Abstract. A manifold is multisymplectic, or more specifically $n$-plectic, if it is equipped with a closed nondegenerate differential form of degree $n+1$. In previous work with Baez and Hoffnung, we described how the ‘higher analogs’ of the algebraic and geometric structures found in symplectic geometry should naturally arise in 2-plectic geometry. In particular, just as a symplectic manifold gives a Poisson algebra of functions, any 2-plectic manifold gives a Lie 2-algebra of 1-forms and functions. Lie $n$-algebras are examples of $L_\infty$-algebras: graded vector spaces equipped with a collection of skew-symmetric multi-brackets that satisfy a generalized Jacobi identity. Here, we generalize our previous result. Given an $n$-plectic manifold, we explicitly construct a corresponding Lie $n$-algebra on a complex consisting of differential forms whose multi-brackets are specified by the $n$-plectic structure. We also show that any $n$-plectic manifold gives rise to another kind of algebraic structure known as a differential graded Leibniz algebra. We conclude by describing the similarities between these two structures within the context of an open problem in the theory of strongly homotopy algebras. We also mention a possible connection with the work of Barnich, Fulp, Lada, and Stasheff on the Gelfand-Dickey-Dorfman formalism.

1. Introduction

Multisymplectic manifolds are smooth manifolds equipped with a closed, nondegenerate differential form. In this paper, we call such a manifold ‘$n$-plectic’ if the form has degree $n+1$. Hence a 1-plectic manifold is a symplectic manifold. Multisymplectic geometry originated in covariant Hamiltonian formalisms for classical field theory, just as symplectic geometry originated in classical mechanics. (See, for example, [11, 18, 19, 22], as well as the review article [31].) More specifically,
in \((n+1)\)-dimensional classical field theory, one can construct a finite-
dimensional \((n+1)\)-plectic manifold known as a ‘multi-phase space’. Particular submanifolds of this space correspond to solutions of the
theory. The data encoded by the submanifolds include the value of
the field as well as the value of its ‘multi-momentum’ at each point in
space-time. The multi-momentum is a quantity that is related to the
time and spacial derivatives of the field via a Legendre transform, in a
manner similar to the relationship between the velocity of a point par-
ticle and its momentum. In fact, a \((0+1)\)-dimensional theory is just the
classical mechanics of point particles, and the corresponding 1-plectic
manifold is the usual extended phase space whose points correspond to
time, position, energy, and momentum.

However, multisymplectic manifolds can be found outside the context
of classical field theory and are interesting from a purely geometric
point of view. For motivation, we provide the following examples:

- An \((n+1)\)-dimensional orientable manifold equipped with a
  volume form is an \(n\)-plectic manifold.
- Given a manifold \(M\), the \(n\)-th exterior power of the cotan-
gent bundle \(\Lambda^n T^* M\) admits a canonical closed non-degenerate
  \((n+1)\)-form and therefore is an \(n\)-plectic manifold. This is
  a generalization of the canonical symplectic structure on the
cotangent bundle.
- Any compact simple Lie group \(G\) is a 2-plectic manifold when
  equipped with the canonical bi-invariant 3-form
  \[ \nu(x, y, z) = \langle x, [y, z] \rangle, \]
  where \(x, y, z \in g\) and \(\langle \cdot, \cdot \rangle\) is the Killing form. The relationship
  between this 2-plectic manifold and the topological group
  \(\text{String}(n)\), which arises in the study of spin structures on loop
  spaces, can be found in our previous work with Baez [5].
- Let \((M, g)\) be a Riemannian manifold which admits two anti-
  commuting, almost complex structures \(J_1, J_2: TM \to TM\), i.e.
  \(J_1^2 = J_2^2 = -\text{id}\) and \(J_1 J_2 = -J_2 J_1\). Then \(J_3 = J_1 J_2\) is also
  an almost complex structure. If \(J_1, J_2, J_3\) preserve the metric
  \(g\), then one can define the 2-forms \(\theta_1, \theta_2, \theta_3\), where
  \(\theta_i(v_1, v_2) = g(v_1, J_i v_2)\). If each \(\theta_i\) is closed, then \(M\) is called a hyper-Kähler
  manifold [36]. Given such a manifold, one can construct the
  4-form:
  \[ \omega = \theta_1 \wedge \theta_1 + \theta_2 \wedge \theta_2 + \theta_3 \wedge \theta_3. \]
  It is straightforward to show \(\omega\) is closed and nondegenerate.
  Hence a hyper-Kähler manifold is a 3-plectic manifold [10].
More examples, as well as the multisymplectic analogs of isotropic submanifolds, co-isotropic submanifolds and real polarizations can be found in the papers by Cantrijn, Ibort, and de León [10] and Ibort [20].

In our previous work with Baez and Hoffnung [4], we described how 2-plectic geometry can be understood as higher or ‘categorified’ symplectic geometry. For example, if a symplectic structure is integral, then it corresponds to the curvature of a principal $U(1)$-bundle. Similarly, in the 2-plectic case, the integrality condition implies that the 2-plectic form is the curvature of a $U(1)$-gerbe, the higher analog of a principal $U(1)$-bundle. Just as a principal bundle can be described as a certain kind of sheaf (its sheaf of sections), a gerbe can be described as a certain kind of sheaf or stack.

From the algebraic point of view, the fundamental object in symplectic geometry is the Poisson algebra of smooth functions whose bracket is induced by the symplectic form. On a 2-plectic manifold, we showed that a 2-plectic structure gives rise to a Lie 2-algebra on a chain complex consisting of smooth functions and certain 1-forms which we call Hamiltonian [4]. Lie $n$-algebras (equivalently, $n$-term $L_{\infty}$-algebras) are higher analogs of differential graded Lie algebras. They consist of a graded vector space concentrated in degrees $0, \ldots, n - 1$ and are equipped with a collection of skew-symmetric $k$-ary brackets, for $1 \leq k \leq n + 1$, that satisfy a generalized Jacobi identity [24, 25]. In particular, the $k = 2$ bilinear bracket behaves like a Lie bracket that only satisfies the ordinary Jacobi identity up to higher coherent homotopy.

One example of a 2-plectic manifold is the multi-phase space for the classical bosonic string [18]. We emphasize that this space is finite-dimensional, and should not be confused with the infinite-dimensional symplectic manifold that is used as a phase space in string field theory [7, 27]. Just as the Poisson algebra of smooth functions represents the observables of a system of particles, we showed that the Lie 2-algebra of Hamiltonian 1-forms contains the observables of the bosonic string [4].

We should mention that there exists other geometric objects, such as Courant algebroids, that also behave like higher symplectic manifolds [26]. Interestingly, Courant algebroids and 2-plectic manifolds have several features in common. In particular, string theory, closed 3-forms and Lie 2-algebras all play important roles in the theory of Courant algebroids [33, 35]. We have discussed some details of the relationship between Courant algebroids and 2-plectic manifolds elsewhere [32]. See also Zambon’s recent work [38] which relates 2-plectic geometry to higher Dirac structures.
In the present work, we generalize our previous result [1] involving 2-plectic manifolds and Lie 2-algebras to \( n \)-plectic manifolds for arbitrary \( n \geq 1 \). Given an \( n \)-plectic manifold, we define a particular space of \((n-1)\)-forms as Hamiltonian, and explicitly construct a Lie \( n \)-algebra on a complex consisting of these forms and arbitrary \( p \)-forms for \( 0 \leq p \leq n - 2 \). The bilinear bracket, as well as all higher \( k \)-ary brackets, are specified by the \( n \)-plectic structure. We then show that any \( n \)-plectic manifold gives rise to another kind of algebraic structure known as a differential graded (dg) Leibniz algebra. A dg Leibniz algebra is a graded vector space equipped with a degree \(-1\) differential and a bilinear bracket that satisfies a Jacobi-like identity, but does not need to be skew-symmetric. There is an interesting relationship between the bilinear bracket on the Lie \( n \)-algebra and the bracket on the corresponding dg Leibniz algebra. We describe some similarities between these two structures within the context of an open problem in the theory of strongly homotopy algebras. Finally, we point out that Barnich, Fulp, Lada, and Stasheff have shown that \( L_\infty \)-algebras naturally arise in the Gelfand-Dickey-Dorfman formalism for classical field theory [6], and that recent work by Bridges, Hydon, and Lawson [8] relating multisymplectic geometry to the variational bicomplex may possibly be used to study the similarities between these \( L_\infty \)-algebras and the Lie \( n \)-algebras constructed here.

2. Notation and preliminaries

2.1. Graded linear algebra. Let \( V \) be a graded vector space. Let \( x_1, \ldots, x_n \) be elements of \( V \) and \( \sigma \in S_n \) a permutation. The Koszul sign \( \epsilon(\sigma) = \epsilon(\sigma; x_1, \ldots, x_n) \) is defined by the equality

\[
x_1 \wedge \cdots \wedge x_n = \epsilon(\sigma; x_1, \ldots, x_n) x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)}
\]

which holds in the free graded commutative algebra generated by \( V \). Given \( \sigma \in S_n \), let \((-1)^\sigma\) denote the usual sign of a permutation. Note that \( \epsilon(\sigma) \) does not include the sign \((-1)^\sigma\).

We say \( \sigma \in S_{p+q} \) is a \((p, q)\)-unshuffle if \( \sigma(i) < \sigma(i + 1) \) whenever \( i \neq p \). The set of \((p, q)\)-unshuffles is denoted by \( \text{Sh}(p, q) \). For example, \( \text{Sh}(2, 1) = \{(1), (23), (123)\} \).

If \( V \) and \( W \) are graded vector spaces, a linear map \( f: V^\otimes n \rightarrow W \) is skew-symmetric if

\[
f(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = (-1)^\sigma \epsilon(\sigma) f(v_1, \ldots, v_n),
\]

for all \( \sigma \in S_n \). The degree of an element \( x_1 \otimes \cdots \otimes x_n \in V^\otimes \ast \) of the graded tensor algebra generated by \( V \) is defined to be \(|x_1 \otimes \cdots \otimes x_n| = \sum_{i=1}^{n} |x_i|\).
2.2. Multivector calculus. In order to aid our computations, we introduce some notation and review the Cartan calculus involving multivector fields and differential forms. We follow the notation and sign conventions found in Appendix A of the paper by Forger, Paufler, and Römer [14]. Let \( \mathfrak{X}(M) \) be the \( C^\infty(M) \)-module of vector fields on a manifold \( M \) and let

\[
\mathfrak{X}^\bullet(M) = \bigoplus_{k=0}^{\dim M} \Lambda^k(\mathfrak{X}(M))
\]

be the graded commutative algebra of multivector fields. On \( \mathfrak{X}^\bullet(M) \) there is a \( \mathbb{R} \)-bilinear map \([\cdot, \cdot] : \mathfrak{X}^\bullet(M) \times \mathfrak{X}^\bullet(M) \to \mathfrak{X}^\bullet(M)\) called the Schouten bracket, which gives \( \mathfrak{X}^\bullet(M) \) the structure of a Gerstenhaber algebra. This means the Schouten bracket is a degree \(-1\) Lie bracket which satisfies the graded Leibniz rule with respect to the wedge product. The Schouten bracket of two decomposable multivector fields \( u_1 \wedge \cdots \wedge u_m, v_1 \wedge \cdots \wedge v_n \in \mathfrak{X}^\bullet(M) \) is

\[
[u_1 \wedge \cdots \wedge u_m, v_1 \wedge \cdots \wedge v_n] = 
\sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{i+j} [u_i, v_j] \wedge u_1 \wedge \cdots \wedge \hat{u}_i \wedge \cdots \wedge u_m \wedge v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_n,
\]

where \([u_i, v_j]\) is the usual Lie bracket of vector fields.

Given a form \( \alpha \in \Omega^\bullet(M) \), the interior product of a decomposable multivector field \( v_1 \wedge \cdots \wedge v_n \) with \( \alpha \) is

\[
\iota(v_1 \wedge \cdots \wedge v_n)\alpha = \iota_{v_n} \cdots \iota_{v_1} \alpha,
\]

where \( \iota_{v_i} \alpha \) is the usual interior product of vector fields and differential forms. The interior product of an arbitrary multivector field is obtained by extending the above formula by \( C^\infty(M) \)-linearity.

The Lie derivative \( \mathcal{L}_v \) of a differential form along a multivector field \( v \in \mathfrak{X}^\bullet(M) \) is defined via the graded commutator of \( d \) and \( \iota(v) \):

\[
\mathcal{L}_v \alpha = d\iota(v)\alpha - (-1)^{|v|}\iota(v)d\alpha,
\]

where \( \iota(v) \) is considered as a degree \(-|v|\) operator.

The last identity we will need involving multivector fields is for the graded commutator of the Lie derivative and the interior product. Given \( u, v \in \mathfrak{X}^\bullet(M) \), it follows from Proposition A3 in [14] that

\[
\iota([u, v])\alpha = (-1)^{|[u] - 1} \iota(v)\mathcal{L}_u \alpha - \iota(v) \mathcal{L}_u \alpha.
\]


3. Multisymplectic geometry

We use the definition of a multisymplectic form given by Cantrijn, Ibort, and de León [10]. Many of the definitions and basic results for \( n \)-plectic structures presented in this section appeared previously in our work with Baez and Hoffnung [4].

**Definition 3.1** ([4, 10]). An \((n+1)\)-form \( \omega \) on a smooth manifold \( M \) is multisymplectic, or more specifically an \( n \)-plectic structure, if it is both closed:

\[
d\omega = 0,
\]

and nondegenerate:

\[
\forall v \in T_x M, \quad \iota_v \omega = 0 \Rightarrow v = 0.
\]

If \( \omega \) is an \( n \)-plectic form on \( M \) we call the pair \((M, \omega)\) a multisymplectic manifold, or \( n \)-plectic manifold.

The name ‘\( n \)-plectic’ was chosen so that a 1-plectic structure is a symplectic structure.

An \( n \)-plectic structure induces an injective map from the space of vector fields on \( M \) to the space of \( n \)-forms on \( M \). This leads us to the following definition:

**Definition 3.2** ([4]). Let \((M, \omega)\) be an \( n \)-plectic manifold. An \((n-1)\)-form \( \alpha \) is Hamiltonian iff there exists a vector field \( v_\alpha \in \mathfrak{X}(M) \) such that

\[
d\alpha = -\iota_{v_\alpha} \omega.
\]

We say \( v_\alpha \) is the Hamiltonian vector field corresponding to \( \alpha \). The set of Hamiltonian \((n-1)\)-forms and the set of Hamiltonian vector fields on an \( n \)-plectic manifold are both vector spaces and are denoted as \( \Omega^{n-1}_{\text{Ham}}(M) \) and \( \mathfrak{X}_{\text{Ham}}(M) \), respectively.

The Hamiltonian vector field \( v_\alpha \) is unique if it exists, but there may be \((n-1)\)-forms having no Hamiltonian vector field. Note that if \( \alpha \in \Omega^{n-1}(M) \) is closed, then it is Hamiltonian and its Hamiltonian vector field is the zero vector field.

An elementary, yet important, fact is that the flow of a Hamiltonian vector field preserves the \( n \)-plectic structure.

**Lemma 3.3** ([4]). If \( v_\alpha \) is a Hamiltonian vector field, then \( \mathcal{L}_{v_\alpha} \omega = 0 \).

**Proof.**

\[
\mathcal{L}_{v_\alpha} \omega = d\iota_{v_\alpha} \omega + \iota_{v_\alpha} d\omega = -dd\alpha = 0
\]

\( \square \)
We now define a bracket on $\Omega^{n-1}_{\text{Ham}}(M)$ that generalizes the Poisson bracket in symplectic geometry. One motivation for considering this bracket comes from its appearance in multisymplectic formulations of classical field theories \cite{19, 22}, in which the usual infinite-dimensional symplectic phase space is replaced with a finite-dimensional ‘multi-phase space’.

**Definition 3.4** \cite{4}. Given $\alpha, \beta \in \Omega^{n-1}_{\text{Ham}}(M)$, the bracket $\{\alpha, \beta\}$ is the $(n-1)$-form given by

$$\{\alpha, \beta\} = \iota_{v_\beta} \iota_{v_\alpha} \omega.$$

When $n = 1$, this bracket is the usual Poisson bracket of smooth functions on a symplectic manifold. These next propositions show that for $n > 1$ the bracket of Hamiltonian forms has several properties in common with the Poisson bracket in symplectic geometry. However, unlike the case in symplectic geometry, we see that the bracket $\{\cdot, \cdot\}$ does not need to satisfy the Jacobi identity for $n > 1$.

**Proposition 3.5** \cite{4}. Let $\alpha, \beta \in \Omega^{n-1}_{\text{Ham}}(M)$ and $v_\alpha, v_\beta$ be their respective Hamiltonian vector fields. The bracket $\{\cdot, \cdot\}$ has the following properties:

1. The bracket is skew-symmetric:
   $$\{\alpha, \beta\} = -\{\beta, \alpha\}.$$

2. The bracket of Hamiltonian forms is Hamiltonian:
   $$d\{\alpha, \beta\} = -\iota_{[v_\alpha, v_\beta]} \omega,$$
   and in particular we have
   $$v_{\{\alpha, \beta\}} = [v_\alpha, v_\beta].$$

*Proof.* The first statement follows from the antisymmetry of $\omega$. To prove the second statement, we use Lemma 3.3

$$d\{\alpha, \beta\} = d\iota_{v_\beta} \iota_{v_\alpha} \omega$$

$$= (\mathcal{L}_{v_\beta} - \iota_{v_\beta} d) \iota_{v_\alpha} \omega$$

$$= \mathcal{L}_{v_\beta} \iota_{v_\alpha} \omega + \iota_{v_\beta} d\iota_{v_\alpha} \omega$$

$$= \iota_{[v_\beta, v_\alpha]} \omega + \iota_{v_\alpha} \mathcal{L}_{v_\beta} \omega$$

$$= -\iota_{[v_\alpha, v_\beta]} \omega.$$
A proof of Proposition 3.6 was given by direct computation in [4]. However, it also follows from the next lemma. We will use this lemma again in the proof of Theorem 5.2 in Section 5.

**Lemma 3.7.** If \((M, \omega)\) is an \(n\)-plectic manifold and \(v_1, \ldots, v_m \in \mathfrak{X}_{\text{Ham}}(M)\) with \(m \geq 2\) then

\[
\begin{align*}
\text{d} \iota(v_1 \wedge \cdots \wedge v_m)\omega &= (-1)^m \sum_{1 \leq i < j \leq m} (-1)^{i+j} \iota([v_i, v_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_m)\omega.
\end{align*}
\]

(5)

**Proof.** We proceed via induction on \(m\). For \(m = 2\):

\[
\begin{align*}
\text{d} \iota(v_1 \wedge v_2)\omega &= d\{\alpha_1, \alpha_2\},
\end{align*}
\]

where \(\alpha_1, \alpha_2\) are any Hamiltonian \((n - 1)\)-forms whose Hamiltonian vector fields are \(v_1, v_2\), respectively. Then Proposition 3.5 implies Eq. 5 holds.

Assume Eq. 5 holds for \(m - 1\). Since \(\iota(v_1 \wedge \cdots \wedge v_m) = \iota(v_1 \wedge \cdots \wedge v_{m-1})\), Eq. 5 implies:

\[
\begin{align*}
\text{d} \iota(v_1 \wedge \cdots \wedge v_m)\omega &= \mathcal{L}_{v_m} \iota(v_1 \wedge \cdots \wedge v_{m-1})\omega - \iota(v_m) \text{d} \iota(v_1 \wedge \cdots \wedge v_{m-1})\omega. \tag{6}
\end{align*}
\]

Consider the first term on the right hand side. Using Eq. 4 we can rewrite it as

\[
\begin{align*}
\mathcal{L}_{v_m} \iota(v_1 \wedge \cdots \wedge v_{m-1})\omega &= \iota([v_m, v_1 \wedge \cdots \wedge v_{m-1}])\omega \\
&\quad + \iota(v_1 \wedge \cdots \wedge v_{m-1}) \mathcal{L}_{v_m}\omega \\
&= \iota([v_m, v_1 \wedge \cdots \wedge v_{m-1}])\omega,
\end{align*}
\]

where the last equality follows from Lemma 3.3.

The definition of the Schouten bracket given in Eq. 1 implies

\[
[v_m, v_1 \wedge \cdots \wedge v_{m-1}] = \sum_{i=1}^{m-1} (-1)^{i+1} [v_m, v_i] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_{m-1}.
\]

Therefore we have

\[
\begin{align*}
\mathcal{L}_{v_m} \iota(v_1 \wedge \cdots \wedge v_{m-1})\omega &= \iota([v_m, v_1 \wedge \cdots \wedge v_{m-1}])\omega \\
&= \sum_{i=1}^{m-1} (-1)^i \iota([v_i, v_m] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_{m-1})\omega.
\end{align*}
\]
Combining this with the second term in Eq. 6 and using the inductive hypothesis gives

\[ d\iota(v_1 \wedge \cdots \wedge v_m)\omega = \sum_{i=1}^{m-1} (-1)^i \iota([v_i, v_m] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_{m-1})\omega \]

\[ - (-1)^{m-1} \sum_{1 \leq i < j \leq m-1} (-1)^{i+j} \iota([v_i, v_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_{m-1})\omega \]

\[ = (-1)^m \left( \sum_{i=1}^{m-1} (-1)^{i+m} \iota([v_i, v_m] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_{m-1})\omega \right) \]

\[ + \sum_{1 \leq i < j \leq m-1} (-1)^{i+j} \iota([v_i, v_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_{m-1})\omega \]

\[ = (-1)^m \sum_{1 \leq i < j \leq m} (-1)^{i+j} \iota([v_i, v_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_{m-1})\omega \].

□

Proof of Proposition 3.6. Apply Lemma 3.7 with \( m = 3 \), and use the fact that \( v_{\{\alpha_i, \alpha_j\}} = [v_{\alpha_i}, v_{\alpha_j}] \). □

4. \( L_\infty \)-algebras

Proposition 3.6 implies that we should not expect \( \Omega_{\text{Ham}}^{n-1}(M) \) to be a Lie algebra unless \( n = 1 \). However, the fact that the Jacobi identity is satisfied modulo boundary terms suggests we consider what are known as strongly homotopy Lie algebras, or \( L_\infty \)-algebras [24, 25].

Definition 4.1. An \( L_\infty \)-algebra is a graded vector space \( L \) equipped with a collection

\[ \{l_k: L^\otimes k \to L | 1 \leq k < \infty\} \]

of skew-symmetric linear maps with \( |l_k| = k - 2 \) such that the following identity holds for \( 1 \leq m < \infty \):

\[ \sum_{i+j=m+1, \sigma \in \text{Sh}(i,m-i)} (-1)^\sigma e(\sigma)(-1)^{(j-1)}l_j(l_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(m)}) = 0. \]

(7)

Definition 4.2. A \( L_\infty \)-algebra \( (L, \{l_k\}) \) is a Lie \( n \)-algebra iff the underlying graded vector space \( L \) is concentrated in degrees 0, \ldots, \( n-1 \).
Note that if \((L, \{l_k\})\) is a Lie \(n\)-algebra, then by degree counting \(l_k = 0\) for \(k > n + 1\).

The identity satisfied by the maps in Definition 4.1 can be interpreted as a ‘generalized Jacobi identity’. Indeed, using the notation \(d = l_1\) and \([\cdot, \cdot] = l_2\), Eq. 7 implies

\[
d^2 = 0
\]

\[
d[x_1, x_2] = [dx_1, x_2] + (-1)^{|x_1||x_2|}[x_1, dx_2].
\]

Hence the map \(l_1: L \to L\) can be interpreted as a differential, while the map \(l_2: L \otimes L \to L\) can be interpreted as a bracket. The bracket is, of course, skew symmetric:

\[
[x_1, x_2] = -(-1)^{|x_1||x_2|}[x_2, x_1],
\]

but does not need to satisfy the usual Jacobi identity. In fact, Eq. 7 implies:

\[
(-1)^{|x_1||x_3|}[x_1, x_2, x_3] + (-1)^{|x_2||x_3|}[x_3, x_1, x_2] + (-1)^{|x_1||x_2|}[x_2, x_3, x_1] = (-1)^{|x_1||x_3|+1}(dl_3(x_1, x_2, x_3) + l_3(dx_1, x_2, x_3)
\]

\[
+ (-1)^{|x_1||x_3|}l_3(x_1, dx_2, x_3) + (-1)^{|x_1||x_2|}l_3(x_1, x_2, dx_3)).
\]

Therefore one can interpret the traditional Jacobi identity as a null-homotopic chain map from \(L \otimes L \otimes L\) to \(L\). The map \(l_3\) acts as a chain homotopy and is referred to as the Jacobiator. Eq. 7 also implies that \(l_3\) must satisfy a coherence condition of its own. From the above discussion, it is easy to see that a Lie 1-algebra is an ordinary Lie algebra, while a \(L_\infty\)-algebra with \(l_k \equiv 0\) for all \(k \geq 3\) is a differential graded Lie algebra.

5. The Lie \(n\)-algebra associated to an \(n\)-plectic manifold

There are several clues that suggest that any \(n\)-plectic manifold gives a \(L_\infty\)-algebra. It was shown in our previous work \[3\] that a Lie 2-algebra can be explicitly constructed from the 2-plectic structure on any 2-plectic manifold. The underlying chain complex of this Lie 2-algebra is

\[
C_\infty(M) \xrightarrow{d} \Omega^1_{\text{Ham}}(M),
\]

where \(d\) is the de Rham differential. This suggests that for an arbitrary \(n\)-plectic manifold, we should look for Lie \(n\)-algebra structures on the chain complex

\[
C_\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-2}(M) \xrightarrow{d} \Omega^{n-1}_{\text{Ham}}(M),
\]

(8)
with the $l_1$ map equal to $d$. We denote this complex as $(L, d)$. It is concentrated in degrees $0, \ldots, n - 1$ with

$$L_i = \begin{cases} \Omega_{\text{Ham}}^{n-1}(M) & i = 0, \\ \Omega^{n-1-i}(M) & 0 < i \leq n - 1. \end{cases}$$

Note that the bracket $\{\cdot, \cdot\}$ given in Definition 3.4 induces a well-defined bracket $\{\cdot, \cdot\}'$ on the quotient

$$\mathfrak{g} = \Omega_{\text{Ham}}^{n-1}(M)/d\Omega^{n-2}(M),$$

where $d\Omega^{n-2}(M)$ is the space of exact $(n - 1)$-forms. This is because the Hamiltonian vector field of an exact $(n - 1)$-form is the zero vector field. It follows from Proposition 3.6 that $(\mathfrak{g}, \{\cdot, \cdot\}')$ is, in fact, a Lie algebra.

If $M$ is contractible, then the homology of $(L, d)$ is

$$H_0(L) = \mathfrak{g},$$
$$H_k(L) = 0 \quad \text{for } 0 < k < n - 1,$$
$$H_{n-1}(L) = \mathbb{R}.$$

Therefore, the augmented complex

$$0 \to \mathbb{R} \hookrightarrow C^\infty(M) \overset{d}{\to} \Omega^1(M) \overset{d}{\to} \cdots \overset{d}{\to} \Omega^{n-2}(M) \overset{d}{\to} \Omega_{\text{Ham}}^{n-1}(M) \quad (9)$$

is a resolution of $\mathfrak{g}$.

Barnich, Fulp, Lada, and Stasheff [6] showed that, in general, if $(C, \delta)$ is a resolution of a vector space $V \cong H_0(C)$ and $C_0$ is equipped with a skew-symmetric map $\tilde{l}_2: C_0 \otimes C_0 \to C_0$ that induces a Lie bracket on $V$, then $\tilde{l}_2$ extends to an $L_\infty$-structure on $(C, \delta)$. Hence we have the following proposition:

**Proposition 5.1.** Given a contractible $n$-plectic manifold $(M, \omega)$, there is a $L_\infty$-algebra $(\tilde{L}, \{l_k\})$ with underlying graded vector space

$$\tilde{L}_i = \begin{cases} \Omega_{\text{Ham}}^{n-1}(M) & i = 0, \\ \Omega^{n-1-i}(M) & 0 < i \leq n - 1, \\ \mathbb{R} & i = n, \end{cases}$$

and $l_1: \tilde{L} \to \tilde{L}$ defined as

$$l_1(\alpha) = \begin{cases} \alpha, & \text{if } |\alpha| = n \\ d\alpha, & \text{if } |\alpha| \neq n, \end{cases}$$
and all higher maps \( \{ l_k : \tilde{L}^k \to \tilde{L} \mid 2 \leq k < \infty \} \) are constructed inductively by using the bracket

\[
\{ \cdot , \cdot \} : \tilde{L}_0 \otimes \tilde{L}_0 \to \tilde{L}_0, \quad \{ \alpha_1, \alpha_2 \} = \iota_{v_{\alpha_2}} \iota_{v_{\alpha_1}} \omega,
\]

where \( v_{\alpha_1}, v_{\alpha_2} \) are the Hamiltonian vector fields corresponding to the Hamiltonian forms \( \alpha_1, \alpha_2 \). Moreover the maps \( \{ l_k \} \) may be constructed so that

\[
l_k(\alpha_1, \ldots, \alpha_k) \neq 0 \quad \text{only if all } \alpha_k \text{ have degree 0},
\]

for \( k \geq 2 \).

Proof. The proposition follows from Theorem 7 in the paper by Barnich, Fulp, Lada, and Stasheff [6]. Since for any n-plectic manifold,

\[
\{ \alpha, d\beta \} = 0 \quad \forall \alpha \in \Omega^{n-1}_{\text{Ham}}(M) \quad \forall \beta \in \Omega^{n-2}(M),
\]

the second remark following Theorem 7 in [6] implies that the maps \( \{ l_k \} \) may be constructed so that they are trivial when restricted to the positive-degree part of the \( k \)-th tensor power of \( \tilde{L} \).

For an arbitrary \( n \)-plectic manifold \( (M, \omega) \), Proposition 5.1 guarantees the existence of \( L_{\infty} \)-algebras locally. We want, of course, a global result in which the higher \( l_k \) maps are explicitly constructed using only the \( n \)-plectic structure. Moreover, in our previous work on 2-plectic geometry, we were able to construct by hand a Lie 2-algebra on a 2-term complex consisting of functions and Hamiltonian 1-forms. We did not need to use a 3-term complex consisting of constants, functions, and Hamiltonian 1-forms. Hence in the general case, we’d expect an \( n \)-plectic manifold to give a Lie \( n \)-algebra whose underlying complex is \( (L, d) \), instead of a Lie \((n+1)\)-algebra whose underlying complex is the \((n+1)\)-term complex used in the above proposition.

We can get an intuitive sense for what the maps \( l_k : L^k \to L \) should be by unraveling the identity given in Definition 4.1 for small values of \( m \) and momentarily disregarding signs and summations over unshuffles. For example, if \( m = 2 \), then Eq. 7 implies that the map \( l_2 : L \otimes L \to L \) must satisfy:

\[
l_1 l_2 + l_2 l_1 = 0.
\]

(10)

Obviously we want \( l_1 \) to be the de Rham differential and \( l_2 \) to be equal to the bracket \( \{ \cdot , \cdot \} \) when restricted to degree 0 elements:

\[
l_2(\alpha_1, \alpha_2) = \pm \iota_{v_{\alpha_2}} \iota_{v_{\alpha_1}} \omega = \{ \alpha_1, \alpha_2 \} \quad \forall \alpha_i \in L_0 = \Omega^{n-1}_{\text{Ham}}(M).
\]

Now consider elements of degree 1. For example, if \( \alpha \in L_0 \) and \( \beta \in L_1 = \Omega^{n-2}(M) \), then \( l_2(\alpha, d\beta) = \{ \alpha, d\beta \} = 0 \). Therefore Eq. (10) implies

\[
dl_2(\alpha, \beta) = l_1 l_2(\alpha, \beta) = 0.
\]
Hence, when restricted to elements of degree 1, \( l_2(\alpha, \beta) \) must be a closed \((n - 2)\)-form. We will choose this closed form to be 0. In fact, we will choose \( l_2 \) to vanish on all elements with degree > 0, since, in general, we want the \( L_\infty \) structure to only depend on the de Rham differential and the \( n \)-plectic structure.

Now suppose \( l_2 \) is defined as above and let \( m = 3 \). Then Eq. 7 implies:

\[
l_1 l_3 + l_2 l_2 + l_3 l_1 = 0.
\]

On degree 0 elements, \( l_1 = 0 \). Therefore it’s clear from Proposition 3.6 that the map \( l_3 : \mathcal{L}^{\otimes 3} \to \mathcal{L} \) when restricted to degree 0 elements must be

\[
l_3(\alpha_1, \alpha_2, \alpha_3) = \pm \iota(v_{\alpha_1} \wedge v_{\alpha_2} \wedge v_{\alpha_3})\omega,
\]

where \( v_{\alpha_i} \) is the Hamiltonian vector field associated to \( \alpha_i \). Now consider a degree 1 element of \( \mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L} \), for example:

\[
\alpha_1 \otimes \alpha_2 \otimes \beta \in \Omega^{n-1}_{\text{Ham}}(M) \otimes \Omega^{n-2}_{\text{Ham}}(M). \quad \text{Since } l_3(\alpha_1, \alpha_2, d\beta) = \pm \iota(v_{\alpha_1} \wedge v_{\alpha_2} \wedge v_{d\beta})\omega = 0,
\]

and \( l_2 \) vanishes on the positive-degree part of the \( k \)-th tensor power of \( \mathcal{L} \), Eq. 11 holds if and only if

\[
dl_3(\alpha_1, \alpha_2, \beta) = 0.
\]

Hence, when restricted to elements of degree 1, \( l_3(\alpha_1, \alpha_2, \beta) \) must be a closed \((n - 2)\)-form. Again, we will choose this closed form to be 0 by forcing \( l_3 \) to vanish on all elements with degree > 0.

Observations like these bring us to our main theorem. In general, we will define the maps \( l_k : \mathcal{L}^{\otimes k} \to \mathcal{L} \) on degree zero elements to be completely specified (up to sign) by the \( n \)-plectic structure \( \omega \):

\[
l_k(\alpha_1, \ldots, \alpha_k) = \pm \iota(v_{\alpha_1} \wedge \cdots \wedge v_{\alpha_k})\omega \quad \text{if } |\alpha_1 \otimes \cdots \otimes \alpha_k| = 0,
\]

and trivial otherwise:

\[
l_k(\alpha_1, \ldots, \alpha_k) = 0 \quad \text{if } |\alpha_1 \otimes \cdots \otimes \alpha_k| > 0.
\]

**Theorem 5.2.** Given a \( n \)-plectic manifold \((M, \omega)\), there is a Lie \( n \)-algebra \( L_\infty(M, \omega) = (L, \{l_k\}) \) with underlying graded vector space

\[
L_i = \begin{cases} 
\Omega^{n-1}_{\text{Ham}}(M) & i = 0, \\
\Omega^{n-1-i}(M) & 0 < i \leq n - 1,
\end{cases}
\]

and maps \( \{l_k : \mathcal{L}^{\otimes k} \to \mathcal{L}|1 \leq k < \infty\} \) defined as

\[
l_1(\alpha) = d\alpha,
\]
if $|\alpha| > 0$ and

$$l_k(\alpha_1, \ldots, \alpha_k) = \begin{cases} 0 & \text{if } |\alpha_1 \otimes \cdots \otimes \alpha_k| > 0, \\ (-1)^{k+1} l(v_{\alpha_1} \wedge \cdots \wedge v_{\alpha_k}) & \text{if } |\alpha_1 \otimes \cdots \otimes \alpha_k| = 0 \text{ and } k \text{ even}, \\ (-1)^k l(v_{\alpha_1} \wedge \cdots \wedge v_{\alpha_k}) & \text{if } |\alpha_1 \otimes \cdots \otimes \alpha_k| = 0 \text{ and } k \text{ odd}, \end{cases}$$

(12)

for $k > 1$, where $v_{\alpha}$ is the unique Hamiltonian vector field associated to $\alpha_i \in \Omega_{\text{Ham}}^{n-1}(M)$.

Proof of Theorem 5.2. We begin by showing the maps $\sigma$ has degree 0, then for all $|\alpha| > 0$ we have skew symmetric maps with $l_{\alpha}^k$ defined. Indeed, if there exists an unshuffle such that the above equality did not hold, then for each $i$ we have

$$\sum \epsilon_{\sigma}(-1)^{\alpha_1 |l_{\alpha_2}(\alpha_1, \alpha_2)|} l_{\alpha_2}(\alpha_1, \alpha_2) = 0,$$

where $\epsilon_{\sigma}$ is odd if $i \in \text{Sh}(i, m - i)$ and even otherwise.

Now we prove the maps satisfy Eq. 7 in Definition 4.1. If $m = 1$, then it is satisfied since $l_1$ is the de Rham differential. If $m = 2$, then a direct calculation shows

$$l_1(l_2(\alpha_1, \alpha_2)) = l_2(l_1(\alpha_1, \alpha_2) + (-1)^{|\alpha_1|} l_2(\alpha_1, l_1(\alpha_2))).$$

Let $m > 2$. We will regroup the summands in Eq. 7 into two separate sums depending on the value of the index $j$ and show that each of these is zero, thereby proving the theorem.

We first consider the sum of the terms with $2 \leq j \leq m - 2$:

$$\sum_{j=2}^{m-2} \sum_{\sigma \in \text{Sh}(i, m - i)} (-1)^{\sigma} \epsilon(\sigma) (-1)^{i(j-1)} l_j(l_1(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(i)}, \alpha_{\sigma(i+1)}, \ldots, \alpha_{\sigma(m)}).$$

(13)

In this case we claim that for all $\sigma \in \text{Sh}(i, m - i)$ we have

$$l_j(l_i(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(i)}, \alpha_{\sigma(i+1)}, \ldots, \alpha_{\sigma(m)} = 0.$$

Indeed, if there exists an unshuffle such that the above equality did not hold, then the definition of $l_j: L^{\otimes j} \to L$ implies

$$|l_i(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(i)} \otimes \alpha_{\sigma(i+1)} \otimes \cdots \otimes \alpha_{\sigma(m)} = 0,$$

which further implies

$$|l_i(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(i)})| = |\alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(i)}| + i - 2 = 0.$$
By assumption, \( l_i(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(i)}) \) must be non-zero and \( j < m - 1 \) implies \( i > 1 \). Hence we must have \(|\alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(i)}| = 0\) and therefore, by Eq. 14, \( i = 2 \). But this implies \( j = m - 1 \), which contradicts our bounds on \( j \). So no such unshuffle could exist, and therefore the sum (13) is zero.

We next consider the sum of the terms \( j = 1, j = m - 1, \) and \( j = m \):

\[
l_1(l_m(\alpha_1, \ldots, \alpha_m)) + \sum_{\sigma \in \text{Sh}(2, m - 2)} (-1)^{\sigma}(\sigma)l_{m-1}(l_2(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}), \alpha_{\sigma(3)}, \ldots, \alpha_{\sigma(m)})
\]

\[
+ \sum_{\sigma \in \text{Sh}(1, m - 1)} (-1)^{\sigma}(\sigma)(-1)^{m-1}l_m(l_1(\alpha_{\sigma(1)}), \alpha_{\sigma(2)}, \ldots, \alpha_{\sigma(m)}).
\]

(15)

Note that if \( \sigma \in \text{Sh}(1, m - 1) \) and \(|l_1(\alpha_{\sigma(1)})| > 0\), then

\[
l_{m-1}(l_1(\alpha_{\sigma(1)}), \alpha_{\sigma(2)}, \ldots, \alpha_{\sigma(m)}) = 0
\]

by definition of the map \( l_m \). On the other hand, if \(|l_1(\alpha_{\sigma(1)})| = 0\), then \( l_1(\alpha_{\sigma(1)}) = d\alpha_{\sigma(1)} \) is Hamiltonian and its Hamiltonian vector field is the zero vector field. Hence the third term in (15) is zero.

Since the map \( l_2 \) is degree 0, we only need to consider the first two terms of (15) in the case when \(|\alpha_1 \otimes \cdots \otimes \alpha_m| = 0\). For the first term we have:

\[
l_1(l_m(\alpha_1, \ldots, \alpha_m)) = \begin{cases} 
(-1)^{\frac{m}{2}+1}dt(v_{\alpha_1} \wedge \cdots \wedge v_{\alpha_m}) \omega & \text{if } m \text{ even}, \\
(-1)^{\frac{m-1}{2}}dt(v_{\alpha_1} \wedge \cdots \wedge v_{\alpha_m}) \omega & \text{if } m \text{ odd}.
\end{cases}
\]

Now consider the second term. If \( \alpha_i, \alpha_j \in \Omega_{\text{Ham}}^{n-1}(M) \) are Hamiltonian \((n - 1)\)-forms then by Definition 3.4, \( l_2(\alpha_i, \alpha_j) = \{\alpha_i, \alpha_j\} \). By Proposition 3.5, \( l_2(\alpha_i, \alpha_j) \) is Hamiltonian and its Hamiltonian vector field is \( v_{\{\alpha_i, \alpha_j\}} = [v_{\alpha_i}, v_{\alpha_j}] \). Therefore for \( \sigma \in \text{Sh}(2, m - 2) \), we have

\[
l_{m-1}(l_2(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}), \alpha_{\sigma(3)}, \ldots, \alpha_{\sigma(m)}) = \begin{cases} (-1)^{\frac{m}{2}}l([v_{\alpha_{\sigma(1)}}, v_{\alpha_{\sigma(2)}}] \wedge \cdots \wedge v_{\alpha_{\sigma(m)}}) \omega & \text{if } m \text{ even}, \\
(-1)^{\frac{m-1}{2}}l([v_{\alpha_{\sigma(1)}}, v_{\alpha_{\sigma(2)}}] \wedge \cdots \wedge v_{\alpha_{\sigma(m)}}) \omega & \text{if } m \text{ odd}.
\end{cases}
\]

Since each \( \alpha_i \) is degree 0, we can rewrite the sum over \( \sigma \in \text{Sh}(2, m - 2) \) as

\[
\sum_{\sigma \in \text{Sh}(2, m - 2)} (-1)^{\sigma}(\sigma)l_{m-1}(l_2(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}), \alpha_{\sigma(3)}, \ldots, \alpha_{\sigma(m)}) = \sum_{1 \leq i < j \leq m} (-1)^{i+j-1}l_{m-1}(l_2(\alpha_i, \alpha_j), \alpha_1, \alpha_2, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_m).
\]
Therefore, if $m$ is even, the sum (15) becomes
\[
(-1)^{m+1} d(u_{\alpha_1} \wedge \cdots \wedge u_{\alpha_m}) \omega + (-1)^m \sum_{1 \leq i < j \leq m} (-1)^{i+j} \iota([u_{\alpha_i}, u_{\alpha_j}] \wedge u_{\alpha_1} \wedge \cdots \wedge \hat{u}_{\alpha_i} \wedge \cdots \wedge \hat{u}_{\alpha_j} \wedge \cdots \wedge u_{\alpha_m}) \omega
\]
and, if $m$ is odd:
\[
(-1)^{m-1} d(u_{\alpha_1} \wedge \cdots \wedge u_{\alpha_m}) \omega + (-1)^{m-1} \sum_{1 \leq i < j \leq m} (-1)^{i+j} \iota([u_{\alpha_i}, u_{\alpha_j}] \wedge u_{\alpha_1} \wedge \cdots \wedge \hat{u}_{\alpha_i} \wedge \cdots \wedge \hat{u}_{\alpha_j} \wedge \cdots \wedge u_{\alpha_m}) \omega.
\]
It then follows from Lemma 3.7 that, in either case, (15) is zero.

It is clear that in the $n = 1$ case, $L_\infty(M, \omega)$ is the underlying Lie algebra of the usual Poisson algebra of smooth functions on a symplectic manifold. In the $n = 2$ case, $L_\infty(M, \omega)$ is the Lie 2-algebra obtained in our previous work with Baez and Hoffnung [4].

Note that the equality
\[
d\{\alpha, \beta\} = -\iota_{[v_\alpha, v_\beta]} \omega
\]
given in Proposition 3.5 implies the existence of a bracket-preserving chain map
\[
\phi: L_\infty(M, \omega) \to \mathfrak{x}_{\text{Ham}}(M),
\]
which in degree 0 takes a Hamiltonian $(n-1)$-form $\alpha$ to its vector field $v_\alpha$. Here we consider the Lie algebra of Hamiltonian vector fields as a Lie 1-algebra whose underlying complex is concentrated in degree 0:
\[
\ldots \to 0 \to 0 \to \mathfrak{x}_{\text{Ham}}(M).
\]
Hence $\phi$ is trivial in all higher degrees. In light of Theorem 5.2, $\phi$ becomes a strict morphism of $L_\infty$-algebras. (See the paper by Lada and Markl [24] for the definition of $L_\infty$-algebra morphisms).

6. The dg Leibniz algebra associated to an n-plectic manifold

In symplectic geometry, every function $f \in C^\infty(M)$ is Hamiltonian. We also have the equality:
\[
\{f, g\} = \iota_{v_f} dg = \mathcal{L}_{v_f} g
\]
for all $f, g \in \Omega^0_{\text{Ham}}(M) = C^\infty(M)$. Hence $\{f, \cdot\}$ is a degree zero derivation on $\Omega^0_{\text{Ham}}(M)$, which makes $(\Omega^0_{\text{Ham}}(M), \{\cdot, \cdot\})$ a Poisson algebra. In general, for $n > 1$, an equality such as Eq. 16 does not hold, and Hamiltonian forms are obviously not closed under wedge product. Therefore,
we shouldn’t expect the Lie n-algebra $L_\infty(M, \omega)$ to behave like a Poisson algebra. But we do have the following simple lemma:

**Lemma 6.1.** Let $(M, \omega)$ be an n-plectic manifold. If $\alpha, \beta \in \Omega_{\text{Ham}}^{n-1}(M)$ are Hamiltonian forms, then

$$L_{v_\alpha} \beta = \{\alpha, \beta\} + d_{v_\alpha} \beta.$$

**Proof.** Definitions 3.2 and 3.4 imply:

$$L_{v_\alpha} \beta = \iota_{v_\alpha} d\beta + d_\iota_{v_\alpha} \beta$$

$$= -\iota_{v_\alpha} \iota_{v_\beta} \omega + d_\iota_{v_\alpha} \beta$$

$$= \{\alpha, \beta\} + d_\iota_{v_\alpha} \beta.$$

Lemma 6.1 suggests that we interpret the $(n-1)$-form $L_{v_\alpha} \beta$ as a type of bracket on $\Omega_{\text{Ham}}^{n-1}(M)$, equal to the bracket given in Definition 3.4 modulo boundary terms. To this end, we consider an algebraic structure known as a differential graded (dg) Leibniz algebra.

**Definition 6.2.** A differential graded Leibniz algebra $(L, \delta, [\cdot, \cdot])$ is a graded vector space $L$ equipped with a degree -1 linear map $\delta : L \to L$ and a degree 0 bilinear map $[\cdot, \cdot] : L \otimes L \to L$ such that the following identities hold:

$$\delta \circ \delta = 0 \quad (17)$$

$$\delta [x, y] = [\delta x, y] + (-1)^{|x|} [x, \delta y] \quad (18)$$

$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|} [y, [x, z]], \quad (19)$$

for all $x, y, z \in L$.

In the literature, dg Leibniz algebras are also called dg Loday algebras. This definition presented here is equivalent to the one given by Ammar and Poncin [11]. Note that the second condition given in the definition above can be interpreted as the Jacobi identity. Hence if the bilinear map $[\cdot, \cdot]$ is skew-symmetric, then a dg Leibniz algebra is a DGLA.

We now show that every n-plectic manifold gives a dg Leibniz algebra.

**Proposition 6.3.** Given an n-plectic manifold $(M, \omega)$, there is a differential graded Leibniz algebra $\text{Leib}(M, \omega) = (L, \delta, [\cdot, \cdot])$ with underlying graded vector space

$$L_i = \begin{cases} 
\Omega_{\text{Ham}}^{n-1}(M) & i = 0, \\
\Omega^{n-1-i}(M) & 0 < i \leq n - 1,
\end{cases}$$
and maps $\delta : L \to L$, $\llbracket \cdot, \cdot \rrbracket : L \otimes L \to L$ defined as
\[
\delta(\alpha) = d\alpha,
\]
if $|\alpha| > 0$ and
\[
\llbracket \alpha, \beta \rrbracket = \begin{cases} 
\mathcal{L}_{v_\alpha} \beta & \text{if } |\alpha| = 0, \\
0 & \text{if } |\alpha| > 0,
\end{cases}
\]
where $v_\alpha$ is the Hamiltonian vector field associated to $\alpha$.

Proof. If $\alpha, \beta \in L_0 = \Omega^{n-1}_{\text{Ham}}(M)$ are Hamiltonian, then Lemma 6.1 implies $d\llbracket \alpha, \beta \rrbracket = d\{\alpha, \beta\} = -\iota_{[v_\alpha, v_\beta]}\omega$. Hence $\llbracket \alpha, \beta \rrbracket$ is Hamiltonian. For $|\beta| > 0$, we have $|\mathcal{L}_{v_\alpha} \beta| = |\beta|$, since the Lie derivative is a degree zero derivation. Hence $\llbracket \cdot, \cdot \rrbracket$ is a bilinear degree 0 map.

We next show that Eq. (18) of Definition 6.2 holds. If $|\alpha| > 1$, then it holds trivially. If $|\alpha| = 1$, then $\llbracket \alpha, \beta \rrbracket = \llbracket \alpha, \delta \beta \rrbracket = 0$ for all $\beta \in L$ by definition, and $\llbracket \delta \alpha, \beta \rrbracket = 0$ since the Hamiltonian vector field associated to $d\alpha$ is zero. If $|\alpha| = 0$ and $|\beta| = 0$, then $\llbracket \alpha, \beta \rrbracket = 0$. Hence all terms in (18) vanish by definition. The last case to consider is $|\alpha| = 0$ and $|\beta| > 0$. We have
\[
\delta \llbracket \alpha, \beta \rrbracket = d\mathcal{L}_{v_\alpha} \beta = \mathcal{L}_{v_\alpha} d\beta = \llbracket \alpha, \delta \beta \rrbracket.
\]

Finally, we show the Jacobi identity (19) holds. Let $\alpha, \beta, \gamma \in L$. Then the left hand side of (19) is $\llbracket \alpha, \llbracket \beta, \gamma \rrbracket \rrbracket$, while the right hand side is $\llbracket \llbracket \alpha, \beta \rrbracket, \gamma \rrbracket + (-1)^{|\alpha||\beta|} \llbracket \beta, \llbracket \alpha, \gamma \rrbracket \rrbracket$. Note equality holds trivially if $|\alpha| > 0$ or $|\beta| > 0$. Otherwise, we use the identity
\[
\mathcal{L}_{[v_1, v_2]} = \mathcal{L}_{v_1} \mathcal{L}_{v_2} - \mathcal{L}_{v_2} \mathcal{L}_{v_1},
\]
and the fact that $d\llbracket \alpha, \beta \rrbracket = -\iota_{[v_\alpha, v_\beta]}\omega$ to obtain the following equalities:
\[
\llbracket \alpha, \llbracket \beta, \gamma \rrbracket \rrbracket = \mathcal{L}_{v_\alpha} \mathcal{L}_{v_\beta} \gamma
= \mathcal{L}_{[v_\alpha, v_\beta]} \gamma + \mathcal{L}_{v_\beta} \mathcal{L}_{v_\alpha} \gamma
= \llbracket \llbracket \alpha, \beta \rrbracket, \gamma \rrbracket + \llbracket \beta, \llbracket \alpha, \gamma \rrbracket \rrbracket.
\]

One interesting aspect of the dg Leibniz structure is that it interprets the bracket of Hamiltonian $(n-1)$-forms geometrically as the change of an observable along the flow of a Hamiltonian vector field. Leibniz algebras, in fact, naturally arise in a variety of geometric settings e.g. in Courant algebroid theory and, more generally, in the derived bracket formalism [23]. It would be interesting to compare Leib$(M, \omega)$ to the Leibniz algebras that appear in these other formalisms.
7. Concluding remarks and open questions

7.1. Applications of Theorem 5.2. We wish to remark that Theorem 5.2 implies that one can assign an $L_\infty$-algebra to each of the multisymplectic manifolds mentioned in the introduction. These algebraic structures may be of interest in their own right. For example, if $(M, \omega)$ is a compact, connected, oriented $(n+1)$-dimensional manifold equipped with a volume form $\omega$, then Zambon \cite{38} showed that the isomorphism class of the Lie $n$-algebra $L_\infty(M, \omega)$ is independent of the choice of $\omega$ and therefore only depends on the manifold $M$.

Another example comes from representation theory and quantum groups. Given a simple finite-dimensional Lie algebra $\mathfrak{g}$ of type $ADE$, one can construct certain hyper-Kähler manifolds known as ‘Nakajima quiver varieties’ \cite{28}. These can be used to study the finite-dimensional representations of the quantum enveloping algebra of the affine Lie algebra corresponding to $\mathfrak{g}$ \cite{29}. As mentioned earlier, every hyper-Kähler manifold is 3-plectic. Therefore we can associate a Lie 3-algebra to any Nakajima quiver variety. It would be interesting to see how these Lie 3-algebras are related to the representation-theoretic structures encoded in these varieties.

7.2. Lie $n$-algebras and dg Leibniz algebras. By extending the work of Baez and Crans \cite{3}, Roytenberg \cite{34} developed what are known as 2-term weak $L_\infty$-algebras, or ‘weak Lie 2-algebras’. In a weak Lie 2-algebra, the skew symmetry condition on the maps given in Definition 4.1 is relaxed. In particular, the bilinear map $l_2 : L \otimes L \to L$ is skew-symmetric only up to homotopy. This homotopy must satisfy a coherence condition, as well as compatibility conditions with the homotopy that controls the failure of the Jacobi identity. Lie 2-algebras in the sense of Definition 4.2 are weak Lie 2-algebras that satisfy skew-symmetry on the nose. They are called ‘semi-strict Lie 2-algebras’ in this context, since the Jacobi identity may still fail to hold. Weak Lie 2-algebras that satisfy a Jacobi identity of the form

$$[x, [y, z]] - [[x, y], z] - [y, [x, z]] = 0,$$

but not necessarily satisfy the skew-symmetry condition, are called ‘hemi-strict Lie 2-algebras’. In fact, any hemi-strict Lie 2-algebra is a 2-term dg Leibniz algebra.

Given an $n$-plectic manifold $(M, \omega)$, it is easy to show that the bracket of degree 0 elements in the dg Leibniz algebra $\text{Leib}(M, \omega)$ is skew-symmetric up to an exact $(n-1)$ form:

$$[[\alpha, \beta] + [\beta, \alpha] = d (\iota_{v_\alpha} \beta + \iota_{v_\beta} \alpha).$$
When \( n = 2 \), we showed in our previous work with Baez and Hoffnung \[4\] that \( \text{Leib}(M, \omega) \) is a hemi-strict Lie 2-algebra, and the map
\[
L_0 \otimes L_0 \to L_1
\]
\[
\alpha \otimes \beta \mapsto (\iota_{v_\alpha} \beta + \iota_{v_\beta} \alpha)
\]
is the relevant homotopy. We then showed that Roytenberg’s definition of morphism is flexible enough to allow the identity map on the underlying chain complexes to lift to an actual isomorphism:
\[
L_\infty(M, \omega) \cong \text{Leib}(M, \omega) \quad \text{for } n = 2,
\]
in the category of weak Lie 2-algebras. In general, we would like to conjecture that some sort of equivalence such as this holds for \( n > 2 \). Unfortunately, it isn’t clear in what category this should occur. Indeed, developing a theory of weak Lie \( n \)-algebras is an open problem. Perhaps by studying the relationships between the structures specifically on \( L_\infty(M, \omega) \) and \( \text{Leib}(M, \omega) \) for arbitrary \( n \) one could get a sense of what explicit coherence conditions would be needed to give a good definition.

On the other hand, there are structures known as ‘Loday-\( \infty \) algebras’ (or sh Leibniz algebras) \[1, 37\] that generalize the definition of an \( L_\infty \)-algebra by, again, relaxing the skew symmetry condition on the maps \( \{l_k\} \). However, this time the skew symmetry is not required to hold up to homotopy. Hence any dg Leibniz algebra is a Loday-\( \infty \) algebra. Any \( L_\infty \)-algebra is as well. Therefore there may be an isomorphism between \( L_\infty(M, \omega) \) and \( \text{Leib}(M, \omega) \) in this category for \( n \geq 2 \).

7.3. Multisymplectic geometry and the Gelfand-Dickey-Dorfman formalism. In many formalisms for classical field theory, fields are considered to be sections of a vector bundle. The observables are represented by ‘local functionals’ which are evaluated on sections with compact support. Local functionals are integrals whose integrands are functions that depend only on the fields and a finite number of their derivatives. Such functions are called ‘local functions’. As usual, one can study the time evolution of the field theory by defining a Poisson bracket on the local functionals. However, there is an advantage to working with local functions directly since they are smooth functions on a finite-dimensional space (specifically, a finite jet bundle). The trade-off with this approach is that there is not a one-to-one correspondence between local functionals and local functions. One has to consider equivalence classes of local functions modulo total divergences \[12\]. Roughly, the formalism developed by Gelfand, Dickey, and Dorfman \[16, 17\] involves considering the Poisson bracket on local functionals as being induced by a skew-symmetric bracket on local
functions which is a Lie bracket only up to a total divergence. Barnich, Fulp, Lada, and Stasheff \cite{6} showed, using the variational bicomplex \cite{2}, that such a bracket gives rise to an \(L_\infty\)-algebra.

There are conceptual similarities between the \(L_\infty\)-algebras arising in multisymplectic geometry and those constructed by Barnich, Fulp, Lada, and Stasheff. For example, they both naturally appear when one attempts to treat the observables of a classical field theory within a finite-dimensional setting. Historically, it appears that multisymplectic geometry has developed, for the most part, independently from those formalisms which use the variational bicomplex. However Bridges, Hydon, and Lawson \cite{8} have recently reinterpreted the multisymplectic formalism using the variational bicomplex and it may be possible to use their results to directly compare the \(L_\infty\)-algebras that arise in multisymplectic geometry with those found in the Gelfand-Dickey-Dorfman formalism.

7.4. Other generalizations of Poisson brackets. In Nambu mechanics \cite{15,30}, one considers a manifold \(M\) equipped with an \(n\)-ary skew-symmetric bracket \(\{\ldots\}\) on the algebra of smooth functions satisfying the identity:

\[
\{f_1, \ldots, f_{n-1}, \{g_1, \ldots, g_n\}\} = \sum_{i=1}^{n} \{g_i, \ldots, \{f_1, \ldots, f_{n-1}, g_i\}, \ldots, g_n\},
\]

for all \(f_i, g_i \in C^\infty(M)\). The vector space \(C^\infty(M)\) equipped with such a bracket is an example of an ‘\(n\)-Lie algebra’ \cite{13}. These structures are quite different from the Lie \(n\)-algebras considered here. There are, however, at least some elementary relationships between the Nambu and \(n\)-plectic formalisms. For example, an \(n\)-plectic form on a manifold of dimension \(n+1\) determines a dual multivector field \(\pi\) of degree \(n+1\). This multivector field gives an \((n+1)\)-Lie bracket:

\[
\{f_1, \ldots, f_{n+1}\} = \pi(df_1, \ldots, df_{n+1}).
\]

(See Theorem 1 in \cite{15}.) The \(n\)-plectic form also determines the graded skew-symmetric map \(l_{n+1}: L^{\otimes n+1} \to L\) defined in Theorem 5.2 as part of the structure of \(L_\infty(M, \omega)\). However, by definition, the restriction of \(l_{n+1}\) to \(C^\infty(M)\) is trivial if \(n > 1\).

The grading of the underlying vector space plays a key role in the theory of \(L_\infty\)-algebras and, in particular, the Lie \(n\)-algebras constructed in the present work. The \(n\)-Lie algebras of the form \((C^\infty(M), \{\ldots\})\), on the other hand, are trivially graded structures. In fact, it has been demonstrated that \(n\)-Lie algebras can be understood as “ungraded” analogues of Lie \(n\)-algebras \cite{21}.
Finally, we mention that other algebraic structures have been considered within the multisymplectic formalism which incorporate Hamiltonian forms of arbitrary degree. (See, for example, [9] and [14].) These forms are related to the n-plectic structure via Hamiltonian multivector fields. It may be worthwhile to investigate whether such Hamiltonian forms can be incorporated into the Lie n-algebra and dg-Leibniz structures considered here.

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REFERENCES

[1] M. Ammar and N. Poncin, Coalgebraic approach to the Loday infinity category, stem differential for 2n-ary graded and homotopy algebras, available as arXiv:0809.4328.
[2] I. M. Anderson, Introduction to the variational bicomplex, in Mathematical aspects of classical field theory (Seattle, WA, 1991), 51–73, Contemp. Math., 132, Amer. Math. Soc., Providence, RI, 1992.
[3] J. Baez and A. Crans, Higher-dimensional algebra VI: Lie 2-algebras, Theory Appl. Categ. 12 (2004), 492–528. Also available as arXiv:math/0307263.
[4] J. Baez, A. Hoffnung, and C. Rogers, Categorified symplectic geometry and the classical string, Comm. Math. Phys. 293 (2010), 701–715. Also available as arXiv:0808.0246
[5] J. Baez, and C. Rogers, Categorified symplectic geometry and the string Lie 2-algebra, Homology Homotopy Appl. 12 (2010), 221–236. Also available as arXiv:0901.4721.
[6] G. Barnich, R. Fulp, T. Lada, and J. Stasheff, The sh Lie structure of Poisson brackets in field theory, Comm. Math. Phys. 191 (1998), 585–601. Also available as arXiv:hep-th/9702176.
[7] M. J. Bowick and S. G. Rajeev, Closed bosonic string theory, Nucl. Phys. B 293 (1987), 348–384.
[8] T. J. Bridges, P. E. Hydon, and J. K. Lawson, Multisymplectic structures and the variational bicomplex, Math. Proc. Cambridge Philos. Soc. 148 (2010), 159–178.
[9] F. Cantrijn, A. Ibort, and M. de León, Hamiltonian structures on multisymplectic manifolds, Rend. Sem. Mat. Univ. Politec. Torino 54 (1996), 225–236.
[10] F. Cantrijn, A. Ibort, and M. de León, On the geometry of multisymplectic manifolds, J. Austral. Math. Soc. (Series A) 66 (1999), 303–330.
[11] J. F. Cariñena, M. Crampin, and L. A. Ibort, On the multisymplectic formalism for first order field theories, Diff. Geom. Appl. 1 (1991), 345–374.
[12] L.A. Dickey, Poisson brackets with divergence terms in field theories: two examples, available as arXiv:solv-int/9703001.
[13] V. T. Filippov, n-Lie algebras, Sibirsk. Mat. Zh. 26 (1985), 126–140.
[14] M. Forger, C. Paufler, and H. Römer, The Poisson bracket for Poisson forms in multisymplectic field theory, *Rev. Math. Phys.* **15** (2003), 705–744. Also available as [arXiv:math-ph/0202043](http://arxiv.org/abs/math-ph/0202043).

[15] P. Gautheron, Some remarks concerning Nambu mechanics, *Lett. Math. Phys.* **37** (1996), 103–116.

[16] I. M. Gelfand and L. A. Dickey, A Lie algebra structure in the formal calculus of variations, *Funktsional. Anal. Priloz.* **10** (1976), 18–25.

[17] I. M. Gelfand and I. Ya. Dorfman, Hamiltonian operators and algebraic structures associated with them, *Funktsional. Anal. Priloz.* **13** (1979), 13–30.

[18] M. Gotay, J. Isenberg, J. Marsden, and R. Montgomery, Momentum maps and classical relativistic fields. Part I: covariant field theory, available as [arXiv:physics/9801019](http://arxiv.org/abs/physics/9801019).

[19] F. Hélein, Hamiltonian formalisms for multidimensional calculus of variations and perturbation theory, in *Noncompact Problems at the Intersection of Geometry*, eds. A. Bahri *et al*, AMS, Providence, Rhode Island, 2001, pp. 127–148. Also available as [arXiv:math-ph/0212036](http://arxiv.org/abs/math-ph/0212036).

[20] A. Ibort, Multisymplectic geometry: generic and exceptional, in *Proceedings of the IX Fall Workshop on Geometry and Physics*, Vilanova i la Geltrú, 2000, eds. X. Gràcia *et al*, Publicaciones de la RSME vol. 3, Real Sociedad Matemática Española, Madrid, 2001, pp. 79–88.

[21] C. Iuliu-Lazaroiu, D. McNamee, C. Sämann, and A. Zejak, Strong homotopy Lie algebras, generalized Nahm equations, and multiple M2-branes, available as [arXiv:0901.3905](http://arxiv.org/abs/0901.3905).

[22] J. Kijowski, A finite-dimensional canonical formalism in the classical field theory, *Commun. Math. Phys.* **30** (1973), 99–128.

[23] Y. Kosmann-Schwarzbach, Derived brackets, *Lett. Math. Phys.* **69** (2004), 61–87.

[24] T. Lada and M. Markl, Strongly homotopy Lie algebras, *Comm. Algebra* **23** (1995), 2147–2161. Also available as [arXiv:hep-th/9406095](http://arxiv.org/abs/hep-th/9406095).

[25] T. Lada and J. Stasheff, Introduction to sh Lie algebras for physicists, *Int. Jour. Theor. Phys.* **32** (7) (1993), 1087–1103. Also available as [hep-th/9209099](http://arxiv.org/abs/hep-th/9209099).

[26] Z.-J. Liu, A. Weinstein and P. Xu, Manin triples for Lie bialgebroids, *J. Differential Geom.* **45** (1997), 547–574.

[27] S. A. Merkulov, On the geometric quantization of bosonic string, *Class. Quant. Grav.* **9** (1992), 2267–2276.

[28] H. Nakajima, Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras, *Duke Math. J.* **76** (1994), 365–416.

[29] H. Nakajima, Quiver varieties and finite dimensional representations of quantum affine algebras, *J. Amer. Math. Soc.* **14** (2001), 145–238.

[30] Y. Nambu, Generalized Hamiltonian dynamics, *Phys. Rev. D* **7** (1973), 2405–2412.

[31] N. Román-Roy, Multisymplectic Lagrangian and Hamiltonian formalisms of classical field theories, *SIGMA* **5** (2009), 100, 25 pages. Also available as [arXiv:math-ph/0506022](http://arxiv.org/abs/math-ph/0506022).

[32] C. Rogers, 2-plectic geometry, Courant algebroids, and categorified prequantization, available as [arXiv:1009.2975](http://arxiv.org/abs/1009.2975 [math-ph]).
[33] D. Roytenberg and A. Weinstein, Courant algebroids and strongly homotopy Lie algebras, *Lett. Math. Phys.* 46 (1998), 81–93. Also available as arXiv:math/9802118

[34] D. Roytenberg, On weak Lie 2-algebras, in: P. Kielanowski et al (eds.) XXVI Workshop on Geometrical Methods in Physics. AIP Conference Proceedings 956, pp. 180-198. American Institute of Physics, Melville (2007). Also available as arXiv:0712.3461

[35] P. Ševera, Letter to Alan Weinstein, available at http://sophia.dtp.fmph.uniba.sk/~severa/letters/

[36] A. Swann, Hyper-Kähler and quaternionic Kähler geometry, *Math. Ann.* 289 (1991), 421–450.

[37] K. Uchino, Derived brackets and sh Leibniz algebras, available as arXiv:0902.0044

[38] M. Zambon, $L_\infty$-algebras and higher analogues of Dirac structures, available as arXiv:1003.1004.

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