Comments on Noncommutative Gauge Theories

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Abstract

We study the gauge theories on noncommutative space. We employ the idea of the covariant position to understand the linear and angular momenta, the center of mass position, and to express all gauge invariant observables including the Wilson line. In addition, we utilize the universality of the $U(1)$ gauge theory, which originates from the underlying matrix theory, to analyze various solitons on $U(N)$ theories, like the unstable static vortex solutions in two dimensions and BPS dyonic fluxon solutions.

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1 Introduction

Recently the classical properties of noncommutative gauge theories have been investigated in many directions[1]-[17]. There are several interesting issues raised recently on the noncommutative gauge theories. First issue is about the gauge invariant observables. Gross, Hashimoto and Itzhaki showed that for any gauge covariant local quantity, one can construct the gauge invariant quantity by taking the trace over space with the open Wilson line[4]. Second concerns about the $U(1)$ universality as noted by Gross and Nekrasov[5]. It seems that it is possible to everything about the $U(N)$ gauge theory can be found in the $U(1)$ theory.

In this letter, we elaborate these issues. We introduce a covariant position operator and show that the gauge invariant operators constructed in Ref. [4] are the “Fourier transformation” of a given operator with respect to the covariant position operator. Then we elaborate the $U(1)$ universality and argue that the underlying matrix theory is the reason behind it. Finally, we explore this universality in some concrete examples. The physics of $M$ D0 branes on $N$ D2 branes is explored. Also we provide the solution of $M$ fluxons in the $U(N)$ theory. Our discussion will be about the classical aspect of the noncommutative gauge theory.

Some of points made in this paper are not quite new, but presented as they illuminate and emphasize the different aspect of the similar ideas. The Wilson line observables have been introduced by Ishibash, Iso, Kawai, and Kitazawa before[3]. GHI elaborated this idea and extended to include other operators[4]. In this paper, we short-circuit this idea by introducing the covariant position operator $X_i = x_i + \theta \epsilon_{ij} A_j$, which is related to the covariant differential operators $D_j$ by $X_i = i\epsilon_{ij} \theta D_j$. Then, the observables constructed by IIKK and GHI are just the Fourier transformation of a given local covariant operator with the covariant position operator. It is much simpler to think this way. Especially one can define even a delta function on noncommutative space, which would lead to almost localized gauge invariant observables.

The idea of the gauge covariant position operator appears also naturally when one considers the conserved linear and angular momenta, the center of mass, etc. The gauge invariant angular momentum density in the standard field theory is $\epsilon_{ij} x_i T_{0j}$. One may wonder what is the corresponding formula in the noncommutative gauge theory as the energy momentum tensor is just gauge covariant, not invariant. The gauge covariant position operator appears naturally instead of ordinary position operators. Similarly, the center of the mass position can be defined with the covariant position operators. When there is no matter in the fundamental representation, the translation and rotational transformations of matter and gauge fields are gauge equivalent to just a transformation of the gauge field. This observation allows to reexpress the linear and angular momenta in only the gauge field or the covariant position.

GN have shown that the $U(1)$ gauge theory on the noncommutative plane with adjoint matters only has many gauge nonequivalent vacua characterized by a natural number $N$, and have argued that they correspond to the vacua of the $U(N)$ gauge theory[6]. In addition, they argue that all $U(N)$ theories are a part of the $U(1)$ theory. There is a good reason behind this universality. We elaborate this point of view more carefully. Basically the degrees of freedom do not change, but one can divide, or regroup the degrees of freedom living on a space into some on a new space and some on internal space. It is a sort of regrouping the degrees of freedom of the $U(1)$ theory. We elaborate this in some detail. Of course the underlying reason for this is that there is a hidden matrix mechanics model, which leads to all $U(N)$ theories. The covariant position operators discussed above are those appear in the matrix theory naturally.
Due to this universality, one can write the solutions of the $U(N)$ gauge theories in the $U(1)$ theory more conveniently. We find the mass of the $M \times N$ tachyonic modes around this solution. The fluxon solution on the $U(1)$ theory is a composite of two BPS monopoles of opposite magnetic charge. We find the $M$ dyonic fluxon solutions in the $U(N)$ gauge theory which are 1/2 BPS.

The plan of this paper is as follows: In Section 2, we review the noncommutative gauge theory and introduce the covariant position operator. In Section 3, we review the $U(1)$ universality. In Section 4, we provide the solutions of the field equation in two and three dimensions.

2 The Covariant Position Operator

We consider the gauge theory on noncommutative plane whose coordinates are $(x, y)$. The coordinates $x, y$ satisfy the relation

$$[x, y] = i\theta$$

with $\theta > 0$. This noncommutative plane has not only the translation symmetry but also the rotational symmetry. One can see that the parity operation $(x, y) \rightarrow (x, -y)$ is broken on noncommutative plane. The classical field on this noncommutative space is an element $\Phi(t, x, y)$ in the algebra $A_\theta$ defined by $x, y$ with the relation (1).

Let us introduce the complex coordinates of the noncommutative space,

$$c = \frac{1}{\sqrt{2\theta}}(x + iy), \quad \bar{c} = \frac{1}{\sqrt{2\theta}}(x - iy),$$

which satisfy $[c, \bar{c}] = 1$. This commutation relation is that of the creation and annihilation operators for a simple harmonic oscillator and so one may use the simple harmonic oscillator Hilbert space $\mathcal{H}$ as a representation of (1). The ground state is $|0\rangle$ such that $c|0\rangle = 0$, and $|n\rangle$ is the usual number eigenstate. The integration over noncommutative two plane becomes the trace over its Hilbert space, which respects the translation symmetry:

$$\int d^2x \mathcal{O}(x) \rightarrow \tilde{\text{Tr}}\mathcal{O}(x) \equiv 2\pi\theta \sum_{n \geq 0} \langle n|\mathcal{O}|n\rangle.$$  

We consider mostly the gauge theory with the matter fields $\phi(t, x, y)$ in the adjoint representation. The gauge fields on the space are $A_\mu(t, x, y)$. We denote a matter field in the fundamental representation $\psi$. Because of the commutation relation, the derivative along the noncommutative coordinate of a function becomes $\partial_i \phi = [\hat{\partial}_i, \phi]$ with

$$\hat{\partial}_i = \frac{i}{\theta} \epsilon_{ij} x_j,$$

where $\epsilon_{12} = 1$, $x^1 = x$ and $x^2 = y$. For the covariant derivatives of the matter fields are defined as $\nabla_i \phi = [D_i, \phi]$, and $\nabla_i \psi = D_i \psi - \bar{\psi} \hat{\partial}_i$, where

$$D_i = \hat{\partial}_i - iA_i = \frac{i}{\theta} \epsilon_{ij} x_j - iA_i.$$
The field strength on the noncommutative space is given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$. The local gauge symmetry is given by a unitary transformation $U(x) \psi \to U \psi$, $\phi \to U \phi U^\dagger$, and $A_i \to U A_i U^\dagger - i[\hat{\partial}_i, U] U^\dagger$. Under this local gauge transformation, the operator $D_i$ transforms covariantly, and so are the covariant derivatives of the fields.

The Lagrangian for the gauge theory with the adjoint matter is

$$L = \frac{1}{e^2} \text{Tr} \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \nabla_\mu \phi^I \nabla^\mu \phi^I + \frac{1}{4} \sum_I [\phi^I, \phi^I]^2 \right\} .$$

(6)

While one can choose the matter potential arbitrarily, we choose the theory so that it is a dimensional reduction of the ten dimensional Yang-Mills theory. The gauge field $A_0$ is the Lagrangian multiplier and implies the Gauss law constraint on the initial data,

$$[D_i, E_i] - i[\phi^I, \nabla_0 \phi^I] = 0 ,$$

(7)

where $E_i = F_{0i} = i \nabla_0 D_i$. The local gauge transformation $U = e^{iA}$ is indeed an $U(\infty)$ operator on the Hilbert space as $\Lambda(x) = \sum_{mn} \Lambda_{mn} |m\rangle \langle n|$. 

This theory has several gauge invariant observables. When there is no matter field in the fundamental representation, we take the trace over all Hilbert space to get the gauge invariant operators.

IIKK introduced the gauge invariant observables by using the open Wilson line[3]. GHI found some generalization by incorporating other covariant quantities[4].

In this paper, we give a new spin to this idea by introducing a gauge covariant position operator

$$X_i = i \theta \epsilon_{ij} D_j = x_i + \theta \epsilon_{ij} A_j .$$

(8)

Clearly in the commutative space limit $\theta = 0$, it is a position operator. We will also see that this operator is the natural operator from the matrix theory point of view. This position operator can be used to identify the position of a given soliton. As $B = F_{12} = -\theta \left(1 + \frac{i}{2} [X^1, X^2] \right)$ and $E_i = F_{0i} = -\frac{i}{\theta} \epsilon_{ij} \left[ \dot{X}^j - i[A_0, X^j] \right]$, the Lagrangian can be written completely with covariant quantities $X^i$ and $\phi^I$. This is one of the most striking aspects of the noncommutative gauge theory.

The covariant open straight Wilson line, which was introduced by IIKK, can be put as

$$W(\alpha) = e^{-\alpha t \hat{\partial}_i} P \exp(i \int_0^\alpha A_i(x + \beta l) t^i d\beta) ,$$

(9)

which transforms under the local gauge transformation as $W(\alpha) \to U(x) W(\alpha) U(x)^\dagger$. From the differential equation satisfied by $W(\alpha)$ and its initial value $W(0)$, we see that it is identical to

$$W(\alpha) = e^{-\alpha t D_i} = e^{ip_i X^i} ,$$

(10)

where $p_i = -\alpha \epsilon_{ij} U^j / \theta$. The trace of the Wilson line with any covariant operator leads to the gauge invariant observable. In our words, it is basically then the Fourier transformation of the covariant local operator $O(x)$ in terms of the covariant position operator,

$$O(p) = \text{Tr} \left( e^{ip_i X^i} O(x) \right) .$$

(11)

In our theory any covariant local operator $O(x)$ would be a function of $X^i(x)$ and $\phi^I(x)$ and so $O(X^i(x), \phi^I(x))$. Any function $f(X^i, \phi^I)$ is covariant and so one can define the weighted invariant
quantities $O_f = \tilde{\text{Tr}} f(x)O(x)$. This provides a much broader class of gauge invariant observables. For example, we define a covariant delta function on the noncommutative plane as

$$\delta^2(X^i - q^i) = \int \frac{d^2p}{(2\pi)^2} e^{ip(X^i - q^i)}. \quad (12)$$

Then one can find a commutative number value by $\tilde{\text{Tr}} \delta^2(X^i - q^i)O(x)$ due to the operator ordering choice there can be several definitions of the delta function which are equivalent in the commutative limit $\theta = 0$. Thus, the above quantity does not measure the localized quantity on the noncommutative space, but a sort of average over a cell of area size $2\pi\theta$ around the point $q$. For the fundamental matter field $\psi$, we can find a gauge invariant operators like $\bar{\psi}(x)O(x)\psi(x)$, which needs no trace over space.

Among the gauge invariant observables, some of the most prominent ones are those related to the symmetries. From the Noether theorem, one can find the conserved quantity for each symmetry. The time translation symmetry leads to the conserved energy $H = \tilde{\text{Tr}} T_{00}$ to be

$$H = \frac{1}{2e^2} \tilde{\text{Tr}} \left\{ E_i^2 + B^2 + (\nabla_0 \phi^I)^2 + (\nabla_i \phi^I)^2 - \sum_{I<J} [\phi^I, \phi^J]^2 \right\}. \quad (13)$$

The space translation symmetry $T = e^{-l^k \hat{\partial}_k}$ which transforms $\phi^I \to T \phi^I \tilde{T}$, $A_i \to T A_i \tilde{T}$, also leads to the conserved linear momentum $P_i = \tilde{\text{Tr}} T_{0i}$ to be

$$P_i = \tilde{\text{Tr}} \left\{ \epsilon_{ij} E_j + \nabla_0 \phi^I \nabla_i \phi^I \right\}. \quad (14)$$

As noted by many, the translation on noncommutative plane is gauge equivalent to the shift of the gauge field only, $\phi^I \to \phi^I$, and $A_i \to A_i - \epsilon_{ij} q^j / \theta$ when only adjoint matters are present. This symmetry leads to the conserved quantity

$$P_i = -\frac{1}{\theta^2} \tilde{\text{Tr}} \nabla_0 X_i, \quad (15)$$

which is identical to the previous one modulo the Gauss law.

The rotational symmetry leads to the conserved angular momentum. However, here the subtlety appears for the noncommutative space. Under the infinitesimal rotational transformation $\delta \phi^I = \epsilon_{jk} x_j [\hat{\partial}_k, \phi^I]$ and $\delta A_i = \epsilon_{jk} x_j [\hat{\partial}_k, A_i] + \epsilon_{ij} A_j$, the Noether theorem leads to the conserved angular momentum $J = \epsilon_{ij} \tilde{\text{Tr}} X^i T_{0j}$ to be

$$J = \epsilon_{ij} \tilde{\text{Tr}} X_i \left( \epsilon_{jk} E_k B + \nabla_0 \phi^I \nabla_j \phi^I \right). \quad (16)$$

Here we have written the density in terms of the gauge covariant quantities, discarding the boundary terms (or the commutator terms). Note that the covariant position operator appears in this expression. Again the rotational symmetry is gauge equivalent to the transformation of the gauge field only, $\delta A_i = \frac{1}{\theta} X_i$. This leads to the conserved angular momentum

$$J = -\frac{1}{\theta^2} \epsilon_{ij} \tilde{\text{Tr}} X_i \nabla_0 X_j, \quad (17)$$

which is identical to the previous one modulo the Gauss law.
For localized field configurations, with the covariantly conserved energy $\mathcal{E}$, not only can we define their energy, but also we can define the moments of the covariant positions. For example, the center of mass position is defined as

$$R_{cm}^i = (\tilde{\text{Tr}} X^i \mathcal{E})/(\tilde{\text{Tr}} \mathcal{E}).$$

The moment of inertia would be $I = \tilde{\text{Tr}} X^i X^i \mathcal{E}$.

### 3 The U(1) Universality

One can easily generalize the $U(1)$ gauge theory Lagrangian to the $U(N)$ theory Lagrangian with the gauge field $A_{N_i}$ and adjoint matter field $\phi_N$, which are $N$ by $N$ hermitian matrices. The spatial variable for the $U(N)$ theory to be $x_N, y_N$. Gross and Nekrasov noticed that this theory is a part of the $U(1)$ theory studied before. In this section, we elaborate this in detail and see the origin of this be the well known matrix theory. The key reason behind this universality is that the Lagrangian can be written in terms of covariant $X^i$ and $\phi^I$.

To start, let us find the vacua of the $U(1)$ gauge theory on the noncommutative space. We rewrite the gauge field $A_i$ in the the complex coordinate

$$A \equiv A_1 + iA_2 = -i\sqrt{\frac{2}{\theta}}(c - C),$$

where $C = (X^1 + iX^2)/\sqrt{2\theta}$. The magnetic field is then $B = ([C, \bar{C}] - 1)/\theta$. The vacua of the theory will have the zero field strength. Especially, the magnetic field should vanish. The most general solution of this constraint has been found. To write that, we regroup the states in the Hilbert space for any natural number $N$ as

$$|p, \alpha\rangle = \sqrt{\sum_{p=0}^{\infty} \sum_{\alpha=0}^{N-1} \sqrt{p+1}|p,\alpha\rangle\langle p+1,\alpha|},$$

where $p = 0, 1, 2, ...$ and $\alpha = 0, 1, ..., N - 1$. The general solution of the zero magnetic field strength modulo gauge transformations is $C = c_N$, where $c_N$ and $\bar{c}_N$ are annihilation and creation operators on index $p$ so that

$$c_N = \sum_{\alpha=0}^{N-1} \sum_{p=0}^{\infty} \sqrt{p+1}|p,\alpha\rangle\langle p+1,\alpha|.$$ (21)

The natural number $N$ counts the dimension of the kernel of the operator $\bar{C}C$. (This is also noted by Gross and Nekrasov recently.) The vacuum gauge field is then

$$A = -i\sqrt{\frac{2}{\theta}}(c - c_N).$$

Thus the natural number $N$ denotes the gauge inequivalent vacua. To understand the meaning of $N$, we reexpress all the covariant quantities, $D_i$, $X^i$, $B$, $E_i$, $\phi^I$ and $A_0$ as $U(N)$ quantities. For example,

$$\phi^I = \sum_{mn} \phi^I_{mn} |m\rangle\langle n| = \sum_{\alpha\beta} \sum_{pq} \phi^I_{pq,\alpha\beta} |p,\alpha\rangle\langle q,\beta|. $$

Now we introduce the complete set of the $N$ by $N$ hermitian matrices $T^a$, $a = 1, ..., N^2$, which are generators of the $U(N)$ group and express all the coefficient of the quantities in terms of these
matrices, e.g. \( \phi_{pa} e^{\beta} = \sum_{a} \phi_{pq} T_{a\beta} \). Also we reexpress the local \( U(1) \) gauge transformation, \( U = e^{i \Lambda(x,y)} \), as \( \Lambda = \sum_{a} \Lambda_{pq} T_{a\beta} |p,\alpha\rangle \langle q,\beta| \). When we regard \( p, q \) indices as those for the noncommutative space and \( \alpha, \beta \) indices as the internal \( U(N) \) gauge symmetry, every covariant expression of the \( U(1) \) theory quantity is reexpressed as one of the \( U(N) \) gauge theory. However the noncommutative coordinates \( x_N, y_N \) of the \( U(N) \) theory should change only \( p, q \) indices not \( \alpha \) and \( \beta \) indices. With \( x_N, y_N \) defined by the \( U(N) \) theory are then

\[
x_N = \sqrt{\frac{\theta}{2}} (c_N + \bar{c}_N), \quad y_N = -i \sqrt{\frac{\theta}{2}} (c_N - \bar{c}_N),
\]

we have \([x_N, y_N] = i \theta\). Letting the covariant position operator \( X^i \) identical in both theories, we define the gauge field \( A_{Ni} \) of the \( U(N) \) theory as

\[
A_{Ni} = \frac{1}{\theta} \epsilon_{ij} (x_N^j - X^j).
\]

The \( N \)-th vacuum \((22)\) of the \( U(1) \) theory becomes the trivial vacuum in the \( U(N) \) theory, indicating that the \( N \)-th vacuum is indeed the conventional vacuum of the \( U(N) \) theory. Its gauge field strength \( F_{N12} \) is covariant and so is identical to that from the \( U(1) \) theory.

Putting them in the Lagrangian and taking the trace over \( |\alpha\rangle \langle \beta| \) states we end up with trace over \( |p\rangle \langle q| \), which we call \( \mathcal{Tr} \) and the \( N \times N \) matrix trace \( tr_N \). Then the Lagrangian becomes

\[
L = \frac{1}{e^2} \mathcal{Tr} \mathcal{Tr}_N \left\{ -\frac{1}{2} F_{\mu\nu} F^\mu_N - \frac{1}{2} \nabla_\mu \phi^I_N \nabla_\nu \phi^I_N + \frac{1}{4} \sum_I \phi^I_N \phi^I_N \right\}.
\]

Thus, the \( U(N) \) theory is a part of the \( U(1) \) theory.

This universality comes about as we regroup the degrees of freedom on the \( x - y \) plane of the \( U(1) \) theory to those for the \( x_N - y_N \) plane of the \( U(N) \) theory and those for the internal space. This regrouping goes both ways. We can reverse the argument given here to construct the \( U(1) \) theory out of the \( U(N) \) theory. Note that the universality works only for gauge theories.

The above universality still holds when there are additional spatial dimensions. One needs only one noncommutative plane for such regrouping. If there are more noncommutative planes, there would be many nonequivalent regrouping of the degrees of freedom.

The \( U(N) \) gauge theory on two dimensional noncommutative space is supposed to describe the \( N \) D2 brane dynamics with the background \( B_{12} \) field in the field theoretic limit \([1]\). Thus, this universality seems to be mysterious. However there is a underlying matrix theory where \( D0 \) branes are basic constituents and \( D2 \) branes are composite \([12, 13]\). The underlying reason for the universality is that there is a matrix theory behind all \( U(N) \) theories. This mechanics model is a theory with \( U(\infty) \) symmetry. The bosonic part of the theory is

\[
L = \frac{1}{2g_s} \mathcal{Tr} \left\{ \sum_{M=1}^{9} (\mathcal{X}^M - i[\mathcal{X}^0, \mathcal{X}^M])^2 + \sum_{M < N} [\mathcal{X}^M, \mathcal{X}^N]^2 \right\},
\]

where \( M, N = 1, \ldots, 9 \) and \( \mathcal{Tr} \) is the ordinary trace.

With the coupling \( g_s = e^\theta/(2\pi\theta) \), the dynamics around a single D2 brane can obtained by \( \mathcal{X}^i = \frac{1}{x_i} \mathcal{X}^i + \epsilon_{ij} A_j = \frac{1}{x_i} X^i \) \((i = 1, 2)\), \( \mathcal{X}^{i+2} = \phi^I \) \((I = 1, 2, \ldots, 7)\), and \( \mathcal{X}^0 = A_0 \). The matrix Lagrangian becomes the \( U(1) \) field theory Lagrangian above plus an infinite constant term and a topological
term. This matrix theory has also the solution for $N$ D2 branes and the dynamics around this background is the noncommutative $U(N)$ gauge theory. Our universality is then the manifestation of the various solutions of the matrix theory. In general for the noncommutative gauge theories in $d$-dimensions, $d > 3$, there exists a corresponding $U(\infty)$ underlying field theory in $d - 2$ dimensions and so the universality holds.

The universality is somewhat larger than what we have described here, as discussed by Aganagic et al.\[14\]. They found the solution which describes the D0 branes not lying on the D2 branes, but separated from the D2 branes. We will see this in more detail in the next section. While this is clear in the matrix theory, it is a something remarkable in the field theory.

When there are fundamental matter fields, there is no relation between the matrix theory and the noncommutative field theory. While the $U(1)$ theory with fundamental matters can be still rewritten in terms of the $U(N)$ theory language i.e. the $N \times N$ matrices, there appears a subtlety to extract a constant from the potential as well as the sense of the regrouping of the degrees of freedom on space does not work fully. In the broken Higgs phase where the matter field has nonzero expectation value, all $N$ vacua with $N \geq 2$ have constant energy density when mapped from the $U(1)$ theory.

4 Examples

There are several examples of the universality. What is convenient about the universality is that one can write the solution of $U(N)$ gauge theory in terms of the $U(1)$ theory variables. Also, the noncommutative $U(1)$ gauge theory on two dimensions describes the dynamics of the D0 branes separated from the D2 branes. This is rather remarkable from the field theory point of view.

For two dimensional system, there exists a static but nontrivial magnetic flux solution, generalizing the solutions found in Ref. \[14\]. (Similarly, one may generalize the exact $U(1)$ vortex solutions\[16\] to $U(N)$ solutions.) This can be interpreted as the $M$ D0 branes in the background of $N$ D2 branes. The general solution is

$$C = \sum_{l=0}^{M-1} \lambda_l |l\rangle\langle l| + \sum_{\alpha,p} \sqrt{p+1}|pN + \alpha + M\rangle\langle (p+1)N + \alpha + M|,$$  \hspace{1cm} (28)

$$\phi^I = \sum_{l=0}^{M-1} \varphi^I_l |l\rangle\langle l| + \sum_{\alpha,p} h^I_\alpha |pN + \alpha + M\rangle\langle pN + \alpha + M|.$$ \hspace{1cm} (29)

The values $\varphi^I_l$ denote the position of the $l$-th D0 brane along the transverse coordinates and $h^I_\alpha$ denotes the position of the $\alpha$-th D2 brane along the same coordinate. The parameters $\lambda_l$ are the coordinates of D0 branes on the noncommutative $x, y$ plane. This fact may be confirmed by computing the moments utilizing the covariant position operators. [See Ref. \[18\] for the detailed computations of vortex positions.] The energy of the configuration does not depend on the transverse positions $\varphi^I_l$ of D0 branes. Thus, the D0 branes are not the deformation of D2 branes but independent entities of the theory. This shows that the noncommutative gauge theory remembers the underlying matrix theory and so is somewhat bigger than the theory of D2 branes.

We did analyze the small fluctuations around the above solution and found that there are $N \times M$
tachyonic modes among the off-diagonal fluctuations and their masses are

\[ m_{l\alpha}^2 = -\frac{1}{\theta}(1 - \sum_l (\varphi_l^* - h_{l\alpha}^*)^2) \],

(30)

where \( l \) and \( \alpha \) are respectively the indices for \( M \) D0 and \( N \) D2. This result confirms that our solution indeed describes \( M \) D0 branes near \( N \) D2 branes. Using the unitary transformation,

\[ U = \sum_{l=0}^{M-1} |l\rangle\langle l, 0| + \sum_p |pN + M\rangle\langle p + M, 0| + \sum_{\alpha \neq 0, p} |pN + \alpha + M\rangle\langle p, \alpha|, \]

(31)

we can put all D0 in the first D2 brane, or \( \alpha = 0 \) brane, which is closer to the \( U(N) \) theory point of view.

A more nontrivial solution is the \( U(N) \) dyonic fluxon solution in three dimensional space, generalizing that of Gross and Nekrasov\[8\]. Here the first two spaces \( x, y \) are noncommutative and the third coordinate \( z \) is commutative. The universality works in this higher dimensional theory which has six adjoint scalar fields. This can be interpreted as the tilted \( M \) D string piercing the \( N \) D3 branes, with some number of the fundamental string bound to \( D3 \) branes. The \( U(1) \) theory has a solution of two BPS Dirac monopoles of opposite charge with the Dirac strings coming out from the monopoles in the opposite directions. The fluxons are the configurations where two BPS magnetic monopoles overlap exactly and so their magnetic charges cancel each other. With a single Higgs \( \phi = \phi^1 \), the 1/2 BPS equations are

\[ E_i = \sin \xi \nabla_i \phi^1, \quad B_i = \cos \xi \nabla_i \phi^1, \]

(32)

where \( i = 1, 2, 3 \). The dyonic fluxon solution which describes the \( M \) D strings plus some F string piercing \( N \) D3 branes is

\[ C = \sum_l \lambda_l |l\rangle\langle l| + \sum_{\alpha, p} \sqrt{p + 1} |pN + M + \alpha\rangle\langle (p + 1)N + M + \alpha|, \]

(33)

\[ \phi^1 = \sum_{l=0}^{M-1} (\varphi_l - \frac{z}{\theta \cos \xi})|l\rangle\langle l| + \sum_{\alpha, p} h_{\alpha} |pN + M + \alpha\rangle\langle pN + M + \alpha|, \]

(34)

with \( A_0 = -\phi^1 \sin \xi \) and \( A_3 = 0 \). The magnetic field is then

\[ B_3 = -\frac{1}{\theta} \sum_{l=0}^{M-1} |l\rangle\langle l|, \]

(35)

and the electric field \( E_3 = B_3 \tan \xi \). The above solution is again given in the matrix theory point of view. From the D3 world volume point of view, we can make the similar gauge transformation as for the D2-D0 case. The values \( h_{\alpha} \) denote the \( \alpha \)-th D3 brane position in the transverse \( x^4 \) direction. From the above solution, we see that the \( l \)-th fluxon is crossing the \( \alpha \)-th \( D3 \) brane at \( z = \theta \cos \xi (\varphi_l - h_{\alpha}) \).

Five additional Higgs fields \( \phi^J \) with \( J = 2, 3, ..., 6 \) can take expectation value without breaking the 1/2 BPS condition. Their energy contribution should be zero. The most general solutions are constant fields,

\[ \phi^J = \sum_{l=0}^{M-1} \varphi_l^J |l\rangle\langle l| + \sum_{\alpha, p} h_{\alpha}^J |pN + M + \alpha\rangle\langle pN + M + \alpha|. \]

(36)
The value $\phi^J$ denotes the position of the $l$-th fluxon along the transverse coordinate $x^{3+J}$ and the value $h^J_\alpha$ denotes the position of the $\alpha$-th D3 brane along the same transverse coordinate. Now there is an interesting possibility that the $l$-th fluxon may not meet any of D3 branes\cite{17}, or may meet only a few of them.

While it is simple to see whether the $l$-th fluxon will meet the $\alpha$-th D3 branes from comparing their transverse positions, it will be more comforting to see this is the case by investigating the zero modes of the solution. When a fluxon goes through two parallel D3 branes, the middle segment can get separated from the whole line without change of energy. Thus, the number of the zero modes will depend on the number of such separable segments.

For the simplest case with a single fluxon and $\phi^J = 0$, the solution can be regarded as a composite of $N+1$ magnetic monopoles, related to the $U(N+2)$ gauge theory broken to $U(1)^{N+2}$, or the D string connecting $N+1$ D3 branes. The first and the last D3 branes are removed to the infinity and so the first and last segment of D strings become infinitely long or their corresponding magnetic monopoles become infinitely heavy and become Dirac monopoles with finite tension Dirac strings on noncommutative space. The rest $N-1$ D string segments have finite mass and appear as finite length sticks whose ends appear as magnetic monopoles on corresponding $U(1)$ group. The $U(N)$ fluxon solution is then one where all $U(N)$ magnetic charges cancel each other exactly. There would be then $4(N-1)$ zero modes, four for each magnetic monopoles. It would be the moduli space of $N+1$ distinct magnetic monopoles such that the first and the last one have infinite mass. For the fluxon case, the position of these two infinitely massive monopoles would be identical. The metric of the moduli space for the zero modes would be given by the Lee-Weinberg-Yi type\cite{18, 2}.

Interesting generalization of these fluxon solutions would be 1/4 BPS and non BPS type of solutions.

Note added: After this paper was posted to hep-th, we became aware of the previous introduction of the covariant position on noncommutative space in Ref.\cite{19}, where some aspects of the issue was discussed.

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