5-dimensional geometries III: the fibered geometries

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Abstract

We classify the 5-dimensional homogeneous geometries in the sense of Thurston. The present paper (part 3 of 3) classifies those in which the linear isotropy representation is nontrivial but reducible. Most of the resulting geometries are products. Some interesting examples include a countably infinite family $L(a; 1) \times S^1 L(b; 1)$ of inequivalent geometries diffeomorphic to $S^3 \times S^2$; an uncountable family $\tilde{S}L_2 \times_{\alpha} S^3$ in which only a countable subfamily admits compact quotients; and the non-maximal geometry $SO(4)/SO(2)$ realized by two distinct maximal geometries.

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1 Introduction

Thurston’s geometries are a family of eight homogeneous spaces that form the building blocks of 3-manifolds in Thurston’s Geometrization Conjecture. Building on Thurston’s classification [Thu97, Thm. 3.8.4] in dimension 3 and Filipkiewicz’s classification [Fil83] in dimension 4, this paper is part of a series carrying out the classification in dimension 5.

In the sense of Thurston, a geometry is a simply-connected Riemannian homogeneous space $M = G/G_p$, with the additional conditions that $M$ has a finite-volume quotient—“model”—and $G$ is as large as possible—“maximality” (details are in Defn. 2.1). Part I [Gen16a] outlines the division of the problem into cases, following the strategy of Thurston and Filipkiewicz, using the action of point stabilizers $G_p$ on tangent spaces $T_p M$ (the “linear isotropy representation”). Part II [Gen16b] performs the classification for the case when this representation is trivial or irreducible, by leveraging other classification results.

The present paper finishes the classification by working out the case when $T_p M$ is nontrivial and reducible. The decomposition of $T_p M$ is used to construct a $G$-invariant fiber bundle structure on $M$—hence the name “fibering geometries”—and the classification of these bundles provides a way to access a classification of geometries. To cope with the increased richness in fiber bundle structures compared to what is possible in lower dimensions, the tools required include conformal geometry, Galois theory, and Lie algebra cohomology in addition to everything used in previous classifications. In particular, extension problems feature much more noticeably than for the analogous cases in lower dimensions. The main result is the following.

**Theorem 1.1 (Classification of 5-dimensional maximal model geometries with nontrivial, reducible isotropy).** Let $M = G/G_p$ be a 5-dimensional maximal model geometry, and let $V$ be an irreducible subrepresentation of $G_p \curvearrowright T_p M$ of maximal dimension.

(i) If $\dim V = 4$ (Section 4), then $M$ is one of the spaces

$$S^4 \times \mathbb{E} \quad \mathbb{H}^4 \times \mathbb{E} \quad \mathbb{C}P^2 \times \mathbb{E} \quad \mathbb{C}H^2 \times \mathbb{E} \quad U(2,1)/U(2) \quad \text{Heis}_5.$$

(ii) If $\dim V = 3$ (Section 5), then $M$ is a product of 2-dimensional and 3-dimensional constant-curvature geometries.

(iii) If $\dim V = 2$ (Section 6), then $M$ is either a product of lower-dimensional geometries or one of the following.

(a) The unit tangent bundles,

$$T^1 \mathbb{H}^3 = \text{PSL}(2,\mathbb{C})/\text{PSO}(2) \quad T^1 \mathbb{E}^{1,2} = \mathbb{R}^3 \times \text{SO}(1,2)^\circ/\text{SO}(2);$$

(b) The associated bundles (see Defn. 6.27 and Table 6.29),

$$\text{Heis}_3 \times \mathbb{R} S^3 \quad \text{SL}_2 \times_\alpha S^3, \quad 0 < \alpha < \infty$$
$$\text{Heis}_3 \times \mathbb{R} \text{SL}_2 \quad \text{SL}_2 \times_\alpha \text{SL}_2, \quad 0 < \alpha \leq 1$$
$$L(a;1) \times S^1 L(b;1), \quad 0 < a \leq b \text{ coprime in } \mathbb{Z};$$
(c) The line bundles over $\mathbb{F}^4$,

$$\mathbb{R}^2 \times \widetilde{\text{SL}_2} \cong (\mathbb{R}^2 \times \widetilde{\text{SL}_2}) \times \text{SO}(2)/\text{SO}(2)$$

$$\mathbb{F}_a^5 = \text{Heis}_3 \times \widetilde{\text{SL}_2}/\{(atz, \gamma(t))\}_{t \in \mathbb{R}}, \quad a = 0 \text{ or } 1;$$

(d) The indecomposable non-nilpotent solvable Lie groups $\mathbb{R}^4 \rtimes \mathbb{R}$, specified by the list of characteristic polynomials of the Jordan blocks of a matrix $A$ where $t \in \mathbb{R}$ acts on $\mathbb{R}^4$ by $e^{tA}$,

$$A_{5,9}^{1,-1,-1} = \mathbb{R}^4 \rtimes \mathbb{R}^{(x-1)^2, x+1, x+1}$$

$$A_{5,7}^{1,-1,-1} = \mathbb{R}^4 \rtimes \mathbb{R}^{x-1, x-1, x+1, x+1}$$

$$A_{5,7}^{1,-1-a, -1+a} = \mathbb{R}^4 \rtimes \mathbb{R}^{x-1, x-1, x-a+1, x+a+1}$$

where $a > 0$, $a \neq 1$, $a \neq 2$,

and $\det(\lambda - e^{tA}) \in \mathbb{Z}[\lambda]$ for some $t > 0$;

(e) and the indecomposable nilpotent Lie groups,

$$A_{5,1} = \mathbb{R}^4 \rtimes \mathbb{R}^{x^2, x^2} \quad A_{5,3} = (\mathbb{R} \times \text{Heis}_3) \times \mathbb{R}.$$ 

Moreover, all of the explicitly named spaces above are indeed maximal model geometries; and each product geometry is a model geometry, and maximal if at most one factor is Euclidean.

The solvable Lie groups $M$ are given with names from [PSWZ76, Table II] for Mubarakzyanov’s classification of 5-dimensional solvable real Lie algebras [Mub63]. Their isometry groups are $M \rtimes G_p$, where the point stabilizer $G_p$ is a maximal compact subgroup of $\text{Aut} M$. Table 1.2 lists these groups $G_p$; more explicit descriptions for the solvable Lie groups can be found in Prop. 6.13 Step 4 and Prop. 6.7(iii) Step 6. The lower dimensional geometries have their usual isometry groups; e.g. $\mathbb{C}P^2$ and $\mathbb{CH}^2$ are as specified in [Fil83, Thm. 3.1.1]. A full list of point stabilizers in dimension 4 can be found in [Wal86, Table 1].

Table 1.2: Non-product fibering geometries by isotropy group $G_p$ (see also Fig. 2.4)

| Isotropy | Geometries |
|----------|------------|
| $U(2)$   | $\text{Heis}_5$ and $U(2,1)/U(2)$ |
| $\text{SO}(2) \times \text{SO}(2)$ | $\mathbb{R}^4 \rtimes \mathbb{R}^{x-1, x-1, x+1, x+1}$ and the associated bundles (Thm. 1.1(iii)(b)) |
| $\text{SO}(2)$ | The remaining solvable groups from Thm. 1.1(iii)(d) |
| $S^1_{1/2}$ | All line bundles over $\mathbb{F}^4$ (Thm. 1.1(iii)(c)) |
| $S^1_1$ | The two unit tangent bundles (Thm. 1.1(iii)(a)) and the nilpotent Lie groups from Thm. 1.1(iii)(e) |

In the spirit of [Mos50, Cor. p. 624], [Gor77], [Ish55], and [Ott09, Thm. 1.0.3], one can also give a classification up to diffeomorphism (Table 1.3). Most of the diffeomorphism types are readily guessed and can be verified by a theorem of Mostow [Mos62a, Thm. A]. The exception is the family of associated bundles $L(a; 1) \times_{S^1} L(b; 1)$; Ottenburger names them $\mathcal{N}_{a b 1}$ in [Ott09, §3.1] and shows that they are all diffeomorphic to $S^3 \times S^2$ in [Ott09, Cor. 3.3.2].
Table 1.3: Non-contractible non-product fibering geometries by diffeomorphism type

| Type       | Geometries                          |
|------------|-------------------------------------|
| $S^2 \times \mathbb{R}^3$ | $T^1 \mathbb{H}^3$                |
| $S^3 \times \mathbb{R}^2$ | $\text{Heis}_3 \times_{\mathbb{R}} S^3$ and $S^3 \times_{\alpha} \tilde{\text{SL}}_2$ |
| $S^3 \times S^2$       | $L(a; 1) \times_{S^1} L(b; 1)$    |

Roadmap. Section 2 lists basic definitions and any external results that need to be used frequently. Then Section 3 uses the decomposition of the linear isotropy representation to establish the existence of invariant fiber bundle structures on geometries (Prop. 3.3), introducing related notations (such as names for invariant distributions) along the way. The remaining sections each deal with base spaces of a single dimension:

- Section 4 handles the case where the fiber bundle has a 4-dimensional base with irreducible isotropy. The geometries are classified by curvature, using a strategy closely following that of Thurston for 3-dimensional geometries over 2-dimensional bases in [Thu97, Thm. 3.8.4(b)].
- Section 5 handles 3-dimensional isotropy irreducible bases. This case produces one non-product geometry that is shown not to be a model geometry by Galois theory (Section 5.2).
- Section 6 handles 2-dimensional bases. This case produces the “associated bundle geometries”, a source of examples such as an uncountable family of geometries (Prop. 6.35) and a geometry whose maximal realization is not unique (Rmk. 6.40).

2 Background

2.1 Geometries, products, and isotropy

Recall the definition of a geometry, following Thurston and Filipkiewicz in [Thu97, Defn. 3.8.1] and [Fil83, §1.1]. This is given in terms of homogeneous spaces, since the upcoming classification will rely heavily on them; [Gen16b, Prop. 2.5] in Part II outlines the equivalence to earlier definitions.

Definition 2.1 (Geometries).

(i) A geometry is a connected, simply-connected homogeneous space $M = G/G_p$ where $G$ is a connected Lie group acting faithfully with compact point stabilizers $G_p$.

(ii) $M$ is a model geometry if there is some lattice $\Gamma \subset G$ that acts freely on $M$. Then the manifold $\Gamma\backslash G/G_p$ is said to be modeled on $M$.

(iii) $M$ is maximal if it is not $G$-equivariantly diffeomorphic to any other geometry $G'/G'_p$ with $G \subset G'$. Any such $G'/G'_p$ is said to subsume $G/G_p$.

Many properties of this definition—including the existence of invariant Riemannian metrics and the correspondence between quotients $\Gamma\backslash G/G_p$ and complete, finite-volume manifolds locally isometric to $M$—are taken for granted here but stated more explicitly in Part II [Gen16b, §2].

Discussion of product geometries was omitted from Part II—but among the fibering geometries are many (28 and one countable family). Some shortcuts are possible with their classification, such as the following.
Proposition 2.2 (Products are models). If $M = G/G_p$ and $M'/G_q'$ are model geometries, then their product $M \times M' = (G \times G')/(G_p \times G_q')$ is a model geometry.

Proof. If $\Gamma \backslash G/G_p$ is modeled on $M$ and $\Gamma' \backslash G'/G_q'$ is modeled on $M'$, then $(\Gamma \times \Gamma') \backslash (G \times G')/(G_p \times G_q')$ is modeled on $M \times M'$.

Maximality is in general a more difficult question; we prove in Prop. 3.12 that the product of two maximal geometries is maximal, but under the assumption that at most one factor admits nonzero invariant vector fields and at most one factor is itself a product with a Euclidean factor. This happens to be enough for our usage in dimension 5 (Prop. 6.37).

There is, however, a shortcut to prove maximality for geometries realized by solvable Lie groups, given by the following rephrasing of a theorem of Gordon and Wilson.

Lemma 2.3 (Maximality for solvable Lie groups). Let $M$ be a simply-connected, unimodular, solvable Lie group whose adjoint representation acts with only real eigenvalues. Then the maximal geometry subsuming $M/\{1\}$ is $M \rtimes K/K$ where $K$ is a maximal compact subgroup of $\text{Aut } M$.

Proof. Under the provided assumptions, in any invariant metric on $M$, there is some $K \subseteq \text{Aut } M$ such that the identity component of the isometry group of $M$ is $\text{Isom}_0 M = M \rtimes K$ [GW88, Thm. 4.3]. Since $K$ is the point stabilizer of the identity, it is compact.

Since the transformation group of a maximal geometry is realizable as the isometry group in some invariant metric [Fil83, Prop. 1.1.2], the maximal geometry subsuming $M/\{1\}$ is of the form $M \rtimes K/K$, with $K$ not in any larger compact group of automorphisms.

When $M$ is nilpotent, the adjoint eigenvalues are always 1; this case is an earlier result by Wilson in [Wil82, Thm. 2(3)].

The existence of non-product geometries forces a classification to confront fiber bundle structures. Section 3 details how the occurrence of these structures is controlled by the action of the point stabilizer $G_p$ on the tangent space $T_p M$. Since $G_p$ is compact, it preserves an inner product, which allows expressing such a representation by a subgroup of $\text{SO}(5)$. Figure 2.4 recalls the classification of such subgroups from Part II in [Gen16b, Prop. 3.1].

Figure 2.4: Closed connected subgroups of $\text{SO}(5)$, with inclusions. $\text{SO}(3)_5$ denotes $\text{SO}(3)$ acting on its 5-dimensional irreducible representation; and $S^1_{m/n}$ acts as on the direct sum $V_m \oplus V_n \oplus \mathbb{R}$ where $S^1$ acts irreducibly on $V_m$ with kernel of order $m$. 

\[
\begin{array}{c}
\text{SO}(5) \\
\text{SO}(4) \quad \text{SO}(3) \times \text{SO}(2) \quad \text{SO}(3)_5 \\
\text{U}(2) \quad \text{SU}(2) \quad \text{SO}(2) \times \text{SO}(2) \\
S^1_1 \quad S^1_{m/n} \quad S^1_{1/2} \\
\{1\}
\end{array}
\]
2.2 Lie algebra extensions

If \( M \) is a \( G \)-invariant fiber bundle over a space \( B \), then the isometry group \( G \) is an extension of the transformation group of \( B \). Passing to Lie algebras permits use of the well-known classification of Lie algebra extensions by second cohomology, which this subsection recalls. For more details, a survey of Lie algebra cohomology in low degrees can be found in [Wag10] or [AMR00, §2–4].

**Definition 2.5 (Lie algebra cohomology), following [Wag10, §2].** Let \( M \) be a module of a Lie algebra \( g \) over a field \( k \). The Chevalley-Eilenberg complex is the cochain complex

\[
C^p(g, M) = \text{Hom}_k(\Lambda^p g, M)
\]

with boundary maps

\[
d_p : C^p \to C^{p+1}
\]

\[
(d_p c)(x_1, \ldots, x_{p+1}) = \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} c([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{p+1})
\]

\[
+ \sum_{1 \leq i \leq p+1} (-1)^{i+1} x_i c(x_1, \ldots, \hat{x}_i, \ldots, x_{p+1})
\]

where \( \hat{x}_i \) means \( x_i \) should be omitted. The cohomology of \( g \) with coefficients in \( M \) is defined to be the cohomology of this complex and denoted \( H^p(g, M) \).

**Theorem 2.6 (Classification of extensions by second cohomology [AMR00, Thm. 8]).** Let \( g \) and \( h \) be Lie algebras and let \( \sigma : g \to \text{out} h = \text{der}(h)/\text{ad}(h) \) be a Lie algebra homomorphism. Then the following are equivalent.

(i) For one (equivalently: any) linear lift \( \alpha : g \to \text{der}(h) \) of \( \sigma \) choose \( \rho : \Lambda^2 g \to h \) satisfying

\[
[\alpha_X, \alpha_Y] - \alpha_{[X,Y]} = \text{ad}_\rho([X,Y])
\]

Then the cohomology class of \( d\rho \) in \( H^3(g, Z(h)) \) vanishes, where the action of \( g \) on \( Z(h) \) is induced by \( \alpha \) and \( d \) is defined by the formula in 2.5 (ignoring that \( h \) may not be a \( g \)-module).

(ii) There exists an extension \( 0 \to h \to e \to g \to 0 \) inducing the homomorphism \( \sigma \).

If either occurs, then the extensions satisfying (ii) are parametrized by \( [\rho] \in H^2(g; Z(h)) \).

2.3 Notations

The naming of Lie groups and Lie algebras requires a number of notations; the following conventions will be used throughout, with scattered reminders.

- \( T_1 G \) denotes the tangent algebra of a Lie group \( G \)—the tangent space at the identity, identified with right-invariant vector fields on \( G \) so that the resulting flows correspond with the action of 1-parameter subgroups by multiplication on the left.

- \( G^0 \) denotes the identity component of a topological group \( G \), and \( \text{Isom}_0 M \) is short for the identity component of the isometry group of \( M \).

- Fraktur letters usually denote Lie algebras, e.g. an occurrence of \( g \) near \( G \) probably means \( g = T_1 G \); and \( \text{isom} M = T_1 \text{Isom} M \).

\(^1\) An almost identical version, generalized to super (i.e. \( Z/2Z \)-graded) Lie algebras, has been published as [AMR05].
3 Fiber bundle structures on geometries

If $M = G/G_p$ admits a $G$-invariant fiber bundle structure $F \to M \to B$, then $F$ and $B$ admit transitive group actions, which affords some hope of reduction to lower-dimensional problems. The aim of this section is to prove that when $\dim M = 5$, reducibility of the isotropy representation $G_p \curvearrowright T_p M$ both implies the existence of such a fibering and constrains some of its properties (Prop. 3.3). This begins with a definition of the structure being sought.

**Definition 3.1 (Fiberings).** A geometry $M = G/G_p$ fibers over a smooth manifold $B$ if $B$ is diffeomorphic to $M/F$ for some $G$-invariant foliation $F$ with closed leaves. (Equivalently, $B \cong G/H$ for some closed subgroup $H \subseteq G$ containing $G_p$.) The fibering is described as isometric if $B$ admits a $G$-invariant Riemannian metric, conformal if $B$ admits a $G$-invariant conformal structure, and essentially conformal (or essential for short) if it is conformal but not isometric.

**Remark 3.2.** Closedness of $H$ ensures that $G/H$ has a natural smooth structure and smooth action by $G$ [Hel78, Thm. II.4.2] and, by the existence of local cross sections for $B \to G$ [Hel78, Lemma II.4.1], that $M$ is a smooth fiber bundle $F \to M \to B$ where $F \cong H/G_p$ is a leaf of $\mathcal{F}$.

This section’s main result, guaranteeing the existence of useful fiberings, is the following Proposition. Its proof will be given in the last subsection, after the first two subsections introduce the $G$-invariant distributions that will be used to work infinitesimally with fiberings.

**Proposition 3.3 (The fibering description).** Let $M = G/G_p$ be a 5-dimensional geometry, and let $d$ be the dimension of the largest irreducible subrepresentation of $G_p \curvearrowright T_p M$.

(i) If $d = 4$ and $M$ is a model geometry then $M$ fibers isometrically over a 4-dimensional simply-connected Riemannian symmetric space.

(ii) If $d = 3$ then $M$ fibers conformally over $S^3$, $E^3$, or $H^3$.

(iii) If $d = 2$ and $M$ is a model geometry then $M$ fibers conformally over $S^2$, $E^2$, or $H^2$.

**Remark 3.4 (Generalizations to higher dimensions).** The first two cases generalize to $n$ dimensions without major modifications of the proof or conclusions. Case (ii) is when the restriction of $T_p M$ to some normal subgroup of $G_p$ decomposes with exactly one nontrivial irreducible subrepresentation, and case (i) is the sub-case when this subrepresentation has codimension 1. In both of these generalizations, the base space is an isotropy irreducible space; and if the fibering is essentially conformal, the base is Euclidean or a sphere.

Other fiberings can be produced from the “natural bundle” if $M$ is compact [GOV93, §II.5.3.2], or by using the Levi decomposition (see e.g. [GOV94, §1.4]) as part of something like [Mos05, Thm. C].

**Example 3.5 (Reducible isotropy does not in general imply fibering).** If $G/G_p$ has an invariant fiber bundle structure with positive-dimensional fiber and base, then the point stabilizer of the base is an intermediate group $G_p \subsetneq H \subsetneq G$. Building on Dynkin’s work classifying maximal proper connected subgroups, Dickinson and Kerr classified compact Riemannian homogeneous spaces with two isotropy summands in [DK08, §6] and found examples $G/G_p$ where $G_p$ is maximal. So these spaces have reducible isotropy but no nontrivial fiberings.

The lowest-dimensional counterexample (that is compact and has two isotropy summands) is $\text{Sp}(3)/\text{Sp}(1)$, in dimension 18 (Example V.10), where the embedding $\text{Sp}(1) \hookrightarrow \text{Sp}(3)$ is given by the irreducible representation of $\text{Sp}(1) \cong \text{SU}(2)$ on $\mathbb{C}^6$. 

7
3.1 Invariant distributions and foliations

This subsection introduces the fixed distributions (Defn. 3.6) that will usually become the vertical distributions—i.e. tangent distributions to the fibers in a fiber bundle. Showing that such a distribution can be integrated to a foliation by submanifolds will require some tools such as the integrability tensor (Lemma 3.8).

The action of $G$ on $TM$ induces the correspondence

$$
\left\{ \begin{array}{c}
G_p\text{-invariant subspaces} \\
\text{(subrepresentations) of } T_pM
\end{array} \right\} \leftrightarrow \left\{ \begin{array}{c}
G\text{-invariant} \\
\text{distributions on } M = G/G_p
\end{array} \right\},
$$

which provides a convenient way to refer to $G$-invariant distributions, including the following.

**Definition 3.6 (Fixed distributions of normal subgroups).** Given a geometry $M = G/G_p$ and a normal subgroup $H \subseteq G_p$, let $TM^H$ be the $G$-invariant distribution defined at $p$ by $T_pM^H$ (the subspace on which $H$ acts trivially). Let $TM^G$ denote the case $H = G_p$.

The most visibly useful property of $TM^H$ is that it can be integrated to produce the fibers of a fiber bundle. That is (see also Rmk. 3.2 above),

**Proposition 3.7.** $TM^H$ is integrable to a $G$-invariant foliation $\mathcal{F}^H$ with closed leaves.

The proof uses a construction known in the context of Riemannian submersions as the “integrability tensor” (see e.g. [O’N66, Lemma 2] and the discussion before [Pet06, Thm. 3.5.5]). Its definition, its tensoriality, and its compatibility with the action of point stabilizers are covered by the following rephrasing of [Fil83, Lemma 3.2.2].

**Lemma 3.8 (Integrability tensor as a representation homomorphism).** Let $D$ be a distribution on a manifold $M$. Then at each point $p$, the Lie bracket of vector fields induces a linear map $\mu : \Lambda^2 D_p \to T_pM/D_p$. If $D$ is $G$-invariant on a homogeneous space $M = G/G_p$, then $\Lambda^2 D_p \to T_pM/D_p$ is a $G_p$-representation homomorphism.

**Proof.** Suppose $\{X_1, \ldots, X_k\}$ is a local basis of $D$, and $Y_1$ and $Y_2$ are vector fields in $D$ with span $(Y_1(p), Y_2(p)) \subseteq \text{span}(X_1(p), X_2(p))$. Pick functions $\{a_i\}$ and $\{b_i\}$ so that

$$
Y_1 = \sum_{i=1}^m a_i X_i, \quad Y_2 = \sum_{i=1}^m b_i X_i
$$
in a neighborhood of $p$. Using the Leibniz rule for Lie brackets,

$$
[Y_1, Y_2] = \sum_{i,j} a_ib_j[X_i, X_j] + a_iX_i(b_j)X_j - b_jX_j(a_i)X_i.
$$

The last two terms are pointwise linear combinations of $X_1, \ldots, X_m$, so they add to some vector field $Z$ in $D$. At $p$, only $a_1, a_2, b_1$, and $b_2$ can be nonzero, so

$$
[Y_1, Y_2](p) = (a_1b_2 - a_2b_1)[X_1, X_2](p) + Z(p),
$$

which ensures that $\mu$ is well-defined. The $G_p$-equivariant version then follows by recalling that diffeomorphisms respect Lie brackets. \qed

**Proof of Prop. 3.7.** The integrability tensor $\mu : \Lambda^2 T_pM^H \to T_pM/(T_pM)^H$ is zero, since the domain is a trivial representation of $H$, and the codomain is a representation with no trivial summands. So by the Frobenius condition, $TM^H$ is integrable.

A $G$-invariant foliation on a geometry $M = G/G_p$ whose tangent distribution contains $TM^G$ has closed leaves [Fil83, Prop. 2.1.1, 2.1.2]. Since $H \subseteq G_p$ implies $T_pM^H \supseteq T_pM^{Gr}$, the foliation integrating $TM^H$ has closed leaves. \qed
3.2 Complementary distributions and conformal actions

If the distribution $TM^H$ is interpreted as vertical—i.e. tangent to the fibers of a fibering $M → B = M/F^H$—then a complementary horizontal distribution should offer some understanding of the target $B$. This subsection introduces such a distribution and its applications to maximality of products (Prop. 3.12) and conformal fiberings (Lemma 3.13).

**Definition 3.9 (Complementary distributions).** Given a geometry $M = G/G_p$ and a normal subgroup $H ≤ G_p$, let $(TM^H)^\perp$ denote the distribution defined at $p$ by the complementary $H$-representation to $(T_pM)^H$ in $T_pM$.

Conveniently, this always agrees with the other distribution deserving the same name—the orthogonal complement—which makes $TM^H$ and $(TM^H)^\perp$ useful together for studying invariant metrics on $M$. Explicitly,

**Lemma 3.10.** $(TM^H)^\perp$ is the orthogonal complement to $TM^H$ in every $G$-invariant metric on $M$.

**Proof.** A $G_p$-invariant metric on $T_pM$ induces a representation isomorphism $T_pM \cong_H T_p^G$. Since $(TM^H)^\perp$ contains no trivial subrepresentation by definition, its image in the trivial representation $(T_p^G)^H$ is zero; so the pairing of $(TM^H)^\perp$ with $(T_pM)^H$ is zero.

**Remark 3.11.** This property makes any isometric fibering $M → M/F^H$ a Riemannian submersion (see e.g. the discussion before [Pet06, Example 1.1.5]) for some invariant metric on $M$ for each invariant metric on $M/F^H$.

Our main purposes in defining $(TM^H)^\perp$ are to understand properties of fiberings given properties of the isotropy representation (Lemma 3.13 below, used to establish the fibering description in Prop. 3.3) and to prove the following statement about maximality of products.

**Proposition 3.12 (Products are often maximal).** Given maximal geometries $M = G/G_p$ and $M' = G'/G'_q$, their product

$$M \times M' = (G × G')/(G_p × G'_q)$$

is maximal provided that both of the following hold.

(i) At most one of $M$ and $M'$ is itself a product with a Euclidean factor.

(ii) At least one of $M$ and $M'$ admits no nonzero invariant vector field.

**Proof.** Hano proved (see e.g. [KN63, Thm. VI.3.5]) that the de Rham decomposition of a complete, simply-connected Riemannian manifold (see e.g. [KN63, Thm. IV.6.2]) also decomposes the isometry group. So if $M$ and $M'$ are complete, connected, simply-connected Riemannian manifolds, one of which is not isometric to any product $M'' × \mathbb{E}^k$, then

$$(\text{Isom}(M × M'))^0 = (\text{Isom} M)^0 × (\text{Isom} M')^0.$$ 

Since the transformation group of a maximal geometry is realizable as the isometry group in some metric [Fil83, Prop. 1.1.2], it suffices to know that all invariant metrics on $M × M'$ are realizable as product metrics. Condition (ii) implies this as follows.

To say that $M$ admits no invariant vector field means that $TM^G$ is zero. This implies that $\left( T_{(p,q)}(M × M') \right)^G = T_q M'$, whose orthogonal complement in every invariant metric is the complementary $G_p$-representation $T_p M$ (Lemma 3.10).
For non-product fiberings, the main use of \((TM^H)^\perp\) is in understanding metrics on \(M/F^H\), using the following slight generalization of part of [Fil83, Thm. 4.1.1].

**Lemma 3.13 (Irreducible + trivial isotropy produces conformal fibering).** Let \(M = G/G_p\) be a geometry and \(H\) a normal subgroup of \(G_p\) whose action on \((TM^H)^\perp_p\) is irreducible. Then there is a metric on \(M/F^H\) with respect to which \(G\) acts conformally.

A key ingredient in similar results such as [Fil83, Thm. 4.1.1] or the start of the proof of [Thu97, Thm. 3.8.4(b)] is the observation that all points of the same fiber \(F\) have the same subgroup of \(G\) for stabilizers. Then one can speak of representation isomorphisms, which the following standard fact turns into conformal maps.

**Lemma 3.14 (Existence [BD85, Thm. II.1.7] and uniqueness [KN63, App. 5 Thm. 1] of invariant inner products).** Every finite-dimensional irreducible representation \(V\) of a compact Hausdorff group \(K\) over \(\mathbb{R}\) has a unique \(K\)-invariant inner product (i.e. positive-definite symmetric bilinear form) up to scaling.

With this control over possible metrics, Filipkiewicz then constructs the appropriate metric using a partition-of-unity argument. Here is the proof, with details of the steps outlined above.

**Proof of the conformal fibering Lemma (3.13).** This proof repeats most of [Fil83, Thm. 4.1.1]; the new material is mostly the first step, which works around the unavailability of the previously mentioned “key ingredient”: points of the same fiber may not have the same stabilizer in \(G\).

**Step 1: Find a group \(G_F\) to replace \(G_p\).** Given \(p\) (thus \(G_p\)) and \(H \trianglelefteq G_p\), let \(F\) be the leaf of \(F^H\) containing \(p\), and let \(G_F\) denote the subgroup of \(G\) that acts as the identity on a leaf \(F\) of \(F^H\). Then \(H \subseteq G_F \subseteq G_p\), so \(G_F\) is a compact group acting irreducibly on \((TM^H)^\perp_q\) for every \(q \in F\).

**Step 2: Isomorphisms of \(G_F\)-irreps become conformal linear maps.** Fix a \(G\)-invariant metric \(\mu\) on \(M\). Its restriction \(\mu|_{(TM^H)^\perp}\) to the (invariant) distribution \((TM^H)^\perp\) is \(G\)-invariant, hence \(G_F\)-invariant for every leaf \(F\) of \(M/F^H\).

The projection \(\pi : M \to M/F^H\) induces \(G_F\)-representation isomorphisms \((TM^H)^\perp_q \to T\pi(q)(M/F^H),\) each producing from \(\mu|_{(TM^H)^\perp}\) a \(G_F\)-invariant inner product on a tangent space to \(M/F^H\). Since invariant inner products on irreducible representations are unique up to scaling (Lemma 3.14 above), all such pushed-forward inner products on the same tangent space are scalar multiples of each other.

**Step 3: Give sufficient conditions for a conformal structure to be \(G\)-invariant.** If a metric on \(M/F^H\) is pointwise a linear combination of the pushforwards described in Step 2, then \(G\) preserves its conformal class since \(G\) preserves \(\mu\), \(F^H\), and \((TM^H)^\perp\). What remains is to construct such a metric—i.e. to show that these pushforwards can be chosen in a smoothly varying way—using the partition-of-unity method from [Fil83, Thm. 4.1.1].

**Step 4: Construct the metric on \(M/F^H\).** Let \(k = \dim(TM^H)^\perp\), and choose a collection of \(k\)-discs \(V_i \subset M\) that

1. are transverse to \(TM^H\),

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2. each map diffeomorphically to $M/F^H$, and

3. cover $M/F^H$ with their images.

The cover of $M/F^H$ can be made locally finite since $M/F^H$ is a manifold; so there is a partition of unity $\{\phi_i\}$ subordinate to $\{\pi(V_i)\}$.

Let $\rho : TM \to (TM^H)\perp$ be the projection with kernel $TM^H$. On each $V_i$, let $\mu_i$ be the metric defined pointwise as the pullback of $\mu|_{(TM^H)\perp}$ by $\rho$. Then the sum

$$\bar{\mu} = \sum_i \phi_i \pi_* \mu_i$$

defines a metric on $M/F^H$ with the property described in Step 3. 

3.3 Proof of the fibering description (Proposition 3.3)

Two more facts will be needed for the proof of Prop. 3.3. First, the base is simply-connected in every case of Prop. 3.3: by definition, a geometry $M$ is simply-connected and a leaf $F$ of $\mathcal{F}$ is connected. So the homotopy exact sequence for $F \to M \to M/F$ implies $M/F$ is simply-connected. Hence the proofs to follow will skip proving simply-connectedness.

The second fact is a lemma extracted from Thurston’s 3-dimensional classification, used to reason about invariant vector fields in cases (i) and (iii).

**Lemma 3.15** (see e.g. [Thu97, Proof of 3.8.4(b)]). A $G$-invariant vector field $X$ on a model geometry $M = G/G_p$ is divergence-free.

**Proof.** Let $\phi_t$ be the $G$-equivariant flow on $M$ integrating $X$. If $N$ is any finite-volume manifold modeled on $M$, then $X$ and $\phi_t$ descend to some $\overline{X}$ and $\overline{\phi_t}$ on $N$, along with any $G$-invariant metric. Since vol $N$ is finite, $\overline{\phi_t}$ is globally volume-preserving; so $\int_N \text{div} \overline{X} \ d\text{vol} = 0$. Since $G$ acts transitively on $M$, the value of $\text{div} X = \text{div} \overline{X}$ is constant—thus 0. 

The proof of the fibering description (Prop. 3.3) can now proceed. The three cases exhibit somewhat different behavior, so they are handled separately. In particular, case (iii) (fiberings over 2-dimensional bases) is a bit more work to prove than the others, due to isotropy representations $G_p \rtimes T_{p_*}M$ in which there is not a canonical choice of a 2-dimensional irreducible summand to form the horizontal distribution.

**3.3.1 Case (i): over dimension 4**

**Proof of 3.3(i).** Let $M = G/G_p$ be a model geometry for which $G_p \rtimes T_{p_*}M$ has irreducible subrepresentations of dimensions 1 and 4. This proof mostly follows the argument in [Thu97, Thm. 3.8.4(b)].

**Case (i), Step 1: $M$ fibers over a 4-dimensional space.** The distribution $TM^G$ is 1-dimensional and integrates to a foliation with closed leaves (Prop. 3.7), so $M$ fibers over a 4-dimensional space $M/F^G$. 

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Case (i), Step 2: *M fibers isometrically.* A nonzero vector in \((T_pM)^G\) pushes forward by the action of \(G\) to a \(G\)-invariant vector field \(X\) on \(M\), with corresponding \(G\)-equivariant flow \(\phi_t\). Then \(\phi_t\) commutes with the action of \(G\), so all points in the same \(\phi_t\)-orbit have the same stabilizer in \(G\). Therefore \(d_p\phi_t : T_pM \to T_{\phi_t(p)}M\) is an isomorphism of \(G_p\)-representations.

Since \(M\) is a model geometry, \(\phi_t\) is volume-preserving (Lemma 3.15). Combined with the fact that \(\phi_t\) preserves the metric (the fiberwise inner product) on \(TM^G\) since it preserves its own velocity field \(X\), this implies \(\phi_t\) preserves volume on the orthogonal complement \((TM^G)^\perp\). Since \((TM^G)^\perp\) is irreducible, \(\phi_t\) preserves the metric on it (Lemma 3.14). So the metric on \((TM^G)^\perp\) descends to a \(G\)-invariant metric on \(M/F^G\).

Case (i), Step 3: \(M/F^G\) is Riemannian symmetric. By the classification of isotropy groups (Fig. 2.4), \(G_p\) is SO(4), U(2), or SU(2). In their standard representations on \(\mathbb{R}^4\), all of these contain the scalar \(-1\), which reverses tangent vectors—and thus geodesics—at the image of \(p\) in \(M/F^G\). □

3.3.2 Case (ii): over dimension 3

*Proof of 3.3(ii).* Suppose \(M = G/G_p\) is a geometry where \(G_p \subset T_pM\) contains an irreducible 3-dimensional subrepresentation. By the classification of isotropy groups (Fig. 2.4), \(G_p\) contains SO(3) as a characteristic subgroup. Therefore \(M\) fibers conformally over \(B = M/F^{SO(3)}\) (Lemma 3.13).

Then \(G\) acts transitively on \(B\), with SO(3) in the point stabilizers. If the fibering \(M \to B\) is isometric, then \(B\) is one of the 3-dimensional constant-curvature spaces. Otherwise,

1. a manifold \(B\) with a transitive\(^2\) essential conformal automorphism group Conf \(B\) is conformally flat [Oba73, Lemma 1]; and

2. if \(B\) is conformally flat and the identity component of Conf \(B\) acts essentially, then \(B\) is conformally equivalent to a sphere or Euclidean space [Laf88, Thm. D.1].

So if the fibering \(M \to B\) is essential, then \(B\) is \(S^3\) or \(E^3\). □

3.3.3 Case (iii): over dimension 2

*Proof of 3.3(iii).* Suppose \(M = G/G_p\) is a model geometry where the irreducible subrepresentations of \(G_p \subset T_pM\) have dimensions 1 and 2.

If \(G_p\) is SO(2) or SO(2) × SO(2), then a strategy like that of case (ii) applies: since \(G_p\) contains SO(2) as a normal subgroup, \(M\) fibers conformally over \(B = M/F^{SO(2)}\) (Lemma 3.13). Since \(B\) is simply-connected by the homotopy exact sequence, the Uniformization Theorem (see e.g. [Ahl10, Thm. 10-3]) implies it is conformally equivalent to \(S^2\), \(\mathbb{E}^2\), or \(\mathbb{H}^2\).

The remaining case is when \(G_p = S^1_{m/n}\) (a 1-parameter subgroup of SO(2) × SO(2)). By the same application of the Uniformization Theorem, it suffices to find a \(G\)-invariant foliation \(F\) of codimension 2 such that \(M\) fibers conformally over \(M/F\). The strategy consists of the following two steps. First, a foliation \(F\) with closed leaves and codimension 2 is found by examining Lie brackets of vector fields tangent to subrepresentations of \(G_p \subset T_pM\). Then if \(M\) does not fiber conformally over \(M/F\), this information is used to find an alternative fibering of \(M\).

---

\(^2\) A stronger version of Obata’s theorem (i.e. without assuming transitivity) could instead be used here, but its proof involves some analytic subtleties—see [Fer96] for an overview. The transitive action of \(G\) provides a way to sidestep these issues.
Case (iii), Step 1: Finding a foliation with closed leaves.  \( G_p = S^1_{m/n} \) means that

\[
T_pM = (T_pM)^G \oplus ((T_pM)^G)^\perp = (T_pM)^G \oplus V_m \oplus V_n,
\]

where \((T_pM)^G\) is a 1-dimensional trivial representation of \( S^1 \), and \( V_m \) is an irreducible representation on which \( S^1 \) acts with kernel of size \( m \).

If \( m \neq n \), then \( V_m \not\cong V_n \). The integrability tensor (Lemma 3.8) is a \( G_p \)-representation homomorphism

\[
\mathbb{R} \oplus V_m \cong \Lambda^2((T_pM)^G \oplus V_m) \to T_pM/((T_pM)^G \oplus V_m) \cong V_n
\]

that agrees with Lie brackets of vector fields. This is zero since \( \mathbb{R} \), \( V_m \), and \( V_n \) are non-isomorphic irreducible representations; so Lie brackets preserve tangency to the 3-plane \((T_pM)^G \oplus V_m\). Hence the corresponding \( G \)-invariant distribution is integrable. The resulting foliation has closed leaves since its tangent distribution contains \( TM^G \), ([Fil83, Prop. 2.1.2]; also used earlier in Prop. 3.7).

If \( m = n \) then \( V_m \) and \( V_n \) are both isomorphic to the standard representation \( V_1 \) of \( SO(2) \). This \( SO(2) \) action induces the structure of a complex vector space on \( V_m \oplus V_n = ((T_pM)^G)^\perp \).

Since \( TM^G \) is 1-dimensional, there is a canonical (up to rescaling) \( G \)-invariant vector field \( v \) along \( TM^G \), with zero divergence (Lemma 3.15). Fixing \( p \in M \), choose \( w \in T_pG \to \text{Vect} M \) such that \( w(p) = v(p) \) (possible since \( G \to M \) is a submersion). Since \( G \) acts by isometries, \( w \) satisfies the following.

- \( w \) is also divergence-free.
- \( \exp tw \) preserves \( v \), so \([v, w] = 0\).
- \( \exp tw \) preserves the \((\exp tv)\)-orbit of \( p \), so it preserves \( G_p \). Conjugation induces a map

\[
t \in \mathbb{R} \to \text{Aut} G_p \cong \{\pm 1\}.
\]

Since \( \mathbb{R} \) is connected, this is the trivial homomorphism; so \( \exp tw \) commutes with \( G_p \).

Then \( v - w \) is divergence-free, \( G_p \)-invariant, and zero at \( p \); so \( \exp t(v - w) \) is a \( G_p \)-equivariant operator on \( T_pM \)—i.e. a \( \mathbb{C} \)-linear operator on \((T_pM)^G\). It has an eigenvector, whose span \( V_p \) (over \( \mathbb{C} \)) is \( G_p \)-invariant. Let \( V \) be the \( G \)-invariant distribution whose 2-plane at \( p \) is \( V_p \).

Since \( V_p \) is an eigenspace of \( \exp t(v - w) \) (thus of \( v - w \)) and \( V \) is \( G \)-invariant,

\[
[v, V](p) \subseteq [v - w, V](p) + [w, V](p) \subseteq V_p + V_p = V_p.
\]

The map \( \Lambda^2V \to T_pM/V \) from Lemma 3.8 lands in a trivial representation since \( \Lambda^2V \) is 1-dimensional; so \([V, V]_p \subseteq (T_pM)^G \). Therefore the \( G \)-invariant distribution of 3-planes \( TM^G \oplus V \) is integrable. As above, it contains \( TM^G \) so the resulting foliation has closed leaves.

Case (iii), Step 2: \( M \) fibers conformally over some space. Let \( \mathcal{F} \) be the foliation tangent to the distribution produced in Step 1, and suppose \( M \) does not fiber conformally over \( B = M/\mathcal{F} \)—i.e. there is no metric on \( B \) with respect to which \( G \) acts by conformal automorphisms.

The subgroup \( G_b \subset G \) fixing \( b \in B \) acts on the tangent space \( T_bB \) by some

\[
G_b \to \text{GL}(T_bB)) \cong \text{GL}(2, \mathbb{R}).
\]

Since \( G_p \subset G_b \) and \( G \) preserves no conformal structure on \( B \), there are two non-coincident copies of \( SO(2) \) acting on \( T_bB \). Together they generate all of \( \text{SL}(2, \mathbb{R}) \) (compare this to rotation groups around
two distinct points of $\mathbb{H}^2$ generating all of $\text{Isom}_0(\mathbb{H}^2)$, so the semisimple part of $G$ contains a subgroup covering $\text{PSL}(2, \mathbb{R})$. Since $G$ is 6-dimensional, the classification of simple Lie groups [Hel78, Ch. X, §6 (p. 516)], implies that $G$ either surjects onto $\text{PSL}(2, \mathbb{R})$ or covers $\text{PSL}(2, \mathbb{C})$.

In both $\text{PSL}(2, \mathbb{R})$ and $\text{PSL}(2, \mathbb{C})$, the maximal torus of the maximal compact subgroup is 1-dimensional; so all copies of $S^1$ are conjugate, and $G_p \cong S^1$ lands in some copy. Thus $M$ fibers isometrically over (or is) $\text{PSL}(2, \mathbb{R})/\text{PSO}(2) \cong \mathbb{H}^2$ or $\text{PSL}(2, \mathbb{C})/\text{PSO}(2)$. The latter fibers conformally over $S^2$, as

$$S^2 \cong \text{Conf}^+ \mathbb{E}^2/\text{Conf}^+ \mathbb{E}^2 \cong \text{PSL}(2, \mathbb{C})/\text{Conf}^+ \mathbb{E}^2.$$

Fibers are closed since the maps used are continuous, and $SO(2)$ is in the point stabilizers on the base since $G$ surjects onto $\text{PSL}(2, \mathbb{R})$ or $\text{PSL}(2, \mathbb{C})$.

Example 3.16. The necessity of Step 2 is demonstrated by the fibering

$$\text{SL}(2, \mathbb{R}) \rightarrow \mathbb{R}^2 \times \text{SL}(2, \mathbb{R}) \rightarrow \mathbb{R}^2.$$

Since $\mathbb{R}^2 \times \text{SL}(2, \mathbb{R})$ acts on $\mathbb{R}^2$ with $\text{SL}(2, \mathbb{R})$ point stabilizers, it cannot preserve any conformal structure. If Step 1 had produced this fibering, then Step 2 would find the conformal fibering

$$\mathbb{R}^2 \times SO(2) \rightarrow \mathbb{R}^2 \times \text{SL}(2, \mathbb{R}) \rightarrow \mathbb{H}^2.$$

4 Geometries fibering over 4D isotropy-irreducible geometries

This section proves part (i) of Theorem 1.1—that is, Proposition 4.1 (Thm. 1.1(i)). The 5-dimensional maximal model geometries $M = G/G_p$ for which the isotropy representation $G_p \curvearrowright T_pM$ contains an irreducible 4-dimensional summand are the product geometries

$$S^4 \times \mathbb{E} \quad \mathbb{H}^4 \times \mathbb{E} \quad \mathbb{CP}^2 \times \mathbb{E} \quad \mathbb{CH}^2 \times \mathbb{E}$$

and the homogeneous spaces

$$\text{U}(2,1)/\text{U}(2) \quad \text{Heis}_5 = \text{Heis}_5 \rtimes \text{U}(2)/\text{U}(2).$$

Our approach to proving Prop. 4.1 closely resembles the classification in [Thu97, Thm. 3.8.4(b)] of 3-dimensional geometries with an irreducible 2-dimensional isotropy summand. Recall that $B = M/\mathcal{F}^G$ is a 4-dimensional Riemannian symmetric space over which $M$ fibers isometrically (Prop. 3.3). Curvature determines $M$ once $B$ is known. That is:

Proposition 4.2 (Base and curvature determine the geometry). A 5-dimensional maximal model geometry $M = G/G_p$ whose isotropy representation contains an irreducible 4-dimensional summand is determined by the following two pieces of information:

1. the geometry $B = M/\mathcal{F}^G$; and

2. whether $(TM^G)\perp$, as a connection on the fiber bundle $M \rightarrow B$, has nonzero curvature.

Moreover, if $G_p = SO(4)$, then $(TM^G)\perp$ has zero curvature.

This key fact, proven in subsection 4.1, reduces the classification problem to listing pairs $(B, x)$ and checking whether each arises from a maximal model geometry. Subsection 4.2 carries this out, finding and then verifying the candidates listed in Table 4.3.
Table 4.3: Candidate geometries with irreducible 4-dimensional isotropy summand

| Base   | Flat (product) | Curved          |
|--------|----------------|-----------------|
| $S^4$  | $S^4 \times E$ |                 |
| $E^4$  | non-maximal $E^5$ |                 |
| $\mathbb{H}^4$ | $\mathbb{H}^4 \times E$ |             |
| $\mathbb{C}P^2$ | $\mathbb{C}P^2 \times E$ | non-maximal $S^5$ |
| $\mathbb{C}H^2$ | $\mathbb{C}H^2 \times E$ | $\text{Heis}_5$ |
|        |                 | $\text{U}(2,1)/\text{U}(2)$ |

4.1 Reconstructing geometries from base and curvature information

This subsection proves Prop. 4.2 in two steps: one recovers a connection from its curvature using some general theory, and one normalizes any nonzero curvature to a single value. The latter interprets $\left( TM^G \right)_p^\perp$ as the quaternions $\mathbb{H}$ and of $\text{SU}(2)$ as the unit quaternions in order to write down and work with the isotropy representation, as follows. Since the action $\tilde{\text{SO}}(4) \cong \text{SU}(2) \times \text{SU}(2) \rtimes \mathbb{H}$ descends to the standard representation of $\text{SO}(4)$, all three of the isotropy representations with a 4-dimensional irreducible summand—$\text{SU}(2) \subset \text{U}(2) \subset \text{SO}(4)$ (Fig. 2.4)—can be written using quaternion multiplication and subgroups of $\text{SU}(2) \times \text{SU}(2)$.

**Proof of Prop. 4.2.** Let $M = G/G_p$ be a 5-dimensional maximal model geometry whose isotropy representation contains an irreducible 4-dimensional summand, and let $B = M/F^G$. Then $M \to B$ is a $G$-invariant principal $S^1$- or $\mathbb{R}$-bundle, with vertical distribution $TM^G$ and a natural connection (horizontal distribution) $\left( TM^G \right)_p^\perp$. The curvature is a $G$-invariant 2-form with values in the Lie algebra of $S^1$ or $\mathbb{R}$, i.e. an element of $(\Omega^2 B \otimes \mathbb{R})^G \cong (\Lambda^2 (TM^G)_p^\perp)^{G_p}$.

**Step 1: Nonzero curvature can be normalized to a single value.** By counting weight vectors, $\Lambda^2 (TM^G)_p^\perp \cong_{\text{SU}(2)} 3\mathbb{R} \oplus 3\mathbb{R}$ (i.e. $\text{SU}(2)$ acts trivially on $3\mathbb{R}$ and as $SO(3)$ on $3\mathbb{R}$). One checks by inspection that the $3\mathbb{R}$ is spanned by

$$1 \wedge i - j \wedge k \quad 1 \wedge j - k \wedge i \quad 1 \wedge k - i \wedge j.$$

Under the action of conjugation by unit quaternions, $(TM^G)_p^\perp$ decomposes instead as $3\mathbb{R}$. In fact one of these copies of $3\mathbb{R}$ is the $3\mathbb{R}$ above, since conjugation by $1 + i$ takes $1 \wedge j - k \wedge i$ to $1 \wedge k - i \wedge j$.

Then if $G_p \cong \text{SU}(2)$, applying some inner automorphism of $\text{SU}(2)$ makes the curvature a scalar multiple of $1 \wedge i - j \wedge k$; and the fibers can be rescaled to make the curvature either 0 or $1 \wedge i - j \wedge k$.

If instead $G_p \cong \text{U}(2)$, take the scalar factor to act as multiplication by $e^{i\theta}$ on the right. This acts on the span of $1 \wedge j - k \wedge i$ and $1 \wedge k - i \wedge j$ by rotation, so $1 \wedge i - j \wedge k$ lies in the only invariant direction.

Finally, if $G_p \cong \text{SO}(4)$, then the action of $\text{SU}(2)$ by conjugation of quaternions factors through $G_p$, so $\Lambda^2 \mathbb{R}^4$ contains no invariant directions—so the curvature is 0.
Step 2: Base and curvature determine a geometry. Let \( \pi \) be the projection \( M \to B \). To \( x \) in a neighborhood \( U \) of \( p \), assign the coordinates \((t, \pi(x))\) where the shortest path from \( \pi(x) \) to \( \pi(p) \) lifts to a horizontal path from \( x \) to \( \phi_t(p) \). (For these coordinates to be well-defined, it suffices to have the radius of \( \pi(U) \) at most the injectivity radius of \( B \).

If \( M \) and \( N \) have matching curvature and base, then choose \( p \in M \) and \( q \in N \) lying over the same point \( b_0 \in B \); and define a map \( f \) from \( U \ni p \) to \( V \ni q \) by \((t, b) \mapsto (t, b)\). Since homogeneous spaces are analytic [KN63, Prop. I.4.2], and the isometry type of a complete, connected, simply-connected, analytic Riemannian manifold is determined by its local isometry type [KN63, Cor. VI.6.4], two geometries are isometric if they contain isometric open sets. Since a maximal geometry is determined by any invariant Riemannian metric [Fil83, Prop. 1.1.2], it suffices to check that \( f \) is an isometry.

Since \( f \) descends to the identity on \( B \), and the metric on \( M \) is the direct sum of its restrictions to \( TM^G \) and \((TM^G)\perp \), it suffices to check that \( f \) takes the horizontal distribution \((TM^G)\perp \) on \( M \) to the horizontal distribution on \( N \). (We’ll say “\( f \) is horizontal”.)

Since \( f \) takes horizontal lifts of geodesics through \( p \) to horizontal lifts of geodesics through \( q \), it’s horizontal at \( p \). At points other than \( p \), since \( M \) and \( N \) have matching curvature and base, it suffices to check that there is only one \( 4 \)-plane distribution with the prescribed \( 4 \)-plane at \( p \) and the prescribed curvature.

Let \( \tilde{S} \) be a circular sector in \( T_{b_0}B \) with its vertex at the origin, and let \( S = \exp \tilde{S} \). The displacement along the fiber \( \mathcal{F}_p \) incurred by traveling around a horizontal lift of \( \partial S \) is the integral of the curvature over \( S \). So if \( s \) denotes the distance along the circular arc in \( \partial S \), then computing \( \frac{dt}{ds} \) in enough directions recovers the slope of the horizontal distribution relative to the coordinates \((t, b)\).

4.2 Classifying and verifying geometries

Having established that the base and curvature determine a 5-dimensional maximal model geometry with irreducible 4-dimensional isotropy (Prop. 4.2), this subsection carries out the classification (i.e. the proof of Prop. 4.1) according to the plan outlined at the start of the section.

Proof of Thm. 1.1(i)/Prop. 4.1. Let \( M = G/G_p \) be a 5-dimensional maximal model geometry whose isotropy representation contains an irreducible 4-dimensional summand. The base \( M/\mathcal{F}^G \) and the curvature of \((TM^G)\perp \) determine \( M \) (Prop. 4.2); so \( M \) occurs in Table 4.3, provided that the list of base spaces is exhaustive (Step 1) and that every entry has the claimed base and curvature (Step 2). Determining which candidates are maximal and model (Step 3) finishes the proof.

Step 1: Classify base spaces for Table 4.3. The base \( M/\mathcal{F}^G \) is a Riemannian symmetric space (Prop. 3.3). The isotropy in \( M \) descends to irreducible isotropy in \( M/\mathcal{F}^G \), so \( M/\mathcal{F}^G \) is either irreducible or Euclidean. Consulting the classification of irreducible Riemannian symmetric spaces in [Hel78, X.6, p.515–518] yields the following non-Euclidean base spaces.

\[
\begin{align*}
\mathbb{H}^4 & \quad S^4 & \quad \mathbb{C}P^2 & \cong SU(3)/S(U(2) \times U(1))
\end{align*}
\]

If \( M/\mathcal{F}^G \) is Euclidean, it may not be a maximal geometry—its isotropy may be some proper but irreducibly-acting subgroup of \( SO(4) \). Hence Table 4.3 also lists \( \mathbb{C}^2 \) as a base, to stand for Euclidean space with \( SU(2) \) or \( U(2) \) isotropy.
Step 2: Verify the candidates in Table 4.3. This step proves that every spot in Table 4.3 is occupied by a space with the correct base and curvature (or empty if no such space exists).

For the product spaces, there is nothing to prove. For the bases with SO(4) isotropy, no non-product geometries occur (Prop. 4.2). For the remaining bases, geometries with nonzero curvature of \((TM^G)\perp\) are realized by the following.

- Over \(\mathbb{CP}^2\) is its tautological circle bundle \(S^5\), since \(\mathbb{CP}^2\) is the quotient of \(S^5 \subset \mathbb{C}^3\) by the action of norm-1 scalars. Its curvature is nonzero since a flat \(S^1\)-bundle over the simply-connected \(\mathbb{CP}^2\) is the product bundle—which, unlike \(S^5\), has nontrivial \(\pi_1\).

- Over \(\mathbb{C}^2\) is \(\text{Heis}_5\), the 5-dimensional Heisenberg group, which can be written as the set \(\mathbb{C}^2 \times \mathbb{R}\) with the product

\[
(v_1, t_1)(v_2, t_2) = (v_1 + v_2, t_1 + t_2 + \Im \langle v_1, v_2 \rangle),
\]

where \(\langle \cdot, \cdot \rangle\) is the standard Hermitian product. Dropping the \(\mathbb{R}\) coordinate is a fibering over \(\mathbb{C}^2\), for which an invariant connection with nonzero curvature is given by the kernel of the invariant contact form

\[
\alpha_{(v, t)} = dt - \Im \langle v, dv \rangle.
\]

- Over \(\mathbb{CH}^2 \cong U(2, 1)/(U(2) \times U(1))\) is the line bundle \(U(2, 1)/U(2)\). That its curvature is nonzero is Prop. 4.4 below.

Step 3: Determine maximal model geometries. Except when multiple Euclidean factors are involved, products of maximal model geometries are maximal model geometries (Prop. 3.12 and surrounding discussion); so it suffices to consider the three non-product geometries.

- \(S^5\) as a circle bundle over \(\mathbb{CP}^2\) is not maximal—it is a homogeneous space \(SU(3)/SU(2)\) or \(U(3)/U(2)\), both of which are subsumed by the geometry \(SO(6)/SO(5)\).

- \(\text{Heis}_5\) is a nilpotent Lie group. Its maximal realization is \(\text{Heis}_5 \times K/K\) where \(K \subset \text{Aut Heis}_5\) is maximal compact (Lemma 2.3). Since \(\text{Aut Heis}_5\) must preserve \(Z(\text{Heis}_5)\), it is block triangular in some basis with blocks of size 4 and 1; so its maximal compact subgroup is conjugate to a subgroup of \(SO(4) \times SO(1)\) [Gen16b, Lemma 5.29]. Moreover, \(\text{Aut Heis}_5\) has to preserve the antisymmetric pairing on \(\text{Heis}_5/Z(\text{Heis}_5)\) used to define \(\text{Heis}_5\) in Step 2. Then

\[
K \subseteq SO(4) \cap \text{Sp}(4, \mathbb{R}) = U(2)
\]

since \(SO(4)\) preserves the real part of a Hermitian form and \(\text{Sp}(4, \mathbb{R})\) preserves the imaginary part. Conversely, \(K \supseteq U(2)\) since \(U(2)\) preserves the Hermitian product used to define \(\text{Heis}_5\). So \(\text{Heis}_5 \times U(2)/U(2)\) is maximal.

- \(\widetilde{U(2, 1)/U(2)}\) is a model geometry since \(\mathbb{CH}^2\) is—the deck group \(\Gamma\) of any compact \(\Gamma \backslash \mathbb{CH}^2\) acts by elements of \(SU(2, 1) \subset U(2, 1)\), so the circle bundle \(U(2, 1)/U(2)\) over \(\mathbb{CH}^2\) descends to a circle bundle \(N\) over \(\Gamma \backslash \mathbb{CH}^2\). To prove maximality it suffices to distinguish \(\widetilde{U(2, 1)/U(2)}\) from geometries with larger isotropy group; consulting Figure 2.4, these are just \(SO(5)\) and \(SO(4)\).

Geometries \(M = G/G_p\) with irreducible 4-dimensional isotropy are distinguished from each other by \(M/F^G\) and the curvature of \((TM^G)^\perp\) (Prop. 4.2), so only the constant-curvature spaces need to be checked.

- \(M = \widetilde{U(2, 1)/U(2)}\) is not \(S^5\) since \(M\) is contractible, being a line bundle over the contractible space \(\mathbb{CH}^2\).
– $M$ is not $E^5$ since $SU(2,1)$ is semisimple and noncompact, whereas every semisimple subgroup of $\text{Isom}_0 E^5$ is compact as its semisimple part is $SO(5)$.

– $M$ is not $H^5$: a circle bundle $N$ over a compact $\mathbb{C}H^2$ manifold has quotients of arbitrarily small volume (by the scalar action of $e^{2\pi i/m}$ for large $m$); whereas for fixed $n \geq 4$, there is a minimum volume for hyperbolic $n$-manifolds by a theorem of Wang [BP92, Thm. E.3.2].

Hence the maximal model geometries in Table 4.3 are the products other than $E^5$ and the two non-product geometries $\text{Heis}_5$ and $U(2,1)/U(2)$.

**Proposition 4.4.** For the fibering of $M = U(1,n)/U(n)$ over $\mathbb{C}H^n$, the curvature of the connection $(TM^{U(n)})^\perp$ is nonzero.

**Proof.** Let $\{e_0, \ldots, e_n\}$ be the standard basis of $\mathbb{C}^{n+1}$. Embed $M$ in $\mathbb{C}^{n+1}$ as the preimage of 1 under the $(1,n)$ Hermitian form

$$\sum_{i=0}^n z_i e_i \mapsto |z_0|^2 - \sum_{i=1}^n |z_i|^2,$$

so that $M$ fibers as a circle bundle over $\mathbb{C}H^2$ where the fibers are the intersections of $M$ with complex lines.

Embed $U(n)$ in $U(1,n)$ as the copy of $U(n)$ fixing $e_0 \in \mathbb{C}^{n+1}$, and let $V$ be the span of $\{e_1, \ldots, e_n\}$. Then $(TM^G)^\perp_{e_0} = V$; and we can find $(TM^G)^\perp$ elsewhere by translating by elements of $U(1,n)$—explicitly, for $p \in M \subset \mathbb{C}^{n+1}$,

$$(TM^G)^\perp_p = p^\perp.$$

If $v \in V$ has unit length, then there is a 1-parameter subgroup of $U(1,n)$ defined by

$$e_0 \mapsto (\cosh t)e_0 + (\sinh t)v$$

$$v \mapsto (\sinh t)e_0 + (\cosh t)v$$

$$w \mapsto w \text{ if } w \in V \text{ and } w \perp v.$$  

Fix a small $t > 0$ and define a path $\gamma(\theta)$ in $M$ by

$$\gamma(\theta) = (\cosh t)e^{i\theta (\tanh t)^2}e_0 + (\sinh t)e^{i\theta}e_1.$$

Then

$$\gamma'(\theta) = i(\tanh t) \left( (\sinh t)e^{i\theta (\tanh t)^2}e_0 + (\cosh t)e^{i\theta}e_1 \right)$$

so $\gamma$ is horizontal. At $\theta = \frac{\pi}{1-(\tanh t)^2}$, both $\gamma(\theta)$ and $\gamma(0)$ are in the same fiber, but $\gamma(0) \neq \gamma(\theta)$. So horizontal lifts of closed paths in $\mathbb{C}H^m$ are not necessarily closed. \qed

### 5 Geometries fibering over 3D isotropy-irreducible geometries

This section proves part (ii) of Theorem 1.1—that is, the 5-dimensional maximal model geometries $M = G/G_p$ for which the isotropy representation $G_p \curvearrowright T_p M$ contains an irreducible 3-dimensional summand are products. Major ingredients of the proof include some reasoning about fiber bundles (set up in Section 3), Lie group extension problems, and some facts about transformation groups of spaces of constant curvature (postponed to Section 5.3).
While the classification seeks only model geometries, it does encounter one geometry that satisfies the weaker condition of having a unimodular isometry group. This geometry is the homogeneous space
\[ \mathbb{R} \rtimes \text{Conf}^+ \mathbb{E}^3 / \text{SO}(3), \]
where the action on \( \mathbb{R} \) is chosen to make \( \mathbb{R} \rtimes \text{Conf}^+ \mathbb{E}^3 \) unimodular. Its failure to be a model geometry is proven using Galois theory; see Section 5.2 for details.

The proof splits into four parts with the following preparation. Since \( M \) is a model geometry, \( G \) must admit a lattice (Defn. 2.1), which requires \( G \) to be unimodular [Fil83, Prop. 1.1.3]. Under the assumption that \( G_p \curvearrowright T_pM \) has an irreducible 3-dimensional summand, \( G_p \) contains a characteristic copy of \( \text{SO}(3) \) (Fig. 2.4). So the fibering \( M \to B = M/\mathcal{F}^{\text{SO}(3)} \) is conformal, with \( B = S^3, \mathbb{E}^3, \) or \( \mathbb{H}^3 \) (Prop. 3.3(ii)). Hence to prove Theorem 1.1(ii) it will suffice to show the following.

**Proposition 5.1.** Let \( M = G/G_p \) be a 5-dimensional maximal geometry where \( G \) is unimodular and \( G_p = \text{SO}(3) \) or \( \text{SO}(3) \times \text{SO}(2) \); and let \( B = M/\mathcal{F}^{\text{SO}(3)} \).

(i) If \( \pi_B : M \to B \) is an isometric fibering, then \( M \) is a product of 2-dimensional geometries and 3-dimensional constant-curvature geometries.

(ii) The products in (i) that are maximal model geometries are those for which both factors have constant curvature and at least one factor is not Euclidean.

(iii) If \( \pi_B \) is essential, then \( M = \mathbb{R} \rtimes \text{Conf}^+ \mathbb{E}^3 / \text{SO}(3) \), where \( A \in \text{Conf}^+ \mathbb{E}^3 \) acts on \( \mathbb{R} \) as dilation by \((\det A)^{-1}\).

(iv) \( \mathbb{R} \rtimes \text{Conf}^+ \mathbb{E}^3 / \text{SO}(3) \) is not a model geometry.

The rest of this section is devoted to proving Prop. 5.1.

### 5.1 Geometries fibering isometrically are products

This section classifies the isometrically fibering geometries, in two steps as delineated by the first two parts of Prop. 5.1. The first is to show that all isometrically fibering geometries are products, and the second is to determine which products are maximal model geometries. Standard facts about groups acting on constant-curvature spaces will be stated where used, with references either to the literature or to proofs deferred to Section 5.3.

**Proof of Prop. 5.1(i).** Let \( M = G/(\text{SO}(3) \times \text{SO}(2)) \) or \( M = G/\text{SO}(3) \) be a 5-dimensional maximal geometry with \( G \) unimodular, and suppose \( \pi_B : M \to B = M/\mathcal{F}^{\text{SO}(3)} \) is isometric.

**Step 1:** \( M \to B \) is a product bundle \( F \times B \). The \( G_p \)-invariant integrability tensor (Lemma 3.8)
\[ \Lambda^2 \left( (TM^{\text{SO}(3)})_p^\perp \right) \to T_pM^{\text{SO}(3)} \]
is zero since the left side is the standard representation of \( \text{SO}(3) \) while the right side is a trivial representation; so \( (TM^{\text{SO}(3)})_p^\perp \) is a flat connection on \( M \to B \). Since \( B \) is simply-connected and flat bundles are classified by the monodromy representation \( \pi_1(B) \to \text{Diff} F \), the fiber bundle \( M \to B \) admits an isomorphism to a product bundle \( F \times B \) taking \( (TM^{\text{SO}(3)})_p^\perp \) to the 3-planes tangent to copies of \( B \). This produces a second projection \( \pi_F : M \to F \).\(^3\)

\(^3\) This alone does not make \( M \) a product geometry, as demonstrated in dimension 3 by the (non-model) geometry \( \text{Conf}^+ \mathbb{E}^2 / \text{SO}(2) \); so there still remains something nontrivial to prove.
Step 2 (some general theory): It suffices to show that \( F \) has a \( G \)-invariant metric. If \( G \)-invariant metrics on \( F \) exist, they correspond one-to-one with invariant inner products on \( TM^\text{SO}(3) \), since both are determined by an inner product on a single 2-plane. The same holds for \( (TM^\text{SO}(3))^\perp \) and \( B \). Then since \( TM^\text{SO}(3) \) and \( (TM^\text{SO}(3))^\perp \) are orthogonal in any invariant metric (Lemma 3.10), every invariant metric on \( M \) is a direct sum of invariant inner products on \( TM^\text{SO}(3) \) and \( (TM^\text{SO}(3))^\perp \). So if both \( \pi_B \) and \( \pi_F \) are both isometric fiberings, then any invariant metric on \( M \) is isometric to some invariant metric on \( F \times B \).

Step 3 (an extension problem): Describe point stabilizers \( G_f \) of \( G \curvearrowright F \). Let \( G_f \subseteq G \) be the subgroup fixing a point \( f \in F \). Identifying \( B \) with \( \pi_F^{-1}(f) \), the image of \( G_f \) in \( \text{Diff} B \) is all of \( \text{Isom}_B \), since it contains \( \text{SO}(3) \) in the point stabilizers and \( G \) acts transitively. If \( G_p \cong \text{SO}(3) \), then counting dimensions shows that \( G_f \cong \text{Isom}_B \).

Otherwise, the kernel of \( G_f \rightarrow \text{Isom}_B \) is the \( \text{SO}(2) \) factor in \( G_p \); so passing to Lie algebras, \( T_1 G_f \) is an extension
\[
0 \rightarrow \mathfrak{so}_2 \mathbb{R} \rightarrow T_1 G_f \rightarrow \text{isom} B \rightarrow 0.
\]
A 1-dimensional representation of \( \text{isom} \ B \) must be trivial for \( B = \mathbb{E}^3 \), \( S^3 \), or \( \mathbb{H}^3 \) (Corollary 5.3(iii)), so \( \text{isom} B \) acts trivially on \( \mathfrak{so}_2 \mathbb{R} \cong \mathbb{R} \). Then since \( H^2(T_1 \text{Isom}_B; \mathbb{R}) = 0 \) (Lemma 5.5), the classification of Lie algebra extensions by second cohomology (Thm. 2.6) implies \( G_f \) is covered by the product \( \mathbb{R} \times \text{Isom}_B \).

Step 4 (a study of transformation groups): \( G_f \) preserves a metric on \( F \). The action of \( G_f \subseteq G \) on \( F \) defines a homomorphism
\[
\phi: \tilde{G}_f \rightarrow \text{Aut} T_f F \cong \text{GL}(2, \mathbb{R}).
\]
Since \( \text{Isom}_B \) has no quotients of dimension 1 or 2 (Corollary 5.3(i)), it acts with determinant 1 on \( \mathbb{R}^2 \)—and thus as a subgroup of \( \text{SL}(2, \mathbb{R}) \) of dimension 0 or 3. In fact \( \text{Isom}_B \) does not admit \( \text{SL}(2, \mathbb{R}) \) as a quotient, since:

1. proper quotients of \( \text{Isom}_B \mathbb{E}^3 \) factor through \( \text{SU}(2) \) (Lemma 5.2);
2. \( \text{Isom}_B S^3 \cong \text{SU}(2) \times \text{SU}(2) \); and
3. \( \text{Isom}_B \mathbb{H}^3 = \text{SO}(3,1) \) is simple.

Therefore \( \phi \left( \text{Isom}_B \right) = \{ 1 \} \). If \( G_p \cong \text{SO}(3) \), then this means \( \phi \) has trivial image. If \( G_p \cong \text{SO}(3) \times \text{SO}(2) \), then this implies \( \phi \) factors through the \( \text{SO}(2) \subseteq G_f \) covered by the \( \mathbb{R} \) factor in \( \tilde{G}_f \cong \mathbb{R} \times \text{Isom}_B \). Either way, the action of \( G_f \) on \( T_f F \) factors through a compact group, so it preserves an inner product on \( T_f F \). Then since \( G \) acts transitively, it preserves a metric on \( F \), which finishes the proof by Step 2.

Completing the classification of geometries fibering isometrically over 3-dimensional constant curvature geometries requires determining which of the products \( F \times B \) from (i) are actually maximal model geometries. So far, \( B \) is known to be \( \mathbb{E}^3 \), \( S^3 \), or \( \mathbb{H}^3 \), while \( F \) is just a geometry—a manifold with a smooth transitive group action with compact point stabilizers. The first step will be to determine all possibilities for \( F \); then general statements such as the maximality of most products (Prop. 3.12) will finish the proof.
Proof of Prop. 5.1(ii) (determination of maximal model geometries). The 2-dimensional geometries with SO(2) isotropy have constant curvature; and those with trivial isotropy are the two simply-connected real Lie groups in dimension 2, namely \( \mathbb{R}^2 \) and Aff\(^+\)\( \mathbb{R} \). Since \( \mathbb{R}^2 \) is a non-maximal \( \mathbb{E}^2 \) and Aff\(^+\)\( \mathbb{R} \) is a non-maximal \( \mathbb{H}^2 \),\(^4\) the only maximal \( F \times B \) from part (i) are those where both factors are maximal model constant-curvature geometries. Since \( \mathbb{E}^2 \times \mathbb{E}^3 \) is a non-maximal \( \mathbb{E}^5 \), only those products with at most one Euclidean factor remain.

Conversely, suppose \( F \times B \) is a product of maximal model constant-curvature geometries of dimensions 2 and 3, at most one of which is Euclidean. Then \( F \times B \) is a model geometry since products are models (Prop. 2.2); and \( F \), having SO(2) isotropy, has no invariant vector fields, so \( F \times B \) is maximal (Prop. 3.12). \( \square \)

5.2 Case study: The geometry fibering essentially conformally

Let \( M = G/G_p \) be a 5-dimensional maximal geometry with \( G \) unimodular, and suppose the fibering \( \pi_B : M \to B = M/\mathcal{F}_{\text{SO}(3)} \) is essentially conformal. This section proves Prop. 5.1(iii)–(iv): that \( M = \mathbb{R} \times \text{Conf}^+ \mathbb{E}^3 / \text{SO}(3) \) and that \( M \) is not a model geometry.

Proof of Prop. 5.1(iii). We will determine \( M \) by solving the following extension problem to find \( G \).

\[
0 \to \mathbb{R} \to T_1 G \to T_1 \text{Conf}^+ \mathbb{E}^3 \to 0
\]

Such an extension is determined by a homomorphism \( \phi : T_1 \text{Conf}^+ \mathbb{E}^3 \to \text{Der} \mathbb{R} \) (realized by lifting to \( g' \) and taking brackets, where \( \text{Der} \mathbb{R} \) denotes the algebra of derivations of \( \mathbb{R} \)) and a class in \( H^2(T_1 \text{Conf}^+ \mathbb{E}^3 ; \mathbb{R}) \).

Step 1: \( G \) is an abelian extension of \( \text{Conf}^+ \mathbb{E}^3 \). In an essential fibering, \( B \) is \( \mathbb{E}^3 \) or \( S^3 \) (since \( \text{Conf}^+ \mathbb{E}^3 = \text{Isom}_0 \mathbb{H}^3 \) [BP92, Thm. A.4.1]). Since the only connected, transitive, essential subgroup of \( \text{Conf}^+ B \) containing \( \text{SO}(3) \) in the point stabilizers is the entire group (Lemma 5.4), the image of \( G \) in \( \text{Diff} B \) is either \( \text{Conf}^+ \mathbb{E}^3 \cong \mathbb{R}^3 \times (\text{SO}(3) \times \mathbb{R}) \) (7-dimensional) or \( \text{Conf}^+ S^3 \cong \text{Isom}_0 \mathbb{H}^4 \) (10-dimensional). Since \( G_p \) is \( \text{SO}(3) \) or \( \text{SO}(3) \times \text{SO}(2) \), the dimension of \( G \) is 8 or 9; so \( B = \mathbb{E}^3 \), and \( G \) is an extension of \( \text{Conf}^+ \mathbb{E}^3 \) by an unimodular group \( H \) of dimension 1 or 2.

Step 2: \( G_p \) is \( \text{SO}(3) \). As a \( G_p \)-representation, \( T_1 G \cong T_1 G_p \oplus T_p M \). Since \( T_1 H \subset T_1 G \) is an ideal, it is also a subrepresentation. Then letting \( V \) denote the standard representation of \( \text{SO}(3) \) and \( \mathbb{R} \) the trivial representation,

\[
T_1 G \cong_{\text{SO}(3)} T_1 \text{Conf}^+ \mathbb{E}^3 \oplus T_1 H \cong_{\text{SO}(3)} 2V \oplus \mathbb{R} \oplus T_1 H.
\]

The \( \mathbb{R} \) is tangent to a group of dilations in \( \text{Conf}^+ \mathbb{E}^3 \); and the two copies of \( V \) are \( \mathfrak{so}_3 \mathbb{R} \subset T_1 G_p \) and the 3-dimensional subspace of \( T_p M \) on which it acts.

If \( G_p = \text{SO}(3) \times \text{SO}(2) \), then \( \mathbb{R} \times T_1 H \) must consist of a 2-dimensional subspace of \( T_p M \) and the \( \mathfrak{so}_2 \mathbb{R} \) acting on it. Then since \( T_1 H \) is an abelian ideal of \( T_1 G \), \( \mathbb{R} \times T_1 H \) is a subalgebra of \( T_1 G \) isomorphic to \( T_1 \text{Isom}_0 \mathbb{E}^2 \) with the \( \mathbb{R} \) corresponding to \( \mathfrak{so}_2 \mathbb{R} \). This cannot occur since this \( \mathbb{R} \) is the tangent algebra to a group of dilations in \( \text{Conf}^+ \mathbb{E}^3 \), while \( \mathfrak{so}_2 \mathbb{R} \) is the tangent algebra to the compact group \( \text{SO}(2) \subset G_p \). Hence \( G_p = \text{SO}(3) \), and \( \dim H = 1 \); so \( T_1 G \) is an extension

\[
0 \to \mathbb{R} \to T_1 G \to T_1 \text{Conf}^+ \mathbb{E}^3 \to 0.
\]

\(^4\) Take \( (x, y) \in \mathbb{R} \times \mathbb{R} \) to the upper-half plane by \( (x, y) \mapsto (x, e^y) \).
Step 3: Unimodularity determines the action of \( \text{Conf}^+ \mathbb{E}^3 \) on \( H \). The above extension problem induces, via Lie brackets, an action \( \phi : T_1 \text{Conf}^+ \mathbb{E}^3 \to \text{Der} \mathbb{R} \). Every ideal in \( T_1 \text{Conf}^+ \mathbb{E}^3 \cong \mathbb{R}^3 \bowtie (\mathfrak{so}_3 \oplus \mathbb{R}) \) either:

- is contained in \( \mathbb{R}^3 \), in which case it’s 0 or all of \( \mathbb{R}^3 \) since \( \mathfrak{so}_3 \) acts irreducibly; or
- contains some \( v \) projecting nontrivially to \( \mathfrak{so}_3 \oplus \mathbb{R} \), in which case it still contains \( \mathbb{R}^3 \) since \( [v, \mathbb{R}^3] \) is a nonzero subspace of \( \mathbb{R}^3 \).

Since \( \mathfrak{so}_3 \) is simple, the ideals of \( T_1 \text{Conf}^+ \mathbb{E}^3 \) are 0, \( \mathbb{R}^3 \), \( \mathbb{R}^3 \bowtie \mathfrak{so}_3 \), \( \mathbb{R}^3 \bowtie \mathbb{R} \), and \( T_1 \text{Conf}^+ \mathbb{E}^3 \), corresponding to the quotients \( T_1 \text{Conf}^+ \mathbb{E}^3 / \mathfrak{so}_3 \oplus \mathbb{R} \), \( \mathbb{R} / \mathfrak{so}_3 \), and 0. Of these, only \( \mathbb{R} \) and 0 can occur as images in \( \text{Der} \mathbb{R} \cong \mathbb{R} \); so \( \phi \) is either zero or of the form

\[
\mathbb{R}^3 \bowtie (\mathfrak{so}_3 \mathbb{R} \oplus \mathbb{R}) \ni (v, r, s) \mapsto ks
\]

for some \( k \in \mathbb{R} \). Since \( \text{ad}(0,0,1) \) is diagonal with three 1s and four 0s, unimodularity of \( G \) requires \( k = -3 \). Then \( G \) is an extension

\[
1 \to \mathbb{R} \to G \to \text{Conf}^+ \mathbb{E}^3 \to 1
\]

where \( A \in \text{Conf}^+ \mathbb{E}^3 \) acts on \( \mathbb{R} \) as dilation by \((\det A)^{-1}\). \( H \cong \mathbb{R} \) since \( S^1 \) admits no dilations.

Step 4: The extension splits—i.e. \( G \) is the semidirect product. A spanning set for \( T_1 \text{Conf}^+ \mathbb{E}^3 \) is given by translations \( t_i \in \mathbb{R}^3 \) \((1 \leq i \leq 3)\), rotations \( r_{ij} \in \mathfrak{so}_3 \) \( [r_{ij}, t_i] = t_j - [-r_{ji}, t_i] \), and a scaling \( s \in \mathbb{R} \). Of these, only \( s \) acts nontrivially; so given a 2-cocycle \( c \), we can subtract a coboundary to make its restriction to \( \mathbb{R}^3 \bowtie \mathfrak{so}_3 \) zero using Lemma 5.5.

Then when we apply the cocycle condition to \( c(t_i, s) = c(r_{ij}, t_j), s \) and \( c(r_{ij}, s) = c([r_{ik}, r_{kj}], s) \), we obtain a sum of terms of the following forms where the blanks are in \( \mathbb{R}^3 \bowtie \mathfrak{so}_3 \):

- \( sc(*,*) \), which is zero since \( c|_{\mathbb{R}^3 \bowtie \mathfrak{so}_3} = 0 \);
- \(*c(*,s)\), which is zero since \( \mathbb{R}^3 \bowtie \mathfrak{so}_3 \) acts trivially; and
- \( c(*,[*,s]) \), which is zero since \( [s, T_1 \text{Conf}^+ \mathbb{E}^3] = \mathbb{R}^3 \).

Thus \( c = 0 \), so \( H^2 = 0 \).

By the classification of abelian extensions by second cohomology (Thm. 2.6), the Lie algebra extension splits. Then \( G \) is covered by \( \mathbb{R} \times \text{Conf}^+ \mathbb{E}^3 \), whose center (consisting of the elements lying over the identity in \( \text{SO}(3) \subset \mathbb{R} \times \text{Conf}^+ \mathbb{E}^3 \)) has order 2. So \( G \) is either \( \mathbb{R} \times \text{Conf}^+ \mathbb{E}^3 \) or \( \mathbb{R} \times \text{Conf}^+ \mathbb{E}^3 \); these deformation retract to their maximal compact subgroups \( \text{SU}(2) \) and \( \text{SO}(3) \), respectively. Since \( G \) must contain \( G_p \cong \text{SO}(3) \) and all maximal compact subgroups are conjugate, the geometry \( M \) is \( \mathbb{R} \times \text{Conf}^+ \mathbb{E}^3 / \text{SO}(3) \).

To show that \( \mathbb{R} \times \text{Conf}^+ \mathbb{E}^3 / \text{SO}(3) \) is not a model geometry, it suffices to show that \( \mathbb{R} \times \text{Conf}^+ \mathbb{E}^3 \) admits no lattice.

**Proof of Prop. 5.1(iv) (\( \mathbb{R} \times \text{Conf}^+ \mathbb{E}^3 / \text{SO}(3) \) is not a model geometry).** Suppose there were a lattice

\[
\Gamma \subset G = \mathbb{R} \times \text{Conf}^+ \mathbb{E}^3 \cong (\mathbb{R} \times \mathbb{R}^3) \rtimes (\text{SO}(3) \times \mathbb{R}).
\]
Step 1: By general theory, produce an integer matrix $A$. Recall the following facts.

1. For a connected Lie group in which every compact semisimple subgroup acts nontrivially on the solvable radical, the intersection of the nilradical with a lattice is a lattice in the nilradical [Mos71, Lemma 3.9].

2. For a closed normal subgroup $H$ of $G$, the group $\Gamma/(\Gamma \cap H)$ is a lattice in $G/H$ if and only if $\Gamma \cap H$ is a lattice in $H$ [OV00, Thm. I.1.4.7].

The nilradical $N$ of $G$ is $\mathbb{R} \times \mathbb{R}^3$; so $\Gamma \cap N \cong \mathbb{Z}^4$, and $\Gamma/(\Gamma \cap N)$ is a lattice in $G/N \cong \text{SO}(3) \times \mathbb{R}$. Then some $g \in \Gamma$ projects nontrivially to the $\mathbb{R}$ factor (the dilation part) in $\text{SO}(3) \times \mathbb{R}$. This $g$ acts by conjugation on $\Gamma \cap N$ as an integer matrix $A$ acting on $\mathbb{Z}^4$.

Step 2: $A$ has a real eigenvalue $\lambda \neq \pm 1$. Since $G$ is unimodular, $\det A = 1$. Then the action of $\text{SO}(3) \times \mathbb{R}$ on $\mathbb{R} \times \mathbb{R}^3$ requires the eigenvalues to be of the form $\lambda$, $\lambda e^{i\theta}$, $\lambda e^{-i\theta}$, and $\lambda^{-3}$ where $\lambda \in \mathbb{R}$. Since Step 1 selected $A$ to act with nontrivial dilation, $\lambda$ is not 1 or $-1$.

Step 3: Using Galois theory, conclude $\lambda = \pm 1$. Since $\lambda$ is real and not $\pm 1$ (and not 0 as $A$ is invertible), the magnitudes of $\lambda$, $\lambda^{-3}$, and $\lambda^9$ are all distinct; so $A$ and $A^{-3}$ share only the eigenvalue $\lambda^{-3}$. Since each eigenvalue of an integer matrix must occur with all of its Galois conjugates, $\lambda^{-3} \in \mathbb{Q}$. Applying the rational root theorem to the characteristic polynomial of $A$ implies $\lambda^{-3} = \pm 1$. Then $\lambda = \pm 1$, which contradicts the conclusion of Step 2. \qed

### 5.3 Groups acting on constant-curvature spaces

This subsection collects some facts used above, concerning groups acting on spaces of constant curvature. First, the classification required some results about low-dimensional representations of $\text{Isom}_0 \mathbb{E}^n$. Their proof begins with the following observation, which will be reused later to classify some extensions of $\text{isom} \mathbb{E}^2$.

**Lemma 5.2.** Every nonzero ideal of $\text{isom} \mathbb{E}^n \cong \mathbb{R}^n \supsetneq \mathfrak{so}_n$ contains the translation subalgebra $\mathbb{R}^n$.

*Proof.* An ideal $\mathfrak{n}$ of $\text{isom} \mathbb{E}^n$ containing no nonzero elements of $\mathbb{R}^n$ acts trivially on $\mathbb{R}^n$—since $\mathbb{R}^n$ is also an ideal,

$$[\mathfrak{n}, \mathbb{R}^n] \subseteq \mathfrak{n} \cap \mathbb{R}^n = 0.$$ 

Since $\mathbb{R}^n$ is a faithful representation of $\mathfrak{so}_n$, any ideal containing no nonzero elements of $\mathbb{R}^n$ is zero. Since $\mathbb{R}^n$ is an irreducible representation of $\mathfrak{so}_n$, any ideal that does contain some nonzero $v \in \mathbb{R}^n$ also contains $[\mathfrak{so}_n, v] \cong \mathbb{R}^n$. \qed

Since $\text{SO}(k)$ ($k \neq 4$) is simple, Lemma 5.2 above implies that for $n \neq 4$, the connected normal subgroups of $\text{Isom}_0 \mathbb{E}^n$ are $\{1\}$, $\mathbb{R}^n$, and $\text{Isom}_0 \mathbb{E}^n$. Since both $\text{Isom}_0 \mathbb{S}^n \cong \text{SO}(n+1)$ and $\text{Isom}_0 \mathbb{H}^n \cong \text{SO}(n,1)$ are simple, statements about quotients and representations of the isometry groups can be made for all of $\mathbb{E}^n$, $\mathbb{S}^n$, and $\mathbb{H}^n$ at once, as follows.

**Corollary 5.3.** Let $G = \text{Isom}_0 \tilde{M}$ where $M$ is $\mathbb{E}^n$, $\mathbb{S}^n$, or $\mathbb{H}^n$.

(i) If $n \geq 3$, then $G$ has no quotient groups of dimension 1 or 2.

(ii) If $n > 4$, then $G$ has no nontrivial quotient groups of dimension less than $\binom{n}{2}$.

(iii) All 1-dimensional real representations of $T_1 G$ are trivial.
The remainder of this section’s facts are only used in the above classification of geometries fibering over $\mathbb{E}^3$, $S^3$, and $\mathbb{H}^3$—starting with the following classification of sufficiently large groups acting by conformal automorphisms on $S^k$ and $\mathbb{E}^k$.

**Lemma 5.4.** Let $B = \mathbb{E}^k$ or $S^k$ for some $k \geq 2$. The only connected, transitive, essential subgroup $H$ of $\text{Conf}^+ B$ with a copy of $\text{SO}(k)$ in its point stabilizer $H_p$ is $\text{Conf}^+ B$.

**Proof.** The two cases are handled separately but with broadly the same strategy: the goal is to show that $H_p$ is the entirety of a point stabilizer of $\text{Conf}^+ B$.

**Case 1:** $B = \mathbb{E}^k$. A point stabilizer of $\text{Conf}^+ \mathbb{E}^k \cong \mathbb{R}^k \rtimes (\text{SO}(k) \times \mathbb{R})$ [BP92, Thm. A.3.7] is $\text{SO}(k) \times \mathbb{R}$. Since $H$ acts essentially, $H_p \subset \text{SO}(k) \times \mathbb{R}$ projects nontrivially to the $\mathbb{R}$ factor. The homotopy exact sequence for $H_p \to H \to B$, along with the assumption that $B$ is simply-connected and $H$ is connected, implies $H_p$ is connected; so $H_p = \text{SO}(k) \times \mathbb{R}$. Then $H$ has the same point stabilizers as $\text{Conf}^+ \mathbb{E}^k$, so $H = \text{Conf}^+ \mathbb{E}^k$.

**Case 2, preparatory claim:** No transitive subgroup $H \subset \text{Conf}^+ S^k$ acts on $S^k$ with point stabilizers $H_p \cong \text{SO}(k) \times \mathbb{R}$. A point stabilizer of $\text{Conf}^+ S^k \rhd S^k$ is $\text{Conf}^+ \mathbb{E}^k$ [BP92, Cor. A.3.8]. Up to conjugacy, $H_p$ is the standard $\text{SO}(k) \times \mathbb{R} \subset \text{Conf}^+ \mathbb{E}^k$, since $\text{SO}(k)$ is maximal compact and $\mathbb{R}$ is its centralizer. This $\text{SO}(k) \times \mathbb{R}$ fixes two points on $S^k$, whose stabilizers in $H$ coincide since point stabilizers of a transitive action are isomorphic. Then $H$ preserves a pairing of points in $S^k$. In particular, if $\text{SO}(k) \times \mathbb{R}$ preserves $p$ and $q$ then it acts transitively on $S^k \setminus \{p, q\}$ while preserving this pairing. So paired points

1. lie in the same $S^{k-1}$ in $S^k \setminus \{p, q\} \cong S^{k-1} \times \mathbb{R}$ since they are exchanged by an order-2 element; and

2. are antipodal in this $S^{k-1}$ since the $\text{SO}(k-1)$ fixing one member of the pair must fix the other.

Interpret $S^k$ as the boundary at infinity of $\mathbb{H}^{k+1}$, following [BP92, Prop. A.5.13(4)]. Any two geodesics in $\mathbb{H}^{k+1}$ joining paired points of $S^k$ must intersect, since $H$ acts transitively and they all intersect the geodesic joining $p$ and $q$. However, only geodesics whose endpoints lie in the same $S^{k-1} \subset S^k \setminus \{p, q\}$ can intersect since each $S^{k-1}$ bounds a totally geodesic $\mathbb{H}^k \subset \mathbb{H}^{k+1}$. Therefore the pairing of points required by an action with $\text{SO}(k) \times \mathbb{R}$ stabilizers cannot be defined on all of $S^k$.

**Case 2:** $B = S^k$. Since $H$ acts essentially, it preserves no Riemannian metric; so some point stabilizer $H_p$ preserves no inner product on the tangent space at $p$. So the quotient map $\pi : \text{Conf}^+ \mathbb{E}^k \to \text{SO}(k) \times \mathbb{R}$ is surjective when restricted to $H_p$. Since $H_p \not\cong \text{SO}(k) \times \mathbb{R}$, it cannot also be injective. Then $H_p$ meets the translation subgroup $\ker \pi \cong \mathbb{R}^k \subset \text{Conf}^+ \mathbb{E}^k$ nontrivially; so it contains all of $\mathbb{R}^k$ since $\text{SO}(k) \rhd \mathbb{R}^k$ is irreducible. Then $H_p = \text{Conf}^+ \mathbb{E}^k$, which implies as in Case 1 that $H = \text{Conf}^+ S^k$.

Finally, the classification of geometries fibering over 3-dimensional isotropy-irreducible geometries required the following computation of second cohomology.

**Lemma 5.5.** $H^2(\text{isom} M; \mathbb{R}) = 0$ for $M = \mathbb{E}^3$, $S^3$, or $\mathbb{H}^3$. 

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Proof. This is a computation using a spanning set, though for $S^3$ and $\mathbb{H}^3$ one could instead appeal to the vanishing of $H^2$ for semisimple algebras [GOV94, Thm. 1.3.2]. The spanning elements of $\text{isom} M$ will be denoted $r_{ij}$ and $t_i$ ($1 \leq i, j \leq n = \dim M$); the linear dependency relations are $r_{ij} = -r_{ji}$, and the nonzero brackets are

\[
[r_{ij}, r_{jk}] = r_{ki} \quad \text{if } i, j, k \text{ distinct}
\]

\[
[r_{ij}, t_i] = t_j \quad \text{if } i \neq j
\]

\[
[t_i, t_j] = K r_{ij} \quad \text{if } i \neq j,
\]

where $K$ is 0 or $\pm 1$ (the sectional curvature of $M$).

The action of $\text{isom} M$ on $\mathbb{R}$ is trivial (Corollary 5.3(iii)); so the cocycle condition for a 2-cocycle $c : \Lambda^2 \text{isom} M \to \mathbb{R}$ becomes

\[
c([x_1, x_2], x_3) + c([x_2, x_3], x_1) + c([x_3, x_1], x_2) = 0,
\]

and the coboundaries are of the form

\[
c(x_1, x_2) = f([x_1, x_2]), \quad f \in (\text{isom} M)^*.
\]

Suppose $c$ is a 2-cocycle, and $i$, $j$, and $k$ are distinct. Applying the cocycle condition to the spanning elements yields

\[
c(r_{ij}, t_k) = c([r_{jk}, r_{ki}], t_k)
\]

\[
= c(r_{jk}, t_i) + c(r_{ki}, t_j)
\]

\[
c(r_{ij}, t_i) = c([r_{jk}, r_{ki}], t_i)
\]

\[
= c(r_{kj}, t_k)
\]

\[
K c(r_{jk}, r_{ki}) = c(K r_{jk}, r_{ki})
\]

\[
= c([t_j, t_k], r_{ki})
\]

\[
= c(t_i, t_j).
\]

A linear combination of cyclic permutations of the first equality is $c(r_{ij}, t_k) = 3c(r_{ij}, t_k)$, so $c(r_{ij}, t_k) = 0$. The second equality implies that $c(r_{ij}, t_i)$ only depends on $i$. Since $n = 3$, distinctness of $i$, $j$, and $k$ implies $c(r_{jk}, r_{ki})$ only depends on $i$ and $j$, which permits defining\(^5\)

\[
f : \text{isom} M \to \mathbb{R}
\]

\[
r_{ij} \mapsto c(r_{jk}, r_{ki})
\]

\[
t_i \mapsto c(r_{ji}, t_j).
\]

This definition and the third equality imply $c(x, y) = f([x, y])$ for all $x$ and $y$ in the spanning set, and therefore on all of $\text{isom} M$. So every cocycle $c$ is a coboundary, i.e. $H^2(\text{isom} M; \mathbb{R}) = 0$. \qed

6 Geometries fibering over 2D geometries

This section carries out the (unfortunately long) task of proving part (iii) of Theorem 1.1. That is, it classifies the 5-dimensional maximal model geometries $M = G/G_p$ in case (iii) of the fibering description (Prop. 3.3)—those for which the irreducible subrepresentations of $G_p \rtimes T_p M$ have dimensions 1 and 2. The first step is to set up an extension problem that can be solved to find $G$.

\(^5\) Independence of $c(r_{jk}, r_{ki})$ from $k$ can be proven for $n > 3$ by computing $c([r_{k\ell}, r_{tj}], r_{ki})$ and using the $n = 3$, $K = 1$ case to show that $c(r_{ij}, r_{k\ell}) = 0$ for distinct $i, j, k, \ell$. This extends Lemma 5.5 to dimensions other than 3.
Lemma 6.1 (The isometry group as an extension). If $M = G/G_p$ is a 5-dimensional model geometry for which $G_p \curvearrowright T_p M$ decomposes into 1-dimensional and 2-dimensional summands, then $G$ is an extension

$$1 \to H \to G \to Q \to 1$$

where:

(i) $Q$ is $\text{Conf}^+ S^2$, $\text{Conf}^+ E^2$, $\text{Isom}_0 S^2$, $\text{Isom}_0 E^2$, or $\text{Isom}_0 \mathbb{H}^2$;

(ii) if $G_p \cong S^1$, then the identity component of $H$ is covered by $\mathbb{R}$, $\mathbb{R}^2$, $S^3$, $\tilde{\text{SL}}(2, \mathbb{R})$, $\text{Sol}^3$, $\text{Isom}_0 E^2$, $\text{Heis}_3$, or $\mathbb{R}^3$; and

(iii) if $G_p = \text{SO}(2) \times \text{SO}(2)$, then the identity component of $H$ is either:

- one of $\text{SO}(3)$, $\text{PSL}(2, \mathbb{R})$, and $\text{Isom}_0 E^2$; or
- covered by $S^3 \times \mathbb{R}$, $\tilde{\text{SL}}(2, \mathbb{R}) \times \mathbb{R}$, $\text{Isom}_0 E^2 \times \mathbb{R}$, or $\text{Isom}_0 \text{Heis}_3$.

To organize the solution of what could be fifty extension problems, the classification proceeds by skimming off classes of geometries until only those with $\tilde{G} = \tilde{H} \times \tilde{Q}$ remain; these either are product geometries or can be described as associated bundles (Section 6.5.1). The plan is illustrated in Figure 6.2.

Figure 6.2: Classification strategy for geometries fibering over 2-D spaces.
6.1 Setting up the extension problem

This section proves Lemma 6.1, which describes the extension problem that will be solved to determine the transformation groups $G$ of the geometries $G/G_p$. Having obtained a fibering over a 2-dimensional space in Prop. 3.3, the bulk of the proof is in establishing the lists of quotients $Q$ and kernels $H$. One lemma is needed, in the form of the following observation about geometries with abelian isotropy—which will also be useful later in recovering a faithfully-acting $G$ from its universal cover $\tilde{G}$.

**Lemma 6.3.** If $M = G/G_p$ be a connected homogeneous space where $G$ is connected and $G_p$ is compact and abelian, then the following are equivalent.

(i) $G$ acts faithfully on $M$ (one of the requirements for a geometry)

(ii) $G_p$ acts faithfully on $T_pM$

(iii) $G_p$ acts faithfully by conjugation on $G$

**Proof.** There are two equivalences to verify.

(i) $\iff$ (ii): Since $G_p$ is compact, $M$ has an invariant Riemannian metric [Thu97, Prop. 3.4.11]. An isometry of a connected Riemannian manifold is determined by its value and derivative at a point [BP92, Prop. A.2.1], so the action of $G_p$ on $M$ is determined by the action of $G_p$ on $T_pM$. Then a nontrivial $g \in G$ acts as the identity on $M$ if and only if it lies in $G_p$ and acts as the identity on $T_pM$; so $G$ acts faithfully on $M$ if and only if $G_p$ acts faithfully on $T_pM$.

(ii) $\iff$ (iii): As a $G_p$-representation, the Lie algebra of $G$ decomposes as $T_1G = T_1G_p \oplus T_pM$.

Since $G_p$ is abelian, $T_1G_p$ is trivial; so $T_1G$ under the adjoint action $T_pM$ is a faithful $G_p$-representation if and only if $T_1G$ under the adjoint action is too. The equivalence for the conjugation action on $G$ follows since a homomorphism from a connected Lie group is determined by its derivative at the identity.

**Remark 6.4.** Condition (iii) is equivalent to $G_p \cap Z(G) = \{1\}$. So an abelian-isotropy homogeneous space $G/G_p$ satisfying all the conditions for a geometry except faithfulness of the $G$-action can be made into a geometry by passing from $G$ to $G/(G_p \cap Z(G))$.

**Proof of Lemma 6.1 (the extension problem for $G$).** Suppose $M = G/G_p$ is a 5-dimensional model geometry for which $G_p \curvearrowright T_pM$ decomposes into 1-dimensional and 2-dimensional summands. Then there is a conformal fibering (Prop. 3.3(iii)) $M \to B$ where $B$ is $S^2$, $\mathbb{E}^2$, or $\mathbb{H}^2$. If $Q$ denotes the image of $G$ in $\text{Conf} B$, then $G$ is an extension

$$1 \to H \to G \to Q \to 1.$$  

**Step 1: The image $Q$ is Isom$_0 B$ or $\text{Conf}^+ B$.** If $G_p = \text{SO}(2)$, then $B = M/F^\text{SO}(2)$ so $Q$ contains a copy of $\text{SO}(2)$. Otherwise, the trivial subrepresentation of $G_p \curvearrowright T_pM$ is 1-dimensional, so $G_p$ acts nontrivially on $B$. In either case, $Q$ acts transitively on $B$ and contains a copy of $\text{SO}(2)$ fixing a point.

Then $Q = \text{Isom}_0 B$ if $B$ admits a $G$-invariant metric; otherwise, $Q$ is a connected, transitive, essential subgroup of $\text{Conf}^+ B$ and must therefore be all of $\text{Conf}^+ B$ (Lemma 5.4). The list in the original statement (Lemma 6.1) omits $\text{Conf}^+ \mathbb{H}^2$ since $\text{Conf}^+ \mathbb{H}^2 = \text{Isom}_0 \mathbb{H}^2$ [BP92, Thm. A.4.1].
Step 2: If $G_p \cong S^1$, then $H$ is unimodular of dimension at most 3. If $G_p \cong S^1$, then $\dim G = 6$. Since $\dim Q \geq 3$ by the previous step, $\dim H \leq 3$. Since $G$ is unimodular [Fil83, Prop. 1.1.3] and $H$ is a closed normal subgroup, $H$ is itself unimodular [Fil83, Prop. 1.1.4]. The explicit list in Lemma 6.1(ii) can be found by consulting Bianchi’s classification of all 3-dimensional real Lie algebras; see e.g. [FH91, Lec. 10], [PSWZ76, Table I], or [Mac99, Table 21.3].

Step 3: If $G_p = SO(2) \times SO(2)$, then some $SO(2) \subset H$ acts faithfully by conjugation. For each $Q$ in Step 1, the maximal compact subgroup has rank 1; so $H \cap G_p$ is a closed subgroup of dimension at least 1. Since $G$ acts with $SO(2)$ stabilizers on $B$, $H \cap G_p$ has dimension at most 1. Therefore the identity component of $H \cap G_p$ is isomorphic to $SO(2)$.

Step 4: Classify possible $H$ when $G_p = SO(2) \times SO(2)$. First, $\dim H \leq 4$ since $\dim Q \geq 3$ and $\dim G = \dim M + \dim(SO(2) \times SO(2)) = 5 + 2 = 7$. If $\dim H \leq 3$, applying the restriction from Step 3 to the list from Step 2 yields the 3-dimensional groups $H$ in Lemma 6.1(iii).

The remainder of the groups occur if $\dim H = 4$. Then $H/(H \cap G_p)$ is a 3-dimensional homogeneous space, with $H \cap G_p = SO(2)$ point stabilizers by Step 3 and unimodular isometry group $H$. The proof of [Thu97, Thm. 3.8.4(b)] (classifying 3-dimensional geometries with $SO(2)$ point stabilizer), finds the spaces listed in Table 6.5. To obtain the final list in Lemma 6.1(iii), eliminate duplicates by observing that semidirect products with inner action are isogenous with direct products.

Table 6.5: 3-dimensional simply-connected homogeneous spaces $H/\mbox{SO}(2)$ with unimodular $H$

| $H/(H \cap G_p)$       | $H$                  |
|------------------------|----------------------|
| $S^3$                  | $S^3 \times S^1$     |
| $\widetilde{\mbox{PSL}}(2, \mathbb{R})$ | $\widetilde{\mbox{PSL}}(2, \mathbb{R}) \times SO(2)$ |
| Heis$_3$               | Heis$_3 \times SO(2)$ |
| $S^2 \times \mathbb{R}$ | SO(3) $\times \mathbb{R}$ |
| $H^2 \times \mathbb{R}$ | $\mbox{PSL}(2, \mathbb{R}) \times \mathbb{R}$ |
| $\mbox{Isom}_0 \mathbb{E}^2$ | $\mbox{Isom}_0 \mathbb{E}^2 \times SO(2)$ |

Remark 6.6. The list of 4-dimensional groups in Step 4 may also be obtained by computing with a convenient basis in the Lie algebra: let $r$ generate $H \cap G_p$, let $x$ and $y$ span the nontrivial $(H \cap G_p)$-subrepresentation of $T_1 H$, and let $z$ (together with $r$) span the trivial subrepresentation. Then one can work out the values of $[x, y]$, $[x, z]$, and $[y, z]$ that satisfy the Jacobi identity (and rescale the basis if it helps).

Alternatively, to carry out Bianchi’s classification from scratch, notice that the only two real semisimple Lie algebras of dimension at most 3 are $so_3 \mathbb{R}$ and $sl_4 \mathbb{R}$. The other 3-dimensional real Lie algebras are therefore solvable. A nilradical in a solvable Lie algebra is at least half the dimension [GOV94, Thm. 2.5.2, attributed to Mubarakzyanov], so these algebras are of the form $\mathbb{R}^2 \times \mathbb{R}$, which can be systematically handled using Jordan forms.
6.2 Geometries fibering essentially

This section handles the first side branch of Figure 6.2: proving the following classification of geometries that fiber essentially over 2-dimensional spaces.

**Proposition 6.7.** Suppose $M = G/G_p$ is a 5-dimensional model geometry for which $G_p \lt T_p M$ decomposes into 1-dimensional and 2-dimensional summands. Furthermore suppose that the fibering $M \rightarrow B$ from Prop. 3.3(iii) is essential.

(i) If $B = S^2$, then $M = T^1\mathbb{H}^3 \cong \text{PSL}(2,\mathbb{C})/\text{PSO}(2)$, with isotropy $G_p = S^1_\mathbb{C}$ in the notation of Figure 2.4.

(ii) $T^1\mathbb{H}^3$ is a maximal model geometry.

(iii) If $B = E^2$, then $M$ is a solvable Lie group of the form $\mathbb{R}^4 \times e\mathfrak{t}_A \mathbb{R}$. Moreover, $M$ is maximal if and only if the multiset of characteristic polynomials of the Jordan blocks of $A$ is one of the following.

(a) $\{x - 1, x - 1, x + 1, x + 1\}$
(b) $\{(x - 1)^2, x + 1, x + 1\}$
(c) $\{x - 1, x - 1, x, x + 2\}$ (This is $\text{Sol}^1_\mathbb{C} \times \mathbb{E}_\mathbb{C}$)
(d) $\{x - 1, x - 1, x - a + 1, x + a + 1\}$; $a > 0$, $a \neq 1$, $a \neq 2$ (This is a family of geometries.)

(iv) The geometries in (iii),(a)-(c) are model geometries; and (iii),(d) is a model geometry if and only if $e\mathfrak{t}_A$ has a characteristic polynomial in $\mathbb{Z}[x]$ for some $t > 0$.

**Remark 6.8.** In (iii), the proof of maximality will reveal that the isotropy consists of an $\text{SO}(2)$ acting by automorphisms on $M$ for every pair of identical Jordan blocks in $A$.

The case when $B = S^2$ is considerably easier than that of $B = E^2$, which has some complexities not immediately apparent in the above statement; we treat these cases separately.

6.2.1 Over the sphere

This section contains the proof that $T^1\mathbb{H}^3$ is a maximal model geometry (ii), and that it is the only one fibering essentially conformally over the sphere (i). The extension problem from Lemma 6.1 is used only to determine that the isometry group covers $\text{PSL}(2,\mathbb{C})$; the rest of the proof is built on properties of $\text{PSL}(2,\mathbb{C})$.

**Proof of Prop. 6.7(i).** If $M = G/G_p$ fibers essentially over $B = S^2$, then $G$ is an extension

$$1 \rightarrow H \rightarrow G \rightarrow \text{Conf}^+ S^2 \rightarrow 1$$

where $H$ is as named in Lemma 6.1. Since $\dim \text{Conf}^+ S^2 = 6$, either $G_p \cong S^1$ and $\dim H = 0$ or $G_p = \text{SO}(2)^2$ and $\dim H = 1$. The list of possible $H$ for $\text{SO}(2)^2$ includes no 1-dimensional entries; so $\dim H = 0$, $G_p \cong S^1$, and $G$ covers $Q$.

As maximal tori in maximal compact subgroups, all copies of $S^1$ in $\text{Conf}^+ S^2 \cong \text{PSL}(2,\mathbb{C})$ are conjugate, as are all copies of $S^1$ in the 2-sheeted universal cover $\text{SL}(2,\mathbb{C})$. Hence $M \cong \text{SL}(2,\mathbb{C})/\text{SO}(2) \cong \text{PSL}(2,\mathbb{C})/\text{PSO}(2)$. Since $\text{PSL}(2,\mathbb{C})$ is centerless, the geometry $M$ expressed with faithful transformation group (Rmk. 6.4) is indeed $\text{PSL}(2,\mathbb{C})/\text{PSO}(2)$.

Since $\text{PSL}(2,\mathbb{C}) \cong \text{Isom}_0 \mathbb{H}^3$, choosing a basepoint in $T^1\mathbb{H}^3$ identifies $\text{PSL}(2,\mathbb{C})/\text{PSO}(2)$ with $T^1\mathbb{H}^3$ (hence the name). The point stabilizers have slope 1 because $\text{PSO}(2)$ rotates $\mathbb{H}^3$ and a tangent space the same way (or one can explicitly decompose $\mathbb{sl}_2\mathbb{C}$ into $S^1$-representations).
Proof of Prop. 6.7(ii). $T^1\mathbb{H}^3$ is a model geometry since it models the unit tangent bundle of any finite-volume hyperbolic 3-manifold.

For maximality, suppose $G$ were a larger connected group of isometries for $T^1\mathbb{H}^3$ under some metric.

If $\dim G = 7$, then $G_p = \text{SO}(2)^2$. From the classification of simple Lie groups [Hel78, Ch. X, §6 (p. 516)], the only connected semisimple Lie group containing $\text{PSL}(2,\mathbb{C})$ of dimension at most 7 is $\text{PSL}(2,\mathbb{C})$. Then using the Levi decomposition [GOV94, §1.4] and the fact that $\text{PSL}(2,\mathbb{C})$ is centerless, $G$ admits $\text{PSL}(2,\mathbb{C})$ as a quotient. Since $\text{PSO}(2)$ is a maximal torus in $\text{PSL}(2,\mathbb{C})$, it contains the image of $G_p$. Then $G/G_p$ fibers essentially over $S^2 \cong \text{Conf}^+ S^2/\text{Conf}^+ \mathbb{E}^2$, which by part (i) is incompatible with $\dim G = 7$.

If $\dim G > 7$, then $\dim G_p > 2$, and previous sections already listed maximal geometries with $\dim G_p > 2$. Of those, only $S^2 \times \mathbb{E}^3$ and $S^2 \times \mathbb{H}^3$ have the same diffeomorphism type. Since $\text{PSL}(2,\mathbb{C})$ admits no nontrivial image in $\text{SO}(3) = \text{Isom}_0 S^2$ (both are simple and the domain has larger dimension), it cannot act transitively by isometries on either of these products.

\subsection{Over the plane}

Suppose $M = G/G_p$ fibers essentially over $\mathbb{E}^2$. The description of $G$ as an extension (Lemma 6.1) is

$$1 \to H \to G \to \text{Conf}^+ \mathbb{E}^2 \to 1$$

where, since $\text{Conf}^+ \mathbb{E}^2$ is 4-dimensional, $H$ is $\mathbb{R}^2$, $\text{SO}(3)$, $\text{PSL}(2,\mathbb{R})$ or $\text{Isom}_0 \mathbb{E}^2$. This section classifies the resulting geometries (Prop. 6.7(iii)–(iv)), all of which will be solvable Lie groups of the form $\mathbb{R}^4 \times_{e^{tA}} \mathbb{R}$. The geometry named in Prop. 6.7(iii).(a) occurs when $H = \text{Isom}_0 \mathbb{E}^2$; the rest will come from $H = \mathbb{R}^2$.

Passing to Lie algebras, we aim to solve the corresponding extension problem

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{q} \to 0.$$ 

The proof relies on the following three computations: the outer derivation algebra of $\mathfrak{h}$ (Lemma 6.9), the action $\mathfrak{q} \to \text{out} \mathfrak{h}$ (Lemma 6.10), and the cohomology $H^2(\mathfrak{q};\mathbb{R}^2)$ with the action in Lemma 6.10 (Lemma 6.11). For the Lie algebra $\mathfrak{q}$ of $\text{Conf}^+ \mathbb{E}^2$, we will use the basis $\{x, y, r, s\}$ where $x$ and $y$ generate the translations, $r$ generates rotations (values $[r, x] = y$), and $s$ generates scaling (values $[s, x] = x$).

\begin{lemma} \textbf{(Outer derivation algebras).} The Lie algebras of the above groups $H$ have the following algebras of outer derivations:

| $\mathfrak{so}_3$ | $\mathfrak{r}^2$ | $\mathfrak{sl}_2$ | $\mathfrak{isom} \mathbb{E}^2$ |
|-----------------|-----------------|-----------------|-----------------|
| $\text{out} \mathfrak{so}_3 = 0$ | $\text{out} \mathfrak{r}^2 = \mathfrak{gl}_2 \mathbb{R}$ | $\text{out} \mathfrak{sl}_2 = 0$ | $\text{out} (\mathfrak{isom} \mathbb{E}^2) = \mathbb{R}$ |
\end{lemma}

\begin{proof}
$\mathbb{R}^2$ is abelian, and the outer derivation algebra of a semisimple Lie algebra is zero [Hum72, §5.3], so only the last one needs any computation.

Let $\{x, y\}$ be a basis for $\mathfrak{isom} \mathbb{E}^2$ where $[r, x] = y$ and $x$ and $y$ generate the translations. For a derivation $d$, the Leibniz rule $d[v, w] = [dv, w] + [v, dw]$ implies that $d$ preserves the lower central series and the derived series—so in this case it takes translations to translations. Subtract inner derivations to ensure $dr = ar$ and $dx = bx$ for some $a$ and $b$; then

$$dy = [dr, x] + [r, dx] = ay + by$$

$$bx = dx = [dy, r] + [y, dr] = (2a + b)x.$$

The second line implies $a = 0$; then $d$ is zero on $r$ and scales by $b$ on the translations.
\end{proof}
Lemma 6.10 (Restrictions on actions on $H$). In the action of $q = T_1 \text{Conf}^+ \mathbb{E}^2$ on $h$, $s$ acts with trace $-2$, $x$ and $y$ act trivially, and $r$ generates a compact subgroup of Out $h$.

In particular, if $H = \mathbb{R}^2$, then up to conjugacy in $\text{GL}(2, \mathbb{R}) = \text{Aut} h$, $s$ acts by one of the following matrices where $a$ is a real parameter:

\[
\begin{pmatrix}
-1 & a \\
-a & -1
\end{pmatrix} \quad \begin{pmatrix}
-1 & 1 \\
0 & -1
\end{pmatrix} \quad \begin{pmatrix}
-1 + a & 0 \\
0 & -1 - a
\end{pmatrix}
\]

And if $r$ acts nontrivially by rotations (or some conjugate thereof), then only the first of these can commute with $r$.

**Proof.** There are three claims to prove in the first sentence of the Lemma statement:

1. Since $s$ acts with trace 2 on $q$ and $g$ is unimodular, $s$ must act with trace $-2$ on $h$.

2. The restriction of $q \to \text{out} h$ to $\text{isom} \mathbb{E}^2$ is either injective or contains the translation ideal (Lemma 5.2). Of the outer derivation algebras (Lemma 6.9), only $\mathfrak{gl}_2 \cong \mathfrak{sl}_2 \oplus \mathbb{R}$ has high enough dimension to contain an injective image of $\text{isom} \mathbb{E}^2$. For the kernel not to contain the translation ideal, the projection to at least one of the summands must be injective; but $\mathbb{R}$ has too low dimension, and $\mathfrak{sl}_2$ contains no subalgebras of dimension 3 other than itself. Hence $\text{isom} \mathbb{E}^2 \to \text{out} h$ always factors through $\mathfrak{so}_2$—i.e. $x$ and $y$ act trivially.

3. Since the Lie algebra extension is induced by a Lie group extension, the homomorphism $q \to \text{out} h$ is induced by some $\text{Conf}^+ \mathbb{E}^2 = Q \to \text{Out} H$, which sends the compact $\text{SO}(2) \subset \text{Conf}^+ \mathbb{E}^2$ generated by $r$ to a compact subgroup of Out $H \subseteq \text{Out} h$.

The claims when $H = \mathbb{R}^2$ then follow from listing $2 \times 2$ Jordan forms with trace $-2$ and the fact that the centralizer of $\text{SO}(2) \actson \mathbb{R}^2$ is generated by itself and the real scalars.

Since $\mathbb{R}^2$ is abelian, $\text{out} \mathbb{R}^2 = \text{der} \mathbb{R}^2$; so the only additional data needed for $H = \mathbb{R}^2$ is second cohomology.

**Lemma 6.11.** The Lie algebra cohomology $H^2(T_1 \text{Conf}^+ \mathbb{E}^2; \mathbb{R}^2)$ has the following values.

- If $s$ acts with an eigenvalue 2, then $H^2$ is 1-dimensional, represented by cocycles $c$ with $c(x, y)$ in the 2-eigenspace.
- If $s$ acts with an eigenvalue 0, then $H^2$ is 1-dimensional, represented by cocycles $c$ with $c(r, s)$ in the 0-eigenspace.
- Otherwise, $H^2 = 0$.

**Proof.** If $\beta$ is a 1-cochain, then the coboundary $d\beta$ has values

\[
\begin{align*}
    d\beta(x, y) &= x\beta(y) - y\beta(x) - \beta([x, y]) = 0 \\
    d\beta(r, x) &= r\beta(x) - x\beta(r) - \beta([r, x]) = r\beta(x) - \beta(y) \\
    d\beta(r, y) &= r\beta(y) - y\beta(r) - \beta([r, y]) = r\beta(y) + \beta(x) \\
    d\beta(s, x) &= s\beta(x) - x\beta(s) - \beta([s, x]) = (s - 1)\beta(x) \\
    d\beta(s, y) &= s\beta(y) - y\beta(s) - \beta([s, y]) = (s - 1)\beta(y) \\
    d\beta(r, s) &= r\beta(s) - s\beta(r) - \beta([s, r]) = r\beta(s) - s\beta(r).
\end{align*}
\]
If \( c \) is a cocycle, then
\[
\begin{align*}
c(x, y) &= -c(x, [y, s]) \\
&= c(y, [s, x]) + c(s, [x, y]) + xc(y, s) + yc(s, x) + sc(x, y) \\
&= c(y, x) + 0 + 0 + 0 + sc(x, y) \\
2c(x, y) &= sc(x, y).
\end{align*}
\]
Thus either \( c(x, y) = 0 \) or \( s \) acts with an eigenvalue 2. Similarly, applying the cocycle condition to each of the equalities
\[
\begin{align*}
c(x, s) &= -c(s, [y, r]) \\
c(y, s) &= -c(s, [r, x]) \\
c(x, y) &= -c(x, [r, s]) \\
c(r, x) &= -c(r, [x, s]) \\
c(r, y) &= -c(r, [y, s])
\end{align*}
\]
yields mostly vacuous equalities, except for the following.
\[
\begin{align*}
c(x, s) &= -rc(y, s) + (1 - s)c(r, y) \\
c(y, s) &= rc(x, s) + (s - 1)c(r, x)
\end{align*}
\]
(1)
Using this information, we will define \( \beta \) to match \( d\beta \) as closely as possible with \( c \).

- If \( r \) acts as 0, then set \( \beta(x) = c(r, y) \) and \( \beta(y) = -c(r, x) \). Then \( c - d\beta \) is zero on \( r \wedge x, r \wedge y, x \wedge s, \text{ and } y \wedge s \) (by equations 1).
  - If \( s \) acts with eigenvalues 0 and -2, then \( c(x, y) = 0 \). Set \( \beta(r) = \frac{1}{2}c(r, s) \), so that the only potentially nonzero value of \( c - d\beta \) is on \( r \wedge s \) and lies in the 0-eigenspace of \( s \).
  - If \( s \) acts with eigenvalues 2 and -4, setting \( \beta(r) = -s^{-1}c(r, s) \) makes \( c - d\beta \) zero on \( r \wedge s \). The only nonzero contribution to \( H^2 \) is from \( c(x, y) \) lying in the 2-eigenspace of \( s \).
  - Otherwise, setting \( \beta(r) = -s^{-1}c(r, s) \) makes \( c - d\beta = 0 \), so \( H^2 = 0 \).

- If \( r \) acts by rotation, then since \( s \) commutes with it, we reinterpret \( \mathbb{R}^2 \) as \( \mathbb{C} \) on which \( r \) acts by \( ik \) for some real \( k \neq 0 \). Defining
  \[
  \begin{align*}
z_s &= c(x, s) + ic(y, s) \\
z_r &= c(r, x) + ic(r, y) \\
w_s &= c(x, s) - ic(y, s) \\
w_r &= c(r, x) - ic(r, y),
\end{align*}
\]
equations 1 become
\[
\begin{align*}
z_s &= -kz_s + i(s - 1)z_r \\
w_s &= kw_s - i(s - 1)w_r.
\end{align*}
\]
Since \( s \neq 1 \) (by having to act with trace -2), \( \beta(x) \) and \( \beta(y) \) can be selected to make \( d\beta \) reproduce \( z_s \) and \( w_s \); then \( z_r \) and \( w_r \) follow dependently.
We further set \( \beta(r) = 0 \) and \( \beta(s) = r^{-1}c(r, s) \) to make \( (c - d\beta)(r, s) = 0 \). Finally, \( c(x, y) = 0 \) since \( s \) has no eigenvalue 2 (both its eigenvalues have real part 1).
Proof of Prop. 6.7(iii). The proof begins by determining the possibilities for $H$.

**Step 1:** $H$ is $\text{Isom}_0\mathbb{E}^2$ or $\mathbb{R}^2$. The outer derivation algebras for the Lie algebras of $\text{SO}(3)$ and $\text{PSL}(2, \mathbb{R})$ are zero [GOV94, Cor. to Thm. 1.3.2]; and being unimodular, they have no inner derivations acting with trace $-2$. Since $T_1\text{Conf}^+\mathbb{E}^2$ contains an element acting on $\mathfrak{h}$ with trace $-2$ (Lemma 6.10), this rules out $\text{SO}(3)$ and $\text{PSL}(2, \mathbb{R})$ as candidates for $H$. Then $G$ is an extension of the solvable group $\text{Conf}^+\mathbb{E}^2$ by either $\mathbb{R}^2$ or $\text{Isom}_0\mathbb{E}^2$.

**Step 2:** If $H = \text{Isom}_0\mathbb{E}^2$, then $M = \left((\mathbb{R}^4 \times \mathbb{R}) \rtimes \text{SO}(2)^{\times 2}\right) / \text{SO}(2)^{\times 2}$. Since $Z(\text{Isom}\mathbb{E}^2) = 0$, the Lie algebra cohomology determining extensions is identically $0$; so every homomorphism

\[
T_1\text{Conf}^+\mathbb{E}^2 \to \text{out}\text{Isom}\mathbb{E}^2 \cong \mathbb{R}
\]

can be realized by an extension, and every such extension splits on the Lie algebra level (Thm. 2.6). Since $r$ has to generate a compact subgroup and the nonzero part of $\text{out}T_1\text{Isom}\mathbb{E}^2$ scales the translation subalgebra, $r$ maps to $0$ in $\text{out}\text{Isom}\mathbb{E}^2$. Since $s$ with trace $-2$, it maps to the scalar $-1$.

Then $\tilde{G} \cong \mathbb{C}^2 \rtimes \mathbb{R}^3$, where $(x, y, z) \in \mathbb{R}^3$ acts on $\mathbb{C}^2$ by the matrix

\[
\begin{pmatrix}
e^{x+iy} & 0 \\ 0 & e^{-x+iz}
\end{pmatrix};
\]

and $Z(\tilde{G}) = \{0\} \times (0 \oplus 2\pi \mathbb{Z} \oplus 2\pi \mathbb{Z})$.

The point stabilizer $G_p$ is compact and acts faithfully by conjugation on $G$ (Lemma 6.3), so it maps injectively to a compact subgroup of $\text{Inn}G = \tilde{G}/Z(\tilde{G}) \cong \mathbb{C}^2 \rtimes (\mathbb{R} \rtimes \text{SO}(2) \times \text{SO}(2))$, in which $\text{SO}(2) \times \text{SO}(2)$ is maximal compact since the quotient by it is contractible. Then $G_p$ covers this $\text{SO}(2) \times \text{SO}(2)$, but not nontrivially due to the faithful conjugation action requirement. So by conjugacy of maximal compact subgroups,

\[
M = \left((\mathbb{R}^4 \times \mathbb{R}) \rtimes \text{SO}(2)^{\times 2}\right) / \text{SO}(2)^{\times 2},
\]

the geometry named in Prop. 6.7(iii)(a).

**Step 3:** Non-split extensions of $\text{Conf}^+\mathbb{E}^2$ by $\mathbb{R}^2$ produce no model geometries. The non-split extensions are with $r$ acting trivially, and with $s$ acting diagonally with an eigenvalue of $0$ or $2$. We’ll show that if $s$ is an eigenvalue of $s$, then $\text{SO}(2)$ fails to extend to a compact point stabilizer; and if $2$ is an eigenvalue of $s$, then $G$ fails to admit lattices.

When $s$ acts with an eigenvalue of $0$, a non-split extension is given by a nonzero value of $c(r, s)$ in the 0-eigenspace; i.e. $[r, s]$ is nonzero in $\mathbb{R}^2 \subset \mathfrak{g}$ (Lemma 6.11). Then $r$—indeed, any element of $\mathfrak{g}$ lying over $r \in T_1\text{Conf}^+\mathbb{E}^2$—acts nontrivially and nilpotently on a subalgebra of $\mathfrak{g}$ and therefore cannot generate a compact subgroup (which must act semisimply). The same holds for an element of the 2-eigenspace of $s$, while an element of the 0-eigenspace generates a central subgroup of $G$; so no nontrivial compact subgroup of $G$ lies over $\text{SO}(2) \subset \text{Conf}^+\mathbb{E}^2$. The point stabilizer $G_p$, being compact, must be such a subgroup; so the group $G$ resulting from this case produces no geometries $G/G_p$.

When $s$ acts with an eigenvalue of $2$, a non-split extension is given by a nonzero value of $c(x, y)$ in the 2-eigenspace. So in $\mathfrak{g}$, the elements $x$ and $y$ generate a copy of the Heisenberg algebra.
Inspecting the actions of the other basis elements shows that this is an ideal, and in fact $\tilde{G}$ can be written as

$$(\text{Heis}_3 \times \mathbb{R}) \times (\mathbb{R} \times \mathbb{R}),$$

where the first $\mathbb{R}$ rotates the $xy$-plane of the Heisenberg group and the second $\mathbb{R}$ scales diagonally with exponents 1, 1, 2, and $-4$.

Since $\mathbb{R} \times \mathbb{R}$ acts non-nilpotently, the nilradical of this is $\text{Heis}_3 \times \mathbb{R}$. Since $G$ is solvable, a theorem of Mostow (see [Fil83, Prop 6.4.2]) ensures that any lattice $\Gamma$ in its universal cover $\tilde{G}$ intersects the nilradical in a lattice and projects to the quotient as a closed cocompact group. Then since $[\text{Heis}_3, \text{Heis}_3] = Z(\text{Heis}_3)$ and $Z(\text{Heis}_3)$ is a copy of $\mathbb{R}$, cocompactness of $\Gamma$ ensures $[\Gamma, \Gamma] \cap Z(\text{Heis}_3)$ is nontrivial. Since $\Gamma$ has cocompact image in $\mathbb{R} \times \mathbb{R}$, some $g \in \Gamma$ acts by nontrivial scaling on $\text{Heis}_3 \times \mathbb{R}$; in particular, this action on $Z(\text{Heis}_3)$ implies $[\Gamma, \Gamma] \cap Z(\text{Heis}_3)$ is not discrete; so $\Gamma$ fails to be a lattice.

**Step 4:** Split extensions $\tilde{G} = \mathbb{R}^2 \rtimes \text{Conf}^+ \mathbb{E}^2$ where $s$ acts by a complex scalar produce non-maximal geometries. In this case, we assume $s$ acts as $\begin{pmatrix} -1 & a \\ -a & -1 \end{pmatrix}$ for some real $a$. Then $\tilde{G} \cong \mathbb{C}^2 \times \mathbb{R}^2$ where, for some real $b$, the action of $(x, y) \in \mathbb{R}^2$ on $\mathbb{C}^2$ is by the matrix

$$\begin{pmatrix} e^{-x+iax+iby} & 0 \\ 0 & e^{x+iy} \end{pmatrix}.$$ 

To have a compact subgroup we can use as the point stabilizer, $0 \oplus \mathbb{R}$ must intersect the center nontrivially; so $G$ is covered by

$$(\mathbb{C}^2 \times \mathbb{R}) \rtimes S^1$$

where the $S^1$ action on each $\mathbb{C}$ may have some degree other than 1. Since this has trivial center, this actually is $G$ (and $G_p$ is its maximal compact subgroup $S^1$). Whatever geometry it produces is subsumed by the geometry

$$[(\mathbb{C}^2 \times \mathbb{R}) \rtimes SO(2)^{\times 2}] / SO(2)^{\times 2}$$

from Step 2.

**Step 5:** Identify the geometries that remain. Similarly to Step 4, the remaining groups $G$ are all described as the semidirect product

$$((\mathbb{R}^4 \rtimes \mathbb{R}) \rtimes S^1$$

(again, to obtain $G$ from the universal cover, we replaced a second $\mathbb{R}$ factor by $S^1$ and noted that the result has trivial center), where $t \in \mathbb{R}$ acts by one of the matrices below (omitted entries are zero) and $S^1$ acts as $SO(2)$ on the last two coordinates.

$$\begin{pmatrix} e^t & e^t \\ e^t & e^{-t} \end{pmatrix}, \quad \begin{pmatrix} e^{(1+a)t} & e^{(1-a)t} \\ e^{(1-a)t} & e^{-t} \end{pmatrix}, \quad \begin{pmatrix} e^{(1+a)t} & e^{(1-a)t} \\ e^{(1-a)t} & e^{-t} \end{pmatrix}$$

Step 4 eliminates $a = 0$; and since $S^1 \subset G$ is maximal compact, $M \cong \mathbb{R}^4 \rtimes \mathbb{R}$ where $t \in \mathbb{R}$ acts by one of the above matrices. The case $a = 1$ is $\text{Sol}_0^1 \times \mathbb{E}$; and the case $a = 2$ is a non-maximal form of $\mathbb{R} \rtimes \text{Conf}^+ \mathbb{E}^3 / SO(3)$ (Section 5.2).
Step 6: Maximal. The maximal geometry realizing a solvable Lie group $M$ with real roots is of the form $M \rtimes K/K$, where $K \subseteq \text{Aut } M$ is maximal compact (Lemma 2.3).

So for these $M = \mathbb{R}^4 \rtimes_{e^t A} \mathbb{R}$, it suffices to determine the maximal compact subgroup of $\text{Aut } M \cong \text{Aut } T_1 M$. An automorphism of the Lie algebra $T_1 M$ must preserve its nilradical $\mathbb{R}^4$, each of the generalized eigenspaces by which $\mathbb{R} \cong T_1 M/\mathbb{R}^4$ acts on $\mathbb{R}^4$, and the filtration by rank on each generalized eigenspace. So in coordinates the matrices in $\text{Aut } T_1 M$ are block upper-triangular, and the maximal compact subgroup $K \subset \text{Aut } T_1 M$ is conjugate into a group whose only nonzero entries are in the diagonal blocks (Part II, [Gen16b, Lemma 5.29]). Since $K$ matches the dimension of the isotropy groups computed above in the classification, we conclude that the maximal geometries are

$$\mathbb{R}^4 \rtimes_{e^t A} \mathbb{R} \rtimes_{(x-1)^2, x+1, x+1} \rtimes_{x-1, x-1, x+1, x+1} \rtimes_{x-1, x-1, x-a+1, x+a+1} \rtimes \text{SO}(2)/\text{SO}(2).$$

Proof of Prop. 6.7(iv) (model geometries). Let $M = G/G_0 \cong \mathbb{R}^4 \rtimes_{e^t A} \mathbb{R}$ be a geometry named in Prop. 6.7. The proof splits into cases according to characteristic polynomials of Jordan blocks of $A$.

Case (a): $x - 1$, $x - 1$, $x + 1$, $x + 1$. This geometry models the solvmanifold $\mathbb{C}^2 \rtimes \mathbb{R}/(\Lambda \rtimes Z)$ where $1 \in Z$ acts by a matrix conjugate to $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $\Lambda \cong \mathbb{Z}^4$ is preserved by this action.

Case (b): $(x - 1)^2$, $x + 1$, $x + 1$. When $A$ is the matrix below, [Fil83, Corollary 6.4.3] ensures the existence of a lattice, if we choose $t$ to be the logarithm of an invertible quadratic integer, e.g. $\ln(3 + 2\sqrt{2})$.

$$\begin{pmatrix} e^t & e^t \\ e^t & e^{-t} \\ e^{-t} & e^t \end{pmatrix}$$

Cases (c, d): $x - 1 - a$, $x - 1 + a$, $x + 1$, $x + 1$. We need a slightly stronger version of the same result; backtracking to Mostow’s theorem [Fil83, Prop 6.4.2], any lattice in $G$ intersects the nilradical $\mathbb{R}^4$ in a lattice and projects to $\mathbb{R} \times \mathbb{R}$ as a closed cocompact subgroup (hence a lattice). So if $G$ admits a lattice $\Gamma$, then it lifts to a lattice in $\tilde{G}$ containing the elements of $\mathbb{R} = \tilde{S}^1$ lying over the identity in $S^1$; furthermore $\Gamma \cap (\mathbb{R}^2 \times \mathbb{C})$ is a lattice in $\mathbb{R}^2 \times \mathbb{C}$ preserved by two independent elements of $\mathbb{R} \times \mathbb{R}$. One of these can be chosen to lie over the identity in $S^1$ so the other just needs to be independent from the $S^1$ direction.

Hence $G$ of the second form admits a lattice if and only if, for some real $\theta$ and $t \neq 0$, the matrix with diagonal entries $e^{(1+a)t}$, $e^{(1-a)t}$, $e^{-t+i\theta}$, and $e^{-t-i\theta}$ has a characteristic polynomial with integer coefficients—i.e. these four numbers are roots of an integer polynomial $p \in \mathbb{Z}[x]$. Then the extension $L$ of $\mathbb{Q}$ containing these roots is Galois. Let $\lambda = e^t$ and $z = e^{i\theta}$, so that these roots can be written as $\lambda^{1+a}$, $\lambda^{1-a}$, $\lambda^{-1}z$, and $\lambda^{-1}z$.

If $z$ is real, then $\lambda$ appears twice but the other roots appear only once, so $\lambda$ has no Galois conjugates—i.e. $\lambda \in \mathbb{Q}$. By Gauss’s lemma, $x - \lambda \in \mathbb{Z}[x]$ so $\lambda \in \mathbb{Z}$. Then $\lambda = 1$, which contradicts $t \neq 0$. Hence $z$ is not real.

If $\lambda^{1-a} \in \mathbb{Q}$, then by the rational root theorem, $\lambda^{1-a} = 1$; so $a = 1$ since $\lambda \neq 1$. Then the resulting geometry is $\text{Sol}_0^4 \times E$ (in Filipkiewicz’s notation, $G_5 \times E$).
If \( \lambda^{1-a} \) has degree 2 over \( \mathbb{Q} \), then its Galois conjugate must be the other real root of \( p \), so \( p \) factors—again in \( \mathbb{Z}[x] \) due to Gauss’s lemma—as

\[
(x^2 - (\lambda^{1+a} + \lambda^{1-a})x + \lambda^2) (x^2 - (2\lambda^{-1} \cos \theta)x + \lambda^{-2}),
\]

which again implies that \( \lambda = 1 \). If \( \lambda^{1-a} \) has degree 3 over \( \mathbb{Q} \), then \( \lambda^{1+a} \) has degree 1 over \( \mathbb{Q} \), again yielding \( \text{Sol}_0^4 \times \mathbb{E} \).

The last case is when \( \lambda^{1-a} \) has degree 4 over \( \mathbb{Q} \). Then

\[
p(x) = x^4 + ax^3 + bx^2 + cx + 1
\]
is irreducible with exactly two real roots \( \phi_1 = \lambda^{1-a} \) and \( \phi_2 = \lambda^{1+a} \), both positive. From the values of \( \phi_1 \) and \( \phi_2 \), we can recover \( a \) by

\[
a = \left| \frac{\ln \phi_1 - \ln \phi_2}{\ln(\phi_1 \phi_2)} \right|.
\]

The condition on \( p \) having exactly two real roots is that its discriminant is negative.

One can use Sturm’s theorem or computer assistance to derive the condition that the two real roots are positive when \( a + c < 0 \).

### 6.3 Geometries requiring non-split extensions

Having handled the essentially-fibering geometries in the previous section, the remaining geometries \( M = G/G_p \) fiber isometrically. So in the description of \( G \) as an extension \( 1 \to H \to G \to Q \to 1 \) (Lemma 6.1), \( Q = \text{Isom}_0 B \) where \( B = S^2, \mathbb{E}^2, \) or \( \mathbb{H}^2 \)—thus reducing the number of extension problems from fifty to thirty.

This section’s contribution is the second branch in Figure 6.2—that is, it proves the following two claims.

**Lemma 6.12.** If \( M = G/G_p \) is a maximal model geometry for which

(i) \( G_p \rtimes T_p M \) decomposes into 1-dimensional and 2-dimensional summands, and

(ii) \( \tilde{G} \) is not a split extension of \( \text{Isom}_0 B \) (\( B = S^2, \mathbb{E}^2, \) or \( \mathbb{H}^2 \)),

then \( G \) is an extension

\[
1 \to \mathbb{R}^3 \to G \to \text{Isom}_0 \mathbb{E}^2 \to 1.
\]

**Proposition 6.13.** The 5-dimensional maximal model geometries \( M = G/G_p \) where \( T_1 G \) is an extension

\[
0 \to \mathbb{R}^3 \to T_1 G \to \text{isom} \mathbb{E}^2 \to 0
\]

are the nilpotent Lie groups

(i) \( M = \mathbb{R}^4 \rtimes_A \mathbb{R} \) with \( A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \)

(ii) \( M \) has Lie algebra with basis \( \{x, y, u, v, w\} \) and

\[
[x, w] = u \quad [y, w] = v \quad [x, y] = w.
\]
On both of these, the point stabilizer is an $S^1$'s worth of Lie group automorphisms at the identity, acting as the diagonal circle in $SO(2) \times SO(2)$; and the full isometry group is $M \rtimes S^1$.

The proofs will set up the tools for solving all thirty extension problems—i.e. for classifying extensions of $Q$ by $H$ for all combinations of the three remaining values of $Q$ and the ten values of $H$ (See Lemma 6.1). The strategy is as follows. Passing to Lie algebras, a (non-split) extension

$$0 \to h \to g \to q \to 0$$

is given by the homomorphism $q \to \text{out} h$ and a (nonzero) class in $H^2(q; Z(h))$ (Thm. 2.6). Since each $Q$ is unimodular and a model geometry’s transformation group is unimodular [Fil83, Prop. 1.1.3], the action on $h$ must be by traceless derivations (Part II, [Gen16b, Lemma 5.9]). Hence a starting point would be to compute $Z(h)$ and the traceless outer derivation algebra $\text{sout}(h)$ for each $h$ (Table 6.14).

This data and the actions of $q$ on $h$ are computed in Section 6.3.1. The reduction to extensions of $\text{isom}\mathbb{E}^2$ by $\mathbb{R}^3$ (Lemma 6.12) is proven in Section 6.3.2). Finally, Section 6.3.3) classifies the geometries resulting from this extension (Prop. 6.13).

Table 6.14: Centers and traceless outer derivation algebras. $T_1\text{Heis}_3$ is written with basis $\{x, y, z\}$ where $[x, y] = z$. See Prop. 6.15 for details.

| $h$             | $Z(h)$ | $\text{sout}(h)$ |
|-----------------|--------|------------------|
| $\mathfrak{so}_3 \oplus \mathbb{R}$ | $0 \oplus \mathbb{R}$ | $0$ |
| $\mathfrak{sl}_2 \oplus \mathbb{R}$ | $0 \oplus \mathbb{R}$ | $0$ |
| $T_1 \text{Isom}_0 \mathbb{E}^2 \oplus \mathbb{R}$ | $0 \oplus \mathbb{R}$ | $\mathbb{R} \oplus \mathbb{R}$ |
| $T_1 \text{Isom}_0 \text{Heis}_3$ | $\mathbb{R}z$ | $\mathbb{R}$ |
| $\mathfrak{so}_3$ | $0$ | $0$ |
| $\mathfrak{sl}_2$ | $0$ | $0$ |
| $T_1 \text{Sol}^3$ | $0$ | $0$ |
| $T_1 \text{Isom}_0 \mathbb{E}^2$ | $0$ | $0$ |
| $T_1 \text{Heis}_3$ | $\mathbb{R}z$ | $\mathfrak{sl}_2$ |
| $\mathbb{R}^3$ | $\mathbb{R}^3$ | $\mathfrak{sl}_3$ |

6.3.1 Data for the extension problem

This section establishes the list of traceless outer derivation algebras in Table 6.14 and classifies the homomorphisms $q \to \text{sout} h$.

Lemma 6.15. The centers and traceless outer derivation algebras for the Lie algebras $h$ are as given in Table 6.14.

Proof. Explicit calculation of the centers is omitted; they are verifiable in any convenient basis.

Calculation of the derivation algebras uses the following shortcut: A derivation $d$ satisfies a Leibniz rule with respect to the Lie bracket; so if $d$ preserves some subset $A$ of $h$, then it also preserves the derived series, the lower central series, and the centralizer of $A$.

Lemma 6.9 provides the derivation algebras for $\mathfrak{so}_3$, $\mathfrak{sl}_2$, and $T_1 \text{Isom}_0 \mathbb{E}^2$. The entry for $\mathfrak{so}_3 \oplus \mathbb{R}$ follows since $\mathfrak{so}_3$ is the first term of the derived series and $\mathbb{R}$ is the center; $\mathfrak{sl}_2 \oplus \mathbb{R}$ is handled likewise. Since $\mathbb{R}^3$ is abelian, $\text{sout}(\mathbb{R}^3) = \mathfrak{sl}(\mathbb{R}^3)$. Only four cases then remain.
Case 1: $T_1 \text{Isom}_0 \mathbb{R}^2 \times \mathbb{R}$. The first term of the derived series is the translation subalgebra, and the center is $\mathbb{R}$. The same calculation as in Lemma 6.9 ensures that an outer derivation can be represented by $d$ which scales the translation subalgebra uniformly and has $dr \in \mathbb{R}$. Then if $d$ acts as the scalar $b$ on the translation subalgebra, tracelessness requires it to act as $-2b$ on $\mathbb{R}$. Writing this algebra as $\mathbb{R} \oplus \mathbb{R}$ is now just writing the matrices by which it acts on the span of $\mathbb{R}$ and $r$.

Case 2: $T_1 \text{Sol}^3$. For the basis $\{x, y, z\}$ of $T_1 \text{Sol}^3$ where $[z, x] = x$ and $[z, y] = -y$, the first term of the derived series is $\mathbb{R}x + \mathbb{R}y$. Subtracting inner derivations, we may assume $dz = az$ for some $a$. Then $dx = d[x, y] = [dz, x] + [z, dx] = ax + [z, dx]$; so $(d - a)x = [z, dx]$, which implies $dx \in \mathbb{R}x$. Then $[z, dx] = dx$ so the above implies $a = 0$, and we can subtract a multiple of $ad z$ to ensure $dx = 0$. Similarly, $dy \in \mathbb{R}y$, and tracelessness then implies $dy = 0$.

Case 3: $T_1 \text{Heis}^3$. (This calculation is outlined quickly in [Fil83, §6.3].) With the basis $\{x, y, z\}$ in which $[x, y] = z$, inner derivations account for the $\mathbb{R}x + \mathbb{R}y \to \mathbb{R}z$ component; so a derivation $d$ can be taken to preserve $\mathbb{R}x + \mathbb{R}y$. Since $\mathbb{R}z$ is central, $d$ preserves $\mathbb{R}z$. Then

$$\text{(tr } d)z = [dx, y] + [x, dy] + dz = d[x, y] + dz = 2dz,$$

so $dz = 0$ and $d$ acts with trace zero on $\mathbb{R}x + \mathbb{R}y$.

Case 4: $T_1 \text{Isom}_0 \text{Heis}^3$. Since $T_1 \text{Isom}_0 \text{Heis}^3$ admits $T_1 \text{Isom}_0 \mathbb{E}^2$ as a quotient, any outer derivation of the former induces an outer derivation of the latter. Hence (consulting Lemma 6.9) any outer derivation of the former can be represented by a derivation $d$ such that

$$dx = bx + c_1 z$$
$$dy = by + c_2 z$$
$$dr = c_3 z$$
$$dz = c_4 z$$

for some real $b$ and $c_i$. Tracelessness implies $c_4 = -2b$, and

$$-2bz = dz = [dx, y] + [x, dy] = 2b[x, y] = 2bz$$

implies $b = 0$. Then

$$dx = d[r, -y] = [dr, -y] + [r, -dy] = 0$$

and similarly $dy = 0$; so all that remains is $c_3$. 

Lemma 6.16. If $0 \to h \to q \to q \to 0$ arises from a 5-dimensional geometry, then $q \to \text{sout } h$ is
nonzero only in the following cases, all of which lift to maps \( q \rightarrow \text{Der} \, h \).

\[
\begin{align*}
\mathfrak{sl}_2 & \sim \text{sout} \, T_1 \text{Heis}_3 \\
\mathfrak{sl}_2 & \hookrightarrow \mathfrak{sl}_3 \cong \text{sout} \, \mathbb{R}^3 \\
\mathfrak{sl}_2 & \cong \mathfrak{so}_{2,1} \hookrightarrow \mathfrak{sl}_3 \cong \text{sout} \, \mathbb{R}^3 \\
\mathfrak{so}_3 & \hookrightarrow \mathfrak{sl}_3 \cong \text{sout} \, \mathbb{R}^3
\end{align*}
\]

\( T_1 \text{Isom}_0 \mathbb{E}^2 \cong \mathbb{R}^2 \not\supset \mathfrak{so}_2 \hookrightarrow \mathfrak{sl}_3 \cong \text{sout} \, \mathbb{R}^3 \) and its negative transpose

\[
\begin{align*}
T_1 \text{Isom}_0 \mathbb{E}^2 & \rightarrow \mathfrak{so}_2 \overset{t \neq 0}{\hookrightarrow} \mathfrak{sl}_2 \cong \text{sout} \, T_1 \text{Heis}_3 \\
T_1 \text{Isom}_0 \mathbb{E}^2 & \rightarrow \mathfrak{so}_2 \overset{t > 0}{\hookrightarrow} \mathfrak{sl}_3 \cong \text{sout} \, \mathbb{R}^3
\end{align*}
\]

Proof. If \( q \) is semisimple then only sout \( T_1 \text{Heis}_3 \cong \mathfrak{sl}_2 \mathbb{R} \) and sout \( \mathbb{R}^3 = \mathfrak{sl}_3 \mathbb{R} \) have high enough dimension to admit nonzero images of \( q \). By counting representations, the only such are the first four maps listed above.

Otherwise, \( q = T_1 \text{Isom}_0 \mathbb{E}^2 \). The proof proceeds similarly to Lemma 6.10, making note of the following facts.

1. Any homomorphism from \( T_1 \text{Isom}_0 \mathbb{E}^2 \) either is faithful or factors through the rotation part \( \mathfrak{so}_2 \) (Lemma 5.2).

2. Since \( q \rightarrow \text{sout} \, h \) comes from the conjugation action \( \text{Isom}_0 \mathbb{E}^2 \rightarrow \text{Out} \, H \), the generator \( r \in q \) of SO(2) generates a compact subgroup of \( \text{Out} \, H \).

Lifting to \( \text{Der} \, h \) will follow from observing that \( \text{sout} \, \mathbb{R}^3 = \text{Der} \, \mathbb{R}^3 \), and \( \text{sout} \, T_1 \text{Heis}_3 \) embeds in \( \text{Der} \, T_1 \text{Heis}_3 \) as the subalgebra preserving \( \mathbb{R}x + \mathbb{R}y \).

Case 1: \( \rho \) is faithful. Only \( \text{sout} \, T_1 \text{Heis}_3 \cong \mathfrak{sl}_2 \) (dimension 3) and \( \text{sout} \, \mathbb{R}^3 \cong \mathfrak{sl}_3 \) (dimension 8) have high enough dimension to admit a faithful image of \( T_1 \text{Isom}_0 \mathbb{E}^2 \). Since \( \mathfrak{sl}_2 \not\cong T_1 \text{Isom}_0 \mathbb{E}^2 \) (only the latter is solvable), a faithful image must land in \( \mathfrak{sl}_3 \).

Let \( T_1 \text{Isom}_0 \mathbb{E}^2 \) have basis \( \{x, y, r\} \) with nonzero brackets \( [r, x] = y \) and \( [r, y] = -x \). An embedding sends \( x \) and \( y \) to commuting, similar matrices.Consulting the list of 2-dimensional abelian subalgebras of \( \mathfrak{sl}_3 \mathbb{R} \) (Part II, Lemma 5.13) yields exactly two embeddings \( T_1 \text{Isom}_0 \mathbb{E}^2 \hookrightarrow \mathfrak{sl}_3 \) up to conjugacy (omitted entries are zero):

\[
ax + by + cr \mapsto \begin{pmatrix}
-c & a \\
c & b \\
0 & 0
\end{pmatrix}
\text{ or } \begin{pmatrix}
-c \\
c \\
-a & -b & 0
\end{pmatrix}.
\]

Case 2: \( \rho \) factors through \( \mathfrak{so}_2 \). From Table 6.14, the volume-preserving parts of \( \text{Out} \,(\text{Isom}_0 \mathbb{E}^2 \times \mathbb{R}) \) and \( \text{Out} \,(\text{Isom}_0 \text{Heis}_3) \) are isomorphic to \( \mathbb{R} \times \mathbb{R} \) and \( \mathbb{R} \) respectively, neither of which has nontrivial compact subgroups. So for a map factoring through \( \mathfrak{so}_2 \) by which \( r \) generates a compact subgroup of \( \text{Out} \, H \), still only \( \mathfrak{sl}_2 \) and \( \mathfrak{sl}_3 \) admit nonzero images. In both of these, all images of \( \mathfrak{so}_2 \) are conjugate, so the only flexibility is in how \( \mathfrak{so}_2 \) maps to this image. \( \square \)
6.3.2 Reduction to one extension problem

Using the data computed in the previous section, this section proves Lemma 6.12—the claim that it suffices to consider extensions of $\text{Isom}_0 \mathbb{E}^2$ by $\mathbb{R}^3$. The idea is to show that the maximal model geometries produced from all other non-split extensions can be produced from split extensions that will be handled in later sections.

Proof of Lemma 6.12. Under the standing assumptions, $G$ is an extension of $Q = \text{Isom}_0 B$ where $B = S^2$, $\mathbb{E}^2$, or $\mathbb{H}^2$; and passing to Lie algebras yields a non-split extension

$$0 \to h \to g \to q \to 0.$$

Step 1: $q$ is $T_1 \text{Isom}_0 \mathbb{E}^2$. The other possibilities for $q$ are semisimple. The first and second cohomology of a semisimple Lie algebra vanish in any coefficient system [GOV94, Thm. 1.3.2]. So if $q$ were semisimple, the classification of these extensions by $H^2(q; Z(h))$ (Thm. 2.6) would imply that $g$ is a split extension of $q$.

Step 2: If $h \neq \mathbb{R}^3$, then $q$ acts trivially on a 1-dimensional $Z(h)$. Since $H^2(q; Z(h))$ must be nonzero for a non-split extension to exist, $Z(h)$ is nonzero. Consulting the list of derivation algebras (Table 6.14) and the nontrivial actions (Lemma 6.16), the dimension of $Z(h)$ is 1, and the action of $q$ on it is trivial. We now aim to show that the resulting geometries are either non-maximal or realized with direct-product transformation groups $\tilde{G}$, which are handled later in Section 6.5.

Step 3: $H^2(q; Z(h)) \cong \mathbb{R}$. With trivial action, the cocycle condition reduces to

$$\sum_{\text{cyc}} c(x_1, [x_2, x_3]) = 0.$$

This expression is trilinear, so to check that it is zero, it suffices to check basis elements. If two of the basis elements are the same, then antisymmetry of $c$ and the Lie bracket imply the value is zero. Since $q$ is 3-dimensional, only one combination then needs to be checked. Letting $q$ have basis $\{x, y, r\}$ where $[r, x] = y$ and $[r, y] = -x$,

$$c(x, [y, r]) + c(y, [r, x]) + c(r, [x, y]) = c(x, x) + c(y, y) + c(r, 0) = 0 + 0 + 0 = 0;$$

so every linear map $\Lambda^2 q \to Z(h)$ is a cocycle.

With trivial action, coboundaries are maps $\Lambda^2 q \to Z(h)$ which factor through the Lie bracket; so coboundaries account for any nonzero values on (and only on) $r \wedge x$ and $r \wedge y$. Therefore $H^2(q; Z(h)) \cong \mathbb{R}$, generated by a cocycle that is nonzero on $x \wedge y$ and zero on $r \wedge x$ and $r \wedge y$.

Step 4: Describe the resulting groups $G$. Since $q \to \text{sout} h$ lifts to $\text{Der} h$ (Lemma 6.16), the Lie algebra structure of $g$ can be recovered from by interpreting the cocycle as a $Z(h)$-valued bracket on $q$, as follows. On the vector space $h \oplus q$, define, following [AMR00, Eqn. 5.5],

$$[h_1 + q_1, h_2 + q_2] = q_1(h_2) - q_2(h_1) + c(q_1, q_2) + [q_1, q_2].$$

If $h$ is one of the direct sums $h' \oplus \mathbb{R}$, then $Z(h)$ is this $\mathbb{R}$ summand. Using the above formula,

$$g \cong h' \oplus (n_3 \cong \mathfrak{so}_2) \cong h' \oplus T_1 \text{Isom}_0 \text{Heis}_3.$$
Then $\tilde{G}$ is one of the direct products below, handled later in Section 6.5.

$$S^3 \times \text{Isom}_0 \text{Heis}_3 \quad \text{PSL}(2, \mathbb{R}) \times \text{Isom}_0 \text{Heis}_3 \quad \text{Isom}_0 \mathbb{E}^2 \times \text{Isom}_0 \text{Heis}_3$$

Otherwise, $\mathfrak{h}$ is $T_1 \text{Heis}_3$ or $T_1 \text{Isom}_0 \text{Heis}_3$. Then rescaling the $x, y$ plane gives an isomorphism of the resulting simply-connected group $\tilde{G}$ with either $\text{Heis}_3 \times \mathbb{R}$ or $\text{Heis}_5 \times \mathbb{R}^2$, with the action by rotations. Point stabilizers $\tilde{G}_p$ act semisimply since they act by some sort of rotations, while the nilradical $\text{Heis}_5$ acts nilpotently; so $\text{Heis}_5$ meets the point stabilizer trivially and therefore acts freely on any resulting geometry $\tilde{G}/\tilde{G}_p$. Then $\tilde{G}/\tilde{G}_p \cong \text{Heis}_5$, which is non-maximal since $\text{Heis}_5 \times \text{U}(2)/\text{U}(2)$ subsumes it.

\[\square\]

### 6.3.3 The geometries

Having established that geometries requiring non-split extensions in the extension problem (Lemma 6.1) are accounted for by extensions of $\text{isom } \mathbb{E}^2$ by $\mathbb{R}^3$ (Lemma 6.12), we now classify the resulting geometries.

**Proof of Prop. 6.13 (Classification when $T_1 G$ is an extension of $\text{isom } \mathbb{E}^2$ by $\mathbb{R}^3$).** The key observation is the first step, that $M$ is a nilpotent Lie group; after that the problem reduces to computing some cohomology (Step 2) and some automorphism groups (Step 3).

**Step 1:** $M$ is the simply-connected nilpotent Lie group whose Lie algebra is the induced extension of $\mathbb{R}^2$ by $\mathbb{R}^3$. The inclusion of $\mathbb{R}^2$ as the translation subalgebra of $\text{isom } \mathbb{E}^2$ induces an inclusion of extensions

$$0 \longrightarrow \mathbb{R}^3 \longrightarrow T_1 G \longrightarrow \text{isom } \mathbb{E}^2 \longrightarrow 0$$

$$\Downarrow \quad J$$

$$0 \longrightarrow \mathbb{R}^3 \longrightarrow \mathfrak{m} \longrightarrow \mathbb{R}^2 \longrightarrow 0.$$

Since $\mathbb{R}^2$ and $\mathbb{R}^3$ are abelian and $\mathbb{R}^2$ acts nilpotently (see the list of actions in Lemma 6.16), $\mathfrak{m}$ is nilpotent. Moreover $\mathbb{R}^2$ is an ideal in $\text{isom } \mathbb{E}^2$, so $\mathfrak{m}$ is an ideal in $T_1 G$, with codimension 1. Since $T_1 G$ is not nilpotent, having the non-nilpotent $\text{isom } \mathbb{E}^2$ as a quotient, $\mathfrak{m}$ is the nilradical of $T_1 G$. Then $T_1 G$ is the extension

$$0 \longrightarrow \mathfrak{m} \longrightarrow T_1 G \longrightarrow \mathfrak{so}_2 \longrightarrow 0,$$

which splits since any linear map from $\mathfrak{so}_2 \cong \mathbb{R}$ is a Lie algebra homomorphism.

Since $G_p$ is compact, it acts semisimply in the adjoint representation on $T_1 G$; while the nilradical of $T_1 G$ acts nilpotently. Then $T_1 G_p \cap \mathfrak{m} = 0$—so $T_1 G_p$ lies over $\mathfrak{so}_2$ in the above extension, and $\mathfrak{m}$ is tangent to a group acting transitively on $M = G/G_p$. Since $\pi_1(M) = 0$ and $\dim M = \dim \mathfrak{m}$, this action is free, which identifies $M$ with the simply-connected group whose Lie algebra is $\mathfrak{m}$ once a basepoint is chosen.

**Step 2:** List candidate nilpotent Lie algebras $\mathfrak{m}$. This step proceeds using the classification of extensions by second cohomology (Thm. 2.6). Since $\mathbb{R}^2$ is 2-dimensional, it has no 3-cocycles; so every $\Lambda^2 \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is a cocycle. Let $\{x, y\}$ be a basis for $\mathbb{R}^2$; then coboundaries take the form $x \wedge y \mapsto x \cdot \beta(y) - y \cdot \beta(x)$, where $\cdot$ denotes the action of $\mathbb{R}^2$ on $\mathbb{R}^3$. Then (like in Part II, Lemma 5.15) coboundaries account for values lying in $x \cdot \mathbb{R}^3 + y \cdot \mathbb{R}^3$, and

$$H^2(\mathbb{R}^2; \mathbb{R}^3) \cong \mathbb{R}^3/(x \cdot \mathbb{R}^3 + y \cdot \mathbb{R}^3).$$

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Using the list of actions of $\text{isom} E^2$ on $\mathbb{R}^3$ in Lemma 6.16), one action has $x$ and $y$ both acting as zero. In that case, every cocycle represents its own class, and $g'$ is either isomorphic to $\mathbb{R}^5$ (zero class) or $R^2 \oplus T_1 \text{Heis}^3$ (nonzero class, normalized by scaling and conjugating in $\text{GL}(3, \mathbb{R}) = \text{Aut} \mathbb{R}^3$).

Letting $\{u, v, w\}$ denote a basis of $\mathbb{R}^3$, the two nontrivial actions are
\[
\begin{align*}
[x, w] &= u \\
[y, w] &= v
\end{align*}
\]
\[
\text{and } \begin{align*}
[x, u] &= w \\
[y, v] &= w.
\end{align*}
\]
On the left, the split extension can be re-expressed as the semidirect sum of $\mathbb{R} w$ acting on the span of the other four basis elements with two $2 \times 2$ Jordan blocks of eigenvalue $0$—call this $\mathbb{R}^4 \oplus x^2, x^2 \mathbb{R}$. On the right, the split extension is the 5-dimensional Heisenberg algebra $T_1 \text{Heis}^5$.

Up to changes of coordinates in the $x,y$ and $u,v$ planes, the non-split extensions are accounted for by, respectively,
\[
[x, y] = w \quad \text{or} \quad [x, y] = u.
\]

**Step 3: Eliminate $m$ whose automorphism group is either too large or too small.** The maximal geometry realizing $M$ is $M \rtimes K/K$ where $K$ is maximal compact in $\text{Aut} M$ (Lemma 2.3). Since the assumptions imply $\dim G = 6$, we require the maximal compact subgroup of $\text{Aut} M$ to be $S^1$.

In three of the groups $M$ produced in Step 2, the maximal compact subgroup of $\text{Aut} M$ is too large, which makes $G/G_p$ non-maximal.

\[
\text{Aut} \mathbb{R}^5 \cong \text{GL}(5, \mathbb{R}) \supset \text{SO}(5)
\]
\[
\text{Aut} (\mathbb{R}^2 \times \text{Heis}^3) \supset \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \supset \text{SO}(2) \times \text{SO}(2)
\]
\[
\text{Aut Heis}^5 \supset U(2) \text{ (see Section 4.2, Step 3)}
\]

If $m$ has basis $\{x, y, u, v, w\}$ with
\[
[x, u] = w \quad [y, v] = w \quad [x, y] = u,
\]
then any automorphism preserves the following filtration of characteristic ideals.

\[
Z(m) = \mathbb{R} w 
\]
\[
[m, m] = \mathbb{R} u + \mathbb{R} w
\]
\[
\text{The preimage of } Z(m) = \mathbb{R} u + \mathbb{R} v + \mathbb{R} w
\]
\[
\text{The centralizer of } [m, m] = \mathbb{R} y + \mathbb{R} u + \mathbb{R} v + \mathbb{R} w
\]

Therefore every automorphism is upper-triangular, so $\text{Aut} m$ contains no compact subgroups of positive dimension. Then $M \rtimes S^1$ cannot be defined with a nontrivial action by $S^1$.

**Step 4: Both remaining geometries are maximal.** What remains are the two possibilities for $M$ claimed in the statement of this Proposition. Their Lie algebras can be expressed with basis $\{x, y, u, v, w\}$ and
\[
[x, w] = u \quad [y, w] = v \quad [x, y] = w \text{ or } 0.
\]
Similarly to Step 3 above,
\[
Z(m) = \mathbb{R} u + \mathbb{R} v
\]
\[
[m, m] = \mathbb{R} u + \mathbb{R} v + \mathbb{R} w \text{ if } [x, y] = w
\]
\[
\{a \mid \text{rank ad } a \leq 1\} = \mathbb{R} x + \mathbb{R} y + \mathbb{R} u + \mathbb{R} v \text{ if } [x, y] = 0.
\]
Either way, every automorphism is block upper-triangular with two $2 \times 2$ blocks and one $1 \times 1$ block. The group of such block upper-triangular matrices has a torus $T^2$ (in each $2 \times 2$ block, something conjugate to a rotation) as its maximal compact subgroup (Part II, [Gen16b, Lemma 5.29]); so $\text{Aut } m$ has maximal compact subgroup inside that.

Up to scaling $x$, $y$, and possibly $w$, any automorphism in this $T^2$ merely rotates the $x, y$ plane. Since $ad w$ takes this plane to the $u, v$ plane, the rotation on the $u, v$ plane is determined; so $K \cong S^1$, acting by rotation at equal rates on these planes.

**Step 5: Both remaining candidates are model geometries.** A nilpotent Lie algebra with rational structure constants in some basis admits a cocompact lattice $\Gamma$ [Rag72, Thm 2.12]. In the bases used above, the structure constants are all integers; so in each case $\Gamma \backslash M / \{1\} \cong \Gamma \backslash M \rtimes S^1 / S^1$ is a compact manifold modeled on $M$.

### 6.4 Line bundles over $F^4$ and $T^1E^{1,2}$ (semidirect-product isometry groups)

Having just classified geometries $G/G_p$ where $T_1G$ is an extension of $\text{isom } E^2$ by $\mathbb{R}^3$ or any non-split extension, this section classifies the next cluster of geometries—those where $T_1G$ is a split extension, i.e. a semidirect sum $h \supset q$ with one of the remaining combinations of $h$ and $q$ that has a nontrivial action. Five combinations remain of the list from Lemma 6.16, producing the five groups in Table 6.19. In terms of which this section’s classification result is stated. We will verify that these are model geometries in Prop. 6.20, but maximality is deferred to Section 6.6 to take advantage of results from the remainder of the classification (Section 6.5).

**Proposition 6.17.** Let $M = G/G_p$ be a maximal model geometry such that $\tilde{G}$ is one of the semidirect products listed in Table 6.19. Let $\gamma: \mathbb{R} \to \tilde{\text{SL}}_2$ send $t$ to a rotation by $2\pi t$ radians, and fix a nontrivial $z \in Z(\text{Heis}_3)$. Then $M$ is one of

$$
\begin{align*}
\text{SAff } \mathbb{R}^2 &= \mathbb{R}^2 \rtimes \tilde{\text{SL}}_2 \cong (\mathbb{R}^2 \rtimes \tilde{\text{SL}}_2) \rtimes \text{SO}(2)/\text{SO}(2) \\
E \times F^4 &= \mathbb{R} \times \mathbb{R}^2 \rtimes \text{SL}(2, \mathbb{R})/\text{SO}(2) \\
F^a_5 &= \text{Heis}_3 \rtimes \tilde{\text{SL}}_2/\{atz, \gamma(t)\}_{t \in \mathbb{R}}, \quad a = 0 \text{ or } 1 \\
T^1E^{1,2} &= \mathbb{R}^3 \rtimes \text{SO}(1, 2)/\text{SO}(2).
\end{align*}
$$

The proof requires knowing the center of the semidirect product $\tilde{G}$; the following formula computes the centers listed in Table 6.19.

**Lemma 6.18.** If $C_x$ is conjugation by $x$, then

$$
Z(A \rtimes_{\phi} B) = \{(x, y) \in A^B \times Z(B) \mid \phi(y) = C_x^{-1}\}.
$$

In particular, if $\phi(B) \cap \text{Inn } A = \{1\}$, then

$$
Z(A \rtimes_{\phi} B) = Z(A)^B \times (Z(B) \cap \ker \phi).
$$

**Proof.** Suppose $xy = (x, y) \in Z(A \rtimes_{\phi} B)$. Then to commute with all $b \in B$,

$$
\begin{align*}
&b = xyb^{-1}x^{-1} \\
&bx = \phi(b)(x)b = xyb^{-1}.
\end{align*}
$$

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so $\phi(b)(x) = x$ and $b = yby^{-1}$ for all $b \in B$, i.e. $x \in A^B$ and $y \in Z(B)$. To commute with all $a \in A$,

$$a = xyay^{-1}x^{-1}$$

$$= x\phi(y)(a)x^{-1}$$

$$= C_x\phi(y)(a).$$

Since $A$ and $B$ generate $A \rtimes_{\phi} B$, these conditions suffice.

If $\phi(B) \cap \text{Inn } A = \{1\}$, then $\phi(y)$ is identity, and the second condition reduces to $x \in Z(A)$. \hfill \square

**Table 6.19:** Nontrivial semidirect products covering candidate isometry groups $G$

| $\tilde{G} = \tilde{H} \rtimes \tilde{Q}$ | center | notes on action |
|------------------------------------------|-------|-----------------|
| $\mathbb{R}^3 \rtimes \widetilde{\text{SO}(3)}$ | $\{0\} \times \mathbb{Z}/2\mathbb{Z}$ | standard representation |
| $\mathbb{R}^3 \rtimes \widetilde{\text{SO}(2,1)}$ | $\{0\} \times \mathbb{Z}$ | standard representation |
| $\text{Heis}_3 \rtimes \widetilde{\text{SL}(2,\mathbb{R})}$ | $\mathbb{R} \times \mathbb{Z}$ | as $\text{SL}(2,\mathbb{R})$ on $x,y$ plane |
| $\text{Heis}_3 \rtimes \text{Isom}_0 \mathbb{E}^2$ | $\mathbb{R} \times \mathbb{Z}$ | family of actions, factors through $\widetilde{\text{SO}(2)}$ |
| $\mathbb{R} \times \mathbb{R}^2 \rtimes \widetilde{\text{SL}(2,\mathbb{R})}$ | $\mathbb{R} \times \mathbb{Z}$ | trivial action on $\mathbb{R}$ |

**Proof of Prop. 6.17 (Classification when the isometry group is a semidirect product).** Suppose that $M = G/G_p$ is such a geometry (a maximal model geometry with $\tilde{G}$ listed in Table 6.19), and let $\tilde{G}_p$ be the preimage in $\tilde{G}$ of $G_p$.

**Preparatory step:** $\tilde{G}_p$ is a 1-parameter subgroup lying over a maximal torus of $\tilde{G}/Z(\tilde{G})$.

Since every candidate $G$ in Table 6.19 is 6-dimensional, $G_p$ and $\tilde{G}_p$ are 1-dimensional. The homotopy exact sequence for $\tilde{G}_p \to \tilde{G} \to M$ and the assumption that $\pi_1(M) \cong \pi_0(M) \cong 0$ imply that $\tilde{G}_p$ is connected. Moreover, $G_p = \tilde{G}_p/(\tilde{G}_p \cap Z(\tilde{G}))$ (Rmk. 6.4); so $G_p$ becomes a copy of $S^1$ in $\tilde{G}/Z(\tilde{G})$. For every candidate $\tilde{G}$ listed, $\tilde{G}/Z(\tilde{G})$ has maximal torus $\text{SO}(2)$, so $S^1$.

For every candidate in Table 6.19, the quotient $\tilde{G}/Z(\tilde{G})$ so in the contractible $\tilde{G}$, where 1-parameter subgroups are isomorphic to $\mathbb{R}$, the intersection of $\tilde{G}_p$ and $Z(\tilde{G})$ is nontrivial.

**Case 1:** If $\dim Z(\tilde{G}) = 0$ then $M = T^1\mathbb{E}^{1,2}$. If $\dim Z(\tilde{G}) = 0$ then $\tilde{G}$ covers $\tilde{G}/Z(\tilde{G})$; so $\tilde{G}_p$ is unique up to conjugacy in $\tilde{G}$. The resulting spaces, and the names chosen for them, are

$$\mathbb{R}^3 \rtimes \text{SO}(3)/\text{SO}(2) = T^1\mathbb{E}^3$$

$$\mathbb{R}^3 \rtimes \text{SO}(2,1)/\text{SO}(2) = T^1\mathbb{E}^{1,2}.$$

The final classification omits $T^1\mathbb{E}^3$ since it is non-maximal, being subsumed by $\mathbb{E}^3 \times S^2$. An interpretation of $T^1\mathbb{E}^{1,2}$, by analogy to the Euclidean (i.e. positive-definite) case, is as one connected component of the sub-bundle of $T\mathbb{R}^3$ consisting of the vectors $v$ where $\langle v, v \rangle = 1$ and $\langle \cdot, \cdot \rangle$ has signature $(+, -, -)$.  

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Case 2: If $\tilde{G} = \text{Heis}_3 \times \text{Isom}_0 \mathbb{E}^2$ then $M$ is a non-maximal $\text{Heis}_3 \times \mathbb{E}^2$. In this case, $\tilde{G}$ is an extension $1 \to \text{Heis}_3 \times \mathbb{R}^2 \to \tilde{G} \to \text{SO}(2) \to 1$, with $\tilde{G}_p$ surjecting onto $\text{SO}(2)$; so

$$G/G_p \cong (\text{Heis}_3 \times \mathbb{R}^2) \times \text{SO}(2)/\text{SO}(2).$$

Since $\text{Heis}_3 \times \mathbb{R}^2$ admits at least a torus’s worth of automorphisms (one $\text{SO}(2)$ acting by rotation on $\mathbb{R}^2$ and another acting by rotation on $\text{Heis}_3$), this geometry is subsumed by $(\text{Heis}_3 \times \mathbb{R}^2) \times \text{SO}(2)^2/\text{SO}(2)^2$.

Case 3: If $\tilde{G} = \text{Heis}_3 \times \text{SL}(2, \mathbb{R})$ then $M$ is $F_0^5$ or $F_1^5$. Let $\gamma : \mathbb{R} \to \SL(2, \mathbb{R}) \subset \tilde{\text{SL}(2, \mathbb{R})}$ send $t$ to a rotation by $2\pi t$, and put coordinates on $\text{Heis}_3$ so that

$$(x, y, z)(x'y'z') = (x + x', y + y', z + z + xy').$$

Since $Z(\tilde{G})$ is 1-dimensional, the groups $\tilde{G}_p$ lying over $\SO(2)$ form a 1-dimensional family: for $a \in \mathbb{R}$, let

$$F_a^5 = \text{Heis}_3 \times \tilde{\text{SL}(2, \mathbb{R})}/\{(0, 0, at), \gamma(t)\}_{t \in \mathbb{R}}.$$  

Conjugating $\tilde{\text{SL}(2, \mathbb{R})}$ by a reflection in $O(2)$ would exchange $\gamma(t)$ with $\gamma(-t)$; so up to this conjugation we may assume $a \geq 0$. If $a > 0$, conjugating $\text{Heis}_3$ by the automorphism

$$(x, y, z) \mapsto \left(xa^{-1/2}, ya^{-1/2}, za^{-1}\right)$$

allows assuming $a = 1$; so $M = G/G_p$ is equivariantly diffeomorphic to either $F_0^5$ or $F_1^5$. The two cases are distinguished by whether a Levi subgroup of $G$ is isomorphic to $\text{SL}(2, \mathbb{R})$ ($a = 0$) or its universal cover ($a = 1$).

Case 4: If $\tilde{G} = \mathbb{R} \times \mathbb{R}^2 \rtimes \tilde{\text{SL}(2, \mathbb{R})}$ then $M$ is $\mathbb{E} \times \mathbb{F}^4$ or $\mathbb{R}^2 \rtimes \tilde{\text{SL}(2, \mathbb{R})}$. As in Case 3, conjugation by an element of $O(2)$ and a rescaling of the $\mathbb{R} = Z(\tilde{G})^0$ factor allow assuming

$$G/G_p \cong \mathbb{R} \times \mathbb{R}^2 \rtimes \tilde{\text{SL}(2, \mathbb{R})}/\{(at, 0, 0), \gamma(t)\}_{t \in \mathbb{R}}$$

with $a = 0$ ($M = \mathbb{E} \times \mathbb{F}^4$) or $a = 1$ ($M = \mathbb{R}^2 \rtimes \tilde{\text{SL}(2, \mathbb{R})}$); and the two are again distinguished by Levi subgroups.

Proposition 6.20. All geometries listed in Prop. 6.17 are model geometries.

Proof. The kernel $\Gamma_3$ of $\text{SL}(2, \mathbb{Z}) \to \text{SL}(2, \mathbb{Z}/3\mathbb{Z})$ is a torsion-free lattice in $\text{SL}(2, \mathbb{R})$ [Mor15, Thm. 4.8.2 Case 1]; so its lift $\tilde{\Gamma}_3$ is a torsion-free lattice in the infinite cyclic cover $\tilde{\text{SL}(2, \mathbb{R})}$. Then $\Gamma_3$ (resp. $\tilde{\Gamma}_3$) acts without fixed points (i.e. freely) anywhere that $\text{SL}(2, \mathbb{R})$ (resp. $\tilde{\text{SL}(2, \mathbb{R})}$) acts faithfully. So finite-volume manifolds modeled on three of the geometries from Prop. 6.17 can be constructed using these groups and are listed in Table 6.21.
Table 6.21: Finite-volume manifolds $\Gamma \backslash G/K$ modeled on $G/K$ fibering over $\mathbb{F}^4$

| $\Gamma$ | $G$ | $K$ | $G/K$ |
|----------|-----|-----|-------|
| $\mathbb{Z}^2 \times \tilde{\Gamma}_3$ | $(\mathbb{R}^2 \times \mathrm{SL}(2, \mathbb{R})) \rtimes \mathrm{SO}(2)$ | $\mathrm{SO}(2)$ | $\mathbb{R}^2 \times \mathrm{SL}_2$ |
| $\mathbb{Z} \times \mathbb{Z}^2 \times \Gamma_3$ | $\mathbb{R} \times \mathbb{R}^2 \times \mathrm{SL}_2$ | $\mathrm{SO}(2)$ | $\mathbb{E} \times \mathbb{F}^4$ |
| $\mathrm{Heis}_3(\mathbb{Z}) \times \Gamma_3$ | $\mathrm{Heis}_3 \times \mathrm{SL}(2, \mathbb{R})$ | $\mathrm{SO}(2)$ | $\mathbb{F}_5^5$ |

**Special case 1:** $\mathbb{F}_5^5$. There is one complication: while $\Lambda = \mathrm{Heis}_3(\mathbb{Z}) \times \tilde{\Gamma}_3$ is a torsion-free lattice in $\tilde{G}$, we must verify that it descends to a torsion-free lattice in $G$. Observe that $\tilde{G}_p = \{(0, 0, t), \gamma(t)\}_{t \in \mathbb{R}}$, where $\gamma(t)$ is rotation by $2\pi t$, meets $\Lambda$ in $\{(0, 0, t), \gamma(t)\}_{t \in \mathbb{Z}} = \tilde{G}_p \cap Z(\tilde{G})$. Then $\Lambda$ remains discrete in $G = \tilde{G}/(\tilde{G}_p \cap Z(\tilde{G}))$.

Only elliptical elements can become torsion in a quotient of $\tilde{\mathrm{SL}}(2, \mathbb{R})$, and the elliptical elements of the torsion-free $\tilde{\Gamma}_3$ all lie over the identity. Then anything in $\Lambda$ that becomes torsion in $G$ lies over the identity of $\mathbb{R}^2 \times \mathrm{SL}(2, \mathbb{R})$—i.e. in $Z(\tilde{G}) \cong \mathbb{R} \times \mathbb{Z}$. Since $Z(\tilde{G})$ has image in $G$ isomorphic to $Z(\tilde{G})/(\tilde{G}_p \cap Z(\tilde{G})) \cong \mathbb{R}$, the image in $G$ of $\Lambda$ is torsion-free.

Then $\Lambda \backslash \mathbb{F}_5^5$ is a manifold, with finite volume since $\Lambda$ is a lattice.

**Special case 2:** $T^1E^{1,2}$. By the classification of representations of $\tilde{\mathrm{SL}}(2, \mathbb{R})$ [FH91, 11.8], the action of $\mathrm{SL}(2, \mathbb{R})$ on the space $\text{Sym}^2 \mathbb{R}^2$ of symmetric $2 \times 2$ matrices given by

$$g \cdot M = gMg^T$$

factors through the standard representation of $\mathrm{SO}(2, 1)$. Since elements $\Gamma_3$ have integer entries, the action by $\Gamma_3$ preserves the subgroup $\Lambda$ consisting of symmetric integer matrices. Then $\Lambda \rtimes \Gamma_3$ descends to a lattice in $\mathbb{R}^3 \rtimes \mathrm{SO}(2, 1)$. This lattice is torsion-free: since $\mathrm{SL}(2, \mathbb{R})$ double covers $\mathrm{SO}(2, 1)$, an $n$-torsion element of $\mathbb{R}^3 \rtimes \mathrm{SO}(2, 1)$ lifts to a $2n$-torsion element of $\mathbb{R}^3 \rtimes \mathrm{SL}(2, \mathbb{R})$, and $\Lambda \rtimes \Gamma$ was constructed to be torsion-free.

Therefore $(\Lambda \rtimes \Gamma_3) \backslash T^1E^{1,2}$ is a finite-volume manifold modeled on $T^1E^{1,2}$. 

**Remark 6.22.** Maximality will need to wait until the classification is more complete, since the proof depends on knowing all geometries with point stabilizer containing $S^1_1$ or $S^1_{1/2}$. The proof is eventually given in Prop. 6.41.

### 6.5 Product geometries and associated bundles

The last piece of the classification, necessarily a sort of catch-all for the leftovers, is this section’s main result.

**Proposition 6.23.** The 5-dimensional maximal model geometries $G/G_p$ for which $\tilde{G}$ occurs in Table 6.24 are

(i) products of 2- and 3-dimensional geometries and
(ii) the following “associated bundle” geometries.

\[
\begin{align*}
\text{Heis}_3 \times \mathbb{R} S^3 & \quad \text{SL}_2 \times \alpha S^3, \quad 0 < \alpha < \infty \\
\text{Heis}_3 \times \mathbb{R} \widetilde{\text{SL}}_2 & \quad \widetilde{\text{SL}}_2 \times \alpha \widetilde{\text{SL}}_2, \quad 0 < \alpha \leq 1 \\
L(a; 1) \times S^1 & \quad L(b; 1), \quad 0 < a \leq b \text{ coprime in } \mathbb{Z}
\end{align*}
\]

After the preceding sections, all that remain are geometries where \( \tilde{G} \) is a split extension with trivial action—i.e. direct products, which Table 6.24 lists with duplicates hidden. We start with a study the associated bundle geometries from several viewpoints in Section 6.5.1, since the vocabulary for describing them is useful in proving the classification in Prop. 6.23 (Section 6.5.2). For completeness, the product geometries are listed in Section 6.5.3. Maximality for non-products is deferred to Section 6.6.

Table 6.24: Groups covering the isometry group \( G \) of geometries in this section

| \( B \) | \( \text{E}^2 \) | \( \text{S}^2 \) | \( \mathbb{H}^2 \) |
|---|---|---|---|
| \( \text{Isom} \mathbb{E}^2 \times \mathbb{R} \) | \( \text{Isom}_0 \mathbb{E}^2 \times \text{Isom}_0 \mathbb{E}^2 \times \mathbb{R} \) | \( \text{Isom}_0 \mathbb{E}^2 \times S^3 \times \mathbb{R} \) | \( \text{Isom}_0 \mathbb{E}^2 \times \text{PSL}(2, \mathbb{R}) \times \mathbb{R} \) |
| \( \text{so}_3 \times \mathbb{R} \) | (duplicate) | \( S^3 \times S^3 \times \mathbb{R} \) | \( S^3 \times \text{PSL}(2, \mathbb{R}) \times \mathbb{R} \) |
| \( \text{sl}_2 \times \mathbb{R} \) | (duplicate) | \( \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R}) \times \mathbb{R} \) |
| \( \text{Isom} \text{Heis}_3 \) | \( \text{Isom}_0 \text{Heis}_3 \times \text{Isom}_0 \mathbb{E}^2 \) | \( \text{Isom}_0 \text{Heis}_3 \times S^3 \) | \( \text{Isom}_0 \text{Heis}_3 \times \text{PSL}(2, \mathbb{R}) \) |
| \( \text{Isom} \mathbb{E}^2 \) | \( \text{Isom}_0 \mathbb{E}^2 \times \text{Isom}_0 \mathbb{E}^2 \) | \( \text{Isom}_0 \mathbb{E}^2 \times S^3 \) | \( \text{Isom}_0 \mathbb{E}^2 \times \text{PSL}(2, \mathbb{R}) \) |
| \( \text{so}_3 \) | (duplicate) | \( S^3 \times S^3 \) | \( S^3 \times \text{PSL}(2, \mathbb{R}) \) |
| \( \text{sl}_2 \) | (duplicate) | \( \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R}) \) |
| \( T_1 \text{Sol}^3 \) | \( \text{Sol}^3 \times \text{Isom}_0 \mathbb{E}^2 \) | \( \text{Sol}^3 \times S^3 \) | \( \text{Sol}^3 \times \text{PSL}(2, \mathbb{R}) \) |
| \( T_1 \text{Heis}_3 \) | \( \text{Heis}_3 \times \text{Isom}_0 \mathbb{E}^2 \) | \( \text{Heis}_3 \times S^3 \) | \( \text{Heis}_3 \times \text{PSL}(2, \mathbb{R}) \) |
| \( \mathbb{R}^3 \) | (Prop. 6.13) | \( \mathbb{R}^3 \times S^3 \) | \( \mathbb{R}^3 \times \text{PSL}(2, \mathbb{R}) \) |

6.5.1 Associated bundle geometries

This section is a study of the associated bundle geometries—the only geometries in this classification that have abelian isotropy and are not products. The most concise names, the classification, and the verification of maximality all use slightly different constructions of these spaces; so this section will provide the notation for and relationships between all three constructions. The first definition motivates the name “associated bundle”.

**Definition 6.25** (Associated bundles, see e.g. [Hus94, Defn. 5.1]). Let \( E \rightarrow B \) be a principal \( G \)-bundle, and let \( \rho : G \rightarrow \text{Aut } F \) be an action of \( G \) on some space \( F \). The \( F \)-bundle \( E \times_\rho F \) associated to \( E \rightarrow B \) is the bundle over \( B \) whose total space is

\[
E \times F/(e, f) \sim (eg, \rho(g^{-1})f) \text{ for } g \in G.
\]
Example 6.26 (Associated bundle geometries). For homogeneous $E = H/H_p$ and $F = K/K_q$, suppose $G$ is a 1-dimensional subgroup of $H$ commuting with $H_p$ by which it acts on $E$ on the right, and $\rho(G)$ is central in $K$. Then

$$E \times_{\rho} F = (H \times K)/(H_p \cdot K_q \cdot \{(g, \rho(g)^{-1}) \mid g \in G\}).$$

In particular we may take $E$ and $F$ each to be one of the homogeneous spaces

- $\text{Heis}_3 \cong \text{Heis}_3 \times \text{SO}(2)/\text{SO}(2)$
- $L(a; 1) \cong S^3 \times S^1/(e^{(2\pi i/a)} \times S^1)$
- $\text{SL}_2 \cong \text{SL}_2 \times \text{SO}(2)/\text{SO}(2)$,

with $G$ or $\rho(G)$ the identity component of the center in each case.

The lens space $L(a; 1)$ is $S^3/e^{(2\pi i/a)}Z$ where $S^3$ is the unit quaternions and $e^{(2\pi i/a)}Z = \mathbb{Z}/a\mathbb{Z} \subset S^1 \subset \mathbb{R} + \mathbb{R}i$. In particular, $L(1; 1) \cong S^3$ and $L(2; 1) \cong \mathbb{R}P^3$.

Our notation is modeled on that in [Sha00, §1.3] surrounding the “Vector Bundles” subsection—we denote such a bundle by $E \times_G F$ when there is a natural choice of $G$-action on $F$, otherwise $E \times_* F$ where $*$ is enough data to specify the $G$-action. Since $G$ is 1-dimensional, a real number suffices to express this data once some conventions are chosen.

The next definition is a description of the associated bundle geometries as homogeneous spaces in terms of data arising from the classification strategy.

Definition 6.27 (Point stabilizers of associated bundle geometries). Let $\hat{G}$ be a connected, simply-connected Lie group, and let $\pi$ denote the quotient map $\hat{G} \to \hat{G}/Z(\hat{G})$. Suppose $Z(\hat{G})$ is 1-dimensional, $\hat{G}/Z(\hat{G})$ has a 2-dimensional maximal torus $T$, and $\tau_{0,0} : \mathbb{R}^2 \to \hat{G}$ is a homomorphism such that $\pi \circ \tau_{a,b}$ has image $T$ and kernel the standard $\mathbb{Z}^2$. For $a$ and $b$ in $\mathbb{R} \cong Z(\hat{G})^0$, define\(^7\)

$$\tau_{a,b} : \mathbb{R}^2 \to \hat{G}$$

$$x, y \mapsto (xa + yb)\tau_{0,0}(x, y).$$

The final definition is a notation for line bundles and circle bundles over products of 2-dimensional geometries. Its resemblance to the 3-dimensional geometries with $\text{SO}(2)$ isotropy may provide some intuition and will be used to distinguish the associated bundle geometries from each other and from other geometries.

Definition 6.28 (Associated bundle geometries as bundles over products). Let $X$ and $Y$ be $\mathbb{E}^2$, $S^2$, or $\mathbb{H}^2$, scaled to have curvature 0 or $\pm 1$; and let $H$ be $\mathbb{R}$ or $S^1 \cong \mathbb{R}/\mathbb{Z}$. Given $a$ and $b$ in $T_1 H$, let $\xi^{a,b} \to X \times Y$ denote the simply-connected principal $H$-bundle over $X \times Y$ with a connection whose curvature form is

$$\Omega_{a,b} = \frac{1}{2\pi} (\text{vol}_X \otimes a + \text{vol}_Y \otimes b).$$

The correspondence between these three definitions is summarized in Table 6.29. The proof of the correspondence for the “associated bundle” column is written out only for the first row for illustrative purposes, since this part of the correspondence is only used in the hope of selecting an evocative name. The proof for the rest of the correspondence is Prop. 6.34.

---

\(^7\) $Z(\hat{G})^0 \cong \mathbb{R}$ follows from computing its $\pi_1$ using the homotopy exact sequence for $Z(\hat{G}) \to \hat{G} \to \hat{G}/Z(\hat{G})$ and the fact that $\pi_2 = 0$ for Lie groups [BD85, Prop. V.7.5].
Proposition 6.30. Let \( L(a; 1) \times_{S^1} L(b; 1) \) denote the associated bundle geometry as defined in Defn. 6.25, where the fiber \( S^1 \) of \( L(a; 1) \to S^2 \) by translating along fibers of \( L(b; 1) \to S^2 \), with kernel \( \mathbb{Z}/d\mathbb{Z} \). Then the homogeneous space

\[
S^3 \times S^3 \times \mathbb{R} / \tau_{a,b}(\mathbb{R}^2) \cong S^3 \times S^3 \times \mathbb{R} / \{ e^{\pi i s}, e^{\pi i t}, a d s + b t \}_{s,t \in \mathbb{R}}.
\]

is a \( \text{gcd}(ad, b) \)-fold cover of \( L(a; 1) \times_{S^1} L(b; 1) \).

The notation interprets \( S^3 \) as the unit quaternions and any complex numbers as lying inside the quaternions. It follows from the above (Prop. 6.30) that

1. \( L(a; 1) \times_{S^1} L(b; 1) \cong L(ad; 1) \times_{S^1} L(b; 1) \);
2. \( L(a; 1) \times_{S^1} L(b; 1) \cong L(b; 1) \times_{S^1} L(a; 1) \); and
3. \( L(a; 1) \times_{S^1} L(b; 1) \) is simply-connected if and only if \( \text{gcd}(a, b) = 1 \). (Use the homotopy exact sequence for \( \tau_{a,b}(\mathbb{R}^2) \to S^3 \times S^3 \times \mathbb{R} \to L(a; 1) \times_{S^1} L(b; 1) \).)

Just one key Lemma is required for its proof.

Lemma 6.31. Let \( \gamma : H \to G \) be a homomorphism, and let \( C : G \to \text{Inn} G \) denote conjugation. Then

\[
G \times H / \{(\gamma(h), h)\}_{h \in H} \cong G \rtimes_{C \gamma} H / H.
\]

Proof. Verify from the definition of a semidirect product that

\[
\phi : G \times H \to G \rtimes_{C \gamma} H
\]

\[
g, h \mapsto g \gamma(h)^{-1}, h.
\]

is an isomorphism sending \( \{(\gamma(h), h)\}_{h \in H} \) onto \( H \subseteq G \rtimes_{C \gamma} H \).

Proof of Prop. 6.30. Interpret \( S^3 \) as the unit quaternions, and all complex numbers as lying in the same copy of \( \mathbb{C} \) in the quaternions. Since semidirect products with inner action are isomorphic to direct products (Lemma 6.31),

\[
L(a; 1) = S^3 \times S^1 / \{ e^{(2\pi i / a)Z}, t \mod 2 \}_{t \in \mathbb{R}}
\]

\[
\cong S^3 \times \mathbb{R} / \{ e^{\pi i t + (2\pi i / a)Z}, t \}_{t \in \mathbb{R}}
\]

\[
\cong S^3 \times \mathbb{R} / \{ e^{\pi i t}, a t + 2\mathbb{Z} \}_{t \in \mathbb{R}}.
\]

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So with the formula for homogeneous associated bundles (Example 6.26),

$$L(a;1) \times_{S^1,d} L(b;1) = S^3 \times \mathbb{R} \times S^3 \times \mathbb{R}/\{e^{\pi i s}, as + 2Z + u, e^{\pi it}, bt + 2Z - du\}_{s,t,u\in \mathbb{R}}.$$  

Since $u$ ranges over all of $\mathbb{R}$, the entire first $\mathbb{R}$ factor in $S^3 \times \mathbb{R} \times S^3 \times \mathbb{R}$ is part of the point stabilizer. Since this $\mathbb{R}$ factor is normal, the homogeneous space is equivariantly diffeomorphic to a coset space of $S^3 \times S^3 \times \mathbb{R}$. To see exactly which coset space, let $u'$ be the coordinate in this $\mathbb{R}$ factor. Then

$$L(a;1) \times_{S^1,d} L(b;1) \cong S^3 \times \mathbb{R} \times S^3 \times \mathbb{R}/\{e^{\pi i s}, u', e^{\pi it}, bt + 2Z - d(u' - as - 2Z)\}_{s,t,u'\in \mathbb{R}}$$

$$\cong S^3 \times S^3 \times \mathbb{R}/\{e^{\pi i s}, e^{\pi it}, bt + 2Z + d(as + 2Z)\}_{s,t\in \mathbb{R}}$$

$$\cong S^3 \times S^3 \times \mathbb{R}/\{e^{\pi i s}, e^{\pi it}, ads + bt + 2Z\}_{s,t\in \mathbb{R}}.$$  

The resemblance to $S^3 \times S^3 \times \mathbb{R}/\tau_{ad,b}(\mathbb{R}^2)$ is now apparent. To finish the proof, observe that since $e^{2\pi i} = 1$,

$$S^3 \times S^3 \times \mathbb{R}/\tau_{ad,b}(\mathbb{R}^2) \cong S^3 \times S^3 \times \mathbb{R}/\{e^{\pi i s}, e^{\pi it}, ads + bt\}_{s,t\in \mathbb{R}}$$

$$\cong S^3 \times S^3 \times \mathbb{R}/\{e^{\pi i s}, e^{\pi it}, ad(s + 2Z) + b(t + 2Z)\}_{s,t\in \mathbb{R}}$$

$$\cong S^3 \times S^3 \times \mathbb{R}/\{e^{\pi i s}, e^{\pi it}, ads + bt + 2\gcd(ad,b)Z\}_{s,t\in \mathbb{R}}.$$  

The rest of this sub-section is mostly devoted to proving the correctness of Table 6.29 other than in the third column. Before doing so, having established some understanding above of the $L(a;1) \times_{S^1} L(b;1)$ family of geometries, we observe one restriction on the parameters $a$ and $b$ that occurs only for this family.

**Proposition 6.32.** $\tau_{a,b}(\mathbb{R}^2)$ is closed in $\tilde{G} = S^3 \times S^3 \times \mathbb{R}$ if and only if $a$ and $b$ are linearly dependent over $\mathbb{Q}$.

**Proof.** Let $H$ be the preimage in $\tilde{G}$ of the maximal torus in $\tilde{G}/Z(\tilde{G})$ over which $\tau_{a,b}(\mathbb{R}^2)$ lies. As the continuous preimage of a closed set, $H$ is closed; so it suffices to consider closedness in $H$.

If $\tilde{G} = S^3 \times S^3 \times \mathbb{R}$, then $H \cong S^1 \times S^1 \times \mathbb{R} \cong \mathbb{R}^2/(\mathbb{Z} \times \{0\})$; and $\tau_{a,b}(\mathbb{R}^2)$ is the image in $H$ of a vector subspace $V \subseteq \mathbb{R}^2$ given in coordinates $(x, y, z)$ by $ax + by - z = 0$. Since $V$ and $\mathbb{Z}^2$ are both closed in $\mathbb{R}^3$, all of the following inclusions are closed if any one of them is.

$$\tau_{a,b}(\mathbb{R}^2) = V/(V \cap \mathbb{Z}^2) \subseteq \mathbb{R}^3/\mathbb{Z}^2 \cong H$$

$$V \mathbb{Z}^2 \subseteq \mathbb{R}^3$$

$$\mathbb{Z}^2/(V \cap \mathbb{Z}^2) \subseteq \mathbb{R}^3/V$$

Consider the last of the above inclusions. Modulo $ax + by - z$, the generators $(1, 0, 0)$ and $(0, 1, 0)$ of $\mathbb{Z}^2$ are equivalent to $(0, 0, -a)$ and $(0, 0, -b)$. These generate a closed subgroup of $\mathbb{R}^3/V \cong \{(0, 0, z) \mid z \in \mathbb{R}\} \subset \mathbb{R}^3$ if and only if $\gcd(a, b)$ exists (i.e. the integer linear combinations of $a$ and $b$ have a smallest positive value rather than accumulating on zero).

**Remark 6.33.** The rational dependence constraint does not appear if the rank $k$ of $\pi_1(H) \cong \mathbb{Z}^k$ is less than 2, since a rank-at-most-one subgroup of a vector space $H/V$ is closed.

In particular, for all $\tilde{G}$ in this part of the classification other than $S^3 \times S^3 \times \mathbb{R}$, the ratio $a/b$ for $\tau_{a,b}$ is allowed to be any real number (really, any point of $\mathbb{R}P^1$).

**Proposition 6.34 (Correctness of Table 6.29).** Let $\tilde{G}$ and $\tau_{a,b}$ be as in one of the rows of Table 6.29. Then
(i) $M = \tilde{G}/\tau_{a,b}(\mathbb{R}^2)$ (Defn. 6.27) fibers over the product $X \times Y$ in the last column.

(ii) For some invariant Riemannian metrics on $X$ and $Y$ with curvature $0$ or $\pm 1$, the connection on $M \to X \times Y$ given by $(TM^{\tau_{a,b}(\mathbb{R}^2)})^\perp$ has the same curvature as the bundle in the last column, provided that both $a$ and $b$ are nonzero.

Proof of Prop. 6.34(i). The quotient map $\pi: \tilde{G} \to \text{Inn}G = \tilde{G}/(\tilde{G})$ sends $\tau_{a,b}(\mathbb{R}^2)$ to a $2$-dimensional maximal torus $T$. Then $(\text{Inn}G)/T$ is a $4$-dimensional geometry with point stabilizer $T$, which is a product of $2$-dimensional geometries $X \times Y$ by [Fil83, Thm. 3.1.1(c)]. Computing $\tilde{G}/(\tilde{G})$ determines $X$ and $Y$.

Proof of Prop. 6.34(ii). Suppose $a, b \in Z(\tilde{G})^0 \cong \mathbb{R}$ are both nonzero. For the connection on $M = \tilde{G}/\tau_{a,b}(\mathbb{R}^2) \to X \times Y$ given by $(TM^{\tau_{a,b}(\mathbb{R}^2)})^\perp$, the curvature $\Omega$ is an invariant $2$-form on $X \times Y$ with values in the Lie algebra of the fiber.

Step 1: Curvature is nonzero only along $X$ and $Y$. If $V$ is the standard representation of $\text{SO}(2)$, then a tangent space to $X \times Y$ decomposes as the $\text{SO}(2) \times \text{SO}(2)$ representation

$$T_xX \oplus T_yY \cong (V \otimes \mathbb{R}) \oplus (\mathbb{R} \otimes V),$$

whose second exterior power is

$$\Lambda^2(V \otimes \mathbb{R}) \oplus (V \otimes V) \oplus \Lambda^2(\mathbb{R} \otimes V) \cong 2\mathbb{R} \oplus (V \otimes V).$$

So the $2$-form $\Omega$, being $\text{SO}(2) \times \text{SO}(2)$-invariant, is determined by its values on the $2\mathbb{R}$ summand—that is, on $\Lambda^2(T_xX)$ and $\Lambda^2(T_yY)$.

Step 2: Express the preimage of $X$ as a homogeneous space. Fix a copy of $X$, and let $E$ be its preimage in $M$. Then choosing a factor of $\tilde{G}$ acting transitively on $X$ and containing $Z(\tilde{G})^0$ expresses $E$ as covered by one of the following homogeneous spaces $H/H_p$.

$$\text{Heis}_3 \times \overset{\sim}{\text{SO}(2)}/\{(0,0,at), \gamma(t)\}_{t \in \mathbb{R}} \cong \text{Heis}_3$$

$$S^3 \times \mathbb{R}/\{e^{\pi it}, at\}_{t \in \mathbb{R}} \cong S^3 \ (\text{or } S^2 \times \mathbb{R} \text{ if } a = 0)$$

$$\overset{\sim}{\text{SL}_2} \times \mathbb{R}/\{\gamma(t), at\}_{t \in \mathbb{R}} \cong \overset{\sim}{\text{SL}_2} \ (\text{or } \mathbb{H}^2 \times \mathbb{R} \text{ if } a = 0)$$

The notation conventions are that:

1. $\text{Heis}_3$ is $\mathbb{R}^3$ with the composition law $(x,y,z)(x',y',z') = (x+x',y+y',z+z'+xy')$;

2. $\gamma: \mathbb{R} \to \overset{\sim}{\text{SO}(2)}$ sends $t \in \mathbb{R}$ to a rotation by $2\pi t$; and

3. $S^3$ and $\mathbb{C}$ are interpreted as subsets of the quaternions.

Step 3: If $X$ has nonzero curvature $K$, then the curvature along $X$ is $\frac{K}{2\pi} \text{vol}_X \otimes a$. The curvature of $(TM^{\tau_{a,b}(\mathbb{R}^2)})^\perp$ restricted to $X$ is the curvature of $TE^H$ as a connection on $E \to X$.

Equivariantly, $S^3$ covers $\text{SO}(3) \cong T^1S^2$ and $\overset{\sim}{\text{SL}_2}$ covers $\text{PSL}(2,\mathbb{R}) \cong T^1\mathbb{H}^2$; which takes $TE^H$ to the connection induced by the Levi-Civita connection on $S^2$ or $\mathbb{H}^2$. If the unit tangent circles are declared to have length $2\pi$, then this connection’s curvature is the surface’s scalar curvature $K$ times its area form $\text{vol}_X$. (This is a version of the Gauss-Bonnet Theorem; see e.g. [DC76, §4.5 p. 274].) In $E = H/H_p$, such a circle is most naturally assigned the length of the interval.
\[0, a) \subset \mathbb{R} \cong Z(\tilde{G})^0 \subset H\] that maps bijectively onto it. Then identifying \(T_0\mathbb{R}\) with \(\mathbb{R} \cong Z(\tilde{G})^0\) by the exponential map gives the expression \(\frac{K}{2\pi} \operatorname{vol} X \otimes a\) for the curvature over \(X\).

The same argument applies to \(Y\), which establishes the claim (ii) for the first three rows of Table 6.29.

**Step 4: Curvature along \(X = \mathbb{E}^2\) is any nonzero multiple of the volume form.** First, observe that
\[
\begin{align*}
\text{Heis}_3 \rtimes \text{SO}(2)/\{(0, 0, at), \gamma(t)\}_{t \in \mathbb{R}} & \cong \text{Heis}_3 \rtimes \text{SO}(2)/\text{SO}(2) \\
x, y, z, \gamma(t) & \mapsto x, y, z - at, \gamma(t).
\end{align*}
\]
Under the usual metrics, \(\text{Heis}_3 \to \mathbb{E}^2\) with the connection \(T\text{Heis}_3^{\text{SO}(2)}\) has curvature 1 (i.e. \(1\) times the area form on \(\mathbb{E}^2\)) [Thu97, discussion after Exercise 3.7.1]. Through an appropriate (possibly orientation-reversing) conformal automorphism of \(\mathbb{E}^2\), the area form can be pulled back to any nonzero constant multiple of itself.

To finish, apply Step 3 to \(Y\), choosing a length scale on \(Z(\tilde{G})^0 = Z(\text{Heis}_3)\) that makes the curvature along \(Y\) equal to \(\frac{1}{2\pi} \operatorname{vol} Y\). Then choose an orientation and scale on \(X = \mathbb{E}^2\) that makes the curvature along \(X\) equal to \(\frac{1}{2\pi} \operatorname{vol} X\). This establishes claim (ii) for the remaining (last two) rows of Table 6.29.

### 6.5.2 Proof of the classification

This section proves Prop. 6.23, the classification of geometries \(G/G_p\) when \(\tilde{G}\) is a direct product listed in Table 6.24. The recurring method is to relate \(G_p\) to a maximal torus of some group; each individual step is merely whatever happens to decrease the number of remaining cases. Maximality is deferred to Section 6.6.

As part of the classification, we also prove that the associated bundle geometry \(\tilde{\text{SL}}_2 \times_{\alpha} S^3\) is a model geometry (Prop. 6.35) but has compact quotients if and only if \(\alpha\) is rational (Prop. 6.36).

**Proof of Prop. 6.23 (except for maximality).** Let \(G/G_p\) be a 5-dimensional maximal model geometry where \(G\) is covered by one of the groups in Table 6.24.

**Step 1:** If \(\dim G = 6\) and \(\tilde{G} \not\cong \ast \times \text{Sol}^3\), then \(G/H\) is non-maximal. We will show that if \(\dim G = 6\) and the maximal torus of \(\text{Aut} \tilde{G}\) has dimension at least 2, then any 5-dimensional geometry with isometry group \(G\) is non-maximal.

Since \(\dim G = 6\), the point stabilizer \(G_p\) is 1-dimensional and therefore isomorphic to \(S^1\). Let \(H\) be the lift of \(G_p\) to \(\tilde{G}\), i.e. the group such that \(\tilde{G}/H \cong G/G_p\). Since \(\text{Aut} \tilde{G}\) has a maximal torus of dimension at least 2, there is an \(S^1 \subset \text{Aut} \tilde{G}\) that commutes with the inner action of \(H\) and is independent—that is, \(H \times S^1\) maps to a 2-dimensional subgroup of \(\text{Aut} \tilde{G}\). Then
\[
(\tilde{G} \times S^1)/(H \times S^1)
\]
is a homogeneous space subsuming \(G/G_p\); and a geometry subsuming \(G/G_p\) is realized by passing to the quotient by \(Z(\tilde{G} \times S^1) \cap (H \times S^1)\) (Rmk. 6.4).

In particular, if \(\tilde{G}\) is a product of any two of the following groups—each with dimension 3 and an \(S^1\) in its automorphism group—then any 5-dimensional geometry \(G/G_p\) is non-maximal.

\[
\begin{array}{cccc}
\text{Isom}_0 \mathbb{E}^2 & S^3 & \text{PSL}(2, \mathbb{R}) & \text{Heis}_3 & \mathbb{R}^3
\end{array}
\]

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Step 2: $G_p$ maps injectively to $\text{Inn} \, G = \tilde{G}/Z(\tilde{G})$ as a maximal torus. In the remaining cases, $\tilde{G}$ is one of the following products, where each $\bullet$ denotes the identity component of the isometry group of $E^2$, $S^2$, or $H^2$.

$$
\bullet \times \bullet \times \mathbb{R} \quad \bullet \times \text{Isom}_0 \text{Heis}_3 \quad \bullet \times \text{Sol}^3
$$

The dimensions of these $\tilde{G}$ are respectively 7, 7, and 6. The corresponding $\text{Inn} \, G$ are

$$
\bullet \times \bullet \quad \bullet \times \text{Isom}_0 \mathbb{E}^2 \quad \bullet \times \text{Sol}^3.
$$

The maximal torus of $\bullet$ has dimension 1, so the maximal torus $T$ of $\text{Inn} \, G$ has dimension 2, 2, or 1, respectively. Since $G_p$ has faithful conjugation action in a geometry (Lemma 6.3), which is equivalent to injectivity of $G_p \to \text{Inn} \, G$, the claim in this step follows from $\dim T = \dim G - 5$.

Step 3: If $\tilde{G} = * \times \text{Sol}^3$ then $G/G_p$ is a product with $\text{Sol}^3$. In this case, $\tilde{G} \to \text{Inn} \, G$ is a covering map; so if $H$ is the identity component of the preimage of $G_p$ in $\tilde{G}$, then $H$ covers a maximal torus of $\text{Inn} \, G$. Then all possible $H$ are conjugate in $\tilde{G}$, so

$$
G/G_p \cong \text{Sol}^3 \times \text{Isom}_0 \mathbb{B}/\text{SO}(2) \cong \text{Sol}^3 \times \mathbb{B}
$$

where $\mathbb{B}$ is $\mathbb{E}^2$, $\mathbb{S}^2$, or $H^2$.

Step 4: $G/G_p \cong \tilde{G}/\tau_{a,b}(\mathbb{R}^2)$ for some $a$ and $b$ in $Z(\tilde{G})^0$. In the remaining cases, $\dim Z(\tilde{G}) = 1$ and $\text{Inn} \, G = \tilde{G}/Z(\tilde{G})$ has 2-dimensional maximal torus $T$. Since $\pi: \tilde{G} \to \text{Inn} \, G$ descends to a map from $G$ sending $G_p$ isomorphically to $T$ (Step 2), the preimage of $G_p$ in $\tilde{G}$ is the image of a homomorphism $\tau : \mathbb{R}^2 \to \pi^{-1}(T)$. Any two such $\tau$ that compose with $\pi$ to the same map $\mathbb{R}^2 \to T$ differ only by some $\mathbb{R}^2 \to Z(\tilde{G})$; so $\tau$ can be identified with some $\tau_{a,b}$ as defined in Defn. 6.27 after choosing $\tau_{0,0}$.

In $\text{Isom}_0 \mathbb{E}^2 \times S^3 \times \mathbb{R}$, choose $\tau_{0,0}(\mathbb{R}^2) = \text{SO}(2) \times S^1 \times \{0\}$. In the three other $\tilde{G}$ that have $\text{Isom}_0 \mathbb{E}^2$ as a factor, let $\tau_{0,0}(\mathbb{R}^2) = \text{SO}(2) \times \text{SO}(2)$. All remaining $\tilde{G}$ occur in Table 6.29, where the choice of $\tau_{0,0}$ is also recorded.

Step 5: If $\tilde{G} = \text{Isom}_0 \mathbb{E}^2 \times K$ for some group $K$, then $G/G_p$ is a product with $\mathbb{E}^2$. Express $\text{Isom}_0 \mathbb{E}^2$ as $\mathbb{C} \times \text{SO}(2)$, i.e. as $\mathbb{C} \times \mathbb{R}$ with the composition law

$$
(x + iy, z)(x' + iy', z') = (x + iy + e^{iz}(x' + iy'), z + z').
$$

Then in $\tilde{G}/\tau_{a,b}(\mathbb{R}^2) \to \mathbb{E}^2 \times Y$, the preimage $E$ of $\mathbb{E}^2$ has a transitive action by a subgroup of $\tilde{G}$ isomorphic to $\text{Isom}_0 \mathbb{E}^2 \times \mathbb{R}$; and there is an equivariant diffeomorphism

$$
E = \text{Isom}_0 \mathbb{E}^2 \times \mathbb{R}/\{(0 + 0i, z), az\}_{z \in \mathbb{R}} \xrightarrow{\sim} \text{Isom}_0 \mathbb{E}^2 \times \mathbb{R}/\text{SO}(2)
$$

$$
x + iy, z, t \mapsto x + iy, z, t - az.
$$

Extending this to $\tilde{G}/\tau_{a,b}(\mathbb{R}^2)$ yields

$$
\tilde{G}/\tau_{a,b}(\mathbb{R}^2) \cong \text{Isom}_0 \mathbb{E}^2 \times K/\tau_{0,0}(\mathbb{R}^2) \cong \mathbb{E}^2 \times (K/K_q).
$$

---

8 That the diffeomorphism extends may be easier to see on the Lie algebra level. It corresponds to an automorphism $\text{isom} \mathbb{E}^2 \times \mathbb{R}$ that sends a basis $\{\hat{x}, \hat{y}, \hat{z}, \hat{t}\}$ to $\{\hat{x}, \hat{y}, \hat{z} - at, \hat{t}\}$. This extends to the appropriate automorphism of $T_1G = \text{isom} \mathbb{E}^2 \oplus \mathfrak{t}$ by the identity on $\mathfrak{t}$. 53
Step 6: If \( \tilde{G} = (\text{Isom \text{Heis} \}_{3})^0 \times K \), then \( G/G_p \) is one of four geometries. There are only two cases remaining for this step to handle: \( K = S^3 \) and \( K = \text{SL}_2 \). Step 4 of Prop. 6.34 showed that the equivariant diffeomorphism type of \( \tilde{G}/\tau_{a,b}(\mathbb{R}^2) \) is independent of \( a \), so set \( a = 0 \). Then if \( \text{Heis}_3 \) is \( \mathbb{R}^3 \) with the composition law

\[
(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy')
\]

and \( \gamma : \mathbb{R} \to K \) is a 1-parameter subgroup of \( K \) with \( \gamma(\mathbb{Z}) = \gamma(\mathbb{R}) \cap Z(K) \),

\[
\tilde{G}/\tau_{0,b}(\mathbb{R}^2) = \text{Heis}_3 \rtimes \text{SO}(2) \times K/\{(0, 0, bt), s, \gamma(t)\}_{s,t \in \mathbb{R}}
\]

is one of

1. a product \( \text{Heis}_3 \times S^2 \) or \( \text{Heis}_3 \times \mathbb{H}^2 \) if \( b = 0 \); or
2. equivariantly diffeomorphic to \( \tilde{G}/\tau_{0,1}(\mathbb{R}^2) \) by a map sending \((x, y, z, r) \in \text{Heis}_3 \times \text{SO}(2) \) to

\[
\left( x|b|^{-1/2}, sy|b|^{-1/2}, z|b|^{-1}, r^a \right)
\]

where \( s = b/|b| \), if \( b \neq 0 \). So the names \( \text{Heis}_3 \times \mathbb{R} \cdot S^3 \) and \( \text{Heis}_3 \times \mathbb{R} \cdot \text{SL}_2 \) from Table 6.29 can be used without ambiguity.

Step 7: Parametrize the isomorphism types of \( \tilde{G}/\tau_{a,b}(\mathbb{R}^2) \) by \([a : b] \in \mathbb{R}P^1 \). In the remaining cases, \( M = \tilde{G}/\tau_{a,b}(\mathbb{R}^2) \) has a canonical fibering over

\[
X \times Y \cong \tilde{G}/(Z(\tilde{G}) \cdot \tau_{a,b}(\mathbb{R}^2)),
\]

where \( X \) and \( Y \) are surfaces of nonzero constant curvature. Decomposing \( G_p \subset T_pM \) into irreducible subrepresentations, every invariant inner product on \( T_pM \) (and hence every invariant metric on \( M \)) is determined by a scale factor along the fiber (i.e. on \( T_pM^{G_p} \)), a scale factor on \( X \), and a scale factor on \( Y \) (Lemmas 3.10 and 3.14). The scale factors on \( X \) and \( Y \) can be chosen by normalizing their curvatures to be \( \pm 1 \).

Given any member of this family of normalized metrics, the ratio of curvatures for the connection \((TM^G)^{-1} \) on \( M \leftrightarrow X \times Y \) (listed in Table 6.29) represents, up to finite choices, an invariant for the family. Explicitly, an invariant number can be recovered as the ratio of displacements along a fiber that result from horizontal lifts of loops enclosing the same small area in \( X \) and \( Y \). The choices that need to be made are

1. assigning \( X \) and \( Y \) to the two factors in the base space, and
2. orientations on \( X \) and \( Y \).

These reflect the following symmetries.

0. Rescaling \( \mathbb{R} \subset \tilde{G} \) induces \( \tilde{G}/\tau_{a,b}(\mathbb{R}^2) \cong \tilde{G}/\tau_{at,bt}(\mathbb{R}^2) \) for nonzero \( t \).
1. If \( X \cong Y \), then exchanging \( X \) and \( Y \) allows assuming \( |a| \leq |b| \).
2. Conjugating by \( j \in S^3 \) reverses the 1-parameter subgroup \( e^{it} \); and conjugating \( \text{SL}_2 \) by \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \)

reverses the 1-parameter subgroup \( \text{SO}(2) \). This allows assuming \( a > 0 \) and \( b > 0 \).
There are two more considerations that affect the parametrization:

3. If \( a = 0 \) or \( b = 0 \), then products result as in Step 6.

4. If \( \hat{G} = S^3 \times S^3 \times \mathbb{R} \), then \( a \) and \( b \) need to be rationally dependent (Prop. 6.32), which allows rescaling them both to be coprime integers. This constraint is not present for other \( \hat{G} \) (Rmk. 6.33).

So excluding products, the geometries remaining to be classified—those fibering over products of \( S^2 \) and \( \mathbb{H}^2 \)—are specified exactly once each by the following.

\[
\begin{align*}
\widetilde{SL}_2 \times_{a/b} \widetilde{SL}_2 &= \widetilde{SL}_2 \times \widetilde{SL}_2 \times \mathbb{R}/\tau_{a,b}(\mathbb{R}^2), & a/b \in (0,1] \\
\widetilde{SL}_2 \times_{a/b} S^3 &= \widetilde{SL}_2 \times S^3 \times \mathbb{R}/\tau_{a,b}(\mathbb{R}^2), & a/b \in (0,\infty) \\
L(a;1) \times_{S^1} L(b;1) &= S^3 \times S^3 \times \mathbb{R}/\tau_{a,b}(\mathbb{R}^2), & 0 < a \leq b \text{ coprime in } \mathbb{Z}.
\end{align*}
\]

**Step 8:** All of the above are model geometries. Products of model geometries are model geometries, since they model products of manifolds modeled on the factors. The \( L(a;1) \times_{S^1} L(b;1) \) geometry is a model geometry since it is already compact. So it only remains to show that bundles associated to \( \text{Heis}_3 \) and \( \widetilde{SL}_2 \) are also model geometries.

The construction is: if \( E \) and \( F \) model compact circle bundles \( M \) and \( N \), then \( M \times_{S^1} N \) is modeled on some \( E \times_{\rho} F \). In particular, \( \text{Heis}_3 \) models the circle bundle \( \text{Heis}_3 / \text{Heis}_3(\mathbb{Z}) \) over a torus modeled on \( \mathbb{E}^2 \cong \text{Heis}_3 / \pi(\text{Heis}_3) \); \( S^3 \) models the Hopf fibration over \( S^2 \); and \( \widetilde{SL}_2 \) models the unit tangent bundle of any compact hyperbolic surface.

However, only some \( E \times_{\rho} F \) can be recovered as the universal cover of an \( M \times_{S^1} N \): combinations of these three bundles are modeled only on \( \hat{G}/\tau_{1,1}(\mathbb{R}^2) \). This is enough for the bundles associated to \( \text{Heis}_3 \). For bundles associated to \( \text{SL}_2 \), compact bundles modeled on \( \hat{G}/\tau_{a,b}(\mathbb{R}^2) \) can be obtained, for integers \( a \) and \( b \), by starting with quotients of the initial circle bundles by rotations of \( 2\pi/a \) and \( 2\pi/b \). This, however, still leaves the case when \( a/b \) is irrational, which is a somewhat different construction, given below in Prop. 6.35.

**Proposition 6.35.** If \( \hat{G} = \widetilde{SL}_2 \times \ast \times \mathbb{R} \), then \( \hat{G}/\tau_{a,b}(\mathbb{R}^2) \) is a model geometry.

**Proof.** More honestly, since a geometry \( M = G/G_p \) must have \( G \) acting faithfully, we need to use \( G = \hat{G}/(Z(\hat{G}) \cap \tau_{a,b}(\mathbb{R}^2)) \cong \hat{G}/\tau_{a,b}(\mathbb{Z}^2) \) (Rmk. 6.4).

**Step 1:** Construct a subgroup \( \hat{\Gamma} \times \hat{\Lambda} \) of \( \hat{G} \). Assume \( b \neq 0 \) since otherwise \( G/G_p \) is a product with \( S^2 \) or \( \mathbb{H}^2 \). Let \( \Gamma = \pi_1(S) \subset \text{PSL}(2,\mathbb{R}) \) for some orientable punctured surface \( S \) of genus at least 2, and let \( \hat{\Gamma} \) be its preimage in \( \text{PSL}_2 \). Then \( \hat{\Gamma} \) is central extension of a free group by \( Z(\text{PSL}_2) \cong \mathbb{Z} \); so it splits as a semidirect product, and \( Z(\text{PSL}_2) \) maps to a copy of \( \mathbb{Z} \) in the abelianization of \( \hat{\Gamma} \). Then there is a homomorphism \( f : \hat{\Gamma} \to \mathbb{R} \) that sends \( Z(\text{PSL}_2) \subset \text{SO}(2) \) to \( \mathbb{Z} \). Let \( \Lambda \) be the fundamental group of \( S^2 \) or a compact orientable surface modeled on \( \mathbb{H}^2 \), let \( \hat{\Lambda} \) be its lift to the group \( \ast \), and define

\[
i : \hat{\Gamma} \times \hat{\Lambda} \to \text{PSL}_2 \times \ast \times \mathbb{R} \\
g,h \mapsto (g,h,af(g)).
\]

9 Requiring orientability permits assuming that \( \pi_1(S) \) embeds in \( \text{PSL}(2,\mathbb{R}) \), the identity component of Isom \( \mathbb{H}^2 \).
Step 2: $\tilde{\Gamma} \times \tilde{\Lambda}$ descends to a discrete subgroup $\Delta$ of $G$. Let $\gamma$ be a 1-parameter subgroup of $\ast$ sending $\mathbb{Z}$ to the center. We first show that the image of $i$ does not accumulate on $\tau_{a,b}(\{0\} \times \mathbb{Z}) = \{1, \gamma(n), nb\}_{n \in \mathbb{Z}}$. By discreteness of $\Gamma \times \Lambda$, some neighborhood $U \times V$ of the identity in $\text{SL}_2 \times \ast$ meets $\Gamma \times \Lambda$ only in the identity. Then with the standing assumption that $b = 0$,

$$\bigcup_{n \in \mathbb{Z}} U \times \gamma(n)V \times ((n - 1)b, (n + 1)b)$$

is an open set containing and preserved by $\tau_{a,b}(\{0\} \times \mathbb{Z})$ in which the only elements $i(g, h) = (g, h, af(g))$ of $i(\tilde{\Gamma} \times \tilde{\Lambda})$ satisfy all of

$$g = 1 \quad h = \gamma(n) \quad 0 = \frac{a}{b}f(1) = \frac{a}{b}f(g) \in (n - 1, n + 1).$$

That is, only the identity in $i(\tilde{\Gamma} \times \tilde{\Lambda})$ lies in this neighborhood of $\tau_{a,b}(\{0\} \times \mathbb{Z})$.

This implies $i(\tilde{\Gamma} \times \tilde{\Lambda})$ remains discrete in $\hat{G}/\tau_{a,b}(\{0\} \times \mathbb{Z})$. Since $i(\tilde{\Gamma} \times \tilde{\Lambda})$ was constructed to contain $\hat{G}/\tau_{a,b}(\mathbb{Z} \times \{0\})$, it remains discrete in $G = \hat{G}/\tau_{a,b}(\mathbb{Z}^2)$.

Step 3: $\Delta$ is a lattice in $G$. With discreteness established, it suffices to show that for the image $\Delta$ of $\tilde{\Gamma} \times \tilde{\Lambda}$ in $G$, the volume of $G/\Delta$ is finite. Let $H = i(\tilde{\Gamma}) \times \tilde{\Lambda} \cdot \tau_{a,b}(\mathbb{Z}^2)$, and observe that

$$\hat{G}/(H \cdot (\ast \times \mathbb{R})) \cong \hat{\text{SL}_2}/\tilde{\Gamma} \cong \hat{S}$$ (chosen at the start of Step 1)

$H \cdot (\ast \times \mathbb{R})/(H \cdot \mathbb{R}) \cong \ast / \tilde{\Lambda}$ (the compact surface chosen at the end of Step 1)

$$H \cdot \mathbb{R}/H \cong \mathbb{R}/\{nb\}_{n \in \mathbb{Z}} \cong S^1.$$

In a situation involving only closed subgroups $E \subseteq F \subseteq G$ of a locally compact $G$, an invariant measure on $G/E$ is constructed as a product of invariant measures on $G/F$ and $F/E$ as in [Mos62b, 2.4 Case 2]; so the volume of $G/H$ is finite since all three intermediate spaces are.

Step 4: $\Delta \setminus G/G_p$ is a finite-volume manifold. The space $\Delta \setminus G/G_p$ has finite volume and is modeled on $G/G_p$ but might be an orbifold—ruling out orbifold points requires checking that $\Delta$ acts freely, i.e. that $\Delta$ meets each point stabilizer in only the identity. Since $\Delta$ is discrete, its intersection with any compact point stabilizer has finite order. So it suffices to check that a subgroup of $\hat{G} = \text{SL}_2 \times \ast \times \mathbb{R}$ surjecting onto $\Delta$—specifically, $i(\tilde{\Gamma} \times \tilde{\Lambda})$—contains no element $(g, h, \tau)$ outside of $\tau_{a,b}(\mathbb{Z}^2)$ with a nonzero power in $\tau_{a,b}(\mathbb{Z}^2)$.

Since $\tau_{a,b}(\mathbb{Z}^2)$ is central, any $i(g, h) = (g, h, af(g))$ with a nonzero power in $\tau_{a,b}(\mathbb{Z}^2)$ has finite-order image in $\hat{G}/Z(\hat{G})$. Since $g$ is in the lift $\tilde{\Gamma}$ of $\pi_1(S)$ where $S$ is a surface (in particular, with no orbifold points), $g$ has finite-order image in $\text{PSL}(2, \mathbb{R})$ only if this image is the identity. Similarly, $h$ lies over the identity of $\text{SO}(3)$ or $\text{PSL}(2, \mathbb{R})$; so some $\tau_{a,b}(m, n)$ has the same first two coordinates $g$ and $h$.

If $\ast = S^1$, then $h = 1$ and $(g, 1, af(g)) = \tau_{a,b}(m, 0)$. If $\ast = \text{SL}_2$, then $Z(\hat{G})$ has no finite-order elements, which makes $n$th roots unique; so $(g, h, af(g))$ has a nonzero power in $\tau_{a,b}(\mathbb{Z}^2)$ if and only if it lies in $\tau_{a,b}(\mathbb{Z}^2)$ itself. Either way, $i(\tilde{\Gamma} \times \tilde{\Lambda})$ contains nothing outside of $\tau_{a,b}(\mathbb{Z}^2)$ with a nonzero power in $\tau_{a,b}(\mathbb{Z}^2)$; so by the first paragraph, $\Delta \setminus G/G_p$ has no orbifold points and is a finite-volume manifold modeled on $G/G_p$.

The above construction always produces noncompact $\Delta \setminus G/G_p$, with fundamental group independent of $a$ and $b$. In compact manifolds, the story is different—for instance, one can prove the following.
Proposition 6.36. Let $\tilde{G} \cong SL_2 \times S^3 \times \mathbb{R}$, If there is a compact manifold $N$ modeled on $\tilde{G}/\tau_{a,b}(\mathbb{R}^2)$ and $b \neq 0$, then $a/b \in \mathbb{Q}$.

Proof. As above in Prop. 6.35, $G = \tilde{G}/\tau_{a,b}(\mathbb{Z}^2)$, in which $\pi_1(N)$ is a cocompact lattice. Its preimage $\tilde{\Gamma}$ in $\tilde{G}$ is also a cocompact lattice. We will study the projection of $\tilde{\Gamma}$ to $\Gamma \subset SL_2 \times (\mathbb{R}/2b\mathbb{Z})$ and show in particular that $\tau_{a,b}(\mathbb{Z} \oplus \{0\})$ projects to a finite subgroup of $\mathbb{R}/2b\mathbb{Z}$.

Step 1: $\Gamma$ is discrete in $SL_2 \times (\mathbb{R}/2b\mathbb{Z})$. Since $\tilde{\Gamma}$ is discrete and $S^3$ is compact, $\tilde{\Gamma}$ cannot accumulate on a coset of $S^3$. Then $\tilde{\Gamma}/S^3$ is closed, so $\tilde{\Gamma}/(\tilde{\Gamma} \cap S^3)$ is discrete in $\tilde{G}/S^3 \cong SL_2 \times \mathbb{R}$. Moreover,

$$\tilde{\Gamma} \supset \tau_{a,b}(\mathbb{Z}^2) \supset \tau_{a,b}(\{0\} \oplus \mathbb{Z}) = \{1, e^{\pi \text{in}}, bn\}_{n \in \mathbb{Z}} \subset \tilde{SL}_2 \times S^3 \times \mathbb{R},$$

so $\tilde{\Gamma}$ contains $2b\mathbb{Z} \subset \mathbb{R}$. Then $\Gamma = \tilde{\Gamma}/(S^3 \times 2b\mathbb{Z})$ is discrete in $SL_2 \times (\mathbb{R}/2b\mathbb{Z})$.

Step 2: $\Gamma$ descends to a cocompact lattice $\Lambda$ in $PSL(2, \mathbb{R})$. This proceeds similarly to Step 1. Let $\Lambda$ be the projection of $\Gamma$ to $SL_2$. Since $\mathbb{R}/2b\mathbb{Z}$ is compact, $\Lambda$ is discrete in $SL_2$. Furthermore, if $k$ is a generator of $Z(\tilde{SL}_2)$, then the image in $\tilde{SL}_2 \times (\mathbb{R}/2b\mathbb{Z})$ of $\tau_{a,b}(\mathbb{Z}^2) \subset \tilde{\Gamma}$ is

$$\{k^n, \text{an mod } 2b\}_{n \in \mathbb{Z}} \subset \Gamma.$$

Then $\Lambda$ contains $Z(\tilde{SL}_2)$, so its image $\Lambda$ in $PSL(2, \mathbb{R})$ is discrete; and $\Lambda$ is cocompact since $\Lambda \setminus PSL(2, \mathbb{R})$ is a quotient space of $\tilde{\Gamma} \setminus \tilde{G}$.

Step 3: $k$ becomes torsion in the abelianization of $\Lambda$. Since $\Lambda \subset PSL(2, \mathbb{R})$ is a cocompact lattice, it is the orbifold fundamental group of a compact orbifold $O$ modeled on $\mathbb{H}^2$; and $\Lambda = \pi_1(T^1O)$ (see e.g. [Thu02, §13.4] for some discussion of unit tangent bundles of orbifolds). Since a hyperbolic 2-orbifold admits a finite cover by a hyperbolic surface [Sco83, Thm. 2.3 and 2.5], $\Lambda$ contains a subgroup isomorphic to the unit tangent bundle of a closed hyperbolic surface $S$. Its center is generated by $k$, and $k^{\chi(S)}$ is a product of commutators [Sco83, discussion surrounding Lemma 3.5]. So $k$ becomes finite order in the abelianization of $\Lambda$.

Step 4: $\tau_{a,b}(\mathbb{Z}^2)$ becomes torsion in the abelianization of $\Gamma$. The intersection of the compact $\mathbb{R}/2b\mathbb{Z}$ with the discrete $\Gamma$ is a finite cyclic group $C$, which makes $\Gamma$ a central extension

$$1 \to C \to \Gamma \to \Lambda \to 1.$$

Using the Stallings exact sequence [Bro82, II.5 Exercise 6(a)], the induced

$$C \to \Gamma^{Ab} \to \Lambda^{Ab}$$

is exact in the middle. Then $(k, a \text{ mod } 2b) \in \Gamma$, which lies over $k \in \Lambda$, becomes finite order in the abelianization; so its image $a$ in the abelian $\mathbb{R}/2b\mathbb{Z}$ has finite order. Therefore $a$ is a rational multiple of $b$. \hfill \Box

6.5.3 Explicit enumeration of product geometries

This section collects a list of the product geometries with nontrivial abelian isotropy—i.e. those with the 2-dimensional base in the fibering description (Prop. 3.3(iii)).

Proposition 6.37. The maximal model product geometries with nontrivial abelian isotropy are:
(i) 4-by-1:

\[ \mathbb{F}^4 \times \mathbb{E} \quad \text{Sol}_0^4 \times \mathbb{E} \]

(ii) 2-by-2-by-1:

\[ S^2 \times S^2 \times \mathbb{E} \quad S^2 \times \mathbb{H}^2 \times \mathbb{E} \quad \mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{E} \]

(iii) 3-by-2:

\[ \text{Heis}_3 \times \mathbb{E}^2 \quad \text{Heis}_3 \times S^2 \quad \text{Heis}_3 \times \mathbb{H}^2 \]

\[ \text{Sol}_3 \times \mathbb{E}^2 \quad \text{Sol}_3 \times S^2 \quad \text{Sol}_3 \times \mathbb{H}^2 \]

\[ \text{SL}_2 \times \mathbb{E}^2 \quad \text{SL}_2 \times S^2 \quad \text{SL}_2 \times \mathbb{H}^2 \]

Proof. The list can be built up by looking up the factors in previous classifications by Thurston [Thu97, Thm. 3.8.4] and Filipkiewicz [Fil83]. Products with multiple Euclidean factors are non-maximal and omitted.

As products of model geometries, these are all model geometries (Prop. 2.2). All are products with at most one Euclidean factor; and except for \( \text{Sol}_0^4 \times \mathbb{E} \), all are products of two maximal geometries where one has no trivial subrepresentation in its isotropy, which makes them maximal (Prop. 3.12). Maximality of \( \text{Sol}_0^4 \times \mathbb{E} \) was proven in Step 6 of Prop. 6.7(iii).

\[ \square \]

6.6 Deferred maximality proofs for isometrically fibering geometries

Having just handled the products in Section 6.5.3 above, maximality remains to be proven only for the five associated bundle geometries or families and the six geometries from semidirect products in Prop. 6.17.

The strategy in many cases (perhaps because the author failed to think of anything less ad-hoc) is to show that \( G'/H' \) cannot subsume \( G/H \) by showing that some subgroup of \( G \) does not appear in \( G' \). The most useful in what follows is a restriction on embeddings of \( \text{Heis}_3 \).

Lemma 6.38. Let \( \mathfrak{g} \) be a direct product of algebras of the form \( T_1 \text{Isom} M \) (\( M = S^k, \mathbb{E}^k, \) or \( \mathbb{H}^k \)). Then \( \mathfrak{g} \) contains no nonabelian nilpotent subalgebra.

Proof. If a subalgebra has an abelian projection to each factor in the product, then it is itself abelian; so it suffices to prove the Lemma when \( \mathfrak{g} = T_1 \text{Isom} M \) for \( M = S^k, \mathbb{E}^k, \) and \( \mathbb{H}^k \).

Case 1: \( M = S^k \). In this case one can in fact show that every solvable subalgebra is abelian. A solvable subalgebra of \( T_1 \text{Isom}_0 S^k \cong \mathfrak{so}_{k+1} \mathbb{R} \) is tangent to a connected solvable group, whose closure in \( \text{SO}(k+1) \) is solvable since solvability is a closed condition. A connected solvable Lie group is abelian if it is compact [Kna02, Cor. IV.4.25].

Case 2: \( M = \mathbb{E}^k \). Suppose \( x \) and \( y \) generate a nonabelian subalgebra \( \mathfrak{g} \) of \( T_1 \text{Isom}_0 \mathbb{E}^k \cong \mathbb{R}^k \oplus \mathfrak{so}_k \). If their images in \( \mathfrak{so}_k \) do not commute, then Case 1 implies \( \mathfrak{g} \) is not solvable.

Otherwise, write \( x = (t_x \in \mathbb{R}^k, r_x \in \mathfrak{so}_k) \) and \( y = (t_y, r_y) \). Then

\[ [x, y] = r_x t_y - r_y t_x \]
\[ [x, [x, y]] = r_x^2 t_y \]
\[ [y, [y, x]] = r_y^2 t_x. \]
Since \([x, y] \neq 0\), at least one of \(r_x t_y\) and \(r_y t_x\) is nonzero. Then since \(r_x\) and \(r_y\) act semisimply on \(\mathbb{R}^k\), at least one of \([x, [x, y]]\) and \([y, [y, x]]\) is nonzero. Recursing, the lower central series of \(\mathfrak{g}\) is never zero, so \(\mathfrak{g}\) is not nilpotent.

**Case 3:** \(M = \mathbb{H}^k\). Suppose \(\mathfrak{g}\) is a nilpotent subalgebra of \(\text{so}_{1,k} \cong T_1 \text{Isom}_0 \mathbb{H}^k\), with corresponding group \(G\). Every connected solvable subgroup of \(\text{Isom}_0 \mathbb{H}^k\) fixes a point either in \(\mathbb{H}^k\) or in its boundary at infinity \(\partial_{\infty} \mathbb{H}^k \cong S^{k-1}\) [Rat06, Thm. 5.5.10]. Then:

- If this fixed point is in \(\mathbb{H}^k\), then \(G \subseteq \text{SO}(k)\), which makes \(G\) abelian by Case 1.
- Otherwise, the fixed point is in \(\partial_{\infty} \mathbb{H}^k\). Since \(\text{Isom}_0 \mathbb{H}^k \cong \text{Conf}^+ S^{k-1}\) [BP92, Prop. A.5.13(4)] acts on \(S^{k-1}\) with point stabilizer \(\text{Conf}^+ \mathbb{E}^{k-1}\) [BP92, Cor. A.3.8], this implies that \(G \subseteq \text{Conf}^+ \mathbb{E}^{k-1} \cong \mathbb{R} \times \text{Isom}_0 \mathbb{E}^{k-1}\), which makes \(G\) is abelian by Case 2.

**Proposition 6.39.** The five associated bundle geometries classified in Prop. 6.23 are maximal.

**Proof.** Let \(M = G/G_p\) be one of the associated bundle geometries.

**Step 1:** The isometry of any subsuming geometry is \(\text{SO}(5)\) or \(\text{SO}(3) \times \text{SO}(2)\). In any geometry \(G'/G_p'\) properly subsuming \(M\), the isogropy \(G_p'\) must contain \(\text{SO}(2)^2\)—so consulting Figure 2.4, \(G_p'\) is one of \(\text{SO}(3) \times \text{SO}(2)\), \(\text{U}(2)\), \(\text{SO}(4)\), or \(\text{SO}(5)\).

In fact, \(G_p'\) cannot be \(\text{U}(2)\) or \(\text{SO}(4)\). If this were the case, \(G'\) would preserve \(TM^G\), inducing a \(G\)-equivariant diffeomorphism \(M/F^G \to M/F^{G'}\). But \(M/F^G\) is a product of 2-dimensional maximal model geometries with \(G\) acting by the isometry group, whereas (consulting the classification in Prop. 4.1) a base space of a geometry with \(\text{U}(2)\) or \(\text{SO}(4)\) isotropy can only be \(S^4\), \(\mathbb{E}^4\), \(\mathbb{C} P^2\), \(\mathbb{C}^2\), and \(\mathbb{C} \mathbb{H}^2\).

**Step 2:** Geometries involving \(\text{Heis}_3\) are maximal. In the isometry group of a constant-curvature geometry, every connected nilpotent subgroup is abelian (Lemma 6.38); so this holds for products of such groups too. Since all geometries with isotropy \(\text{SO}(5)\) or \(\text{SO}(3) \times \text{SO}(2)\) are constant-curvature geometries or products thereof, their isometry groups do not contain subgroups covered by \(\text{Heis}_3\). Therefore the geometries \(\text{Heis}_3 \times \mathbb{R} S^3\) and \(\text{Heis}_3 \times \mathbb{R} \text{SL}_2\) are maximal.

**Step 3:** \(\text{SO}(5)\)-isotropy geometries do not subsume. Since \(\tilde{\text{SL}}_2 \times_{\alpha} S^3\) is an \(S^3\) bundle over \(\mathbb{R}^2\), its nonzero \(\pi_3\) distinguishes it from the \(\text{SO}(5)\)-isotropy geometries. Similarly, \(L(a; 1) \times_{\alpha} L(b; 1)\) is an \(S^3\) bundle over \(S^2\), so it is distinguished from the \(\text{SO}(5)\) geometries by having nontrivial \(\pi_2\).

To it distinguish from \(\mathbb{H}^5\), observe that the image in \(\tilde{\text{SL}}_2 \times_{\alpha} \tilde{\text{SL}}_2\) of \(\tilde{\text{SL}}_2 \times \{1\} \times \mathbb{R} \subseteq \tilde{\text{SL}}_2 \times \tilde{\text{SL}}_2 \times \mathbb{R}\) is a copy of \(\tilde{\text{SL}}_2 \times \text{SO}(2)/\text{SO}(2)\), fixed by an \(\text{SO}(2)\) with trivial projection to the first \(\tilde{\text{SL}}_2\) factor; whereas in \(\mathbb{H}^5\), every group of isometries conjugate to \(\text{SO}(2)\) fixes a copy of \(\mathbb{H}^3\).

**Step 4:** \(\text{SO}(3) \times \text{SO}(2)\) geometries do not subsume. If \(\mathbb{R}\) and \(V\) denote the 1-dimensional trivial and 2-dimensional maximal representations of \(\text{SO}(2)\), then the tangent space \(T_p M\) decomposes into three nonisomorphic representations of \(G_p = \text{SO}(2) \times \text{SO}(2)\) as\(^{10}\)

\[
T_p M \cong (\mathbb{R} \otimes \mathbb{R}) \oplus (\mathbb{R} \otimes V) \oplus (V \otimes \mathbb{R}),
\]

\(^{10}\) Over \(\mathbb{C}\), the irreducible representations of a direct product of groups are the tensor products of their irreducible representations; see e.g. [BD85, Prop. II.4.14]
Table 6.42: 3-dimensional fixed sets of order 2 isometries

| Geometry                      | Fixed set |
|-------------------------------|-----------|
| \( \text{SL}(3, \mathbb{R}) / \text{SO}(3) \) | \( \mathbb{H}^2 \times \mathbb{E} \) |
| \( \mathbb{F}_0^5 \) = Heis_3 \times \text{SL}(2, \mathbb{R}) / \text{SO}(2) \) | \( \mathbb{H}^2 \times \mathbb{E} \) |
| \( \mathbb{F}_1^5 \) | \( \text{SL}_2 \times \text{SL}_2 \) |
| \( \mathbb{R}^2 \times \text{SL}_2 \) | \( \text{SL}_2 \times \text{SL}_2 \) |

which determines two invariant 2-dimensional distributions. From the description of \( M \) as a bundle over 2-by-2 product geometries (Table 6.29, Prop. 6.34), neither of these is integrable. Then \( M \) cannot be subsumed by a 3-by-2 product geometry, since every geometry with \( \text{SO}(3) \times \text{SO}(2) \) isotropy has an invariant integrable 2-dimensional distribution.

Remark 6.40. Proposition 6.39 provides a negative answer to the question raised by Filipkiewicz in the discussion after [Fil83, Prop. 1.1.2]: is every non-maximal geometry subsumed by a unique maximal geometry?

The counterexample is \( T^1 S^3 \cong \text{SO}(4) / \text{SO}(2) \cong S^3 \times S^3 / \Delta(S^1) \) where \( \Delta : S^3 \to S^3 \times S^3 \) is the diagonal map. It is subsumed:

1. by \( S^3 \times S^2 \) since \( S^3 \) is parallelizable by left-invariant vector fields, and
2. by \( L(1; 1) \times_{S^1} L(1; 1) = S^3 \times S^3 \times \mathbb{R} / \tau_{1,1}(\mathbb{R}^2) \) since \( \tau_{1,1}(\mathbb{R}^2) \) meets \( S^3 \times S^3 \) in an antidiagonal (which is conjugate to diagonal) copy of \( S^1 \).

Since \( S^3 \times S^2 \) is maximal as a product and \( L(1; 1) \times_{S^1} L(1; 1) \) is maximal due to Prop. 6.39, a subsuming geometry for \( \text{SO}(4) / \text{SO}(2) \) is not unique.

One may alternatively check that \( S^3 \times S^2 \) does not subsume \( L(1; 1) \times_{S^1} L(1; 1) \) by classifying all homomorphisms

\[
S^3 \times S^3 \times \mathbb{R} \to (S^3)^3 \cong (\text{Isom}(S^3 \times S^2)) \end{equation}

using the fact that \( \text{Aut} S^3 = \text{Inn} S^3 \).

Proposition 6.41. All geometries listed in Prop. 6.17 are maximal.

Proof. Let \( M = G/H \) be one of the geometries named, and suppose \( G'/H' \) subsumes it.

Step 1: List restrictions on subsuming geometries. Since \( G/H \) is contractible, so is \( G'/H' \). Since \( G \) contains a group covered by \( \text{SL}_2 \), so does \( G' \). Finally, \( H' \) contains \( (S^1)^2 \) in the case of \( T^1 \mathbb{E}^{1,2} \) and cannot preserve \( TM^G \)—since if it did, then it would preserve the fibering \( M \to M/\mathcal{F}^G \); but these geometries are the only ones encountered in the classification for which \( M/\mathcal{F}^G \) is \( \mathbb{R}^4 \) or \( T \mathbb{H}^2 \). Consulting Figure 2.4, this last restriction implies \( H \) is \( \text{SO}(5) \), \( \text{SO}(3) \times \text{SO}(2) \), or \( \text{SO}(3)_5 \).

Then by the classification so far, \( G'/H' \) is either \( \text{SL}(3, \mathbb{R}) / \text{SO}(3) \) or a product of a hyperbolic space with zero or more Euclidean or hyperbolic spaces.
Step 2: Find 3-dimensional fixed sets of order 2 isometries. Such a fixed set in \( G/H \) must also appear in \( G'/H' \), so we list these. See Table 6.42 for a summary.

- Using the classification of isometries of \( \mathbb{H}^n \) [BP92, A.5.14] and the fact that \( \text{SO}(k) \) is maximal compact in \( \text{Isom}_0 \mathbb{E}^k \), these fixed sets in products of Euclidean and hyperbolic spaces are also products of Euclidean and hyperbolic spaces.

- In \( \text{SL}(3, \mathbb{R})/\text{SO}(3) \), an order 2 isometry is a \( 3 \times 3 \) matrix with a positive even number of \(-1\) eigenvalues. Its centralizer is conjugate (by diagonalization) to the subgroup of \( 2 + 1 \) block matrices; and the orbit of this subgroup is the isometry’s fixed set,

\[
S(\text{GL}(2, \mathbb{R}) \times \text{GL}(1, \mathbb{R}))^0 / \text{SO}(2) \cong \text{SL}(2, \mathbb{R}) \times \mathbb{R} / \text{SO}(2) \cong \mathbb{H}^2 \times \mathbb{E}.
\]

- In \( \text{Heis}_3 \times \text{SL}(2, \mathbb{R})/\text{SO}(2) \), the order 2 isometries are all conjugate to the order 2 element of \( \text{SO}(2) \). Its centralizer is \( Z(\text{Heis}_3) \times \text{SL}(2, \mathbb{R}) \), with orbit \( \mathbb{E} \times \mathbb{H}^2 \).

- Following the same recipe for \( \mathbb{R}^2 \rtimes \widehat{\text{SL}}_2 \) and \( \mathbb{F}_1^5 \) produces a centralizer of \( \widehat{\text{SL}}_2 \times \text{SO}(2) \), with orbit \( \widehat{\text{SL}}_2 \).

This last result implies \( \mathbb{R}^2 \rtimes \widehat{\text{SL}}_2 \) and \( \mathbb{F}_1^5 \) are maximal, since none of the candidates for \( G'/H' \) have \( \widehat{\text{SL}}_2 \) as a fixed set of an order 2 isometry.

Step 3: \( \text{Heis}_3 \times \text{SL}(2, \mathbb{R})/\text{SO}(2) \) is maximal. If \( G \) contains \( \text{Heis}_3 \), then so does \( G' \). A copy of \( \text{Heis}_3 \) in a direct product must have nonabelian—hence locally injective—image in at least one factor. Since \( \text{Heis}_3 \) covers no subgroup of isometries of \( \mathbb{E}^k \) or \( \mathbb{H}^k \) (Lemma 6.38), only \( \text{SL}(3, \mathbb{R})/\text{SO}(3) \) can subsume.

By the classification of irreducible representations of \( \text{SL}(2, \mathbb{R}) \)—one in each dimension [FH91, 11.8]—all \( \text{SL}(2, \mathbb{R}) \) in \( \text{SL}(3, \mathbb{R}) \) are conjugate to a standard copy. Denoting the standard representation of \( \text{SL}(2, \mathbb{R}) \) by \( V \), counting weights (see e.g. [FH91, §11.2]) produces the decomposition

\[
\mathfrak{sl}_3 \mathbb{R} \cong_{\text{SL}(2,\mathbb{R})} \mathbb{R} \oplus 2V \oplus \mathfrak{sl}_2 \mathbb{R}.
\]

The two copies of \( V \) are \( \text{Hom}(\mathbb{R}^2, \mathbb{R}) \) and \( \text{Hom}(\mathbb{R}, \mathbb{R}^2) \), which are abelian subalgebras of \( \mathfrak{sl}_3 \mathbb{R} \) and therefore cannot generate a copy of \( \text{Heis}_3 \). Therefore \( \text{SL}(3, \mathbb{R})/\text{SO}(3) \) does not subsume \( \text{Heis}_3 \times \text{SL}(2, \mathbb{R})/\text{SO}(2) \).

Step 4: \( T^1\mathbb{E}^{1,2} \) is maximal. Since the isotropy of \( \text{SL}(3, \mathbb{R})/\text{SO}(3) \) does not contain \( S_1^1 \) (Fig. 2.4; see also [Gen16b, Prop. 3.1 footnote]), it cannot subsume \( T^1\mathbb{E}^{1,2} \). So it only remains to eliminate the products of hyperbolic and Euclidean spaces, as follows.

In \( \mathfrak{so}_{1,2} \mathbb{R} \), there is a matrix

\[
A = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix},
\]

which sends the third standard basis vector \( e_3 \) to \( e_1 - e_2 \) and sends \( e_1 - e_2 \) to zero. Then \( A \), \( e_3 \), and \( e_1 - e_2 \) span a copy of the 3-dimensional Heisenberg algebra in \( \mathbb{R}^3 \oplus \mathfrak{so}_{1,2} \mathbb{R} \). Since \( T_1 \text{Isom}_0 \mathbb{H}^k \) and \( T_1 \text{Isom}_0 \mathbb{E}^k \) cannot contain the Heisenberg algebra (Lemma 6.38), neither can a product of them, since the projection to at least one factor would have to be nonabelian and thereby injective. Then none of the products of hyperbolic and Euclidean spaces can subsume \( T^1\mathbb{E}^{1,2} \).

\[\square\]
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