Topological phases in Kitaev chain with imbalanced pairing

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We systematically study a Kitaev chain with imbalanced pair creation and annihilation, which is introduced by non-Hermitian pairing terms. Exact phase diagram shows that the topological phase is still robust under the influence of the conditional imbalance. The gapped phases are characterized by a topological invariant, the extended Zak phase, which is defined by the biorthonormal inner product. Such phases are destroyed at the points where the coalescence of groundstates occur, associating with the time-reversal symmetry breaking. We find that the Majorana edge modes also exist for the open chain within unbroken time-reversal symmetric region, demonstrating the bulk-edge correspondence in such a non-Hermitian system.

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I. INTRODUCTION

The Kitaev model is employed to describe the dynamics of spinless fermions with superconducting p-wave pairing\(^1\). The topological superconducting is demonstrated by unpaired Majorana modes exponentially localized at the ends of open Kitaev chains\(^2,3\). Intimately related to the superconducting phase, the pairing term takes the key role in the Kitaev model, which violates the conservation of fermion number but preserves the parity of the number. It is the fermionized version of the familiar one-dimensional transverse field Ising model\(^4\), which is one of the simplest solvable models exhibiting quantum criticality and demonstrating a spontaneous symmetry breaking driven quantum phase transition\(^5\). So far, most of the investigations on this model focus on the case with the equal magnitude of amplitudes for pair creation and annihilation, i.e., the balanced kitaev model. Intuitively, the balance between pair production and annihilation takes an important role in the existence of the gapped superconducting phase. When the amplitude norm of the pair production is not equal to that of annihilation, the pair state may be destroyed. A natural question is what happens when the balance is broken. Theoretically, the imbalanced pair terms will break the Hermiticity of the Hamiltonian. Nowadays, non-Hermitian Hamiltonian is no longer a forbidden regime in quantum mechanics since the discovery that a certain class of non-Hermitian Hamiltonians could exhibit the entire real spectrum\(^6,7\). The aim of this paper is to answer this question based on the non-Hermitian quantum theory.

The origin of the reality of the spectrum of a non-Hermitian Hamiltonian is the pseudo-Hermiticity of the Hamiltonian operator. Such kinds of Hamiltonians possess a particular symmetry, i.e., it commutes with the combined operator $PT$, but not necessarily with $P$ and $T$ separately. Here $P$ is a unitary operator, such as parity, translation, rotation operators etc., while $T$ is an anti-unitary operator, such as time-reversal operator. The combined symmetry is said to be unbroken if every eigenstate of the Hamiltonian is $PT$-symmetric; then, the entire spectrum is real, while is said to be spontaneously broken if some eigenstates of the Hamiltonian are not the eigenstates of the combined symmetry. The study of non-Hermitian Kitaev model\(^8,9\) and Ising model\(^10,11\) has performed within the pseudo-Hermitian framework. Also, the experimental realization of related non-Hermitian systems is presented in refs.\(^12,13\).

In this paper, we introduce an unequal real amplitude of pairing operator to describe the imbalance pair creation and annihilation in Kitaev chain. It is a non-Hermitian model with time-reversal symmetry, rather than a combined $PT$ symmetry. Exact solution shows a rich phase diagram, including gapped superconducting phases associated time-reversal symmetry and gapless phases associated broken time-reversal symmetry. There are two topological superconducting phases characterized by the extended Zak phase, which is defined in the context of biorthonormal inner product. The boundary of these two phases corresponds to the two-fold degeneracy, which is similar to the Hermitian situation. In contrast, the gapless phase arises from coalescing state in the broken symmetric regions, which indicates that the pairing imbalance in some extent can destroy the superconducting phases. We also investigate the bulk-edge correspondence in such a non-Hermitian system, via the Majorana transformation. We show that there are two types of gapped phases support different kinds of edge modes in the context of biorthonormal inner product, which demonstrate the topological invariants.

This paper is organized as follows. In Section II we present the model Hamiltonian. Based on the solutions, we investigate the phase diagram and analyze the symmetry of the ground state. In Section III we construct the Hermitian counterpart of the model and calculate the Zak phase to show the topological properties of the system. In Section IV we study the corresponding Hamiltonian of the Majorana fermions to show the existence of the bulk-edge correspondence. Finally, we give a summary and discussion in Section V.
II. NON-HERMITIAN KITAEV MODEL

We consider the following fermionic Hamiltonian on a lattice of length \( N \)
\[
\mathcal{H} = -t \sum_{j=1}^{N} (c_j^\dagger c_{j+1} + \text{H.c.}) - \mu \sum_{j=1}^{N} (1 - 2n_j) - \sum_{j=1}^{N} (\Delta_\alpha c_j^\dagger c_{j+1} + \Delta_\beta c_{j+1} c_j),
\]
where \( c_j^\dagger (c_j) \) is a fermionic creation (annihilation) operator on site \( j \), \( n_j = c_j^\dagger c_j \), \( t \) the tunneling rate, \( \mu \) the chemical potential, and \( \Delta_\alpha \) (\( \Delta_\beta \)) the strength of the \( p \)-wave pair creation and annihilation. For a closed chain, we define \( c_{N+1} = c_1 \) and for an open chain, we set \( c_{N+1} = 0 \). It turns out that Hamiltonian (1) has a rich phase diagram in its Hermitian version, i.e., \( \Delta_\alpha = \Delta_\beta \).

Before solving the Hamiltonian, it is profitable to investigate the symmetry of the system and its breaking in the eigenstates. By the direct derivation, we have \([\mathcal{T}, \mathcal{H}] = 0\), i.e., the Hamiltonian is a time reversal \( (\mathcal{T}) \) invariant, where the antilinear time reversal operator \( \mathcal{T} \) has the function \( \mathcal{T} c = -i c \). Before we consider the general non-Hermitian Kitaev model, we first highlight the key idea for a limiting case. When \( t = \mu = 0 \) and \( \Delta_\alpha = -\Delta_\beta \), the Hamiltonian (1) reduces to
\[
\mathcal{H}_0 = \Delta_\beta \sum_{j=1}^{N} (c_j^\dagger c_{j+1} + c_{j+1} c_j),
\]
which is an anti-Hermitian Hamiltonian. Obviously, all the eigenvalues are imaginary and break the \( \mathcal{T} \) symmetry, i.e., for an eigenstate \( |\phi\rangle \) with the nonzero eigenvalue \( E \), \( \mathcal{H}_0 |\phi\rangle = E |\phi\rangle \), we have
\[
\mathcal{H}_0 (\mathcal{T} |\phi\rangle) = -E (\mathcal{T} |\phi\rangle).
\]
It strongly implies that the \( \mathcal{T} \) symmetry in the present model plays the same role as \( \mathcal{PT} \) symmetry in a \( \mathcal{PT} \) pseudo-Hermitian system. It motivates further study of such a model systematically. Taking the Fourier transformation
\[
c_j = \frac{1}{\sqrt{N}} \sum_k e^{ikj} c_k
\]
for the Hamiltonian, with wave vector \( k = 2\pi m/N, m = 0, 1, 2, ..., N - 1 \), we have
\[
H = \sum_k [2 (\mu - t \cos k) c_k^\dagger c_k + i \sin k (\Delta_\beta c_{-k} c_k + \Delta_\alpha c_{-k}^\dagger c_k^\dagger)] - |\mu|,
\]
So far the procedures are the same as those for solving the Hermitian version of \( H \). To diagonalize a non-Hermitian Hamiltonian, we should introduce the Bogoliubov transformation in the complex version:
\[
A_k = i \sqrt{\Delta_\beta/\Delta_\alpha} c_k + \eta_{c_{-k}}^k,
\]
\[
\overline{A}_k = -i \sqrt{\Delta_\alpha/\Delta_\beta} c_k^\dagger + \eta_{c_{-k}}
\]
where
\[ \xi = \text{sgn}(\sin k) \sqrt{\frac{\mu - t \cos k + \xi_k}{2\xi_k}}, \]
\[ \eta = \frac{|\sin k| \sqrt{\Delta_\alpha\Delta_\beta}}{\sqrt{2\xi_k (\xi_k + \mu - t \cos k)}}. \]  
(7)

We would like to point out that this is the crucial step to solve the non-Hermitian Hamiltonian, which essentially establishes the biorthogonal bases. Obviously, complex Bogoliubov modes \( (A_k, \overline{A}_k) \) satisfy the canonical commutation relations
\[ \{A_k, A_{k'}\} = \delta_{k,k'}, \]
\[ \{A_k, \overline{A}_{k'}\} = \{\overline{A}_k, A_{k'}\} = 0, \]  
(8)
which result in the diagonal form of the Hamiltonian
\[ H = \sum_k \epsilon_k (\overline{A}_k A_k - \frac{1}{2}). \]  
(9)
Here the single-particle spectrum in each subspace is
\[ \epsilon_k = 2 \sqrt{(\mu - t \cos k)^2 + \Delta_\alpha\Delta_\beta \sin^2 k}. \]  
(10)
Note that the Hamiltonian \( H \) is still non-Hermitian due to the fact that \( \overline{A}_k \neq A_k^\dagger \). Accordingly, the eigenstates of \( H \) can be written as the form
\[ \prod_{\{k\}} \overline{A}_k |G\rangle, \]  
(11)
which constructs the biorthogonal set associated with the eigenstates
\[ \langle \overline{G} | \prod_{\{k\}} A_k \]  
(12)
of the Hamiltonian \( H^\dagger \), where \( |G\rangle \) (i.e., \( |\overline{G}\rangle \)) is the ground state of \( H \) (\( H^\dagger \)). In the following, we will investigate the phase diagram based on the properties of the solutions.

It is clear that when any one of the momentum \( k \) satisfies
\[ (\mu - t \cos k)^2 + \Delta_\alpha\Delta_\beta \sin^2 k < 0, \]  
(13)
the imaginary energy level appears in single-particle spectrum, which leads to the occurrence of complex energy level for the Hamiltonian \( H^\dagger \), and the \( \mathcal{T} \) symmetry is broken in the corresponding eigenstates. This can be seen from the properties of the single-particle spectrum and the ground states \( |G\rangle \).

Firstly, we focus on the boundary between the broken and unbroken symmetry regions, as well as between two gapped phases. In the thermodynamical limit, the boundary is a 2D surface in 3D parameter space \( (\Delta_\alpha/t, \Delta_\beta/t, \mu/t) \), which is determined by equation \( \epsilon_k = 0 \). A straightforward algebra gives the analytical expression of surfaces as the phase boundaries, which are listed in Table I and is plotted in Fig. 4 as illustration. Note that there exist two kinds of boundaries which consist of exceptional point (EP) and degeneracy point, respectively.

Secondly, according to the non-Hermitian quantum theory, the occurrence of the EP always accomplishes the \( \mathcal{T} \) symmetry breaking of an eigenstate. For the present model, the symmetry of the groundstate \( |G\rangle \) can be an indicator of the phase transition due to the fact that the groundstate energy becomes complex once the system is in the broken region. In the following, we focus on the discussion about the symmetry of \( |G\rangle \) in the different regions.

Applying the \( \mathcal{T} \) operator on the fermion operators and its vacuum state \( |\text{Vac}\rangle \), we have
\[ \mathcal{T} c_{k}^\dagger (\mathcal{T})^{-1} = c_{-k}^\dagger, \mathcal{T} c_k (\mathcal{T})^{-1} = c_{-k}, \]  
(14)
and
\[ \mathcal{T} |\text{Vac}\rangle = |\text{Vac}\rangle, \]  
(15)
which are available in the both regions. However, the coefficients \( \xi \) and \( \eta \) experience a transition as following when the corresponding single-particle level changes from real to imaginary: We have \( \xi^* = \xi \) and \( \eta^* = \eta \) for real levels and \( \xi^* = \text{sgn}(\sin k)\eta \) for the imaginary levels, respectively. This leads to the conclusion that the groundstate is not \( \mathcal{T} \) symmetric in the broken symmetric region, i.e.,
\[ \left\{ \begin{array}{ll} \mathcal{T} |G\rangle = |G\rangle, & \text{Gapped region} \\ \mathcal{T} |G\rangle \neq |G\rangle, & \text{Gapless region} \end{array} \right. \]  
(16)
It shows that the \( \mathcal{T} \) symmetry in the present model plays the same role as \( \mathcal{PT} \) symmetry in a \( \mathcal{PT} \) pseudo-Hermitian system.

As a comparison, it is noted that the phase boundary \( \Delta_\alpha = \Delta_\beta = 0 \) between two gapped phases is not an EP line but the degeneracy line. So across the \( \mu \) axis, the quantum phases transition is a conventional one as that in a Hermitian Kitaev chain. We will demonstrate this point in the next section.

| Regions | Phase | Zak phase |
|---------|-------|-----------|
| \( \Delta_\alpha/t, \Delta_\beta/t > 0, |\mu/t| < 1 \) | Gapped | \( -\pi \) |
| \( \Delta_\alpha/t, \Delta_\beta/t < 0, |\mu/t| < 1 \) | Gapped | \( \pi \) |
| \( \Delta_\alpha\Delta_\beta/t^2 > 0, \mu/t = \pm 1 \) | Gapless | \( 0 \) |
| \( |\mu/t| > 1, \mu^2 + \Delta_\alpha\Delta_\beta > t^2 \) | Gapped | \( 0 \) |
| Otherwise | Coalescing | |
FIG. 2. (Color online) Phase diagrams of the imbalanced kitaev model in $\Delta_\alpha\Delta_\beta$ plane for several representative values of (a) $\mu/t = \pm0.5$, (b) $\mu/t = \pm1$ and (c) $\mu/t = \pm1.1$. The phase descriptions and corresponding Zak phases are indicated in the regions.

III. PHASE DIAGRAM AND TOPOLOGICAL INVARIANT

According to the above analysis, the whole parameter space consists of two kinds of regions, unbroken symmetric one with fully real spectrum and broken symmetric one with complex spectrum, which is originated from the imbalanced pairing process. In this section, we focus on the former region and show that it is further divided into three kinds of sub-regions with different topological invariants. Using the Nambu representation, the Hamiltonian $\mathcal{H}$ can be expressed as

$$\mathcal{H} = \sum_k (c_{k}^\dagger, c_{-k}) \tilde{h}_k \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}$$

(17)

where the core matrix is

$$\tilde{h}_k = \begin{pmatrix} \mu - t \cos k & -i \Delta_\alpha \sin k \\ i \Delta_\beta \sin k & -\mu + t \cos k \end{pmatrix}.$$  

(18)

Matrix $\tilde{h}_k$ contains all the information of the system. (i) In the unbroken symmetric region, $\tilde{h}_k$ is a pseudo-Hermitian matrix, possessing a Hermitian counterpart

$$\tilde{\tilde{h}}_k = \begin{pmatrix} \mu - t \cos k & -i \sqrt{\Delta_\alpha \Delta_\beta} \sin k \\ i \sqrt{\Delta_\alpha \Delta_\beta} \sin k & -\mu + t \cos k \end{pmatrix}.$$  

(19)

which is a Hermitian matrix with the same real eigenvalues of $\tilde{h}_k$. (ii) In the broken symmetric region, there are several $k_c$ satisfying

$$\cos k_c = \frac{\mu t \pm \sqrt{\Delta_\alpha \Delta_\beta (\mu^2 + \Delta_\alpha \Delta_\beta - t^2)}}{t^2 - \Delta_\alpha \Delta_\beta},$$

(20)

at which $\tilde{h}_{k_c}$ is equivalent to a Jordan block. Explicitly we have

$$s \tilde{h}_{k_c} s^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

(21)

where

$$s = \begin{pmatrix} 1 & 0 \\ -\mu - t \cos k_c & i \Delta_\alpha \sin k_c \end{pmatrix}.$$  

(22)

Two eigenvectors of the Jordan block coalesce, resulting coalescing ground state. This is an exclusive feature of the non-Hermitian system. (iii) Along the $\mu$ axis, there are two nodal points at $k_c = \pm \arccos \mu$, within the region $|\mu| \leq 1$, at which the matrix reduces to

$$\tilde{h}_{k_c} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$  

(23)

Then there are two zero-energy eigenvectors which accords with the results obtained directly from the Hermitian Hamiltonian with $\Delta_\alpha = \Delta_\beta = 0$. We will see that it is the line (rather than a surface) that separates two topological phases.

In general, $\tilde{h}_k$ can be expressed as $\tilde{h}_k = d(k) \cdot \sigma$, with a 3D complex vector field

$$d_x(k) = -\frac{i}{2} (\Delta_\alpha - \Delta_\beta) \sin k, \quad d_y(k) = \frac{1}{2} (\Delta_\alpha + \Delta_\beta) \sin k, \quad d_z(k) = \mu - t \cos k,$$

(24)

where $\sigma$ represents 3D Pauli matrices.

Now we focus on the unbroken symmetry region ($\Delta_\alpha \Delta_\beta > 0$). The eigenstates of a pseudo-Hermitian Hamiltonian can construct a complete set of biorthogonal bases in association with the eigenstates of its Hermitian conjugate. For present system, eigenstates $|\psi^k_c\rangle$, $|\psi^k_{-c}\rangle$, $|\eta^k_c\rangle$ and $|\eta^k_{-c}\rangle$ of $\tilde{h}_k^\dagger$ are the biorthogonal bases of the single-particle invariant subspace, which are explicitly expressed as

$$|\psi^k_c\rangle = \begin{pmatrix} \cos \frac{\theta}{2} e^{-i\phi} \\ \sin \frac{\theta}{2} \end{pmatrix}, \quad |\psi^k_{-c}\rangle = \begin{pmatrix} \sin \frac{\theta}{2} e^{-i\phi} \\ -\cos \frac{\theta}{2} \end{pmatrix},$$

(25)

$$|\eta^k_c\rangle = \begin{pmatrix} \cos \frac{\theta}{2} e^{i\phi} \\ \sin \frac{\theta}{2} \end{pmatrix}^*, \quad |\eta^k_{-c}\rangle = \begin{pmatrix} \sin \frac{\theta}{2} e^{i\phi} \\ -\cos \frac{\theta}{2} \end{pmatrix}^*.$$
By rewriting the field $d(k)$ as
\[ d(k) = r (\cos \theta, \sin \theta \sin \phi, \sin \theta \cos \phi) \]
where
\[ r = \sqrt{(\mu - t \cos k)^2 + \Delta_\alpha \Delta_\beta \sin^2 k}, \]
\[ \cos \theta = \frac{i}{2r} (\Delta_\alpha - \Delta_\beta) \sin k, \]
\[ \tan \phi = \frac{(\Delta_\alpha + \Delta_\beta) \sin k}{2(\mu - t \cos k)}. \]

It is ready to check that biorthogonal bases \{ $|\psi^k_\lambda\rangle$, $|\eta^k_\lambda\rangle$ \} $(\lambda = \pm)$ obey the biorthogonal and completeness conditions
\[ \langle \eta^k_{\lambda'} | \psi^k_\lambda \rangle = \delta_{\lambda\lambda'} \delta_{kk'}, \quad \sum_{\lambda,k} |\psi^k_\lambda\rangle \langle \eta^k_\lambda| = 1. \]

These properties are independent of the reality of the spectrum except for the EPs. In unbroken symmetric region, $b_1$ has the fully real spectrum. To characterize the property of the energy band, we introduce the extended Zak phase
\[ Z_\pm = \int_0^{2\pi} A_k dk, \]
where the Berry connection
\[ A_k = \langle \eta^k_{\pm}\frac{\partial}{\partial k} | \psi^k_\pm \rangle = \partial_k \phi A_\phi + \partial_k \theta A_\theta, \]
with
\[ A_\phi = \langle \eta^k_{\pm} | i \partial_k | \psi^k_\pm \rangle, \quad A_\theta = \langle \eta^k_{\pm} | i \partial_\theta | \psi^k_\pm \rangle. \]

It is easy to check that
\[ Z_\pm = \left\{ \begin{array}{ll} -\pi \text{sgn} \left( \frac{\Delta_\alpha + \Delta_\beta}{i} \right), & |\mu| < 1 \\ 0, & |\mu| > 1 \end{array} \right. \]

In Fig. 2 we show the phase diagram described in Table I with several typical values of $\mu/t$, in which the corresponding Zak phases are indicated as topological invariants.

IV. BULK-EDGE CORRESPONDENCE

Based on the above analysis, it turns out that the bulk system exhibits the similar topological feature within the unbroken symmetric regions. In Hermitian systems, the existence of edge modes is intimately related to the bulk topological quantum numbers, which is referred as the bulk-edge correspondence relations. We are interested in the generalization of the bulk-edge correspondence to non-Hermitian systems. Previous works show that when sufficiently weak non-Hermiticity is introduced to topological insulator models, the edge modes can retain some of their original characteristics.

We examine the bulk-edge correspondence in the present model based on the following strategy. It is well known that there is a Hermitian counterpart for a non-Hermitian Hamiltonian within the unbroken symmetric region, in which both two Hamiltonians have the identical fully real spectrum. It allows us to find out the zero modes of the non-Hermitian Hamiltonian with the open boundary condition from its Hermitian counterpart.
mal canonical operators, $D = (D_1, \ldots, D_N, \bar{d}_1, \ldots, \bar{d}_N)$. Here, the canonical operators are defined as

$$d_j = \sqrt{\Delta_\alpha \Delta_\beta} c_j, \quad \bar{d}_j = \sqrt{\Delta_\alpha \Delta_\beta} c_j^\dagger,$$  

which satisfy the canonical relations

$$\{d_j, \bar{d}_l\} = \delta_{jl}, \{d_j, d_l\} = \{\bar{d}_j, \bar{d}_l\} = 0.$$  

Here $h_{BdG}$ is a Hermitian matrix and then one can construct a Hamiltonian

$$H_{cp} = \frac{1}{2} C^\dagger h_{BdG} C,$$  

which represents a Hermitian version of the Kitaev Hamiltonian. $H_{cp}$ possesses the identical spectrum with that of $\mathcal{H}$, and is dubbed as the Hermitian counterpart of $H_{BdG}$. This ensures that one can map some results of $H_{cp}$ to $\mathcal{H}$ directly. For instance, we have the following relation: For an arbitrary compound operator $g(C^\dagger, C)$, which satisfies

$$[g(C^\dagger, C), H_{cp}] = 0,$$  

we always have

$$[g(\bar{D}, D), \mathcal{H}] = 0.$$  

We refer the mapping $[g(C^\dagger, C), H_{cp}] \rightarrow [g(\bar{D}, D), \mathcal{H}]$ as an equivalence relation between $\mathcal{H}$ and $H_{cp}$, which will be used in the subsequent zero-mode analysis.

Now we apply the formalism to the non-Hermitian Kitaev Hamiltonian. In the rest, we only focus on the unbroken regions. Using the transformation in Eq. (34) we rewrite the Hamiltonian with the open boundary condition as

$$\mathcal{H} = -t \sum_{j=1}^{N-1} (\bar{d}_j d_{j+1} + d_{j+1} \bar{d}_j) - \mu \sum_{j=1}^{N} (1 - 2n_j)$$  

$$- \sqrt{\Delta_\alpha \Delta_\beta} \sum_{j=1}^{N-1} (\bar{d}_j \bar{d}_{j+1} + d_{j+1} d_j),$$  

where $n_j = \bar{d}_j d_j$ is the particle number operator in the context of biorthonormal inner product. The Hermitian counterpart of $\mathcal{H}$ can be obtained by replacing $(\bar{d}_j, d_j)$ with $(c_j^\dagger, c_j)$, i.e.,

$$H_{cp} = -t \sum_{j=1}^{N-1} (c_j^\dagger c_{j+1} + \text{H.c.}) - \mu \sum_{j=1}^{N} (1 - 2n_j)$$  

$$- \sqrt{\Delta_\alpha \Delta_\beta} \sum_{j=1}^{N-1} (c_j^\dagger c_{j+1} + \text{H.c.})$$  

with $n_j = c_j^\dagger c_j$. Next we concentrate on the Hamiltonian $H_{cp}$, and then apply the obtained result to $\mathcal{H}$. We induce
the Majorana operators
\[ a_j = c_j^\dagger + c_j, \quad b_j = -i\left(c_j^\dagger - c_j\right), \]
which satisfy the relations
\[ \{a_j, a_t\} = 2\delta_{jt}, \quad \{b_j, b_t\} = 2\delta_{jt}, \]
\[ \{a_j, b_t\} = 0, \quad a_j^2 = b_j^2 = 1. \]  
(41)

Then the Majorana representation of \( H_{\text{cp}} \) is
\[ H_{\text{cp}} = -\frac{1}{4} \sum_{j=1}^{N-1} \left[ (t + \sqrt{\Delta_\alpha \Delta_\beta}) b_j a_{j+1} + i(\sqrt{\Delta_\alpha \Delta_\beta}) b_{j+1} a_j + i2\mu \sum_{j=1}^{N} a_j b_j + \text{H.c.} \right]. \]
(42)

We note that the structure of the Hamiltonian is two coupled SSH chains, which is illustrated in Fig. 3.

We consider a simple case with \( \sqrt{\Delta_\alpha \Delta_\beta} = t \). In this situation the Majorana model reduces to
\[ H_{\text{cp}} = -\frac{i}{2} (t \sum_{j=1}^{N-1} b_j a_{j+1} + \sum_{j=1}^{N} \mu a_j b_j) + \text{H.c.}, \]
(43)

which has the form of the SSH model. For the case \( |\mu| < 1, t = 1 \), there are two combined Majorana fermion operators
\[ f_+ = \frac{1}{\sqrt{\Omega}} \sum_{j=1}^{N} \mu^{j-1} a_j, \quad f_- = \frac{1}{\sqrt{\Omega}} \sum_{j=1}^{N} \mu^{N-j} b_j, \]
with \( \Omega = (\mu^N - 1) / (\mu - 1) \), which obey
\[ \{f_+, f_-\} = 0, \quad f_+^2 = 1. \]
(44)

and
\[ [H_{\text{cp}}, f_\pm] = 0. \]
(45)

We can construct a standard fermionic operator from Majorana operator \( f_\pm \)
\[ f_N = \frac{1}{2} (f_+ - if_-), \]
which obeys
\[ \{f_N, f_N^\dagger\} = 1, \quad (f_N)^2 = 0. \]
(46)

Furthermore, the well-known solution of single-particle SSH chain allows us to construct a set of standard fermionic operators \( \{f_j\} \) \( j \in [1, N] \), in which only \( f_N \) has an analytical expression. The explicit expression for other \( f_j \) is not necessary for the present work, since we only concern the zero-energy mode. Accordingly, the Hamiltonian \( \mathcal{H} \) can be rewritten as
\[ H_{\text{cp}} = \sum_{j=1}^{N-1} \varepsilon_j (f_j^\dagger f_j - \frac{1}{2}) + 0 \times f_N^\dagger f_N, \]
(47)

where \( \varepsilon_j \) corresponds to the spectrum of single-particle SSH chain. Then \( f_N \) represents the zero mode, since it is a zero-energy eigen operator, satisfying
\[ [f_N, H_{\text{cp}}] = 0. \]
(48)

In contrast, there is no zero mode for \( |\mu| \gg 1 \). Similarly, we can get the same conclusion for the case with \( \mu = -1 = -\sqrt{\Delta_\alpha \Delta_\beta} \), which corresponds to another type zero mode illustrated in Fig. 3.

Fig. 4 plots the energy spectrum from Eq. (42), which indicates the coexistence of zero modes and non-zero Zak phases, demonstrating the bulk-edge correspondence.

In order to extend the obtained result to the non-Hermitian variant, we express \( f_N \) by \( \{c_j^\dagger, c_j\} \) as
\[ f_N = \frac{1}{2\sqrt{\Omega}} \sum_{j=1}^{N} \mu^{j-1} (c_j^\dagger + c_j - c_{N-j+1}^\dagger + c_{N-j+1}), \]
(49)

According to the equivalence relation between \( \mathcal{H} \) and \( H_{\text{cp}} \) from Eqs. (50) and (51), we can construct a pair of canonical operators \( \{\mathcal{F}, \mathcal{F}^\dagger\} \)
\[ \mathcal{F}_N = \frac{1}{2\sqrt{\Omega}} \sum_{j=1}^{N} \mu^{j-1} (\bar{a}_j + \bar{d}_j + \bar{a}_{N-j+1} - \bar{d}_{N-j+1}) \]
(50)

which obey
\[ \{\mathcal{F}_N, \mathcal{F}_N^\dagger\} = 1 \]
(51)

and
\[ \mathcal{F}_N = \frac{1}{2\sqrt{\Omega}} \sum_{j=1}^{N} \mu^{j-1} (\bar{a}_j + \bar{d}_j - \bar{a}_{N-j+1} + \bar{d}_{N-j+1}) \]
(52)

It shows that the existence of the zero mode. And the bulk-edge correspondence still holds for the original imbalanced Kitaev model in the unbroken region.

We are interested in the physical picture of the edge modes. We define edge-mode states as
\[ |\psi_L\rangle = \frac{1}{\sqrt{2}} (\mathcal{F} + \mathcal{F}_N^\dagger) |\text{Vac}\rangle, \]
\[ |\psi_R\rangle = \frac{1}{\sqrt{2}} (\mathcal{F} - \mathcal{F}_N^\dagger) |\text{Vac}\rangle, \]
(53)

where \( \mathcal{H} \) and \( \mathcal{F} \) satisfied
which are believed to capture the feature of the zero modes. Obviously, we have
\begin{equation}
|\psi_L\rangle = \sqrt{\frac{\Delta_\beta}{2\Omega\Delta_\alpha}} \sum_{j=1}^{N} \mu^{j-1} |j\rangle,
\end{equation}
\begin{equation}
|\psi_R\rangle = \sqrt{\frac{\Delta_\beta}{2\Omega\Delta_\alpha}} \sum_{j=1}^{N} \mu^{j-1}|N-j+1\rangle,
\end{equation}
where $|j\rangle = c_j^\dagger |\text{Vac}\rangle$ is a position state. We find that $|\psi_L\rangle$ is the reflection of $|\psi_R\rangle$ about the center at $(N-j+1)/2$ and both states are evanescent bound states at two ends. The decay rate is determined by $\mu$, while the asymmetry ratio $\Delta_\beta/\Delta_\alpha$ gives an overall factor. We would like to point out that, state $|\psi_{L,R}\rangle$ is not an eigenstate of the Hamiltonian, but the one related to the Majorana mode.

V. SUMMARY

In summary, we have analyzed a 1D non-Hermitian Kitaev model that exhibits the similar topological features of a Hermitian one, the bulk-edge correspondence within the unbroken time-reversal symmetric regions, in which the Zak phase is defined in the context of biorthonormal inner product and the edge mode is obtained in the aid of the corresponding Hermitian counterpart. It indicates that the imbalance of pair creation and annihilation in some extents, does not destroy the superconducting phases and their topological features. The bulk-edge correspondence is immune to the non-Hermitian effect. For extreme imbalance, spontaneously time-reversal symmetry breaking occurs, corresponding to the emergence of gapless phase arising from coalescing state. We also examined the profile of the edge modes in the spinless fermionic representation. It shows that the Majorana edge modes associate with bound fermionic states at two ends of a Kitaev chain. This may provide a way to detect the existence of Majorana fermions.

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