On positively curved 4-manifolds with $S^1$-symmetry

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Abstract

It is well-known by the work of Hsiang and Kleiner that every closed oriented positively curved 4-dimensional manifold with an effective isometric $S^1$-action is homeomorphic to $S^4$ or $\mathbb{CP}^2$. As stated, it is a topological classification. The primary goal of this paper is to show that it is indeed a diffeomorphism classification for such 4-dimensional manifolds. The proof of this diffeomorphism classification also shows an even stronger statement that every positively curved simply connected 4-manifold with an isometric circle action admits another smooth circle action which extends to a 2-dimensional torus action and is equivariantly diffeomorphic to a linear action on $S^4$ or $\mathbb{CP}^2$. The main strategy is to analyze all possible topological configurations of effective circle actions on simply connected 4-manifolds by using the so-called replacement trick of Pao.

1 Introduction and Main Results

A closed Riemannian $n$-dimensional manifold with positive sectional curvature everywhere is called a positively curved $n$-manifold. These manifolds are very rare and, moreover, satisfy very special topological properties. In particular, a well-known theorem of Synge says that every even dimensional orientable closed positively curved manifold is simply connected, while every such odd dimensional manifold has at most finite first fundamental group (see [4]). Hence it is immediate that in two dimensions two-sphere is only an orientable closed positively curved manifold. In three dimensions, Hamilton has classified all orientable closed positively curved manifolds by using the techniques of Ricci flow [10]. It turns out that they are all diffeomorphic to space forms. On the other hand, in four dimensions the classification of
orientable closed positively curved manifolds is still incomplete. Much interestingly, all the well-known orientable closed positively curved 4-manifolds always admit a positively curved metric with a circle symmetry.

In their paper [11], Hsiang and Kleiner investigated the question which orientable closed positively curved 4-manifolds admit a positively curved metric with an effective isometric $S^1$-action. They showed that if $M$ is a compact oriented positively curved 4-manifold with an effective isometric $S^1$-action, then $M$ is homeomorphic to $S^4$ or $\mathbb{CP}^2$. This classification is remarkable and sparked off many current research activities in this field. In particular, see the works of Grove–Searle and B. Wilking in [8] and [20, 21] respectively. However, it is a topological classification rather than a diffeomorphism one. In fact, Hsiang and Kleiner (and probably many others) asked in the same paper whether or not the following conjecture was true:

**Conjecture 1.1.** An orientable closed positively curved 4-manifold $M$ with an effective isometric $S^1$-action is diffeomorphic to $S^4$ or $\mathbb{CP}^2$.

The main aim of this paper is to affirmatively prove their Conjecture 1.1 (or Conjecture 1 in [11]). What we have done to prove the conjecture is just to add some more remarks to the homeomorphism classification of Hsiang and Kleiner.

Throughout this paper, all $S^1$-actions are assumed to be effective, unless stated otherwise. In order to explain the main ideas, we let $F(S^1, M)$ be the fixed point set of such an $S^1$-action on $M$. It is well known as in [14] that the Euler characteristic of $F(S^1, M)$ is equal to that of $M$ in the presence of an effective $S^1$-action. Thus the Euler characteristic of the fixed point set $F(S^1, M)$ is greater than or equal to 2. One of the main steps in the paper of Hsiang and Kleiner is to prove that the Euler characteristic of the fixed point set $F(S^1, M)$ is at most three. Hence, since each component of $F(S^1, M)$ is a totally geodesic submanifold of $M$, $F(S^1, M)$ has the following four possibilities only:

1. One 2-sphere.
2. The disjoint union of one 2-sphere and one isolated fixed point.
3. Two isolated fixed points.
4. Three isolated fixed points.

Hsiang and Kleiner obtained the above information about the fixed point set $F(S^1, M)$ by essentially using the existence of a positively curved metric
with an effective isometric $S^1$-action. But it seems to need more serious consideration on the topological properties of effective circle actions on simply connected 4-manifolds. Thus our starting point of this paper is to investigate all possible topological configurations of effective circle actions on simply connected 4-manifolds. However, the existence of a positively curved metric with an effective isometric $S^1$-action is also used crucially throughout. Moreover, we need to use the recent resolution of the Poincaré conjecture by Perelman as in [18] and [19]. (See also [15], [23] and [12].) So our proof is differential-topological in nature. We remark that it is inspired by the work [1] of S. Baldridge.

In case that the fixed point set $F(S^1, M)$ is either one 2-sphere or the union of one 2-sphere and one isolated fixed point, the effective isometric $S^1$-action on $M$ has the fixed point set whose codimension is 2. It then follows from Theorem 1.2 of Grove and Searle in [8] that $M$ is diffeomorphic to $S^4$ or $\mathbb{C}P^2$. So it is enough to consider the remaining two cases: $F(S^1, M)$ consists of either two isolated fixed points or three isolated fixed points. In the present paper, we prove that even in this case $M$ is diffeomorphic to $S^4$ or $\mathbb{C}P^2$.

One of the new ingredients of this paper that is not present in the paper of Hsiang and Kleiner is to use the classification results of circle actions on simply connected 4-manifolds by R. Fintushel in [5] and [6]. According to Fintushel, smooth $S^1$-actions on simply connected 4-manifolds can be classified in terms of their legally weighted orbit spaces. Applying this classification to our situation, we will have at least four (resp. three) possibilities for legally weighted orbit spaces, if the fixed point set is three isolated fixed points (resp. two isolated fixed points). By some case-by-case analysis we can finally show the following result.

**Theorem 1.2.** An orientable closed positively curved 4-manifold $M$ with an effective isometric $S^1$-action is diffeomorphic to $S^4$ or $\mathbb{C}P^2$.

This completes the classification of closed oriented positively curved manifolds of dimension 4 with effective isometric $S^1$-actions, up to diffeomorphism. Moreover, as the reader of this paper also pointed out, the proof of Theorem 1.2 in Section 4 actually provides the following even stronger result.

**Theorem 1.3.** A positively curved simply connected 4-manifold $M$ with an isometric circle action admits another smooth circle action which extends to a 2-dimensional torus action and is equivariantly diffeomorphic to a linear action on $S^4$ or $\mathbb{C}P^2$. 

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When a fixed point set contains a 2-dimensional component or there is no weight circle in the orbit space, Theorem 1.3 can be further strengthened to the statement that an isometric circle action on a positively curved simply connected closed 4-manifold is equivariantly diffeomorphic to a linear action on $S^4$ or $\mathbb{CP}^2$ (see also Theorem 2.8 of [9] and Conjecture 3.1 in [7]). So it would be interesting to know whether or not the strengthened statement still holds even in the presence of a weight circle in the orbit space.

We finally remark that the technique of this paper resolves a version of Theorem 1.3 for nonnegatively curved simply connected 4-manifolds with an isometric $S^1$-action as well as the following conjecture in [11] without any significant efforts (e.g., see [13]):

**Conjecture 1.4.** An orientable closed simply connected nonnegatively curved 4-manifold $M$ with an effective isometric $S^1$-action is diffeomorphic to either $S^4$, $\mathbb{CP}^2$, $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$, or $S^2 \times S^2$.

We organized this paper as follows. In Section 2, we review the main ingredients in the classification of circle actions on simply connected 4-manifolds done by Fintushel. This will be largely taken from two papers [5] and [6]. In Section 3, we explain the so-called “replacement trick” by P. Pao in [17]. In Section 4, we give detailed proofs of Theorems 1.2 and 1.3 by considering the various topological configurations of the isolated fixed points.

# 2 Classification of circle actions on simply connected 4-manifolds

The aim of this section is to review the main ingredients in the classification of circle actions on simply connected 4-manifolds by Fintushel in [5] and [6]. These together with the replacement trick by P. Pao in [17] will be our important tools to prove Theorems 1.2 and 1.3.

Throughout this section, let $M$ be a simply connected oriented 4-manifold with an effective $S^1$-action. For any subset $N$ of $M$, let $N^*$ denote the image of $N$ in the orbit space and let $p : M \rightarrow M^*$ be the orbit map. Also if we are given a set $N^*$ in $M^*$, $N$ means the preimage of $N^*$ under the orbit map $p$. Finally let $F$ be the fixed point set of $M$, $E$ the union of the exceptional orbits, and $P$ the union of the principal orbits. Since $M$ is simply connected, $M^*$ is also simply connected. Thus each component of $\partial M^*$ is a 2-sphere by the Poincaré duality and is a subset of $F^*$ of the fixed point set. Then we can assign the orbital data to $M^*$ as follows:
(1) For each boundary component $N^*$ of $M^*$, choose a regular neighborhood $N^* \times [0, 1]$. The restriction of the orbit map $p$ to $N^* \times \{1\}$ is a principal $S^1$-bundle over $N^* \times \{1\}$. Thus we assign to each boundary component $N^*$ of $M^*$ oriented by the outward normal to $N^* \times \{1\}$ the Euler number of this bundle, and $N^*$ is called a weighted 2-sphere.

(2) For each $x^*$ in $F^*$ minus the union of $\partial M^*$ and the closure of $E^*$, let $B^*$ be a 3-disk neighborhood of $x^*$ in the interior of $M^*$. The restriction of $p$ to the boundary $\partial B^* \subset P^*$ is a principal $S^1$-bundle over $\partial B^*$ whose total space is a 3-sphere. With the orientation of $\partial B^*$ by the outward normal of $\partial B^*$, we assign to $x^*$ the Euler number $\pm 1$ of this bundle.

(3) For each simple closed curve $C^*$ in $E^* \cup F^*$, we assign to each component of $E^*$ in $C^*$ the Seifert invariant, once we fix an orientation in $C^*$. The simple closed curve obtained in this way will be called multiply-weighted. If a weighted circle does not contain any fixed points, it is called simply-weighted. But if $M$ is simply connected, then simply-weighted circles in the orbit space $M^*$ do not occur.

(4) For each arc $A^*$ which is a component of $E^* \cup F^*$, we orient $A^*$ and assign the Seifert invariant as in (3). In this case we call $A^*$ a weighted arc and write the weight system as

$$[b'; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n); b'']$$

To each component of (1), (2), (3), and (4), their associated Euler number will be called the index. Thus the weighted arc

$$[b'; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n); b'']$$

has index $b'' - b' = \pm 1$ or 0, an isolated fixed point has index equal to $\pm 1$, and a weighted circle has index equal to 0. Moreover, it is important to note that the sum of the indices of all the components (1), (2), (3), and (4) is always zero. The main result of [5] and [6] says that each simply connected 4-manifold $M$ with an effective $S^1$-action corresponds to a legally weighted 3-manifold $M^*$ which is an oriented compact connected 3-manifold equipped with the following data:

(1) A finite collection of weighted arcs and circles in the interior of $M^*$ as above.
(2) A finite collection of distinguished points in the interior of $M^*$.

(3) Let $S^*$ be the union of $\partial M^*$, the points of (2), and the weight arcs of (1). A class $\chi$ is an element in $H_1(M^*, S^*)$ such that if

$$\partial \chi = (\zeta_1, \ldots, \zeta_m) \in H_0(S^*; \mathbb{Z})$$

then each isolated fixed point has $\zeta_i = \pm 1$ and each weighted arc

$$[b' ; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n) ; b'']$$

has $\zeta_i = b'' - b' = \pm 1$ or 0.

With these preliminaries in place, the following theorem (Theorem 7.1 in [5]) will play a crucial role in this paper.

**Theorem 2.1.** Let $S^1$ act on a simply connected 4-manifold $M$. The action of $S^1$ extends to an action of $T^2 = S^1 \times S^1$ if and only if

1. $M^*$ is not a counterexample to the 3-dimensional Poincaré conjecture and
2. if $M^*$ contains a weighted circle $C^*$ then $M^*$ is homeomorphic to $S^3$, $C^* = E^* \cup F^*$, and $C^*$ is unknotted in $M^*$.

In particular, this theorem can be used to show that if the two conditions of Theorem 2.1 are satisfied then a simply connected 4-manifold with an effective $S^1$-action can be built from $S^4$ with the circle action with two isolated fixed points and equivariantly connect summing $S^4$'s and $\mathbb{CP}^2$'s repeatedly. To be more precise, Fintushel proved the following theorem.

**Theorem 2.2.** Let $S^1$ act on a simply connected 4-manifold $M$, and suppose that the orbit space that is homotopy equivalent to $S^3$ is not a counterexample to the 3-dimensional Poincaré conjecture. Then $M$ is equivariantly homeomorphic to a connected sum of copies of $S^4$, $\pm \mathbb{CP}^2$, and $S^2 \times S^2$.

It is now true by the celebrated works [18] and [19] of Perelman that there is no counterexample to the Poincaré conjecture. Hence we can state the following

**Corollary 2.3.** Let $S^1$ act on a simply connected 4-manifold $M$. Then $M$ is equivariantly homeomorphic to a connected sum of copies of $S^4$, $\pm \mathbb{CP}^2$, and $S^2 \times S^2$. 

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3 Replacement Trick

In this paper we also need the “replacement trick” of P. Pao. This trick makes the given 4-manifold admit many different $S^1$-actions by replacing a weighted arc with a simpler one. To be precise, if the weighted circle $C^*$ contains exactly two isolated fixed points then $C^*$ consists of two exceptional orbit types $E^*_n$ and $E^*_m$ and two fixed points. That is, $C^*$ is just

$$C^* = E^*_n \cup E^*_m \cup \{\text{two isolated fixed points}\}.$$ 

Now choose a tubular neighborhood $N^*$ of the union of $E^*_n$ and two fixed points such that $N^*$ intersect the circle $C^*$ at an open arc containing two fixed points and $E^*_n$. See Figure 1.

Let $N$ be the preimage of $N^*$ under the orbit map. Consider the product $D^2 \times S^2$ of a 2-disk and a 2-sphere. Then represent $D^2 \times S^2$ by $(r, \theta; \rho, \phi, z)$ where $(r, \theta)$ are the polar coordinates of $D^2$ and $(\rho, \phi, z)$ are the cylindrical coordinates of $S^2$. Let $S^1$ act on $D^2 \times S^2$ by

$$(r, \theta; \rho, \phi, z) \mapsto (r, \theta + m\psi; \rho, \phi + n\psi, z).$$

Note that the orbit space of $D^2 \times S^2$ under the above action of $S^1$ is identical to $N^*$. On the other hand, there are many other actions of $S^1$ on $D^2 \times S^2$. For example, if $n' \equiv n \mod m$ then we let $S^1$ act on $D^2 \times S^2$ by

$$(r, \theta; \rho, \phi, z) \mapsto (r, \theta + m\psi; \rho, \phi + n'\psi, z).$$
Now remove $N$ from $M$ and change the $S^1$-action on $N$. Then sew $N$ back to $M$ by an equivariant diffeomorphism. If we require $(n - n')/m$ to be even in addition, then this surgery operation will not change the manifold $M$, but change the orbit structure on $C^*$ by the union of two exceptional orbits $E_n^*$ and $E_m^*$ and two isolated fixed points. This reduction can be repeated with keeping decrease the pair $(m, n)$ of integers to $(1, 1)$ or $(1, 0)$. Hence we have the following theorem of P. Pao in [17] (or Proposition 13.1 in [6]):

**Theorem 3.1.** Let $X$ be a closed oriented 4-manifold with an effective $S^1$-action whose weighted orbit space contains a weighted circle $C^*$ with exactly two isolated fixed points. Then $M$ admits a different $S^1$-action whose weight space is $M^*$ with $C^*$ replaced by two isolated fixed points or $M^*$ minus the interior of 3-ball with $C^*$ removed.

This implies that in the second case the orbit space for the new $S^1$-action will have one more boundary component instead of the weight circle $C^*$.

### 4 Proofs of Theorems 1.2 and 1.3

The goal of this section is to prove Theorems 1.2 and 1.3. As remarked in Section 1, it suffices to prove the theorems only for the following two cases: two isolated fixed points or three isolated fixed points. If the manifold $M$ has two (resp. three) isolated fixed points, then $M$ is equivariantly homeomorphic to $S^4$ (resp. $\mathbb{CP}^2$) by Theorem 2.2.

Assume first that the orbit space $M^*$ does not have any weighted circles, regardless of the number of isolated fixed points. See Figure 2 for some possible configurations of $E^* \cup F^*$. Since there is no counterexample to the Poincaré conjecture by Perelman, it follows from Theorem 2.1 that the $S^1$-action lifts to an action of $T^2 = S^1 \times S^1$. Hence the simply connected positively curved 4-manifold $M$ which is equivariantly homeomorphic to $S^4$ or $\mathbb{CP}^2$ is actually a toric manifold. But then the well-known classification result (or argument) of toric

![Figure 2: Some configurations of $E^* \cup F^*$ for three isolated fixed points](image)
manifolds immediately tells us that $M$ is in fact equivariantly diffeomorphic to $S^4$ or $\mathbb{CP}^2$, depending on the number of isolated fixed points. Or, to be more precise, you may adapt an argument appearing at the end of this section which shows that a simply connected manifold with an action of $T^2$ equivariantly homeomorphic to $\mathbb{CP}^2$ is actually equivariantly diffeomorphic to $\mathbb{CP}^2$.

Next we assume that the orbit space $M^*$ has a weighted circle. Then we need to use the replacement trick of Pao (Theorem 3.1). To do so, first consider the case that there are two isolated fixed points. In this case $E^* \cup F^*$ consists of one weighted circle with two isolated fixed points lying on it. Note that there is no case that $F(S^1, M)$ is the disjoint union of a simply weighted circle and two isolated fixed points, since $M$ is simply connected (see [5] or Section 2). Now applying the replacement trick to the manifold $M$ and its orbit space $M^*$, we obtain either a new manifold $M'$ equivariantly diffeomorphic to $M$ whose $E^* \cup F^*$ consists of two isolated fixed points or a new manifold $M'$ whose fixed point set is a 2-sphere. In the former case $M'$ is equivariantly diffeomorphic to $S^4$ as above. On the other hand, in the latter case there exists an isometric effective $S^1$-action on $M'$ with a positively curved metric whose fixed point is of codimension 2. Thus again by Theorem 2.1 of Grove and Searle in [8] $M'$ (or $M$) should be equivariantly diffeomorphic to $S^4$. In fact, for either case there exists an alternative argument: we can apply Theorem 2.1 to show that $M'$ is equivariantly diffeomorphic to $S^4$, since anyway there is no weighted circle by construction. This finishes the proof of the case of two isolated fixed points.

Finally we consider the case that $E^* \cup F^*$ consists of one weighted circle with three isolated fixed points lying on it. To deal with this case, we decompose the weighted circle into two weighted circles satisfying the following conditions (see [1]):

1. $M$ is equivariantly diffeomorphic to the connected sum of $X$ and $Y$. Here we do not require that $X$ and $Y$ admit a positively curved metric.

2. The weighted orbit space of $X$ is the same as $M$ except that the weighted circle has exactly two isolated fixed points.

3. The weighted orbit space of $Y$ is $S^3$ with a trivially embedded multiply-weighted circle with the original weights which is unknotted in the orbit space $Y^*$ of $Y$. See Figure 3.

By considering the Euler characteristic of the fixed point set, we can know that $X$ (resp. $Y$) is equivariantly homeomorphic to $S^4$ (resp. $\mathbb{CP}^2$).
Figure 3: An example of the decomposition of a weighted circle

Then we again use the replacement trick of P. Pao (Theorem 3.1) to obtain a new manifold \( X' \) equivariantly diffeomorphic to \( X \) either whose \( E^* \cup F^* \) consists of two isolated fixed points or whose orbit space is the orbit space \( X^* \) minus the interior of 3-ball with \( E^* \cup F^* \) removed. But then by Theorem 2.1 the \( S^1 \)-action of \( X' \) lifts to \( S^1 \times S^1 \)-action. Hence \( X' \) should be equivariantly diffeomorphic to \( S^4 \).

Note that by construction \( Y \) is equivariantly homeomorphic to \( \mathbb{C}P^2 \) and has a weighted circle with three isolated fixed points which is unknotted in \( Y^* \) homeomorphic to \( S^3 \). Thus by Theorem 2.1 (Theorem 7.1 in [5]), the action of \( S^1 \) on \( Y \) extends to an action of \( T^2 = S^1 \times S^1 \). Now it is a standard argument to show that \( Y \) is equivariantly diffeomorphic to \( \mathbb{C}P^2 \). To see it more precisely, note first that by construction \( Y \) acted on by \( T^2 \) does not have non-trivial finite isotropy groups. (In fact, this is automatically true: if a closed smooth orientable 4-manifold with an effective \( T^2 \)-action whose first homology group over integer coefficients is trivial then the only finite isotropy group is the identity. See Lemma 5.2 in [16].) Thus it follows from the classification of Orlik and Raymond in [16] that \( Y \) should be \( T^2 \)-equivariantly homeomorphic to \( \mathbb{C}P^2 \). This in turn implies that their orbit spaces are homeomorphic to each other in a weight-preserving way. Since their orbit spaces are all 2-disks, they are indeed weight-preservingly diffeomorphic. Thus by Theorem 4.2 in [16] we can conclude that there exists an equivariant diffeomorphism of \( Y \) onto \( \mathbb{C}P^2 \). This completes the proofs of Theorems 1.2 and 1.3.
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