Time-minimal control of dissipative two-level quantum systems: The integrable case

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September 24, 2008

Abstract

The objective of this article is to apply recent developments in geometric optimal control to analyze the time minimum control problem of dissipative two-level quantum systems whose dynamics is governed by the Lindblad equation. We focus our analysis on the case where the extremal Hamiltonian is integrable.

Keywords. Optimal control, conjugate and cut loci, quantum control

AMS classification. 49K15, 70Q05

1 Introduction

We consider a dissipative two-level quantum system whose dynamics is governed by the Lindblad equation which takes the following form in suitable coordinates $q = (x, y, z)$, i.e., in the coherence vector formulation of density matrix [1, 16]:

\begin{align*}
\dot{x} &= -\Gamma x + u_2 z \\
\dot{y} &= -\Gamma y - u_1 z \\
\dot{z} &= \gamma_+ - \gamma_+ z + u_1 y - u_2 x.
\end{align*}

We refer to [10] and [17] for the details of the model. We recall that $x$ and $y$ are related to off-diagonal terms of the density matrix of the system and $z$ to the difference of population between the two states. In [1, $\Lambda = (\Gamma, \gamma_-, \gamma_+)$ is a set of parameters such that $\Gamma \geq \gamma_+ / 2 > 0$ and $\gamma_+ \geq |\gamma_-|$ which describes the interaction of the two-level system with the environment. More precisely, $\Gamma$ is the dephasing rate and $\gamma_+$ and $\gamma_-$ are respectively equal to $\gamma_{12} + \gamma_{21}$ and $\gamma_{12} - \gamma_{21}$ where the coefficients $\gamma_{12}$ and $\gamma_{21}$ are the population relaxation rates. The control is the complex Rabi frequency $u = u_1 + iu_2$ of the laser field which is assumed to be in resonance with the frequency of the two-level system [11]. The physical state belongs to the Bloch ball, $|q| \leq 1$, which is invariant for the

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dynamics considered. If they are many articles devoted to optimal control of quantum systems in the conservative case (see e.g. [12]), the dissipative case is still an open problem.

The system can be written shortly as a bilinear system

\[
\dot{q} = F_0(q) + u_1 F_1(q) + u_2 F_2(q)
\]  
(2)

and in order to minimize the effect of dissipation, we consider the time minimum control problem for which, up to a rescaling on the set of parameters \(A\), the control bound is \(|u| \leq 1\). The energy minimization problem with the cost \(\int_0^T |u|^2 dt\) where the time \(T\) is fixed but the control bound is relaxed can also be considered and shares similar properties.

A first step in the analysis of such systems is contained in [17]. Assuming \(u\) real, the problem can be reduced to the time-optimal control of a two-dimensional system

\[
\begin{align*}
\dot{y} &= -\Gamma y - u_1 z \\
\dot{z} &= \gamma_- - \gamma_+ z + u_1 y,
\end{align*}
\]  
(3)

with the constraint \(|u_1| \leq 1\). For such a problem, the geometric optimal control techniques for single-input two-dimensional systems presented in [13] succeed to make the time-optimal synthesis for every values of parameters \((\Gamma, \gamma_-, \gamma_+)\).

In order to complete the analysis in the bi-input case, a different methodology has to be applied and we shall make an intensive use of techniques and results developed in a parallel research project to minimize the transfer of a satellite between two elliptic orbits (see [10]). Such techniques are two-fold.

First of all, the maximum principle will select extremal trajectories, candidates as minimizers and solutions of an Hamiltonian equation. A geometric analysis will identify the symmetry group of the system and find suitable co-ordinates to represent the Hamiltonian. A consequence of this analysis is the fact that, in the case \(\gamma_- = 0\), the extremal system is integrable and if \(\Gamma = \gamma_+\), the problem can be in addition reduced to a 2D-almost Riemannian problem on a two-sphere of revolution for which a complete analysis comes from [6]. We take advantage of this property to analyze the general integrable case \(\gamma_- = 0\) using continuation methods on the set of parameters, while the analysis fits in the geometrical framework of Zermelo navigation problems [3]. This case corresponds to the physical situation where the population relaxation rates are equal. The mathematical tools presented in this article and a few numerical simulations are sufficient to complete the analysis for \(\gamma_- = 0\). The generic case where \(\gamma_- \neq 0\) is treated in a forthcoming article [9] combining mathematical analysis and intensive numerical computations.

Secondly, having selected extremal trajectories, second-order conditions using the variational equation and implemented in the cotcot code [7] allow to determine first conjugate points forming the conjugate locus which are points where extremals cease to be locally optimal. Combined with the geometric analysis, we can construct the cut locus which is formed by points where extremals cease to be globally optimal.

The organization of this article is the following. In section 2, we recall the maximum principle and the concept of conjugate points associated to second-order optimality conditions. In section 3, we present the geometric analysis
of the system, followed in section 4 by a thorough analysis of the so-called Grusin problem on a two-sphere of revolution. This problem is generalized into a Zermelo navigation problem suitable to our analysis. In section 5, we study the properties of the extremals to analyze the optimal trajectories, combining analytic and numerical methods.

2 Geometric optimal control

2.1 Maximum principle

We consider the time minimum problem with fixed extremities \( q_0 \) and \( q_1 \). For a smooth system written \( \dot{q} = F(q,u) \), with \( q \in \mathbb{R}^n \) and a control domain \( U \) which is a compact subset of \( \mathbb{R}^m \), we have:

\textbf{Proposition 1.} If \((q,u)\) is an optimal control trajectory pair on \([0,T]\) then there exists an absolutely continuous non-zero vector function \((p,p_0)\in \mathbb{R}^n \times \mathbb{R}\) such that almost everywhere on \([0,T]\), we have:

\[
\dot{q} = \frac{\partial H}{\partial p}(q,p,u), \quad \dot{p} = -\frac{\partial H}{\partial q}(q,p,u) \tag{4}
\]

and

\[
H(q,p,u) = M(q,p) \tag{5}
\]

where \( H(q,p,u) = \langle p,F(q,u) \rangle + p_0 \), \( p_0 \) being a non positive constant; \( M \) is defined by \( M(q,p) = \max_{u \in U} H(q,p,u) \) and \( M \) is zero everywhere.

\textbf{Definition 1.} The mapping \( H \) is called the pseudo-Hamiltonian. A triple \((q,p,u)\) solution of (4) and (5) is called extremal while the component \( q \) is called an extremal trajectory and \( p \) is an adjoint vector.

\textbf{Application:} We consider the time-minimum control problem for a system of the form \( \dot{q} = F_0(q) + \sum_{i=1}^{m} u_i F_i(q) \) with \( m \geq 2 \), \( u = (u_1, \cdots, u_m) \), \( |u| \leq 1 \).

We introduce the Hamiltonian lifts \( H_i = \langle p,F_i(q) \rangle \), \( i = 0, 1, \cdots, m \) of the vector fields \( F_i \) and the set \( \Sigma \) such that \( H_i = 0 \) for \( i = 1, \cdots, m \). Then the maximization condition (5) leads to the following result.

\textbf{Proposition 2.} Outside \( \Sigma \), an extremal control is given by \( u_i = H_i / \sqrt{\sum_{i=1}^{m} H_i^2} \), \( i = 1 \cdots, m \) and extremal pairs \( z = (q,p) \) are solutions of the smooth true Hamiltonian vector field \( \vec{H}_r(z) \) with \( H_r(z) = H_0(z) + (\sum_{i=1}^{m} H_i^2(z))^{1/2} \).

\textbf{Definition 2.} The surface \( \Sigma \) is called the switching surface and the solutions of \( \vec{H}_r(z) \) are called extremals of order zero. To be optimal, they have to satisfy \( H_r(z) \geq 0 \) and those with \( H_r(z) = 0 \) are called abnormal.

An important but straightforward result is the following proposition [8].

\textbf{Proposition 3.} Extremal trajectories of order zero correspond to singularities of the end-point mapping

\[
E^{q_0,T} : u \in L^\infty[0,T] \mapsto q(T,q_0,u) \tag{6}
\]

where \( q(.,q_0,u) \) denotes the response to \( u \) with initial condition \( q_0 \) and such that the control is restricted to the \((m-1)\)-sphere \( |u| = 1 \).
2.2 Second-order optimality conditions

From proposition 3, we can apply the concepts and algorithms presented in [7] to compute second-order optimality conditions in the smooth case, when the control domain $U$ is a manifold, $u$ being restricted to the unit sphere. The framework of this computation is next recalled.

**The concept of conjugate point:**

Since $U$ is a manifold, we may assume locally that $U = \mathbb{R}^{m-1}$ and the maximization condition (5) leads to

$$\frac{\partial H}{\partial u} = 0, \quad \frac{\partial^2 H}{\partial u^2} \leq 0.$$  \hspace{1cm} (7)

Our first assumption is the **strong Legendre-Clebsch condition**:

(H1) The Hessian $\frac{\partial^2 H}{\partial u^2}$ is negative definite along the reference extremal.

From the implicit function theorem, an extremal control can be locally defined as a smooth function of $z = (q, p)$ and plugging $u$ into $H$ defines a smooth true Hamiltonian $H_r$.

Setting $M = \mathbb{R}^n$ and using Hamiltonian formalism, we introduce:

**Definition 3.** Let $z = (q, p)$ be a reference extremal defined on $[0, T]$. The variational equation

$$\delta \dot{z} = dH_r(z(t)) \delta z$$  \hspace{1cm} (8)

is called the Jacobi equation. A Jacobi field is a non trivial solution $\delta z = (\delta q, \delta p)$. It is said to be vertical at time $t$ if $\delta q(t) = 0$.

The following standard result is crucial.

**Proposition 4.** Let $L_0$ be the fiber $T^*_q M$ and let $L_t = \exp[tH_r(L_0)]$ be its image by the one parameter subgroup generated by $H_r$. Then $L_t$ is a Lagrangian manifold whose tangent space at $z(t)$ is spanned by Jacobi fields vertical at $t = 0$.

Moreover, the rank of the restriction to $L_t$ of the projection $\Pi : (q, p) \mapsto q$ is at most $(n - 1)$.

We next formulate the relevant generic assumptions using the end-point mapping.

**Assumptions:**

1. (H2) On each subinterval $[t_0, t_1]$, $0 < t_0 < t_1 \leq T$, the singularity of $E_{q(t_0),t_1-t_0}$ is of codimension one for $u|_{[t_0, t_1]}$.

2. (H3) We are in the normal case $H_r \neq 0$.

As a result, on each subinterval $[t_0, t_1]$ there exists up to a positive scalar an unique adjoint vector $p$ such that $(q, p, u)$ is extremal.

**Definition 4.** We fix $q_0 = q(0)$ and we define the exponential mapping:

$$\exp_{q_0} : (p(0), t) \mapsto \Pi(\exp[tH_r(q(0), p(0))])$$

where $p(0)$ is a $(n - 1)$ dimensional vector, normalized with $H_r = 1$. 
Definition 5. Let \( z = (q, p) \) be the reference extremal on \([0, T]\). Under our assumptions, a time \( t_0 < t_c \leq T \) is called conjugate if the mapping \( \exp_{q_0} \) is not an immersion at \((p(0), t_c)\) and the point \( q(t_c) \) is said to be conjugate to \( q_0 \). We denote \( t_{1c} \) the first conjugate time and \( C(q_0) \) the conjugate locus formed by the set of first conjugate points considering all extremal curves. We get:

Theorem 1. Let \( z(t) = (q(t), p(t)) \) be a reference extremal on \([0, T]\) satisfying assumptions (H1), (H2) and (H3). Then the extremal is optimal in the \( L^\infty \)-norm topology on the set of controls up to the first conjugate time \( t_{1c} \). Moreover, if \( t \mapsto q(t) \) is one-to-one then it can be embedded into a set \( W \), image by the exponential mapping \( \exp_{q(0)} \) of \( N \times [0, T] \), where \( N \) is a conical neighborhood of \( p(0) \). For \( T < t_{1c} \), the reference extremal trajectory is time minimal with respect to all trajectories contained in \( W \).

In order to get global optimality results, it is necessary to glue together such micro-local sets. We need to introduce the following concepts.

Definition 6. Given an extremal trajectory, the first point where it ceases to be optimal is called the cut point and taking all the extremals starting from \( q_0 \), they will form the cut locus \( \text{Cut}(q_0) \). The separating line \( \text{SL}(q_0) \) is formed by the set of points where two minimizers initiating from \( q_0 \) intersect.

3 Geometric analysis of Lindblad equation

3.1 Symmetry of revolution

If we apply to system (1) a change of coordinates defined by a rotation of angle \( \theta \) around the \( z \)-axis:

\[
\begin{align*}
X &= x \cos \theta + y \sin \theta \\
Y &= -x \sin \theta + y \cos \theta \\
Z &= z,
\end{align*}
\]

and a similar feedback transformation on the control:

\[
v_1 = u_1 \cos \theta + u_2 \sin \theta, \quad v_2 = -u_1 \sin \theta + u_2 \cos \theta,
\]

we obtain the system

\[
\begin{align*}
\dot{X} &= -\Gamma X + v_2 Z \\
\dot{Y} &= -\Gamma Y - v_1 Z \\
\dot{Z} &= \gamma - \gamma + Z + v_1 Y - v_2 X.
\end{align*}
\] (9)

Hence, this defines a one dimensional symmetry group and by construction \(|u| = |v|\). Therefore, we deduce that the time-minimum control problem (or the energy minimization problem) are invariant for such an action. Using cylindric coordinates

\[
x = r \cos \theta, \quad y = r \sin \theta, \quad z = z
\]
and the dual variables \( p = (p_r, p_\theta, p_z) \), the Hamiltonian \( H_r \) takes the form

\[
H_r = H_0 + (H_1^2 + H_2^2)^{1/2}
\]

\[
= (-\Gamma r p_r + (\gamma_- - \gamma_+) p_z) + (z^2 p_r^2 + \frac{z^2}{r^2} p_\theta^2 + r^2 p_z^2 - 4 z r p_r p_z)^{1/2}.
\]

In particular, the Bloch ball is foliated by meridian planes \( \theta = \text{constant} \) in which the time-minimum synthesis is the one associated to system (3), where the control is scalar and described in [17]. More precisely, we have:

**Proposition 5.** For the time-minimum control, \( \theta \) is a cyclic coordinate and \( p_\theta \) is a first integral of the motion. The sign of \( \dot{\theta} \) is given by \( p_\theta \) and if \( p_\theta = 0 \) then \( \theta \) is constant and the extremal synthesis for an initial point on the z-axis is up to a rotation given by the synthesis in the plane \( \theta = 0 \). Up to a rotation, the control \( u \) can also be restricted to the single-input control \( (u_1, 0) \).

**Proof.** The proof is a generalization of the geometric situation encountered in [6]. For the Hamiltonian vector field \( \vec{H}_r \), the points on the z-axis correspond to a polar singularity and the extremals starting from the z-axis are contained in meridian planes \( \theta = \theta(0) \). Hence, \( p_\theta \) is constant and extremal curves in the plane \( \theta = \theta(0) \) are solutions of system (3).

### 3.2 Spherical coordinates

More properties can be seen using spherical coordinates:

\[
x = \rho \sin \phi \cos \theta, \; y = \rho \sin \phi \sin \theta, \; z = \rho \cos \phi
\]

and a similar feedback transformation. We obtain the system:

\[
\dot{\rho} = \gamma_- \cos \phi - \rho (\gamma_+ \cos^2 \phi + \Gamma \sin^2 \phi)
\]

\[
\dot{\phi} = -\frac{\gamma_- \sin \phi}{\rho} + \frac{\sin(2\phi)}{2} (\gamma_+ - \Gamma) + v_2
\]

\[
\dot{\theta} = -(\cot \phi)v_1.
\]

and the corresponding Hamiltonian

\[
H_r = \left[ \gamma_- \cos \phi - \rho (\gamma_+ \cos^2 \phi + \Gamma \sin^2 \phi) \right] p_\rho
\]

\[
+ \left[ -\frac{\gamma_- \sin \phi}{\rho} + \frac{\sin(2\phi)}{2} (\gamma_+ - \Gamma) \right] p_\phi + \sqrt{p_\rho^2 + p_\phi^2 \cot^2 \phi}.
\]

In this representation, \((\phi, \theta)\) are the spherical coordinates on the unit sphere of revolution around the z-axis; \( \theta \) is the angle of revolution and \( \phi \in [0, \pi] \) is the angle of the meridian, \( \phi = 0, \pi \) correspond respectively to the north and south poles.

### 3.3 Lie brackets computations

In order to complete the analysis, we immediately compute the Lie brackets up to length 3 for the system written in Cartesian coordinates as:

\[
\dot{q} = (G_0 q + v_0) + u_1 G_1 q + u_2 G_2 q
\]
where the $G_i$’s are the matrices

\[
G_0 = \begin{pmatrix} -\Gamma & 0 & 0 \\ 0 & -\Gamma & 0 \\ 0 & 0 & -\gamma_+ \end{pmatrix}, \quad G_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}
\]

and $^tv_0$ is the vector $(0, 0, \gamma_-)$. It can be lifted into a right-invariant control system on the semi-direct product $GL(3, \mathbb{R}) \ltimes \mathbb{R}^3$ identified to the subgroup of matrices of $GL(4, \mathbb{R})$ of the form:

\[
\begin{pmatrix} 1 & 0 \\ v & g \end{pmatrix}, \quad v \in \mathbb{R}^3, g \in GL(3, \mathbb{R})
\]

acting on the subset of vectors of $\mathbb{R}^4$: \( \begin{pmatrix} 1 \\ q \end{pmatrix}, \quad q \in \mathbb{R}^3 \). To construct affine vector fields, we use the induced action of the Lie algebra \((a, A) \cdot q = Aq + a\) and Lie brackets are given by

\[
[(a, A), (b, B)] = [Ab - Ba, AB - BA].
\]

The control distribution is \( D = \text{Span}\{G_1, G_2\} \) and we have:

\[
[G_1, G_2] = G_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

In particular, we obtain that \( \{G_1, G_2\}_{A.L.} = so(3) \) and hence the system on $SO(3)$:

\[
\frac{dX}{dt} = (u_1 G_1 + u_2 G_2)X
\]

is controllable. For the linear action, it defines a controllable system on the unit sphere. This action has however singularities:

- at 0, the orbit is 0.
- the set on $\mathbb{R}^2$ where $G_1$ and $G_2$ are collinear is the whole plane $z = 0$ and restricted to the unit sphere of revolution, it corresponds to the equator.

To analyze the effect of the drift term associated to dissipation, we use

\[
[G_0, G_1] = (\Gamma - \gamma_+) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad [G_0, G_2] = (\gamma_+ - \Gamma) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}
\]

and \([G_0, G_3] = 0\). Moreover, we have

\[
[G_1, [G_0, G_1]] = 2(\Gamma - \gamma_+) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad [G_2, [G_0, G_2]] = 2(\gamma_+ - \Gamma) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

Those computations reveal the singularity at $\gamma_+ = \Gamma, \gamma_- = 0$ that we describe in the next proposition.
Proposition 6. In the case $\gamma_- = 0$, $\gamma_+ = \Gamma$, the radial component $\rho$ is not controllable and the time-minimum control problem is an almost Riemannian problem on the two-sphere of revolution for the metric in spherical coordinates $g = d\rho^2 + \tan^2 \phi d\theta^2$ with Hamiltonian $H = \frac{1}{2}(p_\phi^2 + p_\theta^2 \cot^2 \phi)$.

Proof. If $\gamma_- = 0$ and $\gamma_+ = \Gamma$ then the first equation of (10) becomes $\dot{\rho} = -\Gamma \rho$ and $\rho$ is not controllable. Hence, the time-minimal control problem reduces to the problem of controlling $(\phi, \theta)$ in minimum time where the associated true Hamiltonian is $\sqrt{p_\phi^2 + p_\theta^2 \cot^2 \phi}$. The time minimum problem is equivalent to minimizing the length for the metric $d\phi^2 + \tan^2 \phi d\theta^2$. According to Maupertuis principle, we can replace the length by the energy with corresponding Hamiltonian $\frac{1}{2}(p_\phi^2 + p_\theta^2 \cot^2 \phi)$.

Definition 7. The almost Riemannian metric $g = d\phi^2 + \tan^2 \phi d\theta^2$ is called the Grusin model on the two-sphere of revolution.

Such a metric appears in quantum control in the conservative case [12] and a similar metric is associated to orbital transfer [3]. It will be analyzed in details in section 4 since it is the starting point of the analysis in the general case using a continuation method on the set of parameters.

Another consequence of the previous computations is the controllability properties of the system and the structure of extremal trajectories.

3.4 Controllability properties

We recall that the Bloch ball $|q| \leq 1$ is invariant. Indeed, introducing $\rho^2 = |q|^2$, we get:

$$\rho \dot{\rho} = -\Gamma (x^2 + y^2) - \gamma_+ z^2 + \gamma_- z \leq 0 \quad (11)$$

which is strictly negative on the unit sphere except if $x^2 + y^2 = 0$, $|z| = 1$ and $\gamma_+ = |\gamma_-|$. Using the representation (10) of the system in spherical coordinates, it is clear that we can control the angular variables $\phi$ and $\theta$ if the controls are not uniformly bounded. If $|u| \leq 1$ then we have restrictions depending upon the set of parameters.

For $\gamma_- = 0$, the system is homogeneous and $q = 0$ is a fixed point. The accessibility set in fixed time is with non empty interior for a non-zero initial point, except in the case $\gamma_+ = \Gamma$ which corresponds to the Grusin model and for which the time and energy minimization problem are equivalent.

The controllability properties for $|u| \leq 1$ are clear in this case. Indeed, if $|\gamma_+ - \Gamma| < 2$ then we can compensate the drift by feedback for the system on the two-sphere of revolution, while it is not the case for $|\gamma_+ - \Gamma| > 2$.

Proposition 7. Let $q_0$ and $q_1$ be two points in the Bloch ball $|q| \leq 1$ such that $q_1$ is accessible to $q_0$. Then there exists a time-minimum trajectory joining $q_0$ to $q_1$. Moreover, every optimal trajectory is

1. either an extremal trajectory with $p_\theta = 0$, contained in a meridian plane, time-optimal solution of the two-dimensional system [3] where $u = (u_1, 0)$.

2. either connection of smooth extremal arcs of order 0, solutions of the Hamiltonian vector field $\overline{H}$, with $p_\theta \neq 0$, while the only possible connections are located in the equatorial plane $\phi = \pi/2$.
Proof. The control domain is convex and the Bloch ball is compact. Hence, we can apply the Filippov existence theorem \[15\]. In order to get a regularity result about optimal trajectories, much more work has to be done. This is due to the existence of a switching surface $\Sigma : H_1 = H_2 = 0$ in which we can connect two extremals arcs of order 0, provided we respect the Erdmann-Weierstrass conditions at the junction, i.e., the adjoint vector remains continuous and the Hamiltonian is constant. The set $\Sigma$ can also contain singular arcs for which $H_1 = H_2 = 0$ holds identically. Hence, we can have intricate behaviors for such systems. In our case, the situation is simplified by the symmetry of revolution.

Indeed, if $p_\theta = 0$ then the singularities are related to the classification of extremals in the single-input case, which is described in \[17\]. We cannot connect an extremal with $p_\theta \neq 0$ where $p_\theta$ is the global first integral $xp_\theta - yp_x$ to an extremal where $p_\theta = 0$ since the adjoint vector has to be continuous.

Hence, the only remaining possibility is to connect two extremals of order 0 with $p_\theta \neq 0$ at a point of $\Sigma$ leading to the conditions $p_\phi = 0$ and $p_\theta \cot \phi = 0$ in spherical coordinates. Since $p_\theta \neq 0$, one gets $\phi = \pi/2$. The result is proved. \[\Box\]

Remark: The classification of extremal trajectories near the equatorial plane is described in proposition \[19\].

4 The Grusin model on a two-sphere of revolution with generalizations to Zermelo navigation problem

The Grusin model $q = d\phi^2 + \tan^2 \phi d\theta^2$ is a special case of metrics of the form $d\phi^2 + G(\phi)d\theta^2$ on a two-sphere of revolution such that:

- (H1) $G'(\phi) \neq 0$ on $[0, \pi/2]$.
- (H2) $G(\pi - \phi) = G(\phi)$ (reflective symmetry with respect to the equator).

They appear in optimal control in the orbit transfer, smooth at the equator or with a polar singularity, in quantum control and in various geometric problems, e.g., Riemannian problems on an ellipsoid of revolution. The importance of this control problem has justified the recent analysis of \[4\] that we complete next, using Hamiltonian formalism in order to make generalizations. We first interpret the Grusin model as a deformation of the round sphere.

Definition 8. The standard homotopy between the Grusin model and the round sphere on the two-sphere of revolution is $g_\lambda = d\phi^2 + G_\lambda(X)d\theta^2$ where $G_\lambda(X) = X = \sin^2 \phi$ and $\lambda \in [0, 1]$.

By construction, the metric is analytic for $\lambda \in [0, 1]$ and for $\lambda = 1$, we have the Grusin model with a pole of order 1 at the equator. The first objective of this section is to show stability results concerning such metrics. We have the following general result \[6\].

Proposition 8. Let $d\phi^2 + G(\phi)d\theta^2$ be a smooth metric on a two-surface of revolution. Then,

1. Extremals are solutions of the Hamiltonian $H = \frac{1}{2}(p_\phi^2 + \frac{p_\theta^2}{G(\phi)})$ and arc-length parametrization amounts to restrict to $H = 1/2$. 
2. If $\psi$ is the angle of an unit-speed extremal with a parallel then $p_\theta = \sqrt{G} \cos \psi$ is a constant and the extremal flow is Liouville integrable with two commuting first integrals $H$ and $p_\theta$.

3. The Gauss curvature is $K = -\frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial \phi^2}$. We next make a complete analysis of the family of metrics $g_\lambda$.

4.1 Curvature analysis

We have the following proposition.

**Proposition 9.** For the family of metrics $g_\lambda$, we have:

1. The Gauss curvature is $K_\lambda = \frac{(1-\lambda) - 2\lambda \cos^2 \phi}{(1-\lambda \sin^2 \phi)^3}$.

2. $K'(\phi) = \frac{\lambda \sin(2\phi)}{(1-\lambda \sin^2 \phi)^4} [5(1-\lambda) - 4\lambda \cos^2 \phi]$.

Hence $K(\phi)$ is non-constant and monotone non-decreasing from the north pole to the equator for $\lambda \in [0, 1/5]$, while for $\lambda \in [1/5, 1]$ it admits a minimum. For $\lambda \in [0, 1]$, the curvature is maximum on the equator. The limit case $\lambda = 1$ corresponds to the Grusin case, for which the curvature is negative everywhere and tends to $-\infty$ when $\phi$ tends to $\pi/2$.

4.2 Geometric properties

We next present the main properties of the extremal flow for a metric on a two-sphere of revolution $g = d\phi^2 + G(\phi)d\theta^2$ satisfying (H1) and (H2) where $g$ is smooth, except may be at the equator where it can admit a pole of order one.

For such a family, we consider the smooth Hamiltonian $H = \frac{1}{2}(p_\phi^2 + \frac{p_\theta^2}{G(\phi)})$ and we restrict extremal curves to the level set $H = 1/2$. Fixing $p_\theta$, the parameterized family of corresponding Hamiltonians described the evolution of the $(p_\phi, \phi)$ variables as solutions of a mechanical system for which $V(\phi) = \frac{p_\phi^2}{G(\phi)}$ plays the role of potential. For $p_\theta = 0$, we get the meridian solutions. Hence, we can assume $p_\theta \neq 0$. Using assumption (H1), the only equilibrium point is for $\phi = \pi/2$ and $p_\phi = 0$. This leads to the equator solution in the regular case.

For the remaining trajectories, the level set $H = 1/2$ is sufficient to analyze the behaviors of $\phi$. Indeed, it is a compact set, symmetric for the two reflections with respect to the $\phi$-axis and the equator $\phi = \pi/2$ and defined respectively by the two transformations: $p_\phi \mapsto -p_\phi$ and $\phi \mapsto \pi - \phi$. Every trajectory is periodic and $\psi = \pi/2 - \phi$ oscillates periodically between $\psi_{\text{max}}$ and $-\psi_{\text{max}}$. There is also a relation between the period of oscillation $T$ and the amplitude $\psi_{\text{max}}$, depending upon $p_\theta$.

By symmetry, every trajectory is defined by its restriction to a quarter of period, that is the sub-arc starting from the equator $\psi = 0$ and reaching $\psi_{\text{max}}$. The trajectory starting from $(\phi(0), p_\phi(0))$ and reaching $\pi - \phi(0)$ after passing $\psi_{\text{max}}$ corresponds to a point rotating on the level set $H = 1/2$ and is chased by a point associated to the trajectory starting from $(\phi(0), -p_\phi(0))$ and reaching $\pi - \phi(0)$. They are distinct if $p_\phi(0) \neq 0$. Moreover, using the assumption (H2), we deduce easily that for fixed $p_\theta$, the extremals starting from $(\phi(0), \theta(0))$ with respectively $p_\phi(0)$ and $-p_\phi(0)$ intersect with equal length on the antipodal
parallel. They are distinct if \( p_\phi(0) \neq 0 \). The case \( p_\phi(0) = 0 \) corresponds for an initial condition not on the equator to tangential arrival and departure at parallels \( \phi(0) \) and \( \pi - \phi(0) \); \( p_\phi(0) = 0 \) gives the equator solution in the non-singular case. For more details see \[5\].

The only difference in the singular case is that the equator is not solution, and for trajectories departing from the equator the extremals are always tangential to the meridian, while the first return to the equator can be arbitrarily closed from the initial point.

Finally, another obvious symmetry is a reflectional symmetry with respect to the meridian obtained by changing \( p_\theta \) into \( -p_\theta \).

As a consequence of this analysis, we deduce:

**Proposition 10.** Let \( g = d\phi^2 + G(\phi)d\theta^2 \) be a metric on a two-sphere of revolution, satisfying (H1) and (H2) and smooth except may be at the equator where it can admit a pole of order one.

1. Then except the meridian and the equator solution in the regular case, every extremal is such that \( \psi = \pi/2 - \phi \) oscillates periodically between two symmetric parallels. The first return mapping to the equator is

\[
R : p_\theta \in [0, \sqrt{G(\pi/2)}] \rightarrow \Delta \theta(p_\theta),
\]

where \( \Delta \theta \) is the corresponding \( \theta \) variation of the extremal.

2. Assume \( p_\theta \neq 0 \) and \( p_\phi(0) \neq 0 \) then
   a) Fixing \( p_\theta \), changing \( p_\phi(0) \) into \( -p_\phi(0) \) gives two distinct extremals with equal length intersecting on the antipodal parallel.
   b) Fixing \( p_\phi(0) \) and changing \( p_\theta \) into \( -p_\theta \) gives two distinct extremals with equal length intersecting on the opposite meridian.

### 4.3 Integrability

For the family of metrics \( g_\lambda \) which fit in the previous geometric framework, we can be more precise and make a complete analysis. For a fixed value of \( \lambda \), the Hamiltonian is:

\[
H_\lambda = \frac{1}{2} \left( p_\phi^2 + \frac{p_\theta^2}{G_\lambda(\phi)} \right), \quad G_\lambda(\phi) = \frac{\sin^2 \phi}{1 - \lambda \sin^2 \phi}
\]

and corresponds for \( \lambda = 1 \) to the Grusin case. Using \( \dot{\phi} = p_\phi \), we get:

\[
H_\lambda = \frac{1}{2} \left( \dot{\phi}^2 + \frac{p_\phi^2(1 - \lambda \sin^2 \phi)}{\sin^2 \phi} \right)
\]

which can be written

\[
H_\lambda = \frac{1}{2} \left[ \dot{\phi}^2 + p_\phi^2(\cot^2 \phi + 1 - \lambda) \right].
\]

Therefore, \( H_\lambda = H_1 + \frac{1}{2} p_\phi^2(1 - \lambda) \) and parameterized by arc-length: \( H_\lambda = 1/2 \), one gets the level set \( H_1 = \frac{1}{2} - \frac{1}{2} p_\phi^2(1 - \lambda) \).

Hence the integration of the Grusin case gives the general solution, and from the homotopy, the corresponding extremals fit not only in the same geometric framework, but also have the same transcendence.
Lemma 1. The family of Hamiltonians $H_\lambda$ admits two first integrals in involution for the Poisson bracket (independent of $\lambda$) $p_\theta$ and $H_1 = \frac{1}{2}p_\phi^2 + p_\theta^2 \cot^2 \phi$.

We next outline the integration method in the Grusin case to provide the computation of the first return mapping to the equator $R$ obtained in [6]. We have $H_1 = \frac{1}{2}(\dot{\phi}^2 + \nu \cot^2 \phi), \; \nu > 0$ and fixing the level set to $1/2$, we get:

$$\left(\frac{d\phi}{dt}\right)^2 = \frac{1 - (\nu + 1)\cos^2 \phi}{\sin^2 \phi}.$$  

Taking the positive branch, we must evaluate the following expression

$$\int \frac{\sin \phi d\phi}{(1 - (\nu + 1)\cos^2 \phi)^{1/2}} = t.$$  

To integrate, we use the relation

$$\int \frac{\cos \phi d\phi}{\sqrt{1 - m^2 \sin^2 \phi}} = \frac{1}{m} \arcsin(m \sin \phi),$$

to deduce the form of the component $\phi$ of the general solution:

$$\phi(t) = \arcsin\left[\frac{1}{m}(\sin(mt + K))\right] + \pi/2.$$  

To complete the integration, we write:

$$\dot{\theta} = \frac{p_\theta}{\sin^2 \phi} - \lambda p_\theta$$

and we use the formula:

$$\int \frac{dx}{1 - a \sin^2 x} = \frac{1}{\sqrt{1 - a}} \arctan\left[\sqrt{1 - a} \tan x\right]$$

for $a < 1$ with the relation $\cos \phi = -\frac{1}{m} \sin(mt + K)$. A straightforward computation then leads to $\theta(t)$.

Proposition 11. For the family of metrics $g_\lambda$, we have:

$$R(p_\theta) = \pi - \frac{\alpha \pi p_\theta}{\sqrt{\alpha + 1} \sqrt{\alpha + 1 + \alpha p_\theta^2}} \quad \alpha = \frac{\lambda}{1 - \lambda}.$$  

In particular, if $\alpha > 0$ then $R'(p_\theta) < 0 < R''(p_\theta)$ on $[0, \sqrt{G(\pi/2)}]$.

This property allows to evaluate conjugate and cut loci for the family of metrics that we next describe [6].

4.4 Conjugate and cut loci

We have:

Theorem 2. • For $\lambda = 0$ (round sphere), the conjugate and cut loci of any point are reduced to the antipodal point.
• For $0 < \lambda < 1$, the conjugate locus of a point different from a pole is diffeomorphic to a standard astroid, while the cut locus is a single branch of the antipodal parallel. Both are symmetric with respect to the opposite meridian.

• For $\lambda = 1$ (Grusin case), the conjugate and cut loci of a point different from a pole and not on the equator are as above. For a point on the equator, the cut locus is the equator minus this point and for the conjugate locus, the cusps on the equator are transformed into folds at this point minus this point.

Geometric interpretation:
For the class of metrics $g_\lambda$, the situation is clear. For the round sphere, all extremals starting from the equator intersect at the same antipodal point and the first return mapping is constant. For $0 < \lambda \leq 1$, the first return mapping is monotone, and in the singular case $R(p) \to 0$ as $p \to +\infty$. Since the cut locus of a point of the equator is formed by intersections with the equator of symmetric extremals, in the homotopy, the cut locus is pinched into a point for $\lambda = 0$, while it is stretched into the whole equator in the case $\lambda = 1$.

4.5 Zermelo navigation problem on the two-sphere of revolution

We introduce the following definition for the Zermelo problem.

**Definition 9.** A Zermelo navigation problem on the two-sphere of revolution is a time-minimum problem of the form:

$$\frac{dq}{dt} = F_0(q) + \sum_{i=1}^{2} u_i F_i(q), \quad |u| \leq 1,$$

where the drift representing the current is of the form $F_0(\phi) \frac{\partial}{\partial \phi} + F_1(\phi) \frac{\partial}{\partial \theta}$ while $F_1$ and $F_2$ form outside the equator an orthonormal frame for a metric of the form $g = d\phi^2 + G(\phi) d\theta^2$. It is called reflectionaly symmetric with respect to the equator if

- (H1) $G'(\phi) \neq 0$ on $[0, \pi/2]$
- (H2) $G(\pi - \phi) = G(\phi)$
- (H3) $F_0^1(\pi - \phi) = -F_0^1(\phi), \quad F_0^2 = 0.$

It defines a Finsler geometric problem if $|F_0| < 1$ for the metric $g$.

According to this classification, we have:

**Proposition 12.** Assume $\gamma_- = 0$ and consider the system (10) restricted to the two-sphere:

$$\dot{\phi} = \frac{\sin(2\phi)(\gamma_+ - \Gamma)}{2} + v_2,$$

$$\dot{\theta} = -(\cot \phi)v_1, \quad |v| \leq 1.$$
Then it defines a Zermelo navigation problem on the two-sphere of revolution where the current is $F_0 = \frac{\sin(2\phi)}{2}(\gamma + \Gamma)$, the metric is $g = d\phi^2 + \tan^2 \phi d\theta^2$ with a singularity at the equator and the assumptions (H1), (H2) and (H3) are satisfied. The drift can be compensated by a feedback when $|\gamma + \Gamma| < 2$, which defines a Finsler geometric problem on the sphere minus the equator.

Controllability analysis
The amplitude of the current is $|\sin(2\phi)(\gamma + \Gamma)/2|$ and is maximum in the upper hemisphere for $\phi = \pi/4$, while it is minimum at the north pole and at the equator. Hence, more generally, we deduce the following proposition.

**Proposition 13.** For $|\gamma + \Gamma| > 2$, the current can be compensated in the north equator except in a band centered at $\phi = \pi/4$, hence defining a Finsler geometric problem near the equator and near the north pole.

The controllability analysis is straightforward and is related in the north hemisphere to the scalar equation:

$$\dot{\psi} = -\frac{\sin(2\psi)(\gamma + \Gamma)}{2} - v_2, \quad \psi = \pi/2 - \phi, \quad |v_2| \leq 1.$$  

Starting at $\psi = 0$ with $v_2 = -1$, to increase $\psi$ we meet a barrier corresponding to the singularity of the vector field. For instance, if $\gamma + \Gamma > 0$ then we have a barrier when $1 = \sin(2\psi)(\gamma + \Gamma)/2$, $\psi \in ]0, \pi/2[.$

5 The integrable case of two-level Lindblad equations

5.1 The program
We now proceed to the analysis of the general case of a two-level Lindblad equation. The method is to start from the Grusin case and then to consider perturbations. This program succeeds only if $\gamma_- = 0$, leading to extremal flows described by a family of 2D-integrable Hamiltonian vector fields.

5.2 The integrable case
We observe that for $\gamma_- = 0$, the true Hamiltonian simplifies into:

$$H_r = -\rho(\gamma + \cos^2 \phi + \Gamma \sin^2 \phi)p_\rho + \frac{\sin(2\phi)(\gamma + \Gamma)}{2}p_\phi + \sqrt{p_\phi^2 + p_\theta^2 \cot^2 \phi}$$

and we immediately deduce:

**Proposition 14.** For $\gamma_- = 0$, using the coordinate $r = \ln \rho$, the Hamiltonian takes the form:

$$H_r = -(\gamma + \cos^2 \phi + \Gamma \sin^2 \phi)p_r + \frac{\sin(2\phi)(\gamma + \Gamma)}{2}p_\phi + \sqrt{p_\phi^2 + p_\theta^2 \cot^2 \phi}.$$  

Hence $r$ and $\theta$ are cyclic coordinates and $p_r, p_\theta$ are first integrals of the motion. The system is Liouville integrable.
We have the following geometric interpretation.

**Proposition 15.** For $\gamma_\rightarrow = 0$, the Hamiltonian $H_r$ is associated if $p_r \leq 0$ to the problem of minimization of $r$, while the case $p_r \geq 0$ corresponds to the maximization of $r$, for $|u| = 1$.

*Proof.* It is a consequence of the maximum principle. Another point of view is to consider the end-point mapping $E : u \mapsto q(t, q_0, u)$. If $u$ is restricted to the sphere $|u| = 1$ then the solutions of $H_r$ parameterize the singularities of the end-point mapping. The case $p_r = 0$ corresponds to singularities of the end-point mapping for the system restricted to the two-sphere. In the extremum problem of $r$, with fixed time, $p_r$ can be normalized to -1, 0 or 1, while the level sets are $H_r = h$. In the extremum problem of time, the Hamiltonian is normalized to 0 or 1 for the minimum case, and 0 or -1 for the maximum one. This is clearly equivalent by homogeneity. Hence, this gives a dual point of view.\[\square\]

In order to indicate the complexity of the problem, we consider first the case of energy, which is equivalent to time from Maupertuis principle, in the Grusin case.

### 5.2.1 The case of energy

In the normal case, the true Hamiltonian is

$$H_r = -p_r(\gamma_+ \cos^2 \phi + \Gamma \sin^2 \phi) + \frac{p_\phi \sin(2\phi)}{2}(\Gamma - \gamma_+) + \frac{1}{2}(p_\theta^2 \cot^2 \phi + p_\phi^2).$$

We fix the level set to $h$ and using the relation $p_\phi = \dot{\phi} + \frac{\sin(2\phi)}{2}(\Gamma - \gamma_+)$, one gets

$$\frac{1}{2}\dot{\phi}^2 + V(\phi) = h$$

where $V(\phi)$ is the potential:

$$V(\phi) = -p_r(\gamma_+ \cos^2 \phi + \Gamma \sin^2 \phi) - \frac{1}{2}\frac{\sin^2(2\phi)}{4}(\Gamma - \gamma_+) + \frac{1}{2}p_\theta^2 \cot^2 \phi.$$

To integrate, we use:

$$\frac{d\phi}{dt} = \pm \sqrt{2(h - V(\phi))},$$

and we must evaluate an integral of the form:

$$\int \frac{d\phi}{\sqrt{2(h - V(\phi))}} = \int \frac{dX}{\sqrt{P(X)}}$$

where

$$P(X) = 4(1-X)[-X^3(\Gamma - \gamma_+)^2 + X^2(\Gamma - \gamma_+)2p_r + (\Gamma - \gamma_+)] + X(2h + 2p_r \gamma_+ - p_\theta^2),$$

with $X = \sin^2 \phi$. This corresponds to an elliptic integral.

### 5.2.2 The time-minimum case

In the time-minimum case, the computations of the extremal curves are more intricate because we cannot reduce the system to a second-order differential equation. The geometric framework is however neat because it is associated to a Zermelo navigation problem.
Computations:
We set \( Q = \sqrt{p_\phi^2 + p_\theta^2 \cot^2 \phi} \) and the Hamiltonian is restricted to a level set \( \varepsilon \), where \( \varepsilon = 0 \) corresponds to the abnormal case and \( \varepsilon = +1 \) to the normal case. This gives the following relation:

\[
-(\gamma_+ \cos^2 \phi + \Gamma \sin^2 \phi)p_r + (\gamma_- - \Gamma)\frac{\sin(2\phi)}{2}p_\phi + Q = \varepsilon \tag{12}
\]

and fixing \( p_r \) and \( p_\theta \), the pair \( \phi, p_\phi \) is solution of the system:

\[
\dot{\phi} = \frac{(\gamma_+ - \Gamma)}{2} \sin(2\phi) + \frac{p_\phi}{Q} \\
\dot{p}_\phi = (\Gamma - \gamma_+) \sin(2\phi)p_r + (\gamma_+ - \Gamma) \cos(2\phi)p_\phi + \frac{p_\theta^2 \cos \phi}{Q \sin^3 \phi}. \tag{13}
\]

Hence \( \dot{\phi} = 0 \) leads to:

\[
\frac{(\gamma_+ - \Gamma)}{2} \sin(2\phi)Q + p_\phi = 0. \tag{14}
\]

Using (12), we deduce that \( p_\phi \) is solution of a polynomial equation of degree 2:

\[
p_\phi^2 \left[ (\gamma_+ - \Gamma)^2 \frac{\sin^2(2\phi)}{4} - 1 \right] - (\gamma_+ - \Gamma) \sin(2\phi)[\varepsilon + (\gamma_+ \cos^2 \phi + \Gamma \sin^2 \phi)p_r]p_\phi + [\varepsilon + (\gamma_+ \cos^2 \phi + \Gamma \sin^2 \phi)p_r]^2 - p_\theta^2 \cot^2 \phi = 0. \tag{15}
\]

The discriminant of this polynomial is:

\[
\Delta = 4[\varepsilon + (\gamma_+ \cos^2 \phi + \Gamma \sin^2 \phi)p_r]^2 + p_\theta^2 \cot^2 \phi[(\gamma_+ - \Gamma)^2 \sin^2(2\phi) - 4]. \tag{16}
\]

From (14), we deduce:

\[
p_\phi \left[ (\gamma_+ - \Gamma)^2 \frac{\sin^2(2\phi)}{4} \right] = -(\gamma_+ - \Gamma)^2 \frac{\sin^2(2\phi)}{4} p_\theta^2 \cot^2 \phi. \tag{17}
\]

Hence the set \( (\dot{\phi} = 0) \cap (H_r = \varepsilon) \) is defined by the relation:

\[
[\varepsilon + (\gamma_+ \cos^2 \phi + \Gamma \sin^2 \phi)p_r]^2 = p_\theta^2 \cot^2 \phi[1 - (\gamma_+ - \Gamma)^2 \sin^2(2\phi)] \tag{18}
\]

Therefore, we have:

**Lemma 2.**
1. If \( (\gamma_+ - \Gamma)^2 \frac{\sin^2(2\phi)}{4} - 1 \neq 0 \) and \( \Delta \geq 0 \) then the level set \( H_r = \varepsilon \) has two real roots \( p_\phi \) which are distinct if \( \Delta > 0 \).

2. The intersection of \( \dot{\phi} = 0 \) with the level set \( H_r = \varepsilon \) is given by \( \Delta = 0 \) which can be written:

\[
[\varepsilon + (\gamma_+(1 - X) + \Gamma X)p_r]^2 = \frac{p_\theta^2 (1 - X)}{X}[1 - (\gamma_+ - \Gamma)^2 X(1 - X)] \tag{19}
\]

where \( X = \sin^2 \phi \).
Extremals analysis:

From the previous analysis, we deduce that there are two types of extremal curves by considering the reduced system \[ (13) \] describing the evolution of \((\phi, p_\phi)\).

**Compact case:**

It corresponds to the situation where the level sets \(H_r = \varepsilon\) define compact surfaces in the 2-plane \((\phi, p_\phi)\). In this case, if the reduced system is without singular point on the level set then the trajectory \(t \mapsto (\phi(t), p_\phi(t))\) is a periodic trajectory with period \(T\).

The following lemma is clear.

**Lemma 3.** The Hamiltonian \(H_r\) is invariant for the transformation \((\phi, p_\phi) \mapsto (\pi - \phi, -p_\phi)\).

A consequence of lemma 3 is the following. Assume that for \((p_\theta, p_r)\) fixed, a level set of \(H_r = \varepsilon\) is such that it is compact without singular points and contains both points \((\phi(0), p_\phi(0))\) and \((\pi - \phi(0), -p_\phi(0))\). In this case, the trajectory starting from \((\phi(0), p_\phi^+(0))\) with \(p_\phi^+(0) > 0\) is periodic of period \(T\) and has a second crossing at the antipodal point \(\pi - \phi(0)\) after a time \(T/2\). It is chased by a trajectory starting from \((\phi(0), p_\phi^-(0))\) where \(p_\phi^+\) and \(p_\phi^-\) are roots of \([13]\) for \(\phi = \phi(0)\) with a time delay of \(T\) and reaches the antipodal point \(\pi - \phi(0)\) at time \(T/2\).

Moreover, since the equations describing the evolutions of the remaining variables \((r, \theta)\) are invariant for the central symmetry: \((\phi, p_\phi) \mapsto (\pi - \phi, -p_\phi)\), we deduce that after half a period \(T/2\), we have, for the two extremals starting respectively from \((\phi(0), p_\phi^+(0))\) and \((\phi(0), p_\phi^-(0))\) while \((r(0), \theta(0))\) are identical, the relations:

\[
r^+(T/2) = r^-(T/2), \quad \theta^+(T/2) = \theta^-(T/2).
\]

This generalizes the analysis of section 4.2 for Riemannian metrics on the two-sphere, except that we replace the reflectional symmetry by a central symmetry, leading to the fact that half a period instead of a quarter of period is necessary to construct the extremals.

We proved:

**Proposition 16.** If for fixed \((p_r, p_\theta)\), the level set \(H_r = \varepsilon\) is compact without singular point and has a central symmetry with respect to \((\phi = \pi/2, p_\phi = 0)\), then it contains a periodic trajectory \((\phi, p_\phi)\) of period \(T\) and if \(p_\phi^+(0)\) are distinct then we have two distinct extremal curves \(q^+(t), q^-(t)\) starting from the same point and intersecting with the same length \(T/2\) at a point such that \(\phi(T/2) = \pi - \phi(0)\).

**Non-compact case:**

Non-compact level sets occur when \(p_\phi \to \infty\). Using \([13]\), the relation \(\dot{\phi} = 0\) gives the set \(S\) of solutions \(\phi_S\) of

\[
(\gamma_+ - \Gamma)^2 \sin^2(2\phi) - 1 = 0
\]

and we must have \(|\Gamma - \gamma_+| \geq 2\). We deduce the following proposition:

**Proposition 17.** If \(|\Gamma - \gamma_+| \geq 2\) then we have extremal trajectories such that \(\phi\) is not periodic, i.e., \(\phi \to 0\), \(\phi \to \phi_S\) and \(p_\phi \to \pm \infty\) when \(t \to +\infty\) while \(\dot{\theta} \to 0\) outside the equator.
Geometric interpretation:
The two types of extremals are related to the Zermelo navigation problem of section 4.5.

- If $|\gamma_+ - \Gamma| < 2$ then the system restricted to the two-sphere defines a Finsler geometric problem for which the extremals of the Grusin case are deformed into extremals described in proposition [16].
- If $|\gamma_+ - \Gamma| \geq 2$ then we have two types of extremals: periodic extremals occur in a band near the equator and non periodic extremals occur when crossing the band around $\phi = \pi/4$ where the current is maximal and asymptotic properties of proposition [17] correspond to the barrier phenomenon.

This will be clarified by the pictures of the final section.

Small time optimal analysis at the equator:
We make a singularity analysis of the extremals near a point of the equator which can be identified to $\phi = \pi/2$ and $\theta = 0$. Following the techniques of [5], the main point is to construct a normal form in order to compute the small time optimal synthesis.

Proposition 18. Near the equator, the small time optimal synthesis is given by the one near $(x, y, z) = (0, 0, 0)$ for the system:

\begin{align*}
\dot{x} &= 1 + \frac{(\gamma_+ - \Gamma)}{\Gamma} y^2 \\
\dot{y} &= (\Gamma - \gamma_+) y + u_2 \\
\dot{z} &= y u_1.
\end{align*}

Proof. The computation is straightforward. We set $\psi = \pi/2 - \phi$ and near $\psi = 0$, we have

\[ \dot{r} \simeq -\Gamma + (\Gamma - \gamma_+) \psi^2, \quad \dot{\psi} \simeq (\Gamma - \gamma_+) \psi - u_2, \quad \dot{\theta} \simeq -\psi u_1. \]

This gives the normal form by setting $x = -r/\Gamma$, $y = \psi$, $z = \theta$ and using a feedback transformation preserving $|u| \leq 1$ and changing $(u_1, u_2)$ into $(-u_1, -u_2)$.

This normal form preserves the integrability property and the optimality analysis amounts to approximate the Hamiltonian $H_r$ near an equatorial point. Moreover, by homogeneity, it describes the extremal behaviors not only near 0 but also along an equatorial line identified to $(t, 0, 0)$.

From this normal form, we can deduce:

Proposition 19. For the system (22), near 0, every small time optimal trajectory is either

1. A trajectory in the meridian plane $z = 0$ with $u_1 = 0$ such that:
   (i) If $\gamma_+ - \Gamma < 0$, an arc of the form BSB, i.e., a concatenation of a bang arc $u_2 = \pm 1$, a singular arc and a bang arc,
   (ii) If $\gamma_+ - \Gamma > 0$, a bang-bang arc BB,
2. or a smooth extremal of order 0 with $p_z \neq 0$. 


Proof. First of all, we analyze extremal curves contained in the plane \( z = 0 \) for which \( u_1 = p_z = 0 \). They correspond to the time-optimal control problem in the 2D-plane \((x, y)\) with \(|u_2| \leq 1\). They are already analyzed in [17] using techniques of [13], but the computed normal form reveals also the properties of the control. The line \( S : t \mapsto (t, 0) \) is a singular trajectory and the corresponding control is \( u_2 = 0 \). Moreover, this line is time minimizing if \( \gamma + \Gamma < 0 \), while it is time maximizing if \( \gamma + \Gamma > 0 \). Other extremals are defined by \( u_2 = \text{sign}(p_y) \).

In case (i), a bang arc can have a connection with the singular arc, with a contact of order two with the switching surface. In contrast, the connection is not possible in case (ii). To complete the analysis, we observe that in case (i), every extremal curve which is of the form BSB is optimal, while in the case (ii), every extremal curve is bang-bang but for optimality, we have at most one switching.

In order to conclude the analysis, we must prove that an extremal of order 0 with \( p_z \neq 0 \) cannot reach the singularity \( y = p_y = 0 \). The Hamiltonian \( H_r \) takes the form

\[
H_r = p_x [1 + \left( \frac{\gamma + \Gamma}{\Gamma} \right) y^2] + p_y (\Gamma - \gamma) y + \sqrt{p_y^2 + p_z^2 y^2}.
\]

\( H_r = \varepsilon \), with \( \varepsilon = 0, 1 \), gives the condition:

\[
p_x \frac{\gamma - \Gamma}{\Gamma} y^2 + p_y (\Gamma - \gamma) y + \sqrt{p_y^2 + p_z^2 y^2} = 0.
\]

This is clearly not possible, taking the Taylor expansions of \( y, p_y \):

\[
y(t) = at + o(t), \quad p_y(t) = bt + o(t)
\]

of an extremal of order 0, with \( p_z \neq 0 \), reaching or departing from the singularity. The result is proved.

As a corollary, we have the following result to determine the cut-locus in the case \( |\gamma + \Gamma| < 2 \) where the time-minimal control problem is related to a Finsler problem.

**Corollary 1.** Let \( q^+(t) \) and \( q^-(t) \) be two distinct extremals of order 0 with non-zero \( p_y \) and intersecting with the same time \( T \). Then they cannot be optimal beyond the intersecting point.

*Proof.* The proof is standard. Assuming optimality beyond the intersecting point, we can construct a broken minimizer which is an extremal of order 0 with non-zero \( p_y \). This contradicts proposition 19.

5.2.3 Numerical computations

We complete the analysis of section 5 by a series of numerical computations on extremal trajectories and on conjugate loci (see section 2). All the computations are done in the case of the time-minimum control problem but could be equivalently done for the energy minimization problem. We use numerical methods described in [7]: an extremal is computed and we calculate along this trajectory the conjugate points.

**Extremal trajectories:**
We compute extremal trajectories for different values of the dissipative parameters $\Gamma$ and $\gamma_+$. We represent in figures 1 and 2 in the coordinates $(\theta, \phi)$ the projection of the trajectories on the sphere of radius 1. The variation of the radial coordinate $\rho$ can be deduced straightforwardly and depends on the value of $\Gamma$ and $\gamma_+$. Extremals can also be plotted in the plane $(\phi, p_\phi)$ by using equation (12). In figure 1 two trajectories intersecting with the same cost on the antipodal parallel are plotted. All other extremal trajectories in the case $|\Gamma - \gamma_+| < 2$ have the same qualitative behavior. For the Grusin model on the sphere, we also recall that the two trajectories which intersect on the cut locus have opposite initial values of $p_\phi$, $\varepsilon$ and $p_\theta$ being fixed. This is no more the case when $\Gamma \neq \gamma_+$ since these initial values depend now on $\Gamma$ and $\gamma_+$. The extremal trajectories for $|\Gamma - \gamma_+| > 2$ are displayed in figure 2. We observe two types of trajectories, the periodic and the aperiodic ones. The periodic extremals have the same behavior as the ones of the case $|\Gamma - \gamma_+| < 2$. When $t \to +\infty$, the aperiodic extremals have an asymptotic limit $\phi_S$ in $\phi$ solution of the equation (21). For given values of $p_\theta$ and $\varepsilon$, this equation has two solutions symmetric with respect to the equator. As can be seen in figure 2, we have found two types of aperiodic extremals: trajectories monotone in $\phi$ which do not cross the parallel $\phi = \phi_S$ and trajectories passing by a maximum or a minimum in $\phi$ different from $\phi_S$ and crossing this parallel. These different types of behaviors can be determined by using equations (17) and (18).

**Conjugate locus:**

![Figure 1: Extremal trajectories for $\Gamma = 2.5$ and $\gamma_+ = 2$. Other parameters are taken to be $p_\theta(0) = -1$ and 2.33, $\phi(0) = \pi/4$, $p_\rho = 1$ and $p_\theta = 2$. Dashed lines represent the equator and the antipodal parallel located at $\phi = 3\pi/4$.](image)

We next determine numerically the conjugate locus for different set of parameters. We restrict the discussion to the case $|\Gamma - \gamma_+| < 2$. A similar study can be done for periodic trajectories if $|\Gamma - \gamma_+| \geq 2$. For aperiodic trajectories, it can be shown that they are locally optimal in the sense that they do not have conjugate points for $t \in [0, +\infty]$. Different conjugate loci are represented in figures 3, 4 and 5 both for the Grusin model and for a deformation of this model with the constraint $|\Gamma - \gamma_+| < 2$. The conjugate locus of the Grusin model on the sphere is given in [6]. Here, in the case $\Gamma = \gamma_+$, the drift vector...
Figure 2: Extremal trajectories for $\Gamma = 4.5$ and $\gamma_+ = 2$. Dashed lines represent the equator and the locus of the fixed points of the dynamics given by equation (21). The solid line corresponds to the antipodal parallel. Numerical values of the parameters are taken to be $\phi(0) = 2\pi/5$, $p_\theta = 8$ and $p_r(0) = 0.25$. The different initial values of $p_\phi$ are -50, -10, 0, 2.637, 3, 5, 10 and 50.

$F_0$ being purely radial, the projection of the conjugate locus is the same as the conjugate locus of the Grusin model on the sphere. In particular, this projection is independent of the value of $p_r$. Figure 3 displays both this conjugate locus and some extremal trajectories. Starting from the Grusin model, we can then modify the difference $\Gamma - \gamma_+$ and determine the corresponding deformation of the conjugate locus. The evolution of the radial component being more complicated (the radial coordinate is no more decoupled from other coordinates if $\Gamma \neq \gamma_+$), the projection of the conjugate locus depends on the value of $p_r$. A first comparison between the two cases is given by figure 4 where it can be seen that the global structure of the extremals is nearly the same. Figure 5 displays the projection of a conjugate locus on the plane $(\theta, \phi)$ corresponding to a particular value of $p_r$. The conjugate locus of the Grusin model has been added for the sake of comparison. We note that this locus is only slightly modified when $\Gamma$ and $\gamma_+$ vary. The trajectories of Fig. 4 are plotted in Fig. 6 up to the first conjugate point.

Acknowledgments

Agence Nationale de la recherche (ANR project CoMoc).

References

[1] C. Altafini, Controllability properties for finite dimensional quantum Markovian master equations, J. Math. Phys. 44, 2357-2372 (2002).

[2] A. Agrachev, U. Boscain and M. Sigalotti, A Gauss-Bonnet like formula on two-dimensional almost Riemannian manifolds, Discrete Contin. Dyn. Syst. 20 (2008), n4, 801-822.
Figure 3: Extremal trajectories for the Grusin model corresponding to $\Gamma = \gamma_+ = 2$. The conjugate and cut loci are represented in dashed lines. The cut locus in this case is the envelope of the extremal curves. Numerical values are taken to be $p_\theta = 2$ and $p_r(0) = 0.5$ and $\phi(0) = \pi/4$.

[3] D. Bao, C. Robles and Z. Shen, Zermelo navigation on Riemannian manifolds, J. Differential Geom. 66 (2004), n3, 377-435.

[4] B. Bonnard and J.-B. Caillau, Riemannian metric of the averaged energy minimization problem in orbital transfer with low thrust, Ann. Inst. Henri Poincaré (Analyse non linéaire) 24, 395-411 (2007)

[5] B. Bonnard and J.-B. Caillau, Geodesic flow of the averaged controlled Kepler equation, Forum Mathematicum (2008).

[6] B. Bonnard, J.-B. Caillau, R. Sinclair and M. Tanaka, Conjugate and cut loci of a two-sphere of revolution with application to optimal control, to be published in Ann. Inst. H. Poincaré, Analyse non linéaire (2008).

[7] B. Bonnard, J.-B. Caillau and E. Trélat, Second-order optimality conditions in the smooth case and applications in optimal control, ESAIM: COCV 13, no. 2, 207-236 (2007)

[8] B. Bonnard and M. Chyba, Singular trajectories and their role in control theory, Math. and Applications 40, Springer-Verlag, Berlin (2003)

[9] B. Bonnard, M. Chyba and D. Sugny, Time-minimal control of dissipative two-level quantum systems: the Generic case, submitted to IEEE TransAC (2008)

[10] B. Bonnard and D. Sugny, Optimal control theory with applications in space and quantum dynamics, submitted (2008).

[11] U. Boscain, T. Chambrion and J.-P. Gauthier, On the $K+P$ problem for a three-level quantum system: Optimality implies resonance, J. Dyn. and Control Syst. 8, 547-572 (2002)
Figure 4: Extremal trajectories for $\Gamma = 2.5$ and $\gamma_+ = 2$. The projection of conjugate locus is represented in dashed lines. The horizontal dashed line is the line where two trajectories intersect with the same length. Numerical values for the parameters are taken to be $\phi(0) = \pi/4$, $p_r = 0.5$ and $p_\theta = 2$.

[12] U. Boscaïn, G. Charlot, J.-P. Gauthier, S. Guérin and H. R. Jauslin, *Optimal control in laser induced population transfer for two and three-level quantum systems*, J. Math. Phys. 43, 2107-2132 (2002)

[13] U. Boscaïn and B. Piccoli, *Optimal syntheses for control systems on 2-D manifolds*, Springer-Verlag, Berlin (2003).

[14] C. Carathéodory, *Calculus of variations and partial differential equations of first order*, Chelsea Publishing Company, New-York (1982).

[15] E. Lee and L. Markus, *Foundations of optimal control theory*, John Wiley, New York 1967.

[16] S. G. Schirmer and T. Zhang and J. V. Leahy, *Orbits of quantum states and geometry of Bloch vectors for N-level systems*, J. Phys. A, 37, 1389-1402 (2004).

[17] D. Sugny, C. Kontz and H. R. Jauslin, *Time-optimal control of a two-level dissipative quantum system*, Phys. Rev. A, 76, 023419 (2007).
Figure 5: Projection of conjugate locus in solid line for $p_r = 0.5$. The conjugate locus of the Grusin model corresponding to $\gamma_+ = \Gamma = 2$ is represented in dashed lines. The horizontal dashed line indicates the position of the cut locus for the Grusin model. Dissipative parameters are taken to be $\Gamma = 2.5$ and $\gamma_+ = 2$, $p_\theta$ is equal to 2.

Figure 6: Same as Fig. 4 but the trajectories are plotted up to the first conjugate point.