ON THE BRAUER CONSTRUCTIONS AND GENERIC JORDAN TYPES OF YOUNG MODULES

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Abstract. Let $p$ be a prime number. We study the dimensions of the Brauer constructions of Young and Young permutation modules with respect to $p$-subgroups of the symmetric groups. They depend only on partitions labelling the modules and the orbits of the action of the $p$-subgroups, and are related to their generic Jordan types. We obtain some reductive formulae and, in the case of two-part partitions, make some explicit calculation.

1. INTRODUCTION

Let $G$ be a finite group and $\mathbb{F}$ be a field of positive characteristic $p$. One of the main tools of studying the $p$-permutation $\mathbb{F}G$-modules via the Brauer construction has been developed by Broué in [2]. For a $p$-subgroup $P$ of $G$, there is a bijection between the set of the isomorphism classes of indecomposable $p$-permutation $\mathbb{F}G$-modules with vertex $P$ and the set of isomorphism classes of indecomposable projective $\mathbb{F}[N_G(P)/P]$-modules defined by the Brauer construction. Furthermore, the Green correspondents of such indecomposable $p$-permutation $\mathbb{F}G$-modules with respect to the subgroup $N_G(P)$ are precisely the inflation of their corresponding indecomposable projective $\mathbb{F}[N_G(P)/P]$-modules. Suppose further that $\mathbb{F}$ is algebraically closed. The generic Jordan type of a module for an elementary Abelian $p$-group as defined by Wheeler [19] is another useful technique to study the $\mathbb{F}G$-modules. For instance, if an indecomposable $\mathbb{F}G$-module $M$ has non-generically free Jordan type upon restriction to an elementary Abelian $p$-subgroup $E$ of $G$ then $E$ is contained in a vertex of $M$.

In this paper, we study the classical objects the Young and Young permutation $\mathbb{F}\mathfrak{S}_n$-modules. Since they are $p$-permutation modules, their stable generic Jordan types (modulo the projectives) restricted to any elementary Abelian $p$-subgroup $E$ of $\mathfrak{S}_n$ have the form $[1]^r$ for some non-negative integers $r$ depending on $E$. We are interested in the numbers $r$ as in the previous sentence. In Section 3, one of our main results shows that the dimension of the Brauer construction $Y^\lambda(E)$ is precisely $r$ where $[1]^r$ is the stable generic Jordan type of $Y^\lambda|_E$ and $\dim_\mathbb{F} Y^\lambda(P)$, for any $p$-subgroup $P$ of $\mathfrak{S}_n$, depend only on the orbit type of $P$ on the set $\{1, \ldots, n\}$. As such, we call $\dim_\mathbb{F} Y^\lambda(P)$ the orbit numbers. For example, when $n = 4$, $P = \langle (1, 2, 3, 4) \rangle$ and $Q = \langle (1, 2)(3, 4), (1, 3)(2, 4) \rangle$, for any partition $\lambda$ of 4, we have $\dim_\mathbb{F} Y^\lambda(P) = \dim_\mathbb{F} Y^\lambda(Q) = r$ where $Y^\lambda|_E$ has stable generic Jordan type $[1]^r$. The orbit numbers

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are interesting in the sense that, when \( P \) is a vertex of \( Y^\lambda \), following [4, Theorem 2], we have \( \dim \ Y^\lambda(P) \) is the product of the dimensions of the projective modules \( Y^{\lambda(0)}, \ldots, Y^{\lambda(s)} \) where \( \lambda = \lambda(0) + p\lambda(1) + \cdots + p^s\lambda(s) \) is the \( p \)-adic expansion of \( \lambda \). It is an open problem to find a closed-form for the dimensions of the indecomposable projective modules for the symmetric groups. In Section 4, we obtain some reductive formulae about orbit numbers. We explicitly calculate the orbit numbers in the case when \( \lambda \) are two-part partitions in Section 5.

2. Preliminaries

Throughout the paper \( F \) is an algebraically closed field of positive characteristic \( p \). For any finite group \( G \), an \( FG \)-module is assumed to be a finitely generated left \( FG \)-module.

2.1. Representation theory of finite groups. For a general background about the modular representation theory of finite groups, we refer readers to [1] or [16].

Let \( G \) be a finite group and let \( M, N \) be two \( FG \)-modules. We write \( N \mid M \) if \( N \) is isomorphic to a direct summand of \( M \), i.e., \( M \cong N \oplus L \) for some \( FG \)-module \( L \). Suppose further that \( N \) is indecomposable. The number of summands in an indecomposable direct sum decomposition of \( M \) that are isomorphic to \( N \) is well-defined by Krull-Schmidt Theorem (see [1, Section 4, Theorem 3]) and is denoted by \( [M : N] \).

Let \( M \) be an indecomposable \( FG \)-module and \( H \) be a subgroup of \( G \). Then \( M \) is said to be relatively \( H \)-projective if there exists some \( FH \)-module \( N \) such that \( M \mid N \uparrow G \), here \( N \uparrow G \) denotes the induction of \( N \) to \( G \). By [10], the minimal (with respect to inclusion of subgroups) subgroups \( H \) of \( G \) subject to the condition such that \( M \mid N \uparrow G \) for some \( FH \)-module \( N \) are \( p \)-subgroups and unique up to \( G \)-conjugation. These \( p \)-subgroups of \( G \) are called the vertices of \( M \). Let \( P \) be a vertex of \( M \). We denote the normalizer of \( P \) in \( G \) by \( N_G(P) \). Then there exists, unique up to isomorphism and \( N_G(P) \)-conjugation, an indecomposable \( FP \)-module \( S \) such that \( M \mid S \uparrow P \). Such an \( FP \)-module is called a source of \( M \).

Let \( M \) be an indecomposable \( FG \)-module, let \( P \) be a vertex of \( M \) and let \( H \) be a subgroup of \( G \) containing \( N_G(P) \). The Green correspondent of \( M \) with respect to the subgroup \( H \) is the unique indecomposable summand \( N \) of \( M \downarrow H \) such that \( N \) has a vertex \( P \).

Let \( E = \langle g_1, \ldots, g_k \rangle \) be an elementary Abelian \( p \)-group of order \( p^k \) and \( M \) be an \( FE \)-module. Let \( K \) be a field extension of \( F \) containing the indeterminates \( \alpha_1, \alpha_2, \ldots, \alpha_k \). Consider the element

\[
u_\alpha := 1 + \sum_{i=1}^k \alpha_i(g_i - 1) \in KE.
\]

Since \( \langle u_\alpha \rangle \) is a cyclic group of order \( p \), the restriction of \( K \otimes_F M \) to the shifted subgroup \( K\langle u_\alpha \rangle \) is isomorphic to a direct sum of \( n_i \) unipotent Jordan blocks of sizes \( i \) where \( i = 1, 2, \ldots, p \). The generic Jordan type of the \( FE \)-module \( M \) is defined as \( [1]^{n_1} [2]^{n_2} \cdots [p]^{n_p} \). By [19], the generic Jordan type is independent of the choice of
the generators of $E$. The stable generic Jordan type of $M$ is $[1]^{n_1}[2]^{n_2} \cdots [p-1]^{n_{p-1}}$.

The module $M$ is called generically free if $n_i = 0$ for all $i = 1, 2, \ldots, p - 1$. The following are the properties we shall need and we refer readers to [6, 7, 19] for more details.

**Lemma 2.1.**

(i) The generic Jordan type of a direct sum of modules is the direct sum of the generic Jordan types of the modules.

(ii) Let $E'$ be a proper subgroup of an elementary Abelian $p$-group $E$ and let $M$ be an $FE'$-module. Then the module $M^{\uparrow E}$ is generically free.

### 2.2. Brauer construction.

One of main techniques that we shall need is the Brauer constructions of $p$-permutation modules introduced by Broué in [2]. An $FG$-module is called a $p$-permutation module if for any $p$-subgroup $P$ of $G$ there exists a basis $B$ that is permuted by $P$, i.e., for each $g \in P$ and $b \in B$, we have $gb \in B$. By [2] (0.4), an indecomposable $p$-permutation $FG$-modules is precisely a module with trivial source. The class of all $p$-permutation $FG$-modules is closed under taking finite direct sum, direct summand and tensor product.

We recall the Brauer construction of a module. Let $M$ be an $FG$-module and $P$ be a $p$-subgroup of $G$. The set of $P$-fixed points in $M$ is

$$M^P := \{m \in M : gm = m \text{ for all } g \in P\}.$$ 

Notice that $M^P$ is an $FN_G(P)$-module on which $P$ acts trivially. Let $Q$ be a proper subgroup of $P$. The relative trace map $\text{Tr}^P_Q: M^Q \rightarrow M^P$ is the linear map defined by

$$\text{Tr}^P_Q(m) := \sum_{g \in P/Q} gm,$$

where $P/Q$ denotes a set of left coset representatives of $Q$ in $P$ and $m \in M^Q$. Observe that $\text{Tr}^P_Q(v)$ is independent of the choice of the set of left coset representatives. Furthermore

$$\text{Tr}^P(M) := \sum Q \text{Tr}^P_Q(M^Q),$$

where $Q$ runs over the set of all proper subgroups of $P$, is an $FN_G(P)$-submodule of $M^P$. One defines the Brauer construction of $M$ with respect to $P$ to be the $F[N_G(P)/P]$-module

$$M(P) := M^P/\text{Tr}^P(M).$$

In general, if $M$ is indecomposable and $M(P) \neq 0$ then $P$ is contained in a vertex of $M$. The converse is true in the case of $p$-permutation modules.

**Theorem 2.2** ([2] Theorem 3.2 (1))). Let $M$ be an indecomposable $p$-permutation $FG$-module, let $Q$ be a vertex of $M$ and let $P$ be a $p$-subgroup of $G$. Then $M(P) \neq 0$ if and only if $P$ is contained in a $G$-conjugate of $Q$.

Suppose further that $M$ is a $p$-permutation $FG$-module. Let $B$ be a basis of $M$ permuted by $P$ and let

$$B(P) := \{b \in B : gb = b \text{ for all } g \in P\}.$$
Notice that $P$ acts trivially on $\mathcal{B}(P)$. As a corollary of Theorem 2.2, we have the following.

**Corollary 2.3.** Let $M$ be a $p$-permutation $\mathbb{F}G$-module and $\mathcal{B}$ be a $p$-permutation basis of $M$ with respect to a $p$-subgroup $P$ of $G$. Then $M_P$ is isomorphic to the $\mathbb{F}$-span of $\mathcal{B}(P)$ as $\mathbb{F}[N_G(P)/P]$-modules. Furthermore, if $M \cong U \oplus V$ then $M_P \cong U_P \oplus V_P$.

We end this subsection with the following well-known result of Broué.

**Theorem 2.4 ([2], Theorems 3.2 and 3.4).** Let $G$ be a finite group and let $P$ be a $p$-subgroup of $G$.

(i) The Brauer construction sending $M$ to $M_P$ is a bijection between the isomorphism classes of indecomposable $p$-permutation $\mathbb{F}G$-modules with vertex $P$ and the isomorphism classes of indecomposable projective $\mathbb{F}[N_G(P)/P]$-modules. Furthermore, the inflation $\text{Inf}^{N_G(P)}_{N_G(P)/P}M_P$ of the $\mathbb{F}[N_G(P)/P]$-module $M_P$ to $N_G(P)$ is the Green correspondent of $M$ with respect to $N_G(P)$.

(ii) Let $N$ be an indecomposable $\mathbb{F}G$-module with a vertex $P$ and $M$ be a $p$-permutation $\mathbb{F}G$-module. Then $N$ is a direct summand of $M$ if and only if $N(M_P)$ is a direct summand of $M_P$. Moreover,

$$[M : N] = [M(M_P) : N(P)].$$

2.3. Composition, partition and orbit. Let $\mathbb{N}$ be the set of nonnegative integers and let $n \in \mathbb{N}$. By a composition $\alpha$ of $n$, we mean a sequence of nonnegative integers $(\alpha_1, \ldots, \alpha_r)$ such that $\sum_{i=1}^{r} \alpha_i = n$. In this case, we write $|\alpha| = n$. By convention, the unique composition of 0 is denoted as $\emptyset$. The composition $\alpha$ is called a partition if $\alpha_1 \geq \cdots \geq \alpha_r$. We write $\mathcal{C}(n)$ and $\mathcal{P}(n)$ for the set of compositions and partitions of $n$ respectively. The set $\mathcal{P}(n)$ is partially ordered by the dominance order $\triangleright$ and totally ordered by the lexicographic order. Notice that the lexicographic order refines the dominance order.

Let $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $\beta = (\beta_1, \ldots, \beta_s)$ be two compositions and let $m$ be a positive integer. We write

$$\alpha + \beta = \beta + \alpha = (\alpha_1 + \beta_1, \ldots, \alpha_r + \beta_r, \beta_{r+1}, \ldots, \beta_s),$$

$$\alpha \cdot \beta = (\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s),$$

$$m\alpha = (m\alpha_1, \ldots, m\alpha_r),$$

if $r \leq s$. A composition $\delta$ is a refinement of $\beta$ if there exist compositions $\delta^{(1)}, \ldots, \delta^{(s)}$ such that $\delta = \delta^{(1)} \cdot \cdots \cdot \delta^{(s)}$ and, for $i = 1, 2, \ldots, s$, we have $|\delta^{(i)}| = \beta_i$.

A partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ is called $p$-restricted if $\lambda_r < p$ and $\lambda_i - \lambda_{i+1} < p$ for all $i = 1, 2, \ldots, r - 1$. We write $\mathcal{R}_p(n)$ for the set of all $p$-restricted partitions of $n$. The $p$-adic expansion of a partition $\lambda$ is the sum

$$\lambda = \sum_{i=0}^{t} p^i \lambda(i)$$
for some nonnegative integer $t$ such that, for each $i = 0, 1, \ldots, t$, $\lambda(i)$ is a $p$-restricted partition. By the proof of [9 Lemma 7.5], there is a way to write down the $p$-adic expansion of $\lambda$ as follows. Let $\lambda_{r+1} = 0$. Suppose that, for each $j = 1, 2, \ldots, r$, we have the $p$-adic sum of the number

$$\lambda_j - \lambda_{j+1} = \sum_{i=0}^{t} a_{i,j}p^i,$$

i.e., $0 \leq a_{i,j} \leq p - 1$. Then, for each $i = 0, 1, \ldots, t$, $\lambda(i)$ is the desired $p$-restricted partition where $\lambda(i)_k = \sum_{j=k}^{r} a_{i,j}$.

For any partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, we denote by $[\lambda]$ the set $\{(i, j) \in \mathbb{N}^2 : 1 \leq i \leq r, 1 \leq j \leq \lambda_i\}$. It is called the Young diagram of $\lambda$. The $p$-core of $\lambda$ is the partition whose Young diagram is obtained by removing all possible rim $p$-hooks from $[\lambda]$ and is denoted by $\kappa_p(\lambda)$. The number of rim $p$-hooks removed from $[\lambda]$ to get $\kappa_p(\lambda)$ is called the $p$-weight of $\lambda$.

Let $A$ be a finite set. The permutation group on the set $A$ is denoted as $\mathfrak{S}_A$. Let $n \in \mathbb{N}$. We denote the set $\{1, \ldots, n\}$ by $[n]$ and let $\mathfrak{S}_n = \mathfrak{S}_{[n]}$. By convention, $[0] = \emptyset$ and $\mathfrak{S}_0$ is the trivial group. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a composition. The Young subgroup $\mathfrak{S}_\lambda$ is identified with the direct product

$$\mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_r},$$

where the first factor $\mathfrak{S}_{\lambda_1}$ acts on the set $\{1, \ldots, \lambda_1\}$, the second factor $\mathfrak{S}_{\lambda_2}$ acts on the set $\{1 + \lambda_1, \ldots, 1 + \lambda_1 + \lambda_2\}$ and so on.

We now discuss the orbits of $p$-subgroups of $\mathfrak{S}_n$ on the set $[n]$. Let

$$\mathcal{P}^{(p)}(n) = \{\mathcal{O} \in \mathcal{C}(n) : \mathcal{O} = (1^{a_0}, p^{a_1}, \ldots, (p^r)^{a_r}) \text{ for some } r\},$$

where, in $\mathcal{O}$, the first $a_0$ entries of $\mathcal{O}$ are 1, the next $a_1$ entries are $p$ and so on. For each $\mathcal{O} = (1^{a_0}, p^{a_1}, \ldots, (p^r)^{a_r}) \in \mathcal{P}^{(p)}(n)$, let

$$p^s \mathcal{O} = ((p^s)^{a_0}, (p^{s+1})^{a_1}, \ldots, (p^{s+r})^{a_r}) \in \mathcal{P}^{(p)}(p^sn).$$

Let $P$ be a $p$-subgroup of $\mathfrak{S}_n$. We denote the set of orbits of the action of $P$ on $[n]$ by $[n]/P$. We say that $[n]/P$ has type $\mathcal{O} = (1^{a_0}, (p)^{a_1}, \ldots, (p^r)^{a_r}) \in \mathcal{P}^{(p)}(n)$ for some $r$ if, for each $i = 0, 1, \ldots, r$, the number of orbits with sizes $p^i$ in $[n]/P$ is exactly $a_i$. We write $[n]/P \simeq [n]/Q$ if $Q$ is another $p$-subgroup of $\mathfrak{S}_n$ such that both $[n]/P$ and $[n]/Q$ have the same type, i.e., there is a permutation $\sigma \in \mathfrak{S}_n$ such that $\sigma A \in [n]/Q$ for all $A \in [n]/P$. It is clear that $[n]/P \simeq [n]/Q$ if $P$ is conjugate to $Q$ in $\mathfrak{S}_n$.

Let $\lambda$ be a partition of $n$, let $\sum_{i=0}^{r} p^i \lambda(i)$ be the $p$-adic expansion of $\lambda$ and let

$$\mathcal{O}_\lambda := (1^{\lambda(0)}, p^{\lambda(1)}, \ldots, (p^r)^{\lambda(r)}) \in \mathcal{P}^{(p)}(n).$$

We fix a Sylow $p$-subgroup of $\mathfrak{S}_{\lambda}$ and denote it by $P_\lambda$. Notice that, since any Sylow $p$-subgroup of $\mathfrak{S}_{p^r}$ acts transitively on the set $[p^r]$, we have that $[n]/P_\lambda$ has type $\mathcal{O}_\lambda$. We end this subsection by the following lemma.
Lemma 2.5. Let $\lambda \in \mathcal{P}(n)$, $O \in \mathcal{P}^{(p)}(n)$ and $P$ be a $p$-subgroup of $\mathfrak{S}_n$ such that $[n]/P$ has type $O$. Then $O$ is a rearrangement of a refinement of $O_{\lambda}$ if and only if $P$ is conjugate to a $p$-subgroup of $P_\lambda$.

Proof. Suppose that $O$ is a rearrangement of a refinement of $O_{\lambda}$. Without loss of generality, since $[n]/P$ has type $O_{\lambda}$, we may assume that each orbit in $[n]/P_\lambda$ is a union of some orbits in $[n]/P$. Let $\sigma \in P$. Then $\sigma$ leaves each orbit in $[n]/P$ invariant, i.e., $\sigma(A) = A$. Therefore, $\sigma$ leaves each orbit in $[n]/P_\lambda$ invariant. This shows that $\sigma \in \mathfrak{S}_{O_{\lambda}}$ and hence $P \subseteq \mathfrak{S}_{O_{\lambda}}$. We conclude that $P$ is conjugate to a subgroup of $P_\lambda$. Conversely, suppose, without loss of generality, that $P$ is a subgroup of $P_\lambda$. Then each orbit in $[n]/P_\lambda$ is a union of some orbits in $[n]/P$. By definition, the type $O$ of $[n]/P$ is a rearrangement of a refinement of the type $O_{\lambda}$ of $[n]/P_\lambda$. □

2.4. Representation theory of symmetric groups. We now turn to the representation theory of symmetric groups. For a general background on this topic, we refer readers to [12] or [14].

For modules of symmetric groups, we assume that readers are familiar with the notion of tableau, tabloid and polytabloid. Let $\mathbb{F}(n)$ be the trivial $\mathfrak{F}\mathfrak{S}_n$-module. For a composition $\lambda$ of $n$, we use $\mathbb{F}(\lambda)$ to denote the restriction of $\mathbb{F}(n)$ to the Young subgroup $\mathfrak{S}_\lambda$. The Young permutation module $M^\lambda$ with respect to $\lambda$ is the induced module $\mathbb{F}(\lambda)^{\mathfrak{S}_n}$. It has a basis consisting of all $\lambda$-tabloids. Notice that $M^\lambda \cong M^\mu$ if $\mu$ can be rearranged to $\lambda$. Suppose now that $\lambda$ is a partition. The Specht module $S^\lambda$ is the submodule of $M^\lambda$ spanned by the $\lambda$-polytabloids. It has a basis given by the standard $\lambda$-polytabloids and dimension given by the hook formula. In the characteristic zero case, the Specht modules are the irreducible $\mathfrak{F}\mathfrak{S}_n$-modules. However, they are usually not irreducible when $p$ is positive. By the Nakayama conjecture, two Specht modules $S^\lambda, S^\mu$ for $\mathfrak{F}\mathfrak{S}_n$ lie in the same block if and only if $\kappa_p(\lambda) = \kappa_p(\mu)$.

The isomorphism classes of indecomposable direct summands of Young permutation modules are called the Young modules and they are parametrized by $\mathcal{P}(n)$ such that the Young module $Y^\lambda$ is a direct summand of $M^\lambda$ with multiplicity one and, if $Y^\mu \mid M^\lambda$, then $\mu \triangleright= \lambda$ (see [13] Theorem 3.1), i.e.,

$$M^\lambda \cong Y^\lambda \oplus \bigoplus_{\mu \triangleright \lambda} k_{\lambda,\mu} Y^\mu,$$

where $k_{\lambda,\mu} = [M^\lambda : Y^\mu]$ are known as the $p$-Kostka numbers. Using the lexicographic order of $\mathcal{P}(n)$, we denote the $p$-Kostka matrix for $\mathfrak{F}\mathfrak{S}_n$ by $K$ whose $(\lambda, \mu)$-entry is $k_{\lambda,\mu}$. Notice that $K$ is upper uni-triangular.

We recall the following reductive formulae for $p$-Kostka numbers proved by Gill.

Theorem 2.6 (see [8] Theorems 13 and 14]). Let $\lambda, \mu \in \mathcal{P}(m)$ and $\nu, \delta \in \mathcal{P}(n)$. We have the following statements.

(i) Let $\lambda_1$ be the first part of $\lambda$ and $\sum_{i=0}^t p^i\mu(i)$ be the $p$-adic expansion of $\mu$. If $s > t$ and $p^s > \lambda_1$ then $k_{\lambda_1+p^s\nu+p^s\delta} = k_{\lambda_1}\cdot k_{\nu,\delta}$.

(ii) Let $\lambda_2$ be the second part of $\lambda$. If $\lambda_2 < p^s$ then $k_{\lambda_1+(p^s r)\nu+(p^s r)} = k_{\lambda_1,\mu}$ for every $r \in \mathbb{N}$.
The regular module $\mathbb{F}S_n$ is the Young permutation module $M^{(1^n)}$. Therefore the projective indecomposable $\mathbb{F}S_n$-modules are Young modules. In fact, the Young module $Y^\lambda$ is projective if and only if $\lambda \in R\mathcal{P}_p(n)$. Let $\lambda \in R\mathcal{P}_p(n)$ and $\text{sgn}(n)$ be the signature representation of $\mathbb{F}S_n$. Since $Y^\lambda \otimes \text{sgn}(n)$ is also projective indecomposable, we have

$$Y^\lambda \otimes \text{sgn}(n) \cong Y^{\mathfrak{m}(\lambda)}$$

for some unique partition $\mathfrak{m}(\lambda) \in R\mathcal{P}_p(n)$. The map $\mathfrak{m} : R\mathcal{P}_p(n) \to R\mathcal{P}_p(n)$ is called the Mullineux map (on $p$-restricted partitions) and is an involution. The $p$-regular version of Mullineux map was conjectured by Mullineux in [15] and proved by Ford and Kleshchev in [5]. In [3], Brundan and Kujawa proved the $p$-restricted version.

We now discuss the Brauer constructions of Young permutation modules and Young modules as in [4].

For each $\lambda \in \mathcal{P}(n)$ and $O \in \mathcal{P}^{(p)}(n)$, let $P$ be a $p$-subgroup of $S_n$ such that $[n]/P$ has type $O$. Let $M_{\lambda,P}$ be the set of all $\lambda$-tabloids $t$ such that each row of $t$ is a union of some orbits in $[n]/P$. Notice that if $Q$ is another $p$-subgroup of $S_n$ such that $[n]/Q \simeq [n]/P$ then $|M_{\lambda,P}| = |M_{\lambda,Q}|$. We write

$$m_{\lambda,O} = |M_{\lambda,P}|.$$

Since, for each $\lambda \in \mathcal{P}(n)$, the Young permutation module $M^\lambda$ has basis the $\lambda$-tabloids permuted by $S_n$, and hence permuted by any $p$-subgroup of $S_n$, Young permutation and Young modules are $p$-permutation $\mathbb{F}S_n$-modules. Let $P$ be a $p$-subgroup of $S_n$. Notice that a $\lambda$-tabloid $t$ is fixed by $P$ if and only if every orbit in $[n]/P$ lies in a row of $t$. By Corollary 2.8 we have the following lemma.

**Lemma 2.7.** Let $\lambda \in \mathcal{P}(n)$, let $P$ be a $p$-subgroup of $S_n$ and suppose that $[n]/P$ has type $O$. Then

$$\dim_{\mathbb{F}} M^\lambda(P) = m_{\lambda,O},$$

i.e., $\dim_{\mathbb{F}} M^\lambda(P)$ is the number of (unordered) ways to insert the orbits in $[n]/P$ into the rows of $\lambda$. In particular, we have $\dim_{\mathbb{F}} M^\lambda(P) = \dim_{\mathbb{F}} M^\lambda(Q)$ if $[n]/P \simeq [n]/Q$.

The precise structure of the Brauer construction $M^\lambda(P)$ when $\lambda \in \mathcal{P}(n)$, $P$ is a Sylow $p$-subgroup of $S_n$ and $O \in \mathcal{P}^{(p)}(n)$ is given in [4] Proposition 1] but we shall not need it here.

Suppose that a normal subgroup $N$ of $G$ acts trivially on an $FG$-module $M$. We write $\text{Def}_{G/N}^G M$ for the deflation of $M$ to the quotient group $G/N$. We now describe the vertices of Young modules and their Brauer constructions with respect to the vertices.

**Theorem 2.8 ([9, 4]).** Let $\lambda \in \mathcal{P}(n)$. Then $Y^\lambda$ is relatively $S_{\mathcal{O}_\lambda}$-projective. If $Y^\lambda$ is also relatively $H$-projective for some Young subgroup $H$ then $S_{\mathcal{O}_\lambda}$ is $S_n$-conjugate to a subgroup of $H$. Furthermore, $Y^\lambda$ has a vertex $P_{\lambda}$.

**Theorem 2.9 ([4]).** Let $\sum_{i=0}^r p^i \lambda(i)$ be the $p$-adic expansion of $\lambda \in \mathcal{P}(n)$ and let $\beta = (a_0, a_1, \ldots, a_r)$ where $a_i = |\lambda(i)|$ for each $i = 0, 1, \ldots, r$. Then $N_{S_{\mathcal{O}_\lambda}}(P_{\beta})/P_{\lambda}$
acts trivially on $Y^\lambda(P_\lambda)$ and
\[
\text{Def}^{N_{\mathfrak{S}_n}(P_\lambda)/P_\lambda}_{N_{\mathfrak{S}_n}(P_\lambda)/N_{\mathfrak{S}_\lambda}(P_\lambda)}(P_\lambda) Y^\lambda(P_\lambda) \cong Y^\lambda(0) \boxtimes Y^\lambda(1) \boxtimes \cdots \boxtimes Y^\lambda(r)
\]
as $\mathfrak{S}_\beta$-modules via the canonical isomorphism
\[
\mathfrak{S}_\beta \cong N_{\mathfrak{S}_n}(P_\lambda)/N_{\mathfrak{S}_\lambda}(P_\lambda) \cong (N_{\mathfrak{S}_n}(P_\lambda)/P_\lambda)/(N_{\mathfrak{S}_\lambda}(P_\lambda)/P_\lambda).
\]

3. ORBIT NUMBERS

In this section, we define the orbit numbers (see Definition 3.3) labelled by $P(n) \times P(p)$ such that $[n]/P \cong [n]E$. The numbers can be simultaneously defined as either the dimensions of the Brauer constructions of Young modules with respect to $p$-subgroups or the nonnegative integers $m$ where $[1]^m$ is the stable generic Jordan types of Young modules restricted to certain elementary Abelian $p$-subgroups.

We begin with the following lemma.

**Lemma 3.1.** Let $M$ be a $p$-permutation $\mathbb{F} G$-module and $E$ be an elementary Abelian $p$-subgroup of $G$. Then the stable generic Jordan type of $M \downarrow_E$ is $[1]^r$ where $r = \dim_{\mathbb{F}} M(E)$.

**Proof.** Let $B$ be a $p$-permutation basis of $M$ with respect to $E$ and suppose that $A_1, \ldots, A_r, B_1, \ldots, B_s$ are the orbits of action of $E$ on $B$ such that $|A_i| = 1$ for $i = 1, 2, \ldots, r$ and $|B_j| > 1$ for $j = 1, \ldots, s$. Then

\[
M \downarrow_E \cong \left( \bigoplus_{i=1}^r \mathbb{F} E \right) \oplus \left( \bigoplus_{j=1}^s \mathbb{F} H_j \uparrow^E \right)
\]
as $\mathbb{F} E$-modules where $H_j$ is the stabiliser of $b_j \in B_j$ for all $j = 1, 2, \ldots, s$ and $\mathbb{F} H_j, \mathbb{F} E$ are the trivial modules for $\mathbb{F} H_j$ and $\mathbb{F} E$ respectively. Since $H_j$ is a proper subgroup of $E$, by Lemma 2.1, $M \downarrow_E$ has stable generic Jordan type $[1]^r$. By Corollary 2.3, $\dim_{\mathbb{F}} M(E) = r$. The result now follows. \hfill \Box

In the case of the Young permutation module $M^\lambda$, Lemmas 2.7 and 3.1 assert that the generic Jordan type of $M^\lambda \downarrow_E$ is $[1]^r$ where

\[
r = \dim_{\mathbb{F}} M^\lambda(E) = m_{\lambda, \mathcal{O}}
\]
and $[n]/E$ has type $\mathcal{O}$.

We now prove the main result of this section.

**Theorem 3.2.** Let $\lambda \in \mathcal{P}(n)$ and $P, E$ be $p$-subgroups of $\mathfrak{S}_n$ such that $[n]/P \cong [n]E$. Then

\[
\dim_{\mathbb{F}} Y^\lambda(P) = \dim_{\mathbb{F}} Y^\lambda(E).
\]

Suppose further that $E$ is elementary Abelian. We have $\dim_{\mathbb{F}} Y^\lambda(E) = m$ where the stable generic Jordan type of $Y^\lambda \downarrow_E$ is $[1]^m$. 
Proof. We prove that \( \dim_F Y^\lambda(P) = \dim_F Y^\lambda(E) \) by using induction on the dominance order of \( \mathcal{P}(n) \). In the base case, since \( Y^{(n)} \cong M^{(n)} \) is the trivial \( \mathbb{F}S_n \)-module, we have \( \dim_F Y^{(n)}(P) = \dim_F Y^{(n)}(E) = 1 \) by Lemma 2.7. Suppose that \( \dim_F Y^\mu(P) = \dim_F Y^\mu(E) \) for all \( \mu \succ \lambda \). By Corollary 2.8 we have

\[
M^\lambda(P) \cong Y^\lambda(P) \oplus \bigoplus_{\mu \succ \lambda} k_{\lambda,\mu} Y^\mu(P),
\]

\[
M^\lambda(E) \cong Y^\lambda(E) \oplus \bigoplus_{\mu \succ \lambda} k_{\lambda,\mu} Y^\mu(E).
\]

Counting the dimensions of the above equations, using Lemma 2.7 and induction on the dominance order, we obtain that \( \dim_F Y^\lambda(P) = \dim_F Y^\lambda(E) \). Suppose further now that \( E \) is elementary Abelian. Since Young modules are \( p \)-permutation as direct summands of Young permutation modules, the second assertion follows from Lemma 3.1.

In the view of Theorem 3.2 we can now define the orbit number.

Definition 3.3. Let \( \lambda \in \mathcal{P}(n) \), \( \mathcal{O} \in \mathcal{P}(p)(n) \) and let \( P, E \) be \( p \)-subgroups of \( S_n \) such that both \( [n]/P \) and \( [n]/E \) have type \( \mathcal{O} \) and \( E \) is elementary Abelian. The orbit number \( y_{\lambda,\mathcal{O}} \) is defined as the following common numbers:

\[
y_{\lambda,\mathcal{O}} := \dim_F Y^\lambda(P) = b,
\]

where \( [1]^b \) is the stable generic Jordan type of \( Y^\lambda \downarrow_E \).

Fix \( n \in \mathbb{N} \). Let both \( \mathcal{P}(n) \) and \( \mathcal{P}(p)(n) \) be ordered by the lexicographic order. Let \( Y, M \) be the \( \mathcal{P}(n) \times \mathcal{P}(p)(n) \)-matrices whose \((\lambda,\mathcal{O})\)-entries of \( Y, M \) are the orbit number \( y_{\lambda,\mathcal{O}} \) and \( m_{\lambda,\mathcal{O}} \) respectively. By Theorem 2.4(ii), we have

\[
m_{\lambda,\mathcal{O}} = m_{\lambda,\mathcal{O}} + \sum_{\mu \succ \lambda} k_{\lambda,\mu} y_{\mu,\mathcal{O}},
\]

or equivalently, \( M = KY \) where, recall that, \( k_{\lambda,\mu} = [M^\lambda : Y^\mu] \) and \( K \) is the \( p \)-Kostka matrix of \( \mathbb{F}S_n \). The \((\lambda,\mathcal{O})\)-entry \( m_{\lambda,\mathcal{O}} \) of \( M \) has a combinatorial description given by Lemma 2.7. Suppose further that \( \mathcal{O} = (1^{a_0}, p^{a_1}, \ldots, (p^r)^{a_r}) \) and let \( \Lambda(\lambda, \mathcal{O}) \) be the set consisting of tuples of compositions \( \alpha = (\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(r)}) \) such that \( \lambda = \sum_{i=0}^r p^i \alpha^{(i)} \) (not necessarily the \( p \)-adic expansion of \( \lambda \)) and \( |\alpha^{(i)}| = a_i \) for all \( i = 0, 1, \ldots, r \). It is easy to see that the number \( m_{\lambda,\mathcal{O}} \) can be described as

\[
m_{\lambda,\mathcal{O}} = \sum_{\alpha \in \Lambda(\lambda, \mathcal{O})} \prod_{i=0}^r \dim_F M^{\alpha^{(i)}}.
\]

To end this section, we give characterisations when an orbit number is nonzero. The following lemma is straightforward following Theorem 2.9.

Lemma 3.4. Let \( \sum_{i=0}^s p^i \lambda(i) \) be the \( p \)-adic expansion of \( \lambda \in \mathcal{P}(n) \). Then

\[
y_{\lambda,\mathcal{O}_\lambda} = \prod_{i=0}^s \dim_F Y^{\lambda(i)} \neq 0.
\]
For example, if \( \lambda \) has a unique \( \lambda(\ell) \) in its \( p \)-adic expansion with more than one part, then
\[
y_{\lambda, O_{\lambda}} = \dim_F Y^{\lambda(\ell)}.
\]
This happens, in particular, when \( Y^\lambda \) is a non-projective periodic Young module (see [11, Corollary 3.3.3]).

Recall that \( P_{\lambda} \) is a fixed Sylow \( p \)-subgroup of \( \mathfrak{S}_{O_{\lambda}} \) as in Subsection 2.3.

**Theorem 3.5.** Let \( \lambda \in \mathcal{P}(n) \), \( O \in \mathcal{P}^{(p)}(n) \) and \( P, E \) be \( p \)-subgroups such that both \( |n|/P \), \( |n|/E \) have type \( O \) and \( E \) is elementary Abelian. Then the following statements are equivalent.

(i) \( y_{\lambda, O} \neq 0 \).

(ii) \( Y^{\lambda} \downarrow E \) is not generically free.

(iii) \( P \) is conjugate to a subgroup of \( P_{\lambda} \).

(iv) \( O \) is rearranged to be a refinement of \( O_{\lambda} \).

In any of the cases above, we have \( y_{\lambda, O} \geq y_{\lambda, O_{\lambda}} \neq 0 \).

**Proof.** The equivalence of parts (i), (ii) and (iii) follows from Definition 3.3 and Theorem 2.2. The equivalence of parts (iii) and (iv) is given by Lemma 2.5. We now prove the last assertion. Let \( Q \) be a conjugate of \( P_{\lambda} \) in \( \mathfrak{S}_n \) such that \( P \) is a \( p \)-subgroup of \( Q \). Let \( \mathcal{B} \) be a \( p \)-permutation basis of \( Y^\lambda \) with respect to \( Q \). Then \( \mathcal{B}(Q) \subseteq \mathcal{B}(P) \).

By Corollary 2.3 and Theorem 3.2, we have
\[
y_{\lambda, O} = \dim_F Y^\lambda(P) \geq \dim_F Y^\lambda(Q) = \dim_F Y^\lambda(P_{\lambda}) = y_{\lambda, O_{\lambda}}.
\]
\( \square \)

### 4. Some Computation

In this section, we present some equalities among the orbit numbers we have defined in Definition 3.3. The main results are Theorems 4.1, 4.5 and 4.9.

We present our first results which follows easily from Lemma 3.4. Recall that \( m \) is the Mullineux map on \( p \)-restricted partitions such that \( Y^\lambda \otimes \text{sgn}(n) \cong Y^{m(\lambda)} \) for all \( \lambda \in \mathcal{R}\mathcal{P}_p(n) \).

**Theorem 4.1.** Let \( \sum_{i=0}^s p^i \lambda(i) \) be the \( p \)-adic expansion of \( \lambda \) and let
\[\mu = \sum_{i=0}^s p^{k_i} m^{\ell_i}(\lambda(i)),\]
where \( k_0, \ldots, k_s \) are mutually distinct nonnegative integers and, for each \( i = 0, 1, \ldots, s \), \( \ell_i \) is either 0 or 1. Then \( y_{\lambda, O_{\lambda}} = y_{\mu, O_{\mu}} \).

**Proof.** Notice that \( \sum_{i=0}^s p^{k_i} m^{\ell_i}(\lambda(i)) \) is the \( p \)-adic expansion of \( \mu \) since \( k_i \)'s are all distinct. By Lemma 3.4, we have
\[
y_{\lambda, O_{\lambda}} = \prod_{i=0}^s \dim_F Y^{\lambda(i)} = \prod_{i=0}^s \dim_F Y^{m^{\ell_i}(\lambda(i))} = y_{\mu, O_{\mu}}.
\]
\( \square \)
Recall that $\beta \bullet \gamma$ denote the concatenation of two compositions $\beta, \gamma$. We need the following lemmas to prove our next result Theorem 4.5.

**Lemma 4.2.** Let $m, n, s \in \mathbb{N}$ such that $p^s > m$. If $\lambda + p^s\mu = \alpha + p^s\beta$ for some $\lambda, \alpha \in \mathcal{C}(m)$ and $\mu, \beta \in \mathcal{C}(n)$ then $\lambda = \alpha$ and $\mu = \beta$.

**Proof.** If $\lambda_i > \alpha_i$ for some $i$ then

$$p^s > \lambda_i - \alpha_i = p^s(\beta_i - \mu_i) \geq p^s,$$

which is a contradiction. Similarly, we must have $\lambda_i \geq \alpha_i$. Therefore $\lambda = \alpha$ and hence $\mu = \beta$. □

**Lemma 4.3.** Let $\lambda \in \mathcal{P}(m), \mu \in \mathcal{P}(n), O \in \mathcal{P}(p^s)(m), O' \in \mathcal{P}(p^s)(n)$ and $s \in \mathbb{N}$ such that $p^s > m$. Then

$$m_{\lambda + p^s\mu, O \bullet p^sO'} = m_{\lambda, O \circ m_{\mu, O'}}.$$

**Proof.** Let $O'' = O \bullet p^sO'$ and let $A = [m]/P$, $B = [n]/Q$ and $C = [m + p^s n]/R$. Furthermore, let $P, Q, R$ be $p$-subgroups of $\mathfrak{S}_m, \mathfrak{S}_n, \mathfrak{S}_{m+p^n}$ such that $A, B, C$ have types $O, O', O''$ respectively. Since $p^s > m$, we may identify the set $C$ with the set $A \cup B$ where an orbit of size $p^i$ in $C$ is identified with an orbit of size $p^i$ in $A$ if $i < s$ and an orbit of size $p^{i-s}$ in $B$ if $i \geq s$. Recall the notation $M_{\lambda, \mu}$ defined in Subsection 2.3. To prove the result, we construct a bijection between the sets $X := M_{\lambda+p^s \mu, R}$ and $Y := M_{\lambda, \mu} \times M_{\mu, Q}$. We define $g : Y \to X$ as follows. For each $(t, s) \in Y$, let $g(t, s) \in X$ be the $(\lambda + p^s \mu)$-tabloid whose $i$th row contains an orbit of $C$ if and only if its corresponding orbit is in $A$ and belongs to the $i$th row of $t$ or it is in $B$ and belongs to the $i$th row of $s$. Conversely, we define $f : X \to Y$ as follows. For each $u \in X$, let $t$ be the $\alpha$-tabloid, for some composition $\alpha$ of $m$, whose $i$th row contains an orbit of $A$ if and only if its corresponding orbit in $C$ belongs to the $i$th row of $u$. Similarly, we obtain an $\beta$-tabloid $s$. Since $\alpha + p^s \beta = \lambda + p^s \mu$, by Lemma 4.2 we have $\alpha = \lambda$ and $\beta = \mu$. Therefore $f(u) = (t, s) \in Y$. Obviously, $f, g$ are inverses of each other. The proof is now complete using Lemma 2.7. □

**Lemma 4.4.** Let $O \in \mathcal{P}(p^s)(m), O' \in \mathcal{P}(p^s)(n)$ and $s, t \in \mathbb{N}$ such that $p^s > m$. If $y_{\tau, O \bullet p^sO'} \neq 0$ then $\tau = \nu + p^s \delta$ for some $\nu \in \mathcal{P}(m)$ and $\delta \in \mathcal{P}(n)$.

**Proof.** Let $\sum_{i=0}^{s-1} p^i \tau(i)$ be the $p$-adic expansion of $\tau$. By Theorem 3.5 $O'' := O \bullet p^sO'$ is a rearrangement of a refinement of $O_{\tau} = (1^{\tau(0)}, \ldots, (p^s)^{\tau(s)})$. Let $\nu := \sum_{i=0}^{s-1} p^i \tau(i)$ and $\delta := \sum_{i=s}^{r} p^{r-s} \tau(i)$. Notice that both $\nu, \delta$ are partitions. We now show that $|\nu| = m$ and $|\delta| = n$. Since $O''$ is a rearrangement of a refinement of $O_{\tau}$, we have $|\delta| \geq n$. On the other hand, we have $m + p^s n = |\nu| + p^s |\delta|$ and hence $0 \leq p^s (|\delta| - n) = m - |\nu| < p^s$. Therefore it forces that $|\delta| = n$ and $|\nu| = m$. □

We are now ready to prove our first inductive formula about orbit numbers.

**Theorem 4.5.** Let $\lambda \in \mathcal{P}(m), \mu \in \mathcal{P}(n), O \in \mathcal{P}(p^s)(m), O' \in \mathcal{P}(p^s)(n)$ and $s, t \in \mathbb{N}$ such that $p^t \geq p^s > m$. Then

$$y_{\lambda + p^t \mu, O \bullet p^sO'} = y_{\lambda + p^t \mu, O \circ p^sO'}.$$
Proof. Let $\mathcal{O}'' := \mathcal{O} \cdot p^s \mathcal{O}'$ and $\mathcal{O}''' := \mathcal{O} \cdot p^t \mathcal{O}'$. We show our statement by using induction on the set

$$X = \{ \nu + p^s \delta : \nu \in \mathcal{P}(m), \delta \in \mathcal{P}(n) \}$$

with respect to the dominance order. When $\nu = (m)$ and $\delta = (n)$, both $Y^{(m+p^n)}$ and $Y^{(m+p' n)}$ are trivial modules. So, by Lemma 4.3, we have

$$y_{(m+p^n),\mathcal{O}''} = m_{(m+p^n),\mathcal{O}''} = 1 = m_{(m+p' n),\mathcal{O}''} = y_{(m+p' n),\mathcal{O}''}.$$ 

Suppose that $y_{\nu+p^s \delta,\mathcal{O}''} = y_{\nu+p^t \delta,\mathcal{O}''}$ for all partitions $\nu + p^s \delta \triangleright \lambda + p^s \mu$ where $\nu \in \mathcal{P}(m)$ and $\delta \in \mathcal{P}(n)$. By Equation 3.1, we have

$$m_{\lambda+p^s \mu,\mathcal{O}''} = \sum_{\tau \triangleright \lambda+p^s \mu} k_{\lambda+p^s \mu,\tau} y_{\tau,\mathcal{O}''}.$$ 

By Lemma 4.4, $y_{\tau,\mathcal{O}''} = 0$ unless $\tau = \nu + p^s \delta$ for some uniquely determined $\nu \in \mathcal{P}(m)$ and $\delta \in \mathcal{P}(n)$ (see Lemma 4.2). By Theorem 2.6(i), $k_{\lambda+p^s \mu,\nu+p^s \delta} = k_{\lambda,\nu}k_{\mu,\delta}$. Using the observation $k_{\lambda,\nu}k_{\mu,\delta} = 0$ unless $\nu \triangleright \lambda$ and $\delta \triangleright \mu$, we deduce that

$$m_{\lambda+p^s \mu,\mathcal{O}''} = y_{\lambda+p^s \mu,\mathcal{O}''} + \sum_{\nu \triangleright \lambda,\delta \triangleright \mu} k_{\lambda,\nu}k_{\mu,\delta} y_{\nu+p^s \delta,\mathcal{O}''}.$$ 

Similarly, we obtain

$$m_{\lambda+p^t \mu,\mathcal{O}''} = y_{\lambda+p^t \mu,\mathcal{O}''} + \sum_{\nu \triangleright \lambda,\delta \triangleright \mu} k_{\lambda,\nu}k_{\mu,\delta} y_{\nu+p^t \delta,\mathcal{O}''}.$$ 

Our result now follows using Lemma 4.3 and inductive hypothesis. 

We obtain the following immediate consequence.

**Corollary 4.6.** Let $t \in \mathbb{N}$, $\mu \in \mathcal{P}(n)$ and $\mathcal{O} \in \mathcal{P}^{(p)}(n)$. Then $y_{p^t \mu, \mathcal{O}} = y_{\mu, \mathcal{O}}$.

Next we prove another reductive formula which depends on the shape of the partition (see Theorem 4.9) instead of on the size of the partition as in Theorem 4.5. The main idea of these two proofs are quite similar but the latter requires slightly different treatment. We need the following two lemmas.

**Lemma 4.7.** Let $\lambda \in \mathcal{P}(m)$, $\mathcal{O} \in \mathcal{P}^{(p)}(m)$, $\mathcal{O}' \in \mathcal{P}^{(p)}(n)$ and $s \in \mathbb{N}$ such that $p^s > \lambda_2$. Then

$$m_{\lambda+(p^s n),\mathcal{O}''} = m_{\lambda,\mathcal{O}},$$

where $\mathcal{O}'' \in \mathcal{P}^{(p)}(m+p^s n)$ is the rearrangement of $\mathcal{O} \cdot p^s \mathcal{O}'$.

**Proof.** Let $\mathcal{O}''$ be the rearrangement of $\mathcal{O} \cdot p^s \mathcal{O}' \in \mathcal{P}^{(p)}(m+p^s n)$. Let $P,Q$ be $p$-subgroups of $\mathcal{S}_m, \mathcal{S}_{m+p^s n}$ such that $[m]/P, [m+p^s n]/Q$ have types $\mathcal{O}, \mathcal{O}''$ respectively. Let $t \in M_{\lambda+(p^s n),Q}$. Since $p^s > \lambda_2$, the orbits in $[m+p^s n]/Q$ with sizes larger than or equal to $p^s$ must be assigned to the first row of $t$. Therefore there is a obvious bijection between $M_{\lambda+(p^s n),Q}$ and $M_{\lambda,P}$. Our claim now follows by using Lemma 2.7. 

\hfill $\Box$
Lemma 4.8. Let $\lambda \in \mathcal{P}(m)$, $\mathcal{O} \in \mathcal{P}(p)(m)$, $\mathcal{O}' \in \mathcal{P}(p)(n)$ and $s \in \mathbb{N}$ such that $p^s > m - \lambda_1$. If $\tau \in \mathcal{P}(m + p^s n)$ such that $\tau \triangleright \lambda + (p^s n)$ and $y_{\tau, \mathcal{O} \bullet p^s \mathcal{O}'} \neq 0$ then $\tau = \nu + (p^s n)$ for some uniquely determined partition $\nu \in \mathcal{P}(m)$.

Proof. Let $\sum_{i=0}^{s-1} p^i \tau(i)$ be the $p$-adic expansion of $\tau$ and let $\alpha = \sum_{i=0}^{s-1} p^i \tau(i)$ and $\beta = \sum_{i=s}^{\infty} p^{i-s} \tau(i)$ such that $\tau = \alpha + p^s \beta$. We claim that $\beta = (b)$ for some $b \geq n$. Since $y_{\tau, \mathcal{O} \bullet p^s \mathcal{O}'} \neq 0$, we have $\mathcal{O} \bullet p^s \mathcal{O}'$ is a rearrangement of a refinement of $\mathcal{O}_r$ and hence $|\beta| \geq |\mathcal{O}'| = n$. Suppose that $\beta_i > 0$ for some $i \geq 2$. Since $\tau \triangleright \lambda + (p^s n)$, we have $\tau_i \geq \lambda_1 + p^s n$. Also, $\tau_i = \alpha_i + p^s \beta_i \geq \alpha_i + p^s$. Therefore

$$p^s \leq \alpha_i + p^s \leq \tau_i \leq (m + p^s n) - \tau_1 \leq m - \lambda_1 < p^s,$$

which is a contradiction. This shows that $\beta$ has at most one part and hence $\beta = (b)$ for some $b \geq n$. Let $\nu = \alpha + p^s (b - n)$. Then $\tau = \nu + (p^s n)$. \(\square\)

We are now ready to prove the second reductive formula about orbit numbers.

Theorem 4.9. Let $\lambda \in \mathcal{P}(m)$ such that $m - \lambda_1 < p^s$ for some $s \in \mathbb{N}$, let $\mathcal{O} \in \mathcal{P}(p)(m)$ and let $\mathcal{O}' \in \mathcal{P}(p)(n)$. Then

$$y_{\lambda + (p^s n), \mathcal{O}''} = y_{\lambda, \mathcal{O}},$$

where $\mathcal{O}'' \in \mathcal{P}(p)(m + p^s n)$ is the rearrangement of $\mathcal{O} \bullet p^s \mathcal{O}'$.

Proof. We argue by induction with respect to the dominance order on the set

$$X = \{\nu \in \mathcal{P}(m) : m - \nu_1 < p^s\}.$$

When $\nu = (m) \in X$, we have $y_{(m) + (p^s n), \mathcal{O}''} = y_{(m), \mathcal{O}}$. Suppose that $y_{\nu + (p^s n), \mathcal{O}''} = y_{\nu, \mathcal{O}}$ for all $\lambda \triangleright \nu \in X$. By Equation 3.1 and Lemma 4.8, we have

$$m_{\lambda + (p^s n), \mathcal{O}''} = y_{\lambda + (p^s n), \mathcal{O}''} + \sum_{\tau \triangleright \lambda + (p^s n)} k_{\lambda + (p^s n), \tau} y_{\tau, \mathcal{O}''}$$

$$= y_{\lambda + (p^s n), \mathcal{O}''} + \sum_{\nu > \lambda} k_{\lambda + (p^s n), \nu + (p^s n)} y_{\nu + (p^s n), \mathcal{O}''}.$$

By Theorem 2.6(ii), since $\lambda_2 \leq m - \lambda_1 < p^s$, we have $k_{\lambda + (p^s n), \nu + (p^s n)} = k_{\lambda, \nu}$. By inductive hypothesis, we have

$$m_{\lambda + (p^s n), \mathcal{O}''} = y_{\lambda + (p^s n), \mathcal{O}''} + \sum_{\nu > \lambda} k_{\lambda, \nu} y_{\nu, \mathcal{O}}.$$

The proof is now complete by using Equation 3.1 $m_{\lambda, \mathcal{O}} = y_{\lambda, \mathcal{O}} + \sum_{\nu > \lambda} k_{\lambda, \nu} y_{\nu, \mathcal{O}}$ and Lemma 4.7. \(\square\)

5. Two-part partitions

In the final section, we provide some explicit calculation about the orbit numbers $y_{\lambda, \mathcal{O}}$ when $\lambda$ is a two-part partition. We begin with the following proposition.
Proposition 5.1. Let $\mathcal{O} = (1^{a_0}, p^{a_1}, \ldots, (p^r)^{a_r}) \in \mathcal{P}(p)(n)$. If $a_0 = 0$, then $y_{(n-1),\mathcal{O}} = 0$. If $a_0 \neq 0$ then
\[
y_{(n-1),\mathcal{O}} = \begin{cases} 
a_0, & \text{if } p \mid n, \\
a_0 - 1, & \text{if } p \nmid n.
\end{cases}
\]

Proof. Let $P$ be a $p$-subgroup such that $[n]/P$ has type $\mathcal{O}$. It is well-known that $M^{(n-1,1)}$ is isomorphic to $Y^{(n-1,1)}$ if $p$ divides $n$ and $Y^{(n)} \oplus Y^{(n-1,1)}$ otherwise. By Lemma 2.7, $\dim_P M^{(n-1,1)}(P)$ is the number of ways to insert the orbits with size one that are in $[n]/P$ into the second row of $(n-1,1)$, i.e., $\dim_P M^{(n-1,1)}(P) = a_0$. If $a_0 = 0$, since $0 \leq y_{(n-1),\mathcal{O}} \leq \dim_P M^{(n-1,1)}(P) = 0$, then $y_{(n-1),\mathcal{O}} = 0$. If $a_0 \neq 0$, then
\[
y_{(n-1),\mathcal{O}} = \dim_P M^{(n-1,1)}(P) - k_{(n),(n-1,1)} \dim_Y Y^{(n)}(P) = a_0 - k_{(n),(n-1,1)},
\]
where $k_{(n),(n-1,1)}$ is 1 if $p \nmid n$ and 0 otherwise. \hfill $\square$

Next, we compute $y_{\lambda,\mathcal{O}_\lambda}$ when $\lambda$ is a two-part partition. We need the following two lemmas.

Lemma 5.2. Any $p$-restricted two-part partition has $p$-weight either 0 or 1. Furthermore, a $p$-restricted partition $(a,b)$ has $p$-weight 1 if and only if $a - b < p - 1$ and $a + 1 \geq p$. In this case, the $p$-core is $(b - 1, a + 1 - p)$.

Proof. Let $\lambda = (a,b)$ which is a $p$-restricted two-part partition, i.e., $a - b < p$ and $b < p$. Suppose that the $p$-weight of $\lambda$ is not zero. It is clear from the Young diagram that it is equivalent to $a - b < p - 1$ and $a + 1 \geq p$. In this case, after removing one $p$-hook from $\lambda$, the remaining partition is $(b - 1, a + 1 - p)$. However, $(b - 1) + 1 < p$, so $(b - 1, a + 1 - p)$ has $p$-weight 0. \hfill $\square$

The weight one blocks of symmetric groups are Morita equivalent by [18, Theorem 1]. For $p \geq 3$, the decomposition matrix of the principal block of $\mathbb{F}S_n$ has been described by Peel in [17] (see also [12, Theorem 24.1]). Let $b$ be the weight one block of $\mathbb{F}S_n$, labelled by a $p$-core $\nu$ and let
\[
\mu^{(0)} \triangleright \mu^{(1)} \triangleright \cdots \triangleright \mu^{(p-1)}
\]
be all the partitions occurring in $b$. Notice that $\mu^{(i)}$ is $p$-restricted for each $i = 1,2,\ldots, p - 1$ and $\mu^{(0)} = \nu + (p)$. Taking the conjugates, we have $(\mu^{(0)})' \triangleright (\mu^{(1)})' \triangleright \cdots \triangleright (\mu^{(p-1)})'$. By the Brauer reciprocity, if $\mu$ is $p$-restricted, we have that the multiplicity of the ordinary irreducible character $\chi^\lambda$ in $\text{ch}(Y^\mu)$ is the decomposition number $d_{\lambda,\mu'}$, i.e., the multiplicity of the simple module $D^\mu'$ in the composition series of $S^\lambda$. In particular, we have
\[
\dim_P Y^{(\mu')} = \dim_P S^{\mu'} + \dim_P S^{\mu^{(i-1)}}.
\]
Suppose that $\lambda = (a,b)$ is a $p$-restricted partition. Suppose first that $p$ is odd. If the $p$-weight of $\lambda$ is zero then $Y^\lambda \cong S^\lambda$. If the $p$-weight of $\lambda$ is one then, by Lemma 5.2, in our discussion above, $\mu^{(1)} = \lambda$ and $\mu^{(0)} = \kappa_p(\lambda) + (p) = (p + b - 1, a + 1 - p)$. Suppose now that $p = 2$. We have that $(a,b)$ is either $(2,1)$ or $(1,1)$. In this case, $Y^{(2,1)} \cong S^{(2,1)}$ and $M^{(1,1)} \cong Y^{(1,1)}$. We have obtained the following lemma.
Lemma 5.3. For a $p$-restricted two-part partition $\lambda$, we have
\[
\dim F Y^\lambda = \begin{cases} 
\dim F S^\lambda, & \text{if } \lambda = \kappa_p(\lambda), \\
\dim F S^\lambda + \dim F S^{\kappa_p(\lambda) + (p)}, & \text{if } \lambda \neq \kappa_p(\lambda).
\end{cases}
\]

Now we can give a description for the orbit numbers $y_{\lambda, O_{\lambda}}$ when $\lambda$ is two-part partition.

Proposition 5.4. Let $\lambda = (a, b)$ be a two-part partition and let the $p$-adic sums of the numbers $a - b$ and $b$ be $\sum_{i \geq 0} x_i p^i$ and $\sum_{i \geq 0} y_i p^i$, respectively. Then
\[
y_{\lambda, O_{\lambda}} = \prod_{i \geq 0} \left( \binom{x_i + 2y_i - 1}{y_i} + \delta(x_i, y_i) \binom{x_i + 2y_i - 1}{x_i + y_i + 1 - p} \right),
\]
where $\delta$ is the function defined as
\[
\delta(x, y) = \begin{cases} 
1, & \text{if } x < p - 1 \text{ and } x + y + 1 \geq p, \\
0, & \text{otherwise}.
\end{cases}
\]

Proof. Notice that the $p$-adic expansion of $(a, b)$ is $\sum_{i \geq 0} p^i (x_i + y_i, y_i)$. By Lemma 5.2 the partition $(x_i + y_i, y_i)$ has $p$-weight 1 if and only if $x_i < p - 1$ and $x_i + y_i + 1 \geq p$. In this case,
\[
\kappa_p(\lambda(i)) + (p) = (p + y_i - 1, x_i + y_i + 1 - p).
\]
Otherwise, the $p$-weight of $(x_i + y_i, y_i)$ is zero. By Lemma 5.3 we have
\[
\dim F Y^{\lambda(i)} = \dim F S^{(x_i + y_i, y_i)} + \delta(x_i, y_i) \dim F S^{(y_i + p - 1, x_i + y_i + 1 - p)}.
\]
The proof is now complete using Lemma 3.4 and Hook Formula for the dimension of a Specht module. \qed

References

[1] J. L. Alperin, Local Representation Theory, Cambridge of Studies in Advanced Mathematics 11, Cambridge University Press, 1986.
[2] M. Broué, On Scott modules and $p$-permutation modules: an approach through the Brauer morphism, Proc. Amer. Math. Soc. 93 (1985), no. 3, 401–408.
[3] J. Brundan, J. Kujawa, A new proof of the Mullineux conjecture, J. Algebraic Combin. 18 (2003), no. 1, 13–39.
[4] K. Erdmann, Young modules for symmetric groups, Special issue on group theory, J. Aust. Math. Soc. 71 (2001), 201–210.
[5] B. Ford, A. Kleshchev, A proof of the Mullineux conjecture, Math. Z. 226 (1997), no. 2, 267–308.
[6] E. M. Friedlander, J. Pevtsova, A. Suslin, Generic and maximal Jordan types, Invent. Math. 168 (2007), no. 3, 485–522.
[7] E. Giannelli, K. J. Lim, M. Wildon, Sylow subgroups of symmetric and alternating groups and the vertex of $S^{(kp-1)}$ in characteristic $p$, J. Algebra 455 (2016), 358–385.
[8] C. Gill, Young module multiplicities and classifying the indecomposable Young permutation modules, J. Algebra Appl. 13 (2014), no. 5, 1350147, 23 pp.
[9] J. Grabmeier, Unzerlegbare Moduln mit trivialer Youngquelle und Darstellungstheorie der Schuralgebra, Bayreuth. Math. Schr. No. 20 (1985), 9–152.
[10] J. A. Green, On the indecomposable representations of a finite group, Math. Z. 70 1958/59 430–445.

[11] D. J. Hemmer, D. K. Nakano, Support varieties for modules over symmetric groups, J. Algebra, 254 (2002), no. 2, 422–440.

[12] G. D. James, The Representation Theory of the Symmetric Groups, Lecture Notes in Mathematics 682. Springer, Berlin, 1978.

[13] G. D. James, Trivial source modules for symmetric groups, Arch. Math. (Basel) 41 (1983), no. 4, 294–300.

[14] G. D. James, A. Kerber, The Representation Theory of the Symmetric Group, Encyclopedia of Mathematics and its Applications 16. Addison-Wesley, 1981.

[15] G. Mullineux, Bijections of $p$-regular partitions and $p$-modular irreducibles of the symmetric groups, J. London Math. Soc. (2) 20 (1979), no. 1, 60–66.

[16] H. Nagao, Y. Tsushima, Representations of Finite Groups, Academic Press, Inc., Boston, MA, 1989.

[17] M. H. Peel, Hook representations of the symmetric groups, Glasgow Math. J. 12 (1971), 136–149.

[18] J. Scopes, Cartan matrices and Morita equivalences for blocks of the symmetric groups, J. Algebra 142 (1991), 441–455.

[19] W. W. Wheeler, Generic module theory, J. Algebra 185 (1996), no. 1, 205–228.

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