Weak field reduction in teleparallel coframe gravity. Vacuum case.

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The teleparallel coframe gravity may be viewed as a generalization of the standard GR. A coframe (a field of four independent 1-forms) is considered, in this approach, to be a basic dynamical variable. The metric tensor is treated as a secondary structure. The general Lagrangian, quadratic in the first order derivatives of the coframe field is not unique. It involves three dimensionless free parameters. We consider a weak field approximation of the general coframe teleparallel model. In the linear approximation, the field variable, the coframe, is covariantly reduced to the superposition of the symmetric and antisymmetric field. We require this reduction to be preserved on the levels of the Lagrangian, of the field equations and of the conserved currents. This occurs if and only if the pure Yang - Mills type term is removed from the Lagrangian. The absence of this term is known to be necessary and sufficient for the existence of the viable (Schwarzschild) spherical-symmetric solution. Moreover, the same condition guarantees the absence of ghosts and tachyons in particle content of the theory. The condition above is shown recently to be necessary for a well defined Hamiltonian formulation of the model. Here we derive the same condition in the Lagrangian formulation by means of the weak field reduction.

I. INTRODUCTION

Einstein’s general relativity (GR) is very successful in describing the long distance (macroscopic) gravity phenomena. This theory, however, encounters serious difficulties on microscopic distances. So far essential problems appear in all attempts to quantize the standard GR (for recent review, see, e.g., [1]). Also, the Lagrangian structure of GR differs, in principle, from the ordinary microscopic gauge theories. In particular, a covariant conserved energy-momentum tensor for the gravitational field cannot be constructed in the framework of GR. Consequently, the study of alternative models of gravity is justified from the physical as well as from the mathematical point of view. Even in the case when GR is unique true theory of gravity, consideration of close alternative models can shed light on the properties of GR itself.

Among various alternative constructions, the Poincaré gauge theory of gravity, see Refs. [2] — [9], is of a special interest. This theory proposes a natural bridge between gauge and geometrical theories. Moreover, it has a straightforward generalization to the metric-affine theory of gravity [5], which involves a wide spectra of spacetime geometries.

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However, it was elucidated recently that even the restriction of the Poincaré gauge theory to the teleparallel model provides a reasonable alternative to GR, see e.g. [23].

A. Coframe (teleparallel) gravity — basic facts and notations

We start with a brief account of the coframe (teleparallel) model of gravity and establish the notations used in this paper. Details, different approaches and additional references can be found in [12] – [28].

Let a 4D differential manifold $\mathcal{M}$ be endowed with two smooth fields: a frame field $e_a$ and a coframe field $\vartheta^a$. In a local coordinate chart,

$$e_a = e_a^\mu(x) \frac{\partial}{\partial x^\mu}, \quad \vartheta^a = \vartheta^a_\mu(x) \, dx^\mu, \quad a, \mu = 0, 1, 2, 3.$$  \hspace{1cm} (1.1)

These fields allow to compare two vectors (more generally, two tensors) attached to different points of the manifold. It is referred to as the teleparallel structure on $\mathcal{M}$. The two basic fields are assumed to fulfill the dual relation: $e_a \vartheta^b = \delta^b_a$. We denote by $\lrcorner$ the interior product operator $X \times \Lambda^p \to \Lambda^{p-1}$ that, for an arbitrary vector field $X \in \mathcal{X}$ and a $p$-form field $w \in \Lambda^p$, $X \lrcorner w := w(X, \cdots)$. So only one of the fields, $e_a$ or $\vartheta^a$, is independent. Thus, two alternative (but, principle, equivalent) representations of the teleparallel geometry are possible.

The frame representation is based on a complex $\{\mathcal{M}, e_a\}$ and applies the tensorial calculus as the main mathematical tool similar to the Einstein tensorial representation of GR.

The coframe representation, which deals with a complex $\{\mathcal{M}, \vartheta^a\}$, applies the exterior form technique. In present paper, we use this approach and call it the coframe gravity, in contrast to the metric gravity of GR.

In a wider context, the coframe field appears as one of the basic dynamical variables in the Poincaré gauge gravity and in the metric-affine gravity. To extract the pure coframe sector, in these theories, one has to require vanishing of the curvature. Here, we treat the coframe field as a self-consistent dynamical variable with its own covariant operators: wedge product, Hodge map and exterior derivative. These two approaches (one with a trivial connection and the other without explicit exhibition of a connection) are principally equivalent.

The indices in (1.1) are basically different. The Greek indices refer to the coordinate space and describe the behavior of tensors under the group of diffeomorphisms of the manifold $\mathcal{M}$. The Roman indices denote different 1-forms of the coframe. The corresponding group of transformations, $SO(1, 3)$, comes together with its natural invariant $\eta_{ab} = diag(1, -1, -1, -1)$.

The metric tensor on $\mathcal{M}$ is expressed via the coframe as

$$g = \eta_{ab} \vartheta^a \otimes \vartheta^b,$$ \hspace{1cm} (1.2)

i.e., the coframe is postulated to be pseudo-orthonormal. The coframe field and all the objects constructed from it are assumed to be global (rigid) covariant. In other words, all the constructions are required to be covariant under the global transformations $\vartheta^a \to A^a_b \vartheta^b$ with a constant matrix $A^a_b \in SO(1, 3)$. The metric tensor (1.2) is invariant under a wider group of transformation: local (pointwise) transformations of the coframe with $A^a_b = A^a_b(x)$. 
Consider a Lagrangian density, which is (i) diffeomorphism invariant, (ii) invariant under global $SO(1, 3)$ transformations of the coframe and (iii) quadratic in the exterior derivatives of the coframe. The most general Lagrangian of this form is a linear combination \[ \mathcal{L} = \frac{1}{2} \sum_{i=1}^{3} \rho_i \mathcal{L}^{(i)}, \] where $\rho_1, \rho_2, \rho_3$ are free dimensionless parameters. The linear independent 4-forms appearing here are expressed via the coframe field strength, $C^a := d\vartheta^a$.

\[(1)\mathcal{L} = C^a \wedge \ast C_a,\]
\[(2)\mathcal{L} = (C_a \wedge \vartheta^a) \wedge \ast (C_b \wedge \vartheta^b),\]
\[(3)\mathcal{L} = (C_a \wedge \vartheta^b) \wedge \ast (C_b \wedge \vartheta^a),\]

The Hodge dual operator $\ast$ is defined by the pseudo-orthonormal coframe $\vartheta^a$ or, equivalently, by the metric (1.2). One may try to include in the Lagrangian some invariant expressions of the second order (similarly to the Hilbert-Einstein Lagrangian). Such terms, however, are reduced to total derivatives and do not affect the field equations and the Noether conserved currents. So (1.3) is the most general Lagrangian that generates the field equations of the second order.

Let us introduce the notion of the field strength

\[\mathcal{F}^a := (1)\mathcal{F}^a + (2)\mathcal{F}^a + (3)\mathcal{F}^a,\] with

\[(1)\mathcal{F}^a := (\rho_1 + \rho_3)C^a,\]
\[(2)\mathcal{F}^a := \rho_2 e^a \mid (\vartheta^m \wedge C_m),\]
\[(3)\mathcal{F}^a := -\rho_3 \vartheta^a \wedge (e_m \mid C^m).\]

Such separation of the strength $\mathcal{F}^a$ involves two scalar-valued forms $\vartheta^a \wedge C_m$ and $e_m \mid C^m$. So some calculations are simplified. For irreducible decomposition of $\mathcal{F}^a$, see [5] and [23].

In the notation (1.8 — 1.10), the coframe Lagrangian (1.3) takes a form similar to the Maxwell Lagrangian,

\[\mathcal{L} = \frac{1}{2} C^a \wedge \ast \mathcal{F}_a,\] (1.11)

The free variation of (1.11) relative to the coframe $\vartheta^a$ has to take into account also the variation of the Hodge dual operator, which implicitly depends on the coframe. It yields the field equation of the form [23]

\[d \ast \mathcal{F}^a = T^a,\] (1.12)

where the 3-form $T^a$ is the energy-momentum current of the coframe field

\[T_a = (e_a \mid C_m) \wedge \ast F^m - e_a \ast \mathcal{L}.\] (1.13)

The conservation law for this 3-form: $dT_a = 0$ is a straightforward consequence of (1.12).
B. Viable models — a problem of physical motivation

A general quadratic coframe model, which is global $SO(1,3)$ invariant, involves three parameters:

$$\rho_1, \quad \rho_2, \quad \rho_3 \quad \text{—— free}.$$  \hfill (1.14)

The ordinary GR is extracted from this family by requiring of the local $SO(1,3)$ invariance, which is realized by the following restrictions of the parameters:

$$\rho_1 = 0, \quad 2\rho_2 + \rho_3 = 0.$$  \hfill (1.15)

The analysis of exact solutions [28] to the field equation (1.12) shows that the Schwarzschild solution appears even for a wider set of parameters (viable set):

$$\rho_1 = 0, \quad \rho_2, \quad \rho_3 \quad \text{—— free}.$$  \hfill (1.16)

Moreover, for $\rho_1 \neq 0$, spherical-symmetric static solutions to (1.12) do not have the Newtonian behavior at infinity [28].

So a problem arises: Which physical motivated requirement extracts the viable set of parameters?

The quantum-theory solution to this problem is known for a long time. In [29] — [33] it was shown that the requirement (1.16) is necessary and sufficient for absence of ghosts and tachyons in particle content of the theory. Another motivation for (1.16) comes from the requirement that the theory has to have a well defined Hamiltonian formulation ( [34]).

In this paper we look for a motivation of (1.16) on a classical Lagrangian level. We deal with linear approximation of the general coframe model. The coframe variable can be treated, in this approximation, as a regular $4 \times 4$ matrix. Consequently, it reduced to a composition of two independent variables: the symmetric and the antisymmetric fields.

Our main result is as follows: Only for (1.16), the coframe model is reduced to two independent models, every one with its own Lagrangian, field equation, and conserved current. In other words, the viable model is exactly this one that approaches the free-field limit, i.e., any interaction between the approximately independent fields appears only in higher orders.

Linear approximation of coframe models was usual applied for study the deviation of teleparallel gravity from the standard GR, and for comparison with the observation data, see [3], [4], [30], [31]. In our approach the reduction of the lower order terms is used as a theoretical device. We show that this condition is enough to distinguish the set of viable models. The relation between these two approaches requires a further consideration.

II. WEAK FIELD REDUCTION

A. Linear approximations

To study the approximate solutions to (1.12), we start with a trivial exact solution, a holonomic coframe, for which,
\[ d\vartheta^a = 0. \]  

(2.1)

Consequently, \[ F^a = C^a = 0, \] so both sides of Eq. (1.12) vanish. By Poincaré’s lemma, the solution of (2.1) can be locally expressed as \[ \vartheta^a = d\tilde{x}^a(x), \] where \( \tilde{x}^a(x) \) is a set of four smooth functions defined in some neighborhood \( U \) of a point \( x \in \mathcal{M} \). The functions \( \tilde{x}^a(x) \), being treated as the components of a coordinate map \( \tilde{x}^a : U \rightarrow \mathbb{R}^4 \), generate a local coordinate system on \( U \). The metric tensor (1.2) reduces, in this coordinate chart, to the flat Minkowskian metric \( g = \eta_{ab}d\tilde{x}^a \otimes d\tilde{x}^b \). Thus the holonomic coframe plays, in the teleparallel background, the same role as the Minkowskian metric in the (pseudo-)Riemannian geometry. Moreover, a manifold endowed with a (pseudo-)orthonormal holonomic coframe is flat. The weak perturbations of the basic solution \( \vartheta^a = dx^a \) are

\[ \vartheta^a = dx^a + h^a = (\delta^a_b + h^a_b)dx^b. \]  

(2.2)

"Weak" means:

\[ ||h^a_b|| = \epsilon = o(1), \quad ||h^a_{b,c}|| = O(\epsilon), \quad ||h^a_{b,c,d}|| = O(\epsilon), \]  

(2.3)

where \( || \cdot \cdot \cdot || \) denotes the maximal tensor norm. We accept that the coframe \( \vartheta^a \) and the holonomic coframe \( dx^a \) have the same physical dimension of \([\text{length}]\). Thus, the components of the matrix \( h^a_b \) and the parameter \( \epsilon \) are dimensionless. Consequently, the approximation conditions (2.3) are invariant under rescaling of the coordinates.

In this paper we will take into account only the first order approximation in the perturbations \( h^a_b \) and in their derivatives (i.e., in the parameter \( \epsilon \)). Note that, in this approximation, the difference between coframe and coordinate indices completely disappears. This justifies our choice, in (2.2) and in the sequel, of the same notation for these (basically different) indices.

In accordance with (2.3), only for weak coordinate transformations are considered. Under a shift

\[ x^a \mapsto x^a + \xi^a(x), \]  

(2.4)

the components of the coframe are transformed as

\[ h^a_b \mapsto h^a_b - \xi^a_b. \]  

(2.5)

Thus, in order to preserve the weakness of the fluctuation, it is necessary to require \( \xi^a_b = O(||h^a_b||) \). We will use the term \textit{approximately covariant} [35] for the expressions which are covariant only to the first order of the perturbations. Observe that this assumption restricts only the amplitudes of the perturbations and of their derivatives. It does not restrict, however, the local freedom to transform the coordinates. An appropriate coordinate system can still be chosen in a small neighborhood of the identity transformation in order to simplify the (local) field equations.

Similarly, in order to be in agreement with the approximation condition (2.3), the global \( \text{SO}(1,3) \) transformations of the coframe field, \( \vartheta^a \mapsto A^a_{b} \vartheta^{b} \), have also to be restricted. It is enough to require the transformations to be in a small neighborhood of the identity

\[ A^a_{b} = \delta^a_{b} + \alpha^a_{b}, \quad ||\alpha^a_{b}|| = o(1). \]  

(2.6)
B. Reduction of the field

In (2.2), \( h_{ab} \) is a perturbation of the flat coframe. Thus

(i) To the first order, the holonomic coframe is expressed by the unholonomic one as

\[
dx^a = (\delta^a_b - h^a_b) \vartheta^b.
\]

(ii) The indices in \( h_{ab} \) can be lowered and raised by the Minkowskian metric

\[
h_{ab} := \eta_{am} h^m_b, \quad h^{ab} := \eta^{bm} h_m^a .
\]

The first operation is exact (covariant to all orders of approximations), while the second is covariant only to the first order, when \( g^{ab} \approx \eta^{ab} \).

(iii) The symmetric and the antisymmetric combinations of the perturbations

\[
\theta_{ab} := h_{(ab)} = \frac{1}{2}(h_{ab} + h_{ba}), \quad \text{and} \quad w_{ab} := h_{[ab]} = \frac{1}{2}(h_{ab} - h_{ba}).
\]

as well as the trace \( \theta := h_{mm} = \theta^m_m \) are covariant to the first order.

(iv) The components of the metric tensor, in the linear approximation, involve only the symmetric combination of the coframe perturbations

\[
g_{ab} = \eta_{ab} + 2\theta_{ab}.
\]

(v) Under the transformations (2.4), two covariant pieces of the fluctuation change as

\[
\theta_{ab} \mapsto \theta_{ab} - \xi_{(a,b)} , \quad \text{and} \quad w_{ab} \mapsto w_{ab} - \xi_{[a,b]} .
\]

Thus the approximately covariant irreducible decomposition of the dynamical variable

\[
h_{ab} = \theta_{ab} + w_{ab}.
\]

is obtained. Thus, instead of one field \( h_{ab} \), we have, in this approximation, two independent fields: a symmetric field \( \theta_{ab} \) and an antisymmetric field \( w_{ab} \).

C. Gauge conditions

The actual values of the components of the fields \( \theta_{ab} \) and \( w_{ab} \) depend on a choice of a coordinate system. Thus four arbitrary relations between the components (equal to the number of coordinates) may be imposed. We require these relations to be Lorentz invariant, i.e., covariant in the first order approximation. Thus the most general form of constraints (gauge conditions) that involve the first order derivatives is
\[ \alpha \theta_{am,m} + \beta \theta_{a,m} + \gamma w_{am} = 0, \quad (2.13) \]

where \( \alpha, \beta, \gamma \) are dimensionless parameters.

Certainly, for some special values of the parameters, these conditions cannot be realized. Indeed, under the coordinate transformations (2.4), Eq. (2.13) changes, in the lowest order, to

\[ \alpha \tilde{\theta}_{am,m} + \beta \tilde{\theta}_{a,m} + \gamma \tilde{w}_{am} = (\alpha \xi_{(a,m)} + \beta \xi_{m,a} + \gamma \xi_{[a,m]}), \quad (2.14) \]

Thus the conditions (2.13) can be realized, by the coordinate transformations (2.4), if and only if the system of PDE (2.14) has a solution \( \xi(x) \) for a given LHS.

Let us check the integrability of this system. Eq. (2.14) results in

\[ (\alpha \xi_{(a,m)\,b} + \beta \xi_{m,a\,b} + \gamma \xi_{[a,m]\,b}) = \alpha \tilde{\theta}_{am\,m} + \beta \tilde{\theta}_{a\,b} + \gamma \tilde{w}_{am\,b}. \quad (2.15) \]

Commuting the indices \( a \) and \( b \), we obtain

\[ (\alpha + \gamma) \Box \xi_{[a,b]} = 2(\alpha \theta_{m[a,b]} - \gamma w_{m[a,b]}). \quad (2.16) \]

Thus, the gauge condition (2.13) with \( \alpha = -\gamma \neq 0 \) cannot be realized by any change of the coordinate system.

Now, take the trace of (2.15)

\[ (\alpha + \beta) \Box \theta_{m\,m} = \alpha \theta_{mn\,m} + \beta \Box \theta. \quad (2.17) \]

Thus \( \alpha = -\beta \neq 0 \) is also forbidden.

We will apply, in the sequel, two separate gauge conditions: for the symmetric field

\[ \theta_{am} - \frac{1}{2} \theta_{a} = 0, \quad (2.18) \]

and for the antisymmetric field

\[ w_{am} = 0. \quad (2.19) \]

Observe, that (2.18) and (2.19) cannot be realized simultaneously by the same coordinate transformation. Indeed, for this, the coordinate functions have to satisfy

\[ \Box \xi_{a} = 2\theta_{am\,m} - \theta_{a}, \quad \text{and} \quad \Box \xi_{a} - (\xi_{m\,m})_{,a} = w_{am\,m}. \quad (2.20) \]

The integrability conditions for these equations yield

\[ \Box \xi_{[a,b]} = 2\theta_{m[a,b]\,m} = -w_{m[a,b]\,m}. \quad (2.21) \]

For arbitrary independent fields \( \theta_{ab} \) and \( w_{ab} \), these conditions are not satisfied.

Certainly, the conditions (2.18) and (2.19) can be realized, separately, by transformation of the coordinates.
D. Reduction of the field strengths

By (2.3), let us decompose the field strengths (1.8 – 1.10). The 2-form $C_a$ is approximated by

$$C_a = h_{ab,c} dx^c \wedge dx^b = -h_{a[b,c]} \vartheta^b \wedge \vartheta^c = -(\theta_{a[b,c]} + w_{a[b,c]}) \vartheta^b \wedge \vartheta^c. \quad (2.22)$$

Consequently, the first part of the field strength, (1.8), takes the form

$$(1) \mathcal{F}_a = -(\rho_1 + \rho_3)(\theta_{a[b,c]} + w_{a[b,c]}) \vartheta^b \wedge \vartheta^c. \quad (2.23)$$

As for the second part, (1.9), it involves only the antisymmetric field,

$$(2) \mathcal{F}_a = -3\rho_2 w_{a[b,c]} \vartheta^b \wedge \vartheta^c. \quad (2.24)$$

The third part, (1.10), takes the form

$$(3) \mathcal{F}_a = \rho_3 \eta_{ac}(h_{mb,m} - h_d) \vartheta^b \wedge \vartheta^c = \rho_3 \eta_{ac}(\theta_{bm,m} - \theta_{db} - w_{bm,m}) \vartheta^b \wedge \vartheta^c. \quad (2.25)$$

Therefore, the field strength is reduced to the sum of two independent strengths — one defined by the symmetric field $\theta_{ab}$ and the second one defined by the antisymmetric field $w_{ab}$

$$\mathcal{F}_a(\theta_{mn}, w_{mn}) = (\text{sym}) \mathcal{F}_a(\theta_{mn}) + (\text{ant}) \mathcal{F}_a(w_{mn}), \quad (2.26)$$

where

$$(\text{sym}) \mathcal{F}_a = -[(\rho_1 + \rho_3)\theta_{a[b,c]} + \rho_3 \eta_{a[b}\theta_{c]m} - \rho_3 \eta_{a[b}\theta_{c]m}] \vartheta^b \wedge \vartheta^c, \quad (2.27)$$

and

$$(\text{ant}) \mathcal{F}_a = -[(\rho_1 + \rho_3)w_{a[b,c]} + 3\rho_2 w_{a[b,c]} - \rho_3 \eta_{a[b} w_{c]m} \vartheta^b \wedge \vartheta^c. \quad (2.28)$$

Hence, for arbitrary values of the parameters $\rho_i$, the field strengths are independent.

E. Reduction of the field equations

The field equation (1.12) includes the second order derivatives of the perturbations in its LHS and the squares of the first order derivatives in both sides. In the linear approximation (2.3), the quadratic terms can be neglected. Thus, (1.12) is approximated by

$$d \ast \mathcal{F}^a = 0. \quad (2.29)$$

The covector valued 2-form $\mathcal{F}_a$ can be expressed in the unholonomic basis as $\mathcal{F}_a = F_{abc} \vartheta^b \wedge \vartheta^c/2$. Accordingly, we derive

$$d \ast \mathcal{F}_a = \frac{1}{2} F_{abc} dx^m \wedge * (\vartheta^b \wedge \vartheta^c) = \frac{1}{2} F_{abc} * [e_m](\vartheta^b \wedge \vartheta^c) = \frac{1}{2} F_{a[b,c]} \cdot \vartheta^b. \quad (2.29)$$
Consequently, Eq. (2.29) reads

\[ F_{abc}^c = 0. \tag{2.30} \]

Applying the antisymmetrization of the corresponding indices to the expression (2.26) we derive the linearized field equation

\[
\begin{align*}
(\rho_1 + \rho_2 + \rho_3)(\Box \theta_{ab} - \theta_{am,b}^m) + \rho_3 \left(\eta_{ab} \Box \theta - \theta_{mb}^m, a + \theta_{a,b} + \eta_{ab} \theta_{mn}^m, n\right) + \\
(\rho_1 + 2\rho_2 + \rho_3)(\Box w_{ab} - w_{am,b}^m) + (2\rho_2 + \rho_3)w_{bm,a}^m = 0.
\end{align*}
\tag{2.31}
\]

Proposition 1: For the case \(\rho_1 = 0\), the linearized coframe field equation (2.31), in arbitrary coordinates, splits into two independent systems

\[
\begin{align*}
\text{(sym)} E_{(ab)}(\theta_{mn}) = 0, & \quad \text{and} \quad \text{(ant)} E_{[ab]}(w_{mn}) = 0.
\end{align*}
\]

If \(\rho_1 \neq 0\), Eq.(2.31) does not split in any coordinate system.

Proof: The equation (2.31) is tensorial to the first order. Thus, by applying symmetrization and antisymmetrization operations, it is reduced covariantly to a system of two independent tensorial (to the first order) equations. The symmetrization yields a system of 10 independent equations

\[
\Box \left[ (\rho_1 + \rho_3)\theta_{ab} - \rho_3 \eta_{ab} \theta \right] - (\rho_1 + 2\rho_3)\theta_{m(a,b)}^m + \rho_3(\theta_{a,b} + \eta_{ab} \theta_{mn}^m, n) + \rho_1 w_{m(a,b)}^m = 0. \tag{2.32}
\]

The antisymmetrization yields a system of 6 independent equations

\[
(\rho_1 + 2\rho_2 + \rho_3)\Box w_{ab} + (\rho_1 + 4\rho_2 + 2\rho_3)w_{m(a,b)}^m - \rho_1 \theta_{m(a,b)}^m = 0. \tag{2.33}
\]

Evidently, the condition \(\rho_1 = 0\) removes the “mixed terms” and yields the separation of the system. Such splitting holds in arbitrary system of coordinates.

Suppose now \(\rho_1 \neq 0\). Thus, the ”mixed terms” remain in both equations — the \(w\)-term in (2.32) and the \(\theta\)-term in (2.33). Let us try to remove these terms by an appropriative choice of a coordinate system. For this we have to require the equations

\[
\begin{align*}
\theta_{m(a,b)}^m = 0, & \quad \text{and} \quad w_{m(a,b)}^m = 0
\end{align*}
\]

to hold simultaneously. These equations can be satisfied only if

\[
\begin{align*}
\theta_{ma}^m = 0, & \quad \text{and} \quad w_{ma}^m = 0.
\end{align*}
\tag{2.34}
\]

The actual values of the variables \(\theta_{ab}\) and \(w_{ab}\) depend on a choice of a coordinate system. Recall that the approximation conditions (2.3) do not restrict the freedom to choose the local coordinate transformations. Therefore, by (2.4), four additional conditions (equal to the number of coordinates), can still be applied to the perturbations in order to satisfy (2.34). We need, however, to eliminate eight independent expressions \(w_{ma}^m\) and \(\theta_{ma}^m\). This cannot be done by four independent functions of the coordinates. Indeed, under the transformations (2.4),
\[ \theta_{ma}^m \mapsto \theta_{ma}^m - \xi_{(m,a)}^m, \]  
\[ w_{ma}^m \mapsto w_{ma}^m - \xi_{[m,a]}^m. \]  

Hence the coordinate transformations have to satisfy

\[ \xi_{(m,a)}^m = \theta_{ma}^m, \quad \text{and} \quad \xi_{[m,a]}^m = w_{ma}^m \]  

simultaneously. Therefore,

\[ \xi_{m,a}^m = h_{ma}^m. \]  

The consistency condition for (2.38) is

\[ h_{ma,b}^m = h_{mb,a}^m, \]

which it is not satisfied in general. ■

Consequently, for \( \rho_1 = 0 \) and generic values of the parameters \( \rho_2, \rho_3 \), the field equation of the coframe field is reduced to two independent field equations for independent field variables.

(i) The symmetric field \( \theta_{ab} \) of 10 independent variables satisfies the system of 10 independent equations

\[ (\text{sym}) \mathcal{E}_{(ab)}(\theta_{mn}) := \rho_3 \left[ \Box (\theta_{ab} - \eta_{ab} \theta) - \theta_{m(a,b)}^m + \theta_{a,b} + \eta_{ab} \theta_{mn}^{,m,n} \right] = 0. \]  

We rewrite it as

\[ \Box (\theta_{ab} - \eta_{ab} \theta) - \left( \theta_{am}^m - \frac{1}{2} \theta_{a} \right)_{,b} - \left( \theta_{bm}^m - \frac{1}{2} \theta_{b} \right)_{,a} + \eta_{ab} \theta_{mn}^{,m,n} = 0. \]

Substituting here the condition (2.18) and its consequence

\[ \theta_{mn}^{,m,n} = \frac{1}{2} \Box \theta \]  

we obtain

\[ \Box \left( \theta_{ab} - \frac{1}{2} \eta_{ab} \theta \right) = 0. \]  

Eq. (2.42) results in \( \Box \theta = 0 \). Then it is equivalent to

\[ \Box \theta_{ab} = 0. \]

Consequently, in the coordinates associated with (2.18), the symmetric field satisfied the wave equation.

(ii) The antisymmetric system of 6 independent equation for 6 independent variables

\[ (\text{ant}) \mathcal{E}_{[ab]}(w_{mn}) := (2 \rho_2 + \rho_3) \left[ \Box w_{ab} + 2w_{m[a,b]}^{,m} \right] = 0. \]  

In the coordinates associated with (2.19) it is reduced to the wave equation

\[ \Box w_{ab} = 0. \]
F. Reduction of the Lagrangian

In the sequel of this paper, we consider the models with parameter $\rho_1 = 0$. Let us examine now the reduction of the Lagrangian (1.3).

**Proposition 2:** For $\rho_1 = 0$, the Lagrangian of the coframe field is reduced, up to a total derivative term, to the sum of two independent Lagrangians

$$\mathcal{L}(\theta_{ab}, w_{ab}) = (\text{sym}) \mathcal{L}(\theta_{ab}) + (\text{ant}) \mathcal{L}(w_{ab}).$$

**(Proof)**: With $\rho_1 = 0$ the term $(1) \mathcal{L}$ does not appear in the Lagrangian. Calculate in the linear approximation (we use the abbreviation $\vartheta^\alpha_{a_1 \ldots a_q} = \vartheta^\alpha_a \wedge \vartheta^\alpha_{b_1} \wedge \ldots$)

$$(2) \mathcal{L} = (d\vartheta^a \wedge \vartheta_a) \wedge *(d\vartheta^b \wedge \vartheta^b) = h^{a_m, n} h_{b_p, q} \vartheta^a_{n, m} \wedge * \vartheta^b_{p, q}.$$  

(2.47)

Applying the formula

$$\vartheta^\alpha_{a b c} \wedge * \vartheta'^{\alpha'}_{a' b' c'} = 6 \delta^\alpha_{a} \delta^\beta_{b} \delta^\gamma_{c} \ast 1$$

we derive

$$(2) \mathcal{L} = 2 w^{a b, c}(w_{a b, c} + w_{c a, b} + w_{b c, a}) \ast 1.$$  

(2.49)

So $(2) \mathcal{L}$ depends only on the antisymmetric field. Consider now the linear approximation to the term $(3) \mathcal{L}$

$$(3) \mathcal{L} = (d\vartheta^a \wedge \vartheta_b) \wedge *(d\vartheta^b \wedge \vartheta^a) = \tilde{h}^{m, n} h_{a_p, q} \vartheta^a_{n, m} \wedge * \vartheta^b_{p, q}.$$  

(2.50)

Use (2.48) to get

$$(3) \mathcal{L} = \tilde{h}^{a b, c}(\tilde{h}^{a b, c} - h^{a c, b}) - \tilde{h}^{a b} h^{a b} \vartheta^a_{a} \vartheta^b_{b} \wedge \vartheta^a_{b}.$$  

(2.51)

Insert here the splitting (2.12). It follows that the Lagrangian (2.51) is reduced to the sum

$$(3) \mathcal{L} = (3) \mathcal{L}(\theta) + (3) \mathcal{L}(w) + (3) \mathcal{L}(\theta, w),$$  

(2.52)

where

$$(3) \mathcal{L}(\theta) = \left[ \theta_{a b, c}(\theta_{a b, c} - \theta_{a c, b}) - \theta_{a b} \theta_{a c} \vartheta^a_{c} + \theta^a (2 \theta_{a b} - \theta_{a b}) \right] \ast 1,$$

(2.53)

$$(3) \mathcal{L}(w) = \left[ w_{a b, c}(w_{a b, c} - w_{a c, b}) - w_{a b} w_{a b} \vartheta^a_{b} \vartheta^b_{b} \right] \ast 1,$$

(2.54)

$$(3) \mathcal{L}(\theta, w) = 2 \left[ - \theta_{a b, c} w_{a c, b} + \vartheta^a w_{b a} \vartheta^b_{b} - \theta_{a b} \vartheta^a_{c} \vartheta^b_{b} \right] \ast 1.$$  

(2.55)

Extracting the total derivatives in the mixed term (2.55) we obtain

$$(3) \mathcal{L}(\theta, w) = \left( \theta_{a b} (w_{a c, b} - w_{b c, a}) \vartheta^a_{c} - \theta w_{b a} \vartheta^a_{b} \right) \ast 1 + \text{exact terms}.$$  

(2.56)
The terms in the brackets vanish identically as a product of symmetric and antisymmetric tensors. Thus the mixed term \( (3) \mathcal{L}(\theta, w) \) is a total derivative. Consequently, desired reduction of the Lagrangian is obtained.

The Lagrangian of the symmetric field \( (\text{sym}) \mathcal{L} = (3) \mathcal{L}(\theta) \) may be rewritten in a more compact form. Observing the identity

\[
\theta_{ab}^a \theta_{cb}^b = \theta_{ab}^c \theta_{cb}^a + \text{exact terms},
\]

and extracting the total derivatives, we obtain

\[
(\text{sym}) \mathcal{L} = \frac{1}{2} \rho_3 \left[ \theta_{ab,c} (\theta_{ac}^b - 2 \theta_{ac}^c) + \theta^{a} (2 \theta_{ab}^b - \theta_{ab}^a) \right] \ast 1. \tag{2.58}
\]

This form of the Lagrangian is acceptable in arbitrary coordinates. In the coordinates associated with the condition (2.18), the last brackets in (2.58) vanish. In the first brackets, we extract the total derivatives and use (2.18) to derive (symbol \( \approx \) used here for equality up to total derivatives)

\[
\theta_{ab,c} \theta_{ac,b} = (\theta_{ab} \theta_{ac,b})_{,c} - \theta_{ab} \theta_{ac,b} \approx - \frac{1}{2} \theta_{ab} \theta_{ac,b,c} \approx 0. \tag{2.59}
\]

Consequently the symmetric field Lagrangian (2.53) is reduced to

\[
(\text{sym}) \mathcal{L} = \frac{1}{2} \kappa \left( \theta_{ab,c} \theta_{ab}^c - \frac{1}{2} \theta_{a,c} \theta_{a}^c \right) \ast 1. \tag{2.59}
\]

Analogously, for the Lagrangian of the antisymmetric field \( (\text{ant}) \mathcal{L} = (2) \mathcal{L} + (3) \mathcal{L}(w) \), we use the identity

\[
w_{ab}^a w_{cb}^b = w_{ab,c} w_{ac,b} + \text{exact terms},
\]

and rewrite it, in an arbitrary system of coordinates, as

\[
(\text{ant}) \mathcal{L} = \frac{1}{2} (2 \rho_2 + \rho_3) \left[ w_{ab,c} (w_{ac}^b - 2 w_{ac}^c) \right] \ast 1, \tag{2.61}
\]

or, equivalently, as

\[
\mathcal{L}(w) = \frac{1}{2} (2 \rho_2 + \rho_3) \left( w_{ab,c} (w_{ac}^b - w_{ac}^c) - w_{ab}^a w_{cb}^c \right) \ast 1. \tag{2.62}
\]

The gauge condition (2.19) removes the last term while the second term is rewritten as

\[
w_{ab,c} w_{ac,b} \approx - w_{ab} w_{ac,b,c} \approx 0.
\]

Thus, the Lagrangian of the antisymmetric field is

\[
\tilde{\mathcal{L}}(w) = \frac{1}{2} (2 \rho_2 + \rho_3) w_{ab,c} w_{ab}^c \ast 1. \tag{2.62}
\]

\section{G. Reduction of the energy-momentum current}

The Lagrangian of the coframe field is decomposed, in the first order approximation, to a sum of two independent Lagrangians for two independent fields. The Noether current expression, being derivable from the Lagrangian, has to have the same splitting.
Proposition 3: The coframe energy-momentum current is reduced, on shell, in the first order approximation, as

\[ \mathcal{T}_a(\theta_{mn}, w_{mn}) = (\text{sym}) \mathcal{T}_a(\theta_{mn}) + (\text{ant}) \mathcal{T}_a(w_{mn}), \]  

(2.63)

up to a total derivative.

Proof: The coframe energy-momentum current is of the form

\[ \mathcal{T}_a = (e_a \lrcorner C_m) \lrcorner \star F^m - e_a \lrcorner \mathcal{L}. \]  

(2.64)

Due to Proposition 2, the second term, in the first order approximation, does not contain the mixed terms \( \theta' \cdot w' \). Hence, it already has the reduced form. To treat the first term, we write the strengths in the component

\[ C_m = C_m_{[bc]} \vartheta^b \wedge \vartheta^c, \quad F^m = F^m_{[pq]} \vartheta^p \wedge \vartheta^q. \]  

(2.65)

Thus, the first term of (2.64) is approximated by

\[ (e_a \lrcorner C_m) \lrcorner \star F^m = C_m_{[bc]} F^m_{[pq]} (e_a \lrcorner) \vartheta^{bc} \lrcorner \star \vartheta^{pq} = 4 C_m_{[an]} F^m_{[bn]} \star \vartheta_b = 4 h_{m[a,n]} F^m_{[bn]} \star \vartheta_b. \]  

(2.66)

The 3-form \( \star \vartheta_b \), in the lowest order approximation, is an exact form. Thus, it is enough to show that the scalar factor, in the RHS of (2.66), has the desired splitting. This expression is a sum of two terms. The first one is proportional to

\[ h_{m[a,n]} F^m_{[bn]} = - h_{ma} F^m_{[bn]} \lrcorner \vartheta^n + \text{total derivatives}, \]

i.e., it is, on shell, an exact form. Now we have to show that the second term, which is proportional to \( h_{m[a,n]} F^m_{[bn]} \), does not involve the mixed products of a type \( \theta \cdot w \). The mixed product expression in the latter term is proportional to

\[ \theta_{mn,a} (w^{mb,n} + 2 \eta^{m[n} w^{b]k} \cdot k) + w_{mn,a} (\eta^{mb,n} + \eta^{mb} \theta^n \cdot k - \eta^{mh} \theta^a). \]  

(2.67)

By recollection of the terms, we rewrite this expression as

\[ (\theta_{mn,a} w^{mb,n} + \theta_{mn} w^{bm,.a}) + (\theta_{a} w^{bn,.m} - \theta_{n} w^{bm,.a}) + (\eta^{mb,n} w_{mn,a} - \theta^{bm,.a} w_{mn,.n}). \]  

(2.68)

The three brackets above are total derivatives, namely,

\[ [(\theta_{mn,.a} w^{mb,.n} + \theta_{mn} w^{bm,.a})] + [(\theta_{a} w^{bn,.m} - \theta_{n} w^{bm,.a})] + [(\eta^{mb,n} w_{mn,.a} - \theta^{bm,.a} w_{mn,.n})]. \]  

(2.69)

Thus, (2.66) and, consequently, (2.64) do not involve the mixed terms. The desired splitting is proved. \( \blacksquare \)

The energy-momentum tensor \( T_a^b \) can be derived from the Noether current \( \mathcal{T}_a \) by applying the relations

\[ \mathcal{T}_a = T_a^b \lrcorner \vartheta_b, \quad T_{ab} = e_b \lrcorner \mathcal{T}_a. \]  

(2.70)

Proposition 4: For the field \( \theta_{ab} \) in the coordinate system associated with the gauge condition

\[ \theta_{am} = \frac{1}{2} \theta_{a} = 0, \]  

(2.71)
The energy-momentum tensor is
\[ T_{ab} = \frac{1}{2} \kappa \left[ \left( \theta_{mn,a} \theta^{mn,b} - \frac{1}{4} \eta_{ab} \theta_{lm,n} \theta^{lm,n} \right) - \frac{1}{2} \left( \theta_{a,b} - \frac{1}{4} \eta_{ab} \theta_{.m,m} \right) \right]. \tag{2.72} \]

This tensor is symmetric and traceless.

**Proof:** We start with the energy-momentum current for the coframe field
\[ \mathcal{T}_a = (e_a | \mathcal{C}_m) \wedge *F^m - e_a | \mathcal{L}. \]

Due to Proposition 3, in the first order approximation, this current is decomposed to two independent currents. Thus we may assume \( w_{ab} = 0 \) in order to derive the expression for \( \mathcal{T}_a(\theta) \).

In the coordinates associated with the gauge condition (2.71), by (2.59)
\[ e_a | \mathcal{L} = \frac{1}{2} \rho_3 \left( \theta_{mn,p} \theta^{mn,p} - \frac{1}{2} \theta_{m,m} \right) \ast \vartheta_a. \]

The first term of \( \mathcal{T}_a \) is derived from (2.64)
\[ (e_a | \mathcal{C}_m) \wedge *F^m = 4 \theta_{m[a,n]}F^{m[bn]} \ast \vartheta_b = 2 \left( \theta_{ma,n}F^{m[bn]} \ast \vartheta_b - \theta_{mn,a}F^{m[bn]} \ast \vartheta_b \right). \]

Observe that, on shell, up to a total derivative
\[ \theta_{ma,n}F^{m[bn]} \approx -\theta_{ma}F^{m[bn],n} = 0. \]

Thus,
\[ (e_a | \mathcal{C}_m) \wedge *F^m = -2 \theta_{mn,a}F^{m[bn]} \ast \vartheta_b. \]

Applying the gauge condition to (2.26) we get
\[ \mathcal{F}_a = -\rho_3 \left[ \theta_{a[b,c]} + \eta_{a[b}(\theta_{c]m,m} - \theta_{c}) \right] \vartheta^{bc} = -\rho_3 \left( \theta_{[a,b,c]} - \frac{1}{2} \eta_{[a[b} \theta_{c]} \right) \vartheta^{bc}. \]

Consequently,
\[ (e_a | \mathcal{C}_m) \wedge *F^m = 2 \rho_3 \theta_{mn,a} \left( \theta^{m[bn]} - \frac{1}{2} \eta^{m[n} \theta^{b]} \right) \ast \vartheta_b \]

Extracting the total derivatives
\[ \theta_{mn,a} \theta^{mb,n} \approx \theta_{mn} \theta^{mb,a} \approx \frac{1}{2} \theta_{m} \theta^{mb,a} \approx \frac{1}{4} \theta_{a} \theta^{b}; \]
\[ \theta_{mn,a} \eta^{mb} \theta^{n,a} \approx \theta_{mn} \theta^{a} \theta^{b}. \]

it follows that
\[ (e_a | \mathcal{C}_m) \wedge *F^m = \rho_3 \left( -2 \theta_{mn,a} \theta^{mn,b} + \theta_{a} \theta^{b} \right) \ast \vartheta_b \]

Collecting the terms into \( \mathcal{T}_a \) and extracting the energy-momentum tensor \( T^a \) from the current \( \mathcal{T}_a \) by \( T_{ab} = e_b | \mathcal{T}_a \)
we get the desired expression. It is clear that energy-momentum tensor is symmetric and traceless. ■
In GR, the behavior of small perturbations of the metric tensor is managed by the wave equation. Thus, for a wave propagating in the positive direction of the $x$-axis, only two independent components of the matrix $\theta_{ab}$ remain.

$$\theta_{23} = \mu(\tau), \quad \theta_{22} = -\theta_{33} = \nu(\tau), \quad \text{where} \quad \tau = t - x. \quad (2.73)$$

The calculation of the energy-momentum tensor for the symmetric field by use of the tensor (2.72) yields

$$T_{ab} = k(\mu_{,a}\mu_{,b} + \nu_{,a}\nu_{,b}). \quad (2.74)$$

The energy flux reads

$$T_{01} = -\rho_3 \left( \dot{\theta}_{23}^2 + \frac{1}{4}(\dot{\theta}_{22} - \dot{\theta}_{33})^2 \right) \quad (2.75)$$

Observe that the expressions (2.74,2.75) are the same as the expressions obtained in GR from the energy-momentum pseudotensors.

Let us turn now to the antisymmetric field.

**Proposition 5**: In the coordinate system associated with the gauge condition

$$w_{am,m} = 0, \quad (2.76)$$

the energy-momentum tensor of the antisymmetric field is

$$T_{ab} = -(2\rho_2 + \rho_3) \left( w_{mn,a}w^{mn,b} - \frac{1}{4}\eta_{ab}w_{mn,p}w^{mn,p} \right). \quad (2.77)$$

This tensor is traceless and symmetric.

**Proof**: The current of the symmetric and of the antisymmetric fields are decoupled. Thus we may assume $\theta_{ab} = 0$. In the coordinates associated with the gauge condition (2.76),

$$e_{a|C} = \frac{1}{2} (2\rho_2 + \rho_3) w_{ab,c} w^{ab,c} \ast \vartheta_b. \quad (2.76)$$

As for the first term of $T_a(u)$ we derive from (2.66)

$$(e_{a|C}^m \wedge \ast F^m = 4w_{m[a,n]}F^{m[bn]}\ast \vartheta_b = 2(w_{ma,n}F^{m[bn]} - w_{mn,a}F^{m[bn]})\ast \vartheta_b.$$\]

The first term vanishes, on shell, up to a total derivative,

$$w_{ma,n}F^{m[bn]} \approx -w_{ma}F^{m[bn]} \ast \vartheta = 0. \quad (2.76)$$

Thus,

$$(e_{a|C}^m \wedge \ast F^m = -2w_{mn,a}F^{m[bn]}\ast \vartheta_b.$$\]

Inserting the gauge condition (2.76) into (2.26) we derive

$$F_a = -(\rho_3 w_{a[c]} + 3\rho_2 w_{[ab,c]} )\vartheta_{bc}. \quad (2.77)$$
Hence,

\[(e_a | C_m) \wedge \star F^m = 2(\rho_3 w_{ma,n} w^{n|b,n} + 3\rho_2 w_{ma,n} w^{[mb,n]}) \star \vartheta^b.\]

Extract the total derivatives and use the gauge condition to get

\[w_{mn,a} w^{mb,n} \approx w_{mn,n} w^{mb,a} \approx 0\]
\[w_{mn,a} w^{bn,m} \approx w_{mn,m} w^{bn,a} \approx 0.\]

Consequently,

\[(e_a | C_m) \wedge \star F^m = -(2\rho_2 + \rho_3) w_{mn,a} w^{mn,b}\]

The desired expression (2.77) is obtained now by collecting the terms. ■

III. THE ROLE OF THE PARAMETERS \(\rho_i\)

The case \(\rho_1 = 0\) is extracted in coframe models by existence of a unique spherical symmetric static solution. Since the exact solution yields the Schwarzschild metric this condition generates a viable subclass of gravity coframe models.

We have involved an independent criteria. Namely, we have shown that only in the case \(\rho_1 = 0\) the weak perturbations of the coframe reduce to two independent fields with their own Lagrangian dynamics. Consequently the models have a free field limit. This effect is correlated to the resent obtained result [34] concerning the Hamiltonian dynamics behavior.

It is interesting to note that in 2D coframe gravity only one term in the Lagrangian preceded by \(\rho_1\) appears. Thus the corresponded reduction of fields is impossible.

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