Geodesics deviation equation approach to chaos.

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Abstract

Geodesics deviation equation (GDE) is introduced. In "adiabatic" approximation exact solution of the GDE is found. Perturbation theory in general case is formulated. Geometrical criterion of local instability which may lead to chaos is formulated.
1 Introduction

Even though classical mechanics is an old subject, the surprising fact is that the mechanisms affecting its dynamical evolution have only recently been qualitatively understood. It is known that chaos in classical mechanical systems emerges when overlapping of resonance zones occurs. The essential problem in study of chaos lies in formulation of a simple mathematical criterion of sensitive dependence on initial conditions.

An example of analytical criterion is proposed by Toda, Duff and Brumer \cite{1} local criterion of the transition from the ordered motion to the chaotic one. The main idea of their approach lies in replacement studies of the behaviour of trajectories in phase space which remain nearby to the selected trajectory by studies of the behaviour of nearby trajectories in vicinity of selected points of the phase space.

The other local criterion of chaos is studied in this article. It is based on observation that Euler - Lagrange equations of motion can be rewritten as the geodesic equation with respect to the Jacobi metric and then on measuring the local tendency of geodesics to converge or to diverge. The idea of this method has been originated in Krylov paper \cite{2}. The use of the deviation equation presupposes that geodesics form a congruence which is true locally in admissible for trajectories region of the configuration space. The property of being a K-system is a global property and therefore deviation equation which is purely local criterion needs some extension so as to obtain a more complete (a kind of Lapunov exponent and study of the influence of the boundary \( \partial D_E \) on global behaviour of the system) characteristic of the system.

The overall plan of this article is as follows. In section 2, we introduce in a new, simplified way the concept of the geodesics deviation and find a simple form of geodesics deviation equation. Section 3 is devoted entirely to searching for solutions of the GDE and to formulation of the perturbation expansion. The last section (4) contains examples and comments.

2 Introduction to geodesics deviation equation
2.1 Dynamics of the system in arbitrary coordinates.

Let us consider classical mechanical system with $N$ degrees of freedom ($i, j = 1, 2, ..., N$) described by the lagrangian

$$
\mathcal{L}(q, \dot{q}) = \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j - V(q),
$$

where $g_{ij} dq^i \otimes dq^j$ is a Riemannian metric on the configuration space $\mathcal{M}$ of the system. Let us assume that the dynamics of this system could be equivalently described in terms of a momentum - position phase space by the Hamiltonian

$$
\mathcal{H}(q, p) = \frac{1}{2} g^{ij}(q) p_i p_j + V(q),
$$

$p_i = g_{ij} \dot{q}^j$ denotes momentum conjugate with $i-th$ coordinate $q^i$, and $" = \frac{d}{dt}"$ differentiation with respect to time. Dynamics of the system is governed by the Hamilton equations of motion:

$$
\dot{p}_i = -\frac{1}{2} \frac{\partial g^{ik}}{\partial q^l} p_j p_k - \frac{\partial V}{\partial q^i} \quad \text{and} \quad \dot{q}^i = g^{ik} p_k.
$$

By eliminating $p_i$ variables and using identity $\frac{\partial g^{ik}}{\partial q^l} g_{jk} = -g^{ik} \frac{\partial g_{jk}}{\partial q^l}$ we can rewrite a system of the first order equations as a system of the second order equations

$$
\ddot{q}^i + \Gamma^i_{kl} \dot{q}^k \dot{q}^l = -g^{ik} \partial_k V,
$$

$\partial_k = \frac{\partial}{\partial q^k}$ and $\Gamma^i_{kl}$ is Levi-Civita connection for metric $g_{ij}$.

2.2 Dynamics on the boundary of an admissible region

An important characteristic of the mechanical system is the shape of region of the configuration space which is admissible for the physical trajectories. It is known that even small deformation of this region could lead from deterministic to chaotic behavior of the system. Well known example of such behavior is the stadium [3]. We will pay some attention to properties of this region.

It is obvious that the total energy of the system is an integral of motion. Therefore for fixed (by the choice of the initial conditions) energy of the system any trajectory in phase space is confined to the hypersurface
\( \frac{1}{2} g^{ij}(q)p_ip_j + V(q) = E. \) The kinetic energy of the system \( \frac{1}{2} g^{ij}p_ip_j \) is positive and therefore (for fixed total energy \( E \)) a projection of the constant energy hypersurface on configuration space provides an admissible for trajectories (movement) region of the configuration space \( \mathcal{D}_E = \{ q \in \mathcal{M} : V(q) \leq E \} \). \( \mathcal{D}_E \) depending on the energy, could be bounded or unbounded, connected or not. In general \( \mathcal{D}_E \) has boundary \( \partial \mathcal{D}_E = \{ q \in \mathcal{M} : V(q) = E \} \). If potential \( V(q) \) has no critical points on the boundary then \( \partial \mathcal{D}_E \) is \( N - 1 \) dimensional submanifold of \( \mathcal{M} \).

We can easy see that point \( q_0 \in \partial \mathcal{D}_E \) in which trajectory reaches the boundary is an isolating point in \( \partial \mathcal{D}_E \).

If trajectory reaches \( q_0 \in \partial \mathcal{D}_E \) then speed \( v^i \) in this point is equal to zero. It is consequence of equality \( \frac{1}{2} v^i v^j g_{ij} + V(q_0) = E = V(q_0) \). We see that in \( \partial \mathcal{D}_E \) exist neighbourhood of \( q_0 \in \partial \mathcal{D}_E \) which does not contain any point of the trajectory. It means that movement along boundary \( \partial \mathcal{D}_E \) is impossible, and that the point \( q_0 \) is isolating point.

In fact, it is easy to show that if potential is smooth function of space coordinates then trajectories approach to \( q_0 \in \partial \mathcal{D}_E \) or depart from \( q_0 \in \partial \mathcal{D}_E \) perpendicularly to the boundary \( \partial \mathcal{D}_E \).

i) Any vector \( \xi^i \) tangent to \( \partial \mathcal{D}_E \) satisfy equation \( \partial_i V(q_0) \xi^i = 0 \) which could be rewritten in the form \( g_{ij}(\text{grad} V)^i \big|_{q=q_0} \xi^j = 0 \). This is just orthogonality condition in the sense of scalar product defined by the matrix \( g_{ij}(q_0) \).

ii) On the other hand if we assume that the trajectory reaches the point \( q_0 \) in instant of time \( t_0 \) then for \( t \) close to \( t_0 \) we have \( q^i(t) = q^i_0 + \dot{q}^i(t_0)(t - t_0) + \frac{1}{2} \ddot{q}^i(t_0)(t - t_0)^2 + O((t - t_0)^2) \). Reminding that \( v^i = \dot{q}^i(t_0) = 0 \) and using equations of motion (4) we obtain \( \dot{q}^i(t) = q^i_0 - \frac{1}{2} g^{ij}(q_0) \partial_j V(q_0)(t - t_0)^2 + O((t - t_0)^2) \). After differentiation of the last formula with respect to \( t \) we evaluate velocity in neighbourhood of the point \( q_0 \), \( \dot{q}^i(t) = (\text{grad} V)^i \big|_{q=q_0} (t_0 - t) \).

From i) and ii) it follows that trajectories are orthogonal to the boundary \( \partial \mathcal{D}_E \).

The stadium (which is not smooth) does not satisfy assumption of the above statement.
2.3 Geometric form of the equations of motion

Let us turn to dynamics of the system in the interior of the admissible region \( \text{Int} \mathcal{D}_E = \{ q \in \mathcal{M} : V(q) < E \} \).

Crucial to our future construction is an observation that geodesic equation in the Riemannian geometry defined by the Jacobi metric \( \hat{g}^\mu_{\nu} = \psi g^\mu_{\nu} \) (with \( \psi(q) = 2(E - V) \) and natural parameter \( s \) so that \( \frac{ds}{dt} = 2(E - V) \)) on the set \( \text{Int} \mathcal{D}_E \) is equivalent to the equations of motion (4).

Geodesic equation for metric \( \hat{g}_{ij} = \psi(q) g_{ij} \) has the well known form

\[
\frac{d^2 q^i}{ds^2} + \hat{\Gamma}^i_{jk} \frac{dq^j}{ds} \frac{dq^k}{ds} = 0,
\]

where \( s \) is a natural parameter in the sense of the Jacobi metric \( (\hat{g}_{ij}(q(s)) \frac{d^2 q^i}{ds^2} = 1) \) and \( \hat{\Gamma}^i_{jk} \) is a Christoffel symbol with respect to the same metric \( \hat{g}_{ij} \). If we express Christoffel symbols of the Jacobi metric by Christoffel symbols \( \Gamma^i_{jk} \) of the metric \( g_{ij} \)

\[
\hat{\Gamma}^i_{jk} = \Gamma^i_{jk} + \frac{1}{2} \left[ \partial_j (\ln \psi) \delta^i_k + \partial_k (\ln \psi) \delta^i_j - \partial_i (\ln \psi) g^{ri} g_{jk} \right],
\]

and exchange the natural parameter \( s \) for time \( t \) then we will obtain the rearranged equation (5)

\[
\frac{d^2 q^i}{dt^2} + \Gamma^i_{jk} \frac{dq^j}{dt} \frac{dq^k}{dt} = \left[ \left( \frac{ds}{dt} \right)^2 \frac{d^2 t}{ds^2} + \frac{dq^i}{dt} \partial_j \ln \psi \right] \frac{dq^i}{dt} + \frac{1}{2\psi} \left( \frac{ds}{dt} \right)^2 g^{ik} \partial_k \ln \psi.
\]

Now we would like to chose \( \psi \) and parameter \( s = s(t) \) so as to geodesic equation (5) be equivalent to equation (4). Let us impose the simplest conditions which eliminate unwanted terms from equation (7)

\[
\left( \frac{ds}{dt} \right)^2 \frac{d^2 t}{ds^2} + \frac{dq^i}{dt} \partial_j \ln \psi = 0,
\]

\[
\frac{1}{2\psi} \left( \frac{ds}{dt} \right)^2 g^{ik} \partial_k \ln \psi = -\partial_j V.
\]
Making further manipulations on parameter derivatives \( \left( \frac{d^2 t}{ds^2} = \frac{dt}{ds} \frac{d}{dt} \left( \frac{1}{
abla t} \right) = -\frac{1}{\left( \frac{dt}{ds} \right)^3} \frac{d^2 s}{dt^2} \right) \), we can simplify the first constrain (8) to the form

\[
\frac{du}{dt} = u \frac{d\ln \psi}{dt},
\]

where \( u(t) = \left( \frac{ds}{dt} \right)^2 \).

The solution of this equation \( \left( \frac{du}{dt} \right)^2 = u(t) = A \psi^2 \) married with condition (9) gives \( \frac{1}{2} \partial_t \psi = -\partial_t V \), which provides the final answer

\[
\psi = 2(M - V).
\]

\( M \) is an integration constant (\( A \) equal one was taken for simplicity).

The only unknown is interpretation of an arbitrary constant. Reminding that \( s \) is a natural parameter for Jacobi Metric

\[
1 = \hat{g}_{ij} \frac{dq^i}{ds} \frac{dq^j}{ds} = 2(M - V) g_{ij} \left( \frac{dt}{ds} \right)^2 \frac{dq^i}{dt} \frac{dq^j}{dt} = \frac{1}{2(M - V)} g_{ij} \frac{dq^i}{dt} \frac{dq^j}{dt},
\]

we find that \( M \) is the total energy of the system

\[
M = \frac{1}{2} g_{ij} \frac{dq^i}{dt} \frac{dq^j}{dt} + V = E.
\]

Therefore

\[
\psi = 2(E - V).
\]

We can also check that \( s \) is a good evolution parameter in the region \( IntD_E \) i.e. \( t \to s(t) \) is monotonically increasing function of time.

In admissible for trajectories region of configuration space \( \frac{ds}{dt}(t) = 2(E - V) \geq 0, \frac{d^2 s}{dt^2}(t) = -2q^i \partial_i V, \) and \( \frac{d^3 s}{dt^3}(t) = -2q^i \partial_i V - 2q^i \partial_i q^j \partial_j V \). Let us fix the origin of the parameter transformation \( s(t = 0) = 0 \). If for some value of the parameters \( s(t_0) = s_0 \) the system reaches the boundary of the admissible region \( q_0 \in \partial D_E \) then

\[
\frac{ds}{dt}(t_0) = \frac{d^2 s}{dt^2}(t_0) = 0, \quad \frac{d^3 s}{dt^3}(t_0) = 2g^{ij}(q_0) \partial_i V(q_0) \partial_j V(q_0) > 0.
\]

In the last inequality we used equations of motion (4). We see that \( s_0 = s(t_0) \) is an inflection point of the considered function \( t \to s(t) \). In the region \( IntD_E \) \( s(t) \) is monotonically increasing function of time.
2.4 Origin of the GDE

In the previous section we discovered that Euler-Lagrange equations (4) could be transformed to the geodesic equation

\[ \hat{\nabla}_u u = 0, \] (12)

where \( u \) is the tangent vector to the geodesic and \( \hat{\nabla} \) is the covariant derivative with respect to the Jacobi metric. Our purpose is to investigate the relative motion of geodesics in a given domain \( \text{Int}D_E \) of the configuration space. We choose a transverse curve \( C \) with respect to the congruence of geodesics, i.e., the curve which only once crosses each of the geodesics belonging to the congruence. We assume that the point at which the curve \( C \) crosses a given geodesic is the zero point of the parameter \( s \) along this geodesic. Now, we find the curve \( C_s \) which is the copy of the curve \( C \). We construct \( C_s \) by transporting each point of the curve \( C \), along the geodesic to the point on this geodesic at which the value of the parameter is \( s \). In this way, in the domain in which the congruence of geodesics is determined, we obtain a 2-dimensional surface. The tangent vector fields to this surface at the point for which the values of parameters are \((\lambda, s)\) will be denoted by \( u(\lambda, s) = \frac{\partial}{\partial s} \big|_{(\lambda, s)} \), and \( \xi(\lambda, s) = \frac{\partial}{\partial \lambda} \big|_{(\lambda, s)} \). The field \( u \) is also tangent to the geodesic. By construction of the surface the Lie bracket of the vector fields vanish \([u, \xi] = 0\). Since the connection \( \hat{\Gamma} \) is torsionfree, one has

\[ \hat{T}(u, \xi) = \hat{\nabla}_u \xi - \hat{\nabla}_\xi u - [u, \xi] = 0, \] (13)

and therefore

\[ \hat{\nabla}_u \xi = \hat{\nabla}_\xi u. \] (14)

Now we would like to find acceleration of the changes of the vector field \( \xi \) along a given geodesic. From the definition of the curvature one could obtain

\[ \hat{R}(u, \xi) u = \left( \hat{\nabla}_u \hat{\nabla}_\xi - \hat{\nabla}_\xi \hat{\nabla}_u - \hat{\nabla}_{[u, \xi]} \right) u = \hat{\nabla}_u \hat{\nabla}_\xi u. \] (15)

With the help of equations (14-15) we can find acceleration of the \( \xi \) field

\[ \hat{\nabla}_u \hat{\nabla}_u \xi = \hat{R}(u, \xi) u. \] (16)

The vector field \( \xi \) informs us about the relative position of points on neighbouring geodesics which start from points on arbitrarily chosen curve \( C \). We
are interested only in the relative motion with respect to geodesics and not in the motion of points along these geodesics. The component of $\xi$ parallel to the vector $u$ contains useless information. Therefore, we will eliminate this component focusing only on orthogonal to $u$ component of $\xi$. Let us decompose $\xi = n + \lambda u$, where $\lambda \in \mathbb{R}$ and $\hat{g}(n, u) = 0$. We also assume unit normalization of the tangent field $\hat{g}(u, u) = 1$. Under above assumption the parameter $\lambda$ is just a projection $u$ on $\xi$, $\hat{g}(\xi, u) = \hat{g}(\xi, u) = \lambda$. The vector field $n$ is called geodesics deviation $n = \xi - \hat{g}(\xi, u)u$. Since $\xi = n + \hat{g}(\xi, u)u$, both sides of the equation (16) could be transformed to the form which is free of parallel (to the vector $u$) component of $\xi$

\[ \hat{\nabla}_u \hat{\nabla}_u \xi = \hat{\nabla}_u \left( \hat{\nabla}_u n + \hat{g}(\hat{\nabla}_u \xi, u)u \right) = \hat{\nabla}_u \left( \hat{\nabla}_u n + \hat{g}(\hat{\nabla}_u \xi, u)u \right) = \hat{\nabla}_u \hat{\nabla}_u n, \]  

(17)

\[ \hat{R}(u, \xi)u = \hat{R}(u, n)u + \hat{g}(\xi, u)\hat{R}(u, u)u = \hat{R}(u, n)u. \]  

(18)

As a result of (17-18) equation (16) can be rewritten in the form known as a geodesics deviation equation

\[ \hat{\nabla}_u \hat{\nabla}_u n = \hat{R}(u, n)u. \]  

(19)

Equation (19) answers the question with which acceleration field $n$ changes along a given geodesic. It should be stressed that equation (19) measures local tendency of geodesics to converge or to diverge and it works if the vector $n$ is small. It also works in a neighbourhood of geodesics which are parallel to each other in some interval.

### 2.5 Newtonian form of the GDE

Let us rewrite the geodesics deviation equation in the form similar to the Newtonian equation of motion. This rearrangement allows us to use the known methods of solving equations of motion.

To this end we shall use the Riemann tensor $\hat{R}(A, B, C, D) \equiv \hat{g}(A, \hat{R}(C, D)B)$ where $A, B, C, D$ are vector fields on configuration space $\mathcal{M}$. The components of this tensor are $\hat{R}_{ijkl} = \hat{R} \left( \frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}, \frac{\partial}{\partial q^k}, \frac{\partial}{\partial q^l} \right) = \hat{g}_{ir} \hat{R}_{jkl}$. Using this tensor we can rearrange the right hand side of the geodesics deviation equation (19)

\[ [\hat{R}(u, n)u]^i = - \frac{1}{2} \hat{g}^{ij} \hat{R}_{klm} (u^k \delta^l_j u^r n^m + u^k n^l u^r \delta^m_j) = - \frac{1}{2} \hat{g}^{ij} \frac{\partial}{\partial n^j}[\hat{R}(u, n, n, n)]. \]  

(20)
Now we introduce the new gradient operator \( [\text{grad}_n]^i = \hat{g}^{ij} \frac{\partial}{\partial n_j} \) and define a "potential" \( V_u(n) = \frac{1}{2} \hat{R}(u, n, u, n) \). This leads to the equation

\[
\frac{D^2 n}{ds^2} \equiv \nabla_u \nabla_u n = -\text{grad}_n V_u(n)
\]

(21)

which looks like Newton equation of motion.

Newtonian nature of equation (21) is even more transparent in the Fermi frame \((E_1, E_2, ..., E_{N-1}, E_N)\), defined by \( \nabla_u E_a = 0 \) and \( \hat{g}(E_a, E_b) = \delta_{ab} \) where \( a, b = 1, 2, ..., N-1, N \).

If we chose the last vector of this frame as a tangent to the geodesic \( E_N = u \) and denote \( \alpha, \beta = 1, 2, ..., N-1 \) then deviation equation (21) takes the form

\[
\frac{d^2 n^\alpha}{ds^2} = -\frac{\partial}{\partial n^\alpha} V_u(n).
\]

(22)

The function \( V \) is the quadratic form with respect to the \( n^\alpha \) variables \( V_u(n) = A_{\alpha\beta} n^\alpha n^\beta \), where \( [A_{\alpha\beta}] \) is \((N-1) \times (N-1)\) matrix defined by the proper components of the Riemann tensor in the Fermi base \( A_{\alpha\beta} = \hat{R}_{\alpha N\beta N} \). Equation (22) describes the system of \( N-1 \) "coupling oscillators" with "time"-dependent frequencies. If we consider the relative motion of geodesics only locally (in the neighbourhood of the point \( q_* = q(s_*) \)) then matrix \( [A_{\alpha\beta}] \) is constant and values of its components are defined by the values of the Riemann tensor in the point \( q_* \). It is clear that if \( V \) is positively defined then motion of geodesics in the neighbourhood of the considered point is stable. In this case geodesics approach each other. If \( V \) is undefined or negatively defined then geodesics diverge i.e. motion is unstable [4] and we have sensitive dependence on initial conditions.

Above analysis has purely local character and does not take into account the influence of the boundary \( \partial D_E \) on the geodesics motion. It should be stressed that it does not determine existence of global chaos in the system. Anyhow, when the admissible region \( D_E \) has no boundary then with the use of (22) new criterion can be formulated which determines the behaviour of the system completely [3].

### 2.6 GDE in terms of an external curvature

The potential \( V \) has a nice geometric interpretation. At a given point of the manifold, for any non-collinear vector fields \( A \) and \( B \) one can define the
sectional curvature $\hat{K}_{AB}$ in the two-direction determined by these fields in the following way $\hat{R}(A, B, A, B) = \hat{K}_{AB}[\hat{g}(A, A)\hat{g}(B, B) - \hat{g}(A, B)\hat{g}(A, B)]$. Therefore, reminding that $\hat{g}(n, u) = 0$ and $\hat{g}(u, u) = 1$ we have

$$V_u(n) = \frac{1}{2} \hat{R}(u, n, u, n) = \frac{1}{2} \hat{K}_{un}\hat{g}(n, n).$$

(23)

Hence, the potential is proportional to the sectional curvature $\hat{K}_{un}$ and square of the deviation field.

Let us confine to the two dimensional case where $(E_1, E_2)$ is Fermi frame. If we chose $E_2 = u$ and $n = xE_1$ then $V_u(n) = \frac{1}{2} \hat{R}(u, n, u, n) = \frac{1}{2} \hat{R}_{1212}x^2$. In two dimensions the sectional curvature has only one independent component which in Fermi frame is equal to the Gauss curvature $\hat{K} = \hat{R}_{1212}$. The deviation equation takes now the simple form

$$\frac{d^2 x}{ds^2} = -\hat{K}(s)x.$$

(24)

It is transparent that when the Gauss curvature is positive then geodesics approaches each other. In opposite case ($\hat{K} < 0$) they diverge.

3 Solutions of the geodesics deviation equation

3.1 Adiabatic approximation

Geodesics deviation equation in two dimensions can be rewritten as a system of first order equations

$$\dot{X} = \mathcal{A}(s)X$$

(25)

where $\cdot' = \frac{d}{ds}$. In the case of two dimensional system the vector of variables has two components $X = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$, and the matrix $\mathcal{A}$ has the form

$$\mathcal{A} = \begin{bmatrix} 0 & 1 \\ -\hat{K}(s) & 0 \end{bmatrix}$$

i) At the beginning let us consider positive curvature case $\hat{K}(s) > 0$. 
Having non-singular matrix \( \mathcal{P}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -\sqrt{\hat{K}(s)} \end{bmatrix} \) we can transform vector \( X, X(s) = \mathcal{P}_1(s)Y(s) \) and the whole equation (25) to the form
\[
\dot{Y} = [\mathcal{P}_1^{-1}\mathcal{A}\mathcal{P}_1 - \mathcal{P}_1^{-1}\dot{\mathcal{P}}_1]Y. \tag{26}
\]
During this transformation only locality of the matrix \( \mathcal{P}_1 \) needs some care. Explicit manipulations on matrices \( \mathcal{A} \) and \( \mathcal{P}_1 \) lead to the equation
\[
\dot{Y} = D_1 \left( I - \begin{bmatrix} 0 & \frac{\dot{\hat{K}}}{2\hat{K}^\frac{3}{2}} \\ 0 & 0 \end{bmatrix} \right) Y, \tag{27}
\]
where \( I \) is identity matrix and
\[D_1 = \mathcal{P}_1^{-1}\mathcal{A}\mathcal{P}_1 = \sqrt{\hat{K}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.\]

We see from (27) and the form of \( D_1 \) that \( \sqrt{\hat{K}} \) has the same dimension as ”s” derivative of \( \hat{K} \). This observation allows us to use a ratio \( \frac{\dot{\hat{K}}}{\hat{K}^\frac{3}{2}} \) as a dimensionless perturbation parameter.

We assume that in the neighbourhood of the considered point of the admissible region Gauss curvature changes very slowly in comparison to its value i.e. \( |\dot{\hat{K}}| \ll \hat{K}^\frac{3}{2} \). Next we keep only zero order term in equation (27)
\[
\dot{Y} = D_1(s)Y. \tag{28}
\]
We call this simplification adiabatic approximation. Note that \( D_1 \) is still curvature \( \hat{K} \) dependent. To make further progress we denote components of the vector \( Y \) in the following way \( Y = \begin{bmatrix} y \\ \tilde{y} \end{bmatrix} \), then equation (28) appears to be a system of coupled first order equations
\[
\dot{y}(s) = -\sqrt{\hat{K}(s)}\tilde{y}(s), \quad \dot{\tilde{y}}(s) = \sqrt{\hat{K}(s)}y(s). \tag{29}
\]
Introducing a complex variable \( z(s) = y(s) + i\tilde{y}(s) \) we rearrange the system (29)
\[
\dot{z}(s) = i\sqrt{\hat{K}}z(s), \tag{30}
\]
so as to easy find solution
\[
z(s) = z(0)exp \left[ i\int_0^s ds' \sqrt{\hat{K}(s')} \right]. \tag{31}
\]
Coming back to the real vector $Y$ we have

$$Y(s) = \begin{bmatrix} \cos \left( \int_0^s ds' \sqrt{K(s')} \right) & - \sin \left( \int_0^s ds' \sqrt{K(s')} \right) \\ \sin \left( \int_0^s ds' \sqrt{K(s')} \right) & \cos \left( \int_0^s ds' \sqrt{K(s')} \right) \end{bmatrix} Y(0). \tag{32}$$

Transforming (32) by the transformation $X(s) = P_1(s)Y(s)$ and then initial values of $Y(0)$ by the transformation $Y(0) = P_1^{-1}(0)X(0)$ we can express the vector $X(s)$ by its initial values

$$X(s) = \begin{bmatrix} \cos \left( \int_0^s ds' \sqrt{K(s')} \right) & \frac{1}{\sqrt{K(0)}} \sin \left( \int_0^s ds' \sqrt{K(s')} \right) \\ -\sqrt{K(s)} \sin \left( \int_0^s ds' \sqrt{K(s')} \right) & \frac{\sqrt{K(s)}}{\sqrt{K(0)}} \cos \left( \int_0^s ds' \sqrt{K(s')} \right) \end{bmatrix} X(0). \tag{33}$$

Therefore adiabatic problem in $\hat{K} > 0$ case has the solution

$$x(s) = x(0) \cos \left( \int_0^s ds' \sqrt{K(s')} \right) + \frac{\dot{x}(0)}{\sqrt{K(0)}} \sin \left( \int_0^s ds' \sqrt{K(s')} \right). \tag{34}$$

We see that equation (34) describes stable relative oscillations of geodesics. The frequency of these oscillations changes with "time" - $s$.

ii) solution of the negative curvature case $\hat{K} < 0$

Construction of the solution for negative Gauss curvature is analogous to the proceeded construction of the positive curvature solution. The only difference lies in the form of the transformation matrix $P_2 = \begin{bmatrix} 1 & 1 \\ -\sqrt{|\hat{K}|} & \sqrt{|\hat{K}|} \end{bmatrix}$.

After this transformation the deviation equation takes similar to the equation (26) form

$$\dot{Y} = [P_2^{-1} A P_2 - P_2^{-1} \dot{P}_2] Y, \tag{35}$$

where $A = \begin{bmatrix} 0 & 1 \\ |\hat{K}(s)| & 0 \end{bmatrix}$. Detailed structure of equation (35) differs significantly from equation (27)

$$\dot{Y} = D_2(s) \left( I - \frac{|\dot{\hat{K}}|}{4 |\hat{K}|^{\frac{3}{2}}} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right) Y, \tag{36}$$
where $D_2 = \mathcal{P}_2^{-1} \mathcal{A} \mathcal{P}_2 = \sqrt{|\hat{K}|} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

For slowly varying curvature i.e. $|\ddot{K}| \ll |\dot{K}|^2$, we can approximate equation (36) by
\begin{equation}
\dot{Y} = D_2(s) Y. \quad (37)
\end{equation}

Fortunately equation (37) describes a system of decoupled first order equations
\begin{equation}
\dot{y}(s) = -\sqrt{|\hat{K}(s)|} y(s), \quad \dot{\tilde{y}}(s) = \sqrt{|\hat{K}(s)|} \tilde{y}(s), \quad (38)
\end{equation}

which has a simple solution
\begin{equation}
y(s) = y(0) \exp \left[ -\int_0^s ds' \sqrt{|\hat{K}(s')|} \right], \quad \tilde{y}(s) = \tilde{y}(0) \exp \left[ \int_0^s ds' \sqrt{|\hat{K}(s')|} \right].
\end{equation}

Coming back to the $X$ vector we have the final answer
\begin{equation}
X(s) = \begin{bmatrix}
cosh \left( \int_0^s ds' \sqrt{|\hat{K}(s')|} \right) & \sqrt{|\hat{K}(0)|} \sinh \left( \int_0^s ds' \sqrt{|\hat{K}(s')|} \right) \\
\sqrt{|\hat{K}(s)|} \sinh \left( \int_0^s ds' \sqrt{|\hat{K}(s')|} \right) & \sqrt{|\hat{K}(s)|} \cosh \left( \int_0^s ds' \sqrt{|\hat{K}(s')|} \right)
\end{bmatrix} X(0). \quad (40)
\end{equation}

First component of this vector
\begin{equation}
x(s) = x(0) \cosh \left( \int_0^s ds' \sqrt{|\hat{K}(s')|} \right) + \frac{\dot{x}(0)}{\sqrt{|\hat{K}(0)|}} \sinh \left( \int_0^s ds' \sqrt{|\hat{K}(s')|} \right) \quad (41)
\end{equation}
describes exponential divergence of the geodesics with "time"-$s$.

### 3.2 Exact solutions of the GDE

i) Let us come back to the geodesics deviation equation (27) in the $\hat{K} > 0$ regime
\begin{equation}
\hat{Y}(s) = [D_1(s) + \varepsilon_1(s) B_1(s)] Y(s), \quad (42)
\end{equation}

where $B_1(s) = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{2} \sqrt{|\hat{K}(s)|} \end{bmatrix}$ and by $\varepsilon_1(s) = \frac{\dot{K}(s)}{K^2}$ we denote the perturbation parameter.

To define perturbation procedure we introduce a kind of interaction picture. Having exact solutions of the adiabatic problem $Y(s)$ we define vector
$U(s)$ which carry part of the evolution of the system which goes beyond the adiabatic approximation solution

$$Y(s) = \begin{bmatrix} \cos \left( \int_0^s ds' \sqrt{|K(s')|} \right) & -\sin \left( \int_0^s ds' \sqrt{|K(s')|} \right) \\ \sin \left( \int_0^s ds' \sqrt{|K(s')|} \right) & \cos \left( \int_0^s ds' \sqrt{|K(s')|} \right) \end{bmatrix} U(s). \quad (43)$$

With the use of the new vector field the equation (42) can be rewritten in the form

$$\dot{U}(s) = \varepsilon_1(s) C_1(s) U(s), \quad U(0) = Y(0), \quad (44)$$

where

$$C_1(s) = -\frac{1}{2} \sqrt{|K|} \begin{bmatrix} \sin^2 \left( \int_0^s ds' \sqrt{|K(s')|} \right) & \frac{1}{2} \sin \left( 2 \int_0^s ds' \sqrt{|K(s')|} \right) \\ \frac{1}{2} \sin \left( 2 \int_0^s ds' \sqrt{|K(s')|} \right) & \cos^2 \left( \int_0^s ds' \sqrt{|K(s')|} \right) \end{bmatrix}. $$

The formal solution of the problem

$$U(s) = \left[ I + \sum_{k=1}^{\infty} \int_0^s ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{k-1}} ds_k \varepsilon_1(s_1) \varepsilon_1(s_2) \ldots \varepsilon_1(s_k) C_1(s_1) C_1(s_2) \ldots C_1(s_k) \right] Y(0) \quad (45)$$

could be treated as a definition of perturbation theory. If we cut the series on arbitrary $k = N$ then we obtain approximate solution of the problem (44). The precision of cutted solution (45) is only limited by the number of considered terms $N$.

ii) In the negative Gauss curvature regime geodesics deviation equation has the form

$$\dot{Y}(s) = [D_2(s) + \varepsilon_2(s) B_2(s)] Y(s), \quad (46)$$

where $B_2(s) = \frac{1}{7} \sqrt{|K|} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ and perturbation parameter $\varepsilon_2(s) = \frac{|\dot{K}(s)|^2}{|K|^2}$. Now we transfer a part of the time dependence of the $Y(s)$ on vector $U(s)$ which carries non-adiabatic part of the evolution

$$Y(s) = \begin{bmatrix} \exp \left( -\int_0^s ds' \sqrt{|K(s')|} \right) & 0 \\ 0 & \exp \left( \int_0^s ds' \sqrt{|K(s')|} \right) \end{bmatrix} U(s). \quad (47)$$
Hence \( U(s) \) satisfy

\[
\dot{U}(s) = \varepsilon_2(s)C_2(s)U(s), \quad U(0) = Y(0),
\]

where 

\[
C_2(s) = \frac{1}{4\sqrt{|K|}} \begin{bmatrix}
-1 & \exp \left( 2\int_0^s ds' \sqrt{|K(s')|} \right) \\
\exp \left( -2\int_0^s ds' \sqrt{|K(s')|} \right) & -1
\end{bmatrix} U(s).
\]

The solution in this case

\[
U(s) = \left[ I + \sum_{k=1}^{\infty} \int_0^s ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{k-1}} ds_k \varepsilon_2(s_1)\varepsilon_2(s_2)\cdots\varepsilon_2(s_k)C_2(s_1)C_2(s_2)\cdots C_2(s_k) \right] Y(0),
\]

could also be used to construct the perturbation expansion.

4 Stability in examples.

In further discussion we will use formula which express Gauss curvature through total energy of the system, potential and its derivative \( s \). Let us find explicit relation between the Riemann curvature tensor \( \hat{R}_{ijkl} \) for Jacobi metric and curvature tensor \( R_{ijkl} \) for metric \( g_{ij} \)

\[
\hat{R}_{ijkl} = R_{ijkl} + \frac{1}{N-2} [\delta^i_k C_{jk} - \delta^i_k C_{jl} + g_{kj} C^i_l - g_{jl} C^i_k],
\]

where auxiliary tensor \( C_{ij} \) is built of total energy and potential in the following way

\[
C_{ij} = -\frac{(N-2)}{4(E-V)^2} \left[ 2(E-V)\nabla_i \nabla_j V + 3(\nabla_i V)(\nabla_j V) - \frac{1}{2} g_{ij} N V^2 \right],
\]

\( N \) is number of dimensions and \( N V^2 \) is square of the gradient of the potential \( N V^2 = g^{ij}(\partial_i V)(\partial_j V) \) which is positive quantity.

Equation (50) allows to express the curvature scalar for Jacobi metric through the curvature scalar for metric \( g_{ij} \)

\[
\hat{R} = \hat{g}^{ij} \hat{R}_{ij} = \hat{g}^{ij} \hat{R}_{ikj} = \frac{1}{2(E-V)} R + \frac{(N-1)}{8(E-V)^3} \left[ 4(E-V)\Delta V + (6-N)N V^2 \right],
\]

15
where $R = g^{ij}R_{ij} = g^{ij}R^{k}_{ikj}$ and $\Delta V$ denotes Laplacian $\Delta V = g^{ij}\nabla_i\nabla_j V$.

Large number of mechanical systems is determined by metric $g_{ij}$ which is just flat metric transformed to curvilinear coordinates. If we confine our interest to this class of systems then first term in equation (51) disappears ($R = 0$). Let us consider two-dimensional case ($N = 2$) where exists extremely simple relation between scalar curvature and Gauss curvature of a given surface $\hat{K} = \frac{1}{2}\hat{R}$. Equation (51) in two dimensions leads to simple form of Gauss curvature

$$\hat{K} = \frac{1}{4(E-V)^3} \left[(E-V)\Delta V + NV^2\right]. \quad (52)$$

In considered region of configuration space $Int D_E$ we have positive $(E-V) > 0$ and the only term which can affect the overall sign of the Gauss curvature is the Laplacian of the potential $\Delta V$.

Now we are ready to test criterion (52) on two-dimensional systems.

**Example 1**

Let us consider lagrangian which describes small oscillations

$$\mathcal{L} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - (\alpha x^2 + 2\beta xy + \gamma y^2).$$

Potential in this case is quadratic form of coordinates. We can check that if $V$ is positively defined (i.e. $\alpha > 0$ and $\alpha \gamma > \beta^2$) then Laplacian of the potential is positive $\Delta V = 2(\alpha + \beta) > 0$ and therefore $\hat{K} > 0$. According to our criterion (52) trajectories do not diverge.

**Example 2**

Another example is two-body problem which is defined by the lagrangian

$$\mathcal{L} = \frac{1}{2}\mu (\dot{r}^2 + r^2 \dot{\phi}^2) - V(r),$$

where $\mu = \frac{m_1m_2}{m_1 + m_2}$ is reduced mass of the system consisting of two masses $m_1$, $m_2$ separated by the distance $r$. This problem is integrable and angular momentum is its integral of motion $L = \mu r^2 \dot{\phi}$.

Jacobi metric for this system is equal $\hat{g}_{ij} = 2(E-V(r))g_{ij}$, where $[g_{ij}] = \mu \left[ \begin{array}{cc} 1 & 0 \\ 0 & r^2 \end{array} \right]$. Gauss curvature of the geometry fixed by Jacobi metric is

$$\hat{K} = \frac{1}{4\mu(E-V)^3} \left[(E-V)\left(V''(r) + \frac{1}{r}V'(r)\right) + (V'(r))^2\right].$$
For particular form of the potential \( V(r) = -\frac{\alpha}{r}, \ (\alpha > 0) \) Gauss curvature is even more simple

\[
\hat{K}(r) = -\frac{\alpha E}{4\mu(\alpha + Er)^3}.
\]

In the Kepler problem we have two cases.

i) We have elliptic motion in the regime \( E < 0 \). In this case \( \hat{K} > 0 \) and trajectories do not diverge.

ii) If total energy is positive \( E > 0 \) then \( \hat{K} < 0 \) and we have hyperbolic motion. Although trajectories diverge the system is integrable.

The exponential instability (i.e. sensitivity to the initial conditions) is one of the important features of the chaotic evolution of dynamical systems and therefore determining of the simple and possibly elegant criterion decisive when this instability will take place, is so important.

Our formula (22) and the comments to it, show the elegant, geometric formulation of the criterion of the local stability of relative movement of geodesics. The chaotic behaviour of dynamical systems is very complicated phenomenon and the exponential instability (the negative sign of Gauss curvature \( \hat{K} \)) is not mostly the sufficient condition in order to the corresponding systems will behave chaotically (as the example of Kepler problem shows).

It is known from Anosov paper [5] that on the compact manifold without boundary the geodesics family corresponding to the geometry with everywhere negative Gauss curvature \( \hat{K} \) behaves chaotically. Unfortunately, space \( D_E \) admissible for movement, usually has non-empty boundary \( \partial D_E \), on which the corresponding geometry, determined by Jacobi metric \( \hat{g}_{ij} \) becomes singular and which has important influence on the geometry of the family of geodesics. Analysis of this influence requires application of more advanced global methods.
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