WHAT MIGHT A HAMILTONIAN DELAY EQUATION BE?

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Abstract. We describe a variational approach to a notion of Hamiltonian delay equations and discuss examples.

1. Introduction

An ordinary differential equation (ODE) on \( \mathbb{R}^d \) is, in the simplest case, of the form
\[
\dot{v}(t) = X(v(t))
\]
where \( X \) is a vector field on \( \mathbb{R}^d \). A delay differential equation (DDE) on \( \mathbb{R}^d \) is, again in the simplest case, of the form
\[
\dot{v}(t) = X(v(t - \tau))
\]
where \( X \) is still a vector field on \( \mathbb{R}^d \) and \( \tau > 0 \) is the time delay. Delay equations therefore model systems in which the instantaneous velocity \( \dot{v}(t) \) depends on the state of the curve \( v \) at a past time. There are very many such systems in science and engineering. We refer to [9] for a foundational text and to [7] for a wealth of examples.

A Hamiltonian differential equation on \( \mathbb{R}^{2n} \) is an ODE of the form
\[
\dot{v}(t) = X_H(v(t))
\]
where the Hamiltonian vector field is of the special form \( X_H = i \nabla H \). Here \( H: \mathbb{R}^{2n} \rightarrow \mathbb{R} \) is a smooth function and \( i \) is the usual complex multiplication on \( \mathbb{R}^{2n} \cong \mathbb{C}^n \). It is now tempting to define a Hamiltonian delay equation to be a DDE of the form
\[
\dot{v}(t) = X_H(v(t - \tau))
\]
with \( \tau > 0 \) and \( X_H \) as before. Such systems where studied by Liu [12], who proved the existence of periodic orbits under natural assumptions on \( H \).

In this paper we take a different approach to Hamiltonian delay equations, or at least to periodic orbits solving what we propose to call a Hamiltonian delay equation. Our approach is through action functionals. Let \( \mathcal{L} = C^\infty(S^1, \mathbb{R}^{2n}) \) be the space of smooth 1-periodic loops in \( \mathbb{R}^{2n} \), and recall from classical mechanics that the 1-periodic solutions of (1.1) are exactly the critical points of the action functional \( \mathcal{A}: \mathcal{L} \rightarrow \mathbb{R} \) given by
\[
\mathcal{A}(v) = \int_0^1 \left[ p(t) \cdot \dot{q}(t) - H(v(t)) \right] dt, \quad v(t) = (q(t), p(t)).
\]
This fact, that interesting solutions can be seen as the critical points of a functional, played a key role in the creation of the modern theory of Hamiltonian dynamics and of symplectic
and we compute the critical point equations of several classes of delay action functionals. If we just take
\[ \mathcal{A}(v) = \int_0^1 [p(t) \cdot \dot{q}(t) - H(v(t - \tau))] \, dt, \]
we get nothing new: The critical point equation is again \( \dot{v}(t) = X_H(v(t)) \). However, if we take two Hamiltonian functions \( H, K \) on \( \mathbb{R}^{2n} \) and the functional
\[ \mathcal{A}(v) = \int_0^1 [p(t) \cdot \dot{q}(t) - H(v(t)) \cdot K(v(t - \tau))] \, dt, \]
then the critical point equation is the honest delay equation
\[ \dot{v}(t) = H(v(t + \tau)) \cdot X_H(v(t)) + K(v(t - \tau)) \cdot X_H(v(t)). \]
Notice that the time shift \(+\tau\) looks like “into the future”, but this does not matter along periodic orbits, along which the future can be identified with the past.

In our approach a Hamiltonian delay equation is thus a delay equation that can be obtained as critical point equation of an action functional.

In Sections 2 and 4 we compute the critical point equations of several classes of delay action functionals on the loop space of \( \mathbb{R}^{2n} \), and more generally of exact symplectic manifolds. As a special case we shall obtain in Section 3 one instance of the delayed Lotka–Volterra equations.

In fact, already in his 1928 paper \[2\] and in his seminal book \[20\] from 1931 Volterra was interested in periodic solutions of delay equations, and formulated the famous Lotka–Volterra equations with and without delay.

**Related works.** General properties of delay action functionals were studied for instance in Chapter VI of \[6\] and in \[16\]. The problem when a delay equation on \( \mathbb{R}^d \) is the critical point equation of a functional is analyzed in \[11\]. A Hamiltonian formalism for certain non-local PDEs on \( \mathbb{R}^{2n} \), that is also based on non-local action functionals, was recently proposed in \[4\].

**Discussion.** In the rest of this introduction we further comment on why we believe that our approach to Hamiltonian delay equations is promising.

1. **Extension to manifolds.** Recall that on \( \mathbb{R}^{2n} \) one possible definition of a Hamiltonian delay equation is
\[ \dot{v}(t) = X_H(v(t - \tau)) \] (1.2)
On a general symplectic manifold \( M \), however, this concept does not make sense, simply because \( v(t) \in T_{v(t)} M \) and \( X_H(v(t - \tau)) \in T_{v(t-\tau)} M \) reside in different tangent spaces. On the other hand, our approach through action functionals readily extends to manifolds: Recall that a symplectic manifold is a manifold \( M \) together with a non-degenerate closed 2-form \( \omega \) on \( M \). For simplicity we assume that \( \omega \) is exact, \( \omega = d\lambda \) for a 1-form \( \lambda \). The Hamiltonian vector field of a smooth function \( H: M \to \mathbb{R} \) is defined by \( \omega(X_H, \cdot) = -dH \), and the 1-periodic solutions of \( X_H \) are exactly the critical point of the action functional
\[ \mathcal{A}(v) = \int_0^1 \left[ \lambda(v(t)) - H(v(t)) \right] \, dt \] (1.3)
on the space of smooth 1-periodic loops in \( M \). Taking \( \lambda = \sum_{j=1}^n p_j dq_j \) on \( \mathbb{R}^{2n} \) we recover the case described above.

Replacing the Hamiltonian term \( H(v(t)) \) in (1.3) by a delay term such as \( H(v(t)) \cdot K(v(t-\tau)) \) or by any of the terms described in Sections 2 and 4, we get as critical point equation a delay equation on \( M \). Thus, if we start from a delay action functional and compute the critical
point equation, then an “accident” as for (1.2) cannot happen, and we always get a meaningful equation.

General DDEs can be readily defined on manifolds, see Section 12.1 of [9] and [14], and there are a few results on periodic orbits of such systems, see [5] and the references therein. In contrast, there seems to be no concept of a Hamiltonian delay equation on manifolds. Our approach at least provides a natural notion of such an equation and a tool for finding periodic solutions.

2. A calculus of variations for Hamiltonian DDEs. While the theory of DDEs is meanwhile quite rich [9], it is nonetheless much less developed than the theory of ODEs. One reason is that for DDEs there is no local flow on the given manifold, and so the whole theory is less geometric and more cumbersome.

In sharp contrast to general ODEs, Hamiltonian ODEs can be studied by variational methods, thanks to the action functional. For one thing, the action functional (even though neither bounded from below nor above, and strongly indefinite) can be used to do critical point theory on the loop space, as was first demonstrated by Rabinowitz [15]. At least as important, one can see symplectic topology as the geometry of the action functional. (This is almost the title and exactly the content of Viterbo’s paper [17], and also in the book [10] the action functional is the main tool.) For instance, a selection of critical values of the action functional by min-max leads to numerical invariants of Hamiltonian systems and symplectic manifolds, that have many applications. The climactic impact of the action functional into symplectic dynamics and topology, however, is Floer homology, which is Morse theory for the action functional on the loop space.

Now, incorporating delays into an action functional, we can try to extend all these constructions to delay action functionals, and thereby create a calculus of variations for Hamiltonian delay equations, that should have many applications at least to questions on periodic orbits of such equations. For instance, a Floer theory in the delay setting should lead to the same lower bounds for the number of periodic orbits of Hamiltonian delay equations as guaranteed in the undelayed case by the solution of the Arnold conjectures. For a special class of Hamiltonian delay equations, these lower bounds are verified in [3] by means of an iterated graph construction and classical (Lagrangian) Floer homology. First steps in the construction of a delay Floer homology were taken in [1] and [2].

2. Delay equations from sums and products of Hamiltonian functions

Let \((M, \omega)\) be a symplectic manifold with exact symplectic form \(\omega = d\lambda\). We choose \(2N + 1\) autonomous Hamiltonian functions

\[ F, H_i, K_i : M \to \mathbb{R}, \quad i = 1, \ldots, N, \]

and define an “action functional” on the free loop space \(\mathcal{L} \equiv \mathcal{L}(M) := C^\infty(S^1, M)\) by

\[ \mathcal{A} : \mathcal{L} \to \mathbb{R}, \]

\[ v \mapsto \int_{S^1} v^* \lambda - \int_0^1 F(v(t)) dt - \sum_{i=1}^N \int_0^1 H_i(v(t)) K_i(v(t - \tau)) dt \quad (2.1) \]
where \( \tau \geq 0 \) is the time delay. To find the critical point equation of \( \mathcal{A} \) we fix \( v \in \mathcal{L} \) and \( \dot{v} \in T_v \mathcal{L} \) and compute

\[
d\mathcal{A}(v)\dot{v} = \int_0^1 \omega(\dot{v}(t), \dot{v}(t)) dt - \int_0^1 dF(v(t))[\dot{v}(t)] dt
- \sum_{i=1}^N \int_0^1 H_i(v(t)) \cdot dK_i(v(t - \tau))[\dot{v}(t - \tau)] dt
- \sum_{i=1}^N \int_0^1 K_i(v(t - \tau)) \cdot dH_i(v(t))[\dot{v}(t)] dt.
\]

Since \( v \) and \( \dot{v} \) are 1-periodic,

\[
\int_0^1 H_i(v(t)) \cdot dK_i(v(t - \tau))[\dot{v}(t - \tau)] dt = \int_0^1 H_i(v(t + \tau)) \cdot dK_i(v(t))[\dot{v}(t)] dt.
\]

Using also the definition \( \omega(X_F, \cdot) = -dF \) of the Hamiltonian vector field \( X_F \), etc., we find

\[
d\mathcal{A}(v)\dot{v} = \int_0^1 \omega(\dot{v}(t), \dot{v}(t)) dt - \int_0^1 \omega(\dot{v}(t), X_F(v(t))) dt
- \sum_{i=1}^N \int_0^1 H_i(v(t + \tau)) \cdot \omega(\dot{v}(t), X_{K_i}(v(t))) dt
- \sum_{i=1}^N \int_0^1 K_i(v(t - \tau)) \cdot \omega(\dot{v}(t), X_{H_i}(v(t))) dt.
\]

The critical point equation is therefore

\[
\dot{v}(t) = X_F(v(t)) + \sum_{i=1}^N \left[ H_i(v(t + \tau)) X_{K_i}(v(t)) + K_i(v(t - \tau)) X_{H_i}(v(t)) \right].
\]

We have proved the following lemma.

**Lemma 2.1.** The critical points of \( \mathcal{A} \) satisfy the Hamiltonian delay equation

\[
\dot{v}(t) = X_F(v(t)) + \sum_{i=1}^N \left[ H_i(v(t + \tau)) X_{K_i}(v(t)) + K_i(v(t - \tau)) X_{H_i}(v(t)) \right]. \tag{2.2}
\]

Using that \( v(t + 1) = v(t) \) we obtain

**Corollary 2.2.** For the time delay \( \tau = \frac{1}{2} \) the Hamiltonian delay equation becomes

\[
\dot{v}(t) = X_F(v(t)) + \sum_{i=1}^N \left[ H_i(v(t - \frac{1}{2})) X_{K_i}(v(t)) + K_i(v(t - \frac{1}{2})) X_{H_i}(v(t)) \right].
\]

3. The Lotka-Volterra equations, with and without delay

In this section we extend the work of Fernandes–Oliva \cite{8} to positive delays. Fix a skew-symmetric \( N \times N \)-matrix \( A = (a_{ij}) \), i.e. \( a_{ji} = -a_{ij} \), and \( N \) real numbers \( b_i \). Take \( M = \mathbb{R}^{2N} \)
with its usual exact symplectic form \( \omega = \sum_i dq_i \wedge dp_i \), and set
\[
F(q, p) := \sum_{i=1}^{N} b_i q_i, \quad H_i(q, p) := -e^{p_i}, \quad K_i(q, p) := e^\frac{1}{2} \sum_{j=1}^{N} a_{ij} q_j.
\]
The Hamiltonian vector fields are
\[
X_F = \sum_{i=1}^{N} b_i \frac{\partial}{\partial p_i}, \quad X_{H_i} = e^{p_i} \frac{\partial}{\partial q_i}, \quad X_{K_i} = \frac{1}{2} \sum_{k=1}^{N} a_{ik} e^{\frac{1}{2} \sum_{j=1}^{N} a_{ij} q_j} \frac{\partial}{\partial p_k}.
\]
Fix \( \tau \geq 0 \). For \( v = (q, p) \in \mathcal{L}(\mathbb{R}^{2N}) \) the Hamiltonian (delay) equation (2.2) becomes
\[
\dot{v}(t) = \sum_{i=1}^{N} b_i \frac{\partial}{\partial p_i} + \sum_{i=1}^{N} \left[ -e^{p_i(t+\tau)} \frac{1}{2} \sum_{k=1}^{N} a_{ik} e^{\frac{1}{2} \sum_{j=1}^{N} a_{ij} q_j(t)} \frac{\partial}{\partial p_k} + e^\frac{1}{2} \sum_{j=1}^{N} a_{ij} q_j(t-\tau) e^{p_i(t)} \frac{\partial}{\partial q_i} \right].
\]
In other words,
\[
\dot{q}_i(t) = e^{p_i(t)+\frac{1}{2} \sum_{j=1}^{N} a_{ij} q_j(t-\tau)} \dot{p}_i(t) + \sum_{j=1}^{N} a_{ij} \dot{q}_j(t-	au)
\]
\[
\dot{p}_i(t) = b_i - \sum_{l=1}^{N} e^{p_i(t+\tau)} \frac{1}{2} \sum_{j=1}^{N} a_{ij} e^{\frac{1}{2} \sum_{j=1}^{N} a_{ij} q_j(t)}
\]
\[
= b_i - \frac{1}{2} \sum_{l=1}^{N} a_{il} \dot{q}_l(t + \tau)
\]
\[
= b_i + \frac{1}{2} \sum_{l=1}^{N} a_{il} \dot{q}_l(t + \tau)
\]
where in the last equation we have used that \( A \) is skew-symmetric. Using these two equations we compute
\[
\ddot{q}_i(t) = \left( \dot{p}_i(t) + \frac{1}{2} \sum_{j=1}^{N} a_{ij} \dot{q}_j(t-\tau) \right) \dot{q}_i(t)
\]
\[
= \left( b_i + \frac{1}{2} \sum_{l=1}^{N} a_{il} \dot{q}_l(t + \tau) + \frac{1}{2} \sum_{j=1}^{N} a_{ij} \dot{q}_j(t - \tau) \right) \dot{q}_i(t).
\]
Now observe that the right hand side only depends on the \( \dot{q}_j \), but not on the \( p_j \). Setting \( x_i(t) := \dot{q}_i(t) \) we thus obtain the first order delay system
\[
\dot{x}_i(t) = b_i x_i(t) + \frac{1}{2} \sum_{j=1}^{N} a_{ij} x_i(t) x_j(t+\tau) + \frac{1}{2} \sum_{j=1}^{N} a_{ij} x_i(t) x_j(t-\tau).
\]
**Case** \( \tau = 0 \). Then (3.1) becomes
\[
\dot{x}_i(t) = b_i x_i(t) + \sum_{j=1}^{N} a_{ij} x_i(t) x_j(t)
\]
with skew-symmetric \( A = (a_{ij}) \). This is one instance of the Lotka–Volterra equations without delay. These equations were proposed by Lotka [13] in his studies of chemical reactions, and independently by Volterra [18] in his studies of predator-prey dynamics.
Case $\tau = \frac{1}{2}$. Then the equations (3.1) for 1-periodic orbits become
\[
\dot{x}_i(t) = b_i x_i(t) + \sum_{j=1}^{N} a_{ij} x_i(t) x_j(t - \frac{1}{2}).
\]
These Hamiltonian delay equations already appeared in Chapter 4 of Volterra’s book [20].

4. More examples

In this section we give three rather special classes of Hamiltonian delay equations, two involving integrals. The reader may invent his own examples.

4.1. More general products of Hamiltonian functions. In (2.1) we may replace the sum by an integral and choose a double time-dependence: Consider functions $H, K : M \times S^1 \times S^1 \to \mathbb{R}$, which we write as $H_{t, \tau}(x)$ and $K_{t, \tau}(x)$ for $x \in M$ and $t, \tau \in S^1$. Then set
\[
\mathcal{A} : \mathcal{L} \to \mathbb{R}
\]
\[
v \mapsto \int_{S^1} v^* \lambda - \int_0^1 \int_0^1 H_{t, \tau}(v(t)) K_{t, \tau}(v(t) - \tau) d\tau dt.
\]
For $v \in \mathcal{L}$ and $\dot{v} \in T_v \mathcal{L}$ we compute
\[
\begin{aligned}
d\mathcal{A}(v) \dot{v} &= \int_0^1 \omega(\dot{v}(t), \dot{v}(t)) dt \\
&\quad - \int_0^1 \left[ \int_0^1 H_{t, \tau}(v(t)) \cdot dK_{t, \tau}(v(t) - \tau) \dot{v}(t - \tau) d\tau \\
&\quad - \int_0^1 K_{t, \tau}(v(t) - \tau) \cdot dH_{t, \tau}(v(t)) \dot{v}(t) d\tau \right] dt.
\end{aligned}
\]
Since $v, \dot{v}$ are 1-periodic and also $H, K$ are periodic in $t$,
\[
\int_0^1 H_{t, \tau}(v(t)) \cdot dK_{t, \tau}(v(t) - \tau) \dot{v}(t - \tau) d\tau = \int_0^1 H_{t+\tau, \tau}(v(t + \tau)) \cdot dK_{t+\tau, \tau}(v(t)) \dot{v}(t) d\tau.
\]
Therefore,
\[
\begin{aligned}
d\mathcal{A}(v) \dot{v} &= \int_0^1 \omega(\dot{v}(t), \dot{v}(t)) dt \\
&\quad - \int_0^1 \left[ \int_0^1 \omega(\dot{v}(t), H_{t+\tau, \tau}(v(t + \tau)) \cdot X_{K_{t+\tau, \tau}}(v(t))) d\tau \\
&\quad - \int_0^1 \omega(\dot{v}(t), K_{t, \tau}(v(t) - \tau) \cdot X_{H_{t, \tau}}(v(t))) d\tau \right] dt.
\end{aligned}
\]
Hence the critical points of $\mathcal{A}$ are the solutions of the Hamiltonian delay equation
\[
\dot{v}(t) = \int_0^1 \left[ H_{t+\tau, \tau}(v(t + \tau)) \cdot X_{K_{t+\tau, \tau}}(v(t)) + K_{t, \tau}(v(t) - \tau) \cdot X_{H_{t, \tau}}(v(t)) \right] d\tau.
\]
In the special case that $H_{t, \tau}$ and $K_{t, \tau}$ are autonomous, equation (4.1) simplifies to
\[
\dot{v}(t) = \int_0^1 H(v(t + \tau)) d\tau \cdot X_{K}(v(t)) + \int_0^1 K(v(t - \tau)) d\tau \cdot X_{H}(v(t)).
\]
If we define the functions $\mathcal{H}, \mathcal{K} : \mathcal{L} \to \mathbb{R}$ by

$$
\mathcal{H}(v) := \int_0^1 H(v(t)) dt \quad \text{and} \quad \mathcal{K}(v) := \int_0^1 K(v(t)) dt
$$

the above equation becomes

$$
\dot{v}(t) = \mathcal{H}(v) X_K(v(t)) + \mathcal{K}(v) X_H(v(t)).
$$

Specializing further to $H = K$ we obtain

$$
\dot{v}(t) = 2\mathcal{H}(v) X_H(v(t)). \quad (4.2)
$$

In this special case, preservation of energy implies that $t \mapsto H(v(t))$ is constant along solutions, and thus we may write (4.2) as

$$
\dot{v}(t) = 2H(v) X_H(v(t)) = X_{H^2}(v(t)). \quad (4.3)
$$

**Remark 4.1.** Of course, this equation can be studied by Floer theory, hence there are (in the Morse–Bott sense) multiplicity results (in terms of cup-length or Betti numbers) for periodic solutions in a certain range of Conley–Zehnder indices. Clearly, (4.3) has many solutions, namely critical points of $H$. However, unless $H$ is $C^2$-small at all critical points, the Morse indices of critical points cannot all agree with their Conley–Zehnder indices, and so Floer theory implies the existence of additional non-constant solutions to (4.3). We expect that also equation (4.1) admits a Floer theory. Note that typically, (4.1) does not have any constant solutions, even when $K = 1$ and $H$ is independent of $\tau$, hence a Floer theory should imply the existence of many interesting periodic solutions.

**4.2. Exponentials of Hamiltonian functions.** We consider yet another incarnation of a Hamiltonian delay equation. Take

$$
\mathcal{A} : \mathcal{L} \to \mathbb{R}
$$

$$
v \mapsto \int_{S^1} v^* \lambda - \int_0^1 \exp \left[ \int_0^1 H_T(v(t - \tau)) d\tau \right] dt
$$

where $H : S^1 \times M \to \mathbb{R}$ is given. We compute

$$
d\mathcal{A}(v) \dot{v} = \int_0^1 \omega(\dot{v}(t), \dot{v}(t)) dt
$$

$$
- \int_0^1 \int_0^1 \exp \left[ \int_0^1 H_T(v(t - \tau)) d\tau \right] dH_\sigma(v(t - \sigma)) \dot{v}(t - \sigma) dt d\sigma.
$$

Substituting $t$ by $t + \sigma$ and changing the order of integration the second summand becomes

$$
- \int_0^1 \int_0^1 \exp \left[ \int_0^1 H_T(v(t + \sigma - \tau)) d\tau \right] dH_\sigma(v(t)) \dot{v}(t) d\sigma dt
$$

$$
= - \int_0^1 \omega(\dot{v}(t), \int_0^1 \exp \left[ \int_0^1 H_T(v(t + \sigma - \tau)) d\tau \right] X_{H_\sigma}(v(t)) d\sigma) dt
$$

The critical point equation is therefore

$$
\dot{v}(t) = \int_0^1 \exp \left[ \int_0^1 H_T(v(t + \sigma - \tau)) d\tau \right] X_{H_\sigma}(v(t)) d\sigma.
$$
4.3. Several inputs. We now consider a function $H: M \times M \to \mathbb{R}$ on the symplectic manifold $(M \times M, \omega \oplus \omega)$, where again $\omega = d\lambda$, and denote by $d_1 H(x, y): T_x M \to \mathbb{R}$ the derivative of $H$ with respect to the first variable and correspondingly by $X^1_H(x, y)$ the Hamiltonian vector field of $H$ with respect to the first variable:

$$d_1 H(x, y) \xi = -\omega_x \left( (X^1_H(x, y), \xi) \right) \quad \forall \xi \in T_x M.$$ 

Further, we consider the action functional

$$\mathcal{A}: \mathcal{L}(M) \to \mathbb{R},$$

$$v \mapsto \int_{S^1} v^* \lambda - \int_0^1 H(v(t), v(t + \tau)) dt .$$

Concrete examples for the function $H$ come for instance from interaction potentials (as in the 2-body problem) or from vortex equations with delay. For $v \in \mathcal{L}(M)$ and $\dot{v} \in T_v \mathcal{L}(M)$ we compute

$$d\mathcal{A}(v) \dot{v} = \int_0^1 \omega_{v(t)} \left( \dot{v}(t), \dot{v}(t) \right) dt$$

$$- \int_0^1 \left( d_1 H(v(t), v(t + \tau)) \dot{v}(t) + d_2 H(v(t), v(t + \tau)) \dot{v}(t + \tau) \right) dt$$

The second summand is equal to

$$- \int_0^1 \left( d_1 H(v(t), v(t + \tau)) \dot{v}(t) + d_2 H(v(t - \tau), v(t)) \dot{v}(t) \right) dt$$

$$= - \int_0^1 \left[ \omega_{v(t)} \left( \dot{v}(t), X^1_H(v(t), v(t + \tau)) \right) + \omega_{v(t)} \left( \dot{v}(t), X^2_H(v(t - \tau), v(t)) \right) \right] dt .$$

The critical point equation for $\mathcal{A}$ is therefore

$$\dot{v}(t) = X^1_H(v(t), v(t + \tau)) + X^2_H(v(t - \tau), v(t)) .$$

(4.4)

We point out that indeed

$$X^1_H(v(t), v(t + \tau)), X^2_H(v(t - \tau), v(t)) \in T_{v(t)} M$$

so that (4.4) makes sense.

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WHAT MIGHT A HAMILTONIAN DELAY EQUATION BE?

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