Two remarks on non-zero constant Jacobian polynomial map of $\mathbb{C}^2^*$

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Abstract

We present some estimations on geometry of the exceptional value sets of non-zero constant Jacobian polynomial maps of $\mathbb{C}^2$ and it’s components.

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1. Introduction. Recall that the exceptional value set $E_h$ of a polynomial mapping $h : \mathbb{C}^m \rightarrow \mathbb{C}^n$ is the smallest subset $E_h \subset \mathbb{C}^n$ such that the restriction $h : h^{-1}(\mathbb{C}^n \setminus E_h) \rightarrow \mathbb{C}^n \setminus E_h$ gives a locally trivial smooth fibration. The mysterious Jacobian conjecture (See [BMW] and [E]), which posed first by Keller in 1939 and still open even for two dimensional case, asserts that a polynomial map of $\mathbb{C}^n$ with non-zero constant Jacobian must be a polynomial bijection. Consider a polynomial map $f = (P,Q) : \mathbb{C}^2_{(x,y)} \rightarrow \mathbb{C}^2_{(u,v)}$ and denote $J(P,Q) := P_xQ_y - P_yQ_x$. It is well-known that if $J(P,Q) \equiv \text{const.} \neq 0$ but $f$ is not bijective, then the sets $E_P$ and $E_Q$ must be non empty finite sets and $E_f$ is a curve so that each of it’s irreducible components is a polynomially parameterized curve (the image of a polynomial map from $\mathbb{C}$ into $\mathbb{C}^2$) (See, for example in [J1, J2]).

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In this note we present the followings.

**Theorem 1.** Suppose $f = (P,Q)$ is a polynomial map with non-zero constant Jacobian. Then, a value $u_0 \in E_P$ if and only if the line $u = u_0$ tangents to an irreducible local branch of $E_f$.

This theorem leads to the following fact that may be used to consider the non-existence of non-zero constant Jacobian polynomial maps with exceptional value curve of given types.

**Theorem 2.** The exceptional value set of a polynomial map of $\mathbb{C}^2$ with non-zero constant Jacobian can not be isomorphic to a curve composed of the images of some polynomial maps of the form $t \mapsto (t^k, q(t))$, $k \in \mathbb{N}$, $q(t) \in \mathbb{C}[t]$.

A special property of the curve with irreducible components given by parameterizations in theorem 2 is that it’s local irreducible branches may tangent to the only line $u = 0$. Simply connected curves are simple examples of such curves. This can be easy deduced from Lin-Zaidenberg’s theorem on simply connected curves (Theorem B in [ZL]).

**Corollary 3.** The exceptional value set of a polynomial map of $\mathbb{C}^2$ with non-zero constant Jacobian can not be a simply connected curve.

Proofs of Theorem 1 and Theorem 2 will be presented in § 3 and §4.

2. Preliminaries. In this section, we present some elementary facts which will be used in the proof of Theorem 1.

i) We will work with finite fractional series with parameter $\xi$ of the form

$$
\varphi(x, \xi) = \sum_{k=1}^{K-1} a_k x^{n_k} + \xi x^{n_K}, 0 \neq a_k \in \mathbb{C}, n_k \in \mathbb{Z}, m \in \mathbb{N}
$$

with $n_1 > n_2 > \ldots > n_K$ and $\gcd\{n_k : k \leq K\} = 1$.

Let $h(x, y)$ be a non-constant primitive polynomial monic in $y$, $\deg_y h = \deg h$. A fractional series $\varphi(x, \xi)$ in (1) is called a *Newton-Puiseux type* of $h$ if

$$
h(x, \varphi(x, \xi)) = h_{\varphi}(\xi) + \text{lower terms in } x, h_{\varphi} \in \mathbb{C}[\xi], \deg h_{\varphi} > 0.
$$

Denote by $\text{mult}(\varphi) := m$ and $\text{ord}(\varphi) := n_K$.  

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The Newton-Puiseux types of \( h(x, y) \) can be constructed from Newton-Puiseux expansions at infinity of the curve \( h(x, y) = 0 \). In fact, if \( y = y(x) \) is such a Newton-Puiseux expansion at infinity, then there is a unique Newton-Puiseux type \( \varphi \) of \( h \) and a number \( \xi_0 \in \mathbb{C} \) such that \( y(x) = \varphi(x, \xi_0) + \text{lower terms in } x \).

For a Newton-Puiseux type \( \varphi \) of \( h \) the rational map \( \Phi: \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C} \mathbb{P}^2 \) defined by
\[
\Phi(t, \xi) := (t - \text{mult}(\varphi), \varphi(t - \text{mult}(\varphi), \xi)).
\]
determines a unbranched \( i \varphi \)-sheeted covering from \( \mathbb{C}^* \times \mathbb{C} \) onto \( \Phi(\mathbb{C}^* \times \mathbb{C}) \subset \mathbb{C}^2 \), where \( i \varphi := \text{mult}(\varphi) / \gcd\{k : a_k \neq 0, \ k < K\} \). Hence, one can use the polynomial \( H_{\varphi}(t, \xi) := h \circ \Phi(t, \xi) \) as a kind of “extension” of \( h(x, y) \).

**Lemma 4.** (See Theorem 2.4, [C]). Suppose \( h(x, y) \) is a primitive polynomial and monic in \( y \). Then, \( c \in E_h \) if and only if \( c \) is either a critical value of \( h \) or a critical value of \( h_{\varphi}(\xi) \) for any Newton-Puiseux type \( \varphi \) of \( h \).

In fact, by Newton’s theorem the polynomial \( h(x, y) - c \) can be factorized as
\[
h(x, y) - c = C \prod_u (y - u(x)),
\]
where the product runs over all Newton-Puiseux expansion at infinity of the curve \( h = c \). Substituting \( y = \varphi(x, \xi) \) into this representation, one can see that a number \( \alpha \in \mathbb{C} \) is a zero point of \( h_{\varphi}(\xi) - c = 0 \) with multiplicity \( n \) if and only if the level \( h = c \) has exactly \( n \) Newton-Puiseux expansions at infinity of the form \( \varphi(x, \alpha) + \text{lower term in } x \). Thus, Puiseux data at infinity of the curves \( h = c \) must be changed when \( c \) is a critical values of \( h_{\varphi} \).

ii) Consider a polynomial map \( f = (P, Q) : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \). By definition the exceptional value set \( E_f \) is composed by the critical value set of \( f \) and the so-called non-proper value set \( A_f \) of \( f \) - the set of all value \( a \in \mathbb{C}^2 \) such that there exists a sequence \( \mathbb{C}^2 \ni p_i \rightarrow \infty \) with \( f(p_i) \rightarrow a \). Following [J1], the non-proper value set \( A_f \), if non-empty, is a curve composed by the images of some polynomial maps from \( \mathbb{C} \) into \( \mathbb{C}^2 \).

A series \( \varphi(x, \xi) \) in (1) is called dicritical series of \( f \) if
\[
f(x, \varphi(x, \xi)) = f_{\varphi}(\xi) + \text{lower terms in } x, \ \deg f_{\varphi} > 0.
\]

**Lemma 5.**
A_f = \bigcup_{\varphi \text{ is a dicritical series of } f} f_{\varphi}(\mathbb{C}).

Proof: Let \( \varphi \) be a dicritical series of \( f \) and \( \Phi(t, \xi) \) be as in (2). The map \( \Phi \) sends \( \mathbb{C}^* \times \mathbb{C} \) to \( \mathbb{C}^2 \) and the line \( \{0\} \times \mathbb{C} \) to the line at infinity of \( \mathbb{C} \mathbb{P}^2 \). Then, the polynomial map \( F_\varphi(t, \xi) := f \circ \Phi(t, \xi) \) maps the line \( \{0\} \times \mathbb{C} \) into \( A_f \subset \mathbb{C}^2 \). Therefore, \( f_{\varphi}(\mathbb{C}) \) is an irreducible component of \( A_f \), since \( \deg f_{\varphi} > 0 \).

Conversely, assume that \( \ell \) is an irreducible component of \( A_f \). By definition we can choose a smooth point \( (u_0, v_0) \) of \( \ell \) and an irreducible branch at infinity \( \gamma \) of the curve \( P = u_0 \) (or the curve \( Q = v_0 \)) such that the image \( f(\gamma) \) is a branch curve intersecting transversally \( \ell \) at \( (u_0, v_0) \). Let \( \varphi(x, \xi) \) be the Newton-Puiseux type of \( P \) constructed corresponding to a Newton-Puiseux expansion at infinity of \( \gamma \). Then, by definitions we can verify that \( \varphi \) is a dicritical series of \( f \) and \( f_{\varphi}(\mathbb{C}) = \ell \). Q.E.D

3. Proof of Theorem 1. We consider a polynomial map \( f : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \) with non-zero constant Jacobian. Fix a suitable affine coordinate \( (x, y) \) so that \( P \) and \( Q \) is monic in \( y \). For a series \( \varphi \) in (1) represent

\[
P(x, \varphi(x, \xi)) = p_{\varphi}(\xi)x^{\frac{a_{\varphi}}{\text{mult}(\varphi)}} + \text{lower terms in } x
\]
\[
Q(x, \varphi(x, \xi)) = q_{\varphi}(\xi)x^{\frac{b_{\varphi}}{\text{mult}(\varphi)}} + \text{lower terms in } x,
\]

Lemma 6. i) Let \( \varphi \) be a Newton-Puiseux type of \( P \). If \( \varphi \) is not a dicritical series of \( f \), then \( \deg p_{\varphi}(\xi) = 1, q_{\varphi}(\xi) \equiv \text{const.} \neq 0 \) and \( b_{\varphi} > 0 \).

ii) A dicritical series of \( f \) must be a Newton-Puiseux type both of \( P \) and \( Q \).

Proof. i) First, we will show that

\[
\deg p_{\varphi} = 1, \quad q_{\varphi} \equiv \text{const.} \neq 0.
\]

Taking differentiation of \( F(t^{-\text{mult}(\varphi)}, \varphi(t^{-\text{mult}(\varphi)}, \xi)) \), as \( a_{\varphi} = 0 \) and \( b_{\varphi} \neq 0 \) we have

\[
\text{mult}(\varphi)J(P, Q)t^{\text{ord}(\varphi)-\text{mult}(\varphi)-1} = -b_{\varphi} \dot{p}_{\varphi}q_{\varphi}t^{-b_{\varphi}-1} + \text{higher terms in } t.
\]

Since \( J(P, Q) \equiv \text{const.} \neq 0 \) and \( \deg p_{\varphi} > 0 \), we get (*).
Now, assume the contrary that \( b_\varphi < 0 \). Then, there exists a Newton-Puiseux root at infinity \( u(x) \) of the curve \( Q = 0 \) such that \( u(x) = \varphi(x, \xi_0) + \text{lower term in } x \). Let \( \psi(x, \xi) \) be the Newton-Puiseux type of \( Q \) constructed corresponding to \( u(x) \). Obviously, \( a_\psi > a_\varphi = 0 \) and hence \( \psi \) is not a dicritical series of \( f \). Furthermore, \( \varphi(x, \xi) = \psi(x, \alpha) + \text{lower terms in } x \) for a zero point \( \alpha \) of \( p_\psi(\xi) \). This is impossible, since \( p_\psi(\xi) \equiv \text{const.} \neq 0 \) by applying (*) to the Newton-Puiseux type \( \psi \) of \( Q \). Thus, we get \( b_\varphi > 0 \).

ii) This is obtained from (i) and definitions. Q.E.D

Proof of Theorem 1. If \( E_f = \emptyset \), then \( f \) is bijective and \( E_P = E_Q = \emptyset \). Hence, we need consider only the situation when \( E_f \neq \emptyset \). In this situation \( E_f = A_f \), since \( f \) has not singularity.

First, suppose the line \( u = u_0 \) tangents to a local irreducible branch curve of an irreducible component \( \ell \) of \( A_f \). By Lemma 5, there is a dicritical series \( \varphi \) of \( f \) such that \( \ell \) is the image of \( f_\varphi := (p_\varphi, q_\varphi) \). By Lemma 6 (ii) the series \( \varphi \) is a Newton-Puiseux type both of \( P \) and \( Q \). Since the line \( u = u_0 \) tangents to \( \ell \), \( u_0 \) must be a critical value of \( p_\varphi \). Hence, by Lemma 4 the number \( u_0 \) is an exceptional value of \( P \).

Conversely, let \( u_0 \in \mathbb{C} \) and denote \( L := \{ u = u_0 \} \). Assume that \( L \) intersects transversally each of local irreducible branches of \( A_f \) located at points in \( L \cap A_f \). We have to show that \( u_0 \) can not be an exceptional value of \( P \). In view of Lemma 4 and Lemma 6 we need only to verify that \( u_0 \) is a regular value of \( p_\varphi(\xi) \) for every dicritical series \( \varphi \) of \( f \).

Let \( \varphi \) be a given dicritical series of \( f \) and \( \ell := f_\varphi(\mathbb{C}) \), which is an irreducible component of \( A_f \) by Lemma 5. Let \( (u_0, v_0) \in \ell \) and \( \xi_0 \in \mathbb{C} \) with \( f_\varphi(\xi_0) = (u_0, v_0) \). We have to show that \( \dot{p}_\varphi(\xi_0) \neq 0 \).

Consider the polynomial map \( F: \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}^2 \)

\[
F(t, \xi) := f \circ (t^{-\text{mult}(\varphi)}, \varphi(t^{-\text{mult}(\varphi)}, \xi)) = (p_\varphi, q_\varphi)(\xi) + \text{higher terms in } t.
\]

For this map \( F(\{0\} \times \mathbb{C}) = \ell \) and

\[
\det DF(t, \xi) = -\text{mult}(\varphi) J(P, Q) t^{\text{ord}(\varphi) - \text{mult}(\varphi) - 1}.
\]

Since \( J(P, Q) \equiv \text{const.} \neq 0 \), \( F \) has singularity only on the line \( t = 0 \).

Let \( \gamma := f_\varphi(\{\xi : |\xi - \xi_0| < \epsilon\}) \) for an enough small \( \epsilon > 0 \). As assumed the line \( L \) intersects transversally \( \gamma \) at \( (u_0, v_0) \). So, we can choose an enough small neighborhood \( U \) of \( (u_0, v_0) \) so that \( \gamma := \ell \cap U \) is a smooth branch.
curve parameterized by \( v = v_0 + h(u - u_0) \) for a holomorphic function \( h, h(0) = 0 \). Define new coordinates \((\bar{u}, \bar{v}) = (u - u_0, v - v_0 - h(u - u_0))\) in \( U \) and \((\bar{t}, \bar{\xi}) = (t, \xi - \xi_0)\) in an enough small neighborhood \( V \) of \((0, \xi_0)\). Let \( \bar{F} = (\bar{F}_1, \bar{F}_2) \) is the representation of \( F \) in these coordinates. Then,

\[
\bar{F}_1(\bar{t}, \bar{\xi}) = p_\varphi(\bar{\xi}) - u_0 + \text{higher terms in } \bar{t}, \tag{4}
\]

\( \bar{F}(0, 0) = (0, 0), \bar{F}(\{\bar{t} = 0\}) \subset \gamma = \{\bar{v} = 0\} \) and \( \det D\bar{F}(\bar{t}, \bar{\xi}) \neq 0 \) for \( \bar{t} \neq 0 \). Then, by examining Newton diagrams of \( \bar{F}_1, \bar{F}_2 \) and \( \det D\bar{F} \) we can verify that

\[
\bar{F}(\bar{t}, \bar{\xi}) = (\bar{\xi}u_1(\bar{t}, \bar{\xi}) + \bar{t}u_2(\bar{t}, \bar{\xi}), \bar{t}^k v_1(\bar{t}, \bar{\xi})), \tag{5}
\]

where \( u_1, u_2 \) and \( v_1 \) are holomorphic functions define in \( V, u_1(0, 0) \neq 0 \) and \( v_1(0, 0) \neq 0 \) (See, for example [O, Lemma 4.1]). From (4) and (5) it follows that \( \bar{p}_\varphi(\xi_0) \neq 0 \). Q.E.D

**Remark.** From Lemma 6 (ii) and Lemma 5 one can easy see that the exceptional value set \( E_f \) of nonsingular polynomial map \( f \) can not contains an irreducible component isomorphic to a line.

**4. Proof of Theorem 2.** Let \( f = (P, Q) \) be representation of \( f \) in the coordinate in which \( E_f \) consists with the images of some polynomial maps of the form \( t \mapsto (t^k, q(t)), k \in \mathbb{N} \). By applying Theorem 1 we have that \( E_P \subset \{0\} \). As \( f \) has not singularity, \( P \) is a nonsingular primitive polynomial. Then, from Suzuki’s equality

\[
\sum_{c \in \mathcal{C}} (\chi_c - \chi) = 1 - \chi
\]

(See in [S]) we get \( \chi_0 = 1 \). Here, \( \chi_c \) and \( \chi \) indicate the Euler-Poincare characteristic of the fiber \( P = c \) and the generic fiber of \( P \), respectively. Since the curve \( P = 0 \) is smooth, it has one connected component \( \ell \) diffeomorphic to \( \mathbb{C} \). This component \( \ell \) must be isomorphic to \( \mathbb{C} \) by the Abhyankar-Moh Theorem [AM] and the restriction of \( f \) on \( \ell \) must be injective. Then, as observed by Gwozdziewicz in [G], \( f \) must be bijective. This is impossible, since \( E_f \neq \emptyset \). Q.E.D

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