Multiple-criteria Heuristic Rating Estimation

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Abstract

One of the most widespread multi-criteria decision-making methods is the Analytic Hierarchy Process (AHP). AHP successfully combines the pairwise comparisons method and the hierarchical approach. It allows the decision-maker to set priorities for all ranked alternatives. But what if, for some of them, their ranking value is known (e.g., it can be determined differently)? The Heuristic Rating Estimation (HRE) method proposed in 2014 tried to bring the answer to this question. However, the considerations were limited to a model that did not consider many criteria. In this work, we go a step further and analyze how HRE can be used as part of the AHP hierarchical framework. The theoretical considerations are accompanied by illustrative examples showing HRE as a multiple-criteria decision-making method.

Keywords: pairwise comparisons, analytic hierarchy process, heuristic ranking estimation, MCDM, AHP, HRE

1. Introduction

The pairwise comparisons (PC) method has a centuries-old history. The first considerations on its use come from a thirteenth-century treatise by Ramon Lull, Ars Electionis \cite{8, 14}. Later, it was the subject of research by Condorcet, Fechner, Thurston and others \cite{9, 13, 39, 32}. Currently, pairwise comparisons are a valuable source of preferential information in many decision-making methods, including TOPSIS \cite{41}, PROMETHEE \cite{5}, MACBETH \cite{2}, BWM \cite{33} or multiple-criteria sorting \cite{18}. One of the best known methods of multiple-criteria decision making based on pairwise comparisons is the Analytic Hierarchy Process (AHP) \cite{37, 26}. The method defined by Saaty in 1977 \cite{35} quickly gained popularity. It became widely used in many areas of application, including performance assessment of employees \cite{30}, country competitiveness \cite{22}, supplier...
segmentation [34], portfolio management [7], or Sustainability [38]. More application examples can be found in the reviews [21, 15, 11]. It also quickly gained opponents who very often rightly pointed out the imperfections of the method [1, 4, 12]. One of the problems is the representation of measurable entities, such as distance, weight or area [40]. In particular, if we can measure a given object and have an exact value, there is no point in relying on experts’ opinions on this entity. However, even if we do not want to ask an expert to compare some alternatives, it may turn out that the priorities after calculating the ranking deviate from the ratios resulting from the measurements. The Heuristic Rating Estimation (HRE) method is an attempt to deal with this problem. The HRE was initially defined as two alternative methods (additive and multiplicative) for calculating the ranking [23, 27]. In this article, we extend this model to include a hierarchy that allows multiple criteria for evaluating alternatives to be considered. The HRE calculation procedure is based on the same concept as EVM (Eigenvalue Method) [35] and GMM (Geometric Mean Method) [10]. That is, the priority of a given alternative is equal to the weighted average value of the priorities of the other alternatives [28]. This compatibility allows the original HRE idea to be easily combined with these methods within one hierarchical model (Example 4). Thus, naturally, HRE becomes a multi-criteria method that complements AHP in cases where some alternatives have measurable or previously known ratings.

The presented work is a follow-up to [23, 27]. The preliminary section introduces the reader to the issues related to pairwise comparisons and priority deriving methods, such as EVM and GMM. The HRE calculation procedure is described in (Sec. 3). The next section (Sec. 4) fits HRE into a multi-criteria hierarchical model used in the context of AHP. The theoretical description is supplemented by practical examples showing the applications of HRE (Ex. 3) and its integration with AHP (Ex. 4).

2. Preliminaries

2.1. Pairwise Comparisons

Let $A = \{a_1, \ldots, a_n\}$ be a set of alternatives and $C = [c_{ij}]$ be a pairwise comparisons (PC) matrix, such that $c_{ij} \in \mathbb{R}_+$ for $i, j \in \{1, \ldots, n\}$. A single element $c_{ij}$ corresponds to a direct comparison of the $i$-th and $j$-th alternatives. A single entry $c_{ij}$ has a quantitative meaning. For example, when an expert decides that the $a_i$ alternative is twice more preferred than $a_j$, then $c_{ij}$ takes the value 2. As a result, the elements of the diagonal are ones, i.e. $C$ takes the form:

$$
C = \begin{bmatrix}
1 & c_{12} & \cdots & c_{1n} \\
c_{21} & 1 & \cdots & c_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
c_{n1} & c_{n2} & \cdots & 1
\end{bmatrix}.
$$
2.1 Pairwise Comparisons

Based on the information stored in \( C \), a priority vector is calculated. For the purpose of the article, we will denote it as \([w(a_1), \ldots, w(a_n)]^T\) where \( w \) is the function \( w : A \rightarrow \mathbb{R}_+ \). We say that \( a_i \) is more preferred than \( a_j \) (denoted as \( a_i > a_j \)) if \( w(a_i) > w(a_j) \). Similarly, \( a_i \) is considered as equally preferred as \( a_j \) (denoted as \( a_i \sim a_j \)) when \( w(a_i) = w(a_j) \).

In the ideal case \( c_{ij} = w(a_i) / w(a_j) \) i.e. the results of comparisons provided by experts perfectly match the computed ranking. In practice, however, we may expect only that \( c_{ij} \approx w(a_i) / w(a_j) \). The reason for this inequality is data inconsistency, which is most often the result of expert errors.

**Definition 1.** A PC matrix \( C \) is called consistent if

\[
c_{ik} = c_{ij}c_{jk}, \quad \text{for every } i, j, k = 1, \ldots, n
\]  

(1)

Every PC matrix is said to be inconsistent if it is not a consistent PC matrix.

**Definition 2.** A PC matrix \( C \) is called reciprocal if

\[
c_{ij} = \frac{1}{c_{ji}} \quad \text{for every } i, j = 1, \ldots, n
\]  

(2)

Although the reciprocity condition is natural, the use of non-reciprocal matrices can also be found in the literature [16, 24, 28]. It is easily seen that every consistent PC matrix is also reciprocal, but not reversely.

One of the essential issues in the PC methods concerns the question of how the priority vector needs to be determined. The answer is not straightforward because the expert’s judgments may not be consistent. There are many methods of priority vector estimation in the literature [17]. Two of the most popular prioritization methods are the Eigenvector Method (EVM) [35] and the Geometric Mean Method (GMM) [10].

In EVM, the priorities of alternatives are determined by the real and positive solution of \( Cw = \lambda_{\text{max}}w_{\text{max}} \), where \( \lambda_{\text{max}} \) is the principal eigenvalue (spectral radius) of \( C \), and \( w_{\text{max}} \) is an appropriate eigenvector. The vector of priorities \( w \) is assumed to be \( w_{\text{max}} \) rescaled so that the sum of its components is 1.

In GMM, the priority of an individual \( a_i \) is calculated as the geometric mean of the \( i \)-th row of \( C \). The sum of components also rescales the created ranking vector. Both EVM and GMM have been repeatedly debated and analyzed, and both methods have their proponents and opponents [3, 36].

Depending on how inconsistent the PC matrix is, calculating the priority vector reflects the real preferences of respondents. To determine the degree to which PC matrices are inconsistent, appropriate indices are used. The most popular one comes from Saaty [35] and is defined as:

\[
CI = \frac{\lambda_{\text{max}} - n}{n - 1},
\]  

(3)

\(^{1}\text{It can be shown that, for a consistent matrix, each ranking method will lead to the same priority vector [26, p. 96].}\)
where \( \lambda_{\text{max}} \) denotes the principal eigenvalue of the \( n \times n \) matrix \( C \). Another interesting inconsistency indicator comes from Koczkodaj \[20\]. For the \( n \times n \) PC matrix \( C \), it is defined as:

\[
K(C)_{df} = \max_{i,j,k \in \{1, \ldots, n\}} \left\{ \min \left\{ \left| 1 - \frac{c_{ij}}{c_{ik}c_{kj}} \right|, \left| 1 - \frac{c_{ik}c_{kj}}{c_{ij}} \right| \right\} \right\}.
\] (4)

In general, the more inconsistent the matrix \( C \), the larger the inconsistency indices. When the value of the inconsistency index is considerable, the ranking results are not credible. Thus, in most cases, it is essential to keep the inconsistency as small as possible. More about different inconsistency indices can be found in \[6, 29, 19\].

### 2.2. Analytic Hierarchy Process

A single PC matrix allows us to compare alternatives with respect to only one criterion at a time. In practice, however, there is often a need to take more than one criterion into account. For example, when buying a car, we do not just choose a "better" vehicle, but also pay attention to its price, terms of warranty, and so on. A framework that allows the use of pairwise comparisons as a multi-criteria method is provided by the Analytic Hierarchy Process (AHP). The method was proposed by Saaty in 1977 \[35\] and it was one of the first to use PC matrices allowing for a large number of criteria. AHP is able to handle multiple criteria by defining a hierarchy (a tree) of alternatives and criteria. Hence, at the bottom of the hierarchical tree defining the decision model, alternatives are compared with respect to the lowest level criteria, then those criteria are compared against each other with respect to the higher level of criteria, and so on. Although the AHP hierarchy can be freely expanded, the most popular models consist of three layers: alternatives \( A = \{a_1, \ldots, a_n\} \), criteria \( S = \{s_1, \ldots, s_m\} \) and the root node of the hierarchy called “goal” or “purpose”.

As an example of a hierarchical AHP decision model, let us consider the problem of selecting a company manager. Suppose there are three candidates for this position: Andrew, Benjamin and Christopher. When considering these alternatives, we take into account their experience in managerial positions, education and interpersonal skills, as well as the ability to work under pressure. We compare the candidates in terms of these criteria. The one with the best results in our comparison will become the CEO of the company.

Going into details, according to the AHP method, we create a separate PC matrix for each criterion. Thus, \( C(s) \) is a PC matrix corresponding to the mutual comparisons of all three candidates with regard to the criterion \( s \). There are four criteria for assessing candidates: experience \((ex)\), education \((ed)\), interpersonal skills \((is)\), and stress resistance \((sr)\). Hence, for example, \( C^{(ex)} \) corresponds to

\[\text{A similar method was proposed J. R. Miller in 1966} \ [31].\]
the comparisons of our candidates in terms of their experience. We assume that
the following four matrices were prepared as a result of the expert assessment.

\[
C^{(\text{ex})} = \begin{bmatrix}
1 & 2 & 4 \\
\frac{1}{2} & 1 & \frac{1}{2} \\
\frac{1}{4} & \frac{1}{2} & 1
\end{bmatrix},
C^{(\text{ed})} = \begin{bmatrix}
1 & \frac{1}{2} & \frac{1}{3} \\
2 & 1 & \frac{1}{3} \\
8 & 4 & 1
\end{bmatrix},
\]

\[
C^{(\text{is})} = \begin{bmatrix}
1 & \frac{1}{2} & 3 \\
2 & 1 & 6 \\
\frac{1}{3} & \frac{1}{6} & 1
\end{bmatrix},
C^{(\text{sr})} = \begin{bmatrix}
1 & \frac{1}{2} & 1 \\
2 & 1 & 2 \\
1 & \frac{1}{2} & 1
\end{bmatrix}.
\]

For each considered matrix \(C^{(s)}\) we calculated the priority vector. These are:

\[
w^{(\text{ex})} = \begin{bmatrix}
4 \\
7 \\
7
\end{bmatrix}^T,
\quad w^{(\text{ed})} = \begin{bmatrix}
1 \\
11 \\
11
\end{bmatrix}^T,
\]

\[
w^{(\text{is})} = \begin{bmatrix}
3 \\
6 \\
\frac{1}{10}
\end{bmatrix}^T,
\quad w^{(\text{sr})} = \begin{bmatrix}
1 \\
2 \\
4
\end{bmatrix}^T.
\]

When calculating the final priority vector, we must take into account the fact that the criteria may contribute to a different extent to achieving the goal. For this reason, we compare them in pairs. These comparisons form one more 4 \times 4 PC matrix \(\hat{C} = [\hat{c}_{ij}]\) where \(\hat{c}_{ij}\) denotes the result of individual comparisons between the criteria \(s_i\) and \(s_j\), where \(s_1, \ldots, s_4\) mean (ex), (ed), (is), and (sr), correspondingly.

\[
\hat{C} = \begin{bmatrix}
1 & 4 & 2 & 8 \\
\frac{1}{2} & 1 & \frac{1}{2} & 2 \\
\frac{1}{4} & \frac{1}{2} & 1 & 4 \\
\frac{1}{8} & \frac{1}{4} & \frac{1}{16} & 1
\end{bmatrix}.
\]

The priority vector \(\hat{w}\) for \(\hat{C}\) is

\[
\hat{w} = \begin{bmatrix}
8 \\
15 \\
15 \\
15
\end{bmatrix}^T.
\]

Thus, according to the expert opinion, the first criterion: experience, has the priority 8/15, education 2/15, interpersonal skills 4/15, and stress resistance – 1/15. The priorities of criteria determine the degree to which the comparisons of alternatives affect the overall score. Thus, they become weights scaling the results of direct comparisons between alternatives (the final priority vector is a linear combination of vectors \(w^{(s)}\) where the scaling factors come from \(\hat{w}\)). The overall ranking \(w\) is given as:

\[
w = \sum_{j=1}^{m} \hat{w}_j w^{(s_j)}.
\]

\footnote{We use EVM \cite{EVM}.}
In our case, we have:

\[ w = \hat{w}_1 w^{(ex)} + \hat{w}_2 w^{(ed)} + \hat{w}_3 w^{(ch)} + \hat{w}_4 w^{(ag)} = \]

\[ = \frac{8}{15} \begin{bmatrix} 4 & 2 & 1 \\ 7 & 7 & 7 \end{bmatrix}^T + \frac{2}{15} \begin{bmatrix} 1 & 2 & 8 \\ 11 & 11 & 11 \end{bmatrix}^T + \frac{4}{15} \begin{bmatrix} 3 & 6 & 1 \\ 10 & 10 & 10 \end{bmatrix}^T + \frac{1}{15} \begin{bmatrix} 1 & 2 & 1 \\ 4 & 4 & 4 \end{bmatrix}^T \approx \]

\[ \approx [0.414, 0.370, 0.216]^T. \]

The above priority vector translates to the observation that Andrew is slightly more preferred to be a company manager than Benjamin, and both are more preferred than Christopher.

3. Heuristic Rating Estimation

When comparing alternatives, sometimes, we already know priorities for some of them. In such a case, we can use them and not ask experts for unnecessary comparisons, saving time and money. This observation gave rise to the Heuristic Rating Estimation (HRE) method [23, 27]. In HRE, the set of alternatives \( A \) is composed of two disjoint subsets: \( A_K \) - alternatives for which the final priorities are known, and \( A_U \) - alternatives for which priority weights need to be estimated (also referred as to unknown alternatives). Knowledge about the priority values of elements in \( A_K \) combined together with pairwise comparisons of all the elements from \( A = A_K \cup A_U \) allow us to calculate priorities for \( A_U \). Hence, \( A_K \) can be considered as the set of references that provide a benchmark for new alternatives in \( A_U \). For the sake of simplicity, in this study we will denote \( A_U = \{a_1, ..., a_k\} \) and \( A_K = \{a_{k+1}, ..., a_n\} \). If \( a_i, a_j \in A_K \), then we will assume that \( w(a_i) \) and \( w(a_j) \) are known. Therefore the value \( c_{ij} \) is also known and equals \( c_{ij} = w(a_i)/w(a_j) \), so querying experts about \( c_{ij} \) is also not necessary. The comparisons matrix \( C \) in HRE takes the form:

\[
C = \begin{bmatrix}
1 & \cdots & c_{1,k} & c_{1,k+1} & \cdots & c_{1,n} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
c_{k,1} & \cdots & 1 & c_{k,k+1} & \cdots & c_{k,n} \\
c_{k+1,1} & \cdots & c_{k+1,k} & 1 & \cdots & w(a_{k+1})/w(a_n) \\
\vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
c_{n,1} & \cdots & c_{n,k} & w(a_n)/w(a_{k+1}) & \cdots & 1
\end{bmatrix}.
\]

The above matrix and vector of reference values \([w(a_{k+1}), ..., w(a_n)]^T\) allow us to calculate the complete ranking for \( A \) (Fig. [1]).

The HRE method has been designed as an extension of AHP where part of the ranking is already known. Therefore, it is based on the same premise that

\[
w(a_i) \approx c_{ij} w(a_j). \quad (5)
\]
Thus, as in AHP, we can request that the priority of the i-th alternative $w(a_i)$ is a weighted average of priorities of all the other alternatives \[28\], i.e.

$$w(a_i) = \frac{1}{n-1} \sum_{j=1, j \neq i}^{n} c_{ij} w(a_j), \text{ for } i = 1, \ldots, k. \quad (6)$$

This leads to a matrix equation

$$Mw = b, \quad (7)$$

where $M$ is given as

$$M = \begin{bmatrix}
1 & -\frac{1}{n-1}c_{1,2} & \cdots & -\frac{1}{n-1}c_{1,k} \\
-\frac{1}{n-1}c_{2,1} & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
-\frac{1}{n-1}c_{k,1} & -\frac{1}{n-1}c_{k,2} & \cdots & 1
\end{bmatrix},$$

the constant term vector,

$$b = \begin{bmatrix}
\frac{1}{n-1}c_{1,k+1}w(a_{k+1}) + \ldots + \frac{1}{n-1}c_{1,n}w(a_n) \\
\frac{1}{n-1}c_{2,k+1}w(a_{k+1}) + \ldots + \frac{1}{n-1}c_{2,n}w(a_n) \\
\vdots \\
\frac{1}{n-1}c_{k,k+1}w(a_{k+1}) + \ldots + \frac{1}{n-1}c_{k,n}w(a_n)
\end{bmatrix}$$

and the ranking vector for unknown alternatives is given as:

$$w = [w(a_1), \ldots, w(a_k)]^T.$$

The solution of $(7)$ - if it exists and is acceptable, forms the desired vector of priorities. It has been shown \[25\] that for a small value of $\kappa(C)$ and sufficiently large set of the reference alternatives $A_K$ the feasible solution of $(7)$ always exists.
In HRE, as in AHP, many methods for calculating the ranking can be defined. The GMM equivalent in AHP is geometric HRE [27]. Thus, starting from (5) we may request
\[ w(a_i) = \left( \prod_{j=1, j \neq i}^{n} c_{ij} w(a_j) \right)^{\frac{1}{n-1}} \text{ for } i = 1, \ldots, n. \] (8)

After raising both sides of (8) to the power \( n-1 \), we obtain the non-linear equation system:
\[
\begin{align*}
    w^{n-1}(a_1) &= c_{1,2} w(a_2) \cdot c_{1,3} w(a_3) \cdot \ldots \cdot c_{1,n} w(a_n) \\
    w^{n-1}(a_2) &= c_{2,1} w(a_1) \cdot c_{2,3} w(a_3) \cdot \ldots \cdot c_{2,n} w(a_n) \\
    &\vdots \\
    w^{n-1}(a_k) &= c_{k,1} w(a_1) \cdot c_{k,2} w(a_2) \cdot \ldots \cdot c_{k,n-1} w(a_{n-1}) \\
\end{align*}
\] (9)

Thanks to the logarithmic transformation, the above is equivalent to the linear equation system
\[ N\mu = d, \] (10)

where \( N \) is a \( k \times k \) auxiliary matrix in the form
\[
N = \begin{bmatrix}
(n-1) & -1 & \cdots & -1 \\
-1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
-1 & -1 & \cdots & (n-1)
\end{bmatrix},
\]
d is a constant term vector
\[
d = \begin{bmatrix}
\lg (c_{1,2} c_{1,3} \cdot \ldots \cdot c_{1,k} c_{1,k+1} w(a_{k+1}) \cdot \ldots \cdot c_{1,n} w(a_n)) \\
\lg (c_{2,1} c_{2,3} \cdot \ldots \cdot c_{2,k} c_{2,k+1} w(a_{k+1}) \cdot \ldots \cdot c_{2,n} w(a_n)) \\
\vdots \\
\lg (c_{k,1} c_{k,2} \cdot \ldots \cdot c_{k,k-1} c_{k,k+1} w(a_{k+1}) \cdot \ldots \cdot c_{k,n} w(a_n))
\end{bmatrix},
\]

and
\[
\mu = \begin{bmatrix}
\mu_1 \\
\vdots \\
\mu_k
\end{bmatrix} = \begin{bmatrix}
\lg w(a_1) \\
\vdots \\
\lg w(a_k)
\end{bmatrix}.
\]

The solution \( \mu \) of (10) induces the weight vector \( w \) – a solution of the original non-linear problem (9)
\[ w = [\exp \mu_1, \ldots, \exp \mu_k]^T. \]

The feasible solution of (10) always exists and is optimal [27].
4. Multiple-criteria Heuristic Rating Estimation Method

Sometimes when we want to compare a few objects or select several items from many others, we follow different criteria with varying degrees of importance. Moreover, we may have preliminary information about specific things. For example, when choosing a school for a child, parents pay attention to the level of education and the percentage of pupils entering the next school, the educational system, location, social group from which the pupils come, or additional activities offered by the school. Specific information about some of the considered schools can be known, although the set of known criteria may differ in each school. As another example, consider a company planning to expand its business by offering some new goods for sale in addition to existing ones. Before introducing new products or services, it is necessary to conduct market research and then select the most profitable products/services. The company has data on the currently offered goods (i.e., sales volume, profitability, and popularity), only the data on the goods considered for introduction to the market are unknown. During pairwise comparisons, the different criteria could be taken into account.

The cases described above have two distinctive features: some of the alternatives considered have initially known priorities, and all the alternatives are compared with respect to more than one criterion. This observation leads us to the idea of the hierarchical HRE method. Such an approach would reduce the multi-criteria problem to a series of simple pairwise comparisons and calculate the weights of concepts related to individual criteria. Finally, the priorities of the alternatives are determined. Moreover, based on previously known estimations, we would be able to estimate actual values of different decision options. Hence, the meaning of the priority is not only relative but real, expressing the current value or unit. In general, using a hierarchy is nothing new. One of the first hierarchical methods - AHP - was introduced by Saaty and is still one of the most popular decision-making methods. The structure of the hierarchical HRE is shown in Fig. 2.

The primary data for the hierarchical HRE method are two sets: a set of criteria $Q = \{q_1, ..., q_s\}$ and a set of alternatives $A = \{a_1, ..., a_n\}$. For each criterion $q_t$ ($t \in \{1, 2, \ldots, s\}$) we construct a PC matrix $C^{(q_t)} = [c^{(q_t)}_{ij}]$ containing comparisons of elements from $A$. For this purpose, we establish a set of reference alternatives (with known priorities) $A^{(q_t)}_K$, and a set of unknown alternatives (i.e. with unknown priorities) $A^{(q_t)}_U$. Obviously, those sets for each $t \in \{1, 2, \ldots, s\}$, must satisfy the following conditions $A = A^{(q_t)}_K \cup A^{(q_t)}_U$, $A^{(q_t)}_K \cap A^{(q_t)}_U = \emptyset$. It is worth noting that the set of initially known alternatives and set of unknown alternatives may differ among criteria.

We construct the PC matrix $C^{(q_t)}$ based on pairwise comparisons of alternatives, where at least one is unknown. Then we apply the HRE prioritization.
Figure 2: The structure of hierarchical HRE

method for the matrix $C^{(q_t)}$. More specifically, we construct the following linear equation:

$$M^{(q_t)}w^{(q_t)} = b^{(q_t)}, \quad (11)$$

where $w^{(q_t)}$ is a priority vector of alternatives $a_i \in A^{(q_t)}_U$, $M^{(q_t)}$ is a $k_t \times k_t$ matrix, where $k_t$ means the number of known alternatives ($k_t = |A^{(q_t)}_K|$) in the form

$$M^{(q_t)} = \begin{bmatrix}
1 & -\frac{1}{n-1}c^{(t)}_{1,2} & \cdots & -\frac{1}{n-1}c^{(t)}_{1,k_t} \\
-\frac{1}{n-1}c^{(t)}_{2,1} & \ddots & \ddots & \\
\vdots & \ddots & \ddots & \\
-\frac{1}{n-1}c^{(t)}_{k_t,1} & -\frac{1}{n-1}c^{(t)}_{k_t,2} & \cdots & 1
\end{bmatrix}$$

and the constant term vector is given as

$$b^{(q_t)} = \begin{bmatrix}
\frac{1}{n-1}c^{(t)}_{1,k_t+1}w^{(q_t)}_K(a^{(t)}_{k_t+1}) + \cdots + \frac{1}{n-1}c^{(t)}_{1,n}w^{(q_t)}_K(a^{(t)}_n) \\
\frac{1}{n-1}c^{(t)}_{2,k_t+1}w^{(q_t)}_K(a^{(t)}_{k_t+1}) + \cdots + \frac{1}{n-1}c^{(t)}_{2,n}w^{(q_t)}_K(a^{(t)}_n) \\
\vdots \\
\frac{1}{n-1}c^{(t)}_{k_t,k_t+1}w^{(q_t)}_K(a^{(t)}_{k_t+1}) + \cdots + \frac{1}{n-1}c^{(t)}_{k_t,n}w^{(q_t)}_K(a^{(t)}_n)
\end{bmatrix}, \quad (12)$$

where $w^{(q_t)}_K$ is a priority vector for alternatives from $A^{(q_t)}_K$. We calculate the priority vector $w^{(q_t)}$ by solving the equation (11).

In the geometric version of the HRE approach [27] for each criterion $q_t$ and the appropriate PC matrix $C^{(q_t)}$ we construct the following linear equation

$$N^{(q_t)}\mu^{(q_t)} = d^{(q_t)}, \quad (13)$$
the HRE equations (7) used must be verified. Note that if an additive approach is used, the existence of a solution to each of
factors come from
The final vector of weights is a linear combination of vectors
\( \hat{w} \) extents to the goal. Thus we construct a PC matrix
is a matrix of dimensions \( k_t \times k_t \), \( k_t = |A^{(q_t)}_K| \) and

\[
N^{(q_t)} = \begin{bmatrix}
(n - 1) & -1 & \cdots & -1 \\
-1 & \ddots & \cdots & -1 \\
\vdots & \ddots & \ddots & \vdots \\
-1 & -1 & \cdots & (n - 1)
\end{bmatrix}
\]

and \( \mu^{(q_t)} \) is a logarithmized priority vector of alternatives \( a_i \in A^{(q_t)}_U \). We compute the vector \( \mu^{(q_t)} \) by solving (13) and then we calculate the vector \( w_U^{(q_t)} = \left( w_U^{(q_t)}(a_1^{(t)}), \ldots, w_U^{(q_t)}(a_{k_t}^{(t)}) \right) \) using the dependency

\[
\mu^{(q_t)} = \left( \log w_U^{(q_t)}(a_1^{(t)}), \ldots, \log w_U^{(q_t)}(a_{k_t}^{(t)}) \right) \quad \text{for every } a_i \in A^{(q_t)}_U.
\]

Based on the priorities given in \( w_U^{(q_t)} \) we create an \( n \) dimensional vector \( \hat{w}^{(q_t)} \) being a catenation of a priori known and just calculated priorities:

\[
\hat{w}^{(q_t)} = \begin{bmatrix}
w_U^{(q_t)} & w_K^{(q_t)}
\end{bmatrix}^T.
\]

Similarly as in AHP, when calculating the final priority vector, we must take into account the fact that the criteria \( q_1, \ldots, q_s \) may contribute at different extents to the goal. Thus we construct a PC matrix \( \tilde{C} \), in which every single entry \( \tilde{C}_{kl} \) corresponds to the result of comparison \( q_k \) against \( q_l \). Then (using the HRE approach) we calculate the priority vector

\[
\tilde{w} = [\tilde{w}(q_1), \ldots, \tilde{w}(q_s)]^T.
\]

The final vector of weights is a linear combination of vectors \( w^{(q_t)} \) where scaling factors come from \( \tilde{w} \), i.e.

\[
w = \sum_{t=1}^{s} \tilde{w}(q_t) w^{(q_t)}.
\]

Note that if an additive approach is used, the existence of a solution to each of the HRE equations (7) used must be verified.
5. Numerical examples

Example 3. A company, managed by Mr. Smith, runs a sports facility consisting of a sports swimming pool, gym and fitness club. The company is developing dynamically. Therefore Mr. Smith intends to expand the facility with two further investments. He considers a bowling alley, a professional massage parlor, a trampoline fitness point, or a recreational pool. He would spend a similar amount of money on each project. However, to increase profitability, Mr. Smith intends to conduct a market survey first. Market research will be carried out based on several criteria: $Q = \{pr\} - average monthly income (profitability), du - durability of the equipment (how many months it will serve), and pop - possible increase in the popularity of a given form of spending free time (also as a result of advertising campaigns). A group of specialists will apply the HRE hierarchical approach to the analysis of the obtained data.

Let $A = \{a_1 - a\ bowling\ alley, a_2 - a\ professional\ massage\ salon, a_3 - a\ trampoline\ fitness\ point, a_4 - a\ recreational\ pool, a_5 - a\ sports\ pool, a_6 - gym, a_7 - fitness\ club\}$. The set of reference alternatives is the same for each criterion $A_K = \{a_5, a_6, a_7\}$.

The first considered criterion is profitability ($pr$). For objects from the reference set, the values $w^{(pr)}(a_5) = 20$ thous., $w^{(pr)}(a_6) = 12$ thous., $w^{(pr)}(a_7) = 9$ thous. represent the average monthly income from the previous year. Considering profitability, experts create a PC matrix $C^{(pr)}$. Each $c_{ij}$ in the matrix $C^{(pr)}$ corresponds to the relative profitability of $a_i$ with respect to $a_j$. We calculate the values $c_{ij}$ for $i, j \in \{5, 6, 7\}, i \neq j$ as $\frac{w(a_i)}{w(a_j)}$, therefore

$$C^{(pr)} = \begin{bmatrix}
1 & \frac{2}{3} & \frac{2}{3} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{3}{7} & 1 & 2 & 2 & 2 & 2 & 2 \\
\frac{3}{7} & \frac{1}{2} & 1 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\
\frac{2}{7} & \frac{2}{2} & 3 & 1 & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} \\
2 & 2 & 4 & 2 & 1 & 20 & 20 & 20 \\
1 & 1 & 1 & \frac{2}{3} & \frac{2}{3} & 1 & \frac{2}{3} & \frac{2}{3} \\
\frac{2}{3} & \frac{1}{2} & \frac{3}{2} & 1 & \frac{9}{20} & \frac{9}{20} & \frac{9}{20} & 1
\end{bmatrix}.$$  

In order to estimate the vector $w^{(pr)}$, we use the HRE method, so according to (??) we construct the matrix $M^{(pr)}$ and vector $b^{(pr)}$

$$M^{(lan)} = \begin{bmatrix}
1 & \frac{1}{n-1}c_{1,2} & \frac{1}{n-1}c_{1,3} & \frac{1}{n-1}c_{1,4} \\
\frac{1}{n-1}c_{2,1} & 1 & \frac{1}{n-1}c_{2,3} & \frac{1}{n-1}c_{2,4} \\
\frac{1}{n-1}c_{3,1} & \frac{1}{n-1}c_{3,2} & 1 & \frac{1}{n-1}c_{3,4} \\
\frac{1}{n-1}c_{4,1} & \frac{1}{n-1}c_{4,2} & \frac{1}{n-1}c_{4,3} & 1
\end{bmatrix}$$

$$b^{(pr)} = \begin{bmatrix}
\frac{1}{n-1}c_{1,5}w^{(pr)}(a_5) + \frac{1}{n-1}c_{1,6}w^{(pr)}(a_6) + \frac{1}{n-1}c_{1,7}w^{(pr)}(a_7) \\
\frac{1}{n-1}c_{2,5}w^{(pr)}(a_5) + \frac{1}{n-1}c_{2,6}w^{(pr)}(a_6) + \frac{1}{n-1}c_{2,7}w^{(pr)}(a_7) \\
\frac{1}{n-1}c_{3,5}w^{(pr)}(a_5) + \frac{1}{n-1}c_{3,6}w^{(pr)}(a_6) + \frac{1}{n-1}c_{3,7}w^{(pr)}(a_7) \\
\frac{1}{n-1}c_{4,5}w^{(pr)}(a_5) + \frac{1}{n-1}c_{4,6}w^{(pr)}(a_6) + \frac{1}{n-1}c_{4,7}w^{(pr)}(a_7)
\end{bmatrix}.$$
After solving the equation, we get the following weights \( w^{(pr)}(a_1) = 11.164, w^{(pr)}(a_2) = 13.667, w^{(pr)}(a_3) = 6.863, w^{(pr)}(a_4) = 17.292 \), the values of which represent the estimated average monthly income for the objects \( a_1, a_2, a_3, a_4 \).

After normalization, the resulting weights are:

\[
\begin{bmatrix}
0.124 & 0.152 & 0.076 & 0.192 & 0.222 & 0.133 & 0.1
\end{bmatrix}^T.
\]

According to the second considered criterion \( (du) \), we compare objects in terms of the durability of the equipment. The values \( c_{ij} \) in the matrix \( C^{(du)} \) indicate how many times the durability of the equipment of the object \( a_i \) is better than the durability of the equipment of the object \( a_j \). For reference alternatives, the values \( w^{(du)}(a_5) = 72, w^{(du)}(a_6) = 24, w^{(du)}(a_7) = 36 \) represent the average number of months the appliance has been used in the last few years.

The PC matrix \( C^{(du)} \) prepared by the experts has the form

\[
\begin{bmatrix}
1 & 3 & 2 & 1 & \frac{1}{7} & 2 & \frac{3}{7} \\
\frac{1}{7} & 1 & 3 & 1 & \frac{1}{7} & 2 & \frac{3}{7} \\
\frac{1}{7} & \frac{2}{3} & 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{2} & \frac{1}{2} \\
1 & 3 & 3 & 1 & \frac{3}{7} & \frac{2}{7} & 1 \\
2 & 4 & 4 & \frac{4}{7} & 1 & \frac{72}{21} & \frac{72}{21} \\
\frac{1}{7} & 2 & \frac{2}{3} & \frac{1}{2} & \frac{36}{36} & \frac{36}{36} & 1 \\
\end{bmatrix}.
\]

Using HRE for the matrix \( C^{(du)} \), similarly to the first criterion (see (11) and (12)), we get the matrix \( M^{(du)} \) and vector \( b^{(du)} \):

\[
M^{(du)} = \begin{bmatrix}
1 & -0.5 & -0.333 & -0.167 \\
-0.056 & 1 & -0.25 & -0.056 \\
-0.083 & -0.111 & 1 & -0.056 \\
-0.167 & -0.5 & -0.5 & 1
\end{bmatrix},
\]

\[
b^{(du)} = \begin{bmatrix}
23 \\
8 \\
8.667 \\
29.6
\end{bmatrix}.
\]

Solving the equation \( M^{(du)}w^{(du)} = b^{(du)} \), gives values that indicate the estimated average durability in months

\[
w^{(du)}(a_1) = 47.183, w^{(du)}(a_2) = 18.119, w^{(du)}(a_3) = 17.688, w^{(du)}(a_4) = 55.367,
\]
Scaling the solution vector gives:

\[ w^{(du)} = \begin{bmatrix} 0.174 & 0.067 & 0.065 & 0.205 & 0.266 & 0.089 & 0.133 \end{bmatrix}^T. \]

Analysis of the third criterion – the possibility of increasing the popularity of activities offered by individual facilities (\( \text{pop} \)) – the values \( c_{ij} \) in the matrix \( C^{(\text{pop})} \) mean how many times the popularity of the object \( a_i \) can increase in relation to the increase in the popularity of the object \( a_j \). The reference alternatives have the values \( w^{(\text{pop})}(a_5) = 5 \), \( w^{(\text{pop})}(a_6) = 20 \), \( w^{(\text{pop})}(a_7) = 25 \), and represent the average percentage increase in popularity in recent years. In this case, the PC - matrix \( C^{(\text{pop})} \) proposed by the experts is

\[
C^{(\text{pop})} = \begin{bmatrix}
1 & 1 & \frac{1}{5} & 3 & 3 & 2 & 2 \\
1 & 1 & \frac{1}{5} & 3 & 3 & 2 & 2 \\
5 & 3 & 1 & 4 & 5 & 3 & 3 \\
\frac{1}{5} & \frac{1}{3} & \frac{1}{5} & 1 & \frac{2}{5} & \frac{1}{5} & \frac{1}{5} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{5} & 2 & 1 & \frac{20}{25} & \frac{20}{25} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{3} & 2 & \frac{25}{25} & \frac{25}{25} & 1
\end{bmatrix}.
\]

The auxiliary matrix \( M^{(\text{pop})} \) and the vector \( b^{(\text{pop})} \) derived from the matrix \( C^{(\text{pop})} \) are

\[
M^{(\text{pop})} = \begin{bmatrix}
1 & -0.167 & -0.033 & -0.5 \\
-0.167 & 1 & -0.056 & -0.5 \\
-0.833 & -0.5 & 1 & -0.667 \\
-0.056 & -0.056 & -0.042 & 1
\end{bmatrix},
\]

\[
b^{(\text{pop})} = \begin{bmatrix}
17.5 \\
17.5 \\
26.667 \\
5
\end{bmatrix}.
\]

Using the HRE method to solve the equation \( M^{(\text{pop})} w^{(\text{pop})} = b^{(\text{pop})} \), we obtain the following values:

\[ w^{(\text{pop})}(a_1) = 31.459, \ w^{(\text{pop})}(a_2) = 32.93, \ w^{(\text{pop})}(a_3) = 77.21, \ w^{(\text{pop})}(a_4) = 11.794, \]

These numbers represent the estimated average growth in popularity of objects \( a_1, a_2, a_3, a_4 \). After normalization, the solution vector of the weights is equal:

\[ w^{(du)} = \begin{bmatrix} 0.155 & 0.162 & 0.38 & 0.058 & 0.025 & 0.098 & 0.123 \end{bmatrix}^T. \]

Since in the considered case the weights of the criteria are known and equal

\[ \tilde{w} = \begin{bmatrix} 0.5 & 0.2 & 0.3 \end{bmatrix}^T. \]
the final priorities of individual objects are calculated according to the schema (15): 
\[ w(a_i) = 0.5 \cdot w^{(pr)}(a_i) + 0.2 \cdot w^{(du)}(a_i) + 0.3 \cdot w^{(pop)}(a_i). \]

So the final weight vector is equal 
\[ w = [\begin{array}{cccccccc} 0.143 & 0.138 & 0.165 & 0.172 & 0.114 & 0.114 \end{array}]^T, \]

which means that investing in a trampoline fitness point and a recreational pool may be slightly more profitable than any of the other two new facilities.

**Example 4.** Mr. Kowalski became interested in collecting porcelain. He paid particular attention to cups. Mr. Kowalski visited a local antique store and found five exciting items. He would like to buy the two of them that are the most valuable. Unfortunately, there is no universal algorithm that allows him to evaluate porcelain step by step. Moreover, the valuation of porcelain is complex and immeasurable due to the specificity of various markets. The identical cups may have different prices depending on the region of the world in which they are offered. For example, Polish collectors are very fond of German products, undervaluing Polish porcelain a bit. At the same time, the English are proud of their products and are eager to buy them; they also value Chinese porcelain. Therefore, when analyzing trends in the porcelain market, we should also remember the purchase location and ask an expert to evaluate various criteria according to which they make a purchase decision. Let us assume that Mr. Kowalski lives in Poland, so we adopt the values of individual criteria according to the Polish market and will help him make the right decision. Our market research shows that the final value of the cups depends here on several factors.

The most straightforward and most obvious rule is that the more difficult it is to get products from a specific series, period, or manufacturer, the higher the price it receives. Due to the fragility of porcelain, whole sets are more expensive than single pieces. Some units have special status: they are almost unattainable on the market, they are valued as "unlimited" and are usually auctioned at exorbitant prices. The products from the first period of operation of the factory in Meissen, Germany, run by the "father of European porcelain," Friedrich Böttger, are a good example. Moreover, products of a particular manufacturer may be more in demand at one time and less at another.

The second criterion is the reputation of the manufacturer. It is closely influenced by the history of a given factory, the quality of the porcelain produced there, or its uniqueness. It’s also worth remembering that reputation is tied to the current fashion - it depends not only on time but also on location. For example, in Germany, Meissen porcelain is the most valued, while in Poland, collectors like Rosenthal more.

Traces, called signatures, can often be found on porcelain. They define its origin, age, model name, and sometimes the project’s author. Some signatures even specify the type of porcelain the item is made of. Thanks to the signature, we can estimate the value of the product. Usually, the older it is, the
more valuable it is. The first characters were discovered on Chinese porcelain from the 14th century. The well-known collector’s rarities include, among others, unique items from the Ming Dynasty. There are no more than a dozen of them in museum collections worldwide. They are almost nonexistent at auctions and, if they do appear, they reach exorbitant prices. In Europe, the first porcelain products were made in Saxony in the 18th century during the reign of Augustus II and were marked with letters and numbers. Soon, other manufacturers started putting their signatures on the bottom of dishes. Depending on the time of creation, the signatures differ in color and how they were made (painting, printing, decal). The markings also changed within one factory, for example, with a change of ownership. On their basis, the experienced collector can accurately determine the creation time.

Ancient porcelain is extremely fragile, which, of course, increases its value. However, in the event of damage - even porcelain from an excellent manufacturer is worthless. Therefore, before buying, you should evaluate the condition of a single piece of porcelain. An ideal copy will be a product that does not have any acquired or primary defects. By primary defects, we mean any flaws visible in the product that are traces of an ineffective porcelain forming process - stains, painting, or decals errors. Reputable manufacturers sold only first-class products. For example, all defective products are considered broken at Rosenthal factories, so they do not end up on the market. Interestingly, primary defects usually do not influence the product’s price, unlike acquired defects. Acquired defects affecting porcelain price include nicks, sticking together, painting with other paint, or visible spider web shape cracks.

The quality of porcelain is primarily determined by the appropriate selection of raw materials, especially a large amount of kaolin. Good porcelain is hard, durable, and scratch-resistant. It requires manual work as machines do not provide a top-quality finish. Therefore, each item is processed precisely and accurately by hand in good factories. This guarantees the highest quality and timeless beauty of the final product. Ornaments on porcelain also determine its quality. The most valuable is hand-decorated porcelain. Often, ornaments are made of noble materials - real gold or, for example, platinum.

Based on the above information, we will compare the value of considered porcelain cups against five criteria: factory, period of creation, uniqueness, state of preservation and quality. Let

\[ Q = \{\text{man, per, un, st, qua}\} \]

 denote a set of significant criteria, where man, per, un, st, qua mean factory, period of creation, uniqueness, state of preservation and quality, respectively.

Let \( A = \{a_1, a_2, a_3, a_4, a_5\} \) denote a considered collection of cups. First, we compare the cups in terms of the first criterion. Depending on the reputation of the factory, each cup is assigned a value from 1 to 10, where 1 denotes the lowest possible score. All cups come from different factories. We know three of them (for the alternatives \( a_3, a_4 \) and \( a_5 \)) and we can assign values to them based on our experience. They are \( w^{(\text{man})}(a_3) = 8.7 \), \( w^{(\text{man})}(a_4) = 4.2 \), \( w^{(\text{man})}(a_5) = 7.2 \)
respectively. As it is the first time that we are dealing with the products of manufacturers from which the cups $a_1$ and $a_2$ come from, we will apply the HRE procedure. As a result, we get the matrix:

$$C^{(\text{man})} = \begin{bmatrix}
1 & 2 & 1/3 & 3 & 1 \\
1/2 & 1 & 1/3 & 1/3 & 1/2 \\
3 & 5 & 1 & 8.7 & 8.7 \\
1/3 & 1 & 4.2 & 1 & 4.2 \\
1 & 2 & 7.2 & 7.2 & 1
\end{bmatrix}.$$  

Since the weight of cups $a_3$, $a_4$ and $a_5$ are known, the pairs $(a_3, a_4)$, $(a_3, a_5)$ and $(a_4, a_5)$ are not evaluated, but $c_{ij}^{(\text{man})}$ is calculated as $w_{ij}^{(\text{man})}/w_{ij}^{(\text{man})}$ ($3 \leq i < j \leq 5$). To estimate the expected values of factories A and B we follow the HRE procedure. Since $A_1^{(\text{man})} = \{a_1, a_2\}$ and $A_2^{(\text{man})} = \{a_3, a_4, a_5\}$, the matrix $M^{(\text{man})}$ and the constant term vector $b^{(\text{man})}$ have the form (see [14])

$$M^{(\text{man})} = \begin{bmatrix}
-1 & -1/Tc_{1.2} \\
1 & 1
\end{bmatrix},$$

$$b^{(\text{man})} = \left[ \frac{1}{n-1}c_{1.3}w^{(\text{man})}(a_3) + \frac{1}{n-1}c_{1.4}w^{(\text{man})}(a_4) + \frac{1}{n-1}c_{1.5}w^{(\text{man})}(a_5), \frac{1}{n-1}c_{2.3}w^{(\text{man})}(a_3) + \frac{1}{n-1}c_{2.4}w^{(\text{man})}(a_4) + \frac{1}{n-1}c_{2.5}w^{(\text{man})}(a_5) \right].$$

Hence, numerically:

$$M^{(\text{man})} = \begin{bmatrix}
1 & -0.476 \\
-0.119 & 1
\end{bmatrix}, \quad b^{(\text{man})} = \begin{bmatrix}
5.675 \\
2.385
\end{bmatrix}.$$  

As $M^{(\text{man})}w^{(\text{man})} = b^{(\text{man})}$ has an admissible solution (i.e. real and positive), the ranking vector $w^{(\text{man})}$ gets the form

$$w^{(\text{man})} = [7.22 \quad 3.244 \quad 8.7 \quad 4.2 \quad 7.2]^T.$$  

Hence, according to experts’ judgments $w^{(\text{man})}(a_1) = 7.22$ and $w^{(\text{man})}(a_2) = 3.244$. After rescaling

$$w^{(\text{man})} = [0.236 \quad 0.106 \quad 0.284 \quad 0.137 \quad 0.235]^T.$$  

The second considered criterion applies to the date the cup was manufactured. Based on the signatures placed on the cups $a_1$, $a_3$ and $a_5$ we can read the years of production and assign the appropriate number of points according to age. The older the cup, the more points it gets. Finally, we receive the results $w^{(\text{per})}(c_1) = 8$, $w^{(\text{per})}(c_3) = 10$, $w^{(\text{per})}(c_5) = 5$. The cup $a_2$ has no signature and the signature of the cup $a_4$ does not include the year, so in order to establish the period of creation for these two cups we ask an expert for help. In the case of this criterion $A_1^{(\text{per})} = \{a_2, a_4\}$ and $A_2^{(\text{per})} = \{a_1, a_3, a_5\}$. As a result of the expert’s assessment, we obtain the matrix:
In order to obtain the values of the priority vector $w^{\text{per}}$ for cups $a_2$ and $a_4$ it is enough to create the matrix $M^{\text{per}}$ and vector $b^{\text{per}}$ using the HRE method and then solve the equation $M^{\text{per}}w^{\text{per}} = b^{\text{per}}$. The matrix $M^{\text{per}}$ can be obtained from the matrix $C^{\text{per}}$ by removing the columns and rows corresponding to known alternatives from the set $A^{\text{per}}_K$ and multiplying values out of a diagonal by $\frac{1}{n-1}$. The values removed only from the columns (but not from the rows) are used to form the vector $b^{\text{per}}$. Thus

$$C^{\text{per}} = \begin{bmatrix} 1 & 2 & \frac{8}{10} & 4 & \frac{8}{5} \\ \frac{1}{2} & 1 & \frac{8}{5} & 2 & 1 \\ \frac{10}{8} & 3 & 1 & 5 & \frac{10}{5} \\ \frac{1}{2} & \frac{1}{2} & \frac{8}{5} & 1 & \frac{1}{2} \\ \frac{5}{8} & 1 & \frac{5}{10} & 2 & 1 \end{bmatrix}.$$ 

$$M^{\text{per}} = \begin{bmatrix} 1 & -\frac{1}{n-1}c_{2,4} \\ -\frac{1}{n-1} & 1 \end{bmatrix},$$

$$b^{\text{per}} = \begin{bmatrix} \frac{1}{n-1}c_{2,1}w^{\text{per}}(a_1) + \frac{1}{n-1}c_{2,3}w^{\text{per}}(a_3) + \frac{1}{n-1}c_{2,5}w^{\text{per}}(a_5) \\ \frac{1}{n-1}c_{4,1}w^{\text{per}}(a_1) + \frac{1}{n-1}c_{4,3}w^{\text{per}}(a_3) + \frac{1}{n-1}c_{4,5}w^{\text{per}}(a_5) \end{bmatrix},$$

and the equation $M^{\text{per}}w^{\text{per}} = b^{\text{per}}$ takes the shape

$$\begin{bmatrix} 1 & -0.498 \\ -0.124 & 1 \end{bmatrix} \begin{bmatrix} w^{\text{per}}(a_2) \\ w^{\text{per}}(a_4) \end{bmatrix} = \begin{bmatrix} 37 \\ 13 \end{bmatrix}.$$ 

The solution of the above equation is the following vector

$$w^{\text{per}} = [ 8 \ 4.15 \ 10 \ 2.14 \ 5 ]^T.$$ 

Thus, in the category “period of creation” the expected number of points on a scale from 1 to 10 for cups $a_2$ and $a_4$ are 4.15 and 2.14 respectively. After normalization

$$w^{\text{per}} = [ 0.273 \ 0.141 \ 0.341 \ 0.073 \ 0.171 ]^T.$$ 

The third considered criterion concerns uniqueness, although this time the set of reference alternatives is empty. Thus, once again, we ask an expert to compare in pairs the profitability of purchasing a given cup in terms of its uniqueness on the market. His assessments form the following PC matrix

$$C^{\text{un}} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 2 & 1 & 2 & 1 & \frac{1}{2} \\ 2 & \frac{1}{2} & 1 & 1 & \frac{1}{2} \\ 2 & 1 & 1 & 1 & \frac{1}{2} \\ 3 & 2 & 2 & 2 & 1 \end{bmatrix}.$$
In such a case, we can use EVM (Sec. 2.1) i.e. solve the equation $C^{(un)}w = \lambda_{\text{max}}w$. Hence, after appropriate rescaling of the vector $w$ we get $w^{(un)} = [0.097 \ 0.214 \ 0.161 \ 0.182 \ 0.345]^T$.

The condition of the cup depends mainly on two factors: primary damage and acquired damage. Therefore, the experts assess all our alternatives against two sub-criteria: level of primary damage and level of acquired damage. As a result of comparing the degree of primary damage, we obtain the following matrix:

$$C^{(pd)} = \begin{pmatrix} 1 & 9 & 9 & \frac{8}{3} & \frac{5}{3} \\ \frac{1}{9} & 1 & \frac{4}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{9} & \frac{9}{4} & 1 & \frac{1}{9} & \frac{1}{8} \\ \frac{1}{9} & 9 & 9 & 1 & 2 \\ \frac{1}{9} & 8 & 8 & \frac{1}{2} & 1 \end{pmatrix}.$$  

The comparisons of acquired damage are as follows:

$$C^{(ad)} = \begin{pmatrix} 1 & \frac{1}{9} & \frac{1}{3} & \frac{4}{5} & \frac{1}{5} \\ \frac{9}{1} & 1 & \frac{4}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{9}{4} & \frac{1}{9} & 1 & \frac{9}{2} & \frac{2}{4} \\ \frac{5}{9} & \frac{1}{9} & \frac{1}{1} & 1 & \frac{1}{1} \\ \frac{9}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{6} & 1 \end{pmatrix}. $$

The priority vectors for the $C^{(pd)}$ and $C^{(ad)}$ matrices calculated using EVM are as follows:

$$w^{(pd)} = [0.416 \ 0.029 \ 0.04 \ 0.289 \ 0.223]^T,$$

and

$$w^{(ad)} = [0.033 \ 0.301 \ 0.39 \ 0.038 \ 0.235]^T.$$

Comparing the significance of both types of damage, the expert found that the acquired damage is two times more important than the primary damage. Hence, the ranking of both sub-criteria is as follows:

$$w^{(st-c)} = \begin{pmatrix} w^{(st-c)}(pd) \\ w^{(st-c)}(ad) \end{pmatrix} = \begin{pmatrix} 0.333 \\ 0.666 \end{pmatrix}.$$

The state of preservation for the i-th cup is calculated as

$$w^{(st)}(a_i) = w^{(st-c)}(pd) \cdot w^{(pd)}(a_i) + w^{(st-c)}(ad) \cdot w^{(ad)}(a_i),$$

for $i = 1, \ldots, 5$, hence, finally the assessment of the state of the cups’ preservation is as follows:
When assessing the quality of porcelain, the expert considered its resistance to scratching. In the case of two cups $a_1$ and $a_2$, it was known and calculated as the average number of significant scratches among a similar class of products appearing at online auctions in the last five years. Thus, $A^{(qua)}_W = \{a_3, a_4, a_5\}$ and $A^{(qua)}_K = \{a_1, a_2\}$, where $w^{(qua)}(a_1) = 5.7$ and $w^{(qua)}(a_2) = 2.4$. As a result of comparing the quality of alternatives by an expert, we get the matrix

$$
C^{(qua)} = \begin{pmatrix}
1 & \frac{57}{24} & 9 & 3 & 7 \\
\frac{7}{4} & 1 & 9 & \frac{3}{8} & \frac{7}{3} \\
\frac{9}{7} & \frac{1}{3} & 1 & 9 & \frac{3}{6} \\
\frac{2}{4} & \frac{3}{3} & \frac{9}{1} & 1 & 9 \\
\frac{5}{6} & 1 & \frac{2}{3} & \frac{9}{2} & 1
\end{pmatrix}.
$$

Based on the HRE approach, the auxiliary matrix and the constant term vector are

$$
M^{(qua)} = \begin{pmatrix}
1 & -0.0259 & -0.0259 \\
-2.104 & 1 & -0.28 \\
-2.104 & -0.194 & 1
\end{pmatrix}, \quad b^{(qua)} = \begin{pmatrix}
0.225 \\
0.808 \\
1.242
\end{pmatrix}.
$$

Solving the HRE equation $M^{(qua)}w = b^{(qua)}$ reveals that the expected number of scratches for $a_3, a_4$ and $a_5$ are 0.344, 2.2 and 2.39. This allows us to form the ranking vector

$$
\tilde{w}^{(qua)} = \begin{pmatrix}
5.7 & 2.4 & 0.344 & 2.2 & 2.39
\end{pmatrix}^T.
$$

However, since the ranking values represent the expected number of scratches, it means that the higher the value, the worse the product. To change this, we will raise each of the elements of $\tilde{w}^{(qua)}$ to the -1 power and then normalize it. As a result, we get the vector:

$$
w^{(qua)} = \begin{pmatrix}
0.04 & 0.0954 & 0.665 & 0.104 & 0.0955
\end{pmatrix}^T.
$$

It is worth noting that, thanks to the last transformation, if the $i$-th alternative is two times worse than the $j$-th one according to $\tilde{w}^{(qua)}$ then according to $w^{(qua)}$ it is two times better than the $j$-th alternative etc. The obtained vector is a ranking of the cups against the porcelain quality criterion.

In order to establish the weights of criteria ($q_1 = man$, $q_2 = per$, $q_3 = un$, $q_4 = st$, $q_5 = qua$) the expert constructed a PC-matrix, in which $c_{ij}$ denotes how many times the criterion $q_i$ is more preferred than the criterion $q_j$.

$$
\hat{C} = \begin{pmatrix}
1 & 1 & \frac{3}{2} & 3 & 3 \\
1 & 1 & \frac{3}{2} & 3 & 3 \\
\frac{2}{3} & 1 & 2 & 2 \\
\frac{3}{2} & \frac{1}{3} & 1 & 1 \\
\frac{2}{3} & \frac{1}{2} & \frac{1}{3} & 1
\end{pmatrix}
$$
Then, using EVM again, we get a priority vector for the criteria:

\[ \hat{w} = [0.3, 0.3, 0.2, 0.1, 0.1]^T. \]

The final weights of individual cups \( a_1, \ldots, a_5 \) are calculated according to the well-known schema:

\[ w(a_i) = 0.3w^{(man)}(a_i) + 0.3w^{(per)}(a_i) + 0.2w^{(un)}(a_i) + 0.1w^{(st)}(a_i) + 0.1w^{(qua)}(a_i). \]

Thus we obtain

\[ w = [0.1922, 0.1478, 0.3138, 0.1222, 0.2237]^T. \]

The recommendation presented to Mr. Kowalski puts in the first place the cup \( a_3 \) with the rank 0.3138, followed by \( a_5 \) with the rank 0.2237, then \( a_1, a_2 \) and \( a_4 \) with the ranking values 0.1922, 0.1478 and 0.1222.

6. Summary

In the work, we show the hierarchical HRE in the context of a multiple-criteria decision-making process. This allows us to look at HRE not only as a procedure for estimating the rating, but as a multiple-criteria decision-making method. The concept of HRE is based on the same assumptions that can be found in EVM or GMM. Thus, this method should not be difficult for all those who know and use AHP. HRE, unlike AHP, does not force the user to abandon the actual measured values associated with given alternatives in favor of the values calculated as a result of the ranking procedure. Once measured or calculated, the selected quantities become reference values in this model and are not subject to further change. If this behavior is desired in a given application, the HRE method will work here.

We believe that HRE fills a specific gap and can prove its usefulness wherever tangible and intangible criteria need to be combined within a single hierarchical model. Therefore, it might interest a range of practitioners who may feel a little awkward when comparing well-measured values such as price, length, or weight, and theoreticians studying the properties of new decision-making methods.

7. Literature

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