Field theories invariant under two-parametric supersymmetry

S.R. Esipova†1, P.M. Lavrov†‡2 and O.V. Radchenko†‡3

†Tomsk State Pedagogical University,
Kievskaya St. 60, 634061 Tomsk, Russia
‡National Research Tomsk State University,
Lenin Av. 36, 634050 Tomsk, Russia

Abstract

We study field models for which a quantum action (i.e. the action appearing in the generating functional of Green functions) is invariant under supersymmetric transformations. We derive the Ward identity which is direct consequence of this invariance. We consider a change of variables in functional integral connected with supersymmetric transformations when its parameter is replaced by a nilpotent functional of fields. Exact form of the corresponding Jacobian is found. We find restrictions on generators of supersymmetric transformations when a consistent quantum description of given field theories exists.

Keywords: supersymmetric invariance, field-dependent supersymmetric transformation, nilpotency

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1E-mail: esipova@tspu.edu.ru
2E-mail: lavrov@tspu.edu.ru
3E-mail: radchenko@tspu.edu.ru
1 Introduction and summary

Field models with quantum action invariant under supersymmetric transformations appear in several ways within modern quantum field theory. The well-known example is the Faddeev-Popov action for Yang-Mills fields [1], which is invariant under nilpotent supersymmetric transformations known as BRST transformations [2, 3]. Recent attempts [4, 5] to formulate Yang-Mills fields in a form being free of the Gribov problem [6, 7, 8] give another examples of actions invariant under some nilpotent supersymmetric transformations. Superextension of sigma models [9] leads to actions again invariant under supersymmetric transformations. Quite recently a new realization of supersymmetry, called scalar supersymmetry, has been proposed in [10] when one meets supersymmetric invariant field models as well. The Curci-Ferrari model of non-abelian massive vector fields [11] possesses supersymmetric invariance connected with the modified BRST and modified anti-BRST transformations. In contrast with the BRST transformations these supersymmetric transformations are not nilpotent. In turn it leads to serious consequences in physical interpretation of the model [12].

In present paper from general point of view we study properties of field theories for which an action appearing in the generating functional of Green functions is invariant under supersymmetric transformations. In turn the supersymmetric transformations can be of three types. The first type is characterized as supersymmetric transformations when there are no any restrictions on generators of these transformations. We derive the Ward identity as a consequence of the supersymmetric invariance and show that there is no a possibility to present this identity in local form. The second type consists of nilpotent supersymmetric transformations. Introducing field-dependent nilpotent supersymmetric transformations we find an exact form of the superdeterminant of the change of variables in the functional integral representing the generating functional of Green functions. We prove that the action appearing after the change of variables is not invariant under supersymmetric transformations and find the source of this non-invariance. We consider the non-invariance as inconsistency in the quantum presentation of given theories. To lift the inconsistency we introduce the third type of nilpotent supersymmetric transformations. In this case the generators are subjected to an additional restriction.

We employ the condensed notation of DeWitt [13]. Derivatives with respect to fields are taken from the right. Left derivatives with respect to fields are labeled by a subscript $l$. The Grassmann parity of a quantity $X$ is denoted as $\varepsilon(X)$. We use the notation $X_{,i}$ for right derivative of $X$ with respect to $\phi^i$.

2 Supersymmetric invariant theories

Our starting point is a theory of fields $\phi = \{\phi^i\}$ with Grassmann parities $\varepsilon(\phi^i) = \varepsilon_i$. We assume a non-degenerate action $S(\phi)$ of the theory so that the generating functional of Green
functions is given by the standard functional integral
\[ Z(J) = \int \mathcal{D}\phi \exp \left\{ \frac{i}{\hbar} [S(\phi) + J\phi] \right\} . \]  
(2.1)

We suppose invariance of \( S(\phi) \)
\[ S(\phi) = S(\varphi(\phi')) = S(\phi') \]  
(2.2)
under supersymmetric two-parametric transformations
\[ \phi^i \mapsto \varphi^i(\phi') , \quad \varphi^i(\phi) = \phi^i + R^{ia}(\phi) \xi_a , \quad a = 1, 2 , \quad \xi_a^b + \xi_b^a = 0 , \]  
(2.3)
so that
\[ S_{,i}R^{ia}(\phi) = 0 , \quad S_{ij}Z^{ji}(\phi) = 0 , \]  
(2.4)
\[ Z^{ij}(\phi) = \frac{1}{2} \epsilon^i - \epsilon^j \delta R^{ia}(\phi) R^{ij}(\phi) - (-1)^{(\epsilon_i+1)(\epsilon_j+1)} \delta R^{ib}(\phi) R^{ja}(\phi) = 0 . \]  
(2.5)

In (2.3) \( \xi_a \) is odd Grassmann parameters and \( R^{ia}(\phi) \) are generators of supersymmetric transformations having the Grassmann parities opposite to fields \( \phi^i \): \( \varepsilon(R^{ia}) = \varepsilon_i + 1 \).

Consider now some consequence of the invariance on quantum level. To this end we make the change of variables (2.3) in the functional integral (2.1). As a result we have
\[ Z(J) = \int \mathcal{D}\phi \ s\text{Det} M(\phi) \exp \left\{ \frac{i}{\hbar} [S(\varphi(\phi)) + J\varphi(\phi)] \right\} \]  
(2.5)
where \( s\text{Det} M \) means the superdeterminant of supermatrix \( M \) with matrix elements
\[ M_{ij}^l(\phi) = \delta^l_j + (-1)^{\varepsilon_i} \frac{\delta R^{ia}(\phi)}{\delta \phi^j} \xi_a , \quad \varepsilon(M^l_j) = \varepsilon_i + \varepsilon_j . \]  
(2.6)

In general, for a theory under consideration this superdeterminant is not equal to unity
\[ s\text{Det} M(\phi) = \exp \left\{ s\text{Tr} \ln M(\phi) \right\} = \exp \left\{ \frac{\delta R^{ia}(\phi)}{\delta \phi^j} \xi_a + \right\} = 1 + \frac{\delta R^{ia}(\phi)}{\delta \phi^j} \xi_a + = 1 + R^{ia}_{,i}(\phi) \xi_a + . \]  
(2.7)

It leads to the following presentation of functional \( Z(J) \)
\[ Z(J) = \int \mathcal{D}\phi \left( 1 + R^i_{,i}(\phi) \xi \right) \exp \left\{ \frac{i}{\hbar} [S(\phi) + J_i \phi^i + J_i R^i(\phi) \xi] \right\} = \]  
\[ = \int \mathcal{D}\phi \left( 1 + R^i_{,i}(\phi) \xi + \frac{i}{\hbar} J_i R^i(\phi) \xi \right) \exp \left\{ \frac{i}{\hbar} [S(\phi) + J\phi] \right\} \]  
(2.8)
from which the identity follows
\[ \int \mathcal{D}\phi \left( R^i_{,i}(\phi) + \frac{i}{\hbar} J_i R^i(\phi) \right) \exp \left\{ \frac{i}{\hbar} [S(\phi) + J\phi] \right\} = 0 . \]  
(2.9)
With the help of usual manipulations this identity can be written in closed form with respect
to $Z(J)$

\[
\left[ J_i R^i \left( \frac{\hbar}{\delta J} \right) - i h R^i_i \left( \frac{\hbar}{\delta J} \right) \right] Z(J) = 0 .
\] (2.10)

This identity is nothing but the Ward identity for generating functional of Green functions. The existence of this identity is direct consequence of supersymmetric invariance of $S(\phi)$. To simplify presentation of the Ward identity we define the extended generating functional of Green functions by introducing additional sources $K_i$ with Grassmann parities opposite to fields $\phi^i$, $\varepsilon(K_i) = \varepsilon_i + 1$

\[
Z(J, K) = \int D\phi \exp \left\{ i \frac{\hbar}{\delta} [S(\phi, K) + J\phi] \right\}.
\] (2.11)

where

\[
S(\phi, K) = S(\phi) + K_i R^i(\phi),
\] (2.12)

In general, the action $S(\phi, K)$ is not invariant under supersymmetric transformation

\[
\hat{s} S(J, K) = K_i \hat{s} R^i(\phi) \neq 0,
\] (2.13)

where the operator $\hat{s}$ of supersymmetric transformation was used. Action of this operator on arbitrary functional $X$ is given by the rule

\[
\hat{s} X = \frac{\delta X}{\delta \phi^i} R^i.
\] (2.14)

It is clear that there is the relation between functionals (2.1) and (2.11)

\[
Z(J, K) \big|_{K=0} = Z(J).
\] (2.15)

In terms of $Z(J, K)$ the Ward identity (2.10) reads

\[
J_i \frac{\delta Z(J, K)}{\delta K_i} = i h R^i_i \left( \frac{\hbar}{\delta J} \right) Z(J, K).
\] (2.16)

Note that the left side of the Ward identity (2.16) has the local form in contrast with corresponding term in (2.10). In turn the right side of (2.16) is a nonlocal.

### 3 Field-dependent supersymmetric transformations

In this section we study more general type of supersymmetric transformations when the parameter $\xi$ in (2.3) is replaced by a field-dependent functional $\xi(\phi)$

\[
\varphi^i(\phi) = \phi^i + R^i(\phi)\xi(\phi), \quad \xi^2(\phi) = 0.
\] (3.1)
We will refer to these transformations as field-dependent supersymmetric transformations. Note that the action \( S = S(\phi) \) remains invariant under transformations (3.1) due to nilpotency of \( \xi(\phi) \)

\[
S(\phi) = S(\varphi(\phi')) = S(\phi') .
\]

Using the technique described in [15] it is not difficult to find the explicit form of the superdeterminant of supermatrix

\[
M^i_j(\phi) = \delta^i_j + R^i_j(\phi)\xi_j(\phi) + (-1)^{\xi_i}R^i_j(\phi)\xi(\phi) ,
\]

corresponding to transformations (3.1) with the result

\[
s\text{Det} M(\phi) = 1 + s\xi(\phi)) = ^1 [1 + R^i_j(\phi)\xi(\phi) - \left(\frac{s^2\xi(\phi))\xi(\phi)}{1 + s\xi(\phi)}\right] .
\]

In (3.4) we took into account that the action of the square operator \( \hat{s} \) on an arbitrary functional \( X \) is given by the relation

\[
\hat{s}^2X = \frac{\delta X}{\delta \phi^i} \frac{\delta R^i_j}{\delta \phi^j} = X_{,i}R^i_j R^j .
\]

In what follows we restrict ourselves to the case when the operator \( \hat{s} \) is nilpotent, \( \hat{s}^2 = 0 \)

\[
\hat{s}^2 = 0 \quad \rightarrow \quad \frac{\delta R^i_j}{\delta \phi^j} = 0 .
\]

In particular, it means

\[
\hat{s}^2 = 0 \quad \rightarrow \quad \hat{s}R^i_j = 0
\]

and we find that the action \( S(\phi, K) \) (2.12), (2.13) is invariant under field-dependent supersymmetric transformations (3.1)

\[
S(\phi, K),_{i} R^i(\phi) = 0 .
\]

The invariance of \( S(\phi, K) \) can be expressed in an unique form

\[
\frac{\delta S(\phi, K)}{\delta \phi^i} \frac{\delta S(\phi, K)}{\delta K^i} = 0 .
\]

The equation (3.9) is nothing but the Zinn-Justin equation appearing for the first time in quantization of non-abelian gauge fields [14].

Performing the change of variables in form of field-dependent supersymmetric transformations (3.1), (3.6) and using (3.4) we have

\[
s\text{Det} M(\phi) = \exp \left\{ R^i_j(\phi)\xi(\phi) - \ln \left(1 + s\xi(\phi)\right)\right\} = \left(1 + s\xi(\phi)\right)^{-1}[1 + R^i_j(\phi)\xi(\phi)]
\]
and arrive at the following presentation of generating functional $Z(J, K)$

$$Z(J, K) = \int \mathcal{D}\phi \exp\left\{ \frac{i}{\hbar} \left[ S(\phi, K) + J(\phi + R(\phi)\xi) - i\hbar R^i_{\phi}(\phi)\xi(\phi) + i\hbar \ln (1 + s\xi) \right] \right\}.$$  \hspace{1cm} (3.11)

We can rewrite the presentation (3.11) in the form

$$Z(J, K) = Z(J, K) + I(J, K) \hspace{1cm} (3.12)$$

where

$$I(J, K) = \int \mathcal{D}\phi (1 + s\xi)^{-1} \left[ \delta\xi(\phi) - R^i_{\phi}(\phi)\xi(\phi) - \frac{i}{\hbar} J_i R^i(\phi)\xi(\phi) \right] \times \exp\left\{ \frac{i}{\hbar} \left[ S(\phi, K) + J\phi \right] \right\}. \hspace{1cm} (3.13)$$

The functional $I(J, K)$ should be zero. Let us prove this property. To this end it is useful to introduce the functional $\Lambda(\phi)$

$$\Lambda(\phi) = \xi(\phi)(1 + \hat{s}\xi(\phi))^{-1}, \hspace{1cm} (3.14)$$

so that

$$\hat{s}\Lambda(\phi) = \frac{\hat{s}\xi(\phi)}{1 + \hat{s}\xi(\phi)}. \hspace{1cm} (3.15)$$

Then we have

$$I(J, K) = \int \mathcal{D}\phi \left[ \frac{\delta\Lambda(\phi)}{\delta\phi^i} R^i(\phi) + \Lambda(\phi) R^i_{\phi}(\phi) + \frac{i}{\hbar} \Lambda(\phi) J_i R^i(\phi) \right] \times \exp\left\{ \frac{i}{\hbar} \left[ S(\phi, K) + J\phi \right] \right\} = \int \mathcal{D}\phi \delta \left[ \Lambda(\phi) R^i(\phi) \exp\left\{ \frac{i}{\hbar} \left[ S(\phi, K) + J\phi \right] \right\} \right] = 0 \hspace{1cm} (3.16)$$

where the invariance of $S(\phi, K)$ (3.8) was used.

From (3.11) it follows that a theory with the action $S(\phi, K)$ invariant under supersymmetric transformation (2.3) or (3.1) admits formulation in term of action $S_\xi(\phi, K)$

$$S_\xi(\phi, K) = S(\phi, K) + i\hbar \ln (1 + \hat{s}\xi(\phi)) - i\hbar R^i_{\phi}(\phi)\xi(\phi) \hspace{1cm} (3.17)$$

In its turn, in general, the action $S_\xi(\phi, K)$ is not invariant under supersymmetric transformations (2.3) or (3.1) due to the third term in rhs (3.17)

$$\hat{s}S_\xi(\phi, K) = -i\hbar \hat{s} \left( R^i_{\phi}(\phi)\xi(\phi) \right) \neq 0. \hspace{1cm} (3.18)$$

We consider this as an indication of the inconsistency in formulation of the model being invariant under supersymmetric transformations. Indeed, it seems strange that a theory with the action invariant under supersymmetric transformations is equivalently presented in the form when this symmetry looks like broken. This inconsistency can be deleted if the additional requirement is fulfilled, $R^i_{\phi}(\phi) = 0$. 
4 Special supersymmetric theories

We will refer special supersymmetric theories for such theories which are invariant under nilpotent supersymmetric transformations when the generators $R^i(\phi)$ are subjected to the restriction

$$R^i_{,i}(\phi) = 0 .$$

(4.1)

Note that the generators of BRST transformations in Yang-Mills theories satisfy this relation. One can easily check that for generators of nilpotent supersymmetric transformations appearing in models introducing in papers [4, 5, 9, 10] the condition (4.1) is valid as well. In case of special supersymmetric theories the superdeterminant of field-dependent supersymmetric transformations reads

$$s\text{Det} M = \frac{1}{1 + \hat{s}\xi}$$

(4.2)

and the action (3.17) reduces to

$$S_\xi(\phi, K) = S(\phi, K) + i\hbar \ln (1 + \hat{s}\xi(\phi)) .$$

(4.3)

Using the nilpotency of $\hat{s}$ we can present the action (4.3) as the modification of initial action $S(\phi)$ by $\hat{s}$-exact term

$$S_\xi(\phi, K) = S(\phi, K) + \hat{s}F(\phi) = S(\phi) + \hat{s}(K\phi + F(\phi)) ,$$

(4.4)

where

$$F = \xi \left[ 1 - \frac{1}{2}(\hat{s}\xi) + \frac{1}{3}(\hat{s}\xi)^2 - \cdots \right] = \xi (\hat{s}\xi)^{-1} \ln (1 + \hat{s}\xi)$$

(4.5)

is a regular function. The presentation (4.4) can be very useful in theories with supersymmetric invariant action. In particular, it was shown [15] that in case of Yang-Mills theories the result of change of variables in vacuum functional with the help of field-dependent BRST transformations can be presented in the form likes (4.4) and interpreted as a modification of gauge condition. This made it possible to prove the independence of the effective action in Yang-Mills theories on the finite increment of gauge on-shell and suggest the formulation of the Gribov-Zwanziger theory [6, 7, 8] free from the problem of gauge dependence (for details, see [17]) of the effective action on-shell [18].

From (4.4) it is clear invariance of $S_\xi(\phi, K$) under supersymmetric transformations

$$\hat{s}S_\xi(\phi, K) = 0 .$$

(4.6)

\footnote{In terms of paper [16] this restriction means that a modular class of a given gauge systems vanishes.}
This invariance can be expressed in the form of Zinn-Justin equation

\[ \frac{\delta S_\xi}{\delta \phi^i} \frac{\delta S_\xi}{\delta K_i} = 0 \]  

(4.7)

As a consequence of the equation (4.7) the generating functional \( Z_\xi(J, K) \) constructed with the help of action \( S_\xi(\phi, K) \) satisfies the Ward identity

\[ J_i \frac{\delta Z_\xi(J, K)}{\delta K_i} = 0 \]  

(4.8)

as the functional \( Z(J, K) \). One can rewrite the Ward identity (4.8) in term of the generating functional of connected Green functions \( W_\xi(J, K) \)

\[ Z_\xi(J, K) = \exp \left\{ \frac{i}{\hbar} W_\xi(J, K) \right\} \]  

(4.9)

as

\[ J_i \frac{\delta W_\xi(J, K)}{\delta K_i} = 0 \]  

(4.10)

Making use the Legendre transformation

\[ \phi^i = \frac{\delta W_\xi(J, K)}{\delta J_i} \]  

(4.11)

and introducing the generating functional of vertex functions \( \Gamma_\xi(\phi, K) \)

\[ \Gamma_\xi(\phi, K) = W_\xi(J, K) - J_i \phi^i, \quad \frac{\delta \Gamma_\xi}{\delta K_i} = \frac{\delta W_\xi}{\delta K_i}, \quad \frac{\delta \Gamma_\xi}{\delta \phi^i} = -J_i, \]  

(4.12)

the Ward identity for \( \Gamma_\xi = \Gamma_\xi(\phi, K) \)

\[ \frac{\delta \Gamma_\xi}{\delta \phi^i} \frac{\delta \Gamma_\xi}{\delta K_i} = 0 \]  

(4.13)

has the form of the Zinn-Justin equation and repeats on quantum level the invariance of a given theory under supersymmetric transformations. It is clear that all relations (4.8)-(4.13) are valid for initial theory \((\xi = 0)\).

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