Minimax Lower Bounds on Dictionary Learning for Tensor Data
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Abstract
This paper provides fundamental limits on the sample complexity of estimating dictionaries for tensor data. The specific focus of this work is on $K$-dimensional tensor data, in which case the underlying dictionary can be expressed in terms of $K$ smaller dictionaries. The data are generated by linear combinations of these structured dictionary atoms and observed through white Gaussian noise. This work first provides a general lower bound on the minimax risk of dictionary learning for tensor data and then adapts the proof techniques for equivalent results in the case of sparse and sparse-Gaussian linear combinations. The results suggest the sample complexity of dictionary learning for tensor data can be significantly lower than that for unstructured data: for unstructured data it scales linearly with the product of the dictionary dimensions, whereas it need only scale linearly with the sum of the product of the dimensions of the (smaller) component dictionaries for tensor data. A partial converse is provided for the case of 2-dimensional tensor data. This involves developing an algorithm for learning of highly-structured dictionaries from noisy tensor data and showing that it achieves one of the minimax lower bounds derived in the paper. Finally, numerical experiments highlight the advantages associated with explicitly accounting for tensor data structure during dictionary learning.

Index Terms
Dictionary learning, Kronecker-structured dictionary, minimax bounds, sparse representations, tensor data.

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I. INTRODUCTION

Dictionary learning is a technique for finding sparse representations of signals or data and has applications in various tasks such as image denoising and inpainting [2], audio processing [3], and classification [4], [5]. Given input training signals \( \{ y_k \in \mathbb{R}^m \}_{k=1}^N \), the goal in dictionary learning is to construct an overcomplete basis, \( D \in \mathbb{R}^{m \times p} \), such that each signal in \( Y = [y_1, \ldots, y_N] \) can be described by a small number of atoms (columns) of \( D \) [6]. This problem can be posed as the following optimization program:

\[
\min_{D, X} \| Y - DX \|_F \quad \text{subject to } \forall k, \| x_k \|_0 \leq s,
\]

where \( x_k \) is the coefficient vector associated with \( y_k \), \( \| \cdot \|_0 \) counts the number of nonzero entries and \( s \) is the maximum number of nonzero elements of \( x_k \). Although existing literature has mostly focused on dictionary learning for one-dimensional data [2]–[6], many real-world signals are multidimensional and have a tensor structure: examples include 2-dimensional images, 3-dimensional videos, and 4-dimensional signals produced via magnetic resonance or computed tomography systems. In traditional dictionary learning literature, multidimensional data are converted into one-dimensional data by vectorization of signals. Such approaches can result in poor sparse representations because they neglect the multidimensional structure of the data [7]. Therefore, it is essential to keep the original tensor structure of multidimensional data for efficient dictionary learning and reliable subsequent processing.

In this paper, our focus is on dictionary learning algorithms that explicitly account for the tensor structure of data: we call such dictionaries Kronecker structured (KS). There have been several algorithms proposed in the literature that can be used to learn KS dictionaries [7]–[13]. In Hawe et al. [9], a Riemannian conjugate gradient method combined with a nonmonotone line search is used for this purpose. Other works rely on various tensor decomposition methods such as the Tucker decomposition [10], [11], [14], the CANDECOMP/PARAFAC (CP) decomposition [8], [15], the HOSVD decomposition [12], [16], the t-product tensor factorization [13], and the tensor-SVD decomposition [7], [17]. To the best of our knowledge, however, none of these works provide an understanding of the sample complexity of KS dictionary learning algorithms. In contrast, our focus in this work is on understanding fundamental limits associated with learning KS dictionaries. To this end, we provide lower bounds on the minimax risk of estimating KS dictionaries from tensor data using any estimator. These bounds not only provide means of quantifying the performance of existing KS dictionary learning algorithms, but they also allude to the potential benefits of explicitly accounting for tensor structure of data during dictionary learning.
A. Our Contributions

Our first result is a general lower bound for the mean squared error (MSE) of estimating dictionaries consisting of $K \geq 2$ coordinate dictionaries that sparsely represent $K$-dimensional tensor data. Here, we define the minimax risk to be the worst-case MSE that is attainable by the best dictionary estimator. Our approach uses the standard procedure for lower bounding the minimax risk in nonparametric estimation by connecting it to the maximum probability of error on a carefully constructed multiple hypothesis testing problem \cite{18}, \cite{19}: the technical challenge is in finding the right hypotheses. In particular, consider a dictionary $\mathbf{D} \in \mathbb{R}^{m \times p}$ consisting of the Kronecker product of $K$ coordinate dictionaries $\mathbf{D}_i \in \mathbb{R}^{m_i \times p_i}$, $i \in \{1, \ldots, K\}$, where $m = \prod_{i=1}^{K} m_i$ and $p = \prod_{i=1}^{K} p_i$, that is generated within the radius $r$ neighborhood (taking the Frobenius norm as the distance metric) of a fixed reference dictionary. Then, our analysis shows that given a sufficiently large $r$ and keeping some other parameters constant, a sample complexity of $N = \Omega(\sum_{i=1}^{K} m_i p_i)$ is necessary for reconstruction of the true dictionary up to a given estimation error. We also provide minimax bounds on the KS dictionary learning problem that hold for different sampling distributions for the coefficient vectors:

- The coefficient vectors are independent and identically distributed (i.i.d.) and can have any distribution with zero mean.
- The coefficient vectors are i.i.d. and sparse.
- The coefficient vectors are i.i.d., sparse, and their non-zero elements follow Gaussian distribution.

Our second contribution is development and analysis of an algorithm to learn dictionaries formed by the Kronecker product of 2 smaller dictionaries that can be used to represent 2-dimensional tensor data. To this end, we show that under certain conditions on the local neighborhood, the proposed algorithm can achieve one of the earlier obtained minimax lower bounds. While not a complete converse, this result suggests our lower bounds may be tight in more general settings.

B. Relationship to Previous Work

In terms of relation to prior work, theoretical insights into the problem of dictionary learning have either focused on existing algorithms for non-KS dictionaries \cite{20}–\cite{26} or lower bounds on minimax risk of dictionary learning for one-dimensional data \cite{27}, \cite{28}. The former works provide sample complexity results for reliable dictionary estimation based on appropriate minimization criteria. Specifically, given a probabilistic model for sparse coefficients and a finite number of samples, these works show that a non-KS dictionary is recoverable within some distance of the true dictionary as a local minimum of some
minimization criterion [24]–[26]. In contrast, Jung et al. [27], [28] provide minimax lower bounds for dictionary learning from one-dimensional data under several coefficient vector distributions and discuss a regime where the bounds are tight for some signal-to-noise (SNR) values. Particularly, for a given dictionary $D$ and sufficiently large neighborhood radius $r$, they show that $N = \Omega(mp)$ samples are required for reliable recovery of the dictionary up to a prescribed MSE within its local neighborhood. However, in the case of tensor data, their approach fails to account for the structure of the data in the dictionary learning problem.

To provide lower bounds on the minimax risk of KS dictionary learning, we adopt the same general approach to minimax bounds as Jung et al. [27], [28] do for the vector case, based on connecting the estimation problem to a multiple-hypothesis testing problem and invoking Fano’s inequality [19]. We construct a family of KS dictionaries which induce similar observation distributions but have a minimum separation from each other. By explicitly taking into account the Kronecker structure of the dictionaries, we show that the sample complexity satisfies a lower bound of $\Omega(\sum_{i=1}^{K} m_i p_i)$ compared to the $\Omega(mp)$ bound from vectorizing the data [28]. Although our general approach is similar to that in [28], there are fundamental differences in the construction of the KS dictionary class and analysis of the minimax risk. This generalizes our earlier work [29] from 2-dimensions to $K$-dimensions and provides a comprehensive analysis of the KS dictionary class construction and minimax lower bounds.

Our results essentially show that the sample complexity depends linearly on the degrees of freedom of a Kronecker structured dictionary, which is $\sum_{i=1}^{K} m_i p_i$, and non-linearly on the SNR and tensor dimension $K$. These lower bounds also depend on the radius of the local neighborhood around a fixed reference dictionary. Our results hold even when some of the coordinate dictionaries are not overcomplete (note that all coordinate dictionaries cannot be undercomplete, otherwise $D$ won’t be overcomplete). Like previous work [28], our analysis is local and our lower bounds depend on the distribution of multidimensional data. Finally, some of our analysis relies on the availability of side information about the signal samples. This suggests that the lower bounds may be improved by deriving them in the absence of such side information.

We next study a KS dictionary learning problem for 2-dimensional tensor data and introduce a corresponding KS dictionary learning algorithm. We show that in this case, one of the provided minimax lower bounds is achievable. We also conduct numerical experiments that demonstrate the empirical performance of the algorithm relative to the MSE upper bound and in comparison to the performance of a non-KS dictionary learning algorithm [28]. The results confirm the effectiveness of learning KS dictionaries for tensor data.
C. Notational Convention and Preliminaries

Underlined bold upper-case, bold upper-case and lower-case letters are used to denote tensors, matrices and vectors, respectively. Lower-case letters denote scalars. The $k$-th column of $X$ is denoted by $x_k$ and its $ij$-th element is denoted by $x_{ij}$. Sometimes we use matrices indexed by multiple letters, such as $X_{(a,b,c)}$, in which case its $j$-th column is denoted by $x_{(a,b,c),j}$. Let $X_T$ be the matrix consisting of columns of $X$ with indices $T$, $X^{(T,T)}$ be the matrix consisting of rows of $X$ with indices $T$ and $I_d$ be the $d \times d$ identity matrix. For a tensor $X \in \mathbb{R}^{p_1 \times \cdots \times p_K}$, its $(i_1, \ldots, i_K)$-th element is denoted as $x_{i_1 \ldots i_K}$. Norms are given by subscripts, so $\|u\|_0$ and $\|u\|_2$ are the $\ell_0$ and $\ell_2$ norms of $u$, respectively, and $\|X\|_2$ and $\|X\|_F$ are the spectral and Frobenius norms of $X$, respectively. We use $\text{vec}(X)$ to denote the vectorized version of matrix $X$, which is a column vector obtained by stacking the columns of $X$ on top of one another. We write $[K]$ for $\{1, \ldots, K\}$. For matrices $X_1$ and $X_2$, we define their distance to be

$$d(X_1, X_2) = \|X_1 - X_2\|_F.$$ 

We define the outer product of two vectors of the same dimension, $u$ and $v$, as $u \odot v = uv^T$ and the inner product between matrices of the same size, $X_1$ and $X_2$, as $\langle X_1, X_2 \rangle = \text{Tr}(X_1^TX_2)$. Furthermore, $P_{B_1}(u)$ denotes the projection of $u$ on the closed unit ball, i.e.

$$P_{B_1}(u) = \begin{cases} u, & \text{if } \|u\|_2 \leq 1, \\ \frac{u}{\|u\|_2}, & \text{otherwise}. \end{cases} \tag{2}$$

We use $f(\varepsilon) = \mathcal{O}(g(\varepsilon))$ and $f(\varepsilon) = \Omega(g(\varepsilon))$ if for sufficiently large $n \in \mathbb{N}$, $f(n) < C_1g(n)$ and $f(n) > C_2g(n)$, respectively, for some positive constants $C_1$ and $C_2$.

We now define some important matrix products. We write $X_1 \otimes X_2$ for the Kronecker product of two matrices $X_1 \in \mathbb{R}^{n \times n}$ and $X_2 \in \mathbb{R}^{p \times q}$, defined as

$$X_1 \otimes X_2 = \begin{bmatrix} x_{1,11}X_2 & x_{1,12}X_2 & \cdots & x_{1,1m}X_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_{1,n1}X_2 & x_{1,n2}X_2 & \cdots & x_{1,nm}X_2 \end{bmatrix}, \tag{3}$$

where the result is an $mp \times nq$ matrix and we have $\|X_1 \otimes X_2\|_F = \|X_1\|_F \|X_2\|_F^{[30]}$. Given matrices $X_1, X_2, Y_1,$ and $Y_2$, where products $X_1 Y_1$ and $X_2 Y_2$ can be formed, we have $[31]$:

$$(X_1 \otimes X_2)(Y_1 \otimes Y_2) = (X_1 Y_1) \otimes (X_2 Y_2). \tag{4}$$

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Given $X_1 \in \mathbb{R}^{m \times n}$ and $X_2 \in \mathbb{R}^{p \times n}$, we write $X_1 \ast X_2$ for their $mp \times n$ Khatri-Rao product \cite{31}, defined by

$$X_1 \ast X_2 = \begin{bmatrix} x_{1,1} \otimes x_{2,1} & x_{1,2} \otimes x_{2,2} & \cdots & x_{1,n} \otimes x_{2,n} \end{bmatrix}. \quad (5)$$

This is essentially the column-wise Kronecker product of matrices $X_1$ and $X_2$. We also use $\otimes_{i \in K} X_i = X_1 \otimes \cdots \otimes X_K$ and $\ast_{i \in K} X_i = X_1 \ast \cdots \ast X_K$.

Next, we review essential properties of $K$-order tensors and the relation between tensors and the Kronecker product of matrices using the Tucker decomposition of tensors.

1) A Brief Review of Tensors: A tensor is a multidimensional array where the order of the tensor is defined as the number of components in the array. A tensor $X \in \mathbb{R}^{p_1 \times p_2 \times \cdots \times p_K}$ of order $K$ can be expressed as a matrix by reordering its elements to form a matrix. This reordering is called unfolding: the mode $n$ unfolding matrix of a tensor is a $p_n \times \prod_{j \neq n} p_j$ matrix, which we denote by $X_{(n)}$. Each column of $X_{(n)}$ consists of the vector formed by fixing all indices of $X$ except the one in the $n$-th dimension. For example, for a 2-dimensional tensor $X$, the mode 1 and mode 2 unfolding matrices are $X$ and $X^T$, respectively. The $n$-rank of a tensor $X$ is defined by $\text{rank}(X_{(n)})$; trivially, $\text{rank}(X_{(n)}) \leq p_n$.

The mode $n$ matrix product of the tensor $X$ and a matrix $A \in \mathbb{R}^{m_n \times p_n}$, denoted by $X \times_n A$, is a tensor of size $p_1 \times \cdots p_{n-1} \times m_n \times p_{n+1} \cdots \times p_K$ whose elements are

$$(X \times_n A)_{i_1 \ldots i_{n-1} j i_{n+1} \ldots i_K} = \sum_{i_n=1}^{p_n} x_{i_1 \ldots i_{n-1} i_n i_{n+1} \ldots i_K} a_{ji_n}. \quad (6)$$

The mode $n$ matrix product of $X$ and $A$ and the matrix multiplication of $X_{(n)}$ and $A$ are related \cite{32}:

$$Y = X \times_n A \Leftrightarrow Y_{(n)} = AX_{(n)}. \quad (7)$$

2) Tucker Decomposition for Tensors: The Tucker decomposition is a powerful tool that decomposes a tensor into a core tensor multiplied by a matrix along each mode \cite{14}, \cite{32}. We take advantage of the Tucker model since the Tucker decomposition can be related to the Kronecker representation of tensors \cite{33}. For the tensor $Y \in \mathbb{R}^{m_1 \times m_2 \times \cdots \times m_K}$ of order $K$, if $\text{rank}(Y_{(n)}) \leq p_n$ holds for all $n \in [K]$ then, according to the Tucker model, $Y$ can be decomposed into:

$$Y = X_1 D_1 \times_2 D_2 \cdots \times_K D_K, \quad (8)$$
where $X \in \mathbb{R}^{p_1 \times p_2 \times \cdots \times p_K}$ denotes the core tensor and $D_i \in \mathbb{R}^{m_i \times p_i}$ are factor matrices. Here, (8) can be interpreted as a form of higher order principal component analysis (PCA):

$$Y = \sum_{i_1 \in [p_1]} \cdots \sum_{i_K \in [p_K]} x_{i_1 \cdots i_K} d_{1,i_1} \odot \cdots \odot d_{K,i_K}, \quad (9)$$

where the $D_i$'s can be interpreted as the principle components in mode $i$. The following is implied by (8) [32]:

$$Y_{(n)} = D_n X_{(n)} (D_K \otimes \cdots \otimes D_{n+1} \otimes D_{n-1} \otimes \cdots \otimes D_1)^\top. \quad (10)$$

Since the Kronecker product satisfies $\text{vec}(BXA^\top) = (A \otimes B) \text{vec}(X)$ [34], (8) is equivalent to [32], [33]

$$\text{vec}(Y) = (D_K \otimes D_{K-1} \otimes \cdots \otimes D_1) \text{vec}(X), \quad (11)$$

where $\text{vec}(Y) \triangleq \text{vec}(Y_{(1)})$ and $\text{vec}(X) \triangleq \text{vec}(X_{(1)})$.

The rest of the paper is organized as follows. We formulate the KS dictionary learning problem and describe the procedure for obtaining minimax risk lower bounds in Section II. Next, we provide a lower bound for general coefficient distribution in Section III and in Section IV we present lower bounds for sparse and sparse Gaussian coefficient vectors. We propose a KS dictionary learning algorithm for 2-dimensional tensor data and analyze its corresponding MSE and empirical performance in Section V. In Section VI, we discuss and interpret the results. Finally, in Section VII we conclude the paper.

### II. Problem Formulation

In the conventional dictionary learning setup, it is assumed that the observations $y_k \in \mathbb{R}^m$ are generated via a fixed dictionary

$$y_k = Dx_k + n_k, \quad (12)$$

in which the dictionary $D \in \mathbb{R}^{m \times p}$ is an overcomplete basis $(m < p)$ with unit-norm columns and rank $m$, $x_k \in \mathbb{R}^p$ is the coefficient vector, and $n_k \in \mathbb{R}^m$ denotes observation noise.

Our focus in this work is on multidimensional signals. We assume the observations are $K$-dimensional tensors $Y_k \in \mathbb{R}^{m_1 \times m_2 \times \cdots \times m_K}$. According to the Tucker model, given coordinate dictionaries $D_i \in \mathbb{R}^{m_i \times p_i}$, coefficient tensor $X_k \in \mathbb{R}^{p_1 \times p_2 \times \cdots \times p_K}$, and noise tensor $N_k$, we can write $y_k \triangleq \text{vec}(Y_k)$ using
\[ y_k = \left( \bigotimes_{i \in [K]} D_i \right) x_k + n_k, \quad (13) \]

where \( x_k \triangleq \text{vec}(X_k) \) and \( n_k \triangleq \text{vec}(N_k) \). Let \( m = \prod_{i \in [K]} m_i \) and \( p = \prod_{i \in [K]} p_i \). Concatenating \( N \) i.i.d noisy observations \( \{y_k\}_{k=1}^N \), which are realizations according to the model (13), into \( Y \in \mathbb{R}^{m \times N} \), we obtain

\[ Y = DX + N, \quad (14) \]

where \( D \triangleq \bigotimes_{i \in [K]} D_i \) is the unknown KS dictionary, \( X \in \mathbb{R}^{p \times N} \) is a coefficient matrix consisting of i.i.d. random coefficient vectors with known distribution that has zero-mean and covariance matrix \( \Sigma_x \), and \( N \in \mathbb{R}^{m \times N} \) is assumed to be additive white Gaussian noise (AWGN) with zero mean and variance \( \sigma^2 \).

Our main goal in this paper is to derive conditions under which the KS dictionary \( D \) can possibly be learned from the noisy observations given in (14). We assume the true KS dictionary \( D \) consists of unit norm columns and we carry out local analysis. That is, the true KS dictionary \( D \) is assumed to belong to a neighborhood around a fixed (normalized) reference KS dictionary

\[ D_0 = \bigotimes_{i \in [K]} D_{(0,i)}, \quad \|d_{(0,i),j}\|_2 = 1 \quad \forall i \in [K], j \in [p_i], \quad (15) \]

and \( D_0 \in \mathcal{D} \):

\[ \mathcal{D} \triangleq \{ D' \in \mathbb{R}^{m \times p} : \|d'_{j}\|_2 = 1 \quad \forall j \in [p], \quad D' = \bigotimes_{i \in [K]} D'_{i}, \quad D'_{i} \in \mathbb{R}^{m_i \times p_i}, \quad \forall i \in [K] \}. \quad (16) \]

We assume the true generating KS dictionary \( D \) belongs to a neighborhood around \( D_0 \):

\[ D \in \mathcal{X}(D_0, r) \triangleq \{ D' \in \mathcal{D} : \|D' - D_0\|_F < r \} \quad (17) \]

for some fixed radius \( r \). Note that \( D_0 \) appears in the analysis as an artifact of our proof technique to construct the dictionary class. In particular, if \( r \) is sufficiently large, then \( \mathcal{X}(D_0, r) \approx \mathcal{D} \) and effectively \( D \in \mathcal{D} \).

A. Minimax Risk

We are interested in lower bounding the minimax risk for estimating \( D \) based on observations \( Y \), which is defined as the worst-case mean squared error (MSE) that can be obtained by the best KS dictionary...
estimator $\hat{\mathcal{D}}(\mathbf{Y})$. That is,

$$
\varepsilon^* = \inf_{\hat{\mathcal{D}}} \sup_{\mathcal{D} \in \mathcal{X}(\mathbf{D}_0, r)} \mathbb{E}_Y \left\{ \| \hat{\mathcal{D}}(\mathbf{Y}) - \mathcal{D} \|_F^2 \right\},
$$

(18)

where $\hat{\mathcal{D}}(\mathbf{Y})$ can be estimated using any KS dictionary learning algorithm. In order to lower bound this minimax risk $\varepsilon^*$, we employ a standard reduction to the multiple hypothesis testing used in the literature on nonparametric estimation [18], [19]. In this case, $\hat{\mathcal{D}}(\mathbf{Y})$ is used to solve a multiple hypothesis testing problem. This approach is equivalent to generating a KS dictionary $\mathcal{D}_l$ uniformly at random from a carefully constructed class $\mathcal{D}_L = \{\mathcal{D}_1, \ldots, \mathcal{D}_L\} \subseteq \mathcal{X}(\mathbf{D}_0, r), L \geq 2$, for a given $(\mathbf{D}_0, r)$. To ensure a tight lower bound, we must construct $\mathcal{D}_L$ such that the distance between any two dictionaries in $\mathcal{D}_L$ is large but the hypothesis testing problem is hard; that is, two distinct dictionaries $\mathcal{D}_l$ and $\mathcal{D}_{l'}$ should produce similar observations. Specifically, for $l, l' \in [L]$, and given error $\varepsilon \geq \varepsilon^*$, we desire a construction such that

$$
\forall l \neq l', \| \mathcal{D}_l - \mathcal{D}_{l'} \|_F \geq 2\sqrt{\gamma \varepsilon} \quad \text{and} \\
D_{KL}(f_{\mathcal{D}_l}(\mathbf{Y}) || f_{\mathcal{D}_{l'}}(\mathbf{Y})) \leq \alpha_L,
$$

(19)

where $D_{KL}(f_{\mathcal{D}_l}(\mathbf{Y}) || f_{\mathcal{D}_{l'}}(\mathbf{Y}))$ denotes the Kullback-Leibler (KL) divergence between the distributions of observations based on $\mathcal{D}_l \in \mathcal{D}_L$ and $\mathcal{D}_{l'} \in \mathcal{D}_L$, while $\gamma, \alpha_L$, and $\varepsilon$ are non-negative parameters. Observations $\mathbf{Y} = \mathbf{D}_l \mathbf{X} + \mathbf{N}$ in this setting can be interpreted as channel outputs that are used to estimate the input $\mathbf{D}_l$ using an arbitrary KS dictionary algorithm that is assumed to achieve the error $\varepsilon$.

Our goal is to detect the correct generating KS dictionary index $l$. For this purpose, a minimum distance detector is used:

$$
\hat{l} = \min_{l' \in [L]} \| \hat{\mathcal{D}}(\mathbf{Y}) - \mathcal{D}_{l'} \|_F.
$$

(20)

Then, we have $\mathbb{P}(\hat{l}(\mathbf{Y}) \neq l) = 0$ for the minimum-distance detector $\hat{l}(\mathbf{Y})$ as long as $\| \hat{\mathcal{D}}(\mathbf{Y}) - \mathcal{D}_l \|_F < \sqrt{\gamma \varepsilon}$. The goal then is to relate $\varepsilon$ to $\mathbb{P}(\| \hat{\mathcal{D}}(\mathbf{Y}) - \mathcal{D}_l \|_F \geq \sqrt{\gamma \varepsilon})$ and $\mathbb{P}(\hat{l}(\mathbf{Y}) \neq l)$ using Fano’s inequality [19]:

$$
(1 - \mathbb{P}(\hat{l}(\mathbf{Y}) \neq l)) \log_2 L - 1 \leq I(\mathbf{Y}; l),
$$

(21)

where $I(\mathbf{Y}; l)$ denotes the mutual information (MI) between the observations $\mathbf{Y}$ and the dictionary $\mathcal{D}_l$. Notice that the smaller $\alpha_L$ is in [19], the smaller $I(\mathbf{Y}; l)$ will be in [21]. Unfortunately, explicitly evaluating $I(\mathbf{Y}; l)$ is a challenging task in our setup because of the underlying distributions. Similar
to [28], we will instead resort to upper bounding \( I(Y; l) \) by assuming access to some side information \( T(X) \) that will make the observations \( Y \) conditionally multivariate Gaussian (recall that \( I(Y; l) \leq I(Y; l|T(X)) \)). A lower bound on the minimax risk in this setting depends not only on problem parameters such as the number of observations \( N \), noise variance \( \sigma^2 \), dimensions \( \{m_i\}_{i=1}^K \) and \( \{p_i\}_{i=1}^K \) of the true KS dictionary, neighborhood radius \( r \), and coefficient covariance \( \Sigma_x \), but also on various aspects of the constructed class \( \mathcal{D}_L \) [18]. Our final results will then follow from the fact that any lower bound for \( \epsilon^* \) given the side information \( T(X) \) will also be a lower bound for the general case [28]. Note that our approach is applicable to the global KS dictionary learning problem, since the minimax lower bounds that are obtained for any \( D \in \mathcal{X}(D_0, r) \) are also trivially lower bounds for \( D \in \mathcal{D} \).

After providing minimax lower bounds for the KS dictionary learning problem, we develop and analyze a simple KS dictionary learning algorithm for \( K = 2 \) order tensor data. Our analysis shows that one of our provided lower bounds is achievable, suggesting the tightness of our lower bounds. Furthermore, we demonstrate the performance of the proposed algorithm via numerical experiments.

\[ B. \text{ Coefficient Distribution} \]

We obtain different lower bounds on the minimax risk under different assumptions on the coefficient distributions of the data generated using the true dictionary. To facilitate comparisons with prior work, we adopt somewhat similar coefficient distributions as in the unstructured case [28]. First, we consider any coefficient distribution and only assume that the coefficient covariance matrix exists. We then specialize our analysis to sparse coefficient vectors and, by adding additional conditions on the reference dictionary \( D_0 \), we obtain a tighter lower bound for the minimax risk for some SNR regimes.

1) \textit{General Coefficients}: First, we consider the general case, where \( x \) is a zero-mean random coefficient vector with covariance matrix \( \Sigma_x = \mathbb{E}_x \{ xx^\top \} \). We make no additional assumption on the distribution of \( x \). We assume side information \( T(X) = X \) to obtain a lower bound on the minimax risk.

2) \textit{Sparse Coefficients}: In the case where the coefficient vector is sparse, additional assumptions on the non-zero entries can be made to obtain a lower bound on the minimax risk using less side information, i.e., \( \text{supp}(x) \), which denotes the support of \( x \) (the set containing indices of the locations of the nonzero entries of \( x \)). We study two cases for the distribution of \( \text{supp}(x) \):

- \textbf{Random Sparsity}. In this case, the random support of \( x \) is distributed uniformly over \( \mathcal{E}_1 = \{ S \subseteq [p] : |S| = s \} \):

\[ \mathbb{P}(\text{supp}(x) = S) = \frac{1}{\binom{p}{s}}, \quad \text{for any } S \in \mathcal{E}_1. \]  \hfill (22)
• **Separable Sparsity.** In this case we sample $s_i$ elements uniformly at random from $[p_i]$, for all $i \in [K]$. The random support of $x$ is $E_2 = \{S_1 \times \cdots \times S_K : S_i \subseteq [p_i], |S_i| = s_i, i \in [K]\}$. The number of non-zero elements in $x$ in this case is $s = \prod_{i \in [K]} s_i$. The random support of $x$ is $\mathbb{E}_2 = \{S_1 \times \cdots \times S_K : S_i \subseteq [p_i], |S_i| = s_i, i \in [K]\}$. The probability of sampling $K$ subsets $\{S_1, \ldots, S_K\}$ is

$$P(\text{supp}(x) = S_1 \times \cdots \times S_K) = \frac{1}{\prod_{i \in [K]} (p_i/s_i)}, \quad \text{for any } S_1 \times \cdots \times S_K \in \mathbb{E}_2. \quad (23)$$

In other words, separable sparsity requires non-zero coefficients to be grouped in blocks. This model arises in the case of processing of images and video sequences [33].

**Remark 1.** If $X$ follows the separable sparsity model with sparsity $(s_1, \ldots, s_K)$, then the columns of the mode $i$ matrix $Y_{(i)}$ of $Y$ have $s_i$-sparse representations with respect to $D_i$, for $i \in [K]$ [33].

For a signal $x$ with sparsity pattern $\text{supp}(x)$, we model the non-zero entries of $x$, i.e., $x_S$, as drawn independently and identically from a probability distribution with known variance $\sigma^2$:  

$$\mathbb{E}_x\{x_S x_S^T | S\} = \sigma^2 I_s. \quad (24)$$

Any $x$ with sparsity model (22) or (23) and nonzero entries satisfying (24) has covariance matrix

$$\Sigma_x = \frac{s}{p} \sigma^2 I_p. \quad (25)$$

### III. LOWER BOUND FOR GENERAL DISTRIBUTION

We now provide our main result for the lower bound for minimax risk of the KS dictionary learning problem for the case of general coefficient distributions.

**Theorem 1.** Consider a KS dictionary learning problem with $N$ i.i.d observations generated according to model (13). Suppose the true dictionary satisfies (17) for some $r$ and fixed reference dictionary $D_0$ satisfying (15). Then for any coefficient distribution with mean zero and covariance $\Sigma_x$, we have the following lower bound on $\varepsilon^*$:

$$\varepsilon^* \geq \min \left\{ \frac{p}{4} \frac{r^2}{8K}, \frac{\sigma^2}{16NK\|\Sigma_x\|_2} \left( c_1 \left( \sum_{i \in [K]} p_i(m_i - 1) \right) - \frac{1}{2} \log_2 K - \frac{5}{2} \right) \right\}, \quad (26)$$

for any $0 < t < 1$ and any $0 < c_1 < \frac{1 - t}{8 \log 2}$.  

The implications of Theorem 1 are examined in Section VI.
Outline of Proof: The idea of the proof is that we construct a set of $L$ distinct KS dictionaries, $\mathcal{D}_L = \{D_1, \ldots, D_L\} \subset \mathcal{X}(D_0, r)$ such that any two distinct dictionaries are separated by a minimum distance. That is, for any pair $l, l' \in [L]$ and any positive $\varepsilon < \frac{I_{4p}}{4} \min \left\{r^2, \frac{r^4}{2Kp} \right\}$:

$$\|D_l - D_{l'}\|_F \geq 2\sqrt{2\varepsilon}, \text{ for } l \neq l'.$$ (27)

In this case, if a dictionary $D_l \in \mathcal{D}_L$ is selected uniformly at random from $\mathcal{D}_L$, then given side information $T(X) = X$, the observations under this dictionary follow a multivariate Gaussian distribution. We can therefore upper bound for the conditional MI by approximating the upper bound for KL-divergence of multivariate Gaussian distributions. This bound depends on parameters $\varepsilon, N, \{m_i\}_{i=1}^K, \{p_i\}_{i=1}^K, \Sigma_x, s, r, K$, and $\sigma^2$.

Assuming (27) holds for $\mathcal{D}_L$, if there exists an estimator achieving the minimax risk $\varepsilon^* \leq \varepsilon$ and the recovered dictionary $\hat{D}(Y)$ satisfies $\|\hat{D}(Y) - D_l\|_F < \sqrt{2\varepsilon}$, the minimum distance detector can recover $D_l$. Then, using the Markov inequality and since $\varepsilon^*$ is bounded, the probability of error $\mathbb{P}(\hat{D}(Y) \neq D_l) \leq \mathbb{P}(\|\hat{D}(Y) - D_l\|_F \geq \sqrt{2\varepsilon})$ can be upper bounded by $\frac{1}{2}$. Further, according to (21), the lower bound for the conditional MI can be obtained using Fano’s inequality [28]. The obtained lower bound is a function of $L$ only. Finally, using the obtained bounds for the conditional MI, we derive a lower bound for the minimax risk $\varepsilon^*$.

Remark 2. We use the constraint in (27) in Theorem 1 for simplicity: the number $2\sqrt{2}$ can be replaced with any arbitrary $\gamma > 0$.

The complete technical proof of Theorem 1 relies on the following lemmas, which are formally proved in the appendix. Although the similarity of our model to that of [28] suggests that the proof should be a simple extension of the proof of Theorem 1 in Jung et al. [28], the hypotheses construction for KS dictionaries is more complex and its analysis requires a different approach, with the exception of Lemma 3 (Lemma 8 of Jung et al. [28]), which connects a lower bound on the Frobenius norm in the construction to a lower bound on the conditional MI used in Fano’s inequality [19].

Lemma 1. Let $\alpha > 0$ and $\beta > 0$. Let $\{A_l \in \mathbb{R}^{m \times p} : l \in [L]\}$ be a set of $L$ matrices where each $A_l$ contains $m \times p$ independent and identically distributed random variables taking values $\pm \alpha$ uniformly. Then we have the following inequality:

$$\mathbb{P} \left( \exists (l, l') \in [L] \times [L], l \neq l' : |\langle A_l, A_{l'} \rangle| \geq \beta \right) \leq 2L^2 \exp \left( -\frac{\beta^2}{4\alpha^4mp} \right).$$ (28)
Lemma 2. Consider the generative model in (13). Fix \( r > 0 \) and a reference dictionary \( D_0 \) satisfying (15). Then there exists a set \( D_L \subseteq X(D_0, r) \) of cardinality \( L = 2^\left\lfloor c_1 \left( \sum_{i \in |K|} (m_i - 1)p_i \right) - \frac{1}{2} \log_2(2K) \right\rfloor \) such that for any \( 0 < t < 1 \), any \( 0 < c_1 < \frac{r^2}{8 \log 2} \), any \( \varepsilon' > 0 \) satisfying

\[
\varepsilon' < r^2 \min \left\{ 1, \frac{r^2}{2Kp} \right\},
\]

all pairs \( l, l' \in [L] \), with \( l \neq l' \), we have

\[
\frac{2p}{r^2} (1 - t) \varepsilon' \leq \| D_l - D_{l'} \|_F^2 \leq \frac{4Kp}{r^2} \varepsilon'.
\]

Furthermore, if \( X \) is drawn from a distribution with mean 0 and covariance matrix \( \Sigma_x \) and assuming side information \( T(X) = X \), we have

\[
I(Y; l|T(X)) \leq \frac{2NKp \| \Sigma_x \|_2^2}{r^2 \sigma^2} \varepsilon'.
\]

Lemma 3 (Lemma 8 [28]). Consider the generative model in (13) and suppose the minimax risk \( \varepsilon^* \) satisfies \( \varepsilon^* \leq \varepsilon \) for some \( \varepsilon > 0 \). If there exists a finite set \( D_L \subseteq D \) with \( L \) dictionaries satisfying

\[
\| D_l - D_{l'} \|_F^2 \geq 8 \varepsilon
\]

for \( l \neq l' \), then for any side information \( T(X) \), we have

\[
I(Y; l|T(X)) \geq \frac{1}{2} \log_2(L) - 1.
\]

Proof of Lemma 3: The proof of Lemma 3 is identical to the proof of Lemma 8 in Jung et al. [28].

Proof of Theorem 1: According to Lemma 2 for any \( \varepsilon' \) satisfying (29), there exists a set \( D_L \subseteq X(D_0, r) \) of cardinality \( L = 2^\left\lfloor c_1 \left( \sum_{i \in |K|} (m_i - 1)p_i \right) - \frac{1}{2} \log_2(2K) \right\rfloor \) that satisfies (31) for any \( 0 < t' < 1 \) and any \( c_1 < \frac{r^2}{8 \log 2} \). Let \( t = 1 - t' \). If there exists an estimator with worst-case MSE satisfying

\[
\varepsilon^* \leq \frac{2tp}{8} \min \left\{ 1, \frac{r^2}{2Kp} \right\},
\]

then, according to Lemma 3 if we set \( \frac{2tp}{r^2} \varepsilon' = 8 \varepsilon^* \), (32) is satisfied for \( D_L \) and (33) holds. Combining (31) and (33) we get

\[
\frac{1}{2} \log_2(L) - 1 \leq I(Y; l|T(X)) \leq \frac{16NKp \| \Sigma_x \|_2^2}{c_2 r^2 \sigma^2} \varepsilon^*.
\]
where $c_2 = \frac{2tp}{r^2}$. We can write (34) as

$$
\epsilon^* \geq \frac{t\sigma^2}{16NK \| \Sigma_x \|_2} \left( c_1 \left( \sum_{i \in [K]} (m_i - 1)p_i \right) - \frac{1}{2} \log_2(K) - \frac{5}{2} \right).
$$

(35)

IV. LOWER BOUND FOR SPARSE DISTRIBUTIONS

Since we are interested in the case where dictionaries yield sparse representations of signals, we now turn our attention to the case of sparse coefficients and obtain lower bounds for the corresponding minimax risk. We first state a corollary of Theorem 1 for sparse coefficients, corresponding to $T(X) = X$.

**Corollary 1.** Consider a KS dictionary learning problem with $N$ i.i.d observations generated according to model (13). Suppose the true dictionary satisfies (17) for some $r$ and fixed reference dictionary $D_0$ satisfying (15). If the random coefficient vector $x$ is selected according to (22) or (23), we have the following lower bound on $\epsilon^*$:

$$
\epsilon^* \geq t \min \left\{ \frac{p}{4 \cdot 8K}, \left( \frac{\sigma}{\sigma_a} \right)^2 \frac{2p \left( c_1 \left( \sum_{i \in [K]} p_i \right) - \frac{1}{2} \log_2(K) - \frac{5}{2} \right)}{16NKs} \right\},
$$

(36)

for any $0 < t < 1$ and any $0 < c_1 < \frac{1 - t}{8 \log 2}$.

This result is a direct consequence of Theorem 1, obtained by substituting the covariance matrix of sparse coefficients given in (25) in (26).

A. Sparse Gaussian Coefficients

In this section, we make an additional assumption on the coefficient vectors generated according to (22) and assume non-zero elements of the vectors follow Gaussian distribution. By additionally assuming the non-zero entries of $x$ are i.i.d Gaussian distributed, we can write $x_S$ as

$$
x_S \sim \mathcal{N}(0, \sigma_a^2 I_s).
$$

(37)

As a result, given side information $T(x_k) = \text{supp}(x_k)$, observations $y_k$ follow a multivariate Gaussian distribution. Part of our forthcoming analysis relies on the restricted isometry property (RIP) for a matrix:
Restricted Isometry Property (RIP): A matrix $\widetilde{D}$ with unit $\ell_2$ norm columns satisfies the RIP of order $s$ with constant $\delta_s$ if

$$(1 - \delta_s) \|x\|_2^2 \leq \|\widetilde{D}x\|_2^2 \leq (1 + \delta_s) \|x\|_2^2,$$

(38)

for all $x$ such that $\|x\|_0 \leq s$.

We now provide a theorem for a lower bound on the minimax risk in the case of coefficient distribution (22) and (37).

**Theorem 2.** Consider a KS dictionary learning problem with $N$ i.i.d observations generated according to model (13). Suppose the true dictionary satisfies (17) for some $r$ and fixed reference dictionary satisfying (15). If the reference coordinate dictionaries $\{D_{0,i}, i \in [K]\}$ satisfy RIP$(s, \frac{1}{2})$ and the random coefficient vector $x$ is selected according to (22) and (37), we have the following lower bound on $\varepsilon^*$:

$$\varepsilon^* \geq t \times \min \left\{ \frac{p}{4s} \frac{r^2}{8K} \left( \frac{\sigma}{\sigma_a} \right)^4 p \left( c_1 \left( \sum_{i \in [K]} p_i (m_i - 1) \right) - \frac{1}{2} \log_2(K) - \frac{5}{2} \right) \right\},$$

(39)

for any $0 < t < 1$ and any $0 < c_1 < \frac{1 - t}{8 \log 2}$.

Note that in Theorem 2 $D$ (or its coordinate dictionaries) need not satisfy the RIP condition. Rather, the RIP is only needed for the coordinate reference dictionaries, $\{D_{0,i}, i \in [K]\}$, which is a significantly weaker (and possibly trivial to satisfy) condition. We state a variation of Lemma 2 necessary for the proof of Theorem 2. The proof of the lemma is provided in the appendix.

**Lemma 4.** Consider the generative model in (13). Fix $r > 0$ and reference dictionary $D_0$ satisfying (15). Then, there exists a set $\mathcal{D}_L \subseteq \mathcal{X}(D_0, r)$ of cardinality $L = 2^{c_1 \left( \sum_{i \in [K]} (m_i - 1) \cdot p_i \left( \frac{r^2}{8 \log 2}, \frac{r^2}{2kp} \right) \right)}$ such that for any $0 < t < 1$, any $0 < c_1 < \frac{r^2}{8 \log 2}$, any $\varepsilon' > 0$ satisfying

$$0 < \varepsilon' \leq r^2 \min \left\{ \frac{1}{s}, \frac{r^2}{2kp} \right\},$$

(40)

and any $l, l' \in [L]$, with $l \neq l'$, we have

$$\frac{2p}{r^2} (1 - t)\varepsilon' \leq \|D_l - D_{l'}\|_F^2 \leq \frac{4Kp}{r^2} \varepsilon'.$$

(41)

Furthermore, assuming the reference coordinate dictionaries $\{D_{0,i}, i \in [K]\}$ satisfy RIP$(s, \frac{1}{2})$, the coefficient matrix $X$ is selected according to (22) and (37), and considering side information $T(X) = \text{supp}(X)$,
we have:

$$ I(Y; l|T(X)) \leq 36(3^{4K}) \left( \frac{\sigma_a}{\sigma} \right)^4 \frac{Ns^2}{r^2} \varepsilon'. $$  \hfill (42)

**Proof of Theorem 2.** According to Lemma 4, for any \( \varepsilon' \) satisfying (40), there exists a set \( \mathcal{D}_L \subseteq \mathcal{X}(D_0, r) \) of cardinality \( L = 2^{c_1(\sum_{i \in [K]} (m_i - 1)p_i) - \frac{1}{2} \log_2(2K)} \) that satisfies (42) for any \( 0 < t' < 1 \) and any \( c_1 < \frac{t'}{8 \log 2} \). Denoting \( t = 1 - t' \) and provided there exists an estimator with worst case MSE satisfying \( \varepsilon^* \leq \frac{2tp}{2r^2} \varepsilon' = 8 \varepsilon^* \), (32) is satisfied for \( \mathcal{D}_L \) and (33) holds. Consequently,

$$ \frac{1}{2} \log_2(L) - 1 \leq I(Y; l|T(X)) \leq \frac{36(3^{4K})}{c_2} \left( \frac{\sigma_a}{\sigma} \right)^4 \frac{Ns^2}{r^2} \varepsilon^*, $$  \hfill (43)

where \( c_2 = \frac{p(1 - t)}{4r^2} \). We can write (43) as

$$ \varepsilon^* \geq \left( \frac{\sigma}{\sigma_a} \right)^4 \frac{tp}{4} \left( \frac{c_1(\sum_{i \in [K]} (m_i - 1)p_i) - \frac{1}{2} \log_2(K) - \frac{5}{2}}{144(3^{4K})Ns^2} \right). $$  \hfill (44)

Focusing on the case where the coefficients follow the separable sparsity model, the next theorem provides a lower bound on the minimax risk for coefficient distribution (23) and (37).

**Theorem 3.** Consider a KS dictionary learning problem with \( N \) i.i.d observations generated according to model (13). Suppose the true dictionary satisfies (17) for some \( r \) and fixed reference dictionary satisfying (15). If the reference coordinate dictionaries \( \{D_{0,i}, i \in [K]\} \) satisfy RIP \((s, \frac{1}{2})\) and the random coefficient vector \( x \) is selected according to (23) and (37), we have the following lower bound on \( \varepsilon^* \):

$$ \varepsilon^* \geq t \times \min \left\{ \frac{p}{4} \cdot \frac{r^2}{8K^2}, \left( \frac{\sigma}{\sigma_a} \right)^4 \frac{p}{4} \left( \frac{c_1(\sum_{i \in [K]} (m_i - 1)p_i) - \frac{1}{2} \log_2(K) - \frac{5}{2}}{144(3^{4K})Ns^2} \right) \right\} $$  \hfill (45)

for any \( 0 < t < 1 \) and any \( 0 < c_1 < \frac{1 - t}{8 \log 2} \).

We state a variation of Lemma 4 necessary for the proof of Theorem 3. The proof of the lemma is provided in the appendix.

**Lemma 5.** Consider the generative model in (13). Fix \( r > 0 \) and reference dictionary \( D_0 \) satisfying (15). Then, there exists a set of dictionaries \( \mathcal{D}_L \subseteq \mathcal{D} \) of cardinality \( L = 2^{c_1(\sum_{i \in [K]} (m_i - 1)p_i) - \frac{1}{2} \log_2(2K)} \) such
that for any $0 < t < 1$, any $0 < c_1 < \frac{t^2}{8 \log 2}$, any $\varepsilon' > 0$ satisfying

$$0 < \varepsilon' \leq r^2 \min \left\{ 1, \frac{r^2}{2Kp} \right\}, \quad (46)$$

and any $l, l' \in [L]$, with $l \neq l'$, we have

$$\frac{2p}{r^2} (1 - t) \varepsilon' \leq \|D_l - D_{l'}\|^2_F \leq \frac{4Kp}{r^2} \varepsilon'.$$  

Furthermore, assuming the coefficient matrix $X$ is selected according to (23) and (37), the reference coordinate dictionaries $\{D_{0,i}, i \in [K]\}$ satisfy RIP$(s_i, \frac{1}{2})$, and considering side information $T(X) = \text{supp}(X)$, we have:

$$I(Y; l|T(X)) \leq 36(3^{4K}) \left(\frac{\sigma_a}{\sigma}\right) 4\frac{Ns^2}{r^2} \varepsilon'.$$  

(48)

**Proof of Theorem 3:** The proof of Theorem 3 follows similar steps as the proof of Theorem 2. The dissimilarity arises in the condition in (46) for Lemma 5, which is different from the condition in (40) for Lemma 4. This changes the range for the minimax risk $\varepsilon^*$ in which the lower bound in (44) holds.

In the next section, we provide a simple KS dictionary learning algorithm for $K = 2$ dimensional tensors and study the corresponding dictionary learning MSE.

V. PARTIAL CONVERSE

In previous sections, we provided lower bounds on the minimax risk for various coefficient vector distributions and corresponding side information. We now study a special case of the problem and introduce an algorithm that achieves the lower bound in Corollary 1 for 2-dimensional tensors. This demonstrates that our obtained lower bounds are tight in some cases.

**Theorem 4.** Consider a dictionary learning problem with $N$ i.i.d observations according to model (13) for $K = 2$ and let the true dictionary satisfy (17) for $D_0 = I_p$ and some $r > 0$. Further, assume the random coefficient vector $x$ is selected according to (22), $x \in \{-1, 0, 1\}^p$ and nonzero entries of $x$ can have any distribution. Next, assume noise standard deviation $\sigma$ and express the KS dictionary as

$$D = (I_{p_1} + \Delta_1) \otimes (I_{p_2} + \Delta_2),$$

(49)
where \( p = p_1 p_2 \), \( \| \Delta_1 \|_F \leq r_1 \) and \( \| \Delta_2 \|_F \leq r_2 \). Then, if the following inequalities are satisfied:

\[
 r_1 \sqrt{p_2} + r_2 \sqrt{p_1} + r_1 r_2 \leq r,
\]

\[
 (r_1 + r_2 + r_1 r_2) \sqrt{8} \leq 0.1
\]

\[
 \max \left\{ \frac{r_1^2}{p_2}, \frac{r_2^2}{p_1} \right\} \leq \frac{1}{2N},
\]

\[
 \sigma \leq 0.4,
\]

there exists a dictionary learning scheme whose MSE satisfies

\[
 \mathbb{E}_Y \left\{ \| \hat{D}(Y) - D \|_F^2 \right\} \leq \frac{8p}{N} \left( \frac{p_1 m_1 + p_2 m_2}{m \text{SNR}} + 3(p_1 + p_2) \right) + 8p \exp \left( -\frac{0.08 p N}{\sigma^2} \right)
\]

for any \( D \in \mathcal{X}(D_0, r) \) that satisfies (49).

To prove Theorem 4, we first introduce an algorithm to learn a KS dictionary for 2-dimensional data. Then, we analyze the performance of the proposed algorithm and obtain an upper bound for the MSE in the proof of Theorem 4. Finally, we provide numerical experiments to validate our obtained results.

A. KS Dictionary Learning Algorithm

We analyze an estimator that thresholds the observations and applies an alternating update rule as follows:

a) Coefficient Update: We utilize a simple thresholding technique for this purpose:

\[
 \hat{x}_k = (\hat{x}_{k,1}, \ldots, \hat{x}_{k,p})^T, \quad \hat{x}_{k,l} = \begin{cases} 
 1 & \text{if } y_{k,l} > 0.5, \\
 -1 & \text{if } y_{k,l} < -0.5, \\
 0 & \text{otherwise}, 
\end{cases} \quad k = [N].
\]

b) Dictionary Update: Denoting \( A \triangleq I_{p_1} + \Delta_1 \) and \( B \triangleq I_{p_2} + \Delta_2 \), we can write \( D = A \otimes B \). We update the columns of \( A \) and \( B \) separately. To learn \( A \), we take advantage of the Kronecker structure of the dictionary and divide each observation \( y_k \in \mathbb{R}^{p_1 p_2} \) into \( p_2 \) observations \( \gamma'_{(k,j)} \in \mathbb{R}^{p_1} \):

\[
 \gamma'_{(k,j)} = \{ y_{k,p_2i+j} \}_{i=0}^{p_1-1}, \quad j = [p_2], \quad k = [N].
\]
This increases the number of observations to $Np_2$. We also divide the original and estimated coefficient vectors:

$$x'_{(k,j)} = \{x_{k,p_2i+j} \}_{i=0}^{p_1-1}, \quad \hat{x}'_{(k,j)} = \{\hat{x}_{k,p_2i+j} \}_{i=0}^{p_1-1}, \quad j = [p_2], \quad k = [N]. \tag{54}$$

Similarly, we define new noise vectors:

$$n'_{(k,j)} = \{n_{k,p_2i+j} \}_{i=0}^{p_1-1}, \quad j = [p_2], \quad k = [N]. \tag{55}$$

To update the columns of $A$, we utilize the following update rule:

$$\tilde{a}_i = \frac{p_1}{Ns} \sum_{k=1}^{N} \sum_{j=1}^{p_2} x'_{(k,j)}, y'_{(k,j)}, \quad i = [p_1]. \tag{56}$$

To update columns of $B$, we follow a different procedure to divide the observations. Specifically, we divide each observation $y_k \in \mathbb{R}^{p_1p_2}$ into $p_1$ observations $y''_j \in \mathbb{R}^{p_2}$:

$$y''_{(k,j)} = \{y_{k,i+p_1(j-1)} \}_{i=1}^{p_2}, \quad j = [p_1], \quad k = [N]. \tag{57}$$

This increases the number of observations to $Np_1$. The coefficient vectors are also divided similarly:

$$x''_{(k,j)} = \{x_{k,i+p_1(j-1)} \}_{i=0}^{p_1-1}, \quad \hat{x}''_{(k,j)} = \{\hat{x}_{k,i+p_1(j-1)} \}_{i=0}^{p_1-1}, \quad j = [p_1], \quad k = [N]. \tag{58}$$

Similarly, we define new noise vectors:

$$n''_{(k,j)} = \{n_{k,i+p_1(j-1)} \}_{i=1}^{p_2}, \quad j = [p_1], \quad k = [N], \tag{59}$$

and the update rule for columns of $B$ is

$$\tilde{b}_i = \frac{p_2}{Ns} \sum_{k=1}^{N} \sum_{j=1}^{p_1} x''_{(k,j)}, y''_{(k,j)}, \quad i = [p_2]. \tag{60}$$

The final update rule for the recovered dictionary is

$$\hat{D} = \hat{A} \otimes \hat{B},$$

$$\hat{A} = (\hat{a}_1, \ldots, \hat{a}_{p_1}), \quad \hat{a}_l = P_{\mathcal{B}_1}(\tilde{a}_l),$$

$$\hat{B} = (\hat{b}_1, \ldots, \hat{b}_{p_2}), \quad \hat{b}_l = P_{\mathcal{B}_2}(\tilde{b}_l), \tag{61}$$

where the projection on the closed unit ball ensures that $\|\tilde{a}_l\|_2 \leq 1$ and $\|\tilde{b}_l\|_2 \leq 1$. Note that although projection onto the closed unit ball does not ensure the columns of $\hat{D}$ to have unit norms, our analysis
Fig. 1: Performance summary of KS dictionary learning algorithm for $p = \{128, 256, 512\}$, $s = 5$ and $r = 0.1$. (a) plots the ratio of the empirical error of our KS dictionary learning algorithm to the obtained error upper bound along with error bars for generated square KS dictionaries, and (b) shows the performance of our KS dictionary learning algorithm compared to the unstructured learning algorithm proposed in [28].

only imposes this condition on the generating dictionary and the reference dictionary, and not on the recovered dictionary.

B. Empirical Comparison to Upper Bound

We are interested in empirically seeing how well does our achievable scheme matches the minimax lower bound when learning KS dictionaries. To this end, we implemented the preceding estimation algorithm for 2-dimensional tensor data.

Figure 1a shows the ratio of the empirical error of the proposed KS dictionary learning algorithm in Section V-A to the obtained upper bound in Theorem 4 for 50 Monte Carlo experiments. This ratio is plotted as a function of the sample size for three choices of the number of columns $p$: 128, 256, and 512. The experiment shows that the ratio is approximately constant as a function of sample size, verifying the theoretical result that the estimator meets the minimax bound in terms of error scaling as a function of sample size. Figure 1b shows the performance of our KS dictionary learning algorithm compared to the unstructured dictionary learning algorithm provided in [28]. It is evident that the error of our algorithm is significantly less than that for the unstructured algorithm for all three choices of $p$. This verifies that taking the structure of the data into consideration can indeed lead to lower dictionary identification error.
TABLE I: Order-wise lower bounds on the minimax risk for various coefficient distributions

| Evolutionary Distribution | Side Information $T(X)$ | Unstructured [28] | Kronecker (this paper) |
|---------------------------|--------------------------|-------------------|-----------------------|
| 1. General                | $X$                      | $\frac{\sigma^2mp}{N\|\Sigma_x\|_2}$ | $\frac{\sigma^2(\sum_{i\in[K]}m_ip_i)}{NK\|\Sigma_x\|_2}$ |
| 2. Sparse                 | $X$                      | $\frac{p^2}{N\text{SNR}}$ | $\frac{p(\sum_{i\in[K]}m_ip_i)}{NKm\text{SNR}}$ |
| 3. Gaussian Sparse        | $\text{supp}(X)$         | $\frac{p^2}{Nm\text{SNR}^2}$ | $\frac{p(\sum_{i\in[K]}m_ip_i)}{3^4KNm^2\text{SNR}^2}$ |

VI. DISCUSSION

We now discuss some of the implications of our results. Table I summarizes the lower bounds on the minimax rates from previous papers and this work. The bounds are given in terms of the number of component dictionaries $K$, the dictionary size parameters ($m_i$’s and $p_i$’s), the coefficient distribution parameters, the number of samples $N$, and SNR, which is defined as

$$\text{SNR} = \frac{\mathbb{E}_X\{\|x\|_2^2\}}{\mathbb{E}_n\{\|n\|_2^2\}} = \frac{\text{Tr}(\Sigma_x)}{m\sigma^2}. \quad (62)$$

These scalings result hold for sufficiently large $p$ and neighborhood radius $r$. Compared to the results for the unstructured dictionary learning problem [28], we are able to decrease the lower bound for various coefficient distributions by reducing the scaling $\Omega(mp)$ to $\Omega(\sum_{i\in[K]}m_ip_i)$ for KS dictionaries. This is intuitively pleasing since the minimax lower bound has a linear relationship with the number of degrees of freedom of the KS dictionary, which is $\sum_{i\in[K]}m_ip_i$.

The results also show that the minimax risk decreases with a larger number of samples, $N$, and increased number of tensor dimensions, $K$. By increasing $K$, we are shrinking the size of the class of dictionaries in which the parameter dictionary lies, thereby simplifying the problem.

Looking at the results for the general coefficient model in the first row of Table I, the lower bound for any arbitrary zero-mean random coefficient vector distribution with covariance $\Sigma_x$ implies an inverse relationship between the minimax risk and SNR due to the fact that $\|\Sigma_x\|_2 \leq \text{Tr}(\Sigma_x)$.

Proceeding to the sparse coefficient vector model in the second row of Table I by replacing $\Sigma_x$ with the expression in (25) in the minimax lower bound for the general coefficient distribution, we obtain the second lower bound given in (36). In this case, we observe a seemingly counter-intuitive increase in the
MSE on the order of $\Omega\left(\frac{p}{s}\right)$ in the lower bound in comparison to the general coefficient model. However, this increase is due to the fact that we do not require coefficient vectors to have constant energy; because of this, SNR decreases to $\frac{s\sigma^2}{m\sigma^2}$ for $s$-sparse coefficient vectors.

Next, looking at the third row of Table I by restricting the class of sparse coefficient vector distributions to the case where non-zero elements of the coefficient vector follow a Gaussian distribution according to (37), we obtain a minimax lower bound that involves less side information than the prior two cases. However, we do make the assumption in this case that reference coordinate dictionaries satisfy RIP$(s, \frac{1}{2})$. This additional assumption has two implications: (1) it introduces the factor of $1/3^4K$ in the minimax lower bound, and (2) it imposes the following condition on the sparsity for the “random sparsity” model: $s \leq \min_{i \in [K]} \{p_i\}$. Nonetheless, considering sparse-Gaussian coefficient vectors, we obtain a minimax lower bound that is tighter than the previous bound for some SNR values. Specifically, in order to compare bounds obtained in (36) and (39) for sparse and sparse-Gaussian coefficient vector distributions, we fix $K$. Then in high SNR regimes, i.e., $\text{SNR} > \frac{1}{m}$, the lower bound in (36) is tighter, while (39) results in a tighter lower bound in low SNR regimes, which correspond to low sparsity settings.

We now focus on our results for the two sparsity pattern models, namely, random sparsity and separable sparsity, for the case of sparse-Gaussian coefficient vector distribution. These results, which are reported in (39) and (45), are almost identical to each other, except for the first term in the minimization. In order to understand the settings in which the separable sparsity model in (23)—which is clearly more restrictive than the random sparsity model in (22)—turns out to be more advantageous, we select the neighborhood radius $r$ to be on the order of $O(\sqrt{p})$; since we are dealing with dictionaries that lie on the surface of a sphere with radius $\sqrt{p}$, this effectively ensures $\mathcal{X}(D, r) \approx D$. In this case, it can be seen from (39) and (45) that if $s = \Omega(K)$ then the separable sparsity model gives a better minimax lower bound. On the hand, the random sparsity model should be considered for the case of $s = O(K)$ because of the less restrictive nature of this model.

Finally, we discuss the achievability of our lower bounds for the minimax risk of learning KS dictionaries. To this end, we provided a simple KS dictionary learning algorithm in Section V for the special scenario of 2-dimensional tensors and analyzed the corresponding MSE, $E\{\|\hat{D}(Y) - D\|_F^2\}$. In terms of scaling, the upper bound obtained for the MSE in Theorem 4 matches the lower bound in Corollary 1 provided $p_1 + p_2 < \frac{m_1p_1 + m_2p_2}{m \text{SNR}}$ holds. This result suggests that more general KS dictionary learning algorithms may be developed to achieve the lower bounds reported in this paper.
VII. CONCLUSION

In this paper we followed an information-theoretic approach to provide lower bounds for the worst-case mean-squared error of Kronecker-structured dictionaries that generate $K$-dimensional tensor data. To this end, we constructed a class of Kronecker-structured dictionaries in a local neighborhood of a fixed reference Kronecker-structured dictionary. Our analysis required studying the mutual information between the observation matrix and the dictionaries in the constructed class. To evaluate bounds on the mutual information, we considered various coefficient distributions and interrelated side information on the coefficient vectors and obtained corresponding minimax lower bounds using these models. In particular, we established that estimating Kronecker-structured dictionaries requires a number of samples that needs only grow linearly with the sum of the sizes of the component dictionaries ($\sum_{i\in[K]} m_i p_i$), which represents the true degrees of freedom of the problem. We also demonstrated that for a special case of $K = 2$ there exists an estimator whose MSE meets the derived lower bounds. While our analysis is local in the sense that we assume the true dictionary belongs in a local neighborhood with known radius around a fixed reference dictionary, the derived minimax risk effectively becomes independent of this radius for sufficiently neighborhood radius.

Future directions of this work include designing general algorithms to learn Kronecker-structured dictionaries that achieve the presented lower bounds. In particular, the analysis in [35] suggests that restricting the class of dictionaries to Kronecker-structured dictionaries may indeed yield reduction in the sample complexity required for dictionary identification by replacing a factor $mp$ in the general dictionary learning problem with the box counting dimension of the dictionary class [26].

VIII. ACKNOWLEDGEMENT

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APPENDIX

Proof of Lemma 1: Fix $L > 0$ and $\alpha > 0$. For a pair of matrices $A_l$ and $A_{l'}$, with $l \neq l'$, consider the vectorized set of entries $a_l = \text{vec}(A_l)$ and $a_{l'} = \text{vec}(A_{l'})$ and define the function

$$f(a_l^T, a_{l'}^T) \equiv |\langle A_l, A_{l'} \rangle|$$

$$= |\langle a_l, a_{l'} \rangle|.$$  \hspace{1cm} (63)
For $\tilde{a} \triangleq (a_l^T, a_{l'}^T) \in \mathbb{R}^{2mp}$, write $\tilde{a} \sim \tilde{a}'$ if $\tilde{a}'$ is equal to $\tilde{a}$ in all entries but one. Then $f$ satisfies the following bounded difference condition:

$$\sup_{\tilde{a} \sim \tilde{a}'} \left| f(\tilde{a}) - f(\tilde{a}') \right| = (\alpha - (-\alpha))\alpha = 2\alpha^2.$$  (64)

Hence, according to McDiarmid’s inequality \cite{McDiarmid}, for all $\beta > 0$, we have

$$\mathbb{P} \left( \left| \langle A_l, A_{l'} \rangle \right| \geq \beta \right) \leq 2 \exp \left( -\frac{2\beta^2}{\sum_{i=1}^{2mp} (2\alpha^2)^2} \right) = 2 \exp \left( -\frac{\beta^2}{4\alpha^4 mp} \right).$$  (65)

Taking a union bound over all pairs $l, l' \in [L], l \neq l'$, we have

$$\mathbb{P} \left( \exists (l, l') \in [L] \times [L], l \neq l' : \left| \langle A_l, A_{l'} \rangle \right| \geq \beta \right) \leq 2L^2 \exp \left( -\frac{\beta^2}{4\alpha^4 mp} \right).$$  (66)

**Proof of Lemma 2**: Fix $r > 0$ and $t \in (0, 1)$. Let $D_0$ be a reference dictionary satisfying (15), and let $\{U_{(i,j)}\}_{j=1}^{p_i} \in \mathbb{R}^{m_i \times m_i}, i \in [K]$, be arbitrary unitary matrices satisfying

$$d_{(i,0),j} = U_{(i,j)} e_1,$$  (67)

where $d_{(i,0),j}$ denotes the $j$-th column of $D_{(i,0)}$.

To construct the dictionary class $D_L \subseteq \mathcal{X}(D_0, r)$, we follow several steps. We consider sets of

$$L_i = 2^{\lceil c_1 (m_i - 1)p_i - \frac{1}{2} - \frac{1}{2} \log_2 K \rceil}$$  (68)

generating matrices $G_{(i,l_i)}$:

$$G_{(i,l_i)} \in \left\{ -\frac{1}{r^{1/K} \sqrt{(m_i - 1)}}, \frac{1}{r^{1/K} \sqrt{(m_i - 1)}} \right\}^{(m_i - 1) \times p_i}$$  (69)

for $i \in [K]$ and $l_i \in [L_i]$. According to Lemma 1 for all $i \in [K]$ and any $\beta > 0$, the following relation is satisfied:

$$\mathbb{P} \left( \exists (l_i, l_i') \in [L_i] \times [L_i], l \neq l' : \left| \langle G_{(i,l_i)}, G_{(i,l_i')} \rangle \right| \geq \beta \right) \leq 2L_i^2 \exp \left( -\frac{r^{A/K} (m_i - 1)\beta^2}{4p_i} \right).$$  (70)

To guarantee a simultaneous existence of $K$ sets of generating matrices satisfying

$$\left| \langle G_{(i,l_i)}, G_{(i,l_i')} \rangle \right| \leq \beta, \quad i \in [K],$$  (71)
we take a union bound of (70) over all $i \in [K]$ and choose parameters such that the following upper bound is less than 1:

$$2KL_i^2 \exp \left( \frac{r^4/K(m_i - 1)\beta^2}{4p_i} \right) = \exp \left( \frac{r^4/K(m_i - 1)\beta^2}{4p_i} + 2 \ln \sqrt{2KL_i} \right),$$

which is satisfied as long as the following inequality holds:

$$\log_2 L_i < \frac{r^4/K(m_i - 1)\beta^2}{8p_i \log 2} - \frac{1}{2} - \frac{1}{2} \log_2 K.$$  \hspace{1cm} (73)

Now, setting $\beta = \frac{pt}{r^{2/K}}$, the condition in (73) holds and there exists a collection of generating matrices that satisfy:

$$\left| \langle G(i,l_i), G(i,l'_i) \rangle \right| \leq \frac{pt}{r^{2/K}}, \quad i \in [K],$$

for any distinct $l_i, l'_i \in [L_i]$, any $t \in (0, 1)$, and any $c_1 > 0$ such that

$$c_1 < \frac{t^2}{8 \log 2}.$$  \hspace{1cm} (75)

We next construct matrices that will be later used for the construction of unit norm column dictionaries. We construct $D_{(i,1,l_i)} \in \mathbb{R}^{m_i \times p_i}$ column-wise using $G(i,l_i)$ and unitary matrices $\{U(i,j)\}_{j=1}^{p_i}$. Let the $j$-th column of $D_{(i,1,l_i)}$ be given by

$$d_{(i,1,l_i),j} = U(i,j) \begin{pmatrix} 0 \\ g_{(i,l_i),j} \end{pmatrix}, \quad i \in [K],$$

for any $l_i \in [L_i]$. Moreover, defining

$$\mathcal{D}_1 \triangleq \left\{ \bigotimes_{i \in [K]} D_{(i,1,l_i)} : l_i \in [L_i] \right\},$$

and denoting

$$\mathcal{L} \triangleq \{ (l_1, \ldots, l_K) : l_i \in [L_i] \},$$

any element of $\mathcal{D}_1$ can be expressed as

$$D_{(1,l)} = \bigotimes_{i \in [K]} D_{(i,1,l_i)}, \forall \ l \in [L],$$

where $L \triangleq \prod_{i \in [K]} L_i$ and we associate an $l \in [L]$ with a tuple in $\mathcal{L}$ via lexicographic indexing. Notice
also that
\[ \| d_{(1, l), j} \|_2^2 \overset{(a)}{=} \prod_{i \in [K]} \| d_{(i, l_i), j} \|_2^2 = \prod_{i \in [K]} \frac{1}{r^2/K} = \frac{1}{r^2}, \]
and thus
\[ \| D_{(1, l)} \|_F^2 = \frac{p}{r^2}, \] (80)
where (a) follows from properties of the Kronecker product. From (76), it is evident that for all \( i \in [K], \)
d\((i, 0), j\) is orthogonal to \( d_{(i, 1, l_i), j} \) and consequently, we have
\[ \langle D_{(i, 0)}, D_{(i, 1, l_i)} \rangle = 0, \quad i \in [K] \] (81)
Also,
\[ \langle D_{(i, 1, l_i)}, D_{(i, 1, l'_i)} \rangle = \sum_{j=1}^{p_i} \langle d_{(i, 1, l_i), j}, d_{(i, 1, l'_i), j} \rangle \]
\[ = \sum_{j=1}^{p_i} \left\langle U_{(i, j)} \begin{pmatrix} 0 \\ g_{(i, l_i), j} \end{pmatrix}, U_{(i, j)} \begin{pmatrix} 0 \\ g_{(i, l'_i), j} \end{pmatrix} \right\rangle \]
\[ \overset{(b)}{=} \sum_{j=1}^{p_i} \langle g_{(i, l_i), j}, g_{(i, l'_i), j} \rangle \]
\[ = \langle G_{(i, l_i)}, G_{(i, l'_i)} \rangle, \] (82)
where (b) follows from the fact that \( \{ U_{(i, j)} \} \) are unitary.

Based on the construction, for all \( i \in [K], l_i \neq l'_i \) s, we have
\[ \| D_{(1, l)} - D_{(1, l')} \|_F^2 = \| D_{(1, l)} \|_F^2 + \| D_{(1, l')} \|_F^2 - 2 \langle D_{(1, l)}, D_{(1, l')} \rangle \]
\[ = \frac{p}{r^2} + \frac{p}{r^2} - 2 \prod_{i \in [K]} \langle D_{(i, 1, l_i)}, D_{(i, 1, l'_i)} \rangle \]
\[ \geq 2 \left( \frac{p}{r^2} - \prod_{i \in [K]} \| D_{(i, 1, l_i)}, D_{(i, 1, l'_i)} \| \right) \]
\[ \overset{(c)}{=} 2 \left( \frac{p}{r^2} - \prod_{i \in [K]} \| G_{(i, l_i)}, G_{(i, l'_i)} \| \right) \]
\[ \overset{(d)}{=} 2 \left( \frac{p}{r^2} - \prod_{i \in [K]} \frac{p_i}{r^2/K} t \right) \]
\[ = \frac{2p}{r^2} (1 - t^K), \] (83)
where (c) and (d) follow from (82) and (74), respectively.

We are now ready to define $\mathcal{D}_L$. The final dictionary class is defined as

$$
\mathcal{D}_L \triangleq \left\{ \bigotimes_{i \in [K]} \mathbf{D}_{(i,l)} : l_i \in [L_i] \right\}
$$

(84)

and any $\mathbf{D}_l \in \mathcal{D}_L$ can be written as

$$
\mathbf{D}_l = \bigotimes_{i \in [K]} \mathbf{D}_{(i,l)};
$$

(85)

where $\mathbf{D}_{(i,l)}$ is defined as

$$
\mathbf{D}_{(i,l)} \triangleq \eta \mathbf{D}_{(i,0)} + \nu \mathbf{D}_{(i,1,l)}, \quad i \in [K],
$$

(86)

and

$$
\eta \triangleq \sqrt{1 - \frac{\epsilon'}{r^2}}, \quad \nu \triangleq \sqrt{\frac{r^2/K \epsilon'}{r^2}}
$$

(87)

for any

$$
0 < \epsilon' < \min \left\{ \frac{r^2}{2Kp}, \frac{r^4}{2K^2p} \right\},
$$

(88)

which ensures that $1 - \frac{\epsilon'}{r^2} > 0$ and $\mathbf{D}_l \in \mathcal{X}(\mathbf{D}_0, r)$. Note that the following relation holds between $\eta$ and $\nu$:

$$
\eta^2 + \frac{\nu^2}{r^2/K} = 1.
$$

(89)

We can expand (85) to facilitate the forthcoming analysis:

$$
\mathbf{D}_l = \sum_{i_1 \in \{0,1\}} \cdots \sum_{i_K \in \{0,1\}} \eta^{(K-\sum_{j=1}^{K} i_j)} \nu^{(\sum_{j=1}^{K} i_j)} \left( \bigotimes_{k \in [K]} \mathbf{D}_{(k,i_k,l_k)} \right),
$$

(90)

where $\mathbf{D}_{(k,0,l_k)} \triangleq \mathbf{D}_{(k,0)}$. To show $\mathcal{D}_L \subseteq \mathcal{X}(\mathbf{D}_0, r)$, we first show that any $\mathbf{D}_l \in \mathcal{D}_L$ has unit norm columns. For any $j \in [p]$, we have

$$
\|\mathbf{d}_{l,j}\|_2^2 = \prod_{i \in [K]} \|\mathbf{d}_{(i,l),j,i}\|_2^2
$$

$$
= \prod_{i \in [K]} \left( \eta^2 \|\mathbf{d}_{(i,0),j,i}\|_2^2 + \nu^2 \|\mathbf{d}_{(i,1,l),j,i}\|_2^2 \right)
$$
\[ = \prod_{i \in [K]} \left( \eta^2 + \nu^2 \left( \frac{1}{\rho^2 / K} \right) \right) \]

\[ \equiv 1, \quad (91) \]

where (e) follows from (89), \( j_i \in [p_i] \) and \( j \) is associated with \((j_1, \ldots, j_K)\) via lexicographic endexing.

Then, we show that \( \|D_l - D_0\|_F \leq \rho \):

\[
\|D_l - D_0\|_F^2 = \left\| D_0 - \sum_{i_1 \in \{0, 1\}} \cdots \sum_{i_K \in \{0, 1\}} \eta^{(K - \sum_{j=1}^K i_j)} \nu^{(\sum_{j=1}^K i_j)} \bigotimes_{k \in [K]} D_{(k, i_k, l_k)} \right\|_F^2
\]

\[
= \left\| (1 - \eta^K)D_0 - \sum_{i_1 \in \{0, 1\}} \cdots \sum_{i_K \in \{0, 1\}} \eta^{(K - \sum_{j=1}^K i_j)} \nu^{(\sum_{j=1}^K i_j)} \bigotimes_{k \in [K]} D_{(k, i_k, l_k)} \right\|_F^2
\]

\[
= (1 - \eta^K)^2 \|D_0\|_F^2 + \sum_{i_1 \in \{0, 1\}} \cdots \sum_{i_K \in \{0, 1\}} \eta^{2(K - \sum_{j=1}^K i_j)} \nu^{2(\sum_{j=1}^K i_j)} \prod_{k \in [K]} \|D_{(k, i_k, l_k)}\|_F^2.
\]

\[ (92) \]

We will bound the two terms in (92) separately. We know

\[
(1 - x^n) = (1 - x)(1 + x + x^2 + \cdots + x^{n-1}). \quad (93)
\]

Hence, we have

\[
(1 - \eta^K)^2 \|D_0\|_F^2 = (1 - \eta^K)^2 p
\]

\[
\leq (1 - \eta^K) p \quad (f)
\]

\[
\leq (1 - \eta^{2K}) p \quad (g)
\]

\[
\equiv (1 - \eta^2) \left( 1 + \eta^2 + \cdots + \eta^{2(K-1)} \right) p
\]

\[
= \frac{\varepsilon'}{\rho^2} \left( 1 + \eta^2 + \cdots + \eta^{2(K-1)} \right) p
\]

\[
\leq \frac{K p \varepsilon'}{\rho^2}, \quad (h)
\]

where (f) and (h) follow from the fact that \( \eta < 1 \) and (g) follows from (93).

Similarly,

\[
\sum_{i_1 \in \{0, 1\}} \cdots \sum_{i_K \in \{0, 1\}} \eta^{2(K - \sum_{j=1}^K i_j)} \nu^{2(\sum_{j=1}^K i_j)} \prod_{k \in [K]} \|D_{(k, i_k, l_k)}\|_F^2
\]

\[ (94) \]
Replacing values for \( \eta \) and \( \nu \) from (87), we can further reduce (95) to get

\[
\begin{align*}
\sum_{i_1 \in \{0,1\}} \cdots \sum_{i_K \in \{0,1\}} \sum_{j=1}^{K-1} \nu^{2(K-\sum_{j=1}^{K-1} i_j)} \nu^{2(\sum_{j=1}^{K-1} i_j)} \prod_{k \in [K]} \left| D_{(k,i_k,k')} \right|^2_F = p \sum_{i=0}^{K-1} \left( \frac{K}{i} \right) \left( 1 - \frac{\epsilon^i}{r^2} \right)^i \left( \frac{\epsilon^i}{r^2} \right)^{K-i} \\
&\leq K p \epsilon^i \frac{1}{r^2},
\end{align*}
\]

(96)

where (i) follows from (93). Adding (94) and (96), we get

\[
\left\| D_t - D_0 \right\|_F^2 \leq \epsilon^t \left( \frac{2Kp}{r^2} \right)
\]

(97)

where (j) follows from the condition in (88). Therefore, (91) and (96) imply that \( D_L \subseteq \mathcal{X}(D_0, r) \).

We now find lower and upper bounds for the distance between any two distinct elements \( D_t, D_{t'} \in D_L \).

1) Lower bounding \( \left\| D_t - D_{t'} \right\|_F^2 \): We define the set \( I_i \subseteq [K] \) where \( |I_i| = i, i \in [K] \). Then, given distinct \( k, k', r, i \in I_i \), we have

\[
\left\| \bigotimes_{k \in I_i} D_{(k,1,k')} - \bigotimes_{k \in I_i} D_{(k,1,k')} \right\|_F^2 \geq \frac{2}{r^{2t/K}} \left( \prod_{k \in I_i} p_k \right) (1 - t^i)
\]

(98)

where (k) follows using arguments similar to those made for (83).

To obtain a lower bound on \( \left\| D_t - D_{t'} \right\|_F^2 \), we emphasize that for distinct \( l, l' \in [L] \), it does not
necessarily hold that $l_i \neq l_i'$ for all $i \in [K]$. In fact, it is sufficient for $\mathbf{D}_l \neq \mathbf{D}_{l'}$ that only one $i \in [K]$ satisfies $l_i \neq l_i'$. Now, assume only $K_1$ out of $K$ coordinate dictionaries are distinct (for the case where all smaller dictionaries are distinct, $K_1 = K$). Without loss of generality, we assume $l_1, \ldots, l_{K_1}$ are distinct and $l_{K_1+1}, \ldots, l_K$ are similar across $\mathbf{D}_l$ and $\mathbf{D}_{l'}$. This is because of the invariance of the Frobenius norm of Kronecker products under permutation, i.e,

$$\left\| \bigotimes_{k \in [K]} \mathbf{A}_k \right\|_F = \prod_{k \in [K]} \| \mathbf{A}_k \|_F = \left\| \bigotimes_{k \in [K]} \mathbf{A}_{\pi(k)} \right\|_F. \quad (99)$$

We then have

$$\| \mathbf{D}_l - \mathbf{D}_{l'} \|_F^2$$

$$= \left\| (\mathbf{D}_{(1,l_1)} \otimes \cdots \otimes \mathbf{D}_{(K_1,l_{K_1})} \otimes \mathbf{D}_{(K_1+1,l_{K_1+1})} \otimes \cdots \otimes \mathbf{D}_{(K,l_K)}) - (\mathbf{D}_{(1,l'_1)} \otimes \cdots \otimes \mathbf{D}_{(K_1,l'_{K_1})} \otimes \mathbf{D}_{(K_1+1,l_{K_1+1})} \otimes \cdots \otimes \mathbf{D}_{(K,l_K)}) \right\|_F^2$$

$$= \left\| \bigotimes_{k \in [K_1]} \mathbf{D}_{(k,l_k)} - \bigotimes_{k \in [K_1]} \mathbf{D}_{(k,l'_k)} \right\|_F^2 \prod_{i = K_1+1}^K \| \mathbf{D}_{(i,l_i)} \|_F^2$$

$$= \left( \prod_{i = K_1+1}^K p_i \right) \left( \sum_{i_1 \in \{0,1\}} \cdots \sum_{i_{K_1} \in \{0,1\}} \eta^{(K_1 - \sum_{j=1}^{K_1} i_j)} \eta^{(\sum_{j=1}^{K_1} i_j)} \left( \bigotimes_{k \in [K_1]} \mathbf{D}_{(k,i_k,l_k)} - \bigotimes_{k \in [K_1]} \mathbf{D}_{(k,i_k,l'_k)} \right) \right)^2$$

$$\geq \left( \prod_{i = K_1+1}^K p_i \right) \left( \sum_{i_1 \in \{0,1\}} \cdots \sum_{i_{K_1} \in \{0,1\}} \eta^{(K_1 - \sum_{j=1}^{K_1} i_j)} \eta^{(\sum_{j=1}^{K_1} i_j)} \left( \prod_{k \in [K_1]} p_k \left( \frac{2}{\eta^{(\sum_{j=1}^{K_1} i_j)/K}} \prod_{k \in [K_1]} p_k \right) (1 - t) \right) \right)$$
\[
(\text{o}) \quad 2p \left(1 - t\right) \sum_{i=0}^{K_i-1} \binom{K_i}{i} \left(1 - \frac{\epsilon'}{r^2}\right)^i \left(\frac{\epsilon'}{r^2}\right)^{K_i-i} \\
(\text{p}) \quad 2p \left(1 - t\right) \left(1 - \left(1 - \frac{\epsilon'}{r^2}\right)^{K_i}\right) \\
\geq 2p \left(1 - t\right) \left(1 - \left(1 - \frac{\epsilon'}{r^2}\right)^{K_i}\right) \\
= \frac{2p}{r^2} \left(1 - t\right) \epsilon',
\]

where (l) follows from the distributive property of Kronecker products, (m) follows the fact that terms in the sum have orthogonal columns (from (4) and (8)), (n) follows from (98), (o) follows from substituting values for \(\eta\) and \(\nu\), and (p) follows from the binomial formula.

2) Upper bounding \(\|D_t - D_{t'}\|^2_F\): In order to upper bound \(\|D_t - D_{t'}\|^2_F\), notice that

\[
\|D_t - D_{t'}\|^2_F = \sum_{i_1 \in \{0,1\}} \cdots \sum_{i_K \in \{0,1\}} \sum_{\sum_{j=1}^{K} i_j \neq 0} \eta^{2(K - \sum_{j=1}^{K} i_j)} \nu^{2(\sum_{j=1}^{K} i_j)} \left(\sum_{k \in [K]} D(k, i_k, l_k)\right)^2 \\
\leq \sum_{i_1 \in \{0,1\}} \cdots \sum_{i_K \in \{0,1\}} \sum_{\sum_{j=1}^{K} i_j \neq 0} \eta^{2(K - \sum_{j=1}^{K} i_j)} \nu^{2(\sum_{j=1}^{K} i_j)} \left(\sum_{k \in [K]} D(k, i_k, l_k)\right)^2 \\
= 4 \sum_{i_1 \in \{0,1\}} \cdots \sum_{i_K \in \{0,1\}} \sum_{\sum_{j=1}^{K} i_j \neq 0} \eta^{2(K - \sum_{j=1}^{K} i_j)} \nu^{2(\sum_{j=1}^{K} i_j)} \left(\sum_{k \in [K]} D(k, i_k, l_k)\right)^2 \\
= 4 \sum_{i_1 \in \{0,1\}} \cdots \sum_{i_K \in \{0,1\}} \sum_{\sum_{j=1}^{K} i_j \neq 0} \eta^{2(K - \sum_{j=1}^{K} i_j)} \nu^{2(\sum_{j=1}^{K} i_j)} \left(\prod_{k \in [K]} D(k, 0)\right)^2 \prod_{k \in [K]} \|D(k, 1, l_k)\|^2_F \\
= 4 \sum_{i_1 \in \{0,1\}} \cdots \sum_{i_K \in \{0,1\}} \sum_{\sum_{j=1}^{K} i_j \neq 0} \eta^{2(K - \sum_{j=1}^{K} i_j)} \nu^{2(\sum_{j=1}^{K} i_j)} \left(\prod_{k \in [K]} p_k\right) \left(\prod_{k \in [K]} p_k^{r^2/K}\right) \\
= 4p \sum_{i=0}^{K-1} \binom{K}{i} \left(1 - \frac{\epsilon'}{r^2}\right)^i \left(\frac{\epsilon'}{r^2}\right)^{K-i} \\
\leq 4Kp \frac{\epsilon'}{r^2},
\]

where (q) follows from the triangle inequality, (r) follows from substituting values for \(\eta\) and \(\nu\), and (s) follows from similar arguments as in (98).
3) **Upper bounding** $I(Y; l|T(X))$: We next obtain an upper bound for $I(Y; l|T(X))$ for the dictionary set $D_L$ according to the general coefficient model and side information $T(X) = X$.

Assuming side information $T(X) = X$, conditioned on the coefficients $x_k$, the observations $y_k$ follow a multivariate Gaussian distribution with covariance matrix $\sigma^2 I$ and mean vector $Dx_k$. From the convexity of the KL divergence [37], following similar arguments as in Jung et al. [28], we have

$$I(Y; l|T(X)) = I(Y; l|X) = \frac{1}{L} \sum_{l \in [L]} \mathbb{E}_X \left\{ D_{KL}(f_{D_l}(Y|X)|| \frac{1}{L} \sum_{l' \in [L]} f_{D_{l'}}(Y|X)) \right\}$$

$$\leq \frac{1}{L^2} \sum_{l,l' \in [L]} \mathbb{E}_X \left\{ D_{KL}(f_{D_l}(Y|X)||f_{D_{l'}}(Y|X)) \right\},$$

(102)

where $f_{D_l}(Y|X)$ is the probability distribution of the observations $Y$, given the coefficient matrix $X$ and the dictionary $D_l$. From Durrieu et al. [38], we have

$$D_{KL}(f_{D_l}(Y|X)||f_{D_{l'}}(Y|X)) = \sum_{k \in [N]} \frac{1}{2\sigma^2} ||(D_l - D_{l'})x_k||_2^2$$

$$= \sum_{k \in [N]} \frac{1}{2\sigma^2} \text{Tr} \left\{ (D_l - D_{l'})^\top (D_l - D_{l'}) x_k x_k^\top \right\}.$$  

(103)

Substituting (103) in (102) results in

$$I(Y; l|T(X)) \leq \mathbb{E}_X \left\{ \sum_{k \in [N]} \frac{1}{2\sigma^2} \text{Tr} \left\{ (D_l - D_{l'})^\top (D_l - D_{l'}) x_k x_k^\top \right\} \right\}$$

$$= \sum_{k \in [N]} \frac{1}{2\sigma^2} \text{Tr} \left\{ (D_l - D_{l'})^\top (D_l - D_{l'}) \Sigma_x \right\}$$

$$\leq \sum_{k \in [N]} \frac{1}{2\sigma^2} \Sigma_x ||D_l - D_{l'}||_F^2$$

$$\leq \sum_{k \in [N]} \frac{1}{2\sigma^2} \Sigma_x ||D_l - D_{l'}||_F^2$$

$$\leq \frac{N}{2\sigma^2} ||\Sigma_x||_2 \left( \frac{4Kp\varepsilon'}{r^2} \right)$$

$$= \frac{2NKp||\Sigma_x||_2}{r^2\sigma^2} \varepsilon',$$

(104)

where (u) follows from (101). To show (t), we use the fact that for any $A \in \mathbb{R}^{p \times p}$ and $\Sigma_x$ with ordered singular values $\sigma_i(A)$ and $\sigma_i(\Sigma_x)$, $i \in [p]$, we have

$$\text{Tr} \left\{ A \Sigma_x \right\} \leq \text{Tr} \left\{ A \Sigma_x \right\}$$

$$\leq \sum_{i=1}^{p} \sigma_i(A) \sigma_i(\Sigma_x)$$

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\begin{align}
\left(\sigma_1(\Sigma_x) \sum_{i=1}^{p} \sigma_i(A) \right) \\
\leq \sigma_1(\Sigma_x) \sum_{i=1}^{p} \sigma_i(A) \\
= \|\Sigma_x\|_2 \text{Tr}\{A\},
\end{align}

(105)

where \((w)\) follows from Von Neumann’s trace inequality \([39]\) and \((w)\) follows from the positivity of the singular values of \(\Sigma_x\). The inequality in \((t)\) follows from replacing \(A\) with \((D_l - D_{l'}^T)(D_l - D_{l'})\) and using the fact that \(\text{Tr}\{(D_l - D_{l'}^T)(D_l - D_{l'})\} = \|D_l - D_{l'}\|_F^2\).

**Proof of Lemma 4** The dictionary class \(D_L\) constructed in Lemma 2 is again considered here. Note that \((40)\) implies \(\varepsilon' < \sigma^2\), as \(s \geq 1\). The first part of Lemma 4 up to \((41)\), thus trivially follows from Lemma 2. In order to prove the second part, notice that in this case the coefficient vector is assumed to be sparse according to \((22)\). Denoting \(x_{S_k}\) as the elements of \(x_k\) with indices \(S_k = \text{supp}(x_k)\), we have observations \(y_k\) as

\[
y_k = D_l,S_k x_{S_k} + n_k.
\]

(106)

Hence conditioned on \(S_k = \text{supp}(x_k)\), observations \(y_k\)'s are zero-mean independent multivariate Gaussian random vectors with covariances

\[
\Sigma_{(k,l)} = \sigma^2_d D_l,S_k D_{l,S_k}^T + \sigma^2 I_s.
\]

(107)

The conditional MI \(I(Y;l|T(X) = \text{supp}(X))\) has the following upper bound \([28]\), \([40]\):

\[
I(Y;l|T(X)) \leq \mathbb{E}_{T(X)} \left\{ \frac{1}{L^2} \text{Tr} \left\{ \left[ \Sigma_{(k,l)}^{-1} - \Sigma_{(k,l')}^{-1} \right] [\Sigma_{(k,l)} - \Sigma_{(k,l')}] \right\} \right\}
\leq \text{rank}\{\Sigma_{(k,l)} - \Sigma_{(k,l')}\} \mathbb{E}_{T(X)} \left\{ \frac{1}{L^2} \sum_{k \in [N]} \left\| \Sigma_{(k,l)}^{-1} - \Sigma_{(k,l')}^{-1} \right\|_2 \left\| \Sigma_{(k,l)} - \Sigma_{(k,l')} \right\|_2 \right\}.
\]

(108)

Since \(\text{rank}(\Sigma_{(k,l)}) \leq s\), \(\text{rank}\{\Sigma_{(k,l)} - \Sigma_{(k,l')}\} \leq 2s\) \([28]\).

Next, note that non-zero elements of the coefficient vector are selected according to \((22)\) and \((37)\), we can write the subdictionary \(D_l,S_k\) in terms of the Khatri-Rao product of matrices:

\[
D_l,S_k = \bigotimes_{i \in [K]} D_{(i,l),S_k},
\]

(109)
where $\mathcal{S}_{k_i} = \{j_k\}_{k_i=1}^s$, $j_k \in [p_i]$, for any $i \in [K]$, denotes the support of $x_k$ according to the coordinate dictionary $D_{(i,j)}$ and $\mathcal{S}_k$ corresponds to the indexing of the elements of $(S_1 \times \ldots \times S_K)$. Note that $D_{l,S_k} \in \mathbb{R}^{\prod_{i \in [K]} m_i \times s}$ and in this case, the $\mathcal{S}_{k_i}$’s can be multisets. We can now write

$$
\Sigma_{(k,l)} = \sigma_a^2 \left( \bigotimes_{i \in [K]} D_{(i,l_i),S_{k_i}} \right) \left( \bigotimes_{j \in [K]} D_{(j,l_j),S_{k_j}} \right)^T + \sigma^2 I_s, \tag{110}
$$

We next write

$$
\frac{1}{\sigma_a} (\Sigma_{(k,l)} - \Sigma_{(k,l')}) = \left( \bigotimes_{i \in [K]} D_{(i,l_i),S_{k_i}} \right) \left( \bigotimes_{j \in [K]} D_{(j,l_j),S_{k_j}} \right)^T - \left( \bigotimes_{i' \in [K]} D_{(i',l'_{i'},S_{k_{i'}})} \right) \left( \bigotimes_{j' \in [K]} D_{(j',l'_{j'},S_{k_{j'}})} \right)^T
$$

$$
= \left( \sum_{i_1 \in \{0,1\}} \cdots \sum_{i_K \in \{0,1\}} \eta^{K-\sum_{k=1}^K i_k} (\sum_{k=1}^K i_k) \times D_{(k_1,i_1,l_1),S_{k_{11}}} \right)^T - \left( \sum_{i'_1 \in \{0,1\}} \cdots \sum_{i'_K \in \{0,1\}} \eta^{K-\sum_{k=1}^K i'_k} (\sum_{k=1}^K i'_k) \times D_{(k_2,i'_1,l'_1),S_{k_{12}}} \right)^T
$$

$$
= \sum_{i_1 \in \{0,1\}} \cdots \sum_{i_K \in \{0,1\}} \left( \sum_{j_1 \in \{0,1\}} \cdots \sum_{j_K \in \{0,1\}} \eta^{(2K-\sum_{k=1}^K i_k-\sum_{k=1}^K j_k)} (\sum_{k=1}^K i_k+\sum_{k=1}^K j_k) \times D_{(k_1,i_1,j_1),S_{k_{11}}} \right)^T
$$

$$
- \sum_{i'_1 \in \{0,1\}} \cdots \sum_{i'_K \in \{0,1\}} \left( \sum_{j'_1 \in \{0,1\}} \cdots \sum_{j'_K \in \{0,1\}} \eta^{(2K-\sum_{k=1}^K i'_k-\sum_{k=1}^K j'_k)} (\sum_{k=1}^K i'_k+\sum_{k=1}^K j'_k) \times D_{(k_2,i'_1,j'_1),S_{k_{12}}} \right)^T.
\tag{111}
$$

We now note that

$$
\|A_1 \ast A_2\|_2 = \|(A_1 \otimes A_2)J\|_2
$$

1Due to the fact that $\mathcal{S}_{k_i}$’s can be multisets, $D_{(i,l_i),S_{k_i}}$’s can have duplicated columns.
\[ \leq \|(A_1 \otimes A_2)\|_2 \|J\|_2 \]
\[ \overset{(a)}{=} \|A_1\|_2 \|A_2\|_2, \]

where \( J \in \mathbb{R}^{p \times s} \) is a selection matrix that selects \( s \) columns of \( A_1 \otimes A_2 \) and \( j_i = e_k \) for \( i \in [s], k \in [p] \).

Here, (a) follows from the fact that \( \|J\|_2 = 1 \) \((J^T J = I)\). From (40), it is apparent that \( \sqrt{\frac{s \varepsilon'}{r^2}} \leq 1 \).

Furthermore,
\[ \|D_{(i,0),S_{k_1}}\|_2 \leq \sqrt{\frac{3}{2}}, \quad \|D_{(i,1,l),S_{k_1}}\|_2 \leq \sqrt{\frac{s}{r^2/K}}, \quad i \in [K], \]

where the fist inequality in (113) follows from the RIP condition for \( \{D_{(0,i)}, i \in [K]\} \) and the second inequality follows from the fact that \( \|A\|_2 \leq \|A\|_F \). We therefore have

\[ \frac{1}{\sigma^2} \|\Sigma(k,l) - \Sigma(k',l')\|_2 \]
\[ \overset{(b)}{\leq} 2 \sum_{i_1 \in \{0,1\}} \cdots \sum_{i_K \in \{0,1\}} \left( \sum_{j_1 \in \{0,1\}} \cdots \sum_{j_K \in \{0,1\}} \eta^{(2K - \sum_{k=1}^{K} i_k - \sum_{k=1}^{K} j_k)} \left( \sum_{k=1}^{K} i_k + \sum_{k=1}^{K} j_k \right) \right) \]
\[ \left( \sum_{j_1 \in \{0,1\}} \cdots \sum_{j_K \in \{0,1\}} \eta^{(K - \sum_{k=1}^{K} j_k)} \left( \sum_{k=1}^{K} j_k \right) \prod_{k_1 \in [K]} \prod_{k \in [K]} \prod_{k_2 \in [K]} \prod_{k \in [K]} \prod_{k \in [K]} \right) \]
\[ + 2 \sum_{i_1' \in \{0,1\}} \cdots \sum_{i_K' \in \{0,1\}} \eta^{(K - \sum_{k=1}^{K} j_k)} \left( \sum_{k=1}^{K} j_k \right) \prod_{k_1 \in [K]} \prod_{k \in [K]} \prod_{k_2 \in [K]} \prod_{k \in [K]} \prod_{k \in [K]} \]

\[ \overset{(c)}{=} 2 \left( \sum_{i=0}^{K-1} \binom{K}{i} \eta^i \nu^{K-i} \left( \sqrt{\frac{3}{2}} \right)^i \left( \sqrt{\frac{s}{r^2/K}} \right)^{K-i} \right) \left( \sum_{j=0}^{K-1} \binom{K}{j} \eta^j \nu^{K-j} \left( \sqrt{\frac{3}{2}} \right)^j \left( \sqrt{\frac{s}{r^2/K}} \right)^{K-j} \right) \]

\[ + 2 \left( \eta \sqrt{\frac{3}{2}} \right)^K \left( \sum_{j=0}^{K-1} \binom{K}{j} \eta^j \nu^{K-j} \left( \sqrt{\frac{3}{2}} \right)^j \left( \sqrt{\frac{s}{r^2/K}} \right)^{K-j} \right) \]

\[ \overset{(c)}{\leq} 2 \left( \sum_{i=0}^{K-1} \binom{K}{i} \left( \sqrt{\frac{3}{2}} \right)^i \left( \sqrt{\frac{s \varepsilon'}{r^2}} \right)^{K-i} \right) \left( \sum_{j=0}^{K-1} \binom{K}{j} \left( \sqrt{\frac{3}{2}} \right)^j \left( \sqrt{\frac{s \varepsilon'}{r^2}} \right)^{K-j} \right) \]
\[ + 2 \left( \frac{3}{2} \right)^K \left( \sum_{j'=0}^{K-1} \left( \frac{3}{2} \right)^{j'} \left( \frac{s\epsilon'}{r^2} \right)^{K-j'} \right) \]

\[ = 2 \sqrt{\frac{s\epsilon'}{r^2}} \left( \sum_{i=0}^{K-1} \left( \frac{3}{2} \right)^i \left( \frac{\sqrt{s\epsilon'}}{r^2} \right)^{K-1-i} \right) \left( \sum_{j=0}^{K} \left( \frac{3}{2} \right)^j \left( \frac{s\epsilon'}{r^2} \right)^{K-j} \right) \]

\[ \leq 2 \sqrt{\frac{s\epsilon'}{r^2}} \left( \frac{3}{2} \right)^{K-1} \sum_{i=0}^{K} \left( \frac{3}{2} \right)^i \left( \frac{s\epsilon'}{r^2} \right)^{K-1} \left( \frac{3}{2} \right)^K \]

\[ \leq 2 \left( \frac{s\epsilon'}{r^2} \right)^{K+1} \left( \frac{3}{2} \right)^K \left( \frac{3}{2} \right)^{2K} \left( \frac{3}{2} \right)^K \]

\[ \leq 3^{2K+1} \sqrt{\frac{s\epsilon'}{r^2}}, \quad (114) \]

where (b) follows from triangle inequality, (c) follows from (112), (d) follows from (113), (e) and (f) follow from replacing the value for \( \nu \) and the fact that \( \eta < 1 \) and \( \frac{s\epsilon'}{r^2} < 1 \) (by assumption). Denoting the smallest eigenvalue of \( \Sigma_{(k,l)} \) as \( \lambda_{\min}(\Sigma_{(k,l)}) \), \( \lambda_{\min}(\Sigma_{(k,l)}) \geq \sigma^2 \) holds; thus, we have \( \|\Sigma_{(k,l)}^{-1}\|_2 \leq \frac{1}{\sigma^2} \) and from (41), we get

\[ \|\Sigma_{(k,l)}^{-1} - \Sigma_{(k,l')}^{-1}\|_2 \leq 2 \|\Sigma_{(k,l)}^{-1}\|_2 \|\Sigma_{(k,l)} - \Sigma_{(k,l')}\|_2 \]

\[ \leq \frac{2}{\sigma^4} \|\Sigma_{(k,l)} - \Sigma_{(k,l')}\|_2. \quad (115) \]

Now (108) can be stated as

\[ I(Y; l|T(X)) \leq \frac{4Ns}{\sigma^4} \sum_{l,l'} \|\Sigma_{(k,l)} - \Sigma_{(k,l')}\|_2^2 \]

\[ \leq \frac{4Ns}{\sigma^4} \|\Sigma_{(k,l)} - \Sigma_{(k,l')}\|_2^2 \]

\[ \leq \frac{4Ns}{\sigma^4} \left( 3^{4K+2} \left( \frac{\sigma_a}{\sigma} \sqrt{\frac{s\epsilon'}{r^2}} \right)^2 \right) \]

\[ = 36 \left( 3^{4K} \right) \left( \frac{\sigma_a}{\sigma} \right)^4 \frac{Ns^2}{r^2} \epsilon', \quad (116) \]

where (g) follow from (114). Thus, the proof is complete.  

**Proof of Lemma 5** Similar to Lemma 4, the first part of this Lemma trivially follows from Lemma 2. Also, in this case the coefficient vector is assumed to be sparse according to (23). Hence, conditioned on \( S_k = \text{supp}(x_k) \), observations \( y_k \)'s are zero-mean independent multivariate Gaussian random vectors with covariances given by (107). Similar to Lemma 4, therefore, the conditional MI has the upper bound given in (108). We now simplify this upper bound further.
When non-zero elements of the coefficient vector are selected according to (23) and (37), we can write the dictionary \( D_{t,S_k} \) in terms of the Kronecker product of matrices:

\[
D_{t,S_k} = \bigotimes_{i \in [K]} D_{(i,l_i),S_{k_i}}, \quad (117)
\]

where \( S_{k_i} = \{ j_{k_i} \}_{k_i=1}^{s_i}, j_{k_i} \in [p_i], \) for all \( i \in [K], \) denotes the support of \( x_k \) on coordinate dictionary \( D_{(i,l_i)} \) and \( S_k \) corresponds to indexing of the elements of \((S_1 \times \cdots \times S_K)\). Note that \( D_{t,S_k} \in \mathbb{R}^{\prod_{i \in [K]} m_i \times s} \).

In contrast to coefficient model (22), in this model the \( S_{k_i} \)’s are not multisets anymore since for each \( D_{(i,l_i)}, i \in [K], \) we select \( s_i \) columns at random and \( D_{(i,l_i),S_{k_i}} \) are submatrices of \( D_{(i,l_i)}. \) Therefore, (107) can be written as

\[
\Sigma_{(k,l)} = \sigma_a^2 \left( \bigotimes_{i \in [K]} D_{(i,l_i),S_{k_i}} \right) \left( \bigotimes_{i \in [K]} D_{(i,l_i),S_{k_i}} \right)^T + \sigma^2 I_s. \quad (118)
\]

In order to find an upper bound for \( \| \Sigma_{(k,l)} - \Sigma_{(k,l')} \|_2 \), notice that the expression for \( \Sigma_{(k,l)} - \Sigma_{(k,l')} \) is similar to that of (111), where \( \otimes \) is replaced by \( \bigotimes \). Using the property of Kronecker product that

\[
\| A \otimes B \|_2 = \| A \|_2 \| B \|_2
\]

and the fact that

\[
\| D_{(i,0),S_{k_i}} \|_2 \leq \sqrt{\frac{3}{2}}, \quad \| D_{(i,1,l_i),S_{k_i}} \|_2 \leq \sqrt{\frac{s_i}{p_i / K}}, \quad \forall i \in [K], \quad (119)
\]

we have

\[
\frac{1}{\sigma_a^2} \| \Sigma_{(k,l)} - \Sigma_{(k,l')} \|_2 \leq 2 \sum_{i_1 \in \{0,1\}} \cdots \sum_{i_{K} \in \{0,1\}} \left( \sum_{j_{1} \in \{0,1\}} \cdots \sum_{j_{K} \in \{0,1\}} \right) \eta^{(K-\sum_{k=1}^{K} i_k - \sum_{k=1}^{K} j_k)} \prod_{k=1}^{K} \left( \sum_{i_k=0 \atop i_k \neq 0} \right) \| D_{(k_1,0),S_{k_{i_k}}}, \|_2 \| D_{(k_{i_1},0),S_{k_{i_1}}}, \|_2 \\
\]

\[
= 2 \sum_{i_1 \in \{0,1\}} \cdots \sum_{i_{K} \in \{0,1\}} \eta^{(K-\sum_{k=1}^{K} i_k)} \prod_{k=1}^{K} \left( \sum_{i_k=0 \atop i_k \neq 0} \right) \| D_{(k_1,0),S_{k_{i_k}}}, \|_2 \| D_{(k_{i_1},0),S_{k_{i_1}}}, \|_2 \\
\]

\[
+ 2 \left( \eta^K \prod_{k=1}^{K} \| D_{(k,0),S_{k_i}} \|_2 \right)
\]
where (a) follows from (119), (b) follows from replacing the value for \( \nu \) and the fact that \( \eta < 1, \frac{\varepsilon'}{r^2} < 1 \) (by assumption), and (c) follows from similar arguments in (114). The rest of the proof follows the same arguments as in Lemma 3 and (116) holds in this case as well.

\textit{Proof of Theorem 4} Any dictionary \( D \in X(I_p, r) \) can be written as

\[
D = A \otimes B = (I_{p_1} + \Delta_1) \otimes (I_{p_2} + \Delta_2),
\]

We have to ensure that \( \|D - I_p\|_F \leq r. \) We have

\[
\|D - I_p\|_F = \|I_{p_1} \otimes \Delta_2 + \Delta_1 \otimes I_{p_2} + \Delta_1 \otimes \Delta_2\|_F \\
\leq \|I_{p_1} \otimes \Delta_2\|_F + \|\Delta_1 \otimes I_{p_2}\|_F + \|\Delta_1 \otimes \Delta_2\|_F \\
= \|I_{p_1}\|_F \|\Delta_2\|_F + \|\Delta_1\|_F \|I_{p_2}\|_F + \|\Delta_1\|_F \|\Delta_2\|_F \\
\leq r_2 \sqrt{p_1} + r_1 \sqrt{p_2} + r_1 r_2 \\
\leq r,
\]

where (a) follows from (50). Therefore, we have

\[
D \in \left\{ A \otimes B = (I_{p_1} + \Delta_1) \otimes (I_{p_2} + \Delta_2) \mid \|\Delta_1\|_F \leq r_1, \|\Delta_2\|_F \leq r_2, r_2 \sqrt{p_1} + r_1 \sqrt{p_2} + r_1 r_2 \leq r, \|a_j\|_2 = 1, j = [p_1], \|b_j\|_2 = 1, j = [p_2] \right\}.
\]

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In this case, the new observation vectors \( y'_{(k,j)} \) can be written as

\[
y'_{(k,j)} = Ax'_{(k,j)} + A_p x_k, \quad j \in [p_2], \quad k \in [N],
\]

where \( A_p \) denotes the matrix consisting of the rows of \((A \otimes \Delta_2)^{T,T_k}\) with indices

\[
T_k \triangleq i p_2 + j, \quad i = \{0\} \cup [p_1 - 1], \quad j = ((k - 1) \mod p_2) + 1.
\]

Similarly, for \( y''_{(k,j)} \) we have

\[
y''_{(k,j)} = Bx''_{(k,j)} + B_p x_k, \quad j \in [p_1], \quad k \in [N],
\]

where \( B_p \) denotes the matrix consisting of the rows of \((\Delta_1 \otimes B)^{T,T_k}\) with indices

\[
I_k \triangleq j p_2 + i, \quad i = \{0\} \cup [p_2 - 1], \quad j = (k - 1) \mod p_1.
\]

Given the fact that \( x_k \in \{-1,0,1\}^p \), \( \sigma_a^2 = 1 \) and \( \|x_k\|_2^2 = s \). After division of the vector coefficient according to (54) and (58), we have

\[
E_{x_k} \{ x_{k,i_1}^2 \} = E_{x'_{(k,j_1)}} \{ x_{(k,j_1),i_2}^2 \} = E_{x''_{(k,j_2),i_3}} \{ x''_{(k,j_2),i_3}^2 \} = \frac{s}{p},
\]

for any \( k \in [N], j_1 \in [p_2], j_2 \in [p_1], i_1 \in [p], i_2 \in [p_1], \) and \( i_3 \in [p_2] \). The SNR is

\[
\text{SNR} = \frac{E_x \{ \|x\|_2^2 \}}{E_n \{ \|n\|_2^2 \}} = \frac{s}{m \sigma^2}.
\]

We are interested in upper bounding \( E_Y \{ \|\tilde{D}(Y) - D\|_F^2 \} \). For this purpose we first upper bound \( E_Y \{ \|\tilde{A}(Y) - A\|_F^2 \} \) and \( E_Y \{ \|\tilde{B}(Y) - B\|_F^2 \} \). We can split these MSEs into the sum of column-wise MSEs:

\[
E_Y \{ \|\tilde{A}(Y) - A\|_F^2 \} = \sum_{l=1}^{p_1} E_Y \{ \|\tilde{a}_l(Y) - a_l\|_2^2 \}.
\]

By construction:

\[
\|\tilde{a}_l(Y) - a_l\|_2^2 \leq 2 \left( \|\tilde{a}_l(Y)\|_2^2 + \|a_l\|_2^2 \right) \leq 4,
\]

We are interested in upper bounding \( E_Y \{ \|\tilde{D}(Y) - D\|_F^2 \} \). For this purpose we first upper bound \( E_Y \{ \|\tilde{A}(Y) - A\|_F^2 \} \) and \( E_Y \{ \|\tilde{B}(Y) - B\|_F^2 \} \). We can split these MSEs into the sum of column-wise MSEs:

\[
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\]

By construction:

\[
\|\tilde{a}_l(Y) - a_l\|_2^2 \leq 2 \left( \|\tilde{a}_l(Y)\|_2^2 + \|a_l\|_2^2 \right) \leq 4,
\]
where (b) follows from the projection step in (61). We define the event $C$ to be

$$C \triangleq \bigcap_{k \in [N]} \bigcap_{i \in [p]} \{ |n_{k,i}| \leq 0.4 \}. \quad (132)$$

In order to find the setting under which $\mathbb{P}\{ \hat{X} = X | C \} = 1$, i.e., when recovery of the coefficient vectors is successful, we observe the original observations and coefficient vectors satisfy:

$$y_{k,i} - x_{k,i} = (I_{p_1} \otimes \Delta_2 + \Delta_1 \otimes I_{p_2} + \Delta_1 \otimes \Delta_2)^{T,i} x_k + n_{k,i} \quad (133)$$

and

$$\begin{align*}
&\left| (I_{p_1} \otimes \Delta_2 + \Delta_1 \otimes I_{p_2} + \Delta_1 \otimes \Delta_2)^{T,i} x_k + n_{k,i} \right| \\
&\leq \left\| (I_{p_1} \otimes \Delta_2 + \Delta_1 \otimes I_{p_2} + \Delta_1 \otimes \Delta_2)^{T,i} \right\|_2 \|x_k\|_2 + |n_{k,i}| \\
&\leq (\|\Delta_1\|_F + \|\Delta_2\|_F + \|\Delta_1\|_F \|\Delta_2\|_F) \|x_k\|_2 + |n_{k,i}| \\
&\leq \left( r_1 + r_2 + r_1 r_2 \right) \sqrt{s} + |n_{k,i}|. \quad (134)
\end{align*}$$

By using the assumption $(r_1 + r_2 + r_1 r_2) \sqrt{s} \leq 0.1$ and conditioned on the event $C$, $|n_{k,i}| \leq 0.4$, we have that for every $k \in [N]$ and $l \in [p_1]$:

$$\begin{cases}
y_{k,i} > 0.5 & \text{if } x_{k,i} = 1, \\
-0.5 < y_{k,i} < 0.5 & \text{if } x_{k,i} = 0, \\
y_{k,i} < -0.5 & \text{if } x_{k,i} = -1,
\end{cases} \quad (135)$$

thus, ensuring correct recovery of coefficients ($\hat{X} = X$) using the thresholding technique when conditioned on $C$. Using standard tail bounds for Gaussian random variables [28, (92)], [42, Proposition 7.5] and taking a union bound over all $pN$ i.i.d. variables $\{n_{k,i}\}, k \in [N], i \in [p]$, we have

$$\mathbb{P}\{ C^c \} \leq \exp\left( -\frac{0.08pN}{\sigma^2} \right). \quad (136)$$

To find an upper bound for $\mathbb{E}_Y \{ \| \hat{a}(Y) - a_l \|_2^2 \}$, we can write it as

$$\begin{align*}
\mathbb{E}_Y \{ \| \hat{a}(Y) - a_l \|_2^2 \} &= \mathbb{E}_{Y,N} \{ \| \hat{a}(Y) - a_l \|_2^2 | C \} \mathbb{P}(C) + \mathbb{E}_{Y,N} \{ \| \hat{a}(Y) - a_l \|_2^2 | C^c \} \mathbb{P}(C^c) \\
&\leq \mathbb{E}_{Y,N} \{ \| \hat{a}(Y) - a_l \|_2^2 | C \} + 4 \exp\left( -\frac{0.08pN}{\sigma^2} \right), \quad (137)
\end{align*}$$
where (c) follows from (131) and (136). To bound $\mathbb{E}_{Y,N}\{\|\hat{a}_t(Y) - a_t\|_2^2|C\}$, we have

$$
\mathbb{E}_{Y,N}\{\|\hat{a}_t(Y) - a_t\|_2^2|C\} \leq \mathbb{E}_{Y,N}\{\|\hat{a}_t(Y) - a_t\|_2^2|C\}
$$

$$(d)$$

$$
= \mathbb{E}_{Y,N}\left\{ \left\| \frac{p_1}{N_s} \sum_{k=1}^{N} \sum_{j=1}^{p_2} \hat{x}'_{(k,j),l}y_{(k,j)} - a_t \right\|_2^2|C\right\}
$$

$$(e)$$

$$
= \mathbb{E}_{X,N}\left\{ \left\| \frac{p_1}{N_s} \sum_{k=1}^{N} \sum_{j=1}^{p_2} x_{(k,j),l}y_{(k,j)} - a_t \right\|_2^2|C\right\}
$$

$$(f)$$

$$
\leq 2\mathbb{E}_{X,N}\left\{ \left\| \frac{p_1}{N_s} \sum_{k=1}^{N} \sum_{j=1}^{p_2} x_{(k,j),l}n_{(k,j)}'\right\|_2^2|C\right\}
$$

$$(g)$$

$$
+ 4\mathbb{E}_{X,N}\left\{ \left\| a_t - \frac{p_1}{N_s} \sum_{k=1}^{N} \sum_{j=1}^{p_2} \sum_{t=1}^{p_1} a_{k,t}x_{(k,j),l} \right\|_2^2|C\right\}
$$

$$(h)$$

$$
+ 4\mathbb{E}_{X,N}\left\{ \left\| \frac{p_1}{N_s} \sum_{k=1}^{N} \sum_{j=1}^{p_2} x_{(k,j),l}p_{t,k,t} \right\|_2^2|C\right\},
$$

(138)

where (d) follows from the fact that $a_t$ belongs to the closed unit ball ($\|a_t\|_2 = 1$), (e) follows from (56), (f) follows from conditioning arguments in (135), (g) follows from (124) and (h) follows from the fact that $\|x_1 + x_2\|_2^2 \leq 2(\|x_1\|_2^2 + \|x_2\|_2^2)$. We bound the three terms in (138) separately. Defining $\nu \triangleq \mathcal{Q}(-0.4/\sigma) - \mathcal{Q}(0.4/\sigma)$, where $\mathcal{Q}(x) \triangleq \int_{z=x}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-\frac{z^2}{2})dz$, we can bound the noise variance conditioned on $C$, $\sigma^2_{n,k,t}$, by (28)

$$
\sigma^2_{n,k,t} \leq \frac{\sigma^2}{\nu}.
$$

(139)

The first expectation in (138) can be bounded by

$$
\mathbb{E}_{X,N}\left\{ \left\| \frac{p_1}{N_s} \sum_{k=1}^{N} \sum_{j=1}^{p_2} x_{(k,j),l}n_{(k,j)}'\right\|_2^2|C\right\}
$$

$$
= \left( \frac{p_1}{N_s} \right)^2 \sum_{k=1}^{N} \sum_{k'=1}^{N} \sum_{j=1}^{p_2} \sum_{j'=1}^{p_2} \mathbb{E}_{X,N}\left\{ x_{(k,j),l}x_{(k',j'),l}n_{(k',j')}^Tn_{(k,j)}|C\right\}
$$

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where (i) follows from the fact that $x_{(k,j)}'$ is independent of the event $C$, (j) follows from (128) and (139), and (k) follows from the fact that $\nu \geq 0.5$ under the assumption that $\sigma \leq 0.4$ [28].

To bound the second expectation in (138), we use similar arguments as in Jung et al. [28]. We can write

$$
\mathbb{E}_\{x_t'(k,j),t \} \mid \mathbb{E}_N \{x_t'(k,j),t \} = \begin{cases}
\left(\frac{s}{p}\right)^2 & \text{if } (k, j) = (k', j') \text{ and } t = t' \neq l, \\
\left(\frac{s}{p}\right)^2 & \text{if } (k, j) = (k', j') \text{ and } t = t' = l, \\
\frac{s}{p} & \text{if } (k, j) = (k', j') \text{ and } t = t' = l, \\
0 & \text{otherwise},
\end{cases}
$$

and we have

$$
\mathbb{E}_N \left\{ \left\| a_t - \frac{p_1}{N} \sum_{k=1}^{N} \sum_{j=1}^{p_2} x_t'(k,j),t \sum_{t=1}^{p_1} a_t x_t'(k,j),t \right\|_2^2 \mid C \right\}
\leq a_t^T a_t - 2\frac{p_1}{N} \sum_{k=1}^{N} \sum_{j=1}^{p_2} \sum_{t=1}^{p_1} a_t^T a_t \mathbb{E}_N \left\{ x_t'(k,j),t x_t'(k,j),t \right\}
+ \left(\frac{p_1}{N}\right)^2 \sum_{k=1}^{N} \sum_{k' \neq 1}^{N} \sum_{j=1}^{p_2} \sum_{j' \neq 1}^{p_2} \sum_{t=1}^{p_1} \sum_{t=1}^{p_1} a_t^T a_t \mathbb{E}_N \left\{ x_t'(k',j'),t x_t'(k',j'),t x_t'(k,j),t x_t'(k,j),t \right\}
= 1 - \frac{2p_1}{Ns} (p_2 N) \left(\frac{s}{p}\right) + \left(\frac{p_1}{N}\right)^2 \left(\frac{p_2 N}{s} + (p_1 - 1) \left(\frac{s}{p}\right)^2 + (p_2 N - 1) \left(\frac{s}{p}\right)^2 \right)
= \frac{p_1}{N} \left(\frac{1}{s} + \frac{1}{p_2} - \frac{2}{p}\right)
\leq \frac{2p_1}{N}.
$$

To upper bound the third expectation in (138), we need to bound the $\ell_2$ norm of columns of $A_p$. We have

$$
\forall t \in [p] : \| a_{p,t} \|_2^2 \leq \| (A \otimes \Delta_2)_t \|_2^2
$$
\[ \|a_i\|_2^2 \|\Delta_2\|_F^2 = r_2^2, \tag{143} \]

where \((A \otimes \Delta_2)_t\) denotes the \(t\)-th column of \((A \otimes \Delta_2)\) and \((\|\|)\) follows from the fact that \(A_p\) is a submatrix of \((A \otimes \Delta_2)\). Moreover, similar to the expectation in (141), we have

\[
\mathbb{E}_x \{ x'_i(k, j), t x'_i(k', j'), t x_k, t x_{k'}, t' \} = \begin{cases} \left( \frac{s}{p} \right)^2 & \text{if } (k, j) = (k', j') \text{ and } t = t' \neq l, \\ \left( \frac{2}{p} \right)^2 & \text{if } (k, j) \neq (k', j') \text{ and } t = t' = l, \\ \frac{s}{p} & \text{if } (k, j) = (k', j') \text{ and } t = t' = l, \\ 0 & \text{Otherwise,} \end{cases} \tag{144} \]

where \(l'\) denotes the index of the element of \(x_k\) corresponding to \(x'_i(k, j), t\). Then, the expectation can be bounded by

\[
\mathbb{E}_x, N \left\{ \left\| \frac{p_1}{Ns} \sum_{k=1}^N \sum_{j=1}^{p_2} x'_i(k, j), t \sum_{t=1}^p a_{p, t} x_k, t \right\|_2^2 \right\} C \right\} \leq r_2^2 \left( \frac{p_1}{Ns} \right)^2 \sum_{k=1}^N \sum_{k'=1}^N \sum_{j=1}^{p_2} \sum_{j'=1}^{p_2} \sum_{t=1}^p \sum_{t'=1}^p a_{p, t}^T a_{p, t} \mathbb{E}_x \{ x'_i(k, j), t x'_i(k', j'), t x_k, t x_{k'}, t' \} \]

\[
\leq 2r_2^2 \left( \frac{p_1}{Ns} \right)^2 N p_2 \left( \frac{s}{p} + (p - 1) \left( \frac{s}{p} \right)^2 + (N p_2 - 1) \left( \frac{s}{p} \right)^2 \right) \]

\[
\leq 2r_2^2 \left( \frac{p_1}{N} \right)^2 \]

\[
\leq 2p_2 \left( \frac{p_1}{N} \right)^2 \tag{145} \]

where \((m)\) follows from (143) and \((n)\) follows from the assumption in (50). Summing up (140), (142), and (145), we have

\[
\mathbb{E}_Y \left\{ \left\| \tilde{a}(Y) - a_i \right\|_2 \right\} \leq 4p_1 \left( \frac{m_1 \sigma^2}{s} + 3 \right) + 4 \exp \left( \frac{-0.08pN}{\sigma^2} \right). \tag{146} \]

Summing up the MSE for all columns, we obtain:

\[
\mathbb{E}_Y \left\{ \left\| \tilde{A}(Y) - A \right\|_F \right\} \leq 4p_1 \left( \frac{m_1 \sigma^2}{s} + 3 \right) + 4p_1 \exp \left( \frac{-0.08pN}{\sigma^2} \right). \tag{147} \]

We can follow similar steps to get

\[
\mathbb{E}_Y \left\{ \left\| \tilde{B}(Y) - B \right\|_F \right\} \leq 4p_2 \left( \frac{m_2 \sigma^2}{s} + 3 \right) + 4p_2 \exp \left( \frac{-0.08pN}{\sigma^2} \right). \tag{148} \]
From (147) and (148),

\[
\mathbb{E}_Y \left\{ \| \hat{D}(Y) - D \|_F^2 \right\} \\
= \mathbb{E}_Y \left\{ \| \hat{A}(Y) \otimes \hat{B}(Y) - A \otimes B \|_F^2 \right\} \\
= \mathbb{E}_Y \left\{ \| (\hat{A}(Y) - A) \otimes \hat{B}(Y) + A \otimes (\hat{B}(Y) - B) \|_F^2 \right\} \\
\leq 2 \left( \mathbb{E}_Y \left\{ \| (\hat{A}(Y) - A) \otimes \hat{B}(Y) \|_F^2 \right\} + \mathbb{E}_Y \left\{ \| A \otimes (\hat{B}(Y) - B) \|_F^2 \right\} \right) \\
\leq 2 \left( p_2 \mathbb{E}_Y \left\{ \| (\hat{A}(Y) - A) \|_F^2 \right\} \mathbb{E}_Y \left\{ \| \hat{B}(Y) \|_F^2 \right\} + \| A \|_F^2 \mathbb{E}_Y \left\{ \| (\hat{B}(Y) - B) \|_F^2 \right\} \right) \\
\leq 2 \left( p_2 \mathbb{E}_Y \left\{ \| (\hat{A}(Y) - A) \|_F^2 \right\} + p_1 \mathbb{E}_Y \left\{ \| (\hat{B}(Y) - B) \|_F^2 \right\} \right) \\
\leq \frac{8p}{N} \left( \frac{\sigma^2}{s} (p_1 m_1 + p_2 m_2) + 3(p_1 + p_2) \right) + 8p \exp \left( -\frac{0.08 p N}{\sigma^2} \right) \\
\leq \frac{8p}{N} \left( \frac{p_1 m_1 + p_2 m_2}{m \text{SNR}} + 3(p_1 + p_2) \right) + 8p \exp \left( -\frac{0.08 p N}{\sigma^2} \right),
\]  
(149)

where \((o)\) follows from (129).

\[\square\]

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