Secular Instability of g-Modes in Rotating Neutron Stars

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ABSTRACT
Gravitational radiation tends to drive gravity modes in rotating neutron stars unstable. For an inviscid star, the instability sets in when the rotation frequency is about 0.7 times the corresponding mode frequency of the nonrotating star. Neutron stars with spin frequencies $\gtrsim 100$ Hz are susceptible to this instability, with growth time of order years. However, it is likely that viscous dissipation suppresses the instability except for a narrow range of temperatures around $10^9$ K. We also show that the viscosity driven instability of g-modes is absent.

Key words: stars: neutron – stars: oscillation – stars: rotation – hydrodynamics – gravitation

1 INTRODUCTION
It is well known that globally nonaxisymmetric instabilities can develop in rapidly rotating neutron stars. Of particular interest is the secular instability driven by gravitational radiation, as first discovered by Chandrasekhar (1970) for the bar-mode ($m = 2$) of an incompressible Maclaurin spheroid, and later shown by Friedman & Schutz (1978a,b) to be a generic feature of rotating stars. The Chandrasekhar-Friedman-Schutz instability (hereafter CFS instability) occurs whenever a backward-going mode (with respect to the rotation) in the corotating frame of the star is “dragged” by rotation to become forward-going in the inertial frame. A CFS unstable mode has negative energy (in the inertial frame), and gravitational radiation makes the energy even more negative.

The CFS instability has been extensively studied for f-modes since it is widely believed that the maximum angular velocities of neutron stars are determined by instabilities of these modes (e.g., Ipser & Lindblom 1990, 1991; Cutler & Lindblom 1992; Lindblom 1995; Yoshida & Eriguchi 1995; Stergioulas & Friedman 1998). The f-mode instabilities occur at rotation frequencies comparable to the maximum “break-up” frequency ($\sim 1000$ Hertz). It was found that viscous dissipation in the neutron star stabilizes almost all f-modes except in certain limited temperature regimes. Recently Andersson (1998) pointed out that r-modes (previously studied by Papaloizou & Pringle 1978, Provost et al. 1981, Saio 1982, etc.) are also destabilized by gravitational radiation (see also Friedman & Morsink 1998). Moreover, the couplings between gravitational radiation and r-modes (mainly through current multipole moments) are so strong that viscous forces present in hot young neutron stars are not sufficient to suppress the instability (Lindblom et al. 1998; Andersson et al. 1998a). It is therefore likely that gravitational radiation can carry away most of the angular momentum of a nascent neutron star through the unstable r-modes, and the gravitational waves generated in this process could potentially be detectable by LIGO (Owen et al. 1998). The implications of the r-mode instability for accreting neutron stars have also been discussed (Andersson et al. 1998b; Bildsten 1998; Levin 1998).

A neutron star also possesses p-modes and g-modes, whose stability properties have not been studied. The p-modes are similar to the f-modes, except that, as compared to the f-modes, they have higher frequencies, weaker couplings to the gravitational radiation and stronger viscous dampings. Thus the p-mode instabilities, if exist, are certainly less important than the f-mode instabilities. The situation with g-modes is not clear, however. Gravity modes in neutron stars arise from composition (proton to neutron ratio) gradient in the stellar core (Reisenegger & Goldreich 1992; Lai 1994), density discontinuities in the crust (Finn 1987; Strohmayer 1993) as well as thermal buoyancy associated with finite temperatures, either due to internal heat (McDermott, Van Horn & Hansen 1988) or due to accretion (McDermott & Taam 1987; Bildsten & Cutler 1995; Strohmayer & Lee 1996; Bildsten & Cumming 1998). For a nonrotating neutron star, the typical frequencies of low-order g-modes are around 100 Hz and are smaller for higher-order modes. Such low frequencies imply that, contrary to f-modes, g-modes may be destabilized by gravitational wave when the star rotates at a rate much smaller than the break-up spin frequency. Moreover, since g-modes possess small, but nonzero mass quadrupole moments, it is not clear a priori whether the driving due to gravitational radiation is stronger or weaker for g-modes than that for r-modes (which have zero mass quadrupole moment). The purpose of this
paper is to address these issues related to the secular instability of g-modes.

Our paper is organized as follows. We start in Section 2 by summarizing the essential properties of g-modes in nonrotating neutron stars and defining notations. In Section 3 we calculate the effect of rotation on the mode frequencies and determine the onset of CFS instabilities of g-modes in invisid stars. In Section 4 we evaluate the driving/damping rates of the g-modes due to gravitational radiation and viscosities. Section 5 considers the possibility of viscosity-driven g-mode instability. In Section 6 we compare the instabilities of g-modes and other modes (f and r-modes) and discuss some possible astrophysical applications of our results.

2 PREPARATIONS: CORE G-MODES OF NEUTRON STARS

In this paper, we shall focus on the core g-modes, with buoyancy provided by the gradient of proton to neutron ratio in the neutron star interior (Reisenegger & Goldreich 1992). These modes exist even in zero-temperature stars. The other types of g-modes (due to crustal density discontinuities or thermal entropy gradient) either have weaker restoring forces or are confined to the outer region of the star and therefore are less susceptible to the CFS instabilities as compared to the core g-modes (see below).

We construct equilibrium neutron star models by solving Newtonian hydrostatic equations. Centrifugal distortion to the stellar structure is neglected as we shall only consider stars with rotation rates far below the break-up value. The equations of state (EOS) used for our neutron star models and g-mode calculations have been described in Lai (1994). The nuclear symmetry energy plays an important role in determining the proton fraction \( x \approx 0.15 \) and thus eliminated the crustal modes (see below). For completeness, Figure 1 also shows the Lamb frequency (or the acoustic cut-off frequency), \( \nu_L = \sqrt{(l+1)c_s/(2\pi)} \), with \( l = 2 \). Gravity wave (with frequency \( \nu \)) can propagate only in the region satisfying \( \nu < \nu_L \). For a typical \( \nu \sim 100 \) Hz, the wave propagation region lies in the range \( 0.5 \lesssim \nu \lesssim 0.9 \). For convenience, the basic properties of the modes for nonrotating stars are summarized in Table 1.

The crustal discontinuity modes have angular frequencies approximately given by \( \omega \sim (g/R)^{1/2}(\Delta \rho/\rho)^{1/2} \). With typical density discontinuity \( \Delta \rho/\rho \sim 1\% \), this gives a frequency of order \( 200 \) Hz (Finn 1987). Although these modes can have frequencies greater than the core g-modes, the core modes are confined in the outer region of the star and therefore have very small mass quadrupole moment. The thermally driven g-modes (McDermott et al. 1988) have frequencies less than \( 50 \) Hz for \( T < 10^9 \) K, indicating smaller restoring forces as compared to the core g-modes. We will not consider the crustal modes and thermal modes in the rest of this paper.

\[ N = \frac{\sqrt{3\rho}}{10^7 T^2} \quad \text{or} \quad \sim (G\rho x)^{1/2} \sim 10^3 \text{ s}^{-1}. \]

Thus the core g-modes typically have frequencies less than \( 100 - 200 \) Hz.

In the following sections, we shall consider two representative models based on a parametrized EOS derived from Wiringa, Fiks & Fabrocini (1988), and labeled as UU and AU in Lai (1994). Figure 1 depicts the profiles of the Brunt-Väisälä frequency, \( \nu_N = N/(2\pi) \), as a function of radius inside the star for the two models. We see that \( \nu_N \sim 0 \) at the center (where \( g = 0 \)) and increases outward because of decreasing sound speed \( c_s \) and increasing \( g \) (de-spite that \( x \) decreases outward). We have set \( \nu_N = 0 \) outside the core (\( \rho \lesssim 10^{14} \text{ g cm}^{-3} \)) and thus eliminated the crustal modes (see below). For completeness, Figure 1 also shows the Lamb frequency (or the acoustic cut-off frequency), \( \nu_L = \sqrt{(l+1)c_s/(2\pi)} \), with \( l = 2 \). Gravity wave (with frequency \( \nu \)) can propagate only in the region satisfying \( \nu < \nu_L \) and \( \nu < \nu_N \). For a typical \( \nu \sim 100 \) Hz, the wave propagation region lies in the range \( 0.5 \lesssim \nu \lesssim 0.9 \). For convenience, the basic properties of the modes for nonrotating stars are summarized in Table 1.

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3 ONSET OF INSTABILITY: INVISCID STARS

For neutron stars consisting of ideal fluid (neglecting viscosity), it is possible to determine the onset of CFS instability of g-modes without calculating the growth/damping rates due to dissipative processes. Consider a mode with azimuthal angular dependence $e^{in\phi}$ and time dependence $e^{i\omega r}$, where $\omega_r (>0)$ is the mode angular frequency in the rotating frame. The rotation angular frequency is $\Omega (= \Omega_0/2)$.

When $\omega_r < 0$, the mode attains a prograde pattern speed, $-(\omega_r/m)$, in the inertial frame, and the CFS instability sets in (Friedman & Schutz 1978a,b).

Since we shall consider $\Omega_s$ comparable to $\omega_r$, a perturbative treatment of the rotational effect is inadequate (see the last paragraph of this section). We shall adopt the so-called “traditional approximation” (Chapman & Lindzen 1970; Unno et al. 1989; Bildsten et al. 1996), where the Coriolis forces associated with the horizontal component of the spin angular velocity are neglected. This allows the mode eigenfunction to be separated into angular and radial components. Note that the traditional approximation is strictly valid only for high-order g-modes for which $\omega_r \ll N$ and $\xi_\theta \gg \xi_r = \xi_\lambda$ (where $\xi_\theta \sim \xi_\lambda \sim \xi_r$ is the horizontal Lagrangian displacement and $\xi_r$ is the radial displacement), and for $\Omega_s \ll N$ (so that the radial component of the Coriolis force is neglected compared to the buoyancy force). While these conditions are not valid everywhere inside the star (e.g., near the center $\xi_r$ is of the same order as $\xi_\theta$, and $\omega_r, \Omega_s \gg \xi_r$), in the region $0.6 \lesssim r/R \lesssim 0.9$ where the mode propagates, they are reasonably satisfied (see below).

Neglecting the perturbation in gravitational potential (Cowling approximation), the radial Lagrangian displacement and Eulerian pressure perturbation can be written as

$$\xi_r (r) = \xi_r (r) H_{jm}(\theta) e^{in\phi}, \quad \delta P(r) = \delta P(r) H_{jm}(\theta) e^{in\phi},$$

where $H_{jm}(\theta)$ is the Hough function ($jm$ are the indices analogous to $lm$ in the spherical harmonic $Y_{lm}$), satisfying the Laplace tidal equation (Chapman & Lindzen 1970; Bildsten et al. 1996):

$$\frac{\partial}{\partial \mu} \left( \frac{1 - \mu^2}{1 - q^2 \mu^2} \frac{\partial H_{jm}}{\partial \mu} \right) - \frac{m^2 H_{jm}}{(1 - \mu^2)(1 - q^2 \mu^2)} = \frac{q \mu (1 + q^2 \mu^2) H_{jm}}{(1 - q^2 \mu^2)^2} - \lambda H_{jm},$$

Table 1. Core g-modes of nonrotating neutron stars

| Model | $M (M_\odot)$ | $R$ (km) | $\rho_c$ (g cm$^{-3}$) | mode | $\nu$ (Hz) | $|\delta D_{22}|$ |
|-------|----------------|---------|------------------------|------|------------|----------------|
| UU    | 1.4            | 13.48   | $6.95 \times 10^{14}$  | $g_1$| 148        | $6.2 \times 10^{-4}$ |
| AU    | 1.4            | 12.37   | $8.18 \times 10^{14}$  | $g_1$| 72         | $1.1 \times 10^{-4}$ |

$^a$ The subscript $n$ in $g_n$ specifies the radial order of the mode. The angular order is $l = 2$.

$^b$ Note that the values of $|\delta D_{22}|$ quoted here differ from those in Lai (1994) by up to 80%. This is because we adopt Cowling approximation in this paper while in Lai (1994) the full equations are solved; see discussion at the end of §3.

with $\mu = \cos \theta$ and $q = 2\Omega_s/\omega_r$. The eigenvalue, $\lambda$, depends on $m$ and $q$. For $q \rightarrow 0$ (the nonrotating case), the function $H_{jm}(\theta) e^{im\phi}$ becomes $Y_{jm} (\theta, \phi)$ while $\lambda$ degenerates into $l(l+1)$. The other eigenfunctions of the mode are related to $\delta P$ and $\xi_r$ via

$$\delta \rho (r) = \delta \rho (r) H_{jm} e^{in\phi} = \left( \frac{\delta P}{c_s^2} + \frac{q \mu N^2 g}{\xi_r} \right) H_{jm} e^{im\phi},$$

$$\xi_\phi (r) = \frac{\xi_\phi (r)}{1 - q^2 \mu^2} \left( \frac{\partial H_{jm}}{\partial \theta} + \frac{mq \mu}{\sin \theta} H_{jm} \right) e^{im\phi},$$

$$\xi_\phi (r) = \frac{iq \xi_\phi (r)}{1 - q^2 \mu^2} \left( \frac{q \mu \partial H_{jm}}{\sin \theta} + \frac{mq \mu}{\sin \theta} H_{jm} \right) e^{im\phi},$$

where $c_s$ is the sound speed and $\xi_\phi (r) = \delta P(r)/(\rho c_s^2$).

Separating out the angular dependence, the fluid continuity equation and Euler equation (in the rotating frame) reduce to a set of coupled radial equations:

$$\frac{d}{dr} \delta P = - \frac{q}{c_s^2} \delta P + \rho \omega_r^2 - N^2 \xi_r,$$

$$\frac{d}{dr} (r^2 \xi_r) = \frac{q}{c_s^2} (r^2 \xi_r) - \frac{q}{c_s^2} \delta P + \frac{\lambda}{\omega_r^2} \delta P.$$

The properties of the Hough function have been extensively studied (Longuet-Higgins 1967). We have adopted the numerical approach of Bildsten et al. (1996) in our calculations. For given $q$ and $m$, the eigenvalue $\lambda$ is obtained by solving equation (5). To obtain the eigen-frequency $\omega_r$, we solve the radial equations (9)-(10) subjected to the following boundary conditions: (i) At the surface $r = R$, we have the standard requirement $\delta P = \delta P = - \rho \xi_r = 0$; (ii) At a small $r$ close to the center, regularity requires $\delta P/\rho = Y_0 r^\beta$, $\xi_r = (\beta/\omega_r^2) Y_0 r^{\beta - 1}$ ($Y_0$ is a constant), where $\beta$ is determined from $\beta (1 + \lambda) = \beta$, or $\beta = (\sqrt{1 + 4\lambda} - 1)/2$. Once $\omega_r$ is found, the actual rotation rate $\Omega_s$ is recovered from $\Omega_s = q \omega_r/2$.

Figure 2 shows the frequencies (in the inertial frame) of the $j = m = 2$ g modes for model AU and UU as functions of the spin frequencies (These modes have $l = 2$ in the $\Omega_s = 0$ limit). Note that as $\Omega_s$ increases, $\omega_r$ increases while $\omega_r$ decreases.

An approximate solution to the radial equation can be obtained by the WKB analysis. Substituting $r^2 \xi_r \propto e^{ikr}$ and $\delta P/\rho \propto e^{ikr}$ into the radial equations, we obtain the dispersion relation for g-modes:

$$k^2 \sim \frac{\lambda}{\omega_r^2} (N^2 - \omega_r^2).$$

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This indicates that \( \omega_r \propto \sqrt{\lambda} \). In fact, with the condition \( \int_{r_1}^{r_2} k \, dr = (n + C) \pi \) \((n)\) is the number of radial node, \( C \) is a constant of order unity, \( r_1 \) and \( r_2 \) are the inner and outer turning points where \( \omega_r = N \), we can write
\[
\nu_r \simeq \frac{234 \text{ Hz}}{n + C} \left( \frac{\lambda}{6} \right)^{1/2} \left( \frac{\mu \, d \ln r}{300 \text{ Hz}} \right).
\] (12)

Thus we have, for the \( j = 2 \) modes, \( \omega_r/\omega^{(0)} = \sqrt{\lambda/6} \) and \( \Omega_s/\omega^{(0)} = q \sqrt{\lambda/24} \), where \( \omega^{(0)} \) is the corresponding mode frequency at zero rotation. The WKBJ results are also plotted in Fig. 2, showing remarkable agreement with the actual numerical result. Clearly, \( \omega_i = 0 \) occurs at \( q = 1 \), which corresponds to \( \lambda = 11.13 \) from the solution of the \( m = 2 \) Laplace equation (eq. [3]). Thus, the onset of instability occurs when the ratio of spin frequency \( \Omega_s \) and the nonrotating mode frequency \( \omega^{(0)} \) satisfies
\[
\frac{\Omega_s}{\omega^{(0)}} = \left( \frac{11.13}{24} \right)^{1/2} = 0.68
\] (13)

The actual numerical calculations give slightly smaller critical spin frequency (see Fig. 2). For \( \Omega_s \equiv \Omega_s/\omega^{(0)} < 1 \), in the WKBJ limit, the \( j = m = 2 \) mode frequency can be fitted to the following analytic expression to within 1\%:
\[
\frac{\omega_i}{\omega^{(0)}} = 1 - \frac{5}{3} \Omega_s + \frac{13}{42} \Omega_s^2 - 0.064 \Omega_s^3.
\] (14)

Figure 3 gives two examples of the radial eigenfunctions. We see that the mode is concentrated in the outer region of the stellar core \((0.6 \leq r/R \leq 0.9)\), and rotation tends to “squeeze” the mode into that region. This is because rotation increases \( \omega_r \) and narrows the region in which the wave can propagate \((\omega_r < N)\). The displacements \( \xi_r \) and \( \xi_\perp \) can be quite significant near the surface, but the associated energy density (proportional to \( \rho \xi_\perp^2 \)) is significant only in the propagation zone.

We can similarly calculate the frequencies of higher-order g-modes. The results are even closer to the WKBJ limit (as expected). These modes tend to be unstable at lower spin frequencies. Similarly, the higher-\( m \) modes are also destabilized by gravitational waves. We shall not be concerned with these higher-order modes since they have smaller growth rates due to gravitational radiation and higher damping rates due to viscosities as compared to the \( m = 2, g_1 \) mode (see Section 4).

Finally, we discuss the validity of the two approximations adopted in our calculations. As far as the mode frequency is concerned, the Cowling approximation is an excellent approximation for g-modes: For nonrotating stars, our mode frequencies agree with those of Lai (1994) (where the full equations are solved) to within 2\%. The Cowling approximation does introduce an error (up to a factor of 80\%) to the quadrupole moment of the mode (see §4). However, the uncertainty in the nuclear equation of state (particularly the nuclear symmetry energy which determines the g-mode properties) is presumably larger. As noted before, the conditions \( \omega_{\nu} \ll N \) and \( \Omega_s \ll N \), needed for the validity of the traditional approximation, are satisfied in the propagation zone of the mode, although they are clearly violated near the stellar center and surface. Our numerical results indicate that for the \( g_1 \) mode, the inequality \( |\xi_\perp| > |\xi_r| \) is achieved only for \( r/R \gtrsim 0.6 - 0.7 \) (depending on the rotation rate), and \(|\xi_\perp| \gg |\xi_r| \) is achieved only near the stellar surface [indeed \( \xi_\perp/\xi_r = GM/(R^3 \omega_r^2) \gg 1 \) at \( r = R \)]. Thus the traditional approximation is not rigorously justified for the \( g_1 \) mode, although it becomes increasingly valid for higher-order modes. Clearly, without an exact, full cal-

\[\text{Figure 2.}\] The frequencies of the \( j = m = 2 \) \( g_1 \) modes in the inertial frame for model AU and UU as functions of the spin frequency. The onset of instability occurs at \( \nu_i,\text{mode} = 0 \). The solid lines are numerical results obtained by solving the radial equations; The dashed lines correspond to the WKBJ limits.

\[\text{Figure 3.}\] Density perturbation \( \delta \rho \) (in arbitrary unit) as a function of radius for the \( j = m = 2 \) \( g_1 \) mode of model UU. The solid line is for \( \nu_s = 0 \), and the dashed line is for \( \nu_s \simeq 140 \text{ Hz} \) (corresponding to \( q = 1.5 \)).

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4 DRIVING AND DAMPING OF THE MODES

The condition that \( \omega \) changes sign is only a necessary condition for the CFS instability. For the instability to be of interest, it must grow sufficiently fast and that the growth rate must overcome the possible damping rates due to viscosities. In this section we determine the mode growth rate and damping rate.

The timescale \( \tau \) associated with any dissipative process is given by

\[
\frac{1}{\tau} = \frac{1}{2E} \left( \frac{dE}{dt} \right),
\]

where \( E \) is the canonical mode energy in the rotating frame (Friedman & Schutz 1978a):

\[
E = \frac{1}{2} \int d^3r \left[ \rho \omega^2 \mathbf{\hat{\xi}} \cdot \mathbf{\hat{\xi}} + \left( \frac{\delta P}{\rho} - \delta \Phi \right) \delta \rho^* \right. \\
\left. + (\nabla \cdot \mathbf{\hat{\xi}}) \mathbf{\hat{\xi}}^* \cdot (\nabla \rho - \mathbf{\hat{\xi}} \nabla \rho) \right].
\]

We normalize the mode amplitude according to

\[
\int d^3r \rho \mathbf{\hat{\xi}} \cdot \mathbf{\hat{\xi}} = \int_0^R dr \rho (\mathbf{\hat{\xi}} \cdot \mathbf{\hat{\xi}} + \mathbf{\hat{\xi}} \nabla \mathbf{\hat{\xi}}) = 1,
\]

where \( \Lambda \) is evaluated using

\[
\Lambda = 2\pi \int_{-1}^1 \frac{d\mu}{(1 - q^2\mu^2)^2} \left[ (1 + q^2\mu^2)(1 - \mu^2) \right] \frac{dH_{\mu m}}{d\mu} \left[ (1 - q^2\mu^2)H_{\mu m} - 4mqH_{\mu m} \right] \left[ \frac{\partial H_{\mu m}}{\partial \mu} \right],
\]

and the Hough function is normalized via \( 2\pi \int d\mu H_{\mu m}^2 = 1 \). The g-mode energy is dominated by the first term in eq. (19), which gives

\[
E \approx \frac{1}{2} \omega_0^2 (1 + \varepsilon),
\]

with

\[
\varepsilon = \frac{1}{\omega_0^2} \int dr \rho \frac{\delta P(r)}{\rho} \delta \rho(r).
\]

The dissipation due to gravitational waves is calculated from the multipole radiation formula derived by Lindblom et al. (1998) using the formalism of Thorne (1980):

\[
\left( \frac{dE}{dt} \right)_{gw} = -\omega_0 \sum_{l=2}^\infty N_l \omega_l^{l+1} |(\delta D_{lm})|^2 + |\delta J_{lm}|^2.
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and the Hough function is normalized via \( 2\pi \int d\mu H_{\mu m}^2 = 1 \). The g-mode energy is dominated by the first term in eq. (19), which gives

\[
E \approx \frac{1}{2} \omega_0^2 (1 + \varepsilon),
\]
The gravitational radiation damping rate \( \tau_{gr}^{-1} \) as a function of spin frequency for the two g-modes depicted in Fig. 2. Positive \( \tau_{gr}^{-1} \) (for small \( \nu_s \)) implies damping, while negative \( \tau_{gr}^{-1} \) implies driving of the mode by gravitational radiation. At \( \nu_s = 0 \), we have \( \tau_{gr}^{-1} \simeq 0.04 \text{ yr}^{-1} \) for model AU and \( \tau_{gr}^{-1} \simeq 6 \times 10^{-5} \text{ yr}^{-1} \) for model UU.

The dissipation of mode energy due to viscosities can be calculated from (Ipser & Lindblom 1991)

\[
\left( \frac{dE}{dt} \right)_{\text{visc}} = - \int d^3x \left( 2\eta \delta \sigma_{ab} \delta \sigma_{ab}^* + \zeta |\delta \sigma|^2 \right),
\]

where \( \delta \sigma_{ab} \) and \( \delta \sigma \) are the shear and expansion of the perturbation, respectively:

\[
\delta \sigma_{ab} = \frac{i \omega_c}{2} \left( \nabla_a \xi_b + \nabla_b \xi_a - \frac{2}{3} \delta_{ab} \nabla \cdot \xi \right),
\]

\[
\delta \sigma = i \omega_c \nabla \cdot \xi.
\]

The shear viscosity resulting from neutron-neutron scattering in the normal liquid core is given by (Flowers & Itoh 1979; Cutler & Lindblom 1987):

\[
\eta = 1.1 \times 10^{13} T_g^{-2} \left( \frac{\rho}{\rho_{\text{nuc}}} \right)^{9/4} \text{g/(cm s)},
\]

where \( T_g \) is the temperature in units of 10⁹ K, and \( \rho_{\text{nuc}} = 2.8 \times 10^{14} \text{ g cm}^{-3} \). The bulk viscosity results from the tendency to achieve chemical equilibrium via the modified URCA process during the perturbation (Sawyer 1989):

\[
\zeta = 4.8 \times 10^{24} T_g^2 \left( \frac{\rho}{\rho_{\text{nuc}}} \right)^2 \left( \frac{\omega_c}{1 \text{ s}^{-1}} \right)^2 \text{g/(cm s)}.
\]

We have not been able to derive an explicit expression for \( (dE/dt)_{\text{visc}} \) from the g-mode wavefunctions to facilitate convenient numerical computation of the viscous damping rate. We shall therefore be contented with order of magnitude estimates. The mode damping rate due shear viscosity is approximately given by the rate at which momentum diffuses across a mode wavelength:

\[
\frac{1}{\tau_{\text{shear}}} \sim \frac{\eta}{\rho L^2} \simeq 10^{-2} L_1^{-2} T_g^{-2} \left( \frac{\rho}{\rho_{\text{nuc}}} \right)^{5/4} \text{yr}^{-1},
\]

where \( L = 10 L_1 \) km is the characteristic wavelength of the g-mode. Similarly, the mode damping rate due to bulk viscosity is estimated as

\[
\frac{1}{\tau_{\text{bulk}}} \sim 0.6 L_1^{-2} T_g^6 \left( \frac{\rho}{\rho_{\text{nuc}}} \right) \frac{\nu_c}{150 \text{Hz}}^{-2} \text{yr}^{-1},
\]

(35)

(where \( \nu_c \) is the mode frequency in the rotating frame). The length scale of the mode is somewhat less than 10 km, as comparison with the numerical results for nonrotating g-modes (Lai 1994) indicates that eq. (34) (with \( L_1 = 1 \)) underestimates \( 1/\tau_{\text{shear}} \) by a factor of ten, while eq. (35) underestimates \( 1/\tau_{\text{bulk}} \) by a factor of three. When the star rotates faster than the mode frequency at zero rotation, the transverse wavelength of the mode decreases as \( \sim 2 R_f/q \) (for \( q \gg 1 \)), and the radial wavelength also tends to be smaller (see Fig. 3). For the rotation rate of interest in this paper (\( \Omega_b \) is at most a few times \( \omega_c \)), the mode wave function is only modestly changed by rotation, and we expect that rotation modifies the viscous damping rates by at most a factor of a few. Also note that the mode frequency (in the rotating frame) \( \nu_s \) increases with increasing spin frequency (e.g., for model AU, we have \( \nu_s = 296 \text{Hz} \) when \( \nu_s = 300 \text{Hz} \)), and this tends to reduce our estimate for \( \tau_{\text{bulk}} \). In any case, the uncertainty in our estimate of viscous damping rate is probably comparable to the uncertainty in our understanding of the microscopic viscosity of neutron star matter.

The total damping rate of the g-mode is given by

\[
\frac{1}{\tau} = \frac{1}{\tau_{gr}} + \frac{1}{\tau_{\text{shear}}} + \frac{1}{\tau_{\text{bulk}}},
\]

(36)

The mode is unstable when \( \tau^{-1} < 0 \). Despite the uncertainty in our estimates for \( \tau_{\text{shear}} \) and \( \tau_{\text{bulk}} \), by comparing Fig. 4 with eqs. (24) and (25), we can readily conclude that for \( T \gtrsim 10^9 \text{ K} \) the g-mode instability is suppressed by the bulk viscosity, while for \( T \lesssim 10^9 \text{ K} \) it is likely to be suppressed by the shear viscosity. For a narrow range of neutron star temperatures, around a few times \( 10^8 \text{ K} \) to \( 10^9 \text{ K} \) (depending on the detail of viscous dissipation), the g₁ mode may be unstable due to gravitational radiation. To achieve this instability, the rotation rate of the star must be at least a factor of two to three greater than the mode frequency at zero rotation (see Fig. 4).

So far we have only focused on the \( j = m = 2 \) g₁ mode. As the order of the g-modes increases, both the mode frequency and \( \delta D_{lm} \) decreases, which tends to reduce \( |\tau_{gr}^{-1}| \). Also, the viscous mode damping rate increases with increasing mode order. Therefore we expect that the CFS instabilities of higher-order g-modes are completely suppressed by viscous dissipations.

### 5 Viscosity Driven Instability?

For an incompressible rotating Maclaurin spheroid, the \( l = -m = 2 \) f-mode becomes unstable due to viscosity at the same point as the CFS instability of the \( l = m = 2 \) mode (Chandrasekhar 1987). The \( l = -m > 2 \) viscous instability occurs at higher rotation rates. Recent studies indicate the instability occurs only for neutron stars with very stiff equation of state (Bonazzola et al. 1997).

The viscous instability sets in when the mode frequency \( \omega_c \) in the rotating frame goes through zero. Figure 5 shows

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of changing the orbital decay rate (Reisenegger & Goldreich 1994; Lai 1994). Rotation may significantly change the strength of the resonance (Lai 1997), giving rise to measurable effect in the gravitational wave form (Ho & Lai 1999). In another situation, when the neutron star is surrounded by a disk, tidal coupling between the stellar g-modes and disk density waves can drive the the unstable modes, which may affect the stellar rotation. By contrast, r-modes possess no mass quadrupole moment, and the coupling with disk is mediated only through higher order multipole moments and thus is expected to be weak. We wish to study some of these issues in a future paper.

6 DISCUSSION

We have shown in this paper that g-modes in an inviscid neutron star are susceptible to the CFS instability when the rotation rate is comparable to or greater than the mode frequencies. However, viscous dissipation tends to suppress the instability except possibly in a narrow range of temperatures around 10^{10} K.

Comparing to f-modes, the g-mode instability has the advantage that it sets in at a lower rotation rate. Comparing to r-modes, the g-mode instability is much weaker. Calculations (Lindblom et al. 1998; Andersson et al. 1998a) indicate that at spin frequency of 300 Hz, the growth rate of r-mode is approximately 10^3 yr^{-1}, more than three orders of magnitude larger than 1/\tau_{\nu_{s}} for the g-modes studied in this paper. Thus for isolated neutron stars, the r-mode instability plays an much more important role than the g-mode in determining the spin evolution of the stars (Owen et al. 1998; Andersson et al. 1998b).

However, one should not dismiss possible importance of neutron star g-modes and their stabilities in other situations. In a merging neutron star binary, g-modes can be resonantly excited by the companion star, with the consequence

$\omega \propto \sqrt{\lambda}$

We therefore conclude that viscosity-driven instability is absent for g-modes.

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Note that as the rotation rate approaches the maximum value, our calculation of the mode frequency breaks down since the rotational distortion becomes significant.
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