Witten’s Invariant of 3-Dimensional Manifolds: Loop Expansion and Surgery Calculus.

L. Rozansky

Theory Group, Department of Physics, University of Texas at Austin
Austin, TX 78712-1081, U.S.A.

Abstract

We review two different methods of calculating Witten’s invariant: a stationary phase approximation and a surgery calculus. We give a detailed description of the 1-loop approximation formula for Witten’s invariant and of the technics involved in deriving its exact value through a surgery construction of a manifold. Finally we compare the formulas produced by both methods for a 3-dimensional sphere $S^3$ and a lens space $L(p, 1)$.

1 to be published in the volume Knots and Applications.

2Work supported by NSF Grant 9009850 and R. A. Welch Foundation.
1 Introduction

A quantum field theory based on Chern-Simons action has been developed by E. Witten in his paper [1]. Consider a connection $A_\mu$ of a $G$ bundle $E$ on a 3-dimensional manifold $M$, $G$ being a simple Lie group. If the bundle is trivial, then an integral

$$S_{CS} = \frac{1}{2} \epsilon^{\mu\nu\rho} \text{Tr} \int_M (A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho) \, d^3x. \quad (1.1)$$

defines a Chern-Simons action as a function of $A_\mu$. A manifold invariant $Z(M)$ is a path integral

$$Z(M, k) = \int [DA_\mu] e^{\frac{i}{\hbar} S_{CS}[A_\mu]}, \quad \hbar = \frac{\pi}{k}, \quad (1.2)$$

here $k \in \mathbb{Z}$, and the brackets in $[DA_\mu]$ mean that we integrate over the gauge equivalence classes of connections. The action (1.1) does not depend on the choice of local coordinates on $M$, neither does it depend on the metric of the manifold $M$. Therefore the integral $Z(M, k)$, also known to physicists as a partition function, is a topological invariant of the manifold (modulo the possible metric dependence of the integration measure $[DA_\mu]$).

Witten considered two different methods of calculating $Z(M, k)$. He first applied a stationary phase approximation to the integral (1.2). This is a standard method of quantum field theory. It expresses $Z(M, k)$ as asymptotic series in $k^{-1}$. The first term in this series contains such ingredients as Chern-Simons action of flat connections and Reidemeister torsion. The other method of calculating the invariant, which we call “surgery calculus” is based upon a construction of $M$ as a surgery on a link in $S^3$ (or in $S^1 \times S^2$). It presents $Z(M, k)$ as a finite sum, however the number of terms in it grows as a power of $k$. Reshetikhin and Turaev used the surgery calculus formula in [2] as a definition of Witten’s invariant and proved its invariance (i.e. independence of the choice of surgery to construct a given manifold $M$) without referring to the path integral (1.2).

A systematic comparison between both methods of calculating Witten’s invariant has been initiated in [3]. D. Freed and R. Gompf compared the numeric values of the invariants of some lens spaces and homology spheres for large values of $k$ as given by the two methods.
The full analytic comparison has been carried out in [4] and [5] for lens spaces and mapping tori. It was extended further to Seifert manifolds in [6]. A complete agreement between the stationary phase approximation and surgery calculus has been found in all these papers.

In this paper we will review both methods of calculating Witten’s invariant and compare their results. In section 2 we explain the stationary phase approximation method. Section 3 contains the basics of surgery calculus. In section 4 we apply both methods to the calculation of Witten’s invariant of the sphere $S^3$ and lens space $L(p, 1)$.

# 2 Stationary Phase Approximation

## 2.1 Finite Dimensional Integrals

Let us start with a simple example of the stationary phase approximation. Consider a finite dimensional integral

$$Z(\hbar) = \int \frac{d^nX}{(2\pi\hbar)^{n/2}} \exp \left[ \frac{i}{\hbar} S(X_1, \ldots, X_n) \right]$$

for some function $S$. Here $\hbar$ is an arbitrary small constant, called Planck’s constant in quantum theory. The integral (2.1) is a finite dimensional version of the path integral (1.2). Note that a path integral measure $[DA]$ includes implicitly a factor $(2\pi\hbar)^{-1/2} = \pi^{-1}(k/2)^{1/2}$ for each of the one-dimensional integrals comprising the full path integral.

In the limit of small $\hbar$ the dominant contribution to $Z(\hbar)$ comes from the extrema of $S$, i.e. from the points $X_i^a$ such that

$$\left. \frac{\partial S}{\partial X_i} \right|_{X_i = X_i^a} = 0, \quad 1 \leq i \leq n$$

(2.2)

If we retain only the quadratic terms in Taylor expansion of $S$ in the vicinity of these points, then

$$Z(\hbar) = \sum_a e^{iS(X^a)} \int d^nX \exp \left[ i\pi \sum_{i,j=1}^n x_i x_j \left. \frac{\partial^2 S}{\partial X_i \partial X_j} \right|_{X_i = X_i^a} \right]$$

$$= \sum_a e^{iS(X^a)} \det^{-1/2} \left( -i \left. \frac{\partial^2 S}{\partial X_i \partial X_j} \right|_{X_i = X_i^a} \right).$$

(2.3)
A phase of this expression requires extra care. The matrix $\frac{\partial^2 S}{\partial X_i \partial X_j}$ is hermitian. It has only real eigenvalues $\lambda$, but they can be both positive and negative. Each positive eigenvalue contributes a phase factor $(-i)^{-1/2} = e^{i\pi/4}$ to the inverse square root of the determinant in eq. (2.3), while each negative eigenvalue contributes $i^{-1/2} = e^{-i\pi/4}$. Therefore a refined version of the formula (2.3) is

$$Z(\bar{\hbar}) = \sum_a e^{\bar{\hbar} S(X^{(a)})} e^{i\frac{\pi}{4} \eta^a} \left| \det \left( \frac{\partial^2 S}{\partial X_i \partial X_j} \bigg|_{X_i = X^{(a)}_i} \right) \right|^{-1/2},$$

(2.4)

where

$$\eta^a = \# \text{ positive } \lambda - \# \text{ negative } \lambda$$

(2.5)

In the context of quantum field theory the integral (2.1) becomes an infinite dimensional path integral, however the stationary phase approximation method remains the same if we can make sense of infinite dimensional determinants. Physicists call the formula (2.3) a 1-loop approximation, because it can be derived by summing up all 1-loop Feynman diagrams.

### 2.2 Gauge Invariant Theories

The integral (1.2) presents a special challenge, because the action (1.1) is invariant under a gauge transformation (i.e. under a local change of basis in the fibers)

$$A_\mu \rightarrow A^g_\mu = g^{-1} A_\mu g + g^{-1} \partial_\mu g.$$

(2.6)

The integral over the gauge equivalence classes of connections is equal to the integral over all connections divided by the volume of the group of gauge transformations. However the latter integral can not be calculated through eq. (2.4), because its stationary phase points are not isolated. They form the orbits of the gauge action (2.6). Therefore we should rather integrate over the submanifold in the space of all connections, which is transversal to gauge orbits, multiply the terms in the sum (2.4) by the volumes of those orbits and divide the whole sum by the volume of the group of gauge transformations.
The problem of reducing an integral of a function invariant under the action of a group, to an integral over a factor manifold, is not unfamiliar to mathematicians. For example, an integral of the product of characters over a simple Lie group can be reduced to its maximal torus at a price of adding an extra factor which accounts for the volume of the orbits of adjoint action. This factor is equal to the square of denominator in the Weyl character formula and appears as a Jacobian of a certain coordinate transformation.

A similar trick was developed for gauge invariant path integrals by Faddeev and Popov. Consider a Lie algebra valued functional $\Phi[A_\mu]$ such that each gauge orbit intersects transversally the set of its zeros

$$\Phi[A_\mu] = 0 \quad (2.7)$$

For a constant function $g(x) = g = \text{const}$, the second term in eq. (2.6) vanishes. If $g$ also belongs to the center $Z(G)$ of $G$, then, obviously, $A^g = A$. Therefore a general gauge orbit intersects the set (2.7) at the same point $\text{Vol}(Z(G))$ times. $\text{Vol}(Z(G))$ denotes the number of elements in $Z(G)$. We use this notation to make connection with the formula (2.33), which, as we will see, works also for the case when the tangent spaces of the manifold (2.7) and a gauge orbit intersect along a finite dimensional space.

A path integral generalization of a simple formula

$$\int_{-\infty}^{+\infty} \delta(f(x)) \frac{dx}{\sqrt{2\pi\hbar}} = \sum_{x_i: f(x_i) = 0} \left| \sqrt{2\pi\hbar} f'(x_i) \right|^{-1} \quad (2.8)$$

can be used to derive the following identity:

$$1 = \frac{1}{\text{Vol}(Z(G))} \left| \det \left( \frac{\delta(\sqrt{2\pi\hbar}\Phi[A^g])}{\delta g} \bigg|_{\Phi[A^g]=0} \right) \right| \int Dg \delta(\Phi[A^g]). \quad (2.9)$$

A path integral $\delta$-function which will reduce the integral over all connections to a submanifold (2.7) is called “gauge fixing”.

A multiplication of the integral (1.2) by the r.h.s. of the identity (2.9) and a subsequent change of variables $A^g_\mu \to A_\mu$ allows us to factor the volume of the group of gauge
transformations out of the integral over all gauge connections:

\[ Z(h) = (Dg)^{-1} \int DA_{\mu} e^{i S_{CS}[A_{\mu}]} \]

\[ = (Dg)^{-1} \frac{1}{\text{Vol}(Z(G))} \int DA_{\mu} e^{i S_{CS}[A_{\mu}]} \left| \text{det} \left( \frac{\delta(\sqrt{2\pi h}\Phi[A])}{\delta g} \right)_{g=1} \right| \int Dg \delta(\Phi[A]) \]

\[ = \frac{1}{\text{Vol}(Z(G))} \int DA_{\mu} e^{i S_{CS}[A_{\mu}]} \delta(\Phi[A]) \left| \text{det} \left( \frac{\delta(\sqrt{2\pi h}\Phi[A])}{\delta g} \right)_{g=1} \right| . \quad (2.10) \]

### 2.3 Chern-Simons Path Integral

Let us apply a stationary phase approximation to the integral (2.10). We first look for the stationary phase points:

\[ \frac{\delta S}{\delta A_{\mu}} \sim \epsilon^{\mu\nu\rho} F_{\nu\rho} = 0 \quad (2.11) \]

These points are flat connections, i.e. connections for which \( F_{\mu\nu} = 0 \). The gauge equivalence classes of flat connections are in one-to-one correspondence with the homomorphisms

\[ \pi_{1}(\mathcal{M}) \xrightarrow{\Delta} G, \ A : x \mapsto g(x) \in G \quad (2.12) \]

up to a conjugacy, that is, the homomorphisms \( x \to g(x) \) and \( x \to h^{-1}g(x)h \) are considered equivalent.

The next step is to expand the action (1.1) up to the terms quadratic in gauge field variation \( a_{\mu} \) around a particular flat connection \( A_{\mu}^{(a)} \):

\[ S_{CS}[A_{\mu}^{(a)}] + \pi \sqrt{2 k a_{\mu}} \approx S_{CS}[A_{\mu}^{(a)}] + \frac{\pi^{2}}{k} \epsilon^{\mu\nu\rho} \text{Tr} \int a_{\mu} D_{\nu} a_{\rho} d^{3}x. \quad (2.13) \]

Eq. (2.11) can be used to verify the gauge invariance of the action (1.1) under small gauge transformations. The infinitesimal version of eq. (2.6) is

\[ \delta_{\omega} A_{\mu} = D_{\mu} \omega, \]

so that

\[ \delta_{\omega} S = \int \frac{\partial S}{\partial A_{\mu}} D_{\mu} \omega d^{3}x \sim \epsilon^{\mu\nu\rho} \text{Tr} \int F_{\nu\rho} D_{\mu} \omega d^{3}x \]

\[ = -\text{Tr} \int \omega \epsilon^{\mu\nu\rho} D_{\mu} F_{\nu\rho} d^{3}x = 0 \]

because of Bianchi identity.
Here $D_\nu$ is a covariant derivative with respect to the “background field” $A^{(a)}_\mu$:

$$D_\nu = \partial_\nu + [A^{(a)}_\mu, *]. \quad (2.14)$$

A gauge fixing condition should be imposed on the fluctuation field $a_\mu$. Witten suggested a covariant (with respect to $A^{(a)}_\mu$) choice:

$$\Phi[a_\mu] = D_\mu a_\mu. \quad (2.15)$$

According to eq. (2.6), a change of $a_\mu$ under an infinitesimal gauge transformation $g(x) \approx 1 + \omega(x)$ is

$$\delta_\omega a_\mu = \frac{1}{\pi} \sqrt{\frac{k}{2}} D_\mu \omega. \quad (2.16)$$

Therefore the operator $\delta(\sqrt{2\pi h A})/\delta g|_{g=1}$ is a covariant Laplacian $\Delta = D_\mu D_\mu$ acting on 0-forms. As for the $\delta$-function of $\Phi[a_\mu]$, it can be presented as a path integral over a Lie algebra valued scalar field:

$$\delta(\Phi[a_\mu]) = \int \mathcal{D}a_\mu \exp \left[ 2\pi i \operatorname{Tr} \int d^3x \left( \epsilon_{\mu\nu\rho} D_\nu a_\rho + 2\phi D_\mu a_\mu \right) \right]. \quad (2.17)$$

As a result, at the 1-loop level

$$Z(M, k) \approx \frac{1}{\operatorname{Vol}(Z(G))} \sum_a e^{i\frac{\pi}{h} S_{CS}[A^{(a)}_\mu]} |\det \Delta| \times \int \mathcal{D}a_\mu \mathcal{D}\phi \exp \left[ i\pi \operatorname{Tr} \int d^3x \left( \epsilon_{\mu\nu\rho} a_\mu D_\nu a_\rho + 2\phi D_\mu a_\mu \right) \right]$$

$$= \frac{1}{\operatorname{Vol}(Z(G))} \sum_a e^{i\frac{\pi}{h} S_{CS}[A^{(a)}_\mu]} |\det \Delta| (\det L_-)^{1/2}. \quad (2.18)$$

Here $L_-$ is the operator of the quadratic form in the exponential of the path integral. $L_-$ acts on the direct sum of 0-forms and 1-forms on $M$:

$$L_-(\phi, a_\mu) = (D_\mu a_\mu, \epsilon_{\mu\nu\rho} D_\nu a_\rho - D_\mu \phi). \quad (2.19)$$

If we use 3-forms instead of 0-forms, then $L_- = *D + D*$, $*$ is the Hodge operator. Note that the integration measures of the fluctuation fields $\mathcal{D}\phi$ and $\mathcal{D}a_\mu$ do not contain any implicit factors in contrast to $\mathcal{D}g$ and $\mathcal{D}A_\mu$.\footnote{Note that $\Phi[a_\mu]$ depends on the metric of $M$. This dependence ultimately results in a framing dependence of $Z(M, k)$.}
2.4 $\eta$-Invariant

A. Schwartz observed in [7] that the absolute value of the ratio of determinants in eq. (2.18) was equal to the square root of the Reidemeister-Ray-Singer analytic torsion. The phase of the ratio is equal to the $\eta$-invariant of Atiyah, Patodi and Singer. Similarly to eq. (2.3) it is a difference between the number of positive and negative eigenvalues of $L_-$. Thus the formula for the ratio of determinants is

$$\frac{|\det \Delta|}{(\det L_-)^{1/2}} = R e^{i\pi \frac{\eta}{4}}.$$  \hspace{1cm} (2.20)

Actually $L_-$ has infinitely many eigenvalues, so a regularization is needed to define $\eta$. To get some idea of how $\eta$ might depend on the background connection $A_{\mu}^{(a)}$ consider the following simple problem. Let the eigenvalues be $\lambda_m = m + a$, $m \in \mathbb{Z}$ and let us calculate the number of positive $\lambda_m$ minus the number of negative ones as a function of $a$. The simplest regularization is

$$\eta_a = \lim_{\epsilon \to 0} \left[ \sum_{\lambda_m > 0} e^{-\lambda_m \epsilon} - \sum_{\lambda_m < 0} e^{\lambda_m \epsilon} \right].$$ \hspace{1cm} (2.21)

Suppose that $0 < a < 1$, then

$$\eta_a = \lim_{\epsilon \to 0} \left[ \frac{e^{-a \epsilon}}{1 - e^{-\epsilon}} - \frac{e^{-(1-a) \epsilon}}{1 - e^{-\epsilon}} \right] = 1 - 2a.$$ \hspace{1cm} (2.22)

In particular $\eta_{1/2} = 0$ because of the symmetry between the positive and negative $\lambda$ for $a = 1/2$. A dependence of $\eta_a$ on $a$ is a nontrivial consequence of the infinity of the number of eigenvalues, since naively the number of positive and negative $\lambda$ is the same for any $a \in (0, 1)$.

Obviously, $\eta_a$ is a periodic function of $a$: $\eta_{a+n} = \eta_a$, $n \in \mathbb{Z}$, because $a$ and $a + n$ define the same set of eigenvalues $\lambda$. Therefore eq. (2.22) requires a modification to work for all $a$. Indeed, when $a$ moves through an integer number $n$, an eigenvalue $\lambda_{-n}$ changes the sign and the value of $\eta_a$ jumps by two units. Define $I_a$ to be a number of positive eigenvalues becoming negative minus a number of negative eigenvalues becoming positive when the parameter goes from 1/2 to $a$. Then

$$\eta_a = 1 - 2a - 2I_a.$$ \hspace{1cm} (2.23)
If we define \( \eta_0 \) to be equal to 1 as if \( \lambda_0 = 0 \) is counted as positive, then we arrive to the formula

\[
\eta_a = \eta_0 - 1 + (1 - 2a) - 2I_a. \tag{2.24}
\]

A similar formula for the \( \eta \)-invariant of \( L_- \) was derived in \[3\]:

\[
\eta_a = \eta_0 - (1 + b^1(M)) \dim G + \frac{4}{\pi^2} c_V S_{CS} [A^{(a)}_\mu] - 2I_a, \tag{2.25}
\]

here \( \eta_0 \) is the \( \eta \)-invariant of the trivial connection, \( b^1(M) \) is the first Betti number of \( M \), \( I_a \) is a spectral flow of \( L_- \) and \( c_V \) is a dual Coxeter number of the group \( G \) (e.g. \( c_V = N \) for \( SU(N) \)). The operator \( L_- \) for a trivial connection has \( \dim G \) 0-form zero modes which are constant Lie algebra valued functions and \( b^1(M) \dim G \) 1-form zero modes which are Lie algebra valued closed 1-forms. All these modes are counted as positive in \( \eta_0 \), hence the term \((1 + b^1(M)) \dim G \). The role of the smooth function \( 1 - 2a \) is played by \( \frac{4}{\pi^2} c_V S_{CS} [A^{(a)}_\mu] \).

The metric of \( M \) enters the gauge fixing functional (2.15) as well as the operators \( \Delta \) and \( L_- \). We could naively assume that this dependence would cancel out from the ratio of determinants in eq. (2.20). However the phase \( \eta \) has an “anomalous” dependence on the metric. It can be compensated by multiplying \( Z(M, k) \) by an extra phase factor

\[
\exp \left[ -i \dim G \frac{1}{96\pi} \int_M \Tr (\omega \wedge d\omega + \frac{2}{3} \omega \wedge \omega \wedge \omega) d^3x \right], \tag{2.26}
\]

here \( \omega \) is a Levi-Chivita connection on \( M \) and the integral in the exponent is its Chern-Simons invariant. This invariant is defined relative to the choice of basis in the tangent space at each point of \( M \). The local change in that basis is the analog of the gauge transformation. The exponent of eq. (2.26) is invariant under the transformations which are homotopic to identity. The choice of basis modulo such transformations is called “framing”. The change in framing by \( n \) units shifts the phase of the factor (2.26) by \( \pi n \dim G/12 \). Actually the whole invariant (2.1) with a compensated metric dependence would be multiplied by a factor

\[
\exp \left[ i \frac{\pi}{12} n \dim G \frac{k}{k + c_V} \right]. \tag{2.27}
\]
Physicists call the exponent of eq. (2.26) a 1-loop counterterm. It converts the metric dependence of $\eta$ into a framing dependence of the invariant $Z(M, k)$. According to [3],

$$\eta_0 = 0$$

in the special framing of $M$ called canonical.

### 2.5 Zero Modes

To complete the study of the stationary phase approximation we have to consider the flat connections for which the operators $L_-$ and $\Delta$ have zero modes. The 0-form zero modes of these operators satisfy the same equation

$$D_\mu \omega = 0,$$

so they are the elements of a cohomology $H_0^a$ built upon a covariant derivative (2.14). A 1-form $a_\mu$ which is a zero mode of $L_-$, satisfies two equations

$$\epsilon^{\mu\nu\rho} D_\nu a_\rho = 0, \quad D_\mu a_\mu = 0$$

The first equation means that $a_\mu$ is a closed form with respect to $D$, the second one means that it is not exact: if $a_\mu = D_\mu \omega$, then $D_\mu^2 \omega = 0$, hence $D_\mu \omega = 0$. As a result, the 1-form zero modes are the elements of the cohomology $H_1^a$.

Let us remove the zero modes from the operators $L_-$ and $\Delta$. The absolute value of the ratio of their determinants is still equal to the square root of the Reidemeister torsion, which, as noted in [4], becomes an element of $\Lambda^\text{max} H_0^a \otimes (\Lambda^\text{max} H_1^a)^*$. As for the phase $\eta$, it can be obtained by a simple correction of eq. (2.25) presented in [3]:

$$\eta_a = \eta_0 - (1 + b^1(M))\text{dim}G - (\text{dim}H_0^a + \text{dim}H_1^a) + \frac{4}{\pi^2} c_V S_{CS} [A_\mu^{(a)}] - 2I_a.$$ (2.31)

The zero modes of $L_-$ are counted as positive in the spectral flow $I_a$. Therefore their number had to be subtracted from $\eta_a$ since they are removed from the l.h.s. of eq. (2.20) and do not affect its phase.
According to eq. (2.29), the 0-form zero modes are the infinitesimal gauge transformations that do not change the background field $A^{(a)}_\mu$. The group of gauge transformations which is a symmetry of $A^{(a)}_\mu$, is isomorphic to a subgroup $H_a \subset G$ which commutes with the image of the homomorphism (2.12). Therefore $H^0_a$ is isomorphic to a Lie algebra of $H_a$.

The 1-form zero modes $a_\mu$ are the deformations of a flat connection $A^{(a)}_\mu$ which preserve its flatness in the linear order in $a_\mu$:

$$F_{\mu\nu}[A^{(a)}_\rho + a_\rho] \approx \epsilon_{\mu\nu\lambda} \epsilon^{\lambda\sigma\rho} D_\sigma a_\rho = 0.$$ (2.32)

In most cases these infinitesimal deformations can be extended up to the finite flatness preserving deformations. Then $H^1_a$ is a tangent space of the moduli space $M_a$ of flat connections at the point $A^{(a)}_\mu$.

A removal of the 0-form zero modes from the determinants of the r.h.s. of eq. (2.20) amounts to “forgetting” about $H_a$ as a part of the group of gauge transformations. In other words, the symmetry under the global $H_a$ gauge transformations is not fixed, as it is demonstrated on a simple finite dimensional example in the Appendix of [6]. As a result, we have to divide the integrals of eq. (2.10) by the volume of $H_a$ “by hands”. A square root of the Reidemeister torsion as an element of $\Lambda^{\text{max}} H^0_a \otimes (\Lambda^{\text{max}} H^1_a)^*$ defines a “ratio” of the volume forms on $M_a$ and $H_a$. Therefore $\sqrt{\tau_R/\text{Vol}(H_a)}$ is a volume form on $M_a$ and it is quite natural to supplement a sum in eq. (2.18) by an integral over the components of the moduli space.

The volume forms for $H_a$ and $M_a$ being the part of the path integral measure, contain the factors $(2\pi \hbar)^{-1/2} = \pi (k/2)^{1/2}$. After extracting these factors from the integration measures we obtain the following 1-loop formula:

$$Z(M, k) = \sum_a e^{\frac{i}{2}(k + c_v)S_{CS}[A^{(a)}_\mu]} e^{-\frac{i\pi}{4}[1+b^1(M)] \text{dim} G + \text{dim} H^0_a + \text{dim} H^1_a + 2I_a} \left( \frac{k}{2\pi^2} \right)^{\frac{\text{dim} H^1_a - \text{dim} H^0_a}{2}} \times \frac{1}{\text{Vol}(H_a)} \int_{M_a} \tau_R^{1/2}.$$ (2.33)
The sum goes over the connected components of the moduli space of flat connections on $M$. The Chern-Simons action $S_{CS}[A^{(a)}]$ is constant within those components, because its derivative is zero due to eq. (2.11).

The formula (2.33) is not the end of the story, because sometimes $\dim \mathcal{M}_a < \dim H_1^a$. In other words, not all the 1-form zero modes of $L_-$ can be extended to finite deformations of the flat connection $A^{(a)}_\mu$. This means that the Chern-Simons action is not constant in the direction of these modes, rather its expansion around $A^{(a)}_\mu$ starts with the terms of order $m > 2$. The corresponding piece of the path integral has a form

$$
\int d^n x \exp \left[ 2\pi i (2\pi \hbar)^{\frac{m-2}{m}} \frac{\partial^{(m)} S}{\partial X_{i_1} \cdots \partial X_{i_m}} \bigg|_{X_{i_1}=X^{(a)}_{i_1} \cdots x_{i_m}} \right] \sim e^{i \pi \frac{m}{2} (2\pi \hbar)^{\frac{m(2-m)}{2m}}} (2.34)
$$

here $n = \dim H_1^a - \dim \mathcal{M}_a$ and $\hbar = \pi/k$ as defined in eq. (1.2). Therefore if $\dim \mathcal{M}_a < \dim H_1^a$, then we should substitute $\dim \mathcal{M}_a$ instead of $\dim H_1^a$ in the r.h.s. of eq. (2.33) and multiply it by the factor (2.34).

### 3 Surgery Calculus

#### 3.1 Multiplicativity in Quantum Theory

The surgery calculus uses one of the basic principles of quantum field theory: a multiplicativity of the path integral (1.2). We are going to describe briefly what this multiplicativity means. Suppose that a 3-dimensional manifold $M$ has a boundary $\partial M$. Let us impose a boundary condition on a connection $A_\mu$. For example, we choose a tangent vector field $v_\mu$ on $\partial M$ and demand that $A_\mu v_\mu$ is equal to some fixed function $A$ on $\partial M$:

$$
A_\mu v_\mu = A. \quad (3.1)
$$

Then a path integral (1.2) taken over all the connections on $M$ satisfying this condition becomes a functional $\Psi[A]$. Such functional is called a wave function or a state in quantum theory. All possible functionals $\Psi[A]$ for a given manifold $\partial M$ form a Hilbert space $\mathcal{H}_{\partial M}$. A
scalar product in it is defined by the path integral over all functions $A$ on $\partial M$:

$$\langle \Psi_2|\Psi_1 \rangle = \int \bar{\Psi}_2[A]\Psi_1[A]DA. \quad (3.2)$$

Suppose now that two manifolds $M_1$ and $M_2$ have diffeomorphic boundaries (with opposite orientations): $\partial M_1 = \partial M_2$. We can glue them together to form a single manifold $M$. An integration over connections $A_\mu$ on $M$ can be split into an integration over connections $A_\mu^{(1)}$ on $M_1$ and connections $A_\mu^{(2)}$ on $M_2$ satisfying the same condition (3.1) and an integration over all boundary conditions $A$. If $A_\mu^{(1)}v_\mu|_{\partial M_1} = A_\mu^{(2)}v_\mu|_{\partial M_2}$, then the Chern-Simons action is additive:

$$S_M[A_\mu] = S_{M_1}[A_\mu^{(1)}] + S_{M_2}[A_\mu^{(2)}]. \quad (3.3)$$

Since the exponential is multiplicative, the integrals over $A_\mu^{(1)}$ and $A_\mu^{(2)}$ can be calculated separately yielding the wave functions $\Psi_{1,2}[A]$. The whole integral is a product $\Psi_1[a]\Psi_2[A]$ ($\Psi_2$ is complex conjugated because $\partial M_1$ and $\partial M_2$ have opposite orientations). The final integral over $A$ gives a scalar product:

$$Z(M,k) = \langle \Psi_2|\Psi_1 \rangle. \quad (3.4)$$

To summarize, multiplicativity means that gluing the manifolds is achieved by taking a scalar product of the states appearing on their boundaries.

We adopt the strategy of [2]. Each 3-dimensional manifold can be constructed by a surgery on a link in $S^3$. The tubular neighborhoods of the link components are cut out, the modular transformations on their boundaries are performed and then they are glued back. So if we find the wave functions on both sides of the boundaries of tubular neighborhoods, then we can use eq. (3.4) to find Witten’s invariant. The boundaries of the tubular neighborhoods are 2-dimensional tori $T^2$, so we start by describing the Hilbert space $\mathcal{H}_{T^2}$. We use canonical quantization as described in [8], where it was called “first constraining, then quantizing”.

---

6 In fact, the action (1.1) on a manifold with a boundary should be corrected by a certain boundary term which guarantees that a derivative transversal to $\partial M$ does not act on a tangential component of $A_\mu$ which is not fixed by condition (3.1) and hence is not necessarily continuous after the gluing.
3.2 Canonical Quantization

Consider a manifold \( M = \mathbb{R}^1 \times T^2 = \mathbb{R}^1 \times S^1 \times S^1 \) with coordinates \( t \) along \( \mathbb{R}^1 \) and \( x_{1,2} \) along both circles, \( 0 \leq x_{1,2} < 1 \). The Chern-Simons action \( (1.1) \) can be cast in the form (up to a total derivative in \( t \) that can be removed by adding appropriate boundary terms):

\[
S_{CS} = \text{Tr} \int dt d^2x (A_2 \partial_0 A_1 + A_0 F_{12}).
\]

The 1-form \( A_\mu \) takes values in the Lie algebra of \( G \). Lie algebra elements are antihermitean matrices in the adjoint representation. Quantum field theory deals usually with hermitean objects, so we introduce hermitean forms

\[
\tilde{A}_\mu = -iA_\mu, \quad \tilde{F}_{\mu\nu} = -iF_{\mu\nu}.
\]

Now

\[
S_{CS} = -\text{Tr} \int dt d^2x (\tilde{A}_2 \partial_0 \tilde{A}_1 + \tilde{A}_0 \tilde{F}_{12}).
\]

Compare this with the action of a constrained mechanical system

\[
S = \int dt \left[ p_i \dot{q}_i + h(p_i, q_i) + \lambda_\alpha \phi_\alpha(p_i, q_i) \right],
\]

here \( q_i \) are coordinates, \( p_i \) are conjugate momenta, \( h(p_i, q_i) \) is a hamiltonian, \( \phi_\alpha(p_i, q_i) \) are constraints and \( \lambda_\alpha \) are Lagrange multipliers. We see that \( \tilde{A}_1 \) and \( -\tilde{A}_2 \) are conjugate coordinates and momenta. The hamiltonian is zero as it happens in diffeomorphism invariant theories.

A path integral over \( A_0 \) in eq. \((1.2)\) produces a \( \delta \)-function of the constraint \( F_{12} \), so we should, in fact, study only flat 2-dimensional connections as coordinates in the phase space. A gauge transformation can make both \( A_1 \) and \( A_2 \) constant. Moreover, since \( \pi_1(T^2) \) is commutative, \( A_1 \) and \( A_2 \) will belong to the same Cartan subalgebra (e.g. they will be made diagonal simultaneously for \( G = SU(N) \)). The action \((3.7)\) becomes simply

\[
S_{CS} = \int dt \tilde{A}_2^a \dot{\tilde{A}}_1^a,
\]

14
an index $a$ runs over the orthonormal basis of Cartan subalgebra. After a quantization the fields $\tilde{A}^a_i$ become hermitian operators $\hat{\tilde{A}}^a_i$ satisfying the Heisenberg commutation relation:

$$[\hat{\tilde{A}}^a_i, \hat{\tilde{A}}^b_j] = \iota \hbar \delta^{ab} \equiv i \frac{\pi}{k} \delta^{ab}. \quad (3.10)$$

This algebra can be represented in a space of functions $\psi(\tilde{A}^a_i)$:

$$\hat{\tilde{A}}^a_i \psi(\tilde{A}^b_j) = \tilde{A}^a_i \psi(\tilde{A}^b_j), \quad \hat{\tilde{A}}^a_2 \psi(\tilde{A}^b_1) = \iota \hbar \frac{\partial}{\partial \tilde{A}^a_1} \psi(\tilde{A}^b_1). \quad (3.11)$$

The eigenfunctions of $\hat{\tilde{A}}^a_1$ are $\delta$-functions, while the eigenfunctions of $\hat{\tilde{A}}^a_2$ are exponentials

$$|\alpha_i; 2 \rangle \sim e^{i\alpha_i \tilde{A}^a_1}, \quad (3.12)$$

here we use a standard quantum mechanical notation for eigenstates:

$$\hat{\tilde{A}}^a_i |\alpha_i; i \rangle = \hbar \alpha^a_i |\alpha_i; i \rangle. \quad (3.13)$$

However a construction of a representation for the algebra $\hat{\tilde{A}}^a_i$ should reflect the fact that a Cartan subalgebra is not an appropriate configuration space for the torus $T^2$.

### 3.3 $U(1)$ Theory

Let us study carefully the simplest case of $G = U(1)$. The constant field components appearing in the action (3.15) are equal to the contour integrals along the periods $C_{1,2}$ of $T^2$

$$\tilde{A}_i = \oint_{C_i} \tilde{A}_j(x) dx^j \quad (3.14)$$

of any connection $\tilde{A}_j(x)$ which can be reduced to a constant one by a homotopically trivial gauge transformation. A homotopically nontrivial gauge transformation

$$g(x) = e^{2\pi i (m_1 x_1 + m_2 x_2)} \quad (3.15)$$

is well defined if $m_{1,2} \in \mathbb{Z}$. Eq. (2.7) shows that $\hat{\tilde{A}}_1$ and $\hat{\tilde{A}}_2$ remain constant under this transformation, but their values are shifted:

$$\hat{\tilde{A}}_i \rightarrow \hat{\tilde{A}}_i + 2\pi m_i. \quad (3.16)$$
Thus both coordinate $\tilde{A}_1$ and momentum $\tilde{A}_2$ are periodic with a period of $2\pi$. The phase space is compact (it is $S^1 \times S^1$), its volume is $(2\pi)^2$ and, according to the WKB approximation, the dimension of the Hilbert space should be approximately

$$\dim \mathcal{H}_{T^2}^{U(1)} \approx \frac{(2\pi)^2}{2\pi \hbar} \equiv 2k$$

in the limit of large $k$. In fact, as we will see, eq. (3.17) is exact.

A periodicity in $\tilde{A}_1$ leads to a quantization of the eigenvalues of $\hat{A}_2$: $\alpha$ should be integer to make the eigenfunctions (3.12) periodic. On the other hand, since $\tilde{A}_2$ is also periodic, we should limit the number of independent values of $\alpha$, e.g.

$$-k \leq \alpha < k, \quad \alpha \in \mathbb{Z}$$

(3.18)

This procedure is self-consistent, because for integer $k$ the period of $\tilde{A}_2$ ($2\pi$) is a multiple of the spacing of its eigenvalues ($\hbar = \pi/k$).

The $2k$ values of $\alpha$ determine the momentum eigenstates $|\alpha; 2\rangle$, which form an orthonormal basis of $\mathcal{H}_{T^2}^{U(1)}$. Another basis is formed by the coordinate eigenstates $|\alpha; 1\rangle$ with the same range (3.18) of possible values of $\alpha$. These two bases are related by a finite dimensional version of the Fourier transform (which also provides a relation between coordinate and momentum eigenstates in quantum mechanics of a particle on a line):

$$|\alpha; 2\rangle = \frac{1}{\sqrt{2k}} \sum_{\beta = -k}^{k-1} e^{-i\frac{\pi}{k} \alpha \beta} |\beta; 1\rangle.$$  

(3.19)

3.4 Modular Transformations

A unimodular transformation of cycles $C_1, 2$ in eq. (3.14) generates a canonical transformation of our system:

$$C_i \overset{U}{\rightarrow} C_i' = U_{ij} C_j, \quad \tilde{A}_i \overset{U}{\rightarrow} \tilde{A}_i' = U_{ij} \tilde{A}_j, \quad U \in SL(2, \mathbb{Z}).$$  

(3.20)

Therefore $SL(2, \mathbb{Z})$ can be represented in $\mathcal{H}_{T^2}^{U(1)}$. This group is generated by two elements

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(3.21)
satisfying a relation

\[(ST)^3 = S^2\]  \hspace{1cm} (3.22)

Each matrix

\[U^{(p,q)} = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in SL(2, \mathbb{Z})\]  \hspace{1cm} (3.23)

can be presented as their product

\[U^{(p,q)} = T^{a_t} S \ldots T^{a_1} S.\]  \hspace{1cm} (3.24)

The integer numbers \(a_i\) form a continued fraction expansion of \(p/q\):

\[\frac{p}{q} = a_t - \frac{1}{a_{t-1} - \frac{1}{\ldots - \frac{1}{a_1}}}.\]  \hspace{1cm} (3.25)

We denote as \(\hat{U}^{(p,q)}\) an action of \(U^{(p,q)}\) in \(\mathcal{H}^{U(1)}_{T^2}\). According to eq. (3.24), it is determined by choosing \(\hat{S}\) and \(\hat{T}\).

The matrix \(S\) interchanges coordinate and momentum operators:

\[\hat{S}\hat{A}_1\hat{S}^{-1} = \hat{A}_2, \quad \hat{S}\hat{A}_2\hat{S}^{-1} = -\hat{A}_1.\]  \hspace{1cm} (3.26)

The same is achieved by the matrix of eq. (3.19), so

\[\hat{S}_{\alpha\beta} = \frac{1}{\sqrt{2k}} e^{-i\frac{\pi}{4} \delta_{\alpha\beta}}\]  \hspace{1cm} (3.27)

in the coordinate basis \(|\alpha; 1\rangle\). We use a formula

\[\exp \left[ \frac{ik}{2\pi} \hat{A}_2^2 \right] \hat{A}_2 \exp \left[ -\frac{ik}{2\pi} \hat{A}_2^2 \right] = \hat{A}_2 + \hat{A}_1,\]  \hspace{1cm} (3.28)

which is easy to check by using a representation (3.11), in order to find \(\hat{T}\) in the coordinate basis:

\[\hat{T}_{\alpha\beta} = e^{-i\frac{\pi}{4} \delta_{\alpha\beta}}\]  \hspace{1cm} (3.29)

The phase of \(\hat{T}\) is chosen to comply with eq. (3.22).
3.5 \( SU(2) \) Theory

Let us turn now to the case of \( G = SU(2) \). Its Cartan subalgebra (an algebra of diagonal traceless antihermitian \( 2 \times 2 \) matrices) is isomorphic to that of \( U(1) \). A new feature is the Weyl reflection. A global gauge transformation

\[
g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

(3.30)

changes the signs of \( \tilde{A}_1 \) and \( \tilde{A}_2 \):

\[
g \tilde{A}_i g^{-1} = -\tilde{A}_i, \quad i = 1, 2.
\]

(3.31)

The phase space should be factored by this transformation, its volume becoming half of that for \( U(1) \). Therefore we expect a dimension of the Hilbert space \( \mathcal{H}_{SU(2)}^{T_2} \) also to be approximately half of that of \( \mathcal{H}_{U(1)}^{T_2} \).

The space \( \mathcal{H}_{SU(2)}^{T_2} \) is isomorphic to a subspace of \( \mathcal{H}_{U(1)}^{T_2} \) which is antisymmetric under the Weyl reflection (3.31), if we make an identification

\[
k_{U(1)} = k_{SU(2)} + 2 = K_{SU(2)}.
\]

(3.32)

In other words, the basis of \( \mathcal{H}_{SU(2)}^{T_2} \) is formed by the antisymmetric combinations

\[
|\alpha; i\rangle_{SU(2)} = \frac{1}{\sqrt{2}} \left( |\alpha; i\rangle_{U(1)} - | -\alpha; i\rangle_{U(1)} \right), \quad 0 < \alpha < K, \quad \dim \mathcal{H}_{SU(2)}^{T_2} = K - 1.
\]

(3.33)

As a result, the matrices \( \hat{S}_{\alpha\beta} \) and \( \hat{T}_{\alpha\beta} \) for \( SU(2) \) are obtained (after a minor change in phase factors) by restricting the matrices (3.27) and (3.29) to the subspace (3.33):

\[
\hat{S}_{\alpha\beta} = \sqrt{\frac{2}{K}} \sin \frac{\pi \alpha \beta}{K}, \quad \hat{T}_{\alpha\beta} = e^{-i\pi} e^{i\pi \frac{\alpha^2}{2K}} \delta_{\alpha\beta}
\]

(3.34)

Eq. (3.24) was used in [4] in order to derive a formula for \( \hat{U}(p,q) \):

\[
\hat{U}^{(p,q)}_{\alpha\beta} = -i \frac{\text{sign}(q)}{\sqrt{2K|q|}} e^{-\frac{i\pi}{4} \Phi(M^{(p,q)})} \sum_{\mu=\pm 1} \sum_{n=0}^{q-1} \mu \exp \frac{i\pi}{2Kq} \left[ p\alpha^2 + 2\mu\alpha(\beta + 2K\mu) + s(\beta + 2K\mu) \right]
\]

(3.35)
Here $\Phi(M)$ is a Rademacher phi function defined as follows
\begin{equation}
\Phi \left[ \begin{array}{cc} p & r \\ q & s \end{array} \right] = \begin{cases} 
\frac{p+s}{q} - 12s(s, q) & \text{if } q \neq 0 \\
\frac{r}{s} & \text{if } q = 0
\end{cases}, \tag{3.36}
\end{equation}
a function $s(s, q)$ being a Dedekind sum:
\begin{equation}
s(m, n) = \frac{1}{4n} \sum_{j=1}^{n-1} \cot \frac{\pi j}{n} \cot \frac{\pi m j}{n}. \tag{3.37}
\end{equation}

### 3.6 A General Simple Lie Group

Consider now a general simple Lie group $G$. A gauge transformation can make the fields $A_i$ constant and belonging to a Cartan subalgebra of the Lie algebra associated with $G$. The homotopically nontrivial gauge transformations like (3.15) force the eigenvalues $\alpha$ of the eigenvectors $|\alpha; i \rangle$ of $\hat{A}_i$ to belong to the weight lattice $\Lambda_w$ of $G$ factored by the root lattice $\Lambda_R$ magnified $K$ times (here $K = k + c_V$). The Weyl reflections similar to (3.31) require us to take only the Weyl antisymmetric combinations
\begin{equation}
\sum_{w \in \tilde{W}} (-1)^{|w|} |w(\alpha); i \rangle, \tag{3.38}
\end{equation}
here $\tilde{W}$ is the Weyl group and $|w|$ denotes a determinant of the transformation $w$. As a result, the basis of $\mathcal{H}_T^G$ is formed by the states (3.38) ($i$ is either 1 or 2), for which $\alpha \in \Lambda_w$ belongs to the fundamental domain of the affine Weyl group $\tilde{W}$. This group is a semidirect product of the Weyl group $W$ and a group of translations by the elements of the lattice $K \Lambda_R$. The walls of the fundamental domain should be excluded, that is, we require $\tilde{w}(\alpha) \neq \alpha$ for any $\tilde{w} \in \tilde{W}$.

A scalar product $\langle \alpha; i | \beta; i \rangle$ of the basis elements of $\mathcal{H}_T^G$ is equal to 1 if $\alpha$ and $\beta$ are the shifted highest weights of conjugate representations, otherwise it is zero.

The formulas for $\hat{S}$ and $\hat{T}$ matrices of the simply laced Lie group $G$, as presented in \[1\] (see also \[4\]), are
\begin{align}
\hat{S}_{\alpha \beta} &= i^{|\Delta_+|} \sqrt{\frac{\text{Vol } \Lambda_w}{\text{Vol } K \Lambda_R}} \sum_{w \in W} (-1)^{|w|} \exp \left( - \frac{2\pi i}{K} \langle w(\alpha), \beta \rangle \right), \tag{3.39} \\
\hat{T}_{\alpha \beta} &= \delta_{\alpha \beta} \exp \left( \frac{i\pi}{K} \langle \alpha, \alpha \rangle - \frac{i\pi}{c_V} \langle \rho, \rho \rangle \right). \tag{3.40}
\end{align}
here $\Delta_+$ is a set of positive roots of $G$, $|\Delta_+|$ is their number, $\langle \ , \ \rangle$ is a Cartan scalar product normalized so that the length of all roots of $G$ is equal to $\sqrt{2}$, $\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$. Note that $\hat{S}_{\alpha \beta}$ is proportional to the numerator of the Weyl character formula.

Now we should explain how to produce the states $|\alpha; i\rangle$ by taking a path integral over a 3-dimensional manifold $M$ with a boundary $\partial M = T^2$. First we extend the path integral (1.2) by adding the so-called Wilson lines. Consider a closed manifold $M$ with a link $L$ inside it. Let us attach representations $V_{\alpha_i}$ of $G$ to its components $L_i$. Here $V_{\alpha}$ denotes a representation of $G$ with the shifted highest weight $\alpha$ (i.e. the highest weight of $V_{\alpha}$ is $\alpha - \rho$). For any connection $A_\mu$ on $M$ the trace of its holonomy $O_i = \text{Tr}_{V_{\alpha_i}} P \exp \oint A_\mu dx^\mu$ (3.41)
is invariant under gauge and coordinate transformations. Therefore the path integral

$$Z^{\alpha_1, \ldots, \alpha_n}(M, L, k) = \int [DA_\mu] e^{i \frac{k}{\pi} S_{CS}(A_\mu)} \prod_i O_i$$ (3.42)
is an invariant of the link $L$ in $M$. Witten used the methods of conformal field theory to prove that it satisfies skein relations. Thus he proved that this invariant is equal to the Jones polynomial up to a normalization constant.

Take the integral (3.42) for a solid torus $M = S^1 \times D^2$ ($\partial M = S^1 \times \partial D^2 = T^2$) and $L$ consisting of one component $S^1 \times P$, $P$ being the center of the disk $D^2$. Let $C_1$ be the cycle which is contractible through $M$. Witten claimed in [1], that if we attach a representation $V_{\alpha}$ to the only component of $L$, then the integral (3.42) produces a state $|\alpha; 1\rangle$ in $H^G_{T^2}$. In particular, a solid torus without any link inside it is equivalent to a torus with a link carrying a trivial representation, hence it has a state $|\rho; 1\rangle$ on its boundary.

Let us cut out a tubular neighborhood of an $n$-component link $L$ in $S^3$. A remaining piece $S^3 \setminus L$ has a boundary $(T^2)^n$. Therefore a path integral (1.2) taken over $S^3 \setminus L$ produces a state $|L\rangle$ in $(H^G_{T^2})^\otimes n$. Suppose that we glue a tubular neighborhood back after putting a Wilson line (3.41) inside each of its components. Then according to the multiplicativity law (3.4),

$$Z^{\alpha_1, \ldots, \alpha_n}(M, L, k) = \langle L| \bigotimes_{i=1}^n |\alpha_i; 1\rangle$$ (3.43)
Therefore the state $|L\rangle$ can be identified with the tensor $Z^{\alpha_1,\ldots,\alpha_n}$ belonging to the dual Hilbert space $\left[\mathcal{H}_{T^2}^G\right]^{\otimes n}$. The latter can be calculated by using cabling (for the link components which carry representations of $G$ other than the fundamental one) and skein relations.

Let us glue the components of the tubular neighborhood of $L$ back after performing modular transformations $U^{(p_i,q_i)}$ on their boundaries. Any 3-dimensional manifold $M$ can be constructed in this way. A multiplicativity law (3.4) leads to the following expression for its invariant:

$$Z(M,K) = \sum_{\alpha_1,\ldots,\alpha_n} Z^{\alpha_1,\ldots,\alpha_n}(S^3,L,k)\hat{U}^{(p_1,q_1)}_{\alpha_1\rho} \ldots \hat{U}^{(p_n,q_n)}_{\alpha_n\rho}. \quad (3.44)$$

The sum here goes, of course, over $\alpha_i$ belonging to the fundamental domain of the affine Weyl group $\tilde{W}$. Different surgeries on different knots in $S^3$ can produce the same manifold $M$. Reshetikhin and Turaev proved in [2] that the value of the r.h.s. of eq. (3.44) is the same for all such surgeries.

4 Some Examples

4.1 A Gluing Formula

We are going to use the surgery calculus in order to calculate the $U(1)$ and $SU(2)$ invariants of some simple 3-dimensional manifolds $M$. We will construct these manifolds by gluing together 2 solid tori after performing a modular transformation $U \in SL(2,\mathbb{Z})$ on the surface of one of them. Since neither of the tori has a Wilson line (3.41) inside it, then they have a state $|\rho;1_i\rangle \in \mathcal{H}_{T^2}^G$ corresponding to a trivial representation, on their boundary. An index $i = 1,2$ refers to the fact that there are two contractible cycles $C^{(i)}_1$ on the common boundary $T^2$: $C^{(1)}_1$ is contractible through the first solid torus while $C^{(2)}_1$ is contractible through the second one. Let us use the basis in the Hilbert space $\mathcal{H}_{T^2}^G$ corresponding to the cycles $C^{(1)}_1$ of the first solid torus, then

$$|\rho,1_2\rangle = \sum_{\alpha} \hat{U}_{\rho\alpha}|\alpha;1_1\rangle. \quad (4.1)$$
According to the multiplicativity law (3.4), Witten’s invariant of the manifold constructed by gluing the tori, is a scalar product

$$\langle \rho; 1_1 | \rho; 1_2 \rangle = \hat{U}_{\rho\rho}. \quad (4.2)$$

We will calculate the matrix element $U_{\rho\rho}$ with the help of eqs. (3.27), (3.29) and (3.34). The gluing induces a particular framing of the manifold, which may differ from the canonical one, so we will supplement $\hat{U}_{\rho\rho}$ with a correction factor (2.27). Then we will compare its large $k$ limit with the stationary phase approximation formula (2.33). Since $U(1)$ is abelian, its Chern-Simons action is purely quadratic. Therefore the path integral (1.2) is gaussian and the formula (2.33) should be exact for the $U(1)$ invariant.

### 4.2 3-Dimensional Sphere

We start with the 3-dimensional sphere $S^3$. The two solid tori that form it are glued through a modular transformation $S$. The induced framing is canonical, so the invariants are

$$Z_{U(1)}(S^3, k) = \hat{S}_{00} = \frac{1}{\sqrt{2k}}, \quad (4.3)$$

$$Z_{SU(2)}(S^3, k) = \hat{S}_{11} = \sqrt{2} \sin \frac{\pi}{K} k \to \infty \sqrt{2\pi K}^{-3/2}. \quad (4.4)$$

To determine the invariants entering eq. (2.33) we note that $\pi_1(S^3)$ is trivial and so is the only flat connection on $S^3$. Its Chern-Simons action is zero and $\tau_R = 1$. All the phase factors of eq. (2.33) can be dropped due to eq. (2.28). For any $U(1)$ flat connection on a manifold with $b_1(M) = 0$

$$H_a = U(1), \quad \dim H_a^0 = \dim H_a = 1, \quad \dim H_a^1 = 0, \quad (4.5)$$

while for the trivial $SU(2)$ connection

$$H_a = SU(2), \quad \dim H_a^0 = \dim H_a = 3, \quad \dim H_a^1 = 0 \quad (4.6)$$

As a result eq. (4.3) and the r.h.s. of eq. (4.4) coincide with the 1-loop formula (2.33) if we assume that

$$\text{Vol} (U(1)) = 2\pi, \quad \text{Vol} (SU(2)) = 2\pi^2. \quad (4.7)$$
Both volumes are perfectly consistent since $SU(2)$ is a 3-dimensional sphere and $U(1)$ is its big circle.

### 4.3 A Lens Space $L(p, 1)$

A less trivial example of a manifold is a lens space $L(p, 1)$. It is constructed by gluing two solid tori through a modular transformation

$$U(-p, 1) = ST^{-p} S.$$ (4.8)

According to [3] and [4], the induced framing differs from the canonical one by $p - 3$ units, so in canonical framing

$$Z_{U(1)}(L(p, 1), k) = e^{-i\pi/p} (ST^{-p} S)_{00} = e^{i\pi} \frac{1}{2k} \sum_{\alpha=0}^{2k-1} \exp \left( -\frac{i\pi}{2k} p\alpha^2 \right).$$ (4.9)

We changed here the range of summation over $\alpha$ from (3.18) to an equivalent one $0 \leq \alpha < 2k$.

The fundamental group of $L(p, 1)$ is $\mathbb{Z}_p$, so there are $p$ flat $U(1)$ connections corresponding to different homomorphisms (2.12). Therefore our objective is to transform the r.h.s. of eq. (4.8) into a sum of $p$ terms. We are going to use a Poisson resummation formula, which relates a sum over integer numbers of a function and its Fourier transform:

$$\sum_{\alpha \in \mathbb{Z}} f(\alpha) = \sum_{m \in \mathbb{Z}} \int_{-\infty}^{+\infty} e^{2\pi i m \alpha} f(\alpha) d\alpha.$$ (4.10)

The sum in eq. (4.9) has a finite range, but we can extend it by using a periodicity of its summand as a function of integer numbers:

$$\exp \left[ -\frac{i\pi}{2k} p(\alpha + 2kn)^2 \right] = \exp \left[ -\frac{i\pi}{2k} p\alpha^2 \right], \text{ for } \alpha, n \in \mathbb{Z}.$$ (4.11)

A “regularization” formula

$$\sum_{\alpha=0}^{T-1} f(\alpha) = \lim_{\epsilon \to 0} (Te^{1/2}) \sum_{\alpha \in \mathbb{Z}} e^{-\pi \epsilon \alpha^2} f(\alpha), \text{ if } f(\alpha + T) = f(\alpha) \text{ for } \alpha, T \in \mathbb{Z}.$$ (4.12)

together with eq. (4.10) allow us to reexpress the invariant (4.8)

$$Z_{U(1)}(L(p, 1), k) = e^{i\pi} \frac{1}{2k} \lim_{\epsilon \to 0} (2ke^{1/2}) \sum_{m \in \mathbb{Z}} \int_{-\infty}^{+\infty} e^{-\pi \epsilon \alpha^2} \exp i\pi \left[ -\frac{1}{2k} p\alpha^2 + 2m\alpha \right].$$ (4.13)
The integral over $\alpha$ is gaussian, it is exactly equal to the contribution of the stationary phase point

$$\alpha_m = \frac{2k}{p} m$$

(4.14)
determined by eq. (2.3). The only effect of the prefactor $e^{-\pi \alpha^2}$ to the leading order in $\epsilon$ is to suppress that contribution by the factor $e^{-\pi \alpha^2 m}$:

$$Z_{U(1)}(L(p, 1), k) = \lim_{\epsilon \to 0} (2k\epsilon^{1/2}) \sum_{m \in \mathbb{Z}} e^{-\pi \alpha^2_m} \frac{1}{\sqrt{2kp}} \exp \left( \frac{i\pi}{2k} p \alpha^2_m \right).$$

(4.15)

The stationary phase points $\alpha_m$ and their contributions exhibit the same symmetry under the action of the affine Weyl group, as the original summand in eq. (4.9). This means that if we add $p$ to $m$, then $\alpha_m$ is shifted by $2k$, while the last exponential of eq. (4.15) remains unchanged. Therefore we can roll eq. (4.12) backwards in order to limit the summation range of $m$ to its fundamental domain $0 \leq m < p$:

$$Z_{U(1)}(L(p, 1), k) = \sum_{m=0}^{p-1} \frac{1}{\sqrt{2kp}} \exp \left( \frac{2\pi ikm^2}{p} \right).$$

(4.16)

Thus we conclude that the 1-loop contributions of stationary points (4.14) appear to be exact. We just have to limit the sum to those $\alpha_m$ which belong to the fundamental domain of $\alpha$:

$$0 \leq \alpha_m < 2k.$$  

(4.17)

We achieved our goal of resumming eq. (4.9). Now a comparison with the stationary phase formula (2.33) is straightforward. The Chern-Simons action of the flat $U(1)$ connection is known to be

$$S_m = 2\pi^2 m^2$$

(4.18)

its Reidemeister torsion is $1/p$. We can again drop all the phase factors of eq. (2.33) since in this case all $\eta_m = 0$. Eqs. (1.5) and (4.7) complete the picture: the 1-loop formula (2.33) is really exact.
A calculation of the $SU(2)$ invariant of the lens space $L(p, 1)$ goes along the similar lines. The invariant in the canonical framing is equal to

$$Z_{SU(2)}(L(p, 1), k) = \exp \left[ -i \frac{\pi}{4} (p - 3) \frac{K - 2}{K} \right] \sum_{\alpha=1}^{K-1} \hat{S}_{\alpha} \hat{T}_{\alpha}^{-p} \hat{S}_{\alpha}$$

$$= -\frac{1}{2K} \exp \left[ \frac{\pi}{4K} [2p + 3(K - 2)] \sum_{\alpha=1}^{K-1} \sum_{\mu_1, \mu_2=\pm 1} \mu_1 \mu_2 \exp \left( -i \frac{\pi}{2K} [p\alpha^2 + 2\alpha(\mu_1 + \mu_2)] \right) \right].$$

Again we apply a Poisson resummation formula \( (4.10) \). We limit the sum over the stationary phase points \( (4.14) \) to those which belong to the $SU(2)$ affine Weyl group fundamental domain:

$$0 \leq \alpha_m \leq K.$$ \( (4.20) \)

Recall that it is twice as small as that of $U(1)$, because the $SU(2)$ affine Weyl group includes a reflection $\alpha \rightarrow -\alpha$. A contribution of the stationary phase points which lie on the boundaries of the domain \( (4.20) \) should be cut in half (in the case of $U(1)$ we could avoid this by excluding the point $\alpha = 2K$ from the domain \( (4.17) \)). If $p$ is odd, then there is only one such point $\alpha_0 = 0$, and the resummed expression \( (4.19) \) is

$$Z_{SU(2)}(L(p, 1), k) = -i \sqrt{2\pi} \exp \left( \frac{2\pi}{Kp} (p - 3) \right) \times \left[ \frac{1}{2} \left( e^{2\pi i p} - 1 \right) + \sum_{m=1}^{p-1} \left( e^{2\pi i p} \cos \frac{4\pi m}{p} - 1 \right) \exp \left( 2\pi i \frac{m^2}{p} \right) \right]$$

$$\overset{k \rightarrow \infty}{\longrightarrow} \sqrt{2\pi} (Kp)^{-3/2} + \sum_{m=1}^{p-1} \frac{i}{\sqrt{2Kp}} \left( 2 \sin \frac{2\pi m}{p} \right)^2 \exp \left( 2\pi i \frac{m^2}{p} \right).$$

The number of terms in this equation is approximately half of that in the $U(1)$ formula \( (4.16) \). The number of flat $SU(2)$ connections is also approximately twice as small as that of $U(1)$, because the Weyl reflection (a conjugation by $g \in SU(2)$ of eq. \( (3.30) \)) makes the nontrivial homomorphisms $\pi_1(L(p, 1)) = Z_p \rightarrow U(1)$ pairwise equivalent.

The first term of the r.h.s. of eq. \( (4.21) \) is a contribution of the trivial connection. Indeed, the Reidemeister torsion of the trivial connection is $p^{-3}$, so we get an agreement with eq. \( (2.33) \). The sum in the r.h.s. of eq. \( (4.21) \) goes over nontrivial flat connections. Their Chern-Simons action is again given by eq. \( (1.18) \). Since $\pi_1(L(p, 1)) = Z_p$ is abelian,
it is mapped by the homomorphism (2.12) into $U(1) \subset SU(2)$, so that its image commutes with the group $H_m = U(1)$. Therefore

$$\dim H_m^0 = \dim H_m = 1.$$  \hfill (4.22)

It is also known that $\dim H_m^1 = 0$. According to \[3\] and \[4\],

$$\exp \left(-i \frac{\pi}{2} l_m \right) = -i.$$  \hfill (4.23)

Combining all the pieces we see that the formula (2.33) coincides with the r.h.s. of eq. (4.21). The 1-loop formula is again demonstrated to work properly.

### 5 Discussion

Despite an obvious progress in calculating and understanding Witten’s invariant, many open questions still remain. The $1/k$ expansion of knot invariants carried through the Feynman diagram technics in \[10\], \[11\] and \[12\] was very successful. The terms in this expansion appeared to be Vassiliev knot invariants, and they are expressed as integrals generalizing in a certain way the gaussian linking number. However, a systematic loop expansion of the manifold invariants started in \[13\], proved to be technically hard. At the same time, the exact resummed formulas for lens spaces (like the middle expression in eq. (4.21)), which are the sums over flat connections, look abnormally simple and nice from the quantum field theory point of view. Moreover, the contributions of the irreducible flat connections on Seifert manifolds, extracted from the surgery formulas in \[6\], are finite loop exact (that is, the corrections to the terms of eq. (2.33) go only up to a finite order in $1/k$ expansion). All these facts require a genuine 3-dimensional explanation.

Another possible development of the quantum Chern-Simons theory was suggested by Witten in \[14\]. He noted that if the fermionic gauge fields were properly added to the action (1.1) (in other words, if the Chern-Simons theory was based on an appropriate supergroup), then their determinant might cancel the bosonic one (i.e. a square root of the Reidemeister torsion) in eq. (2.33) up to a sign. A resulting invariant would be a sum over
flat connections, each taken with a certain sign. Witten conjectured that Casson invariant might be obtained in that way. A Chern-Simons invariant based on a supergroup $U(1|1)$ was studied in [15]. It is related to the Alexander polynomial and also produces a “$U(1)$ Casson invariant” which is simply the order of the homology group. It is possible that a generalization of this theory to other supergroups, such as $U(2|2)$ may produce Casson invariant and provide a quantum field theory explanation for its calculation through surgery construction (see e.g. [10]).

Acknowledgements

I am thankful to C. DeWitt-Morette, D. Freed, L. Kauffman and A. Vaintrob for inviting me to give review talks at their seminars and for encouraging me to write these notes. I am also indebted to H. Saleur for many useful discussions.

References

[1] E. Witten, *Commun.Math.Phys.* **121** (1989) 351.

[2] N. Reshetikhin, V. Turaev, *Invent.Math.* **103** (1991) 547.

[3] D. Freed, R. Gompf, *Commun.Math.Phys.* **141** (1991) 79.

[4] L. Jeffrey, *Commun.Math.Phys.* **147** (1992) 563.

[5] S. Garoufalidis, *Relations Among 3-Manifold Invariants*, University of Chicago preprint, 1991.

[6] L. Rozansky, *A Large k Asymptotics of Witten’s Invariant of Seifert Manifolds*, preprint UTTG-06-93, [hep-th/9303099](https://arxiv.org/abs/hep-th/9303099).

[7] A. Schwarz, *Lett.Math.Phys.* **2** (1978) 247.
[8] S. Elitzur, G. Moore, A. Schwimmer, N. Seiberg, *Nucl. Phys.* **B326** (1989) 108.

[9] V. Kac, *Infinite Dimensional Lie Algebras*, (Progress in Mathematics, vol.44), Basel: Birkhäuser 1983.

[10] L. Alvarez-Gaume, J. Labastida, A. Ramallo, *Nucl. Phys.* **B334** (1990) 103.

[11] E. Guadagnini, M. Martellini, M. Mintchev, *Phys. Lett.* **277B** (1989) 111.

[12] D. Bar-Natan, *Perturbative Chern-Simons Theory*, Princeton Preprint, August 23, 1990.

[13] S. Axelrod, I. Singer, *Chern-Simons Perturbation Theory*, Proceedings of XXth Conference on Differential Geometric Methods in Physics, Baruch College, C.U.N.Y., NY, NY.

[14] E. Witten, *Nucl. Phys.* **B323** (1989) 113.

[15] L. Rozansky, S. Saleur, *Nucl. Phys.* **B376** (1992) 461;  *Nucl. Phys* **B389** (1993) 365;  *Reidemeister Torsion, the Alexander Polynomial and U(1,1) Chern-Simons Theory*, Preprint YCTP-P35-1992.

[16] K. Walker, *An extension of Casson’s Invariant to Rational Homology Spheres*, preprint.