SUMMATION FROM THE VIEWPOINT OF DISTRIBUTIONS

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Abstract. Let \( \{a_1, a_2, \ldots, a_n, \ldots\} \) be a sequence of complex numbers. In this paper, inspired by a recent work of Sasane, we give an explanation of the sum
\[
a_1 + 2a_2 + 3a_3 + \cdots + na_n + \cdots,
\]
and more generally, for any \( k \in \mathbb{N} \), the sum
\[
1^k a_1 + 2^k a_2 + 3^k a_3 + \cdots + n^k a_n + \cdots,
\]
from the viewpoint of distributions. As applications, we explain the following summation formulas
\[
1^k - 2^k + 3^k - \cdots = \frac{E_k(0)}{2},
\]
\[
1^k + 2^k + 3^k + \cdots = \frac{B_{k+1}}{k+1},
\]
\[
\epsilon^1 1^k + \epsilon^2 2^k + \epsilon^3 3^k + \cdots = -\frac{B_{k+1}(\epsilon)}{k+1},
\]
where \( E_k(0), B_k \) and \( B_k(\epsilon) \) are the Euler polynomials at 0, the Bernoulli numbers and the Apostol–Beroulli numbers, respectively.

1. Introduction

In the classical analysis, it is well known that
\[
N = 1 + 2 + 3 + \cdots = +\infty.
\]
But in the quantum field theory, the Casimir effect indicates an absurd formula
\[
(1.1) \quad N = 1 + 2 + 3 + \cdots = -\frac{1}{12}.
\]
There are several explanations of the above summation, including the Abel summation by using the power series ([6, Sec. 8.2] or [5, p. 54]) and the analytic continuation of zeta functions ([6, Sec. 8.4]). Recently, Sasane [4] gave a new explanation ([1]) based on the Fourier series of periodic distributions.

His approach is as follows. Denote by the alternating series
\[
(1.2) \quad A = 1 - 2 + 3 - 4 + \cdots.
\]
By noticing the following relationship between (1.1) and (1.2):
\[
(1.3) \quad A = (1 - 2^2)N = -3N,
\]

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he investigated the following $2\pi$-periodic distribution

$$D = e^{it} - 2e^{2it} + 3e^{3it} - 4e^{4it} + \cdots \in \mathcal{D}'(\mathbb{R}),$$

which corresponds to (1.2), and showed that

$$D = \text{Pf} \frac{e^{it}}{(1 + e^{it})^2} + i\pi \sum_{n \in \mathbb{Z}} \delta_{(2n+1)\pi},$$

where \( \text{Pf} \frac{e^{it}}{(1 + e^{it})^2} \in \mathcal{D}'(\mathbb{R}) \) is the $2\pi$-periodic distribution given by

$$\langle \text{Pf} \frac{e^{it}}{(1 + e^{it})^2}, \varphi \rangle = \lim_{\epsilon \to 0} \left( \int_{(-\delta, \pi-\epsilon) \cup (\pi+\epsilon, 2\pi+\delta)} \varphi(t) \frac{e^{it}}{(1 + e^{it})^2} dt - \frac{\varphi(\pi)}{\tan(\epsilon/2)} \right)$$

for \( \varphi \in \mathcal{D}(\mathbb{R}) \) with support \( \text{supp}(\varphi) \subset (-\delta, 2\pi + \delta) \), where \( \delta \in (0, \pi) \).

Then Sasane introduced the following generalized summation method by “evaluating a distribution at a point”.

**Definition 1.1** (see [4, Definition 2.4]). For a $2\pi$-periodic distribution $T$, let

$$\sum_{n \in \mathbb{Z}} c_n(T) e^{int} = T$$

in \( \mathcal{D}'(\mathbb{R}) \).

If there exists a $\sigma \in \mathbb{C}$ such that for any approximate identity \( \{ \varphi_m \}_{m=1}^\infty \), the limit $\lim_{m \to \infty} \langle T, \varphi_m \rangle$ exists, and $\lim_{m \to \infty} \langle T, \varphi_m \rangle = \sigma$, then we say the series

$$\sum_{n \in \mathbb{Z}} c_n(T)$$

is summable, or the Fourier series of $T$ is summable at $t = 0$, and we define

$$\sum_{n \in \mathbb{Z}} c_n(T) = \sigma.$$

Now let \( \{ \varphi_m \}_{m=1}^\infty \) be any approximate identity, that is,

$$\lim_{m \to \infty} \varphi_m = \delta_0$$

in \( \mathcal{D}'(\mathbb{R}) \). From (1.2), he proved

$$\lim_{m \to \infty} \langle D, \varphi_m \rangle = \frac{1}{4},$$

which leads to the summable of (1.2) in the sense of Def. 1.1 and the sum is $A = \frac{1}{4}$; then from (1.3) we have

$$N = -\frac{1}{3}A = -\frac{1}{12},$$

which is the same as (1.1) predicated by the Casimir effect.

In this paper, we extend Sasane’s method to a more general setting. That is, we consider the summable of the generalized series

$$a_1 + 2a_2 + 3a_3 + \cdots + na_n + \cdots$$
in the sense of distributions (see Def. 1.1 above), where \( \{a_n\}_{n=1}^{\infty} \) is a sequence grows at most polynomially. Here the term “grow at most polynomially” means that, for some \( M > 0 \) and \( k > 0 \) we have
\[
(1.5) \quad \text{for all } n \in \mathbb{N}, \quad |a_n| \leq M(1+n)^k.
\]
Under the above condition, the Fourier series
\[
D = a_1e^{it} + 2a_2e^{2it} + 3a_3e^{3it} + \cdots + na_ne^{int} + \cdots
\]
converges in the sense of distributions, that is, \( D \in D'(\mathbb{R}) \). Let
\[
f(z) = a_1z + a_2z^2 + a_3z^3 + \cdots + a_nz^n + \cdots
\]
be the corresponding power series. From the Cauchy-Hadamard theorem, it will converge in the domain
\[
D_R = \{ z \in \mathbb{C} \mid |z| < R \},
\]
where
\[
(1.7) \quad \frac{1}{R} = \lim_{n \to \infty} \sqrt[n]{|a_n|}.
\]
By the polynomial growth condition (1.5), we have
\[
\frac{1}{R} = \lim_{n \to \infty} \sqrt[n]{|a_n|} \leq \lim_{n \to \infty} \sqrt[n]{M(1+n)^k} = 1
\]
and \( R \geq 1 \). Let \( g(t) = f(e^{it}) \) be the corresponding 2\( \pi \)-periodic function on \( \mathbb{R} \). Then we have \( g'(t) = if'(e^{it})e^{it} \).

If \( R > 1 \), then the power series \( f(z) \) is analytic on the closed disc \( \overline{D_1} = \{ z \in \mathbb{C} \mid |z| \leq 1 \} \). We will show that
\[
D = a_1e^{it} + 2a_2e^{2it} + 3a_3e^{3it} + \cdots + na_ne^{int} + \cdots = \text{Pf}(f'(e^{it})e^{it}),
\]
where \( \text{Pf}(f'(e^{it})e^{it}) \in D'(\mathbb{R}) \) is the 2\( \pi \)-periodic distribution given by
\[
(1.8) \quad \langle \text{Pf}(f'(e^{it})e^{it}), \varphi \rangle = \int_0^{2\pi} f'(e^{it})e^{it}\varphi(t)dt
\]
for \( \varphi \in D((0,2\pi)) \) (see Theorem 2.2 below).

If \( R = 1 \), then \( f(z) \) is analytic in the open disc \( D_1 = \{ z \in \mathbb{C} \mid |z| < 1 \} \).

It may be analytic continued to some larger area in the complex plane contained \( D_1 \). In this case, there may have some singular points on the unit circle \( C_1 = \{ z \in \mathbb{C} \mid |z| = 1 \} \). Here we assume that there exists only one pole \( z_0 \neq 1 \) with order 1 on \( C_1 \) and \( f(z) \) has the following Laurent series expansion
\[
(1.9) \quad f(z) = \frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots
\]
in the annular
\[
D_{z_0} = \{ z \in \mathbb{C} \mid 0 < |z-z_0| < r \}
\]
and \( 2 < r < +\infty \). Denote by
\[
g(z) = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots
\]
its analytic part. Then we have
\[ f(z) = \frac{c-1}{z-z_0} + g(z). \]

So
\[ f'(z) = \frac{-c-1}{(z-z_0)^2} + g'(z). \]

For simplification of the notations, let \( d_{-1} = -c_{-1} \), we have
\[ \tag{1.10} f'(z) = \frac{d_{-1}}{(z-z_0)^2} + g'(z). \]

Denote by \( z_0 = e^{it_0} \) for some \( t_0 \neq 0 \) (since \( z_0 \neq 1 \) by our assumption). In this case, we show that
\[ \tag{1.11} D = a_1 e^{it} + 2a_2 e^{2it} + 3a_3 e^{3it} + \cdots + na_n e^{int} + \cdots \]
where \( \text{Pf}(f'(e^{it})e^{it}) \) is the \( 2\pi \)-periodic distribution given by
\[ \langle \text{Pf}(f'(e^{it})e^{it}), \varphi \rangle = \lim_{\epsilon \to 0} \left( \int_{(t_0-\pi,t_0+\epsilon)\cup(t_0+\epsilon,t_0+\pi)} f'(e^{it})e^{it} \varphi(t) dt + \frac{d_{-1} e^{-it_0}}{\tan(\epsilon/2)} \varphi(t_0) \right) \]
for \( \varphi \in \mathcal{D}(\mathbb{R}) \) with support \( \text{supp}(\varphi) \subset (t_0-\pi-\delta, t_0+\pi+\delta) \), where \( \delta \in (0, \pi) \) (see Theorem 3.2 below).

In both cases \((R > 1 \text{ and } R = 1)\), we will show the summation
\[ a_1 + 2a_2 + 3a_3 + \cdots + na_n + \cdots = f'(1) \]
in the sense of distributions. More generally, for any \( k \in \mathbb{N} \), we have
\[ 1^k a_1 + 2^k a_2 + 3^k a_3 + \cdots + n^k a_n + \cdots = \left. \frac{1}{i^k} \left( \frac{d}{dt} \right)^k f(e^{it}) \right|_{t=0} \]
(see Theorem 4.1 below).

As applications, in the last section, we shall explain the following summation formulas
\[ 1^k - 2^k + 3^k - \cdots = -\frac{E_k(0)}{2}, \]
\[ 1^k + 2^k + 3^k + \cdots = -\frac{B_{k+1}}{k+1}, \]
\[ \epsilon^1 1^k + \epsilon^2 2^k + \epsilon^3 3^k + \cdots = -\frac{B_{k+1}(\epsilon)}{k+1}, \]
where \( E_k(0) \), \( B_k \) and \( B_k(\epsilon) \) are the Euler polynomials at 0, the Bernoulli numbers and the Apostol–Bernoulli numbers, respectively.
2. $R > 1$

In this case, we first prove the following proposition which shows that $\text{Pf}(f'(e^it)e^it)$ (see [L8]) defines a distribution on the open interval $(0, 2\pi)$.

**Proposition 2.1.** For $\varphi \in \mathcal{D}((0, 2\pi))$, define

$$
\langle \text{Pf}(f'(e^it)e^it), \varphi \rangle = \int_0^{2\pi} f'(e^it)e^it\varphi(t)dt.
$$

Then $\text{Pf}(f'(e^it)e^it) \in \mathcal{D}'((0, 2\pi))$.

**Proof.** It only needs to prove the continuity. Since $f(z)$ is analytic, $f'(e^it)$ is continuous and bounded on $\mathbb{R}$, that is, there exists a constant $m > 0$, such that

$$
|f'(e^it)| \leq m \quad \text{for } t \in [0, 2\pi].
$$

Let $\{\varphi_n\}_{n=1}^\infty$ be a sequence converges to 0 in $\mathcal{D}((0, 2\pi))$, then in particular, $\{\varphi_n\}_{n=1}^\infty$ converges to 0 uniformly on $(0, 2\pi)$. Thus

$$
\left| \langle \text{Pf}(f'(e^it)e^it), \varphi_n \rangle \right| = \left| \int_0^{2\pi} f'(e^it)e^it\varphi_n(t)dt \right|
$$

$$
\leq \int_0^{2\pi} |f'(e^it)||\varphi_n(t)|dt
$$

$$
\leq 2\pi m\|\varphi_n\|_{\infty}
$$

and

$$
\lim_{n \to \infty} \left| \langle \text{Pf}(f'(e^it)e^it), \varphi_n \rangle \right| \leq \lim_{n \to \infty} 2\pi m\|\varphi_n\|_{\infty} = 0.
$$

So $\text{Pf}(f'(e^it)e^it) \in \mathcal{D}'((0, 2\pi))$. 

Now we calculate the Fourier coefficients of the distribution $D = \text{Pf}(f'(e^it)e^it)$ and show that

**Theorem 2.2.** The following Fourier series expansion is valid in $\mathcal{D}'(\mathbb{R})$:

$$
D = \text{Pf}(f'(e^it)e^it) = a_1e^it + 2a_2e^{2it} + 3a_3e^{3it} + \cdots + na_ne^{int} + \cdots.
$$

**Proof.** For each $\delta > 0$, let $\rho_\delta \in \mathcal{D}'(\mathbb{R})$ be the test function constructed as in [4] p. 489 such that $\rho_\delta|_{(\delta, 2\pi - \delta)} = 1$. Define $\varphi_\delta(t) \in \mathcal{D}(\mathbb{R})$ by

$$
\varphi_\delta(t) = \rho_\delta(t)e^{-int}.
$$

So as $\delta \searrow 0$, $\varphi_\delta$ converges pointwise to 1 on $(0, 2\pi)$, and to 0 on $\mathbb{R} \setminus [0, 2\pi]$. Then we have

$$
\varphi_{\delta, \text{circle}}(e^it) = \sum_{m \in \mathbb{Z}} \varphi_\delta(t - 2\pi m) = \sum_{m \in \mathbb{Z}} \rho_\delta(t - 2\pi m)e^{-int} = e^{-int}.
$$

Hence

$$
c_n(D) = \frac{1}{2\pi} \langle D_{\text{circle}}, e^{-int} \rangle = \frac{1}{2\pi} \langle D_{\text{circle}}, \varphi_{\delta, \text{circle}} \rangle = \frac{1}{2\pi} \langle D, \varphi_\delta \rangle
$$
Consider the interval $O = (-\frac{\pi}{2}, \frac{3\pi}{2})$. Since

$$\langle D, \varphi_\delta \rangle = \langle \operatorname{Pf}(f'(e^{it})e^{it}), \varphi_\delta \rangle$$

$$= \lim_{\delta \searrow 0} \int_O f'(e^{it})e^{it}\varphi_\delta(t)dt$$

$$= \lim_{\delta \searrow 0} \int_O f'(e^{it})e^{it}\rho_\delta(t)e^{-int}dt$$

$$= \int_0^{2\pi} f'(e^{it})e^{it}e^{-int}dt,$$ 

the $n$th Fourier coefficient of the distribution $D$ is given by

$$c_n(D) = \frac{1}{2\pi} \langle D, \varphi_\delta \rangle$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f'(e^{it})e^{it}e^{-int}dt$$

$$= \frac{1}{2\pi i} \oint_{C_1} f'(z)z^{-n}dz,$$

where $C_1$ be the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$. Because

$$f'(z) = a_1 + 2a_2z + 3a_3z^2 + \cdots + na_nz^{n-1} + \cdots,$$

we have

$$c_n(D) = \frac{1}{2\pi i} \oint_{C_1} f'(z)z^{-n}dz = \operatorname{Res}_{z=0}(f'(z)z^{-n}) = na_n,$$

as desired. \( \square \)

3. $R=1$

In this case, as the previous discussion, first we also show that $\operatorname{Pf}(f'(e^{it})e^{it})$ (see [1,12]) defines a distribution on the open interval $(t_0 - \pi, t_0 + \pi)$.

**Proposition 3.1.** For any $\varphi \in \mathcal{D}((t_0 - \pi, t_0 + \pi))$, we have

(3.1) $$\langle \operatorname{Pf}(f'(e^{it})e^{it}), \varphi \rangle = d_{-1} \int_{t_0 - \pi}^{t_0 + \pi} \frac{(t - t_0)^2}{(e^{it} - e^{it_0})^2} e^{it} \int_0^1 (1 - \theta)\varphi''(t_0 + \theta(t - t_0))d\theta dt$$

$$+ \int_{t_0 - \pi}^{t_0 + \pi} g'(e^{it})e^{it}\varphi(t)dt$$

for some $\theta \in (0, 1)$ and $\operatorname{Pf}(f'(e^{it})e^{it}) \in \mathcal{D}'((t_0 - \pi, t_0 + \pi))$. 
Proof. We extend the argument of [4, Proposition 3.1] to our case. By (1.12) and (1.10), we have

\[
\langle Pf(f'(e^{it})e^{it}), \varphi \rangle = \lim_{\varepsilon \to 0} \left( \int_{\Omega_{\varepsilon}} f'(e^{it})e^{it}\varphi(t)dt + \frac{d_{-1}e^{-it_0}}{\tan(\varepsilon/2)}\varphi(t_0) \right)
\]

(3.2) \[
= d_{-1} \lim_{\varepsilon \to 0} \int_{\Omega_{\varepsilon}} \frac{e^{it}}{(e^{it} - e^{it_0})^2} \varphi(t)dt \quad \text{(I)} + \lim_{\varepsilon \to 0} \int_{\Omega_{\varepsilon}} g'(e^{it})e^{it}\varphi(t)dt \quad \text{(II)} + \lim_{\varepsilon \to 0} \frac{d_{-1}e^{-it_0}}{\tan(\varepsilon/2)}\varphi(t_0),
\]

where \( \Omega_{\varepsilon} = (t_0 - \pi, t_0 - \varepsilon) \cup (t_0 + \varepsilon, t_0 + \pi) \).

First we calculate (I). By Taylor’s formula, we have

\[
\varphi(t) = \varphi(t_0) + (t - t_0)\varphi'(t_0) + (t - t_0)^2 \int_0^1 (1 - \theta)\varphi''(t_0 + \theta(t - t_0))d\theta
\]

for some \( \theta \in (0, 1) \). Thus

(3.3) \[
\int_{\Omega_{\varepsilon}} \frac{e^{it}}{(e^{it} - e^{it_0})^2} \varphi(t)dt = \varphi(t_0) \int_{\Omega_{\varepsilon}} \frac{e^{it}}{(e^{it} - e^{it_0})^2}dt \quad \text{(A)} + \varphi'(t_0) \int_{\Omega_{\varepsilon}} \frac{(t - t_0)e^{it}}{(e^{it} - e^{it_0})^2}dt \quad \text{(B)} \]

\[
+ \int_{\Omega_{\varepsilon}} \frac{(t - t_0)^2e^{it}}{(e^{it} - e^{it_0})^2} \int_0^1 (1 - \theta)\varphi''(t_0 + \theta(t - t_0))d\theta dt \quad \text{(C)}.
\]

To compute A, note that

\[
\frac{d}{dt} \frac{1}{e^{it} - e^{it_0}} = -\frac{ie^{it}}{(e^{it} - e^{it_0})^2},
\]
we have

\[ A = \int_{\Omega} \frac{e^{it}}{(e^{it} - e^{it_0})^2} dt \]

\[ = \frac{1}{i} \left( \frac{1}{e^{it} - e^{it_0}} \right)_{t_0 - \epsilon}^{t_0 + \epsilon} + \frac{1}{e^{it} - e^{it_0}} \left( \epsilon \right)_{t_0 - \pi}^{t_0 + \pi} \]

\[ = \frac{1}{i} \left( \frac{e^{it_0} e^{-i\epsilon} - e^{it_0}}{e^{it_0} e^{-i\epsilon} - e^{it_0}} \right) - \frac{1}{e^{it_0} e^{i\epsilon} - e^{it_0}} - \frac{1}{e^{it_0} e^{-i\epsilon} - e^{it_0}} \]

\[ = \frac{i}{e^{it_0}} \left( \frac{1}{e^{-i\epsilon} - 1} - \frac{1}{e^{i\epsilon} - 1} \right) \]

\[ = -\frac{i}{e^{it_0} e^{-i\epsilon} - e^{it_0}} - \frac{i}{e^{it_0} e^{i\epsilon} - e^{it_0}} \]

\[ = -\frac{i}{e^{it_0} e^{i\epsilon} - e^{it_0}} - \frac{1}{e^{it_0} e^{i\epsilon} - e^{it_0}} \]

\[ = -\frac{1}{e^{it_0} \tan(\epsilon/2)}. \]

For B, we have

\[ B = \int_{\Omega} \frac{(t - t_0)e^{it}}{(e^{it} - e^{it_0})^2} dt \]

\[ = \int_{t_0 - \pi}^{t_0 - \epsilon} \frac{(t - t_0)e^{it}}{(e^{it} - e^{it_0})^2} dt + \int_{t_0 + \epsilon}^{t_0 + \pi} \frac{(t - t_0)e^{it}}{(e^{it} - e^{it_0})^2} dt \]

\[ = \int_{t_0 - \pi}^{t_0 - \epsilon} \frac{(t - t_0)e^{it}}{(e^{it} - e^{it_0})^2} dt + \int_{t_0 + \epsilon}^{t_0 + \pi} \frac{(t_0 - \tau)e^{it_0} e^{i\tau} - e^{-i\tau}}{(e^{it_0} e^{i\tau} - e^{it_0})^2} (-1) d\tau \]

(letting \( \tau = 2t_0 - t \))

\[ = \int_{t_0 - \pi}^{t_0 - \epsilon} \frac{(t - t_0)e^{it}}{(e^{it} - e^{it_0})^2} dt + \int_{t_0 - \pi}^{t_0 + \epsilon} \frac{(t_0 - \tau)e^{it_0} e^{i\tau} - e^{-i\tau}}{(e^{it_0} e^{i\tau} - e^{it_0})^2} d\tau \]

\[ = 0. \]

Then we consider the integral C. By Taylor’s formula

\[ e^{it} = e^{it_0} + i(t - t_0)e^{it_0} + O((t - t_0)^2), \]

we have

\[ \lim_{t \to t_0} \left| \frac{e^{it} - e^{it_0}}{t - t_0} \right| = 1. \]
So by the continuity, there exists a constant $m_1 > 0$ such that
\[
\left| \frac{e^{it} - e^{it_0}}{t - t_0} \right| \geq m_1
\]
for $t \in [t_0 - \pi, t_0 + \pi]$, thus
\[
\lim_{\epsilon \to 0} \int_{\Omega_\epsilon} \frac{(t-t_0)^2}{(e^{it} - e^{it_0})^2} e^{it} \int_0^1 (1-\theta)\varphi''(t_0 + \theta(t - t_0))d\theta dt.
\]
Substituting the above computations of $A$, $B$ and $C$ into (3.3), we have
\[
(3.6) \quad (I) = \lim_{\epsilon \to 0} \int_{\Omega_\epsilon} e^{it} \varphi(t)dt \quad = \int_{t_0 + \pi}^{t_0 - \pi} \frac{(t-t_0)^2}{(e^{it} - e^{it_0})^2} e^{it} \int_0^1 (1-\theta)\varphi''(t_0 + \theta(t - t_0))d\theta dt.
\]
For (II), since $g'(e^{it})$ is continuous on the interval $[t_0 - \pi, t_0 + \pi]$, we have
\[
(3.7) \quad (II) = \lim_{\epsilon \to 0} \int_{\Omega_\epsilon} g'(e^{it})e^{it} \varphi(t)dt = \int_{t_0 - \pi}^{t_0 + \pi} g'(e^{it})e^{it} \varphi(t)dt.
\]
Then substitute (3.6) and (3.7) into (3.2) we get
\[
\langle \text{Pf}(f'(e^{it})e^{it}), \varphi \rangle = d_{-1} \int_{t_0 - \pi}^{t_0 + \pi} \frac{(t-t_0)^2}{(e^{it} - e^{it_0})^2} e^{it} \int_0^1 (1-\theta)\varphi''(t_0 + \theta(t - t_0))d\theta dt
\]
\[
\quad + \int_{t_0 + \pi}^{t_0 - \pi} g'(e^{it})e^{it} \varphi(t)dt.
\]
To show the continuity of $\langle \text{Pf}(f'(e^{it})e^{it}), \varphi \rangle$, let $\{\varphi_n\}_{n=1}^\infty$ be a sequence converges to $0$ in the topology of $D((t_0 - \pi, t_0 + \pi))$. Then, in particular, $\{\varphi_n\}_{n=1}^\infty$ and $\{\varphi'_n\}_{n=1}^\infty$ converge uniformly to $0$ on $(t_0 - \pi, t_0 + \pi)$. Since $g'(e^{it})$ is continuous on the closed interval $[t_0 - \pi, t_0 + \pi]$, there exists a constant $m_2 \geq 0$, such that
\[
|g'(e^{it})| \leq m_2, \text{ for } t \in [t_0 - \pi, t_0 + \pi].
\]
So
\[
|\langle \text{Pf}(f'(e^{it})e^{it}), \varphi_n \rangle| \leq |d_{-1}| \left| \int_{t_0 - \pi}^{t_0 + \pi} \frac{(t-t_0)^2}{(e^{it} - e^{it_0})^2} e^{it} \int_0^1 (1-\theta)\varphi''(t_0 + \theta(t - t_0))d\theta dt \right|
\]
\[
\quad + \left| \int_{t_0 - \pi}^{t_0 + \pi} g'(e^{it})e^{it} \varphi_n(t)dt \right|
\]
\[
\leq |d_{-1}| \cdot 2\pi \cdot \frac{1}{m_1^2} \cdot ||\varphi''_n||_\infty + 2\pi \cdot m_2 \cdot ||\varphi_n||_\infty
\]
and
\[
\lim_{n \to \infty} \langle \text{Pf}(f'(e^{it})e^{it}), \varphi_n \rangle = 0.
\]
In conclusion, \( \text{Pf}(f'(e^{it})e^{it}) \in D'((t_0 - \pi, t_0 + \pi)). \)

Now we calculate the Fourier coefficients of the distribution
\[
D = \text{Pf}(f'(e^{it})e^{it}) - d_{-1}i\pi e^{-it_0} \sum_{n \in \mathbb{Z}} \delta_{t_0 + 2n\pi},
\]
and we have the following result.

**Theorem 3.2.** The following Fourier series expansion is valid in \( D'(\mathbb{R}) \):
\[
D = a_1e^{it} + 2a_2e^{2it} + 3a_3e^{3it} + \cdots + na_ne^{int} + \cdots.
\]

**Proof.** We extend the argument of [4, Theorem 5.1] to our case. For each \( \delta > 0 \), let \( \rho_\delta \in D(\mathbb{R}) \) be the test function such that \( \rho_\delta \big|_{(t_0 - \pi, t_0 + \pi - \delta)} = 1 \).

Thus
\[
\sum_{n \in \mathbb{Z}} \rho_\delta(t + 2\pi n) = 1 \quad (t \in \mathbb{R}).
\]

Similar with the arguments in [4, p. 489], here \( \rho_\delta \) can be constructed as follows. For \( \delta > 0 \) being small, consider any symmetric, nonnegative test function \( \varphi \) with support in \([t_0 - \pi - \delta, t_0 - \pi + \delta]\) and such that
\[
\int_{t_0 - \pi - \delta}^{t_0 - \pi + \delta} \varphi(t)dt = 1.
\]
Define the function
\[
\Phi(t) := \int_{(-\infty,t]} \varphi(\tau)d\tau.
\]
Then it can be seen that for all \( t \in \mathbb{R} \), \( \Phi \) satisfies \( \Phi(t_0 - \pi + t) + \Phi(t_0 - \pi - t) = 1 \). So \( \rho_\delta \) can be defined by
\[
\rho_\delta(t) = \Phi(t) \cdot \Phi(2t_0 - t).
\]
Then we define \( \varphi_\delta \in D(\mathbb{R}) \) by
\[
\varphi_\delta(t) = \rho_\delta(t)e^{-int}
\]
and
\[
\varphi_{\delta, \text{circle}}(e^{it}) = \sum_{m \in \mathbb{Z}} \varphi_\delta(t - 2\pi m) = \sum_{m \in \mathbb{Z}} \rho_\delta(t - 2\pi m)e^{-int} = e^{-int}.
\]
Hence the \( n \)th Fourier coefficients of \( D \) can be calculated by
\[
c_n(D) = \frac{1}{2\pi} \langle \varphi_{\delta, \text{circle}}, e^{-int} \rangle = \frac{1}{2\pi} \langle D_{\text{circle}}, \varphi_{\delta, \text{circle}} \rangle = \frac{1}{2\pi} \langle D, \varphi_\delta \rangle
\]
(see [4, p. 490]).

Let
\[
O_\epsilon = \left( t_0 - \frac{3}{2}\pi, t_0 - \epsilon \right) \cup \left( t_0 + \epsilon, t_0 + \frac{3}{2}\pi \right)
\]
and we partition it as
\[ O_\epsilon = \left( t_0 - \frac{3}{2} \pi, t_0 - \frac{\pi}{2} \right] \cup \left( t_0 + \frac{\pi}{2}, t_0 + \frac{3}{2} \pi \right) \]
\[ \cup \left( t_0 - \frac{\pi}{2}, t_0 - \epsilon \right) \cup \left( t_0 + \epsilon, t_0 + \frac{\pi}{2} \right) . \]

Then by (1.12) we have
\[
\langle \text{Pf} \left( f'(e^{it})e^{it} \right), \varphi_\delta \rangle = \lim_{\epsilon \searrow 0} \left( \int_{O_\epsilon} f'(e^{it})e^{it} \varphi_\delta(t) dt + \frac{d_{-1}\varphi_\delta(t_0)}{e^{it_0} \tan(\epsilon/2)} \right)
\]
\[ = \lim_{\epsilon \searrow 0} \int_{V} f'(e^{it})e^{it} \rho_\delta(t) e^{-int} dt \]
\[ + \lim_{\epsilon \searrow 0} \left( \int_{V_\epsilon} f'(e^{it})e^{it} e^{-int} dt + \frac{d_{-1}e^{-int_0}}{e^{it_0} \tan(\epsilon/2)} \right) \]
\[ = \int_{(t_0 - \pi, t_0 - \frac{\pi}{2}) \cup (t_0 + \frac{\pi}{2}, t_0 + \pi)} f'(e^{it})e^{it} e^{-int} dt \]
\[ + \lim_{\epsilon \searrow 0} \left( \int_{V_\epsilon} f'(e^{it})e^{it} e^{-int} dt + \frac{d_{-1}e^{-int_0}}{e^{it_0} \tan(\epsilon/2)} \right) \]
\[ = \lim_{\epsilon \searrow 0} \left( \int_{\Omega_\epsilon} f'(e^{it})e^{it} e^{-int} dt + \frac{d_{-1}e^{-int_0}}{e^{it_0} \tan(\epsilon/2)} \right), \]

here we recall that \( \Omega_\epsilon = (t_0 - \pi, t_0 - \epsilon) \cup (t_0 + \epsilon, t_0 + \pi) \). Also
\[
\left\langle \sum_{n \in \mathbb{Z}} \delta_{t_0 + 2n\pi}, \varphi_\delta \right\rangle = -\varphi_\delta'(t_0) = -(e^{-int})' \bigg|_{t=t_0} = ine^{-int_0}.
\]

Hence
\[
c_n(D) = \frac{1}{2\pi} \langle D, \varphi_\delta \rangle
\]
\[ = \frac{1}{2\pi} \left( \text{Pf} \left( f'(e^{it})e^{it} \right) - d_{-1}i\pi e^{-it_0} \sum_{n \in \mathbb{Z}} \delta_{t_0 + 2n\pi}, \varphi_\delta \right) \]
\[ = \frac{1}{2\pi} \left( \lim_{\epsilon \searrow 0} \left( \int_{\Omega_\epsilon} f'(e^{it})e^{it} e^{-int} dt + \frac{d_{-1}e^{-int_0}}{e^{it_0} \tan(\epsilon/2)} \right) \right.
\]
\[ + d_{-1}n\pi e^{-i(n+1)t_0} \right) . \]

In what follows, we shall calculate the integral \( \frac{1}{2\pi} \int_{\Omega_\epsilon} f'(e^{it})e^{it} e^{-int} dt \).

Let
\[ C^c_\epsilon = \{ z = e^{it} \mid t \in \Omega_\epsilon \} \]
and \( L_\epsilon \) be the line connected the points \( e^{i(t_0 - \epsilon)} \) and \( e^{i(t_0 + \epsilon)} \). Let
\[ C^c_\frac{1}{2} = \left\{ z \in \mathbb{C} \mid |z| = \frac{1}{2} \right\} . \]
Since $f'(z)$ is analytic in the open disc $\{z \in \mathbb{C} \mid |z| < 1\}$, by Cauchy’s theorem, we have

\[
\frac{1}{2\pi i} \int_{\Omega} f'(e^{it})e^{it}e^{-i\alpha t} dt = \frac{1}{2\pi i} \int_{C_{\epsilon}^{\pm}} f'(z)z^{-n} dz
\]

(see Fig. 1).

By (1.6), we have

\[
f'(z) = a_1 + 2a_2 z + \cdots + na_n z^{n-1} + \cdots
\]

for $|z| < 1$, thus

\[
\frac{1}{2\pi i} \int_{C_{\frac{1}{2}}} f'(z)z^{-n} dz = \text{Res}_{z=0} \left( f'(z)z^{-n} \right) = na_n.
\]

Substitute into (3.9), we have

\[
(3.10) \quad \frac{1}{2\pi} \int_{\Omega} f'(e^{it})e^{it}e^{-i\alpha t} dt = na_n - \frac{1}{2\pi i} \int_{L_{\epsilon}} f'(z)z^{-n} dz.
\]

Now we calculate the integral $\frac{1}{2\pi i} \int_{L_{\epsilon}} f'(z)z^{-n} dz$ in the above equation. If $n \neq 0$, let

\[
z^{-n} = z_0^{-n} + (-n)z_0^{-n-1}(z - z_0) + O((z - z_0)^2))
\]
be the Taylor expansion of \( z^{-n} \) around \( z_0 \), then
\[
\frac{1}{2\pi i} \int_{L_\epsilon} f'(z) z^{-n} dz = z_0^{-n} \frac{1}{2\pi i} \int_{L_\epsilon} f'(z) dz + (-n)z_0^{-n-1} \frac{1}{2\pi i} \int_{L_\epsilon} f'(z) (z - z_0) dz
\]
\[
\triangleq \int_{L_\epsilon} f'(z) (z - z_0)^2 dz.
\]
Since by (1.10)
\[
f'(z) = \frac{d_{-1}}{(z - z_0)^2} + g'(z)
\]
for \( z \in D_{z_0} = \{ z \in \mathbb{C} \mid 0 < |z - z_0| < r \} \) and \( 2 < r < +\infty \), \( f'(z) (z - z_0)^2 \) is analytic on the line \( L_\epsilon \) for any \( \epsilon > 0 \), we have
\[
\lim_{\epsilon \searrow 0} \int_{L_\epsilon} f'(z) (z - z_0)^2 dz = 0.
\]
So by (3.11) we have
\[
\frac{1}{2\pi i} \int_{L_\epsilon} f'(z) z^{-n} dz = z_0^{-n} \lim_{\epsilon \searrow 0} \frac{1}{2\pi i} \int_{L_\epsilon} f'(z) dz + (-n)z_0^{-n-1} \lim_{\epsilon \searrow 0} \frac{1}{2\pi i} \int_{L_\epsilon} f'(z) (z - z_0) dz.
\]
For the integral \( \int_{L_\epsilon} f'(z) z^{-n} dz \), by (1.10), we have
\[
\int_{L_\epsilon} f'(z) dz = d_{-1} \int_{L_\epsilon} \frac{1}{(z - z_0)^2} dz + \int_{L_\epsilon} g'(z) dz.
\]
By Cauchy’s theorem,
\[
\int_{L_\epsilon} \frac{1}{(z - z_0)^2} + \int_{\Omega_\epsilon} \frac{1}{(z - z_0)^2} = 0,
\]
thus
\[
\int_{L_\epsilon} \frac{1}{(z - z_0)^2} = -\int_{\Omega_\epsilon} \frac{1}{(z - z_0)^2}
\]
\[
= -i \int_{\Omega_\epsilon} \frac{e^{it} - e^{i\theta_0}}{(e^{it} - e^{i\theta_0})^2} dt
\]
\[
= \frac{i}{e^{i\theta_0} \tan(\epsilon/2)}.
\]
the last equality follows from (3.4). Since \( g'(z) \) is analytic on the line \( L_\epsilon \) for any \( \epsilon > 0 \), we have
\[
\lim_{\epsilon \searrow 0} \int_{L_\epsilon} g'(z) dz = 0.
\]
Substitute (3.14) and (3.15) into (3.13), we get
\[
\lim_{\epsilon \searrow 0} \int_{L_\epsilon} f'(z) dz = \lim_{\epsilon \searrow 0} d_{-1} \frac{i}{e^{i\theta_0} \tan(\epsilon/2)}.
\]
Now we consider the integral $\int_{L_\epsilon} f'(z)(z - z_0)dz$ in (3.13). Again by (1.10), we have

$$\int_{L_\epsilon} f'(z)(z - z_0)dz = d_{-1} \int_{L_\epsilon} \frac{1}{z - z_0}dz + \int_{L_\epsilon} g'(z)(z - z_0)dz.$$  

Since $L_\epsilon$ be the line connected the points $e^{i(t_0 - \epsilon)}$ and $e^{i(t_0 + \epsilon)}$ and $e^{it} = \cos(t) + i \sin(t)$, we have the parametrized equation

$$L_\epsilon := \begin{cases} 
  z = x + iy \\
  x = \cos(t_0 - \epsilon) + it_0(\cos(t_0 - \epsilon) + i \sin(t_0 - \epsilon)) 
\end{cases},$$

thus

$$\int_{L_\epsilon} \frac{1}{z - z_0}dz = \ln(z - z_0) \bigg|_{z = e^{i(t_0 + \epsilon)}}^{z = e^{i(t_0 - \epsilon)}} = \ln(e^{i(t_0 + \epsilon)} - e^{it_0}) - \ln(e^{i(t_0 - \epsilon)} - e^{it_0}).$$

Note that

$$\ln(e^{i(t_0 + \epsilon)} - e^{it_0}) = \ln|e^{i(t_0 + \epsilon)} - e^{it_0}| + i \arg(e^{i(t_0 + \epsilon)} - e^{it_0})$$

$$= \ln|e^{i\epsilon} - 1| + i \arg(e^{i(t_0 + \epsilon)} - e^{it_0}).$$

Since

$$e^{i(t_0 + \epsilon)} - e^{it_0} = e^{it_0}(e^{i\epsilon} - 1),$$

we have

$$\arg(e^{i(t_0 + \epsilon)} - e^{it_0}) = \arg(e^{it_0}) + \arg(e^{i\epsilon} - 1)$$

$$= t_0 + \arg(e^{i\epsilon} - 1).$$

Note that

$$e^{i\epsilon} - 1 = \cos(\epsilon) - 1 + i \sin(\epsilon)$$

$$= -2 \sin^2(\epsilon/2) + i2 \sin(\epsilon/2) \cos(\epsilon/2),$$

we have

$$\arg(e^{i\epsilon} - 1) = -\tan^{-1}\left(\frac{\cos(\epsilon/2)}{\sin(\epsilon/2)}\right) = -\left(\frac{\pi}{2} - \frac{\epsilon}{2}\right).$$

Combining (3.19), (3.20) and (3.21), we get

$$\ln(e^{i(t_0 + \epsilon)} - e^{it_0}) = \ln|e^{i\epsilon} - 1| - i\left(\frac{\pi}{2} - t_0 - \frac{\epsilon}{2}\right).$$

Similarly, we have

$$\ln(e^{i(t_0 - \epsilon)} - e^{it_0}) = \ln|e^{i\epsilon} - 1| + i\left(\frac{\pi}{2} + t_0 - \frac{\epsilon}{2}\right).$$

The substitute (3.22) and (3.23) into (3.18), we get

$$\int_{L_\epsilon} \frac{1}{z - z_0}dz = -i(\pi - \epsilon).$$
and
\begin{equation}
(3.24) \quad \lim_{\epsilon \to 0} \int_{L_{\epsilon}} \frac{1}{z - z_0} \, dz = -i\pi.
\end{equation}
Since \( g'(z) \) is analytic on the line \( L_{\epsilon} \) for any \( \epsilon > 0 \), we have
\begin{equation}
(3.25) \quad \lim_{\epsilon \to 0} \int_{L_{\epsilon}} g'(z)(z - z_0) \, dz = 0.
\end{equation}
Then substitute (3.24) and (3.25) into (3.17), we get
\begin{equation}
(3.26) \quad \lim_{\epsilon \to 0} \int_{L_{\epsilon}} f'(z)(z - z_0) \, dz = -d_{-1}i\pi.
\end{equation}
So substitute (3.16) and (3.26) into (3.12), we get
\begin{equation}
(3.27) \quad \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{L_{\epsilon}} f'(z) z^{-n} \, dz = \frac{1}{2\pi} \left( \lim_{\epsilon \to 0} \frac{d_{-1} e^{-int_0}}{e^{i\epsilon/2} \tan(\epsilon/2)} + d_{-1} \pi e^{-i(n+1)t_0} \right),
\end{equation}
and substitute the above result into (3.10) we get
\begin{equation}
(3.28) \quad \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{\Omega_{\epsilon}} f'(e^{it}) e^{i\epsilon t} e^{-i\epsilon t} \, dt = n a_n - \frac{1}{2\pi} \left( \lim_{\epsilon \to 0} \frac{d_{-1} e^{-int_0}}{e^{i\epsilon/2} \tan(\epsilon/2)} + d_{-1} \pi e^{-i(n+1)t_0} \right).
\end{equation}
If \( n = 0 \), by (3.10) and (3.16) we have
\begin{equation}
(3.29) \quad \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{\Omega_{\epsilon}} f'(e^{it}) e^{i\epsilon t} \, dt = -\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{L_{\epsilon}} f'(z) \, dz = -\frac{1}{2\pi} \lim_{\epsilon \to 0} \frac{d_{-1} e^{-int_0}}{e^{i\epsilon/2} \tan(\epsilon/2)},
\end{equation}
thus (3.28) is also applicable in this case. Finally, substitute (3.28) into (3.8), we have the nth Fourier coefficient \( c_n(D) = na_n \) for \( n \in \mathbb{N} \), as desired. 

\section{The Fourier series of \( D \) is summable at \( t = 0 \)}

In this section, we consider the summable of the Fourier series of \( D \) at \( t = 0 \) and prove the following theorem.

\textbf{Theorem 4.1.} For \( k \in \mathbb{N} \), let \( D^{(k-1)} \) be the \((k-1)\)th derivative of \( D \), that is,
\[ D^{(k-1)} = i^{k-1} \left( 1^k a_1 e^{it} + 2^k a_2 e^{2it} + 3^k a_3 e^{3it} + \cdots + n^k a_n e^{nit} + \cdots \right) \in \mathcal{D}'(\mathbb{R}). \]
Then \( D^{(k-1)} \) is summable at \( t = 0 \) in the sense of distributions (see Def. 1.1) and we have the sum
\[ 1^k a_1 + 2^k a_2 + 3^k a_3 + \cdots + n^k a_n + \cdots = \frac{1}{i^k} \left( \frac{d}{dt} \right)^k f(e^{it}) \bigg|_{t=0}, \]
where \( f(z) \) is the corresponding power series (see (1.4)).
Proof. In the case of \( R > 1 \), the proof goes the same as [1, Proposition 8.1]. So we modify this proof to suitable for \( R = 1 \). In this case, by our assumption, the singular point \( z_0 = e^{it_0} \neq 1 \), that is \( t_0 \neq 0 \). Let \( \{ \varphi_m \}_{m=1}^{\infty} \) be any approximate identity. For all large enough \( m \), the support of \( \varphi_m \) is contained inside \((-\delta, \delta)\) for some small delta such that \( t_0 \notin (-\delta, \delta) \). Then we have

\[
\lim_{m \to \infty} \langle D^{(k-1)}, \varphi_m \rangle = \lim_{m \to \infty} (-1)^{k-1} \langle D, \varphi_m^{(k-1)} \rangle \\
= \lim_{m \to \infty} (-1)^{k-1} \int_{-\delta}^{\delta} f'(e^{it}) e^{it} \varphi_m^{(k-1)}(t) dt \\
(\text{by (1.11) and note that } t_0 \notin (-\delta, \delta)) \\
= \lim_{m \to \infty} (-1)^{k-1} \langle \varphi_m^{(k-1)}, f'(e^{it})e^{it} \rangle \\
= \lim_{m \to \infty} \left\langle \varphi_m, \left( \frac{d}{dt} \right)^{k-1} f'(e^{it})e^{it} \right\rangle \\
= \left\langle \delta_0, \left( \frac{d}{dt} \right)^{k-1} f'(e^{it})e^{it} \right\rangle \\
= \left. \left( \frac{d}{dt} \right)^{k-1} f'(e^{it})e^{it} \right|_{t=0} \\
= \frac{1}{i} \left. \left( \frac{d}{dt} \right)^{k} f(e^{it}) \right|_{t=0}.
\]

This completes the proof. \( \square \)

5. Applications

In this section, as applications of our constructions, we give some examples.

Example 5.1. For \( k \in \mathbb{N} \), the alternating series is given by

\[
A_k = 1^k - 2^k + 3^k - \cdots + (-1)^{n-1} n^k + \cdots.
\]

By Theorem 4.1 we consider the power series

\[
f(z) = z - z^2 + z^3 - \cdots + (-1)^{n-1} z^n + \cdots
\]

for \(|z| < 1\). Since

\[
z - z^2 + z^3 - \cdots + (-1)^{n-1} z^n + \cdots = \frac{z}{z+1}
\]

for \(|z| < 1\), \( f(z) \) can be analytic continued to the complex plane as a meromorphic function

\[
f(z) = \frac{z}{z+1}
\]
with a single pole \( z_0 = -1 \) on the unit circle \( C_1 \). So by Theorem 4.1, we have

\[
1^k - 2^k + 3^k - \cdots + (-1)^{n-1}n^k + \cdots = \left. \frac{1}{ik} \left( \frac{d}{dt} \right)^k \frac{e^{it}}{e^{it} + 1} \right|_{t=0} 
\]

(5.1)

\[
= \left. \frac{1}{ik} \left( \frac{d}{dt} \right)^k \left( 1 - \frac{1}{e^{it} + 1} \right) \right|_{t=0}
\]

\[
= \left. -\frac{1}{ik} \left( \frac{d}{dt} \right)^k \frac{1}{e^{it} + 1} \right|_{t=0}.
\]

Recall that Euler polynomials are defined by the generating function

\[
\frac{2e^{xt}}{e^t + 1} = \sum_{m=0}^{\infty} E_m(x) \frac{t^m}{m!}
\]

(see (3)), thus

(5.2) \[
\frac{2}{e^t + 1} = \sum_{m=0}^{\infty} E_m(0) \frac{t^m}{m!}.
\]

Substitute into (5.1), we have

\[
A_k = 1^k - 2^k + 3^k - \cdots + (-1)^{n-1}n^k + \cdots
\]

(5.3)

\[
= \left. -\frac{1}{2ik} \left( \frac{d}{dt} \right)^k \frac{2}{e^{it} + 1} \right|_{t=0}
\]

\[
= \left. -\frac{1}{2ik} \left( \frac{d}{dt} \right)^k \left( \sum_{m=0}^{\infty} E_m(0) (it)^m \right) \right|_{t=0}
\]

\[
= \left. -\frac{E_k(0)}{2} \right|_{t=0}.
\]

Since \( E_1(0) = -\frac{1}{2} \), in particular, we have

\[
A = 1 - 2 + 3 - \cdots + (-1)^{n-1}n + \cdots = \frac{1}{4}
\]

and by (1.3)

\[
N = 1 + 2 + 3 + \cdots + n + \cdots = -\frac{1}{3}A = -\frac{1}{12},
\]

which is the same as (1.1) predicated by the Casimir effect.

**Example 5.2.** Furthermore, let the Bernoulli numbers \( B_m \) be defined by the following generating function

(5.4) \[
\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}.
\]
Comparing the generating function (5.2 and (5.4), we have a relation between $B_k$ and $E_k(0)$:

\[(5.5)\]

\[E_k(0) = 2(1 - 2^{k+1}) \frac{B_{k+1}}{k+1}\]

for $k \in \mathbb{N}$. Denote by

\[N_k = 1^k + 2^k + 3^k - \ldots + n^k + \ldots .\]

It is easily seen that

\[A_k = (1 - 2^{k+1})N_k,\]

thus

\[N_k = \frac{1}{1 - 2^{k+1}} A_k.\]

Then substitute (5.3) to the above equality and notice (5.5), we have

\[(5.6)\]

\[N_k = \frac{1}{1 - 2^{k+1}} \left(-\frac{1}{2}E_k(0)\right) = -\frac{B_{k+1}}{k+1} .\]

This result was first obtained by Euler in 1740 (see [6, p. 203]).

**Example 5.3.** For $k \in \mathbb{N}$ and a complex number $\epsilon$ with $|\epsilon| \leq 1$, but $\epsilon \neq 1$, we consider the sum

\[\epsilon^1 1^k + \epsilon^2 2^k + \epsilon^3 3^k + \ldots + \epsilon^n n^k + \ldots .\]

Then the corresponding power series is

\[f(z) = \epsilon z + \epsilon^2 z^2 + \ldots + \epsilon^n z^n + \ldots\]

for $|z| < 1$. It is easy to see that

\[f(z) = \sum_{n=1}^{\infty} \epsilon^n z^n = \frac{1}{1 - \epsilon z} - 1\]

for $|z| < 1$, thus it satisfies the conditions of Sec. 1. Then by Theorem 4.1, we have

\[(5.7)\]

\[\epsilon^1 1^k + \epsilon^2 2^k + \epsilon^3 3^k + \ldots + \epsilon^n n^k + \ldots = \frac{1}{\epsilon^k} \left(\frac{d}{dt}\right)^k \left(\frac{1}{1 - \epsilon e^{it}} - 1\right) \bigg|_{t=0} = \frac{1}{\epsilon^k} \left(\frac{d}{dt}\right)^k \left(\frac{1}{1 - \epsilon e^{it}}\right) \bigg|_{t=0} .\]

Recall that the Apostol–Bernoulli numbers $B_m(\epsilon)$ are defined by the following generating function

\[(5.8)\]

\[\frac{t}{\epsilon e^t - 1} = \sum_{m=0}^{\infty} B_m(\epsilon) \frac{t^m}{m!}\]

\[= \sum_{m=1}^{\infty} B_m(\epsilon) \frac{t^m}{m!}\]

(since by our assumption $\epsilon \neq 1$, we have $B_0(\epsilon) = 0$)
(see [1, Eq. (3.1)]) or [2, Eq. (1.3)]). So substitute into (5.7), we have

$$
\epsilon^1 1^k + \epsilon^2 2^k + \epsilon^3 3^k + \cdots + \epsilon^n n^k + \cdots
$$

$$
= \frac{1}{i^k} \left( \frac{d}{dt} \right)^k \left( \frac{1}{1 - \epsilon e^{it}} \right) \bigg|_{t=0}
$$

$$
= -\frac{1}{i^k} \left( \frac{d}{dt} \right)^k (it)^{-1} \left( \frac{it}{\epsilon e^{it} - 1} \right) \bigg|_{t=0}
$$

$$
= -\frac{1}{i^k} \left( \frac{d}{dt} \right)^k (it)^{-1} \left( \sum_{m=1}^{\infty} B_m(\epsilon) \frac{(it)^m}{m!} \right) \bigg|_{t=0}
$$

$$
= -\frac{1}{i^k} \left( \frac{d}{dt} \right)^k \sum_{m=0}^{\infty} B_{m+1}(\epsilon) \frac{(it)^m}{(m+1)!} \bigg|_{t=0}
$$

$$
= -\frac{B_{k+1}(\epsilon)}{k + 1}.
$$

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