Massive Complex Scalar Field in a Kerr-Sen Black Hole Background:
Exact Solution of Wave Equation and Hawking Radiation

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The separated radial part of a massive complex scalar wave equation in the Kerr-Sen geometry is shown to satisfy the generalized spheroidal wave equation which is, in fact, a confluent Heun equation up to a multiplier. The Hawking evaporation of scalar particles in the Kerr-Sen black hole background is investigated by the Damour-Ruffini-Sanna's method. It is shown that quantum thermal effect of the Kerr-Sen black hole has the same character as that of the Kerr-Newman black hole.

**Key Words**: Klein-Gordon equation, Generalized spheroidal wave equation, confluent Heun equation, Hawking effect

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### I. INTRODUCTION

Exact solutions to various wave equations such as Klein-Gordon, Dirac and Maxwell equations in a special black hole background are of considerable importance in mathematical physics as well as in black hole physics. Knowing an explicit expression for some exact solutions is undoubtedly very helpful to construct Feynman Green function inside a Schwarzschild black hole [1], to do second quantization in the Kerr metric [2], to study quasinormal modes of Kerr black hole and Kerr-Newman black hole [3], to check stability of Kerr black hole and Kerr-de Sitter spacetime [4], to calculate absorption rate of Kerr-de Sitter black hole and Kerr-Newman-de Sitter black hole [5] and to investigate black hole perturbation [6], etc. Effort to seek such exact solutions has been undertaken all the time. It is instructive to present a brief review on studies about exact solution of a quantum field equation in some well-known black hole spacetimes.

At 1982, Birrell and Davies [7] stated in their classical masterpiece *Quantum Fields in Curved Space* that “It is not possible to write the solutions \( R_{\omega}(r) \) to the radial equation (8.2) in terms of known functions, though the properties of the solutions have been extensively investigated.” (see also [8]). By the radial equation, they referred to the separated radial part of a massless Klein-Gordon equation in the Schwarzschild space-time. Prior to that time, exact solution to the angular part of a wave equation on a black hole background had already received a large mount of studies and named as the spin-weighted spheroidal wave function [9], but solution to its radial part [10] was unknown to almost all of those researchers who studied black hole physics. The radial wave equation was thought of as not being related to any known differential equation of mathematical physics before then [11].

This situation changed in the next year. Blandin, Pons and Marcilhacy [12] showed that the spin-weighted spheroidal functions may be obtained by an elementary transformation from Heun’s confluent function. They also noted that both the angular part and the radial part of Teukolsky’s master equation [12] are particular forms of a single linear ordinary differential equation. By a private communication, G. W. Gibbons notified Jensen and Candelas [13] that the radial part for a massless scalar wave equation in the Schwarzschild spacetime is, up to a multiplier, a confluent Heun function.

The next important development started from Leaver’s work [14]. He showed that the radial and angular parts of Teukolsky master equation [12] are equations of the same type, namely, the generalized spheroidal wave equation (GSWE) [14,15]. Especially, he also expanded the solution of a GSWE in terms of hypergeometry and confluent hypergeometry functions. Recently, we demonstrate that both the radial part and the angular part of a massive charged scalar field equation on the Kerr-Newman black hole background can be transformed into a GSWE [16]. The most remarkable progress comes from recent researches on revealing the relations among the generalized Teukolsky master equation, GSWE and Heun equation. Suzuki, Takasugi and Umetsu [17] have studied perturbations of Kerr-de Sitter black holes and analytic solutions of Teukolsky equation in the Kerr-de Sitter and Kerr-Newman-de Sitter geometries, and shown that this generalized Teukolsky master equation has a close relation to Heun’s equation [18] (and references therein) which is introduced by Heun [19] as early as 1889. Also, they have proved that a GSWE is a confluent Heun equation when cosmological constant approaches to zero. Analytic solutions of the Teukolsky equation were studied in Ref. [18]. The integral equations of fields on the rotating black hole had been investigated in Ref. [20]. The integral equations for Heun functions were presented in Ref. [21].

Through these researches, it may be generally accepted that the generalized Teukolsky master equation in the Kerr-Newman-de Sitter spacetimes can be recast into the form of a generalized spheroidal wave equation which is,
in fact, a Heun equation. But it is not clear until now whether the generalized Teukolsky equation in the general type-D vacuum backgrounds with cosmological constant (namely, stationary C-metric solutions) can be transformed into a Heun equation. Here we also mention Couch’s work that relates a series of exact solutions by transforming the separated radial equation into a modified Whittaker-Hill equation under some special conditions.

In a recent paper, we have investigated exact solution of a massive complex scalar field equation in the Kerr-Newman black hole background. Previous work on solution of a massive scalar wave equation in the Kerr-Newman spacetime had been completed in Ref. [25]. It is interesting to extend our analysis to solution of a scalar wave equation in a Kerr-Sen black hole background. The Kerr-Sen solution arising in the low energy effective string field theory is a rotating charged black hole generated from the Kerr solution. The thermodynamic property of this twisted Kerr black hole was discussed in Ref. [26] by using separation of the Hamilton-Jacobi equation of a test particle. The aim of this paper is to find some exact solutions to a massive charged scalar wave equation and to investigate quantum thermal effect of scalar particles on the Kerr-Sen spacetime.

The paper is organized as follows: In Sec. 2, we separate a massive charged scalar field equation on the Kerr-Sen black hole background into the radial and angular parts. Sec. 3 is devoted to transforming the radial part into a generalized spheroidal wave equation and to relating it to the confluent Heun equation. Then, we investigate quantum thermal effect of scalar particles in the Kerr-Sen spacetime. Finally, we summarize our discussions in the conclusion section.

II. SEPARATING VARIABLES OF KLEIN-GORDON EQUATION ON KERR-SEN BLACK HOLE BACKGROUND

Constructed from the charge neutral rotating (Kerr) black hole solution, the Kerr-Sen solution [26] is an exact classical four dimensional black hole solution in the low energy effective heterotic string field theory. In the Boyer-Lindquist coordinates, the Kerr-Sen metric can be rewritten as [27]:

\[ ds^2 = -\frac{\Delta}{\Sigma} (dt - a \sin^2 \theta d\varphi)^2 + \frac{\sin^2 \theta}{\Sigma} (dr^2 + 2a \cos^2 \theta d\varphi d\theta) - \left(\Sigma + a^2 \sin^2 \theta\right) d\varphi^2 + \Sigma \left(\frac{dr^2}{\Delta} + d\theta^2\right) , \]

where \( \Delta = r^2 + 2(b - M)r + a^2 = (r - r_+)(r - r_-), \Sigma = r^2 + 2br + a^2 \cos^2 \theta \) and \( r_\pm = M - b \pm \epsilon \) with \( \epsilon = \sqrt{(M - b)^2 - a^2} \). The electromagnetic field vector potential is concisely chosen as [27]:

\[ A = -\frac{Qr}{\Sigma}(dt - a \sin^2 \theta d\varphi) . \]

This metric describes a black hole carrying mass \( M \), charge \( Q \), angular momentum \( J = Ma \), and magnetic dipole moment \( Qa \). The twist parameter \( b \) is related to the Sen’s parameter \( \alpha \) via \( b = Q^2 / 2M = M \tan^2(\alpha / 2) \). Because \( M \geq b \geq 0 \), \( r = r_- \) is a new singularity in the region \( r \leq 0 \), the event horizon of the Kerr-Sen black hole is located at \( r = r_+ \). The area of the out event horizon of the twisted Kerr solution is given by \( A = 4\pi(r_+^2 + 2br + a^2) = 8\pi Mr_+ \).

We consider the solution of a massive charged test scalar field on the Kerr-Sen black hole background (in Planck unit system \( G = \hbar = c = k_B = 1 \)). The complex scalar field \( \Phi \) with mass \( \mu \) and charge \( q \) in such a space-time satisfies the following Klein-Gordon equation (KGE):

\[ -\frac{1}{\Delta} \left[ (r^2 + 2br + a^2)\partial_t + a \partial_\varphi + iqQr \right]^2 \Phi + \partial_r (\Delta \partial_r \Phi) + \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta \Phi) + \left( a \sin \theta \partial_t + \frac{1}{\sin \theta} \partial_\varphi \right)^2 \Phi = \mu^2 \Sigma \Phi . \]

The wave function \( \Phi \) of KGE has a solution of variables separable form \( \Phi(t, r, \theta, \varphi) = R(r)S(\theta)e^{i(m\varphi - \omega t)} \), in which the separated radial and angular parts of KGE can be given as follows:

\[ \frac{1}{\sin \theta} \partial_\theta [\sin \theta \partial_\theta S(\theta)] + \left[ \lambda - \frac{m^2}{\sin^2 \theta} \right] S(\theta) = 0 , \]

\[ \partial_r [\Delta \partial_r R(r)] + \left[ \frac{K^2(r)}{\Delta} - \mu^2 (r^2 + 2br + a^2) \right] - \lambda + 2m\omega R(r) = 0 , \]

where \( \lambda \) is a separation constant, \( K(r) = \omega(r^2 + 2br + a^2) - qQR - ma \).

The general solution \( S^l_{m,0}(ka, \theta) \) to the angular equation is an ordinary spheroidal angular wave functions with spin-weight \( s = 0 \), while the radial equation can be reduced to

\[ \partial_r [\Delta \partial_r R(r)] + \left[ \frac{(Ar - ma)^2}{\Delta} + k^2 \Delta \right] + 2Dr - \lambda R(r) = 0 , \]

where we have put \( A = 2M\omega - qQ, D = A\omega - M\mu^2, k = \sqrt{\omega^2 - \mu^2} \) (supposed that \( \omega > \mu \)). For later convenience, we also denote \( \epsilon B = A(M - b - ma) \) and introduce \( W_\pm = (A \pm B) / 2 \).

Eq. (6) has two regular singular points \( r = r_\pm \) with indices \( \pm iW_+ \) and \( \pm iW_- \), respectively. The radial wave function \( R(r) \) has asymptotic behaviors when \( r \to r_\pm \):

\[ R(r) \sim \begin{cases} \frac{(r - r_+)^{\pm iW_+}}{(r - r_-)^{\pm iW_-}} \quad (r \to r_+) \\ \frac{(r - r_-)^{\pm iW_-}}{(r - r_+)^{\pm iW_+}} \quad (r \to r_-) \end{cases} \]
Making substitution $R(r) = (r - r_+)^{(A + B)/2}(r - r_-)^{(A - B)/2}F(r)$, we can transform Eq. (3) for $R(r)$ into a modified generalized spheroidal wave equation with imaginary spin-weight $iA$ and boost-weight $iB$ for $F(r)$ [44]:

$$\Delta \partial^2_r F(r) + 2[i e B + (1 + i A)(r - M + b)] \partial_r F(r) + [k^2 \Delta + 2Dr + iA - \lambda] F(r) = 0. \quad (8)$$

Eq. (8) has indices $\rho_+ = 0, -2iW_+$ and $\rho_- = 0, -2iW_-$ at two singularities $r = r_{\pm}$, respectively. Thus there have two linear independent solutions at each point. The infinity is an irregular singularity of Eq. (8) or (3). The functions $R(r)$ and $F(r)$ at infinity have asymptotic form $R(r) \sim e^{\pm ikr}$. Eq. (8) has the same form as the radial part of the massive complex scalar wave equation in the Kerr-Newman geometry [44] with its solution [44] (when $\mu = 0$) named as the generalized spheroidal wave function. It is interesting to note that a special solution of function $F(r)$ satisfies the Jacobi equation of imaginary index when $\omega = \pm \mu = qQ/M$ (namely, $k = D = 0$).

III. GENERALIZED SPHEROIDAL WAVE FUNCTION AND HEUN EQUATION

In this section, we shall show that the generalized spheroidal wave equation (3) of imaginary number order is, in fact, a confluent form of Heun equation [44]. To this end, let us make a coordinate transformation $r = M - b + e\varepsilon$ and substitute $R(r) = (z - 1)^{(A + B)/2}(z + 1)^{(A - B)/2}F(z)$ into Eq. (3), then we can reduce it to the following standard forms of a generalized spheroidal wave equation [44][45]:

$$(z^2 - 1)R''(z) + 2zR'(z) + \left( (ek)^2(z^2 - 1) + 2D\varepsilon \right) z^2 - 1 + 2D(M - b) - \lambda) R(z) = 0, \quad (9)$$

and

$$(z^2 - 1)F''(z) + 2[i e B + (1 + i A)z]F'(z) + [(ek)^2(z^2 - 1) + 2D\varepsilon + 2D(M - b) + iA - \lambda] F(z) = 0, \quad (10)$$

where a prime denote the derivative with respect to its argument.

The spin-weighted spheroidal wave function $F(z)$ is symmetric under the reflect $k \rightarrow -k$. Letting $F(z) = e^{i\varepsilon k z}G(z)$ without loss of generality, we can transform Eq. (10) to

$$(z^2 - 1)G''(z) + 2[i e B + (1 + i A)z + i\varepsilon k(z^2 - 1)]G'(z) + [2i\varepsilon k(1 + i A - iD/k)z - 2ekB + iA + 2D(M - b) - \lambda]G(z) = 0. \quad (11)$$

By means of changing variable $z = 1 - 2x$, we arrange the singularities $r = r_+$ ($z = 1$) to $x = 0$ and $r = r_-$ ($z = -1$) to $x = 1$, respectively, and reduce Eq. (11) to a confluent form of Heun’s equation [11][15]

$$G''(x) + \left( \frac{\beta + \gamma}{x} + \frac{\delta}{x - 1} \right) G'(x) + \frac{\alpha \beta x - h}{x(x - 1)} G(x) = 0, \quad (12)$$

with $\gamma = 1 + 2iW_+, \delta = 1 + 2iW_-, \beta = 4ie\varepsilon, \alpha = -(1 + iA) + iD/k$, $h = \lambda - 2ek - iA + 4ekW_+ - 2Dr_+$.

This confluent Heun equation (12), with $h$ its accessory parameter, has two regular singular points at $x = 0, 1$ with exponents $(0, 1 - \gamma)$ and $(0, 1 - \delta)$, respectively, as well as an irregular singularity at the infinity point. The power series solution in the vicinity of the point $x = 0$ for Eq. (12) can be written as

$$G(\alpha, \beta, \gamma, \delta, h; x) = \sum_{n=0}^{\infty} g_n x^n, \quad (13)$$

and the coefficient $g_n$ satisfies a three-term recurrence relation [11][18]

$$g_0 = 1, \quad g_1 = -h/\gamma , \quad (n + 1)(n + \gamma)g_{n+1} - \beta(n - 1 + \alpha)g_{n-1} = [n(n - 1 - \beta + \gamma + \delta) - h]g_n. \quad (14)$$

It is not difficult to deduce the exponent $1 - \gamma$ solution for $x = 0$ [11] and obtain the power series solution in the vicinity of the point $x = 1$ by a linear transformation interchanging the regular singular points $x = 0$ and $x = 1$: $x \rightarrow 1 - x$. Expansion of solutions to the confluent Heun’s equation in terms of hypergeometric and confluent hypergeometric functions has been presented in [18][20][14]. The confluent Heun’s functions can be orthonormalized to constitute a group of orthogonal complete functions [13]. It should be noted that Heun’s confluent equation also admits quasi-polynomial solutions for particular values of the parameters [11][18]. It follows from the three-term recurrence relation that $G(\alpha, \beta, \gamma, \delta, h; x)$ is a polynomial solution if

$$\alpha = -N, \quad \text{with integer } N \geq 0, \quad (15)$$

$g_{N+1}(h) = 0$, $g_{N+1}$ being a polynomial of degree $N + 1$ in $h$, that is, therefore $N + 1$ eigenvalues $h_i$ for $h$ such as $g_{N+1}(h_i) \equiv 0$.

IV. HAWKING RADIATION OF SCALAR PARTICLES

Now we investigate the Hawking evaporation [29] of scalar particles in the Kerr-Sen black hole by using the Damour-Ruffini-Sammam’s (DRS) method [10]. This approach only requires the existence of a future horizon and is completely independent of any dynamical details of the process leading to the formation of this horizon.
The DRS method assumes analyticity properties of the wave function in the complexified manifold.

In the following, we shall consider a wave outgoing from the event horizon $r_+$ over interval $r_+ < r < \infty$. According to the DRS method, a correct outgoing wave $\Phi^\text{out} = \Phi^\text{out}(t, r, \theta, \varphi)$ is an adequate superposition of functions $\Phi^\text{out}_{r>r_+}$ and $\Phi^\text{out}_{r<r_+}$:

$$
\Phi^\text{out} = C [\eta(r-r_+)\Phi^\text{out}_{r>r_+} + \eta(r_+ - r)\Phi^\text{out}_{r<r_+}e^{2\pi W_+}],
$$

(16)

where $\eta$ is the conventional unit step function, $C$ is a normalization factor.

In fact, components $\Phi^\text{out}_{r>r_+}$ and $\Phi^\text{out}_{r<r_+}$ have asymptotic behaviors:

$$
\Phi^\text{out}_{r>r_+} = \Phi^\text{out}_{r>r_+}(t, r, \theta, \varphi) \rightarrow c_1 (r-r_+)^{iW_+} \times S^t_{m,0}(ka, \theta)e^{i(m\varphi - \omega t)}, \quad (r \to r_+) \quad (17)
$$

$$
\Phi^\text{out}_{r<r_+} = \Phi^\text{out}_{r<r_+}(t, r, \theta, \varphi) \rightarrow c_2 (r-r_+)^{-iW_+} \times S^t_{m,0}(ka, \theta)e^{i(m\varphi + \omega t)}, \quad (r \to r_+) \quad (18)
$$

when $r \to r_+$. Clearly, the outgoing wave $\Phi^\text{out}_{r>r_+}$ can't be directly extended from $r_+ < r < \infty$ to $r_- < r < r_+$, but it can be analytically continued to an outgoing wave $\Phi^\text{out}_{r<r_+}$ that inside event horizon $r_+$ by the lower half complex $r$-plane around unit circle $r = r_+ - i0$:

$$
(r - r_+ \to (r_+ - r)e^{-i\pi}).
$$

By this analytical treatment, we have

$$
\Phi^\text{out}_{r<r_+} \sim c_2 (r-r_+)^{-iW_+} S^t_{m,0}(ka, \theta)e^{i(m\varphi + \omega t)}. \quad (19)
$$

As a difference factor $(r-r_+)^{-2iW_+}$ emerges, then $\Phi^\text{out}_{r>r_+}$ differs $\Phi^\text{out}_{r<r_+}$ by a factor $e^{2\pi W_+}$, thus we can derive the relative scattering probability of the scalar wave at the event horizon

$$
\frac{\Phi^\text{out}_{r>r_+}}{\Phi^\text{out}_{r<r_+}} = e^{-4\pi W_+}, \quad (20)
$$

and obtain the thermal radiation spectrum with the Hawking temperature $T = \kappa/2\pi$.

$$
\langle \mathcal{N} \rangle = |C|^2 = \frac{1}{e^{4\pi W_+} - 1}, \quad (21)
$$

$$
W_+ = \frac{Ar_+ - ma}{2\epsilon} = \frac{\omega - m\Omega - q\Phi}{2\kappa}, \quad (22)
$$

where the angular velocity at the horizon is $\Omega = a/2Mr_+$, the electric potential is $\Phi = Q/2M = b/Q$, the surface gravity at the pole is $\kappa = (r_+ - M + b)/2Mr_+ = \epsilon/2Mr_+$.

The black body radiation spectrum (21) demonstrates that the thermal property of Kerr-Sen black hole is similar to that of Kerr-Newman black hole though its geometry character likes that of the Kerr solution [27]. Correspondingly, there exist four thermodynamical laws of the Kerr-Sen black hole, similar to those of Kerr-Newman black hole thermodynamics.

### V. Conclusion

In this paper, we have shown that the separation of variables of the scalar wave equation in the Kerr-Newman black hole background [23] can apply completely to the case of the twisted Kerr solution. The separated radial part can be recast into the generalized spheroidal wave equation, which is, in fact, a confluent form of Heun equation.

In addition, we find that the thermal property of the twisted Kerr black hole resembles that of Kerr-Newman black hole though its geometry character likes that of the Kerr solution. The Kerr-Sen solution shares similar four black hole thermodynamical laws and quantum thermal effect as the Kerr-Newman spacetime does.

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