A NEW APPROACH TO THE REAL NUMBERS

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Abstract. In this paper we provide a complete approach to the real numbers via decimal representations. Construction of the real numbers by Dedekind cuts, Cauchy sequences of rational numbers, and the algebraic characterization of the real number system by the concept of complete ordered field are also well explained in the new setting.

1. Dedekind cuts, Cauchy sequences, axiomatic approach and decimal representations

“The discovery of incommensurable quantities was a severe blow to the Pythagorean program of understanding nature by means of numbers. (The slogan of the Pythagoreans was 'All is number'.) The Greeks developed a sophisticated theory of ratios, presumably the work of Eudoxus, to work around the problem that certain quantities, even certain lengths, could not be reduced to numbers, as then understood. This theory anticipates the development of the real number system by Dedekind and Cantor in the nineteenth century.” ([7])

In popular literatures there are mainly three approaches to the real numbers such as construction by Dedekind cuts (see e.g. [7, 9, 11, 18, 20, 22, 27, 32, 35, 43, 47]), Cauchy sequences of rational numbers (see e.g. [6, 9, 11, 12, 18, 29, 43, 45, 52, 53]), and an axiomatic definition (see e.g. [4, 6, 7, 9, 17, 19, 20, 33, 36, 39, 42, 43, 45, 46, 47, 49, 55]). We first make a brief summary.

Let \( \mathbb{Q} \) be the set of rational numbers endowed with the standard additive operation +, multiplicative operation \( \cdot \), also denoted by \( \times \), and total order \( \leq \). A Dedekind cut is a pair of nonempty subsets \( A, B \) of \( \mathbb{Q} \), denoted by \( (A \mid B) \), such that

1. \( A \cap B = \emptyset, A \cup B = \mathbb{Q} \),
2. \( a \in A, b \in B \Rightarrow a < b \),
3. \( A \) contains no greatest element.

The set of all Dedekind cuts is denoted by \( \mathbb{R} \). In 1872, Richard Dedekind ([15, 16]) introduced an operation \( \oplus \) called “addition”, an operation \( \otimes \) called “multiplication” and a total order \( \preceq \) on \( \mathbb{R} \), getting the so-called real number system \( (\mathbb{R}, \oplus, \otimes, \preceq) \). This work profoundly influenced subsequent studies of the foundations of mathematics.

A sequence of rational numbers \( \{x^{(n)}\}_{n \in \mathbb{N}} \) is called Cauchy if for every rational \( \epsilon > 0 \), there exists an \( N \in \mathbb{N} \) such that \( \forall m, n \geq N, |x^{(m)} - x^{(n)}| < \epsilon \). The set of all Cauchy sequences of rational numbers is denoted by \( \mathbb{C} \). Two Cauchy sequences of rational numbers \( \{x^{(n)}\}_{n \in \mathbb{N}}, \{y^{(n)}\}_{n \in \mathbb{N}} \) are said to be equivalent, denoted by \( \{x^{(n)}\}_{n \in \mathbb{N}} \approx \{y^{(n)}\}_{n \in \mathbb{N}} \), if for every rational \( \epsilon > 0 \), there exists an \( N \in \mathbb{N} \) such that \( \forall n \geq N, |x^{(n)} - y^{(n)}| < \epsilon \). It is easy to prove \( \approx \) is an equivalence relation, thus yields a quotient space \( \mathbb{C}/\approx \) denoted

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by $\mathbb{CR}$ for consistency. Around 1869–1872, Charles Méray ([41]), Georg Cantor ([10]) and Eduard Heine ([28]) independently introduced basically the same additive operation $\bigoplus$, multiplicative operation $\bigotimes$ and total order $\preceq$ on $\mathbb{CR}$, getting yet another number system ($\mathbb{CR}, \bigoplus, \bigotimes, \preceq$). This approach has the advantage of providing a standard way for completing an abstract metric space.

Both systems are isomorphic in the sense that one can find a bijection $\omega : \mathbb{DR} \rightarrow \mathbb{CR}$ such that for any two elements $x, y \in \mathbb{DR}$, $\omega(x \bigoplus y) = \omega(x) \bigoplus \omega(y)$, $\omega(x \bigotimes y) = \omega(x) \bigotimes \omega(y)$, $x \preceq y \Leftrightarrow \omega(x) \preceq \omega(y)$. Detailed studies of a dozen or so properties of $\bigoplus, \bigotimes, \preceq$ lead to the basic concept of “complete ordered field” and an algebraic-axiomatic approach to the real number system. Later on, when to verify a new system is isomorphic to those of Dedekind and Méray-Cantor-Heine, it suffices to prove that it is a complete ordered field. We should also note several severe criticisms of the algebraic-axiomatic approach:

“The necessary axioms should come as a byproduct of the construction process and not be predetermined.” ([38])

“... the algebraic-axiomatic definition of a real number is simply appalling and abhorrent ... to define a real number via a cold and boring list of a dozen or so axioms for a complete ordered field is like replacing life by death or reading an obituary column.” ([3])

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“Few mathematical structures have undergone as many revisions or have been presented in as many guises as the real numbers. Every generation reexamines the reals in the light of its values and mathematical objectives.” ([21])

Next we introduce the real numbers through a rather old geometric approach. Given a point $x$ in an “axis”, the decimal representation of it is obtained step by step as follows.

- **Step 0:** Partition this “axis” into countably many disjoint unions $\bigcup_{z \in \mathbb{Z}} [z, z + 1)$, then find a unique integer $x_0 \in \mathbb{Z}$ such that $x \in [x_0, x_0 + 1)$.
- **Step 1:** Partition $[x_0, x_0 + 1)$ into ten disjoint unions $\bigcup_{i=0}^{9} [x_0 + \frac{i}{10}, x_0 + \frac{i+1}{10})$, then find a unique element $x_1 \in \mathbb{Z}_{10}$ such that $x \in [x_0 + \frac{x_1}{10}, x_0 + \frac{x_1+1}{10})$, here and afterwards we denote $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ by $\mathbb{Z}_{10}$ for simplicity.
- **Step 2:** Partition $[x_0 + \frac{x_1}{10}, x_0 + \frac{x_1+1}{10})$ into ten disjoint unions $\bigcup_{j=0}^{9} [x_0 + \frac{x_1}{10} + \frac{j}{10^2}, x_0 + \frac{x_1+1}{10} + \frac{j+1}{10^2})$, then find a unique element $x_2 \in \mathbb{Z}_{10}$ such that $x \in [x_0 + \frac{x_1}{10} + \frac{x_2}{10^2}, x_0 + \frac{x_1+1}{10} + \frac{x_2+1}{10^2})$.
- **Step $\vdash$: ...$

Then we say $x$ has decimal representation $x_0.x_1x_2x_3 \cdots$, and call $x_k$ the $k$-th digit of $x$. A thorough consideration leads to a natural question: can any point of this “axis” have decimal representation $0.999999 \cdots$? and if not, why can we expel its existence? This is not a silly question at all. To correctly answer it, we should first know what an “axis” it is, or what on earth a “real number” it is!

Although constructing real numbers via decimals has been known since Simon Stevin ([1, 2]) in the 16-th century, developed also by Karl Weierstrass ([13]), Otto Stolz ([50, 51]) in the 19-th century, and many other modern mathematicians (see e.g. [3, 5, 8, 9, 14, 21, 24, 26, 30, 34, 37, 38, 40, 44, 54]), almost all popular Mathematical Analysis books didn’t choose this approach. The decimal construction hasn’t got the attention it should deserve:
“Perhaps one of the greatest achievements of the human intellect throughout the entire
history of the human civilization is the introduction of the decimal notation for the purpose
of recording the measurements of various magnitudes. For that purpose the decimal notation
is most practical, most simple, and in addition, it reflects most outstandingly the profound
subtleties of the human analytic mind. In fact, decimal notation reflects so much of the
Arithmetic and so much of the Mathematical Analysis…” ([3])

To the author’s opinion, at least one of the reasons behind this phenomenon is most of
the authors only gave outlines or sketches. We note an interesting phenomenon happened
in popular Mathematical Analysis books: many authors (see e.g. [4, 7, 11, 22, 31, 35,
36, 42, 43, 45, 47, 48, 49, 52, 55]) first chose one of the other three approaches discussed
before, then proved that every real number has a suitable decimal representation. But
if without using the algebraic intuition that the real number system is nothing but a
complete ordered field which is not so easy to grasp for beginners, we could find basically
no literature reversing this kind of discussion. No matter how fundamental sets and
sequences are in mathematics, conflicting with our primary, secondary and high school
education that a number is a string of decimals, skepticism over explicit construction of
the real numbers by either Dedekind cuts or Cauchy sequences never ends:

“The degeometrization of the real numbers was not carried out without skepticism. In
his opus Mathematical Thought from Ancient to Modern Times, mathematics historian Morris
Klein quotes Hermann Hankel who wrote in 1867: Every attempt to treat the irrational numbers
formally and without the concept of [geometric] magnitude must lead to the most abstruse
and troublesome artificialities, which, even if they can be carried through complete rigor, as we
have every right to doubt, do not have a right scientific value.” ([24])

“The definition of a real number as a Dedekind cut of rational numbers, as well as a Cauchy
sequence of rational numbers, is cumbersome, impractical, and ..., inconsequential for the
development of the Calculus or the Real Analysis.” ([3])

Based on the fundamental concept of “order” and its derived operations, in this paper
we will provide a complete approach to the real numbers via decimals, and some of our
ideas are new to the existing literatures. Also in this new setting, construction of the
real numbers by Dedekind cuts, Cauchy sequences of rational numbers, and the algebraic
characterization of the real number system by the concept of complete ordered field can
be well explained. The general strategy of our approach is as follows.

The starting point is to choose

\[ \mathcal{R} \triangleq \{ x_0.x_1x_2x_3 \ldots \mid x_0 \in \mathbb{Z}, x_k \in \mathbb{Z}_{10}, k \in \mathbb{N} \}. \]

as our ambient space, which was already discussed before. Once adopted this decimal
notation, we suggest you imagine it in mind as a series \( \sum_{k=0}^{\infty} \frac{x_k}{10^k} \). There are mainly two
reasons why we choose this notation, one is instead of discussing “subtraction”, we shall
focus on “additive inverse” which will be introduced in an elegant manner, the other comes
from the next paragraph.

Since as sets \( \mathcal{R} \) is basically the same as \( \mathbb{Z} \times \mathbb{Z}_{10}^\mathbb{N} \), we can introduce a lexicographical order
\( \preceq \) on \( \mathcal{R} \), then prove the least upper bound property and the greatest lower bound property
for \( (\mathcal{R}, \preceq) \) from the same properties for \( (\mathbb{Z}, \leq) \) as soon as possible. As experienced readers
should know, this would mean that \( (\mathcal{R}, \preceq) \) is “complete”. So in this complete setting,
we can derive five basic operations such as the supremum operation \( \text{sup}(\cdot) \), the infimum
operation \( \text{inf}(\cdot) \), the upper limit operation \( \text{LIMIT}(\cdot) \), the lower limit operation \( \text{LIMIT}(\cdot) \) and the limit operation \( \text{LIMIT}(\cdot) \).
It is very natural to define
\[ x \oplus y \triangleq \text{LIMIT} \left( \left\{ [x]_k + [y]_k \right\}_{k \in \mathbb{N}} \right) \]
for any two elements \( x, y \in \mathcal{R} \), where \([x]_k \triangleq x_0.x_1x_2 \cdots x_k = \frac{\sum_{i=0}^{k} x_i \cdot 10^{k-i}}{10^k} \in \mathbb{Q}\) is the truncation of \( x \) up to the \( k \)-th digit, so is \([y]_k\). As for the definition of \([x]_k + [y]_k\), even a primary school student may know how to do it, that is, for example,
\[
\begin{align*}
(−15).3456 \\
+ (−18).6789 \\
= (−32).0245
\end{align*}
\]

To define a multiplicative operation in a succinct way we need some preparation. First we introduce a signal map \( \text{sign} : \mathcal{R} \to \{0, 1\} \) by
\[
\text{sign}(x) \triangleq \begin{cases} 
0 & \text{if } x \succeq 0.000000 \ldots, \\
1 & \text{if } x \preceq (−1).999999 \ldots .
\end{cases}
\]
This map partitions \( \mathcal{R} \) into two parts, one is \( \text{sign}^{-1}(0) \), understood as the positive part of \( \mathcal{R} \), the other is \( \text{sign}^{-1}(1) \), understood as the negative part of \( \mathcal{R} \). Both parts are closed connected through an “additive inverse” map
\[
\Psi(x_0.x_1x_2x_3 \cdots) \triangleq (−1 − x_0) \cdot (9 − x_1) \cdot (9 − x_2) \cdot (9 − x_3) \cdots,
\]
which turns a positive element into a negative one, and vice versa. The absolute value of \( x \), denoted by \( ||x|| \), is defined to be the maximum over \( x \) and \( \Psi(x) \). Because of the wonderful formula \( x = \Psi(\text{sign}(x))(||x||) \), the author likes to call them three golden flowers.

Now for any two elements \( x, y \in \mathcal{R} \), we define their multiplication by
\[
x \otimes y \triangleq \Psi(\text{sign}(x)+\text{sign}(y)) \left( \text{LIMIT} \left( \left\{ ||x||_k \cdot ||y||_k \right\}_{k \in \mathbb{N}} \right) \right).
\]
In primary school we have already learnt how to define \( ||x||_k \cdot ||y||_k \).

We remark that motivated by the above definitions of addition and multiplication on \( \mathcal{R} \), similar operations will be introduced on \( \mathbb{DR} \) in a highly consistent way.

At this stage, we have introduced a rough system \( (\mathcal{R}, \preceq, \oplus, \otimes) \) without any pain. But unfortunately, several well-known properties generally a standard addition and a standard multiplication should have don’t hold for \( \oplus \) and \( \otimes \). To overcome these difficulties, we will introduce an equivalence relation \( \sim \) identifying \( 0.999999 \cdots \) with \( 1.000000 \cdots \), and the same like. No matter adopting decimal, binary or hexadecimal notation, no matter introducing such relations earlier or later, we cannot avoid doing it.

With these preparations, we then verify the commutative, associate and distribute laws, the existences of additive (multiplicative) unit and inverse, and so on. Almost all the verification work depend only on a pleasant Lemma 3.8. Finally we define the set \( \mathbb{R} \) of real numbers to be the set of equivalent classes \( \mathcal{R}/\sim \) with derived operations \( \oplus, \otimes, \preceq \) from \( \oplus, \otimes \), \( \preceq \) respectively, thus yields our desired number system \( (\mathbb{R}, \oplus, \otimes, \preceq) \).

Now in the new setting \( (\mathcal{R}, \preceq) \) with derived operations such as \( \text{sup}(\cdot) \), \( \text{inf}(\cdot) \) for subsets, and \( \text{LIMIT}(\cdot) \) and \( \text{LIMIT}(\cdot) \) for sequences of \( \mathcal{R} \), we can further explain the construction of the real numbers by Dedekind cuts and Cauchy sequences of rational numbers. Given a Dedekind cut \( (A|B) \), we obviously have \( \text{sup } A \preceq \text{inf } B \). It would be very nice if \( \text{sup } A \sim \text{inf } B \), thus one can derive a map \( \tau \) from \( \mathbb{DR} \) to \( \mathbb{R} \) by sending \( (A|B) \) to \( [\text{sup } A] = [\text{inf } B] \).
 Later on we shall prove that this is indeed the case. Given a Cauchy sequence of rational
numbers \( \{x^{(n)}\}_{n \in \mathbb{N}} \), we obviously have \( \overline{\text{LIMIT}}(\{x^{(n)}\}_{n \in \mathbb{N}}) \leq \text{LIMIT}(\{x^{(n)}\}_{n \in \mathbb{N}}) \). It would be great if \( \overline{\text{LIMIT}}(\{x^{(n)}\}_{n \in \mathbb{N}}) \sim \text{LIMIT}(\{x^{(n)}\}_{n \in \mathbb{N}}) \), thus one can derive a map \( \kappa \) from \( \text{CR} \) to \( \mathbb{R} \) by sending \( \{x^{(n)}\}_{n \in \mathbb{N}} \) to \( \overline{\text{LIMIT}}(\{x^{(n)}\}_{n \in \mathbb{N}}) = [\text{LIMIT}(\{x^{(n)}\}_{n \in \mathbb{N}})] \). Later on we shall prove that this is also indeed the case. Motivated by these observations, we will continue to prove that our number system is isomorphic to those of Dedekind and Méray-Cantor-Heine.

We can explain Cauchy sequences of rational numbers in another way. Given a Cauchy sequence of rational numbers \( \{x^{(n)}\}_{n \in \mathbb{N}} \), if it could represent a “real number”, then we have no doubt that any subsequence of it, say for example a monotonically increasing or a monotonically decreasing subsequence, should represent the same “real number”. Then we put such a monotone subsequence of rational numbers in \( (\mathcal{R}, \preceq) \) whose existence is taken for granted at this time, from either the least upper bound property or the greatest lower bound property in the new setting, the interested readers definitely know which decimal representation should be understood as the “real number” the original sequence represents. Simply speaking, it would be great if we can discuss Dedekind cuts or Cauchy sequences of rational numbers in a constructive, complete setting.

Still in the setting \( (\mathcal{R}, \preceq) \), we will give the traditional characterization of irrational numbers, which makes our approach matching what we have learnt in high school. We also explain how can we come to the basic concept of complete ordered field.

To develop this approach, the author owed a lot to Loo-Keng Hua’s masterpiece [30]. He also thanks Rong Ma for helpful discussions. This work was partially supported by the Natural Science Foundation of China (Grant Number 11001174).

Some notations used throughout this paper:

- \( \mathbb{N} \) is the set of natural numbers
- \( \mathbb{Z} \) is the set of integers
- Denote \( \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \) by \( \mathbb{Z}_{10} \)
- \( \lfloor \cdot \rfloor \) is the floor function, \( \lceil \cdot \rceil \) is the ceil function
- For any nonempty bounded above (below) subset \( E \) of \( \mathbb{Z} \), denote by \( \max E \) (\( \inf E \)) the unique least upper (greatest lower) bound for \( E \)
- For any binary operation \( \ast \) on a set \( G \), denote by \( A \ast B \) the set \( \{a \ast b| a \in A, b \in B\} \), where \( A, B \) are nonempty subsets of \( G \)
- Rational number system: \( (\mathbb{Q}, +, \times, \leq) \)
- Decimal system: \( (\mathcal{R}, \oplus, \otimes, \preceq) \)
- Dedekind cut system: \( (\text{DR}, \sqcup, \sqcap, \preceq) \)
- Cauchy sequence system: \( (\text{CR}, \sqcup, \sqcap, \preceq) \)
- Real number system: \( (\mathcal{R}, \oplus, \otimes, \leq) \)

2. Least upper bound and greatest lower bound properties for \( (\mathcal{R}, \preceq) \)

2.1. Ambient space, lexicographical order and its derived operations.

**Definition 2.1.** As in Section 1 we define the ambient space of this paper to be

\[
\mathcal{R} \triangleq \{x_0.x_1x_2x_3 \cdots | x_0 \in \mathbb{Z}, x_k \in \mathbb{Z}_{10}, k \in \mathbb{N}\}.
\]

For any \( x = x_0.x_1x_2x_3 \cdots \in \mathcal{R} \) and \( k \in \mathbb{N} \), let \( [x]_k \triangleq x_0.x_1 \cdots x_k = \sum_{i=0}^{k} x_i 10^{k-i} \in \mathbb{Q} \) be the truncation of \( x \) up to the \( k \)-th digit.
Remark 2.2. For any \( \alpha \in \mathbb{Q} \), there exist two integers \( p \in \mathbb{N} \) and \( q \in \mathbb{Z} \) such that \( \alpha = \frac{q}{p} \). Via the long division algorithm \( \alpha \) has a decimal representation \( \alpha_0 \alpha_1 \alpha_2 \alpha_3 \cdots \), where
\[
\alpha_0 \triangleq \frac{q}{p}, \quad \beta_0 \triangleq q - \alpha_0 p, \quad \alpha_{k+1} \triangleq \frac{10\beta_k}{p}, \quad \beta_{k+1} \triangleq 10\beta_k - \alpha_{k+1} p.
\]
Obviously, this representation is independent of the choices of \( p \) and \( q \), so from now on we can always view \( \mathbb{Q} \) as a subset of \( \mathbb{R} \). For example,
\[
\frac{5}{2} = 2.500000 \cdots, \quad -\frac{40}{3} = (-14).666666 \cdots.
\]

Definition 2.3. Let \( \preceq \) be the lexicographical order on \( \mathbb{R} \), that is, \( x \preceq y \) if and only if \( \forall k \in \mathbb{N}, [x]_k \leq [y]_k \).

As usual, we may write \( y \succeq x \) when \( x \preceq y \), and \( x \prec y \) when \( x \preceq y, x \neq y \). Obviously, as elements of \( \mathbb{Q} \), \( x \preceq y \) if and only if \( x \leq y \), which is self-evident since we can write \( x, y \) by a common denominator, then use the long division algorithm.

Definition 2.4. A nonempty subset \( W \) of \( \mathbb{R} \) is called bounded above (below) if there exists an \( M \in \mathbb{R} \) such that \( \forall w \in W, w \leq M \) (\( w \succeq M \)). A nonempty subset of \( \mathbb{R} \) is called bounded if it is bounded both above and below. A sequence \( \{x^{(n)}\}_{n \in \mathbb{N}} \) of \( \mathbb{R} \) is called bounded above (below), bounded if the set \( \{x^{(n)} \mid n \in \mathbb{N}\} \) is of the corresponding property.

Theorem 2.5. Every nonempty bounded above subset of \( (\mathbb{R}, \preceq) \) has a least upper bound, every nonempty bounded below subset of \( (\mathbb{R}, \succeq) \) has a greatest lower bound.

Proof. This theorem follows from the least upper bound property and the greatest lower bound property for \( (\mathbb{Z}, \leq) \). We shall only prove the first part of this theorem, and leave the second one to the interested readers. Let \( W \) be a nonempty bounded above subset of \( \mathbb{R} \). Denote
\[
M_0 \triangleq \max\{x_0 \mid x_0.x_1x_2x_3 \cdots \in W\},
\]
\[
M_k \triangleq \max\{x_k \mid x_0.x_1x_2x_3 \cdots \in W \text{ with } x_i = M_i, i = 0, 1, \ldots, k - 1\} \quad (\forall k \in \mathbb{N}).
\]
Obviously, \( M \triangleq M_0, M_1, M_2, M_3 \cdots \) is an upper bound for \( W \). Let \( \widetilde{M} \) be an arbitrary upper bound for \( W \). \( \forall k \in \mathbb{N} \), by the definition of \( M_k \) one can find an \( x = x_0.x_1x_2x_3 \cdots \in W \) such that \( x_i = M_i \) for \( i = 0, 1, \ldots, k \). Thus \( [\widetilde{M}]_k \geq [x]_k = x_0 \cdot \cdots \cdot x_k = M_0 \cdot M_1 \cdot \cdots \cdot M_k = [M]_k \), which means \( \widetilde{M} \geq M \). This proves \( M \) is the (unique) least upper bound for \( W \).

Definition 2.6. The least upper bound, also known as the supremum, for a nonempty bounded above subset \( W \) of \( \mathbb{R} \) is denoted by \( \sup W \), while the greatest lower bound, known as the infimum, for a nonempty bounded below subset \( W \) of \( \mathbb{R} \) is denoted by \( \inf W \).

Definition 2.7. Generally given a sequence \( \{x^{(n)}\}_{n \in \mathbb{N}} \), we should pay attention to its asymptotic behavior. Obviously for any \( n \in \mathbb{N} \), \( \sup\{x^{(k)} \mid k \geq n\} \) can be understood as an “upper bound” for the sequence \( \{x^{(n)}\}_{n \in \mathbb{N}} \) if we don’t care about several of its initial terms. Thus if we want to obtain an asymptotic “upper bound” that is as least as possible, then we are naturally led to the concept of upper limit of a bounded sequence, that is,
\[
\text{LIMIT}(\{x^{(n)}\}_{n \in \mathbb{N}}) \triangleq \inf \{ \sup\{x^{(k)} \mid k \geq n\} \mid n \in \mathbb{N} \}.
\]
Similarly, we define the lower limit of a bounded sequence \( \{x^{(n)}\}_{n \in \mathbb{N}} \) to be
\[
\text{LIMIT}(\{x^{(n)}\}_{n \in \mathbb{N}}) \triangleq \sup \{ \inf\{x^{(k)} \mid k \geq n\} \mid n \in \mathbb{N} \}.
\]
When it happens that $\lim\{x^{(n)}\}_{n \in \mathbb{N}} = \lim\{x^{(n)}\}_{n \in \mathbb{N}} = L$, we shall simply write $\lim\{x^{(n)}\}_{n \in \mathbb{N}}$ for their common value $L$, and say the sequence $\{x^{(n)}\}_{n \in \mathbb{N}}$ has limit $L$.

**Remark 2.8.** It is no hard to prove that any bounded above monotonically increasing sequence, that is, $x^{(1)} \leq x^{(2)} \leq x^{(3)} \leq x^{(4)} \leq \cdots \leq M$, or any bounded below monotonically decreasing sequence, that is, $x^{(1)} \geq x^{(2)} \geq x^{(3)} \geq x^{(4)} \geq \cdots \geq M$, has a limit. For example, suppose $x^{(1)} \leq x^{(2)} \leq x^{(3)} \leq x^{(4)} \leq \cdots \leq M$. Then

$$\lim\{x^{(n)}\}_{n \in \mathbb{N}} = \inf\{ \sup\{x^{(k)}| k \geq n\} | n \in \mathbb{N}\}$$

$$= \inf\{ \sup\{x^{(k)}| k \in \mathbb{N}\} | n \in \mathbb{N}\}$$

$$= \sup\{x^{(k)}| k \in \mathbb{N}\}$$

$$= \inf\{ \inf\{x^{(n)}| n \geq k\} | k \in \mathbb{N}\}$$

$$= \lim\{x^{(n)}\}_{n \in \mathbb{N}}.$$

We state below some elementary properties of $\sup(\cdot)$, $\inf(\cdot)$, $\lim(\cdot)$, $\lim(\cdot)$ and $\lim(\cdot)$ in a lemma, and leave the proofs to the interested readers.

**Lemma 2.9.** (1) $\inf W \leq \sup W$,
(2) $W_1 \subset W_2 \Rightarrow \sup W_1 \leq \sup W_2$,
(3) $W_1 \subset W_2 \Rightarrow \inf W_1 \geq \inf W_2$,
(4) $\lim\{x^{(n)}\}_{n \in \mathbb{N}} \leq \lim\{x^{(n)}\}_{n \in \mathbb{N}}$,
(5) $x^{(n)} \leq y^{(n)} \Rightarrow \lim\{x^{(n)}\}_{n \in \mathbb{N}} \leq \lim\{y^{(n)}\}_{n \in \mathbb{N}}$,
(6) $x^{(n)} \leq y^{(n)} \Rightarrow \lim\{x^{(n)}\}_{n \in \mathbb{N}} \leq \lim\{y^{(n)}\}_{n \in \mathbb{N}}$,
(7) $\lim\{x^{(n)}\}_{n \in \mathbb{N}} \leq \lim\{x^{(n)}\}_{n \in \mathbb{N}}$,
(8) $\lim\{x^{(n)}\}_{n \in \mathbb{N}} \geq \lim\{x^{(n)}\}_{n \in \mathbb{N}}$,
(9) $\lim\{x^{(n)}\}_{n \in \mathbb{N}} = L \Rightarrow \lim\{x^{(n)}\}_{n \in \mathbb{N}} = L$,
(10) $\lim\{x_k\}_{k \in \mathbb{N}} = x$.

2.2. An equivalence relation.

**Definition 2.10.** Two elements $x, y$ of $\mathcal{R}$ are said to have no gap, denoted by $x \sim y$, if it does not exist an element $z \in \mathcal{R}\{x, y\}$ lying exactly between $x$ and $y$.

**Definition 2.11.** Let $\mathcal{R}_9, \mathbb{Q}_F$ be respectively the sets of all decimal representations ending in an infinite string of nines, and zeros.

**Lemma 2.12.** $x \prec y, x \sim y \Rightarrow x \in \mathcal{R}_9, y \in \mathbb{Q}_F$.

**Proof.** Suppose $x = x_0.x_1x_2x_3\cdots < y = y_0.y_1y_2y_3\cdots$. Let $k$ be the minimal non-negative integer such that $x_k < y_k$. Note

$$x \preceq x_0.x_1x_2x_3\cdots 9999999\cdots < y_0.y_1\cdots y_k000000 \cdots \leq y.$$ 

Since $x \sim y$, we must have $x = x_0.x_1\cdots x_k999999\cdots$, else we get a contradiction $x < x_0.x_1\cdots x_k999999\cdots < y$. Similarly, $y = y_0.y_1\cdots y_k000000\cdots$. This finishes the proof. \qed

**Remark 2.13.** As a corollary of Lemma 2.12, it is easy to prove that $\sim$ is an equivalence relation on $\mathcal{R}$, and every equivalent class has at most two elements of $\mathcal{R}$. As we know in real life, one can find a medium of any two different points in a straight line, but at this time it is hard for us to define the medium between two points having no gap. So to express a straight line from $\mathcal{R}$, it is absolutely necessary to module something from $\mathcal{R}$, which we shall discuss in detail at a very late stage of this paper.
Next we prepare a useful characterization lemma. An enhanced Lemma 3.8 will be given in the next section.

Lemma 2.14. $x \sim y \iff \forall k \in \mathbb{N}, \|[x]_k - [y]_k\| \leq \frac{1}{10^k}$.

Proof. The necessary part follows immediately from Lemma 2.12, so we need only prove the sufficient one, and suppose $\forall k \in \mathbb{N}, \|[x]_k - [y]_k\| \leq \frac{1}{10^k}$. Without loss of generality we may assume that $x \preceq y$. Thus $\forall k \in \mathbb{N}, [x]_k \leq [y]_k \leq [x]_k + \frac{1}{10^k}$.

Case 1: Suppose $x \in \mathcal{R}_g$. There exists an $m \in \mathbb{N}$ such that $\forall k > m, x_k = 9$. Obviously, $x \sim x_0.x_1x_2\cdots x_m + \frac{1}{10^m}$. Note $\forall k > m$,

$$[x]_k \leq [y]_k \leq [x]_k + \frac{1}{10^k} = x_0.x_1x_2\cdots x_m + \frac{1}{10^m}. $$

Letting $k \to \infty$ gives $x \preceq y \preceq x_0.x_1x_2\cdots x_m + \frac{1}{10^m}$, here we have used the fifth and tenth parts of Lemma 2.9. This naturally implies $x \sim y$.

Case 2: Suppose $x \not\in \mathcal{R}_g$. There exists a sequence of natural numbers $m_1 < m_2 < m_3 < \cdots$ such that $\forall i \in \mathbb{N}, x_{m_i} < 9$. Noting

$$[x]_{m_i} \leq [y]_{m_i} \leq [x]_{m_i} + \frac{1}{10^{m_i}},$$

and

$$[[x]_{m_i}]_{m_{i-1}} = [[x]_{m_i} + \frac{1}{10^{m_i}}]_{m_{i-1}} = [x]_{m_{i-1}},$$

we must have $[y]_{m_{i-1}} = [[y]_{m_i}]_{m_{i-1}} = [x]_{m_{i-1}}$. By the ninth and tenth parts of Lemma 2.9, we have $x = y$.

This concludes the whole proof of the lemma. $\square$

3. ADDITIVE OPERATIONS

3.1. Additive operations.

Definition 3.1 (Addition). For any two elements $x, y \in \mathcal{R}$, let

$$x \oplus y \triangleq \text{LIMIT}(\{[x]_k + [y]_k\}_{k \in \mathbb{N}}).$$

This is well-defined since the sequence $\{[x]_k + [y]_k\}_{k \in \mathbb{N}}$ is monotonically increasing with an upper bound $x_0 + y_0 + 2$, where $x = x_0.x_1x_2x_3\cdots, y = y_0.y_1y_2y_3\cdots$.

Example 3.2. For any $x \in \mathcal{R}$, $x \oplus 0.000000\cdots = x$. This means $(\mathcal{R}, \oplus)$ has a unit.

Remark 3.3. Given two elements $x, y \in \mathbb{Q}_F \subset \mathcal{Q}$, it is easy to verify that $x \oplus y = x + y$. Therefore from now on we can abuse the uses of $+$ and $+$ if the summands lie in $\mathbb{Q}_F$.

Theorem 3.4. For any two elements $x, y \in \mathcal{R}$, we have $x \oplus y = y \oplus x$.

Proof. $x \oplus y = \text{LIMIT}(\{[x]_k + [y]_k\}_{k \in \mathbb{N}}) = \text{LIMIT}(\{[y]_k + [x]_k\}_{k \in \mathbb{N}}) = y \oplus x$. $\square$

Theorem 3.5. Given $x, y, z, w \in \mathcal{R}$ with $x \preceq z$ and $y \preceq w$, we have $x \oplus y \preceq z \oplus w$.

Proof. $\forall k \in \mathbb{N}$ we have $[x]_k \leq [z]_k$ and $[y]_k \leq [w]_k$, which yields $[x]_k + [y]_k \leq [z]_k + [w]_k$. By Lemma 2.9, \text{LIMIT}(\{[x]_k + [y]_k\}_{k \in \mathbb{N}}) \preceq \text{LIMIT}(\{[z]_k + [w]_k\}_{k \in \mathbb{N}})$. $\square$

Lemma 3.6. For any element $x \in \mathcal{R}$ and $k \in \mathbb{N}$, we have $[x]_k \preceq x \preceq [x]_k + \frac{1}{10^k}$. 


Remark 3.11. Given two elements $x, y \in \mathcal{R}$, and $k \in \mathbb{N}$, we have

$$[x]_k + [y]_k \leq [x \oplus y]_k \leq x \oplus y \leq [x]_k + [y]_k + \frac{2}{10^k}.$$ 

Proof. To prove the first inequality, we need only note

$$[x]_n + [y]_n \leq ([x]_k + [y]_k) + [x]_n + [y]_n \leq [x]_k + [y]_k + \frac{2}{10^k}.$$ 

Letting $n \to \infty$ yields $x \oplus y \leq [x]_k + [y]_k + \frac{2}{10^k}$. This proves the second inequality. The third one follows from the previous lemma, so we finish the whole proof. 

Lemma 3.8. If there exists an $M \in \mathbb{N}$ such that $\forall k \in \mathbb{N}, |[x]_k - [y]_k| \leq \frac{M}{10^k}$, then $x \sim y$.

Proof. For any $k \in \mathbb{N}$, $y \leq [y]_k + \frac{1}{10^k} \leq [x]_k + \frac{M+1}{10^k} \leq x \oplus \frac{M+1}{10^k}$. For any $m \in \mathbb{N}$, we can find a sufficiently large $k$ such that $\frac{M+1}{10^k} \leq \frac{1}{10^m}$. Consequently, $y \leq x \oplus \frac{1}{10^m}$, which yields $|[y]_m - [x]_m| \leq \frac{1}{10^m}$. By symmetry, we can also have $|[x]_m - [y]_m| \leq \frac{1}{10^m}$. Thus $|[x]_m - [y]_m| \leq \frac{3}{10^m}$, and this finishes the proof simply by applying Lemma 2.14. 

Theorem 3.9. Given $x, y, z, w \in \mathcal{R}$ with $x \sim z$ and $y \sim w$, we have $x \oplus y \sim z \oplus w$.

Proof. For any $k \in \mathbb{N}$,

$$[x \oplus y]_k \leq [x]_k + [y]_k + \frac{2}{10^k} \leq [z]_k + [w]_k + \frac{4}{10^k} \leq [z \oplus w]_k + \frac{4}{10^k},$$

here we have used Lemma 2.14 for $x \sim z$ and $y \sim w$. By symmetry we can also have $[z \oplus w]_k \leq [x \oplus y]_k + \frac{1}{10^k}$. Finally by Lemma 3.8, we are done. 

Theorem 3.10. For any three elements $x, y, z \in \mathcal{R}$, we have $(x \oplus y) \oplus z \sim x \oplus (y \oplus z)$.

Proof. For any $k \in \mathbb{N}$,

$$[x]_k + [y]_k + [z]_k \leq [x \oplus y]_k + [z]_k \leq [(x \oplus y) \oplus z]_k \leq (x \oplus y) \oplus z \leq ([x]_k + [y]_k + [z]_k + \frac{3}{10^k}.$$

Similarly, we can also have $[x]_k + [y]_k + [z]_k \leq [(y \oplus z) \oplus x]_k \leq [x]_k + [y]_k + [z]_k + \frac{3}{10^k}$. Finally by Lemma 3.8, $(x \oplus y) \oplus z \sim (y \oplus z) \oplus x = x \oplus (y \oplus z)$. This concludes the proof. 

Remark 3.11. Given $n$ elements $\{x(i)\}_{i=1}^n$ of $\mathcal{R}$ and a permutation $\tau$ on the index set $\{1, 2, \ldots, n\}$, according to Theorems 3.4, 3.9 and 3.10, it is easy to verify that

$$(\cdots(((x(1) \oplus x(2)) \oplus x(3)) \oplus x(4)) \cdots) \oplus x(n) \sim \cdots(((x(\tau_1) \oplus x(\tau_2)) \oplus x(\tau_3)) \oplus x(\tau_4)) \cdots) \oplus x(\tau_n).$$

As usual, we may simply write

$$x(1) \oplus x(2) \oplus x(3) \oplus x(4) \cdots \oplus x(n) \sim x(\tau_1) \oplus x(\tau_2) \oplus x(\tau_3) \oplus x(\tau_4) \cdots \oplus x(\tau_n)$$

since it does not matter where the parentheses lie.
3.2. Additive inverses.

**Definition 3.12.** For any element \( x = x_0x_1x_2x_3 \cdots \in \mathcal{R} \), let
\[
\Psi(x) \triangleq (-1 - x_0)(9 - x_1)(9 - x_2)(9 - x_3) \cdots,
\]
understood as the “additive inverse” of \( x \). The absolute value of \( x \) to is defined to be
\[
\|x\| \triangleq \max\{x, \Psi(x)\}.
\]

Let sign : \( \mathcal{R} \to \{0, 1\} \) be the signal map
\[
\text{sign}(x) \triangleq \begin{cases} 
0 \text{ if } x \geq 0.000000 \cdots, \\
1 \text{ if } x \leq (-1).999999 \cdots.
\end{cases}
\]

Some elementary properties on \( \Psi(\cdot), \|\cdot\| \) and sign(\( \cdot \)) are collected below without proofs. The interested readers can easily provide the details without much difficulty.

1. \( x \oplus \Psi(x) = (-1).999999 \cdots; \)
2. \( \Psi(\Psi(x)) = x; \)
3. \( x \preceq y \iff \Psi(x) \succeq \Psi(y); \)
4. \( x \sim y \iff \Psi(x) \sim \Psi(y); \)
5. \( x \sim y \implies \|x\| \sim \|y\|; \)
6. \( x \sim \Psi(x) \iff x \sim 0.000000 \cdots; \)
7. \( \|x\| \succeq 0.000000 \cdots; \)
8. \( \|\Psi(x)\| = \|x\|; \)
9. \( \text{sign}(x) + \text{sign}(\Psi(x)) = 1; \)
10. \( \Psi(\text{sign}(x))(x) = \|x\|; \)
11. \( \Psi(\text{sign}(x))(\|x\|) = x; \)
12. \( \Psi(\sup W) = \inf \Psi(W); \)
13. \( \Psi(\text{LIMIT}\{\{x^{(n)}\}_{n \in \mathbb{N}}\}) = \text{LIMIT}\{\Psi(x^{(n)})\}_{n \in \mathbb{N}}\).

**Theorem 3.13.** Given \( x, y, z \in \mathcal{R} \) with \( x \oplus z \sim y \oplus z \), we have \( x \sim y \).

**Proof.** By Theorems 3.9 and 3.10,
\[
x = x \oplus 0.000000 \cdots \\
\sim x \oplus z \oplus \Psi(z) \\
\sim y \oplus z \oplus \Psi(z) \\
\sim y \oplus 0.000000 \cdots \\
= y.
\]

\[ \square \]

**Theorem 3.14.** For any two elements \( x, y \in \mathcal{R} \), we have \( \Psi(x \oplus y) \sim \Psi(x) \oplus \Psi(y) \).

**Proof.** By Theorems 3.4, 3.9 and 3.10,
\[
\Psi(x \oplus y) = \Psi(x \oplus y) \oplus 0.000000 \cdots \oplus 0.000000 \cdots \\
\sim \Psi(x \oplus y) \oplus (x \oplus \Psi(x)) \oplus (y \oplus \Psi(y)) \\
\sim \Psi(x \oplus y) \oplus (x \oplus y) \oplus (\Psi(x) \oplus \Psi(y)) \\
\sim \Psi(x) \oplus \Psi(y).
\]

\[ \square \]
4. Multiplicative operations

4.1. Multiplicative operations.

Definition 4.1 (Multiplication). For any two elements \( x, y \in \mathcal{R} \), let
\[
x \otimes y \triangleq \Psi(\text{sign}(x)+\text{sign}(y)) \left( \text{LIMIT} \left( \left\{ \|x\|_k \cdot \|y\|_k \right\}_{k \in \mathbb{N}} \right) \right),
\]
where \( \Psi^{(k)} \) is the \( k \)-times composites of \( \Psi \).

Example 4.2. For any \( x \in \mathcal{R} \),
\[
x \otimes 1.000000 \cdots = \Psi(\text{sign}(x)) \left( \text{LIMIT} \left( \left\{ \|x\|_k \right\}_{k \in \mathbb{N}} \right) \right) = \Psi(\text{sign}(x))(\|x\|) = x.
\]
This means \((\mathcal{R}, \otimes)\) has a unit.

Example 4.3. For any \( x \in \mathcal{R} \),
\[
x \otimes (-1).999999 \cdots = \Psi(\text{sign}(x)+1) \left( \text{LIMIT} \left( \left\{ \|x\|_k \cdot 0 \right\}_{k \in \mathbb{N}} \right) \right)
\]
\[
= \Psi(\text{sign}(x)+1)(0.000000 \cdots)
\]
\[
\sim 0.000000 \cdots.
\]

Example 4.4. Given a calculator with sufficiently long digits, we could observe that
\[
1 < 1.4^2 < 1.99 < 2 < 1.5^2
\]
\[
1.9 < 1.41^2 < 1.9999 < 2 < 1.42^2
\]
\[
1.99 < 1.414^2 < 1.999999 < 2 < 1.415^2
\]
\[
\vdots
\]
\[
\underbrace{1.99 \cdots 99}_n < (a_0.a_1a_2a_3 \cdots a_n)^2 < \underbrace{1.999 \cdots 999}_{2n} < 2 < (a_0.a_1a_2a_3 \cdots a_n + \frac{1}{10^n})^2
\]
\[
\vdots
\]

For thousands of years \( a \triangleq a_0.a_1a_2a_3 \cdots \) has been understood as the positive square root of 2, so what is the reason behind? According to Definition 4.1, \( a \otimes a = 1.999999 \cdots \). Also from the above formulas, it is no hard to observe (see also [14, 23, 25]) that there is no element \( z \in \mathcal{R} \) such that \( z \otimes z = 2.000000 \cdots \). So if we want to define the positive square root of 2, except \( a_0.a_1a_2a_3 \cdots \), which else could be?

Remark 4.5. Given \( x, y \in \mathbb{Q}_F \subset \mathbb{Q} \) with \( x, y \preceq 0.000000 \cdots \), it is easy to verify that \( x \otimes y = x \cdot y \). Therefore from now on we can abuse the uses of \( \otimes \) and \( \cdot \) if the summands lie in \( \mathbb{Q}_F \) with signs zero.

Theorem 4.6. For any two elements \( x, y \in \mathcal{R} \), we have \( x \otimes y = y \otimes x \).
Proof.

\[x \otimes y = \Psi(\text{sign}(x) + \text{sign}(y)) \left( \text{LIMIT}(\{[|x||k] \cdot [|[y]|]|k \in \mathbb{N}\}) \right) = \Psi(\text{sign}(y) + \text{sign}(x)) \left( \text{LIMIT}(\{[|y||k] \cdot [|[x]|]|k \in \mathbb{N}\}) \right) = y \otimes x.\]

\[\square\]

**Theorem 4.7.** For any two elements \(x, y \in \mathcal{R}\), we have

\[\Psi(x \otimes y) = \Psi(x) \otimes y = x \otimes \Psi(y).\]

**Proof.** This is the twin theorem of Theorem 3.14. By Theorem 4.6 it suffices to prove \(\Psi(x \otimes y) = \Psi(x) \otimes y\). To this aim we note

\[\Psi(x) \otimes y = \Psi(\text{sign}(\Psi(x)) + \text{sign}(y)) \left( \text{LIMIT}(\{[|\Psi(x)||k] \cdot [|[y]|]|k \in \mathbb{N}\}) \right) = \Psi(1 + \text{sign}(x) + \text{sign}(y)) \left( \text{LIMIT}(\{[|x||k] \cdot [|[y]|]|k \in \mathbb{N}\}) \right) = \Psi \left( \Psi(\text{sign}(x) + \text{sign}(y)) \right) \left( \text{LIMIT}(\{[|x||k] \cdot [|[y]|]|k \in \mathbb{N}\}) \right) = \Psi(x \otimes y),\]

here we have used the fact that \(\Psi^{(2)}\) is the identity map. This proves the theorem.

\[\square\]

**Theorem 4.8.** Given \(x, y, z, w \in \mathcal{R}\) with \(0.000000 \leq x \leq z \leq w \leq y \leq 0.000000\), we have \(x \otimes y \leq z \otimes w\).

**Proof.** This is the twin theorem of Theorem 3.5. \(\forall k \in \mathbb{N}\) we have \(0 \leq [x]_k \leq [z]_k\) and \(0 \leq [y]_k \leq [w]_k\), which yields \([x]_k \cdot [y]_k \leq [z]_k \cdot [w]_k\). By Lemma 2.9, \(\text{LIMIT}(\{[x]_k \cdot [y]_k \}_{k \in \mathbb{N}}) \leq \text{LIMIT}(\{[z]_k \cdot [w]_k \}_{k \in \mathbb{N}})\). This proves the theorem.

\[\square\]

**Theorem 4.9.** Given \(x, y, z, w \in \mathcal{R}\) with \(x \sim z \leq y \sim w\), we have \(x \otimes y \sim z \otimes w\).

**Proof.** Obviously if two equivalent elements \(x\) and \(z\) have different signs, then we must have \(\{x, z\} = \{0.000000, \ldots, (-1), 0.000000, \ldots\}\). From Example 4.3, \(x \otimes y \sim 0.000000 \cdots \sim z \otimes w\). Thus to prove this theorem, we may assume that \(\text{sign}(x) = \text{sign}(z)\), \(\text{sign}(y) = \text{sign}(w)\), which yields \(\text{sign}(x) + \text{sign}(y) = \text{sign}(z) + \text{sign}(w)\). By Theorem 4.7,

\[\Psi(\text{sign}(x) + \text{sign}(y)) (x \otimes y) = \Psi(\text{sign}(x)) (x) \otimes \Psi(\text{sign}(y)) (y) = ||x|| \otimes || y||,\]

\[\Psi(\text{sign}(z) + \text{sign}(w)) (z \otimes w) = \Psi(\text{sign}(z)) (z) \otimes \Psi(\text{sign}(w)) (w) = ||z|| \otimes || w||.\]

Consequently, to prove this theorem we may further assume below \(x, y, z, w \geq 0.000000 \cdots\), by which we shall make use of Theorem 4.8. Let \(M \in \mathbb{N}\) be an upper bound for \(\{x, y, z, w\}\).

For any \(k \in \mathbb{N}\),

\[|x \otimes y|_k \leq |x| \otimes |y| \leq ([x]_k + \frac{1}{10^k}) \cdot ([y]_k + \frac{1}{10^k}) \leq ([z]_k + \frac{2}{10^k}) \cdot ([w]_k + \frac{2}{10^k}) \leq [z]_k \cdot [w]_k + \frac{5M}{10^k},\]

\[\leq (z \otimes w) + \frac{5M}{10^k} \leq |z \otimes w|_k + \frac{5M + 1}{10^k}.\]
here we have used Lemma 2.14 for $x \sim z$ and $y \sim w$. By symmetry we can also have $[z \otimes w]_k \leq [x \otimes y]_k + \frac{5M+1}{10k}$. Finally by Lemma 3.8, $x \otimes y \sim z \otimes w$. This concludes the whole proof.

\[\square\]

**Theorem 4.10.** For any three elements $x, y, z \in \mathcal{R}$, we have $(x \otimes y) \otimes z \sim x \otimes (y \otimes z)$.

*Proof.* By Theorem 4.7, to prove this theorem we may assume that $x, y, z \geq 0.000000 \cdots$, by which we shall also make use of Theorem 4.8. Let $M \in \mathbb{N}$ be an upper bound for \{x, y, z\}. For any $k \in \mathbb{N}$,

\[(x \otimes y) \otimes z \leq ([x]_k + \frac{1}{10k}) \cdot ([y]_k + \frac{1}{10k}) \cdot ([z]_k + \frac{1}{10k}) \leq [x]_k \cdot [y]_k \cdot [z]_k + \frac{4M^2}{10k} \leq w \oplus \frac{4M^2}{10k},\]

where $w \triangleq \text{LIMIT}(\{[x]_n \cdot [y]_n \cdot [z]_n\}_{n \in \mathbb{N}})$. On the other hand,

\[[x]_n \cdot [y]_n \cdot [z]_n \leq (x \otimes y) \otimes [z]_n \leq (x \otimes y) \otimes z.\]

Letting $n \to \infty$ yields $w \leq (x \otimes y) \otimes z$. With these preparations in hand, for any $k \in \mathbb{N}$ we have $[w]_k \leq |(x \otimes y) \otimes z|_k \leq [w \oplus \frac{4M^2}{10k}]_k = [w]_k + \frac{4M^2}{10k}$. Similarly, we can also have $[w]_k \leq |(y \otimes z) \otimes x|_k \leq [w]_k + \frac{4M^2}{10k}$. Finally by Lemma 3.8, $(x \otimes y) \otimes z \sim (y \otimes z) \otimes x = x \otimes (y \otimes z)$. This concludes the whole proof. \[\square\]

**Theorem 4.11.** For any three elements $x, y, z \in \mathcal{R}$, we have $(x \otimes y) \otimes z \sim (x \otimes z) \oplus (y \otimes z)$.

*Proof.* By Theorems 3.14 and 4.7, to prove this theorem we may assume $	ext{sign}(z) = 0$, and suppose this is the case.

Case 1: Suppose $	ext{sign}(x) = \text{sign}(y)$. By Theorems 3.14 and 4.7 again, we may further assume $	ext{sign}(x) = \text{sign}(y) = 0$. Let $M \in \mathbb{N}$ be an upper bound for \{x, y, z\}. For any $k \in \mathbb{N}$,

\[(x \oplus y) \otimes z \leq (x \oplus y) \otimes z \leq ([x \oplus y]_k + \frac{1}{10k}) \cdot ([z]_k + \frac{1}{10k}) \leq ([x]_k + [y]_k) \cdot ([z]_k + \frac{1}{10k}) \leq ([x]_k \cdot [y]_k \cdot [z]_k + \frac{6M}{10k}) \leq ([x]_k \cdot [y]_k \cdot [z]_k + \frac{6M}{10k}) \leq ([x \otimes y]_k \oplus (y \otimes z))_k + \frac{6M}{10k}.\]

On the other hand,

\[(x \otimes z) \oplus (y \otimes z) \leq (x \otimes z) \oplus (y \otimes z) \leq ([x \otimes z]_k \oplus [y \otimes z])_k \leq ([x]_k + [y]_k) \cdot ([z]_k + \frac{1}{10k}) \leq ([x]_k \cdot [y]_k \cdot [z]_k + \frac{6M}{10k}) \leq ([x \otimes y]_k \oplus (y \otimes z))_k + \frac{6M}{10k},\]

Thus by Lemma 3.8, $(x \oplus y) \otimes z \sim (x \otimes z) \oplus (y \otimes z)$.

Case 2: Suppose $x, y$ have different signs. Thus $x \oplus y$ has the same sign as one of $x, y$, say for example, has the same sign as $x$. Thus $\Psi(y)$ and $x \oplus y$ have the same sign.
According to the analysis in Case 1,

\((\Psi(y) \oplus (x \oplus y)) \otimes z \sim (\Psi(y) \otimes z) \oplus ((x \oplus y) \otimes z)\).

Adding \(y \otimes z\) to the both sides of the above formula yields

\((y \otimes z) \oplus (x \otimes z) \sim (\Psi(y) \otimes z) \oplus ((x \oplus y) \otimes z)\).

So we are left to prove that \((y \otimes z) \oplus (\Psi(y) \otimes z) \sim 0.000000 \cdots\). To this aim, we note

\[\Psi((y \otimes z) \oplus (\Psi(y) \otimes z)) \sim (\Psi(y) \otimes z) \oplus (y \otimes z) = (y \otimes z) \oplus (\Psi(y) \otimes z),\]

which naturally implies \((y \otimes z) \oplus (\Psi(y) \otimes z) \sim 0.000000 \cdots\).

This concludes the whole proof of the theorem. \(\square\)

4.2. Multiplicative inverses.

**Definition 4.12** (Inverse of multiplication). For any \(x \in \mathcal{R}\) with \(\|x\| > 0.000000 \cdots\), let

\[x^{-1} \triangleq \Psi(\text{sign}(x)) \left(\text{LIMIT}\left(\left\{\frac{1}{\|x\|_k}\right\}_{k \in \mathbb{N}}\right)\right)\].

For some small \(k\), \(\|x\|_k\) might be equal to zero. So it is possible that \(\frac{1}{\|x\|_k}\) is meaningless, or has the meaning of positive infinity. But since we are mainly concerned with the greatest lower bound for the set \(\left\{\frac{1}{\|x\|_k}\right\}_{k \in \mathbb{N}}\), it does not matter whether the initial items of the sequence \(\left\{\frac{1}{\|x\|_k}\right\}_{k \in \mathbb{N}}\) are equal to positive infinity or not.

**Theorem 4.13.** For any \(x \in \mathcal{R}\) with \(\|x\| > 0.000000 \cdots\), we have \(\Psi(x)^{-1} = \Psi(x^{-1})\).

**Proof.** Since \(\|\Psi(x)\| = \|x\| > 0.000000 \cdots\), by definition we have

\[\Psi(x)^{-1} = \Psi(\text{sign}(\Psi(x))) \left(\text{LIMIT}\left(\left\{\frac{1}{\|\Psi(x)\|_k}\right\}_{k \in \mathbb{N}}\right)\right)
= \Psi(1+\text{sign}(x)) \left(\text{LIMIT}\left(\left\{\frac{1}{\|x\|_k}\right\}_{k \in \mathbb{N}}\right)\right) = \Psi(x^{-1}).\]

\(\square\)

**Theorem 4.14.** For any \(x \in \mathcal{R}\) with \(\|x\| > 0.000000 \cdots\), we have \(x \otimes x^{-1} \sim 1.000000 \cdots\).

**Proof.** According to Theorem 4.13, to prove this theorem we may assume \(x > 0.000000 \cdots\), and suppose this is the case. Since \(\|x\| > 0.000000 \cdots\), one can find a sufficiently large \(m \in \mathbb{N}\) such that \(\frac{1}{10^m} \leq x\). For any naturals number \(k \geq n \geq m\), \([x]_n \leq [x]_k \leq [x]_n + 10^{-n}\). Hence taking reciprocals in each part we get \(\frac{1}{[x]_n + 10^{-n}} \leq \frac{1}{[x]_k} \leq \frac{1}{[x]_n}\). Letting \(k \to \infty\) yields \(\frac{1}{[x]_n + 10^{-n}} \leq x^{-1} \leq \frac{1}{[x]_n}\). Combining this with Lemma 3.6 we have

\[([x]_n \cdot \frac{1}{[x]_n + 10^{-n}} \leq x \otimes x^{-1} \leq ([x]_n + 10^n) \cdot \frac{1}{[x]_n}].\]

Since \(\frac{1}{10^m} \leq x\), we have \(\frac{1}{10^m} \leq [x]_m \leq [x]_n\), and consequently, \([x]_n \cdot \frac{1}{[x]_n + 10^{-n}} \geq 1 - 10^{-m-n}\), \(([x]_n + 10^n) \cdot \frac{1}{[x]_n} \leq 1 + 10^{m-n}\). Finally for any \(s \in \mathbb{N}\), let us take \(n = m + s\). Then
\[1 - \frac{1}{10^m} \leq x \otimes x^{-1} \leq 1 + \frac{1}{10^m},\]
which yields \(1 - \frac{1}{10^m} \leq [x \otimes x^{-1}]_s \leq 1 + \frac{1}{10^m}\). Note also \(1 - \frac{1}{10^m} \leq [1.000000 \cdots]_s \leq 1 + \frac{1}{10^m}\). Thus a standard application of Lemma 2.14 gives the desired result. We are done.

\(\square\)
5. Construction of real numbers

5.1. Motivation. In mathematics, since \( \mathbb{R} \) is a set, there is no doubt that 0.999999\( \cdots \) and 1.000000\( \cdots \) are two different elements. But in the real world, if we still imagine 0.999999\( \cdots \) and 1.000000\( \cdots \) as two different objects, then it will cause some trouble. For example, how can we define and understand

\[
\frac{0.999999\cdots + 1.000000\cdots}{2}
\]

Since there is no element \( z \in \mathbb{R} \) such that 0.999999\( \cdots \) \( < \) \( z \) \( < \) 1.000000\( \cdots \), we may regard \( \mathbb{R} \) having “holes” in many places. Have you ever seen a bunch of sunlight having lots of holes? It is too terrible!

Noting the sequence \( \{[0.999999\cdots : k]\}_{k=1}^{\infty} \) is getting “closer and closer” to 1.000000\( \cdots \) as \( k \) approaches to infinity, the best way we think to understand the relation between 0.999999\( \cdots \) and 1.000000\( \cdots \) both in mathematics and in the real world, is to view them as the same object, which leads to the invention of the real “real numbers” below.

5.2. Methodology. Define the set of real numbers \( \mathbb{R} \) to be the set of equivalent classes \( \mathbb{R}/\sim \). For those readers who are not too familiar with the algebraic representation \( \mathbb{R}/\sim \), the definition of \( \mathbb{R} \) is not too abstract at all: if the equivalent class has exactly two elements, you may simply discard one element, say for example discard those elements in \( \mathbb{R}_0 \) as we have done in Section 1, and leave the other one remained.

For any two equivalent classes \([x], [y]\), we introduce an induced operation \( \hat{\oplus} \) on \( \mathbb{R} \) by

\[
[x] \hat{\oplus} [y] \triangleq [x \oplus y].
\]

By Theorems 3.9, \( \hat{\oplus} \) is well-defined. By Theorem 3.4, \( \hat{\oplus} \) is commutative. By Theorem 3.10, \( \hat{\oplus} \) is associative. From Example 3.2, \((\mathbb{R}, \hat{\oplus})\) has a unit \([0.000000\cdots]\). Since \( x \oplus \Psi(x) = (-1).999999\cdots \sim 0.000000\cdots \), every element of \((\mathbb{R}, \hat{\oplus})\) has an inverse. In the language of algebra, \((\mathbb{R}, \hat{\oplus})\) is an Abelian group.

For any two equivalent classes \([x], [y]\), we introduce an induced operation \( \hat{\otimes} \) on \( \mathbb{R} \) by

\[
[x] \hat{\otimes} [y] \triangleq [x \otimes y].
\]

By Theorem 4.9, \( \hat{\otimes} \) is well-defined. By Theorem 4.6, \( \hat{\otimes} \) is commutative. By Theorem 4.10, \( \hat{\otimes} \) is associative. From Example 4.2, \((\mathbb{R}, \hat{\otimes})\) has a unit \([1.000000\cdots]\). By Theorem 4.14, every element of \((\mathbb{R}^*, \hat{\otimes})\) has an inverse, where \( \mathbb{R}^* \triangleq \mathbb{R} \setminus [0.000000\cdots] \). By Theorem 4.11, \( \hat{\otimes} \) is distribute over \( \hat{\oplus} \). Hence we have derived a number system \((\mathbb{R}, \hat{\oplus}, \hat{\otimes})\), which in the language of algebra is a field.

We can also introduce an order \( \hat{\geq} \) on \( \mathbb{R} \) from the total order \( \leq \) on \( \mathbb{R} \), that is, we say \([x] \hat{\geq} [y]\) if \( y \oplus \Psi(x) \geq (-1).999999\cdots \). The interested readers can verify that \( \hat{\geq} \) is well-defined and is a total order. Finally, we state the least upper and greatest lower bounds properties for \((\mathbb{R}, \hat{\geq})\) below without proofs, which is a simple corollary of Theorem 2.5.

**Theorem 5.1.** Every nonempty bounded above (below) subset of \((\mathbb{R}, \hat{\geq})\) has a least upper (greatest lower) bound.

As experienced readers should know, from the least upper bound property, after introducing Weierstrass’s \( \epsilon - N \) definition of convergence of sequences, we can deduce many other theorems such as the monotone convergence theorem, Cauchy’s convergence principle, the Bolzano-Weierstrass theorem, the Heine-Borel theorem and so on in a few pages.
6. FROM DECIMALS TO DEDEKIND CUTS

6.1. A natural bijection.

**Definition 6.1.** A Dedekind cut is a pair of nonempty subsets \( A, B \) of \( \mathbb{Q} \), denoted by \((A|B)\), such that

- \( A \cap B = \emptyset, A \cup B = \mathbb{Q} \),
- \( a \in A, b \in B \Rightarrow a < b \),
- \( A \) contains no greatest element.

The set of all Dedekind cuts is denoted by \( \text{DR} \).

Given a Dedekind cut \((A|B)\), we obviously have \( \sup A \preceq \inf B \). To go even further, we can have the following theorem.

**Theorem 6.2.** For any Dedekind cut \((A|B)\), we have \( \sup A \sim \inf B \).

**Proof.** Fix arbitrarily \( k \in \mathbb{N}, a \in A \) and \( b \in B \). Step 1: Let \( c(1) \triangleq \frac{a + b}{2} \). If \( c(1) \in A \), then we let \( a(1) \triangleq c(1) \), \( b(1) \triangleq b \); else if \( c(1) \in B \), then we let \( a(1) \triangleq a, b(1) \triangleq c(1) \). Obviously \( b(1) - a(1) = \frac{b - a}{2} \).

Step 2: Let \( c(2) \triangleq \frac{c(1) + b(1)}{2} \). If \( c(2) \in A \), then we let \( a(2) \triangleq c(2), b(2) \triangleq b(1) \); else if \( c(2) \in B \), then we let \( a(2) \triangleq a(1), b(2) \triangleq c(2) \). Obviously \( b(2) - a(2) = \frac{b(1) - a(1)}{2} = \frac{b - a}{4} \).

We can repeat this procedure to the \( 4k \)-th step to get \( a^{(4k)} \in A \) and \( b^{(4k)} \in B \) so that \( b^{(4k)} - a^{(4k)} = \frac{b - a}{16^k} \). Note \( a^{(4k)} \preceq \sup A \preceq \inf B \preceq b^{(4k)} \), thus

\[
0 \leq [\inf B]_k - [\sup A]_k \leq [b^{(4k)}]_k - [a^{(4k)}]_k \leq b^{(4k)} - a^{(4k)} - \left( \frac{1}{10^k} \right) \leq \frac{b - a}{16^k} + \frac{1}{10^k} \leq \frac{2b - a}{10^k},
\]

here we have used Lemma 3.6 at the third inequality. By Lemma 3.8, \( \sup A \sim \inf B \). \( \square \)

Now we can derive a map \( \tau \) from \( \text{DR} \) to \( \mathcal{R}/\sim \) by sending \((A|B)\) to \([\sup A] = [\inf B]\). It would be great if \( \tau \) is bijective. In the following we shall prove this is indeed the case.

**Lemma 6.3.** We have \( \mathcal{R}_q \cap \mathbb{Q} = \emptyset \).

**Proof.** Suppose the contrary one can find an element \( \alpha \in \mathcal{R}_q \cap \mathbb{Q} \). Let us choose the corresponding element \( \beta \in \mathbb{Q}_F \) so that \( \alpha < \beta, \alpha \sim \beta \). Obviously as elements of \( \mathbb{Q}, \alpha < \beta \).

Now choose a sufficiently large \( k \in \mathbb{N} \) so that \( \beta - \alpha \geq \frac{3}{10^k} \), then by Lemma 3.6,

\[
[\beta]_k - [\alpha]_k \geq \frac{b - a}{10^k} \sim \beta - \frac{1}{10^k} - \alpha \geq \frac{2}{10^k}.
\]

According to Lemma 2.14, this is impossible. We are done. \( \square \)

**Theorem 6.4.** The map \( \tau \) from \( \text{DR} \) to \( \mathcal{R}/\sim \) is bijective.

**Proof.** We first prove that \( \tau \) is surjective. Fix arbitrarily \( x \in \mathcal{R} \). Let

\[
A \triangleq \{ q \in \mathbb{Q} \mid q < x \},
B \triangleq \{ q \in \mathbb{Q} \mid q \geq x \}.
\]

Obviously to verify that \((A|B)\) is a Dedekind cut, it suffices to prove that \( A \) contains no greatest element. Suppose the contrary, then \( \sup A \in A \), which by the definitions of \( A \)
and $B$ leads to $\sup A < x \leq \inf B$. By Theorem 6.2, $\sup A \sim \inf B$. By Lemma 2.12, $\sup A \in \mathcal{R}_9$, which contradicts to Lemma 6.3. Hence $(A|B)$ must be a Dedekind cut. Note $\sup A \leq x \leq \inf B$, by Theorem 6.2 again, we have $\tau((A|B)) = [x]$. Thus $\tau$ is surjective.

Next we prove that $\tau$ is injective. Given two different Dedekind cuts $(A|B)$, $(C|D)$, without loss of generality we may assume there exists an element $\alpha \in C \setminus A$, which by the definition of Dedekind cut yields $\sup A \leq \alpha < \sup C$. Our aim is to prove that $\sup A$ and $\sup C$ are not equivalent. Suppose this is the contrary, then $\sup A \sim \sup C$. By Lemma 2.12, $\alpha = \sup A \in \mathcal{R}_9$, which contradicts to Lemma 6.3. Thus $\tau$ is injective.

This concludes the whole proof of the theorem.

\[ \Box \]

6.2. **Isomorphisms between operations.**

**Definition 6.5.** A generalized Dedekind cut is a pair of nonempty subsets $A, B$ of $\mathbb{Q}$, denoted by $[A \mathbin{\uparrow} B]$, such that

- $A \cap B = \emptyset$, $A \cup B = \mathbb{Q}$,
- $a \in A, b \in B \Rightarrow a < b$.

**Remark 6.6.** To form a Dedekind cut $(\tilde{A}\tilde{B})$ from a generalized Dedekind cut $[A \mathbin{\uparrow} B]$, we can simply move the greatest element of $A$ if it exists, from $A$ to $B$. In this case it is no hard to prove that $\sup A \sim \inf B \sim \sup \tilde{A} \sim \inf \tilde{B}$, so we can abuse the uses of $[A \mathbin{\uparrow} B]$ and $(\tilde{A}\tilde{B})$. For example, let $\tilde{\Psi} : \mathbb{D} \mathbb{R} \rightarrow \mathbb{D} \mathbb{R}$ be the additive inverse map defined by sending $(A|B)$ to $[-B \mathbin{\uparrow} -A]$.

**Definition 6.7.** Let $\Theta : \mathbb{D} \mathbb{R} \rightarrow \{0, 1\}$ be the signal map

$$\Theta((A|B)) \triangleq \max_{x \in B} \text{sign}(x).$$

The absolute value of a Dedekind cut $(A|B)$ is defined by

$$\|A|B\| \triangleq \tilde{\Psi}(\Theta((A|B))) ((A|B)).$$

**Definition 6.8.** For any two Dedekind cuts $(A|B)$, $(C|D)$, define the addition of both cuts by

$$(A|B) \boxplus (C|D) = [\mathbb{Q}\setminus (B + D) \mathbin{\uparrow} B + D].$$

If the Dedekind cuts $(A|B)$, $(C|D)$ are of signs zero, define the multiplication of them by

$$(A|B) \boxtimes (C|D) = [\mathbb{Q}\setminus (B \cdot D) \mathbin{\uparrow} B \cdot D].$$

Generally for any two Dedekind cuts $(A|B)$, $(C|D)$, define their multiplication by

$$(A|B) \boxtimes (C|D) = \tilde{\Psi}(\Theta((A|B)) + \Theta((C|D))) \big(\|A|B\| \boxtimes \|C|D\|\big).$$

**Theorem 6.9.** For any two Dedekind cuts $(A|B)$, $(C|D)$, we have

$$\tau((A|B) \boxplus (C|D)) = \tau((A|B)) \oplus \tau((C|D)).$$

**Proof.** To prove this theorem, it suffices to prove that $\inf(B + D) \sim \inf B \oplus \inf D$. To this aim we note $\sup(A + C) \leq \sup A \oplus \sup C \leq \inf B \oplus \inf D \leq \inf(B + D)$. Hence we need only to prove that $\sup(A + C) \sim \inf(B + D)$.

Fix arbitrarily $k \in \mathbb{N}$. We can repeat the procedures in the proof of Theorem 6.2 from any common starting points $a \in A \cap C$ and $b \in B \cap D$ to get $a^{(4k)} \in A$, $b^{(4k)} \in B$, $c^{(4k)} \in C$
and \(d^{(4k)} \in D\) so that \(b^{(4k)} - a^{(4k)} = d^{(4k)} - c^{(4k)} = \frac{b - a}{16^k}\). Thus
\[
a^{(4k)} + c^{(4k)} \leq \sup(A + C) \leq \inf(B + D) \leq b^{(4k)} + d^{(4k)},
\]
and consequently,
\[
0 \leq [\inf(B + D)]_k - [\sup(A + C)]_k \leq [b^{(4k)} + d^{(4k)}]_k - [a^{(4k)} + c^{(4k)}]_k \leq \frac{2[b - a]}{16^k} + \frac{1}{10^k}.
\]
By Lemma 3.8, we are done. □

**Theorem 6.10.** For any two Dedekind cuts \((A|B), (C|D)\), we have
\[
\tau((A|B) \otimes (C|D)) = \tau((A|B)) \otimes \tau((C|D)).
\]

**Proof.** We shall only prove this theorem for the special case that the Dedekind cuts \((A|B), (C|D)\) are of signs zero, and leave the general case to the interested readers. Suppose the Dedekind cuts \((A|B), (C|D)\) are of signs zero. To our aim, it suffices to prove that \(\inf(B \cdot D) \sim \inf B \otimes \inf D\). If either \(0 \in B\) or \(0 \in D\), then it is easy to observe that \(\inf(B \cdot D) = \inf B \otimes \inf D = 0.000000 \cdots\). Hence we may further assume that \(0 \in A \cap C\). Let us choose a fixed natural number \(z \in B \cap D\).

Fix arbitrarily \(k \in \mathbb{N}\). We can repeat the procedures in the proof of Theorem 6.2 from the starting points \(0 \in A \cap C\) and \(z \in B \cap D\) to get \(a^{(4k)} \in A, b^{(4k)} \in B, c^{(4k)} \in C\) and \(d^{(4k)} \in D\) so that \(b^{(4k)} - a^{(4k)} = d^{(4k)} - c^{(4k)} = \frac{z - 0}{16^k} = \frac{z}{16^k}\). Thus
\[
0.000000 \cdots \leq a^{(4k)} \cdot c^{(4k)} \leq \sup A \otimes \sup C \leq \inf B \otimes \inf D \leq \inf(B \cdot D) \leq b^{(4k)} \cdot d^{(4k)},
\]
and consequently,
\[
0 \leq [\inf(B \cdot D)]_k - [\inf B \otimes \inf D]_k \leq [b^{(4k)} \cdot d^{(4k)}]_k - [a^{(4k)} \cdot c^{(4k)}]_k \leq \frac{2z^2}{16^k} + \frac{z^2}{256^k} + \frac{1}{10^k}.
\]
By Lemma 3.8, we are done. □

Given two Dedekind cuts \((A|B), (C|D)\), we say \((A|B)\) is less than or equal to \((C|D)\), denoted by \((A|B) \preceq (C|D)\), if \(\sup A \preceq \sup C\). It is a piece of cake to prove that
\[
(A|B) \preceq (C|D) \iff \tau((A|B)) \preceq \tau((C|D)).
\]
Thus with Theorems 6.4, 6.9 and 6.10, \((\mathbb{R}, \oplus, \odot, \preceq)\) is isomorphic to \((\mathbb{D}

7. From decimals to Cauchy sequences

7.1. A natural surjection.

**Definition 7.1.** A sequence of rational numbers \(\{x^{(n)}\}_{n \in \mathbb{N}}\) is called Cauchy if for every rational \(\epsilon > 0\), there exists an \(N \in \mathbb{N}\) such that \(\forall m, n \geq N, |x^{(m)} - x^{(n)}| < \epsilon\). The set of all Cauchy sequences of rational numbers is denoted by \(\text{CR}\). Two Cauchy sequences of rational numbers \(\{x^{(n)}\}_{n \in \mathbb{N}}, \{y^{(n)}\}_{n \in \mathbb{N}}\) are said to be equivalent, denoted by \(\{x^{(n)}\}_{n \in \mathbb{N}} \approx \{y^{(n)}\}_{n \in \mathbb{N}}\), if for every rational \(\epsilon > 0\), there exists an \(N \in \mathbb{N}\) such that \(\forall n \geq N, |x^{(n)} - y^{(n)}| < \epsilon\).

It is easy to prove \(\approx\) is an equivalence relation, thus we can get a quotient space \(\text{CR}/\approx\), denoted by \(\overline{\text{CR}}\) for consistency. Given a Cauchy sequence of rational numbers \(\{x^{(n)}\}_{n \in \mathbb{N}}\), we obviously have \(\overline{\text{LIMIT}}(\{x^{(n)}\}_{n \in \mathbb{N}}) \leq \overline{\text{LIMIT}}(\{x^{(n)}\}_{n \in \mathbb{N}})\). To go even further, we can have the following theorem.
**Theorem 7.2.** \( \lim_{n \to \infty} \{x^{(n)}\}_{n \in \mathbb{N}} \sim \lim_{n \to \infty} \{y^{(n)}\}_{n \in \mathbb{N}} \) holds for any Cauchy sequence of rational numbers \( \{x^{(n)}\}_{n \in \mathbb{N}} \).

**Proof.** Fix arbitrarily \( k \in \mathbb{N} \), there exists an \( N \in \mathbb{N} \) depending on \( k \) such that \( \forall n, m \geq N, |x^{(n)} - x^{(m)}| < \frac{1}{10^k} \). Thus \( \forall n \geq N, x^{(N)} - \frac{1}{10^k} \leq x^{(n)} \leq x^{(N)} + \frac{1}{10^k} \). Letting \( n \to \infty \) gives

\[
x^{(N)} - \frac{1}{10^k} \leq \lim_{n \to \infty} \{x^{(n)}\}_{n \in \mathbb{N}} \leq \lim_{n \to \infty} \{x^{(n)}\}_{n \in \mathbb{N}} \leq x^{(N)} + \frac{1}{10^k},
\]

and consequently

\[
0 \leq [\lim_{n \to \infty} \{x^{(n)}\}_{n \in \mathbb{N}}]_k - [\lim_{n \to \infty} \{x^{(n)}\}_{n \in \mathbb{N}}]_k \leq [x^{(N)} + \frac{1}{10^k}]_k - [x^{(N)} - \frac{1}{10^k}]_k = \frac{2}{10^k}.
\]

By Lemma 3.8, \( \lim_{n \to \infty} \{x^{(n)}\}_{n \in \mathbb{N}} \sim \lim_{n \to \infty} \{y^{(n)}\}_{n \in \mathbb{N}} \). This finishes the proof.

Now we can derive a map \( \kappa \) from \( \mathbb{CR} \) to \( \mathbb{R}/ \sim \) by sending Cauchy sequence \( \{x^{(n)}\}_{n \in \mathbb{N}} \) to

\[
[\lim_{n \to \infty} \{x^{(n)}\}_{n \in \mathbb{N}}] = [\lim_{n \to \infty} \{x^{(n)}\}_{n \in \mathbb{N}}].
\]

Obviously, \( \kappa \) is surjective since for any \( x \in \mathbb{R} \), we have a typical Cauchy sequence \( \{[x]_k\}_{k \in \mathbb{N}} \) such that \( x = \lim_{n \to \infty} \{[x]_k\}_{k \in \mathbb{N}} \).

**Theorem 7.3.** For any two Cauchy sequences of rational numbers \( \{x^{(n)}\}_{n \in \mathbb{N}}, \{y^{(n)}\}_{n \in \mathbb{N}} \), \( \{x^{(n)}\}_{n \in \mathbb{N}} \approx \{y^{(n)}\}_{n \in \mathbb{N}} \iff \kappa(\{x^{(n)}\}_{n \in \mathbb{N}}) = \kappa(\{y^{(n)}\}_{n \in \mathbb{N}}) \).

**Proof.** Suppose \( \{x^{(n)}\}_{n \in \mathbb{N}} \approx \{y^{(n)}\}_{n \in \mathbb{N}} \). Fix arbitrarily \( k \in \mathbb{N} \), there exists an \( N \in \mathbb{N} \) such that \( \forall n \geq N, |x^{(n)} - y^{(n)}| < \frac{1}{10^k} \). Equivalently, \( \forall n \geq N \) we have \( x^{(n)} \leq y^{(n)} + \frac{1}{10^k} \) and \( y^{(n)} \leq x^{(n)} + \frac{1}{10^k} \). Letting \( n \to \infty \) yields \( \lim_{n \to \infty} \{x^{(n)}\}_{n \in \mathbb{N}} \leq \lim_{n \to \infty} \{y^{(n)}\}_{n \in \mathbb{N}} + \frac{1}{10^k} \) and \( \lim_{n \to \infty} \{y^{(n)}\}_{n \in \mathbb{N}} \leq \lim_{n \to \infty} \{x^{(n)}\}_{n \in \mathbb{N}} + \frac{1}{10^k} \), which further implies

\[
|[\lim_{n \to \infty} \{x^{(n)}\}_{n \in \mathbb{N}}]_k - [\lim_{n \to \infty} \{y^{(n)}\}_{n \in \mathbb{N}}]_k | \leq \frac{1}{10^k}.
\]

By Lemma 3.8, \( \lim_{n \to \infty} \{x^{(n)}\}_{n \in \mathbb{N}} \sim \lim_{n \to \infty} \{y^{(n)}\}_{n \in \mathbb{N}} \). This proves the necessary part.

Suppose \( \kappa(\{x^{(n)}\}_{n \in \mathbb{N}}) = \kappa(\{y^{(n)}\}_{n \in \mathbb{N}}) \). We argue by contraposition the sufficient part and suppose \( \{x^{(n)}\}_{n \in \mathbb{N}}, \{y^{(n)}\}_{n \in \mathbb{N}} \) are not equivalent. Then there exist a positive rational number \( \varepsilon_0 \) and a sequence of natural numbers \( n_1 < n_2 < n_3 < \ldots \) such that \( \forall i \in \mathbb{N} \), \( |x^{(n_i)} - y^{(n_i)}| \geq \varepsilon_0 \). It may happen either \( x^{(n_i)} \geq y^{(n_i)} + \varepsilon_0 \) or \( y^{(n_i)} \geq x^{(n_i)} + \varepsilon_0 \). Without loss of generality, we may assume there are infinitely many terms \( x^{(n_i)} \geq y^{(n_i)} + \varepsilon_0 \). Now we first choose a \( k \in \mathbb{N} \) such that \( \frac{2}{10^k} \leq \varepsilon_0 \), then choose a subsequence \( \{n_{i_j}\}_{j \in \mathbb{N}} \) from \( \{n_i\}_{i \in \mathbb{N}} \) such that \( x^{(n_{i_j})} \geq y^{(n_{i_j})} + \frac{2}{10^k} \). Thus

\[
\lim_{n \to \infty} \{x^{(n_{i_j})}\}_{j \in \mathbb{N}} \geq \lim_{n \to \infty} \{y^{(n_{i_j})}\}_{j \in \mathbb{N}} + \frac{2}{10^k},
\]

which yields \( [\lim_{n \to \infty} \{x^{(n_{i_j})}\}_{j \in \mathbb{N}}]_k \geq [\lim_{n \to \infty} \{y^{(n_{i_j})}\}_{j \in \mathbb{N}}]_k + \frac{2}{10^k} \), and consequently by Lemma 3.8, \( \lim_{n \to \infty} \{x^{(n_{i_j})}\}_{j \in \mathbb{N}} \) and \( \lim_{n \to \infty} \{y^{(n_{i_j})}\}_{j \in \mathbb{N}} \) are not equivalent in \( \mathbb{R} \). But this is impossible since by Lemma 2.9, Theorem 7.2 and our assumptions,

\[
\begin{align*}
\lim_{n \to \infty} \{x^{(n)}\}_{n \in \mathbb{N}} &\leq \lim_{n \to \infty} \{x^{(n_{i_j})}\}_{j \in \mathbb{N}} \\
\lim_{n \to \infty} \{y^{(n)}\}_{n \in \mathbb{N}} &\leq \lim_{n \to \infty} \{y^{(n_{i_j})}\}_{j \in \mathbb{N}} \\
\lim_{n \to \infty} \{x^{(n)}\}_{n \in \mathbb{N}} &\sim \lim_{n \to \infty} \{x^{(n_{i_j})}\}_{j \in \mathbb{N}} \\
\lim_{n \to \infty} \{y^{(n)}\}_{n \in \mathbb{N}} &\sim \lim_{n \to \infty} \{y^{(n_{i_j})}\}_{j \in \mathbb{N}}.
\end{align*}
\]
which naturally implies that \( \text{LIMIT}(\{x^{(n)}\}_{j \in \mathbb{N}}) \sim \text{LIMIT}(\{y^{(n)}\}_{j \in \mathbb{N}}) \). This proves the sufficient part, also concludes the whole proof of the theorem. \( \square \)

7.2. Homomorphisms between operations.

**Definition 7.4.** Given two Cauchy sequences of rational numbers \( \{x^{(n)}\}_{n \in \mathbb{N}}, \{y^{(n)}\}_{n \in \mathbb{N}} \), it is easy to verify that \( \{x^{(n)} + y^{(n)}\}_{n \in \mathbb{N}} \) and \( \{x^{(n)} \cdot y^{(n)}\}_{n \in \mathbb{N}} \) are also Cauchy sequences, so we can define the addition and multiplication respectively by

\[
\begin{align*}
\{x^{(n)}\}_{n \in \mathbb{N}} \oplus \{y^{(n)}\}_{n \in \mathbb{N}} &= \{x^{(n)} + y^{(n)}\}_{n \in \mathbb{N}}, \\
\{x^{(n)}\}_{n \in \mathbb{N}} \otimes \{y^{(n)}\}_{n \in \mathbb{N}} &= \{x^{(n)} \cdot y^{(n)}\}_{n \in \mathbb{N}}.
\end{align*}
\]

**Theorem 7.5.** For any two Cauchy sequences of rational numbers \( \{x^{(n)}\}_{n \in \mathbb{N}}, \{y^{(n)}\}_{n \in \mathbb{N}} \),

\[
\kappa(\{x^{(n)}\}_{n \in \mathbb{N}} \oplus \{y^{(n)}\}_{n \in \mathbb{N}}) = \kappa(\{x^{(n)}\}_{n \in \mathbb{N}}) \otimes \kappa(\{y^{(n)}\}_{n \in \mathbb{N}}).
\]

**Proof.** Fix arbitrarily \( k \in \mathbb{N} \), there exists a common starting index \( N \in \mathbb{N} \) such that \( \forall m, n \geq N, |x^{(m)} - x^{(n)}| < \frac{1}{10^k}, |y^{(m)} - y^{(n)}| < \frac{1}{10^k} \). Thus \( \forall n \geq N, \)

\[
\begin{align*}
x^{(N)} - \frac{1}{10^k} &\leq x^{(n)} \leq x^{(N)} + \frac{1}{10^k}, \\
y^{(N)} - \frac{1}{10^k} &\leq y^{(n)} \leq y^{(N)} + \frac{1}{10^k}, \\
x^{(N)} + y^{(N)} - \frac{2}{10^k} &\leq x^{(n)} + y^{(n)} \leq x^{(N)} + y^{(N)} + \frac{2}{10^k}.
\end{align*}
\]

Letting \( n \to \infty \) gives

\[
\begin{align*}
(7.1) &\quad x^{(N)} - \frac{1}{10^k} \leq \text{LIMIT}(\{x^{(n)}\}_{n \in \mathbb{N}}) \leq x^{(N)} + \frac{1}{10^k}, \\
(7.2) &\quad y^{(N)} - \frac{1}{10^k} \leq \text{LIMIT}(\{y^{(n)}\}_{n \in \mathbb{N}}) \leq y^{(N)} + \frac{1}{10^k}, \\
(7.3) &\quad x^{(N)} + y^{(N)} - \frac{2}{10^k} \leq \text{LIMIT}(\{x^{(n)} + y^{(n)}\}_{n \in \mathbb{N}}) \leq x^{(N)} + y^{(N)} + \frac{2}{10^k}.
\end{align*}
\]

Applying Theorem 3.5 to (7.1) and (7.2) yields

\[
(7.4) \quad x^{(N)} + y^{(N)} - \frac{2}{10^k} \leq \text{LIMIT}(\{x^{(n)}\}_{n \in \mathbb{N}}) \oplus \text{LIMIT}(\{y^{(n)}\}_{n \in \mathbb{N}}) \leq x^{(N)} + y^{(N)} + \frac{2}{10^k}.
\]

Finally a standard application of Lemma 3.8 to (7.3) and (7.4) gives the desired result, we are done. \( \square \)

**Theorem 7.6.** For any two Cauchy sequences of rational numbers \( \{x^{(n)}\}_{n \in \mathbb{N}}, \{y^{(n)}\}_{n \in \mathbb{N}} \),

\[
\kappa(\{x^{(n)}\}_{n \in \mathbb{N}} \oplus \{y^{(n)}\}_{n \in \mathbb{N}}) = \kappa(\{x^{(n)}\}_{n \in \mathbb{N}}) \otimes \kappa(\{y^{(n)}\}_{n \in \mathbb{N}}).
\]

**Proof.** By Formula (13) in Subsection 3.2 and Theorem 4.7, to prove this theorem we may assume both \( \text{LIMIT}(\{x^{(n)}\}_{n \in \mathbb{N}}) \geq 0.000000 \cdots \) and \( \text{LIMIT}(\{y^{(n)}\}_{n \in \mathbb{N}}) \geq 0.000000 \cdots \), which easily implies that \( \{x^{(n)}\}_{n \in \mathbb{N}} \) and \( \{y^{(n)}\}_{n \in \mathbb{N}} \) are eventually positive, that is, there is an starting index \( M \in \mathbb{N} \) such that \( \forall n \geq M, x^{(n)} \geq 0.000000 \cdots, y^{(n)} \geq 0.000000 \cdots \).
A NEW APPROACH TO THE REAL NUMBERS

Similar to the proof of the previous theorem, for arbitrary \( k \in \mathbb{N} \) one can find a starting index \( N \geq M \) such that \( \forall m, n \geq N, |x^{(m)} - x^{(n)}| < \frac{1}{10^k}, |y^{(m)} - y^{(n)}| < \frac{1}{10^k} \), and also

\[
(7.5) \quad x^{(N)} - \frac{1}{10^k} \leq \lim \{x^{(n)}\}_{n \in \mathbb{N}} \leq x^{(N)} + \frac{1}{10^k},
\]

\[
(7.6) \quad y^{(N)} - \frac{1}{10^k} \leq \lim \{y^{(n)}\}_{n \in \mathbb{N}} \leq y^{(N)} + \frac{1}{10^k},
\]

\[
(7.7) \quad x^{(N)} \cdot y^{(N)} - \frac{2z}{10^k} + \frac{1}{100^k} \leq \lim \{x^{(n)} \cdot y^{(n)}\}_{n \in \mathbb{N}} \leq x^{(N)} \cdot y^{(N)} + \frac{2z}{10^k} + \frac{1}{100^k},
\]

where \( z \in \mathbb{N} \) is any common upper bound for the sequences \( \{|x^{(n)}|\}_{n \in \mathbb{N}}, \{|y^{(n)}|\}_{n \in \mathbb{N}} \).

**Case 1:** Suppose \( x^{(N)} - \frac{1}{10^k} \geq 0 \) and \( y^{(N)} - \frac{1}{10^k} \geq 0 \). Then applying Theorem 4.8 to (7.5) and (7.6) yields

\[
(7.8) \quad s - \frac{2z}{10^k} + \frac{1}{100^k} \leq \lim \{x^{(n)}\}_{n \in \mathbb{N}} \otimes \lim \{y^{(n)}\}_{n \in \mathbb{N}} \leq s + \frac{2z}{10^k} + \frac{1}{100^k},
\]

where \( s \) stands for \( x^{(N)} \cdot y^{(N)} \) purely for the sake of simplicity. Next a standard application of Lemma 3.8 to (7.7) and (7.8) gives the desired result.

**Case 2:** Suppose \( x^{(N)} - \frac{1}{10^k} < 0 \). Now we revise (7.5)–(7.7) slightly to

\[
(7.9) \quad 0 \leq \lim \{x^{(n)}\}_{n \in \mathbb{N}} \leq \frac{2}{10^k},
\]

\[
(7.10) \quad 0 \leq \lim \{y^{(n)}\}_{n \in \mathbb{N}} \leq z + \frac{1}{10^k},
\]

\[
(7.11) \quad 0 \leq \lim \{x^{(n)} \cdot y^{(n)}\}_{n \in \mathbb{N}} \leq \frac{3z}{10^k} + \frac{1}{100^k}. \]

Then applying Theorem 4.8 to (7.9) and (7.10) yields

\[
(7.12) \quad 0 \leq \lim \{x^{(n)}\}_{n \in \mathbb{N}} \otimes \lim \{y^{(n)}\}_{n \in \mathbb{N}} \leq \frac{2z}{10^k} + \frac{2}{100^k},
\]

Finally a standard application of Lemma 3.8 to (7.11) and (7.12) gives the desired result.

**Case 3:** Suppose \( y^{(N)} - \frac{1}{10^k} < 0 \). The proof is fully identical to that in Case 2.

This concludes the whole proof of the theorem.

Given two Cauchy sequences of rational numbers \( \{x^{(n)}\}_{n \in \mathbb{N}}, \{y^{(n)}\}_{n \in \mathbb{N}} \), we say \( \{x^{(n)}\}_{n \in \mathbb{N}} \) is less than or equal to \( \{y^{(n)}\}_{n \in \mathbb{N}} \), denoted by \( \{x^{(n)}\}_{n \in \mathbb{N}} \preceq \{y^{(n)}\}_{n \in \mathbb{N}} \), if either \( \{x^{(n)}\}_{n \in \mathbb{N}} \approx \{y^{(n)}\}_{n \in \mathbb{N}} \) or the sequence \( \{y^{(n)} - x^{(n)}\}_{n \in \mathbb{N}} \) is eventually positive, that is, there exists an \( n \in \mathbb{N} \) such that \( \forall n \geq N, y^{(n)} - x^{(n)} > 0 \). It is also very easy to verify that

\[
\{x^{(n)}\}_{n \in \mathbb{N}} \preceq \{y^{(n)}\}_{n \in \mathbb{N}} \iff \kappa(\{x^{(n)}\}_{n \in \mathbb{N}}) \preceq \kappa(\{y^{(n)}\}_{n \in \mathbb{N}}).
\]

Thus with Theorem 3.9, Theorem 4.9, Theorem 7.3, Theorem 7.5 and Theorem 7.6, \( (\mathbb{R}, \preceq, \preceq, \lesssim) \) is isomorphic to \( (\mathbb{CR}, \preceq, \preceq, \lesssim) \).

We note a noteworthy difference. When constructing the real numbers via decimals as before, we introduced a total order as earlier as possible, then based on derived operations of this order, we introduced additive and multiplicative operations. But when people work on the Cauchy sequence approach, total order is not so important a concept as we regard. What they prefer most is another fundamental concept called “distance”. It is amazing to see different approaches lead to the same structure, mathematics is so harmonious!
8. Rationals v.s. Irrationals

According to Lemma 6.3, there are generally three types of elements in $\mathcal{R}$, that is,

- Type 1: $\mathbb{Q}$
- Type 2: $\mathcal{R}_9$
- Type 3: The others remained

According to Lemma 2.12, every element of second type will be identified with an element of first type in $\mathbb{R}$. Thus no matter in $\mathcal{R}$ or in $\mathbb{R}$, we should pay attention to the third type of elements, which has deserved not so better a name in history.

**Definition 8.1.** An element of $\mathcal{R}$ is called irrational if it does not belong to the first two types of elements.

**Theorem 8.2.** An element $x_0.x_1x_2x_3\cdots \in \mathcal{R}$ is irrational if and only if it cannot end with infinite recurrence of a block of digits.

**Proof.** By the pigeonhole principle, it is easy to prove that any rational number $\alpha = \frac{a}{b}$ with $a, b \in \mathbb{N}$ and $q \in \mathbb{Z}$ must end with infinite recurrence of a block of length $\leq p$. Obviously, any element of $\mathcal{R}_9$ is also of such property. Thus to prove this theorem, it suffices to show any element ends with infinite recurrence of a block of digits must belong to the first two types. To this aim, without loss of generality let us suppose

$$x = x_0.x_1\cdots x_k y_1\cdots y_s y_1\cdots y_s y_1\cdots y_s \cdots \in \mathcal{R}$$

with $y_8 < 9$ (please think why we can impose this condition), then we need only prove $x$ belongs to the first type, that is, $x \in \mathbb{Q}$. Now we define a rational number

$$z = [x]_k + \frac{1}{10^k} \sum_{j=1}^{s} y_j \cdot 10^{s-j} = \left(\sum_{i=0}^{k} x_i \cdot 10^{k-i}\right) \cdot (10^s - 1) + \sum_{j=1}^{s} y_j \cdot 10^{s-j}.$$ 

Then in the following we devise two sequences of rational numbers with left hand $\{\mu_n\}_{n \in \mathbb{N}}$, and right hand $\{\omega_n\}_{n \in \mathbb{N}}$ satisfying the desired inequalities\(^1\)

\[
\begin{align*}
x_0.x_1\cdots x_k y_1\cdots y_s & \leq z \leq x_0.x_1\cdots x_k y_1\cdots y_{s-1}(y_s + 1) \\
x_0.x_1\cdots x_k y_1\cdots y_s y_1\cdots y_s & \leq z \leq x_0.x_1\cdots x_k y_1\cdots y_{s-1}(y_s + 1) \\
x_0.x_1\cdots x_k y_1\cdots y_s y_1\cdots y_s y_1\cdots y_s & \leq z \leq x_0.x_1\cdots x_k y_1\cdots y_{s-1}(y_s + 1) \\
& \cdots \leq z \leq \cdots 
\end{align*}
\]

Note $\{\mu_n\}_{n \in \mathbb{N}}$ is monotonically increasing with limit $x$, while $\{\omega_n\}_{n \in \mathbb{N}}$ is monotonically decreasing with limit also $x$, thus by Lemma 2.9 we must have $x = z$. This proves $x$ is a rational number, we are done. \(\square\)

\(^1\)Let us see from a special example to know why these inequalities are true, from which one can easily prove the general cases. For example, suppose $x = 0.232323\cdots$ and $z = \frac{23}{99}$. Take the third of this sequence of inequalities for example, multiplying each parts by $\frac{23}{99}$ gives

$$0.999999 \leq 0.999999 + 0.000001 \cdot \frac{99}{23},$$

which is obviously true.
9. Algebraic abstraction: Complete ordered field

After detailed studies of three explicit approaches to the real number system, let us explain how can we come to its algebraic characterization by the so-called concept of complete ordered field.

If there exists only one real number system in the mathematical world, then it should be a system $(\mathbb{R}, +, \times, \leq)$ with three binary operations $+, \times, \leq$ endowed on the same ambient set $\mathbb{R}$ satisfying at least a few conditions:

- $\mathbb{R}$ is infinite (reason: should contain $\mathbb{Q}$ as a subset)
- $(\mathbb{R}, +, \times)$ is a field (reason: $(\mathbb{R}, \oplus, \otimes)$ is a field)
- $\leq$ is a total order on $\mathbb{R}$ and $(\mathbb{R}, \leq)$ has both the least upper bound property and the greatest lower bound property (reason: $(\mathbb{R}, \preceq)$ is of such properties)

An algebraic system satisfies the above conditions can not characterize the real number system yet. The main reason is we can not sense any difference between the positive and the negative parts of our desired real number system, so we need more observation. To mention a few: addition of elements of $\mathbb{C}$ preserves order $\preceq$, so are multiplication by positive elements of $\mathbb{C}$. Is an algebraic system $(\mathbb{R}, +, \times, \leq)$ satisfies the following conditions isomorphic to those systems of Dedekind and Méray-Cantor-Heine?

- $(\mathbb{R}, +, \times)$ is an infinite field
- $\leq$ is a total order on $\mathbb{R}$ and $(\mathbb{R}, \leq)$ has both the least upper bound property and the greatest lower bound property
- $x < y, z \in \mathbb{R} \Rightarrow x + z < y + z$
- $x < y, z > 0 \Rightarrow xz < yz$

Yes, it is! Such a system is called a complete ordered field. It does not so matter whether some of the conditions are superfluous or not, at least we derived an algebraic characterization of the well-known real number system. How to prove it? Decimal representations! (Imagine it first, prove it second!)

(Remark: The infinite cardinality assumption follows from the third assumption and one item of the field axioms, that is, the additive unit is not equivalent to the multiplicative unit, while the greatest lower bound property follows from the least lower bound property and the other three assumptions, so they are unnecessary.)

10. Epilog: Analysis v.s. Algebra or Analysis plus Algebra?

When to teach mathematical majors Mathematical Analysis, instructors need to make a serious choice.

Dedekind cuts

“Although the definition is geometrically motivated, and is based on a ‘natural idea’, the definition is actually far too abstract to be efficient at the beginning of a calculus course.” ([38]) We also don’t prefer this approach. The main reason is, once the real number system is established through this approach, we can find almost no use of the language of Dedekind cuts in the subsequent study of the Mathematical Analysis course.

Cauchy sequences
We believe this approach is too abstract to be acceptable for beginners, who have not received adequate mathematical training yet. We also have a sense that when to construct the real number system explicitly, people would prefer more the Dedekind cut approach than the abstract Cauchy sequence one. Introducing it in the Functional Analysis course would be a better choice.

**Axiomatic Definition**

As said in the Introduction, we found many authors liked to choose, say for example the algebraic-axiomatic definition, then proved that every real number has a suitable decimal representation. But how can we persuade students believe that a real number is an element of an algebraic system satisfying a dozen or so properties at the beginning of a Mathematical Analysis course?

**Suggestion: Decimal (First) plus Axiomatic (Second) approaches**

This paper provides a complete approach to the real number system via rather old decimal representations, which perfectly matches what we have learnt a number is in high school. Once the real number system is established through this approach, the least upper bound property, the greatest lower bound property, the supremum, infimum, upper limit and lower limit operations are immediately, also naturally obtained. This shows huge advantages over the Dedekind cut and the Cauchy sequence approaches. It would be great if after the decimal approach, one can give an algebraic characterization of the real number system, which help students grasp the fundamental properties of the real numbers instead of heavily relying on the decimal representations, also help them understand there is only one real number system in the mathematical world when they consult other Mathematical Analysis books.

Although we have not reviewed many other construction of the real number system, we show full respect to any such kind of work. We conclude this paper with a saying by Jonathan Bennett: “You can have lots of good ideas while walking in a straight line.”

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