Solutions for the quasi-linear elliptic problems involving the critical Sobolev exponent

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Abstract
In this article, we study the existence and multiplicity of positive solutions for the quasi-linear elliptic problems involving critical Sobolev exponent and a Hardy term. The main tools adopted in our proofs are the concentration compactness principle and Nehari manifold.

Keywords: quasi-linear elliptic problems; Nehari manifold; positive solution; best Sobolev constant

1 Introduction
In this article, we consider the following quasi-linear elliptic problem:

\begin{align}
-\Delta_p u - \mu \frac{|u|^{p-2} u}{|x|^p} = & \ |u|^{p^*-2} u + \beta |x|^{q-p} |u|^{p-2} u + \lambda |x|^{q-2} u \quad \text{in } \Omega, \\
\quad & u = 0 \quad \text{on } \partial \Omega,
\end{align}

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with the smooth boundary $\partial \Omega$ such that $0 \in \Omega$. $-\Delta_p u = \text{div}(\nabla |u|^{p-2} \nabla u)$ is the $p$-Laplace operator of $u$, $1 < p < N$, $\lambda > 0$ is a positive real number, $0 \leq \mu < \mu^* (\mu^* = \frac{(N-p)p}{p})$ is the best Hardy constant, $1 < q < p$ and $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent, $0 < \alpha < p - 1$, $0 < \beta < \beta_1$ ($\beta_1$ is the first eigenvalue that $-\Delta_p u - \mu \frac{|u|^{p-2} u}{|x|^p} = |x|^{p^*-2} |u|^{p-2} u$ under Dirichlet boundary condition).

Definition 1.1 The function $u \in W_0^{1,p}(\Omega)$ is called a weak solution of (1.1) if $u$ satisfies

\begin{align}
\int_{\Omega} \left( |\nabla u|^{p-2} \nabla u \cdot \nabla v - \mu \frac{|u|^{p-2} u v}{|x|^p} \right) dx \\
= \int_{\Omega} \left( |u|^{p^*-2} u v + \beta |x|^{q-p} |u|^{p-2} u v + \lambda |x|^{q-2} u v \right) dx
\end{align}

for all $v \in W_0^{1,p}(\Omega)$.

In this paper, we use the following norm of $W_0^{1,p}(\Omega)$:

$$
\|u\| = \left( \int_{\Omega} \left( |\nabla u|^{p} - \mu \frac{|u|^{p}}{|x|^p} \right) dx \right)^{\frac{1}{p}}.
$$

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By the Hardy inequality (see [1, 2])

$$\int_{\Omega} \frac{|u|^p}{|x|^p} \, dx \leq \frac{1}{\mu} \int_{\Omega} |\nabla u|^p \, dx, \quad \forall u \in W_0^{1,p}(\Omega),$$

so this norm is equivalent to $\left( \int_{\Omega} |\nabla u|^p \, dx \right)^{\frac{1}{p}}$, the usual norm in $W_0^{1,p}(\Omega)$.

The norm in $L^p(\Omega)$ is represented by $\|u\|_p = \left( \int_{\Omega} |u|^p \, dx \right)^{\frac{1}{p}}$. According to Hardy inequality, the following best Sobolev constant is well defined for $1 < p < N$, and $0 \leq \mu < \frac{N}{p}$:

$$S_{\mu,0} = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{f_{\Omega}(\nabla u|^p - \mu \frac{|u|^p}{|x|^p}) \, dx}{\left( \int_{\Omega} |u|^p \, dx \right)^{\frac{1}{p}}}.$$

The quasi-linear problems on Hardy inequality have been studied extensively, either in the smooth bounded domain or in the whole space $\mathbb{R}^N$. More and more excellent results have been obtained, which provide us opportunities to understand the singular problems. However, compared with the semilinear case, the quasi-linear problems related to Hardy inequality are more complicated [3–16]. Abdellaoui, Felli and Peral [3] considered the extremal function which achieves the best constant $S_{\mu,0}$, and gave the properties of the extremal functions. The conclusions obtained in [3] can be applied in the problems with critical Sobolev exponent and Hardy term.

Wang, Wei and Kang [10] investigated the following problem:

$$\begin{cases}
-\Delta_p u - \lambda \frac{|u|^{p-2}u}{|x|^p} = \mu f(x)|u|^{p-2}u + g(x)|u|^{p^*-2}u, & x \in \Omega, \\
u(x) = 0, & x \in \partial\Omega,
\end{cases}$$

(1.4)

where $1 < q < p$, $\mu > 0$, $f$ and $g$ are non-negative functions and $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent. The property of the Nehari manifold was used to prove the existence of multiple positive solutions for (1.4). Furthermore, Hsu [11, 12] improved and complemented the main results obtained in [10]. Recently, Goyal and Sreenadh [13] investigated a class of singular $N$-Laplacian problems with exponential nonlinearities in $\mathbb{R}^N$. Very recently, Xiang [14] established the asymptotic estimates of weak solutions for $p$-Laplacian equation with Hardy term and critical Sobolev exponent. We should mention that Liu, Guo and Lei [17] studied the existence and multiplicity of positive solutions of Kirchhoff equation with critical exponential nonlinearity. Inspired by [17, 18], we study the problem (1.1) on critical Sobolev exponent. Comparing with the main results obtained in [4, 6, 10–12], in this paper, on the one hand, we will analysis the effect of $\beta |x|^{q-p} |u|^{p-2} u$, and the more careful estimates are needed. On the other hand, we establish an lower bound for $\lambda_*$ ($\lambda_*$ is defined in Theorem 1.1).

Define the energy functional associated to problem (1.1) as follows:

$$I_\lambda(u) = \frac{1}{p} \|u\|^p - \frac{\beta}{p} \int_{\Omega} |u|^p |x|^{q-p} \, dx - \frac{1}{p^*} \int_{\Omega} |u|^{p^*} \, dx - \frac{\lambda}{q} \int_{\Omega} |u|^q \, dx.$$  

(1.5)

We obtain the following result.

**Theorem 1.1** Suppose that $1 < q < p$, $0 < \alpha < p-1$. Then there exists $\lambda_* > 0$ such that problem (1.1) admits at least two solutions and one of the solutions is a ground state solution for all $\lambda \in (0, \lambda_*)$. 


2 Preliminaries

Firstly, we introduce the Nehari manifold

\[ \mathcal{N}_\lambda = \{ u \in W_0^{1,p}(\Omega) \setminus \{0\} : J'_\lambda(u), u) = 0 \}. \]

Furthermore \( u \in \mathcal{N}_\lambda \) if and only if

\[ \| u \|^p - \int_\Omega |u|^p \, dx - \beta \int_\Omega |u|^q \, dx - \lambda \int_\Omega |u|^q \, dx = 0. \]  \hfill (2.1)

Let

\[ \psi(u) := \| u \|^p - \beta \int_\Omega |u|^q \, dx - \beta \int_\Omega |u|^{q-p} \, dx - \beta \int_\Omega |u|^q \, dx, \]

then

\[ \langle \psi'(u), u \rangle = p \| u \|^p - p\beta \int_\Omega |u|^p \, dx - p\beta \int_\Omega |u|^{q-p} \, dx - q \int_\Omega u^q \, dx. \]

\( \mathcal{N}_\lambda \) can be divided into the following three parts:

\[ \mathcal{N}_\lambda^* = \left\{ u \in \mathcal{N}_\lambda : p \| u \|^p - p\beta \int_\Omega |u|^{q-p} \, dx > 0 \right\}, \]

\[ \mathcal{N}_\lambda^0 = \left\{ u \in \mathcal{N}_\lambda : p \| u \|^p - p\beta \int_\Omega |u|^{q-p} \, dx = 0 \right\}, \]

\[ \mathcal{N}_\lambda^- = \left\{ u \in \mathcal{N}_\lambda : p \| u \|^p - p\beta \int_\Omega |u|^{q-p} \, dx < 0 \right\}. \]

Applying the Hölder inequality and the Sobolev inequality, for all \( u \in W_0^{1,p}(\Omega) \setminus \{0\} \) we have

\[ \int_\Omega |u|^q \, dx \leq \left( \int_\Omega |u|^\frac{q}{r} \, dx \right)^\frac{r}{q} \left( \int_\Omega 1 \, dx \right)^{1 - \frac{r}{q}} = |\Omega|^{\frac{p-q}{p}} \left( \int_\Omega |u|^p \, dx \right)^\frac{q}{p}. \]  \hfill (2.5)

Lemma 2.1 Assume that \( \lambda \in (0, T_1) \) with

\[ T_1 = \frac{(\beta-\beta_p)|\Omega|^{\frac{p}{q}}}{\beta_p |\Omega|^{\frac{p}{q}}} \left( \frac{p-1}{p} \right)^\frac{p-1}{p} \left( \frac{p}{p-p} \right)^\frac{p}{p} \left( \frac{p}{p-p} \right)^\frac{p}{p} \left( \frac{p}{p-p} \right)^\frac{p}{p} \left( \frac{p}{p-p} \right)^\frac{p}{p}. \]

Then (i) \( \mathcal{N}_\lambda^+ \neq \emptyset \), and (ii) \( \mathcal{N}_\lambda^- = \emptyset \).
Proof (i) We define a function $\Phi \in C(\mathbb{R}^+, \mathbb{R})$ by

$$
\Phi(s) = \left(1 - \frac{\beta}{\beta_1}\right)s^{p-p^*}\|u\|^p - \lambda s^{q-p^*} \int_{\Omega} |u|^q \, dx.
$$

Let $\Phi'(s) = 0$, that is,

$$
\Phi'(s) = \left(1 - \frac{\beta}{\beta_1}\right)(p-p^*)s^{p-p^*-1}\|u\|^p - \lambda (q-p^*)s^{q-p^*-1} \int_{\Omega} |u|^q \, dx = 0.
$$

We can deduce that

$$
s_{\text{max}} := s = \left[\frac{\lambda \beta_1 (q-p^*)}{\beta_1 (q-p^*) + \lambda \int_{\Omega} |u|^q \, dx}\right]^{\frac{p-p^*}{q-p^*}}.
$$

It is easy to check that $\Phi'(s) > 0$ for all $0 < s < s_{\text{max}}$ and $\Phi'(s) < 0$ for all $s > s_{\text{max}}$. Consequently, $\Phi(s)$ attains its maximum at $s_{\text{max}}$, that is,

$$
\Phi(s_{\text{max}}) = \left(1 - \frac{\beta}{\beta_1}\right)\left[\frac{\lambda \beta_1 (q-p^*)}{\beta_1 (q-p^*) + \lambda \int_{\Omega} |u|^q \, dx}\right]^{\frac{p-p^*}{q-p^*}} \|u\|^p
$$

$$
- \lambda \left[\frac{\lambda \beta_1 (q-p^*)}{\beta_1 (q-p^*) + \lambda \int_{\Omega} |u|^q \, dx}\right]^{\frac{p-p^*}{q-p^*}} \int_{\Omega} |u|^q \, dx
$$

$$
= \left(\frac{\beta_1 - \beta}{\beta_1 (q-p^*)}\right)^{\frac{p-p^*}{q-p^*}} \left(\frac{q-p}{p-p^*}\right) \|u\| \frac{\|u\|^{q-p^*}}{(\lambda \int_{\Omega} |u|^q \, dx)^{\frac{q-p^*}{q-p^*}}}.
$$

Since

$$
\tilde{\Phi}(s) := s^{p-p^*}\|u\|^p - \beta s^{q-p^*} \int_{\Omega} |u|^q \, dx - \lambda s^{q-p^*} \int_{\Omega} |u|^q \, dx
$$

$$
\geq s^{p-p^*} \left(1 - \frac{\beta}{\beta_1}\right)\|u\|^p - \lambda s^{q-p^*} \int_{\Omega} |u|^q \, dx.
$$

By (1.3) and (2.5), we have

$$
\tilde{\Phi}(s_{\text{max}}) - \int_{\Omega} |u|^{p^*} \, dx
$$

$$
\geq \Phi(s_{\text{max}}) - \int_{\Omega} |u|^{p^*} \, dx
$$

$$
= \left(\frac{\beta_1 - \beta}{\beta_1 (q-p^*)}\right)^{\frac{p-p^*}{q-p^*}} \left(\frac{q-p}{p-p^*}\right) \|u\| \frac{\|u\|^{q-p^*}}{(\lambda \int_{\Omega} |u|^q \, dx)^{\frac{q-p^*}{q-p^*}}} - \int_{\Omega} |u|^{p^*} \, dx
$$

$$
> \left(\frac{\beta_1 - \beta}{\beta_1 (q-p^*)}\right)^{\frac{p-p^*}{q-p^*}} \left(\frac{q-p}{p-p^*}\right) \frac{\|u\|^{q-p^*}}{(\lambda \int_{\Omega} |u|^q \, dx)^{\frac{q-p^*}{q-p^*}}} - \int_{\Omega} |u|^{p^*} \, dx
$$

$$
= \left\{\left(\frac{\beta_1 - \beta}{\beta_1 (q-p^*)}\right)^{\frac{p-p^*}{q-p^*}} \left(\frac{q-p}{p-p^*}\right) \frac{1}{(\lambda \int_{\Omega} |u|^q \, dx)^{\frac{q-p^*}{q-p^*}}} - 1\right\}.
$$
\[
\times \int_{\Omega} |u|^{p^*} \, dx \\
\geq \left\{ \left( \frac{\beta_1 - \beta}{\beta_1 (q - p^*)} \right)^{\frac{p^* - p}{p}} \left( \frac{q - p}{p - p^*} \right) \frac{1}{[\lambda, \Omega]^{\frac{p - q}{p}}} S_{\mu, 0}^{\frac{p^*}{p}} - 1 \right\} \int_{\Omega} |u|^{p^*} \, dx \\
> 0,
\]

where \(0 < \lambda < T_1\). Thus, there exist constants \(s^+\) and \(s^-\) such that

\[
0 < s^+ = \lambda < s^- = s^-(u), \quad s^+ u \in \mathcal{N}_{\lambda}^+ \text{ and } s^- u \in \mathcal{N}_\lambda^-.
\]

(ii) We prove that \(\mathcal{N}_{\lambda}^0 = \emptyset\) for all \(\lambda \in (0, T_1)\). By contradiction, assume that there exists \(u_0 \neq 0\) such that \(u_0 \in \mathcal{N}_{\lambda}^0\). From (2.1), we have

\[
\|u_0\|^p - \int_{\Omega} |u_0|^{p^*} \, dx - \beta \int_{\Omega} |u_0|^p |x| ^{\alpha - p} \, dx - \lambda \int_{\Omega} |u_0|^q \, dx = 0,
\]

combining with (2.3), we obtain

\[
(p - p^*) \|u_0\|^p = (p - p^*) \beta \int_{\Omega} |u_0|^p |x| ^{\alpha - p} \, dx + (p^* - q) \beta \int_{\Omega} |u_0|^q \, dx.
\]

Equations (2.6) and (2.7) imply that

\[
(p - q) \|u_0\|^p - (p - q) \beta \int_{\Omega} |u_0|^p |x| ^{\alpha - p} \, dx = (p^* - q) \beta \int_{\Omega} |u_0|^q \, dx,
\]

that is,

\[
\int_{\Omega} |u_0|^p \, dx \geq \frac{p - q}{p^* - q} \left( 1 - \frac{\beta}{\beta_1} \right) \|u_0\|^p.
\]

Similarly,

\[
(p - p^*) \|u_0\|^p - (p - p^*) \beta \int_{\Omega} |u_0|^p |x| ^{\alpha - p} \, dx = \lambda (q - p^*) \beta \int_{\Omega} |u_0|^q \, dx,
\]

that is,

\[
\lambda \int_{\Omega} |u_0|^q \, dx \geq \frac{p - p^*}{q - p^*} \left( 1 - \frac{\beta}{\beta_1} \right) \|u_0\|^p.
\]

Note that (1.3) holds for \(u \in \mathcal{N}_{\lambda}^0 \setminus \{0\}\). Then

\[
\Theta := \left( \frac{\Omega^{\frac{p - q}{p}}}{\lambda^{\frac{p - q}{p}}} \right) \frac{\|u_0\|^{p^* \alpha - p}}{S_{\mu, 0}^{\frac{p^*}{p}}} \left( \frac{1}{\int_{\Omega} (u_0)^q \, dx} \right)^{\frac{p^*}{p}} - \int_{\Omega} |u_0|^{p^*} \, dx
\]

\[
\geq \left[ \frac{1}{S_{\mu, 0}^{\frac{p^*}{p}}} \left( \frac{\|u_0\|^p}{\int_{\Omega} |u_0|^{p^*} \, dx} \right)^{\frac{p^*}{p}} - 1 \right] \int_{\Omega} |u_0|^{p^*} \, dx \geq 0.
\]
It follows from (2.8) and (2.9) that

\[
\Theta = \frac{(||\Omega||_{\mathcal{S}_{\mu,0}})^{\frac{q_p^*}{q_p} - \frac{q_p}{p}}}{} \varpi^{\frac{q_p}{p}} \frac{\|u_0\|^{\frac{p-q}{p-q}}}{(\lambda \int_\Omega (u_0)^q dx)^{\frac{p-q}{p}} - \int_\Omega |u_0|^{p^*} dx}
\]

\[
\leq \frac{(||\Omega||_{\mathcal{S}_{\mu,0}})^{\frac{q_p^*}{q_p} - \frac{q_p}{p}}}{} \varpi^{\frac{q_p}{p}} \frac{\|u_0\|^{p}}{(\frac{p-q}{p-q}) (1 - \frac{p}{p_1})} - \left( \frac{p-q}{p-q} \left( 1 - \frac{\beta}{\beta_1} \right) \right) \|u_0\|^{p^*}
\]

\[
= \left[ \frac{(||\Omega||_{\mathcal{S}_{\mu,0}})^{\frac{q_p^*}{q_p} - \frac{q_p}{p}}}{} \varpi^{\frac{q_p}{p}} \frac{\|u_0\|^{p}}{(\frac{p-q}{p-q}) (1 - \frac{p}{p_1})} - \left( \frac{p-q}{p-q} \left( 1 - \frac{\beta}{\beta_1} \right) \right) \right] \|u_0\|^{p^*}
\]

< 0,

for 0 < \lambda < \lambda_1. This is a contradiction. \(\square\)

**Lemma 2.2** \(I_\lambda\) is coercive and bounded below on \(N_\lambda\).

**Proof** For \(u \in N_\lambda\), we can deduce from (1.3) and (2.5) that

\[
I_\lambda(u) = \frac{1}{p} |u|^p - \frac{\beta}{p} \int_\Omega |u|^p |x|^{\alpha - p} \, dx - \frac{1}{p^*} \int_\Omega |u|^{p^*} \, dx - \frac{\lambda}{q} \int_\Omega |u|^q \, dx
\]

\[
= \left( \frac{1}{p} - \frac{1}{p^*} \right) |u|^p - \frac{1}{p} \int_\Omega |u|^p |x|^{\alpha - p} \, dx - \frac{1}{p^*} \int_\Omega |u|^{p^*} \, dx - \frac{1}{q} \int_\Omega |u|^q \, dx
\]

\[
\geq \left( \frac{1}{p} - \frac{1}{p^*} \right) \left( 1 - \frac{\beta}{\beta_1} \right) |u|^p - \lambda \left( \frac{1}{q} - \frac{1}{p^*} \right) |\Omega| \varpi^{\frac{q_p^*}{q_p}} \mathcal{S}_{\mu,0} \|u\|^{q^*}
\]

Note that 1 < \(q < p\) and 0 < \(\beta < \beta_1\), we see that \(I_\lambda\) is coercive and bounded below on \(N_\lambda\). \(\square\)

From Lemma 2.1, we know that \(N_\lambda^+\) and \(N_\lambda^-\) are nonempty. Furthermore, taking into account Lemma 2.2, we define

\[
\kappa_\lambda = \inf_{u \in N_\lambda^+} I_\lambda(u), \quad \kappa_\lambda^* = \inf_{u \in N_\lambda^-} I_\lambda(u), \quad \kappa_\lambda^- = \inf_{u \in N_\lambda^+} I_\lambda(u).
\]

**Lemma 2.3** \(\kappa_\lambda \leq \kappa_\lambda^* < 0\).

**Proof** For \(u \in N_\lambda^+\), using (2.1) and (2.2), we have

\[
(p-q) |u|^p - (p-q) \beta \int_\Omega |u|^p |x|^{\alpha - p} \, dx > (p^* - q) \int_\Omega |u|^{p^*} \, dx
\]

and

\[
(p-q) |u|^p \left( 1 - \frac{\beta}{\beta_1} \right) > (p^* - q) \int_\Omega |u|^{p^*} \, dx,
\]

that is,

\[
\int_\Omega |u|^{p^*} \, dx < \left( \frac{p-q}{p^* - q} \left( 1 - \frac{\beta}{\beta_1} \right) \right) \|u\|^p.
\]

(2.10)
By (2.10), we get
\[
I_\lambda(u) = \left( \frac{1}{p} - \frac{1}{q} \right) \|u\|^p - \left( \frac{1}{p} - \frac{1}{q} \right) \beta \int_\Omega |u|^p |x|^{\alpha - p} \, dx - \left( \frac{1}{p^*} - \frac{1}{q} \right) \int_\Omega |u|^p \, dx
\]
\[
< \left( \frac{1}{p} - \frac{1}{q} \right) \left( 1 - \frac{\beta}{\beta_1} \right) \|u\|^p - \left( \frac{1}{p} - \frac{1}{q} \right) \left( \frac{p - q}{p^* - q} \right) \beta \int_\Omega |u|^p \, dx
\]
\[
= \left( 1 - \frac{\beta}{\beta_1} \right) (q - p) \left( \frac{1}{qp} - \frac{1}{qp^*} \right) \|u\|^p
\]
\[
< 0.
\]
Therefore, we have \( \kappa_\lambda \leq \kappa_\lambda^* < 0 \).

**Lemma 2.4** For \( u \in \mathcal{N}_\lambda \), there exist \( \varepsilon > 0 \) and a differentiable function \( \tilde{f} = \tilde{f}(\omega) : B(0, \varepsilon) \subset W_0^{1,p}(\Omega) \rightarrow \mathbb{R}^+ \) such that
\[
\tilde{f}(0) = 1, \quad \tilde{f}(\omega)(u + \omega) \in \mathcal{N}_\lambda, \quad \forall \omega \in B(0, \varepsilon).
\]

**Proof** Define
\[
\tilde{F} : \mathbb{R} \times W_0^{1,p}(\Omega) \rightarrow \mathbb{R}
\]
as follows:
\[
\tilde{F}(s, \omega) = s^{p-q} \int_\Omega \left( |\nabla (u + \omega)|^p - \mu \frac{|u + \omega|^p}{|x|^p} \right) \, dx - s^{p-q} \beta \int_\Omega |u + \omega|^p |x|^{\alpha - p} \, dx
\]
\[
- s^{p-q} \int_\Omega |u + \omega|^p \, dx + \lambda \int_\Omega |u + \omega|^q \, dx, \quad u \in \mathcal{N}_\lambda.
\]
It is clear that
\[
\tilde{F}(1, 0) = \int_\Omega \left( |\nabla u|^p - \mu \frac{|u|^p}{|x|^p} \right) \, dx - \beta \int_\Omega |u|^p |x|^{\alpha - p} \, dx - \int_\Omega |u|^p \, dx - \lambda \int_\Omega |u|^q \, dx
\]
and
\[
\tilde{F}_\lambda(s, \omega) = (p - q)s^{p-q-1} \int_\Omega \left( |\nabla (u + \omega)|^p - \mu \frac{|u + \omega|^p}{|x|^p} \right) \, dx
\]
\[
- (p - q)s^{p-q-1} \beta \int_\Omega |u + \omega|^p |x|^{\alpha - p} \, dx
\]
\[
- (p^* - q)s^{p-q-1} \int_\Omega |u + \omega|^p \, dx,
\]
which implies that
\[
\tilde{F}_\lambda(1, 0) = (p - q) \int_\Omega \left( |\nabla u|^p - \mu \frac{|u|^p}{|x|^p} \right) \, dx - (p - q) \beta \int_\Omega |u|^p |x|^{\alpha - p} \, dx
\]
\[
- (p^* - q) \int_\Omega |u|^p \, dx.
\]
Lemma 2.1 tells us that $\hat{f}(1,0) \neq 0$. Thus, by the implicit function theorem at the point $(0,1)$, there exist $\varepsilon > 0$, and a differentiable function

$$\hat{f}: B(0,\varepsilon) \subset W^{1,p}_0(\Omega) \rightarrow \mathbb{R}^+$$

such that

$$\hat{f}(0) = 1, \quad \hat{f}(\omega) > 0 \quad \text{and} \quad \hat{f}(\omega)(u + \omega) \in N_\lambda, \quad \forall \omega \in B(0,\varepsilon).$$

Lemma 2.5 For $u \in N_\lambda$, there exist $\varepsilon > 0$ and a differentiable function $\tilde{f} = \tilde{f}(v): B(0,\varepsilon) \subset W^{1,p}_0(\Omega) \rightarrow \mathbb{R}^+$ such that

$$\tilde{f}(0) = 1 \quad \text{and} \quad \tilde{f}(v)(u + v) \in N_\lambda - \lambda, \quad \forall v \in B(0,\varepsilon).$$

Proof The proof is similar to that of Lemma 2.4, and we omit it here.

Lemma 2.6 If $\{u_n\} \subset N_\lambda$ is a minimizing sequence of $I_\lambda$, for every $\phi \in W^{1,p}_0(\Omega)$, then

$$-\frac{|f_n'(0)||u_n| + \|\phi\|}{n} \leq \langle I_\lambda'(u_n), \phi \rangle \leq \frac{|f_n'(0)||u_n| + \|\phi\|}{n}. \quad (2.11)$$

Proof It follows from Lemma 2.2 that $I_\lambda$ is coercive on $N_\lambda$. Using the Ekeland variational principle [19], we can find a minimizing sequence $\{u_n\} \subset N_\lambda$ of $I_\lambda$ satisfying

$$I_\lambda(u_n) < \kappa_\lambda + \frac{1}{n}, \quad I_\lambda(u_n) \leq I_\lambda(w) + \frac{1}{n}\|w - u_n\| \quad \forall w \in N_\lambda. \quad (2.12)$$

Without loss of generality, we can assume that $u_n \geq 0$. By Lemma 2.2, we know that $\{u_n\}$ is bounded in $W^{1,p}_0(\Omega)$. As a consequence, there exist a subsequence (still denoted by $\{u_n\}$) and $u_* \in W^{1,p}_0(\Omega)$ such that

$$\begin{cases}
u_n \rightharpoonup u_* \quad \text{weakly in } W^{1,p}_0(\Omega), \\
u_n \rightarrow u_* \quad \text{strongly in } L^p(\Omega) \quad (1 \leq p < p^*), \\
u_n(x) \rightarrow u_*(x) \quad \text{a.e. in } \Omega. \quad (2.13)
\end{cases}$$

From Lemma 2.4, for $s > 0$ sufficiently small and $\phi \in W^{1,p}_0(\Omega)$, and set $u = u_n, \omega = s\phi \in W^{1,p}_0(\Omega)$, we can find that $f_n(s) = f_n(s\phi)$ such that $f_n(0) = 1$ and $f_n(s)(u_n + s\phi) \in N_\lambda$. Since

$$\|u_n\| = \int_{\Omega} |u_n|^p dx - \beta \int_{\Omega} |u_n|^p |x|^{\alpha-p} dx - \lambda \int_{\Omega} |u_n|^q dx = 0. \quad (2.14)$$

By (2.12), we obtain

$$\frac{1}{n} \left[ f_n(s) - 1 \|u_n\| + s f_n(s)\|\phi\| \right] \geq \frac{1}{n} \left[ f_n(s)(u_n + s\phi) - u_n \right] \geq I_\lambda(u_n) - I_\lambda[f_n(s)(u_n + s\phi)]. \quad (2.15)$$
Notice that
\[
I_1 \left[ f_n(s)(u_n + s\phi) \right] = \frac{1}{p} \left\| f_n(s)(u_n + s\phi) \right\|^p - \frac{\beta}{p} \int_{\Omega} |x|^{a-p} |f_n(s)(u_n + s\phi)|^p \, dx
- \frac{1}{p^*} \int_{\Omega} \left| f_n(s)(u_n + s\phi) \right|^p \, dx - \frac{\lambda}{q} \int_{\Omega} |f_n(s)(u_n + s\phi)|^q \, dx
= \frac{f_n^p(s)}{p} \left\| u_n + s\phi \right\|^p - \frac{\beta}{p^*} \int_{\Omega} |x|^{a-p} |u_n + s\phi|^p \, dx
- \frac{f_n^q(s)}{q} \int_{\Omega} |u_n + s\phi|^q \, dx.
\]

Therefore
\[
I_1(u_n) - I_1 \left[ f_n(s)(u_n + s\phi) \right]
= \frac{1}{p} \left\| u_n \right\|^p - \frac{f_n^p(s)}{p} \left\| u_n \right\|^p + \frac{f_n^p(s)}{p^*} \int_{\Omega} \left| u_n + s\phi \right|^p \, dx - \frac{1}{p^*} \int_{\Omega} \left| u_n + s\phi \right|^p \, dx
+ \frac{\lambda}{q} \int_{\Omega} \left| u_n + s\phi \right|^q \, dx - \frac{\lambda}{q} \int_{\Omega} 
\int_{\Omega} \left| u_n + s\phi \right|^q \, dx
- \frac{\beta}{p^*} \int_{\Omega} |x|^{a-p} |u_n + s\phi|^p \, dx
- \frac{f_n^q(s)}{q} \int_{\Omega} |u_n + s\phi|^q \, dx
+ \frac{\beta}{p^*} \int_{\Omega} |x|^{a-p} |u_n + s\phi|^p \, dx
+ \frac{\beta}{p} \int_{\Omega} |u_n + s\phi|^p \, dx - \frac{\beta}{p} \int_{\Omega} |u_n|^p |x|^{a-p} \, dx
= \frac{1 - f_n^p(s)}{p} \left\| u_n \right\|^p + \frac{f_n^p(s) - 1}{p^*} \int_{\Omega} \left| u_n + s\phi \right|^p \, dx + \frac{\lambda}{q} \left( f_n^q(s) - 1 \right) \int_{\Omega} \left| u_n + s\phi \right|^q \, dx
+ \frac{\beta}{p} \int_{\Omega} \left[ |u_n + s\phi|^p - |u_n|^p \right] |x|^{a-p} \, dx.
\]

Dividing by \( s > 0 \) and taking the limit for \( s \to 0 \), combining with (2.14) and (2.15), we have
\[
\frac{f_n'(0)||u_n|| + \|\phi\|}{n}
\geq -f_n'(0)||u_n||^p + f_n'(0) \int_{\Omega} |u_n|^p \, dx + f_n'(0) \int_{\Omega} |u_n|^q \, dx
+ \beta f_n'(0) \int_{\Omega} |u_n|^p |x|^{a-p} \, dx - \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla \phi \, dx
+ \mu \int_{\Omega} \frac{|u_n|^{p-2} u_n \phi}{|x|^p} \, dx + \int_{\Omega} |u_n|^{p-1} \phi \, dx
+ \lambda \int_{\Omega} |u_n|^{q-1} \phi \, dx + \beta \int_{\Omega} |u_n|^{p-1} |x|^{a-p} \, dx
\]
\[\begin{align*}
&= -f_n'(0) \left[ \|u_n\|^p - \int_\Omega |u_n|^p \, dx - \lambda \int_\Omega |u_n|^q \, dx - \beta \int_\Omega |u_n|^p |x|^{\alpha - p} \, dx \right] - \langle I_{\lambda}', \phi \rangle \\
&= -\langle I_{\lambda}', \phi \rangle.
\end{align*}\]

Consequently

\[\frac{-|f_n'(0)|}{n} \|u_n\| + \|\phi\| \leq \langle I_{\lambda}', \phi \rangle \quad (2.16)\]

for every \( \phi \in W_0^{1,p}(\Omega) \). Note that (2.16) holds equally for \(-\phi\), we see that (2.11) holds. \( \square \)

**Lemma 2.7** (see [8, 10]) Set \( D^{1,p}(\mathbb{R}^N) = \{ u \in L^p(\mathbb{R}^N) : |\nabla u| \in L^p(\mathbb{R}^N) \} \). Assume that \( 1 < p < N \) and \( 0 \leq \mu < \frac{p}{N} \).

Then the limiting problem

\[\begin{align*}
\begin{cases}
-\Delta_p u - \mu \frac{|u|^{p-1}}{|x|^{p-1}} = \frac{u}{|x|^{p-1}} \quad \text{in } \mathbb{R}^N \setminus \{0\}, \\
u > 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}, \\
u \in D^{1,p}(\mathbb{R}^N)
\end{cases}
\end{align*}\]

has radially symmetric ground states

\[V_\epsilon(x) = \epsilon^{\frac{N}{p}} U_{\mu,\epsilon}(\frac{x}{\epsilon}) = \epsilon^{\frac{N}{p}} U_{\mu,\epsilon}(\frac{|x|}{\epsilon}) \quad \forall \epsilon > 0,
\]

such that

\[\int_{\mathbb{R}^N} \left( |\nabla V_\epsilon(x)|^p - \mu \frac{|V_\epsilon(x)|^p}{|x|^p} \right) \, dx = \int_{\mathbb{R}^N} |V_\epsilon(x)|^p \, dx = S_{\mu,0}^N.
\]

where the function \( U_{p,\mu}(x) = U_{p,\mu}(|x|) \) is the unique radial solution of the above limiting problem with

\[U_{p,\mu}(1) = \left( \frac{N(N - \mu)}{N - p} \right)^{\frac{1}{N-p}}.
\]

In the following, we define \( \Lambda = \frac{1}{N} S_{\mu,0}^N \).

**Lemma 2.8** Let \( \{u_n\} \subset N^- \) be a minimizing sequence for \( I_{\lambda} \) with \( \kappa^-_\lambda < \Lambda - D^{1,p} \), where

\[D = \frac{p - q}{p} \left[ \frac{p^*-q}{p^*q} S_{\mu,0}^{-\frac{2}{p}} \left( \frac{\beta_1 - \beta}{N\beta_1} \right)^{-\frac{2}{p}} \right]^{\frac{p}{p-q}}.
\]

Then there exists \( u \in W_0^{1,p}(\Omega) \) such that \( u_n \to u \) in \( L^p(\Omega) \).

**Proof** Since

\[I_{\lambda}(u_n) \to \kappa^-_\lambda \quad \text{as } n \to +\infty.
\]

(2.19)
By Lemma 2.2, we know that \( \{u_n\} \) is bounded in \( W^{1,p}_0(\Omega) \). In fact, we can deduce from (1.3) and (2.19) that

\[
1 + \kappa_\lambda^{-1} + o(\|u_n\|) \\
\geq I_\lambda(u_n) - \frac{1}{p^*} \langle F_{\pi}(u_n), u_n \rangle \\
= \frac{1}{p} \|u_n\|^{p^*} - \frac{\beta}{p} \int_\Omega |u_n|^{p^*} dx - \frac{1}{p^*} \int_\Omega |u_n|^{p^*} dx - \lambda \int_\Omega |u_n|^q dx \\
+ \frac{\beta}{p^*} \left( \|u_n\|^{p^*} - \frac{1}{p^*} \int_\Omega |u_n|^{p^*} dx \right) \\
\geq \left( \frac{1}{p} - \frac{1}{p^*} \right) \|u_n\|^{p^*} - \left( \frac{1}{p^*} - \frac{1}{q} \right) \lambda \int_\Omega |u_n|^q dx \\
\geq \left( \frac{1}{p} - \frac{1}{p^*} \right) \|u_n\|^{p^*} + \left( \frac{1}{p^*} - \frac{1}{q} \right) \lambda \int_\Omega |u_n|^q dx \\
\geq \left( \frac{1}{p} - \frac{1}{p^*} \right) \|u_n\|^{p^*} + \left( \frac{1}{p^*} - \frac{1}{q} \right) \lambda |\Omega|^{p^* - q} S_{\mu,0} \|u_n\|^q,
\]

where \( 0 < \beta < \beta_1, 1 < q < p \), we see that \( \{u_n\} \) is bounded in \( W^{1,p}_0(\Omega) \). We can choose a subsequence (still denoted by \( \{u_n\} \)) and \( u \in W^{1,p}_0(\Omega) \) satisfying

\[
\begin{aligned}
\{u_n\} &\rightharpoonup u \text{ weakly in } W^{1,p}_0(\Omega), \\
\{u_n\} &\rightarrow u \text{ strongly in } L^p(\Omega) \text{ (1 \leq p < p^*)}, \\
\{u_n(x)\} &\rightarrow u(x) \text{ a.e. in } \Omega.
\end{aligned}
\]

In term of the concentration compactness principle, going if necessary to a subsequence, there exist an at most countable set \( J \), a set of points \( \{x_j\}_{j \in J} \subset \Omega \setminus \{0\} \), and real numbers \( \mu_j, v_j, \tilde{\chi}_0 \) such that

\[
|\nabla u_n|^p \rightarrow d\mu = |\nabla u|^p + \sum_{j \in J} \mu_j \delta_{x_j} + \mu_0 \delta_0, \\
|u_n|^p \rightarrow dv = |u|^p + \sum_{j \in J} v_j \delta_{x_j} + v_0 \delta_0, \\
\frac{|u_n|^p}{|x|^p} \rightarrow d\tilde{\chi} = \frac{|u|^p}{|x|^p} + \tilde{\chi}_0 \delta_0,
\]

where \( \delta_{x_j} \) is the Dirac mass at \( x_j \).

Let \( \epsilon \) be sufficient small satisfying \( 0 \notin B(x_j, \epsilon) \) and \( B(x_j, \epsilon) \cap B(x_i, \epsilon) = \emptyset \) for \( i \neq j, i, j = 1, 2, \ldots, k \). Let \( \psi_{\epsilon,j}(x) \) be a smooth cut-off function centered at \( x_j \) such that \( 0 \leq \psi_{\epsilon,j}(x) \leq 1, \)
\( \psi_{\epsilon,j}(x) = 1 \) for \( x \in B(x_j, \frac{\epsilon}{2}) \), \( \psi_{\epsilon,j}(x) = 0 \) for \( x \in \Omega \setminus B(x_j, \epsilon) \) and \( |\nabla \psi_{\epsilon,j}(x)| \leq \frac{1}{\epsilon} \). Note that

\[
|f_j(u_n), u_n \psi_{\epsilon,j}(x)|
= \int_{\Omega} |\nabla u_n|^p \psi_{\epsilon,j}(x) \, dx + \int_{\Omega} u_n |\nabla u_n|^{p-2} \nabla u_n \nabla \psi_{\epsilon,j}(x) \, dx - \mu \int_{\Omega} \frac{|u_n|^p}{|x|^p} \psi_{\epsilon,j}(x) \, dx
- \int_{\Omega} |u_n|^p \psi_{\epsilon,j}(x) \, dx - \lambda \int_{\Omega} |u_n|^q \psi_{\epsilon,j}(x) \, dx - \beta \int_{\Omega} |u_n|^p |x|^{p-q} \psi_{\epsilon,j}(x) \, dx.
\]

Furthermore, we have

\[
\lim_{n \to \infty} \int_{\Omega} |\nabla u_n|^p \psi_{\epsilon,j}(x) \, dx = \int_{\Omega} \psi_{\epsilon,j}(x) \, d\mu \geq \int_{\Omega} |\nabla u|^p \psi_{\epsilon,j}(x) \, dx + \mu_j,
\]
\[
\lim_{n \to \infty} \int_{\Omega} |u_n|^p \psi_{\epsilon,j}(x) \, dx = \int_{\Omega} \psi_{\epsilon,j}(x) \, dv = \int_{\Omega} |u|^p \psi_{\epsilon,j}(x) \, dx + v_j,
\]
\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \int_{\Omega} u_n |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \psi_{\epsilon,j}(x) = 0,
\]
\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \int_{\Omega} \frac{|u_n|^p}{|x|^p} \psi_{\epsilon,j}(x) = 0.
\]

By (1.3), we deduce that

\[
\left| \int_{\Omega} |u_n|^q \psi_{\epsilon,j} \, dx \right| \leq \int_{B(x_j, \epsilon)} |u_n|^q \, dx
\leq \left( \int_{B(x_j, \epsilon)} |u_n|^q \, dx \right)^{\frac{p}{p-\alpha}} \left( \int_{B(x_j, \epsilon)} |x|^{\frac{\alpha(p-q)}{p}} \, dx \right)^{\frac{p-\alpha}{p}}
\leq S_{\mu,0}^{\frac{p}{p-\alpha}} |u_n|^q \left( \int_{\Omega} \frac{r^{N-1}}{2} \, dr \right)^{\frac{p-\alpha}{p}} \|u_n\|_q
\leq \left( \frac{1}{N} \right)^{\frac{p-\alpha}{p}} S_{\mu,0}^{\frac{p}{p-\alpha}} \frac{N \|u_n\|_{\mu}^q}{\|u_n\|_q}
\]

and

\[
\left| \int_{\Omega} \frac{|u_n|^p}{|x|^{p-q}} \psi_{\epsilon,j} \, dx \right| \leq \left( \int_{B(x_j, \epsilon)} |u_n|^p \, dx \right)^{\frac{p}{p-\alpha}} \left( \int_{B(x_j, \epsilon)} |x|^{p-q} \, dx \right)^{\frac{p-\alpha}{p}}
\leq \left( \int_{B(x_j, \epsilon)} |u_n|^p \, dx \right)^{\frac{p}{p-\alpha}} \left( \int_{B(x_j, \epsilon)} |x-x_j|^{p-q} \, dx \right)^{\frac{p-\alpha}{p}}
\leq S_{\mu,0}^{-1} \|u_n\|^p \left( \int_{\Omega} r^{N-1} \frac{r^{p-q}}{p} \, dr \right)^{\frac{p-\alpha}{p}}
\leq S_{\mu,0}^{-1} \|u_n\|^p \left( \frac{p}{N\epsilon} \right)^{\frac{p-\alpha}{p}}.
Since \([u_n]\) is bounded in \(W_0^{1,p}(\Omega)\), and \(u_n \rightharpoonup u\) weakly in \(L^p(\Omega)\), we conclude that

\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \int_{\Omega} |u_n|^q \psi_{\epsilon,j}(x) \, dx = 0
\]

and

\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \int_{\Omega} |u_n|^p |x|^{a-p} \psi_{\epsilon,j}(x) \, dx = 0.
\]

By (2.11), we have

\[
0 = \lim_{\epsilon \to 0} \lim_{n \to \infty} \{I_j'(u_n), u_n \psi_{\epsilon,j}(x)\} \geq \mu_j - v_j.
\]

Since \(S_0 v_j^\frac{p}{p-1} \leq \mu_j\), we have \(\mu_j = v_j = 0\) or \(\mu_j \geq (S_0)^\frac{p}{p-1}\).

On the other hand, let \(\epsilon > 0\) be sufficiently small satisfying \(x_j \notin B(0, \epsilon), \forall j \in J\). Let \(\psi_{\epsilon,0}(x)\) a smooth cut-off function centered at the origin such that \(0 \leq \psi_{\epsilon,0}(x) \leq 1, \psi_{\epsilon,0}(x) = 1\) for \(|x| \leq \frac{\epsilon}{2}, \psi_{\epsilon,0}(x) = 0\) for \(|x| \geq \epsilon\) and \(|\nabla \psi_{\epsilon,0}(x)| \leq \frac{1}{\epsilon}\). Hence, we have

\[
\lim_{n \to \infty} \int_{\Omega} |\nabla u_n|^p \psi_{\epsilon,0}(x) \, dx = \int_{\Omega} \psi_{\epsilon,0}(x) \, d\mu \geq \int_{\Omega} |\nabla u|^p \psi_{\epsilon,0}(x) \, dx + \mu_0,
\]

\[
\lim_{n \to \infty} \int_{\Omega} |u_n|^p \psi_{\epsilon,0}(x) \, dx = \int_{\Omega} \psi_{\epsilon,0}(x) \, dv = \int_{\Omega} |u|^p \psi_{\epsilon,0}(x) \, dx + v_0,
\]

\[
\lim_{n \to \infty} \int_{\Omega} |u_n|^p |x|^{a-p} \psi_{\epsilon,0}(x) \, dx = \int_{\Omega} \psi_{\epsilon,0}(x) \, d\overline{S}_0 = \int_{\Omega} |u|^p |x|^{a-p} \psi_{\epsilon,0}(x) \, dx + \overline{S}_0,
\]

\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \left| \int_{\Omega} u_n |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \psi_{\epsilon,0}(x) \, dx \right| = 0,
\]

\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \int_{\Omega} |u_n|^q \psi_{\epsilon,0}(x) \, dx = 0
\]

and

\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \int_{\Omega} |u_n|^p |x|^{a-p} \psi_{\epsilon,0}(x) \, dx = 0.
\]

Therefore

\[
0 = \lim_{\epsilon \to 0} \lim_{n \to \infty} \{I_j'(u_n), u_n \psi_{\epsilon,0}(x)\} \geq \mu_0 - \mu \overline{S}_0 - v_0.
\]

Combining the definition of \(S_{\mu,0}\), we get that \(S_{\mu,0} v_0^\frac{p}{p-1} \leq \mu_0 - \mu \overline{S}_0 \leq v_0\), which implies that \(v_0 = 0\) or \(v_0 \geq (S_{\mu,0})^\frac{p}{p-1}\). Now, we prove that \(\mu_j \geq (S_{\mu,0})^\frac{p}{p-1}\) and \(v_0 \geq (S_{\mu,0})^\frac{p}{p-1}\) are not true. If not, we have

\[
\kappa_j^- = \lim_{n \to \infty} \left[ I_j(u_n) - \frac{1}{p^*} I_j'(u_n, u_n) \right] \\
\geq \lim_{n \to \infty} \left[ \left( \frac{1}{p} - \frac{1}{p^*} \right) \|u_n\|^p + \left( \frac{1}{p^*} - \frac{1}{q} \right) \lambda |\Omega| S_{\mu,0}^{\frac{p}{p-1}} \|u_n\|^q \right]
\]

where $D$ is defined in (2.18). Hence, we conclude that $\Lambda - D, \frac{p^*}{q} \leq k_\lambda < \Lambda - D, \frac{p^*}{q}$, which is a contradiction. It follows that $v_j = 0$ for $j \in \{0\} \cup J$, which means that $\int_\Omega |u_n|^p \, dx \to \int_\Omega |u|^p \, dx$ as $n \to \infty$. The proof is completed. \hfill \Box

In the following, we need some estimates for the extremal function $v_\epsilon$ defined in Lemma 2.7. Given $R > 0$, let $\psi(x) \in W_0^{1,p}(\Omega)$, $0 \leq \psi(x) \leq 1$, $\psi(x) = 1$ for $|x| \leq R$, $\psi(x) = 0$ for $|x| \geq 2R$. Set $v_\epsilon(x) = \psi(x) V_\epsilon(x)$. For $1 < p < N$ and $1 < q < p^*$, we have the following estimates (see [4, 6]):

\begin{align*}
\|v_\epsilon\| &= (S_{\mu,0})^{\frac{N}{p}} + O(e^{b(\mu)p-N}), \\
\int_\Omega |v_\epsilon|^p \, dx &= (S_{\mu,0})^{\frac{N}{p}} + O(e^{b(\mu)p-N}),
\end{align*}

then

\begin{align*}
\int_\Omega |v_\epsilon|^q \, dx &= \begin{cases} 
C e^{N q (1 - \frac{N}{p})} & \frac{N}{b(\mu)} < q < p, \\
C e^{N q (1 - \frac{N}{p}) |\ln \epsilon|} & q = \frac{N}{b(\mu)}, \\
C e^{b(\mu) q (1 - \frac{N}{p})} & 1 < q < \frac{N}{b(\mu)},
\end{cases}
\end{align*}

where $b(\mu)$ is the zero of the function

\[ f(\xi) = (p-1)\xi^p - (N-p)\xi^{p-1} + \mu, \quad \xi \geq 0, 0 \leq \mu < \overline{\mu}, \]

satisfying $0 < \frac{N-p}{p} < b(\mu) < \frac{N-p}{p-1}$.

**Lemma 2.9** There exists $\lambda_0 > 0$ such that

\[ \sup_{\epsilon > 0} I_\epsilon(sv_\epsilon) < \Lambda - D, \frac{p^*}{q}, \quad \text{for} \ \lambda \in (0, \lambda_0), \]

where $\Lambda$ and $D$ are defined in 2.8.

**Proof** For two positive constants $s_0$ and $s_1$ (independent of $\epsilon$, $\lambda$), we show that there exists $s_\epsilon > 0$ with $0 < s_0 \leq s_\epsilon \leq s_1 < \infty$ such that $\sup_{\epsilon > 0} I_\epsilon(sv_\epsilon) = I_\epsilon(s_\epsilon v_\epsilon)$. In fact, since \[ \lim_{\epsilon \to +\infty} I_\epsilon(sv_\epsilon) = -\infty, \]
we can deduce that

\begin{equation}
\frac{s_\epsilon^{p-1}}{p} |v_\epsilon|^p - \frac{p^{p-1}}{p} \int_\Omega |v_\epsilon|^p |x|^{q-p} \, dx - \frac{s_\epsilon^{q-1}}{q} \int_\Omega |v_\epsilon|^q \, dx - \lambda s_\epsilon^{q-1} \int_\Omega |v_\epsilon|^q \, dx = 0
\end{equation}

\[ \sup_{\epsilon > 0} I_\epsilon(sv_\epsilon) < \Lambda - D, \frac{p^*}{q}, \quad \text{for} \ \lambda \in (0, \lambda_0), \]

where $\Lambda$ and $D$ are defined in 2.8.
and

\[(p-1)s_\epsilon^{p-2}\|v_\epsilon\|^p - (p-1)\beta s_\epsilon^{p-2} \int_\Omega |v_\epsilon|^p |x|^{a-p} \, dx \]
\[- (p^* - 1)s_\epsilon^{p^*-2} \int_\Omega |v_\epsilon|^{p^*} \, dx - (q-1)\lambda s_\epsilon^{p-2} \int_\Omega |v_\epsilon|^q \, dx < 0. \quad (2.25)\]

Equations (2.24) and (2.25) imply that

\[(p-q)s_\epsilon^{p-2}\|v_\epsilon\|^p - (p-q)\beta s_\epsilon^{p-2} \int_\Omega |v_\epsilon|^p |x|^{a-p} \, dx - (p^* - q)\beta s_\epsilon^{p^*-2} \int_\Omega |v_\epsilon|^{p^*} \, dx < 0. \]

That is,

\[(p-q)s_\epsilon^{p-2}\|v_\epsilon\|^p - (p-q)\beta s_\epsilon^{p-2} \int_\Omega |v_\epsilon|^p |x|^{a-p} \, dx < (p^* - q)\beta s_\epsilon^{p^*-2} \int_\Omega |v_\epsilon|^{p^*} \, dx. \quad (2.26)\]

Hence, we can obtain from (2.26) that \(s_\epsilon\) is bounded below. Moreover, it is clear to see from (2.24) that \(s_\epsilon\) is bounded above for all \(\epsilon > 0\) small enough. Therefore, our claim holds.

Set

\[h(s, v_\epsilon) = \frac{s^p}{p} \|v_\epsilon\|^p - \frac{s^{p^*}}{p^*} \int_\Omega |v_\epsilon|^{p^*} \, dx.\]

In the following, we prove that

\[h(s, v_\epsilon) \leq \Lambda + O(\epsilon^{p(b(\mu)-\frac{N}{p}+1)}). \quad (2.27)\]

Let

\[\tilde{h}(s) = \frac{s^p}{p} \|v_\epsilon\|^p - \frac{s^{p^*}}{p^*} \int_\Omega |v_\epsilon|^{p^*} \, dx.\]

Direct computations give us that \(\lim_{s \to \infty} \tilde{h}(s) = -\infty\) and \(\tilde{h}(0) = 0\). Thus \(\sup_{s \geq 0} \tilde{h}(s)\) is obtained at some \(S_\epsilon > 0\), and

\[S_\epsilon = \left( \frac{\|v_\epsilon\|^p}{\int_\Omega |v_\epsilon|^{p^*} \, dx} \right)^{\frac{1}{p^*-p}}.\]

Since \(\tilde{h}(s)|_{S_\epsilon} = 0\), that is,

\[S_\epsilon^{p^*-1}\|v_\epsilon\|^p - S_\epsilon^{p^*-1} \int_\Omega |v_\epsilon|^{p^*} \, dx = 0,\]

It is easy to check that \(h(s)\) is increasing in \([0, S_\epsilon]\), according to (2.21) and (2.22), we have

\[h(s, v_\epsilon) \leq \tilde{h}(S_\epsilon) = \left( \frac{1}{p} - \frac{1}{p^*} \right) \left( \frac{\|v_\epsilon\|^p}{\int_\Omega |u_\epsilon|^{p^*} \, dx} \right)^{\frac{p^*}{p^*-p}}\]
Now, we consider the following cases:

\[
\frac{1}{p} \leq \frac{1}{p'} \leq \frac{N}{p'N-1} = \frac{\frac{N}{p'N-1}}{\frac{N}{p'N-1} - 1} = \frac{N}{p'N-1}.
\]

Therefore, by (2.27), we have

\[
I_\lambda(s_i v_i) = h(s_i v_i) - \frac{\beta s_i^p}{p} \int_\Omega |v_i|^{p} |x|^\alpha - \frac{\lambda}{q} \int_\Omega |v_i|^q \, dx
\leq \Lambda + C e^{p(b(\mu) - \frac{\lambda}{p} - 1)} - \frac{\beta p}{p} \int_\Omega |v_i|^{p} |x|^\alpha \, dx - \frac{\lambda q}{q} \int_\Omega |v_i|^q \, dx.
\]

Now, we consider the following cases:

(i) \( \frac{N}{b(p)} < q < p \). Choose \( \epsilon = \lambda^{-\frac{1}{(p-q)(b(\mu) - \frac{\lambda}{p} - 1)}} \), for \( \lambda < \lambda_1 := \left( \frac{C_1 + D}{C_3} \right)^{\frac{1}{(p-q)(b(\mu) - \frac{\lambda}{p} - 1)}} \), we have

\[
C_1 e^{p(b(\mu) - \frac{\lambda}{p} - 1)} - \lambda C_2 e^{N+b(1-\frac{\lambda}{p})} = C_1 \lambda^{-\frac{p}{p-q}} - \lambda C_2 \lambda^{\frac{N+b(1-\frac{\lambda}{p})}{(p-q)(b(\mu) - \frac{\lambda}{p} - 1)}}
\]

\[
= C_1 \lambda^{-\frac{p}{p-q}} - \lambda C_2 \lambda^{\frac{N+b(1-\frac{\lambda}{p})}{(p-q)(b(\mu) - \frac{\lambda}{p} - 1)}} + 1
\]

\[
= \lambda^{-\frac{p}{p-q}} (C_1 - C_2 \lambda^{\frac{N+b(1-\frac{\lambda}{p})}{(p-q)(b(\mu) - \frac{\lambda}{p} - 1)}})
\]

\[
< -D \lambda^{-\frac{p}{p-q}}.
\]

(ii) \( q = \frac{N}{b(p)} \). Put \( \epsilon = \lambda^{-\frac{1}{(p-q)(b(\mu) - \frac{\lambda}{p} - 1)}} \), for \( \lambda < \lambda_2 := e^{-\frac{C_1 + D}{C_3}} \), we have

\[
C_1 e^{p(b(\mu) - \frac{\lambda}{p} - 1)} - \lambda C_2 e^{N+b(1-\frac{\lambda}{p})} |\ln \epsilon| = C_1 \lambda^{-\frac{p}{p-q}} - \lambda C_2 \lambda^{\frac{N+b(1-\frac{\lambda}{p})}{(p-q)(b(\mu) - \frac{\lambda}{p} - 1)}} |\ln \lambda|
\]

\[
= C_1 \lambda^{-\frac{p}{p-q}} - \lambda C_2 \lambda^{\frac{N+b(1-\frac{\lambda}{p})}{(p-q)(b(\mu) - \frac{\lambda}{p} - 1)}} + 1 |\ln \lambda|
\]

\[
< \lambda^{-\frac{p}{p-q}} (C_1 - C_2 |\ln \lambda|)
\]

\[
< -D \lambda^{-\frac{p}{p-q}},
\]

where \( C_3 = \frac{C_3}{(p-q)(b(\mu) - \frac{\lambda}{p} - 1)} \).

(iii) \( 1 < q < \frac{N}{b(p)} \). Put \( \epsilon = \lambda^{\frac{N+b(1-\frac{\lambda}{p})}{(p-q)(b(\mu) - \frac{\lambda}{p} - 1)}} \), for \( \lambda < \lambda_3 := \left( \frac{C_1 + D}{C_3} \right)^{\frac{1}{(p-q)(b(\mu) - \frac{\lambda}{p} - 1)}} \) with \( C_2 > D \), we have

\[
C_1 e^{p(b(\mu) - \frac{\lambda}{p} - 1)} - \lambda C_2 e^{q(\mu) + 1 - \frac{\lambda}{p}} = C_1 \lambda^{-\frac{p}{p-q}} - \lambda C_2 \lambda^{\frac{q}{p} - \frac{1}{q}}
\]

\[
= \lambda^{-\frac{p}{p-q}} (C_1 \lambda^{\frac{p}{p-q}} - C_2)
\]

\[
< -D \lambda^{-\frac{p}{p-q}}.
\]
Consequently, for $\lambda < \lambda_0 := \min\{\lambda_1, \lambda_2, \lambda_3\}$, we deduce that

$$I_\lambda(s, v) < \Lambda - D\nu_{\nu/q}.$$

\[\square\]

### 3 Proof of main result

We can find a constant $\delta > 0$ such that $\Lambda - D\nu_{\nu/q} > 0$ for $\lambda < \delta$. Let $\lambda_\ast = \min\{T_\delta, \delta, \lambda_0\}$. For $\lambda \in (0, \lambda_\ast)$, Lemmas 2.1-2.4, 2.6 and 2.7 hold.

Let $\{u_n\} \subset N_\lambda$ be a minimizing sequence of $I_\lambda$. It is easy to see that $\{u_n\}$ is bounded in $W^{1,p}_0(\Omega)$ and there exist a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and $u_\ast \in W^{1,p}_0(\Omega)$ such that

$$\begin{cases}
u_n \to u_\ast & \text{weakly in } W^{1,p}_0(\Omega), \\
u_n \to u_\ast & \text{strongly in } L^r(\Omega) (1 \leq s < p^\ast), \\
u_n(x) \to u_\ast(x) & \text{a.e. in } \Omega,
\end{cases} \quad (3.1)$$

as $n \to \infty$.

Firstly, by Lemma 2.4, we can know that $f'_n(0)$ is bounded with respect to $n \in \mathbb{N}$. Letting $n \to \infty$ in (2.11), we deduce that

$$\int_\Omega |\nabla u_n|^p - \nu_\ast \cdot \nabla \phi - \mu \int_\Omega |u_n|^{p-2}u_n \phi = \int_\Omega |u_n|^{p-1} \phi + \lambda \int_\Omega |u_n|^{q-1} \phi + \beta \int_\Omega |u_n|^{p-1}|x|^{\alpha-p} \phi \quad (3.2)$$

for all $\phi \in W^{1,p}_0(\Omega)$. Equation (3.2) implies that $u_\ast$ is a solution of (1.1). We claim that $u_\ast \not\equiv 0$. If not, $u_\ast = 0$, since $u_n \in N_\lambda$, we have

$$\|u_n\|^p - \int_\Omega |u_n|^{p-1} \phi + \lambda \int_\Omega |u_n|^{q-1} \phi - \mu \int_\Omega |u_n|^q = 0.$$

Note that

$$\lim_{n \to \infty} \int_\Omega |u_n|^p |x|^{\alpha-p} \, dx = 0, \quad \lim_{n \to \infty} \int_\Omega |u_n|^q \, dx = 0.$$

Put $\lim_{n \to \infty} \|u_n\| = m$, we conclude that $m \geq S_{\nu/q}^{p^\ast p - p}$. By Lemma 2.8, we obtain

$$k_\lambda = \lim_{n \to \infty} I_\lambda(u_n)$$

$$= \lim_{n \to \infty} \left[\frac{1}{p} \|u_n\|^p - \frac{\beta}{p} \int_\Omega |u_n|^{p-1} \phi + \frac{1}{p^\ast} \int_\Omega |u_n|^{p^\ast} \, dx - \frac{\lambda}{q} \int_\Omega |u_n|^q \, dx \right]$$

$$\geq \lim_{n \to \infty} \left(\frac{1}{p} - \frac{1}{p^\ast}\right) \|u_n\|^p$$

$$\geq \frac{p^\ast - p}{pp^\ast} S_{\nu/q}^{p^\ast p - p}$$

$$= \frac{1}{S_{\nu/q}^p} \frac{p^\ast}{p^\ast} \frac{p^\ast}{pp^\ast}$$

$$= \frac{p^\ast}{pp^\ast} \frac{1}{S_{\nu/q}^p},$$

which contradicts with $k_\lambda < \Lambda - D\nu_{\nu/q}$ (from Lemma 2.9).
Secondly, we prove that $u_\lambda \in \mathcal{N}_\lambda^+$. Suppose that this is not true, i.e., $u_\lambda \in \mathcal{N}_\lambda^-$. From Lemma 2.1, we can find positive numbers $s^*$ and $s^-$ with $s^* < s_{\text{max}} < s^- = 1$ such that $s^* u_\lambda \in \mathcal{N}_\lambda^+$, $s^- u_\lambda \in \mathcal{N}_\lambda^-$ and

$$
\kappa_\lambda < I_\lambda(s^* u_\lambda) < I_\lambda(s^- u_\lambda) = I_\lambda(u_\lambda) = \kappa_\lambda,
$$

which is a contradiction. Hence $u_\lambda \in \mathcal{N}_\lambda^+$. Furthermore, combining with Lemma 2.3, we can obtain

$$
I_\lambda(u_\lambda) = \kappa_\lambda^+ = \kappa_\lambda < 0.
$$

Therefore, we see that $u_\lambda$ is a non-negative ground state solution of problem (1.1).

In the following, we prove that problem (1.1) has a second solution $v_\lambda$ with $v_\lambda \in \mathcal{N}_\lambda^-$. Since $I_\lambda$ is coercive on $\mathcal{N}_\lambda^-$, according to the Ekeland variational principle and Lemma 2.9, there exists a minimizing sequence $\{v_n\} \subset \mathcal{N}_\lambda^-$ of $I_\lambda$ such that

(i) $I_\lambda(v_n) \leq \kappa_\lambda^+ + \frac{1}{n}$;
(ii) $I_\lambda(u) \geq I_\lambda(v_n) - \frac{1}{n} \|u - v_n\|$ for all $u \in \mathcal{N}_\lambda^-$.  

Note that $\{v_n\}$ is bounded in $W^{1,p}(\Omega)$, there exist a subsequence (still denoted by $\{v_n\}$) and $v_\lambda \in W^{1,p}(\Omega)$ such that

$$
\begin{cases}
 v_n \rightharpoonup v_\lambda \text{ weakly in } W^{1,p}(\Omega), \\
v_n \to v_\lambda \text{ strongly in } L^s(\Omega) \ (1 \leq s < p^*), \\
v_n(x) \to v_\lambda(x) \text{ a.e. in } \Omega,
\end{cases}
$$

as $n \to \infty$.

Similar to the above discussion, we can deduce that $v_n \to v_\lambda$ in $W^{1,p}_0(\Omega)$ and $v_\lambda$ is a non-negative solution of (1.1). Thirdly, we show that $v_\lambda \not= 0$ in $\Omega$. According to $v_n \in \mathcal{N}_\lambda^-$, we obtain

$$
(p - q)\|v_n\|^p = (p^* - q) \int_\Omega |v_n|^{p^*} \, dx + (p - q)\beta \int_\Omega |v_n|^p |x|^{\alpha - p} \, dx < (p^* - q) S_{\mu,0}^p \|v_n\|^{p^*} + (p - q) \frac{\beta}{\beta_1} \|v_n\|^p,
$$

hence

$$
\|v_n\| > \left[ \frac{(p - q)(1 - \frac{\alpha}{p}) S_{\mu,0}^p}{p^* - q} \right]^{\frac{1}{p^* - p}}, \quad \forall v_n \in \mathcal{N}_\lambda^-,
$$

(3.4)

together with $v_n \to v_\lambda$ in $W^{1,p}_0(\Omega)$ means that $v_\lambda \not= 0$.

Lastly, we show that $v_\lambda \in \mathcal{N}_\lambda^-$. We only need to prove that $\mathcal{N}_\lambda^-$ is closed. In fact, for $\{v_n\} \subset \mathcal{N}_\lambda^-$, it follows from Lemmas 2.8 and 2.9 that

$$
\lim_{n \to \infty} \int_\Omega |v_n|^{p^*} \, dx = \int_\Omega |v_\lambda|^{p^*} \, dx.
$$
In addition

\[(p - q)\|v_n\|^p - (p^* - q) \int_{\Omega} |v_n|^{p^*} \, dx - (p - q)\beta \int_{\Omega} |v_n|^p |x|^{-p} \, dx < 0.\]

Thus

\[(p - q)\|v_{\lambda}\|^p - (p^* - q) \int_{\Omega} |v_{\lambda}|^{p^*} \, dx - (p - q)\beta \int_{\Omega} |v_{\lambda}|^p |x|^{-p} \, dx \leq 0,

which means that \(v_{\lambda} \in \mathcal{N}_{\lambda}^0 \cup \mathcal{N}_{\lambda}^{-}\). Combining with Lemma 2.1 and \(v_{\lambda} \not \equiv 0\), we see that \(\mathcal{N}_{\lambda}^{-}\) is closed. Note that \(\mathcal{N}_{\lambda}^0 \cap \mathcal{N}_{\lambda}^{-} = \emptyset\), we know that \(u_{\lambda}\) and \(v_{\lambda}\) are different.

4 Conclusions

In this paper, we study the existence and multiplicity of positive solutions for the quasi-linear elliptic problem which consists of critical Sobolev exponent and a Hardy term.

The main conclusions of this work:

1. Adding a linear perturbation in the nonlinear term of elliptic equation.
2. The main challenge of this study is the lack of compactness of the embedding \(W_0^{1,p} \hookrightarrow L^{p^*}\). We overcome it by the concentration compactness principle.
3. We apply the Ekeland variational principle to obtain a minimizing sequence with good properties.

5 Discussion

In the future, a natural question is whether the multiplicity of positive solutions for (1.1) can be established with negative exponent \(\frac{1}{p^*}\) \((0 < \gamma < 1)\).

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Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

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