NESTED FIBRE BUNDLES IN BOTT-SAMELSON VARIETIES

VLADIMIR SHCHIGOLEV

Abstract. We give a topological explanation of the main results of V. Shchigolev, Categories of Bott-Samelson Varieties, Algebras and Representation Theory, 23 (2), 349–391, 2020. To this end, we consider some subspaces of Bott-Samelson varieties invariant under the action of the maximal compact torus $K$ and study their topological and homological properties. Moreover, we describe multiplicative generators of the equivariant cohomologies of Bott-Samelson varieties.

1. Introduction

Let $G$ be a semisimple complex algebraic group, $B$ be its Borel subgroup and $T$ be a maximal torus contained in $B$. For any simple reflection $t$, we have the minimal parabolic subgroup $P_t = B \cup BtB$. In this paper, we investigate how Bott-Samelson varieties $BS(s) = P_{s_1} \times \cdots \times P_{s_n}/B^n$ are related on the levels of topology and cohomology for different sequences of simple reflections $s = (s_1, \ldots, s_n)$. In the formula above, the power $B^n$ acts on the product $P_{s_1} \times \cdots \times P_{s_n}$ on the right by the rule

$$(g_1, g_2, \ldots, g_n)(b_1, b_2, \ldots, b_n) = (g_1 b_1, b_1^{-1} g_2 b_2, \ldots, b_{n-1}^{-1} g_n b_n).$$

Moreover, the torus $T$ acts on $BS(s)$ via the first factor

$$t(g_1, g_2, \ldots, g_n)B^n = (tg_1, g_2, \ldots, g_n)B^n.$$ We have the $T$-equivariant map $\pi(s) : BS(s) \to G/B$ that takes an orbit $(g_1, g_2, \ldots, g_n)B^n$ to $g_1 g_2 \cdots g_n B$.

Let $s = (s_1, \ldots, s_n)$ be a subsequence of a sequence $\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_{\tilde{n}})$, that is, there exists a strictly increasing map $p : \{1, \ldots, n\} \to \{1, \ldots, \tilde{n}\}$ such that $\tilde{s}_{p(i)} = s_i$ for any $i = 1, \ldots, n$. Then we have a $T$-equivariant embedding $BS(s) \to BS(\tilde{s})$ defined by the rule

$$(g_1, \ldots, g_n)B^n \mapsto (\tilde{g}_1, \ldots, \tilde{g}_{\tilde{n}})B^{\tilde{n}},$$

where $\tilde{g}_{p(i)} = g_i$ for any $i = 1, \ldots, n$ and $\tilde{g}_j = 1$ for $j$ not in the image of $p$.

Hence we get the restriction map $H^*_T(BS(\tilde{s}), k) \to H^*_T(BS(s), k)$ for any ring of coefficients $k$. It turns out that there exist many other such maps not necessarily when $s$ is a subsequence of $\tilde{s}$. These additional morphisms constitute the so called folding categories $\bar{\text{Seq}}$, see $[S2]$ Section 3.3]. Until now this was a purely cohomological phenomenon proved with the help of M. Härtherich’s criterion for the image of the localization $[H]$.

The main aim of this paper is to provide a topological explanation of the existence of the folding categories and to develop the corresponding topology (fibre bundles, affine pavings, etc.) that may be interesting in its own right. Moreover, in a subsequent paper, the author plans to obtain some tensor product theorems for cohomologies of Bott-Samelson varieties, applying the topological spaces introduced here in full generality.

As was noted in $[S2]$, the existence of the folding categories can not be explained on the level of $T$-equivariant topology. The main idea to overcome this obstacle that we pursue in this paper is to use the equivariant cohomology with respect to the maximal compact
torus $K$. Using this torus and certain compact subgroups $C_s$ (defined in Section 2.1), we introduced the \textit{compactly defined Bott-Samelson variety}

$$BS_c(s) = C_{s_1} \times \cdots \times C_{s_n}/K^n$$

for any sequence $s = (s_1, \ldots, s_n)$ of reflections not necessarily simple ones. Here $K^n$ acts on $C_{s_1} \times \cdots \times C_{s_n}$ on the right by the same formula as $B^n$ above. The torus $K$ acts continuously on $BS_c(s)$ via the first factor and we can consider the $K$-equivariant cohomology of this space. Note that topologically $BS_c(s)$ is nothing more than a Bott tower $G/K$. However, we consider this space together with the natural projection $\pi_c(s) : BS_c(s) \to C/K$, where $C$ is the maximal compact subgroup of $G$, given by $(c_1, \ldots, c_n)K^n \mapsto c_1 \cdots c_nK$. This makes the situation more complicated.

From the Iwasawa decomposition it follows that $BS_c(s)$ and $BS(s)$ are isomorphic as $K$-spaces if all reflections of $s$ are simple $[GK]$. Therefore, we can identify $H^*_K(\mathcal{B}_c(s), k)$ and $H^*_c(\mathcal{B}_c(s), k)$. The space $BS_c(s)$ has enough $K$-subspaces $BS_c(s, v)$, see Section 3.1 for the definitions, to recover the main results of $[S2]$ by purely topological arguments without resorting to localizations and M.Härterich’s criterion. The corresponding constructions are described in Section 4.4.

The spaces $BS_c(s, v)$ are the main technical tool of this paper, where we need only a very special type of them. The reason why we introduce them in such a generality is that they will be used latter to prove the tensor product decompositions for cohomologies $[S3]$. However, the definition of the spaces $BS_c(s, v)$ looks natural, see [1], and we would like to study their structure already here. More exactly, we prove that these spaces can be represented as twisted products (that is, iterations of fibre bundles, see Theorem 7) of the elementary factors, which are the spaces $BS_c(t)$ and $\pi_c(t)^{-1}(wB)$ for some sequences of reflections $t$ and elements of the Weyl group $w$. This decomposition does not allow us to reconstruct $BS_c(s, v)$ from the elementary factors in a unique way (because of the twist) but in some cases allows us to compute its $K$-equivariant cohomology.

One of the cases in which this computation is possible is the case when the space $BS_c(s, v)$ has a paving by spaces homeomorphic to $C^d$ for some $d$. We say then that this space has an \textit{affine} paving. In this case, the ordinary odd cohomologies vanish and we can apply spectral sequences for the calculation of $H^*_K(\mathcal{B}_c(s, v), k)$. It is still an open question whether every space $BS_c(s, v)$ has an affine paving. But we can guarantee this property in one special case when the pair $(s, v)$ is of gallery type (Definition 11). This notion can be considered as a generalization of the notion of a morphism in the categories of Bott-Samelson varieties $[S2]$. It is still unclear why pairs of gallery type emerge but there are probably plenty of them. For example, if the root system of $G$ has type $A_1$ or $A_2$, then any pair is of gallery type.

The paper is organized as follows. In Section 2, we fix the notation for simisimple complex algebraic groups, their compact subgroups and Bott-Samelson varieties. To avoid renumeration, we consider sequences as maps defined on totally ordered sets, see Section 2.2. In Section 3, we define the space $BS_c(s, v)$ for a sequence of reflections $s$ and a function $v$ from a nested structure $R$ (Section 3.1) to the Weyl group of $G$. The function $v$ may be considered as a set of not overlapping restrictions fixing the products over segments. For example,

- if $R = \varnothing$, then $BS_c(s, v) = BS_c(s)$;
- if $R = \{(\min I, \max I)\}$, where $s$ is defined on $I$, then $BS_c(s, v) = \pi_c(s)^{-1}(wB)$, where $w$ is the value of $v$;
- if $R$ consists only of diagonal pairs $(i, i)$, then $BS_c(s, v)$ is either empty or is isomorphic to $BS_c(t)$ for some $t$. This result is used to give a topological prove of the main results of $[S2]$. 
Of course, one can combine the above cases. We believe that nested structures and the spaces $BS_c(s, v)$ arising from them will be quite useful in establishing relations between Bott-Samelson varieties. An example of such a relation is given in Section 3.6.

Equivariant cohomology is considered in Section 4.4. We have different choices for principal bundles. To get an action of the Weyl group on equivariant cohomologies, we consider Bott-Samelson varieties. An example of such a relation is given in Section 3.6.

To prove some general facts about cohomologies, for example, Lemmas 15 and 23. We also use some standard notation like $k$ for the constant sheaf on a topological space $X$, GL$(V)$ for the group of linear isomorphisms $V \to V$, $M_{m,n}(k)$ the set of $m \times n$-matrices with entries in $k$ and $U(n)$ for the group of unitary $n \times n$-matrices. We also apply spectral sequences associated with fibre bundles. All of them are first quadrant sequences — the fact that we tacitly keep in mind.

2. Bott-Samelson varieties

2.1. Compact subgroups of complex algebraic groups. Let $G$ be a semisimple complex group with root system $\Phi$. We assume that $\Phi$ is defined with respect to the Euclidean space $E$ with the scalar product $(\cdot, \cdot)$. This scalar product determines a metric on $E$ and therefore a topology. We will denote by $\overline{A}$ the closure of a subset $A \subset E$.

We consider $G$ as a Chevalley group generated by the root elements $x_\alpha(t)$, where $\alpha \in \Phi$ and $t \in \mathbb{C}$ (see [St]). We also fix the following elements of $G$:

$$w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t), \quad \omega_\alpha = w_\alpha(1), \quad h_\alpha(t) = w_\alpha(t)\omega_\alpha^{-1}.$$  

For any $\alpha \in \Phi$, we denote by $s_\alpha$ the reflection of $E$ through the hyperplane $L_\alpha$ of the vectors perpendicular to $\alpha$. We call $L_\alpha$ a wall perpendicular to $\alpha$. These reflections generate the Weyl group $W$. We choose a decomposition $\Phi = \Phi^+ \sqcup \Phi^-$ into positive and negative roots and write $\alpha > 0$ (resp. $\alpha < 0$) if $\alpha \in \Phi^+$ (resp. $\alpha \in \Phi^-$). Let $\Pi \subset \Phi^+$ be the set of simple roots corresponding to this decomposition. We denote

$$T(W) = \{s_\alpha \mid \alpha \in \Phi\}, \quad S(W) = \{s_\alpha \mid \alpha \in \Pi\}.$$  

and call these sets the set of reflections and the set of simple reflections respectively. We use the standard notation $(\alpha, \beta) = 2(\alpha, \beta)/(|\beta|^2)$.

There is the analytic automorphism $\sigma$ of $G$ such that $\sigma(x_\alpha(t)) = x_{-\alpha}(-t)$ [St] Theorem 16. Here and in what follows $\overline{t}$ denotes the complex conjugate of $t$. We denote by $C = G_{\sigma}$ the set of fixed points of this automorphism. We also consider the compact torus $K = T_{\sigma} = B_{\sigma}$, where $T$ is the subgroup of $G$ generated by all $h_\alpha(t)$ and $B$ is the subgroup generated by $T$ and all root elements $x_\alpha(t)$ with $\alpha > 0$.

For $\alpha \in \Phi$, we denote by $G_\alpha$ the subgroup of $G$ generated by all root elements $x_\alpha(t)$ and $x_{-\alpha}(t)$ with $t \in \mathbb{C}$. There exists the homomorphism $\varphi_\alpha : SL_2(\mathbb{C}) \to G$ such that

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mapsto x_\alpha(t), \quad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mapsto x_{-\alpha}(t).$$  

Therefore, we will denote by $\overline{K}$ the closure of a subset $K \subset G$. We will denote by $\overline{T}$ the closure of a subset $T \subset G$. We use the standard notation $\langle \phi \rangle$ for the group of unitary $n \times n$-matrices.
The kernel of this homomorphism is either \( \{1\} \) or \( \{\pm 1\} \). Let \( \sigma' \) be the automorphism of \( \text{SL}_2(\mathbb{C}) \) given by \( \sigma'(M) = (M^T)^{-1} \). Here and in what follows, we denote by \( M^T \) the transpose of \( M \). The we get \( \varphi_\alpha \sigma' = \sigma \varphi_\alpha \). As the equation \( \sigma'(M) = -M \) does not have a solution, we get

\[
C \cap G_\alpha = \varphi_\alpha(\text{SU}_2).
\]

This fact is proved, for example, in [St, Lemma 45] and is true for any root \( \alpha \) not necessarily simple. It follows from this formula that \( \omega_\alpha \in C \cap G_\alpha \).

Let \( \mathcal{N} \) be the subgroup of \( G \) generated by the elements \( \omega_\alpha \) and the torus \( K \). Clearly \( \mathcal{N} \subset C \). The arguments of [St, Lemma 22] prove that there exists an isomorphism \( \varphi : W \to \mathcal{N}/K \) given by \( s_\alpha \mapsto \omega_\alpha K \). We choose once and for all a representative \( \bar{w} \in \varphi(w) \) for any \( w \in W \). Abusing notation, we denote \( wK = \bar{w}K = \varphi(w) \).

Now consider the product \( C_\alpha = \varphi_\alpha(\text{SU}_2)_K \). It is the image of the compact space \( \varphi_\alpha(\text{SU}_2) \times K \) under the multiplication \( G \times G \to G \). Therefore, it is compact and closed in \( G \). It is easy to check that \( C_\alpha \) is a group. Note that \( G_\alpha = G_{-\alpha} \). Therefore \( C_\alpha = C_{-\alpha} \) and we can denote \( C_{s_\alpha} = C_\alpha \). For any \( w \in W \), we have \( \bar{w}C_{s_\alpha} \bar{w}^{-1} = C_{w\alpha}w^{-1} = C_{\alpha w} \).

### 2.2. Sequences and products.

Let \( I \) be a finite totally ordered set. We denote by \( \leq \) (\( < \), \( \geq \), etc.) the order on it. We also add two additional elements \(-\infty \) and \( +\infty \) with the natural properties: \(-\infty < i \) and \( i < +\infty \) for any \( i \in I \) and \(-\infty < +\infty \). For any \( i \in I \), we denote by \( i+1 \) (resp. \( i-1 \)) the next (resp. the previous) element of \( I \cup \{-\infty, +\infty\} \). We write \( \min I \) and \( \max I \) for the minimal and the maximal elements of \( I \) respectively and denote \( I' = I \setminus \{\max I\} \) if \( I \neq \emptyset \). Note that \( \min \emptyset = -\infty \) and \( \max \emptyset = +\infty \). For \( i,j \in I \), we set \( [i,j] = \{k \in I \mid i \leq k \leq j\} \).

Any map \( s \) from \( I \) to an arbitrary set is called a sequence on \( I \). We denote by \( |s| \) the cardinality of \( I \) and call this number the length of \( s \). For any \( i \in I \), we denote by \( s_i \) the value of \( s \) on \( i \). Finite sequences in the usual sense are sequences on the initial intervals of the natural numbers, that is, on the sets \( \{1,2,\ldots,n\} \). For any sequence \( s \) on a nonempty set \( I \), we denote by \( s' = s|_I \) its truncation.

We often use the following notation for Cartesian products:

\[
\prod_{i=1}^n X_i = X_1 \times X_2 \times \cdots \times X_n
\]

and denote by \( p_{i_1i_2\cdots i_m} \) the projection of this product to \( X_{i_1} \times X_{i_2} \times \cdots \times X_{i_m} \). If we have maps \( f_j : Y \to X_i \) for all \( i = 1,\ldots,n \), then we denote by \( f_1 \boxtimes \cdots \boxtimes f_n \) the map that takes \( y \in Y \) to the \( n \)-tuple \((f_1(y),\ldots,f_n(y))\). To avoid too many superscripts, we denote

\[
X(I) = \prod_{i \in I} X.
\]

We consider all such products with respect to the product topology if all factors are topological spaces. Suppose that \( X \) is a group. Then \( X(I) \) is a topological group with respect to the componentwise multiplication. In that case, for any \( i \in I \) and \( x \in X \), we consider the indicator sequence \( \delta_i(x) \in X(I) \) defined by

\[
\delta_i(x)_j = \begin{cases} x & \text{if } j = i; \\ 1 & \text{otherwise} \end{cases}
\]

For any \( i \in I \cup \{-\infty\} \) and a sequence \( \gamma \) on \( I \), we use the notation \( \gamma^i = \gamma_{\min I} \gamma_{\min I+1} \cdots \gamma_i \).

Obviously \( \gamma^{-\infty} = 1 \). We set \( \gamma^{\max I} = \gamma^{\max I} \).

We will also use the following notation. Let \( G \) be a group. If \( G \) acts on the set \( X \), then we denote by \( X/G \) the set of \( G \)-orbits of elements of \( X \). Suppose that \( \varphi : X \to Y \) is a
G-equivariant map. Then \( \varphi \) maps any \( G \)-orbit to a \( G \)-orbit. We denote the corresponding map from \( X/G \) to \( Y/G \) by \( \varphi/G \).

2.3. **Galleries.** As \( L_{\alpha} = L_{\alpha} \) for any root \( \alpha \in \Phi \), we can denote \( L_{s_{\alpha}} = L_{\alpha} \). For any \( w \in W \), we get \( wL_{s_{\alpha}} = wL_{\alpha} = L_{w_{\alpha}} = L_{s_{w_{\alpha}^{-1}}} \).

A chamber is a connected component of the space \( E \setminus \bigcup_{\alpha \in \Phi} L_{\alpha} \). We denote by \( \text{Ch} \) the set of chambers and by \( L \) the set of walls \( L_{s_{\alpha}} \). We say that a chamber \( \Delta \) is *attached* to a wall \( L \) if the intersection \( \Delta \cap L \) has dimension \( \text{dim}E - 1 \). The *fundamental* chamber is defined by

\[
\Delta_{+} = \{v \in E \mid (v, \alpha) > 0 \text{ for any } \alpha \in \Pi \}.
\]

A *labelled gallery* on a finite totally ordered set \( I \) is a pair of maps \( (\Delta, L) \), where \( \Delta : I \cup \{-\infty\} \rightarrow \text{Ch} \) and \( L : I \rightarrow L \) are such that for any \( i \in I \), both chambers \( \Delta_{i-1} \) and \( \Delta_{i} \) are attached to \( L_{i} \). The Weyl group \( W \) acts on the set of labelled galleries by the rule \( w(\Delta, L) = (w\Delta, wL) \), where \( (w\Delta)_{i} = w\Delta_{i} \) and \( (wL)_{i} = wL_{i} \).

For a sequence \( s : I \rightarrow T(W) \), we define the set of *generalized combinatorial galleries*

\[
\Gamma(s) = \{ \gamma : I \rightarrow W \mid \gamma_{i} = s_{i} \text{ or } \gamma_{i} = 1 \text{ for any } i \in I \}.
\]

If \( s \) maps \( I \) to \( S(W) \), then the elements of \( \Gamma(s) \) are called *combinatorial galleries*. To any combinatorial gallery \( \gamma \in \Gamma(s) \) there corresponds the following labelled gallery:

\[
i \mapsto \gamma^{i}_{\Delta^{+}}, \quad i \mapsto L_{\gamma^{i}_{s_{i}(\gamma^{i})^{-1}}}(1).
\]

Moreover, any labelled gallery \( (\Delta, L) \) on \( I \) such that \( \Delta_{-\infty} = \Delta_{+} \) can be obtained this way (for the corresponding \( s \) and \( \gamma \)). If we reflect all chambers \( \Delta_{j} \) and hyperplanes \( L_{j} \) for \( j > i \) through the hyperplane \( L_{i} \), where \( i \in I \), then we will get a labelled gallery again. This new gallery begins at the same chamber as the old one. We use the following notation for this operation on the level of combinatorial galleries: for any \( \gamma \in \Gamma(s) \) and index \( i \in I \), we denote by \( f_{i}^{s} \gamma \) the combinatorial gallery of \( \Gamma(s) \) such that \( (f_{i}^{s} \gamma)_{i} = \gamma_{i}s_{i} \) and \( (f_{i}^{s} \gamma)_{j} = \gamma_{j} \) for \( j \neq i \).

Let \( \gamma \in \Gamma(s) \), where \( s : I \rightarrow T(W) \). We consider the sequence \( s^{(\gamma)} : I \rightarrow T(W) \) defined by

\[
s_{i}^{(\gamma)} = \gamma_{i}s_{i}(\gamma_{i})^{-1}.
\]

Moreover for any \( w \in W \), we define the sequence \( s^{w} : I \rightarrow T(W) \) by \( (s^{w})_{i} = w_{s_{i}}w^{-1} \) and the generalized combinatorial gallery \( \gamma^{w} \in \Gamma(s^{w}) \) by \( \gamma_{i}^{w} = w_{\gamma_{i}}w^{-1} \).

**Definition 1.** A sequence \( s : I \rightarrow T(W) \) is of *gallery type* if there exists a labelled gallery \( (\Delta, L) \) on \( I \) such that \( L_{i} = L_{s_{i}} \) for any \( i \in I \).

In this case, there exist an element \( x \in W \), a sequence of simple reflections \( t : I \rightarrow S(W) \) and a combinatorial gallery \( \gamma \in \Gamma(t) \) such that \( x\Delta_{-\infty} = \Delta_{+} \) and \( x(\Delta, L) \) corresponds to \( \gamma \) by \( \Pi \). We call \( (x, t, \gamma) \) a *galleralification* of \( s \). This fact is equivalent to \( t^{(\gamma)} = s^{x} \).

Note that for any \( w \in W \) a sequence \( s \) is of gallery type if and only if the sequence \( s^{w} \) is of gallery type.

**Example 2.** If \( \Phi \) is of type either \( A_{1} \) or \( A_{2} \), then every sequence \( s : I \rightarrow T(W) \) is of gallery type. In the first case, this fact is obvious, as there is only one wall.

Let us consider the second case. We will prove by induction on the cardinality of \( I \neq \emptyset \) that for any sequence \( s : I \rightarrow T(W) \) and a chamber \( \Delta_{+\infty} \) attached to \( L_{s_{\text{max} I}} \), there exists a labelled gallery \( (\Delta, L) \) such that \( L_{i} = L_{s_{i}} \) for any \( i \in I \) and \( \Delta_{\text{max} I} = \Delta_{+\infty} \). The case \( |I| = 1 \) is obvious: we can take, for example, the labelled gallery \( (\Delta, L) \) on \( I \), where \( \Delta_{-\infty} = \Delta_{\text{max} I} = \Delta_{+\infty} \) and \( \Delta_{\text{max} I} = L_{s_{\text{max} I}} \). Now suppose that \( |I| > 1 \) and the claim is true for smaller sets. If \( s_{\text{max} I'} = s_{\text{max} I} \), then we define \( \Delta_{+\infty} = \Delta_{+\infty} \). Otherwise let \( \Delta_{+\infty} \) be a
chamber equal to $\Delta_{+\infty}$ or obtained from $\Delta_{+\infty}$ by the reflection through $L_{s_{\max}I}$ such that $\Delta_{+\infty}$ is attached to both $L_{s_{\max}I}$ and $L_{s_{\max}I'}$. The picture below shows how to do it.

By the inductive hypothesis, there exists a gallery $(\Delta, \mathcal{L})$ on $I'$ such that $\mathcal{L}_i = L_{s_i}$ for any $i \in I'$ and $\Delta_{\max}I' = \Delta_{+\infty}$. It suffices to extend this gallery to $I$ by defining $\Delta_{\max}I = \Delta_{+\infty}$ and $\mathcal{L}_{\max}I = L_{s_{\max}I}$.

### 2.4. Definition via the Borel subgroup

For any simple reflection $t \in S(W)$, we consider the minimal parabolic subgroup $P_t = B \cup BtB$. Note that $C_t$ is a maximal compact subgroups of $P_t$ and $C_t = P_t \cap C$. Let $s : I \to S(W)$ be a sequence of simple reflections. We consider the space

$$P(s) = \prod_{i \in I} P_{s_i}$$

with respect to the product topology. The group $B(I)$ acts on $P(s)$ on the right by $(bp)_i = b_{-1}^{-1}p_b$. Here and in what follows, we assume $b_{-\infty} = 1$. Let

$$BS(s) = P(s)/B(I).$$

This space is compact and Hausdorff. The Borel subgroup $B$ acts continuously on $P(s)$ by

$$(bp)_i = \begin{cases} bp_i & \text{if } i = \min I; \\ p_i & \text{otherwise.} \end{cases}$$

As this action commutes with the right action of $B(I)$ on $P(s)$ described above, we get the left action of $B$ on $BS(s)$. Note that $BS(s)$ is a singleton with the trivial action of $B$ if $I = \emptyset$.

We also have the map $\pi(s) : BS(s) \to G/B$ that maps $pB(I)$ to $p^{\max}B$. This map is invariant under the left action of $B$. However, we need here only the action of the torus $T$. We have $BS(s)^T = \{\gamma B(I) \mid \gamma \in \Gamma(s)\}$, where $\gamma$ is identified with the sequence $i \mapsto \gamma_i$ of $P(s)$. We will identify $BS(s)^T$ with $\Gamma(s)$. For any $w \in \Gamma(s)$, we set $BS(s, w) = \pi(s)^{-1}(wB)$.

### 2.5. Definition via the compact torus

Let $s : I \to T(W)$ be a sequence of reflections. We consider the space

$$C(s) = \prod_{i \in I} C_{s_i}$$

with respect to the product topology. The group $K(I)$ acts on $C(s)$ on the right by $(ck)_i = k_{i_{-1}}^{-1}c k_i$. Here and in what follows, we assume $k_{-\infty} = 1$. Let

$$BS_c(s) = C(s)/K(I)$$

and $\nu_c(s) : C(s) \to BS_c(s)$ be the corresponding quotient map. We denote by $[c]$ the right orbit $cK(I) = \nu_c(s)(c)$ of $c \in C(s)$. The space $BS_c(s)$ is compact and Hausdorff (as the quotient of a compact Hausdorff space by a continuous action of a compact group).
The action of $K(I)$ on $C(s)$ described above commutes with the following left action of $K$:

$$(kc)_i = \begin{cases} 
kc_i & \text{if } i = \min I; \\
$c_i$ & \text{otherwise}.
\end{cases}$$

Therefore $K$ acts continuously on $\text{BS}_c(s)$ on the left: $k[c] = [kc]$.

We also have the map $\pi_c(s) : \text{BS}_c(s) \to C/K$ that maps $[c]$ to $c^{\text{max}}K$. This map is obviously invariant under the left action of $K$. We have $\text{BS}_c(s)^K = \{[\gamma] | \gamma \in \Gamma(s)\}$, where $\gamma$ is identified with the sequence $i \mapsto \gamma_i$ of $C(s)$. We also identify $\text{BS}_c(s)^K$ with $\Gamma(s)$. For any $w \in W$, we set $\text{BS}_c(s, w) = \pi_c(s)^{-1}(wK)$. The set of $K$-fixed points of this space is

$$\text{BS}_c(s, w)^K = \{\gamma \in \Gamma(s) | \gamma^\text{max} = w\} = \Gamma(s, w).$$

For any $w \in W$, let $l_w : C/K \to C/K$ and $r_w : C/K \to C/K$ be the maps given by $l_w(cK) = \hat{w}cK$ and $r_w(cK) = cwK$. They are obviously well-defined and $l_w^{-1} = l_{w^{-1}}$, $r_w^{-1} = r_{w^{-1}}$. Moreover, $l_w$ and $r_{w'}$ commute for any $w, w' \in W$. Let $d_w : \text{BS}_c(s) \to \text{BS}_c(s^w)$ be the map given by $d_w([c]) = [c^w]$, where $([c^w])_i = \hat{w}c_i\hat{w}^{-1}$. This map is well-defined and a homeomorphism. Moreover, $d_w(ka) = \hat{w}k\hat{w}^{-1}a$ for any $k \in K$ and $a \in \text{BS}_c(s)$. We have the commutative diagram

$$\text{BS}_c(s) \xrightarrow{\sim} \text{BS}_c(s^w) \quad \pi_c(s) \downarrow \quad \pi_c(s^w)$$

$$\text{C/K} \quad \xrightarrow{\sim} \quad \text{C/K} \quad l_w r_w^{-1} \quad \downarrow$$

Hence for any $x \in W$, the map $d_w$ yields the isomorphism

$$\text{BS}_c(s, x) \cong \text{BS}(s^w, wxw^{-1}). \quad (2)$$

There is a similar construction for a generalized combinatorial gallery $\gamma \in \Gamma(s)$. We define the map $D_\gamma : \text{BS}_c(s) \to \text{BS}_c(s^\gamma)$ by

$$D_\gamma([c]) = [c^{\gamma}] = c^{\gamma_1} \cdots c^{\gamma_{\min I}}^{-1} \gamma_{\min I}^{-1} \cdots \gamma_{\min I}^{-1}.$$ 

This map is well-defined. Indeed let $[c] = [\tilde{c}]$ for some $c, \tilde{c} \in C(s)$. Then $\tilde{c} = ck$ for some $k \in K(I)$. Then $c^{\gamma} = c^{\gamma}k'$. Then $k' = (\gamma_{\min I}^{-1} \cdots \gamma_{\min I}^{-1})^{-1}k^{\gamma_1} \gamma_{\min I}^{-1} \cdots \gamma_{\min I}^{-1}$. The map $D_\gamma$ is obviously continuous and a $K$-equivariant homeomorphism. We have the following commutative diagram:

$$\text{BS}_c(s) \xrightarrow{\sim} \text{BS}_c(s^\gamma) \quad \pi_c(s) \downarrow \quad \pi_c(s^\gamma)$$

$$\text{C/K} \quad \xrightarrow{\sim} \quad \text{C/K} \quad r_{\text{max} I}^{-1} \quad \downarrow$$

Hence for any $x \in W$, the map $D_\gamma$ yields the isomorphism

$$\text{BS}_c(s, x) \cong \text{BS}_c(s^\gamma, x(\gamma^{\text{max}})^{-1}). \quad (3)$$

### 2.6. Isomorphism of two constructions.

Consider the Iwasawa decomposition $G = CB$ (see, for example, [St, Theorem 16]). Hence for any simple reflection $t$, the natural embedding $C_t \subset P_t$ induces the isomorphism of the left cosets $C_t/K \simeq P_t/B$. Therefore, for any sequence of simple reflections $s : I \to S(W)$, the natural embedding $C(s) \subset P(s)$ induces the homeomorphism $\text{BS}_c(s) \simeq \text{BS}(s)$. This homeomorphism is clearly $K$-equivariant.
Moreover, the following diagram is commutative:

\[
\begin{array}{ccc}
BS_c(s) & \rightarrow & BS(s) \\
\downarrow & & \downarrow \\
C/K & \rightarrow & G/B
\end{array}
\]

Note that \( BS_c(s)^K \) is mapped to \( BS(s)^K = BS(s)^T \). These sets are identified with \( \Gamma(s) \).

Any isomorphism of finite totally ordered set \( \iota : J \rightarrow I \) induces the obvious homeomorphisms \( BS(s) \rightarrow BS(s\iota) \) and \( BS_c(s) \rightarrow BS_c(s\iota) \). They obviously respect the isomorphisms described above.

3. Nested fibre bundles

3.1. Nested structures. Let \( I \) be a finite totally ordered set. We denote the order on it by \( \leq \) (respectively, \( \geq, <, \) etc.). Let \( R \) be a subset of \( I^2 \). For any \( r \in R \), we denote by \( r_1 \) and \( r_2 \) the first and the second component of \( r \) respectively. So we have \( r = (r_1, r_2) \). We will also use the notation \( [r] = [r_1, r_2] \) for the intervals. We say that \( R \) is a nested structure on \( I \) if

\[
\begin{align*}
\bullet & \quad r_1 \leq r_2 \text{ for any } r \in R; \\
\bullet & \quad \{r_1, r_2\} \cap \{r'_1, r'_2\} = \emptyset \text{ for any } r, r' \in R; \\
\bullet & \quad \text{for any } r, r' \in R, \text{ the intervals } [r] \text{ and } [r'] \text{ are either disjoint or one of them is contained in the other.}
\end{align*}
\]

Abusing notation, we will write

\[
\begin{align*}
\bullet & \quad r \subset r' \text{ (resp. } r \subsetneq r') \text{ to say that } [r] \subset [r'] \text{ (resp. } [r] \subsetneq [r']); \\
\bullet & \quad r < r' \text{ to say that } r_2 < r'_1.
\end{align*}
\]

Let \( s : I \rightarrow T(W) \) and \( v : R \rightarrow W \) be arbitrary maps. We define

\[
BS_c(s, v) = \{ [c] \in BS_c(s) \mid \forall r \in R : c_r c_{r_1+1} \cdots c_{r_2} \in v_r K \}.
\]

We consider this set with respect to the subspace topology induced by its embedding into \( BS_c(s) \).

Consider the maps \( \xi_r : C(s) \rightarrow C/K \) defined by \( \xi_r(c) = c_{r_1} c_{r_1+1} \cdots c_{r_2} K \). Then \( BS_c(s, v) \) is the image of the intersections

\[
C(s, v) = \bigcap_{r \in R} \xi_r^{-1}(v_r K)
\]

under the projection \( \nu_c(s) : C(s) \rightarrow BS_c(s) \). As \( C(s, v) \) is compact, the space \( BS_c(s, v) \) is also compact and thus is closed in a Hausdorff space \( BS_c(s) \).

Let \( \nu_c(s, v) \) denote the restriction of \( \nu_c(s) \) to \( C(s, v) \). We have a Cartesian diagram

\[
\begin{array}{ccc}
C(s, v) & \xrightarrow{\nu_c(s, v)} & BS_c(s, v) \\
\downarrow & & \downarrow \\
C(s) & \xrightarrow{\nu_c(s)} & BS_c(s)
\end{array}
\]

Clearly the subspace topology on \( BS_c(s, v) \) coincides with the quotient topology induced by the upper arrow. As \( \nu_c(s) \) is a fibre bundle, \( \nu_c(s, v) \) is also a fibre bundle as a pull-back. The fibres of both bundles are \( K(I) \).
3.2. Projection. Let $F \subset R$ be a nonempty subset such that the intervals $[f]$, where $f \in F$, are pairwise disjoint. In what follows, we use the notation

$$ F = \{ f_1, \ldots, f_n \} $$

with elements written in the increasing order: $f_1 < f_2 < \cdots < f_n$. We set

$$ I^F = I \setminus \bigcup_{f \in F} [f], \quad R^F = R \setminus \{ r \in R \mid \exists f \in F : r < f \}. $$

Obviously, $R^F$ is a nested structure on $I^F$.

We also introduce the following auxiliary notation. Let $i \in I^F$. We choose $m = 0, \ldots, n$ so that $f_{2m}^i < i < f_{m+1}^i$. In these inequalities and in what follows, we assume that $f_0^i = -\infty$ and $f_n^i = +\infty$. We set

$$ v^i = v_{f_1} v_{f_2} \cdots v_{f_m}, \quad \dot{v}^i = \dot{v}_{f_1} \dot{v}_{f_2} \cdots \dot{v}_{f_m}. $$

Clearly, $v^i$ is image of $\dot{v}^i$ under the quotient homomorphism $N \to W$. We define the sequences $s^F : I^F \to \mathcal{T}(W)$ and $v^F : R^F \to W$ by

$$ s_i^F = v_i s_i(v^i)^{-1}, \quad v_i^F = v_i r(v_i^2)^{-1}. $$

The main aim of this section is to define the map $p^F : BS_c(s, v) \to BS_c(s^F, v^F)$, which we call the projection along $F$.

**Definition 3.** A sequence $c \in C(s, v)$ is called $F$-balanced if

$$ c_{f_1} c_{f_{i+1}} \cdots c_{f_2} = \dot{v}_f $$

for any $f \in F$.

It is easy to prove that for any $a \in BS_c(s, v)$, there exists an $F$-balanced $c \in C(s, v)$ such that $a = [c]$ (see the third part of the proof of Lemma 34). For an $F$-balanced $c \in C(s, v)$, we define the sequence $c^F \in C(s^F)$ by

$$ c_i^F = \dot{v}_i c_i(\dot{v}^i)^{-1}. $$

Then we set $p^F([c]) = [c^F]$.

**Lemma 4.** The map $p^F : BS_c(s, v) \to BS_c(s^F, v^F)$ is well-defined, $K$-equivariant and continuous.

**Proof.** Part 1: $p^F$ is well-defined. First, we prove that $c^F \in C(s^F, v^F)$ for an $F$-balanced $c$. Let $r \in R^F$ and $l$ and $m$ be the numbers such that

$$ f_2^l < r_1 < f_{l+1}^l, \quad f_2^m < r_2 < f_{m+1}^1. $$

We get

$$ (c_{r_1}^F \cdots c_{f_{l+1}^l-1}^F)(c_{f_{l+1}^l}^F \cdots c_{f_{l+2}^l-1}^F) \cdots (c_{f_{m}^l}^F \cdots c_{r_2}^F) $$

$$ = \dot{v}_{f_1} \dot{v}_{f_2} \cdots \dot{v}_{f_l} c_{r_1} \cdots c_{f_{l+1}^l} (\dot{v}_{f_1} \dot{v}_{f_2} \cdots \dot{v}_{f_l})^{-1} \times $$

$$ \times \dot{v}_{f_1} \dot{v}_{f_2} \cdots \dot{v}_{f_{l+1}^l} c_{f_{l+1}^l} \cdots c_{f_{l+2}^l-1} (\dot{v}_{f_1} \dot{v}_{f_2} \cdots \dot{v}_{f_{l+1}^l})^{-1} \times $$

$$ \times \dot{v}_{f_1} \dot{v}_{f_2} \cdots \dot{v}_{f_{l+1}^l} \cdots \dot{v}_{f_m} c_{f_{l+1}^l} \cdots c_{r_2} (\dot{v}_{f_1} \dot{v}_{f_2} \cdots \dot{v}_{f_m})^{-1} $$

$$ = \dot{v}_{f_1} \dot{v}_{f_2} \cdots \dot{v}_{f_l} c_{r_1} \cdots c_{f_{l+1}^l} \dot{v}_{f_1} c_{f_{l+1}^l} \cdots c_{f_{l+2}^l-1} \dot{v}_{f_{l+2}} \times $$

$$ \times \dot{v}_{f_1} \dot{v}_{f_2} \cdots \dot{v}_{f_{l+1}^l} \cdots \dot{v}_{f_m} c_{f_{l+1}^l} \cdots c_{r_2} (\dot{v}_{f_1} \dot{v}_{f_2} \cdots \dot{v}_{f_m})^{-1} $$

$$ = \dot{v}_{f_1} \dot{v}_{f_2} \cdots \dot{v}_{f_l} c_{r_1} \cdots c_{r_2} (\dot{v}_{f_1} \dot{v}_{f_2} \cdots \dot{v}_{f_m})^{-1} = \dot{v}_{r_1} c_{r_2} (\dot{v}^r)^{-1} K \in v^F K. $$
Now let us prove that \([c^F]\) is independent of the choice of an \(F\)-balanced \(c\). Suppose that \([d] = [c]\) for another \(F\)-balanced sequence \(d\). We get \(d = ck\) for some \(k \in K(I)\). Comparing (5) and the similar equalities for \(d\), we get

\[
\dot{v}_f k_{f_2} = k_{f_1-1} \dot{v}_f
\]

for any \(f \in F\). As usually, we assume that \(k_{-\infty} = 1\).

We define \(k^F \in K(I^F)\) by \(k^F_i = \dot{v}_i k_i(\dot{v}i)^{-1}\). We claim that \(c^F k^F = d^F\) and thus \([c^F] = [d^F]\).

Indeed, let \(i \in I^F\). We choose \(m\) so that \(f_2^m < i < f_1^{m+1}\). If \(f_2^m < i - 1\), then \(\dot{v}i = \dot{v}i^{-1}\) and we get

\[
(c^F k^F)_i = (k^F_{i-1})^{-1} c^F k^F_i = (\dot{v}i k_{i-1}(\dot{v}i)^{-1})^{-1} \dot{v}i c_i(\dot{v}i)^{-1} \dot{v}i k_i(\dot{v}i)^{-1} = \dot{v}i k_{i-1} c_i k_i(\dot{v}i)^{-1} = \dot{v}i d_i(\dot{v}i)^{-1} = d_i^F.
\]

Now suppose that \(i - 1 = f_2^m\). Let \(l = 1, \ldots, m\) be the greatest number such that \(f_1^l > f_2^{-l} + 1\). This number is well-defined, as we assumed that \(f_2^0 = -\infty\). First consider the case \(l = 1\). In this case, \(i\) is the minimal element of \(I^F\). We claim that \(k_{f_2^j} = 1\) for any \(h = 0, \ldots, m\). The case \(h = 0\) is true by definition. Now suppose that \(h < m\) and \(k_{f_2^j} = 1\).

By (6), we get

\[
\dot{v}_{f_{h+1}} k_{f_2^j} = k_{f_2^j-1} \dot{v}_{f_{h+1}} = k_{f_2^j-1} \dot{v}_{f_{h+1}} = \dot{v}_{f_{h+1}}.
\]

Cancelling out \(\dot{v}_{f_{h+1}}\), we get \(k_{f_2^j} = 1\). We have proved that \(k_{i-1} = k_{f_2^m} = 1\). We get

\[
(c^F k^F)_i = (k^F_{i-1})^{-1} c^F k^F_i = (\dot{v}i k_{i-1}(\dot{v}i)^{-1})^{-1} \dot{v}i c_i(\dot{v}i)^{-1} \dot{v}i k_i(\dot{v}i)^{-1} = \dot{v}i k_{i-1} c_i k_i(\dot{v}i)^{-1} = \dot{v}i d_i(\dot{v}i)^{-1} = d_i^F.
\]

Now consider the case \(l > 1\). In this case, \(j = f_1^l - 1\) immediately precedes \(i\) in the set \(I^F\). We claim that

\[
(\dot{v}_{f_l} \cdots \dot{v}_{f_1})^{-1} k_{f_2^j} \dot{v}_{f_l} \cdots \dot{v}_{f_1} = k_{f_2^j} \dot{v}_{f_1} k_{f_2^j} \dot{v}_{f_1} = k_{f_2^j} \dot{v}_{f_1} k_{f_2^j} \dot{v}_{f_1} = k_{f_2^j+1}.
\]

Applying (7) for \(h = m\), we get

\[
(c^F k^F)_i = (k^F_j)^{-1} c^F k^F_i = (\dot{v}i k_j(\dot{v}i)^{-1})^{-1} \dot{v}i c_i(\dot{v}i)^{-1} \dot{v}i k_i(\dot{v}i)^{-1} = \dot{v}i (\dot{v}i)^{-1} k_{i-1}^{(\dot{v}i)^{-1}} c_i k_i(\dot{v}i)^{-1} = \dot{v}i k_{i-1}^{(\dot{v}i)^{-1}} c_i k_i(\dot{v}i)^{-1} = d_i^F.
\]

**Part 2:** \(p^F\) is \(K\)-equivariant. Let \(c \in C(s, v)\) be \(F\)-balanced. If \(min I\) belongs to \(I^F\), then \(kc\) is also \(F\)-balanced and \((kc)^F = kc^F\) for any \(k \in K\). Thus \(p^F(k[c]) = [(kc)^F] = [k[c^F] = kp^F([c])\) for any \(k \in K\).

So we consider the case \(min I / I^F = I\) which is \(f_1^1\). We can also assume that \(I^F \neq \emptyset\), as otherwise BS\(_n(s^F, v^F)\) is a singleton.

Let \(m = 1, \ldots, n\) be the smallest number such that \(f_2^m + 1 \in I^F\). This element is obviously the minimal element of \(I^F\). Let

\[
d = k c \delta_{f_2}((\dot{v}_{f_1}^{-1} k^{-1} v_{f_1}) \delta_{f_2}((\dot{v}_{f_1} k_{f_2} v_{f_2})^{-1} k_{f_1}^{-1} v_{f_1} v_{f_2}) \times 
\]

\[
\cdots \times \delta_{f_2}((\dot{v}_{f_1} \cdots \dot{v}_{f_m})^{-1} k^{-1} v_{f_1} \cdots v_{f_m}).
\]
Here we multiplied on the right by the indicator sequences on $I$ defined in Section 2.2. It is easy to check that $d$ is $F$-balanced. Indeed, for any $h = 1, \ldots, m$, we get

$$
\begin{align*}
d_{1} d_{f_{1} + 1} \cdots d_{f_{h}} &= (\hat{v}_{f_{1}} \cdots \hat{v}_{f_{h-1}})^{-1} k \hat{v}_{f_{1}} \cdots \hat{v}_{f_{h-1}} c_{f_{h}} c_{f_{h} + 1} \cdots c_{f_{h}} (\hat{v}_{f_{1}} \cdots \hat{v}_{h})^{-1} k^{-1} \hat{v}_{f_{1}} \cdots \hat{v}_{f_{h}} \\
&= (\hat{v}_{f_{1}} \cdots \hat{v}_{f_{h-1}})^{-1} k \hat{v}_{f_{1}} \cdots \hat{v}_{f_{h-1}} \hat{v}_{f_{h}} (\hat{v}_{f_{1}} \cdots \hat{v}_{h})^{-1} k^{-1} \hat{v}_{f_{1}} \cdots \hat{v}_{f_{h}} = \hat{v}_{f_{h}}.
\end{align*}
$$

Therefore $p^{F}([k c]) = [d^{F}]$. So it remains to prove that $k c^{F} = d^{F}$. Let $i \in I^{F}$. If $i > f_{m}^{m} + 1$, we get $c_{i} = d_{i}$ and thus

$$(k c^{F})_{i} = \hat{v}_{i} c_{i} \hat{v}_{i}^{-1} = \hat{v}_{i} d_{i} (\hat{v}_{i}^{-1})^{-1} = d_{i}^{F}.$$ 

Now let $i = f_{m}^{m} + 1$. Then $\hat{v}_{i} = \hat{v}_{f_{1}} \cdots \hat{v}_{f_{m}}$ and we get

$$(k c^{F})_{i} = \hat{v}_{i} d_{i} (\hat{v}_{i}^{-1})^{-1} = \hat{v}_{i} (\hat{v}_{f_{1}} \cdots \hat{v}_{f_{m}})^{-1} k \hat{v}_{f_{1}} \cdots \hat{v}_{f_{m}} c_{i} (\hat{v}_{i})^{-1} = k \hat{v}_{i} c_{i} (\hat{v}_{i})^{-1} = (k c^{F})_{i}.$$ 

Part 3: $p^{F}$ is continuous. We describe the above mentioned process of bringing an element $c \in C(s, v)$ to its $F$-balanced form in more detail. For any $m = 1, \ldots, n$, let $b_{i} : C(s, v) \to C(s, v)$ be the following map:

$$b_{m}(c) = c \delta_{f_{m}^{m}}((c_{f_{1}^{m}} c_{f_{1}^{m} + 1} \cdots c_{f_{m}^{m}})^{-1} \hat{v}_{f_{m}})$$

This map is continuous and has the following property: if $c \in \{ f_{1}, \ldots, f_{m} - 1 \}$-balanced, then $b_{m}(c)$ is $\{ f_{1}, \ldots, f_{m} \}$-balanced. Thus $b_{m} \cdots b_{1}(c)$ is $F$-balanced for any sequence $c \in C(s, v)$ and moreover $[c] = [b_{m} \cdots b_{1}(c)]$.

It remains to consider the continuous map $\tilde{p}^{F}(c) : C(s, v) \to \text{BS}_{c}(s, v^{F})$ given by $\tilde{p}^{F}(c) = [(b_{m} \cdots b_{1}(c))^{F}]$. As is proved before, this map factors through $\text{BS}_{c}(s, v)$ and thus yields $\tilde{p}^{F}$.

### 3.3. Closed nested structures

A nested structure $R$ on $I$ is called closed if $I \neq \emptyset$ and it contains the pair $(\min I, \max I)$, which we denote by $\text{span}(I)$. In this case, we consider the following set

$$\tilde{C}(s, v) = \xi^{-1}(\hat{v}_{\text{span}(I)} \cap \bigcap_{r \in R \setminus \{\text{span}(I)\}} \xi_{r}^{-1}(v_{r} K),$$

where $\hat{\xi} : C(s) \to C$ is the map given by $\hat{\xi}(c) = c_{\min I} c_{\min I + 1} \cdots c_{\max I}$. We obviously have

$$\tilde{C}(s, v) \subset C(s, v).$$

The right set is invariant under right action of $K(I)$. However, the left set is not. To get the invariance, let us consider the embedding $\alpha : K(I') \to K(I)$ given by

$$\alpha(k)_{i} = \begin{cases} 
  k_{i} & \text{if } i \neq \max I \\
  1 & \text{otherwise}
\end{cases}$$

Then we set

$$c * k = c \alpha(k)$$

for any $c \in \tilde{C}(s, v)$ and $k \in K(I')$. It is easy to check that $\tilde{C}(s, v)$ is invariant under this action of $K(I')$. Let $\tilde{\nu}_{c}(s, v) : \tilde{C}(s, v) \to \text{BS}_{c}(s, v)$ denote the restriction of $\nu_{c}(s, v)$.

**Lemma 5.** The map $\tilde{\nu}_{c}(s, v)$ is surjective. Its fibres are exactly the orbits of the *-action of $K(I')$. The subspace topology on $\text{BS}_{c}(s, v)$ induced from its embedding into $\text{BS}_{c}(s)$ coincides with the quotient topology induced by $\tilde{\nu}_{c}(s, v)$. 

Proof. Any $K(I')$-orbit is contained in a fibre, as $[c * k] = [c * \alpha(k)] = [c]$ for any $c \in \tilde{C}(s, v)$ and $k \in K(I')$.

Now suppose that $c, \tilde{c} \in \tilde{C}(s, v)$ and $[c] = [\tilde{c}]$. Then $\tilde{c} = c k$ for some $k \in K(I)$. We get

$$\hat{v}_{\text{span}} I = \hat{c}_{\min} r c_{\min} I_{+1} \cdots \hat{c}_{\max} I = c_{\min} r c_{\min} I_{+1} \cdots c_{\max} I k_{\max} I = \hat{v}_{\text{span}} I k_{\max} I.$$  

Hence $k_{\max} I = 1$. Thus $\alpha(k') = k$ and $\tilde{c} = c \alpha(k') = c * k'$.

Finally, let us prove the statement about the surjectivity and the topologies. Let $\iota : \tilde{C}(s, v) \to C(s, v)$ denote the natural embedding and $\beta : C(s, v) \to \tilde{C}(s, v)$ be the map defined by

$$\beta(c) = c \delta_{\max} I \left( \hat{\xi}(c)^{-1} \hat{v}_{\text{span}} I \right).$$

We get the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{C}(s, v) & \xrightarrow{\iota} & C(s, v) \\
\downarrow \tilde{\nu}_c(s, v) & & \downarrow \nu_c(s, v) \\
BS_c(s, v) & \xrightarrow{\beta} & \tilde{C}(s, v)
\end{array}
\]

The surjectivity of $\tilde{\nu}_c(s, v)$ follows from the surjectivity of $\nu_c(s, v)$ and the commutativity of the right triangle.

For any subset $U \subset BS_c(s, v)$, we have

$$(\nu_c(s, v))^{-1}(U) \text{ is open} \iff (\tilde{\nu}_c(s, v))^{-1}(U) \text{ is open},$$

as follows from the continuity of $\iota$ and $\beta$ and the equalities

$$(\tilde{\nu}_c(s, v))^{-1}(U) = \iota^{-1} \nu_c(s, v)^{-1}(U), \quad \nu_c(s, v)^{-1}(U) = \beta^{-1} \tilde{\nu}_c(s, v)^{-1}(U).$$

We have thus proved that the quotient topology on $BS_c(s, v)$ induced by $\tilde{\nu}_c(s, v)$ coincides with the quotient topology induced by $\nu_c(s, v)$. The latter topology coincides with the subspace topology induced by the embedding of $BS_c(s, v)$ to $BS_c(s)$. \qed

Remark. It follows from this lemma that $\tilde{\nu}_c(s, v)$ is an open map.

3.4. Fibres. Let $f \in F$. For any sequence $\gamma$ on $I$, we denote by $\gamma_f$ the restriction of $\gamma$ to $[f]$. We also set

$$R_f = \{ r \in R \mid r \subset f \}, \quad v_f = v|_{R_f}.$$  

From this construction, it is obvious that $R_f$ is a closed nested structure on $[f]$.

We consider the following continuous map:

$$\theta : C(s^F, v^F) \times \tilde{C}(s_{f_1}, v_{f_1}) \times \cdots \times \tilde{C}(s_{f_n}, v_{f_n}) \to C(s)$$

given by

$$\theta(c, e^{(1)}, \cdots, e^{(n)})_i = \begin{cases} 
(\hat{v}^i)^{-1} c_i \hat{v}^i & \text{if } i \in I^F; \\
c^{(m)}_i & \text{if } i \in [f^m].
\end{cases}$$

We claim that the image of $\theta$ is actually contained in $C(s, v)$. Indeed, let $r \in R \setminus R^F$. Then there exists $m = 1, \ldots, n$ such that $r \subset f^m$. We get

$$\xi_r \theta(c, e^{(1)}, \cdots, e^{(n)}) = c^{(m)}_{r_1} c^{(m)}_{r_1 + 1} \cdots c^{(m)}_{r_2} K = v_r K.$$

Now we take $r \in R^F$. Let $l$ and $m$ be the numbers such that

$$f^l_2 < r_1 < f^{l+1}_1, \quad f^m_2 < r_2 < f^{m+1}_1.$$
We get
\[ \xi, \theta(c, c^{(1)}, \ldots, c^{(n)}) = (\dot{v}_{f_1} \cdots \dot{v}_{f_L})^{-1} c_{r_1} \cdots c_{r_{L+1}} (\dot{v}_{f_1} \cdots \dot{v}_{f_L}) \times \]
\[ \times \left( I_{f_1}^{(L+1)} I_{f_1}^{(L+1)} I_{f_1}^{(L+1)} (\dot{v}_{f_1} \cdots \dot{v}_{f_L})^{-1} c_{r_1} \cdots c_{r_{L+1}} (\dot{v}_{f_1} \cdots \dot{v}_{f_L}) \times \right. \]
\[ \times \left. \cdots c_{I_{f_1}^{(m-1)}} c_{I_{f_1}^{(m)}} (\dot{v}_{f_1} \cdots \dot{v}_{f_m})^{-1} c_{I_{f_1}^{(m)}} c_{I_{f_1}^{(m)}} (\dot{v}_{f_1} \cdots \dot{v}_{f_m}) K \right) \]
\[ = (\dot{v}_{f_1} \cdots \dot{v}_{f_L})^{-1} c_{r_1} \cdots c_{r_{L+1}} c_{r_{L+2}} \cdots c_{r_{L+1}} (\dot{v}_{f_1} \cdots \dot{v}_{f_m}) K \]
\[ = (\dot{v}_{f_1} \cdots \dot{v}_{f_L})^{-1} v_{f_1} \cdots (\dot{v}_{f_1} \cdots \dot{v}_{f_m}) K = v_{f_1} K. \]

The composition \( \nu_c(s) \theta \) obviously factors through the \( * \)-actions of each group \( K([f^m]) \) on the \( m + 1 \)th factor. Thus we get the continuous map
\[ \Theta : C(s, v^F) \times BS_c(s_{f_1}, v_{f_1}) \times \cdots \times BS_c(s_{f_m}, v_{f_m}) \to BS_c(s, v) \]

This map however does not factor in general through the action on \( K(I^F) \) on the first factor.

**Lemma 6.**

1. \( p^F \Theta(c, a^{(1)}, \ldots, a^{(n)}) = [c] \).

2. For any \( c \in C(s, v^F) \), the map \( \Theta(c, \underline{, \ldots, ,}) \) maps \( BS_c(s_{f_1}, v_{f_1}) \times \cdots \times BS_c(s_{f_m}, v_{f_m}) \) homeomorphically to \( (p^F)^{-1}([c]) \).

**Proof.**

1. By Lemma 5, we get for any \( m = 1, \ldots, n \) that \( a^{(m)} = [c^{(m)}] \) for some \( c^{(m)} \in C(s_{f_m}, v_{f_m}) \). Note that by definition the sequence \( \theta(c, c^{(1)}, \ldots, c^{(n)}) \) is \( F \)-balanced. Thus
\[ p^F \Theta(c, a^{(1)}, \ldots, a^{(n)}) = p^F([\theta(c, c^{(1)}, \ldots, c^{(n)})]) = [\theta(c, c^{(1)}, \ldots, c^{(n)})]^F = [c]. \]

2. As our map is continuous and between compact Hausdorff spaces, it remains to prove that it is bijective.

The first part proves that the image of \( \Theta(c, \underline{, \ldots, ,}) \) is indeed contained in \( (p^F)^{-1}([c]) \). Let us prove the surjectivity. Let \( a \in (p^F)^{-1}([c]) \) be an arbitrary point. We can write \( a = [d] \) for some \( F \)-balanced \( d \in C(s, v) \). Then we get \( [d^F] = p^F([d]) = [c] \). Therefore \( d^F k = c \) for some \( k \in K(I^F) \). Let us define \( k_F \in K(I) \) as follows. We set \( (k_F)_i = (\dot{v}^i)^{-1} k_i \dot{v}^i \) if \( i \in I^F \). If \( i = f_2^{m} \) for some \( m = 1, \ldots, n \), then we denote by \( j \) the element of \( I^F \) immediately preceding \( i \). Then we define
\[ (k_F)_f = (\dot{v}_{f_1} \cdots \dot{v}_{f_m})^{-1} k_j \dot{v}_{f_1} \cdots \dot{v}_{f_m} \]

For all other \( i \), we set \( (k_F)_i = 1 \).

We claim that \( d k_F \) is also \( F \)-balanced. We get
\[ (d k_F)_f \cdots (d k_F)_f = (\dot{v}_{f_1} \cdots \dot{v}_{f_m})^{-1} k_j \dot{v}_{f_1} \cdots \dot{v}_{f_m} \]
\[ = (\dot{v}_{f_1} \cdots \dot{v}_{f_m})^{-1} k_j \dot{v}_{f_1} \cdots \dot{v}_{f_m} \]
\[ = (\dot{v}_{f_1} \cdots \dot{v}_{f_m})^{-1} k_j \dot{v}_{f_1} \cdots \dot{v}_{f_m} \]

Let us prove that \( (d k_F)_i = (\dot{v}^i)^{-1} c_i \dot{v}^i \) for \( i \in I^F \). If \( i - 1 \notin \{ f_2^{m} \} \), then we get \( \dot{v}^{i-1} = \dot{v}^i \) and
\[ (d k_F)_i = (k_F)_{i-1} d_i (k_F)_i = (c_i - 1) k_i c_i^{-1} d_i (\dot{v}^i)^{-1} k_i \dot{v}_i \]
\[ = (\dot{v}^i)^{-1} k_i^{-1} d_i (\dot{v}^i)^{-1} k_i \dot{v}_i = (\dot{v}^i)^{-1} (d^F k_i) \dot{v}_i = (\dot{v}^i)^{-1} c_i \dot{v}_i. \]

If \( i = f_2^{m} + 1 \) for some \( m \), then \( \dot{v}^i = \dot{v}_{f_1} \cdots \dot{v}_{f_m} \). Let \( j \) be the element of \( I^F \) immediately preceding \( i \). We get
\[ (d k_F)_i = (k_F)_{i-1} d_i (k_F)_i = (\dot{v}_{f_1} \cdots \dot{v}_{f_m})^{-1} k_j \dot{v}_{f_1} \cdots \dot{v}_{f_m} d_i (\dot{v}^i)^{-1} k_i \dot{v}^i \]
\[ = (\dot{v}^i)^{-1} k_j \dot{v}_i d_i (\dot{v}^i)^{-1} k_i \dot{v}_i = (\dot{v}^i)^{-1} k_j^{-1} d_i k_i^{-1} \dot{v}_i = (\dot{v}^i)^{-1} (d^F k_i) \dot{v}_i = (\dot{v}^i)^{-1} c_i \dot{v}_i. \]
Now it is easy to check that
\[ \theta(c, (dk_F)_{f^1}, \ldots, (dk_F)_{f^m}) = dk_F. \]

Hence
\[ \Theta(c, [(dk_F)_{f^1}], \ldots, [(dk_F)_{f^m}]) = [dk_F] = [d] = a. \]

Finally, let us prove the injectivity. Let \( a^{(m)}, b^{(m)} \in BS_c(s_{f^{m}}, v_{f^{m}}) \), where \( m = 1, \ldots, n \), be points such that \( \Theta(c, a^{(1)}, \ldots, a^{(n)}) = \Theta(c, b^{(1)}, \ldots, b^{(n)}) \). We write \( a^{(m)} = [e^{(m)}] \) and \( b^{(m)} = [d^{(m)}] \) for some sequences \( e^{(m)}, d^{(m)} \in C(s_{f^{m}}, v_{f^{m}}) \). Then we get \( [\theta(c, e^{(1)}), \ldots, e^{(n)}]) = [\theta(c, d^{(1)}), \ldots, d^{(n)}]) \). Thus there exists \( k \in K(I) \) such that
\[ \theta(c, d^{(1)}, \ldots, d^{(n)}) = \theta(c, e^{(1)}, \ldots, e^{(n)})k. \] (8)

We will prove by induction that \( k_i = 1 \) for any \( i \in I^F \cup \{f^1_2, \ldots, f^n_2\} \cup \{-\infty\} \). We start with the obvious case \( i = -\infty \). Now suppose that \( i \in I^F \) and the claim holds for smaller indices. We have obviously \( i - 1 \in I^F \cup \{f^1_2, \ldots, f^n_2\} \cup \{-\infty\} \), whence \( k_{i-1} = 1 \). Evaluating (8) at \( i \), we get
\[ (\hat{\varphi})^{-1}c_i\hat{\varphi} = k_{i-1}^{-1}(\hat{\varphi})^{-1}c_i\hat{\varphi}k_i = (\hat{\varphi})^{-1}c_i\hat{\varphi}k_i. \]
Hence we get \( k_i = 1 \).

Consider the case \( i = f^m_2 \) for some \( m \). Then we have \( f^1_1 - 1 \in I^F \cup \{f^1_2, \ldots, f^n_2\} \cup \{-\infty\} \), whence \( k_{f^1_1 - 1} = 1 \). We get from (8) that
\[ \hat{\varphi}_{f^m_2} = d_{f^1_1}^{(m)}d_{f^1_1 + 1}^{(m)} \cdots d_{f^2_2}^{(m)} = k_{f^1_1 - 1}^{(m)}c_{f^1_1}^{(m)}c_{f^1_1 + 1}^{(m)} \cdots c_{f^2_2}^{(m)}k_{f^2_2} = \hat{\varphi}_{f^m_2}k_{f^m_2}. \]
Hence we get \( k_{f^m_2} = 1 \).

From (8), it is obvious now that \( d^{(m)}k_{f^m_2} = c^{(m)} \). Hence \( a^{(m)} = b^{(m)} \).

3.5. \( p^F \) as a fibre bundle. From the results of the previous section, we get the following result.

**Theorem 7.** The map \( p^F : BS_c(s, v) \to BS_c(s^F, v^F) \) is a fibre bundle with fibre \( BS_c(s_{f^1}, v_{f^1}) \times \cdots \times BS_c(s_{f^n}, v_{f^n}) \).

**Proof.** For brevity, we set \( D = BS_c(s_{f^1}, v_{f^1}) \times \cdots \times BS_c(s_{f^n}, v_{f^n}) \). Let \( b \in BS_c(s^F, v^F) \) be an arbitrary point. As
\[ \nu_c(s^F, v^F) : C(s^F, v^F) \to BS_c(s^F, v^F) \]
is a fibre bundle, there exists an open neighbourhood \( U \) of \( b \) and a section \( \tau : U \to C(s^F, v^F) \) of this bundle. Now we define the map
\[ \varphi : U \times D \to BS_c(s, v) \]
by
\[ \varphi(u, a^{(1)}, \ldots, a^{(n)}) = \Theta(\tau(u), a^{(1)}, \ldots, a^{(n)}). \]

By part (1) of Lemma 6, we get the following commutative diagram:
\[
\begin{array}{ccc}
U \times D & \xrightarrow{\varphi} & (p^F)^{-1}(U) \\
\downarrow p_i & & \downarrow p^F \\
U & \xrightarrow{p^F} & U
\end{array}
\]

Here the right arrow should have been labelled by the restriction \( p^F|_{(p^F)^{-1}(U)} \) rather than by the map \( p^F \) itself. We will use similar abbreviations in the sequel.

Part (2) of the same lemma proves that the top arrow is bijective. Let \( V \) be an open neighbourhood of \( b \) such that \( \overline{V} \subset U \). The the restriction of \( \varphi \) to \( \overline{V} \) is a homeomorphism.
from $\overline{V} \times D$ to $(p^F)^{-1}(V)$, as both spaces are compact. Then the restriction of $\varphi$ to $V \times D$ is a homeomorphism from $V$ to $(p^F)^{-1}(V)$ and the following diagram is commutative:

$$
\begin{array}{c}
V \times D \\
\downarrow \varphi \\
V
\end{array}
\xymatrix{
\ar[r]^{p_1} \\
\ar[r]^{p^F} \\
V}
$$

3.6. Example. Suppose that $G = \text{SL}_5(\mathbb{C})$ with the following Dynkin diagram:

```
α_1 ---- α_2 ---- α_3 ---- α_4
```

The simple reflections are $\omega_i = s_{\alpha_i}$. Let us consider the following sequence and the element of the Weyl group

$$
\begin{align*}
\omega & = (\omega_4, \omega_1, \omega_2, \omega_1, \omega_3, \omega_4, \omega_3, \omega_4), \\
w & = \omega_3 \omega_4.
\end{align*}
$$

We would like to study the space $BS_c(s, w)$. Computing directly with matrices of $\text{SL}_5(\mathbb{C})$, we can prove that $BS_c(s, w) = BS_c(s, v)$, where $v$ is the map from $R = \{(1, 10), (2, 6)\}$ to $W$ given by $v(1, 10) = w$ and $v(2, 6) = \omega_2$. By Theorem 7 applied for $F = \{(2, 6)\}$, we get the fibre bundle $p^F : BS_c(s, w) \rightarrow BS_c(s, v)$ with fibre $BS_c((\omega_1, \omega_2, \omega_1, \omega_2, 1), \omega_2)$. By definition, we get $s^F = (\omega_4, \omega_2 \omega_3 \omega_2, \omega_4, \omega_2 \omega_3 \omega_2, \omega_4) = (\omega_4, \omega_3, \omega_4, \omega_3, \omega_4)\omega_2$. Hence by (2), we get $BS_c(s^F, w \omega_2) \cong BS_c((\omega_4, \omega_3, \omega_4, \omega_3, \omega_4), \omega_3 \omega_4)$. Replacing compactly defined Bott-Samelson varieties with the usual ones as described in Section 2.6 we get the following fibre bundle:

$$
\xymatrix{
BS((\omega_1, \omega_2, \omega_1, \omega_2, 1), \omega_2) \ar[r]^{p^F} & BS(s, w) \ar[r]^{p^F} & BS((\omega_4, \omega_3, \omega_4, \omega_3, \omega_4), \omega_3 \omega_4).}
$$

By [S2, Corollary 5], all three spaces above are smooth.

3.7. Affine pavings. We say that a topological space $X$ has an affine paving if there exists a filtration

$$
\varnothing = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_{n-1} \subset X_n = X
$$

such that all $X_i$ are closed and each difference $X_i \setminus X_{i-1}$ is homeomorphic to $\mathbb{C}^{m_i}$ for some integer $m_i$. As any fibre bundle over a Euclidian space is trivial, we get the following result.

**Proposition 8.** Let $Y \rightarrow X$ be a fibre bundle with fibre $Z$. If $X$ and $Z$ have affine pavings, then $Y$ also has.

For the next lemma, remember Definition 1.

**Lemma 9.** Let $s : I \rightarrow T(W)$ be a sequence of gallery type. For any $w \in W$, the space $BS_c(s, w)$ has an affine paving.

**Proof.** Let $(x, t, \gamma)$ be a gallerification of $s$. We have $t^{(\gamma)} = s^\gamma$. Applying (2) and (3), we get the homeomorphisms

$$
BS_c(s, w) \cong BS_c(s^\gamma, xw x^{-1}) = BS_c(t^{(\gamma)}, x w x^{-1}) \cong BS_c(t, x w x^{-1} \gamma^{\text{max}}) \cong BS(t, x w x^{-1} \gamma^{\text{max}}).
$$

The last space has an affine paving by [11, Proposition 2.1].

Let $r \in R$ and $r^1, \ldots, r^q$ be the maximal (with respect to the inclusion) elements of those elements of $R$ that are strictly contained in $r$. We write them in the increasing order $r^1 < \cdots < r^q$. We set

$$
I(r, R) = [r] \setminus \bigcup_{r^i \subset r} [r^i] = [r] \setminus \bigcup_{m=1}^q [r^m].
$$
We define the sequence $s^{(r,v)} : I(r, R) \to \mathcal{T}(W)$ by

$$s_i^{(r,v)} = v_{r_1} \cdots v_{r_m} s_i (v_{r_1} \cdots v_{r_m})^{-1},$$

where $r_2^m < i < r_1^{m+1}$. Here we suppose as usually that $r_2^0 = -\infty$ and $r_1^{q+1} = +\infty$.

**Definition 10.** Let $s$ be a sequence of reflections on $I$ and $v$ be a map from a nested structure $R$ on $I$ to $W$. We say that the pair $(s,v)$ is of gallery type if for any $r \in R$, the sequence $s^{(r,v)}$ is of gallery type.

Note that if $R = \emptyset$, then $(s,v)$ is always of gallery type, and if $R = \{\text{span} I\}$, then $(s,v)$ is of gallery type if and only if $s$ is so.

**Lemma 11.** If $(s,v)$ is of gallery type, then $(s^F, v^F)$ and $(s_{f_1}, v_{f_1}), \ldots, (s_{f_p}, v_{f_p})$ are also of gallery type.

**Proof.** First let us prove that $(s^F, v^F)$ is of gallery type. Let $r \in R^F$, the pairs $r^1, \ldots, r^q$ be chosen as above and $I$ and $m$ be numbers such that

$$f_2^l < r_1 < f_1^{l+1}, \quad f_2^m < r_2 < f_1^{m+1}.$$

Let $r^i_1, \ldots, r^i_p$, where $i_1 < \cdots < i_p$, be those elements of $r^1, \ldots, r^q$ that do not belong to $F$. They are all maximal elements of $R^F$ strictly contained in $r$. It is convenient to set $i_0 = 0$ and $i_{p+1} = q + 1$.

For any $t = 1, \ldots, p$, we choose the numbers $a_t$ and $b_t$ so that

$$f_2^{a_t} < r^{i_t}_1 < f_1^{a_t+1}, \quad f_2^{b_t} < r^{i_t}_2 < f_1^{b_t+1}.$$

Note that $I(r, R) = I^F(r, R^F)$ and

$$(f_1^{l+1}, \ldots, f_1^{n}, f_2^{b_1+1}, \ldots, f_2^{b_2}, f_2^{b_2+1}, \ldots, f_2^{b_p}, f_2^{b_p+1}, \ldots, f_2^{m}) = (r^1, \ldots, r^q),$$

where $a_1 = m$ if $p = 0$. Now we take any $i \in I^F(r, R^F)$. First choose $h = 0, \ldots, p$ so that $r^{i_h}_2 < i < r^{i_{h+1}}_1$. Then let $j = 0, \ldots, n$ be such that $f_1^j < i < f_1^{j+1}$. Note that $b_h \leq j \leq a_{h+1}$. We get

$$(s_i^{F})^{(r,v^F)} = v_{r_1}^{F} \cdots v_{r_h}^{F} s_i(v_{r_1}^{F} \cdots v_{r_h}^{F})^{-1}. $$

We can compute the needed ingredients of this formula as follows:

$$v_{r_1}^F \cdots v_{r_h}^F = v_{r_1}^F \cdots v_{r_h} (v_{f_1}^F \cdots v_{f^p_1})^{-1} v_{f_1}^F \cdots v_{f^p_2} v_{r_{i_1}} (v_{f_1}^F \cdots v_{f^p_2})^{-1} \times$$

$$\cdots v_{f_1}^F \cdots v_{f^p_2} v_{r_{i_1}} (v_{f_1}^F \cdots v_{f^p_2})^{-1} \times v_{f_1}^F \cdots v_{f^p_1} v_{r_{i_1}} (v_{f_1}^F \cdots v_{f^p_1})^{-1} \times$$

$$v_{f_1}^F \cdots v_{f^p_1} v_{r_{i_1}} (v_{f_1}^F \cdots v_{f^p_1})^{-1}. $$

As a result, we get

$$(s_i^{F})^{(r,v^F)} = v_{f_1}^F \cdots v_{f_l} s_i^F (v_{f_1}^F \cdots v_{f_l})^{-1}. $$

This proves that $(s^F)^{(r,v^F)}$ is of gallery type.

Now let us prove that each $(s_{f_m}, v_{f_m})$ is of gallery type. Note that for any $r \in R_{f_m}$ the pairs $r^1, \ldots, r^q$ are all maximal pairs of $R_{f_m}$ strictly contained in $r$ and $[f^m](r, R_{f_m}) = I(r, R)$. Hence we get $(s_{f_m})^{(r,v_{f_m})} = s^{(r,v)}$ and this sequence is of gallery type.

**Corollary 12.** If $(s,v)$ is of gallery type, then $B_{c}(s,v)$ has an affine paving.
Proof. Arguing by induction on the cardinality of $R$, applying Theorem 7, Proposition 8 and Lemma 11 it suffices to consider the cases $R = \emptyset$ and $R = \{\text{span } I\}$. It the first case $BS_c(s, v) = BS_c(s)$. This space has an affine paving as a Bott tower [GK, Proposition 3.10]. In the second case, we have $BS_c(s, v) = BS_c(s, \text{span } I)$. This space has an affine paving by Lemma 9.

4. Equivariant cohomology and categories of Bott-Samelson varieties

4.1. Definitions. Let $p_T : E_T \to B_T$ be a universal principal $T$-bundle. For any $T$-space $X$, we consider the Borel construction $X \times_T E_T = (X \times E_T)/T$, where $T$ acts on the Cartesian product diagonally: $t(x, e) = (tx, te)$. Then we define the $T$-equivariant cohomology of $X$ with coefficients $k$ by

$$H^*_T(X, k) = H^*(X \times_T E_T, k).$$

This definition a priori depends on the choice of a universal principal $T$-bundle. However, all these cohomologies are isomorphic. Indeed let $p'_T : E'_T \to B'_T$ be another universal principal $T$-bundle. We consider the following diagram (see [J, 1.4]):

\[
\begin{array}{ccc}
X \times_T E_T & \stackrel{p_{12}/T}{\longrightarrow} & (X \times E_T \times E'_T)/T \\
p_{13}/T & & \searrow \quad p_{13}/T \\
& X \times_T E'_T &
\end{array}
\]

where the $T$ acts diagonally on $X \times E_T \times E'_T$. As $p_{12}/T$ and $p_{13}/T$ are fibre bundles with fibres $E'_T$ and $E_T$ respectively, we get by the Vietoris-Begle mapping theorem (see Lemma 15) the following diagram for cohomologies:

\[
\begin{array}{ccc}
H^*(X \times_T E_T, k) & \stackrel{(p_{12}/T)^*}{\longrightarrow} & H^*((X \times E_T \times E'_T)/T, k) \\
& \searrow \quad (p_{13}/T)^* & \nearrow \\
& H^*(X \times_T E'_T, k) &
\end{array}
\]

It allows us to identify the cohomologies $H^*(X \times_T E_T, k)$ and $H^*(X \times_T E'_T, k)$. The reader can easily check that this identification respects composition and is identical if both universal principal bundles are equal.

Similar constructions are possible for the compact torus $K$: for a $K$-space $X$, we define

$$H^*_K(X, k) = H^*(X \times_K E_K, k),$$

where $p_K : E_K \to B_K$ is a universal principal $K$-bundle. There is the following choice of universal principal $T$- and $K$-bundles that allows us to identify $T$- and $K$-equivariant cohomologies. Let $T \cong (\mathbb{C}^\times)^d$ for the corresponding $d$. Then $K \cong (S^1)^d$. We consider the space $E_T = (\mathbb{C}^\infty \setminus \{0\})^d$ and its subspace $\mathcal{E}_K = (S^\infty)^d$, where

$$\mathbb{C}^\infty = \lim \mathbb{C}^n, \quad S^\infty = \lim S^n$$

($S^n$ denotes the $n$-dimensional sphere). Let us also consider the space

$$\mathcal{B} = (\mathbb{C}P^\infty)^d = \lim (\mathbb{C}P^n)^d$$

together with the natural maps $\mathcal{E}_T \to \mathcal{B}$ and $\mathcal{E}_K \to \mathcal{B}$. They are universal principal $T$- and $K$-bundles respectively. For each $T$-space $X$, the spaces $X \times_T \mathcal{E}_T$ and $X \times_K \mathcal{E}_K$ are homeomorphic [J, 1.6]. Hence $H^*_T(X, k) = H^*_K(X, k)$.
4.2. **Stiefel manifolds.** The universal principal $K$-bundle $\mathcal{E}_K \to \mathcal{B}$ considered above is a classical choice for calculating $K$-equivariant cohomologies. However, it has the following disadvantage: we do not know how to extend the action of $K$ on $\mathcal{E}_K$ to a continuous action of the maximal compact subgroup $C$.

The solution to this problem is to consider an embedding of $C$ into a unitary group and take a universal principal for this group. We explain this construction in a little more detail.

Being a Chevalley group, the group $G$ admits an embedding $G \leq \text{GL}(V)$ for some faithful representation $V$ of the Lie algebra of $G$. Let $V = V_1 \oplus \cdots \oplus V_k$ be a decomposition into a direct sum of irreducible $G$-modules. By [St, Chapter 12], there exist for each $i$ a positive definite Hermitian form $\langle \cdot, \cdot \rangle_{V_i}$ on $V_i$ such that $\langle gu, v \rangle_{V_i} = \langle u, g^{-1}v \rangle_{V_i}$ for any $x \in G$ and $u, v \in V_i$, where $\sigma$ is the automorphism of $G$ defined in Section 2.1. Their direct sum $\langle \cdot, \cdot \rangle_V$ is a positive definite Hermitian form on $V$ satisfying the same property. We consider the unitary group

$$U(V) = \{ g \in \text{GL}(V) \mid \langle gu, gv \rangle_V = \langle u, v \rangle_V \ \forall u, v \in V \},$$

for which we have $U(V) \cap G = C$. Choosing an orthonormal basis $(v_1, \ldots, v_k)$ of $V$, we get an isomorphism $U(V) \cong U(\mathfrak{t})$ taking an operator of $U(V)$ to its matrix in this basis.

For any natural number $N \geq r$, we consider the *Stiefel manifold* $E^N = \{ A \in M_{r,N}(\mathbb{C}) \mid A \bar{A}^T = I_r \}$, where $M_{r,N}(\mathbb{C})$ is the space of $r \times N$ matrices with respect to metric topology and $I_r$ is the identity matrix. The group $U(\mathfrak{t})$ of unitary $\mathfrak{t} \times \mathfrak{t}$ matrices acts on $E^N$ on the left by multiplication. Similarly, $U(N)$ acts on $E^N$ on the right. The last action is transitive and both actions commute. The quotient space $\text{Gr}^N = E^N / U(\mathfrak{t})$ is called a Grassmanian and the corresponding quotient map $E^N \to \text{Gr}^N$ is a principal $U(\mathfrak{t})$-bundle. Note that the group $U(N)$ also acts on $\text{Gr}^N$ by the right multiplication. For $N' > N$, we get the embedding $E^N \to E^{N'}$ by adding $N' - N$ zero columns to the right.

Taking the direct limits

$$E^\infty = \lim_{\to} E^N, \quad \text{Gr} = \lim_{\to} \text{Gr}^N,$$

we get a universal principal $U(\mathfrak{t})$-bundle $E^\infty \to \text{Gr}$.

We need the spaces $E^N$ to get the principal $K$-bundles $E^N \to E^N / K$. It is easy to note that this bundle for $N = \infty$ is the direct limit of the bundles for $N < \infty$.

4.3. **Equivariant cohomology of a point.** We denote by $S = H^*_T(pt, k)$ the equivariant cohomology of a point. It is well known that $S$ is a polynomial ring with zero odd degree component. More exactly, let $\mathfrak{X}(T)$ be the group of all continuous homomorphisms $T \to \mathbb{C}^\times$. For each $\lambda \in \mathfrak{X}(T)$, let $\mathbb{C}_\lambda$ be the $\mathbb{C}T$-module that is equal to $\mathbb{C}$ as a vector space and has the following $T$-action: $tc = \lambda(t)c$. Then we have the line bundle $\mathbb{C}_\lambda \times_T E_T \to B_T$ denoted by $\mathcal{L}_T(\lambda)$, where $E_T \to B_T$ is a universal principal $T$-bundle. We get the map $\mathfrak{X}(T) \to H^2_T(pt) = H^2(B_T,k)$ given by $\lambda \mapsto c_1(\mathcal{L}_T(\lambda))$, where $c_1$ denotes the first Chern class, which extends to the isomorphism with the symmetric algebra:

$$\text{Sym}(\mathfrak{X}(T) \otimes_{\mathbb{Z}} \mathbb{C}) \cong S. \quad (11)$$

Similarly, let $\mathfrak{X}(K)$ be the group of continuous homomorphisms $K \to \mathbb{C}^\times$. For each $\lambda \in \mathfrak{X}(K)$, we have the bundle $\mathcal{L}_K(\lambda)$ similar to $\mathcal{L}_T(\lambda)$. Therefore, we have the isomorphism

$$\text{Sym}(\mathfrak{X}(K) \otimes_{\mathbb{Z}} \mathbb{C}) \cong S \quad (12)$$

\footnote{This space is usually denoted by $V_\mathfrak{t}(\mathbb{C}^N)$ or $\mathbb{C}V_{N,r}$. We also transpose matrices, as we want to have a left action of $U(\mathfrak{t})$.}
induced by $\lambda \mapsto c_1(\mathcal{L}_K(\lambda))$. In what follows, we identify $\mathfrak{X}(T)$ with $\mathfrak{X}(K)$ via the restriction. Then both isomorphisms (11) and (12) become equal. Note that the Weyl group $W$ acts on $\mathfrak{X}(K)$ and $\mathfrak{X}(T)$ by $(w\lambda)(t) = \lambda(\check{w}^{-1}tw)$.

We are free to choose a universal principal $K$-bundle $E_K \to B_K$ to compute $S = H^*(B_K, k)$. We assume that the $K$-action on $E_K$ can be extended to a continuous $C$-action. The quotient map $E^\infty \to E^\infty / K$ is an example of such a bundle. The map $\rho_u : E_K \to E_K$ defined by $\rho_u(e) = we$ factors through the action of $K$ and we get the map $\rho_u / K : E_K / K \to E_K / K$. This map induces the ring homomorphism $(\rho_u / K)^* : S \to S$. It is easy to check that the pullback of $\mathcal{L}_K(\lambda)$ along $\rho_u / K$ is $\mathcal{L}(w^{-1}\lambda)$. Therefore under identification (12), we get

$$ (\rho_u / K)^*(u) = w^{-1}u \tag{13} $$

for any $u \in S$.

For a finite space $X$ with the discrete topology and trivial action of $K$, we identify $H^*_T (X, \mathfrak{k})$ with $S(X)$. More exactly, let $x \in X$ be an arbitrary point. Consider the map $j_x : E_K / K \to X \times_K E_K$ given by $j_x(Ke) = K(x, e)$. Then any element $h \in H^*_T (X, \mathfrak{k})$ is identified with the function $x \mapsto j_x^* h$. A similar identification is possible for a finite discrete space $X$ with the trivial action of $T$.

4.4. Categories of Bott-Samelson varieties. We are going to recall the definition of the folding category $\text{Seq}$ from [S2]. The objects of this category are sequences (on the initial intervals of natural numbers) of simple reflections. Each morphism $(s_1, \ldots, s_n) \to (\tilde{s}_1, \ldots, \tilde{s}_n)$ is a triple $(p, w, \varphi)$ such that

1. $p : \{1, \ldots, n\} \to \{1, \ldots, \tilde{n}\}$ is a monotone embedding;
2. $w \in W$;
3. $\varphi : \Gamma(s) \to \Gamma(\tilde{s})$ is a map such that

$$ \varphi(\gamma)^{p(i)} \tilde{s}_{p(i)}(\varphi(\gamma)^{p(i)})^{-1} = w\gamma^is_i(\gamma^i)^{-1}w^{-1}, \quad \varphi(f_i\gamma) = f_{p(i)}\varphi(\gamma) \tag{14} $$

for any $\gamma \in \Gamma(s)$ and $i = 1, \ldots, n$.

In the last property, we used the folding operators $f_i$ introduced in Section 2.3.

Under some restrictions on the ring of coefficients, each morphism $(p, w, \varphi)$ induces the map $\tilde{H}((p, w, \varphi))$ between $T$-equivariant cohomologies of Bott-Samelson varieties, see [S2 Section 3.5]. The existence of the contravariant functor $\tilde{H}$ followed in [S2] from Theorem 1 proved in that paper with the help of M. Härterich's criterion for the image of the localization [H]. We are going to show how this theorem and thus the existence of $\tilde{H}$ naturally follow from the results obtained in this paper.

We consider here the following special case of the constructions of Section 3. $F = R$ and for any $r \in R$, we have $r_1 = r_2$ and $v_r = 1$ or $v_r = s_{r_1}$. In this case, it follows from Theorem 7 that the map

$$ p^R : \text{BS}_c(s, v) \to \text{BS}_c(s^R, v^R) = \text{BS}_c(s^R) $$

is a homeomorphism, as each space $\text{BS}_c(s_{f_1^m}, v_{f_1^m})$ consists of the only point $[s_{f_1^m}]$.

**Lemma 13.** Let $(p, w, \varphi)$ be a morphism from $s = (s_1, \ldots, s_n)$ to $\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_n)$ in the category $\text{Seq}$. Then there exists a continuous map $\psi : \text{BS}_c(s) \to \text{BS}_c(\tilde{s})$ such that $\psi^K = \varphi$ and $\psi(ka) = wkw^{-1}a$ for any $k \in K$ and $a \in \text{BS}_c(s)$.

**Proof.** We set

$$ I = \{1, 2, \ldots, \tilde{n}\}, \quad R = \{(i, i) \mid i \in I \setminus \text{im} p\}. $$
We get $I^R = \text{im} \, p$. We define the map $v : R \to W$ by $v_{(i,i)} = \varphi(\gamma)_i$ where $\gamma$ is an arbitrary element of $\Gamma(s)$. This definition makes no confusion, as by the second formula of (14), the element $\varphi(\gamma)_i$ does not depend on the choice of $\gamma$ if $i \not\in \text{im} \, p$.

Let us compute the sequence $\tilde{s}^R$. Let $\tilde{\gamma} \in \Gamma_s$ be such that $\varphi(\tilde{\gamma})_i = 1$ for any $i \in \text{im} \, p$. This gallery exists and is unique. Then we have $\varphi(\tilde{\gamma})_i = v_i$ for any $i \in \text{im} \, p$. By the first equation of (14), we get

$$\tilde{s}^R p = (s(\tilde{\gamma})^w).$$

Thus $\tilde{s}^R p = (s(\tilde{\gamma})^w)$.

We have the following homeomorphisms:

$$\text{BS}_c(\tilde{s}, v) \xrightarrow{p_R} \text{BS}_c(\tilde{s}^R) \xhookrightarrow{\text{BS}_c(\tilde{s}^R p)} = \text{BS}_c(s(\tilde{\gamma})^w) \xleftarrow{d_w} \text{BS}_c(s(\tilde{\gamma})) \xrightarrow{D_s} \text{BS}_c(s). \quad (15)$$

As $\text{BS}_c(\tilde{s}, v) \subset \text{BS}_c(\tilde{s})$, we obtain the embedding $\psi : \text{BS}_c(s) \to \text{BS}_c(\tilde{s})$, reading the above diagram from right to left. Clearly, $\psi(ka) = \tilde{w}k\tilde{w}^{-1}\psi(a)$ for any $a \in \text{BS}_c(s)$ and $k \in K$, as the same property holds for $d_w$ and the other isomorphisms are $K$-equivariant. So we can consider the restriction $\psi^K : \Gamma(s) \to \Gamma(\tilde{s})$. Let us prove that $\psi^K = \varphi$. Let $\gamma \in \Gamma(s)$. Then by the definition of $v$, we get that $\varphi(\gamma)$ belongs to $\text{BS}_c(s,v)^K$ and is $F$-balanced (under the identifications of Section 2.5). Let us write how the galleries $\varphi(\gamma)$ and $\gamma$ are mapped towards each other in diagram (15) as follows:

$$\varphi(\gamma) \longmapsto \delta \longmapsto \lambda, \quad \mu \longmapsto \rho \longmapsto \gamma.$$  

Our aim is obviously to prove that $\lambda = \mu$. To this end, take an arbitrary $i = 1,\ldots,n$. Then

$$\lambda_i = \delta_{p(i)} = \varphi(\gamma)_{p(i)}^F = v_{p(i)} \varphi(\gamma)_{p(i)}(v_{p(i)})^{-1} = \varphi(\tilde{\gamma})_{p(i)} \varphi(\gamma)_{p(i)}(\varphi(\tilde{\gamma})_{p(i)})^{-1}. $$

and

$$\mu_i = w_{p(i)} w^{-1} = w_{\tilde{\gamma}_i}(\tilde{\gamma}_i)^{-1} w^{-1}. $$

If $\varphi(\gamma)_{p(i)} = 1$, then $\gamma_i = \tilde{\gamma}_i$ and therefore $\mu_i = 1 = \lambda_i$. Suppose now that $\varphi(\gamma)_{p(i)} = \tilde{s}_{p(i)}$. Then $\gamma_i = \tilde{\gamma}_i s_i$. We get

$$\lambda_i = \varphi(\tilde{\gamma})_{p(i)} \tilde{s}_{p(i)}(\varphi(\tilde{\gamma})_{p(i)})^{-1}, \quad \mu_i = w_{\tilde{\gamma}_i} s_i(\tilde{\gamma}_i)^{-1} w^{-1}. $$

These elements are equal by the first equation of (14). \hfill \Box

Now let us apply this lemma to the computation of the equivariant cohomologies. The map $\psi \times \rho_w : \text{BS}_c(s) \times E^\infty \to \text{BS}_c(\tilde{s}) \times E^\infty$ maps $K$-orbits to $K$-orbits. Indeed, for any $k \in K$, $a \in \text{BS}_c(s)$ and $e \in E^\infty$, we get

$$(\psi \times \rho_w)(k(a,e)) = (\psi \times \rho_w)((ka,ke)) = (\psi(ka),\rho_w(ke)) = (\tilde{w}k\tilde{w}^{-1} \psi(a),\tilde{w}k\tilde{w}^{-1} \psi(a),\tilde{w}k\tilde{w}^{-1} \psi(ka),\tilde{w}k\tilde{w}^{-1} (\psi \times \rho_w)((a,e)).$$

Hence we get the quotient map $\psi \times_K \rho_w : \text{BS}_c(s) \times_K E^\infty \to \text{BS}_c(\tilde{s}) \times_K E^\infty$. Similarly, we get the map $\varphi \times_K \rho_w : \Gamma(s) \times_K E^\infty \to \Gamma(\tilde{s}) \times_K E^\infty$. We have the following commutative diagram:

$$H_K^\bullet(\text{BS}_c(s),\mathbb{k}) \xleftarrow{(\psi \times_K \rho_w)^*} H_K^\bullet(\text{BS}_c(\tilde{s}),\mathbb{k}) \downarrow \quad \downarrow$$

$$H_K^\bullet(\Gamma(s),\mathbb{k}) \xleftarrow{(\varphi \times_K \rho_w)^*} H_K^\bullet(\Gamma(\tilde{s}),\mathbb{k})$$

where the vertical arrow are restrictions. In the rest of this section, we identify $K$- and $T$-equivarinat cohomologies.
We can now give a different proof of Theorem 1 from [S2]. To do it, we need to compute the map \((\varphi \times_K \rho_w)^*(g)\) in the bottom arrow:

\[
(\varphi \times_K \rho_w)^*(g)(\gamma) = j^*_\gamma(\varphi \times_K \rho_w)^*(g) = ((\varphi \times_K \rho_w)j^*_\gamma)^*(g) = (j_{\varphi(\gamma)}(\rho_w/K))^*(g) = (\rho_w/K)^*(\rho_w/K)^*(g(\varphi(\gamma))) = w^{-1}g(\varphi(\gamma)),
\]

where we applied (13) to obtain the last equality. Thus in the notation of [S2, Theorem 1], we get

\[
(\varphi \times_K \rho_w)^*(g) = g_{(p,w,\varphi)}.
\]

We have just given another prove of this theorem: if \(g\) is in the image of the right vertical arrow of (16), then \(g_{(p,w,\varphi)}\) is in the image of the left vertical arrow. Note that we did not impose in this prove any restrictions on the commutative ring \(k\).

We can identify the top arrow of (16) with the map \(\tilde{H}((p,w,\varphi))\) from [S2, Section 3.5] under some conditions on \(k\) and the root system implying the localization theorem.

The category \(\textbf{Seq}_f\) is defined similarly to the category \(\textbf{Seq}\), see [S2, Section 5.3]. The main difference is that the objects are pairs \((s,x)\), where \(s\) is a sequence of simple reflections and \(x \in W\). Our preceding arguments give the following topological proof of Theorem 5 from [S2]. Let \((p,w,\varphi): (s,x) \to (\tilde{s},\tilde{x})\) be a morphism of the category \(\textbf{Seq}_f\). We assume that \(\Gamma(s,x) \neq \emptyset\). Then by [S2, Lemma 19], there exists a map \(\tilde{\varphi}: \Gamma(s) \to \Gamma(\tilde{s})\) such that \((p,w,\varphi)\) is a morphism of \(\textbf{Seq}\) and the restriction of \(\tilde{\varphi}\) to \(\Gamma(s,x)\) is \(\varphi\). Choosing \(\tilde{\gamma} \in \Gamma(s)\) so that \(\tilde{\varphi}(\tilde{\gamma}) = 1\) for any \(i \in \text{im}p\) and reading (13) from right to left, we construct the map \(\psi: BS_c(s) \to BS_c(\tilde{s})\). It is easy to check that it takes \(BS_c(s,x)\) to \(BS_c(\tilde{s},\tilde{x})\). Indeed, looking at (15), we conclude that we have to prove the equality \(\tilde{x}(\tilde{\varphi}(\gamma)^{\text{max}})^{-1} = wx(\gamma^{\text{max}})^{-1}w^{-1}\).

This can be done by induction, that is, we are going to prove that

\[
\tilde{x}(\tilde{\varphi}(\gamma)^{\text{max}})^{-1} = wx(\gamma^{\text{max}})^{-1}w^{-1} \tag{17}
\]

for any \(\gamma \in \Gamma(s)\). First, we take any \(\gamma \in \Gamma(s,x)\). Then \(\tilde{\varphi}(\gamma) = \varphi(\gamma) \in \Gamma(\tilde{s},\tilde{x})\) and equality (17) is reduced to \(1 = 1\). Now suppose that (17) is true for some \(\gamma \in \Gamma(s)\). Applying (14), we get

\[
\tilde{x}(\tilde{\varphi}(\Gamma)^{\text{max}})^{-1} = \tilde{x}((\Gamma)^{\text{max}})^{-1} = \tilde{x}(\tilde{\varphi}(\Gamma)^{\text{max}})^{-1}\varphi(\gamma)^{\text{max}} = \tilde{x}(\tilde{\varphi}(\gamma^{\text{max}})^{-1}w^{-1} = wx(\gamma^{\text{max}})^{-1}w^{-1} = \varphi(\Gamma)^{\text{max}})^{-1}w^{-1}.
\]

As we can reach any combinatorial gallery of \(\Gamma(s)\) from any other combinatorial galley of this set by successively applying the folding operators, we get (17) for all \(\gamma \in \Gamma(s)\). Now we can prove Theorem 5 from [S2] exactly in the same way as we proved Theorem 1 above.

5. APPROXIMATION BY COMPACT SPACES

5.1. Vietoris-Begle theorem. First, remember the following classical result.

**Proposition 14** ([1, Theorem IV.1.6]). Let \(f: X \to Y\) be a fibre bundle whose fibre is homotopic to a compact Hausdorff space. Then for each \(y \in Y\), the canonical map

\[
(R^n_{f_*})(y) \to H^n(f^{-1}(y), X)
\]

is an isomorphism for any \(n\).

From this proposition, we get the following version of the Vietoris-Begle mapping theorem.
Lemma 15. Let $f : X \to Y$ be a fibre bundle whose fibre $F$ is connected and homotopic to a compact Hausdorff space. Suppose that there exists an integer $N$ or $N = \infty$ such that $H^n(F, k) = 0$ for any $0 < n < N$. Then the canonical map

$$H^n(Y, k) \to H^n(X, k)$$

is an isomorphism for any $n < N$.

Proof. We generally follow the lines of the proof [I, Theorem III.6.4]. First we note that the morphism $a : k_Y \to f_*k_X$, given by the adjunction unit is an isomorphism. Indeed, for any open $V \subset Y$ and $s \in k_Y(V)$, the map $a(V)(s)$ is the composition $f^{-1}(V) \xrightarrow{f} V \xrightarrow{s} k$. If we restrict $a$ to a point $y \in Y$ and compose it with (18) for $n = 0$:

$$k \to (f_*k_X)_y = (R^0f_*k_X)_y \xrightarrow{\sim} H^0(f^{-1}(y), k),$$

then we get the map that takes $\gamma \in k$ to the function on $f^{-1}(y)$ taking constantly the value $\gamma$. As $f^{-1}(y)$ is connected, this map is an isomorphism. Hence the restriction of $a$ to $y$ is also an isomorphism.

Now let $k_X \to I^\bullet$ be an injective resolution. Applying $f_*$, we get the sequence

$$0 \to f_*k_X \to f_*I^0 \to f_*I^1 \to \cdots \to f_*I^N,$$

which is exact by Proposition 14 (see the proof of [I, Theorem III.6.4]). This sequence can be completed to an injective resolution $f_*k_X \to J^\bullet$, where $J^n = f_*I^n$ for $n \leq N$. Composing with $a$, we also get an injective resolution $k_Y \to J^\bullet$.

Now we want to describe (19) applying the definition from [I, II.5]. The comparison theorem for injective resolutions, yields (dashed arrows) a commutative diagram with exact rows

\[
\begin{array}{cccccccc}
0 & \to & f^*k_Y & \to & f^*f_*k_X & \to & f^*f_*I^0 & \to & f^*f_*I^1 & \to & \cdots & \to & f^*f_*I^N & \to & f^*J^{N+1} & \to & f^*J^{N+2} & \to & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \cdots \\
0 & \to & k_Y & \to & I^0 & \to & I^1 & \to & \cdots & \to & I^N & \to & I^{N+1} & \to & I^{N+2} & \to & \cdots
\end{array}
\]

where the solid (not dashed) vertical arrows come from the counit of the adjunction. The zigzag identity shows that the map

$$\Gamma(Y, J^n) \to \Gamma(X, f^*J^n) \to \Gamma(X, I^n) = \Gamma(Y, f_*I^n)$$

is the identity map for $n \leq N$. Hence (19) is an isomorphism for $n < N$. \qed

5.2. Approximation. The following result is well known.

Proposition 16. (1) $E^N$ is simply-connected if $N > \tau$.

(2) $H^n(E^N, k) = 0$ for $0 < n < 2(N - \tau) + 1$.

(3) $H^n(E^N, k)$ is free for any $n$.

To compute the modules $H^n_k(X, k)$, one can approximate $E^\infty$ by $E^N$ with $N$ big enough as follows.

Lemma 17. Let $X$ be a $K$-space. Then $H^n_k(X, k) \cong H^n(X \times_K E^N, k)$ for $n < 2(N - \tau) + 1$. These isomorphisms are natural along continuous $K$-equivariant maps and with respect to with the cup product.
Proof. For \( n < 2(N - r) + 1 \), have the diagram similar to (10)

\[
\begin{array}{ccc}
H^n((X \times E^\infty \times E^N)/K, k) & \sim & H^n(X \otimes K, k) \\
(p_{12}/K)^* & \sim & (p_{13}/K)^* \\
\end{array}
\]

Both arrows are isomorphisms by Proposition 16(2) and Lemma 15. The two remaining assertions can be checked routinely.

□

Lemma 18. Let \( X_1, \ldots, X_m \) be \( K \)-spaces. Then

\[
H^n\left(\prod_{i=1}^{m} X_i \times_K E^\infty, k\right) \cong H^n\left(\prod_{i=1}^{m} X_i \times_K E^N, k\right)
\]

for \( n < 2(N - r) + 1 \). These isomorphisms are natural with respect to the cup product and the projections to factors. In particular, we get the following commutative diagram:

\[
\begin{array}{ccc}
\bigotimes_{n_1+\cdots+n_m=n} H^n\left(\prod_{i=1}^{m} X_i \times_K E^\infty, k\right) & \longrightarrow & H^n\left(\prod_{i=1}^{m} X_i \times_K E^\infty, k\right) \\
\bigotimes_{n_1+\cdots+n_m=n} H^n\left(\prod_{i=1}^{m} X_i \times_K E^N, k\right) & \longrightarrow & H^n\left(\prod_{i=1}^{m} X_i \times_K E^N, k\right)
\end{array}
\]

where the horizontal arrow are given by the cross product.

Proof. The result follows if we consider the following direct products of fibre bundles:

\[
\prod_{i=1}^{m} X_i \times_K E^\infty \quad \text{and} \quad \prod_{i=1}^{m} X_i \times_K E^N
\]

The left bundle has fibre \((E^\infty)^m\) and the right one fibre \((E^N)^m\). The last space is contractible and \(H^n((E^N)^m, k) = 0\) for \(0 < n < 2(N - r) + 1\) by the Künneth formula. It remains to take cohomologies and apply Lemma 15.

□

We want to prove the results similar to Proposition 16 for the quotient \( E^N/K \).

Lemma 19. (1) \( E^N/K \) is simply-connected if \( N > r \).

(2) Suppose that \( N > r \). Then for \( n < 2(N - r) + 1 \), the \( k \)-module \( H^n(E^N/K, k) \) is free of finite rank and equals zero if \( n \) is odd.

Proof. (1) As \( K \) is connected, Part (1) follows from Proposition 16(1) and the long exact sequence of homotopy groups:

\[
\{1\} = \pi_1(E^N) \to \pi_1(E^N/K) \to \pi_0(K) = \{1\}.
\]

(2) By Lemma 17, we get \( H^n(E^N/K, k) \cong H^n_k(\text{pt}, k) = S \). As the latter module vanishes in odd degrees, the result follows.

□
Lemma 20. Let $X$ be a $K$-space having an affine paving. Then for $N < \infty$ and any $n \leq 2(N - r) - 1$, the $k$-module $H^p_c(X \times_K E^N, k)$ is free of finite rank and is zero if $n$ is odd.

Proof. It suffices to consider only the case $n \geq 0$. Then we have $N > r$. Let us consider the Leray spectral sequence with compact support for the canonical projection $X \times_K E^N \to E^N/K$. As $E^N/K$ is simply connected, it has the following second page:

$$E^2_{p,q} = H^p(E^N/K, H^q_c(X, k)).$$

By Lemma [19(2)], we get that $E^2_{p,q} = 0$ except the following cases: $p \geq 2(N - r) + 1$; both $p$ and $q$ are even. Moreover, $E^2_{p,q}$ is free of finite rank for $p < 2(N - r) + 1$.

The differentials coming to and starting from $E^p_{a,q}$ are thus zero if $a \geq 2$ and $p + q + 1 \leq 2(N - r)$. So we get $E^p_{\infty,q} = E^p_{2,q}$ for $p + q + 1 \leq 2(N - r)$. It follows from the spectral sequence that $H^p_c(X \times_K E^N, k) = 0$ for odd $n \leq 2(N - r) - 1$. For even $n \leq 2(N - r) - 1$, the module $H^p_c(X \times_K E^N, k)$ is filtered by the free of finite rank modules $E^p_{2,q}$ with even nonnegative $p$ and $q$ such that $p + q = n$. As Ext$^1(k, k) = 0$, the module $H^p_c(X \times_K E^N, k)$ is also free of finite rank. □

6. Cohomology of BS$_c(s)$

In this section, we are going to describe a set of multiplicative generators of $H^*_c(BS_c(s), k)$.

6.1. Twisted actions of $S$. Suppose that $E_K \to B_K$ is a universal principal $K$-bundle such that the $K$-action on $E_K$ can be extended to a continuous $C$-action. An example of such a bundle is the quotient map $E^\infty \to E^\infty/K$ (see, Section 4.2). Let $s : I \to T(W)$ be a sequence and $i$ be an element of $I \cup \{-\infty\}$. We define the map $\Sigma(s, i, w) : BS_c(s) \times_K E_K \to E_K/K$ by

$$K([e], e) \mapsto K(c^i w)^{-1} e.$$ (20)

The reader can easily check that this map is well-defined and continuous. Taking cohomologies, we get the map

$$\Sigma(s, i, w)^* : S \to H^*_c(BS_c(s), k).$$

This map induces the action of $S$ on $H^*_c(BS_c(s), k)$ by $u \cdot h = \Sigma(s, i, w)^*(u) \cup h$. For $w = 1$ and $i = -\infty$, we get the canonical action of $S$. Note that these actions are independent of the choice of the universal principal $K$-bundle and of the action of $C$.

We also consider the finite dimensional version of these maps. Let $\Sigma^N(s, i, w) : BS_c(s) \times_K E^N_K \to E^N_K/K$ be the map given by (20). Here $N$ may be an integer greater than or equal to $r$ or $\infty$. Note that $\Sigma^\infty(s, i, w)$ is a representative of $\Sigma(s, i, w)$.

6.2. Embeddings of Borel constructions. We are going to prove the following result, which use for induction. Remember that we denote truncated sequences by the prime.

Let $r \leq N$, $s : I \to T(W)$ be a nonempty sequence, $w \in W$ and $\text{tr} : BS_c(s) \to BS_c(s')$ be the truncation map $\text{tr}([c]) = [c']$. We consider the map

$$\chi^N_w = (\text{tr} \times_K \text{id}) \otimes \Sigma^N(s, \max I, w)$$

from $BS_c(s) \times_K E^N$ to $L^N = (BS_c(s') \times_K E^N) \times (E^N/K)$. We also abbreviate $L = L^\infty$ and $\chi_w = \chi^\infty_w$.

Lemma 21. For $N < \infty$, the map $\chi^N_w$ is a topological embedding. Its image consists of the pairs $(K([d], e), K\tilde{e})$ such that

$$C_{w^{-1}s_{\max}I}w\tilde{e} = C_{w^{-1}s_{\max}I}w\Sigma^N(s', \max I', w)(K([d], e)).$$ (21)
Proof. We denote $i = \max I$ for brevity. To prove the claim about the topological embedding, it suffices to prove that the above map is injective. Suppose that two orbits $K([c], e)$ and $K([\tilde{c}], \tilde{e})$ are mapped to the same pair. As $K([c], e) = K([\tilde{c}], \tilde{e})$, we can assume that $[c] = [\tilde{c}]$ and $e = \tilde{e}$. Therefore, without generality we can assume that $c' = \tilde{c}'$.

We get $\Sigma^N(s, i, w)(K([c], e)) = \Sigma^N(s, i, w)(K([\tilde{c}], e))$. Let us write this equality as follows:

$$\dot{w}^{-1}c_i^1(c^{i-1})^{-1}e = k\dot{w}^{-1}\tilde{c}_i^1(c^{i-1})^{-1}e$$

for some $k \in K$. As $C$ acts freely on $E^N$, we get $\dot{w}^{-1}c_i^1 = k\dot{w}^{-1}\tilde{c}_i^1$, whence $\dot{c}_i = c_i\dot{w}k\dot{w}^{-1}$. As $\dot{w}k\tilde{w}^{-1} \in K$, we get $[\tilde{c}] = [c]$.

Let us check that any element of the image of $\chi_w^N$ satisfies \eqref{21}. Let $K([c], e)$ be an element of $\text{BS}_c(s) \times_K E^N$. It is mapped to $(K([c'], e), K(c'\dot{w})^{-1}e)$. We get

$$(c'\dot{w})^{-1}e = \dot{w}^{-1}c_i^1(c^{i-1})^{-1}e = (\dot{w}^{-1}c_i^1\dot{w})(c^{i-1}\dot{w})^{-1}e = \dot{w}^{-1}c_i^1\dot{w}\Sigma^N(s', i - 1, w)(K([c], e)).$$

It remains to note that $\dot{w}^{-1}c_i^1\dot{w} \in \dot{w}^{-1}C_s\dot{w} = C_{w^{-1}s_iw}$.

Conversely, suppose that a pair $(K([d], e), K\tilde{e})$ of $L^N$ satisfies \eqref{21}. Then there exists an element $c \in \dot{w}^{-1}C_s\dot{w}$ such that $c\tilde{e} = (d^{-1}\dot{w})^{-1}e$. We define the sequence $c : I \to \mathcal{T}(W)$ by

$$c_j = \begin{cases} d_j & \text{if } j < i; \\ \dot{w}c_iw^{-1} & \text{otherwise.} \end{cases}$$

We get

$$(\text{tr} \times_K \text{id})(K([c], e)) = K([c'], e) = K([d], e).$$

On the other hand

$$\Sigma^N(s, i, w)(K([c], e)) = K(c'\dot{w})^{-1}e = K\dot{w}^{-1}c_i^1(c^{i-1})^{-1}e = Kc^{-1}(d^{-1}\dot{w})^{-1}e = K\tilde{e}.$$

\hfill \Box

6.3. The difference $L^N \setminus \text{im } \chi_w^N$. We study the cohomology of this difference by considering it as the total space of a fibre bundle. Here and in what follows $\chi_w^N$ and $L^N$ are as in Lemma \eqref{21}.

**Lemma 22.** Let $r \leq N < \infty$. The projection to the first component $\omega : L^N \setminus \text{im } \chi_w^N \to \text{BS}_c(s') \times_K E^N$ is a fibre bundle.

**Proof.** We denote $i = \max I$ and $\Sigma = \Sigma^N(s', i - 1, w)$ for brevity. Let $b$ be an arbitrary element of $\text{BS}_c(s') \times_K E^N$. The right action of the unitary group $U(N)$ on $E^N$ induces the right action of $U(n)$ on $E^N/K$. We denote this action by $\cdot$. Let $t : U(N) \to E^N/K$ be the map $t(g) = \Sigma(b) \cdot g$. As $U(n)$ acts transitively on $E^N$, it acts transitively on $E^N/K$ and $t$ is a fibre bundle. Therefore, there exists an open neighbourhood of $V$ of $\Sigma(b)$ in $E^N/K$ and a continuous section $g : V \to U(N)$ of $t$. Hence for any $u \in V$, we get

$$u = t(g(u)) = \Sigma(b) \cdot g(u). \quad (22)$$

We define $H = \Sigma^{-1}(V)$. It is an open subset of $\text{BS}_c(s') \times_K E^N$ containing $b$. We construct the map $\varphi : H \times \omega^{-1}(b) \to L^N$ by

$$(h, (b, Ke)) \mapsto \varphi \left( h, Ke \cdot g(\Sigma(h)) \right).$$

Suppose that the right-hand side of the above formula belongs to $\text{im } \chi_w^N$. By Lemma \eqref{21} we get

$$C_{w^{-1}s_iw}e \cdot g(\Sigma(h)) = C_{w^{-1}s_iw}\Sigma(h).$$

Thus

$$C_{w^{-1}s_iw}e = C_{w^{-1}s_iw}\Sigma(h) \cdot g(\Sigma(h))^{-1}$$

and the projection is a fibre bundle. \hfill \Box
By (22) with \( u = \Sigma(h) \), we get
\[ C_{w^{-1} \eta, \nu} = C_{w^{-1} \eta, \nu} \Sigma(b). \]
By Lemma [21], this contradicts the fact that \( (b, K \epsilon) \notin \text{im } \chi_w^N \). Thus we actually have the map \( \varphi : H \times \omega^{-1}(b) \to \omega^{-1}(H) \). It is easy to see that \( \varphi \) is a homeomorphism. Indeed the inverse map \( \omega^{-1}(H) \to H \times \omega^{-1}(b) \) is given by
\[ (h, K \epsilon) \mapsto \left( h, (b, K \epsilon \cdot g(\Sigma(h))^{-1}) \right). \]
We get the following commutative diagram:
\[
\begin{array}{ccc}
H \times \omega^{-1}(b) & \xrightarrow{\varphi} & \omega^{-1}(H) \\
| & p_i & | \\
\downarrow & & \downarrow \\
H & & \omega
\end{array}
\]
Finally note that \( \omega^{-1}(b) \) are homeomorphic for different \( b \), as the space \( \text{BS}_e(s') \times K E^N \) is connected and compact.

6.4. Compliment to a fibre. We are going now to study the cohomology of the difference \( \overline{L^N \setminus \text{im } \chi_w^N} \). To this end, we need the following general result.

**Lemma 23.** Let \( X \) be a locally compact Hausdorff space, \( \pi : X \to Y \) be a fibre bundle with fibre \( Z \), \( y \in Y \) be a point and \( k \) be a commutative ring. Suppose that \( Y \) is compact, Hausdorff, connected and simply connected, all \( H_c^n(Z, k) \) are free \( k \)-modules of finite rank, \( H_c^n(Z, k) = 0 \) for odd \( n \) and \( H_c^n(Y, k) = 0 \) for odd \( n \leq N \).

1. The restriction map \( H_c^n(X, k) \to H_c^n(\pi^{-1}(y), k) \) is surjective for all \( n < N \).
2. \( H_c^n(X \setminus \pi^{-1}(y), k) = 0 \) for odd \( n < N \).
3. If \( H_c^n(Y, k) \) are free of finite rank for \( n \leq N \), then the modules \( H_c^n(X \setminus \pi^{-1}(y), k) \) are also free of finite rank for \( n < N \).

**Proof.** [1] Consider the following Cartesian diagram:
\[
\begin{array}{ccc}
\pi^{-1}(y) & \xrightarrow{i} & X \\
\downarrow & & \downarrow \pi \\
\{y\} & \xrightarrow{i} & Y
\end{array}
\]
For the map \( \pi \), we consider two Leray spectral sequences with compact support, one for the complex \( \mathbb{R}^q \pi_! k_X \) and the other one for the complex \( \mathbb{R}^q \pi_! k_{\pi^{-1}(y)} \). Their second pages are
\[
E_2^{p,q} = H_c^p(Y, \mathbb{R}^q \pi_! k_X),
\]
\[
\tilde{E}_2^{p,q} = H_c^p(Y, \mathbb{R}^q \pi_! \mathbb{R}^q \pi_! k_{\pi^{-1}(y)})
\]
respectively. As \( \pi \) is a fibre bundle, \( \mathbb{R}^q \pi_! k_X \) is a locally constant sheaf with stalk \( H_c^q(Z, k) \). As \( Y \) is simply connected, we get \( \mathbb{R}^q \pi_! k_X = H_c^q(Z, k) = k_{Y_0}^{m_q} \), where \( m_q \) is the rank of the \( k \)-module \( H_c^q(Z, k) \). Hence \( E_2^{p,q} = H_c^p(Y, k_{Y_0}^{m_q}) \). This module is zero except the following cases: both \( p \) and \( q \) are even; \( p > N \). As the differentials coming to and starting from \( E_2^{p,q} \) are zero if \( a \geq 2 \) and \( p + q + 1 \leq N \), we get \( E_{\infty}^{p,q} = E_2^{p,q} \) for \( p + q + 1 \leq N \).

Now let us compute \( \tilde{E}_2^{p,q} \). As \( i_! \) is exact, we get
\[
\mathbb{R}^q (i_! k_{\pi^{-1}(y)}) = \mathbb{R}^q (\pi_! \mathbb{R}^q i_! k_{\pi^{-1}(y)}) = \mathbb{R}^q (\pi_! k_{\pi^{-1}(y)}) = h \mathbb{R}^q \pi_! k_{\pi^{-1}(y)}.
\]
The last sheaf is isomorphic to \( i_! \mathcal{H}_c^g(\pi^{-1}(y), k)(y) \simeq i_! \mathcal{H}_c^g(Z, k)(y) \simeq i_! \mathcal{E}^{i+1}_{p,q}(y) \). Therefore,

\[
\tilde{\mathcal{E}}_{p,q}^n \simeq H^p_c(Y, i_! \mathcal{E}^{i+1}_{p,q}(y)) = H^p_c(\{y\}, k)^{\oplus m_q} = H^p(\{y\}, k)^{\oplus m_q}.
\]

Hence \( \tilde{\mathcal{E}}_{p,q}^n = 0 \) unless \( p = 0 \) and \( q \) is even. So the Leray spectral sequence with compact support for \( \tilde{i}_! \mathcal{E}^{i+1}_{n-1}(y) \) collapses at the second page, whence \( \tilde{\mathcal{E}}_{p,q}^n = \tilde{\mathcal{E}}_{p,q}^2 \).

We would like to use the functoriality of the Leray spectral sequence with compact support for \( \pi \) along the map \( \tilde{n}_\pi X : \tilde{k}_X \to \tilde{i}_! \mathcal{E}^{i+1}_{n-1}(y) \), where \( \tilde{n} : \mathrm{id} \to \tilde{i}_! \mathcal{E}^{i+1} \) is the unit of adjunction. Note that this map induces the restriction map \( \tilde{\mathcal{E}}_{p,q}^{i+1}(\pi^{-1}(y), k) \) of the restriction map is \( \tilde{\mathcal{E}}_{p,q}^{i+1} \).

Let us compute the induced map between the second pages. Applying \( R\pi_! \), we get the morphism

\[
R\pi_! \tilde{n}_\pi X : R\pi_! \tilde{k}_X \to R\pi_! \tilde{i}_! \mathcal{E}^{i+1}_{n-1}(y).
\]

Taking the \( q \)th cohomology, we get the morphism of sheaves

\[
\mathbb{R}^q R\pi_! \tilde{n}_\pi X : \mathbb{R}^q R\pi_! \tilde{k}_X \to \mathbb{R}^q R\pi_! \tilde{i}_! \mathcal{E}^{i+1}_{n-1}(y).
\]

Now taking the \( p \)th cohomologies with compact support on \( Y \), we get the map

\[
H^p_c(\{Y, \mathbb{R}^q R\pi_! \tilde{n}_\pi X\}) : E^p_{2,q} \to \tilde{\mathcal{E}}^p_{2,q}.
\]

This map is zero for \( p \neq 0 \). We claim that it is an isomorphism for \( p = 0 \). To prove it, let us write the morphism \( \mathbb{R}^q R\pi_! \tilde{n}_\pi X \), applying the proper base change, as follows:

\[
R\pi_! \tilde{n}_\pi X : R\pi_! \tilde{k}_X \to R\pi_! \tilde{i}_! \mathcal{E}^{i+1}_{n-1}(y) = \tilde{i}_! R\pi_! \tilde{i}_! \mathcal{E}^{i+1}_{n-1}(y) = i_* i^* \pi_! \pi^* \mathcal{E}^{i+1}_{n-1}(y).
\]

It follows from the construction of the base change isomorphism (for example, [KS, the Proof of Proposition 2.5.11]) that this morphism is exactly \( \eta R\pi_! \tilde{k}_X \), where \( \eta : \mathrm{id} \to i_* i^* \) is the unit of adjunction. Hence the morphism of sheaves

\[
\mathbb{R}^q R\pi_! \tilde{n}_\pi X : \mathbb{R}^q R\pi_! \tilde{k}_X \to i_* i^* \mathbb{R}^q R\pi_! \tilde{k}_X
\]

is conjugate to the identity morphism \( i_* \mathbb{R}^q R\pi_! \tilde{k}_X \to i_* \mathbb{R}^q R\pi_! \tilde{k}_X \). Hence the resulting map

\[
H^p_c(\{Y, \mathbb{R}^q R\pi_! \tilde{n}_\pi X\}) : E^p_{2,q} \to \tilde{\mathcal{E}}^p_{2,q} = H^p_c(\{Y, k\}) \oplus H^p(\{y\}, k) \oplus m_q = \tilde{\mathcal{E}}^p_{2,q}.
\]

is the restriction map. It is clearly an isomorphism for \( p = 0 \), as \( Y \) is connected.

Let us fix an integer \( n \leq N - 1 \). We have some decreasing filtrations

\[
H^0_c(X, k) = F^0 H^0_c(X, k) \supset F^1 H^0_c(X, k) \supset \cdots \supset F^{n+1} H^0_c(X, k) = 0,
\]

\[
H^0_c(\pi^{-1}(y), k) = F^0 H^0_c(\pi^{-1}(y), k) \supset F^1 H^0_c(\pi^{-1}(y), k) \supset \cdots \supset F^{n+1} H^0_c(\pi^{-1}(y), k) = 0
\]

coming from the corresponding Leray spectral sequences. Here

\[
F^i H^0_c(X, k) / F^{i+1} H^0_c(X, k) \cong E^{i,n-i}_0 = E^{i,n-i}_2,
\]

\[
F^i H^0_c(\pi^{-1}(y), k) / F^{i+1} H^0_c(\pi^{-1}(y), k) \cong \tilde{E}^{i,n-i}_0 = \tilde{E}^{i,n-i}_2.
\]

We get \( \tilde{\mathcal{E}}^i_{n} = 0 \) for \( i > 0 \). The restriction map \( H^0_c(X, k) \to H^0_c(\pi^{-1}(y), k) \) respects filtrations and is surjective as the map \( \tilde{E}^{i,n-i}_2 \to \tilde{E}^{i,n-i}_2 \) defined above is so. The kernel of the restriction map is \( H^0_c(X, k) \to H^0_c(\pi^{-1}(y), k) \) is \( F^i H^0_c(X, k) \).

[2] This claim follows from the following exact sequence for odd \( n \leq N - 1 \):

\[
H^{n-1}_c(X, k) \to H^{n-1}_c(\pi^{-1}(y), k) \to H^n_c(X \setminus \pi^{-1}(y), k) \to H^n_c(X, k) = 0.
\]

Here the last equality follows from the first of the above spectral sequences.
Let \( n \) be even and \( n \leq N - 1 \). Note that under the assumption of this case all \( E_2^{i,n-i} \) are free of finite rank for any \( i = 0, \ldots, n \). We have an exact sequence

\[
0 = H^{n-1}_c(\pi^{-1}(y), \mathbb{k}) \rightarrow H^n_c(X \setminus \pi^{-1}(y), \mathbb{k}) \rightarrow H^n_c(X, \mathbb{k}) \rightarrow H^n_c(\pi^{-1}(y), \mathbb{k})
\]

Hence \( H^n_c(X \setminus \pi^{-1}(y), \mathbb{k}) \cong F^1 H^n_c(X, \mathbb{k}) \) as the kernel of the restriction \( H^n_c(X, \mathbb{k}) \rightarrow H^n_c(\pi^{-1}(y), \mathbb{k}) \). This module is free of finite rank, as it has a finite filtration with such quotients. \( \square \)

6.5. Generators. First, we consider restrictions of cohomologies.

**Lemma 24.** Suppose that 2 is invertible in \( \mathbb{k} \).

1. If \( \tau < N < \infty \), then \( H^n_c(L^N \setminus \text{im} \chi_w, \mathbb{k}) = 0 \) for odd \( n < 2(N - \tau) - 1 \).
2. If \( \tau < N < \infty \), then the restriction map \( H^n_c(L^N, \mathbb{k}) \rightarrow H^n_c(\text{BS}_c(s) \times_K E^N, \mathbb{k}) \) induced by \( \chi_w \) is surjective for \( n < 2(N - \tau) - 2 \).
3. For any \( n \), the restriction map \( H^n_c(L, \mathbb{k}) \rightarrow H^k_c(\text{BS}_c(s), \mathbb{k}) \) induced by \( \chi_w \) is surjective.

**Proof.** (1) We denote \( i = \max I \) for brevity. We write the Leray spectral sequence for cohomologies with compact support for the fibre bundle \( \omega \) as in Lemma 22. It has the following second page:

\[
E_2^{p,q} = H^p_c(\text{BS}_c(s') \times_K E^N, H^q_c(\omega^{-1}(b), \mathbb{k})),
\]

where \( b \) is an arbitrary point of the space \( \text{BS}_c(s') \times_K E^N \), which is simply connected. To prove the last statement, first note that \( \text{BS}_c(s') \) is simply connected as a Bott tower and then consider the long exact sequence of homotopy groups for the fibre bundle \( \text{BS}_c(s') \times_K E^N \rightarrow E^N/K \) with fibre \( \text{BS}_c(s') \).

By Lemma 21, the space \( \omega^{-1}(b) \) is homeomorphic to the space \( (E^N/K) \setminus \tau^{-1}(y) \), where \( \tau : E^N/K \rightarrow E^N/C_w \rightarrow \text{im} \chi_w \) is the natural quotient map and \( y \) is an arbitrary point of \( E/C_w \).

We know that \( \tau \) is a fibre bundle with fibre \( Kc \setminus C_w^{-1}s_w = \{Kc \mid c \in C_w^{-1}s_w \} \cong \mathbb{C}P^1 \). Let us apply the Gysin sequence to this fibre bundle as in [McC, Example 5.C]. As \( \mathbb{C}P^1 \) is homeomorphic to the 2-sphere, we have an exact sequence

\[
H^n_3(E^N/C_w^{-1}s_w, \mathbb{k}) \xrightarrow{z \downarrow} H^n(E^N/C_w^{-1}s_w, \mathbb{k}) \xrightarrow{\tau^*} H^n(E^N/K, \mathbb{k})
\]

where \( z \) is some element of \( H^3(E^N/C_w^{-1}s_w, \mathbb{k}) \) such that \( 2z = 0 \). As 2 is invertible in \( \mathbb{k} \), we get \( z = 0 \). If \( n \) is less than \( 2(N - \tau) + 1 \) and is odd, then \( H^n(E^N/K, \mathbb{k}) = 0 \) by Lemma 19(2). Hence and from the above exact sequence, we get \( H^n(E^N/C_w^{-1}s_w, \mathbb{k}) = 0 \) for such \( n \). Note that \( E^N/C_w^{-1}s_w \) is simply connected and connected by an argument similar to Lemma 19(1). Applying Lemma 23 to \( \tau \), we get that \( H^3_3(\omega^{-1}(b), \mathbb{k}) = 0 \) for odd \( q < 2(N - \tau) \).

By Lemma 20, we get that \( E_2^{p,q} \) is zero except the following cases: \( p \geq 2(N - \tau); q \geq 2(N - \tau); p \) and \( q \) are both even. It is easy to note that the differentials coming to and starting from \( E_2^{p,q} \) are zero if \( a \geq 2 \) and \( p + q < 2(N - \tau) - 1 \). Hence \( E_\infty^{p,q} = E_2^{p,q} \) for \( p + q < 2(N - \tau) - 1 \) and the claim immediately follows.

(2) Let \( n < 2(N - \tau) - 2 \). If \( n \) is even, then by the first part of this lemma, we get an exact sequence

\[
H^n(L^N, \mathbb{k}) \rightarrow H^n(\text{BS}_c(s) \times_K E^N, \mathbb{k}) \rightarrow H^{n+1}_c(L^N \setminus \text{im} \chi_w, \mathbb{k}) = 0.
\]

If \( n \) is odd, then the restriction under consideration is surjective as \( H^n(\text{BS}_c(s) \times_K E^N, \mathbb{k}) = 0 \) by Lemma 20.
This result follows from the previous part and Lemma 18 and the following commutative diagram

\[
\begin{align*}
H^n(L^N, \mathbb{k}) & \xrightarrow{(\chi^N_w)^*} H^n(BS_c(s) \times_K E^N, \mathbb{k}) \\
\big| | & \\
H^n(L, \mathbb{k}) & \xrightarrow{\chi^w} H^n(BS_c(s), \mathbb{k})
\end{align*}
\]

which holds for \( n < 2(N - r) + 1 \).

Lemma 25. Let \( \mathbb{k} \) be a commutative ring of finite global dimension with invertible 2. Let \( I \) be a finite totally ordered set and \( s : I \to T(W) \) and \( w : I \cup \{-\infty\} \to W \) be arbitrary maps. Then all elements \( \Sigma(s, i, w_i)^*(h) \), where \( i \in I \cup \{-\infty\} \) and \( h \in S \), generate \( H^*_K(BS_c(s), \mathbb{k}) \) as a ring with respect to addition and the cup product.

Proof. Let us apply the induction on the length of \( s \). If \( I = \emptyset \), then \( BS_c(s) \) is a singleton and \( \Sigma(s, -\infty, w_{-\infty})^*(h) = w_{-\infty}^{-1}h \) by (13). Obviously any element of \( H^*_K(BS_c(s), \mathbb{k}) = S \) has this form.

Now suppose that \( I \neq \emptyset \). We denote \( i = \max I \) for brevity. By the Künneth formula and Lemma 18 we get

\[
H^*(L, \mathbb{k}) \cong H^*_K(BS_c(s'), \mathbb{k}) \otimes_{\mathbb{k}} S.
\]

By the inductive hypothesis, \( H^*(L, \mathbb{k}) \) is generated as a ring by elements \( p_i^j \Sigma(s', j, w_j)^*(h) \) and \( p_i^j(h) \), where \( j \in I' \cup \{-\infty\} \), \( h \in S \) and \( p_1 \) and \( p_2 \) are projections of \( L \) to its first and second component respectively.

Hence by Lemma 24(3) applied to \( \chi_{w_i} \), we get that \( H^*_K(BS_c(s), \mathbb{k}) \) is generated as a ring by elements

\[
\chi_{w_i}^* p_i^j \Sigma(s', j, w_j)^*(h) = (\Sigma(s', j, w_j) p_i \chi_{w_i})^*(h) = \Sigma(s, j, w_j)^*(h),
\]

where \( j \in I' \cup \{-\infty\} \), and \( \chi_{w_i}^* p_i^j(h) = (p_2 \chi_{w_i})^*(h) = \Sigma(s, i, w_i)^*(h) \).

\[\square\]

6.6. Copy and concentration operators. The results we have proved allow us to explain the existence of the operators of copy and concentration defined in [S1] at least for a principal ideal domain \( \mathbb{k} \) with invertible 2. Let \( \mathcal{X}_c \) denote the image of the restriction \( H^*(BS_c(s), \mathbb{k}) \to H^*(\Gamma(s), \mathbb{k}) \). We suppose that \( s = (s_1, \ldots, s_n) \) is a nonempty sequence. For any \( g \in \mathcal{X}_c(s') \), we consider the function \( \Delta g : \Gamma(s) \to S \) defined by \( \Delta g(\gamma) = g(\gamma') \). We call this element the copy of \( g \), see [S1] Section 4.2. Let \( h \in H^*_K(BS_c(s'), \mathbb{k}) \) be an element whose restriction to \( \Gamma(s') \) is \( g \). Then it is easy to understand that \( \Delta g \) is the restriction of \( \text{tr}^*(h) \) to \( \Gamma(s) \), where \( \text{tr} : BS_c(s) \to BS_c(s') \) is the truncation map defined at the beginning of Section 6.2. Indeed, let \( \gamma \in \Gamma(s) \) and \( j_\gamma : E^\infty/K \to BS_c(s) \times_K E^\infty \) and \( j_\gamma' : E^\infty/K \to BS_c(s') \times_K E^\infty \) be the maps as in Section 1.3. We get

\[
\text{tr}^*(h)(\gamma) = j_\gamma^* \text{tr}^*(h) = (\text{tr} j_\gamma)^*(h) = j_\gamma'^*(h) = g(\gamma') = \Delta g(\gamma).
\]

On the other hand, for any \( t \in \{1, s_n\} \), we consider the function \( \nabla_t g \) called the concentration of \( g \) at \( t \), see Section [S1] Section 4.2. It is defined by

\[
\nabla_t g(\gamma) = \begin{cases} 
\gamma^n(-\alpha_n)g(\gamma') & \text{if } \gamma_n = t; \\
0 & \text{otherwise,}
\end{cases}
\]

where \( \alpha_n \) is a positive root such that \( s_n = s_{\alpha_n} \). Let \( c \in S \). Then by (13), the restriction of \( \Sigma(s, i, 1)^*(c) \) to \( \gamma \in \Gamma(s) \) is given by

\[
\Sigma(s, i, 1)^*(c)(\gamma) = j_\gamma^* \Sigma(s, i, 1)^*(c) = (\Sigma(s, i, 1) j_\gamma)^*(c) = (\rho(\gamma)^{-1}/K)^*(c) = \gamma^i c.
\]
Hence we get

\[
\nabla_t g = -\frac{\sum(s, n-1, 1)^*(t\alpha_n) + \sum(s, n, 1)^*(\alpha_n)}{2}\bigg|_{\Gamma(s)} \Delta g.
\]

Finally note that to get exactly the operators from [S1], we need to consider only sequences \( s \) of simple reflections and identify compactly defined and usual Bott-Samelson varieties as well as \( K \)- and \( T \)-equivariant cohomologies.

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Financial University under the Government of the Russian Federation, 49 Leningradsky Prospekt, Moscow, Russia

E-mail address: shchigolev_vladimir@yahoo.com