HIGHER AUSLANDER CORRESPONDENCE FOR DUALIZING $R$-VARIETIES

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Abstract. Let $R$ be a commutative artinian ring. We extend higher Auslander correspondence from Artin $R$-algebras of finite representation type to dualizing $R$-varieties. More precisely, for a positive integer $d$, we show that a dualizing $R$-variety is $d$-abelian if and only if it is a $d$-Auslander dualizing $R$-variety if and only if it is equivalent to a $d$-cluster-tilting subcategory of the category of finitely presented modules over a dualizing $R$-variety.

1. Introduction

Throughout this article we fix a commutative artinian ring $R$. Recall that an Artin $R$-algebra is an $R$-algebra which is a finitely generated $R$-module. An Artin $R$-algebra $A$ is an Auslander algebra if

$$\text{gl. dim } A \leq 2 \leq \text{dom. dim } A,$$

where $\text{dom. dim } A$ is the dominant dimension of $A$ in the sense of Tachikawa [Tac64]. The notion of dominant dimension was further developed by Auslander’s school in their beautiful theory [AB69, FGR75, AR94, AR96] of Auslander–Goldman rings. One of the most important aspects of Auslander–Goldman rings was already given in [Aus71], where Auslander established a one-to-one correspondence between Auslander algebras and Artin algebras of finite representation type up to Morita equivalence.

It is natural to extend this correspondence to arbitrary Artin algebras. This extension is better expressed in the language of dualizing $R$-varieties, which are additive $R$-categories $A$ enjoying a certain duality between finitely presented $A$-modules and finitely presented $A^{op}$-modules. One can think of a dualizing $R$-variety as an analog of the category of finitely generated projective modules over an Artin algebra, but with possibly infinitely many indecomposable objects up to isomorphism. Auslander correspondence can be extended to the following characterization of categories of finitely presented modules over dualizing $R$-varieties. Although this characterization should be well known to specialists, we did not find it in the literature. We refer the reader to Section 2 for the definitions of dualizing $R$-variety and Auslander dualizing $R$-variety.

Theorem 1.1. Let $A$ be a dualizing $R$-variety. Then, the following statements are equivalent.

(a) The category $A$ is abelian.
(b) There exist a dualizing $R$-variety $B$ and an equivalence $A \cong \text{mod} \ B$.

c) The category $A$ is an Auslander dualizing $R$-variety.

In fact, Theorem 1.1 is a particular case of a more general result, Theorem 1.2 below, which gives a characterization of $d$-cluster-tilting subcategories of categories of finitely presented modules over dualizing $R$-varieties. This result is motivated by a “higher dimensional” version of Auslander correspondence which we briefly recall.

From the viewpoint of higher dimensional Auslander–Reiten theory, the first author introduced in [Iya07a] the class of $d$-Auslander algebras, which are the Artin $R$-algebras $A$ such that

$$\text{gl.dim} \ A \leq d + 1 \leq \text{dom.dim} \ A,$$

where $d$ is a positive integer. Thus, if $d = 1$, then one recovers the classical Auslander algebras. Moreover, a one-to-one correspondence was established in [Iya07a, Thm. 02] between Morita equivalence classes of $d$-Auslander algebras and equivalence classes of $d$-cluster-tilting subcategories with additive generators of categories of finitely presented modules over Artin algebras. One can think of a $d$-cluster-tilting subcategory as a higher analog of the module category. The notion of $d$-abelian category was introduced in [Jas14] in order to make precise part of this analogy. Indeed, $d$-cluster-tilting subcategories are typical examples of $d$-abelian categories.

The following natural generalization of [Iya07a, Thm. 02] to the setting of dualizing $R$-varieties is one of the main results of this article. We refer the reader to Section 2 for the definitions of $d$-abelian category, $d$-Auslander dualizing $R$-variety, and $d$-cluster-tilting subcategory.

**Theorem 1.2** (Auslander correspondence). Let $A$ be a dualizing $R$-variety and $d$ a positive integer. Then, the following statements are equivalent.

(a) The category $A$ is $d$-abelian.

(b) There exist a dualizing $R$-variety $B$ and a fully faithful functor $F : A \to \text{mod} \ B$ such that $FA$ is a $d$-cluster-tilting subcategory of $\text{mod} \ B$.

(c) The category $A$ is a $d$-Auslander dualizing $R$-variety.

(d) We have $\text{gl.dim} \ A \leq d + 1 \leq \text{dom.dim} \ A$, $\perp A \subset A^{-d}$ and $A^{\oplus \perp} \subset A^{\perp \oplus}$.

Note that the implication (b)$\Rightarrow$(a) in Theorem 1.2 is shown in [Jas14, Thm. 3.16].

Also, we mention that a partial version of Theorem 1.2, which does not involve the relationship with $d$-abelian categories (one of the main points of our result), is shown in [Bel15, Thm. 8.4] in a more general setting than that of dualizing $R$-varieties.

As a consequence of Theorem 1.2, we obtain a characterization of $d\mathbb{Z}$-cluster-tilting subcategories, that is those $d$-cluster-tilting subcategories which are closed under $d$-syzygies and $d$-cosyzygies. We refer the reader to Section 2 for definitions of $d\mathbb{Z}$-cluster-tilting subcategory and of $d$-abelian categories having $d$-syzygies (resp. $d$-cosyzygies).

**Theorem 1.3** (Homological Auslander correspondence). Let $A$ be a dualizing $R$-variety and $d$ a positive integer. Then, the following statements are equivalent.

(a) The category $A$ is $d$-abelian and has $d$-cosyzygies.

(b) There exist a dualizing $R$-variety $B$ and a fully faithful functor $F : A \to \text{mod} \ B$ such that $FA$ is a $d\mathbb{Z}$-cluster-tilting subcategory of $\text{mod} \ B$.

(c) The category $A$ is a $d$-Auslander dualizing $R$-variety satisfying

$$\Omega^{-d}(A_A) \subset \Omega(A_A).$$

Moreover, the three equivalent statements above are equivalent to the following statements:
(a) The category $\mathcal{A}$ is $d$-abelian and has $d$-syzygies.

(b) There exist a dualizing $R$-variety $\mathcal{B}$ and a fully faithful functor $\mathcal{F}: \mathcal{A}^{\text{op}} \to \text{mod} \mathcal{B}$ such that $\mathcal{F} \mathcal{A}^{\text{op}}$ is a $d\mathbb{Z}$-cluster-tilting subcategory of $\text{mod} \mathcal{B}$.

(c) The category $\mathcal{A}$ is a $d$-Auslander dualizing $R$-variety satisfying

$$\Omega^d(D \mathcal{A}_A) \subseteq \Omega^{-1}((D \mathcal{A}_A)^{\perp}).$$

Note that the implication (b)⇒(a) in Theorem 1.3 is shown in [Jas14, Thm. 5.16].

Conceptually, Theorems 1.2 and 1.3 enhance our understanding of the relationship between $d$-abelian categories and $d$-cluster-tilting subcategories.

Finally, let us describe the structure of the article. In Section 2 we recall the definitions and results needed in the sequel. In Section 3 we give a proof of Theorem 1.2. In Section 4 we give a proof Theorem 1.3. Finally, in Section 5 we provide examples illustrating our results.

**Conventions.** We fix a commutative artinian ring $R$ together with a positive integer $d$. We denote by $\text{mod} \ R$ the category of finitely presented $R$-modules. Let $I$ be the injective envelope of $R/\text{rad} R$ and $D := \text{Hom}_R(-, I)$ the usual duality. All categories we consider are assumed to be additive and Krull–Schmidt, that is every object decomposes as a finite direct sum of objects whose endomorphism rings are local. Moreover, all categories are assumed to be $R$-linear and all functors are assumed to be $R$-linear and additive. Let $\mathcal{A}$ be an $R$-category and $a, a' \in \mathcal{A}$. We denote the $R$-module of morphisms $a \to a'$ by $\mathcal{A}(a, a')$. We denote the Yoneda embedding by $a \mapsto P_a := \mathcal{A}(-, a)$. By subcategory we mean full subcategory which is closed under isomorphisms. Under our standing assumptions, for an object $a \in \mathcal{A}$ we denote its additive closure by $\text{add} a$. Recall that $\text{add} a$ is the smallest subcategory of $\mathcal{A}$ containing $a$ and which is closed under finite direct sums and direct summands.

## 2. Preliminaries

In this section we recall the notion of dualizing $R$-variety introduced by Auslander and Reiten in [AR74]. We also recall the definitions of $d$-cluster-tilting subcategory and $d$-abelian category as well as technical results which are needed in the proofs of the main theorems.

### 2.1. Dualizing $R$-varieties

We begin by recalling the basics on functor categories and finitely presented modules. We refer the reader to [Aus66] for a thorough development of these concepts.

Let $\mathcal{A}$ be an essentially small category. We assume that $\mathcal{A}$ is a $\text{Hom}$-finite $R$-category, that is for all $a, a' \in \mathcal{A}$ the $R$-module $\mathcal{A}(a, a')$ is finitely generated and the composition

$$\mathcal{A}(a, a') \otimes \mathcal{A}(a', a'') \longrightarrow \mathcal{A}(a, a'')$$

is $R$-bilinear. A (right) $\mathcal{A}$-module is a contravariant $R$-linear additive functor $\mathcal{A} \to \text{Mod} R$; a morphism $M \to N$ between $\mathcal{A}$-modules $M$ and $N$ is a natural transformation. Thus, we obtain an abelian category of $\mathcal{A}$-modules denoted by $\text{Mod} \mathcal{A}$. If $M, N$ are $\mathcal{A}$-modules, we denote the set $(\text{Mod} \mathcal{A})(M, N)$ of natural transformations $M \to N$ by $\text{Hom}_{\mathcal{A}}(M, N)$, which is an $R$-module in a natural way.

An $\mathcal{A}$-module $M$ is finitely generated if there exist an epimorphism $P_a \to M$ for some $a \in \mathcal{A}$. Via the Yoneda embedding we identify $\mathcal{A}$ with the full subcategory $\mathcal{A}_A$ of $\text{Mod} \mathcal{A}$ of finitely generated projective $\mathcal{A}$-modules. With some abuse of notation, we write $\mathcal{A}_{A^{\text{op}}} := \mathcal{A}_{A^{op}}^{\text{op}}$ for the category of finitely generated projective $\mathcal{A}^{\text{op}}$-modules, which is a subcategory of $\text{Mod}(\mathcal{A}^{\text{op}})$. An $\mathcal{A}$-module $M$ is finitely
presented} if there exist a morphism \( f : a \to a' \) in \( \mathcal{A} \) and an exact sequence
\[
P_a \xrightarrow{P_f} P_{a'} \to M \to 0.
\]

We denote the full subcategory of \( \text{Mod} \mathcal{A} \) of finitely presented \( \mathcal{A} \)-modules by \( \text{mod} \mathcal{A} \). Note that \( \text{mod} \mathcal{A} \) is closed under cokernels and extensions (by the Horseshoe Lemma) in \( \text{Mod} \mathcal{A} \), and that it is closed under kernels in \( \text{Mod} \mathcal{A} \) if and only if \( \mathcal{A} \) has weak kernels, see [Aus66]. Therefore, \( \text{mod} \mathcal{A} \) is an exact abelian subcategory of \( \text{Mod} \mathcal{A} \) if and only if \( \mathcal{A} \) has weak kernels.

**Definition 2.1.** [AR74] An essentially small \( R \)-linear additive Krull–Schmidt category \( \mathcal{A} \) is a **dualizing \( R \)-variety** if the contravariant functor \( \text{Mod} \mathcal{A} \to \text{Mod}(\mathcal{A}^{\text{op}}) \) given by \( M \mapsto D \circ M \) induces a duality \( D: \text{mod} \mathcal{A} \to \text{mod}(\mathcal{A}^{\text{op}}) \).

Let \( \mathcal{A} \) be a dualizing \( R \)-variety. Since \( \text{mod} \mathcal{A} \) and \( \text{mod}(\mathcal{A}^{\text{op}}) \) have cokernels, it follows from the existence of a duality \( \text{mod} \mathcal{A} \to \text{mod}(\mathcal{A}^{\text{op}}) \) that \( \text{mod} \mathcal{A} \) and \( \text{mod}(\mathcal{A}^{\text{op}}) \) are closed under kernels in \( \text{Mod} \mathcal{A} \) and \( \text{Mod}(\mathcal{A}^{\text{op}}) \) respectively. Therefore \( \text{mod} \mathcal{A} \) and \( \text{mod}(\mathcal{A}^{\text{op}}) \) are abelian categories with enough projectives and injectives, see [AR74, Thm. 2.4].

The most basic examples of dualizing \( R \)-varieties arise from Artin algebras. Recall that if \( \mathcal{A} \) is an Artin algebra, then there is an equivalence \( \text{mod} \mathcal{A} \cong \text{mod}(\text{proj} \mathcal{A}) \).

**Proposition 2.2.** [AR74, Prop. 2.5] Let \( \mathcal{A} \) be an Artin algebra. Then, the category \( \text{proj} \mathcal{A} \) of finitely generated projective \( \mathcal{A} \)-modules is a dualizing \( R \)-variety.

The importance of the concept of dualizing \( R \)-variety comes from the following result which was instrumental in the first proof of the existence theorem of almost-split sequences. It allows us to investigate dualizing \( R \)-varieties and their categories of finitely presented modules using the same representation-theoretic methods.

**Proposition 2.3.** [AR74, Prop. 2.6] Let \( \mathcal{A} \) be a dualizing \( R \)-variety. Then, \( \text{mod} \mathcal{A} \) is a dualizing \( R \)-variety.

Let \( \mathcal{A} \) be a dualizing \( R \)-variety. Recall that a subcategory \( \mathcal{B} \) of \( \mathcal{A} \) is **contravariantly finite** if for all \( a \in \mathcal{A} \) there exist \( b \in \mathcal{B} \) and a morphism \( f : b \to a \) such that the sequence
\[
\mathcal{A}(\cdot, b)|_{\mathcal{B}} \xrightarrow{P_f} \mathcal{A}(\cdot, a)|_{\mathcal{B}} \to 0
\]
is exact. Such a morphism \( f \) is called a **right \( \mathcal{B} \)-approximation of \( a \)**. **Covariantly finite** subcategories of \( \mathcal{A} \) are defined dually. The subcategory \( \mathcal{B} \) is **functorially finite** if it is both contravariantly finite and covariantly finite. The following well known result is a basic tool for constructing dualizing \( R \)-varieties, see [AS81, Thm. 2.3] and [Iya07a, Prop. 1.2] for a general statement.

**Proposition 2.4.** Let \( \mathcal{A} \) be a dualizing \( R \)-variety and \( \mathcal{B} \) a functorially finite subcategory of \( \mathcal{A} \). Then, \( \mathcal{B} \) is a dualizing \( R \)-variety.

**Lemma 2.5.** Let \( \mathcal{A} \) be a dualizing \( R \)-variety. Then, \( \mathcal{A}_{\mathcal{A}} \) and \( (\text{DA})_{\mathcal{A}} \) are functorially finite in \( \text{mod} \mathcal{A} \).

**Proof.** By duality, it suffices to show that \( \mathcal{A}_{\mathcal{A}} \) is functorially finite in \( \text{mod} \mathcal{A} \). Contravariantly finiteness is clear. Fix \( M \) in \( \text{mod} \mathcal{A} \). Since \( M^* \) is in \( \text{mod}(\mathcal{A}^{\text{op}}) \), we can take a surjection \( P_a^* \to M^* \). It is easy to check that the composition \( M \to M^{**} \to P_a \) is a left \( \mathcal{A}_{\mathcal{A}} \)-approximation of \( M \).

We recall the following fundamental property of dualizing \( R \)-varieties.
Proposition 2.6. [AR74, Prop. 3.4] Let \( \mathcal{A} \) be a dualizing \( R \)-variety. Then every finitely presented \( \mathcal{A} \)-module has a minimal projective (resp. injective) presentation (resp. copresentation). In particular, \( \text{mod} \mathcal{A} \) has projective covers and injective envelopes.

Let \( \mathcal{A} \) be a dualizing \( R \)-variety. We recall the construction of the Auslander–Bridger transpose of an \( \mathcal{A} \)-module. Firstly, Yoneda’s lemma implies that the contravariant left exact functor

\[
(-)^*: \text{Mod} \mathcal{A} \rightarrow \text{Mod}(\mathcal{A}^{\text{op}})
\]

defined by

\[
M^*: = \text{Hom}_{\mathcal{A}}(M, -)|_{\mathcal{A}}
\]

induces a duality \((-)^*: \mathcal{A}_{\mathcal{A}} \rightarrow \mathcal{A}_{\mathcal{A}^{\text{op}}} \) which satisfies \( P_a^* \cong \mathcal{A}(a, -) \). We call this duality the \( \mathcal{A} \)-duality. Secondly, let \( M \in \text{mod} \mathcal{A} \) and choose a projective presentation

\[
P_a \xrightarrow{f} P_{a0} \rightarrow M \rightarrow 0
\]

and set \( \text{Tr} M := \text{coker}(P_a^f) \). Finally, in order to extend \( \text{Tr} \) to a functor, denote by \( \overline{\text{mod}} \mathcal{A} \) the quotient of the category \( \text{mod} \mathcal{A} \) by the ideal of morphisms which factor through a finitely generated projective \( \mathcal{A} \)-module. Using the lifting property of projective \( \mathcal{A} \)-modules it is easy to see that this association induces a well defined functor \( \overline{\text{Tr}}: \overline{\text{mod}} \mathcal{A} \rightarrow \overline{\text{mod}}(\mathcal{A}^{\text{op}}) \) which is called the Auslander–Bridger transpose.

The following result is well known in the case of Artin algebras, see [Aus 66, Prop. 6.3]. We need it in the more general setting of dualizing \( R \)-varieties. Recall that Heller’s syzygy functor \( \Omega: \text{mod} \mathcal{A} \rightarrow \text{mod} \mathcal{A} \) is defined by a short exact sequence

\[
0 \rightarrow \Omega M \rightarrow P_{a0} \rightarrow M \rightarrow 0
\]

The cosyzygy functor \( \Omega^{-1}: \overline{\text{mod}} \mathcal{A} \rightarrow \overline{\text{mod}} \mathcal{A} \) is defined dually.

Proposition 2.7 (Auslander–Bridger sequence). Let \( \mathcal{A} \) be a dualizing \( R \)-variety. For each \( M \in \text{mod} \mathcal{A} \) there exists an exact sequence

\[
0 \rightarrow \text{Ext}^1_{\mathcal{A}^{\text{op}}}(\text{Tr} M, -)|_{\mathcal{A}^{\text{op}}} \rightarrow M \rightarrow M^{**} \rightarrow \text{Ext}^2_{\mathcal{A}^{\text{op}}}(\text{Tr} M, -)|_{\mathcal{A}^{\text{op}}} \rightarrow 0
\]

Moreover, \( M^{**} \in \Omega^2(\overline{\text{mod}} \mathcal{A}) \).

Proof. The proof of [Aus 66, Prop. 6.3] carries over. We give a direct proof for the convenience of the reader. Let \( P_a \rightarrow P_{a'} \rightarrow M \rightarrow 0 \) be a projective presentation of \( M \). By definition, the \( \mathcal{A} \)-duality yields is an exact sequence

\[
0 \rightarrow M^* \rightarrow P_{a'^*} \rightarrow P_{a^*} \rightarrow \text{Tr} M \rightarrow 0.
\]

Let \( P_{b^*} \rightarrow P_{b'} \rightarrow M^* \rightarrow 0 \) be a projective presentation of \( M^* \). Thus, we obtain a commutative diagram

\[
\begin{array}{cccccc}
P_{a^*} & \rightarrow & P_{a'^*} & \rightarrow & P_{b'^*} & \rightarrow & P_{b^*} \\
\downarrow & & \downarrow & & \uparrow & & \uparrow \\
0 & \rightarrow & M & \rightarrow & M^{**} & \rightarrow & 0 \\
\downarrow & & \uparrow & & & & \\
0 & & 0 & & & & 
\end{array}
\]

in which the sequences \( P_{a'^*} \rightarrow P_{a^*} \rightarrow M \rightarrow 0 \) and \( 0 \rightarrow M^{**} \rightarrow P_{b'^*} \rightarrow P_{b^*} \) are exact. Moreover, it is readily seen that the kernel of \( M \rightarrow M^{**} \) is isomorphic to the cohomology of the top row at \( P_{a'^*} \) which is isomorphic to \( \text{Ext}^1_{\mathcal{A}^{\text{op}}}(\text{Tr} M, -)|_{\mathcal{A}^{\text{op}}} \);
similarly, the cokernel of \( M \to M^{**} \) is isomorphic to the cohomology of the top row at \( P_k \) which is isomorphic \( \operatorname{Ext}_{\mathcal{A}^{op}}^2(\mathcal{T}r M, -) \). This yields the required exact sequence. The second claim follows immediately from the construction of the Auslander–Bridger transposition. □

For each \( k \geq 1 \) we consider the functor \( \mathcal{T}r_k := \mathcal{T}r \Omega^{k-1}: \mod \mathcal{A} \to \mod (\mathcal{A}^{op}) \). These functors are instrumental in higher Auslander–Reiten theory, see [Iya07b].

We need the following well-known property.

**Proposition 2.8.** Let \( \mathcal{A} \) be a dualizing \( R \)-variety. Then, for each \( M \in \mod \mathcal{A} \) and for each \( k \geq 1 \) there is an exact sequence

\[
0 \to \operatorname{Ext}_{\mathcal{A}}^k(M, -) \|_{\mathcal{A}} \to \mathcal{T}r_k M \to \Omega \mathcal{T}r_{k+1} M \to 0.
\]

such that \( \phi^* = 0 \).

**Proof.** We include a proof for the convenience of the reader. The first part of the proof is analogous to the proof of Proposition 2.7. Let \( M \in \mod \mathcal{A} \) and \( P_* \to M \) a projective resolution of \( M \). For each \( k \geq 1 \) the \( \mathcal{A} \)-duality yields an exact sequence

\[
0 \to (\Omega^k M)^* \to P^*_k \to P^*_{k+1} \to \mathcal{T}r_{k+1} M \to 0
\]

It is readily verified that there exist a commutative diagram with exact rows and columns

\[
\begin{array}{ccccccc}
P^*_{k-1} & \to & (\Omega^k M)^* & \to & \operatorname{Ext}_{\mathcal{A}}^1(\Omega^{k-1} M, -) \|_{\mathcal{A}} & \to & \operatorname{Ext}_{\mathcal{A}}^1(P^*_{k-1}, -) \|_{\mathcal{A}} = 0 \\
\downarrow & & \downarrow & & \phi & & \downarrow \\
P^*_{k-1} & \to & P^*_k & \to & \mathcal{T}r_k M & \to & \Omega \mathcal{T}r_{k+1} M \\
\downarrow & & \downarrow & & \psi & & \downarrow \\
0 & \to & \Omega \mathcal{T}r_{k+1} M & \to & \Omega \mathcal{T}r_{k+1} M
\end{array}
\]

where the dotted column can be seen to be exact by applying the Snake Lemma to the leftmost two columns. The first claim follows since \( \operatorname{Ext}_{\mathcal{A}}^1(\Omega^{k-1} M, -) \|_{\mathcal{A}} \cong \operatorname{Ext}_{\mathcal{A}}^k(M, -) \|_{\mathcal{A}} \).

It remains to show that \( \phi^* = 0 \). Thus, we need to show that every morphism \( \mathcal{T}r_k M \to P^* \) where \( P \) is a projective \( \mathcal{A} \)-module factors through \( \psi \). Equivalently, we need to show that every morphism \( P^*_k \to P^* \) such that the composition with \( P^*_{k-1} \to P^*_k \) vanishes factors through \( P^*_k \to \Omega \mathcal{T}r_{k+1} M \). Indeed, let \( f^*: P^*_k \to P^* \) be such a morphism. Then, since \( P \) is projective and the complex \( P_* \to M \) is exact, there is a commutative diagram

\[
\begin{array}{ccc}
P & \to & 0 \\
\downarrow & & & & \downarrow & & & & \downarrow \\
P^*_{k+1} & \to & P^*_k & \to & P^*_{k-1}
\end{array}
\]

By applying the \( \mathcal{A} \)-duality to this diagram we deduce that \( f^* \) factors through \( P^*_k \to P^*_{k+1} \), which implies the required factorization. This shows that \( \phi^* = 0 \). □

Let \( \mathcal{A} \) be a dualizing \( R \)-variety. We recall from [AR74] that there is a unique bifunctor \( \otimes_{\mathcal{A}} -: \mod \mathcal{A} \times \mod(\mathcal{A}^{op}) \to \mod \mathcal{B} \), called of course the **tensor product**, characterized by the following properties:

(a) Let \( M \in \mod \mathcal{A} \). The functor \( M \otimes_{\mathcal{A}} -: \mod(\mathcal{A}^{op}) \to \mod \mathcal{B} \) is right exact, commutes with direct sums and for each \( a \in \mathcal{A} \) there is an equality \( M \otimes_{\mathcal{A}} P^*_a = \operatorname{Hom}_{\mathcal{A}}(P_a, M) \).
(b) Let \( N \in \text{Mod}(A^{\text{op}}) \). The functor \(- \otimes_A N : \text{Mod} A \to \text{Mod} R\) is right exact, commutes with direct sums and for each \( a \in A \) there is an equality \( P_a \otimes_A N = \text{Hom}_{A^{\text{op}}}(P_a^*, N) \).

For an arbitrary \( A^{\text{op}}\)-module \( N \), the functors \( \text{Tor}_k^A(-, N) : \text{mod} A \to \text{mod} R \) are defined as usual, that is as the left derived functors of \( - \otimes_A N \). We need the following well known isomorphism from homological algebra, cf. [CE99, Prop. 5.3].

**Lemma 2.9.** Let \( A \) be a dualizing \( R \)-variety, \( M \in \text{mod} A \) and \( I \) an injective \( A^{\text{op}}\)-module. Then, for each \( k \geq 0 \) there is a natural isomorphism

\[
\text{Tor}_k^A(M, I) \cong \text{Hom}_{A^{\text{op}}}(\text{Ext}_k^A(M, -)|_A, I).
\]

**Proof.** We give a proof for the convenience of the reader. Let \( P_* \to M \) be a projective resolution of \( M \). On one hand, for each \( k \geq 0 \) the homology of the complex

\[
\cdots \to P_k \otimes_A I \to \cdots \to P_1 \otimes_A I \to P_0 \otimes_A I \to 0
\]

at \( P_k \otimes_A I \) is isomorphic to \( \text{Tor}_k^A(M, I) \).

On the other hand, for each \( k \geq 0 \) the homology of the complex

\[
0 \to P_0^* \to P_1^* \to \cdots \to P_k^* \to \cdots
\]

at \( P_k^* \) is isomorphic to \( \text{Ext}_k^A(M, -)|_A \). Since \( I \) is injective, the contravariant functor \( \text{Hom}_{A^{\text{op}}}(\cdot, I) \) is exact, hence it preserves homology. Thus, for each \( k \geq 0 \) the homology of the complex

\[
\cdots \to \text{Hom}_{A^{\text{op}}}(P_k^*, I) \to \cdots \to \text{Hom}_{A^{\text{op}}}(P_1^*, I) \to \text{Hom}_{A^{\text{op}}}(P_0^*, I) \to 0.
\]

Finally, by the definition of the tensor product, the complexes (2.1) and (2.2) are isomorphic, therefore they have isomorphic homologies. This finishes the proof. □

Let \( A \) be a dualizing \( R \)-variety. Recall that the \textit{global dimension} of \( A \), denoted by \( \text{gl. dim} A \), is the supremum of all the projective dimensions of finitely presented \( A \)-modules. The duality \( D : \text{mod} A \to \text{mod}(A^{\text{op}}) \) implies \( \text{gl. dim} A = \text{gl. dim} A^{\text{op}} \). We also recall the definition of the \textit{dominant dimension} of a dualizing \( R \)-variety. Let \( A \) be a dualizing \( R \)-variety and \( d \) a positive integer. We say that \( A \) is a \( d \)-\textit{Auslander dualizing} \( R \)-variety if

\[
\text{gl. dim} A \leq d + 1 \leq \text{dom. dim} A.
\]

If \( d = 1 \), then we simply say that \( A \) is an \textit{Auslander dualizing} \( R \)-variety.
2.2. \(d\)-cluster-tilting subcategories. We now recall the definition of \(d\)-cluster-tilting subcategory. For convenience, we introduce the following notation. Let \(A\) be a dualizing \(R\)-variety. Given a subcategory \(X\) of \(\text{mod} A\), we define the subcategories 
\[ \perp X := \{ M \in \text{mod} A \mid \text{Hom}_A(M, X) = 0 \}, \]
and, for \(d \geq 1\),
\[ \perp^{d-1} X := \{ M \in \text{mod} A \mid \forall k \in \{1, \ldots, d-1\} \text{ Ext}^k_A(M, X) = 0 \}. \]
The subcategories \(X^{\perp}\) and \(X^{\perp,d-1}\) are defined dually. Note that \(\perp^0 X = \text{mod} A\), hence \(\perp X\) and \(\perp^0 X\) are different in general. The subcategory \(X\) is \(d\)-rigid if \(X \subseteq \perp^{d-1} X\).

**Definition 2.11.** [Iya07b, Def. 2.2] Let \(A\) be a dualizing \(R\)-variety, \(M \subseteq \text{mod} A\) a functorially finite subcategory and \(d \geq 1\). We say that \(M\) is \(d\)-cluster-tilting if the equalities \(M = \perp^{d-1} M = \perp M\) hold.

**Remark 2.12.** Let \(A\) be a dualizing \(R\)-variety. Then, \(\text{mod} A\) has a unique 1-cluster-tilting subcategory, namely \(\text{mod} A\) itself.

We need the following characterization of \(d\)-cluster-tilting subcategories.

**Proposition 2.13.** [Iya07b, Prop. 2.2.2] Let \(A\) be a dualizing \(R\)-variety and \(M \subseteq \text{mod} A\) a functorially finite subcategory. Then, the following statements are equivalent.

(a) The subcategory \(M\) is \(d\)-cluster-tilting.
(b) There is an equality \(M = \perp^{d-1} M\) and \(M\) contains all injective \(A\)-modules.
(c) There is an equality \(M = \perp M\) and \(M\) contains all projective \(A\)-modules.

We recall the following property of \(d\)-cluster-tilting subcategories, which exposes their higher homological nature.

**Proposition 2.14.** [Iya11, Lemma 3.5] Let \(A\) be a dualizing \(R\)-variety and \(M \subseteq \text{mod} A\) a \(d\)-cluster-tilting subcategory. Then, for each \(X \in M\) and for each
\[ 0 \rightarrow L \rightarrow M^1 \rightarrow \cdots \rightarrow M^d \rightarrow N \rightarrow 0 \]
exact sequence in \(\text{mod} A\) whose terms lie in \(M\) there are exact sequences
\[ 0 \rightarrow \text{Hom}_A(X, L) \rightarrow \text{Hom}_A(X, M^1) \rightarrow \cdots \rightarrow \text{Hom}_A(X, M^d) \rightarrow \text{Hom}_A(X, N) \rightarrow \text{Ext}_A^d(X, L) \rightarrow \text{Ext}_A^d(X, M^1) \]
and
\[ 0 \rightarrow \text{Hom}_A(N, X) \rightarrow \text{Hom}_A(M^d, X) \rightarrow \cdots \rightarrow \text{Hom}_A(M^1, X) \rightarrow \text{Hom}_A(L, X) \rightarrow \text{Ext}_A^d(N, X) \rightarrow \text{Ext}_A^d(M^d, X). \]

**Proof.** The proof of [Iya11, Lemma 3.5] carries over. \(\square\)

In view of Proposition 2.14, it is natural to consider the following class of \(d\)-cluster-tilting subcategories which are better behaved from the viewpoint of higher homological algebra.

**Definition-Proposition 2.15.** Let \(A\) be a dualizing \(R\)-variety and \(M \subseteq \text{mod} A\) a \(d\)-cluster-tilting subcategory. Then, we say that \(M\) is \(d\mathbb{Z}\)-cluster-tilting if it satisfies the following equivalent conditions.

(a) \(\text{Ext}_A^k(M, M) \neq 0\) implies that \(k \in d\mathbb{Z}\).
(b) \(\Omega^d(M) \subseteq M\).
(c) \(\Omega^{-d}(M) \subseteq M\).
(d) For each $X \in \mathcal{M}$ and for each

$$0 \rightarrow L \rightarrow M^1 \rightarrow \cdots \rightarrow M^d \rightarrow N \rightarrow 0$$

exact sequence in $\mathsf{mod}\mathcal{A}$ whose terms lie in $\mathcal{M}$ there is an exact sequence

$$0 \rightarrow \text{Hom}_{\mathcal{A}}(X, L) \rightarrow \text{Hom}_{\mathcal{A}}(X, M^1) \rightarrow \cdots \rightarrow \text{Hom}_{\mathcal{A}}(X, M^d) \rightarrow \text{Hom}_{\mathcal{A}}(X, N) \rightarrow 0 \tag{1}$$

$$\rightarrow \text{Ext}^1_{\mathcal{A}}(X, L) \rightarrow \text{Ext}^1_{\mathcal{A}}(X, M^1) \rightarrow \cdots \rightarrow \text{Ext}^1_{\mathcal{A}}(X, M^d) \rightarrow \text{Ext}^1_{\mathcal{A}}(X, N) \rightarrow 0 \tag{2}$$

$$\rightarrow \text{Ext}^{2d}_{\mathcal{A}}(X, L) \rightarrow \text{Ext}^{2d}_{\mathcal{A}}(X, M^1) \rightarrow \cdots \rightarrow \text{Ext}^{2d}_{\mathcal{A}}(X, M^d) \rightarrow \text{Ext}^{2d}_{\mathcal{A}}(X, N) \rightarrow \cdots . \tag{3}$$

(e) For each $X \in \mathcal{M}$ and for each

$$0 \rightarrow L \rightarrow M^1 \rightarrow \cdots \rightarrow M^d \rightarrow N \rightarrow 0$$

exact sequence in $\mathsf{mod}\mathcal{A}$ whose terms lie in $\mathcal{M}$ there is an exact sequence

$$0 \rightarrow \text{Hom}_{\mathcal{A}}(N, X) \rightarrow \text{Hom}_{\mathcal{A}}(M^1, X) \rightarrow \cdots \rightarrow \text{Hom}_{\mathcal{A}}(M^d, X) \rightarrow \text{Hom}_{\mathcal{A}}(L, X) \rightarrow 0 \tag{4}$$

$$\rightarrow \text{Ext}^1_{\mathcal{A}}(N, X) \rightarrow \text{Ext}^1_{\mathcal{A}}(M^1, X) \rightarrow \cdots \rightarrow \text{Ext}^1_{\mathcal{A}}(M^d, X) \rightarrow \text{Ext}^1_{\mathcal{A}}(L, X) \rightarrow 0 \tag{5}$$

$$\rightarrow \text{Ext}^{2d}_{\mathcal{A}}(N, X) \rightarrow \text{Ext}^{2d}_{\mathcal{A}}(M^1, X) \rightarrow \cdots \rightarrow \text{Ext}^{2d}_{\mathcal{A}}(M^d, X) \rightarrow \text{Ext}^{2d}_{\mathcal{A}}(L, X) \rightarrow \cdots . \tag{6}$$

Proof. First we show the equivalence between conditions (a), (b) and (c). Let $k \in \{1, \ldots, d - 1\}$ and $m \geq 0$. Then, there is a sequence of isomorphisms

$$\text{Ext}^k_{\mathcal{A}}(\Omega^m \mathcal{M}, \mathcal{M}) \cong \text{Ext}^k_{\mathcal{A}}(\mathcal{M}, \Omega^{-m} \mathcal{M}) \cong \text{Ext}^{dm+k}_{\mathcal{A}}(\mathcal{M}, \mathcal{M}).$$

Since the obstructions for the required sequences to be exact are precisely extension groups in degrees which are not multiples of $d$ between $\mathcal{A}$-modules in $\mathcal{M}$, the claim follows immediately from the equalities $\mathcal{M} = \mathcal{M}^{-d+1} = \mathcal{M}^{d+1}$ and the assumption that $\Omega^d(\mathcal{M}) \subset \mathcal{M}$ and $\Omega^{-d}(\mathcal{M}) \subset \mathcal{M}$. The equivalence between (d) (resp. (e)) and conditions (a)–(c) follows from Proposition 2.14 and the existence of isomorphisms $\text{Ext}^{dm}_{\mathcal{A}}(X, -) \cong \text{Ext}^{d(m-1)}_{\mathcal{A}}(X, -)$ for all $m \geq 1$. We leave the details to the reader. \hfill $\square$

Remark 2.16. We note that the equivalent conditions in Definition-Proposition 2.15 appear in the construction of Geiß, Keller and Oppermann of $(d + 2)$-angulated categories from $d$-cluster-tilting subcategories of triangulated categories, see [GKO13].

2.3. $d$-abelian categories. The class of $d$-abelian categories is meant to abstract the intrinsic properties of $d$-cluster-tilting subcategories. Before giving the definition we recall the definition of $d$-exact sequence in an additive category, a higher analog of the classical notion of short exact sequence.

Definition 2.17. [Jas14,Defs. 2.2 and 2.4] Let $\mathcal{A}$ be an additive category. A sequence of morphisms in $\mathcal{A}$

$$0 \rightarrow a_{d+1} \rightarrow a_d \rightarrow \cdots \rightarrow a_1 \rightarrow a_0$$

is called left $d$-exact\footnote{We borrow this terminology from [Lin14].} if the induced sequence of functors

$$0 \rightarrow \mathcal{A}(-, a_{d+1}) \rightarrow \mathcal{A}(-, a_d) \rightarrow \cdots \rightarrow \mathcal{A}(-, a_1) \rightarrow \mathcal{A}(-, a_0)$$

is exact. We define right $d$-exact sequences dually. A sequence is $d$-exact if it is both left $d$-exact and right $d$-exact.
Remark 2.18. Note that 1-exact sequences are nothing but short exact sequences in the usual sense.

Definition 2.19. [Jas14, Def. 3.1] Let \( \mathcal{A} \) be an additive category with split idempotents. We say that \( \mathcal{A} \) is \( d \)-abelian if the following properties are satisfied.

(A1) For every morphism \( f : a_1 \to a_0 \) in \( \mathcal{A} \) there exists a left \( d \)-exact sequence
\[
0 \to a_{n+1} \to a_n \to \cdots \to a_1 \xrightarrow{f} a_0.
\]
(A1)\textsuperscript{op} For every morphism \( f : a_{d+1} \to a_d \) in \( \mathcal{A} \) there exists a right \( d \)-exact sequence
\[
a_{d+1} \xrightarrow{f} a_d \to \cdots \to a_1 \to a_0 \to 0.
\]

(A2) For every epimorphism \( f : a_1 \to a_0 \) in \( \mathcal{A} \) there exists a \( d \)-exact sequence
\[
0 \to a_{n+1} \to a_n \to \cdots \to a_1 \xrightarrow{f} a_0 \to 0.
\]
(A2)\textsuperscript{op} For every monomorphism \( f : a_{d+1} \to a_d \) in \( \mathcal{A} \) there exists a \( d \)-exact sequence
\[
0 \to a_{d+1} \xrightarrow{f} a_d \to \cdots \to a_1 \to a_0 \to 0.
\]

Axioms (A1) and (A1)\textsuperscript{op} immediately imply the following statement.

Proposition 2.20. Let \( \mathcal{A} \) be a \( d \)-abelian category. Then, \( \text{gl.dim} \mathcal{A} \leq d + 1 \) and \( \text{gl.dim} \mathcal{A}^\text{op} \leq d + 1 \).

The following result gives a connection between \( d \)-cluster-tilting subcategories and \( d \)-abelian categories. In fact, it is the main motivation for the investigation of \( d \)-abelian categories.

Theorem 2.21. [Jas14, Thm. 3.16] Let \( \mathcal{A} \) be a dualizing \( R \)-variety and \( M \subseteq \text{mod} \mathcal{A} \) a \( d \)-cluster-tilting subcategory. Then, \( M \) is a \( d \)-abelian category.

We need the following definition in the statement of Theorem 1.3.

Definition 2.22. Let \( \mathcal{A} \) be a \( d \)-abelian category. As usual, we say that an object \( a \in \mathcal{A} \) is projective if for every epimorphism \( f : b \to c \) the induced morphism \( \mathcal{A}(a, b) \to \mathcal{A}(a, c) \) is surjective. We say that \( \mathcal{A} \) has \( d \)-syzygies if for every \( a \in \mathcal{A} \) there exist a \( d \)-exact sequence
\[
0 \to b \to p_{d-1} \to \cdots \to p_0 \to a \to 0
\]
where \( p_0, p_1, \ldots, p_{d-1} \) are projective objects in \( \mathcal{A} \). With some abuse of notation, we sometimes denote \( b \) by \( \Omega^d a \). The notion of \( \mathcal{A} \) having \( d \)-cosyzygies and \( \Omega^{-d} a \) are defined dually.

We recall the following result.

Theorem 2.23. [Jas14, Thm. 5.16]. Let \( \mathcal{A} \) be an abelian category with enough projectives and \( M \subseteq \mathcal{A} \) a \( d \)-cluster-tilting subcategory. If \( \Omega^d(M) \subseteq M \), then \( M \) is a \( d \)-abelian category with \( d \)-syzygies.

3. Auslander correspondence

In this section we give a proof of Theorem 1.2. For readability purposes we divide the proof in two parts. Note that the implication \( (b) \Rightarrow (a) \) is shown in Theorem 2.21 since \( d \)-cluster-tilting subcategories are functorially finite, hence dualizing \( R \)-varieties by Proposition 2.4.
3.1. **Proof of** (a)⇒(d) **in Theorem 1.2.** In this subsection, we fix a d-abelian dualizing R-variety A. By Proposition 2.20, the inequality \( \text{gl.dim} A \leq d + 1 \) holds. We begin with the following general lemma.

**Lemma 3.1.** Let \( M \in \text{mod} A \) and

\[
P_a \xrightarrow{P} P_{a_0} \rightarrow M \rightarrow 0
\]

a projective presentation of \( M \). Then, \( M \in \perp A \) if and only if \( f \) is an epimorphism in \( A \).

**Proof.** Let \( g : a_0 \rightarrow a \) be a morphism in \( A \). Then, \( f g = 0 \) if and only if \( P_f P_g = 0 \) if and only if there is a commutative diagram

\[
P_a \xrightarrow{P} P_{a_0} \xrightarrow{P_g} P_a \xrightarrow{g} M.
\]

From this diagram, and since the Yoneda embedding is faithful, it is clear that \( M \in \perp A \) if and only if \( f \) is an epimorphism. □

**Proof of** (a)⇒(d) **in Theorem 1.2.** Let \( M \in \perp A \) and

\[
P_a \xrightarrow{P} P_{a_0} \rightarrow M \rightarrow 0
\]

a projective presentation of \( M \). By Lemma 3.1 the morphism \( f \) is an epimorphism. Since \( A \) is a d-abelian category, there exists a d-exact sequence

\[
0 \rightarrow a_{d+1} \rightarrow a_d \rightarrow \cdots \rightarrow a_1 \rightarrow a_0 \rightarrow 0.
\]

By Yoneda’s lemma, for each \( a \in A \) there is an isomorphism between the complex

\[
(3.1) \quad 0 \rightarrow \mathcal{A}(P_{a_0}, P_a) \rightarrow \mathcal{A}(P_{a_1}, P_a) \rightarrow \cdots \rightarrow \mathcal{A}(P_{a_d}, P_a) \rightarrow \mathcal{A}(P_{a_{d+1}}, P_a)
\]

and the acyclic complex

\[
0 \rightarrow \mathcal{A}(a_0, a) \rightarrow \mathcal{A}(a_1, a) \rightarrow \cdots \rightarrow \mathcal{A}(a_d, a) \rightarrow \mathcal{A}(a_{d+1}, a).
\]

Finally, since for each \( k \in \{0, \ldots, d\} \) the R-module \( \text{Ext}^k_A(M, P_a) \) is isomorphic to the homology of the complex (3.1) at \( \text{Hom}_A(P_{a_k}, P_a) \), we conclude that \( M \in \perp A \). Therefore \( \perp A \subseteq \perp A \subseteq \perp A \). Dually, we have \( \perp A^\vee \subseteq \perp A \).

3.2. **Proof of** (d)⇒(c) **in Theorem 1.2.**

**Proposition 3.2.** The subcategory \( \perp A \) is a Serre subcategory of \( \text{mod} A \). In particular, \( \perp A \) is an abelian category.

**Proof.** It is clear that \( \perp A \) is closed under extensions and quotients. Let \( 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \) be a short exact sequence in \( \text{mod} A \) such that \( M \in \perp A \). Condition (d) implies that \( N \in \perp A \subseteq \perp A \). Hence for each \( a \in A \) the functor \( \text{Hom}_A(-, P_a) \) induces an exact sequence

\[
0 = \text{Hom}_A(M, P_a) \rightarrow \text{Hom}_A(L, P_a) \rightarrow \text{Ext}^1_A(N, P_a) = 0.
\]

Therefore \( L \in \perp A \), which is what we needed to show. □

The following result is necessary to establish a certain Gorenstein property of \( A \) in Proposition 3.4.
Lemma 3.3. The contravariant functor $\text{mod}\mathcal{A} \to \text{mod} \mathcal{A}^{\text{op}}$ given by

$$M \mapsto \text{Ext}_\mathcal{A}^{d+1}(M, -)|_\mathcal{A}$$

induces a duality between abelian categories $\perp \mathcal{A} \to \perp (\mathcal{A}^{\text{op}})$.

Proof. This can be shown as in the proof of [Iya05, 6.2(1)] since $\perp \mathcal{A} \subset \perp \mathcal{A}$ by condition (d). We leave the details to the reader. □

We can now establish the following ($d + 1$)-Gorenstein property of $\mathcal{A}$, cf. [Iya05, Def. 0.1].

Proposition 3.4. Let $S$ be a simple $\mathcal{A}$-module of projective dimension $d + 1$. Then, the following statements hold.

(a) The $\mathcal{A}$-module $S$ belongs to $\perp \mathcal{A}$.

(b) The $\mathcal{A}^{\text{op}}$-module $\text{Ext}_\mathcal{A}^{d+1}(S, -)|_\mathcal{A}$ is simple and has projective dimension $d + 1$.

Proof. Since $\text{gl.dim} \mathcal{A} \leq d + 1$, every submodule of $\mathcal{A}$ has projective dimension at most $d$. Therefore $S \in \perp \mathcal{A}$. Lemma 3.3 implies that $\text{Ext}_\mathcal{A}^{d+1}(S, -)|_\mathcal{A}$ is a simple $\mathcal{A}^{\text{op}}$-module which belongs to $\perp \mathcal{A}^{\text{op}} \subset \perp \mathcal{A}^{\text{op}}$, and therefore has projective dimension $d + 1$. □

We denote by $I^0(\mathcal{A})$ the additive closure of the full subcategory of $\text{mod}\mathcal{A}$ given by the injective $\mathcal{A}$-modules $I$ such that there exist $a \in \mathcal{A}$ and an injective envelope $P_a \to I$.

Lemma 3.5. Let $a \in \mathcal{A}$ and

$$(P_a \to I^*) = (0 \to P_a \to I^0 \to I^1 \to \cdots \to I^d \to I^{d+1} \to 0)$$

a minimal injective coresolution of $P_a$. Then, the following statements hold.

(a) For all $k \in \{0, 1, \ldots, d\}$ the $\mathcal{A}$-module $I^k$ belongs to $I^0(\mathcal{A})$.

(b) The equality $I^0(\mathcal{A}) \cap (\text{add} I^{d+1}) = 0$ is satisfied.

Proof. (a) Let $k \in \{0, 1, \ldots, d\}$ and $I$ an indecomposable direct summand of $I^k$. Let $S$ be the socle of $I$. Since $S$ is a simple $\mathcal{A}$-module and $P_a \to I^*$ is a minimal injective coresolution, there is an isomorphism $\text{Ext}_\mathcal{A}^k(S, P_a) \cong \text{Hom}_\mathcal{A}(S, I^k) \neq 0$. Then, condition (d) implies that $S \notin \perp \mathcal{A}$. Hence, there exists $a' \in \mathcal{A}$ such that $\text{Hom}_\mathcal{A}(S, P_{a'}) \neq 0$. Therefore $I$ is a direct summand of the injective envelope of $P_a$. This shows that $I \in \text{add} I^0(\mathcal{A})$.

(b) It is enough to show that for each simple submodule $S$ of $I^{d+1}$ there is an equality $\text{Hom}_\mathcal{A}(S, I^0(\mathcal{A})) = 0$. Since $\text{Ext}_\mathcal{A}^{d+1}(S, P_a) \neq 0$, the $\mathcal{A}$-module $S$ has projective dimension $d + 1$. Given that every submodule of a projective $\mathcal{A}$-module has projective dimension at most $d$, we deduce that $S \in \perp \mathcal{A}$. Therefore $\text{Hom}_\mathcal{A}(S, \mathcal{A}) = \text{Hom}_\mathcal{A}(S, I^0(\mathcal{A})) = 0$ as required. □

We need one more technical lemma.

Lemma 3.6. Let $a \in \mathcal{A}$ and

$$(P_a \to I^*) = (0 \to P_a \to I^0 \to I^1 \to \cdots \to I^d \to I^{d+1} \to 0)$$

a minimal injective coresolution of $P_a$. Then, for each $k \in \{0, 1, \ldots, d\}$ the inequality $\text{proj.dim} I^k \leq d$ is satisfied.
Proof. The proof of [Iya05, Prop. 6.3(1)] carries over. We reproduce it here for the convenience of the reader. Recall that \( \text{gl.dim} \, A \leq d + 1 \), see Proposition 2.20. Let \( k \in \{0, \ldots, d\} \) and suppose that \( \text{proj.dim} \, I^k = d + 1 \). Since minimal projective resolutions in \( \text{mod} \, A \) exist, see Proposition 2.6, it readily follows that there exist a simple \( A \)-module \( S \) such that \( \text{Tor}^A_{d+1}(S, I^k) \neq 0 \). Note that this implies that \( S \) has projective dimension \( d + 1 \). Lemma 2.9 then implies that

\[
\text{Hom}_A(- \mathcal{O}(\text{Ext}^d_{A^o}(S, -)|_A, I^k)) \neq 0.
\]

Given that \( \text{Ext}^d_{A^o}(S, -)|_A \) is a simple \( A^o \)-module of projective dimension \( d + 1 \), see Proposition 3.4, and \( P_0 \rightarrow I^* \) is a minimal injective coresolution, there is an isomorphism

\[
\text{Ext}^k_{A^o}(\text{Ext}^{d+1}_{A^o}(S, -)|_A, P_0) \cong \text{Hom}_A(- \mathcal{O}(\text{Ext}^d_{A^o}(S, -)|_A, I^k)) \neq 0.
\]

This contradicts the fact that \( \text{Ext}^d_{A^o}(S, -)|_A \) belongs to \( \mathcal{L}(A^o) \subset \mathcal{L}(A^o) \), see Proposition 3.4. Therefore \( I^k \) has projective dimension at most \( d \). \( \square \)

As we shall see, the proof of the implication \((d) \Rightarrow (c)\) in Theorem 1.2 follows from the following result.

**Proposition 3.7.** Let \( A \) be a dualizing \( R \)-variety of global dimension \( d + 1 \) such that \( \mathcal{L} \subset \mathcal{L} \). Then,

\[
\Omega(\text{mod} \, A) = \{ M \in \text{mod} \, A \mid \text{proj.dim} \, M \leq d \}.
\]

Proof. Since \( \text{gl.dim} \, A \leq d + 1 \), see Proposition 2.20, it is clear that

\[
\Omega(\text{mod} \, A) \subset \{ M \in \text{mod} \, A \mid \text{proj.dim} \, M \leq d \}.
\]

We now show that the opposite inclusion holds.

Let \( M \) be an \( \mathcal{A} \)-module of projective dimension at most \( d \). By Proposition 2.7 there exist an exact sequence

\[
0 \rightarrow \text{Ext}^1_{\mathcal{A}^o}(\text{Tr} \, M, -)|_A \rightarrow M \rightarrow M^{**} \rightarrow \text{Ext}^2_{\mathcal{A}^o}(\text{Tr} \, M, -)|_A \rightarrow 0
\]

and \( M^{**} \in \mathcal{L}(\text{mod} \, A) \). Therefore it is enough to show that \( \text{Tr} \, M \in \mathcal{L} \) since \( \Omega(\text{mod} \, A) \) is closed under submodules in \( \text{mod} \, A \). Using backwards induction on \( k \), we show that \( \text{Tr} \, M \in \mathcal{L} \) for \( k \in \{1, \ldots, d\} \).

Let \( 0 \rightarrow P_d \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0 \)

be a projective resolution of \( M \). Then we have an exact sequence \( P_{d-1} \rightarrow P_d^* \rightarrow \text{Tr} \, M \rightarrow 0 \). Applying the \( \mathcal{A} \)-duality it readily follows that \( \text{Tr} \, M \in \mathcal{L} \), that is \( \text{Tr} \, M \in \mathcal{L} \subset \mathcal{L} \).

We need to show that \( \text{Tr} \, M \in \mathcal{L} \) implies \( \text{Tr} \, M \in \mathcal{L} \). By Proposition 2.8 there exists a short exact sequence

\[
0 \rightarrow \text{Ext}^2_{\mathcal{A}}(\text{Tr} \, M, -)|_A \rightarrow \mathcal{O} \rightarrow \text{Tr} \, M \rightarrow 0
\]

such that \( \mathcal{O}^* = 0 \). Put \( E_k := \text{Ext}^2_{\mathcal{A}}(\text{Tr} \, M, -)|_A \). Applying \( (-)^* \) to this sequence yields an exact sequence

\[
(\text{Tr} \, M)^* \rightarrow \mathcal{O} \rightarrow \text{Ext}^1_{\mathcal{A}}(\text{Tr} \, M, -)|_A \cong \text{Ext}^2_{\mathcal{A}}(\text{Tr} \, M, -)|_A.
\]

Since \( \mathcal{O}^* = 0 \) and, by assumption, \( \text{Ext}^2_{\mathcal{A}}(\text{Tr} \, M, -)|_A = 0 \), we conclude that \( (E_k)^* = 0 \), that is \( E_k \in \mathcal{L} \subset \mathcal{L} \). Finally, for each \( j \in \{1, \ldots, k\} \) there is an exact sequence

\[
0 = \text{Ext}^j_{\mathcal{A}}(\text{Tr} \, M, -)|_A \rightarrow \text{Ext}^j_{\mathcal{A}}(\text{Tr} \, M, -)|_A \rightarrow \text{Ext}^j_{\mathcal{A}}(E_k, -)|_A = 0.
\]
Therefore $\text{Tr}_k M \in \frac{1}{k} A$. This finishes the proof. \hfill \Box

We can now give the proof of the implication \((d) \Rightarrow (c)\) in Theorem 1.2.

**Proof of** \((d) \Rightarrow (c)\) in Theorem 1.2. By Proposition 2.20 the inequality $\text{gl. dim } A \leq d + 1$ holds. Moreover, since $A$ is a $d$-abelian dualizing $R$-variety if and only if so is $A^{\text{op}}$, it is enough to show the one-sided condition $\text{dom. dim } A \geq d + 1$. Let $a \in A$ and

$$0 \to P_a \to I^0 \to I^1 \to \cdots \to I^d \to I^{d+1} \to 0$$

a minimal injective coresolution of $P_a$. Then, by Lemma 3.6 and Proposition 3.7 for each $k \in \{0, 1, \ldots, d\}$ the $A$-module $I^k$ belongs to $\Omega(\text{mod } A)$. In particular, there exist $b \in A$ and a monomorphism $I^k \to P_b$, which splits since $I^k$ is injective. Therefore for all $k \in \{0, 1, \ldots, d\}$ the $A$-module $I^k$ is projective. This shows that $\text{dom. dim } A \geq d + 1$, whence $A$ is a $d$-Auslander dualizing $R$-variety. \hfill \Box

3.3. **Proof of** \((c) \Rightarrow (b)\) in Theorem 1.2. We follow closely the proof [Iya08, Thm. 2.6]. In this subsection, we fix a $d$-Auslander dualizing $R$-variety $A$. We denote by $\Omega = \Omega_A$ the full subcategory of $A \subseteq \text{mod } A$ of projective-injective $A$-modules and define

$$\mathcal{B} := \{ b \in A \mid D(P_b^\ast) \in \Omega_A \}.$$

Then we have an equivalence $D(-)^\ast : \mathcal{B} \to \Omega$. Consider the functors

$$F : \text{mod } A \to \text{mod } \mathcal{B} \quad \text{and} \quad G : \text{mod } \mathcal{B} \to \text{mod } A.$$

defined by $FM := M|_\mathcal{B}$ and $GX := \text{Hom}_A(P,-|_\mathcal{B}, X)$. Note that $F$ is an exact functor and $(F, G)$ is an adjoint pair. We shall show that $\mathcal{B}$ is a dualizing $R$-variety (Lemma 3.8) and $M := FA$ is a $d$-cluster-tilting subcategory of $\text{mod } \mathcal{B}$, thus proving the implication \((c) \Rightarrow (b)\) in Theorem 1.2.

**Lemma 3.8.** The category $\mathcal{B}$ is a dualizing $R$-variety.

**Proof.** By Proposition 2.4 it is enough to show that $\Omega_A$ is functorially finite in $A_A$.

Let $P_a \in A_A$. Given that $\text{dom. dim } A \geq d + 1$ there exist a projective-injective $A$-module $P_q$ and a monomorphism $P_a \to P_q$. Moreover, every morphism $P_a \to P_q$ such that $P_a$ is projective-injective factors through $P_a \to P_q$. This shows that $\Omega_A$ is covariantly finite in $A_A$. In order to show that $\Omega_A$ is contravariantly finite in $A_A$, note that $D(\Omega_A) = \Omega_{A^{\text{op}}}$. Since $\text{dom. dim } A^{\text{op}} \geq d + 1$, by what we have shown above $D(\Omega_A)$ is covariantly finite in $A_{A^{\text{op}}}$. Since $D$ is a duality, $\Omega_A$ is contravariantly finite in $(DA)_A$ and hence in $A_A$ by Lemma 2.5. This shows that $\Omega_A$ is functorially finite in $A_A$. \hfill \Box

In the following lemma, note that $D(\mathcal{B}_{A^{\text{op}}})$ is precisely the full subcategory of $\text{mod } \mathcal{B}$ of all injective $\mathcal{B}$-modules.

**Lemma 3.9.** The following statements hold.

(a) There is a natural isomorphism $FG \cong 1_{\text{mod } \mathcal{B}}$.

(b) The functors $F$ and $G$ induce mutually quasi-inverse equivalences

$$\Omega_A \xrightarrow{\sim} D(\mathcal{B}_{A^{\text{op}}}) \quad \text{and} \quad D(\mathcal{B}_{A^{\text{op}}}) \xrightarrow{\sim} \Omega_A.$$

**Proof.** (a) Let $X \in \text{mod } \mathcal{B}$. By Yoneda’s lemma, for each $b \in \mathcal{B}$ there are functorial isomorphisms

$$(FGX)(b) = (GX)(b) \cong \text{Hom}_A(P_b|_\mathcal{B}, X) \cong X(b).$$

It follows that $FG \cong 1_{\text{mod } \mathcal{B}}$. 

(b) Let \( b \in \mathcal{B} \); hence \( D(P_b^*) \in \mathcal{Q} \). There are functorial isomorphisms
\[
FD(P_b^*) = D \text{Hom}_{\mathcal{A}}(P_b, -)|_{\mathcal{B}} \cong DA(b, -)|_{\mathcal{B}} = DB(b, -).
\]
On the other hand, for each \( a \in \mathcal{A} \) there are functorial isomorphisms
\[
(GDB(b, -))(a) \cong \text{Hom}_{\mathcal{B}}(P_a|_{\mathcal{B}}, DB(b, -)) \cong \text{Hom}_{\mathcal{B}^a}(B(b, -), DPa|_{\mathcal{B}}) \cong DPa(b).
\]
Therefore \( GDB(b, -) \cong DA(b, -) \cong D(P_b^*) \). The claim follows. \( \square \)

As a first consequence of Lemma 3.9, we obtain the following result.

**Lemma 3.10.** The following statements hold.

(a) The functor \( F : \mathcal{A}_{\mathcal{A}} \to \text{mod } \mathcal{B} \) is fully faithful.

(b) The subcategory \( \mathcal{M} = FA \subseteq \text{mod } \mathcal{B} \) is \( d \)-rigid, that is \( \mathcal{B} \subseteq \mathcal{A} - 1 \mathcal{B} \).

**Proof.** Let \( a \in \mathcal{A} \) and
\[
0 \to P_a \to I^0 \to I^1 \to \cdots \to I^d
\]
be part of a minimal injective coresolution of \( P_a \). Since \( \text{dom.dim } \mathcal{A} \geq d + 1 \), the injective \( \mathcal{A} \)-modules \( I^0, I^1, \ldots, I^d \) are also projective. Then, by Lemma 3.9(b), we have an injective coresolution
\[
0 \to FP_a \to FI^0 \to FI^1 \to \cdots \to FI^d.
\]
By definition, for each \( k \in \{1, \ldots, d\} \) the homology of the complex
\[
0 \to GF^{\mathcal{P}}_a \to GF^I^0 \to GF^I^1 \to \cdots \to GF^I^d
\]
at \( GF^I^k \) is isomorphic to \( \text{Ext}^k(\mathcal{M}, \mathcal{M}) \). Finally, by Lemma 3.9(b), the complex (3.3) is isomorphic to the acyclic complex (3.2). This shows that \( GF^{\mathcal{P}}_a = P_a \), which means that \( F|_{\mathcal{A}} \) is fully faithful, and that \( \mathcal{M} \) is a \( d \)-rigid subcategory of \( \text{mod } \mathcal{B} \). \( \square \)

We now give the proof of the implication (c) \( \Rightarrow \) (b) in Theorem 1.2.

**Proof of** (c) \( \Rightarrow \) (b) in Theorem 1.2. By Lemmas 3.9 and 3.10 and Proposition 2.13 it only remains to show that if \( X \) is a \( \mathcal{B} \)-module such that \( X \in \mathcal{M}^{\perp \mathcal{A}} \), then \( X \in \mathcal{M} \) (note that \( \mathcal{M} \) contains \( B_{\mathcal{A}} \) by construction). Let
\[
0 \to X \to I^0 \to I^1 \to \cdots \to I^d \to \cdots
\]
be a minimal injective coresolution of \( X \). By assumption, applying \( G \) to (3.4) yields an exact sequence
\[
0 \to GX \to GI^0 \to GI^1 \to \cdots \to GI^d.
\]
in which \( GI^0, GI^1, \ldots, GI^d \) are projective. Since \( \text{gl.dim } \mathcal{A} \leq d + 1 \), the \( \mathcal{A} \)-module \( GX \) is projective. Thus, \( X \cong FGX \) belongs to \( \mathcal{M} \), see Lemma 3.9(a). \( \square \)

As a consequence of Theorem 1.2 we obtain a characterization of injective objects in \( d \)-abelian dualizing \( R \)-varieties. It should be compared with [Jas14, Thm. 3.12].

**Corollary 3.11.** Let \( \mathcal{A} \) be a \( d \)-abelian dualizing \( R \)-variety and \( q \in \mathcal{A} \). Then, the following statements are equivalent.

(a) The object \( q \) is injective in \( \mathcal{A} \), that is for every monomorphism \( f : b \to c \) in \( \mathcal{A} \) the induced morphism \( A(c, q) \to A(b, q) \) is surjective.

(b) The \( \mathcal{A} \)-module \( P_q \) is injective.
For every left $d$-exact sequence

$$0 \rightarrow a_{d+1} \rightarrow a_d \rightarrow \cdots \rightarrow a_1 \rightarrow a_0$$

the sequence

$$\mathcal{A}(a_0, q) \rightarrow \mathcal{A}(a_1, q) \rightarrow \cdots \rightarrow \mathcal{A}(a_{d+1}, q) \rightarrow 0$$

is exact.

Proof.  $(a) \Rightarrow (b)$ Let $q \in \mathcal{A}$ be an injective object. Since $\mathcal{A}$ is a $d$-Auslander dualizing $R$-variety, the injective hull of $P_q$ is of the form $P_q \rightarrow P_q'$; this is induced by a monomorphism $q \rightarrow q'$, which splits. Thus, $P_q$ is a direct summand of $P_q'$ and therefore an injective $\mathcal{A}$-module.

$(b) \Leftrightarrow (c)$ The $\mathcal{A}$-module $P_q$ is injective if and only if for every $M \in \text{mod } \mathcal{A}$ there is an equality $\text{Ext}^>_{\mathcal{A}}(M, P_q) = 0$. Equivalently, noting that $\text{gl.dim } \mathcal{A} \leq d + 1$, the $\mathcal{A}$-module $P_q$ is injective if and only if for every exact sequence

$$0 \rightarrow P_{a_{d+1}} \rightarrow \cdots \rightarrow P_{a_1} \rightarrow P_{a_0} \rightarrow M \rightarrow 0$$

the sequence

$$0 \rightarrow (M, P_q) \rightarrow (P_{a_0}, P_q) \rightarrow (P_{a_1}, P_q) \rightarrow \cdots \rightarrow (P_{a_{d+1}}, P_q) \rightarrow 0$$

is exact. The claim follows immediately from Yoneda’s embedding.

$(c) \Rightarrow (a)$ This is clear, since every monomorphism in the $d$-abelian category $\mathcal{A}$ is the left-most morphism in a $d$-exact sequence. \qed

4. Homological Auslander Correspondence

In this section we give a proof of Theorem 1.3. We only deal with the first sequence of equivalences since the second sequence follows by duality.

Proof of Theorem 1.3. $(a) \Rightarrow (c)$ Let $\mathcal{A}$ be a $d$-abelian dualizing $R$-variety with $d$-cosyzygies. The fact that $\mathcal{A}$ is $d$-Auslander dualizing $R$-variety follows from Theorem 1.2. Let $a \in \mathcal{A}$. By assumption, there exists a $d$-exact sequence

$$0 \rightarrow a \rightarrow q^0 \rightarrow \cdots \rightarrow q^{d-1} \rightarrow \Omega^{-d}a \rightarrow 0$$

where $q^0, \ldots, q^{d-1}$ are injective objects in $\mathcal{A}$. By definition, there is an exact sequence in $\text{mod } \mathcal{A}$ of the form

$$0 \rightarrow P_a \rightarrow P_{q^0} \rightarrow \cdots \rightarrow P_{q^{d-1}} \rightarrow P_{\Omega^{-d}a} \rightarrow M \rightarrow 0$$

where $P_{q^0}, \ldots, P_{q^{d-1}}$ are injective $\mathcal{A}$-modules by Corollary 3.11. Finally, Lemma 3.1 implies $M \in \mathcal{A}$. Therefore $\Omega^{-d}(\mathcal{A}_A) \subseteq \Omega(\mathcal{A}_A)$.

$(c) \Rightarrow (b)$ Let $\mathcal{A}$ be a $d$-Auslander dualizing $R$-variety satisfying $\Omega^{-d}(\mathcal{A}_A) \subseteq \Omega(\mathcal{A}_A)$. We use the notation of Section 3. Thus, we denote by $\Omega = \mathcal{Q}_A$ the full subcategory of $\mathcal{A}_A \subseteq \text{mod } \mathcal{A}$ of projective-injective $\mathcal{A}$-modules and define

$$\mathcal{B} := \{ b \in \mathcal{A} \mid D(P_b^\ast) \in \mathcal{Q}_A \}.$$
defined by \( \mathbb{F}M := M \mid_\mathcal{B} \) induces an equivalence \( \Omega_\mathcal{A} \to D(\mathcal{B}_\text{proj}) \), see Lemma 3.9. Moreover, by (the proof of) Theorem 1.2, we know that \( \mathcal{M} := \mathbb{F}\mathcal{A} \) is a \( d \)-cluster-tilting subcategory of \( \text{mod} \mathcal{B} \). It remains to show that \( \Omega^{-d}(\mathcal{M}) \subset \mathcal{M} \). Let \( a \in \mathcal{A} \).

By assumption, there exists an exact sequence in \( \text{mod} \mathcal{A} \) of the form

\[
0 \to P_a \to P_{q^0} \to \cdots \to P_{q^{d-1}} \to P_{\Omega^{-d}a} \to M \to 0
\]

where \( P_{q^0}, \ldots, P_{q^{d-1}} \) are injective \( \mathcal{A} \)-modules and \( M \in \perp \mathcal{A} \). Applying the exact functor \( \mathbb{F} \) yields an exact sequence

\[
0 \to \mathbb{F}P_a \to \mathbb{F}P_{q^0} \to \cdots \to \mathbb{F}P_{q^{d-1}} \to \mathbb{F}P_{\Omega^{-d}a} \to \mathbb{F}M \to 0
\]

where \( \mathbb{F}P_{q^0}, \ldots, \mathbb{F}P_{q^{d-1}} \) are injective \( \mathcal{B} \)-modules. Thus, \( \mathbb{F}\Omega^{-d}P_a \cong \Omega^{-d}\mathbb{F}P_a \). We claim that \( \mathbb{F}M = 0 \), hence \( \Omega^{-d}\mathbb{F}P_a \cong \Omega^{-d}P_a \cong \mathbb{F}P_{\Omega^{-d}a} \in \mathcal{M} \). Equivalently, we show that \( \mathbb{F}P_{q^{d-1}} \to \mathbb{F}P_{\Omega^{-d}a} \) is an epimorphism. We need to show that for every \( b \in \mathcal{B} \) each morphism \( P_b \to P_{\Omega^{-d}a} \) factors through \( P_{q^{d-1}} \to P_{\Omega^{-d}a} \). The situation can be visualized in the commutative diagram

\[
P_b \quad \overset{\exists}{\longrightarrow} \quad P_{q^{d-1}} \to P_{\Omega^{-d}a} \to M \to 0
\]

where the bottom row is exact. Applying the \( \mathcal{A} \)-duality to this diagram yields the commutative diagram

\[
P_b^* \quad \overset{\exists}{\longrightarrow} \quad P_{q^{d-1}}^* \quad \overset{\exists}{\longleftarrow} \quad P_{\Omega^{-d}a}^* \quad \overset{\exists}{\longleftarrow} \quad M^* = 0
\]

where the bottom row is exact (recall that \( M \in \perp \mathcal{A} \)). Thus, it is enough to show that \( P_b^* \) is an injective \( \mathcal{A}^{\text{op}} \)-module. Indeed, \( P_b^* = D\mathbb{Q}_\mathcal{A} = \mathbb{Q}_{\mathcal{A}^{\text{op}}} \) consists of projective-injective \( \mathcal{A}^{\text{op}} \)-modules.

The implication (b)\( \Rightarrow \) (a) follows from Theorem 2.23 since \( d \)-cluster-tilting subcategories are functorially finite, hence dualizing \( R \)-varieties by Proposition 2.4.

This finishes the proof of the theorem. \( \square \)

5. Examples

In this section we provide a fundamental class of examples of \( d\mathbb{Z} \)-cluster-tilting subcategories.

Let \( \mathcal{R} \) be a field and \( \mathcal{A} \) a \( d \)-representation-finite algebra in the sense of [IO11]. Thus, \( \mathcal{A} \) is a finite dimensional \( \mathcal{R} \)-algebra of global dimension \( d \) and there exist a finite dimensional (right) \( \mathcal{A} \)-module \( M \) such that \( \mathcal{M} := \text{add} M \) is a \( d \)-cluster-tilting subcategory of \( \text{mod} \mathcal{A} \). We let \( \Gamma := \text{End}_{\mathcal{A}}(M) \) be the projectively-stable endomorphism algebra of \( M \). It is shown in [Iya11, Thm. 1.21] that

\[
U_d(\mathcal{A}) := \text{add} \{ X[d\ell] \in D^b(\text{mod} \mathcal{A}) \mid X \in \mathcal{M} \text{ and } \ell \in \mathbb{Z} \}.
\]
is a $d$-cluster-tilting subcategory of $\mathsf{D}^b(\mod A)$. It is known that $\mathsf{U}_d(A)$ is a dualizing $R$-variety and that $\mathsf{mod} \ U_d(A)$ is a Frobenius abelian category such that there is an equivalence of triangulated categories

$$\mathsf{mod} \ U_d(A) \xrightarrow{\sim} \mathsf{D}^b(\mod \Gamma),$$

see [IY08, Props. 2.10 and 2.11] and [IO13, Coros. 3.7 and 4.10]. Thus, if $\Gamma$ is a $(d+1)$-representation finite algebra, then $U_{d+1}(\Gamma)$ induces a $(d+1)$-cluster-tilting subcategory of $\mathsf{mod} \ U_d(A)$ which is readily seen to be $d\mathbb{Z}$-cluster-tilting. This is the case, for example, if $A$ is a $d$-representation-finite algebra of type $A$ in the sense [IO11, Sec. 5].

These examples are used in [DI15] for constructing selfinjective finite dimensional algebras with $d$-cluster-tilting subcategories with additive generators by extending the methods of Riedtmann [Rie80]. We also refer the reader to [JK15] where families of $d\mathbb{Z}$-cluster-tilting subcategories are constructed based partly on the methods of [DI15].

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