Effect of Delay and Control on a Predator-prey Ecosystem with Hassell-Varley Functional Response of Generalist Predator

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Abstract

Generalist predators are an important component of an ecosystem which may act as a biocontrol agent and influence the dynamics significantly. In this paper, we have studied the effect of delayed logistic growth of the prey species with Hassell-Varley functional response of generalist predator. The Lyapunov stability criteria for the interior equilibrium point is derived. Conditions for alternative stable solutions are obtained using the Kolmogorov theorem and hence we get a stable system. Also, the condition of Hopf-bifurcation and the point of bifurcation are obtained. The length of the delay is also estimated for the system to preserve stability. Numerical simulations are performed and illustrated to support the obtained analytical results. It is observed that by the application of an indirect feedback control mechanism, the system can be brought back to a stable state which was earlier unstable. Latin Hypercube Sampling/Partial Rank Correlation Coefficient (LHS/PRCC) sensitivity analysis, which is an efficient tool often employed in uncertainty analysis, is used to explore the entire parameter space of a model.

Keywords: Logistic Delay, Generalist Predator, Hassell-Varley Functional Response, Leslie-Gower scheme, Indirect Feedback Control

1. Introduction

Management of natural resources through preservation and restoration using biological control agents are drawing the attention of ecologists nowadays [1]. Generalist predators have the potential to act as biological control agent. Biological control methods help to protect the flora and fauna of an ecosystem, are used in many recovery plans [2].

One way to successfully deploy biological control is by introducing a population species that preys upon the invasive species. The interaction can be then modelled by a predator-prey model. There are various factors by which the predator-prey population dynamics are affected. One of the components among those is the predators per capita kill rate, which has significant contributions towards the change in species densities in the ecosystem. Functional responses reflect this characteristic in a population model. Holling [3], in 1959, proposed three functional responses based upon some characteristics of types of predation. Later various other types of functional responses have been introduced by Beddington-DeAngelis [4, 5], Arditi-Ginzburg [6], Hassel-Varley [7], etc. Several authors studied these functional responses

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in various ecosystems. Tian and Xu [8] studied the global dynamics of a predator-prey system using Holling type II functional response. Liu et al. [9] analyzed a host-parasitoid model using Holling type III functional response in the presence of Allee effect. With Beddington-DeAngelis functional response Li et al. [10] studied a stage-structured plant-pollinator model. Gakkhar [11] studied chaos in a food chain model with the ratio-dependent functional response.

Forming groups is one of the most fascinating behaviours seen in diverse animal species. These characteristics are registered by the functional responses while modelling predator-prey dynamics. There are evidence of both predators and prey forming groups [12]. In 1969, Hassell-Varley [7] introduced a functional response to incorporate the grouping behaviour in species. Predators often form groups to maximize their predation rate, while prey form groups to reduce the predation rate. In 1999, Cosner [13] proposed a theory on the structure of a functional response for modelling the grouping of species. According to Cosner, a functional response will be predator and prey density dependent if predators are assumed to form dense colonies. In contrast, a functional response will be only prey density dependent if prey stay in groups and predators are assumed to have homogeneous spatial distribution. Among numerous articles that studied the predator grouping in species, Hsu et al. [14] studied the global dynamics of a two-species model considering the Hassell-Varley functional response. Kim and Baek [15] studied the impulsive effect on such a system. Wu and Li [16] studied the global attractivity of a discrete predator-prey system with this functional response. Du et al. [17] considered a stochastic predator-prey model to study the effect of Hassell-Varley functional response in a stochastic environment. Grouping in prey often justified as group defence or group vigilance. Batabyal et al. [18] obtained the blow-up criterion with mutualistic prey considering prey dependent Hassel-Varley functional response.

Logistic growth puts a bound on the growth of a species as it incorporates the carrying capacity of the environment. Hence, the logistic equation is very useful in single species population models as well as interacting population models. However, due to the physiology of breeding, organic insusceptible reaction and several other factors, the growth rate of the species does not respond immediately. So, the delayed logistic equation, introduced by Hutchinson [19], is considered to model this feature. The logistic delay can represent the maturation time of an individual among a species [20]. Delay differential equations are often used while modelling natural population dynamics. The presence of delay can bring severe change in the stability of a system like destabilization of the system, large oscillations [20], the occurrence of chaotic behaviour [21, 22] etc.

As the growth of a species relies on the food source, hence food habit of a species also plays a prominent role in population density. While several species rely upon a particular food source, generalist predators can feed on a wide range of food varieties and survive a severe change in environmental condition, which distinguishes them from others. Hence, these can switch preys when their favourite food is not abundant. North American raccoons are a good example of generalist predators [23]. They are found in a wide variety of environments like forests, mountains and cities. These omnivore species can feed on almost everything from fruit and nuts to insects, frogs, eggs etc.

In a predator-prey model, Leslie [24] first proposed that the carrying capacity of the predators increases as the prey population increases. Moreover, the carrying capacity is a multiple of the prey population. Later Aziz-Alaoui [25] modified the functional response to incorporate the predators preference for alternate food source when their favourite food is scarce, which is the phenomenon of prey switching. It is introduced to the response term with
the addition of an extra parameter which measures the quantity of alternative food source
available in the environment. This is called the modified Leslie-Gower scheme. Hence, the
functional response of generalist predators is modelled by modified Leslie-Gower functional
response.

Using sensitivity analysis, one can determine which parameters influence the model output
the most or the least. Consequently, influential parameters on the model output need to be assigned accurate values while less influential parameters suffice to have a rough estimate
In this study, partial rank correlation coefficient (PRCC), a global sensitivity analysis technique proven to be most reliable and efficient among sampling-based methods, is utilized. The PRCC addresses the effect of changes in a specific parameter (linearly discounting the influences over the other parameters) on the reference model output

The paper is organized as follows. Section 2 shows the formulation of the mathematical model and section 3 contains some preliminary results which include the positivity and the boundedness of the system. Section 4 contains the Lyapunov stability analysis and section 5 contains the Kolmogorov analysis of the model. Hopf-bifurcation is shown in section 6 in the presence of the delay. In section 7 the length of the delay parameter is obtained to preserve stability. Section 8 illustrates the numerical simulations. In section 9 the stability of the controlled model is studied by the method of indirect control. Finally, section 10 represents the conclusions and discussions.

2. The Model

We propose a two-dimensional mathematical model which includes the following enlisted
assumptions for modelling our predator-prey system. The assumptions are as follows:

(a) the environment has a carrying capacity,
(b) growth in prey species involves maturation delay,
(c) functional response in prey equation is prey dependent Hassell-Varley type
(d) the predator species reproduces sexually,
(e) the functional response of the generalist predator species is modelled by the modified
Leslie-Gower scheme.

Though in a real ecological system, there may be many preys and predators: among predators
some are specialists and generalists, yet to capture the effect of a generalist predator on a
particular prey we have taken a two-dimensional system where there is alternate food to the
predators.

Incorporating the above assumptions, the predator-prey model can be represented as
follows:

\[
\frac{dX}{dT} = RX \left(1 - \frac{X(T - \tau)}{K}\right) - \frac{MY}{X^p + C}Y
\]
\[
\frac{dY}{dT} = \left(D - \frac{E}{X + A}\right)Y^2,
\]  

(1)

with initial conditions:

\[
X(\Theta) = \psi_1(\Theta) > 0,
Y(\Theta) = \psi_2(\Theta) > 0, \quad \Theta \in [-\tau, 0); \psi_i(0) > 0, i = 1, 2,
\]  

(2)
where $X(T)$ and $Y(T)$ are respectively the densities of the prey and predator species at time $T$. The parameters used in the system (1) bear the following meanings:

- $R$ = Intrinsic growth rate of the prey species $X$
- $\tau$ = Maturation time delay/Logistic delay
- $K$ = Environmental carrying capacity for the preys
- $M$ = Maximum predation rate
- $C$ = The protection provided to the prey population by the environment
- $D$ = Reproduction rate of the generalist predator by sexual reproduction
- $E$ = Maximum rate of death of predator population
- $A$ = Measures the other food sources available for the predator species
- $p$ = The Hassell-Varley constant

Now, let us consider, the non-dimensional variables

$$x = \frac{X}{K}, \quad y = \frac{Y}{K} \quad \text{and} \quad t = RT.$$

We use the above transformations on the system (1) to reduce the system to non-dimensionalised form and the reduced predator-prey model is given by,

$$\begin{align*}
\frac{dx}{dt} &= x(1 - x(t - \varrho)) - \frac{mxy}{x^p + c}, \\
\frac{dy}{dt} &= \left( d - \frac{e}{x + a} \right) y^2,
\end{align*}$$

subject to the initial conditions:

$$\begin{align*}
x(\theta) &= \phi_1(\theta) > 0, \\
y(\theta) &= \phi_2(\theta) > 0, \quad \theta \in [-\varrho, 0); \phi_i(0) > 0, i = 1, 2,
\end{align*}$$

where $\varrho$ is the logistic delay, and

$$\varrho = R\tau, \quad m = \frac{M}{RK^{p-1}}, \quad c = \frac{C}{K^p}, \quad d = \frac{DK}{R}, \quad e = \frac{E}{R} \quad \text{and} \quad a = \frac{A}{K}.$$

3. Preliminary Results

In this section, we shall discuss the positivity and boundedness of the solutions of the system (3) and also the permanence criteria of that system.

The system (3) has one interior equilibrium point which is $(x^*, y^*)$, where

$$x^* = \frac{e}{d} - a \quad \text{and} \quad y^* = \frac{1}{m}(1 - x^*)(x^{*p} + c).$$
3.1. Positivity

**Lemma 3.1.** The positive quadrant \( \text{int}(R^2_+) \) is invariant for system (3).

**Proof.** Here we want to show that \( \forall t \in [0, A) \) and \( A \in (0, +\infty) \), \( x(t) > 0 \) and \( y(t) > 0 \). Now solving the system (3) we have

\[
\begin{align*}
x(t) &= x(0) \exp \left[ \int_0^t \left( 1 - x(s - p) \frac{my}{xp + c} \right) ds \right], \\
y(t) &= y(0) \exp \left[ \int_0^t \left( d - \frac{e}{x + a} \right) yds \right].
\end{align*}
\]  

(5)

∴ \( x(t) > 0 \) and \( y(t) > 0 \) as \( x(0) > 0 \) and \( y(0) > 0 \).

Thus we see that the system (3) has positive solution with the positive initial condition given in (4). Thus the positive quadrant \( \text{int}(R^2_+) \) is invariant [32].  

3.2. Permanence

Now we show that the system (3) has permanence. Before proving, we first state the definition of permanence and a lemma which will be used to prove our statement given below in Theorem 3.4.

**Lemma 3.2** ([31, 33]). If for \( t \geq 0 \) and \( x(0) \geq 0 \) we have \( \dot{x} \geq x(c - dx) \) where \( c > 0, d > 0 \) then

\[
\lim_{t \to \infty} \inf x(t) \geq \frac{c}{d}
\]

and if for \( t \geq 0 \) and \( x(0) \geq 0 \) we have \( \dot{x} \leq x(c - dx) \) where \( c > 0, d > 0 \) then

\[
\lim_{t \to \infty} \sup x(t) \leq \frac{c}{d}.
\]

**Definition 3.3** ([31, 33]). A system is said to have permanence if for positive constant ‘m’ and ‘M’, we have

\[
m \leq \lim_{t \to \infty} \inf x(t) \leq \lim_{t \to \infty} \sup x(t) \leq M
\]

for all positive solution \( x(t) \) of the system.

**Theorem 3.4.** Let \( m_1, m_2, M_1 \) and \( M_2 \) be positive constants and independent of the initial solution of system (3), if the following conditions hold:

\[
m_1 \leq \lim_{t \to \infty} \inf x(t) \leq \lim_{t \to \infty} \sup x(t) \leq M_1
\]

\[
m_2 \leq \lim_{t \to \infty} \inf y(t) \leq \lim_{t \to \infty} \sup y(t) \leq M_2
\]

with positive initial condition, i.e., \( x(0) > 0, y(0) > 0 \), then we say that the system (3) has permanence [32].

**Proof.** With the positive initial condition \( (x(0), y(0)) \), it is easy to see that the solution \( (x(t), y(t)) \) of the system (3) is positive. From the first equation of system (3) we can write

\[
\frac{dx}{dt} \leq x.
\]  

(6)
Integrating both sides of (6) within \( t - \varrho \) to \( t \), we get
\[
\ln \frac{x(t)}{x(t-\varrho)} \leq \int_{t-\varrho}^{t} \, dt = \varrho,
\implies x(t-\varrho) \geq x(t)e^{-\varrho}. \tag{7}
\]
From the first equation of (3) and using (7), we get
\[
\frac{dx}{dt} \leq x(t) \left\{ 1 - e^{-\varrho}x(t) \right\}. \tag{8}
\]
Following Lemma 3.2 we can say
\[
\lim_{t \to \infty} \sup x(t) \leq e^\varrho = M_1, \tag{9}
\]
i.e., for all \( t \geq 0 \), \( \exists T_1 > 0 \) such that, \( x(t) \leq M_1 \), \( \forall t > T_1 \). Now, from the second equation of system (3), we can write
\[
\lim_{t \to \infty} \sup y(t) \leq \frac{1}{1-d} = M_2, \quad \text{provided} \quad d < 1, \tag{10}
\]
i.e., for all \( t \geq 0 \), \( \exists T_2 > 0 \) such that, \( y(t) \leq M_2 \), \( \forall t > T_2 \).
Again from the first equation (3), we can write
\[
\frac{dx}{dt} \geq x \left\{ 1 - M_1 - \frac{mM_2}{M_1^p + c} \right\}. \tag{11}
\]
Now integrating both sides of (10) within \( [t - \varrho, t] \), we get
\[
x(t-\varrho) \geq x(t) \exp \left( - \left\{ 1 - M_1 - \frac{mM_2}{M_1^p + c} \right\} \varrho \right). \tag{12}
\]
Therefore, from the first equation of (3) we can write
\[
\frac{dx}{dt} \geq x \left[ 1 - \exp \left( - \left\{ 1 - M_1 - \frac{mM_2}{M_1^p + c} \right\} \varrho \right) \right] x(t) - \frac{mM_2}{M_1^p + c}. \tag{13}
\]
Using Lemma 3.2, we can write
\[
\lim_{t \to \infty} x(t) \geq m_1, \tag{14}
\]
where, \( m_1 = \left( 1 - \frac{mM_2}{M_1^p + c} \right) \exp \left( \left\{ 1 - M_1 - \frac{mM_2}{M_1^p + c} \right\} \varrho \right) \). Similarly from the second equation of system (3), we get
\[
\lim_{t \to \infty} \inf y(t) \geq m_2, \tag{15}
\]
where, \( m_2 = \frac{M_1 + a}{1-d\left( (M_1 + a) - d \right)} \), provided \( (1-d)(M_1 + a) - d > 0 \).
Hence from (12) & (14) and (10) & (15), we have
\[
m_1 \leq \lim_{t \to \infty} \inf x(t) \leq \lim_{t \to \infty} \sup x(t) \leq M_1, \tag{16}
\]
\[
m_2 \leq \lim_{t \to \infty} \inf y(t) \leq \lim_{t \to \infty} \sup y(t) \leq M_2, \tag{17}
\]
with positive initial condition, i.e. \( x(0) > 0 \), \( y(0) > 0 \).
Thus, we conclude that the system (3) has permanence. \( \square \)
4. Lyapunov Stability Analysis

Here, we have studied the stability of \((3)\) by using a suitable Lyapunov functional as done in [28]. The system \((6)\) has positive equilibrium point \(E = (x^*, y^*)\). Introducing the new set of variables \(\bar{x} = x - x^*\) and \(\bar{y} = y - y^*\) in system \((3)\), we get the following linearised system as:

\[
\begin{align*}
\frac{du}{dt} &= a_{11}\bar{x} + a_{12}\bar{y}, \\
\frac{d\bar{y}}{dt} &= a_{22}\bar{x},
\end{align*}
\]

(16)

where, \(u = \bar{x} - x^* \int_{t-\varrho}^t \bar{x}(s)ds\), \(a_{11} = \frac{max^* \varrho}{(x^{*2} + c)^2}\), \(a_{12} = -\frac{max^*}{x^{*2} + c}\) and \(a_{22} = \frac{dy^*}{e}\).

Now following the steps as in [28], we shall check the stability of the system by assuming a suitable Lyapunov function \(w(v)(t)\) as follows:

\[
w(v)(t) = k_1w_1(v)(t) + k_2w_2(v)(t) + k_3w_3(v)(t),
\]

(17)

where,

\[
egin{align*}
w_1(v)(t) &= u^2 + x^*(a_{11} + a_{12}) \int_{t-\varrho}^t \bar{x}^2(l)dlds, \\
w_2(v)(t) &= \bar{y}^2, \\
w_3(v)(t) &= u\bar{y} + \frac{a_{22}x^*}{2} \int_{t-\varrho}^t \bar{x}^2(l)dlds,
\end{align*}
\]

and

\[
\begin{align*}
k_1 &= 2a_{12}\varrho - a_{12}, \\
k_2 &= a_{22}\left(1 + \frac{\varrho}{2}\right), \\
k_3 &= (a_{11} + a_{22})\varrho.
\end{align*}
\]

As all the parameters are assumed positive so, \(k_1 > 0, k_2 > 0, k_3 > 0\) and \(w(v)(t) > 0\). Taking the derivative of \((17)\), we get

\[
\frac{d}{dt}w(v)(t) \leq \Lambda_1\bar{x}^2 + \Lambda_2\bar{y}^2,
\]

(18)

where,

\[
\begin{align*}
\Lambda_1 &= k_1\{2a_{11} - a_{11}x^* - x^*(a_{11} + a_{12})\varrho\} + k_3\frac{a_{22}x^*}{2}(\varrho - 1), \\
\Lambda_2 &= a_{12}(k_3 - k_1 x^*).
\end{align*}
\]

**Theorem 4.1.** If the value of the delay \(\varrho\) satisfies the conditions \(\Lambda_1 < 0, \Lambda_2 < 0\) then the interior equilibrium point \(E(x^*, y^*)\) of \((3)\) is locally asymptotically stable.

**Proof.** Following the steps as done in [28], one can easily prove Theorem 4.1. \(\square\)
5. Kolmogorov analysis

Here, we shall discuss the stability of the system using the Kolmogorov theorem. It not only shows the existence of a limit cycle but also discusses its stability. It strongly suggests that those natural ecosystems which seem to have a persistent pattern of reasonably regular oscillations are in fact stable limit cycles.

**Theorem 5.1** (Kolmogorov Theorem). Let a dynamical system is given by

\[ \dot{x} = x f(x, y), \quad \dot{y} = y g(x, y). \]  

If the system satisfies the following five conditions:

(i) \( \frac{\partial f}{\partial y} < 0 \),  
(ii) \( \frac{\partial g}{\partial y} < 0 \),  
(iii) \( f(0, 0) > 0 \),  
(iv) \( x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} < 0 \),  
(v) \( x \frac{\partial g}{\partial x} + y \frac{\partial g}{\partial y} > 0 \),

and four requirements: \( \exists A, B, C > 0 \) s.t.

(i) \( f(0, A) = 0 \),  
(ii) \( f(B, 0) = 0 \),  
(iii) \( g(C, 0) = 0 \),  
(iv) \( B > C \),

then the system will have either a stable limit cycle or a stable equilibrium point.

For system \( f(x, y) = 1 - x(t - \varrho) - \frac{my}{xp + c} \) and \( g(x, y) = (d - \frac{e}{x + a})y \).

Here, we use \( x(t - \varrho) = x(t)e^{-\lambda \varrho} \). Now,

\[ \frac{\partial f}{\partial y} < 0 \Rightarrow -m < 0, \text{ which is always true.} \]

\[ \frac{\partial g}{\partial y} < 0 \Rightarrow d - \frac{e}{x + a} < 0 \Rightarrow d - \frac{e}{M_1 + a} < 0. \]

\[ x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} < 0 \Rightarrow -xe^{-\lambda \varrho} + \frac{pxmymy}{(xp + c)^2} - \frac{my}{xp + c} < 0 \]

\[ \Rightarrow \min \left\{ \frac{pxmymy}{(xp + c)^2} \right\} - \max \left\{ xe^{-\lambda \varrho} + \frac{my}{xp + c} \right\} < 0 \]

\[ \Rightarrow \frac{pmnym_2}{(M_1^p + c)^2} - M_1 e^{-\lambda \varrho} - \frac{mM_2}{m_1^2 + c} > 0 \]

(23)

\[ f(0, 0) = 1 > 0. \]

\[ x \frac{\partial g}{\partial x} + y \frac{\partial g}{\partial y} > 0 \Rightarrow \frac{-ex}{(x + a)^2} + 2y^2 \left( d - \frac{e}{x + a} \right) > 0 \]

\[ \Rightarrow \min \left\{ 2y^2 \left( d - \frac{e}{x + a} \right) \right\} - \max \left\{ \frac{ex}{(x + a)^2} \right\} > 0 \]

\[ \Rightarrow 2m_2^2 \left( d - \frac{e}{m_1 + a} \right) - \frac{eM_1}{(m_1 + a)^2} > 0. \]

(24)

Again,

\[ f(0, A) = 0 \Rightarrow A = \frac{c}{m} > 0 \]

\[ f(B, 0) = 0 \Rightarrow B = e^{\lambda \varrho} > 0. \]

\[ g(C, 0) = 0 \Rightarrow C = \frac{e}{d} - a > 0. \]

\( B > C, \) by suitably choosing \( \varrho \) and corresponding \( \lambda \) value.
Theorem 5.2. The system (3) will possess a stable limit cycle or a stable equilibrium point when the inequalities (22), (23) and (24) hold.

6. Bifurcation analysis

In this section we shall discuss the Hopf-bifurcation for the system (3). Let the equilibrium point be $E(x^*, y^*)$. Letting $\bar{x} = x - x^*$ and $\bar{y} = y - y^*$ and substituting into Eq. (3) we get the linearised form as:

$$\frac{d\bar{x}}{dt} = a_{11}\bar{x} - x^*e^{-\lambda_0}\bar{x} + a_{12}\bar{y},$$
$$\frac{d\bar{y}}{dt} = a_{22}\bar{x},$$  \hspace{1cm} \text{(25)}$$

where the expressions for $a_{11}$, $a_{12}$ and $a_{22}$ as mentioned in section 4. The characteristic equation for the system (25) as given below

$$\left\{\lambda^2 - a_{11}\lambda + a_{11}a_{22}\right\} + e^{-\lambda_0}\left\{x^*\lambda - x^*a_{12}\right\} = 0.$$  \hspace{1cm} \text{(26)}$$

Let $\lambda = i\omega (> 0)$ then from (26) separating real and imaginary part we get

$$x^*\omega \sin(\omega\bar{\rho}) - x^*a_{12}\cos(\omega\bar{\rho}) = \omega^2 - a_{11}a_{22},$$
$$x^*a_{12}\sin(\omega\bar{\rho}) + x^*\omega \cos(\omega\bar{\rho}) = \omega(a_{11} + a_{22}),$$  \hspace{1cm} \text{(27)}$$

which gives

$$\omega^4 + \omega^2\left(a_{11}^2 + a_{22}^2 - x^*^2\right) + a_{12}^2(a_{11}^2 - x^*^2) = 0.$$  \hspace{1cm} \text{(28)}$$

The equation (28) will have positive root if

$$a_{11}^2 > x^*^2.$$  \hspace{1cm} \text{(29)}$$

Now we eliminate $\sin(\omega\bar{\rho})$ from (27) we have

$$\cos(\omega\bar{\rho}) = \frac{\omega^2(a_{11} + 2a_{22}) - a_{11}a_{22}^2}{x^*(\omega^2 - a_{12}^2)}.$$  \hspace{1cm} \text{(30)}$$

Let, $\omega = \omega_0$ be a positive root of (28), then

$$\varrho_n^* = \frac{1}{\omega_0}\left[\arccos\frac{\omega_0^2(a_{11} + 2a_{22}) - a_{11}a_{22}^2}{x^*(\omega_0^2 - a_{12}^2)} + 2n\pi\right], \hspace{1cm} n = 0, 1, 2, .....$$  \hspace{1cm} \text{(31)}$$

We define the function $\theta(\varrho) \in [0, 2\pi)$, such that $\cos \theta(\varrho)$ is given by the right hand side of (31). Then solving

$$S_n(\varrho) = \varrho - \varrho_n^*$$

we get the $\varrho$, at which stability switches occur. If $\lambda(\varrho)$ be the root of the characteristic equation (26) satisfying $Re\lambda(\varrho_n^*) = 0$ and $Im\lambda(\varrho_n^*) = \omega_0$, we get

$$\frac{d}{d\tau}Re\lambda \neq 0.$$  

Therefore, we see that Hopf-bifurcation occurs for $\varrho = \varrho_0^*$.  

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7. Estimation of the length of delay to preserve stability

Here, we consider the system \([3]\) and the set of all continuous real functions defined on the domain \([-\varrho, \infty)\) with the positive initial conditions defined in \([1]\) on \([-\varrho, 0]\). Linearizing the system \([3]\) around the origin after substituting \(v_1 = x - x^*\) and \(v_2 = y - y^*\), we get,

\[
\begin{align*}
\frac{dv_1}{dt} &= -b_{11}v_1(t - \varrho) + b_{12}v_1 - b_2v_2 \\
\frac{dv_2}{dt} &= b_3v_1
\end{align*}
\]  

(32)

where,

\[
\begin{align*}
b_{11} &= x^*, & b_{12} &= \frac{mpx^*y^*}{(x^*p + c)^2}, & b_2 &= \frac{mx^*}{x^*p + c}, & b_3 &= \frac{dy^*}{e}.
\end{align*}
\]

Applying Laplace transformation to the system \([32]\), we get,

\[
\begin{align*}
(s + b_{11}e^{-s\varrho} + b_{12})V_1 &= v_1(0) - b_{11} \mathcal{L}(s) e^{-s\varrho} - b_2V_2 \\
sV_2 &= v_2(0) + b_3V_1
\end{align*}
\]  

(33)

where, \(\mathcal{L}(s) = \int_0^s e^{-st}v_1(t) \, dt\), and \(V_1\) and \(V_2\) are Laplace transforms of \(v_1\) and \(v_2\) respectively. Following along the lines of \([32]\) and by using Nyquist criterion \([32]\), we can have the conditions for local asymptotic stability for the interior equilibrium point of the system \([3]\) are given by

\[
\begin{align*}
\text{Re } H(i\eta_0) &= 0 \quad (34) \\
\text{Im } H(i\eta_0) &> 0 \quad (35)
\end{align*}
\]

where \(H(s) = s^2 + p_1s + p_2 + e^{-s\varrho}(q_1s + q_2)\) is the characteristic equation and \(\eta_0\) is the smallest positive root of \(\text{Re } H(i\eta_0) = 0\). Hence, equations \([34]\) and \([35]\) can be rewritten as

\[
\begin{align*}
q_2 \cos \eta_0 \varrho + q_1\eta_0 \sin \eta_0 \varrho &= \eta_0^2 - p_2 \quad (36) \\
q_1\eta_0 \cos \eta_0 \varrho - q_2 \sin \eta_0 \varrho &> -p_1\eta_0 \quad (37)
\end{align*}
\]

The stability of the system is guaranteed if the conditions \([36]\) and \([37]\) hold simultaneously. We first need to find upper bound \(\eta_+\) of \(\eta_0\) independent of \(\varrho\) such that the system will be stable for all such \(\eta_0\) that lie in \([0, \eta_+]\). We will use this \(\eta_+\) to estimate the delay length \(\varrho\).

From equation \([36]\), as \(|\sin \eta_0 \varrho| \leq 1 \& |\cos \eta_0 \varrho| \leq 1|\),

\[
\eta_0^2 - p_2 \leq |q_2| + |q_1|\eta_0.
\]

Hence, from \([38]\), if

\[
\eta_+ \leq \frac{1}{2} \left[ |q_1| + \sqrt{q_1^2 + 4(p_2 - |q_2|)} \right] \quad (39)
\]

then we have \(\eta_0 \leq \eta_+\).

From equations \([36]\) & \([37]\),

\[
\frac{q_1\eta_0}{q_2} \left[ \eta_0^2 - p_2 - q_1\eta_0 \sin \eta_0 \varrho \right] - q_2 \sin \eta_0 \varrho > -p_1\eta_0 \quad (40)
\]
simplifying which we get,
\[ \left[ \frac{q_1^2 \eta_0^2}{q_2} + q_2 \right] \sin \eta_0^0 \leq \frac{q_1 \eta_0}{q_2} \left( \eta_0^2 - p_2 \right) + p_1 \eta_0. \] (41)

Now, we can have,
\[ \left[ \frac{q_1^2 \eta_0^2}{q_2} + q_2 \right] \sin \eta_0^0 \leq \frac{1}{q_2} (q_1^2 \eta_0^2 + q_2^2) \eta_0^+ \] (42)

and
\[ \frac{q_1 \eta_0}{q_2} (\eta_0^2 - p_2) + p_1 \eta_0 \leq \frac{q_1 (\eta_0^2 - p_2) + |p_1 q_2|}{|q_2|} \eta_0^+. \] (43)

Now, from (41), (42) and (43),
\[ 0 \leq \varrho < \varrho^+. \] (44)

where
\[ \varrho^+ = \frac{q_1 (\eta_0^2 - p_2) + |p_1 q_2|}{q_1^2 \eta_0^+ + q_2^2}. \] (45)

Thus when (44) holds for \( \varrho \), the system preserves stability.

8. Numerical Simulation

In the previous section, the conditions for stability and Hopf-bifurcation of the system (3) have been derived analytically. Now we do some numerical computations of the system (3) to understand our results obtained in previous sections by choosing suitable values of the parameters. For different values of delays, we have obtained different scenarios with \( E(x^*, y^*) \) as the interior equilibrium point. The values of the parameters are taken as: \( m = 1.2, p = 0.7, c = 0.3, d = 0.4, e = 0.25, a = 0.2 \). In Fig. 1 a stable solution for the species (3) has been plotted by taking the delay as bifurcation parameter. The interior equilibrium point \( E(x^*, y^*) \) is seen to be stable for less values of the delay i.e., \( \varrho < \varrho_0^* = 1.623 \). Now, with the increased values of delays, the oscillation of the population also increases. Therefore, at the critical value of delay \( (\varrho = \varrho_0^*) \) we get a stable periodic solution where the Hopf-bifurcation occurs and there we get a stable limit cycle around \( E(x^*, y^*) \) (Fig. 2). For large values of delay \( (\varrho > \varrho_0^*) \) the system becomes unstable (Fig. 3).

![Figure 1](image1.png)

Figure 1: Time versus population have been plotted by taking delay as a bifurcation parameter, which represents a stable solution for \( \varrho < \varrho^* \).
Figure 2: Time versus population have been plotted by taking delay as a bifurcation parameter, which represents a periodic solution for $\hat{\rho} = \hat{\rho}^*$. 

Figure 3: Time versus population have been plotted by taking delay as a bifurcation parameter, which represents an unstable solution for $\hat{\rho} > \hat{\rho}^*$. 

The Hopf-bifurcation in system parameters $p$, $c$, $d$, $e$ and $a$ are illustrated in the figures below for zero delay in the system (3) i.e. $\hat{\rho} = 0$. The parameter $m$ doesn’t show Hopf-bifurcation. The parameter set taken are $m = 1.2$, $p = 0.7$, $c = 0.3$, $d = 0.4$, $e = 0.2$ and $a = 0.2$. Each time when performing bifurcation on a single parameter, other parameters are taken constant whose values are taken from the above parameter set. 

Figure 4 represents Hopf-bifurcation in $p$ for the system (3). The bifurcation point is $p^* = 0.7431$ for $m = 1.2$, $c = 0.3$, $d = 0.4$, $e = 0.2$ and $a = 0.2$. The figure also shows system dynamics for $p = 0.7 < p^*$ and $p = 0.8 > p^*$. 

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Figure 4: Hopf-bifurcation in $p$ for system (3) when $\rho = 0$

Figure 5 represents Hopf-bifurcation in $p$ for the system (3). The bifurcation point is $c^* = 0.2727$ for $m = 1.2$, $p = 0.7$, $d = 0.4$, $e = 0.2$ and $a = 0.2$. The figure also shows system dynamics for $c = 0.2 < c^*$ and $c = 0.35 > c^*$.

Figure 6: Hopf-bifurcation in $d$ for system (3) when $\rho = 0$

Figure 6 represents Hopf-bifurcation in $p$ for the system (3). The bifurcation point is $d^* = 0.4081$ for $m = 1.2$, $p = 0.7$, $c = 0.3$, $e = 0.2$ and $a = 0.2$. The figure also shows system dynamics for $d = 0.35 < d^*$ and $d = 0.45 > d^*$.
Figure 7 represents Hopf-bifurcation in $e$ for the system (3). The bifurcation point is $e^* = 0.196$ for $m = 1.2$, $p = 0.7$, $c = 0.3$, $d = 0.4$ and $a = 0.2$. The figure also shows system dynamics for $e = 0.15 < e^*$ and $e = 0.25 > e^*$.

![Figure 7: Hopf-bifurcation in $e$ for system (3) when $g = 0$](image)

Figure 8 represents Hopf-bifurcation in $a$ for the system (3). The bifurcation point is $a^* = 0.2099$ for $m = 1.2$, $p = 0.7$, $c = 0.3$, $d = 0.4$ and $e = 0.2$. The figure also shows system dynamics for $a = 0.18 < a^*$ and $a = 0.22 > a^*$.

![Figure 8: Hopf-bifurcation in $a$ for system (3) when $g = 0$](image)

The corresponding non-delayed system of the system (3) does not show Hopf-bifurcation with respect to the parameter $m$.

The PRCC deals with the impact of changes in a particular parameter on the model output [26]. So as to get the PRCC estimations, Latin Hypercube Sampling (LHS) is picked for the input parameters where stratified sampling without substitution is performed. In the present study, uniform dissemination is assigned to each model parameter and sampling is done autonomously. The range for every parameter is at first set to ±25% of the nominal values (Fig. 9). It is seen that PRCC values lie between −0.8 to 0.7 for the non-delayed model. A positive value of PRCC indicates a positive correlation of the parameter with the model output, where negative values indicate the negative correlation. A positive correlation implies that a positive change in the parameter will increase the model output. Similarly, the negative correlation implies that a negative change in the parameter will decrease the model output. The larger the absolute value of the PRCC is similar as the greater the correlation.
of the parameter with the output. The PRCC values for the non-delayed model is depicted as bar graphs in Fig. 9a and its time evolution are illustrated in Fig. 9b respectively.

Figure 9: In this figure the partial rank correlation coefficients (PRCCs) of the model parameters at different time points has been plotted in absence of delay. In Figure (a), PRCC of the model parameters has been shown and in Figure (b) PRCC has been plotted over time.

9. Indirect Feedback Control

In this section, we will introduce indirect control to the system to control instability in the system. Let \( u_1 \) and \( u_2 \) be two feedback control functions. Incorporating these to the system (3),

\[
\begin{align*}
\frac{dx}{dt} &= x(1 - x(t - \varrho)) - \frac{mxy}{x^p + c} - u_1(t - \varrho) \\
\frac{dy}{dt} &= \left(d - \frac{e}{x + a}\right)y^2 - u_2(t - \varrho) \\
\frac{du_1}{dt} &= \alpha_1x - \beta_1u_1 \\
\frac{du_2}{dt} &= \alpha_2y - \beta_2u_2
\end{align*}
\]  

(46)

where \( \alpha_1, \alpha_2, \beta_1 \) and \( \beta_2 \) are control parameters.

Let \( (x^*, y^*, u_1^*, u_2^*) \) be interior fixed point of the system (46). Then

\[
\begin{align*}
\frac{\alpha_1}{\beta_1} x^* &= u_1^*, \\
\frac{\alpha_2}{\beta_2} y^* &= u_2^*, \\
y^* &= \frac{1}{m} \left(x^p + c\right) \left(1 - \frac{\alpha_1}{\beta_1} - x^*\right) = \frac{\alpha_2}{\beta_2} \left(d - \frac{e}{x^* + a}\right)
\end{align*}
\]

and \( x^* \) is the solution of the following equation

\[
\frac{\beta_2}{m} \left(d - \frac{e}{x^* + a}\right) \left(x^p + c\right) \left(1 - \frac{\alpha_1}{\beta_1} - x^*\right) - \alpha_2 = 0.
\]

The equilibrium point \( (x^*, y^*, u_1^*, u_2^*) \) is feasible if each component is defined and non-zero. Hence, the following conditions must hold:

\[
\beta_1 \neq 0, \quad \beta_2 \neq 0, \quad x^* \neq \frac{e}{d - a} \quad \text{and} \quad x^* \neq 1 - \frac{\alpha_1}{\beta_1}.
\]
Now translating the system to origin and linearizing around it, let $A$ be the obtained Jacobian matrix, then

$$A = \begin{bmatrix}
A_{11} & A_{12} & -e^{-\lambda \theta} & 0 \\
A_{21} & A_{22} & 0 & -e^{-\lambda \theta} \\
\alpha_1 & 0 & -\beta_1 & 0 \\
0 & \alpha_2 & 0 & -\beta_2
\end{bmatrix} \tag{47}$$

where

$$A_{11} = 1 - x^*(1 + e^{-\lambda \theta}) - \frac{my^* (c + (1 - p)x^{sp})}{(x^{sp} + c)^2}, \quad A_{12} = -\frac{mx^*}{x^{sp} + c},$$

$$A_{21} = \frac{ey^*^2}{(x^* + a)^2} \quad \& \quad A_{22} = 2 \left( d - \frac{e}{x^* + a} \right) y^*.$$

Hence, its characteristic equation is given by,

$$\lambda^4 + a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0 + (b_3 \lambda^3 + b_2 \lambda^2 + b_1 \lambda + b_0)e^{-\lambda \theta} + (c_1 \lambda + c_0)e^{-2\lambda \theta} = 0$$ \tag{48}
When controlled system.

From (49), the minors of the Hurwitz matrix are given by

where

\[
a_3 = \beta_1 + \beta_2 - 1 + x^* + \frac{my^* (c + (1 - p)x^p)}{(x^* + c)^2} - 2dy^* + \frac{2ey^*}{x^* + a},
\]

\[
a_2 = \beta_1 \beta_2 + (\beta_1 + \beta_2) \left\{-1 + x^* + \frac{my^* (c + (1 - p)x^p)}{(x^* + c)^2} - 2dy^* + \frac{2ey^*}{x^* + a} \right\}
+ \left\{-1 + x^* + \frac{my^* (c + (1 - p)x^p)}{(x^* + c)^2} \right\} \left\{-2dy^* + \frac{2ey^*}{x^* + a} \right\}
+ \frac{mex^* y^2}{(x^* + c)(x^* + a)^2},
\]

\[
a_1 = (\beta_1 + \beta_2) + \left\{-1 + x^* + \frac{my^* (c + (1 - p)x^p)}{(x^* + c)^2} \right\} \left\{-2dy^* + \frac{2ey^*}{x^* + a} \right\}
+ \beta_1 \beta_2 \left\{-1 + x^* + \frac{my^* (c + (1 - p)x^p)}{(x^* + c)^2} - 2dy^* + \frac{2ey^*}{x^* + a} \right\}
+ (\beta_1 + \beta_2) \frac{mex^* y^2}{(x^* + c)(x^* + a)^2},
\]

\[
a_0 = \beta_1 \beta_2 \left\{-1 + x^* + \frac{my^* (c + (1 - p)x^p)}{(x^* + c)^2} \right\} \left\{-2dy^* + \frac{2ey^*}{x^* + a} \right\}
+ \frac{\beta_1 \beta_2 mex^* y^2}{(x^* + c)(x^* + a)^2},
\]

\[b_3 = x^*,\]

\[b_2 = \alpha_1 + \alpha_2 + (\beta_1 + \beta_2)x^* + 2x^* y^* \left( \frac{e}{x^* + a} - d \right),\]

\[b_1 = \alpha_1 (\beta_2 - 2dy^* + \frac{2ey^*}{x^* + a}) + \alpha_2 (\beta_2 - 1 + x^* + \frac{my^* (c + (1 - p)x^p)}{(x^* + c)^2})
+ 2x^* y^* \beta_1 \beta_2 \left( \frac{e}{x^* + a} - d \right),\]

\[b_0 = 2(\alpha_1 + x^* \beta_1) y^* \beta_2 \left( \frac{e}{x^* + a} - d \right) + \alpha_2 \beta_1 \left\{-1 + x^* + \frac{my^* (c + (1 - p)x^p)}{(x^* + c)^2} \right\},\]

\[c_1 = \alpha_2 x^*,\]

\[c_0 = (\alpha_1 + \beta_1 x^*) \alpha_2\]

and studying the nature of the roots of (48), we can have the stability property of the controlled system.

When \( \varphi = 0 \), the characteristic equation reduces to

\[
\lambda^4 + (a_3 + b_3) \lambda^3 + (a_2 + b_2) \lambda^2 + (a_1 + b_1 + c_1) \lambda + (a_0 + b_0 + c_0) = 0. \quad (49)
\]

From (49), the minors of the Hurwitz matrix are given by

\[
\Delta_1 = a_3 + b_3, \quad \Delta_2 = \begin{vmatrix} a_3 + b_3 & a_1 + b_1 + c_1 \\ 1 & a_2 + b_2 \end{vmatrix},
\]

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Hence, by Routh-Hurwitz criteria, \((x^*, y^*, u_1^*, u_2^*)\) is stable iff
\[
\Delta_i > 0, \quad \text{for } i = 1, 2, 3, 4.
\]

Now, let \(\varrho \neq 0\). Let \(\lambda = \omega i, i = \sqrt{-1}\), then from (48),
\[
\omega^4 - a_2 \omega^2 + a_0 = (b_0 \omega^2 - b_0) \cos \omega \varrho + (-b_3 \omega^3 + b_1 \omega) \sin \omega \varrho - c_0 \cos 2 \omega \varrho + c_1 \omega \sin 2 \omega \varrho
\]
\[
a_3 \omega^3 - a_1 \omega = (-b_3 \omega^3 + b_1 \omega) \cos \omega \varrho + (b_0 - b_2 \omega^2) \sin \omega \varrho + c_1 \omega \cos 2 \omega \varrho + c_0 \sin 2 \omega \varrho.
\]

From [34] and Nyquist criterion [35], the conditions of local asymptotic stability are given by
\[
\begin{align*}
\text{Re } H(i \omega_0) &= 0 \\
\text{Im } H(i \omega_0) &= 0
\end{align*}
\]

where, \(H(\lambda) = 0\) is the characteristic equation given in (48) and \(\omega_0\) is the smallest positive root of equation (51). Therefore, in the presence of delay, the asymptotic stability of \((x^*, y^*, u_1^*, u_2^*)\) is guaranteed when
\[
\omega_0^4 - a_2 \omega_0^2 + a_0 = (b_0 \omega_0^2 - b_0) \cos \omega_0 \varrho + (-b_3 \omega_0^3 + b_1 \omega_0) \sin \omega_0 \varrho - c_0 \cos 2 \omega_0 \varrho + c_1 \omega_0 \sin 2 \omega_0 \varrho
\]
\[
a_3 \omega_0^3 - a_1 \omega_0 < (-b_3 \omega_0^3 + b_1 \omega_0) \cos \omega_0 \varrho + (b_0 - b_2 \omega_0^2) \sin \omega_0 \varrho + c_1 \omega_0 \cos 2 \omega_0 \varrho + c_0 \sin 2 \omega_0 \varrho
\]

hold simultaneously.

10. Conclusion and Discussion

In this paper, a two species competition model is considered and investigated along with the assumption that the prey species shows delayed logistic growth and the predator species is of generalist type. Food switching, when prey is scarce, is one of the most common behaviours of the generalist predators. The predator’s functional response is modelled by the Hassell-Varley functional response which generally shows the grouping mechanism of a population species. The stability and bifurcation viewpoints may influence the decision related to the management of ecological resources.

Throughout the paper, we have discussed the stability of the positive equilibrium state \(E(x^*, y^*)\) of the system (3) as this represents the co-existence of both prey and predator populations. We observed that the system attains permanence irrespective of the parameters and the delay length. Hence, the overall population of the ecosystem is always bounded. By constructing suitable Lyapunov function \(w(v)(t)\), it is observed that the interior equilibrium state is locally asymptotically stable if the conditions
\[
\begin{align*}
\Lambda_1 &= k_1 \{2a_{11} - a_{11} x^* - x^*(a_{11} + a_{12}) \varrho\} + k_3 \frac{a_{22} x^*}{2} (\varrho - 1) < 0, \\
\Lambda_2 &= a_{12}(k_3 - k_1 x^*) < 0
\end{align*}
\]

are satisfied by the parameters in the system (16).
Using Kolmogorov theorem, we have obtained the following sufficient conditions so that the system (3) will have a stable equilibrium point or a stable limit cycle around that positive equilibrium point.

\[ d - \frac{e}{M_1 + a} < 0 \]

\[ \frac{pmm_1^2m_2}{(M_1^2 + c)^2} - M_1e^{-\lambda e} - \frac{mM_2}{m_1^2 + c} > 0 \]

\[ 2m_2^2 \left( d - \frac{e}{m_1 + a} \right) - \frac{eM_1}{(m_1 + a)^2} > 0 \]

Hence, the global stability of the populations is achieved by means of asymptotic stability or periodic oscillations. This also signifies that the populations will never explode or go extinct.

The bifurcation point \( \varrho = \varrho_0^* \) for Hopf-bifurcation are obtained by considering logistic delay \( \varrho \) as the bifurcation parameter. For parameter values \( m = 1.2, p = 0.7, c = 0.3, d = 0.4, e = 0.25 \) and \( a = 0.2 \), the positive equilibrium is stable when \( \varrho < \varrho_0^* \) whereas for \( \varrho > \varrho_0^* \), the equilibrium point becomes unstable, and stable periodic solutions appear for \( \varrho = \varrho_0^* \).

As the presence of delay has the ability to destabilize a stable ecosystem, so estimation of the length of the delay is very important for system stability. We observed that the delay length is estimated to be within \( 0 \leq \varrho \leq \varrho_+ \) for the system (3) to possess local asymptotic stability.

Along with the time series and phase portrait plots for different \( \varrho \) values, numerical simulations on bifurcation in the parameters of the system (3) for \( \varrho = 0 \) are also presented in the numerical analysis section in 8. The Hopf-bifurcation diagrams for each parameter are plotted. The original parameter set taken to study the bifurcation is \( m = 1.2, p = 0.7, c = 0.3, d = 0.4, e = 0.25 \) and \( a = 0.2 \). Each time a parameter is considered for bifurcation, the parameter is perturbed and the system dynamics are studied for each parameter value. Considering the parameter \( p \), when the \( p \)-value is increased, it is observed that the system changes stability as the stable interior equilibrium point becomes unstable and a stable limit cycle arises around it when \( p \) passes \( p^* = 0.7431 \). Hence, the bifurcation point is \( p^* = 0.7431 \). Similarly for other parameters, the bifurcation values are obtained at \( c^* = 0.2727, e^* = 0.196, d^* = 0.4081 \) and \( a^* = 0.2099 \). Time series and phase portraits are also presented. The parameter \( m \) does not show Hopf-bifurcation.

The study of the system through a change in the quantity of alternate food source available to the predator shows that it must be in sufficient amount (at least \( a = 0.2099 \) in our case as shown in figure 5) to produce regular periodic oscillations in both species. Similarly, keeping other parameters constant, the growth rate must be above a threshold limit (\( d = 0.4081 \) in our case as shown in figure 5) to produce periodic oscillations in the population densities. In each of the above cases, the complementary values of the parameters give rise to a situation where the population species, over the long term, will have net increment or loss negligible.

Delayed feedback control functions \( u_1 \) and \( u_2 \) are incorporated, along with several control parameters, by the method of indirect control. This is considered by the assumption that the feedback mechanism takes \( \varrho \) units of time to respond to the changes in population density due to the control function. The objective is to make both the species exist simultaneously by using natural or external control factors without losing the general characteristics of the original system when the system dynamics show instability. This makes an ecosystem insusceptible to a high amplitude parameter variation. The stability conditions are derived
for both the presence and absence of delay situations. The necessary and sufficient stability conditions of the positive equilibrium for $\rho = 0$ are given by $\Delta_i > 0$, $(i = 1, 2, 3, 4)$ as in (50). Moreover, in the presence of delay, the asymptotic stability of the positive equilibrium is assured when both conditions in (53) hold.

Time series are designed to examine how the changes in the parameter affect the model output. Therefore, the PRCCs of the model output at a particular instance are obtained with respect to each parameter.

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