Spectral Subspace Sparsification

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Abstract

We introduce a new approach to spectral sparsification that approximates the quadratic form of the pseudoinverse of a graph Laplacian restricted to a subspace. We show that sparsifiers with a near-linear number of edges in the dimension of the subspace exist. Our setting generalizes that of Schur complement sparsifiers. Our approach produces sparsifiers by sampling a uniformly random spanning tree of the input graph and using that tree to guide an edge elimination procedure that contracts, deletes, and reweights edges. In the context of Schur complement sparsifiers, our approach has two benefits over prior work. First, it produces a sparsifier in almost-linear time with no runtime dependence on the desired error. We directly exploit this to compute approximate effective resistances for a small set of vertex pairs in faster time than prior work (Durfee-Kyng-Peebles-Rao-Sachdeva ’17). Secondly, it yields sparsifiers that are reweighted minors of the input graph. As a result, we give a near-optimal answer to a variant of the Steiner point removal problem.

A key ingredient of our algorithm is a subroutine of independent interest: a near-linear time algorithm that, given a chosen set of vertices, builds a data structure from which we can query a multiplicative approximation to the decrease in the effective resistance between two vertices after identifying all vertices in the chosen set to a single vertex with inverse polynomial additional additive error in near-constant time.

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1 Introduction

Graph sparsification has had a number of applications throughout algorithms and theoretical computer science. In this work, we loosen the requirements of spectral sparsification and show that this loosening enables us to obtain sparsifiers with fewer edges. Specifically, instead of requiring that the Laplacian pseudoinverse quadratic form is approximated for every vector, we just require that the sparsifier approximates the Laplacian pseudoinverse quadratic form on a subspace:

**Definition 1.1 (Spectral subspace sparsifiers).** Consider a weighted graph $G$, a vector space $S \subseteq \mathbb{R}^{V(G)}$ that is orthogonal to $1^{V(G)}$, and $\epsilon \in (0, 1)$. For a minor $H$ of $G$ with contraction map $\phi : V(G) \to V(H)$, let $P \in \mathbb{R}^{V(H) \times V(G)}$ be a matrix with $P_{uv} = 1[u = \phi(v)]$ for all $u \in V(H), v \in V(G)$. A reweighted minor $H$ of $G$ is called an $(S, \epsilon)$-spectral subspace sparsifier if for all vectors $x \in S$,

$$(1 - \epsilon)x^T L_G^+ x \leq x_H^T L_H^+ x_H \leq (1 + \epsilon)x^T L_G^+ x$$

where $x_H := Px$.

[KMST10] also considers a form of specific form of subspace sparsification related to controlling the $k$ smallest eigenvalues of a spectral sparsifier for $S = \mathbb{R}^{V(G)}$. When $S$ is the dimension $|S| - 1$ subspace of $\mathbb{R}^{|S|} \times 0^{|V(G) - |S||}$ that is orthogonal to $1^{V(G)}$, a $(S, \epsilon)$-spectral subspace sparsifier is a sparsifier for the Schur complement of $G$ restricted to the set of vertices $S$. Schur complement sparsifiers are implicitly constructed in [KS16] and [KLP+16] by an approximate Gaussian elimination procedure and have been used throughout spectral graph theory. For example, they are used in algorithms for random spanning tree generation [DKP+17], approximate maximum flow [MP13], and effective resistance computation [GHP18, GHP17, DKP+17].

Unlike the existing construction of Schur complement sparsifiers [DKP+17], our algorithm (a) produces a sparsifier with vertices outside of $S$ and (b) produces a sparsifier that is a minor of the input graph. While (a) is a disadvantage to our approach, it is not a problem in applications, in which the number of edges in the sparsifier is the most relevant feature for performance, as illustrated by our almost-optimal algorithm for $\epsilon$-approximate effective resistance computation. (b) is an additional benefit to our construction and connects to the well-studied class of Steiner point removal problems [CGH16, EGK+14].

In the Approximate Terminal Distance Preservation problem [CGH16], one is given a graph $G$ and a set of $k$ vertices $S$. One is asked find a reweighted minor $H$ of $G$ with size $\text{poly}(k)$ for which

$$d_G(u, v) \leq d_H(u, v) \leq \alpha d_G(u, v)$$

for all $u, v \in S$ and some small distortion $\alpha > 1$. The fact that $H$ is a minor of $G$ is particularly useful in the context of planar graphs. One can equivalently phrase this problem as a problem of finding a minor $H$ in which the $\ell_1$-norm of the $\ell_1$-minimizing flow between any two vertices $s, t \in S$ is within an $\alpha$-factor of the $\ell_1$ norm of the $\ell_1$-minimizing $s - t$ flow in $G$. The analogous problem for $\ell_\infty$ norms is the problem of constructing a flow sparsifier (with non-$s - t$ demands as well). Despite much work on flow sparsifiers [Moi09, LM10, CLLM10]
[MM10, EGK+14, Chu12, AGK14, RST14], it is still not known whether $\alpha = (1 + \epsilon)$-flow sparsifiers with size $\text{poly}(k, 1/\epsilon)$ exist, even when the sparsifier is not a minor of the original graph.

1.1 Our Results

Our main result is the following:

**Theorem 1.2.** Consider a weighted graph $G$, a $d$-dimensional vector space $S \subseteq \mathbb{R}^{V(G)}$, and $\epsilon \in (0, 1)$. Then an $(S, \epsilon)$-spectral subspace sparsifier for $G$ with $O\left(\frac{d \log d}{\epsilon^2}\right)$ edges exists.

When $S$ is the maximal subspace of $\mathbb{R}^S \times \mathbf{0}^{V(G) \setminus S}$ orthogonal to $\mathbf{1}^{V(G)}$ for some set of vertices $S \subseteq V(G)$, $(S, \epsilon)$-spectral subspace sparsifiers satisfy the same approximation guarantee as Schur complement sparsifiers. The approximation guarantee of a spectral subspace sparsifier $H$ of $G$ is equivalent to saying that for any demand vector $d \in S$, the energy of the $\ell_2$-minimizing flow for $d$ in $H$ is within a $(1 + \epsilon)$ factor of the energy for the $\ell_2$-minimizing flow for $d$ in $G$. This yields an near-optimal (up to a $\log d$ factor) answer to the $(1 + \epsilon)$-approximate Steiner vertex removal problem for the $\ell_2$ norm. The $\ell_2$ version is substantially different from the $\ell_1$ problem, in which there do not exist $o(k^2)$-size minors that $2$-approximate all terminal distances [CGH16].

Unlike Schur complement sparsifiers, $(\mathbb{R}^S, \epsilon)$-spectral subspace sparsifiers may contain “Steiner nodes;” i.e. vertices outside of $S$. This is generally not relevant in applications, as we illustrate in Section 6. Allowing Steiner nodes allows us to obtain sparsifiers with fewer edges, which in turn allows us to obtain faster constructions. Specifically, we show the following result:

**Theorem 1.3.** Consider a weighted graph $G$, a set of vertices $S \subseteq V(G)$, and $\epsilon \in (0, 1)$. Let $T_{\text{rst}}(G)$ denote the time it takes to generate a random spanning tree from a distribution with total variation distance at most $1/m^{10}$ from the uniform distribution. Then $(\mathbb{R}^S \times \mathbf{0}^{V(G) \setminus S}, \epsilon)$-spectral subspace sparsifier for $G$ with $\min(m, O\left(|S|^{\log \log(n)} \epsilon^2\right))$ edges can be constructed in $T_{\text{rst}}(G) + O(m \text{polylog}(n)) \leq m^{1+o(1)}$ time.

This sparsifier has as many edges as the Schur complement sparsifier given in [DKP+17] but improves on their $\tilde{O}(m + n/\epsilon^2)$ runtime. An important ingredient in the above construction is an algorithm for multiplicatively approximating changes in effective resistances due to certain modifications of $G$. This algorithm is called with $\delta = \Theta(1)$ in this paper:

**Lemma 1.4.** Consider a weighted graph $G$, a set of vertices $S \subseteq V(G)$, and $\delta_0, \delta_1 \in (0, 1)$. There is an $O(\text{mpolylog}(n) \log(m/\delta_1)/\delta_0^2)$-time algorithm $\text{DiffApx}(G, S, \delta_0, \delta_1)$ that outputs numbers $\nu_e$ for all $e \in E(G)$ with the guarantee that

$$(1 - \delta_0)\nu_e - \delta_1 \leq \frac{b_e^T L_G^+ b_e}{r_e} - \frac{b_e^T L_G^+ L_{G/S} b_e}{r_e} \leq (1 + \delta_0)\nu_e + \delta_1$$

Finally, we replace the use of Theorem 6.1 in [DKP+17] with our Theorem 1.3 in their improvement to Johnson-Lindenstrauss to obtain a faster algorithm.
Corollary 1.5. Consider a weighted graph $G$, a set of pairs of vertices $P \subseteq V(G) \times V(G)$, and an $\epsilon \in (0,1)$. There is an $m^{1+o(1)} + \tilde{O}(|P|/\epsilon^2)$-time algorithm $\text{ResApx}(G, P, \epsilon)$ that outputs $(1 + \epsilon)$-multiplicative approximations to the quantities

$$b^T_{uv} L^+_G b_{uv}$$

for all pairs $(u, v) \in P$.

This directly improves upon the algorithm in [DKP+17], which takes $O((m + (n + |P|)/\epsilon^2)\text{polylog}(n))$-time.

1.2 Technical Overview

To construct Schur complement sparsifiers, [DKP+17] eliminates vertices one-by-one and sparsifies the cliques resulting from those eliminations. This approach is fundamentally limited in that each clique sparsification takes $\Omega(1/\epsilon^2)$ time in general. Furthermore, in the $n+1$ vertex star graph with $n$ vertices $v_1, v_2, \ldots, v_n$ connected to a single vertex $v_{n+1}$, a $(1+\epsilon)$-approximate Schur complement sparsifier without Steiner vertices for the set $\{v_1, v_2, \ldots, v_n\}$ must contain $\Omega(n/\epsilon^2)$ edges. As a result, it seems difficult to obtain Schur complement sparsifiers in time less than $\tilde{O}(m + n/\epsilon^2)$ time using vertex elimination.

Instead, we eliminate edges from a graph by contracting or deleting them. Edge elimination has the attractive feature that, unlike vertex elimination, it always reduces the number of edges. Start by letting $H := G$. To eliminate an edge $e$ from the current graph $H$, sample $X_e \sim \text{Ber}(p_e)$ for some probability $p_e$ depending on $e$, contract $e$ if $X_e = 1$, and delete $e$ if $X_e = 0$.

To analyze the sparsifier produced by this procedure, we set up a matrix-valued martingale and reduce the problem to bounding the maximum and minimum eigenvalues of a random matrix with expectation equal to the identity matrix. The right value for $p_e$ for preserving this matrix in expectation turns out to be the probability that a uniformly random spanning tree of $H$ contains the edge $e$. To bound the variance of the martingale, one can use the Sherman-Morrison rank one update formula to bound the change in $L^+_H$ due to contracting or deleting the edge $e$. When doing this, one sees that the maximum change in eigenvalue is at most a constant times

$$\max_{x \in S} \frac{(x^T L^+_H b_e)^2}{r_e \min(\text{lev}_H(e), 1 - \text{lev}_H(e)) (x^T L^+_G x)}$$

where $\text{lev}_H(e)$ is the probability that $e$ is in a uniformly random spanning tree of $H$. This quantity is naturally viewed as the quotient of two quantities:

(a) The maximum fractional energy contribution of $e$ to any demand vector in $S$’s electrical flow.

(b) The minimum of the probabilities that $e$ is in or is not in a uniformly random spanning tree of $H$. 

4
We now make the edge elimination algorithm more specific to bound these two quantities. Quantity (a) is small on average over all edges in \( e \) (see Proposition 3.9), so choosing the lowest-energy edge yields a good bound on the maximum change. To get a good enough bound on the stepwise martingale variance, it suffices to sample an edge uniformly at random from the half of edges with lowest energy. Quantity (b) is often not bounded away from 0, but can be made so by modifying the sampling procedure. Instead of contracting or deleting the edge \( e \), start by splitting it into two parallel edges with double the resistance or two series edges with half the resistance, depending on whether or not \( 1 \text{ev}_\mathcal{H}(e) \leq 1/2 \). Then, pick one of the halves \( e_0 \), contract it with probability \( p_{e_0} \), or delete it otherwise. This produces a graph in which the edge \( e \) is either contracted, deleted, or reweighted. This procedure suffices for proving our main existence result (Theorem 1.2). This technique is similar to the technique used to prove Lemma 1.4 of [Sch17].

While the above algorithm does take polynomial time, it does not take almost-linear time. We can accelerate it by batching edge eliminations together using what we call steady oracles. The contraction/deletion/reweight decisions for edges in \( \mathcal{H} \) during each batch can be made by sampling just one \( 1/m^{10} \)-approximate uniformly random spanning tree, which takes \( m^{1+o(1)} \) time. The main remaining difficulty is finding a large set of edges for which quantity (a) does not change much over the course of many edge contractions/deletions. To show the existence of such a set, we exploit electrical flow localization [SRS17]. To find this set, we use matrix sketching and a new primitive for approximating the change in leverage score due to the identification of some set of vertices \( S \) (Lemma 1.4), which may be of independent interest. The primitive for approximating the change works by writing the change in an Euclidean norm, reducing the dimension by Johnson-Lindenstrauss Lemma, and then computing the embedding by Fast Laplacian Solvers in near-linear time.

We conclude by briefly discussing why localization is relevant for showing that quantity (a) does not change over the course of many iterations. The square root of the energy contribution of an edge \( e \) to \( x \)'s electrical flow after deleting an edge \( f \) is

\[
\frac{x^T L^+_\mathcal{H} b_e}{\sqrt{\text{ev}(e)}} = \frac{x^T L^+_\mathcal{H} b_e}{\sqrt{\text{ev}(e)}} + \frac{(x^T L^+_\mathcal{H} b_f)(b^T_f L^+_\mathcal{H} b_e)}{(r_f - b^T_f L^+_\mathcal{H} b_f)\sqrt{\text{ev}(e)}}
\]

\[
= \frac{x^T L^+_\mathcal{H} b_e}{\sqrt{\text{ev}(e)}} + \frac{1}{1 - \text{ev}_\mathcal{H}(f)} \frac{x^T L^+_\mathcal{H} b_f b^T_f L^+_\mathcal{H} b_e}{\sqrt{r_f} \sqrt{r_f} \sqrt{\text{ev}(e)}}
\]

\[
\leq \frac{x^T L^+_\mathcal{H} b_e}{\sqrt{\text{ev}(e)}} + \frac{1}{1 - \text{ev}_\mathcal{H}(f)} \frac{x^T L^+_\mathcal{H} b_f b^T_f L^+_\mathcal{H} b_e}{\sqrt{r_f} \sqrt{r_f} \sqrt{\text{ev}(e)}}
\]

by Sherman-Morrison. In particular, the new energy on \( e \) is at most the old energy plus some multiple of the energy on the deleted edge \( f \). By [SRS17], the average value of this multiplier over all edges \( e \) and \( f \) is \( \tilde{O}(1/|E(\mathcal{H})|) \), which means that the algorithm can do \( \tilde{O}(|E(\mathcal{H})|) \) edge deletions/contractions without seeing the maximum energy on edges \( e \) change by more than a factor of 2.

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2 Preliminaries

2.1 Graphs and Laplacians

For a graph \( G \) and a subset of vertices \( S \), let \( G/S \) denote the graph obtained by identifying \( S \) to a single vertex \( s \). Specifically, for any edge \( e = \{u, v\} \) in \( G \), replace each endpoint \( u, v \in S \) with \( s \) and do not change any endpoint not in \( S \). Then, remove all self-loops to obtain \( G/S \).

Let \( G = (V(G), E(G)) \) be a weighted undirected graph with \( n \) vertices, \( m \) edges, and edge weights \( \{w_e\}_{e \in E(G)} \). The Laplacian of \( G \) is an \( n \times n \) matrix given by:

\[
(L_G)_{u,v} := \begin{cases} 
-w_{(u,v)} & \text{if } u \neq v \text{ and } (u, v) \in E(G), \\
\sum_{(u,w) \in E(G)} w_{(u,w)} & \text{if } u = v, \\
0 & \text{otherwise.}
\end{cases}
\]

We define edge resistances \( \{r_e\}_{e \in E(G)} \) by \( r_e = 1/w_e \) for all \( e \in E(G) \).

If we orient every edge \( e \in E(G) \) arbitrarily, we can define the signed edge-vertex incidence matrix \( B_G \) by

\[
(B_G)_{e,u} := \begin{cases} 
1 & \text{if } u \text{ is } e \text{'s head,} \\
-1 & \text{if } u \text{ is } e \text{'s tail,} \\
0 & \text{otherwise.}
\end{cases}
\]

Then we can write \( L_G \) as

\[
L_G = B_G^T W_G B_G,
\]

where \( W_G \) is a diagonal matrix with \( (W_G)_{e,e} = w_e \).

For vertex sets \( S, T \subseteq V \), \( (L_G)_{S,T} \) denotes the submatrix of \( L_G \) with row indices in \( S \) and column indices in \( T \).

\( L_G \) is always positive semidefinite, and only has one zero eigenvalue if \( G \) is connected. For a connected graph \( G \), let \( 0 = \lambda_1(L_G) < \lambda_2(L_G) \leq \ldots \leq \lambda_n(L_G) \) be the eigenvalues of \( L_G \). Let \( u_1, u_2, \ldots, u_n \) be the corresponding set of orthonormal eigenvectors. Then, we can diagonalize \( L_G \) and write

\[
L_G = \sum_{i=2}^{n} \lambda_i(L_G) u_i u_i^T.
\]

The pseudoinverse of \( L_G \) is then given by

\[
L_G^+ = \sum_{i=2}^{n} \frac{1}{\lambda_i(L_G)} u_i u_i^T.
\]

In the rest of the paper, we will write \( \lambda_{\min}(\cdot) \) to denote the smallest eigenvalue and \( \lambda_{\max}(\cdot) \) to denote the largest eigenvalue. We will also write \( \sigma_{\max}(\cdot) \) to denote the largest singular value, which is given by

\[
\sigma_{\max}(A) = \sqrt{\lambda_{\max}(A^T A)}
\]

for any matrix \( A \).

We will also need to use Schur complements which are defined as follows:
**Definition 2.1** (Schur Complements). The Schur complement of a graph $G$ onto a subset of vertices $S \subset V(G)$, denoted by $SC(G, S)$ or $SC(L_G, S)$, is defined as

$$SC(L_G, S) = (L_G)_{S,S} - (L_G)_{S,T}(L_G)_{T,T}^{-1}(L_G)_{T,S},$$

where $T := V(G) \setminus S$.

The fact below relates Schur complements to the inverse of graph Laplacian:

**Fact 2.2** (see, e.g., Fact 5.4 in [DKP+17]). For any graph $G$ and $S \subset V(G)$,

$$
(I - \frac{1}{|S|} J)(L_G)_{S,S}^{-1}(I - \frac{1}{|S|} J) = (SC(L_G, S))^+,
$$

where $I$ denotes the identity matrix, and $J$ denotes the matrix whose entries are all 1.

### 2.2 Leverage scores and rank one updates

For a graph $G$ and an edge $e \in E(G)$, let $b_e \in \mathbb{R}^{V(G)}$ denote the signed indicator vector of the edge $e$; that is the vector with $-1$ on one endpoint, $1$ on the other, and $0$ everywhere else. Define the **leverage score** of $e$ to be the quantity

$$\text{lev}_G(e) := \frac{b_e^T L_G^+ b_e}{r_e}.$$ 

Let $d_1, d_2 \in \mathbb{R}^n$ be two vectors with $d_1, d_2 \perp 1^{V(G)}$. Then the following results hold by the Sherman-Morrison rank 1 update formula:

**Proposition 2.3.** For a graph $G$ and an edge $f$, let $G \setminus f$ denote the graph with $f$ deleted. Then

$$d_1^T L_{G \setminus f}^+ d_2 = d_1^T L_G^+ d_2 + \frac{(d_1^T L_G^+ b_f)(b_f^T L_G^+ d_2)}{r_f - b_f^T L_G^+ b_f}.$$ 

**Proposition 2.4.** For a graph $G$ and an edge $f$, let $G/f$ denote the graph with $f$ contracted. Then

$$d_1^T L_{G/f}^+ d_2 = d_1^T L_G^+ d_2 - \frac{(d_1^T L_G^+ b_f)(b_f^T L_G^+ d_2)}{b_f^T L_G^+ b_f}.$$ 

### 2.3 Random spanning trees

We use the following result on uniform random spanning tree generation:

**Theorem 2.5** (Theorem 1.2 of [Sch17]). Given a weighted graph $G$ with $m$ edges, a random spanning tree $T$ of $G$ can be sampled from a distribution with total variation distance at most $1/m^{10}$ from the uniform distribution in time $m^{1+o(1)}$.

Let $T \sim G$ denote the uniform distribution over spanning trees of $G$. We also use the following classic result:

**Theorem 2.6** ([Kir47]). For any edge $e \in E(G)$, $\Pr_{T \sim G}[e \in T] = \text{lev}_G(e)$.

For an edge $e \in E(G)$, let $G[e]$ denote a random graph obtained by contracting $e$ with probability $\text{lev}_G(e)$ and deleting $e$ otherwise.
2.4 Some useful bounds and tools

We now describe some useful bounds/tools we will need in our algorithms. In all the following bounds, we define the quantities $w_{\text{max}}$ and $w_{\text{min}}$ as follows:

$$w_{\text{max}} := \max \left\{ 1, \max_{e \in E(G)} \frac{1}{r_e} \right\},$$
$$w_{\text{min}} := \min \left\{ 1, \min_{e \in E(G)} \frac{1}{r_e} \right\}.$$

The following lemma bounds the range of eigenvalues for Laplacians and SDDM matrices:

**Lemma 2.7.** For any Laplacian $L_G$ and $S \subset V(G)$,

1. $\lambda_2(L_G) \geq \frac{w_{\text{min}}}{n^2},$
2. $\lambda_{\text{min}}((L_G)_{S,S}) \geq \frac{w_{\text{min}}}{n^2},$
3. $\lambda_{\text{max}}((L_G)_{S,S}) \leq \lambda_{\text{max}}(L_G) \leq n w_{\text{max}}.$

**Proof.** Deferred to Appendix A.

The lemma below gives upper bounds on the largest eigenvalues/singular values for some useful matrices:

**Lemma 2.8.** The following upper bounds on the largest singular values/eigenvalues hold:

1. $\sigma_{\text{max}}(W_1^{1/2} B_G) \leq \sqrt{n w_{\text{max}}},$
2. $\lambda_{\text{max}}(SC(L_G, S)) \leq n w_{\text{max}},$
3. $\sigma_{\text{max}}((L_G)_{S,T}) = \sigma_{\text{max}}((L_G)_{T,S}) \leq n w_{\text{max}},$

where $T := V(G) \setminus S$.

**Proof.** Deferred to Appendix B.

We will need to invoke Fast Laplacian Solvers to apply the inverse of a Laplacian of an SDDM matrix. The following lemma characterizes the performance of Fast Laplacian Solvers:

**Lemma 2.9 (Fast Laplacian Solver [ST14, CKM+14]).** There is an algorithm $\tilde{x} = \text{LaplSolve}(M, b, \epsilon)$ which takes a matrix $M_{n \times n}$ either a Laplacian or an SDDM matrix with $m$ nonzero entries, a vector $b \in \mathbb{R}^n$, and an error parameter $\epsilon > 0$, and returns a vector $\tilde{x} \in \mathbb{R}^n$ such that

$$\|x - \tilde{x}\|_M \leq \epsilon \|x\|_M$$

holds with high probability, where $\|x\|_M := \sqrt{x^T M x}$, $x := M^{-1} b$, and $M^{-1}$ denotes the pseudoinverse of $M$ when $M$ is a Laplacian. The algorithm runs in time $O(m \text{polylog}(n) \log(1/\epsilon))$.

The following lemmas show how to bound the errors of Fast Laplacian Solvers in terms of $\ell_2$ norms, which follows directly from the bounds on Laplacian eigenvalues in Lemma 2.7.
Lemma 2.10. For any Laplacian $L_G$, vectors $x, \tilde{x} \in \mathbb{R}^n$ both orthogonal to 1, and real number $\epsilon > 0$ satisfying
\[ \|x - \tilde{x}\|_{L_G} \leq \epsilon \|x\|_{L_G}, \]
the following statement holds:
\[ \|x - \tilde{x}\| \leq \epsilon n^{1.5} \left( \frac{w_{\max}}{w_{\min}} \right)^{1/2} \|x\|. \]

Proof. Deferred to Appendix C.

Lemma 2.11. For any Laplacian $L_G$, $S \subset V$, vectors $x, \tilde{x} \in \mathbb{R}^{|S|}$, and real number $\epsilon > 0$ satisfying
\[ \|x - \tilde{x}\|_M \leq \epsilon \|x\|_M, \]
where $M := (L_G)_{S,S}$, the following statement holds:
\[ \|x - \tilde{x}\| \leq \epsilon n^{1.5} \left( \frac{w_{\max}}{w_{\min}} \right)^{1/2} \|x\|. \]

Proof. Deferred to Appendix C.

When computing the changes in effective resistances due to the identification of a given vertex set (i.e. merging vertices in that set and deleting any self loops formed), we will need to use Johnson-Lindenstrauss lemma to reduce dimensions:

Lemma 2.12 (Johnson-Lindenstrauss Lemma [JL84, Ach01]). Let $v_1, v_2, \ldots, v_n \in \mathbb{R}^d$ be fixed vectors and $0 < \epsilon < 1$ be a real number. Let $k$ be a positive integer such that $k \geq 24 \log n/\epsilon^2$ and $Q_{k \times d}$ be a random $\pm 1$ matrix. With high probability, the following statement holds for any $1 \leq i, j \leq n$:
\[ (1 - \epsilon) \|v_i - v_j\|^2 \leq \|Qv_i - Qv_j\|^2 \leq (1 + \epsilon) \|v_i - v_j\|^2. \]

3 Existence of sparsifiers

In this section, we reduce the construction of spectral subspace sparsifiers to an oracle that outputs edges that have low energy with respect to every demand vector in the chosen subspace $S$. We prove it by splitting and conditioning on edges being present in a uniformly random spanning tree one-by-one until $\tilde{O}(d/\epsilon^2)$ edges are left. This construction is a high-dimensional generalization of the construction given in Section 10.1 of [Sch17]. We use the following matrix concentration inequality:
Theorem 3.1 (Matrix Freedman Inequality applied to symmetric matrices [Tro11]). Consider a matrix martingale $(Y_k)_{k \geq 0}$ whose values are symmetric matrices with dimension $s$, and let $(X_k)_{k \geq 1}$ be the difference sequence $X_k := Y_k - Y_{k-1}$. Assume that the difference sequence is uniformly bounded in the sense that

$$\lambda_{\text{max}}(X_k) \leq R$$

almost surely for $k \geq 1$. Define the predictable quadratic variation process of the martingale:

$$W_k := \sum_{j=1}^{k} E[X_j^2 | Y_{j-1}]$$

Then, for all $t \geq 0$ and $\sigma^2 > 0$,

$$\Pr[\exists k \geq 0 : \lambda_{\text{max}}(Y_k - Y_0) \geq t \text{ and } \lambda_{\text{max}}(W_k) \leq \sigma^2] \leq s \exp\left(\frac{-t^2/2}{\sigma^2 + Rt/3}\right)$$

Now, we give an algorithm $\text{SubspaceSparsifier}(G, S, \varepsilon)$ that proves Theorem 1.2. The algorithm simply splits and conditions on the edge that minimizes the martingale difference repeatedly until there are too few edges left. For efficiency purposes, $\text{SubspaceSparsifier}(G, S, \varepsilon)$ receives martingale-difference-minimizing edges from a steady oracle $\mathcal{O}$ with the additional guarantee that differences remain small after many edge updates. This oracle is similar to the stable oracles given in Section 10 of [Sch17].

Definition 3.2 (Steady oracles). A $(\rho, K(z))$-steady oracle is a function $Z \leftarrow \mathcal{O}(I, S)$ that takes in a graph $I$ and a subspace $S \subseteq \mathbb{R}^{V(I)}$ that satisfy the following condition:

- (Leverage scores) For all $e \in E(I)$, $\text{lev}_I(e) \in [3/16, 13/16]$.

and outputs a set $Z \subseteq E(I)$. Let $I_0 = I$ and for each $i > 0$, obtain $I_i$ by picking a uniformly random edge $f_{i-1} \in Z$, arbitrarily letting $I_i \leftarrow I_{i-1} \setminus f_{i-1}$ or $I_i \leftarrow I_{i-1}/f_{i-1}$, and letting $Z \leftarrow Z \setminus \{f_{i-1}\}$. $\mathcal{O}$ satisfies the following guarantees with high probability for all $i < K(|E(I)|)$:

- (Size of $Z$) $|Z| \geq |E(I)|/\rho$
- (Leverage score stability) $\text{lev}_{I_i}(f_i) \in [1/8, 7/8]$
- (Martingale change stability) $\max_{x \in S} \frac{(x_i^T L_i^k b_{i,k})^2}{r_{i,k}(x^T L_i x)} \leq \rho \frac{\text{dim}(S)}{|E(I)|}$

We now state the main result of this section:

Lemma 3.3. Consider a weighted graph $G$, a $d$-dimensional vector space $S \subseteq \mathbb{R}^{V(G)}$, and $\varepsilon \in (0, 1)$. There is an algorithm $\text{SubspaceSparsifier}(G, S, \varepsilon)$ that, given access to a $(\rho, K(z))$-steady-oracle $\mathcal{O}$, computes a $(S, \varepsilon)$-spectral subspace sparsifier for $G$ with

$$O\left(\frac{\rho^2 d \ln d}{\varepsilon^2}\right)$$
edges in time

\[ O \left( (\log n)( \max_{z \leq |E(G)|} z/K(z)) (T_{rst} + T_O + m) \right) \leq O \left( (\log n)( \max_{z \leq |E(G)|} z/K(z))(m^{1+o(1)} + T_O) \right) \]

where \( T_{rst} \) is the time required to generate a spanning tree of \( G \) from a distribution with total variation distance \( \leq n^{-10} \) from uniform and \( T_O \) is the runtime of the oracle.

The algorithm will use two simple subroutines that modify the graph by splitting edges. \( \text{Split} \) replaces each edge with approximate leverage score less than 1/2 with a two-edge path and each edge with approximate leverage score greater than 1/2 with two parallel edges. \( \text{Unsplit} \) reverses this split for all pairs that remain in the graph. We prove the following two results about this subroutine in the appendix:

**Proposition 3.4.** There is a linear-time algorithm \((I, P) \leftarrow \text{Split}(H)\) that, given a graph \( H \), produces a graph \( I \) with \( V(H) \subseteq V(I) \) and a set of pairs of edges \( P \) with the following additional guarantees:

- (Electrical equivalence) For all \( x \in \mathbb{R}^{V(I)} \) that are supported on \( V(H) \), \( x^T L_I^+ x = x^T_L L_H^+ x_H \).
- (Bounded leverage scores) For all \( e \in E(I) \), \( \text{lev}(e) \in [3/16, 13/16] \)
- (P description) Every edge in \( I \) is in exactly one pair in \( P \). Furthermore, there is a bijection between pairs \((e_0, e_1) \in P \) and edges \( e \in E(H) \) for which either (a) \( e_0, e_1 \) and \( e \) have the same endpoint pair or (b) \( e_0 = \{u, w\}, e_1 = \{w, v\} \), and \( e = \{u, w\} \) for some degree 2 vertex \( w \).

**Proposition 3.5.** There is a linear-time algorithm \( H \leftarrow \text{Unsplit}(I, P) \) that, given a graph \( I \) and a set of pairs \( P \) of edges in \( I \), produces a minor \( H \) with \( V(H) \subseteq V(I) \) and the following additional guarantees:

- (Electrical equivalence) For all \( x \in \mathbb{R}^{V(I)} \) that are supported on \( V(H) \), \( x^T L_I^+ x = x^T_H L_H^+ x_H \).
- (Edges of \( H \)) There is a surjective map \( \phi : E(I) \to E(H) \) from non-self-loop, non-leaf edges of \( I \) such that for any pair \((e_0, e_1) \in P \), \( \phi(e_0) = \phi(e_1) \). Furthermore, for each \( e \in E(H) \), either (a) \( \phi^{-1}(e) = e \), (b) \( \phi^{-1}(e) = \{e_0, e_1\} \), with \((e_0, e_1) \in P \) and \( e_0, e_1 \) having the same endpoints as \( e \) or (c) \( \phi^{-1}(e) = \{e_0, e_1\} \), with \((e_0, e_1) \in P \) and \( e_0 = \{u, w\}, e_1 = \{w, v\} \), and \( e = \{u, v\} \) for a degree 2 vertex \( w \).
Let $S_3$. κ from the uniform distribution rather than from a distribution with total variation distance and let $H^{10000}$ loop iterations. Let $I$ all $i$ so it suffices to show that $f$ be the vector with $v$ $k$ $t > 0$ and $H^{10000}$ $G$. We start by defining the first family, which just consists of one martingale. Let $H_0 := G$ and let $H_k$ be the graph $H$ between iterations $k$ and $k+1$ of the while loop of SubspaceSparsifier. Let $d = \dim(S)$. Since $S$ is orthogonal to $1^{V(G)}$, $\dim((L_G^+)^{1/2}S) = \dim(S) = d$, which means that $S$ has a basis $\{y_i\}_{i=1}^d$ for which $y^T_i L^+_G y_j = 0$ for all $i \neq j \in [d]$ and $y^T_i L^+_G y_i = 1$ for all $i \in [d]$. Let $Y_k$ be the $|V(H_k)| \times d$ matrix with $i$th column $(y_i)_{H_k}$ and let $Y := Y_0$. Let $M_k := Y_T^T L^+_G H_k Y = Y^T L^+_G Y$. Since the $y_i$s form a basis of $S$, there is a vector $a_x$ for which $x = Y a_x$ for any $x \in S$. Furthermore, $x_{H_k} = Y_k a_x$ for any $k \geq 0$. In particular,  

$$\frac{x^T_{H_k} L^+_G H_k x_{H_k}}{x^T L^+_G x} - 1 = \frac{|a_x^T M_k a_x|}{||a_x||^2}$$

so it suffices to show that $\lambda_{\max}(M_k) \leq \epsilon$ for all $k \leq k_{\text{final}}$, where $k_{\text{final}}$ is the number of while loop iterations.

In order to bound the change between $M_k$ and $M_{k+1}$, we introduce a second family of martingales consisting of one martingale for each while loop iteration. Let $I_{k,0} := I$ during the $k$th iteration of the while loop in SubspaceSparsifier. Generate $Z'$ in $Z$ during iteration $k$ of the while loop by sampling a sequence of edges $f_{k,0}, f_{k,1}, \ldots, f_{k,K(|E(I)|)}$ without replacement from $Z$. Let $I_{k,t} = I_{k,t-1}[f_{k,t-1}]$ for all $t > 0$. For a vector $v \in \mathbb{R}^{V(G)}$, let $v_{I_{k,0}, p} = v_{H_k}(p)$ for $p \in V(H_k)$ and $v_{I_{k,0}, p} = 0$ for $p \in V(I_{k,0}) \setminus V(H_k)$. For $t > 0$ and $v \in \mathbb{R}^{V(G)}$, let $v_{I_{k,t}, p} := (v_{I_{k,0}, p})_{I_{k,t}}$. Let $Y_{k,t}$ be the $|V(I_{k,t})| \times d$ matrix with $i$th column $(y_i)_{I_{k,t}}$. Let $N_{k,t} := Y_{k,t}^T L^+_G Y_{k,t} - Y^T L^+_G Y$. For any $x \in S$, $t \geq 0$, and $k \geq 0,$

---

**Algorithm 1: SubspaceSparsifier(G, S, ε)**

**Input:** A weighted graph $G$, a vector space $S \subseteq \mathbb{R}^{V(G)}$, $ε \in (0, 1)$, and (implicitly) a $(ρ, K(z))$-steady oracle $O$

**Output:** A $(S, ε)$-spectral subspace sparsifier for $G$

1. $H \leftarrow G$
2. while $|E(H)| \geq 10000ρ^2(\dim(S) \log(\dim(S)))/ε^2$ do
3. \hspace{1em} $(I, P) \leftarrow \text{Split}(H)$
4. \hspace{1em} $Z \leftarrow O(I, S)$
5. \hspace{1em} $Z' \leftarrow$ a uniformly random subset of $Z$ with size $K(|E(I)|)$
6. \hspace{1em} $T \leftarrow$ a spanning tree of $I$ drawn from a distribution with TV distance $\leq κ_0 := 1/n^{10}$ from uniform
7. \hspace{1em} $I' \leftarrow$ the graph with all edges in $E(T) \cap Z'$ contracted and all edges in $Z' \setminus E(T)$ deleted
8. \hspace{1em} $H \leftarrow \text{Unsplit}(I', P)$
9. end
10. return $H$
\[ x_{k,t} = Y_{k,t}a_x. \] In particular,
\[
\left| \frac{x^T I_{k,t}^+ x_{k,t}}{x^T L_G^+ x} - 1 \right| = \frac{a_x^T N_{k,t} a_x}{||a_x||^2}
\]
Next, we write an equivalent formulation for the steady oracle “Martingale change stability” guarantee that is easier to analyze:

**Proposition 3.6.**
\[
\max_{x \in \mathcal{S}} \left( \frac{x^T I_{k,t}^+ b_f}{r_f (x^T L_G^+ x)} \right)^2 = \frac{b_f^T L_{k,t}^+ Y_{k,t} Y_{k,t}^T L_{k,t}^+ b_f}{r_f}
\]

**Proof.** Notice that
\[
\max_{x \in \mathcal{S}} \left( \frac{x^T I_{k,t}^+ b_f}{r_f (x^T L_G^+ x)} \right)^2 = \max_{x \in \mathcal{S}} \frac{(a_x^T Y_{k,t}^T L_{k,t}^+ b_f)(b_f^T L_{k,t}^+ Y_{k,t} a_x)}{r_f ||a_x||^2}
\]
\[
= \max_{a \in \mathbb{R}^d} \frac{a_x^T Y_{k,t}^T I_{k,t} L_{k,t}^+ b_f b_f^T L_{k,t}^+ Y_{k,t} a_x}{r_f ||a_x||^2}
\]
\[
= \lambda_{\max} \left( \frac{Y_{k,t}^T L_{k,t}^+ Y_{k,t} Y_{k,t}^T L_{k,t}^+ b_f}{r_f} \right)
\]
\[
= \frac{b_f^T L_{k,t}^+ Y_{k,t} Y_{k,t}^T L_{k,t}^+ b_f}{r_f}
\]

as desired. \( \square \)

Now, we analyze the inner family of matrices \( N_{k,t} \). Let \( Z_{k,t} \) denote the set \( Z \) during iteration \( k \) of the while loop after sampling \( t \) edges without replacement.

**Proposition 3.7.** \( Y_t := N_{k,t} \) for fixed \( k \geq 0 \) and varying \( t \geq 0 \) is a matrix martingale. Furthermore, if
\[
\frac{x^T I_{k,s}^+ x_{k,s}}{x^T L_G^+ x} \leq 10
\]
for all \( x \in \mathcal{S}, k \geq 0, \) and \( s \leq t \) for some \( t \geq 0 \), \( \lambda_{\max}(X_{t+1}) \leq \frac{90d}{E(T_{k,t})} \) and \( \lambda_{\max}(E[X_{t+1}^2|Y_t]) \leq \frac{25600\rho^2d}{|E(I_{k,0})|^2} \), where \( X_{t+1} \) is defined based on the \( Y_t \)s as described in Theorem 3.1.

**Proof.** We compute the conditional expectation of \( X_{t+1} = Y_{t+1} - Y_t \) given \( Y_t \) using Sherman-Morrison:
\( \mathbf{E}[X_{t+1} | Y_t] = \mathbf{E}[N_{k,t+1} - N_{k,t} | N_{k,t}] \)
\[ = \frac{1}{|Z_{k,t}|} \sum_{f \in Z_{k,t}} b_f^T L_{k,t}^+ L_{k,t}^+ Y_{k,t} \cdot \frac{b_f^T L_{k,t}^+ b_f}{r_f} \]
\[ + \frac{1}{|Z_{k,t}|} \sum_{f \in Z_{k,t}} \left( 1 - \frac{b_f^T L_{k,t}^+ b_f}{r_f} \right) \frac{b_f^T L_{k,t}^+ L_{k,t}^+ Y_{k,t}}{r_f - b_f^T L_{k,t}^+ b_f} \]
\[ = 0 \]

Therefore, \( (Y_t)_{t \geq 0} \) is a martingale. Since \( I_{k,0} \) is the output of \texttt{Split}, all edges in \( I_{k,0} \) have leverage score between \( 3/16 \) and \( 13/16 \) by Proposition 3.4. In particular, the input condition to \( \mathcal{O} \) is satisfied. Furthermore,

\[ \lambda_{\max}(X_{t+1}) \leq \lambda_{\max}(N_{k,t+1} - N_{k,t}) \]
\[ \leq \frac{1}{|Z_{k,t}|} \sum_{f \in Z_{k,t}} \lambda_{\max} \left( \frac{b_f^T L_{k,t}^+ L_{k,t}^+ Y_{k,t}}{\min(b_f^T L_{k,t}^+ b_f, r_f - b_f^T L_{k,t}^+ b_f)} \right) \]
\[ \leq 8 \max_{f \in Z_{k,t}} \lambda_{\max} \left( \frac{(x^T I_{k,t}^+ L_{k,t}^+ b_f)^2}{r_f (x^T L_{k,t}^+ G x)} \right) \]
\[ \leq 80 \max_{f \in Z_{k,t}, x \in \mathcal{S}} \frac{(x^T I_{k,t}^+ L_{k,t}^+ b_f)^2}{r_f (x^T I_{k,0}^+ L_{k,0} x_{I_{k,0}})} \]
\[ \leq \frac{90 d}{|\mathbf{E}(I_{k,0})|} \]

where the third inequality follows from “Leverage score stability,” the equality follows from Proposition 3.6, the fourth inequality follows from the input condition, and the last inequality follows from “Martingale change stability.” Also,
\[
\lambda_{\text{max}}(\mathbb{E}[X_{t+1}^2|Y_t]) = \lambda_{\text{max}}(\mathbb{E}[(N_{k,t+1} - N_{k,t})^2|Y_t]) \\
\leq \frac{256}{|Z_{k,t}|} \lambda_{\text{max}} \left( \sum_{f \in Z_{k,t}} \frac{1}{r_f} Y^T_{k,t} L_{k,t} b_f b_f^T L_{k,t} + Y_{k,t}^T Y_{k,t} \right) \\
\leq \frac{256d}{|Z_{k,t}|} \left( \max_{f \in Z_{k,t}} \frac{1}{r_f} Y^T_{k,t} L_{k,t} b_f b_f^T L_{k,t} \right) \lambda_{\text{max}} \left( \sum_{f \in Z_{k,t}} \frac{1}{r_f} Y^T_{k,t} L_{k,t} b_f b_f^T L_{k,t} \right) \\
\leq \frac{2560 \rho d}{|Z_{k,t}| |E(I_{k,0})|} \lambda_{\text{max}} \left( Y^T_{k,t} L_{k,t} + Y_{k,t} \right) \\
\leq 2560 \rho d \frac{\max_{x \in \mathcal{S}} x^T L^+_G x}{|E(I_{k,0})|^2} \\
\leq 2560 \rho d \frac{|E(I_{k,0})|^2}{|E(I_{k,0})|^2} 
\]

where the second inequality follows from Sherman-Morrison and “Leverage score stability,” the fourth follows from “Martingale change stability,” and the last follows from “Size of Z” and the input condition.

Now, consider the sequence of matrices \(((N_{k,t+1} - N_{k,t})^2)_{t=0}^K\) obtained by concatenating the \((N_{k,t})_t\) martingales for each \(k\). We now analyze this sequence:

**Proposition 3.8.** The sequence of matrices \((Y_{k,t})_{k,t}\) ordered lexicographically by \((k,t)\) pairs defined by \(Y_{k,t} := N_{k,t}\) is a matrix martingale. Furthermore, if for any \(k \geq 0, t \geq 0\), any pairs \((l,s)\) lexicographically smaller than \((k,t)\), and any \(x \in \mathcal{S}\),

\[
\frac{x^T_{l,s} L^+_{l,s} x_{l,s}}{x^T L^+_G x} \leq 10
\]

then

\[
\lambda_{\text{max}}(X_{k',t'}) \leq \frac{90d}{|E(I_{k',t'})|}
\]

\[
\lambda_{\text{max}}(\mathbb{E}[X_{k',t'}^2|Y_{k,t}]) \leq \frac{2560 \rho d}{|E(I_{k',t'})|^2},
\]

and

\[
\lambda_{\text{max}}(W_{k,t}) \leq \sum_{(l,s) \geq (k,t)} 2560 \rho d \frac{|E(I_{l,s})|^2}{|E(I_{l,s})|^2}
\]

where \((k',t')\) is the lexicographic successor to \((k,t)\) and \(X_{k',t'} = Y_{k',t'} - Y_{k,t}\) as described in Theorem 3.7.

**Proof.** Consider a pair \((k',t')\) with lexicographic predecessor \((k,t)\). If \(k = k'\), then \(t' = t + 1\), which means that

\[
\mathbb{E}[Y_{k',t'}|Y_{k,t}] = \mathbb{E}[N_{k,t+1}|N_{k,t}] = N_{k,t} = Y_{k,t}
\]
, \lambda_{\max}(X_{k,t'}) \leq \frac{90d}{|\mathcal{E}(I_{k,t'})|}, \text{ and } \lambda_{\max}(\mathbb{E}[X_{k,t'}^2 | Y_{kt}]) \leq \frac{256000 \rho^2 d}{|E(I_{k,t})|^2} \text{ by Proposition 3.7}.

If \( k = k' - 1 \), then \( t' = 0 \). As a result, \( Y_{kt} = N_{k,t} = M_k \) by the “Electrical equivalence” guarantee of Proposition 3.8 and \( M_k = \rho N_{k,t'} = Y_{k,t'} \) by the “Electrical equivalence” guarantee of Proposition 3.4. In particular, \( X_{k,t'} = 0 \) and satisfies the desired eigenvalue bounds. The bound on \( \lambda_{\max}(W_{kt}) \) follows directly from the stepwise bound \( \lambda_{\max}(\mathbb{E}[X_{k,t'}^2 | Y_{kt}]) \) and the definition of \( W_{kt} \) as the predictable quadratic variation.

Now, we are ready to prove Lemma 3.3.

**Proof of Lemma 3.3.** \( H \) a minor of \( G \). It suffices to show that for every \( k \geq 0 \), \( H_{k+1} \) is a minor of \( H_k \). \( I' \) is a minor of \( I \), as \( I \) is only modified by deletion and contraction. Now, we show that the unweighted version of \( H_{k+1} \) can be obtained from \( H_k \) by contracting each edge \( e \in E(H_k) \) with an \( I' \)-self-loop in its pair \((e_0, e_1) \in P \) and deleting each edge \( e \in E(H_k) \) with an \( I' \)-leaf edge in its pair. Let \( H_{k+1}' \) be the result of this procedure.

We show that \( H_{k+1}' = H_{k+1} \) without weights. We start by showing that \( V(H_{k+1}) = V(H_{k+1}') \). Each vertex \( v \in V(H_{k+1}) \) corresponds to a set of vertices in \( V(H_k) \) that were identified, as the “Edges of \( H \)” requirement ensures that \( H_{k+1} \) contains no vertices that were added to \( H_k \) by \text{Split}. Since \( T \) is a tree, each vertex \( v \in V(H_{k+1}) \) corresponds to a subtree of \( H_k \). Since \( Z \) only contains one edge for each pair in \( \mathcal{P} \), the self-loop edges in \( I' \) match the edges contracted to form the subtree for \( v \), which means that \( V(H_{k+1}) = V(H_{k+1}') \). \( E(H_{k+1}) \subseteq E(H_{k+1}') \) because for every \( e \in E(H_{k+1}) \), \( \phi^{-1}(e) \) does not contain an \( I' \) self-loop or leaf by the “Edges of \( H \)” and “\( \mathcal{P} \) description” guarantees. \( E(H_{k+1}') \subseteq E(H_{k+1}) \) because each \( e \in E(H_{k+1}) \) does not map to a self-loop or leaf in \( I' \), which means that \( \phi^{-1}(e) \) exists by surjectivity of \( \phi \). Therefore, \( H_{k+1}' = H_{k+1} \). Since \( H_{k+1}' \) is a minor of \( H_k \), \( H_{k+1} \) is also a minor of \( H_k \), as desired.

**Number of edges.** This follows immediately from the while loop termination condition.

**Approximation bound.** Let \((k_t, t_r)\) be the final martingale index pair that the while loop encounters before termination. We start by obtaining a high-probability bound on \( W_{k_t,t_r} \), given that \( T \) is drawn from the exact uniform distribution on spanning trees of \( I \). By Proposition 3.8,

\[
W_{k_t,t_r} \leq \sum_{(k,t) \leq (k_t,t_r)} \frac{256000 \rho^2 d}{|E(I_{k,t})|^2}.
\]

The process of generating \( I_{k_t+1} \) from \( I_{k_t} \) does not increase the number of edges and decreases the number of edges by 1 with probability at least 1/8, by “Leverage score stability.” Therefore, by Azuma’s Inequality, \( |E(I_{k,t})| \leq 2|E(G)| - c_{k,t}/8 + 10 \sqrt{\log d \sqrt{c_{k,t}} \rho} \) with probability at least \( 1 - 1/d^3 \), where \( c_{k,t} \) is the number of pairs that are lexicographically less than \((k,t)\). Therefore, as long as \( |E(G)| > 20 \log d \), which is true when \( d > 1000000 = \Theta(1) \),

\[
|E(I_{k,t})| \leq 2|E(G)| - c_{k,t}/16
\]

with probability at least \( 1 - 1/d^3 \) for all pairs \((k,t)\). This means that

\[
c_{k_t,t_r} \leq 32000000|E(G)|
\]

and that

\[
W_{k_t,t_r} \leq \frac{32000000000 \rho^2 d}{|E(I_{k_t,t_r})|^2} \leq \epsilon^2/(16 \log d)
\]

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with probability at least $1 - 1/d^3$.

Now, we apply the Matrix Freedman Inequality (Theorem 3.1). Apply it to the martingale $(Y_{k,t})_{k,t}$ to bound $\lambda_{\text{max}}(Y_{k_{\tau},t} - Y_{00})$. By Proposition 3.8 and the termination condition for the while loop, we may set $R \leftarrow \epsilon/(10 \log d) \geq 90d|E_{(k_{\tau},t)}|$. By Theorem 3.1,

$$\Pr[\lambda_{\text{max}}(Y_{k_{\tau},t}) \geq \epsilon \text{ and } \lambda_{\text{max}}(W_{k_{\tau},t}) \leq \epsilon^2/(10 \log d)] \leq d \exp\left(-\frac{\epsilon^2/2}{(10 \log d)} + \frac{\epsilon^2}{(30 \log d)}\right) \leq 1/d^2$$

Therefore,

$$\Pr_T[\lambda_{\text{max}}(Y_{k_{\tau},t}) \geq \epsilon] \leq 1/d^2 + 1/d^3 \leq 2/d^2$$

Now, switch uniform spanning tree sampling to $\kappa_0$-approximate random spanning tree sampling. The total number of iterations is at most $m$, so the total TV distance of the joint distribution sampled throughout all iterations is at most $m\kappa_0$. Therefore,

$$\Pr_T[\kappa_0\text{-uniform } \lambda_{\text{max}}(Y_{k_{\tau},t})] \leq 2/d^2 + m\kappa_0 \leq 3/d^2$$

In particular, with probability at least $1 - 3/d^2$,

$$\left| \frac{x_{H_k}^T L^+_{H_k} x_{H_k}}{x^T L^+_{G} x} - 1 \right| \leq \lambda_{\text{max}}(M_{k_{\tau}}) = \lambda_{\text{max}}(Y_{k_{\tau},t}) \leq \epsilon$$

for all $x \in S$, as desired.

**Runtime.** By Azuma’s Inequality,

$$|E(H_k)| \leq |E(H_{k-1})| - K(|E(H_{k-1})|)/32 \leq (1 - \min_{z \geq 0} K(z)/32z)|E(H_{k-1})|$$

for all $k \leq k_{\tau}$ with probability at least $1 - 1/d^2$. Therefore,

$$|E(H_k)| \leq (1 - \min_{z \geq 0} K(z)/(32z))^k|E(G)|$$

which means that the termination condition is satisfied with high probability after

$$O((\log n) \max_{z \leq |E(G)|} z/K(z))$$

iterations with high probability. Each iteration samples one spanning tree, calls the oracle once, and does a linear amount of additional work, yielding the desired runtime.

### 3.1 Slow oracle and proof of existence

In this section, we prove Theorem 1.2 by exhibiting a $(2,1)$-steady oracle $\text{SlowOracle}(I,S)$. The oracle just returns all edges in the bottom half by maximum energy fraction:

**Algorithm 2:** $\text{SlowOracle}$, never executed

**Input:** A graph $I$ and a subspace $S \subseteq \mathbb{R}^{|V(I)|}$

**Output:** A set $Z$ of edges satisfying the steady oracle definition

1. return all edges $e \in E(I)$ with $\max_{x \in S} \frac{(x^T L^+_{I_e} b_e)^2}{r_e(x^T L^+_{I_e} x)} \leq \frac{2 \text{dim}(S)}{|E(I)|}$

To lower bound the number of edges added to $Z$, we use the following result and Markov’s Inequality:
Proposition 3.9.

$$\sum_{f \in E(I)} \max_{x \in S} \frac{(x^T L_I^+ b_f)^2}{r_f(x^T L_I^+ x)} = \dim(S)$$

Proof. Let $Y_I$ be a $V(I) \times \dim(S)$-matrix consisting of a basis $(y_i)_{i=1}^d$ for $S$ with $y_i^T L_I^+ y_j = 0$ for all $i \neq j \in [\dim(S)]$ and $y_i^T L_I^+ y_i = 1$ for all $i \in [\dim(S)]$. By Proposition 3.6,

$$\sum_{f \in E(I)} \max_{x \in S} \frac{(x^T L_I^+ b_f)^2}{r_f(x^T L_I^+ x)} = \sum_{f \in E(I)} \frac{b_f^T L_I^+ Y_I Y_I^T L_I^+ b_f}{r_f}$$

$$= \sum_{f \in E(I)} \text{trace} \left( \frac{b_f^T L_I^+ Y_I Y_I^T L_I^+ b_f}{r_f} \right)$$

$$= \sum_{f \in E(I)} \text{trace} \left( \frac{L_I^+ Y_I Y_I^T L_I^+ b_f b_f^T}{r_f} \right)$$

$$= \text{trace} \left( L_I^+ Y_I Y_I^T \right)$$

$$= \sum_{i=1}^d y_i^T L_I^+ y_i$$

$$= \dim(S)$$

as desired. \qed

Now, we prove Theorem 1.2.

Proof of Theorem 1.2 By Lemma 3.3 it suffices to show that $\texttt{SlowOracle}$ is a $(2,1)$-steady oracle.

Size of $Z$. By Markov’s Inequality and Proposition 3.9, $|Z| \geq |E(I)|/2$.

Leverage score stability. We are only interested in $i = 0$, for which the “Leverage score” input condition immediately implies the “Leverage score stability” guarantee.

Martingale change stability. We are only interested in $i = 0$. The return statement specifies the “Martingale change stability” guarantee for $\rho = 2$. \qed

4 Fast oracle

In this section, we give a $(O(\log^3 n), \Omega(z/\log^3 n))$-steady oracle $\texttt{FastOracle}$ that proves Theorem 1.3 when plugged into $\texttt{SubspaceSparsifier}$. To do this, we use localization [SRS17] to find a set of edges whose leverage scores and martingale changes do not change much over time. We use sketching and Lemma 1.4 to find these edges efficiently. This section can be described using the flexible function framework given in [Sch17], but we give a self-contained treatment here.
4.1 Efficient identification of low-change edges

FastOracle needs to find a large collection of edges whose electrical energies do not change over the course of many iterations. This collection exists by the following result:

**Theorem 4.1** (Theorem 1 of [SRS17]). Let \( I \) be a graph. Then for any vector \( w \in \mathbb{R}^{E(I)} \),

\[
\sum_{e,f \in E(I)} w_e w_f \frac{|b^T L^+_f b_f|}{\sqrt{r_e} \sqrt{r_f}} \leq c_{\text{local}} (\log^2 n) ||w||_2^2
\]

for some constant \( c_{\text{local}} \).

Plugging in \( w \leftarrow 1^{E(I)} \) shows that at least half of the edges \( e \in E(I) \),

\[
\sum_{f \in E(I)} \frac{|b^T L^+_f b_f|}{\sqrt{r_e} \sqrt{r_f}} \leq 2c_{\text{local}} \log^2 n
\]

We decrease this bound by subsampling the edges in \( I \) to obtain \( Z \). To identify the edges with low sum, we use matrix sketching:

**Theorem 4.2** (Theorem 3 of [Ind06] stated for \( \ell_1 \)). An efficiently computable, polylog\((d)\)-space linear sketch exists for \( \ell_1 \) norms. That is, given a \( d \in \mathbb{Z}_{\geq 1} \), \( \delta \in (0,1) \), and \( \epsilon \in (0,1) \), there is a matrix \( C = \text{SketchMatrix}(d, \delta, \epsilon) \in \mathbb{R}^{l \times d} \) and an algorithm \( \text{RecoverNorm}(s, d, \delta, \epsilon) \) with the following properties:

- **(Approximation)** For any vector \( v \in \mathbb{R}^d \), with probability at least \( 1 - \delta \) over the randomness of \( \text{SketchMatrix} \), the value \( r = \text{RecoverNorm}(Cv, d, \delta, \epsilon) \) is as follows:

  \[
  (1 - \epsilon)||v||_1 \leq r \leq (1 + \epsilon)||v||_1
  \]

- \( l = c/\epsilon^2 \log(1/\delta) \)

- **(Runtime)** \( \text{SketchMatrix} \) and \( \text{RecoverNorm} \) take \( O(ld) \) and poly\( (l) \) time respectively.

4.1.1 Approximation of column norms

Consider a graph \( I \) and a set \( W \subseteq E(I) \). We can obtain multiplicative approximations the quantities \( \sum_{f \in W} \frac{|b^T L^+_f b_f|}{\sqrt{r_e} \sqrt{r_f}} \) for all \( e \in W \) in near-linear time using Theorem 4.2. However, we actually need to multiplicatively approximate the quantities \( \sum_{f \in W, f \neq e} \frac{|b^T L^+_f b_f|}{\sqrt{r_e} \sqrt{r_f}} \). In particular, we need to estimate the \( \ell_1 \) norm of the rows of the matrix \( M \) with \( M_{ef} := \frac{|b^T L^+_f b_f|}{\sqrt{r_e} \sqrt{r_f}} \) with the diagonal left out. To do this, we tile the matrix as described in Section 11.3.2 of [Sch17]:

- Do \( \Theta(\log n) \) times:
  - Pick a random balanced partition \( (W_0, W_1) \) of \( W \)
  - For each \( e \in W_0 \), approximate \( a_e \leftarrow \sum_{f \in Z_1} \frac{|b^T L^+_f b_f|}{\sqrt{r_e} \sqrt{r_f}} \) using sketching
• For each $e \in W$, average the $a_e$s together and scale up the average by a factor of 4 to obtain an estimate for $\sum_{f \neq e \in W} \frac{|b^T_L L^+_f b_f|}{\sqrt{r_e r_f}}$.

The expected contribution of each off-diagonal entry is 1, while no diagonal entry can contribute. After $\Theta(\log n)$ trials, the averages concentrate by Chernoff and a union bound. Now, we formally implement this idea:

**Proposition 4.3.** There is a near-linear time algorithm $(a_e)_{e \in W} \leftarrow \text{ColumnApx}(I, W)$ that takes a graph $I$ and a set of edges $W \subseteq E(I)$ and returns estimates $a_e$ for which

$$a_e/2 \leq \sum_{f \neq e \in W} \frac{|b^T_L L^+_f b_f|}{\sqrt{r_e r_f}} \leq 3a_e/2$$

for all $e \in W$.

**Algorithm 3: ColumnApx$(I, W)$**

- **Input:** a graph $I$ and $W \subseteq E(I)$
- **Output:** approximations to the values $\{\sum_{f \neq e \in W} \frac{|b^T_L L^+_f b_f|}{\sqrt{r_e r_f}}\}_{e \in W}$

```plaintext
1. $K \leftarrow 100 \log n$
2. $\kappa_e \leftarrow 0$ for each $e \in W$
3. foreach $k \leftarrow 1, 2, \ldots, K$ do
   4. $W_0, W_1 \leftarrow$ partition of $W$ with $e \in W_0$ or $e \in W_1$ i.i.d. with probability $1/2$
   5. $C \leftarrow \text{SketchMatrix}(|W_1|, 1/n^6, 1/4)$
   6. $D \leftarrow V(I) \times |W_1|$ matrix of columns $b_f/\sqrt{r_f}$ for $f \in W_1$
   7. $U \leftarrow L^+_f DC^T$
   8. foreach $e \in W_0$ do
      9. Increase $\kappa_e$ by $\text{RecoverNorm}(U^T(b_e/\sqrt{r_e}), |W_1|, 1/n^6, 1/4)$
   10. end
11. end
12. return $(4\kappa_e/K)_{e \in W}$
```

**Proof of Proposition 4.3.** Approximation. Let $Y^{(k)}_{ef}$ be the indicator variable of the event $\{e \in W_0$ and $f \in W_1$ in iteration $k\}$. By the “Approximation” guarantee of Theorem 4.2 at the end of the foreach loop in ColumnApx,

$$\kappa_e \in [3/4, 5/4] \left( \sum_{f \in W} \frac{|b^T_L L^+_f b_f|}{\sqrt{r_e r_f}} \left( \sum_{k=1}^{K} Y^{(k)}_{ef} \right) \right)$$

for each $e \in W$. Since $Y^{(k)}_{ee} = 0$ for all $k$ and $e \in W$,

$$\sum_{f \in W} \frac{|b^T_L L^+_f b_f|}{\sqrt{r_e r_f}} \left( \sum_{k=1}^{K} Y^{(k)}_{ef} \right) = \sum_{f \neq e \in W} \frac{|b^T_L L^+_f b_f|}{\sqrt{r_e r_f}} \left( \sum_{k=1}^{K} Y^{(k)}_{ef} \right)$$
Notice that for $e \neq f$, $\{Y^{(k)}_{ef}\}_k$ is a family of independent Bernoullis with mean 1/4. Therefore, by Chernoff bounds and our choice of $K$, $K(1/4)(7/8) \leq \sum_{k=1}^K Y^{(k)}_{ef} \leq K(1/4)(9/8)$ for all $e \neq f$ with probability at least $1 - 1/n^5$. As a result,

$$\kappa_e \in \frac{K}{4}[1/2, 3/2] \left( \sum_{f \neq e \in W} \frac{|b^T_i L_i^+ b_f|}{\sqrt{r_e \sqrt{r_f}}} \right)$$

with high probability, as desired.

**Runtime.** Lines 5 and 9 contribute at most $\tilde{O}(|E(I)|)$ to the runtime of ColumnApx by the “Runtime” guarantee of Theorem 4.2. Line 7 only takes $\tilde{O}(|E(I)|)$ time to compute $U$ because $C^T$ only has $O(\log n)$ columns. All other lines take linear time, so ColumnApx takes near-linear time.

### 4.1.2 Construction of concentrated edges

Now, we subsample localized sets:

**Proposition 4.4.** Given a graph $I$ and $\gamma \in (0, 1)$, there is a set of edges $W \subseteq E(I)$ with two properties:

- (Size) $|W| \geq (\gamma/4)|E(I)|$
- (Value) For all $e \in W$, $\sum_{f \neq e \in W} \frac{|b^T_i L_i^+ b_f|}{\sqrt{r_e \sqrt{r_f}}} \leq \psi$ for all $e \in W$, where $\psi := 100c_{\text{local}} \gamma (\log^2 n)$

Furthermore, there is an $\tilde{O}(|E(I)|/\gamma)$-expected time algorithm Subsample($I, \gamma$) that produces $W$.

**Algorithm 4: Subsample($I, \gamma$)**

1. while $W$ does not satisfy Proposition 4.4 do
2.   $W_0 \leftarrow$ random subset of $E(I)$, with each edge of $e \in E(I)$ included i.i.d. with probability $2\gamma$
3.   $(a_e)_{e \in W_0} \leftarrow \text{ColumnApx}(I, W_0)$
4.   $W \leftarrow$ set of edges $e \in W_0$ with $a_e \leq \psi/2$
5. end
6. return $W$

**Proof.** We show that each iteration of the while loop terminates with probability at least $1/\text{polylog}(n)$. As a result, only polylog($n$) iterations are required to find the desired set. We do this by setting up an intermediate family of subsets of $E(I)$ to obtain $W$.

**Size.** Let $X_1 \subseteq E(I)$ be the set of edges $e$ with $\sum_{f \in E(I)} \frac{|b^T_i L_i^+ b_f|}{\sqrt{r_e \sqrt{r_f}}} \leq 2c_{\text{local}} \log^2 n$. By Theorem 4.1, $|X_1| \geq |E(I)|/2$.

Let $W_1 := X_1 \cap W_0$. $W_1$ can alternatively be sampled by sampling $W_1$ from $X_1$, including each element of $X_1$ in $W_1$ i.i.d. with probability $2\gamma$. Furthermore,
\[
E_{W_1} \left[ \sum_{f \neq e \in W_0} \frac{|b_e^T L_e^+ b_f|}{\sqrt{T_e \sqrt{T_f}}} \right] = E_{W_1} \left[ \sum_{f \neq e \in W_0} \frac{|b_e^T L_e^+ b_f|}{\sqrt{T_e \sqrt{T_f}}} \right] = 2\gamma \sum_{f \neq e \in E(I)} \frac{|b_e^T L_e^+ b_f|}{\sqrt{T_e \sqrt{T_f}}} \leq 4\gamma c_{\text{local}}(\log^2 n)
\]

By the approximation upper bound for \(a_e\) and Markov’s Inequality,

\[
\Pr_{W_1}[e \notin W|e \in W_1] \leq \Pr_{W_1}[a_e > \psi/2|e \in W_1] \\
\leq \Pr_{W_1} \left[ \sum_{f \neq e \in E(I)} \frac{|b_e^T L_e^+ b_f|}{\sqrt{T_e \sqrt{T_f}}} > \psi/4|e \in W_1 \right] \\
\leq \frac{4\gamma c_{\text{local}}(\log^2 n)}{\psi/4} \leq 1/2
\]

for every \(e \in X_1\). Therefore,

\[
E[|W|] > (1/2)E[|W_1|] = \gamma|X_1| \geq \gamma|E(I)|/2
\]

Since \(0 \leq |W| \leq |E(I)|\), \(|W| \geq \gamma|E(I)|/4\) with probability at least \(\gamma/4\), as desired.

**Value.** By the upper bound on \(a_e\) due to Proposition 4.3, all edges \(e \in W\) have the property that

\[
\sum_{f \neq e \in W} \frac{|b_e^T L_e^+ b_f|}{\sqrt{T_e \sqrt{T_f}}} \leq \sum_{f \neq e \in W_0} \frac{|b_e^T L_e^+ b_f|}{\sqrt{T_e \sqrt{T_f}}} \leq \psi
\]

as desired.

**Runtime.** Each iteration of the while loop succeeds with probability at least \(\gamma/4\), as discussed in the “Size” analysis. Each iteration takes \(O(|E(I)|)\) time by the runtime guarantee for ColumnApx. Therefore, the expected overall runtime is \(O(|E(I)|/\gamma)\).

\[\square\]

### 4.2 FastOracle

We now implement the \((\Theta(\log^3 n), \Theta(z/\log^3 n))\)-steady oracle **FastOracle**. It starts by finding a set \(W\) guaranteed by Proposition 4.4 with \(\gamma = \Theta(1/\log^3 n)\). It then further restricts \(W\) down to the set of edges satisfying “Martingale change stability” for \(I_0\) and returns that
set. The “Value” guarantee of Proposition 4.4 ensures that these edges continue to satisfy the “Martingale change stability” guarantee even after conditioning on edges in $Z$.

**Algorithm 5: FastOracle($I, S$)**

**Input:** a graph $I$ with leverage scores in $[3/16, 13/16]$ and a subspace $S \subseteq V(I)$ with $S := \mathbb{R}^S \times 0^{V(I)\setminus S}$ for some $S \subseteq V(I)$

**Output:** a set $Z \subseteq E(I)$ satisfying the steady oracle guarantees

1 $W \leftarrow \text{Subsample}(I, \gamma)$, where $\gamma := 1/(100000000c_{\text{local}}(\log^3 n)))$
2 $\{\nu_e\}_{e \in E(I)} \leftarrow \text{DiffApx}(I, S, 1/4, 1/m^5)$
3 return all $e \in W$ for which $\nu_e \leq \frac{4|S|}{|W|}$

To ensure that $\text{DiffApx}$ is applicable, note the following equivalence to what its approximating and the quantity in the “Martingale change stability” guarantee:

**Proposition 4.5.**

$$\max_{x \in \mathbb{R}^S} \frac{(x^T L_H^+ b_f)^2}{r_f(x^T L_H^+ x)} = \max_{c \in \mathbb{R}^{|S|-1}} \frac{c^T C^T L_H^+ b_f c}{r_f(c^T C^T L_H^+ C c)}$$

**Proof.** Define an $n \times (|S| - 1)$ matrix $C$ with signed indicator vectors of edges in a star graph on $C$. For every $x \in \mathbb{R}^S$ with $x^T 1 = 0$, $x = C c_x$ for some unique $c_x \in \mathbb{R}^{|S|-1}$. Therefore,

$$\max_{x \in \mathbb{R}^S} \frac{(x^T L_H^+ b_f)^2}{r_f(x^T L_H^+ x)} = \max_{c \in \mathbb{R}^{|S|-1}} \frac{c^T C^T L_H^+ b_f c}{r_f(c^T C^T L_H^+ C c)}$$

$$= \frac{1}{r_f} \lambda_{\text{max}}((C^T L_H^+ C)^{-1/2} C^T L_H^+ b_f C(C^T L_H^+ C)^{-1/2})$$

$$= \frac{1}{r_f} \frac{(b_f^T L_H^+ C(C^T L_H^+ C)^{-1} C^T L_H^+ b_f)}{r_f}$$

$$= \frac{b_f^T L_H^+ b_f}{r_f} - \frac{b_f^T L_H^+ / S b_f}{r_f}$$

where the last equality follows from the Woodbury formula.

To analyze $\text{FastOracle}$, we start by showing that any set of localized edges remains localized under random edge modifications:

**Proposition 4.6.** Consider a graph $I$ and a set of edges $Z \subseteq E(I)$ that satisfy the following two initial conditions:

- (Initial leverage scores) $\text{lev}_I(e) \in [3/16, 13/16]$ for all $e \in Z$.
- (Initial localization) $\sum_{f \in Z} \frac{|b_f^T L_H^+ b_f|}{\sqrt{r_f}} \leq \tau$ for all $e \in Z$, where $\tau = \frac{1}{10000 \log n}$.

Sample a sequence of minors $\{I_k\}_{k \geq 0}$ of $I$ and sets $Z_k \subseteq E(I_k)$ by letting $I_0 := I$ and for each $k \geq 0$, sampling a uniformly random edge $e_k \in Z_k$, letting $I_{k+1} := I_k \setminus e_k$ or $I_{k+1} := I_k / e_k$ arbitrarily, and letting $Z_{k+1} := Z_k \setminus e_k$. Then with probability at least $1 - 1/n^2$, the following occurs for all $i$:  

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• (All leverage scores) \( \hat{\lambda}(v_i) \in [1/8, 7/8] \) for all \( v_i \in Z_k \).

• (All localization) \( \sum_{i \in Z_k, i \neq v} \frac{|b_i^T L^+_i b_i|}{\sqrt{e_i} \sqrt{r_i}} \leq \tau' \) for all \( v_i \in Z_k \), where \( \tau' = 2\tau \).

To prove this result, we cite the following submartingale inequality:

**Theorem 4.7** (Theorem 27 of [CL06] with \( a_i = 0 \) for all \( i \)). Let \((Y_i)_{i \geq 0}\) be a submartingale with difference sequence \( X_i := Y_i - E[Y_i|Y_{i-1}] \) and \( W_i := \sum_{j=1}^i E[X_j^2|Y_{j-1}] \). Suppose that both of the following conditions hold for all \( i \geq 0 \):

- \( W_i \leq \sigma^2 \)
- \( X_i \leq M \)

Then

\[
\Pr[Y_i - Y_0 \geq \lambda] \leq \exp\left(-\frac{\lambda^2}{2(\sigma^2 + M\lambda/3)}\right)
\]

**Proof of Proposition 4.0** We prove this result by induction on \( k \). For \( k = 0 \), “Initial leverage scores” and “Initial localization” imply “All leverage scores” and “All localization” respectively. For \( k > 0 \), we use submartingale concentration to show the inductive step. For any edge \( e \in Z_k \), define two random variables \( U_e^{(k)} := \hat{\lambda}(v_i|v_j) \) and \( V_e^{(k)} := \sum_{f \in Z_k, f \neq e} \frac{|b_i^T L^+_i b_i|}{\sqrt{e_i} \sqrt{r_i}} \).

Let

\[
\hat{U}_e^{(k)} := U_e^{(k)} - \sum_{l=0}^{k-1} E[U_{e,l}^{(l+1)} - U_{e,l}^{(l)}|U_{e,l}^{(l)}]
\]

and

\[
\hat{V}_e^{(k)} := V_e^{(k)} - \sum_{l=0}^{k-1} E[V_{e,l}^{(l+1)} - V_{e,l}^{(l)} + \frac{|b_i^T L^+_i b_i|}{\sqrt{e_i} \sqrt{r_i}}|V_{e,l}^{(l)}]
\]

\((\hat{U}_e^{(k)})_{k \geq 0}\) is a martingale and \((\hat{V}_e^{(k)})_{k \geq 0}\) is a submartingale for all \( v_i \in Z_k \). Let

\[
\bar{X}_e^{(k)} := \hat{U}_e^{(k)} - E[\hat{U}_e^{(k)}|\hat{U}_e^{(k-1)}] = U_e^{(k)} - U_{e,k-1}^{(k-1)} - E[U_{e,k}^{(k)} - U_{e,k-1}^{(k-1)}|U_{e,k-1}^{(k-1)}]
\]

\[
\bar{X}_V_e^{(k)} := \hat{V}_e^{(k)} - E[\hat{V}_e^{(k)}|\hat{V}_e^{(k-1)}] = V_e^{(k)} - V_{e,k-1}^{(k-1)} - E[V_{e,k}^{(k)} - V_{e,k-1}^{(k-1)}|V_{e,k-1}^{(k-1)}]
\]

\[
\bar{W}_U_e^{(k)} := \sum_{j=1}^k E[(\bar{X}_e^{(j)})^2|\hat{U}_e^{(j-1)}]
\]

and

\[
\bar{W}_V_e^{(k)} := \sum_{j=1}^k E[(\bar{X}_V_e^{(j)})^2|\hat{V}_e^{(j-1)}]
\]

By Sherman-Morrison and the inductive assumption applied to the edges \( e, e_{k-1} \in Z_{k-1} \),
\[
\left| \hat{X} U_e^{(k)} \right| \leq |U_e^{(k)} - U_e^{(k-1)}| + \mathbb{E}[|U_e^{(k)} - U_e^{(k-1)}||U_e^{(k-1)}|] \\
\leq 2 \left( \frac{(b_e^T L^+_{k-1} b_{ek-1})^2}{r_e \min(b_e^T L^+_{k-1} b_{ek-1}, r_e - b_e^T L^+_{k-1} b_{ek-1})} \right) \\
\leq 16 (\tau')^2
\]

\[
\hat{X} V_e^{(k)} = \left( V_e^{(k)} - \left( V_e^{(k-1)} - \frac{|b_e^T L^+_{k-1} b_{ek-1}|}{\sqrt{r_e \sqrt{T_{ek-1}}}} \right) \right) - \mathbb{E} \left[ V_e^{(k)} - \left( V_e^{(k-1)} - \frac{|b_e^T L^+_{k-1} b_{ek-1}|}{\sqrt{r_e \sqrt{T_{ek-1}}}} \right) \right] \\
\leq 2 \sum_{g \in Z_{k-1}, g \neq e} \left| \frac{|b_e^T L^+_{k-1} b_{ek-1}|}{\sqrt{r_e \sqrt{T_g}}} - \frac{|b_e^T L^+_{k-1} b_{ek-1}|}{\sqrt{r_e \sqrt{T_g}}} \right| \\
\leq 2 \sum_{g \in Z_{k-1}, g \neq e} \frac{|b_e^T L^+_{k-1} b_{ek-1}|}{\sqrt{r_e \sqrt{T_{ek-1}}}} \min(1 - \text{lev}_{k-1}(e_{k-1}), \text{lev}_{k-1}(e_{k-1})) \sqrt{T_g} \\
\leq 16 \frac{|b_e^T L^+_{k-1} b_{ek-1}|}{\sqrt{r_e \sqrt{T_{ek-1}}}} \sum_{g \in Z_{k-1}, g \neq e} \frac{|b_e^T L^+_{k-1} b_{ek-1}|}{\sqrt{T_g \sqrt{T_{ek-1}}}} \\
\leq 16 (\tau')^2
\]

\[
\mathbb{E}[|\hat{X} U_e^{(k)}|^2 | U_e^{(k-1)}, e \neq e_{k-1}, \ldots, e \neq e_0] \\
\leq 4 \mathbb{E}[|U_e^{(k)} - U_e^{(k-1)}|^2 | U_e^{(k-1)}, e \neq e_{k-1}, \ldots, e \neq e_0] \\
\leq 4 \mathbb{E}_{ek-1} \left[ \frac{(b_e^T L^+_{k-1} b_{ek-1})^4}{r_e^2 \sqrt{T_{ek-1}} \min(1 - \text{lev}_{k-1}(e_{k-1}), \text{lev}_{k-1}(e_{k-1}))^2} \right] \\
\leq 256 \mathbb{E}_{ek-1} \left[ \frac{(b_e^T L^+_{k-1} b_{ek-1})^4}{r_e^2 \sqrt{T_{ek-1}}} \right] \\
\leq 256 \frac{256}{|Z_{k-1}| - 1} \left( \sum_{f \in Z_{k-1}, f \neq e} \frac{|b_e^T L^+_{k-1} b_f|}{\sqrt{r_e \sqrt{T_f}}} \right)^4 \\
\leq 256 (\tau')^4
\]
\[
\begin{align*}
\mathbb{E}[(\hat{XV}_e^{(k)})^2|\hat{V}_e^{(k-1)}, e \neq e_{k-1}, \ldots, e \neq e_0] & \leq 4\mathbb{E} \left[ \left( V_e^{(k)} - \left( V_e^{(k-1)} - \frac{|b_e^T L_{I_{k-1}}^+ b_{e_{k-1}}|}{\sqrt{r_e r_{e_{k-1}}}} \right) \right)^2 \left| V_e^{(k-1)}, e \neq e_{k-1}, \ldots, e \neq e_0 \right] \\
& \leq 4\mathbb{E}_{e_{k-1}} \left[ \left( \sum_{g \in Z_{k-1}, g \neq e} \frac{|b_e^T L_{I_{k-1}}^+ b_{e_{k-1}}|}{\sqrt{r_e r_{e_{k-1}}}} \min(1 - \text{lev}_{I_{k-1}}(e_{k-1}), \text{lev}_{I_{k-1}}(e_{k-1})) \sqrt{r_g}} \right)^2 \left| e \neq e_{k-1} \right] \\
& \leq 256\mathbb{E}_{e_{k-1}} \left[ \left( \frac{(b_e^T L_{I_{k-1}}^+ b_{e_{k-1}})^2}{r_e r_{e_{k-1}}} \right) \left( \sum_{g \in Z_{k-1}, g \neq e} \frac{|b_{e_{k-1}}^T L_{I_{k-1}}^+ b_g|}{\sqrt{r_g}} \right)^2 \left| e \neq e_{k-1} \right] \\
& \leq \frac{256}{|Z_{k-1}| - 1} \max_{f \in Z_{k-1}} \left( \sum_{g \in Z_{k-1}, g \neq f} \frac{|b_{e_{k-1}}^T L_{I_{k-1}}^+ b_g|}{\sqrt{r_f \sqrt{r_g}}} \right)^4 \\
& \leq \frac{256(\tau')^4}{|Z_{k-1}| - 1}
\end{align*}
\]

Therefore, for all \(k \leq |Z|/2\), \(|\hat{WU}_e^{(k)}| \leq 256(\tau')^4\) and \(|\hat{WV}_e^{(k)}| \leq 256(\tau')^4\) given the inductive hypothesis. By Theorem 4.7,

\[
\Pr[|\hat{U}_k - \hat{U}_0| > 2000(\log n)(\tau')^2] \leq \exp \left( -\frac{(2000(\log n)(\tau')^2)^2}{512(\tau')^4 + 512(\tau')^2(2000(\log n)(\tau')^2/3)} \right) \leq \frac{1}{n^5}
\]

and

\[
\Pr[|\hat{V}_k - \hat{V}_0| > 2000(\log n)(\tau')^2] \leq \exp \left( -\frac{(2000(\log n)(\tau')^2)^2}{512(\tau')^4 + 512(\tau')^2(2000(\log n)(\tau')^2/3)} \right) \leq \frac{1}{n^5}
\]

Now, we bound \(U_k - \hat{U}_k\) and \(V_k - \hat{V}_k\). By Sherman-Morrison and the inductive assumption for \(Z_{k-1}\),

\[
\begin{align*}
\mathbb{E}[(U_e^{(k)} - U_e^{(k-1)})|U_e^{(k-1)}, e \neq e_{k-1}] & \leq \mathbb{E}_{e_{k-1}} \left[ \frac{(b_e^T L_{I_{k-1}}^+ b_{e_{k-1}})^2}{r_e \min(1 - \text{lev}_{I_{k-1}}(e_{k-1}), \text{lev}_{I_{k-1}}(e_{k-1})) r_{e_{k-1}}} \left| e \neq e_{k-1} \right] \\
& \leq \frac{8(\tau')^2}{|Z_{k-1}| - 1}
\end{align*}
\]

and

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Therefore, by Markov’s Inequality,

\[ \nu \leq \frac{E\left[V^{(k)}_e - V^{(k-1)}_e + \left| \frac{b^T L^+_{I_{k-1}} b_{e_{k-1}}}{\sqrt{r_e}} \right| V^{(k-1)}_e, e \neq e_{k-1} \right]}{\nu} \]

\[ \leq \frac{\nu}{\min(1 - \text{lev}_{I_{k-1}}(e_{k-1}), \text{lev}_{I_{k-1}}(e_{k-1})) \sqrt{T_g}} \]

\[ \leq \frac{8(\tau')^2}{|Z_{k-1}| - 1} \]

so for \( k \leq |Z|/2, |U_k - \hat{U}_k| \leq 8(\tau')^2 \) and \( V_k - \hat{V}_k \leq 8(\tau')^2 \). In particular, with probability at least \( 1 - 2/n^5 \),

\[ |\text{lev}_{I_k}(e) - \text{lev}_{I_0}(e)| = |U^{(k)}_e - U^{(0)}_e| \]

\[ \leq |U^{(k)}_e - U^{(k)}_e| + |U^{(0)}_e - U^{(0)}_e| \]

\[ \leq 8(\tau')^2 + 2000(\log n)(\tau')^2 + 0 \]

\[ \leq 1/16 \]

Therefore, \( \text{lev}_{I_k}(e) \in [1/8, 7/8] \) with probability at least \( 1 - 2/n^5 \) for all \( e \in Z_k \). Furthermore,

\[ \sum_{g \in Z_k, g \neq e} \frac{|b^T L^+_{I_k} b_g|}{\sqrt{r_e \sqrt{T_g}}} = V^{(k)}_e \]

\[ = (V^{(k)}_e - \hat{V}^{(k)}_e) + (\hat{V}^{(k)}_e - \hat{V}^{(0)}_e) + (\hat{V}^{(0)}_e - V^{(0)}_e) + V^{(0)}_e \]

\[ \leq 8(\tau')^2 + 2000(\log n)(\tau')^2 + 0 + \tau \]

\[ \leq 2\tau = \tau' \]

This completes the inductive step and the proof of the proposition.

Now, we prove Theorem 1.3. By Lemma 3.3, it suffices to show that FastOracle is a \( (O(\log^3 n), \Omega(z/\log^2 n)) \)-steady oracle with runtime \( O(|E(I)|) \).

Proof of Theorem 1.3 Size of \( Z \). By Proposition 3.9 Proposition 4.5 and the approximation upper bound for \( \nu_e \),

\[ \sum_{e \in E(I)} \nu_e \leq (5/4)|S| + 1/m^4 \leq (3/2)|S| \]

Therefore, by Markov’s Inequality, \( |Z| \geq 5|W|/8 \). By the “Size” guarantee of Proposition 4.4 \( |W| \geq \Omega(1/(\log n)^3)|E(I)| \), so \( |Z| \geq \Omega(1/(\log n)^3)|E(I)| \), as desired.
**Leverage score stability.** We start by checking that the input conditions for Proposition 4.6 are satisfied for $Z$. The “Initial leverage scores” condition is satisfied thanks to the “Leverage scores” input guarantee for steady oracles. The “Initial localization” condition is satisfied because of the “Value” output guarantee for Subsample applied to $W$. Therefore, Proposition 4.6 applies. The “All leverage scores” guarantee of Proposition 4.6 is precisely the “Leverage score stability” guarantee of steady oracles, as desired.

**Martingale change stability.** Let $(y_i)_{i=1}^d$ be a basis of $S$ for which $y_i L_i^+ y_j = 0$ for $i \neq j$ and $y_i L_i^+ y_i$ for all $i \in \dim(S)$. Let $Y_t$ be a $V(I_t) \times \dim(S)$ matrix with columns $(y_i)_{I_t}$. By Proposition 3.6 applied with $G \leftarrow I$, we now bound this quantity over the course of deletions and contractions of the edges $e_t$ by setting up a martingale. For all $t \geq 0$ and $f \in E(I_t)$, let

$$A_f := \frac{b_f^T L_i^+ Y_i Y_i^T L_i^+ b_f}{r_f}$$

and

$$\hat{A}_f := A_f - \sum_{s=0}^{t-1} [A_f^{(s+1)} - A_f^{(s)}] A_f^{(s)}$$

For each $f \in E(I)$, $(A_f^{(t)})_{t \geq 0}$ is a martingale. Let

$$\hat{X}A_f := \hat{A}_f - \hat{A}_f^{(t-1)} = A_f^{(t)} - A_f^{(t-1)} - E[A_f^{(t)} - A_f^{(t-1)} | A_f^{(t-1)}]$$

and

$$\hat{W}A_f := \sum_{s=1}^{t} E[(\hat{X}A_f^{(s)})^2 | A_f^{(s-1)}]$$

We now inductively show that for all $f \in Z_t$ (which includes $f_t$),

$$A_f^{(t)} \leq \frac{\xi' |S|}{|E(I)|}$$

where $\xi := 8\frac{E(I)}{|W|} \leq O(\log^3 n)$ and $\xi' := 2\xi$. Initially,

$$A_f^{(0)} \leq \frac{8|S|}{|W|} = \frac{\xi |S|}{|W|}$$

for all $f \in Z$ by the approximation lower bound for $\nu_e$, completing the base case. For $t > 0$, we bound $A_f^{(t)}$ for $f \in Z_t$ by using martingale concentration. We start by bounding differences using the “All leverage scores” guarantee of Proposition 4.6, Sherman-Morrison, Cauchy-Schwarz, and the inductive assumption:
\[ |A_f^{(t)} - A_f^{(t-1)}| = \left| \frac{b_f^T L^{+}_{t-1} Y_{t-1} Y_{t-1}^T L^{+}_{t-1} b_f}{r_f} \right| \]
\[
\leq 2 \left| \frac{b_f^T L^{+}_{t-1} b_{f_{t-1}} b_f^T L^{+}_{t-1} Y_{t-1} Y_{t-1}^T L^{+}_{t-1} b_f}{r_{f_{t-1}} \min(1 - \text{lev}_{f_{t-1}}(f_{t-1}), \text{lev}_{f_{t-1}}(f_{t-1}))) r_f} \right| \\
+ \left| \frac{b_f^T L^{+}_{t-1} b_{f_{t-1}} b_f^T L^{+}_{t-1} Y_{t-1} Y_{t-1}^T L^{+}_{t-1} b_f}{r_{f_{t-1}}^2 \min(1 - \text{lev}_{f_{t-1}}(f_{t-1}), \text{lev}_{f_{t-1}}(f_{t-1}))) r_f} \right|^2 r_f \\
\leq 16 \left| \frac{b_f^T L^{+}_{t-1} b_{f_{t-1}}}{r_{f_{t-1}} r_f} \right| \\
+ 64 \left( \frac{b_f^T L^{+}_{t-1} b_{f_{t-1}}}{r_{f_{t-1}} r_f} \right)^2 A^{(t-1)}_{f_{t-1}} \\
\leq 80 \left| \frac{b_f^T L^{+}_{t-1} b_{f_{t-1}}}{r_{f_{t-1}} r_f} \right| \xi' |S| \frac{\tau'}{|E(I)|} \\
\leq 80 \left| \frac{b_f^T L^{+}_{t-1} b_{f_{t-1}}}{r_{f_{t-1}} r_f} \right| \xi' |S| \frac{\tau'}{|E(I)|}
\]

By the “All localization” guarantee of Proposition 4.6,
\[ |\hat{X} A_f^{(t)}| \leq |A_f^{(t)} - A_f^{(t-1)}| + \mathbb{E}[|A_f^{(t)} - A_f^{(t-1)}| |A_f^{(t-1)}] \leq 160 \tau' \xi' |S| \frac{\tau'}{|E(I)|} \]

and
\[ \mathbb{E}[(\hat{X} A_f^{(t)})^2 |\hat{A}_f^{(t-1)}] \leq 4 \mathbb{E}_{f_{t-1}}[(A_f^{(t)} - A_f^{(t-1)})^2 |A_f^{(t-1)}, f \neq f_{t-1}] \leq 6400(\tau')^2 \left( \frac{\xi' |S|}{|E(I)|} \right)^2 \\
\frac{6400(\tau')^2}{|Z_{t-1}|} - 1 \left( \frac{\xi' |S|}{|E(I)|} \right)^2 \]

Since \( K(|Z|) \leq |Z|/2, |\hat{W} A_f^{(t)}| \leq 6400(\tau')^2 \left( \frac{\xi' |S|}{|E(I)|} \right)^2 \). Therefore, by Theorem 4.7 applied to the submartingales \((\hat{A}_f^{(t)})_{t \geq 0}\) and \((-\hat{A}_f^{(t)})_{t \geq 0}\),
\[ \Pr \left[ |\hat{A}_f^{(t)} - \hat{A}_f^{(0)}| > \frac{\xi' |S|}{5|E(I)|} \right] \leq \exp \left( -\frac{((\xi' |S|)/5|E(I)|))^2}{(6400(\tau')^2 + 160(\tau')(\xi' |S|)(|E(I)|))^2} \right) \leq 1/n^5 \]
Since $\tilde{A}_{f}^{(0)} = A_{f}^{(0)}$, we just need to bound $|A_{f}^{(t)} - \tilde{A}_{f}^{(t)}|$. We do this by bounding expectations of differences:

$$
E[|A_{f}^{(t)} - A_{f}^{(t-1)}| | A_{f}^{(t-1)}, f \neq f_{t-1}] \leq 80 \mathbb{E} \left[ \frac{|\mathbf{b}_{f}^{T} L_{I_{t-1}} b_{f_{t-1}}|}{\sqrt{r_{f_{t-1}}} \sqrt{r_{f}}} |E(I)| \right] f \neq f_{t-1}
$$

$$
\leq \frac{80 \tau'}{|Z_{t-1}| - 1 |E(I)|}
$$

Therefore, $|A_{f}^{(t)} - \tilde{A}_{f}^{(t)}| \leq \sum_{s=0}^{t-1} E[|A_{f}^{(s+1)} - A_{f}^{(s)}| | A_{f}^{(s)}, f \neq f_{s}] \leq \frac{\xi' |S|}{5 |E(I)|}$. This means that

$$
A_{f}^{(t)} \leq |A_{f}^{(t)} - \tilde{A}_{f}^{(t)}| + |A_{f}^{(t)} - \tilde{A}_{f}^{(0)}| + \tilde{A}_{f}^{(0)} \leq \frac{2 \xi' |S|}{5 |E(I)|} + \frac{\xi' |S|}{|E(I)|} \leq \frac{\xi' |S|}{|E(I)|}
$$

with probability at least $1 - 1/n^5$, which completes the inductive step.

Therefore, by a union bound and the fact that $f_{t} \in Z_{t}$ for all $t \geq 0$,

$$
\max_{x \in S} \frac{(x_{I_{t}}^{T} L_{I_{t}}^{+} b_{f_{t}})^{2}}{r_{f_{t}}(x^{T} L_{I_{t}}^{+} x)} = A_{f_{t}}^{(t)} \leq \frac{\xi' |S|}{|E(I)|} \leq \frac{O(\log^{3} n) |S|}{|E(I)|}
$$

completing the “Martingale change stability” proof.

\[\square\]

## 5 Efficient approximation of differences

In this section, we show how to approximate changes in effective resistances due to the identification of a given vertex set $S$, and thus prove Lemma 1.4. Namely, given a vertex set $S \subset V$, we need to approximate the following quantity for all edges $e \in E(G)$:

$$
(b_{e}^{T} L_{G}^{+} b_{e}) - (b_{e}^{T} L_{G/S}^{+} b_{e}).
$$

By a proof similar to that of Proposition 4.5, this quantity equals

$$
\max_{x \perp 1, x \in \mathbb{R}^{S}} \frac{(x^{T} L_{G}^{+} b_{e})^{2}}{x^{T} L_{G}^{+} x}, \quad (7)
$$

where $1$ denotes the all-one vector.

**Lemma 5.1.** The decrease in the effective resistance of an edge $e \in E(G)$ due to the identification of a vertex set $S \subset V$ equals

$$
(b_{e}^{T} L_{G}^{+} b_{e}) - (b_{e}^{T} L_{G/S}^{+} b_{e}) = \max_{x \perp 1, x \in \mathbb{R}^{S}} \frac{(x^{T} L_{G}^{+} b_{e})^{2}}{x^{T} L_{G}^{+} x}.
$$
Proof. Let $C$ be the $n \times (|S| - 1)$ matrix with signed indicator vectors of edges in a star graph supported on $S$. Then we have

\[
(b^T L^+ G b_e) - (b^T L^+_{G/S} b_e) = b^T L^+_H C (C^T L^+_H C)^{-1} C^T L^+_H b_e \quad \text{by Woodbury}
\]

\[
= \lambda_{\max}((C^T L^+_H C)^{-1/2} C^T L^+_H b_e b^T_L L^+_H C (C^T L^+_H C)^{-1/2})
\]

\[
= \max_{c \in \mathbb{R}^{|S|-1}} \frac{c^T C^T L^+_H b_e b^T_L L^+_H C c}{c^T C^T L^+_H C c}
\]

\[
= \max_{x \in \mathbb{R}^S} \frac{(x^T L^+_G b_e)^2}{x^T L^+_G x},
\]

where the last equality follows from that the columns of $C$ form a basis of the subspace of $\mathbb{R}^S$ orthogonal to the all-ones vector.

Let $k := |S|$, and suppose without loss of generality that $S$ contains the first $k$ vertices in $G$. We construct a basis (plus an extra vector) of the subspace of $\mathbb{R}^S$ orthogonal to the all-ones vector by letting

\[
C_{n \times k} := \begin{pmatrix} I_{k \times k} - \frac{1}{k} J_{k \times k} & \end{pmatrix}, \quad \text{(8)}
\]

where $I$ denotes the identity matrix, and $J$ denotes the matrix whose entries are all 1. Let $P_{n \times k} := (I_{k \times k} \ 0)^T$ be the projection matrix taking the first $k$ coordinates, and let $\Pi_{k \times k} := I_{k \times k} - \frac{1}{k} J_{k \times k}$. Now we can write (7) as

\[
\max_{x \in \mathbb{R}^S} \frac{(x^T L^+_G b_e)^2}{x^T L^+_G x} = \max_{c \in \mathbb{R}^k} \frac{(c^T \Pi_{k \times k}) C^T L^+_G b_e b^T_L L^+_G C (\Pi_{k \times k} c)}{(c^T \Pi_{k \times k}) C^T L^+_G C (\Pi_{k \times k} c)} \quad \text{by } C \Pi_{k \times k} = C
\]

\[
= \max_{c \in \mathbb{R}^k} \frac{(c^T (C^T L^+_G C)^{+/2}) C^T L^+_G b_e b^T_L L^+_G C ((C^T L^+_G C)^{+/2} c)}{(c^T (C^T L^+_G C)^{+/2}) C^T L^+_G C ((C^T L^+_G C)^{+/2} c)}
\]

since $(C^T L^+_G C)^{+/2}$ and $\Pi_{k \times k}$ have the same column space

\[
= \lambda_{\max}((C^T L^+_G C)^{+/2} C^T L^+_G b_e b^T_L L^+_G C (C^T L^+_G C)^{+/2})
\]

\[
= b^T L^+_G C (C^T L^+_G C)^{+} C^T L^+_G b_e
\]

\[
= b^T L^+_G C (\Pi_{k \times k} P^T L^+_G \Pi_{k \times k})^+ C^T L^+_G b_e \quad \text{by } P \Pi_{k \times k} = C
\]

\[
= b^T L^+_G C \ SC(L_G, S) \ C^T L^+_G b_e \quad \text{by Fact 2.2} \quad \text{(9)}
\]
To approximate \((9)\), we further write it as
\[
\begin{align*}
 b^T_e L_G^+ C (SC(L_G, S)) C^T L_G^+ b_e \\
= b^T_e L_G^+ C SC(L_G, S)(SC(L_G, S))^+ SC(L_G, S) C^T L_G^+ b_e \\
= b^T_e L_G^+ C SC(L_G, S) C^T L_G^+ C SC(L_G, S) C^T L_G^+ b_e \\
= b^T_e L_G^+ C SC(L_G, S) C^T L_G^+ (B_G^T W_G B_G) L_G^+ C SC(L_G, S) C^T L_G^+ b_e,
\end{align*}
\]
where the last equality follows from \(L_G^+ = B_G^T W_G B_G\) and \(L_G = B_G^T W_G B_G\).

We now write the change in the effective resistance of an edge \(e\) in a square of an Euclidean norm as
\[
b^T_e L_G^+ b_e - b^T_e L_{G/S}^+ b_e = \left\| W_G^{1/2} B_G L_G^+ C (SC(L_G, S)) C^T L_G^+ b_e \right\|^2.
\]
We then use Johnson-Lindenstrauss Lemma to reduce dimensions. Let \(Q_{k \times m}\) be a random \(\pm 1\) matrix where \(k \geq 24 \log n / \epsilon^2\). By Lemma 2.12, the following statement holds for all \(e\) with high probability:
\[
\left\| W_G^{1/2} B_G L_G^+ C (SC(L_G, S)) C^T L_G^+ b_e \right\|^2 \approx_{1 + \epsilon} \left\| Q W_G^{1/2} B_G L_G^+ C (SC(L_G, S)) C^T L_G^+ b_e \right\|^2. \tag{10}
\]

To compute the matrix on the rhs, we note that \(C\) is easy to apply by applying \(I\) and \(J\), and \(L_G^+\) can be applied to high accuracy by Fast Laplacian Solvers. Thus, we only need to apply the Schur complement \(SC(L_G, S)\) to high accuracy fast. We recall Definition 2.1 of Schur complements:
\[
SC(L_G, S) := (L_G)_{S,S} - (L_G)_{S,T} (L_G)^{-1}_{T,T} (L_G)_{T,S},
\]
where \(T := V \setminus S\). Since \((L_G)_{T,T}\) is a principle submatrix of \(L_G\), it is an SDDM matrix and hence its inverse can be applied also by Fast Laplacian Solvers to high accuracy.

### 5.1 The subroutine and proof of Lemma 1.4

We give the algorithm for approximating changes in effective resistances due to the identification of \(S\) as follows:

**Algorithm 6: DiffApx\((G, S, \delta_0, \delta_1)\)**

**Input:** A weighted graph \(G\), a set of vertices \(S \subseteq V(G)\), and \(\delta_0, \delta_1 \in (0, 1)\)  
**Output:** Estimates \(\{\nu_e\}_{e \in E(G)}\) to differences in effective resistances in \(G\) and \(G/S\)

1. Let \(Q_{k \times m}\) be a random \(\pm 1\) matrix where \(k \geq 24 \log n / \delta_0^2\).
2. Compute each row of \(Y_{k \times n} := Q W_G^{1/2} B_G L_G^+ C (SC(L_G, S)) C^T L_G^+\) by applying \(L_G^+\) and \(L_{V \setminus S, V \setminus S}^+\) to accuracy
\[
\epsilon = \frac{\delta_1}{48 \sqrt{k} \cdot n^{8.5} \cdot w_{\max}^{2.5} w_{\min}^{-3}}.
\]
3. \(\nu_e \leftarrow \|Y b_e\|^2\) for all \(e \in E(G)\)  
4. **return** \(\{\nu_e\}_{e \in E(G)}\)

To prove the approximation ratio for DiffApx, we first track the errors for applying Schur complement in the following lemma:
Lemma 5.2. For any Laplacian \( L_G \), \( S \subset V(G) \), vector \( b \in \mathbb{R}^n \), and \( \epsilon > 0 \), the following statement holds:

\[
\|x - \tilde{x}\| \leq \epsilon n^{3.5} w_{\text{max}}^{2.5} w_{\text{min}}^{-0.5} \|b\|,
\]

where

\[
x := ((L_G)_{S,S} - (L_G)_{S,T}(L_G)^{-1}_{T,T}(L_G)_{T,S}) b,
\]

\[
\tilde{x} := (L_G)_{S,S} b - (L_G)_{S,T} \tilde{x}_1,
\]

\[
\tilde{x}_1 := \text{LaplSolve}((L_G)_{T,T}, (L_G)_{T,S}, \epsilon).
\]

Using Lemma 5.2, we track the errors for computing the embedding in (10) as follows:

Lemma 5.3. For any Laplacian \( L_G \), \( S \subset V(G) \), vector \( q \in \mathbb{R}^n \) with entries \( \pm 1 \), and \( 0 < \epsilon < 1/\left(4n^6 \cdot w_{\text{max}}^{2.5} w_{\text{min}}^{-1.5}\right) \),

\[
\|x - \tilde{x}\| \leq \epsilon \cdot 8n^8 \cdot \left(\frac{w_{\text{max}}}{w_{\text{min}}}\right)^{2.5},
\]

where

\[
x := (q^T W_G^{1/2} B_G L_G^+ C(SC(L_G, S)) C^T L_G^+) T,
\]

\[
\tilde{x} := \text{LaplSolve}(L_G, C \tilde{x}_1, \epsilon),
\]

\[
\tilde{x}_1 := (L_G)_{S,S} (C^T \tilde{x}_2) - (L_G)_{S,T} \text{LaplSolve}((L_G)_{T,T}, (L_G)_{T,S} (C^T \tilde{x}_2), \epsilon),
\]

\[
\tilde{x}_2 := \text{LaplSolve}(L_G, B_G^T W_G^{1/2} q, \epsilon).
\]

Before proving the above two lemmas, we show how they imply Lemma 1.4.

Proof of Lemma 1.4. The running time follows directly from the running time of \( \text{LaplSolve} \). Let \( X_{k \times n} := Q W_G^{1/2} B_G L_G^+ C(SC(L_G, S)) C^T L_G^+ \). The multiplicative approximation follows from Johnson-Lindenstrauss Lemma. To prove the additive approximation, we write the difference between \( \|X b_e\|^2 \) and \( \|Y b_e\|^2 \) as

\[
\|X b_e\|^2 - \|Y b_e\|^2 = \|X b_e\| - \|Y b_e\| \cdot (\|X b_e\| + \|Y b_e\|).
\]
Let $u, v$ be the endpoints of $e$. We upper bound $\|Xb_e - Yb_e\|$ by

$$\|Xb_e\| - \|Yb_e\| \leq \|(X - Y)b_e\| = \|(X - Y)(e_u - e_v)\|$$
by triangle ineq.

$$\leq \|(X - Y)e_u\| + \|(X - Y)e_v\|$$
by triangle ineq.

$$\leq \sqrt{2} \left( \|(X - Y)e_u\|^2 + \|(X - Y)e_v\|^2 \right)^{1/2}$$
by Cauchy-Schwarz

$$\leq \sqrt{2} \|X - Y\|_F = \sqrt{2} \left( \sum_{i=1}^{k} \| (X - Y)^T e_i \|^2 \right)^{1/2}$$

$$\leq \sqrt{2k} \cdot \epsilon \cdot 8n^8 \cdot \left( \frac{w_{\text{max}}}{w_{\text{min}}} \right)^{2.5}$$
by Lemma 5.3

$$\leq \frac{\delta_1}{3\sqrt{2}n^{1/2}w_{\text{min}}^{-1/2}}$$

and upper bound $\|Xb_e\| + \|Yb_e\|$ by

$$\|Xb_e\| + \|Yb_e\| \leq 2\|Xb_e\| + \|Xb_e\| - \|Yb_e\|$$

$$\leq 2 \left( 1 + \delta_0 \right) \left( b_e^T L_G^+ b_e - b_e^T L_G^+ T_e \right) \right)^{1/2} + \|Xb_e\| - \|Yb_e\|$$
by Lemma 2.12

$$\leq 2 \left( 1 + \delta_0 \right) n/w_{\text{min}}^{1/2} + \|Xb_e\| - \|Yb_e\|$$
upper bounding $b_e^T L_G^+ b_e$

$$\leq 3\sqrt{2} n^{1/2} w_{\text{min}}^{-1/2}$$
by $\delta_0 < 1$

Combining these two upper bounds gives

$$\|Xb_e\|^2 - \|Yb_e\|^2 \leq \delta_1,$$

which proves the additive error.

\[\square\]

### 5.2 Analysis of additional errors

We now prove Lemma 5.2 and 5.3

**Proof of Lemma 5.2** We upper bound $\|x - \tilde{x}\|$ by

$$\|x - \tilde{x}\| = \| (L_G)_{s,T} (L_G)_{t,T}^{-1} (L_G)_{t,s}b - \tilde{x}_1 \|$$

$$\leq n w_{\text{max}} \| (L_G)_{t,T}^{-1} (L_G)_{t,s}b - \tilde{x}_1 \|$$
by (6)

$$\leq \epsilon n^{2.5} w_{\text{max}} w_{\text{min}}^{-0.5} \| (L_G)_{t,s}b \|$$
by Lemma 2.11

$$\leq \epsilon n^{3.5} w_{\text{max}} w_{\text{min}}^{-0.5} \| b \|$$
by (6).

\[\square\]

**Proof of Lemma 5.3** We first bound the norm of vector $L_G^+ B_G T W_{G}^{1/2} q$ by

$$\| L_G^+ B_G T W_{G}^{1/2} q \| \leq \frac{n^2}{w_{\text{min}}} \| q \|$$
by $\sigma_{\text{max}}(L_G^+ B_G T W_{G}^{1/2}) = \lambda_{\text{max}}(L_G^+)$ and (11)

$$= \frac{n^{2.5}}{w_{\text{min}}}$$
since $q$’s entries are $\pm 1$,

(12)
and upper bound the norm of vector $SC(L_G, S)C^T L_G^+ B_G^T W^{1/2}_G q$ by
\[
\| SC(L_G, S)C^T L_G^+ B_G^T W^{1/2}_G q \| \leq n w_{\text{max}} \left\| L_G^+ B_G W^{1/2}_G q \right\| \leq n^{3.5} \frac{w_{\text{max}}}{w_{\text{min}}}.
\]
(13)

The error of $\tilde{x}_2$ follows by
\[
\left\| L_G^+ B_G^T W^{1/2}_G q - \tilde{x}_2 \right\| \leq \epsilon n^{1.5} \left( \frac{w_{\text{max}}}{w_{\text{min}}} \right)^{1/2} \left\| L_G^+ B_G^T W^{1/2}_G q \right\| \leq \epsilon n^4 w_{\text{max}}^{1/2} w_{\text{min}}^{-1.5}
\]
(14) by Lemma 2.10

The norm of $\tilde{x}_2$ can be upper bounded by
\[
\left\| \tilde{x}_2 \right\| \leq \left\| L_G^+ B_G^T W^{1/2}_G q \right\| + \left\| L_G^+ B_G^T W^{1/2}_G q - \tilde{x}_2 \right\| \leq 2n^{2.5} \frac{w_{\text{min}}}{w_{\text{min}}}
\]
by triangle inequality (12) and (11).

The error of $\tilde{x}_1$ follows by
\[
\left\| SC(L_G, S)C^T L_G^+ B_G^T W^{1/2}_G q - \tilde{x}_1 \right\| \leq \left\| SC(L_G, S)C^T \left( L_G^+ B_G^T W^{1/2}_G q - \tilde{x}_2 \right) \right\| + \left\| SC(L_G, S)C^T \tilde{x}_2 - \tilde{x}_1 \right\| \leq \epsilon n^5 w_{\text{max}}^{1.5} w_{\text{min}}^{-1.5} + \| SC(L_G, S)C^T \tilde{x}_2 - \tilde{x}_1 \| \leq \epsilon n^5 w_{\text{max}}^{1.5} w_{\text{min}}^{-1.5} + \epsilon \cdot \left\| C^T \tilde{x}_2 \right\| \cdot n^{3.5} \cdot w_{\text{max}}^{2.5} w_{\text{min}}^{-0.5} \leq \epsilon n^5 w_{\text{max}}^{1.5} w_{\text{min}}^{-1.5} + \epsilon \cdot 2n^6 \cdot w_{\text{max}}^{2.5} w_{\text{min}}^{-1.5}
\]
by Lemma 5.2

The norm of $\tilde{x}_1$ can be upper bounded by
\[
\left\| \tilde{x}_1 \right\| \leq \left\| SC(L_G, S)C^T L_G^+ B_G^T W^{1/2}_G q \right\| + \left\| SC(L_G, S)C^T L_G^+ B_G^T W^{1/2}_G q - \tilde{x}_1 \right\| \leq 2n^{3.5} \frac{w_{\text{max}}}{w_{\text{min}}}
\]
by triangle inequality (13) and (11).

Finally, the error of $\tilde{x}$ follows by
\[
\| x - \tilde{x} \| \leq \left\| L_G^+ C \left( SC(L_G, S)C^T L_G^+ B_G^T W^{1/2}_G q - \tilde{x}_1 \right) \right\| + \left\| L_G^+ C \tilde{x}_1 - \tilde{x} \right\| \leq \epsilon \cdot 4n^8 \cdot w_{\text{max}}^{2.5} w_{\text{min}}^{-2.5} + \| L_G^+ C \tilde{x}_1 - \tilde{x} \| \leq \epsilon \cdot 4n^8 \cdot w_{\text{max}}^{2.5} w_{\text{min}}^{-2.5} + \epsilon n^{1.5} w_{\text{max}}^{0.5} w_{\text{min}}^{0.5} \| C \tilde{x}_1 \| \leq \epsilon \cdot 4n^8 \cdot w_{\text{max}}^{2.5} w_{\text{min}}^{-2.5} + 2\epsilon n^5 w_{\text{max}}^{1.5} w_{\text{min}}^{-1.5}
\]
by Lemma 2.10

(18)
6 Better effective resistance approximation

In this section, we use divide-and-conquer based on Theorem 1.3 to $\epsilon$-approximate effective resistances for a set of pairs of vertices $P \subseteq V(G) \times V(G)$ in time $O(m^{1+o(1)} + (|P|/\epsilon^2)\text{polylog}(n))$. The reduction we use is the same as in [DKP+17]. We give the algorithm \texttt{ResApx} as follows:

\begin{algorithm}
\caption{\texttt{ResApx}(G, P, \epsilon), never executed}
\begin{algorithmic}
\State \textbf{Input:} A weighted graph $G$, a set of pairs of vertices $P$, and an $\epsilon \in (0,1)$
\State \textbf{Output:} Estimates $\{\tilde{r}_{u,v}\}_{(u,v) \in P}$ to effective resistances between vertex pairs in $P$
\State \textbf{if} $|P| = 1$ \textbf{then}
\State \quad Compute the Schur complement $H$ of $G$ onto $P$ with error $\epsilon$
\State \quad \textbf{return} $\{\tilde{r}_{u,v} := b^T_{u,v}L_H^+b_{u,v}\}$ for the only $(u,v) \in P$
\State \textbf{end}
\State \textbf{Let} $\epsilon_1 := \frac{1}{2} \cdot \epsilon \cdot (1/\log |P|)$ and $\epsilon_2 := \epsilon \cdot (1 - 1/\log |P|)$.
\State \textbf{Divide} $P$ into subsets $P^{(1)}$ and $P^{(2)}$ with equal sizes.
\State \textbf{Let} $V^{(1)}$ and $V^{(2)}$ be the respective set of vertices in $P^{(1)}$ and $P^{(2)}$.
\State \textbf{Compute} the Schur complement $H^{(1)}$ of $G$ onto $V^{(1)}$ with error $\epsilon_1$
\State \textbf{Compute} the Schur complement $H^{(2)}$ of $G$ onto $V^{(2)}$ with error $\epsilon_2$
\State \quad $\tilde{r} \leftarrow \texttt{ResApx}(H^{(1)}, P^{(1)}, \epsilon_2) \cup \texttt{ResApx}(H^{(2)}, P^{(2)}, \epsilon_2)$
\State \textbf{return} $\tilde{r}$
\end{algorithmic}
\end{algorithm}

\textit{Proof of Corollary 1.3} The approximation guarantees follows from

$$
\tilde{r}_{u,v} \geq \left(1 - \frac{1}{2} \cdot \epsilon/\log |P|\right)^{\log |P| - 1} \cdot (b^T_{u,v}L_H^+b_{u,v})
$$

and

$$
\tilde{r}_{u,v} \leq \left(1 + \frac{1}{2} \cdot \epsilon/\log |P|\right)^{\log |P| - 1} \cdot (b^T_{u,v}L_H^+b_{u,v})
$$

We then prove the running time. Let $T(p, \epsilon)$ denote the running time of $\texttt{ResApx}(G, P, \epsilon)$ when $|P| = p$ and $|E(G)| = O((p/\epsilon^2)\text{polylog}(n))$. Clearly, the total running time of $\texttt{ResApx}(G, P, \epsilon)$ for any $G$ with $m$ edges is at most

$$
2 \cdot T(|P|/2, \epsilon \cdot (1 - 1/\log |P|)) + O \left(m^{1+o(1)} + (|P|/\epsilon^2)\text{polylog}(n)\right),\quad (19)
$$

since the first step of $\texttt{ResApx}$ will divide the graph into two Schur complements with $O((|P|/\epsilon^2)\text{polylog}(n))$ edges each. Furthermore, we can write $T(p, \epsilon)$ in a recurrence form as

$$
T(p, \epsilon) = 2 \cdot T(p/2, \epsilon \cdot (1 - 1/\log p)) + O \left(p^{1+o(1)} + (p/\epsilon^2)\text{polylog}(n)\right),
$$

which gives

$$
T(p, \epsilon) = O \left(p^{1+o(1)} + (p/\epsilon^2)\text{polylog}(n)\right).
$$
Combining this with (19) gives the overall running time

\[ O \left( m^{1+o(1)} + (|P|/\epsilon^2)\text{polylog}(n) \right). \]

\[ \square \]

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A Bounds on eigenvalues of Laplacians and SDDM matrices

We first give upper bounds on the traces of the inverses of Laplacians and their submatrices.

Lemma A.1. For any Laplacian $L_G$ and $S \subset V(G)$,
\[ \text{Tr} \left( L_G^+ \right) \leq \frac{n^2}{w_{\text{min}}} \]
\[ \text{Tr} \left( (L_G)^{-1}_{S,S} \right) \leq \frac{n^2}{w_{\text{min}}} \]

Proof of Lemma A.1. Let $T := V(G) \setminus S$. The first upper bound follows by
\[ \text{Tr} \left( L_G^+ \right) = \frac{1}{n} \sum_{u,v \in V} b^T_{u,v} L_G^+ b_{u,v} \leq \frac{1}{n} \left( n^2 \frac{1}{w_{\text{min}}} \right) \leq \frac{n^2}{w_{\text{min}}} \]
The second upper bound follows by
\[
\text{Tr} \left( (LG)_{S,S}^{-1} \right) = \sum_{u \in S} b_{u,T}^T L_{G,T}^+ b_{u,T} \leq n \cdot \frac{n}{w_{\min}} \leq \frac{n^2}{w_{\min}}.
\] (23)

The first inequalities of (22) and (23) both follow from the fact that the effective resistance is at most the shortest path.

**Lemma 2.7.** For any Laplacian \( LG \) and \( S \subset V(G) \),
\[
\lambda_2(L_G) \geq \frac{w_{\min}}{n^2};
\] (1)
\[
\lambda_{\min}((LG)_{S,S}) \geq \frac{w_{\min}}{n^2};
\] (2)
\[
\lambda_{\max}((LG)_{S,S}) \leq \lambda_{\max}(LG) \leq n w_{\max}.
\] (3)

**Proof of Lemma 2.7.** For the upper bounds, we have
\[
\lambda_{\max}((LG)_{S,S}) \leq \lambda_{\max}(\mathcal{L}_G) \leq n w_{\max},
\]
where the first inequality follows from Cauchy interlacing, and \( K_n \) denotes the complete graph of \( n \) vertices.

For the lower bounds, we have
\[
\lambda_2(L_G) \geq \frac{1}{\text{Tr} \left( L_G^{-1} \right)} \geq \frac{w_{\min}}{n^2},
\]
\[
\lambda_{\min}(LS,S) \geq \frac{1}{\text{Tr} \left( (LG)_{S,S}^{-1} \right)} \geq \frac{w_{\min}}{n^2}.
\]

**B Bounds on 2-norms of some useful matrices**

**Lemma 2.8.** The following upper bounds on the largest singular values/eigenvalues hold:
\[
\sigma_{\max}(W_{G}^{1/2} B_G) \leq (n w_{\max})^{1/2};
\] (4)
\[
\lambda_{\max}(SC(L_G, S)) \leq n w_{\max};
\] (5)
\[
\sigma_{\max}((LG)_{S,T}) = \sigma_{\max}((LG)_{T,S}) \leq n w_{\max},
\] (6)

where \( T := V(G) \setminus S \).

**Proof of Lemma 2.8.** The largest singular value of \( W_{G}^{1/2} B_G \) follows by
\[
\sigma_{\max}(W_{G}^{1/2} B_G) \leq (\lambda_{\max}(L_G))^{1/2} \leq (n w_{\max})^{1/2} \quad \text{by (3)}.
\]

The largest eigenvalue of Schur complements follows by
\[
\lambda_{\max}(SC(L_G, S)) \leq \lambda_{\max}((LG)_{S,S}) \leq n w_{\max} \quad \text{by (3)}.
\]

The largest singular value of \((LG)_{S,T}\) follows by
\[
\sigma_{\max}((LG)_{S,T}) \leq \left( \lambda_{\max} \left( (LG)_{S,T}^T (LG)_{T,S} \right) \right)^{1/2}
\leq \left( n w_{\max} \cdot \lambda_{\max} \left( (LG)_{S,T}^T (LG)_{T,T}^T (LG)_{T,S} \right) \right)^{1/2} \quad \text{by (3)}
\leq (n w_{\max})^{1/2} \quad \text{since } SC(L_G, S) \text{ is positive semi-definite}
\leq n w_{\max} \quad \text{by (3)}.
\]
C  Bounds on errors of LaplSolve using $\ell_2$ norms

**Lemma 2.10.** For any Laplacian $L_G$, vectors $x, \tilde{x} \in \mathbb{R}^n$ both orthogonal to 1, and real number $\epsilon > 0$ satisfying

$$\|x - \tilde{x}\|_{L_G} \leq \epsilon \|x\|_{L_G},$$

the following statement holds:

$$\|x - \tilde{x}\| \leq \epsilon n^{1.5} \left( \frac{w_{\max}}{w_{\min}} \right)^{1/2} \|x\|.$$

**Proof of Lemma 2.10.** The error follows by

$$\|x - \tilde{x}\| \leq n w_{\min}^{-1/2} \|x - \tilde{x}\|_{L_G} \leq n w_{\min}^{-1/2} \epsilon \|x\|_{L_G} \leq n^{1.5} \left( \frac{w_{\max}}{w_{\min}} \right)^{1/2} \|x\| \text{ by (3)}$$

\[ \Box \]

**Lemma 2.11.** For any Laplacian $L_G$, $S \subset V$, vectors $x, \tilde{x} \in \mathbb{R}^{|S|}$, and real number $\epsilon > 0$ satisfying

$$\|x - \tilde{x}\|_M \leq \epsilon \|x\|_M,$$

where $M := (L_G)_{S,S}$, the following statement holds:

$$\|x - \tilde{x}\| \leq \epsilon n^{1.5} \left( \frac{w_{\max}}{w_{\min}} \right)^{1/2} \|x\|.$$

**Proof of Lemma 2.11.** The error follows by

$$\|x - \tilde{x}\| \leq n w_{\min}^{-1/2} \|x - \tilde{x}\|_M \leq n w_{\min}^{-1/2} \epsilon \|x\|_M \leq n^{1.5} \left( \frac{w_{\max}}{w_{\min}} \right)^{1/2} \|x\| \text{ by (3)}$$

\[ \Box \]

D  Split subroutines

**Proposition 3.4.** There is a linear-time algorithm $(I, P) \leftarrow \text{Split}(H)$ that, given a graph $H$, produces a graph $I$ with $V(H) \subseteq V(I)$ and a set of pairs of edges $P$ with the following additional guarantees:

- (Electrical equivalence) For all $x \in \mathbb{R}^{V(I)}$ that are supported on $V(H)$, $x^T L_I^+ x = x_H^T L_H^+ x_H$. 

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• (Bounded leverage scores) For all $e \in E(I)$, $\text{lev}_I(e) \in [3/16, 13/16]$

• ($\mathcal{P}$ description) Every edge in $I$ is in exactly one pair in $\mathcal{P}$. Furthermore, there is a bijection between pairs $(e_0, e_1) \in \mathcal{P}$ and edges $e \in E(H)$ for which either (a) $e_0, e_1$ and $e$ have the same endpoint pair or (b) $e_0 = \{u, w\}$, $e_1 = \{w, v\}$, and $e = \{u, w\}$ for some degree 2 vertex $w$.

Algorithm 8: $\text{Split}(H)$

Input: a graph $H$
Output: a graph $I$ with a pair of edges for each edge in $H$ and a set of paired edges in $\mathcal{P}$

1. $I \leftarrow H$
2. $\mathcal{P} \leftarrow \emptyset$
3. foreach edge $e \in E(H)$ do
   4. if $1/16$-JL-approximation to $\text{lev}_H(e) \geq 1/2$ then
      5. Replace $e = \{u, v\} \in E(I)$ with two edges $e_0 = \{u, v\}$ and $e_1 = \{u, v\}$ with $r_{e_0} = r_{e_1} = 2r_e$
      6. Add the pair $(e_0, e_1)$ to $\mathcal{P}$
   7. else
      8. Add a vertex $w$ to $V(I)$
      9. Replace $e = \{u, v\} \in E(I)$ with two edges $e_0 = \{u, w\}$ and $e_1 = \{w, v\}$ with $r_{e_0} = r_{e_1} = r_e/2$
     10. Add the pair $(e_0, e_1)$ to $\mathcal{P}$
    11. end
4. end
12. return $(I, \mathcal{P})$

Proof. Electrical equivalence. Two parallel edges with resistance $2r_e$ are electrically equivalent to one edge with resistance $r_e$. Two edges with resistance $r_e/2$ in series are equivalent to one edge with resistance $r_e$. Therefore, both ways of replacing edges in $H$ with pairs of edges in $I$ result in an electrically equivalent graph.

Bounded leverage scores. For an edge $e$ that is replaced with two series edges $e_0$ and $e_1$,

$$\text{lev}_I(e_0) = \text{lev}_I(e_1) = \frac{1}{2} + \frac{\text{lev}_H(e)}{2} \in [1/2, 3/4]$$

since $\text{lev}_H(e) \in [0, 1/2(1 + 1/16)]$. For an edge $e$ that is replaced with two parallel edges $e_0$ and $e_1$,

$$\text{lev}_I(e_0) = \text{lev}_I(e_1) = \frac{1}{2} \text{lev}_H(e) \in [1/4, 1/2]$$

since $\text{lev}_H(e) \in [1/2(1 - 1/16), 1]$. Since all edges in $I$ result from one of these operations, they all have leverage score in $[3/16, 13/16]$, as desired.

$\mathcal{P}$ description. (a) describes edges resulting from parallel replacements, while (b) describes edges resulting from series replacements.

Runtime. Estimating the leverage scores takes near-linear time [SSOS]. Besides this, the algorithm just does linear scans of the graph. Therefore, it takes near-linear time.
Proposition 3.5. There is a linear-time algorithm \( H \leftarrow \text{Unsplit}(I, \mathcal{P}) \) that, given a graph \( I \) and a set of pairs \( \mathcal{P} \) of edges in \( I \), produces a minor \( H \) with \( V(H) \subseteq V(I) \) and the following additional guarantees:

- (Electrical equivalence) For all \( x \in \mathbb{R}^{V(I)} \) that are supported on \( V(H) \), \( x^T L_I^+ x = x^T L_H^+ x_H \).
- (Edges of \( H \)) There is a surjective map \( \phi : E(I) \to E(H) \) from non-self-loop, non-leaf edges of \( I \) such that for any pair \( (e_0, e_1) \in \mathcal{P} \), \( \phi(e_0) = \phi(e_1) \). Furthermore, for each \( e \in E(H) \), either (a) \( \phi^{-1}(e) = e \), (b) \( \phi^{-1}(e) = \{e_0, e_1\} \), with \( (e_0, e_1) \in \mathcal{P} \) and \( e_0, e_1 \) having the same endpoints as \( e \) or (c) \( \phi^{-1}(e) = \{e_0, e_1\} \), with \( (e_0, e_1) \in \mathcal{P} \) and \( e_0 = \{u, w\}, e_1 = \{w, v\} \), and \( e = \{u, v\} \) for a degree 2 vertex \( w \).

Algorithm 9: \text{Unsplit}(I, \mathcal{P})

\[
\begin{align*}
\text{Input:} & \quad \text{a graph } I \text{ and a set of nonintersecting pairs of edges } \mathcal{P} \\
\text{Output:} & \quad \text{a graph } H \text{ with each pair unsplit to a single edge} \\
1 & H \leftarrow I \\
2 & \text{foreach } (e_0, e_1) \in \mathcal{P} \text{ do} \\
3 & \quad \text{if } e_0 \text{ and } e_1 \text{ have the same endpoints } \{u, v\} \text{ and } e_0, e_1 \in E(I) \text{ then} \\
4 & \quad \quad \text{Replace } e_0 \text{ and } e_1 \text{ in } H \text{ with one edge } e = \{u, v\} \text{ with } r_e = 1/(1/r_{e_0} + 1/r_{e_1}) \\
5 & \quad \text{else if } e_0 = \{u, w\}, e_1 = \{w, v\}, w \text{ has degree 2, and } e_0, e_1 \in E(I) \text{ then} \\
6 & \quad \quad \text{Replace } e_0 \text{ and } e_1 \text{ in } H \text{ with one edge } e = \{u, v\} \text{ with } r_e = r_{e_0} + r_{e_1} \\
7 & \text{end} \\
8 & \text{end}
\end{align*}
\]

Proof. **Electrical equivalence.** Two parallel edges with resistance \( r_{e_0} \) and \( r_{e_1} \) are electrically equivalent to one edge with resistance \( 1/(1/r_{e_0} + 1/r_{e_1}) \). Two edges with resistance \( r_{e_0} \) and \( r_{e_1} \) in series are equivalent to one edge with resistance \( r_{e_0} + r_{e_1} \). Therefore, both ways of replacing pairs of edges in \( I \) with single edges in \( H \) result in an electrically equivalent graph.

**Edges of \( H \).** Since the pairs in \( \mathcal{P} \) do not intersect, the map \( \phi(e_i) = e \) that maps an edge \( e_i, i \in \{0, 1\} \) to the \( e \) as described in the foreach loop is well-defined. Since each \( (e_0, e_1) \in \mathcal{P} \) pair is assigned to the same edge \( e \), \( \phi(e_0) = \phi(e_1) = e \). Each edge in the output graph \( H \) originates from the initialization of \( H \) to \( I \), the if statement, or the else statement. These are type (a),(b), and (c) edges respectively. Therefore, \( \phi \) satisfies the required conditions.

**Runtime.** The algorithm just requires a constant number of linear scans over the graph. 

\( \square \)