Braided Hom-Lie bialgebras

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Abstract

We introduce the new concept of braided Hom-Lie bialgebras which is a generalization of Sommerhäuser-Majid’s braided Lie bialgebras and Yau’s Hom-Lie bialgebras. Using this concept we give the unified product construction for Hom-Lie bialgebras which can be seen as a Hom-Lie version of Bespalov-Drabant’s cocycle cross product bialgebras. Some special cases of unified products such as crossed product and matched pair of braided Hom-Lie bialgebras are investigated. As an application, we solve the Agore-Militaru extending problem for Hom-Lie bialgebras by using some non-abelian cohomology theory. Furthermore, one dimensional flag extending structures for Hom-Lie bialgebras are also investigated.

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1 Introduction

In [24], Y. Sommerhäuser introduced the concept of braided Lie bialgebras (he call it Yetter-Drinfeld Hom-Lie algebra) to give a construction of symmetrizable Kac-Moody algebras. The theory of braided Lie bialgebras was also developed further by S. Majid in [18], where the bosonisation theorem for braided Lie bialgebras are proved. For the theory of Yetter-Drinfel’d Hopf algebra, see [8, 9, 23, 25, 28]. The notion of Hom-Lie bialgebras was introduced by D. Yau in [27] which is a generalization of Drinfeld’s classical Lie bialgebra.

On the other hand, the theory of unified product and extending structure for many types of algebras were well developed by Agore and Militaru in [1, 2, 3, 4, 5, 6]. Let \( A \) be a Lie (associative, Leibniz, Poisson, Jordan, etc.) algebra and \( E \) a vector space containing \( A \) as a subspace. The extending problem is to describe and classify all Lie (associative, Leibniz, etc.) algebras structures on \( E \) such that \( A \) is a subalgebra of \( E \). Until now, extending structures for Hom-algebra and Hom-bialgebra structures were not developed as well as the above mentioned algebra. Recently, extending structures for 3-Lie algebras, Lie bialgebras, infinitesimal bialgebras and Lie conformal superalgebras were studied in [30, 13, 31, 32, 34].

It is a natural question whether can we develop a unified product theory to solve the extending problem for Hom-Lie bialgebras. In this paper, we give a affirmation answer to this question. The first step is to give a well definition of braided Hom-Lie bialgebras.

The organization of this paper is as follows. In section 2 we review some basic facts and notations about Hom-Lie bialgebras and braided Hom-Lie bialgebras. In section 3 the definition of unified product for braided Hom-Lie bialgebras is introduced. We give the necessary and sufficient conditions for a unified product to form Hom-Lie bialgebras. In the last section we study some applications of unified products, which include extending problem for braided Hom-Lie bialgebras and Hom-Lie bialgebras. We also study the flag extending systems.

Throughout this paper, all Hom-Lie algebras are assumed to be over an algebraically closed field \( k \) of characteristic different from 2 and 3. The space of linear maps from \((V, \alpha_V)\) to \( W \) is denoted by \( \text{Hom}(V, W) \). The identity map of a Hom-vector space \((V, \alpha_V)\) is denoted by \( \text{id}_V : V \to V \) or simply by \( \text{id} : V \to V \). The twisting maps \( \tau : V \otimes V \to V \otimes V \), \( \tau_{12}, \tau_{23} : V \otimes V \otimes V \to V \otimes V \otimes V \) are denoted by

\[
\tau(x \otimes y) = y \otimes x, \quad \tau_{12}(x \otimes y \otimes z) = y \otimes x \otimes z, \quad \tau_{23}(x \otimes y \otimes z) = x \otimes z \otimes y.
\]

2 Preliminaries

Definition 2.1 ([12]). A Hom-Lie algebra is a triple \((H, [\cdot, \cdot], \alpha_H)\) where \( H \) a vector space equipped with a skew-symmetric bracket \([\cdot, \cdot] : H \otimes H \to H\) and a linear map \( \alpha_H : H \to H \) satisfying the following Hom-Jacobi identity:

\[
[x, y] = -[y, x], \quad (1) \\
[\alpha_H(x), [y, z]] + [\alpha_H(y), [z, x]] + [\alpha_H(z), [x, y]] = 0, \quad (2)
\]
for all \(x, y, z \in H\). A Hom-Lie algebra is called a multiplicative Hom-Lie algebra if \(\alpha_H\) is an algebra homomorphism, i.e., for all \(x, y \in H\), \(\alpha_H([x, y]) = [\alpha_H(x), \alpha_H(y)]\). A Hom-Lie algebra is called a regular Hom-Lie algebra if \(\alpha_H\) is an algebra automorphism.

By skew-symmetry of the bracket map, the Hom-Jacobi identity is equivalent to

\[
[\alpha_H(x), [y, z]] = [[x, y], \alpha_H(z)] + [\alpha_H(y), [x, z]].
\]

(3)

For an element \(x\) in a Hom-Lie algebra \((H, [\cdot, \cdot], \alpha_H)\) and \(n \geq 2\), define the adjoint map \(\text{ad}_x: H^\otimes n \to H^\otimes n\) by

\[
\text{ad}_x(y_1 \otimes \cdots \otimes y_n) := \sum_{i=1}^n \alpha_H(y_1) \otimes \cdots \otimes \alpha_H(y_{i-1}) \otimes [x, y_i] \otimes \alpha_H(y_{i+1}) \cdots \otimes \alpha_H(y_n).
\]

(4)

**Definition 2.2.** A Hom-Lie coalgebra is a Hom-vector space \((H, \alpha_H)\) equipped with a linear map \(\delta: H \to H \otimes H\), called a cobracket, satisfying the co-anticommutativity and the co-Jacobi identity:

(\(\text{CL}1\)) \(\delta(x) = -\tau \delta(x)\),

(\(\text{CL}2\)) \((\alpha_H \otimes \delta)\delta(x) = (\delta \otimes \alpha_H)\delta(x) + \tau_{12}(\alpha_H \otimes \delta)\delta(x)\).

We would like to use the sigma notation \(\delta(x) := \sum x_1 \otimes x_2\) for all \(x \in H\) to denote the cobracket. We will often omit the summation sign \(\sum\) to simplify the typography. The above conditions can be also written as:

(\(\text{CL}1\)) \(\sum x_1 \otimes x_2 = -\sum x_2 \otimes x_1\),

(\(\text{CL}2\)) \(\sum \alpha_H(x_1) \otimes \delta(x_2) = \sum \delta(x_1) \otimes \alpha_H(x_2) + \sum \tau_{12}(\alpha_H(x_1) \otimes \delta(x_2))\).

**Definition 2.3 (\([27]\)).** A Hom-Lie bialgebra \((H, \alpha_H)\) is a Hom-vector space equipped simultaneously with a Hom-Lie algebra structure \((H, [\cdot, \cdot], \alpha)\) and a Hom-Lie coalgebra \((H, \delta, \alpha)\) structure such that the following compatibility condition is satisfied,

\[
\delta([x, y]) = [\alpha_H(x), \delta(y)] + [\delta(x), \alpha_H(y)] = \text{ad}_{\alpha_H(x)} \delta(y) - \text{ad}_{\alpha_H(y)} \delta(x).
\]

(5)

We denote it by \((H, [\cdot, \cdot], \delta, \alpha_H)\).

Using the sigma notation, the above equation (5) is equivalent to

\[
\delta([x, y]) = \sum [\alpha_H(x), y_1] \otimes \alpha_H(y_2) + \sum \alpha_H(y_1) \otimes [\alpha_H(x), y_2] \\
+ \sum \alpha_H(x_1) \otimes [x_2, \alpha_H(y)] + \sum [x_1, \alpha_H(y)] \otimes \alpha_H(x_2) \\
= \sum [\alpha_H(x), y_1] \otimes \alpha_H(y_2) + \sum \alpha_H(y_1) \otimes [\alpha_H(x), y_2] \\
- \sum [\alpha_H(y), x_1] \otimes \alpha_H(x_2) - \sum \alpha_H(x_1) \otimes [\alpha_H(y), x_2].
\]
A homomorphism of Hom-Lie bialgebras \( \varphi : (H, [\cdot, \cdot], \delta) \to (H', [\cdot, \cdot]', \delta') \) is both a homomorphism of Hom-Lie algebras and a homomorphism of Hom-Lie coalgebras, i.e. for all \( a, b \in H \),

\[
\varphi \circ \alpha_H = \alpha_H \circ \varphi, \quad \varphi([x, y]) = [\varphi(x), \varphi(y)]', \quad \delta' \circ \varphi(x) = (\varphi \otimes \varphi) \circ \delta(x).
\]

Let \( A, H \) be both Hom-Lie algebras and Hom-Lie coalgebras. For \( a, b \in A, x, y \in H \), we denote maps

\[
\triangleright : H \otimes A \to A, \quad \triangleleft : H \otimes A \to H, \quad \phi : A \to H \otimes A, \quad \psi : H \to H \otimes A
\]

by

\[
\triangleright(x \otimes a) = x \triangleright a, \quad \triangleleft(x \otimes a) = x \triangleleft a, \\
\phi(a) = \sum a_{(-1)} \otimes a_{(0)}, \quad \psi(x) = \sum x_{(0)} \otimes x_{(1)}.
\]

We now fix some notations. For a Hom-Lie algebra \( (H, \alpha_H) \) and a linear map \( \triangleright : H \otimes A \to A \) such that

\[
[x, y] \triangleright \alpha_A(a) = \alpha_H(x) \triangleright (y \triangleright a) - \alpha_H(y) \triangleright (x \triangleright a),
\]

for all \( x, y \in H, a \in A \), then \((A, \alpha)\) is called a left \( H \)-Hom-Lie module. For a Hom-Lie coalgebra \((H, \alpha_H)\) and a linear map \( \phi : A \to H \otimes A \) such that

\[
\sum \delta_H(a_{(-1)}) \otimes \alpha_A(a_{(0)}) = \sum \alpha_H(a_{(-1)}) \otimes \phi(a_{(0)}) - \sum \tau_{12} (\alpha_H(a_{(-1)}) \otimes \phi(a_{(0)})),
\]

then \((A, \phi)\) is called a left \( H \)-Hom-Lie comodule. If \((H, \alpha_H)\) and \((A, \alpha_A)\) are Hom-Lie algebras, \((A, \alpha_A)\) is a left \( H \)-Hom-Lie module and

\[
\alpha_H(x) \triangleright [a, b] = [x \triangleright a, \alpha_A(b)] + [\alpha_A(a), x \triangleleft b],
\]

then \((A, [\cdot, \cdot], \alpha)\) is called a left \( H \)-Hom-Lie module algebra. If \((H, \alpha_H)\) is a Hom-Lie coalgebra and \((A, \alpha_A)\) is a Hom-Lie algebra, \((A, \alpha_A)\) is a left \( H \)-Hom-Lie comodule and

\[
\phi([a, b]) = \sum a_{(-1)} \otimes [a_{(0)}, b] + \sum b_{(-1)} \otimes [a, b_{(0)}],
\]

then \((A, \alpha_A)\) is called a left \( H \)-Hom-Lie comodule algebra. Right Hom-Lie (co)module and Hom-Lie (co)module algebra can be defined similarly.

**Definition 2.4.** Let \((A, [\cdot, \cdot])\) be a given Hom-Lie algebra (Hom-Lie coalgebra, Hom-Lie bialgebra), \((E, \alpha_E)\) a Hom-vector space. An extending system of \((A, \alpha_A)\) through \((V, \alpha_V)\) is a Hom-Lie algebra (Hom-Lie coalgebra, Hom-Lie bialgebra) on \((E, \alpha_E)\) such that \((V, \alpha_V)\) a complement subspace of \((A, \alpha_A)\) in \((E, \alpha_E)\), the canonical injection map \( i : A \to E, a \mapsto (a, 0) \) or the canonical projection map \( p : E \to A, (a, x) \mapsto a \) is a Hom-Lie algebra (Hom-Lie coalgebra, Hom-Lie bialgebra) homomorphism. The extending problem is to describe and classify up to an isomorphism the set of all Hom-Lie algebra (Hom-Lie coalgebra, Hom-Lie bialgebra) structures that can be defined on \((E, \alpha_E)\).
We remark that our definition of extending system of \((A, \alpha_A)\) through \((V, \alpha_V)\) contains not only extending structure in [1] [2] [3] but also the global extension structure in [5]. The reason is that when we consider extending problem for Hom-Lie bialgebras, both of them are necessarily used. Note that in our extending system we do not demand \((A, \alpha_A)\) to be a subalgebra of \((E, \alpha_E)\) although \((A, \alpha_A)\) is always a Hom-Lie algebra. In fact, the canonical injection map \(i : A \to E\) is a Lie (co)algebra homomorphism if and only if \((A, \alpha_A)\) is a Lie sub-(co)algebra of \((E, \alpha_E)\).

**Definition 3.1.** Let \(\alpha_A\) be a Hom-Lie algebra (Hom-Lie coalgebra, Hom-Lie bialgebra), \((E, \alpha_E)\) be a Hom-Lie algebra (Hom-Lie coalgebra, Hom-Lie bialgebra) such that \(A\) is a subspace of \((E, \alpha_E)\) and \((V, \alpha_V)\) a complement of \(A\) in \((E, \alpha_E)\). Let \((E, \alpha_E)\) and \((E, \alpha_E')\) be two Hom-Lie algebra (Hom-Lie coalgebra, Hom-Lie bialgebra) structures on \((E, \alpha_E)\). For a linear map \(\varphi : E \to E\) we consider the diagram:

\[
\begin{array}{cccc}
0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & V & \longrightarrow & 0 \\
\downarrow \text{id}_A & & \downarrow i & & \downarrow \pi & & \downarrow \text{id}_V & & \downarrow \varphi & & \downarrow \pi' & & \downarrow 0 \\
0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & V & \longrightarrow & 0.
\end{array}
\]

where \(\pi : E \to V\) is the canonical projection of \(E = A \oplus V\) onto \((V, \alpha_V)\) and \(i : A \to E\) is the inclusion map. We say that \(\varphi : E \to E\) stabilizes \((A, \alpha_A)\) if the left square of the diagram is commutative. Two extending system \((E, \alpha_E)\) and \((E, \alpha_E')\) are called equivalent, and we denote this by \((E, \alpha_E) \equiv (E, \alpha_E')\), if there exists a Hom-Lie algebra (Hom-Lie coalgebra, Hom-Lie bialgebra) isomorphism \(\varphi : (E, \alpha_E) \to (E, \alpha_E')\) which stabilizes \(A\). Denote by \(\text{Ext}(E, A)\) \((\text{CExt}(E, A), \text{BExt}(E, A))\) the set of equivalent classes of Hom-Lie algebra (Hom-Lie coalgebra, Hom-Lie bialgebra) structures on \((E, \alpha_E)\).

3 Braided Hom-Lie bialgebras

3.1 Hom-Yetter-Drinfeld modules and braided Hom-Lie bialgebras

**Definition 3.1.** Let \((H, \alpha_H)\) be simultaneously a Hom-Lie algebra and a Hom-Lie coalgebra. If \((V, \alpha_V)\) is a left \(H\)-Hom-Lie module and left \(H\)-Hom-Lie comodule, and satisfying the following condition:

\[[\text{YD1}]\] \(\phi(x \triangleright v) = [\alpha_H(x), v_{(-1)}] \otimes \alpha (v_{(0)}) + \alpha (v_{(-1)}) \otimes \alpha_H(x) \triangleright v_{(0)} + \alpha_H(x_1) \otimes x_2 \triangleright \alpha(v),\)

then \((V, \alpha_V)\) is called a left Hom-Yetter-Drinfeld module over \((H, \alpha_H)\).

We denote the category of Hom-Yetter-Drinfeld modules over \((H, \alpha_H)\) by \(\mathcal{H}_H^M\).

**Definition 3.2.** Let \((H, \alpha_H)\) be a Hom-Lie bialgebra. If \((A, \alpha_A)\) be a Hom-Lie algebra, a Hom-Lie coalgebra and a left Hom-Yetter-Drinfeld module over \((H, \alpha_H)\), then we call \((A, \alpha_A)\) to be a braided Hom-Lie bialgebra in \(\mathcal{H}_H^M\), if the following condition is satisfied

\[[\text{LBSa}]\] \(\delta([a, b]) = [\alpha_A(a), \delta(b)] + [\delta(a), \alpha_A(b)] - s(a \otimes b),\)
where

\[ s(a \otimes b) = b_{(-1)} \triangleright a_A(a) \otimes a_A(b_{(0)}) + \alpha_A(a_{(0)}) \otimes a_{(-1)} \triangleright a_A(b) \]

\[ -a_{(-1)} \triangleright a_A(b) \otimes a_A(a_{(0)}) - \alpha_A(b_{(0)}) \otimes b_{(-1)} \triangleright a_A(a) \]

is called the infinitesimal braiding.

When \( \alpha = \text{id} \), this is exactly the Yetter-Drinfeld Lie algebra introduced by Sommerhauser in [24] and braided Lie bialgebra studied by Majid in [18]. Thus our braided Hom-Lie bialgebra is a generalization of braided Lie bialgebra.

Now we construction Hom-Lie bialgebra from braided Hom-Lie bialgebra via biproduct. Let \((H, \alpha_H)\) be an Hom-Lie bialgebra, \((A, \alpha_A)\) be a Hom-Lie algebra and a Hom-Lie coalgebra in \( H \mathcal{M} \). We define bracket and cobracket on the direct sum vector space \( E = A \oplus H \) by

\[ \alpha_E(a, x) := (\alpha_A(a), \alpha_H(x)), \]

\[ [(a, x), (b, y)] := ([a, b] + x \triangleright b - y \triangleright a, [x, y]), \]

\[ \delta_E(a, x) := \delta_A(a) + \phi(a) - \tau\phi(a) + \delta_H(x). \]

This is called biproduct of \((A, \alpha_A)\) and \((H, \alpha_H)\) which will be denoted by \( A \bowtie H \).

**Remark 3.3.** In sigma notation, the above equation (9) can be written as

\[ \delta_E(a, x) = a_1 \otimes a_2 + a_{(-1)} \otimes a_{(0)} - a_{(0)} \otimes a_{(-1)} + x_1 \otimes x_2. \]

The elements in the right hand side of this equation should be seen as in \( E \otimes E = (A \oplus H) \otimes (A \oplus H) \cong A \otimes A \oplus H \otimes A \oplus A \otimes H \oplus H \otimes H \).

**Theorem 3.4.** Let \((H, \alpha_H)\) is a Hom-Lie bialgebra and \((A, \alpha_A)\) is a left Hom-Lie module and a left Hom-Lie comodule in \( H \mathcal{M} \). Then the biproduct \( A \bowtie H \) form a Hom-Lie bialgebra if and only if \((A, \alpha_A)\) is braided Hom-Lie bialgebra in \( H \mathcal{M} \).

**Proof.** It is easy to verify that \( A \bowtie H \) is Hom-Lie algebra and Hom-Lie coalgebra under the above bracket \( [\text{S}] \) and cobracket \( [\text{I}] \). We are left to check that compatibility condition for Hom-Lie bialgebra \( A \bowtie H \):

\[ \delta_E([a, x], (b, y)] = [\alpha_E(a, x), \delta_E(b, y)] + [\delta_E(a, x), \alpha_E(b, y)]. \]

The left hand side is equal to

\[ \delta((a, x), (b, y)] = \delta \left( [a, b] + x \triangleright b - y \triangleright a, [x, y] \right) \]

\[ = \delta_A([a, b]) + \delta_A(x \triangleright b - \delta_A(y \triangleright a) \]

\[ + \phi([a, b]) + \phi(x \triangleright b) - \phi(y \triangleright a) \]

\[ - \tau\phi([a, b]) - \tau\phi(x \triangleright b) + \tau\phi(y \triangleright a) + \delta_H([x, y]). \]
From (6)–(9) we have that
\[ A(a), b_1] \otimes A(b_2) + \alpha_H(x) \triangleright b_1 \otimes A(b_2) \]
\[ + \alpha_A(b_1) \otimes [A(a), b_2] + \alpha_A(b_1) \otimes \alpha_H(x) \triangleright b_2 \]
\[ - b_{(-1)} \triangleright A(a) \otimes A(b_{(0)}) + [\alpha_H(x), b_{(-1)}] \otimes A(b_{(0)}) \]
\[ + \alpha_H(b_{(-1)}) \otimes [A(a), b_{(0)}] + \alpha_H(b_{(-1)}) \otimes \alpha_H(x) \triangleright b_{(0)} \]
\[ - [A(a), b_{(0)}] \otimes \alpha_H(b_{(-1)}) - \alpha_H(x) \triangleright b_{(0)} \otimes \alpha_H(b_{(-1)}) \]
\[ + \alpha_A(b_{(0)}) \otimes b_{(-1)} \triangleright \alpha_A(a) - \alpha_A(b_{(0)}) \otimes [\alpha_H(x), b_{(-1)}] \]
\[ - y_1 \triangleright \alpha_A(a) \otimes \alpha_H(y_2) + [\alpha_H(x), y_1] \otimes \alpha_H(y_2) \]
\[ - \alpha_H(y_1) \otimes y_2 \triangleright \alpha_A(a) + \alpha_H(y_1) \otimes [\alpha_H(x), y_2] \]
\[ + [a_1, \alpha_A(b)] \otimes \alpha_A(a_2) - \alpha_H(y) \triangleright a_1 \otimes \alpha_A(a_2) \]
\[ + \alpha_A(a_1) \otimes [a_2, \alpha_A(b)] - \alpha_A(a_1) \otimes \alpha(y) \triangleright a_2 \]
\[ + a_{(-1)} \triangleright \alpha_A(b) \otimes \alpha_A(a_{(0)}) + [a_{(-1)}, \alpha_H(y)] \otimes \alpha_A(a_{(0)}) \]
\[ + \alpha_H(a_{(-1)}) \otimes [a_{(0)}, \alpha_A(b)] - \alpha_H(a_{(-1)}) \otimes \alpha(y) \triangleright a_{(0)} \]
\[ - [a_{(0)}, \alpha_A(b)] \otimes \alpha_H(a_{(-1)}) + \alpha_H(y) \triangleright a_{(0)} \otimes \alpha_H(a_{(-1)}) \]
\[ - \alpha_A(a_{(0)}) \otimes a_{(-1)} \triangleright \alpha_A(b) - \alpha_A(a_{(0)}) \otimes [a_{(-1)}, \alpha_H(y)] \]
\[ - x_1 \triangleright \alpha_A(b) \otimes \alpha(x_2) + [x_1, \alpha_H(y)] \otimes \alpha(x_2) \]
\[ + \alpha(x_1) \otimes x_2 \triangleright \alpha_A(b) + \alpha(x_1) \otimes [x_2, \alpha_H(y)] \]

Thus the two sides are equal to each other if and only if the following conditions hold.

1. \[ \delta_A ([a, b]) = [\alpha_A(a), b_1] \otimes A(b_2) + \alpha_A(b_1) \otimes [\alpha_A(a), b_2] \]
\[ + [a_1, \alpha_A(b)] \otimes \alpha_A(a_2) + \alpha_A(a_1) \otimes [a_2, \alpha_A(b)] \]
\[ + a_{(-1)} \triangleright \alpha_A(b) \otimes \alpha_A(a_{(0)}) + \alpha_A(a_{(0)}) \otimes b_{(-1)} \triangleright \alpha_A(a) \]
\[ - b_{(-1)} \triangleright \alpha_A(a) \otimes \alpha_A(b_{(0)}) - \alpha_A(a_{(0)}) \otimes a_{(-1)} \triangleright \alpha_A(b), \]

2. \[ \delta_A (x \triangleright b) = \alpha_H(x) \triangleright b_1 \otimes \alpha_A(b_2) + \alpha_A(b_1) \otimes \alpha_H(x) \triangleright b_2, \]

3. \[ \delta_A (y \triangleright a) = \alpha_H(y) \triangleright a_1 \otimes \alpha_A(a_2) + \alpha_A(a_1) \otimes \alpha(y) \triangleright a_2, \]

4. \[ \phi ([a, b]) = \alpha_H(b_{(-1)}) \otimes [\alpha_A(a), b_{(0)}] + \alpha_H(a_{(-1)}) \otimes [a_{(0)}, \alpha_A(b)] \]

5. \[ \phi (x \triangleright b) = [\alpha_H(x), b_{(-1)}] \otimes \alpha_A(b_{(0)}) + \alpha_H(b_{(-1)}) \otimes \alpha_H(x) \triangleright b_{(0)} + \alpha_H(x_1) \otimes x_2 \triangleright \alpha_A(b), \]

6. \[ \phi (y \triangleright a) = [\alpha_H(y), a_{(-1)}] \otimes \alpha_A(a_{(0)}) + \alpha_H(a_{(-1)}) \otimes \alpha(y) \triangleright a_{(0)} + \alpha_H(y_1) \otimes y_2 \triangleright \alpha_A(a), \]

7. \[ \tau \phi ([a, b]) = [\alpha_A(a), b_{(0)}] \otimes \alpha_H(b_{(-1)}) + [a_{(0)}, \alpha_A(b)] \otimes \alpha_H(a_{(-1)}) \]

8. \[ \tau \phi (x \triangleright b) = \alpha_A(b_{(0)}) \otimes [\alpha_H(x), b_{(-1)}] + \alpha_H(x) \triangleright b_{(0)} \otimes \alpha_H(b_{(-1)}) + x_2 \triangleright \alpha_A(b) \otimes \alpha(x_1), \]

9. \[ \tau \phi (y \triangleright a) = \alpha_A(a_{(0)}) \otimes [\alpha_H(y), a_{(-1)}] + \alpha_H(y) \triangleright a_{(0)} \otimes \alpha_H(a_{(-1)}) + y_1 \triangleright \alpha_A(a) \otimes \alpha_H(y_1). \]

From (6)–(9) we have that \( A \) is a left \( H \)-module Hom-Lie coalgebra and \( H \)-comodule Hom-Lie algebra, from (2)–(5) we get that \( A \) is a left Yetter-Drinfel module over \( H \), and (1) is the condition for \( A \) to be a braided Hom-Lie bialgebra. The proof is completed. \[\square\]
3.2 From quasitriangular Hom-Lie bialgebras to braided Hom-Lie bialgebras

Let \((A, [\cdot, \cdot], \alpha)\) be a Hom-Lie algebra equipped with \(r = r_1 \otimes r_2 \in A \otimes A\) such that \((\alpha \otimes \alpha)(r) = r\). In what follows, for an element \(r = \sum r_1 \otimes r_2\), we denote \(\tau(r) = \sum r_2 \otimes r_1\). We define the following elements in \(A \otimes A \otimes A\)

\[
[r_{12}, r_{13}] = [r_1, r'_1] \otimes \alpha(r_2) \otimes \alpha(r'_2), \\
[r_{12}, r_{23}] = \alpha(r_1) \otimes [r_2, r'_1] \otimes \alpha(r'_2), \\
[r_{13}, r_{23}] = \alpha(r_1) \otimes \alpha(r'_1) \otimes [r'_2, r_2],
\]

where \(r' = \sum r'_1 \otimes r'_2\) is a copy of \(r\). We consider solutions \(r \in A \otimes A\) to the equation

\[
C(r) = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0
\]

which is called the classical Hom-Yang-Baxter equation.

Definition 3.5. \([27]\) Let \((A, [\cdot, \cdot], \alpha)\) be a Hom-Lie algebra and \(r \in A \otimes A\) be an element such that \((\alpha \otimes \alpha)(r) = r\). Define the linear map \(\delta_r : A \to A \otimes A\) by

\[
\delta_r(x) = [x, r_1] \otimes \alpha(r_2) + \alpha(r_1) \otimes [x, r_2]
\]

Then Furthermore we have \((\alpha \otimes \alpha) \circ \delta_r = \delta_r \circ \alpha \circ \alpha\) and the following identity holds:

\[
\delta_r([x, y]) = [\alpha(x), \delta_r(y)] + [\delta_r(x), \alpha(y)]
\]

Thus we obtain \((A, [\cdot, \cdot], \delta_r, \alpha)\) is a Hom-Lie bialgebra. This is called a coboundary Hom-Lie bialgebra. A coboundary Hom-Lie bialgebra for which \(C(r) = 0\) is called a quasitriangular Hom-Lie bialgebra.

We remark that \(r\) is not assumed to be anti-symmetric \(r + \tau(r) = 0\) in a coboundary Hom-Lie bialgebra. Note that the skew symmetry of \(\delta_r\) is equivalent to \(r + \tau(r)\) is adjoint invariant, that is, \([x, r + \tau(r)] = 0\) for every \(x \in A\):

\[
[x, r_1] \otimes \alpha(r_2) + \alpha(r_1) \otimes [x, r_2] + [x, r_2] \otimes \alpha(r_1) + \alpha(r_2) \otimes [x, r_1] = 0. \tag{10}
\]

Lemma 3.6. \([27]\) Let \((A, [\cdot, \cdot], \alpha, r)\) be a quasitriangular Hom-Lie bialgebra. Then the following statements are true:

\[
(\alpha \otimes \delta)(r) = [r_{13}, r_{12}], \tag{11}
\]

\[
(\delta \otimes \alpha)(r) = [r_{13}, r_{23}]. \tag{12}
\]

Theorem 3.7. Let \((A, [\cdot, \cdot], \alpha, r)\) be a quasitriangular Hom-Lie bialgebra and \(V\) be a left Hom-Lie module of \(A\). Then we obtain that \(V\) is a Yetter-Drinfeld module over \(A\) via the left Hom-Lie comodule map \(\phi : V \to A \otimes V\) defined by

\[
\phi(v) = r_2 \otimes r_1 \triangleright v. \tag{13}
\]
Proof. First, we prove that $\phi$ is indeed a left Hom-Lie comodule:

$$\sum \delta(a_{(-1)}) \otimes \alpha_A(a_{(0)}) = \sum \alpha(a_{(-1)}) \otimes \phi(a_{(0)}) - \sum \tau_{12} \left( \alpha(a_{(-1)}) \otimes \phi(a_{(0)}) \right).$$

Using equation (11) in the above Lemma 3.6, the left hand side is equal to

$$\delta(a_{(-1)}) \otimes \alpha_A(a_{(0)}) = \delta_r(r_2) \otimes \alpha(r_1) \triangleright \alpha(v) = \alpha(\delta_r(r_2) \otimes [r_1, r_1']) \triangleright \alpha(v),$$

and the hand side is equal to

$$\alpha(a_{(-1)}) \otimes \phi(a_{(0)}) - \tau_{12} \left( \alpha(a_{(-1)}) \otimes \phi(a_{(0)}) \right) = \alpha(r_2) \otimes \phi(r_1 \triangleright v) - \tau_{12} \left( \alpha(r_2) \otimes \phi(r_1 \triangleright v) \right) = \alpha(r_2) \otimes [r_1, r_1'] \triangleright \alpha(v).$$

Thus the two sides are equal to each other.

Next, we verify the Yetter-Drinfeld module condition:

$$\phi(x \triangleright v) = [\alpha(x), v_{(-1)}] \otimes \alpha(v_{(0)}) + \alpha(v_{(-1)}) \otimes \alpha(x) \triangleright v_{(0)} + \alpha(x_1) \otimes x_2 \triangleright \alpha(v).$$

Since $\alpha(r) = r$, we obtain that the left hand side of the above equation is equal to

$$\phi(x \triangleright v) = r_2 \otimes r_1 \triangleright (x \triangleright v) = \alpha(r_2) \otimes \alpha(r_1) \triangleright (x \triangleright v),$$

and the hand side is equal to

$$[\alpha(x), v_{(-1)}] \otimes \alpha(v_{(0)}) + \alpha(v_{(-1)}) \otimes \alpha(x) \triangleright v_{(0)} + \alpha(x_1) \otimes x_2 \triangleright \alpha(v) = [\alpha(x), r_2] \otimes \alpha(r_1 \triangleright v) + \alpha(r_2) \otimes \alpha(x) \triangleright (r_1 \triangleright v) - \alpha(x_2) \otimes x_1 \triangleright \alpha(v) = \alpha(r_2) \otimes \alpha(x) \triangleright (r_1 \triangleright v) - [\alpha(x), r_2] \otimes \alpha(r_1) \triangleright \alpha(v) = \alpha(r_2) \otimes \alpha(x) \triangleright (r_1 \triangleright v) - \alpha(r_2) \otimes [x, r_1] \triangleright \alpha(v).$$

Therefore the two sides are equal to each other because $V$ is a Hom-Lie module over $A$: $[x, r_1] \triangleright \alpha(v) = \alpha(x) \triangleright (r_1 \triangleright v) - \alpha(r_1) \triangleright (x \triangleright v)$.

\[\square\]

Theorem 3.8. For every quasitriangular Hom-Lie bialgebra $(A, [\cdot, \cdot], \alpha, r)$, we obtain a braided Hom-Lie bialgebra $A$ with the new cobracket defined by

$$\delta(x) = \alpha(r_1) \otimes [x, r_2] + \alpha(r_2) \otimes [x, r_1].$$

This braided Hom-Lie bialgebra is called a transmutation of the quasitriangular Hom-Lie bialgebra $(A, [\cdot, \cdot], \alpha, \delta_r)$. 

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Proof. We give a proof by direct computations. By the definition of $\hat{\delta}(x)$ we get

$$(\alpha \otimes \delta)\hat{\delta}(x)$$

$$= (\alpha \otimes \delta)(\alpha(r_1) \otimes [x, r_2] + \alpha(r_2) \otimes [x, r_1])$$

$$= \alpha(r_1) \otimes \alpha(r'_1) \otimes [[x, r_2], r'_2] + \alpha(r_1) \otimes \alpha(r'_2) \otimes [[x, r_2], r'_1]$$

$$+ \alpha(r_2) \otimes \alpha(r'_1) \otimes [[x, r_1], r'_2] + \alpha(r_2) \otimes \alpha(r'_2) \otimes [[x, r_1], r'_1]$$

$$= \alpha(r_1) \otimes \alpha(r'_1) \otimes [x, [r_2, r'_2]] + \alpha(r_1) \otimes \alpha(r'_1) \otimes [[x, r_2], r_2]$$

$$+ \alpha(r_1) \otimes \alpha(r'_2) \otimes [x, [r_2, r'_1]] + \alpha(r_1) \otimes \alpha(r'_2) \otimes [[x, r_2], r_2]$$

$$+ \alpha(r_2) \otimes \alpha(r'_1) \otimes [x, [r_1, r'_2]] + \alpha(r_2) \otimes \alpha(r'_1) \otimes [[x, r_1], r_1]$$

$$+ \alpha(r_2) \otimes \alpha(r'_2) \otimes [x, [r_1, r'_1]] + \alpha(r_2) \otimes \alpha(r'_2) \otimes [[x, r_1], r_1]$$

$$= -\alpha(r_1) \otimes \alpha(r'_2) \otimes [x, [r_2, r'_1]] - \alpha(r_1) \otimes [r_2, r'_1] \otimes [x, r'_2]$$

$$- \alpha(r_1) \otimes [r_2, r'_2] \otimes [x, r'_1] + \alpha(r_1) \otimes \alpha(r'_1) \otimes [[x, r_2], r_2]$$

$$+ \alpha(r_1) \otimes \alpha(r'_2) \otimes [x, [r_2, r'_1]] + \alpha(r_1) \otimes \alpha(r'_2) \otimes [[x, r_2], r_2]$$

$$- \alpha(r_2) \otimes \alpha(r'_2) \otimes [x, [r_1, r'_1]] - \alpha(r_2) \otimes [r_1, r'_1] \otimes [x, r'_2]$$

$$- \alpha(r_2) \otimes [r_1, r'_2] \otimes [x, r'_1] + \alpha(r_2) \otimes \alpha(r'_1) \otimes [[x, r_2], r_1]$$

$$+ \alpha(r_2) \otimes \alpha(r'_2) \otimes [x, [r_1, r'_1]] + \alpha(r_2) \otimes \alpha(r'_2) \otimes [[x, r_1], r_1]$$

$$= -\alpha(r_1) \otimes [r_2, r'_1] \otimes [x, r'_2] - \alpha(r_1) \otimes [r_2, r'_2] \otimes [x, r'_1]$$

$$+ \alpha(r_1) \otimes \alpha(r'_1) \otimes [x, r'_2], r_2] + \alpha(r_1) \otimes \alpha(r'_2) \otimes [[x, r'_1], r_2]$$

$$- \alpha(r_2) \otimes [r_1, r'_1] \otimes [x, r'_2] - \alpha(r_2) \otimes [r_1, r'_2] \otimes [x, r'_1]$$

$$+ \alpha(r_2) \otimes \alpha(r'_1) \otimes [[x, r'_2], r_1] + \alpha(r_2) \otimes \alpha(r'_2) \otimes [[x, r'_1], r_1]$$

where we use the Hom-Jacobi identity in the third equality and (10) in the fourth equality. In the meantime, we have

$$\tau_{12}(\alpha \otimes \delta)\hat{\delta}(x) + (\hat{\delta} \otimes \text{id})\hat{\delta}(x)$$

$$= \tau_{12}(\alpha \otimes \delta)(\alpha(r_1) \otimes [x, r_2] + \alpha(r_2) \otimes [x, r_1])$$

$$+ (\hat{\delta} \otimes \alpha)(\alpha(r_1) \otimes [x, r_2] + \alpha(r_2) \otimes [x, r_1])$$

$$= \alpha(r'_1) \otimes \alpha(r_1) \otimes [[x, r_2], r'_2] + \alpha(r'_2) \otimes \alpha(r_1) \otimes [[x, r_2], r'_1]$$

$$+ \alpha(r'_1) \otimes \alpha(r_2) \otimes [[x, r_1], r'_2] + \alpha(r'_2) \otimes \alpha(r_2) \otimes [[x, r_1], r'_1]$$

$$+ \alpha(r'_1) \otimes [r_1, r'_2] \otimes [x, r_2] + \alpha(r'_2) \otimes [r_1, r'_1] \otimes [x, r_2]$$

$$+ \alpha(r'_1) \otimes [r_2, r'_2] \otimes [x, r_1] + \alpha(r'_2) \otimes [r_2, r'_1] \otimes [x, r_1]$$

$$= \alpha(r_1) \otimes \alpha(r'_1) \otimes [[x, r'_2], r_2] + \alpha(r_2) \otimes \alpha(r'_1) \otimes [[x, r'_2], r_1]$$

$$+ \alpha(r_1) \otimes \alpha(r'_2) \otimes [[x, r'_1], r_2] + \alpha(r_2) \otimes \alpha(r'_2) \otimes [[x, r'_1], r_1]$$

$$+ \alpha(r_1) \otimes [r_1, r'_2] \otimes [x, r_2] + \alpha(r_2) \otimes [r_1, r'_1] \otimes [x, r_2]$$

$$+ \alpha(r_1) \otimes [r_2, r'_2] \otimes [x, r_1] + \alpha(r_2) \otimes [r_2, r'_1] \otimes [x, r_1]$$

Thus the two sides are equal to each other by comparing term by term using skew-symmetry.
of Hom-Lie algebra \(A\).

**Example 3.9.** Let \(A = \mathfrak{sl}(2) = \text{span}\{H, X, Y\}\) be the three dimensional Hom-Lie algebra with \(\alpha : \mathfrak{sl}(2) \to \mathfrak{sl}(2)\) given by
\[
\alpha(H) = H, \quad \alpha(X) = kX, \quad \alpha(Y) = k^{-1}Y,
\]
the Lie bracket given by
\[
[H, X] = 2kX, \quad [H, Y] = -2k^{-1}Y, \quad [X, Y] = H
\]
and the Lie cobracket given by
\[
\delta(H) = 0, \quad \delta(X) = kX \wedge H, \quad \delta(Y) = k^{-1}Y \wedge H.
\]
Then \((\mathfrak{sl}(2), \alpha)\) is a quasi-triangular Hom-Lie bialgebra with \(r = X \otimes Y + \frac{1}{4}H \otimes H\). By the above Theorem 3.8, we obtain a braided Hom-Lie bialgebra with the new cobracket by
\[
\hat{\delta}(H) = 2k^2(X \otimes Y + Y \otimes X), \quad \hat{\delta}(X) = kX \wedge H, \quad \hat{\delta}(Y) = k^{-1}Y \wedge H.
\]

### 4 Unified product for braided Hom-Lie bialgebras

In this section, we will construct unified product for braided Hom-Lie bialgebras. First, we review the notion of matched pairs of Hom-Lie algebras and Hom-Lie coalgebras.

#### 4.1 Unified product of braided Hom-Lie bialgebras

In the following definitions, we introduce some new concept of cocycle Hom-Lie algebras and cycle Hom-Lie coalgebras, which are in fact not really ordinary Hom-Lie algebras and Hom-Lie coalgebras, but generalized ones. Using these new algebraic structures, we will construct the unified product of braided Hom-Lie bialgebras. Denote maps
\[
\sigma : H \otimes H \to A, \quad \theta : A \otimes A \to H, \quad P : A \to H \otimes H, \quad Q : H \to A \otimes A
\]
by
\[
\sigma(x, y) \in A, \quad \theta(a, b) \in H, \quad P(a) = \sum a_{[1]} \otimes a_{[2]} \in H \otimes H, \quad Q(x) = \sum x_{<1>} \otimes x_{<2>} \in A \otimes A.
\]

An antisymmetric bilinear map \(\sigma : H \otimes H \to A\) is called a cocycle on \((H, \alpha_H)\) if
\[
\text{(CC1)} \quad \alpha_H(x) \triangleright \sigma(y, z) + \alpha_H(y) \triangleright \sigma(z, x) + \alpha_H(z) \triangleright \sigma(x, y) = \sigma([x, y], \alpha_H(z)) + \sigma([y, z], \alpha_H(x)) + \sigma([z, x], \alpha_H(y)).
\]

An antisymmetric bilinear map \(\theta : A \otimes A \to H\) is called a cocycle on \((A, \alpha_A)\) if
\[
\text{(CC2)} \quad \theta(a, b) \triangleleft \alpha_A(c) + \theta(b, c) \triangleleft \alpha_A(a) + \theta(c, a) \triangleleft \alpha_A(b) = \theta(\alpha_A(a), [b, c]) + \theta(\alpha_A(b), [c, a]) + \theta(\alpha_A(c), [a, b]).
\]
A co-antisymmetric linear map $P : A \to H \otimes H$ is called a cycle on $(A, \alpha_A)$ if
\begin{equation}
\alpha_H(a_{(-1)}) \otimes P(a_{(0)}) + \tau_{12} \tau_{23} \left( \alpha_H(a_{(-1)}) \otimes P(a_{(0)}) \right) + \tau_{23} \tau_{12} \left( \alpha_H(a_{(-1)}) \otimes P(a_{(0)}) \right)
= \delta(a_{[1]}) \otimes \alpha_H(a_{[2]}) + \tau_{12} \tau_{23} \left( \delta(a_{[1]}) \otimes \alpha_H(a_{[2]}) \right) + \tau_{23} \tau_{12} \left( \delta(a_{[1]}) \otimes \alpha_H(a_{[2]}) \right).
\end{equation}

A co-antisymmetric linear map $Q : H \to A \otimes A$ is called a cycle on $(H, \alpha_H)$ if
\begin{equation}
Q(x_{(0)}) \otimes \alpha_A(x_{(1)}) + \tau_{12} \tau_{23} \left( Q(x_{(0)}) \otimes \alpha_A(x_{(1)}) \right) + \tau_{23} \tau_{12} \left( Q(x_{(0)}) \otimes \alpha_A(x_{(1)}) \right)
= \alpha_A(x_{<1>}) \otimes \delta(x_{<2>}) + \tau_{12} \tau_{23} \left( \alpha_A(x_{<1>}) \otimes \delta(x_{<2>}) \right) + \tau_{23} \tau_{12} \left( \alpha_A(x_{<1>}) \otimes \delta(x_{<2>}) \right).
\end{equation}

**Definition 4.1.** (i): Let $\sigma : H \otimes H \to H$ be a cocycle on $(H, \alpha_H)$ equipped with an antisymmetric bilinear map $[\cdot, \cdot] : H \otimes H \to H$, satisfying the the following cocycle Jacobi identity:
\begin{equation}
[[x, y], \alpha_H(z)] + [[y, z], \alpha_H(x)] + [[z, x], \alpha_H(y)] = \alpha_H(x) \triangleright \sigma(y, z) + \alpha_H(y) \triangleright \sigma(z, x) + \alpha_H(z) \triangleright \sigma(x, y).
\end{equation}

Then $(H, \alpha_H)$ is called a $\sigma$-Hom-Lie algebra.

(ii): Let $\theta : A \otimes A \to H$ be a cocycle on $(A, \alpha_A)$ equipped with antisymmetric bilinear map $[\cdot, \cdot] : A \otimes A \to A$, satisfying the the following cocycle Jacobi identity:
\begin{equation}
[[a, b], \alpha_A(c)] + [[b, c], \alpha_A(a)] + [[c, a], \alpha_A(b)] = \theta(a, b) \triangleright \alpha_A(c) + \theta(b, c) \triangleright \alpha_A(a) + \theta(c, a) \triangleright \alpha_A(b).
\end{equation}

Then $(A, \alpha_A)$ is called a $\theta$-Hom-Lie algebra.

(iii): Let $P : A \to H \otimes H$ be a cycle on $(H, \alpha_H)$ equipped with a co-antisymmetric linear map $\delta : H \to H \otimes H$, satisfying the the following cycle co-Jacobi identity:
\begin{equation}
\delta(x_{(0)}) \otimes P(x_{(1)}) + \tau_{12} \tau_{23} \left( \delta(x_{(0)}) \otimes P(x_{(1)}) \right) + \tau_{23} \tau_{12} \left( \delta(x_{(0)}) \otimes P(x_{(1)}) \right)
= \alpha_H(x_{(0)}) \otimes P(x_{(1)}) + \tau_{12} \tau_{23} \left( \alpha_H(x_{(0)}) \otimes P(x_{(1)}) \right) + \tau_{23} \tau_{12} \left( \alpha_H(x_{(0)}) \otimes P(x_{(1)}) \right).
\end{equation}

Then $(H, \alpha_H)$ is called a $P$-Hom-Lie coalgebra.

(iv): Let $Q : H \to A \otimes A$ be a cycle on $(A, \alpha_A)$ equipped with a co-antisymmetric linear map $\delta : A \to A \otimes A$, satisfying the the following cycle co-Jacobi identity:
\begin{equation}
\delta(a_{(-1)}) \otimes \alpha_A(a_{(0)}) + \tau_{12} \tau_{23} \left( \delta(a_{(-1)}) \otimes \alpha_A(a_{(0)}) \right) + \tau_{23} \tau_{12} \left( \delta(a_{(-1)}) \otimes \alpha_A(a_{(0)}) \right)
= Q(a_{(-1)}) \otimes \alpha_A(a_{(0)}) + \tau_{12} \tau_{23} \left( Q(a_{(-1)}) \otimes \alpha_A(a_{(0)}) \right) + \tau_{23} \tau_{12} \left( Q(a_{(-1)}) \otimes \alpha_A(a_{(0)}) \right).
\end{equation}

Then $(A, \alpha_A)$ is called a $Q$-Hom-Lie coalgebra.

First we consider the Hom-Lie algebra structures on $E = A \oplus H$.

**Theorem 4.2.** Let $(A, \alpha_A)$ be a $\theta$-Hom-Lie algebra and $(H, \alpha_H)$ be a $\sigma$-Hom-Lie algebra. Then $E = A \oplus H$ is a Hom-Lie algebra with bracket given by:
\begin{align}
\alpha_E(a, x) &= (\alpha_A(a), \alpha_H(x)), \\
[(a, x), (b, y)] &= ([a, b] + x \triangleright b - y \triangleright a + \sigma(x, y), [x, y] + x \triangleright b - y \triangleright a + \theta(a, b))
\end{align}

if and only if the following compatibility conditions hold:
(TM1) \[ [x,y] \triangleright \alpha_A(a) + [\sigma(x,y), \alpha_A(a)] = \alpha_H(x) \triangleright (y \triangleright a) - \alpha_H(y) \triangleright (x \triangleright a) + \sigma(\alpha_H(x), y \triangleright a) + \sigma(x \triangleright a, \alpha_H(y)), \]

(TM2) \[ \alpha_H(x) \triangleleft [a,b] + [\alpha_H(x), \theta(a,b)] = (x \triangleleft a) \triangleleft \alpha_A(b) - (x \triangleleft b) \triangleleft \alpha_A(a) + \theta(x \triangleright a, \alpha_A(b)) + \theta(\alpha_A(a), x \triangleright b), \]

(TBB1) \[ \alpha_H(x) \triangleright [a,b] + \sigma(\alpha_H(x), \theta(a,b)) = [x\triangleright a, \alpha_A(b)] + [\alpha_A(a), x\triangleright b] + (x \triangleleft a) \triangleright \alpha_A(b) - (x \triangleleft b) \triangleright \alpha_A(a), \]

(TBB2) \[ [x,y] \triangleleft \alpha_A(a) + \theta(\sigma(x,y), \alpha_A(a)) = \alpha_H(x), y \triangleleft a] + [x \triangleleft a, \alpha_H(y)] + \alpha_H(x) \triangleleft (y \triangleright a) - \alpha_H(y) \triangleleft (x \triangleright a). \]

In this case, \((A,H)\) is called a cocycle cross product system. This Hom-Lie algebra will be denoted by \(A_{\psi,\theta} \#_{\alpha} \sigma H\).

Proof. We have to check when

\[ [\alpha_H(x), [a,b]_E]_E + [\alpha_A(a), [b,x]_E]_E + [\alpha_A(b), [x,a]_E]_E = 0. \tag{16} \]

In fact,

\[ [\alpha_H(x), [a,b]_E]_E = \alpha_H(x) \triangleright [a,b] + \alpha_H(x) \triangleleft [a,b] + [\alpha_H(x), \theta(a,b)] + \sigma(\alpha_H(x), \theta(a,b)), \]

\[ [\alpha_A(a), [b,x]_E]_E = -[\alpha_A(a), x \triangleright b] - \theta(\alpha_A(a), x \triangleright b) + (x \triangleright b) \triangleright \alpha_A(a) + (x \triangleleft b) \triangleleft \alpha_A(a), \]

\[ [\alpha_A(b), [x,a]_E]_E = [\alpha_A(b), x \triangleright a] + \theta(\alpha_A(b), x \triangleright a) - (x \triangleleft a) \triangleright \alpha_A(b) - (x \triangleleft a) \triangleleft \alpha_A(b). \]

Thus equation \((16)\) holds if and only if (TM2) and (TBB1) hold. Similarly, one can verify that \([\alpha_A(a), [x,y]_E]_E + [\alpha_H(x), [y,a]_E]_E + [\alpha_H(y), [a,x]_E]_E = 0\) holds if and only if (TM1) and (TBB2) hold.

Next we consider the Hom-Lie coalgebra structure on \(E = A \oplus H\).

**Theorem 4.3.** Let \((A, \alpha_A)\) be a Q-Hom-Lie coalgebra and \((H, \alpha_H)\) be a P-Hom-Lie coalgebra. If we define \(E = A^{\phi,P} \#^{\psi,Q} H\) as the vector space \(A \oplus H\) with the Lie cobracket

\[ \delta_E(a) = \delta_A(a) + \phi(a) - \tau \phi(a) + P(a), \quad \delta_E(x) = \delta_H(x) + \psi(x) - \tau \psi(x) + Q(x), \tag{17} \]

then \(A^{\phi,P} \#^{\psi,Q} H\) is a Hom-Lie coalgebra if and only if the following compatibility conditions hold:

(TM3) \[ \delta_H(a_{(-1)} \otimes \alpha_A(a_{(0)}) + P(a_1) \otimes \alpha_A(a_2)) = \alpha_H(a_{(-1)} \otimes \phi(a_{(0)}) - \tau_2 (\alpha_H(a_{(-1)} \otimes \phi(a_{(0)}))), \]

\[ + \alpha_H(a_{(1)} \otimes \psi(a_{(2)})) + \tau_3 (\psi(a_{(1)}) \otimes \alpha_H(a_{(2)}))), \]

(TM4) \[ \alpha_H(x_{(0)}) \otimes \delta_A(x_{(1)}) + \alpha_H(x_1) \otimes Q(x_2)
= \psi(x_{(0)}) \otimes \alpha_A(x_{(1)}) - \tau_2 (\psi(x_{(0)}) \otimes \alpha_A(x_{(1)}))
+ \psi(x_{<1>}) \otimes \alpha_A(x_{<2>}) + \tau_2 (\alpha_A(x_{<1>}) \otimes \psi(x_{<2>)))), \]

\[ + \psi(x_{<1>}) \otimes \alpha_A(x_{<2>}) + \tau_2 (\alpha_A(x_{<1>}) \otimes \psi(x_{<2>}))), \]

\[ + \psi(x_{<1>}) \otimes \alpha_A(x_{<2>}) + \tau_2 (\alpha_A(x_{<1>}) \otimes \psi(x_{<2>}))), \]

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In this case, \((A, H)\) is called a cycle cross coproduct system.

The proof Theorem 13 is dual to Theorem 12 so we omit the details.

Combing the two Theorem 12 is dual to Theorem 13 together, we obtain an ordinary Hom-Lie bialgebra from two cocycle braided Hom-Lie bialgebras.

**Theorem 4.4.** Let \((A, H)\) be a cocycle cross product system and a cycle cross coproduct system. Then the Hom-Lie algebra \(A \otimes \delta \# \alpha \# H\) and Hom-Lie coalgebra \(A \circ \beta \# \psi \# Q \# H\) fit together to form an ordinary Hom-Lie bialgebra if and only if the following compatibility conditions hold:

\[(TBB5) \quad \delta_H(a_1 \otimes a_0) = \alpha_H(a_1) \otimes \alpha_H(a_0) + \alpha_H(a_1) \otimes \alpha_H(a_0) + \alpha_H(a_1) \otimes \alpha_H(x) \otimes a_2
\]

\[(TBB6) \quad \delta_H(x \otimes a) = \alpha_H(x) \otimes a_0 \otimes \alpha_A(a_1) + \alpha_A(a_0) \otimes \alpha_H(x) \otimes a_2
\]

\[(TBB7) \quad \phi([a, b]) + \psi(\theta(a, b), a_0) = \alpha_H(a_1) \otimes [a_0, \alpha_A(b)] + \alpha_H(b_1) \otimes \alpha_A(a_0) \otimes [a_0, b_0]
\]

\[(TBB8) \quad \psi([x, y]) + \phi(x, y) = [\alpha_H(x), \alpha_H(y)] \otimes \alpha_A(x_1) + [\alpha_H(x), \alpha_H(y)] \otimes \alpha_A(x_2)
\]

\[(TLB1) \quad \delta_H \theta(a, b) + \theta(a_1, b_1) = \alpha_H(a_1) \otimes \theta(a_0, b_0) + \alpha_H(b_1) \otimes \theta(a_0, b_0) - \theta(a_1, b_1) \otimes \alpha_A(a_0) \otimes \alpha_H(b_0)
\]

\[(TLB2) \quad \delta_H a(x, y) + Q([x, y]) = \sigma(x_1, y_1) \otimes \alpha_A(y_2) + \sigma(x_1, y_1) \otimes \alpha_A(y_2) - \sigma(x_1, y_1) \otimes \alpha_A(x_2, y_2)
\]
Proof. We investigate the case of (LB) on $A \otimes A$:

$$\delta_E([a, b]_E) = \delta_E([a, b] + \theta(a, b)) = \delta([a, b]) + \phi([a, b]) - \tau\phi([a, b]) + P([a, b])$$

$$+ \delta\theta(a, b) + \psi(a, b) - \tau\psi(a, b) + Q\theta(a, b).$$

Denote the right hand side terms by $(a), (b), \cdots, (h)$.

$$[\alpha_A(a), \delta_E(b)] + [\delta_E(a), \alpha_A(b)]$$

$$= [\alpha_A(a), (\delta + \phi - \tau\phi + P)(b)] + [(\delta + \phi - \tau\phi + P)(a), \alpha_A(b)]$$

$$= [\alpha_A(a), b_1] \otimes \alpha_A(b_2)(1) + \alpha_A(b_1) \otimes [\alpha_A(a), b_2](2) - b_{(-1)} \triangleright \alpha_A(a) \otimes \alpha_A(b_0)(3)$$

$$- b_{(-1)} \triangleleft \alpha_A(a) \otimes \alpha_A(b_0)(4) + \alpha_H(b_{(-1)}) \otimes [\alpha_A(a), b_0](5) - [\alpha_A(a), b_0] \otimes \alpha_H(b_{(-1)})(6)$$

$$+ \alpha_H(b_0) \otimes [a_1, \alpha_A(b)](7) + [a_1, \alpha_A(b)] \otimes b_{(-1)} \triangleright \alpha_A(a)(8)$$

$$+ \alpha_H(b_0) \otimes \theta(\alpha_A(a), b_0)(19) - \theta(\alpha_A(a), b_0) \otimes \alpha_H(b_{(-1)})(20) - b_{[1]} \triangleright \alpha_A(a) \otimes \alpha_H(b_{[2]})(21)$$

$$- b_{[1]} \triangleleft \alpha_A(a) \otimes \alpha_H(b_{[2]})(22) - \alpha_H(b_{[1]}) \otimes b_{[2]} \triangleright \alpha_A(a)(23) - \alpha_H(b_{[1]}) \otimes b_{[2]} \triangleleft \alpha_A(a)(24)$$

$$+ \alpha_A(a_1) \otimes [a_2, \alpha_A(b)](9) + [a_1, \alpha_A(b)] \otimes \alpha_A(a_2)(10)$$

$$+ \alpha_H(a_1) \otimes [a_0, \alpha_A(b)](11) + a_{(-1)} \triangleright \alpha_A(b) \otimes \alpha_A(a_0)(12) + a_{(-1)} \triangleleft \alpha_A(b) \otimes \alpha_A(a_0)(13)$$

$$- \alpha_A(a_0) \otimes a_{(-1)} \triangleright \alpha_A(b)(14) - \alpha_A(a_0) \otimes a_{(-1)} \triangleleft \alpha_A(b)(15)$$

$$- [a_0, \alpha_A(b)] \otimes \alpha_H(a_{(-1)})(16) + \theta(\alpha_A(a), b_1) \otimes [\alpha_A(a), b_2](17) + \alpha_A(b_1) \otimes \theta(\alpha_A(a), b_2)(18)$$

$$+ \alpha_A(a_1) \otimes \theta(\alpha_2, \alpha_A(b))(25) + [\alpha_1, \alpha_A(b)] \otimes \alpha_A(a_2)(26) + \alpha_H(b_{(-1)}) \otimes \theta(a_0, \alpha_A(b))(27)$$

$$- \theta(a_0, \alpha_A(b) \otimes \alpha_H(b_{(-1)})(28) + \alpha_H(a_{[1]}) \otimes a_{[2]} \triangleright \alpha_A(b)(29) + \alpha_H(a_{[1]}) \otimes a_{[2]} \triangleleft \alpha_A(b)(30)$$

$$+ a_{[1]} \triangleright \alpha_A(b) \otimes \alpha_H(a_{[2]})(31) + a_{[1]} \triangleleft \alpha_A(b) \otimes \alpha_H(a_{[2]})(32).$$

Then by (TLB3) we get $(a) + (h) = (1) + (2) + (9) + (10) - (3) - (14) + (12) + (7)$; by (TBB7) we get $(b) + (f) = (11) + (5) + (13) - (4) + (17) + (26) + (29) - (23)$, $(c) + (g) = (16) + (6) + (8) - (13) - (18) - (25) - (31) + (21)$; by (TLB1) we get $(d) + (e) = (27) + (19) - (20) - (28) + (30) + (32) - (22) - (24)$.
We investigate the case of (LB) on $H \otimes H$:

$$
\delta_E([x, y]_E) = \delta_E([x, y] + \sigma(x, y)) =
\delta([x, y]) + \psi([x, y]) - \tau \psi([x, y]) + Q([x, y])
+ \delta \sigma(x, y) + \phi \sigma(x, y) - \tau \phi \sigma(x, y) + P \sigma(x, y)
$$

Denote the right hand side terms by $(a), (b), \ldots, (h)$.

$$
[a_H(x), \delta_E(y)] + [\delta_E(x), a_H(y)]
$$

by (TLB4) we get $(a) + (h) = (1) + (2) + (9) + (10) - (12) - (7) + (5) + (16)$; by (TBB8) we get $(b) + (f) = (3) + (13) + (4) - (11) + (25) + (18) + (22) - (32)$, $(c) + (g) = (8) + (14) + (6) - (15) - (26) - (17) - (24) + (30)$; by (TLB2) we get $(d) + (e) = (27) + (19) - (20) - (28) + (21) + (23) - (29) - (31)$.

We now check the axiom (LB) on $H \otimes A$. For $x \in H, a \in A$, we get the equality below:

$$
\delta_E([x, a]_E) = \delta_E(x \triangleright a) + \delta_E(x \triangleleft a) =
\delta_A(x \triangleright a) + \phi(x \triangleright a) - \tau \phi(x \triangleright a) + P(x \triangleright a)
+ \delta_H(x \triangleleft a) + \psi(x \triangleleft a) - \tau \psi(x \triangleleft a) + Q(x \triangleleft a)
$$

Denote the right hand side terms by $(a), (b), \ldots, (h)$.

$$
[a_H(x), \delta_E(a)] + [\delta_E(x), a_A(a)]
$$

by (TLB4) we get $(a) + (h) = (1) + (2) + (9) + (10) - (12) - (7) + (5) + (16)$; by (TBB8) we get $(b) + (f) = (3) + (13) + (4) - (11) + (25) + (18) + (22) - (32)$, $(c) + (g) = (8) + (14) + (6) - (15) - (26) - (17) - (24) + (30)$; by (TLB2) we get $(d) + (e) = (27) + (19) - (20) - (28) + (21) + (23) - (29) - (31)$.
$$+\alpha_A(a_1) \otimes \alpha_H(x) \triangleright a_2(4) + [\alpha_H(x), a_{(-1)}] \otimes \alpha_A(a_{(0)}) \otimes \alpha_H(x) \triangleright a_{(0)}(6)$$

$$\alpha_H(a_{(-1)}) \otimes \alpha_H(x) \triangleright a_{(0)}(7) - \alpha_H(x) \triangleright a_{(0)}(8) - \alpha_H(x) \triangleright a_{(0)}(9)$$

$$-\alpha_A(a_{(0)}) \otimes [\alpha_H(x), a_{(-1)}](10)$$

$$+\sigma(\alpha_H(x), a_{(-1)}) \otimes \alpha_A(a_{(0)})(21) - \alpha_A(a_{(0)}) \otimes \sigma(\alpha_H(x), a_{(-1)})(22)$$

$$+[\alpha_H(x), a_{[1]}] \otimes \alpha_H(a_{[2]})(23) + \sigma(\alpha_H(x), a_{[1]} \otimes \alpha_H(a_{[2]})(24)$$

$$+\alpha_H(a_{[1]} \otimes [\alpha_H(x), a_{[2]}] + \alpha_H(a_{[1]} \otimes \sigma(\alpha_H(x), a_{[2]})(26)$$

$$+\alpha_H(x_1) \otimes x_2 \triangleright \alpha_A(a)(11) + \alpha_H(x_1 \otimes x_2 \triangleleft \alpha_A(a)(12)$$

$$+x_1 \triangleright \alpha_A(a) \otimes \alpha_H(x_2)(13) + x_1 \triangleleft a \otimes \alpha_H(x_2)(14)$$

$$+\alpha_H(x_{(0)} \otimes [x_{(1)}, \alpha_A(a)](15) + x_{(0)} \triangleright \alpha_A(a) \otimes \alpha_A(x_{(1)})(16) + x_{(0)} \triangleleft a \otimes \alpha_A(x_{(1)})(17)$$

$$-\alpha_A(x_{(1)} \otimes x_{(0)} \triangleright \alpha_A(a)(18) - \alpha_A(x_{(1)} \otimes x_{(0)} \triangleleft \alpha_A(a)(19) - [x_{(1)}, \alpha_A(a)] \otimes \alpha_H(x_{(0)})(20)$$

$$+\alpha_H(x_{(0)} \otimes \theta(x_{(1)}, \alpha_A(a))(27) - \theta(x_{(1)}, \alpha_A(a)) \otimes \alpha_H(x_{(0)})(28)$$

$$+\alpha_A(x_{<1>}) \otimes \theta(x_{<2>}, \alpha_A(a))(29) + \alpha_A(x_{<1>}) \otimes [x_{<2>}, \alpha_A(a)](30)$$

$$+[x_{<1>}, \alpha_A(a)] \otimes \alpha_A(x_{<2>})(31) + \theta(x_{<1>}, \alpha_A(a)) \otimes \alpha_A(x_{<2>})(32)$$

Then by (TBB5) we get ($a + (h) = (1) + (3) + (16) - (18) + (29) + (31) + (21) - (22)$; by (TBB6) we get ($d + (e) = (12) + (14) + (7) - (8) + (23) + (25) + (27) - (28)$; by (TYB) we get ($b + (f) = (5) + (6) + (11) + (15) + (17) + (2) + (26) + (32)$, ($c + (g) = (10) + (9) - (13) + (20) + (19) - (4) - (26) - (30)$).

In the case $\theta = 0, P = 0$, then $(A, [\cdot, \cdot])$ is a Hom-Lie algebra and $(A, \delta_A)$ is a Hom-Lie coalgebra and by (TLB3) we obtain that $(A, [\cdot, \cdot], \delta_A)$ is a braided Hom-Lie bialgebra in $\mathcal{H}_{\mathcal{H}} \mathcal{M}$. In the case $\sigma = 0, Q = 0$, then $(H, [\cdot, \cdot], \triangleright)$ is a Hom-Lie algebra and $(H, \delta_H)$ is a Hom-Lie coalgebra and by (TLB4) we obtain that $(H, [\cdot, \cdot], \delta_H)$ is really a braided Hom-Lie bialgebra in $\mathcal{M}_A^A$. That is why in Theorem 4.4 we call $A^{\phi, P}_{\psi, Q} H$ the unified product for braided Hom-Lie bialgeras.

Put $\theta = 0, Q = 0$, then from (TLB3) we get that $(A, \alpha_A)$ is a braided Hom-Lie bialgebra. By the above Theorem 4.3 we obtain:

**Theorem 4.5.** Let $(A, \alpha_A)$ be a braided Hom-Lie bialgebra and $(V, \alpha_V)$ be a Hom-vector space. An extending datum of $(A, \alpha_A)$ by $(V, \alpha_V)$ is $\Omega^b(A, V) = \{\triangleright, \triangleleft, \sigma, [\cdot, \cdot], \psi, Q, \delta_V\}$ consisting of eight linear maps

$$\triangleright : V \times A \rightarrow A, \quad \triangleleft : V \times A \rightarrow V, \quad \sigma : V \times V \rightarrow A, \quad [\cdot, \cdot] : V \times V \rightarrow V,$$

$$\phi : A \rightarrow V \otimes A, \quad \psi : V \otimes A \rightarrow A, \quad P : A \rightarrow V \otimes V, \quad \delta_V : V \rightarrow V \otimes V.$$

Then the unified product $A^{\phi, P}_{\psi, Q} V$ with bracket

$$[(a, x), (b, y)] := ([a, b] + x \triangleright b - y \triangleright a + \sigma(x, y), [x, y] + x \triangleleft b - y \triangleleft a) \quad (18)$$

and cobracket

$$\delta_E(a) = \delta_A(a) + \phi(a) - \tau \phi(a) + P(a), \quad \delta_E(x) = \delta_V(x) + \psi(x) - \tau \psi(x) \quad (19)$$
form a Hom-Lie bialgebra if and only if $A_\circ\#_{\sigma,\theta}V$ form a Hom-Lie algebra, $A_\otimes\#^\psi,QV$ form a Hom-Lie coalgebra and the following conditions are satisfied:

(TBB5) $\delta_A(x \triangleright a) = \alpha_H(x) \triangleright a_1 \otimes \alpha_A(a_2) + \alpha_A(a_1) \otimes \alpha_H(x) \triangleright a_2 + x(0) \triangleright \alpha_A(a) \otimes \alpha_A(x(1)) - \alpha_A(x(1)) \otimes x(0) \triangleright \alpha_A(a) + \sigma(x(0), a(1)) \otimes \alpha_A(a(0)) - \alpha_A(a(0)) \otimes \sigma(x(0), a(0)),

(TBB6) $\delta_H(x \triangleleft a) + P(x \triangleright a) = \alpha_H(x_1) \otimes x_2 \triangleleft \alpha_A(a) + x_1 \triangleleft \alpha_A(a) \otimes x_2 + \alpha_H(a(1)) \otimes \alpha_H(x) \triangleleft a(0) - \alpha_H(x) \triangleleft a(0) \otimes \alpha_H(a(1)) + [\alpha_H(x), P(a)],$

(TBB7) $\phi([a, b]) = \alpha_H(a(1)) \otimes [a(0), \alpha_A(b)] + \alpha_H(b(1)) \otimes [\alpha_A(a), b(0)] + a(-1) \triangleright \alpha_A(b) \otimes \alpha_A(a(0)) - b(-1) \triangleleft \alpha_A(a) \otimes \alpha_A(b(0)) + \alpha_H(a(1)) \otimes a(2) \triangleright \alpha_A(b) - \alpha_H(b(1)) \otimes a(2) \triangleleft \alpha_A(a),$

(TBB8) $\psi(x, y) + \phi(\psi(x, y)) = [\alpha_H(x), y(0)] \otimes \alpha_A(y(1)) + [x(0), \alpha_H(y)] \otimes \alpha_A(x(1)) + \alpha(y(0)) \otimes \alpha_H(x) \triangleright y(1) - \alpha_H(x(0)) \otimes \alpha_H(y) \triangleright x(1) + \alpha_H(x(1)) \otimes \sigma(x_2, \alpha_H(y)) + \alpha_H(y(1)) \otimes \sigma(\alpha_H(x), y_2),$

(TLB1) $P([a, b]) = \alpha_H(a(1)) \otimes a(2) \triangleleft \alpha_A(b) + a(1) \triangleleft \alpha_A(b) \otimes \alpha_A(a(2)) - b(1) \triangleleft \alpha_A(a) \otimes \alpha_A(b(2)) - \alpha_H(b(1)) \otimes a(2) \triangleleft \alpha_A(a),$

(TLB2) $\delta_A\sigma(x, y) = \sigma(x(0), y) \otimes \alpha_A(x(1)) + \sigma(\alpha_H(x), y(0)) \otimes \alpha_A(y(1)) - \alpha_A(y(1)) \otimes \sigma(\alpha_H(x), y(0)) - \alpha_A(x(1)) \otimes \sigma(x(0), y),$

(TLB3) $\delta_A([a, b]) = [\alpha_A(a), \delta_A(b)] + [\delta_A(a), \alpha_A(b)] - b(-1) \triangleright \alpha_A(a) \otimes \alpha_A(b(0)) - \alpha_A(a(0)) \otimes a(-1) \triangleright \alpha_A(b) + a(-1) \triangleright \alpha_A(b) \otimes \alpha_A(a(0)) + \alpha_A(b(0)) \otimes b(-1) \triangleright \alpha_A(a),$

(TLB4) $\delta_H([x, y]) + P\sigma(x, y) = [\delta_H(x), \alpha_H(y)] + [\alpha_H(x), \delta_H(y)] - y(-1) \triangleright \alpha(x) \otimes \alpha_H(y(0)) - \alpha_H(x(0)) \otimes x(-1) \triangleright \alpha_H(y) + x(-1) \triangleright \alpha_H(y) \otimes \alpha_H(x(0)) + \alpha_H(y(0)) \otimes y(-1) \triangleright \alpha_H(x),$

(TYD) $\phi(x \triangleright a) + \psi(x \triangleleft a) = [\alpha_H(x), a(1)] \otimes \alpha_A(a(0)) + \alpha_H(x(1)) \otimes x(0) \triangleright \alpha_A(a) + \alpha_H(x) \triangleleft a(0) + \alpha_H(x) \triangleleft a_1 \otimes \alpha_A(a_2) + \alpha_H(a(1)) \otimes \sigma(\alpha_H(x), a(2)).$

4.2 Matched pairs of braided Hom-Lie bialgebras

In the following of this section, we investigate the spatial case when the cocycle maps $\sigma, \theta$ are zero. In this case, we obtain Hom-Lie bialgebras from two braided Hom-Lie bialgebras. The other cases for which $(A, \alpha_A)$ is a Hom-Lie bialgebra will be given in the next sections.

**Definition 4.6.** Assume that $(A, \alpha)$ and $(H, \sigma)$ are Hom-Lie algebras. If $(A, \triangleright)$ is a left $H$-Hom-Lie module, $(H, \triangleleft)$ is a right $A$-module, and the following condition (M1) and (M2) hold, then $(A, H, \triangleright, \triangleleft)$ (or $(A, H)$) is called a matched pair of Lie algebras:
(BB1) \( \alpha_H(x) \triangleright [a, b] = [x \triangleright a, \alpha_A(b)] + [\alpha_A(a), x \triangleright b] + \alpha_A(a) \triangleright (x \triangleleft b) - (x \triangleleft b) \triangleright \alpha_A(a), \)

(BB2) \( [x, y] \triangleleft \alpha_A(a) = [\alpha_H(x), y \triangleleft a] + [x \triangleleft a, \alpha_H(y)] + \alpha_B(x) \triangleright (y \triangleright a) - \alpha_B(y) \triangleleft (x \triangleright a). \)

**Lemma 4.7.** Let \((A, H)\) be a matched pair of Hom-Lie algebras, then we obtain a new Hom-Lie algebra on the vector space \( E = A \oplus H \) with bracket given by

\[
\alpha_E(a, x) = (\alpha_A(a), \alpha_H(x)),
\]

\[
[(a, x), (b, y)] = ([a, b] + x \triangleright b - y \triangleright a, [x, y] + x \triangleleft b - y \triangleleft a).
\]

We will denote it by \( A \triangleright \triangleleft H \).

The dual version is the matched pair of Hom-Lie coalgebras.

**Definition 4.8.** Two Hom-Lie coalgebras \((A, H)\) form a matched pair of Hom-Lie coalgebras if \((A, \phi)\) is a left \( H \)-Hom-Lie comodule and \((H, \psi)\) is a right \( A \)-comodule, obeying the conditions

(BB3) \( (\alpha \otimes \delta)\phi = (\phi \otimes \alpha)\delta + (\tau \otimes \alpha)(\alpha \otimes \phi)\delta + (\psi \otimes \alpha)\phi + (\alpha \otimes \tau)(\psi \otimes \alpha)\phi, \)

(BB4) \( (\delta \otimes \alpha)\psi = (\alpha \otimes \psi)\delta + (\alpha \otimes \tau)(\alpha \otimes \psi)\delta + (\alpha \otimes \phi)\psi + (\tau \otimes \alpha)(\alpha \otimes \psi)\).

In sigma notation, the above conditions are

(BB3) \[
\sum \alpha_H(a_{-1}) \otimes \delta_A(a_{0}) = \sum \phi(a_1) \otimes \alpha_A(a_2) + \sum \tau_{12}(\alpha_A(a_1) \otimes \phi(a_2)) + \sum \psi(a_{-1}) \otimes \alpha_A(a_{0}) - \sum \tau_{23}(\psi(a_{-1}) \otimes \alpha_A(a_{0})),
\]

(BB4) \[
\sum \delta_H(x_{0}) \otimes \alpha_A(x_{1}) = \sum \alpha_H(x_1) \otimes \psi(x_2) + \sum \tau_{23}(\psi(x_1) \otimes \alpha_H(x_2)) + \sum \alpha_H(x_{0}) \otimes \psi(x_{1}) - \sum \tau_{12}(\alpha_H(x_{0}) \otimes \psi(x_{1})).
\]

**Lemma 4.9.** Let \((A, H)\) be a matched pair of Hom-Lie coalgebras. We define \( E = A \triangleright \triangleleft H \) as the vector space \( A \oplus H \) with Lie cobracket

\[
\delta_E(a) = (\delta_A + \phi - \tau \phi)(a), \quad \delta_E(x) = (\delta_H + \psi - \tau \psi)(x),
\]

that is

\[
\delta_E(a) = \sum a_1 \otimes a_2 + \sum a_{-1} \otimes a_{0} - \sum a_{0} \otimes a_{-1},
\]

\[
\delta_E(x) = \sum x_1 \otimes x_2 + \sum x_{0} \otimes x_{1} - \sum x_{1} \otimes x_{0}.
\]

Then \( A \triangleright \triangleleft H \) is a Hom-Lie coalgebra.

**Definition 4.10.** Let \((A, H)\) be matched pair of Hom-Lie algebras and Hom-Lie coalgebras. If the following conditions hold:

(BB5) \[
\delta_A(x \triangleright a) = \alpha_H(x) \triangleright a_1 \otimes \alpha_A(a_2) + \alpha_A(a_1) \otimes \alpha_H(x) \triangleright a_2 + x_{(0)} \triangleright \alpha_A(a) \otimes \alpha_A(x_{(1)}) - \alpha_A(x_{(1)}) \otimes x_{(0)} \triangleright \alpha_A(a),
\]
\( \psi ([x, y]) = [\alpha (x, y_0)] \otimes \alpha (y_1) + [x_0, \alpha (y)] \otimes \alpha (x_1) \\
+ \alpha (y_0) \otimes \alpha (x_0) \triangleright y_1 - \alpha (x_0) \otimes \alpha (y_0) \triangleright x_1, \)
\( \psi ([x, y]) = \psi (x \triangleright a) + \psi (x \triangleright a) = [\alpha (x, a_{(-1)})] \otimes \alpha (a_0) + \alpha (a_{(-1)}) \otimes \alpha (a_0) \triangleright a_0 + \alpha (x_1) \otimes x_2 \triangleright \alpha (a) \\
+ \alpha (x_0) \otimes [x_1, \alpha (a)] + x_0 \triangleright \alpha (a) \otimes \alpha (x_1) + \alpha (x) \triangleright a_1 \otimes \alpha (a_2), \)
then \((A, H)\) is called a double matched pair.

\textbf{Theorem 4.11.} Let \((A, H)\) be matched pair of Hom-Lie algebras and Hom-Lie coalgebras. If we define the double biproduct of \((A, \alpha_A)\) and \((H, \alpha_H)\), denoted by \(A \triangleright \triangleleft H\), \(A \triangleright \triangleleft H = A \otimes H\) as Lie algebra, \(A \triangleright \triangleleft H = A \triangleright \triangleleft H\) as Hom-Lie coalgebra, then \(A \triangleright \triangleleft H\) become a Hom-Lie bialgebra if and only if \((A, H)\) is a double matched pair.

In particular, the condition \(\text{YDB}\) can be splitting into
\( \phi (x \triangleright a) = [\alpha (x, a_{(-1)})] \otimes \alpha (a_0) + \alpha (a_{(-1)}) \otimes \alpha (a_0) \triangleright a_0 + \alpha (x_1) \otimes x_2 \triangleright \alpha (a) \)
and
\( \psi (x \triangleleft a) = \alpha (x_0) \otimes [x_1, \alpha (a)] + x_0 \triangleright \alpha (a) \otimes \alpha (x_1) + \alpha (x) \triangleright a_1 \otimes \alpha (a_2). \)

In this case, \((A, \alpha_A)\) is a left Hom-Yetter-Drinfeld module in \(\mathcal{H} \otimes \mathcal{M}\) and \((H, \alpha_H)\) is a right Hom-Yetter-Drinfeld module in \(\mathcal{M} \otimes \mathcal{H}\). Together with \(\text{BB8}\) and \(\text{BB9}\), we obtain that \((A, \alpha_A)\) is a braided Hom-Lie bialgebra in \(\mathcal{H} \otimes \mathcal{M}\), \((H, \alpha_H)\) is a braided Hom-Lie bialgebra in \(\mathcal{M} \otimes \mathcal{H}\).

\section{5 Applications}

In this section, we will study the extending problem and non-abelian extension problem for Hom-Lie bialgebra. We will find some special cases when the braided Hom-Lie bialgebra \((A, [], \delta_A)\) is reduced to an ordinary Hom-Lie bialgebra. It is proved that these problems can be solved by using the non-abelian cohomology theory based on our unified product for braided Hom-Lie bialgebras in last section. The proof of most result of this section is by direct computation, so we omit them. For more details, the reader could see \[\text{31}].
5.1 Extending structures for Hom-Lie algebras

There are two cases for $\mathcal{A}(\alpha_A)$ to be a Hom-Lie algebra. The first case is when $\triangleright = 0, \theta \neq 0$, from (TBB1) we get $\sigma(x, \theta(a, b)) = 0$, since $\theta \neq 0$ we assume $\sigma = 0$ for simplicity, thus we obtain the following type (a1) unified product for Hom-Lie algebras.

**Corollary 5.1.** Let $(\mathcal{A}, [\cdot, \cdot])$ be a Hom-Lie algebra and $(V, \alpha_V)$ a Hom-vector space. An extending datum of $(\mathcal{A}, \alpha_A)$ by $(V, \alpha_V)$ of type (a1) is $\Omega^{(1)}(\mathcal{A}, V) = (\beta, \theta, [\cdot, \cdot]_V)$ consisting of bilinear maps

$$\theta : \mathcal{A} \times \mathcal{A} \rightarrow V, \quad \triangleright : V \times \mathcal{A} \rightarrow V, \quad [\cdot, \cdot]_V : V \times V \rightarrow V.$$  

Denote by $A_\theta \#_\triangleright V$ the vector space $E = A \oplus V$ with bracket $[\cdot, \cdot] : E \times E \rightarrow E$ given by

$$[(a, x), (b, y)] := ([a, b], x \triangleright y - y \triangleright a + [x, y] + \theta(a, b)),$$

for all $a, b \in \mathcal{A}, x, y \in V$. Then $A_\theta \#_\triangleright V$ is a Hom-Lie algebra if and only if the following compatibility conditions hold for all $a, b \in \mathcal{A}, x, y, z \in V$:

(A1) $\theta(a, a) = 0$, $\ [x, x] = 0$,

(A2) $[[x, y], \alpha_V(z)] + [[y, z], \alpha_V(x)] + [[z, x], \alpha_V(y)] = 0$,

(A3) $[x, y] \triangleright \alpha_A(a) = [\alpha_V(x), y \triangleright a] + [x \triangleright a, \alpha_V(y)]$,

(A4) $\alpha_V(x) \triangleright [a, b] + [\alpha_V(x), \theta(a, b)] = (x \triangleright a) \triangleright \alpha_A(b) - (x \triangleright b) \triangleright \alpha_A(a)$,

(A5) $\theta(a, b) \triangleright \alpha_A(c) + \theta(b, c) \triangleright \alpha_A(a) + \theta(c, a) \triangleright \alpha_A(b) = \theta(\alpha_A(a), [b, c]) + \theta(\alpha_A(b), [c, a]) + \theta(\alpha_A(c), [a, b])$.

Note that in this case by (A2) we obtain that $(V, \alpha_V)$ is a Hom-Lie algebra. Furthermore, $(V, \alpha_V)$ is in fact a subalgebra of $A_\theta \#_\triangleright V$ but $(\mathcal{A}, \alpha_A)$ is not. Instead $(\mathcal{A}, \alpha_A)$ is a quotient algebra of $A_\theta \#_\triangleright V$.

Denote the set of all Hom-Lie algebra extending datum of $(\mathcal{A}, \alpha_A)$ by $(V, \alpha_V)$ of type (a1) by $\mathcal{A}^{(1)}(\mathcal{A}, V)$.

Note that $A_\theta \#_\triangleright V$ is a Hom-Lie algebra containing $V$ as a subalgebra. In fact any Hom-Lie algebraic structure on $(E, \alpha_E)$ containing $(\mathcal{A}, \alpha_A)$ as subspace and $V$ as subalgebra is isomorphic to such a unified product of this type.

In the following, we always assume that $(\mathcal{A}, \alpha_A)$ is a subspace of a Hom-vector space $(E, \alpha_E)$, there exists a projection map $p : E \rightarrow \mathcal{A}$ such that $p(a) = a$, for all $a \in \mathcal{A}$. Then the kernel space $V := \ker(p)$ is also a subspace of $(E, \alpha_E)$ and a complement of $(\mathcal{A}, \alpha_A)$ in $(E, \alpha_E)$.

**Lemma 5.2.** Let $(\mathcal{A}, [\cdot, \cdot])$ be a Hom-Lie algebra and $(E, \alpha_E)$ a Hom-vector space containing $(\mathcal{A}, \alpha_A)$ as a subspace. Suppose that there is a Hom-Lie algebraic structure $(E, [\cdot, \cdot]_E)$ on $(E, \alpha_E)$ such that $(V, \alpha_V)$ is a Lie subalgebra of $(E, \alpha_E)$ and the canonical projection map $p : E \rightarrow \mathcal{A}$ is a Hom-Lie algebra homomorphism. Then there exists a Hom-Lie algebra extending datum $\Omega^{(1)}(\mathcal{A}, V)$ of $(\mathcal{A}, \alpha_A)$ by $(V, \alpha_V)$ such that $(E, [\cdot, \cdot]_E) \cong A_\theta \#_\triangleright V$. 

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Proof. Since $(V, \alpha_V)$ is a subalgebra of $(E, \alpha_E)$, we have $[x, y]_E \in V$. We define the extending datum of $(A, \alpha_A)$ through $(V, \alpha_V)$ by the following formulas:

$$
\langle : V \times A \to V, \quad x \triangleleft a := [x, a]_E,
$$

$$
\theta : A \times A \to V, \quad \theta(a, b) := [a, b]_E - p([a, b]_E),
$$

$$
[\cdot, \cdot]_V : V \times V \to V, \quad [x, y]_V := [x, y]_E.
$$

for any $a, b \in A$ and $x, y \in V$. It is easy to see that the above maps are well defined and $\Omega^{(1)}(A, V) = (\theta, \triangleleft, [\cdot, \cdot]_V)$ is an extending system of $A$ through $(V, \alpha_V)$ and

$$
\varphi : A_\#_\triangleleft V \to E, \quad \varphi(a, x) := a + x
$$

is an isomorphism of Hom-Lie algebras.

□

Lemma 5.3. Let $\Omega^{(1)}(A, V) = (\varphi, \triangleleft, [\cdot, \cdot]_V)$ and $\Omega^{(1)}(A, V) = (\theta', \triangleleft', [\cdot, \cdot]_{V}')$ be two Hom-Lie algebraic extending datums of $(A, \alpha_A)$ by $(V, \alpha_V)$ of type (a1) and $A_\#_\triangleleft V, A_{\theta'}\#_{\triangleleft'} V$ be the corresponding unified products. Then there exists a bijection between the set of all homomorphisms of Hom-Lie algebras $\varphi : A_\#_\triangleleft V \to A_{\theta'}\#_{\triangleleft'} V$ whose restriction on $(A, \alpha_A)$ is the identity map and the set of pairs $(r, s)$, where $r : V \to A$ and $s : V \to V$ are two linear maps satisfying

$$
r(x \triangleleft a) = [r(x), a],
$$

$$
[a, b]' = [a, b] + r\theta(a, b),
$$

$$
r([x, y]) = [r(x), r(y)]',
$$

$$
s(x) \triangleleft' a + \theta'(r(x), a) = s(x \triangleleft a),
$$

$$
\theta'(a, b) = s\theta(a, b),
$$

$$
s([x, y]) = [s(x), s(y)]' + s(x) \triangleleft' r(y) - s(y) \triangleleft' r(x) + \theta'(r(x), r(y)),
$$

for all $a, b \in A$ and $x, y \in V$.

Under the above bijection the homomorphism of Hom-Lie algebras $\varphi = \varphi_{r, s} : A_\#_\triangleleft V \to A_{\theta'}\#_{\triangleleft'} V$ to $(r, s)$ is given by $\varphi(a, x) = (a + r(x), s(x))$ for all $a \in A$ and $x \in V$. Moreover, $\varphi = \varphi_{r, s}$ is an isomorphism if and only if $s : V \to V$ is a linear isomorphism.

The second case is when $\theta = 0, \triangleright \neq 0$, we obtain the following type (a2) unified product for Hom-Lie algebras.

Corollary 5.4. Let $(A, \alpha_A)$ be a Hom-Lie algebra and $(V, \alpha_V)$ a Hom-vector space. An extending datum of $(A, \alpha_A)$ by $(V, \alpha_V)$ of type (a2) is $\Omega^{(2)}(A, V) = (\triangleright, \triangleleft, [\cdot, \cdot])$ consisting of four bilinear maps

$$
\triangleright : V \times A \to A, \quad \triangleleft : V \times A \to V, \quad \sigma : V \times V \to A, \quad [\cdot, \cdot] : V \times V \to V.
$$

Denote by $A_{\triangleright, \triangleleft, \sigma, \#} H$ the vector space $E = A \oplus V$ with the bilinear map $[\cdot, \cdot]_E : E \times E \to E$ given by

$$
[(a, x), (b, y)] := ([a, b] + x \triangleright b - y \triangleright a + \sigma(x, y), x \triangleleft b - y \triangleleft a + [x, y]),
$$

(27)
for all \( a, b \in A, x, y \in V \). Then \( A_{\sigma}#_{\alpha}V \) is a Hom-Lie algebra if and only if the following compatibility conditions hold for all \( a, b \in A, x, y, z \in V \):

\begin{align*}
(B1) \quad &\sigma(x, x) = 0, \quad [x, x] = 0, \\
(B2) \quad &\alpha_V(x) \triangleleft [a, b] = (x \triangleleft a) \triangleleft \alpha_A(b) - (x \triangleleft b) \triangleleft \alpha_A(a), \\
(B3) \quad &\alpha_V(x) \triangleright [a, b] = [x \triangleright a, \alpha_A(b)] + [\alpha_A(a), x \triangleright b] + (x \triangleleft a) \triangleright \alpha_A(b) - (x \triangleleft b) \triangleright \alpha_A(a), \\
(B4) \quad &[x, y] \triangleleft \alpha_A(a) = [\alpha_V(x), y \triangleleft a] + [x \triangleleft a, \alpha_V(y)] + \alpha_V(x) \triangleleft (y \triangleright a) - \alpha_V(y) \triangleleft (x \triangleright a), \\
(B5) \quad &[x, y] \triangleright \alpha_A(a) = \alpha_V(x) \triangleright (y \triangleright a) - \alpha_V(y) \triangleright (x \triangleright a) + \sigma(\alpha_V(x), y \triangleleft a) + \sigma(x \triangleleft a, \alpha_V(y)) + [\alpha_A(a), \sigma(x, y)], \\
(B6) \quad &\alpha_V(x) \triangleright \sigma(y, z) + \alpha_V(y) \triangleright \sigma(z, x) + \alpha_V(z) \triangleright \sigma(x, y) = \sigma([x, y], \alpha_V(z)) + \sigma([y, z], \alpha_V(x)) + \sigma([z, x], \alpha_V(y)), \\
(B7) \quad &[[x, y], \alpha_V(z)] + [[y, z], \alpha_V(x)] + [[z, x], \alpha_V(y)] = \alpha_V(x) \triangleleft \sigma(y, z) + \alpha_V(y) \triangleleft \sigma(z, x) + \alpha_V(z) \triangleleft \sigma(x, y).
\end{align*}

In this case, \((A, \alpha_A)\) is a subalgebra of \((E, \alpha_E)\) and \((V, \alpha_V)\) is in fact a \(\sigma\)-Hom-Lie algebra acting on \((A, \alpha_A)\).

Denote the set of all Hom-Lie algebra extending datum of \((A, \alpha_A)\) by \((V, \alpha_V)\) of type \((a2)\) by \(A^{(2)}(A, V)\).

Note that \(A_{\sigma}#_{\alpha}H\) is a Hom-Lie algebra containing \((A, \alpha_A)\) as a subalgebra. In fact, any Hom-Lie algebraic structure on \((E, \alpha_E)\) containing \((A, \alpha_A)\) as a subalgebra is isomorphic to such a unified product.

**Lemma 5.5.** Let \((A, \cdot, \cdot)\) be a Hom-Lie algebra and \((E, \alpha_E)\) a Hom-vector space containing \((A, \alpha_A)\) as a subspace. Suppose that there is a Hom-Lie algebraic structure \((E, \cdot, \cdot)\) on \((E, \alpha_E)\) such that \((A, \cdot, \cdot)\) is a Lie subalgebra of \((E, \alpha_E)\). Then there exists a Hom-Lie algebraic extending system \(\Omega^{(2)}(A, V)\) of \((A, \alpha_A)\) by \((V, \alpha_V)\) such that \((E, \cdot, \cdot, \delta_E) \cong A_{\sigma}#_{\alpha}V\).

**Lemma 5.6.** Let \(\Omega^{(2)}(A, V) = (\triangleright, \triangleleft, \alpha, \cdot, \cdot)\) and \(\Omega^{(2)}(A, V) = (\triangleright', \triangleleft', \alpha', \cdot', \cdot')\) be two Hom-Lie algebraic extending datums of \((A, \alpha_A)\) by \((V, \alpha_V)\) of type \((a2)\) and \(A_{\sigma}#_{\alpha}V, A_{\sigma'}#_{\alpha'}V\) be the corresponding unified products. Then there exists a bijection between the set of all homomorphisms of Hom-Lie algebras \(\varphi : A_{\sigma}#_{\alpha}V \to A_{\sigma'}#_{\alpha'}V\) whose restriction on \((A, \alpha_A)\) is the identity map and the set of pairs \((r, s)\), where \(r : V \to A\) and \(s : V \to V\) are two linear maps satisfying

\begin{align*}
s(x) \triangleleft a &= s(x \triangleleft a), \quad (28) \\
r(x \triangleleft a) &= [r(x), a] - x \triangleright a + s(x) \triangleright' a, \quad (29) \\
s([x, y]) &= [s(x), s(y)]' + s(x) \triangleleft' r(y) - s(y) \triangleleft' r(x), \quad (30) \\
r([x, y]) &= [r(x), r(y)] + s(x) \triangleright' r(y) - s(y) \triangleright' r(x) + \sigma'(s(x), s(y)) - \sigma(x, y)
\end{align*}
for all \( a \in A \) and \( x, y \in V \).

Under the above bijection the homomorphism of Hom-Lie algebras \( \varphi = \varphi_{r,s} : A \#_{a,s} V \to A \#_{a,s} V \) to \( (r, s) \) is given by \( \varphi(a,x) = (a + r(x), s(x)) \) for all \( a \in A \) and \( x \in V \). Moreover, \( \varphi = \varphi_{r,s} \) is an isomorphism if and only if \( s : V \to V \) is a linear isomorphism.

Let \( (A, \alpha_A) \) be a Hom-Lie algebra and \( (V, \alpha_V) \) a Hom-vector space. Two Hom-Lie algebra extending systems \( \Omega(i)(A, V) \) and \( \Omega(i)(A, V) \) are called equivalent if \( \varphi_{r,s} \) is an isomorphism. We denote it by \( \Omega(i)(A, V) \equiv \Omega(i)(A, V) \). From the above lemmas, we obtain the following result.

**Theorem 5.7.** Let \( (A, [\cdot, \cdot]) \) be a Hom-Lie algebra, \( (E, \alpha_E) \) a Hom-vector space containing \( (A, \alpha_A) \) as a subspace and \( (V, \alpha_V) \) be a complement of \( (A, \alpha_A) \) in \( (E, \alpha_E) \). Denote \( \mathcal{H}A(V, A) := A^{(1)}(A, V) \sqcup A^{(2)}(A, V)/ \equiv \). Then the map

\[
\Psi : \mathcal{H}A(V, A) \to \text{Ext}(E, A), \quad \Omega^{(1)}(A, V) \mapsto A \#_{a,s} V, \quad \Omega^{(2)}(A, V) \mapsto A \#_{a,s} V
\]

is bijective, where \( \Omega(i)(A, V) \) is the equivalence class of \( \Omega(i)(A, V) \) under \( \equiv \).

**5.2 Extending structures for Hom-Lie coalgebras**

Next we consider the Hom-Lie coalgebra structures on \( E = A^{\phi,P} \#_{\psi,Q} V \).

There are two cases for \( (A, \delta_A) \) to be a Hom-Lie coalgebra. The first case is when \( \phi \neq 0, Q = 0 \), we obtain the following type (c1) unified product for Hom-Lie coalgebras.

**Corollary 5.8.** Let \( (A, \delta_A) \) be a Hom-Lie coalgebra and \( (V, \alpha_V) \) a Hom-vector space. An extending datum of \( (A, \alpha_A) \) by \( (V, \alpha_V) \) of type (c1) is \( \Omega^{(1)}(A, V) = (\phi, \psi, P, \delta_V) \) with linear maps

\[
\phi : A \to V \otimes A, \quad \psi : V \to V \otimes A, \quad P : A \to V \otimes V, \quad \delta_V : V \to V \otimes V.
\]

Denote by \( A^{\phi,P} \#_{\psi,Q} V \) the vector space \( E = A \oplus V \) with the linear map \( \delta_E : E \to E \otimes E \) given by

\[
\delta_E(a) = \delta_A(a) + \phi(a) - \tau \phi(a) + P(a), \quad \delta_E(x) = \delta_V(x) + \psi(x) - \tau \psi(x).
\]

Then \( A^{\phi,P} \#_{\psi,Q} V \) is a Hom-Lie coalgebra with the Lie cobracket given by \([34]\) if and only if the following compatibility conditions hold:

\[(C1) \quad P(a) = -\tau P(a), \quad \delta_V(x) = -\tau \delta_V(x), \]

\[(C2) \quad \delta_V(a_{(-1)}) \otimes \alpha_A(a_{(0)}) + P(a_1) \otimes \alpha_A(a_2) = \alpha_V(a_{(-1)}) \otimes \phi(a_{(0)}) - \tau_{12} \left( a_{(-1)} \otimes \phi(a_{(0)}) \right)
\]
\[\quad + \alpha_V(a_{[1]}) \otimes \psi(a_{[2]}) + \tau_{23} \left( \psi(a_{[1]}) \otimes \alpha_V(a_{[2]}) \right),\]

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(C3) $\alpha_V(x(0)) \otimes \delta_A(x(1)) = \psi(x(0)) \otimes \alpha_A(x(1)) - \tau_{23} \left( \psi(x(0)) \otimes \alpha_A(x(1)) \right)$,

(C4) $\alpha_V(a_{(-1)}(0)) \otimes \delta_A(a(0)) = \phi(a(1)) \otimes \alpha_A(a(2)) + \tau_{12} \left( \alpha_A(a(1)) \otimes \phi(a(2)) \right)$

$+ \psi(a_{(-1)}(0)) \otimes \alpha_A(a(0)) - \tau_{23} \left( \psi(a_{(-1)}(0)) \otimes \alpha_A(a(0)) \right)$,

(C5) $\delta_V(x(0)) \otimes \alpha_A(x(1)) = \alpha_V(x(1)) \otimes \psi(x(2)) + \tau_{23} \left( \psi(x(1)) \otimes \alpha_V(x(2)) \right)$

$+ \alpha_V(x(0)) \otimes \psi(x(1)) - \tau_{12} \left( \alpha_V(x(0)) \otimes \psi(x(1)) \right)$,

(C6) $\alpha_V(a_{(-1)}(0)) \otimes P(a(0)) + \tau_{12} \tau_{23} \left( \alpha_V(a_{(-1)}(0)) \otimes P(a(0)) \right) + \tau_{23} \tau_{12} \left( \alpha_V(a_{(-1)}(0)) \otimes P(a(0)) \right)$

$= \delta(a_{(1)}(0)) \otimes \alpha_V(a_{(2)}(0)) + \tau_{12} \tau_{23} \left( \delta(a_{(1)}(0)) \otimes \alpha_V(a_{(2)}(0)) \right)$

$+ \tau_{23} \tau_{12} \left( \delta(a_{(1)}(0)) \otimes \alpha_V(a_{(2)}(0)) \right)$,

(C7) $\delta(x_{(1)}) \otimes \alpha_V(x_{(2)}) + \tau_{12} \tau_{23} \left( \delta(x_{(1)}) \otimes \alpha_V(x_{(2)}) \right) + \tau_{23} \tau_{12} \left( \delta(x_{(1)}) \otimes \alpha_V(x_{(2)}) \right)$

$= \alpha_V(x_{(0)}) \otimes P(x_{(1)}) + \tau_{12} \tau_{23} \left( \alpha_V(x_{(0)}) \otimes P(x_{(1)}) \right) + \tau_{23} \tau_{12} \left( \alpha_V(x_{(0)}) \otimes P(x_{(1)}) \right)$.

Denote the set of all Hom-Lie coalgebra extending datum of $(A, \alpha_A)$ by $(V, \alpha_V)$ of type (c1) by $C^{(1)}(A, V)$.

In this case, although $(A, \alpha_A)$ is a Hom-Lie coalgebra but it is not a subcoalgebra of $A^{\delta, P} \#^\psi V$. The characterization of this type of Hom-Lie coalgebra $(A, \alpha_A)$ is as follows.

**Lemma 5.9.** Let $(A, \delta_A)$ be a Hom-Lie coalgebra and $(E, \alpha_E)$ a Hom-vector space containing $(A, \alpha_A)$ as a subspace. Suppose that there is a Hom-Lie coalgebra structure $(E, \delta_E)$ on $(E, \alpha_E)$ such that $p : E \rightarrow A$ is a Hom-Lie coalgebra homomorphism. Then there exists a Hom-Lie coalgebra extending system $\Omega^{(1)}(A, V)$ of $(A, \delta_A)$ by $(V, \alpha_V)$ such that $(E, \delta_E) \cong A^{\delta, P} \#^\psi V$.

**Proof.** Let $p : E \rightarrow A$ and $\pi : E \rightarrow V$ be the projection maps and $V = \ker(p)$. Then the extending datum of $(A, \delta_A)$ by $(V, \alpha_V)$ is defined as follows:

$\phi : A \rightarrow V \otimes A, \quad \phi(x) = (\pi \otimes p) \delta_E(a)$,

$\psi : V \rightarrow V \otimes A, \quad \psi(x) = (\pi \otimes p) \delta_E(x)$,

$\delta_V : V \rightarrow V \otimes V, \quad \delta_V(x) = (\pi \otimes \pi) \delta_E(x)$,

$P : A \rightarrow V \otimes V, \quad P(a) = (\pi \otimes \pi) \delta_E(a)$.

One check that $\varphi : A^{\delta, P} \#^\psi V \rightarrow E$ given by $\varphi(a, x) = a + x$ for all $a \in A, x \in V$ is a Hom-Lie coalgebra isomorphism. \hfill \Box

**Lemma 5.10.** Let $\Omega^{(1)}(A, V) = (\phi, \psi, P, \delta_V)$ and $\Omega^{(1)}(A, V) = (\phi', \psi', P', \delta'_V)$ be two Hom-Lie coalgebra extending datums of $(A, \delta_A)$ by $(V, \alpha_V)$. Then there exists a bijection between the set of Hom-Lie coalgebra homomorphisms $\varphi : A^{\delta, P} \#^\psi V \rightarrow A^{\delta', P'} \#^\psi V$ whose restriction on $(A, \alpha_A)$ is the identity map and the set of pairs $(r, s)$, where $r : V \rightarrow A$ and $s : V \rightarrow V$ are two linear maps satisfying

$P'(a) = s(a_{(1)}) \otimes s(a_{(2)}), \quad \tag{35}$

$\phi'(a) = s(a_{(1)}) \otimes a_{(0)} + s(a_{(1)}) \otimes r(a_{(2)}), \quad \tag{36}$

$\delta'_A(a) = \delta_A(a) + r(a_{(1)}) \otimes a_{(0)} - a_{(0)} \otimes r(a_{(1)}) + r(a_{(1)}) \otimes r(a_{(2)}) \quad \tag{37}$

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\[ \delta'_{V}(s(x)) = (s \otimes s)\delta_{V}(x), \quad (38) \]
\[ \psi'(s(x)) = s(x_1) \otimes r(x_2) + s(x_0) \otimes x_1, \quad (39) \]
\[ \delta'_A(r(x)) = r(x_1) \otimes r(x_2) + r(x_0) \otimes (x_1 - x_1 \otimes r(x_0)). \quad (40) \]

Under the above bijection the Hom-Lie coalgebra homomorphism \( \varphi = \varphi_{r,s} : A^\phi.P^\#V \rightarrow A^\delta'.P^\#V \) to \((r,s)\) is given by \( \varphi(a,x) = (a + r(x), s(x)) \) for all \( a \in A \) and \( x \in V \). Moreover, \( \varphi = \varphi_{r,s} \) is an isomorphism if and only if \( s : V \rightarrow V \) is a linear isomorphism.

In the case \( \phi = 0, Q \neq 0 \), then from (TBB3) we get that \( a_{[1]} \otimes Q(a_{[2]}) = 0 \), since \( Q \neq 0 \) we assume \( P = 0 \) for simplicity, thus we obtain the following type (c2) unified product for Hom-Lie coalgebras.

**Corollary 5.11.** Let \((A, \delta_A)\) be a Hom-Lie coalgebra and \((V, \alpha_V)\) a Hom-vector space. An extending datum of \((A, \delta_A)\) by \((V, \alpha_V)\) of type (c2) is \(\Omega^{(2)}(A, V) = (\psi, Q, \delta_V)\) with linear maps
\[ \psi : V \rightarrow V \otimes A, \quad Q : V \rightarrow A \otimes A, \quad \delta_V : V \rightarrow V \otimes V. \]

Denote by \(A^\#QV\) the vector space \(E = A \oplus V\) with the linear map \(\delta_E : E \rightarrow E \otimes E\) given by
\[ \delta_E(a) = \delta_A(a), \quad \delta_E(x) = \delta_V(x) + \psi(x) - \tau\psi(x) + Q(x). \quad (41) \]

Then \(A^\#QV\) is a Hom-Lie coalgebra with the Lie cobracket given by \(\{41\}\) if and only if the following compatibility conditions hold:

\[ (D1) \ Q(x) = -\tau Q(x), \quad \delta_V(x) = -\tau\delta_V(x), \]
\[ (D2) \ \delta_V(x_1) \otimes \alpha_V(x_2) + \tau_1 \tau_2 (\delta_V(x_1) \otimes \alpha_V(x_2)) + \tau_2 \tau_3 (\delta_V(x_1) \otimes \alpha_V(x_2)) = 0, \]
\[ (D3) \ \alpha_V(x_{[0]}) \otimes \delta_A(x_{[1]}) + \alpha_V(x_{[1]}) \otimes \psi(x_2) = \psi(x_{[0]}) \otimes \alpha_A(x_{[1]}) - \tau_2 (\psi(x_{[0]}) \otimes \alpha_A(x_{[1]})) \]
\[ \quad + \psi(x_{[1]}) \otimes \alpha_A(x_{[2]}) + \tau_1 (\alpha_A(x_{[1]}) \otimes \psi(x_{[2]})), \]
\[ (D4) \ \delta_V(x_{[0]}) \otimes \alpha_A(x_{[1]}) = \alpha_V(x_{[1]}) \otimes \psi(x_2) + \tau_2 (\psi(x_{[1]}) \otimes \alpha_V(x_2)), \]
\[ (D5) \ Q(x_{[0]}) \otimes \alpha_A(x_{[1]}) + \tau_1 \tau_2 (Q(x_{[0]}) \otimes \alpha_A(x_{[1]})) + \tau_2 \tau_3 (Q(x_{[0]}) \otimes \alpha_A(x_{[1]})) \]
\[ = \alpha_A(x_{[1]}) \otimes \delta_A(x_{[2]}) + \tau_1 \tau_2 (\alpha_A(x_{[1]}) \otimes \delta_A(x_{[2]})) + \tau_2 \tau_3 (\alpha_A(x_{[1]}) \otimes \delta_A(x_{[2]})). \]

Denote the set of all Hom-Lie coalgebra extending datum of \((A, \alpha_A)\) by \((V, \alpha_V)\) of type (c2) by \(C^{(2)}(A, V)\).

Similar as Hom-Lie algebra case, one show that any Hom-Lie coalgebra structure on \((E, \alpha_E)\) containing \((A, \alpha_A)\) as a subcoalgebra is isomorphic to such a unified coproduct.

**Lemma 5.12.** Let \((A, \delta_A)\) be a Hom-Lie coalgebra and \((E, \alpha_E)\) a Hom-vector space containing \((A, \alpha_A)\) as a subspace. Suppose that there is a Hom-Lie coalgebra structure \((E, \delta_E)\) on \((E, \alpha_E)\) such that \((A, \delta_A)\) is a Lie subcoalgebra of \((E, \alpha_E)\). Then there exists a Hom-Lie coalgebra extending system \(\Omega^{(2)}(A, V)\) of \((A, \delta_A)\) by \((V, \alpha_V)\) such that \((E, \delta_E) \cong A^\#QV\).
Proof. Let $p : E \to A$ and $\pi : E \to V$ be the projection map and $V = \ker(p)$. Then the extending datum of $(A, \delta_A)$ by $(V, \alpha_V)$ is defined as follows:

$\psi : V \to V \otimes A, \quad \phi(x) = (\pi \otimes p)\delta_E(x),$

$\delta_V : V \to V \otimes V, \quad \delta_V(x) = (\pi \otimes \pi)\delta_E(x),$

$Q : V \to A \otimes A, \quad Q(x) = (p \otimes p)\delta_E(x).$

One check that $\varphi : A \#^{\psi,Q} V \to E$ given by $\varphi(a, x) = a + x$ for all $a \in A, x \in V$ is a Hom-Lie coalgebra isomorphism.

Lemma 5.13. Let $\Omega^{(2)}(A, V) = (\psi, Q, \delta_V)$ and $\Omega^{(2)}(A, V) = (\psi', Q', \delta_V')$ be two Hom-Lie coalgebra extending datums of $(A, \delta_A)$ by $(V, \alpha_V)$. Then there exists a bijection between the set of Hom-Lie coalgebra homomorphisms $\varphi : A \#^{\psi,Q} V \to A \#^{\psi',Q'} V$ whose restriction on $(A, \alpha_A)$ is the identity map and the set of pairs $(r, s)$, where $r : V \to A$ and $s : V \to V$ are two linear maps satisfying

$$\psi'(s(x)) = s(x_1) \otimes r(x_2) + s(x_0) \otimes x_1, \quad (42)$$

$$\delta_V'(s(x)) = (s \otimes s)\delta_V(x), \quad (43)$$

$$\delta_V'(r(x)) + \delta_V'(s(x)) = r(x_1) \otimes r(x_2) + r(x_0) \otimes x_1 - x_1 \otimes r(x_0) + Q(x). \quad (44)$$

Under the above bijection the Hom-Lie coalgebra homomorphism $\varphi = \varphi_{r,s} : A \#^{\psi,Q} V \to A \#^{\psi',Q'} V$ to $(r, s)$ is given by $\varphi(a, x) = (a + r(x), s(x))$ for all $a \in A$ and $x \in V$. Moreover, $\varphi = \varphi_{r,s}$ is an isomorphism if and only if $s : V \to V$ is a linear isomorphism.

Let $(A, \delta_A)$ be a Hom-Lie coalgebra and $(V, \alpha_V)$ a Hom-vector space. Two Hom-Lie coalgebra extending systems $\Omega^{(i)}(A, V)$ and $\Omega^{(i)}(A, V)$ are called equivalent if $\varphi_{r,s}$ is an isomorphism. We denote it by $\Omega^{(i)}(A, V) \equiv \Omega^{(i)}(A, V)$. From the above lemmas, we obtain the following result.

Theorem 5.14. Let $(A, \delta_A)$ be a Hom-Lie coalgebra, $(E, \alpha_E)$ a Hom-vector space containing $(A, \alpha_A)$ as a subspace and $(V, \alpha_V)$ be a $A$-complement in $(E, \alpha_E)$. Denote $\mathcal{HC}(V, A) := C^{(1)}(A, V) \sqcup C^{(2)}(A, V) / \equiv$. Then the map

$$\Psi : \mathcal{HC}_A(V, A) \to C\mathcal{Ext}(E, A), \quad (45)$$

$$\Omega^{(1)}(A, V) \mapsto A^{\phi,F} \#^{\psi} V, \quad \Omega^{(2)}(A, V) \mapsto A \#^{\psi,Q} V \quad (46)$$

is bijective, where $\overline{\Omega^{(i)}(A, V)}$ is the equivalence class of $\Omega^{(i)}(A, V)$ under $\equiv$.

5.3 Extending structures for Hom-Lie bialgebras

There are two special cases for which $(A, [\cdot, \cdot], \delta_A)$ is reduced to a Hom-Lie bialgebra. The first case is when $\triangleright = 0, \sigma = 0, Q = 0$ in the above Theorem 4.3. In this case we obtain the following result.
Theorem 5.15. Let \((A,\lbrack \cdot, \cdot \rbrack, \delta_A)\) be a Hom-Lie bialgebra and \((V, \alpha_V)\) a Hom-vector space. An extending datum of \((A, \alpha_A)\) by \((V, \alpha_V)\) of type (I) is \(\Omega^{(1)}(A, V) = (\varphi, \phi, \psi, \Pi, \lbrack \cdot, \cdot \rbrack, \vartheta, \delta_V)\) consisting of linear maps

\[ \varphi : V \times A \to V, \quad \theta : A \times A \to V, \quad \lbrack \cdot, \cdot \rbrack : V \times V \to V, \]
\[ \phi : A \to V \otimes A, \quad \psi : V \to V \otimes A, \quad \Pi : A \to V \otimes V, \quad \delta_V : V \to V \otimes V. \]

Then the unified product \(A_{\varphi, \psi}^{P} \#_{\delta} V\) with bracket

\[ [(a, x), (b, y)] := ([a, b], [x, y] + x \triangleleft b - y \triangleleft a + \theta(a, b)) \] (47)

and cobracket

\[ \delta_E(a) = \delta_A(a) + \phi(a) - \tau(a) + P(a), \quad \delta_E(x) = \delta_V(x) + \psi(x) - \tau(x) \] (48)

form a Hom-Lie bialgebra if and only if \(A_{\varphi, \psi}^{P} \#_{\delta} V\) form a Hom-Lie coalgebra and the following conditions are satisfied:

(E1) \[ \delta_V(x \triangleleft a) = \alpha_V(x_1) \otimes x_2 \triangleleft \alpha_A(a) + x_1 \triangleleft \alpha_A(a) \otimes x_2 \]
\[ + \alpha_V(a_{(-1)}) \otimes \alpha_V(x) \triangleleft a_{(0)} - \alpha_V(x) \triangleleft a_{(0)} \otimes \alpha_V(a_{(-1)}) \]
\[ + [\alpha_V(x), P(a)] + \alpha_V(x_{(0)}) \otimes \theta(x_{(1)}, a_{(1)}) - \theta(x_{(1)}, a_{(1)}) \otimes \alpha_V(x_{(0)}), \]

(E2) \[ \phi([a, b]) + \psi(a, b) = \alpha_V(a_{(-1)}) \otimes [a_{(0)}, \alpha_A(b)] + \alpha_H(b_{(-1)}) \otimes [\alpha_A(a), b_{(0)}] \]
\[ + [a_{(-1)} \triangledown \alpha_A(b) \triangledown \alpha_A(a_{(0)}) - b_{(-1)} \triangledown \alpha_A(a) \triangledown \alpha_A(b_{(0)}) \]
\[ + \theta(a_{(1)}, a_{(0)}) \triangledown \alpha_A(b_1) \triangledown \alpha_A(a_{(1)}) \triangledown \alpha_A(a_{(2)}), \]

(E3) \[ \psi([x, y]) = [\alpha_V(x), y_{(0)}] \triangledown \alpha_A(y_{(1)}) + [x_{(0)}, \alpha_V(y)] \triangledown \alpha_A(x_{(1)}), \]

(E4) \[ \delta_V([a, b]) = \alpha_V(a_{(-1)}) \otimes \theta(a_{(0)}, \alpha_A(b)) + \alpha_H(b_{(-1)}) \otimes \theta(\alpha_A(a), b_{(0)}) \]
\[ - \theta(\alpha_A(a), b_{(0)}) \otimes \alpha_H(b_{(-1)}) - \theta(a_{(0)}, \alpha_A(b)) \otimes \alpha_V(a_{(-1)}) \]
\[ + [\alpha_V(a_{(1)}) \otimes a_{(2)} \triangledown \alpha_A(b) + a_{(1)} \triangledown \alpha_A(b) \otimes \alpha_V(a_{(2)}) \]
\[ - b_{(1)} \triangledown \alpha_A(a) \otimes \alpha_V(b_{(2)}) - \alpha_V(b_{(1)}) \otimes b_{(2)} \triangledown \alpha_A(a), \]

(E5) \[ \delta_V([x, y]) = [\delta_V(x), \alpha_V(y)] + [\alpha_V(x), \delta_V(y)] \]
\[ - x_{(0)} \otimes \alpha_V(y) \triangledown x_{(1)} - \alpha_V(x) \triangledown y_{(1)} \otimes y_{(0)} \]
\[ + y_{(0)} \otimes \alpha_V(x) \triangledown y_{(1)} + \alpha_V(y) \triangledown x_{(1)} \otimes x_{(0)}, \]

(E6) \[ \psi(x \triangleleft a) = \alpha_V(x_{(0)}) \otimes [x_{(1)}, \alpha_A(a)] + x_{(0)} \triangledown \alpha_A(a) \triangledown \alpha_A(x_{(1)}) + \alpha_V(x) \triangledown a_1 \otimes \alpha_A(a_2), \]

Conversely, any Hom-Lie bialgebra structure on \((E, \alpha_E)\) with the canonical projection map \(p : E \to A\) both a Hom-Lie algebra homomorphism and a Hom-Lie coalgebra homomorphism is of this form.

Note that in this case, although \((A, \lbrack \cdot, \cdot \rbrack, \delta_A)\) is not a Lie sub-bialgebra of \(A_{\varphi, \psi}^{P} \#_{\delta} V\), but it is indeed a Hom-Lie bialgebra and a subspace \(A_{\varphi, \psi}^{P} \#_{\delta} V\). Denote the set of all Hom-Lie bialgebra extending datum of type (I) by \(\mathcal{CB}^{(1)}(A, V)\).

The second case is when \(\theta = 0, P = 0, \phi = 0\) in the above Theorem 5.14. In this case we obtain the following result.
Theorem 5.16. Let \((A, \alpha_A)\) be a Hom-Lie bialgebra and \((V, \alpha_V)\) a Hom-vector space. An extending datum of \((A, \alpha_A)\) by \((V, \alpha_V)\) of type (II) is \(\Omega^{(2)}(A, V) = \langle >, \triangleleft, \psi, Q, [, , ]_V, \delta_V \rangle\) consisting of linear maps

\[
\begin{align*}
\triangleright & : V \times A \to A, \\
\triangleleft & : V \times A \to V, \\
\sigma & : V \times V \to A, \\
[, , ]_V & : V \times V \to V,
\end{align*}
\]

\[
\psi : V \to V \otimes A, \quad Q : V \to A \otimes A, \quad \delta_V : V \to V \otimes V.
\]

Then the unified product \(A_v \#_{\triangleleft, \psi, Q} V\) with bracket

\[
[(a, x), (b, y)] := ([a, b] + x \triangleright b - y \triangleright a + \sigma(x, y), [x, y] + x \triangleleft b - y \triangleleft a)
\]

and cobracket

\[
\delta_E(a) = \delta_A(a), \quad \delta_E(x) = \delta_V(x) + \psi(x) - \tau\psi(x) + Q(x)
\]

form a Hom-Lie bialgebra if and only if \(A_v \#_{\triangleleft, \psi, Q} V\) form a Hom-Lie algebra, \(A^{\#_{\psi, Q}} V\) form a Hom-Lie coalgebra and the following conditions are satisfied:

\[(F1) \quad \delta_A(x \triangleright a) + Q(x \triangleleft a) = \alpha_V(x) \triangleright a_1 \otimes \alpha_A(a_2) + \alpha_A(a_1) \otimes \alpha_V(x) \triangleright a_2
\]

\[
+ x(0) \triangleright \alpha_A(a) \otimes \alpha_A(x(1)) - \alpha_A(x(1)) \otimes x(0) \triangleright \alpha_A(a) + [Q(x), \alpha_A(a)],
\]

\[(F2) \quad \delta_V(x \triangleleft a) = \alpha_V(x_1) \otimes x_2 \triangleleft \alpha_A(a) + x_1 \triangleleft \alpha_A(a) \otimes x_2,
\]

\[(F3) \quad \psi([x, y]) = [\alpha_V(x), y(0)] \otimes \alpha_A(y(1))\]

\[
+ [x(0), \alpha_V(y)] \otimes \alpha_A(x(1))
\]

\[
+ \alpha_V(x) \triangleright y(1) - \alpha_V(x(0)) \triangleright \alpha_V(y) \triangleright x(1)
\]

\[
+ \alpha_V(x) \triangleleft x(2) \otimes \alpha_V(y) + \alpha_V(y) \otimes x(2)
\]

\[
+ \alpha_V(x) \triangleleft y(2) \otimes \alpha_A(x < 2>) - \alpha_V(y) \triangleleft x < 2> \otimes \alpha_A(x < 2>),
\]

\[(F4) \quad \delta_A \sigma(x, y) + Q([x, y]) = \sigma(x(0), y) \otimes \alpha_A(x(1)) + \sigma(\alpha_V(x), y(0)) \otimes \alpha_A(y(1))
\]

\[- \alpha_A(y(1)) \otimes \sigma(\alpha_V(x), y(0)) - \alpha_A(x(1)) \otimes \sigma(x(0), y)
\]

\[
+ \alpha_V(x) \triangleright y < 2> \otimes \alpha_A(y < 2>) + \alpha_A(y < 2>) \otimes \alpha_V(x) \triangleright y < 2>
\]

\[- \alpha_A(x < 2>) \otimes \alpha_V(y) \triangleright x < 2> - \alpha_V(y) \triangleright x < 2> \otimes \alpha_A(x < 2>)],
\]

\[(F5) \quad \delta_V([x, y]) = [\delta_V(x), \alpha_V(y)] + [\alpha_V(x), \delta_V(y)]
\]

\[- y(-1) \triangleright \alpha(x) \otimes \alpha_V(y(0)) - \alpha_V(x(0)) \otimes x(-1) \triangleright \alpha_V(y)
\]

\[
+ x(-1) \triangleright \alpha_V(y) \otimes \alpha_V(x(0)) + \alpha_V(y(0)) \otimes y(-1) \triangleright \alpha_V(x),
\]

\[(F6) \quad \psi(x \triangleleft a) = \alpha_V(x_1) \otimes x_2 \triangleright \alpha_A(a)
\]

\[
+ \alpha_V(x(0)) \otimes [x(1), \alpha_A(a)] + x(0) \triangleleft \alpha_A(a) \otimes \alpha_A(x(1)) + \alpha_V(x) \triangleleft a_1 \otimes \alpha_A(a_2).
\]

Conversely, any Hom-Lie bialgebra structure on \((E, \alpha_E)\) with the canonical injection map \(i : A \to E\) both a Hom-Lie algebra homomorphism and a Hom-Lie coalgebra homomorphism is of this form.
Denote the set of all Hom-Lie bialgebra extending datum of type (II) by $\mathcal{LB}^{(2)}(A, V)$.

Note that $A^\phi_P \#_{\psi,\sigma}^V V$ and $A^\psi_Q \#_{\phi,\sigma}^V V$ are all Hom-Lie bialgebra structures on $(E, \alpha_E)$. Conversely, any Hom-Lie bialgebra extending system $(E, \alpha_E)$ of $(A, \alpha_A)$ through $(V, \alpha_V)$ is isomorphic to such a unified products of the two types. Now from Theorem 5.7, Theorem 5.14 in last section and Theorem 5.15 Theorem 5.16 we obtain the main result of in this section, which solve the extending problem for Hom-Lie bialgebra.

**Theorem 5.17.** Let $(A, [-, -], \delta_A)$ be a Hom-Lie bialgebra, $(E, \alpha_E)$ a Hom-vector space containing $(A, \alpha_A)$ as a subspace and $(V, \alpha_V)$ be a complement of $(A, \alpha_A)$ in $(E, \alpha_E)$. Denote by

$$\mathcal{HLB}(V, A) := \mathcal{LB}^{(1)}(A, V) \sqcup \mathcal{LB}^{(2)}(A, V)/\equiv.$$ 

Then the map

$$\Upsilon : \mathcal{HLB}(V, A) \to BExtd(E, A),$$

$$\Omega^{(1)}(A, V) \mapsto A^\phi_P \#_{\psi,\sigma}^V V, \quad \Omega^{(2)}(A, V) \mapsto A^\psi_Q \#_{\phi,\sigma}^V V$$

is bijective, where $\Omega^{(i)}(A, V)$ is the equivalence class of $\Omega^{(i)}(A, V)$ under $\equiv$.

### 5.4 Flag extending structures

In this section, we study the case when $(V, \alpha_V)$ is a 1-dimensional vector space. This will be called flag extending system. Since $(V, \alpha_V)$ is a 1-dimensional vector space, then the bracket and cobracket of $(V, \alpha_V)$ is given by $[x, y] = 0$ and $\delta_V(x) = 0$ for all $x, y \in V$.

**Lemma 5.18.** Let $(A, [-, -], \delta_A)$ be a braided Hom-Lie bialgebra and $V = k\{x\}$ be a 1-dimensional vector space. A flag datum consists of

$$\lambda : A \to k, \quad D : A \to A, \quad T : A \to A, \quad a_0 \in A$$

satisfying the following compatibility conditions:

$$\lambda([a, b]) = \lambda(\alpha_A(a))\lambda(b) - \lambda(\alpha_A(b))\lambda(a),$$

$$D([a, b]) = [D(a), \alpha_A(b)] + [\alpha_A(a), D(b)] + \lambda(a)D(b) - \lambda(b)D(a),$$

$$T([a, b]) = [T(a), \alpha_A(b)] + [\alpha_A(a), T(b)] + \lambda(b)T(a) - \lambda(a)T(b),$$

$$T(D(a)) = D(T(a)) + [a_0, \alpha_A(a)] + \lambda(a_1)\alpha(a_2).$$

The corresponding the extending datum $\Omega(A, V)$ is given by:

$$x \triangleright a = D(a), \quad x \triangleleft a = \lambda(a)x, \quad \phi(a) = x \otimes T(a), \quad \psi(x) = x \otimes a_0,$$

$$\sigma(x, x) = 0, \quad [x, x] = 0, \quad P(a) = 0, \quad \delta_V(x) = 0.$$ 

The unified product associated to this flag extending system is given by

$$[(a, x), (b, y)] = \left([a, b] + D(a)y - D(b)x, \lambda(a)y - \lambda(b)x\right).$$
and

\[
\delta_E(a) = \delta_A(a) + x \otimes T(a) - T(a) \otimes x, \quad \delta_E(x) = x \otimes a_0 - a_0 \otimes x. \tag{60}
\]

Denote the set of all flag datums of braided Hom-Lie bialgebra by \( FB(A) \).

**Definition 5.19.** Two flag datums \((\lambda, D, T, a_0)\) and \((\lambda', D', T, a'_0)\) \(\in FB(A)\) are called equivalent if \(\lambda' = \lambda, a'_0 = a_0\) and there exist some element \(r_0 \in A\) such that

\[
\lambda(a)r_0 = [r_0, a] - D(a) + sD'(a), \tag{61}
\]

\[
\delta'_A(r_0) = r_0 \otimes a_0 - a_0 \otimes r_0, \tag{62}
\]

\[
\delta'_A(a) = \delta_A(a) + r_0 \otimes T(a) - T(a) \otimes r_0. \tag{63}
\]

By the above lemma, we have

**Theorem 5.20.** Let \((A, [\cdot, \cdot], \delta_A)\) be a braided Hom-Lie bialgebra and \((V, \alpha_V)\) be a 1-dimensional vector space. Then there is a bijection between the set \(BLB(A, V)\) of all Hom-Lie bialgebra extending systems of \((A, \alpha_A)\) by \((V, \alpha_V)\) and \(FB(A)\).

Next, we consider flag extending systems for Hom-Lie bialgebras.

**Lemma 5.21.** Let \((A, [\cdot, \cdot], \delta_A)\) be a Hom-Lie bialgebra. A flag datum of type \((I)\) consists of

\[
\lambda : A \to k, \quad T : A \to A, \quad a_0 \in A
\]

satisfying the following compatibility conditions:

\[
\lambda([a, b]) = \lambda(\alpha_A(a)) \lambda(b) - \lambda(\alpha_A(b)) \lambda(a), \tag{64}
\]

\[
T([a, b]) = [T(a), \alpha_A(b)] + [\alpha_A(a), T(b)], \tag{65}
\]

\[
[a_0, \alpha_A(a)] + \lambda(a_1)\alpha(a_2) = 0. \tag{66}
\]

The corresponding the extending datum \(\Omega^{(1)}(A, V)\) of type \((I)\) is given by:

\[
x \triangleleft a = \lambda(a)x, \quad \phi(a) = x \otimes T(a), \quad \psi(x) = x \otimes a_0, \tag{67}
\]

\[
\theta(a, b) = 0, \quad [x, x] = 0, \quad P(a) = 0, \quad \delta_V(x) = 0. \tag{68}
\]

The unified product \(A^{(1)}# V\) associated to the flag extending system is given by

\[
[[a, x], (b, y)] = \left( [a, b], \lambda(b)x - \lambda(a)y \right), \tag{69}
\]

and

\[
\delta_E(a) = \delta_A(a) + x \otimes T(a) - T(a) \otimes x, \quad \delta_E(x) = x \otimes a_0 - a_0 \otimes x. \tag{70}
\]

Denote the set of all flag datums of type \((I)\) by \(\mathcal{F}^{(1)}(A)\).
Lemma 5.22. Let \((A, [\cdot, \cdot], \delta_A)\) be a Hom-Lie bialgebra. A flag datum of type (II) consists of \(\lambda : A \to k, \ D : A \to A, \ a_0 \in A, \ Q \in A \wedge A\) satisfying the following compatibility conditions:

\[
\lambda([a, b]) = \lambda(\alpha_A(a))\lambda(b) - \lambda(\alpha_A(b))\lambda(a), \tag{71}
\]

\[
D([a, b]) = [D(a), \alpha_A(b)] + [\alpha_A(a), D(b)] + \lambda(a)D(b) - \lambda(b)D(a), \tag{72}
\]

\[
[a_0, \alpha_A(a)] + \lambda(a_1)\alpha(a_2) = 0, \tag{73}
\]

\[
\delta_A(D(a)) + \lambda(a)Q = [\alpha_A(a), Q] + qD(a_1) \otimes \alpha(a_2) + q\alpha(a_1) \otimes D(a_2)
\]

\[
+ D(\alpha_A(a)) \otimes \alpha_A(a_0) - \alpha_A(a_0) \otimes D(\alpha_A(a)), \tag{74}
\]

\[
\alpha(a_0) \otimes Q - \tau_{12}(\alpha(a_0) \otimes Q) + Q \otimes \alpha(a_0) = (\alpha \otimes \delta - \tau_{12}(\alpha \otimes \delta) - \delta \otimes \alpha) Q. \tag{75}
\]

The corresponding is the extending datum \(\Omega^{(2)}(A, V)\) of type (II) given by:

\[
x \triangleleft a = \lambda(a)x, \ x \triangleright a = D(a), \ \omega(x, x) = 0, \tag{76}
\]

\[
\psi(x) = x \otimes a_0, \ Q(x) = Q. \tag{77}
\]

The unified product \(A \#^{(2)}V\) is given

\[
[(a, x), (b, y)] = \left([a, b] + D(a)y - D(b)x, \lambda(a)y - \lambda(b)x\right). \tag{78}
\]

and

\[
\delta_E(a) = \delta_A(a), \ \delta_E(x) = x \otimes a_0 - a_0 \otimes x + Q. \tag{79}
\]

Denote the set of all flag datums of type (II) by \(\mathcal{F}^{(2)}(A)\).

By the above two lemmas, we have

Theorem 5.23. Let \((A, [\cdot, \cdot], \delta_A)\) be a Hom-Lie bialgebra and \(V = k\{x\}\) be a 1-dimensional vector space. Then there is a bijection between the set \(\mathcal{LB}(A, V)\) of all Hom-Lie bialgebra extending systems of \((A, \alpha_A)\) by \((V, \alpha_V)\) and \(\mathcal{F}(A) = \mathcal{F}^{(1)}(A) \sqcup \mathcal{F}^{(2)}(A)\).

Definition 5.24. Two flag datums \((\lambda, T, a_0)\) and \((\lambda', T', a'_0)\in \mathcal{F}^{(1)}(A)\) are called equivalent if \(\lambda' = \lambda, \ a'_0 = a_0\) and there exist some element \(r_0 = r(x) \in A\) such that

\[
\lambda(a)r_0 = [r_0, a], \tag{80}
\]

\[
\delta'_A(r_0) = r_0 \otimes a_0 - a_0 \otimes r_0, \tag{81}
\]

\[
\delta'_A(a) = \delta_A(a) + r_0 \otimes T(a) - T(a) \otimes r_0. \tag{82}
\]

Definition 5.25. Two flag datums \((\lambda, D, a_0, Q)\) and \((\lambda', D', a'_0, Q') \in \mathcal{F}^{(2)}(A)\) are called equivalent if \(\lambda' = \lambda, \ a'_0 = a_0\) and there exist some element \(r_0 = r(x) \in A\) and \(s \in k^*\) such that

\[
\lambda(a)r_0 = [r_0, a] - D(a) + sD'(a), \tag{83}
\]

\[
\delta_A(r_0) + sQ' = r_0 \otimes a_0 - a_0 \otimes r_0 + Q. \tag{84}
\]

From the above discussion, we obtain:

Theorem 5.26. Let \((A, [\cdot, \cdot], \delta_A)\) be a Hom-Lie bialgebra of codimension one in a Hom-vector space \((E, \alpha_E)\). Then we have \(BExtd(E, A) \cong \mathcal{LB}(V, A) \cong \mathcal{F}(A)/\equiv\).
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