Quantum Algebras
and
Quantum Physics

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Abstract

In Quantum Mechanics operators must be hermitian and, in a direct product space, symmetric. These properties are saved by Lie algebra operators but not by those of quantum algebras. A possible correspondence between observables and quantum algebra operators is suggested by extending the definition of matrix elements of a physical observable, including the eventual projection on the appropriate symmetric space. This allows to build in the Lie space of representations one-parameter families of operators belonging to the enveloping Lie algebra that satisfy an approximate symmetry and have the properties required by physics.

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1 Introduction

Quantum groups arose in the work of the Leningrad school related to the inverse scattering method [1]. Their interest in mathematics is indisputable, and their physical applications cover integrable models, quantum conformal field theories, quantum field theories, quantum gravity, spin chains, etc. However, our hopes to rewrite all countless applications of Lie algebras, with a free parameter inside, has had a limited success up to now.

The reasons are essentially two. First of all, the essential point of applications of Lie–Hopf algebras to quantum physics is the one-to-one correspondence among physical observables from one side and hermitian operators on a Hilbert space from the other. In the case of standard deformations [2, 3] such correspondence cannot be extended to quantum algebras for all values of the deformation parameter $q \in \mathbb{C}$. As it is well known, we have to require $q \in \mathbb{R}$ or $|q| = 1$ to obtain hermitian irreducible representations. For non-standard quantum algebras [4] the situation is worse since raising and lowering operators have a completely different behaviour for any value of $q$.

The second reason that stops applications of quantum algebras to physics is related with the concept of composed system: such an object is nothing else that the set of two (or more) sub-systems that, in some approximation, can be considered as independent. The fundamental assumption is that the Hilbert space is the direct product of those of the elementary systems and the interaction Hamiltonian, when not disregarded, is such that it does not change this basic structure modifying only the transition matrix elements. Physics is indeed described in direct product spaces —at least we have to pick up the observed system from the rest of the laboratory— and the scheme must be such that the observables of the composed systems are determined by the observables of their constituents. In physics all systems have bosonic or fermionic behaviour, i.e., they exhibit well defined properties under interchange of their identical constituents. This property is obviously translated on the direct product space and cannot be modified by the operators working on it; so, they must be symmetric. For non-identical constituents this property of symmetry must be also preserved since physics is independent of the order taken in the direct product of their wavefunctions.

When we have a Lie symmetry, observables are additive (as for the angular momentum described in $su(2)$, that for two particles is simply the sum of the two angular momenta) and, thus, symmetric. For quantum algebra operators this property of symmetry is not verified since global observables are determined by the coalgebra, which is never symmetric (and seldom hermitian) whatever kind of deformation (standard or non-standard) and value of the deformation parameter are considered.

The point is that both, physics and mathematics, give strong prescriptions on operators structure and these prescriptions seem to be in contradiction. It looks that nobody can hope to find a solution of this consistency problem inside the well established rules of quantum mechanics or quantum algebras where not contrastable results forbid any possibility. The only way is to work on the correlation between quantum observables and quantum algebra operators.

In other words, the application of Lie algebras to physics is based on two invariances:
the first one is the invariance of the roots of a Lie algebra under the Weyl group which originates the symmetry between raising and lowering operators. And the second one is the invariance of the operators acting on $\mathcal{H}^\otimes n$ under the permutation group $S_n$, i.e., they carry the trivial representation of $S_n$. Quantum algebra operators have not these symmetries and, for this reason they cannot, in a direct way, describe physical observables. To avoid these difficulties, we shall consider the projection of the quantum algebra operators on an appropriate space in order to define suitable physical matrix elements.

To illustrate our approach we develop in a detailed way the simple case of the standard and non-standard deformations of $su(2)$ because of the physical relevance of their applications and their computational simplicity. The generalization to higher dimension algebras is only a technical matter.

## 2 Standard $su_q(2)$

A quantum algebra, like $su_q(2)$, is characterized by two algebraic structures: one at the level of the Lie algebra but with ‘deformed’ commutators and a second one at the level of the coalgebra [5].

Let $H$, $X_\pm (= X_1 \pm iX_2)$ be the generators of $su_q(2)$. The deformed commutators are

$$[H, X_\pm] = \pm 2X_\pm, \quad [X_+, X_-] = \frac{\sinh zH}{\sinh z},$$

(2.1)

with $z = \log q$. Note that when $z \to 0$ we recover $su(2)$.

The $(2j + 1)$–dimensional irreducible representations of $su_q(2)$, $D^q_j$, are given by

$$H \mid z,j,m \rangle = m \mid z,j,m \rangle,$$

$$C_q \mid z,j,m \rangle = [j^q][j + 1]^q \mid z,j,m \rangle,$$

$$X_\pm \mid z,j,m \rangle = \sqrt{[j \pm m]^q[j \pm m + 1]^q} \mid z,j,m \pm 1 \rangle,$$

(2.2)

where $[n]^q = \sinh(zn)/\sinh z$, $2j \in Z^\geq 0$, $m = -j, -j + 1, \ldots, j$, and $C_q$ is the deformed Casimir operator

$$C_q \equiv X_-X_+ + [H]^q[H + 1]^q.$$  

(2.3)

At the level of the irreducible representations of $su_q(2)$ both for $z$ real or imaginary (with $|z|/\pi$ rational and irrational, i.e. with $q$ root of unity or not) the matrix representations of the generators $H$, $X_1$ and $X_2$ as well as the Casimir (2.3) are hermitian. Indeed, in both cases the scheme is similar to the nondeformed case, and a complete set of commuting observables is composed by $H$ and $C_q$. Since matrix elements depends on $z^2$, which is real in both cases, one has

$$\langle z,j,n | X_\pm | z,j,m \rangle = \langle z,j,m | X_\mp | z,j,n \rangle,$$

(2.4)

and the usual scalar product $\langle z,j',n | z,j,m \rangle = \delta_{j',j} \delta_{n,m}$ is sufficient to define a $*$–representation [3] with $H$, $X_1$ and $X_2$ hermitian operators and $X_\pm = X_\mp$. 

3
We mentioned in the introduction that most of the difficulties appear when composed systems are considered. The structure of Hopf algebra, characteristic of a quantum algebra, includes in a natural way the composed systems in the coalgebra. As it is well known, the coalgebra is basically determined by the coproduct $\Delta$. For $su_q(2)$ we have

$$\Delta H = H \otimes 1 + 1 \otimes H, \quad \Delta X_{\pm} = X_{\pm} \otimes e^{\pm H} + e^{-\pm H} \otimes X_{\pm}. \quad (2.5)$$

Note that $\Delta H$ remains additive and symmetric like in the nondeformed case, while the expressions of $\Delta X_{\pm}$, imposed by the commutation relations (2.1) on the composed systems (i.e. $[\Delta [\cdot, \cdot] = [\Delta \cdot, \Delta \cdot]$) become deformed. As we stressed before, physical requirements imply that the operators of a composed system are hermitian and symmetric. In the present case for $z$ real they are hermitian but non symmetric as it is obvious by inspection of expression (2.5). For $z$ imaginary the ‘naive adjoint’ could look physically acceptable, as the change of two component spaces correspond to turn clockwise or anticlockwise in the complex plane. However, this is not more true for systems with more that two components.

Thus, also our pragmatic approach cannot escape to the result established in general [5]: there is an involution for quantum algebras such that $\Delta X_1$ and $\Delta X_2$ are hermitian for $q$ real only. However, this involution is unsatisfactory for a physicist since does not preserve the symmetry between the factor spaces.

In order to restore this symmetry, let us observe that the Lie algebra elements $\Delta (X) = X \otimes 1 + 1 \otimes X$ (X is called primitive element) carry the trivial representation of the symmetry group $S_2$, i.e.

$$\sigma \Delta (X) \sigma^{-1} = \Delta (X), \quad (2.6)$$

where $\sigma$ is the permutation operator ($\sigma(a \otimes b) = b \otimes a$). This property (2.6) is essential for defining a one-to-one correspondence between operators and physical observables. The crucial point is to improve (2.6) in the deformed case.

We propose, thus, a new definition of the matrix elements of an operator $O$ for two particles

$$\langle \phi | O | \psi \rangle_{\text{phys}} := \langle \phi | \frac{1}{2} (O + \sigma O \sigma^{-1}) | \psi \rangle. \quad (2.7)$$

Notice that if the operator is symmetric this expression is equivalent to the usual one. We will denote in general

$$\langle \phi | \tilde{O} | \psi \rangle \equiv \langle \phi | O | \psi \rangle_{\text{phys}}, \quad (2.8)$$

where

$$\tilde{O} := \frac{1}{2!} \sum_{\sigma \in S_2} \sigma O \sigma^{-1}. \quad (2.9)$$

It is worthy to note that $\tilde{O}$ carries the trivial representation of $S_2$ as required and that, because we are projecting on the symmetric subspace, the physical matrix element of the product $O_1 O_2$ is related to $\tilde{O}_1 \tilde{O}_2$ and not to $\tilde{O}_1 \tilde{O}_2$.

For systems with more that two ‘particles’ the coproduct of high order is build by iteration of the coproduct. So

$$\Delta^{(3)} : A \rightarrow A \otimes A \otimes A \quad (2.10)$$
is defined by
\[ \Delta^{(3)} := (\text{id} \otimes \Delta^{(2)}) \circ \Delta^{(2)} = \Delta^{(2)} \otimes \text{id} \circ \Delta^{(2)}, \]
where \( \Delta^{(2)} = \Delta \). Following this iteration procedure we can obtain \( \Delta^{(n)} \)
\[ \Delta^{(n)} := (\text{id} \otimes \Delta^{(n-1)}) \circ \Delta^{(2)} = \Delta^{(n-1)} \otimes \text{id} \circ \Delta^{(2)}. \]

In general, for a system of \( n \) particles we consider the operator
\[ \tilde{O}^{(n)} := \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \mathcal{O}^{(n)}(\sigma^{-1}), \]
which commutes with any permutation of \( S_n \) as it is easy to see using the rearrangement lemma. Remark that this definition is consistent with the usual one for symmetric operators.

Returning to the case of \( n = 2 \) we obtain from (2.7) that
\[ \tilde{\Delta} H = H \otimes 1 + 1 \otimes H \quad \Delta \tilde{X}_\pm = X_\pm \otimes \cosh(\frac{\tilde{z}}{2}H) + \cosh(\frac{\tilde{z}}{2}H) \otimes X_\pm. \]

For \( n = 3 \) we get from (2.13) that
\[ \tilde{\Delta}^{(3)} H = H \otimes 1 \otimes 1 + 1 \otimes H \otimes 1 + 1 \otimes 1 \otimes H \quad \Delta^{(3)} \tilde{X}_\pm = \frac{1}{3}\{X_\pm \otimes [2 \cosh(\frac{\tilde{z}}{2}H) \cosh(\frac{\tilde{z}}{2}H) + \cosh(1 \otimes \frac{\tilde{z}}{2}H + \frac{\tilde{z}}{2}H \otimes 1)] \]
\[ + [1 \otimes X_\pm \otimes 1] [2 \cosh(\frac{\tilde{z}}{2}H) \otimes 1 \otimes \cosh(\frac{\tilde{z}}{2}H) \]
\[ + \cosh(1 \otimes 1 \otimes \frac{\tilde{z}}{2}H + \frac{\tilde{z}}{2}H \otimes 1 \otimes 1)] \]
\[ + [2 \cosh(\frac{\tilde{z}}{2}H) \cosh(\frac{\tilde{z}}{2}H) + \cosh(1 \otimes \frac{\tilde{z}}{2}H + \frac{\tilde{z}}{2}H \otimes 1)] \otimes X_\pm \}. \]

Note that expressions (2.14) and (2.15) satisfies all the physical requirements for \( z \) real as well as imaginary.

### 3 Non-standard \( su_\omega(2) \)

The non-standard quantum algebra \( su_\omega(2) \) has the following Hopf algebra structure: deformed commutators
\[ [H, X_+] = \frac{2}{\omega} \sinh \omega X_+, \]
\[ [H, X_-] = -X_-(\cosh \omega X_+) - (\cosh \omega X_+)X_-, \]
\[ [X_+, X_-] = H; \]

and coalgebra
\[ \Delta H = H \otimes e^{\omega X_+} + e^{-\omega X_+} \otimes H, \]
\[ \Delta X_+ = X_+ \otimes 1 + 1 \otimes X_+, \]
\[ \Delta X_- = X_- \otimes e^{\omega X_+} + e^{-\omega X_+} \otimes X_- \]

\[ 5 \]
Notice the different roles played by $X_+$ and $X_-$ in comparison with the standard deformation case. That requires to ‘symmetrize’ also the irreducible representations.

The fundamental representation ($j = 1/2$) is always for quantum algebras like in the non-deformed case. In Ref. [7] a 3–dimensional ($j = 1$) irreducible representation of $su_\omega(2)$ is presented, and in Ref. [8] representations for $j = 1$ and $j = 3/2$ are displayed with the matrix associated to $H$ diagonal. Here we consider equivalent representations for $j = 1$ and $j = 3/2$ such that, in the limit of $\omega$ going to zero, the usual representations of $su(2)$ are recovered.

For $j = 1$ the matrix representation for the generators is

$$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & -\frac{\omega^2}{2\sqrt{2}} & 0 \\ \sqrt{2} & 0 & -\frac{\omega^2}{2\sqrt{2}} \\ 0 & \sqrt{2} & 0 \end{pmatrix}; \quad (3.3)$$

and in the case of $j = 3/2$

$$H = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & \sqrt{3} & 0 & \frac{\omega^2}{\sqrt{3}} \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.4)$$

$$X_- = \begin{pmatrix} 0 & -\frac{3\omega^2}{2} & 0 & \frac{3\omega^4}{8} \\ \sqrt{3} & 0 & -\frac{3\omega^2}{2} & 0 \\ 0 & 2 & 0 & -\frac{\sqrt{3}\omega^2}{2} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}. \quad (3.7)$$

Representations for higher dimensions are similar.

Physical hermiticity compels us to restore the symmetry between raising and lowering operators. To achieve it, let us consider the quantum algebra, $su'_\omega(2)$, isomorphic to $su_\omega(2)$, obtained by the transformation

$$H \rightarrow H' = H^\dagger, \quad X_\pm \rightarrow X'_\pm = X_\pm^\dagger. \quad (3.5)$$

Let $\mathcal{H}_j$ and $\mathcal{H}'_j$ be the carrier spaces of the $(2j+1)$–dimensional irreducible representations of $su_\omega(2)$ and $su'_\omega(2)$, respectively. The physical matrix elements of an operator $O$ are defined by the mean of the matrix elements in each carrier space, i.e.,

$$\hat{O} \equiv \langle j, n|O|j, m \rangle := \frac{1}{2}(\langle j, n|O|j, m \rangle + \langle j, n|O'|j, m \rangle). \quad (3.6)$$

Hermiticity requires that the parameter $\omega$ has to be only real or imaginary like in the case of standard deformations.

In this way, for the representations of $su_\omega(2)$ we obtain

$$\hat{H} = \frac{1}{2}(H + H^\dagger), \quad \hat{X}_\pm = \frac{1}{2}(X_\pm + X_\pm^\dagger). \quad (3.7)$$

6
In the representations (3.3) and (3.4) we have considered $H = H^\dagger$ and, hence, $\hat{H} = H$.

For composed systems the ‘symmetrization’ procedure starts from the symmetrization like the standard case and finish with the above reported ‘hermitianization’. In particular, for systems with two particles the procedure is:

1).- To symmetrize $su_\omega(2)$

\[
\begin{align*}
\tilde{\Delta} H &= H \otimes \cosh(\omega X_+) + \cosh(\omega X_+) \otimes H, \\
\Delta X_+ &= X_+ \otimes 1 + 1 \otimes X_+, \\
\Delta X_- &= X_- \otimes \cosh(\omega X_+) + \cosh(\omega X_+) \otimes X_-, 
\end{align*}
\]

(3.8)

2).- To symmetrize $su'_\omega(2)$

\[
\begin{align*}
\tilde{\Delta} H' &= H \otimes \cosh(\omega X_+^\dagger) + \cosh(\omega X_+^\dagger) \otimes H, \\
\Delta X'_+ &= X_+^\dagger \otimes \cosh(\omega X_+^\dagger) + \cosh(\omega X_+^\dagger) \otimes X_+^\dagger, \\
\Delta X'_- &= X_+^\dagger + 1 \otimes 1 \otimes X_+^\dagger. 
\end{align*}
\]

(3.9)

3).- To consider the mean of both symmetrization results

\[
\begin{align*}
\tilde{\Delta} H' &= H \otimes \frac{1}{2}[\cosh(\omega X_+) + \cosh(\omega X_+^\dagger)] + \frac{1}{2}[\cosh(\omega X_+) + \cosh(\omega X_+^\dagger)] \otimes H, \\
\Delta X'_+ &= \frac{1}{2}[X_+ \otimes 1 + X_+^\dagger \otimes \cosh(\omega X_+^\dagger) + 1 \otimes X_+ + \cosh(\omega X_+^\dagger) \otimes X_+^\dagger], \\
\Delta X'_- &= \frac{1}{2}[X_+^\dagger \otimes 1 + X_- \otimes \cosh(\omega X_+) + 1 \otimes X_+^\dagger + \cosh(\omega X_+) \otimes X_-]. 
\end{align*}
\]

(3.10)

We do not write the expression for the case of three particles since its computation is straightforward.

## 4 Conclusions

We can sum up the problem as follows: in quantum physics all the observables must have real eigenvalues and, hence, they must be hermitian. Moreover, they must be symmetric since, as we mention before, physical results are independent of the order in the direct product spaces.

On the other hand, quantum algebra generators are seldom hermitian and never symmetric. Thus, there is not a physical theory whose observables belong to a quantum algebra. We propose to construct physical observables in terms of quantum algebra operators enlarging the representation space and later projecting on an appropriate subspace.

We have chosen to present our proposal in an informal way. So, we have simply symmetrized and hermitianized quantum algebra operators. A formal description would require the introduction of a density matrix to be inserted in a reducible representation at the moment of evaluating the matrix elements. It should be more clear in this case that
the algebraic structure of the set of quantum algebra operators has not been modified
anyway, and that the correct expression for the product of operators is $\tilde{O}_1\tilde{O}_2$ because the
density matrix is used only in the last step of the procedure.

We have considered the standard deformation as well as the non-standard one of $su(2)$. The
procedure is generalizable to any other Lie algebra with one or other deformation, but also to quantum algebras with hybrid deformations, i.e, quantum algebras combining both kinds of deformations [3, 10].

Symmetry in real world is always approximate, but we can find in the literature only one procedure for describing an approximate symmetry: the spontaneous symmetry breaking that is strange in the sense that is an exact symmetry but manifested as broken. We propose here an algebraic approach to broken symmetry that is between the enormous freedom we have to build physical quantities in Lie universal enveloping algebras and the too rigid scheme to identify them with the generators. One-parameter families inside Lie universal enveloping algebras are in this way built that go with continuity from the exact symmetry ($q = 1$) to a catastrophic breaking ($q \to \infty$ or in the neighborhood of a root of unity).

It is interesting to note that, while an exact symmetry implies that physical quantities are additive, an approximate symmetry seems to describe objects with correlations between their constituents.

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