Lifespan estimates via Neumann heat kernel

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Abstract

This paper studies the lower bound of the lifespan $T^*$ for the heat equation $u_t = \Delta u$ in a bounded domain $\Omega \subset \mathbb{R}^n (n \geq 2)$ with positive initial data $u_0$ and a nonlinear radiation condition on partial boundary: the normal derivative $\partial u/\partial n = u^q$ on $\Gamma_1 \subseteq \partial \Omega$ for some $q > 1$, while $\partial u/\partial n = 0$ on the other part of the boundary. Previously, under the convexity assumption of $\Omega$, the asymptotic behaviors of $T^*$ on the maximum $M_0$ of $u_0$ and the surface area $|\Gamma_1|$ of $\Gamma_1$ were explored. In this paper, without the convexity requirement of $\Omega$, we will show that as $M_0 \to 0^+$, $T^*$ is at least of order $M_0^{(q-1)}$ which is optimal. Meanwhile, we will also prove that as $|\Gamma_1| \to 0^+$, $T^*$ is at least of order $|\Gamma_1|^{-1}$ for $n \geq 3$ and $|\Gamma_1|^{-1}/\ln(|\Gamma_1|^{-1})$ for $n = 2$. The order on $|\Gamma_1|$ when $n = 2$ is almost optimal. The proofs are carried out by analyzing the representation formula of $u$ in terms of the Neumann heat kernel.

1 Introduction

1.1 Problem and Results

In this paper, $\Omega$ represents a bounded open subset in $\mathbb{R}^n (n \geq 2)$ with $C^2$ boundary $\partial \Omega$. $\Gamma_1$ and $\Gamma_2$ denote two disjoint relatively open subsets of $\partial \Omega$ such that $\Gamma_1 \neq \emptyset$ and $\overline{\Gamma_1} \cup \overline{\Gamma_2} = \partial \Omega$. Moreover, the interface $\Gamma$, defined by $\Gamma = \overline{\Gamma_1} \cap \overline{\Gamma_2}$, is the common boundary of $\Gamma_1$ and $\Gamma_2$. We assume $\Gamma$ is $C^1$ as the boundary of $\Gamma_1$ or $\Gamma_2$. For example (see Figure 1), if $\Omega$ is a ball, and $\Gamma_1$ and $\Gamma_2$ are the open upper and lower hemispheres,

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{interface.png}
\caption{Interface $\Gamma$}
\end{figure}

then the interface $\Gamma$ is the equator.

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We study the following problem:

\[
\begin{cases}
(\partial_t - \Delta_x)u(x, t) = 0 & \text{in } \Omega \times (0, T], \\
\frac{\partial u(x, t)}{\partial n(x)} = u^q(x, t) & \text{on } \Gamma_1 \times (0, T], \\
\frac{\partial u(x, t)}{\partial n(x)} = 0 & \text{on } \Gamma_2 \times (0, T], \\
u(x, 0) = u_0(x) & \text{in } \Omega,
\end{cases}
\]

where \( q > 1, u_0 \in C^1(\Omega), u_0(x) \geq 0, u_0(x) \not\equiv 0. \) (1.2)

The normal derivative on the boundary is understood in the classical way: for any \((x, t) \in \partial \Omega \times (0, T],\)

\[
\frac{\partial u(x, t)}{\partial n(x)} \triangleq \lim_{h \to 0^+} \frac{u(x, t) - u(x - h \vec{n}(x), t)}{h},
\]

where \( \vec{n}(x) \) denotes the exterior unit normal vector at \( x. \) \( \partial \Omega \) being \( C^2 \) ensures that \( x - h \vec{n}(x) \) belongs to \( \Omega \) when \( h \) is positive and sufficiently small.

Throughout this paper, we write

\[ M_0 = \max_{x \in \Omega} u_0(x) \] (1.4)

and denote \( M(t) \) to be the supremum of the solution \( u \) to (1.1) on \( \Omega \times [0, t]: \)

\[ M(t) = \sup_{(x, \tau) \in \Omega \times [0, t]} u(x, \tau). \] (1.5)

\( |\Gamma_1| \) represents the surface area of \( \Gamma_1, \) that is

\[ |\Gamma_1| = \int_{\Gamma_1} dS, \]

where \( dS \) means the surface integral. \( \Phi \) refers to the heat kernel of \( \mathbb{R}^n: \)

\[ \Phi(x, t) = \frac{1}{(4\pi t)^{n/2}} \exp \left( -\frac{|x|^2}{4t} \right), \quad \forall (x, t) \in \mathbb{R}^n \times (0, \infty). \] (1.6)

In addition, \( C = C(a, b, \ldots) \) and \( C_i = C_i(a, b, \ldots) \) stand for positive and finite constants which only depend
on the parameters \( a, b, \ldots. \) One should also note that \( C \) and \( C_i \) may represent different constants in different
places.

The recent paper \[35\] studied (1.1) systematically and the motivation was the disaster of the Space Shuttle Columbia (see Figure 2 in 2003, we refer the reader to that paper for the detailed discussion of the background. As a summary of its conclusions, \[35\] first established the local existence and uniqueness theory for (1.1) in the following sense: there exist \( T > 0 \) and a unique solution \( u \) in \( C^{2,1}(\Omega \times (0, T]) \cap C(\bar{\Omega} \times [0, T]) \) which satisfies (1.1) pointwisely and also satisfies

\[
\frac{\partial u(x, t)}{\partial n(x)} = \frac{1}{2} u^q(x, t), \quad \forall (x, t) \in \bar{\Gamma} \times (0, T].
\]

Moreover, it is shown that this unique solution \( u \) is strictly positive when \( t > 0. \) We want to remark here
that the solution constructed in \[35\] through the heat potential technique automatically satisfies (1.7) due to
a generalized jump relation (See Theorem A.3 in [35]). The purpose of imposing this additional restriction (1.7) to the local solution is to ensure the uniqueness through the Hopf’s lemma, it is not clear whether the uniqueness will still hold without this restriction. After the local existence and uniqueness theory was set up, [35] also studied the blow-up phenomenon of (1.1). If \( T^* \) denotes the lifespan (maximal existence time) of the local solution \( u \), then it is proved that \( T^* < \infty \) and

\[
\lim_{t \uparrow T^*} M(t) = \infty. \tag{1.8}
\]

So the lifespan \( T^* \) is exactly the blow-up time of \( u \). Moreover, if \( \min \Omega_0 > 0 \), then \( T^* \) has the following upper bound:

\[
T^* \leq \frac{1}{(q-1)|\Gamma_1|} \int_{\Omega} u_0^{1-q}(x) \, dx. \tag{1.9}
\]

Meanwhile, [35] also provides a lower bound for \( T^* \).

Later in [36], it improves the lower bound as below.

\[
T^* \geq \frac{C}{q-1} \ln \left( 1 + (2M_0)^{-4(q-1)} |\Gamma_1|^{-\frac{2}{q-1}} \right). \tag{1.10}
\]

where \( C = C(n, \Omega) \). Based on (1.9) and (1.10), if \( q \to 1^+ \), then both the upper and lower bounds of \( T^* \) tends to infinity at the order \( (q-1)^{-1} \), which implies the order of \( T^* \) is exactly \( (q-1)^{-1} \). On the other hand, if \( |\Gamma_1| \to 0^+ \), then the order of the upper bound is \( |\Gamma_1|^{-1} \) while the lower bound only has a logarithmic order of \( |\Gamma_1|^{-1} \). Similarly, if the initial maximum \( M_0 \to 0^+ \), then the order of the upper bound (by assuming \( u_0 \) is comparable to \( M_0 \)) is \( M_0^{-(q-1)} \) while the lower bound only has a logarithmic order of \( M_0^{-1} \). So it is natural to ask that as \( |\Gamma_1| \to 0^+ \) (resp. \( M_0 \to 0^+ \)), whether the lower bound can be improved to be of order \( |\Gamma_1|^{-\alpha} \) (resp. \( M_0^{-\alpha} \)) for some \( \alpha > 0 \)? In [36], it gives an affirmative answer to this question under the convexity assumption of the domain \( \Omega \). However, in many situations, the domain \( \Omega \) may not be convex. For example, as the motivation of the problem (1.1) illustrated in [35], the Space Shuttle Columbia (see Figure 2) is not convex. So the main goal of this paper is to remove the convexity assumption, and we will apply a new approach which takes advantage of the Neumann heat kernel.

Previously, the methods used in [35] and [36] based on the representation formula (see Corollary 3.9 in [35]) of \( u \) in terms of the heat kernel \( \Phi \) of \( \mathbb{R}^n \). More precisely, for any boundary point \( x \in \partial \Omega \) and
\( t \in [0, T^*) \),

\[
  u(x, t) = 2 \int_{\Omega} \Phi(x - y, t) u_0(y) \, dy - 2 \int_0^t \int_{\partial \Omega} \frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} u(y, \tau) \, dS(y) \, d\tau \\
  + 2 \int_0^t \int_{\Gamma_1} \Phi(x - y, t - \tau) u^q(y, \tau) \, dS(y) \, d\tau,
\]

(1.11)

where

\[
  \frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} = -(D\Phi)(x - y, t - \tau) \cdot \vec{n}(y).
\]

We want to remark that there also exists a representation formula for the inside point \( x \in \Omega \) and \( t \in [0, T^*) \), see Theorem 3.8 in [35]. That is,

\[
  u(x, t) = \int_{\Omega} \Phi(x - y, t) u_0(y) \, dy - \int_0^t \int_{\partial \Omega} \frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} u(y, \tau) \, dS(y) \, d\tau \\
  + \int_0^t \int_{\Gamma_1} \Phi(x - y, t - \tau) u^q(y, \tau) \, dS(y) \, d\tau,
\]

(1.12)

The formula (1.12) is different from (1.11) in that the coefficients 2’s do not appear in front of the integrals on the right hand side. The existence of the coefficients 2’s in (1.11) is due to the jump relation of the single-layer heat potential when \( x \in \partial \Omega \) (see e.g. Corollary Appendix A.2 in [32] or Theorem 9.5, Sec. 2, Chap. 9 in [15]). The drawback of the formula (1.11) is the uncertainty of the sign of the term \( \frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} \).

It is this reason that demands the convexity of \( \Omega \) in [36] to improve the lower bound of \( T^* \) to a power order of \(|\Gamma_1|^{-1} \) (resp. \( M_0^{-1} \)) as \(|\Gamma_1| \to 0^+ \) (resp. \( M_0 \to 0^+) \).

In order to avoid the integral containing \( \frac{\partial \Phi(x - y, t - \tau)}{\partial n(y)} \), it motivates us to consider the Green’s function \( G(x, t, y, s) \) of the heat operator in \( \Omega \) with the Neumann boundary condition whose normal derivative vanishes (see (2.6) in Lemma 2.2). As a convention, \( G(x, t, y, s) \) is also called the Neumann Green’s function. In addition, since the coefficients of the heat operator are constants, the Neumann Green’s function is invariant under time translation (see part(f) in Lemma 2.2). So it is more convenient to consider the corresponding Neumann heat kernel \( N(x, y, t) \) of \( \Omega \) (see Definition 2.3). Unlike the heat kernel \( \Phi \) of \( \mathbb{R}^n \), the Neumann heat kernel \( N(x, y, t) \) of \( \Omega \) does not have an explicit formula in general which makes it difficult to quantify. Fortunately, for small time \( t \), \( N(x, y, t) \) can be bounded in terms of \( \Phi \) (see Lemma 2.3). This property will help us to justify the representation formula (2.12) via \( N(x, y, t) \) and further analyze it. In fact, if the solution \( u \) to (1.1) is smooth, then it is straightforward to obtain (2.12) based on the properties of \( N(x, y, t) \) in Corollary 2.4. Now although \( u \) is not smooth near the boundary \( \partial \Omega \) or near the initial time \( t = 0 \) by taking advantage of Lemma 2.5 we are able to verify (2.12) in a way similar to the proof for (1.12) in [35]. In contrast to (1.12), the formula (2.12) does not contain any term that may cause the jump relation along the boundary \( \partial \Omega \). As a result, the formula (2.12) holds for both \( x \in \Omega \) and \( x \in \partial \Omega \). In addition, due to Lemma 2.5 again, for small time \( t \), the estimate on the term \( \int_0^t \int_{\Gamma_1} N(x, y, t - \tau) u^q(y, \tau) \, dS(y) \, d\tau \) boils down to the estimate on \( \int_0^t \int_{\Gamma_1} \Phi(x - y, 2(t - \tau)) u^q(y, \tau) \, dS(y) \, d\tau \) which has been treated in [36]. Finally, noticing that the method in [36] analyzes the representation formula discretely and in each step the time is indeed small, so we can combine that method with (2.12) and Lemma 2.5 to achieve our goal. The following are the main results of this paper.

**Theorem 1.1.** Assume (2.2) and let \( T^* \) be the lifespan for (1.1). Then there exists a constant \( C = C(n, \Omega) \) such that

\[
  T^* \geq \frac{C}{q - 1} \ln \left( 1 + (2M_0)^{-4(q - 1)} |\Gamma_1|^{-\frac{q}{2(q - 1)}} \right).
\]

(1.13)
This theorem is not new and it has appeared in Theorem 1.1 of [36]. But its proof in this paper, as mentioned above, is different and based on the representation formula (2.12) involving the Neumann heat kernel \( N(x,y,t) \).

**Theorem 1.2.** Assume (1.2) and let \( T^* \) be the lifespan for (1.1). Denote \( M_0 \) as in (1.4) and define

\[
Y = \begin{cases} 
M_0^{q-1} |\Gamma_1|^{\frac{n}{n-1}}, & \text{if } n \geq 3, \\
M_0^{q-1} |\Gamma_1| \ln \left(1 + \frac{1}{|\Gamma_1|}\right), & \text{if } n = 2.
\end{cases}
\]

Then there exist constants \( Y_0 = Y_0(n,\Omega) \) and \( C = C(n,\Omega) \) such that if \( Y \leq \frac{Y_0}{q} \), then

\[
T^* \geq C \left( \frac{q}{q-1} \right) Y. \tag{1.14}
\]

The lower bound (1.14) was also obtained in [36] under the convexity assumption of \( \Omega \), so the significance of Theorem 1.2 is the removal of the convexity requirement. The method in the proof again relies on the representation formula (2.12).

**Remark 1.3.** From Theorem 1.2, we draw two conclusions.

1. Relation between \( T^* \) and \( M_0 \): if \( M_0 \rightarrow 0^+ \) and other factors are fixed, then (1.14) implies

\[
T^* \geq C_1 M_0^{-(q-1)}.
\]

This order is optimal since if the initial data \( u_0 \) is a constant function, then it follows from (1.9) that

\[
T^* \leq C_2 M_0^{-(q-1)}.
\]

2. Relation between \( T^* \) and \( |\Gamma_1| \): if \( |\Gamma_1| \rightarrow 0^+ \) and other factors are fixed, then it follows from (1.3) that \( T^* \) is at most of order \( |\Gamma_1|^{-1} \). On the other hand, (1.14) implies that \( T^* \) is at least of order \( |\Gamma_1|^{-1} / \ln (|\Gamma_1|^{-1}) \) for \( n \geq 3 \) and \( |\Gamma_1|^{-1} / \ln (|\Gamma_1|^{-1}) \) for \( n = 2 \). In particular when \( n = 2 \), the order of the lower bound is almost optimal (within a logarithmic order to the upper bound).

### 1.2 Historical Works

#### 1.2.1 Blow-up phenomenon for the heat equation with nonlinear Neumann conditions

Since the pioneering papers by Kaplan [17] and Fujita [10], the blow-up phenomenon of parabolic type has been extensively studied in the literature for the Cauchy problem as well as the boundary value problems. We refer the readers to the surveys [4, 21], the books [11, 31] and the references therein.

One of the typical problems is the heat equation with Neumann boundary conditions in a bounded domain \( \Omega \):

\[
\begin{cases}
(\partial_t - \Delta_x)u(x,t) = 0 & \text{in } \Omega \times (0, T], \\
\frac{\partial u(x,t)}{\partial n(x)} = F(u(x,t)) & \text{on } \partial \Omega \times (0, T], \\
u(x,0) = \psi(x) & \text{in } \Omega.
\end{cases} \tag{1.15}
\]

Here, the initial data \( \psi \) is not assumed to be nonnegative. It is well-known that there are two ways to construct the classical solution to (1.15) depending on the smoothness of \( \partial \Omega \), \( F \) and \( u_0 \) (see Theorem 1.1 and 1.3 in [25], also see the books [9][19][24]).
(a) The first way is by Schauder estimate. Assume $\partial \Omega$ is $C^{2+\alpha}$, $F \in C^{1+\alpha}(\mathbb{R})$, $\psi \in C^{2+\alpha}(\overline{\Omega})$ and the compatibility condition

$$\frac{\partial \psi(x)}{\partial n(x)} = F(\psi(x)), \quad \forall x \in \partial \Omega.$$ 

Then there exist $T > 0$ and a unique function $u$ in $C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega} \times [0, T])$ which satisfies (1.15) pointwise.

(b) The second way is by the heat potential technique. The requirements on the data can be relaxed and in particular, the compatibility condition is no longer needed, but accordingly the conclusion is also weaker. More precisely, assume $\partial \Omega$ is $C^{1+\alpha}$, $F \in C^1(\mathbb{R})$ and $\psi \in C^1(\overline{\Omega})$. Then there exist $T > 0$ and a unique function $u$ in $C^2(\Omega \times (0, T)) \cap C(\overline{\Omega} \times [0, T])$ which satisfies (1.15) pointwise.

In most papers, the assumptions will fall into or similar to either case (a) or case (b). In the following statements, we will ignore their distinctions and just refer them to be the local (classical) solutions.

It has been already known that if $F$ is bounded on $\mathbb{R}$, then the local solutions can be extended globally. But if $F$ is unbounded, then the finite-time blowup may occur. The first result on the blow-up phenomenon for (1.15) is due to Levine and Payne [22]. They used a concavity argument to conclude that any classical solution blows up in finite time under the two assumptions below.

- First,

$$F(z) = |z|^q h(z), \quad (1.16)$$

for any constant $q > 1$ and any differentiable, non-decreasing function $h(z)$.

- Secondly,

$$\frac{1}{|\partial \Omega|} \int_{\partial \Omega} \left( \int_0^{\psi(x)} F(z) \, dz \right) \, dS(x) > \frac{1}{2} \int_{\Omega} |D\psi(x)|^2 \, dx. \quad (1.17)$$

**Remark 1.4.** As a corollary of the result in [22], if $h(z)$ in (1.16) is also positive and $\psi$ is a positive constant function, then (1.17) is satisfied and therefore the solution blows up in finite time. Combining this fact with the comparison principle, it implies that for any positive $h(z)$ in (1.16) and for any positive $\psi$, the solution blows up in finite time.

Later, Walter [34] gave a more complete characterization for the blow-up phenomenon by introducing some comparison functions. More precisely, if $F(z)$ is positive, increasing and convex for $z \geq z_0$ with some constant $z_0$, then there are exactly two possibilities.

- First, if $\int_{z_0}^{\infty} \frac{1}{F(z)} \, dz = \infty$, then the solution exists globally for any initial data $\psi$.

- Secondly, if $\int_{z_0}^{\infty} \frac{1}{F(z)} \, dz < \infty$, then the solution blows up in finite time for large initial data $\psi$.

The result was further generalized by Rial and Rossi [32] (also see [25]). In [32], by assuming $F$ to be $C^2$, increasing and positive in $\mathbb{R}_+$, and also assuming $1/F$ to be locally integrable near $\infty$ (that is $\int_{\infty}^{\infty} \frac{1}{F(z)} \, dz < \infty$), it is shown that for any positive initial data $\psi$, the classical solution blows up in finite time. The success of their method was due to a clever choice of an energy function which made the proof short and elementary.

Applying these earlier results to the simpler model (that is (1.1) with $\Gamma_2 = \emptyset$)

$$\begin{cases}
(\partial_t - \Delta_x)u(x, t) = 0 \quad &\text{in} \quad \Omega \times (0, T), \\
\frac{\partial u(x, t)}{\partial n(x)} = u^\theta(x, t) \quad &\text{on} \quad \partial \Omega \times (0, T), \\
u(x, 0) = u_0(x) \quad &\text{in} \quad \Omega,
\end{cases} \quad (1.18)$$
where \( q > 1 \) and the initial data \( u_0 \geq 0 \) and \( u_0 \neq 0 \), it can be shown that any solution to (1.18) blows up in finite time. In fact, by the maximum principle, the solution \( u \) becomes positive as soon as \( t > 0 \). Then either Remark 1.4 or the result in [32] (also see [12]) implies the finite time blowup of the solution. However, when the nonlinear radiation condition is only imposed on partial boundary (that is when \( \Gamma_2 \neq \emptyset \) in (1.1)), additional difficulties appear due to the discontinuity of the normal derivative along the interface \( \tilde{\Gamma} \) between \( \Gamma_1 \) and \( \Gamma_2 \). To our knowledge, [35] and [36] were the first papers that dealt with this problem and quantified both upper and lower bounds of the lifespan (or equivalently the blow-up time).

1.2.2 Lower bound estimate for the lifespan

When considering the bounds of the lifespan, the upper bound is usually related to the nonexistence of the global solutions and various methods on this issue have been developed (see [20] for a list of six methods). The lower bound was not studied as much in the past and not many methods have been explored. However, the lower bound may be more useful in practice since it serves as the safe time. In the existing literature, the most common ideas are the comparison argument and the differential inequality techniques.

The first work on the lower bound estimate of the lifespan was due to Kaplan [17]. Later, Payne and Schaefer developed a very robust method on this issue. For example, they derived the lower bound of the lifespan for the nonlinear heat equation with homogeneous Dirichlet or Neumann boundary conditions in [28, 29]. Later this idea was also applied to the problem (1.15) (see [30]) and many other types of problems (see e.g. [1, 2, 5, 7, 23, 26, 27, 33]). However, this method requires the domain to be convex. In addition, it is not effective to deal with the partial nonlinear boundary conditions like the one in (1.1).

Recently, in order to obtain the lower bound of the lifespan for the problem (1.1), the authors of this paper developed a new method in [36] by discretely analyzing the representation formula of the solution in terms of \( \Phi \). Firstly, without the convexity assumption, [36] obtained a lower bound for \( T^* \) which was logarithmic order of \( |\Gamma_1|^{-1} \) (resp. \( M_0^{-1} \)) as \( |\Gamma_1| \to 0^+ \) (resp. \( M_0 \to 0^+ \)). On the other hand, by assuming \( \Omega \) to be convex, it improved the lower bound to be of power order of \( |\Gamma_1|^{-1} \) (resp. \( M_0^{-1} \)) when \( |\Gamma_1| \to 0^+ \) (resp. \( M_0 \to 0^+ \)) as in Theorem 1.2.

In the current paper, without the convexity assumption on \( \Omega \), by combining the method in [36] with the new representation formula in terms of the Neumann heat kernel \( N(x, y, t) \) (see (2.7)), we are able to show that \( T^* \) is at least the same power order of \( |\Gamma_1|^{-1} \) (resp. \( M_0^{-1} \)) when \( |\Gamma_1| \to 0^+ \) (resp. \( M_0 \to 0^+ \)) as in [36].

1.3 Organization

The organization of this paper is as follows. Section 2 will introduce the definitions and the basic properties of the Neumann Green’s function and the Neumann heat kernel of \( \Omega \). In addition, it will discuss the representation formula of the solution and provide two crucial estimates on the boundary-time integrals of the heat kernel \( \Phi \) of \( \mathbb{R}^n \). Section 3 and Section 4 will prove Theorem 1.1 and Theorem 1.2 respectively. Section 5 will demonstrate the sharpness of Lemma 2.10 which plays an essential role in Section 4. Finally in the Appendix A, a rigorous proof will be given to the representation formula mentioned in Section 2 which is the key tool in this paper.
2 Preliminaries

2.1 Neumann Green’s Function and Neumann Heat Kernel

Given a bounded domain $\Omega$ in $\mathbb{R}^n$ and a parabolic operator $L$ on $\Omega$, similar to the elliptic case, one can define the fundamental solution associated to $L$ on $\Omega$ (see e.g. [6, 8, 13]). If in addition the boundary conditions are considered, one can also study the fundamental solution adapted to the boundary conditions (see e.g. [14–16]). Such a fundamental solution with the boundary condition is usually called the Green’s function. In particular, if the boundary condition is of Neumann type, then the associated fundamental solution is called the Neumann Green’s function. When the coefficients of the parabolic operator $L$ are independent of the time $t$, the Neumann Green’s function is invariant under the time translation. Consequently, it automatically generates a Neumann heat kernel which has a simpler form but captures the essential properties of the Neumann Green’s function. The operator considered in this paper is just the heat operator $L = \partial_t - \Delta_x$ whose coefficients are constants, so we will first state the precise definitions of the associated Neumann Green’s function and the Neumann heat kernel, and then collect some classical properties which are needed later.

Roughly speaking, for the heat operator

$$L_{tx} = \partial_t - \Delta_x$$

(2.1)

with the Neumann boundary condition, the associated Green’s function on $\Omega$, which is also called the Neumann Green’s function on $\Omega$, is a function $G(x,t,y,s)$ defined on $\{(x,t,y,s) : x,y \in \Omega, t,s \in \mathbb{R}, s < t\}$ such that for any fixed $s \in \mathbb{R}$ and $y \in \Omega$,

$$\begin{cases}
(\partial_t - \Delta_x)G(x,t,y,s) = 0, & \forall x \in \Omega, t > s, \\
\frac{\partial G(x,t,y,s)}{\partial n(x)} = 0, & \forall x \in \partial \Omega, t > s, \\
\lim_{t \to s^+} G(x,t,y,s) = \delta(x-y), & \text{in distributional sense.}
\end{cases}$$

In other words, for any fixed $s \in \mathbb{R}$ and for any test function $\psi$ that satisfies

$$\psi \in C(\Omega) \quad \text{and} \quad \frac{\partial \psi(x)}{\partial n(x)} = 0, \quad \forall x \in \partial \Omega,$$

(2.2)

the function $v(x,t)$ defined as

$$v(x,t) = \int_{\Omega} G(x,t,y,s) \psi(y) \, dy, \quad \forall x \in \overline{\Omega}, t > s,$$

solves the following initial-boundary value problem:

$$\begin{cases}
(\partial_t - \Delta_x)v(x,t) = 0 & \text{in } \Omega \times (s, \infty), \\
\frac{\partial v(x,t)}{\partial n(x)} = 0 & \text{on } \partial \Omega \times (s, \infty), \\
v(x,s) = \psi(x) & \text{in } \Omega.
\end{cases}$$

(2.3)

For the formal definition of the Neumann Green’s function associated to the heat operator (2.1) on $\Omega$, we follow ( [14], Page 171).

**Definition 2.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with $C^2$ boundary $\partial \Omega$. Then we define the Neumann
Green’s function for the heat operator in $\Omega$ to be a continuous function $G(x, t, y, s)$ on $\{(x, t, y, s) : x, y \in \Omega, t, s \in \mathbb{R}, s < t\}$ such that for any fixed $s \in \mathbb{R}$ and for any $\psi$ in (2.2), the function $v(x, t)$ defined as

$$v(x, t) = \int_{\Omega} G(x, t, y, s) \psi(y) \, dy$$

(2.4)

belongs to $C^{2,1}(\overline{\Omega} \times (s, \infty))$ and solves (2.3) in the following sense:

$$\begin{cases}
(\partial_t - \Delta_x) v(x, t) = 0, & \forall x \in \overline{\Omega}, t > s, \\
\frac{\partial v(x, t)}{\partial n(x)} = 0, & \forall x \in \partial \Omega, t > s, \\
\lim_{t \to s^+} v(x, t) = \psi(x), & \text{uniformly in } x \in \overline{\Omega}.
\end{cases}$$

(2.5)

Lemma 2.2. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with $C^2$ boundary $\partial \Omega$. Then there exists a unique Neumann Green’s function $G(x, t, y, s)$ for the heat operator in $\Omega$ as in Definition 2.1. In addition, it has the following properties.

(a) $G(x, t, y, s)$ is $C^2$ in $x$ and $y$ ($x, y \in \overline{\Omega}$), and $C^1$ in $t$ and $s$ ($s < t$).

(b) For fixed $s \in \mathbb{R}$ and $y \in \overline{\Omega}$, as a function in $x$ and $t$ ($x \in \overline{\Omega}$ and $t > s$), $G(x, t, y, s)$ satisfies

$$\begin{cases}
(\partial_t - \Delta_x) G(x, t, y, s) = 0, & \forall x \in \overline{\Omega}, t > s, \\
\frac{\partial G(x, t, y, s)}{\partial n(x)} = 0, & \forall x \in \partial \Omega, t > s.
\end{cases}$$

(2.6)

(c) For any $s \in \mathbb{R}$ and $\psi$ in (2.2), the function $v(x, t)$ defined in (2.4) is the unique function in $C^{2,1}(\overline{\Omega} \times (s, \infty))$ that satisfies (2.3).

(d) $G(x, t, y, s) \geq 0$ for any $x, y \in \overline{\Omega}$ and $s < t$.

(e) $\int_{\Omega} G(x, t, y, s) \, dy = 1$ for any $x \in \overline{\Omega}$ and $s < t$.

(f) For any $x, y \in \overline{\Omega}$ and $s < t$,

$$G(x, t, y, s) = G(x, t - s, y, 0) \quad \text{and} \quad G(x, t, y, s) = G(y, t, x, s).$$

Proof. The existence and uniqueness of the Neumann Green’s function, and part (a)–(d) follow from Theorem 1–Theorem 4 in [14]. We just want to remark that although the regularity of $\partial \Omega$ in [14] is required to be $C^{4, \gamma}$ for the general manifold and the general second order parabolic operators, here in the case of the Euclidean space and the heat operator, $\partial \Omega$ being $C^2$ is enough.

- For part (e), let $\psi \equiv 1$. Then $v \equiv 1$ obviously satisfies (2.3). Combining this fact with part (c) concludes (e).

- For part (f), since we are considering the heat equation whose coefficients are constants, it follows from Theorem 3 in [16] that there exists a function $N(x, y, t)$ such that $G(x, t, y, s) = N(x, y, t - s)$. In addition, Theorem 4 in [16] claims that $N(x, y, t) = N(y, x, t)$. Consequently, part (f) is justified.

\[\square\]
As we have seen from the above proof that there exists a function \( N(x, y, t) \) such that \( N(x, y, t - s) = G(x, t, y, s) \). In particular, choosing \( s = 0 \) leads to

\[
N(x, y, t) = G(x, t, y, 0).
\]  

This function \( N(x, y, t) \) is called the Neumann heat kernel of \( \Omega \).

**Definition 2.3.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with \( C^2 \) boundary \( \partial \Omega \). A function \( N(x, y, t) \) on \( \Omega \times \Omega \times (0, \infty) \) is called a Neumann heat kernel if the function \( G(x, t, y, s) \) defined by

\[
G(x, t, y, s) = N(x, y, t - s)
\]  
is the Neumann Green’s function in Definition 2.1.

Combining (2.7) and Lemma 2.2, we list some properties of the Neumann heat kernel.

**Corollary 2.4.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with \( C^2 \) boundary \( \partial \Omega \). Then there exists a unique Neumann heat kernel \( N(x, y, t) \) of \( \Omega \) as in Definition 2.3. In addition, it has the following properties.

(a) \( N(x, y, t) \) is \( C^2 \) in \( x \) and \( y \) (\( x, y \in \Omega \)), and \( C^1 \) in \( t \) (\( t > 0 \)).

(b) For fixed \( y \in \Omega \), as a function in \( (x, t) \), \( N(x, y, t) \) satisfies

\[
\left\{ \begin{array}{ll}
(\partial_t - \Delta_x)N(x, y, t) = 0, & \forall x \in \Omega, \ t > 0, \\
\frac{\partial N(x, y, t)}{\partial n(x)} = 0, & \forall x \in \partial \Omega, \ t > 0.
\end{array} \right.
\]  

(2.8)

(c) For fixed \( \psi \) in (2.2), the function \( w(x, t) \) defined by

\[
w(x, t) = \int_\Omega N(x, y, t)\psi(y) \, dy
\]  

is the unique function in \( C^{2,1}(\Omega \times (0, \infty)) \) that satisfies the following equations.

\[
\left\{ \begin{array}{ll}
(\partial_t - \Delta_x)w(x, t) = 0, & \forall x \in \Omega, \ t > 0, \\
\frac{\partial w(x, t)}{\partial n(x)} = 0, & \forall x \in \partial \Omega, \ t > 0, \\
\lim_{t \to 0^+} w(x, t) = \psi(x), & \text{uniformly in } x \in \Omega.
\end{array} \right.
\]  

(2.10)

(d) \( N(x, y, t) \geq 0 \) and \( N(x, y, t) = N(y, x, t) \) for any \( x, y \in \Omega \) and \( t > 0 \).

(e) \( \int_\Omega N(x, y, t) \, dy = 1 \) for any \( x \in \Omega \) and \( t > 0 \).

Proof. These are direct consequences of Definition 2.1, Lemma 2.2 and Definition 2.3.

Unlike the heat kernel \( \Phi \) of \( \mathbb{R}^n \) in (1.6), the Neumann heat kernel \( N(x, y, t) \) of \( \Omega \) usually does not have an explicit formula. Nevertheless, when \( t \) is small, \( N(x, y, t) \) can be bounded in terms of \( \Phi \).

**Lemma 2.5.** There exists \( C = C(n, \Omega) \) such that for any \( x, y \in \Omega \) and \( t \in (0, 1] \),

\[
0 \leq N(x, y, t) \leq C \Phi(x - y, 2t),
\]  

where \( \Phi \) is defined as in (1.6).
Proof. Applying Theorem 3.2.9 on Page 90 of \cite{3} with \(\lambda = 1, \mu = 1\) and \(\delta = 1/2\) to the heat operator \(\mathcal{A}\).

Then there exists \(C_1 = C_1(\Omega)\) such that

\[
0 \leq N(x, y, t) \leq C_1 \max\{t^{-n/2}, 1\} \exp\left(\frac{-|x - y|^2}{6t}\right).
\]

In particular, when \(t \in (0, 1]\), we have

\[
0 \leq N(x, y, t) \leq C_1 t^{-n/2} \exp\left(\frac{-|x - y|^2}{6t}\right) \leq 2^{n/2}C_1 \Phi(x - y, 2t).
\]

\(\square\)

2.2 Representation Formula By the Neumann Heat Kernel

One of the applications of the Neumann heat kernel is the representation formula of the solution to the heat equation with Neumann boundary conditions. As a heuristic argument, let’s fix any \(x \in \Omega\) and \(t > 0\) and pretend the solution \(u\) to (1.1) is sufficiently smooth. Then it follows from part (b) and (d) of Corollary 2.4 that

\[
(\partial_t - \Delta_y)N(x, y, t - \tau) = (\partial_t - \Delta_y)N(y, x, t - \tau) = 0, \quad \forall y \in \overline{\Omega}, 0 < \tau < t.
\]

As a result,

\[
\int_0^t \int_\Omega (\partial_t - \Delta_y)N(x, y, t - \tau) u(y, \tau) \, dy \, d\tau = 0.
\]

Equivalently,

\[
\int_0^t \int_\Omega (-\partial_\tau - \Delta_y)N(x, y, t - \tau) u(y, \tau) \, dy \, d\tau = 0.
\]

Now formally integrating by parts and taking advantage of (b), (c) and (d) in Corollary 2.4 we obtain

\[
u(x, t) = \int_0^t \int_\Omega N(x, y, t - \tau) (\partial_\tau - \Delta_y)u(y, \tau) \, dy \, d\tau + \int_\Omega N(x, y, t) u(y, 0) \, dy
\]

\[
+ \int_0^t \int_{\partial\Omega} N(x, y, t - \tau) \frac{\partial u(y, \tau)}{\partial n(y)} \, dS(y) \, d\tau.
\]

Keeping in mind that \(u\) is the solution to (1.1), so

\[
u(x, t) = \int_\Omega N(x, y, t)u_0(y) \, dy + \int_0^t \int_{\Gamma_1} N(x, y, t - \tau)u^0(y, \tau) \, dS(y) \, d\tau.
\]

This is the representation formula that is desired, but we still need to justify it rigorously. Since the proof is standard but tedious, we decide to put it into Appendix \(A\). Here we will just present the formal statement of the representation formula.

Lemma 2.6. Let \(u\) be the solution to (1.1) and denote \(T^*\) to be its lifespan. Then for any \((x, t) \in \overline{\Omega} \times (0, T^*)\),

\[
u(x, t) = \int_\Omega N(x, y, t)u_0(y) \, dy + \int_0^t \int_{\Gamma_1} N(x, y, t - \tau)u^0(y, \tau) \, dS(y) \, d\tau.
\]

(2.12)

Proof. See Appendix \(A\) \(\square\)
The total number $K$ and $C$

**Proof.** Since (2.11) and (2.15).

3.1 and Lemma 4.1, while these two lemmas are the key ingredients in the proofs of Theorem 1.1 and 2.3 Boundary-Time Integral of the Heat Kernel of $\mathbb{R}^n$

This section will provide the crucial estimates (2.15) and (2.19) that will be used in the proofs of Lemma 2.6 and (2.19). The three statements in this subsection have already essentially appeared in [35, 36]. But for completeness and preciseness, we still include their proofs here.

In the following, for any $\tilde{x} \in \mathbb{R}^{n-1}$, we denote

$$B(\tilde{x}, \rho) = \{ \tilde{y} \in \mathbb{R}^{n-1} : |\tilde{y} - \tilde{x}| < \rho \}$$

(2.14)

to be the ball centered at $\tilde{x}$ in $\mathbb{R}^{n-1}$ with radius $\rho$.

**Lemma 2.8.** There exists $C = C(n, \Omega)$ such that for any $t > 0$ and $x \in \Omega$,

$$t^{1/2} \int_{\partial \Omega} \Phi(x - y, t) dS(y) \leq C.$$  

**Proof.** Since $\partial \Omega$ in this paper is assumed to be $C^2$, there exist finitely many balls $\bar{B}_i \equiv B(\tilde{z}_i, r_i) \subset \mathbb{R}^{n-1}$ and $C^2$ mappings $\varphi_i : \bar{B}_i \to \mathbb{R} (1 \leq i \leq K)$ such that

$$\partial \Omega = \bigcup_{i=1}^{K} \{ (\tilde{y}, \varphi_i(\tilde{y})) : \tilde{y} \in \bar{B}_i \}.$$  

The total number $K$ only depends on $\Omega$. As a result, by writing $x = (\tilde{x}, x_n)$ and parametrizing $\partial \Omega$, then

$$t^{1/2} \int_{\partial \Omega} \Phi(x - y, t) dS(y) = C t^{-\frac{n-1}{2}} \int_{\partial \Omega} \exp \left( -\frac{|x - y|^2}{4t} \right) dS(y)$$

$$\leq C t^{-\frac{n-1}{2}} \sum_{i=1}^{K} \int_{\bar{B}_i} \exp \left( -\frac{|\tilde{x} - \tilde{y}|^2 + |x_n - \varphi_i(\tilde{y})|^2}{4t} \right) d\tilde{y}$$

$$\leq C \sum_{i=1}^{K} t^{-\frac{n-1}{2}} \int_{\bar{B}_i} \exp \left( -\frac{|\tilde{y}|^2}{4t} \right) d\tilde{y}$$

$$\leq C \sum_{i=1}^{K} t^{-\frac{n-1}{2}} \int_{\mathbb{R}^{n-1}} \exp \left( -\frac{|\tilde{z}|^2}{4t} \right) d\tilde{z}$$

$$= CK$$

where the last equality is because $t^{-\frac{n-1}{2}} \int_{\mathbb{R}^{n-1}} \exp \left( -\frac{|\tilde{z}|^2}{4t} \right) d\tilde{z}$ is a universal constant. 

Lemma 2.9. For any $\alpha \in [0, \frac{1}{n-1})$, there exists $C = C(n, \Omega, \alpha)$ such that for any $x \in \Omega$ and $t \geq 0$,

$$\int_{0}^{t} \int_{\Gamma_1} \Phi(x - y, \tau) dS(y) d\tau \leq C |\Gamma_1|^\alpha t^{1-(n-1)\alpha}/2.$$  

(2.15)
Proof. By Holder’s inequality,
\[
\int_{\Gamma_1} \Phi(x-y, \tau) dS(y) \leq |\Gamma_1|^\alpha \left( \int_{\Gamma_1} \left[ \Phi(x-y, \tau) \right]^{1-\alpha} dS(y) \right)^{1-\alpha} = C |\Gamma_1|^\alpha \left( \int_{\Gamma_1} \tau^{1-n\alpha/2} \Phi(x-y, (1-\alpha)\tau) dS(y) \right)^{1-\alpha}.
\] (2.16)

By applying Lemma 2.8,
\[
\int_{\Gamma_1} \Phi(x-y, (1-\alpha)\tau) dS(y) \leq \int_{\partial \Omega} \Phi(x-y, (1-\alpha)\tau) dS(y) \leq C \tau^{-1/2}.
\] (2.17)

Plugging (2.17) into (2.16),
\[
\int_{\Gamma_1} \Phi(x-y, \tau) dS(y) \leq C |\Gamma_1|^\alpha \tau^{1-(n-1)\alpha/2}.
\]

Now integrating \( \tau \) from 0 to \( t \) yields (2.15). \( \square \)

In the above lemma, \( \alpha \) is strictly less than \( \frac{1}{n-1} \), so it is natural to ask what will happen when \( \alpha = \frac{1}{n-1} \)? If we simply plug \( \alpha = \frac{1}{n-1} \) into (2.15), then it leads to
\[
\int_0^t \int_{\Gamma_1} \Phi(x-y, \tau) dS(y) d\tau \leq C |\Gamma_1|^{1/(n-1)}
\] (2.18)
whose right hand side is an upper bound independent of \( t \). However, (2.18) is not true in general. For example, in the simplest case when \( \Gamma_1 \) is flat and ball-shaped in a hypersurface and \( x \in \Gamma_1 \) (see Figure 3), then (2.18) fails when either \( t \to \infty \) or \( n = 2 \) and \(|\Gamma_1| \to 0^+\).

![Figure 3: Flat \( \Gamma_1 \)](image)

- First, when \( t \to \infty \), by parametrizing \( \Gamma_1 \) as what we will do in the proof for Proposition 5.1, it follows from Lemma 5.2 or Lemma 5.3 that the left hand side in (2.18) tends to infinity.
- Secondly, when \( n = 2 \) and \(|\Gamma_1| \to 0^+\), it follows similarly from Lemma 5.3 that
\[
\int_0^t \int_{\Gamma_1} \Phi(x-y, \tau) dS(y) d\tau \geq C|\Gamma_1| \left[ 1 + \ln \left( \frac{t}{|\Gamma_1|^2} \right) \right],
\]
which cannot be bounded by \( C|\Gamma_1| \).
Consequently, we can expect (2.18) to hold only if \( t \) is bounded and the right hand side of (2.18) adds an extra logarithmic term of \(|\Gamma_1|^{-1}\) when \( n = 2 \), see the following lemma.

**Lemma 2.10.** There exists \( C = C(n, \Omega) \) such that for any \( x \in \bar{\Omega} \),

\[
\int_0^2 \int_{\Gamma_1} \Phi(x - y, \tau) \, dS(y) \, d\tau \leq \begin{cases} 
C |\Gamma_1|^{\frac{1}{n-1}}, & \text{if } n \geq 3, \\
C |\Gamma_1| \ln \left(1 + \frac{1}{|\Gamma_1|}\right), & \text{if } n = 2.
\end{cases} \tag{2.19}
\]

**Proof.** When \( n \geq 3 \), we refer the readers to Lemma 2.7 in [36]. When \( n = 2 \), it has been shown in Lemma 2.10 in [36] that

\[
\int_1^2 \int_{\Gamma_1} \Phi(x - y, \tau) \, dS(y) \, d\tau \leq C |\Gamma_1| \ln \left(1 + \frac{1}{|\Gamma_1|}\right).
\]

So we only need to estimate \( \int_1^2 \int_{\Gamma_1} \Phi(x - y, \tau) \, dS(y) \, d\tau \). For \( \tau \geq 1 \),

\[
\Phi(x - y, \tau) \leq \frac{1}{(4\pi \tau)^{n/2}} \leq C.
\]

Therefore,

\[
\int_1^2 \int_{\Gamma_1} \Phi(x - y, \tau) \, dS(y) \, d\tau \leq C|\Gamma_1| \leq C|\Gamma_1| \ln \left(1 + \frac{1}{|\Gamma_1|}\right),
\]

where the last inequality is due to

\[
\ln \left(1 + \frac{1}{|\Gamma_1|}\right) \geq \ln \left(1 + \frac{1}{|\partial \Omega|}\right).
\]

The order of the right hand side in (2.19) on \(|\Gamma_1|\) is optimal as \(|\Gamma_1| \to 0^+\), see Proposition 5.1 in Section 5.

### 3 Proof of Theorem 1.1

In order to derive a lower bound for \( T^* \), it is important to investigate how fast the solution \( u \) can grow.

**Lemma 3.1.** Let \( u \) be the solution to (1.1). Define \( M(t) \) as in (1.2). For any \( \alpha \in [0, \frac{1}{n-1}] \), there exists \( C = C(n, \Omega, \alpha) \) such that for any \( T \geq 0 \) and \( 0 \leq t < \min\{1, T^* - T\} \),

\[
\frac{M(T + t) - M(T)}{M^q(T + t)} \leq C |\Gamma_1|^\alpha t^{1-\alpha}/2. \tag{3.1}
\]

**Proof.** It is equivalent to prove

\[
M(T + t) \leq M(T) + C M^q(T + t) |\Gamma_1|^\alpha t^{1-\alpha}/2. \tag{3.2}
\]

For any \( \sigma \in [0, T] \) and \( x \in \bar{\Omega} \), it follows from the definition of \( M(t) \) that

\[
u(x, \sigma) \leq M(T). \tag{3.3}
\]
In the rest of the proof, we assume \( \sigma \in (T, T + t) \) and \( x \in \overline{\Omega} \). By the representation formula (2.13) with \( t = \sigma - T \),

\[
    u(x, \sigma) = \int_{\Omega} N(x, y, \sigma - T)u(y, T) \, dy + \int_{\Gamma_1} N(x, y, \sigma - T)u(y, T + \tau) \, dS(y) \, d\tau \\
\leq M(T) \int_{\Omega} N(x, y, \sigma - T) \, dy + M^q(\sigma) \int_{\Gamma_1} N(x, y, \sigma - T) \, dS(y) \, d\tau.
\]

Applying part (e) in Corollary 2.4 and a change of variable in \( \tau \), we get

\[
    u(x, \sigma) \leq M(T) + M^q(\sigma) \int_{\Gamma_1} N(x, y, \tau) \, dS(y) \, d\tau.
\]

Combining the nonnegativity of \( N(x, y, \tau) \) in (2.11) and the fact that \( \sigma \leq T + t \), we obtain

\[
    u(x, \sigma) \leq M(T) + M^q(T + t) \int_{\Gamma_1} N(x, y, \tau) \, dS(y) \, d\tau. \quad (3.4)
\]

Since \( t \leq 1 \), it follows from Lemma 2.5 that

\[
    \int_{\Gamma_1} N(x, y, \tau) \, dS(y) \, d\tau \leq C \int_{\Gamma_1} \Phi(x - y, 2\tau) \, dS(y) \, d\tau = C \int_{\Gamma_1} \Phi(x - y, \tau) \, dS(y) \, d\tau.
\]

Combining the above inequality with Lemma 2.9 we get

\[
    \int_{\Gamma_1} N(x, y, \tau) \, dS(y) \, d\tau \leq C |\Gamma_1|^\alpha t^{1 - (n+1)\alpha}/2. \quad (3.5)
\]

Plugging the above inequality into (3.4),

\[
    u(x, \sigma) \leq M(T) + CM^q(T + t) |\Gamma_1|^\alpha t^{1 - (n+1)\alpha}/2. \quad (3.6)
\]

Finally, (3.3) and (3.6) together lead to (3.2).

In order to elaborate the proof of Theorem 1.1 more clearly, we give an elementary result as below.

**Lemma 3.2.** Let \( A > 0 \) and \( 0 < \lambda < 1 \) be two constants. Then

\[
    \sum_{k=1}^{\infty} \min\{1, \lambda^k A\} \geq \frac{\ln(1 + \lambda A)}{2 \ln(\lambda^{-1})}. \quad (3.7)
\]

**Proof.** If \( A \leq 1/\lambda \), then

\[
    \sum_{k=1}^{\infty} \min\{1, \lambda^k A\} = \sum_{k=1}^{\infty} \lambda^k A = \frac{\lambda A}{1 - \lambda} \geq \frac{\ln(1 + \lambda A)}{\ln(\lambda^{-1})},
\]

which implies (3.7).
If $A > \frac{1}{\lambda}$, then there exists $K \geq 1$ such that
\[ \lambda^K A > \frac{1}{2} \quad \text{and} \quad \lambda^{K+1} A \leq \frac{1}{2}. \] (3.8)

Thus,
\[ \sum_{k=1}^{\infty} \min\{1, \lambda^k A\} \geq \sum_{k=1}^{K} \min\{1, \lambda^k A\} \geq \frac{K}{2}. \] (3.9)

Since $\lambda^{K+1} A \leq \frac{1}{2}$, then
\[ K + 1 \geq \frac{\ln(2A)}{\ln(\lambda^{-1})}. \] (3.10)

Noticing $\lambda A > 1$, so
\[ K \geq \frac{\ln(2A)}{\ln(\lambda^{-1})} - 1 \geq \frac{\ln(1 + \lambda A)}{\ln(\lambda^{-1})}. \] (3.11)

Plugging (3.10) into (3.9) also yields (3.7).

Now we start to prove Theorem 1.1.

**Proof of Theorem 1.1** For $k \geq 0$, define
\[ M_k = 2^k M_0. \] (3.12)

Consider the function $M(t)$ defined in (1.5). Denote $T_k$ to be the first time that $M(t)$ reaches $M_k$. That is
\[ T_k = \inf\{t \geq 0 : M(t) = M_k\}. \] (3.13)

Since the solution $u$ to (1.1) is continuous on $\overline{\Omega} \times [0, T^*),$
\[ T_k = \min\{t \geq 0 : M(t) = M_k\}. \] (3.14)

In particular, $T_0 = 0.$

Now for any $k \geq 1$, denote
\[ t_k = T_k - T_{k-1}. \]

If $t_k < 1$, then by Lemma 3.1 (choose $T = T_{k-1}, t = t_k$ and $\alpha = \frac{1}{2(n-1)}),$
\[ \frac{M_k - M_{k-1}}{M_k^q} \leq C |\Gamma_1|^{-1} t_k^{1/4}. \] (3.15)

Plugging $M_k = 2^k M_0$ and simplifying, we obtain
\[ t_k \geq C (2^k M_0)^{-4(q-1)}|\Gamma_1|^{-\frac{2}{n-1}}. \] (3.16)

Keeping in mind that (3.15) is valid under the assumption that $t_k < 1$. Thus,
\[ t_k \geq \min\left\{1, C (2^k M_0)^{-4(q-1)}|\Gamma_1|^{-\frac{2}{n-1}}\right\} \geq C \min\left\{1, 2^{-4(q-1)k} M_0^{-4(q-1)}|\Gamma_1|^{-\frac{2}{n-1}}\right\}. \]
Applying Lemma 3.2 with 

$$
\lambda = 2^{−4(q−1)} \quad \text{and} \quad A = M_0^{−4(q−1)}|\Gamma_1|^{−\frac{4}{q−1}},
$$

then

$$
T^* = \sum_{k=1}^{\infty} t_k \geq C \sum_{k=1}^{\infty} \min\left\{1, 2^{−4(q−1)k}M_0^{−4(q−1)}|\Gamma_1|^{−\frac{4}{q−1}}\right\}
$$

$$
\geq \frac{C}{(q−1)} \ln \left(1 + (2M_0)^{−4(q−1)}|\Gamma_1|^{−\frac{4}{q−1}}\right).
$$

\[\square\]

4 Proof of Theorem 1.2

Let’s first state an analogous result as Lemma 3.1 concerning the growth rate of the solution. But this time it pushes to the critical power on $|\Gamma_1|$.

Lemma 4.1. Let $u$ be the solution to (1.1). Define $M(t)$ as in (1.5). Then there exists $C = C(n, \Omega)$ such that for any $T \geq 0$ and $0 \leq t < \min\{1, T^* − T\}$,

$$
\frac{M(T+t) − M(T)}{M^q(T+t)} \leq \begin{cases} 
C|\Gamma_1|^{\frac{1}{q−1}}, & \text{if } n \geq 3, \\
C|\Gamma_1|\ln\left(1 + \frac{1}{|\Gamma_1|}\right), & \text{if } n = 2.
\end{cases}
$$

(4.1)

**Proof.** Noticing that $t < 1$, so this proof is exactly the same as that of Lemma 3.1, except that Lemma 2.10 is needed instead of Lemma 2.9 to estimate $\int_0^T \int_{\Gamma_1} N(x, y, \tau) \, dS(y) \, d\tau$ in (3.5) for $t < 1$.

In the rest of this section, we denote

$$
E_q = (q−1)^{q−1}/q^q, \quad \forall q > 1.
$$

(4.2)

By elementary calculus,

$$
\frac{1}{3q} < E_q < \min\left\{\frac{1}{q}, \frac{1}{(q−1)e}\right\} < 1.
$$

(4.3)

**Lemma 4.2.** Fix any $q > 1$ and $m > 0$, denote $E_q$ as in (4.2) and define $g : (m, \infty) \to \mathbb{R}$ by

$$
g(\lambda) = \frac{\lambda − m}{\lambda^q}, \quad \forall \lambda > m.
$$

(4.4)

Then the following two claims hold.

1. For any $y \in (0, m^{1−q}E_q]$, there exists a unique $\lambda \in (m, \frac{q−1}{q−1}m]$ such that $g(\lambda) = y$.

2. For any $y > m^{1−q}E_q$, there does not exist $\lambda > m$ such that $g(\lambda) = y$.

**Proof.** This is elementary, so we omit the proof. One can also see Lemma 4.1 in [36].

**Proof of Theorem 1.2** We will demonstrate the detailed proof for the case $n \geq 3$ and briefly mention the case $n = 2$ at the end since they are similar. In this proof, $C$ denote the constants which only depend on $n$ and $\Omega$, the values of $C$ may be different in different places. But $C^*$ will represent fixed constants which also
only depend on \( n \) and \( \Omega \). \( M(t) \) represents the same function as in (1.5). The strategy of the proof is to find an appropriate finite increasing sequence \((M_k)_{0 \leq k \leq L}\) such that

\[
T_k - T_{k-1} > 1, \quad \forall 1 \leq k \leq L, \tag{4.5}
\]

where \( T_k \) is defined as

\[
T_k = \min\{t \geq 0 : M(t) = M_k\}.
\]

After such a sequence is found, we will derive a lower bound for \( L \) which is also a lower bound for \( T^* \) due to (4.5).

Let \( n \geq 3 \). Based on Lemma 4.1, there exists a constant \( C^* = C^*(n, \Omega) \) such that for any \( T \geq 0 \) and \( 0 \leq t < \min\{1, T^* - T\} \),

\[
\frac{M(T + t) - M(T)}{M^q(T + t)} \leq C^* |\Gamma_1|^\frac{1}{q-1}. \tag{4.6}
\]

Define

\[
\delta_1 = 2C^*|\Gamma_1|^\frac{1}{q-1}. \tag{4.7}
\]

Then we will use induction to construct a sequence \( (M_k) \) as below.

- Define \( M_0 \) as in (1.4).
- Suppose \( M_{k-1} \) has been constructed for some \( k \geq 1 \).
  - If \( M_k \geq \frac{q}{q-1} M_{k-1} \delta_1 \), then according to Lemma 4.2, we define \( M_k \) to be the unique solution such that
    \[
    M_{k-1} < M_k \leq \frac{q}{q-1} M_{k-1} \tag{4.8}
    \]
    and
    \[
    \frac{M_k - M_{k-1}}{M_k^{q-1}} = \delta_1. \tag{4.9}
    \]
  - If \( M_k \geq \frac{q}{q-1} M_{k-1} \delta_1 \), then we do not define \( M_k \) and stop the construction.

In the following, we will first show that the above construction stops after finite steps.

In fact, if the above construction continuous forever, then \( M_k \geq \frac{q}{q-1} M_{k-1} \delta_1 \) for any \( k \geq 1 \). In addition, both (4.8) and (4.9) hold. As a result, \((M_k)_{k \geq 0}\) is a strictly increasing sequence and

\[
M_k = M_{k-1} + M_k \delta_1 \geq M_{k-1} + M_{k-1} M_{k-1}^{q-1} \delta_1 = (1 + M_{k-1}^{q-1} \delta_1)M_{k-1}.
\]

Hence,

\[
M_k \geq (1 + M_0^{q-1} \delta_1)^k M_0 \to \infty \quad \text{as} \quad k \to \infty,
\]

which contradicts to the fact that \( M_k \delta_1 \leq E_q \). Thus, the inductive construction stops after finite steps and we denote the last term to be \( M_L \).

For any \( 1 \leq k \leq L \), write

\[
t_k = T_k - T_{k-1}.
\]
If \( t_k < 1 \), then plugging \( T = T_{k-1} \) and \( t = t_k \) into (4.10) yields

\[
\frac{M_k - M_{k-1}}{M_k^q} \leq C^* |\Gamma_1|^{1/(q-1)}. \tag{4.10}
\]

But this contradicts to the choice of \( M_k \) in (4.9) due to the definition (4.7) for \( \delta_1 \). Hence, \( t_k \geq 1 \). As a result,

\[
T^* \geq \sum_{k=1}^{L} t_k > L. \tag{4.11}
\]

The rest of the proof will provide a lower bound for \( L \).

In fact, we will prove that

\[
L > \frac{1}{10(q-1)} \left( \frac{1}{M_0^{q-1} \delta_1} - 9q \right). \tag{4.12}
\]

But before justifying this lower bound, let us first admit it and finish the proof of Theorem 1.2. Combining (4.11) and (4.12),

\[
T^* > \frac{1}{10(q-1)} \left( \frac{1}{M_0^{q-1} \delta_1} - 9q \right). \tag{4.13}
\]

Recalling the definition of \( \delta_1 \),

\[
T^* > \frac{1}{10(q-1)} \left( \frac{1}{2C^* M_0^{q-1} |\Gamma_1|^{1/(q-1)}} - 9q \right). \tag{4.14}
\]

Denote \( Y = M_0^{q-1} |\Gamma_1|^{1/(q-1)} \). Then

\[
T^* > \frac{1}{10(q-1)} \left( \frac{1}{2C^* Y} - 9q \right). \tag{4.15}
\]

If

\[
Y \leq \frac{1}{36C^* q},
\]

then \( q \leq 1/(36C^* Y) \) and it follows from (4.15) that

\[
T^* \geq \frac{1}{40C^* (q-1) Y}. \tag{4.16}
\]

Hence, we finish the proof of Theorem 1.2.

The remaining task is to verify (4.12). If \( M_0^{q-1} \delta_1 > \frac{1}{9q} \), then (4.12) holds automatically. So from now on, we assume \( M_0^{q-1} \delta_1 \leq \frac{1}{9q} \). Recalling (4.3), we have

\[
M_0^{q-1} \delta_1 \leq \min \left\{ \frac{1}{2}, E_q \right\}. \tag{4.17}
\]

On the other hand, since the construction stops at \( M_L \), then

\[
M_L^{q-1} \delta_1 > E_q. \tag{4.18}
\]

Comparing (4.17) and (4.18), we conclude that \( L \) is at least 1. As a result, there exists \( 1 \leq L_0 \leq L \) such that

\[
M_{L_0-1}^{q-1} \delta_1 \leq \min \left\{ \frac{1}{2}, E_q \right\} \quad \text{and} \quad M_L^{q-1} \delta_1 > \min \left\{ \frac{1}{2}, E_q \right\}. \tag{4.19}
\]

The reason of considering \( \min \left\{ \frac{1}{2}, E_q \right\} \) here instead of \( E_q \) is because later we need the upper bound \( \frac{1}{2} \) to
justify (4.19). According to (4.9),

$$M_{k-1} = M_k - M_k^{q-1} \delta_1 = M_k (1 - M_k^{q-1} \delta_1).$$

Raising both sides to the power $q - 1$ and multiplying by $\delta_1$,

$$M_k^{q-1} \delta_1 = M_k^{q-1} (1 - M_k^{q-1} \delta_1)^{q-1} \delta_1.$$

Define $x_k = M_k^{q-1} \delta_1$. Then $x_0 = M_0^{q-1} \delta_1$ and

$$x_k = x_k (1 - x_k)^{q-1}, \quad \forall \, 1 \leq k \leq L.$$ (4.17)

Moreover, it follows from (4.16) that

$$x_{L_0-1} \leq \min \left\{ \frac{1}{2}, E_q \right\} \quad \text{and} \quad x_{L_0} > \min \left\{ \frac{1}{2}, E_q \right\}.$$

Now we claim the following inequality:

$$\frac{1}{x_0} \leq \frac{1}{x_{L_0-1}} + 10(q - 1)(L_0 - 1).$$ (4.18)

In fact, if $L_0 = 1$, then (4.18) automatically holds. If $L_0 \geq 2$, then for any $1 \leq k \leq L_0 - 1$, we have

$$0 < x_k \leq x_{L_0-1} \leq 1/2$$

and therefore,

$$x_{k-1} = x_k (1 - x_k)^{q-1} \geq x_k (1 - 2(q - 1)x_k).$$ (4.19)

Recalling the fact $x_k \leq x_{L_0-1} \leq E_q$ and the estimate $E_q < \frac{1}{(q-1)e}$ in (4.3), then

$$1 - 2(q - 1)x_k \geq 1 - 2(q - 1)E_q \geq \frac{1}{5}.$$

Hence, taking the reciprocal in (4.19) yields

$$\frac{1}{x_{k-1}} \leq \frac{1}{x_k [1 - 2(q - 1)x_k]}$$

$$= \frac{1}{x_k} + \frac{2(q - 1)}{1 - 2(q - 1)x_k}$$

$$\leq \frac{1}{x_k} + 10(q - 1).$$ (4.20)

Summing up (4.20) for $k$ from 1 to $L_0 - 1$ yields (4.18).

Finally, since (4.8) implies $M_{L_0} \leq \frac{q}{q - 1} M_{L_0-1}$, then

$$x_{L_0} = \left( \frac{M_{L_0}}{M_{L_0-1}} \right)^{q-1} x_{L_0-1} \leq \left( \frac{q}{q - 1} \right)^{q-1} E_q = \frac{1}{q}.$$

Thus,

$$\frac{1}{3q} < \min \left\{ \frac{1}{2}, E_q \right\} < x_{L_0} \leq \frac{1}{q},$$
Recalling (4.17), then

\[ x_{L_0 - 1} = x_{L_0} (1 - x_{L_0})^{q - 1} > \frac{1}{3q} \left( 1 - \frac{1}{q} \right)^{q - 1} = \frac{E_q}{3}. \]

Plugging the above inequality and \( x_0 = M_0^{q - 1} \delta_1 \) into (4.18),

\[ \frac{1}{M_0^{q - 1} \delta_1} < \frac{3}{E_q} + 10(q - 1)(L_0 - 1) < 9q + 10(q - 1)(L_0 - 1). \]

Rearranging this inequality yields

\[ L_0 > \frac{1}{10(q - 1)} \left( \frac{1}{M_0^{q - 1} \delta_1} - 9q \right) + 1. \]

Hence, (4.12) follows from \( L \geq L_0 \).

The proof for the case \( n = 2 \) is almost the same except (due to Lemma 4.1) changing \( |\Gamma_1|^{\frac{1}{n - 1}} \) to be \( |\Gamma_1| \ln \left( 1 + \frac{1}{|\Gamma_1|} \right) \) in the above arguments.

5 Sharpness of the Key Estimate

The key estimate Lemma 2.10 played an essential role in the proof of Lemma 4.1 (and therefore Theorem 1.2). Actually, the order on \( |\Gamma_1| \) in Lemma 2.10 determines the order on \( |\Gamma_1| \) of the lower bound of \( T^* \) in Theorem 1.2 as \( |\Gamma_1| \to 0^+ \). So it is desired to explore whether the order on \( |\Gamma_1| \) in Lemma 2.10 is optimal as \( |\Gamma_1| \to 0^+ \)? The goal of this section is to give an affirmative answer to this question when \( \Gamma_1 \), as a partial boundary of \( \Omega \), is flat and ball-shaped (see e.g. Figure 3). As notation conventions, we denote

\[ \widetilde{B}(\rho) = \{(\tilde{x}, 0) : \tilde{x} \in \mathbb{R}^{n - 1}, |	ilde{x}| < \rho \} \]

to be the flat ball centered at the origin on the hypersurface \( \{ x \in \mathbb{R}^n : x_n = 0 \} \) with radius \( \rho \). In addition, we use \( \bar{0} \) to represent the origin in \( \mathbb{R}^{n - 1} \) and define \( B(\bar{x}, \rho) \) as (2.14) to be the ball in \( \mathbb{R}^{n - 1} \).

Proposition 5.1. There exists \( C = C(n) \) such that for any \( \Gamma_1 = \widetilde{B}(\rho) \) with \( 0 < \rho \leq 1 \) and for any \( x \in \bar{\Gamma}_1 \),

\[
\int_{0}^{T} \int_{\Gamma_1} \Phi(x - y, t) dS(y) dt \geq \begin{cases} C |\Gamma_1|^{\frac{1}{n - 2}}, & \text{if } n \geq 3, \\ C |\Gamma_1| \ln \left( 1 + \frac{1}{|\Gamma_1|} \right), & \text{if } n = 2. \end{cases}
\]

(5.1)

In order to prove Proposition 5.1, we need two elementary results, Lemma 5.2 and Lemma 5.3 which may be of independent interest. In this section, for any positive integer \( n \geq 2 \), we define the function \( \phi_n : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) by

\[
\phi_n(T, R) = \int_{0}^{T} \int_{0}^{R} \frac{r^{n-2}}{t^{n/2}} \exp \left( -\frac{r^2}{4t} \right) dr dt, \quad \forall T > 0, R > 0.
\]

(5.2)
Lemma 5.2. Let \( n \geq 3 \) and define \( \phi_n \) as (5.2). Then there exist \( C_1 = C_1(n) \) and \( C_2 = C_2(n) \) such that

\[
C_1 \leq \frac{\phi_n(T, R)}{\min\{\sqrt{T}, R\}} \leq C_2, \quad \forall T > 0, \ R > 0.
\] (5.3)

Lemma 5.3. Define \( \phi_2 \) as (5.2) with \( n = 2 \). Then there exist two universal constants \( C_1 \) and \( C_2 \) such that

\[
C_1 \leq \frac{\phi_2(T, R)}{\sqrt{T}} \leq C_2, \quad \forall 0 < T < R^2,
\] (5.4)

and

\[
C_1 \leq \frac{\phi_2(T, R)}{R[1 + \ln \left(\frac{1}{R^2}\right)]} \leq C_2, \quad \forall 0 < R^2 \leq T.
\] (5.5)

We will first prove Proposition 5.1 by admitting Lemma 5.2 and Lemma 5.3, and then justify these two lemmas at the end of this section.

Proof of Proposition 5.1. Noticing that the surface area of \( \Gamma_1 \) is

\[
|\Gamma_1| = C \rho^{n-1},
\]

so it is equivalent to prove

\[
\int_0^2 \int_{B(\rho)} \Phi(x - y, t) \, dS(y) \, dt \geq \begin{cases} 
C \rho, & \text{if } n \geq 3, \\
C \rho \ln \left(1 + \frac{1}{\rho}\right), & \text{if } n = 2.
\end{cases}
\] (5.6)

Since \( x \in \Gamma_1 \), we write \( x = (\tilde{x}, 0) \), where \( \tilde{x} \in \mathbb{R}^{n-1} \) with \( |\tilde{x}| \leq \rho \). Then by the parametrization \( y = (\tilde{y}, 0) \) on \( \tilde{B}(\rho) \), we have

\[
\int_0^2 \int_{\tilde{B}(\rho)} \Phi(x - y, t) \, dS(y) \, dt = C \int_0^2 t^{-n/2} \int_{\tilde{B}(\tilde{0}, \rho)} \exp \left(-\frac{|\tilde{x} - \tilde{y}|^2}{4t}\right) \, d\tilde{y} \, dt
\]

\[
= C \int_0^2 t^{-n/2} \int_{\tilde{B}(\tilde{x}, \rho)} \exp \left(-\frac{|\tilde{y}|^2}{4t}\right) \, d\tilde{y} \, dt.
\]

Since \( |\tilde{x}| \leq \rho \), the overlap between \( B(\tilde{x}, \rho) \) and \( B(\tilde{0}, \rho) \) is comparable to \( B(\tilde{0}, \rho) \). For example, in Figure 4 which shows the case when \( n = 3 \), the overlap is at least one-third of \( B(\tilde{0}, \rho) \). Also noticing that the function

\[
\exp \left(-\frac{|\tilde{y}|^2}{4t}\right)
\]

is radial in \( \tilde{y} \), so there exists a constant \( C = C(n) \) such that

\[
\int_{\tilde{B}(\tilde{x}, \rho)} \exp \left(-\frac{|\tilde{y}|^2}{4t}\right) \, d\tilde{y} \geq C \int_{\tilde{B}(\tilde{0}, \rho)} \exp \left(-\frac{|\tilde{y}|^2}{4t}\right) \, d\tilde{y}.
\]
By polar coordinates,
\[
\int_{B(\hat{0}, \rho)} \exp \left( -\frac{|\hat{y}|^2}{4t} \right) d\hat{y} = C \int_0^\rho r^{n-2} \exp \left( -\frac{r^2}{4t} \right) dr.
\]

Thus,
\[
\int_0^2 \int_{\tilde{B}(\rho)} \Phi(x - y) dS(y) dt \geq C \int_0^2 t^{-n/2} \int_0^\rho r^{n-2} \exp \left( -\frac{r^2}{4t} \right) dr = C \phi_n(2, \rho),
\]
(5.7)

where \(\phi_n\) is defined as in (5.2).

- If \(n \geq 3\), then it follows from \(0 < \rho \leq 1\) and Lemma 5.2 that
  \[
  \phi_n(2, \rho) \geq C\rho.
  \]
  (5.8)

- If \(n = 2\), then it follows from \(0 < \rho \leq 1\) and Lemma 5.3 that
  \[
  \phi_2(2, \rho) \geq C\rho \left[ 1 + \ln \left( \frac{2}{\rho^2} \right) \right] \\
  \geq C\rho \ln \left( 1 + \frac{1}{\rho^2} \right).
  \]
  (5.9)

Combining (5.7), (5.8) and (5.9) together yields (5.6).

Now we will prove Lemma 5.2 and Lemma 5.3 which have been used in the above argument.

**Proof of Lemma 5.2.** We will verify the conclusion by considering two cases \(0 < T < R^2\) and \(0 < R^2 \leq T\).

- Case 1: Let \(0 < T < R^2\) (see Figure 5).

  \[
  \phi_n(T, R) = \left( \int_I + \int_{II} \right) r^{n-2} \exp \left( -\frac{r^2}{4t} \right) dr dt \\
  \triangleq g_1(T, R) + g_2(T, R).
  \]

  ![Figure 5: Case 1](image)

  Based on Figure 5

  \[
  g_1(T, R) = \int_0^T \int_0^{\sqrt{T}r^{n-2} \frac{1}{t^{n/2}}} \exp \left( -\frac{r^2}{4t} \right) dr dt.
  \]
In this region, 
\[ e^{-1/4} \leq \exp \left( -\frac{r^2}{4t} \right) \leq 1. \]

Also notice that 
\[ \int_0^T \int_0^{\sqrt{T}} \frac{r^{n-2}}{t^{n/2}} \, dr \, dt = \frac{2\sqrt{T}}{n-1}. \]

Therefore, 
\[ \frac{2\sqrt{T}}{e^{1/4(n-1)}} \leq g_1(T, R) \leq \frac{2\sqrt{T}}{n-1}. \] (5.10)

Again according to Figure 5, 
\[ g_2(T, R) = \int_0^T \int_0^R \frac{r^{n-2}}{t^{n/2}} \exp \left( -\frac{r^2}{4t} \right) \, dr \, dt. \]

Consider the inner integral and use the change of variable \( \rho = \frac{r}{2\sqrt{t}} \) for \( r \), 
\[ \int_0^R \frac{r^{n-2}}{t^{n/2}} \exp \left( -\frac{r^2}{4t} \right) \, dr \leq \int_0^\infty \frac{r^{n-2}}{t^{n/2}} \exp \left( -\frac{r^2}{4t} \right) \, dr \]
\[ = 2^{n-1} t^{-1/2} \int_0^\infty \rho^{n-2} e^{-\rho^2} \, d\rho \]
\[ = C t^{1/2}. \]

As a result, 
\[ 0 \leq g_2(T, R) \leq C \sqrt{T}. \] (5.11)

Combining (5.10) and (5.11), the estimate (5.3) is justified.

- Case 2: Let \( T \geq R^2 \) (see Figure 6).

\[
\phi_n(T, R) = \left( \int_I + \int_{II} \right) \frac{r^{n-2}}{t^{n/2}} \exp \left( -\frac{r^2}{4t} \right) \, dr \, dt
\]
\[ \triangleq h_1(T, R) + h_2(T, R). \]

Figure 6: Case 2
Based on Figure 6

\[ h_2(T, R) = \int_0^R \int_0^{r^2} \frac{r^{n-2}}{t^{n/2}} \exp \left( - \frac{r^2}{4t} \right) dt \, dr. \]

Consider the inner integral and use a change of variable \( y = \frac{r^2}{t} \) for \( t \),

\[ \int_0^{r^2} \frac{r^{n-2}}{t^{n/2}} \exp \left( - \frac{r^2}{4t} \right) dt = \int_1^\infty y^{n/2-2} e^{-y/4} dy = C. \]

Therefore,

\[ h_2(T, R) = CR. \] (5.12)

Again according to Figure 6

\[ h_1(T, R) = \int_0^R \int_{r^2}^T \frac{r^{n-2}}{t^{n/2}} \exp \left( - \frac{r^2}{4t} \right) dt \, dr. \] (5.13)

In this integral region,

\[ e^{-1/4} \leq \exp \left( - \frac{r^2}{4t} \right) \leq 1. \] (5.14)

On the other hand, by direct calculation,

\[ \int_0^R \int_r^T \frac{r^{n-2}}{t^{n/2}} dt \, dr = \frac{2R}{n-2} \left[ 1 - \frac{1}{n-1} \left( \frac{R^2}{T} \right)^{n/2-1} \right]. \] (5.15)

Since \( T \geq R^2 \), then

\[ \frac{2R}{n-1} \leq \int_0^R \int_r^T \frac{r^{n-2}}{t^{n/2}} dt \, dr \leq \frac{2R}{n-2}. \] (5.16)

Combining (5.13), (5.14) and (5.16), we obtain

\[ \frac{2R}{e^{1/4(n-1)}} \leq h_1(T, R) \leq \frac{2R}{n-2}. \] (5.17)

Hence, (5.17) and (5.12) together justifies (5.3).

Proof of Lemma 5.3. The proof for (5.4) follows the same argument as Case 1 in the above proof for Lemma 5.2. The proof for (5.5) also follows the same argument as Case 2 in the above proof except the estimate on the term \( h_1(T, R) \) in (5.13). Let’s rewrite \( h_1(T, R) \) when \( n = 2 \) as below.

\[ h_1(T, R) = \int_0^R \int_{r^2}^T t^{-1} \exp \left( - \frac{r^2}{4t} \right) dt \, dr. \] (5.18)

By direct calculations,

\[ \int_0^R \int_{r^2}^T t^{-1} dt \, dr = \int_0^R \ln(t) - 2 \ln(r) \, dr = R \left[ \ln \left( \frac{T}{R^2} \right) + 2 \right]. \] (5.19)

Combining (5.18), (5.19) and (5.14), we get

\[ e^{-1/4} R \left[ \ln \left( \frac{T}{R^2} \right) + 2 \right] \leq h_1(T, R) \leq R \left[ \ln \left( \frac{T}{R^2} \right) + 2 \right]. \] (5.20)
Hence, (5.20) and (5.12) together justifies (5.5).

Appendix

A Proof of the Representation Formula

The goal of this appendix is to justify the representation formula (2.12) in Lemma 2.6. First, notice that in part (c) of Corollary 2.4, it claims that for any \( \psi \) in (2.2),

\[
\lim_{t \to 0^+} \int_{\Omega} N(x, y, t) \psi(y) \, dy = \psi(x), \quad \text{uniformly in } x \in \overline{\Omega}.
\]

(A.1)

However, in many situations, the compatibility condition \( \partial \psi / \partial n = 0 \) in (2.2) may not be satisfied. So next, we will provide a convergence result for all the functions \( \psi \) in \( C(\Omega) \). But the convergence will only be pointwise and only valid for the inside point \( x \in \Omega \).

Lemma A.1. Let \( N(x, y, t) \) be the Neumann heat kernel of \( \Omega \) as in Definition 2.3. Then for any \( \psi \in C(\Omega) \) and for any \( x \in \Omega \),

\[
\lim_{t \to 0^+} \int_{\Omega} N(x, y, t) \psi(y) \, dy = \psi(x).
\]

(A.2)

Proof. Fix any \( x \in \Omega \). Then the distance \( d_x \) from \( x \) to the boundary \( \partial \Omega \) is a fixed positive number. Let \( M = \max_{y \in \Omega} |\psi(y)| \).

Then \( M \) is finite. Choose any mollifier \( \eta(y) \in C^\infty(\mathbb{R}^n) \) with support in the unit ball and \( \int_{\mathbb{R}^n} \eta(y) \, dy = 1 \). For any \( j \geq 1 \), denote

\[
\Omega_{1/j} = \left\{ y \in \Omega : \operatorname{dist}(y, \partial \Omega) > \frac{1}{j} \right\}
\]

and define \( \varphi_j \) on \( \mathbb{R}^n \) as

\[
\varphi_j(y) = \begin{cases} 
\psi(y), & y \in \Omega_{1/j}, \\
0, & y \notin \Omega_{1/j}.
\end{cases}
\]

In addition, denote \( \eta_j(y) = j^n \eta(jy) \) and define \( \psi_j(y) = (\eta_{2j} * \varphi_j)(y) \). Then we know

(a) \( \psi_j \in C^\infty(\mathbb{R}^n) \) with support in \( \Omega_{1/(2j)} \).

(b) \( |\psi_j(y)| \leq M \) for any \( j \geq 1 \) and \( y \in \mathbb{R}^n \).

(c) For any compact subset \( K \) in \( \Omega \), \( \psi_j \) uniformly converges to \( \psi \) in \( K \) as \( j \to \infty \).

Now given any \( \epsilon > 0 \), we choose a compact subset \( K \subset \Omega \) such that \( x \in K \), \( \operatorname{dist}(x, \partial K) > d_x/2 \) and the volume \( |\Omega \setminus K| < \epsilon \) (see Figure 7). Fix this domain \( K \). Then \( \psi_j \) converges to \( \psi \) uniformly on \( K \). Thus, there exists \( J \) such that

\[
|\psi_j(y) - \psi(y)| < \epsilon, \quad \forall j \geq J, y \in K.
\]

(A.3)

Fix this \( J \). Since \( \psi_j \) satisfies (2.2), then it follows from (A.1) that

\[
\lim_{t \to 0^+} \int_{K} N(x, y, t) \psi_j(y) \, dy = \psi_j(x).
\]

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So there exists $0 < \delta < 1$ such that for any $t \in (0, \delta)$,

$$\left| \int_{\Omega} N(x, y, t) \psi_J(y) \, dy - \psi_J(x) \right| < \epsilon.$$ 

Combining all the above results, we have that for any $t \in (0, \delta)$,

$$\left| \int_{\Omega} N(x, y, t) \psi(y) \, dy - \psi(x) \right| \leq \int_{K} N(x, y, t) \psi_J(y) \, dy + \int_{\Omega \setminus K} N(x, y, t) \psi_J(y) \, dy - \psi_J(x) \left| \int_{\Omega} \psi(y) \, dy - \psi(x) \right| + \left| \psi_J(x) - \psi(x) \right| \leq 2CM \int_{\Omega \setminus K} \Phi(x - y, 2t) \, dy.$$ 

In order to estimate the first integral term, we split $\Omega$ into $K$ and $\Omega \setminus K$. On $K$, noticing (A.3) and part (e) in Corollary 2.4, so

$$\int_{K} N(x, y, t) \psi_J(y) \, dy \leq \epsilon \int_{K} N(x, y, t) \, dy \leq \epsilon.$$ 

On $\Omega \setminus K$, since both $|\psi|$ and $|\psi_J|$ are bounded by $M$,

$$\int_{\Omega \setminus K} N(x, y, t) \psi(y) - \psi_J(y) \, dy \leq 2M \int_{\Omega \setminus K} N(x, y, t) \, dy.$$ 

Then by Lemma 2.5, there exists a constant $C = C(n, \Omega)$ such that

$$\int_{\Omega \setminus K} N(x, y, t) \psi(y) - \psi_J(y) \, dy \leq 2CM \int_{\Omega \setminus K} \Phi(x - y, 2t) \, dy.$$ 

Since $\text{dist}(x, \partial K) > d_x/2$, then $|y - x| > d_x/2$ for any $y \in \Omega \setminus K$. Consequently,

$$\Phi(x - y, 2t) \leq \frac{C}{(\pi/2)^n} \exp \left( - \frac{d_x^2}{32t} \right) \leq \frac{C}{d_x^n}.$$ 

Hence,

$$\int_{\Omega \setminus K} N(x, y, t) \psi_J(y) \, dy \leq \frac{CM}{d_x^n} |\Omega \setminus K| \leq \frac{CM}{d_x^n}.$$ 

Based on (A.4), (A.5) and (A.6), we conclude the proof. \qed
Before justifying the representation formula, it is helpful to discuss an auxiliary result.

**Lemma A.2.** Let $N(x, y, t)$ be the Neumann heat kernel of $\Omega$ as in Definition 2.3. Let $\Gamma_1 \subset \partial \Omega$ be any part on the boundary. Then for any $x \in \overline{\Omega}$, $t > 0$ and any continuous function $g$ on $\Gamma_1 \times [0, t]$,

$$\lim_{\epsilon \to 0^+} \int_0^t \int_{\Gamma_1} N(x, y, t + \epsilon - \tau) g(y, \tau) \, dS(y) \, d\tau = \int_0^t \int_{\Gamma_1} N(x, y, t - \tau) g(y, \tau) \, dS(y) \, d\tau.$$  

Proof. For any $0 < \delta < \min \left\{ \frac{1}{2}, \frac{1}{4} \right\}$, we split the integral $\int_0^t$ into $\int_0^{t-\delta}$ and $\int_{t-\delta}^t$. On $\int_0^{t-\delta}$, due to the continuity of $N$ and $g$, it is obvious that

$$\lim_{\epsilon \to 0^+} \int_0^{t-\delta} \int_{\Gamma_1} N(x, y, t + \epsilon - \tau) g(y, \tau) \, dS(y) \, d\tau = \int_0^{t-\delta} \int_{\Gamma_1} N(x, y, t - \tau) g(y, \tau) \, dS(y) \, d\tau.$$  

Hence, it suffices to prove

$$\lim_{\delta \to 0^+} \int_{t-\delta}^t \int_{\Gamma_1} N(x, y, t + \epsilon - \tau) g(y, \tau) \, dS(y) \, d\tau = 0 \quad \text{uniformly for } 0 \leq \epsilon \leq \frac{1}{4}. \quad (A.7)$$  

In fact, let

$$M = \max_{\Gamma_1 \times [0, t]} |g|.$$  

Then for $0 \leq \epsilon \leq 1/4$ and $0 < \delta < \min \left\{ \frac{1}{2}, \frac{1}{4} \right\}$, it follows from Lemma 2.8 that

$$\left| \int_{t-\delta}^t \int_{\Gamma_1} N(x, y, t + \epsilon - \tau) g(y, \tau) \, dS(y) \, d\tau \right| \leq CM \int_{t-\delta}^t \int_{\Gamma_1} \Phi(x - y, 2(t + \epsilon - \tau)) \, dS(y) \, d\tau$$

for some constant $C = C(n, \Omega)$. Applying Lemma 2.8 then

$$\int_{\Gamma_1} \Phi(x - y, 2(t + \epsilon - \tau)) \, dS(y) \leq C (t + \epsilon - \tau)^{-1/2} \leq C (t - \tau)^{-1/2}.$$  

As a result,

$$\int_{t-\delta}^t \int_{\Gamma_1} \Phi(x - y, 2(t + \epsilon - \tau)) \, dS(y) \, d\tau \leq C \int_{t-\delta}^t (t - \tau)^{-1/2} \, d\tau = 2C \delta^{1/2},$$

which justifies $(A.7)$. \(\square\)

**Proof of Lemma 2.6.** We will first consider the case $x \in \Omega$ and then the case $x \in \partial \Omega$.

- Fix any $x \in \Omega$, $t \in (0, T^*)$ and $\epsilon > 0$. We define $\phi^\epsilon : \overline{\Omega} \times [0, t] \to \mathbb{R}$ by
  
  $$\phi^\epsilon(y, \tau) = N(x, y, t + \epsilon - \tau).$$

  One can see that $\phi^\epsilon$ is $C^2$ in $y$ ($y \in \overline{\Omega}$) and $C^1$ in $\tau$ ($\tau \in [0, t]$). In addition,

  $$(\partial_\tau + \Delta_y) \phi^\epsilon(y, \tau) = (-\partial_t + \Delta_y) N(x, y, t + \epsilon - \tau) = (-\partial_t + \Delta_y) N(y, x, t + \epsilon - \tau) = 0.$$  

  On the other hand, since $u$ is the classical solution to (1.1), it is also the weak solution according to
Then by choosing $\phi = \phi^\epsilon$ in Definition 3.4 in [35], we have

$$0 = \int_{\Omega} \phi^\epsilon(y,t) u(y,t) - \phi^\epsilon(y,0) u_0(y) \, dy - \int_0^t \int_{\Gamma_1} \phi^\epsilon(y,\tau) u^\alpha(y,\tau) \, dS(y) \, d\tau$$

$$+ \int_0^t \int_{\partial \Omega} u(y,\tau) \frac{\partial \phi^\epsilon(y,\tau)}{\partial n(y)} \, dS(y) \, d\tau.$$

Plugging $\phi^\epsilon(y,\tau) = N(x,y,t + \epsilon - \tau)$ into the above equality and noticing that

$$\frac{\partial \phi^\epsilon(y,\tau)}{\partial n(y)} = \frac{\partial N(x,y,t + \epsilon - \tau)}{\partial n(y)} = \frac{\partial N(y,x,t + \epsilon - \tau)}{\partial n(y)} = 0,$$

we obtain

$$0 = \int_{\Omega} N(x,y,\epsilon) u(y,t) - N(x,y,t + \epsilon) u_0(y) \, dy$$

$$- \int_0^t \int_{\Gamma_1} N(x,y,t + \epsilon - \tau) u^\alpha(y,\tau) \, dS(y) \, d\tau.$$

Sending $\epsilon \to 0^+$ and applying Lemma A.1 and Lemma A.2 yields (2.12).

• Fix any $x \in \partial \Omega$ and $t \in (0,T^*)$. We choose a sequence $(x_j)_{j \geq 1}$ such that $x_j \in \Omega$ and $x_j \to x$. Then from the above argument, it follows from (2.12) that for each $j \geq 1$,

$$u(x_j,t) = \int_{\Omega} N(x_j,y,t) u_0(y) \, dy + \int_0^t \int_{\Gamma_1} N(x_j,y,t - \tau) u^\alpha(y,\tau) \, dS(y) \, d\tau.$$

Since $t > 0$, $N(x,y,t)$ is $C^2$ in $x$ on $\Omega$. Sending $j \to \infty$, then $u(x_j,t)$ converges to $u(x,t)$ and $\int_{\Omega} N(x_j,y,t) u_0(y) \, dy$ converges to $\int_{\Omega} N(x,y,t) u_0(y) \, dy$. Finally, by similar argument as in the proof of Lemma A.2, we have

$$\int_0^t \int_{\Gamma_1} N(x_j,y,t - \tau) u^\alpha(y,\tau) \, dS(y) \, d\tau \to \int_0^t \int_{\Gamma_1} N(x,y,t - \tau) u^\alpha(y,\tau) \, dS(y) \, d\tau.$$

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