Lambda^2 = (1 + \delta_{1B}^2)(1 \pm 0.01\% \mp 0.07\% \pm 0.07\%)^2 \times 7.02, \tag{71} \]

\[ = (1 \pm 0.01\% \mp 0.07\% \pm 0.07\%)^2 \times 6.93, \tag{72} \]

where again the errors correspond to the uncertainties in \( \left| \alpha_C \right|, r_C \), and \( \rho_{Q} \) (or \( A_S \)) in this order. The only possible sources of theoretical error, which are due to uncertainties in the Argonne potential, are in \( \delta_{1B} \) (eq. [65]) and \( \delta_{1B}^{C2,N} \) (eq. [69]). If we were to assign conservatively a \( \sim 50\% \) error to the two quantities there, the resulting theoretical error on the reduced matrix element would then be \( \sim 0.1\% \).

5.2. Calculation of \( \mathcal{M}_{1B}^{C+N} \) in Effective Field Theory

In \$2\$, we argued that the calculation of the single-particle matrix element of the electroweak current using the Argonne wave functions is equivalent to a "first-principle" calculation using effective field theory of QCD. How this comes out in the case of the \( np \) capture was discussed in Park et al. (1997b). Here we briefly report the result of a first-principle calculation for the leading Gamow-Teller matrix element equation (57). Such a calculation involving the Coulomb potential is interesting on its own merit, so we shall describe it in detail elsewhere (Park et al. 1998).

As in Park et al. (1997b), we shall integrate out all fields other than the matter field for the nucleon and treat to next to leading order (NLO). The potential has then the contact interactions used for the \( np \) channel plus the Coulomb interaction

\[ V(q) = \frac{4\pi}{m_N} \left[ C_0 + (C_2 \delta^{ij} + D_2 \sigma^{ij} q^i q^j) + Z_1 Z_2 \frac{\alpha}{q^2} \right], \tag{73} \]

\[ \sigma^{ij} = \frac{3}{2\sqrt{8}} \left( \sigma_1^i \sigma_2^j + \sigma_2^i \sigma_1^j - \frac{\delta^{ij}}{3} \sigma_1 \cdot \sigma_2 \right). \tag{74} \]

In our case, of course, \( Z_1 Z_2 = 1 \) for the \( pp \) channel and 0 for the \( np \) channel. Since the Coulomb interaction is not separable, we use the Gaussian regulator as

\[ V(r) = \int \frac{d^3q}{(2\pi)^3} S_{\Lambda}(q^2) e^{iq \cdot r} V(q) \tag{75} \]

where \( \Lambda \) is the cutoff. The coefficients \( C_{0,2} \) for the spin singlet and triplet channels and \( D_2 \) are fixed by \( np \) scattering and the properties of the deuteron. The cutoff \( \Lambda \) defines a regime in which the effective theory is applicable. It is not a free parameter. By the rule of (cutoff) effective field theory, physical quantities should not depend sensitively on the exact form of the cutoff nor on the exact value of \( \Lambda \). Since the pion is integrated out, the cutoff should be \( \sim m_\pi \). Indeed as we found in Park et al. (1997b), all the static properties of the deuteron are well reproduced for the range of cutoff (140–200) MeV. For this range of cutoff, the one-body matrix element for the \( np \) capture comes out to be 4.00–3.96, which agrees well with the result of the Argonne wave function ("experiment"), 3.98. The same formalism turns out to give

\[ \mathcal{M}_{1B}^{C+N}(\text{EFT}) = 4.89 \pm 0.85. \tag{76} \]

As in the case of the \( np \) capture, this agrees closely with what we might refer to as the "experimental" value, 4.86, with an error of about 0.5\%. This validates our assertion made in \$2\$. In our approach, the corrections from the \( V_s \) potential are found to be

\[ \delta_{1B}^{VP} = -0.45\%, \quad \delta_{1B}^{C2,C} = 0.03\%, \quad \delta_{1B}^{C2,N} = (-0.19 \pm 0.15)\% . \tag{77} \]

These values validate the former calculation, equation (69), where we used Argonne wave function corresponding to the all order of contributions. Both \( \delta_{1B}^{VP} \) and \( \delta_{1B}^{C2,C} \) are found to be extremely insensitive to the cutoff. Even \( \delta_{1B}^{C2,N} \) turns out

11 If one calculates with the Argonne wave functions instead of using the experimental values, one finds \( \Lambda_{\text{Argonne}} = (1 + \delta_{1B})^2 \times 7.03. \)

12 The numerical values of \( \Lambda \) figuring here could well be slightly different from the value in Park et al. (1997b) because of the slightly different regularizations involved. The range of \( \Lambda \) found there to give correct static properties of the deuteron was about (160–220) MeV.

13 Although small in magnitude from the viewpoint of the next to leading order EFT, the error bar of about 0.5\% is bigger than that with the Argonne potential, \( \sim 0.1\% \). This difference may be accounted for by the next-order contribution, namely, the effective volume term, which we have not considered.
to be only mildly dependent on $\Lambda$. This is somewhat surprising, for the effect of $V_{C2} \propto (1/r)V_n(r)$ is expected to be amplified at small $r$ by the nuclear potential.

5.3. $O(Q^3)$ Contribution: The Exchange-Current Matrix Element

The matrix element of the two-body current equation (22) can be separated into three pieces,

$$\mathcal{M}_1 = \mathcal{M}_{2B}^{\text{finite}} + \mathcal{M}_{2B}^{\text{kin}} + \mathcal{M}_{2B}^{\delta},$$

where $\mathcal{M}_{2B}^{\text{finite}}$ is the contribution from the “kinetic term,” $\mathcal{M}_{2B}^{\text{kin}}$ is the zero-ranged contributions, and $\mathcal{M}_{2B}^{\delta}$ is the remaining finite-range contribution. From equations (54) and (22)

$$\mathcal{M}_{2B}^{\text{finite}} = \frac{2}{3} \left( \hat{c}_3 + 2\hat{c}_4 + \frac{1}{2} \right) \frac{m_n^2}{m_N f^2} \int_0^\infty dr u_2(r) y_0(m_n r)$$

$$\times u_{pp}(r) - \frac{2\sqrt{2}}{3} \left( \hat{c}_3 - \hat{c}_4 - \frac{1}{4} \right) \frac{m_n^3}{m_N f^2} \int_0^\infty dr u_2(r) y_0^2(m_n r) u_{pp}(r),$$

$$\mathcal{M}_{2B}^{\text{kin}} = \frac{m_n^3}{3m_N f^2} \int_0^\infty dr \frac{y_1(m_n r)}{2m_n^2 r} \left[ u_{pp}(r) u_{pp}^*(r) - u(r) \right]$$

$$\times u_{pp}(r) - \frac{\sqrt{2}m_n^3}{3m_N f^2} \int_0^\infty dr \frac{y_1(m_n r)}{2m_n^2 r}$$

$$\times \left[ w_2(r) u_{pp}^*(r) - w_2(r) u_{pp}(r) - \frac{3}{r} w_2(r) u_{pp}(r) \right],$$

$$\mathcal{M}_{2B}^{\delta} = -2 \left( \hat{d}_1 - 2\hat{d}_2 + \frac{1}{3} \hat{c}_3 + \frac{2}{3} \hat{c}_4 + \frac{1}{6} \right) \frac{1}{m_N f^2}$$

$$\times \int_0^\infty dr u(r) \frac{\delta(r)}{4\pi r^2} u_{pp}(r).$$

It is straightforward to calculate the matrix elements with the Argonne wave functions. The results are

$$\mathcal{M}_{2B}^{\text{finite}} = 0.246 \text{ fm},$$

$$\mathcal{M}_{2B}^{\text{kin}} = 0.007 \text{ fm},$$

$$\mathcal{M}_{2B}^{\delta} = -(\hat{d}_1 - 2\hat{d}_2 + 0.78) \times (0.001 \text{ fm}).$$

There are two important points to remark here. The first is that the finite-range term equation (82) receives a dominant contribution ($\sim 93\%$) from the deuteron $D$ state. This means that the matrix element is somewhat sensitive to the shorter range part of the current, i.e., $y_2(m_n r)$. We will come back to this later. The second point is that the zero-range contribution equation (84) is strongly suppressed insofar as the magnitudes of the (unknown) constants $\hat{d}_{1,2}$ are “natural,” that is $O(1)$, as required by chiral symmetry. Indeed, in the way formulated in Park et al. (1995, 1996), the zero-range operators should be killed by the short-range correlation associated with the very short-distance physics (i.e., degree of freedom) that is “integrated out” from the low-energy effective chiral Lagrangian. We will therefore drop the zero-range term equation (84) and evaluate the finite-range terms equation (82) and equation (83) with a short-range correlation function incorporated as in Park et al. (1995, 1996). As a short-range correlation function we use a radial wave function cutoff of the form $\theta(r - r_c)$ with $r_c \sim (2m_n)^{-1} \sim 0.7$ fm.

In Figure 2 is shown the ratio

$$\delta_{2B} \equiv \frac{\mathcal{M}_{2B}^{\delta}}{\mathcal{M}_{2B}^{\text{finite}}}$$

as a function of $r_c$.

Taking $0.55 \text{ fm} \leq r_c \leq 0.8 \text{ fm}$, our results may be summarized as

$$\delta_{2B} = (4.0 \pm 0.5)\%.$$  

This is in the range of the values quoted in the literature (Blin-Stoyle & Papageorgiou 1965; Gari & Huffman 1972; Dautry, Rho, & Riska 1976; Gari 1978). We also present the results with other types of short-range correlation function $\hat{g}(r)$ that are frequently used in the literature (Towner 1987):

$$\delta_{2B} = 4.84\%, \quad \hat{g}(r) = 1 - j_0(q_c r) = 1 - \frac{\sin (q_c r)}{q_c r},$$

$$\delta_{2B} = 4.01\%, \quad \hat{g}(r) = 1 - e^{-ar}(1 - br^2),$$

where $q_c = 3.93 \text{ fm}^{-1}, a = 1.1 \text{ fm}^{-2}$, and $b = 0.68 \text{ fm}^{-2}$.

As mentioned, the nonnegligible $r_c$ dependence in the region $r_c \gtrsim (2m_n)^{-1}$ comes from the fact that a short-range (or high-order) current and the $D$-state component of the deuteron wave function are involved. This is due to the fact that because of the symmetry involved, $O(Q)$ and $O(Q^3)$ terms relative to the single-particle operator are suppressed, an aspect closely related to the chiral filter mechanism. This means that some terms higher order than the $O(Q^3)$ ones calculated here—starting at $O(Q^4)$ and probing shorter length scales—may not be negligible compared with the terms that are retained.14

14 For instance, if one adopts a phenomenological approach and calculates $\rho$-meson–exchange diagrams involving a $\rho N\Lambda$ vertex (with form factors appended to all the vertices), the one-pion–exchange contribution is found to be considerably quenched (Barsholz 1979; Carlson et al. 1991). From the chiral perturbation point of view, however, these diagrams come at $O(Q^4)$ or higher, and therefore they need to be calculated together with all other diagrams of the same chiral order. We will not attempt such a calculation here.
5.4. Cross Section

For comparison with KB, we focus on the low-energy cross section factor \( S_{pp}(E) \) (Bahcall & May 1969),

\[
S_{pp}(E) = \frac{\sigma(E)Ee^{-\eta}}{\sigma(E)Ee^{-\eta}},
\]

where \( \sigma(E) \) is the cross section for the process (1) and \( \eta = m_x/2p \). Explicitly, it is of the form

\[
S_{pp}(E) = \left(1 + \frac{\delta_{2B}}{2}\right) \frac{6}{\pi} m_x \varepsilon G_F^2 g_A^2 A^2 \frac{m_e^2 f(E_0 + E)}{\gamma},
\]

where \( f(E_0 + E) \) with \( E_0 \equiv 2m_p - m_d \) is the phase volume defined in Bahcall (1966) with, however, the WM term contribution as well as the deuteron recoil taken into account. These two effects turn out to be very small.\(^{15}\) Setting \( E \approx 0 \), we have

\[
S^{\text{PT}}_{pp}(0) = 4.05 \left(1 + \frac{\delta_{2B}}{1.04}\right) \frac{g_A^2}{1.2601} \left(\frac{\Lambda^2}{6.93}\right)^2 (10^{-25} \text{ MeV barn}),
\]

where we have substituted \( G_F = 1.1356 \times 10^{-5} \text{ GeV}^{-2} \) and the phase volume \( f(E_0) = 0.1421 \). It is worth noting, in passing, that our result for the above \( S \) factor is a precise prediction based upon the standard model, that is, without neutrino mass (oscillation). The corresponding expression given by KB is

\[
S^{KB}_{pp}(0) = 3.89 \left(1 + \frac{\delta_{2B}}{1.01}\right) \frac{g_A^2}{1.2573} \left(\frac{\Lambda^2}{6.92}\right)^2 (10^{-25} \text{ MeV barn}).
\]

By plugging in the numbers obtained above, we arrive at

\[
S^{KB}_{pp}(0) = 4.05 \times 10^{-25}(1 \pm 0.15\% \pm 0.48\%) \text{ MeV barn},
\]

where the first error arises from the one-body matrix element and the second from the two-body term due to the uncertainty in the short-range correlation. (The latter does not include the possible contribution of the next order \( \text{[i.e., } O(Q^4)\text{] term; see later]. This is to be compared with that of KB}

\[
S^{KB}_{pp}(0) = 3.89 \times 10^{-25}(1 \pm 1.1\%) \text{ MeV barn}. \quad (93)
\]

We are now ready to make a detailed comparison between our result and that of KB:

1. The weak-interaction constants used by KB are \( g_A = 1.2575 \) and \( G_F = 1.151 \times 10^{-5} \text{ GeV}^{-2} \) (Freedman 1990).\(^{16}\) Ours are \( g_A = 1.2601 \) and \( G_F = 1.136 \times 10^{-5} \text{ GeV}^{-2} \).
2. KB gave \( \Lambda^2 = 6.92 \) and \( \delta_{2B} = 1\% \), whereas we have obtained \( \Lambda^2 = 6.93 \) and \( \delta_{2B} = 4\% \).

Apart from small differences in the constants used, the major difference is in the exchange-current contribution. In the next section, we will discuss whether this is a genuine difference and to what extent our calculation will be affected by higher chiral-order terms.

Although, as noted by KB, the corrections due to the long-range potential \( V_\tau \) are small, it may be of some interest to make a few remarks on its role. Our result is that the VP potential decreases the pp rate by about 0.9% in agreement with (Bohannon & Heller 1977) (where the decrease was found to be about 0.8%-1.2%) as well as with KB, who found a 0.9% decrease.\(^{17}\) Our result is slightly at variance with that of (Gould 1990), wherein with the use of the WKB approximation a 1.3% decrease was found. In addition the C2 potential lowers the pp rate by about 0.4%. In total, therefore, the downward correction due to \( V_\tau \) is somewhat larger than that of KB. Since the \( \Lambda^2 \) is about the same, this means that our single-particle matrix element \( A_{1B} \) is somewhat larger than that of KB.

6. DISCUSSION

Following the procedure of chiral perturbation theory proven to be highly successful for the thermal np capture process, we have calculated the pp fusion rate to \( O(Q^4) \) in chiral counting relative to the leading single-particle Gamow-Teller matrix element. To the order considered, the error involved in the calculation is small, \( \leq 1\% \). This result, given a justification from a cutoff effective field theory of low-energy QCD as in the case of the np capture (Park et al. 1997b), supports the canonical value of Bahcall and his coworkers and does not support the "relativistic field theory model" result of Ivanov et al. (1997) and Oberhummer et al. (1998).

The main caveat in this calculation is in the meson-exchange contribution that comes out to be about 4% when calculated to the chiral-order \( O(Q^4) \). At the next order, \( O(Q^4) \), loops and higher order counter terms enter, so that there is no reason to believe that they are negligible compared with the \( O(Q^4) \) tree contributions. (For instance, it could be lowered to about 1%-2% instead of \( \approx 4\% \) found here.) This aspect is different from the case of the np capture where the chiral filter mechanism assures the dominance of the tree-order pion exchange-current contribution. Here the absence of the chiral filter phenomenon can allow higher order (loop) terms to figure equally importantly as the tree-order terms. From this viewpoint it is not surprising that a model calculation of the terms of \( O(Q^4) \) and higher based on the vector-meson exchange and form factors (Bargholtz 1979) indicates that there can be a considerable suppression of the tree-order correction. Although such a reduction in the exchange-current contribution seems to go in the right direction for beta decays of higher mass nuclei (Carlson et al. 1991), it is probably unsafe to import the result of such model calculations into our work. To go to \( O(Q^4) \) or above in our theory at which heavy mesons and form factors come in, a large number of Feynman graphs of the same chiral order have to be computed on the same footing to assure chiral symmetry, and such a calculation has not been done yet. In the absence of consistent calculations, our attitude is that we should not attach any error estimates on terms not accounted for to the chiral order computed. Calculating the higher order terms will be left for a future exercise.

\(^{15}\) The WM term increases \( f \) by 0.02%, whereas the deuteron recoil correction decreases it by 0.06%, so the net effect is a 0.04% decrease.

\(^{16}\) The value of \( G_F \) is not given explicitly in KB, but we can reproduce eq. (94) only with the use of this value of \( G_F \), a value that was also adopted in Carlson et al. (1991).

\(^{17}\) As mentioned, despite the similarity in the numerical results, our calculational framework is different from that of KB. This difference will be analyzed in detail elsewhere (Park et al. 1998).
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