A FINITELY PRESENTED GROUP WHOSE WORD PROBLEM HAS SAMPLEABLE HARD INSTANCES

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Abstract. Hard instances of natural computational problems are often elusive. In this note we present an example of a natural decision problem, the word problem for a certain finitely presented group, whose hard instances are easy to find. More precisely the problem has a complexity core sampleable in linear time.

1. Introduction

In 1975 Nancy Lynch [8] proved that for every computable decision problem not decidable in polynomial time there exists an infinite computable set of instances, X, such that the problem cannot be decided in polynomial time on any infinite subset of X. Such an X is called a complexity core for the decision problem.

Lynch’s result attracted the attention of several other authors, who considered decision problems in the form of membership problems for subsets \( S \subseteq \{0, 1\}^\ast \). Cores of non-sparse density with membership decidable in subexponential time are investigated in [5, 9]. If a core exists, then so does a proper core [6], i.e., \( X \subseteq S \). Proper cores are necessarily \( P \)-immune, and \( \{0, 1\}^\ast \) is itself a core if and only if \( S \) is \( P \)-bi-immune [2]. Generalizations to complexity classes beyond \( P \) are given in [3, 4, 6]. These results are reviewed in [1, Chapter 6].

Lynch’s construction of cores involves enumeration of all Turing machines, and in general the membership problem for cores is superpolynomial. In [9] the authors observe that as all known cores are more or less artificially constructed (when the core is \( \{0, 1\}^\ast \), it is the \( P \)-bi-immune set \( S \) which is artificially constructed), it would be extremely interesting to find natural examples of cores. Theorem 1 exhibits such a core, albeit with respect to a slight variation of Lynch’s original definition of cores.

Notation. For any finite set \( \Sigma \), \( \Sigma^\ast \) is the set of all words over \( \Sigma \), and \( \hat{\Sigma} \) is the union of \( \Sigma \) with a disjoint set of formal inverses.

Theorem 1. There exists a finitely presented group \( G = \langle \Sigma \mid R \rangle \) and a nonempty subset \( \Delta \subset \Sigma \) such that if \( D \) is the domain of convergence for any (correct) partial algorithm deciding the word problem, then

\[
\lim_{n \to \infty} \frac{|D \cap \hat{\Delta}^n|}{|\hat{\Delta}^n|} = 0
\]

where \( \hat{\Delta}^n \) denotes the set of all words of length at most \( n \) in \( \hat{\Delta}^\ast \).

Date: February 9, 2016.
Partially supported by NSF Grant 1318716.
The word problem for $G$ (with respect to the given presentation) is to decide whether an arbitrary word over $\hat{\Sigma}$ represents the identity in $G$. Theorem 1 says that every partial algorithm for the word problem fails on virtually all words from $\hat{\Delta}^*$. Thus $\hat{\Delta}^*$ is a readily available set of provably hard instances. Clearly membership in $\hat{\Delta}^*$ is decidable in linear time, and $\hat{\Delta}^*$ can be sampled in linear time. In addition $\hat{\Delta}^*$ is a complexity core in the sense of Lynch except that the set inputs from $\hat{\Delta}^*$ on which a partial algorithm succeeds is not finite, as in Lynch’s definition of a core, but rather of asymptotic density zero in the sense of Equation 1.

Theorem 1 is an immediate consequence of Theorem 3, a recent result from combinatorial group theory.

2. Background and Proof

Recall that in a finite presentation $\langle \Sigma \mid R \rangle$, $\Sigma$ is a finite set of generators and $R$ is a finite set of relators, i.e., of words over $\hat{\Sigma}$. Also an arbitrary word over $\hat{\Sigma}$ represents the identity of $G$ if and only if it can be reduced to the empty word by inserting and deleting words from $R$ and their inverses, along with the trivial words $aa^{-1}, a^{-1}a$ for $a \in \Sigma$.

**Definition 2** ([7]). A finitely generated group $H$ is algorithmically finite if every infinite computably enumerable subset of words in the generators and their inverses contains two words which represent the same element of $H$.

All finite groups are algorithmically finite. The interesting fact is that infinite algorithmically finite groups exist.

**Theorem 3** ([7] Theorems 1.1 and 1.3). Infinite recursively presented algorithmically finite groups exist. Any partial algorithm for the word problem of such a group converges only on a set of asymptotic density zero.

**Proof of Theorem 1**. Let $H$ be an infinite finitely generated recursively presented algorithmically finite group. By the well known Higman embedding Theorem $H$ is a subgroup of a finitely presented group $G$. Without loss of generality the generators $\Sigma$ of $G$ can be augmented to include generators, $\Delta$, of $H$. By Theorem 3 any partial algorithm for the word problem of $G$ fails everywhere on $\hat{\Delta}^*$ except on a subset of asymptotic density 0 in $\hat{\Delta}^*$.

3. Conclusion

Infinite algorithmically finite groups are a new kind of group with unsolvable word problem. The proof of Theorem 1 does not employ Turing machines; instead, Golod–Shafarevich presentations and analogs of simple sets from computability theory are used.

The construction of $H$ is not natural in our sense, as it involves enumeration of all recursively enumerable subsets of $\hat{\Sigma}^*$. However $G$ itself is specified by a straightforward finite presentation. Since the proof of the Higman Embedding Theorem is constructive [10], as is the construction of the recursive presentation for $H$, one could in principle compute this finite presentation.

The convergence of the limit in Theorem 1 can be made to occur exponentially fast. See [7] Corollary 1.4.
Sampleability of hard instances is of interest in crytography. The word problem of the group $G$ from Theorem 1 is unsolvable and thus not useful as a cryptoprimitive. It seems unlikely that our approach could produce a useful cryptoprimitive, but examples of sampleable cores at lower complexity levels might provide some useful insights.

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