Weak approximations of nonlinear SDEs with non-globally Lipschitz continuous coefficients †

Xiaojie Wang\textsuperscript{a}, Yuying Zhao\textsuperscript{a,b} *

\textsuperscript{a} School of Mathematics and Statistics, HNP-LAMA, Central South University, Changsha, Hunan, P. R. China

\textsuperscript{b} Department of Mathematics Sciences, Worcester Polytechnic Institute, Worcester, MA 01609 USA

April 11, 2022

Abstract

As opposed to an overwhelming number of works on strong approximations, weak approximations of stochastic differential equations (SDEs), sometimes more relevant in applications, are less studied in the literature. Most of the weak error analysis among them relies on a fundamental weak approximation theorem originally proposed by Milstein in 1986, which requires the coefficients of SDEs to be globally Lipschitz continuous. However, SDEs from applications rarely obey such a restrictive condition and the study of weak approximations in a non-globally Lipschitz setting turns out to be a challenging problem. This paper aims to carry out the weak error analysis of discrete-time approximations for SDEs with non-globally Lipschitz coefficients. Under certain broad assumptions on the analytical and numerical solutions of SDEs, a general weak convergence theorem is formulated for one-step numerical approximations of SDEs. Explicit conditions on coefficients of SDEs are also offered to guarantee the aforementioned broad assumptions, which allows coefficients to grow superlinearly. As applications of the obtained weak convergence theorems, we prove the expected weak convergence rate of two well-known types of schemes such as the tamed Euler method and the backward Euler method, in the non-globally Lipschitz setting. Numerical examples are finally provided to confirm the previous findings.

AMS subject classification: 60H35, 65C30, 60H10.

Keywords: SDEs with non-globally Lipschitz coefficients, fundamental weak convergence theorem, tamed Euler method, backward Euler method, weak convergence rate

1 Introduction

Stochastic differential equations (SDEs) are widely used to mathematically model random phenomena arising from scientific fields of physics, chemistry, biology, finance and many other branches of

\textsuperscript{†}This work was supported by Natural Science Foundation of China (12071488, 11971488), Natural Science Foundation of Hunan Province (2020JJ2040) and the innovative project of graduate students of Central South University (2021zzts0040).

*E-mail addresses: x.j.wang7@csu.edu.cn, y.y.zhao68@csu.edu.cn.
science. Nevertheless, the analytic solutions of nonlinear SDEs are rarely available and one often resorts to their numerical approximations. Under the classical global Lipschitz conditions, the corresponding numerical analysis is well-understood \cite{19,27}. However, coefficients of most models in applications do not obey the classical but restrictive conditions. For example, the coefficients might behave super-linearly and violate the globally Lipschitz assumption. A natural question thus arises as to whether the commonly used schemes in the global Lipschitz setting are still able to perform well in the non-globally Lipschitz setting. As already asserted by \cite{15}, the commonly used Euler-Maruyama method produces strongly and weakly divergent numerical approximations when used to solve a large class of SDEs with super-linearly growing coefficients (see also the early reference \cite{13}). Thereafter, a rich variety of convergent numerical methods have been constructed and analyzed for SDEs with super-linearly growing coefficients, see, e.g., \cite{6,12,16,18,23,31,34,36} and references therein.

It is worthwhile to point out that an overwhelming majority of existing works are devoted to developing and analyzing schemes for strong approximations of SDEs in non-globally Lipschitz settings, where the approximation errors are measured in the sense of $\|X(T) - Y(T)\|_{L^q(\Omega, \mathbb{R}^d)}$. Here $X(T)$ and $Y(T)$ represent the exact solution and the numerical solution of SDEs, respectively. In particular, Tretyakov and Zhang \cite{36} derived a fundamental strong convergence theorem in a non-globally Lipschitz setting, extending the counterpart in the globally Lipschitz setting \cite{27}. But in many applications, e.g., in financial engineering, it is only required that the average $E[\varphi(Y(T))]$ is close to $E[\varphi(X(T))]$ for a class of (payoff) functions $\varphi$. More precisely, the weak approximations are interested in the approximation of the law, with the error measured in the sense of

$$\|E[\varphi(X(T))] - E[\varphi(Y(T))]\| \leq C h^p$$

(1.1)

for a class of test functions $\varphi$, where $h$ denotes the discretization step size and the constant $C$ does not depend on $h$. Constructing weak schemes and deriving their optimal weak convergence rates $p \geq 1$ in (1.1) is not an easy task, even in the globally Lipschitz setting \cite{1,2,4,9,11,21,25,27,29,35}, to just mention a few. Most of the weak error analysis in the literature relies on a fundamental weak convergence theorem originally introduced by Milstein \cite[Theorem 2]{25}. Such a fundamental weak convergence theorem is powerful as one can infer the global weak convergence order based on the analysis of the local weak error. When the global Lipschitz condition is violated, the weak error analysis of the numerical approximations is more challenging and less studied in the literature \cite{3,5,7,20,21,26,28,32,33,39}. In 2005, Milstein and Tretyakov \cite{26} introduced a new concept of discarding the approximate trajectories that leave a sufficiently large sphere $S_R := \{x : |x| < R\}$. This strategy allows them to use any usual weak scheme for solving a broad class of SDEs with non-globally Lipschitz coefficients, resulting arbitrarily small weak errors by increasing the radius of the sphere. But the explicit weak convergence rates cannot be identified. Recently in 2021, the authors of \cite{7} proposed an exponential Euler scheme for one-dimensional SDE with superlinearly growing coefficients and recovered a rate of weak convergence of order one for the scheme, by means of the associated backward Kolmogorov PDE. When the non-globally Lipschitz coefficients grow (sub-)linearly (e.g., the Heston model), several scholars analyzed weak rates of some weak approximation schemes \cite{8,21,39}. Moreover, a good progress was recently made on weak approximations of SDEs with irregular coefficients, see, e.g., \cite{5,32,33}.

To the best of our knowledge, there are so far no general weak convergence results for SDEs with non-globally Lipschitz conditions. More accurately, a fundamental weak convergence theorem in the non-globally Lipschitz setting is still missing, which motivates the present work. In this article
we concentrate on weak approximations of SDEs with non-globally Lipschitz coefficients. At first we establish a general weak convergence theorem for general one-step numerical approximations under certain board assumptions on the analytical and numerical solutions of SDEs (Theorem 2.4). Explicit conditions on coefficients of SDEs are also offered to guarantee the aforementioned board assumptions, which allows coefficients to grow super-linearly (Corollary 2.15). As applications of the obtained weak convergence theorems, we prove the a rate of weak convergence of order one for two well-known types of numerical methods such as a tamed Euler method (Theorem 3.7) and the backward Euler method (Theorem 4.4) in the non-globally Lipschitz setting.

The remainder of this article is structured as follows. In the forthcoming section, a setting is set up and some fundamental weak convergence theorems are stated and proved. In sections 3 and 4, we obtain weak convergence rates for two well-known types of schemes, e.g., a tamed Euler method and the backward Euler method in the non-globally Lipschitz setting, with the aid of the fundamental weak convergence theorem. Some numerical experiments are finally presented in section 5.

2 Settings and fundamental weak convergence theorems

Throughout this paper the following notation is frequently used. Let $|\cdot|$ and $\langle \cdot, \cdot \rangle$ be the Euclidean norm and the inner product of vectors in $\mathbb{R}^d$, respectively. By $A^T$ we denote the transpose of vector or matrix $A$. Given a matrix $A$, we use $\|A\| := \sqrt{\text{trace}(A^T A)}$ to denote the trace norm of $A$. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we use $\mathbb{E}$ to mean expectation and $L^r(\Omega; \mathbb{R}^{d \times m})$, $r \in \mathbb{N}$, to denote the family of $\mathbb{R}^{d \times m}$-valued variables with the norm defined by $\|\xi\|_{L^r(\Omega; \mathbb{R}^{d \times m})} = (\mathbb{E}[\|\xi\|^r])^{1/r} < \infty$.

Next, we give some notation for partial derivatives of functions, frequently used throughout this paper. A vector $\alpha = (\alpha_1, \ldots, \alpha_d)$ is called a multiindex of order $|\alpha| = \alpha_1 + \cdots + \alpha_d$, where each component $\alpha_i \geq 0$ is a nonnegative integer. For a multiindex $\alpha$, we define the partial derivatives of $v: \mathbb{R}^d \to \mathbb{R}^l$ as

$$D^\alpha v(x) := \frac{\partial^{|\alpha|} v(x)}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} v(x).$$

(2.1)

For a nonnegative integer $k$, we use

$$D^k v(x) := \{D^\alpha v(x) : |\alpha| = k\}$$

(2.2)

to denote the set of all partial derivatives of order $k$. Assigning some ordering to the various partial derivatives, we can also regard $D^k v(x)$ as a point in $\mathbb{R}^d$ in the particular case $l = 1$. For example, in the special case $l = 1$, $Dv(x)$ is the gradient vector:

$$Dv := \left( \frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, \ldots, \frac{\partial v}{\partial x_d} \right),$$

(2.3)

and $D^2 v(x)$ is the Hessian matrix:

$$D^2 v := \left( \frac{\partial^2 v}{\partial x_i \partial x_j} \right)_{d \times d}.$$  

(2.4)

For $l > 1$ but $k = 1$, $Dv$ denotes a Jacobi matrix of $v$. Moreover, we define the norm

$$|D^k v(x)| = \left( \sum_{|\alpha| = k} |D^\alpha v(x)|^2 \right)^{1/2}.$$  

(2.5)
Let $d, m \in \mathbb{N}, T > 0$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \in [0,T]}$ satisfying the usual conditions. In this article we are interested in the weak approximations of autonomous SDEs in the Itô form of

$$
\begin{aligned}
\begin{cases}
dX(t) = f(X(t)) \, dt + g(X(t)) \, dW(t), & t \in (0,T], \\
X(0) = X_0 \in \mathbb{R}^d,
\end{cases}
\end{aligned}
$$

(2.6)

where $f = (f^1, f^2, \ldots, f^d)^T: \mathbb{R}^d \to \mathbb{R}^d$ is the drift coefficient function, and $g: \mathbb{R}^d \to \mathbb{R}^{d \times m}$ is the diffusion coefficient function, frequently written as $g = (g^{ij})_{d \times m} = (g^1, g^2, \ldots, g^m) = (g_1, g_2, \ldots, g_d)^T: \mathbb{R}^d \to \mathbb{R}^{d \times m}$. Here and below $f^i: \mathbb{R}^d \to \mathbb{R}$ is a real-valued function, $g^j: \mathbb{R}^d \to \mathbb{R}^d$ is a $d \times 1$ vector function and $g_i: \mathbb{R}^d \to \mathbb{R}^m$ is a $1 \times m$ vector function. Moreover, $W: [0,T] \times \Omega \to \mathbb{R}^m$ stands for the $\mathbb{R}^m$-valued standard Brownian motions with respect to $\{\mathcal{F}_t\}_{t \in [0,T]}$ and the initial data $X_0 \in \mathbb{R}^d$ is assumed to be deterministic for simplicity.

This section aims to establish a fundamental weak convergence theorem for general one-step approximation. This theorem plays the same role in the theory of weak approximation as the main convergence theorem does in the theory of mean-square approximation. In order to approximate the general SDEs (2.6), we construct a uniform mesh on $[0,T]$ with $h = \frac{T}{N}$ being the step size, for any $N \in \mathbb{N}$. Here we introduce a new notation $X_{t,x}(s)$ or $X(t,x;s)$ for $0 \leq s \leq T$, denoting the solution of (2.6) satisfying the initial condition $X_{t,x}(t) = X(t,x;t) = x$. When we write $X(t)$, $t \in [0,T]$, We mean a solution to the SDEs (2.6) with the initial value $X(0) = X_0$. For $x \in \mathbb{R}^d$, $t \in [0,T]$, $h > 0$, $0 < t + h \leq T$, we introduce the general one-step approximation $Y(t,x;t+h)$ for $X(t,x;t+h)$, in the form of

$$
Y(t,x;t+h) = x + \Phi(t,x,h;\xi_t),
$$

(2.7)

where $\xi_t$ is a random variable (in general, a vector) defined on $(\Omega, \mathcal{F}, \mathbb{P})$, having moments of a sufficiently high order, and $\Phi$ is a function from $[0,T] \times \mathbb{R}^d \times (0,T] \times \mathbb{R}^m$ to $\mathbb{R}^d$. Similarly, by $Y(t,x;t+h)$ or $Y_{t,x}(t+h)$ we denote an approximation of the solution at $t + h$ with initial value $Y(t,x;t) = Y_{t,x}(t) = x$. Then one can use $Y_{n+1} = Y(t_n,Y_n;t_{n+1})$ to recurrently construct numerical approximations $\{Y_n\}_{0 \leq n \leq N}$ on the uniform mesh grid $\{t_n = nh, n = 0, 1, \ldots, N\}$, given by

$$
Y_0 = X(0) = X_0, \quad Y_{n+1} = Y_n + \Phi(t_n,Y_n,h;\xi_n), \quad n = 0, 1, \ldots, N - 1,
$$

(2.8)

where the $\xi_n$ for $n \geq 0$ is independent of $Y_0, Y_1, \ldots, Y_n, \xi_0, \xi_1, \ldots, \xi_{n-1}$. Alternatively, one can write

$$
Y_{n+1} = Y(t_n,Y_n;t_{n+1}) = Y(t_0,Y_0;t_{n+1}), \quad Y_0 = X_0 \quad n = 0, 1, \ldots, N - 1.
$$

(2.9)

For notational simplicity, the letter $C$ is used to denote a generic positive constant independent of time step size and may vary for each appearance. In order to carry out the weak error analysis, we make some assumptions as follows.

**Assumption 2.1.** Assume $X_0 \in \mathcal{D} \subset \mathbb{R}^d$ and the SDE problem (2.6) admits a unique $\{\mathcal{F}_t\}_{t \in [0,T]}$-adapted $\mathcal{D}$-valued solution $X: [0,T] \times \Omega \to \mathcal{D} \subset \mathbb{R}^d$, given by

$$
X(t) = X_0 + \int_0^t f(X(s)) \, ds + \int_0^t g(X(s)) \, dW(s), \quad a.s.
$$

(2.10)
In order to carry out the weak error analysis, it is crucial to introduce \(u: [0, T] \times \mathbb{R}^d \to \mathbb{R}\) for a measurable function \(\varphi: \mathbb{R}^d \to \mathbb{R}\), defined by
\[
u(t, x) = \mathbb{E}\left[\varphi(X(t, x; T))\right],
\]
where \(X(t, x; T)\) denotes the solution to (2.6) at \(T\), starting from the initial value \(x\) at \(t\), namely,
\[
X(t, x; s) = x + \int_t^s f(X(t, x; r)) \, dr + \int_t^s g(X(t, x; r)) \, dW(r), \quad 0 \leq t \leq s \leq T.
\]
It is easy to see that \(u(T, x) = \varphi(x)\). Under certain regularity assumptions, \(u\) defined by (2.11) can be the unique classical solution of the associated Kolmogorov equations [10, Theorem 1.6.2].

**Assumption 2.2.** Let \(u(t, x) < \infty\) given by (2.11) be well-defined for any \(t \in [0, T]\) and \(x \in \mathcal{D}\). There exists a function \(\nu: \mathcal{D} \to \mathbb{R}\) such that
\[
|\mathbb{E}\left[u(t + h, X(t, x; t + h))\right] - \mathbb{E}\left[u(t + h, Y(t, x; t + h))\right]| \leq \nu(x)h^{p+1}, \quad \forall x \in \mathcal{D} \subset \mathbb{R}^d \tag{2.13}
\]
holds uniformly with respect to \(t \in [0, T - h]\).

**Assumption 2.3.** The numerical approximations \(\{Y_n\}_{0 \leq n \leq N}\) uniquely determined by (2.8) always take values in \(\mathcal{D} \subset \mathbb{R}^d\) and there exists a function \(\Upsilon: \mathcal{D} \to [0, \infty)\) such that
\[
\sup_{N \in \mathbb{N}} \sup_{0 \leq n \leq N} \mathbb{E}[|\nu(Y_n)|] \leq \Upsilon(Y_0) < \infty, \tag{2.14}
\]
where the function \(\nu\) comes from Assumption 2.2.

Under these assumption, one can establish a fundamental weak convergence theorem.

**Theorem 2.4 (A general fundamental weak convergence theorem).** Let Assumptions 2.1, 2.2, 2.3 be fulfilled. Then the numerical solution produced by (2.8) has a weak convergence rate of order \(p\) in the sense that
\[
|\mathbb{E}\left[\varphi(X(t_0, X_0; T))\right] - \mathbb{E}\left[\varphi(Y(t_0, Y_0; t_N))\right]| \leq \Upsilon(X_0)T^p h^p, \tag{2.15}
\]
where the functions \(\varphi: \mathcal{D} \to \mathbb{R}\) and \(\Upsilon: \mathcal{D} \to [0, \infty)\) come from (2.11) and (2.14), respectively.

We would like to point out that the above theorem generalizes the famous fundamental weak convergence theorem due to Milstein [25] (see also [27, Chapter 2], [26]) proved under the global conditions on the SDEs coefficients.

**Proof of Theorem 2.4.** Recall \(Y_0 = X_0\) and
\[
X(t, x; T) = X(s, X_{t,x}(s); T), \quad 0 \leq t \leq s \leq T. \tag{2.16}
\]
Noting \(t_N = T\) and recalling the definition (2.11) yield
\[
\mathbb{E}\left[\varphi(X(t_0, X_0; T))\right] - \mathbb{E}\left[\varphi(Y(t_0, Y_0; t_N))\right] = \mathbb{E}[u(t_0, X_0)] - \mathbb{E}[u(t_N, Y_N)]. \tag{2.17}
\]
Again, by the definition \((2.11)\) and \((2.16)\),

\[
u(t, x) = \mathbb{E} [\varphi(X(t, x; T))] = \mathbb{E} [\varphi(X(s, X_{t,x}(s); T))]
= \mathbb{E} [\mathbb{E} (\varphi(X(s, X_{t,x}(s); T))|\sigma(X_{t,x}(s)))]
= \mathbb{E} [u(s, X_{t,x}(s))], \quad \text{for } 0 \leq t \leq s \leq T.
\]  

(2.18)

By taking \(t = t_i\), \(s = t_{i+1}\), a conditional version of such an equality

\[
u(t_i, Y_i) = \mathbb{E} [u(t_{i+1}, X_{t_i,Y_i}(t_{i+1})|\sigma(Y_i))],
\]  

(2.19)

implies

\[
\mathbb{E} [\mathbb{E} [u(t_i, Y_i)]] = \mathbb{E} [u(t_{i+1}, X(t_i, Y_i; t_{i+1}))], \quad i = 0, 1, ..., N - 1.
\]  

(2.20)

Using this repeatedly helps one to arrive at

\[
\mathbb{E} [u(t_0, X_0)] = \mathbb{E} [u(t_1, X(t_1))] - \mathbb{E} [u(t_1, Y_1)] + \mathbb{E} [u(t_2, X(t_1, Y_1; t_2))]
= \mathbb{E} [u(t_1, X(t_0, X_0; t_1))] - \mathbb{E} [u(t_1, Y_1)]
+ \mathbb{E} [u(t_2, X(t_1, Y_1; t_2))] - \mathbb{E} [u(t_2, Y_2)] + \mathbb{E} [u(t_3, X(t_2, Y_2; t_3))]
= \ldots
\]  

(2.21)

\[
= \sum_{i=0}^{N-2} \left( \mathbb{E} [u(t_{i+1}, X(t_i, Y_i; t_{i+1}))] - \mathbb{E} [u(t_{i+1}, Y_i; t_{i+1})] \right) + \mathbb{E} [u(t_{N-1}, Y_{N-1})].
\]

In view of \((2.20)\), one can infer

\[
\mathbb{E} [u(t_{N-1}, Y_{N-1})] - \mathbb{E} [u(t_N, Y_N)] = \mathbb{E} [u(t_N, X(t_{N-1}, Y_{N-1}; t_N))] - \mathbb{E} [u(t_N, Y_N)],
\]  

(2.22)

and thus one can derive from \((2.21)\) that

\[
\mathbb{E} [u(t_0, X_0)] - \mathbb{E} [u(t_N, Y_N)] = \sum_{i=0}^{N-1} \left( \mathbb{E} [u(t_{i+1}, X(t_i, Y_i; t_{i+1}))] - \mathbb{E} [u(t_{i+1}, Y_i; t_{i+1})] \right)
\]  

\[
= \sum_{i=0}^{N-1} \left( \mathbb{E} [u(t_{i+1}, X(t_i, Y_i; t_{i+1}))] - \mathbb{E} [u(t_{i+1}, Y(t_i, Y_i; t_{i+1}))] \right).
\]  

(2.23)

Finally, combining a conditional version of \((2.13)\) with \((2.23)\) yields

\[
|\mathbb{E} [\varphi(X(t_0, X_0; T))] - \mathbb{E} [\varphi(Y(t_0, Y_0; t_N))]| \leq \sum_{i=0}^{N-1} |\mathbb{E} [u(t_{i+1}, X(t_i, Y_i; t_{i+1}))] - \mathbb{E} [u(t_{i+1}, Y(t_i, Y_i; t_{i+1}))]| \leq \sum_{i=0}^{N-1} \mathbb{E} [|\nu(Y_i)|] h^{p+1} \leq \gamma(X_0) T h^p,
\]  

(2.24)
which validates the desired assertion (2.15).

It is worthwhile to point out that, the above fundamental weak convergence theorem, Theorem 2.4, is very general in the sense that it is formulated in a domain $D \subset \mathbb{R}^d$ and the test function $\phi$ is not required to be smooth. This setting enables Theorem 2.4 to work for SDE models with low regular coefficients or evolving in particular domains and also to work for discontinuous payoff functions $\phi$. Nevertheless, verifying Assumption 2.2 in the general setting is not an easy task and we shall give sufficient conditions for Assumption 2.2 to hold, which is sometimes restrict but convenient in practice. More precisely, we restrict ourselves into smooth test functions $\phi$ with polynomial growth, which fall into function spaces given by the following definition.

**Definition 2.5.** A function $\psi: \mathbb{R}^d \to \mathbb{R}$ is said to belong to the class $H$, written as $\psi \in H$, if there exist constants $L > 0, l > 0$ such that the following inequality holds

$$|\psi(x)| \leq L(1 + |x|^l).$$ (2.25)

For each integer $k \geq 1$ we also use $H^k$ to denote a subset of $H$, consisting of $k$-times continuously differentiable functions which, together with its partial derivatives up to and including order $k$, belong to $H$.

In addition, we give the following definition.

**Definition 2.6.** Let $\Psi: \Omega \times \mathbb{R}^d \to \mathbb{R}$ and $\phi_i: \Omega \times \mathbb{R}^d \to \mathbb{R}$ be random functions satisfying

$$\lim_{\tau \to 0} \mathbb{E}\left[\frac{1}{\tau} \left(\Psi(x + \tau e_i) - \Psi(x)\right) - \phi_i(x)\right]^2 = 0, \quad \forall i \in \{1, 2, \ldots, d\},$$ (2.26)

where $e_i \in \mathbb{R}^d$ is a unit vector in $\mathbb{R}^d$, with the $i$-th element being 1. Then $\Psi$ is called to be mean-square differentiable, with $\phi = (\phi_1, \phi_2, \ldots, \phi_d)$ being the derivative (in the mean-square differentiable sense) of $\Psi$ and we also write $\partial_i \Psi = \phi_i$.

We mention that, the above two definitions can be generalized to vector-valued functions in a component-wise manner.

**Assumption 2.7.** Assume that the SDE (2.12) admits a unique $\{\mathcal{F}_s\}_{s \in [t,T]}$-adapted $D$-valued solution $X(t, x; s), 0 \leq t \leq s \leq T$ such that for sufficiently large $q_0 \geq 2$

$$\sup_{0 \leq t \leq s \leq T} \mathbb{E}[|X(t, x; s)|^{q_0}] \leq C(1 + |x|^{q_0}).$$ (2.27)

Moreover, the solution $X(t, x; T)$ is $2p + 2$ times mean-square differentiable with respect to the initial data $x$ and for sufficiently large $q_1 \geq 2$ and for any $j = 1, \ldots, 2p + 2$ it holds that

$$\sup_{x \in D} \mathbb{E}[|D^j X(t, x; T)|^{q_1}] \leq C(T, q_1, j) < \infty,$$ (2.28)

where $C(T, q_1, j)$ is a constant depending on $T, q_1, j$. 


Assumption 2.8. There exist measurable functions $K_i : \mathcal{D} \to [0, \infty)$ for $i \in \{0, 1, 2\}$ such that, for any $i_j \in \{1, 2, \cdots, d\}$,
\[
\left| \mathbb{E} \left[ \prod_{j=1}^{s} (\delta_{X,x})^{i_j} \right] - \mathbb{E} \left[ \prod_{j=1}^{s} (\delta_{Y,x})^{i_j} \right] \right| \leq K_0(x) h^{p+1}, \quad s = 1, \ldots, 2p + 1, \quad (2.29)
\]
\[
\left\| \prod_{j=1}^{2p+2} (\delta_{X,x})^{i_j} \right\|_{L^2(\Omega; \mathbb{R})} \leq K_1(x) h^{p+1},\quad \left\| \prod_{j=1}^{2p+2} (\delta_{Y,x})^{i_j} \right\|_{L^2(\Omega; \mathbb{R})} \leq K_2(x) h^{p+1}, \quad (2.30)
\]

where we denote
\[(\delta_{X,x})^{i_j} := X^{i_j}(t, x; t + h) - x^{i_j}, \quad (\delta_{Y,x})^{i_j} := Y^{i_j}(t, x; t + h) - x^{i_j}, \quad i_j \in \{1, 2, \cdots, d\}. \quad (2.31)\]

Assumption 2.9. The numerical approximations $\{Y_n\}_{0 \leq n \leq N}$ produced by (2.8) always take values in $\mathcal{D} \subset \mathbb{R}^d$ and for some $q_2 \geq 0$, $\beta \geq 1$ it holds that
\[
\sup_{N \in \mathbb{N}} \sup_{0 \leq n \leq N} \mathbb{E} \|Y_n\|^{q_2} \leq C(1 + |Y_0|^\beta q_2). \quad (2.32)
\]

Lemma 2.10. Let $\varphi : \mathcal{D} \to \mathbb{R}$ be the test function with $\varphi \in H^{2p+2}$ and there exist constants $L > 0$ and $\chi \geq 0$ such that
\[
|D^j \varphi(x)| \leq L(1 + |x|^\chi), \quad j \in \{1, 2, \ldots, 2p + 2\}. \quad (2.33)
\]
Let Assumption 2.7 hold with $q_0 = 2\chi$, $q_1 = 4p + 4$, let Assumption 2.7 be fulfilled with $q_2 = 2\chi$, $\beta \geq 1$ and let Assumption 2.9 be satisfied. Then $u$ defined by (2.11) fulfills Assumption 2.2 with $\nu(x) = C(1 + |x|^\beta \chi)[K_0(x) + K_1(x) + K_2(x)]$.

Proof of Lemma 2.10. Since $\varphi(x)$ is differentiable and the solution $X(t, x; T)$ of the SDEs is mean-square differentiable with respect to the initial data $x$ by assumption, $u(t, x)$ is differentiable with respect to $x$ and
\[
\frac{\partial u(t, x)}{\partial x_i} = \frac{\partial \mathbb{E} \varphi(X(t, x; T))}{\partial x_i} = \mathbb{E} \left[ D \varphi(X(t, x; T)) \frac{\partial X(t, x; T)}{\partial x_i} \right] = \mathbb{E} \left[ \sum_{j=1}^{d} \partial_j \varphi(X(t, x; T)) \frac{\partial X_j(t, x; T)}{\partial x_i} \right]. \quad (2.34)
\]
Noting further that the partial derivative of $\varphi$ belongs to $H$ by assumption, one can use (2.28), (2.33) and the H"older inequality to deduce
\[
|\frac{\partial u(t, x)}{\partial x_i}| \leq \|D \varphi(X(t, x; T))\|_{L^2(\Omega; \mathbb{R}^d)} \|\frac{\partial X(t, x; T)}{\partial x_i}\|_{L^2(\Omega; \mathbb{R}^d)} \leq C(1 + |x|^\chi). \quad (2.35)
\]
For higher order derivatives, we recall that $\varphi$'s partial derivatives of order up to $2p + 2$ inclusively, belong to $H$ and the solution of the SDE is $2p + 2$ times mean-square differentiable with respect to the initial data $x$. These facts together with the chain rule enable us to similarly show that $u(t, x)$ is $2p + 2$ times differentiable with respect to $x$ and
\[
|\frac{\partial^k u(t, x; h)}{\partial x_{i_1} \cdots \partial x_{i_k}}| \leq C(1 + |x|^\chi), \quad k = 2, 3, \cdots, 2p + 2. \quad (2.36)
\]
By virtue of the Taylor expansion [14], we get
\[
\mathbb{E} \left[ u(t + h, X(t, x; t + h)) - u(t, h, Y(t, x; t + h)) \right] = \sum_{k=1}^{2p+1} \sum_{i_1, \ldots, i_k=1}^{d} \frac{1}{k!} \frac{\partial^k u(t + h, x)}{\partial x_{i_1} \cdots \partial x_{i_k}} \mathbb{E} \left[ \prod_{j=1}^{k} (\delta_{X,x})^{i_j} - \prod_{j=1}^{k} (\delta_{Y,x})^{i_j} \right] + \mathbb{E}[R_{2p+2}],
\]
where we used the notation (2.31) and denote
\[
(\delta_{X,x})^\alpha := \prod_{j=1}^{2p+2} (\delta_{X,x})^{i_j}, \quad (\delta_{Y,x})^\alpha := \prod_{j=1}^{2p+2} (\delta_{Y,x})^{i_j}, \quad \alpha = (i_1, i_2, \ldots, i_{2p+2})
\]
and
\[
R_{2p+2} := \sum_{|\alpha|=2p+2} \left| \frac{\alpha!}{\alpha!} \left( \int_0^1 (1-s)^{|\alpha|} - 1 \right) D^\alpha u(t + h, x + s\delta_{X,x}) ds \delta_{X,x}^\alpha \right|
\]
Combining Theorem 2.4 with Lemma 2.10 gives the following corollary.

Using the H"{o}lder inequality, (2.27), (2.32) and (2.36) one can derive
\[
|\mathbb{E}[R_{2p+2}]| \leq \sum_{|\alpha|=2p+2} C \int_0^1 \left\| D^\alpha u(t + h, (1-s)x + sX(t, x; t + h)) \right\|_{L^2(\Omega; \mathbb{R})} ds \left\| \delta_{X,x}^\alpha \right\|_{L^2(\Omega; \mathbb{R})}
\]
\[
+ \sum_{|\alpha|=2p+2} C \int_0^1 \left\| D^\alpha u(t + h, (1-s)x + sY(t, x; t + h)) \right\|_{L^2(\Omega; \mathbb{R})} ds \left\| \delta_{Y,x}^\alpha \right\|_{L^2(\Omega; \mathbb{R})}
\]
\[
\leq \sum_{|\alpha|=2p+2} C \left[ (1 + |x|^\chi + \|X(t, x; t + h)\|_{L^{2\chi}(\Omega; \mathbb{R})}) K_1(x) h^{p+1} \right]
\]
\[
+ \sum_{|\alpha|=2p+2} C \left[ (1 + |x|^\chi + \|Y(t, x; t + h)\|_{L^{2\chi}(\Omega; \mathbb{R})}) K_2(x) h^{p+1} \right]
\]
\[
\leq C \left[ (1 + |x|^\beta \chi) [K_1(x) + K_2(x)] h^{p+1}, \right]
\]
where we used the assumption \(q_0 = 2\chi, q_1 = 4p + 4, \) and \(q_2 = 2\chi, \beta \geq 1.\) Plugging (2.39) and (2.40) into (2.37) enables us to get
\[
|\mathbb{E}[\varphi(X(t, x; t + h))] - \mathbb{E}[\varphi(Y(t, x; t + h))]| \leq C(1 + |x|^{\beta \chi}) [K_0(x) + K_1(x) + K_2(x)] h^{p+1}
\]
(2.41)

The proof of the lemma is thus complete.

Combining Theorem 2.4 with Lemma 2.10 gives the following corollary.
Corollary 2.11. Let \( \varphi : \mathcal{D} \to \mathbb{R} \) be the test function with \( \varphi \in \mathbb{H}^{2p+2} \) and there exist constants \( L > 0 \) and \( \chi \geq 0 \) such that

\[
|D^j \varphi(x)| \leq L(1 + |x|^{\chi}), \quad j \in \{1, 2, \ldots, 2p + 2\}, \forall x \in \mathcal{D}.
\] (2.42)

Let Assumptions 2.7, 2.8, 2.9 be satisfied with \( q_0 = 2\chi, q_1 = 4p + 4, q_2 = 2\chi \) and some \( \beta \geq 1 \) and let Assumption 2.3 be satisfied with \( \nu(x) = C(1 + |x|^{\beta})[K_0(x) + K_1(x) + K_2(x)] \) and some positive function \( \Upsilon \). Then the numerical solution produced by (2.8) has a global weak convergence rate of order \( p \), namely,

\[
|\mathbb{E}[\varphi(X(t_0, X_0; T))] - \mathbb{E}[\varphi(Y(t_0, Y_0; t_N))]| \leq \Upsilon(X_0)Th^p.
\] (2.43)

We remark that Assumption 2.9 always remains true when derivatives of the test function \( \varphi \) are bounded, i.e., \( \chi = 0 \) in (2.42). So in this case Assumption 2.9 is not needed. In Assumptions 2.7, it is assumed that the underlying SDEs are well-posed with bounded moments and that the solution is mean-square differentiable with respect to the initial data. However, it is not clear how such assumptions can be satisfied. In what follows, we take \( \mathcal{D} = \mathbb{R}^d \) and put some assumptions on the drift and diffusion coefficients, to ensure that conditions in Assumptions 2.7 are all fulfilled, allowing for super-linearly growing coefficients.

Assumption 2.12. Suppose that

(1) the drift coefficient function \( f \in C^{2p+2}(\mathbb{R}^d; \mathbb{R}^d) \) and there exists \( r \geq 0 \) such that

\[
\sup_{x \in \mathbb{R}^d} \frac{|D^\alpha f(x)|}{1 + |x|^{2r+1-\gamma}} < \infty, \quad |\alpha| = j, \quad j \in \{0, 1, \ldots, 2p + 2\};
\] (2.44)

(2) the diffusion coefficient function \( g \in C^{2p+2}(\mathbb{R}^d; \mathbb{R}^{d \times m}) \) and there exists \( \rho \leq r \) such that

\[
\sup_{x \in \mathbb{R}^d} \frac{\|D^\alpha g(x)\|}{1 + |x|^{2\rho}} < \infty, \quad |\alpha| = j, \quad j \in \{0, 1, \ldots, 2p + 2\};
\] (2.45)

(3) for all \( \lambda > 0 \), there exists \( C_\lambda \in \mathbb{R} \) such that

\[
\langle Df(x)y, y \rangle + \lambda \|Dg(x)y\|^2 \leq C_\lambda |y|^2, \quad \forall x, y \in \mathbb{R}^d;
\] (2.46)

(4) there exist \( a_1 > 0 \) and \( \gamma, c_1 > 0 \) such that for any \( x, y \in \mathbb{R}^d \) it holds:

\[
\langle f(x + y) - f(x), y \rangle \leq -a_1 |y|^{2\gamma + 2} + c_1 (|x|^{\gamma} + 1).
\] (2.47)

The condition (2.44) in Assumption 2.12 immediately implies

\[
|f(x) - f(y)| \leq C(1 + |x|^{2\rho} + |y|^{2\rho})|x - y|, \quad \forall x, y \in \mathbb{R}^d,
\] (2.48)

which gives the polynomial growth

\[
|f(x)| \leq C(1 + |x|^{2\rho + 1}), \quad \forall x \in \mathbb{R}^d.
\] (2.49)

Similarly, (2.45) immediately yields

\[
\|g(x) - g(y)\| \leq C(1 + |x|^{\rho - 1} + |y|^{\rho - 1})|x - y|, \quad \forall x, y \in \mathbb{R}^d,
\] (2.50)

and thus

\[
\|g(x)\| \leq C(1 + |x|^\rho), \quad \forall x \in \mathbb{R}^d.
\] (2.51)

Additionally, we would like to add some comments here.
Remark 2.13. The condition (2.46) in Assumption 2.12 implies that for any \( \lambda > 0 \) and \( x, y \in \mathbb{R}^d \),
\[
\langle x - y, f(x) - f(y) \rangle + \lambda \|g(x) - g(y)\|^2 \leq C_\lambda |x - y|^2.
\] (2.52)
Moreover, by virtue of (2.52), for any \( \lambda > 0 \), one can obtain that
\[
\langle x, f(x) \rangle + \lambda \|g(x)\|^2 \leq C(1 + |x|^2) \quad \forall x \in \mathbb{R}^d.
\] (2.53)

By [10, Theorem 1.3.6], one can get the following regularity results.

Lemma 2.14. Let Assumption 2.12 be fulfilled. Then the SDEs (2.12) admits a unique adapted solution \( X(t, x; s), 0 \leq t \leq s \leq T \) in \( \mathcal{D} = \mathbb{R}^d \), which is \( 2p + 2 \) times mean-square differentiable with respect to the initial data \( x \in \mathbb{R}^d \) and for any \( q \geq 1 \) and \( j = 1, \ldots, 2p + 2 \) it holds that
\[
\sup_{x \in \mathbb{R}^d} \mathbb{E}[|D^jX(t, x; s)|^q] \leq \phi_{j,q}(s - t) \leq C(T, q, j),
\] (2.54)
for suitable continuous increasing functions \( \phi_{j,q} \), where \( C(T, q, j) \) is a constant that depend on \( T, q, j \). Moreover, for any \( q \geq 1 \) it holds that
\[
\sup_{0 \leq t \leq s \leq T} \mathbb{E}[|X(t, x; s)|^q] \leq C(T, q)(1 + |x|^q),
\] (2.55)
and for any \( t, s_1, s_2 \in [0, T], \gamma \geq 1 \), we have
\[
\|X(t, x; s_2) - X(t, x; s_1)\|_{L^\gamma(\Omega; \mathbb{R}^d)} \leq C(1 + |x|^{2r+1})|s_2 - s_1|^\frac{1}{\gamma}.
\] (2.56)

As a direct consequence of (2.48), (2.50), (2.55) and (2.56), we have, for any \( t, s_1, s_2 \in [0, T] \),
\[
\|f(X(t, x; s_2)) - f(X(t, x; s_1))\|_{L^\gamma(\Omega; \mathbb{R}^d)} \leq C(1 + |x|^{4r+1})|s_2 - s_1|^\frac{1}{\gamma}\\
\|g(X(t, x; s_2)) - g(X(t, x; s_1))\|_{L^\gamma(\Omega; \mathbb{R}^{d \times m})} \leq C(1 + |x|^{2r+\rho})|s_2 - s_1|^\frac{1}{\gamma}.
\] (2.57)
Moreover, the above lemma guarantees that all conditions in Assumption 2.7 hold for any \( q_0, q_1 \geq 2 \). As a direct consequence of Theorem 2.4 and Lemma 2.14 one can immediately get the following weak convergence result.

Corollary 2.15. Let \( \mathcal{D} = \mathbb{R}^d \) and let \( \varphi : \mathbb{R}^d \to \mathbb{R} \) be the test function with \( \varphi \in \mathbb{H}^{2p+2} \) and there exist constants \( L > 0 \) and \( \chi \geq 0 \) such that
\[
|D^j\varphi(x)| \leq L(1 + |x|^\chi), \quad j \in \{1, 2, \ldots, 2p + 2\}, x \in \mathbb{R}^d.
\] (2.58)

Let Assumption 2.7 be satisfied with \( q_2 = 2\chi \), \( \beta \geq 1 \) and let Assumption 2.8 be satisfied with \( \nu(x) = C(1 + |x|^{\beta \chi})[K_0(x) + K_1(x) + K_2(x)] \) and some positive function \( \Upsilon \). Let Assumptions 2.8, 2.12 hold. Then the numerical solution produced by (2.8) has a global weak convergence rate of order \( p \), namely,
\[
|\mathbb{E}[\varphi(X(t_0, X_0; T))] - \mathbb{E}[\varphi(Y(t_0, Y_0; t_N))]| \leq \Upsilon(X_0)Th^p.
\] (2.59)
3 Weak convergence rate of a tamed Euler method

In this section, we shall apply the previously obtained weak convergence theorem to derive the weak convergence rate of a tamed Euler scheme under Assumption 2.12. Given a uniform mesh on \([0, T]\) with \(h = \frac{T}{N} > 0\) being the uniform step-size, the tamed Euler scheme considered here is given by

\[
Y_{n+1} = Y_n + \frac{1}{1 + h|f(Y_n)|^2} \left[ f(Y_n) h + g(Y_n) \Delta W_n \right], \quad Y_0 = X_0,
\]

(3.1)

where \(\Delta W_n := W(t_{n+1}) - W(t_n), \ n \in \{0, 1, 2, \ldots, N - 1\}\). Such a type of tamed Euler scheme is originally proposed by Hutzenthaler, Jentzen and Kloeden [17], to numerically solve SDEs with superlinearly growing coefficients. Thereafter, various variants of tamed schemes were extensively developed and analyzed in the literature [16, 17, 30, 31]. We mention that the scheme (3.1) is different from the tamed Euler schemes in [30, 31], where the taming factor only allows for a presence of \(h^\alpha\) with \(\alpha \leq \frac{1}{2}\) (see, e.g., (5.2)). To apply the fundamental weak convergence theorem, the first step is to obtain the boundedness of the moments of the numerical approximations.

3.1 Bounded moments of the tamed Euler method

This part is devoted to the boundedness of high-order moments of the tamed Euler method. Before coming to the bounded moments of numerical solutions, we introduce the following functions

\[
\bar{f}_h(x) := \frac{f(x)}{1 + h|f(x)|^2}, \quad \bar{g}_h(x) := \frac{g(x)}{1 + h|f(x)|^2}, \quad \forall x \in \mathbb{R}^d,
\]

(3.2)

which satisfy the following properties.

**Lemma 3.1.** Under the condition (2.53), the functions \(\bar{f}_h\) and \(\bar{g}_h\) obey

\[
|\bar{f}_h(x)|^2 \leq h^{-1}, \quad \forall x \in \mathbb{R}^d,
\]

(3.3)

\[
\|\bar{g}_h(x)\|^4 \leq C(1 + |x|^4 + h^{-1}|x|^2), \quad \forall x \in \mathbb{R}^d.
\]

(3.4)

**Proof of Lemma 3.1.** From the definition of the function \(\bar{f}_h(x)\), one can deduce that

\[
|\bar{f}_h(x)|^2 = \frac{|f(x)|^2}{(1 + h|f(x)|^2)^2} \leq \frac{h^{-1} + |f(x)|^2}{1 + h|f(x)|^2} = h^{-1},
\]

(3.5)

which validates (3.3). To treat (3.4), we first use (2.53) and the Cauchy-Schwarz inequality to get

\[
\|g(x)\|^2 \leq C(1 + |x|^2 + |x||f(x)|).
\]

(3.6)

This helps one to infer that

\[
\|\bar{g}_h(x)\|^4 \leq \frac{\|g(x)\|^4}{1 + h|f(x)|^2} \leq \frac{C(1 + |x|^4 + |x||f(x)|^2)}{1 + h|f(x)|^2} \leq C(1 + |x|^4 + h^{-1}|x|^2),
\]

(3.7)

as required.

In addition, we introduce the notation

\[
[t]_h := t_i, \quad \text{for } t \in \{t_i, t_{i+1}\}, \quad i \in \{0, 1, \ldots, N - 1\},
\]

(3.8)
and define a continuous version of the tamed Euler scheme (3.1) as,

\[ Y_t = Y_0 + \int_0^t \tilde{f}_h(Y_{[s]}) \, ds + \int_0^t \tilde{g}_h(Y_{[s]}) \, dW(s), \quad t \in [0, T]. \]  

(3.9)

According to (3.3) and (3.4), one can easily show \( \sup_{0 \leq t \leq T} E[(1 + |Y_t|^2)^{\frac{q}{2}}] < \infty \) for any \( q \geq 2 \) by iteration. But the moment bounds depend on the inverse of the step size \( h \). Next, we shall improve the moment bounds for the tamed Euler scheme.

**Theorem 3.2.** Let Assumption 2.12 be satisfied with \( p = 1 \). Then for any \( q \geq 4 \) the approximation process \( Y_t \) produced by (3.9) obeys

\[ \sup_{t \in [0, T]} E[|Y_t|^q] \leq C(1 + |X_0|^q). \]  

(3.10)

**Proof of Theorem 3.2.** Applying the Itô formula yields

\[ (1 + |Y_t|^2)^{\frac{q}{2}} = (1 + |Y_0|^2)^{\frac{q}{2}} + q \int_0^t (1 + |Y_s|^2)^{\frac{q}{2} - 1} \langle Y_s, \tilde{f}_h(Y_{[s]}) \rangle \, ds \]

\[ + \frac{q}{2} \int_0^t (1 + |Y_s|^2)^{\frac{q}{2} - 1} \| \tilde{g}_h(Y_{[s]}) \|^2 \, ds \]

\[ + \frac{q(q-2)}{2} \int_0^t (1 + |Y_s|^2)^{\frac{q}{2} - 2} |Y_s^T \tilde{g}_h(Y_{[s]})|^2 \, ds \]

\[ + q \int_0^t (1 + |Y_s|^2)^{\frac{q}{2} - 1} \langle Y_s, \tilde{g}_h(Y_{[s]}) \rangle \, dW(s), \]  

(3.11)

which straightforwardly implies

\[ (1 + |Y_t|^2)^{\frac{q}{2}} \leq (1 + |Y_0|^2)^{\frac{q}{2}} + q \int_0^t (1 + |Y_s|^2)^{\frac{q}{2} - 1} \langle Y_s, \tilde{f}_h(Y_{[s]}) \rangle \, ds \]

\[ + \frac{q(q-1)}{2} \int_0^t (1 + |Y_s|^2)^{\frac{q}{2} - 1} \| \tilde{g}_h(Y_{[s]}) \|^2 \, ds \]

\[ + q \int_0^t (1 + |Y_s|^2)^{\frac{q}{2} - 1} \langle Y_s, \tilde{g}_h(Y_{[s]}) \rangle \, dW(s). \]  

(3.12)

After taking expectation, we make a further decomposition as follows:

\[ E[(1 + |Y_t|^2)^{\frac{q}{2}}] \leq (1 + |Y_0|^2)^{\frac{q}{2}} + qE\left[ \int_0^t (1 + |Y_s|^2)^{\frac{q}{2} - 1} \langle Y_s - Y_{[s]}, \tilde{f}_h(Y_{[s]}) \rangle \, ds \right] \]

\[ + qE\left[ \int_0^t (1 + |Y_s|^2)^{\frac{q}{2} - 1} \langle Y_s - Y_{[s]}, \tilde{f}_h(Y_{[s]}) \rangle \, ds \right] \]

\[ + \frac{q(q-1)}{2}E\left[ \int_0^t (1 + |Y_s|^2)^{\frac{q}{2} - 1} \| \tilde{g}_h(Y_{[s]}) \|^2 \, ds \right] \]

\[ = (1 + |Y_0|^2)^{\frac{q}{2}} + qE\left[ \int_0^t (1 + |Y_s|^2)^{\frac{q}{2} - 1} \langle Y_s - Y_{[s]}, \tilde{f}_h(Y_{[s]}) \rangle \, ds \right] \]

\[ + \frac{q}{2}E\left[ \int_0^t (1 + |Y_s|^2)^{\frac{q}{2} - 1} (2 \langle Y_{[s]}, \tilde{f}_h(Y_{[s]}) \rangle + (q-1) \| \tilde{g}_h(Y_{[s]}) \|^2) \, ds \right] \]

\[ = (1 + |Y_0|^2)^{\frac{q}{2}} + J_1 + J_2. \]
In what follows, we bound these two terms $J_1, J_2$ separately. By the definition of $Y_t$ and using the Cauchy-Schwarz inequality and \((3.3)\), the term $J_1$ can be treated in the following way:

\[
J_1 = q\mathbb{E}\left[\int_0^t (1 + |Y_s|^2)^{\frac{q}{2} - 1} \left(\int_{|s|_h}^{s_0} \tilde{f}_h(Y_{[r],h}) \, dr, \tilde{f}_h(Y_{[s],h}) \right) \, ds \right]
\]

\[
+ q\mathbb{E}\left[\int_0^t (1 + |Y_s|^2)^{\frac{q}{2} - 1} \left(\int_{|s|_h}^{s_0} \tilde{g}_h(Y_{[r],h}) \, dW(r), \tilde{f}_h(Y_{[s],h}) \right) \, ds \right]
\]

\[
\leq q\mathbb{E}\left[\int_0^t (1 + |Y_s|^2)^{\frac{q}{2} - 1} \left|\tilde{f}_h(Y_{[r],h})\right| \, d\left|\tilde{f}_h(Y_{[s],h})\right| \, ds \right]
\]

\[
+ q\mathbb{E}\left[\int_0^t (1 + |Y_s|^2)^{\frac{q}{2} - 1} \left(\int_{|s|_h}^{s_0} \tilde{g}_h(Y_{[r],h}) \, dW(r), \tilde{f}_h(Y_{[s],h}) \right) \, ds \right]
\]

\[
\leq q\mathbb{E}\left[\int_0^t (1 + |Y_s|^2)^{\frac{q}{2} - 1} (s - |s|_h)h^{-1} \, ds \right]
\]

\[
+ q\mathbb{E}\left[\int_0^t ((1 + |Y_s|^2)^{\frac{q}{2} - 1} - (1 + |Y_{[s],h}|^2)^{\frac{q}{2} - 1}) \left(\int_{|s|_h}^{s_0} \tilde{g}_h(Y_{[r],h}) \, dW(r), \tilde{f}_h(Y_{[s],h}) \right) \, ds \right].
\]

where by Itô’s formula we have

\[
(1 + |Y_s|^2)^{\frac{q}{2} - 1} = (1 + |Y_{[s],h}|^2)^{\frac{q}{2} - 1} + (q - 2) \int_{|s|_h}^{s_0} (1 + |Y_r|^2)^{\frac{q}{2} - 2} \langle Y_r, \tilde{f}_h(Y_{[r],h}) \rangle \, dr
\]

\[
+ (q - 2) \int_{|s|_h}^{s_0} (1 + |Y_r|^2)^{\frac{q}{2} - 2} \langle Y_r, \tilde{g}_h(Y_{[r],h}) dW(r) \rangle
\]

\[
+ \frac{(q-2)(q-4)}{2} \int_{|s|_h}^{s_0} (1 + |Y_r|^2)^{\frac{q}{2} - 3} |Y_r^T \tilde{g}_h(Y_{[r],h})|^2 \, dr
\]

\[
+ \frac{(q-2)}{2} \int_{|s|_h}^{s_0} (1 + |Y_r|^2)^{\frac{q}{2} - 2} \|\tilde{g}_h(Y_{[r],h})\|^2 \, dr.
\]

Plugging \((3.15)\) into \((3.14)\) helps us to split $J_1$ into four additional terms:

\[
J_1 \leq C\mathbb{E}\left[\int_0^t (1 + |Y_s|^2)^{\frac{q}{2}} \, ds \right]
\]

\[
+ q(q - 2)\mathbb{E}\left[\int_0^t \int_{|s|_h}^{s_0} (1 + |Y_r|^2)^{\frac{q}{2} - 2} \langle Y_r, \tilde{f}_h(Y_{[r],h}) \rangle \, dr \left\langle\int_{|s|_h}^{s_0} \tilde{g}_h(Y_{[r],h}) \, dW(r), \tilde{f}_h(Y_{[s],h}) \right\rangle \, ds \right]
\]

\[
+ q(q - 2)\mathbb{E}\left[\int_0^t \int_{|s|_h}^{s_0} (1 + |Y_r|^2)^{\frac{q}{2} - 2} \langle Y_r, \tilde{g}_h(Y_{[r],h}) \, dW(r) \rangle \left\langle\int_{|s|_h}^{s_0} \tilde{g}_h(Y_{[r],h}) \, dW(r), \tilde{f}_h(Y_{[s],h}) \right\rangle \, ds \right]
\]

\[
+ \frac{q(q-2)(q-3)}{2}\mathbb{E}\left[\int_0^t \int_{|s|_h}^{s_0} (1 + |Y_r|^2)^{\frac{q}{2} - 2} \|\tilde{g}_h(Y_{[r],h})\|^2 \, dr \left\langle\int_{|s|_h}^{s_0} \tilde{g}_h(Y_{[r],h}) \, dW(r), \tilde{f}_h(Y_{[s],h}) \right\rangle \, ds \right]
\]

\[
= J_{11} + J_{12} + J_{13} + J_{14}.
\]

\[(3.16)\]
Next we estimate $J_{12}, J_{13}, J_{14}$ separately. Using the Cauchy-Schwarz inequality, \((3.3)\), the Young inequality, the moment inequality [22, Theorem 7.1] and \((3.4)\) leads to

\[
J_{12} \leq q(q - 2) \mathbb{E} \left[ \int_{0}^{t} \int_{s}^{t} (1 + |Y_r|^2)^{\frac{4}{2q-2}} |r_{Y_r}| \, dr \left| \int_{s}^{t} \bar{g}_h(Y_{r|h}) \, dW(r) \right| \, ds \right] \\
\leq q(q - 2) h^{-1} \mathbb{E} \left[ \int_{0}^{t} \int_{s}^{t} (1 + |Y_r|^2)^{\frac{4}{2q-2}} \, dr \left| \int_{s}^{t} \bar{g}_h(Y_{r|h}) \, dW(r) \right| \, ds \right] \\
= q(q - 2) \mathbb{E} \left[ \int_{0}^{t} h^{-\frac{4}{2q-2}} \int_{s}^{t} (1 + |Y_r|^2)^{\frac{4}{2q-2}} \, dr \cdot h^{-\frac{4}{2q-2}} \left| \int_{s}^{t} \bar{g}_h(Y_{r|h}) \, dW(r) \right| \, ds \right] \\
\leq (q - 1)(q - 2) h^{-\frac{3q}{4(q-1)}} \mathbb{E} \left[ \int_{0}^{t} (1 + |Y_r|^2)^{\frac{4}{2q-2}} \, dr \right]^{2q-2} \, ds \\
+ (q - 2) h^{-\frac{4}{2q-2}} \mathbb{E} \left[ \int_{0}^{t} \int_{s}^{t} \bar{g}_h(Y_{r|h}) \, dW(r) \right]^{q} \, ds \\
\leq (q - 1)(q - 2) h^{-\frac{3q}{4(q-1)}} \mathbb{E} \left[ \sup_{0 \leq r \leq s} \left| (1 + |Y_r|^2)^{\frac{4}{2q-2}} \right|^{\frac{q}{2q-2}} \, ds \right] \\
+ (q - 2) \left( \frac{4(q-1)}{2q-2} \right) h^{-\frac{3q}{4(q-1)}} \mathbb{E} \left[ \int_{0}^{t} \int_{s}^{t} \bar{g}_h(Y_{r|h}) \, dW(r) \right]^{q} \, ds \\
\leq (q - 1)(q - 2) h^{-\frac{3q}{4(q-1)}} \mathbb{E} \left[ \sup_{0 \leq r \leq s} \left| (1 + |Y_r|^2)^{\frac{4}{2q-2}} \right|^{\frac{q}{2q-2}} \, ds \right] \\
+ C(q - 2) \left( \frac{4(q-1)}{2q-2} \right) h^{-\frac{3q}{4(q-1)}} \mathbb{E} \left[ \sup_{0 \leq r \leq s} \left( (1 + |Y_r|^4 + h^{-1}|Y_r|^2)^{\frac{4}{2q-2}} \right) \, ds \right] \\
\leq C \int_{0}^{t} \sup_{0 \leq r \leq s} \mathbb{E} \left[ (1 + |Y_r|^2)^{\frac{4}{2q-2}} \right] \, ds.
\]

(3.17)

Also, using the generalized Itô isometry (see, e.g., [38, Theorem 2.3.4], [37, P144]), \((3.3)\), \((3.4)\) and
Finally, we similarly utilize (3.3), the Young inequality, the Hölder inequality, the Burkholder-

\[ J_{13} = q(q - 2) \mathbb{E} \left[ \int_0^t \int_{[s]_h} (1 + |Y_r|^2)^{\frac{q}{2} - 2} \left< Y_r, \tilde{g}_h(Y_{[r]_l}) \right> \left< \int_{[s]_h} \tilde{g}_h(Y_{[r]_l}) dW(r), \tilde{f}_h(Y_{[s]_l}) \right> ds \right] 
\]
\[ = q(q - 2) \mathbb{E} \left[ \int_0^t \int_{[s]_h} (1 + |Y_r|^2)^{\frac{q}{2} - 2} \left< Y_r, \sum_{j=1}^m \tilde{g}_h(Y_{[r]_l}) \right> dW^j(r) \right] 
\times \left< \int_{[s]_h} \sum_{l=1}^m \tilde{g}_h(Y_{[r]_l}) dW^l(r), \tilde{f}_h(Y_{[s]_l}) \right> ds 
\]
\[ = q(q - 2) \int_0^t \sum_{j=1}^m \mathbb{E} \left[ \int_{[s]_h} (1 + |Y_r|^2)^{\frac{q}{2} - 2} \left< Y_r, \tilde{g}_h(Y_{[r]_l}) \right> dW^j(r) \right] ds 
\times \int_{[s]_h} \left< \tilde{g}_h(Y_{[r]_l}), \tilde{f}_h(Y_{[s]_l}) \right> dW^j(r) ds 
\]
\[ = q(q - 2) \int_0^{t} \sum_{j=1}^m \mathbb{E} \left[ \int_{[s]_h} (1 + |Y_r|^2)^{\frac{q}{2} - 2} \left< Y_r, \tilde{g}_h(Y_{[r]_l}) \right> \left< \tilde{g}_h(Y_{[r]_l}), \tilde{f}_h(Y_{[s]_l}) \right> dr \right] ds 
\leq q(q - 2) \int_0^t \mathbb{E} \left[ \int_{[s]_h} (1 + |Y_r|^2)^{\frac{q}{2} - 2} |Y_r||\tilde{g}_h(Y_{[r]_l})|^2 |\tilde{f}_h(Y_{[s]_l})| dr \right] ds 
\leq q(q - 2) h^{-\frac{1}{2}} \int_0^t \mathbb{E} \left[ \int_{[s]_h} (1 + |Y_r|^2)^{\frac{q}{2} - 2} |\tilde{g}_h(Y_{[r]_l})|^2 dr \right] ds 
\]
\[ = q(q - 2) \int_0^t \mathbb{E} \left[ \int_{[s]_h} h^{-\frac{q-2}{2}} (1 + |Y_r|^2)^{\frac{q}{2} - 2} h^{\frac{q-4}{2}} |\tilde{g}_h(Y_{[r]_l})|^2 dr \right] ds 
\leq (q - 2)^2 \int_0^t \mathbb{E} \left[ \int_{[s]_h} h^{-1} (1 + |Y_r|^2)^{\frac{q}{2}} dr \right] ds + \frac{q-2}{2} \int_0^t \mathbb{E} \left[ \int_{[s]_h} h^{\frac{q-4}{2}} |\tilde{g}_h(Y_{[r]_l})|^q dr \right] ds 
\leq (q - 2)^2 \int_0^t \sup_{0 \leq r \leq s} \mathbb{E} \left[ (1 + |Y_r|^2)^{\frac{q}{2}} \right] ds 
+ C \int_0^t \mathbb{E} \left[ \int_{[s]_h} h^{\frac{q-4}{2}} (1 + |Y_{[r]_l}|^4 + h^{-1}|Y_{[r]_l}|^2)^{\frac{q}{2}} dr \right] ds 
\leq C \int_0^t \sup_{0 \leq r \leq s} \mathbb{E} \left[ (1 + |Y_r|^2)^{\frac{q}{2}} \right] ds, 
\]

(3.18)
Davis-Gundy-type inequality and (3.4) to bound $J_{14}$ as follows,

$$
J_{14} \leq \frac{q(q-2)(q-3)}{2} h^{-\frac{1}{2}} \mathbb{E}\left[ \int_0^t \left( \int_{[s,t]} \left( 1 + |Y_r|^2 \right)^{\frac{q-3}{2}} \left\| \bar{g}_h(Y_{[r,t]}) \right\|^2 dr \right) \int_{[s,t]} \bar{g}_h(Y_{[r,t]}) dW(r) \right] ds
$$

$$
= \frac{q(q-2)(q-3)}{2} h^{-\frac{1}{2}} \mathbb{E}\left[ \int_0^t h^{-\frac{1}{2}} \left( \int_{[s,t]} \left( 1 + |Y_r|^2 \right)^{\frac{q-3}{2}} \left\| \bar{g}_h(Y_{[r,t]}) \right\|^2 dr \right) \frac{1}{h^{-\frac{1}{2}}} \int_{[s,t]} \bar{g}_h(Y_{[r,t]}) dW(r) \right] ds
$$

$$
\leq \frac{(q-2)^2(q-3)^2}{2} h^{-\frac{1}{2}} \mathbb{E}\left[ \int_0^t \left( \int_{[s,t]} \left( 1 + |Y_r|^2 \right)^{\frac{q-3}{2}} \left\| \bar{g}_h(Y_{[r,t]}) \right\|^2 dr \right) \frac{1}{h^{-\frac{1}{2}}} ds \right]
$$

$$
+ C_{q} h^{-\frac{3}{2}} \mathbb{E}\left[ \int_0^t \left( \int_{[s,t]} \left\| \bar{g}_h(Y_{[r,t]}) \right\|^2 dr \right) \frac{1}{h^{-\frac{1}{2}}} ds \right]
$$

$$
\leq \frac{(q-2)^2(q-3)}{2} h^{-\frac{1}{2}} \mathbb{E}\left[ \int_0^t \left( \int_{[s,t]} \left( 1 + |Y_r|^2 \right)^{\frac{q-3}{2}} \left\| \bar{g}_h(Y_{[r,t]}) \right\|^2 dr \right) \frac{1}{h^{-\frac{1}{2}}} ds \right]
$$

$$
+ C_{q} h^{-\frac{3}{2}} \mathbb{E}\left[ \int_0^t \left( \int_{[s,t]} \left( 1 + |Y_r|^4 + h^{-1}|Y_r|^2 \right) \frac{1}{h^{-\frac{1}{2}}} dr \right) \frac{1}{h^{-\frac{1}{2}}} ds \right]
$$

$$
\leq \frac{(q-2)^2(q-3)}{2} h^{-\frac{1}{2}} \mathbb{E}\left[ \int_0^t \left( \int_{[s,t]} \left( 1 + |Y_r|^4 + h^{-1}|Y_r|^2 \right) \frac{1}{h^{-\frac{1}{2}}} dr \right) \frac{1}{h^{-\frac{1}{2}}} ds \right]
$$

$$
+ (q-2)(q-3) \int_0^t \sup_{0 \leq r \leq s} \mathbb{E}\left[ (1 + |Y_r|^4 + h^{-1}|Y_r|^2) \frac{1}{h^{-\frac{1}{2}}} \right] ds
$$

$$
+ C_{q} \int_0^t \sup_{0 \leq r \leq s} \mathbb{E}\left[ (1 + |Y_r|^4 + h^{-1}|Y_r|^2) \frac{1}{h^{-\frac{1}{2}}} \right] ds
$$

$$
\leq C \int_0^t \sup_{0 \leq r \leq s} \mathbb{E}\left[ (1 + |Y_r|^2) \frac{1}{h^{-\frac{1}{2}}} \right] ds.
$$

(3.19)

This together with (3.17), (3.18) enables us to arrive at

$$
J_1 \leq C \int_0^t \sup_{0 \leq r \leq s} \mathbb{E}\left[ (1 + |Y_r|^2) \frac{1}{h^{-\frac{1}{2}}} \right] ds.
$$

(3.20)
With regard to $J_2$, due to \((2.33)\) one can easily show that
\[
J_2 = \frac{q}{2} \mathbb{E} \left[ \int_0^t (1 + |Y_s|^2)^{\frac{q}{2} - 1} \left( 2\langle Y_{[s]}, f(Y_{[s]}) \rangle + (q - 1) \frac{g(Y_{[s]})}{1 + |f(Y_{[s]})|^2h} \right) ds \right]
\]
\[
\leq \frac{q}{2} \mathbb{E} \left[ \int_0^t (1 + |Y_s|^2)^{\frac{q}{2} - 1} \frac{2\langle Y_{[s]}, f(Y_{[s]}) \rangle + (q - 1)||g(Y_{[s]})||^2}{1 + |f(Y_{[s]})|^2h} ds \right]
\]
\[
\leq C \mathbb{E} \left[ \int_0^t (1 + |Y_s|^2)^{\frac{q}{2} - 1}(1 + |Y_{[s]}|^2) ds \right]
\]
\[
\leq C \int_0^t \sup_{0 \leq r \leq s} \mathbb{E} \left[ (1 + |Y_r|^2)^{\frac{q}{2}} \right] ds
\]
(3.21)

Taking these two estimates into account, we derive from \((3.13)\) that
\[
\sup_{0 \leq s \leq t} \mathbb{E}(1 + |Y_s|^2)^{\frac{q}{2}} \leq (1 + |Y_0|^2)^{\frac{q}{2}} + C \int_0^t \sup_{0 \leq r \leq s} \mathbb{E}(1 + |Y_r|^2)^{\frac{q}{2}} ds
\]
(3.22)

for any $t \in [0, T]$. Applying the Gronwall inequality thus completes the proof. □

### 3.2 Analysis of the weak convergence rate of the tamed Euler method

In light of the fundamental weak convergence theorem, Corollary \((2.15)\), we need to verify Assumption \((2.8)\) with the one-step weak convergence rate $p$ in \((2.29)\) and \((2.30)\) determined. To this aim, let us consider the one-step approximation of the tamed Euler method \((3.1)\), given by
\[
Y(t, x; t + h) = x + \int_t^{t+h} \frac{f(x)}{1 + |f(x)|^2h} ds + \int_t^{t+h} \frac{g(x)}{1 + |f(x)|^2h} dW(s).
\]
(3.23)

For the purpose of weak error analysis, we also introduce an auxiliary one-step approximation,
\[
Y_E(t, x; t + h) = x + \int_t^{t+h} f(x) ds + \int_t^{t+h} g(x) dW(s),
\]
(3.24)

which can be viewed as a one-step approximation of the Euler-Maruyama method. Similarly as above, we denote
\[
(\delta_{Y,E,i}) := Y_E^{(i)}(t, x; t + h) - x^{(i)}, \quad i \in \{1, 2, \ldots, d\},
\]
(3.25)

where by $x^{(i)}$ we mean the $i$-th coordinate of the vector $x$. We will present some lemmas before showing the one-step error of the tamed Euler scheme \((3.1)\). For the purpose of the weak error analysis, we start from the following easy lemma.

**Lemma 3.3.** Let Assumption \((2.12)\) be satisfied with $p = 1$, for any $q \geq 1$ we have
\[
\|\delta_{X,E} - \delta_{Y,E}\|_{L^2(\Omega, \mathbb{R}^d)} \leq C(1 + |x|^{2r+1})h,
\]
(3.26)
\[
\|\delta_{Y,E}\|_{L^2(\Omega, \mathbb{R}^d)} \leq C(1 + |x|^{2r+1})h^{\frac{q}{2}},
\]
(3.27)
\[
\|\delta_{X,E}\|_{L^2(\Omega, \mathbb{R}^d)} \leq C(1 + |x|^{2r+1})h^{\frac{q}{2}}.
\]
(3.28)
Proof of Lemma 3.3 According to (2.10) and (3.24), one can use the Hölder inequality, the moment inequality and (2.57), to acquire
\[
\mathbb{E}[|\delta_{X,x} - \delta_{Y,E,x}|^{2q}] \\
= \mathbb{E}\left[\left|\int_t^{t+h} f(X(t, x; s)) - f(x) \, ds + \int_t^{t+h} g(X(t, x; s)) - g(x) \, dW(s)\right|^{2q}\right] \\
\leq C\mathbb{E}\left[\left|\int_t^{t+h} f(X(t, x; s)) - f(x) \, ds\right|^{2q}\right] + C\mathbb{E}\left[\left|\int_t^{t+h} g(X(t, x; s)) - g(x) \, dW(s)\right|^{2q}\right] \\
\leq Ch^{2q-1} \int_t^{t+h} ||f(X(t, x; s)) - f(x)||_{L^{2q}(\Omega,\mathbb{R}^d)} \, ds + Ch^{q-1} \int_t^{t+h} ||g(X(t, x; s)) - g(x)||_{L^{2q}(\Omega,\mathbb{R}^{d\times m})} \, ds \\
\leq C(1 + |x|^{2q(4r+1)})h^{2q},
\]
which implies (3.26), as required. In the same way, we get
\[
\mathbb{E}[|\delta_{Y,E,x}|^{2q}] = \mathbb{E}\left[\left|\int_t^{t+h} f(x) \, ds + \int_t^{t+h} g(x) \, dW(s)\right|^{2q}\right] \\
\leq C h^{2q} ||f(x)||_{L^{2q}(\Omega,\mathbb{R}^d)} + C h^q ||g(x)||_{L^{2q}(\Omega,\mathbb{R}^{d\times m})}^2 \leq C(1 + |x|^{2q(2r+1)})h^q,
\]
so that (3.27) is validated. The estimate of \(\delta_{X,x}\) as (3.28) can be done similarly.

The above estimates help us to identify the one-step error for the Euler approximation (3.24).

Lemma 3.4. Let Assumption 2.12 be fulfilled with \(p = 1\). Then for \(i_j \in \{1, 2, \ldots, d\}\) we have
\[
\left|\mathbb{E}\left[\prod_{j=1}^{s} (\delta_{X,x})^{i_j} - \prod_{j=1}^{s} (\delta_{Y,E,x})^{i_j}\right]\right| \leq C(1 + |x|^{8r+3})h^2, \quad s = 1, 2, 3.
\]

Proof of Lemma 3.4 We treat the case \(s = 1\) first. Using the multi-dimensional Itô formula, the Cauchy-Schwarz inequality, (2.44) and (2.45) shows
\[
\left|\mathbb{E}\left[(\delta_{X,x})^{i_1} - (\delta_{Y,E,x})^{i_1}\right]\right| \\
= \left|\mathbb{E}\left[\int_t^{t+h} f^{i_1}(X(s)) - f^{i_1}(x) \, ds\right]\right| \\
= \left|\mathbb{E}\left[\int_t^{t+h} \int_t^{s} \langle Df^{i_1}(X(r)), f(X(r))\rangle \, dr \, ds\right] + \mathbb{E}\left[\int_t^{t+h} \int_t^{s} \langle Df^{i_1}(X(r)), g(X(r)) \, dW(r)\rangle \, dr \, ds\right]\right| \\
+ \frac{1}{2} \left|\mathbb{E}\left[\int_t^{t+h} \int_t^{s} \text{trace}(g^T(X(r))D^2 f^{i_1}(X(r))g(X(r))) \, dr \, ds\right]\right| \\
\leq \left|\mathbb{E}\left[\int_t^{t+h} \int_t^{s} |Df^{i_1}(X(r))||f(X(r))| \, dr \, ds\right]\right| \\
+ C \left|\mathbb{E}\left[\int_t^{t+h} \int_t^{s} |D^2 f^{i_1}(X(r))||g(X(r))|^2 \, dr \, ds\right]\right| \\
\leq C(1 + |x|^{4r+1})h^2 + C(1 + |x|^{2r+2^p-1})h^2 \\
\leq C(1 + |x|^{4r+1})h^2.
\]
For $s = 2$, we first derive
\[
\begin{align*}
|\mathbb{E}[(\delta_{X,s})^{i_1}(\delta_{X,s})^{i_2} - (\delta_{Y,s})^{i_1}(\delta_{Y,s})^{i_2}]| &
\leq |\mathbb{E}[(\delta_{X,s})^{i_1} - (\delta_{Y,s})^{i_1}](\delta_{Y,s})^{i_2}]| + |\mathbb{E}[(\delta_{X,s})^{i_1}((\delta_{X,s})^{i_2} - (\delta_{Y,s})^{i_2})]| =: I_1 + I_2.
\end{align*}
\]
Before proceeding further with the estimate of $I_1$, we decompose it as follows:
\[
I_1 = \mathbb{E}\left[\left(\int_t^{t+h} f^{i_1}(X(s)) - f^{i_1}(x) \, ds + \int_t^{t+h} g^{i_1}(X(s)) - g^{i_1}(x) \, dW(s)\right) \times \left(\int_t^{t+h} f^{i_2}(x) \, ds + \int_t^{t+h} g^{i_2}(x) \, dW(s)\right)\right]
\leq \mathbb{E}\left[\left(\int_t^{t+h} f^{i_1}(X(s)) - f^{i_1}(x) \, ds\right) \cdot \int_t^{t+h} f^{i_2}(x) \, ds\right]
+ \mathbb{E}\left[\left(\int_t^{t+h} g^{i_1}(X(s)) - g^{i_1}(x) \, dW(s)\right) \cdot \int_t^{t+h} f^{i_2}(x) \, ds\right]
+ \mathbb{E}\left[\left(\int_t^{t+h} f^{i_1}(X(s)) - f^{i_1}(x) \, ds\right) \cdot \int_t^{t+h} g^{i_2}(x) \, dW(s)\right]
+ \mathbb{E}\left[\left(\int_t^{t+h} g^{i_1}(X(s)) - g^{i_1}(x) \, dW(s)\right) \cdot \int_t^{t+h} g^{i_2}(x) \, dW(s)\right]
= : I_{1,1} + I_{1,2} + I_{1,3}.
\]
Since $I_{1,1}$ is an easy term, we treat it first. In view of (2.49) and (2.57), we get
\[
I_{1,1} \leq h|f^{i_2}(x)| \int_t^{t+h} |\mathbb{E}[f^{i_1}(X(s)) - f^{i_1}(x)]| \, ds \leq C(1 + |x|^{6r + 2})h^{\frac{7}{2}}.
\]
Regarding $I_{1,2}$, with the aid of the Hölder inequality and the Itô isometry, one employs (2.45) and (2.57) to obtain
\[
I_{1,2} \leq \int_t^{t+h} \|f^{i_1}(X(s)) - f^{i_1}(x)\|_{L^2(\Omega, \mathbb{R})} \, ds \cdot \left\|\int_t^{t+h} g^{i_2}(x) \, dW(s)\right\|_{L^2(\Omega, \mathbb{R})}
\leq C(1 + |x|^{4r+1})h^{\frac{7}{2}} \left(\int_t^{t+h} \mathbb{E}[|g^{i_2}(x)|^2] \, ds\right)^{\frac{1}{2}}
\leq C(1 + |x|^{4r+2})h^2.
\]
Now we come to the estimate of $I_{1,3}$, which requires more careful arguments. In light of the
multi-dimensional Itô formula, one can separate the considered term $I_{1,3}$ as follows:

$$I_{1,3} \leq \mathbb{E} \left[ \left( \int_t^{t+h} \left\langle \int_t^s Dg_{i_1}(X(r)) \cdot g(X(r)) \, dW(r), \, dW(s) \right\rangle \right) \int_t^{t+h} g_{i_2}(x) \, dW(s) \right]$$

$$+ \mathbb{E} \left[ \left( \int_t^{t+h} \left\langle \int_t^s Dg_{i_1}(X(r)) \cdot f(X(r)) \, dr, \, dW(s) \right\rangle \right) \int_t^{t+h} g_{i_2}(x) \, dW(s) \right]$$

$$+ \frac{1}{2} \sum_{l=1}^m \left| \mathbb{E} \left[ \int_t^{t+h} \left\langle \text{trace} \left( g^T(X(r)) D^2 g_{i_1}(X(r)) g(X(r)) \right) \right\rangle \, dr \, dW^l(s) \right] \int_t^{t+h} g_{i_2}(x) \, dW(s) \right|$$

$$=: I^{(1)}_{1,3} + I^{(2)}_{1,3} + I^{(3)}_{1,3}.$$  \hspace{2cm} (3.37)

The three terms $I^{(1)}_{1,3}, I^{(2)}_{1,3}, I^{(3)}_{1,3}$ will be estimated separately. Since the number of the Wiener processes participating in $I^{(1)}_{1,3}$ is odd, we have

$$I^{(1)}_{1,3} = 0.$$  \hspace{2cm} (3.38)

In order to properly handle $I^{(2)}_{1,3}$, we utilize the Hölder inequality, \((2.44)\) and \((2.45)\) to deduce

$$I^{(2)}_{1,3} \leq \left\| \int_t^{t+h} \left\langle \int_t^s Dg_{i_1}(X(r)) \cdot f(X(r)) \, dr, \, dW(s) \right\rangle \right\|_{L^2(\Omega, \mathbb{R})} \left\| \int_t^{t+h} g_{i_2}(x) \, dW(s) \right\|_{L^2(\Omega, \mathbb{R})}$$

$$\leq C(1 + |x|^{2r+\rho}) h^{\frac{3}{2}} \cdot (1 + |x|^\rho) h^{\frac{3}{2}}$$

$$\leq C(1 + |x|^{2r+2\rho}) h^2.$$  \hspace{2cm} (3.39)

For the term $I^{(3)}_{1,3}$, similar techniques used in \((3.39)\) help us to show

$$I^{(3)}_{1,3} \leq C(1 + |x|^{4\rho-2}) h^2.$$  \hspace{2cm} (3.40)

Substituting the above estimates for $I^{(1)}_{1,3}, I^{(2)}_{1,3}$ and $I^{(3)}_{1,3}$ into \((3.37)\) gives

$$I_{1,3} \leq C(1 + |x|^{2r+2\rho}) h^2.$$  \hspace{2cm} (3.41)

Collecting the estimates of the three parts of the $I_1$, we thus arrive at

$$I_1 \leq C(1 + |x|^{6r+2}) h^2.$$  \hspace{2cm} (3.42)

By proceeding in the same way as before, one can prove

$$I_2 \leq C(1 + |x|^{6r+2}) h^2.$$  \hspace{2cm} (3.43)

This together with \((3.42)\) helps us to derive from \((3.33)\) that

$$\mathbb{E} \left[ (\delta_{X,x})^{i_1} (\delta_{X,x})^{i_2} - (\delta_{Y_{E,x}})^{i_1} (\delta_{Y_{E,x}})^{i_2} \right] \leq C(1 + |x|^{6r+2}) h^2.$$  \hspace{2cm} (3.44)
To handle the case $s = 3$, we first infer that
\[
\left| \mathbb{E}[(\delta_{X,x})^{i_1}(\delta_{X,x})^{i_2}(\delta_{X,x})^{i_3} - (\delta_{Y_{E,x}})^{i_1}(\delta_{Y_{E,x}})^{i_2}(\delta_{Y_{E,x}})^{i_3})] \right| \\
\leq \left| \mathbb{E}[(\delta_{X,x})^{i_1}(\delta_{X,x})^{i_2}(\delta_{X,x})^{i_3} - (\delta_{Y_{E,x}})^{i_2}(\delta_{Y_{E,x}})^{i_3})] \right| + \left| \mathbb{E}[(\delta_{Y_{E,x}})^{i_1}(\delta_{Y_{E,x}})^{i_2}(\delta_{Y_{E,x}})^{i_3})] \right| \\
\leq \left| \mathbb{E}[(\delta_{X,x})^{i_1}(\delta_{X,x})^{i_2}(\delta_{X,x})^{i_3}] \right| + \left| \mathbb{E}[(\delta_{Y_{E,x}})^{i_1}(\delta_{Y_{E,x}})^{i_2}(\delta_{Y_{E,x}})^{i_3})] \right| \\
+ \left| \mathbb{E}[(\delta_{X,x})^{i_1} - (\delta_{Y_{E,x}})^{i_1})(\delta_{Y_{E,x}})^{i_2}(\delta_{Y_{E,x}})^{i_3})] \right|. 
\] \hspace{1cm} (3.45)

Combining the Hölder inequality, (3.26), (3.27), and (3.28) shows
\[
\left| \mathbb{E}[(\delta_{X,x})^{i_1}(\delta_{X,x})^{i_2}(\delta_{X,x})^{i_3} - (\delta_{Y_{E,x}})^{i_1}(\delta_{Y_{E,x}})^{i_2}(\delta_{Y_{E,x}})^{i_3})] \right| \leq C(1 + |x|^{8s+3})h^2. \hspace{1cm} (3.46)

Putting these estimates with $s = 1, 2, 3$ together gives the desired assertion. □

In addition, we need the following lemma.

**Lemma 3.5.** Let Assumption [2.12] be fulfilled with $p = 1$, Then for any $q \geq 1$ it holds that
\[
\|\delta_{Y_{E,x}} - \delta_{Y_{x}}\|_{L^2(\Omega,\mathbb{R}^d)} \leq C(1 + |x|^{6q+3})h^q, \hspace{1cm} (3.47)
\]
\[
\|\delta_{Y_{x}}\|_{L^2(\Omega,\mathbb{R}^d)} \leq C(1 + |x|^{2q+1})h^q. \hspace{1cm} (3.48)
\]

**Proof of Lemma 3.5.** Using (3.23), (3.24), (3.26), (3.27), (3.28) and the moment inequality gives
\[
\mathbb{E}[|\delta_{Y_{E,x}} - \delta_{Y_{x}}|^{2q}] \\
\leq C\left| f(x)h \frac{|f(x)|^2h}{1+|f(x)|^2h} \right|^{2q} + C\mathbb{E}\left[ |g(x)(W(t+h) - W(t))| \frac{|f(x)|^2h}{1+|f(x)|^2h} \right]^{2q} \\
\leq C|f(x)|^{6q}h^{3q} + C|f(x)|^{4q}\|g(x)\|^{2q}h^2\mathbb{E}\left[ |W(t+h) - W(t)|^{2q} \right] \\
\leq C(1 + |x|^{2q(6q+3)})h^{3q},
\]
which implies the first assertion. The second one can be estimated in a similar way, Using (3.24), (3.25) and the moment inequality yields
\[
\mathbb{E}[|\delta_{Y_{x}}|^{2q}] \\
\leq C\left| f(x)h \frac{|f(x)|}{1+|f(x)|^2h} \right|^{2q} + C\mathbb{E}\left[ \frac{|g(x)|}{1+|f(x)|^2h} (W(t+h) - W(t)) \right]^{2q} \\
\leq C|f(x)|^{2q}h^{2q} + C\|g(x)\|^{2q}\mathbb{E}\left[ |W(t+h) - W(t)|^{2q} \right] \\
\leq C(1 + |x|^{2q(2q+1)})h^{q}.
\] (3.50)

This gives the second assertion and finishes the proof. □

The next lemma gives estimates for the one-step error of the tamed Euler scheme (3.23).

**Lemma 3.6.** Under Assumption [2.12] with $p = 1$, the one-step tamed Euler scheme (3.23) satisfies
\[
\left| \mathbb{E}\left[ \prod_{j=1}^{s}(\delta_{X,x})^{i_j} - \prod_{j=1}^{s}(\delta_{Y_{x}})^{i_j} \right] \right| \leq C(1 + |x|^{10q+5})h^2, \hspace{1cm} s = 1, 2, 3, \hspace{1cm} (3.51)
\]
\[
\left\| \prod_{j=1}^{4}(\delta_{X,x})^{i_j} \right\|_{L^2(\Omega,\mathbb{R})} \leq C(1 + |x|^{8q+4})h^2, \hspace{1cm} (3.52)
\]
\[
\left\| \prod_{j=1}^{4}(\delta_{Y_{x}})^{i_j} \right\|_{L^2(\Omega,\mathbb{R})} \leq C(1 + |x|^{8q+4})h^2. \hspace{1cm} (3.53)
\]
Proof of Lemma 3.6. A triangle inequality yields
\[
\left| \mathbb{E} \left[ \prod_{j=1}^{s} (\delta_{X,x})_{ij} - \prod_{j=1}^{s} (\delta_{Y,x})_{ij} \right] \right| \leq \left| \mathbb{E} \left[ \prod_{j=1}^{s} (\delta_{X,x})_{ij} - \prod_{j=1}^{s} (\delta_{Y,E,x})_{ij} \right] \right| + \left| \mathbb{E} \left[ \prod_{j=1}^{s} (\delta_{Y,E,x})_{ij} - \prod_{j=1}^{s} (\delta_{Y,x})_{ij} \right] \right| =: A_1 + A_2, \quad s = 1, 2, 3.
\]

(3.54)

Thanks to (3.51), for any \( s = 1, 2, 3 \) we have
\[
A_1 \leq C(1 + |x|^{8r+3})h^2. \quad (3.55)
\]

For \( s = 1 \), the second term \( A_2 \) can be estimated as follows:
\[
\left| \mathbb{E} \left[ (\delta_{Y,E,x})_{i1}((\delta_{Y,E,x})_{i2}^{1} - (\delta_{Y,x})_{i2}) \right] \right| = \left| \mathbb{E} \left[ \int_{t}^{t+h} f_{i1}(x) - \frac{f_{i1}(x)}{1+|f(x)|^2} \, ds + \int_{t}^{t+h} g_{i1}(x) - \frac{g_{i1}(x)}{1+|f(x)|^2} \, dW(s) \right] \right| \leq \frac{|f(x)|^2 h^2}{1+|f(x)|^2} \leq C(1 + |x|^{6r+3})h^2. \quad (3.56)
\]

For \( s = 2 \), by the Hölder inequality, (3.27) and Lemma 3.5, one can get
\[
\left| \mathbb{E} \left[ ((\delta_{Y,E,x})_{i1}^{1})^{2} + ((\delta_{Y,x})_{i1})^{2} \right] \right| \leq \left\| (\delta_{Y,E,x})_{i1}^{1} \right\|_{L^2(\Omega,\mathbb{R})} \left\| (\delta_{Y,E,x})_{i1}^{2} \right\|_{L^2(\Omega,\mathbb{R})} \leq C(1 + |x|^{6r+3})h^2.
\]

Likewise, for \( A_2 \) with \( s = 3 \) one can utilize the Hölder inequality, (3.27) and Lemma 3.5 to derive
\[
\left| \mathbb{E} \left[ \prod_{j=1}^{3} (\delta_{Y,E,x})_{ij} - \prod_{j=1}^{3} (\delta_{Y,x})_{ij} \right] \right| \leq \left| \mathbb{E} \left[ ((\delta_{Y,E,x})_{i1}^{1})^{2} (\delta_{Y,E,x})_{i3}^{2} - (\delta_{Y,x})_{i1}^{2} (\delta_{Y,x})_{i3}^{2} \right] \right| + \left| \mathbb{E} \left[ ((\delta_{Y,E,x})_{i1}^{1})^{2} (\delta_{Y,E,x})_{i2}^{2} (\delta_{Y,x})_{i3}^{2} \right] \right| \leq C(1 + |x|^{10r+5})h^2. \quad (3.58)
\]

This together with (3.56), (3.57) gives
\[
A_2 \leq C(1 + |x|^{10r+5})h^2, \quad s = 1, 2, 3. \quad (3.59)
\]

This combined with (3.55) promises (3.51). Using the Hölder inequality and (3.28) yields
\[
\left\| \prod_{j=1}^{4} (\delta_{X,x})_{ij} \right\|_{L^2(\Omega,\mathbb{R})} \leq \prod_{j=1}^{4} \left\| (\delta_{X,x})_{ij} \right\|_{L^2(\Omega,\mathbb{R})} \leq C(1 + |x|^{8r+4})h^2. \quad (3.60)
\]

23
Armed with (3.48), we can derive (3.53) in a similar way.

Finally, with the help of Theorem 3.2, Lemma 3.6 and Corollary 2.15, we are able to identify the global weak convergence rate of the tamed Euler method (3.1).

Theorem 3.7. Let \( \varphi : \mathbb{R}^d \to \mathbb{R} \) be the test function with \( \varphi \in \mathcal{H}^4 \) and there exist constants \( L > 0 \) and \( \chi \geq 0 \) such that
\[
|D^j \varphi(x)| \leq L(1 + |x|^\chi), \quad j \in \{1, 2, 3, 4\}, \ x \in \mathbb{R}^d.
\]  
(3.61)

Suppose Assumption 2.12 holds with \( p = 1 \). Then the tamed Euler method (3.1) has a global weak convergence rate of order 1, namely,
\[
|E[\varphi(X(t_0, X_0; T))] - E[\varphi(Y(t_0, Y_0; T))]| \leq C(1 + |X_0|^{(10r+5+\chi)})h.
\]  
(3.62)

4 Weak convergence rate of the backward Euler method

As another application of the fundamental weak convergence theorem, we analyze the weak convergence rate of the well-known backward Euler approximation in this section. The backward Euler method applied to SDEs (2.6) takes the following form:
\[
Y_{n+1} = Y_n + f(Y_{n+1})h + g(Y_n)\Delta W_n,
\]  
(4.1)

where \( \Delta W_n := W(t_{n+1}) - W(t_n) \) for \( n \in \{0, 1, 2, \ldots, N-1\} \). Then the one-step approximation of (4.1) is denoted by
\[
Y(t, x; t+h) = x + \int_t^{t+h} f(Y(t, x; t+h)) \, ds + \int_t^{t+h} g(x) \, dW(s).
\]  
(4.2)

Similarly as before, we denote \( \delta_{X,x} := X(t, x; t+h) - x \), \( \delta_{Y,x} := Y(t, x; t+h) - x \). Subtracting (3.24) from this yields
\[
Y(t, x; t+h) = Y_E(t, x; t+h) + R_f, \quad R_f := \int_t^{t+h} f(Y(t, x; t+h)) - f(x) \, ds.
\]  
(4.3)

As already established in [16, Corollary 2.27], \( q \)-th moments of the backward Euler approximations are uniformly bounded under Assumption 2.12.

Proposition 4.1. Let Assumption 2.12 hold with \( p = 1 \) and let \( \{Y_n\}_{0 \leq n \leq N} \) be given by (4.1). Then there exists \( C > 0 \), independent of \( h \) such that
\[
\sup_{N \in \mathbb{N}} \sup_{0 \leq n \leq N} \mathbb{E}[|Y_n|^q] \leq C(1 + |X_0|^q), \quad \forall q \geq 0.
\]  
(4.4)

Equipped with the moment bound, we can continue to analyze the weak convergence rate of the backward Euler method by means of Corollary 2.15. To begin with, we present the following lemma, which is useful in the analysis.
Lemma 4.2. Let Assumption 2.12 hold with \( p = 1 \). Then for any \( q \geq 1 \) the following inequalities hold for the backward Euler scheme (4.1):

\[
\|\delta_{Y,x}\|_{L^{2q}(\Omega,\mathbb{R}^d)} \leq C(1 + |x|^{2r+1})h^\frac{q}{2},
\]

\( (4.5) \)

\[
\|Rf\|_{L^{2q}(\Omega,\mathbb{R}^d)} \leq C(1 + |x|^{4r+1})h^\frac{q}{2},
\]

\( (4.6) \)

\[
\|\delta_{X,x} - \delta_{Y,x}\|_{L^{2q}(\Omega,\mathbb{R}^d)} \leq C(1 + |x|^{4r+1})h.
\]

\( (4.7) \)

**Proof of Lemma 4.2.** Repeating the same lines in the proof of (3.30) and (4.1) results in

\[
\mathbb{E}\left[ (\delta_{Y,x})^{2q} \right] = \mathbb{E}\left[ \int_t^{t+h} f(Y(t,x; t + s)) \, ds + \int_t^{t+h} g(x) \, dW(s) \right]^{2q}
\]

\[
\leq Ch^{2q-1} \int_t^{t+h} \|f(Y(t,x; t + s))\|_{L^{2q}(\Omega,\mathbb{R}^d)}^{2q} \, ds + Ch^{q-1} \int_t^{t+h} \|g(x)\|_{L^{2q}(\Omega,\mathbb{R}^d \times \mathbb{R}^d)}^{2q} \, ds
\]

\[
\leq C(1 + |x|^{2q(2r+1)})h^q,
\]

\( (4.8) \)

which confirms (4.5). To prove (4.6), we use the Hölder inequality, (2.57) and (4.5) to derive that

\[
\mathbb{E}\left[ |Rf|^{2q} \right] \leq Ch^{2q-1} \int_t^{t+h} \|f(Y(t,x; t + s)) - f(x)\|_{L^{2q}(\Omega,\mathbb{R}^d)}^{2q} \, ds
\]

\[
\leq C(1 + |x|^{2q(4r+1)})h^{2q}.
\]

\( (4.9) \)

Also, one can follow the same lines in the proof of (3.26) to show that

\[
\mathbb{E}\left[ |\delta_{X,x} - \delta_{Y,x}|^{2q} \right]
\]

\[
= \mathbb{E}\left[ \int_t^{t+h} f(X(s)) - f(Y(t,x; t + s)) \, ds + \int_t^{t+h} g(X(s)) - g(x) \, dW(s) \right]^{2q}
\]

\[
\leq Ch^{2q-1} \int_t^{t+h} \|f(X(s)) - f(x)\|_{L^{2q}(\Omega,\mathbb{R}^d)}^{2q} \, ds + Ch^{q-1} \int_t^{t+h} \|f(x) - f(Y(t,x; t + s))\|_{L^{2q}(\Omega,\mathbb{R}^d)}^{2q} \, ds
\]

\[
+ Ch^{q-1} \int_t^{t+h} \|g(X(s)) - g(x)\|_{L^{2q}(\Omega,\mathbb{R}^d \times \mathbb{R}^d)}^{2q} \, ds
\]

\[
\leq C(1 + |x|^{2q(4r+1)})h^{2q},
\]

\( (4.10) \)

as required. This thus finishes the proof of the lemma.

At this stage, we shall detect the one-step weak error of the backward Euler method.

Lemma 4.3. Let Assumption 2.12 hold with \( p = 1 \). Then for the one-step backward Euler scheme (4.2) we have

\[
\left| \mathbb{E}\left[ \prod_{j=1}^{s}(\delta_{X,x})_{i_j} - \prod_{j=1}^{s}(\delta_{Y,x})_{i_j} \right] \right| \leq C(1 + |x|^{8r+3})h^2, \quad s = 1, 2, 3.
\]

\( (4.11) \)

\[
\left\| \prod_{j=1}^{4}(\delta_{Y,x})_{i_j} \right\|_{L^{2}(\Omega,\mathbb{R})} \leq C(1 + |x|^{8r+4})h^2.
\]

\( (4.12) \)
Proof of Lemma 4.3. For $s = 1$, we first note that
\[
\mathbb{E}[(\delta_{X,x})^{i_1} - (\delta_{Y,x})^{i_1}] = \mathbb{E}[(\delta_{X,x})^{i_1} - (\delta_{Y_x})^{i_1} - R^{i_1}_j] \\
\leq \mathbb{E}[(\delta_{X,x})^{i_1} - (\delta_{Y_x})^{i_1}] + \mathbb{E}\left[ \left| \int_t^{t+h} f^{i_1}(Y(t,x; t+h)) - f^{i_1}(x) \, ds \right| \right]. \tag{4.13}
\]
The estimate for the first term has been done in (3.32), namely,
\[
\mathbb{E}[(\delta_{X,x})^{i_1} - (\delta_{Y_x})^{i_1}] \leq C(1 + |x|^{4r+1}) h^2. \tag{4.14}
\]
Next we bound the second term. Employing Taylor’s expansion in the form of
\[
f^{i_1}(Y(t,x; t+h)) = f^{i_1}(x) + \sum_{j_1=1}^d \frac{\partial f^{i_1}(x)}{\partial x_{j_1}} (Y^{j_1}(t,x; t+h) - x^{j_1}) \\
+ \sum_{j_1,j_2=1}^d \int_0^1 \frac{\partial^2 f^{i_1}(x + \theta(Y(t,x; t+h) - x))}{\partial x_{j_1} \partial x_{j_2}} (1 - \theta) \, d\theta
\cdot (Y^{j_1}(t,x; t+h) - x^{j_1})(Y^{j_2}(t,x; t+h) - x^{j_2}),
\]
and using the assumption on the function $f$ in (2.44) and the Hölder inequality, one can infer
\[
\left| \mathbb{E}\left[ \int_t^{t+h} f^{i_1}(Y(t,x; t+h)) - f^{i_1}(x) \, ds \right] \right| \\
\leq \sum_{j_1=1}^d \left| \mathbb{E}\left[ \int_t^{t+h} \frac{\partial f^{i_1}(x)}{\partial x_{j_1}} (Y^{j_1}(t,x; t+h) - x^{j_1}) \, ds \right] \right| \\
+ \sum_{j_1,j_2=1}^d \left| \mathbb{E}\left[ \int_t^{t+h} \int_0^1 \frac{\partial^2 f^{i_1}(x + \theta(Y(t,x; t+h) - x))}{\partial x_{j_1} \partial x_{j_2}} (1 - \theta) \, d\theta \, ds \right] \right|
\cdot (Y^{j_1}(t,x; t+h) - x^{j_1})(Y^{j_2}(t,x; t+h) - x^{j_2}) \\
\leq C(1 + |x|^{2r}) \mathbb{E}\left[ \int_t^{t+h} (Y^{j_1}(t,x; t+h) - x^{j_1}) \, ds \right] \\
+ C(1 + |x|^{2r-1}) h \| (\delta_{Y,x})^{j_1} \|_{L^2(\Omega, \mathcal{F})} \| (\delta_{Y,y})^{j_2} \|_{L^2(\Omega, \mathcal{F})} \\
\leq C(1 + |x|^{6r+1}) h^2.
\]
This together with (4.14) implies
\[
\mathbb{E}[(\delta_{X,x})^{i_1} - (\delta_{Y_x})^{i_1}] \leq C(1 + |x|^{6r+1}) h^2. \tag{4.17}
\]
For $s = 2$, we first note that
\[
\left| \mathbb{E}\left[ (\delta_{X,x})^{i_1}(\delta_{X,x})^{i_2} - (\delta_{Y_x})^{i_1}(\delta_{Y_x})^{i_2} \right] \right| \\
\leq \mathbb{E}[(\delta_{X,x})^{i_1}(\delta_{X,x})^{i_2} - (\delta_{Y,x})^{i_1}(\delta_{Y,x})^{i_2}] + \mathbb{E}\left[ \left| (\delta_{X,x})^{i_1} - (\delta_{Y_x})^{i_1} \right| (\delta_{Y,x})^{i_2} \right] \\
= \mathbb{E}\left[ (\delta_{X,x})^{i_1}(\delta_{X,x})^{i_2} - (\delta_{Y,x})^{i_1}(\delta_{Y,x})^{i_2} \right] + \mathbb{E}\left[ \left| (\delta_{X,x})^{i_1} - (\delta_{Y_x})^{i_1} \right| \left( (\delta_{Y,x})^{i_2} - R^{i_2}_j \right) \right] \\
\leq \mathbb{E}\left[ (\delta_{X,x})^{i_1}(\delta_{X,x})^{i_2} - (\delta_{Y,x})^{i_2} \right] + \mathbb{E}\left[ (\delta_{X,x})^{i_1} - (\delta_{Y_x})^{i_1} \right] (\delta_{Y,x})^{i_2} \\
+ \mathbb{E}\left[ (\delta_{X,x})^{i_1} R^{i_2}_j \right] + \mathbb{E}\left[ R^{i_2}_j \right]. \tag{4.18}
\]
By (3.43), it is easy to see that
\[
\mathbb{E}[|\delta_{X,x}^{i_1}(\delta_{X,x}^{i_2} - \delta_{Y,x}^{i_2})|] \leq C(1 + |x|^{6r+2})h^2.
\] (4.19)
The estimate for the second term in (4.18) is obtained in the same way. Utilizing the Hölder inequality, (3.28) and (4.6) yields
\[
\mathbb{E}[|\delta_{X,x}^{i_1}|^2] \leq \|\delta_{X,x}^{i_1}\|_{L^2(\Omega,\mathbb{R})}\|R_f^{i_2}\|_{L^2(\Omega,\mathbb{R})} \leq C(1 + |x|^{6r+2})h^2.
\] (4.20)

Following a similar way as in (4.20), we acquire
\[
\mathbb{E}[|\delta_{Y,x}^{i_2}|^2] \leq C(1 + |x|^{6r+2})h^2.
\] (4.21)

Collecting (4.19), (4.20), and (4.21) enables us to arrive at
\[
\mathbb{E}[|\delta_{X,x}^{i_1}(\delta_{X,x}^{i_2} - \delta_{Y,x}^{i_2})|] \leq C(1 + |x|^{6r+2})h^2.
\] (4.22)

Finally, for \(s = 3\) we make a decomposition as follows:
\[
\mathbb{E}[|\delta_{X,x}^{i_1}(\delta_{X,x}^{i_2} - \delta_{Y,x}^{i_2})|] \leq \mathbb{E}[|\delta_{X,x}^{i_1}| |\delta_{X,x}^{i_2} - \delta_{Y,x}^{i_2}|] + \mathbb{E}[|\delta_{X,x}^{i_1} - \delta_{Y,x}^{i_1}| |\delta_{X,x}^{i_2} - \delta_{Y,x}^{i_2}|] \leq \mathbb{E}[|\delta_{X,x}^{i_1}|^2] + \mathbb{E}[|\delta_{X,x}^{i_1} - \delta_{Y,x}^{i_1}|^2] \leq C(1 + |x|^{6r+2})h^2.
\] (4.23)

Using (3.28), (4.5), and Hölder inequality yields
\[
\mathbb{E}[|\delta_{X,x}^{i_1}(\delta_{X,x}^{i_2} - \delta_{Y,x}^{i_2})|] \leq C(1 + |x|^{8r+3})h^2
\] (4.24)
and
\[
\mathbb{E}[|\delta_{X,x}^{i_1} - \delta_{Y,x}^{i_1}|^2] \leq C(1 + |x|^{8r+3})h^2.
\] (4.25)

All together, we thus obtain
\[
\mathbb{E}\left[\prod_{j=1}^s (\delta_{X,x}^{i_j})\right] - \mathbb{E}\left[\prod_{j=1}^s (\delta_{Y,x}^{i_j})\right] \leq C(1 + |x|^{8r+3})h^2, s = 1, 2, 3.
\] (4.27)

Similarly as before, one can employ the Hölder inequality and (4.5) to deduce
\[
\left\|\prod_{j=1}^4 (\delta_{Y,x}^{i_j})\right\|_{L^2(\Omega,\mathbb{R})} \leq \prod_{j=1}^4 \left\|\delta_{Y,x}^{i_j}\right\|_{L^2(\Omega,\mathbb{R})} \leq C(1 + |x|^{8r+4})h^2,
\] (4.28)
which finishes the proof of Lemma 4.3.

With the aid of Corollary 2.15, one can immediately derive the global weak convergence rate of the backward Euler method (4.11).

**Theorem 4.4.** Let \(\varphi : \mathbb{R}^d \to \mathbb{R}\) be the test function with \(\varphi \in \mathcal{H}^4\) and there exist constants \(L > 0\) and \(\chi \geq 0\) such that
\[
|D^j\varphi(x)| \leq L(1 + |x|^\chi), \quad j \in \{1, 2, 3, 4\}, x \in \mathbb{R}^d.
\] (4.29)

Let Assumption 2.12 hold with \(p = 1\). Then the backward Euler method (4.11) has a global weak convergence rate of order 1, namely,
\[
\mathbb{E}[\varphi(X(t_0, X_0; T))] - \mathbb{E}[\varphi(Y(t_0, Y_0; T))] \leq C(1 + |X_0|^{8r+4+\chi})h.
\] (4.30)
5 Numerical experiments

In this section, some numerical experiments are performed to illustrate the previous theoretical findings. More accurately, we test the weak convergence rates of numerical schemes for two stochastic models. The 'true' solutions of these two models are identified with the numerical ones using a small stepsize \( h_{\text{exact}} = 2^{-13} \) and the expectations are approximated by averages over 10^5 Brownian paths in the following simulations.

**Example 5.1.** As the first test model, we consider the stochastic Ginzburg-Landau (GL) equation [19]

\[
\begin{align*}
\frac{dX(t)}{dt} &= (\alpha X(t) - \delta X(t)^3)dt + \beta X(t) \, dW(t), & t \in (0, T], \\
X(0) &= X_0,
\end{align*}
\]

(5.1)

where we assign \( T = 1, X_0 = 1, \alpha = \delta = 1 \) and \( \beta = 0.5 \). Since the drift is a cubic polynomial, one can easily check that all conditions in Assumption 2.12 are fulfilled with \( r = 1, \rho = 1 \). Therefore, Theorems 3.7 and 4.4 are applicable here.

In Figure 1 and Figure 2, we plot weak approximation errors of three different numerical methods against six different stepizes \( h = 2^{-k}, k = 3, 4, \cdots, 8 \) on a log-log scale. In addition to the tamed Euler scheme (3.1) (TEM for short) and the backward Euler method (4.1) (BEM for short), we also implement a tamed Euler method from [31], which applied to Example 5.1 reads,

\[
Y_{n+1} = Y_n + \frac{1}{1+h^{1/2}|Y_n|^2} \left[ f(Y_n)h + g(Y_n) \Delta W_n \right], \quad Y_0 = X_0.
\]

(5.2)

From Figure 1 and Figure 2, the expected weak convergence rate of order 1 of the TEM and BEM is numerically confirmed, but for the tamed Euler method (5.2) (D TEM for abbreviation) we only observe a weak convergence rate of order \( \frac{1}{2} \). All weak errors are also presented in Table 1 and Table 2, where one can clearly see that the weak errors of the TEM and BEM are significantly better than the DTEM (5.2).

| \begin{tabular}{c|c|c|c|c|c|c} 
Weak errors with & Weak errors with & \\
& \( \varphi(x) = x \) & \( \varphi(x) = x^2 \) \\
\hline
h & TEM & DTEM & BEM & TEM & DTEM & BEM \\
\hline
2^{-8} & 0.00011 & 0.01023 & 0.00024 & 0.00068 & 0.01729 & 0.00037 \\
2^{-7} & 0.00018 & 0.01373 & 0.00043 & 0.00127 & 0.02317 & 0.00067 \\
2^{-6} & 0.00037 & 0.01816 & 0.00087 & 0.00253 & 0.03061 & 0.00134 \\
2^{-5} & 0.00076 & 0.02350 & 0.00175 & 0.00501 & 0.03959 & 0.00270 \\
2^{-4} & 0.00134 & 0.02960 & 0.00332 & 0.00954 & 0.04987 & 0.00515 \\
2^{-3} & 0.00224 & 0.03614 & 0.00607 & 0.01814 & 0.06115 & 0.00946 \\
\end{tabular} |

Table 1: Weak errors for different stepsizes (h) with \( \varphi(x) = x \) and \( \varphi(x) = x^2 \).

**Example 5.2.** Consider the stochastic FitzHugh-Nagumo (FHN) model [8] in the form of:

\[
\begin{pmatrix}
\frac{dX_1(t)}{dt} \\
\frac{dX_2(t)}{dt}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\epsilon} (X_1(t) - X_1^3(t) - X_2(t)) \\
\gamma X_1(t) - X_2(t) + \beta
\end{pmatrix} dt + \begin{pmatrix}
c_1 X_1(t) + d_1 \\
c_2 X_2(t) + d_2
\end{pmatrix} dW(t)
\]

(5.3)

for \( t \in (0, T] \) and \( X(0) = X_0 \), with a solution \( X(t) := (X_1(t), X_2(t))^T \) for \( t \in [0, T] \). Here \( W(t) := (W_1(t), W_2(t))^T \) is a two dimensional Brownian motion. We take \( X_0 = (0, 0)^T \) and other
Figure 1: Weak convergence rates for the stochastic GL equation \((5.1)\) for \(\varphi(x) = x\) (Left) and \(\varphi(x) = x^2\) (Right).

Figure 2: Weak rates for stochastic GL equation for \(\varphi(x) = x^4\) (Left) and \(\varphi(x) = \cos(x)\) (Right).
Figure 3: Weak rates for stochastic FHN model with $\varphi(x) = x$ (Left) and $\varphi(x) = x^2$ (Right).

Figure 4: Weak rates for stochastic FHN model with $\varphi(x) = x^4$ (Left) and $\varphi(x) = \cos(x)$ (Right).
model parameters are set to 1. Similarly, all conditions in Assumption 2.12 are fulfilled with $r = 1$, $\rho = 1$ and Theorems 3.7 and 4.4 are applicable.

Next we test the weak convergence rates of the TEM, DTEM and BEM for six different stepsizes $h = 2^{-k}$, $k = 3, 4, \ldots, 8$. From Figures 3 and 4, it is clear to observe that weak convergence rate of the TEM and BEM is order 1. For the DTEM, we only observe a weak convergence of order $\frac{1}{2}$. All weak errors are also listed in Table 3 and Table 4 where it is observed that the TEM and BEM schemes perform much better than the DTEM (5.2).

Table 3: Weak errors for different stepsizes with $\varphi(x) = x$ and $\varphi(x) = x^2$.

| $h$   | TEM   | DTEM   | BEM   | TEM   | DTEM   | BEM   |
|-------|-------|--------|-------|-------|--------|-------|
| $2^{-8}$ | 0.00932 | 0.01782 | 0.00225 | 0.01714 | 0.01091 | 0.00559 |
| $2^{-7}$ | 0.01655 | 0.02541 | 0.00477 | 0.02869 | 0.01570 | 0.01129 |
| $2^{-6}$ | 0.02812 | 0.03638 | 0.00974 | 0.04558 | 0.02303 | 0.02257 |
| $2^{-5}$ | 0.04620 | 0.05279 | 0.01981 | 0.06829 | 0.03445 | 0.04411 |
| $2^{-4}$ | 0.07388 | 0.07849 | 0.03950 | 0.09658 | 0.05282 | 0.08239 |
| $2^{-3}$ | 0.11632 | 0.12065 | 0.07452 | 0.13121 | 0.08465 | 0.14599 |

References

[1] Assyr Abdulle, David Cohen, Gilles Vilmart, and Konstantinos C Zygalakis. High weak order methods for stochastic differential equations based on modified equations. *SIAM Journal on Scientific Computing*, 34(3):A1800–A1823, 2012.

[2] Assyr Abdulle, Gilles Vilmart, and Konstantinos C Zygalakis. Second weak order explicit stabilized methods for stiff stochastic differential equations. *SIAM Journal on Scientific Computing*, 2012.

[3] Martin Altmayer and Andreas Neuenkirch. Discretising the Heston model: an analysis of the weak convergence rate. *IMA Journal of Numerical Analysis*, 37(4):1930–1960, 2017.
Table 4: Weak errors for different stepsizes with $\varphi(x) = x^4$ and $\varphi(x) = \cos(x)$.

| h    | TEM  | DTEM | BEM  | TEM  | DTEM | BEM  |
|------|------|------|------|------|------|------|
| $2^{-8}$ | 0.07723 | 0.03627 | 0.01486 | 0.00579 | 0.00409 | 0.00224 |
| $2^{-7}$ | 0.12081 | 0.04898 | 0.02930 | 0.00995 | 0.00600 | 0.00454 |
| $2^{-6}$ | 0.17821 | 0.06707 | 0.05796 | 0.01624 | 0.00897 | 0.00911 |
| $2^{-5}$ | 0.24684 | 0.09259 | 0.10995 | 0.02499 | 0.01370 | 0.01791 |
| $2^{-4}$ | 0.33274 | 0.12981 | 0.19493 | 0.03594 | 0.02145 | 0.03379 |
| $2^{-3}$ | 0.49859 | 0.19455 | 0.32898 | 0.04792 | 0.03489 | 0.06043 |

[4] Vlad Bally and Denis Talay. The law of the Euler scheme for stochastic differential equations I. Convergence rate of the distribution function. *Probability theory and related fields*, 104(1):43–60, 1996.

[5] Jianhai Bao, Xing Huang, and Shao-Qin Zhang. Convergence rate of EM algorithm for SDEs under integrability condition. *arXiv preprint arXiv:2009.04781*, 2020.

[6] W. Beyn, E. Isaak, and R. Kruse. Stochastic C-stability and B-consistency of explicit and implicit Euler-type schemes. *Journal of Scientific Computing*, 67(3):955–987, 2016.

[7] Mireille Bossy, Jean-François Jabir, and Kerlyns Martinez. On the weak convergence rate of an exponential Euler scheme for SDEs governed by coefficients with superlinear growth. *Bernoulli*, 27(1):312–347, 2021.

[8] Evelyn Buckwar, Adeline Samson, Massimiliano Tamborrino, and Irene Tubikanec. Splitting methods for SDEs with locally Lipschitz drift. An illustration on the FitzHugh-Nagumo model. *arXiv preprint arXiv:2101.01027*, 2021.

[9] Evelyn Buckwar and Tony Shardlow. Weak approximation of stochastic differential delay equations. *IMA journal of numerical analysis*, 25(1):57–86, 2005.

[10] S. Cerra. *Second Order PDE’s in Finite and Infinite Dimension: A Probabilistic Approach*. Springer Science & Business Media, 2001.

[11] Kristian Debrabant and Anne Kværnø. B–Series Analysis of Stochastic Runge–Kutta Methods that use an iterative scheme to compute their internal stage values. *SIAM journal on numerical analysis*, 47(1):181–203, 2009.

[12] Wei Fang and Michael Bryce Giles. Adaptive Euler–Maruyama method for SDEs with non-globally Lipschitz drift. *Ann. Appl. Probab.*, 30(2):526–560, 2020.

[13] Desmond J Higham, Xuerong Mao, and Chenggui Yuan. Almost sure and moment exponential stability in the numerical simulation of stochastic differential equations. *SIAM journal on numerical analysis*, 45(2):592–609, 2007.

[14] L. Hörmander. *Linear Partial Differential Operators I*. Springer, 1963.
[15] M. Hutzenthaler, A. Jentzen A, and P. E. Kloeden. Strong and weak divergence in finite time of Euler’s method for stochastic differential equations with non-globally Lipschitz continuous coefficients. The Royal Society, 467:1563–1576, 2011.

[16] M. Hutzenthaler and A. Jentzen. Numerical approximations of stochastic differential equations with non-globally Lipschitz continuous coefficients, volume 236. American Mathematical Society, 2015.

[17] M. Hutzenthaler, A. Jentzen, and P. E. Kloeden. Strong convergence of an explicit numerical method for SDEs with non-globally Lipschitz continuous coefficients. The Annals of Applied Probability, 22(4), 2012.

[18] C. Kelly and G. J. Lord. Adaptive time-stepping strategies for nonlinear stochastic systems. IMA Journal of Numerical Analysis, 38(3):1523–1549, 2017.

[19] Peter E Kloeden and Eckhard Platen. Numerical solution of stochastic differential equations, volume 23. Springer Science & Business Media, 1992.

[20] Arturo Kohatsu-Higa, Antoine Lejay, and Kazuhiro Yasuda. Weak rate of convergence of the Euler–Maruyama scheme for stochastic differential equations with non-regular drift. Journal of Computational and Applied Mathematics, 326:138–158, 2017.

[21] Yoshio Komori, David Cohen, and Kevin Burrage. Weak second order explicit exponential Runge–Kutta methods for stochastic differential equations. SIAM Journal on Scientific Computing, 39(6):A2857–A2878, 2017.

[22] X. Mao. Stochastic Differential Equations and Applications. Elsevier, 2007.

[23] Xuerong Mao. The truncated Euler–Maruyama method for stochastic differential equations. J. Comput. Appl. Math., 290:370–384, 2015.

[24] Annalena Mickel and Andreas Neuenkirch. The weak convergence order of two Euler-type discretization schemes for the log-Heston model. arXiv preprint arXiv:2106.10926, 2021.

[25] G. N. Milstein. Weak approximation of solutions of systems of stochastic differential equations. Theory of Probability Its Applications, 30(4), 1986.

[26] G. N. Milstein and M. V. Tretyakov. Numerical integration of stochastic differential equations with nonglobally Lipschitz coefficients. SIAM journal on numerical analysis, 43(3):1139–1154, 2005.

[27] G. N. Milstein and M. V. Tretyakov. Stochastic Numerics for Mathematical Physics. Springer Science & Business Media, 2013.

[28] Hoang-Long Ngo and Dai Taguchi. Approximation for non-smooth functionals of stochastic differential equations with irregular drift. Journal of Mathematical Analysis and Applications, 457(1):361–388, 2018.
[29] Andreas Rößler. Rooted tree analysis for order conditions of stochastic Runge-Kutta methods for the weak approximation of stochastic differential equations. *Stochastic analysis and applications*, 24(1):97–134, 2006.

[30] Sotirios Sabanis. A note on tamed euler approximations. *Electronic Communications in Probability*, 18:1–10, 2013.

[31] Sotirios Sabanis. Euler approximations with varying coefficients: the case of superlinearly growing diffusion coefficients. *The Annals of Applied Probability*, 26(4):2083–2105, 2016.

[32] Jinghai Shao. Weak convergence of Euler-Maruyama’s approximation for SDEs under integrability condition. *arXiv preprint arXiv:1808.07250*, 2018.

[33] Yongqiang Suo, Chenggui Yuan, and Shao-Qin Zhang. Weak convergence of Euler scheme for SDEs with low regular drift. *Numerical Algorithms*, pages 1–17, 2021.

[34] Łukasz Szpruch and Xîlíng Zhîng. V-integrability, asymptotic stability and comparison property of explicit numerical schemes for non-linear SDEs. *Mathematics of Computation*, 87(310):755–783, 2018.

[35] Denis Talay and Luciano Tubaro. Expansion of the global error for numerical schemes solving stochastic differential equations. *Stochastic analysis and applications*, 8(4):483–509, 1990.

[36] M. V. Tretyakov and Z. Zhang. A fundamental mean-square convergence theorem for SDEs with locally Lipschitz coefficients and its applications. *SIAM J. Numer. Anal.*, 51(6):3135–3162, 2013.

[37] E Weinan, Tiejun Li, and Eric Vanden-Eijnden. *Applied stochastic analysis*, volume 199. American Mathematical Soc., 2021.

[38] Zhongqiang Zhang and George Karniadakis. *Numerical methods for stochastic partial differential equations with white noise*. Springer, 2017.

[39] Chao Zheng. Weak convergence rate of a time-discrete scheme for the Heston stochastic volatility model. *SIAM Journal on Numerical Analysis*, 55(3):1243–1263, 2017.