ON CONGRUENCES MOD $p^m$ BETWEEN EIGENFORMS AND THEIR ATTACHED GALOIS REPRESENTATIONS.

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ABSTRACT. Given a prime $p$ and cusp forms $f_1$ and $f_2$ on some $\Gamma_1(N)$ that are eigenforms outside $Np$ and have coefficients in the ring of integers of some number field $K$, we consider the problem of deciding whether $f_1$ and $f_2$ have the same eigenvalues mod $p^m$ (where $p$ is a fixed prime of $K$ over $p$) for Hecke operators $T_\ell$ at all primes $\ell \nmid Np$.

When the weights of the forms are equal the problem is easily solved via an easy generalization of a theorem of Sturm. Thus, the main challenge in the analysis is the case where the forms have different weights. Here, we prove a number of necessary and sufficient conditions for the existence of congruences mod $p^m$ in the above sense.

The prime motivation for this study is the connection to modular mod $p^m$ Galois representations, and we also explain this connection.

1. Introduction

Let $N \in \mathbb{N}$ and let $p$ be a fixed prime number.

Suppose that we are given cusp forms $f_1 = \sum a_n(f_1)q^n$ and $f_2 = \sum a_n(f_2)q^n$ (where $q := e^{2\pi i z}$) on $\Gamma_1(N)$ of weights $k_1$ and $k_2$, respectively, and with coefficients in $\mathcal{O}_K$ where $K$ is some number field. We will assume in all that follows that $f_1$ and $f_2$ are normalized, i.e., that $a_1(f_1) = a_1(f_2) = 1$.

We say that $f_1$ and $f_2$ are eigenforms outside $Np$ if they are (normalized) eigenforms for all Hecke operators $T_\ell$ for primes $\ell \nmid Np$. The corresponding eigenvalues for such $T_\ell$ acting on $f_i$ are then exactly the coefficients $a_\ell(f_i)$.

Now fix a prime $p$ of $K$ over $p$. If $f_i$ is an eigenform outside $Np$, and if $m \in \mathbb{N}$, there is attached to $f_i$ a ‘mod $p^m$’ Galois representation:

$$\rho_{f_i,p^m} : G_\mathbb{Q} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathcal{O}_K/p^m)$$

obtained by making the $p$-adic representation attached to $f_i$ integral with coefficients in $\mathcal{O}_K$ and then reducing modulo $p^m$. The representation $\rho_{f_i,p^m}$ is unramified outside $Np$ and we have:

$$(*) \quad \text{tr} \rho_{f_i,p^m}(\text{Frob}_\ell) = (a_\ell(f_i) \mod p^m)$$

for primes $\ell \nmid Np$.

By a theorem of Carayol, cf. Théorème 1 of [2], combined with the Chebotarev density theorem, the representation $\rho_{f_i,p^m}$ is determined up to isomorphism by the property $(*)$ for primes $\ell \nmid Np$ if we additionally suppose that the mod $p$ representation $\rho_{f_i,p}$ is absolutely irreducible.

Motivated by a study of the arithmetic properties of modular mod $p^m$ Galois representations [3], we found it natural to prepare the ground for numerical experimentation with these representations. As is obvious from the above, the key to this
is to obtain a computationally decidable criterion for when we have \( a_\ell(f_1) \equiv a_\ell(f_2) \quad (p^m) \) for all primes \( \ell \nmid Np \), if \( f_1 \) and \( f_2 \) as above are given cusp forms that are eigenforms outside \( Np \).

Now, for the case \( m = 1 \), and if the weights \( k_1 \) and \( k_2 \) are equal, there is a well-known theorem of Sturm that gives a necessary and sufficient condition for the forms to be congruent mod \( p \) in the sense that all their Fourier coefficients are congruent mod \( p \). It turns out to be very easy to generalize Sturm’s theorem to the cases \( m > 1 \) provided that we still have \( k_1 = k_2 \). Then, still under the assumption that the weights are equal, a simple twisting argument allows us to discuss the case of eigenforms outside \( Np \).

For various reasons we are interested in also considering cases where the weights are distinct and this turns out to present a genuinely new challenge.

We study two distinct approaches to this challenge. Under favorable circumstances these approaches both result in computable necessary and sufficient conditions for the forms to be ‘congruent mod \( p^m \) outside \( Np \)’ in the above sense.

The first approach is to generalize a theorem of Serre-Katz on \( p \)-adic modular forms, cf. Cor. 4.4.2 of [5] which – under certain restrictions on the levels of the forms – gives a necessary congruence between the weights for the forms to be congruent mod \( p^m \). In the Serre-Katz theorem one needs to assume that the prime \( p \) of the field \( K \) of coefficients is unramified relative to \( p \) in \( \mathbb{Q} \). We are able to generalize this theorem to cases where \( p \) is ramified over \( p \).

Under certain technical restrictions, in particular that the ramification index relative to \( p \) of the Galois closure of the field \( K \) of coefficients is not divisible by \( p \), and that \( p \) is odd, our Theorem 1 results in the desired computable necessary and sufficient conditions. See Corollary 1 below.

The second approach is via a study of the determinants of the attached mod \( p^m \) representations. Again under certain technical restrictions, here notably a restriction on the nebentypus characters of the forms, our Theorem 2 leads to the desired computable necessary and sufficient conditions. Cf. Corollary 2 below.

It is remarkable that these two rather distinct approaches result – under the technical restrictions alluded to above – in necessary and sufficient conditions that are close to being equivalent.

We illustrate the results by a few numerical examples.

Finally, let us mention that Kohnen has considered similar questions, but only in the mod \( p \) case, see [6], and our results can also be regarded as a generalization of Kohmen’s results to the mod \( p^m \) setting.

1.1. Notation. To formulate our results, let us introduce the following notation:

Define

\[
N' := \begin{cases} 
N \cdot \prod_{q \mid N} q^{-1} , & \text{if } p \mid N \\
N \cdot p^2 \cdot \prod_{q \mid N} q^{-1} , & \text{if } p \nmid N
\end{cases}
\]

where the products are over prime divisors \( q \) of \( N \). Put:

\[
\mu := [\text{SL}_2(\mathbb{Z}) : \Gamma_1(N)] , \quad \mu' := [\text{SL}_2(\mathbb{Z}) : \Gamma_1(N')]
\]
and fix the following notation:
\[\begin{align*}
m & : \text{ a natural number,} \\
k & : \max\{k_1, k_2\}, \\
p & : \text{ a fixed prime of } K \text{ over } p, \\
e & : e(p/p), \text{ the ramification index of } p \text{ over } p, \\
L & : \text{ Galois closure of } K/\mathbb{Q}, \\
e(L, p) & : \text{ the ramification index of } L \text{ relative to } p \text{ in } \mathbb{Q}, \\
r & : \text{ largest power of } p \text{ dividing the ramification index } e(L, p), \\
\ell & : \text{ a (not fixed) prime number.}
\end{align*}\]

For a natural number \(a\) and a modular form \(f = \sum c_n q^n\) on some \(\Gamma_1(M)\) and coefficients \(c_n\) in \(\mathcal{O}_K\) we define:

\[\ord_p h = \inf\{n \mid p^n \nmid (c_n)\},\]

with the convention that \(\ord_p h = \infty\) if \(p^n \mid (c_n)\) for all \(n\).

We say that \(f_1\) and \(f_2\) are congruent modulo \(p^a\) if \(\ord_p(f_1 - f_2) = \infty\), and we denote this by \(f_1 \equiv f_2 \mod{p^a}\).

1.2. Results. The following proposition is the first, basic observation, and is an easy generalization of a well-known theorem of Sturm, cf. [10].

**Proposition 1.** Suppose that \(N\) is arbitrary, but that \(f_1\) and \(f_2\) are forms on \(\Gamma_1(N)\) of the same weight \(k = k_1 = k_2\) and coefficients in \(\mathcal{O}_K\).

Then \(\ord_p (f_1 - f_2) > k\mu/12\) implies \(f_1 \equiv f_2 \mod{p^m}\).

Part (i) of the following theorem is a slight generalization of theorems of Serre and Katz, cf. [9], Théorème 1, [5], Corollary 4.4.2.

**Theorem 1.** Assume \(N \geq 3\), and let \(f_1\) and \(f_2\) be normalized cusp forms on \(\Gamma_1(N)\) of weights \(k_1\) and \(k_2\), respectively, and with coefficients in \(\mathcal{O}_K\).

(i) Assume additionally that \(p \nmid N\) and that \(f_1\) and \(f_2\) are forms on \(\Gamma_1(N) \cap \Gamma_0(p)\).

Then if \(f_1 \equiv f_2 \mod{p^m}\) we have \(k_1 \equiv k_2 \mod{p^s(p-1)}\) with the non-negative integer \(s\) defined as follows:

\[s := \begin{cases} 
\max\{0, \left\lceil \frac{m}{e} \right\rceil - 1 - r\}, & \text{if } p \geq 3 \\
\max\{0, \alpha\left(\left\lceil \frac{m}{e} \right\rceil - r\)\}, & \text{if } p = 2 
\end{cases}\]

with \(\alpha(u)\) defined for \(u \in \mathbb{Z}\) as follows:

\[\alpha(u) := \begin{cases} 
u - 1, & \text{if } u \leq 2 \\
u - 2, & \text{if } u \geq 3. 
\end{cases}\]

(ii) Let \(N\) be arbitrary, but assume \(3 \mid N\) if \(p = 2\), and \(2 \mid N\) if \(p = 3\).

Suppose that \(k_1 \equiv k_2 \mod{p^s(p-1)}\).

Then, if \(a_\ell(f_1) \equiv a_\ell(f_2) \mod{p^m}\) for all primes \(\ell \leq k\mu/12\) with \(\ell \nmid \text{NP}\), we have

\[a_\ell(f_1) \equiv a_\ell(f_2) \mod{p^{\min\{\ell \cdot (s+1), m\}}}
\]

for all primes \(\ell \nmid \text{NP}\).

In particular, if either \(m \leq e\) or \((p > 2 \text{ and } r = 0)\) or \((p = 2, r = 0, \text{ and } m \leq 2e)\), we have

\[a_\ell(f_1) \equiv a_\ell(f_2) \mod{p^m}\]
for all primes $\ell \nmid Np$.

The following corollary is an immediate consequence of Theorem 1.

**Corollary 1.** Retain the setup and notation of Theorem 1, and assume that $p$ is odd, $r = 0$, that $N$ is prime to $p$, that $3 \mid N$ if $p = 2$, and $2 \mid N$ if $p = 3$, and that $f_1$ and $f_2$ are forms on $\Gamma_1(N) \cap \Gamma_0(p)$.

Then we have $a_{\ell}(f_1) \equiv a_{\ell}(f_2) \left( p^m \right)$ for all primes $\ell \nmid Np$ if and only if this congruence holds for all primes $\ell \leq k\mu'/12$ with $\ell \nmid Np$ and we have the congruence

$$k_1 \equiv k_2 \pmod{p^s(p-1)}$$

between the weights.

**Theorem 2.** Suppose that $N$ is arbitrary, but assume that $p$ is odd and that $f_1$ and $f_2$ are normalized cusp forms on $\Gamma_1(N)$ of weights $k_1$ and $k_2$ and with nebentypus characters $\psi_1$ and $\psi_2$, respectively.

Suppose that $f_1$ and $f_2$ are eigenforms outside $Np$ and have coefficients in $O_K$, and that the mod $p$ Galois representation attached to $f_1$ is absolutely irreducible.

View the nebentypus characters $\psi_i$ as finite order characters on $G_F$, and let the order of the character $\left( \psi_2\psi_1^{-1} \mod p^m \right)_{|I_p}$

where $I_p$ is an inertia group at $p$, be $p^s \cdot d$ with $d$ a divisor of $p - 1$.

(i) If we have $a_{\ell}(f_1) \equiv a_{\ell}(f_2) \left( p^m \right)$ for all primes with $\ell \nmid Np$ then $\delta \leq \left[ \frac{m}{\ell} \right] - 1$ and we have:

$$k_1 \equiv k_2 \pmod{p^{\left[ \frac{m}{\ell} \right] - \delta} \cdot (p - 1)/d}$$

so that in particular, $k_1 \equiv k_2 \pmod{p^{\left[ \frac{m}{\ell} \right] - 1} \cdot (p - 1)/d}$ if $\delta = 0$.

(ii) Suppose that

$$k_1 \equiv k_2 \pmod{p^{\left[ \frac{m}{\ell} \right]} - 1} \cdot (p - 1)/d).$$

Then, if $a_{\ell}(f_1) \equiv a_{\ell}(f_2) \left( p^m \right)$ for all primes $\ell \leq k\mu'/12$ with $\ell \nmid Np$ we have this congruence for all primes $\ell \nmid Np$.

The following corollary follows immediately from Theorem 2.

**Corollary 2.** Retain the setup and notation of Theorem 2, and assume that $\delta = 0$.

Then we have $a_{\ell}(f_1) \equiv a_{\ell}(f_2) \left( p^m \right)$ for all primes $\ell \nmid Np$ if and only if this congruence holds for all primes $\ell \leq k\mu'/12$ with $\ell \nmid Np$ and we have the congruence

$$k_1 \equiv k_2 \pmod{p^{\left[ \frac{m}{\ell} \right]} - 1} \cdot (p - 1)/d)$$

between the weights.

**Remark:** Obtaining results like those in the corollaries, but in more general situations, for instance with $r$ and $\delta$ not necessarily 0, are obvious problems for future work. We suspect such questions will be more involved.
2. Proofs

Let us first prove Proposition 1 that turns out to be an easy generalization of a theorem by Sturm, cf. [10].

Proof of Proposition 1: We prove this by induction on \( m \). It will be convenient to prove a slightly more general statement, namely that the proposition holds for forms with coefficients in \((\mathcal{O}_K)_p\), the localization of \( \mathcal{O}_K \) w.r.t. \( p \): If \( h \) is such a form we can define \( \text{ord}_p(h) \) in the same manner as above, and the claim is then that \( \text{ord}_p(h) > k\mu/12 \) implies \( \text{ord}_p(h) = \infty \).

This statement for \( m = 1 \) follows immediately from a theorem of Sturm, cf. Theorem 1 of [10]: If \( h \) is a form on \( \Gamma_1(N) \) of weight \( k \) and coefficients in \((\mathcal{O}_K)_p\) then there is a number \( \alpha \in \mathcal{O}_K \setminus p \) such that \( \alpha \cdot h \) has coefficients in \( \mathcal{O}_K \); this follows from the ‘bounded denominators’ property for modular forms. Then, if \( \text{ord}_p(h) > k\mu/12 \) we have \( \text{ord}_p(\alpha \cdot h) > k\mu/12 \) and by Sturm this implies \( \text{ord}_p(\alpha \cdot h) = \infty \) and so also \( \text{ord}_p(h) = \infty \).

Assume that \( m > 1 \), and that the proposition in the above slightly more general form is true for powers \( p^n \) of \( p \) with \( a < m \). Consider then forms \( f_1 \) and \( f_2 \) on \( \Gamma_1(N) \) of weight \( k \) with coefficients in \((\mathcal{O}_K)_p\) such that \( \text{ord}_p(f_1 - f_2) > k\mu/12 \). Let \( \varphi = f_1 - f_2 \). By assumption we have \( \text{ord}_p \varphi > k\mu/12 \), and therefore also \( \text{ord}_{p^{-1}} \varphi > k\mu/12 \), and hence the induction hypothesis gives \( \text{ord}_{p^{-1}} \varphi = \infty \). Choose a uniformizer \( \pi \) for \( p \), i.e., an element \( \pi \in p \setminus p^2 \).

We see that the form
\[
\psi := \frac{1}{\pi^{m-1}} \cdot \varphi
\]
is a form on \( \Gamma_1(N) \) of weight \( k \) with coefficients in \((\mathcal{O}_K)_p\).

Since \( \text{ord}_p \varphi > k\mu/12 \), we must have \( \text{ord}_p \psi > k\mu/12 \), so that \( \text{ord}_p \psi = \infty \) by the induction hypothesis for \( m = 1 \). From this we conclude that \( \text{ord}_p \varphi = \infty \), as desired.

In subsequent arguments we occasionally need the following simple and probably well-known lemma.

Lemma 1. Let \( F'/F \) be a finite extension of number fields. Let \( q \) be a prime ideal of \( F \) and let \( \mathfrak{q} \) be a prime ideal of \( F' \) over \( q \) of ramification index \( \epsilon \). Let \( b \) be a positive integer.

Then
\[
\mathfrak{q}^b \cap F = q^{\lceil \frac{b}{\epsilon} \rceil}.
\]

Proof. There is a non-negative integer \( a \) such that \( a\epsilon < b \leq (a + 1)\epsilon \), and then we have
\[
\mathfrak{q}^{(a+1)\epsilon} \subseteq \mathfrak{q}^b \subseteq \mathfrak{q}^{a\epsilon}.
\]
From this we get that
\[
q^{a+1} = \mathfrak{q}^{(a+1)\epsilon} \cap F \subseteq \mathfrak{q}^b \cap F \subseteq \mathfrak{q}^{a\epsilon} \cap F = q^a,
\]
and so \( \mathfrak{q}^b \cap F \) is either \( q^a \) or \( q^{a+1} \).

Assume that \( \mathfrak{q}^b \cap F = q^a \). Then \( q^a \subseteq \mathfrak{q}^b \), i.e., \( \mathfrak{q}^{a\epsilon} \subseteq \mathfrak{q}^b \), and so \( a\epsilon \geq b \), a contradiction. We conclude that \( \mathfrak{q}^b \cap F = q^{a+1} \), and since \( a + 1 = \lceil \frac{b}{\epsilon} \rceil \) by the definition of \( a \), we are done.
Part (i) of Theorem 1 can be seen as a generalization of a theorem of Serre and Katz, cf. Cor. 4.4.2 of [5], and Katz’ theorem is also the main point of the proof.

Proof of part (i) of Theorem 1: Recall that $L$ denotes the Galois closure of $K$. Let us fix a prime $\mathfrak{P}$ over $\mathfrak{p}$ in the Galois closure $L$ of $K$. Thus, the ramification index $e(L, p)$ is the ramification index of $e(\mathfrak{P}/p)$ of $\mathfrak{P}$ relative to $p$ in $\mathbb{Q}$. Recall that we denote the ramification index $e(\mathfrak{p}/p)$ by $e$.

Let $L_0$ be the subfield of $L$ corresponding to the inertia group $I(\mathfrak{P}/p)$. Let $\mathfrak{p}_0$ be the prime of $L_0$ under $\mathfrak{P}$.

We now let $I(\mathfrak{P}/p)$ act on the $f_i$ by acting on their Fourier coefficients. Since $f_1 \equiv f_2 (p^n)$ we have $\sigma(f_1) \equiv \sigma(f_2) (\mathfrak{P}^{m-e(\mathfrak{P}/p)})$ for all $\sigma \in I(\mathfrak{P}/p)$. Letting

$$F_1 = \sum_{\sigma} \sigma(f_1) \quad \text{and} \quad F_2 = \sum_{\sigma} \sigma(f_2)$$

with the sums taken over all $\sigma \in I(\mathfrak{P}/p)$, we therefore obtain

$$F_1 \equiv F_2 (\mathfrak{P}^{m-e(\mathfrak{P}/p)}) .$$

Now, since $F_1$ and $F_2$ are invariant under the action of $I(\mathfrak{P}/p)$ they actually have coefficients in $L_0$, and we therefore have

$$F_1 \equiv F_2 (\mathfrak{P}_0^{[\frac{m}{n}]}),$$

since $\mathfrak{P}^b \cap L_0 = \mathfrak{p}_0^{[\frac{b}{n}]}$ for non-negative integers $b$, cf. Lemma 1, and because

$$e(L, p) = e(\mathfrak{p}/p) e(\mathfrak{P}/p) = e \cdot e(\mathfrak{P}/p) .$$

Now, the extension $(L_0)_{\mathfrak{p}_0}/\mathbb{Q}_\mathfrak{p}$ of local fields is unramified, and so $(L_0)_{\mathfrak{p}_0}$ is the field of fractions of the ring $W = W(\mathbb{F}_{p^f})$ of Witt vectors over $\mathbb{F}_{p^f}$ for some $f$. Since the $F_i$ have integral coefficients in $L_0$, we can view them as having coefficients in $W$.

Now let $a$ be the largest non-negative integer such that all Fourier coefficients of $F_1$ and $F_2$ are divisible by $p^a$. Then the forms $p^{-a}F_1$ and $p^{-a}F_2$ are cusp forms on $\Gamma_1(N) \cap \Gamma_0(p)$ of weights $k_1$ and $k_2$, respectively, and with coefficients in $W$. At least one of these forms has a $q$-expansion that does not reduce to 0 identically modulo $p$. Their $q$-expansions are congruent modulo

$$p_0^{\max\{0, \frac{m}{n} - a\}}$$

and hence also modulo

$$p_0^{\max\{0, \frac{m}{n} - r\}}$$

since certainly $a \leq r$ because the coefficients of $q$ for both forms $F_i$ equals $\#I(\mathfrak{P}/p)$ which is just $e(L, p)$.

By a theorem of Katz, see Cor. 4.4.2 as well as Theorem 3.2 of [5], we then deduce that

$$k_1 \equiv k_2 \ (p^s(p - 1))$$

where $s$ is given as in the theorem. Notice that we need our hypothesis $N \geq 3$ because of this reference to [5].

To prepare for the proof of part (ii) of Theorem 1 we need the following lemma.

Let us say that a cusp form $h = \sum c_n q^n$ on $\Gamma_1(N)$ and coefficients in $O_K$ is an eigenform mod $\mathfrak{p}^m$ outside $Np$ if it is normalized and we have $T_\ell h \equiv \lambda_\ell h (\mathfrak{p}^m)$ for
all primes \( \ell \nmid Np \) with certain \( \lambda_\ell \in \mathcal{O}_K \). The same argument as in characteristic 0 shows that in that case, the mod \( p^m \) eigenvalues \( \lambda_\ell \) are congruent mod \( p^m \) to the Fourier coefficients \( c_\ell \).

**Lemma 2.** Let \( N \) be arbitrary and let \( f_1 \) and \( f_2 \) be normalized forms of the same weight \( k \) on \( \Gamma_1(N) \) and with coefficients in \( \mathcal{O}_K \).

Suppose that \( f_1 \) and \( f_2 \) are eigenforms mod \( p^m \) outside \( Np \) such that

\[
a_\ell(f_1) \equiv a_\ell(f_2) \pmod{p^m}
\]

for all primes \( \ell \leq k\mu'/12 \) with \( \ell \nmid Np \).

Then \( a_\ell(f_1) \equiv a_\ell(f_2) \pmod{p^m} \) for all primes \( \ell \nmid Np \).

**Proof.** We first apply Lemma 4.6.5 of Miyake [7]: By that lemma we obtain from the \( f_i \) forms \( f'_i \) of weight \( k \) on \( \Gamma_1(N') \) by putting:

\[
f'_i := \sum_{\gcd(n,Np)=1} a_n(f_i) \cdot q^n.
\]

Here, \( N' \) is as defined in the notation section. The forms \( f'_i \) obviously still have coefficients in \( \mathcal{O}_K \).

Now, since the \( f_i \) are eigenforms mod \( p^m \) outside \( Np \), the forms \( f'_i \) are also eigenforms mod \( p^m \) outside \( Np \), with the same eigenvalues \( (a_\ell(f_i) \bmod{p^m}) \).

On the other hand, all Fourier coefficient of the forms \( f'_i \) at any index \( n \) not prime to \( Np \) vanishes. By our hypotheses we can thus conclude that

\[
\text{ord}_{p^m}(f'_1 - f'_2) > k\mu'/12
\]

and by Proposition 1 this implies \( f'_1 \equiv f'_2 \pmod{p^m} \).

But then \( a_\ell(f_1) \equiv a_\ell(f_2) \pmod{p^m} \) for all primes \( \ell \nmid Np \). \( \square \)

**Proof of part (ii) of Theorem 1.** Assume without loss of generality that \( k_2 \geq k_1 \).

We can then write:

\[
k_2 = k_1 + t \cdot p^s(p-1)
\]

where \( t \) is a non-negative integer.

Now, we have an Eisenstein series \( E \) of weight \( p-1 \) on \( \Gamma_1(N) \) with coefficients in \( \mathbb{Z} \) and such that \( E \equiv 1 \pmod{p} \): If \( p \geq 5 \) we can take \( E := E_{p-1} \) the standard Eisenstein series of weight \( p-1 \) on \( \text{SL}_2(\mathbb{Z}) \). If \( p = 2 \) there is, cf. [4] chap. 4.8 for instance, an Eisenstein series of weight 1 on \( \Gamma_1(3) \):

\[
E := 1 - \frac{2}{B_{1,\psi}} \cdot \sum_{n=1}^{\infty} \left( \sum_{d|n} \psi(d) \right) \cdot q^n;
\]

here, \( \psi \) is the primitive Dirichlet character of conductor 3, and \( B_{1,\psi} \) is the first Bernoulli number of \( \psi \). One computes \( B_{1,\psi} = -\frac{1}{4} \), so that in fact \( E \) has coefficients in \( \mathbb{Z} \) and reduces to 1 modulo 2. Also, \( E \) is a modular form on \( \Gamma_1(N) \) as we have assumed \( 3 \mid N \) if \( p = 2 \).

If \( p = 3 \) we choose

\[
E := 1 - 24 \cdot \sum_{n=1}^{\infty} \left( \sum_{d|n} d \right) \cdot q^n;
\]

this is a modular form of weight 2 on \( \Gamma_1(2) \) and hence also on \( \Gamma_1(N) \) as we have \( 2 \mid N \) if \( p = 3 \). Again, cf. for instance [4], chap. 4.6.
With the above choice of $E$ we have in all cases that $E$ is a modular form of weight $p - 1$ on $\Gamma_1(N)$ with coefficients in $\mathbb{Z}$ that reduces to 1 modulo $p$. By induction on $j$ we see that $E^{p^j} \equiv 1 \pmod{p^{j+1}}$ for all non-negative integers $j$, and hence also:

$$E^{t \cdot p^j} \equiv 1 \pmod{p^{j+1}}$$

that we write as $E^{t \cdot p^j} \equiv 1 \pmod{p^{(s+1)}}$. Consequently, the form

$$\tilde{f} := E^{t \cdot p^j} \cdot f_1$$

satisfies $\tilde{f} \equiv f_1 \pmod{p^{(s+1)}}$. If we call $\tilde{a}_n$ the Fourier coefficients of $\tilde{f}$ we have then

$$\tilde{a}_n \equiv a_n(f_1) \pmod{p^{(s+1)}}$$

and thus consequently:

$$\tilde{a}_n \equiv a_n(f_1) (p^{\min\{e \cdot (s+1), m\}})$$

for all primes $\ell \leq k\mu/12$ with $\ell \nmid Np$, because of our hypothesis on $f_1$ and $f_2$.

Now, $\tilde{f}$ and $f_2$ are both forms on $\Gamma_1(N)$ of weight $k = k_2$. As $f_1$ is an eigenform mod $p^m$ outside $Np$, we have that $\tilde{f}$ and $f_2$ are both eigenforms mod $p^{\min\{e \cdot (s+1), m\}}$ outside $Np$. Thus, Lemma 2 implies that

$$\tilde{a}_n \equiv a_n(f_1) (p^{\min\{e \cdot (s+1), m\}})$$

and hence also

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{p^{\min\{e \cdot (s+1), m\}}}$$

for all primes $\ell \nmid Np$.

Using the definition of $s$ one checks that if either $m \leq e$ or $(p > 2$ and $r = 0)$ or $(p = 2$, $r = 0$, and $m \leq 2e$) then we have $e \cdot (s + 1) \geq m$. In each of those cases we thus have

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{p^m}$$

for all primes $\ell \nmid Np$. \hfill \Box

Proof of Theorem 2: Proof of part (i): Consider the representations $\rho_{f_1, p^m}$ attached to the forms $f_1$.

Since $a_\ell(f_1) \equiv a_\ell(f_2) \pmod{p^m}$ for all primes $\ell \nmid Np$ we can conclude by Chebotarev’s density theorem that the representations $\rho_{f_1, p^m}$ and $\rho_{f_2, p^m}$ have the same traces. As $\rho_{f_1, p^m}$ is assumed absolutely irreducible, Théorème 1 of Carayol [2] then implies that $\rho_{f_1, p^m}$ and $\rho_{f_2, p^m}$ are isomorphic. Hence, the determinants of these representations are also isomorphic. These determinants are:

$$\det \rho_{f_1, p^m} = (\psi_i \cdot \chi^{k_1-1} \pmod{p^m})$$

where $\chi$ denotes the $p$-adic cyclotomic character $\chi : G_Q \to \mathbb{Z}_p^\times$, and the nebentypus characters $\psi_i$ are now seen as finite order characters on $G_Q$. Observe that the characters $\psi_i$ take values in $O_K$ so that it makes sense to reduce them mod $p^m$. Also, reducing $\chi$ mod $p^m$ is to be taken in the obvious sense.

We can now deduce that

$$(\psi_2 \psi_1^{-1} \pmod{p^m})|_{I_p} = (\chi \pmod{p^m})^{k_1-k_2}_{|_{I_p}}.$$ 

Now let us view via local class field theory the character $(\chi \pmod{p^m})|_{I_p}$ as a character on $\mathbb{Z}_p^\times$. As such it factors through $(\mathbb{Z}/p^m)^{\times}$ and has order

$$p^{\left(\frac{m}{p-1}\right)} - 1, (p-1);$$
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By definition, the character $(\psi_2 \psi_1^{-1} \mod p^m)_{\mu_p}$ has order $p^\delta \cdot d$ with $d$ a divisor of $p - 1$. Hence, first we see that $p^\delta \cdot d$ is a divisor of $p^{\lceil \frac{m}{e} \rceil - 1} \cdot (p - 1)$ which implies that $\delta \leq \lceil \frac{m}{e} \rceil - 1$. Secondly, we then conclude that $k_1 - k_2$ is divisible by $p^{\lceil \frac{m}{e} \rceil - 1 - \delta} \cdot (p - 1)/d$ as desired.

**Proof of part (ii):** Observe first the following. If $p \nmid N$ then upon replacing $N$ by $Np$ and then calculating $\mu'$ we end up with the same number $\mu'$ as had we calculated it from $N$. And of course our forms are also forms on $\Gamma_1(Np)$.

This means that we may well assume that $N$ is divisible by $p$, – our hypotheses remain unchanged when $N$ is replaced by $Np$ if $N$ is not divisible by $p$.

In particular, we may assume that the group $\Gamma_1(N)$ is contained in $\Gamma_1(p)$.

Now assume without loss of generality that $k_2 \geq k_1$. Our hypotheses imply that we can then write:

$$k_2 = k_1 + t \cdot p^{\lceil \frac{m}{e} \rceil - 1} \cdot (p - 1)/d$$

with $t$ a non-negative integer.

Since $p$ is odd there is a certain Eisenstein series $E$ on $\Gamma_1(p)$ of weight

$$\kappa := (p - 1)/d$$

and $p'$-adically integral coefficients in the field $\mathbb{Q}(\mu_{p-1})$ of $(p - 1)$’st roots of unity with $p'$ a prime of $\mathbb{Q}(\mu_{p-1})$ over $p$, and which reduces to 1 modulo $p'$: $E$ is the form derived from

$$G := L(1 - \kappa, \omega^{-\kappa})/2 + \sum_{n=1}^{\infty} \left( \sum_{d|n} \omega^{-\kappa}(d) \cdot d^{\kappa-1} \right)$$

by scaling so that the constant term is 1. Here, $\omega$ is the character that becomes the Teichmüller character when viewed as taking values in $\mathbb{Z}_p^\times$. Cf. Serre, [9], Lemme 10, and Ribet, [8], §2.

Now view $E$ as having coefficients in the compositum $M$ of $K$ and $\mathbb{Q}(\mu_{p-1})$. Pick a prime $p_1$ of $M$ over $p$ and $p'$. Then the ramification index of $p_1$ relative to $p$ is $e$. We deduce that

$$E^{\lceil \frac{m}{e} \rceil - 1} \equiv 1 \mod p_1^{m}$$

and so $\tilde{f} := f_1 \cdot E^{\lceil \frac{m}{e} \rceil - 1} \equiv f_1 \mod p_1^{m}$. As now $\tilde{f}$ is a form on $\Gamma_1(N)$ (as $N$ is divisible by $p$ and $E$ is on $\Gamma_1(p)$) of weight

$$k_1 + p^{\lceil \frac{m}{e} \rceil - 1} \cdot t \cdot \kappa = k_2$$

we can finish the argument in the same way as in the proof of part (ii) of Theorem 1.

**3. Examples**

We used the mathematics software program MAGMA [1] to find examples illustrating Theorem 1. We looked for examples of higher congruences and where $p$ is ramified in the field of coefficients. In the notation of this paper, what we are looking for are situations where $e > 1$ and $s \geq 1$. Here are 2 such examples.

We start with

$$f_1 = q - 8q^4 + 20q^7 + \cdots,$$
the (normalized) cusp form on $\Gamma_0(9)$ of weight 4 with integral coefficients, and look for congruences of the coefficients of $f_1$ and $f_2$ modulo powers of a prime above 5, for a form $f_2$ of weight $k_2$ satisfying $k_2 \equiv 4 \pmod{5 \cdot (5 - 1)}$.

The smallest possible choice of weight for $f_2$ is $k_2 = 24$. There is a newform $f_2$ on $\Gamma_0(9)$ of weight 24 with coefficients in the number field $K = \mathbb{Q}(\alpha)$ with $\alpha$ a root of $x^4 - 29258x^2 + 97377280$. The prime 5 is ramified in $K$ and has the decomposition $5O_K = p^2p_2$.

We have $k = 24$, $N = 9$, $N' = 675$ and $\mu' = 1080$, and we find that $a_\ell(f_1) \equiv a_\ell(f_2) \pmod{p^3}$ for primes $\ell \leq k\mu'/12 = 2160$ with $\ell \not\equiv 3, 5$.

Since $[K : \mathbb{Q}] = 4$, the Galois closure $L$ of $K$ satisfies $[L : \mathbb{Q}] \mid 24$ (in fact $[L : \mathbb{Q}] = 8$ in this case). This shows that $5 \nmid e(L, 5)$, i.e., $r = 0$. Since we also have $m = 3$ and $e = e(p) = 2$, we get $s = 1$ as desired. By Theorem 1 we conclude that $a_\ell(f_1) \equiv a_\ell(f_2) \pmod{p^3}$ for all primes $\ell \not\equiv 3, 5$.

Similarly we find a newform $f_3$ on $\Gamma_0(9)$ of weight $k_3 = 44$ with coefficients in a number field $K' = \mathbb{Q}(\beta)$ with $\beta$ a root of $x^8 - 438896x^6 + 60873718294x^4 - 2968020622607040x^2 + 40426030666768772025$.

As before 5 is ramified in $K'$ and has the decomposition $5O_{K'} = p^4p_2^2p_3^2$, and thus $e = 4$. One finds that $a_\ell(f_1) \equiv a_\ell(f_3) \pmod{p^5}$ for primes $\ell \leq k_3\mu'/12 = 3960$ with $\ell \not\equiv 3, 5$. The Galois closure $L'$ of $K'$ satisfies $[L' : \mathbb{Q}] = 384 \not\equiv 0 \pmod{5}$, which again implies $r = 0$. With $m = 5$ we have $s = 1$ and conclude by Theorem 1 that $a_\ell(f_1) \equiv a_\ell(f_3) \pmod{p^5}$ for all primes $\ell \not\equiv 3, 5$.

We are developing a larger database of similar examples. This will be reported on elsewhere.

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