Entropy dimensions and a class of constructive examples

Sébastien Ferenczi
Institut de Mathématiques de Luminy
CNRS - UMR 6206
Case 907, 163 av. de Luminy
F13288 Marseille Cedex 9 (France)
and Fédération de Recherche des Unités de Mathématiques de Marseille
CNRS - FR 2291
ferenczi@iml.univ-mrs.fr

Kyewon Koh Park
Department of Mathematics
Ajou University
Suwon 442-729
Korea
kkpark@madang.ajou.ac.kr

December 9, 2004

Abstract
Motivated by the study of actions of $\mathbb{Z}^d$ and more general groups, and their non-cocompact subgroup actions, we investigate entropy-type invariants for deterministic systems. In particular, we define a new isomorphism invariant, the entropy dimension, and look at its behaviour on examples. We also look at other natural notions suitable for processes.

AMS subject classification: 37A35, 37A15.

Keywords: Ergodic theory, entropy, examples.

Abbreviated title: Entropy dimensions.

*The authors were supported in part by a joint CNRS/KOSEF Cooperative Research Grant between Korea and France. The second author was also supported in part by grant BK 21. The second author would like to thank Korea Institute for Advanced Study for the pleasant stay while this work was completed.
Let \((X, B, \mu, \sigma, P)\) be a process, where \(\sigma\) denotes an action of a group \(G\), and \(P = \{P_0, \ldots, P_{k-1}\}\) denotes a (finite, measurable) partition of \(X\). In the study of a general group action, subgroup actions play an important role: if a \(G\)-action has positive entropy, it is not hard to see that every non-cocompact subgroup action has infinite entropy (see for example [3]). In the case of a \(\mathbb{Z}^2\)-action generated by two commuting maps, say \(T\) and \(S\), if either \(h(T)\) or \(h(S)\) is finite, the entropy of the \(\mathbb{Z}^2\)-action is 0. Hence it is increasingly important to study systems of entropy zero, as they may give rise to interesting subgroup actions, and to classify them up to measure-theoretic isomorphism. One way to achieve this goal is to look at the amount of determinism in the system, in a more precise way that is given by the mere knowledge of the entropy. Several refinements of the notion of entropy have been introduced by various authors, such as the slow entropy [5], the measure-theoretic complexity [4], the entropy convergence rates [1]; following a suggestion of J. Milnor, we propose a new notion, the entropy dimension; though it seems most promising for actions of groups like \(\mathbb{Z}^p\), for simplicity we develop here the basic definitions and examples in the case of \(\mathbb{Z}\)-actions.

1 Growth rates and names

A first tentative way to define an entropy dimension would be to define

\[
\bar{D}^H(P) = \sup \{0 \leq \alpha \leq 1; \limsup \frac{1}{n^\alpha} H(\vee_{i=0}^{n-1} T^{-i} P) > 0\}.
\]

This can be generalized to \(\mathbb{Z}^k\) by taking a joint on a suitable part \(G_n\) instead of the interval \([0, n]\), and letting \(\alpha\) vary from 0 to \(k\). However, this does not define an isomorphism invariant, as the following proposition implies that \(\sup_P \bar{D}^H(P) = 1\):

**Proposition 1** For any given \(P\) with \(\bar{D}^H(P) < 1\) and any \(\delta > 0\), there exists \(\tilde{P}\) such that \(|P - \tilde{P}| < \delta\), and \(\bar{D}^H(\tilde{P}) > \bar{D}^H(P)\).

**Proof**

Let \(a_0 = \bar{D}^H(P)\). We choose \(a_0 < \alpha < 1\). We build a Rokhlin stack of height \(n_1\) such that \(n_1^{a_0-1} \leq 2^{-L}\) for a very large \(L\). We may ensure that the distribution of the columns on the base level \(B_0\) of the stack is the same as the distribution of \(\vee_{i=0}^{n_1-1} T^i P\). We divide each column into \(2^{n_1}\) subcolumns of equal measure and change the partition \(P\) into \(\tilde{P}^1\) on the first \(n_1\) levels from the bottom, so that each subcolumn has a different \(\tilde{P}^1\)-\([0, n_1]\)-name. For \(x\) and \(y\) in \(B_0\), their \(\tilde{P}^1\)-\([0, n_1]\)-names may agree if their \(\tilde{P}^1\)-\([n_1^2, n_1]\)-names are the same, so the number of different \(\tilde{P}^1\)-\([0, n_1]\)-names may be smaller than \(2^{n_1}\) times the number of columns. However, there are at least \(2^{n_1}\) different \(\tilde{P}^1\)-\([0, n_1]\)-names, each one of measure at most \(2^{-n_1}\). Hence

\[
\frac{1}{n_1^2} H(\vee_{i=0}^{n_1-1} T^i \tilde{P}^1) \geq \frac{1}{n_1^2} \sum_{i=0}^{2^{n_1}-1} \frac{1}{2^{n_1} \log \frac{1}{2^{n_1}}} \pm \epsilon
\]

\[
= \log 2 \pm \epsilon
\]
where $\epsilon$ comes from the error set. Also
\[
\frac{1}{n_1^\alpha}H(\bigvee_{i=0}^{n_1-1}T^i\tilde{P}) \leq \frac{1}{n_1^\alpha} \sum_{\lambda} \sum_{i=0}^{2^{n_1^\alpha} - 1} \lambda \log \frac{\lambda}{2^{n_1^\alpha}} \pm \epsilon
\]
\[
eq \frac{1}{n_1^\alpha} (-\sum_{\lambda} \lambda \log \lambda + \sum_{\lambda} n_1^\alpha \log 2) \pm \epsilon = \log 2 \pm \epsilon
\]
where $\lambda$ denotes the measure of a column and we sum over all columns.

We note that $|P - \tilde{P}| < 2^{-L}$. Let $E_i$ denote the $n_1^\alpha$ levels where $P$ and $\tilde{P}$ may differ. We repeat this for Rokhlin stacks of height $n_k$ where $n_k^{\alpha-1} < 2^{-L-k}$ for $k = 2, 3, \ldots$. In the $k$-th Rokhlin stack, we choose $n_k^\alpha$ many levels in each column and change $\tilde{P}^{k-1}$ to $\tilde{P}^k$ on these levels so that there are at least $2^{n_k^\alpha}$ many different names for each column. We choose these levels so that their union $E_k$ is disjoint from $\bigcup_{i=1}^{k-1} E_i$. Thus $|\tilde{P}^k - \tilde{P}^{k-1}| < 2^{-L-k}$, and we can define $\tilde{P} = \lim \tilde{P}^k$. And we have $D^H(\tilde{P}) \geq \alpha$. Note also that $|P - \tilde{P}| < 2^{-L+1}$, thus $\tilde{P}$ can be chosen arbitrarily close to $P$. And since each $\tilde{P}_k$ is measurable with respect to the $\sigma$-algebra generated by $P$, so is $\tilde{P}$; if $\tilde{P}$ generates a factor $\sigma$-algebra, we can modify it further so that it generates the whole $\sigma$-algebra. ♣

Remark
It is possible to define $D$ using lower instead of upper limits. Note that if $\alpha = 1$ the construction of $\tilde{P}$ is possible.

For a point $x$ in $X$, the $P$-name of $x$ is the sequence $P(x)$ where $P_i(x) = l$ whenever $\sigma^i(x)$ is in $P_l$; we denote by $P_{[0,n)}(x)$ the sequence $P_0(x) \ldots P_{n-1}(x)$. Between $P_{[0,n)}(x)$ and $P_{[0,n)}(y)$, there is the natural Hamming distance, counting the ration of different coordinates in the names: for two sequences $a = (a_1, \ldots a_k)$ and $b = (b_1, \ldots b_k)$ over a finite alphabet, we recall that
\[
\overline{d}(a, b) = \frac{1}{k} \# \{i; a_i \neq b_i\}.
\]

We can define a complexity dimension for a process by
\[
\overline{D}_0(P) = \sup \{0 \leq \alpha \leq 1; \limsup \frac{1}{n^\alpha} \log \# \{\text{different } P - [0, n) - \text{names} \} > 0\},
\]
\[
\underline{D}_0(P) = \sup \{0 \leq \alpha \leq 1; \liminf \frac{1}{n^\alpha} \log \# \{\text{different } P - [0, n) - \text{names} \} > 0\}.
\]

However, it is easy to see, as in the previous case, that this is not an isomorphism invariant. Hence, instead of counting names, we should use the number of $\overline{d}$-balls around names.

\section{Entropy dimensions and subgroup actions}

\begin{definition}
For a point $x \in X$, we define
\[
B(x, n, \epsilon) = \{y \in X; \overline{d}(P_{[0,n)}(x), P_{[0,n)}(y)) < \epsilon\}.
\]

And let $K(n, \epsilon)$ be the smallest number $K$ such that there exists a subset of $X$ of measure at least $1 - \epsilon$ covered by at most $K$ balls $B(x, n, \epsilon)$. Then
\[
\overline{D}(P, \epsilon) = \sup \{0 \leq \alpha \leq 1; \limsup \frac{1}{n^\alpha} \log K(n, \epsilon) > 0\},
\]
\end{definition}
\[ \overline{D}(P) = \lim_{\epsilon \to 0} \overline{D}(P, \epsilon), \]
\[ \overline{D} = \sup_P \overline{d}(P). \]

Similarly
\[ \underline{D}(P, \epsilon) = \sup\{0 \leq \alpha \leq 1; \lim \inf \frac{1}{n^\alpha} \log K(n, \epsilon) > 0\}, \]
\[ \underline{D}(P) = \lim_{\epsilon \to 0} \underline{D}(P, \epsilon), \]
\[ \underline{D} = \sup_P \underline{d}(P). \]

We call \( \overline{D} \), resp. \( \underline{D} \), the upper, resp. lower, entropy dimension of the system \((X, \mathcal{B}, \mu, \sigma)\). If \( \overline{D} = \underline{D} \), we just call it the entropy dimension and denote it by \( D \).

Note that for a \( \mathbb{Z} \)-action, the entropy dimension may be 1 while the entropy is 0.

It is a straightforward consequence of our definition, proved by the same proof as Corollary 1 in [4], that \( \overline{D}(P) = \overline{D} \) when \( P \) is a generating partition.

We want to investigate the relation between the entropy dimension and the entropy of subgroup actions, particularly in the case of \( \mathbb{Z}^2 \): if one of the directions has positive entropy, then \( K(n, \epsilon) \) grows at least at the rate of \( e^{cn} \) and the lower entropy dimension is at least one. Hence, if \( \overline{D} < 1 \), then \( h(v) = 0 \) for every direction \( v \), and, moreover, the cone entropy [2] has the property that \( h_c(v) = h(v) = 0 \). The converse is not true: Katok and Thouvenot [5] provide an example where the upper entropy dimension is arbitrarily close to 2 while the directional entropy is 0 for almost all directions; note that in this example the upper and lower entropy dimensions do not agree.

We recall that there exists an example in [7] where \( h(\sigma^{(1,0)}) > 0 \) while all the remaining directional entropies (including the irrational directions) are 0; this \( \mathbb{Z}^2 \)-action has clearly entropy dimension equal to 1. In the well-known example of Ledrappier ([6]), the entropy dimension is 1 and every directional entropy is positive. By making a direct product of countably many copies of that example, we can build a \( \mathbb{Z}^2 \)-action whose entropy dimension is 1 and every direction has infinite entropy, because of the following lemma, which holds also for countable products:

**Lemma 3**
\[ \overline{D}(\sigma \times \tau) = \max(\overline{D}(\sigma), \overline{D}(\tau)). \]

**Proof**
The entropy dimension of \( \sigma \times \tau \) may be computed by taking only partitions of the form \( P \times Q \). But then for these partitions \( B((x, y), n, 2\epsilon) \) contains \( B(x, n, \epsilon) \times B(y, n, \epsilon) \) (respectively for \( P \) and \( Q \)) and is included in \( B(x, n, 2\epsilon) \times B(y, n, 2\epsilon) \), which yields the result. ♦
3 Examples of entropy dimensions

We define inductively a family of blocks $B_{n,i}$, $1 \leq i \leq b_n$, in the following way; given two sequences of positive integers $e_n$ and $r_n$:

- $b_0 = k$, $B_{0,i} = i - 1$, $1 \leq i \leq k$,
- $b_{n+1} = (b_n)^{e_n}$,
- the $B'_{n,i}$, $1 \leq i \leq b_{n+1}$ are all the possible concatenations of $e_n$ blocks $B_{n,i}$,
- for each $1 \leq i \leq b_{n+1}$, $B_{n+1,i}$ is a concatenation of $r_n$ blocks $B_{n,i}$.

Let $h_n$ be the length of the $B_{n,i}$, $h'_n$ be the length of the $B'_{n,i}$.

We can thus define a topological system as the shift on the set of sequences $\{x_n, n \in \mathbb{Z}\}$ such that for every $s < t$ there exists $n$ and $i$ such that $x_s \ldots x_t$ is a subword of $B_{n,i}$. We put an invariant measure on it by giving to each block $B_{n,i}$ the measure $\frac{1}{b_n}$. We denote by $P$ the natural partition in $k$ sets given by the zero coordinate.

The above construction is well known to ergodic theory specialists, and a generalization of it to $\mathbb{Z}^2$-actions is used in [5]; however, even its one-dimensional version can yield new types of counter-examples. This system will be referred in the sequel as the standard example.

**Proposition 4** There is a choice of parameters such that the standard example satisfies

$$\overline{D} = 1,$$

$$\underline{D} = 0.$$

**Proof**

**Lower limit.**

For any $\epsilon$, $K(h_{n+1}, \epsilon)$ is smaller than the total number of $P - [0, h_{n+1})$-names. The possible names of length $h_{n+1}$ are all the $W_{n+1}(a, i, j)$ where, for $0 \leq a \leq h_{n+1}$, $1 \leq i \leq b_{n+1}$, $1 \leq j \leq b_{n+1}$, $W_{n+1}(a, i, j)$ is the suffix of length $a$ of $B_{n+1,j}$ followed by the prefix of length $h_{n+1} - a$ of $B_{n+1,j}$. Hence their numbers is at most $h_{n+1} b_{n+1}^2$, with $b_{n+1}$ as above and $h_{n+1} = e_0 \ldots e_{n-1} r_0 \ldots r_n$. If, $e_0, \ldots, e_n, r_0, \ldots, r_{n-1}$ being fixed, we choose $r_n$ large enough, we shall have $\log K(h_{n+1}, \epsilon) \leq (h_{n+1})^{b_n}$ for any given sequence $\delta_n$.

**Upper limit.**

Let $L'_n(\epsilon)$ be the number of $\epsilon$-$d$-balls than can be made with blocks $B'_{n,i}$. Note that, on an alphabet of $k$ letters, for a given word $w$ of length $m$, the number of words $w'$ with $d(w, w') < \epsilon$ is at most $\left( \frac{m}{m\epsilon} \right)^{k m \epsilon} \leq k^{m g(\epsilon)}$ for some $g(\epsilon) \to 0$ when $\epsilon \to 0$. In the above construction, the number of different blocks $B'_{n,i}$ is $b_{n+1} = k^{e_0 \ldots e_n}$. As in every of these blocks the repetitions occur exactly at the same places, for a given word $B'_{n,j}$, the number of words $B'_{n,j}$ with $d(B'_{n,i}, B'_{n,j}) < \epsilon$ is at most $\left( \frac{e_0 \ldots e_n}{e_0 \ldots e_n \epsilon} \right) k^{e_0 \ldots e_n \epsilon}$. Hence

$$L'_n(\epsilon) \geq k^{e_0 \ldots e_n - g(\epsilon)}.$$

4
As all different blocks are given the same measure, we have

\[ K(h'_n, \epsilon) \geq (1 - \epsilon)L_n(\epsilon). \]

As \( h'_n = e_0 \ldots e_{n-1}r_0 \ldots r_{n-1} \), if, \( e_0, \ldots, e_{n-1}, r_0, \ldots, r_{n-1} \) being fixed, we choose \( e_n \) large enough, we shall have \( \log K(h'_n, \epsilon) \geq (h'_n)^{1 - \delta_n} \) for any given sequence \( \delta_n \).

**Proposition 5** For any \( 0 < \alpha < 1 \), there is a choice of parameters such that the standard example satisfies \( D = \alpha \).

**Proof**

We make the proof for \( \alpha = \frac{1}{2} \). We define a sequence \( l_n \) by choosing a very large \( l_1 \), then \( l_n = \left\lfloor \frac{\sqrt{l_{n-1}^{n-1}}}{2} \right\rfloor \), then, in the standard construction, starting from the two 0-blocks 0 and 1, we put \( e_0 = r_0 = \left\lfloor \sqrt{l_1} \right\rfloor \), and, for \( n \geq 1 \), \( e_n = r_n = \left\lfloor \frac{\sqrt{l_{n-1}^{n-1}}}{2} \right\rfloor \). \( \lfloor x \rfloor \) denotes the integer part of \( x \), but in the following computations we shall assimilate a large enough \( x \) with its integer part.

Then, the lower limit is reached along the sequence \( \{h_{n+1} = l_n\} \) and the upper limit along the sequence \( h'_n = \{l_{n-1}^{n-1} \} \).

**Lower limit**

As in the second part of the proof of the last proposition, \( K(h_{n+1}, \epsilon) \) is at least \((1 - \epsilon)\) times the number \( L_{n+1}(\epsilon) \) of \( \epsilon \)-\( \bar{d} \)-balls than can be made with blocks \( B_{n+1} \). Because of the repetitions, and the computation in the last proposition

\[ L_{n+1}(\epsilon) \geq L'_n(\epsilon) \geq 2^{e_0 \ldots e_n - g(\epsilon)}. \]

So we have only to compute

\[
\lim_{n \to +\infty} \frac{1}{\sqrt{l_n}} \log(2^{e_0 \ldots e_n}) =
\lim_{n \to +\infty} \frac{1}{\sqrt{l_n}} \log 2\sqrt{n} \sqrt{n}^{1/2} \sqrt{n}^{1/2} - \sqrt{n}^{1/2} =
\lim_{n \to +\infty} \frac{1}{\sqrt{l_n}} \sqrt{l_n} \log 2 = \log 2.
\]

Hence \( D \geq \frac{1}{2} \).

**Upper limit**

As in the first part of the proof of the last proposition, \( K(h'_n, \epsilon) \) is smaller than the total number of \( P - \{0, h'_n\} \)-names, and this is at most \( b_{n+1}^2 h'_n \). We take some \( b > \frac{1}{2} \);

\[
\lim_{n \to +\infty} \frac{1}{(h'_n)^b} \log b_{n+1}^2 h'_n =
2 \lim_{n \to +\infty} \frac{1}{(h_n)^b} \log b_{n+1} =
\]
\[
2 \limsup_{n \to +\infty} \frac{1}{(l_{n-1})^b} \log \sqrt[2]{l_1^2} \sqrt[3]{l_2^3} \cdots \sqrt[n]{l_n^n} = \\
2 \limsup_{n \to +\infty} \frac{(l_{n-1})^{1+1}}{(l_{n-1})^b} = 0.
\]

Hence \(D \leq \frac{1}{2}\), which gives what we claimed.

The general case (for a given \(\alpha\)) follows with the same proof, by taking \(l_n = l_{n-1} \lfloor \frac{\alpha}{n-1} \rfloor \lfloor \frac{1-\alpha}{n-1} \rfloor\), \(e_n = \lfloor \frac{2}{n-1} \rfloor\), \(r_n = \lfloor \frac{1-n}{n-1} \rfloor\).

The above examples can be generalized to \(\mathbb{Z}^2\)-actions; by alternating repetitions and independent stacking, we can build an example whose entropy dimension is any given \(0 \leq \alpha \leq 2\).

In [4], where the rate of growth of \(K(n, \epsilon)\) is used to define the so-called measure-theoretic complexity, it is asked whether this growth rate can be unbounded but smaller than \(O(n)\) (its topological version for symbolic systems, the symbolic complexity has to be bounded if it is smaller than \(n\)). Our class of examples allows to answer this question; note that the proofs are slightly more involved as we are dealing with sub-exponential growths:

**Proposition 6** For any given function \(\phi\) growing to infinity with \(n\), there is a choice of parameters such that the standard example satisfies, for every fixed \(\epsilon\) small enough,

\[K(n, \epsilon) \to +\infty\]

with \(n\), but

\[K(n, \epsilon) \leq \phi(n)\]

for all \(n\).

**Proof**

**Upper bounds**

We give upper bounds for \(K' \geq k\), where \(K'(n, \epsilon)\) is the smallest number of \(\epsilon\)-\(\bar{d}\)-balls of names of length \(n\) necessary to cover a proportion of the space of measure 1. We look at \(K'\) at the end of its times of maximal growth, namely \(K'(h_n', \epsilon)\). The possible words of length \(h_n'\) are all the \(W_n'(a, i, j)\) where, for \(0 \leq a \leq h_n' - 1\), \(1 \leq i \leq b_{n+1}\), \(1 \leq j \leq b_{n+1}\), \(W_n'(a, i, j)\) is the suffix of length \(a\) of \(B_{n,i}'\) followed by the prefix of length \(h_n' - a\) of \(B_{n,j}'\). Each one of these words is at a \((\bar{d})\) distance at most \(\epsilon\) of some \(W_n'(s, i, j)\) for \(1 \leq s \leq K'(h_n', \epsilon)\).

We look now at words of length \(h_{n+1}\); they are all the \(W_n(a, i, j)\) where, for \(0 \leq a \leq h_{n+1} - 1\), \(1 \leq i \leq b_{n+1}\), \(1 \leq j \leq b_{n+1}\), \(W_n(a, i, j)\) is the suffix of length \(a\) of \(B_{n+1,i}'\) followed by the prefix of length \(h_{n+1} - a\) of \(B_{n+1,j}'\). Hence for \(0 \leq t \leq r_n - 1\), \(0 \leq a \leq h_{n+1} - 1\), \(1 \leq i \leq b_{n+1}\), \(1 \leq j \leq b_{n+1}\),

\[W_n(a + th_n', i, j) = W(a, i, i)W(a, i, j)W(a, j, j)^{r_n-t-1}.
\]
Each one of these will be at a distance at most $\epsilon + \frac{1}{r_n}$ of some $W(a_s, i_s, j_s) + W(a_s, j_s, j_s)$ and, for fixed $s$, $W(a_s, i_s, i_s) + W(a_s, j_s, j_s)$ and $W(a_s, i_s, i_s) + W(a_s, j_s, j_s)$ are at a distance at most $\frac{|t-t'|}{r_n}$. Hence, for a given sequence $v_n$, we have

$$K'(h_{n+1}, \epsilon(1 + \frac{1}{r_n} + v_n)) \leq \frac{K'(h'_n, \epsilon)^2}{\epsilon v_n}.$$ 

Then, during the stage of independent stacking, a straightforward computation gives that

$$K'(h_{n+1}, \epsilon) \leq K'(h_{n+1}, \epsilon)^{v_n}.$$ 

If we fix the sequence $e_n$, and suppose $\sum \frac{1}{r_n} < +\infty$; we choose any sequence $v_n$ such that $\sum v_n < +\infty$; then, if we choose $r_n$ large enough in terms of $K'(h'_n, \epsilon)$, $h'_n$, and $e_n$, we get that $K'(h'_{n+1}, 2\epsilon)$ is smaller than $\phi(h'_{n+1})$, and this is true a fortiori for other values.

**Lower bounds**

We shall show that $K(n, \epsilon) \to +\infty$ with $n$. For this, let $L_n(\epsilon)$ be the number of $\epsilon$-balls than can be made with blocks $B_n$, and $L'_n(\epsilon)$ be the number of $\epsilon$-balls that can be made with $B'_n$. During the repetition stage, we have

$$L_n(\epsilon) \geq L'_{n-1}(\epsilon).$$

Then, during the independent stage, we start from $L = L_n(\epsilon)$ blocks which are $\epsilon$-separated; we call them $B_{n,s_1}, \ldots, B_{n,s_L}$. Then, if $e_n$ is a multiple of $L$, the $2L$ blocks $B'_{n,s_i}$, $1 \leq i \leq L$, and $B_{n,s_1} \ldots B_{n,s_L}$, $1 \leq i \leq L$, are $\epsilon(1 - \frac{1}{L})$-separated.

Thus whenever $e_n$ is large compared to $L_n(\epsilon)$ we have

$$L_{n+1}(\epsilon(1 - \frac{1}{L_n(\epsilon)})) \geq 2L_n(\epsilon)$$

and hence $L_n(\frac{\epsilon}{L})$ tends to infinity with $n$; and, because of the structure of the names and the fact that each block $B_{n,i}$ has the same measure for fixed $n$, we get that $K(h_n, \epsilon)$ tends to infinity with $n$. ♦

**Remarks**

To make our examples weakly mixing, it is enough to place a spacer between two consecutive blocks at each repetition stage.

It is easy to see that all our examples satisfy a form of Shannon-McMillan-Breiman theorem (indeed, all atoms have the same measure); in a forthcoming paper, we shall give examples which do not satisfy it.

**References**

[1] F. BLUME: Possible rates of entropy convergence, *Ergodic Theory Dynam. Systems* 17 (1997), no. 1, 45–70.
[2] R. BURTON, K. K. PARK: Spatial determinism for a $\mathbb{Z}^2$-action, preprint.

[3] J.-P. CONZE: Entropie d’un groupe abélien de transformations, (in French), Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 25 (1972/73), 11–30.

[4] S. FERENCZI: Measure-theoretic complexity of ergodic systems, Israël J. Math. 100 (1997), 189-207.

[5] A. KATOK, J.-P. THOUVENOT: Slow entropy type invariants and smooth realization of commuting measure-preserving transformations, Ann. Inst. H. Poincaré Probab. Statist. 33 (1997), no. 3, 323–338.

[6] F. LEDRAPPIER: Un champ markovien peut être d’entropie nulle et mélangeant, (in French), C. R. Acad. Sci. Paris Sr. A-B 287 (1978), no. 7, A561–A563.

[7] K. K. PARK: On directional entropy functions, Israël J. Math. 113 (1999), 243–267.