GLOBAL SOLUTIONS OF A SURFACE QUASI-GEOSTROPHIC FRONT EQUATION

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Abstract. We consider a nonlinear, spatially-nonlocal initial value problem in one space dimension on \( \mathbb{R} \) that describes the motion of surface quasi-geostrophic (SQG) fronts. We prove that the initial value problem has a unique local smooth solution under a convergence condition on the multilinear expansion of the nonlinear term in the equation, and, for sufficiently smooth and small initial data, we prove that the solution is global.

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1. Introduction

In this paper, we prove the existence of global small, smooth solutions of the following initial value problem

$$\begin{cases} 
\varphi_t(x,t) + \int_{\mathbb{R}} \left[ \varphi_x(x,t) - \varphi_x(x + \zeta, t) \right] \left\{ \frac{1}{|\zeta|} - \frac{1}{\sqrt{\zeta^2 + [\varphi(x,t) - \varphi(x + \zeta,t)]^2}} \right\} d\zeta = 2 \log |\partial_x \varphi(x,t)|, \\
\varphi(x,0) = \varphi_0(x),
\end{cases}$$

(1.1)

where $\varphi: \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is defined for $x \in \mathbb{R}, t \in \mathbb{R}_+$, and $\log |\partial_x|$ is the Fourier multiplier operator with symbol $\log |\xi|$. Our main result is stated in Theorem 5.1.

This initial value problem describes front solutions of the surface quasi-geostrophic (SQG) equation

$$\theta_t + u \cdot \nabla \theta = 0, \quad u = (-\Delta)^{-1/2} \nabla^\perp \theta,$$

(1.2)

where $(-\Delta)^{-1/2}$ is a fractional inverse Laplacian on $\mathbb{R}^2$, and $\nabla^\perp = (-\partial_y, \partial_x)$. The SQG equation arises as a description of quasi-geostrophic flows confined to a surface [42, 48]. After the incompressible Euler equation, it is the most physically important member of a family of two-dimensional active scalar problems for $\theta$ with a divergence-free transport velocity $u = (-\Delta)^{-\alpha/2} \nabla^\perp \theta$ and $0 < \alpha < 2$. The case $\alpha = 2$ gives the vorticity-stream function formulation of the incompressible Euler equation [45], while $\alpha = 1$ gives the SQG equation.

The SQG equation is also of interest from an analytical perspective because it has similar features to the three-dimensional incompressible Euler equation [11]; in both cases, the question of singularity formation in smooth solutions remains open. The SQG equation has global weak solutions [46, 49], and, as for the Euler equation, non-unique weak solutions of the SQG initial value problem may be constructed by convex integration [4, 37]. The SQG equation also has a nontrivial family of global smooth solutions [7].

By SQG front solutions, we mean piecewise-constant solutions of (1.2) with

$$\theta(x, y, t) = \begin{cases} 
\theta_+ & \text{if } y > \varphi(x,t), \\
\theta_- & \text{if } y < \varphi(x,t),
\end{cases}$$

where $\theta_+$ and $\theta_-$ are distinct constants, in which $\theta$ has a jump discontinuity across a front located at $y = \varphi(x,t)$ with $x \in \mathbb{R}$; in (1.1), the jump is normalized to $\theta_+ - \theta_- = 2\pi$. We assume that the front is a graph and do not consider questions related to the breaking or filamentation of the front.

We contrast these front solutions with SQG patches, in which

$$\theta(x, y, t) = \begin{cases} 
\theta_+ & \text{if } (x, y) \in \Omega(t), \\
0 & \text{if } (x, y) \notin \Omega(t),
\end{cases}$$

where $\Omega(t) \subset \mathbb{R}^2$ is a bounded, simply-connected region with smooth boundary. Contour dynamics equations for the motion of patches in SQG, Euler, and generalized SQG (with arbitrary values of $0 < \alpha < 2$) are straightforward to write down, although they require an appropriate regularization of a locally non-integrable singularity in the Green’s function of $(-\Delta)^{\alpha/2}$ when $0 < \alpha \leq 1$. Local well-posedness of the contour dynamics
equations for SQG and generalized SQG patches is proved in \[12, 22\], and generalized SQG patches in the more locally singular regime \(0 < \alpha < 1\) are studied in \[8, 35, 39\].

The boundary of a vortex patch in the Euler equation remains globally smooth in time \([1, 9, 10]\), but this question remains open for SQG patches. Splash singularities cannot occur in a smooth boundary of an SQG patch \([23]\), while numerical results suggest the formation of complex, self-similar singularities in a single patch \([51, 52]\), and a curvature blow up when two patches touch \([14]\). Singularity formation in the boundary of generalized SQG patches has been proved in the presence of a rigid boundary when \(\alpha\) is sufficiently close to 2 \([40, 41]\), and a class of nontrivial global smooth solutions for SQG patches is constructed in \([5, 6, 27]\).

When \(0 < \alpha < 1\), it is straightforward to derive contour dynamics equations for fronts in the same way as one does for patches. In that case, Córdoba et al. \([15]\) prove the global well-posedness of the initial-value problem on \(\mathbb{R}\) for small, smooth generalized SQG fronts.

When \(1 \leq \alpha \leq 2\), additional problems arise in the formulation of contour dynamics equations for fronts as a result of the slow decay of the Green’s function and the lack of compact support of \(\theta\). Front equations, including \([11]\), are derived by a regularization procedure in \([29]\), and a detailed derivation of \([11]\) from the SQG equation is given in \([31]\). Unlike the front equations with \(\alpha \neq 1\), the SQG front equation requires both ‘ultraviolet’ and ‘infrared’ regularization in the front equation to account for the failure of both local and global integrability of the SQG Green’s function \(G(r) = 1/r\) on \(\mathbb{R}\). This failure leads to the logarithmic derivatives in \([11]\), rather than the fractional derivatives that occur for generalized SQG fronts with \(\alpha \neq 1\).

In the case of spatially periodic fronts with \(x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}\), one can write down front equations directly by using the Green’s function of \((-\Delta)^{\alpha/2}\) on the cylinder \(\mathbb{T} \times \mathbb{R}\). Local well-posedness for spatially-periodic SQG front-type equations is proved in \([50]\) for \(C^\infty\)-solutions by a Nash-Moser method, and in \([19]\) for analytic solutions by a Cauchy-Kowalewski method. Almost sharp fronts, across which \(\theta\) is continuous, are studied in \([13, 18, 20, 21]\).

The local well-posedness in Sobolev spaces of a cubically nonlinear approximation of \([11]\) for spatially periodic solutions is proved in \([30]\). In this paper, we consider the fully nonlinear equation \([11]\) on \(\mathbb{R}\). The problem on \(\mathbb{R}\) differs from the problem on \(\mathbb{T}\) in two respects. First, the logarithmic multiplier \(\log|\xi|\) is unbounded at low frequencies, which does not occur on \(\mathbb{T}\) when \(\xi \in \mathbb{Z}\backslash\{0\}\) is discrete and nonzero. Second, the linearized equation on \(\mathbb{R}\) provides dispersive decay, which allows us to get global solutions for sufficiently small, smooth initial data. In this paper, we do not attempt to obtain a sharp regularity result for these solutions.

The general strategy for proving the global existence of small solutions of dispersive equations is to prove an energy estimate together with a dispersive decay estimate. Energy estimates for \([11]\) in the usual \(H^s\)-Sobolev spaces lead to a logarithmic loss of derivatives \([20]\). However, as shown in \([30]\) for spatially periodic solutions of the cubic approximation, we can obtain good energy estimates in suitably weighted \(H^s\)-spaces by para-linearizing the equation and using the linear dispersive term to control the logarithmic loss of derivatives from the nonlinear term.

The proof of the dispersive estimates is more delicate. The linear part of the equation provides \(t^{-1/2}\) decay for the \(L^2\)-norm of the solution, but this is not sufficient to close the global energy estimates for the full equation, since the \(O(t^{-1})\) contribution from the cubically nonlinear term is not integrable in time. We therefore need to analyze the nonlinear dispersive behavior in more detail. We do this by the method of space-time resonances introduced by Germain, Masmoudi and Shatah \([24, 25, 26]\), together with estimates for weighted \(L^2_\xi\)-norms — the so-called \(Z\)-norms — developed by Ionescu and his collaborators \([15, 16, 17, 33, 34, 35, 36]\).

Our \(Z\)-norm estimates in Section \([8]\) involve a detailed frequency-space analysis. The most difficult part is the estimate of the cubically nonlinear terms. In most regions of frequency space, these terms are nonresonant, and we can use integration-by-parts in either the spatial or temporal frequency variables to estimate
Throughout this paper, we use \( A \) to mean there is a constant \( C \). Products can be found in \([2, 10, 28, 53]\). that follow from the Weyl para-differential calculus. Further discussion of the Weyl calculus and para-differential calculus.

In Section 2, we state several lemmas for Fourier multiplier operators that follow from the Weyl para-differential calculus. Further discussion of the Weyl calculus and para-products can be found in \([2, 10, 28, 63]\).

We denote the Fourier transform of \( f : \mathbb{R} \to \mathbb{C} \) by \( \hat{f} : \mathbb{R} \to \mathbb{C} \), where \( \hat{f} = \mathcal{F} f \) is given by

\[
\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x} \, dx.
\]

For \( s \in \mathbb{R} \), we denote by \( H^s(\mathbb{R}) \) the space of Schwartz distributions \( f \) with \( \| f \|_{H^s} < \infty \), where

\[
\| f \|_{H^s} = \left( \int_{\mathbb{R}} \left( 1 + |\xi|^2 \right)^s |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2}.
\]

Throughout this paper, we use \( A \leq B \) to mean there is a constant \( C \) such that \( A \leq CB \), and \( A \gtrsim B \) to mean there is a constant \( C \) such that \( A \geq CB \). We use \( A \approx B \) to mean that \( A \leq B \) and \( B \leq A \).

Let \( \chi : \mathbb{R} \to \mathbb{R} \) be a smooth function such that

\[
\chi \text{ is supported in the interval } \{ \xi \in \mathbb{R} \mid |\xi| \leq 1/10 \}, \quad \chi(\xi) = 1 \text{ on } \{ \xi \in \mathbb{R} \mid |\xi| \leq 3/40 \}.
\]

If \( f \) is a Schwartz distribution and \( a : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \) is a symbol, then we define a Weyl paraproduct \( T_a f \) by

\[
\mathcal{F} [T_a f](\xi) = \int_{\mathbb{R}} \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \hat{a} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f}(\eta) \, d\eta,
\]

where \( \hat{a}(\xi, \eta) \) denotes the partial Fourier transform of \( a(x, \eta) \) with respect to \( \eta \). For \( r_1, r_2 \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), we define a normed symbol space by

\[
\mathcal{M}_{(r_1, r_2)} = \{ a : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \mid \| a \|_{\mathcal{M}_{(r_1, r_2)}} < \infty \},
\]

\[
\| a \|_{\mathcal{M}_{(r_1, r_2)}} = \sup_{(x, \xi) \in \mathbb{R}^2} \left\{ \sum_{\alpha = 0}^{r_1} \sum_{\beta = 0}^{r_2} (1 + |\xi|)^\beta |D_x^\alpha \partial_\xi^\beta a(x, \xi)| \right\}.
\]

The following lemma is proved in Appendix \([h]\).

**Lemma 2.1.** Let \( s \in \mathbb{R} \). If \( a \in \mathcal{M}_{(1,1)} \) and \( f \in H^s(\mathbb{R}) \), then \( T_a f \in H^s(\mathbb{R}) \) and

\[
\| T_a f \|_{H^s} \lesssim \| a \|_{\mathcal{M}_{(1,1)}} \| f \|_{H^s}.
\]
Next, we prove some commutator estimates. We denote by \( \log_+ |\partial_x| \) the Fourier multiplier with symbol
\[
\log_+ |\xi| = \begin{cases} 
\log |\xi| & \text{for } |\xi| > 1, \\
0 & \text{for } |\xi| \leq 1.
\end{cases}
\]

**Lemma 2.2.** Let \( s \in \mathbb{R} \). Suppose that \( f \in H^s(\mathbb{R}) \), \( a \in \mathcal{M}_{(2,1)} \), and \( b \in \mathcal{M}_{(1,2)} \). Then
\[
\begin{align*}
\| |\log_+ |\partial_x|, T_a| f \|_{H^s} & \leq \| a \|_{\mathcal{M}_{(2,1)}} \| f \|_{H^{s-1}}, \\
\| [x, \log_+ |\partial_x|] f \|_{H^s} & \leq \| f \|_{H^{s-1}}, \\
\| [x, T_b] f \|_{H^s} & \leq \| b \|_{\mathcal{M}_{(1,2)}} \| f \|_{H^s}, \\
\| xT_b f - T_b x f \|_{H^s} & \leq \| b \|_{\mathcal{M}_{(1,2)}} \| f \|_{H^s}.
\end{align*}
\]

**Proof.** 1. By the definition (2.22) of the Weyl para-product, we have for \( \xi \neq 0 \) that
\[
\mathcal{F} \left[ |\log_+ |\partial_x| T_a | f \right] (\xi) = \log_+ |\xi| \int_{\mathbb{R}} \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \hat{a} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f}(\eta) \, d\eta
\]
\[
= \int_{\mathbb{R}} \log_+ |\xi - \eta + \eta| \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \hat{a} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f}(\eta) \, d\eta. \tag{2.8}
\]
If \((\xi, \eta)\) belongs to the support of \( \chi(|\xi - \eta|^2/(1 + |\xi + \eta|^2)) \), then we claim that
\[
\left| \frac{\xi - \eta}{\eta} \right| \leq \frac{17}{18} \text{ when } |\eta| \geq 2. \tag{2.9}
\]
To prove this claim, we observe that
\[
\frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \leq \frac{1}{10}
\]
implies that
\[
9 \left| \frac{\xi - \eta}{\eta} - \frac{2}{9} \right|^2 \leq \frac{40}{9} + \frac{1}{\eta^2} \leq \frac{169}{36},
\]
and it follows that
\[
\left| \frac{\xi - \eta}{\eta} \right| \leq \left| \frac{\xi - \eta}{\eta} - \frac{2}{9} \right| + \frac{2}{9} \leq \frac{17}{18}.
\]

We introduce a smooth cutoff function \( \iota(\eta) \) supported in \(|\eta| \leq 3\) with \( \iota(\eta) = 1 \) on \(|\eta| \leq 2\). In view of (2.9), when \( |\eta| > 2 \) we can use
\[
\log \left| \xi - \eta + \eta \right| = \log |\eta| + \log \left| 1 + \frac{\xi - \eta}{\eta} \right|,
\]
and we obtain from (2.8) that, for \( |\xi| > 1 \),
\[
\mathcal{F} \left[ |\log_+ |\partial_x| T_a | f \right] (\xi) = \int_{\mathbb{R}} (1 - \iota(\eta)) \left[ \log |\eta| + \log \left| 1 + \frac{\xi - \eta}{\eta} \right| \right] \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \hat{a} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f}(\eta) \, d\eta
\]
\[
+ \log_+ |\xi| \int_{\mathbb{R}} \iota(\eta) \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \hat{a} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f}(\eta) \, d\eta.
\]
We also have
\[
\mathcal{F} \left[ T_a |\log_+ |\partial_x| f \right] (\xi) = \int_{\mathbb{R}} \log_+ |\eta| \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \hat{a} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f}(\eta) \, d\eta.
\]
By taking the difference of the previous two equations, we get
\[
\mathcal{F} \left[ \log_+ |\hat{\varphi}_x | T_{\alpha} f \right](\xi) - \mathcal{F} \left[ T_{\alpha} \log_+ |\hat{\varphi}_x | f \right](\xi)
\]
\[
= \int_{\mathbb{R}} \left( 1 - \iota(\eta) \right) \left[ \log \left| 1 + \frac{\xi - \eta}{\eta} \right| \right] \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \hat{a} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f}(\eta) \, d\eta \tag{2.10}
\]
\[
+ \int_{\mathbb{R}} \iota(\eta) \left( \log_+ |\xi| - \log_+ |\eta| \right) \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \hat{a} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f}(\eta) \, d\eta.
\]

The integrand in the first integral on the right-hand side of (2.10) is supported on \( \{(\xi, \eta) \mid |\eta| > 2, |\xi - \eta|/|\eta| < 17/18\} \).

Thus, if \( \mathcal{P}(\xi, \eta) \) is a smooth cut-off function supported in a small neighborhood of this set and equal to 1 on the set, then the first integral can be written as
\[
\int_{\mathbb{R}} \mathcal{P}(\xi, \eta) \left[ \frac{\eta}{\xi - \eta} \log \left| 1 + \frac{\xi - \eta}{\eta} \right| \right] \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \left[ 1 - \iota(\eta) \right] \hat{f}(\eta) \, d\eta.
\]

We define
\[
\tilde{A}(\zeta_1, \zeta_2) = \frac{2\zeta_2 - \zeta_1}{2\zeta_1} \log \left| 1 + \frac{2\zeta_1}{2\zeta_2 - \zeta_1} \right| \tilde{a} \left( \zeta_1, \zeta_2 \right) \mathcal{P} \left( \zeta_2 + \frac{\zeta_1}{2}, \zeta_2 - \frac{\zeta_1}{2} \right),
\]
so that
\[
A(x, \zeta_2) = \frac{1}{2} \partial_x^{-1} (2\zeta_2 + i\partial_x) \log \left| 1 - 2i\partial_x (2\zeta_2 + i\partial_x)^{-1} \right| \mathcal{P} \left( \zeta_2 - \frac{i\partial_x}{2}, \zeta_2 + \frac{i\partial_x}{2} \right) \tilde{a} \left( x, \zeta_2 \right).
\]

Then the first integral on the right-hand-side of (2.10) can be written in terms of a para-differential operator with symbol \( A \) as
\[
\int_{\mathbb{R}} \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \tilde{A} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \left[ 1 - \iota(\eta) \right] \hat{f}(\eta) \, d\eta = \mathcal{F} \left[ T_{\tilde{A}} g \right](\xi), \quad g = \mathcal{F}^{-1} \left[ \frac{1 - \iota}{\eta} \hat{f} \right].
\]

By Lemma 2.1 we have
\[
|T_{\tilde{A}} g| \lesssim \|A\|_{\mathcal{M}(1,1)} \|g\|_{H^{\alpha}} \lesssim \|A\|_{\mathcal{M}(1,1)} \|f\|_{H^{\alpha-1}}.
\]

Because of the cut-off function \( \mathcal{P} \), we see that the support of \( \tilde{A}(\zeta_1, \zeta_2) \) is contained in
\[
|2\zeta_2 - \zeta_1| > 4, \quad \left| \frac{2\zeta_1}{2\zeta_2 - \zeta_1} \right| < \frac{17}{18},
\]
so \( \tilde{a} \left( \cdot, \zeta_2 \right) \rightarrow A \left( \cdot, \zeta_2 \right) \) is a zeroth order pseudo-differential operator. By carrying out a dyadic decomposition and using Bernstein’s inequality [2], we obtain that
\[
\|A\|_{\mathcal{M}(1,1)} \lesssim \|a\|_{\mathcal{M}(2,1)}.
\]

It follows that the first term on the right-hand side of (2.10) satisfies the estimate (2.1).

For the second term on the right-hand side of (2.10), the cutoff functions \( \chi, \iota \) ensure that \( |\xi| < 6, |\eta| < 3 \).

Therefore we have the \( H^{\alpha} \)-estimate
\[
\left\| (1 + |\xi|^2)^{s/2} \int_{\mathbb{R}} \iota(\eta) \left( \log_+ |\xi| - \log_+ |\eta| \right) \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \hat{a} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f}(\eta) \, d\eta \right\|_{L^2_{\xi}}
\]
\[
\lesssim \left\| (1 + |\xi|^2)^{s/2} \log_+ |\xi| \int_{\mathbb{R}} \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \hat{a} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \left[ \iota(\eta) \hat{f}(\eta) \right] \, d\eta \right\|_{L^2_{\xi}}
\]
\[
+ \left\| (1 + |\xi|^2)^{s/2} \int_{\mathbb{R}} \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \hat{a} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \left[ \iota(\eta) \log_+ |\eta| \hat{f}(\eta) \right] \, d\eta \right\|_{L^2_{\xi}}.
\]
where $1$ is the indicator function for the set $\{\xi > 1\}$.

2. Taking Fourier transforms, we get that

$$
\hat{\mathcal{F}} \left[ (x, \log_+ |\partial_x|) f \right] = i \partial_\xi \left[ \log_+ |\xi| \hat{f}(\xi) \right] - \log_+ |\xi| (i \partial_\xi \hat{f}(\xi)) = \frac{i}{\xi} 1_{|\xi|>1} \hat{f}(\xi),
$$

where $1_{|\xi|>1}$ is the indicator function for the set $\{|\xi| > 1\}$. Then (2.5) follows.

3. To prove (2.6), we compute that

$$
\hat{\mathcal{F}} \left[ (x, T_b) f \right] (\xi) = i \partial_\xi \left( \hat{T}_b f (\xi) - \hat{T}_b (xf) (\xi) \right)
= i \int_R \partial_\xi \left[ \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right] \hat{b} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f}(\eta) \, d\eta
$$

We rewrite the first integral above as

$$
\int_R \partial_\xi \left[ \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right] \hat{b} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f}(\eta) \, d\eta
= \int_R (\partial_{\xi_1} + \partial_{\xi_2}) \left[ \frac{|\xi_1 - \eta|^2}{1 + |\xi_2 + \eta|^2} \right] \hat{b} \left( \xi_1 - \eta, \frac{\xi_2 + \eta}{2} \right) \hat{f}(\eta) \, d\eta
$$

$$
= \int_R 2 \partial_{\xi_2} - \partial_{\eta} \left[ \frac{|\xi_1 - \eta|^2}{1 + |\xi_2 + \eta|^2} \right] \hat{b} \left( \xi_1 - \eta, \frac{\xi_2 + \eta}{2} \right) \hat{f}(\eta) \, d\eta
$$

$$
= \int_R 2 \partial_{\xi_2} \left[ \frac{|\xi_1 - \eta|^2}{1 + |\xi_2 + \eta|^2} \right] \hat{b} \left( \xi_1 - \eta, \frac{\xi_2 + \eta}{2} \right) \hat{f}(\eta) \, d\eta
$$

It follows that

$$
\hat{\mathcal{F}} [x, T_b] f = 2i \int_R \partial_{\xi_2} \left[ \frac{|\xi_1 - \eta|^2}{1 + |\xi_2 + \eta|^2} \right] \hat{b} \left( \xi_1 - \eta, \frac{\xi_2 + \eta}{2} \right) \hat{f}(\eta) \, d\eta
$$

$$
= 2i \int_R \frac{2 |\xi - \eta|^2 (\xi + \eta)}{1 + |\xi + \eta|^2} \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \hat{b} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f}(\eta) \, d\eta
$$

$$
+ i \int_R \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \partial_2 b \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f}(\eta) \, d\eta. \quad (2.11)
$$
From (2.9), in the support of the cut-off function \( \chi \) we have
\[
\frac{1}{18} |\eta| \leq |\xi| \leq \frac{35}{18} |\eta| \quad \text{when } |\eta| > 2, \quad \text{and } |\xi| < 6 \quad \text{when } |\eta| < 2.
\]
Thus, the first integral on the right-hand-side of (2.11) satisfies
\[
\left\| \left(1 + |\xi|^2\right)^{s/2} \int_{\mathbb{R}} \frac{2|\xi - \eta|^2(\xi + \eta)}{1 + |\xi + \eta|^2} \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \hat{b} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f} (\eta) \, d\eta \right\|_{L^2_\xi}
\leq \left\| \int_{\mathbb{R}} \frac{2|\xi - \eta|^2(\xi + \eta)}{1 + |\xi + \eta|^2} \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \hat{b} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \left( \left(1 - \iota(\eta) \right) + \left(1 + |\eta|^2\right)^{s/2} \hat{f} (\eta) \right) \, d\eta \right\|_{L^2_\xi}.
\]
These terms can be expressed in terms of a Weyl pseudo-differential operator \( B^w \) in (3.1) with symbol
\[
B(x, \xi) = \frac{4\xi \xi_x}{(1 + 4\xi^2)^2} \chi' \left( \frac{-\xi_x^2}{1 + 4\xi^2} \right) b(x, \xi).
\]
Using Theorem (B.2) and Bernstein’s inequality, we then get that
\[
\left\| \left(1 + |\xi|^2\right)^{s/2} \int_{\mathbb{R}} \frac{2|\xi - \eta|^2(\xi + \eta)}{1 + |\xi + \eta|^2} \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \hat{b} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f} (\eta) \, d\eta \right\|_{L^2_\xi} \leq \| B \|_{M_{1,1}} \| f \|_{H^r}
\leq \| b \|_{M_{1,1}} \| f \|_{H^r}.
\]
The second integral on the right-hand-side of (2.11) is the paraproduct \( \mathcal{F}[T_{\partial_y} f] \). By using Lemma (2.7) and the previous estimate, we then obtain (2.6).

4. We compute that
\[
\mathcal{F}(xT_b f - T_{xb} f) = i \int \partial_\xi \left[ \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \hat{b} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f} (\eta) \, d\eta - i \int \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \partial_\xi \hat{b} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f} (\eta) \, d\eta \right].
\]
\[
= i \int \partial_\xi \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \hat{b} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) + \frac{1}{2} \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \partial_2 \hat{b} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f} (\eta) \, d\eta.
\]
The first term satisfies
\[
\left\| \int \partial_\xi \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \hat{b} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f} (\eta) \, d\eta \right\|_{H^r} \leq \| b \|_{M_{1,1}} \| f \|_{H^r},
\]
and the second term satisfies
\[
\left\| \int \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \hat{b} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f} (\eta) \, d\eta \right\|_{H^r} \leq \| b \|_{M_{1,2}} \| f \|_{H^r},
\]
which proves (2.7).
Finally, we give an expansion of $|D|$ acting on para-products (cf. [43]).

**Lemma 2.3.** Let $s \in \mathbb{R}$, $s \geq 2$. If $a \in \mathcal{M}_{(3,1)}$ and $f \in H^s(\mathbb{R})$, then

$$|D|^s T_a f = T_a |D|^s f + s T_{Da} |D|^{s-2} D f + \mathcal{R},$$

where $\mathcal{R}$ satisfies

$$\|\mathcal{R}\|_{L^2} \leq \|a\|_{\mathcal{M}_{(3,1)}} \|f\|_{H^{s-2}(\mathbb{R})},$$

and $Da$ means that the differential operator $D$ acts on the function $x \mapsto a(x, \xi)$ for fixed $\xi$.

**Proof.** By the definition of the Weyl paraproduct

$$\mathcal{F}(|D|^s T_a f) (\xi) = |\xi|^s \int_\mathbb{R} \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \hat{a} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f}(\eta) \, d\eta$$

and using Bernstein’s inequality, we see that

$$r^s \in \mathcal{R}.$$  

By the definition of the Weyl paraproduct

$$|D|^s T_a f = T_a |D|^s f + s T_{Da} |D|^{s-2} D f + \mathcal{R},$$

where $\mathcal{R}$ denotes the partial Fourier transform of $a$ in the first variable. The low frequency part satisfies the reminder estimate, so it can be absorbed into $\mathcal{R}$, and we only need to consider the high frequency part with $|\eta| > 2$. In that case, (2.1) is satisfied on the support of $\chi(|\xi - \eta|^2/(1 + |\xi + \eta|^2))$. Define $b(x) = (1 + x)^s - 1 - sx$. Then

$$|\xi - \eta + \eta|^s = |\eta|^s \left[ 1 + \frac{\xi - \eta}{\eta} \right] = |\eta|^s \left[ 1 + s \frac{\xi - \eta}{\eta} + \frac{\xi - \eta}{\eta} \right].$$

In the expression for $\mathcal{F}[|D|^s T_a f]$, we get

$$\mathcal{F}[|D|^s T_a f](\xi)$$

$$= \int_\mathbb{R} |\eta|^s \left[ 1 + s \frac{\xi - \eta}{\eta} + \frac{\xi - \eta}{\eta} \right] \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \hat{a} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f}(\eta) \, d\eta.$$  

Then we only need to estimate

$$\int_\mathbb{R} |\eta|^2 b \left( \frac{\xi - \eta}{\eta} \right) \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \hat{a} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f}(\eta) \, d\eta.$$

Define the symbol $A$ by

$$\hat{A}(\zeta_1, \zeta_2) = \left( \frac{2 \zeta_2 - \zeta_1}{2} \right)^2 \left( 1 - \left( \frac{2 \zeta_2 - \zeta_1}{2} \right) \right) b \left( \frac{2 \zeta_1}{2 \zeta_2 - \zeta_1} \right) \hat{a}(\zeta_1, \zeta_2).$$

Then (2.12) can be viewed as a para-differential operator with symbol $A$. By considering the supports of $\chi$, $\zeta$ and using Bernstein’s inequality, we see that

$$\|A\|_{\mathcal{M}_{(1,1)}} \leq \|a\|_{\mathcal{M}_{(3,1)}}.$$

The result then follows by applying Lemma 2.1 to (2.12).  

\[ \square \]

2.2. **Fourier multipliers.** Let $\psi : \mathbb{R} \rightarrow [0, 1]$ be a smooth function supported in $[-8/5, 8/5]$ and equal to 1 in $[-5/4, 5/4]$. For any $k \in \mathbb{Z}$, we define

$$\psi_k(\xi) = \psi(\xi/2^k) - \psi(\xi/2^{k-1}), \quad \psi_{\leq k}(\xi) = \psi(\xi/2^k), \quad \psi_{\geq k}(\xi) = 1 - \psi(\xi/2^{k-1}),$$

$$\tilde{\psi}_k(\xi) = \psi_{k-1}(\xi) + \psi_k(\xi) + \psi_{k+1}(\xi),$$

and denote by $P_k$, $P_{\leq k}$, $P_{\geq k}$, and $\tilde{P}_k$ the Fourier multiplier operators with symbols $\psi_k, \psi_{\leq k}, \psi_{\geq k}$, and $\tilde{\psi}_k$, respectively. Notice that $\psi_k(\xi) = \psi_0(\xi/2^k)$, $\tilde{\psi}_k(\xi) = \tilde{\psi}_0(\xi/2^k)$.
It is easy to check that
\[ \|\psi_k\|_{L^2} \approx 2^{k/2}, \quad \|\psi_k\|_{L^2} \approx 2^{-k/2}. \] (2.14)

We will need the following interpolation lemma, whose proof can be found in [36].

**Lemma 2.4.** For any \( k \in \mathbb{Z} \) and \( f \in L^2(\mathbb{R}) \), we have
\[ \|\tilde{P}_k f\|_{L^\infty} \leq \|P_k f\|_{L^1} \leq 2^{-k} \|\hat{f}\|_{L^\infty} \left[ 2^k \|\hat{\phi}_k\|_{L^2} + \|\hat{f}\|_{L^2} \right]. \]

We will also use an estimate for multilinear Fourier multipliers proved in [35]. Before stating the estimate, we introduce some notation.

We define a norm on symbols \( \kappa : \mathbb{R}^d \to \mathbb{C} \) by
\[ \|\kappa\|_{S^{\infty}} = \|F^{-1}\kappa\|_{L^1}, \]
and define the symbol class
\[ S^{\infty} = \left\{ \kappa : \mathbb{R}^d \to \mathbb{C} \mid \kappa \text{ continuous and } \|\kappa\|_{S^{\infty}} < \infty \right\}. \] (2.15)

Given \( \kappa \in S^{\infty} \), we define a multilinear operator \( M_{\kappa} \) acting on Schwartz functions \( f_1, \ldots, f_m \in S(\mathbb{R}) \) by
\[ M_{\kappa}(f_1, \ldots, f_m)(x) = \int_{\mathbb{R}^m} e^{ix(\xi_1 + \cdots + \xi_m)} \kappa(\xi_1, \ldots, \xi_m) \hat{f}_1(\xi_1) \cdots \hat{f}_m(\xi_m) \, d\xi_1 \cdots d\xi_m. \]

**Lemma 2.5.** (i) If \( \kappa_1, \kappa_2 \in S^{\infty} \), then \( \kappa_1 \kappa_2 \in S^{\infty} \).

(ii) Suppose that \( 1 \leq p_1, \ldots, p_m \leq \infty, \) \( 1 \leq p \leq \infty \), satisfy
\[ \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = \frac{1}{p}. \]

If \( \kappa \in S^{\infty} \), then
\[ \|M_{\kappa}\|_{L^{p_1} \times \cdots \times L^{p_m} \to L^p} \lesssim \|\kappa\|_{S^{\infty}}. \]

(iii) Assume \( p, q, r \in [1, \infty] \) satisfy \( 1/p + 1/q + 1/r = 1 \), and \( m \in S^{\infty}_{\eta_1, \eta_2} L^\infty \). Then, for any \( f \in L^p(\mathbb{R}) \), \( g \in L^q(\mathbb{R}) \), and \( h \in L^r(\mathbb{R}) \),
\[ \left\| \int_{\mathbb{R}^2} m(\eta_1, \eta_2, \xi) \hat{f}(\eta_1) \hat{g}(\eta_2) \hat{h}(\xi - \eta_1 - \eta_2) \, d\eta_1 \, d\eta_2 \right\|_{L^\infty} \lesssim \|m\|_{S^{\infty}_{\eta_1, \eta_2} L^\infty} \|\hat{f}\|_{L^p} \|\hat{g}\|_{L^q} \|\hat{h}\|_{L^r}. \]

In particular, using interpolation, we can estimate the \( S^{\infty} \)-norm of a symbol \( m(\eta_1, \eta_2) \) in \( C^{\infty}_c \) by
\[ \|m\|_{S^{\infty}} \lesssim \|m\|_{L^2}^{1/4} \|\hat{c}_n^2 m\|_{L^1}^{1/2} \|\hat{c}_n^2 \hat{c}_{2n} m\|_{L_1}^{1/4} \quad \text{where } i = 1, 2. \] (2.16)

3. **Reformulation of the equation.** In this section, we expand the nonlinearity in the SQG front equation
\[ \varphi_t(x, t) + \int_{\mathbb{R}} \left[ \varphi_x(x, t) - \varphi_x(x + \zeta, t) \right] \left\{ \frac{1}{|\zeta|} - \frac{1}{\sqrt{\zeta^2 + [\varphi(x, t) - \varphi(x + \zeta, t)]^2}} \right\} \, d\zeta = 2 \log |\hat{c}_x| \varphi_x(x, t) \] (3.1)
for fronts with small slopes \( |\varphi_x| \ll 1 \). As we will show, (3.1) can be rewritten as
\[ \varphi_t(x, t) - \sum_{n=1}^{\infty} \frac{c_n}{2n+1} \frac{\partial}{\partial x} \int_{\mathbb{R}^{2n+1}} T_n(\eta_n) \hat{\varphi}(\eta_1, t) \hat{\varphi}(\eta_2, t) \cdots \hat{\varphi}(\eta_{2n+1}, t) e^{i(\eta_1 + \eta_2 + \cdots + \eta_{2n+1})x} \, d\eta_n \]
\[ = 2 \log |\hat{c}_x| \varphi_x(x, t), \] (3.2)
where \( \eta_n = (\eta_1, \eta_2, \ldots, \eta_{2n+1}) \), and

\[
T_n(\eta_n) = \int_{\mathbb{R}} \frac{\prod_{j=1}^{n+1} (1 - e^{i\eta_j})}{|\zeta|^{2n+1}} \, d\zeta, \quad c_n = \frac{\sqrt{n}}{\Gamma\left(\frac{1}{2} - n\right) \Gamma(n+1)}.
\]  

We remark that \( c_n = O(n^{-1/2}) \) as \( n \to \infty \).

In fact, if we expand the nonlinearity in (3.1) around \( \varphi(x, t) = 0 \), we find that

\[
\int_{\mathbb{R}} \left[ \frac{\varphi_x(x, t) - \varphi_x(x + \zeta, t)}{|\zeta|} \frac{\varphi(x, t) - \varphi(x + \zeta, t)}{\sqrt{\zeta^2 + (\varphi(x, t) - \varphi(x + \zeta, t))^2}} \right] \, d\zeta = -\sum_{n=1}^{\infty} c_n \int_{\mathbb{R}} \frac{\varphi(x, t) - \varphi(x + \zeta, t)}{|\zeta|} \, d\zeta.
\]

\[
= -\sum_{n=1}^{\infty} \frac{c_n}{2n+1} \int_{\mathbb{R}} \left[ \frac{\varphi(x, t) - \varphi(x + \zeta, t)}{|\zeta|} \right]^{2n+1} \, d\zeta.
\]

Writing

\[
f_n(x) = \int_{\mathbb{R}} \left[ \frac{\varphi(x) - \varphi(x + \zeta)}{|\zeta|} \right]^{2n+1} \, d\zeta, \quad \varphi(x) = \int_{\mathbb{R}} \varphi(\eta)e^{in\eta} \, d\eta,
\]

we have

\[
f_n(x) = \int_{\mathbb{R}^{2n+1}} T_n(\eta_n) \varphi(\eta_1) \varphi(\eta_2) \cdots \varphi(\eta_{2n+1}) e^{i(\eta_1 + \eta_2 + \cdots + \eta_{2n+1})x} \, d\eta_n,
\]

which gives (3.2).

Isolating the lowest degree nonlinear term in (3.2), which is cubic, we can also write (3.1) as

\[
\varphi_t + \varphi_t \int_{\mathbb{R}} T_1(\eta_1, \eta_2, \eta_3) \varphi(\eta_1, t) \varphi(\eta_2, t) \varphi(\eta_3, t) e^{i(\eta_1 + \eta_2 + \eta_3)x} \, d\eta_1 \, d\eta_2 \, d\eta_3
\]

\[
+ N_{\leq 5}(\varphi)(x, t) = 2 \log |\partial_x|^2 \varphi(x, t),
\]

where \( N_{\leq 5}(\varphi) \) denotes the nonlinear terms of quintic degree or higher

\[
N_{\leq 5}(\varphi)(x, t) = -\sum_{n=2}^{\infty} c_n \int_{\mathbb{R}^{2n+1}} T_n(\eta_n) \varphi(\eta_1, t) \varphi(\eta_2, t) \cdots \varphi(\eta_{2n+1}, t) e^{i(\eta_1 + \eta_2 + \cdots + \eta_{2n+1})x} \, d\eta_n.
\]  

Equation (3.4) will be used in Section 5 in order to carry out nonlinear dispersive estimates, where the main difficulty is controlling the slowest decay in time caused by the lowest degree, cubic nonlinearity.

In the appendix, we evaluate the integrals in (3.3) and show that we can write (3.2) in the alternative form

\[
\varphi_t + \partial_x \left\{ \sum_{n=1}^{\infty} \sum_{\ell=1}^{n} (-1)^{\ell+1} d_{n, \ell} \varphi^{2n-\ell+1} \partial_x^{2n} \log |\partial_x| \varphi^{\ell} \right\} = 2 \log |\partial_x| \varphi_x,
\]

where the constants \( d_{n, \ell} \) are given in (A.4). We will not use (3.6) in this paper since it makes sense classically only for \( C^\infty \)-solutions and does not make explicit the fact that, owing to a cancelation of derivatives, the nonlinear flux in (3.6) involves at most logarithmic derivatives of \( \varphi \). However, we remark that if the quintic and higher-order terms in (3.6) are neglected, then the equation becomes

\[
\varphi_t + \frac{1}{2} \partial_x \left\{ \varphi^2 \log |\partial_x| \varphi_x - \varphi \log |\partial_x| (\varphi^2)_{xx} + \frac{1}{3} \log |\partial_x| (\varphi^3)_{xx} \right\} = 2 \log |\partial_x| \varphi_x,
\]

which is the cubic approximation for the front equation that is derived in (29) and analyzed in [30].
3.2. Para-linearization of the equation. In this section, we para-linearize the SQG front equation (3.2) and put it in a form that allows us to make weighted energy estimates. This form extracts a nonlinear term \( L(T_{B^{\text{disp}}[\varphi]} \varphi) \) from the flux that is responsible for the logarithmic loss of derivatives in the dispersionless equation.

We use Weyl para-differential calculus to decompose the nonlinearity in (3.1). In the following, we use \( C(n, s) \) to denote a positive constant depending only on \( n \) and \( s \), which may change from line to line.

**Proposition 3.1.** Suppose that \( \varphi(\cdot, t) \in H^{s}(\mathbb{R}) \) with \( s \geq 4 \) and \( \|\varphi_x\|_{W^{2, \infty}} + \|L\varphi_x\|_{W^{2, \infty}} \) is sufficiently small. Then (3.1) can be written as

\[
\varphi_t + \partial_x T_{B^0[\varphi]} \varphi + \mathcal{R}(\varphi) = L[(2 - T_{B^{\text{disp}}[\varphi]}) \varphi],
\]

where the symbols \( B^0[\varphi] \) and \( B^{\text{log}}[\varphi] \) are given by the following multilinear expansions in \( \varphi_x \):

\[
\begin{align*}
B^0[\varphi] &= \sum_{n=1}^{\infty} B^0_n[\varphi] \cdot \xi, \\
B^{\text{log}}[\varphi] &= \sum_{n=1}^{\infty} B^{\text{log}}_n[\varphi] \cdot \xi,
\end{align*}
\]

\[
B^0_n[\varphi] = \frac{1}{2c_n} \int_{\mathbb{R}^{2n+1}} \delta \left( \frac{\sum_{j=1}^{2n} \eta_j}{\xi} \right) \prod_{j=1}^{2n} \left( i \eta_j \varphi(\eta_j) \chi \left( \frac{(2n+1)\eta_j}{\xi} \right) \right) d\eta_n,
\]

\[
B^{\text{log}}_n[\varphi] = \frac{1}{2c_n} \int_{\mathbb{R}^{2n+1}} \delta \left( \frac{\sum_{j=1}^{2n} \eta_j}{\xi} \right) \prod_{j=1}^{2n} \left( i \eta_j \varphi(\eta_j) \chi \left( \frac{(2n+1)\eta_j}{\xi} \right) \right) \cdot \log \left( \sum_{j=1}^{2n} \eta_j s_j \right) d\eta_n.
\]

Here, \( c_n \) is given by (3.3), \( \delta \) is the delta-distribution, \( \chi \) is the cutoff function in (2.1), \( \eta_n = (\eta_1, \eta_2, \ldots, \eta_{2n}) \), and \( S_n = (s_1, \ldots, s_{2n}) \). The operators \( T_{B^{\text{log}}[\varphi]} \) and \( T_{B^0[\varphi]} \) are self-adjoint and their symbols satisfy the estimates

\[
\|B^0[\varphi]\|_{\mathcal{M}_{j, 2}} \approx \sum_{n=1}^{\infty} C(n, s)|c_n|\|\varphi_x\|_{W^{2, \infty}}, \quad j = 2, 3,
\]

\[
\|B^{\text{log}}[\varphi]\|_{\mathcal{M}_{j, 2}} \approx \sum_{n=1}^{\infty} C(n, s)|c_n|\left( \|L\varphi_x\|_{W^{2, \infty}} + \|\varphi_x\|_{W^{2, \infty}} \right),
\]

while the remainder term \( \mathcal{R} \) satisfies

\[
\|\mathcal{R}(\varphi)\|_{H^s} \lesssim \|\varphi\|_{H^s} \left( \sum_{n=1}^{\infty} C(n, s)|c_n|\left( \|\varphi_x\|_{W^{2, \infty}} + \|L\varphi_x\|_{W^{2, \infty}} \right) \right),
\]

where the constants \( C(n, s) \) have at most exponential growth in \( n \).

**Proof.** We define

\[
f_n(x) = \int_{\mathbb{R}^{2n+1}} T_n(\eta_n) \varphi(\eta_1) \varphi(\eta_2) \cdots \varphi(\eta_{2n+1}) e^{i(\eta_1 + \eta_2 + \cdots + \eta_{2n+1})x} d\eta_n.
\]

In view of (3.2) and the commutator estimate (2.4), we only need to prove that

\[
- \sum_{n=1}^{\infty} \frac{c_n}{2n+1} \partial_x f_n(x) = \partial_x T_{B^0[\varphi]} \varphi + \partial_x [(T_{B^{\text{disp}}[\varphi]} \varphi)],
\]

where \( \mathcal{R} \) satisfies (3.10), and to do this it suffices to prove for each \( n \) that

\[
\frac{c_n}{2n+1} \partial_x f_n(x) = -\partial_x T_{B^0[\varphi]} \varphi - \partial_x [(T_{B^{\text{disp}}[\varphi]} \varphi)] + \mathcal{R}_n,
\]
Thus, the lower frequency part satisfies the estimate (3.10), and this term can be absorbed in $\mathcal{R}$. Then

$$
\frac{c_n}{2n+1} \hat{c}_x f_n(x) = c_n \hat{c}_x \int \mathbf{T}_n(\eta_n) \hat{\varphi}(\eta_1) \hat{\varphi}(\eta_2) \cdots \hat{\varphi}(\eta_{2n+1}) e^{i(\eta_1 + \eta_2 + \cdots + \eta_{2n+1})x} \, d\eta_n
$$

where

$$
\mathbf{T}_n(\eta_n) \equiv \frac{1}{|\eta_n|} \int_{|\eta_j| \leq |\eta_{2n+1}|} \mathbf{S}_{S_{\eta_{2n+1}}^\infty} \mathbf{L}_{W^{1,\infty}} \|\varphi\|^2 \, d\eta_n
$$

By symmetry, we can assume that $|\eta_{2n+1}|$ is the largest frequency in the expression of $f_n$. Then

$$
= c_n \hat{c}_x \int_{|\eta_j| \leq |\eta_{2n+1}|} \int_{\text{for all } j=1,\ldots,2n} \mathbf{T}_n(\eta_n) \hat{\varphi}(\eta_1) \hat{\varphi}(\eta_2) \cdots \hat{\varphi}(\eta_{2n}) e^{i(\eta_1 + \eta_2 + \cdots + \eta_{2n})x} \, d\eta_n \hat{\varphi}(\eta_{2n+1}) e^{ix\eta_{2n+1}} \, d\eta_{2n+1}.
$$

(3.11)

To proceed, we split the above integral into two parts corresponding to the lower and higher frequencies of $\eta_{2n+1}$. Define $\mathbf{U}_n(\eta_n) = \mathbf{T}_n(\eta_n) \chi(\eta_{2n+1})$, and $\mathbf{A}_n(\eta_n) = \mathbf{T}_n(\eta_n) - \mathbf{U}_n(\eta_n)$. For the lower frequency part, we have

$$
\hat{c}_x \int_{|\eta_j| \leq |\eta_{2n+1}|} \mathbf{U}_n(\eta_n) \hat{\varphi}(\eta_1) \hat{\varphi}(\eta_2) \cdots \hat{\varphi}(\eta_{2n}) e^{i(\eta_1 + \eta_2 + \cdots + \eta_{2n})x} \, d\eta_n \hat{\varphi}(\eta_{2n+1}) e^{ix\eta_{2n+1}} \, d\eta_{2n+1}
$$

where the symbol $\mathbf{A}_n$ is defined by

$$
\mathbf{A}_n(x, \eta_{2n+1}) = \chi(\eta_{2n+1}) \int_{|\eta_j| \leq |\eta_{2n+1}|} \mathbf{S}_{S_{\eta_{2n+1}}^\infty} \mathbf{L}_{W^{1,\infty}} \|\varphi\|^2 \, d\eta_n
$$

Using an $L^2$-boundedness theorem for pseudo-differential operators (Theorem 1.1 of [3]), Lemma 2.3, and the compact support of the cutoff functions, we obtain

$$
\left\| \hat{c}_x \int_{\mathbb{R}} \mathbf{A}_n(x, \eta_{2n+1}) \hat{\varphi}(\eta_{2n+1}) e^{ix\eta_{2n+1}} \, d\eta_{2n+1} \right\|_{L^2} \lesssim \sum_{i,j \leq 1} \left\| \hat{c}_x \hat{\varphi} \mathbf{A}_n \right\|_{L^2} \|\varphi\|_{L^2}
$$

where

$$
\mathbb{I}_n(\eta_n, \eta_{2n+1}) = \begin{cases} 
1 & \text{if } |\eta_j| \leq |\eta_{2n+1}| \text{ for } j = 1, \ldots, 2n, \\
0 & \text{otherwise}.
\end{cases}
$$

(3.12)

Thus, the lower frequency part satisfies the estimate (3.10), and this term can be absorbed in $\mathcal{R}$ in (3.7).
Next, we consider the higher frequency part in (3.11), which we write as

\[
\begin{align*}
\int_{\mathbb{R}} \left( \prod_{j=1}^{2n} \Lambda_n(\eta_j) \chi \left( \frac{(2n+1)\eta_j}{\eta_{2n+1}} \right) + \left[ 1 - \chi \left( \frac{(2n+1)\eta_j}{\eta_{2n+1}} \right) \right] \hat{\varphi}(\eta_j) \right) e^{i(\eta_1 + \eta_2 + \cdots + \eta_{2n})x} d\eta_{2n+1}. 
\end{align*}
\]  

(3.13)

We expand the product in the above integral, and consider two cases depending on whether a term in the expansion contains only factors of \(\chi\) or contains at least one factor \(1 - \chi\). In the first case, the frequency \(\eta_{2n+1}\) is much larger than all of the other frequencies, and we can extract a logarithmic derivative acting on the highest frequency; in the second case at least one other frequency is comparable to \(\eta_{2n+1}\), and we get a remainder term by distributing derivatives on comparable frequencies.

**Case I.** When we take only factors of \(\chi\) in the expansion of the product in (3.13), we get the integral

\[
\int_{\mathbb{R}} \left( \prod_{j=1}^{2n+1} \Lambda_n(\eta_j) \chi \left( \frac{(2n+1)\eta_j}{\eta_{2n+1}} \right) \hat{\varphi}(\eta_j) e^{i(\eta_1 + \eta_2 + \cdots + \eta_{2n})x} d\eta_{2n+1} \right) d\eta_{2n+1}. 
\]  

(3.14)

From (3.3), we can write \(\Lambda_n = [1 - \chi(\eta_{2n+1})]T_n\) as an integral with respect to \(s_n = (s_1, s_2, \ldots, s_{2n+1})\),

\[
\Lambda_n(\eta_n) = \int_{\mathbb{R}} sgn \zeta \left( \prod_{j=1}^{2n+1} i\eta_j e^{i\eta_j s_j} \right) d\eta_{2n+1} d\zeta 
\]

\[
= 2(-1)^n [1 - \chi(\eta_{2n+1})] \left( \prod_{j=1}^{2n+1} \eta_j \right) \int_{[0,1]^{2n+1}} \frac{1}{\sum_{j=1}^{2n+1} \eta_j s_j} d\eta_{2n+1} 
\]

\[
= 2(1 - \chi(\eta_{2n+1})) \left( \prod_{j=1}^{2n} (i\eta_j) \right) \int_{[0,1]^{2n}} \text{log} \left| 1 + \sum_{j=1}^{2n} \eta_j s_j \right| d\eta_{2n} - \log \left| \sum_{j=1}^{2n} \eta_j \right| d\eta_{2n+1} 
\]

\[
+ (1 - \chi(\eta_{2n+1})) \left( \prod_{j=1}^{2n} (i\eta_j) \right) \int_{[0,1]^{2n}} \text{log} \left| 1 + \sum_{j=1}^{2n} \eta_j s_j \right| d\eta_{2n}. 
\]

Substitution of this expression into (3.13) gives the following three terms

\[
\int_{\mathbb{R}} \left( \prod_{j=1}^{2n} \Lambda_n(\eta_j) \chi \left( \frac{(2n+1)\eta_j}{\eta_{2n+1}} \right) \hat{\varphi}(\eta_j) e^{i(\eta_1 + \eta_2 + \cdots + \eta_{2n})x} d\eta_{2n+1} \right) d\eta_{2n+1}. 
\]  

(3.15)

\[
\int_{\mathbb{R}} \left( \prod_{j=1}^{2n} \Lambda_n(\eta_j) \chi \left( \frac{(2n+1)\eta_j}{\eta_{2n+1}} \right) \hat{\varphi}(\eta_j) e^{i(\eta_1 + \eta_2 + \cdots + \eta_{2n})x} d\eta_{2n+1} \right) d\eta_{2n+1}. 
\]  

(3.16)
\[ c_n \partial_x \int_{\mathbb{R}} \int_{|\eta_j| \leq |\eta_{2n+1}|} \Lambda_n^{\leq -1}(\eta_n) \prod_{j=1}^{2n} \chi \left( \frac{(2n+1)\eta_j}{\eta_{2n+1}} \right) \hat{\varphi}(\eta_j) e^{i(\eta_1 + \eta_2 + \cdots + \eta_{2n})x} \, d\tilde{\eta}_n \hat{\varphi}(\eta_{2n+1}) e^{ix\eta_{2n+1}} \, d\eta_{2n+1}, \]

(3.17)

where

\[ \Lambda_n^{\log}(\eta_n) = 2(1 - \chi(\eta_{2n+1})) \log |\eta_{2n+1}| \cdot \prod_{j=1}^{2n} (i\eta_j), \]

\[ \Lambda_n^0(\eta_n) = -2(1 - \chi(\eta_{2n+1})) \left( \prod_{j=1}^{2n} (i\eta_j) \right) \int_{[0,1]^{2n}} \log \left| \sum_{j=1}^{2n} \eta_j s_j \right| \, ds_n, \]

\[ \Lambda_n^{\leq -1}(\eta_n) = 2(1 - \chi(\eta_{2n+1})) \left( \prod_{j=1}^{2n} (i\eta_j) \right) \int_{[0,1]^{2n}} \log \left| 1 + \sum_{j=1}^{2n} \frac{\eta_j}{\eta_{2n+1}} s_j \right| \, ds_n. \]

We claim that the terms (3.15) and (3.10) can be rewritten as

\[- \partial_x T^{\log}_{B^\bullet \varphi}[\varphi] \log + |\partial_x| \varphi + \mathcal{R}_1 \quad \text{and} \quad - \partial_x T^0_{B^\bullet \varphi}[\varphi] \varphi + \mathcal{R}_2, \]

(3.18)

where \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) satisfy the estimate (3.10). Indeed,

\[ \mathcal{F} \left[ \partial_x T^{\log}_{B^\bullet \varphi}[\varphi] \log + |\partial_x| \varphi \right] (\xi) = -2c_n i\xi \int_{\mathbb{R}} \chi \left( \frac{|\xi - \eta_n|^2}{1 + |\xi + \eta_n|^2} \right) \log |\eta| \int_{\mathbb{R}^{2n}} \delta \left( \xi - \eta - \sum_{j=1}^{2n} \eta_j \right) \cdot \prod_{j=1}^{2n} (i\eta_j) \hat{\varphi}(\eta_j) \chi \left( \frac{2(2n+1)\eta_j}{\xi + \eta} \right) \, d\tilde{\eta}_n, \]

while the Fourier transform of (3.15) is

\[ 2c_n i\xi \int_{\mathbb{R}} \int_{|\eta_j| \leq |\eta_{2n+1}|} \delta \left( \xi - \sum_{j=1}^{2n+1} \eta_j \right) (1 - \chi(\eta_{2n+1})) \log |\eta_{2n+1}| \]

\[ \cdot \prod_{j=1}^{2n} \chi \left( \frac{(2n+1)i\eta_j}{\eta_{2n+1}} \right) \hat{\varphi}(\eta_j) \, d\tilde{\eta}_n, \]

(3.19)

The difference of the above two integrals is

\[ 2c_n i\xi \int_{\mathbb{R}^{2n+1}} \delta \left( \xi - \sum_{j=1}^{2n+1} \eta_j \right) \log |\eta_{2n+1}| \cdot \left[ I_n(\hat{\eta}_n, \eta_{2n+1}) \prod_{j=1}^{2n} \chi \left( \frac{(2n+1)i\eta_j}{\eta_{2n+1}} \right) \hat{\varphi}(\eta_j)(1 - \chi(\eta_{2n+1})) \right] \]

\[ - \chi \left( \frac{|\xi - \eta_{2n+1}|^2}{1 + |\xi + \eta_{2n+1}|^2} \right) 1_{|\eta_{2n+1}| > \sum_{j=1}^{2n} \prod_{j=1}^{2n} \left( i\eta_j \hat{\varphi}(\eta_j) \chi \left( \frac{2(2n+1)i\eta_j}{\xi + \eta_{2n+1}} \right) \right) \, d\tilde{\eta}_n, \]

(3.19)

where \( I_n \) is given by (3.12).

When \( \eta_n \) satisfies

\[ |\eta_j| \leq \frac{1}{40} \frac{1}{2n+1} |\eta_{2n+1}| \quad \text{for} \quad j = 1, 2, \ldots, 2n, \]

(3.20)
we have $I_n = 1$ and $\chi \left( \frac{(2n+1)\eta_j}{\eta_{2n+1}} \right) = 1$. In addition, since $\xi = \sum_{j=1}^{2n+1} \eta_j$, we have

$$\frac{|\xi - \eta_{2n+1}|^2}{1 + |\xi + \eta_{2n+1}|} \leq \frac{2\sum_{j=1}^{2n} \eta_j}{|\xi + \eta_{2n+1}|} \leq \frac{1}{(2 - \frac{1}{40})|\eta_{2n+1}|} = \frac{1}{79} < \frac{3}{40},$$

so

$$\chi \left( \frac{|\xi - \eta_{2n+1}|^2}{1 + |\xi + \eta_{2n+1}|} \right) = 1, \quad \chi \left( \frac{2(2n+1)|\eta_j|}{\xi + \eta_{2n+1}} \right) = 1.$$

Therefore the integrand of (3.19) is supported outside the set (3.20), and there exists $j_1 \in \{1, \ldots, 2n\}$, such that $|\eta_{j_1}| > \frac{1}{40} \sum_{j=1}^{2n} |\eta_{j}|$. Since $|\eta_{2n+1}|$ is the largest frequency, we see that $|\eta_{j_1}|$ and $|\eta_{2n+1}|$ are comparable in the error term. Therefore, the $H^r$-norm of (3.19) is bounded by

$$\|\varphi\|_{H^r C(n, s)} c_n \left( \|\varphi\|_{W^{2n, \infty}} + \|L\varphi\|_{W^{2n, \infty}} \right).$$

It follows that (3.15) can be written as in (3.18). A similar calculation applies to (3.16).

Next, we estimate the symbols $B_{n}^{\text{loc}}[\varphi]$ and $B_{n}^{\text{loc}}[\varphi]$. First, we notice that they are real-valued, so that $T_{B_{n}^{\text{loc}}[\varphi]}$ and $T_{B_{n}^{\text{loc}}[\varphi]}$ are self-adjoint. Again, without loss of generality, we assume $|\eta_{2n}| = \max_{1 \leq j \leq 2n} |\eta_j|$ and observe that

$$\int_{[0,1]^{2n}} \log \left( \sum_{j=1}^{2n} \eta_j s_j \right) \, ds_n = \log |\eta_{2n}| + \int_{[0,1]^{2n-1}} \left( \sum_{j=1}^{2n-1} \frac{\eta_j s_j}{\eta_{2n}} \right) \log \left( 1 + \sum_{j=1}^{2n-1} \frac{1}{\eta_{2n} s_j} \right) \, ds_{n-1} = \log |\eta_{2n}| + O(1).$$

Thus, using Young’s inequality, we obtain from (3.8) the estimate (3.9), where the constants $C(n, s)$ have at most exponential growth in $n$.

To estimate the third term (3.17), we observe that on the support of the functions $\chi \left( \frac{(2n+1)\eta_j}{\eta_{2n+1}} \right)$, we have

$$\frac{|\eta_j|}{|\eta_{2n+1}|} \leq \frac{1}{10(2n+1)}.$$

Since $s_j \in [0, 1]$, a Taylor expansion gives

$$\left\| A_{n}^{-1}(\eta_n) \right\| \leq \left\| \sum_{j=1}^{2n} |\eta_j| \right\| \left\| \sum_{j=1}^{2n} |\eta_j| \right\| \left\| \eta_{2n+1} \right\| \leq \frac{1}{10(2n+1)}.$$

Therefore the $H^s$-norm of (3.17) is bounded by $C(n, s) c_n \|\varphi\|_{H^s \|\varphi\|_{W^{2n, \infty}}}$, where $C(n, s)$ has at most exponential growth in $n$. 
Case II. When there is at least one factor of the form $1 - \chi$ in the expansion of the product in the integral $\tilde{f} \equiv \int_{\mathbb{R}} \int_{|\eta_j| < \eta_{2n+1}} \Lambda_n(\eta_n) \prod_{k=1}^n \left[ 1 - \chi \left( \frac{(2n+1)\eta_{jk}}{\eta_{2n+1}} \right) \right] \chi \left( \frac{(2n+1)\eta_{jk}}{\eta_{2n+1}} \right) \prod_{k=\ell+1}^{2n} \chi \left( \frac{(2n+1)\eta_{jk}}{\eta_{2n+1}} \right) \frac{2^n}{\eta_{2n+1}},$

where $1 \leq \ell \leq 2n$ is an integer, and \{\{j_k : k = 1, \ldots, 2n \} is a permutation of \{1, \ldots, 2n\}. $\chi$ is compactly supported on $\frac{|\eta_{j_1}|}{|\eta_{2n+1}|} \geq \frac{3}{40(2n+1)}$.

By assumption, $\eta_{2n+1}$ has the largest absolute value, so $\frac{3}{40(2n+1)} |\eta_{2n+1}| \leq |\eta_{j_1}| \leq |\eta_{2n+1}|$, meaning that the frequencies $|\eta_{j_1}|$ and $|\eta_{2n+1}|$ are comparable.

Without loss of generality, we assume that $|\eta_{j_1}| \leq |\eta_{j_2}| \leq \cdots \leq |\eta_{j_{2n+1}}| \leq |\eta_{2n+1}|$, define $\eta_{j_{2n+1}} = \eta_{2n+1}$, and, using (3.3), split the integral for $\Lambda_n$ into three parts:

$$\Lambda_n(\eta_n) = \Lambda_{n, \text{low}}(\eta_n) + \sum_{k=1}^{2n} \Lambda_{n, \text{med},(k)}(\eta_n) + \Lambda_{n, \text{high}}(\eta_n),$$

where

$$\Lambda_{n, \text{low}}(\eta_n) = \left[ 1 - \chi(\eta_{2n+1}) \right] \int_{|\eta_{2n+1}| < 2} \frac{\prod_{j=1}^{2n+1} \left( 1 - e^{i\eta_j \zeta} \right)}{\zeta^{2n+1}} \text{sgn} \zeta \, d\zeta,$$

(3.22)

$$\Lambda_{n, \text{med},(k)}(\eta_n) = \left[ 1 - \chi(\eta_{2n+1}) \right] \int \left[ 2/|\eta_{j_k+1}| \leq |\zeta| \leq 2/|\eta_{j_k}| \right] \frac{\prod_{j=1}^{2n+1} \left( 1 - e^{i\eta_j \zeta} \right)}{\zeta^{2n+1}} \text{sgn} \zeta \, d\zeta,$$

(3.23)

$$\Lambda_{n, \text{high}}(\eta_n) = \left[ 1 - \chi(\eta_{2n+1}) \right] \int \left[ |\eta_{j_k+1}| > 2 \right] \frac{\prod_{j=1}^{2n+1} \left( 1 - e^{i\eta_j \zeta} \right)}{\zeta^{2n+1}} \text{sgn} \zeta \, d\zeta.$$
Considering the support of the cut-off functions, we therefore have in (3.21) by

\[
|A_n^{med,(k)}(\eta_n)| \leq 2 \prod_{k=1}^{2n} |\eta_{jk}| \cdot \int_{2/|\eta_{2n+1}| < \xi \leq 2/|\eta_{2n}|} \frac{1}{|\xi|} \, d\xi \\
= 4 \prod_{k=1}^{2n} |\eta_{jk}| \cdot \log \left| \frac{\eta_{2n+1}}{\eta_{2n}} \right| \leq C(n, s) \prod_{k=1}^{2n} |\eta_{jk}|,
\]

where the last line follows from the fact that \( |\eta_{2n}| \) and \( |\eta_{2n+1}| \) are comparable.

As for (3.24), we have

\[
|A_n^{high}(\eta_n)| \leq |\eta_{j1}| \int_{|\eta_{j1}| > 2} \left( \prod_{k=2}^{2n+1} \left| 1 - e^{i\eta_{jk} \xi} \right| \right) \cdot \left| 1 - e^{i\eta_{j1} \xi} \right| \, d\xi \\
\leq 2^{2n} |\eta_{j1}| \int_{|\eta_{j1}| > 2} \frac{d\xi}{|\xi|^{2n}} \\
\leq \frac{4}{2n - 1} \left( \prod_{k=1}^{2n} |\eta_{jk}| \right).
\]

Collecting these estimates, we find that

\[
|A_n(\eta_n)| \leq C(n, s) \left( \prod_{k=1}^{2n} |\eta_{jk}| \right).
\]

Using the \( L^2 \)-boundedness theorem for pseudo-differential operators, we can bound the \( H^s \)-norm of \( f(n) \) in (3.21) by

\[
\|f_n\|_{H^s} \lesssim \sum_{j,k=0,1} \|\partial^j_x \partial^k_x [P_{n} T_{n, \eta_{2n+1}}] \| \varphi \|_{H^s},
\]

where

\[
P_{n}(x, \eta_{2n+1}) = \left( \sum_{j=1}^{2n+1} \eta_j \right) \int_{\mathbb{R}^{2n}} \Pi_n(\eta_n, \eta_{2n+1}) T_n(\eta_n) \prod_{k=1}^{\ell} \left[ 1 - \chi \left( \frac{(2n + 1) \eta_{jk}}{\eta_{2n+1}} \right) \right] \\
\cdot \prod_{k=\ell+1}^{2n} \chi \left( \frac{(2n + 1) \eta_{jk}}{\eta_{2n+1}} \right) \prod_{j=1}^{2n} \hat{\varphi}(\eta_j) e^{i(\eta_1 + \cdots + \eta_{2n}) x} \, d\eta_n.
\]

Considering the support of the cut-off functions, we therefore have

\[
\|f_n\|_{H^s} \lesssim \|\varphi\|_{H^s} \left( \sum_{n=1}^{N} C(n, s) \|\varphi\|_{W^2, x} \right).
\]

So we have proved that the equation can be written as

\[
\varphi + \partial_x T_{B^{0}([\varphi])} \varphi + R(\varphi) = 2L \varphi - \left[ (T_{B^{0}([\varphi])}) \log |\partial_x \varphi| \right] \varphi.
\]

Then the proposition follows by the commutator estimate (2.4) and the fact that \( 1_{|\xi| < \epsilon} \partial_x \log |\partial_x \varphi| \) is bounded from \( H^s(\mathbb{R}) \) to \( H^s(\mathbb{R}) \).
4. Energy estimates and local well-posedness

In this section, we prove a local existence and uniqueness result for the initial value problem \((1.1)\), which is stated in Theorem 4.6.

As noted in the introduction, standard \(H^s\)-estimates do not close, so we introduce a weighted energy \(E^{(s)}\). The solutions we construct satisfy \(\|T_{B^{k\infty}}\|_{L^2 \to L^2} < 2\), and then \((2 - T_{B^{k\infty}})\) is a positive, self-adjoint operator on \(L^2\). We can therefore define homogeneous and nonhomogeneous weighted energies that are equivalent to the \(H^s\)-energies by

\[
E^{(s)}(t) = \int_\mathbb{R} |D|^s \varphi(x, t) \cdot \left(2 - T_{B^{k\infty}}\right)^{2s+1} |D|^s \varphi(x, t) \, dx, \quad \tilde{E}^{(s)}(t) = \sum_{j=0}^{s} E^{(j)}(t). \tag{4.1}
\]

For simplicity, we consider only integer norms with \(s \in \mathbb{N}\).

We begin by stating an \textit{a priori} energy estimate. In the following, we use \(F\) to denote an increasing, continuous, non-negative function, which might change from line to line.

Proposition 4.1. Let \(s \geq 5\) be an integer and \(\varphi_0 \in H^s(\mathbb{R})\). Then there exist constants \(\tilde{C} > 0\) depending only on \(s\), and \(T > 0\) such that the following statement holds: If \(\varphi_0\) satisfies

\[
\|T_{B^{k\infty}}[\varphi_0]\|_{L^2 \to L^2} \leq C, \quad \sum_{n=1}^{\infty} \tilde{C}^n |c_n| \left(\|\hat{\varphi}_x\varphi_0\|_{W^{3,\infty}} + \|L \hat{\varphi}_x\varphi_0\|_{W^{2,\infty}}\right) < \infty
\]

for some constant \(0 < C < 2\), where \(c_n\) is defined in \((3.3)\), then a solution \(\varphi \in C([0, T]; H^s(\mathbb{R}))\) of \((1.1)\) with initial data \(\varphi(0) = \varphi_0\) satisfies

\[
\tilde{E}^{(s)}(t) \leq \tilde{E}^{(s)}(0) + \int_0^t \left(\|\varphi_x(\tau)\|_{W^{3,\infty}} + \|L \varphi_x(\tau)\|_{W^{2,\infty}}\right)^2 F\left(\|\varphi_x(\tau)\|_{W^{3,\infty}} + \|L \varphi_x(\tau)\|_{W^{2,\infty}}\right) \tilde{E}^{(s)}(\tau) \, d\tau, \tag{4.2}
\]

\[
\|T_{B^{k\infty}}[\varphi(t)]\|_{L^2 \to L^2} < 2, \quad \sum_{n=1}^{\infty} \tilde{C}^n |c_n| \left(\|\varphi_x(t)\|_{W^{3,\infty}} + \|L \varphi_x(t)\|_{W^{2,\infty}}\right) < \infty,
\]

for all \(t \in [0, T]\). In \((4.2)\), \(\tilde{E}^{(s)}\) is defined in \((4.1)\), and \(F(\cdot)\) is an increasing, continuous, non-negative function such that

\[
F\left(\|\varphi_x\|_{W^{3,\infty}} + \|L \varphi_x\|_{W^{2,\infty}}\right) \approx \sum_{n=0}^{\infty} \tilde{C}^n |c_n| \left(\|\varphi_x\|_{W^{2,\infty}}^2 + \|L \varphi_x\|_{W^{1,\infty}}^2\right). \tag{4.3}
\]

This result follows from the more general result in Proposition 4.3. In order to prove that proposition, we first state a lemma.

Lemma 4.2. Suppose that \(s \geq 5\) is an integer. If \(\varphi \in H^s(\mathbb{R})\) is a solution of \((3.3)\) and \(\psi \in C^1_t L^2_x\), then

\[
\hat{\psi}_t(2 - T_{B^{k\infty}})^s \hat{\psi} = (2 - T_{B^{k\infty}})^s \hat{\psi}_t - s(2 - T_{B^{k\infty}})^{s-1} T_{\hat{\psi}_t B^{k\infty}} \hat{\psi} + R_2(\psi),
\]

where the remainder term satisfies

\[
\|R_2(\psi)\|_{H^1} \leq \|\psi\|_{L^2} \left(\sum_{n=1}^{\infty} C(n, s) |c_n| \left(\|\varphi_x\|_{W^{2,\infty}}^2 + \|\varphi_x\|_{W^{1,\infty}}^2\right)\right)
\]

for constants \(C(n, s)\) with at most exponential growth in \(n\).

Proof. Since \(s\) is an integer, we can calculate the time derivative as

\[
\hat{\psi}_t(2 - T_{B^{k\infty}})^s \hat{\psi} = T_{\hat{\psi}_t B^{k\infty}}(2 - T_{B^{k\infty}})^s \hat{\psi} + (2 - T_{B^{k\infty}})^{s-1} T_{\hat{\psi}_t B^{k\infty}}(2 - T_{B^{k\infty}})^{s-2} \hat{\psi} + \cdots + (2 - T_{B^{k\infty}})^{s-1} T_{\hat{\psi}_t B^{k\infty}}(2 - T_{B^{k\infty}})^{s-2} \hat{\psi}.
\]
By Lemma \([2.3,1]\)
\[
\left\| \left[ T_{\partial_x B^{0,[u]}}, (2 - T_{B^{0,[\varphi]}}) \right] f \right\|_{L^2} \leq \left\| f \right\|_{L^2} \left( \sum_{n=1}^{\infty} C(n, s) |c_n| \left( \|\varphi_x\|_{W^{1,\infty}} + \|\varphi_x\|_{W^{1,\infty}} \right) \right).
\]

By taking \(f = (2 - T_{B^{0,[\varphi]}})^{s-2} \psi, (2 - T_{B^{0,[\varphi]}})^{s-3} \psi, \ldots, (2 - T_{B^{0,[\varphi]}}) \psi \) and applying the above estimate repeatedly, we obtain the conclusion. \(\square\)

We then have the following estimate for a linearization of \((3.7)\).

**Proposition 4.3.** Assume \(s \geq 5\), \(\varphi_0 \in H^s(\mathbb{R})\), and \(\Upsilon \in C([0, T]; H^s(\mathbb{R}))\). Suppose that \(u_x \in C([0, T]; W^{3,\infty}(\mathbb{R}))\), \(u_t \in C([0, T]; W^{3,\infty}(\mathbb{R}))\), and \(L u_x \in C([0, T]; W^{3,\infty}(\mathbb{R}))\) satisfy
\[
\| T_{B^{0,[u(t)]}} \|_{L^2 \rightarrow L^2} < 2, \quad \sum_{n=1}^{\infty} \hat{C}^n |c_n| \left( \|u_x(t)\|_{W^{3,\infty}} + \|Lu_x(t)\|_{W^{2,\infty}} + \|u_{xx}(t)\|_{W^{1,\infty}} \right) < \infty \quad \text{for all } t \in [0, T].
\]

Consider the initial value problem
\[
\varphi_t + \partial_x T_{B^{0,[u]}} \varphi + \Upsilon(x, t) = L[(2 - T_{B^{0,[u]}}) \varphi]_x, \quad \varphi(x, 0) = \varphi_0(x).
\]
Then this linear problem has a unique solution \(\varphi \in C([0, T]; H^s(\mathbb{R}))\), and the linearized energy
\[
E^{(s)}_u = \int \left| D^s \varphi(x, t) \right| \cdot \left( 2 - T_{B^{0,[u]}} \right)^{2s+1} \left| D^s \varphi(x, t) \right| \, dx, \quad \bar{E}^{(s)}_u(t) = \sum_{j=0}^{s} E^{(j)}_u(t), \quad (4.4)
\]
satisfies
\[
\bar{E}^{(s)}_u(t) \leq \bar{E}^{(s)}_u(0) + \int_0^t F \left( \|u_x(\tau)\|_{W^{3,\infty}} + \|u_{xx}(\tau)\|_{W^{1,\infty}} + \|Lu_x(\tau)\|_{W^{2,\infty}} \right) \left( \|\varphi(\tau)\|_{H^s} + \|\Upsilon(\tau)\|_{H^s} \right) \, d\tau, \quad (4.5)
\]
where \(F\) is an increasing, continuous, non-negative function that satisfies
\[
F \left( \|u_x(\tau)\|_{W^{3,\infty}} + \|u_{xx}(\tau)\|_{W^{1,\infty}} + \|Lu_x(\tau)\|_{W^{2,\infty}} \right) \approx \sum_{n=0}^{\infty} \hat{C}^n |c_n| \left( \|u_x\|_{W^{3,\infty}} + \|u_{xx}\|_{W^{1,\infty}} + \|Lu_x(\tau)\|_{W^{2,\infty}} \right), \quad (4.6)
\]
for some constant \(\hat{C} > 0\) depending only on \(s\).

**Proof.** We apply the operator \(|D|^s\) to equation \((3.7)\) to get
\[
|D|^s \varphi_t + \partial_x |D|^s T_{B^{0,[u]}} \varphi + |D|^s \Upsilon = \partial_x |D|^s \left( 2 - T_{B^{0,[u]}} \right) \varphi.
\]
Using Lemma \([2.8]\) we find that
\[
|D|^s \left( 2 - T_{B^{0,[u]}} \right) \varphi = 2 |D|^s \varphi - |D|^s (T_{B^{0,[u]}} \varphi)
\]
\[
= 2 |D|^s \varphi - T_{B^{0,[u]}} |D|^s \varphi - sT_{\partial_x B^{0,[u]}} |D|^s - 2 \varphi_x + R_2,
\]
where
\[
\left\| \partial_x R_2 \right\|_{L^2} \leq \left( \sum_{n=1}^{\infty} C(n, s) |c_n| \|u_x\|_{W^{3,\infty}} \right) \|\varphi\|_{H^{s-1}}.
\]
Thus, we can write the right-hand side of (4.7) as
\[ \partial_x L|D|^s \left[ (2 - T_{B^{\psi[u]}})\varphi \right] \]
\[ = \partial_x L \left[ (2 - T_{B^{\psi[u]}})|D|^s \varphi - sT_{\partial_x B^{\psi[u]}}|D|^{s-2} \varphi_x \right] + R_3 \]
\[ = L \left[ (2 - T_{B^{\psi[u]}})|D|^s \varphi_x - T_{\partial_x B^{\psi[u]}}|D|^s \varphi + sT_{\partial_x B^{\psi[u]}}|D|^s \varphi \right] + R_3 \]
\[ = L \left[ (2 - T_{B^{\psi[u]}})|D|^s \varphi_x + (s - 1)T_{\partial_x B^{\psi[u]}}|D|^s \varphi \right] + R_3, \]
where
\[ \|R_3\|_{L^2} \lesssim \left( \sum_{n=1}^\infty C(n, s)|c_n| \|u_x\|^2 \|D_{x}\|^2 \right) \|\varphi\|^2. \]

Applying \((2 - T_{B^{\psi[u]}})^s\) to (4.7) and commuting \((2 - T_{B^{\psi[u]}})^s\) with \(L\) up to remainder terms, we obtain that
\[ (2 - T_{B^{\psi[u]}})^s|D|^s \varphi_x + (2 - T_{B^{\psi[u]}})^s \partial_x |D|^s \varphi_x + (2 - T_{B^{\psi[u]}})^s|D|^s \varphi + (2 - T_{B^{\psi[u]}})^s|D|^s \varphi \]
\[ = L \left( (2 - T_{B^{\psi[u]}})^s \partial_x |D|^s \varphi \right) + (s - 1)(2 - T_{B^{\psi[u]}})^s T_{\partial_x B^{\psi[u]}}|D|^s \varphi \]
\[ = \partial_x L \left( (2 - T_{B^{\psi[u]}})^s \partial_x |D|^s \varphi \right) + R_5, \]
where \(\|R_5\|_{L^2} \lesssim \left( \sum_{n=1}^\infty C(n, s)|c_n| \|u_x\|^2 \|D_{x}\|^2 \right) \|\varphi\|^2. \]

By Lemma 4.2 with \(\psi = |D|^s \varphi\), the time derivative of \(E_u^{(s)}(t)\) in (4.4) is
\[ \frac{d}{dt} E_u^{(s)}(t) = - \int R (2s + 1)|D|^s \varphi \cdot (2 - T_{B^{\psi[u]}})^s T_{\partial_t B^{\psi[u]}}|D|^s \varphi \ dx \]
\[ + 2 \int R |D|^s \varphi \cdot (2 - T_{B^{\psi[u]}})^{2s+1} |D|^s \varphi_1 \ dx + \int R_2 \cdot |D|^s \varphi_1 \ dx. \]

We will estimate each of the terms on the right-hand side of (4.9), where the second term requires the most work.

The first term on the right-hand side of (4.9) can be estimated by
\[ \left| \int R (2s + 1)|D|^s \varphi \cdot (2 - T_{B^{\psi[u]}})^s T_{\partial_t B^{\psi[u]}}|D|^s \varphi \ dx \right| \]
\[ \lesssim \left( \sum_{n=1}^\infty C(n, s)|c_n| \left( \|u_x\|^2 \|D_{x}\|^2 \right) \right) \|\varphi\|^2. \]

Using Lemma 4.2 we can estimate the third term on the right-hand side of (4.9) by
\[ \int R_2 \cdot |D|^s \varphi_1 \ dx \lesssim \left( \sum_{n=1}^\infty C(n, s)|c_n| \left( \|u_x\|^2 \|D_{x}\|^2 \right) \right) \|\varphi\|^2. \]

To estimate the second term on the right-hand side (4.9), we multiply (4.8) by \((2 - T_{B^{\psi[u]}})^{s+1}|D|^s \varphi\), integrate the result with respect to \(x\), and use the self-adjointness of \((2 - T_{B^{\psi[u]}})^{s+1}|D|^s \varphi\), which gives
\[ \int R |D|^s \varphi \cdot (2 - T_{B^{\psi[u]}})^{2s+1} |D|^s \varphi_1 \ dx = I + II + III + IV, \]
where
\[ I = - \int R |D|^s \varphi \cdot (2 - T_{B^{\psi[u]}})^{2s+1} |D|^s \partial_x T_{B^{\psi[u]}} \varphi \ dx, \]
II = \int_\mathbb{R} (2 - T_{B^0[u]})^{s+1} |D|^s \varphi \cdot \partial_x L (2 - T_{B^0[u]})^{s+1} |D|^s \varphi \, dx,

III = \int_\mathbb{R} (2 - T_{B^0[u]})^{s+1} |D|^s \varphi \cdot R_5 \, dx,

IV = \int_\mathbb{R} |D|^s \varphi \cdot (2 - T_{B^0[u]})^{2s+1} |D|^s \varphi \, dx.

We have II = 0, since \( \partial_x L \) is skew-symmetric, and

\[
III \leq \left( \sum_{n=1}^{\infty} (n,s) |c_n| \left( \|u_x\|_{W^{2n,\infty}}^2 \right) \|\varphi\|_{H^s}^2, \quad IV \leq F(\|u_x\|_{W^{2n,\infty}}) \|\varphi\|_{H^s} \|Y\|_{H^s},
\]

since \( \|R_5\|_{L^2} \leq \left( \sum_{n=1}^{\infty} (n,s) |c_n| \left( \|u_x\|_{W^{2n,\infty}}^2 \right) \|\varphi\|_{H^s}^2 \) and \( (2 - T_{B^0[u]})^{s+1} \) is bounded on \( L^2 \).

**Term I estimate.** We write I = \( -I_a + I_0 \), where

\[
I_a = \int_\mathbb{R} |D|^s \varphi \cdot (2 - T_{B^0[u]})^{2s+1} \partial_x T_{B^0[u]} |D|^s \varphi \, dx,

I_b = \int_\mathbb{R} |D|^s \varphi \cdot (2 - T_{B^0[u]})^{2s+1} \partial_x [T_{B^0[u]} |D|^s \varphi \, dx.
\]

By a commutator estimate and (3.9), the second integral satisfies

\[
|I_b| \leq \left( \sum_{n=1}^{\infty} (n,s) |c_n| \left( \|u_x\|_{W^{2n,\infty}}^2 + \|L u_x\|_{W^{2n,\infty}} \right) \|\varphi\|_{H^s}^2.
\]

To estimate the first integral, we write it as

\[
I_a = I_{a_1} - I_{a_2},
\]

where

\[
I_{a_1} = \int_\mathbb{R} |D|^s \varphi \cdot (2 - T_{B^0[u]})^{2s+1} \partial_x T_{B^0[u]} |D|^s \varphi \, dx,

I_{a_2} = \int_\mathbb{R} |D|^s \varphi \cdot (2 - T_{B^0[u]})^{2s+1} (T_{B^0[u]} |D|^s \varphi) \, dx.
\]

**Term I_{a_1} estimate.** A Kato-Ponce type commutator estimate and (3.9) gives

\[
|I_{a_1}| \leq \left( \sum_{n=1}^{\infty} (n,s) |c_n| \left( \|u_x\|_{W^{2n,\infty}}^2 + \|L u_x\|_{W^{2n,\infty}} \right) \|\varphi\|_{H^s}^2.
\]

**Term I_{a_2} estimate.** We have

\[
I_{a_2} = \int_\mathbb{R} \left( T_{B^0[u]} |D|^s \varphi \right) \cdot \partial_x \left( (2 - T_{B^0[u]})^{2s+1} |D|^s \varphi \right) - \left( \partial_x, (2 - T_{B^0[u]})^{2s+1} |D|^s \varphi \right) \, dx

= - \int_\mathbb{R} \partial_x \left( T_{B^0[u]} |D|^s \varphi \right) \cdot (2 - T_{B^0[u]})^{2s+1} |D|^s \varphi \, dx

- \int_\mathbb{R} \left( T_{B^0[u]} |D|^s \varphi \right) \cdot \left( \partial_x, (2 - T_{B^0[u]})^{2s+1} |D|^s \varphi \right) \, dx

= - \int_\mathbb{R} \left( T_{B^0[u]} |D|^s \varphi + \partial_x, T_{B^0[u]} |D|^s \varphi \right) \cdot (2 - T_{B^0[u]})^{2s+1} |D|^s \varphi \, dx

- \int_\mathbb{R} \left( T_{B^0[u]} |D|^s \varphi \right) \cdot \left( \partial_x, (2 - T_{B^0[u]})^{2s+1} |D|^s \varphi \right) \, dx.
\]
Using commutator estimates and (3.9), we get that
\[
\left\| \dot{\partial}_x T_{B^0[u]} D^n \varphi \right\|_{L^2} \lesssim \left( \sum_{n=1}^{\infty} C(n,s) |c_n| \left( \|u_x\|_{W^{2,n}}^2 + \|L u_x\|_{W^{2,n}}^2 \right) \right) \|\varphi\|_{H^n},
\]
\[
\left\| \partial_x \left( 2 - T_{B^{n,s}[u]} \right)^{2s+1} D^n \varphi \right\|_{L^2} \lesssim \left( \sum_{n=1}^{\infty} C(n,s) |c_n| \left( \|u_x\|_{W^{2,n}}^{2n} \right) \right) \|\varphi\|_{H^n},
\]
\[
\left\| \dot{\partial}_x \left( 2 - T_{B^{n,s}[u]} \right)^{2s+1}, T_{B^0[u]} \right\| D^n \varphi \right\|_{L^2} \lesssim \left( \sum_{n=1}^{\infty} C(n,s) |c_n| \left( \|u_x\|_{W^{2,n}}^{2n} + \|L u_x\|_{W^{2,n}}^n \right) \right) \|\varphi\|_{H^n}.
\]
Since \( T_{B^0[\varphi]} \) is self-adjoint, we can rewrite (4.10) as
\[
I_{a_2} = -I_{a_2} + R_6,
\]
with
\[
|R_6| \lesssim \left( \sum_{n=1}^{\infty} C(n,s) |c_n| \left( \|u_x\|_{W^{2,n}}^{2n} + \|L u_x\|_{W^{2,n}}^n \right) \right) \|\varphi\|_{H^n}^2,
\]
and we conclude that
\[
|I_{a_2}| \lesssim \left( \sum_{n=1}^{\infty} C(n,s) |c_n| \left( \|u_x\|_{W^{2,n}}^{2n} + \|L u_x\|_{W^{2,n}}^n \right) \right) \|\varphi\|_{H^n}^2.
\]
This completes the estimate of the terms on the right hand side of (4.9). Collecting the above estimates and using the interpolation inequalities for \( E_u^{(0)} \) and \( E_u^{(s)} \), we obtain (4.15).

We observe that there exists a constant \( \bar{C}(s) > 0 \) such that \( C(n,s) \leq \bar{C}(s)^n \). The series in (4.3) then converges whenever \( \|u_x\|_{W^{3,\infty}} + \|u_{tx}\|_{W^{1,\infty}} + \|L u_x(t)\|_{W^{2,\infty}} \) is sufficiently small, and we can choose \( F \) to be an increasing, continuous, non-negative function that satisfies (4.6).

For \( s \geq 5 \), define a map
\[
G : C([0,T];H^n(\mathbb{R})) \to C([0,T];H^n(\mathbb{R}))
\]
by \( G(u) = \varphi \) where
\[
\varphi_t + \partial_x T_{B^0[u]} \varphi + R(u) = L (2 - T_{B^{n,s}[u]} \varphi)_x, \quad \varphi(x,0) = \varphi_0(x),
\]
with the same \( R(\cdot) \) as the one in (3.7). We will prove that \( G \) is a contraction mapping for sufficiently small \( T > 0 \), which implies the existence and uniqueness of local solutions of the initial value problem for (3.7).

We first prove the following proposition.

**Proposition 4.4 (Boundedness).** Assume \( \varphi_0 \) satisfies the assumptions of Proposition 4.2 and \( \|\varphi_0\|_{H^s} \leq \bar{C} \) for some \( C > 0 \). Define
\[
X_T = \left\{ u \in C([0,T];H^n(\mathbb{R})) \mid \|u\|_{L^\infty_T(0,T;H^n(\mathbb{R}))} \leq 2\bar{C}, \quad \|T_{B^{n,s}[u(t)]}\|_{L^2} < 2, \quad \sum_{n=1}^{\infty} C_n |c_n| \left( \|u_x(t)\|_{W^{3,\infty}}^2 + \|L u_x(t)\|_{W^{2,\infty}}^2 + \|u_{tx}(t)\|_{W^{1,\infty}}^2 \right) < \infty, \forall t \in [0,T] \right\}.
\]
Then there exists \( T > 0 \), such that \( G : X_T \to X_T \).

**Proof.** Taking \( Y = R(u) \) in Proposition 4.3 and using (3.10), we obtain that
\[
\dot{E}_u^{(s)}(t) \leq E_u^{(s)}(0) + \int_0^t \left( \|u_x(t)\|_{W^{3,\infty}}^2 + \|u_{tx}(t)\|_{W^{1,\infty}}^2 + \|L u_x(t)\|_{W^{2,\infty}}^2 \right)^2 \cdot F \left( \|u_x(t)\|_{W^{3,\infty}} + \|u_{tx}(t)\|_{W^{1,\infty}} + \|L u_x(t)\|_{W^{2,\infty}} \right) \left( \|\varphi(t)\|_{H^n}^2 + \|u(t)\|_{H^n}^2 \right) dt,
\]
(4.11)
where $F$ is a positive continuous function. Since $\|\varphi_0\|_{H^s} \leq C$, and $\|\varphi(\cdot, t)\|_{H^s} \approx \left[ E_u(s)(t) \right]^{1/2}$, $\|u(t)\|_{H^s}$ are continuous in time, there exists $T > 0$ such that $G(u) = \varphi \in X_T$.

Now, we prove Proposition \[4.1\]

**Proof of Proposition \[4.1\]** We take $u = \varphi$ and $\Upsilon = \mathcal{R}(\varphi)$ in Proposition \[4.3\]. Since $\|2 - T_{B^{\infty}[\varphi]}\|_{L^2 \to L^2} \geq 2 - C$, and $\|B^{\infty} \varphi(\cdot, t)\|_{M_{(2, 1)}}$ and $F(\|\varphi_x\|_{W^{3, \infty}} + \|L \varphi_x\|_{W^{2, \infty}})$ are continuous in time, there exist $T > 0$ and $m > 0$, depending only on the initial data, such that

$$\|2 - T_{B^{\infty}[\varphi(t)]}\|_{L^2 \to L^2} \geq m \quad \text{for } 0 \leq t \leq T.$$ 

We therefore obtain that

$$m^{2s+1}\|\varphi\|_{H^s}^2 \leq \tilde{E}(s) \leq m^{2s+1}\|\varphi\|_{H^s}^2,$$

so, using the remainder estimate \[3.11\], we get \[4.2\] and \[4.3\] from \[4.3\] and \[4.4\].

We construct a sequence of approximate solutions by

$$\varphi^{(0)}(x, t) = \varphi_0(x), \quad \varphi^{(i)} = G(\varphi^{(i-1)}) \quad \text{for } i \in \mathbb{N}.$$ 

By Proposition \[4.1\] the set $\{\varphi^{(i)} \mid i \in \mathbb{N}\}$ is bounded in $X_T$. We prove the convergence of this sequence by showing that $G$ is a contraction mapping with respect to a low norm.

**Proposition 4.5** (Contraction). For sufficiently small $T > 0$, $G : X_T \to X_T$ defined above is a contraction mapping with respect to $\| \cdot \|_{L^p_t H^s}$.

**Proof.** For $u, v \in W$, let $\varphi$ and $\psi$ be solutions of the equations

$$\varphi_t + \partial_x T_{B^0[u]} \varphi + \mathcal{R}(u) = L\left[ (2 - T_{B^0[u]}) \varphi \right],$$

$$\psi_t + \partial_x T_{B^0[v]} \psi + \mathcal{R}(v) = L\left[ (2 - T_{B^0[v]}) \psi \right],$$

with the same initial data. Taking their difference, we have

$$(\varphi - \psi)_t + \partial_x T_{B^0[u]}(\varphi - \psi) = L\left[ (2 - T_{B^0[u]}) (\varphi - \psi) \right] + \partial_x (T_{B^0[u]} - T_{B^0[v]}) \psi - L\left[ (T_{B^0[u]} - T_{B^0[v]}) \psi \right] + \mathcal{R}(v) - \mathcal{R}(u).$$

Applying Proposition \[4.3\] with

$$T = -\partial_x (T_{B^0[u]} - T_{B^0[v]}) \psi + L\left[ (T_{B^0[u]} + T_{B^0[v]}) \psi \right] - \mathcal{R}(v) + \mathcal{R}(u),$$

we obtain that, for $k \leq 3$,

$$\frac{d}{dt} \int_x |D|^k (\varphi - \psi)(2 - T_{B^0[u]})^{2k+1} |D|^k (\varphi - \psi) \, dx$$

$$\leq (\|u_x\|_{W^{2, \infty}} + \|u_{xx}\|_{W^{1, \infty}} + \|L u_x\|_{W^{2, \infty}}) F_1(\|u\|_{W^{3, \infty}} + \|u_{xx}\|_{W^{1, \infty}} + \|L u_x\|_{W^{2, \infty}}) + F_2(\|u_x\|_{W^{3, \infty}} + \|u_{xx}\|_{W^{1, \infty}} + \|L u_x\|_{W^{2, \infty}}) + \|v_x\|_{W^{3, \infty}} + \|v_{xx}\|_{W^{1, \infty}} + \|L v_x\|_{W^{2, \infty}}) |u - v|_{H^s} \varphi - \varphi_{H^s} \psi_{H^s} \psi_{H^{s+2}}$$

where $F_1, F_2$ are two positive continuous functions. Since $\varphi = \psi$ at $t = 0$, we have, by Grönwall’s inequality,

$$\| (\varphi - \psi)(t) \|_{H^s} \leq \int_0^t e^{\int_0^s F_1(\varphi - \psi) \, ds} F_2(\varphi - \psi)(\tau) \| (u - v)(\cdot, \tau) \|_{H^s} \psi(\cdot, \tau) \|_{H^{s+2}} \, d\tau,$$

where $\varphi' \psi$ are the same arguments as above. By the $H^s$-energy estimate with $s \geq 5$, the function

$$e^{\int_0^s F_1(\varphi - \psi) \, ds} F_2(\varphi - \psi)(\tau) \| \psi(\cdot, \tau) \|_{H^{s+2}}$$

is bounded on $[0, T]$ for $k \leq 3$. Thus, by taking $T$ small enough, we deduce that

$$\| \varphi - \psi \|_{L^p_t H^s} \leq \lambda \| u - v \|_{L^p_t H^s}$$

for some $0 < \lambda < 1$. □
So for any \( s \geq 5 \) and \( \varphi_0 \in H^s(\mathbb{R}) \), \( \{ \varphi^{(0)}, \varphi^{(1)}, \ldots, \varphi^{(i)}, \ldots \} \) is a bounded sequence with respect to \( \| \cdot \|_{L^p_tH^s_x} \). In fact, for any \( 0 < \varepsilon < 1 \), there is a positive integer \( N = \log \frac{1-\lambda}{\varepsilon} \) such that, for any \( i > j > N \),

\[
\| \varphi^{(i)} - \varphi^{(j)} \|_{L^p_tH^s_x} \leq \| \varphi^{(i)} - \varphi^{(i-1)} \|_{L^p_tH^s_x} + \cdots + \| \varphi^{(j+1)} - \varphi^{(j)} \|_{L^p_tH^s_x} \\
\leq (\lambda^{i-N-1} + \cdots + \lambda^{j-N}) \| \varphi^{(N+1)} - \varphi^{(N)} \|_{L^p_tH^s_x} \\
\leq (\lambda^{i-N-1} + \cdots + \lambda^{j-N}) \lambda^N \| \varphi^{(1)} - \varphi^{(0)} \|_{L^p_tH^s_x} \\
\leq \frac{3\tilde{C}}{1-\lambda} \lambda^N < \varepsilon.
\]

So \( \lim_{j \to \infty} \varphi^{(j)} \) exists and is unique. The regularity of the solution follows from the \textit{a priori} estimate.

Therefore, we obtain the following existence theorem with a blow-up criterion.

**Theorem 4.6.** Let \( s \geq 5 \) be an integer. Suppose that \( \varphi_0 \in H^s(\mathbb{R}) \) satisfies

\[
\| T_{B^{10,s}[\varphi_0]} \|_{L^2 \to L^2} \leq C, \quad \sum_{n=1}^{\infty} \tilde{C}^n |c_n| \left( \| \partial_x \varphi_0 \|_{W^{2n,\infty}_x}^2 + \| L \partial_x \varphi_0 \|_{W^{2n,2}_x}^2 \right) < \infty
\]

for some constant \( 0 < C < 2 \), where \( \tilde{C} \) is the same constant as the one in Proposition 4.1 and the symbol \( B^{10,s}[\varphi_0] \) is defined in (5.8). Then there exists a maximal time of existence \( 0 < T_{\text{max}} \leq \infty \) depending only on \( \| \varphi_0 \|_{H^s_x}, C \), and \( \tilde{C} \) such that the initial value problem (1.1) has a unique solution with \( \varphi \in C([0,T_{\text{max}}); H^s(\mathbb{R})) \).

If \( T_{\text{max}} < \infty \), then either

\[
\lim_{t \uparrow T_{\text{max}}} \sum_{n=1}^{\infty} \tilde{C}^n |c_n| \left( \| \varphi_x(t) \|_{W^{2n,\infty}_x}^2 + \| L \varphi_x(t) \|_{W^{2n,2}_x}^2 \right) = \infty \quad \text{or} \quad \lim_{t \uparrow T_{\text{max}}} \left\| T_{B^{10,s}[\varphi(\cdot,t)]} \right\|_{L^2 \to L^2} = 2.
\]

We remark that, by interpolation, one can also replace \( \| \varphi_x(t) \|_{W^{2n,\infty}_x}^2 + \| L \varphi_x(t) \|_{W^{2n,2}_x}^2 \) by \( \| \partial_x \varphi(t) \|_{L^\infty}^2 + \| L \varphi(t) \|_{L^2}^2 \).

The front equation is invariant under \( (x,t) \mapsto (-x,-t) \), so the same result holds backward in time. One could use a Bona-Smith argument, as in [30], to prove that the solution depends continuously on the initial data, but we will not carry out the details here.

5. Global solution for small initial data

Beginning with this section, we address the global well-posedness of (1.1) with small initial data. From now on, we fix the following parameter values

\[
s = 1200, \quad r = 1, \quad p_0 = 10^{-4}.
\]

The front equation (1.1) is invariant under the transformation

\[
x \mapsto \lambda(x + 2 \log |\lambda| t), \quad t \mapsto \lambda t, \quad \varphi \mapsto \lambda \varphi.
\]

The scaling-Galilean part of this transformation is generated by the vector-field

\[
\mathcal{S} = (x + 2t) \partial_x + t \partial_t,
\]

and the linearized equation \( \varphi_t = 2 \log |\partial_x| \varphi \) commutes with \( \mathcal{S} \) (cf. Lemma 7.1). We also introduce the notation

\[
h(x,t) = e^{-2it \partial_x \log |\partial_x|} \varphi(x,t), \quad \hat{h}(\xi,t) = e^{-2it \xi \log |\xi|} \hat{\varphi}(\xi,t)
\]

for the function \( h \) obtained by removing the action of the linearized evolution group on \( \varphi \). When convenient, we write \( h(\cdot,t) = h(t), \varphi(\cdot,t) = \varphi(t) \).

Our global existence theorem is as follows.
Theorem 5.1. Let \( s, r, p_0 \) be defined as in (5.1). There exists a constant \( 0 < \varepsilon \ll 1 \), such that if \( \varphi_0 \in H^s(\mathbb{R}) \) satisfies
\[
\|\varphi_0\|_{H^r} + \|x\partial_x \varphi_0\|_{H^r} \leq \varepsilon_0
\]
for some \( 0 < \varepsilon_0 \ll \varepsilon \), then there exists a unique global solution \( \varphi \in C([0, \infty); H^s(\mathbb{R})) \) of (1.1). Moreover, this solution satisfies
\[
\|\varphi(t)\|_{H^r} + \|S\varphi(t)\|_{H^r} \leq \varepsilon_0(t + 1)^{p_0},
\]
where \( S \) is the vector field in (5.2).

Given local existence, we only need to prove the global \textit{a priori} bound. In order to do this, we introduce the \( Z \)-norm of a function \( f \in L^2(\mathbb{R}) \), defined by
\[
\|f\|_Z = \left\|(|\xi| + |\xi|^{r+4})\hat{f}(\xi)\right\|_{L^\xi},
\]
and prove the global bound by use of the following bootstrap argument.

Proposition 5.2 (Bootstrap). Let \( T > 1 \) and suppose that \( \varphi \in C([0, T]; H^s) \) is a solution of (1.1), where the initial data satisfies
\[
\|\varphi_0\|_{H^r} + \|x\partial_x \varphi_0\|_{H^r} \leq \varepsilon_0
\]
for some \( 0 < \varepsilon_0 \ll 1 \). If there exists \( \varepsilon_1 \leq \varepsilon_0^{1/3} \) such that the solution satisfies
\[
(t + 1)^{-\rho_0} (\|\varphi(t)\|_{H^r} + \|S\varphi(t)\|_{H^r}) + \|\varphi\|_Z \leq \varepsilon_1
\]
for every \( t \in [0, T] \), then the solution satisfies an improved bound
\[
(t + 1)^{-\rho_0} (\|\varphi(t)\|_{H^r} + \|S\varphi(t)\|_{H^r}) + \|\varphi\|_Z \leq \varepsilon_0.
\]

Theorem 5.1 then follows from combining this bootstrap proposition with the local existence and blow-up result in Theorem 1.6. We call the assumptions in Proposition 5.2 the \textit{bootstrap assumptions}. To prove Proposition 5.2, we need the following lemmas, some of whose proofs are deferred to the next sections.

Lemma 5.3 (Sharp pointwise decay). Under the bootstrap assumptions,
\[
\|\partial_x^{r+2} \varphi_x(t)\|_{L^\infty} + \|L\varphi_x(t)\|_{L^\infty} \leq \varepsilon_1(t + 1)^{-1/2}.
\]

Lemma 5.4 (Scaling-Galilean estimate). Under the bootstrap assumptions,
\[
(t + 1)^{-\rho_0} |S\varphi(t)|_{H^r} \leq \varepsilon_0.
\]

Lemma 5.5. Under the bootstrap assumptions,
\[
(t + 1)^{-\rho_0}(\|\varphi(t)\|_{H^r} + \|x\partial_x \varphi(t)\|_{H^r}) \leq \varepsilon_0.
\]

Proof. Recall the energy estimate (1.2)
\[
\tilde{E}^{(s)}(t) \leq \tilde{E}^{(s)}(0)e^{\int_0^t F(\|\varphi_x(\tau)\|_{W^{3,\infty}} + |L\varphi_x(\tau)|_{W^{3,\infty}})\|\varphi_x(\tau)\|_{W^{3,\infty}} + |L\varphi_x(\tau)|_{W^{2,\infty}}^2 d\tau}.
\]
From Lemma 5.3, we have
\[
F(\|\varphi_x(\tau)\|_{W^{3,\infty}} + |L\varphi_x(\tau)|_{W^{3,\infty}}) \leq 1,
\]
which implies that
\[
\tilde{E}^{(s)}(t) \leq \varepsilon_0^2(t + 1)^{C\varepsilon_1^2}
\]
for some constant \( C \), so once \( \varepsilon_1^2 \ll p_0 \), we have
\[
(t + 1)^{-\rho_0}\|\varphi\|_{H^r} \leq \varepsilon_0.
\]
Next, we observe that we can use \( \| \mathcal{S} \varphi \|_{H^r} \) to control \( \| x \partial_x h \|_{H^r} \). It follows from (5.3), the definition of \( \mathcal{S} \), and (6.2) that

\[
\mathcal{F}_x [x \partial_x h] (\xi) = -\partial_\xi (\xi \hat{h}(\xi)) = -\hat{h}(\xi) - \xi \partial_\xi \hat{h}(\xi),
\]

\[
\mathcal{F}_x [x \partial_x h] (\xi, t) = \xi e^{-2it\xi \log |\xi|} \left( -2it(\log |\xi| + 1) \hat{\varphi}(\xi, t) + \partial_\xi \hat{\varphi}(\xi, t) \right)
\]

\[
= e^{-2it\xi \log |\xi|} \left[ \xi \partial_\xi \hat{\varphi}(\xi, t) - (2it\xi - 1) \hat{\varphi}(\xi, t) - t \hat{\varphi}_t(\xi, t) - t \hat{N}(\xi, t) - \hat{\varphi}(\xi, t) \right]
\]

(5.5)

where \( \mathcal{N} \) denotes the nonlinear term in (3.2), which satisfies the estimate

\[
\| \partial_x \mathcal{N} \|_{L^2} \lesssim \sum_{n=1}^\infty (\| f \|_{W^{n,\infty}}^n + \| L_f \|_{W^{n,\infty}}^n) \| \varphi \|_{H^{j+2}} \quad \text{for all } j = 0, \ldots, r.
\]

(5.6)

By the bootstrap assumptions, Lemma 5.3 and Lemma 5.4 we then find that

\[
(t + 1)^{-p_0} \| x \partial_x h(t) \|_{H^r} \lesssim \varepsilon_0,
\]

and the same estimate holds for \( \varphi \) in view of (5.3).

Lemma 5.6 (Nonlinear dispersive estimate). Under the bootstrap assumptions, the solution of (1.1) satisfies

\[
\| \varphi(t) \|_{Z} \lesssim \varepsilon_0.
\]

Proposition 5.2 then follows by combining Lemmas 5.3–5.6.

6. Linear dispersive estimate

In this section, we prove a dispersive estimate for the linearized evolution operator \( e^{2t \partial_x \log |\partial_x|} \) defined in (5.3) and use it to prove Lemma 5.3. We recall that \( P_k \) and \( \tilde{P}_k \) are the frequency-localization operators with symbols \( \psi_k \) and \( \tilde{\psi}_k \), respectively (see (2.13)).

Lemma 6.1. For \( t > 0 \) and \( f \in L^2 \), we have the linear dispersive estimates

\[
\| e^{2t \partial_x \log |\partial_x|} P_k f \|_{L^\infty} \lesssim (t + 1)^{-1/2} 2^{-2k/4} \| P_k f \|_{L^\infty} + (t + 1)^{-3/4} 2^{-k/4} \left[ \| P_k (x \partial_x f) \|_{L^2} + \| \tilde{P}_k f \|_{L^2} \right].
\]

(6.1)

Proof. Using the inverse Fourier transform, we can write the solution as

\[
e^{2t \partial_x \log |\partial_x|} P_k f = \int_{\mathbb{R}} e^{ix\xi + 2it(\log |\xi|)t} \psi_k(\xi) \hat{f}(\xi) \, d\xi.
\]

Since

\[
\partial_\xi e^{ix\xi + 2it(\log |\xi|)t} = [ix + 2it(\log |\xi| + 1)]e^{ix\xi + 2it(\log |\xi|)t},
\]

(6.2)

we can integrate by parts to get

\[
\| e^{2t \partial_x \log |\partial_x|} P_k f \|_{L^\infty} = \left\| \int_{\mathbb{R}} e^{ix\xi + 2it(\log |\xi|)t} \hat{f}(\xi) \psi_k(\xi) \, d\xi \right\|_{L^\infty}
\]

\[
= \left\| \int_{\mathbb{R}} \frac{1}{ix + 2it(\log |\xi| + 1)} \partial_\xi e^{ix\xi + 2it(\log |\xi|)t} \hat{f}(\xi) \psi_k(\xi) \, d\xi \right\|_{L^\infty}
\]

\[
= \left\| \int_{\mathbb{R}} e^{ix\xi + 2it(\log |\xi|)t} \partial_\xi \left( \frac{1}{ix + 2it(\log |\xi| + 1)} \hat{f}(\xi) \psi_k(\xi) \right) \, d\xi \right\|_{L^\infty}
\]
Proof of Lemma 5.3. Using (6.2) and integration by parts, we have

\[ = \left\| \int_{\mathbb{R}} e^{ix\xi + 2it(\xi \log |\xi|)t} \left( \frac{-2it}{\xi(2it + \log |\xi| + 1)^2} \hat{f}(\xi)\psi_k(\xi) + \frac{1}{ix + 2it(\log |\xi| + 1)} \hat{f}(\xi) \psi_k'(\xi) \right) \right\|_{L^\infty}. \]

1. If \(|ix + 2it(\log |\xi| + 1)| \geq (t + 1)|\), we use (2.14) and get

\[ \|e^{2\xi \log |\xi|} P_k f\|_{L^\infty} \lesssim \frac{1}{t + 1} \left\| \xi^{-1} \hat{f}(\xi) \psi_k(\xi) + \psi_k(\xi) \hat{f}(\xi) \psi_k(\xi) + \hat{f}(\xi) \psi_k'(\xi) \right\|_{L^\infty} \lesssim \frac{1}{t + 1} \left[ 2^{-k} \|\hat{P}_k f\|_{L^2}^2 + 2^{-k/2} \|P_k F^{-1}(\xi \hat{f})\|_{L^2}^2 + 2^{-k/2} \|\hat{P}_k f\|_{L^2}^2 \right]. \]

Then (6.1) follows when \((t + 1)^{-1} \leq 2^k\). Otherwise, when \(t + 1 \leq 2^{-k}\), we have

\[ \|e^{2\xi \log |\xi|} P_k f\|_{L^\infty} \lesssim 2^k \|\hat{P}_k f\|_{L^\infty} \lesssim (t + 1)^{-1/2} 2^{k/2} \|\hat{P}_k f\|_{L^\infty}. \]

2. Next we prove estimates for the case when \(|ix + 2it(\log |\xi| + 1)| \leq (t + 1)|\). Let \(\eta_0^\pm = \pm e^{-1-2t/2n}\)

be the solutions of \(x + 2it(\log |\xi| + 1) = 0\). Since \(\psi_k\) is supported in an annulus with radius around \(2^k\), we only need to consider the case when \(|\xi_0^\pm| \approx 2^k\) and \(\psi_k\) is supported in the neighborhood of the stationary phase point \(\eta_0^\pm\). We decompose the integral and estimate it as

\[ \left\| \int_{\mathbb{R}} e^{ix\xi + 2it(\xi \log |\xi|)t} \hat{f}(\xi) \psi_k(\xi) \right\|_{L^\infty} \lesssim \sum_{l \in k+N} |J_l^+| + |J_l^-|, \]

with

\[ J_l^+ = \int_{\mathbb{R}} e^{ix\xi + 2it(\xi \log |\xi|)t} \hat{f}(\xi) \psi_k(\xi) \chi_+(\xi) \psi_l(\xi - \eta_0^\pm) \right\|_{L^\infty} \]

where \(\chi_\pm\) is the indicator function supported on \(\mathbb{R}_\pm\) and \(N\) is large enough that the support of \(\psi_k\) is covered by the set \(\bigcup_{l \in k+N} \{\xi | \psi_l(\xi - \eta_0^\pm) = 1\}\).

When \(2^l \leq 2^{k/2}(t + 1)^{-1/2}\), we have

\[ \sum_{2^{l} \leq 2^{k/2}(t + 1)^{-1/2}} |J_l^+| \lesssim \sum_{2^{l} \leq 2^{k/2}(t + 1)^{-1/2}} 2^l \|\hat{P}_k f\|_{L^\infty} \lesssim 2^{k/2}(t + 1)^{-1/2} \|\hat{P}_k f\|_{L^\infty}. \]

When \(2^{k/2}(t + 1)^{-1/2} \leq 2^l \leq 2^{k+N}\), since \(|\xi - \xi_0| \approx 2^l\) and \(|\xi_0| \approx 2^k\), we get the estimate

\[ x + 2it(\log |\xi| + 1) = 2t \log \left| \frac{\xi}{\xi_0} \right| \approx 2t \log \left| 1 \pm \frac{2^l}{2^k} \right|. \]

Using (6.2) and integration by parts, we have

\[ |J_l^+| \lesssim \frac{2^{k-l}}{(t + 1)} \int_{\mathbb{R}} \left( |\hat{\psi}_l(\xi)| + 2^{-l} |\hat{f}(\xi)| \right) \psi_l(\xi - \eta_0^\pm) d\xi \lesssim \frac{2^{k-l}}{(t + 1)} \|\hat{f}\|_{L^\infty} + \frac{2^{k-l/2}}{(t + 1)} \|\hat{\psi}_l\|_{L^2}. \]

Then we take the sum of \(J_l\) over \(2^l \geq 2^{k/2}(t + 1)^{-1/2}\) to get the estimates (6.1). \(\square\)

**Proof of Lemma 5.5** After splitting into high-frequency and low-frequency parts, it suffices to bound the terms

\[ \left\| \sum_{k \leq 0} P_k \varphi \right\|_{L^\infty}, \quad \left\| \sum_{k > 0} P_k \psi_{k+\varepsilon} \varphi \right\|_{L^\infty}. \]
Take the function \( f \) in Lemma 6.1 to be \( L\partial_x h \). Since \( e^{2t\partial_x \log |\partial_x|} \) and \( P_k \) commute, and
\[
x \partial_x^2 Lh = \partial_x(x \partial_x Lh) - \partial_x Lh = \partial_x[x \partial_x, L]h + \partial_x L(x \partial_x h) - \partial_x Lh = -\partial_x h + \partial_x L(x \partial_x h) - \partial_x Lh,
\]
we have that
\[
\| P_k L\partial_x \varphi \|_{L^\infty} \leq (t + 1)^{-1/2} \| F(P_k L) [\partial_x] \tilde{\varphi} \|_{L^\infty} + (t + 1)^{-3/4} \| 2^{\frac{3k}{4}} P_k L(x \partial_x \varphi) \|_{L^2} + 2^{\frac{1}{4}} (1 + |k|) \| P_k h\|_{L^2} + \| \tilde{P}_k (|\partial_x|^2 \varphi) \|_{L^2}.
\]
It follows from (5.3) that
\[
\| P_k (x \partial_x h) \|_{L^2} \leq \| P_k \varphi \|_{L^2} + \| P_k S \varphi \|_{L^2} + t \| P_k N \|_{L^2}.
\]
We first observe that \( k \leq 0 \) automatically leads to \( (t + 1)^{-1/4 + \rho_0} 2^{\frac{3k}{4}} |k| \leq 1 \), and then we have
\[
\| P_k L\partial_x \varphi \|_{L^\infty} \leq (t + 1)^{-1/2} \| F(P_k \partial_x \varphi) \|_{L^\infty} + (t + 1)^{-3/4} [2^{\frac{3k}{4}} \| P_k (x \partial_x \varphi) \|_{L^2} + \| \tilde{P}_k (|\partial_x|^2 \varphi) \|_{L^2}].
\]
(6.3)
Summing over \( k \leq 0 \), using (5.6), the bootstrap assumptions, and (6.3) in the corresponding range of \( k \), and we obtain that
\[
\left\| \sum_{k \leq 0} P_k L\partial_x \varphi \right\|_{L^\infty} \leq \varepsilon_1 (t + 1)^{-1/2}.
\]
To estimate \( \| P_k |\partial_x|^{-2} \varphi_x \|_{L^\infty} \), we take the function \( f \) in Lemma 6.1 to be \( |\partial_x|^{-2} h \) and obtain
\[
\| P_k |\partial_x|^{-2} \varphi_x \|_{L^\infty} \leq (t + 1)^{-1/2} \| F(P_k |\partial_x|^{-2} \varphi) \|_{L^\infty} + (t + 1)^{-3/4} [2^{\frac{1}{4}} \| P_k (x |\partial_x|^{-2} h) \|_{L^2} + \| \tilde{P}_k (|\partial_x|^{-2} \varphi) \|_{L^2}].
\]
Using
\[-x |\partial_x|^{-4} h = [x \partial_x, \partial_x |\partial_x|^{-2} h + \partial_x |\partial_x|^{-2} (x \partial_x h)] = -(r + 3) \partial_x |\partial_x|^{-2} h + \partial_x |\partial_x|^{-2} (x \partial_x h),
\]
and (5.5), we get that
\[
\| P_k |\partial_x|^{-2} \varphi_x \|_{L^\infty} \leq (t + 1)^{-1/2} \| \psi_k (\xi) \|_{L^\infty} + (t + 1)^{-3/4} [2^{\frac{1}{4}} \| P_k \tilde{\varphi} \|_{L^2} + \| \tilde{P}_k (|\partial_x|^{-2} \varphi) \|_{L^2}.
\]
For \( k \in \mathbb{Z}_+ \) and \( (t + 1)^{-1/4 + \rho_0} 2^{(r + 1/2)k} \leq 1 \), we have
\[
\| P_k |\partial_x|^{-2} \varphi_x \|_{L^\infty} \leq (t + 1)^{-1/4} \| \psi_k (\xi) \|_{L^\infty} + (t + 1)^{-3/4} [2^{\frac{1}{4}} \| \tilde{P}_k \varphi \|_{L^2} + \| P_k S \varphi \|_{L^2} + t \| P_k N \|_{L^2}.
\]
(6.4)
Finally, for \( k \in \mathbb{Z}_+ \) and \( (t + 1)^{-1/4 + \rho_0} 2^{(r + 1/2)k} \geq 1 \), we have
\[
\| P_k \tilde{\varphi} \|_{L^\infty} \leq \| \tilde{\varphi} \|_{L^\infty} + \| \tilde{P}_k \varphi \|_{H^*} \leq 2^{(r + 3 - s + \frac{1}{2})k} \| \tilde{P}_k \varphi \|_{H^*} \leq (t + 1)^{-s} \| \tilde{P}_k \varphi \|_{H^*}.
\]
(6.5)
Summing over \( k \in \mathbb{Z}_+ \), using (5.6), the bootstrap assumptions, and (6.5) in the corresponding range of \( k \), we obtain that
\[
\left\| \sum_{k > 0} P_k |\partial_x|^{-2} \varphi_x \right\|_{L^\infty} \leq \varepsilon_1 (t + 1)^{-1/2},
\]
which completes the proof.
7. Scaling-Galilean estimate

In this section, we prove the scaling-Galilean estimate in Lemma 5.4.

First, we summarize some commutator identities for the scaling-Galilean operator $S$ defined in (5.2) and $L = \log |\partial_x|$. The straightforward proofs follow by use of the Fourier transform and are omitted.

**Lemma 7.1.** Let $\varphi(x,t)$ be a Schwartz distribution on $\mathbb{R}^2$ such that $L\varphi(x,t)$ is a Schwartz distribution. Then

\[
[S, \partial_x] \varphi = -\partial_x \varphi, \quad [S, L] \varphi = -\varphi, \quad [S, L \partial_x] \varphi = -\varphi_x - L \partial_x \varphi, \quad [S, \partial_t] \varphi = -2\partial_t \varphi - \partial_t \varphi, \quad [S, \partial_t - 2L \partial_x] \varphi = -\partial_t \varphi + 2L \partial_x \varphi.
\]

Next, we prove a weighted energy estimate for $S\varphi$.

**Proof of Lemma 7.1.** Applying $S$ to equation (3.7) and using Lemma 7.1, we get

\[
(S\varphi)_t - 2L \partial_x (S\varphi) + \partial_x T_{B^0[\varphi]} S\varphi + L[T_{B^0[\varphi]}] S\varphi|_x + SR = \text{commutators},
\]

where the commutators are

\[
\partial_x [S, T_{B^0[\varphi]}] \varphi, \quad [S, \partial_x] T_{B^0[\varphi]} \varphi, \quad [S, L \partial_x] (T_{B^0[\varphi]} \varphi), \quad L \partial_x \left([S, T_{B^0[\varphi]}] \varphi\right).
\]

The first commutator can be written as

\[
[S, T_{B^0[\varphi]}] \varphi = [(x + 2t) \partial_x + t \partial_t, T_{B^0[\varphi]}] \varphi
\]

\[
= (x + 2t) \partial_x T_{B^0[\varphi]} \varphi - T_{B^0[\varphi]} [(x + 2t) \partial_x \varphi] + t \partial_t T_{B^0[\varphi]} [(x + 2t) \varphi]
\]

\[
= (x + 2t) \partial_x T_{B^0[\varphi]} \varphi + [(x + 2t), T_{B^0[\varphi]}] \partial_x \varphi + t \partial_t T_{B^0[\varphi]} \varphi
\]

\[
= T_{(x+2t)\partial_x T_{B^0[\varphi]} \varphi} + \left((x \partial_x \partial_x T_{B^0[\varphi]} \varphi - T_{x \partial_x T_{B^0[\varphi]} \varphi} \varphi + [x, T_{B^0[\varphi]}] \partial_x \varphi + T_{t \partial_t T_{B^0[\varphi]} \varphi}.\right)
\]

By the commutator estimates in Lemma 2.2 and Theorem 3.3, we obtain for $0 \leq k \leq r$ that

\[
\|\partial_x [S, T_{B^0[\varphi]}] \varphi\|_{H^k} \lesssim \|x, T_{B^0[\varphi]} \partial_x \varphi\|_{H^{k+1}} + \|x, T_{B^0[\varphi]} T_{B^0[\varphi]} \varphi\|_{H^{k+1}} + \|T_{B^0[\varphi]} \partial_x \varphi\|_{H^k}
\]

\[
\lesssim \|B^0[\varphi]\|_{M_{1,2}} \|\varphi\|_{H^{k+2}} + \|BB[\varphi]\|_{M_{1,2}} \|\varphi\|_{H^{k+1}} + \|SB^0[\varphi]\|_{L^1} \|\varphi\|_{W^{k+1,2}}.
\]

Using (3.3), together with Lemma 2.2 and similar estimates for $[SB^0[\varphi]]_{C^2}$, we find that

\[
\|\partial_x [S, T_{B^0[\varphi]}] \varphi\|_{H^k} \lesssim F(\|L \varphi\|_{W^{2,2}} + \|\varphi\|_{W^{2,2}})(\|L \varphi\|_{W^{2,2}} + \|\varphi\|_{W^{2,2}})(\|\varphi\|_{W^{r+1,2}}(\|S\varphi\|_{H^r} + \|\varphi\|_{H^r}).
\]

Similarly, we have

\[
\|L \partial_x \left([S, T_{B^0[\varphi]}] \varphi\right)\|_{H^k} \lesssim F(\|L \varphi\|_{W^{2,2}} + \|\varphi\|_{W^{2,2}})(\|L \varphi\|_{W^{2,2}} + \|\varphi\|_{W^{2,2}})(\|\varphi\|_{W^{r+1,2}}(\|S\varphi\|_{H^r} + \|\varphi\|_{H^r}).
\]

By Lemma 2.2, Lemma 2.2, and (3.3), the second and third commutators satisfy

\[
\|S, \partial_x T_{B^0[\varphi]} \varphi\|_{H^k} = \|T_{B^0[\varphi]} \varphi\|_{H^{k+1}} \lesssim F(\|L \varphi\|_{W^{2,2}} + \|\varphi\|_{W^{2,2}})(\|L \varphi\|_{W^{2,2}} + \|\varphi\|_{W^{2,2}})^2 \|\varphi\|_{H^{k+1}},
\]

\[
\|S, \partial_x \left(T_{B^0[\varphi]} \varphi\right)\|_{H^k} \lesssim F(\|L \varphi\|_{W^{1,2}} + \|\varphi\|_{W^{1,2}})(\|L \varphi\|_{W^{1,2}} + \|\varphi\|_{W^{1,2}})^2 \|\varphi\|_{H^{k+1} + \|L \varphi\|_{H^{k+1}}}
\]

Thus, the evolution equation for $S\varphi$ can be written as

\[
(S\varphi)_t + \partial_x T_{B^0[\varphi]} S\varphi + R_S = L\left[2 - T_{B^0[\varphi]}\right] S\varphi,
\]

where the remainder $R_S$ satisfies

\[
R_S \lesssim (\|\varphi\|_{W^{2,2}} + \|L \varphi\|_{W^{2,2}})^2 (\|S\varphi\|_{H^r} + \|\varphi\|_{H^{r+1}} + \|L \varphi\|_{H^{r+1}}).
\]
As in [4, 1], we define a weighted energy for $S\varphi$ by

$$E^{(j)}_S(t) = \int_{\mathbb{R}} |D|^{j} S\varphi(x, t) \cdot \left(2 - T_{B^{\infty}[\varphi]}\right)^{2j+1} |D|^{j} S\varphi(x, t) \, dx, \quad j = 0, 1, \ldots, r,$$

and repeat similar estimates to the ones in the proof of Proposition 4.1 to get

$$\frac{d}{dt} E^{(j)}_S(t) \lesssim F\left(\|L\varphi_x\|_{W^{2,\infty}} + \|\varphi_x\|_{W^{2,\infty}}\right)^2 \|S\varphi\|_{H^j}^2 + (\|\varphi_x\|_{W^{2,\infty}} + \|L\varphi_x\|_{W^{2,\infty}})^2 \|S\varphi\|_{H^{j+1}}^2 \|\varphi\|_{H^{j+1}} \|S\varphi\|_{H^j}.$$

Using Lemma 5.3 and the equivalence of $\tilde{E}^{(r)}_S$ and $\|S\varphi\|_{H^j}^2$, when $\|2 - T_{B^{\infty}[\varphi]}\|_{L^2-L^2}$ is bounded away from zero, we find by integrating in $t$ that

$$\tilde{E}^{(r)}_S(t) \lesssim \epsilon_0^2 (t+1)^{2p_0},$$

which proves the lemma.

8. Nonlinear dispersive estimate

In this section, we prove the estimate in Lemma 5.6 for the $Z$-norm $\|\varphi\|_Z$ defined in (5.4).

When $|\xi| < (t+1)^{-p_0}$, Lemma 2.3 and the bootstrap assumptions give

$$|(|\xi| + |\xi|^{r^4}) \hat{\varphi}(\xi, t)|^2 \lesssim (|\xi| + |\xi|^{r^4}) \|\varphi\|_{L^1_t(\|\varphi\|_{L^2} + \|\varphi\|_{L^2})} \lesssim (|\xi| + |\xi|^{r^4}) \|\varphi\|_{L^1_t(\|S\varphi\|_{L^2} + \|\varphi\|_{L^2})} \lesssim \varepsilon_0^2.$$

Let $p_1 = 10^{-6}$. When $|\xi| \geq (t+1)^{p_1}$, Lemma 2.3 and the bootstrap assumptions, with the parameter values (5.1), give

$$|(|\xi| + |\xi|^{r^4}) \hat{\varphi}(\xi, t)|^2 \lesssim \frac{(\|\xi| + |\xi|^{r^4})^2}{|\xi|^{r^4+1}} \|\varphi\|_{H^{-1}}(\|S\varphi\|_{L^2} + \|\varphi\|_{L^2}) \lesssim |\xi|^{2r^4+7-s} \varepsilon_0^2 (t+1)^{2p_0} \lesssim \varepsilon_0^2.$$

Thus, we only need to consider the frequency range

$$(t+1)^{-p_0} \leq |\xi| \leq (t+1)^{p_1}. \quad (8.1)$$

In the following, we fix $\xi$ in this range, and denote by $\rho(\xi, t)$ a smooth cutoff function compactly supported on a small neighborhood of $\{(\xi, t) : (t+1)^{-p_0} < |\xi| < (t+1)^{p_1}\}$.

Taking the Fourier transform of (3.4), we obtain that

$$\hat{\varphi}_1(\xi) + \frac{1}{6} \alpha \int_{\mathbb{R}^2} T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) \hat{\varphi}(\xi - \eta_1 - \eta_2) \hat{\varphi}(\eta_1) \hat{\varphi}(\eta_2) \, d\eta_1 \, d\eta_2 + \hat{\mathcal{N}}_{2s5}(\varphi)(\xi) = 2\alpha \log |\xi| \hat{\varphi}(\xi), \quad (8.2)$$

where $\mathcal{N}_{2s5}(\varphi)$ is given by (3.5). From (5.3),

$$T_1(\eta_1, \eta_2, \eta_3) = -\eta_1^2 \log |\eta_1| - \eta_2^2 \log |\eta_2| - \eta_3^2 \log |\eta_3| - (\eta_1 + \eta_2 + \eta_3)^2 \log |\eta_1 + \eta_2 + \eta_3|$$

$$+ \left\{(\eta_1 + \eta_2)^2 \log |\eta_1 + \eta_2| + (\eta_1 + \eta_3)^2 \log |\eta_1 + \eta_3| + (\eta_2 + \eta_3)^2 \log |\eta_2 + \eta_3| \right\}.$$
with
\[\hat{\varphi}_{\eta_1}(T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)) = -2\left\{\eta_1 \log |\eta_1| - (\eta_1 + \eta_2) \log |\eta_1 + \eta_2| \right\}
+ (\xi - \eta_1) \log |\xi - \eta_1| - (\xi - \eta_1 - \eta_2) \log |\xi - \eta_1 - \eta_2|,\]
\[\hat{\varphi}_{\eta_2}(T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)) = -2\left\{\eta_2 \log |\eta_2| - (\eta_1 + \eta_2) \log |\eta_1 + \eta_2| \right\}
+ (\xi - \eta_2) \log |\xi - \eta_2| - (\xi - \eta_1 - \eta_2) \log |\xi - \eta_1 - \eta_2|\right\}.
\]

(8.3)

8.1. **Modified scattering.** Nonlinearity leads to a cumulative frequency shift in the long-time behavior of the Fourier components of the solution due to space-time resonances of the form \(\xi + \xi = \xi\). To account for this effect, we use the method of modified scattering and introduce a phase correction

\[\Theta(\xi, t) = -2t \xi \log |\xi| + \xi \int_0^t \left[\beta_1(t) T_1(\xi, -\xi) + \beta_2(t) T_1(\xi, -\xi, \xi) + \beta_3(t) T_1(\xi, -\xi, \xi)\right]|\hat{\varphi}(\xi, \tau)|^2 \, d\tau,
\]

where \(\beta_1(t), \beta_2(t),\) and \(\beta_3(t)\) are real-valued functions of \(t\) to be determined later. We then let

\[\hat{\varphi}(\xi, t) = e^{\Theta(\xi, t)} \hat{\varphi}(\xi, t).
\]

Using (8.2), we find that

\[\hat{\varphi}(\xi, t) = e^{\Theta(\xi, t)} \left\{ -\frac{1}{6} \xi^2 \int_{\mathbb{R}^2} T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) \hat{\varphi}(\xi - \eta_1 - \eta_2, t) \hat{\varphi}(\eta_1, t) \hat{\varphi}(\eta_2, t) \, d\eta_1 \, d\eta_2
+ i \xi \left[\beta_1(t) T_1(\xi, -\xi) + \beta_2(t) T_1(\xi, -\xi, \xi) + \beta_3(t) T_1(\xi, -\xi, \xi)\right]|\hat{\varphi}(\xi, t)|^2 \hat{\varphi}(\xi, t) \right\},
\]

\[U_2(\xi, t) = \hat{\varphi}(\xi, t) \left\{ \xi \int_0^t \left[\beta_1(t) T_1(\xi, -\xi) + \beta_2(t) T_1(\xi, -\xi, \xi) + \beta_3(t) T_1(\xi, -\xi, \xi)\right]|\hat{\varphi}(\xi, \tau)|^2 \, d\tau \right\}.
\]

The coefficient of \(\hat{\varphi}\) in the term \(U_2\) is purely imaginary, so it leads to a phase shift in \(\hat{\varphi}\) that does not affect its norm, and we get from (8.3) that

\[\|\varphi\|_{L^2} = \left\|(|\xi| + |\xi|^{r+4}) \hat{\varphi}(\xi, t)\right\|_{L^2} = \left\|(|\xi| + |\xi|^{r+4}) \hat{\varphi}(\xi, t)\right\|_{L^2}
\leq \int_0^t \left\|(|\xi| + |\xi|^{r+4}) U_1(\xi, \tau)\right\|_{L^2} + \left\|(|\xi| + |\xi|^{r+4}) \hat{\varphi}(\xi, \tau)\right\|_{L^2} \, d\tau.
\]

We will estimate the cubic terms involving \(U_1\) in Sections 8.2 and 8.5 and the higher-degree terms involving \(\hat{\varphi}(\xi, \tau)\) in Section 8.6. We do not need to consider the terms in \(U_1\) that involve the \(\beta_j\) until we come to an analysis of the space-time resonances in Section 8.5.

To begin with, we recall that \(h = e^{-2it \xi \log |\hat{\varphi}|} \varphi\) is defined in (8.3). From (8.2), we find that \(\hat{h}\) satisfies

\[\hat{h}(\xi, t) + \frac{1}{6} i \xi \int_{\mathbb{R}^2} T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) e^{i\Theta(\xi, \eta_1, \eta_2)} \hat{h}(\xi - \eta_1 - \eta_2, t) \hat{h}(\eta_1, t) \hat{h}(\eta_2, t) \, d\eta_1 \, d\eta_2
+ e^{-2it \xi \log |\xi|} \hat{\varphi}(\xi, \tau) = 0,
\]

(8.5)

where

\[\Phi(\xi, \eta_1, \eta_2) = 2(\xi - \eta_1 - \eta_2) \log |\xi - \eta_1 - \eta_2| + 2\eta_1 \log |\eta_1| + 2\eta_2 \log |\eta_2| - 2\xi \log |\xi|.
\]

(8.6)
Suppressing the dependence on the time variable $t$, we can write the integral in $U_1$ involving $\varphi$ in terms of $h$ as

$$
\iint_{\mathbb{R}^2} T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) \hat{\varphi}(\xi - \eta_1 - \eta_2) \hat{\varphi}(\eta_1) \hat{\varphi}(\eta_2) \, d\eta_1 \, d\eta_2
= \iint_{\mathbb{R}^2} T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) e^{i\Phi(\xi, \eta_1, \eta_2)} \hat{h}(\xi - \eta_1 - \eta_2) \hat{h}(\eta_1) \hat{h}(\eta_2) \, d\eta_1 \, d\eta_2.
$$

(8.7)

Carrying out a dyadic decomposition, with $h_j = P_j h$ and $\varphi_j = P_j \varphi$ where $P_j$ is the Fourier multiplier with symbol $\psi_j$ defined in (2.13), we rewrite this integral in each dyadic block as

$$
\iint_{\mathbb{R}^2} T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) e^{i\Phi(\xi, \eta_1, \eta_2)} \hat{h}_{j_1}(\eta_1) \hat{h}_{j_2}(\eta_2) \hat{h}_{j_3}(\xi - \eta_1 - \eta_2) \, d\eta_1 \, d\eta_2.
$$

(8.7)

In the following subsections, we estimate this integral in various regions of frequency-space.

In Section 8.2, we estimate the integral for high frequencies (large $j_1$, $j_2$, and $j_3$). In Section 8.3, we estimate the integral for nonresonant frequencies, using oscillatory integral estimates with respect to the frequency variables together with multilinear estimates to get sufficient time decay.

In Section 8.4, we consider frequencies that are close to the resonant frequencies. In that case, the bounds for the multilinear symbols are worse, so we cannot obtain sufficient time decay by the method used for the nonresonant frequencies. We resolve this issue by an additional dyadic decomposition centered at each resonant point and a refinement of the symbol estimates.

Finally, in Section 8.5, we consider frequencies that are at the space resonance or space-time resonances. For the space resonance, we estimate the integral in a region about the space resonance point that shrinks in time, using an oscillatory integral estimate with respect to time and the equation to eliminate the time-derivative of the solution. For the space-time resonances, we take advantage of the modified scattering phase correction and estimate the integral on shrinking regions about the space-time resonance points.

8.2. High frequencies. When $\max\{j_1, j_2, j_3\} \gtrsim 10^{-3} \log_2 |t + 1| > 0$, we can estimate the nonlinear terms (8.7) by using Lemma 2.5 with the $L^2$-norm placed on the lowest derivative term. There are, in total, $r + 6 = 13$ derivatives shared by three factors of $\varphi$. Thus, we can ensure that the term with least derivatives has at most four derivatives, with or without a logarithmic derivative.

To be more specific, using Hölder's inequality, Sobolev embedding, and the bootstrap assumptions, we obtain the estimate

$$
\left\| \xi(|\xi| + |\xi|^{r+4}) \vartheta(\xi, t) \iint_{\mathbb{R}^2} T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) e^{i\Phi(\xi, \eta_1, \eta_2)} \hat{h}_{j_1}(\eta_1) \hat{h}_{j_2}(\eta_2) \hat{h}_{j_3}(\xi - \eta_1 - \eta_2) \, d\eta_1 \, d\eta_2 \right\|_{L^2_{\xi}}
\leq (t + 1)^{(r+8-s)10^{-3}} \left\| \varphi_{\min} \right\|_{L^2} \left( \left\| \varphi_{\med} \right\|_{L^\infty} + \left\| L\partial_x \varphi_{\med} \right\|_{W^{r, \infty}} \right) \left\| \varphi_{\max} \right\|_{H^r}
\leq (t + 1)^{(r+8-s)10^{-3}} \left\| \varphi_{j_1} \right\|_{H^r} \left\| \varphi_{j_2} \right\|_{H^r} \left\| \varphi_{j_3} \right\|_{H^r},
$$

where max, med, min represent the maximum, median, and the minimum of $j_1$, $j_2$, $j_3$, and $\vartheta$ is the cutoff function for the frequency range (8.4). From (8.4), we have $(r + 8 - s)10^{-3} < -1.1$, so the right-hand-side is summable over $j_1$, $j_2$, $j_3$, and the sum is integrable for $t \in (0, \infty)$.

8.3. Nonresonant frequencies. We now only need to consider when $\max\{j_1, j_2, j_3\} < 10^{-3} \log_2(t + 1)$. The regions $|j_1 - j_3| > 1$ or $|j_2 - j_3| > 1$ correspond to nonresonant frequencies. Without loss of generality, we assume $|j_1 - j_3| > 1$.
Notice that by (8.4), we have
\[ \partial_{\eta_1}\Phi = 2\log|\eta_1| - 2\log|\xi - \eta_1 - \eta_2|. \] (8.8)
Since $|\eta_1|$ and $|\xi - \eta_1 - \eta_2|$ are in different dyadic blocks, we have $||\eta_1| - |\xi - \eta_1 - \eta_2|| \gtrsim \max\{|\eta_1|, |\xi - \eta_1 - \eta_2|\}$. Therefore, $|\partial_{\eta_1}\Phi| \gtrsim 1$.

After integrating by parts, we have
\[
\int \int_{\mathbb{R}^2} T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)e^{it\Phi(\xi, \eta_1, \eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)'\hat{h}_{j_3}(\xi - \eta_1 - \eta_2)\,d\eta_1
d\eta_2
\]
\[= \int \int_{\mathbb{R}^2} T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)i\partial_{\eta_1}\Phi(\xi, \eta_1, \eta_2)\partial_{\eta_1}e^{it\Phi(\xi, \eta_1, \eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)'\hat{h}_{j_3}(\xi - \eta_1 - \eta_2)\,d\eta_1
d\eta_2
\]
\[= -W_1 - W_2 - W_3,
\]
where
\[
W_1(\xi, t) = \int \int_{\mathbb{R}^2} \partial_{\eta_1}\left[T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)\partial_{\eta_1}\Phi(\xi, \eta_1, \eta_2)\right]e^{it\Phi(\xi, \eta_1, \eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)'\hat{h}_{j_3}(\xi - \eta_1 - \eta_2)\,d\eta_1
d\eta_2,
\]
\[
W_2(\xi, t) = \int \int_{\mathbb{R}^2} \left[T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)\partial_{\eta_1}\Phi(\xi, \eta_1, \eta_2)\right]e^{it\Phi(\xi, \eta_1, \eta_2)}\partial_{\eta_1}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)'\hat{h}_{j_3}(\xi - \eta_1 - \eta_2)\,d\eta_1
d\eta_2,
\]
\[
W_3(\xi, t) = \int \int_{\mathbb{R}^2} \partial_{\eta_1}\left[T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)\partial_{\eta_1}\Phi(\xi, \eta_1, \eta_2)\right]e^{it\Phi(\xi, \eta_1, \eta_2)}\partial_{\eta_1}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)'\hat{h}_{j_3}(\xi - \eta_1 - \eta_2)\,d\eta_1
d\eta_2.
\]

**Estimate of $W_1$.** Since
\[
\|W_1\|_{L^1_\xi} \lesssim \|F^{-1}(W_1)\|_{L^1_t},
\] (8.9)
it suffices to estimate the $L^1_\xi$ norm of
\[
\int \int_{\mathbb{R}^3} e^{it\xi} \partial_{\eta_1}\left[T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)\partial_{\eta_1}\Phi(\xi, \eta_1, \eta_2)\right]e^{it\Phi(\xi, \eta_1, \eta_2)}\hat{h}_{j_1}(\eta_1)\hat{h}_{j_2}(\eta_2)'\hat{h}_{j_3}(\xi - \eta_1 - \eta_2)\,d\eta_1
d\eta_2
d\xi.
\]

Notice that by (8.5)
\[
\partial_{\eta_1}\left[T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)\partial_{\eta_1}\Phi(\xi, \eta_1, \eta_2)\right] = \kappa_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) - \frac{\kappa_2(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{2},
\]
where
\[
\kappa_1(\eta_1, \eta_2, \eta_3) = \frac{\partial_{\eta_1}T_1(\eta_1, \eta_2, \eta_3) - \partial_{\eta_2}T_1(\eta_1, \eta_2, \eta_3)}{\log|\eta_1| - \log|\eta_3|},
\]
\[
\kappa_2(\eta_1, \eta_2, \eta_3) = T_1(\eta_1, \eta_2, \eta_3)\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)\left(\log|\eta_1| - \log|\eta_3|\right)^2.
\]

Making a change of variable $\eta_3 = \xi - \eta_1 - \eta_2$, we need to estimate the trilinear form
\[
\frac{1}{it} \int \int_{\mathbb{R}^3} e^{it\xi} \left[\kappa_1(\eta_1, \eta_2, \eta_3) + \kappa_2(\eta_1, \eta_2, \eta_3)\right]\hat{\varphi}_{j_1}(\eta_1)\hat{\varphi}_{j_2}(\eta_2)\hat{\varphi}_{j_3}(\eta_3)\,d\eta_1
d\eta_2
d\eta_3,
\]
with symbol
\[
\left[\kappa_1(\eta_1, \eta_2, \eta_3) + \kappa_2(\eta_1, \eta_2, \eta_3)\right] \hat{\psi}_{j_1}(\eta_1)\hat{\psi}_{j_2}(\eta_2)\hat{\psi}_{j_3}(\eta_3).
\]
According to Lemma 2.5 this trilinear operator is bounded on $L^2 \times L^2 \times L^\infty \to L^1$ by
\[
\left\| \left[ \kappa_1(\eta_1, \eta_2, \eta_3) + \kappa_2(\eta_1, \eta_2, \eta_3) \right] \psi_{j_1}(\eta_1) \psi_{j_2}(\eta_2) \psi_{j_3}(\eta_3) \right\|_{S^\infty} \\
\lesssim \left( \left\| \partial_{\eta_1} T_1(\eta_1, \eta_2, \eta_3) \tilde{\psi}_{j_1}(\eta_1) \tilde{\psi}_{j_2}(\eta_2) \tilde{\psi}_{j_3}(\eta_3) \right\|_{S^\infty} \\
+ \left\| \partial_{\eta_2} T_1(\eta_1, \eta_2, \eta_3) \tilde{\psi}_{j_1}(\eta_1) \tilde{\psi}_{j_2}(\eta_2) \tilde{\psi}_{j_3}(\eta_3) \right\|_{S^\infty} \\
+ \left\| \frac{T_1(\eta_1, \eta_2, \eta_3)}{\eta_1} \tilde{\psi}_{j_1}(\eta_1) \tilde{\psi}_{j_2}(\eta_2) \tilde{\psi}_{j_3}(\eta_3) \right\|_{S^\infty} \right) \cdot \left\| \frac{\psi_{j_1}(\eta_1) \psi_{j_2}(\eta_2) \psi_{j_3}(\eta_3)}{\log |\eta_1| - \log |\eta_3|} \right\|_{S^\infty} \tag{8.10} \]
\]

Lemma 8.1. Suppose that $|j_1 - j_3| > 1$. Then for any $m \in \mathbb{Z}_+$,
\[
\left\| \frac{1}{(\log |\eta_1| - \log |\eta_3|)^m} \psi_{j_1}(\eta_1) \psi_{j_2}(\eta_2) \psi_{j_3}(\eta_3) \right\|_{S^\infty} \lesssim 1.
\]

Proof. By the definition of the $S^\infty$-norm (2.13) and the definition of $\psi_k$ (2.13), we have that
\[
\left\| \frac{\psi_{j_1}(\eta_1) \psi_{j_2}(\eta_2) \psi_{j_3}(\eta_3)}{\log |\eta_1| - \log |\eta_3|} \right\|_{S^\infty} = \left\| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \psi_0(2^{-j_1} \eta_1) \psi_0(2^{-j_2} \eta_2) \psi_0(2^{-j_3} \eta_3) e^{i(\eta_1 \eta_1 + \eta_2 \eta_2 + \eta_3 \eta_3)} \, d\eta_1 \, d\eta_2 \, d\eta_3 \tag{8.10} \right\|_{L^1} \]
\[
\lesssim 1,
\]

where the last inequality comes from oscillatory integral estimates, using the fact that $|j_1 - j_3| > 1$ and the support of $\psi_0$ is $(-\frac{s}{5}, -\frac{s}{5}) \cup (\frac{s}{5}, \frac{s}{5})$.

For the estimates of other symbols in (8.10), we have the following lemma.

Lemma 8.2. For any $j_1, j_2, j_3 \in \mathbb{Z}$, we have
\[
\left\| \partial_{\eta_1} T_1(\eta_1, \eta_2, \eta_3) \tilde{\psi}_{j_1}(\eta_1) \tilde{\psi}_{j_2}(\eta_2) \tilde{\psi}_{j_3}(\eta_3) \right\|_{S^\infty} \lesssim 2^{\max\{j_2, j_3\}}, \tag{8.11} \]
\[
\left\| T_1(\eta_1, \eta_2, \eta_3) \tilde{\psi}_{j_1}(\eta_1) \tilde{\psi}_{j_2}(\eta_2) \tilde{\psi}_{j_3}(\eta_3) \right\|_{S^\infty} \lesssim 2^{\max\{j_1, j_2, j_3\} + \min\{j_1, j_2, j_3\}}, \tag{8.12} \]
and
\[
\left\| \frac{T_1(\eta_1, \eta_2, \eta_3)}{\eta_1} \tilde{\psi}_{j_1}(\eta_1) \tilde{\psi}_{j_2}(\eta_2) \tilde{\psi}_{j_3}(\eta_3) \right\|_{S^\infty} \lesssim 2^{\max\{j_2, j_3\}}. \tag{8.13} \]
Furthermore, since $T_1$ is symmetric, we also have
\[
\left\| \partial_{\eta_2} T_1(\eta_1, \eta_2, \eta_3) \tilde{\psi}_{j_1}(\eta_1) \tilde{\psi}_{j_2}(\eta_2) \tilde{\psi}_{j_3}(\eta_3) \right\|_{S^\infty} \lesssim 2^{\max\{j_1, j_2\}}, \]
\[
\left\| T_1(\eta_1, \eta_2, \eta_3) \tilde{\psi}_{j_1}(\eta_1) \tilde{\psi}_{j_2}(\eta_2) \tilde{\psi}_{j_3}(\eta_3) \right\|_{S^\infty} \lesssim 2^{\max\{j_1, j_2\}}, \]
\[
\left\| \frac{T_1(\eta_1, \eta_2, \eta_3)}{\eta_3} \tilde{\psi}_{j_1}(\eta_1) \tilde{\psi}_{j_2}(\eta_2) \tilde{\psi}_{j_3}(\eta_3) \right\|_{S^\infty} \lesssim 2^{\max\{j_1, j_2\}}.
\]
Proof. 1. We prove (8.11) first. Using inverse Fourier transform in \((\eta_1, \eta_2, \eta_3)\), we obtain
\[
\mathcal{F}^{-1}[\hat{\mathcal{E}}_{\eta_1} \mathbf{T}_1(\eta_1, \eta_2, \eta_3) \hat{\psi}_{\eta_1}(\eta_1) \hat{\psi}_{\eta_2}(\eta_2) \hat{\psi}_{\eta_3}(\eta_3)]
\]
\[
= \iint_{\mathbb{R}^3} e^{i(y_1 \eta_1 + y_2 \eta_2 + y_3 \eta_3)} \hat{\psi}_{y_1}(\eta_1) \hat{\psi}_{y_2}(\eta_2) \hat{\psi}_{y_3}(\eta_3) \frac{1}{|\zeta|^3} \prod_{j=1}^3 (1 - \frac{e^{i\eta_j \zeta}}{|\zeta|^3}) d\zeta \hat{\psi}_{\eta_1}(\eta_1) \hat{\psi}_{\eta_2}(\eta_2) \hat{\psi}_{\eta_3}(\eta_3) d\eta_1 d\eta_2 d\eta_3
\]
\[
= \iiint_{\mathbb{R}^3} \int_{\mathbb{R}} -i\zeta \hat{\psi}_{\eta_1}(\eta_1) (e^{i\eta_1 \zeta} - e^{i\eta_2 (\zeta + y_2) (e^{i\eta_3 (\zeta + y_3)})} - \hat{\psi}_{\eta_1}(\eta_1) \hat{\psi}_{\eta_2}(\eta_2) \hat{\psi}_{\eta_3}(\eta_3) d\eta_1 d\eta_2 d\eta_3
\]
\[
= \int_{\mathbb{R}} \frac{-i\zeta}{|\zeta|^3} \left[ \mathcal{F}^{-1}[\hat{\psi}_{\eta_1}](y_1 + \zeta) \right] \left[ \mathcal{F}^{-1}[\hat{\psi}_{\eta_2}](y_2) - \mathcal{F}^{-1}[\hat{\psi}_{\eta_2}](\zeta + y_2) \right] \left[ \mathcal{F}^{-1}[\hat{\psi}_{\eta_3}](y_3) - \mathcal{F}^{-1}[\hat{\psi}_{\eta_3}](\zeta + y_3) \right] d\zeta.
\]
Notice that
\[
\left| \mathcal{F}^{-1}[\hat{\psi}_{\eta_1}](y_1 + \zeta) \right| = 2^{j_1} \left| \mathcal{F}^{-1}[\hat{\psi}_{\eta_0}](2^{j_1} (y_1 + \zeta)) \right|,
\]
\[
\left| \mathcal{F}^{-1}[\hat{\psi}_{\eta_2}](y_2) - \mathcal{F}^{-1}[\hat{\psi}_{\eta_2}](\zeta + y_2) \right| = 2^{j_2} \left| \mathcal{F}^{-1}[\hat{\psi}_{\eta_0}](2^{j_2} y_2) - \mathcal{F}^{-1}[\hat{\psi}_{\eta_0}](2^{j_2} (\zeta + y_2)) \right|,
\]
\[
\left| \mathcal{F}^{-1}[\hat{\psi}_{\eta_3}](y_3) - \mathcal{F}^{-1}[\hat{\psi}_{\eta_3}](\zeta + y_3) \right| = 2^{j_3} \left| \mathcal{F}^{-1}[\hat{\psi}_{\eta_0}](2^{j_3} y_3) - \mathcal{F}^{-1}[\hat{\psi}_{\eta_0}](2^{j_3} (\zeta + y_3)) \right|,
\]
and that
\[
\int_{\mathbb{R}} \left| \mathcal{F}^{-1}[\hat{\psi}_{\eta_0}](2^{j_1} (y_1 + \zeta)) \right| dy_1 \lesssim 2^{-j_1},
\]
\[
\int_{\mathbb{R}} \left| \mathcal{F}^{-1}[\hat{\psi}_{\eta_2}](2^{j_2} y_2) - \mathcal{F}^{-1}[\hat{\psi}_{\eta_2}](2^{j_2} (\zeta + y_2)) \right| dy_2 \lesssim \min \{2^{-j_2}, |\zeta|\},
\]
\[
\int_{\mathbb{R}} \left| \mathcal{F}^{-1}[\hat{\psi}_{\eta_3}](2^{j_3} y_3) - \mathcal{F}^{-1}[\hat{\psi}_{\eta_3}](2^{j_3} (\zeta + y_3)) \right| dy_3 \lesssim \min \{2^{-j_3}, |\zeta|\}.
\]
Therefore, we have
\[
\left\| \mathcal{F}^{-1}[\hat{\mathcal{E}}_{\eta_1} \mathbf{T}_1(\eta_1, \eta_2, \eta_3) \hat{\psi}_{\eta_1}(\eta_1) \hat{\psi}_{\eta_2}(\eta_2) \hat{\psi}_{\eta_3}(\eta_3)] \right\|_{L^1} \lesssim \int_{\mathbb{R}} \frac{1}{|\zeta|^2} 2^{2j_1 + j_2} \min \{2^{-j_2}, |\zeta|\} \min \{2^{-j_3}, |\zeta|\} d\zeta
\]
\[
= 2^{j_1 + j_2} \left( \int_{|\zeta| > \max \{2^{-j_2}, 2^{-j_3}\}} \frac{1}{|\zeta|^2} 2^{2j_2 - j_3} d\zeta + \int_{\min \{2^{-j_2}, 2^{-j_3}\} < |\zeta| < \max \{2^{-j_2}, 2^{-j_3}\}} \frac{1}{|\zeta|} \min \{2^{-j_2}, 2^{-j_3}\} d\zeta \right)
\]
\[
+ \int_{|\zeta| < \min \{2^{-j_2}, 2^{-j_3}\}} d\zeta \lesssim 2^{\max \{j_1, j_2, j_3\}}.
\]

2. Next, we prove (8.12) and (8.13). The estimate of (8.12) is similarly to (8.11). We first use inverse Fourier transform and write
\[
\mathcal{F}^{-1} \left[ \mathbf{T}_1(\eta_1, \eta_2, \eta_3) \hat{\psi}_{\eta_1}(\eta_1) \hat{\psi}_{\eta_2}(\eta_2) \hat{\psi}_{\eta_3}(\eta_3) \right]
\]
\[
= \iiint_{\mathbb{R}^3} e^{i(y_1 \eta_1 + y_2 \eta_2 + y_3 \eta_3)} \int_{\mathbb{R}} \frac{1}{|\zeta|^3} \prod_{j=1}^3 (1 - \frac{e^{i\eta_j \zeta}}{|\zeta|^3}) d\zeta \hat{\psi}_{\eta_1}(\eta_1) \hat{\psi}_{\eta_2}(\eta_2) \hat{\psi}_{\eta_3}(\eta_3) d\eta_1 d\eta_2 d\eta_3
\]
\[
= \iiint_{\mathbb{R}^3} \left( \int_{\mathbb{R}} e^{i(y_2 \eta_2 - y_1 \eta_1 (\zeta + y_1))} e^{i(y_2 \eta_2 - y_1 \eta_1 (\zeta + y_2))} e^{i(y_2 \eta_2 - y_1 \eta_1 (\zeta + y_3))} \right) \frac{1}{|\zeta|^3} \prod_{j=1}^3 (1 - \frac{e^{i\eta_j \zeta}}{|\zeta|^3}) d\zeta \hat{\psi}_{\eta_1}(\eta_1) \hat{\psi}_{\eta_2}(\eta_2) \hat{\psi}_{\eta_3}(\eta_3) d\eta_1 d\eta_2 d\eta_3
\]
Then it follows from the support of $\psi$ and the fact that $\psi$ forms a partition of unity that

$$\frac{T_1(\eta_1, \eta_2, \eta_3)}{\eta_1} \tilde{\psi}_j(\eta_1) \tilde{\psi}_j(\eta_2) \tilde{\psi}_j(\eta_3) = \left[ \frac{1}{\eta_1} \tilde{\psi}_j(\eta_1) \tilde{\psi}_j(\eta_2) \tilde{\psi}_j(\eta_3) \right].$$

By Lemma 2.3, we have

$$\frac{T_1(\eta_1, \eta_2, \eta_3)}{\eta_1} \tilde{\psi}_j(\eta_1) \tilde{\psi}_j(\eta_2) \tilde{\psi}_j(\eta_3) \leq \frac{T_1(\eta_1, \eta_2, \eta_3)}{\eta_1} \tilde{\psi}_j(\eta_1) \tilde{\psi}_j(\eta_2) \tilde{\psi}_j(\eta_3) = \frac{T_1(\eta_1, \eta_2, \eta_3)}{\eta_1} \tilde{\psi}_j(\eta_1) \tilde{\psi}_j(\eta_2) \tilde{\psi}_j(\eta_3).$$

In view of (8.12), we only need to estimate the second term. To this end, we have

$$\left\| \frac{1}{\eta_1} \tilde{\psi}_j(\eta_1) \tilde{\psi}_j(\eta_2) \tilde{\psi}_j(\eta_3) \right\|_{S^\infty} \leq \left\| \frac{T_1(\eta_1, \eta_2, \eta_3)}{\eta_1} \tilde{\psi}_j(\eta_1) \tilde{\psi}_j(\eta_2) \tilde{\psi}_j(\eta_3) \right\|_{S^\infty} \leq 2^{-j_3}. \tag{8.14}$$

Therefore, by (8.14) and considering all the possible relations between $j_1$, $j_2$, and $j_3$, we obtain (8.13). \qed

Applying the above lemmas to (8.10) and (8.39), we obtain

$$\|W_j\|_{L^\infty} \leq (t + 1)^{-1} \left[ \|\partial_x \varphi_{\max}\|_{L^\infty} \|\varphi_{j_1}\|_{L^2} \|\varphi_{j_2}\|_{L^2} \|\varphi_{j_3}\|_{L^2} + \|\partial_x \varphi_{\max}\|_{L^\infty} \|\varphi_{j_1}\|_{L^2} \|\varphi_{j_2}\|_{L^2} \|\varphi_{j_3}\|_{L^2} \right].$$

Since the two terms are symmetric in $j_1$ and $j_3$, it suffices to estimate one of them, as the other one is similar. We use lemma (8.1) and get

$$\|\partial_x \varphi_{\max}\|_{L^\infty} \leq (t + 1)^{-1/2} \|\xi\|_{L^\infty} \|\varphi_{\max}\|_{L^2} \|\varphi_{j_1}\|_{L^2} \|\varphi_{j_2}\|_{L^2} \|\varphi_{j_3}\|_{L^2}.$$
Therefore,

\[ \| W_1 \|_{L^\infty} \lesssim (t + 1)^{-1.5} \left( 1_{\max(j_1, j_2) \leq 0} 2^{0.5 \max\{j_1, j_2\}} \| \xi \| \hat{h}_{\max\{j_1, j_2\}} \| L^\infty \right) \\
+ 1_{\max(j_1, j_2) > 0} 2^{(-1.5 - r) \max\{j_1, j_2\}} \| \xi \| \hat{h}_{\max\{j_1, j_2\}} \| L^\infty \| \varphi_{\min\{j_1, j_2\}} \| L^2 \\
+ (t + 1)^{-1.75} \left[ \| \partial_x^0 P_{\max\{j_1, j_2\}} (x \partial_x h) \| L^2 + \| \partial_x^0 \hat{h}_{\max\{j_1, j_2\}} \| L^2 \right] \| \varphi_{\min\{j_1, j_2\}} \| L^2 \\
+ (t + 1)^{-1.5} \left( 1_{\max(j_2, j_3) \leq 0} 2^{(-1.5 - r) \max\{j_2, j_3\}} \| \xi \| \hat{h}_{\max\{j_2, j_3\}} \| L^\infty \right) \| \varphi_{\min\{j_1, j_2\}} \| L^2 \\
+ (t + 1)^{-1.75} \left[ \| \partial_x^0 P_{\max\{j_2, j_3\}} (x \partial_x h) \| L^2 + \| \partial_x^0 \hat{h}_{\max\{j_2, j_3\}} \| L^2 \right] \| \varphi_{\min\{j_1, j_2\}} \| L^2. \]

\textbf{Estimate of } W_2 \text{ and } W_3. \text{ We rewrite } W_2 \text{ as}

\[ \int_{\mathbb{R}^2} \left[ \frac{T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{it\partial_{\eta_1} \Phi(\xi, \eta_1, \eta_2)(\xi - \eta_1 - \eta_2)} \right] e^{it\Phi(\xi, \eta_1, \eta_2)} \hat{h}_{\xi_1}(\eta_1) \hat{h}_{\xi_2}(\eta_2) \left[ (\xi - \eta_1 - \eta_2) \partial_{\eta_1} \hat{h}_{\xi_1}(\xi - \eta_1 - \eta_2) \right] d\eta_1 d\eta_2. \]

In view of the multilinear estimate Lemma 2.3, we need to estimate the $S^\infty$-norm of the symbol

\[ \frac{T_1(\eta_1, \eta_2, \eta_3)}{\log |\eta_1| - \log |\eta_3|}) \psi_{\xi_1}(\eta_1) \psi_{\xi_2}(\eta_2) \psi_{\xi_3}(\eta_3). \]

Using Lemma 8.1 and Lemma 8.2 as in the estimates of $W_1$, we obtain

\[ \| W_2 \|_{L^\infty} \lesssim (t + 1)^{-1} \| \partial_x^0 \varphi_{\max\{j_1, j_2\}} \| L^\infty \| \xi \| \dot{h}_{\xi_1} \| L^2 \| \varphi_{\min\{j_1, j_2\}} \| L^2. \]

Using Lemma 6.1 we have

\[ \lesssim (t + 1)^{-1.5} \left( 1_{\max(j_1, j_2) \leq 0} 2^{0.5 \max\{j_1, j_2\}} + 1_{\max(j_1, j_2) > 0} 2^{(-1.5 - r) \max\{j_1, j_2\}} \right) \| \varphi_{\min\{j_1, j_2\}} \| L^2 \]

Similarly, we have

\[ \| W_3 \|_{L^\infty} \lesssim (t + 1)^{-1.5} \left( 1_{\max(j_2, j_3) \leq 0} 2^{0.5 \max\{j_2, j_3\}} + 1_{\max(j_2, j_3) > 0} 2^{(-1.5 - r) \max\{j_2, j_3\}} \right) \| \varphi_{\min\{j_2, j_3\}} \| L^2. \]
In conclusion, for nonresonant frequencies,

\[
\left\| \xi (|\xi| + |\xi|^{-4}) \hat{\partial} (\xi, t) \int_{\mathbb{R}^2} T_1 (\eta_1, \eta_2, \xi - \eta_1 - \eta_2) e^{i \Phi (\xi, \eta_1, \eta_2)} \hat{h}_{j_1} (\eta_1) \hat{h}_{j_2} (\eta_2) \hat{h}_{j_3} (\xi - \eta_1 - \eta_2) \, d\eta_1 \, d\eta_2 \right\|_{L^\infty_x} \\
\lesssim (t + 1)^{-r+5} \left( \| W_1 \|_{L^\infty_x}^2 + \| W_2 \|_{L^\infty_x}^2 + \| W_3 \|_{L^\infty_x} \right)
\]

\[
\lesssim (t + 1)^{-1.5+r+(r+5)} \left( \| \varphi_{j_1} \|_{L^2} \| \varphi_{\min(j_2,j_3)} \|_{L^2} + \| \xi \hat{\partial}_j \hat{h}_{j_3} \|_{L^2} \| \varphi_{\min(j_2,j_3)} \|_{L^2} \right)
\]

\[
\cdot \left( 1_{\max(j_2,j_3)} \leq 0^2 \right) + \left( \| \hat{h}_{j_1} \|_{L^2} \| \varphi_{\min(j_1,j_2,j_3)} \|_{L^2} \right)
\]

\[
+ (t + 1)^{-1.5+(r+5)} \left( \| \varphi_{j_2} \|_{L^2} \| \varphi_{\min(j_1,j_2)} \|_{L^2} + \| \xi \hat{\partial}_j \hat{h}_{j_3} \|_{L^2} \| \varphi_{\min(j_1,j_2)} \|_{L^2} \right)
\]

\[
\cdot \left( 1_{\max(j_1,j_2,j_3)} \leq 0^2 \right) + \left( \| \hat{h}_{j_1} \|_{L^2} \| \varphi_{\min(j_1,j_2,j_3)} \|_{L^2} \right)
\]

\[
+ (t + 1)^{-1.75+(r+5)} \left( \| \partial_x \|_{0.75} \| \varphi_{\max(j_2,j_3)} \|_{L^2} \| \partial_x \|_{0.75} \| \varphi_{\min(j_1,j_2)} \|_{L^2} \right)
\]

\[
+ (t + 1)^{-1.0+(r+5)} \left( \| \partial_x \|_{0.75} \| \varphi_{\max(j_1,j_2)} \|_{L^2} \| \partial_x \|_{0.75} \| \varphi_{\min(j_1,j_2)} \|_{L^2} \right)
\]

\[
+ (t + 1)^{-1.0+(r+5)} \left( \| \partial_x \|_{0.75} \| \varphi_{\max(j_1,j_2,j_3)} \|_{L^2} \| \partial_x \|_{0.75} \| \varphi_{\min(j_1,j_2,j_3)} \|_{L^2} \right)
\]

\[
+ (t + 1)^{-1.0+(r+5)} \left( \| \partial_x \|_{0.75} \| \varphi_{\max(j_1,j_2,j_3)} \|_{L^2} \| \partial_x \|_{0.75} \| \varphi_{\min(j_1,j_2,j_3)} \|_{L^2} \right)
\].

By the bootstrap assumptions and Lemma 5.5, the right-hand-side is summable for \( j_1, j_2, j_3 \) and the sum is integrable for \( t \in (0, \infty) \).

8.4. Close to the resonance. When

\[
\max\{j_1, j_2, j_3\} < 10^{-3} \log_2 (t + 1), \quad |j_3 - j_2| \leq 1, \quad |j_3 - j_1| \leq 1,
\]

we need to consider the following two cases:

(i) Frequencies \( \eta_1, \eta_2 \) and \( \xi - \eta_1 - \eta_2 \) have the same sign.

By the definition of cutoff function \( \psi \), we have

\[
\frac{5}{8} 2^{j_1} \leq |\eta_1| \leq \frac{5}{8} 2^{j_1}, \quad \frac{5}{8} 2^{j_2} \leq |\eta_2| \leq \frac{5}{8} 2^{j_2}, \quad \frac{5}{8} 2^{j_3} \leq |\xi - \eta_1 - \eta_2| \leq \frac{5}{8} 2^{j_3},
\]

and thus,

\[
\frac{5}{8} (2^{j_1} + 2^{j_2} + 2^{j_3}) \leq |\xi| \leq \frac{5}{8} (2^{j_1} + 2^{j_2} + 2^{j_3}).
\]

This corresponds to the region near the space resonance \( \eta_1 = \eta_2 = \xi - \eta_1 - \eta_2 = \xi/3 \).

(ii) Frequencies \( \eta_1, \eta_2 \) and \( \xi - \eta_1 - \eta_2 \) do not have the same sign.

This corresponds to the region near the space-time resonances \( (\eta_1, \eta_2) = (\xi, \xi), (\xi, -\xi), \) or \( (-\xi, \xi) \) separately. Since the symbol \( T'_1 (\eta_1, \eta_2, \eta_3) \) is symmetric in \( \eta_1, \eta_2, \) and \( \eta_3, \) it suffices to (8.7) in the region near \( (\xi, \xi) \).

To estimate (8.7) in the region (8.15), we decompose the region further. Denoting \( (\xi_1, \xi_2, \xi_3) = (\xi, \xi, -\xi) \) or \( (\xi, \xi, \xi) \), we decompose (8.15) using the new cutoff functions \( \psi_{k_1} \) and \( \psi_{k_2} \). Using the fact that

\[
\sum_{(k_1, k_2) \in \mathbb{Z}^2} \psi_{k_1} (\eta_1 - \xi_1) \psi_{k_2} (\eta_2 - \xi_2) = 1,
\]

we write the integral (8.7) as
\[ \iiint_{\mathbb{R}^2} T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) e^{i \Phi(\xi, \eta_1, \eta_2)} \hat{h}_{j_1}(\eta_1) \hat{h}_{j_2}(\eta_2) \hat{h}_{j_3}(\xi - \eta_1 - \eta_2) \]
\[ \cdot \left[ \max_{k_1=-\infty}^{k_1=\infty} \psi_{k_1}(\eta_1 - \xi_1) \right] \left[ \max_{k_2=-\infty}^{k_2=\infty} \psi_{k_2}(\eta_2 - \xi_2) \right] d\eta_1 d\eta_2, \]

where
\[ \left[ \max_{k_1=-\infty}^{k_1=\infty} \psi_{k_1}(\eta_1 - \xi_1) \right] \left[ \max_{k_2=-\infty}^{k_2=\infty} \psi_{k_2}(\eta_2 - \xi_2) \right] = 1 \]
on the support of \( \hat{h}_{j_1}(\eta_1) \hat{h}_{j_2}(\eta_2) \hat{h}_{j_3}(\xi - \eta_1 - \eta_2) \). Thus, we need to consider
\[ \iiint_{\mathbb{R}^2} T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) e^{i \Phi(\xi, \eta_1, \eta_2)} \hat{h}_{j_1}(\eta_1) \hat{h}_{j_2}(\eta_2) \hat{h}_{j_3}(\xi - \eta_1 - \eta_2) \]
\[ \cdot \psi_{k_1}(\eta_1 - \xi_1) \psi_{k_2}(\eta_2 - \xi_2) d\eta_1 d\eta_2. \]

In this subsection, we restrict our attention to
\[ k_1 \geq \log_2[\varrho(t)] \quad \text{or} \quad k_2 \geq \log_2[\varrho(t)], \]
where
\[ \varrho(t) = (t + 1)^{-0.49}. \]

The case of \( k_1 < \log_2[\varrho(t)] \) and \( k_2 < \log_2[\varrho(t)] \), related to the resonant frequencies, will be discussed in Section 5.5.

Since these expressions are symmetric in \( \eta_1 \) and \( \eta_2 \), we assume without loss of generality that \( j_1 \geq k_1 \geq k_2 \geq \log_2[\varrho(t)] \). The other case can be discussed in the familiar way.

Using integrating by parts, we can write (8.10) as
\[ \iiint_{\mathbb{R}^2} \frac{T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{2i \log[|\eta_1| - \log[|\xi - \eta_1 - \eta_2|] e^{i \Phi(\xi, \eta_1, \eta_2)} \hat{h}_{j_1}(\eta_1) \hat{h}_{j_2}(\eta_2) \hat{h}_{j_3}(\xi - \eta_1 - \eta_2) \]
\[ \cdot \psi_{k_1}(\eta_1 - \xi_1) \psi_{k_2}(\eta_2 - \xi_2) d\eta_1 d\eta_2 \]
\[ = \frac{i}{2t} (V_1 + V_2 + V_3 + V_4), \]
where
\[ V_1(\xi, t) = \iiint_{\mathbb{R}^2} \hat{c}_{\eta_1} \left[ \frac{T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{\log[|\eta_1| - \log[|\xi - \eta_1 - \eta_2|] e^{i \Phi(\xi, \eta_1, \eta_2)} \hat{h}_{j_1}(\eta_1) \hat{h}_{j_2}(\eta_2) \hat{h}_{j_3}(\xi - \eta_1 - \eta_2) \]
\[ \cdot \psi_{k_1}(\eta_1 - \xi_1) \psi_{k_2}(\eta_2 - \xi_2) d\eta_1 d\eta_2, \]
\[ V_2(\xi, t) = \iiint_{\mathbb{R}^2} \left[ \frac{T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{\log[|\eta_1| - \log[|\xi - \eta_1 - \eta_2|] e^{i \Phi(\xi, \eta_1, \eta_2)} \hat{c}_{\eta_1} \hat{h}_{j_1}(\eta_1) \hat{h}_{j_2}(\eta_2) \hat{h}_{j_3}(\xi - \eta_1 - \eta_2) \]
\[ \cdot \psi_{k_1}(\eta_1 - \xi_1) \psi_{k_2}(\eta_2 - \xi_2) d\eta_1 d\eta_2, \]
\[ V_3(\xi, t) = \iiint_{\mathbb{R}^2} \left[ \frac{T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{\log[|\eta_1| - \log[|\xi - \eta_1 - \eta_2|] e^{i \Phi(\xi, \eta_1, \eta_2)} \hat{h}_{j_1}(\eta_1) \hat{h}_{j_2}(\eta_2) \hat{c}_{\eta_1} \hat{h}_{j_3}(\xi - \eta_1 - \eta_2) \]
\[ \cdot \psi_{k_1}(\eta_1 - \xi_1) \psi_{k_2}(\eta_2 - \xi_2) d\eta_1 d\eta_2, \]
\[ V_4(\xi, t) = \iiint_{\mathbb{R}^2} \left[ \frac{T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{\log[|\eta_1| - \log[|\xi - \eta_1 - \eta_2|] e^{i \Phi(\xi, \eta_1, \eta_2)} \hat{h}_{j_1}(\eta_1) \hat{h}_{j_2}(\eta_2) \hat{h}_{j_3}(\xi - \eta_1 - \eta_2) \]
\[ \cdot \psi_{k_1}(\eta_1 - \xi_1) \psi_{k_2}(\eta_2 - \xi_2) d\eta_1 d\eta_2, \]
\[ V_4(\xi, t) = \int_{\mathbb{R}^2} \left[ \frac{T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{\log |\eta_1| - \log |\xi - \eta_1 - \eta_2|} \right] e^{i\Phi(\xi, \eta_1, \eta_2)} \hat{h}_{j_1}(\eta_1) \hat{h}_{j_2}(\eta_2)(\xi - \eta_1 - \eta_2) \cdot \partial_{\eta_1} \psi_{k_1}(\eta_1 - \xi_1) \psi_{k_2}(\eta_2 - \xi_2) d\eta_1 d\eta_2. \]

**Estimate of** $V_4$. We first denote the symbol for $V_4$ as

\[ m(\eta_1, \eta_2, \xi) = -\frac{2}{\log |\eta_1| - \log |\xi - \eta_1 - \eta_2|} \cdot \left[ \eta_1 \log |\eta_1| - (\eta_1 + \eta_2) \log |\eta_1 + \eta_2| \right. \]
\[ \left. + (\xi - \eta_1) \log |\xi - \eta_1| - (\xi - \eta_1 - \eta_2) \log |\xi - \eta_1 - \eta_2| \right] \]
\[ - \frac{\eta_1^{-1} + (\xi - \eta_1 - \eta_2)^{-1}}{\log |\eta_1| - \log |\xi - \eta_1 - \eta_2|} \cdot \left[ - \eta_1^2 \log |\eta_1| - \eta_2^2 \log |\eta_2| - \eta_1^2 \log |\eta_1| - (\eta_1 + \eta_2 + \eta_3)^2 \log |\eta_1 + \eta_2 + \eta_3| \right. \]
\[ \left. + (\eta_1 + \eta_2)^2 \log |\eta_1 + \eta_2| + (\eta_1 + \eta_3)^2 \log |\eta_1 + \eta_3| + (\eta_2 + \eta_3)^2 \log |\eta_2 + \eta_3| \right]. \]

Denote $v_i = \eta_i - \xi_i$, $i = 1, 2$, and it suffices to estimate

\[ \left\| \int_{\mathbb{R}^2} m(v_1 + \xi_1, v_2 + \xi_2) e^{i\Phi(v_1 + \xi_1, v_2 + \xi_2)} \hat{h}_{j_1}(v_1 + \xi_1) \hat{h}_{j_2}(v_2 + \xi_2) \hat{h}_{j_3}(\xi_3 - v_1 - v_2) \cdot \psi_{k_1}(v_1) \psi_{k_2}(v_2) dv_1 dv_2 \right\|_{L^\infty_\xi}. \]

Using Lemma 2.5 we have

\[ \|V_4\|_{L^\infty_\xi} \lesssim \|\chi_{j_1,j_3}^{k_1,k_2}(v_1, v_2, \xi) m(v_1 + \xi_1, v_2 + \xi_2, \xi)\|_{S^0_{v_1, v_2, L^\infty_\xi}} \cdot \|\hat{\phi}_{j_1}(v_1 + \xi_1) \psi_{k_1}(v_1)\|_{L^\infty_\xi} \cdot \|\hat{\phi}_{j_2}(v_2 + \xi_2) \psi_{k_2}(v_2)\|_{L^\infty_\xi} \cdot \|\hat{\phi}_{j_3}(\xi_3 - v_1 - v_2)\|_{L^\infty_\xi}. \]

where

\[ \chi_{j_1,j_3}^{k_1,k_2}(v_1, v_2, \xi) = \hat{\psi}_{k_1}(v_1) \hat{\psi}_{k_2}(v_2) \hat{\phi}_{j_1}(v_1 + \xi_1) \hat{\psi}_{j_2}(v_2 + \xi_2) \hat{\psi}_{j_3}(\xi_3 - v_1 - v_2) \chi(\xi). \]

(i) If $(\xi_1, \xi_2, \xi_3) = (\xi/3, \xi/3, \xi/3)$, since $S^\infty$-norm is rotational and scaling invariant, setting $w_1 = v_1$, $w_2 = -2v_1 - v_2$, and using (2.19), we have

\[ \|\chi_{j_1,j_3}^{k_1,k_2}(v_1, v_2, \xi) m(v_1 + \xi_1, v_2 + \xi_2, \xi)\|_{S^0_{v_1, v_2, L^\infty_\xi}} \lesssim \|\chi_{j_1,j_3}^{k_1,k_2}(w_1, -2w_1 - w_2, \xi) m(w_1 + \xi_1, -2w_1 - w_2 + \xi_2, \xi)\|_{S^0_{w_1, w_2, L^\infty_\xi}} \]
\[ \lesssim \|\chi_{j_1,j_3}^{k_1,k_2}(w_1, -2w_1 - w_2, \xi) m(w_1 + \xi_1, -2w_1 - w_2 + \xi_2, \xi)\|_{L^1_{w_1,w_2} L^\infty_\xi} \cdot \|\hat{\phi}_{w_1}(w_1, -2w_1 - w_2, \xi) m(w_1 + \xi_1, -2w_1 - w_2 + \xi_2, \xi)\|_{L^1_{w_1,w_2} L^\infty_\xi} \]
\[ \cdot \|\hat{\phi}_{w_1}(w_1, -2w_1 - w_2, \xi) m(w_1 + \xi_1, -2w_1 - w_2 + \xi_2, \xi)\|_{L^1_{w_1,w_2} L^\infty_\xi} \]
\[ \lesssim (1 + |j_1|) (2^j + k_1)^{1/2} (2^{-j_1} + k_1)^{1/2} (2^{-j_1} - k_1) (2^{j_1})^{1/2} \]
\[ = (1 + |j_1|) \cdot 2^{(j_1 + k_1)/2}, \]

where we have used the estimate

\[ \left| \frac{1}{\log |w_1 + \xi_1| - \log |w_1 + w_2|} \right| \lesssim 2^{-k_1}. \]
\[
\begin{align*}
\left| \chi_{j_1,j_3} \frac{\gamma^2}{w_1} \frac{1}{\log |w_1 + \xi| - \log |\xi + w_1 + w_2|} \right| & \leq 2^{3(j_1-k_1)j_2(2j_1+k_1)} = 2^{-j_1-k_1}, \\
\left| \chi_{j_1,j_3} \frac{\gamma^2}{w_1} \frac{1}{\log |w_1 + \xi| - \log |\xi + w_1 + w_2|} \right| & \leq 2^{5(j_1-k_1)2j_1(2j_1+k_1)} = 2^{-j_1-3k_1}.
\end{align*}
\]

Therefore, using (2.14), (3.1), and (8.15), we obtain
\[
\| V_1 \|_{L^\infty} \leq (1 + |j_1|)2^{(j_1+k_1)/2-j_1(t+1)} \\| \tilde{\phi}_{j_1}(v_1 + \xi_1)\psi_{j_3}(v_1) \|_{L^\infty} \| \tilde{\phi}_{j_2}(v_2 + \xi_2)\psi_{j_3}(v_2) \|_{L^\infty} \| \tilde{\sigma}_x \rho_{j_3} \|_{L^\infty}
\leq (1 + |j_1|)2^{0.5j_1+k_1+0.5k_2} \| \psi_{j_1}\tilde{\phi}_{j_1} \|_{L^\infty} \| \psi_{j_2}\tilde{\phi}_{j_2} \|_{L^\infty} \left\{ (t+1)^{-1.5} \| \xi \|_{L^2}^{j_1-k_1,k_2} \right\}
+ (t+1)^{-1.75} \left\{ \| \tilde{\sigma}_x \|_{L^2}^{3/4} P_{j_3}^{1/4} (x\tilde{\sigma}_x h) \right\}
\leq (1 + |j_1|)2^{(j_1+k_1)/2-1/4} \left( 2^{-(j_1+k_1)/2} \right)^{1/4} \left( 2^{-j_1+1/2} \right)^{1/4}
= (1 + |j_1|) \cdot 2^{1/2 j_1+k_1-k_2},
\]
where we have used the estimates
\[
\left| \chi_{j_1,j_3} \frac{k_1,k_2}{\log |v_1 + \xi| - \log |\xi - v_1 - v_2|} \right| \leq 2^{j_1-k_2},
\left| \chi_{j_1,j_3} \frac{k_1,k_2}{\log |v_1 + \xi| - \log |\xi - v_1 - v_2|} \right| \leq 2^{3(j_1-k_1)j_2(2j_1+k_1)} = 2^{-j_1-k_2},
\left| \chi_{j_1,j_3} \frac{k_1,k_2}{\log |v_1 + \xi| - \log |\xi - v_1 - v_2|} \right| \leq 2^{5(j_1-k_1)2j_1(2j_1+k_1)} = 2^{-j_1-3k_2}.
\]

Therefore, using (2.14), (3.1), and (8.15)
\[
\| V_1 \|_{L^\infty} \leq (1 + |j_1|)2^{(j_1+k_1)/2-j_1(1+0.5k_2)} \left( t+1 \right)^{-1} \| \tilde{\phi}_{j_1}(v_1 + \xi_1)\psi_{j_3}(v_1) \|_{L^\infty} \| \tilde{\phi}_{j_2}(v_2 + \xi_2)\psi_{j_3}(v_2) \|_{L^\infty} \| \tilde{\sigma}_x \rho_{j_3} \|_{L^\infty}
\leq (1 + |j_1|)2^{0.5j_1+k_1+0.5k_2} \| \psi_{j_1}\tilde{\phi}_{j_1} \|_{L^\infty} \| \psi_{j_2}\tilde{\phi}_{j_2} \|_{L^\infty} \left\{ (t+1)^{-1.5} \| \xi \|_{L^2}^{j_1-k_1,k_2} \right\}
+ (t+1)^{-1.75} \left\{ \| \tilde{\sigma}_x \|_{L^2}^{3/4} P_{j_3}^{1/4} (x\tilde{\sigma}_x h) \right\}
\leq (1 + |j_1|)2^{1.5j_1+0.5k_1}.
\]

Estimates of $V_2-V_4$. The estimates for $V_2-V_4$ are similar to $V_1$. We omit the details here. The resulting estimates are as follows.

(i) If $(\xi_1,\xi_2,\xi_3) = (\xi,\xi,\xi)$, the symbol can be estimated as
\[
\left| T_j \left( (\xi_1 + v_1, \xi_2 + v_2, \xi_3 - v_1 - v_2) \right) \right| \leq 2^{1.5j_1+0.5k_1}.
\]
(ii) If \((\xi_1, \xi_2, \xi_3) = (\xi, \xi, -\xi)\), the symbol can be estimated as

\[
\left\| \psi_{k_1, k_2}(v_1, v_2, \xi) \frac{\mathbf{T}_i^\dagger(\xi_1 + v_1, \xi_2 + v_2, \xi_3 - v_1 - v_2)}{\log |\xi_1 + v_1| - \log |\xi_3 - v_1 - v_2|} \right\|_{S_{\xi_1 = -2}^{\xi} L_{\xi}^b} \\
\lesssim \left(2^{j_1 + k_1} 2^{2j_1}/2 \right)^{1/4} \left(2^{-j_1 + k_1} 2^{2j_1}/2 \right)^{1/4} \left(2^{-j_1 - 2k_2 + k_1} 2^{2j_1}/2 \right)^{1/4} \\
= (1 + |j_1|) \cdot 2^{1.5j_1 + k_1 - 0.5k_2}.
\]

In either case, we have the following estimates

\[
\begin{align*}
\| V_2 \|_{L_{\xi}^b} &\lesssim (1 + |j_1|) 2^{-0.5j_1 + k_1} \| \eta_1 \partial_\eta_1 \dot{\hat{\varphi}}_{j_1}(\eta_1) \|_{L_{\eta_1}^{2}} \| \psi_{k_2} \dot{\hat{\varphi}}_{j_2} \|_{L_{\xi}^b} \\
&\cdot \left\{ (t + 1)^{-1.5} \| \xi \|_{\mathcal{H}_{j_1}} L_{\xi}^b + (t + 1)^{-1.75} \| \| \partial_\xi \|_{3/4} p_{j_1}(x \partial_\xi) h \|_{L^2} + \| \| \partial_\xi \|_{3/4} h_{j_3} \|_{L^2} \right\}, \\
\| V_3 \|_{L_{\xi}^b} &\lesssim (1 + |j_1|) 2^{-0.5j_1 + k_1} \| \eta_1 \partial_\eta_1 \dot{\hat{\varphi}}_{j_1}(\eta_1) \|_{L_{\eta_1}^{2}} \| \psi_{k_2} \dot{\hat{\varphi}}_{j_2} \|_{L_{\xi}^b} \\
&\cdot \left\{ (t + 1)^{-1.5} \| \xi \|_{3/2} \hat{h}_{j_1} \|_{L_{\xi}^b} + (t + 1)^{-1.75} \| \| \partial_\xi \|_{3/4} p_{j_1}(x \partial_\xi) h \|_{L^2} + \| \| \partial_\xi \|_{3/4} h_{j_3} \|_{L^2} \right\}, \\
\| V_4 \|_{L_{\xi}^b} &\lesssim (1 + |j_1|) 2^{-0.5j_1 + 0.5k_1} \| \psi_{k_1} \dot{\hat{\varphi}}_{j_1} \|_{L_{\xi}^b} \| \psi_{k_2} \dot{\hat{\varphi}}_{j_2} \|_{L_{\xi}^b} \\
&\cdot \left\{ (t + 1)^{-1.5} \| \xi \|_{3/2} \hat{h}_{j_1} \|_{L_{\xi}^b} + (t + 1)^{-1.75} \| \| \partial_\xi \|_{3/4} p_{j_1}(x \partial_\xi) h \|_{L^2} + \| \| \partial_\xi \|_{3/4} h_{j_3} \|_{L^2} \right\}.
\end{align*}
\]

Now we take the summation over \(\log_2(q(t))\) \(\leq k_1, k_2 \leq \max\{j_1, j_3\} + 1\), and combine the estimates (S.18)–(S.22) to get

\[
\| \xi |\xi| + |\xi|^{r+})d(\xi, t) \int_{\mathbb{R}^2} T_1^\dagger(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) e^{i\lambda t}(\xi, \eta_1, \eta_2) \hat{h}_{j_1}(\eta_1) \hat{h}_{j_2}(\eta_2) \hat{h}_{j_3}(\xi - \eta_1 - \eta_2) \\
\cdot \left[ \sum_{k_1 = \log_2(q(t))}^{\max\{j_1, j_3\} + 1} \psi_{k_1}(\eta_1 - \xi_1) \cdot \sum_{k_2 = \log_2(q(t))}^{\max\{j_2, j_3\} + 1} \psi_{k_2}(\eta_2 - \xi_2) \right] \sum_{\eta_1, \eta_2} d\eta_1 d\eta_2 \\
\lesssim (1 + |j_1|) \left[ \max\{j_1, j_3\} \log_2(q(t)) \right]^2 (t + 1)^{(r+4)p_1} \\
\cdot \left\{ \| |\xi| \|_{\mathcal{H}_{j_1}} L_{\xi}^b \| \| \psi_{k_1} \dot{\hat{\varphi}}_{j_1} \|_{L_{\xi}^b} \| \psi_{k_2} \dot{\hat{\varphi}}_{j_2} \|_{L_{\xi}^b} \| \eta_1 \partial_\eta_1 \dot{\hat{\varphi}}_{j_1}(\eta_1) \|_{L_{\eta_1}^{2}} \| \psi_{k_2} \dot{\hat{\varphi}}_{j_2} \|_{L_{\xi}^b} \| \eta_2 \partial_\eta_2 \dot{\hat{\varphi}}_{j_2}(\eta_2) \|_{L_{\eta_2}^{2}} \right\} \\
\cdot \left\{ (t + 1)^{-1.5} \| \xi \|_{3/2} \hat{h}_{j_1} \|_{L_{\xi}^b} + (t + 1)^{-1.75} \| \| \partial_\xi \|_{3/4} p_{j_1}(x \partial_\xi) h \|_{L^2} + \| \| \partial_\xi \|_{3/4} h_{j_3} \|_{L^2} \right\}.
\]

The right-hand-side is summable with respect to \(j_1, j_2, j_3\) under \(|j_3 - j_2| \leq 1\) and \(|j_3 - j_1| \leq 1\), since we can write

\[
\| |\xi| \|_{\mathcal{H}_{j_1}} L_{\xi}^b \lesssim (1_{j_1 < 2/3} + 1_{j_1 > 2/3}) \| h_{j_1} \|_{L^2},
\]

and the resulting sum is integrable for \(t \in (0, \infty)\).

### 8.5. Resonant frequencies.

In this section, we estimate (S.16) in the region

\[
|j_1 - j_3| \leq 1, \quad |j_2 - j_3| \leq 1, \quad k_1 < \log_2(q(t)), \quad k_2 < \log_2(q(t)),
\]

and then sum over \(k_1, k_2 < \log_2(q(t))\). After taking the sum, the cutoff function of the integrand is

\[
\mathbf{b}(\xi, \eta_1, \eta_2, t) := \psi \left( \frac{\eta_1 - \xi_1}{\vartheta(t)} \right) \cdot \psi \left( \frac{\eta_2 - \xi_2}{\vartheta(t)} \right).
\]

The support of this cutoff function is

\[
\left\{ (\eta_1, \eta_2) \in \mathbb{R}^2 \mid |\eta_1 - \xi_1| < \frac{8}{5} \vartheta(t), \ |\eta_2 - \xi_2| < \frac{8}{5} \vartheta(t) \right\},
\]
which can be written as the union of four disjoint sets $A_1 \cup A_2 \cup A_3 \cup A_4$, where

\[
A_1 = \left\{ (\eta_1, \eta_2) \mid |\eta_1 - \frac{\xi}{3}| < \frac{8}{5} \varphi(t), \quad |\eta_2 - \frac{\xi}{3}| < \frac{8}{5} \varphi(t) \right\},
\]

\[
A_2 = \left\{ (\eta_1, \eta_2) \mid |\eta_1 - \xi| < \frac{8}{5} \varphi(t), \quad |\eta_2 - \xi| < \frac{8}{5} \varphi(t) \right\},
\]

\[
A_3 = \left\{ (\eta_1, \eta_2) \mid |(\eta_1 - \xi)| < \frac{8}{5} \varphi(t), \quad |\eta_2 - (-\xi)| < \frac{8}{5} \varphi(t) \right\},
\]

\[
A_4 = \left\{ (\eta_1, \eta_2) \mid |\eta_1 - (-\xi)| < \frac{8}{5} \varphi(t), \quad |\eta_2 - \xi| < \frac{8}{5} \varphi(t) \right\}.
\]

We notice that the regions $A_1, A_2, A_3, A_4$ are discs centered at $(\xi/3, \xi/3), (\xi, \xi), (\xi, -\xi)$, and $(-\xi, \xi)$, respectively. The region $A_1$ corresponds to space resonances $\xi = \xi/3 + \xi/3 + \xi/3$, while $A_2, A_3, A_4$ correspond to space-time resonances $\xi = \xi + \xi - \xi$.

8.5.1. Space resonances. When $(\eta_1, \eta_2) \in A_1$, we can expand $T_1/\Phi$ around $(\xi, \xi/3, \xi/3)$ as

\[
\frac{T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{\Phi(\xi, \eta_1, \eta_2)} = \left( \frac{1}{2} - \frac{2 \log 2}{3 \log 3} \right) \xi + O\left( \frac{\xi^2}{3} + \frac{\eta_1 - \xi^2}{3} \right).
\]

(8.23)

After writing

\[
e^{i\tau \Phi(\xi, \eta_1, \eta_2)} = \frac{1}{i\Phi(\xi, \eta_1, \eta_2)} \left[ \partial_\tau e^{i\tau \Phi(\xi, \eta_1, \eta_2)} \right],
\]

and integrating by parts with respect to $\tau$, we get that

\[
\int_1^t \int_{\mathbb{R}^2} T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) e^{i\tau \Phi(\xi, \eta_1, \eta_2)} \hat{h}_{j_1}(\eta_1, \tau) \hat{h}_{j_2}(\eta_2, \tau) \hat{h}_{j_3}(\xi - \eta_1 - \eta_2, \tau) b(\xi, \eta_1, \eta_2, \tau) d\eta_1 d\eta_2 d\tau
\]

\[
= \int_1^t \int_{\mathbb{R}^2} \frac{T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{\Phi(\xi, \eta_1, \eta_2)} \partial_\tau e^{i\tau \Phi(\xi, \eta_1, \eta_2)} \hat{h}_{j_1}(\eta_1, \tau) \hat{h}_{j_2}(\eta_2, \tau) \hat{h}_{j_3}(\xi - \eta_1 - \eta_2, \tau) b(\xi, \eta_1, \eta_2, \tau) d\eta_1 d\eta_2 d\tau
\]

\[
= J_1 - \int_1^t J_2(\tau) + J_3(\tau) d\tau,
\]

where

\[
J_1 = \int_{\mathbb{R}^2} \int_1^t \frac{T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{\Phi(\xi, \eta_1, \eta_2)} \hat{h}_{j_1}(\eta_1, \tau) \hat{h}_{j_2}(\eta_2, \tau) \hat{h}_{j_3}(\xi - \eta_1 - \eta_2, \tau) e^{i\tau \Phi(\xi, \eta_1, \eta_2)} b(\xi, \eta_1, \eta_2, \tau) d\eta_1 d\eta_2 |^{\tau = t}_{\tau = 1},
\]

\[
J_2(\tau) = \int_{\mathbb{R}^2} \frac{T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{\Phi(\xi, \eta_1, \eta_2)} e^{i\tau \Phi(\xi, \eta_1, \eta_2)} \partial_\tau \hat{h}_{j_1}(\eta_1, \tau) \hat{h}_{j_2}(\eta_2, \tau) \hat{h}_{j_3}(\xi - \eta_1 - \eta_2, \tau) b(\xi, \eta_1, \eta_2, \tau) d\eta_1 d\eta_2,
\]

\[
J_3(\tau) = \int_{\mathbb{R}^2} \partial_\tau b(\xi, \eta_1, \eta_2, \tau) \frac{T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{\Phi(\xi, \eta_1, \eta_2)} e^{i\tau \Phi(\xi, \eta_1, \eta_2)} \hat{h}_{j_1}(\eta_1, \tau) \hat{h}_{j_2}(\eta_2, \tau) \hat{h}_{j_3}(\xi - \eta_1 - \eta_2, \tau) d\eta_1 d\eta_2.
For $J_1$, for any $\tau \geq 1$, we have from (8.23) that

$$
\left| \left| \left( |\xi| + |\xi|^{r-4} \right) \int_{\mathbb{R}^2} b(\xi, \eta_1, \eta_2, t) \xi \Phi(\xi, \eta_1, \eta_2) \hat{h}_{jj}(\eta_1, \eta_2) \hat{h}_{jj}(\xi - \eta_1 - \eta_2) e^{i\tau \Phi(\xi, \eta_1, \eta_2)} \, d\eta_1 \, d\eta_2 \right| \right|
\leq \left| \left| \left( |\xi| + |\xi|^{r-4} \right) \int_{\mathbb{R}^2} b(\xi, \eta_1, \eta_2, t) \xi \Phi(\xi, \eta_1, \eta_2) \hat{h}_{jj}(\eta_1, \eta_2) \hat{h}_{jj}(\xi - \eta_1 - \eta_2) e^{i\tau \Phi(\xi, \eta_1, \eta_2)} \, d\eta_1 \, d\eta_2 \right| \right| + \left( |\xi| + |\xi|^{r-4} \right) \int_{\mathbb{R}^2} |b(\xi, \eta_1, \eta_2, t)| \left( |\xi| \hat{h}_{jj}(\eta_1, \eta_2) \hat{h}_{jj}(\xi - \eta_1 - \eta_2) \right) \, d\eta_1 \, d\eta_2
\leq (\tau + 1)^{2p_0+(r+3)p_1} \left( \left| \left| \left| \hat{h}_{jj}(\eta_1, \eta_2) \hat{h}_{jj}(\xi - \eta_1 - \eta_2) \right| \right| \right| L^\infty_{\xi} \left( \left| \left| \left| \hat{h}_{jj}(\eta_1, \eta_2) \right| \right| \right| L^\infty_{\xi} \left( \left| \left| \left| \hat{h}_{jj}(\xi - \eta_1 - \eta_2) \right| \right| \right| L^\infty_{\xi} \right) \right) + \left( \left| \left| \left| \hat{h}_{jj}(\eta_1, \eta_2) \hat{h}_{jj}(\xi - \eta_1 - \eta_2) \right| \right| \right| L^\infty_{\xi} \right) \right).$$

Notice that in $A_1$, the number of summations of $j_1$, $j_2$, and $j_3$ is or order $\log(t+1)$, and therefore, the right-hand-side is bounded for $\tau \geq 1$ after the summation in $j_1$, $j_2$, and $j_3$.

After taking the time derivative, the term $J_2$ can be written as a sum of three terms.

$$
\xi \int_{\mathbb{R}^2} \frac{T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{\Phi(\xi, \eta_1, \eta_2)} e^{i\tau \Phi(\xi, \eta_1, \eta_2)} \left[ \partial_x \hat{h}_{jj}(\eta_1, \eta_2) \hat{h}_{jj}(\xi - \eta_1 - \eta_2) \right] b(\xi, \eta_1, \eta_2, \tau) \, d\eta_1 \, d\eta_2,
\xi \int_{\mathbb{R}^2} \frac{T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{\Phi(\xi, \eta_1, \eta_2)} e^{i\tau \Phi(\xi, \eta_1, \eta_2)} \left[ \partial_x \hat{h}_{jj}(\eta_1, \eta_2) \hat{h}_{jj}(\xi - \eta_1 - \eta_2) \right] b(\xi, \eta_1, \eta_2, \tau) \, d\eta_1 \, d\eta_2,
\xi \int_{\mathbb{R}^2} \frac{T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2)}{\Phi(\xi, \eta_1, \eta_2)} e^{i\tau \Phi(\xi, \eta_1, \eta_2)} \left[ \partial_x \hat{h}_{jj}(\eta_1, \eta_2) \hat{h}_{jj}(\xi - \eta_1 - \eta_2) \right] b(\xi, \eta_1, \eta_2, \tau) \, d\eta_1 \, d\eta_2.
$$

Notice that by (8.5), and the bootstrap assumptions and Lemma 5.3, we have

$$
\left\| \partial_x \hat{h} \right\|_{L^\infty_{\xi}} \leq \left\| \xi \int_{\mathbb{R}^2} T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) e^{i\tau \Phi(\xi, \eta_1, \eta_2)} \hat{h}_{jj}(\xi - \eta_1 - \eta_2) \hat{h}(\eta_1) \hat{h}(\eta_2) \, d\eta_1 \, d\eta_2 \right\|_{L^\infty_{\xi}} + \left\| \hat{N}_{g_5}(\varphi) \right\|_{L^\infty_{\xi}} \leq \left\| \partial_x \right\|_{L^1} \left( \left\| \varphi^2 \log |\varphi| \varphi_{xx} - \varphi \log |\varphi| \varphi_{xx} \right\|_{L^1} + \left\| \hat{N}_{g_5}(\varphi) \right\|_{L^1} \right) \leq \varepsilon_1^2(t+1)^{2p_0-\frac{r}{2}}.
$$

Therefore, we obtain

$$
\left| \left| \left( |\xi| + |\xi|^{r-4} \right) J_2(\tau) \right| \right| \leq \sum_{\ell_1, \ell_2, \ell_3} \left\| \partial_{\ell_1} \hat{h}_{\ell_2} \hat{h}_{\ell_3} \right\|_{L^\infty_{\xi}} \left\| \hat{h}_{\ell_1} \right\|_{L^\infty_{\xi}} \left\| \hat{h}_{\ell_2} \hat{h}_{\ell_3} \right\|_{L^2_{\xi}} \left( |\varphi(t)|^2 \right) \leq \varepsilon_1^2(t+1)^{p_0-\frac{r}{2}} \left( |\varphi(t)|^2 \right) \sum \left\| \hat{h}_{\ell_1} \right\|_{L^\infty_{\xi}} \left\| \hat{h}_{\ell_2} \hat{h}_{\ell_3} \right\|_{L^2_{\xi}},
$$

where we sum over all permutations $(\ell_1, \ell_2, \ell_3)$ of $(j_1, j_2, j_3)$ in the space resonance region $A_1$. Again, we notice that the number of summations is of order $\log(t+1)$, and the resulting sum is integrable for $\tau \in (1, \infty)$.

As for the term $J_3$, by the definition of the cutoff function, we have

$$
\left\| \partial_x \left[ \psi_{g_k}(\varphi(t)) \left( |\eta| \left| |\eta| - |\eta_1| - |\eta_2| \right| \cdot \psi_{g_k}(\varphi(t)) \left( |\eta| \left| |\eta| - |\eta_1| - |\eta_2| \right| \right) \right] \right\|_{L^\infty_{\xi}} \leq \varepsilon(t+1)^{-1} \leq \frac{1}{t+1}.
$$

The area of its support is of the order of $|\varphi(t)|^2$. Then, using (8.23), we get that

$$
\left| \left| \left( |\xi| + |\xi|^{r-4} \right) J_3(\tau) \right| \right| \leq (t+1)^{2p_0+(r+3)p_1-1} \left( |\varphi(t)|^2 \right) \sum \left\| \hat{h}_{\ell_1} \right\|_{L^\infty_{\xi}} \left\| \hat{h}_{\ell_2} \hat{h}_{\ell_3} \right\|_{L^2_{\xi}} \left\| \hat{h}_{\ell_4} \right\|_{L^\infty_{\xi}},
$$

where we sum over all permutations $(\ell_1, \ell_2, \ell_3, \ell_4)$ of $(j_1, j_2, j_3, j_4)$ in the space resonance region $A_1$.
where the summation is taken over permutations \((\ell_1, \ell_2, \ell_3)\) of \((j_1, j_2, j_3)\), so the sum converges and is integrable for \(\tau \in (1, \infty)\).

### 8.5.2. Space-time resonances

We now use modified scattering to consider the term
\[
\frac{1}{6} \int_{A_2 \cup A_3 \cup A_4} i \xi b(\xi, \eta_1, \eta_2, t) T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) \tilde{\varphi}_{j_1}(\eta_1) \tilde{\varphi}_{j_2}(\eta_2) \tilde{\varphi}_{j_3}(\xi - \eta_1 - \eta_2) \, d\eta_1 \, d\eta_2 \\
- i \xi [\beta_1(t) T_1(\xi, \xi, -\xi) + \beta_2(t) T_1(\xi, -\xi, \xi) + \beta_3(t) T_1(-\xi, -\xi, 0)] |\varphi(\tau, \xi)|^2 \varphi(\tau, \xi).
\]

For \(A_2\), we take
\[
\beta_1(t) = \frac{1}{6} \int_{A_2} b(\xi, \eta_1, \eta_2, t) \, d\eta_1 \, d\eta_2.
\]

Therefore, using a Taylor expansion and \((8.3)\), we obtain
\[
\left| (|\xi| + |\xi|^{r+4}) \frac{1}{6} i \xi \int_{A_2} b(\xi, \eta_1, \eta_2, t) \right| \\
\left[ T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) \tilde{\varphi}_{j_1}(\eta_1) \tilde{\varphi}_{j_2}(\eta_2) \tilde{\varphi}_{j_3}(\xi - \eta_1 - \eta_2) - T_1(\xi, \xi, -\xi) |\varphi(\tau, \xi)|^2 \varphi(\tau, \xi) \right] \, d\eta_1 \, d\eta_2 \\
\lesssim (|\xi| + |\xi|^{r+4}) (i \xi) \int_{A_2} \partial_{\eta_1} \left[ T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) \tilde{\varphi}_{j_1}(\eta_1) \tilde{\varphi}_{j_2}(\eta_2) \tilde{\varphi}_{j_3}(\xi - \eta_1 - \eta_2) \right] \bigg|_{\eta_1 = \eta_1'} (\xi - \eta_1) \\
+ \partial_{\eta_2} \left[ T_1(\eta_1, \eta_2, \xi - \eta_1 - \eta_2) \tilde{\varphi}_{j_1}(\eta_1) \tilde{\varphi}_{j_2}(\eta_2) \tilde{\varphi}_{j_3}(\xi - \eta_1 - \eta_2) \right] \bigg|_{\eta_2 = \eta_2'} (\xi - \eta_2) \, d\eta_1 \, d\eta_2 \\
\lesssim (t + 1)^{(r+3)p_1} |\xi| \tilde{\varphi}_{j_1} \|_{L^p} |\xi| \tilde{\varphi}_{j_2} \|_{L^p} |\xi| \tilde{\varphi}_{j_3} \|_{L^p} \| \varphi(t) \|_3 + \sum \| \xi \tilde{\varphi}_{\ell_1} \|_{L^p} \| \xi \tilde{\varphi}_{\ell_2} \|_{L^p} \| S \varphi_{\ell_0} \|_{H^r} \| \varphi(t) \|^{5/2},
\]

where \(\eta_1'\) (or \(\eta_2'\)) in the first inequality is some number between \(\xi\) and \(\eta_1\) (or \(\eta_2\)), and the summation in the second inequality is over permutations \((\ell_1, \ell_2, \ell_3)\) of \((j_1, j_2, j_3)\). The estimates for \(A_3\) and \(A_4\) follow by a similar argument.

Taking a summation over \(j_1, j_2, j_3\) and using the estimates in the above subsections together with the time-decay of \(\varphi(t)\) in \((8.17)\), we conclude that
\[
\int_0^\infty \| (|\xi| + |\xi|^{r+4}) U_t \|_{L^p} \, dt \lesssim \varepsilon_0.
\]

### 8.6. Higher-degree terms

In this subsection, we prove that
\[
\left\| (|\xi| + |\xi|^{r+4}) \mathcal{N}_{\geq 5}(\varphi) \right\|_{L^p}
\]
is integrable in time. We begin by proving an estimate for the symbol \(T_n\). We have
\[
\mathcal{F}^{-1} \left[ T_n(\eta_1, \eta_2, \ldots, \eta_{2n+1}) \psi_{j_1}(\eta_1) \psi_{j_2}(\eta_2) \cdots \psi_{j_{2n+1}}(\eta_{2n+1}) \right] \\
= \int_{\mathbb{R}^{2n+1}} e^{i\eta_1 y_1 + i\eta_2 y_2 + \cdots + i\eta_{2n+1} y_{2n+1}} \left[ \int_{\mathbb{R}} \prod_{j=1}^{2n+1} \left( 1 - e^{i\eta_j \zeta} \right) \, d\eta_j \right] \psi_{j_1}(\eta_1) \psi_{j_2}(\eta_2) \cdots \psi_{j_{2n+1}}(\eta_{2n+1}) \, d\eta_n
\]
\[
= \int_{\mathbb{R}^{2n+1}} \left[ \int_{\mathbb{R}} \left( e^{i\eta_1 y_1 - i\eta_1 (\zeta + y_1)} \cdots e^{i\eta_{2n+1} y_{2n+1} - i\eta_{2n+1} (\zeta + y_{2n+1})} \right) \, d\zeta \right] \psi_{j_1}(\eta_1) \psi_{j_2}(\eta_2) \cdots \psi_{j_{2n+1}}(\eta_{2n+1}) \, d\eta_n
\]
coefficients of the monomials (A.2) are zero. Let 1
\[ \alpha \]
where
\[ A \]
A general term in the expansion of left-hand-side of (A.1) is proportional to
\[ F \]
Proof.
This completes the proof of Theorem 5.1.
\[ \ell \]
We first prove an algebraic identity that will be used in deriving (3.6). Therefore, by Lemma 2.5, we have
\[ F \]
\[ \alpha \]
\[ \eta \]
\[ \phi \]
\[ \eta \]
\[ \phi \]
Therefore, by Lemma 2.5, we have
\[ \alpha \]
\[ \eta \]
\[ \phi \]
\[ \eta \]
Using the dispersive estimate Lemma 5.3 we see that the right-hand-side is integrable in \( t \), which leads to
\[ \alpha \]
This completes the proof of Theorem 5.1.

**Appendix A. Alternative formulation of the SQG front equation**

We first prove an algebraic identity that will be used in deriving (3.6).

**Lemma A.1.** Let \( N \geq 2 \) be an integer. Then for any integer \( 1 \leq p \leq N - 1 \) and any \( \eta \in \mathbb{R} \), \( j = 1, 2, \ldots, N \)
\[ \sum_{\ell=1}^{N} \sum_{1 \leq m_1 < m_2 < \cdots < m_\ell \leq N} (-1)^\ell (\eta m_1 + \eta m_2 + \cdots + \eta m_\ell)^p = 0. \]  
(A.1)

**Proof.** A general term in the expansion of left-hand-side of (A.1) is proportional to
\[ \eta_{\alpha_1}^{\alpha_2} \cdots \eta_{\alpha_N}^{\alpha_N} \]
where \( \alpha_1, \alpha_2, \ldots, \alpha_N \) are nonnegative integers such that \( \alpha_1 + \alpha_2 + \cdots + \alpha_N = p \). It suffices to show that the coefficients of the monomials (A.2) are zero. Let \( 1 \leq M \leq N - 1 \) denote the number of nonzero terms in the list \( \alpha_1, \alpha_2, \ldots, \alpha_N \). Using the multinomial theorem, we see that the coefficient of (A.2) is
\[ \left( \frac{p}{\alpha_1, \ldots, \alpha_N} \right) \sum_{j=0}^{N-M} (-1)^{M+j} \binom{N-M}{j} (-1)^M (1 - 1)^{N-M} = 0. \]  
\[ \square \]
To compute $T_n(\eta_n)$ in (3.3), we first expand the product
\[
\Re \prod_{j=1}^{2n+1} (1 - e^{i\eta_n \zeta}) = 1 + \sum_{\ell=1}^{2n+1} \sum_{1 \leq m_1 < m_2 < \cdots < m_\ell \leq 2n+1} (-1)^\ell \cos \left( (\eta_{m_1} + \eta_{m_2} + \cdots + \eta_{m_\ell}) \zeta \right)
\]
\[
= \sum_{\ell=1}^{2n+1} \sum_{1 \leq m_1 < m_2 < \cdots < m_\ell \leq 2n+1} (-1)^{\ell+1} \left[ 1 - \cos \left( (\eta_{m_1} + \eta_{m_2} + \cdots + \eta_{m_\ell}) \zeta \right) \right].
\]

We replace the integral over $\Re$ in (3.3) by an integral over $\Re \setminus (-\epsilon, \epsilon)$, where $\epsilon < 1$, and decompose the expression for $T_n$ into a sum of terms of the form
\[
\int_{|\epsilon|<|\zeta|<\infty} \frac{1 - \cos(\eta \zeta)}{|\zeta|^{2n+1}} \, d\zeta = \int_{|\epsilon|<|\zeta|<1/|\eta|} \frac{1 + \sum_{j=1}^{n} \frac{(-1)^j (\eta \zeta)^{2j}}{(2j)!} - \cos(\eta \zeta)}{|\zeta|^{2n+1}} \, d\zeta + \int_{|\zeta|>1/|\eta|} \frac{1 - \cos(\eta \zeta)}{|\zeta|^{2n+1}} \, d\zeta
\]
\[
- \sum_{j=1}^{n} \frac{(-1)^j \eta^{2j}}{(2j)!} \int_{|\epsilon|<|\zeta|<1/|\eta|} \frac{1}{|\zeta|^{2n-2j+1}} \, d\zeta
\]
\[
= C_{n,1} \eta^{2n} - \sum_{j=1}^{n} \frac{(-1)^j \eta^{2j}}{(2j)!} \int_{|\epsilon|<|\zeta|<1/|\eta|} \frac{1}{|\zeta|^{2n-2j+1}} \, d\zeta + o(1),
\]
where
\[
C_{n,1} = \int_{\theta \leq 1} (1 + \sum_{j=1}^{n} \frac{(-1)^j (\theta)^{2j}}{(2j)!} - \cos(\theta)) \frac{1}{|\theta|^{2n+1}} \, d\theta + \int_{|\theta|>1} \frac{1 - \cos(\theta)}{|\theta|^{2n+1}} \, d\theta
\]
is some constant that depends only on $n$.

We have
\[
\sum_{j=1}^{n} \frac{(-1)^j \eta^{2j}}{(2j)!} \int_{|\epsilon|<|\zeta|<1/|\eta|} \frac{1}{|\zeta|^{2n-2j+1}} \, d\zeta = C_{n,2} \eta^{2n} + \sum_{j=1}^{n-1} C_{n,3} \eta^{2j} + C_{n,4} \eta^{2n} \log |\eta|,
\]
where
\[
C_{n,2} = \sum_{j=1}^{n-1} \frac{(-1)^j + 1}{(n-j)(2j)!} + 2 \frac{(-1)^{n+1} \log \epsilon}{(2n)!}, \quad C_{n,3} = \frac{(-1)^j \epsilon^{2j-2n}}{(n-j)(2j)!}, \quad C_{n,4} = 2 \frac{(-1)^{n+1}}{(2n)!}.
\]

Thus, we conclude that
\[
\int_{|\epsilon|<|\zeta|<1/|\eta|} \frac{1 - \cos(\eta \zeta)}{|\zeta|^{2n+1}} \, d\zeta = (C_{n,1} - C_{n,2}) \eta^{2n} - \sum_{j=1}^{n-1} C_{n,3} \eta^{2j} - C_{n,4} \eta^{2n} \log |\eta|.
\]

We use these results in the expression for $T_n$ and take the limit as $\epsilon \to 0^+$. The singularity at $\epsilon = 0$ does not enter into the final result because of the cancelation in Lemma A.1, and we find that
\[
T_n(\eta_n) = 2 \frac{(-1)^{n+1}}{(2n)!} \sum_{\ell=1}^{2n+1} \sum_{1 \leq m_1 < m_2 < \cdots < m_\ell \leq 2n+1} (-1)^\ell (\eta_{m_1} + \cdots + \eta_{m_\ell})^{2n} \log |\eta_{m_1} + \eta_{m_2} + \cdots + \eta_{m_\ell}|. \quad (A.3)
\]

It follows that
\[
f_n = 2 \frac{(-1)^n}{(2n)!} \sum_{\ell=1}^{2n+1} \binom{2n+1}{\ell} (-1)^\ell \varphi^{2n-\ell+1} \epsilon^{2n} \log |\varphi| (\varphi^\ell).
\]
Therefore, we conclude that
\[
\int_{\mathbb{R}} \left[ \frac{\varphi_{x}(x,t) - \varphi_{x}(x + \zeta, t)}{\zeta} - \frac{\varphi_{x}(x, t) - \varphi_{x}(x + \zeta, t)}{\sqrt{\zeta^2 + (\varphi(x, t) - \varphi(x + \zeta, t))^2}} \right] \, d\zeta
\]
\[
= - \sum_{n=1}^{\infty} 2n \frac{(-1)^{n}}{1 + 2n} \hat{\partial}_{x} \left\{ \frac{2n + 1}{2} \left( \frac{2n + 1}{\ell} \right) (-1)^{\ell} \varphi^{2n-\ell+1} \varphi'(x, t) \right\} \log |\varphi'(x, t)|
\]
\[
= \sum_{n=1}^{\infty} \sum_{\ell=1}^{2n+1} (-1)^{\ell+1} d_{n, \ell} \hat{\partial}_{x} \left\{ \varphi^{2n-\ell+1} \varphi'(x, t) \right\},
\]
where
\[
d_{n, \ell} = \frac{2\sqrt{n}}{\Gamma \left( \frac{1}{2} - n \right) \Gamma(\ell + 1) \Gamma(2n + 2 - \ell) \Gamma(n + 1)} > 0. \quad (A.4)
\]
Using this expansion in \(3.1\), we get \(3.6\).

**Appendix B. Para-differential calculus**

In this appendix, we use the Weyl calculus \[44\] to prove some estimates for Weyl paraproducts.

### B.1. Weyl operators

The Weyl quantization of a symbol \(a: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}\) is the operator \(a^{w}\) defined by
\[
(a^{w} f)(x) = \frac{1}{2\pi} \int_{\mathbb{R}^{2}} e^{i(x-y)\xi} a \left( \frac{x+y}{2}, \xi \right) f(y) \, dy \, d\xi = \int_{\mathbb{R}} F^{-1}_{2} a \left( \frac{x+y}{2}, x-y \right) f(y) \, dy,
\]
where \(F_{1} a\) denotes the Fourier transform of \(a(x_{1}, x_{2})\) with respect to with the \(i\)-th variable \((i = 1, 2)\). The Fourier transform of \(a^{w} f\) can be written as
\[
\mathcal{F}(a^{w} f)(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^{2}} e^{i(x-y)\eta - i\xi \eta} a \left( \frac{x+y}{2}, \eta \right) f(y) \, dy \, d\eta \, dx = \int_{\mathbb{R}} F_{1}^{-1} a \left( \xi - \eta, \frac{x+y}{2} \right) \hat{f}(\eta) \, d\eta. \quad (B.1)
\]

For \(m \in \mathbb{R}\), we have the symbol class
\[
\mathcal{S}_{1,0}^{m} = \left\{ a(x, \xi) \in C^{m} (\mathbb{R} \times \mathbb{R}) \mid \sup_{\xi \in \mathbb{R}} ||\partial_{\xi}^{\alpha} \partial_{x}^{\beta} a(\cdot, \xi)||_{L^{\infty}} \leq C_{\alpha, \beta} (1 + |\xi|)^{m-|\alpha|}, \ \forall \alpha, \beta \in \mathbb{N}_{0} \right\}.
\]

For integers \(r_{1}, r_{2} \geq 0\), we define a symbol norm by
\[
M_{r_{1}, r_{2}}^{m} (a) = \max_{0 \leq \alpha \leq r_{2}} \sup_{\xi} \left\| (1 + |\xi|)^{m-\alpha} \partial_{\xi}^{\alpha} a(\cdot, \xi) \right\|_{W_{r_{1}, \infty}}^{r_{1}},
\]
and introduce a class of symbols with finite regularity
\[
\Gamma_{r_{1}, r_{2}}^{m} = \left\{ a: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C} \mid M_{r_{1}, r_{2}}^{m} (a) < \infty \right\}.
\]

We note that if \(M_{r_{1}, r_{2}}\) is the symbol class defined in \(2.2\), then
\[
\left\| (1 + |\xi|)^{-m} a(x, \xi) \right\|_{M_{r_{1}, r_{2}}} \approx M_{r_{1}, r_{2}}^{m} (a).
\]

In particular, \(M_{r_{1}, r_{2}} \approx \Gamma_{r_{1}, r_{2}}^{0}\).
B.2. Para-differential operators. Recall from Section 2 that $\chi : \mathbb{R} \to \mathbb{R}$ is a smooth function supported in the interval $\{ \xi \in \mathbb{R} \mid |\xi| \leq 1/10 \}$ and equal to 1 on $\{ \xi \in \mathbb{R} \mid |\xi| \leq 3/40 \}$. If $f : \mathbb{R} \to \mathbb{C}$ and $a : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is a symbol, then the Weyl paraproduct $T_a f$ in (2.2) is defined by

$$
\mathcal{F} [T_a f] (\xi) = \int_\mathbb{R} \chi \left( \frac{|\xi - \eta|^2}{1 + |\xi + \eta|^2} \right) \hat{a} (\xi - \eta, \frac{\xi + \eta}{2}) \hat{f} (\eta) \, d\eta.
$$

Introducing the notation

$$
\sigma_a (\cdot, \zeta_2) = \mathcal{F}_1 \left[ \chi \left( \frac{|\zeta_1|^2}{1 + 4|\zeta_2|^2} \right) \hat{a} (\zeta_1, \zeta_2) \right],
$$

we can also write

$$
\mathcal{F} [T_a f] (\xi) = \int_\mathbb{R} \mathcal{F}^{-1} \sigma_a (\xi - \eta, \frac{\xi + \eta}{2}) \hat{f} (\eta) \, d\eta.
$$

Comparing this result with (B.1), we see that $T_a = \sigma_a^w$.

Lemma B.1. If $a \in \Gamma_{r_1, r_2}^m$, then $\sigma_a \in \Gamma_{r_1, r_2}^m$ and $M_{r_1, r_2}^m (\sigma_a) \lesssim M_{r_1, r_2}^m (a)$.

Proof. To prove that $\sigma_a \in \Gamma_{r_1, r_2}^m$, we write

$$
\begin{align*}
\partial_{\zeta_2}^\alpha \partial_x^\beta \sigma_a (x, \zeta_2) &= \sum_{i_1 + i_2 = \alpha} c_{i_1, i_2, \alpha} \mathcal{F}_{\zeta_1} \left[ \partial_{\zeta_1}^{i_1} \chi \left( \frac{|\zeta_1|^2}{1 + 4|\zeta_2|^2} \right) \partial_{\zeta_2}^{i_2} \sigma_x (\zeta_1, \zeta_2) \right],
\end{align*}
$$

where the $c_{i_1, i_2, \alpha}$ are multinomial coefficients.

For each term, by Young’s inequality,

$$
\begin{align*}
&\left| (1 + |\zeta_2|)^\alpha \mathcal{F}_{\zeta_1} \left[ \partial_{\zeta_1}^{i_1} \chi \left( \frac{|\zeta_1|^2}{1 + 4|\zeta_2|^2} \right) \partial_{\zeta_2}^{i_2} \sigma_x (\zeta_1, \zeta_2) \right] \right| \\
&= \left| \mathcal{F}_{\zeta_1} \left[ (1 + |\zeta_2|)^{i_1} \partial_{\zeta_1}^{i_1} \chi \left( \frac{|\zeta_1|^2}{1 + 4|\zeta_2|^2} \right) \right] \ast \left[ (1 + |\zeta_2|)^{i_2} \partial_{\zeta_2}^{i_2} \sigma_x (-x, \zeta_2) \right] \right| \\
&\lesssim \left\| \mathcal{F}_{\zeta_1} \left[ (1 + |\zeta_2|)^{i_1} \partial_{\zeta_1}^{i_1} \chi \left( \frac{|\zeta_1|^2}{1 + 4|\zeta_2|^2} \right) \right] \right\| \left( 1 + |\zeta_2| \right)^{i_2} \left\| \partial_{\zeta_2}^{i_2} \sigma_x (x, \zeta_2) \right\|_{L_x^\infty}.
\end{align*}
$$

Using the Faà di Bruno’s formula, a general term of $(1 + |\zeta_2|)^{i_1} \partial_{\zeta_2}^{i_1} \chi \left( \frac{|\zeta_1|^2}{1 + 4|\zeta_2|^2} \right)$ is a linear combination of the terms of the form

$$
(1 + |\zeta_2|)^{i_1} \chi^{m_1 + \cdots + m_i} \left( \frac{|\zeta_1|^2}{1 + 4|\zeta_2|^2} \right) \prod_{\ell=1}^{i_1} \left[ \partial_{\zeta_2}^{\ell} \left( \frac{|\zeta_1|^2}{1 + 4|\zeta_2|^2} \right) \right]^{m_\ell},
$$

where $m_\ell \in \mathbb{N}_0$ satisfies $\sum_{\ell=1}^{i_1} \ell m_\ell = i_1$.

Bernstein’s inequality implies that

$$
\left\| \mathcal{F}_{\zeta_1} \left[ (1 + |\zeta_2|)^{i_1} \partial_{\zeta_2}^{i_1} \chi \left( \frac{|\zeta_1|^2}{1 + 4|\zeta_2|^2} \right) \right] \right\|_{L_x^\infty} \lesssim \left\| \int_\mathbb{R} e^{-ix\zeta_1} \left( 1 + 4|\zeta_2|^2 \right)^{\frac{i_1}{2}} \partial_{\zeta_2}^{i_1} \chi \left( \frac{|\zeta_1|^2}{1 + 4|\zeta_2|^2} \right) \, d\zeta_1 \right\|_{L_x^\infty} \lesssim 1,
$$

since the middle term in this inequality is supported on the set $\{ (\zeta_1, \zeta_2) \mid |\zeta_1| \leq \sqrt{1 + 4|\zeta_2|^2} \}$. Therefore, we have that

$$
\sum_{\alpha \leq r_2, \beta \leq r_1} \left\| (1 + |\zeta_2|)^\alpha \partial_{\zeta_2}^\beta \sigma_x (x, \zeta_2) \right\|_{L_x^\infty} \lesssim \sum_{\alpha \leq r_2, \beta \leq r_1} \sum_{\Gamma_{r_2, r_1}^m} \left( 1 + |\zeta_2| \right)^{i_1} \partial_{\zeta_2}^{i_1} \sigma_x (-x, \zeta_2) \right\|_{L_x^\infty} \lesssim (1 + |\zeta_2|)^m,
$$

so $M_{r_1, r_2}^m (\sigma_a) \lesssim M_{r_1, r_2}^m (a)$, which completes the proof. □
B.4. $H^s$ estimates. The next theorem follows from Theorem 1.2 in Boukhemair [3].

Theorem B.2. Let $m \in \mathbb{R}$. If $a \in \Gamma_{1,1}^m$, then the Weyl operator $a^w : H^s(\mathbb{R}) \to H^{s-m}(\mathbb{R})$ with symbol $a$ is bounded and its operator-norm is bounded by $M_{1,1}^m(a)$.\[ \|T_a f\|_{H^{s-m}} \lesssim M_{1,1}^m(a) \|f\|_{H^s}. \]

Using Lemma 2.1 and the fact that $T_a = \sigma_a^w$, we then get the following estimate for Weyl paraproducts.

Theorem B.3. If $a \in \Gamma_{1,1}^m$, then the Weyl paraproduct operator $T_a : H^s(\mathbb{R}) \to H^{s-m}(\mathbb{R})$ is bounded for all $m, s \in \mathbb{R}$, and \[ \|T_a f\|_{H^{s-m}} \lesssim M_{1,1}^m(a) \|f\|_{H^s}. \]

In particular, setting $m = 0$ and using the fact that $M_{1,1}^0(a) \approx \|a\|_{\mathcal{M}_{1,1}}$, we get Lemma 2.1.

B.4. $L^\infty$-to-$L^2$ estimates. We also need some estimates in which we bound $\|T_a f\|_{L^2}$ by $\|f\|_{L^\infty}$.

Theorem B.4. Let $p(\xi) = |\xi|^k, k \geq 0$ or $p(\xi) = |\xi|^k \log |\xi|, k \geq 1$. Assume $f \in L^\infty(\mathbb{R})$ with $p(\partial_x)\partial_x f \in L^\infty(\mathbb{R})$, and $a(x, \xi)$ is a function such that $\|a\|_{L^\infty} < \infty$, where \[ \|a\|_{L^\infty} := \sup_{\xi} \|a(\cdot, \xi)\|_{L^2} + \|\partial_x a(\cdot, \xi)\|_{L^2}. \]

Then we have \[ \|p(\partial_x)T_a f\|_{L^2} \lesssim (\|f\|_{L^\infty} + \|p(\partial_x)\partial_x f\|_{L^\infty}) \|a\|_{L^\infty}. \]

Proof. Recall that \[ T_a f(x) = \sigma_a^w f(x) = \int_{\mathbb{R}} F_2^{-1} \sigma_a \left( \frac{x+y}{2}, x-y \right) f(y) \, dy = \int_{\mathbb{R}} \chi \left( \frac{|\xi - \eta|}{1 + |\xi + \eta|^2} \right) \hat{a} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f}(\eta) \, d\eta. \]

We split $T_a f$ into a low-frequency part \[ \int_{\mathbb{R}} F_2^{-1} \sigma_a \left( \frac{x+y}{2}, x-y \right) [i(\partial_y) f(y)] \, dy = \int_{\mathbb{R}} \chi \left( \frac{|\xi - \eta|}{1 + |\xi + \eta|^2} \right) \hat{a} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f}(\eta) \, d\eta, \]

and a high-frequency part \[ \int_{\mathbb{R}} F_2^{-1} \sigma_a \left( \frac{x+y}{2}, x-y \right) [(1 - i(\partial_y)) f(y)] \, dy = \int_{\mathbb{R}} (1 - \chi(\eta)) \chi \left( \frac{|\xi - \eta|}{1 + |\xi + \eta|^2} \right) \hat{a} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f}(\eta) \, d\eta. \]

Here, the cutoff function $\chi$ is the same as the one defined in the proof of Lemma 2.2.

The integrand in the low-frequency part is supported in $|\xi| < 6, |\eta| < 2$. Thus, $|\xi + \eta| < 10$ and $|\xi - \eta| < 10$ on its support, so we can put a cut-off function $\chi(\frac{\xi + \eta}{5})\chi(\frac{\xi - \eta}{5})$ into the integral without changing its value: \[ \int_{\mathbb{R}} F_2^{-1} \sigma_a \left( \frac{x+y}{2}, x-y \right) [i(\partial_y) f(y)] \, dy = \int_{\mathbb{R}} \chi(\eta) \chi \left( \frac{|\xi - \eta|}{1 + |\xi + \eta|^2} \right) \hat{a} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f}(\eta) \, d\eta. \]

Therefore, defining $b(x, \xi) = i(\partial_x)\chi(\eta)2(\xi/5)a(x, \xi)$, we have \[ \int_{\mathbb{R}} F_2^{-1} \sigma_a \left( \frac{x+y}{2}, x-y \right) [i(\partial_y) f(y)] \, dy = \int_{\mathbb{R}} \chi(\eta) \chi \left( \frac{|\xi - \eta|}{1 + |\xi + \eta|^2} \right) \hat{b} \left( \xi - \eta, \frac{\xi + \eta}{2} \right) \hat{f}(\eta) \, d\eta \]

So we obtain \[ \|p(\partial_x) \int_{\mathbb{R}} F_2^{-1} \sigma_b \left( \frac{x+y}{2}, x-y \right) [i(\partial_y) f(y)] \, dy \|_{L^2} \lesssim \|\int_{\mathbb{R}} F_2^{-1} \sigma_b \left( \frac{x+y}{2}, x-y \right) [i(\partial_y) f(y)] \, dy \|_{L^2}. \]
\[ \| \sigma_b \|_{L^2_\nu} \leq \left\| (i \partial_y f)(y) \right\|_{L^2_\nu} \left\| F_{-1}^{\sigma_b} \left( \frac{x+y}{2}, x-y \right) \right\|_{L^1_\nu L^2_\nu} \]

By the H"older inequality and a change of coordinates,

\[ \left\| \sigma_b \|_{L^2_\nu} \right\| = \left\| F_{-1}^{\sigma_b} \left( \frac{|\zeta_1|^2}{1 + 4|\xi|^2} \right) b(\zeta_1, \xi) \right\|_{L^1_\nu L^2_\nu} \]

where the last term satisfies

\[ \left\| F_{-1}^{\sigma_b} \left( \frac{x-z}{2}, z \right) \right\|_{L^1_\nu L^2_\nu} \]

For the high frequency part, we make a dyadic decomposition of \( f \), after which we mainly need to estimate

\[ \int_{\mathbb{R}} F_{-1}^{\sigma_a} \left( \frac{x+y}{2}, x-y \right) \left[ (1 - \iota (i \partial_y f_k)(i \partial_y f) \right] dy = \int_{\mathbb{R}} \left( 1 - \iota (\eta) \chi \left( \frac{|\xi-\eta|^2}{1 + |\xi+\eta|^2} \right) \right) A(\chi - \eta, \xi) \right] \psi_k(\eta) d\eta. \]

When \( |\eta| > 2 \), we have

\[ \frac{1}{18} |\eta| \leq |\xi| \leq \frac{35}{18} |\eta|, \quad \frac{1}{2} |\eta| \leq |\xi + \eta| \leq \frac{40}{9} |\eta| \]

on the support of the cut-off function \( \chi \left( \frac{|\xi-\eta|^2}{1 + |\xi+\eta|^2} \right) \). Therefore \( |\eta| \approx |\xi + \eta| \approx |\xi| \approx 2^k \) on the support, and, since \( |\eta| > 2 \), we only need to consider \( k \geq 0 \).

By the H"older inequality and a change of coordinates,

\[ \left\| \sigma_a \|_{L^2_\nu} \right\| = \left\| F_{-1}^{\sigma_b} \left( \frac{x+y}{2}, x-y \right) \left[ (1 - \iota (i \partial_y f_k)(i \partial_y f) \right] dy \right\|_{L^2_\nu} \]

and

\[ \left\| \sigma_b \|_{L^2_\nu} \right\| = \left\| F_{-1}^{\sigma_b} \left( \frac{|\zeta_1|^2}{1 + 4|\xi|^2} \right) b(\zeta_1, \xi) \right\|_{L^1_\nu L^2_\nu} \]

The last term satisfies

\[ \left\| F_{-1}^{\sigma_b} \left( \frac{|\zeta_1|^2}{1 + 4|\xi|^2} \right) b(\zeta_1, \xi) \psi_k(\xi) \right\|_{L^1_\nu L^2_\nu} \]

where the last term satisfies

\[ \left\| F_{-1}^{\sigma_b} \left( \frac{x-z}{2}, z \right) \right\|_{L^1_\nu L^2_\nu} \]

For the high frequency part, we make a dyadic decomposition of \( f \), after which we mainly need to estimate

\[ \int_{\mathbb{R}} F_{-1}^{\sigma_a} \left( \frac{x+y}{2}, x-y \right) \left[ (1 - \iota (i \partial_y f_k)(i \partial_y f) \right] dy = \int_{\mathbb{R}} \left( 1 - \iota (\eta) \chi \left( \frac{|\xi-\eta|^2}{1 + |\xi+\eta|^2} \right) \right) A(\chi - \eta, \xi) \right] \psi_k(\eta) d\eta. \]
Theorem B.6. Let
\[ \sum \text{inequalities over } t \]
where
\[ \text{Theorem B.5.} \]

The following theorem is from [44] (see Theorem 2.3.7).

Theorem B.5 (Composition). Let \( a_1 \in S^{m_1}_{1,0} \) and \( a_2 \in S^{m_2}_{1,0} \). Then
\[ a_1 \# b(x, \xi) = \int_{\mathbb{R}^d} e^{-i y \cdot a(x, \xi)} b(y + x, \xi) \, dy \, d\eta. \]

The following theorem is from [44] (see Theorem 2.3.7).

Theorem B.5 (Composition). Let \( a_1 \in S^{m_1}_{1,0} \) and \( a_2 \in S^{m_2}_{1,0} \). Then
\[ a_1 \# b(x, \xi) = \int_{\mathbb{R}^d} e^{-i y \cdot a(x, \xi)} b(y + x, \xi) \, dy \, d\eta. \]

Theorem B.6. Let \( a \in \Gamma^{m_1}_{3,3}, b \in \Gamma^{m_2}_{3,3}, \) and \( f \in H^{s}(\mathbb{R}) \). Then
\[ T_a T_b f = T_{ab} f + \frac{1}{24} T_{[a,b]} f + \mathcal{R}, \]
where \( \{a, b\} = \partial_x a \partial_x b - \partial_x b \partial_x a \) is the Poisson bracket.

Using Theorem B.3 we therefore obtain the following estimate.

Theorem B.6. Let
\[ \partial_x \int_{\mathbb{R}} F^{-1}_\eta \mathcal{A}_\eta \left( \frac{x + y}{2}, x - y \right) [1 - \nu(\iota \partial_y)] f(y) \, dy \right|_{L^2_x} \lesssim \| p(\partial_x f) \|_{L^2_x} \| a \|_{L^2_x}. \]
The theorem then follows by combining the low and high frequency estimates. \( \square \)

B.5. Composition. Finally, we state a commutator estimate for Weyl paraproducts. The composition of two symbols \( a \) and \( b \) is defined by
\[ a \# b(x, \xi) = \int_{\mathbb{R}^d} e^{-i y \cdot a(x, \xi)} b(y + x, \xi) \, dy \, d\eta. \]

The following theorem is from [44] (see Theorem 2.3.7).

Theorem B.5 (Composition). Let \( a_1 \in S^{m_1}_{1,0} \) and \( a_2 \in S^{m_2}_{1,0} \). Then
\[ a_1 \# a_2 = a_1 a_2 - \frac{1}{24} [a_1, a_2] \in S^{m_1+m_2-2}_{1,0}, \]
where \( \{a_1, a_2\} = \partial_x a_1 \partial_x a_2 - \partial_x a_2 \partial_x a_1 \) is the Poisson bracket.

Using Theorem B.3 we therefore obtain the following estimate.

Theorem B.6. Let \( a \in \Gamma^{m_1}_{3,3}, b \in \Gamma^{m_2}_{3,3}, \) and \( f \in H^{s}(\mathbb{R}) \). Then
\[ T_a T_b f = T_{ab} f + \frac{1}{24} T_{[a,b]} f + \mathcal{R}, \]
where \( \{a, b\} = \partial_x a \partial_x b - \partial_x b \partial_x a \) is the Poisson bracket of \( a \) and \( b \), and the remainder \( \mathcal{R} \) satisfies
\[ \| \mathcal{R} \|_{H^{s-(m_1+m_2-2)}} \lesssim M^{m_1}_{k,3}(a) M^{m_2}_{k,3}(b) \| f \|_{H^{s}}. \]

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