Raining Rocks: An analytical formulation for collision timescales in planetary systems

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ABSTRACT
The dynamical interaction of minor bodies (such as comets or asteroids) with planets plays an essential role in the planetary system’s architecture and evolution. As a result of these interactions, structures like the Kuiper belt and the Oort cloud can be created. In particular, the collision of minor bodies with planets can drastically change the planet’s internal and orbital evolution. We present an analytic formulation to determine the collision timescale for a minor body to impact a planet for arbitrary geometry. By comparing with a suite of detailed N-body simulations and an analytical method for collision timescales in the solar system, we confirmed the accuracy of our analytic formulation. As a proof of concept, we focused on the collision rate of minor bodies randomly distributed around a Jupiter-like planet, emulating a Kuiper belt-like disk. We show that our analytical method yields in good agreement with the numerical simulations. The formalism presented here thus provides a succinct and accurate alternative to numerical calculations.

Key words: orbital dynamics, comets and asteroids, cometary impacts

1 INTRODUCTION
Minor bodies such as comets and asteroids in the solar system are remnants of the planet formation process (Kokubo & Ida 2002; Kenyon & Bromley 2006; Wyatt 2008; Johansen & Lambrechts 2017). These objects play an important role in the evolution of their planetary system (Nesvorný 2018; Torres et al. 2019; Cai et al. 2019). In particular, the dynamical evolution of these bodies in any planetary system is dominated by the gravitational interaction with major bodies such as the planets. As the comets come close to planetary regions, planets become the main influence for these objects. Thus, gravitational interactions with planets, such as close encounters and collisions, may have influenced the planets’ history, composition, structure, and evolution (Asphaug et al. 2006; Brasser et al. 2020; Morgan et al. 2021). Examples of these processes include cometary impacts that may be responsible for the dawn-dusk asymmetry of Mercury’s exosphere (e.g., Benz et al. 1988; Pokorný et al. 2017), the changes of the surface and atmosphere on Mars (e.g., Carr 1989; Melosh & Vickery 1989; Woo et al. 2019), and the dynamical evolution of the gas giants and trans-neptunian objects (Gomes et al. 2005; Muñoz-Gutiérrez et al. 2021), for more examples see Stern (1995); Marov & Rickman (2001); Charnoz & Morbidelli (2003). Furthermore, collisions with these remnants may have a dramatic effect on a planet’s orbit. For example, repeated collisions may have resulted in the tilt of Uranus (e.g., Brunini 1995; Parisi & Brunini 1997; Rogoszinski & Hamilton 2021). Lastly, the Chicxulub impact on Earth is suspected to be the main cause of the extinction of the dinosaurs (e.g., Alvarez et al. 1980; Schulte et al. 2010).

Cometary (and other minor body) impacts in the solar system have been extensively studied in the literature (e.g., Opik 1951; Kessler 1981; Greenberg et al. 1988; Bottke & Greenberg 1993; Marov & Rickman 2001; Muinonen et al. 2001; Valsecchi 2005; Rickman et al. 2014). Of particular interest is the impact rate of comet collisions with planets. These calculations are often based on the Öpik’s analytic method (Opik 1951). However, these methods are often tuned to model cometary impacts in the inner part of the solar system. As a result, it is challenging to calculate collision rates in other planetary systems with different architectures, arbitrary configurations and geometries than the solar system.

Here we present a succinct and accurate model to calculate the collision rate and timescale for a minor body to impact a planet. Our methodology is applicable for all geometries and configurations and is consistent with direct numerical calculations. In Sect. 2, we present our model for collisional timescales. In Sect. 3 we test our model by comparing our predictions with a well known analytic method for collision rates in the solar system and detailed N-body simulations. Finally, we discuss our results in Sect. 4.

2 COLLISIONAL TIMESCALES FOR PARTICLE IMPACTS ON PLANETS
Here we present a general analytical approach, to calculate the collision rate of a minor body with a planet for arbitrary geometry of interaction. Hereafter we refer to a minor body as a particle to highlight the wide range of application.
Consider the collision rate, \( \Gamma_{\text{coll}} \) (Eq. (1)), of a particle with a planet. This rate can be calculated by assuming a population of planet-orbital-crossing particles with number density \( n \) that will eventually collide with the planet. The relative velocity between the planet and the particle \( v_{\text{rel}} \), and the cross section of interaction \( \sigma \). We assume an arbitrary configuration for the planet and particle; see Fig. 1 for illustration. Thus, the collision rate can be approximated (e.g., Binney & Tremaine 2008; Nesvorný et al. 2020; Rose et al. 2020)

\[
\Gamma_{\text{coll}} = n v_{\text{rel}} \sigma. \tag{1}
\]

The number density is then simply \( n = N_c/V \), where \( N_c \) is the number of colliding particles and \( V \) is the volume where collision can take place, and is given by,

\[
V = 2\pi^2 a_p R_p^2 \sin i_c, \tag{2}
\]

where \( a_p \) and \( R_p \) are the semi-major axis and radius of the planet respectively and \( i_c \) is the inclination of the particle.

The relative velocity magnitude between the particle and the planet is given by the magnitude difference between the velocity vector of the particle \( v_c \) and the particle \( v_p \), in other words, \( v_{\text{rel}} = |v_{\text{rel}}| = |v_p - v_c| \). To obtain the velocity vectors of the particle and the planet, we first need to establish the geometry of the encounter. We considered a reference plane in which the planet is at the center. The orbital motion of the particle about the star with respect to the center is in three-dimensional space.

The position vector of the particle in the frame of its bound orbit about the host star is: \( r_p = x \hat{\mathbf{x}} + y \hat{\mathbf{y}} + z \hat{\mathbf{z}} \), where \( x = r_c \cos f_c, y = r_c \sin f_c, z = 0 \). Where \( f_c \) is the true anomaly of the particle, and \( r_c \) is given by:

\[
r_c = a_c \frac{1 - e_c^2}{1 + e_c \cos f_c}. \tag{3}
\]

Then, the position vector projected to the plane of the planet is given by \( R_1 R_2 R_3 r_c \), where \( R_j, j = 1, 2, 3 \) are the rotation matrices given by (e.g., Murray & Dermott 2000),

\[
R_1 = \begin{pmatrix} \cos \omega & -\sin \omega & 0 \\ \sin \omega & \cos \omega & 0 \\ 0 & 0 & 1 \end{pmatrix},
R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{pmatrix},
R_3 = \begin{pmatrix} \cos \Omega & -\sin \Omega & 0 \\ \sin \Omega & \cos \Omega & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4}
\]

Therefore, the components of the position vector of the particle projected on the planet’s can be calculated by,

\[
\begin{pmatrix} X' \\ Y' \end{pmatrix} = R_1 R_2 R_3 \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{5}
\]

Consequently, the velocity vectors of the particle \( (v_c) \) and the planet \( (v_p) \) in their own individual orbital planes at any given time are given by

\[
v_{c,p} = \begin{pmatrix} -h_{c,p} \sin f_{c,p} \\ h_{c,p} (e_{c,p} + \cos f_{c,p}) \end{pmatrix} = \begin{pmatrix} -\frac{h_{c,p}}{a_{c,p} (1 - e_{c,p}^2)} \\ \frac{h_{c,p} (e_{c,p} + \cos f_{c,p})}{a_{c,p} (1 - e_{c,p}^2)} \end{pmatrix}, \tag{6}
\]

where, the subscript \( c \) and \( p \) stands for the particle and the planet respectively. The specific angular momentum of the particle (planet) \( h_c \) (\( h_p \)) is given by,

\[
h_c = \sqrt{G (M_s + M_p) a_{c,p} (1 - e_{c,p}^2)}, \tag{7}
\]

where \( a_c (a_p), e_c (e_p) \) and \( f_c (f_p) \) are the semi-major axis, eccentricity and true anomaly of the particle (planet). Additionally, \( M_s \) and \( M_p \) are the mass of the star and the planet respectively. Thus, the relative velocity is given by:

\[
v_{\text{rel}} = |v_p - R_1 R_2 R_3 v_c|. \tag{8}
\]

Where recall that we are rotating the particle velocity vector to the planet frame using \( R_1 R_2 R_3 \). The relative velocity between the particle and the planet is calculated at the moment of collision. The collision of the particle with the planet takes place at the node, where the cometary orbital plane and the planet’s plane coincide.

Finally, the cross section \( \sigma \), enhanced by gravitational focusing, is given by:

\[
\sigma = \pi \left( R_p^2 + \frac{2G M_p}{v_{\text{rel}}^2} \right). \tag{9}
\]

Substituting equations 2, 8, and 9 into Eq. (1) we obtain a final expression for the collision rate per year as a function of the particle’s orbital elements \( \Gamma_{\text{coll}}(a_c, e_c, i_c, \Omega_c, \omega_c, f_c) \):

\[
\Gamma_{\text{coll}} = \frac{N_c}{2\pi^2 a_p R_p^2 \sin i_c} \left( v_{\text{rel}} \pi R_p^2 + \pi R_p \frac{2G M_p}{v_{\text{rel}}} \right). \tag{10}
\]

Equation 10 represents the most general expression for a collision between a minor body and a planet in an arbitrary geometry.

### 3 COMPARISON WITH ANALYTIC AND NUMERICAL METHODS

In this section we tested our model by comparing it with the often use Öpik method for collisions in the solar system (see also Appendix B) and with detailed N-body simulations (Sec.3.3).
3.1 The Öpik Method

The Öpik method (Opik 1951) in its original form provides an expression for the collision rate of particles (asteroids or comets) with planets. In this 1951 method, the planet is assumed to be fixed in space in a circular orbit and the colliding particle on an arbitrary orbit. The collision happened when the orbit of the two bodies intersect. The Öpik method assumes a restricted 3-body problem, considering the small body massless and moving on an unperturbed heliocentric Keplerian orbit.

The Öpik method considers two main parts to calculate the collision rate: the relative velocity between the particle and the planet and the collisional area or cross-section. The Öpik method often uses units \( G = 1 \) and assumes the star’s mass \( M_S = 1 \, M_\odot \).

The reference frame is set so that the particle is at one of the nodes of its orbit when the encounter with the planet occurs. Therefore, the relative velocity \( U_{\text{opik}} \) can be expressed in terms of the Tisserand parameter with respect to the planet and is given by (e.g., Opik 1951; Carusi et al. 1990),

\[
U_{\text{opik}} = \sqrt{3 - T},
\]

(11)

where \( T \), the Tisserand parameter which is defined as,

\[
T = \frac{a_p}{a_c} + 2 \cos i_c \sqrt{\frac{a_c}{a_p} (1 - e_c^2)}.
\]

(12)

The relative velocity components, \( U_x, U_y, \) and \( U_z \) are given by (see e.g., Carusi et al. 1990),

\[
U_x = \pm \sqrt{2 - 1/a_c - a_c(1 - e_c^2)},
\]

\[
U_y = \sqrt{a_c(1 - e_c^2)} \cos i_c - 1,
\]

\[
U_z = \pm \sqrt{a_c(1 - e_c^2)} \sin i_c.
\]

(13)

When a particle reaches the Hill’s sphere of a planet, the Sun’s perturbations can be neglected, and the trajectory of the particle can be modeled as a planetocentric. Once inside of the Hill’s region, a particle can collide with the planet if the pericenter distance of the particle \( q_c \) is smaller or similar to the radius of the planet \( R_P \), i.e., \( q_c \ll R_P \). Therefore the the cross-section for interaction can be expressed by \( \sigma_{\text{opik}} \) (e.g., Opik 1951, 1976),

\[
\sigma_{\text{opik}} = R_P \sqrt{1 + \frac{2 G M_P}{U_{\text{opik}}^2}}.
\]

(14)

With the expression for the encounter velocity \( U_{\text{opik}} \) and the cross-section \( \sigma_{\text{opik}} \), the collision rate per year as a function of the particle’s orbital elements \( \Gamma_{\text{opik}}(a_c, e_c, i_c) \) is calculated as followed.

The Öpik method considers a particle in a heliocentric orbit, which crosses two times an sphere of radius \( r = 1 \). The radial velocity of the particle is \( U_x \). Then the time spent of the particle in the sphere is \( dt = 2dr/U_x \). The number of particles \( N_c \) in the sphere per orbital revolution can be calculated as \( N_c = dt/P \), where \( P \) is the particle’s orbital period. Opik (1951) showed that a particle could only be found inside a band with two parallel latitudes \( \pm \) and a volume \( dV = 4\pi \sin i \, dr \). Thus, the collision rate of a particle can be expressed by (see e.g., Opik 1976, for a detailed derivation),

\[
\Gamma_{\text{opik}} = \frac{\sigma_{\text{opik}}^2 U_{\text{opik}}}{\pi \sin i_c |U_x|}.
\]

(15)

Despite the simplicity, the Öpik’s method yields consistent results for Jupiter-family comets (e.g., Greenberg et al. 1988; Nakamura & Kurahashi 1998; Dones et al. 1999). However, it fails to accurately model the collision rate for those particles with Tisserand parameter \( T \geq 3 \).

Many improvements in Öpik original theory have been done by several authors (e.g., Nakamura & Kurahashi 1998; Manley et al. 1998; Dones et al. 1999; Levison et al. 2000; Zahnle 2001; Vokrouhlický et al. 2012; Pokorný & Vokrouhlický 2013; Rickman et al. 2014; JeongAhn & Malhotra 2017; Vokrouhlický et al. 2019; Abedin et al. 2021), creating a variety of Öpik-like models that address some of the existing issues of the classic method. These Öpik-like methods represent a quick (but at times less accurate) alternative to more robust numerical simulations. However, these expansions are mainly tuned for objects in the inner parts of the solar systems and they lack the flexibility to model minor bodies in exo-planetary systems for a wide range of configurations.

3.2 Numerical Method

We used the N-body package REBOUND (Rein & Liu 2012) with the WH-Fast integrator (Rein & Liu 2012) to calculate the collisional history of minor bodies with a planet. We considered a system composed by a solar mass star and a Jupiter-like planet with semi-major axis \( a_p = 5.2 \, \text{au} \), eccentricity \( e_p = 0.05 \), mass \( M_p = 0.001 \, M_\odot \). We used an inflated collisional radius \( R_p = (M_p/3M_\odot)^{1/3} \, a_p \) (to assure more collisions in shorter time). We added 5000 test particles representing the minor bodies. To compare the numerical simulation with the analytical calculation we construct two representative runs (see Appendix C). In one, \( \text{R-inc} \), we vary only the initial inclination but keep all of the other orbital parameters constant, and in the other, \( \text{R-ecce} \) we vary only the initial eccentricity of the particle. The full set of initial conditions are described in Table 1. We model the collision of the particles as inelastic encounters. For simplicity, every particle that collided with the planet was removed from the simulation. The simulation was run up to \( 10^4 \) yrs. We note that the number of collisions does not converge on this timescale. As a function of time, the number of particles that undergo collisions increases, as expected. We performed a series of tests using a simulation time of \( 10^3 \) yrs, and we did not find qualitatively change our results. In Appendix C we show the results of the simulations.

### Table 1. Input orbital elements of the particle and planet in the numerical simulations: \( a_c, e_c, i_c, \Omega_c, \omega_c, f_c \), \( f_p \) represents the semi-major axis, eccentricity, inclination, longitude of the ascending node, argument of periaipsis, and true anomaly of the particle and the particle, respectively.

| name | \( e_c \) | \( a_c \) | \( i \) | \( \Omega_c \) | \( \omega_c \) | \( f_c \) | \( f_p \) |
|------|--------|--------|-----|--------|--------|-----|-----|
| R-inc | 0.5    | 5.5    | 0 – 180 | 0  | 0  | 0  | 0  |
| R-ecce | 0 – 1 | 5.5    | 27.5 | 0  | 0  | 0  | 0  |
The inconsistency between our model and the Öpik method are differences with the Öpik method. Therefore, we focus on a simple comparison with the Öpik theory for collisions to any exo-planetary system. Therefore, we focus on a simple comparison with the backbone of the theory, the classic Öpik method (Öpik 1976). Our formulation $\Gamma_{\text{coll}}$ provides a succinct solution to determine the collision rate of a particle as a function of its orbital elements ($\Gamma_{\text{coll}}(a_c, e_c, i_c, \Omega_c, \omega_c, f_c)$). These allowed us to model the collision of particles with planets for any encounter geometry and orbit, providing an accurate alternative to costly N-body simulations.

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DATA AVAILABILITY

The python scripts used to generate the data for this work can be accessed here: https://santiago-torres.com/Research

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Figure 2. Comparison between the N-body simulation (solid orange line), the Öpik method (dotted blue line) and the $\Gamma_{\text{coll}}$ (dash-dot red line). The top panel shows the collision rate as a function of the inclination using the values for $R_{\text{inc}}$ listed in Table 1. While the bottom panel shows the distribution for $R_{\text{ecc}}$ and the collision rate as a function of the eccentricity.

As depicted in Fig. 2, our analytical rate calculation, $\Gamma_{\text{coll}}$ is consistent with the N-body rate in both its functional form and value.

Note that $\Gamma_{\text{opik}}$ is at times few orders of magnitude different than the numerical results. Furthermore, as clearly seen in Fig. 2, $\Gamma_{\text{opik}}$ estimated higher collision rate for circular orbits, at odds with the numerical and $\Gamma_{\text{coll}}$ estimations.

4 SUMMARY AND DISCUSSION

Here we present an analytical model to determine the collision rate of a minor body (particle) with a planet for any type encounter geometry and orbit (Eq. (10)). We tested our formulation by comparing with the Öpik method (Sect. 3.1) and detailed N-body simulations (Sect. 3.2). As a proof of concept we choose two representative examples, one for which we vary the colliding particles eccentricities, and the other, by varying their inclinations. Our prediction for the collision rate of those particles with small values of inclination and $|U_x|$, producing a singularity since Eq. (A4) goes to infinity (see Fig. B1). Therefore bodies similar to the centaurs, nearly isotropic and long-period comets, cannot be accurately model following Eq. (A4). These objects are expected to be abundant in exo-planetary systems as a consequence of planet formation (e.g., Wyatt 2008; Johansen & Lambrechts 2017). On the other hand, $\Gamma_{\text{coll}}$ produces better estimations for the collision rate for all varieties of orbital elements. This is because we allow for arbitrary geometry. As a result, the function $\Gamma_{\text{coll}}$ allows us to model any type of minor body orbits given the flexibility to estimate collisional rates in exo-planetary systems accurately.

We note that the Öpik-like methods might provide better estimations than the original one (e.g., Valsecchi 2005; JeongAhn & Malhotra 2017; Vokrouhlický et al. 2019; Abedin et al. 2021). However, a detailed comparison with the variations of the method is beyond the scope of our paper. We omit these comparisons because our intention is not to adapt or extend the Öpik theory for collisions to any exo-planetary system. Therefore, we focus on a simple comparison with the expression of the theory, the classic Öpik method (Öpik 1976).
parameter (Eq. 12) and the relative velocity (Eq. 11) can be rewritten

\[ \frac{GM}{a_p^2} + 2GM \frac{a_c(1-e_c^2)}{a_p^2} \cos i_c \]

\[ T = \frac{GM}{a_p} + 2GM \frac{a_c(1-e_c^2)}{a_p^2} \cos i_c \]

\[ \Gamma = \frac{1}{P} \frac{a_p^2}{2\pi} \int_0^{2\pi} \frac{U_{\text{opik}}}{\sin i_c} U_x \, \frac{d\omega_c}{\omega_c} \]

\[ U_x = \pm \sqrt{GM} \frac{2 - 1 - a_c(1-e_c^2)}{r^2} \]

\[ U_y = \sqrt{GM} \frac{a_c(1-e_c^2)}{r} \cos i_c - 1 \]

\[ U_z = \pm \sqrt{GM} \frac{a_c(1-e_c^2)}{r} \sin i_c \]

\[ \Gamma_{\text{opik}} = \frac{1}{P} \frac{a_p^2}{2\pi} \frac{U_{\text{opik}}}{\sin i_c} U_x \]

\[ \text{Table B1. Input orbital elements of the particle: } a_c, e_c, i_c, \Omega_c, \omega_c, \text{ and } f_c \text{ represents the semi-major axis, eccentricity, inclination, longitude of the ascending node, argument of perigee, and true anomaly of the particle with respect to it's orbit about the star.} \]

| name | \( e_c \) | \( a_c \) | \( i_c \) | \( \Omega_c \) | \( \omega_c \) | \( f_c \) |
|------|--------|--------|--------|--------|--------|--------|
| T-inc | 0.1    | 6      | 0–180  | 0      | 0      | 0      |
| T-ecc | 0–1    | 6      | 2.5    | 0      | 0      | 0      |

\[ \begin{align*}
\Gamma_{\text{opik}} &= \frac{1}{P} \frac{a_p^2}{2\pi} \frac{U_{\text{opik}}}{\sin i_c} U_x \quad \text{(A4)}
\end{align*} \]

where \( P \) is the planet’s orbital period.

APPENDIX B: COMPARISON WITH THE ÖPIK METHOD

In this section, we compare our method for collision rates with the classic Öpik theory to highlight the flexibility of our formulation. We choose two representative examples, comparing the rates behavior as a function of inclination (T-inc) and eccentricity (T-ecc) while keeping all other parameters constant. In both cases we choose the semi-major axis, eccentricity and true anomaly of the planet \( a_p = 5.2 \text{ au}, e_p = 0.05, \text{ and } f_c = 0^\circ \) respectively, while for the particles we choose the values shown in Table B1. For consistency with Sect. 3.2, we used an inflated radius of collision, \( R_p = (M_p/3M_\star)^{1/3} a_p \), where \( M_\star \) is the mass of the host star, taken to be one solar mass. We note that the original formulation of \( \Gamma_{\text{opik}} \) (Eq. 15) uses Jacobi normalized units. Therefore, in order to proper compare with \( \Gamma_{\text{coll}} \) (Eq. 10), we used \( \Gamma_{\text{opik}} \) with the proper units (Eq. A4).

In Fig. B1 top panel, we show the collision rates as a function of the particle inclination with respect to the planet at the onset of collision for T-inc. The two collision rates exhibit similar functional form as a function of the mutual inclination, \( i \) since both are dominated by a similar volume dependency on the mutual inclination \( V \sim \sin i \).

In Fig. B1 bottom panel, we show the collision rates as a function of the particle eccentricity at the onset of collision for T-ecc. As depicted in Fig. B1 bottom panel, our collision rate \( \Gamma_{\text{coll}} \) is qualitatively
different from $\Gamma_{\text{opik}}$. The Opik method predicts higher collision rates for eccentricities less than $\sim 0.85$, and it monotonically decreases from $e = 0$ to $e = 1$. In contrast, $\Gamma_{\text{coll}}$ predicts an increasing distribution with two maximum. The first one for eccentricities within $0$ and $0.1$. While the second for $e_c$ within $0.4 - 1$.

**APPENDIX C: N-BODY SIMULATIONS**

Following the method described in Sect. 3.2 and using the input parameters shown in Table 1, we perform two sets of simulations, **Run-inc** and **Run-ecc**. In **Run-inc** after 10,000 years $\sim 2,123$ particles collided with the planet ($\sim 42.46\%$ of the initial particles). In Fig. C1 we show the distribution of the collided particles, for the semi-major axis (first row), the eccentricity (second row) and the inclination (third row). We find that the particles with inclinations between $0 - 50$ and $130 - 160$ degrees are the most probable for collision. These particles have eccentricities within $0.4 - 0.6$ (see Fig. C1 third column second panel). Additionally, the particles shown a bi-modal distribution in $f_c$, $\omega_c$ and $M_c$, having their maximums around $100^\circ$ and $250^\circ$ (Fig. C2).

In Figures C3 and C4 we show the results of the simulation **Run-ecc**. We find that $\sim 74.84\%$ of the particles remained in the system after 10,000 years. The eccentricity of the collided particles ($\sim 1,258$) formed a distribution with three peaks with maximums around $0.1$, $0.4$ and $0.7$ (second row Fig. C3). Particles with eccentricity $\sim 0.6$ did not collide. The longitude of the ascending node $\Omega_c$ of the particles has a preferred angle within $-100 - 0$ degrees (second-row Fig. C4). These differed from the distribution of $\Omega_c$ in **Run-inc**, where the particles have Gaussian distribution with maximum $\sim 0$. Overall, we find that the collided particles in **Run-inc** and **Run-ecc** have a strong dependency in the initial orbital elements, in particular the inclination and eccentricity.

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Figure C1. Run-inc. Density map of semi-major axis, eccentricity and inclination of the collided particles.
Figure C2. Run-inc. Orbital elements density map of the collided particles. First row, shows the true anomaly $f_c$ as function of the semi-major axis $a_c$, eccentricity $e_c$ and inclination $i_c$. Second row, shows the longitude of the ascending node $\Omega_c$ as function of $a_c$, $e_c$, and $i_c$. Third row, shows the argument of periapsis $\omega_c$ as function of $a_c$, $e_c$, and $i_c$. Finally, last row shows the mean anomaly $M_c$ as function of $a_c$, $e_c$, and $i_c$. 

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Figure C3. Run-ecc. Density map of Semi-major axis, eccentricity and inclination of the collided particles.
Figure C4. Run-ecc. Orbital elements density map of the collided particles. First row, shows the true anomaly $f_c$ as function of the semi-major axis $a_c$, eccentricity $e_c$ and inclination $i_c$. Second row, shows the longitude of the ascending node $\Omega_c$ as function of $a_c$, $e_c$, and $i_c$. Third row, shows the argument of periaxis $\omega_c$ as function of $a_c$, $e_c$, and $i_c$. Finally, last row shows the mean anomaly $M_c$ as function of $a_c$, $e_c$, and $i_c$. 