An approach for the calculation of one-loop effective actions and vacuum energies: scattering methods

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ABSTRACT: An approach for calculating one-loop effective actions and vacuum energies is suggested. Spectral functions are functions of the eigenvalue spectrum of an operator. The starting point is to regard one-loop effective actions and vacuum energies in quantum field theory and scattering phase shifts and amplitudes in quantum mechanics as spectral functions of an operator. Different spectral functions of the same operator may belong to different fields of physics. Methods for calculating various spectral functions can be interconverted through transform relations among spectral functions. In this paper, we convert the method for calculating scattering phase shifts and amplitudes to a method for calculating one-loop effective actions and vacuum energies. In principle, all methods for the calculation of scattering phase shifts and amplitudes can be converted to methods for the calculation of the one-loop effective actions and vacuum energies. As an example, we convert the Born approximation in quantum-mechanical scattering theory to a method for calculating one-loop effective actions and vacuum energies. In appendices, we provide integral representations of the Bessel function for calculating the sum encountered.

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1 Introduction

The physical information is embedded in physical operators, such as the Hamiltonian. By the same Hamiltonian, one can construct both classical theory, the Hamiltonian equation, and quantum theory, the Schrödinger equation. From a given physical operator, classical mechanics or classical field theory extracts classical information, while quantum mechanics or quantum field theory extracts quantum information. The difference among various theories is the way of extracting information.

The Hamiltonian $H$ is a differential operator, and all the information of $H$ is embedded in the eigenproblem:

$$H\phi_n = \lambda_n \phi_n.$$  \hspace{1cm} (1.1)

If the eigenproblem is solved completely, i.e., the eigenvalue $\lambda_n$ and eigenfunction $\phi_n$ are obtained, then all the information about the operator $H$ is known. In the spectral representation of operators, the operator can be reconstructed by its eigenvalues and eigenfunctions:

$$H(x', x) = \sum_n \lambda_n \phi_n^*(x') \phi_n(x).$$

That is, the eigenvalues and eigenfunctions contain all the information of an operator.

An interesting problem is that if we know all the eigenvalues but do not know the eigenfunctions, what information can be extracted. This problem can also be expressed as: from the eigenvalues of a Hamiltonian, what physical quantities can be constructed and what information can be extracted.

Physical quantities constructed from eigenvalues are called spectral functions. For example, the global heat kernel $K(t) = \sum_n e^{-\lambda_n t}$, also known as the partition function, is a spectral function, which is defined by the eigenvalue spectrum $\{\lambda_n\}$. The scattering phase shift, the one-loop effective action, and the vacuum energy are also spectral functions.

"How much information can be extracted from an eigenvalue spectrum" is a famous mathematical problem, known as the problem "Can one hear the shape of a drum?", raised by Kac [1]. This question is essentially asking whether we can reconstruct the Hamiltonian from its eigenvalues. The answer is, of course, no [2]. (Note that in the original Kac problem, the information of the Hamiltonian is reflected in the boundary condition, namely the shape of the drum.)

The eigenvalue spectrum does not contain all the information of an operator, so the problem turns to “what information can be extracted from eigenvalues”. From eigenvalues, we can obtain various spectral functions, such as scattering phase shifts, one-loop effective actions, vacuum energies, global heat kernels (partition functions), and so on. Various spectral functions can be transformed into each other. Starting from a known spectral function, we can obtain other spectral functions by transforms, such as the transform among global heat kernels, one-loop effective actions, and vacuum energies [3], the transform between global heat kernels (partition functions) and spectral counting functions [4, 5], and the transform between scattering phase shifts and global heat kernels [6, 7].

The one-loop effective action and the vacuum energy are quantities in quantum field theory. The scattering phase shift and the scattering amplitude are quantities in quantum mechanics. They are all spectral functions. The aim of this paper is to provide an approach for calculating the one-loop effective action and the vacuum energy from the scattering phase
shift and the scattering amplitude. This approach converts a quantum-field-theory problem to a quantum mechanical problem, or, in other words, converts a quantum mechanical method to a quantum-field-theory method.

Concretely, we find a relation between the global heat kernel and the spectral counting function which counts the eigenstates whose eigenvalues are less than a certain number \(4, 5\). This relation is the starting point for finding the relation between the global heat kernel and the scattering phase shift. In the discussion of the relation between the heat kernel method [3] and the spectral method [8] in quantum field theory, we find a transform relation between the partial-wave scattering phase shift and the partial-wave global heat kernel [6]. Then we find a transform relation between the scattering phase shift and the local heat kernel [7]. These previous works are the basis of the present paper. In this paper, we suggest an approach for calculating the one-loop effective action and the vacuum energy: calculating these quantum-field-theory quantities from the scattering phase shift and amplitude in quantum mechanics. In other words, this approach calculates the one-loop effective action and the vacuum energy by quantum mechanical methods. Various methods for calculating scattering phase shifts in quantum mechanics can be converted into methods for calculating the one-loop effective action and the vacuum energy in quantum field theory. In the following, as an example, we convert the Born approximation, a quantum-mechanical method, into a method for calculating the one-loop effective action and the vacuum energy.

The heat kernel acts as a bridge in our method, which bridges one-loop effective actions, vacuum energies, and scattering phase shifts. The heat kernel expansion is an important method in quantum field theory. There are two heat kernel expansions: the covariant perturbation theory [9–15] and the Schwinger-DeWitt technique [3, 16]. Various methods are developed for heat kernel approaches, such as calculating heat kernel traces by the path integral [17], the Green function approach [18], the technique of labeled operators [19], and heat kernel diagrammatic equations [20]. The heat kernel of higher-order differential operators is considered [21]. Heat kernel expansions are very important, such as the Schwinger-DeWitt expansion in the induced gravity on the AdS background [22]. The heat kernel method applies to calculate effective actions [23], such as the effective field theory in curved spacetime [24, 25], the heat kernel expansion and the one-loop effective action in QCD [26], the Seeley-DeWitt expansion for the one-loop effective action in the Einstein-Maxwell theory [27], the one-loop effective action for the modified Gauss–Bonnet gravity [28] and in dS\(_2\) and AdS\(_2\) spacetime [29], \(\varphi^4\)-fields [30], and various operators [31–33]. The heat kernel method also applies to calculate vacuum energies, such as Casimir energies in curved spacetime [34–37] and in spherically symmetric backgrounds [38]. The vacuum energy is also calculated by the heat kernel method [39] and by the spectral functions [40]. Applying scattering theory to calculate vacuum energy is pioneered in Refs. [8, 41–45]. Various methods in scattering theory can be found in Refs. [46–49].

In section 2, we give a brief review on various spectral functions. In section 3, we calculate global heat kernels, one-loop effective actions, and vacuum energies from scattering phase shifts. In sections 4 and 5, we convert the Born approximation method to a method for calculating one-loop effective actions and vacuum energies in three and \(n\) dimensions, respectively. In section 6, we calculate global heat kernels, one-loop effective actions, and
vacuum energies from scattering amplitudes. The conclusion is given in section 7. In Appendix A, we give some integral representations for the Bessel function.

2 Heat kernels, one-loop effective actions, vacuum energies, and scattering phase shifts: a brief review

In this section, we first briefly introduce the scattering phase shift, the heat kernel, the one-loop effective action, and the vacuum energy. They are spectral functions of an operator. Once the operator is given, the above spectral functions are determined simultaneously.

2.1 Scattering phase shifts

We consider elastic scattering of a plane wave on a spherically symmetric potential.

If there is no scattering, namely $V(r) = 0$, the wave function is a plane wave [50, 51]

$$
\psi_0(r, \theta) = e^{ikr \cos \theta} = \sum_{l=0}^{\infty} (2l + 1) \frac{1}{2} \left[ h_l^{(2)}(kr) + h_l^{(1)}(kr) \right] P_l(\cos \theta);
$$

(2.1)

if there exists scattering, namely $V(r) \neq 0$, the wave function becomes [50, 51]

$$
\psi(r, \theta) = \sum_{l=0}^{\infty} (2l + 1) \frac{1}{2} \left[ h_l^{(2)}(kr) + e^{2i\delta_l} h_l^{(1)}(kr) \right] P_l(\cos \theta),
$$

(2.2)

where $h_l^{(1)}(kr)$, the first kind spherical Hankel function, describes the outgoing wave, and $h_l^{(2)}(kr)$, the second kind spherical Hankel function, describes the incoming wave. Comparing $\psi_0(r, \theta)$ and $\psi(r, \theta)$, we can see that the effect of scattering is to multiply the outgoing wave by a phase factor $e^{2i\delta_l}$. Scattering causes a phase shift $\delta_l$ on the outgoing wave function, called the scattering phase shift.

If the observer is far from the target, one can take the large-distance asymptotic approximation [50]:

$$
h_l^{(1,2)}(kr) \sim \pm (\mp i)^{l+1} e^{\pm ikr}.
$$

(2.3)

Under the large-distance asymptotic approximation, the incident plane wave (2.1) becomes

$$
\psi_0(r, \theta) = \sum_{l=0}^{\infty} (2l + 1) \frac{1}{2} \sin \left( kr - \frac{l\pi}{2} \right) P_l(\cos \theta)
$$

(2.4)

and the scattering wave (2.2) becomes

$$
\psi(r, \theta) = \sum_{l=0}^{\infty} (2l + 1) \frac{1}{2} e^{i\delta_l} \sin \left( kr - \frac{l\pi}{2} + \delta_l \right) P_l(\cos \theta).
$$

(2.5)

The corresponding radial wave function, under the large-distance asymptotic approximation, is

$$
R_{n,l}^0(r) \sim \frac{\sin(kr - l\pi/2)}{kr},
$$

(2.6)

and

$$
R_{n,l}(r) \sim \frac{\sin(kr - l\pi/2 + \delta_l(k))}{kr}.
$$

(2.7)
2.2 Global Heat kernels, one-loop effective actions, and vacuum energies

For a differential operator $D$, the spectral functions are defined by its eigenvalue spectrum $\{\lambda_n\}$. Formally, the global heat kernel is \[ K(t) = \sum_n e^{-\lambda_n t}, \] (2.8)

the one-loop effective action can be formally expressed as \[ W = \sum_n \ln \sqrt{\lambda_n}, \] (2.9)

and the vacuum energy can be formally expressed as \[ E_0 = \frac{1}{2} \sum_n \lambda_n. \] (2.10)

Physical operators, e.g., the Hamiltonian, are lower-bounded. The global heat kernel (2.8) is well-defined. Nevertheless, the one-loop effective action (2.9) and the vacuum energy (2.10) are obviously not well-defined, for they diverge for upper unbounded spectra. In order to obtain finite one-loop effective actions and vacuum energies, one introduces the regularized one-loop effective action and the regularized vacuum energy [53].

By inspection of the formal expressions of global heat kernels, one-loop effective actions, and vacuum energies, Eqs. (2.8), (2.9), and (2.10), we can see that they are related formally by the relations $W = -\frac{1}{2} \int_0^\infty K(t) \frac{1}{t} dt$ and $E_0 = \frac{1}{2} \frac{1}{\Gamma(-\frac{1}{2})} \int_0^\infty K(t) t^{-2} dt$. However, it is obvious that these two relations also diverge. To remove divergence, by use of the well-defined global heat kernel, one introduces the regularized one-loop effective action [3, 52]

\[ W(s) = -\frac{1}{2} \tilde{\mu}^{2s} \int_0^\infty K(t) t^{s-1} dt \] (2.11)

and the regularized vacuum energy

\[ E_0(\epsilon) = \frac{1}{2} \tilde{\mu}^{2\epsilon} \frac{1}{\Gamma(-\frac{1}{2} + \epsilon)} \int_0^\infty K(t) t^{-\frac{1}{2} + \epsilon - 1} dt, \] (2.12)

where $\tilde{\mu}$ is a constant of the dimension of mass introduced to keep proper dimension [3]. When $s = 0$ and $\epsilon = -1/2$, the regularized one-loop effective action and vacuum energy, $W(s)$ and $E_0(\epsilon)$, recover one-loop effective action and vacuum energy, $W$ and $E_0$, but, of course, such a substitution must undergo a reorganization process.

In this paper, we suggest an approach for calculating the heat kernel, the regularized one-loop effective action, and the regularized vacuum energy from the scattering phase shift.

3 Calculating global heat kernels, one-loop effective actions, and vacuum energies from scattering phase shifts

In the section, we calculate the global heat kernel, the regularized one-loop effective action, and the regularized vacuum energy from the scattering phase shift.
The local heat kernel $K(t; \mathbf{r}, \mathbf{r}')$ is the Green function of the initial-value problem for the differential operator $D = -\nabla^2 + V(r)$. The local partial-wave heat kernel $K^l_i(t; \mathbf{r}, \mathbf{r}')$ is the Green function of the initial-value problem for the radial operator $D_l = -\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) + \frac{l(l+1)}{r^2} + V(r)$. The global heat kernel $K(t)$ is the trace of $K(t; \mathbf{r}, \mathbf{r}')$ and the global partial-wave heat kernel $K^l_i(t)$ is the trace of $K_l(t; \mathbf{r}, \mathbf{r}')$.

The global partial-wave heat kernel can be divided into three parts: $K_l(t) = K^s_l(t) + K^r_l(t) + K^b_l(t)$, where $K^s_l(t)$ is the scattering-state global partial-wave heat kernel, $K^r_l(t)$ is the free-state global partial-wave heat kernel, and $K^b_l(t)$ is the bound-state global partial-wave heat kernel. The $n$-dimensional free-state global partial-wave heat kernel $K^r_l(t) = R/\sqrt{4\pi t} - (l + 1/2)/2$. The bound-state global partial-wave heat kernel $K^b_l(t)$ needs to work out the sum over all discrete bound states. In this paper, we care only about the scattering states, for most quantum-field-theory problems are related to scattering states. In Ref. [6], as a by-product of the discussion of the relation between the heat kernel method and the spectral method in quantum field theory, we find a relation between the partial-wave phase shift and the global partial-wave heat kernel of scattering states:

$$K^s_l(t) = \frac{2}{\pi} t \int_0^\infty \delta_l(k) e^{-k^2 t} k dk - \frac{\delta_l(0)}{\pi}. \quad (3.1)$$

Here, according to the Levinson theorem [54, 55], $\delta_l(0) = n_l \pi$ with $n_l$ the number of bound states with the angular momentum $l$. Note that if there exists the half-bound state (a half-bound state may occur only when $l = 0$), $\delta_l(0) = (n_l + \frac{1}{2}) \pi$. If only considering scattering states, $\delta_l(0)$ does not contribute. Often, the partial-wave heat kernel has no physical meaning but just an intermediate quantity.

The relation between the global heat kernel $K^s(t)$ and the global partial-wave heat kernel $K^s_l(t)$ of scattering states is

$$K^s(t) = \sum_{l=0}^\infty D_l K^s_l(t), \quad (3.2)$$

where $D_l$ is the degeneracy. Then, by Eqs. (3.1) and (3.2), we can obtain the relation between the global heat kernel and the scattering phase shift:

$$K^s(t) = \frac{t}{\pi} \sum_{l=0}^\infty D_l \int_0^\infty e^{-k^2 t} dk^2 \delta_l(k). \quad (3.3)$$

The relation between the regularized one-loop effective action and the global heat kernel can be obtained by Eqs. (2.11) and (3.3):

$$W^s(s) = -\frac{1}{2\pi} \tilde{\mu}^{2s} \int_0^\infty t^s dt \int_0^\infty e^{-k^2 t} dk^2 \sum_{l=0}^\infty D_l \delta_l(k)$$

$$= -\frac{1}{2\pi} \tilde{\mu}^{2s} \Gamma(s + 1) \sum_{l=0}^\infty D_l \int_0^\infty dk^2 \frac{\delta_l(k)}{(k^2)^{s+1}}. \quad (3.4)$$
The relation between the regularized vacuum energy and the global heat kernel can be obtained by Eqs. (2.12) and (3.3):

\[
E_0^\varepsilon = \frac{1}{2\pi} \hat{\mu}^2 \frac{1}{\Gamma \left(-\frac{1}{2} + \varepsilon\right)} \int_0^\infty t^{-\frac{1}{2} + \varepsilon} dt \int_0^\infty e^{-k^2 t} dk^2 \sum_{l=0}^\infty D_l \delta_l (k)
\]

\[
= \frac{1}{2\pi} \hat{\mu}^2 \Gamma \left(\frac{3}{2} + \varepsilon\right) \frac{\Gamma \left(-\frac{1}{2} + \varepsilon\right)}{\Gamma \left(-\frac{1}{2} + \varepsilon\right)} \sum_{l=0}^\infty D_l \int_0^\infty \frac{dk^2}{(k^2)^{1/2 + \varepsilon}}.
\]

The relations (3.4) and (3.5) convert the method for calculating scattering phase shifts in quantum mechanics to a method for calculating one-loop effective actions and vacuum energies in quantum field theory. That is, it converts a quantum-field-theory problem to a quantum-mechanical problem.

In Eqs. (3.3), (3.4), and (3.5), there is a sum should be worked out. In order to deal with this sum, in Appendix (A) we give some integral representations for the Bessel function.

4 The Born approximation: three-dimensional cases

The integral equation method is a fundamental approach for the scattering problem in quantum mechanics and for the perturbation theory in quantum field theory. It can also be applied to the scattering problem in curved spacetime [56, 57]. By using the Green function, the integral equation method converts the differential equation defined by the operator \( D \), e.g., the eigenequation of the Hamiltonian, into an integral equation. This integral equation can be expanded into a perturbation series by an iterative method. The leading-order contribution of the perturbation series is the Born approximation. The Born approximation is the most important and mature method in the perturbation theory of scattering.

In the following, we convert the Born approximation of calculating scattering phase shifts in quantum mechanics to a method of calculating global heat kernels, one-loop effective actions, and vacuum energies.

4.1 The first-order Born approximation

The first-order Born approximation of the scattering phase shift for a spherically symmetric potential \( V(r) \) in three dimensions is [58]

\[
\delta_l^{(1)} = -\frac{\pi}{2} \int_0^\infty J_{l+1/2}^2 (kr) V(r) r dr,
\]

where \( J_{l} (z) \) is the Bessel function.

4.1.1 Heat kernels

The first-order approximation of the heat kernel can be obtained by substituting Eq. (4.1) and the degeneracy in three dimensions, \( D_l = 2l + 1 \), into Eq. (3.3):

\[
K^{s(1)} (t) = -\frac{t}{2} \int_0^\infty V(r) r dr \int_0^\infty e^{-k^2 t} dk^2 \sum_{l=0}^\infty (2l + 1) J_{l+1/2}^2 (kr).
\]

\[
= 1 - \frac{1}{\pi} \int_0^\infty \frac{J_{l+1/2}^2 (kr) V(r) r dr}{\Gamma \left(-\frac{1}{2} + \varepsilon\right)} \sum_{l=0}^\infty D_l \delta_l (k).
\]
By the sum rule [59]

\[
\sum_{l=0}^{\infty} \frac{(2q + 2l) \Gamma(2q + l)}{\Gamma(l + 1)} J_{q+l}^2(z) = \frac{\Gamma(2q + 1)}{\Gamma(q + 1)^2} \left( \frac{z}{2} \right)^{2q},
\]

we arrive at

\[
\sum_{l=0}^{\infty} (2l + 1) J_{l+1/2}^2(kr) = \frac{2}{\pi} kr.
\]

Thus, we have

\[
K^{s(1)}(t) = -\frac{1}{2} \int_0^\infty V(r) r dr \int_0^\infty e^{-k^2 t} dk \frac{2kr}{\pi}.
\]

Working out the integral which is in fact a Laplace transform gives the first-order Born approximation for heat kernels:

\[
K^{s(1)}(t) = -\frac{1}{\sqrt{4\pi t}} \int_0^\infty V(r) r^2 dr.
\]

### 4.1.2 One-loop effective actions

The first-order one-loop effective action can be obtained by substituting the first-order phase shift (4.1) into Eq. (3.4):

\[
W^{(1)}(s) = \frac{1}{4} \tilde{\mu}^{2s} \Gamma(s + 1) \int_0^\infty V(r) r dr \int_0^\infty (k^2)^{-s-1} \left[ \sum_{l=0}^{\infty} (2l + 1) J_{l+1/2}^2(kr) \right] dk^2.
\]

Using the sum rule (4.4), we arrive at

\[
W^{(1)}(s) = \frac{1}{2\pi} \tilde{\mu}^{2s} \Gamma(s + 1) \int_0^\infty V(r) r^2 dr \int_0^\infty (k^2)^{-s-1} k dk^2.
\]

Here the integral of \( k \) may diverge. According to Ref. [30], we rewrite \( (k^2)^{-s-1} \) as \( (k^2 + m^2)^{-s-1} \):

\[
W^{(1)}(s) = \frac{1}{2\pi} \tilde{\mu}^{2s} \Gamma(s + 1) \int_0^\infty V(r) r^2 dr \int_0^\infty \left( k^2 + m^2 \right)^{-s-1} k dk^2.
\]

Working out the integral, we obtain the first-order Born approximation of the one-loop effective action:

\[
W^{(1)}(s) = \frac{\tilde{\mu}^{2s} \Gamma \left( s - \frac{3}{2} \right) (m^2)^{\frac{1}{2}-s}}{4\sqrt{\pi}} \int_0^\infty V(r) r^2 dr.
\]

### 4.1.3 Vacuum energies

The first-order vacuum energy can be obtained by substituting the first-order phase shift (4.1) into Eq. (3.5):

\[
E_0^{s(1)}(\epsilon) = -\frac{1}{4} \tilde{\mu}^{2s} \frac{\Gamma \left( \frac{1}{2} + \epsilon \right)}{\Gamma \left( -\frac{1}{2} + \epsilon \right)} \int_0^\infty V(r) r dr \int_0^\infty (k^2)^{-1/2-\epsilon} \left[ \sum_{l=0}^{\infty} (2l + 1) J_{l+1/2}^2(kr) \right] dk^2.
\]
Using the sum rule (4.4), we arrive at

$$E_0^{(1)} (\epsilon) = -\frac{1}{2\pi} \mu^2 \frac{\Gamma \left( \frac{1}{2} + \epsilon \right)}{\Gamma \left( \frac{1}{2} + \epsilon \right)} \int_0^\infty V (r) r^2 dr \int_0^\infty (k^2)^{-1/2-\epsilon} k dk. \quad (4.12)$$

The integral of $k$ may diverge. According to Ref. [30], we rewrite $(k^2)^{-1/2-\epsilon}$ as $(k^2 + m^2)^{-1/2-\epsilon}$:

$$E_0^{(1)} (\epsilon) = -\frac{1}{2\pi} \mu^2 \frac{\Gamma \left( \frac{1}{2} + \epsilon \right)}{\Gamma \left( \frac{1}{2} + \epsilon \right)} \int_0^\infty V (r) r^2 dr \int_0^\infty (k^2 + m^2)^{-1/2-\epsilon} k dk. \quad (4.13)$$

Working out the integral, we obtain the first-order Born approximation of the vacuum energy:

$$E_0^{(1)} (\epsilon) = -\frac{\mu^2}{4\sqrt{\pi} \Gamma \left( \frac{1}{2} + \epsilon \right)} (m^2)^{1-\epsilon} \int_0^\infty V (r) r^2 dr. \quad (4.14)$$

### 4.2 The second-order Born approximation

The second-order Born approximation of scattering phase shifts for a spherically symmetric potential $V (r)$ is [58]

$$\delta_l^{(2)} = -\int_0^\infty r^2 dr k j_l (kr) n_l (kr) V (r) \int_0^\infty r^2 dr' k j_{l'} (kr') V (r')$$

$$-\int_0^\infty r^2 dr k j_l (kr) V (r) \int_r^\infty r'^2 dr' k j_{l'} (kr') n_l (kr') V (r'), \quad (4.15)$$

where $j_{\nu} (z)$ and $n_{\nu} (z)$ are the spherical Bessel functions of the first kind and of the second kind, respectively.

#### 4.2.1 Heat kernels

The second-order approximation of the heat kernel can be obtained by substituting the second-order phase shift (4.15) into Eq. (3.3):

$$K^{(2)} (t) = -\frac{t}{\pi} \int_0^\infty e^{-k^2 t} k^2 dk \int_0^\infty r^2 dr V (r) \int_0^\infty r'^2 dr' V (r') \Sigma_1 (k; r, r')$$

$$-\frac{t}{\pi} \int_0^\infty e^{-k^2 t} k^2 dk \int_0^\infty r^2 dr V (r) \int_r^\infty r'^2 dr' V (r') \Sigma_2 (k; r, r'), \quad (4.16)$$

where

$$\Sigma_1 (k; r, r') = \sum_{l=0}^{\infty} (2l + 1) j_l (kr) n_l (kr) j_{l'}^2 (kr'), \quad (4.17)$$

$$\Sigma_2 (k; r, r') = \sum_{l=0}^{\infty} (2l + 1) j_l^2 (kr) j_l (kr') n_l (kr'). \quad (4.18)$$

In the following, we deal with the above sums.

To perform these sums, we give an integral representation of $j_l^2 (kr)$ in Appendix A.1:

$$j_l^2 (kr) = \frac{1}{2} \int_0^\pi \sin qr P_l (\cos \theta) d \cos \phi \quad (4.19)$$
and an integral representation of $j_l(kr) n_l(kr)$ in Appendix A.2:

$$j_l(kr) n_l(kr) = -\frac{1}{2} \int_0^\pi \frac{\cos qr}{qr} P_l(\cos \theta) \, d\cos \phi. \quad (4.20)$$

Substituting the above two integral representations into Eq. (4.17) gives

$$\Sigma_1 (k; r, r') = -\frac{1}{4} \int_0^\pi \frac{\cos qr}{qr} d\cos \theta \int_0^\pi \frac{\sin qr'}{qr'} d\cos \theta' \sum_{l=0}^\infty (2l + 1) P_l(\cos \theta) P_l(\cos \theta') , \quad (4.21)$$

where $q = 2k \sin \frac{\theta}{2}$ and $q' = 2k \sin \frac{\theta'}{2}$. Using the relation [60]

$$\sum_{l=0}^\infty (2l + 1) P_l(\cos \theta) P_l(\cos \theta') = 2 \delta (\cos \theta - \cos \theta') \quad (4.22)$$

and performing the integral, we have

$$\Sigma_1 (k; r, r') = -\frac{1}{2} \int_0^\pi \frac{\cos qr}{qr} d\cos \theta \int_0^\pi \frac{\sin qr'}{qr'} d\cos \theta' \delta (\cos \theta - \cos \theta')$$

$$= -\frac{1}{2} \int_0^\pi \frac{\cos qr \sin qr'}{qr'} d\cos \theta$$

$$= \text{Si} (2kr - 2kr') - \text{Si} (2kr + 2kr') \frac{4k^2rr'}{4k^2rr'}, \quad (4.23)$$

where $\text{Si} (z)$ is the Sine integral function.

Similarly, we obtain

$$\Sigma_2 (k; r, r') = -\frac{\text{Si} (2kr + 2kr') + \text{Si} (2kr - 2kr')}{4k^2rr'}. \quad (4.24)$$

Substituting Eqs. (4.23) and (4.24) into Eq. (4.16) gives the second-order global heat kernel:

$$K^{(2)}(t) = -\frac{t}{4\pi} \int_0^\infty rdrV(r) \int_0^\infty r' dr' V(r') \int_0^\infty e^{-k^2t} dk^2 [\text{Si} (2kr - 2kr') - \text{Si} (2kr + 2kr')]$$

$$+ \frac{t}{4\pi} \int_0^\infty rdrV(r) \int_r^\infty r' dr' V(r') \int_0^\infty e^{-k^2t} dk^2 [\text{Si} (2kr + 2kr') + \text{Si} (2kr - 2kr')]. \quad (4.25)$$

The integral of $k^2$ is a Laplace transform. Performing the Laplace transform gives the second-order Born approximation of heat kernels

$$K^{(2)}(t) = -\frac{1}{8} \int_0^\infty rdrV(r) \left\{ \int_0^r r' dr' V(r') \left[ \text{erf} \left( \frac{r - r'}{\sqrt{t}} \right) - \text{erf} \left( \frac{r + r'}{\sqrt{t}} \right) \right] \right. \right.$$  

$$- \left. \int_r^\infty r' dr' V(r') \left[ \text{erf} \left( \frac{r'}{\sqrt{t}} \right) + \text{erf} \left( \frac{r + r'}{\sqrt{t}} \right) \right] \right\}, \quad (4.26)$$

where $\text{erf} (z)$ is the Error function [60].
4.2.2 One-loop effective actions

We next calculate the second-order Born approximation of the one-loop effective action by directly using the relation between heat kernels and one-loop effective actions, Eq. (2.11), though in principle we can also obtain this result by substituting the second-order phase shift (4.15) into Eq. (3.4).

Substituting Eq. (4.26) into Eq. (2.11) gives the second-order Born approximation of the one-loop effective action

\[
W^{(2)} (s) = \frac{1}{16} \tilde{\mu}^2 \int_0^\infty r V (r) \left\{ \int_0^r r' dr' V (r') \left[ \operatorname{erf} \left( \frac{r - r'}{\sqrt{t}} \right) - \operatorname{erf} \left( \frac{r + r'}{\sqrt{t}} \right) \right] t^{s-1} dt \right. \\
\left. - \int_r^\infty r' dr' V (r') \int_0^\infty \left[ \operatorname{erf} \left( \frac{r - r'}{\sqrt{t}} \right) + \operatorname{erf} \left( \frac{r + r'}{\sqrt{t}} \right) \right] t^{s-1} dt \right\}.
\]

Performing the integral, we have

\[
W^{(2)} (s) = \frac{\tilde{\mu}^2 \Gamma \left( \frac{1}{2} - s \right)}{16 \sqrt{\pi s}} \int_0^\infty r V (r) \left\{ \int_0^r r' dr' V (r') \left[ (r - r')^{2s} - (r + r')^{2s} \right] \right. \\
\left. - \int_r^\infty r' dr' V (r') \left[ (r - r')^{2s} + (r + r')^{2s} \right] \right\}.
\]

4.2.3 Vacuum energies

Similarly, using the relation between the heat kernel and the vacuum energy, Eq. (2.12), we can obtain the second-order Born approximation of the vacuum energy. Substituting Eq. (4.26) into Eq. (2.12) gives

\[
E_0^{(2)} (\epsilon) = -\frac{\mu^2 \Gamma \left( 1 - \epsilon \right)}{16 \pi \Gamma \left( \epsilon + \frac{1}{2} \right)} \int_0^\infty r V (r) \left\{ \int_0^r r' dr' V (r') \left[ (r - r')^{2\epsilon - 1} + (r + r')^{2\epsilon - 1} \right] \right. \\
\left. - \int_r^\infty r' dr' V (r') \left[ (r - r')^{2\epsilon - 1} + (r + r')^{2\epsilon - 1} \right] \right\}.
\]

5 The Born approximation: n-dimensional cases

In this section, we consider the n-dimensional Born approximation. The n-dimensional results can be used to do the dimensional renormalization. For example, the dimensional renormalization can be used to remove the divergence in the Born approximation [50].

5.1 The first-order Born approximation

The first-order Born approximation of scattering phase shifts in n dimensions is [59]

\[
\delta_l^{(1)} (k) = -\frac{\pi}{2} \int_0^\infty J_{\frac{2}{2} + l-1}^2 (kr) V (r) r dr.
\]

5.1.1 Heat kernels

The first-order approximation of the heat kernel can be obtained by substituting Eq. (5.1) into Eq. (3.3):

\[
K^{(1)} (t) = -\frac{t}{2} \int_0^\infty V (r) r dr \int_0^\infty e^{-k^2 t} dk^2 \sum_{l=0}^\infty D_l J_{\frac{2}{2} + l-1}^2 (kr).
\]
The $n$-dimensional degeneracy for spherically symmetric potentials is [59]

$$D_l = \frac{(n + 2l - 2) \Gamma(n + l - 2)}{\Gamma(n - 1) \Gamma(l + 1)}.$$  \hspace{1cm} (5.3)

Taking $q = \frac{n}{2} - 1$ in the sum rule [59]

$$\sum_{l=0}^{\infty} \frac{(2q + 2l) \Gamma(2q + l)}{\Gamma(l + 1)} J_{2+l}^2(z) = \frac{\Gamma(2q + 1)}{\Gamma(q + 1)} \left(\frac{z}{2}\right)^{2q},$$  \hspace{1cm} (5.4)

gives

$$\sum_{l=0}^{\infty} \frac{\Gamma(n + l - 2) (n + 2l - 2)}{\Gamma(n - 1) \Gamma(l + 1)} J_{\frac{n}{2}+l-1}^2(kr) = \frac{1}{\Gamma \left(\frac{n}{2}\right)^2} \left(\frac{kr}{2}\right)^{n-2}. \hspace{1cm} (5.5)$$

Substituting Eq. (5.5) into Eq. (5.2) gives the first-order Born approximation of the heat kernel:

$$K_{s(1)}^n(t) = -\frac{t^2}{4} \int_0^\infty V(r) \Gamma(s + 1) \int_0^\infty \Gamma \left(\frac{n}{2}\right)^{2-s} \sum_{l=0}^{\infty} D_l J_{\frac{n}{2}+l-1}^2(kr). \hspace{1cm} (5.6)$$

Performing the Laplace transform, we arrive at the first-order Born approximation of the $n$-dimensional heat kernel:

$$K_{s(1)}^n(t) = -\frac{2^1-n}{\Gamma \left(\frac{n}{2}\right)^{\frac{n}{2}-1}} \int_0^\infty V(r) r^{n-1} dr. \hspace{1cm} (5.7)$$

### 5.1.2 One-loop effective actions

The first-order $n$-dimensional one-loop effective action can be obtained by substituting the first-order $n$-dimensional phase shift (5.1) into Eq. (3.4):

$$W_{s(1)}^n(s) = \frac{1}{4} \mu^2 \Gamma \left(\frac{s + 1}{2}\right) \int_0^\infty V(r) r dr \int_0^\infty d(k^2) \left(k^2\right)^{-s-1} \sum_{l=0}^{\infty} D_l J_{\frac{n}{2}+l-1}^2(kr). \hspace{1cm} (5.8)$$

Using the sum rule (5.5), we arrive at

$$W_{s(1)}^n(s) = \frac{1}{2^n \Gamma \left(\frac{n}{2}\right)^2} \mu^2 \Gamma \left(\frac{s + 1}{2}\right) \int_0^\infty V(r) r^{n-1} dr \int_0^\infty d(k^2) \left(k^2 + m^2\right)^{-s-1} k^{n-2}. \hspace{1cm} (5.9)$$

Here the integral of $k$ may diverge. According to Ref. [30], we rewrite \((k^2)^{-s-1}\) as \((k^2 + m^2)^{-s-1}\):

$$W_{s(1)}^n(s) = \frac{1}{2^n \Gamma \left(\frac{n}{2}\right)^2} \mu^2 \Gamma \left(\frac{s + 1}{2}\right) \int_0^\infty V(r) r^{n-1} dr \int_0^\infty d(k^2) \left(k^2 + m^2\right)^{-s-1} k^{n-2}. \hspace{1cm} (5.10)$$

Working out the integral, we obtain the first-order Born approximation of the one-loop effective action:

$$W_{s(1)}^n(s) = \frac{\mu^2 \Gamma \left(\frac{s + 1}{2}\right)}{2^n \Gamma \left(\frac{n}{2}\right)^2} \left(m^2\right)^{\frac{n}{2}-1-s} \int_0^\infty V(r) r^{n-1} dr. \hspace{1cm} (5.11)$$

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5.1.3 Vacuum energies

The first-order $n$-dimensional vacuum energy can be obtained by substituting the first-order $n$-dimensional phase shift (5.1) into Eq. (3.5):

$$E^{(1)}_0 (\epsilon) = -\frac{1}{4} \mu^2 \Gamma \left( \frac{1}{2} + \epsilon \right) \int_0^\infty V (r) \, dr \int_0^\infty dk^2 (k^2)^{-1/2-\epsilon} \sum_{l=0}^\infty D_l J_{2+l-1}^2 (kr). \quad (5.11)$$

Using the sum rule (5.5), we arrive at

$$E^{(1)}_0 (\epsilon) = -\frac{1}{2^n \Gamma \left( \frac{n}{2} \right)^2 \mu^2} \Gamma \left( \frac{1}{2} + \epsilon \right) \int_0^\infty V (r) \, dr \int_0^\infty dk^2 (k^2)^{-1/2-\epsilon} k^{n-2}. \quad (5.12)$$

The integral of $k$ may diverge. According to Ref. [30], we rewrite $(k^2)^{-1/2-\epsilon}$ as $(k^2 + m^2)^{-1/2-\epsilon}$:

$$E^{(1)}_0 (\epsilon) = -\frac{1}{2^n \Gamma \left( \frac{n}{2} \right)^2 \mu^2} \Gamma \left( \frac{1}{2} + \epsilon \right) \int_0^\infty V (r) \, dr \int_0^\infty dk^2 (k^2 + m^2)^{-1/2-\epsilon} k^{n-2}. \quad (5.13)$$

Working out the integral, we obtain the first-order Born approximation of the $n$-dimensional vacuum energy:

$$E^{(1)}_0 (\epsilon) = -\frac{1}{2^n \Gamma \left( \frac{n}{2} \right)^2 \mu^2} \Gamma \left( \epsilon + \frac{1}{2} - \frac{n}{2} \right) \int_0^\infty V (r) \, dr \int_0^\infty dk^2 (m^2)^{-\epsilon} k^{n-2}. \quad (5.14)$$

It should be noted that in performing the integral in the Born approximation, one may encounter divergences which are not coming from the usual divergence in quantum field theory. Such divergences can be removed by the procedure given in Ref. [50].

5.2 The second-order Born approximation

The $n$-dimensional second-order Born approximation of the scattering phase shift for a spherically symmetric potential $V (r)$ is [58]

$$\delta^{(2)}_l = -\frac{\pi^2}{4} \int_0^\infty J_{2+l-1}^2 (kr) Y_{2+l-1}^2 (kr) V (r) \, dr \int_0^r J_{2+l-1}^2 (kr') V (r') \, dr' \left( J_{2+l-1}^2 (kr') \right)^2$$

$$-\frac{\pi^2}{4} \int_0^\infty J_{2+l-1}^2 (kr) V (r) \, dr \int_r^\infty J_{2+l-1}^2 (kr') Y_{2+l-1} (kr') V (r') \, dr', \quad (5.15)$$

where $Y_{\nu} (z)$ is the Bessel function of the second kind.

5.2.1 Heat kernels

The second-order approximation of the $n$-dimensional heat kernel can be obtained by substituting the second-order phase shift (5.15) into Eq. (3.3):

$$K^{s(2)} (t) = -\frac{\pi t}{4} \int_0^\infty e^{-k^2t} dk^2 \int_0^\infty V (r) \, dr \int_0^r V (r') \, dr' \Sigma_1 (k; r, r')$$

$$-\frac{\pi t}{4} \int_0^\infty e^{-k^2t} dk^2 \int_0^\infty V (r) \, dr \int_r^\infty V (r') \, dr' \Sigma_2 (k; r, r'). \quad (5.16)$$
where

\[ \Sigma_1 (k; r, r') = \sum_{l=0}^{\infty} \frac{(n + 2l - 2) \Gamma (n + l - 2)}{\Gamma (n - 1) \Gamma (l + 1)} J_{l+1}^2 (kr) Y_{l+1}^2 (kr) J_{l+1}^2 (kr') , \quad (5.17) \]

\[ \Sigma_2 (k; r, r') = \sum_{l=0}^{\infty} \frac{(n + 2l - 2) \Gamma (n + l - 2)}{\Gamma (n - 1) \Gamma (l + 1)} J_{l+1}^2 (kr) J_{l+1}^2 (kr') Y_{l+1}^2 (kr') . \quad (5.18) \]

To perform these sums, we give an integral representation of \( J_{l+1}^2 (kr) \) in Appendix A.3:

\[ J_{l+1}^2 (kr) = \frac{2^{\mu-1} \Gamma (l + 1) \Gamma (\mu)}{\pi \Gamma (2\mu + l)} (kr)^{2\mu} \int_0^\pi J_{\mu} (qr) C_{2}^{\mu} (\cos \theta) \sin^{2\mu} \theta d\theta \]

(5.19)

and an integral representation of \( J_{l+1}^2 (kr) Y_{l+1}^2 (kr) \) in Appendix A.4:

\[ J_{l+1}^2 (kr) Y_{l+1}^2 (kr) = \frac{2^{\mu-1} \Gamma (l + 1) \Gamma (\mu)}{\pi \Gamma (2\mu + l)} (kr)^{2\mu} \int_0^\pi J_{\mu} (qr') C_{2}^{\mu} (\cos \theta') \sin^{2\mu} \theta' d\theta' \]

(5.20)

where \( C_{2}^{\mu} (\cos \theta) \) is the Gegenbauer polynomial. Substituting the above two integral representations into Eq. (5.17) gives

\[ \Sigma_1 (k; r, r') = \sum_{l=0}^{\infty} \frac{(2\mu + 2l) \Gamma (2\mu + l)}{\Gamma (2\mu + 1) \Gamma (l + 1)} \int_0^{\pi} J_{\mu} (qr) C_{2}^{\mu} (\cos \theta) \sin^{2\mu} \theta d\theta \]

\[ \times \frac{2^{\mu-1} \Gamma (l + 1) \Gamma (\mu)}{\pi \Gamma (2\mu + l)} (kr)^{2\mu} \int_0^\pi J_{\mu} (qr') C_{2}^{\mu} (\cos \theta') \sin^{2\mu} \theta' d\theta' \]

\[ = \frac{2^{2(\mu-1)} \Gamma^2 (\mu)}{\pi^2 \Gamma (2\mu + 1)} (kr)^{2\mu} (kr')^{2\mu} \int_0^{\pi} J_{\mu} (qr) C_{2}^{\mu} (\cos \theta) \sin^{2\mu} \theta d\theta \int_0^{\pi} J_{\mu} (qr') C_{2}^{\mu} (\cos \theta') \sin^{2\mu} \theta' d\theta' \]

\[ \times \sum_{l=0}^{\infty} \frac{(2\mu + 2l) \Gamma (2\mu + l)}{\Gamma (2\mu + 1) \Gamma (l + 1)} C_{l}^{\mu} (\cos \theta) C_{l}^{\mu} (\cos \theta') , \quad (5.21) \]

where \( \mu = \frac{\nu}{2} - 1 \), \( q = 2k \sin \frac{\theta}{2} \), and \( q' = 2k \sin \frac{\theta'}{2} \). Using the relation [60]

\[ \sum_{l=0}^{\infty} \frac{\Gamma (l + 1) (2\mu + 2l)}{\Gamma (2\mu + l)} C_{l}^{\mu} (\cos \theta) C_{l}^{\mu} (\cos \theta') = \frac{2^{2-2\mu} \Gamma (\mu)}{\Gamma^2 (\mu)} (\sin \theta)^{\frac{1-2\mu}{2}} (\sin \theta')^{\frac{1-2\mu}{2}} \delta (\cos \theta - \cos \theta') , \]

(5.22)

we arrive at

\[ \Sigma_1 (k; r, r') = \frac{(kr)^{2\mu} (kr')^{2\mu}}{\pi \Gamma (2\mu + 1)} \int_0^{\pi} J_{\mu} (qr) \sin^{2\mu-1} \theta d\cos \theta \int_0^{\pi} J_{\mu} (qr') \sin^{2\mu-1} \theta' d\cos \theta' \]

\[ = \frac{(kr)^{2\mu} (kr')^{2\mu}}{\pi \Gamma (2\mu + 1)} \int_0^{\pi} J_{\mu} (qr) J_{\mu} (qr') \sin^{2\mu-1} \theta d\cos \theta . \quad (5.23) \]

Similarly, we have

\[ \Sigma_2 (k; r, r') = \frac{(kr)^{2\mu} (kr')^{2\mu}}{\pi \Gamma (2\mu + 1)} \int_0^{\pi} J_{\mu} (qr) J_{\mu} (qr') \sin^{2\mu-1} \theta d\cos \theta . \quad (5.24) \]
Substituting Eqs. (5.23) and (5.24) into Eq. (5.16) gives the n-dimensional second-order global heat kernel:

\[
K^{(2)}(t) = -\frac{t}{4\pi(n-1)} \int_0^\infty e^{-\frac{r^2}{4t}} dr V(r) (kr)^{n-2} \times \left\{ \int_0^r r'dr' V(r') \left[ (kr')^{n-2} \int_0^\pi Y_{n-1}^{\frac{n-1}{2}}(qr') J_{\frac{n-1}{2}}(kr') \frac{\sin \theta \cos \theta}{(q r')^{\frac{n-1}{2}-1}} \right] + \int_r^\infty r'dr' V(r') \left[ (kr')^{n-2} \int_0^\pi Y_{n-1}^{\frac{n-1}{2}}(qr') J_{\frac{n-1}{2}}(kr') \frac{\sin 2\mu \theta \cos \theta}{(q r')^{\frac{n-1}{2}-1}} \right] \right\}.
\]  

(5.25)

The integral encountered in Eq. (5.25) is difficult. The odd-dimensional case and even-dimensional case are very different [61], for the odd-dimensional Bessel polynomial is a polynomial but the even-dimensional case is not [50]. In the following we only consider the odd-dimensional case.

For odd-dimensional cases, the integral representation given in Appendix A.4 with \( \mu = \frac{1}{2} \) and \( l = \frac{n}{2} - \frac{1}{2} \) \( (n = 3, 5, 7, \ldots) \), becomes

\[
Y_{\frac{n}{2}-1}(qr) J_{\frac{n}{2}-1}(qr') = \frac{q\sqrt{rr'}}{\sqrt{2\pi}} \int_0^\pi \frac{Y_{1/2}}{(q \sqrt{r^2 + r'^2 - 2rr' \cos \phi})^{1/2}} P_{\frac{n}{2}-\frac{1}{2}}(\cos \phi) d\cos \phi = -\frac{q\sqrt{rr'}}{\pi} \int_0^\pi \frac{\cos (q \sqrt{r^2 + r'^2 - 2rr' \cos \phi})}{P_{\frac{n}{2}-\frac{1}{2}}(\cos \phi)} P_{\frac{n}{2}-\frac{1}{2}}(\cos \phi) d\cos \phi.
\]

(5.26)

Then the integral over \( \theta \) in Eq. (5.25) reads

\[
\int_0^\pi \frac{Y_{\frac{n}{2}-1}(2kr \sin \frac{\theta}{2}) J_{\frac{n}{2}-1}(2kr' \sin \frac{\theta}{2})}{(2kr \sin \frac{\theta}{2})^{\frac{n-1}{2}}(2kr' \sin \frac{\theta}{2})^{\frac{n-1}{2}}} \sin^{n-3} \theta d\cos \theta
= \int_0^\pi \frac{1}{(2kr \sin \frac{\theta}{2})^{n-1}(2kr' \sin \frac{\theta}{2})^{\frac{n-1}{2}}}
\times \left\{ -\frac{q\sqrt{rr'}}{\pi} \int_0^\pi \frac{\cos (q \sqrt{r^2 + r'^2 - 2rr' \cos \phi})}{P_{\frac{n}{2}-\frac{1}{2}}(\cos \phi)} P_{\frac{n}{2}-\frac{1}{2}}(\cos \phi) d\cos \phi \right\} \sin^{n-3} \theta d\cos \theta
\]

(5.27)

with \( q = 2k \sin \frac{\theta}{2} \). We first perform the integral over \( \theta \):

\[
\int_0^\pi \frac{\cos (2k \sin \frac{\theta}{2} \sqrt{r^2 + r'^2 - 2rr' \cos \phi})}{(\sin \frac{\theta}{2})^{n-2}} \sin^{n-3} \theta d\cos \theta
= -2^{n-2} \sqrt{\pi} \Gamma \left( \frac{n}{2} - \frac{1}{2} \right) \frac{J_{\frac{n}{2}-1}(2k \sqrt{r^2 + r'^2 - 2rr' \cos \phi})}{(k \sqrt{r^2 + r'^2 - 2rr' \cos \phi})^{\frac{n}{2}-1}}.
\]

(5.28)
Then we have

\[
\int_0^\infty \frac{Y_{n-1}^2 (qr') J_{n-1}^2 (qr)}{(qr)^{n-1} (qr')^{n-1}} \sin^{n-3} \theta d\cos \theta
\]

\[
= \frac{2^\frac{n}{2} \Gamma \left( \frac{n}{2} - \frac{1}{2} \right)}{\pi^{\frac{n}{2} - \frac{1}{2}} (rr')^{\frac{n}{2} - \frac{1}{2}}} \int_0^\pi \frac{J_{n-1}^2 \left( 2k \sqrt{r^2 + r'^2 - 2rr' \cos \phi} \right)}{\left( 2k \sqrt{r^2 + r'^2 - 2rr' \cos \phi} \right)^{\frac{n}{2}} \frac{P_{\frac{n}{2} - \frac{1}{2}} \left( \cos \phi \right)}{d\cos \theta}}.
\] (5.29)

Eq. (5.25), by Eq. (5.29), becomes

\[
K^{(2)} \left( t \right) = -\frac{t}{4\sqrt{\pi} \Gamma \left( \frac{n}{2} - \frac{1}{2} \right)} \int_0^\infty \frac{r^{n+1}}{dV (r)}
\times \left\{ \int_0^r \left( r' \right)^{n+1} dV (r') \int_0^\pi \left[ \int_0^\infty e^{-k^2 t k n-1} d k^2 \frac{J_{n-1} \left( 2k \sqrt{r^2 + r'^2 - 2rr' \cos \phi} \right)}{\left( k \sqrt{r^2 + r'^2 - 2rr' \cos \phi} \right)^{\frac{n}{2}}} \right] \right\}
\times P_{\frac{n}{2} - \frac{1}{2}} \left( \cos \phi \right) \cos \theta.
\] (5.30)

Performing the integral over \( k \), which is indeed a Laplace transform,

\[
K^{(2)} \left( t \right) = -\frac{1}{2 \sqrt{2\pi t^{\frac{n}{2} - \frac{1}{2}}} \Gamma \left( \frac{n}{2} - \frac{1}{2} \right)} \int_0^\infty \frac{r^{n+1}}{dV (r)}
\times \left\{ \int_0^r \left( r' \right)^{n+1} dV (r') \int_0^\pi K_{1/2} \left( \frac{r^2 + r'^2 - 2rr' \cos \phi}{t} \right) P_{\frac{n}{2} - \frac{1}{2}} \left( \cos \phi \right) \cos \phi \right\}.
\] (5.31)

By \( \sqrt{r^2 + r'^2 - 2rr' \cos \phi} = |r - r'| \), we rewrite

\[
K^{(2)} \left( t \right) = -\frac{1}{2 \sqrt{2\pi t^{\frac{n}{2} - \frac{1}{2}}} \Gamma \left( \frac{n}{2} - \frac{1}{2} \right)} \int_0^\infty \frac{r^{n+1}}{dV (r)}
\times \left\{ \int_0^r \left( r' \right)^{n+1} dV (r') \int_0^\pi K_{1/2} \left( \frac{(r - r')^2}{t} \right) P_{\frac{n}{2} - \frac{1}{2}} \left( \cos \phi \right) \cos \phi \right\}.
\] (5.32)
or, equivalently,

\[
K_{s(2)}(t) = -\frac{1}{4\sqrt{\pi t^{n-1}}} \frac{\Gamma \left( \frac{n}{2} - \frac{1}{2} \right)}{\Gamma(n-1)} \int_0^\infty \frac{r^{n-1}}{\sqrt{\pi t^{n-1}}} dr V(r) \\
\times \left\{ \int_0^r (r')^{n-1} dr' V(r') \int_0^\pi \frac{\exp \left( -\frac{(r-r')^2}{2t} \right)}{\sqrt{t}} P_{\frac{n}{2} - \frac{3}{2}} (\cos \phi) d\cos \phi \right. \\
+ \int_r^\infty (r')^{n-1} dr' V(r') \int_0^\pi \frac{\exp \left( -\frac{(r-r')^2}{2t} \right)}{\sqrt{t}} P_{\frac{n}{2} - \frac{3}{2}} (\cos \phi) d\cos \phi \left\} ,
\]

(5.33)

where \( K_v(z) \) is the modified Bessel function of the second kind [60].

In the heat-kernel theory, one often concentrates on the small \( t \) case, e.g., the Seeley-DeWitt expansion [3]. For small \( t \), we have the following expansion:

\[
K_{1/2} \left( \frac{(r-r')^2}{t} \right) \sim \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \phi}} + \cdots .
\]

(5.34)

Substituting into Eq. (5.32) gives

\[
K_{s(2)}(t) = -\frac{1}{4\sqrt{\pi t^{n-1}}} \frac{\Gamma \left( \frac{n}{2} - \frac{1}{2} \right)}{\Gamma(n-1)} \int_0^\infty \frac{r^{n-1}}{\sqrt{\pi t^{n-1}}} dr V(r) \\
\times \left\{ \int_0^r (r')^{n-1} dr' V(r') \int_0^\pi \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \phi}} P_{\frac{n}{2} - \frac{3}{2}} (\cos \phi) d\cos \phi \right. \\
+ \int_r^\infty (r')^{n-1} dr' V(r') \int_0^\pi \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \phi}} P_{\frac{n}{2} - \frac{3}{2}} (\cos \phi) d\cos \phi \left\} .
\]

(5.35)

By using [60]

\[
\begin{align*}
\frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \phi}} &= \frac{1}{r} \sum_{l=0}^\infty \left( \frac{r'}{r} \right)^l P_l (\cos \phi) , \quad r > r', \\
\frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \phi}} &= \frac{1}{r'} \sum_{l=0}^\infty \left( \frac{r}{r'} \right)^l P_l (\cos \phi) , \quad r < r'.
\end{align*}
\]

(5.36)

Note that in Eq. (5.35), the first term corresponds to \( r > r' \) and the second term corresponds to \( r < r' \). Then

\[
\int_0^\pi \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \phi}} P_{\frac{n}{2} - \frac{3}{2}} (\cos \phi) d\cos \phi = \frac{1}{r} \sum_{l=0}^\infty \left( \frac{r'}{r} \right)^l \int_0^\pi P_l (\cos \phi) P_{\frac{n}{2} - \frac{3}{2}} (\cos \phi) d\cos \phi \\
= \frac{1}{r} \sum_{l=0}^\infty \left( \frac{r'}{r} \right)^l \frac{2}{2l + 1} \delta_{l, \frac{n}{2} - \frac{3}{2}} \\
= \frac{2}{n - 2} \frac{1}{r} \left( \frac{r'}{r} \right)^{\frac{n}{2} - \frac{3}{2}}, \quad (r > r').
\]

(5.37)
Similarly,
\[
\int_{0}^{\pi} \frac{1}{\sqrt{r'^2 + r^2 - 2rr' \cos \phi}} P_{\frac{n}{2} - \frac{1}{2}} (\cos \phi) \, d \cos \phi = \frac{2}{n - 2} r' \left( \frac{r}{r'} \right)^{\frac{n}{2} - \frac{1}{2}}, \quad (r < r').
\] (5.38)

The second-order heat kernel in odd dimensions, by (5.35), reads
\[
K^{(2)} (t) = -\frac{1}{2(n - 2)} \sqrt{\pi t^{n-1}} \frac{\Gamma \left( \frac{n}{2} - \frac{1}{2} \right)}{\Gamma(n - 1)} \times \left[ \int_{0}^{r} r' \, dr' \, V (r') \int_{0}^{\pi} (r'^n - 1) \, dr' \, V (r') + \int_{0}^{r'} r'^n - 1 \, dr' \, V (r) \int_{r}^{\infty} r' \, dr' \, V (r') \right].
\] (5.39)

### 5.2.2 One-loop effective actions

Next by using the relation between the global heat kernel and the one-loop effective action, Eq. (2.11), and substituting Eq. (5.25) into Eq. (2.11), we obtain the second-order \(n\)-dimensional one-loop effective action:
\[
W^{(2)} (s) = \frac{\pi}{8} n^{2s} \int_{0}^{\infty} r \, dr \, V (r)
\times \left\{ \int_{0}^{r} r' \, dr' \, V (r') \int_{0}^{\infty} dk^2 \left( \frac{(kr)^{n-2} (kr')^{n-2}}{\pi \Gamma (2 \mu + 1)} \int_{0}^{\pi} \frac{Y_{\frac{n}{2} - 1} (qr)}{(qr')^{\frac{n}{2} - 1}} \sin^{n-3} \theta d \cos \theta \right) \right. \right.
\times \left. \int_{0}^{\infty} e^{-k^2 t \epsilon} \, dt \right.
\times \left. \int_{r}^{\infty} r' \, dr' \, V (r') \int_{0}^{\infty} dk^2 \left( \frac{(kr)^{n-2} (kr')^{n-2}}{\pi \Gamma (2 \mu + 1)} \int_{0}^{\pi} \frac{Y_{\frac{n}{2} - 1} (qr)}{(qr')^{\frac{n}{2} - 1}} \sin^{n-3} \theta d \cos \theta \right) \right.
\times \left. \int_{0}^{\infty} e^{-k^2 t \epsilon} \, dt \right\}. \] (5.40)

Performing the integral over \(t\), we have
\[
W^{(2)} (s) = \frac{\pi}{8} n^{2s} \Gamma (s + 1) \int_{0}^{\infty} r \, dr \, V (r)
\times \left\{ \int_{0}^{r} r' \, dr' \, V (r') \int_{0}^{\infty} (k^2)^{-s-1} \, dk^2 \left( \frac{(kr)^{n-2} (kr')^{n-2}}{\pi \Gamma (2 \mu + 1)} \int_{0}^{\pi} \frac{Y_{\frac{n}{2} - 1} (qr)}{(qr')^{\frac{n}{2} - 1}} \sin^{n-3} \theta d \cos \theta \right) \right. \right.
\times \left. \int_{0}^{\infty} e^{-k^2 t \epsilon} \, dt \right.
\times \left. \int_{r}^{\infty} r' \, dr' \, V (r') \int_{0}^{\infty} (k^2)^{-s-1} \, dk^2 \left( \frac{(kr)^{n-2} (kr')^{n-2}}{\pi \Gamma (2 \mu + 1)} \int_{0}^{\pi} \frac{Y_{\frac{n}{2} - 1} (qr)}{(qr')^{\frac{n}{2} - 1}} \sin^{n-3} \theta d \cos \theta \right) \right.
\times \left. \int_{0}^{\infty} e^{-k^2 t \epsilon} \, dt \right\}. \] (5.41)

For odd-dimensional cases, substituting Eq. (5.33) into Eq. (2.11) gives the second-
order Born approximation of the one-loop effective action:

\[
W^{(2)} (s) = \frac{\tilde{\mu}^{2s}}{8\sqrt{\pi}} \frac{\Gamma \left( \frac{n}{2} - \frac{1}{2} \right)}{\Gamma (n-1)} \int_0^\infty r^{n+1} r^\frac{1}{2} dr V (r) \\
\times \left\{ \int_0^r (r') \frac{n+1}{2} dr' V (r') \int_0^\pi P_{\frac{n}{2} - \frac{3}{2}} (\cos \phi) d \cos \phi \\
\times \int_0^\infty \frac{dtt^s-1}{t} \exp \left( -\frac{1}{4} \left( r^2 + r'^2 - 2rr' \cos \phi \right) \right) \int_0^\pi P_{\frac{n}{2} - \frac{3}{2}} (\cos \phi) d \cos \phi \\
+ \int_r^\infty (r') \frac{n+1}{2} dr' V (r') \int_0^\pi P_{\frac{n}{2} - \frac{3}{2}} (\cos \phi) d \cos \phi \\
\times \int_0^\infty \frac{dtt^s-1}{t} \exp \left( -\frac{1}{4} \left( r^2 + r'^2 - 2rr' \cos \phi \right) \right) \right\}. \tag{5.42}
\]

Performing the integral over \(t\) gives

\[
W^{(2)} (s) = \frac{\tilde{\mu}^{2s}}{8\sqrt{\pi}} \frac{\Gamma \left( \frac{n}{2} - s \right) \Gamma \left( \frac{n}{2} - \frac{1}{2} \right)}{\Gamma (n-1)} \int_0^\infty r^{n+1} r^\frac{1}{2} dr V (r) \\
\times \left\{ \int_0^r (r') \frac{n+1}{2} dr' V (r') \int_0^\pi P_{\frac{n}{2} - \frac{3}{2}} (\cos \phi) d \cos \phi \\
+ \int_r^\infty (r') \frac{n+1}{2} dr' V (r') \int_0^\pi P_{\frac{n}{2} - \frac{3}{2}} (\cos \phi) d \cos \phi \right\}. \tag{5.43}
\]

Around \(s = 0\), for \(n \neq 1\), we have

\[
W^{(2)} = \frac{1}{16} \frac{\Gamma \left( \frac{n}{2} - \frac{1}{2} \right)}{\Gamma (n-1)} \int_0^\infty r^{n+1} r^\frac{1}{2} dr V (r) \left\{ \int_0^r (r') \frac{n+1}{2} dr' V (r') \int_0^\pi P_{\frac{n}{2} - \frac{3}{2}} (\cos \phi) \right\} \\
\times \int_0^\infty \frac{dtt^s-1}{t} \exp \left( -\frac{1}{4} \left( r^2 + r'^2 - 2rr' \cos \phi \right) \right) d \cos \phi \\
+ \int_r^\infty (r') \frac{n+1}{2} dr' V (r') \int_0^\pi P_{\frac{n}{2} - \frac{3}{2}} (\cos \phi) \right\} d \cos \phi. \tag{5.44}
\]

Next we perform the angle integral. Rewriting the angle integral as

\[
\int_0^\pi \frac{P_{\frac{n}{2} - \frac{3}{2}} (\cos \phi)}{(r^2 + r'^2 - 2rr' \cos \phi)^2} d \cos \phi = \frac{1}{4 (rr')^2} \int_0^\pi \frac{P_{\frac{n}{2} - \frac{3}{2}} (\cos \phi)}{(R - \cos \phi)^2} d \cos \phi, \tag{5.45}
\]

where \(R = \frac{r^2 + r'^2}{2rr'}\). It can be checked that

\[
\frac{1}{(R - \cos \phi)^2} = -\frac{d}{dR} \frac{1}{R - \cos \phi}, \tag{5.46}
\]

\[\text{--- 19 ---}\]
so the integral (5.45) becomes

\[
\int_0^{\pi} \frac{P_{\frac{n}{2} - \frac{3}{2}}(\cos \phi)}{(r^2 + r'^2 - 2rr' \cos \phi)^2} d\cos \phi
\]
\[
= -\frac{1}{4(r_\perp r_\parallel)^2} \int_0^{\pi} \frac{P_{\frac{n}{2} - \frac{3}{2}}(\cos \phi)}{R - \cos \phi} d\cos \phi
\]
\[
= -\frac{1}{2(r_\perp r_\parallel)^2} \frac{d}{dR} Q_{\frac{n}{2} - \frac{3}{2}}(R)
\]
\[
= -2 \frac{n - \frac{3}{2}}{r_\perp r_\parallel} Q_{\frac{n}{2} - \frac{1}{2}} \left( \frac{r^2 + r'^2}{2rr'} \right) - \frac{r^2 + r'^2}{2rr'} \left( \frac{n - \frac{1}{2}}{r_\perp r_\parallel} \right) Q_{\frac{n}{2} - \frac{3}{2}} \left( \frac{r^2 + r'^2}{2rr'} \right),
\]

(5.47)

where \( Q_n(z) \) is the Legendre function of the second kind: \( \mathbb{P} \int_0^{\pi} \frac{1}{r - \cos \phi} P_{\frac{n}{2} - \frac{3}{2}}(\cos \phi) d\cos \phi = 2Q_{\frac{n}{2} - \frac{3}{2}}(R) \) [60]. Then the second-order one-loop effective action reads

\[
W^{(2)} = -\frac{1}{8} \frac{\Gamma \left( \frac{n}{2} - \frac{3}{2} \right)}{\Gamma (n - 1)} \int_0^{\infty} r^{\frac{n-1}{2}} dr V(r)
\]
\[
\times \left\{ \int_0^{r} (r')^{\frac{n-1}{2}} dr' V(r') \left( \frac{n - \frac{1}{2}}{r_\perp r_\parallel} Q_{\frac{n}{2} - \frac{1}{2}} \left( \frac{r^2 + r'^2}{2rr'} \right) - \frac{r^2 + r'^2}{2rr'} \left( \frac{n - \frac{1}{2}}{r_\perp r_\parallel} \right) Q_{\frac{n}{2} - \frac{3}{2}} \left( \frac{r^2 + r'^2}{2rr'} \right) \right) \right.
\]
\[
+ \left. \int_r^{\infty} (r')^{\frac{n-1}{2}} dr' V(r') \left( \frac{n - \frac{1}{2}}{r_\perp r_\parallel} Q_{\frac{n}{2} - \frac{1}{2}} \left( \frac{r^2 + r'^2}{2rr'} \right) - \frac{r^2 + r'^2}{2rr'} \left( \frac{n - \frac{1}{2}}{r_\perp r_\parallel} \right) Q_{\frac{n}{2} - \frac{3}{2}} \left( \frac{r^2 + r'^2}{2rr'} \right) \right) \right\}.
\]

(5.48)

5.2.3 Vacuum energies

Similarly, using the relation between the heat kernel and the vacuum energy, Eq. (2.12), we can obtain the second-order \( n \)-dimensional Born approximation of the vacuum energy. Substituting Eq. (5.25) into Eq. (2.12) gives

\[
E_0^{(2)}(\epsilon) = -\frac{\pi \mu^2}{8 \Gamma (-\frac{1}{2} + \epsilon)} \int_0^{\infty} r dr V(r)
\]
\[
\times \left\{ \int_0^{r} r' dr' V(r') \int_0^{\infty} dk^2 \left[ \frac{(kr)_n^{-2}(kr')_n^{-2}}{\pi \Gamma (2\mu + 1)} \int_0^{\pi} \frac{Y_{\frac{n}{2} - 1} (qr) \frac{J_{\frac{n}{2} - 1} (qr')}{(qr')^{\frac{1}{2} - 1}} \sin^{n-3} \theta d\cos \theta \right] \right.
\]
\[
\times \int_0^{\infty} e^{-k^2 t - \frac{1}{2} + \epsilon} dt
\]
\[
+ \int_r^{\infty} r' dr' V(r') \int_0^{\infty} dk^2 \left[ \frac{(kr)_n^{-2}(kr')_n^{-2}}{\pi \Gamma (2\mu + 1)} \int_0^{\pi} \frac{Y_{\frac{n}{2} - 1} (qr) \frac{J_{\frac{n}{2} - 1} (qr')}{(qr')^{\frac{1}{2} - 1}} \sin^{n-3} \theta d\cos \theta \right] \right.
\]
\[
\times \int_0^{\infty} e^{-k^2 t - \frac{1}{2} + \epsilon} dt \right\}.
\]

(5.49)
Performing the integral, we have

\[
E_0^{(2)} (\epsilon) = - \frac{\pi \tilde{\mu}^{2 \epsilon}}{8 \sqrt{\pi} \Gamma(\frac{1}{2} + \epsilon) \Gamma(\frac{1}{2} - \epsilon)} \int_0^\infty r dr V(r) \frac{1}{(k^2)^{\frac{1}{2} - \epsilon}} \int_0^\infty (kr)^{n-2} dk^2 \left[ \frac{(kr)^{n-2} (kr')^{n-2}}{\pi \Gamma(2 \mu + 1)} \frac{Y_{\mu-1} (qr) J_{\mu-1} (qr')}{(qr)^{\frac{n}{2}-1}} \sin^{n-3} \theta d \cos \theta \right] \times \left\{ \int_0^r (r')^{\frac{n+1}{2}} dr' V(r') \int_0^\infty \frac{P_{\frac{n}{2}-\frac{1}{2}} (\cos \phi)}{(r^2 + r'^2 - 2rr' \cos \phi)^{\frac{n}{2}}} \right. \\
+ \int_r^\infty (r')^{\frac{n+1}{2}} dr' V(r') \int_0^\infty \frac{P_{\frac{n}{2}-\frac{1}{2}} (\cos \phi)}{(r^2 + r'^2 - 2rr' \cos \phi)^{\frac{n}{2}}} \left\}.
\]

For odd-dimensional cases, substituting Eq. (5.33) into Eq. (2.12) gives the odd-dimensional second-order Born approximation of the vacuum energy:

\[
E_0^{(2)} (\epsilon) = - \frac{\mu^{2 \epsilon}}{8 \sqrt{\pi} \Gamma(\frac{1}{2} + \epsilon) \Gamma(\frac{1}{2} - \epsilon)} \int_0^\infty r^{\frac{n+1}{2}} dr V(r) \frac{1}{(k^2)^{\frac{1}{2} - \epsilon}} \int_0^\infty (kr)^{n-2} dk^2 \left[ \frac{(kr)^{n-2} (kr')^{n-2}}{\pi \Gamma(2 \mu + 1)} \frac{Y_{\mu-1} (qr) J_{\mu-1} (qr')}{(qr)^{\frac{n}{2}-1}} \sin^{n-3} \theta d \cos \theta \right] \times \left\{ \int_0^r (r')^{\frac{n+1}{2}} dr' V(r') \int_0^\infty \frac{P_{\frac{n}{2}-\frac{1}{2}} (\cos \phi)}{(r^2 + r'^2 - 2rr' \cos \phi)^{\frac{n}{2}}} \right. \\
+ \int_r^\infty (r')^{\frac{n+1}{2}} dr' V(r') \int_0^\infty \frac{P_{\frac{n}{2}-\frac{1}{2}} (\cos \phi)}{(r^2 + r'^2 - 2rr' \cos \phi)^{\frac{n}{2}}} \left\}.
\]
A similar treatment gives

\[
\int_0^\pi \frac{P_{\frac{n}{2} + \frac{3}{2}} (\cos \phi)}{(r^2 + r'^2 - 2rr' \cos \phi)^2} d\cos \phi
\]

\[
= \frac{1}{8 (rr')^2} dR^2 \int_0^\pi \frac{P_{\frac{n}{2} + \frac{3}{2}} (\cos \phi)}{R - \cos \phi} d\cos \phi
\]

\[
= \frac{1}{8 (rr')^2} dR^2 Q_{\frac{n}{2} + \frac{3}{2}} (R)
\]

\[
= \frac{1}{8 (rr')^2} \left\{ \left( \frac{n}{2} + \frac{1}{2} \right) \left[ \left( \frac{r^2 + r'^2}{2rr'} \right)^2 \left( \frac{n}{2} + \frac{3}{2} \right) + 1 \right] Q_{\frac{n}{2} - \frac{1}{2}} \left( \frac{r^2 + r'^2}{2rr'} \right) \left[ \left( \frac{r^2 + r'^2}{2rr'} \right)^2 - 1 \right]^2 \right.
\]

\[
- \left. \left( \frac{n}{2} + \frac{1}{2} \right) (n + 4) \left( \frac{r^2 + r'^2}{2rr'} \right)^2 Q_{\frac{n}{2} + \frac{3}{2}} \left( \frac{r^2 + r'^2}{2rr'} \right) + \left( \frac{n}{2} + \frac{1}{2} \right) \left( \frac{n}{2} + \frac{3}{2} \right) Q_{\frac{n}{2} + \frac{3}{2}} \left( \frac{r^2 + r'^2}{2rr'} \right) \right\}. \quad (5.54)
\]

Then the second-order vacuum energy reads

\[
E_0^{(2)} = -\frac{3\mu^{-1} \Gamma \left( \frac{n}{2} - \frac{1}{2} \right)}{256 \Gamma (n-1)} \times \left\{ \int_0^\infty r^{-\frac{n-3}{2}} dr V(r) \int_0^r (r')^{-\frac{n-3}{2}} dr' V(r') \left\{ \left( \frac{n}{2} + \frac{1}{2} \right) \left[ \left( \frac{r^2 + r'^2}{2rr'} \right)^2 \left( \frac{n}{2} + \frac{3}{2} \right) + 1 \right] Q_{\frac{n}{2} - \frac{1}{2}} \left( \frac{r^2 + r'^2}{2rr'} \right) \left[ \left( \frac{r^2 + r'^2}{2rr'} \right)^2 - 1 \right]^2 \right.
\]

\[
- \left. \left( \frac{n}{2} + \frac{1}{2} \right) (n + 4) \left( \frac{r^2 + r'^2}{2rr'} \right)^2 Q_{\frac{n}{2} + \frac{3}{2}} \left( \frac{r^2 + r'^2}{2rr'} \right) + \left( \frac{n}{2} + \frac{1}{2} \right) \left( \frac{n}{2} + \frac{3}{2} \right) Q_{\frac{n}{2} + \frac{3}{2}} \left( \frac{r^2 + r'^2}{2rr'} \right) \right\} \right. + \int_r^\infty (r')^{-\frac{n-3}{2}} dr' V(r') \left\{ \left( \frac{n}{2} + \frac{1}{2} \right) \left[ \left( \frac{r^2 + r'^2}{2rr'} \right)^2 \left( \frac{n}{2} + \frac{3}{2} \right) + 1 \right] Q_{\frac{n}{2} - \frac{1}{2}} \left( \frac{r^2 + r'^2}{2rr'} \right) \left[ \left( \frac{r^2 + r'^2}{2rr'} \right)^2 - 1 \right]^2 \right.
\]

\[
- \left. \left( \frac{n}{2} + \frac{1}{2} \right) (n + 4) \left( \frac{r^2 + r'^2}{2rr'} \right)^2 Q_{\frac{n}{2} + \frac{3}{2}} \left( \frac{r^2 + r'^2}{2rr'} \right) + \left( \frac{n}{2} + \frac{1}{2} \right) \left( \frac{n}{2} + \frac{3}{2} \right) Q_{\frac{n}{2} + \frac{3}{2}} \left( \frac{r^2 + r'^2}{2rr'} \right) \right\}. \quad (5.55)
\]
Calculating global heat kernels, one-loop effective actions, and vacuum energies from scattering amplitudes

In the above, we calculate the heat kernel, the vacuum energy, and the one-loop effective action from the scattering phase shift. In this section, we suggest a method which calculates them from the scattering amplitude.

The scattering wave function is [50]

\[
\psi (r, \theta) = e^{ikr \cos \theta} + \sum_{l=0}^{\infty} a_l (\theta) h_l^{(1)} (kr),
\]

where

\[
a_l (\theta) = (2l + 1)^{\frac{1}{2}} \left( e^{2i\delta_l} - 1 \right) P_l (\cos \theta)
\]

is the partial scattering amplitude.

Under the large-distance approximation, by the asymptotic of the Hankel function \( h_l^{(1)} (kr) \), Eq. (2.3), we have

\[
\psi (r, \theta) = e^{ikr \cos \theta} + f (\theta) \frac{e^{ikr}}{r},
\]

where

\[
f (\theta) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l + 1) P_l (\cos \theta) \left( e^{2i\delta_l} - 1 \right)
\]

is the scattering amplitude, for the differential scattering cross section \( \sigma (\theta) = |f (\theta)|^2 \).

For small phase shifts, we approximate \( e^{2i\delta_l} \simeq 1 + 2i\delta_l \) in Eq. (6.4):

\[
f (\theta) \simeq \frac{1}{k} \sum_{l=0}^{\infty} (2l + 1) \delta_l P_l (\cos \theta).
\]

Then the forward-scattering amplitude, the scattering amplitude in the direction \( \theta = 0 \), is

\[
f (0) = \frac{1}{k} \sum_{l=0}^{\infty} (2l + 1) \delta_l.
\]

Noting that \( D_l = 2l + 1 \) is the degeneracy, we have

\[
\sum_{l=0}^{\infty} (2l + 1) \delta_l = kf (0).
\]

Substituting into Eq. (3.3) gives

\[
K (t) = \frac{t}{\pi} \int_0^{\infty} e^{-k^2 t} dk \sum_{l=0}^{\infty} (2l + 1) \delta_l
\]

\[
= \frac{2t}{\pi} \int_0^{\infty} f (0) e^{-k^2 t} k^2 dk.
\]
The heat kernel now is expressed by the forward scattering amplitude.

Similarly, by Eq. (2.11) we can express the one-loop effective action by the forward scattering amplitude,

\[ W(s) = -\frac{1}{2} \tilde{\mu}^2 s \int_0^{\infty} dt t^{s-1} \left[ \frac{2}{\pi} \int_0^{\infty} f(0) e^{-k^2 t} dk \right] \]

\[ = -\tilde{\mu}^2 s \Gamma(s + 1) \int_0^{\infty} f(0) (k^2)^{-s} dk. \quad (6.9) \]

By Eq. (2.12), we can express the vacuum energy by the forward scattering amplitude,

\[ E_0(\epsilon) = \frac{1}{2} \tilde{\mu}^2 \epsilon \int_0^{\infty} dt t^{\epsilon - \frac{1}{2} - 1} \left[ \frac{2}{\pi} \int_0^{\infty} f(0) e^{-k^2 t} dk \right] \]

\[ = \tilde{\mu}^2 \epsilon \Gamma(\epsilon + \frac{1}{2}) \int_0^{\infty} f(0) (k^2)^{\frac{1}{2} - \epsilon} dk. \quad (6.10) \]

As a verification, substituting the first-order Born approximation for the scattering amplitude \[ f^{(1)}_{\text{Born}}(\theta) = -\frac{1}{q} \int_0^{\infty} dr V(r) \sin(qr), \]

where \( q = 2k \sin \frac{\theta}{2} \), into Eq. (6.8) gives

\[ K(t) = \frac{2}{\pi} t \int_0^{\infty} \left[ -\frac{1}{q} \int_0^{\infty} dr V(r) \sin(qr) \right]_{\theta=0} e^{-k^2 t} \]

\[ = -\frac{1}{\sqrt{4\pi t}} \int_0^{\infty} V(r) r^2 dr. \quad (6.12) \]

Expanding \( \frac{\sin(qr)}{q} \) around \( \theta = 0 \), \( \frac{\sin(qr)}{q} = r + \cdots \), and substituting into Eq. (6.12) give

\[ K(t) = \frac{2}{\pi} t \int_0^{\infty} \int_0^{\infty} V(r) e^{-k^2 t} \]

\[ = -\frac{1}{\sqrt{4\pi t}} \int_0^{\infty} V(r) r^2 dr. \quad (6.13) \]

This agrees with the result given by Eq. (4.6).

Moreover, similar calculations give the one-loop effective action,

\[ W(s) = \frac{\tilde{\mu}^2 s - \frac{1}{2}}{4\sqrt{\pi}} (m^2)^{\frac{1}{2} - s} \int_0^{\infty} V(r) r^2 dr \quad (6.14) \]

and the vacuum energy

\[ E_0(\epsilon) = -\frac{\tilde{\mu}^2 \epsilon}{4\sqrt{\pi \Gamma(\epsilon - \frac{1}{2})}} (m^2)^{1-\epsilon} \int_0^{\infty} V(r) r^2 dr. \quad (6.15) \]

7 Conclusions

By regarding one-loop effective actions, vacuum energies, and scattering phase shifts as various spectral functions of a differential operator, we convert the quantum-field-theory problem into a quantum-mechanical problem by the relation among spectral functions. As
an example, we convert the Born approximation method, a method in quantum mechanics, to a method for calculating the one-loop effective action and the vacuum energy in quantum field theory. In principle, all methods for calculating scattering phase shifts and scattering amplitude in quantum mechanics, such as the WKB method and the eikonal approximation, can be converted to approaches for calculating effective actions and vacuum energies in this way.

Generally, all spectral functions can be calculated by this approach no matter what fields of physics it belongs to. For example, the spectral function in quantum field theory (such as effective actions and vacuum energies), in quantum mechanics (such as scattering phase shifts and scattering amplitudes), or in statistical mechanics (such as partition functions and various thermodynamic quantities) can be converted to each other by the relation among spectral functions. The method in a certain physical area can be converted to a method in other physical areas. Let us see an example in statistical mechanics. The energy spectrum of an interacting many-body system is a fundamental problem in statistical mechanics. Eigenvalues are the simplest spectral functions, and we can calculate them from other spectral functions. In Ref. [5], the energy eigenvalue spectrum of an interacting many-body system is calculated from the partition function. In statistical mechanics, there are many methods developed for calculating partition functions and grand partition functions, such as the cluster expansion method, the field theory method [63], and some mathematical methods [64, 65]. The eigenvalue and the partition function are both spectral functions. The method for calculating the partition function, e.g., the cluster expansion method, can be converted to a method for calculating the energy spectrum of interacting many-body systems.

In quantum-mechanical scattering theory, one almost only concentrates on short-range scattering, though the Born approximation can deal with some long-range potentials, e.g., the Coulomb potential. The scattering phase shift in long-range potential scattering can be uniformly treated by the tortoise coordinate [66], which we will discuss in future works.

In classical and quantum mechanics and in field theory, there exists a duality [67]. In quantum mechanics, this duality relates different eigenproblems and in field theory this duality relates different fields. The present paper provides a connection between spectral functions in quantum mechanics and in quantum field theory. In future works, we will discuss the relation between the duality in quantum mechanics and the duality in field theory.

To sum up, in this paper, we suggest an approach to convert a certain spectral function problem to another spectral function problem by the transform between spectral functions. This approach converts the methods in various physical areas, such as quantum field theory, quantum mechanics, and statistical mechanics, to each other.

A Appendix

In this appendix, we give some integral representations for the Bessel function.
A.1 An integral representation of \( j_\nu^2 (kr) \)

Taking \(|u| = |v| = kr\) in the expansion [68]

\[
\sin \frac{w}{w} = \sum_{l=0}^{\infty} (2l + 1) j_l (v) j_l (u) P_l (\cos \theta), \tag{A.1}
\]

where \( w = \sqrt{u^2 + v^2 - 2uv \cos \theta} \) and \( \theta \) is the angle between \( u \) and \( v \), gives

\[
\sin \frac{qr}{qr} = \sum_{l=0}^{\infty} (2l + 1) j_l^2 (kr) P_l (\cos \theta). \tag{A.2}
\]

with \( w = qr = 2kr \sin \frac{\theta}{2} \). Multiplying \( P'_\nu (\cos \theta) \) on both sides of Eq. (A.2) and then integrating from 0 to \( \pi \) give

\[
\int_{0}^{\pi} \sin \frac{qr}{qr} P'_\nu (\cos \theta) \sin \theta d\theta = \int_{0}^{\pi} \sum_{l=0}^{\infty} (2l + 1) j_l^2 (kr) P_l (\cos \theta) P'_\nu (\cos \theta) \sin \theta d\theta. \tag{A.3}
\]

By

\[
\int_{0}^{\pi} P_l (\cos \theta) P'_\nu (\cos \theta) \sin \theta d\theta = \frac{2}{2l + 1} \delta_{l\nu}, \tag{A.4}
\]

we have

\[
\int_{0}^{\pi} \sin \frac{qr}{qr} P'_\nu (\cos \theta) \sin \theta d\theta = 2 j_{\nu}^2 (kr). \tag{A.5}
\]

This gives an integral representation of \( j_\nu^2 (kr) \):

\[
j_\nu^2 (kr) = \frac{1}{2} \int_{0}^{\pi} \sin \frac{qr}{qr} P_l (\cos \theta) \sin \theta d\theta. \tag{A.6}
\]

A.2 An integral representation of \( j_l (kr) n_l (kr) \)

Taking \(|u| = |v| = kr\) in the expansion [68]

\[
\cos \frac{w}{w} = -\sum_{l=0}^{\infty} (2l + 1) j_l (v) n_l (u) P_l (\cos \theta), \tag{A.7}
\]

where \( w = \sqrt{u^2 + v^2 - 2uv \cos \theta} \) and \( \theta \) is the angle between \( u \) and \( v \), gives

\[
\cos \frac{qr}{qr} = -\sum_{l=0}^{\infty} (2l + 1) j_l (kr) n_l (kr) P_l (\cos \theta). \tag{A.8}
\]

with \( w = qr = 2kr \sin \frac{\theta}{2} \). Multiplying \( P'_\nu (\cos \theta) \) on both sides of Eq. (A.8) and then integrating from 0 to \( \pi \) give

\[
\int_{0}^{\pi} \cos \frac{qr}{qr} P'_\nu (\cos \theta) \sin \theta d\theta
\]

\[= -\int_{0}^{\pi} \sum_{l=0}^{\infty} (2l + 1) j_l (kr) n_l (kr) P_l (\cos \theta) P'_\nu (\cos \theta) \sin \theta d\theta
\]

\[= -2 j_{\nu} (kr) n_{\nu} (kr). \tag{A.9}
\]

This gives an integral representation of \( j_l (kr) n_l (kr) \):

\[
j_l (kr) n_l (kr) = -\frac{1}{2} \int_{0}^{\pi} \cos \frac{qr}{qr} P_l (\cos \theta) \sin \theta d\theta. \tag{A.10}
\]
A.3 An integral representation of $J_{l+\mu}^2 (kr)$

Taking $|u| = |v| = kr$ in the expansion [68]

$$
\frac{J_\mu (w)}{w^\mu} = 2^\mu \Gamma (\mu) \sum_{l=0}^{\infty} (l + \mu) \frac{J_{l+\mu} (u) J_{l+\mu} (v)}{u^\mu v^\mu} C_l^\mu (\cos \theta), \quad (u > v), \quad (A.11)
$$

where $w = \sqrt{u^2 + v^2 - 2uv \cos \theta} = qr = 2kr \sin \frac{\theta}{2}$ and $C_l^\mu (z)$ is the Gegenbauer polynomial [60], multiplying $C_l^\mu (\cos \theta)$ on both sides of Eq. (A.11), and integrating from 0 to $\pi$ give

$$
\int_0^\pi J_\mu (qr) C_l^\mu (\cos \theta) \sin 2\mu \theta d\theta = 2^\mu \Gamma (\mu) \sum_{l=0}^{\infty} (l + \mu) \frac{J_{l+\mu}^2 (kr)}{(kr)^{2\mu}} \int_0^\pi C_l^\mu (\cos \theta) C_l^\mu (\cos \theta) \sin 2\mu \theta d\theta. \quad (A.12)
$$

By [60]

$$
\int_0^\pi C_l^\mu (\cos \theta) C_l^\mu (\cos \theta) \sin 2\mu \theta d\theta = \frac{2^{1-2\mu} \pi \Gamma (l+2\mu)}{\Gamma (l+1) (l+\mu) \Gamma^2 (\mu)} \delta_{ll'}, \quad (A.13)
$$

we arrive at an integral representation of $J_{l+\mu}^2 (kr)$:

$$
J_{l+\mu}^2 (kr) = \frac{2^{\mu-1} \Gamma (l+1) \Gamma (\mu)}{\pi \Gamma (2\mu + l)} \frac{J_{l+\mu}^2 (kr)}{(kr)^{2\mu}} \int_0^\pi J_\mu (qr) C_l^\mu (\cos \theta) \sin 2\mu \theta d\theta. \quad (A.14)
$$

A.4 An integral representation of $J_{l+\nu} (kr) Y_{l+\nu} (kr)$

Taking $|u| = |v| = kr$ in the expansion [68]

$$
\frac{Y_\mu (w)}{w^\mu} = 2^\mu \Gamma (\mu) \sum_{l=0}^{\infty} (l + \mu) \frac{Y_{l+\mu} (u) Y_{l+\mu} (v)}{u^\mu v^\mu} C_l^\mu (\cos \theta), \quad (u > v), \quad (A.15a)
$$

where $w = \sqrt{u^2 + v^2 - 2uv \cos \theta} = qr = 2kr \sin \frac{\theta}{2}$ and, multiplying $C_l^\mu (\cos \theta)$ on both sides of Eq. (A.15a) and integrating from 0 to $\pi$ give

$$
\int_0^\pi Y_\mu (qr) C_l^\mu (\cos \theta) \sin 2\mu \theta d\theta = 2^\mu \Gamma (\mu) \sum_{l=0}^{\infty} (l + \mu) \frac{J_{l+\mu} (kr) Y_{l+\mu} (kr)}{(kr)^{2\mu}} \int_0^\pi C_l^\mu (\cos \theta) C_l^\mu (\cos \theta) \sin 2\mu \theta d\theta. \quad (A.16)
$$

By [60]

$$
\int_0^\pi C_l^\mu (\cos \theta) C_l^\mu (\cos \theta) \sin 2\mu \theta d\theta = \frac{2^{1-2\mu} \pi \Gamma (l+2\mu)}{\Gamma (l+1) (l+\mu) \Gamma^2 (\mu)} \delta_{ll'}, \quad (A.17)
$$

we arrive at an integral representation of $J_{l+\mu} (kr) Y_{l+\mu} (kr)$:

$$
J_{l+\mu} (kr) Y_{l+\mu} (kr) = \frac{2^{\mu-1} \Gamma (l+1) \Gamma (\mu)}{\pi \Gamma (2\mu + l)} \frac{J_{l+\mu} (kr) Y_{l+\mu} (kr)}{(kr)^{2\mu}} \int_0^\pi Y_\mu (qr) C_l^\mu (\cos \theta) \sin 2\mu \theta d\theta. \quad (A.18)
$$
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