MONOIDAL RING AND CORING STRUCTURES OBTAINED FROM
WREATHS AND COWREATHS

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ABSTRACT. Let \( A \) be an algebra in a monoidal category \( \mathcal{C} \), and let \( X \) be an object in \( \mathcal{C} \). We study \( A \)-(co)ring structures on the left \( A \)-module \( A \otimes X \). These correspond to (co)algebra structures in \( \text{EM}(\mathcal{C})(A) \), the Eilenberg-Moore category associated to \( \mathcal{C} \) and \( A \). The ring structures are in bijective correspondence to wreaths in \( \mathcal{C} \), and their category of representations is the category of representations over the induced wreath product. The coring structures are in bijective correspondence to cowreaths in \( \mathcal{C} \), and their category of corepresentations is the category of generalized entwined modules. We present several examples coming from (co)actions of Hopf algebras and their generalizations. Various notions of smash products that have appeared in the literature appear as special cases of our construction.

INTRODUCTION

Let \( A \) be an algebra, and let \( C \) be a coalgebra, and suppose that we have an entwining map \( \psi \). It is well-known that the vector space \( A \otimes C \) carries a coring structure, such that the category of entwined modules is isomorphic to the category of comodules over this coring. Examples of entwining structures come from Doi-Koppinen data over a bialgebra. Doi-Koppinen data can be defined over quasi-bialgebras; we have similar results: \( A \otimes C \) is still an \( A \)-coring; however, there is one major difference from the classical theory: \( C \) is no longer a \( k \)-coalgebra. Otherwise stated, we can build an \( A \)-coring structure on \( A \otimes C \) although \( C \) is not an ordinary \( k \)-coalgebra.

The aim of this paper is to describe all possible \( A \)-(co)ring structures of the form \( A \otimes X \). We have chosen to present our results in the language of \( \mathcal{C} \)-categories, also known as module categories. The motivation for this choice was twofold. On one hand, the generality of this approach allows us to cover many constructions that are known for Hopf algebras and their generalizations. On the other hand, the naturality of the involved categorical arguments allows us to simplify some of the computations. We use the machinery developed in \( [25, 30] \), slightly improved in \( [13] \). Schauenburg \( [30] \) has observed that \( A \)-ring structures on \( A \otimes X \) (with left \( A \)-module structure given by multiplication of \( A \)) depends on two morphisms, which we will call \( \zeta \) and \( \sigma \). These morphisms have to satisfy certain conditions; these are not given in \( [30] \). We will work them out in Section 3, and we will see that they are similar to conditions that appear in the Brzeziński crossed product \( [9] \). We also discuss the dual question, namely we discuss \( A \)-coring structures on \( A \otimes X \), and show that they are determined by two morphisms \( \delta : X \to A \otimes X \otimes X \) and \( f : X \to A \) satisfying a list of compatibility conditions. In Section 5 we will restate the conditions on \( \delta, f \) (respectively \( \zeta, \sigma \) in the ring case). Actually (co)ring structures on \( A \otimes X \) correspond to (co)algebra structure on \( X \) in a suitable monoidal category \( \mathcal{T}_A^\# \).

Our notation is inspired by the well-known result that entwining structures of the form \( (A, C) \psi \), with fixed \( A \), correspond bijectively to coalgebras in Tambara’s category \( \mathcal{T}_A \). During a visit of the first author to the Wigner Research Centre for Physics in Budapest, Gabriella Bőhm pointed out that \( \mathcal{T}_A^\# \) is the monoidal category \( \text{EM}(\mathcal{C})(A) \) of endomorphisms of \( A \), viewed as a 0-cell in the Eilenberg-Moore category \( \text{EM}(\mathcal{C}) \), where \( \mathcal{C} \) - being a monoidal category - is viewed as a 2-category with one 0-cell. A similar result is that \( \mathcal{T}_A = \text{Mnd}(\mathcal{C})(A) \), where \( \text{Mnd}(\mathcal{C}) \) is the 2-category of monads.

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in $\mathcal{C}$, as introduced in [27]. A second categorial interpretation is presented in Section 4 $A$-(co)ring structures on $A \otimes X$ are in bijective correspondence to (co)wreath structures in $\mathcal{C}$, regarded again as a 2-category with one object. We also show that the category of representations of an $A$-ring of the form $A \otimes X$ is isomorphic to the category of representations of the corresponding wreath product, see [21]. The category of corepresentations of an $A$-coring of the form $A \otimes X$ is isomorphic to the category of generalized entwined modules, as introduced in [15].

We present some applications in Section 5. Using the theory of actions and coactions over a quasi-bialgebra we give examples of (co)wreaths $(A, X)$ with $X$ regarded as an object in $\mathcal{T}_A^*$ rather than $\mathcal{T}_A$. As a consequence we obtain that the generalized-(quasi) smash product algebra defined in [11] is an example of a wreath product. Also the crossed product algebra built within the monoidal category of corepresentations over a dual quasi-Hopf algebra [2] is a wreath product.

Quasi-Hopf bimodules over a quasi-bialgebra $H$ can be applied to construct a cowreath $(\mathfrak{A}, C)$ in $H \mathcal{M}$, the monoidal category of left $H$-representations. We remark that $C$ is viewed as an object in $\mathcal{T}_\mathfrak{A}$ and not in $\mathcal{T}_\mathfrak{A}$, and that the category of corepresentations over the resulting coring is isomorphic to $H \mathcal{M}_\mathfrak{C}$, the category of quasi-Hopf bimodules associated to $(\mathfrak{A}, C, H)$.

More examples can be obtained from actions and coactions of a bialgebroid. We propose an alternative way to define the crossed product algebra over a bialgebroid [5], the underlying idea is to describe this algebra as a wreath product. We also construct a coring from a Doi-Koppinen datum over a bialgebroid, compatible with a module category structure, and recover the isomorphism between the category of Doi-Koppinen modules over a bialgebroid and the category of corepresentations over a suitable coring [7]. These examples can be specified to bialgebroids coming from weak bialgebras.

Our theory can be applied to braided bialgebras; this will be the topic of the forthcoming paper [15].

After an earlier version of this paper was finished, we were informed about the following possible alternative approach, based on the description of the (co)wreath structures in a certain bicategory, leading to Theorem 4.8. Theorem 4.3 is actually a special case of Theorem 4.8. We have investigated this, and we could give a proof of Theorem 6.8, but to this end we needed Theorem 3.3, so that the two results are basically equivalent. Details are given in Section 6.

1. Preliminaries

1.1. Module categories. We assume that the reader is familiar with the basic theory of monoidal categories, and refer to [10, 20, 22] for more detail. Throughout this paper, $\mathcal{C}$ will be a monoidal category with tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and unit object $1$. We denote the identity morphism of an object $X \in \mathcal{C}$ by $1_X$. We will assume implicitly that the monoidal category $\mathcal{C}$ is strict, that is, the associativity $a$ and unit constraints $l, r$ are all identity morphisms in $\mathcal{C}$. Our results will remain valid in arbitrary monoidal categories, since every monoidal category is monoidal equivalent to a strict one, see for example [10, 21].

A right $\mathcal{C}$-category is a quadruple $(\mathcal{D}, \circ, \Psi, r)$, where $\mathcal{D}$ is a category, $\circ : \mathcal{D} \times \mathcal{C} \to \mathcal{D}$ is a functor, and $\Psi : \circ \circ (\circ \times 1) \to \circ \circ (1 \times \circ)$ and $r : \circ \circ (1 \times 1) \to 1$ are natural isomorphisms such that $(1 \circ \tt) \circ 1 = r \circ 1 \circ 1$ and the diagrams

\[
\begin{array}{ccc}
(\mathfrak{M} \circ X) \circ Y & \overset{\Psi_{\circ,Y,Z}}{\longrightarrow} & (\mathfrak{M} \circ X) \circ (Y \otimes Z) \\
\Psi_{\circ,X,Y} \circ \tt & & \tt \circ \Psi_{\circ,X,Y,Z} \\
(\mathfrak{M} \circ (X \otimes Y)) \circ Z & \overset{\Psi_{\circ,(X \otimes Y),Z}}{\longrightarrow} & \mathfrak{M} \circ ((X \otimes Y) \otimes Z)
\end{array}
\]

commute, for all $\mathfrak{M} \in \mathcal{D}$ and $X, Y, Z \in \mathcal{C}$. Obviously $\mathcal{C}$ is itself a right $\mathcal{C}$-category, with $\circ = \circ$, and $\Psi$ and $r$ the natural identities (recall that we assumed that $\mathcal{C}$ is strict). In fact, the above mentioned coherence theorem can be extended to $\mathcal{C}$-categories, and this enables us to assume throughout that $\Psi$ and $r$ are natural identities, without loss of generality. In the literature, $\mathcal{C}$-categories are also named module categories.
Let $\mathcal{D}$ be a right $\mathcal{C}$-category, and consider an algebra $A$ in $\mathcal{C}$. A right module in $\mathcal{D}$ over $A$ is an object $\mathfrak{M} \in \mathcal{D}$ together with a morphism $\nu_{\mathfrak{M}} : \mathfrak{M} \circ A \to \mathfrak{M}$ such that $\nu_{\mathfrak{M}} \circ (\text{Id}_{\mathfrak{M}} \circ \eta_A) = \rho_{\mathfrak{M}}$ and the diagram

\[
(M \circ A) \circ A \xrightarrow{\nu_{\mathfrak{M}} \circ \text{Id}_A} M \circ A \xrightarrow{\nu_{\mathfrak{M}}} M
\]

commutes. Let $\mathcal{D}_A$ be the category of right modules and right linear maps in $\mathcal{D}$ over $A$. The right module structure on $\mathfrak{M} \in \mathcal{D}_A$ will be written symbolically as

\[
\nu_{\mathfrak{M}} = \frac{M}{\mathfrak{M}} A.
\]

We can also define the dual notion of right comodule in a right $\mathcal{C}$-category $\mathcal{D}$ over a coalgebra $C$ in $\mathcal{C}$. The category of right comodules and right coilinear maps in $\mathcal{D}$ over $C$ will be denoted as $\mathcal{D}^C$.

The right comodule structure on $\mathfrak{M} \in \mathcal{D}^C$ will be written as

\[
\rho_{\mathfrak{M}} = \frac{\mathfrak{M}}{\mathfrak{M}} C.
\]

### 1.2. Rings and corings in monoidal categories

The notion of ring and coring in a monoidal category is essentially due to Pareigis [25] and Schauenburg [30]. We present a brief survey on the topic, following terminology and notation as in [13].

It is well-known that the category $\mathcal{A} \mathcal{M}_A$ of bimodules over a $k$-algebra $A$ is monoidal. A (co)algebra $A$ is called an $A$-(co)ring, see [22] for the original definition.

Let $\mathcal{D}$ be a right $\mathcal{C}$-category, and assume that both $\mathcal{C}$ and $\mathcal{D}$ have coequalizers. Take an algebra $A$ in $\mathcal{C}$, $\mathfrak{M} \in \mathcal{D}_A$ and $X \in \mathcal{A} \mathcal{C}$, with structure morphism $\mu_X : A \otimes X \to X$. We consider the coequalizer $(M \circ A, \mu_{\mathfrak{M},X})$ of the parallel morphisms $\nu_{\mathfrak{M}} \circ \text{Id}_X$ and $(\text{Id}_{\mathfrak{M}} \circ \mu_X)\Psi_{\mathfrak{M},A,X}$ in $\mathcal{D}$:

\[
(M \circ A) \circ X \xrightarrow{\nu_{\mathfrak{M}} \circ \text{Id}_X} M \circ X \xrightarrow{\mu_{\mathfrak{M},X}} M \circ_A X.
\]

For a left $A$-linear morphism $f : X \to Y$ in $\mathcal{C}$, let $\widetilde{f} : \mathfrak{M} \circ_A X \to \mathfrak{M} \circ_A Y$ the unique morphism in $\mathcal{D}$ satisfying the equation

\[
\widetilde{f} q_{\mathfrak{M},X}^A = q_{\mathfrak{M},Y}^A (\nu_{\mathfrak{M}} \circ \text{Id}_f).
\]

Take $X \xrightarrow{f} Y \xrightarrow{\widetilde{f}} Z$ in $\mathcal{A} \mathcal{C}$. It is easily verified that $\tilde{g} \tilde{f} = \tilde{g} \tilde{f}$.

Now let $g : \mathfrak{N} \to \mathfrak{M}$ in $\mathcal{D}_A$ and $Y \in \mathcal{A} \mathcal{C}$. $g : \mathfrak{M} \circ_A Y \to \mathfrak{M} \circ_A Y$ is the unique morphism in $\mathcal{D}$ such that

\[
\overline{g} q_{\mathfrak{M},Y}^A = q_{\mathfrak{N},Y}^A (g \circ \text{Id}_Y).
\]

For $M \xrightarrow{f} N \xrightarrow{g} \mathfrak{P}$ in $\mathcal{D}_A$, we have that $\overline{g} \overline{f} = \overline{g} \overline{f}$.

For $\mathfrak{M} \in \mathcal{D}_A$, $X \in \mathcal{C}$ and $Y \in \mathcal{A} \mathcal{C}$, we have canonical isomorphisms $\Upsilon_{\mathfrak{M}}, \Upsilon_{\mathfrak{M},X}$ and $\Upsilon_Y$:

- $\Upsilon_{\mathfrak{M}} : \mathfrak{M} \circ_A A \xrightarrow{\nu_{\mathfrak{M}}} \mathfrak{M}$, uniquely determined by the property $\Upsilon_{\mathfrak{M}} q_{\mathfrak{M},A}^A = \nu_{\mathfrak{M}}$;
- $\Upsilon_{\mathfrak{M},X} : \mathfrak{M} \circ_A (A \otimes X) \xrightarrow{\nu_{\mathfrak{M}}} \mathfrak{M} \circ A$, uniquely determined by the property $\Upsilon_{\mathfrak{M},X} q_{\mathfrak{M},A \otimes X}^A = (\nu_{\mathfrak{M}} \circ \text{Id}_X)\Psi_{\mathfrak{M},A,X}^{-1}$;
- $\Upsilon_Y : A \otimes_A Y \xrightarrow{\nu_{\mathfrak{M}}} Y$, uniquely determined by the property $\Upsilon_Y q_{\mathfrak{M},Y}^A = \mu_Y$.

The following properties are now easily verified:

\[
\Upsilon_{\mathfrak{M}}^{-1} = q_{\mathfrak{M},A}^A (\nu_{\mathfrak{M}} \circ \eta_A) \quad ; \quad \Upsilon_Y^{-1} = q_{\mathfrak{M},Y}^A (\eta_A \circ \text{Id}_Y);
\]

\[
\Upsilon_{\mathfrak{M},X}^{-1} = q_{\mathfrak{M},A \otimes X}^A (\nu_{\mathfrak{M}} \circ \eta_{A \otimes X} \circ \text{Id}_X).
\]

Before we are able to introduce the associativity constraint on $\mathcal{A} \mathcal{C}_A$, we need the following concepts.
**Definitions 1.1.** Let $\mathcal{D}$ be a right $C$-category and $A$ an algebra in $C$. 

(i) An object $X \in \mathcal{C}$ is called (left) $\mathcal{D}$-coflat if the functor $- \otimes X : \mathcal{D} \to \mathcal{D}$ preserves coequalizers. If $\mathcal{D} = \mathcal{C}$ then we simply say that $X$ is left coflat.

(ii) An object $Y \in A\mathcal{C}$ is called (left) $\mathcal{D}$-robust if for any $\mathfrak{M} \in \mathcal{D}$ and $X \in A\mathcal{C}_A$, the morphism $\Theta_{\mathfrak{M},X,Y} : (\mathfrak{M} \otimes X) \circ A Y \to \mathfrak{M} \circ (X \otimes_A Y)$ defined by the commutativity of the diagram

\[
\begin{array}{c}
((\mathfrak{M} \otimes X) \circ A) \circ Y & \xrightarrow{\nu_{\mathfrak{M},X,Y} \circ \text{Id}_Y} & (\mathfrak{M} \otimes X) \circ Y \\
(Id_\mathfrak{M} \otimes \text{Id}_X) \circ \psi & \xrightarrow{\Sigma_{\mathfrak{M},X,Y}} & (Id_\mathfrak{M} \otimes X) \circ \psi \\
\end{array}
\]

is an isomorphism. Here $\xi_{\mathfrak{M},X,Y} := (\text{Id}_\mathfrak{M} \otimes \psi_A) \circ \psi_{\mathfrak{M},X,Y}$. If $\mathcal{D} = \mathcal{C}$ then we simply say that $Y$ is left robust and write $\Theta_{\mathfrak{M},X,Y} := \theta_{\mathfrak{M},X,Y}$.

We denote by $\mathcal{D}_A\mathcal{C}$ the category of left $A$-modules in $\mathcal{C}$ which are left $\mathcal{D}$-robust, left robust, left $\mathcal{D}$-coflat and left coflat. We write $\mathcal{D}_A\mathcal{C} = \mathcal{A}_A\mathcal{C}$.

Note that left $\mathcal{D}$-coflatness of $A$ and left $\mathcal{D}$-robustness of $Y \in A\mathcal{C}$ are needed in order to define:

- a right module structure on $\mathfrak{M} \circ A X$ in $\mathcal{D}$ over $A$, for all $\mathfrak{M} \in \mathcal{D}_A$ and $X \in A\mathcal{C}_A$;
- a left $A$-module structure on $X \otimes_A Y$ in $\mathcal{C}$, for any $X \in A\mathcal{C}_A$;
- “canonical” inverse isomorphisms

\[
\begin{array}{c}
(\mathfrak{M} \circ A X) \circ A Y & \xrightarrow{\gamma_{\mathfrak{M},X,Y}} & \mathfrak{M} \circ (X \otimes_A Y), \\
\end{array}
\]

for $\mathfrak{M} \in \mathcal{D}_A$ and $X \in A\mathcal{C}_A$.

Assume that $A$ is left coflat and left $\mathcal{D}$-coflat. Then the category $\mathcal{D}_A\mathcal{C}_A$ is monoidal with tensor product $\otimes_A$, associativity constraint $\Sigma_{\cdot,\cdot,\cdot}$, unit object $A$, and left and right unit constraints $\Upsilon_-$ and $\Upsilon_+$. Here $\Sigma$, $\Upsilon_-$, and $\Upsilon_+$ have to be specialized to the case $\mathcal{D} = \mathcal{C}$. Moreover, the functor $\circ_A$ defines a right $\mathcal{D}_A\mathcal{C}_A$-category structure on $\mathcal{D}_A$. More details about all these concepts and results can be found in [25, 30, 13].

We are now able to define the notions of ring and coring in a monoidal category.

**Definition 1.2.** Let $\mathcal{D}$ be a right $C$-category and $A$ an algebra in $C$ which is left $\mathcal{D}$-coflat and left coflat. An $A$-(co)ring in $C$ compatible with $D$ is a (co)algebra in the monoidal category $\mathcal{D}_A\mathcal{C}_A$. For such a (co)ring a right (co)module in $\mathcal{D}$ is a (co)module over it in the right $\mathcal{D}_A\mathcal{C}_A$-category $\mathcal{D}_A$.

A (co)ring in $C$ compatible with $C$ is simply termed a (co)ring in $C$. Then a right (co)module over a (co)ring in $C$ is exactly a (co)module over it in the right $\mathcal{A}_A\mathcal{C}_A$-category $\mathcal{C}_A$.

2. $A$-(CO)RINGS OF THE FORM $A \otimes -$
Let $\mathcal{C} \in \mathcal{A}_\mathcal{C}$, as in Lemma \[\text{2.1}\]. Since $A$ is left coflat, $\mathcal{C} \otimes_A \mathcal{C}$ is a right $A$-module via $\nu_\mathcal{C} \otimes_A \mathcal{C}$, the unique morphism in $\mathcal{C}$ making the triangle in the diagram

\[
\begin{align*}
\mathcal{C} \otimes A \otimes \mathcal{C} \otimes A & \xrightarrow{\nu_\mathcal{C} \otimes \text{Id}_A} \mathcal{C} \otimes A \otimes \mathcal{C} \otimes A \\
& \xrightarrow{\text{Id}_A \otimes \nu_\mathcal{C} \otimes \text{Id}_A} \mathcal{C} \otimes A \otimes \mathcal{C} \otimes A \\
& \xrightarrow{\text{Id}_A \otimes \mu_\mathcal{C}} \mathcal{C} \otimes A \otimes \mathcal{C} \\
& \xrightarrow{\nu_\mathcal{C} \otimes A} \mathcal{C} \otimes A \otimes \mathcal{C} \\
& \xrightarrow{\text{Id}_A \otimes \mu_\mathcal{C}} \mathcal{C} \otimes A \otimes \mathcal{C} \\
& \xrightarrow{\nu_\mathcal{C} \otimes A} \mathcal{C} \otimes A \otimes \mathcal{C}
\end{align*}
\]

commutative. Since $\mathcal{C}$ is left robust, $\mathcal{C} \otimes_A \mathcal{C}$ is a left $A$-module via $\mu_\mathcal{C} \otimes_A \mathcal{C} = \text{Id}_A \otimes \mathcal{C}$, where $\theta'$ is defined in Definitions \[\text{2.2}\] and $\mu_\mathcal{C} \otimes_A \mathcal{C}$ is the unique morphism in $\mathcal{C}$ making the triangle below commutative:

\[
\begin{align*}
A \otimes \mathcal{C} \otimes \mathcal{C} & \xrightarrow{\text{Id}_A \otimes \nu_\mathcal{C} \otimes \text{Id}_A} A \otimes \mathcal{C} \otimes A \\
& \xrightarrow{\nu_\mathcal{C} \otimes A \otimes \text{Id}_A} A \otimes \mathcal{C} \otimes \mathcal{C} \\
& \xrightarrow{\text{Id}_A \otimes \mu_\mathcal{C}} A \otimes \mathcal{C} \otimes \mathcal{C} \\
& \xrightarrow{\nu_\mathcal{C} \otimes A} A \otimes \mathcal{C} \otimes \mathcal{C} \\
& \xrightarrow{\text{Id}_A \otimes \mu_\mathcal{C}} A \otimes \mathcal{C} \otimes \mathcal{C} \\
& \xrightarrow{\nu_\mathcal{C} \otimes A} A \otimes \mathcal{C} \otimes \mathcal{C}.
\end{align*}
\]

Our next aim is to characterize left $A$-linear morphisms that define a (co)multiplication on $\mathcal{C}$.

**Lemma 2.2.** Assume that $\mathcal{C} = A \otimes X \in \mathcal{A}_\mathcal{C}$, as in Lemma \[\text{2.1}\].

(i) There is a bijective correspondence between left $A$-linear morphisms $m_\mathcal{C} : \mathcal{C} \otimes A \rightarrow \mathcal{C}$ and morphisms $\zeta : X \otimes X \rightarrow A \otimes X$ in $\mathcal{C}$.

(ii) There is a bijective correspondence between left $A$-linear morphisms $\Delta_\mathcal{C} : \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C}$ in $\mathcal{C}$ and morphisms $\delta : X \rightarrow A \otimes X \otimes X$.

**Proof.** (i) The morphism corresponding to a left $A$-linear $m_\mathcal{C} : \mathcal{C} \otimes A \rightarrow \mathcal{C}$ is $\zeta := m_\mathcal{C} \otimes X^{-1}(\eta_X \otimes \text{Id}_A \otimes \text{Id}_A)$. The left $A$-linear morphism corresponding to $\zeta$ is $m_\mathcal{C} := (m_A \otimes \text{Id}_X)(\text{Id}_A \otimes \zeta) \gamma_{X,X}$. For the sake of completeness, observe that the left $A$-linearity of $m_\mathcal{C}$ is equivalent to $m_\mathcal{C} = \mu_\mathcal{C}(m_A \otimes \mu_{\mathcal{C}})$. (ii) A left $A$-linear morphism $\Delta_\mathcal{C} : \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C}$ in $\mathcal{C}$ has the form $\Delta_\mathcal{C} = T_{\mathcal{C},X}^{-1}(m_A \otimes \text{Id}_X^2 \otimes \text{Id}_X)(\text{Id}_A \otimes \delta_X)$, for some $\delta_X : X \rightarrow A \otimes X \otimes X$. $\delta_X$ is obtained from $\Delta_\mathcal{C}$ as $\delta_X = T_{\mathcal{C},X}(\eta_X \otimes \text{Id}_A)$. \[\square\]

**Lemma 2.3.** A left $A$-linear $m_\mathcal{C} : \mathcal{C} \otimes A \rightarrow \mathcal{C}$ is right $A$-linear if and only if

\[
(\mathcal{C} \otimes A)(\text{Id}_A \otimes \psi)(\zeta \otimes \text{Id}_A) = (\mathcal{C} \otimes A)(\text{Id}_A \otimes \zeta)(\psi \otimes \text{Id}_X)(\text{Id}_X \otimes \psi).
\]

A left $A$-linear $\Delta_\mathcal{C} : \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C}$ is right $A$-linear if and only if

\[
(\mathcal{C} \otimes A)(\text{Id}_A \otimes \psi)(\delta_X \otimes \text{Id}_A) = (\mathcal{C} \otimes A)(\text{Id}_A \otimes \psi)(\delta_X \otimes \text{Id}_A).
\]

**Proof.** We only prove the second assertion, the first one is left to the reader. It is not hard to see that

\[
(\text{Id}_A \otimes T_{\mathcal{C},X})(\theta'_{A,\mathcal{C},X} A_{\mathcal{C},\mathcal{C},A} = \text{Id}_A \otimes T_{\mathcal{C},X} q_{A,\mathcal{C},X}^A,
\]

and since $q_{A,\mathcal{C},X}^A$ is a coequalizer it follows that

\[
(\text{Id}_A \otimes T_{\mathcal{C},X})\theta'_{A,\mathcal{C},X} = T_{\mathcal{C},X} \text{Id}_A.
\]

In order to investigate when $\Delta_\mathcal{C}$ is right $A$-linear, we compute

\[
\nu_{\mathcal{C} \otimes A} \otimes (\text{Id}_A \otimes A) = \nu_{\mathcal{C} \otimes A} \otimes (\text{Id}_A \otimes A)(\text{Id}_A \otimes \delta_X \otimes \text{Id}_A)
\]

\[
= \nu_{\mathcal{C} \otimes A} \otimes (\text{Id}_A \otimes \delta_X \otimes \text{Id}_A)(\text{Id}_A \otimes \delta_X \otimes \text{Id}_A)(\text{Id}_A \otimes \delta_X \otimes \text{Id}_A)
\]

\[
= \nu_{\mathcal{C} \otimes A} \otimes (\text{Id}_A \otimes \delta_X \otimes \text{Id}_A)(\text{Id}_A \otimes \delta_X \otimes \text{Id}_A)(\text{Id}_A \otimes \delta_X \otimes \text{Id}_A)
\]

\[
= \nu_{\mathcal{C} \otimes A} \otimes (\text{Id}_A \otimes \delta_X \otimes \text{Id}_A)(\text{Id}_A \otimes \delta_X \otimes \text{Id}_A)(\text{Id}_A \otimes \delta_X \otimes \text{Id}_A)
\]

\[
= \nu_{\mathcal{C} \otimes A} \otimes (\text{Id}_A \otimes \delta_X \otimes \text{Id}_A)(\text{Id}_A \otimes \delta_X \otimes \text{Id}_A)(\text{Id}_A \otimes \delta_X \otimes \text{Id}_A)
\]
Lemma 2.4. Then
\[
\Delta e \nu e = \Upsilon^{-1}_{e,X} (m_A \otimes \delta_X)(m_A \otimes \delta_X)(m_A \otimes \delta_X)
\]
and
\[
\Upsilon_{e,X} \varphi_{A,e}^2 (m_A \otimes \Delta e)(m_A \otimes \delta_X)(m_A \otimes \delta_X)(m_A \otimes \delta_X)(m_A \otimes \delta_X)(m_A \otimes \delta_X)
\]
where \(m_A^2 := m_A(m_A \otimes \Delta A) = m_A(m_A \otimes m_A)\), we obtain that \(\Delta e\) is right \(A\)-linear if and only if
\[
(m_A^2 \otimes \Delta e)(m_A \otimes (m_A \otimes m_A)) = (m_A^2 \otimes \Delta e)(m_A \otimes (m_A \otimes m_A)) = (m_A \otimes \Delta e)(m_A \otimes (m_A \otimes m_A))
\]
composed both sides of (2.6) with \(\eta \otimes \Delta A \otimes \Delta e\), we find (2.4). Consequently (2.4) is equivalent to (2.0), and the result follows.

Now we have to investigate when \(m\) is associative and when \(\Delta\) is coassociative.

**Lemma 2.4.** (i) Assume that \(m : C \otimes A \rightarrow C\) is left and right \(A\)-linear, that is, \(\varphi_{C,A}^1\) is satisfied. Then \(m\) is associative if and only if
\[
(m_A \otimes C_A)(m_A \otimes C_A)(m_A \otimes C_A)(m_A \otimes C_A)(m_A \otimes C_A)(m_A \otimes C_A)
\]
and
\[
(m_A \otimes C_A)(m_A \otimes C_A)(m_A \otimes C_A)(m_A \otimes C_A)(m_A \otimes C_A)(m_A \otimes C_A)
\]
(ii) Assume that \(\Delta : C \rightarrow C \otimes A \otimes C\) is left and right \(A\)-linear, that is, \(\varphi_{C,A}^2\) is satisfied. Then \(\Delta\) is coassociative if and only if
\[
(m_A \otimes C_A)(m_A \otimes C_A)(m_A \otimes C_A)(m_A \otimes C_A)(m_A \otimes C_A)(m_A \otimes C_A)
\]
*Proof.* We will only prove assertion (ii). Recall from [25, 30, 13] that the morphism \(\gamma_{C,A}^1 : \mathcal{M} \otimes (X \otimes A) \rightarrow (\mathcal{M} \otimes A) \otimes A\) is determined by the commutativity of the following diagram:
\[
\begin{array}{c}
\mathcal{M} \otimes A \otimes (X \otimes A) \\
\xrightarrow{\varphi_{\mathcal{M},A}^1} \\
\mathcal{M} \otimes (X \otimes A) \otimes A \\
\xrightarrow{\varphi_{\mathcal{M},A}^1} \\
\mathcal{M} \otimes A \\
\xrightarrow{\varphi_{\mathcal{M},A}^1} \\
\mathcal{M} \otimes A \\
\xrightarrow{\varphi_{\mathcal{M},A}^1} \\
\mathcal{M} \otimes A \\
\xrightarrow{\varphi_{\mathcal{M},A}^1} \\
\mathcal{M} \otimes A \\
\end{array}
\]
We also have to make the observation that
\[
\Lambda_{\mathcal{M},X} := \varphi_{\mathcal{M},A}^1(\gamma_{\mathcal{M},X}^{-1} \otimes \mathcal{I}_X)(\gamma_{\mathcal{M},A}^{-1} \otimes \mathcal{I}_X)
\]
is an isomorphism in \(D\). Now we can compute that
\[
\begin{array}{c}
\Gamma_{e,e,C}^1 \varphi_{A,e}^1(\gamma_{A,e}^{-1} \otimes \mathcal{I}_X)(\gamma_{A,e}^{-1} \otimes \mathcal{I}_X)(\gamma_{A,e}^{-1} \otimes \mathcal{I}_X)(\gamma_{A,e}^{-1} \otimes \mathcal{I}_X)(\gamma_{A,e}^{-1} \otimes \mathcal{I}_X)(\gamma_{A,e}^{-1} \otimes \mathcal{I}_X)
\end{array}
\]
Similarly, we have that

\[
\Delta_\epsilon \Delta_\epsilon = \Delta_\epsilon q_{\epsilon,\epsilon}^A(\text{Id}_A \otimes \eta_A) = \Delta_\epsilon q_{\epsilon,1}^A(\text{Id}_A \otimes \eta_A) \text{Id}_X \otimes \text{Id}_X = \Delta_\epsilon q_{1,1}^A(\text{Id}_A \otimes \eta_A) \text{Id}_X \otimes \text{Id}_X, \]

where \(q_{\epsilon,\epsilon}^A, q_{\epsilon,1}^A, q_{1,\epsilon}^A, q_{1,1}^A : \text{Id}_A \otimes \text{Id}_A \rightarrow \text{Id}_A \otimes \text{Id}_A \otimes \text{Id}_A \otimes \text{Id}_A \) are the coproducts for \(\text{Id}_A \otimes \text{Id}_A \). Now, since

\[
\Delta_\epsilon \Delta_\epsilon = \Delta_\epsilon q_{\epsilon,\epsilon}^A(\text{Id}_A \otimes \eta_A) \text{Id}_X \otimes \text{Id}_X = \Delta_\epsilon q_{\epsilon,1}^A(\text{Id}_A \otimes \eta_A) \text{Id}_X \otimes \text{Id}_X, \]

it follows that \(\Delta_\epsilon\) is coassociative if and only if

\[
(\text{Id}_A \otimes \text{Id}_X^2)(\text{Id}_A \otimes \text{Id}_A \otimes \text{Id}_X \otimes \text{Id}_X)(\text{Id}_A \otimes \text{Id}_X \otimes \text{Id}_X \otimes \text{Id}_X),
\]

and this is equivalent to \(\Delta_\epsilon\).

Finally, we have to discuss when \(m_\epsilon\) has a unit and when \(\Delta_\epsilon\) has a counit. To this end, we first observe that left \(A\)-linear morphisms \(\eta_\alpha : A \rightarrow \mathcal{C}\) corresponds bijectively to morphisms \(\sigma : 1 \rightarrow A \otimes X\). Actually, \(\sigma\) can be obtained from \(\eta_\alpha\) as \(\eta_\alpha \eta_A^*\), while \(\eta_\alpha\) can be reconstructed from \(\sigma\) using the formula \((m_A \otimes \text{Id}_X)(\text{Id}_A \otimes \sigma) : A \rightarrow A \otimes X\).

In a similar way, left \(A\)-linear morphisms \(\varepsilon_\alpha : \mathcal{C} \rightarrow A\) are in bijective correspondence to morphisms \(f : X \rightarrow A\). Indeed, \(f\) can be obtained from \(\varepsilon_\alpha\) as \(f := \varepsilon_\alpha(\text{Id}_A \otimes \text{Id}_X)\). Conversely, \(\varepsilon_\alpha\) can be obtained from \(f\) using the formula \(\varepsilon_\alpha := m_\alpha(\text{Id}_A \otimes f)\) from \(\mathcal{C}\) to \(A\).

**Lemma 2.5.** (i) Let \(m_\alpha : A \rightarrow \mathcal{C}\) be a left and right \(A\)-linear associative map, as in Lemma 2.4, and \(\eta_\alpha : A \rightarrow \mathcal{C}\) left \(A\)-linear.

\(\eta_\alpha : A \rightarrow \mathcal{C}\) is right \(A\)-linear if and only if

\[
(m_A \otimes \text{Id}_X)(\text{Id}_A \otimes \sigma) = (m_A \otimes \text{Id}_X)(\text{Id}_A \otimes \psi)(\sigma \otimes \text{Id}_A),
\]

In this situation, \(\eta_\alpha\) is a unit for \(m_\alpha\) if and only if

\[
(m_A \otimes \text{Id}_X)(\text{Id}_A \otimes \zeta)(\psi \otimes \text{Id}_X)(\text{Id}_X \otimes \sigma) = \eta_\alpha \otimes \text{Id}_X,
\]

\[
(m_A \otimes \text{Id}_X)(\text{Id}_A \otimes \zeta)(\sigma \otimes \text{Id}_X) = \eta_\alpha \otimes \text{Id}_X.
\]

(ii) Let \(\Delta_\alpha : \mathcal{C} \rightarrow A\) be a left and right \(A\)-linear coassociative map, as in Lemma 2.4, and \(\varepsilon_\alpha : \mathcal{C} \rightarrow A\) a left \(A\)-linear.

\(\varepsilon_\alpha\) is a counit for \(\Delta_\alpha\) if and only if

\[
\text{Id}_A \otimes \varepsilon_\alpha(f) = \text{Id}_A \otimes f,
\]

In this situation, \(\varepsilon_\alpha\) is a counit for \(\Delta_\alpha\) if and only if

\[
(m_A \otimes \text{Id}_X)(\text{Id}_A \otimes f \otimes \text{Id}_X)(\text{Id}_X \otimes \sigma) = \eta_A \otimes \text{Id}_X,
\]

\[
(m_A \otimes \text{Id}_X)(\text{Id}_A \otimes f \otimes \text{Id}_X)(\text{Id}_X \otimes \sigma) = \eta_A \otimes \text{Id}_X,
\]

\[
(m_A \otimes \text{Id}_X)(\text{Id}_A \otimes f \otimes \text{Id}_X)(\text{Id}_X \otimes \sigma) = \eta_A \otimes \text{Id}_X,
\]
Certain monoidal category. In an earlier version of this paper we have constructed this category by

\begin{equation}
(2.11) \quad (m_A \otimes \text{Id}_X)(\text{Id}_A \otimes \psi)(\text{Id}_e \otimes f)\delta_X = \eta_A \otimes \text{Id}_X.
\end{equation}

\textit{Proof}. We will prove the second statement, and leave the first one to the reader. \( \varepsilon \) is right \( A \)-linear if and only if

\begin{equation}
(2.15) \quad m_A^2(\text{Id}_A \otimes (\text{Id}_A \otimes f))\psi = m_A^2(\text{Id}_A \otimes f \otimes \text{Id}_A).
\end{equation}

Clearly \eqref{2.15} is equivalent to \eqref{2.12}. Note that \eqref{2.12} follows from \eqref{2.15}, after we compose both sides with \( \eta_A \otimes \text{Id}_X \otimes \text{Id}_A \). Now

\[ \Upsilon_{A,X} \varepsilon \cong \delta = \Upsilon_{A,X} \varepsilon \cdot \delta \cdot \varepsilon, \]

and we obtain that \( \Upsilon_{A,X} \varepsilon \cdot \delta \cdot \varepsilon = \text{Id}_e \) if and only if \eqref{2.13} holds. In a similar way, we have that \( \Upsilon_{A,X} \varepsilon \cdot \delta \cdot \varepsilon = (m_A^2 \otimes \text{Id}_X)(A \otimes (\text{Id}_A \otimes \psi)(\text{Id}_e \otimes f)\delta_X), \)

and so \( \Upsilon_{A,X} \varepsilon \cdot \delta \cdot \varepsilon = \text{Id}_e \) if and only if \eqref{2.14} holds. \hfill \Box

\textbf{Proposition 2.6.} Let \( A \) be an algebra in a monoidal category \( C \) which is left coflat and \( X \) a left \( A \)-module in \( C \) via the multiplication \( m_A \) of \( A \). Then

(i) \( A \)-ring structures on \( A \otimes X \) in \( C \) correspond bijectively to morphisms \( \psi : X \otimes A \to A \otimes X \), \( \zeta_X : X \otimes X \to A \otimes X \) and \( \sigma : 1 \to A \otimes X \) in \( C \) such that \eqref{2.1}, \eqref{2.2}, \eqref{2.4}, \eqref{2.7}, \eqref{2.9}, \eqref{2.10} and \eqref{2.11} are satisfied.

(ii) \( A \)-coring structures on \( C = A \otimes X \) correspond bijectively to morphisms \( \psi : X \otimes A \to A \otimes X \), \( \delta_X : X \to A \otimes X \) and \( f : X \to A \) in \( C \) such that \eqref{2.1}, \eqref{2.2}, \eqref{2.4}, \eqref{2.8}, \eqref{2.12}, \eqref{2.13} and \eqref{2.14} hold.

For further use, record that \( C \) is a (co)ring in \( C \) compatible with a right \( C \)-category \( D \) if, in addition, \( A \) and \( X \) are also (left) \( D \)-coflat objects of \( C \). In both cases we refer to the \( A \)-(co)ring \( C \) as a quadruple \( C = (A \otimes X, \psi, (\delta_X)_{C}, \sigma) \) with \( \psi, \sigma, (\delta_X)_{C} \) satisfying all the conditions in Proposition \ref{2.6}.

\section{The categories \( \text{EM}(C)(A) \) and \( \text{Mnd}(C)(A) \)}

The main goal of this section is to restate the necessary and sufficient conditions for \( A \otimes - \) to be an \( A \)-(co)ring in terms of monoidal categories. More precisely, we will show that \( A \otimes X \) admits an \( A \)-(co)ring structure with the given left \( A \)-module structure if and only if \( X \) is a (co)algebra in a certain monoidal category. In an earlier version of this paper we have constructed this category by hand, inspired by the structures of \( X \) that endow \( A \otimes X \) with an \( A \)-(co)ring structure. Afterwards Gabriella Böhm pointed us that our monoidal category should be related to the Eilenberg-Moore category associated to \( C \), viewed as a 2-category in the canonical way. After some more investigations we obtained the results of this Section.

Let \( \mathcal{K} \) be a 2-category; its objects (or 0-cells) will be denoted by capital letters. 1-cell between two 0-cells \( U \) and \( V \) will be denoted as \( U \xrightarrow{f} V \), the identity morphism of a 1-cell \( f \) by \( 1_f \) and more generally, a 2-cell by \( f \xrightarrow{\beta} f' \). We also denote by \( \circ \) the vertical composition of 2-cells \( f \xrightarrow{\rho} f' \xrightarrow{\gamma} f'' \) in \( \mathcal{K}(U, V) \), by \( \circ \) the horizontal composition of 2-cells

\begin{equation}
U \xrightarrow{g} V \xrightarrow{f} W, \quad g f \xrightarrow{\varphi \circ} g f',
\end{equation}
and by $(U \xrightarrow{1_U} U, 1_U \xrightarrow{1_U} 1_U)$ the pair defined by the image of the unit functor from 1 to $K(U, U)$, where 1 is the terminal object of the category of small categories. For more detail on 2-categories, we refer the reader to [3 Ch. 7] or [29, Ch. XII].

To a 2-category $K$, we can associate a new 2-category $EM(K)$, called the Eilenberg-Moore category associated to $K$. We sketch the definition, following [21].

- 0-cells are monads in $K$, that is quadruples $(A, t, \mu, \eta)$ consisting in an object $A$ of $K$, a 1-cell $A \xrightarrow{t} A$ in $K$ and 2-cells $t \circ t \xrightarrow{\mu} t$ and $1_A \xrightarrow{\eta} t$ in $K$ such that

$$
\mu \circ (t \circ 1_A) = \mu \circ (1_t \circ t) , \mu \circ (1_t \circ \eta) = 1_t = \mu \circ (\eta \circ 1_t) ;
$$

- 1-cells are the monad morphisms, i.e., if $\mathbb{A} = (A, t, \mu, \eta)$ and $\mathbb{B} = (B, s, \mu_s, \eta_s)$ are monads in $K$ then a monad morphism between $\mathbb{A}$ and $\mathbb{B}$ is a pair $(f, \psi)$ with $A \xrightarrow{f} B$ a 1-cell in $K$ and $sf \xrightarrow{\psi} ft$ a 2-cell in $K$ such that the following equalities hold:

$$
(1_f \circ \mu_t) \circ (\psi \circ 1_t) \circ (1_s \circ \psi) = \psi \circ (\mu_s \circ 1_f) , \psi \circ (\eta_s \circ 1_f) = 1_f \circ \eta_f ;
$$

- 2-cells $(f, \psi) \xrightarrow{\theta} (g, \phi)$ are 2-cells $f \xrightarrow{\theta} g t$ in $K$ obeying the equality

$$
(1_g \circ \mu_t) \circ (\rho \circ 1_t) \circ \psi = (1_g \circ \mu_t) \circ (\phi \circ 1_t) \circ (1_s \circ \rho) ;
$$

- the vertical composition of two 2-cells $(f, \psi) \xrightarrow{\rho} (g, \phi) \xrightarrow{\rho'} (h, \gamma)$ is given by

$$
(f, \psi) \xrightarrow{\rho \circ \rho'} (h, \gamma) , \rho' \circ \rho := (1_h \circ \mu_t) \circ (\rho' \circ 1_t) \circ \rho ;
$$

- the horizontal composition of two cells

$$
\xymatrix{ \mathbb{A} 
\ar@<2pt>[rr]^-{(f, \psi)} 
\ar@<-2pt>[rr]_-{(g, \phi)} 
\ar@<2pt>[dr]^-{\rho} 
\ar@<-2pt>[dr]_-{\rho'} 
\ar@<2pt>@/_1pc/[rrr]^-{C} 
\ar@<-2pt>@/_1pc/[rrr]_-{\rho' \circ \rho} 
\ar@{}[r]|-\| & \mathbb{B} 
\ar@<2pt>^{|-|}[rr]^-{(g', \phi')} }
$$

is defined by $(g, \phi)(f, \psi) = (gf, (1_g \circ \psi) \circ (\phi \circ 1_f))$, etc. and $gf \xrightarrow{\rho \circ \rho'} g'f't$ given by

$$
\rho' \circ \rho := (1_{g'} \circ 1_f \circ \mu_t) \circ (1_{g'} \circ \rho \circ 1_t) \circ (1_{g'} \circ \psi) \circ (\rho' \circ 1_f) ;
$$

- The identity morphism of the 1-cell $(f, \psi)$ is $1_f \circ \eta_f$, and for any monad $\mathbb{A} = (A, t, \mu, \eta)$ in $K$ we have $(1_A, i_A) = (1_A, i_A)$.

It is well-known that strict monoidal categories can be viewed as 2-categories with one 0-cell. The 1-cells are the objects of the monoidal category, and the 2-cells are its morphisms. So we can consider the Eilenberg-Moore category associated to a strict monoidal category. This will be described in Proposition 3.1.

**Proposition 3.1.** The Eilenberg-Moore category $EM(C)$ of a strict monoidal category $C$ can be described as follows.

- 0-cells: algebras in $C$;

- 1-cells: $A \xrightarrow{(X, \psi)} B$ with $X$ an object of $C$ and $\psi : X \otimes B \to A \otimes X$ morphism in $C$ compatible with the algebra structure of $A$ and $B$, in the sense that

$$
\xymatrix{ X & B 
\ar@{=}[r] & B 
\ar@{=}[r] & X 
\ar@{=}[r] & X 
A & X \
A & X 
} = \xymatrix{ X & B 
\ar@{=}[r] & B 
\ar@{=}[r] & X 
\ar@{=}[r] & X 
A & X \
A & X 
} ,

where $\psi = \xymatrix{ X & B 
\ar@{=}[r] & B 
\ar@{=}[r] & X 
\ar@{=}[r] & X 
A & X \
A & X 
}$ is our diagrammatic notation for a morphism $\psi : X \otimes B \to A \otimes X$;
• 2-cells: (X, ψ) \xrightarrow{\rho} (Y, \phi) is a morphism \( \rho : X \to A \otimes Y \) in \( C \) such that

\[
(\mathsf{m}_A \otimes \text{Id}_Y)(\text{Id}_A \otimes \rho)\psi = (\mathsf{m}_A \otimes \text{Id}_Y)(\text{Id}_A \otimes \phi)(\rho \otimes \text{Id}_B);
\]

• the vertical composition of two 2-cells \( (X, \psi) \xrightarrow{\rho} (Y, \phi) \xrightarrow{\rho'} (Z, \gamma) \) is the following composition in \( C \):

\[
\rho' \circ \rho := (\mathsf{m}_A \otimes \text{Id}_Z)(\text{Id}_A \otimes \rho')\rho : X \to A \otimes Z;
\]

• the horizontal composition of 2-cells

\[
\begin{array}{c}
\xymatrix{ A \ar[r]^{Y, \rho} & B \ar[r]^{Y', \rho'} & C } \\
(X, \psi) \ar@{|->}[rr] & & (X', \psi')
\end{array}
\]

is defined by \( (Y, \phi)(X, \psi) = \begin{pmatrix} X \otimes Y, \rho \otimes \text{Id} & X \times Y \\ A \times X \end{pmatrix} \), where \( \rho = \begin{pmatrix} 1_X & \text{Id} \\ \text{Id} & 1_Y \end{pmatrix} \) etc. and \( \rho' \circ \rho : X \otimes Y \to A \otimes X' \otimes Y' \) given by

\[
\begin{array}{c}
\xymatrix{ X \ar[r]^X & Y \\
A & B \ar[l]_Y \\
A \times X' \ar[u]_\rho & , \rho' \ar[u]_\eta \ar[l]_Y \\
A \times Y' \ar[u]_\rho \ar[u]_\rho & \ar[r]_Y \\
A \times X' \ar[u]_\rho \ar[u]_\eta & 
\end{array}
\]

• For any algebra \( A \) in \( C \) we have \( 1_{A} = (1, \text{Id}_A) \) and \( i_A = 1_{\mathsf{m}_A} \), and for any 1-cell \( A \xrightarrow{(X, \psi)} A \) we have \( 1_{(X, \psi)} = 1_{\mathsf{m}_A} \otimes \text{Id}_X \).

**Proof.** The starting point is the identification of a monad in \( EM(C) \) with an algebra in \( C \). It can be easily checked that a monad \( \mathcal{A} = (\mathsf{m}_A, \eta_A, \epsilon_A) \) consisting of an object \( A \in C \) together with two morphisms \( \mathsf{m}_A : A \otimes A \to A \) and \( \eta_A : 1 \to A \) in \( C \) such that

\[
\mathsf{m}_A(\mathsf{m}_A \otimes \text{Id}_A) = \mathsf{m}_A(\text{Id}_A \otimes \mathsf{m}_A) \quad \text{and} \quad \mathsf{m}_A(\eta_A \otimes \mathsf{m}_A) = \text{Id}_A = \mathsf{m}_A(\text{Id}_A \otimes \eta_A).
\]

Thus the 0-cells of \( EM(C) \) are the algebras in \( C \). Then a monad morphism between two algebras \( A \) and \( B \) in \( C \) is a pair \( (\mathsf{m}_A, \eta_A) \) with \( X \in C \) and \( \psi : BX = X \otimes B \to XA = A \otimes X \) morphism in \( C \) such that

\[
(\mathsf{m}_A \otimes \text{Id}_X)(\text{Id}_A \otimes \psi)(\psi \otimes \text{Id}_B) = \psi(\text{Id}_X \otimes \mathsf{m}_B) \quad \text{and} \quad \psi(\text{Id}_X \otimes \eta_B) = \eta_A \otimes \text{Id}_X.
\]

In diagrammatic notation these equalities read as \( (3.2) \), so they are the required conditions for \( (X, \psi) \) to be a 1-cell in \( EM(C) \).

Now, a 2-cell \( (X, \psi) \xrightarrow{\rho} (Y, \phi) \) is a morphism \( \rho : X \to YA = A \otimes Y \) satisfying the condition

\[
\begin{array}{c}
\xymatrix{ X \ar[r]^X & B \\
A \ar[ur]_\rho & , \phi \ar[ur]_\eta \\
A \ar[u]_\rho \ar[u]_\rho & \ar[r]_Y \\
A \times Y \ar[u]_\rho \ar[u]_\eta & 
\end{array}
\]

If we rewrite this formula as a composition of maps, then we obtain \( (3.3) \). The proof of all the remaining assertions is similar. We point out that the proof of \( (3.5) \) is based on \( (3.6) \). \( \square \)
Let $U$ be 0-cell in a 2-category $K$. Then $K(U) := K(U, U)$ is a monoidal category. The objects are 1-cells $K \rightarrow K$, morphisms are 2-cells, and the tensor product is given by vertical composition of 2-cells. The unit is $1_U$, the unit 1-cell on $U$. We can apply this construction to $EM(C)$. In this way, we obtain a monoidal category $EM(C)(A)$, for any algebra $A$ in $C$. In Corollary 3.2 we provide an explicit description of this category.

**Corollary 3.2.** Let $A$ be an algebra in a monoidal category $C$. Then $EM(C)(A)$ is the following monoidal category.

- Objects are pairs $(X, \psi)$ with $X$ object in $C$ and $\psi : X \otimes A \rightarrow A \otimes X$ morphism in $C$ such that

\[ (X, \psi) \in EM(C)(A) \]

- Morphisms $\rho : (X, \psi) \rightarrow (Y, \phi)$ are morphisms $\rho : X \rightarrow A \otimes Y$ satisfying (3.6), and the composition of two morphisms $\rho$ and $\rho'$ is as in (3.4). The identity morphism $\text{Id}_{(X, \psi)}$ is $\eta_A \otimes \text{Id}_X$.

- The tensor product is defined by

\[ (X, \psi) \otimes (Y, \phi) = (X \otimes Y, \psi_{X \otimes Y}), \text{ with } \psi_{X \otimes Y} := X \otimes Y \]

and the unit object is $(1, \text{Id}_A)$. The tensor product of two morphisms $\rho$ and $\rho'$ is $\rho \otimes \rho' := \rho' \otimes \rho$ as it is defined by (3.5).

We are now ready to state and prove the main result of this Section.

**Theorem 3.3.** Let $C$ be a monoidal category, $A$ an algebra in $C$, $D$ a right $C$-category and $X$ an object of $C$. Suppose that $A, X$ are (left) $D$-coflat and left coflat objects of $C$. Then $C = A \otimes X$ has an $A$-(co)ring structure in $C$ compatible with $D$ if and only if $X$ has a (co)algebra structure in $EM(C)(A)$.

**Proof.** We show that the conditions in Proposition 2.6 are equivalent to $X$ being a (co)algebra in $EM(C)(A)$. Denoting $\psi = \left[ \begin{array}{c} X \\ A \\ A \\ X \\ X \end{array} \right]$, in diagrammatic notation, (2.1, 2.2) reduce to (3.7). Hence (2.1, 2.2) are equivalent to the fact that $(X, \psi)$ is an object of $EM(C)(A)$. Now we write

\[ \delta_X = \left[ \begin{array}{c} X \\ A \\ X \\ X \\ X \end{array} \right] : \quad f = \left[ \begin{array}{c} X \\ A \\ X \\ X \end{array} \right] : \quad \zeta_X = \left[ \begin{array}{c} X \\ A \\ X \\ X \end{array} \right] : \quad \sigma = \left[ \begin{array}{c} 1 \\ A \\ X \end{array} \right]. \]

(2.3) and (2.7) take the form

\[ (X, \psi) \otimes (Y, \phi) = (X \otimes Y, \psi_{X \otimes Y}), \text{ with } \psi_{X \otimes Y} := X \otimes Y \]

and

\[ (X, \psi) \otimes (Y, \phi) = (X \otimes Y, \psi_{X \otimes Y}), \text{ with } \psi_{X \otimes Y} := X \otimes Y \]
(2.4) and (2.8) reduce to

\[ (3.10) \]

\[
\begin{align*}
X & \xrightarrow{\mathbb{E}} A \\
A & \xrightarrow{\mathbb{E}} X \\
A & \xrightarrow{\mathbb{E}} X \\
\end{align*}
\]

and

\[
\begin{align*}
X & \xrightarrow{\mathbb{E}} A \\
X & \xrightarrow{\mathbb{E}} X \\
A & \xrightarrow{\mathbb{E}} X \\
A & \xrightarrow{\mathbb{E}} X \\
\end{align*}
\]

In a similar way, (2.9), (2.10) and (2.11) read as

\[ (3.11) \]

\[
\begin{align*}
A & \xrightarrow{\mathbb{E}} X \\
A & \xrightarrow{\mathbb{E}} X \\
A & \xrightarrow{\mathbb{E}} X \\
\end{align*}
\]

while (2.12), (2.13) and (2.14) can be written as

\[ (3.12) \]

\[
\begin{align*}
X & \xrightarrow{\mathbb{E}} A \\
X & \xrightarrow{\mathbb{E}} X \\
A & \xrightarrow{\mathbb{E}} X \\
A & \xrightarrow{\mathbb{E}} X \\
\end{align*}
\]

It can be easily checked that the first equality in (3.9) is equivalent to the fact that the map \( \mathbf{m}_X : (X, \psi) \otimes (X, \psi) \rightarrow (X, \psi) \) defined by \( \mathbf{m}_X = \zeta_X : X \otimes X \rightarrow A \xrightarrow{\otimes} X \) is a morphism in \( \mathbf{EM}(\mathbf{C})(A) \), and that (3.10) is equivalent to the fact that \( \Delta_X : (X, \psi) \rightarrow (X, \psi) \otimes (X, \psi) \) defined by \( \Delta_X = \delta_X : X \rightarrow A \xrightarrow{\otimes} X \otimes X \) is a morphism in \( \mathbf{EM}(\mathbf{C})(A) \).

We now show that \( \Delta_X \) is coassociative if and only if the second equality in (3.10) holds:

\[
\begin{align*}
\text{Id}_{(X, \psi) \otimes (X, \psi)} \circ \Delta_X & = \\
\text{Id}_{(X, \psi) \otimes (X, \psi)} \circ \Delta_X & = \\
\text{Id}_{(X, \psi) \otimes (X, \psi)} \circ \Delta_X & = \\
\end{align*}
\]

so

\[
\begin{align*}
\text{Id}_{(X, \psi) \otimes (X, \psi)} \circ \Delta_X & = \\
\text{Id}_{(X, \psi) \otimes (X, \psi)} \circ \Delta_X & = \\
\text{Id}_{(X, \psi) \otimes (X, \psi)} \circ \Delta_X & = \\
\end{align*}
\]
In a similar way, we have that

\[ \Delta_X \otimes \text{Id}_{(X, \psi)} = \begin{array}{c} X \otimes X \otimes A \otimes X \otimes X \otimes X \\ A \otimes X \otimes X \otimes X \otimes X \otimes X \end{array}, \text{ so } \Delta_X \circ (\text{Id}_{(X, \psi)} \otimes \Delta_X) = \begin{array}{c} X \otimes X \otimes A \otimes X \otimes X \otimes X \\ A \otimes X \otimes X \otimes X \otimes X \otimes X \end{array}. \]

Comparing these two equalities, we obtain that coassociativity of \( \Delta_X \) is equivalent to the second equality in (3.10). In a similar way, associativity of \( m_X \) is equivalent to the second equality in (3.9).

Finally, assume that \( \eta : (1, \text{Id}_A) \to (X, \psi) \) and \( \xi_X : (X, \psi) \to (1, \text{Id}_A) \) correspond respectively to \( \sigma : 1 \to A \otimes X \) and \( f : X \to A \) in \( C \), as in Lemma 2.5 Then \( \eta_X \) is a morphism in \( EM(C)(A) \) if and only if the first equality in (3.11) is satisfied, and \( \xi_X \) is a morphism in \( EM(C)(A) \) if and only if the first equality in (3.12) holds. Composing the equalities

\[ \xi_X \otimes \text{Id}_{(X, \psi)} = \begin{array}{c} X \otimes X \otimes A \otimes X \otimes X \otimes X \\ A \otimes X \otimes X \otimes X \otimes X \otimes X \end{array} \text{ and } \text{Id}_{(X, \psi)} \otimes \xi_X = \begin{array}{c} X \otimes X \otimes A \otimes X \otimes X \otimes X \\ A \otimes X \otimes X \otimes X \otimes X \otimes X \end{array} \]

with \( \Delta_X \), we obtain that

\[ (\xi_X \otimes \text{Id}_{(X, \psi)}) \circ \Delta_X = \begin{array}{c} X \otimes X \otimes A \otimes X \otimes X \otimes X \\ A \otimes X \otimes X \otimes X \otimes X \otimes X \end{array} \text{ and } (\text{Id}_{(X, \psi)} \otimes \xi_X) \circ \Delta_X = \begin{array}{c} X \otimes X \otimes A \otimes X \otimes X \otimes X \\ A \otimes X \otimes X \otimes X \otimes X \otimes X \end{array}. \]

Therefore, \( (\xi_X \otimes \text{Id}_{(X, \psi)}) \circ \Delta_X = \text{Id}_{(X, \psi)} \) if and only if the second equality in (3.12) holds, and \( (\text{Id}_{(X, \psi)} \otimes \xi_X) \circ \Delta_X = \text{Id}_{(X, \psi)} \) if and only if the third equality in (3.12) holds. In a similar manner we can show that the unit property for \( \eta_X = \sigma \) is equivalent to the second and the third equality in (3.11). \( \square \)

It is well-known fact in classical Hopf algebra theory that particular examples of \( A \)-(co)ring structures of the form \( A \otimes X \) can be obtained in the situation when \( X \) is a \( (co)\)algebra entwined with \( A \). We end this section by showing that this situation occurs in the particular case when \( \zeta_X = \eta_A \otimes m_X \), \( \sigma = \xi_A \otimes \xi_X \), \( f = \zeta_A \) for some \( m_X : X \otimes X \to X \), \( \eta_X : 1 \to X \) in \( C \), respectively when \( \delta_X = \xi_A \otimes \Delta_X \), \( f = \xi_A \xi_X \) for some morphism \( \Delta_X : X \to X \otimes X \), \( \xi_X : X \to 1 \) in \( C \). Actually, if this is the case then

\[ (3.13) \quad m_X = (\text{Id}_A \otimes m_X)\gamma_{e,X} : \eta_X = \text{Id}_A \otimes \eta_X, \]

and respectively

\[ (3.14) \quad \Delta_X = \gamma_{e,X}^{-1}(\text{Id}_A \otimes \Delta_X) : \xi_X = \text{Id}_A \otimes \xi_X. \]

**Proposition 3.4.** Let \( A, X \) be left coflat objects in \( C \) such that \( A \) carries an algebra structure in \( C \) and \( \eta_X : \text{Id}_X \) is a monomorphism, for any \( Y \in C \).

(i) \( (\xi_X, m_X, \eta_X) \), with \( m_X \) and \( \eta_X \) defined by (3.13), is an \( A \)-ring if and only if \( (X, m_X, \Delta_X) \) is an algebra in \( C \) and there exists \( \psi : X \otimes A \to A \otimes X \) such that

\[ \psi(\text{Id}_X \otimes m_A) = (\psi \otimes \text{Id}_A)(\text{Id}_A \otimes \psi)(\eta_A \otimes \text{Id}_X) ; \quad \psi(\text{Id}_X \otimes \eta_A) = \eta_A \otimes \text{Id}_X ; \quad \psi(m_X \otimes \text{Id}_A) = (\text{Id}_X \otimes \psi)(\psi \otimes \text{Id}_X)(\text{Id}_A \otimes m_X) ; \quad \psi(\eta_X \otimes \text{Id}_A) = \text{Id}_A \otimes \eta_X. \]
(ii) \((\mathcal{C}, \Delta_X, \varepsilon_X)\), with \(\Delta_X\) and \(\varepsilon_X\) as in \((\ref{3.11})\), is an \(A\)-co-ring if and only if \((X, \Delta_X, \varepsilon_X)\) is a coalgebra in \(\mathcal{C}\) and there exists a morphism \(\psi : X \otimes A \to A \otimes X\) in \(\mathcal{C}\) such that

\[
\psi(Id_X \otimes m_A) = (\psi \otimes Id_A)(Id_A \otimes \psi)(m_A \otimes Id_X), \quad \psi(Id_X \otimes \eta_A) = \eta_A \otimes Id_X,
\]

\[
(\psi \otimes \Delta_X)(\Delta_X \otimes Id_A) = (\Delta_A \otimes \Delta_X)\psi, \quad (\Delta_X \otimes \varepsilon_A)\psi = \varepsilon_X \otimes Id_A.
\]

**Proof.** This is a consequence of Proposition \((\ref{2.6})\). The first two equalities in (i) and (ii) coincide, and are imposed by Lemma \((\ref{2.4})\). Then \((\ref{2.3})\), the condition that \(\Delta_X\) is right \(A\)-linear, simplifies to the third condition in (i). Right \(A\)-linearity of \(\Delta_X\) is equivalent to \((\ref{2.4})\), which reduces to the third equality in (i). The fourth equality in (i) is equivalent to \((\ref{2.9})\), which is equivalent to right \(A\)-linearity of \(\eta_A\). Similarly, the fourth equality in (ii) is equivalent to \((\ref{2.12})\), the condition that is needed to make \(\varepsilon_X\) right \(A\)-linear.

If \(m_X\) is given by \((\ref{3.13})\), then \(m_X\) is associative if and only if

\[
(\eta_A \otimes Id_X)m_X(m_X \otimes Id_X) = (\eta_A \otimes Id_X)m_X(Id_X \otimes m_X).
\]

Using the assumption that \(\eta_A \otimes Id_X\) is monic, we obtain that \(m_X\) is associative if and only if \(m_X\) is associative.

Similarly, if \(\Delta_X\) is defined by \((\ref{3.13})\), then \((\ref{3.8})\) reduces to

\[
Id_A \otimes (Id_X \otimes \Delta_X)\Delta_X = Id_A \otimes (\Delta_X \otimes Id_X)\Delta_X,
\]

and therefore the coassociativity of \(\Delta_X\) implies the coassociativity of \(\Delta_X\). Conversely, if \(\Delta_X\) is coassociative then the above equality and the naturality of \(\otimes\) imply that

\[
(\eta_A \otimes Id_{X^{op}})(Id_X \otimes \Delta_X)\Delta_X = (\eta_A \otimes Id_{X^{op}})(\Delta_X \otimes Id_X)\Delta_X.
\]

By assumption, \(\eta_A \otimes Id_{X^{op}}\) is monic, so \(\Delta_X\) is coassociative.

Finally, it is easy to see that the counit property of \(\varepsilon_X\) (with respect to \(\Delta_X\)) implies the counit property for \(\Delta_X\) (with respect to \(\Delta_X\)). Conversely, the assumption that \(\eta_A \otimes Id_X\) is monic implies that the counit property of \(\Delta_X\) follows from the counit property of \(\Delta_X\).

The equivalence of the unit property of \(\eta_X\) with respect to \(m_X\) and the unit property of \(\eta_X\) with respect to \(m_X\) follows using similar arguments. \(\square\)

**Remark 3.5.** The assumption that \(\eta_A \otimes Id_X\) is monic is not needed in the two converse implications in Proposition \((\ref{3.3})\) if \(X\) is a (co)algebra in \(\mathcal{C}\) satisfying the four conditions in (i) and (ii), then \(\mathcal{C}\) is always an \(A\)-(co)ring. For the direct implications, it suffices to assume that \(\eta_A \otimes Id_X\) and \(\eta_A \otimes Id_{X^{op}}\) are monic.

We finish this Section with a monoidal interpretation of Proposition \((\ref{3.3})\). First we recall from \((\ref{27})\) that we can associate a 2-category \(\text{Mnd}(\mathcal{K})\) to any 2-category \(\mathcal{K}\). The 0-cells and 1-cells of \(\text{Mnd}(\mathcal{K})\) are the same as those of \(\text{EM}(\mathcal{K})\), that is, 0-cells are monads in \(\mathcal{K}\) and 1-cells are monad morphisms.

The 2-cells are defined in a different way: if \((f, \psi)\) and \((g, \phi)\) are 1-cells \(A = (A, t, \mu, \eta) \to B = (B, s, \mu_s, \eta_s)\), then a 2-cell \((f, \psi) \overset{\rho}{\rightarrow} (g, \phi)\) is a so-called monad transformation, that is a 2-cell \(f \overset{\rho}{\rightarrow} g\) in \(\mathcal{K}\) such that

\[
(\rho \circ 1_f) \circ \psi = \phi \circ (1_g \circ \rho).
\]

The vertical composition of 2-cells is given by vertical composition in \(\mathcal{K}\) and the horizontal composition of 2-cells

\[
\begin{array}{ccc}
A & \overset{(f, \psi)}{\longrightarrow} & B \\
\downarrow{\rho} & & \downarrow{(g, \phi)} \\
(f', \psi') & \overset{\rho'}{\longrightarrow} & (g', \phi')
\end{array}
\]

is

\[
gf \overset{\rho' \circ \rho}{\Longrightarrow} g'f', \quad \rho' \circ \rho := (1_{g'} \circ \rho) \circ (\rho' \circ 1_f),
\]

where \((g, \phi)(f, \psi) = (gf, (1_g \circ \psi) \circ (\phi \circ 1_f))\), etc. We also have that \(1_{(f, \psi)} = 1_f\) and \((1_A, i_A) = (1_A, 1_f, i_A)\), for any monad \(A = (A, t, \mu, \eta)\) in \(\mathcal{K}\). In Proposition \((\ref{3.4})\) we describe the 2-category \(\text{Mnd}(\mathcal{C})\) corresponding to a 2-category with one 0-cell, that is a monoidal category \(\mathcal{C}\). We omit the proof, as it is similar to the proof of Proposition \((\ref{3.1})\).
Proposition 3.6. Let $C$ be a monoidal category. The 0-cells and 1-cells of $\text{Mnd}(C)$ coincide with those of $EM(C)$, see Proposition [77]. A 2-cell $(X,\phi) \xrightarrow{\rho} (Y,\phi)$ is a morphism $\rho : X \rightarrow Y$ such that $\phi(\rho \otimes \text{Id}_Y) = (\text{Id}_A \otimes \rho)\psi$. Vertical composition of 2-cells is given by composition of morphisms in $C$. The horizontal composition of two 2-cells

$$
\begin{array}{c}
A \xrightarrow{(X,\phi)} B \xrightarrow{(Y,\phi)} C
\end{array} = \begin{array}{c}
\begin{array}{c}
X \otimes Y \\
A \\
Y \\
X
\end{array}
\end{array},
$$

and $\rho' \circ \rho := \rho \circ \rho' : X \otimes Y \rightarrow X' \otimes Y'$. Furthermore, for any algebra $A$ in $C$ we have $1_A = (1, \text{Id}_A)$ and $i_A = \text{Id}_A$, and for any 1-cell $A \xrightarrow{(X,\psi)} A$ we have $1_{(X,\psi)} = \text{Id}_X$.

As the reader might expect, the A-(co)ring structures in Proposition 3.4 are related to the monoidal structure of $\text{Mnd}(C)(A)$. These structures can be also viewed in $EM(C)(A)$ via the monoidal functor $F : \text{Mnd}(C)(A) \rightarrow EM(C)(A)$ that acts as the identity on objects and sends a morphism $f$ in $\text{Mnd}(C)(A)$ to $F(f) = \eta_{\text{Id}_A} \circ f$. $F$ comes from the 2-functor $E : \text{Mnd}(K) \rightarrow EM(K)$ which is the identity on objects and 1-cells and sends a 2-cell $(f,\psi) \xrightarrow{\rho} (g,\phi)$ in $\text{Mnd}(C)$ to $(f,\psi)^{1(1\otimes\eta)}_{X,\psi} \xrightarrow{\rho} (g,\phi)$, a 2-cell in $EM(K)$. Now the proof of Proposition 3.7 is left to the reader as a straightforward exercise.

Proposition 3.7. Let $A, X$ be left coflat objects in $C$, and assume that $A$ carries an algebra structure in $C$ such that $\eta_{\text{Id}_A}$ is monic, for any $Y \in C$.

(i) $(A, m_A, \eta_A)$, with $m_A$ and $\eta_A$ defined by (3.15), is an $A$-ring if and only if $(X, m_X, \eta_X)$ is an algebra in $\text{Mnd}(C)(A)$.

(ii) $(A, \Delta_A, \xi_A)$, with $\Delta_A$ and $\xi_A$ as in (3.17), is an $A$-coring if and only if $(X, \Delta_X, \xi_X)$ is a coalgebra in $\text{Mnd}(C)(A)$.

4. Wreath products and categories of (co)representations

The monoidal category $\text{Mnd}(C)(A)$ appears already in the work of Tambara [35], where it is termed the category of transfer morphisms. Inspired by this terminology, and by the paper [30], we introduce the notation $\mathcal{T}_A = \text{Mnd}(C)(A)$, and, by analogy, $\mathcal{T}^F_A = EM(C)(A)$. It is known that algebras in $\mathcal{T}_A$ are in bijective correspondence with cross algebra product structures, see for example [14, Proposition 2.1]. The aim of this Section is to show that algebras in $\mathcal{T}^F_A$ are in bijective correspondence with the so-called wreath products. We will also discuss coalgebras in $\mathcal{T}^F_A$. Recall that a wreath in a 2-category $K$ is a monad in $EM(K)$. According to [21], a wreath is a monad $\mathcal{A} = (A, t, \mu, \eta)$ in $K$ together with a 1-cell $A \xrightarrow{s} A$ and 2-cells $\xrightarrow{\psi} st$, $1_A \xrightarrow{\sigma} st$ and $ss \xrightarrow{\zeta} st$ satisfying the following conditions:

\begin{align*}
(1.1) & \quad (1_s \otimes \mu) \circ (\psi \circ 1_t) \circ (1_t \circ \psi) = \psi \circ (\mu \circ 1_s), \quad \psi \circ (\eta \circ 1_s) = 1_s \circ \eta; \\
(1.2) & \quad (1_s \otimes \mu) \circ (\psi \circ 1_t) \circ (1_t \circ \sigma) = (1_s \otimes \mu) \circ (\sigma \circ 1_t); \\
(1.3) & \quad (1_s \otimes \mu) \circ (\psi \circ 1_t) \circ (1_t \circ \psi) = (1_s \circ \mu) \circ (\psi \circ 1_s); \\
(1.4) & \quad (1_s \otimes \mu) \circ (\psi \circ 1_t) \circ (1_t \circ \psi) = (1_s \circ \mu) \circ (\psi \circ 1_s); \\
(1.5) & \quad (1_s \otimes \mu) \circ (\psi \circ 1_t) \circ (1_t \circ \sigma) = 1_s \circ \eta; \\
(1.6) & \quad (1_s \otimes \mu) \circ (\psi \circ 1_t) \circ (1_t \circ \psi) \circ (\sigma \circ 1_s) = 1_s \circ \eta.
\end{align*}

Let us introduce the dual notion. A triple $(C, C \xrightarrow{t} C, tt \xrightarrow{\delta} t, C \xrightarrow{f} 1_C)$ is called a comonad if $1_t \circ \delta \circ \delta = (\delta \circ 1_t) \circ \delta$ and $(f \circ 1_t) \circ \delta = 1_t = (1_t \circ f) \circ \delta$.

A comonad in $EM(K)$ is called a covreath. It can be described as a monad $\mathcal{A} = (A, t, \mu, \eta)$ in $K$ equipped with a 1-cell $A \xrightarrow{s} A$ and 2-cells $\xrightarrow{\psi} st$, $\xrightarrow{\delta} st$ and $\xrightarrow{\zeta} ts$ such that
A bijective correspondence between right representations of $A$ in the definition of a wreath are morphisms as we have seen in the proof of Proposition 3.1, giving a monad in $\mathcal{C}$. Proof. Since $\Upsilon$ is a (co)algebra in $\mathcal{T}_A^\#$. Consequently, (co)ring structures of the form $A \otimes X$ studied in Sections 2 and 3 are in bijective correspondence with (co)wreath structures in $\mathcal{C}$. Proposition 4.1. A (co)wreath in a monoidal category $\mathcal{C}$ is a pair $(A, X)$, where $A$ is an algebra in $\mathcal{C}$, an $X$ is a (co)algebra in $\mathcal{T}_A^\#$. Consequently, (co)ring structures of the form $A \otimes X$ studied in Sections 2 and 3 are in bijective correspondence with (co)wreath structures in $\mathcal{C}$.

Proof. As we have seen in the proof of Proposition 3.1 giving a monad in $\mathcal{C}$ is equivalent to giving an algebra $A$ in $\mathcal{C}$. Moreover, a $1$-cell $\circlearrowright$ is an object $X$ of $\mathcal{C}$ and the required 2-cells $\psi$, $\sigma$, $\zeta$ in the definition of a wreath are morphisms $\psi : X \otimes A \rightarrow A \otimes X$, $\sigma : 1 \rightarrow A \otimes X$, $\zeta : X \otimes X \rightarrow A \otimes X$ in $\mathcal{C}$. It then comes out that (4.11) is $\psi$ of $\mathcal{C}$, (4.12) is the first equality in (4.11), (4.3) is the first equality in (4.9), (4.4) is the second equality in (4.9), (4.5) is the last equality in (4.11) and (4.6) is the second equality in (4.11). The dual statement about cowreaths can be proved in a similar way, we leave the details to the reader.

Now we discuss (co)representations of (co)algebras defined by (co)wreaths. Since the computations are rather lengthy, we decided to divide them into several lemma.

If $\mathcal{D}$ is a right $\mathcal{C}$-category then a right module over an $A$-ring $\mathcal{C}$ is an algebra for the monad $\otimes A$ on the category $\mathcal{D}_A$. Explicitly, it is a right module $M$ in $\mathcal{D}$ over $A$ together with a morphism $\nu_M : \mathcal{M} \otimes A \mathcal{C} \rightarrow \mathcal{M}$ in $\mathcal{D}_A$ which is associative and unital modulo the $\mathcal{C}$-category structure of $\mathcal{D}$ and the ring structure of $\mathcal{C}$.

Lemma 4.2. Let $\mathcal{D}$ be a right $\mathcal{C}$-category and $(A, X)$ a wreath in $\mathcal{C}$ such that the $A$-coring $\mathcal{C} = A \otimes X$ is compatible with the $\mathcal{C}$-category structure of $\mathcal{D}$. Then a right $\mathcal{C}$-module is an object $M$ in $\mathcal{D}_A$ (with structure morphism $\nu_A^M : \mathcal{M} \otimes A \rightarrow \mathcal{M}$) equipped with a right $X$-action $\nu_X^M : \mathcal{M} \otimes X \rightarrow \mathcal{M}$ such that

\[
\begin{align*}
(4.12) & \quad \nu_X^M (\nu_A^M \circ \text{Id}_X) (\text{Id}_\mathcal{M} \circ \psi) = \nu_A^M (\nu_X^M \circ \text{Id}_A) ; \\
(4.13) & \quad \nu_X^M (\nu_X^M \circ \text{Id}_X) = \nu_X^M (\nu_X^M \circ \text{Id}_A) (\text{Id}_\mathcal{M} \circ \zeta) ; \\
(4.14) & \quad \nu_X^M (\nu_A^M \circ \text{Id}_A) (\text{Id}_\mathcal{M} \circ \sigma) = \text{Id}_\mathcal{M}.
\end{align*}
\]

Proof. Since $\mathcal{T}_{\mathcal{M}, X} : \mathcal{M} \otimes_A (A \otimes X) \rightarrow \mathcal{M} \otimes X$ is an isomorphism, any morphism $\nu_M^\mathcal{C} : \mathcal{M} \otimes_A (A \otimes X) \rightarrow \mathcal{M}$ in $\mathcal{D}$ is completely determined by a morphism $\nu_M^A : \mathcal{M} \otimes X \rightarrow \mathcal{M}$ in $\mathcal{D}$. Then $\nu_M^A$ is right $A$-linear if and only if (4.12) holds. Moreover $\nu_M^\mathcal{C}$ is associative if and only if (4.13) is satisfied, and $\nu_M^\mathcal{C}$ is unital if and only if (4.14) is fulfilled. Verification of the details is left to the reader.

This description of right representations of the ring $A \otimes X$ will allow us to show that there is a bijective correspondence between right representations of $A \otimes X$ and representations of the wreath product $A \#_{\psi, \zeta} X$, see Theorem 4.3. Recall from [21] that if $(A, s, \psi, \sigma, \zeta)$ is a wreath in a 2-category $\mathcal{C}$ then $st$ with

\[
\begin{align*}
st & \quad 1_s \otimes \psi, \zeta, \sigma \quad stt \quad 1_s \otimes \psi, \zeta, \sigma \quad stt \quad \psi \quad stt \quad 1_s \otimes \psi, \zeta, \sigma \quad stt \quad 1_s \otimes \psi, \zeta, \sigma
\end{align*}
\]

and $1_A \circlearrowright st$ is a monad in $\mathcal{K}$, called the wreath product of $A$ and $X$. If $\mathcal{K} = \mathcal{C}$ is a monoidal category, and $(A, X)$ is a wreath in $\mathcal{C}$ then $A \#_{\psi, \zeta} X$, the wreath product of $A$ and $X$, is $A \otimes X$
with multiplication

\[ (4.15) \]

and unit \( \sigma : 1 \to A \otimes X \).

**Remarks 4.3.** (i) Observe that Brzeziński products [9] are particular examples of wreath products. Namely, they are wreath products for which the unit morphism \( \sigma \) has the form \( \eta_A \otimes \iota \), for some \( \iota : 1 \to X \) morphism in \( C \).

(ii) Let \( A\#_{\psi, \zeta, \sigma} X \) be a wreath product in a monoidal category \( C \). It is easy to show that \( \psi \) and \( \zeta \) can be recovered from the multiplication

\[ (4.16) \]

in \( A\#_{\psi, \zeta, \sigma} X \), namely

\[ (4.17) \]

Furthermore, in \( A\#_{\psi, \zeta, \sigma} X \) we have the identities

\[ (4.17) \]

A long but straightforward computation shows that if \( A \otimes X \) carries an algebra structure in \( C \) with unit \( \sigma \) then \( A \otimes X \) is a wreath product with the same unit as \( A \otimes X \) if and only if the two conditions above are fulfilled. To this end, we define \( \psi, \zeta \) as in (4.16) and then show that (3.7), (3.9) and (3.11) are fulfilled, and that the original multiplication of \( A \otimes X \) coincides with the one on \( A\#_{\psi, \zeta, \sigma} X \). We leave all these details to the reader.

(iii) This characterization of a wreath product algebra allows us to show that there exist algebras of the form \( A \otimes X \) which are not wreath product algebras.

Let \( H \) be a \( k \)-bialgebra, let \( A \otimes H \) be an algebra in the monoidal category of \( H \)-bimodules. Now we consider the right version of the \( L-R \) smash product \( A \otimes H \), as introduced in [24, Proposition 2.1]. As a vector space, \( A \otimes H = \mathcal{A} \otimes \mathcal{A} \), with multiplication

\[(u \otimes \varphi)(u' \otimes \varphi') = u_{(0)}u'_{(0)} \otimes (\varphi \cdot u'_{(1)})u_{(-1)} \cdot \varphi'), \]
for \( u, u' \in \mathcal{A} \) and \( \varphi, \varphi' \in \mathcal{A} \), and unit \( 1_{\mathcal{A}} \). Then \( \mathcal{A} \) is an associative unital \( k \)-algebra, and a simple inspection shows that (4.17) is not satisfied. We can conclude that \( \mathcal{A} \otimes \mathcal{A} \) is not a wreath product in the category of \( k \)-vector spaces, \( k \mathcal{M} \).

Now we can prove the main result of this Section.

**Theorem 4.4.** Let \( \mathcal{D} \) be a right \( \mathcal{C} \)-category and \((\mathcal{A}, X)\) a wreath in \( \mathcal{C} \) such that the \( \mathcal{A} \)-ring \( \mathcal{E} = A \otimes X \) is compatible with the \( \mathcal{C} \)-category structure of \( \mathcal{D} \). Then the category of right representations in \( \mathcal{D} \) over the \( \mathcal{A} \)-coring \( \mathcal{C} \) is isomorphic to the category of right modules in \( \mathcal{D} \) over the wreath product \( A \#_{\psi, \zeta, \sigma} X \).

**Proof.** We use the characterization of a right representation over \( \mathcal{E} \) presented in Lemma 4.2. First observe that a right \( \mathcal{E} \)-module \( \mathfrak{M} \) is a right module in \( \mathcal{D} \) over \( A \#_{\psi, \zeta, \sigma} X \) with

\[
\begin{array}{c}
\mathfrak{M} A X = \mathfrak{M} A X
\end{array}
\]

Indeed, we have that

\[
\begin{array}{c}
\mathfrak{M} A X = \mathfrak{M} A X
\end{array}
\]

as needed, cf. (4.16). Moreover, it follows from (4.14) that this action is unital. Conversely, if \( \mathfrak{M} \) is a right module in \( \mathcal{D} \) over \( A \#_{\psi, \zeta, \sigma} X \) via \( \nu_{\mathfrak{M}} \) then the actions on \( \mathfrak{M} \) defined by

\[
\nu_{\mathfrak{M}}^A := \nu_{\mathfrak{M}}(Id_{\mathfrak{M}} \otimes (m_A \otimes Id_X)(Id_A \otimes \sigma)) : \mathfrak{M} \otimes A \to \mathfrak{M}
\]

and

\[
\nu_{\mathfrak{M}}^X := \nu_{\mathfrak{M}}(Id_A \otimes \eta_A \otimes Id_X) : \mathfrak{M} \otimes X \to \mathfrak{M}
\]

satisfy (4.12)(4.14), so \( \mathfrak{M} \) is a right \( \mathcal{E} \)-module. We leave it to the reader to verify that these constructions are inverse to each other. \( \square \)

Now we will focus on the dual situation. We need some preliminary results first.

**Lemma 4.5.** Let \( \mathfrak{M} \in \mathcal{D}_A \) and let \( \rho_{\mathfrak{M}}^X : \mathfrak{M} \to \mathfrak{M} \otimes X \) and \( \rho_{\mathfrak{M}}^\mathcal{E} : \mathfrak{M} \to \mathfrak{M} \otimes \mathcal{A} \mathcal{C} \) be morphisms in \( \mathcal{D} \) such that \( \rho_{\mathfrak{M}}^X = \Upsilon_{\mathfrak{M}, X} \rho_{\mathfrak{M}}^\mathcal{E} \). Then \( \rho_{\mathfrak{M}}^\mathcal{E} \) is right \( \mathcal{A} \)-linear if and only if

\[
(4.18) \quad \rho_{\mathfrak{M}}^\mathcal{E} \nu_{\mathfrak{M}} = (\nu_{\mathfrak{M}} \otimes Id_X)(Id_{\mathfrak{M}} \otimes \psi)(\rho_{\mathfrak{M}}^\mathcal{E} \otimes Id_A).
\]

**Proof.** Recall from [25, 30, 13] that \( \mathfrak{M} \otimes \mathcal{A} \mathcal{C} \) has a right module structure in \( \mathcal{D} \) over \( \mathcal{A} \) given by the unique morphism \( \nu_{\mathfrak{M} \otimes \mathcal{A} \mathcal{C}} : (\mathfrak{M} \otimes \mathcal{A} \mathcal{C}) \otimes \mathcal{A} \to \mathfrak{M} \otimes \mathcal{A} \mathcal{C} \) in \( \mathcal{D} \) satisfying \( \nu_{\mathfrak{M} \otimes \mathcal{A} \mathcal{C}}(\rho_{\mathfrak{M} \otimes \mathcal{A} \mathcal{C}}^\mathcal{E} \otimes Id_A) = \rho_{\mathfrak{M} \otimes \mathcal{A} \mathcal{C}}^A(Id_{\mathfrak{M}} \otimes \nu_{\mathcal{C}}). \)

We then have

\[
\nu_{\mathfrak{M} \otimes \mathcal{A} \mathcal{C}}^\mathcal{E}(\rho_{\mathfrak{M} \otimes \mathcal{A} \mathcal{C}}^\mathcal{E} \otimes Id_A) = \nu_{\mathfrak{M} \otimes \mathcal{A} \mathcal{C}}^\mathcal{E}(\Upsilon_{\mathfrak{M}, X}^{-1} \otimes Id_A))(\rho_{\mathfrak{M} \otimes \mathcal{A} \mathcal{C}}^\mathcal{E} \otimes Id_A)
\]

\[
\begin{align*}
\nu_{\mathfrak{M} \otimes \mathcal{A} \mathcal{C}}^\mathcal{E}(\rho_{\mathfrak{M} \otimes \mathcal{A} \mathcal{C}}^\mathcal{E} \otimes Id_A) & = \nu_{\mathfrak{M} \otimes \mathcal{A} \mathcal{C}}^\mathcal{E}((\rho_{\mathfrak{M} \otimes \mathcal{A} \mathcal{C}}^\mathcal{E} \otimes Id_A)((Id_{\mathfrak{M}} \otimes \eta_A \otimes Id_X) \otimes Id_A)(\rho_{\mathfrak{M} \otimes \mathcal{A} \mathcal{C}}^\mathcal{E} \otimes Id_A)) \\
& = \rho_{\mathfrak{M} \otimes \mathcal{A} \mathcal{C}}^A(Id_{\mathfrak{M}} \otimes \nu_{\mathcal{C}})(Id_{\mathfrak{M}} \otimes (\eta_A \otimes Id_X \otimes Id_A))(\rho_{\mathfrak{M} \otimes \mathcal{A} \mathcal{C}}^\mathcal{E} \otimes Id_A)
\end{align*}
\]

\[
\begin{align*}
\rho_{\mathfrak{M} \otimes \mathcal{A} \mathcal{C}}^\mathcal{E}(\rho_{\mathfrak{M} \otimes \mathcal{A} \mathcal{C}}^\mathcal{E} \otimes Id_A) & = \rho_{\mathfrak{M} \otimes \mathcal{A} \mathcal{C}}^A(Id_{\mathfrak{M}} \otimes \nu_{\mathcal{C}})((\Upsilon_{\mathfrak{M}, X}^{-1} \otimes Id_A))(\rho_{\mathfrak{M} \otimes \mathcal{A} \mathcal{C}}^\mathcal{E} \otimes Id_A).
\end{align*}
\]

Since \( \rho_{\mathfrak{M}}^\mathcal{E} \nu_{\mathfrak{M}} = \Upsilon_{\mathfrak{M}, X}^{-1} \rho_{\mathfrak{M}}^\mathcal{E} \nu_{\mathfrak{M}} \) it follows that \( \nu_{\mathfrak{M} \otimes \mathcal{A} \mathcal{C}}^\mathcal{E} \) is right \( \mathcal{A} \)-linear if and only if

\[
\rho_{\mathfrak{M} \otimes \mathcal{A} \mathcal{C}}^\mathcal{E} \nu_{\mathfrak{M}} = \Upsilon_{\mathfrak{M}, X}^{-1} \rho_{\mathfrak{M}}^\mathcal{E} \nu_{\mathfrak{M}}(\rho_{\mathfrak{M}}^\mathcal{E} \otimes Id_A).
\]

It is clear that this condition is equivalent to (4.15). \( \square \)
Lemma 4.6. Under the assumptions of Lemma 4.5, assume that $\rho_{2n}^X$ is right $A$-linear, so that $\hat{\rho}_{2n}^X$ is coassociative if and only if

\begin{equation}
(\rho_{2n}^X \circ \text{Id}_X)(\rho_{2n}^X) = (\nu_{2n} \circ \text{Id}_X)(\rho_{2n}^X \circ \text{Id}_X).
\end{equation}

Proof. Consider $\Gamma_{2n,X,Y}$ and $\Lambda_{2n,X}$ as defined in the proof of Lemma 2.4

On one hand, we compute

$$
\begin{align*}
\hat{\rho}_{2n}^X \circ \text{Id}_X &= \rho_{2n}^X \circ \text{Id}_X = q_{2n,A,X}(\rho_{2n}^X \circ \text{Id}_X)(\rho_{2n}^X \circ \text{Id}_X) \\
&= q_{2n,A,X}(\rho_{2n}^X \circ \text{Id}_X)((\rho_{2n}^X \circ \text{Id}_X)(\rho_{2n}^X \circ \text{Id}_X)) \\
&= q_{2n,A,X}(\rho_{2n}^X \circ \text{Id}_X)((\rho_{2n}^X \circ \text{Id}_X)(\rho_{2n}^X \circ \text{Id}_X)) \\
&= \Lambda_{2n,X}(\rho_{2n}^X \circ \text{Id}_X)(\rho_{2n}^X \circ \text{Id}_X) = \Gamma_{2n,X,Y}(\Gamma_{2n,X,Y}^{-1} \circ \text{Id}_X)(\rho_{2n}^X \circ \text{Id}_X).
\end{align*}
$$

On the other hand, we have

$$
\begin{align*}
\Gamma_{2n,X,Y}^{-1} \circ \text{Id}_X &= \Gamma_{2n,X,Y}^{-1} \circ \text{Id}_X \\
&= \gamma_{2n,A,X}(\rho_{2n}^X \circ \text{Id}_X)(\rho_{2n}^X \circ \text{Id}_X) \\
&= \gamma_{2n,A,X}(\rho_{2n}^X \circ \text{Id}_X)((\rho_{2n}^X \circ \text{Id}_X)(\rho_{2n}^X \circ \text{Id}_X)) \\
&= \gamma_{2n,A,X}(\rho_{2n}^X \circ \text{Id}_X)((\rho_{2n}^X \circ \text{Id}_X)(\rho_{2n}^X \circ \text{Id}_X)) \\
&= \Gamma_{2n,X,Y}(\Gamma_{2n,X,Y}^{-1} \circ \text{Id}_X)(\rho_{2n}^X \circ \text{Id}_X).
\end{align*}
$$

It follows that $\rho_{2n}^X$ is coassociative if and only if

$$
(\rho_{2n}^X \circ \text{Id}_X)(\rho_{2n}^X) = (\Gamma_{2n,X,Y}(\Gamma_{2n,X,Y}^{-1} \circ \text{Id}_X)(\rho_{2n}^X \circ \text{Id}_X)).
$$

Now

$$
(\Gamma_{2n,X,Y}(\Gamma_{2n,X,Y}^{-1} \circ \text{Id}_X)(\rho_{2n}^X \circ \text{Id}_X)) = \nu_{2n} \circ \text{Id}_X \circ \text{Id}_X,
$$

so we can conclude that $\rho_{2n}^X$ is coassociative if and only if (4.19) holds.

The above results lead us to our next definition.

Definition 4.7. Let $(A, X)$ be a cowreath in a monoidal category $C$ and let $D$ be a right $C$-category. A right generalized entwined module in $D$ over $(A, X)$ is a right module $M$ in $D$ over $A$ together with a morphism $\rho_{2n}^X : M \to M \otimes X$ in $D$ satisfying (4.18), (4.19) and

$$
\nu_{2n}(\text{Id}_A \otimes f) \rho_{2n}^X = \text{Id}_M.
$$

$D(\psi, \delta, f)^X$ is the category of right generalized entwined modules in $D$ over $(A, X)$ with morphisms $\theta : M \to M$ in $D_A$ satisfying

$$
\rho_{2n}^X \theta = (\nu_{2n} \circ \text{Id}_X)(\theta \circ \text{Id}_X).
$$

Lemmas 4.5 and 4.6 provide the following description of the category corepresentations over a coring of the form $A \otimes X$.

Theorem 4.8. Let $D$ be a right $C$-category and $(A, X)$ a cowreath in $C$ such that $A$ and $X$ are (left) $D$-coflat and left coflat objects of $C$. Then the category $D^C$ of right corepresentations in $D$ over the A-coring $C = A \otimes X$, is isomorphic to $D(\psi, \delta, f)^X$.

Proof. The isomorphism is given the functor $F : D(\psi, \delta, f)^X \to D^C$ defined as follows. For $M \in D(\psi, \delta, f)^X$, $F(M) = M$ as modules in $D$ over $A$, with right $C$-comodule structure $\rho_{2n}^X := \Gamma_{2n,X,Y}^{-1} \circ \text{Id}_X$. Note that $\Gamma_{2n,X,Y}^{-1} \circ \text{Id}_X$ is $\text{Id}_M$ if and only if (4.20) is satisfied.
5. Examples of wreaths and cowreaths arising from actions and coactions of Hopf algebras and their generalizations

In this final Section, we discuss a series of examples, coming from quasi-bialgebras, dual quasi-bialgebras, bialgebroids and weak bialgebras.

5.1. Quasi-bialgebras. Our first aim is to provide examples of (co)wreaths of the form \((A,X)\) with \(X\) as a 0-cell in \(T_A^\#\) rather than in \(T_A\). Such examples can be produced using actions and coactions of quasi-bialgebras and their duals. For the definition of a quasi-bialgebra \(H\) we invite the reader to consult [20, 22]. We note that we adopt the following convention: the tensor components of the reassociator \(\Phi\) of \(H\) are denoted by capital letters,

\[
\Phi = X^3 \otimes X^1 \otimes X^2 \in H \otimes H \otimes H
\]

and the components of the inverse \(\Phi^{-1}\) are denoted by small letters,

\[
\Phi^{-1} = x^1 \otimes x^2 \otimes x^3 \in H \otimes H \otimes H.
\]

Let \(H\) be a quasi-bialgebra, and let \((\mathfrak{A}, \rho, \Phi_\rho)\) be a right \(H\)-comodule algebra, as defined in [18].

For the right \(H\)-coaction \(\rho\) on \(\mathfrak{A}\), we use the notation \(\rho(a) = a_{(0)} \otimes a_{(1)} \in \mathfrak{A} \otimes H\). Let \(A\) be an \(H\)-bimodule algebra, that is an algebra in the monoidal category of \(H\)-bimodules. According to [11], \(\mathfrak{A}^\# \mathfrak{A}\) is a left \(H\)-module algebra, that is an algebra in the monoidal category of left \(H\)-modules \(\mathcal{M}\). The multiplication is given by the formula

\[
(a^\# \varphi)(a^\# \varphi') = a^\#_{(0)} \varphi x^3 \rho \rho (\varphi' \cdot \varphi^\#),
\]

where \(\Phi_\rho^{-1} = \xi \cdot x^3 \otimes x^2 \otimes x^1\) is the inverse of \(\Phi_\rho\) in the tensor product algebra \(\mathfrak{A} \otimes H \otimes H\) and denotes the right action of \(H\) on \(\mathfrak{A}\), the unit is \(1_{\mathfrak{A}}^\# 1_A\), and the action of \(H\) on \(A = \mathfrak{A}^\# \mathfrak{A}\) is given by \(h \cdot (a^\# \varphi) = a^\# h \cdot \varphi\). \(\mathfrak{A}^\# \mathfrak{A}\) is called the generalized quasi-smash product of \(\mathfrak{A}\) and \(A\). In the case where \(\mathcal{A} = B\) is a right \(H\)-module algebra, considered as an \(H\)-bimodule algebra with trivial left \(H\)-action obtained by restriction of scalars via the counit \(\varepsilon\), \(\mathfrak{A}^\# B\) reduces to the right generalized smash product of \(\mathfrak{A}\) and \(B\), as introduced in [11].

**Proposition 5.1.** Let \(H\) be a quasi-bialgebra, \((\mathfrak{A}, \rho, \Phi_\rho)\) a right \(H\)-comodule algebra and \(A\) an \(H\)-bimodule algebra. Then \((A, \mathfrak{A})\) is a wreath in \(\mathcal{M}\) with

- \(\mathfrak{A}\) considered as a left \(H\)-module in a trivial way;
- \(\psi : A \otimes \mathfrak{A} \to \mathfrak{A} \otimes A\), \(\psi(\varphi \otimes a) = a_{(0)} \otimes \varphi \cdot a_{(1)}\);
- \(\zeta : A \otimes A \to \mathfrak{A} \otimes \mathfrak{A}\), \(\zeta(\varphi \otimes \varphi') = \varphi x^3 \otimes (\varphi' \cdot \varphi^\#);\)
- \(\sigma : k \to \mathfrak{A} \otimes A\), \(\sigma(1) = 1_{\mathfrak{A}} \otimes 1_A\).

Furthermore, the resulting wreath product is the generalized quasi-smash product \(\mathfrak{A}^\# A\).

**Proof.** The associativity constraint on \(\mathcal{M}\) is given by left action by the tensor components of \(\Phi\). In several situations below, we have left \(H\)-modules with trivial action, and then we can freely omit the parentheses. With this observation in mind, we have the following.

1) \((\mathfrak{A}, \rho, \Phi_\rho)\) follows from the fact that \(\rho\) is an algebra morphism and \(A\) is a right \(H\)-module;

2) the first equality in (5.9) comes out as

\[
\xi x^3 \rho = a_{(0)} \otimes (\varphi \cdot \varphi^\# a_{(1)}) = a_{(0)} \otimes (\varphi \cdot a_{(1)} \varphi^\#),
\]

and follows from the coassociativity of the \(H\)-coaction \(\rho\) on \(\mathfrak{A}\);

3) the second equality in (5.9) reduces to

\[
(\varphi x^3 \rho)_{(0)} \varphi x^3 \rho \otimes (\varphi' \cdot \varphi^\# (\xi \cdot x^3 \rho)_{(1)}) = (\varphi \cdot \varphi' \cdot \varphi^\# x^3 \rho)_{(0)} \varphi x^3 \rho \otimes (\varphi' \cdot \varphi^\# x^3 \rho)_{(1)},
\]

and follows from the coassociativity of the multiplication on \(A\) in \(H \otimes \mathcal{M}_H\) (which is modulo the conjugation by \(\Phi\)), and from the 3-cocycle condition on \(\Phi_\rho\);

4) \((\mathfrak{A}, \rho, \Phi_\rho)\) is trivially satisfied.

Using (1.15), we can easily verify that the corresponding wreath product \(\mathfrak{A}^\# \psi \zeta \sigma A\) is precisely the generalized quasi-Hopf smash product \(\mathfrak{A}^\# A\). \(\square\)
Corollary 5.2. Let $H$ be a quasi-bialgebra, let $(\mathfrak{A}, \rho, \Phi_\rho)$ be a right $H$-comodule algebra and let $B$ be a right $H$-module algebra. Then $(\mathfrak{A}, B)$ is a wreath in $\mathfrak{k}M$ and the corresponding wreath product is the right generalized smash product of $\mathfrak{A}$ and $B$.

Proof. This follows from the comments preceding Proposition 5.1. Observe that the wreath structure of $(\mathfrak{A}, B)$ is precisely as in the statement of Proposition 5.1 and that the resulting wreath product is a $k$-algebra since the left $H$-action on $B$ is trivial.

In [21] Example 3.3 it is shown that Sweedler’s crossed product of Hopf algebras [33] is a particular example of a wreath product. For quasi-Hopf algebra we don’t have yet such a construction. Nevertheless, it can be considered in the dual case [22], and as we will next see it is a particular case of a wreath product as well.

We end this subsection with two examples of cowreaths $(A, X)$ for which $X$ has to be considered in $\mathcal{T}_{A}^H$.

Proposition 5.3. Let $H$ be a quasi-bialgebra, let $(\mathfrak{A}, \rho, \Phi_\rho)$ be a right $H$-comodule algebra and let $C$ be an $H$-bimodule coalgebra, that is a coalgebra in the monoidal category of $H$-bimodules. Then $(\mathfrak{A}, C)$ is a cowreath in $\mathfrak{k}M$ via the following structure:

- $\mathfrak{A}$ is a left $H$-module via the counit of $H$;
- $\psi : C \otimes \mathfrak{A} \rightarrow \mathfrak{A} \otimes C$, $\psi(c \otimes a) = a_{(0)} \otimes c \circ a_{(1)}$;
- $\delta : C \rightarrow \mathfrak{A} \otimes C \otimes C$, $\delta(c) = \bar{X}_\rho^1 \otimes c \bar{X}_\rho^2 \otimes c \bar{X}_\rho^3$, where $\Delta(c) = c_{\underline{1}} \otimes c_{\underline{2}}$ is our Sweedler notation for the comultiplication on $C$;
- $f : C \rightarrow A$, $f(c) = \underline{\varepsilon}(c) 1$, where $\underline{\varepsilon}$ is the counit of $C$.

The category of right corepresentations of the induced $\mathfrak{A}$-coring $\mathfrak{A} \otimes C$ in $\mathfrak{k}M$ is isomorphic to the category $\mathfrak{H}_{\mathcal{M}_{\mathcal{A}}}$ of quasi-Hopf $(H, \mathfrak{A})$-bimodules over $C$.

Proof. $\psi$ is left $H$-linear since $C$ is an $H$-bimodule. (2.1, 2.2) follow from the fact that $\rho : \mathfrak{A} \rightarrow \mathfrak{A} \otimes H$ is an algebra morphism in $\mathfrak{k}M$. The map $\delta$ is left $H$-linear because $\underline{\Delta}$ is left $H$-linear and the left $H$-action on $\mathfrak{A}$ is trivial.

Write $\Phi_\rho = \bar{X}_\rho^1 \otimes \bar{X}_\rho^2 \otimes \bar{X}_\rho^3$. The first equation in (5.1) comes out as

\begin{equation}
\bar{X}_\rho^1 a_{(0)} \otimes c \bar{X}_\rho^2 a_{(1)} \otimes c_2 \bar{X}_\rho^3 = a_{(0)} \bar{X}_\rho^1 \otimes c_{\underline{1}} \cdot a_{\underline{1}} \cdot \bar{X}_\rho^2 \otimes c_2 \cdot a_{\underline{1}} \cdot \bar{X}_\rho^3,
\end{equation}

for all $a \in \mathfrak{A}$ and $c \in C$, and follows because of the coassociativity of $\rho$.

The second equation in (5.1) takes the form

\begin{equation}
\bar{X}_\rho^1 \bar{Y}_\rho^1 \otimes \left( X^1 \cdot c_{(\underline{2})} \cdot (\bar{X}_\rho^2)^1 \bar{Y}_\rho^2 \otimes (X^2 \cdot c_{(\underline{2})} \cdot (\bar{X}_\rho^3)^1 \bar{Y}_\rho^3 \otimes X^3 \cdot c_2 \cdot \bar{X}_\rho^3) \right) = \bar{X}_\rho^1 (\bar{Y}_\rho^1)^{\underline{1}} \otimes \left( c_{\underline{1}} \cdot \bar{X}_\rho^2 (\bar{Y}_\rho^1)^{\underline{1}} \otimes (c_{(\underline{2})} \cdot (\bar{X}_\rho^3)^1 \bar{Y}_\rho^3 \otimes c_{(\underline{2})} \cdot (\bar{X}_\rho^3)^1 \bar{Y}_\rho^3) \right),
\end{equation}

where $\Phi_\rho = \bar{Y}_\rho^1 \otimes \bar{Y}_\rho^2 \otimes \bar{Y}_\rho^3$ is a second copy of $\Phi_\rho$, and follows by applying the coassociativity of $\underline{\Delta}$ and the 3-cocycle condition on $\Phi_\rho$.

Finally, it follows from the fact that $\underline{\Delta}$ is left $H$-linear that $f$ is a morphism in $\mathfrak{k}M$. (5.11) follows immediately from the normality of $\Phi_\rho$ and the counit property of $\underline{\Delta}$, we leave the verification of these details to the reader.

We now prove the second assertion. The monoidal category $\mathfrak{k}M$ of left modules over a quasi-bialgebra $H$ has coequalizers: the coequalizers of two parallel morphisms $f, g : M \rightarrow N$ in $\mathfrak{k}M$ is the pair $(\text{Coker}(f - g), q)$, where $q : N \rightarrow \text{Coker}(f - g)$ is the canonical surjection. Let $A$ be an algebra in $\mathfrak{k}M$, and take $M \in (\mathfrak{k}M)_A$ and $N \in A(\mathfrak{k}M)$. The tensor product $M \otimes_A N$ in $\mathfrak{k}M$ is the quotient of $M \otimes N$ over the subobject of $M \otimes N$ spanned by the elements of the form

\begin{equation}
m \cdot a \otimes n - X^1 \triangleright m \otimes (X^2 \cdot a) \triangleright (X^3 \triangleright n)
\end{equation}

with $m \in M$, $a \in A$ and $n \in N$. It is also clear that all objects of the category $\mathfrak{k}M$ are left and right coflat.

Note that giving a left $\mathfrak{A}$-module in $\mathfrak{k}M$ is equivalent to giving a left $H$-module $M \in \mathfrak{k}M$ which is also a left $\mathfrak{A}$-module in $\mathfrak{M}$, and such that $\mathfrak{A}(hm) = h(\mathfrak{A}m)$, for all $a \in \mathfrak{A}$, $h \in H$ and $m \in M$.

In a similar way, a right $\mathfrak{A}$-module in $\mathfrak{k}M$ is a left $H$-module that is also a right $\mathfrak{A}$-module such that $h(\mathfrak{A}a) = (hm)a$, for all $a \in \mathfrak{A}$, $h \in H$ and $m \in M$. It follows that a module with the structure
of a left and right $\mathfrak{A}$-module in $H\mathcal{M}$ is an $\mathfrak{A}$-bimodule in $H\mathcal{M}$ if it is an $\mathfrak{A}$-bimodule in $k\mathcal{M}$. In other words, $M$ is an $\mathfrak{A}$-bimodule in $H\mathcal{M}$ if and only if $M$ is an $(\mathfrak{A} \otimes H, \mathfrak{A})$-bimodule in $k\mathcal{M}$, where $\mathfrak{A} \otimes H$ has the tensor product algebra structure in $k\mathcal{M}$. It is also easy to see that the tensor product over $\mathfrak{A}$ in $H\mathcal{M}$ is nothing else than the usual tensor product over $\mathfrak{A}$ in $k\mathcal{M}$ endowed with the $H$-module structure given by the comultiplication $\Delta$ of $H$. Summarizing our results, we have that $\mathfrak{A}(H\mathcal{M})_\mathfrak{A} \cong \mathfrak{A} \otimes H\mathcal{M}_\mathfrak{A}$ are monoidal categories with tensor product taken over $\mathfrak{A}$ in $k\mathcal{M}$.

Applying Theorem 1.3, we obtain that the categories $(H\mathcal{M})^\mathfrak{A}$ and $H\mathcal{M}(\psi, \delta, f)^\mathfrak{A}$ are isomorphic. The objects of $H\mathcal{M}(\psi, \delta, f)^\mathfrak{A}$ are $(H, \mathfrak{A})$-bimodules $\mathfrak{M}$ together with a left $H$-linear map $\rho_{\mathfrak{M}}^\mathfrak{A} : \mathfrak{M} \ni m \mapsto m_{[0]} \otimes m_{[1]} \in \mathfrak{M} \otimes \mathfrak{C}$ satisfying (4.18) and $\varepsilon(m_{[i]})m_{[0]} = m$. More precisely, the left $H$-linearity means that $\gamma^\mathfrak{A}(h \cdot m) = h_1 \cdot m_{[0]} \otimes h_2 \cdot m_{[1]}$, for all $h \in H$ and $m \in \mathfrak{M}$. (4.18) says that

$$\rho_{\mathfrak{M}}^\mathfrak{A}(m \cdot a) = m_{[0]} \cdot a_{[0]} \otimes m_{[1]} \cdot a_{[1]},$$

for all $m \in \mathfrak{M}$ and $a \in \mathfrak{A}$. Keeping the monoidal structure of $H\mathcal{M}$ in mind, (4.19) reduces to

$$X^1 \cdot m_{[0]} \otimes X^2 \cdot m_{[1]} = m_{[0]} \cdot \tilde{X}^1_{\rho} \otimes m_{[1]} \cdot \tilde{X}^2_{\rho} \otimes X^3,$$

for all $m \in \mathfrak{M}$. The morphisms in $H\mathcal{M}(\psi, \delta, f)^\mathfrak{A}$ are the $(H, \mathfrak{A})$-bimodule morphisms that are right $C$-colinear, so $H\mathcal{M}(\psi, \delta, f)^\mathfrak{A}$ is the category of quasi-Hopf $(H, \mathfrak{A})$-bimodules over $C$, $H\mathcal{M}_\mathfrak{A}^C$. 

Every right $H$-module coalgebra is an $H$-bimodule coalgebra, with trivial left $H$-action. If we apply Proposition 5.3 to a right $H$-module coalgebra, then we obtain the following result.

**Corollary 5.4.** Let $H$ be a quasi-Hopf algebra, let $(\mathfrak{A}, \rho, \Phi)$ be a right $H$-comodule algebra and let $C$ be a right $H$-module coalgebra. Then, with the structure as in Proposition 7.5 (\(\mathfrak{A}, C\)) is a corepresentation over the induced $\mathfrak{A}$-coring is isomorphic to the category of right Doi-Hopf modules $M(H)^\mathfrak{A}_C$ as introduced in [12].

**5.2. Dual quasi-Hopf algebras.** It was shown in [21, Example 3.3] that the Sweedler crossed product of Hopf algebras $[33]$ is a particular example of wreath product. An obvious question is to extend this result to the context of quasi-bialgebras. As far as we know, the Sweedler crossed product construction has not been extended to quasi-bialgebras. Nevertheless, it has been introduced for dual quasi-bialgebras by A. Bălan in [2]. We will see that Bălan’s construction is a particular case of a wreath product.

For the definition of a dual quasi-bialgebra, we refer to [22]. A dual quasi-bialgebra is a coassociative counital $k$-coalgebra $H$ equipped with a unital multiplication which is associative up to conjugation by a convolution invertible element $\phi : H \otimes H \otimes H \to k$, called the reassociator. The definition is designed in such a way that $\mathcal{M}^H$, the category of corepresentations of $H$ is monoidal.

**Proposition 5.5.** Let $H$ be a dual quasi-Hopf algebra with reassociator $\phi$ and $A$ a $k$-algebra on which $H$ acts from the left, and consider it as a right $H$-comodule via the trivial coaction $A \ni a \mapsto a \otimes 1 \in A \otimes H$. Consider a $k$-linear map $\tau : H \otimes A \to A$ and the following morphisms in $\mathcal{M}^H$:

- $\psi : H \otimes A \to A \otimes H$, $\psi(h \otimes a) = h_1 \cdot a \otimes h_2$;
- $\zeta : H \otimes H \to A \otimes H$, $\zeta(h \otimes h') = \tau(h_1 \otimes h'_2) \otimes h_2 h'_3$;
- $\sigma : k \to A \otimes H$, $\sigma(1) = 1 \otimes 1$.

Then $(A, H)$ is a wreath in $\mathcal{M}^H$ if and only if $(A, \tau)$ is an $H$-crossed system in the sense of [2]. Furthermore, the corresponding wreath product is the crossed product algebra $A \mathcal{M}_H$, an algebra in $\mathcal{M}^H$.

**Proof.** It can be seen easily that $\psi$, $\zeta$ and $\sigma$ are right $H$-colinear. The equalities in [34,37] hold if and only if

$$h \cdot (aa') = (h_1 \cdot a)(h_2 \cdot a')$$

for all $h \in H$ and $a, a' \in A$, that is, $H$ is measuring $A$. In a similar way, the first equality in [34,37] takes the form

$$[h_1 \cdot (h'_1 \cdot a)] \tau(h_2 \otimes h'_2) \otimes h_3 h'_3 = \tau(h_1 \otimes h'_1)(h_2 h'_2 \cdot a) \otimes h_3 h'_3,$$

and is clearly equivalent to

$$[h_1 \cdot (h'_1 \cdot a)] \tau(h_2 \otimes h'_2) = \tau(h_1 \otimes h'_1)(h_2 h'_2 \cdot a),$$

where $\cdot$ denotes the $H$-action on $A$.
for all \( h, h' \in H \) and \( a \in A \), the twisted module condition. The second equality in (3.9) comes down to
\[
\left( h_1 \cdot (h'_1 \otimes h''_1) \right) \tau(h_2 \otimes h'_2 h''_2) = \tau(h_1 \otimes h'_1) \tau(h_2 h'_2 \otimes h''_1) \otimes (h_3 h'_3 h''_3) \cdot \phi^{-1}(h_4, h'_4, h''_4).
\]
Using the quasi-associativity of \( H \), we see that this condition is equivalent to
\[
\left( h_1 \cdot (h'_1 \otimes h''_1) \right) \tau(h_2 \otimes h'_2 h''_2) = \tau(h_1 \otimes h'_1) \tau(h_2 h'_2 \otimes h''_1) \phi^{-1}(h_3, h'_3, h''_2),
\]
for all \( h, h', h'' \in H \). This is the cocycle condition in the dual quasi-bialgebra case. Finally, it can be easily checked that (3.11) is equivalent to
\[
\tau(1 \otimes h) = \tau(h \otimes 1) = \varepsilon(h) 1,
\]
for all \( h \in H \), which means that \( \tau \) is normal.

This shows that \((A, H)\) is a wreath in \( \mathcal{M}^H \) if and only if \((A, \tau)\) is an \( H \)-crossed system. Moreover, the multiplication on the wreath product is given by the formula
\[
(a \otimes h)(a' \otimes h') = a(h_1 \cdot a') \tau(h_2 \otimes h'_1) \otimes h_3 h'_2,
\]
and the unit of the wreath product is \( 1 \otimes 1 \). This is precisely the crossed product algebra \( \overline{A \#_\tau H} \) in \( \mathcal{M}^H \). Note that \( \overline{A \#_\tau H} \) is a right \( H \)-comodule via \( a \mapsto a \mapsto h_1 \otimes h_2 \).

5.3. Bialgebroids. Let \( B \) be a \( k \)-algebra. Throughout this Section, \( H \) is a left \( L \)-bialgebroid with source map \( s : L \to \mathbb{H} \) and target map \( t : L^{op} \to \mathbb{H} \). Recall from [5] that \( H \) considered as an \( L \)-bimodule via \( l \cdot h \cdot l' = (l')s(l)h = s(l)t(l')h \) admits an \( L \)-coring structure \((H, \Delta, \varepsilon)\) such that \( \text{Im}(\Delta) \subseteq \mathbb{H} \times_L \mathbb{H} \), where
\[
\mathbb{H} \times_L \mathbb{H} = \left\{ \sum_i x_i \otimes_L y_i \in \mathbb{H} \otimes_L \mathbb{H} \mid \sum_i x_i t(l) \otimes_L y_i = \sum_i x_i \otimes_L y_i s(l) , \forall \ l \in L \right\}
\]
is the Takeuchi product [31], and
\[
\varepsilon(1_H) = 1_L , \quad \varepsilon(h h') = \varepsilon(h s(\varepsilon(h'))) = \varepsilon(h t(\varepsilon(h'))),
\]
for all \( h, h' \in H \). Via the component-wise multiplication \( H \times_L \mathbb{H} \) is an \( L \)-ring with unit \( 1_H \otimes_L 1_H \).

\( \Delta \) is an algebra map, by definition.

We use the notation \( \Delta(h) = h_1 \otimes_L h_2 \). The axioms above imply that
\begin{align}
\Delta(s(l)) &= s(l) \otimes_L 1_H , \quad \Delta(t(l)) = 1_H \otimes_L t(l) , \\
\varepsilon(\Delta(h_1))h_2 &= h = t(\varepsilon(h_2))h_1 \quad \text{and} \quad h_1 t(l) \otimes_L h_2 = h_1 \otimes_L h_2 s(l) ,
\end{align}
for all \( l \in L \) and \( h \in H \). Our main aim is to define a wreath \((\mathbb{H}, B)\) within the monoidal category \((L, \mathcal{M}_L, \otimes_L, L)\) of \( L \)-bimodules such that the associated wreath product generalizes the Sweedler crossed product \([32]\) to bialgebroids. We work in a context that is different from the one considered in [5]; however, in the end we obtain the same algebra structure on the space \( B \otimes_L \mathbb{H} \) as in [5].

Let \( i : L \to B \) be a \( k \)-algebra morphism; then \( B \) is an \( L \)-ring, and \( B \) is an \( L \)-bimodule by restriction of scalars. In a similar way, \( s : L \to \mathbb{H} \) is a \( k \)-algebra morphism, making \( \mathbb{H} \) into an \( L \)-bimodule. This new \( L \)-bimodule structure is given by the formula \( l \triangleright h \triangleleft l' = s(l) h s(l') \). Note that it is different from the \( L \)-bimodule structure given above.

**Definition 5.6.** Let \( \mathbb{H} \) be a left \( L \)-bialgebroid, and let \( B \) be an \( L \)-ring. \( \mathbb{H} \) measures \( B \) if there exists a \( k \)-linear map \( \cdot : \mathbb{H} \otimes B \to B \) satisfying the following conditions, for all \( l \in L \), \( h \in \mathbb{H} \) and \( b, b' \in B \).
\begin{align}
(a) \quad h \cdot (i(l)b) &= (hs(l)) \cdot b , \quad h \cdot (bi(l)) = (ht(l)) \cdot b , \\
(b) \quad h \cdot 1_B &= i(\varepsilon(h)) , \quad 1_H \cdot b = b , \\
(c) \quad h \cdot (bb') &= (h_1 \cdot b)(h_2 \cdot b').
\end{align}
We present an alternative characterization of measurements.
Proposition 5.7. With notation as above, $\mathbb{H}$ measures $B$ if and only if $1_B \cdot b = b$, for all $b \in B$, (5.4)(c) holds and

\begin{equation}
7 \cdot i(l) = i(\tilde{e}(hs(l))) = i(\tilde{e}(ht(l)),
\end{equation}

for all $l \in L$ and $h \in \mathbb{H}$.

Proof. The direct implication is immediate, note only that in any left bialgebroid $\mathbb{H}$ we have that

\begin{equation}
\tilde{e}(hs(l)) = \tilde{e}(ht(l)),
\end{equation}

for all $l \in L$ and $h \in \mathbb{H}$, and so the equalities in (5.5) are not contradictory. For the converse, observe first that $h \cdot 1_B = h \cdot i(1_L) = i(\tilde{e}(h))$, for all $h \in \mathbb{H}$. Now, for all $l \in L$, $h \in \mathbb{H}$ and $b \in B$ we have

\begin{equation}
(7 \cdot i(l)b = (h_1 \cdot i(l))(h_2 \cdot b) = i(\tilde{e}(hs(l)))(h_2 \cdot b)
\end{equation}

\begin{equation}
= i(\tilde{e}(hs(l)))(\tilde{e}(ht(l))_2 \cdot b) = (\tilde{e}(hs(l))_1 \cdot 1_B)((\tilde{e}(ht(l))_2) \cdot b)
\end{equation}

\begin{equation}
= (\tilde{e}(ht(l))_1 \cdot b)(\tilde{e}(ht(l))_2) = ((ht(l))_1 \cdot b)((ht(l))_2 \cdot 1_B)
\end{equation}

\begin{equation}
= (ht(l)) \cdot b,
\end{equation}

as needed. This completes the proof. □

The definition of a measuring is designed in such a way that we have the following result.

Lemma 5.8. Let $B$ be an $L$-ring and $\mathbb{H}$ a left $L$-bialgebroid measuring $B$. Then

\begin{equation}
\psi : \mathbb{H} \otimes_L B \to B \otimes_L \mathbb{H}, \quad \psi(h \otimes_L b) = h \cdot b \otimes_L h_2
\end{equation}

is a well-defined morphism in $L\mathcal{M}_L$ satisfying (3.7).

Proof. The map $\psi$ is well-defined because of the second equality in (5.3) and the first condition in (5.4)(a). $\psi$ is left $L$-linear because of the second equality in (5.3).

To see that $\psi$ is right $L$-linear we compute, $l \in L$, $h \in \mathbb{H}$, $b \in B$,

\begin{equation}
\psi((h \otimes_L b)l) = \psi(h \otimes_L b \cdot l) = h_1 \cdot (b \cdot l) \otimes_L h_2 = (h_1 \cdot l) \otimes_L h_2
\end{equation}

\begin{equation}
= h_1 \cdot b \otimes_L h_2 \cdot l = \psi(h \otimes_L b)l.
\end{equation}

Finally, (5.7) follows from (5.7) and

\begin{equation}
\psi(h \otimes_L 1_B) = h_1 \cdot 1_B \otimes_L h_2 = i(\tilde{e}(h)) \otimes_L h_2 = 1_B \otimes_L \tilde{e}(h) \cdot h_2
\end{equation}

\begin{equation}
= 1_B \otimes_L \tilde{e}(h) \cdot h = 1_B \otimes_L h,
\end{equation}

as needed. □

For a left $L$-linear morphism $\tau : \mathbb{H} \otimes_L \mathbb{H} \to B \otimes_L \mathbb{H}$, we have another left $L$-linear morphism $\zeta : \mathbb{H} \otimes_L \mathbb{H} \to B \otimes_L \mathbb{H}$, defined by the formula

\begin{equation}
\zeta(h \otimes_L h') = \tau(h \otimes_L h') \otimes_L h_2 h_2', \quad \forall h, h' \in H.
\end{equation}

It is easy to show that $\zeta$ is well-defined and left $L$-linear. $\zeta$ is also right $L$-linear if $\tau$ satisfies the extra condition

\begin{equation}
\tau(h \otimes_L h' \cdot l(l)) = \tau(h \otimes_L h' \cdot l(l)),
\end{equation}

for all $h, h' \in \mathbb{H}$ and $l \in L$.

Proposition 5.9. Let $B$ be an $L$-coring, $\mathbb{H}$ a left $L$-bialgebroid measuring $B$ and $\tau : \mathbb{H} \otimes_L \mathbb{H} \to B$ a left $L$-linear map satisfying (5.4). Consider $\psi$ as in Lemma 5.8, $\zeta$ defined by (5.6) and $\sigma : L \to B \otimes_L \mathbb{H}$ given by $\sigma(l) = i(l) \otimes_L 1_B$, for all $l \in L$. If the equalities

\begin{equation}
(\tau(h_1 \otimes_L h'_1)(h_2 \cdot b) = [h_1 \cdot (h'_1 \cdot b)](h_2 \otimes_L h'_1),
\end{equation}

\begin{equation}
(h_1 \cdot (h'_1 \otimes_L h'_1))(h_2 \cdot b) = \tau(h_1 \otimes_L h'_1)(h_2 h'_2 \otimes_L h'_1),
\end{equation}

\begin{equation}
\tau(h \otimes_L 1_B) = i(\tilde{e}(h)) = \tau(1_B \otimes_L h),
\end{equation}

then $\zeta$ is a well-defined morphism in $\tau$. □
hold for all $h, h', h'' \in \mathbb{H}$ and $b \in B$ then $(B, \mathbb{H})$ is a wreath in $L\mathcal{M}_L$. Furthermore, if $\tau$ satisfies the condition
\begin{equation}
(5.8) \quad \tau(h_1 \otimes_L h'_1)i(\bar{\tau}(h_2h'_2)) = \tau(h \otimes_L h'),
\end{equation}
for all $h, h' \in \mathbb{H}$, then the converse is also true. The corresponding wreath product is $B \otimes_L \mathbb{H}$ with multiplication
\[(b \otimes_L h)(b' \otimes_L h') = b(h_1 \cdot h_1')\tau(h_2 \otimes_L h'_1) \otimes_L h_3h'_2\]
and unit $1_B \otimes_L 1_\mathbb{H}$. It is an $L$-ring via $L \ni l \mapsto i(l) \otimes_L 1_H = 1_B \otimes_L s(l) \in B \otimes_L \mathbb{H}$, and therefore a unital associative $k$-algebra.

**Proof.** Similar to the one of Proposition 5.9, see also the proof of [5, Prop. 4.3]. \hfill \square

The wreath product obtained in Proposition 5.9 is called the crossed product of $B$ and $\mathbb{H}$, and it is denoted as $B \# \mathbb{H}$. We will now show that it generalizes the smash product algebra from [19]. Szlachányi [25] has presented a reformulation of the definition of bialgebroid in terms of monoidal categories. Let $\mathbb{H}$ be an $L \otimes L^{op}$-ring. Left $L$-bialgebroid structures on $\mathbb{H}$ correspond bijectively to monoidal structures on $\mathbb{H}\mathcal{M}$, such that the restriction of scalars functor $\mathbb{H}\mathcal{M} \to L\mathcal{M}_L$ is strict monoidal. The monoidal structure on $\mathbb{H}\mathcal{M}$ is defined by $\otimes$ with $h \cdot (m \otimes n) = h_1 \cdot m \otimes h_2 \cdot n$, for all $m \in M \in \mathbb{H}\mathcal{M}$ and $n \in N \in \mathbb{H}\mathcal{M}$, and the unit is $L$ considered as a left $\mathbb{H}$-module via the action $h \triangleright l = \bar{\varepsilon}(hs(l)) = \varepsilon(ht(l))$, for all $h \in \mathbb{H}$ and $l \in L$.

**Corollary 5.10.** Let $\mathbb{H}$ be a left $L$-bialgebroid and $B$ and algebra in $\mathbb{H}\mathcal{M}$, that is, a left $\mathbb{H}$-module algebra. Then $\mathbb{H} \otimes B \xrightarrow{can} \mathbb{H} \otimes_L B \rightarrow B$ defines a measuring on $B$. Furthermore, the map $\tau : \mathbb{H} \otimes_L \mathbb{H} \rightarrow B$ given by $\tau(h \otimes_L h') = (hh') \cdot 1_B$, for all $h, h' \in \mathbb{H}$, is well defined, left $L$-linear, obeys (5.7) and is such that the induced map $\zeta$ from (5.4) fulfills all the conditions in Proposition 5.9. Consequently, $(B, \mathbb{H})$ is a wreath in $L\mathcal{M}_L$ with the corresponding wreath product given by
\[(b \otimes_L h)(b' \otimes_L h') = b(h_1 \cdot b) \otimes_L h_2h',\]
for all $b, b' \in B$ and $h, h' \in \mathbb{H}$.

**Proof.** By the strict monoidality of the forgetful functor $\mathbb{H}\mathcal{M} \to L\mathcal{M}_L$, any left $\mathbb{H}$-module algebra $B$ has a canonical $L$-ring structure. Its unit is the map $i : L \ni l \mapsto s(l) \cdot 1_B = t(l) \cdot 1_B \in B$. Thus $B$ inherits a $L$-bimodule structure from the $\mathbb{H}$-action: $l \cdot b \cdot l' = s(l)t(l') \cdot b = t(l')s(l) \cdot b$, for all $l, l' \in L$ and $b \in B$. In addition, the fact that $i : L \rightarrow B$ is left $\mathbb{H}$-linear means that
\[h \cdot i(l) = i(h \triangleright l) = i(\varepsilon(h)(s(l))) = i(\varepsilon(h)(t(l))),\]
for all $h \in \mathbb{H}$ and $l \in L$. Since $B$ is an algebra in $\mathbb{H}\mathcal{M}$ it follows that $1_H \cdot b = b$, for all $b \in B$, and that (5.5) (c) is satisfied. From Proposition 5.7 we obtain that $\mathbb{H}$ measures $B$. Consider $\tau : \mathbb{H} \otimes_L \mathbb{H} \rightarrow B$, $\tau(h \otimes L h') = hh' \cdot 1_B = i(\varepsilon(hh'))$. It is easy to see that $\tau$ is well-defined, and left $\mathbb{H}$-linear via $\varepsilon$ defined by s. (5.8) is satisfied since
\[\tau(h_1 \otimes_L h'_1)i(\varepsilon(h_2h'_2)) = (h_1h'_1 \cdot 1_B)(hh'_2 \cdot 1_B) = hh' \cdot 1_B = \tau(h \otimes_L h'),\]
for all $h, h' \in \mathbb{H}$. Similar computations guarantee us that the three equalities in the statement of Proposition 5.9 hold, and $(B, \mathbb{H})$ is a wreath product in $L\mathcal{M}_L$. We end the proof by noting that the resulting $L$-ring is the so called smash product of $B$ and $\mathbb{H}$ introduced in [19, Def. 2.4]. \hfill \square

**5.4. Weak bialgebras.** Particular examples of left bialgebroids are given by weak bialgebras, see [17, 7]. Recall from [6] that a weak bialgebra $H$ is a $k$-algebra and a $k$-coalgebra such that the comultiplication $\Delta$ is multiplicative, the counit $\varepsilon$ respects the units,
\[(\Delta \otimes \text{Id}_H)(\Delta(1)) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1),\]
and $\varepsilon(hh') = \varepsilon(h_1)\varepsilon(h_2) = \varepsilon(h_3)\varepsilon(h_4)$, for all $h, h' \in H$. To a weak bialgebra $H$, we can associate four projections $\varepsilon_t, \varepsilon_s, \tau_t, \tau_s : H \rightarrow H$, given by the formulas $\varepsilon_t(h) = \varepsilon(1_h)1_2$, $\varepsilon_s(h) = \varepsilon(1_2)h_1$,
ε(h12)1, τs(h) = ε(h1)12, and τs(h) = ε(12h)11, for all h ∈ H. Let L = Im(εi) and define s, t : L → H by
\[ s(z) = z, t(z) = τs(z). \]

Then H is a left L-bialgebroid with source and target morphisms s and t, comultiplication \( \tilde{\Delta} : H \rightarrow H \otimes H \rightarrow H \otimes_L H \) and counit \( \tilde{\varepsilon} = \varepsilon_i \), see for example [7] Prop. 3.1]. Specializing Proposition 5.9 to weak Hopf algebras we obtain the following result.

**Corollary 5.11.** Let H be a weak bialgebra, let B be a k-algebra and let \( i : L \rightarrow B \) be a k-algebra map. Assume B is a left H-module, the left H-action of h ∈ H on b ∈ B is denoted by \( h \cdot b \). We assume that
\[ h \cdot i(εt(hg)) = i(εt(hg)), 1_H \cdot b = b \text{ and } h \cdot (bb') = (h_1 \cdot b)(h_2 \cdot b'), \]
for all h, g ∈ H and b, b' ∈ B. H is an L-bimodule by restriction of scalars via s. Suppose that we have a well defined left L-linear morphism \( \tau : H \otimes_L H \rightarrow B \) such that,
\[ \tau(h \otimes_L h') = \varepsilon(h'_1)\tau(h \otimes_L h'_2), \]
\[ \tau(h_1 \otimes_L h'_1)(h_2h'_2 \cdot b) = [h_1 \cdot (h'_1 \cdot b)] \tau(h_2 \otimes_L h'_2), \]
\[ (h_1 \cdot \tau(h'_1 \otimes_L k'_1)) \tau(h_2 \otimes_L h'_2) = \tau(h_1 \otimes_L h'_1) \tau(h_2h'_2 \otimes_L h'_3'), \]
\[ \tau(h \otimes_L 1) = i(εt(h)), \]
for all h, h', h'', g ∈ H. With \( \psi, \zeta \) and \( \sigma \) defined as in Proposition 5.9, \( (B, H) \) is a wreath in the monoidal category of L-bimodules.

**Proof.** It has to be shown that \( \tau(h \otimes_L h'1_2 \varepsilon(1_g2_g) = \tau(h \otimes_L h'1_2 \varepsilon(1_g) \) is equivalent to the first of the four conditions imposed on \( \tau \). This follows easily from the formulas \( \varepsilon_1(h_1) \otimes h_2 = 1 \otimes h_1 \), \( \varepsilon_2(h_1) \otimes h_2 = h_1 \otimes \varepsilon_1(h_2) \) and \( \varepsilon_3(h_1) \otimes h_2 = \varepsilon(h_2) \), see [6].

We can apply Corollary 5.10 to the left L-bialgebroid associated to a weak bialgebra H. An algebra in \( hM \) is a left H-module algebra A, as introduced in [4]. The associated smash product is the smash product that was introduced in [23].

### 5.5. Doi-Koppinen data over right bialgebroids

Now we look at the dual situation. More precisely, we will construct a coring in a category of bimodules from a Doi-Koppinen datum over a bialgebroid. Since we have to deal with right actions and coactions we need to work over a right bialgebroid \( \mathbb{H} \) over a k-algebra \( R \). As in the left handed case, \( s : R \rightarrow \mathbb{H} \) and \( t : R^\text{op} \rightarrow \mathbb{H} \) will be the source and target algebra morphisms, and endow \( \mathbb{H} \) with the \( R \)-bimodule structure given by \( r \cdot h \cdot r' := h \cdot t(r)s(r') = h \cdot s(r')t(r) \). Then, by definition, \( \mathbb{H} \) is an \( R \)-coring such that the image of the comultiplication \( \tilde{\Delta} \) is included in
\[ \mathbb{H} \times_R \mathbb{H} = \left\{ \sum_i x_i \otimes_R y_i \in \mathbb{H} \otimes_R \mathbb{H} \mid \sum_i s(r)x_i \otimes_R t(r)y_i = \sum_i x_i \otimes_R t(r)y_i, \forall r \in R \right\}. \]
\( \mathbb{H} \times_R \mathbb{H} \) is an \( R \)-ring under the component-wise multiplication, with unit \( 1 \mathbb{H} \otimes_R 1 \mathbb{H} \). It is required that \( \tilde{\Delta} \) is an algebra morphism. The counit \( \tilde{\varepsilon} \) has to respect the unit and has to satisfy the condition
\[ \tilde{\varepsilon}(t(\tilde{\varepsilon}(h))h') = \tilde{\varepsilon}(hh') = \tilde{\varepsilon}(s(\tilde{\varepsilon}(h))h'), \]
for all h, h' ∈ H. A right corepresentation of a right R-bialgebroid \( \mathbb{H} \) is a right \( R \)-module \( \mathfrak{M} \) together with a right \( R \)-module map \( \rho_{\mathfrak{M}} : \mathfrak{M} \otimes_R \mathbb{H} \rightarrow \mathfrak{M} \otimes_R \mathfrak{M} \) which is coassociative and counital. Although \( \mathfrak{M} \) is not a left \( R \)-module, \( \mathfrak{M} \otimes_R \mathbb{H} \) is a left \( R \)-module, with left \( R \)-action given by \( r \cdot (m \otimes_R h) = m \otimes_R s(r)h \). By [1] Prop. 1.1], \( \mathfrak{M} \otimes_R \mathbb{H} \) has a unique left \( R \)-module structure, given by
\[ r \cdot m := m_{(0)} \cdot \tilde{\varepsilon}(s(r)m_{(1)}), \text{ where } \rho_{\mathfrak{M}}(m) = m_{(0)} \otimes_R m_{(1)} \in \mathfrak{M} \otimes_R \mathbb{H} \text{, making } \mathfrak{M} \text{ into an } \mathbb{H} \text{-module, and such that } \rho_{\mathfrak{M}} \text{ is an } \mathbb{H} \text{-bimodule map and } \text{Im}(\rho_{\mathfrak{M}}) \text{ is included in the Takeuchi product} \]
\[ \mathfrak{M} \times_R \mathbb{H} := \left\{ \sum_i x_i \otimes_R y_i \in \mathfrak{M} \otimes_R \mathbb{H} \mid \sum_i r \cdot x_i \otimes_R y_i = \sum_i x_i \otimes_R t(r)y_i, \forall r \in R \right\}. \]
This key result allows us to define a monoidal structure on $\mathcal{M}_R^H$, the category of right $H$-corepresentations and right $H$-colinear maps. The tensor product of $\mathcal{M}, \mathcal{N} \in \mathcal{M}_R^H$ is $\mathcal{M} \otimes_R \mathcal{N}$, together with the structure map $\rho_{\mathcal{M} \otimes_R \mathcal{N}}$ given by

$$\rho_{\mathcal{M} \otimes_R \mathcal{N}}(m \otimes_R n) = m_{(0)} \otimes_R n_{(0)} \otimes_R m_{(1)} n_{(1)}.$$  

The unit object is $R$ together with the structure map $\rho_R$ given by $\rho_R(r) = 1_R \otimes_R s(r) \in R \otimes_R H$. A right $H$-comodule algebra is by definition an algebra in $\mathcal{M}_R^H$. $\mathcal{M}_R$, the category of right $H$-representations, is also a monoidal category. The tensor product over $R$ of two right $H$-modules is a right $H$-module by restriction of scalars via $\Delta$. The unit object is $R$, which is a right $H$-module via

$$r \mapsto h = \bar{\varepsilon}(t(r)h) = \bar{\varepsilon}(s(r)h),$$

for all $r \in R$ and $h \in H$. A coalgebra in $\mathcal{M}_R$ is called a right $H$-module coalgebra. A right $H$-module coalgebra $C$ is an $R$-coring; $C$ is as an $R$-bimodule via $r \cdot c \cdot r' = c \cdot s(r)t(r') = c \cdot t(r)s(r')$. $C$ is a right $H$-module, we denote the action by $\cdot : C \otimes_R H \to C$. Then

$$\Delta_C(c \cdot h) = c_1 \cdot h_1 \otimes_R c_2 \cdot h_2$$

and $\varepsilon_C(c \cdot h) = \varepsilon_C(c) \cdot h$, 

for all $c \in C$ and $h \in H$.

**Definition 5.12.** [7] A right Doi-Koppinen datum is a triple $(\mathbb{H}, A, C)$ where $\mathbb{H}$ is a right $R$-bialgebroid, $A$ is a right $H$-comodule algebra and $C$ is a right $H$-module coalgebra. A right $(\mathbb{H}, A, C)$-module is a vector space $\mathcal{M}$ with the following structure:

- $\mathcal{M}$ is a right $A$-module, and therefore a right $R$-module by restriction of scalars via $i : R \to A$, $i(r) = r \cdot 1_A = 1_A \cdot r$;
- $\mathcal{M}$ is a right $C$-comodule, with structure map $\rho_{\mathcal{M}}^C : M \to \mathcal{M} \otimes_R C$, $\rho_{\mathcal{M}}^C(m) = m_{(0)} \otimes_R m_{(1)}$;
- for all $a \in A$ and $m \in \mathcal{M}$, we have $\rho_{\mathcal{M}}^C(m \cdot a) = m_{(0)} \cdot a_{(0)} \otimes_R m_{(1)} \cdot a_{(1)}$.

$\mathcal{D}(\mathbb{H})_A$ is the category with right $(\mathbb{H}, A, C)$-modules as objects and right $A$-linear right $C$-colinear maps as morphisms.

Now we will show that the (right-handed version of) [7] Prop. 4.1 is a consequence of Theorem 4.8.

**Proposition 5.13.** For a right Doi-Koppinen datum $(\mathbb{H}, A, C)$, we have the following assertions.

(i) $(A, C)$ is a cowreath in $R \mathcal{M}_R$, via $\psi : C \otimes_R A \to A \otimes_R C$, $\psi(c \otimes_R a) = a_{(0)} \otimes_R c \cdot a_{(1)}$ and $\zeta : A \otimes_R C \otimes_R C \to A \otimes_R C \otimes_R C$, $\zeta = i \otimes_R \Delta_C$, where $i : R \to A$ is as above.

(ii) The bifunctor $\otimes_R$ defines a right $R \mathcal{M}_R$-category structure on $\mathcal{M}_R$ and the category of right corepresentations in $\mathcal{M}_R$ over the $A$-coring $\mathcal{E} = A \otimes_C \mathcal{C}$ corresponding to the cowreath $(A, C)$ from (i) can be identified with the category of right corepresentations over the $A$-coring $\mathcal{E}$ viewed now in $\mathcal{k} \mathcal{M}$, and is isomorphic to the category $\mathcal{D}(\mathbb{H})_A^\mathcal{E}$.

**Proof.** (i) We first show that $\psi$ is well-defined. For all $c \in C$, $r \in R$ and $a \in A$ we have that

$$\psi(c \otimes_R r \cdot a) = \psi(c \otimes_R a_{(0)} \cdot \bar{s}(s(r)a_{(1)})) = \bar{\varepsilon}(a_{(0)} \otimes_R c \cdot a_{(1)}), s(\bar{\varepsilon}(s(r)a_{(1)})) = a_{(0)} \otimes_R c \cdot (s(r)a_{(1)})_1 s(\bar{\varepsilon}(s(r)a_{(1)})) = a_{(0)} \otimes_R c \cdot s(r)a_{(1)} = a_{(0)} \otimes_R c \cdot a_{(1)} = \psi(c \cdot r \cdot a),$$

At $(\ast)$, we used that $\rho_A$ is right $R$-linear. The left $R$-linearity of $\psi$ follows from the fact that $\text{Im}(\rho_A) \subseteq A \times_R \mathbb{H}$:

$$\psi(r \cdot c \otimes_R a) = \psi(c \cdot t(r) \otimes_R a) = a_{(0)} \otimes_R c \cdot t(r)a_{(1)} = r \cdot a_{(0)} \otimes_R c \cdot a_{(1)}.$$

The right $R$-linearity of $\psi$ follows immediately from the right $R$-linearity of $\rho_A$. All the other conditions that are needed to make $(A, C, \psi)$ to be a cowreath in $R \mathcal{M}_R$ with $(C, \psi) \in \mathcal{T}_A$ are straightforward, and are left to the reader.

(ii) Clearly $\otimes_R : \mathcal{M}_R \times R \mathcal{M}_R \to \mathcal{M}_R$ yields a right $R \mathcal{M}_R$-category structure on $\mathcal{M}_R$. Furthermore, if $A$ is an algebra in $R \mathcal{M}_R$, then any right module $\mathcal{M}$ in $\mathcal{M}_R$ over $A$ has the right $R$-module structure inherited from the right $A$-action, since $(m \cdot r) \cdot a = m \cdot (r \cdot a)$, for all $m \in \mathcal{M}$, $r \in R$ and $a \in A$, and therefore $m \cdot r = m \cdot (r \cdot 1_A) = m \cdot i(r)$, for all $m \in \mathcal{M}$ and $r \in R$. So $\mathcal{M}$ is a right $A$-module in $\mathcal{k} \mathcal{M}$, considered as a right $R$-module via $i$. 

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In a similar way, if \( \mathcal{R} \) is an \( A \)-bimodule in \( \mathcal{R} \mathcal{M}_R \) then the \( R \)-bimodule structure on \( \mathcal{R} \) is induced by the \( A \)-module structure on \( \mathcal{R} \), that is, \( r \cdot n \cdot r' = i(r) \cdot n \cdot i(r') \), for all \( r, r' \in R, n \in \mathcal{R} \). This tells us that the tensor product over \( A \) in \( \mathcal{R} \mathcal{M}_R \) is precisely the tensor product over \( A \) in \( \mathcal{M}_A \). Thus, if \( \mathcal{C} \) is an \( A \)-coring in \( \mathcal{R} \mathcal{M}_R \) then it is actually an \( A \)-coring in \( \mathcal{M}_A \), viewed as an \( R \)-bimodule via \( i \), and a right corepresentation in \( \mathcal{M}_R \) over \( \mathcal{C} \) is a usual corepresentation of the \( A \)-coring \( \mathcal{C} \) in \( \mathcal{M}_A \). In other words, \( (\mathcal{M}_R)^\mathcal{C} \cong \mathcal{M}_A^{\mathcal{C}} \).

Now consider \( \mathcal{C} = A \otimes C \), the \( A \)-coring in \( \mathcal{R} \mathcal{M}_R \) determined by the cowreath \( (A, C) \) in \( \mathcal{R} \mathcal{M}_R \) described in (i). From the above comments and Theorem 6.8 it follows that \( \mathcal{M}_A^{\mathcal{C}} \cong (\mathcal{M}_R)^\mathcal{C} \cong \gamma D(\mathcal{H})F \).

A weak bialgebra is a left bialgebroid. The base \( k \)-algebra \( R \) is the image of the idempotent morphism \( \varepsilon : H \to H \) defined in Section 5.4. The source map \( s : R \to H \) is the inclusion, and \( t : R \to H \) is the restriction of \( \varepsilon \). The comultiplication is defined as in the left-handed case, and the counit is \( \varepsilon \).

Applying Proposition 6.13 to the case where \( \mathcal{H} \) is a weak bialgebra, we obtain that the category of right weak Doi-Koppinen modules as defined in [4] is isomorphic to the category of corepresentations over a coring in \( \mathcal{M}_A \). To this end we have to use the right handed version of [7, Theorem 3.11].

Our theory applies also to Doi-Koppinen data in braided monoidal categories. This will be explained in full detail in the forthcoming paper [15].

### 6. Appendix

After an earlier version of this paper was circulating we were informed that Theorem 6.6 holds in a more general setting, see Theorem 6.8, leading to a different proof based on 2-categorical arguments. We will now explore this idea; it will turn out that Theorem 6.6 holds, but that we need Theorem 6.8 in the proof, so that it cannot be used to give an alternative proof of Theorem 6.6.

The starting point for the generalization of Theorem 6.6 is an explicit description of \( EM(\mathcal{K}) \) in the case where the 2-category \( \mathcal{K} \) admits the Eilenberg-Moore (EM for short) construction for monads. If \( \mathcal{K} \) is a 2-category then by \( \mathcal{Mn}d(\mathcal{K}) \) we denote the 2-category of monads, monad morphisms, and monad transformations, see Section 3 for detail. An object \( X \) of \( \mathcal{K} \) gives rise to a monad \( X^\Delta = (X, X \rightrightarrows X, i_X, i_X) \), i.e., to an object \( X^\Delta \) in \( \mathcal{Mn}d(\mathcal{K}) \). Furthermore, any 1-cell \( X \overset{f}{\to} Y \) defines a 1-cell \( X \overset{f}{\to} Y \) in \( \mathcal{Mn}d(\mathcal{K}) \) and any 2-cell \( f \overset{\beta}{\Rightarrow} g \) in \( \mathcal{K} \) becomes a 2-cell \( (f, 1_f) \overset{\beta}{\Rightarrow} (g, 1_g) \) in \( \mathcal{Mn}d(\mathcal{K}) \). These correspondences produce a 2-functor \( \text{Inc}_\mathcal{K} : \mathcal{K} \to \mathcal{Mn}d(\mathcal{K}) \), called the inclusion 2-functor of \( \mathcal{K} \). Conversely, we have the so-called underlying 2-functor \( \text{Und}_\mathcal{K} : \mathcal{Mn}d(\mathcal{K}) \to \mathcal{K} \) that maps \((A, t)\) to \( A \), \((f, \psi)\) to \( f \), and \( \rho \) to \( \rho \). From [31, Theorem 1] we know that the underlying 2-functor is a left 2-adjoint for the inclusion 2-functor of \( \mathcal{K} \) in \( \mathcal{Mn}d(\mathcal{K}) \). For more detail on 2-functors and 2-adjunctions we invite the reader to consult [3, Ch. 7].

#### Definition 6.1

A 2-category \( \mathcal{K} \) admits the EM construction for monads if the inclusion 2-functor \( \text{Inc}_\mathcal{K} \) has a right 2-adjoint.

Let us explain this more explicitly. Let \( F : \mathcal{Mn}d(\mathcal{K}) \to \mathcal{K} \) be a right 2-adjoint functor for \( \text{Inc}_\mathcal{K} \). If \( \varepsilon : \text{Inc}_\mathcal{K} F \to 1_{\mathcal{Mn}d(\mathcal{K})} \) is the counit of the 2-adjunction then for any monad \( A = (A, t, \mu, \eta) \) in \( \mathcal{K} \) we have that \( \varepsilon_A = (u^\Delta, v^\Delta) : A^\Delta \to A \) is a monad morphism, where \( A^\Delta = F(A) \) is the so-called EM object of \( A \). Moreover, for any monad \( A \) in \( \mathcal{K} \) and any object \( X \) of \( \mathcal{K} \) we have a category isomorphism, natural in both arguments,

\[
\tilde{\mathcal{F}}_{X,A} : \mathcal{K}(X, A^\Delta) \cong \mathcal{Mn}d(\mathcal{K})(X, A)
\]

defined as follows. \( \tilde{\mathcal{F}}_{X,A} \) sends a 1-cell \( f \overset{\beta}{\Rightarrow} \) in \( \mathcal{K} \) in the monad morphism \( (u^\Delta f, v^\Delta f \circ 1_f) : X \to A \), while a 2-cell \( X \overset{f}{\Rightarrow} A^\Delta \) in \( \mathcal{K} \) is mapped by \( \tilde{\mathcal{F}}_{X,A} \) in the 2-cell \( (u^\Delta f, v^\Delta f \circ 1_f) : X \overset{(u^\Delta f, v^\Delta f \circ 1_f)}{\Rightarrow} A \) of \( \mathcal{Mn}d(\mathcal{K}) \).
To a monad \( A = (A,t,\mu_t,\eta_t) \) in \( K \) we can associate a monad morphism \( A \xrightarrow{(t,\mu_t)} A \). Consequently, there exists a unique 1-cell \( A \xrightarrow{u^t} A^t \) in \( K \) such that
\[
(u^t \cdot \chi^t, \chi^t \circ 1_{v^t}) = (t, \mu_t).
\]
In a similar way, an easy computation shows that \((tu^t = u^t \cdot \psi^t, \mu_t \circ 1_{u^t}) \xrightarrow{\psi^t} (u^t, \chi^t)\) is a 2-cell in \( \text{Mnd}(K) \), and therefore there exists a unique 2-cell \( A^t \xrightarrow{\psi^t} A^t \) such that \(1_{u^t} \circ \rho^t = \chi^t\). By \[31\]
\[\xi^1 \]
\( A^t \xrightarrow{\psi^t} A \) defines an adjoint pair of 1-cells, in the sense that the 2-cells \( 1_A \xrightarrow{\eta^t} u^t \cdot \psi^t = t \) and \( u^t \cdot \psi^t \xrightarrow{\rho^t} 1_A \) satisfy the equalities
\[
\left(1_{u^t} \circ \rho^t\right)(\eta_t \circ 1_{u^t}) = u^t \text{ and } (\rho^t \circ 1_{v^t})(1_{u^t} \circ \eta_t) = v^t.
\]
For any monad morphism \( A = (A,t,\mu_t,\eta_t) \xrightarrow{(f,\psi)} B = (B,s,\mu_s,\eta_s) \) denote \( F(f,\psi) \) by \( \overline{f} \), a 1-cell between \( A^t \) and \( B^s \). By the 2-naturality of the counit we have a commutative diagram
\[
\begin{array}{ccc}
A^t & \xrightarrow{(u^t,\chi^t)} & A \\
\overline{f} \downarrow & & \downarrow \overline{f} \\
B^s & \xrightarrow{(u^t,\chi^t)} & B
\end{array}
\]
Otherwise stated, \( \overline{f} : A^t \rightarrow B^s \) is a 1-cell in \( K \) such that
\[
f^t = u^t \overline{f} \text{ and } \chi^t \circ 1_{\overline{f}} = (1_f \circ \chi^t)(\psi \circ 1_{u^t}).
\]
Then by \[21, \xi^2\] and \[26, Theorem 3.10\] to give a 1-cell in \( \text{Mnd}(K) \) is equivalent to give a pair \((f,\overline{f})\) of 1-cells in \( K \) such that the first equation in \((6.3)\) holds. Note that the converse of this assertion has the following meaning.

**Lemma 6.2.** If \( A, B \) are monads in \( K \) and \((f,\overline{f})\) is a pair of 1-cells in \( K \) obeying \( fu^t = u^s \overline{f} \) then there exists \( \psi : sf \Rightarrow ft \) a 2-cell in \( K \) such that \((f,\psi) : A \rightarrow B \) is a monad morphism. Moreover, \( \psi \) satisfies the second equality in \((6.3)\).

**Proof.** The explicit definition of \( \psi \) was given in the proof of \[26, Theorem 3.10\] (as well as the fact that \((f,\psi)\) is a monad morphism and \( F(f,\psi) = \overline{f} \)). Namely, \( \psi \) is defined by the following vertical composition of 2-cells,
\[
\psi : sf \xrightarrow{1_f \circ \eta^t} sfu^t = sf \overline{f}u^t = \psi^t = \chi^t \circ 1_{\overline{f}u^t} = f^t.
\]
To see that \( \psi \) satisfies the second equality in \((6.3)\) observe first that
\[
(1_f \circ \chi^t)(\chi^s \circ 1_\chi^t) = (1_{fu^t} \circ \rho^t)(1_{u^t} \circ \rho^t \circ 1_{\overline{f}u^t}) = 1_{u^t} \circ (1_{\overline{f}u^t} \circ \rho^t) = (\chi^s \circ 1_{\overline{f}u^t} \circ \chi^t) = (1_{sf} \circ \chi^t),
\]
so that
\[
(1_f \circ \chi^t)(\psi \circ 1_{u^t}) = (1_f \circ \chi^t)(\chi^s \circ 1_\chi^t)(1_{sf} \circ \eta_t \circ 1_{u^t}) = (\chi^s \circ 1_{\overline{f}u^t} \circ \rho^t)(\eta_t \circ 1_{u^t}) = \chi^t \circ 1_{\overline{f}u^t},
\]
as stated. \(\square\)
Conversely, let\( \tau \) be a 2-cell in \( \text{Mnd}(\mathcal{K}) \), and fortiﬁer the 1-cells in \( EM(\mathcal{K}) \), can be identiﬁed with commutative diagrams of the form

\[
\begin{array}{ccc}
A' & \xrightarrow{\tau} & B' \\
\downarrow{u'} & & \downarrow{u'} \\
A & \xrightarrow{f} & B
\end{array}
\]

(6.6)

It was pointed out in [21 §2.2] that, via this identiﬁcation, 2-cells in \( EM(\mathcal{K}) \) correspond to 2-cells in \( \mathcal{K} \). We discuss this in more detail in Proposition 6.3. In the proof, we focus on the parts that are missing in [21].

**Proposition 6.3.** Let \( \mathcal{K} \) be a 2-category admitting \( EM \) constructions for monads. Then \( EM(\mathcal{K}) \) is isomorphic to the following 2-category:

- 0-cells are monads in \( \mathcal{K} \);
- a 1-cell from \( \mathcal{K} = (A, t) \) to \( \mathbb{B} = (B, s) \) is a pair \( (f, \bar{\tau}) \) of 1-cells in \( \mathcal{K} \) making the diagram (6.6) commutative;
- a 2-cell from \( (f, \bar{\tau}) \) to \( (g, \bar{\sigma}) \) is a 2-cell \( \tau : \bar{\tau} \Rightarrow \bar{\sigma} \) in \( \mathcal{K} \).

**Proof.** Let \( \rho : (f, \psi) \mapsto (g, \phi) \) be a 2-cell in \( EM(\mathcal{K}) \), that is, \( \rho : f \Rightarrow g \) is a 2-cell in \( \mathcal{K} \) such that (6.4) is satisﬁed. We identify \( (f, \psi) \) with \( (f, \bar{\tau}) \) and \( (g, \phi) \) with \( (g, \bar{\sigma}) \), as in Lemma 6.2, and we deﬁne

\[
\lambda : u^T = fu^t \rho \circ 1_{u^t} \xrightarrow{\gamma u^t} gu^t = u^\mathcal{G}.
\]

We claim that \( \lambda : (u^T, \chi^s \circ 1_{\bar{T}}) \Rightarrow (u^\mathcal{G}, \chi^s \circ 1_{\bar{\sigma}}) \) is a monad transformation, that is a 2-cell in \( \text{Mnd}(\mathcal{K})(A', \mathbb{B}) \). Indeed, on one hand we have

\[
\lambda(\chi^s \circ 1_{\bar{T}}) = (1_g \circ \chi^t)(\rho \circ 1_{u^t})(\chi^s \circ 1_{\bar{T}})
\]

(6.4)

\[
= (1_g \circ \chi^t)(\rho \circ 1_{u^t})(1_f \circ \chi^t)(\psi \circ 1_{u^t})
\]

\[
= (1_g \circ \chi^t(1_t \circ \chi^s))(\rho \circ 1_{u^t} \psi \circ 1_{u^t})
\]

(6.3)

\[
= (1_g \circ \chi^t)(1_g \circ \mu_t)(\rho \circ 1_{u^t} \psi \circ 1_{u^t})(1_s \circ \rho) \circ 1_{u^t}.
\]

On the other hand,

\[
(\chi^s \circ 1_{\bar{T}})(1_s \circ \lambda)
\]

(6.4)

\[
= (1_g \circ \chi^t)(\phi \circ 1_{u^t})(1_s \circ \rho \circ 1_{u^t})
\]

\[
= (1_g \circ \chi^t(1_t \circ \chi^s))(\phi \circ 1_{u^t})(1_s \circ \rho) \circ 1_{u^t}.
\]

Consequently \( \lambda(\chi^s \circ 1_{\bar{T}}) = (\chi^s \circ 1_{\bar{T}})(1_s \circ \lambda) \), proving that \( \lambda \) is a monad morphism. It follows from (6.1) that there exists a unique 2-cell \( \tau_\rho : \bar{T} \Rightarrow \bar{\sigma} \) in \( \mathcal{K}(A', B') \) such that

\[
1_{u^t} \circ \tau_\rho = (1_g \circ \chi^t)(\rho \circ 1_{u^t}).
\]

Conversely, let \( \tau : \bar{T} \Rightarrow \bar{\sigma} \) be a 2-cell in \( \mathcal{K} \). We claim that \( \rho_\tau : (f, \psi) \rightarrow (g, \phi) \) deﬁned as the composition

\[
\rho_\tau : f \xrightarrow{1_{u^t} \circ \tau} ft = fu^t \psi^t = u^T \psi^t \xrightarrow{1_{u^t} \circ \tau \circ 1_{u^t}} u^\mathcal{G} \psi^t = gu^t \psi^t = gt
\]

is a 2-cell in \( EM(\mathcal{K}) \). To this end observe ﬁrst that the diagram below is commutative

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{ccc}
A' & \xrightarrow{\bar{T}} & B' \\
\downarrow{u'} & & \downarrow{u'} \\
A & \xrightarrow{f} & B
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
u^T & \xrightarrow{u^t \circ \tau} & gu^t \\
\downarrow{1_{u^t} \circ \tau} & & \downarrow{1_{u^t} \circ \tau}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
u^\mathcal{G} & \xrightarrow{u^t \circ \tau \circ 1_{u^t}} & gu^t
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
u^T & \xrightarrow{u^t \circ \tau} & gu^t \\
\downarrow{1_{u^t} \circ \tau \circ 1_{u^t}} & & \downarrow{1_{u^t} \circ \tau \circ 1_{u^t}}
\end{array}
\end{array}
\]
Indeed, by an isomorphism in \( \tau \) and \( \psi \) we can define \( u(t) \). Giving a wreath in \( K \) on \( u(t) \) we can specialize our results above to the bicategory Bim(\( C \)). We are now ready to state and prove the result announced at the beginning of this Appendix. From Corollary 6.4, see [21, p. 257] and Proposition 6.3. This completes the proof. □

Proposition 6.3 allows us to provide an alternative description for wreaths and cowreaths in \( K \). For the proof of Corollary 6.4 see [21, p. 257] and Proposition 6.3.

Corollary 6.4. Let \( K \) be a 2-category admitting the EM constructions for monads.

(i) Giving a wreath in \( K \) is equivalent to giving the following data:

(i) A monad \( k = (A, l, \mu, \eta) \) in \( K \);

(ii) A pair \( (A \xrightarrow{\sigma} A, A \xrightarrow{\tau} A) \) of 1-cells in \( K \) satisfying \( \sigma \circ \tau = \mu \);

(iii) A monad structure \( (A, A \xrightarrow{\mu} A, \tau : \sigma \rightarrow \eta) \) on \( \sigma \) in \( K \).

(ii) Giving a cowreath in \( K \) is equivalent to giving the following data:

(i) A monad \( h = (A, t, \mu, \eta) \) in \( K \);

(ii) A pair \( (A \xrightarrow{\delta} A, A \xrightarrow{\sigma} A) \) of 1-cells in \( K \) satisfying \( \sigma \circ \delta = \eta \);

(iii) A cowreath structure \( (A, A \xrightarrow{\sigma} A, \delta : \sigma \rightarrow \tau, \xi : \tau \rightarrow \mu) \) on \( \sigma \) in \( K \).

We are now ready to state and prove the result announced at the beginning of this Appendix. From now on, we specialize our results above to the bicategory Bim(\( C \)). Let \( M \) be a left module over an algebra \( A \) in \( C \); \( M \) is called right robust if for any \( X \in A \) and \( Y \in C \) the canonical morphism (see [13, 30])

\[
\theta_{M, X, Y} : M \otimes_A (X \otimes Y) \rightarrow (M \otimes_A X) \otimes Y
\]

is an isomorphism in \( C \).

By \( A \) we denote the full subcategory of \( A \) whose objects are right robust. By [13, 30] the category \( A \) is monoidal, the monoidal structure being similar to that of \( A \).
Definition 6.5. If \( \mathcal{C} \) is a monoidal category with coequalizers then Bim(\( \mathcal{C} \)) is the bicategory that has as objects coflat algebras in \( \mathcal{C} \), as 1-cells right robust bimodules and as 2-cells bimodule morphisms in \( \mathcal{C} \), respectively. The vertical composition of 2-cells in Bim(\( \mathcal{C} \)) is the morphisms composition in \( \mathcal{C} \) and the horizontal one is given by the monoidal structure of \( \mathcal{C} \).

In the sequel, we will regard Bim(\( \mathcal{C} \)) as a 2-category. Before giving a description of wreaths and cowreaths, we need a description of monads in Bim(\( \mathcal{C} \)).

Lemma 6.6. Giving a monad in Bim(\( \mathcal{C} \)) is equivalent to giving a pair \((T, T)\) of coflat algebras in \( \mathcal{C} \), together with an algebra morphism \( i : A \to T \) in \( \mathcal{C} \) such that \( T \) is right robust when it is considered as an \( A \)-bimodule via \( i \).

Proof. A monad in Bim(\( \mathcal{C} \)) is defined by the following data:

- A 0-cell \( A \) in Bim(\( \mathcal{C} \)), that is, a coflat algebra \( A \) in \( \mathcal{C} \);
- A 1-cell \( T \to A \) in Bim(\( \mathcal{C} \)), that is, a right robust \( A \)-bimodule \( T \) in \( \mathcal{C} \);
- 2-cells \( T T \to T \) and \( 1_A \to T \) satisfying coherence conditions. Explicitly, we have \( A \)-bimodule morphisms \( \mu : T \otimes_A T \to T \) and \( \eta : A \to T \) such that \( (T, \mu, \eta) \) is an algebra in \( \mathcal{C} \).

It is well known that such a data can provide an algebra morphism \( i : A \to T \) in \( \mathcal{C} \). The algebra structure of \( T \) in \( \mathcal{C} \) is given my \( \mu_T = \mu_{T,T} \) and \( \eta_T = \eta_T \), and so \( \eta_T \) becomes an algebra morphism in \( \mathcal{C} \). Conversely, \( T \) is an \( A \)-bimodule in \( \mathcal{C} \) via the restriction of scalars functor defined by \( i \), and its algebra structure in \( \mathcal{C} \) determines an algebra structure in \( \mathcal{C} \). We leave the verification of the details to the reader.

The category Bim(\( \mathcal{C} \)) admits the EM constructions for monads. If \( i : A \to T \) is a monad in Bim(\( \mathcal{C} \)) then \( A^T = T \) as algebras, i.e., objects in Bim(\( \mathcal{C} \)). Furthermore, \( T^T \to A \) is \( T \) regarded as a \((T, A)\)-bimodule in \( \mathcal{C} \) via the multiplication of \( T \) and \( i \). \( T \to T \) in \( \mathcal{C} \), \( \chi^T : T \to T \otimes_A T \to T = T \) is the \((T, A)\)-bimodule morphism in \( \mathcal{C} \) uniquely determined by \( q_{T, T}^A \to \tau^T \). Finally, \( A^T \to A^T \) is again \( T \), viewed now as an \((A, T)\)-bimodule in \( \mathcal{C} \) via \( i \) and \( T \), while \( T \to \mathcal{C}_A \), where \( T \to \mathcal{C}_A \) is the unique morphism in \( \mathcal{C} \) determined by \( q_{T, T}^A \to \tau^T \) but considered now as a \( T \)-bimodule morphism in the usual way.

Lemma 6.7. Let \( i : A \to T \) and \( j : B \to S \) be monads in Bim(\( \mathcal{C} \)). Give a monad morphism between \( A \to T \) and \( B \to S \) is equivalent to giving a pair \((X, \psi)\) consisting of a right robust \((A, B)\)-bimodule \( X \) and an \((A, B)\)-bimodule morphism \( \psi : X \otimes_B S \to T \otimes_A X \) in \( \mathcal{C} \) such that the following diagrams commute:

\[
\begin{array}{ccc}
X \otimes_B S & \xrightarrow{\tilde{m}_S} & T \otimes_A X \\
\psi & \downarrow & \\ \\
X & \xrightarrow{\tilde{m}_B} & T \otimes_A X
\end{array}
\]

\[
\begin{array}{ccc}
X \otimes_B S & \xrightarrow{\tilde{m}_B} & T \otimes_A X \\
\psi & \downarrow & \\ \\
X & \xrightarrow{\tilde{m}_S} & T \otimes_A X
\end{array}
\]

Here \( \tilde{m}_A \) and \( \tilde{m}_B \) stand for the multiplication of the algebras \( T \) and \( S \) in \( \mathcal{C} \). The diagram below commutative:

\[
\begin{array}{ccc}
X \otimes_B S & \xrightarrow{\tilde{m}_S} & T \otimes_A Y \\
\psi & \downarrow & \\ \\
T \otimes_A X & \xrightarrow{\tilde{m}_A} & T \otimes_A Y
\end{array}
\]

Proof. We know that Bim(\( \mathcal{C} \)) admits the EM constructions for monads, hence giving a monad morphism from \( A \to T \) to \( B \to S \) is equivalent to giving a pair \((s : A \to B, \pi : A^T \to B^S)\) of 1-cells in Bim(\( \mathcal{C} \)) such that \( s^T \pi = s u^T \). Clearly, this is equivalent to giving a pair \((X, \chi)\) with \( X \in \mathcal{C}_B \) and \( \chi \in \tau \mathcal{C}_S \) such that \( \chi \otimes S = T \otimes_A X \). This forces \( \chi = T \otimes_A X \) in \( \tau \mathcal{C}_S \), and therefore...
we should have a right \( S \)-module structure on \( T \otimes_A X \) when \( X \) is an \((A, B)\)-bimodule in \( C \). We next show that this is equivalent to the existence of an \((A, B)\)-bimodule morphism \( \psi : X \otimes_B S \to T \otimes_A X \) in \( C \) such that the first two diagrams in the statement are commutative.

Let \( \phi_{T,A} : (T \otimes_A X) \otimes S \to T \otimes_A X \) be a right \( S \)-action on \( X = T \otimes_A X \); we then define \( \psi : X \otimes_B S \to T \otimes_A X \) as follows. First consider \( \psi_0 : X \otimes S \to T \otimes_A X \) as the composition
\[
\psi_0 = \left( X \otimes S \xrightarrow{\mu_T \otimes \text{Id}_S} T \otimes X \otimes S \xrightarrow{\phi_{T,A} \otimes \text{Id}_S} (T \otimes_A X) \otimes S \xrightarrow{\mu_T} T \otimes_A X \right).
\]
We can easily show that \( \psi_0 \) behaves well with respect to the universality property of the coequalizer
\[
X \otimes B \otimes S \xrightarrow{\mu_T \otimes \text{Id}_S} X \otimes S \xrightarrow{\mu_T} X \otimes_B S,
\]
and so there exists a unique morphism \( \psi : X \otimes_B S \to T \otimes_A X \) in \( C \) such that \( \psi q^B_{X,S} = \psi_0 \). We leave it to the reader to show that \( \psi \) is left \( A \)-linear and right \( B \)-linear, and that it satisfies the two equations in the statement.

Conversely, if we know \( \psi \) then \( T \otimes_A X \) becomes a right \( S \)-module via the structure morphism
\[
(T \otimes_A X) \otimes S \xrightarrow{\mu_T \otimes \text{Id}_S} T \otimes (X \otimes S) \xrightarrow{\phi_{T,A} \otimes \text{Id}_S} T \otimes_A X \otimes B \otimes S \xrightarrow{\nu_T} T \otimes_A X \otimes_B S \xrightarrow{\mu_T} T \otimes_A X.
\]
Notice that these two correspondences are the counterparts of the ones defined in Lemma 6.7.

Finally it follows from Proposition 6.3 that giving a 2-cell in \( EM(Bim(C)) \) is equivalent to giving a \((T, S)\)-bimodule morphism \( \tau : T \otimes_A X \to T \otimes_A Y \) in \( C \). Also, it is immediate that \( \tau \) is completely determined by an \((A, B)\)-bilinear morphism \( \bar{\tau} : X \to T \otimes_A Y \). Since \( \tau \) can be recovered from \( \bar{\tau} \) it follows that \( \psi \) is right \( S \)-linear if and only if the third diagram in the statement is commutative. This finishes the proof.

The above explicit description of \( EM(Bim(C)) \) allows us to prove our final result Theorem 6.8. Although Theorem 3.3 follows from Theorem 6.8 this does not lead to a new proof, since the proof of Theorem 6.8 is based on Proposition 4.1 which is itself based on Theorem 3.3.

**Theorem 6.8.** Let \( C \) be a monoidal category with coequalizers. Then there exists a bijective correspondence between the (co)wreath structures in \( Bim(C) \) and pairs \((A, X)\) consisting of a coflat algebra \( A \) in \( C \) and a (co)wreath \( X \) in \( AC^A_A \).

**Proof.** We specialize Corollary 6.4 to \( Bim(C) \) and use the descriptions in Lemma 6.7 to conclude that a (co)wreath in \( Bim(C) \) consists of a triple \((A \to T, X, \psi)\), where
\[
\begin{align*}
(\text{a}) & \quad A, T \text{ are coflat algebras in } C, \ i \text{ is an algebra morphism, and } T \text{ considered as an } A\text{-bimodule via } i \text{ is right robust}; \\
(\text{b}) & \quad X \text{ is a right robust } A\text{-bimodule}; \\
(\text{c}) & \quad \psi : X \otimes_A T \to T \otimes_A X \text{ is an } A\text{-bimodule morphism in } C \text{ such that } T \otimes_A X \text{ becomes a } T\text{-bimodule in } C \text{ when it is considered as a left } T\text{-module via the multiplication } \mu_T \text{ of } T \text{ and as a } T\text{-right module via } \psi \text{ and } \mu_T; \\
(\text{d}) & \quad T \otimes_A X \text{ with the } T\text{-bimodule structure described in (c) admits a } T\text{-}(co)ring structure in } C.
\end{align*}
\]

The conditions (a-d) can be restated as follows:
\[
\begin{align*}
(\text{a'}) & \quad A \text{ is a coflat algebra in } C \text{ and } T \in AC^A_A \text{ is an algebra}; \\
(\text{b'}) & \quad X \text{ is an object of } AC^A_A; \\
(\text{c'}) & \quad \psi : X \otimes_A T \to T \otimes_A X \text{ is a morphism in } AC^A_A \text{ that endows } T \otimes_A X \text{ with a } T\text{-bimodule structure in } AC^A_A; \text{ providing that } T \otimes_A X \text{ has the left } T\text{-module structure given by the multiplication } \mu_T \text{ of the algebra } T \text{ in } AC^A_A; \\
(\text{d'}) & \quad T \otimes_A X \text{ admits a } T\text{-}(co)ring structure in } AC^A_A.
\end{align*}
\]

We now apply Proposition 4.1 to the monoidal category \( AC^A_A \). Then we find that triples \((A \to T, X, \psi)\) satisfying (a’-d’) are in bijective correspondence with pairs \((A, X)\) consisting of a coflat algebra \( A \) and a (co)wreath \( X \) in \( AC^A_A \). This finishes our proof.

\(\square\)
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