Nondeterministic One-Tape Off-Line Turing Machines and Their Time Complexity

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Abstract

In this paper we consider the time and the crossing sequence complexities of one-tape off-line Turing machines. We show that the running time of each nondeterministic machine accepting a nonregular language must grow at least as \( n \log n \), in the case all accepting computations are considered (accept measure). We also prove that the maximal length of the crossing sequences used in accepting computations must grow at least as \( \log n \). On the other hand, it is known that if the time is measured considering, for each accepted string, only the faster accepting computation (weak measure), then there exist nonregular languages accepted in linear time. We prove that under this measure, each accepting computation should exhibit a crossing sequence of length at least \( \log \log n \). We also present efficient implementations of algorithms accepting some unary nonregular languages.

Keywords: Turing machine, Space complexity, Time complexity, Crossing sequence, Unary language

1 Introduction

One of the main problems in the design of computer algorithms and in their implementation is that of producing efficient programs, under the restrictions given by the available resources.

For example, up to the first ‘80 years, when the central memories of the computers were very small and the operating systems of personal computers did not provide virtual memory functionalities, one of the critical topics in the implementation of applications using significative amounts of data was that of choosing data structures requiring only a small amount of extra memory, besides the memory needed by the actual data represented. Many times, computer programmers, in order to cope with a restricted space availability, were forced to choose data structures not efficient from the point of view of the time used by the algorithms manipulating them.

In the last two decades, we assisted to a continue and enormous increasing of the capacity of memory chips, together with a reduction of their costs. As an effect, now

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we can write very large computer programs, implementing sophisticated algorithms, and, at the same time, computer programs can use all this memory capacity to efficiently manipulate huge amount of data. However, in many situations, the space is still a critical resource. For example, the memories of portable devices, like mobile phones, usually are very limited; also the programs for embedded systems have strong memory restrictions. This is the main reason for which the designers of the Java programming language decided to provide, among the primitive data types representing integer numbers, the types byte (8 bits), short (16 bits) for the representation of “small” integers, besides the “standard” types int (32 bits) and long (64 bits).

Thus, the investigation of computing with restricted resources is always an interesting and important topic. From a theoretical point of view, this subject has been considered by studying and comparing the power of machine models using certain amounts of resources. A special area of interest is the border between finite state devices (described by finite automata) and more powerful devices.

Since early '60, many researches have been done in order to discover the minimal amount of computational resources needed by a machine model in order to be more powerful than finite state devices. This kind of investigation can be formalized in the realm of formal languages, by studying the minimal amount of resources needed to recognize nonregular languages.

Among these researches, the most famous are those ones concerning space and time lower bounds.

The investigation of the space requirements for the recognition of nonregular languages started with the works of Hartmanis, Lewis and Stearns, Hopcroft and Ullman. (For surveys see, e.g., and . The last paper presents also some new and recent results.) The first works concerning time requirements have been done by and . We now discuss more in details the time resource, because it is the main subject of this paper.

In his seminal paper, Hennie considered a very restricted computational model: the one-tape off-line Turing machine. This machine has just one semi-infinite tape that at the beginning of the computation contains the input string (for this reason it is called “off-line”). The content of the tape can be modified during the computation. The original model was deterministic. The time is measured by taking into account the time used by the machine on all input strings. Hennie proved that if a machine of this kind works in linear time then the accepted language is regular. Hence, the capability of storing data on the tape is useless under this time restriction. A stronger bound on the time was independently obtained by and by Hartmanis: they proved that in order to recognize nonregular languages the time must grow at least as \( n \log n \).

A similar investigation has been done, later, for the nondeterministic version of one-tape off-line machines. In 1986, Wagner and Wechsung provided a counterexample.
showing that the $n \log n$ time lower bound cannot hold in the nondeterministic case. Subsequently, in 1991, Michel \cite{16} proved the existence of NP-complete languages accepted in linear time\cite{2}. However, these results were obtained by measuring the time in an “optimistic” way, namely by considering, among all accepting computations on a string, the faster one (this measure is called \textit{weak}). On the other hand, in 2004, Tadaki, Yamakami, and Lin \cite{19} proved that, by taking into account, for each input string, all the computations (this measure is called \textit{strong}), the $n \log n$ lower bound holds even for the nondeterministic case.

In this paper, we further refine these results. Besides the \textit{strong} and the \textit{weak} measures, we consider an intermediate measure called \textit{accept}. This measure considers the costs of all accepting computations. (It has been already used in the literature, mainly for the space complexity, and differences with other measures have been shown, see \cite{15}.) First, we extend the $n \log n$ lower bound for nonregular language acceptance in the nondeterministic case, from the \textit{strong} measure to the \textit{accept} measure. Concerning the \textit{weak} measure, we are able to prove that in order to accept a nonregular language, some tape cells must be visited at least $\log \log n$ times, even if the language is accepted in linear time. This will be proved using the concept of \textit{crossing sequence}\cite{3}.

We will also present some examples of languages accepted by these machines, obtaining fast implementations of some recognition algorithms. All these examples are \textit{unary} of \textit{tally} languages, namely languages defined over a one-letter alphabet. So, the only information that the machine can use to accept or reject an input string is its length. We think that the techniques used to implement these algorithms are interesting examples of programming with very restricted resources. Furthermore, different implementations of the same recognition algorithms, using restricted amounts of space, have been presented in the literature.

\section{Preliminary notions and definitions}

As usual, we denote by $\Sigma^*$ the set of all strings over a finite alphabet $\Sigma$, by $\epsilon$ the empty string, and by $|x|$ the length of a string $x \in \Sigma^*$.

A \textit{language} $L$ is a subset of $\Sigma^*$. $L$ is said to be a \textit{unary} or \textit{tally language} if $\Sigma$ contains only one element. In this case we stipulate $\Sigma = \{a\}$.

Given a language $L$ and an integer $n$, we denote by $L_{\leq n}$ the set of strings of length at most $n$ belonging to $L$.

In the paper we will consider \textit{one-tape off-line Turing machines} \cite{9}. Besides a finite state control, these devices are equipped with a semi-infinite tape. At the beginning of the computation the input string is written on the tape starting from the leftmost tape square, while the remaining squares contain the blank symbol. At each step of the computation, the machine writes a symbol on the currently scanned square of the tape (possibly changing its content), and moves its head to the left, to the right, or keeps it stationary, according to the transition function. The machine never writes the blank symbol. Hence, it is able to work and to modify the part of the tape that, at the beginning, was containing the input.

\footnote{It is trivial to observe that there are regular languages requiring linear time.}

\footnote{Two tables summarizing the already known lower bounds and the new results proved in this paper can be found in the last section.}
string, as the part of the tape to the right of the input string. However, after a square has been visited, it will never contain the blank symbol. Special states are designed as accepting and rejecting states. We assume that in these states the computation stops.

To study the behavior of this kind of machines, it is useful the notion of **crossing sequences**, that we now recall [9]. Let us consider a deterministic or nondeterministic one-tape off-line Turing machine \( M \). Given a computation \( C \) of \( M \) and a boundary \( b \) between two squares of the tape, the crossing sequence of \( C \) at \( b \) is the sequence of the states \( (q_1, q_2, \ldots, q_k) \), where \( q_i \), \( i = 1, \ldots, k \), is the state of \( M \) when in the computation \( C \) the head crosses for the \( i \)th time the boundary \( b \). Note that for odd values of \( i \) the state \( q_i \) corresponds to a move of the head from left to right, while for even values of \( i \) to a move in the opposite direction. Hence, if the length \( k \) of the crossing sequence \( c \) is odd then the computation \( C \) must end with the head scanning a square to the right of the boundary \( b \), while if \( k \) is even then \( C \) must end with the head scanning a square to the left of \( b \). A nonending computation could have crossing sequences of infinite length. However, all the crossing sequences we will have to consider in our proofs are finite.

We also point out that given two finite sequences \( c', c'' \) of states, it is possible to verify whether or not they are “compatible” with respect to a tape symbol \( a \), namely, if \( c' \) and \( c'' \) can be, respectively, the crossing sequence at the left and at the right boundary of a tape square initially containing \( a \). This test can be done “locally”, i.e., without knowing the rest of the tape content. We now sketch a nondeterministic procedure performing this test.

Let \( c' = (q'_1, \ldots, q'_{k'}) \), \( c'' = (q''_1, \ldots, q''_{k''}) \) be the two crossing sequences. The head of the machine will reach the tape square under consideration for the first time with a move from the square to its left side. Hence, the state during the first visit of the cell is \( q'_1 \). The next move depends on \( q'_1 \) and on the original content \( a \) of the square under consideration. The procedure selects, in a nondeterministic way, one transition among the finitely many possible.

- **If the selected transition moves the head to the left**, in the state \( q'_2 \), writing a symbol \( b \) on the tape square, then the next visit to the tape square should be from the left side, in the state \( q'_3 \). Hence, the procedure continues in a similar way, after replacing \( c' \) with the shorter sequence \( (q'_3, \ldots, q'_{k'}) \), and by considering \( b \) as content of the tape square.

- **If the selected transition moves to the right**, in the state \( q''_1 \), writing a symbol \( b \) on the tape square, then the next visit to the tape square should be from the right side, in the state \( q''_2 \). In this case the verification procedure continues expecting the next move from the right side. The crossing sequences \( c' \) and \( c'' \) are replaced, respectively, by the shorter sequences \( (q'_2, \ldots, q'_{k'}) \) and \( (q''_2, \ldots, q''_{k''}) \). The square content considered will be the new symbol \( b \). Hence, at the next step, a transition from the state \( q''_2 \) with the symbol \( b \) should be considered.

- **If the transition does not move the input head** (stationary move) a new move from the reached configuration is selected and the process is repeated (after updating the symbol on the square) until to select either a move to the left or to the right, or to reach a configuration where the computation stops. Notice that if a finite sequence of stationary moves reaches a same configuration twice, then there exists a shorter
sequence of stationary moves ending as the original sequence. The maximal number of stationary moves, before a repetition, is given by the product of the number of the states by the number of the possible symbols. Hence, if the procedure exceeds this number, it can stop and reject.

- In the other cases the procedure stops and reject.

The procedure accepts if it is able to end with the two crossing sequences empty. (Many technical details have been omitted. For a more extended discussion about this topic, in the context of the reduction of two-way automata to one-way ones we address the reader to [10].)

Given four strings \( u, v, u', v' \in \Sigma^* \), and two computations \( C \) and \( C' \) on inputs \( uv \) and \( u'v' \), respectively, suppose that the crossing sequence \( c \) of \( C \) at the boundary between \( u \) and \( v \) coincides with the crossing sequence of \( C' \) at the boundary between \( u' \) and \( v' \). Thus, there is a computation \( \hat{C} \) on the input \( u'v \) which exhibits the same crossing sequence \( c \) at the boundary between \( u' \) and \( v \). Furthermore, \( \hat{C} \) behaves as \( C' \) on the prefix \( u' \) and as \( C \) on the suffix \( v \). Note that if the length of \( c \) is odd, then \( u'v \) is accepted or rejected by \( \hat{C} \) if and only \( uv \) if it is, respectively, accepted or rejected by \( C \). In a similar way, if the length of \( c \) is even, then \( u'v \) is accepted or rejected by \( \hat{C} \) if and only \( u'v' \) if it is, respectively, accepted or rejected by \( C' \). Because \( \hat{C} \) is obtained by pasting together pieces of \( C \) with pieces of \( C' \), according to a “cut-and-paste” of the inputs, we will call this method cut-and-paste.

We consider also the standard Turing machine model, having a finite state control, a two-way read-only input tape and one separate semi-infinite worktape (see, e.g., [10]).

For each computation \( C \) of a deterministic or nondeterministic, one-tape off-line or standard Turing machine \( M \), we consider the following resources:

- The time, denoted as \( t(C) \), is the number of moves in the computation \( C \).

- The space, denoted as \( s(C) \), is the number of cells used in the computation \( C \). In the case of standard machines, the space is measured by keeping into account only the worktape. (For one-tape off-line machines we do not consider this resource in the paper.)

- The length of the crossing sequences, denoted as \( c(C) \), for a one-tape off-line machine is the number of the states in the longest crossing sequence used by \( C \). (It is possible to give a similar measure even for standard machines see, e.g., [3], but we will do not make use of it in the paper.)

Given an input string \( x \), we want to consider how much of the above resources is used by the machine \( M \), having \( x \) as input. In the case of deterministic machines, there is only one computation associated with each string \( x \). Hence the use of each resource could be trivially defined referring to such a computation. On the other hand, in the case of nondeterministic machines, we can have several computations on a same string. This leads to several measures. We now present and briefly discuss those ones considered in the paper.

We say that machine \( M \) uses \( r(x) \) of a resource \( r \in \{ t, s, c \} \) (time, space, length of crossing sequences, resp.), on an input \( x \) if and only if
• **strong measure:**
  \[ r(x) = \max \{ r(C) \mid C \text{ is a computation on } x \} \]

• **accept measure:**
  \[ r(x) = \begin{cases} \max \{ r(C) \mid C \text{ is an accepting computation on } x \} & \text{if } x \in L \\ 0 & \text{otherwise} \end{cases} \]

• **weak measure:**
  \[ r(x) = \begin{cases} \min \{ r(C) \mid C \text{ is an accepting computation on } x \} & \text{if } x \in L \\ 0 & \text{otherwise} \end{cases} \]

The **weak** measure corresponds to an optimistic view related to the idea of nondeterminism: a nondeterministic machine, besides choosing an accepting computation, if any, is able to chose that one of minimal cost. On the opposite side, the **strong** measure keeps into account the costs of all possible computations. Between these two measures, the **accept** one keeps into account the costs of all accepting computations. (For rejected inputs, for technical reasons, it is suitable to set the **accept** and the **weak** measure to 0). These notions have been proved to be different, for example in the context of space bounded computations [15].

As usual, we will mainly define complexities with respect to input lengths. This is done by considering the worst case among all possible inputs of the same length. Hence, under the **strong**, **accept**, and **weak** measures, for \( r \in \{ t, s, c \} \), we define

\[ r(n) = \max \{ r(x) \mid x \in \Sigma^*, |x| = n \}. \]

All the logarithms are in bases 2. We assume that the reader is familiar with the asymptotic notations, in particular with **big-Oh** and **little-Oh** (denoted, respectively, as \( O(\cdot) \) and \( o(\cdot) \)), and with basic techniques concerning Turing machines, in particular with the use of tape tracks.

### 3 Simple bounds

In this section, we present some simple lower bounds for the recognition of nonregular languages. Probably they are already known as “folklore”, but we include them for the sake of completeness and also because some of the arguments used to prove them will be interesting later in the paper. We formulate the bounds for the **weak** measure. As a consequence, they hold for all the measures considered in the paper.

First of all, considering crossing sequences, we prove the following:

**Theorem 1** If \( L \) is accepted by a nondeterministic one-tape off-line Turing machine \( M \) with \( c(n) = O(1) \), under the **weak** measure, then \( L \) is regular.

**Proof:** Let \( M \) be a nondeterministic one-tape off-line Turing machine accepting \( L \) and \( c \) a constant such that, for each \( x \in L \), \( M \) has an accepting computation on \( x \) using only
crossing sequences of length bounded by \( c \). Without loss of generality, we can suppose that when \( M \) accepts, its head is positioned on the leftmost square of the tape containing the blank symbol (namely, besides the right end of the input and of the tape portion used during the computation).

There exists a nondeterministic finite automaton \( N \), whose set of states is the set of all possible sequences at most \( c \) states of \( M \), whose initial state is the sequence containing only the initial state of \( M \), whose transition function is defined according to the compatibility relation between crossing sequences described in Section 2, and whose set of final states contains the sequences which, at the right boundary of the input, lead to the acceptance. (By definition, the machine \( M \) during the computation can also use a portion of the tape to the right of the input string. At the beginning of the computation, this portion always contains blank symbols, hence, it does not depend on the input. The possible behaviors of \( M \) on this portion, completely depend on the crossing sequence at its left boundary. In particular, for a given machine, it is possible to determine the set of crossing sequences that, to the left of a blank portion of the tape, lead to the acceptance.)

It can be easily observed that each accepting computation of \( N \) simulates an accepting computation of \( M \). Since each \( x \in L \) has an accepting computation using crossing sequences of length at most \( c \), we can conclude that \( N \) accepts exactly \( L \).

We now consider the time. We can observe that in each computation \( C \) on an input \( w \), ending in less than \( |w| \) steps, the head cannot reach the right end of the input. This implies that the same computation can be performed on each string having \( w \) as a prefix. Using this observation, it is not difficult to prove that if a language \( L \) is accepted in time \( t(n) = o(n) \) under the strong measure, then \( L \) is regular and, actually, it can be accepted in constant time. This result can be proved also for the weak measure:

**Theorem 2** Let \( M \) be a nondeterministic (one-tape off-line or standard) Turing machine accepting a language \( L \) in time \( t(n) = o(n) \), under the weak measure. Then \( t(n) = O(1) \) and \( L \) is regular.

**Proof:** Since \( t(n) = o(n) \), it should exists an integer \( n_0 \) such that \( t(n) < n \) for each \( n \geq n_0 \). We prove that each string belonging to \( L \) is accepted by a computation of less than \( n_0 \) steps, thus implying \( t(n) = O(1) \).

Given \( x \in L \) with \( |x| \geq n_0 \), there exists a computation accepting \( x \) in \( t(x) < |x| \) steps. Hence, such a computation can read at most the first \( t(x) \) symbols of \( x \) and, so, it will accept even the prefix \( x_0 \) of \( x \) of length \( t(x) \). Hence, \( t(x_0) \leq t(x) = |x_0| \).

If \( t(x_0) < |x_0| \), then there exists a computation which accepts \( x_0 \) just reading a proper prefix of \( x_0 \). Because \( x_0 \) is a prefix of \( x \), the same computation should accept also \( x \). Since the length of such a computation is less than \( t(x) \), this is a contradiction.

This permit us to conclude that \( t(x_0) = t(x) = |x_0| \). Because \( t(n) < n \), for \( n \geq n_0 \), we can conclude that \( |x_0| < n_0 \). Hence, \( x \) is accepted in less than \( n_0 \) steps.

We finally point out that this result does not depend on the machine model: it holds for one-tape off-line Turing machines as well as for standard machines. \( \square \)
4 Lower bounds for the accept measure

The problem of proving lower bounds for the length of crossing sequences and for the running time of one-way off-line Turing machines accepting nonregular languages was investigated by several authors. Considering deterministic machines, in the paper of Hennie [9] a logarithmic lower bound for the length of the crossing sequences was proved. Furthermore, it was shown that each language accepted in linear time is regular. A better lower bound (of the order of $n\log n$) for the time needed to recognize nonregular languages was independently proved by Trakhtenbrot [20], Hartmanis [6], and Kobayashi [13].

For the nondeterministic case, Wagner and Wechsung [21] showed that under the weak measure the same does not hold. In particular, Michel [16] gave examples of nonregular languages accepted in linear time. However, considering the strong measure, recently Tadaki, Yamakami and Lin [19] extended the argument used by Kobayashi to nondeterministic machines, by proving that in such a case, the logarithmic lower bound on the length of the crossing sequences and the $n\log n$ lower bound for the time needed to accept nonregular languages still hold. In this section, we further deepen such an investigation by showing that the lower bounds for the deterministic case are true even for nondeterministic machines, if we restrict our attention only to accepting computations, namely if we replace the strong measure considered in [19] with the accept measure. Our proof is obtained by suitable extending that one presented in the book of Wagner and Wechsung [21 Th.s 8.15, 8.17] for the deterministic case.

Theorem 3 Let $M$ be a nondeterministic one-tape off-line Turing machine, using crossing sequences of length $c(n)$ and working in time $t(n)$ under the accept measure. Then:

1. $c(n) = o(\log n)$ implies $c(n) = O(1)$,
2. $t(n) = o(n\log n)$ implies $t(n) = O(n)$.

Furthermore, $c(n) = O(1)$ if and only if $t(n) = O(n)$. In this case the accepted language is regular.

Proof: As in the proof of Theorem 11 without loss of generality we can suppose that $M$ accepts with the head to the right of the portion of the tape used during the computation.

For each integer $k \geq 1$, let $L[k]$ be the set of strings having an accepting computation whose longest crossing sequence consists of exactly $k$ states.

For $L[k] \neq \emptyset$, let $w_k$ be a shortest word belonging to $L[k]$ and $n_k = |w_k|$. Hence, $c(n_k) \geq k$. Consider a computation $C$ accepting $w_k$ and using a crossing sequence $c_0$, occurring at a tape boundary $b_0$, with $|c_0| = k$.

If a same crossing sequence $c$ occurs in the computation $C$ at three boundaries $b_1, b_2, b_3$ of the input zone, with $1 \leq b_1 < b_2 < b_3 \leq |w_k|$, then $w_k$ can be decomposed as $xyzt$, where $x$ is the prefix of $w_k$ which ends at the boundary $b_1$, $y \neq \epsilon$ and $z \neq \epsilon$ are the factors delimited, respectively, by the boundaries $b_1$ and $b_2$, and $b_2$ and $b_3$, $t$ is the suffix which starts at the boundary $b_3$. We can use the “cut-and-paste” method discussed in Section 2 combining in several ways these strings and the associated computation slices. In particular, if $b_0 > b_2$ then we can get an accepting computation on the string $xzt$ that still uses (in the part $zt$) the crossing sequence $c_0$. In a similar way, if $b_0 \leq b_2$ then we can get an accepting computation on the string $xyt$ which uses $c_0$. Hence, in both cases, we get a string $w'$ shorter than $w_k$ and belonging to $L[k]$. This is a contradiction to our
choice of \( w_k \). This permit us to conclude that every crossing sequence used in the input zone of \( w_k \) can occur at most twice.

Consequently, \( w_k \) has at least \( \left\lfloor \frac{n_k - 1}{q} \right\rfloor \) different crossing sequences. If \( q \) is the number of the states of \( M \), this implies \( \left\lfloor \frac{n_k - 1}{q} \right\rfloor \leq q^{k+1} \), and hence \( c(n_k) \geq k \geq \log_q \left\lfloor \frac{n_k - 1}{q} \right\rfloor - 1 \).

If \( c(n) \) is unbounded, then \( L[k] \neq \emptyset \) for infinitely many \( k \), thus implying \( c(n) \geq d \log_2 n \), for some constant \( d > 0 \) and infinitely many integers \( n \), i.e., \( c(n) \neq o(\log_2 n) \).

We can easily observe that \( c(n) = O(1) \) implies \( t(n) = O(n) \). Hence, if \( M \) does not work in linear time, then \( c(n) \) is unbounded and, according to the previous part of the proof, there exists a sequence \( w_1, w_2, \ldots \) of strings with \( |w_1| < |w_2| < \ldots \) such that \( w_i \) has an accepting computation \( C_i \) using at least \( \left\lfloor \frac{|w_i| - 1}{q} \right\rfloor \) different crossing sequences. On the other hand, the number of different crossing sequences of \( C_i \) of length less than \( \frac{\log |w_i|}{\log q} \) is at most \( \left\lfloor \frac{|w_i| + 1}{q} \right\rfloor \). As a consequence, the accepting computation \( C_i \) has at least \( \left\lfloor \frac{|w_i| - 1}{q} \right\rfloor \) different crossing sequences of length at least \( \frac{\log |w_i|}{\log q} \) and, hence, it consists of at least \( \left\lfloor \frac{|w_i| - 1}{q} \right\rfloor \cdot \frac{\log |w_i| + 1}{\log q} \geq d|w_i|\log(|w_i|) \) steps, for some constant \( d > 0 \). This permits us to conclude that \( t(n) \geq d n \log n \), for infinitely many integers \( n \). Hence, \( t(n) = o(n \log n) \) implies \( t(n) = O(n) \) and \( c(n) = O(1) \).

By summarizing the implications we have proved, we can also conclude that \( c(n) = O(1) \) if and only if \( t(n) = O(n) \). In this case, by Theorem 3 it follows that the language accepted by \( M \) is regular.

\( \square \)

5 Lower bound for the weak measure

While the strong measure takes into account the costs of all computations and the accept measure the costs of all accepting computations, the weak measure restricts only to accepting computations of minimal costs. This is closely related to the notion of nondeterminism: a machine \( M \), among all possible computations, selects the accepting one of minimal cost.

Under the weak measure, the time lower bound presented in Theorem 3 for nonregular language acceptance does not hold. In fact, as proved by Michel [16], it is possible to accept in linear time some NP-complete languages.

On the other hand, as proved in Theorem 1 a constant bound on the length of the crossing sequences implies the regularity of the accepted language. Hence, we can ask if it is still possible to state a lower bound on the maximal length of the crossing sequences for the recognition of nonregular languages, under the weak measure. The next theorem gives a positive solution to this problem, stating a \( \log \log n \) lower bound (hence, smaller than the bound for the accept measure). In the next section we will show that this lower bound is reachable, even in the case of languages defined over a one letter alphabet.

**Theorem 4** Let \( M \) be a nondeterministic one-tape offline Turing machine, using crossing sequences of length at most \( c(n) \) under the weak measure. If \( c(n) = o(\log \log n) \) then \( M \) accepts a regular language.
Proof: Using the same technique as in the proof of Theorem 1, we can show that for each integer \( n \geq 1 \) there exists a nondeterministic finite automaton \( N_n \), whose states are the sequences of at most \( c(n) \) states of \( M \), behaving as \( M \) on strings of length at most \( n \).

Hence, denoting by \( L \) the language accepted by \( M \), we get that \( L(N_n) \leq n = L \leq n = L(A_n) \leq n \), where \( A_n \) is a deterministic automaton equivalent to \( N_n \).

If \( q \) is the number of the states of \( M \), then the number of the states of \( N_n \) is bounded by \( q^{c(n)+1} \), and that of \( A_n \) by \( 2^{2^{c(n)+1} \log q} \). By a result of Karp [12], if \( L \) is nonregular, such a number must be at least \( \frac{n+3}{2} \), for infinitely many \( n \). Hence, we get that \( c(n) \geq d \log \log n \), for some constant \( d \), infinitely often.

In the proof of Theorem 4, for each integer \( n \) we simulate the machine \( M \) with a finite automaton, which agrees with \( M \) on strings of length at most \( n \). Using a similar idea, we can simulate the machine \( M \) with an equivalent standard machine \( M' \) that uses a one-way read-only input tape. The machine \( M' \) works essentially as the automaton \( N_n \), scanning the input from the left to the right and checking the compatibility between crossing sequences. However, the machine \( M' \) does not have any limitation on the length of the crossing sequences. This is achieved using two tracks of the worktape. The first track contains a crossing sequence at the left boundary of the current input square. The machine guesses on the second track another crossing sequence and verifies whether or not it is compatible with that on the first track, with respect to the symbol under the input head. If the outcome of this test is negative then the machine stops and reject, otherwise the machine continues the computation. When the right end of the input tape is reached, the machine \( M' \) can continue by simulating the crossing sequences on the blank part of the tape of \( M \). Hence, we get the following:

**Theorem 5** Each (deterministic or nondeterministic) one-tape off-line Turing machine \( M \) can be simulated by an equivalent one-way nondeterministic standard Turing machine \( M' \) such that:

- If \( M \) uses crossing sequences of length at most \( c(n) \) under the weak measure, then \( M' \) works in space \( c(n) \) under the weak measure.
- If \( M \) uses crossing sequences of length at most \( c(n) \) under the strong or accept measures, then \( M' \) works in space \( c(n) \) under the accept measure.

Proof: From the above outlined construction, we can observe that to each accepting computation of \( M \) using crossing sequences of length at most \( k \) corresponds an accepting computation of \( M' \) using space \( O(k) \). Hence, the two statements of the theorem easily follows.

Space lower bound for Turing machines accepting nonregular languages have been extensively investigated in the literature. In particular, in the case of one-way nondeterministic machines, it is known that in order to accept a nonregular language the space used by the machine must grow at least as \( \log \log n \) under the weak measure [1] and as

\[ \frac{n+3}{2} \]

\[ \log \log n \]
log \( n \) under the accept measure \[15\]. Combining these space lower bounds with the result given in Theorem \[5\] we get an alternative proof of the lower bounds on the length of the crossing sequences presented in Theorem \[3\] and in Theorem \[4\].

6 Fast recognition of unary languages

In this section we present some interesting examples of nonregular languages accepted by one-tape off-line Turing machines, using small amounts of time. All the languages we consider are defined over a one letter alphabet and have been previously presented in the literature because they have small space complexities on standard machines.

Before introducing the examples, we briefly recall a technique presented in \[16, proof of Th. 2.2\] which will be extensively used in our recognition algorithms. This technique is useful to count input factors of length \( k \), using \( O(k \log k) \) moves, on a one-tape off-line machine:

- Three tape tracks are used: track 1, which is left unchanged, contains the input string; track 2 and track 3 are used to count.

- The computation starts with the head scanning the first symbol of the input that must be counted, and 0 written on track 2 (in the same cell scanned by the head). At each step\(^5\) the counter on track 2 is incremented. However, it is also shifted one position to the right in such a way that, after \( j \) steps, the representation of the number \( j \) on the track 2 is not “too far” from the counted position \( j \). More precisely:

  - After counting \( j \leq k \) input symbols, \( j \) must be written in a base \( d \) on track 2, starting from the tape square under the \( j \)th position.

  - To increment from position \( j \) to position \( j+1 \), the counter is copied and incremented, one position to the right on track 3. (Then, it is copied back on track 2 for the next step, or the roles of tracks 2 and 3 are switched.)

The increment step from \( j \) to \( j+1 \) uses at most \( c_1 \log_d j + 1 \) moves, for some constant \( c_1 \). Since, during this process, the counter is incremented from 0 to \( k \), the total number of moves is bounded by \( k(c_1 \log_d k + 1) \leq \frac{\log k}{\log \log k} k \log k = O(k \log k) \) for sufficiently large \( k \). Hence, a factor of length \( k \) of the input can be counted in time \( O(k \log k) \), by a deterministic procedure. It can be also observed that, by suitably choosing the base \( d \), the time can be bounded by \( \epsilon k \log k \), for any given \( \epsilon > 0 \).

We are now ready to present our first example. We denote by \( p_1, p_2, \ldots \) the sequence of prime numbers in increasing order. We consider the language

\[
L_0 = \{ a^n \mid \exists t \geq 1 \text{ s.t. } n \text{ is divisible by } p_1, p_2, \ldots, p_t \text{ but not by } p_1^2, p_2^2, \ldots, p_t^2 \text{ and } p_{t+1} \}.
\]

Hartmanis and Berman \[7\] proved that \( L_0 \) is a nonregular language accepted in \( O(\log \log n) \) space by a deterministic standard machine under the strong measure. Hence, \( L_0 \) matches...
the space lower bound for this kind of machines. We now show that this language matches also the time lower bound proved in Theorem 3. More precisely:

**Theorem 6** $L_0$ is accepted by a deterministic one-tape off-line Turing machine in time $O(n \log n)$, under the strong measure.

**Proof:** In order to verify membership to $L_0$, the following immediate algorithm can be used:

input $a^n$

$i \leftarrow 1$

while $p_i$ divides $n$ and $p_i^2$ does not divide $n$ do

$i \leftarrow i + 1$

if $p_i$ does not divide $n$ then accept

else reject

Now, we explain how the steps of this algorithm can be implemented in the machine model we are considering and we evaluate the time used by such an implementation:

- **Testing divisibility of $n$ by an integer $k$:**
  Given $k$ written on a track $T$ of the tape, we can adapt the above described procedure, in order to count the input length modulo $k$. The procedure is modified in such a way that at each step, the value of the counter is compared with the original value of $k$, resetting the counter when it reaches $k$. In order to avoid the use of extra moves, at each step, when the counter is shifted and incremented, also the representation of $k$ on the track $T$ is moved one position to the right. In this way, when the right side of the input is reached, the value contained in the counter will be $n \mod k$. Hence, the machine can finally verify whether or not such a value is 0.

  The time used to count a factor of length $k$ is $O(k \log k)$. Because this is repeated $\left\lceil \frac{n}{k} \right\rceil$ times, the overall time used to test the divisibility is $O(n \log k)$.

- **Testing divisibility of $n$ by $k^2$, for a given integer $k$:**
  This can be done, while checking the divisibility of $n$ by $k$, just introducing another counter modulo $k$ (making use of extra tracks). The new counter is shifted each time an input position is counted (like the first counter used to test the divisibility by $k$). However, it is incremented only when the first counter is reset, i.e., each time a block of $k$ input symbols has been scanned.

  We point out that this test can be performed in the above time bound.

- **Computing the sequence of primes $p_1, p_2, \ldots$:**
  This is done by using the Eratostene sieve algorithm, implemented as a “coroutine” of the main program. A special track, called $P$, is used to designate the positions of non prime numbers, i.e., the cell in position $j$ of $P$ will be marked with a special symbol $X$ after discovering that $j$ is not a prime number. At the beginning of the computation, all the cells are unmarked. When the machine tests the divisibility of $n$ by a prime $p_i$, using the above described procedure, it can also mark on the track $P$ the positions corresponding to multiples of $p_i$ (different from $p_i$). Hence, also the generation of the primes can be done within the previous time bound.
Preparing the next iteration: $i ← i + 1$

Actually, we do not explicitly need the value of $i$, but we have to switch from a prime $p_i$ to the next prime $p_{i+1}$. To this aim, on the track $P$, the first unmarked position to the right of the position $p_i$ must be searched. We remind that the prime $p_i$ was kept on the track $T$ (it has been shifted to the right side, to test the divisibility of $n$ by $p_i$, but after that it can be moved back within $O(n \log p_i)$ steps). Starting from the left side of the tape, the position $p_i$ is reached (counting until $p_i$). Hence, the positions to its right side are scanned (by continuing to count), until to reach the first square which on the track $P$ is unmarked. The position of such a square (available in the counter) gives the next prime $p_{i+1}$. Finally, the representation of $p_{i+1}$ is shifted back, in order to have it at the beginning of track $T$, to be used in the next iteration.

The number of steps used to get $p_{i+1}$ from $p_i$ by this procedure is $O(p_{i+1} \log p_{i+1})$, which is bounded by $O(n \log p_i)$ because the primes we have to consider do not exceed $n$ and $p_{i+1} < 2p_i$ [5].

Hence, the total number of steps used to prepare the iteration $i + 1$ is $O(n \log p_i)$.

We can now evaluate the overall time used by this procedure. Given an integer $n$, let $x$ be the first prime not dividing $n$. The above described algorithm accepts $a^n$, if it belongs to $L_0$, ending with $p_i = x$. On the other hand, if $a^n \notin L_0$ then the algorithm rejects with $p_i < x$. In both cases, the time used is bounded by:

$$\sum_{p_i \leq x} cn \log p_i = cn \sum_{p_i \leq x} \log p_i,$$

for some constant $c$. Since the sum on the (natural) logarithms of all primes not exceeding an integer $x$ is asymptotic to $x$ and the smaller prime not dividing an integer $n$ is $O(\log n)$ [5, 7], the value of the last sum is $O(\log n)$. Hence, we can finally conclude that the total running time is $O(n \log n)$. \hfill $\square$

We give a further example of language recognized in time $O(n \log n)$ by a deterministic machine under the strong measure. Such a language was presented by Alt and Mehlhorn [2], that showed that it is accepted by a deterministic standard machine in space $O(\log \log n)$ (minimal amount of space needed to recognize nonregular languages). That result was improved in [3], where it has been shown how to recognize the same language with a deterministic standard machine that uses the same amount of space and $O(\frac{\log n}{\log \log n})$ input head reversals (minimum amount needed for the recognition of nonregular languages by machines using the minimum amount of space $O(\log \log n)$).

Given a nonnegative integer $n$, let us denote by $q(n)$ the smallest integer non dividing $n$. We consider the language

$$L_{AM} = \{a^n \mid q(n) \text{ is a power of } 2\}$$

**Theorem 7** $L_{AM}$ is accepted by a deterministic one-tape off-line Turing machine in time $O(n \log n)$, under the strong measure.

**Proof:** In [3], it is proved that for each integer $n$, the number $q(n)$ is a power of a prime number. As a consequence, an algorithm was described that divides $n$ by all prime powers, considered in the increasing order, until to find $q(n)$. 

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We can implement such an algorithm by a one-tape off-line Turing machine, along the same ideas used in the proof of Theorem 6. The main difference is that in this case we have to test divisibility of the input length not only for prime numbers, but also for their powers. To this aim, we modify the routine implementing the Eratostene sieve as we now explain. We use two different symbols $X$ and $Y$ to mark the positions on the track $P$: when we have to mark the square at position $k$, because we discovered that $k$ is a multiple of a prime number, we first check the content of the square. If the square is unmarked, then we mark it with the symbol $X$ (this means that just one prime divisor of $k$ has been found), otherwise, we mark it with $Y$ (this means that at least two different primes dividing $k$ have been found). In this way, up to a certain value, the prime powers correspond to the positions that in the track $P$ are unmarked or are marked with $X$.

Finally, when $q(n)$ has been computed, the machine must verify whether or not it is a power of 2. This can be trivially done if the counters used in the computation are represented in base 2.

We can now estimate the time used by this implementation of the algorithm. For each prime power $k$, $O(n \log k)$ steps are used to check the divisibility of $n$ by $k$. Furthermore, the number of prime powers not exceeding an integer $m$ is $O\left(\frac{m}{\log m}\right)$ [3], and $q(n) = O(\log n)$ [2]. Hence, for some constant $c$, the time is at most
$$
\sum_{\substack{k \leq q(n) \text{ prime power}}} cn \log k = cn \log q(n) \cdot \frac{q(n)}{\log q(n)} = cn \cdot q(n) = O(n \log n).
$$

The languages $L_0$ and $L_{AM}$ are interesting examples of languages accepted with a minimal amount of resources. Besides to be examples of languages accepted using a small amount of space by standard machines (under the strong and accept measures) [15], they use also a minimal amount of time (under the same measures) on one-tape off-line Turing machines, as proved in Theorems 6 and 7, showing in this way that the lower bound stated in Theorem 3 is tight even in the case of unary languages.

The complement of $L_{AM}$ seems to be even more interesting. In fact, as shown [15], under the weak measure it can be accepted in space $O(\log \log n)$ even by a standard one-way machine, proving the optimality of the corresponding space lower bound in the unary case (a similar result it is not known for $L_0$ and $L_{AM}$). Concerning one-tape off-line Turing machines accepting the complement of $L_{AM}$, we can prove the following:

**Theorem 8** The complement of $L_{AM}$ is accepted by a nondeterministic one-tape off-line Turing machine in time $O(n \log \log n)$, using crossing sequences of length $O(\log \log n)$, under the weak measure.

**Proof:** It can be observed that for each integer $n \geq 1$, $q(n)$ is not a power of 2 if and only if there are two positive integer $s$ and $t$ such that $2^s < t < 2^{s+1}$, $n \mod 2^s = 0$, and $n \mod t \neq 0$. Using such a property, in [15] the following nondeterministic algorithm for the recognition of the complement $L_{AM}$ was presented:

input $a^n$
guess an integer $s$, $s > 1$
guess an integer \( t, 2^s < t < 2^{s+1} \)
if \( n \mod 2^s = 0 \) and \( n \mod t \neq 0 \) then accept
else reject

The above algorithm can be implemented as follows: one track of the tape is used to guess a power of 2 in binary notation, while another track is used to guess the integer \( t \). By making use of the technique described in the proof of Theorem 6, the machine can check in time \( O(n \log t) \) the divisibility of \( n \) by \( 2^s \) and by \( t \), and, hence, accept or reject the input. If \( a^n \notin L_{AM} \) then there is an accepting computation such that \( t = q(n) \). Furthermore, \( q(n) = O(\log n) \) [2]. This permit us to conclude that the time used by the machine, under the weak measure, is \( O(n \log \log n) \).

Theorem 8 proves the optimality of the lower bound given in Theorem 4 for the length of the crossing sequences in the case of the weak measure. Concerning the time, we recall that in [16], the existence of NP-complete languages accepted in linear time, under the weak measure, have been proved. Such languages are obtained using a padding technique which relies on the use of an input alphabet with more than one symbol. We conjecture that, in the unary case, the recognition of unary nonregular languages by nondeterministic one-tape off-line Turing machines requires, under the weak measure, more than linear time. More precisely, in the light of Theorem 8, we strongly believe that each unary language accepted in time \( o(n \log \log n) \) under the weak measure is regular.

7 Conclusion

In the paper the lower bounds obtained by Trakhtenbrot [20], Hennie [9], and Hartmanis [6] in the case of deterministic machines have been mentioned several times. Those results refer to the strong measure, which is usually considered in the case of deterministic computations [21]. However, even in the deterministic case, it can be interesting to know what happens by considering only the costs of accepting computations. Because a deterministic machine cannot have more than one computation on each input string, it turns out that, in the deterministic case, the weak and the accept measures coincide. Hence, the lower bounds proved in Theorem 8 in the case of nondeterministic machines under the accept measure hold even for deterministic machines under the weak measure. Because they coincide with the lower bound for the strong measure, which is known to be optimal, they cannot be increased, i.e., they are also optimal.

In Table 1 the time lower bounds for the recognition of nonregular languages by one-tape off-line Turing machines are summarized. The table should be read as follows: a row \( r \) denotes a type of machine while a column \( c \) a measure. If the element at the position \((r, c)\) of the table is the function \( f(n) \), then \( t(n) \notin o(f(n)) \) for each one-tape off-line Turing machine of type \( r \) recognize a nonregular language in time \( t(n) \) under the measure corresponding to column \( c \). Table 2 gives a similar overview for the lower bounds on the maximal length \( c(n) \) of the crossing sequences, used in nonregular language recognition.

The numbers in the tables refer to the following list, where a short justification of each result is given:

1. Proved by Trakhtenbrot [20] and Hartmanis [6]. Hennie [9] proved the same lower bound for \( c(n) \) and a smaller lower bound for the time.
2. Proved by Tadaki, Yamakami, and Lin [19].

3. Theorem 3

4. Consequence of 3.

5. Consequence of 4.

6. Theorem 2 and Theorem 4

|                      | Deterministic machines | Nondeterministic machines |
|----------------------|------------------------|---------------------------|
| strong               | $n \log n$             | $n \log n$                |
| accept               | $n \log n$             | $n \log n$                |
| weak                 | $n \log n$             | $n$                       |

Table 1: Lower bounds for $t(n)$

|                      | Deterministic machines | Nondeterministic machines |
|----------------------|------------------------|---------------------------|
| strong               | $\log n$               | $\log n$                  |
| accept               | $\log n$               | $\log n$                  |
| weak                 | $\log n$               | $\log \log n$             |

Table 2: Lower bounds for $c(n)$

I like to complete the paper with some final remarks, with the hope to stimulate the reader, and myself, for future researches. As in the classical studies in complexity, in the paper I made a large use of asymptotic notations, tape tracks (that, in real implementations, mean large alphabets), and so on. I believe that researches about computing with restricted resources deserve closer investigations and refinements. It should be useful, for instance, to study what is hidden in the big-Oh’s and how other parameters (for example the cardinality of the alphabet used by the machine) influence the complexity. A recent example of interesting research in this direction is [8]. Furthermore, I think that the relationships with the descriptional complexity should be investigated, for example studying how the succinctness of the description of a machine can affect its computation time or the used space.

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