ON THE DEVELOPMENT OF NONLINEAR OPERATOR THEORY

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Abstract. The basic results for nonlinear operators are given. These results include nonlinear versions of classical uniform boundedness theorem and Hahn-Banach theorem. Furthermore, the mappings from a metrizable space into another normed space can fall in some normed spaces by defining suitable norms. The results for the mappings on the metrizable spaces can be applied to the operators on the space of bounded linear functionals corresponding to the Dirac’s delta function.

1. Introduction

Operator theory has been at the heart of research in analysis (see [1]; [3], Chapter 4). Moreover, as implied by [2], considering nonlinear case should be essential. Developing useful results for the operators holds the promise for the wide applications of nonlinear functional analysis to a variety of scientific areas.

In classical functional analysis, the space of bounded linear operators is a normed space endowed with a sensible norm. In next section, several classes of normed functions are defined and some set including certain nonlinear operators from a normed space into a normed space turns out to be a normed space.

For the bounded linear operators between the normed spaces, the Hahn-Banach theorem and the uniform boundedness theorem are basic theorems. It is sensible to develop the nonlinear counterparts of these theorems. The nonlinear uniform boundedness theorem and the nonlinear Hahn-Banach theorem are given in Section 3 and Section 4, respectively.

Some mappings, for example, the distributions on some metrizable spaces, do not have the ”normed” values because the metrizable spaces are not normable. To resolve the problem, the mappings from a metrizable space into a normed space can have the normed values by defining a suitable norm depending on the metric of the metrizable space. The results for the mappings on the metrizable spaces along with some examples are given in the last section.

2010 Mathematics Subject Classification. Primary 47H99; Secondary 46H30.

Key words and phrases. Banach algebra, Dirac’s delta function, Nonlinear Hahn-Banach theorem, Nonlinear operators, Nonlinear uniform boundedness theorem.
2. Nonlinear functional spaces

Let $X$ and $Y$ be the normed spaces over the field $K$ with some sensible norms $\| \cdot \|_X$ and $\| \cdot \|_Y$, respectively, where $K$ is either a real field $\mathbb{R}$ or a complex field $\mathbb{C}$. Note that the vector spaces and the normed spaces in this article are assumed to be not trivial, i.e., not only including the zero element. Let $V(X,Y)$ be the set of all operators from $X$ into $Y$, i.e., the set of arbitrary maps from $X$ into $Y$. Note that the operators in $V(X,Y)$ are not assumed to be continuous. Let the algebraic operations of $F_1, F_2 \in V(X,Y)$ be the operators from $X$ into $Y$ with $(F_1 + F_2)(x) = F_1(x) + F_2(x)$ and $(\alpha F_1)(x) = \alpha F_1(x)$ for $x \in X$, where $\alpha \in K$ is a scalar. Also let the zero element in $V(X,Y)$ be the operator with the image equal to the zero element in $Y$. $V(X,Y)$ is a vector space over the field $K$. Define a non-negative extended real-valued function $p$, i.e., the range of $p$ including $\infty$, on $V(X,Y)$ by

$$p(F) = \max \left( \sup_{x \neq 0, x \in X} \frac{\|F(x)\|_Y}{\|x\|_X}, \|F(0)\|_Y \right)$$

for $F \in V(X,Y)$. The non-negative extended real-valued function $p$ is a generalization of the norm for the linear operators. Let $B(X,Y)$, the subset of $V(X,Y)$, consist of all operators with $p(F)$ being finite. Note that $p$ is a norm on $B(X,Y)$ and $[B(X,Y), p]$ is a normed space.

**Remark 1.** For $F \in B(X,Y)$, another norm equivalent to $p$ is

$$p^*(F) = \sup_{x \neq 0, x \in X} \frac{\|F(x)\|_Y}{\|x\|_X} + \|F(0)\|_Y.$$ 

In addition, two classes of norms related to $p$ and $p^*$ are

$$q(F) = \max \left( \sup_{x \neq 0, x \in X} \frac{\|F(x)\|_Y}{\|x\|_X}, s\|F(0)\|_Y \right)$$

and

$$q^*(F) = \sup_{x \neq 0, x \in X} \frac{\|F(x)\|_Y}{\|x\|_X} + s\|F(0)\|_Y,$$

$s > 0$. Furthermore, the other class of the non-negative extended real-valued functions on $V(X,Y)$ which includes the function $p$ is

$$p_k(F) = \max \left( \sup_{x \neq 0, x \in X} \frac{\|F(x)\|_Y}{\|x\|_X^k}, \|F(0)\|_Y \right),$$

where $k$ is a positive number. Note that

$$\|F(x)\|_Y \leq p_k(F) \|x\|_X^k; x \neq 0,$$

i.e., the normed value of $F(x)$ being dominated by the power of the normed value of $x \neq 0$, if $p_k(F)$ is finite.
In addition, let $V(S,Y)$ be the set of all operators from the set $S \subset X$ into $Y$ and $0 \in S$. Also let the zero element in $V(S,Y)$ be the operator of which image equal to the zero element in $Y$. Then $V(S,Y)$ is a vector space. Let $B(S,Y)$ be the subset of $V(S,Y)$ with the property that $\|F\|_{B(S,Y)}$ is finite for all $F \in B(S,Y)$, where

$$
\|F\|_{B(S,Y)} = \max \left( \sup_{x \neq 0, x \in S} \frac{\|F(x)\|_Y}{\|x\|_X}, \|F(0)\|_Y \right).
$$

Note that $B(S,Y)$ is a normed space, i.e., $\|\cdot\|_{B(S,Y)}$ being a normed function on $B(S,Y)$.

Hereafter the norm $p$ is used, i.e., $\|F\|_{B(X,Y)} = p(F)$ for $F \in B(X,Y)$. Note that the bounded linear operators fall in $B(X,Y)$. However, unlike a linear operator, a continuous nonlinear operator might not fall in the space $B(X,Y)$. As $X = Y$, the notation $B(X) = B(X,X)$ is used. If the other norm is used, for example, $p_k$, the notation $[B(X,Y), p_k]$ specifying the norm is used.

Let the notation of the composition of two operators be $\circ$ in the following lemma.

**Lemma 1.** Let $F_1 \in B(X,Y)$ and $F_2 \in B(Y,Z)$, where $X, Y,$ and $Z$ are normed spaces. Then

$$
\|F_1(x)\|_Y \leq \|F_1\|_{B(X,Y)} \|x\|_X, x \neq 0.
$$

If $F_1(0) = F_2(0) = 0$, then

$$
\|F_2 \circ F_1\|_{B(X,Z)} \leq \|F_1\|_{B(X,Y)} \|F_2\|_{B(Y,Z)},
$$

i.e., $F_2 \circ F_1 \in B(X,Z)$.

**Proof.** As $x \neq 0$,

$$
\frac{\|F_1(x)\|_Y}{\|x\|_X} \leq \sup_{x \neq 0, x \in X} \frac{\|F_1(x)\|_Y}{\|x\|_X} \leq \|F_1\|_{B(X,Y)}
$$

and hence $\|F_1(x)\|_Y \leq \|F_1\|_{B(X,Y)} \|x\|_X$. In addition, as $F_1(0) = F_2(0) = 0$,

$$
\|F_2 \circ F_1\|_{B(X,Z)} = \sup_{x \neq 0, x \in X} \frac{\|F_2 \circ F_1(x)\|_Z}{\|x\|_X}
\leq \sup_{x \neq 0, x \in X} \frac{\|F_2\|_{B(Y,Z)} \|F_1(x)\|_Y}{\|x\|_X}
= \|F_2\|_{B(Y,Z)} \left( \sup_{x \neq 0, x \in X} \frac{\|F_1(x)\|_Y}{\|x\|_X} \right)
= \|F_2\|_{B(Y,Z)} \|F_1\|_{B(X,Y)}.
$$

\[\Box\]
3. NONLINEAR UNIFORM BOUNDEDNESS THEOREM

For a family of bounded linear operators, the pointwise boundedness implies the uniform boundedness. Theorem 3, the main theorem in this section and considered as the nonlinear version of the uniform boundedness theorem, gives the analogous result for certain nonlinear operators.

The bounded linear operators or the bounded nonlinear operators between normed spaces map bounded sets into bounded sets. The class of operators mapping the bounded sets into the bounded sets is defined as follows.

**Definition 1.** An operator $F : X \to Y$ is topology bounded if and only if $F$ maps bounded sets in the normed space $X$ into bounded sets in the normed space $Y$.

The operator $F \in B(X, Y)$ is also called the norm bounded operator. A linear operator is norm bounded if and only if it is topology bounded. It might not be true for the nonlinear operator. A nonlinear operator is topology bounded might not be norm bounded. However, given some sufficient condition, a topology bounded nonlinear operator can be norm bounded, as indicated by the following theorem.

**Theorem 1.** If $F \in B(X, Y)$, then $F$ is topology bounded. On the other hand, if $F$ is topology bounded and

$$||F(kx)||_Y \leq M ||k|| ||F(x)||_Y$$

for some positive constant $M$, any scalar $k \neq 0$, and any $x \neq 0$, then $F \in B(X, Y)$.

**Proof.** If $F$ is norm bounded, then for any bounded set $V \in X$, there exists an open ball $B_r \in X$ with a radius $r > 0$ and a center at 0 such that $V \subset B_r$ and $\sup_{x \in B_r} ||F(x)||_Y \leq ||F||_{B(X, Y)} \max(r, 1)$ by Lemma 1, i.e., $F(V)$ being bounded. Conversely, let $\partial B_1$ be the boundary of the unit closed ball centered at 0 in $X$. Because $F$ is topology bounded, there exists some nonnegative number $\bar{k}$ such that $\bar{k} = \sup_{x \in \partial B_1} ||F(x)||_Y$. Then for any $x \in X, x \neq 0$,

$$\frac{||F(x)||_Y}{||x||_X} \leq \frac{M ||x||_X ||F(x/||x||_X)||_Y}{||x||_X} \leq M \bar{k}.$$

Therefore, $||F||_{B(X, Y)} \leq \max(M \bar{k}, ||F(0)||_Y)$ and $F \in B(X, Y)$.

The inequality in the above theorem, referred to as the $M$-contraction property, turns out to be crucial for the uniform boundedness of a certain family of nonlinear operators.

**Definition 2.** An operator $F$ from the normed space $X$ into the normed space $Y$ is called a $M$-contraction operator if and only if

$$||F(kx)||_Y \leq M ||k|| ||F(x)||_Y$$
for some positive constant $M$, any scalar $k \neq 0$, and any $x \neq 0$. The set of all topology bounded $M$-contraction operators is denoted as $B^M(X,Y)$.

By Theorem 1, a $M$-contraction operator is topology bounded if and only if it is norm bounded.

**Theorem 2.** $B^M(X,Y)$ is a closed subset of $B(X,Y)$.

**Proof.** By Theorem 1, a topology bounded $M$-contraction operator is norm bounded, i.e., $B^M(X,Y) \subseteq B(X,Y)$. Suppose $F_n \rightarrow F$, where $F_n \in B^M(X,Y)$ and $F \in B(X,Y)$. Then there exists $m$ such that

$$
\|F(kx)\|_Y \\
\leq \|F_m(kx) - F(kx)\|_Y + \|F_m(kx)\|_Y \\
\leq \epsilon + M |k| (\|F_m(x) - F(x)\|_Y + \|F(x)\|_Y) \\
\leq (M |k| + 1) \epsilon + M |k| \|F(x)\|_Y
$$

for every scalar $k \neq 0$, $x \neq 0$ and any $\epsilon > 0$. Therefore, $F \in B^M(X,Y)$ and $B^M(X,Y)$ is closed.

Note that the space of the bounded linear operators is the subset of $B^M(X,Y)$ for any $M \geq 1$. The uniform boundedness property of the nonlinear operators of interest is described as follows.

**Definition 3.** Let $\{F_\alpha\}$, a subset of $V(X,Y)$, be a family of operators, where $\alpha \in I$ and $I$ is an index set. $\{F_\alpha\}$ is uniformly topology bounded if and only if for any bounded set $U \subseteq X$ there exists a bounded set $V \subseteq Y$ satisfying $F_\alpha(U) \subseteq V$ for every $\alpha \in I$. $\{F_\alpha\}$ is uniformly norm bounded if and only if $\|F_\alpha\|_{B(X,Y)} \leq c$ for every $\alpha \in I$ and some positive constant $c$.

The uniform boundedness theorem holds for certain nonlinear operators which have the $M$-contraction property.

**Theorem 3.** Let $\{F_\alpha\}$ be a family of continuous $M$-contraction operators from a Banach space $X$ into a normed space $Y$, where $\alpha \in I$ and $I$ is an index set. $\{F_\alpha\}$ is uniformly norm bounded if the following conditions hold.

(a) $\{\|F_\alpha(x)\|_Y : \alpha \in I\}$ is bounded for $x \in X$, i.e., $\|F_\alpha(x)\|_Y \leq c_x$ for every $\alpha \in I$, where $c_x$ depending on $x$ is a positive number.

(b) There exists a positive constant $L$ such that

$$
\|F_\alpha(x_1 + x_2)\|_Y \leq L \|F_\alpha(x_1) + F_\alpha(x_2)\|_Y
$$

for every $\alpha \in I$ and $x_1, x_2 \in X$.

**Proof.** Let $A_k = \{x : \|F_\alpha(x)\|_Y \leq k, x \in X, \alpha \in I\}, k = 1, 2, \ldots$. $A_k$ is closed by the continuity of $F_\alpha$ and $X = \cup_{k=1}^\infty A_k$. Since $X$ is of second category, there exists $A_{k_0}$ such that $B(x_0, r) \subseteq A_{k_0}$, where $k_0$ is some positive integer and $B(x_0, r)$ is an open ball with the center $x_0 \in X$. \[\diamondsuit\]
and the radius \( r > 0 \). For any \( x \neq 0 \), there exists a constant \( c > 1 \) such that \( x = cr^{-1}||x||_X(z - x_0) \), where \( z \in B(x_0, r) \) and \( z \neq x_0 \). Then

\[
\begin{align*}
\|F_\alpha(x)\|_Y & \leq Mcr^{-1}||x||_X \|F_\alpha(z - x_0)\|_Y \\
& \leq MLCr^{-1}||x||_X \|F_\alpha(z) + F_\alpha(-x_0)\|_Y \\
& \leq MLCr^{-1}||x||_X (k_0 + M \|F_\alpha(x_0)\|_Y) \\
& \leq M(M + 1)Lcr^{-1}k_0||x||_X.
\end{align*}
\]

Therefore,

\[
\begin{align*}
\|F_\alpha\|_{B(X,Y)} & \leq \max \left[ M(M + 1)Lcr^{-1}k_0, c_0 \right]
\end{align*}
\]

for every \( \alpha \in I \) and \( \{F_\alpha\} \) is uniformly norm bounded.

\( \diamond \)

A bounded linear operator is a \( M \)-contraction operator and condition (b) in the above theorem also holds for the bounded linear operator, i.e., Theorem 3 being a generalization of the classical uniform boundedness theorem.

**Corollary 1.** Let \( F : X \to Y \), \( \{F_n\} \) be a sequence of continuous \( M \)-contraction operators from a Banach space \( X \) into a normed space \( Y \) and \( \{F_n(x)\} \) converges to \( F(x) \) with respect to the norm topology \( \| \cdot \|_Y \) for every \( x \in X \). If there exists a positive constant \( L \) such that

\[
\|F_n(x_1 + x_2)\|_Y \leq L \|F_n(x_1) + F_n(x_2)\|_Y
\]

for every \( n \) and \( x_1, x_2 \in X \), then the sequence \( \{F_n\} \) is uniformly norm bounded and \( F \in B(X,Y) \).

**Proof.** Because

\[
\|F_n(x)\|_Y \leq \|F_n(x) - F(x)\|_Y + \|F(x)\|_Y
\]

and \( \|F_n(x) - F(x)\|_Y \xrightarrow{n \to \infty} 0 \) for every \( x \in X \), hence \( \{F_n(x)\} \) is bounded for every \( x \in X \). By Theorem 3, \( \{F_n\} \) is uniformly norm bounded, i.e., \( \|F_n\|_{B(X,Y)} \leq c \) for every \( n \) and some positive constant \( c \). Finally, for \( x \neq 0 \),

\[
\|F(x)\|_Y = \lim_{n \to \infty} \|F_n(x)\| \leq c \|x\|_X
\]

by Lemma 1 and

\[
\|F(0)\|_Y = \lim_{n \to \infty} \|F_n(0)\| \leq c,
\]

hence \( \|F\|_{B(X,Y)} \leq c \), i.e., \( F \in B(X,Y) \).

\( \diamond \)
4. NONLINEAR HAHN-BANACH THEOREMS

The Hahn-Banach theorem states that a linear functional on a subspace of a vector space $X$ can be extended to the whole space with two properties preserved, linearity and the inequality for the linear functional and a sublinear functional. It turns out that a nonlinear functional on a subset of a vector space can be extended to the whole space with a certain inequality preserved, as given in Theorem 4. Further, as $X$ is a separable Hilbert space, Theorem 5 states that the extension to the whole space also holds with both the inequality similar to the one in Theorem 4 and the continuity of the nonlinear functional preserved. Theorem 6, the last theorem in this section, is concerned with the extension results for the nonlinear functionals with some specific forms. Hereafter, let $\text{Dom}(F)$ be the domain of $F$ which is the operator or the functional.

**Theorem 4.** Let $X$ be a vector space and $p$ be a sub-additive functional on $X$. Let $F : S \to \mathbb{R}$ be a functional on $S$ and

$$F(s_1) + F(s_2) \leq p(s_1 + s_2), s_1, s_2 \in S, s_1 \neq s_2,$$

where $S$ is a proper subset of $X$. $F$ has an extension $\hat{F} : X \to \mathbb{R}$ satisfying

$$\hat{F}(s) = F(s), s \in S,$$

and

$$\hat{F}(x_1) + \hat{F}(x_2) \leq p(x_1 + x_2), x_1, x_2 \in X, x_1 \neq x_2.$$

**Proof.** By Zorn’s lemma, the set of all extensions of $F$ satisfying the inequality has a maximal element by defining the partial ordering as the inclusion of the domains of the extensions. The maximal element $\hat{F}$ being defined on the whole space $X$ is proved next.

Suppose that $\text{Dom}(\hat{F})$ is a proper subset of $X$. Since

$$\hat{F}(x_1) + \hat{F}(x_2) \leq p(x_1 + x_2) \leq p(x_1 - y) + p(x_2 + y),$$

$$\hat{F}(x_2) - p(x_2 + y) \leq p(x_1 - y) - \hat{F}(x_1)$$

thus, where $y \in X/\text{Dom}(\hat{F})$, i.e., $y$ falling in the intersection of $X$ and the complement of $\text{Dom}(\hat{F})$, and $x_1, x_2 \in \text{Dom}(\hat{F}), x_1 \neq x_2$. Define $F_1 : \text{Dom}(\hat{F}) \cup \{y\} \to \mathbb{R}$ by $F_1(x) = \hat{F}(x)$ for $x \in \text{Dom}(\hat{F})$ and $F_1(y) = -c$, where

$$c = \sup_{x \in \text{Dom}(\hat{F})} \left[ \hat{F}(x) - p(x + y) \right]$$

and the existence of $c$, i.e., the supremum being finite, is due to

$$c \leq \max \left[ p(x_1 - y) - \hat{F}(x_1), p(x_2 - y) - \hat{F}(x_2) \right].$$

Then for $x \in \text{Dom}(\hat{F})$,

$$F_1(x) + F_1(y) = \hat{F}(x) - c \leq p(x + y).$$
Thus, \( F_1 \) satisfying the inequality is an extension of \( \hat{F} \), i.e., a contradiction.

\[ \Box \]

The Hahn-Banach extension theorem for the linear functionals is a special case of the following corollary.

**Corollary 2.** Let \( X \) be a vector space, \( S \) be a proper subset of \( X \), \( 0 \in S \), \( F : S \to \mathbb{R} \) be a functional on \( S \), \( F(0) = 0 \), \( p \) be a sub-linear functional on \( X \), and

\[ F(s_1) + F(s_2) \leq p(s_1 + s_2), s_1, s_2 \in S. \]

Then \( F \) has an extension \( \hat{F} : X \to \mathbb{R} \) satisfying

\[ \hat{F}(s) = F(s), s \in S, \]

and

\[ \hat{F}(x_1) + \hat{F}(x_2) \leq p(x_1 + x_2), x_1, x_2 \in X. \]

**Proof.** \( S \) can be assumed to have at least two elements since \( F \) can be extended to have the domain including the zero element and the other element and to satisfy the required inequality otherwise. By Theorem 4, there exists an extension \( \hat{F} \) of \( F \) such that

\[ \hat{F}(x_1) + \hat{F}(x_2) \leq p(x_1 + x_2), x_1, x_2 \in X, x_1 \neq x_2. \]

Further, because \( p \) is sub-linear,

\[ \hat{F}(x) + \hat{F}(x) \leq p(x) + p(x) = p(x + x) \]

for \( x \in X. \)

\[ \Box \]

Let \( F^+ \) and \( F^- \) be the positive and negative parts of \( F : S \to \mathbb{R} \) defined by \( F^+(s) = \max[F(s), 0] \) and \( F^-(s) = \max[-F(s), 0] \) for \( s \in S \), respectively, where \( S \) is a subset of the vector space \( X \). By the nonlinear Hahn-Banach theorem, there exists a bounded extension to the whole space for the functional having the bounded positive and negative parts on the subset of the vector space.

**Corollary 3.** Let \( F \in B(S, \mathbb{R}) \), \( 0 \in S \), \( F(0) = 0 \), and for \( s_1, s_2 \in S \),

\[ F^+(s_1) + F^+(s_2) \leq M_1 ||s_1 + s_2||_X, \]

and

\[ F^-(s_1) + F^-(s_2) \leq M_2 ||s_1 + s_2||_X, \]

where \( S \) is a proper subset of a normed space \( X \), \( M_1 \) and \( M_2 \) are some positive constants, and \( F^+ \) and \( F^- \) are the positive and negative parts of \( F \), respectively. Then there exists an extension \( \hat{F} \in B(X, \mathbb{R}) \) of \( F \) satisfying

\[ \hat{F}(s) = F(s), s \in S, \]
and the inequality
\[ |\hat{F}(x_1) + \hat{F}(x_2)| \leq (M_1 + M_2) \|x_1 + x_2\|_X \]
for \( x_1, x_2 \in X \).

**Proof.** By Corollary 2, there exist \( \hat{F}^+ \) and \( \hat{F}^- \) such that \( \hat{F}^+(s) = F^+(s) \), \( \hat{F}^-(s) = F^-(s) \), \( s \in S \),
\[ \hat{F}^+(x_1) + \hat{F}^+(x_2) \leq M_1 \|x_1 + x_2\|_X, \]
and
\[ \hat{F}^-(x_1) + \hat{F}^-(x_2) \leq M_2 \|x_1 + x_2\|_X \]
for \( x_1, x_2 \in X \). Let \( \hat{F} = \hat{F}^+ - \hat{F}^- \). Then \( \hat{F}(s) = F(s) \) and
\[
\begin{align*}
|\hat{F}(x_1) + \hat{F}(x_2)| &= |\hat{F}^+(x_1) + \hat{F}^+(x_2) - \hat{F}^-(x_1) - \hat{F}^-(x_2)| \\
&\leq |\hat{F}^+(x_1) + \hat{F}^+(x_2)| + |\hat{F}^-(x_1) + \hat{F}^-(x_2)| \\
&\leq (M_1 + M_2) \|x_1 + x_2\|_X.
\end{align*}
\]
Finally, \( \hat{F} \in B(X, R) \) because
\[ |\hat{F}(x_1)| \leq (M_1 + M_2) \|x_1\|_X. \]

\( \diamond \)

The complex version of the nonlinear Hahn-Banach theorem can be established by applying Theorem 4, as stated by the following corollary.

**Corollary 4.** Let \( F : S \to C \) and \( F = F_r + iF_c \), where \( S \) is a proper subset of a vector space \( X \) and both \( F_r \) and \( F_c \) are real-valued functionals defined on \( S \). If for \( s_1, s_2 \in S \) and \( s_1 \neq s_2 \),
\[ F_r(s_1) + F_r(s_2) \leq p_r(s_1 + s_2), \]
and
\[ F_c(s_1) + F_c(s_2) \leq p_c(s_1 + s_2), \]
then \( F \) has an extension \( \hat{F} : X \to C, \hat{F} = \hat{F}_r + i\hat{F}_c \) satisfying
\[ \hat{F}(s) = F(s), \ s \in S, \]
and for \( x_1, x_2 \in X \) and \( x_1 \neq x_2 \),
\[ \hat{F}_r(x_1) + \hat{F}_r(x_2) \leq p_r(x_1 + x_2), \]
and
\[ \hat{F}_c(x_1) + \hat{F}_c(x_2) \leq p_c(x_1 + x_2), \]
where both \( \hat{F}_r \) and \( \hat{F}_c \) are real-valued functionals defined on \( X \) and \( p_r \) and \( p_c \) are sub-additive functionals on \( X \).
If \( 0 \in S \), \( F(0) = 0 \), \( p_r \) and \( p_c \) are sub-linear functionals on \( X \), and the above inequalities for \( F_r \) and \( F_c \) hold for any \( s_1, s_2 \in S \), then the above extension result holds and the above inequalities for \( \hat{F}_r \) and \( \hat{F}_c \) also hold for any \( x_1, x_2 \in X \).

The following theorem indicates that both the continuity and the inequality can be preserved as extending a nonlinear functional on a closed subspace of a separable Hilbert space to the whole space.

**Theorem 5.** Let \( Z \) with the orthonormal basis \( \{e_j\} \) be a proper closed subspace of a separable Hilbert space \( X \) with the orthonormal basis \( \{e_j\} \cup \{e_i^*\} \), \( i = 1, \ldots, j = 1, \ldots \), and \( F : Z \rightarrow \mathbb{R} \) be a continuous functional satisfying \( F(0) = 0 \) and

\[
F(z_1) + F(z_2) \leq p(z_1 + z_2)
\]

for orthogonal vectors \( z_1 \) and \( z_2 \) falling in the union of one-dimensional spaces each spanned by \( e_j \), where \( p \) is a uniformly continuous sub-additive functional on \( X \) with \( p(0) = 0 \). Then there exists an extension \( \hat{F} \) of \( F \) such that \( \hat{F} : X \rightarrow \mathbb{R} \) is continuous,

\[
\hat{F}(z) = F(z), \quad z \in Z,
\]

and

\[
\hat{F}(x_1) + \hat{F}(x_2) \leq p(x_1 + x_2)
\]

for orthogonal vectors \( x_1 \) and \( x_2 \) falling in the union of one-dimensional spaces each spanned by \( e_j \) or \( e_i^* \).

**Proof.** Similar to the proof of Theorem 4, the existence of the maximal element \( \hat{F} \) can be proved by Zorn’s lemma and by defining the partial ordering as the inclusion of the domains \( X_m, m = 1, \ldots \), the space spanned by \( \{e_j\} \) and \( \{e_i^* : i \leq m\} \), or \( X_0 \) of the extensions, where \( X_0 = Z \). It remains to prove that \( \hat{F} \) is defined on the whole space \( X \).

Suppose that \( \text{Dom}(\hat{F}) = X_{m-1} \) is a proper closed subspace of \( X \). Let \( E_{m-1} \subset X_{m-1} \) be the union of one-dimensional spaces each spanned by \( \{e_j\} \) and \( \{e_i^* : i \leq m-1\} \), \( e_0^* = 0 \), and \( F_1 : X_m \rightarrow \mathbb{R} \) defined by

\[
F_1(x + te_m^*) = \hat{F}(x) - r(t),
\]

where \( x \in \text{Dom}(\hat{F}), t \in K, \) and the real-valued function \( r : K \rightarrow \mathbb{R} \) is defined by

\[
r(t) = \sup_{x \in E_{m-1}} \left[ \hat{F}(x) - p(x + te_m^*) \right].
\]

Then, for \( x \in E_{m-1}, \)

\[
\hat{F}(x) - p(x + te_m^*) \leq r(t)
\]

and hence

\[
F_1(x) + F_1(te_m^*) = \hat{F}(x) - r(t) \leq p(x + te_m^*).
\]
If $F_1$ is continuous, then $F_1$ satisfying the required inequality is an extension of the continuous functional $\hat{F}$, i.e., a contradiction. It remains to prove the continuity of $F_1$. For $z_n = x_n + t_n e_m, x_n \in \text{Dom}(\hat{F}), z_n \to z = x + te_m, x \in \text{Dom}(\hat{F}),$ implies that $x_n \to x$ and $t_n \to t$ owing to 

$$||z_n - z||_X^2 = ||x_n - x||_X^2 + |t_n - t|^2,$$

where the norm $||x||_X = (\langle x, x \rangle_X)^{1/2}$ and $\langle \cdot, \cdot \rangle_X$ is the inner product on $X$. Hence, if $r$ is a continuous function of $t$, then

$$|F_1(z_n) - F_1(z)| \leq \hat{F}(x_n) - \hat{F}(x) + |r(t_n) - r(t)|$$

and thus $F_1(z_n) \to F_1(z)$ as $z_n \to z$, i.e., $F_1$ being continuous. For every $\epsilon > 0$, there exists an $N$ such that

$$r(t_n) = \sup_{x \in E_{m-1}} [\hat{F}(x) - p(x + te_m) + p(x + te_m) - p(x + t_ne_m)]$$

$$\leq \sup_{x \in E_{m-1}} [\hat{F}(x) - p(x + te_m)] + \sup_{x \in E_{m-1}} [p(x + te_m) - p(x + t_ne_m)]$$

$$\leq r(t) + \epsilon$$

for $n \geq N$ by the uniform continuity of $p$ and similarly $r(t) \leq r(t_n) + \epsilon$, i.e., $|r(t_n) - r(t_n)| \leq \epsilon$, and hence $r$ is continuous.

By Theorem 5, the continuity of the nonlinear functional and the boundedness on the basis can be extended from the subspace to the whole space, as indicated by the following corollary.

**Corollary 5.** Let $Z$ with the orthonormal basis $\{e_j\}$ be a proper closed subspace of a separable Hilbert space $X$ with the orthonormal basis $\{e_j\} \cup \{e_j^*\}, i = 1, \ldots, j = 1, \ldots$ and the norm induced by the inner product, $F : Z \to R$ be a continuous functional, $F(0) = 0$, and for orthogonal vectors $z_1$ and $z_2$ falling in the union of one-dimensional spaces each spanned by $e_j$,

$$F^+(z_1) + F^+(z_2) \leq M_1 ||z_1 + z_2||_X,$$

and

$$F^-(z_1) + F^-(z_2) \leq M_2 ||z_1 + z_2||_X,$$

where $M_1$ and $M_2$ are some positive constants and $F^+$ and $F^-$ are the positive and negative parts of $F$, respectively. Then there exists an extension $\hat{F} : X \to R$ of $F$ which is continuous and satisfies

$$\hat{F}(z) = F(z), z \in Z,$$

and the inequality

$$|\hat{F}(x_1) + \hat{F}(x_2)| \leq (M_1 + M_2)||x_1 + x_2||_X$$
for orthogonal vectors $x_1$ and $x_2$ falling in the union of one-dimensional spaces each spanned by $e_j$ or $e_i^*$.

The complex version of Theorem 5 is stated by the following corollary.

**Corollary 6.** Let $Z$ with the orthonormal basis \{e_j\} be a proper closed subspace of a separable Hilbert space $X$ with the orthonormal basis \{e_j\} \cup \{e_i^*\}, $i = 1, \ldots, j = 1, \ldots$, $F : Z \to C$, $F(0) = 0$, $F = F_r + iF_c$, and for orthogonal vectors $z_1$ and $z_2$ falling in the union of one-dimensional spaces each spanned by $e_j$,

$$F_r(z_1) + F_r(z_2) \leq p_r(z_1 + z_2),$$

and

$$F_c(z_1) + F_c(z_2) \leq p_c(z_1 + z_2),$$

where both $F_r$ and $F_c$ are real-valued continuous functionals defined on $Z$ and $p_r$ and $p_c$ are uniformly continuous sub-additive functionals on $X$ with $p_r(0) = 0$ and $p_c(0) = 0$. Then there exists an extension $\hat{F} : X \to C$ such that

$$\hat{F}(z) = F(z), z \in Z,$$

$\hat{F} = \hat{F}_r + i\hat{F}_c$ and for orthogonal vectors $x_1$ and $x_2$ falling in the union of one-dimensional spaces each spanned by $e_j$ or $e_i^*$,

$$\hat{F}_r(x_1) + \hat{F}_r(x_2) \leq p_r(x_1 + x_2),$$

and

$$\hat{F}_c(x_1) + \hat{F}_c(x_2) \leq p_c(x_1 + x_2),$$

where both $\hat{F}_r$ and $\hat{F}_c$ are real-valued continuous functionals defined on $X$.

The following theorem is a direct application of classic Hahn-Banach theorem to the nonlinear case. The specific form of the nonlinear functional can be preserved as extending from a subspace of a vector space to the whole space. Let $R^+ = \{x : x \geq 0, x \in R\}$.

**Theorem 6.** Let $F : Z \to R$ be a functional defined by

$$F(z) = f[|T(z)|]$$

and

$$F(z) \leq f[p(z)], z \in Z,$$

where $Z$ is a proper subspace of a vector space $X$, $T : Z \to K$ is a linear functional, $p$ is a semi-norm defined on $X$, and $f : R \to R$ is increasing on $R^+$. Then $F$ has an extension $\hat{F} : X \to R$ with the form

$$\hat{F}(x) = f[|\hat{T}(x)|]$$

and

$$\hat{F}(x) \leq f[p(x)], x \in X,$$
where \( \hat{T} : X \to K \) is a linear extension of \( T \), i.e., \( \hat{T}(z) = T(z) \) for \( z \in Z \).

**Proof.** Because \( f \) is increasing on \( R^+ \), then \( F(z) = f(|T(z)|) \leq f(p(z)) \) implies that \( |T(z)| \leq p(z) \) for \( z \in Z \). By the Hahn-Banach extension theorem (see [4], Theorem 3.3) there exists a linear extension \( \hat{T} \) of \( T \) such that \( \hat{T}(z) = T(z) \) for \( z \in Z \) and \( |\hat{T}(x)| \leq p(x) \) for \( x \in X \). Therefore, let \( \hat{F}(x) = f(|\hat{T}(x)|) \). Then \( \hat{F}(z) = F(z) \) for \( z \in Z \) and \( \hat{F}(x) = f(|\hat{T}(x)|) \leq f(p(x)) \) for \( x \in X \) owing to \( f \) being increasing on \( R^+ \).

\( \diamond \)

The above theorem can be applied to the nonlinear functionals associated with the powers of the linear functional.

**Corollary 7.** Let \( F : Z \to R \) be a functional defined by

\[
F(z) = |T(z)|^k
\]

and

\[
F(z) \leq [p(z)]^k, z \in Z,
\]

where \( k \) is a positive number, \( Z \) is a proper subspace of a vector space \( X \), \( T : Z \to K \) is a linear functional, and \( p \) is a semi-norm defined on \( X \). Then \( F \) has an extension \( \hat{F} : X \to R \) with the form

\[
\hat{F}(x) = |\hat{T}(x)|^k
\]

and

\[
\hat{F}(x) \leq [p(x)]^k, x \in X,
\]

where \( \hat{T} : X \to K \) is a linear extension of \( T \).

**Proof.** Let \( f(x) = x^k \) and hence the results hold by Theorem 6.

\( \diamond \)

The above nonlinear functionals associated with the bounded linear functionals can be bounded as certain norms are employed. As the linear functional \( T \) is bounded, the following corollary indicates that the functionals in Corollary 7 fall in \([B(Z,R),p_k]\) (see Remark 1) and the associated extended functionals are in \([B(X,R),p_k]\).

**Corollary 8.** Let \( F : Z \to R \) be a functional defined by

\[
F(z) = |T(z)|^k, z \in Z,
\]

where \( k \) is a positive number, \( Z \) is a proper subspace of a normed space \( X \), and \( T : Z \to K \) is a bounded linear functional. Then \( F \in [B(Z,R),p_k] \) and there exists an extension \( \hat{F} : X \to R \) of \( F \) such that \( \hat{F} \in [B(X,R),p_k] \),

\[
\|\hat{F}\|_{[B(X,R),p_k]} = \|F\|_{[B(Z,R),p_k]},
\]

and

\[
\hat{F}(x) = |\hat{T}(x)|^k, x \in X,
\]

where the bounded linear functional \( \hat{T} : X \to K \) is an extension of \( T \).
Proof.

\[ ||F||_{[B(Z,R),p_k]} = \max \left[ \sup_{z \neq 0, z \in Z} \frac{|F(z)|}{||z||_X^k}, |F(0)| \right] = ||T||_{B(Z,K)}^k \]

is finite, i.e., \( F \in [B(Z,R),p_k] \). Furthermore, by the Hahn-Banach theorem, there exists a linear extension \( \hat{T} : X \to K \) of \( T \) such that \( ||T||_{B(Z,K)} = ||\hat{T}||_{B(X,K)} \). Let \( \hat{F}(x) = |\hat{T}(x)|^k, x \in X \), then \( \hat{F}(z) = F(z), z \in Z \), and

\[ ||\hat{F}||_{[B(X,R),p_k]} = \left( \sup_{x \neq 0, x \in X} \frac{|\hat{T}(x)|}{||x||_X} \right)^k = ||\hat{T}||_{B(X,K)}^k = ||F||_{[B(Z,K),p_k]}^k, \]

i.e., \( \hat{F} \in [B(X,R),p_k] \). \( \diamond \)

The following corollary is the complex version of Theorem 6.

**Corollary 9.** Let \( F : Z \to C \) and \( F = F_r + iF_c \), where both \( F_r \) and \( F_c \) are real-valued functionals on \( Z \) defined by

\[ F_r(z) = f_r[|T_r(z)|] \]

and

\[ F_c(z) = f_c[|T_c(z)|] \]

and satisfying

\[ F_r(z) \leq f_r[p_r(z)] \]

and

\[ F_c(z) \leq f_c[p_c(z)] \]

for \( z \in Z \), and where \( Z \) is a proper subspace of a vector space \( X \), \( T_r : Z \to K \) and \( T_c : Z \to K \) are linear functionals defined on \( Z \), both \( p_r \) and \( p_c \) are seminorms defined on \( X \), and \( f_r : R \to R \) and \( f_c : R \to R \) are increasing on \( R^+ \). Then \( F \) has an extension \( \hat{F} : X \to C \) having the form \( \hat{F} = \hat{F}_r + i\hat{F}_c \),

\[ \hat{F}_r(x) = f_r[|\hat{T}_r(x)|] \]

and

\[ \hat{F}_c(x) = f_c[|\hat{T}_c(x)|] \]

and satisfying

\[ \hat{F}_r(x) \leq f_r[p_r(x)] \]

and

\[ \hat{F}_c(x) \leq f_c[p_c(x)] \]

for \( x \in X \), where both \( \hat{F}_r \) and \( \hat{F}_c \) are real-valued functionals defined on \( X \) and \( \hat{T}_r : X \to K \) and \( \hat{T}_c : X \to K \) are linear extensions of \( T_r \) and \( T_c \), respectively.
5. Nonlinear mappings on metrizable spaces

The operators of interest in previous sections are defined on the vector spaces or the normed spaces. In this section, the mappings on a metrizable space can have normed values by defining a norm. Thus, some linear functional such as the Dirac’s delta function considered as the linear functional falls in certain normed spaces. The basic facts about the mappings of interest are given in next subsection, while several examples of the mappings on the metrizable spaces are presented in the second subsection.

5.1. Bounded mappings on metrizable spaces. In this subsection, the main results that the set of mappings between certain metrizable spaces being a translation invariant metrizable vector space and some mappings from a metrizable space into a normed space falling in a normed space are proved in Theorem 10. Thus some linear functionals corresponding to commonly used distributions fall in some normed spaces and are given in next subsection.

In this subsection, let $X$ and $Y$ be the metrizable vector spaces over the field $K$. Let $V(X, Y)$ be the vector space of all mappings from $X$ into $Y$ with the algebraic operations of $F_1, F_2 \in V(X, Y)$ being the mappings from $X$ into $Y$ defined by $(F_1 + F_2)(x) = F_1(x) + F_2(x)$ and $(\alpha F_1)(x) = \alpha F_1(x)$ for $x \in X$ and $\alpha \in K$. Note that the zero element in $V(X, Y)$ is the mapping of which image equal to the zero element in $Y$. Define a nonnegative extended real-valued function $d$ on $V(X, Y)$ by

$$d(F_1, F_2) = \max \left\{ \sup_{x \neq 0, x \in X} \frac{d_Y[F_1(x), F_2(x)]}{d_X(x, 0)}, d_Y[F_1(0), F_2(0)] \right\},$$

where $d_X$ and $d_Y$ are the metrics on $X$ and $Y$, respectively. Let $B_d(X, Y)$, containing the zero element of $V(X, Y)$ and the subset of $V(X, Y)$, have the property that $d(F_1, F_2) < \infty$ for any $F_1, F_2 \in B_d(X, Y)$. If $F$ is a mapping from $X$ into $Y$ and $d(F, 0)$ is finite, then $F \in B_d(X, Y)$ because for any $G \in B_d(X, Y)$,

$$d(F, G) \leq d(F, 0) + d(G, 0)$$

is finite.

The following theorem indicates that $[B_d(X, Y), d]$ is a metric space. The routine proof is not presented.

**Theorem 7.** $d$ is a metric on $B_d(X, Y)$ and $[B_d(X, Y), d]$ is a metric space.

It is well known that the space of all bounded linear operators from a normed space $X$ to a Banach space $Y$ is complete. The following corollary can be considered as the generalization of the completeness result for the bounded linear operators to the possibly nonlinear mappings on the metrizable spaces.

**Corollary 10.** If $Y$ is complete, then $[B_d(X, Y), d]$ is a complete metric space.
Proof. Let \( \{F_n\} \) be a Cauchy sequence in \( B_d(X,Y) \). Then for any positive \( \epsilon \), there exists an \( N \) such that for \( m,n > N \), \( d(F_n, F_m) < \epsilon \). Then as \( x \neq 0 \),

\[
d_y [F_n(x), F_m(x)] \leq d(F_n, F_m)d_X(x, 0) < \epsilon d_X(x, 0)
\]

and

\[
d_y [F_n(0), F_m(0)] \leq d(F_n, F_m) < \epsilon.
\]

Thus, \( \{F_n(x)\} \) is Cauchy in \( Y \) for \( x \in X \) and \( F_n(x) \xrightarrow{n \to \infty} y, y \in Y \) owing to the completeness of \( Y \). Define a mapping \( F : X \to Y \) by \( F(x) = y \). For \( x \neq 0 \),

\[
d_y [F_n(x), F(x)] = d_y[F_n(x), \lim_{m \to \infty} F_m(x)] = \lim_{m \to \infty} d_y[F_n(x), F_m(x)] \\
\leq \epsilon d_X(x, 0).
\]

In addition,

\[
d_y [F_n(0), F(0)] = \lim_{m \to \infty} d_y[F_n(0), F_m(0)] \leq \epsilon.
\]

Thus, \( F \in [B_d(X,Y), d] \) owing to \( d(F,F_n) \) being finite and \( \{F_n\} \) converges to \( F \) because of \( d(F_n, F) \leq \epsilon \).

\( \diamond \)

The following theorem gives the characterization of the mappings falling in \( B_d(X,Y) \).

**Theorem 8.** If \( F \in B_d(X,Y) \), then for any \( \epsilon_1 > 0 \) there exists a \( \epsilon_2 > 0 \) such that \( F[B_X(\epsilon_1)] \subset B_Y(\epsilon_2) \), where \( B_X(\epsilon_1) = \{x : d_X(x, 0) < \epsilon_1, x \in X\} \) and \( B_Y(\epsilon_2) = \{y : d_Y(y, 0) < \epsilon_2, y \in Y\} \). On the other hand, if the limit

\[
\lim_{x \to 0} \frac{d_y[F(x), 0]}{d_X(x, 0)}
\]

exists and is finite, there exists a bounded ball \( B_X(\epsilon) \) such that \( F \) maps the complement of \( B_X(\epsilon) \) into a bounded subset of some bounded ball \( B_Y(\epsilon^*) \), and for any \( \epsilon_1 > 0 \) there exists a \( \epsilon_2 > 0 \) such that \( F[B_X(\epsilon_1)] \subset B_Y(\epsilon_2) \), then \( F \in B_d(X,Y) \), where \( \epsilon, \epsilon^* > 0 \).

**Proof.** If \( F \in B_d(X,Y) \), then \( d(F, 0) \) is finite. Hence for any \( x \in X \),

\[
d_y[F(x), 0] \leq d(F, 0) \max \{d_X(x, 0), 1\}
\]

and thus

\[
d_y[F(x^*), 0] \leq d(F, 0) \max (\epsilon_1, 1)
\]

for any \( x^* \in B_X(\epsilon_1) \).

On the other hand, if the given conditions hold, there exist positive numbers \( \delta < \epsilon, m, \) and \( M \) such that for \( d_X(x_1, 0) < \delta, x_1 \neq 0, \) and \( d_X(x_2, 0) > \epsilon \),

\[
\frac{d_y[F(x_1), 0]}{d_X(x_1, 0)} < m
\]
and
\[ \frac{d_Y [F(x_2), 0]}{d_X (x_2, 0)} < M. \]

Therefore,
\[
\begin{align*}
d(F, 0) \\
= \max \left\{ \sup_{x \neq 0, x \in X} \frac{d_Y [F(x), 0]}{d_X (x, 0)}, d_Y [F(0), 0] \right\} \\
= \max \left\{ \sup_{d_X (x, 0) < \delta, x \neq 0, x \in X} \frac{d_Y [F(x), 0]}{d_X (x, 0)}, \sup_{\delta \leq d_X (x, 0) \leq \epsilon, x \in X} \frac{d_Y [F(x), 0]}{d_X (x, 0)} \right\} \\
\leq \max \left\{ m, \frac{\epsilon}{\delta}, M, d_Y [F(0), 0] \right\},
\end{align*}
\]

and thus \( F \in B_d(X, Y) \), where \( \epsilon \) is some positive number.

It is natural to ask when \( B_d(X, Y) \) can be a vector space. It turns out that the following property plays a crucial role.

**Definition 4.** A translation invariant metric \( d_X \) is scale bounded on a subset \( S \) of a metric vector space \( [X, d_X] \) if and only if
\[ d_X (\alpha s, 0) \leq C_{\alpha} d_X (s, 0), s \in S, \]
where \( \alpha \in K \) and \( C_{\alpha} \) is a positive number depending on \( \alpha \).

Note that the metric induced by the normed function is scale bounded. A linear operator from a normed space into another normed space is continuous if and only if it is bounded. In addition, if the linear operator is continuous at one point, then it is a bounded operator. The continuity of a linear mapping implying the boundedness of the linear mapping relies on the scale boundedness of the metric, as indicated by the following theorem.

**Theorem 9.** Let \( F : X \to Y \) be a linear mapping and the metric \( d_Y \) is translation invariant, where \( X \) and \( Y \) are metric vector spaces. Then:
(a) If \( F \in B_d(X, Y) \), then \( F \) is continuous. On the other hand, if \( F \) is continuous, the translation invariant metric \( d_X \) is scale bounded with \( C_{\alpha} = M(\alpha)|\alpha|, Y \) is a normed space, and the metric on \( Y \) is the one induced by the norm, then \( F \in B_d(X, Y) \), where \( C_{\alpha} \) is considered as a positive function defined on \( K \) and \( M \) is a positive bounded function defined on \( K \).
(b) If \( F \) is continuous at a single point and \( d_X \) is translation invariant, then \( F \) is continuous.

**Proof.** (a): If \( F \in B_d(X, Y) \), then
\[ d_Y [F(x_n), F(x)] = d_Y [F(x_n - x), 0] \leq d(F, 0)d_X (x_n - x, 0) \]
and thus \( F(x_n) \rightarrow F(x) \) as \( x_n \rightarrow x \) in \( X \), i.e., \( F \) being continuous. On the other hand, if \( F \) is continuous and \( Y \) is a normed space with the metric induced by the norm, then for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that \( ||F(x)||_Y < \epsilon \) for \( x \in X \) satisfying \( d_X(x, 0) < \delta \). Further, \( d_X[x/d_X(x, 0), 0] \) is bounded for \( x \neq 0 \) by the condition imposed on \( d_X \) and hence there exists an \( \alpha_0 \neq 0 \) such that \( d[(\alpha_0 x)/d_X(x, 0), 0] < \delta \). Thus, for \( x \neq 0 \), \( ||F[(\alpha_0 x)/d_X(x, 0)]||_Y = |\alpha_0|||F[x/d_X(x, 0)]||_Y < \epsilon \) and hence \( ||F[x/d_X(x, 0)]||_Y < \epsilon/|\alpha_0| \). Finally, \( F \in B_d(X, Y) \) because

\[
d(F, 0) = \sup_{x \neq 0, x \in X} \left| \frac{d_X(x, 0)F \left[ \frac{x}{d_X(x, 0)} \right]}{d_X(x, 0)} \right|_Y < \epsilon/|\alpha_0|.
\]

(b): Assume that \( F \) is continuous at \( x_0 \). Thus, for every \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that \( d_Y[F(x^*), F(x_0)] < \epsilon \) for any \( x^* \in X \) satisfying \( d_X(x^*, x_0) < \delta \). Then, by the translation invariance property of the metrics, for any \( x \in X \) and any \( x^{**} \in X \) satisfying \( d_X(x^{**}, x) = d_X(x_0 + x^{**} - x, x_0) < \delta \), \( d_Y[F(x^{**}), F(x)] = d_Y[F(x_0 + x^{**} - x), F(x_0)] < \epsilon \), i.e., \( F \) being continuous.

\[ \diamond \]

The scale boundedness of the translation invariant metric \( d_Y \) on \( Y \) is the key for \( B_d(X, Y) \) being a vector space. Furthermore, if \( Y \) is a normed space, it turns out that \( B_d(X, Y) \) can be a normed space, as given in the following theorem.

**Theorem 10.** If the translation invariant metric \( d_Y \) is scale bounded on \( Y \), then \( [B_d(X, Y), d] \) is a translation invariant metric vector space. Furthermore, if \( Y \) is a normed space and \( d_Y \) is the metric induced by the norm on \( Y \), then \( B_d(X, Y) \) is a normed space with the norm \( ||F||_{B_d(X, Y)} = d(F, 0) \).

**Proof.** To prove that \( d \) is translation invariant, for \( F_1, F_2, F_3 \in B_d(X, Y) \),

\[
d(F_1 + F_3, F_2 + F_3) = \max \left\{ \sup_{x \neq 0, x \in X} \frac{d_Y[F_1(x) + F_3(x), F_2(x) + F_3(x)]}{d_X(x, 0)}, \right. \]

\[
d_Y[F_1(0) + F_3(0), F_2(0) + F_3(0)] \right\} \]

\[
= \max \left\{ \sup_{x \neq 0, x \in X} \frac{d_Y[F_1(x), F_2(x)]}{d_X(x, 0)}, d_Y[F_1(0), F_2(0)] \right\} \]

\[
= d(F_1, F_2)
\]

owing to \( d_Y \) being translation invariant. To prove that \( B_d(X, Y) \) is a vector space, for \( F_1, F_2 \in B_d(X, Y) \) and \( \alpha \in K \),

\[
d(\alpha F_1, 0) = \max \left\{ \sup_{x \neq 0, x \in X} \frac{d_Y[\alpha F_1(x), 0]}{d_X(x, 0)}, d_Y[\alpha F_1(0), 0] \right\} \]

\[
\leq C_\alpha d(F_1, 0)
\]
is finite, where $C_\alpha$ is a positive number depending on $\alpha$. Then by the translation invariance of $d$,

$$d(\alpha F_1 + F_2, 0) \leq d(\alpha F_1, 0) + d(F_2, 0)$$

gives that $d(\alpha F_1 + F_2, 0)$ is finite, i.e., $\alpha F_1 + F_2 \in B_d(X, Y)$ and $B_d(X, Y)$ being a vector space.

Next is to prove that $B_d(X, Y)$ is a vector space. Since $Y$ is a normed space, $d_Y$ induced by the norm is translation invariant and scale bounded. Thus, $B_d(X, Y)$ is a vector space. It remains to prove that $d(F, 0)$ is a norm.

For $F_1, F_2 \in B_d(X, Y)$ and $\alpha \in K$, by the properties of the translation invariant metric $d$,

$$||\alpha F_1||_{B_d(X,Y)} = d(F_1, 0) \geq 0, \quad d(F_1, 0) = ||\alpha F_1||_{B_d(X,Y)} = 0$$

if and only if $F_1 = 0$,

$$||F_1 + F_2||_{B_d(X,Y)} \leq d(F_1 + F_2, F_2) + d(F_2, 0) = d(F_1, 0) + d(F_2, 0) = ||F_1||_{B_d(X,Y)} + ||F_2||_{B_d(X,Y)},$$

and finally

$$||\alpha F_1||_{B_d(X,Y)} = \max \left\{ \sup_{x \neq 0, x \in X} \frac{||\alpha F_1(x)||_Y}{d_X(x, 0)}, ||\alpha F_1(0)||_Y \right\}$$

$$= \max \left\{ \sup_{x \neq 0, x \in X} \frac{||\alpha F_1(x)||_Y}{d_X(x, 0)}, ||\alpha|| ||F_1(0)||_Y \right\}$$

$$= ||\alpha|| ||F_1||_{B_d(X,Y)}.$$

\[\Diamond\]

**Remark 2.** The space $B(X, Y)$ is one example of the space $B_d(X, Y)$. Let $X$ and $Y$ be the normed spaces, Then the space $B(X, Y)$ is the space $B_d(X, Y)$ as $d_x$ and $d_y$ are the metrics induced by the norms on $X$ and $Y$, respectively, i.e., $d$ being the metric induced by the norm on $B(X, Y)$.

By Corollary 10 and Theorem 10, the following result holds.

**Corollary 11.** If $Y$ is a Banach space and $d_Y$ is the metric induced by the norm on $Y$, then $B_d(X, Y)$ is a Banach space with the norm $||F||_{B_d(X,Y)} = d(F, 0)$.

The normed space of all bounded linear operators from a normed space $X$ into $X$ is a normed algebra with the multiplication being the composition of the operators. The normed space $B(X)$ is not a normed algebra with the multiplication being the composition of the operators. However, $B(X)$ can be a normed algebra or a Banach algebra depending on $X$ being a normed algebra or a Banach algebra as the multiplication of the operators is defined properly. Similarly, the normed space $B(X, Y)$ can be a normed algebra.
or a Banach algebra depending on $Y$ being a normed algebra or a Banach algebra. The following corollary indicates that $B_d(X, Y)$ can be a normed algebra or a Banach algebra by defining an operation of multiplication for two mappings and hence the normed space $B(X, Y)$ can be a normed algebra or a Banach algebra (see Remark 2).

**Corollary 12.** Let $Y$ be a normed (Banach) algebra and $d_Y$ is the metric induced by the norm on $Y$. Define the multiplication of two mappings $F_1 : X \to Y$ and $F_2 : X \to Y$ by

$$(F_1 * F_2)(x) = \frac{F_1(x)F_2(x)}{d_X(x, 0)}, \quad x \neq 0, \quad x \in X,$$

and

$$(F_1 * F_2)(0) = F_1(0)F_2(0).$$

Then $B_d(X, Y)$ is a normed (Banach) algebra with the norm $|||F|||_{B_d(X, Y)} = d(F, 0)$. If $Y$ is unital, then $B_d(X, Y)$ is unital.

**Proof.** By Theorem 10 or Corollary 11, $B_d(X, Y)$ is a normed space or a Banach space depending on $Y$ being a normed space or a Banach space. Next is to prove that $B_d(X, Y)$ is a normed algebra. Let $F_1, F_2, F_3 \in B_d(X, Y)$. First, as $x \neq 0$,

$$[(F_1 * F_2) * F_3](x) = \frac{(F_1 * F_2)(x)F_3(x)}{d_X(x, 0)} = \frac{F_1(x)F_2(x)F_3(x)}{[d_X(x, 0)]^2} = \frac{F_1(x)(F_2 * F_3)(x)}{d_X(x, 0)} = [F_1 * (F_2 * F_3)](x).$$

As $x = 0$,

$$[(F_1 * F_2) * F_3](0) = (F_1 * F_2)(0)F_3(0) = F_1(0)F_2(0)F_3(0) = F_1(0)(F_2 * F_3)(0) = [F_1 * (F_2 * F_3)](0).$$

Secondly, as $x \neq 0$,

$$[F_1 * (F_2 + F_3)](x) = \frac{F_1(x)(F_2 + F_3)(x)}{d_X(x, 0)} = \frac{F_1(x)F_2(x) + F_1(x)F_3(x)}{d_X(x, 0)} = (F_1 * F_2)(x) + (F_1 * F_3)(x).$$
As $x = 0$,
\[
[F_1 \ast (F_2 + F_3)](0) = F_1(0)(F_2 + F_3)(0) = F_1(0)F_2(0) + F_1(0)F_3(0) = (F_1 \ast F_2)(0) + (F_1 \ast F_3)(0).
\]

\[
[(F_1 + F_2) \ast F_3](x) = (F_1 + F_3)(x) + (F_2 + F_3)(x) \quad \text{and} \quad [\alpha(F_1 \ast F_2)](x) = [(\alpha F_1) \ast F_2](x) = [F_1 \ast (\alpha F_2)](x) \quad \text{for} \ x \in X \text{ and } \alpha \in K, \text{ can be proved analogously. Further,}
\]
\[
\|F_1 \ast F_2\|_{B_d(X,Y)} = \max \left( \sup_{x \neq 0, x \in X} \frac{\|F_1(x)\|_Y}{d_X(x,0)} \|F_2(x)\|_Y, \|F_1(0)F_2(0)\|_Y \right)
\]
\[
\leq \max \left( \sup_{x \neq 0, x \in X} \frac{\|F_1(x)\|_Y}{d_X(x,0)} \sup_{x \neq 0, x \in X} \frac{\|F_2(x)\|_Y}{d_X(x,0)}, \|F_1(0)\|_Y \|F_2(0)\|_Y \right)
\]
\[
\leq \max \left( \sup_{x \neq 0, x \in X} \frac{\|F_1(x)\|_Y}{d_X(x,0)}, \|F_1(0)\|_Y \right)
\]
\[
\|F_1\|_{B_d(X,Y)} \|F_2\|_{B_d(X,Y)}
\]

As 1 is the unit element in $Y$, the unit element $e$ in $B_d(X,Y)$ is given by $e(x) = d_X(x,0)1$ for $x \neq 0$ and $e(0) = 1$. Then
\[
(F \ast e)(x) = \frac{F(x)e(x)}{d_X(x,0)} = F(x)1 = F(x)
\]
\[
= 1F(x) = \frac{e(x)F(x)}{d_X(x,0)} = (e \ast F)(x)
\]
for $x \neq 0$,
\[
(F \ast e)(0) = F(0)e(0) = F(0)1 = F(0)
\]
\[
= 1F(0) = e(0)F(0) = (e \ast F)(0),
\]
and
\[
\|e\|_{B_d(X,Y)} = \max \left( \sup_{x \neq 0, x \in X} \frac{\|e(x)\|_Y}{d_X(x,0)}, \|e(0)\|_Y \right)
\]
\[
= \|1\|_Y
\]
\[
= 1,
\]
where the last 1 is the unit element in the scalar field.
\diamond
5.2. **Examples.** The following examples are the applications of the results in previous subsection. The first example is concerned with the linear functionals corresponding to the distributions. By defining a translation invariant metric, commonly used linear functionals such as the distributions corresponding to a Lebesgue integrable function and the Dirac’s delta function fall in some Banach space by Corollary 11. The second example is concerned with the possibly nonlinear operators defined on the Banach space which is one of the Banach spaces given in the first example. The operators of interest are associated with the position operator and the momentum operator in quantum mechanics. Finally, the Fourier transform and the Fourier-Plancherel transform defined as the bounded linear operators and the bounded linear mapping, respectively, are shown in the third example.

**Example 1.** Let the subspace \( D(\Omega) \) of \( C^\infty(\Omega) \) be the space of all complex valued functions defined on the nonempty open subset \( \Omega \) of \( \mathbb{R}^n \) with compact supports, where \( C^\infty(\Omega) \) is the space of all complex valued infinitely differentiable functions defined on \( \Omega \). Define a translation invariant metric on \( C^\infty(\Omega) \) by

\[
d_{C^\infty(\Omega)}(f, g) = \max \left[ \sum_{i=1}^{\infty} a_i p_i(f - g), b_i p_i(f - g) \right]
\]

for \( f, g \in C^\infty(\Omega) \), where \( N \) is a positive integer, \( a > 1 \), \( b > 0 \),

\[
p_i(f) = \max \left\{ \left| \frac{\partial^{\alpha} f}{\partial x_{j_1}^{\alpha_1} \cdots \partial x_{j_r}^{\alpha_r}} \right|(x) : x = (x_1, \ldots, x_n)^t \in K_i, |\alpha| \leq i \right\},
\]

and where \( \alpha = (\alpha_1, \ldots, \alpha_r), |\alpha| = \sum_{k=1}^{r} \alpha_k \) are non-negative integers, \( \{j_1, \ldots, j_r\} \subset \{1, \ldots, n\}, K_i \) are compact sets satisfying \( K_i \) lies in the interiors of \( K_{i+1} \) and \( \bigcup_{i=1}^{\infty} K_i = \Omega \). As \( |\alpha| = 0 \), \( \partial^{\alpha} f / \partial x_{j_1}^{\alpha_1} \cdots \partial x_{j_r}^{\alpha_r} = f \). Endowed with this metric topology, \( C^\infty(\Omega) \) is a locally convex space with a complete translation invariant metric, i.e., a Fréchet space, and \([D(\Omega), d_{C^\infty(\Omega)}]\) is a translation invariant metric vector space.

\( B_d[D(\Omega), C] \) is a Banach space and a unital Banach algebra by Corollary 11 and Corollary 12. Let \( L_p(\Omega), 1 \leq p < \infty \), be the spaces of complex-valued functions \( x \) defined on \( \Omega \) satisfying that \( |x|^p \) is integrable with respect to the Lebesgue measure. Let \( m \) of which support is a subset of \( K_N \) be a measurable complex function defined on \( \Omega \). Then, if \( m \) is a Lebesgue integrable complex function, then \( \Lambda_m \), the corresponding linear functional, i.e., a distribution with respect to another topology on \( D(\Omega) \), falls in the space \( B_d[D(\Omega), C] \)
owing to
\[ \|\Lambda_m\|_{B_d[D(\Omega), C]} = \sup_{f \neq 0, f \in D(\Omega)} \frac{|\Lambda_m(f)|}{d_{C^\infty}(\Omega)(f, 0)} \leq \sup_{f \neq 0, f \in D(\Omega)} \frac{p_N(f) \|m\|_{L^1(\Omega)}}{d_{C^\infty}(\Omega)(f, 0)} \leq \|m\|_{L^1(\Omega)}. \]

If \( m \in L_\infty(\Omega) \), the corresponding linear functional \( \Lambda_m \) also falls in the space \( B_d[D(\Omega), C] \), where \( L_\infty(\Omega) \) is the space of all essentially bounded functions on \( \Omega \). If \( m \in L_p(\Omega), 1 < p < \infty \), and the Lebesgue measure on the set \{ \( x : m(x) \leq 1, x \in \Omega \} \) is finite, the corresponding linear functional \( \Lambda_m \) falls in the space \( B_d[D(\Omega), C] \).

In addition to the linear functionals corresponding to the "ordinary" functions, consider \( \delta \), the Dirac delta function as a linear functional on \( D(\Omega) \). Then \( \delta_c, c \in K_N \),
\[ d(\delta_c, 0) = \sup_{f \neq 0, f \in D(\Omega)} \frac{|f(c)|}{d_{C^\infty}(\Omega)(f, 0)} \leq 1, \]
i.e., \( \delta_c \in B_d[D(\Omega), C] \), where \( \delta_c(f) = f(c) \) for \( f \in D(\Omega) \). Analogously, as \( 0 \in K_N \),
\[ d(\delta^{(k)}, 0) = \max \left\{ \sup_{f \neq 0, f \in D(\Omega)} \frac{\|\delta^{(k)}(f)\|}{d_{C^\infty}(\Omega)(f, 0)}, \|\delta^{(k)}(0)\| \right\} \leq 1, \]
i.e., \( \delta^{(k)} \in B_d[D(\Omega), C] \), where \( k = (k_1, k_2, \ldots, k_r) \), \( \delta^{(k)}(f) = (-1)^k f^{(k)}(0) \) for \( f \in D(\Omega), f^{(k)} = \partial^{|k|} f / \partial x_{j_1}^{k_1} \cdots \partial x_{j_r}^{k_r}, \) and \( |k| \leq N \).

If the metric \( d_{D(\Omega)} \) imposed on \( D(\Omega) \) (not on \( C^\infty(\Omega) \)) has the same form as \( d_{C^\infty(\Omega)} \) with \( p_i \) modified to
\[ p_i(f) = \max \left\{ \left| \frac{\partial^{|\alpha|} f}{\partial x_{j_1}^{\alpha_1} \cdots \partial x_{j_r}^{\alpha_r}} (x) \right| : x = (x_1, \ldots, x_n)^t \in \Omega, |\alpha| \leq i \right\}, \]
then \( \delta_c \) with \( c \in \Omega \) and \( \delta^{(k)} \) with \( 0 \in \Omega, |k| \leq N \), fall in the space \( B_d[D(\Omega), C] \) with respect to this metric. Furthermore, without the assumption that the support of \( m \) is a subset of \( K_N \), \( \Lambda_m \in B_d[D(\Omega), C] \) still holds for \( m \in L_1(\Omega) \).
and for $m \in L_p(\Omega)$ with the Lebesgue measure on the set $\{x : m(x) \leq 1, x \in \Omega\}$ being finite, $1 < p < \infty$. As the volume of $\Omega$ is finite and $m \in L_\infty(\Omega)$, the corresponding linear functional $\Lambda_m \in B_d[D(\Omega), C]$ with respect to the metric $d_D(\Omega)$.

The above linear functionals falling in the space $B_d[D(\Omega), C]$ are also continuous by Theorem 9. In addition, as $\Lambda_m, \delta_c, \alpha \delta^{(k)}$ fall in $B_d[D(\Omega), C]$, the square of these mappings, i.e., $\Lambda_m \Lambda_m, \delta_c \delta_c, \alpha \delta^{(k)} \alpha \delta^{(k)}$ with the multiplication operation given in Corollary 12, also fall in $B_d[D(\Omega), C]$ and are not linear.

**Example 2.** Let $\Omega \subset \mathbb{R}$ be a nonempty open set. Then, $B_d[D(\Omega), C]$ (also see Example 1) is a Banach space and a unital Banach algebra by Corollary 11 and Corollary 12.

(a): Let $F : \text{Dom}(F) \to B_d[D(\Omega), C]$ be the operator defined by

$$F(\Lambda)(\phi) = \Lambda(x \phi)$$

for $\Lambda \in \text{Dom}(F)$, where $\text{Dom}(F) \subset B_d[D(\Omega), C]$ consists of some functionals $\Lambda$ in $B_d[D(\Omega), C]$ satisfying that $F(\Lambda) \in B_d[D(\Omega), C]$, $\phi \in D(\Omega)$, and $x$ is a real-valued function on $\Omega$ defined by $x(t) = t$ for $t \in \Omega$. As $\Lambda$ are the linear functionals corresponding to the Dirac’s delta function or the square integrable functions in $L_2(\Omega)$, $F$ is associated with the multiplication operator, i.e., being associated with the position operator in quantum mechanics. However, $F$ might not be associated with the multiplication operator as $\Lambda$ are some other nonlinear functionals in $B_d[D(\Omega), C]$. If $\text{Dom}(F) \subset \{\delta_c : c \in \mathbb{K}_N\}$, then $F \in B\{\text{Dom}(F), B_d[D(\Omega), C]\}$ is bounded with respect to the metric $d_{C^\infty(\Omega)}$ owing to $F(\delta_c) = c \delta_c$ and thus

$$||F||_{B\{\text{Dom}(F), B_d[D(\Omega), C]\}} = \sup_{\delta_c \in \text{Dom}(F)} \frac{||F(\delta_c)||_{B_d[D(\Omega), C]}}{||\delta_c||_{B_d[D(\Omega), C]}} \leq \sup_{c \in \mathbb{K}_N} |c| .$$

However, as the metric on $D(\Omega)$ is $d_D(\Omega)$, $\{\delta_c : c \in \Omega\} \subset \text{Dom}(F)$, and $\Omega$ is unbounded, then $F$ is not bounded.

(b): Let $F : \text{Dom}(F) \to B_d[D(\Omega), C]$ defined by $F(\Lambda)(\phi) = (-1)^\alpha \Lambda(\phi(\alpha))$, where $\text{Dom}(F) \subset B_d[D(\Omega), C]$ consists of some functionals $\Lambda$ in $B_d[D(\Omega), C]$ satisfying that $F(\Lambda) \in B_d[D(\Omega), C]$, $\phi \in D(\Omega)$, and $\alpha$ is a positive integer. $F$ is the differential operator on the ”bounded” linear functionals. Note that $F$ is associated the momentum operator in quantum mechanics as $\alpha = 1$. Further, if $\alpha = 1$, $F$ is unbounded because

$$\frac{||F(\Lambda_n)||_{B_d[D(\Omega), C]}}{||\Lambda_n||_{B_d[D(\Omega), C]}} \geq 2n,$$

where $\Lambda_n \in \text{Dom}(F)$ are the linear functionals, i.e., the distributions on $D(\Omega)$ endowed with another topology, corresponding to the functions $\lambda_n(t) = n(t - c), c \leq t \leq c + 1/n$ and 0 otherwise, $n \geq N_0, c \in \Omega$, and where $N_0$ is some positive integer.
Example 3. (a): Let $F : L_1(R^n) \to C_0(R^n)$ be the Fourier transform defined by
\[
[F(f)](t) = \hat{f}(t) = (2\pi)^{-n/2} \int_{R^n} f e^{-it \cdot x} \, dx
\]
for $f \in L_1(R^n)$, where $C_0(R^n)$ endowed with the supremum norm is the Banach space of all complex continuous functions on $R^n$ that vanish at infinity. Since $\|\hat{f}\|_{C_0(R^n)} \leq \|f\|_{L_1(R^n)}$ (see [4], Theorem 7.5),
\[
\|F\|_{B[L_1(R^n), C_0(R^n)]} = \sup_{f \neq 0, f \in L_1(R^n)} \frac{\|\hat{f}\|_{C_0(R^n)}}{\|f\|_{L_1(R^n)}} \leq 1
\]
and hence $F \in B[L_1(R^n), C_0(R^n)]$.

As $F$ is the Fourier-Plancherel transform, i.e., $F : L_2(R^n) \to L_2(R^n)$, $F \in B[L_2(R^n), L_2(R^n)]$ since it is a linear isometry.

(b): Let $\mathcal{L}(R^n) \subset C^\infty(R^n)$ (see Example 1 and Example 2) be the Fréchet space with the metric
\[
d(f, g) = \max \left( \sum_{k=0}^\infty \frac{b^{-k} \|f - g\|_k}{a + \|f - g\|_k}, \|f - g\|_N \right)
\]
for $f, g \in \mathcal{L}(R^n)$, where $N$ is a positive integer, $a > 0$, $b > 1$,
\[
\|f\|_k = \sup_{|\alpha| \leq k} \sup_{x \in R^n} \left( 1 + ||x||_R^n \right)^k \left| \frac{\partial^{|\alpha|} f}{\partial x_{j_1}^{\alpha_1} \cdots \partial x_{j_r}^{\alpha_r}}(x) \right|
\]
and where $||\cdot||_R^n$ is the usual Euclidean norm on $R^n$. Note that $\|f\|_k$ is finite for $f \in \mathcal{L}(R^n)$. $B_d[\mathcal{L}(R^n), C]$ is a Banach space and a unital Banach algebra by Corollary 11 and Corollary 12. The linear functionals corresponding to $\delta_c$, $c \in R^n$, $\delta^{(k)}$, $|k| \leq N$, and the Lebesgue integrable complex function $m$, i.e., the tempered distributions (see [4], Definition 7.11) with respect to some topology on $\mathcal{L}(R^n)$, fall in $B_d[\mathcal{L}(R^n), C]$. If $m \in L_p(R^n)$ with the Lebesgue measure on the set $\{x : m(x) \leq 1, x \in R^n\}$ being finite, $1 < p < \infty$, the corresponding linear functional $\Lambda_m$ falls in the space $B_d[\mathcal{L}(R^n), C]$.

As $F : \mathcal{L}(R^n) \to \mathcal{L}(R^n)$ is the Fourier transform, $F$ is bounded, i.e., mapping bounded sets in $\mathcal{L}(R^n)$ into bounded sets in $\mathcal{L}(R^n)$, because $F$ is a continuous linear mapping from $\mathcal{L}(R^n)$ into $\mathcal{L}(R^n)$ (see [1], Theorem 7.4).

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