A Liouville type theorem for some conformally invariant fully nonlinear equations

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Following the approach in our earlier paper [2] and using the gradient estimates developed in [2] and [3], we give another Liouville type theorem for some conformally invariant fully nonlinear equations. Various Liouville type theorems for conformally invariant equations have been obtained by Obata, Gidas-Ni-Nirenberg, Caffarelli-Gidas-Spruck, Viaclovsky, Chang-Gursky-Yang, and Li-Li. For these, as well as for related works, see [2] and the references therein.

For \( n \geq 3 \), let \( S_{n \times n} \) be the set of \( n \times n \) real symmetric matrices, \( S_{n \times n}^+ \subset S_{n \times n} \) be the set of positive definite matrices, and let \( O(n) \) be the set of \( n \times n \) real orthogonal matrices.

For \( 1 \leq k \leq n \), let
\[
\sigma_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}, \quad \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n,
\]
denote the \( k \)-th symmetric function, and let \( \Gamma_k \) denote the connected component of \( \{ \lambda \in \mathbb{R}^n \mid \sigma_k(\lambda) > 0 \} \) containing the positive cone \( \{ \lambda \in \mathbb{R}^n \mid \lambda_1, \ldots, \lambda_n > 0 \} \). It is known that
\[
\Gamma_n = \{ \lambda \in \mathbb{R}^n \mid \lambda_1, \ldots, \lambda_n > 0 \}, \quad \Gamma_1 = \{ \lambda \in \mathbb{R}^n \mid \lambda_1 + \cdots + \lambda_n > 0 \},
\]
\[
\Gamma_k = \{ \lambda \in \mathbb{R}^n \mid \sigma_1(\lambda) > 0, \ldots, \sigma_k(\lambda) > 0 \},
\]

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Γ_k is a convex cone with its vertex at the origin with the properties

\[ \Gamma_n \subset \cdots \subset \Gamma_2 \subset \Gamma_1, \]

\[ \frac{\partial \sigma_k}{\partial \lambda_i} > 0 \text{ in } \Gamma_k, \ 1 \leq i \leq n, \]

\[ \sigma_k^+ \text{ is concave in } \Gamma. \]

For a positive \( C^2 \) function \( u \), let

\[ A^u := -\frac{2}{n-2}u^{-\frac{n+2}{2}} \nabla^2 u + \frac{2n}{(n-2)^2}u^{-\frac{2n}{n-2}} \nabla u \otimes \nabla u - \frac{2}{(n-2)^2}u^{-\frac{2n}{n-2}}|\nabla u|^2 I, \]

where \( I \) is the \( n \times n \) identity matrix.

Assume \( U \subset S^{n \times n} \) is an open set satisfying

\[ O^{-1}UO = U, \quad \forall \ O \in O(n), \quad (1) \]

and

\[ U \cap \{ M + tN \mid 0 < t < \infty \} \text{ is convex} \quad \forall \ M \in S^{n \times n}, N \in S_+^{n \times n}, \quad (2) \]

and

\[ \Gamma_U := \{ \lambda(M) \mid M \in U \} \subset \Gamma_k, \quad \text{for some } k > \frac{n+1}{2}, \quad (3) \]

where \( \lambda(M) \) denotes the eigenvalues of \( M \).

Let \( F \in C^2(U) \) satisfy

\[ F(O^{-1}MO) = F(M), \quad \forall \ M \in U, \ O \in O(n), \quad (4) \]

\[ 0 \text{ does not belong to } F^{-1}(1), \quad (5) \]

\[ (F_{ij}(M)) > 0, \quad \forall \ M \in U, \quad (6) \]

\[ F \text{ is locally concave in } U, \quad (7) \]

and, for some \( 0 < \gamma \leq 1, \)

\[ \sum_{i,j=1}^{n} F_{ij}(M)M_{ij} \leq \frac{1}{\gamma}|M|^{1-\gamma} \sum_{i=1}^{n} F_{ii}(M), \quad \forall \ M \in U, F(M) = 1, |M| \geq 1, \quad (8) \]

where \( F_{ij}(M) := \frac{\partial F}{\partial M_{ij}}(M). \)

We establish in this paper the following Liouville type theorem.
Theorem 1 For \( n \geq 3 \), let \( U \subset S^{n \times n} \) be an open set satisfying (2), (3) and (4), and let \( F \in C^2(U) \) satisfy (4), (5), (6) and (8). Let \( u \in C^4(\mathbb{R}^n) \) be a positive solution of
\[
F(A^u) = 1, \quad A^u \in U, \quad \text{on } \mathbb{R}^n.
\]
Then for some \( \bar{x} \in \mathbb{R}^n \), and some positive constants \( a \) and \( b \) satisfying
\[
2b^2a^{-2}I \in U \quad \text{and} \quad F(2b^2a^{-2}I) = 1,
\]
\[
\tag{9}
\]
\[
\]
Remark 1 In Theorem 1, if \( F \) is in \( C^{2,\beta}(U) \) for some \( \beta \in (0, 1) \), then, since the equation is elliptic, any positive \( C^2 \) solution \( u \) is in fact in \( C^{4,\beta} \).

We give a consequence of Theorem 1.

Let \( \Gamma \subset \mathbb{R}^n \) be an open convex cone with its vertex at the origin such that
\[
\Gamma_n \subset \Gamma \subset \Gamma_k, \quad \text{for some } k > \frac{n+1}{2},
\]
and
\[
\Gamma \text{ is symmetric in the } \lambda_i.
\]
Let
\[
f \in C^2(\Gamma) \cap C^0(\overline{\Gamma}) \text{ be concave and symmetric in the } \lambda_i.
\]
In addition, we assume that
\[
f = 0 \text{ on } \partial \Gamma; \quad f_{\lambda_i} > 0 \text{ on } \Gamma \forall 1 \leq i \leq n,
\]
and
\[
\lim_{s \to \infty} f(s\lambda) = \infty, \quad \forall \lambda \in \Gamma.
\]
By (14) and (13), there exists a unique \( \bar{b} > 0 \) such that
\[
\tag{16}
\]
where \( e = (1, \cdots, 1) \).
Corollary 1  For $n \geq 3$, let $(f, \Gamma)$ satisfy (10), (11), (12), (13), (14) and (15), and let $u \in C^4(\mathbb{R}^n)$ be a positive solution of

$$f(\lambda(Au)) = 1, \quad \lambda(Au) \in \Gamma, \quad \text{on} \ \mathbb{R}^n.$$ 

Then for some $\bar{x} \in \mathbb{R}^n$, and some positive constant $a$,

$$u(x) \equiv \left(1 + \frac{a}{\frac{1}{2}a^2b|x - \bar{x}|^2}\right)^{-\frac{n-2}{2}}, \quad \forall \ x \in \mathbb{R}^n.$$ 

Proof of Theorem 1. Since (3) implies the superharmonicity of the positive function $u$ on $\mathbb{R}^n$, we have $\liminf_{|x| \to \infty} |x|^{n-2}u(x) > 0$. Let $w(x) = \frac{1}{|x|^{n-2}}u(x)$ for $x \in \mathbb{R}^n \setminus \{0\}$. Then $w$ is regular at $\infty$, $\liminf_{|x| \to 0} w(x) > 0$, and $w$ satisfies

$$F(A^w) = 1, \quad A^w \in U, \quad \text{on} \ \mathbb{R}^n \setminus \{0\}.$$ 

Let $\xi(x) = \frac{n-2}{2}w(x)^{-\frac{n-2}{2}}$. Then, for some positive constant $C_1$,

$$0 < \xi < C_1 \quad \text{on} \ B_2 \setminus \{0\}. \quad (17)$$ 

By (3) and lemma 6.3 in [4], $\lambda(D^2 \xi(x)) \in \Gamma_k$ for $x \in B_2 \setminus \{0\}$. Let $P$ be any hyperplane which intersects $B_1$ but does not pass through the origin, and let $\xi_P$ be the restriction of $\xi$ on $P$. Then

$$\lambda(D^2 \xi_P) \in \Gamma_{k-1} \subset \mathbb{R}^{n-1}, \quad \text{on} \ P \cap B_2,$$

where $D^2 \xi_P$ denotes $(n - 1) \times (n - 1)$ Hessian of $\xi_P$, and $\lambda(D^2 \xi_P)$ denotes the eigenvalues of $D^2 \xi_P$. Here we have used the following property of $\Gamma_k$: If $\lambda(M) \in \Gamma_k \subset \mathbb{R}^n$, then $\lambda(M) \in \Gamma_{k-1} \subset \mathbb{R}^{n-1}$ where $\hat{M}_{ij} = M_{ij}$ for $i \leq j \leq n - 1$. Since $k \geq \frac{n+1}{2}$, we have $k - 1 > \frac{n-1}{2}$. As in [4], by using theorem 2.7 in [4], we have, for some constants $\alpha \in (0, 1)$ (depending only on $n$ and $k$) and $C > 0$ (depending only on $n$, $k$ and $C_1$), that

$$\|\xi\|_{C^{\alpha}(P \cap B_2)} \leq C. \quad (18)$$

For any $x, y \in B_1 \setminus \{0\}$, we pick $z_i \in \mathbb{R}^n$ such that $z_i \to 0$ and the line going through $x$ and $y + z_i$ does not go through the origin. Then $x$ and $y + z_i$ lies on some hyperplane $P_i$ which does not go through the origin. Thus, by (18),

$$|\xi(x) - \xi(y + z_i)| \leq C|x - (y + z_i)|^\alpha.$$
for some constant $C$ depending only on $n$, $k$ and $C_1$. Sending $i$ to infinity, we have

$$|\xi(x) - \xi(y)| \leq C|x - y|^\alpha.$$ 

Therefore $\xi$ can be extended to a function in $C^\alpha(B_1)$.

We distinguish into two cases.

**Case 1.** $\xi(0) = 0$.

**Case 2.** $\xi(0) > 0$.

In Case 1, $\lim_{|x| \to \infty} (|x|^{n-2}u(x)) = \infty$. For every $x \in \mathbb{R}^n$, as in the proof of lemma 2.1 in [4], there exists $\lambda_0(x) > 0$ such that $u_{x,\lambda}(y) := (\frac{\lambda}{|y - x|})^{n-2}u(x + \frac{\lambda^2(y - x)}{|y - x|^2}) \leq u(y), \forall 0 < \lambda < \lambda_0(x), |y - x| \geq \lambda$.

Set, for $x \in \mathbb{R}^n$,

$$\bar{\lambda}(x) = \sup\{\mu > 0 \mid u_{x,\lambda}(y) \leq u(y), \text{ for all } |y - x| \geq \lambda, 0 < \lambda \leq \mu\}.$$

**Lemma 1** $\bar{\lambda}(x) = \infty$ for all $x \in \mathbb{R}^n$.

**Proof of Lemma** [4]. If $\bar{\lambda}(\bar{x}) < \infty$ for some $\bar{x} \in \mathbb{R}^n$. Making a translation, we may assume without loss of generality that $\bar{x} = 0$, and we still have $\lim_{|x| \to \infty} (|x|^{n-2}u(x)) = \infty$. Thus, there exists some $R > \bar{\lambda} + 9$ (we use notation $\bar{\lambda} = \bar{\lambda}(0)$ such that $u_\lambda(y) < u(y), \forall 0 < \lambda \leq \bar{\lambda} + 2, |y| \geq R$,

where we have used notation $u_\lambda = u_{0,\lambda}$.

By the definition of $\bar{\lambda}$,

$$u_\lambda(y) \leq u(y), \quad \forall |y| \geq \bar{\lambda}.$$

Let $w(t) := tu + (1 - t)u_\lambda, 0 \leq t \leq 1$. Then, as in the proof of lemma 2.1 in [4],

$$L(u - u_\lambda) = 0, \quad \text{in } \mathbb{R}^n \setminus B_\lambda,$$

where

$$L = a_{ij}(y)\partial_{ij} + b_i(y)\partial_i + c(y),$$

$$a_{ij} = -\frac{2}{n-2} \int_0^1 w_t^{\frac{n+2}{n}} F_{ij}(A^{w_t})dt,$$

and $b_i$ and $c$ are continuous functions.
Using the Hopf Lemma and the strong maximum principle as in the proof of lemma 2.1 in [2], we have
\[(u - u_{\lambda})(y) > 0, \quad \text{in } \mathbb{R}^n \setminus \overline{B}_{\lambda},\]
and
\[\frac{\partial(u - u_{\lambda})}{\partial r} \bigg|_{\partial B_{\lambda}} > 0,\]
where \(\frac{\partial}{\partial r}\) denotes the outer normal differentiation.

The following argument is similar to the one used in the proof of lemma 2.2 in [4]. Since \(\partial B_{\lambda}\) is compact, \(\frac{\partial(u - u_{\lambda})}{\partial r} \bigg|_{\partial B_{\lambda}}\) has a positive lower bound. Using the \(C^1\) regularity of \(u\), we can find some \(0 < \delta < 1\) such that
\[\frac{\partial(u - u_{\lambda})}{\partial r}(y) > 0, \quad \forall \, \lambda \leq \lambda \leq \lambda + \delta, \lambda \leq |y| \leq \lambda + \delta.\]
Since \((u - u_{\lambda})(y) = 0\) for \(|y| = \lambda\), the above implies
\[u_{\lambda}(y) \leq u(y), \quad \forall \, \lambda \leq \lambda \leq \lambda + \delta, \lambda \leq |y| \leq \lambda + \delta.\]
Since \((u_{\lambda} - u)(y) < 0\) for \(\lambda + \delta \leq |y| \leq R\), and since the set is compact, there exists \(\epsilon \in (0, \delta)\) such that
\[u_{\lambda}(y) < u(y), \quad \forall \, \lambda \leq \lambda \leq \lambda + \epsilon, \lambda + \delta \leq |y| \leq R.\]
Here we have used the the continuity of \(u\).

We have proved, for the \(\epsilon\) above, that
\[u_{\lambda}(y) \leq u(y), \quad \forall \, \lambda \leq \lambda \leq \lambda + \epsilon, \, |y| \geq \lambda.\]
This violates the definition of \(\lambda\). Lemma 1 is established.

It follows from Lemma 1 that
\[u_{x,\lambda}(y) \leq u(y), \quad \forall \, x \in \mathbb{R}^n, \, 0 < \lambda < \infty, \, |y - x| \geq \lambda.\]
This, together with some calculus lemma (see, e.g., lemma 11.2 in [4]), implies that \(u\) is a constant on \(\mathbb{R}^n\), thus \(A u \equiv 0\). This is impossible because of (5). We have ruled out Case 1.

In Case 2, there exists some constant \(0 < \delta < \frac{1}{20}\) such that
\[\delta \leq w \leq \frac{1}{\delta}, \quad \text{on } B_{10\delta}.\]  
(20)
Lemma 2

\[ \limsup_{|x| \to 0} (|x| |\nabla w(x)|) < \infty. \]

Proof of Lemma 2. For any \(0 < r < 5\delta\), let \(v(y) := w(ry)\) for \(0 < |y| < 2\). Then \(v\) satisfies

\[ F(r^{-2} A^v) = 1, \quad A^v \in U, \quad \text{on } B_2 \setminus \{0\}. \]  \tag{21}

For any \(x \in B_\frac{3}{2} \setminus B_\frac{1}{2}\), as in the proof of lemma 2.1 in [4], there exists \(\lambda_0(x) \in (0, \frac{1}{5})\) such that

\[ v_{x,\lambda}(y) := \left( \frac{\lambda}{|y - x|} \right)^{n-2} v(x + \frac{\lambda^2(y - x)}{|y - x|^2}) \leq v(y), \quad \forall \ y \in (B_2 \setminus B_\frac{2}{3}) \setminus B_\lambda(x), \ 0 < \lambda \leq \lambda_0(x). \]

Set, for \(x \in B_\frac{3}{2} \setminus B_\frac{1}{2}\),

\[ \bar{\lambda}(x) = \sup \{ \mu > 0 \mid v_{x,\lambda}(y) \leq v(y), \ \forall \ y \in (B_2 \setminus B_\frac{2}{3}) \setminus B_\lambda(x), \ 0 < \lambda \leq \mu \}. \]

Using the Hopf Lemma and the strong maximum principle as in the proof of lemma 2.1 in [3], and using the argument in Lemma 1, we know that stopping at \(\bar{\lambda}(x)\) is due to a boundary touching, i.e., there exists some \(y_0 \in \partial (B_2 \setminus B_\frac{1}{2})\) such that \(v_{x,\bar{\lambda}(x)}(y_0) = v(y_0)\), i.e.,

\[ \left( \frac{\bar{\lambda}(x)}{|y_0 - x|} \right)^{n-2} w(rx + \frac{\lambda^2 r(y_0 - x)}{|y_0 - x|^2}) = w(r y_0), \]

from which we deduce, using (21), that

\[ \bar{\lambda}(x)^{n-2} = |y_0 - x|^{n-2} \frac{w(r y_0)}{w(r x + \frac{\lambda^2 r(y_0 - x)}{|y_0 - x|^2})} \geq \delta^2 |y_0 - x|^{n-2} \geq 4^{2-n} \delta^2. \]

Thus we have shown that for any \(x \in B_\frac{3}{2} \setminus B_\frac{1}{2}\) and any \(0 < \lambda < \frac{1}{4} \delta^{-2}\) we have

\[ v_{x,\lambda}(y) \leq v(y), \quad \forall \ y \in B_2 \setminus B_\frac{1}{2}, \ |y - x| \geq \lambda. \]

This and some calculus lemma (see lemma 1 in [3]) imply, for some constant \(C\) depending only on \(\delta\), that

\[ |\nabla v(y)| \leq C v(y) \quad \forall \ |y| = 1, \]
i.e.,

$$|\nabla w(ry)| \leq C \frac{w(ry)}{r}, \quad \forall |y| = 1.$$  

Since this holds for all $0 < r < 5\delta$, we have

$$|z||\nabla w(z)| \leq Cw(z), \quad \forall 0 < |z| < 5\delta.$$  

Lemma 2 is established. □

Our next lemma provides estimates of the second derivatives of $w$ near the origin.

**Lemma 3**

$$\limsup_{|x| \to 0} \left( |x|^2 |\nabla^2 w(x)| \right) < \infty.$$  

**Proof of Lemma 3.** Let $\delta$ be as in the proof of Lemma 2, $0 < r < 5\delta$, and $v(y) := w(ry)$. Then $v$ satisfies (21), i.e.,

$$\tilde{F}(A^v) = r^2, \quad A^v \in \tilde{U}, \quad \text{on } B_2 \setminus \{0\},$$  

where $\tilde{U} := r^2U$ and $\tilde{F}(M) := r^2F(r^{-2}M), M \in \tilde{U}$. Clearly, $(\tilde{F}, \tilde{U})$ satisfies (1), (2), (3), (4), (6), (7) (with $(F, U)$ replaced by $(\tilde{F}, \tilde{U})$), and

$$\sum_{i,j=1}^n \tilde{F}_{ij}(M)M_{ij} \leq \frac{1}{\gamma} |M|^{1-\gamma} \sum_{i=1}^n \tilde{F}_{ii}(M), \quad \forall M \in \tilde{U}, \tilde{F}(M) = r^2, |M| \geq 1.$$  

We know from (20) and Lemma 2 that

$$v + |\nabla v| \leq C \quad \text{on } B_{\frac{3}{2}}^r \setminus B_{\frac{1}{2}}^r$$  

for some constant $C$ independent of $r$.

Following, with minor modification, the computation in the proof of theorem 1.6 in [2] (with $F$ there replaced by our $\tilde{F}$, $v$ there replaced by $-\frac{2}{n-2} \log v$ with our $v$, and keep in mind that $h$ there is a constant $r^2$; for some earlier works on second derivative estimates, see remark 1.13 in [2]), we obtain

$$|\nabla^2 v| \leq C \quad \text{on } \partial B_1$$  

for some constant $C$ independent of $r$. Lemma 3 follows immediately.
Since $w \in C^\alpha(B_1)$ and since we have proved that
\[
\limsup_{|x| \to 0}(|x||\nabla w(x)| + |x|^2|\nabla^2 w(x)|) < \infty,
\]
we can apply lemma 6.4 in [2] to obtain $\limsup_{|x| \to 0}(|x|^{1-\frac{\alpha}{2}}|\nabla w(x)|) < \infty$. In particular,
\[
\lim_{|x| \to 0}(|x||\nabla w(x)|) = 0.
\]

Now we are in a position to apply theorem 1.2 in [2] (with $u_{0,1}$ there being our $w$) to conclude that $u$ must be of the form (9). Theorem 1 is established.

\[\square\]

**Proof of Corollary 1.** Let
\[
U := \{M \in S^{n \times n} \mid \lambda(M) \in \Gamma\},
\]
and
\[
F(M) := f(\lambda(M)), \quad M \in U.
\]
To establish Corollary 1, we only need to verify that $(F, U)$ satisfies the hypothesis of Theorem 1. These are well known to people in the field, but for convenience of the reader, we provide some details. Since $\Gamma$ is an open subset of $\mathbb{R}^n$, $U$ is an open subset of $S^{n \times n}$. Since orthogonal conjugation does not change the set of eigenvalues and since $\Gamma$ is symmetric in the $\lambda_i$, we know that $U$ satisfies (1) and $F$ satisfies (4). Since $\Gamma_n \subset \Gamma$ and $\Gamma$ satisfies (11), we know that $\lambda + \mu = 2(\frac{\lambda + \mu}{2}) \in \Gamma$ for all $\lambda \in \Gamma$ and $\mu \in \Gamma_n$. For $M \in U$ and $N \in S^{n \times n}$, let $\lambda_n(M) \geq \cdots \geq \lambda_1(M)$ denote the eigenvalues of $M$, we know that
\[
\lambda_i(M) = \inf_{\text{dim } K = i, x \in X, ||x|| = 1} \sup (x'Mx), \quad 1 \leq i \leq n.
\]
Similar formula holds for $M + N$. Thus $\lambda_i(M + N) \geq \lambda_i(M)$ for all $1 \leq i \leq n$. Write $\lambda = (\lambda_1(M), \cdots, \lambda_n(M))$ and $\mu = (\lambda_1(M + N) - \lambda_1(M), \cdots, \lambda_n(M + N) - \lambda_n(M))$, then $\lambda \in \Gamma$ and $\mu \in \Gamma_n$, thus $\lambda + \mu = (\lambda_1(M + N), \cdots, \lambda_n(M + N)) \in \Gamma$, i.e., $M + N \in U$. So $U$ satisfies (2). Since $\Gamma_U = \Gamma$, (3) follows from (11). Clearly, (4) follows from $f(0) = 0$. Property (5) and (6) can be deduced from the concavity of $f$ in $\Gamma$ and the fact that $f_{\lambda_i} > 0$ in $\Gamma$ for every $1 \leq i \leq n$, see e.g., [1]. For all $\lambda \in \Gamma$ satisfying $f(\lambda) = 1$, we have, using the concavity of $f$ in $\Gamma$ and the convexity of $\Gamma$,
\[
1 = f(\bar{b}e) \leq f(\lambda) + \sum_i f_{\lambda_i}(\lambda)(\bar{b} - \lambda_i) = 1 + \sum_i f_{\lambda_i}(|\bar{b} - \lambda_i|).
\]
i.e.,
\[ \sum_i f_{\lambda_i}(\lambda) \lambda_i \leq \bar{b} \sum_i f_{\lambda_i}(\lambda). \]

This, after diagonalizing $M$ by an orthogonal conjugation, implies (8). Corollary 1 is established.

\[ \square \]

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