Lamé polynomials, hyperelliptic reductions and Lamé band structure

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The band structure of the Lamé equation, viewed as a one-dimensional Schrödinger equation with a periodic potential, is studied. At integer values of the degree parameter $\ell$, the dispersion relation is reduced to the $\ell = 1$ dispersion relation, and a previously published $\ell = 2$ dispersion relation is shown to be partly incorrect. The Hermite–Krichever Ansatz, which expresses Lamé equation solutions in terms of $\ell = 1$ solutions, is the chief tool. It is based on a projection from a genus-$\ell$ hyperelliptic curve, which parametrizes solutions, to an elliptic curve. A general formula for this covering is derived, and is used to reduce certain hyperelliptic integrals to elliptic ones. Degeneracies between band edges, which can occur if the Lamé equation parameters take complex values, are investigated. If the Lamé equation is viewed as a differential equation on an elliptic curve, a formula is conjectured for the number of points in elliptic moduli space (elliptic curve parameters space) at which degeneracies occur. Tables of spectral polynomials and Lamé polynomials, i.e., band edge solutions, are given. A table in the older literature is corrected.

Keywords: Lamé equation, Lamé polynomial, dispersion relation, band structure, hyperelliptic reduction, Hermite–Krichever Ansatz

1. Introduction

(a) Background

The term ‘Lamé equation’ refers to any of several closely related second-order ordinary differential equations [Whittaker & Watson 1927; Erdélyi 1953–55; Arscott 1964]. The first such equation was obtained by Lamé, by applying the method of separation of variables to Laplace’s equation in an ellipsoidal coordinate system. Lamé equations arise elsewhere in theoretical physics. Recent application areas include (i) the analysis of preheating after inflation, arising from parametric amplification [Bogolyubov et al. 1996; Greene et al. 1997; Kaiser 1998; Ivanov 2001]; (ii) the stability analysis of critical droplets in bounded spatial domains [Maier & Stein 2001]; (iii) the stability analysis of static configurations in Josephson junctions [Caputo et al. 2000]; (iv) the computation of the distance–redshift relation in inhomogeneous cosmologies [Kantowski & Thomas 2001], and (v) magnetostatic problems in triaxial ellipsoids [Dobner & Ritter 1998].

In some versions of the Lamé equation, elliptic functions appear explicitly, and in others (the algebraic versions) they appear implicitly. The version that appears...
most often in the physics literature is the Jacobi one,

$$\left[-\frac{d^2}{d\alpha^2} + \ell(\ell+1)m \text{sn}^2(\alpha|m)\right] \Psi = E \Psi,$$

(1.1)

which is a one-dimensional Schrödinger equation with a doubly periodic potential, parametrized by $m$ and $\ell$. Here $\text{sn}(\cdot|m)$ is the Jacobi elliptic function with modular parameter $m$. $m$ is often restricted to $(0,1)$, though in general $m \in \mathbb{C} \setminus \{0,1\}$ is allowed. When $m \in (0,1)$, the function $\text{sn}^2(\cdot|m)$ has real period $2K := 2K(m)$ and imaginary period $2iK' := 2iK(1-m)$, with $K(m)$ the first complete elliptic integral.

If $\alpha$ is restricted to the real axis and $m$ and $\ell$ are real, (1.1) becomes a real-domain Schrödinger equation with a periodic potential, i.e., a Hill’s equation. Standard results on Hill’s equation apply (Magnus & Winkler 1979; McKean & van Moerbeke 1976). Equipping (1.1) with a quasi-periodic boundary condition

$$\Psi(\alpha + 2K) = e^{ik(2K)}\Psi(\alpha) := \xi \Psi(\alpha),$$

(1.2)

where $k \in \mathbb{R}$ is fixed, defines a self-adjoint boundary value problem. For any real $k$ (i.e., for any Floquet multiplier $\xi$ with $|\xi| = 1$), there will be an infinite discrete set of energies $E \in \mathbb{R}$ for which this problem has a solution, called a Bloch solution with crystal momentum $k$. Each such $E$ will lie in one of the allowed zones, which are intervals delimited by energies corresponding to $\xi = \pm 1$, i.e., to periodic and anti-periodic Bloch solutions. These form a sequence $E_0 < E_1 < E_2 < E_3 < E_4 < \cdots$, where $E_0$ is a ‘periodic’ eigenvalue, followed by alternating pairs of anti-periodic and periodic eigenvalues (each pair may be coincident). The allowed zones are the intervals $[E_{2j}, E_{2j+1}]$. The complementary intervals $(E_{2j+1}, E_{2j+2})$ are forbidden zones, or lacunae. Any solution of Hill’s equation with energy in a lacuna is unstable: its multiplier $\xi$ will not satisfy $|\xi| = 1$, and its crystal momentum $k$ will not be real.

It is a celebrated result of Ince (1940) that if the degree $\ell$ is an integer, which without loss of generality may be chosen to be non-negative, the Lamé equation (1.1) will have only a finite number of nonempty lacunae. A converse to this statement holds as well (Gesztesy & Weikard 1995a). If $\ell$ is an integer, the Bloch spectrum consists of the $\ell+1$ bands $[E_0, E_1], [E_2, E_3], \ldots, [E_{2\ell}, \infty)$, and $\ell(\ell+1)m \text{sn}^2(\cdot|m)$ is said to be a finite-band or algebro-geometric potential. The $2\ell+1$ band edges $E_0, \ldots, E_{2\ell}$ are algebraic functions of the parameter $m$. That is, they are the roots of a certain polynomial, the coefficients of which are polynomial in $m$. The corresponding periodic and anti-periodic Bloch solutions are called Lamé polynomials: they are polynomials in the Jacobi elliptic functions $\text{sn}(\alpha|m)$, $\text{cn}(\alpha|m)$, and $\text{dn}(\alpha|m)$. The double eigenvalues embedded in the topmost band $[E_{2\ell}, \infty)$ (the ‘conduction’ band), namely $E_{2j} = E_{2j+1}$, $j > \ell$, are loosely called transcendental eigenvalues. For $\ell = 1$ at least, they are known to be transcendental functions of $m$ (Chudnovsky & Chudnovsky 1980).

There has been much work on algebraizing the integer-$\ell$ Lamé equation, to facilitate the computation of the band edges and the coefficients of the Lamé polynomials (Alhassid et al. 1983; Turbiner 1983; Li & Kusnezov 1994; Li et al. 2000; Finkel et al. 2000). Such schemes have been extended to the case when $\ell$ is a half-odd-integer, in which there are an infinite number of lacunae. In this case, certain ‘mid-band’ Bloch functions, namely ones with $\xi = \pm i$ and real period $8K$, are algebraic functions of $\text{sn}(\alpha|m)$. Certain rational values of $\ell$ with $2\ell \notin \mathbb{Z}$ also yield

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algebraic Bloch functions, provided the parameters $m$ and $E$ are chosen appropriately [Maier 2001].

An algebraic understanding of band edges is useful, but it is also desirable to have a closed-form expression for the dispersion relation: $k$ as a function of $E$. The value of $k$ is not unique, since it can be negated (equivalently, $\xi \mapsto 1/\xi$), and any integer multiple of $\pi/K$ can be added. However, each branch has the property that $k \sim E^{1/2}$ or $k \sim -E^{1/2}$ to leading order as $E \to +\infty$. Also, $k \in \mathbb{R}$ in each band.

The goal of this paper is the efficient computation of the dispersion relation when $\ell$ is an integer. The following are illustrations of why this is of importance in theoretical physics. In application (i) above (preheating after inflation), particle production is due to parametric amplification: a solution having a multiplier $\xi$ with $|\xi| > 1$. This corresponds to the energy $E$ not being at a band edge, or even in a band, but in a lacuna. In application (ii) (the stability analysis of a critical droplet), the analysis includes an imposition of Dirichlet rather than quasi-periodic boundary conditions on an $\ell = 2$ Lamé equation [Maier & Stein 2001]. The resulting Bloch solution is not a Lamé polynomial, but rather a mid-band solution.

When $\ell$ is an integer, the Lamé equation is integrable, and the general integral of (1.1) can be expressed in terms of Jacobi theta functions. The dispersion relations $k = k_1(E|m)$, $\ell \geq 1$, can in principle be computed in terms of elliptic integrals. The case $\ell = 1$ is by far the easiest. If $\ell = 1$, the solution space of (1.1), except when $E$ is at a band edge, will be spanned by the pair of Hermite–Halphen solutions

$$\Phi(\alpha; \pm \alpha_0|m) := \frac{H(\alpha \mp \alpha_0|m)}{\Theta(\alpha|m)} \exp \left[ \alpha Z(\pm \alpha_0|m) \right].$$

(1.3)

Here $H, \Theta, Z$ are the Jacobi eta, theta and zeta functions, with periods $4K, 2K, 2K$ respectively, and $\alpha_0 \in \mathbb{C}$ is defined up to sign by $\text{dn}^2(\alpha_0|m) = E - m$. So

$$k_1(E|m) = -iZ(\alpha_0|m) + \pi/2K(m), \quad \text{dn}^2(\alpha_0|m) = E - m,$$

(1.4)

up to multi-valuedness. This is a parametric dispersion relation. It has been exploited in a study of Wannier–Stark resonances with non-real $E$ and $k$ [Grecchi & Sacchetti 1997; Sacchetti 1997]. However, the extension to $\ell > 1$ is numerically nontrivial. $k_1(E|m)$ turns out to equal $\sum_{r=0}^{\ell-1}[-iZ(\alpha_r|m) + \pi/2K(m)]$, where $\{\alpha_r\}_{r=0}^{\ell-1}$ satisfy coupled transcendental equations involving $E$ and $m$ [Whittaker & Watson 1927, §23.71]. Li, Kusnezov & Iachello calculated and graphed $k = k_2(E|m)$ as well as $k = k_1(E|m)$ in the ‘lemniscatic’ case $m = 1/2$ [Li & Kusnezov 1999; Li et al. 2000]. Unfortunately their graph of $k = k_2(E|1/2)$ is incorrect, as will be shown.

(b) Overview of results

When $\ell > 1$, we abandon the traditional Hermite–Halphen solutions, and examine instead the implications for the Lamé dispersion relations of what is now called the Hermite–Krichever Ansatz. This is an alternative way of generating closed-form solutions of the Lamé solution at arbitrary energy $E$; for small integer values of $\ell$, at least. Until the 1980s, the only reference for the Ansatz was the classic work of Halphen [1888, chapitre XII], who applied it to the cases $\ell = 2, 3, 4$, and in part to $\ell = 5$. Krichever [1980] revived it as an aid in the construction of elliptic solutions of the Korteweg–de Vries and other integrable evolution equations. Belokolos et al. [1986] and Belokolos & Enol’skii [2000] summarize early and recent developments.

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The Hermite–Krichever Ansatz is easy to explain, even in the context of the Jacobi form of the Lamé equation, which is not the most convenient for symbolic manipulation. It asserts that for any integer \( \ell \geq 1 \), fundamental solutions of (1.1) can be constructed from the \( \ell = 1 \) solutions \( \tilde{\Phi}(\alpha; \pm \alpha_0|m) \) as finite series of the form

\[
\left[ \sum_{j=0}^{N_\ell} C_j^{(\ell)} \left( \frac{d}{d\alpha} \right)^j \tilde{\Phi}(\alpha; \pm \alpha_0|m) \right] \exp(\pm \kappa_\ell \alpha),
\]

where the parameter \( \alpha_0 \) is now computed from a reduced energy \( \mathcal{E}_\ell \) by the formula \( \text{dn}^2(\alpha_0|m) = \mathcal{E}_\ell - m \). The reduced energy \( \mathcal{E}_\ell \), the exponent \( \kappa_\ell \), and the coefficients \( \{C_j^{(\ell)}\} \) will depend on \( E \) and \( m \). \( \mathcal{E}_\ell(E|m) \) may be chosen to be rational in \( E \) and \( m \), and \( \pm \kappa_\ell \) to be of the form \( \hat{\kappa}_\ell(E|m) \) times \( \pm i \sqrt{\hat{L}_\ell(E|m)} \), where \( \hat{\kappa}_\ell(E|m) \) is also rational, and \( \hat{L}_\ell(E|m) \) is the spectral polynomial \( \prod_{s=0}^{2\ell}(E - E_s(m)) \), a degree-(2\( \ell \)+1) polynomial in \( E \) the coefficients of which, as noted, are rational in \( m \).

If the Lamé equation can be integrated in the framework of the Hermite–Krichever Ansatz, it follows from (1.5) that up to sign, etc.,

\[
k_\ell(E|m) = k_1(\mathcal{E}_\ell(E|m)|m) + \hat{\kappa}_\ell(E|m) \sqrt{\hat{L}_\ell(E|m)}. \tag{1.6}
\]

The dispersion relation for any integer \( \ell \) can be expressed in terms of the \( \ell = 1 \) relation. To compute \( k_\ell \), only one transcendental function (i.e., \( k_1 \)) needs to be evaluated, since the other functions in (1.6) are elementary. The only difficult matter is choosing the relative sign of the two terms, since each is defined only up to sign.

The functions \( \mathcal{E}_\ell(E|m), \hat{\kappa}_\ell(E|m) \) are rational with integer coefficients, but working them out when \( \ell \) is large is a lengthy task. In principle one can write down a recurrence relation for the coefficients \( \{C_j\} \), and work out \( \mathcal{E}_\ell(E|m), \hat{\kappa}_\ell(E|m) \) from the condition that the series terminate. However, their numerator and denominator degrees grow quadratically as \( \ell \) increases. This explains why Halphen’s treatment of the \( \ell = 5 \) case was only partial. In a series of papers, Kostov, Enol’skii, and collaborators used computer algebra systems to perform a full analysis of the cases \( \ell = 2, 3, 4, 5 \) (Gerdt & Kostov 1989; Kostov & Enol’skii 1993; Enol’skii & Kostov 1994; Eilbeck & Enol’skii 1994). When \( \ell = 5 \), using Mathematica to compute the integer coefficients of rational functions equivalent to \( \mathcal{E}_5(E|m), \hat{\kappa}_5(E|m) \) required seven hours of time on a Sparc-1, a Unix workstation of that era (Eilbeck & Enol’skii 1994). Until now, their analysis has not been extended to higher \( \ell \).

While performing extensive symbolic computations, we recently made a discovery which is formalized in Theorem L below. For all integer \( \ell \geq 2 \), the degree-\( \ell \) Lamé equation can be integrated in the framework of the Hermite–Krichever Ansatz, and the rational functions that perform the reduction to the \( \ell = 1 \) case can be computed by simple formulas from certain spectral polynomials of the degree-\( \ell \) equation, which are relatively easy to work out. These are the ordinary spectral polynomial \( \prod_{s=0}^{2\ell}(E - E_s(m)) \) associated with the band-edge solutions, and the spectral polynomials associated with two other types of closed-form solution that have not previously been studied in the literature. We call them twisted and theta-twisted Lamé polynomials. In the context of the Jacobi form, the former are polynomials in \( \text{sn}(\alpha|m), \text{cn}(\alpha|m) \) and \( \text{dn}(\alpha|m) \), multiplied by a factor \( \exp(\kappa \alpha) \). (If \( \kappa \in \mathbb{R} \), ‘canted’ would be better than ‘twisted’.) The latter contain a factor resembling (1.3).
Theorem L follows from modern finite-band integration theory: specifically, from the Baker–Akhiezer uniformization of the relation between the energy and the crystal momentum. This uniformization is closely tied to classical work on the Lamé equation (the parametrized Baker–Akhiezer solutions of the integer-ℓ Lamé equation are in fact equivalent to the Hermite–Halphen solutions). Theorem L greatly simplifies the computation of higher-ℓ dispersion relations. It also has implications for the theory of hyperelliptic reduction: the reduction of hyperelliptic integrals to elliptic ones [Belokolos et al. 1986]. This is on account of the following. For any ℓ ≥ 1 and m, the solutions of the Lamé equation, both the Hermite–Halphen solutions and those derived from the Hermite–Krichever Ansatz, are single-valued functions not of E, but rather of a point (E, ˜ν) on the ℓth Lamé spectral curve: a hyperelliptic curve comprising all (E, ˜ν) satisfying

\[ \tilde{\nu}^2 = \prod_{s=0}^{2\ell} |E - E_s(m)|. \]  

The fact that ˜ν = ˜ν(E) is two-valued (except at a band edge) is responsible for the two-valuedness of, e.g., the parameter α₀ = α₀(E) of (1.3), and in general, for the uncertainty in the sign of k. The ℓth spectral curve generically has genus ℓ and may be denoted ˜Γ_ℓ := Γ_ℓ(m). For any integer ℓ ≥ 2, the E → Eℓ(m) reduction map of the Hermite–Krichever Ansatz induces a covering πℓ : ˜Γ_ℓ → Γ_1. The first known covering of an elliptic curve by a higher-genus hyperelliptic curve was constructed by Legendre and generalized by Jacobi [Belokolos et al. 1986 §4]. But it is difficult to enumerate such coverings, or even work out explicit examples. Those generated by the Ansatz applied to the Lamé equation are a welcome exception.

The integral with respect to E of any rational function of E and \sqrt{Π(E)}, where Π is a polynomial, is a line integral on the algebraic curve defined by ˜ν^2 = Π(E). The covering πℓ : ˜Γ_ℓ → Γ_1, a formula for which is provided by Theorem L, reduces certain such hyperelliptic integrals to elliptic ones. In modern language, the theorem specifies how certain holomorphic differentials on hyperelliptic curves can arise as pullbacks of holomorphic differentials on elliptic curves.

We investigate the degeneracies of the band edges \{E_s(m)\}_{s=0}^{2\ell} that can occur when the modular parameter m is non-real. Such level-crossings were first considered by Cohn [1888] in a dissertation that seems not to have been followed up, though it was later cited by Whittaker & Watson (1927 §23.41). We conjecture a formula for the ℓ-dependence of the number of values of m ∈ \mathbb{C} \setminus \{0, 1\}, or equivalently the number of values of the Klein invariant J ∈ \mathbb{C}, at which two band edges coincide. When this occurs, the genus of the hyperelliptic spectral curve ˜Γ_ℓ is reduced from ℓ to ℓ − 1, though its arithmetic genus remains equal to ℓ. Band-edge degeneracies are responsible for a fact discovered by Turbiner (1989): if ℓ ≥ 2, the complex curve comprising all points \{(m, E_s(m))\}_{s=0}^{2\ell} in \mathbb{C} \setminus \{0, 1\} × \mathbb{C} has only four, rather than 2ℓ + 1, connected components.

This paper is organized as follows. Section 2 introduces the Lamé equation in its elliptic-curve form and relates the Hermite–Halphen solutions to the Baker–Akhiezer function. In §3, Lamé polynomials in the context of the elliptic-curve form are classified. In §4 the Hermite–Krichever Ansatz is introduced, and Theorem L is stated and proved. The application to hyperelliptic reduction is covered in §5. In §§6 and 7 dispersion relations are worked out and the abovementioned dispersion
relation for the case \( \ell = 2 \) is corrected. The \( \ell = 3 \) dispersion relation is graphed as well. Finally, an area for future investigation is mentioned in §8.

2. The elliptic-curve algebraic form

In §§3 through 6 we use exclusively what we call the elliptic-curve algebraic form of the Lamé equation, which is the most convenient for symbolic computation. In this section we derive it, and also define a fundamental multi-valued function \( \Phi \), which appears in the elliptic-curve version of both the Hermite–Halphen solutions and the Hermite–Krichever Ansatz.

(a) An elliptic-curve Schrödinger equation

Many algebraic forms can be obtained from (1.1) by changing to new independent variables which are elliptic functions of \( \alpha \), such as \( \text{sn}(\alpha|m) \). (See, e.g., Arscott & Khabaza [1962, §1.1] and Arscott [1964, pp 192–3]). A form in which the domain of definition is explicitly a cubic algebraic curve of genus 1, i.e., a cubic elliptic curve, can be obtained as follows. First, the Lamé equation is restated in terms of the Weierstrassian function \( \wp(u; g_2, g_3) \). This is the canonical elliptic function with a double pole at \( u = 0 \), satisfying \((\wp'')^2 = f(\wp)\) where \( f(x) := 4x^3 - g_2x - g_3 = 4 \prod_{\gamma=1}^{3}(x - e_\gamma) \). For ellipticity the roots \( \{e_\gamma\}_{\gamma=1}^{3} \) must be distinct, which is equivalent to the condition that the modular discriminant \( \Delta := g_3^2 - 27g_2^3 \) be non-zero. Either of \( g_2, g_3 \in \mathbb{C} \) may equal zero, but not both.

The relation between the Jacobi and Weierstrass elliptic functions is well-known (Abramowitz & Stegun 1965, §18.9). Choose \( \{e_\gamma\}_{\gamma=1}^{3} \) according to

\[
(e_1, e_2, e_3) = A^2 \left( \frac{2 - m}{3}, \frac{2m - 1}{3}, \frac{-(m + 1)}{3} \right),
\]

where \( A \in \mathbb{C} \setminus \{0\} \) is any convenient proportionality constant. Then

\[
g_2 = A^4 \frac{4(m^2 - m + 1)}{3}, \quad g_3 = A^6 \frac{4(m - 2)(2m - 1)(m + 1)}{27},
\]

and the dimensionless (\( A \)-independent) Klein invariant \( J := g_3^2/\Delta \) will be given by

\[
J = \frac{4}{27} \frac{(m^2 - m + 1)^3}{m^2(1 - m)^2}.
\]

The two sorts of elliptic function will be related by, e.g.,

\[
\text{sn}^2(Az|m) = \frac{e_1 - e_3}{\wp(z) - e_3}, \quad \text{ns}^2(Az|m) = \frac{\wp(z) - e_3}{e_1 - e_3},
\]

and the periods of \( \wp \), denoted \( 2\omega, 2\omega' \), will be related to those of \( \text{sn}^2 \) by

\[
2\omega = 2K/A, \quad 2\omega' = 2iK'/A.
\]

The case when \( 2K, 2K' \) are real, or equivalently \( \omega \in \mathbb{R}, \omega' \in i\mathbb{R} \) (we assume \( A \in \mathbb{R} \)), is the case when \( g_2, g_3 \in \mathbb{R} \) and \( \Delta > 0 \) (Abramowitz & Stegun 1965, §18.1).
Choosing for simplicity \( A = 1 \), so that \( e_1 - e_3 = A^2 = 1 \), and rewriting the Lamé equation \((2.1)\) with the aid of \((2.4)\), yields the Weierstrassian form

\[
\left\{ \frac{d^2}{du^2} - \ell(\ell + 1)\psi(u; g_2, g_3) + B \right\} \Psi = 0,
\]

(2.6)

where \( u := \alpha + iK' \). (The translation of \((2.1)\) by \( iK' \) replaces \( m \sin^2 \psi \) by \( \sin^2 \psi \).) Here \( B := -E(e_1 - e_3) - \ell(\ell + 1)e_3 \), i.e.,

\[
B := -E + \frac{1}{2} \ell(\ell + 1)(m + 1),
\]

(2.7)

is a transformed energy parameter. Changing to the new independent variable \( x := \varphi(u; g_2, g_3) \) converts \((2.6)\) to the commonly encountered algebraic form

\[
\left\{ \frac{d^2}{dx^2} + \frac{1}{2} \sum_{\gamma=1}^{3} \frac{1}{x - e_{\gamma}} \frac{d}{dx} \frac{\ell(\ell + 1)x + B}{4 \prod_{\gamma=1}^{3}(x - e_{\gamma})} \right\} \Psi = 0.
\]

(2.8)

This is a differential equation on the Riemann sphere \( \mathbb{P}^1 := \mathbb{C} \cup \{ \infty \} \) with regular singular points at \( x = e_1, e_2, e_3, \infty \). Any solution of the original Lamé equation \((2.1)\) or the Weierstrassian form \((2.6)\), which is quasi-periodic in the sense that it is multiplied by \( \xi, \xi' \) in the context of the algebraic form, the dispersion relation is still a relation between \( C \) quasi-periodicity on \( \mathbb{C} \). Equation \((2.8)\) follows directly from \((2.6)\), since \( d/u = \varphi' d/\varphi = y d/x \). It is a differential equation on \( E_{g_2, g_3} \) with a single singular point: a regular one at \( (x, y) = (\infty, \infty) \). Note that the 2-to-1 covering map \( \pi : E_{g_2, g_3} \to \mathbb{P}^1 \) defined by \( \pi(x, y) = x \) has \( \{(e_\gamma, 0)\}_{\gamma=1}^{3} \) and \( (\infty, \infty) \) as simple critical points. One reason why \((2.8)\) is more fundamental than \((2.6)\) is that the singular points of \((2.8)\) at \( x = e_1, e_2, e_3 \) can be regarded as artifacts: consequences of \( \{(e_\gamma, 0)\}_{\gamma=1}^{3} \) being critical points of \( \pi \).

The complex-analytic differential geometry of the elliptic curve \( E_{g_2, g_3} \) takes a bit of getting used to. Both \( x \) and \( y \) are meromorphic \( \mathbb{P}^1 \)-valued functions on \( E_{g_2, g_3} \), and the only pole that either has on \( E_{g_2, g_3} \) is at the point \( O := (\infty, \infty) \). In a
neighborhood of any generic point \((x, y)\) other than \(O\) and the three points \((e_\gamma, 0)\),
either \(x\) or \(y\) will serve as a local coordinate. However, near each \((e_\gamma, 0)\) only \(y\) will
be a good local coordinate, since \(dy/dx\) diverges at \(x = e_\gamma\). Also, \(x\) has a double
and \(y\) has a triple pole at \(O\), so the appropriate local coordinate near \(O\) is the
quotient \(x/y\). The 1-form \(dx/y\) is not merely meromorphic but holomorphic, with
no poles on \(E_{g_2, g_3}\). Its dual is the vector field (or directional derivative) \(y\,d/dx\).

Elliptic functions, i.e., doubly periodic functions, of the original variable \(u \in \mathbb{C}\)
correspond to single-valued functions on \(E_{g_2, g_3}\). These are rational functions of \(x, y\)
and may be written as \(R_0(x) + R_1(x)y\), i.e., \(R_0(\wp(u)) + R_1(\wp(u))\wp'(u)\). The formula
\((y\,d/dx)y = 6x^2 - \frac{3}{2}g_2\) allows such functions to be differentiated algebraically. In
a similar way, quasi-doubly periodic functions of \(u\) (sometimes called elliptic functions
of the second kind), which are multiplied by \(\xi\) when \(u \leftarrow u + 2\omega\) and by \(\xi'\) when
\(u \leftarrow u + 2\omega'\), correspond to multiplicatively multi-valued functions on \(E_{g_2, g_3}\).

\(E_{g_2, g_3}\) has genus 1 and is topologically a torus. A fundamental pair of loops
that cannot be shrunk to a point may be chosen to be a loop that extends between
\((e_2, 0)\) and \((e_3, 0)\), and one that extends between \((e_1, 0)\) and \((e_2, 0)\), with (if \(g_2, g_3\)
are real, at least) half of each loop passing through positive values of \(y\), and the
other half through negative values. One way of constructing an elliptic function
of the second kind is to anti-differentiate a rational function \(R(x, y)\). Here ‘anti-
differentiate’ means to compute \(\int R(x, y)\,dx/y\), its indefinite integral against the
holomorphic 1-form \(dx/y\). The resulting function will typically have a non-zero
modulus of periodicity associated with each loop. If so, exponentiating it will yield
a multiplicatively multi-valued function on \(E_{g_2, g_3}\), with non-unit multipliers \(\xi, \xi'\).

(b) The Hermite–Halphen solutions

The fundamental multi-valued function \(\Phi\) on the elliptic curve will now be defined.
It is an elliptic-curve version of Halphen’s l’élément simple [Halphen 1888].

**Definition.** On the elliptic curve \(E_{g_2, g_3}\), the multi-valued meromorphic function \(\Phi\),
parametrized by \((x_0, y_0) \in E_{g_2, g_3} \setminus \{\infty, \infty\}\), is defined up to a constant factor by
a formula containing an indefinite elliptic integral,

\[
\Phi(x, y; x_0, y_0) = \exp \left[ \frac{1}{2} \int \left( \frac{y + y_0}{x - x_0} \right) \frac{dx}{y} \right].
\] (2.11)

Its multi-valuedness, which is multiplicative, arises from the path of integration
winding around \(E_{g_2, g_3}\) in any combination of the two directions. Each branch
of \(\Phi\) has a simple zero at \((x, y) = (x_0, y_0)\) and a simple pole at \((x, y) = (\infty, \infty)\).

To motivate the definition of \(\Phi\), a brief sketch will now be given of the construction
of the Hermite–Halphen solutions of the elliptic-curve algebraic Lamé equation
(2.9) for integer \(\ell \geq 1\). The standard published exposition is not fully algebraic,
being framed largely in the context of the Weierstrassian form [Whittaker & Watson
1927, §23.7]. This sketch will relate the Hermite–Halphen solutions to modern
finite-band integration theory and the concept of a Baker–Akhiezer function [Gesztesy & Holden
2003; Treibich 2001]. The starting point is the differential equation

\[
\left\{ \left( y \frac{d}{dx} \right)^3 - 4q(x) + B \right\} \left( y \frac{d}{dx} \right) - 2 \left[ \left( y \frac{d}{dx} \right) q(x) \right] \right\} \mathcal{F} = 0.
\] (2.12)

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The differential operator in (2.12) is the ‘symmetric square’ of the elliptic-curve Schrödinger operator of (2.10), so the solutions of (2.12) include the product of any pair of solutions of (2.10). If the potential function \( q(x) \) is rational, it is known that the solution space of (2.12) contains a function \( \mathcal{F}(x; B) \) which is (i) meromorphic in \( x \) (the only poles being at the poles of \( q(x) \)) and (ii) monic polynomial in \( B \), if and only if (2.10) is a finite-band Schrödinger equation on \( E_{g_2, g_3} \), i.e., a finite-band Schrödinger equation with a doubly periodic potential (Its & Matveev 1974). This is an alternative to the characterization of Gesztesy & Weikard (1996), according to which (2.10) is finite-band if and only if for all \( B \in \mathbb{C} \), every solution \( \Psi \) is meromorphic on \( E_{g_2, g_3} \) (multi-valuedness being allowed).

For example, when \( q(x) = \ell (\ell + 1)x \), there is a polynomial solution \( \mathcal{F}_\ell(x; B; g_2, g_3) \) of (2.12) that satisfies conditions (i) and (ii), of degree \( \ell \) in both \( x \) and \( B \). So the integer-\( \ell \) Lamé equation is a finite-band Schrödinger equation. \( \mathcal{F}_\ell(x; B; g_2, g_3) \) is called the \( \ell \)th Hermite–Halphen polynomial. It may be written \( \mathcal{F}_\ell(x; B; g_2, g_3) \) when it is normalized to be monic in \( x \), rather than in \( B \) (see table 1).

In the case of a general rational potential \( q(x) \) which is finite-band, let \( \mathcal{F}(x; B) \) denote the specified solution of (2.12), and let its degree in \( B \) be denoted \( \ell \). It follows from manipulations parallel to those of Whittaker & Watson that the function \( \Psi \) on \( E_{g_2, g_3} \) defined by a formula containing an indefinite integral,

\[
\Psi(x, y; B, \nu) := \exp \left( \int \frac{F'(x; B) y - \nu}{F(x; B)} \, dx \right), \tag{2.13}
\]

will be a solution of the Schrödinger equation (2.10). Here \( B \in \mathbb{C} \) and \( \nu \) is a \( B \)-dependent but position-independent quantity, determined only up to sign, that can be computed by what Whittaker & Watson term an “interesting formula,”

\[
\nu^2 = -\frac{1}{2} F \left( y \frac{d}{dx} \right)^2 \mathcal{F} + \left[ \frac{1}{2} \left( y \frac{d}{dx} \right)^2 \right] F + [ q(x) + B ] \mathcal{F}_\ell^2. \tag{2.14}
\]

(It is not obvious that the right-hand side is independent of the point \((x, y) \in E_{g_2, g_3}\). It is widely known (Smirnov, 2002) that \( \nu \) is identical to the coordinate \( \nu \) on the spectral curve \( \Gamma_\ell \) defined by \( \nu^2 = \prod_{s=0}^{2\ell} (B - B_s) \), where \( \{ B_s \}_{s=0}^{2\ell} \) are the band edges of the Schrödinger operator; though no really simple proof of this fact seems to have been published. A consequence of this is that the formula (2.13) parametrizes solutions of the elliptic-curve Schrödinger equation (2.10) by \( (B, \nu) \in \Gamma_\ell \setminus \{ (\infty, \infty) \} \). As defined, \( \Psi \) is called a Baker–Akhiezer function (Krichever, 1990).

Consider now the special case of the integer-\( \ell \) Lamé equation (2.14). In this case the function \( \Psi \) computed by (2.13) from the Hermite–Halphen polynomials \( \mathcal{F} = \mathcal{F}_\ell(x; B; g_2, g_3) \) is in fact an Hermite–Halphen solution of the Lamé equation, reexpressed in terms of the elliptic curve coordinates \((x, y)\). One can write \( \Psi = \Psi^\pm(x, y; B; g_2, g_3) \), where the superscript ‘\( \pm \)’ refers to the ambiguity in the sign of

\[\text{Table 1. Hermite–Halphen polynomials (can der Waal, 2002, table A.2)}\]

| \( \ell \) | \( \mathcal{F}_\ell(x; B; g_2, g_3) \) |
|---|---|
| 1 | \( x - B \) |
| 2 | \( x^2 - \frac{1}{2} B x + \left( \frac{1}{4} B^2 - \frac{1}{4} g_2 \right) \) |
| 3 | \( x^3 - \frac{1}{3} B x^2 + \left( g_3 + \frac{1}{3} B^2 - \frac{1}{3} g_2 \right) x + \left( -\frac{1}{3} B^3 + \frac{1}{3} B g_2 - \frac{1}{3} g_3 \right) \) |
\(\nu = \nu(B)\). If \(\nu \neq 0\), the two solutions \(\Psi^\pm_\ell\) are distinct. They are path-multiplicative, since they are exponentials of anti-derivatives of rational functions on \(E_{g_2, g_3}\).

It should be noted that the Hermite–Halphen polynomials are not merely a tool for generating the solutions \(\Psi^\pm_\ell\) of the Lamé equation. They are algebraically interesting in their own right. [Klein 1892, figures 1, 2] supplies a sketch of the real portion of the curve \(F_\ell(x; B) = 0\) when \(\ell = 5, 6\), showing how when \(\ell \geq 4\), \(B\) is a band edge only if \(F_\ell(x; B)\), regarded as a polynomial in \(x\), has a double root.

The relevance of the fundamental multi-valued function \(\Phi\) can now be explained. It follows from (2.13), and the fact that \(F_1 = B - x\) (see table I), that

\[
\Psi^\pm_1(x, y; B; g_2, g_3) = \Phi(x, y; B, \pm \sqrt{4B^3 - g_2B - g_3}). \tag{2.15}
\]

That is, if \((x_0, y_0) \in E_{g_2, g_3}\) is ‘above’ \(x_0 = B\), then \(\Phi(\cdot, \cdot; x_0, y_0)\) will be a solution of the \(\ell = 1\) Lamé equation in the form (2.9). There are two such points, related by \(y_0\) being negated, unless \(4B^3 - g_2B - g_3 = 0\), i.e., unless \(B = e_1, e_2, e_3\), in which case \(y_0 = 0\) is the only possibility. These are the three band-edge values of \(B\) for \(\ell = 1\).

It is not difficult to show that the \(\ell = 1\) Hermite–Halphen solutions (2.15) are identical to the solutions (1.3), though they are expressed as functions of the variable \((x, y) \in E_{g_2, g_3}\) rather than the original independent variable \(\alpha \in \mathbb{C}\). The parametrizing point \((x_0, \pm y_0) \in E_{g_2, g_3}\) corresponds to the parameter \(\pm x_0 \in \mathbb{C}\) of (1.3). These solutions are clearly easier to formulate in the elliptic curve context.

For any integer \(\ell\), the Lamé dispersion relation can be computed numerically from (2.13) by calculating the multiplier arising from the path of integration winding around \(E_{g_2, g_3}\). However, (2.13) is not adapted to symbolic computation. By expanding the integrand in partial fractions one can derive the remarkable formula

\[
\Psi^\pm_\ell(x, y; B; g_2, g_3) = \prod_{r=1}^\ell \Phi(x, y; x_r, y^\pm_r), \tag{2.16}
\]

where \(\{(x_r, y_r^\pm)\}_{r=1}^\ell\) are points on \(E_{g_2, g_3}\) above \(\{x_r\}_{r=1}^\ell\), the \(B\)-dependent roots of the degree-\(\ell\) polynomial \(F_\ell(x; B; g_2, g_3)\). (Cf. Whittaker & Watson (1927, § 23.7).) Unfortunately, when \(\ell \geq 5\) the roots \(\{x_r\}_{r=1}^\ell\) cannot be computed in terms of radicals. This reduction to degree-1 solutions is less computationally tractable than the one that will be provided by the Hermite–Krichever Ansatz.

3. Finite families of Lamé equation solutions

The solutions of the integer-\(\ell\) Lamé equation include the Lamé polynomials, which are the traditional band-edge solutions. In the Jacobi-form context they are periodic or anti-periodic functions on \([0, 2K]\), with Floquet multiplier \(\xi = \pm 1\), respectively. There are exactly \(2\ell+1\) values of the spectral parameter \(B \in \mathbb{C}\), i.e., of the energy \(E\), for which a Lamé polynomial may be constructed, the counting being up to multiplicity. By definition these are the roots of the spectral polynomial \(L_\ell(B; g_2, g_3)\).

As functions on the curve \(E_{g_2, g_3}\), the Lamé polynomials are single or double-valued and are essentially polynomials in the coordinates \(x, y\). (In the Weierstrassian context \(\wp, \wp'\) substitute for \(x, y\).) However, no fully satisfactory table of the Lamé polynomials or the Lamé spectral polynomials has yet been published. Whittaker & Watson (1927, § 23.42) refer to a list of Guerritore (1909) that covers \(\ell \leq 10\). Sadly, although
he produced it as a *dissertazione di laurea* at the University of Naples, most of his results on \( \ell \geq 5 \) are incorrect. This has long been known \cite{Strutt1967}, but his paper is still occasionally cited for completeness \cite{Gesztesy1998}. Arscott \cite[§ 9.3.2]{Ahmad1964} gives a brief table of the Jacobi-form Lamé polynomials, covering only \( \ell = 1, 2, 3 \). His table is correct, with a single misprint \cite{Fernandez1998, Finkel2000}. But its brevity has been misinterpreted. An erroneous belief has arisen that when \( \ell \geq 4 \), the Lamé polynomial coefficients and band edge energies cannot be expressed in terms of radicals. This sets in only when \( \ell \geq 8 \).

Due to these confusions, in this section we tabulate the Lamé polynomials and the spectral polynomials \( L(B; g_2, g_3) \). Both are computed from coefficient recurrence relations. We supply such relations and tables of spectral polynomials for the twisted and theta-twisted Lamé polynomials, as well. The number of values of \( B \in \mathbb{C} \) for which the latter two sorts of solution exist, i.e., the degrees of their spectral polynomials, will be given. All three sorts of solution will play a role in Theorem L. In fact, all will be special cases of the solutions constructed for arbitrary \( B \) by the Hermite–Krichever Ansatz.

When \( \ell \geq 2 \), many of the spectral polynomials will have degenerate roots if \( g_2, g_3 \in \mathbb{C} \) are appropriately chosen. This means that, for example, a pair of the \( 2\ell + 1 \) band edge energies can be made to coincide by moving the modular parameter \( m \in \mathbb{C} \setminus \{0, 1\} \) to one of a finite set of complex values. We indicate how to calculate these, or the corresponding values of Klein’s absolute invariant \( J = g_2^3/(g_2^2 - 27g_3^2) \in \mathbb{C} \).

\( J \) is the more fundamental parameter, in algebraic geometry at least, since two elliptic curves are isomorphic (birationally equivalent) if and only if they have the same value of \( J \). The \( m \mapsto J \) correspondence \cite{Whittaker1927} maps \( \mathbb{C} \setminus \{0, 1\} \) onto \( \mathbb{C} \), and it also maps \( m \in (0, \frac{1}{2}) \) onto \( J \in [1, \infty) \). Formally it is 6-to-1. Each value of \( J \) corresponds to six values of \( m \), with the exception of \( J = 0 \) (i.e., \( g_2 = 0 \)), which corresponds to \( m = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i \), and \( J = 1 \) (i.e., \( g_3 = 0 \)), which corresponds to \( m = -1, \frac{1}{2}, 2 \). Elliptic curves with \( J = 0, 1 \) are called *equianharmonic* and *lemniscatic*, respectively \cite[§§ 18.13 and 18.15]{Abramowitz1964}. Any equianharmonic curve has a triangular period lattice, with \( \omega'/\omega = e^{\pm 2\pi i/3} \), and any lemniscatic curve has a square period lattice, with \( \omega'/\omega = \pm i \).

(a) Lamé polynomials

The Lamé polynomials are classified into species 1, 2, 3, 4 \cite[§ 23.2]{Whittaker1927}. This is appropriate for some forms of the Lamé equation, but for the elliptic-curve algebraic form, a more structured classification scheme is better.

**Definition.** A solution of the Lamé equation \cite{Whittaker1927} on the elliptic curve \( E_{g_2, g_3} \) is said to be a Lamé polynomial of Type I if it is single-valued and of the form \( C(x) \) or \( D(x) \), where \( C, D \) are polynomials. A solution is said to be a Lamé polynomial of Type II, associated with the branch point \( e_\gamma \) of the curve \( \gamma = 1, 2, 3 \), if it is double-valued and of the form \( E(x)/\sqrt{x - e_\gamma} \) or \( F(x)\sqrt{x - e_\gamma} \), where \( E, F \) are polynomials. The subtypes of Types I and II are species 1, 4 and 2, 3, respectively.

To determine necessary conditions on \( \ell \) and \( B \) for there to be a nonzero Lamé polynomial of each subtype, one may substitute the corresponding expression \( (C(x), \text{etc.}) \) into the Lamé equation \cite{Whittaker1927}, and work out a recurrence for the polynomial coefficients. This is similar to the approach of expanding in integer or half-integer
coefficients of powers of $x$. (Whittaker & Watson 1927, §23.41), though it leads to four-term rather than three-term recurrences. For the Type I solutions at least, the present approach seems more natural, since they are not associated with any singular point $c_\gamma$.

If $C(x) = \sum_j c_j x^j$, $D(x) = \sum_j d_j x^j$, $E(x) = \sum_j e_j x^j$, and $F(x) = \sum_j f_j x^j$, substituting the expression for each species of solution into (2.3) and equating the coefficients of powers of $x$ leads to the recurrence relations

\begin{align}
(2j - \ell)(2j + \ell + 1) c_j - B c_{j+1} \\
- (j + 2)(j + 3) c_j + (j + 3) c_{j+3} = 0, \tag{3.1}
\end{align}

\begin{align}
(2j - \ell - 3)(2j + \ell + 4) d_j - B d_{j+1} \\
- (j + 2)(j + 3) d_j + (j + 3) d_{j+3} = 0, \tag{3.2}
\end{align}

\begin{align}
(2j - \ell + 1)(2j + \ell + 2) e_j + [(4j + 5)e_\gamma - B] e_{j+1} \\
+ [(j + 2) g_2 + 4e_\gamma^2](j + 2) e_{j+2} - (j + 2)(j + 3) g_3 e_{j+3} = 0, \tag{3.3}
\end{align}

\begin{align}
(2j - \ell + 2)(2j + \ell + 3) f_j + [(4j + 7)e_\gamma - B] f_{j+1} \\
+ [(j + 2) g_2 + 4e_\gamma^2](j + 2) f_{j+2} - (j + 2)(j + 3) g_3 f_{j+3} = 0. \tag{3.4}
\end{align}

It is easy to determine the integers $\ell$ for which $C, D, E, F$ may be a polynomial.

Proposition 3.1. If $\ell \geq 1$ is odd, nonzero Type I Lamé polynomials of the fourth species and Type II ones of the second species can in principle be constructed from these recurrence relations, with $\deg D = (\ell - 3)/2$ and $\deg E = (\ell - 1)/2$ respectively. (The former assumes $\ell \geq 3$.) If $\ell \geq 2$ is even, nonzero Type I Lamé polynomial of
Table 3. Lamé spectral polynomials of Types I, II

(Most of the ones with \(\ell \geq 5\) disagree with those published by Gueritorte [1909].)

| \(\ell\) | \(L_{\ell}^I(B; g_2, g_3)\) |
|-------|-----------------|
| 1     | \(B - e_\gamma\) |
| 2     | \(B + 3e_\gamma\) |
| 3     | \(B^2 - 6e_\gamma B + (45e_\gamma^2 - 15g_2)\) |
| 4     | \(B^2 + 10e_\gamma B - (35e_\gamma^2 - 7g_2)\) |
| 5     | \(B^3 - 15e_\gamma B^2 + (315e_\gamma^2 - 132g_2)B + (\frac{1050}{7}e_\gamma g_2 + \frac{280}{7}g_3)\) |
| 6     | \(B^3 + 21e_\gamma B^2 - (189e_\gamma^2 - 84g_2)B + (\frac{3630}{7}e_\gamma g_2 + \frac{1440}{7}g_3)\) |
| 7     | \(B^4 - 28e_\gamma B^3 + (1134e_\gamma^2 - 574g_2)B^2 + (3409e_\gamma g_2 + 8525g_1)B\) + \(\frac{292383}{4}e_\gamma^2 g_2 - \frac{175175}{4}e_\gamma g_3 + 22113g_2^2\) |
| 8     | \(B^4 + 36e_\gamma B^3 + (594e_\gamma^2 - 414g_2)B^2 + (-9855e_\gamma g_2 + 12285g_1)B\) + \(\frac{245025}{4}e_\gamma^2 g_2 + \frac{552825}{4}e_\gamma g_3 + 7425g_2^2\) |

The first species and Type II ones of the third species can be constructed similarly, with \(\deg C = \ell/2\) and \(\deg F = (\ell - 2)/2\) respectively.

The coefficients in each Lamé polynomial are computed from the appropriate recurrence relation by setting the coefficient of the highest power of \(x\) to unity, and working downward. Unless \(B\) is specially chosen, the coefficients of negative powers of \(x\) may be nonzero. But by examination, they will be zero if the coefficient of \(x^{-1}\) equals zero.

**Definition.** The Type-I Lamé spectral polynomial \(L_{\ell}^I(B; g_2, g_3)\) is the polynomial monic in \(B\) which is proportional to the coefficient \(d_{-1}\) if \(\ell\) is odd and \(c_{-1}\) if \(\ell\) is even. (The former assumes \(\ell \geq 3\); by convention \(L_{1}^{I} := 1\).) The Type-II Lamé spectral polynomial \(L_{\ell}^{II}(B; e_\gamma, g_2, g_3)\) is similarly obtained from the coefficient \(e_{-1}\) if \(\ell\) is odd and \(f_{-1}\) if \(\ell\) is even. Each spectral polynomial may be regarded as \(\prod s[B - B_s(g_2, g_3)]\), resp. \(\prod s[B - B_s(e_\gamma, g_2, g_3)]\), where the roots \(\{B_s\}\) are the values of \(B\) for which a Lamé equation solution of the indicated type exists, counted with multiplicity.

By examination, \(N_{\ell}^I := \deg L_{\ell}^{I}\) is \((\ell - 1)/2\) if \(\ell\) is odd and \(\ell/2 + 1\) if \(\ell\) is even; and \(N_{\ell}^{II} := \deg L_{\ell}^{II}\) is \((\ell + 1)/2\) if \(\ell\) is odd and \(\ell/2\) if \(\ell\) is even. So as expected,

\[
L_{\ell}(B; g_2, g_3) := L_{\ell}^{I}(B; g_2, g_3) \prod_{\gamma=1}^{3} L_{\ell}^{II}(B; e_\gamma, g_2, g_3), \tag{3.5}
\]

the full Lamé spectral polynomial, has degree \(N_{\ell}^I + 3N_{\ell}^{II} = 2\ell + 1\) in \(B\).

It should be noted that \(\prod_{\gamma=1}^{3} L_{\ell}^{II}(B; e_\gamma, g_2, g_3)\), the full Type-II Lamé spectral polynomial, is a function only of \(B; g_2, g_3\), since any symmetric polynomial...
in $e_1, e_2, e_3$ can be written in terms of $g_2, g_3$. For example, $e_1 e_2 e_3 = g_3/4$. This is why $e_γ$ is absent on the left-hand side of (3.3). When using the recurrences (3.4)–(3.5), one should also note that $4e_γ^3 - g_2 e_γ - g_3 = 0$, so $e_γ^3 = \frac{1}{4}(g_2 e_γ + g_3)$. Any polynomial in $e_γ, g_2, g_3$ can be reduced to one which is of degree at most 2 in $e_γ$, much as any polynomial in $x, y$ can be reduced to one of degree at most 1 in $y$.

The Lamé polynomials of Types I and II are listed in table 2 and the corresponding spectral polynomials in table 3. They replace the table of Guerritore (1909), with its many unfortunate errors. The spectral polynomials with $\ell \leq 7$ were recently computed by a different technique (van der Waall 2002, table A.3). The table of van der Waall displays the full Type II spectral polynomials, rather than the more fundamental $e_γ$-dependent polynomials $L_1^1(B; e_γ, g_2, g_3)$. The spectral polynomials can also be computed by the technique of Gesztesy & Weikard (1995a), which employs the Weierstrassian counterpart of the Ansatz used by Hermite in his solution of the Jacobi-form Lamé equation (Whittaker & Watson 1927, § 23.71).

The roots of the spectral polynomials are the energies $B$ for which the Lamé polynomials are solutions of the Lamé equation. It is clear that when $\ell \geq 9$, the Type II energies cannot be expressed in terms of radicals, since the degree of the spectral polynomial will be 5 or above. When $\ell = 8$ or $\ell \geq 10$, the Type I energies cannot be so expressed. These statements apply also to the coefficients of the Lamé polynomials, which depend on $B$. So when $\ell \geq 10$, the symbolic computation of the Lamé polynomials is impossible, and when $\ell = 8$ or 9, it is possible only in part. But when $g_2, g_3$ take on special values, what would otherwise be impossible may become possible. For instance, when $g_3 = 0$ (the lemniscatic case, including $m = \frac{1}{2}$), the

| $\ell$ | Type I Cohn polynomial |
|--------|------------------------|
| 1      | $J$                    |
| 2      | $J$                    |
| 3      | $J$                    |
| 4      | $2^2 3^2 J + 5^2 7^2$  |
| 5      | $J^2$                  |
| 6      | $2^2 3^4 2^2 J^2 + 11 \cdot 37 \cdot 59 J - 2^2 3^7$ |
| 7      | $2^2 3^5 5^2 J + 11^3 13^2$ |
| 8      | $J (2^3 5^2 7^2 J^3 + 2^3 3^3 3664447 J^2 - 2^4 3^2 \cdot 397 \cdot 364069 J + 11^3 5^2)$ |

| $\ell$ | Type II Cohn polynomial |
|--------|-------------------------|
| 1      | $J$                     |
| 2      | $J$                     |
| 3      | $2^2 J + 1$             |
| 4      | $2^3 3^2 J - 5^3$       |
| 5      | $2^3 3^2 J^3 + 2^3 3^2 5^2 109 J^2 - 2^2 5^4 17 \cdot 151 J + 5^6 7^3$ |
| 6      | $2^3 3^2 J^3 + 2^3 3^2 5^2 109 J^2 - 2^2 5^4 17 \cdot 151 J + 5^6 7^3$ |
| 7      | $2^3 3^2 J^3 + 2^3 3^2 5^2 109 J^2 - 2^2 5^4 17 \cdot 151 J + 5^6 7^3$ |
| 8      | $2^3 3^2 J^3 + 2^3 3^2 5^2 109 J^2 - 2^2 5^4 17 \cdot 151 J + 5^6 7^3$ |
quintic spectral polynomial $L_5(B; g_2, g_3)$ reduces to $B^5 - 1044g_2B^3 + 112320g_2^2B$, the roots of which can obviously be expressed in terms of radicals.

In the context of the Jacobi form, the $2\ell + 1$ values \{E_{\ell}(m)\}_{\ell=0}^2$ for which a Lamé polynomial solution exists can be thought of as the $2\ell + 1$ branches of a spectral curve that lies over the triply-punctured sphere $P^1 \setminus \{0, 1, \infty\}$, the space of values of the modular parameter $m$. Turbiner (1989) showed that if $\ell \geq 2$, this curve has only four connected components, not $2\ell + 1$. It is now clear why. These are the Type I component and the three Type II components, one associated with each point $e_\gamma$. Since each of the four is defined by a polynomial in $E$ and $m$, each can be extended to an algebraic curve over $P^1$. At the values $m = 0, 1, \infty$, the four curves may touch one another. (See, e.g., (2000, figure 3), for the behavior of the real portions of the $\ell = 1, 2$ curves as $m \to 0, 1$, and (1983) for $\ell = 3$.) These three values of $m$ correspond to two of $e_1, e_2, e_3$ coinciding, and the elliptic curve $y^2 = 4 \prod_{\gamma=1}^3(x - e_\gamma)$ becoming rational rather than elliptic. Level crossings of this sort are perhaps less interesting than ‘intra-curve’ ones.

In the present context, $E$ is replaced by the transformed energy $B$, and $m$ by the pair $g_2, g_3$ or the Klein invariant $J$, with $J = \infty$ corresponding to $m = 0, 1, \infty$. It is easy to determine which finite values of $J$ yield coincident values of $B$. One simply computes the discriminants of the Type I and full Type II spectral polynomials, $L_{\ell}(B; g_2, g_3)$ and $\prod_{\gamma=1}^2 L_{\ell}(B; e_\gamma, g_2, g_3)$. Each discriminant is zero if and only if there is a double root. By using $J = g_2^3/(g_3^3 - 27g_3^2)$, one can eliminate $g_2, g_3$ and obtain a polynomial equation for $J$. For each of Types I and II, there are coincident values of $B$ if and only if $J$ is a root of what may be called a Cohn polynomial.

In Table 2 the Cohn polynomials are listed. Since the coefficients are rather large integers that may have number-theoretic significance, each is given in a fully factored form. An interesting feature of these polynomials is that none has a zero on the real half-line $[1, \infty)$. Since $J \in (1, \infty)$ corresponds to $m \in (0, 1)$, the existence of such a zero would imply that for some $m \in (0, 1)$, two of the $2\ell + 1$ band edges become degenerate. That this cannot occur follows from a Sturmian argument [Whittaker & Watson 1927, §23.41]. It also follows from the analysis of Gesztesy & Weikard (1995a, §3).

**Proposition 3.2.** For any integer $\ell \geq 1$, the degeneracies of the algebraic spectrum of the Lamé operator, which comprises the $2\ell + 1$ roots (up to multiplicity) of the spectral polynomial $L_{\ell}(B; g_2, g_3)$, are fully captured by the Cohn polynomials of Types I and II. As the parameters $g_2, g_3$ are varied, a pair of roots will coincide, reducing the number of distinct roots from $2\ell + 1$ to $2\ell$, if and only if the Klein invariant $J$ is a root of one of the two Cohn polynomials, and there are no multiple coincidences.

This proposition will be proved in §4. The following conjecture is based on a close examination of the spectral and Cohn polynomials, for all $\ell \leq 25$.

**Conjecture 3.3.**

1. As a polynomial in $J$ with integer coefficients, no Cohn polynomial has a nontrivial factor, except for the Type I Cohn polynomials with $\ell \equiv 2 \pmod{3}$, each of which is divisible by $J$. (These factors of $J$ are visible in Table 2.)

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2. If $N^I_\ell$ and $N^{II}_\ell$ denote the degrees of the spectral polynomials of Types I, II, which are given above, then the Cohn polynomials of Types I, II have degrees

$$\left\lfloor \left( N^I_\ell^2 - N^I_\ell + 4 \right) / 6 \right\rfloor$$

and $N^{II}_\ell (N^{II}_\ell - 1) / 2$, respectively.

The conjectured degree formulas constitute a conjecture as to the number of points in elliptic moduli space (elliptic curve parameter space), labelled by $J$, at which the $2\ell + 1$ distinct energies in the algebraic spectrum are reduced to $2\ell$. For example, $N^I_3, N^{II}_3$ equal 1, 2, so when $\ell = 3$ there is no Type-I polynomial, and the Type-II one is linear in the invariant $J$. According to table 4 it equals $4J + 1$. A nondegeneracy condition equivalent to the linear condition $4J + 1 \neq 0$ was previously worked out by Treibich (1994, § 6.6), namely $\prod_{3=1}^3 (5g_2 - 12c_2^2) \neq 0$.

The remarks regarding extra $J$ factors amount to a conjecture that in the equianharmonic case $m = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ (i.e., $J = 0$ or $g_2 = 0$), there are only $2\ell$ distinct energies if and only if $\ell \equiv 2 \pmod{3}$. For those values of $\ell$, the double energy eigenvalue is evidently located at $B = 0$. It should be mentioned that when $J = 0$ and $\ell \equiv 0 \pmod{3}$, there is also an eigenvalue at $B = 0$, but it is a simple one.

A periodicity of length 3 in $\ell$ is present in the equianharmonic case of a third-order equation resembling (2.12), now called the Halphen equation (Halphen 1888, pp 571–4). By the preceding, a similar periodicity appears to be present in the equianharmonic case of the Lamé equation. This was not previously realized.

(b) Twisted Lamé polynomials

The twisted Lamé polynomials are exponentially modified Lamé polynomials. They will play a major role in Theorem L and in the hyperelliptic reductions following from the Hermite–Krichever Ansatz, but they are of independent interest.

**Definition.** A solution of the Lamé equation (2.9) on the elliptic curve $E_{g_2,g_3}$ is said to be a twisted Lamé polynomial, of Type I or Type II associated with the point $e_\gamma$, if it has one of the two forms

$$\left\{ \begin{array}{l}
C(x) + D(x) y \\
E(x) \sqrt{x - e_\gamma} + F(x) y / \sqrt{x - e_\gamma}
\end{array} \right\} \times \exp \left[ \kappa \int \frac{dx}{y} \right],$$

with $\kappa \in \mathbb{C}$ nonzero. Here $C, D, E, F$ are polynomials.

On the level of Fuchsian differential equations, there is little to distinguish between twisted Lamé polynomials and ordinary Lamé polynomials, which are simply twisted polynomials with $\kappa = 0$. A function $\Psi(x,y) = \hat{\Psi}(x,y) \exp [\kappa \int dx/y]$ will be a solution of the Lamé equation (2.9) if and only if $\hat{\Psi}$ satisfies

$$\left\{ \left( \frac{d}{dy} \right)^2 + 2\kappa \left( \frac{d}{dy} \right) - \left[ \ell(\ell + 1) x + B - \kappa^2 \right] \right\} \hat{\Psi} = 0. \quad (3.6)$$

This is a Fuchsian equation on $E_{g_2,g_3}$ that generalizes but strongly resembles (2.9). It has a single regular singular point, $(x,y) = (\infty,\infty)$, and its characteristic exponents there are $\{-\ell, \ell + 1\}$. Like $B$, $\kappa$ is an accessory parameter that does not
and equating the coefficients of powers of $x$ yields the coupled pairs of recurrences

\begin{align}
(2j - \ell)(2j + \ell + 1) c_j + (\kappa^2 - B) c_{j+1} & - (j + 1)(j + 3)g_2 c_{j+2} - (j + 2)(j + 3)g_3 c_{j+3} \\
& + 2\kappa (4j + 2) d_{j-1} - 2\kappa (j + 3)g_2 d_{j+1} \\
& - 2\kappa (j + 2)g_3 d_{j+2} = 0, \\
(2j - \ell + 1)(2j + \ell + 2) e_j + [(4j + 5)e_\gamma + \kappa^2 - B] e_{j+1} & - [(j + 1)g_2 + 4\kappa^2 e_\gamma](j + 2) e_{j+2} - (j + 3)(j + 3)g_3 e_{j+3} \\
& + 2\kappa (4j + 4) f_j + 2\kappa (4j + 6)e_\gamma f_{j+1} \\
& + 2\kappa (4j + 8)(e_\gamma^2 - \frac{1}{4}g_2) f_{j+2} = 0, \\
(2j - \ell + 2)(2j + \ell + 3) f_j + [(4j + 7)e_\gamma + \kappa^2 - B] f_{j+1} & - [(j + 1)g_2 - 4\kappa^2 e_\gamma](j + 2) f_{j+2} + (j + 3)(j + 3)g_3 f_{j+3} \\
& + 2\kappa (j + 2)e_\gamma f_{j+1} - 2\kappa (j + 2)e_\gamma e_{j+2} = 0.
\end{align}

It is easy to determine the maximum value of the exponent $j$ in each of $C, D, E, F$.

**Proposition 3.4.** Nonzero twisted Lamé polynomials of Type I (if $\ell \geq 3$) and of Type II (if $\ell \geq 2$) can in principle be constructed from these recurrence relations. If $\ell$ is odd, resp. even, then $\deg C, D; E, F$ are $(\ell - 1)/2, (\ell - 3)/2; (\ell - 1)/2, (\ell - 3)/2$, resp. $\ell/2, (\ell - 4)/2; \ell/2 - 1, \ell/2 - 1$. The coefficients are computed by setting the coefficient of the highest power of $x$ to unity in $D$ and $E$, resp. $C$ and $F$, and working downward.

Unless $B, \kappa$ are specially chosen, the coefficients of negative powers of $x$ may be nonzero. But by examination, they will be zero if the coefficients of $x^{-1}$ in $C$ and $D$ (for Type I), or $E$ and $F$ (for Type II), equal zero. $c_{-1} = 0, d_{-1} = 0,$ and $e_{-1} = 0, f_{-1} = 0,$ are coupled polynomial equations in $B, \kappa,$ and their solutions may be computed by polynomial elimination, e.g., by computing resultants. A minor problem is the proper handling of the case $\kappa = 0$, in which relations reduce to $3.1-3.4$. If $\ell$ is odd, resp. even, then $c_{-1}$ and $f_{-1},$ resp. $d_{-1}$ and $e_{-1},$ turn out to be divisible by $\kappa.$ By dividing the appropriate equations by $\kappa$ before solving each pair of coupled equations, the spurious $\kappa = 0$ solutions can be eliminated.

**Definition.** The Type-I twisted Lamé spectral polynomial $Lt^1_\ell(B; g_2, g_3)$ is the polynomial monic in $B$ which is proportional to the resultant of $c_{-1}, d_{-1}$ with respect to $\kappa$, with $\kappa$ factors removed as indicated. (This assumes $\ell \geq 3$; by convention...
$Lt^1_l = Lt^2_l := 1$. The Type-II twisted Lamé spectral polynomial $Lt^1_l(B; e_1, g_2, g_3)$ is similarly obtained from $e_{-1}, f_{-1}$. (This assumes $\ell \geq 2$; by convention $Lt^1_1 := 1$.) Each twisted spectral polynomial may be regarded as $\prod_s (B - B_s(e_1, g_2, g_3))$, resp. $\prod_s (B - B_s(e_2, g_2, g_3))$, where the roots $B_s$ are the values of $B$ for which a Lamé equation solution of the specified type exists, counted with multiplicity. The twisted Lamé spectral polynomials for $\ell \leq 8$ are listed in Table 5. The polynomials $Lt^1_l$ and $Lt^2_l$ are omitted on account of lack of space (their respective degrees are 12 and 16). The following proposition will be proved in §4.

**Proposition 3.5.**

1. For any integer $\ell \geq 3$, resp. $\ell \geq 2$, there is a nontrivial twisted spectral polynomial of Type I, resp. Type II.

2. For any integer $\ell \geq 1$, $Nt^1_l := \deg Lt^1_l$ is $(\ell^2 - 1)/4$ if $\ell$ is odd and $\ell^2/4 - 1$ if $\ell$ is even; and $Nt^2_l := \deg Lt^2_l$ is $(\ell^2 - 1)/4$ if $\ell$ is odd and $\ell^2/4 - 1$ if $\ell$ is even. So the full twisted Lamé spectral polynomial $Lt_{\ell}(B; g_2, g_3)$, which by definition equals $Lt^1_l(B; g_2, g_3) \prod_{s=1}^{\ell} Lt^1_l(B; e_s, g_2, g_3)$, will be of degree $Nt^1_\ell + 3Nt^2_\ell = \ell^2 - 1$ in the spectral parameter $B$. Like the ordinary degree-$(2\ell + 1)$ spectral polynomial $L_{\ell}$, $Lt_{\ell}$ is a function of $B; g_2, g_3$ only, because any symmetric polynomial in $e_1, e_2, e_3$ can be written in terms of the invariants $g_2, g_3$.

(c) **Theta-twisted Lamé polynomials**

Lamé equation solutions of a third sort can be constructed for certain values of the spectral parameter $B$. These are linear combinations, over polynomials in the coordinate $x$, of (i) the multi-valued meromorphic function $\Phi(x, y; x_0, y_0)$ parametrized by the point $(x_0, y_0) \in E_{g_2, g_3} \setminus \{\infty, \infty\}$, and (ii) its derivative

$$\Phi^{(1)}(x, y; x_0, y_0) := \left(\frac{y}{dx} \right) \Phi(x, y; x_0, y_0) = \frac{1}{2} \left( \frac{y + y_0}{x - x_0} \right) \Phi(x, y; x_0, y_0).$$

One way of seeing that $\Phi, \Phi^{(1)}$ are a natural basis is to note that when $(x_0, y_0) = (e_\gamma, 0)$, they reduce to $\sqrt{x - e_\gamma}, \frac{1}{2} y/\sqrt{x - e_\gamma}$. So the class of functions constructed from them will include the Lamé polynomials of Type II.

**Definition.** A solution of the Lamé equation [290] on the elliptic curve $E_{g_2, g_3}$ is said to be a theta-twisted Lamé polynomial if it is of the form $A(x) \Phi(x, y; x_0, y_0) + 2B(x)\Phi^{(1)}(x, y; x_0, y_0)$, with $(x_0, y_0) \neq (e_\gamma, 0)$ for $\gamma = 1, 2, 3$. Here $A, B$ are polynomials, and the innocuous ‘2’ factor compensates for the $\frac{1}{2}$ factor of (3.11).

If $A(x) = \sum_j a_j x^j$ and $B(x) = \sum_j b_j x^j$, substituting this expression into (2.9) and equating the coefficients of powers of $x$ yields the coupled pair of recurrences

$$\begin{align*}
(2j - \ell + 1)(2j + \ell + 2) a_j + [(4j + 5)x_0 - B] a_{j+1} \\
+ [- (j + \frac{3}{2})g_2(4x_0^2(j + 2) a_{j+2} - (j + 2)(j + 3)g_3) a_{j+3} \\
- 2y_0(4j + 6) b_{j+1} - 4x_0y_0(j + 2) b_{j+2} = 0, \\
(2j - \ell + 2)(2j + \ell + 3) b_j \\
+ [- (j + \frac{3}{2})g_2(4x_0^2(j + 2) b_{j+2} - (j + 2)(j + 3)g_3) b_{j+3} \\
y_0(j + 2) a_{j+2} = 0. \\
\end{align*}
$$
Lamé polynomials, hyperelliptic reductions and Lamé band structure

Table 5. Twisted Lamé spectral polynomials of Types I,II

| $\ell$ | $L_{\ell}^I(B; g_1, g_3)$ |
|-------|-------------------|
| 1     | $g_3$             |
| 2     | $g_3 + g_3$       |
| 3     | $B^2 - \frac{75}{4} g_2$ |
| 4     | $B^3 - \left(\frac{171}{2} g_3 + \frac{945}{2} g_2 B + 3 \right) g_3 B^2 + \frac{945}{2} g_2 B$ |
| 5     | $B^6 - \left(\frac{171}{2} g_3 + \frac{3 \cdot 2^3 \cdot 5^2 \cdot 7}{2} g_2 B + \right) g_2 B^2 + \frac{945}{2} g_2 B$ |
| 6     | $B^9 - \left(\frac{171}{2} g_3 + \frac{3 \cdot 2^3 \cdot 5^2 \cdot 7}{2} g_2 B + \right) g_2 B^2 + \frac{945}{2} g_2 B$ |

| $\ell$ | $L_{\ell}^I(B; c_1, g_2, g_3)$ |
|-------|-------------------|
| 1     | $g_3$             |
| 2     | $g_3 + g_3$       |
| 3     | $B^2 - 15 c_1 B + \left(\frac{225 e^2}{2} + \frac{225 e}{2} g_2 \right) B$ |
| 4     | $B^3 - \left(\frac{171}{2} g_3 + \frac{945}{2} g_2 B + \right) g_3 B^2 + \frac{945}{2} g_2 B$ |
| 5     | $B^6 - \left(\frac{171}{2} g_3 + \frac{3 \cdot 2^3 \cdot 5^2 \cdot 7}{2} g_2 B + \right) g_2 B^2 + \frac{945}{2} g_2 B$ |
| 6     | $B^9 - \left(\frac{171}{2} g_3 + \frac{3 \cdot 2^3 \cdot 5^2 \cdot 7}{2} g_2 B + \right) g_2 B^2 + \frac{945}{2} g_2 B$ |
Proposition 3.6. If $\ell \geq 4$, nonzero theta-twisted Lamé polynomials can in principle be computed from these recurrences. If $\ell$ is odd, resp. even, then $\deg \mathcal{A}, \mathcal{B}$ are $(\ell - 1)/2$, $(\ell - 5)/2$, resp. $\ell/2 - 2$, $\ell/2 - 1$. The coefficients are computed by setting the coefficient of the highest power of $x$ in $\mathcal{A}$, resp. $\mathcal{B}$, to unity, and working downward.

Unless $B$ and the point $(x_0, y_0)$ are specially chosen, the coefficients of negative powers of $x$ may be nonzero. But by examination, they will be zero if the coefficients of $x^{-1}$ in $\mathcal{A}$ and $\mathcal{B}$ are both zero. $a_{-1} = 0$, $b_{-1} = 0$ are equations in $B; x_0, y_0$. Together with the identity $y_0^2 = 4x_0^3 - g_2x_0 - g_3$, they make up a set of three polynomial equations for these three unknowns. This system may be solved by polynomial elimination. For example, to obtain a single polynomial equation for $B$ (involving $g_2, g_3$ of course), one may eliminate $y_0$ from $a_{-1} = 0$, $b_{-1} = 0$ by computing their resultants against the third equation; and then eliminate $x_0$. Alternatively, a Gröbner basis calculation may be performed (Brezhnev 2004).

Irrespective of which procedure is followed, there is a minor problem: the handling of the improper case $(x_0, y_0) = (e_1, 0)$, in which (3.12)–(3.13) reduce to (3.3)–(3.4). If $\ell$ is odd, resp. even, then the left-hand side of the equation $b_{-1} = 0$, resp. $a_{-1} = 0$, turns out to be divisible by $y_0$. By dividing the appropriate equation by $y_0$ before eliminating $x_0, y_0$, the spurious solutions with $y_0 = 0$ can be eliminated.

Definition. The theta-twisted Lamé spectral polynomial $L\theta_\ell(B; g_2, g_3)$ is the polynomial monic in $B$ which is obtained by eliminating $x_0, y_0$ from the equations $a_{-1} = 0$, $b_{-1} = 0$, with $y_0$ factors removed as indicated. (This assumes $\ell \geq 4$; by convention $L\theta_1 = L\theta_2 = L\theta_3 := 1$.) Each theta-twisted spectral polynomial may be regarded as $\prod_i [B - B_i(g_2, g_3)]$, where the roots $\{B_i\}$ are the values of $B$ for which a theta-twisted Lamé polynomial exists, counted with multiplicity.

The theta-twisted Lamé spectral polynomials for $\ell \leq 8$ are listed in Table 6. The following proposition will be proved in §4.

**Table 6. Theta-twisted Lamé spectral polynomials**

| $\ell$ | $L\theta_\ell(B; g_2, g_3)$ |
|--------|-----------------------------|
| 1      | $L\theta_1(B; g_2, g_3)$   |
| 2      | $L\theta_2(B; g_2, g_3)$   |
| 3      | $L\theta_3(B; g_2, g_3)$   |
| 4      | $L\theta_4(B; g_2, g_3)$   |
| 5      | $L\theta_5(B; g_2, g_3)$   |
| 6      | $L\theta_6(B; g_2, g_3)$   |
| 7      | $L\theta_7(B; g_2, g_3)$   |
| 8      | $L\theta_8(B; g_2, g_3)$   |

**Proposition 3.7.**
1. For any integer \( \ell \geq 4 \) there is a nontrivial theta-twisted spectral polynomial \( L\theta_\ell \).

2. For any integer \( \ell \geq 2 \), \( N\theta_\ell := \deg L\theta_\ell \) is \( (\ell + 1)(\ell - 3)/4 \) if \( \ell \) is odd and \( \ell(\ell - 2)/4 \) if \( \ell \) is even.

4. The Hermite–Krichever Ansatz

The Hermite–Krichever Ansatz is a tool for solving any Schrödinger-like differential equation, not necessarily of second order, with coefficient functions that are elliptic. Such an equation should ideally have one or more independent solutions which, according to the Ansatz, are expressible as finite series in the derivatives of an elliptic Baker–Akhiezer function, including an exponential factor. (Cf. (1.5).)

In the context of (2.9), the elliptic-curve algebraic form of the Lamé equation, this means that one hopes to be able to construct a solution on the curve \( E_{g_2,g_3} \), except at a finite number of values of the spectral parameter \( B \in \mathbb{C} \), as a finite series in the functions \( \Phi(j) := (y d/dx)^j \Phi \), \( j \geq 0 \), multiplied by a factor \( \exp \left[ \kappa \int dx/y \right] \).

Here \( \Phi(x, y) \) is the fundamental multi-valued meromorphic function on \( E_{g_2,g_3} \) introduced in \( \S 2 \). Actually, a different but equivalent sort of series is easier to manipulate symbolically. By examination \( \Phi(2) = (2x + x_0)\Phi \), from which it follows by induction on \( j \) that any finite series in \( \Phi(j), j \geq 0 \), is a combination (over polynomials in \( x \)) of the basis functions \( \Phi, \Phi(1) \). This motivates the following definition.

**Definition.** A solution of the Lamé equation (2.9) on the elliptic curve \( E_{g_2,g_3} \) is said to be an Hermite–Krichever solution if it is of the form

\[
\left[ A(x)\Phi(x, y; x_0, y_0) + 2B(x)\Phi^{(1)}(x, y; x_0, y_0) \right] \exp \left[ \kappa \int dx/y \right] = \left[ A(x) + B(x) \left( y + y_0 \right) \right] \Phi(x, y; x_0, y_0) \exp \left[ \kappa \int dx/y \right],
\]

for some \((x_0, y_0) \in E_{g_2,g_3} \setminus \{(\infty, \infty)\} \) and \( \kappa \in \mathbb{C} \). Here \( A, B \) are polynomials.

As defined, Hermite–Krichever solutions subsume most of the solutions explored in \( \S 3 \). If \( \kappa = 0 \), they reduce to theta-twisted Lamé polynomials. If \((x_0, y_0) = (e_\gamma, 0) \) for \( \gamma = 1, 2, 3 \), in which case \( \Phi, \Phi^{(1)} \) degenerate to \( \sqrt{x - e_\gamma}, \frac{1}{2} y/\sqrt{x - e_\gamma} \), they reduce to twisted Lamé polynomials of Type II. If both specializations are applied, they reduce to ordinary Lamé polynomials of Type II. The Lamé polynomials of Type I, both ordinary and twisted, are not of the Hermite–Krichever form, but they can be viewed as arising from a passage to the \((x_0, y_0) \to (\infty, \infty) \) limit.

If \( A(x) = \sum_j a_j x^j \) and \( B(x) = \sum_j b_j x^j \), substituting (4.1) into (2.9) and equat-
ing the coefficients of powers of $x$ yields the coupled pair of recurrences

\begin{align}
(2j - \ell + 1)(2j + \ell + 2) a_j + [(4j + 5)x_0 + \kappa^2 - B] a_{j+1} & \\
+ [- (j + 3/2)g_2 + 4x_0^2 - 2\kappa y_0](j + 2) a_{j+2} - (j + 2)(j + 3)g_3 a_{j+3} & \\
+ 8\kappa(j + 1) b_j + 4(\kappa x_0 - y_0)(2j + 3) b_{j+1} & \\
+ 2[\kappa(4x_0^2 - g_2) - 2x_0 y_0](j + 2) b_{j+2} = 0, & \\
(2j - \ell + 2)(2j + \ell + 3) b_j + [- (j + 7)x_0 + \kappa^2 - B] b_{j+1} & \\
+ [- (j + 3/2)g_2 - 4x_0^2 + 2\kappa y_0](j + 2) b_{j+2} - (j + 2)(j + 3)g_3 b_{j+3} & \\
+ \kappa(2j + 3) a_{j+1} - (2\kappa x_0 - y_0)(j + 2) a_{j+2} = 0. & 
\end{align}

If $\kappa = 0$, \(4.2\) and \(4.3\) reduce to \(3.1\) and \(3.3\), and if \((x_0, y_0) = (e_\gamma, 0)\), they reduce to \(3.1\). If both specializations are applied, they reduce to \(3.1\).

**Proposition 4.1.** For all $\ell \geq 2$, Hermite–Krichever solutions can in principle be computed from these recurrences. If $\ell$ is odd, resp. even, then deg $A, B$ are $(\ell - 1)/2$, $(\ell - 3)/2$, resp. $\ell/2$–1, $\ell/2$–1. The coefficients are computed by setting the coefficient of the highest power of $x$ in $A$, resp. $B$, to unity, and working downward.

Unless $B, \kappa$ and the point $(x_0, y_0)$ are specially chosen, the coefficients of negative powers of $x$ may be nonzero. But by examination, they will be zero if the coefficients of $x^{-1}$ in $A$ and $B$ are both zero. $a_{-1} = 0$, $b_{-1} = 0$ are equations in $B; x_0, y_0$. They are ‘compatibility conditions’ similar to those that appear in other applications of the Hermite–Krichever Ansatz. Together with the identity $y_0^2 = 4x_0^3 - g_2x_0 - g_3$, they make up a set of three equations for these four unknowns.

Informally, one may eliminate any two of $B; x_0, y_0$, and derive an algebraic relation between the remaining two unknowns (involving $g_2, g_3$ of course). A rigorous investigation must be more careful. For example, if the ideal generated by the three equations contained a polynomial involving only $B$ (and $g_2, g_3$), then a solution of the Hermite–Krichever form would exist for very few values of $B$ \cite{Brezhnev2004}. In practice, this problem does not arise: except at a finite number of values of $B$ (at most), the Ansatz can be employed \cite{Gesztesy1998}. In fact, in previous work an algebraic curve in $(B, \kappa)$ has been derived for each $\ell \leq 5$. The $(B, \kappa)$-curve is the one with the most physical significance, since $B$ is a transformed energy and $\kappa$ is related to the crystal momentum.

Solving for $(x_0, y_0)$ as functions of $(B, \kappa)$ reveals that $x_0$ is a rational function of $B$, and that if $\kappa$ is not identically zero, $y_0$ is a rational function of $B$, times $\kappa$. These facts can be interpreted in terms of the following seemingly different curve.

**Definition.** The $\ell$th Lamé spectral curve $\Gamma_\ell := \Gamma_\ell(g_2, g_3)$ is the hyperelliptic curve over $\mathbb{P}^1 \ni B$ comprising all $(B, \nu)$ satisfying $\nu^2 = L_\ell(B; g_2, g_3)$, where $L_\ell$ is the full Lamé spectral polynomial, of degree $2\ell + 1$ in $B$. $\Gamma_\ell$ was informally introduced as $\bar{\Gamma}_\ell$ in \cite{McKean1979}, where the original energy parameter $E$ was used. $\Gamma_\ell$ will have genus $\ell$ unless two roots of $L_\ell(\cdot; g_2, g_3)$ coincide, i.e., unless the Klein invariant $J$ is a root of one of the two Cohn polynomials of table \cite{McKean1979} in which case the genus equals $\ell - 1$.

For each $\ell$, there must exist a parametrization of Lamé equation solutions by a point $(B, \nu)$ on the punctured curve $\Gamma_\ell \setminus \{(\infty, \infty)\}$, by the general theory of Hill’s equation on $\mathbb{R} \setminus \mathbb{R}$ \cite{McKean1979}. For any finite-band Schrödinger
equation on an elliptic curve, including the integer-ℓ Lamé equation, the Baker–Akhiezer function [24,3] provides such a parametrization of solutions. In the general theory, the parametrizing hyperelliptic curve Γ for any finite-band Hill’s equation arises from differential-difference bispectrality: as a uniformization of the relation between the energy parameter B and the crystal momentum k [Treibich 2001]. This curve Γ ⊃ (B, ν) is defined by an irrationality of the form ν² = L(B), and B and k are meromorphic functions on it; the former single-valued, and the latter additively multi-valued. The energy is computed from the degree-2 map B : Γ → ℙ¹ given by (B, ν) ↦ B, and the crystal momentum from the formula

\[ k = -i \oint \left[ \frac{1}{2} \left( \frac{y + y_0(B, \nu)}{x - x_0(B, \nu)} \right) + \kappa(B, \nu) \right] \frac{dx}{y} \]  

in which the line integrals on \( E_{g_2,g_3} \) are taken over the appropriate fundamental loop. Here (B, ν) ↦ (x₀, y₀) is a certain projection \( \pi : \Gamma \rightarrow E_{g_2,g_3} \), and \( \kappa : \Gamma \rightarrow \mathbb{P}^1 \) is a certain auxiliary meromorphic function. These two morphisms of complex manifolds are ‘odd’ under the involution (B, ν) ↦ (B, −ν), i.e., \( x₀(B, −ν) = x₀(B, ν) \) and \( y₀(B, −ν) = −y₀(B, ν) \), and \( \kappa(B, −ν) = −\kappa(B, ν) \). So \( x₀ \) must be a rational function of \( B \), and each of \( y₀ \) and \( \kappa \) must be a rational function of \( B \), times \( ν \).

In the general theory of finite-band equations, the Baker–Akhiezer uniformization is viewed as more fundamental than the Hermite–Krichever Ansatz. However, in the case of the integer-ℓ Lamé equation one can immediately identify the curve in \( (B, \kappa) \), derived from the Ansatz as explained above, with the ℓᵗʰ Lamé spectral curve \( \gamma \), π : \( \Gamma \rightarrow E_{g_2,g_3} \). It is isomorphic to it by a birational equivalence of a simple kind: the ratio \( \kappa/ν \) is a rational function of \( B \).

This interpretation makes possible a geometrical understanding of each of the types of Lamé spectral polynomial worked out in §14. Due to oddness, each of the finite Weierstrass points \( \{(B_s, 0)\}_{s=0}^{2ℓ} \) on Λ, which correspond to band edges, must be mapped by the projection \( \pi_{s} : \Gamma_{s} \rightarrow E_{g_2,g_3} \) to one of the finite Weierstrass points \( \{(ε_γ, 0)\}_{γ=1}^{3} \) or to \((∞, ∞)\). Bearing in mind that Type I Lamé polynomials, both ordinary and twisted, are not of the Hermite–Krichever form, on account of \( (x₀, y₀) \) formally equalling \((∞, ∞)\) and \( \kappa \) equalling \( ∞ \), one has the following proposition.

Proposition 4.2.

1. The roots of the Type-I Lamé spectral polynomial \( L^I(B) \) are the B-values of the finite Weierstrass points \( \{(B_s, 0)\}_{s=0}^{2ℓ} \) that are projected by \( \pi_{s} : \Gamma_{s} \rightarrow E_{g_2,g_3} \) to the infinite Weierstrass point \((∞, ∞)\). Moreover, the roots of the Type-I twisted Lamé spectral polynomial \( L^I_{τ}(B) \) include the B-values of the finite non-Weierstrass points that are projected to \((∞, ∞)\), and do not include the B-value of any finite point that is not projected to \((∞, ∞)\).

2. For \( γ = 1, 2, 3 \), the roots of the Type-II Lamé spectral polynomial \( L^{II}(B; e_{γ}) \) are the B-values of the finite Weierstrass points \( \{(B_s, 0)\}_{s=0}^{2ℓ} \) that are projected by \( \pi_{s} : \Gamma_{s} \rightarrow E_{g_2,g_3} \) to the finite Weierstrass point \((ε_{γ}, 0)\). Moreover, the roots of the Type-II twisted Lamé spectral polynomial \( L^{II}_{τ}(B; e_{γ}) \) include the B-values of the finite non-Weierstrass points that are projected to \((ε_{γ}, 0)\), and do not include the B-value of any finite point that is not projected to \((ε_{γ}, 0)\).
3. The roots of the theta-twisted Lamé spectral polynomial $L\theta_\ell(B)$ include the $B$-values of the finite non-Weierstrass points that are zeroes of $\kappa_\ell : \Gamma_\ell \to \mathbb{P}^1$, and do not include the $B$-value of any finite point that is not a zero.

Remark. The phrasing of the proposition leaves open the possibilities that (1) a root of $Lt_\ell^I(B)$ may be a root of $L\gamma_\ell(B)$, (2) a root of $Lt_\ell^I(B, e_\gamma)$ may be a root of $L\gamma_\ell^I(B, e_\gamma)$, and (3) a root of $L\theta_\ell(B)$ may be the $B$-value of a finite Weierstrass point, i.e., a band edge. Generically these three types of coincidence do not occur, but instances are not difficult to find. One is the case $\ell \equiv 0 \pmod{3}$ and $g_2 = 0$ (i.e., $J = 0$), in which by examination $Lt_\ell^I$ and $L\gamma_\ell^I$ have the common root $B = 0$.

To proceed beyond the proposition, a significant result from finite-band integration theory is needed. For the integer-$\ell$ Lamé equation, the covering $\pi_\ell : \Gamma_\ell \to E_{g_2, g_3}$ and the auxiliary map $\kappa_\ell : \Gamma_\ell \to \mathbb{P}^1$ are both of degree $\ell(\ell + 1)/2$, irrespective of the choice of elliptic curve $E_{g_2, g_3}$. From this fact, supplemented by proposition 4.2, the two propositions 3.3 and 3.7 which were left unproved in 3.6 immediately follow.

The fibre over any point $(x_0, y_0) \in E_{g_2, g_3}$ must comprise $\ell(\ell + 1)/2$ points of $\Gamma_\ell$, the counting being up to multiplicity. Consider the fibre over $(\infty, \infty)$, which by examination includes with unit multiplicity the point $(B, \nu) = (\infty, \infty)$. It also includes each finite Weierstrass point of $\Gamma_\ell$ that corresponds to a Type-I Lamé polynomial. As was shown in 3.3, the number of these up to multiplicity, $N_\ell^I := \deg Lt_\ell^I$, equals $(\ell - 1)/2$ if $\ell$ is odd and $\ell/2 + 1$ if $\ell$ is even. So the number of additional points above $(\infty, \infty)$ is $\ell(\ell + 1)/2 - 1 - (\ell - 1)/2 = (\ell^2 - 1)/2$ if $\ell$ is odd and $\ell(\ell + 1)/2 - 1 - (\ell/2 + 1) = \ell^2/2 - 2$ if $\ell$ is even. Since the projection $\pi_\ell$ is odd under the involution $(B, \nu) \mapsto (B, -\nu)$, these occur in pairs. So $Nt_\ell^I := \deg Lt_\ell^I$ must equal $(\ell^2 - 1)/4$ if $\ell$ is odd and $\ell^2/4 - 1$ if $\ell$ is even; which is the formula for $Nt_\ell^I$ stated in proposition 3.3.

A similar computation applied to the fibre above any finite Weierstrass point $(e_\gamma, 0)$ yields the formula for $Nt_\ell^{II}$ given in that proposition.

The formula for $N\theta_\ell := \deg L\theta_\ell$ stated in proposition 3.3.5 can also be derived with the aid of proposition 3.2. Since the map $\kappa : \Gamma_\ell \to \mathbb{P}^1$ has degree $\ell(\ell + 1)/2$, the fibre above $0$ comprises that number of points, up to multiplicity. It includes each finite Weierstrass point of $\Gamma_\ell$ that corresponds to a Type-II Lamé polynomial (but not the Weierstrass points corresponding to Type-I Lamé polynomials, since those are not of the Hermite–Krichever form and formally have $\kappa = \infty$). The number of these up to multiplicity is three times $N_\ell^{II} := \deg Lt_\ell^{II}$, which equals $3(\ell + 1)/2$ if $\ell$ is odd and $3\ell/2$ if $\ell$ is even. So the number of additional points above $0$ is $\ell(\ell + 1)/2 - 3(\ell + 1)/2 = (\ell + 1)(\ell - 3)/2$ if $\ell$ is odd and $\ell(\ell + 1)/2 - 3\ell/2 = (\ell - 2)/2$ if $\ell$ is even. They come in pairs, and division by two yields the formula for $N\theta_\ell$ given in proposition 3.3.5.

A geometrized version of proposition 3.3.5 can also be proved, with the aid of an additional result that goes beyond the Hermite–Krichever Ansatz. The Lamé spectral curve $\Gamma_\ell(g_2, g_3)$ is nonsingular with genus $\ell$ for generic values of the Klein invariant $J = J(g_2, g_3)$, and when it degenerates to a singular curve $\Gamma_\ell^I := \Gamma_\ell(g_2^3, g_3^3)$, the singular curve has genus $\ell - 1$, with singularities that are limits as $(g_2, g_3) \to (g_2^3, g_3^3)$ of Weierstrass points of $\Gamma_\ell(g_2, g_3)$. It follows that the $\ell$th Type-I and Type-II Cohn polynomials, which characterize the pairs $(g_2, g_3)$ for which the $\ell$th Lamé operator on $E_{g_2, g_3}$ has degenerate algebraic spectrum of the specified type, i.e., for which $\Gamma_\ell(g_2, g_3)$ has a pair of degenerate finite Weierstrass points of the specified type, in fact do more: they characterize the $(g_2, g_3)$ for which $\Gamma_\ell(g_2, g_3)$
is singular. Due to the reduction of the genus by at most unity, there can be, as
proposition 3.2 states, no multiple coincidences of the algebraic spectrum.

The problem of explicitly constructing an Hermite–Krichever solution of the
integer-ℓ Lamé equation, of the form 4.4, will now be considered. What are needed
are the quantities \(x_0, y_0, \kappa,\) or equivalently \(x_0, y_0/\nu, \kappa/\nu.\) Each of the latter is a
rational function of the spectral parameter \(B.\)

One way of deriving these functions is to eliminate variables from the system
of three polynomial equations in \(B; \nu, x_0, y_0,\) as explained above. Coupled with
the spectral equation \(\nu^2 = L_\ell(B),\) this yields explicit expressions for the three desired
functions. By the standards of polynomial elimination algorithms, this procedure
is singular. Due to the reduction of the genus by at most unity, there can be, as
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Theorem L. For all integer \(\ell \geq 1,\) the covering map \(\pi_\ell : \Gamma_\ell \to E_{g_2, g_3}\)
appearing in the Ansatz maps \((B, \nu)\) to \((x_0, y_0)\) according to

\[
\begin{align*}
x_0(B; g_2, g_3) &= e_\gamma + \frac{4}{\ell(\ell + 1)^2} \frac{L^I_{\ell 1}((B; e_\gamma, g_2, g_3)[L^I_{\ell 1}(B; e_\gamma, g_2, g_3)]^2}{L^I_{\ell 1}(B; g_2, g_3)[L^I_{\ell 1}(B; g_2, g_3)]^2} \\
y_0(B, \nu; g_2, g_3) &= \frac{16}{\ell(\ell + 1)^2} \left\{ \frac{\prod_{\gamma=1}^3 L^I_{\ell 1}(B; e_\gamma, g_2, g_3)}{L^I_{\ell 1}(B; g_2, g_3)[L^I_{\ell 1}(B; g_2, g_3)]^3} \right\} \nu,
\end{align*}
\]

with \(\gamma\) in \(\begin{bmatrix} 1, 2, 3 \end{bmatrix}\) being any of \(1, 2, 3.\) The auxiliary function \(\kappa_\ell : \Gamma_\ell \to \mathbb{P}^1\) is given by

\[
\kappa(B, \nu; g_2, g_3) = -\frac{\ell - 1}{\ell(\ell + 1)} \left[ \frac{L_{\theta_\ell}(B; g_2, g_3)}{L^I_{\ell 1}(B; g_2, g_3) L^I_{\ell 1}(B; g_2, g_3)} \right] \nu.
\]

Proof. With the exception of the three \(\ell\)-dependent prefactors, such as \(4/\ell(\ell + 1)^2,\)
the formulas \(4.5, 4.7\) follow uniquely from proposition 4.2 regarded as a list of
properties that \(\pi_\ell\) and \(\kappa_\ell\) must satisfy.

\(\pi_\ell\) must map each point \((B_s, 0),\) where \(B_s\) is a root of \(L^I_{\ell 1}(B),\) singly to \((\infty, \infty),\nand each point \((B_s, 0),\) where \(B_s\) is a root of \(L^I_{\ell 1}(B; e_\gamma),\) singly to \((e_\gamma, 0).\) It must
also map each point \((B_0, \pm \nu),\) where \(B_0\) is a root of \(L^1_\ell(B),\) singly to \((\infty, \infty),\nand each point \((B_0, \pm \nu),\) where \(B_0\) is a root of \(L^I_{\ell 1}(B; e_\gamma),\) singly to \((e_\gamma, 0).\) In all these
statements, the counting is up to multiplicity.

Also, \(B \mapsto \kappa/\nu\) must map each point \((B', \pm \nu'),\) where \(B'\) is a root of \(L_{\theta_\ell}(B),\n\) singly to zero, and must map each point \((B_s, 0),\) where \(B_s\) is a root of \(L^I_{\ell 1}(B),\nand each point \((B_0, \pm \nu),\) where \(B_0\) is a root of \(L^I_{\ell 1}(B; e_\gamma),\) singly to \(\infty.\) In these statements
as well, the counting is up to multiplicity.
The \( \ell \)-dependent prefactors in (4.5)–(4.7) can be deduced from the leading-order asymptotic behavior of \( x_0 \) and \( \kappa/\nu \) as \( B \to \infty \).

The remarkably simple formulas of the theorem permit Hermite–Krichever solutions of the form (4.1) to be constructed for quite large values of \( \ell \), since the Lamé spectral polynomials \( L, L_t, L_\theta \) (ordinary, twisted and theta-twisted) are relatively easy to work out, as made clear. Tables 3 and 5 may be consulted.

It should be stressed that in the formula (4.5) for \( x_0 \), the same right-hand side results, irrespective of which of the three values of \( \gamma \) is chosen. All terms explicitly involving \( e_\gamma \) will cancel. Of course, all powers of \( \gamma \) higher than the second must first be rewritten in terms of \( g_2, g_3 \) by using the identity \( e_\gamma^3 = \frac{B}{3}(g_2 e_\gamma + g_3) \). In the same way, it is understood that the numerator of the right-hand side of (4.6), the terms of which are symmetric in \( e_1, e_2, e_3 \), should be rewritten in terms of \( g_2, g_3 \).

(This can always be done: for example, \( e_1^3 e_2^3 + e_2^3 e_3^3 + e_3^3 e_1^3 \) equals \( g_2^3/16 \).)

The application of Theorem L to the cases \( \ell = 1, 2, 3 \) may be illuminating.

- If \( \ell = 1 \), then \((x_0, y_0) = (B, 2\nu)\) and \( \kappa = 0 \). The map \( \pi_1 : \Gamma_1 \to E_{g_2,g_3} \) is a mere change of normalization, since \( \Gamma_1 \) is isomorphic to \( E_{g_2,g_3} \); cf. (2.15).

- If \( \ell = 2 \), then

\[
x_0 = e_\gamma + \frac{1}{9} \frac{(B + 3e_\gamma)(B - 6e_\gamma)^2}{B^2 - 3g_2} = \frac{B^3 + 27g_3}{9(B^2 - 3g_2)},
\]

\[
y_0 = \frac{2}{27} \frac{\prod_{\gamma=1}^3 (B - 6e_\gamma)}{(B^2 - 3g_2)^2} \nu = \frac{2(B^3 - 9g_2B - 54g_3)}{27(B^2 - 3g_2)^2} \nu,
\]

and \( \kappa = -\left\{2 / [3(B^2 - 3g_2)]\right\} \nu \).

- If \( \ell = 3 \), then

\[
x_0 = e_\gamma + \frac{1}{36} \frac{(B^2 - 6e_\gamma B + 45e_\gamma^2 - 15g_2)(B^2 - 15e_\gamma B - 225e_\gamma^2 + \frac{75}{4}g_2)^2}{B(B^2 - \frac{75}{4}g_2)^2}
= \frac{(16B^6 + 360g_2B^4 + 27000g_3B^3 - 3375g_2^2B^2 - 303750g_2g_3B - 84375g_3^2 + 2278125g_2^2)}{36B(4B^2 - 75g_2)^2},
\]

\[
y_0 = \frac{1}{108} \frac{\prod_{\gamma=1}^3 (B^2 - 15e_\gamma B - 225e_\gamma^2 + \frac{75}{4}g_2)}{B^2(B^2 - \frac{75}{4}g_2)^3} \nu
= \frac{(16B^6 - 1800g_2B^4 - 54000g_3B^3 - 16875g_2^2B^2 + 421875g_3^2 + 11390625g_2^3)}{27B^2(4B^2 - 75g_2)^3} \nu,
\]

and \( \kappa = -\left\{10 / [3B(4B^2 - 75g_2)]\right\} \nu \).

The formulas for \( \ell = 2, 3 \) were essentially known to Hermite. Setting \( \ell = 4, 5 \) in Theorem L yields the less familiar and more complicated formulas which Enol’skii and Kostov (1994) and Eilbeck & Enol’skii (1994) derived by eliminating variables from compatibility conditions. Theorem L readily yields the covering map \( \pi_{\ell} \) and auxiliary function \( \kappa_{\ell} \) for far larger \( \ell \).

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5. Hyperelliptic reductions

The cover \( \pi_\ell : \Gamma_\ell(g_2, g_3) \to E_{g_2, g_3} \) introduced as part of the Hermite–Krichever Ansatz, i.e., the map \((B, \nu) \mapsto (x_0, y_0)\), is of independent interest, since explicit examples of coverings of elliptic curves by higher-genus algebraic curves are few, and the problem of determining which curves can cover \( E_{g_2, g_3} \), for either specified or arbitrary values of the invariants \( g_2, g_3 \), remains unsolved. \( \Gamma_\ell(g_2, g_3) \) generically has genus \( g = \ell \), as noted, and the cover will always be of degree \( N = \ell(\ell + 1)/2 \).

The formula for \( x_0 = x_0(B; g_2, g_3) \) given in Theorem L is consistent with this, since \( N \) equals \( \max(N_{\ell}^{I}, 2N_{\ell}^{III}, N_{\ell}^{II} + 2N_{\ell}^{III}) \), the maximum of the degrees in \( B \) of the numerator and denominator of \( x_0 \). The degrees \( N_{\ell}^{I}, N_{\ell}^{II} \) were computed in \( \S 3 \) and the twisted degrees \( N_{\ell}^{III}, N_{\ell}^{I/2} \) were also (see proposition \( 3.3 \)).

Since \( \Gamma_\ell(g_2, g_3) \) is hyperelliptic (defined by the irrationality \( \nu^2 = L_\ell(B; g_2, g_3) \)) and \( E_{g_2, g_3} \) is elliptic (defined by the irrationality \( y_0^2 = 4x_0^3 - g_2 x_0 - g_3 \)), the map \( \pi_\ell \) enables certain hyperelliptic integrals to be reduced to elliptic ones. Just as \( E_{g_2, g_3} \) is equipped with the canonical holomorphic 1-form \( dx_0/y_0 \), so can \( \Gamma_\ell(g_2, g_3) \) be equipped with the homomorphic 1-form \( dB/\nu \). Any integral of a function in the function field of a hyperelliptic curve (here, any rational function \( R(B, \nu) \)) against its canonical 1-form is called a hyperelliptic integral. Hyperelliptic integrals are classified as follows (Belokolos et al. 1986). The linear space of meromorphic 1-forms of the form \( R(B, \nu) dB/\nu \), i.e., of Abelian differentials, is generated by 1-forms of the first, second and third kinds. These are (i) holomorphic 1-forms, with no poles; (ii) 1-forms with one multiple pole; and (iii) 1-forms with a pair of simple poles, the residues of which are opposite in sign. The indefinite integrals of (i)–(iii) are called hyperelliptic integrals of the first, second and third kinds. They generalize the three kinds of elliptic integral (Abramowitz & Stegun 1965, chapter 17).

Hyperelliptic integrals of the first kind are the easiest to study, since the linear space of holomorphic 1-forms is finite-dimensional and is spanned by \( B^{r-1} dB/\nu \), \( r = 1, \ldots, g \), where \( g \) is the genus. So there are only \( g \) independent integrals of the first kind. A consequence of the map \( \pi_\ell : \Gamma_\ell(g_2, g_3) \to E_{g_2, g_3} \) is that on any hyperelliptic curve of the form \( \Gamma_\ell(g_2, g_3) \) there are really only \( g - 1 \) independent integrals of the first kind, modulo elliptic integrals (considered trivial by comparison). Changing variables in \( \int dx_0/y_0 \), the elliptic integral of the first kind, yields

\[
\int \left[ \frac{y_0}{\nu} \right]^{-1} dx_0 dB/\nu = \int \frac{dx_0}{y_0}.
\]

(5.1)

The quantity in square brackets is rational in \( B \), and in fact is guaranteed to be a polynomial in \( B \) of degree less than or equal to \( g - 1 \), since the left-hand integrand is a pulled-back version of the right-hand one, and must be a holomorphic 1-form. Equation (5.1) is a linear constraint relation on the \( g \) basic hyperelliptic integrals of the first kind. It reduces the number of independent integrals from \( g \) to \( g - 1 \).

The cases \( \ell = 2, 3 \) of (5.1) may be instructive. The maps \((B, \nu) \mapsto (x_0, y_0)\) were given in \( \S 3 \), and the degree-\((2\ell + 1)\) spectral polynomials \( L_\ell(\cdot; g_2, g_3) \) follow from table \( 6 \), if \( \ell = 2 \), one obtains the hyperelliptic-to-elliptic reduction

\[
\int \frac{\frac{3}{2} B dB}{\sqrt{(B^2 - 3g_2)(B^3 - \frac{3}{2} g_2 B + \frac{27}{4} g_3)}} = \int \frac{dx_0}{\sqrt{4x_0^3 - g_2 x_0 - g_3}},
\]

(5.2)
Table 7. Polynomials specifying the holomorphic 1-forms pulled back from $E_{g_2,g_3}$

| $\ell$ | $P_\ell(B;g_2,g_3)$ |
|-------|----------------------|
| 1     | 1                    |
| 2     | $B$                  |
| 3     | $B^2 - \frac{15}{7} g_2$ |
| 4     | $B^3 - \frac{61}{7} g_2 B + \frac{172}{7} g_3$ |
| 5     | $B^4 - \frac{321}{7} g_2 B^2 + \frac{284}{7} g_3 B + \frac{891}{7} g_3^2$ |
| 6     | $B^5 - \frac{901}{7} g_2 B^3 + \frac{284}{7} g_3 B^2 + \frac{42}{7} g_3^2 B - \frac{280665}{49} g_3 g_3^2$ |
| 7     | $B^6 - \frac{91}{7} g_2 B^4 + 10813 g_3 B^3 + \frac{693175}{16} g_3^2 B^2 - \frac{2145114}{16} g_3^2 g_3 B + \frac{54971877}{16} g_3^3 - \frac{5417685}{16} g_3^4$ |
| 8     | $B^7 - \frac{153}{7} g_2 B^5 + 29916 g_3 B^4 + \frac{3120305}{16} g_3^2 B^3 - \frac{51102410}{16} g_3^2 g_3 B^2$ |

where the change of variables is performed by (1.8). If $\ell = 3$, one obtains

$$\int \frac{[3(B^2 - \frac{15}{7} g_2)] dB}{\sqrt{L_3(B;g_2,g_3)}} = \int \frac{dx_0}{\sqrt{4x_0^3 - g_2 x_0 - g_3}}.$$  \tag{5.3}$$

where the full spectral polynomial $L_3(B;g_2,g_3)$ is

$$B \left( B^6 - \frac{63}{2} g_2 B^4 + \frac{297}{2} g_3 B^3 + \frac{4185}{16} g_3^2 B^2 - \frac{18225}{8} g_3^2 g_3 B - \frac{3775}{16} g_3^3 + \frac{91125}{16} g_3^2 \right)$$

and the change of variables is performed by (4.10). These reductions were known to Hermite [Königsberger 1872; Belokolos et al. 1986]. More recently, the reductions induced by the $\ell = 4, 5$ coverings were worked out [Enol’skii & Kostov 1994; Eilbeck & Enol’skii 1994]. But the reductions with $\ell > 5$ proved too difficult to compute. Theorem L makes possible the computation of many such higher reductions.

The following proposition specifies the normalization of the pulled-back 1-form. It follows from the known leading-order asymptotic behavior of $x_0, y_0/\nu$ as $B \to \infty$.

**Proposition 5.1.** For all integer $\ell \geq 1$, the polynomial function $P_\ell(B;g_2,g_3) := [(y_0/\nu)^{-1} dx_0/dB](B;g_2,g_3)$ in the hyperelliptic-to-elliptic reduction formula

$$\int \frac{P_\ell(B;g_2,g_3) dB}{\sqrt{L_\ell(B;g_2,g_3)}} = \int \frac{dx_0}{\sqrt{4x_0^3 - g_2 x_0 - g_3}}$$

where the change of variables $x_0 = x_0(B;g_2,g_3)$ is given by Theorem L, equals $\ell(\ell + 1)/4$ times a polynomial $\hat{P}_\ell(B;g_2,g_3)$ which is monic and of degree $\ell - 1$ in $B$. The polynomials $\hat{P}_\ell$ are listed in Table 7; $\hat{P}_4, \hat{P}_5$ agree with those found by Enol’skii et al., if allowance is made for a difference in normalization conventions.

A complete analysis of Lamé-derived elliptic covers will need to consider exceptional cases of several kinds. The covering curve $\Gamma_{\ell}(g_2,g_3)$ generically has genus $g = \ell$, but if the Klein invariant $J = g_2^3/(g_2^2 - 27g_3^2)$ is a root of one of the two Cohn polynomials of Table 4, the genus will be reduced to $\ell - 1$. According to conjecture 5.3, this will happen, for instance, if $\ell \equiv 2 \pmod 3$ and $g_2 = 0$ (i.e., $J = 0$), so that the base curve $E_{g_2,g_3}$ is equianharmonic. When $g$ is reduced to $\ell - 1$ in this way, the linear space of holomorphic 1-forms will be spanned by $B^{r-1}(B-B_0) dB/\nu$, $r = 1, \ldots, \ell - 1$, where $B_0$ is the degenerate root of the spectral polynomial; but 5.4 will still provide a linear constraint on the associated hyperelliptic integrals.
Another sort of degeneracy takes place when the modular discriminant \( \Delta := g_2^3 - 27g_3^2 \) equals zero, i.e., when \( J = \infty \). In this case \( E_{g_2, g_3} \) will degenerate to a rational curve, due to two or more of \( e_1, e_2, e_3 \) being coincident. The Lamé-derived reduction formulas, such as (5.2)–(5.3), continue to apply. (They are valid though trivial even in the case \( e_1 = e_2 = e_3 \), in which \( g_2 = g_3 = 0 \).) So these formulas include as special cases certain hyperelliptic-to-rational reductions.

Subtle degeneracies of the covering map \( \pi_\ell \) can occur, even in the generic case when \( \Gamma_\ell \) has genus \( \ell \) and \( E_{g_2, g_3} \) has genus 1. The branching structure of \( \pi_\ell \) is determined by the polynomial \( P_\ell \) of table 4 which is proportional to \( dx_0/dB \). If \( P_\ell \) has distinct roots \( \{B^{(i)}\}_{i=1}^{\ell-1} \), then \( \pi_\ell \) will normally have \( 2\ell - 2 \) simple critical points on \( \Gamma_\ell \), of the form \( \{(B^{(i)}, \pm \nu^{(i)})\}_{i=1}^{\ell-1} \). However, if any \( B^{(i)} \) is located at a band edge, i.e., at a branch point of the hyperelliptic \((B, \nu)\)-curve, then \( (B^{(i)}, 0) \) will be a double critical point. This appears to happen when \( \ell \equiv 0 \pmod{3} \) and the base curve \( E_{g_2, g_3} \) is equianharmonic; the double critical point being located at \( (B, \nu) = (0, 0) \). Even if no root of \( P_\ell \) is located at a band edge, it is possible for it to have a double root, in which case each of a pair of points \( (B^{(i)}, \pm \nu^{(i)}) \) will be a double critical point. By examination, this happens when \( \ell = 4 \) and \( J = -2^{25}3^5 53 \).

A few hyperelliptic-to-elliptic reductions, similar to the quadratic \((N = 2)\) reduction of Legendre and Jacobi, can be found in handbooks of elliptic integrals (Byrd & Friedman 1954, §§575 and 576). The Lamé-derived reductions, indexed by \( \ell \), should certainly be included in any future handbook. It is natural to wonder whether they can be generalized in some straightforward way. The problem of finding the genus-2 covers of an elliptic curve was intensively studied in the nineteenth century, by Weierstrass and Poincaré among many others, and one may reason by analogy with results on \( \ell = 2 \). One expects that for all \( \ell \geq 2 \) and for arbitrarily large \( N \), a generic \( E_{g_2, g_3} \) can be covered by some genus-\( \ell \) curve via a covering map of degree \( N \). Each Lamé-derived covering \( \pi_\ell : \Gamma_\ell(g_2, g_3) \to E_{g_2, g_3} \) has \( N = \ell(\ell + 1)/2 \) and may be only a low-lying member of an infinite sequence of coverings. Generalizing the Lamé-derived coverings may be possible even if one confines oneself to \( N = \ell(\ell + 1)/2 \). One can of course pre-compose with an automorphism of \( \Gamma_\ell(g_2, g_3) \) and post-compose with an automorphism of \( E_{g_2, g_3} \) (a modular transformation). But when \( \ell = 2 \) a quite different covering map with the same degree is known to exist (Belokolos et al. 1986). \( \pi_2 \) has two simple critical points on \( \Gamma_2 \), but the other degree-3 covering map has a single double critical point on its analogue of \( \Gamma_2 \). Both can be generalized to include a free parameter (Burnside 1892; Belokolos & Enol’skii 2000). It seems possible that when \( \ell > 2 \), similar alternatives to the Lamé-derived coverings may exist, with degree \( \ell(\ell + 1)/2 \) but different branching structures.

6. Dispersion relations

It is now possible to introduce dispersion relations, and determine the way in which the Hermite–Krichever Ansatz reduces higher-\( \ell \) to \( \ell = 1 \) dispersion relations. The starting point is the fundamental multi-valued function \( \Phi \) introduced in §2. As noted, if the parametrization point \( (x_0, y_0) \) on the punctured elliptic curve \( E_{g_2, g_3} \setminus \{(\infty, \infty)\} \) is over \( x_0 = B \in \mathbb{C} \), then \( \Phi(\cdot, \cdot ; x_0, y_0) \) will be a solution of the \( \ell = 1 \) case of the Lamé equation (5.1). \( E_{g_2, g_3} \) is defined by \( y^2 = 4x^3 - g_2x - g_3 \), so the hypothesis here is that \( (x_0, y_0) \) should equal \( (B, \pm \sqrt{4B^3 - g_2B - g_3}) \).
In the Jacobi form (with independent variable $\alpha$), resp. the Weierstrassian form (with independent variable $u$), the crystal momentum $k$ characterizes the behavior of a solution of the Lamé equation under $\alpha \mapsto \alpha + 2K$, resp. $u \mapsto u + 2\omega$. Both shifts correspond to motion around $E_{g_2, g_3}$, along a fundamental loop that passes between $(x, y) = (e_1, 0)$ and $(\infty, \infty)$, and cannot be shrunk to a point. (If $e_1, e_2, e_3$ are defined by (2.1), this will be because $y$ is positive on one-half of the loop, and negative on the other.) By definition, the solution will be multiplied by $\xi := \exp[i k(2K)] = \exp[i k(2\omega)]$. It follows from the definition (2.11) of $\Phi$ that when $\ell = 1$,

$$\xi = \exp[i k(2\omega)] = \exp\left[\frac{1}{2} \int \left(\frac{y + y_0}{x - x_0}\right) \frac{dx}{y}\right]. \quad (6.1)$$

That is, when $\ell = 1$ the crystal momentum is given by a complete elliptic integral.

In the context of finite-band integration theory, this is a special case of (4.4).

It was pointed out in §4 that the spectral curve $\Gamma_1$ that parametrizes $\ell = 1$ solutions can be identified with $E_{g_2, g_3}$ itself, via the identification $(B, \nu) = (x_0, y_0/2)$. This suggests a subtle but important reinterpretation of $k$. In §4 it was introduced as a function of the energy parameter, here $B$, which is determined only up to integer multiples of $\pi/K = \pi/\omega$, and which is also undetermined as to sign. If the presence of $y_0 = 2\nu$ in (6.1) is taken into account, it is clear that the $\ell = 1$ crystal momentum, called $k_1$ henceforth, should be regarded as a function not on $\mathbb{P}^1 \setminus \{\infty\}$ but rather on $\Gamma_1 \setminus \{(\infty, \infty)\} \equiv (B, \nu)$. In this interpretation, the indeterminacy of sign disappears. The additive indeterminacy, on account of which $k_1$ is an elliptic function of the second kind, remains but can be viewed as an artifact: it is due to $k_1 \propto \log \xi$, where $\xi$ is the Floquet multiplier. The behavior of $k_1$ near the puncture $(B, \nu) = (\infty, \infty)$ is easily determined. It follows from (6.1) that as $(B, \nu) \to (\infty, \infty)$, i.e., $(x_0, y_0) \to (\infty, \infty)$, each branch of $k_1$ is asymptotic to $-\nu/B$ to leading order. Since $B = x_0, \nu = 2y_0$ have double and triple poles there, respectively, it follows that each branch of $k_1$ has a simple pole at the puncture.

The multiplier $\xi$ is a true single-valued function on the punctured spectral curve $\Gamma_1 \setminus \{(\infty, \infty)\}$, and moreover is entire. One can write $\xi : \Gamma_1 \setminus \{(\infty, \infty)\} \to \mathbb{P}^1 \setminus \{0, \infty\}$, since the multiplier is never zero. Like $k_1$, this function is not algebraic: it necessarily has an essential singularity at the puncture. The $((B, \nu), \xi)$-curve over $\Gamma_1 \setminus \{(\infty, \infty)\} \equiv (B, \nu)$, which is a single cover, and the $(B, \nu)$-curve over $\mathbb{C} \supset B$, which is a double cover, are both transcendental curves.

The crystal momentum for each integer $\ell \geq 1$ may similarly be viewed as an additively multi-valued function on the punctured spectral curve $\Gamma_\ell \setminus \{(\infty, \infty)\}$. It will be written as $k_\ell(B, \nu; g_2, g_3)$, with the understanding that for this to be well-defined, $B, \nu$ must be related by the spectral curve relation $\nu^2 = L_\ell(B; g_2, g_3)$. The quantity $k_\ell(B, \nu; g_2, g_3)$ will not be undetermined as to sign. Suppose now that the projections $\pi_\ell : \Gamma_\ell \to E_{g_2, g_3}$ of the Hermite–Krichever Ansatz are regarded as maps $\pi_\ell : \Gamma_\ell \to \Gamma_1$. That is, $\pi_\ell$ maps $(B, \nu) \in \Gamma_\ell$ to the point $(B', \nu') := (x_0, y_0/2) \in \Gamma_1$. The reductions $(B, \nu) \mapsto (B', \nu')$ for $\ell = 2, 3$, for example, follow from (5.3–5.4).

**Proposition 6.1.** If the integration of the Lamé equation on the elliptic curve $E_{g_2, g_3}$, for integer $\ell \geq 1$, can be accomplished in the framework of the Hermite–Krichever Ansatz by maps $\pi_\ell : \Gamma_\ell \to \Gamma_1$ and $\kappa_\ell : \Gamma_\ell \to \mathbb{P}^1$, where $\pi_\ell$ and $\kappa_\ell$ map the point $(B, \nu)$ to $(B(B; g_2, g_3), \nu(B, \nu; g_2, g_3))$ and $\kappa_\ell(B; g_2, g_3), \nu$, respectively, then the dispersion relation for the Hermite–Krichever solutions will be $k = k_\ell(B, \nu; g_2, g_3)$.
in which $k_t$ can be expressed in terms of $k_1$ by

$$k_t(B, \nu; g_2, g_3) = k_1(B_t(B, \nu; g_2, g_3), \nu(B, \nu; g_2, g_3)) - ik_t(B, \nu; g_2, g_3)\nu. \quad (6.2)$$

This proposition follows from the form of the Hermite–Krichever solutions (4.1.1). The first term in (6.2) arises from the $\Phi, \Phi'$ factors, and the second from the exponential. The factors $A, B$ in (4.1.1) do not contribute to the crystal momentum. Equation (6.2) could also be derived from the general theory of finite-band integration, specifically from the formula (4.1.3). However, a derivation from the Hermite–Krichever Ansatz seems more natural in the present context.

The effort expended in replacing two-valuedness by single-valuedness is justified by the following observation. As a function of $u$, the Lamé–periodic coefficient, rather than an equation on the curve $C$. To derive dispersion relations that can be compared with previous work, the formulation of §6 must be converted to the language of the Jacobi form.

The relationships among the several forms were sketched in §2. In the Weierstrassian and Jacobi forms, the Lamé equation is an equation on $\mathbb{C}$ with doubly periodic coefficients, rather than an equation on the curve $E_{g_2, g_3}$. In the conversion to the Jacobi form the invariants $g_2, g_3$ are expressed in terms of the modular parameter $m$ by (2.2), with $A = 1$ by convention. The coordinate $x$ on $E_{g_2, g_3}$ is interpreted as the function $\wp(u; g_2, g_3)$, i.e., as $m \sin^2(\alpha|m) - \frac{1}{2}(m + 1)$, where $u \in \mathbb{C}$ and $\alpha := u - iK'(m) \in \mathbb{C}$ are the respective independent variables of the Weierstrassian and Jacobi forms. The holomorphic differential $dx/y$ corresponds to $du$ or $do$, and the derivative $y d/dx$ to $d/du$ or $d/do$. The coordinate $y = (y d/dx)x$ on $E_{g_2, g_3}$ is interpreted as $\wp'(u; g_2, g_3)$, i.e., as the doubly periodic function $2m \sin(\alpha|m) \csc(\alpha|m) \sin(\alpha|m)$ on the complex $\alpha$-plane. The functions

$$\sqrt{x - \delta_\gamma}, \gamma = 1, 2, 3, \text{ correspond to } -i \sin(\alpha|m), -im^{1/2}\sin(\alpha|m) \text{ and } m^{1/2} \sin(\alpha|m).$$

relates the accessory parameters $B, E$ of the different forms.

With these formulas, it is easy to convert the Lamé polynomials of table 2 to polynomials in $\sin(\alpha|m), \csc(\alpha|m), \sin(\alpha|m)$, and the spectral polynomials of table 3 to polynomials in $E$, for comparison with the list given by Arscott (1964, §9.3.2).
Definition. The Jacobi-form spectral polynomial \( \tilde{L}_\ell(E|m) \) is the negative of the spectral polynomial \( L_\ell(B; g_2, g_3) \), when \( B; g_2, g_3 \) are expressed in terms of \( E, m \). It is a monic degree-\((2\ell + 1)\) polynomial in \( E \) with coefficients polynomial in \( m \), and can be regarded as \( \prod_{\nu = 0}^{2\ell}[E - E_\nu(m)] \), where the roots \( \{E_\nu\} \) are the values of the energy \( E \) for which there exists a Lamé polynomial solution of the Lamé equation, counted with multiplicity. (The negation is due to the relative minus sign in the \( B \leftrightarrow E \) correspondence.)

Definition. The \( \ell \)th Jacobi-form spectral curve \( \tilde{\Gamma}_\ell := \tilde{\Gamma}_\ell(m) \) is the hyperelliptic curve over \( \mathbb{P}^1 \) \( \ni E \) comprising all \( (E, \nu) \) satisfying \( \nu^2 = \tilde{L}_\ell(E|m) \). In the usual case when \( m \) is real, \( \nu \) will be real if \( E \) is in a band, and non-real if \( E \) is in a lacuna. In both cases it is determined only up to negation. By convention, the correspondence between the curve \( \tilde{\Gamma}_\ell \ni (E, \nu) \) and the previously introduced curve \( \Gamma_\ell \ni (B, \nu) \), which was defined by \( \nu^2 = L_\ell(B; g_2, g_3) \), is given by \( \nu = \pm \tilde{\nu} \).

The following cases are examples. When \( \ell = 1, 2, 3 \), the spectral polynomial factors over the integers into polynomials at most quadratic in \( E \). In full,

\[
\tilde{L}_1(E|m) = (E - 1)(E - m)(E - m - 1) 
\]

(7.2)

\[
\tilde{L}_2(E|m) = [E^2 - 4(m + 1)E + 12m] (E - m - 1)(E - 4m - 1)(E - m - 4)
\]

(7.3)

\[
\tilde{L}_3(E|m) = (E - 4m - 4) \left[ E^2 - 2(2m + 5)E + 3(8m + 3) \right] 
\]

\[
\times \left[ E^2 - 2(5m + 2)E + 3(3m^2 + 8m) \right] 
\]

\[
\times \left[ E^2 - 10(m + 1)E + 3(3m^2 + 26m + 3) \right].
\]

(7.4)

In (7.2) and (7.3), the first factor arises from \( L_1(B; g_2, g_3) \) and the remaining three from the factors \( L_1^H(B; \gamma, g_2, g_3), \gamma = 1, 2, 3 \). In (7.2) there is no Type I factor. The polynomials (7.2) \( \rightarrow \) (7.4) agree with those obtained by Arscott.

The derivation of the Jacobi-form spectral polynomial \( \tilde{L}_\ell(E|m) \) from \( L_\ell(B; g_2, g_3) \) sheds light on a regularity noticed by Ince (1940a \( \S \) 7), which arises in the lemniscatic case \( m = \frac{1}{2} \). Ince observed that if \( \ell \leq 6 \), at least, then \( \tilde{L}_\ell(E|\frac{1}{2}) \) has an integer root, namely \( E = \ell(\ell + 1)/2 \). In fact, this is the case for all integer \( \ell \). By (7.2), the presence of this root is equivalent to the full spectral polynomial \( \tilde{L}_\ell(B; g_2, 0) \) having \( B = 0 \) as a root. But if \( m = \frac{1}{2} \), it follows from (7.4) that \( c_2 = 0 \). A glance at the pattern of coefficients in table (7) reveals that if \( g_3 = 0 \) and a singular point \( e_\gamma \) also equals zero, then either the Type-I spectral polynomial \( L_1^H(B; g_2, g_3) \) (if \( \ell \equiv 0, 3 \) mod 4) or one of the three Type-II spectral polynomials \( L_1^H(B; e_\gamma, g_2, g_3) \) (if \( \ell = 1, 2 \) mod 4) will necessarily have \( B = 0 \) as a root.

Dispersion relations in their Jacobi form can now be investigated. Recall that if \( \ell = 1 \), the Jacobi-form Lamé equation (7.1) has \( \Phi(\cdot; \alpha_0|m) \) as a solution, where the theta quotient \( \Phi \) (the Jacobi-form version of Halphen’s l’élément simple) is defined in (1.3), and the multi-valued parameter \( \alpha_0 \) is defined by \( \text{dn}^2(\alpha_0|m) = E - m \). This solution has crystal momentum \( k = k_1 \) equal to \(-iZ(\alpha_0|m) + \pi/2K(m)\), which is indetermined up to sign, and is also determined only up to integer multiples of \( \pi/K(m) \). The sign indeterminacy is due to \( \text{dn}^2(-m) \) being even. This causes \( \alpha_0 \) to be indetermined up to sign, and \( k_1 \) as well, since the function \( Z(-m) \) is odd.

The parametrization point \( \alpha_0 \), or the equivalent point \( u_0 := \alpha_0 + iK'(m) \) of the Weierstrassian form, corresponds to the parametrization point \( (x_0, y_0) \) of the funda-
mental function \( \Phi \) on the elliptic curve \( E_{g_2,g_3} \). The correspondence is the usual one: 
\[ x_0 = \varphi(u_0; g_2, g_3), \quad y_0 = \varphi'(u_0; g_2, g_3). \]
The first of these two equations says that \( x_0 = m \text{sn}^2(\alpha_0|m) - \frac{1}{4}(m + 1) \), and the latter that \( y_0 = 2m \text{sn}(\alpha_0|m) \text{cn}(\alpha_0|m) \text{dn}(\alpha_0|m) \).
The formula which computes \( \alpha_0 \) from \( E \), namely \( \text{dn}^2(\alpha_0|m) = E - m \), is readily seen to be a translation to the Jacobi form of the familiar condition \( x_0 = B \), which simply says that the parametrization point \((x_0, y_0)\) must be ‘over’ \( B \subset \mathbb{C} \).

The correspondence between the Jacobi and elliptic-curve forms motivates the following reinterpretation of the crystal momentum of the fundamental solution \( \tilde{\Phi} \), which is modelled on the reinterpretation of the last section. \( k_1 \) should be viewed as a function not of the energy \( E \in \mathbb{C} \), but rather of a point \((E, \tilde{\nu})\) on the punctured Jacobi-form spectral curve \( \Gamma_1(m) \setminus \{(\infty, \infty)\} \). There are two such points for each energy \( E \), except when \( E \) is a band edge. This is the source of the sign ambiguity in the parameter \( \alpha_0 \). Since \( y_0 = 2\nu = 2i\tilde{\nu} \), the equation
\[ m \text{sn}(\alpha_0|m) \text{cn}(\alpha_0|m) \text{dn}(\alpha_0|m) = i\tilde{\nu} \tag{7.5} \]
determines a unique sign for \( \alpha_0 \), provided that \( \tilde{\nu} \) is specified in addition to \( E \). \( k_1 \) will be written as \( k_1(E, \tilde{\nu}|m) \), with the understanding that for this to be well-defined, the pair \((E, \tilde{\nu})\) must be related by the spectral curve relation \( \tilde{\nu}^2 = \mathcal{L}_\ell(E|m) \). The additively undetermined quantity \( k_1(E, \tilde{\nu}|m) \) will not be undetermined as to sign. It is easily checked that on each branch, \( k_1 \sim \tilde{\nu}/E \) as \((E, \tilde{\nu}) \to (\infty, \infty)\).

**Definition.** A solution of the Jacobi-form Lamé equation \((\ref{eq:JacobiFormLame})\) is said to be an Hermite–Krichever solution if it is of the form
\[ \left[ \mathcal{A} \left( \text{sn}^2(\alpha|m) \right) \mathcal{\Phi}(\alpha; \alpha_0|m) + 2\mathcal{B} \left( \text{sn}^2(\alpha|m) \right) \mathcal{\Phi}'(\alpha; \alpha_0|m) \right] \exp(\kappa \alpha), \tag{7.6} \]
for some \( \alpha_0 \in \mathbb{C} \) and \( \kappa \in \mathbb{C} \). Here \( \mathcal{A}, \mathcal{B} \) are polynomials, and \( \mathcal{\Phi}' := (d/d\alpha)\mathcal{\Phi} \).

The expression \((\ref{eq:JacobiFormLame})\) is a replacement for the original Jacobi-form expression \((\ref{eq:JacobiFormLame})\), to which it is equivalent. Regardless of which is used, it is easy to compute the crystal momentum of an Hermite–Krichever solution. The momentum computed from \((\ref{eq:JacobiFormLame})\) will be \([-iZ(\alpha_0|m) + \pi/2K(m)] - i\kappa \), up to additive multi-valuedness. The first term arises from the \( \mathcal{\Phi} \), \( \mathcal{\Phi}' \) factors, and the second from the exponential. The factors \( \mathcal{A}, \mathcal{B} \) do not contribute, since \( \text{sn}^2(\alpha|m) \) is periodic in \( \alpha \) with period \( 2K(m) \).

The Jacobi form of the Hermite–Krichever Ansatz asserts that for all integer \( \ell \) and \( m \in \mathbb{C} \setminus \{0,1\} \), there is an Hermite–Krichever solution for all but a finite number of values of the energy \( E \). On the elliptic curve \( E_{g_2,g_3} \), these solutions were constructed from two maps: a projection \( \pi_\ell : \Gamma_\ell \to E_{g_2,g_3} \) and an auxiliary function \( \kappa_\ell : \Gamma_\ell \to \mathbb{P}^1 \). But \( \pi_\ell \) should really be regarded as a map from \( \Gamma_\ell \) to \( \Gamma_1 \), on account of the correspondence between \( E_{g_2,g_3} \) and \( \Gamma_1 \) provided by \((x_0, y_0) = (B, 2\nu) \). The following is the Jacobi-form version of proposition \((\ref{prop:JacobiFormLame})\).

**Proposition 7.1.** Suppose that the integration of the Lamé equation on the elliptic curve \( E_{g_2,g_3} \), for integer \( \ell \geq 1 \), can be accomplished in the framework of the Hermite–Krichever Ansatz by the maps \( \pi_\ell : \Gamma_\ell \to \Gamma_1 \) and \( \kappa_\ell : \Gamma_\ell \to \mathbb{P}^1 \), where \( \pi_\ell \) and \( \kappa_\ell \) map the point \((B, \nu)\) to \((B; g_2, g_3), \nu(B, \nu; g_2, g_3) \) and \( \kappa_\ell(B; g_2, g_3)\nu \), respectively. Then the dispersion relation for the solutions of the Jacobi form of the Lamé equation will be \( k = k_\ell(E, \tilde{\nu}|m) \), where up to additive multi-valuedness
\[ k_\ell(E, \tilde{\nu}|m) = k_1(E|m), \nu_\ell(E, \tilde{\nu}|m) \big| m + \hat{k}_\ell(E|m), \tilde{\nu}, \tag{7.7} \]

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in which

\[ E_\ell(E|m) := -B_\ell(-E + \frac{\ell}{4}(\ell + 1)(m + 1); g_2(m), g_3(m)) + \frac{\ell}{4}(m + 1), \quad (7.8) \]
\[ \tilde{\nu}_\ell(E, \tilde{\nu}|m) := -i\nu_\ell(-E + \frac{\ell}{4}(\ell + 1)(m + 1), \tilde{\nu}; g_2(m), g_3(m)), \quad (7.9) \]
\[ \hat{\kappa}_\ell(E|m) := \hat{\kappa}_\ell(-E + \frac{\ell}{4}(\ell + 1)(m + 1); g_2(m), g_3(m)). \quad (7.10) \]

The formula (7.7) follows by inspection. The projection \( \pi_\ell \) reduces the integration of the Lamé equation to the integration of an \( \ell = 1 \) equation, the ‘\( B' \) parameter of which equals \( B_\ell(B; g_2, g_3) \). By (7.1), the ‘\( E' \) parameter of the \( \ell = 1 \) equation will be the right-hand side of (7.7). The two terms of (7.7) are simply the two terms of \( [-i Z(\alpha_0|m) + \pi/2K(m)] - i\kappa \). The equality \( \nu = i\tilde{\nu} \) has been used.

It is straightforward to apply proposition 7.1 to the cases \( \ell = 2, 3 \), since the coverings \( \pi_2, \pi_3 \) and auxiliary functions \( \kappa_2, \kappa_3 \) were worked out in \( 8.4 \). A brief discussion of the \( \ell = 2 \) case should suffice. After some algebra, one finds

\[ E_2(E|m) = \frac{E^3 - 12(m + 1)^2E - 4(m + 1)(4m^2 - 19m + 4)}{9[E^2 - 4(m + 1)E + 12m]}, \quad (7.11) \]
\[ \tilde{\nu}_2(E, \tilde{\nu}|m) = -\frac{(E + 2m - 4)(E - 4m + 2)(E - 4m - 4)}{27[E^2 - 4(m + 1)E + 12m]^2} \tilde{\nu}, \quad (7.12) \]
\[ \hat{\kappa}_2(E|m) = -\frac{2}{3[E^2 - 4(m + 1)E + 12m]^2}, \quad (7.13) \]

from which \( k_2(E, \tilde{\nu}|m) \) may be computed by (7.7). Like \( k_1, k_2 \) is determined only up to integer multiples of \( \pi/K := \pi/K(m) \). Each branch of \( k_2 \) has the property that \( k_2(E, \tilde{\nu}|m) \sim \pm E^{1/2}, E \to +\infty \), with ‘\( \pm \)’ determined by the sign of \( \nu = \tilde{\nu}(E) \).

This is a special case of a general fact: for all integer \( \ell \geq 1 \), \( k_\ell(E, \tilde{\nu}|m) \sim \pm E^{1/2}, E \to +\infty \), since each branch of \( k_{\ell} \) is asymptotic to \( (-)^{\ell-1}\tilde{\nu}/E^{\ell} \) as \( (E, \tilde{\nu}) \to (\infty, \infty) \).

The real portions of the dispersion relations \( k = k_1(E, \tilde{\nu}|\frac{1}{2}), k_2(E, \tilde{\nu}|\frac{1}{2}) \) and \( k_3(E, \tilde{\nu}|\frac{1}{2}) \) are graphed in figure 1. For ease of viewing, each crystal momentum is regarded as lying in the interval \( [0, \pi/2K] \); which is equivalent to choosing the sign of \( \nu = \tilde{\nu}(E) \) in an \( E \)-dependent way. As (7.2)–(7.4) imply, the two \( \ell = 1 \) bands are \([\frac{1}{2}, \frac{3}{2}], [\frac{1}{2}, \infty)\), the three \( \ell = 2 \) bands are \([3 - \sqrt{3}, \frac{3}{2}], [3, \frac{9}{2}], [3 + \sqrt{3}, \infty)\), and the four \( \ell = 3 \) bands are \([\frac{9}{2} - \sqrt{6}, 6 - \sqrt{15}], [\frac{15}{2} - \sqrt{6}, 6], [\frac{9}{2} + \sqrt{6}, 6 + \sqrt{15}], [\frac{15}{2} + \sqrt{6}, \infty)\).
The $\ell = 1$ graph agrees with that of Li et al. (2000, figure 6), and for confirmation, with that of Sutherland (1973, figure 1). Unfortunately, the $\ell = 2$ graph disagrees with that of Li et al. in the placement or direction of curvature of each of the two upper bands. The algorithm they used for reducing $\ell = 2$ to $\ell = 1$, which was based on Hermite’s solution of the Jacobi-form Lamé equation (Whittaker & Watson 1927, § 23.71), evidently yielded incorrect results for these bands. It appears that for the middle band, at least, the discrepancy can be traced to an incorrect choice of relative sign for the two terms of $k = k_2$. The reinterpretation of the crystal momentum as a function on the spectral curve, rather than a function of the energy, eliminates such sign ambiguities.

8. Summary and final remarks

A new approach to the closed-form solution of the Lamé equation has been introduced. Theorem L provides a formula for the covering map of the Hermite–Krichever Ansatz in terms of certain polynomials which are of independent interest, namely twisted spectral polynomials. The theorem permits an efficient computation of Lamé dispersion relations, of a mixed symbolic–numerical kind. Cohn polynomials, which are a new concept, have also been introduced. The roots of such a polynomial are the points in elliptic moduli space at which a Lamé spectral polynomial has a double root, so that the Lamé spectral curve becomes singular. Twisted and theta-twisted Cohn polynomials could be defined, as well.

The approach of this paper can be extended from the Lamé equation to the Heun equation, which as a differential equation on the elliptic curve $E_{g_2, g_3}$ has up to four regular singular points, positioned at the finite Weierstrass points $\{(e_\gamma, 0)\}_{\gamma=1}^3$ as well as at $(\infty, \infty)$. Its Weierstrassian form is called the Treibich–Verdier equation (Smirnov 2002), and its Jacobi form, at least when only two of the Weierstrass points are singular points, the associated Lamé equation (Magsms & Winkler 1979, § 7.3).

The ‘four triangular numbers’ condition for the Heun equation to have the finite-band property, due to Treibich & Verdier (1992) and Gesztesy & Weikard (1995b), is now well known. In the finite-band case, the number of points in the algebraic spectrum has been computed up to multiplicity (Gesztesy & Weikard 1995). The corresponding band-edge solutions are Heun polynomials. Applying the Hermite–Krichever Ansatz to the finite-band Heun equation leads to a greater variety of coverings of $E_{g_2, g_3}$ than arise in the solution of the integer-$\ell$ Lamé equation: for example, coverings by a genus-2 hyperelliptic curve that have degrees 3, 4, 5 (Belokolos & Enol’skii 2000). These coverings play a role in the construction of elliptic soliton solutions of certain nonlinear evolution equations that occur in fibre optics (Christiansen et al. 2000). A treatment of the Heun equation along the lines of this paper will appear elsewhere.

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References

Abramowitz, M. & Stegun, I. A. (eds) 1965 Handbook of mathematical functions. New York: Dover.

Article submitted to Royal Society
R. S. Maier

Alhassid, Y., Gürsey, F. & Iachello, F. 1983 Potential scattering, transfer matrix, and group theory. *Phys. Rev. Lett.* **50**, 873–876.

Arscott, F. M. 1964 *Periodic differential equations*. New York: Macmillan.

Arscott, F. M. & Khabsa, I. M. 1962 *Tables of Lamé polynomials*, vol. 17 of Mathematical Tables Series. Oxford: Pergamon.

Belokolos, E. D. & Enol’skii, V. Z. 2000 Reductions of Abelian functions and algebraically integrable systems, II. *Izvestiya: Mathematics* **64**(1), 1–49 (1992).

Belokolos, E. D., Bobenko, A. I., Matveev, V. B. & Enol’skii, V. Z. 1986 Algebraic-geometric principles of superposition of finite-zone solutions of integrable non-linear equations. *Uspekhi Mat. Nauk* **41**(2), 3–42. (Transl. *Russian Math. Surveys* **41**(2), 1–49 (1986).)

Boyanovsky, D., de Vega, H. J., Holman, R. & Salgado, J. F. J. 1996 Analytic and numerical study of preheating dynamics. *Phys. Rev. D* **54**, 7570–7598. Available as arXiv:hep-ph/9608205.

Brezhnev, Y. V. 2001 Elliptic solitons and Gröbner bases. *J. Math. Phys.* **45**, 696–712. Available as arXiv:nlin.SI/0007028.

Burnside, W. 1892 Über Lamé’sche Funktionen mit komplexen Parametern. Ph.D. dissertation, University of Königsberg, Germany.

Dobner, H.-J. & Ritter, S. 1998 Reliable computation of eigenvalues of the magnetostatic integral operator. *Math. Comput. Modelling* **27**, 1–10.

Eilbeck, J. C. & Enol’skii, V. Z. 1994 Elliptic Baker–Akhiezer functions and an application to an integrable dynamical system. *J. Phys. A* **35**, 1192–1201.

Enol’skii, V. Z. & Kostov, N. A. 1994 On the geometry of elliptic solitons. *Acta Appl. Math.* **36**, 57–86.

Finkel, F., González-López, A. & Rodríguez, M. A. 2001 Redesigned postprocessors for the construction of multi-peak soliton solutions of the *mKdV* equation. *Appl. Math. Comput.* **121**, 239–251.

Gesztesy, F. & Holden, H. 2003 *Soliton equations and their algebro-geometric solutions*, vol. I. Cambridge, UK: Cambridge University Press.

Gesztesy, F. & Weikard, R. 1995a Lamé potentials and the stationary (m)KdV hierarchy. *Math. Nachr.* **176**, 73–91.
Gesztesy, F. & Weikard, R. 1995b Treibich–Verdier potentials and the stationary (m)KdV hierarchy. Math. Z. 219, 451–476.

Gesztesy, F. & Weikard, R. 1996 Picard potentials and Hill’s equation on a torus. Acta Math. 176, 73–107.

Gesztesy, F. & Weikard, R. 1998 Elliptic algebro-geometric solutions of the KdV and AKNS hierarchies—analytic approach. Bull. Amer. Math. Soc. 35, 271–317.

Grecchi, V. & Sacchetti, A. 1997 Lifetime of the Wannier–Stark resonances and perturbation theory. Commun. Math. Phys. 185, 359–378.

Greene, P. B., Kofman, L., Linde, A. & Starobinsky, A. A. 1997 Structure of resonance in preheating after inflation. Phys. Rev. D 56, 6175–6192. Available as arXiv:hep-ph/9705347.

Guerritore, G. 1909 Calcolo delle funzioni di Lamè fino a quelle di grado 10. Giorn. Mat. Battaglini 47, 164–172.

Halphen, G.-H. 1888 Traité des fonctions elliptiques et de leurs applications, vol. II. Paris: Gauthier–Villars.

Ince, E. L. 1940a The periodic Lamè functions. Proc. Roy. Soc. Edinburgh 60, 47–63.

Ince, E. L. 1940b Further investigations into the periodic Lamè functions. Proc. Roy. Soc. Edinburgh 60, 83–99.

Its, A. R. & Matveev, V. B. 1974 Schrödinger operators with finite-gap spectrum and N-soliton solutions of the Korteweg–de Vries equation. Teoret. Mat. Fiz. 23(1), 51–68. (Transl. Theor. and Math. Phys. 23(1), 343–355 (1975).)

Ivanov, P. 2001 On Lamè’s equation of a particular kind. J. Phys. A 34, 8145–8150. Available as arXiv:math-ph/0008008.

Kaiser, D. I. 1998 Resonance structures for preheating with massless fields. Phys. Rev. D 57, 702–711. Available as arXiv:hep-ph/970516.

Kantowski, R. & Thomas, R. C. 2001 Distance-redshift in inhomogeneous Ω0 = 1 Friedmann–Lemaître–Robertson–Walker cosmology. Astrophys. J. 561, 491–495. Available as arXiv:astro-ph/0011176.

Klein, F. 1892 Über den Hermite’schen Fall der Lamè’schen Differentialgleichungen. Math. Ann. 40, 123–129.

Königsberger, L. 1878 Über die Reduction hyperelliptischer Integrale auf elliptische. J. Reine Angew. Math. 85, 273–294.

Kostov, N. A. & Enol’skii, V. Z. 1993 Spectral characteristics of elliptic solitons. Mat. Zametki 53(3), 62–71. (Transl. Math. Notes 55(3–4), 287–293 (1993).)

Krichever, I. M. 1980 Elliptic solutions of the Kadomtsev–Petviashvili equation and integrable systems of particles. Funktsional. Anal. i Prilozhen 14(4), 45–54, 95. (Transl. Funct. Anal. Appl. 14, 282–290 (1980).)

Krichever, I. M. 1990 Generalized elliptic genera and Baker–Akhiezer functions. Mat. Zametki 47(2), 34–45, 158. (Transl. Math. Notes 47(1–2), 132–142 (1990).)

Li, H. & Kusnezov, D. 1999 Group theory approach to band structure: Scarf and Lamé Hamiltonians. Phys. Rev. Lett. 83, 1283–1286. Available as arXiv:cond-mat/9907202.

Li, H., Kusnezov, D. & Iachello, F. 2000 Group-theoretical properties and band structure of the Lamé Hamiltonian. J. Phys. A 33, 6413–6429.

Magnus, W. & Winkler, S. 1979 Hill’s equation, revised edn. New York: Dover.

Maier, R. S. 2004 Algebraic solutions of the Lamé equation, revisited. J. Differential Equations 198, 16–34. Available as arXiv:math.CA/0206285.

Maier, R. S. & Stein, D. L. 2001 Droplet nucleation and domain wall motion in a bounded interval. Phys. Rev. Lett. 87, 270601-1–270601-4. Available as arXiv:cond-mat/0108217.

McKeen, H. P. & van Moerbeke, P. 1979 The spectrum of Hill’s equation. Invent. Math. 30, 217–274.

Article submitted to Royal Society
Sacchetti, A. 1997 Band functions for the Lamé equation. *MapleTech* 4(1), 28–33.
Smirnov, A. O. 2002 Elliptic solitons and Heun’s equation. In *The Kowalevski Property* (ed. V. B. Kuznetsov), CRM Proc. Lecture Notes, no. 32, pp. 287–305. Providence, RI: American Mathematical Society. Available as arXiv: [math.CA/0109149](http://arxiv.org/abs/math.CA/0109149).
Strutt, M. J. O. 1967 *Lamésche- Mathieusche- und verwandte Funktionen in Physik und Technik*. New York: Chelsea Publishing Co. Reprint of the 1932 Berlin edition.
Sutherland, B. 1973 Some exact results for one-dimensional models of solids. *Phys. Rev. A* 8, 2514–2516.
Treibich, A. 1994 New elliptic potentials. *Acta Appl. Math.* 36, 27–48.
Treibich, A. 2001 Hyperelliptic tangential covers and finite-gap potentials. *Uspekhi Mat. Nauk* 56(6), 89–136. (Transl. *Russian Math. Surveys* 56, 1107–1151 (2001).)
Treibich, A. & Verdier, J.-L. 1992 Revêtements exceptionnels et sommes de quatre nombres triangulaires. *Duke Math. J.* 68, 217–236.
Turbiner, A. V. 1989 Lamé equation, sl(2) algebra and isospectral deformations. *J. Phys. A* 22, L1–L3.
vander Waall, A. 2002 Lamé equations with finite monodromy. Ph.D. dissertation, University of Utrecht, The Netherlands. Currently available at [http://www.library.uu.nl/digiarchief/dip/diss/2002-0530-113355/inhoud.htm](http://www.library.uu.nl/digiarchief/dip/diss/2002-0530-113355/inhoud.htm).
Whittaker, E. T. & Watson, G. N. 1927 *A course of modern analysis*, 4th edn. Cambridge, UK: Cambridge University Press.