Localization and fluctuations of local spectral density on tree-like structures with large connectivity: Application to the quasiparticle line shape in quantum dots

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We study fluctuations of the local density of states (LDOS) on a tree-like lattice with large branching number $m$. The average form of the local spectral function (at given value of the random potential in the observation point) shows a crossover from the Lorentzian to semicircular form at $\alpha \sim 1/m$, where $\alpha = (V/W)^2$, $V$ is the typical value of the hopping matrix element, and $W$ is the width of the distribution of random site energies. For $\alpha > 1/m^2$ the LDOS fluctuations (with respect to this average form) are weak. In the opposite case, $\alpha < 1/m^2$, the fluctuations get strong and the average LDOS ceases to be representative, which is related to the existence of the Anderson transition at $\alpha_c \sim 1/(m^2 \log^2 m)$. On the localized side of the transition the spectrum is discrete, and LDOS is given by a set of $\delta$-like peaks. The effective number of components in this regime is given by $1/P$, with $P$ being the inverse participation ratio. It is shown that $P$ has in the transition point a limiting value $P_c$ close to unity, $1 - P_c \sim 1/\log m$, so that the system undergoes a transition directly from the deeply localized to extended phase. On the side of delocalized states, the peaks in LDOS get broadened, with a width $\sim \exp\{-\text{const} \log m [(\alpha - \alpha_c)/\alpha_c]^{-1/2}\}$ being exponentially small near the transition point. We discuss application of our results to the problem of the quasiparticle line shape in a finite Fermi system, as suggested recently by Altshuler, Gefen, Kamenev, and Levitov.

I. INTRODUCTION.

The recent experimental observation of single-level peaks in excitation spectra of quantum dots [12] has motivated theoretical interest in the problem of the quasiparticle line shape induced by the Coulomb interaction in mesoscopic samples [4-6]. It was found that the single particle levels with excitation energy $E < E_T$ have widths less than the one-particle mean level spacing $\Delta$, and thus can be resolved. Here $E_T$ is the Thouless energy, which has the physical meaning of the inverse time of spreading of a wave packet over the system, the ratio $g = E_T/\Delta$ being the dimensionless conductance of the dot. Thus, the theory predicts that approximately the first $g$ single-particle levels can be resolved, in agreement with experimental findings [4-6]. In a more recent paper [4], Altshuler, Gefen, Kamenev and Levitov (AGKL) mapped the problem onto a tree-like tight-binding model in the Fock space. This model is very close to the Anderson tight-binding model on the Bethe lattice, which was found [13] to undergo the localization transition. This allowed AGKL to conclude that there is a localization threshold $E_c$ ($\Delta < E_c < E_T$) in the problem, so that the states well below $E_c$ are strongly localized, i.e. given by the bare single-particle states with small perturbative admixture of many-particle states.

The qualitative arguments used by AGKL are not sufficient, however, for a more precise determination of the behavior of the spectral shape of the quasiparticle peak in the vicinity of the localization threshold. On the other hand, the critical behavior of the Anderson model on the Bethe lattice was studied in detail in [8]. The critical behavior found there is completely analogous to that obtained earlier for the Bethe lattice version of the $\sigma$-models [4-7] which can be derived from Wegner’s $n$-orbital model with $n \gg 1$ states per lattice site [11] (or, equivalently, considering a system of weakly coupled metallic granules [13]). Exactly the same type of the transition and critical behavior on the Bethe lattice has been found [14] in the “toy” version of the supersymmetric $\sigma$-model introduced in [4], which is much simpler technically than the “true” $\sigma$-models of localization. In Ref. [15] fluctuations of the local density of states (LDOS) near the Anderson transition were studied; as we will see, this question is closely related to the subject under discussion.

The aim of this paper is to study the model derived by AGKL via the supersymmetry technique. In fact, since many results for the critical behavior on the Bethe lattice are already known, we have just to apply them to the present model, properly identifying parameters and quantities of interest. There are, however, new features, which require an additional investigation. First of all, the model of AGKL has a large connectivity (branching number) $m \gg 1$. Secondly, we will be interested in the structure of the local spectral function in energy space; the question, which was not addressed in Refs. [8,15].

The outline of the paper is as follows. In Sec. II we formulate the model as derived by AGKL and identify the parameters and quantities to be studied. Sec. III is devoted to calculation of the average shape of the local spectral function on a Bethe lattice with large branching number. In Sec. IV we evaluate the magnitude of the LDOS fluctuations in the region where these fluctuations are weak. In Sec. V we study the correlations of amplitudes of eigenstates on a tree-like lattice in the critical region. This information is used in Sec. VI, where we study fluctuations of the local spectral function and its typical shape in the vicinity of the Anderson transition, both in the phases of localized and extended states. In
As a result of interaction, the exact eigenstates of a generic many-body Hamiltonian of $N$ pairwise interacting particles:

$$
\mathcal{H} = \sum_p \mathcal{E}_p a_p^\dagger a_p + \sum_{p,q,r,s} V_{p,q}^r s a_p^\dagger a_q^\dagger a_r a_s \quad (1)
$$

where $\mathcal{E}_p$ are energies of the corresponding single particle states. When one neglects the interaction $V$ the complete set of eigenstates is provided by Slater determinants obtained from the filled Fermi sea $|0_N\rangle$ as

$$
|\Psi^{(p)}_N\rangle = a_{p_{2m}}^\dagger \cdots a_{p_{m+1}}^\dagger a_{p_m} \cdots a_{p_1}|0_N\rangle \quad (2)
$$

As a result of interaction, the exact eigenstates $|\alpha\rangle$ for interacting particles are superpositions of the Slater determinants $|\alpha\rangle = \sum_{\{p\}} A_{\{p\}}^\alpha |\Psi^{(p)}_N\rangle$.

To analyze the form of the line for single-particle excitations AGKL suggested to consider the hierarchy of the many-body states in the Fock space. Let us consider the system with $N$ particles in its Hartree-Fock ground state $|0_N\rangle$. The first level of hierarchy is formed by states with one particle: $|1_p\rangle = a_p^\dagger |0_N\rangle$, second by the states with two particles and one hole (which we will denote as three-particle states for brevity): $|3_{pqr}\rangle = a_p^\dagger a_q^\dagger a_r^\dagger |0_N\rangle$, the third one by the states with three particles and two holes (to be denoted as five-particle states) and so forth. Neither of these states (which we denote generally as $|\nu\rangle$) is an eigenstate of $\mathcal{H}$. Therefore if an extra particle is added to the system in such a way that a single-particle state $|1\rangle$ on the first level of hierarchy is formed, this state will decay to the states on the second level of hierarchy, which in turn may spread further in the Fock space. The process of the spread may be looked at as a quantum diffusion of a fictitious particle on a graph whose vertices $i$ are formed by the states $|\nu\rangle$ at different levels of hierarchy. As such, it can be described by an effective one-particle tight-binding model characterized by the Hamiltonian:

$$
\hat{H} = \sum_i \epsilon_i c_i^\dagger c_i + \sum_{\langle i,j \rangle} V_{ij} c_i^\dagger c_j \quad , (3)
$$

with "site energies" $\epsilon_i = \langle i | \mathcal{H} | i \rangle$ and hopping constants $V_{ij} = \langle i | \mathcal{H} | j \rangle$.

Let us note that a problem of decay of an isolated level to a dense band in a situation when the level is directly coupled only to a small fraction of the states of the continuum was studied by Akulin and Dykhne [14]. Though the model considered in [16] is somewhat different from that studied in the present paper, many physical features are actually similar for both models. In particular, it was stressed in [15] that the dynamics of level decay can by affected by the phenomenon of Anderson localization.

The spectral shape of a quasiparticle peak is obtained by projecting the initial single-particle state $|1\rangle$ onto exact eigenstates $|\nu\rangle$:

$$
\rho_1(E) = \sum_\alpha |\langle \nu | \alpha \rangle|^2 \delta(E - E_\alpha) \quad (4)
$$

If we now consider the state $|1\rangle$ as a site of the tight-binding model, we see that $\rho_1(E)$ is nothing else but the local density of states (LDOS) in the point 1 at energy $E$, i.e. the Fourier-transform

$$
\rho_1(E) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{iEt} \langle 1 | e^{-i\mathcal{H}t} | 1 \rangle \quad (5)
$$

of the overlap of the spreading wave function with the initial state $|1\rangle$. In another context this quantity is called the strength function and is frequently used in nuclear and atomic physics to characterize spectra of complex systems, see e.g. [17] for recent applications and further information.

The interaction matrix elements couple the states of each generation to those of the preceding and of the following generations. In particular, a single-particle state is connected with three-particles states, but not with five-, seven-, . . . -particle ones. The density of one-particle states is equal to $\nu_1 = 1/\Delta$. Since we are interested in the regime, where one-particle states can be resolved or, in other words, the width is less than $\Delta$, we can keep only the states with energies within a window $(E - \Delta/2, E + \Delta/2)$ containing just one single-particle state. The density of three-particle states at given energy $E \gg \Delta$ is given by

$$
\nu_3(E) \simeq \frac{1}{2\Delta^3} \int_0^{E} dE_1 \int_0^{E_1} dE_2 = \frac{E^2}{4\Delta^3} \quad , (6)
$$

the factor $1/2$ taking care of the identity of the quasiparticles. The ratio $m = \nu_3(E)/\nu_1 = E^2/4\Delta^2$ defines the effective branching number of the problem and is considered as a big parameter [18]. The important fact observed by AGKL is that this branching number stays parametrically the same for the following levels of hierarchy. For example, the density of the five-particle states $\nu_5(E) \sim E^4/\Delta^5$ and each of these states is connected with of order of one three-particle state (in the chosen energy window of the width $\Delta$), whereas each three-particle state is connected to $\sim E^2/\Delta^2 \sim m$ five-particle states. Following AGKL, we will neglect the change in numerical coefficients, and consider a tree structure with constant branching number $m \gg 1$ (Bethe lattice) [19].

The Hamiltonian of the Anderson model on the Bethe lattice is determined essentially by the following three parameters: the branching number $m$, the characteristic magnitude $V$ of the hopping matrix elements, and the
width $W$ of the distribution of the random site energies $\epsilon_i$. They are related to the parameters of the original (quantum dot) problem as follows. The connectivity $m$ is determined by the proliferation of the density of states from one level of the hierarchy to the next one, as discussed above:

$$m \sim \nu_3(E)/\nu_1 \sim \nu_5(E)/\nu_3(E) \sim \ldots \sim E^2/\Delta^2$$  \hspace{1cm} (7)

The characteristic value of the hopping matrix element is equal to [3]

$$V \sim \Delta^2/E_T \sim \Delta/g$$  \hspace{1cm} (8)

Finally, $W$ can be determined by equating the mean spacing of the states in the next level of the hierarchy connected to a given state, $1/\nu_3(E) \sim \Delta^3/E^2$, to the corresponding quantity in the Anderson model on the Bethe lattice, $W/m$. This yields:

$$W \sim m\Delta^3/E^2 \sim \Delta$$  \hspace{1cm} (9)

In order to check that restriction to the states in the energy interval $\Delta$ is justified, we can modify the model by including all states from a broader energy window $W_n = n\Delta$, with $1 \ll n \ll E/\Delta$. Then each three-particle state will be coupled to $n$ one-particle states; each five-particle state to of order of $n$ three-particle states etc. This modified model thus resembles the $n$-orbital model on the Bethe lattice. It has branching number given by [31]. It is indeed an irrelevant parameter.

On the Bethe lattice. It has branching number given by

$$\alpha \sim (nV/W_n)^2 = (V/\Delta)^2 \sim 1/g^2$$, independent of $n$, which confirms that $n$ is indeed an irrelevant parameter.

Equivalence of the $\alpha$ which confirms that $n\alpha \ll 1$ will be discussed in more detail elsewhere [3].

It is well known [4,5] that the Anderson model on the Bethe lattice exhibits the Anderson localization transition when one of control parameters $W, V, m$ varies. In our case, the change in the quasiparticle excitation energy $E$ implies, see eq.(1), the change of the branching number $m$. Such a phase transition is reflected in a drastic change of statistical properties of eigenfunctions. In our preceding publications [8,11] we have discussed the distribution function of LDOS and its typical spatial shape (for given energy) near the Anderson transition. Now, our interest is concentrated on a typical shape of $\rho_1(E)$ in energy space, for a given “point” [1].

**III. AVERAGE LOCAL SPECTRAL FUNCTION.**

In this section, we study the average form of the local spectral density $\rho_1(E)$ in a site [1] of a Bethe lattice with large connectivity (branching number) $m \gg 1$. We consider for definiteness a model with site-diagonal disorder

$$\hat{H} = \sum_i \epsilon_i c_i^\dagger c_i + V \sum_{(ij)} c_i^\dagger c_j$$  \hspace{1cm} (10)

The averaging is performed over the random energies $\epsilon_i$ in all sites except the site $|1\rangle$, where the spectral density is studied. Thus, we calculate the average form of the LDOS in a site with given value $\epsilon_1$ of the random potential. The same object was studied previously in a random banded matrix model with strongly fluctuating diagonal elements [28]. We will compare the results for the present case to those of the Ref. [28] in the end of the section.

The average value of the LDOS is given by:

$$\rho_1(E) = \frac{1}{\pi} \text{Re} \int dR R F^{m+1}(R^2) e^{i(E - \epsilon_1)R^2/2}$$,  \hspace{1cm} (11)

where $F(R^2)$ is the solution to the following non-linear integral equation:

$$F(S^2) = 1 - S \int_0^\infty dR \int de \times \gamma(e) e^{i(E - \epsilon)R^2/2} F^{m}(R^2) V J_1(VRS)$$  \hspace{1cm} (12)

Here $\gamma(e)$ is the distribution of the random site energies $\epsilon_i$, and $J_1(x)$ is the Bessel function. We look for an approximate solution of eq.(12) in a Gaussian form

$$F(R^2) = \exp\{-g_0 R^2\}$$  \hspace{1cm} (13)

Substituting eq.(13) in the r.h.s. of eq.(12), we get

$$\text{r.h.s.}(12) = \int d\gamma(e) \exp\left\{-\frac{1}{4} V^2 S^2 \left[ \frac{1}{mg_0 - i(E - \epsilon)/2} \right] \right\} \times [1 + g_1 S^4 + \ldots]$$,  \hspace{1cm} (14)

where

$$g_1 = \frac{1}{32} V^4 \left\{ \int d\gamma(e) \left[ \frac{1}{mg_0 - i(E - \epsilon)/2} \right]^2 \right\} - \left\{ \int d\gamma(e) \left[ \frac{1}{mg_0 - i(E - \epsilon)/2} \right] \right\}^2$$  \hspace{1cm} (15)

From eqs.(12), (13), and (14) we get an equation for $g_0$:

$$g_0 = \frac{V^2}{4} \int d\gamma(e) \frac{1}{mg_0 - i(E - \epsilon)/2}$$  \hspace{1cm} (16)

The Ansatz (13) is justified if the $O(S^4)$ correction in eq.(14) can be neglected. Since

$$F^{m}(R^2) = \exp\{-mg_0 R^2\}(1 + mg_1 R^4 + \ldots)$$,  \hspace{1cm} (17)

the integrals in the r.h.s. of eqs.(11), (12) are dominated by the region $R^2 \sim 1/mg_0$. Furthermore, according to eq.(14), $g_1 \sim g_0$. Thus, the $R^4$ term in eq.(17) leads to a relative correction of order of $1/m$, which is small in the case of large connectivity, $m \gg 1$. Therefore, eq.(13) is indeed the proper approximation for $m \gg 1$. 

3
Now we will analyze eq.(10) in the two regimes: (i) \((V/W)^2 \ll 1/m\), and (ii) \((V/W)^2 \gg 1/m\), where \(W\) is a typical magnitude of the random energies \(\epsilon_i\). In the case (i) one can neglect \(g_0\) in denominator of \((10)\). This yields

\[
\text{Reg}_0 = \frac{\pi}{2} V^2 \gamma(E) \equiv \frac{u}{2}
\]

\[
\text{Im}g_0 = \frac{1}{2} V^2 \int d\gamma(\epsilon) \frac{1}{E - \epsilon} \equiv \frac{u}{2}
\]

(18)

Since a typical scale for \(\gamma(E)\) is \(\gamma(E) \sim W^{-1}\) we find \(g_0 \sim V^2/W\). We see that under the condition \((V/W)^2 \ll 1/m\) the quantity \(m\gamma_0\) can be indeed neglected in the denominator of eq.(10) as compared to \(E - \epsilon \sim W\). Then the LDOS, eq.(11), takes the Breit–Wigner form

\[
\langle \rho_1(E) \rangle = \frac{1}{2\pi} \frac{\text{Re} \frac{1}{(m + 1)g_0 - i(E - \epsilon_1)/2}}{mv} \sim \frac{1}{\pi} \frac{mv}{(mv)^2 + (E - \epsilon_1 - u)^2}
\]

(19)

(we neglected the difference between \(m + 1\) and \(m\) in the second line), i.e. it is a Lorentzian with a width \(\Gamma/2 = mv = m\pi V^2 \gamma(E) \sim mV^2/W \ll W\)

(20)

The center of this Lorentzian is determined by the local random energy \(\epsilon_1\) (slightly shifted by \(u \sim \Gamma\)). The width \(\Gamma\) is exactly the quasiparticle decay rate one would get by applying the Fermi Golden Rule: it is equal to \(2\pi\) times matrix element \(V\) squared times density of states into which the particle decays \((m\gamma(\epsilon_1) \simeq m\gamma(E))\).

In the opposite case, \((V/W)^2 \gg 1/m\), we can neglect \(\epsilon\) in the denominator of eq.(10), which reduces then to

\[
g_0 = \frac{V^2}{4} \frac{1}{mg_0 - iE/2}
\]

(21)

We find therefore

\[
g_0 = \frac{1}{4m} (iE + \sqrt{4mV^2 - E^2})
\]

(22)

and

\[
\langle \rho_1(E) \rangle = \frac{1}{2\pi} \frac{\text{Re} \frac{1}{mg_0 - iE/2}}{\pi V^2} \text{Re}g_0 = \left\{ \begin{array}{ll} \frac{1}{2\pi m V^2} \sqrt{4m V^2 - E^2} & , \quad |E| \leq 2\sqrt{mV} \\ 0 & , \quad |E| \geq 2\sqrt{mV} \end{array} \right.
\]

(23)

Thus, the LDOS in this limit has a semicircular form. The width of the semicircle is of order of \(\sqrt{mV} \gg W\) justifying omission of \(\epsilon \lesssim W\) in the denominator of eq.(10).

The crossover from the Lorentzian to the semicircular shape of local DOS is completely analogous to that found for the model of random banded matrices with strongly fluctuating diagonal elements [28]. The physical reason for this crossover is the same in both cases. When the disorder is sufficiently strong, an eigenstate is not spread uniformly over all the sites of the lattice, but rather is concentrated on a small fraction of lattice, formed by the sites which satisfy a kind of resonance condition with a given site [1]. This leads to a Lorentzian form of the LDOS with a width \(\Gamma\) much less than the width \(W\) of the distribution of diagonal matrix elements (random energies). Position of this Lorentzian for a site \((i)\) is determined essentially by the random energy \(\epsilon_i\). The weaker is the disorder the larger is \(\Gamma\). The crossover point is determined by the condition \(\Gamma \sim W\). For a still weaker disorder the LDOS acquires the semicircular form, one and the same for all sites of the lattice. In this regime eigenstates are spread over all the lattice like in the Gaussian ensemble.

Finally, we note that when the Bethe lattice model is used to describe excitations in a quantum dot only the first (Lorentzian) regime is relevant. Indeed, in this case one is interested in the regime \(\Gamma < \Delta\) when the single-particle levels are resolved, whereas the energies \(\epsilon_i\) of many-particle states included in the Bethe lattice model are distributed in the interval of the order of \(\Delta\), see Eq.(1). The Lorentzian form of the local spectral function holds, of course, also for higher energies, \(E \gtrsim E_T\), where the discreteness of the spectrum is unimportant and the Fermi Golden Rule is certainly applicable. To see this in our approach, one has to include in the model the states from a broader energy interval \(n\Delta > \Gamma\), as discussed in the end of Sec.II. However, we do not consider this region, where the single-particle states are not resolved and all the results (say, for the lifetimes) can be simply obtained in perturbation theory.

IV. FLUCTUATIONS OF LDOS: REGIME OF WEAK FLUCTUATIONS

In the preceding section, we have calculated the local spectral function \(\rho_1(E)\) at fixed value of the random potential \(\epsilon_1\) in a given site [1] but averaged over all the other random energies \(\epsilon_i, i \neq 1\). The main aim of the remaining part of the paper is to find out how a typical (rather than averaged) LDOS looks like. For this purpose, we study fluctuations of \(\rho_1(E)\) in various regimes. It turns out that the parameter which distinguishes between the regimes (at fixed value of the branching number \(m\)) is

\[
\alpha \sim (V/W)^2
\]

(24)

Our consideration in this section will be similar to that of the preceding one. This will allow us to describe fluctuations of the LDOS in the regime where they are weak, namely \(\alpha \gg 1/m^2\).

To study the fluctuations of the LDOS, it is not sufficient to consider the integral equation (12) for the function \(F(S^2)\) determining the average LDOS. Instead a more complicated equation on a function \(F(S^1, S_2^2)\) of two variables has to be considered [3]:

\[
F(S_1^2, S_2^2) = \int dS_3^2 \frac{1}{S_3^2} F(S_1^2, S_2^2, S_3^2)
\]

(25)
\[ F(S_1^2, S_2^2) = 4 \int R_1 dR_1 \int R_2 dR_2 J_0(V R_1 S_1) J_0(V R_2 S_2) \]
\[ \times \frac{\partial^2}{\partial (R_1^2) \partial (R_2^2)} F^m(R_1^2, R_2^2) \int d\epsilon \gamma(\epsilon) \]
\[ \times \exp[i(E - \epsilon)(R_1^2 - R_2^2)/2] \]  
(25)

Let us look again for its approximate solution in the Gaussian form:

\[ F(S_1^2, S_2^2) = \exp \left\{ \frac{i}{2} S_1^2 (u - iv) + \frac{i}{2} S_2^2 (u + iv) \right\} \]  
(26)

Substituting (26) in the r.h.s. of eq.(25), we get

\[ \text{r.h.s.}(25) = \int d\epsilon \gamma(\epsilon) \exp \left\{ \frac{V^2 S_1^2}{2i} \frac{1}{-m(u - iv) + E - \epsilon} \right\} \]  
\[ + \frac{V^2 S_2^2}{2i} \frac{1}{m(u + iv) - (E - \epsilon)} \]  
(27)

Expanding this expression up to the terms of the order of \( S_1^2 \) and \( S_2^2 \) one finds that \( (v + iv)/2 \equiv g_0 \) satisfies the same equation (13) obtained in Sec. III where the average LDOS was considered.

The function \( F(S_1^2, S_2^2) \) determines the fluctuations of the one-site Green’s function

\[ G_1(E) = \sum_{\alpha} \vert \langle \alpha | 1 \rangle \vert^2 (E - E_{\alpha} + i0)^{-1} \]  

( imaginary part of which is equal to \(-\pi \rho_1(E)\) ) via the following equation (15)

\[ F^{m+1}(R_1^2, R_2^2) \exp\{i(E - \epsilon_1)(R_1^2 - R_2^2)/2\} \]
\[ = \int_{-\infty}^{\infty} d\tilde{u} \int_0^{\infty} d\tilde{v} P(\tilde{u}, \tilde{v}) \exp\{i\tilde{u}(R_1^2 - R_2^2) - \tilde{v}(R_1^2 + R_2^2)\} \]  
(28)

Here \( \tilde{u} \) and \( \tilde{v} \) are the real and imaginary part of the inverse Green’s function,

\[ [G_1(E)]^{-1} = \tilde{u} - i\tilde{v} \]  
(29)

and \( P(\tilde{u}, \tilde{v}) \) is the joint probability distribution of \( \tilde{u} \) and \( \tilde{v} \).

Substituting eq.(24) into (25), we find

\[ P(\tilde{u}, \tilde{v}) = \delta(\tilde{u} - \tilde{u}_0) \delta(\tilde{v} - \tilde{v}_0) \]  
(30)

where (neglecting the difference between \( m + 1 \) and \( m \) as in Sec.V)

\[ \tilde{u}_0 = \frac{1}{2}(E - \epsilon_1 - m\nu) \]  
\[ \tilde{v}_0 = \frac{1}{2} \nu \]  
(31)

(32)

We see that eqs.(31), (32) together with the definition (24) reproduce precisely the average LDOS found in the preceding section. Therefore, the Gaussian approximation (24) corresponds to absence of any LDOS fluctuations, as is clearly seen from eq.(23). To calculate corrections to the Gaussian approximation we make the next iteration and expand eq.(27) up to \( O(S^4) \) terms. Assuming that we are in the Lorentzian regime, \((V/W)^2 \ll 1/m\), we get after some algebra

\[ F(S_1^2, S_2^2) = \exp \left\{ -\frac{i}{2} S_1^2 (u - iv) + \frac{i}{2} S_2^2 (u + iv) \right\} \]
\[ \times \left[ 1 + \frac{\pi \gamma(E - mu)}{4m} V^4 S_1^2 S_2^2 + \ldots \right] \]  
(33)

Substituting this in the l.h.s. of eq.(28) and expanding both sides of this equation in a power series in \((R_1^2 - R_2^2)\) and \((R_1^2 + R_2^2)\), we find the average values \( \langle \tilde{v} \rangle = \tilde{v}_0 \) (as given by eqs.(31), (32)), and the variances

\[ \text{var}(\tilde{v}) = \text{var}(\tilde{v}_0) = \frac{\pi}{8} V^4 \gamma(E - mu) / v \simeq \frac{V^2}{8} \]  
\[ x \]  
(34)

The parameters \( \tilde{u}_0 \) and \( \tilde{v}_0 \) determine the position and the width of the averaged (Lorentzian) LDOS whereas eq.(34) describes its fluctuations. The relative strength of the fluctuations is characterized by the ratio

\[ \frac{\text{var}(\tilde{v})}{(\tilde{v})^2} = \frac{\pi}{4} V^3 \gamma(E - mu) / v^2 \simeq \frac{1}{4m\pi \gamma(E) V} \simeq \frac{W}{mV} \]  
\[ x \]  
(35)

Thus, the LDOS fluctuations are weak when \( \alpha \gg 1/m^2 \). In the opposite case the fluctuations are strong and the Gaussian Ansatz is not a good approximation any longer. Correspondingly, the expansion around it employed in this section breaks down.

V. EIGENFUNCTIONS CORRELATIONS NEAR THE LOCALIZATION THRESHOLD

Now we want to study a typical structure of LDOS in the vicinity of the Anderson transition point. For this purpose, we will need, however, information concerning the correlations in amplitudes of two different eigenfunctions with energies close to the mobility edge in the same spatial point. We will not use the fact that \( m \gg 1 \) here and so our calculation will be valid for arbitrary branching number. Moreover, we believe that the results of this section are of even more general validity and are not restricted to tree-like lattices (see the discussion in the end of the section).

The quantity we want to calculate here is the correlation function \( \sigma(r, E, \omega) \) representing the overlap of the eigenfunctions,

\[ \sigma(r, E, \omega) = \Delta^2 \hat{R}^{-1}(\omega) \sum_{\alpha \neq \beta} \vert \Psi_{\alpha}(r) \vert^2 \vert \Psi_{\beta}(r) \vert^2 \]
\[ \times \delta(E - E_{\alpha}) \delta(E + \omega - E_{\beta}) \]  
(36)

where \( \hat{R}(\omega) \) is the level correlation function,
\[ \hat{R}(\omega) = \Delta^2 \left( \sum_{\alpha \neq \beta} \delta(E - E\alpha)\delta(E + \omega - E\beta) \right). \]  

(37)

We are going to study it for energies \( E, E+\omega \) close to the mobility edge (in the phase of delocalized states) so that the correlation length \( \xi \) is large. We will assume however that the system size is larger than the correlation length, so that the system is in the critical regime, but not exactly in the critical point. We will compare \( \sigma(r,E,\omega) \) to the analogous correlation function containing a single eigenfunction,

\[ \pi(r,E) = \Delta \left( \sum_{\alpha} |\Psi\alpha(r)|^4 \delta(E - E\alpha) \right), \]

which defines the inverse participation ratio,

\[ P(E) = \sum_r \pi(r,E). \]

(39)

In fact, the Bethe lattice is not a proper model to study quantities like \( \pi(r,E) \) since these quantities are well-defined only for finite systems. However, a finite Bethe lattice has a peculiar feature: most of its sites are located at the boundary. As the result, the properties may depend crucially on boundary conditions even in the thermodynamic limit. There exists, however, another model – that of sparse random matrices (SRM) – which describes a lattice having locally a tree-like structure similar to that of the Bethe lattice but possessing also large-scale loops ensuring that all sites are essentially equivalent. As a consequence, this model is free from the problem of boundary conditions. At the same time it is known that the localization transition in the SRM model is equivalent to that on the Bethe lattice \[24\]. The inverse participation ratio \[23\] was studied in the delocalized phase of the SRM model near the localization threshold in Ref. \[2\]. It was shown there that its critical behavior is determined by an exponentially large scale \( C(E) \) emerging in the non-compact sector of the effective supersymmetric model,

\[ P(E) \propto N^{-1} C(E) \]

(40)

\[ C(E) \propto \exp\{\text{const}|E - E_c|^{-1/2}\} \]

(41)

The scale \( C(E) \) has a physical meaning of the “correlation volume” \[2\] and determines also the critical behavior of conductivity in the delocalized phase. We extend now the calculation of Ref. \[22\] to find also the correlation function \([50]\).

To calculate the overlap function defined in Eq.\((36)\) we follow \[22\] and use the identity relating \( \pi(r,E) \) and \( \sigma(r,E,\omega) \) to advanced and retarded Green functions

\[ G^{R,A}(r,E) = \sum_{\alpha=1}^N |\Psi\alpha(r)|^2 (E \pm i0 - E\alpha)^{-1}: \]

\[ 2\pi^2 \left[ \Delta^{-1} \pi(r,E)\delta(\omega) + \Delta^{-2} \hat{R}_2(\omega)\sigma(r,E,\omega) \right] \]

\[ = \text{Re} \left[ \langle G^R(r,E)G^A(r,E+\omega) - G^R(r,E)G^R(r,E+\omega) \rangle \right] \]

(42)

Let us consider for definiteness the ensemble of real symmetric SRM, corresponding to systems with unbroken time-reversal invariance. For any site index \( r = 1,\ldots,N \) we introduce one eight-component supervector \( \Phi^\dagger = (\Phi^\dagger_{R1},\Phi^\dagger_{A1}) \) consisting of two four-component supervectors \( \Phi^\dagger_{R1} = (\phi_{\sigma,b1},\phi_{\sigma,b2},\phi_{\sigma,f1}^\dagger - \phi_{\sigma,f}) \), where indices \( \sigma = R,A \) and \( b,f \) are used to label advanced-retarded and boson-fermion subspaces, respectively. The ensemble-averaged products \( \langle \Phi^\dagger \Phi^* \rangle \) for RSM model in the limit \( N \gg 1 \) can be extracted from the paper \[20\] and the Appendix D of the paper \[23\] and is given by:

\[ \langle G^\sigma(r,E)G^{\alpha\sigma}(r,E+\omega) \rangle = \left( 1 - \frac{4}{3} \delta_{\sigma,\alpha} \right) \times \int D\Omega \langle \phi_{\sigma,b1}\phi_{\sigma,b1}\phi_{\alpha,s},\phi_{\sigma,f1} \rangle_{g_z} \exp \left( \frac{i\pi\rho\omega N}{4} \text{Str} Q\Lambda \right); \]

\[ \langle \cdots \rangle_{g_z} = \int d\Phi \langle \cdots \rangle \exp \left[ \frac{i}{2} E\Phi^\dagger L\Phi + mg_T(\Phi) \right]. \]

(43)

The function \( g_T(\Phi) \equiv g_0(\Phi^\dagger T^1T^1\Phi) \) satisfies the integral equation:

\[ g_T(\Psi) = \langle [h_F(\Phi^\dagger T^1\Psi) - 1] \rangle_{g_T}, \]

(44)

where \( h_F(t) = \int dz e^{it\rho(z)} \) is the Fourier-transform of the distribution of nonzero elements of the SRM. The 8 \times 8 supermatrices \( T \) satisfy the condition \( T^1LT = L \) where \( L = \text{diag}(1,1,1,1,-1,-1,-1,1) \) and belong to a graded coset space whose explicit parametrization can be found in \[24,25\]. The supermatrices \( Q \) are expressed in terms of \( T \) as \( Q = T^{-1}LT \). At last, the matrix \( \Lambda = \text{diag}(1,1,1,1,-1,-1,-1,1) \), and the density of states \( \rho \) is expressed in terms of the solution of the equation Eq.\((14)\) as \( \rho(E) = -2g_{00}/(\pi B_2) \), where \( B_2 = \int dz h(z)z^2 \) and \( g_{00} = \partial g_0(0,y)/\partial x |_{x=y=0} \). It is assumed that \( \Phi^\dagger = \Phi^\dagger L\Phi \).

When deriving Eq.\((13)\), evaluation of a functional integral by the saddle-point method has been employed, see details in \[24,25\]. An accurate consideration shows that such a procedure is legitimate as long as: i) the matrix size \( N \) (playing in our model the role of the volume) is large enough (much larger than the coefficient \( C(E) \) determining the size dependence of IPR, see above); and ii) the energy difference \( \omega \) is small enough (much smaller than \( C^{-1}(E) \)). Though \( C(E) \) is exponentially large near the transition point, it depends on the energy \( E \) only, so that when we keep \( E \) fixed and increase the system size \( N \), the number of levels in the interval \( C^{-1}(E) \) gets arbitrarily large, since the level spacing scales as \( 1/N \).

Expanding both sides of Eq.\((14)\) over \( \Psi \), one can express \( \langle \phi_{\sigma,b1}\phi_{\sigma,b1}\phi_{\alpha,s},\phi_{\sigma,f1}\rangle_{g_T} \) in terms of the matrix \( Q \) as

\[ \langle \phi_{\sigma,b1}\phi_{\sigma,b1}\phi_{\sigma,b1}\phi_{\sigma,b1}\rangle_{g_T} = 4! \left[ \frac{1}{2} g_{0,xx} Q^{\sigma\sigma}_{b1,b1} Q^{\sigma\sigma}_{b1,b1} + g_{0,xy} Q^{\sigma\sigma}_{b1,b1} + g_{0,yy} \right]; \]

\[ \langle \phi_{R,b1}\phi_{R,b1}\phi_{A,b1}\phi_{A,b1}\rangle_{g_T} \]
the mobility edge everywhere in the phase of extended eigenstates, up to $(\text{system volume})$ exceeds the correlation volume. In contrast, $\psi(r)$ is Gaussian white-noise component fluctuating in space on the scale of lattice constant. It fills in the “smooth” component $\Phi_E(r)$ in an individual way for each eigenfunction, but is not critical, i.e. is not sensitive to the vicinity of the Anderson transition. These Gaussian fluctuations are responsible for the factor $\beta/(\beta + 2)$ (which is the same as in the corresponding Gaussian Ensemble) in Eq. (49).

As was already mentioned, this picture is valid in the energy window $\delta E \sim C^{-1}(E)$ around the energy $E$; the number of levels in this window being large as $\delta E/\Delta \sim NC^{-1}(E) \gg 1$ in the thermodynamic limit $N \to \infty$. These states form a kind of Gaussian Ensemble on a spatially non-uniform (multifractal for $E \to E_c$) background $\Phi_E(r)$. Since the eigenfunction correlations are described by the formula (49), which has exactly the same form as in the Gaussian Ensemble, it is not surprising that the level statistics has the WD form everywhere in the extended phase [24].

Let us make a side remark here. We believe on physical grounds that the same picture should hold for a conventional $d$-dimensional conductor. First of all, the general mechanism of the transition is the same in $d < \infty$ and $d = \infty$ models. Furthermore, the sparsity (multifractality) of eigenstates near the transition point takes its extreme form for $d = \infty$ models [15], so that since the strong correlations (49) take place at $d = \infty$ it would be very surprising if they do not hold at finite $d$ as well. Finally, Eq.(49) was proven by an explicit calculation in the weak localization regime [24], where $\sigma(r, E, \omega) = \frac{2}{\beta + 2} \pi(r, E) = V^{-2}[1 + \Pi(r, r)]$, with $V$ being the system volume and $\Pi(r, r)$ the diffusion propagator. Therefore, we believe that also for a $d$-dimensional conductor the WD statistics applies everywhere in the delocalized phase, if the system is large enough. Furthermore, this implies that exactly in the critical point the level repulsion has a conventional $\omega^2$ form on a scale $\omega \sim \Delta$. We refer the reader to a separate publication [27], where these questions are discussed in more detail.

VI. FLUCTUATIONS OF THE LDOS: CRITICAL REGION.

Now we are prepared to study the structure of the LDOS in the vicinity of the localization transition. We note first that the parameter $\alpha$ defined by Eq.(24) is nothing else but the coupling constant of the corresponding $\sigma$-model. In fact, to derive $\sigma$-model rigorously one has to introduce $n \gg 1$ states per lattice site [1,12], whereas...
the Anderson model corresponds to \( n = 1 \). However, the both models have essentially equivalent critical behavior, see below. The \( \sigma \)-model is defined by the action (we consider the unitary symmetry case for definiteness):

\[
S\{Q\} = \frac{\alpha}{2} \sum_{\langle ij \rangle} \text{Str} Q_i Q_j + \varepsilon \sum_i \text{Str} \Lambda Q_i ,
\]

(50)

where the summation in the first term goes over the pairs of nearest neighbor lattice sites, \( Q_i = T_i^{-1} A T_i \) is a \( 4 \times 4 \) supermatrix belonging to certain coset space and satisfying the constraint \( Q^2 = 1 \), \( \varepsilon = -i \pi \nu \omega / 2 \), \( \nu \) is the density of states, \( \Lambda = \text{diag}\{1, 1, -1, -1\} \), and \( \text{Str} \) denotes the supertrace. The reader is referred to Refs. [24–26] for more detailed information on the supersymmetry formalism.

While considering the vicinity of the Anderson transition point we prefer to work within the \( \sigma \)-model formalism. The \( \sigma \)-model formulation has a considerable advantage: its behavior is determined by a single coupling constant \( \alpha \). This should be contrasted with the original Anderson model controlled by energy \( E \) and the whole distribution function \( \gamma(\epsilon) \). However, the mechanism underlying the transition and all the essential features of the critical behavior are exactly the same for both the \( \sigma \)-model and the Anderson model on the Bethe lattice, see Refs. [8,10,13].

When formulated on a Bethe lattice the nonlinear \( \sigma \)-model can be reduced to a certain non-linear integral equation for a function \( Y(Q) \) on the coset space defined as

\[
Y(Q_1) = \int \prod_{i \neq 1} DQ_i e^{-S(Q)}
\]

(51)

Because of the invariance properties the function \( Y(Q) \) depends only on the two scalar variables (eigenvalues) \( 1 \leq \lambda_1 < \infty \) and \(-1 \leq \lambda_2 \leq 1 \). Moreover, in the localized phase (as well as close to the transition point in the delocalized phase) only the dependence on the “non-compact” variable \( \lambda_1 \) is essential. In particular, in the localized phase the integral equation Eq. (51) in the limit \( \eta \to 0 \) takes the form

\[
y(t) = \int_{-\infty}^{\infty} dt' L_\alpha(t - t') \exp\{-2e^{t'}\} y^m(t') ,
\]

(52)

where \( t = \varepsilon \lambda_1 \) and

\[
L_\alpha(t) = \left( \frac{\alpha}{2\pi} \right)^{1/2} e^{t/2} e^{-\alpha \cosh t} \left[ \sinh \alpha \cosh t \right. \\
\left. + \left( \cosh \alpha - \frac{\sinh \alpha}{2\alpha} \right) \right]
\]

(53)

In the localized regime \((\alpha < \alpha_c)\), eq. (52) has a solution in the form of a kink with the asymptotics \( y(t) \approx 1 \) at \( t \to -\infty \) and \( y(t) \approx 0 \) at \( t \to +\infty \). In contrast, in the delocalized phase \((\alpha < \alpha_c)\), eq. (52) has a trivial solution \( y(t) = 0 \) only. The Anderson transition point \( \alpha_c \) is thus determined by a condition of the stability of the kink solution, which has the form

\[
mw_\alpha(1/2) = 1 ;
\]

(54)

\[
w_\alpha(\theta) = \int dt e^{-\theta t} L_\alpha(t)
\]

(55)

For the nonlinear \( \sigma \)-model with orthogonal symmetry (corresponding to the case of unbroken time reversal invariance) the underlying structure is completely analogous, except for a more complicated form of the function \( L_\alpha(t) \) [10]. However, all the gross properties and the results are essentially the same as in the unitary case.

The technically simplest (but still non-trivial) "toy" variant of the supersymmetric nonlinear \( \sigma \)-model was introduced by Zirnbauer in [14] and called "hyperbolic superplane" (HSP). When formulated on the Bethe lattice it has the critical behavior completely equivalent to that of the true nonlinear \( \sigma \)-models of Anderson localization [14]. In particular, the "invariant" phase (which should be associated with the localized states regime) is described by the same integral equation (52), with the kernel \( L_\alpha(t) \) given by

\[
L_\alpha(t) = \frac{1}{2K_{1/2}(\alpha)} \exp\left( \frac{t}{2} - \alpha \cosh t \right)
\]

(56)

The kernel (56) has the same gross features as (53). Moreover, we are able to show that the position of the mobility edge in the Anderson model on the large connectivity Bethe lattice is exactly given by Eqs. (54), (55) with the same kernel Eq. (56). Derivation of this result, and more generally, demonstration of the equivalence of the \( \sigma \)-model and the Anderson model in the regime \( \alpha \ll 1 \) will be published elsewhere [31]. Let us stress, that this is exactly the region, where the transition for \( m \gg 1 \) happens, see Eq. (58) below.

Substituting eq. (56) into eqs. (54), (55), one gets the following equation for the transition point [13]:

\[
m \frac{K_0(\alpha_c)}{K_{1/2}(\alpha_c)} = 1 ,
\]

(57)

where \( K_\nu(z) \) is the modified Bessel function. Using now \( K_{1/2}(z) = (\pi / 2z)^{1/2} e^{-z} \) and the small-\( z \) behavior of \( K_0(z) \), \( K_0(z) \approx \log(z/2) \), we find

\[
\alpha_c \approx \frac{\pi}{8m^2 \log^2 m} , \quad m \gg 1
\]

(58)

Let us first analyze the structure of the LDOS \( \rho_1(E) \) in the phase of localized states, \( \alpha < \alpha_c \). In this regime, the spectrum is discrete, so that \( \rho_1(E) \) is given by a sum of \( \delta \)-functions with positions in \( E = E_0 \) and residues \( \left| \langle \alpha | 1 \rangle \right|^2 \). Only those eigenstates \( |\alpha \rangle \), which are localized close to the site \( |1\rangle \), give appreciable contributions to this sum. The effective number of such states is measured by \( P^{-1}(\epsilon) \), where \( P(E) \) is the inverse participation ratio,
\[ P(E) = \sum_\alpha |\langle \alpha | 1 \rangle|^4 \simeq \nu^{-1} \left( \sum_\alpha |\langle \alpha | 1 \rangle|^4 \delta(E_\alpha - E) \right) \]  

(59)

Rewriting \( P(E) \) in the second form (more convenient for an analytical study), we used the fact that the states \( \langle \alpha | 1 \rangle \) contributing appreciably to \( \rho_1(E) \) have energies \( E_\alpha \) close to \( \epsilon_1 \). The latter statement follows immediately from the results of Sec. I, where the average LDOS was shown to be a narrow Lorentzian around \( \epsilon_1 \), see eq. (10).

The IPR takes values between 1 and \( 1/N \) corresponding to complete localization and delocalization of eigenfunctions respectively. In the localized phase, \( P(E) \) is finite, whereas in the delocalized one it is proportional to \( \tau_2 \) (in the thermodynamic limit). In a \( d \)-dimensional conductor, when \( E \) approaches the transition point from the phase of localized states, \( P(E) \) vanishes continuously: \( P(E) \propto |E - E_c|^\gamma \), where \( \gamma \) is one of the set of the exponents determining the multifractal structure of eigenfunctions [32]. In contrast, on a Bethe lattice this happens in an abrupt manner [37,40]: \( P(E) \) has a finite limiting value \( P_c \) as \( E \) approaches \( E_c \) from the side of localized states (\( \alpha \to \alpha_c, \epsilon_0 \to 0 \) in terms of the \( \sigma \)-model) and then jumps discontinuously to zero in the extended phase (\( \alpha = \alpha_c + 0 \)). Therefore, the effective number of \( \delta \)-peaks in \( \rho_1(E) \) in the localized phase stays finite as we approach the Anderson transition point.

This result was obtained in Refs. [37,40,42], where the lattice connectivity was considered as a number of order of unity. Let us study now what happens with the limiting value \( P_c \) for large connectivity \( m \gg 1 \). The IPR is given by the following expression [41,40]

\[ P = 2 \int_{-\infty}^{\infty} dt e^t \exp\{-2e^t\} y^{m+1}(t) \]  

(60)

where \( y(t) \) is the solution of eq. (62). For \( m \gg 1 \) the transition happens at \( \alpha_c \ll 1 \), see eq. (58). It is easy to see that the function \( L_\alpha(t) \), eq. (54), has in this regime a sharp maximum around \( t = \log(1/\alpha) \gg 1 \) with a width of the order of unity. Taking into account the normalization property, \( \int dt L_\alpha(t) = 1 \), we have thus \( L_\alpha(t) \approx \delta(t - \log(1/\alpha)) \). Substituting this in eq. (62), we find that the factor \( \exp(-2e^t) \) in the r.h.s. implies that the kink in the function \( y(t) \) is located at \( t \simeq \log(1/\alpha) \). We know from the analysis of Refs. [41,40,42,43] that the small-\( t \) behavior of \( y(t) \) near the transition point is the following:

\[ y_c(t) \simeq 1 - e^{(t-t_0)/2}, \quad t < t_0, \]  

(61)

where \( t_0 \) is a constant determining the position of the kink. We find thus \( e^{t_0} \sim 1/\alpha_c \), up to a factor of the order of unity. Substituting this in eq. (62), we get

\[ P_c \simeq 2 \int_{-\infty}^{\infty} dt e^t \exp\{-2e^t\} \left[ 1 - me^{(t-t_0)/2} \right], \]  

(62)

so that

\[ 1 - P_c = 2me^{-t_0/2} \int dt \exp\{3t/2 - 2e^t\} \]

\[ \simeq m \sqrt{\frac{t}{8}} e^{-t_0/2} \sim m \alpha_c^{1/2} \sim \frac{1}{\log m} \]  

(63)

Therefore, at \( m \gg 1 \) not only IPR has a non-zero limit at the transition point but this limiting value is close to unity, i.e. to the upper bound corresponding to extremely localized states concentrated on a single site. Roughly speaking, the system undergoes a transition directly from the deeply localized phase (\( P \) close to unity) to the extended phase (\( P = 0 \)).

The above results, eqs. (61)-(63) can be also reproduced by solving iteratively the self-consistency equation [42]. Numerical results obtained in Refs. [10,30] show a clear tendency of increase of \( P_c \) with \( m \), in full agreement with eq. (63). Therefore, when the system approaches the Anderson transition point from the localized side, the strength function \( \rho_1(E) \) has a form of the sum of \( \delta \)-like peaks with almost the whole of the spectral weight (except a small fraction of the order of \( 1/\log m \)) concentrated in one peak.

When the system is driven through the critical point into the delocalized phase, this picture evolves gradually. Namely, the peaks get broadened with a width depending on the distance to the critical point and vanishing (in the thermodynamic limit) at the critical point. It is clear that the width is given by the scale \( C^{-1}(E) \), within which the eigenfunctions are fully correlated (see sec. II). This scale is exponentially small in the vicinity of the critical point, see eq. (11), or in terms of the \( \sigma \)-model coupling constant \( \alpha \) [41,44,33],

\[ C^{-1}(\alpha) \propto \exp \left\{ -c_1 \left| \frac{\alpha - \alpha_c}{\alpha_c} \right|^{-1/2} \right\}; \quad \alpha > \alpha_c \]  

(64)

Such a critical behavior is a peculiar feature of the tree-like lattices, and is replaced by a conventional power-law critical behavior for a \( d \)-dimensional systems with \( d < \infty \) [33]. Therefore, in the critical vicinity of the transition point (on the delocalized side), the peaks in \( \rho_1(E) \) have an exponentially small width of the order of \( C(\alpha) \), eq. (64). This is also confirmed by the behavior [41] of the IPR in this region.

Let us study now the behavior of the coefficient \( c_1 \) of the critical behavior [44] in the limit of large connectivity \( m \gg 1 \). This can be again the most easily done for the HSP model. The condition on the transition point is given by eq. (58), and the scale \( C(\alpha) \) is given by [33]

\[ C(\alpha) \simeq e^{\pi/\eta}, \]  

(65)

where \( \eta \) is determined from the equation

\[ m \frac{K_{m}(\alpha)}{K_{1/2}(\alpha)} = 1 \]  

(66)
Expanding eq.(66) in $\eta$ and $\alpha - \alpha_c$ and using the asymptotic behavior of the modified Bessel function,

$$K_{\eta}(\alpha) \simeq \frac{1}{2} \left[ \Gamma(i\eta) \left( \frac{\alpha}{2} \right)^{-i\eta} + \Gamma(-i\eta) \left( \frac{\alpha}{2} \right)^{i\eta} \right], \quad \alpha \ll 1$$

$$\Gamma(i\eta) = \frac{1}{i\eta} \Gamma(1 + i\eta) \simeq \frac{1}{i\eta} (1 - i\eta C), \quad \eta \ll 1$$

($C$ being the Euler’s constant), we reduce eq.(66) to the following form:

$$\eta^2 = \frac{6}{-\log^2(\alpha_c/2)} \left[ \frac{\sqrt{\pi/2}}{2m\alpha_c^{3/2}} - \frac{1}{\alpha_c} \right] (\alpha - \alpha_c)$$

Using now the formula (58) for $\alpha_c$, we find

$$\eta \simeq \sqrt{\frac{6}{\pi} m(\alpha - \alpha_c)^{1/2}}, \quad (67)$$

so that according to eq.(58),

$$C(\alpha) \simeq \exp \left\{ \frac{\sqrt{\pi^3}}{6m} (\alpha - \alpha_c)^{-1/2} \right\}$$

$$= \exp \left\{ \pi \sqrt{\frac{4}{3}} \log m \frac{\alpha - \alpha_c}{\alpha_c}^{-1/2} \right\} \quad (69)$$

Therefore, the coefficient $c_1$ in eq.(64) behaves as $c_1 \simeq c_2 \log m$, with a factor $c_2$ of the order of unity. Analyzing the derivation of eq.(68), we find that the critical behavior in the form (69), (64) is valid for $\alpha - \alpha_c \ll \alpha_c$, where the fluctuations are exponentially strong. At $\alpha - \alpha_c \sim \alpha_c$, the quantity $C(\alpha)$ as given by eq.(64) is still large, $C(2\alpha_c) \simeq e^{\pi/2} = m^{c_2}$, in view of $m \gg 1$. However, $C(\alpha)$ is not exponentially large anymore and ceases to be the leading factor determining the critical behavior of most of the relevant quantities, such as the LDOS moments and the conductivity. For this reason, it turns out to be difficult to extend the above consideration onto the region $\alpha - \alpha_c \gtrsim \alpha_c$. We know, however, from Sec.IV that the region of relatively strong LDOS fluctuations extends up to $\alpha \sim 1/m^2 \sim \alpha_c \log m$.

VII. APPLICATION TO THE PROBLEM OF A QUASIPARTICLE LINE SHAPE IN QUANTUM DOTS

Let us now translate the obtainer results into the context of the problem of the one-particle excitations in a quantum dot. This can be straightforwardly done by using the relations (3), (4), (5), and (2) between the parameters of the two problems. The localization transition then corresponds to the energy $E_c \sim (g/\log g)^{3/2}$ (5). In the localized region (excitation energy $E$ below $E_c$) the bare states forming the basis of the Fock space mix only weakly to each other, so that the exact eigenstates are close to the bare ones. In particular, admixture of many-particle states to a single particle one is weak. Therefore, only one exact eigenstate will contribute essentially to the spectral decomposition (3) of a single-particle state, see fig 1a. In the delocalized domain ($E$ above $E_c$) exact eigenstates are superpositions of many bare ones. This is in full analogy with a delocalized state of the tight-binding model, which covers many (infinitely many in the thermodynamic limit) sites of the lattice. As a result, there are many exact eigenstates contributing to the strength function (4). The corresponding envelope is of irregular (strongly fluctuating) shape in the intermediate (critical) regime $E_c < E < E_c'$, with $E_c' \sim \Delta g^{1/2}$, and acquires a Breit-Wigner form at $E \gg E_c'$. The width of this Breit-Wigner envelope (spreading width of the one-particle state) is given by eq.(24) (Golden Rule), yielding $\Gamma \sim \Delta(E/\Delta g)^2$.

Finally, let us remind that what we considered throughout the paper was a tree-like model with a constant coordination number. In reality, however, the number of states to which a state of the $n$-th generation is coupled, decreases with increase of $n$. Let us briefly discuss how this is expected to modify the results. The transition will be now smeared into a crossover, since the transition point gets “generation-dependent” (see also recent preprint [33]). Furthermore, in the constant coordination number approximation all states on the energy shell (i.e. with energies within the spreading width $\sim \Gamma$ around $E$) get mixed at $E \gg E_c'$. In contrast, now only first few generation will get mixed under this condition (that will be however sufficient to produce the Breit-Wigner envelope of the spectral function (4)). Admixture of higher generations will require higher energies. To estimate, when the complete mixing of the states on the energy shell happens, we note that the density of states of the generation $n$ is equal to

$$\nu_{2n+1}(E) = \frac{1}{n!(n!)(2n!)(E/\Delta)^{2n}}, \quad (70)$$

which is a direct generalization of eq.(4). Maximizing this expression, we find that a typical many-particle state belongs to a generation with $n \sim (E/2\Delta)^{1/2}$, with typical energies of quasiparticles $E \sim (E/\Delta)^{1/2}$. The level spacing of the states to which this one is coupled is $\sim \Delta(\Delta/E)^{3/2}$. Comparing this to the typical value of the matrix element, $V \sim \Delta/g$, we conclude that the full mixing of the states on the energy shell (ergodicity) will be reached at $E > E_c'$, with $E_c' \sim \Delta g^{3/2}$. The same estimate for the “chaotization border” was obtained very recently by Jacquod and Shepelyansky [34].

VIII. SUMMARY

In this paper we have studied in detail the structure of the average and typical local density of states in a tight-binding model on a tree-like lattice with a large branching number $m \gg 1$. In the framework of the mapping
recently suggested in Ref. [3] this local density of states describes the shape of the quasiparticle excitation line in a finite Fermi system (e.g., quantum dot). We have exploited the supersymmetry approach to the problem developed previously, see Refs. [2, 10, 13, 30], and some of the results obtained in these papers.

The results depend on the relation between two dimensionless parameters: the branching number $m$ and the coupling constant $\alpha \sim (V/W)^2$ ($V$ being the typical magnitude of the hopping matrix element in Fock space and $W$ the width of the distribution of random site energies $\epsilon_i$). The relation of these parameters to those of the original quantum dot model can be found in Sec. [11]. When $\alpha \gg 1/m$ the LDOS in a given site $i$ averaged over the distribution of all random energies of other sites $\epsilon_j$, $j \neq i$ has a semicircular form. In the opposite case (relevant to the quantum dot model), $\alpha \ll 1/m$, the averaged LDOS has a Lorentzian form with a width given by the Golden Rule, see eq. [22]. However, the typical LDOS is close to its average value only for $\alpha \gtrsim 1/m^2$. In the opposite case, $\alpha \ll 1/m^2$, the LDOS fluctuations are strong, and the averaged LDOS is not representative, see Fig. 1. This is related to the existence of the Anderson localization transition at the point $\alpha = \alpha_c \sim 1/(m^2 \log^2 m)$.

On the insulating side of the transition, $\alpha < \alpha_c$, the LDOS is given by a discrete sum of $\delta$-function peaks. The effective number of such peaks is characterized by $1/P$, $P$ being the inverse participation ratio. It is known [3, 10, 30] that on the Bethe lattice the inverse participation ratio has a finite limiting value $P_c$ when the system approaches the transition point from the localized phase. We have shown that in the limit $m \gg 1$ this limiting value is close to unity, $1 - P_c \sim 1/\log m \ll 1$, so that almost all the spectral weight [4] (except a small part of the order of $1/\log m$) is concentrated in a single $\delta$-peak. Roughly speaking, the system undergoes a transition directly from the deeply localized phase to the extended one.

When the system is driven through the critical point into the phase of extended states the peaks get broadened, with their width depending on the distance to the critical point and vanishing (in the thermodynamic limit) at the critical point. The width is determined by the scale $C^{-1}(E)$ such that for energy separations smaller than $C^{-1}(E)$ different eigenfunctions are fully correlated (see sec. [V]). As a result, the width is exponentially small in the vicinity of the critical point, see eqs. [41, 43]. This is also confirmed by the behavior [11] of the inverse participation ratio in this region.

Our results by and large confirm the picture presented by AGKL [3]. We have, however, quantified many features of the problem by using the supersymmetry approach and some of the results obtained earlier in the framework of this method.

We disagree with AGKL concerning their statement that in the delocalized phase the level correlation function of the tree-like model with fixed coordination number is not of the Wigner-Dyson form close to the transition point because exact eigenstates are very sparse and “do not talk to each other” [3]. We have shown in Sec. IV that the eigenfunctions of the sparse random matrix model which are close in energy are strongly correlated in this regime. This explains the Wigner-Dyson form of the level correlation function proven in one of our earlier publications [4].

For the quantum dot problem the obtained results imply the Fock-space delocalization of a single-particle excitation at excitation energies $E > E_c \sim \Delta (g/\log g)^{1/2}$ and formation of regular-shaped Breit-Wigner envelopes at $E > E'_c \sim \Delta g^{1/2}$. Taking into account decrease of the coordination number in higher generations allows to estimate an energy, at which the complete ergodicity on the energy shell is restored as $E'' \sim \Delta g^{2/3}$.

Finally, we would like to mention that an extensive study of statistical properties of eigenstates of a finite system of interacting fermions was undertaken recently by Flambaum, Izrailev, Casati, and Gribakin [35, 36]. In particular, these authors discuss crossover from the regime of strongly fluctuating local spectral density to that characterized by small fluctuations, which is similar to the questions addressed in the present paper [37].

IX. ACKNOWLEDGMENTS.

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FIG. 1. Schematic view of the local density of states in various regimes depending on the relation between the coupling constant $\alpha$ and the connectivity $m$. The dotted line represents the average local spectral density, which has a Lorentzian form, see eq. (19). a) $\alpha$ approaches the transition point $\alpha_c$ from the the phase of localized states $\alpha < \alpha_c$. The spectral density is given by a sum of $\delta$-functions, with almost all the spectral weight (except a small fraction of order of $1/m$) concentrated in one $\delta$-peak; b) delocalized phase ($\alpha > \alpha_c$), critical region. The $\delta$-peaks get broadened, with a width being exponentially small, see eq. (64), for $\alpha - \alpha_c \ll \alpha_c$; c) the region of weak fluctuations of local spectral density, $\alpha \gg 1/m^2$. 

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