Currents on locally conformally Kähler manifolds

Alexandra Otiman

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Abstract We characterize the existence of a locally conformally Kähler metric on a compact complex manifold in terms of currents, adapting the celebrated result of Harvey and Lawson for Kähler metrics.

1 Introduction

A locally conformally Kähler manifold (LCK for short) is a Hermitian manifold \((M, J, g)\) for which the fundamental two-form \(\omega(X, Y) = g(JX, Y)\) satisfies

\[d\omega = \theta \wedge \omega, \quad d\theta = 0\]  

for some one-form \(\theta\) called the Lee form.

There are many examples of compact LCK and non-Kähler manifolds, among them the Hopf manifolds, see [DO], [OV].

As \(d\theta = 0\), the twisted differential \(d_\theta := d - \theta \wedge\) defines a twisted cohomology which is the Morse-Novikov cohomology of \(X\). The LCK condition simply means that the fundamental form of \((X, J, g)\) is \(d\theta\)-closed.

The aim of this note is to obtain an analogue of the intrinsic characterization in [HL] for Kähler manifolds in the context of LCK geometry.

2 LCK condition in terms of currents

Our main result is the following:

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Theorem 2.1: Let $X$ be a compact, complex manifold of complex dimension $n \geq 2$, and let $\theta$ be a closed one-form on $X$. Then $X$ admits a LCK metric with Lee form $\theta$ if and only if there are no non-trivial positive currents which are $(1,1)$-components of $d_\theta$-boundaries.

Remark 2.2: Suppose $X$ is a compact complex manifold, admitting a LCK metric, $\omega$, with Lee form $\theta$. Then any closed 1-form $\eta \in [\theta]_{dR}$ will be a Lee form for a conformal metric of $\omega$ and moreover, any conformal change of $\omega$ will be LCK with a Lee form in the same de Rham cohomology class as $\theta$. Therefore, we need not fix $\theta$, we can directly use its cohomology class, $[\theta]_{dR}$. By this observation, the theorem above can be stated as:

Let $X$ be a compact, complex manifold of complex dimension $n \geq 2$, and let $[\theta]_{dR}$ a cohomology class in $H^1_{dR}(X)$. Then $X$ admits a LCK metric with Lee form $\theta$ if and only if there are no non-trivial positive currents which are $(1,1)$-components of $d_\eta$-boundaries, for any closed one-form $\eta$ belonging to $[\theta]_{dR}$.

The rest of Section 2 is devoted to the proof, which follows the lines in [HL]. We use the same results and intermediate steps as [HL], the difficult part being that of finding some proper analogues in LCK geometry for the Kähler notions used in the original article. Each following subsection is a step of the proof.

2.1 Range of $d_\theta$ is closed

Associated with $d_\theta$ are the following operators:

$$
\partial_\theta = \partial - \theta^{1,0} \wedge, \quad \bar{\partial}_\theta = \bar{\partial} - \theta^{0,1} \wedge, \quad d_\bar{\theta} = i(\partial_\theta - \bar{\partial}_\theta)
$$

Definition 2.3: A smooth function is called $\theta$-pluriharmonic if it is locally the real part of a smooth $\partial_\theta$-closed function.

We let $\mathcal{H}_\theta$ be the sheaf of germs of $\theta$-pluriharmonic functions on $X$.

Lemma 2.4: $\mathcal{H}_\theta$ is the kernel of the sheaves morphism $\mathcal{E}_\mathbb{R}^{d_\theta d_\bar{\theta}} \rightarrow \mathcal{E}_\mathbb{R}^{1,1}$, where the subscript $\mathbb{R}$ denotes the germs of real valued forms.

Proof: The proof is based on the following easy observation

$$
\bar{\partial}_\theta f = 0 \iff \frac{1}{2i}(\bar{\partial}_\theta f - \partial_\theta \bar{f}) = 0
$$
Let now \( f = u + iv \). One obviously has
\[
\overline{\partial}_\theta f = 0 \iff \overline{\partial}_\theta (u + iv) = \partial_\theta (u - iv) = 0 \iff d_\theta v + d_\bar{\theta} u = 0 \quad (2.1)
\]

Let \( g \) for which a \( f' \) exists such that \( f = g + if' \) is \( \overline{\partial}_\theta \)-closed. It follows from (2.1) that \( d_\theta f' + d_\bar{\theta} g = 0 \), which implies \( d_\theta d_\bar{\theta} g = 0 \).

Conversely, if \( g \) satisfies \( d_\theta d_\bar{\theta} g = 0 \), finding a \( f' \) such that \( f = g + if' \) is \( \overline{\partial}_\theta \)-closed is equivalent to solving the equation \( d_\theta f' = -d_\bar{\theta} g \).

Since \( \theta \) is locally exact, let \( \theta = dh \) on a contractible open set. Then \( e^{-h} d_\bar{\theta} g \) is closed and by Poincaré lemma there exists a function \( h' \) such that \( e^{-h} d_\bar{\theta} g = dh' \). Then \( f' = -e^h h' \) which completes the proof.

The above result shows that the following is an exact sequence of sheaves:
\[
0 \rightarrow H^0 \rightarrow E \rightarrow \mathcal{E}_{1,1} \rightarrow \mathcal{E}_{1,2} \oplus \mathcal{E}_{2,1} \rightarrow \cdots
\]

Since \( [\mathcal{E}^{p,q}]_\mathbb{R} \) are acyclic, the above is a resolution which computes the cohomology groups of \( H_\theta \).

We now prove that \( H^i(X, H_\theta) \) are finite dimensional for all \( i \geq 0 \).

Let \( O_\theta \) denote the sheaf of germs of smooth functions satisfying \( \overline{\partial}_\theta f = 0 \) and let \( F \) be the kernel of the sheaves morphism \( \text{Re} : O_\theta \rightarrow H_\theta \):
\[
0 \rightarrow F \rightarrow O_\theta \xrightarrow{\text{Re}} H_\theta \rightarrow 0 \quad (2.2)
\]

**Proposition 2.5:** \( O_\theta \) is locally free of rank 1 over the sheaf of germs of holomorphic functions, \( O_X \), and \( F \) is locally constant.

**Proof:** To prove that \( F \) is locally constant, we characterize the non-zero \( \overline{\partial}_\theta \)-closed real valued functions.

Let \( h \) be a (unique up to addition with constants) real valued smooth function on a contractible neighbourhood such that \( \theta^{0,1} = \overline{\partial} h \). Then
\[
\overline{\partial}_\theta f = 0 \iff \overline{\partial} f - f\theta^{0,1} = 0 \iff \overline{\partial} f = f\overline{\partial} h
\]
Since both \( f \) and \( h \) are real valued, the above last equality gives, by conjugation, \( \partial f = f\overline{\partial} h \).

Summing up, we obtain \( df = f dh \), which yields
\[
d \log f = dh, \quad \text{and hence} \quad f = e^h \cdot c, \quad c \in \mathbb{R}
\]
This proves that on the neighbourhood where \( \theta^{0,1} \) is \( \overline{\partial} \)-exact, the sheaves \( F \) and \( \mathbb{R} \) are isomorphic.
We use a similar argument for $\mathcal{O}_\theta$. Let $h$ be as above. Then $e^h$ is $\overline{\partial}_\theta$-closed. Let $f \in \mathcal{O}_{x,X}$ defined on an open set contained in the domain of $h$. Let $\lambda := fe^{-h}$. As $\overline{\partial}_\theta f = 0$, we have $\overline{\partial}_\theta \lambda = 0$ which is equivalent to $\lambda \in \mathcal{O}_{x,X}$. Hence $\mathcal{O}_{\theta x,X} \cong \mathcal{O}_{x,X}$, proving that $\mathcal{O}_\theta$ is locally free of rank 1.

Corollary 2.6: $\mathcal{F}$ and $\mathcal{O}_\theta$ have finite dimensional cohomology groups.

Proof: By proving that $\mathcal{O}_\theta$ is locally free of rank 1, we have proved its coherence. Using now the Cartan-Serre theorem for coherent sheaves on compact complex manifolds [T], we obtain the finite dimension of its cohomology groups. As for $\mathcal{F}$, the compactness of $X$ assures the existence of a finite covering of contractible sets, on which $\mathcal{F}$ is isomorphic to $\mathbb{R}$. However, $\mathbb{R}$ has vanishing cohomology groups on contractible sets. Thus, by Leray theorem [D], we find a covering which computes via the Čech complex the cohomology of $\mathcal{F}$. But every term in the Čech complex associated to this covering is a real finite dimension vector space, hence the finite cohomological dimension of $\mathcal{F}$ is obvious.

Splitting the long exact sequence in cohomology associated to (2.2) into short exact sequences and using the above Corollary proves:

Corollary 2.7: $\mathcal{H}_\theta$ has finite dimensional cohomology groups.

Remark 2.8: One usually proves the finite dimensionality of the cohomology groups of a complex by means of elliptic operators, the most famous example being that of the Hodge isomorphism theorem stating that $H^\bullet(X, \mathbb{R}) \cong \text{Ker}\Delta$. Examples of elliptic operators dealing with twisted differentials such as $d_\theta$ are given in [AK]. However, we would be interested in an elliptic operator such that its kernel is given by the cohomology groups of the sheaf $\mathcal{H}_\theta$ and this case is not covered by the results in [AK]. More specifically, we are interested only in the second cohomology group of this sheaf, as we will see in the following sections.

Corollary 2.9: The operator $d_\theta : \mathcal{E}^{1,1}_\mathbb{R} \longrightarrow [\mathcal{E}^{1,2} \oplus \mathcal{E}^{2,1}]_\mathbb{R}$ has closed range.

Proof: Since $H^2(X, \mathcal{H}_\theta)$ is finite dimensional, $\text{Im} d_\theta$ has finite codimension in $Z(X) = \{\psi \in [\mathcal{E}^{1,2} \oplus \mathcal{E}^{2,1}]_\mathbb{R}, d_\theta \psi = 0\}$. $Z(X)$ remains a Fréchet space, as every closed subset of a Fréchet space does, so $d_\theta$ is a continuous linear function between two Fréchet spaces, whose codimension is finite (i.e.
$Z(X)/\text{Im}d_\theta$ is finite dimensional). We will invoke now the open mapping theorem for Fréchet spaces \cite[p.170]{TE}, which states that every surjective continuous and linear map between two Fréchet spaces is open, in order to prove the following general result, which is an elementary functional analysis lemma. If a linear continuous map between two Fréchet spaces has finite codimension, its range is closed.

We include its proof for the sake of completeness. Let $T : A \to B$ be such a map. Let $\omega_1, \ldots, \omega_n$ be elements in $B$ that give a basis for $B/\text{Im}T$ and let $C = \langle \omega_1, \ldots, \omega_n \rangle \subset B$. We consider now the map $F : A \oplus C \to B$ given by $F(x + y) = T(x) + y$. It is a simple observation that $F$ is surjective. $F$ is also continuous and linear, hence by the open mapping theorem an open map. We may assume that $T$ is actually injective, otherwise we factorize by its kernel and thus, $F$ becomes a bijective open continuous map, hence a homeomorphism. $T(A) = F(A \oplus \{0\})$, which is a closed set.

So the open mapping theorem was crucial for proving that $\text{Im}d_\theta$ is closed in $Z(X)$ and hence closed in $[E^{1,2} \oplus E^{2,1}]_\mathbb{R}$.

2.2 Extension of $d_\theta$ to currents

We follow the definitions and conventions in \cite{D} for currents.

Let $[\mathcal{E}'_\star(X)]_\mathbb{R}$ denote the dual space of $[\mathcal{E}'_\star(X)]_\mathbb{R}$. Recall that the differential $d : [\mathcal{E}'_\star(X)]_\mathbb{R} \to [\mathcal{E}'_{\star-1}(X)]_\mathbb{R}$ acts by

$$\langle dT, \eta \rangle := \langle T, d\eta \rangle, \quad \eta \in \mathcal{E}'^{\star-1}(X)$$

and the exterior product of a current and a 1-form $\cdot \wedge \xi : [\mathcal{E}'_\star(X)]_\mathbb{R} \to [\mathcal{E}'_{\star-1}(X)]_\mathbb{R}$ is defined by

$$\langle T \wedge \xi, \eta \rangle = \langle T, \xi \wedge \eta \rangle$$

We then define $d_\theta : [\mathcal{E}'_\star(X)]_\mathbb{R} \to [\mathcal{E}'_{\star-1}(X)]_\mathbb{R}$ as follows:

$$\langle d_\theta T, \eta \rangle = \langle T, d_\theta \eta \rangle, \quad \eta \in \mathcal{E}'^{\star-1}(X) \quad (2.3)$$

Let $T \in [\mathcal{E}'_{p,q}(X)]_\mathbb{R}$. In particular, $T$ is a $p + q$ current which vanishes on all $(i, j)$ forms with $(i, j) \neq (p, q)$, $d_\theta T \in \mathcal{E}'_{p+q-1}$ and decomposes as:

$$d_\theta T = \sum_{i+j=p+q-1} (d_\theta T)_{i,j}$$
\[ \langle (d_\theta T)_{i,j}, \eta \rangle = \langle T, d_\theta \eta \rangle, \quad \eta \in \mathcal{E}^{i,j}(X), i + j = p + q - 1 \]

But since
\[ \langle T, \eta \rangle = \sum_{i+j=p+q-1} \langle T_{i,j}, \eta_{ij} \rangle, \quad \eta_{ij} = \text{the (i,j) part of } \eta, \]
we obtain
\[ (d_\theta T)_{i,j} = 0 \text{ for all } (i, j) \notin \{(p, q-1), (p-1, q)\} \]

Let now \( T \in [\mathcal{E}'_{p,q+1}(X) \oplus \mathcal{E}'_{p,q}(X)]_R \). Then \( d_\theta T \in \mathcal{E}'_{p+q}(X) \). By (2.3), the only possibly non-zero components \( d_\theta T \) are \( (d_\theta T)_{p,q}, (d_\theta T)_{p+1,q-1}, (d_\theta T)_{p-1,q+1} \), as only the differential of \((p, q), (p+1, q-1)\) and \((p-1, q+1)\) forms can have non-trivial \((p, q+1)\) and \((p+1, q)\) parts. We have proved:

**Claim 2.10:** \( \langle d_\theta T, \eta \rangle = \langle (d_\theta T)_{p,q}, \eta \rangle \), for any \( \eta \in \mathcal{E}^{p,q}(X) \).

As in [HL], let \( \Pi_{p,q} : \mathcal{E}'_{p,q+1}(X) \rightarrow \mathcal{E}'_{p,q}(X) \) be the projector associating the \((p,q)\) part of a \((p,q+1)\) current. Let also
\[ (d^\theta_{p,q} T)^{\text{not.}} = (d_\theta T)_{p,q} = \Pi_{p,q} \circ d_\theta [\mathcal{E}'_{p,q+1}(X) \oplus \mathcal{E}'_{p,q}(X)]_R(T) \]
Denote \( B^\theta_{p,q} = \text{Im}(d^\theta_{p,q}) \). We prove:

**Lemma 2.11:** Let \( \eta \in \mathcal{E}^{1,1}(X) \). Then \( d\eta = \theta \wedge \eta \) if and only if \( \langle T, \eta \rangle = 0 \) for any \( T \in B^\theta_{1,1} \).

**Proof:** If \( \eta = d_\theta \) - closed, then \( \langle T, d_\theta \eta \rangle = 0 \) for all \( T \in [\mathcal{E}'_{2,1}(X) \oplus \mathcal{E}'_{1,2}(X)]_R \) and hence \( \langle d^\theta_{1,1} T, \eta \rangle = 0 \), yielding \( \langle T, \eta \rangle = 0 \) for all \( T \in B^\theta_{1,1} \).

Conversely, if \( \langle d^\theta_{1,1} T, \eta \rangle = 0 \) for \( T \in [\mathcal{E}'_{2,1}(X) \oplus \mathcal{E}'_{1,2}(X)]_R \), then \( \langle T, d\eta \rangle = 0 \), equality which is attained even for all \( T \in [\mathcal{E}'_{1,1}(X)]_R \), since a \((3,0)\) current vanishes on \( d\eta \), and thus \( d\eta = 0 \).

We finally prove:

**Proposition 2.12:** The operator \( d^\theta_{1,1} : [\mathcal{E}'_{1,2}(X) \oplus \mathcal{E}'_{2,1}(X)]_R \rightarrow [\mathcal{E}'_{1,1}(X)]_R \) has closed range. In other words, \( B^\theta_{1,1} \) is closed in \([\mathcal{E}^{1,1}(X)]_R \).
Proof: From [Claim 2.10] we know that $d_{1,1}^θ$ is the adjoint of $d_θ : [E^{1,1}(X)]_R → [E^{1,2}(X) ⊕ E^{2,1}(X)]_R$, which, by [Corollary 2.9] has closed range. Since both $[E^{1,1}(X)]_R$ and $[E^{1,2}(X) ⊕ E^{2,1}(X)]_R$ are Fréchet spaces, we may apply the closed range theorem, as in [S, chap. IV, section 7.7] to conclude that $d_{1,1}^θ$ has closed range too.

2.3 Positive currents

We collect here, mainly without proof, several facts we shall need about positive currents. The reference is [D].

Let $T$ be a $(p,p)$ current. It can be written locally as

$$ T = \sum_{|I|=n-p} T_{I,J} dz_I ∧ d\bar{z}_J, $$

where $T_{I,J}$ is a distribution.

For a positive current, $T_{I,J}$ is a complex measure that satisfies $T_{I,J} = T_{J,I}$ and $T_{I,I} > 0$. We denote by $\|T\| := \sum |T_{I,J}|$ the mass measure of $T$.

Since $|T_{I,J}|$ is absolutely continuous with respect to $\|T\|$, Radon-Nykodim theorem applies and hence there exists a measurable function $f_{I,J}$ such that

$$ T_{I,J} = \int f_{I,J} d\|T\|, \quad \eta ∈ E^{p,p}(X) $$

(2.4)

If $\overrightarrow{T}$ is defined by $\eta_x(\overrightarrow{T}_x) = \eta_x ∧ f_x$, $x ∈ X$, (2.4) can be rewritten as

$$ \langle T, η \rangle = \int_X \eta ∧ f d\|T\|, \quad η ∈ E^{p,p}(X) $$

(2.5)

Proposition 2.13: The current $T$ is positive if and only if the function

$$ f(x) = \sum_{I,J} f_{I,J}(x) dz_I ∧ d\bar{z}_J ∈ Λ^{n-p,n-p}T^*_x X $$

is positive $\|T\|$ a.e.

We apply the above considerations for a $(1,1)$ current $T$ (in which case $\overrightarrow{T}$ is a bivector). Then

$$ T \text{ is positive } ⇔ f = \sum_{|I|=n-1} f_{I,J} dz_I ∧ d\bar{z}_J \text{ is positive } \|T\| \text{ a.e.} $$

(2.6)

$$ ⇔ \overrightarrow{T} ∈ \text{conv}(G_C(1, T_x X)) $$
2.4 Wirtinger inequality on LCK manifolds

**Theorem 2.14:** Let $\omega$ be a LCK form on $X$ with Lee form $\theta$. Then $\omega(\xi) \leq 1$ for any $\xi \in \text{conv}(G_R(2, T_X X))$, with equality if and only if $\xi$ lies in $\text{conv}(G_C(1, T_X X))$.

**Remark 2.15:** Although the inequality is often stated for Kähler forms, the condition $d\omega = 0$ is not used in the proof. The only property of the Kähler form which is used is that $\omega^n$ is a volume form, and this holds on LCK manifolds too (and more generally, whenever $\omega$ is strictly positive).

Using also (2.6), it then follows that for a positive $(1,1)$ current $T$,

$$\langle T, \omega \rangle = \int_X \omega_x(\overline{T_x}) \|T\| = \int_X \|T\|$$

But $\int_X \|T\| = \|T\|(X) > 0$ and hence $\langle T, \omega \rangle > 0$ for any positive non-zero $(1,1)$ current $T$.

2.5 Proof of Theorem 2.1

We adjust the proof in [HL].

Denote by $P_{1,1}(X)$ the space of positive currents on the compact LCK manifold $X$. Recall that we proved the following facts:

$$\langle T, \omega \rangle = 0, \text{ for } T \in B_{1,1}^\theta \text{ and } \langle T, \omega \rangle > 0, \text{ for } T \in P_{1,1}(X) \setminus \{0\}$$

and hence we have

$$B_{1,1}^\theta \cap P_{1,1} = \{0\} \quad \text{(2.7)}$$

The difficult task is to prove the converse.

We let $X$ be complex, compact and fix a closed one form $\theta$. Assuming (2.7), we look for a positive $(1,1)$ form which is $d_\theta$-closed (it will define the LCK metric).

We choose an arbitrary Hermitian metric $h$ on $X$ and we let $\psi = - \text{Im}(h)$. Then $\psi \in [E_{1,1}^1(X)]_R$. Using $\psi$ we define the set $K = \{T \in P_{1,1}(X) : \langle T, \psi \rangle = 1\}$ which is a compact base for $P_{1,1}(X)$ and is weakly compact in $[E_{1,1}^1(X)]_R$, as a consequence of Banach-Alaoglu theorem [D]. As $B_{1,1}^\theta(X)$ is closed, we may apply the Hahn-Banach separation theorem [S], stating there is a closed real hyperplane separating a closed set and a compact set in a locally convex space, as long as they are disjoint. The space of real $(1,1)$-currents, $E_{1,1}^{1,1}(X)$, is locally convex and so is the quotient space $E_{1,1}^{1,1}(X)/B_{1,1}^\theta(X)$ [DR].
Applying now the Hahn-Banach theorem for the locally convex space \( \mathcal{E}^{1,1}_R(X)/\mathcal{B}^{1,1}_\theta(X) \), the closed set \( \{0\} \) and the compact set \( K \) (which does not contain the 0 current), we get a hyperplane that separates \( K \) from 0. Thus, we obtain a continuous linear functional \( f : \mathcal{E}^{1,1}_R(X)/\mathcal{B}^{1,1}_\theta(X) \to \mathbb{R} \), which takes only strictly positive or negative values on \( K \) and by a change of sign we can assume the values are strictly positive. \( f \) provides a functional \( \tilde{f} \) on the whole \( \mathcal{E}^{1,1}_R(X) \), which vanishes on \( \mathcal{B}^{1,1}_\theta(X) \) and is positive on \( K \).

We define the real (1, 1)-form, \( \omega \), as \( \langle T, \omega \rangle = \tilde{f}(T) \), for any (1, 1)-current \( T \). This holds as definition since the pairing between a current and a form given by the evaluation \( \langle T, \omega \rangle \) is nondegenerate. This real (1, 1)-form will vanish on \( \mathcal{B}^{1,1}_\theta(X) \) and will be strictly positive on \( K \).

Since the condition of vanishing on \( \mathcal{B}^{1,1}_\theta(X) \) is equivalent to \( \omega = 0 \), we already obtain a \( d_\theta \)-closed form. As \( \omega \) is strictly positive on \( K \) and \( K \) is a compact base for \( \mathcal{P}^{1,1}(X) \), we also obtain the positivity on \( \mathcal{P}^{1,1}(X) \).

What remains is to show that \( \omega \) is a non-degenerate, positive form.

We shall prove that

\[
\omega_x(v \wedge \overline{v}) > 0 \text{ for any } v \in T^{1,0}_x X.
\]

Let \( \overrightarrow{T}_x = v \wedge \overline{v} \in G^C_{(1, n)} \subset \Lambda^{1,1}T_x X \).

By now, we associated to each (1, 1) - current a smooth collection of bivectors \( \{\overrightarrow{T}_x\} \) and now we go the other way around, by defining the (1, 1) current \( T = \delta_x \overrightarrow{T} \), where \( \delta_x \) is the Dirac measure concentrated in \( x \). Then \( T \) is a positive current since \( \overrightarrow{T} \) was chosen from \( G^C_{(1, n)} \) and hence \( \langle T, \omega \rangle > 0 \). This is equivalent to \( \omega(v \wedge \overline{v}) > 0 \), concluding that \( \omega \) is a positive \( d_\theta \)-closed (1, 1) form, thus producing a LCK metric.

3 Transverse \((p, p)\)-forms

In \[AA\], Alessandrini and Andreatta extend Theorem 14 in \[HL\] p. 176] to transverse closed \((p, p)\)-forms. As a byproduct of adapting to \( d_\theta \) the usual operations on currents, as presented in Section 2.2, we give an analogue of Theorem 1.17 in \[AA\] p. 188] by considering the existence of a transverse \( d_\theta \)-closed \((p, p)\)-form instead of a usual transverse closed \((p, p)\)-form. The particular case \( p = 1 \) recovers precisely Theorem 2.1 of the present note.

**Definition 3.1:** A transverse \((p, p)\)-form is a form which at any point belongs to the interior of the cone of strongly positive forms.

It is proved in \[AA\] that given a complex compact manifold \( X \), there exists a transverse closed \((p, p)\)-form if and only if there are no positive
currents which are \((p,p)\)-components of boundaries. The same steps and techniques can be used in order to prove the following result:

**Proposition 3.2:** Let \(X\) be a complex, compact manifold and \(\theta\) a real closed 1-form. There exists a transverse \((p,p)\)-closed form if and only if there are no positive \((p,p)\)-currents which are \(d\theta\)-boundaries.

In order to prove this result we need first to present some intermediate facts.

Let \(B^\theta_{p,p}\) denote the space of currents which are \((p,p)\)-components of \(d\theta\)-boundaries and \(\Omega^0_{\theta}\) the kernel of the following sheaf morphism:

\[
\overline{\partial}_\theta : \mathcal{E}^{p,0} \longrightarrow \mathcal{E}^{p,1}
\]

We have this exact sequence of sheaves:

\[
0 \longrightarrow \mathcal{H}_\theta \xrightarrow{f} \mathcal{L}_\theta^0 \xrightarrow{f_0} \mathcal{L}_\theta^{p-1} \xrightarrow{g_0} \mathcal{B}_\theta^p \xrightarrow{g_0} \mathcal{B}_\theta^{p+1} \xrightarrow{g_{p+1}} \cdots
\]

\[
\cdots \longrightarrow \mathcal{B}_\theta^{2p} \xrightarrow{g_{2p-1}} \mathcal{E}^{p,p}_{\mathbb{R}} \xrightarrow{d\theta^p_{\mathbb{R}}} \mathcal{E}^{p+1,p+1}_{\mathbb{R}} \xrightarrow{d\theta} \mathcal{E}^{p+1,p+2}_{\mathbb{R}} \xrightarrow{d\theta} \cdots
\]

where:

\[
\mathcal{L}_\theta^k = \overline{\Omega}_{\theta k}^{k+1} \oplus \mathcal{E}^{0,k}_{\mathbb{R}} \oplus \mathcal{E}^{1,k-1}_{\mathbb{R}} \oplus \cdots \oplus \mathcal{E}^{p,k-p}_{\mathbb{R}} \oplus \Omega_{\theta k}^{k+1}
\]

for \(0 \leq k \leq p - 1\);

\[
\mathcal{B}_\theta^k = \mathcal{E}^{k-p,p}_{\mathbb{R}} \oplus \cdots \oplus \mathcal{E}^{p,k-p}_{\mathbb{R}}
\]

for \(p \leq k \leq 2p - 1\);

\[
f : \mathcal{H}_\theta \rightarrow \mathcal{L}_\theta^0
\]

\[
f(\varphi) = (-\overline{\partial}_\theta \varphi, \varphi, -\partial \theta \varphi)
\]

\[
f_k : \mathcal{L}_\theta^k \rightarrow \mathcal{L}_\theta^{k+1}
\]

\[
f_k(\varphi, a^{0,k}, a^{1,k-1}, \ldots, a^{k-1,1}, a^{k,0}, \eta) = (-\overline{\partial}_\theta \varphi, \varphi + \overline{\partial} a^{0,k}, \partial \theta a^{0,k} + \overline{\partial} a^{1,k-1}, \ldots, \partial \theta a^{i-1,j} + \overline{\partial} a^{i,j-1}, \ldots, \eta + \partial \theta a^{k,0}, -\partial \theta \eta)
\]

\[
g : \mathcal{L}_\theta^{p-1} \rightarrow \mathcal{B}_\theta^p
\]

\[
g(\varphi, a^{0,p-1}, a^{1,p-1}, \ldots, a^{p-1,1}, a^{p,0}, \eta) =
\]
\((\varphi + \partial_\theta a^{0,p}, \partial_\theta a^{0,p} + \bar{\partial} a^{1,p-1}, \ldots, \partial_\theta a^{i-1,j} + \bar{\partial}_\theta a^{i,j-1}, \ldots, \eta + \partial_\theta a^{p,0}, -\partial_\theta \eta)\)

\[ g_k : B^k_\theta \to B^{k+1}_\theta \]

\[ g_k(a^{k-p,p} + \ldots + a^{p,k-p}) = \]

\((\partial_\theta a^{k-p,p} + \bar{\partial}_\theta a^{k-p+1,p-1}, \ldots, \partial_\theta a^{p-1,k-p} + \bar{\partial}_\theta a^{p,k-p-1})\)

\[ h : B^{2p-1}_\theta \to E^{p,p}_\mathbb{R} \]

\[ h(a^{p-1,p}, a^{p-1,p}) = \partial_\theta a^{p-1,p} + \bar{\partial}_\theta a^{p,p-1} \]

**Remark 3.3:** The sequence considered above is not a resolution for \(H_\theta\) since \(L^k_\theta\) are not acyclic.

**Proposition 3.4:** \(L^k_\theta\) has finite dimensional cohomology groups.

**Proof:** The sheaves \(E^{k,q}_\mathbb{R}\) are acyclic, therefore \(H^i(L^k_\theta) = H^i(\Omega^k_\theta \oplus \bar{\Omega}^k_\theta)\), for any \(i > 0\). But \(\Omega^k_\theta\) and its conjugate have both finite dimensional cohomology groups, since \(\Omega^k_\theta\) is locally isomorphic to \(\Omega^k\) via an argument similar to the coherence of the sheaf \(O_\theta\).

Let \(Z\) be the kernel of \(g_p\). By splitting the sequence into short exact sequences and by using the proposition above, we obtain that the connecting morphism

\[ H^k(X, Z) \to H^{k+p}(X, \mathcal{H}_\theta) \]

has finite dimensional kernel and cokernel. We may now use the finite dimensionality of the cohomology of \(\mathcal{H}_\theta\) and obtain the finite dimensionality of the cohomology groups of \(Z\).

Since the resolution:

\[
\begin{array}{c}
0 \longrightarrow Z \longrightarrow B^p_\theta \longrightarrow B^{p+1}_\theta \longrightarrow \cdots \longrightarrow B^{2p-1}_\theta \longrightarrow E^{p,p}_\mathbb{R} \longrightarrow E^{p+1,p+1}_\mathbb{R} \longrightarrow \cdots
\end{array}
\]

computes the cohomology of \(Z\), we conclude that \(B^p_{\theta,p,p}\) is closed, since \(H^{p+2}(X, Z)\) is finite dimensional.
**Remark 3.5:** It is easy to see that a $(p,p)$-form is $d_\theta$-closed if and only if it vanishes on $B_{\theta,p,p}^\theta$.

The proof of Proposition 3.2 is now identical to the proof of Theorem 1.17 in [AA, p. 188], by replacing $B_{p,p}$ with $B_{\theta,p,p}^\theta$.

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University of Bucharest,
Faculty of Mathematics
and Computer Science,
14 Academiei str., Bucharest, Romania.
alexandra_otiman@yahoo.com