LEARNING QUANTUM ENTANGLEMENT DISTILLATION WITH NOISY CLASSICAL COMMUNICATIONS

Hari Hara Suthan Chittoor  Osvaldo Simeone

KCLIP Lab, Department of Engineering, King’s College London, UK

ABSTRACT

An important primitive for quantum networking is entanglement distillation, whose goal is to enhance the fidelity of entangled qubits through local operations and classical communication (LOCC). Existing distillation protocols assume the availability of ideal, noiseless, communication channels. In this paper, we study the case in which communication takes place over noisy binary symmetric channels. We propose to implement local processing through parameterized quantum circuits (PQCs) that are optimized to maximize the average fidelity, while accounting for communication errors. The introduced approach, Noise Aware-LOCCNet (NA-LOCCNet), is shown to have significant advantages over existing protocols designed for noiseless communications.

Index Terms— Quantum machine learning, entanglement distillation, parameterized quantum circuits

1. INTRODUCTION

Quantum networking, and with it the quantum Internet, rely on the management and exploitation of entanglement [1, 2, 3]. In fact, entangled qubits enable fundamental quantum communication primitives such as teleportation and superdense coding [4, 5]. Practical sources of entangled qubits, such as single-photon detection [6, 7], are imperfect, producing mixed states with reduced fidelity as compared to ideal, fully entangled, Bell pairs. In order to enhance the fidelity of entangled qubits available at distributed parties, entanglement distillation protocols leverage local operations and classical communication (LOCC). While existing solutions assume ideal classical communications, this paper studies the case in which communications between the parties holding imperfectly entangled qubits are noisy. As illustrated in Fig. 1, to address this more challenging scenario, we propose the use of quantum machine learning (QML) via parameterized quantum circuits (PQCs) [8, 9].

In entanglement distillation protocols, a source produces a number of imperfectly entangled qubit pairs. Each qubit of a pair is made available at one of two parties, conventionally referred to as Alice and Bob. The goal is to leverage LOCC to produce qubit pairs that have a higher degree of fidelity with respect to a fully entangled Bell pair. As seen in Fig. 2, Alice and Bob start with two qubit pairs, and output either one qubit pair or a declaration of failure at the end of the process.

Traditionally, entanglement distillation protocols have been designed by hand, targeting specific mixed states as the input of the protocol [10, 2, 11]. Specific examples include the DEJmps protocol, which targets the so-called S-state [11]. These methods rely on local operations via unitaries; on the measurement of one qubit at Alice and Bob; and on classical communication of the measurement outputs on a noiseless channel. Based on the measurement outputs, Alice and Bob decide whether to keep the unmeasured pair of qubits or to declare a distillation failure.

Recently, a QML framework was introduced in [12] for the design of LOCC protocols. The approach, termed LOCC-Net, prescribes the use of PQCs for the local unitaries applied by Alice and Bob. PQCs have been widely investigated in recent years as means to program small-scale noisy quantum computers via classical optimization, with applications ranging from combinatorial optimization to generative modelling [8]. A PQC typically consists of a sequence of one- and two-qubit rotations, whose parameters can be optimized, as well as of fixed entangling gates.

The design of LOCCNet in [12] assumes ideal, noiseless, classical communications. In contrast, in this paper, we study the case in which communication takes place over noisy binary symmetric channels as seen in Fig. 1. We propose to optimize a specific PQC architecture (see Fig. 2) with the
goal of maximizing the average fidelity when accounting for the randomness caused by communication errors. The introduced approach, Noise-Aware-LOCCNet (NA-LOCCNet), is shown to have significant advantages over existing protocols designed for noiseless communications.

Notations: The Kronecker product is denoted by $\otimes$; $I_d$ represents the $d \times d$ identity matrix; $M^\dagger$ represents the complex conjugate transpose of the matrix $M$; $\text{tr}(M)$ represents trace of the matrix $M$. We adopt standard notations for quantum states, computational basis, and quantum operations [4].

2. PROBLEM FORMULATION

In this section, we formulate the problem of entanglement distillation in the presence of a noisy classical communication channel, and we describe the performance metrics of interest.

2.1. Setting

As illustrated in Fig. 1, we consider a system consisting of two main parties – Alice and Bob – aided by a third party – Charlie. Alice and Bob have local quantum processing capability, while Charlie is not equipped with quantum computing devices. Alice and Bob can communicate to Charlie over a noisy classical channel. Alice and Bob through local operations (LO) at Alice and Bob, as well as through classical communication (CC) to Charlie.

The quantum entanglement generator produces $k$ pairs of noisy ebits. The state of each qubit pair is described by a $4 \times 4$ density matrix $\rho_{AB}$. Throughout the paper, we use subscript $A$ to denote the qubits available at Alice, while the subscript $B$ is used for the qubits at Bob. As in [12], we specifically focus on the noisy, i.e., mixed, ebit state described by the density matrix

$$\rho_{AB} = F|\phi^+\rangle\langle\phi^+| + (1 - F)|00\rangle\langle00|,$$

where $F \in [0, 1]$ represents the input fidelity and

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

is a maximally entangled Bell state. The noisy ebit state in (1) is also known as $S$-state [12], and it describes a situation in which the two qubits are in the maximally entangled state, $|\phi^+\rangle$, with probability $F$, and in the separable, i.e., non-entangled, state $|00\rangle$ with probability $1 - F$. This type of noisy state arise in the protocols for entanglement generation that use single-photon detection in the presence of photon loss [13, 14, 7]. Furthermore, the S state is known to be more challenging to “denoise” than other mixed states in which the separable state, occurring with probability $1 - F$, is orthogonal to $|\phi^+\rangle$ [13].

As in [12], we focus on the standard case in which $k = 2$ identical pairs of S-states $\rho_{A_0B_0}$ and $\rho_{A_1B_1}$ are generated. The goal is to distill the two noisy ebits pairs to obtain a single pair of less noisy ebits. Following standard terminology [2], the qubits $A_0$ and $B_0$ are referred to as the preserved pair, and the qubits $A_1$ and $B_1$ as the sacrificial pair. As shown in Fig. 1, Alice and Bob process the respective qubits – $A_1$ and $A_0$ for Alice, and $B_1$ and $B_0$ for Bob – via local quantum operations defined by unitaries $U_A(\theta)$ and $U_B(\theta)$ respectively.

As detailed in the next sections, the operation of the units generally depend on a vector $\theta$ of classical parameters. Then, the qubits $A_1$ and $B_1$ are measured in the computational basis at Alice and Bob respectively, and the measurement outcomes (0 or 1) are communicated to Charlie using noisy classical channels. We specifically assume that communication to Charlie occurs over independent binary symmetric channels with bit flip probability $p$.

If Charlie receives message 0 from both Alice and Bob, it declares that the distillation is successful, and Alice and Bob retain the pair of qubits $A_0$ and $B_0$. Instead, if Charlie receives the pairs of messages $(0, 1)$, $(1, 0)$ or $(1, 1)$ from Alice and Bob, it declares a failure. In this case, Alice and Bob discard the qubits $A_0$ and $B_0$.

We remark that most conventional entanglement distillation protocols [10, 11] use decision rules in which either pair of messages $(0, 0)$ or $(1, 1)$ is considered as success. Here we follow the approach in [12] of treating $(0, 0)$ as the only case in which Charlie declares success. This design choice facilitates the optimization of the unitaries $U_A(\theta)$ and $U_B(\theta)$ through vector $\theta$.

The goal of this work is to design the unitaries $U_A(\theta)$ and $U_B(\theta)$ at Alice and Bob such that the output state of qubits $A_0$ and $B_0$, upon successful distillation, is as close as possible in terms of fidelity to the ideal ebit state $|\phi^+\rangle$.

2.2. Performance Metrics

The performance of entanglement distillation is measured in this paper, as in [12, 15], in terms of fidelity and probability of success. The fidelity of a state $\rho_{AB}$ with respect to the ebit state $|\phi^+\rangle$ is defined as

$$F(\rho_{AB}) = \langle \phi^+ | \rho_{AB} | \phi^+ \rangle,$$

while probability of success is the probability of receiving the pair of messages $(0, 0)$ at Charlie.

Let $U(\theta)$ be the $16 \times 16$ unitary operation corresponding to the separate application of the $4 \times 4$ local unitaries $U_A(\theta)$ and $U_B(\theta)$ to their respective qubit pairs $(A_0, A_1)$ and $(B_0, B_1)$, respectively. We order the qubits as $(A_0, B_0, A_1, B_1)$ to facilitate the derivations below. The state of the four qubits after the local operations is given by the density matrix

$$\rho_{\text{out}}(\theta) = U(\theta)(\rho_{A_0B_0} \otimes \rho_{A_1B_1})U(\theta)^\dagger,$$

where we have made explicit dependence on the model parameter vector $\theta$. 

Authorized licensed use limited to the terms of the applicable license agreement with IEEE. Restrictions apply.
Accordingly, the measurement returns output \((\Pi^y)_{xy}\rangle\langle xy|\), with \((x, y) \in \{0, 1\}^2\), where \(I_4\) is the 4 \times 4 identity matrix. Accordingly, the measurement returns output \((x, y) \in \{0, 1\}^2\) with probability \(P^{xy}(\theta) = \text{tr}(\Pi^y \rho_{out}(\theta)),\) and the corresponding post-measurement state for the qubits \((A_0, B_0)\) is
\[
\rho_{A_0B_0}^{xy}(\theta) = \frac{(I_4 \otimes |xy\rangle\langle xy|)(I_4 \otimes |xy\rangle\langle xy|)}{P^{xy}(\theta)}.
\]
Conditioned on the measurement outcome being \((x, y) \in \{0, 1\}^2\), the fidelity (3) of the state \(\rho_{A_0B_0}^{xy}(\theta)\) with respect to the ebit state \(|\phi^+\rangle\) is hence
\[
F^{xy}(\theta) = \langle \phi^+ | \rho_{A_0B_0}^{xy}(\theta) | \phi^+ \rangle.
\]

3. EXISTING DISTILLATION PROTOCOLS

In this section, we review current state-of-the-art distillation protocols. We focus on the DEJmps protocol [11] and on the LOCCN protocol [12] as applied to \(k\) copies of the S-state (1). We emphasize that all the existing distillation protocols are designed for noiseless classical communication channels to Charlie, i.e., assuming \(p = 0\).

3.1. DEJMPs Protocol

In the DEJMPs protocol, the local unitaries \(U_A(\theta)\) and \(U_B(\theta)\) applied by Alice and Bob do not have free parameters, and are hence denoted as \(U_A\) and \(U_B\), dropping the dependence on the model parameter vector \(\theta\). Specifically, the unitary \(U_A\) at Alice is given by Pauli X-rotation \(R_X(\pi/2)\) applied on both qubits, followed by a controlled NOT (CNOT) gate with the qubit \(A_0\) as control and the qubit \(A_1\) as target. Similarly, the unitary \(U_B\) at Bob is defined by the cascade of Pauli X-rotations \(R_X(\pi/2)\) on the two qubits and of a CNOT gate with the qubit \(B_0\) as control and the qubit \(B_1\) as target. If Charlie receives messages \((0, 0)\) or \((1, 1)\) from Alice and Bob, it declares that distillation is successful, and the qubit pair \((A_0, B_0)\) is retained.

3.2. LOCCN

In [12], a QML-based entanglement distillation protocol, known as LOCCN, is introduced that uses parameterized quantum circuits (PQCs) for unitaries \(U_A(\theta)\) and \(U_B(\theta)\) at Alice and Bob. The PQC \(U_A(\theta)\) consists of a CNOT gate, with the qubit \(A_1\) as control and the qubit \(A_0\) as target, followed by a Pauli Y-rotation \(R_Y(\theta)\); while the PQC \(U_B(\theta)\) is given by two CNOT gates followed by \(R_Y(\theta)\), where the first CNOT gate has the qubit \(B_1\) as control and the qubit \(B_0\) as target and the second CNOT gate has the qubit \(B_0\) as control and the qubit \(B_1\) as target. The rotation angle \(\theta\) of the Pauli Y-rotation is subject to optimization. If Charlie receives messages \((0, 0)\) from Alice and Bob through noiseless channels, i.e., \(p = 0\), a success is declared and the pair \((A_0, B_0)\) of qubits is retained. Model parameter vector \(\theta\) is optimized with the goal of maximizing the fidelity \(F^{00}(\theta)\) in (8).

4. NOISE AWARE-LOCCNET

In this section, we propose Noise Aware-LOCCNet (NA-LOCCNet), which distills two qubit pairs, each in the S-state (1), in the presence of noisy classical channels from Alice and Bob to Charlie as shown in Fig. 1. The key innovation as compared to LOCCNet is that we explicitly target the performance in terms of average fidelity by accounting for the impact of channel errors. We first describe the design objective, and then introduce the assumed structure for the PQCs \(U_A(\theta)\) and \(U_B(\theta)\).

![Fig. 2: Proposed Noise Aware-LOCCNet (NA-LOCCNet) circuit for distilling two S states.](image-url)

4.1. Design Objective

NA-LOCCNet aims at maximizing the average conditional fidelity of a retained pair \((A_0, B_0)\) in case of success. As explained in Section 2.1, Charlie declares a success if it receives the pair of messages \((0, 0)\) from Alice and Bob through the respective binary symmetric channels with bit flip probability \(p\). LOCCNet assumes a noiseless channel \((p = 0)\), and hence it targets the objective \(F^{00}(\theta)\), that is, the fidelity conditioned on measurement \((0, 0)\) being produced by Alice and Bob. In contrast, NA-LOCCNet accounts for the fact that, where Charlie declares a success as it receives messages \((0, 0)\), the actual measurement outcomes may be different due to channel errors.

In fact, messages \((0, 0)\) are received at Charlie with probability \(P^{00} = (1 - p)^2\) if the measurement outcomes are \((x, y) = (0, 0)\); with probability \(P^{01} = (1 - p)p\) if the measurement outcomes are \((x, y) = (0, 1)\); with probability \(P^{10} = p(1 - p)\) if the measurement outcomes are \((x, y) = (1, 0)\); and with probability \(P^{11} = p^2\) if the measurement outcomes are \((x, y) = (1, 1)\). Hence, the average fidelity conditioned on the reception of messages \((0, 0)\) is computed as
\[
\bar{F}(\theta) = \sum_{x,y} P^{xy} F^{xy}(\theta) P_{\text{succ}}(\theta),
\]
where
\[
P_{\text{succ}}(\theta) = \sum_{x,y} P^{xy} F^{xy}(\theta).
\]
is the probability of success, i.e., of receiving messages \((0,0)\), and we have used definitions (6) and (8). The proposed protocol NA-LOCCNet addresses the problem

\[
\max_{\theta} F(\theta). \tag{11}
\]

4.2. Architecture of the PQCs

For the PQCs \(U_A(\theta)\) and \(U_B(\theta)\) at Alice and Bob, respectively, we adopt the architecture shown in Fig. 2. Unlike the LOCCNet architecture [12], we introduce a parameterized two-qubit gate, namely the Pauli \(ZY\)-rotation [16]. This is defined by the unitary

\[
R_{ZY}(\theta) = \exp \left( -\frac{i}{2} (Z \otimes Y) \theta \right), \tag{12}
\]

which is parameterized by angle \(\theta\). Two-qubit rotation gates [16] were shown to provide performance advantages as gates in PQCs for various QML applications. In our work, the choice of the parameterized two-qubit gate (12) was dictated by extensive experiments with alternative architectures. As an example, in Section 5, we will compare the performance obtained by the architecture in Fig. 2 with the original LOCCNet system in [12], when addressing problem (11).

4.3. Optimization

Addressing problem (11) with PQCs characterized by a single scalar parameter \(\theta\), as for the LOCCNet architecture [12] and the architecture in Fig. 2, requires a one-dimensional search over the limited domain \([0, 2\pi]\). This can be carried out using standard optimization techniques like gradient descent.

5. EXPERIMENTS AND CONCLUSIONS

In this section, we evaluate the performance of NA-LOCCNet in the presence of noisy communication channels from Alice and Bob to Charlie\(^1\). We consider the benchmark schemes DEJMPS (Section 3.1) and LOCCNet (Section 3.2). For the latter, we consider two designs: the original optimization in [12] of the fidelity \(F^{\text{LOCC}}(\theta)\) in (8) and the optimization of the conditional average fidelity \(F(\theta)\) in (11).

Fig. 3 plots the average output fidelity, conditioned on a successful distillation, as a function of the bit flip probability \(p\) of the noisy classical channels by fixing the input fidelity of the S-state (1) to \(F = 0.6\); while Fig. 4 plots the same quantity as a function of the input fidelity \(F\) by fixing the bit flip probability to \(p = 0.25\). Note that the conditional average fidelity is given by (9) for LOCCnet and NA-LOCCNet, while for DEJMPS one needs to consider both received messages \((0,0)\) and \((1,1)\) as indicating successful distillation.

Fig. 3 shows that, as the bit flip probability \(p\) increases, the average fidelity of both DEJMPS and LOCCNet decreases significantly, reaching the minimum fidelity of 0.5 when the channels are maximally noisy, i.e., with \(p = 0.5\). Note that this fidelity level is smaller than the input fidelity \(F = 0.6\). Interestingly, the performance of the LOCCNet architecture [12] does not improve noticeably when optimized via the channel-aware criterion (11), as opposed to the noise-agnostic fidelity criterion considered in [12]. In contrast, the proposed NA-LOCCNet with PQC architecture in Fig. 2 exhibits a significantly milder decrease in fidelity as \(p\) grows, yielding the average output fidelity level of \(F = 0.8\) for \(p = 0.5\).

The advantages of NA-LOCCNet are further validated by Fig. 4, which shows gains at all values of the input fidelity \(F\). In particular, unlike the other schemes, NA-LOCCNet never yields an output fidelity lower than the input fidelity \(F\). Future work may involve the integration of the proposed scheme into a network protocol for entanglement distillation [17].

---

\(^1\)We simulated all the experiments on a laptop with i7 processor and 16 GB RAM. The PyTorch code for regenerating the results of this paper is available at <https://github.com/kclip/Noise-Aware-LOCCNet>.
6. REFERENCES

[1] P.P. Rohde, *The Quantum Internet: The Second Quantum Revolution*, Cambridge University Press, 2021.

[2] Rodney Van Meter, *Quantum Networking*, Wiley-IEEE Press, 2014.

[3] Angela Sara Cacciapuoti, Marcello Caleffi, Francesco Tafuri, Francesco Saverio Cataliotti, Stefano Gherardini, and Giuseppe Bianchi, “Quantum internet: Networking challenges in distributed quantum computing,” *IEEE Network*, vol. 34, pp. 137–143, 2020.

[4] Michael A. Nielsen and Isaac L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, 2010.

[5] Mark M. Wilde, *Quantum Information Theory*, Cambridge University Press, 2013.

[6] Matthew D Eisaman, Jingyun Fan, Alan Migdall, and Sergey V Polyakov, “Invited review article: Single-photon sources and detectors,” *Review of scientific instruments*, vol. 82, no. 7, pp. 071101, 2011.

[7] Earl T. Campbell and Simon C. Benjamin, “Measurement-based entanglement under conditions of extreme photon loss,” *Phys. Rev. Lett.*, vol. 101, pp. 130502, Sep 2008.

[8] Maria Schuld and Francesco Petruccione, *Machine Learning with Quantum Computers*, Springer Cham, 2021.

[9] Osvaldo Simeone, *An Introduction to Quantum Machine Learning for Engineers*, https://osimeone.wordpress.com/, 2022.

[10] Charles H. Bennett, Gilles Brassard, Sandu Popescu, Benjamin Schumacher, John A. Smolin, and William K. Wootters, “Purification of noisy entanglement and faithful teleportation via noisy channels,” *Phys. Rev. Lett.*, vol. 76, pp. 722–725, Jan 1996.

[11] David Deutsch, Artur Ekert, Richard Jozsa, Chiara Macchiavello, Sandu Popescu, and Anna Sanpera, “Quantum privacy amplification and the security of quantum cryptography over noisy channels,” *Phys. Rev. Lett.*, vol. 77, pp. 2818–2821, Sep 1996.

[12] Xuanqiang Zhao, Benchi Zhao, Zihe Wang, Zhixin Song, and Xin Wang, “Practical distributed quantum information processing with locnet,” *Quantum Information*, vol. 7, no. 1, pp. 1–7, 2021.

[13] Filip Rozpedek, Thomas Schiet, Le Phuc Thinh, David Elkouss, Andrew C. Doherty, and Stephanie Wehner, “Optimizing practical entanglement distillation,” *Phys. Rev. A*, vol. 97, pp. 062333, Jun 2018.

[14] Naomi H. Nickerson, Joseph F. Fitzsimons, and Simon C. Benjamin, “Freely scalable quantum technologies using cells of 5-to-50 qubits with very lossy and noisy photonic links,” *Phys. Rev. X*, vol. 4, pp. 041041, Dec 2014.

[15] W Dür and H J Briegel, “Entanglement purification and quantum error correction,” *Reports on Progress in Physics*, vol. 70, no. 8, pp. 1381–1424, Jul 2007.

[16] Jia-Bin You, Dax Enshan Koh, Jian Feng Kong, Wen-Jun Ding, Ching Eng Png, and Lin Wu, “Exploring variational quantum eigensolver ansatzes for the long-range xy model,” *arXiv preprint arXiv:2109.00288*, 2021.

[17] Jessica Illiano, Marcello Caleffi, Antonio Manzalini, and Angela Sara Cacciapuoti, “Quantum internet protocol stack: A comprehensive survey,” *Computer Networks*, vol. 213, pp. 109092, 2022.