THE SET OF JUMPING CONICS OF A LOCALLY
FREE SHEAF OF DIMENSION 2 ON $P^2$

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Abstract. We consider a locally free sheaf $F$ of dimension 2 on $P^2$. A conic $q$ on $P^2$ is called a jumping conic if the restriction of $F$ to $q$ is not the generic one. We prove that the set of jumping conics is the maximal determinantal variety of a skew form. Some properties of this skew form are found. Translation from Russian; the original is published in: "Constructive algebraic geometry". Yaroslavl, 1981. N.194, p. 79 – 82.

Let $V$ be a vector space of dimension 3 over $\mathbb{C}$, $P^2 = P(V)$ its projectivization, $F$ a locally free sheaf of dimension 2 on $P^2$ satisfying $c_1(F) = 0$, $c_2(F) = n$, $H^0(P^2, F(1)) = 0$. Let $q$ be a non-singular conic in $P^2$. We have $F|_q = O(d) \oplus O(-d)$. For a generic $q$ we have $d = 0$, let us call these conics regular. If $d > 0$ we shall call them jumping conics, the number $d$ is called the multiplicity of jump of $F$ at $q$.

The main result of the present paper is

Theorem 1. The set of jumping conics is the maximal determinantal variety of a skew form on $H^1(F \otimes \Omega(-1))$.

For the reader’s convenience, before giving a proof we give firstly the main construction not at all variety of conics on $P^2$ but at a given fixed conic. The solution of the analogous problem in [B] uses a difference between the values of $h^1(F(-1)|_l)$ for an ordinary straight line $l$ and a jumping line. It is easy to see that $\forall l$, $l$ the numbers $h^i(F(k)|_q)$ coincide for an ordinary conic and a conic of simple jump. There is no sheaf $O(\frac{1}{2})$, so we use the sheaf $\Omega(1)$. Let us denote $E_k := H^1(F \otimes \Omega(k))$. We have $\dim E_{-1} = \dim E_1 = 2n$, and let $t : E_{-1} \otimes S^2V^* \rightarrow E_1$ be the $\cup$-multiplication.

Proposition 2. $\forall q \in S^2V^*$ the map $t_q : E_{-1} \rightarrow E_1$ is a skew symmetric bilinear form with respect to a duality $\langle , , \rangle : E_{-1} \otimes E_1 \rightarrow \mathbb{C}$.

Proof. The duality is defined as follows. Formulas $\lambda^2(F) = O$, $\lambda^2(\Omega) = O(-3)$ imply existence of maps $\varepsilon_1 : F \otimes F \rightarrow O$, $\varepsilon_2 : \Omega \otimes \Omega \rightarrow O(-3)$ which are skew symmetric in the following meaning: the diagrams

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σ are transposition maps, i.e. \( \sigma(a \otimes b) = b \otimes a \) on local sections) are anti-commutative. Let us construct a map \( \varepsilon : F \otimes \Omega \otimes F \otimes \Omega \rightarrow O(-3) \) as follows: 
\[
\varepsilon(f_1 \otimes \omega_1 \otimes f_2 \otimes \omega_2) = \varepsilon_1(f_1 \otimes f_2) \cdot \varepsilon_2(\omega_1 \otimes \omega_2),
\]
where \( f_i, \omega_i \) are sections. The diagram

\[
\begin{array}{ccc}
F \otimes F & \xrightarrow{\varepsilon_1} & \Omega \otimes \Omega \\
\downarrow \sigma_1 & & \downarrow \sigma_2 \\
F \otimes F & \xrightarrow{\varepsilon_1} & \Omega \otimes \Omega \\
\end{array}
\]

(\( \sigma_i \) are transposition maps, i.e. \( \sigma_i(a \otimes b) = b \otimes a \) on local sections) are anti-commutative.

To complete the proof of Proposition 2, we shall find now \( < e_{-1}, e_1 > = H^2(\varepsilon)(e_{-1} \cup e_1) \in H^2(O(-3)) = \mathbb{C} \).

**Lemma 3.** Skew symmetry of \( t_q \) is equivalent to the condition \( \forall e_{-1}, e'_{-1} \in E_{-1} \) we have \( < e_{-1}, t(e'_{-1} \otimes q) > = < e'_{-1}, t(e_{-1} \otimes q) > \).

**Proof.** We have \( < e_{-1}, t(e'_{-1} \otimes q) > = H^2(\varepsilon)(e_{-1} \cup e'_{-1} \cup q) \) and \( < e'_{-1}, t(e_{-1} \otimes q) > = H^2(\varepsilon)(e'_{-1} \cup e_{-1} \cup q) \). Cup-product in odd dimensions is anti-commutative, i.e. \( H^2(\sigma(-2))(e_{-1} \cup e'_{-1}) = -(e'_{-1} \cup e_{-1}) \), hence

\[
H^2(\sigma)(e_{-1} \cup e'_{-1} \cup q) = -(e'_{-1} \cup e_{-1} \cup q) \quad \text{and}
\]

\[
H^2(\varepsilon)(e_{-1} \cup e'_{-1} \cup q) = H^2(\varepsilon) \circ H^2(\sigma)(e_{-1} \cup e'_{-1} \cup q) = -H^2(\varepsilon)(e'_{-1} \cup e_{-1} \cup q) \quad \square
\]

Let us consider the exact sequence corresponding to the inclusion \( q \hookrightarrow P^2 \)

\[
0 \rightarrow O_{P^2}(-2) \rightarrow O_{P^2} \rightarrow i_*O_q \rightarrow 0 \quad (4)
\]
multiply it by \( F \otimes \Omega(1) \) and take cohomology:

\[
0 \rightarrow H^0(F \otimes \Omega(1)|_q) \rightarrow E_{-1} \xrightarrow{t_q} E_1 \rightarrow H^1(F \otimes \Omega(1)|_q) \rightarrow 0
\]

For regular \( q \) (resp. for \( q \) of simple jump) we have: \( F \otimes \Omega(1)|_q = O(-1)^{\oplus 4} \), resp.
\( F \otimes \Omega(1)|_q = O(-2)^{\oplus 2} \oplus O^{\oplus 2} \), hence the dimension of both first and fourth terms of (4) are 0, resp. 2. This means that the set of jumping conics is the intersection of \( P(S^2(V^*)) \) with the maximal determinantal variety of \( \text{Hom}_{\text{skew}}(E_{-1}, E_1) \). Its degree is \( n \).

**Proof of Theorem 1.** Let \( P^5 := P(S^2(V^*)) \) be the set of conics in \( P^2 \) and \( D \hookrightarrow P^5 \cdot P^2 \) a flag variety defined as follows: \( (q, t) \in D \iff t \in q \). We have diagrams
where $s$ is a point of $P^5$, $q_s$ is the corresponding conic, it is the fibre of $\pi_5 \circ i$ at $s$, and $(\pi_2 \circ i) \circ u'_s$ is simply the inclusion of $q_s$ in $P^2$.

The exact sequence corresponding to $D$ is

$$0 \to \pi_5^* O_{P^5}(-1) \otimes \pi_2^* O_{P^2}(-2) \to O_{P^5 \times P^2} \to i_* O_D \to 0$$

We multiply it by $\pi_2^*(F \otimes \Omega(1))$

$$0 \to \pi_5^* O_{P^5}(-1) \otimes \pi_2^* (F \otimes \Omega(-1)) \to \pi_2^*(F \otimes \Omega(1)) \to \pi_2^*(F \otimes \Omega(1))|_D \to 0$$

and apply $\pi_{5*}$:

$$0 \to \pi_{5*}(\pi_2^*(F \otimes \Omega(1))|_D) \to E_{-1} \otimes O_{P^5}(-1) \xrightarrow{\varphi} E_1 \otimes O_{P^5} \to$$

$$\to \pi_{5*1}(\pi_2^*(F \otimes \Omega(1))|_D) \to 0$$

Since the functor of restriction to a fibre is right exact, we get that the support of the sheaf $\pi_{5*1}(\pi_2^*(F \otimes \Omega(1))|_D)$ is the set of jumping conics. The restriction of this sheaf to the set of jumping conics is an analog of the sheaf $\theta(1)$ where $\theta$ is the theta-characteristic sheaf for the case of restriction to straight lines ([T]). Its dimension is 2 at conics of simple jump. The sheaf $\pi_{5*}(\pi_2^*(F \otimes \Omega(1))|_D)$ is obviously 0, and the map $\varphi$ comes from the $\cup$-multiplication $t : E_{-1} \otimes S^2 V^* \to E_1$. □

Let us consider some properties of this map. Its composition with the epimorphism $V^* \otimes V^* \to S^2 V^*$ gives us a map $t' : E_{-1} \otimes V^* \otimes V^* \to E_1$ which is the composition of two $\cup$-multiplications $E_{-1} \otimes V^* \to E_0$, $E_0 \otimes V^* \to E_1$. The map $t'' : E_{-1} \otimes V^* \to E_1 \otimes V$ — obtained from $t'$ by moving $V^*$ to the right hand side — can be factored via $E_0$ and hence has the rank $\dim E_0 = 2n + 2$. By analogy with [T] we can choose a basis $\{e_i\}$ of $E_1$ and a basis $\{v_i\}$ of $V^*$ such that the matrix of the map $t''$ in the basis $\{e_i \otimes v_j\}$ of $E_{-1} \otimes V^*$ is given by the block matrix $A = (A_{ij})$ where $i, j = 1, 2, 3$, $A_{ij}$ is a skew symmetric matrix of size $2n$ and $A_{ij} = A_{ji}$, hence $A$ is skew symmetric.

References

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