Effective Gauge Field Theory of the t-J Model in the Charge-Spin Separated State and its Transport Properties

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We study the slave-boson t-J model of cuprates with high superconducting transition temperatures, and derive its low-energy effective field theory for the charge-spin separated state in a self-consistent manner. The phase degrees of freedom of the mean field for hoppings of holons and spinons can be regarded as a U(1) gauge field, \( A_i \). The charge-spin separation occurs below certain temperature, \( T_{CSS} \), as a deconfinement phenomenon of the dynamics of \( A_i \). Below certain temperature \( T_{SG}(< T_{CSS}) \), the spin-gap phase develops as the Higgs phase of the gauge-field dynamics, and \( A_i \) acquires a mass \( m_A \). The effective field theory near \( T_{SG} \) takes the form of Ginzburg-Landau theory of a complex scalar field \( \lambda \) coupled with \( A_i \), where \( \lambda \) represents \( d \)-wave pairings of spinons. Three dimensionality of the system is crucial to realize a phase transition at \( T_{SG} \). By using this field theory, we calculate the dc resistivity \( \rho \). At \( T > T_{SG} \), \( \rho \) is proportional to \( T \). At \( T < T_{SG} \), it deviates downward from the \( T \)-linear behavior as \( \rho \propto T(1 - c(T_{SG} - T)^d) \). When the system is near (but not) two dimensional, due to the compactness of the phase of the field \( \lambda \), the exponent \( d \) deviates from its mean-field value 1/2 and becomes a nonuniversal quantity which depends on temperature and doping. This significantly improves the comparison with the experimental data.

I. INTRODUCTION

In many physical systems, one may identify their microscopic Hamiltonians, but calculations of physical quantities starting from these Hamiltonians may not be straightforward. One promising way to overcome this difficulty is to derive an effective theory that is appropriate for the energy scale in question and use it for calculations.

The discovery of cuprates with high superconducting transition temperatures \( T_c \) has stimulated condensed-matter physicists for one and a half decades. Here the t-J model serves as a canonical model of cuprates, and is expected to have potentiality to explain their various inter-esting properties observed in experiments. The t-J model itself can be regarded as a low-energy effective model of the Hubbard model or the d-p model at the energy scales of the hopping amplitude of electrons \( t(\approx 0.3eV) \), and the antiferromagnetic (AF) spin couplings \( J(\approx 0.1eV) \), which are much lower than the energy scales of the strong onsite Coulomb repulsion \( U(\approx 3eV) \). The t-J model allows us to understand some basic physical properties of strongly-correlated electrons. For example, the origin of the superconductivity is attributed to the formation and condensation of hole pairs at nearest-neighbor (NN) sites in the background of short-range AF spin order. Furthermore, the mean-field theory (MFT) in the slave-boson (SB) representation predicts an interesting phase diagram in the doping \( (\delta) \)-temperature \( (T) \) plane such as the spin-gap state in which the NN AF spin pairings develop. However, it is still not easy to calculate physical quantities such as dc resistivity analytically from the t-J model itself. Thus it is welcome to derive an effective theory of the t-J model at lower energies. It is preferable that this effective theory takes a form of local field theory so that many established techniques are applicable.

Many quasi-two-dimensional cuprates exhibit certain anomalous metallic behavior above \( T_c \) in various quantities such as dc resistivity, Hall coefficient, magnetic susceptibility. For example, the dc resistivity \( \rho \) at fixed \( \delta \) shows the \( T \)-linear dependence on \( T \) above certain temperature. These anomalies call for a new theoretical explanation, probably in a framework beyond the conventional Fermi-liquid theory. Anderson pointed out that the charge-spin separation (CSS) phenomenon may explain them.

The CSS is a phenomenon in which charge and spin degrees of freedom of strongly-correlated electrons behave independently. In the SB representation, an electron is viewed as a composite of two constituents, a holon and a spinon, which bear charge and spin degrees of freedom of electron respectively. The CSS is “naturally” described by the SB (or slave-fermion) MFT, in which the holons and spinons are treated as quasifree particles having no correlations among them. When one incorporates fluctuations of phase degrees of freedom of MFs, these phases behave as gauge fields coupled to holons and spinons, and the system possesses a local U(1) gauge symmetry.

In the previous paper (hereafter we call them Pa-
pers I), we have studied the possibility of the CSS in the t-J model using gauge-theoretical methods. By introducing auxiliary fields, the system can be viewed as a lattice gauge model. The corresponding gauge dynamics has two general possibilities in its realization: (i) the confining phase in which holons and spinons are confined to electrons and the gauge fields fluctuate strongly, or (ii) the deconfining phase in which holons and spinons appear as almost independent quasi-free particles and the gauge-field fluctuations are weak and can be treated by the usual perturbation theory. We interpreted the CSS as the second possibility above; CSS is a deconfinement phenomenon of gauge theory. Furthermore, we obtained the result that the system exhibits a confinement-deconfinement phase transition (CDPT) at certain transition temperature $T_{\text{CSS}}$, below which the deconfinement phase takes place, i.e., the CSS takes place below $T_{\text{CSS}}$. The CDPT is of second order when the system has three-dimensional couplings whatever small they are, while it is infinite order of Kosterlitz-Thouless type when the system is purely two dimensional.

Very recently, study of the CSS was revived, and the confinement-deconfinement problem of gauge theories of nonrelativistic fermions is addressed. Unfortunately, most of these studies do not quote the previous works in which important results on that problem were obtained. Moreover, some of the recent arguments are in apparent contradictions to the previous results, but these authors do not discuss the origins of these contradictions. For example, in his recent paper, Nayak discussed the above problem and concluded that the slave particles (holons and spinons) are always confined by $U(1)$ gauge field. This result is obviously in sharp contrast to our result in Papers I which predicts a deconfinement phase at low $T$.

How does this contradiction come out? In Nayak’s paper, dynamics of the gauge field is not discussed at all and it is simply concluded that the infinitely strong gauge coupling in the bare Lagrangian necessarily leads to confinement. On the other hand, in Papers I, we studied the gauge-field dynamics by using nonperturbative methods. Even if the gauge coupling is infinite in the original lattice model, the gauge field can acquire nontrivial dynamics at low energies such as a deconfinement phenomenon due to the couplings to matter fields. In short, due to ample pair creations of holons and/or spinons, fluxes of electric-like field connecting external charges which would cause the linear-rising confining potential are truncated into short segments, giving rise to a deconfining potential. In lattice gauge theory such as multi-flavor QCD, this fact is now well-established and verified by numerical studies. We showed that a similar deconfinement phenomenon occurs in the gauge theory of the slave-particle t-J model.

Above is the main point of our comment to Nayak’s paper. Another point we argued is that the Elitzur’s theorem does not exclude the possibility of deconfinement phase. There are some misunderstandings that the Elitzur’s theorem prohibits the existence of the deconfinement phase, since the theorem states that the averages of gauge-noninvariant quantities should vanish. However, MFT, for example, can describe the deconfinement phase in accord with the Elitzur’s theorem by averaging over the gauge-rotated copies of a MF solution.

In his reply, Nayak admitted that these two points are certainly correct. There he also posed an argument that there are ambiguities in assigning EM charges to slave particles, which makes the possibility of CSS quite doubtful. This is a misunderstanding. Our explanation is as follows; This ambiguity is related to the choice of reference state from which the EM charges are measured. If measured from the vacuum, the EM charge of a holon is zero and the charge of a spinon is the same as the charge of an electron, $e(<0)$. Similarly, if they are measured from the half-filled state, the holon charge is $-e$ and the spinon charge is zero. Since this is an important point to understand the slave particle approach, we present our explanation in Sect.6 of the present paper after deriving the relevant formulae in Sect.5A.

In the present paper, we focus on the CSS state in the SB t-J model and derive the low-energy effective field theory. By using the lattice model suggested by this field theory, we confirm that the nature of the transition into the spin-gap state is not a crossover but a genuine phase transition for the three-dimensional system. We apply this effective field theory to calculate the dc resistivity. Previously, Lee and Nagaosa and Ioffe et al. showed that $\rho \propto T$ for fermions and bosons interacting with a massless gauge field. Their system has some relation to the uniform RVB MFT of the SB t-J model. Recent experiments on YBaCuO by Ito et al. and others reported that $\rho$ deviates downward from the $T$-linear behavior below certain temperature $T_{\text{SG}}(>T_c)$. This $T_{\text{SG}}$ coincides with the temperature determined by NMR and neutron experiments at which a spin gap starts to develop. So it is quite interesting to derive an effective theory in the spin-gap state of the SB t-J model, and calculate the spin-gap effect on $\rho$ as a function of $\delta$ and $T$. The effective theory used in Ref. is inadequate for this purpose, since it assumes no spin gap. In Sect.3, we shall see that the spin gap generates a mass of gauge field via the Anderson-Higgs mechanism. Thus the effect of scatterings of holons and spinons by gauge fields is reduced. This gives rise to a downward deviation of $\rho$ from the $T$-linear behavior. Our explicit result of $\rho$ in Sect.5C is consistent with the experiments.

The structure of the present paper is as follows: In Sect.2, we start with the MFT of the SB t-J model, and review the results of Papers I for the $U(1)$ gauge dynamics of the t-J model in a compact but selfcontained manner. Various analytical and numerical results which are relevant for the present system are consulted in order to obtain the correct phase diagram. When $T$ is higher than the CDPT temperature $T_{\text{CSS}}(\delta)$ which depends on the holon density $\delta$, $T > T_{\text{CSS}}(\delta)$, the gauge dynamics is realized in the confineinent phase and the holons and spinons are confined to electrons. When $T < T_{\text{CSS}}(\delta)$, the gauge dynamics is realized in the
deconfinement phase and the CSS takes place as mentioned. The MFT predicts another phase transition at \( T_{SG} < T_{CSS} \), below which the spin gap develops. This transition is sometimes claimed to be a crossover rather than a genuine phase transition. We shall discuss this problem in Sect.4 in detail, and show that it is a genuine phase transition when the system has weak but finite three dimensionality.

In Sect.3, we shall derive the low-energy and low-temperature effective field theory in the CSS state of the SB t-J model. The result of Sect.2 assures that this derivation is self consistent, since the gauge dynamics is deconfining at \( T < T_{CSS} \). The effective theory is a Ginzburg-Landau (GL) theory \( L(\lambda, A_i) \) coupled with a gauge field \( A_i \), where the complex scalar \( \lambda \) represents the \( d \)-wave spinon pairing and \( A_i \) is the phase of the hopping amplitude of holons and spinons. Halperin, Lubensky, and Ma considered a similar system, where \( A_i \) corresponds to the electromagnetic (EM) field and \( \lambda \) to the Cooper-pair field. They calculated the effect of \( A_i \) on \( \lambda \) to conclude that it converts the second-order phase transition to a first-order one. Then it was pointed out that the radial fluctuations of \( \lambda \) may keep the transition second-order in some parameter region. This is supported by numerical simulations. More recently, Ubbens and Lee calculated the one-loop effect of \( A_i \) in the SB MFT of the t-J model, and concluded again that the pairing transition at \( T_{SG} \) becomes first order. (However, their \( T_{SG} \) appears below the superconducting transition temperature \( T_c \), so they concluded that the spin-gap phase is completely destroyed by gauge-field fluctuations.) In the present study, we take the compactness of the phase of \( \lambda \) into account. Even in the CSS state, the compactness generates nontrivial interaction vertices that are missing in the previous treatments. We find that the periodic interaction above stabilizes the system at \( T_{SG} \), which is higher than \( T_c \) at low \( \delta \). This allows us to use this effective theory to study the spin-gap state at \( T_c < T < T_{SG} \). We find that the critical exponents (such as \( d \)) that characterize the gauge field mass \( m_A \) through \( m_A \propto (T_{SG} - T)^\delta \) become \( T \) and \( \delta \)-dependent, that is, they are not universal anymore.

In Sect.4, we study the phase transition into the spin-gap state for the 3D system in detail by using the lattice Abelian Higgs model which is a natural extension of the effective field theory of Sect.3 to a lattice model. Below certain on-set temperature \( T_{SG} \) (which is shown to be lower than \( T_{CSS} \)), the spin-gap phase develops as a Higgs phase of the gauge-field dynamics, in which the gauge field acquires a finite mass \( m_A \). Recent numerical studies on 3D U(1) gauge Higgs models show a genuine phase transition and existence of the Higgs phase which supports our discussion on the spin-gap phase. In the region \( T_{SG} < T < T_{CSS} \), the gauge dynamics is realized in the so-called Coulomb phase in which \( m_A = 0 \). At the end of Sect.4, we also comment on the recent dual vortex theory of bosonized fermions in pure two dimensions by Balents et al.

In Sect.5, we calculate the dc resistivity \( \rho \) as an example of calculations of physical quantities in the spin-gap state by using the effective field theory obtained in Sect.3. First, we obtain the expression of \( \rho \) in the random-phase approximation (RPA), which depends crucially on the \( T \)-dependence of \( m_A \). Then we substitute \( m_A \) calculated in Sect.3 to this \( \rho \).

The results on these problems of Sect.3-5 have been reported in part briefly in our previous two short letters. In Sect.3-5 of the present paper, we account for the details of these results in a self-contained manner.

Sect.6 is devoted to discussion. We present a couple of comments related to the problem of observing holons and spinons in the CSS state and ambiguity of charge assignment for holon and spinon. These comments and remarks should clarify certain misunderstandings on the confinement-deconfinement problem in the slave-particle studies of strongly-correlated electron systems.

In Appendix, we consider the U(1) Higgs model in the 3D continuum that is tightly related to the lattice model we studied in Sect.4. We study its phase structure in terms of dual variables, and criticize the recent result of Nagaosa and Lee on this model.

II. PHASE STRUCTURE OF THE GAUGE DYNAMICS

In Sect.2A, we review the MFT of the SB t-J model and show the phase diagram in the \( \delta-T \) plane. In Sect.2B, we study the effect of fluctuations of MFs in a general viewpoint of gauge theory. A CDPT takes place at \( T_{CSS} \), below which the CSS occurs. In Sect.2C, we briefly explain the spin gap state as a “dual” Higgs mechanism of lattice gauge theory with multiple gauge fields.

A. MFT of the SB t-J Model

Let us start with the t-J model Hamiltonian,

\[
H = \sum_{x,i} \left[ -t \left( C_{x+i,\sigma}^\dagger C_{x\sigma} + \text{H.c.} \right) + J \left( \vec{S}_{x+i} \cdot \vec{S}_x - \frac{1}{4} n_{x+i} n_x \right) \right],
\]

\[
\vec{S}_x = \frac{1}{2} \sum_{\sigma,\sigma'} C_{x\sigma}^\dagger \vec{\sigma}_{\sigma \sigma'} C_{x\sigma'}, \quad n_x = \sum_{\sigma} C_{x\sigma}^\dagger C_{x\sigma},
\]

where \( C_{x\sigma} \) is the electron annihilation operator at the site \( x \) of a 2D or 3D lattice with the \( z \)-component of spin \( \sigma = \uparrow, \downarrow \). We use \( i(=1,2,3) \) both as the direction index and as the unit vector in the \( i \)-th direction. The SB representation of \( C_{x\sigma} \) is written as

\[
C_{x\sigma} = b_i^x f_{i\sigma},
\]
where \( b_x \) is the bosonic annihilation operator of holon, and \( f_{x\sigma} \) is the fermionic annihilation operator of spinon. They satisfy the local constraint
\[
b^\dagger_x b_x + \sum_{\sigma} f^\dagger_{x\sigma} f_{x\sigma} = 1, \tag{2.3}
\]
so that only the three states for each \( x \), \( \langle \text{vac} \rangle, C^\dagger_{x\uparrow}\langle \text{vac} \rangle, C^\dagger_{x\downarrow}\langle \text{vac} \rangle, (C_{x\sigma}\langle \text{vac} \rangle = 0) \) are allowed as physical states and the double-occupancy state \( C^\dagger_{x\uparrow}C^\dagger_{x\downarrow}\langle \text{vac} \rangle \) is excluded due to the large on-site Coulomb energy \( U \). These three states are expressed in the SB representation as \( b^\dagger_x |0\rangle, f^\dagger_{x\uparrow} |0\rangle, f^\dagger_{x\downarrow} |0\rangle, (b_x |0\rangle = f_{x\sigma} |0\rangle = 0) \), respectively. The Hamiltonian \( (2.3) \) is rewritten for the physical states in terms of \( b_x, f_{x\sigma} \) as
\[
H = -t \sum_{x, i, \sigma} \left( b^\dagger_{x+i} f_{x\sigma} f_{x+i\sigma} b_x + \text{H.c.} \right) - \frac{U}{2} \sum_{x, i} \left| f^\dagger_{x\uparrow} f^\dagger_{x\downarrow} - f_{x\uparrow} f_{x\downarrow} \right|^2. \tag{2.4}
\]
This \( H \) preserves the total number of holons and spinons at each \( x \), the L.H.S. of \( (2.3) \), and so maps a physical state to another physical state. In path integral formalism, the partition function \( Z(\beta) = \text{Tr} \exp(-\beta H) \) \( \beta \equiv (k_B T)^{-1} \) is given by
\[
Z = \int [db][df][d\chi][d\lambda] \exp(-S), \quad S = \int_0^\beta d\tau \left[ \sum_x \left( b^\dagger_x \frac{\partial b_x}{\partial \tau} + \sum_{\sigma} f^\dagger_{x\sigma} \frac{\partial f_{x\sigma}}{\partial \tau} \right) + H + H_{\text{LC}} \right], \quad H_{\text{LC}} = t \sum_x \alpha_x (b^\dagger_x b_x + \sum_{\sigma} f^\dagger_{x\sigma} f_{x\sigma} - 1), \tag{2.5}
\]
where \( \tau \) \( (0 \leq \tau \leq \beta) \) is the imaginary time and \([db] \equiv \prod_x db^\dagger_x (\tau) db_x (\tau) \), etc. Here \( b_x (\tau) \) (we omitted the argument \( \tau \) in \( (2.3) \)) is regarded as a complex variable at \( x \) and \( \tau \), and \( f_{x\sigma} (\tau) \) is an anticommuting Grassmann variable. \( H_{\text{LC}} \) respects the constraint \( (2.3) \) via the integration over the Lagrange multiplier \( \alpha_x (\tau) \).

To set up the MFT, we introduce two complex auxiliary fields \( \chi_{x\uparrow} \) and \( \chi_{x\downarrow} \) on the link \((x, x+i)\) to decouple both \( t \) and \( J \) terms as
\[
H_{\text{MF}} = \sum_{x, i} \left[ \frac{3J}{8} |\chi_{x\uparrow}|^2 + \frac{2}{3J} |\chi_{x\downarrow}|^2 - \left\{ \chi_{x\uparrow} \left( \frac{3}{8} J \sum_{\sigma} f^\dagger_{x+i\sigma} f_{x\sigma} + t b^\dagger_{x+i} b_x \right) + \text{H.c.} \right\} - \frac{1}{2} \left\{ \chi_{x\downarrow} \left( f^\dagger_{x\uparrow} f^\dagger_{x+i\downarrow} - f_{x\uparrow} f_{x+i\downarrow} \right) + \text{H.c.} \right\} \right]. \tag{2.6}
\]
In the MFT, the local constraint \( (2.3) \) is relaxed to the following global constraints for averages:
\[
\langle b^\dagger_x b_x \rangle = \delta, \quad \sum_{\sigma} \langle f^\dagger_{x\sigma} f_{x\sigma} \rangle = 1 - \delta, \tag{2.7}
\]
where \( \delta \) \( (\in [0, 1]) \) is the doping parameter. This modification is validated when the system is in the CSS state a posteriori because the CSS implies that the local constraint becomes irrelevant for holons and spinons. This is the subject of Sect.2B, and some supplementary discussion is also given in Sect.6. Then, \( Z(\beta) \) is written as
\[
Z = \int [db][df][d\chi][d\lambda] \exp(-S'), \quad S' = \int_0^\beta d\tau \left[ \sum_x \left( b^\dagger_x \frac{\partial b_x}{\partial \tau} + \sum_{\sigma} f^\dagger_{x\sigma} \frac{\partial f_{x\sigma}}{\partial \tau} \right) + H_{\text{MF}} + H_\mu \right], \quad H_\mu = -\sum_x \left( \mu_b b^\dagger_x b_x + \mu_F \sum_{\sigma} f^\dagger_{x\sigma} f_{x\sigma} \right), \tag{2.8}
\]
where we replaced \( H_{\text{LC}} \) by \( H_\mu, \mu_b, \mu_F \) in which are the chemical potentials to enforce \( (2.7) \). By differentiating the integrand in \( (2.4) \) w.r.t. \( \chi_{x\uparrow} \) and \( \chi_{x\downarrow} \), we obtain the following relations (Schwinger-Dyson equations) among averages:
\[
\langle \chi_{x\uparrow} \rangle = \sum_{\sigma} f^\dagger_{x+i\uparrow} f_{x\sigma} + \frac{8t}{3J} b^\dagger_{x+i} b_x, \quad \langle \chi_{x\downarrow} \rangle = \frac{3J}{2} \left( f^\dagger_{x\uparrow} f^\dagger_{x+i\downarrow} - f_{x\uparrow} f_{x+i\downarrow} \right), \tag{2.9}
\]
which show that \( \chi_{x\uparrow} \) describes hopplings of holons and spinons, while \( \chi_{x\downarrow} \) describes the resonating-valence-bond (RVB) (NN singlet spin-pair) amplitude.

The MFT can be set up first by parametrizing the auxiliary fields as
\[
\chi_{x\uparrow} (\tau) = \chi_1 U_{x\uparrow} (\tau), \quad U_{x\uparrow} (\tau) = \exp(iA_{x\uparrow} (\tau)), \quad \lambda_{x\uparrow} (\tau) = \lambda V_{x\uparrow} (\tau), \quad V_{x\uparrow} (\tau) = \exp(iB_{x\uparrow} (\tau)). \tag{2.10}
\]
by ignoring the site- and \( \tau \)-dependence of their amplitudes, and then ignoring the phase fluctuations by setting \( A_{x\uparrow} = 0, B_{x\uparrow} = 0 \). The effects of these phases are considered in Section 2B. Numerical studies were performed for this MFT in 2D with various patterns of the MF’s \( \chi_1 \) and \( \lambda_1 \). One of the most interesting case is the so-called uniform RVB, in which \( \chi_1 \) are uniform, \( \chi_1 \equiv \chi \), and \( \lambda_1 \) are the d-wave configuration, \( \lambda_1 = -\lambda_2 \equiv \lambda \). In Sect.3C, we shall study the general pattern of \( \lambda_1 \) in terms of the GL theory and show the stability of this d-wave configuration. The MF phase diagram in \( \delta-T \) plane can be calculated in a straightforward manner. In Fig.4 we show the result of Ref.4. There appears several critical temperatures. Below \( T_X \), \( \chi \) develops. \( \chi \) is estimated at small \( \delta \)’s as
\[
\chi \simeq \frac{4\pi}{7} \sin^2 \left( \frac{\pi}{2} \sqrt{1-\delta} \right) + \frac{8t}{3J} \delta, \tag{2.11}
\]
at \( T_{SG} < T \) where \( \lambda_{x\uparrow} = 0 \). From \( (2.6) \) and \( (2.9) \), one expects holons and spinons may hop independently for \( \chi \neq 0 \). Thus, in the level of MFT, the CSS state is realized below \( T_X \). However, as we shall see in Sect.2B,
the effect of phase fluctuations of $\chi_{x_i}$ reduces this CSS onset temperature down to $T_{CSS}$ which is much lower than $T_K$. Below $T_{SG}$, $\lambda$ develops, so the system enters into the spin-gap state. There is another transition temperature $T_{BC}$ below which bose condensation of holons, $\langle b_x \rangle \neq 0$, takes place. To obtain a nonvanishing $T_{BC} > 0$, a weak three-dimensional coupling $\chi = \alpha \chi$ is included in Ref. In the SB MFT, the superconducting phase is described by the simultaneous condensations of $\lambda$ and $\langle b_x \rangle$, because the order parameter of superconductivity, the pairing amplitude of NN hole states, is expressed as follows:

$$
\begin{align*}
(C_{x\downarrow}C_{x+\uparrow\downarrow} - C_{x\uparrow}C_{x+\downarrow\uparrow}) & = (f_{x\downarrow}^\dagger f_{x+\uparrow\downarrow} - f_{x\uparrow}^\dagger f_{x+\downarrow\uparrow}) \times \langle b_x b_{x+i} \rangle \\
& = \lambda \langle b_x \rangle^2.
\end{align*}
$$

Thus $T_c(\delta)$ is the lower temperature of $T_{\lambda}(\delta)$ and $T_{BC}(\delta)$.

### B. Beyond MFT: Phase Fluctuations and U(1) Gauge Theory

In any MFT, it is indispensable to examine the stability of its solution. Especially, the SB t-J model possesses a time-dependent U(1) local gauge symmetry,

$$
\begin{align*}
b_x(\tau) & \to e^{i\theta_x(\tau)} b_x(\tau), \\
f_{x\sigma}(\tau) & \to e^{i\theta_x(\tau)} f_{x\sigma}(\tau), \\
\alpha_x(\tau) & \to \alpha_x(\tau) + \partial_\tau \theta_x(\tau).
\end{align*}
$$

The Lagrange multiplier $\alpha_x(\tau)$ is regarded as a time component of the gauge field.

In the partition function of the decoupled MFT, we fixed the gauge to the temporal gauge by setting $\alpha_x(\tau) = 0$. A careful and precise treatment of this gauge fixing is given in Ref. which assures us to redervies the results of Papers I. After this gauge fixing, there still remains in the decoupled Hamiltonian a time-independent residual gauge symmetry (with $\tau$-independent $\theta_x$):

$$
\begin{align*}
b_x(\tau) & \to e^{i\theta_x} b_x(\tau), \\
f_{x\sigma}(\tau) & \to e^{i\theta_x} f_{x\sigma}(\tau), \\
\chi_{x\uparrow}(\tau) & \to e^{-i\theta_x} \chi_{x\uparrow}(\tau) e^{i\theta_{x+i}}, \\
\lambda_{x\uparrow}(\tau) & \to e^{i\theta_x} \lambda_{x\uparrow}(\tau) e^{i\theta_{x+i}}.
\end{align*}
$$

The last two relations, which are naturally understood when one recalls show that the phase degrees of freedom of the auxiliary fields $\chi_{x\uparrow}$ and $\lambda_{x\uparrow}$ can be regarded as two kinds of gauge fields:

$$
\begin{align*}
A_{x\uparrow}(\tau) & \to A_{x\uparrow}(\tau) - \theta_x + \theta_{x+i}, \\
B_{x\uparrow}(\tau) & \to B_{x\uparrow}(\tau) + \theta_x + \theta_{x+i},
\end{align*}
$$

both of which are associated with the common U(1) gauge symmetry. Thus, the problem of stability around MFT, $A_{x\uparrow} = B_{x\uparrow} = 0$, reduces to the dynamics of the resultant gauge theory $A_{x\uparrow}$ and $B_{x\uparrow}$.

Let us first consider the region $T > T_{SG}$, where $\lambda_i = 0$ and one needs to consider only $A_{x\uparrow}(\tau)$ or $U_{x\uparrow}(\tau)$. The region $T < T_{SG}$, where both $A_{x\uparrow}(\tau)$ and $B_{x\uparrow}(\tau)$ appear, shall be considered in Sect.2C. The dynamics of $U_{x\uparrow}(\tau)$ is governed by an effective action that is obtained by integrating out the holon and spinon variables. This idea is similar to that for the $O(N)$ nonlinear $\sigma$-model, in which an effective action or a potential of the auxiliary (Lagrange multiplier) field is calculated by integrating out the original scalar fields $n_{\sigma a}$ satisfying the local constraint $\sum_{\alpha=1}^{N} n_{\sigma a} n_{\sigma a} = 1$. The successive large $N$ analysis for 3D system reveals the existence of a phase in which the local constraint is irrelevant. This phase corresponds to the CSS phase of the t-J model. What is important in integrating over $b_x$ and $f_{x\sigma}$ is to intact the dynamics of $U_{x\uparrow}$. For example, if one reduces the lattice system into a simple continuum field theory at this stage, the action may contain only quadratic terms of $A_{x\uparrow}$ such as the kinetic term, $|\partial_{\tau} b(x)|^2$ and $|\partial_{\tau} - i A_{j}(x) b(x)|^2$, and the Maxwell term, $\partial_{\tau} A_{j} - \partial_{x} A_{j} \partial_{x}^2$. This action fits well to a perturbative analysis in which $A_{i}$ is assumed as a small quantity, but it necessarily loses the possibility to describe the confinement phase in which $A_{i}$ is large, although the integration over $b(x)$ and $f_{x\sigma}(x)$ can be done exactly. As we proposed in Papers I, we can perform the integration over $b_x$ and $f_{x\sigma}$ by an approximate but a nonperturbative method, i.e., the hopping expansion in powers of $U_{x\uparrow}$, which gives rise to an action that is capable to describe both confinement and deconfinement phases. This action $SU$ generally contains two different kinds of terms; the electric one $SU_E$ that controls the $\tau$-dependence of $U_{x\uparrow}(\tau)$, and the magnetic one $SU_M$ that controls the spatial-variation of $U_{x\uparrow}(\tau)$. Typical forms of them are given as

$$
\begin{align*}
SU & = SU_E + SU_M, \\
SU_E & = \frac{1}{g_E(T)} \sum_{x\uparrow} \int_0^\beta d\tau \partial_\tau U^\dagger_{x\uparrow}(\tau) \partial_\tau U_{x\uparrow}(\tau), \\
SU_M & = \frac{1}{g_M(T)} \sum_{pl} \prod_{\ell=1}^4 \int_0^\beta d\tau \prod_{pl} U_{x\uparrow}(\tau),
\end{align*}
$$

where $\prod_{pl} U_{x\uparrow}$ denotes the product of four $U_{x\uparrow}$‘s along an oriented plaquette; $U_{x\uparrow, y\uparrow} U^\dagger_{x+\downarrow, y+\uparrow} U^\dagger_{x, y}$, $g_E(T)$ and $g_M(T)$ are the T-dependent effective coupling constants. In Papers I, their T-dependences are calculated explicitly as

$$
\begin{align*}
g_E^2(T) & \propto T^3, \\
g_M^2(T) & \propto const,
\end{align*}
$$

by using the hopping expansion w.r.t. $U_{x\uparrow}$.

In the conventional lattice gauge theory the action in the temporal gauge $A_{\tau 0} = 0$ is given by

$$
S^{LGT} = S^{LGT}_E + S^{LGT}_M,
$$
\[ S_{E}^{LGT} = \frac{1}{e^2} \sum_{x,i} \int_{0}^{\beta} d\tau \partial_{\tau} U_{xi}^\dagger(\tau) \partial_{\tau} U_{xi}(\tau), \]
\[ S_{M}^{LGT} = -\frac{1}{e^2} \sum_{p} \int_{0}^{\beta} d\tau \prod_{i} U_{xi}(\tau) \]
\[ = -\frac{1}{e^2} \sum_{p} \int_{0}^{\beta} d\tau \prod_{i} \cos F_{xi}(\tau), \]  
(2.18)

where \( F_{xi}(\tau) \equiv A_{x_i} + A_{x+i,j} - A_{x+j,i} - A_{x_j} \) is the magnetic field penetrating the plaquette. In the Hamiltonian formalism, the corresponding Hamiltonian \( H^{LGT} \) is derived as

\[ H^{LGT} = H_{E}^{LGT} + H_{M}^{LGT}, \]
\[ H_{E}^{LGT} = e^2 \sum_{x,i} E_{xi}^2, \]
\[ H_{M}^{LGT} = -\frac{1}{e^2} \sum_{x,i,j} (U_{x_i} U_{x+i,j} U_{x+j,i} U_{x_j}^\dagger + \text{H.c.}). \]  
(2.19)

Here, the electric field \( E_{xi} \equiv \partial_{\tau} A_{xi} \) is the variable canonically conjugate to \( A_{xi} \), so the following uncertainty principle holds;

\[ [E_{xi}, A_{yj}] = i\delta_{xy}\delta_{ij}, \quad \Delta E_{xi} \Delta A_{xi} > 1. \]  
(2.20)

Since \( A_{xi}(\in [0, 2\pi]) \) is an angle variable, \( E_{xi} \) takes an integer in its diagonal representation. When one applies \( U_{xi} \) to a state, a segment of electric flux is created on \( (x, x+i) \); \( U_{xi} \) is a creation operator of electric flux. The time-independent residual gauge symmetry restricts the physical states so as to satisfy the Gauss’s law,

\[ \sum_{i} \nabla_{i} E_{xi} \equiv \sum_{i} (E_{x+i,i} - E_{xi}) = 0, \]  
(2.21)

where \( \nabla_{i} \) is the lattice difference operator; \( \nabla_{i} f_{x} \equiv f_{x+i} - f_{x} \).

The system (2.18 - 2.22) is known to exhibit a CDPT at finite \( T \). This was shown by Polyakov and Susskind by mapping the system (2.19) to a XY spin model. Monte Carlo simulations confirm the CDPT. Below we present a general explanation why the CDPT takes place. For this purpose, let us first ignore the magnetic term \( H_{M}^{LGT} \) in (2.19), the reason of which shall be explained later. The partition function is then written as

\[ Z_{LGT} = \prod_{x,i} \sum_{E_{xi}} \prod_{x} \delta_{x}, \nabla_{x}, E_{xi}=0 \exp(-\beta e^2 \sum_{x,i} E_{xi}^2). \]  
(2.22)

For large coupling \( e^2 \) and/or at low \( T \) such as \( e^2 \beta \gg 1 \), configurations with small \( |E_{xi}| \) are favored. To investigate the gauge dynamics, let us introduce two test charges with \( Q = \pm e \) at \( x_1, x_2 \). Then the Gauss’s law is modified to \( \nabla_{i} E_{xi} = Q_{x} \equiv e \delta_{x,x_1} - e \delta_{x,x_2} \), and the configuration with the lowest energy is the one with electric flux \( E_{xi} = \pm 1 \) formed along the straight line connecting \( x_1 \) and \( x_2 \), and \( E_{xi} = 0 \) elsewhere. See Fig.3a.

The potential energy of this configuration is linear rising, \( V(x_1, x_2) = e^2 |x_1 - x_2| \), so the two charges are confined. In this confinement state, \( \Delta E_{xi} = 0 \), so the relation (2.20) implies \( \Delta A_{xi} = \infty \), that is, \( A_{xi} \) fluctuates violently. On the other hand, for small coupling \( e^2 \) and/or at high \( T \) such as \( e^2 \beta \ll 1 \), \( E_{xi} \) fluctuates violently, \( \Delta E_{xi} \gg 1 \), which implies \( \Delta A_{xi} \ll 1 \). Thus, the fluctuations of \( A_{xi} \) are small, and can be treated by usual perturbation theory. The potential energy can be evaluated via exchange of a massless gauge boson, leading to the Coulomb potential, \( V(x_1, x_2) = e^2 |x_1 - x_2|^{-1} \) in 3D space. This is the deconfinement state. The electric fluxes are attached isotropically to each external charge. See Fig.3b. These considerations lead us to a certain finite temperature \( T \) at which the CDPT takes place. For fixed \( e^2 \), the string tension vanishes and charged particles are deconfined above \( T \). The quark-gluon plasma in QCD at high \( T \) is such a state. Below \( T \), particles with nonvanishing charges that couple to \( A_{xi} \) are confined to form charge-neutral objects. Mesons and baryons in QCD are such objects.

Let us estimate the effect of the magnetic term \( H_{M}^{LGT} \). Let us consider a state in the confinement phase, \( E_{xi} \) of which takes definite values for all links. When \( H_{M}^{LGT} \) is applied to this state, each segment of electric flux at \( (x, x+i) \) is deformed to the one lying on a detour \( (x, x+j) \), \( (x+j, x+i) \), \( (x+i, x+j) \). Thus \( H_{M}^{LGT} \) lets \( E_{xi} \) fluctuate and increases \( \Delta E_{xi} \), and so acts to favor the deconfinement phase. Thus \( H_{M}^{LGT} \) may decrease \( T \) certainly, but \( T \) does not vanish, as the strong coupling expansion w.r.t. \( e^2 \gg 1 \) at \( T = 0 \) assures us the existence of the confinement phase [34].

Let us return to the present case (2.19) of the t-J model. The condition that the deconfinement state occurs may be estimated as \( \beta g_{F}^2 < 1 \). Due to the \( T \)-dependence of \( g_{F}^2(T) \) as shown in (2.17), the deconfinement state, hence the CSS, takes place below certain critical temperature \( T_{CSS} \):

\[ \beta g_{F}^2 \simeq C \frac{T}{T^3} \simeq \left( \frac{T}{T_{CSS}} \right)^2 < 1. \]  
(2.23)

For \( T < T_{CSS} \), the MFT works as a first approximation because \( \Delta A_{xi} \) is small there. We stress that the reason why our result \( T < T_{CSS} \) is for the CSS state is opposite to that of (2.16), \( T > T_{CDPT} \) as in QCD, due to the nontrivial \( T \)-dependence of coupling constants (2.17), which are generated by integrating out holons and spions. In Ref.4, Nagaosa studied certain dissipative gauge theory of fermions and obtained the result that a deconfinement state appears above certain temperature or the strength of dissipation is strong enough. However, we note that he does not discuss the possibility that the parameters of his model may be effective and \( T \)-dependent.

At first, the above conclusion may seem rather strange
since the original gauge coupling is infinite (the coefficient of Maxwell term is zero). Our study shows the possibility that a coupling to matter-fields strongly influences the phase structure. In fact, this fact is now well-established in the elementary particle physics, especially in the lattice gauge theory. A good example is the non-Abelian gauge model without matter fields, which is always in the confinement phase in 3D at $\delta > 0$. Even at infinite-gauge-coupling limit, this quark-gluon system is in a deconfinement phase for $N_f > 7$ at $T = 0$. This example is closely related with the present $U(1)$ gauge-theory model for the t-J model. First, Polyakov showed that the pure compact $U(1)$ lattice gauge system on a 3D lattice without matter fields is always in a confinement phase. (This system corresponds to the 2D Hamiltonian at $T = 0$. In certain approximation, it can be also regarded as an effective system of a quantum system in 3D space at finite $T$. For more details, see the paragraph above (4.3).) Next, inclusion of nonrelativistic fermions (the spinons in the SB representation) can be regarded as inclusion of many-flavor light Dirac fermions because of the Fermi surface (line) instead of the “Fermi point” of the Dirac fermion. Thus, what we showed for the t-J model in Papers I is nothing but the similar deconfining phenomenon induced by ample light fermions.

Let us make a couple of comments on the explicit curve of $T_{CSS}(\delta)$ in Fig. 4. As mentioned in Sect.2.A, it is much lower than the MF value $T_X$. This implies that the effect of gauge fluctuations are significant. In contrast with Fig. 4, the observed $T_{CSS}(\delta)$ seems decreasing as $\delta$ is increased. In Ref. 8, we observe that the two-body interactions introduced there tend to lower $T_{CSS}$ for larger $\delta$. We also note that the value of $T_{CSS}(\delta)$ in Fig. 4 is not reliable at very small $\delta$, because the long-range AF order in 3D at $\delta \leq 0.04$ upsets the validity of the hopping expansion we used. (This point is related with the argument given in Sect.2D below.)

**C. Spin Gap State**

At low temperatures below $T_{SG}$, the MFT of Sect.2 predicts that the NN spin-pairing amplitude $\lambda$ develops in addition to $\chi$. This phase is called the spin-gap phase, because the spinon excitations in 2D acquire the following excitation energy $E(k)$, i.e., an energy gap;

$$E^2(k) = \left( \frac{3J\chi}{4} \sum_i \cos k_i - 2\mu_F \right)^2 + \left( \lambda \sum_i (-)^i \cos k_i \right)^2.$$  

(2.24)

We note that there are several gapless points in $k$ space, at which $\cos k_1 = \cos k_2 = 2\mu_F/(3J\chi)$. The mechanism to generate this gap is similar to that of the famous energy gap in the BCS model of conventional superconductivity. Since the spinon part of the MF Hamiltonian (2.6) has the structure, $H_f \sim f^\dagger f + \lambda f^\dagger f_i f_i$, diagonalization of $H_f$ by a Bogoliubov transformation gives rise to $H_f \sim \sum_{k,\sigma} E(k)\alpha^\dagger_{k\sigma}\alpha_{k\sigma}$, $E(k) \simeq (\epsilon^2 + \lambda^2)^{1/2}$.

In this phase, there appear two kinds of gauge fields $A_{xi}$ and $B_{xi}$ of (2.10) and (2.13). Their gauge dynamics can be studied by the action $S_{UV}$ that is obtained by integrating over $b_x$ and $f_x$ by the hopping expansion w.r.t. $U_{xi}$, $V_{xi}$, $S_{UV}$ is calculated in Papers I. It takes the form;

$$S_{UV} = S_{UVE} + S_{UVM},$$

$$S_{UVE} = \int_0^\beta d\tau \sum_{xi} \left[ \frac{1}{g_{EUV}(T)} \partial_\tau U_{xi}(\tau) \partial_\tau U_{xi}(\tau) \right.$$  

$$+ \frac{1}{g_{EUV}(T)} \partial_\tau U_{xi}(\tau) \partial_\tau V_{xi}(\tau) \right],$$

$$S_{UVM} = -\int_0^\beta d\tau \sum_{x,i<j} \left[ \frac{1}{g_{MUV}(T)} U_{x+i,j} V_{x+i,j}^\dagger U_{x+i,j} V_{x+i,j}^\dagger \right.$$  

$$+ \frac{1}{g_{MUV}(T)} V_{x+i,j} U_{x+i,j} V_{x+i,j}^\dagger U_{x+i,j} V_{x+i,j}^\dagger + \frac{1}{g_{MUV}(T)} \right],$$  

(2.25)

Nature of gauge excitations of this system may be drawn by calculating the quadratic parts of $S_{UVM}$ in terms of $A_{xi}$, $B_{xi}$. The $g_{MUV}^2UUU^\dagger V^\dagger V$ term gives rise to the Maxwell term of $A_{xi}$. If this term alone is kept, it describes a massless gauge field $A_{xi}$. The term $g_{MUV}^2VV^\dagger V^\dagger V$ of itself also describes another massless excitations $B_{xi}$. However, the mixed term $g_{MUV}^2UU^\dagger VU^\dagger V^\dagger V$ generates the term $m_A^2A_{xi}^2 + m_B^2B_{xi}^2$, which acts as mass terms of $A_{xi}$ and $B_{xi}$ with $m_A^2 \propto \chi^2\lambda^2$. Thus the excitations $A_{xi}$ and $B_{xi}$ become massive simultaneously at $T < T_{SG}$. This may be called dual Anderson-Higgs mechanism for two kinds of gauge fields; one gauge field acts as a Higgs field for the other gauge field and vise versa. So we call this phase a Higgs phase, in which the potential energy $V(x_1, x_2)$ between two external charges falls off exponentially due to exchange of massive gauge bosons. In Table 1, we compare $V(x_1, x_2)$ for each phase. The lattice gauge model (2.25) is complicated, and no systematic study has been done. Instead, in Sect.4, we shall study the nature of phase transition at $T_{SG}$ in detail using the lattice Abelian Higgs model that is derived from the effective field theory of Sect.3.

**D. Lower Dimensional Cases**

In Sect.2B, we explained that the CSS can be understood as a deconfinement phenomenon of the dynamical
gauge field $A_{\vec{x}i}$. We do not rely explicitly upon the fact that the t-J model of our interest has a weak but finite three-dimensionality. Then one may ask whether the same argument can be applied for systems in lower spatial dimensions. We first consider the 1D case. Then we comment on the 2D case.

For example, the supersymmetric t-J model ($t = J$) in one-dimension is known to show the CSS. However, a straightforward application of the arguments in Papers I, i.e., the hopping expansion and the Polyakov-Susskind theory of CDPT to the 1D t-J model leads the result that only the confinement phase exists. Technically speaking, the Polyakov-Susskind method maps the system to an asymmetric XY spin model in 1D, and this spin model is easily shown to have only a disordered phase, which corresponds to the confinement phase. This result seems rather natural because in 1D the Coulomb force itself generates a linear-rising confining potential. There is no possibility for a deconfinement phase to appear in 1D if the effective action of the dynamical gauge field is local. However, if the hopping expansion is not a good approximation or is not convergent, a deconfinement phase may exist. This is in sharp contrast to the system in spatial 3D in which the existence of a deconfinement phase can be predicted in the local gauge action and the hopping expansion.

In order to gain insight into the above problem in 1D and to show that a deconfinement phase really exists, we consider gauge dynamics at low $T$ and move to the momentum space instead of the real space assuming the fluctuation of the dynamical gauge field is small. This is legitimate since at low $T$ the fluctuation of the gauge field becomes small in the higher-dimensional cases. Even in this case, the gauge interaction gives rise to linear-rising confining potential. The potential energy between two slave particles with the opposite charges can be ignored. We shall focus on the spinon contribution for simplicity. For $U_{\vec{x}i} = 1$, there exist two Fermi points at $\pm k_F$ whose value depends on the hole concentration. Then the left and right moving modes can be defined in the usual way. The continuum field theory is easily obtained and that is nothing but the 2-flavor massless Schwinger model. This model can be exactly solved. Especially and that is nothing but the 2-flavor massless Schwinger model.

The MC simulation leases us from this problem. The Coulomb phase is less realized as a Higgs phase. The hopping expansion in the real space cannot uncover this possibility.

Next, let us consider the 2D case. The method of Paper I gives rise to an asymmetric XY spin model in 2D. The MC simulation and the analytical method show that this model has an order-disorder phase transition, which corresponds to a CDPT. These analyses show that the CDPT occurs in higher $T$ than the spin-gap generation temperature $T_{SG}$ as in 3D (See Fig. 3). Thus the effective model is nothing but the 2D (symmetric) XY model, and the CDPT is the Kosterlitz-Thouless transition of infinite-order. Although the validity of hopping expansion has not yet been clarified, very existence of a CDPT may survive in the correct treatment. There is a special point in 2D. The potential energy between two slave particles with the opposite charges in the Coulomb phase is logarithmic $V(r) \sim \ln(r)$ due to the 2D Coulomb force. Thus the slave particles are “confined” even in the Coulomb phase in the sense that $\lim_{r \to \infty} V(r) = \infty$. This requires a careful study of the nature of quasiparticles in the Coulomb phase of the pure 2D system. As mentioned, inclusion of a finite three dimensionality releases us from this problem.

### III. EFFECTIVE FIELD THEORY OF THE SLAVE-BOSON T-J MODEL

#### A. Hamiltonian of the Continuum Field Theory and Integration over Holons and Spinons

Let us consider the low temperature region $T < T_{CSS}$, where the fluctuations of gauge field $A_{\vec{x}i}$ is small, as explained in Sect.2B. Thus it is reasonable to derive the low-energy effective theory there in the form of local gauge field theory, and apply it the usual perturbation theory to evaluate the effect of $A_{\vec{x}i}$. Let us start with the Hamiltonian $H_{MF}$ of (2.5) and translate the lattice variables to the continuum fields such as $f_{\vec{x}i} = a f_{\sigma}(\vec{x})$ by using the lattice constant $a$. For $U_{\vec{x}i}$ we set

$$U_{\vec{x}i} = \exp(i a A_{\vec{x}}(\vec{x})) \simeq 1 + i a A_{\vec{x}}(\vec{x}) - \frac{1}{2} a^2 A_{\vec{x}}(\vec{x})^2.$$  

(3.1)

The mass term comes from a “massless mode” in the particle-anti-particle channel, which is a direct consequence of the kinematics in 1D and behaves like a Higgs boson. It is obvious that $\Pi_{\mu\nu}(k)$ is finite as $k \to 0$ and so the higher-order terms of the hopping expansion in the real space are to give the main contribution to it.

The above consideration in momentum space assuming small and smooth fluctuations of the gauge field gives a consistent solution of the deconfinement phase which is realized as a Higgs phase. The hopping expansion in the real space cannot uncover this possibility.

Let us consider the 2D case. The method of Paper I gives rise to a symmetric XY spin model in 2D. The MC simulation and the analytical method show that this model has an order-disorder phase transition, which corresponds to a CDPT. These analyses show that the CDPT occurs in higher $T$ than the spin-gap generation temperature $T_{SG}$ as in 3D (See Fig. 3). Thus the effective model is nothing but the 2D (symmetric) XY model, and the CDPT is the Kosterlitz-Thouless transition of infinite-order. Although the validity of hopping expansion has not yet been clarified, very existence of a CDPT may survive in the correct treatment. There is a special point in 2D. The potential energy between two slave particles with the opposite charges in the Coulomb phase is logarithmic $V(r) \sim \ln(r)$ due to the 2D Coulomb force. Thus the slave particles are “confined” even in the Coulomb phase in the sense that $\lim_{r \to \infty} V(r) = \infty$. This requires a careful study of the nature of quasiparticles in the Coulomb phase of the pure 2D system. As mentioned, inclusion of a finite three dimensionality releases us from this problem.

The Hamiltonian $H_{c}$ of the continuum field theory is then calculated as

$$H_{c} = \int d^2x \left[ \frac{3}{4a^2} \chi^2 + m_F \chi \left( |\lambda_{s}(\vec{x})|^2 + |\lambda_{d}(\vec{x})|^2 \right) \right] + \int d^2x \left[ \frac{1}{2m_B} \sum_{\vec{x}} |D_{\vec{x}} b(\vec{x})|^2 - \mu_B |b(\vec{x})|^2 \right].$$
\[ + \int d^2x \left[ \frac{1}{2m_F} \sum_i \left| D_i f_\sigma(x) \right|^2 - \mu_F \left| f_\sigma(x) \right|^2 \right] \]
\[ + \int \frac{d^2k}{(2\pi)^2} \left[ \Delta_{SG}(k) f^\dagger_i(k) f^\dagger_j(-k) + \text{H.c.} \right], \quad (3.2) \]

where \( \mu_{B,F} \) denote the chemical potentials for the continuum theory, \( f_\sigma(k) \) is the Fourier transform of \( f_\sigma(x) \),
\[ f_\sigma(x) = \int \frac{d^2k}{(2\pi)^2} f_\sigma(k) \exp(ikx), \quad (3.3) \]
and
\[ \lambda_+ = \frac{1}{2} \lambda_1(x) + \lambda_2(x), \]
\[ \lambda_d = \frac{1}{2} \lambda_1(x) - \lambda_2(x), \]
\[ \frac{1}{2m_B} = \frac{1}{2m_F} = \frac{3}{2} J \chi a^2, \]
\[ D_i = \partial_i - iA_i, \quad k_F^2 = \frac{2\pi}{a^2}(1 - \delta) = 2m_F \mu_F. \quad (3.4) \]

\( D_i \) is the covariant derivative w.r.t. the gauge field \( A_i(x) \), and \( k_F \) is the Fermi momentum of spinons. The last term of \( H_c \), in (3.2) describes the coupling of the spin-gap amplitude to spinon pairs. Originally, it has the form of
\[ \int \frac{d^2k}{(2\pi)^2} \left[ \Delta_{SG}(k, q) f^\dagger_i(k + \frac{q}{2}) f^\dagger_j(-k + \frac{q}{2}) + \text{H.c.} \right], \]
\[ \Delta_{SG}(k, q) \simeq 2(1 - \delta) \left( \frac{k_1^2 - k_2^2}{k_F^2} \right) \lambda_d(q) - 2\delta \lambda_+(q). \quad (3.5) \]

\( \Delta_{SG}(k, q) \) consists of the first d-wave term and the second s-wave term. At half-filling, \( \delta = 0 \), the s-wave term vanishes. In \( H_c \) of (3.2), we simplify \( \Delta_{SG} \) by keeping only its d-wave term and ignoring its q-dependence as
\[ \Delta_{SG}(k) = 2(1 - \delta) \left( \frac{k_1^2 - k_2^2}{k_F^2} \right) \lambda_d, \quad (3.6) \]
where \( \lambda_d = \lambda_d(0) \) (\( \lambda_i \simeq -\delta \lambda_d \)). We also set \( q = 0 \) in \( f^\dagger_j \) of (3.2).

To obtain the effective action \( L_{eff}(A_i, \lambda_i) \) of \( \lambda_i(x, \tau) \) and \( A_i(x, \tau) \), we integrate over \( b, f_\sigma \) by the standard bilinear integrations in the partition function,
\[ Z = \int [db][df][d\lambda_i][dA_i] \exp(-\int d\tau L_c), \]
\[ = \int [d\lambda][dA_i] \exp(-\int d\tau L_{eff}(A_i, \lambda_i)), \]
\[ L_c = \int d^2x (b^\dagger \partial_\tau b + \sum_\sigma f^\dagger_\sigma \partial_\tau f_\sigma) + H_c. \quad (3.7) \]

**B. Gauge-Field Propagator**

The propagator of the gauge field, \( D_{ij}(x, \tau) \), is generated by fluctuations (vacuum polarization) of spinons and holons, i.e.,
\[ D_{ij}(x, \tau) \equiv \langle A_i(x, \tau) A_j(0, 0) \rangle \]
\[ = \left[ \Pi_F + \Pi_B \right]^{-1}_{ij}, \]
\[ \Pi_{F,B} ij(x, \tau) \equiv -\langle J_{F,B} i(x, \tau) J_{F,B} j(0, 0) \rangle_{\text{PI}} + \delta_{ij} \delta(x) \delta(\tau) n_{F,B}, \quad (3.8) \]

where
\[ n_F = \frac{1 - \delta}{a^2}, \quad n_B = \frac{\delta}{a^2}, \quad (3.9) \]
are the holon and spinon density and \( \Pi_{F,B} ij \) represent one-particle-irreducible (1PI) diagrams of spinon and holon loops made of the following currents coupled to \( A_i \):
\[ J_{Fi} \equiv \frac{1}{2m_F} \sum_\sigma (i\partial_\mu f_\sigma + \text{H.c.}), \]
\[ J_{Bi} \equiv \frac{1}{2m_B} (ib\partial \tau b + \text{H.c.}). \quad (3.10) \]

In the Coulomb gauge, \( \partial_\tau A_i = 0 \), the propagator at momentum \( q \) and Matsubara frequency \( \epsilon_i \equiv 2\pi i/\beta \) is written as
\[ D_{ij}(q, \epsilon_i) \equiv \int \frac{d^2q}{(2\pi)^2} \frac{d\tau}{\beta} e^{i(qx + \omega \tau)} D_{ij}(x, \tau) \]
\[ = \left( \delta_{ij} - \frac{q_i q_j}{q^2} \right) D(q, \epsilon_i), \]
\[ D(q, \epsilon_i) = \left[ \Pi_F(q, \epsilon_i) + \Pi_B(q, \epsilon_i) \right]^{-1}. \quad (3.11) \]

Since we shall need \( D \) later in calculating \( \rho \), we calculate \( D \) below in the random-phase approximation as
\[ D \sim \left[ \Pi_F^R + \Pi_B^R \right]^{-1}. \quad (3.12) \]

When the spin gap is sufficiently small, its effect to \( \Pi_B^R \) is evaluated by perturbation theory. We obtain
\[ \Pi_B^R(q, \epsilon_i) \simeq \left\{ \frac{\epsilon_i^2 + \sqrt{\epsilon_i^2 + \frac{\epsilon_i^2}{2m_F}}} {2m_F}, \frac{\epsilon_i}{q} \ll v_F q, |\epsilon_i| \gg v_F q \right\}, \quad (3.13) \]
where we used the relation, \( \mu_F \simeq \pi n_F/m_F \) and \( v_F \equiv k_F/m_F \). The second term \( \propto |\epsilon_i|/q \) in the first line represents the dissipation due to fermions. This term appears also in the studies of non-Fermi liquids and \( \nu = 1/2 \) quantum Hall effect. The superfluid density of spinons and is calculated for small \( \Delta_{SG}(k)/(k_F^2 T) \) as
\[ n_F^S \simeq \frac{n_F}{2\pi} \int d\phi \left| \frac{\Delta_{SG}(k)}{2k_B T} \right|^2 \]
\[ = \frac{n_F}{2} \left( 1 - \delta \right)^2 \left( \frac{\pi \lambda}{2k_B T} \right)^2 (3.14) \]
with \( k_1/k_2 = \tan \phi \) and \(|k| = k_F\). We note that the last constant term of (3.13) represents a mass term of \( A_1(x, \tau) \). \( \Pi_B^\lambda \) is calculated as

\[
\frac{\Pi_B^\lambda(g, \epsilon_t)}{g^2f_B(|\mu_B|)} \simeq \left\{ \frac{q^2}{24\pi m_B} + \sqrt{\frac{\tilde{n}_B(T) |\epsilon_t|}{\pi \frac{\tilde{n}_B(T)}{m_B} |\epsilon_t|}} \right\},
\]

\[
f_B(\epsilon) = \frac{1}{\exp(\beta \epsilon) - 1},
\]

\[
\tilde{n}_B(T) = \frac{n_B}{f_B(|\mu_B|)} , \quad \tilde{v}_B(T) = \frac{\sqrt{4\pi n_B(T)}}{m_B}.
\]

Eqs.(3.13) and (3.15) are obtained for small \( q \). For large \( q \)'s, they should be replaced by anisotropic expressions due to \( \Delta_{SG}(k) \). These anisotropy can be ignored as long as the spin gap is sufficiently small.

C. Ginzburg-Landau Theory with Gauge Field

The most dominant contributions to \( Z \) from the integrations over \( \lambda_i \) and \( A_i \) come from their static (\( \tau \)-independent) modes. It is seen for \( A_i \) by (3.13,3.15). So we keep only the static (\( \tau \)-independent) modes in the effective Lagrangian density \( L_{\text{eff}} \). The partition function is then written as

\[
Z = \int \prod_x \prod_i d\lambda_i(x) dA_i(x) \exp(-\beta \int d^2x L_{\text{eff}}).
\]

\( L_{\text{eff}} \) is given up to the fourth-order in fields and derivatives by

\[
L_{\text{eff}} = c \sum_i \left( 2 \delta^2 |D_i \lambda_i|^2 + (1 - \delta)^2 |D_i A_i|^2 \right) + c \delta(1 - \delta) \left( \bar{D}_i A_i D_i \lambda_i - D_i A_i D_i \lambda_i + \text{H.c.} \right) + \frac{1}{12\pi m} \sum_{ij} \frac{1}{4} F_{ij} F_{ij} + V(\lambda_s, \lambda_d),
\]

\[
V(\lambda_s, \lambda_d) = a_s |\lambda_s|^2 + a_d |\lambda_d|^2 + 4b \delta^2 |\lambda_s|^4 + \frac{3b}{2} (1 - \delta)^4 |\lambda_d|^4 + 2b \delta^2 (1 - \delta)^2 \left( 4|\lambda_s|^2 |\lambda_d|^2 + \lambda_s^2 \lambda_d^2 \right),
\]

(3.17)

where the bars represents complex conjugate quantities.

The GL coefficients, etc. are given by

\[
\begin{align*}
\frac{1}{m} & = \frac{1}{m_F} + \frac{f_B(|\mu_B|)}{2m_B}, \\
a_s & = m_F \chi - \frac{2}{m_F} \delta^2 \ln \left( \frac{\bar{\epsilon}}{\beta \omega_\lambda} \right), \\
ad & = m_F \chi - \frac{m_F}{\pi} (1 - \delta)^2 \ln \left( \frac{\bar{\epsilon}}{\beta \omega_\lambda} \right), \\
b & = \frac{7\zeta(3)}{8\pi^3} \beta^2 m_F, \\
c & = \frac{7\zeta(3)}{32\pi^3} \beta^2 k_F^2, \\
D_i & = \partial_i - 2iA_i, \\
F_{ij} & = \partial_i A_j - \partial_j A_i,
\end{align*}
\]

(3.18)

and \( \gamma \) is the Euler number. \( \omega_\lambda \) is the cutoff of the spinon energy \( [\delta \equiv k^2/(2m_F) - \mu_F] \) in the one-loop integrals representing spinon pairings, and is estimated as \( \omega_\lambda \sim O(\mu_F) \).

The system favors the d-wave state at small \( \delta \)'s, and the s-wave state at large \( \delta \)'s. Actually, the transition temperature \( T_{\lambda(d)} \), at which \( \lambda_{d(s)} \) starts to condense is estimated from the potential energy \( V(\lambda_s, \lambda_d) \) by setting \( a_{\lambda(d)} = 0 \), and one sees that \( T_{\lambda(d)} > T_\lambda \) (\( T_{\lambda(d)} < T_\lambda \)) for small (large) \( \delta \). Let us consider the region of small \( \delta \)'s and focus on the d-wave component by setting \( \lambda_s = 0 \) and parameterizing as

\[
\lambda_1(x) = \lambda \exp[i\phi(x)], \quad \lambda_2(x) = -\lambda \exp[i\phi(x)].
\]

(3.19)

Here we introduced the MF \( \lambda \), the d-wave spin-gap amplitude, for the radial parts of \( \lambda_1(x) \), ignoring their fluctuations. Then we have the effective Lagrangian density,

\[
L_{\text{eff}} = L_\lambda + L_A,
\]

\[
L_A = a_d \lambda^2 + \frac{3}{2} b (1 - \delta)^4 \lambda^4,
\]

\[
L_A = c (1 - \delta)^2 \lambda^2 \left( \partial_i \phi - 2A_i \right)^2 + \frac{1}{12\pi m} \sum_{ij} \frac{1}{4} F_{ij} F_{ij}.
\]

(3.20)

The MF result is obtained by minimizing the above \( L_\lambda \) w.r.t. \( \lambda \) as

\[
k_B T_{SG} = \frac{2e\gamma}{\pi} \omega_\lambda \exp \left[ -\frac{\pi \chi}{(1 - \delta)^2} \right]
\]

\[
c (1 - \delta)^2 \lambda^2 \simeq \frac{k_F^2}{12\pi m_F} \left( \frac{T}{T_{SG}} \right)
\]

(3.21)

where \( T_{SG}(\delta) \) is the critical temperature below which \( \lambda \) develops. (See \( T_{SG} \) of Fig.1.) The second result holds for \( T \) near \( T_{SG} \). We assume a small but finite threedimensionality to stabilize these MFT results. This point and other details of the transition at \( T_{SG} \) are studied in Sect.4.

D. Gauge-Field Mass in Variational Method

Let us consider the effect of \( L_A \) upon \( L_\lambda \) by integrating over \( A_i \). We have treated \( A_i \) as a noncompact gauge field in spite of being originally compact. This procedure is appropriate in respect of the kinetic term of \( A_i \), because we consider the region \( T < T_{CSS} \). In short, the compactness of the gauge-field kinetic term should be respected in studying the CDPT at \( T_{CSS} \), while the compactness of the coupling to the Higgs field is crucial to study the transition into Higgs phase at \( T_{SG} \). So we respect the compactness of the mass term by replacing \( (A_i - \partial_i \phi/2)^2 \) in \( L_A \) to the periodic one. Then the new Lagrangian \( L_B \) which replaces \( L_A \) is expressed as
\begin{align}
L_B &= \frac{1}{12\pi\bar{m}} \sum_{ij} \frac{1}{4} F_{ij} F_{ij} \\
&+ c(1 - \delta)^2 \lambda^2 \cdot \frac{1}{a^2} \left[ 4 - \sum_i 2 \cos(2a B_i) \right], \quad (3.22)
\end{align}

where we introduced the Proca (massive vector) field \( B_i \),
\[ B_i = A_i - \frac{1}{2} \partial_i \phi, \]
\[ F_{ij} = \partial_i B_j - \partial_j B_i. \quad (3.23) \]

Noe that if the cosine term in \( L_B \) is expanded up to the second order, \( L_A \) is recovered. Let us take the unitary gauge \( \phi = 0 \). Then the path integrals reduce as
\[ \prod_x \! d\phi \prod_{x,i} dA_i = \prod_x \! d\phi \prod_{x,i} dB_i \rightarrow \prod_{x,i} dB_i. \quad (3.24) \]

Let us estimate the gauge-field mass \( m_A \) by the variational method. We choose the variational Lagrangian \( L_B^\prime \) for \( L_B \) as
\[ L_B^\prime = \frac{1}{12\pi\bar{m}} \left( \sum_{ij} \frac{1}{4} F_{ij} F_{ij} + \sum_i m_A^2 B_i B_i \right), \quad (3.25) \]

where \( m_A \) is introduced here as a variational parameter. There holds the Jensen-Peierl inequality for the free energy density \( F_B^\text{var} \) of \( L_B \),
\[ F_B^\text{var} \leq F_B^\text{exact} \equiv F_B + \frac{1}{V} \int d^2x (L_B - L_B^\prime), \]
\[ \exp(-\beta F_B^\text{exact} V) = \int \prod_{x,i} dB_i \exp\left( -\beta \int d^2x L_B \right), \]
\[ \exp(-\beta F_B^\prime V) = \int \prod_{x,i} dB_i \exp\left( -\beta \int d^2x L_B^\prime \right), \]
\[ \langle O \rangle = \int \prod_{x,i} dB_i O \exp\left( -\beta \int d^2x L_B^\prime + \beta F_B^\prime V \right), \]
\[ (3.26) \]

where \( V = \int d^2x \) is the system volume. By taking the definition of the propagator at the same point and the trace of a functional operator \( \hat{O} \) as
\[ \langle B_i(x) B_j(x) \rangle \equiv \lim_{y \to x} \langle B_i(x) B_j(y) \rangle, \]
\[ \text{Tr} \hat{O} = \int d^2x \lim_{y \to x} \langle \hat{O} | y \rangle, \quad (3.27) \]

the variational free energy density \( F_B^\text{var} \) is calculated by integrating over \( B_i \) as follows;
\[ F_B^\text{var}(m_A) = F_B(0) + \frac{k_B T}{8\pi} m_A^2 \]
\[ - \frac{4\epsilon (1 - \delta^2 \lambda^2) \left( m_A^2 / q_c^2 \right)}{a^2}, \]
\[ k_B \Theta = \frac{1}{3a^2 \bar{m}} = \chi \left[ \frac{J}{4} + \frac{t}{3} f_B(-\mu_B) \right], \quad (3.28) \]

where \( q_c \) is the momentum cutoff of the \( B_i \) modes. Its explicit value is estimated later in (5.21). \( \Theta(\delta, T) \) is a function \( \Theta(\delta, T) \) of \( \delta \) and \( T \). In \( F_B^\text{var} \) of (3.22) we have omitted higher-order terms of \( O(\epsilon m_A^2 / q_c^2) \) since we are interested in small \( m_A^2 / q_c^2 \). This assumption is justified a posteriori by the result itself.

By minimizing \( F_B^\text{var}(m_A) \) w.r.t. \( m_A \), we obtain
\[ m_A^2 = q_c^2 \left[ 96\pi \bar{m} \epsilon (1 - \delta^2 \lambda^2) \right]^{2d}, \]
\[ d = \frac{1}{2 \Theta - T}. \quad (3.29) \]

We note that the fluctuations of \( A_i \) do not affect the MF result (3.21) as long as \( d > 1 \) since the corrections then becomes higher-order than \( \lambda^4 \). Thus the gauge-field mass \( m_A \) at fixed \( \delta \) starts to develop continuously at \( T_{SG}(\delta) \) as
\[ m_A(\delta, T) \propto (T_{SG}(\delta) - T)^{\delta(\delta, T_{SG}(\delta))}. \quad (3.30) \]

That is the exponent \( d \) is neither 1/2 nor a constant, and drastically changes especially when \( T_{SG} \sim T_A \), where \( T_A \) is a root of the equation \( T_A = \Theta(\delta, T_A) \), at which \( d(\delta, T) \) diverges. This is in strong contrast with the noncompact case, in which the effect of \( A_i \) upsets the MF result of GL Lagrangian \( L_A \) as stated. Actually, in the calculation of Ref., \( L_A \) in 3D generates a notorious \( \lambda^3 \) term.

IV. PHASE TRANSITION INTO THE SPIN GAP STATE: LATTICE ABELIAN HIGGS MODEL

In this section, we clarify the nature of the phase transition at \( T_{SG} \) described by the effective field theory (3.20) of Sect.3C. The reader who is interested in the practical applications of the effective field theory may skip this section, and directly go to Sect.5.

To study the transition at \( T_{SG} \), it is useful to introduce a lattice model that is suggested by the effective field theory (3.20), because ample methods of analysis are available for lattice models. The most natural lattice model for this purpose is a noncompact version of the so called lattice Abelian Higgs model, which is defined by
\[ Z_{AH} = \prod_x \int_{-\pi}^\pi d\phi_x \prod_{x,i} \int_{-\infty}^\infty dA_{xj} \exp(-S_{AH}), \]
\[ S_{AH} = -\rho \sum_{i,j} \cos[\phi_x - A_{xj}] + \frac{\kappa}{2} \sum_{i,j} (\epsilon_{ij} \nabla_i A_{xj})^2, \]
\[ \rho \propto \lambda^2, \quad \kappa \propto 1 / \bar{m}. \quad (4.1) \]

where \( \phi_x \in [-\pi, \pi] \) is the phase of a Higgs field on \( x \) corresponding to \( \phi(x) \) of (3.19) and \( A_{xj} \in [-\infty, \infty] \) is a noncompact gauge field corresponding to \( A_i(x) \). The two parameters \( \rho \) and \( \kappa \) are determined by the parameters of the effective model of Sect.3. (This \( \rho \) should not be
confused with the dc resistivity although we use the same latter.) $S_{AH}$ has the local U(1) gauge symmetry under
\[
\phi_x \rightarrow \phi_x + \theta_x,
\]
\[
A_{x} \rightarrow A_{x} + \nabla \theta_x,
\quad (4.2)
\]
Its naive continuum limit reduces to (3.20).

A similar but better studied model is the compact version of (4.1) defined by
\[
Z_{AH} = \prod_x \int_{-\pi}^{\pi} d\phi_x \prod_{x,i} \int_{-\pi}^{\pi} dA_{x,i} \exp(-S_{AH}^c),
\]
\[
S_{AH}^c = -\rho \sum_{x,i<\j} \cos[\nabla_i \phi_x - q A_{x,i}] - \kappa \sum_{x,i<\j} \cos(\epsilon_{ij} \nabla_i A_{x,j}).
\quad (4.3)
\]
It is obtained from (4.2) by (i) letting $A_{x,j} \in [-\pi, \pi]$, (ii) replacing the quadratic Maxwell term $(\epsilon_{ij} \nabla_i A_{x,j})^2$ by the periodic (compact) one, $\cos(\epsilon_{ij} \nabla_i A_{x,j})$, and (iii) introducing $q$. $q$ is the charge of the Higgs field, and the phase structure may differ for each $q$. On the contrary, in the noncompact case of (4.1), we have set $q = 1$ since $q$ can be eliminated by rescaling $A_i \rightarrow q^{-1} A_i$, $\kappa \rightarrow q^2 \kappa$ and so irrelevant there.

The main difference between these two models on 3D lattice is that monopoles (which are nothing but instantons in the 3D lattice system) are allowed in (4.3), while not in (4.1). A monopole is a particular configuration of $A_i$,
\[
\nabla_i H_i = 4\pi \delta_{x,0}, \quad H_i \equiv \epsilon_{ijk} \nabla_j A_k,
\quad (4.4)
\]
which is a solution of the field equation and has the non-vanishing divergence of “magnetic field” $H_i$. Condensation of these monopoles let $A_i$ random, $\Delta A_i \equiv \infty$, and drive the system into the conﬁnement phase. Since we are interested in $T$ around $T_{SG} \approx T_{CSS}$ which is in the deconfinement phase, we consider the noncompact model (4.1). On the other hand, we respect the periodicity (compactness) of the first term in $S_{AH}$ of (4.1), which plays the important role in the transition at $T_{SG}$. This point was already pointed out in Sect.3D. At the end of this section, we shall mention the compact model (4.3).

Let us ﬁrst consider the isotropic 3D model (4.1) with $i = 1, 2, 3$, because the phase structure of the quasi-2D t-J model in which we are interested is governed by the three dimensionality of its effective lattice model. (The effect of anisotropy is discussed later in the paragraph containing eq. (4.14) below.) Here we comment on the dimensionality of an effective lattice model. It is well known that the path integral quantization maps a quantum system in $D$-dimensional space to a classical system in $D + 1$ dimensional space, where the extra one dimension represents the imaginary time, $\tau \in (0, \beta)$. In many cases, this $D + 1$ classical system is approximated by an effective model in $D$-dimensional space. A good example is nothing but our model, $L_{\text{eff}}$ of (3.10), where only the $\tau$-independent modes appear. As long as $T \neq 0$, the extra dimension has a ﬁnite extension $\beta$, and so it should not affect the long-range properties and the phase structure. (The case $T = 0$ is different of course.) The $T$ dependence is incorporated into the coefﬁcients of the effective action. The reduction to the $D$-dimensional effective model is reasonable in this sense. Because the Abelian Higgs model is originated from $L_{\text{eff}}$ of (3.10), both models should be considered in the same spatial dimensionality.

Let us modify the gauge-Higgs term of $S_{AH}$ to the following Villain (periodic Gaussian) form:
\[
\exp \left[ \beta \cos(\nabla_i \phi_x - A_{x,i}) \right] \rightarrow e^\beta \sum_{n_{x,i} = -\infty}^{\infty} \exp \left[ -\frac{\beta}{2} (\nabla_i \phi_x - A_{x,i} + 2\pi n_{x,i})^2 \right]
\]
\[
= e^\beta \sum_{m_{x,i} = -\infty}^{\infty} \exp \left[ -\frac{2}{\beta} m_{x,i}^2 + im_{x,i} (\nabla_i \phi_x - A_{x,i}) \right].
\quad (4.5)
\]
This replacement keeps the periodicity (compactness) in $\phi_x$. It is then straightforward to perform a duality transformation to reach the following dual representation:
\[
Z_{AH} = \prod_i \sum'_{J_{x,i}} \int_{-\infty}^{\infty} dC_{x,i} \times \exp \left[ -\sum_i \frac{1}{4\beta} \left( \epsilon_{ij} \nabla_j C_i \right)^2 + m^2 C_i^2 \right] + 2\pi i \sum_i J_{x,i} C_i \]
\[
= Z_0 \prod_i \sum'_{J_{x,i}} \exp \left[ -4\pi^2 \rho \sum_i \sum_{j} J_{x,i} D_{ij}(x - y; m^2) J_{y,j} \right],
\quad (4.6)
\]
where $m^2 = \rho/\kappa$. The dual variables $J_{x,i} ( = 0, \pm 1, \pm 2, \ldots)$ and $C_{x,i} ( \in [-\infty, \infty] )$ are deﬁned on the link $(x, x + i)$ of the dual lattice (which is obtained from the original lattice by shifting each site $x$ to $x + (1 + 2 + 3)/2$). From the saddle point equations, one obtains the relation,
\[
\langle J_{x,i} \rangle \propto \langle \epsilon_{ijk} \nabla_j (\nabla_k \phi_x - A_{x,k}) \rangle.
\quad (4.7)
\]
Thus $J_{x,i}$ represents the gauge-invariant ”vortex current”, because $\bar{J} \propto \nabla \times \nabla \phi$ for $A = 0$. (Recall that the vorticity in the continuum is deﬁned by $\oint d\phi$ along a closed line. On the lattice, a vortex with a phase change $2\pi$ along a plaquette in the 12 plane is expressed by $J_{x,3} = \pm 1$, and so on.) From the gauge invariance, $J_{x,i}$ can be chosen so as to conserve at each site,
\[
\sum_i \nabla_i J_{x,i} = 0.
\quad (4.8)
\]
The prime on $\sum_{x,i}$ of (4.6) implies this condition. The vector ﬁeld $C_{x,i}$ mediates the force between two vortex current elements, $J_{x,i}$, and generates the potential $D_{ij}(x - y; m^2)$ as shown in the last line of (4.6).
$Z_0$ is the partition function of the vector field $C_x$ with mass $m$, the Green’s function of which is given as

$$Z_0 = \exp(-\frac{1}{2}\text{TrLog}D_{ij}(x-y;m^2)),
(C_xC_y) = D_{ij}(x-y;m^2)$$

$$= \left[\delta_{ij} - \frac{\nabla_i\nabla_j}{m^2}\right]D(x-y;m^2),$$

$$(-\nabla^2 + m^2)D(x-y;m^2) = \delta_{xy}. \quad (4.9)$$

The system (4.7) is characterized by the density of vortices. If there are no vortices, (4.7) shows that every $\phi_x$ is stabilized to a fixed value, say $\phi_x = 0$, with $\Delta\phi_x = 0$, and gives rise to a coherent condensation of the Higgs field with a long-range order, $(\lambda_x) \neq 0$. Appearance of vortices let $\phi_x$ fluctuate, and the proliferation of vortices, i.e., a vortex condensation, leads to destruction of the long-range order of the Higgs field. Thus one expects a phase transition between these two states: (i) the Higgs phase where the Higgs field condenses and (ii) the Coulomb phase where vortices condense and no Higgs condensation, $(\lambda_x) = 0$. A balance between the energy and the entropy determines the critical line. From (4.7), the energy of a vortex current with the strength $|J_{xi}| = 1$ and the length $L$ is estimated as

$$E(L) \sim 4\pi^2 \rho D(0;m^2)L,$$

whereas its entropy $S(L)$ is estimated as

$$S(L) \sim \ln \mu L,$$

with some numerical constant $\mu \sim 4.7$ for the simple cubic lattice. Thus we estimate the transition temperature $T^*$ (which should be identified with $T_{SG}$) as

$$F = E(L) - TS(L) \simeq 0,
T^* \simeq 4\pi^2 \rho D(0;m^2) \ln \mu. \quad (4.12)$$

$\rho D(0;m^2)$ in $T^*$ is estimated as

$$\rho D(0;m^2) = \rho \int_{-\pi}^\pi d^3p \frac{1}{\sum_i \sin^2 \beta_i + m^2}
\simeq \left\{\begin{array}{ll}
\rho, & \rho/\kappa < 1
\rho/\kappa \gg 1 & .
\end{array}\right. \quad (4.13)$$

The duality between the Higgs field $\lambda(x)$ and the vortex density $v(x)$ is illustrated in Table 2.

Detailed studies of the above system have been performed by numerical simulations. There the radial degrees of freedom of the Higgs field are treated also as dynamical variables. It is found that the phase transition really occurs, and the order of the transition depends on the parameters of the model. For a small Higgs-four-point interaction, it is of first-order, whereas for large four-point interaction, it becomes a “continuous” transition. The mass of the gauge boson was also calculated in detail near the continuous phase transition as a function of the Higgs mass. In the isotropic 3D case, the gauge boson mass develops as $(T_{SG} - T)^{1/2}$ as predicted by the MFT of $\langle 3\rangle$, i.e., $m_A \propto \lambda$ and $d = 1/2$.

For the quasi-2D case, a similar phase transition between the Higgs phase and the Coulomb phase is expected to occur, i.e., the phase structure of the quasi-2D system is governed by its three dimensionality. This expectation is quite natural from a general view point of universality in renormalization group. The effect of the three dimensionality may be described by introducing anisotropy parameters in the Hamiltonian such as

$$H = -\sum_{x,i} t_i \left(b_i^\dagger f_{x+i} f_{x-i} b_x + \text{H.c.}\right) - \sum_{x,i} J_i \left|f_i^\dagger \sigma^\dagger f_{x+i}^\dagger - f_i^\dagger \sigma f_{x+i}\right|^2,$$

$$t_1 = t_2 \equiv t, \quad t_3 = \alpha t, \quad J_1 = J_2 \equiv J, \quad J_3 = \gamma \alpha J. \quad (4.14)$$

The parameter $\alpha$ controls the quasi-two-dimensionality, i.e., $\alpha = 1$ corresponds to the isotropic 3D case and $\alpha = 0$ to the pure 2D case. Another parameter $\gamma$ respects the difference in $t_i$ and $J_i$. If we use the relation $J = 4t^2/U$ obtained in the derivation of the t-J model from the Hubbard model, we have $\gamma = \alpha$.

As an explicit example of the related quasi-2D models, we have studied the quasi-2D antiferromagnetic Heisenberg model at finite $T$ [4]. By using the quasi-2D O(3) nonlinear $\sigma$ model, we showed that the critical temperature $T_{AF}$ behaves as $T_{AF} \sim 1/|\log \alpha|$ for small $\alpha$, which vanishes for 2D as expected, and the correlation functions exhibit 3D critical behavior in the vicinity of the transition point $T_{AF}$ with the interval of $\Delta T \sim O(1/(\log \alpha)^2)$.

Yet another example is the quasi-2D XY spin model. There the critical temperature in the 2D case remains finite, at which a phase transition of Kosterlitz-Thouless type takes place. We note that the application of the Polyakov-Susskind theory to the effective lattice gauge theory of Sect.2B in 2D case maps the system into the 2D XY spin model, hence predicts a phase transition of Kosterlitz-Thouless type at finite $T_{SG}$ for 2D. This point is different from the Heisenberg model, for which the critical temperature vanishes as $\alpha \rightarrow 0$.

Returning to the present quasi-2D model (1.1), a phase transition should occur at $T = T_{SG} > 0$, which may be either continuous or discontinuous. In the very narrow interval $\Delta T$ around $T_{SG}$, the system exhibits a genuine 3D critical properties. This interval, however, may be too small to access by the experiments. Outside of this $\Delta T$, the system should exhibit the 2D behaviors, which can be studied well by using the results obtained in Sect.2D. Analytical studies and the MC simulations of the quasi-2D case of (4.3) are welcome to confirm these expectations.

Let us now comment on the compact $U(1)$ Abelian Higgs model (4.3) in 3D. The duality transformation can be performed as in the noncompact case [4]. The resultant partition function is given by the same expression.
as \( \rho = 1, 2 \) but the vortex current \( J_{ex} \) can terminate at locations of instantons (monopoles) and anti-instantons. Thus eq. (4.8) is modified to

\[
\nabla J_{ex} = gQ_x, \tag{4.15}
\]

where \( Q_x (= 0, \pm 1, \pm 2, \ldots) \) is the instanton density. The partition function is obtained by summing over all the instanton configurations \( \sum_{Q_x} \) in addition to \( \sum_{J_{ex}} \). Existence of sufficient instantons converts the Coulomb phase into the confinement phase.

Recently, this 3D compact \( U(1) \) gauge-Higgs model was examined in the continuum field theory by Nagaosa and Lee\(^\text{[8]}\) to study the competition of Bose condensation versus gauge-field fluctuations. They employed a duality transformation similar to the one above on a lattice. They concluded that, in 3D the vortex condensation of sufficient instantons converts the Coulomb phase to an instanton. However, their conclusion contradicts the numerical studies\(^\text{[84, 95]}\) which exhibit a genuine phase transition at \( T_c > T_{SG} \). On the other hand, in \( 2D \) and \( \rho \)-plane, this line terminates at some point.

Finally, we mention the recent work of Balent et al.\(^\text{[22]}\). They studied strongly-correlated electron systems in spatial \( 2D \) and argued its phase structure by mapping the electrons into bosons by using the Chern-Simons gauge theory and then by applying the duality transformation. They also introduced two composite fields for spin and charge degrees of freedom as in the bosonization in \( 1D \). One may expect a similar method may be used for the spinons in the SB t-J model. However, their way to derive the low-energy theory of the composite bosons is accompanied with certain assumptions and not straightforward. Anyway, we should stress here that their approach to the strongly-correlated \( 2D \) system is very close to that to the \( 1D \) system for which the bosonization separates the spin and charge degrees of freedom quite easily, but loose the ability to describe the Higgs phase (i.e., \( T_{SG} = 0 \)). On the other hand, in our approach, we take the three dimensionality of the high-\( T_c \) cuprates seriously, which is crucial to predict the realistic phase structure with \( T_{SG} > 0 \) as we explained above.

In the following sections, we shall calculate the conductivity in the spin-gap phase. There we assume the quasi-two-dimensionality and the genuine phase transition at \( T_{SG} \). However, we checked that the kinematical formulae appearing in our calculations, which we obtained assuming \( T_{SG} > 0 \), have smooth and weak dependence on \( \alpha \) near \( \alpha = \frac{1}{4} \). From this observation and the estimation \( \alpha \sim 10^{-5} \), we set \( \alpha = 0 \) in the calculations in Sect.5 for simplicity. The errors caused by this simplification do not modify the qualitative conclusion there.

### V. Resistivity

In Sect.5A, we rederive the Ioffe-Larkin formula for conductivity with a general assignment of EM charges of holons and spinons. In Sect.5B, the detailed expression of the conductivity is obtained. In Sect.5C, the spin gap effect on the conductivity is calculated by using the gauge-boson mass \( m_A \) of Sect.3D.

#### A. Linear Response Theory and Ioffe-Larkin Formula

Here we shall consider the response of the system to the external EM field \( A_{ex}^i \) \( (i = 1, 2, 3) \). We shall work in the temporal gauge, where \( A_0 = 0 \), \( A_0^x = 0 \). In principle, the effective action of \( A_{ex}^i \) is obtained by integrating out all the quantum fields (spinon, holon, dynamical gauge filed \( A_j \)) and one can calculate the response to \( A_{ex}^i \) from that action. There is an ambiguity in the way how \( A_{ex}^i \) couples to spinons and holons. Namely, one may assign the EM charge of a holon, \( Q_h \) and the charge of a spinon, \( Q_s \) arbitrarily as long as the charge of an electron \( e \) is expressed as

\[
 e = Q_s - Q_h, \tag{5.1}
\]
as suggested by the relation \( \Delta Q \). This problem was investigated first by Ioffe and Larkin\(^\text{[22]}\) and the result is that this ambiguity does not affect the expectation values of gauge-invariant physical quantities. For the conductivity, this can be seen explicitly in the formula given below. Let us recall that the relevant part of the Hamiltonian with general charges \( Q_h, Q_s \) is written as

\[
H_{em} = \frac{1}{2m_B} \int d^2 x \sum_i \left( \partial_i - iQ_h A_{ex}^i - iA_i \right)b(x) \left( \partial_i - iQ_s A_{ex}^i - iA_i \right)b(x) \left( \partial_i - iQ_h A_{ex}^i - iA_i \right)f_\sigma(x) \left( \partial_i - iQ_s A_{ex}^i - iA_i \right)f_\sigma(x) \left[ \Delta_{SG}(k)f_\sigma^1(k)f_{-\sigma}^1(-k) + \text{H.c.} \right]. \tag{5.2}
\]

Note that the last line acquires no explicit \( A_{ex}^i \) dependence since its expression in coordinate space \( \sim \int dx^2 \Delta_{ex}^i f_{\sigma}^1 f_{-\sigma}^1 \exp \left( i \theta_{ex}^i \right) \) remains unchanged under the EM gauge transformation, \( \Delta_{ex} \rightarrow \Delta_{ex} \exp \left( i \theta_{ex}^i \right) \). By integrating out the spinon and holon fields, we obtain the following effective Lagrangian;

\[
L_{\text{eff}}[A_i, A_{ex}^i] = (A_i + Q_s A_{ex}^i)\Pi_B^\sigma(A_j + Q_s A_{ex}^j) + (A_i + Q_h A_{ex}^i)\Pi_B^h(A_j + Q_h A_{ex}^j) + O(3). \tag{5.3}
\]
where $O(3)$ represents terms higher than quadratic in $A_i$ and/or $A_i^{\text{ex}}$. In order to integrate $A_i$ concretely, the above Lagrangian needs to be approximated by a quadratic one. However, the $O(3)$ terms are very important, because they renormalize the spinon propagator, the holon propagator, and the vertices. This effect is partially included by replacing $\Pi_{F(B)}^{ij}$ with their renormalized quantities, $\tilde{\Pi}_{F(B)}^{ij}$, represented by Fig.\ref{fig:4}E. With this replacement, the Lagrangian is expressed as

$$L_{\text{eff}}[A_i, A_i^{\text{ex}}] = (A_i + Q_i A_i^{\text{ex}}) \tilde{\Pi}_{F}^{ij}(A_j + Q_j A_j^{\text{ex}}) + (A_i + Q_h A_i^{\text{ex}}) \tilde{\Pi}_{B}^{ij}(A_j + Q_h A_j^{\text{ex}}).$$

After integrating over $A_i$, we obtain

$$L_{\text{eff}}[A_i^{\text{ex}}] = A_i^{\text{ex}} \tilde{\Pi}_{ij} A_j^{\text{ex}}$$

$$\tilde{\Pi} = Q_h^2 \tilde{\Pi}_B + Q_s^2 \tilde{\Pi}_F$$

$$- (Q_h \tilde{\Pi}_B + Q_s \tilde{\Pi}_F)(\tilde{\Pi}_B + \tilde{\Pi}_F)^{-1}(Q_h \tilde{\Pi}_B + Q_s \tilde{\Pi}_F)$$

$$= (Q_h - Q_s)^2 \left[ \tilde{\Pi}_B^{-1} + \tilde{\Pi}_F^{-1} \right]^{-1}$$

$$= e^2 \left[ \tilde{\Pi}_B^{-1} + \tilde{\Pi}_F^{-1} \right]^{-1}, \quad (5.4)$$

where $\tilde{\Pi}$ is nothing but the response function of electrons. Finally, from the linear-response theory and $L_{\text{eff}}[A_i^{\text{ex}}]$, the dc conductivity $\sigma(=\sigma_{11} = \sigma_{22})$ is expressed as

$$\sigma_{ij} = \lim_{\epsilon \to 0} \frac{1}{-i\epsilon} \tilde{\Pi}_{ij}(q, -i\epsilon),$$

$$\tilde{\Pi}_{ij}(q, \epsilon) = e^2 \left[ \tilde{\Pi}_B^{-1}(q, \epsilon) + \tilde{\Pi}_F^{-1}(q, \epsilon) \right]^{-1}_{ij}. \quad (5.5)$$

So one arrives at the Ioffe-Larkin formula,

$$\sigma^{-1} = \sigma_B^{-1} + \sigma_F^{-1}, \quad (5.6)$$

where $\sigma_B(F)$ is the conductivity of holons (spinons).

The last expression of Eq.\ref{eq:5.3} is symmetric w.r.t. the holon contribution $\tilde{\Pi}_B^{-1}$ and the spinon contribution $\tilde{\Pi}_F^{-1}$, and depends only on the difference $\tilde{\Pi}_B^{ij}$. The reason why the ambiguity of charge assignment

$$Q_h \to Q_h + c e, \quad Q_s \to Q_s + c e,$$

disappears in the final expression of the conductivity is that one can make a shift of the integration variable

$$A_i \to A_i - c e A_i^{\text{ex}} \quad (5.8)$$

so as to eliminate the $c$-dependence from $\tilde{\Pi}_{F}^{ij}$. By a shift of integration variable, the partition function and physical quantities are of course unchanged.

**B. Expression of Resistivity**

In the spin-gap state, the spinon conductivity diverges $\sigma_F \to \infty$ due to a superflow generated by the spin-gap condensation $\langle \lambda_{\sigma} \rangle \neq 0$. This is an analog of the well-known fact in the BCS theory that the electron resistivity vanishes below $T_c$ due to a superflow generated by a Cooper-pair condensation. Actually, $A_i^{\text{ex}}$ has the same structure as the BCS model. Thus the total resistivity $\rho$ in the spin-gap state is equal to the resistivity of holons,

$$\rho = \sigma_1^{-1} = \sigma_B^{-1} + \sigma_F^{-1} \to \sigma_B^{-1}. \quad (5.9)$$

Effects of spinons to $\rho$ is indirect, but certainly exist and show up through the dressed propagator $D(q, \epsilon)$ in calculating $\tilde{\Pi}_B$.

Now we calculate the response function $\tilde{\Pi}_B$. By solving the Schwinger-Dyson equation approximately according to Ref.\ref{ref:5.3}, we obtain

$$\tilde{\Pi}_B ij(0, \epsilon_l) \simeq - \frac{1}{\beta} \sum_n \int \frac{d^2 q}{(2\pi)^2} \frac{q_i q_j}{m_B^2} R_B(q, \epsilon_n; \epsilon_l)$$

$$\times \frac{i\epsilon_l}{i\epsilon_l - i\epsilon_l \Gamma_B(q, \epsilon_n; \epsilon_l) - \Delta \Sigma_B(q, \epsilon_n; \epsilon_l)},$$

$$R_B(q, \epsilon_n; \epsilon_l) \equiv G_B(q, \epsilon_n) G_B(q, \epsilon_n + \epsilon_l),$$

$$G_B(q, \epsilon_n) \equiv \left[ i\epsilon_n - \frac{q^2}{2m_B} - \mu_B \right]^{-1}. \quad (5.10)$$

$$\Delta \Sigma_B(q, \epsilon_n; \epsilon_l),$$

representing diagrams containing self-energy of holons $\Sigma_B(q, \epsilon_n)$, is necessary to keep gauge invariance,

$$\Delta \Sigma_B(q, \epsilon_n; \epsilon_l) \equiv \frac{1}{\beta} \sum_{n'} \int \frac{d^2 q'}{(2\pi)^2} \times \left[ \Sigma_B(q, \epsilon_n) G_B^2(q, \epsilon_n) - \Sigma_B(q, \epsilon_n + \epsilon_l) G_B^2(q, \epsilon_n + \epsilon_l) \right], \quad (5.11)$$

However, in the perturbative calculation, this combination vanishes in the dc limit by the symmetry under summations. We expect this term does not contribute to the dc resistivity, and neglect it hereafter.

$$\Gamma_B(q, \epsilon_n; \epsilon_l)$$

represents vertex diagrams, and contributes to $\sigma_B$. It is given by

$$\Gamma_B(q, \epsilon_n; \epsilon_l) \equiv \left( \frac{q}{m_B} \right)^2 \frac{1}{\beta} \sum_{n'} \int \frac{d^2 q'}{(2\pi)^2} \times \left[ q \cdot (q' - q) \right]$$

$$\times D(|q' - q|, \epsilon_n' - \epsilon_n) R_B(q', \epsilon_n'; \epsilon_l), \quad (5.12)$$

where $q \times q' \equiv q_x q'_y - q_y q'_x$. We fix the length of $q$ in $\Gamma_B$ to a typical length $\tilde{q}_B$,

$$\tilde{q}_B^2 \equiv 4\pi \tilde{n}_B(T). \quad (5.13)$$

This $\tilde{q}_B$ is determined so that the similar integral, $\int |q| \Gamma_B$ with an relaxation time in the holon propagator and $i\epsilon_l \Gamma_B + \Delta \Sigma_B \to 0$, being evaluated by this approximation, gives rise to the correct result. Furthermore, $\tilde{q}_B$
behaves as $\tilde{q}_B \sim \sqrt{2m_B k_B T}$ at the temperature region, $k_B T \gg n_B/m_B$. So this is a natural choice for the typical momentum scale of holons. We also fix the length of $q'$ of $D$ in eq.(5.12) to $\tilde{q}_B$ to obtain

$$
\Gamma_B(\tilde{q}_B, \epsilon_n; \epsilon_l) \simeq \frac{\alpha g' B}{2\pi^2 \sqrt{m_B}} \int d\phi \sin^2 \phi \times D \left( \frac{\sqrt{2(1 - \cos \phi)}}{\sqrt{\tilde{n}}} \right) \times \int_{|\mu_B|} dE \frac{1}{\bar{\epsilon}_n' - E} \times \frac{1}{i\epsilon_n' + i\epsilon_l - E}. \quad (5.14)
$$

We consider the underdoped region, $n_B \ll n_F$ ($\delta \ll 1$), and temperatures around $\beta^{-1} \sim n_B/m_B$. Assuming that $D(q, \epsilon_l)$ in $\Gamma_B$ dominates in the region near the static limit $\epsilon_l = 0$, we use the upper expressions in eqs. (3.13) and (3.15). In the denominator of $D$, the dissipation term is larger than the $q'$ term,

$$
\sqrt{\bar{n}} \equiv \sqrt{\bar{n}_F + f_B(\mu_B)} \sqrt{|\bar{n}_B(T)/2|}, \quad (5.15)
$$

as long as $\epsilon_l \neq 0$, since their ratio is $(\tilde{q}_B/\bar{n})/(\sqrt{|\bar{n}/(\beta B)|}) \sim \mathcal{O}(\sqrt{|\bar{n}_B(T)/n_F|})$ and small. So the $n'$-sum is dominated at $\epsilon_n' = \epsilon_n$. Then we get

$$
\Gamma_B(\tilde{q}_B, \epsilon_n; \epsilon_l) \simeq \frac{3\bar{m}}{4\pi m_B \beta} \int d\phi \sin^2 \phi \times D \left( \frac{\tilde{q}_B}{\sqrt{2(1 - \cos \phi)}} \right) \times \frac{3\bar{m}n^3_B(T)}{2m_F \bar{n}_B(T)} \times \frac{1}{\epsilon_l (\epsilon_n + \epsilon_l) - \epsilon_n}
$$

$$
\simeq \frac{1}{2\epsilon_l \tau(T)} \left( \frac{\epsilon_n + \epsilon_l}{\epsilon_n} \right) \mathcal{O}(\epsilon_l^2) \quad (5.16)
$$

where

$$
\frac{1}{\tau(T)} \equiv \frac{3\bar{m}n^3_B(T)}{2m_F \bar{n}_B(T)} \left[ \frac{1}{1 + \frac{3\bar{m}n^3_B(T)}{2m_F \bar{n}_B(T)}} - 1 \right]. \quad (5.17)
$$

To calculate $\tilde{\pi}_B(0, \epsilon_l)$ we insert eq.(5.14) into eq.(5.10) and do the $q$-integral and $n$-sum as in eq.(5.12) to get

$$
\tilde{\pi}_B(0, \epsilon_l) \simeq \delta_{ij} \frac{n_B}{m_B} \frac{i\epsilon_l}{\epsilon_l \mathcal{C}(\epsilon_l) \epsilon_l + i\tau^{-1}(T) \text{sgn}(\epsilon_l)}. \quad (5.18)
$$

where $\lim_{\epsilon_l \to 0} \mathcal{C}(\epsilon_l)$ is finite. After analytic continuation $\epsilon_l > 0 \to -i\epsilon$ and using eq.(3.14), we finally obtain the resistivity as

$$
\rho \simeq \frac{m_B}{e^2 n_B \tau(T)} \times T \left[ 1 - \frac{3f_B(\mu_B)\bar{m}(1 - \delta)}{2m_F \delta} \frac{\pi(1 - \delta)\lambda}{2k_B T} \right]. \quad (5.19)
$$

For $T_{SG} < T < T_{CSS}$, $\lambda = 0$ and this result reproduces the $T$-linear behavior of Ref.[14]. For $T$ near and below $T_{SG}$, one expects the following behavior;

$$
\lambda \propto (T_{SG} - T)^d, \quad \rho \propto T \left[ 1 - c(T_{SG} - T)^d \right]. \quad (5.20)
$$

where $d(\delta, T)$ is a “critical exponent” in 2D, which depends both on $T$ and $\delta$. The downward deviation of $\rho$ from the $T$-linear behavior is reduced with increasing the doping $\delta$. In Fig.[1], we plot $\rho$ of eq.(5.14) with various values of $d$. The MFT value $d = 1/2$ of (3.21) is not consistent with the experiment which gives rise to smooth deviation from the $T$-linear behavior (no discontinuity in $d\rho/dT$ at $T = T_{SG}$). To achieve such a behavior, one needs $d > 1$. This implies that the fluctuation effect of phases of $\lambda_{xi}$ beyond the MFT is important to obtain a realistic curve of $\rho$.

The data of Ref.[12] show that one may fit $\rho$ in a form $C_0 + C_1 T$ for $T_{SG} < T$. This implies spinon contribution to $\rho$, calculated in Refs.[14,15] as $\sigma_F^{-1} \sim (k_B T/\mu_F)^4/3$, is negligibly small compared with $\sigma_B^{-1} \sim m_B k_B T/n_B$ due to higher power in $T$ and a small coefficient. $\sigma_F^{-1} = 0$ for $T < T_{SG}$ as explained, but the discontinuity at $T_{SG}$ in $\sigma_F^{-1}$ is not observable due to its smallness.

The constant part $C_0$, surviving below $T_{SG}$, may be attributed to scatterings of charged holons with impurities. They may be described by $H_{\text{imp}} = \sum V_x b_x^\dagger b_x$, where $V_x$ is a random potential. Actually, standard calculations show that it generates $T$-independent contribution to $\rho$, $\Delta \sigma_B^{-1} \propto 1/n_B$ at intermediate $T_{SG}$.

C. Spin-Gap Effect on Resistivity

Let us estimate the momentum cutoff $q_c$ of the gauge boson. Since the gauge boson is viewed as a composite particle made of two spinons and of two holons, it is natural to estimate it by using the cutoff $k_F$ of spinons as

$$
q_c = 2k_F \quad (5.21)
$$

for small $\delta$ where the spinons dominate over holons. The qualitative behavior of $\rho$ near $T_{SG}$ is not sensitive to the choice of $q_c$. In fact, the exponent $d(T_{SG})$ is independent of $q_c$.

Next, let us consider the renormalization effect of the hopping parameter $t$. We assume that the three-dimensional system exhibits Bose condensation at the temperature scale of
\[ T_B \approx 2\pi \frac{n_B}{m_B} = 4\pi t \chi \delta, \quad (5.22) \]

\( T_B \) as the observed \( T_c \) in the lightly-doped region. Since \( t \sim 0.3 \) eV gives rise to \( T_B \sim 3000 \) K at \( \delta \sim 0.15 \), one needs to use an effective hopping parameter \( t' \sim 1.01 \) eV in place of \( t \) so as to obtain a realistic \( T_c \sim 100 \)K.

In Fig.3 we present the phase diagram calculated by using these parameters, where \( T_{SG} \) is the spin-gap on-set temperature, \( T_A \) is the temperature at which the mass exponent \( d \) diverges, and \( T_B \) is the bose condensation temperature. In Fig.4 we plot the \( T \)-dependence of the exponent \( d(T) \). In Fig.5 we plot \( \rho \) as a function of \( T \) for several \( \delta \)'s. As explained, the curves reproduce the experimental data much better than those with \( d = 1/2 \) of the MF result, showing smooth departures from the \( T \)-linear curves, i.e., \( d(T_{SG}) > 1 \) for the region of interest, \( 0.05 \lesssim \delta \lesssim 0.15 \).

VI. DISCUSSION

In this paper, we studied the effective gauge theory and the phase structure of the slave-boson t-J model. We reviewed our analytical studies which indicates the CSS at low temperatures as supported by experiments. We also explained that the spin-gap state corresponds to the Higgs phase in the effective gauge theory. Both analytical and numerical studies show that there is a phase transition to the Higgs phase in 3D. In the second half of this paper, we summarized the calculation of the resistivity especially in the spin-gap phase. The results of \( \rho(\delta, T) \) are consistent with the experimental observations. This may support that our treatment of gauge-field fluctuations in a quasi-2D system by the variational treatment of compactness is suitable to describe the spin-gap state in the t-J model.

Concerning to the dimensionality, we cite the work of Su et al.24 They consider the pure 2D SB t-J model, and apply the dualty transformation for bosonized spinons as in Ref.24. They calculate \( \rho \) in the spin-gap state using the disorder parameter, the density of vortex condensation, \( \nu(x) \), which is nonvanishing in the spin-gap state as explained in Table 2. (For more details, see Ref.24.) After fitting several parameters to the experimental data, they plotted \( \rho \). We stress that their approach is dual to ours, but fails to locate \( T_{SG} \). This makes their parameter fitting rather obscure.

Finally, we address here on the observability of holons and spinons in the CSS state because it often brings some misunderstandings. In Sect.2B, we explained that the gauge dynamics is in the deconfinement phase or the Coulomb phase below \( T_{CSS} \). Then one may ask if the holon and spinon appear as quasiparticles, though physical quantities are all gauge-invariant. Deconfinement of the slave particles does not necessarily imply that these particles are “observed” each by each. This point is often misunderstood.

To see this, it is useful to recall QED. In QED in 3+1-dimensions, genuine asymptotic states are constructed by the gauge-invariant operators13 like

\[ \psi^\dagger(x) \sum_P w(P) \exp(i\epsilon \int_P dx_i A_{iEM}^P), \quad (6.1) \]

where \( \psi(x) \) is the electron operator and \( A_{iEM}^P \) is the photon operator. \( \sum_P \) implies superposition over different paths \( P \) starting from \( x \) to infinity with a suitable weight \( w(P) \). \( w(P) \) characterizes the configuration of electric field since the exponential factor acts as a creation operator of electric flux along the path \( P \). In the deconfinement phase like the Coulomb phase, the correlation functions of the above operator can be expanded in powers of \( e \) and at each order of \( e \) the infrared singularities are cancelled out with each other. As the electrons in QED, the holons and spinons in the CSS state should accompany clouds of the (soft) gauge bosons and genuine quasiparticles should be described by gauge-invariant operators like (6.1). However, explicit construction of such operators is a future problem.

Here we should mention that there is a point that distinguishes the quasiparticles of the t-J model from those of QED. Since the t-J model is the model of electrons, any physical quantities such as conductivity are expressed by the electron operators, and therefore gauge-invariant (here the gauge invariance means the invariance under a local rotation of the phases of slave particles, and nothing to do with the EM gauge symmetry). The deconfinement of slave particles means that these physical quantities that are expressed by slave particles are (approximately) calculated by perturbative calculation with respect to the auxiliary gauge field which couples to the slave particles beacuse fluctuations of the gauge field is not large. In this sense, the holons, the spinons and the gauge bosons can be regarded as (weakly interacting) quasiexcitations at low energies that live only inside of the material. This point is different from the well known gauge theories in particle physics such as QED, in which one can prepare quasiparticles at low energies as observable incoming and outgoing particles at infinitely separated positions in space. Thus, we think that one needs further studies to propose a clever method to directly measure the charges of the slave particles.

Associated with the above problem of observability of the slave particles, the following question may arise: Since the electron operator is constructed as Eq.(2.2) one may assign the EM charges of holons and spinons freely as long as the electron has the known EM charge \( e(< 0) \). Because deconfinement implies that the EM charges of holons and spinons can be observed separately, how does this freedom be fixed? If this freedom survives, it causes nonsensical result like fractional charges of holons and spinons. A similar question has been addressed in Sect.5A for the conductivity, and the answer there was that the conductivity is independent of this freedom.
Let us consider this question in detail here. This question makes sense if the EM charges, $Q_h$, $Q_s$, which one may assign to slave particles, were the absolute charges that are measured from the vacuum (the state with no electrons). However, the ambiguity in assigning the charges of slave particles just reflects our freedom to select the reference state from which their charges are measured. For example, if we measure the charges of a holon, $Q_h$, and a spinon, $Q_s$, from the half-filled state, we have $Q_h = -e$, $Q_s = 0$, while if they are measured from the vacuum, then $Q_h = 0$, $Q_s = e$. More generally, if they are measured from the state of average absolute charge $-c e$ per site ($c$ is a real number), one has $Q_h = 0 - (-c e) = c e$, $Q_s = e - (-c e) = (1 + c) e$. These expressions are nothing but what we explained at (5.7) and (5.8) in Sect.5A. Thus they are physically equivalent. It is also instructive to see that the electron charge per site,

$$Q_C = e \sum_\sigma C^{\dagger}_\sigma C_\sigma = e \sum_\sigma f^{\dagger}_\sigma f_{i\sigma} = e(1 - b_i^\dagger b_i), \quad (6.2)$$

depends on the state itself. For the half-filled state, $\prod_i f^\dagger_i |0\rangle$ ($|0\rangle$ is the state with no slave particles), we have $Q_C = e$, while for the vacuum state, $\prod_i b_i^\dagger |0\rangle$, $Q_C = 0$. The EM charge is an additive quantity, and what we actually measure as “charges” in the experiments are always the differences of the charges. For example, by applying the electron variable, $C_{\sigma} = b_i^\dagger f_{i\sigma}$, to a state, the EM charge changes by $Q_h - Q_s = ce - (1 + c) e = -e$. This difference is of course independent of the reference state itself and has a physical meaning.

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**APPENDIX A: PHASE STRUCTURE OF THE COMPACT ABELIAN HIGGS MODEL**

In Sect.4, we derived the dual-variable representation of the noncompact Abelian Higgs model (1.1). Similar analysis is possible in the continuum field theory of the compact Abelian Higgs model (1.3). Nagaosa and Lee studied this model in 3D continuum and concluded that the system is always in the confinement phase. In this Appendix, we point out that their argument is insufficient and the phase transition into the deconfining-Higgs phase (i.e., the superconducting phase with a massive gauge field in their context) certainly exists.

The action is given by

$$S = \int d^3 x \left[ \frac{\rho}{2} (\nabla \phi(x) - \vec{A}(x))^2 + \frac{\kappa}{2} (\nabla \times \vec{A})^2 \right], \quad (A.1)$$

where we consider the 3D continuum system, and $\phi$ is the phase of the Higgs field and $\vec{A}$ is the vector potential. Duality transformation for (A.1) can be performed as in the lattice model. We take the U(1) gauge group as compact and allow singular configurations of $\vec{A}$ as

$$\nabla \cdot \nabla \times \vec{A} = 4\pi Q(x), \quad (A.2)$$

where $Q(x)$ is the instanton (monopole) density as (4.13) in the lattice model. The phase $\phi(x)$ of the Higgs field is also compact, so singular configurations with nonvanishing vorticity are allowed:

$$\nabla \times \nabla \phi = 2\pi \vec{J}, \quad (A.3)$$

$\vec{J}$ is the vortex current as (4.7) in the lattice model. If the radial degrees of freedom of the Higgs field are introduced, vortices can appear as regular solitons of the field equations, which are called the Nielsen-Olesen vortex. In any way, it is standard to introduce a complex scalar “vortex field” $\psi_V$, whose world lines are identified with $\vec{J}$. The field $\psi_V(x)$ is viewed as the creation (annihilation) operator of a vortex. Then $\psi^\dagger_V(x) \psi_V(x)$ measures the vortex density. $\psi_V$ interacts with each other via a short-range interaction as in (1.6). One can sum up instantons by the dilute gas approximation. To do this, one introduces a phase field $\phi(x)$. In fact, the relation (A.2) is respected by the Lagrange multiplier field $\phi(x)$ as

$$\int_{-\pi}^\pi d\phi(x) \exp \left( i \phi(x) \left( \frac{1}{4\pi} \nabla \cdot \nabla \times \vec{A}(x) - Q(x) \right) \right), \quad (A.4)$$

and the summation over the instanton $Q(x) = 0, \pm 1$ gives rise to a factor,

$$\sum_{Q(x)=0,\pm 1} e^{-i\phi(x)Q(x)} z^{|Q(x)|} [\psi_V(x)^\dagger Q(x)]^{|Q(x)|} = 1 + z \left( e^{-i\phi(x)Q(x)} + e^{i\phi(x)Q(x)} \right) \simeq \exp \left( z (e^{-i\phi(x)Q(x)} + e^{i\phi(x)Q(x)}) \right), \quad (A.5)$$

where $z$ is the fugacity of instantons, and the first line describes that each instanton $Q(x)$ is connected with the vortex current generated by $\psi_V(x)$. $\psi_V(x)^\dagger Q(x) \equiv (\psi_V(x)^\dagger)^{-Q(x)}$ for $Q(x) < 0$. As the factor $\exp(i\phi Q)$ indicates, the angle $\phi(x)$ is conjugate to the integer $Q(x)$. Thus, there holds the uncertainty principle,

$$\Delta \phi(x) \Delta Q(x) \geq 1. \quad (A.6)$$

This implies that the following two phases are possible:

(i) Confinement : $\Delta \phi(x) = 0$; $\Delta Q(x) = \infty$,

(ii) Deconfinement : $\Delta \phi(x) = \infty$; $\Delta Q(x) = 0. \quad (A.7)$

(i) is the confinement phase where the instanton condenses, hence $A_i$ fluctuates violently $\Delta A_i = \infty$ as seen
by \( \Delta_2 \). (ii) is the deconfinement phase where no instantons appear \( Q(x) = 0, \Delta Q(x) = 0 \), so \( \Delta A_1 = 0 \).

Next, we note that the Higgs field \( \exp(i\bar{\phi}) \) and the vortex field \( \psi_v \) are “conjugate” each other in the sense that the following two cases are possible:

(a) Disorder – Vortex : \( \langle \psi_v \rangle \neq 0 \); \( \langle \exp(i\bar{\phi}) \rangle = 0 \),

(b) Order – Higgs : \( \langle \psi_v \rangle = 0 \); \( \langle \exp(i\bar{\phi}) \rangle \neq 0 \). (A.8)

Actually, when vortices proliferate with finite density \( \langle \psi_v(x)\psi_v(x) \rangle \neq 0 \), while \( \Delta \phi = \infty \) and so \( \langle \exp(i\bar{\phi}) \rangle = 0 \). This is the case (a), which one may call the disorder phase, \( \Delta \phi = \infty \), or the vortex phase. On the other hand, when vortices are rare, \( \langle \psi_v(x) \rangle = 0 \), while \( \langle \exp(i\bar{\phi}) \rangle \neq 0 \) due to \( \Delta \phi = 0 \). This is the case (b), which on may call the ordered phase, \( \Delta \phi = 0 \), or the Higgs phase. This Higgs-vortex duality is summarized in Table 2, where the vortex density \( v(x) = \langle \psi_v(x)\psi_v(x) \rangle \).

The continuum action can be expressed in terms of \( \varphi \) and the dual variables as follows:

\[
S = \int d^3x \left[ \frac{1}{2\rho} (\nabla \times c)^2 + \frac{1}{2\kappa} (c_0^2 + 1)(\nabla \varphi)^2 + \frac{1}{2\kappa} (-K(\nabla + i\overline{c})^2 + M^2)|\psi_v + g(\psi_v^\dagger\psi_v)^2 \right.
- z(\psi_v^\dagger e^{i\varphi} + \psi_v e^{-i\varphi}) \right].
\] (A.9)

By integrating over \( \psi_v \), its world lines (the vortex current) are generated, and with each line segment the current \( \bar{c} \) is coupled. Integration over the force-mediating field \( \vec{c} \) produces interactions among the vortex currents. The mass of the vortex field is given as \( M^2 = 4\pi^2 \rho D(0;m^2) – \ln \mu \) as in the lattice model. There is also a short-range repulsive interaction \( (g > 0) \) between vortices.

To study the phase structure, Nagaosa and Lee introduced a new field \( \psi_v' = \psi_ve^{-iz}\) by a field redefinition or a “gauge transformation”. The last line of the action (A.9) then generates a linear term of \( \psi_v' \). Due to this term, they simply concluded that \( \langle \psi_v' \rangle \neq 0 \) regardless of the value of \( M^2 \). From this, they argued that the system is always in the vortex condensed phase of (a) of (A.8), and there occurs no phase transition into the Higgs phase (b) of (A.8).

However, their argument above using the field redefinition or gauge transformation is not correct. A good counter example is the gauge-Higgs model itself. In the original variables, the gauge-Higgs coupling is given by

\[ e^{i\phi(x+i)} e^{iA_1(x)} e^{-i\phi(x)} + H.c. \] (A.11)

We can fix the gauge to the unitary gauge, i.e., \( \exp(i\bar{\phi}(x)) = 1 \) and then the linear term of the exponentiated gauge field \( \exp(iA_1(x)) \) appears in the action, i.e., \( S \sim UUUU + U \). But this linear term does not necessarily lead to \( \langle \exp(iA_1(x)) \rangle \neq 0 \) which is the condition that the system is in a nonconfining phase, like the Coulomb phase or the Higgs phase. For example, a MFT of this model with a variational action of single-link form \( S \sim U \) offers us two possibilities. (i) \( \langle U \rangle \neq 0 \) for the Coulomb or the Higgs phase or (ii) \( \langle U \rangle = 0 \) for the confinement phase, and predicts a phase transition between (i) and (ii).

To confirm this point, let us analyse the system (A.9) in a more straightforward manner referring to Polyakov’s result. He considered the dynamics of the pure compact U(1) gauge model in 3D\( [2 + 1)D \), and showed that the system is always in the confinement phase. After integrating over the gauge field (instantons), the action \( S_{U(1)} \) is given by

\[ S_{U(1)} = \int d^3x \left[ \frac{1}{2\kappa} (\nabla \varphi)^2 - z \cos \varphi \right]. \] (A.12)

where the field \( \varphi \) here again conjugates to instanton density and mediates the force among instantons. One can approximate the relevant potential term \( \cos \varphi \sim \varphi^2 \) to find that the vacuum is given by \( \varphi = 0 \) (mod 2\( \pi \)). This obviously means \( \Delta \varphi < 1 \), and so a condensation of instantons; the state (i) of (A.7) above. Thus the system is in the confinement phase. This is the result of Polyakov.

Now let us return to the present model (A.9) and consider two cases, \( M^2 < 0 \) and \( M^2 > 0 \), separately. We assume \( z \) is sufficiently small, so that the renormalization effect of \( M^2 \) is negligible, although the last \( z \) term gives rise to a negative renormalization to \( M^2 \).

For \( M^2 < 0 \), the vortex field condenses \( \langle \psi_v \rangle = \psi_0 \neq 0 \) because the effective action of \( \psi_v \) has a negative curvature at the origin as in the standard Ginzburg-Landau theory with the global U(1) symmetry. Then the field \( \varphi \) acquires a potential term like \( z\psi_0 \cos \varphi \) where we assumed that \( \psi_0 \) is real without loss of generality. The instanton action \( S_\varphi \) reduces to that of the pure U(1) gauge model (A.13), and the Polyakov’s result applies; condensation of instantons occurs and the confining phase is realized. This phase corresponds to the confinement-vortex phase (ii) of (A.7) and (A.8).

For \( M^2 > 0 \) we can safely integrate out the vortex field because there are only short-range interactions between vortices. Then the resultant action describes the dynamics of instantons or of their conjugate variable \( \varphi(x) \). The action of \( \varphi(x) \) is given as

\[ S_\varphi = \frac{1}{2\kappa} \int d^3x(\nabla \varphi)^2 - z^2 \int d^3xd^3ye^{i\varphi(x)} e^{-M|\varphi(x)|} e^{i\varphi(y)}. \] (A.13)

The second term in (A.13) originates from the fact that an instanton and an anti-instanton is connected by a vortex flux tube via (A.13). We recognize that, for large \( M \), the second term of (A.13) is approximated by a derivative term, hence
\[ S_\varphi \rightarrow \frac{1}{2\kappa} \int d^3 x (\nabla \varphi)^2 + z^2 \int d^3 x (\nabla \varphi)^2. \] 

(A.14)

There is no potential term of \( \varphi \) in (A.14) and \( \varphi \) does not have a definite value, but fluctuates randomly (for sufficiently large \( \kappa \) and small \( z^2 \)). This is the state (ii) above, leading to a deconfinement phase. Here, we have no a definite value, but fluctuates randomly (for

\[ \langle \psi \rangle = 0 \text{ because the large fluctuation of } \varphi \text{ makes the last term in } \phi \varphi(x) \text{ irrelevant}; \langle \exp(\mathcal{I}) \rangle = 0. \] 

From (A.3), this means that the original Higgs field has a coherent phase. This phase corresponds to the deconfinement-Higgs phase (iib) of (A.7) and (A.8).

The above arguments indicate that there is a phase transition from the confining to the Higgs phases. This is actually found now in the numerical studies. On the other hand, for the compact gauge-Higgs model with fixed radius of the Higgs field, Fradkin and Shenker show that the confinement and Higgs phases are connected if the Higgs charge is fundamental \( q = \pm 1 \). The above two observations can be reconciled if the critical line terminates at some point in the \((\kappa - \rho)\) plane. If the radius of the Higgs field is also a dynamical variable, the critical line does not terminate as the parameter space is now three dimensional instead of two.

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4 See, e.g., I.Ichinose and T.Matsui, Mod.Phys.Lett.B 955(1990); Phys.Rev. B 45 9976 (1992). G.Tatara and T.Matsui, Phys.Rev. B 44 2867 (1991). In these papers, the hole pairing is studied in the slave-fermion representation of the t-J model.
5 See, e.g., T.Nishikawa et al., J. Phys. Soc. Jpn. 63, 1441 (1994); J.Takeda et al., Physica C 231, 293 (1994); H.Y.Hwang et al., Phys. Rev. Lett. 72, 2636 (1994).
6 M.Gurvitch and A.T.Fiory, Phys. Rev. Lett. 59, 1337 (1987).
7 P.W.Anderson, Phys. Rev. Lett. 64, 1839 (1990). Some authors use the term “spin-charge separation”, but we use CSS according to Anderson’s original naming.
8 I.Ichinose and T.Matsui, Nucl. Phys. B 394, 281 (1993); Phys. Rev. B 51, 11860 (1995).
9 The gauge theoretical approach to the t-J model may be traced back to G.Baskaran and P.W.Anderson, Phys.Rev. 37, 580 (1988); A.Nakamura and T.Matsui, Phys.Rev.37, 7940 (1988).
10 C.Nayak, Phys. Rev. Lett. 85, 178 (2000).
11 Y.Iwasaki, K.Kanaya, S.Sakai and T.Yoshie, Phys.Rev.Lett. 69, 21 (1992).
12 I.Ichinose and T.Matsui, Phys. Rev. Lett. 86, 942 (2001).
13 P.W.Anderson, Phys.Rev. D 12,3978 (1975).
14 J.M.Drouffe, Nucl.Phys.B 170, 211 (1980).
15 C.Nayak, Phys. Rev. Lett. 86, 943 (2001).
16 We wrote this explanation in a letter to the editor. The Referee recommended to publish, but it was not included in the comment reply.
17 P.A.Lee and N.Nagaosa, Phys. Rev. B 46, 5621 (1992). N. Nagaosa and P.A. Lee: Phys. Rev. Lett. 64 (1990) 2450; Phys. Rev. B 43 (1991) 1233.
18 L. Ioffe and P. Wiegmann: Phys. Rev. Lett. 65 (1990) 653. I. Ioffe and G. Kotliar: Phys. Rev. B 42 (1990) 10348.
19 T.Ito, K.Takenaka and S.Uchida, Phys. Rev. Lett. 70, 3995 (1993).
20 H.Yasuoka et al., “Strong Correlation and Superconductivity”, ed. H.Fukuyama et al. (Springer Series, Berlin, 1989) p.254; M.Takigawa et al., Phys. Rev. B 43, 247 (1991); J.Rossat-Mignot et al., Physica 185-189C, 86 (1991); J.M.Tranquada et al., Phys. Rev. B 46, 5561 (1992). For a review of an experimentanl survey of the spin gap, see T.Timusk and B.Statt, Rep. Prog. Phys. 62, 61 (1999).
21 In the pure 2D system, N. Nagaosa and P. A. Lee, Phys. Rev. B 45, 966 (1992) argued that vortices put the second-order transition into a crossover.
22 B. I. Halperin, T. C. Lubensky, and S. Ma, Phys. Rev. Lett. 32, 292 (1974). See also S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973).
23 H. Kleiner, Lett. Nuovo Cimento 35, 405 (1982).
24 M. U. Ubbens and P. A. Lee, Phys. Rev. B 49, 6853 (1994); See also G.Kotliir and Liu, Phys. Rev. B 38, 5142 (1988); Y.Suzumura, Hasegawa, and H.Fukuyama, J.Phys.Soc.Jpn.57, 2768 (1988).
25 K.Kajantie, M.Karjalainen, M.Laine and J.Peisa, Phys.Rev. B 57, 3011 (1998); K.Kajantie, et al, Nucl.Phys. B 546, 351 (1999) and references cited therein.
26 L.Balents, M.P.A.Fisher and C.Nayak, Phys.Rev.B 61, 6307 (2000).
27 M. Onoda, I.Ichinose, and T.Matsui, J. Phys. Soc. Jpn. 67, 2606 (1998); 69, 3497 (2000)
28 N.Nagaosa and P.A.Lee, Phys.Rev.B 61, 9166 (2000).
29 P.W.Anderson, Science 235,1196 (1987).
30 For a more realistic Hamiltonian with anisotropic couplings reflecting quasi-two-dimensionality, see (4.14). We present the isotropic expressions below because most of the argument does not require the explicit anisotropic formula.
31 A careful treatment of this constraint in a framework of discretized imaginary time is studied in I.Ichinose and T.Matsui (to appear).
32 This decoupling is proposed in Ref[2], but ignores two-body interactions. These terms must be added in order to achieve the exact equivalence to the t-J model. The relevance of these interactions are studied by I.Ichinose, T.Matsui and K.Sakakibara (to appear).
33 I.Ichinose, T.Matsui, and K.Sakakibara, J.Phys.Soc.Jpn 67, 543 (1998).
34 The first term of Eq.(2.11) is obtained by modifying the form of Fermi curve by the square of volume \( \pi^2 (1 - \delta) N^{-2} \). This is a good approximation with errors below 1% up to \( \delta = 0.3 \)
35 Because the carriers of superflow are NN hole pairs, the electromagnetic charge of each carrier \( Q_{\text{super}} \) should be \(-2\epsilon(>0)\). However, the present SB MFT ignores the pair-
ing of two holons, predicting \( Q_{\text{super}} = -e \) incorrectly. On the other hand, the SF approach describes holon pairings and predicts \( Q_{\text{super}} = -2e \) correctly.

33 See, e.g., B.Sakita, “Quantum Theory of Many Variable Systems and Fields”, (World Scientific, Singapore, 1985). An \( \epsilon \) expansion for a \( 2 + \epsilon \) D system also confirms the existence of such a constraint-free phase. In this sense, the \( O(N) \) nonlinear \( \sigma \) offers us yet another established example where the local constraint is dynamically irrelevant.

34 K.Wilson, Phys.Rev.D 10, 2445 (1974); J.B.Kogut and L.Susskind, Phys.Rev.D 11, 395 (1975).

35 A.Polyakov, Phys.Lett.B 72, 477 (1978); L.Susskind, Phys.Rev.D 20, 2610 (1979).

36 N. Nagaosa, Phys. Rev. Lett. 71 (1993) 4210.

37 A.M.Polyakov, Phys.Lett.B 59, 82 (1975); Nucl.Phys.B 120,429 (1977).

38 See, e.g., B.I.Halperin, P.A.Lee, and N.Read, Phys.Rev. B 47, 7312 (1993); Nayak and Wilczek, Nucl.Phys.B 441, 483 (1995); M.Onoda, I.Ichinose, and T.Matsui, Nucl.Phys.B 446,353 (1995); S.Chakravarty,R.E.Norton, and O.F.Syljuasen, Phys. Rev. Lett. 74, 1423 (1995).

39 The asymmetric XY spin model for XY spins \( (S^1_x, S^2_y) \equiv (\cos \theta_x, \sin \theta_x) \) has the action \( \sum x, i [c_1 S^1_x S^1_{x+i} + c_2 S^2_y S^2_{y+i}] \), where the asymmetry is caused by the spin gap and implies \( c_1 \neq c_2 \). If no spin gap exists, the model reduces to the usual (symmetric) XY model (with \( c_1 = c_2 \)).

40 C.Holm, W.Janke, T.Matsui and K.Sakakibara, Physica A246, 633-645 (1997).

41 B.I.Halperin, P.A.Lee, and N.Read, Phys.Rev. B 47, 7312 (1993); Nayak and Wilczek, Nucl.Phys.B 417,359 (1994); B430,534 (1994);

42 R. P. Feynman, “Statistical Mechanics, A set of Lectures”, Chap.8, W. A. Benjamin (1972).

43 R.Savit, Phys. Rev. Lett. 39, 55 (1977).

44 M.B.Einhorn and R.Savit, Phys.Rev.D 17, 2583 (1978); 19, 1198 (1979).

45 H.Yamamoto, G.Tatara, I.Ichinose and T.Matsui, Phys.Rev.B 44, 7654 (1991).

46 W.Janke and T.Matsui, Phys.Rev.B 42, 10673 (1990).

47 L.B.Ioffe and I.Larkin, Phys. Rev. B 39,8988 (1989).

48 M. Onoda, I.Ichinose, and T.Matsui, Phys. Rev. B 54, 13674 (1996).

49 See also P.A.Lee and N.Nagaosa, Phys. Rev. Lett. 79, 3755 (1997).

50 See, e.g., G.Baskaran, cond-mat/0008424

51 The two-body interactions in Ref.29 renormalize \( t \) to smaller values at low temperatures. This presents a possible explanation why \( t^* \) enters into the effective theory.

52 X.Dai, Y.Yu,T.Xiang, and Z.B.Su, Phys.Rev.B 61, 8683 (2000).

53 See for example, P.P.Kulish and L.D.Faddeev, Theor.Math.Phys. 4, 745 (1971).

54 H.Nielsen and P.Olesen, Nucl.Phys.B 61, 45 (1973).

55 There appear massless Nambu-Goldstone bosons as a result of the spontaneous breaking of the global U(1) symmetry. Then one may wonder about the stability of the “ground state” with \( \varphi = 0 \). However it can be verified that the massless boson does not contribute to the mass of \( \varphi \) at \( O(z^2) \).

56 E.Fradkin and S.Shenker, Phys.Rev.D 19, 3682 (1979).
| Phase   | $T$       | $V(r)$          | $Coulomb$ | $SG$ | $CSS$ | $T$ | $T_{SG}$ |
|---------|-----------|-----------------|-----------|------|-------|-----|---------|
| Confinement | $T_{CSS} < T$ | $\propto r$ | $\propto r$ |      |       |     |         |
| Coulomb      | $T_{SG} < T < T_{CSS}$ | $\propto 1/r$ |           | $T$ | $T_{CSS}$ |
| Higgs        | $T < T_{SG}$   | $\propto \exp(-m_A r)/r$ |           | $T$ | $T_{CSS}$ |

Table 1. Phases of the gauge dynamics of the SB t-J model. $V(r)$ is the potential energy between two external charges separated by a distance $r$.

| Phase   | $T$       | $\lambda(x)$ $v(x)$ | $T_{SG}$ | $T_{CSS}$ |
|---------|-----------|----------------------|-----------|-----------|
| Coulomb | $T_{SG} < T < T_{CSS}$ | $0 \neq 0$ | $T_{SG}$ | $T_{CSS}$ |
| Higgs   | $T < T_{SG}$   | $\neq 0$ 0         |           | $T_{CSS}$ |

Table 2. Comparison of the ordinary representation and the dual representation of the 3D (quasi-2D) non-compact Abelian Higgs model for the transition at $T_{SG}$ into the spin-gap state. The order parameter of the ordinary representation is the spin-gap amplitude $\lambda(x)$, while the disorder parameter in the dual representation is the vortex density $v(x)$. In the pure 2D system, $T_{SG} = 0$ and only the Coulomb phase exists, in which $v(x)$ is always nonvanishing.

**FIG. 1.** Phase diagram of the SB t-J model in the $\delta - T$ plane. Along $T_{\chi}$ and $T_{V/B}$, $\chi$ and $D$ vanish, respectively. $T_{BC}$ is the onset $T$ of Bose condensation, $\langle b_x \rangle \neq 0$. $T_{CSS}$ is the critical temperature below which the CSS takes place. There are five phases: (i) Strange Metal Phase; Deconfinement-Coulomb; (ii) Spin-Gap Phase; Deconfinement-Higgs with $D \neq 0$; (iii) Fermi-Liquid Phase; Deconfinement-Higgs with $\langle b_x \rangle \neq 0$; (iv) Superconducting Phase; Deconfinement-Higgs with $D, \langle b_x \rangle \neq 0$; (v) Electron Phase; Confinement above $T_{CSS}$.

**FIG. 2.** Illustration of electric fluxes connecting two external charges. (a) Confinement state. (b) Deconfinement state.

**FIG. 3.** Graphical representation of the Schwinger-Dyson equation for $\tilde{\Pi}_{F(B)}$.

**FIG. 4.** Plot of the resistivity $\rho$ divided by $\rho_0 \equiv 2\pi m_F k_B T_{SG} / (e^2 n_F)$ for $\lambda_s(T) \simeq \lambda_0 (1 - T/T_{SG})^2$ ($d = 1/2, 1, 2$). For definiteness we chose $\delta = 0.05$, $\lambda_0 = 2k_B T^* / \pi$, $2\pi n_B / m_B = 4\delta k_B T_{SG} / (1 - \delta)$ and $J/t = 0.35$. 

$\delta = 0.05$
FIG. 5. Mean-field phase diagram of the t-J model. $T_{BC}$ is the Bose condensation temperature. $T_{\lambda}$ is the spin-gap onset temperature. $T_A$ is the root of $T = \Theta(T)$ at which $d(T)$ diverges. We chose $t^* = 0.01$ eV, $J = 0.15$ eV, and $\omega_{\lambda} = \pi J\chi/(2e^\gamma)$.

FIG. 6. Exponent $d(T)$ vs $T$ for $\delta = 0.04$ and 0.1.

FIG. 7. Resistivity $\rho(\delta, T)$ in $h/e^2$ for several $\delta$’s with the parameters chosen in Fig.2. The dotted lines represent the case of $X(T) = 0$ in (1). The exponent $d(T_{SC})$ decreases as $16.4$, $4.8$, $2.8$, $2.0$, as $\delta$ increases.