Astrophysical Systems:
A model based on the self-similarity scaling postulates.

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In the present work, it is developed a formalism to deal with the macroscopic study of the astrophysical systems, which is based on the consideration of the exponential self-similarity scaling laws that these systems exhibit during the realization of the thermodynamic limit. Due to their scaling laws, these systems are pseudoextensive, since although they are nonextensive in the usual sense, they can be studied by the Boltzmann-Gibbs Statistics if an appropriate representation of the integrals of motion of the macroscopic description is chosen. As example of application, it is analyzed the system of classical identical particles interacting via Newtonian interaction. A renormalization procedure is used in order to perform a well-defined macroscopic description of this system in quasi-stationary states, since it can not be in a real thermodynamic equilibrium. Our analysis showed that the astrophysical systems exhibit self-similarity under the following thermodynamic limit: $E \to \infty$, $L \to 0$, $N \to \infty$, keeping $E/N^2 = \text{const}$, $LN^\gamma = \text{const}$, where $L$ is the characteristic linear dimension of the system. It is discussed the effect of these scaling laws in the dynamical properties of the system. In a general way, our solution exhibits the same features of the Antonov problem: the existence of the gravitational collapse at low energies as well as a region with a negative heat capacity.

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I. INTRODUCTION

Traditional thermodynamics is not able to describe the astrophysical systems. They do not fulfill the additivity and homogeneity conditions, indispensable requirements for the good performance of this formulation.

In the last years it has been devoted so much effort to the extension of the Thermodynamics to the study of the nonextensive systems. In this context, the astrophysical systems have received a special attention.

Among the remarkable new results obtained in the present frame, it can be mentioned the application of the so popular Tsallis’ nonextensive statistics $\mathcal{Q}$ to the analysis of astrophysical systems. In this approach it has been justified the application of the polytropic models $\mathcal{Q}$, which have been extensively used in the descriptions of such a systems $\mathcal{Q}$:

$$p = C \rho^\gamma, \quad \gamma > 1,$$

where $p$ is the pressure, $\rho$, the particles density, and $\gamma$, the polytropic index.

However, in our opinion, in those works it has been put off the microscopic justification of the polytropic models to the applicability of the Tsallis’ theory. The nonextensive statistics is not a completely satisfactory formulation, due to the all theory dependence on the entropic index, $q$, which is a parameter representing a measure of the degree of nonextensivity of the system. In spite of the attractiveness of this formulation, the theory is not able to determine univocally the value of the entropic index, at least, in the context of the equilibrium Thermodynamics, so that, it must be appealed to the experiment or computational simulations in order to precise it. Some evidences aim that the entropy index could be obtained throughout the sensitivity of the system to the initial conditions and the relaxation properties towards equilibrium $\mathcal{Q}$.

Alternative approaches have been proposed using the microcanonical ensemble. Although it cannot be assured its application to any Hamiltonian system $\mathcal{Q}$, the Thermostatistics could be justified starting from this ensemble $\mathcal{Q}$.

1. Apparently the microcanonical ensemble is only well-defined statistical ensemble, whose justification ordinarily is based on the chaotic properties of the trajectories for a generic non-integrable system when it is overcome a few tens of degrees of freedom. However, there are some computational evidences in dynamical studies of some nonextensive systems in which the mixing time is extremely long, and diverges...
without invoking anything outside the Mechanics. This description is applicable to many situations in which the canonical description fails, allowing us to determine the necessary conditions for the applicability of any generalized canonical ensemble.

Starting from this viewpoint, in the ref. [12] it was addressed a generalization of the extensive postulates of the traditional Thermodynamics in order to extend its application to the nonextensive systems. According to our proposition, this objective could be carried out taking into consideration the self-similarity scaling postulates: the equivalence of the microcanonical ensemble with a generalized canonical one during the realization of the thermodynamic limit throughout of the self-similarity scaling properties of the system fundamental physical observables: the behavior of integrals of motion, the external parameters and the accessible volume of the microcanonical description with the increasing of the system degrees of freedom.

So far, these postulates have been applied to the analysis of the necessary conditions for the validity of two statistical formulations: in the ref. [13], to the microcanonical thermostatistics of D. H. E. Gross [14]15, as well as in the ref. [16], to the Tsallis’ nonextensive statistics [17].

The first is a theory based on the consideration of the microcanonical ensemble with the assumption of the Boltzmann’s definition of entropy:

\[
S_B = \ln W, \quad (2)
\]

his famous gravestone epithaph in Vienna. Since the Boltzmann’s entropy does not demand the realization of the thermodynamic limit this formulation is applicable to some small and mesoscopic systems. The thermodynamic formalism of this theory has been defined in order to be equivalent to the traditional one when it is applied to extensive systems. That is the reason why this theory is appropriate to the macroscopic description of those systems becoming extensive when the thermodynamic limit is invoked although they are not found in the thermodynamic limit.

In the ref. [13] it was shown that the Gross’s theory is also applicable to all those Hamiltonian systems exhibiting exponential self-similarity scaling laws in the thermodynamic limit. We called these system as pseudoextensive when \( N \to \infty \), since in spite of they are nonextensive in the usual sense, they can be tried by means of the Boltzmann-Gibbs’ statistics if an appropriate representation of the integrals of motion is chosen. Many of the systems found in the real world belong to the class of the pseudoextensive systems, since it is enough an additive kinetic part in the Hamiltonian of the system for exhibiting this kind of scaling laws in the thermodynamic limit.

In the ref. [16] was shown that the Tsallis’ statistics is appropriate to the macroscopic description of those Hamiltonian nonextensive systems exhibiting a potential self-similarity scaling laws in the thermodynamic limit. In our approach many details of this formalism naturally appear, starting only from the Mechanics under the consideration of this kind of scaling laws, i.e., the \( q \)-expectation values, the \( q \)-generalization of the Legendre’s Transformation (see for example in refs. [17,18]).

The above reasons allow us to consider that the astrophysical systems belong to the class of the pseudoextensive systems, and therefore, it is also justified the application of the Boltzmann-Gibbs’ statistics to the study of these systems, at least, when the phase transitions are not present.

The Tsallis’ statistics is expected to describe systems exhibiting potential distributions, that is, systems with fractal characteristics. In the ref. [13] it is suggested that the Tsallis’ potential distribution fitted very well the differential energy distribution of the dark matter in halos obtained by a numerical simulation. Similar analysis have been carried out in ref. [19] by Fa & Pedron for the elliptical galaxies. However, the fractals properties can be also found in the context of Boltzmann-Gibbs’ statistics, for example, in the Michie-King models for globular clusters [20] (see ref. [21] for review), or the models proposed by Stiavelli & Bertin [22], Hjorth & Madsen [23] for elliptical galaxies, and others [24]. These models lead to composite configurations with an isothermal core and a polytropic envelope and can not be justified by the Tsallis’ generalized Thermodynamics. Many authors state that these models are more appropriate for the description of those astrophysical objects (see for example in ref. [25]).

In the present work it is pretended to reconsider the thermo-statistical description of the astrophysical system, but this time, taking into consideration the self-similarity scaling postulates [26] in order to find the necessary conditions for the validity of the generalized canonical ensemble (Boltzmann-Gibbs) in the thermodynamic limit.
II. MICROCANONICAL DESCRIPTION.

A. Microcanonical Mean Field Approximation.

Let \( S \) be a Hamiltonian system composed by a huge number \( N \) of identical particles, which interact among them by means of short-range and long-range forces simultaneously. Let us also consider that the characteristic linear dimension of the system is comparable with the effective radio of the long-range interactions, but it is extremely large in comparison with the effective radio of the short-range interactions.

The above conditions allow us to speak about two small scales in the system:

- **Microscopic scale**: linear dimensions comparable with the effective radio of the short-range interactions.
- **Local scale**: very large linear dimensions in regard to the microscopic scale, but extremely short in comparison with the effective radio of the long-range interactions.

Let us consider a partition of accessible physical space in not overlapped cells, \( \{ c_k \} \), whose characteristic linear dimensions correspond to a local scale. It is easy to show that the \( N \)-body phase space integration can be decomposed in the following way:

\[
\frac{1}{N!} \int dX_N \equiv \prod_{k}^{N} \hat{\mathcal{O}} \left[ q_{n_k}^{(k)} \right] \delta^{(c)} \left( N - \sum_{k} n_k \right), \quad (3)
\]

where it have been taken into account all the possible configurations for distributing \( N \) identical particles in the cells. In this expression, \( X_N \) represents the \( N \)-body phase space, while \( q_{n_k}^{(k)} \) represents the \( n_k \)-body phase space whose accessible physical space have been limited to the \( k \)-th cell, \( c_k \). In addition, it has been introduced the following integral operator \( \hat{\mathcal{O}} \left[ q_{n_k}^{(k)} \right] \):

\[
\hat{\mathcal{O}} \left[ q_{n_k}^{(k)} \right] = \begin{cases} \left\{ \frac{1}{n!} \int dX_n \right\}, & \text{if } n \neq 0, \\ 1, & \text{if } n = 0, \end{cases} \quad (4)
\]

as well as the function \( \delta^{(c)} (n) \equiv \delta_{0n} \), which assures the particles number conservation. A derivation of the Eq. (4) appears in the appendix A.

The enormous scale separation among the effects of the short and long-range interactions supports the validity of the spatial adiabatic approximation. Let us consider those configurations in which each cell \( c_k \) contains a very large number of particles \( n_k \) and let us denote this subsystem of particles by \( S_k \). The physical quantities characterizing the long-range interactions, that is, the long-range interacting fields \( \phi (r) \), almost do not vary at the spatial region occupied by the cell due to its linear dimension. It means that the long-range interactions almost do not distinguish the internal structure of the subsystem enclosed by the cell, so that, these interactions are only effective for the subsystem as a whole.

Therefore, the contribution of long-range interactions to the system total energy can be approximated by functional terms of the mean values of the long-range interacting fields at the region of each cell, \( \phi_k \), as well as of certain collective quantities \( q_k \) characterizing the subsystems \( \{ S_k \} \), like the mean values of particles density, magnetization density, etc. Similarly, the mean values of the long-range interacting fields can be determined from the mean values of some collective quantities of the subsystems \( \{ S_k \} \). On the other hand, due also to the linear dimensions of the cells, it can be neglected the short-range interactions among the particles belonging to different cells.

Taking into account all the above exposed, the system Hamiltonian can be approximately expressed as follows:

\[
H \simeq \sum_{k} h_{\text{int}}^{(n_k)} (X_{n_k}; \phi_k) + V_{\text{loc}} (\phi_k, q_k), \quad (5)
\]

where \( h_{\text{int}}^{(n_k)} (X_{n_k}; \phi_k) \) is the internal energy of the \( n_k \)-body subsystem \( S_k \) enclosed in the cell \( c_k \), which only involves the kinetic energy of the particles as well as the contribution of the short-range interactions among them, being \( X_{n_k} \) their microscopic degrees of freedom. Here, it is also included a parametric dependence of the internal energy of the fields \( \phi_k \) in order to take into account a possible influence of these fields on the internal configurations of the subsystem \( S_k \). On the other hand, \( V_{\text{loc}} \) is a local term of energy containing the contribution of the subsystem \( S_k \) as a whole, which only involves its effective interactions with the long-range interacting fields \( \phi_k \). As already mentioned, the fields \( \phi_k \) are determined through a determined functionals of certain set local quantities, \( q \equiv \{ q_k \} \), which characterize the local subsystems:

\[
\phi_k = \mathcal{F}_k (q). \quad (6)
\]

As it could be seen, the effective long-range interacting fields \( \phi_k \) at the \( k \)-th cell can be considered as external parameters for the subsystem \( S_k \). Therefore, each subsystem \( S_k \) can be considered as locally extensive. Due to the presence of the long-range interactions, the system \( S = \bigcup_{k} S_k \) is nonextensive: the quantities characterizing the subsystems as a whole vary during the continuous passage among the cells.

There are systems in which the energy is not the only one integral of motion determining their macroscopic description (for example: the astrophysical systems), so that, it could be considered other integrals of motion, like total angular momentum \( \{ \mathcal{I} \} \). In such as cases, it is well-known that the total angular momentum admits a decomposition similar to the expression of the energy in the Eq. (3):
\[ M = \sum_k \mathbf{m}^{(n_k)}_{\text{int}} \left( \bar{X}_{n_k} \right) + \mu_k \mathbf{r}_k \times \mathbf{v}_k, \quad \text{(7)} \]

where \( \mathbf{r}_k \) and \( \mathbf{v}_k \) are the position and velocity of the mass center of the subsystem \( S_k \), \( \mu_k \) is its total mass, while \( \mathbf{m}^{(n_k)}_{\text{int}} \left( \bar{X}_{n_k} \right) \) and \( \bar{X}_{n_k} \) are the internal angular momentum and the microscopic degrees of freedom of the subsystem in its own mass center frame. The consideration of the collective motion of the subsystem \( S_k \) leads to add a kinetic term \( \frac{1}{2} \mu_k \mathbf{v}_k^2 \) to the sum in the Eq.(7) as well as to substitute \( X_{n_k} \) by \( \bar{X}_{n_k} \).

Thus, the integrals of motion involved in the macroscopic description of the system can be represented in the following form after the considerations assumed above:

\[ I \simeq \sum_k \mathcal{I}^{(n_k)}_{\text{internal}} (X_{n_k}; \phi_k) + \mathcal{J}_{\text{collective}} (\phi_k, q), \quad \text{(8)} \]

where \( \mathcal{I}^{(n_k)}_{\text{internal}} (X_{n_k}; \phi_k) \) represents the internal contribution from the microscopic degrees of freedom of the subsystem \( S_k \), while \( \mathcal{J}_{\text{collective}} (\phi_k, q_k) \) is the contribution of its collective degrees of freedom at local level.

According with what was exposed above, the accessible volume of the system in the microcanonical ensemble is approximately given by:

\[ W(I, N) \simeq \prod_k \int d\phi_k \left( \sum_{n_k=0}^N \tilde{\mathcal{C}}^{(n_k)}_k \right) \delta [\phi_k - \mathcal{F}_k (q)] \times \delta^{(c)} \left( N - \sum_k n_k \right) \times \delta \left[ I - \sum_k \mathcal{I}^{(n_k)}_{\text{internal}} (\bar{X}_{n_k}; \phi_k) + \mathcal{J}_{\text{collective}} (\phi_k, q_k, \mathbf{v}_k) \right], \quad \text{(9)} \]

where it has been explicitly introduced the dependence of the collective term of the mass center velocity, in order to take into account the collective motion of the subsystems. Using the identity:

\[ \int d\mathbf{i}_k d\mathbf{v}_k \delta \left[ \mathbf{i}_k - \mathcal{I}^{(n_k)}_{\text{internal}} (\bar{X}_{n_k}; \phi_k) \right] \delta \left[ \mathbf{v}_k - \frac{1}{\mu_k} \mathbf{P}_k \right] = 1, \quad \text{(10)} \]

where \( \mathbf{P}_k \) is the total linear momentum of the subsystem \( S_k \), the Eq.(9) can be rewritten introducing the Boltzmann’s entropy for each local subsystems \( S_k \), \( S_B (i_k, n_k; \phi_k) \):

\[ \exp \left[ S_B (i_k, n_k; \phi_k) \right] = \frac{1}{n_k!} \int dX_{n_k} \delta \left[ \mathbf{i}_k - \mathcal{I}^{(n_k)}_{\text{internal}} (X_{n_k}; \phi_k) \right] \delta \left[ \mathbf{P}_k \right], \quad \text{(11)} \]

where \( i_k \) is the internal contribution of the subsystem to the total integrals of motion of the system \( S \). Thus, the accessible volume of the system can be computed from

\[ W(I, N) \simeq \prod_k \sum_{n_k=0}^N \int d\mathbf{i}_k d\phi_k d\mathbf{v}_k \delta [\phi_k - \mathcal{F}_k (q)] \times \exp \left[ \sum_k S_B (i_k, n_k; \phi_k) \right] \delta^{(c)} \left( N - \sum_k n_k \right) \times \delta \left[ I - \sum_k i_k + \mathcal{J}_{\text{collective}} (\phi_k, q_k, \mathbf{v}_k) \right], \quad \text{(12)} \]

where the physical quantities \( q_k \) are determined from the \( i_k \) by means of certain functional dependencies:

\[ q_k = f_q (i_k, n_k, \mathbf{v}_k). \quad \text{(13)} \]

Developing the continuum limit:

\[ N \to \infty, \quad \text{and} \quad \mu (c_j) / \left( \sum_k \mu (c_k) \right) \to 0, \quad \text{(14)} \]

where \( \mu (c_k) \) is the physical volume of the \( k \)-th cell, the subsystem \( S_k \) is locally perceived as a fluid. Thus, it is obtained finally the microcanonical mean field approximation (MFA):

\[ W(I, N) \simeq W_{\text{MFA}} (I, N) = \]

\[ C \int \mathcal{D}\phi (r) \mathcal{D}\rho (r) \mathcal{D}\phi (r) \mathcal{D}\mathbf{v} (r) \delta \left\{ \phi (r) - \mathcal{F}_\phi [r; \rho, \phi, \mathbf{v}] \right\} \times \exp \left[ \int d^3r s [\rho (r), \rho (r), \phi (r)] \delta \left( N - \int d^3r \rho (r) \right) \times \delta \left[ I - \int d^3r \rho (r) + \mathcal{D} [r; \phi (r), \rho (r), \mathbf{v} (r)] \right], \quad \text{(15)} \]

where \( \rho (r), \phi (r), \mathcal{D} [r; \phi (r), \rho (r), \mathbf{v} (r)] \) are respectively: the particles density, the densities of the internal and collective contributions to the total integrals of motions of the system, and the entropy density of the fluid at the neighborhood of the point \( r \). \( C \) is an unimportant constant which appears as consequence of the continuous limit and can be ignored.

### B. The case of the astrophysical systems.

Let us apply the microcanonical mean field approximation to the analysis of an astrophysical system composed
by a static fluid of identical particles, whose macroscopic state is only determined by its total mass (number of particles) and the energy. Let $\varepsilon (r)$, $\rho (r)$ and $s [\varepsilon , \rho ; \phi]$ be respectively the internal energy, particle, and entropy densities of the system neighborhood of the point $r$, where $\phi$ is the self-gravitating Newtonian potential. In this case the microcanonical mean field approximation is written as follows:

$$W_{MFA} (E, N) = \int \mathcal{D} \varepsilon (r) \mathcal{D} \rho (r) \mathcal{D} \phi (r) \delta \{ \phi (r) - \mathcal{L} [r; \rho] \}$$

$$\times \exp \left[ \int d^3 r \ s [\varepsilon (r), \rho (r); \phi (r)] \delta \left( N - \int d^3 r \ \rho (r) \right) \right]$$

$$\times \delta \left( E - \int d^3 r \ \mathcal{H} [\varepsilon (r), \rho (r), \phi (r)] \right),$$

(16)

where $\mathcal{H} [\varepsilon (r), \rho (r), \phi (r)]$ is the Hamiltonian density of the system, which is given by:

$$\mathcal{H} [\varepsilon, \rho, \phi] = \frac{1}{8 \pi G} (\nabla \phi)^2 + m \rho \phi + \varepsilon.$$  

(17)

In the above definition: $G$ is the Newton’s constant and $m$ is the particles mass. Finally, $\mathcal{L} [r; \rho]$ is the functional:

$$\mathcal{L} [r; \rho] = m \int d^3 r' \ g (r, r') \rho (r'),$$

(18)

which determines the spacial configuration of the Newtonian potential $\phi (r)$ at a given spacial configuration of the particles density, $\rho (r)$, where $g (r, r')$ is the Green’s function of the Poisson’s problem:

$$\Delta G (r, r') = 4 \pi G \delta (r - r').$$

(19)

In the tridimensional case:

$$g (r, r') = - \frac{G}{|r - r'|}.$$  

(20)

Using the Fourier’s representation of the Dirac’s delta function:

$$\delta (x - x') = \int \frac{dk}{2 \pi} \exp \left[ i k (x - x') \right],$$  

(21)

where $z = \beta + i k$, with $\beta \in \mathbb{R}$, the Eq. (16) can be rewritten as follows:

$$W_{MFA} (E, N) = \int_{-\infty}^{+\infty} \frac{dk}{(2 \pi)^2} \mathcal{Z} [z_1, N] \exp \left[ z_1 E \right].$$

(22)

The functional $\mathcal{Z} [z_1, N]$, with argument $z_1 = \beta + i k$, is given by:

$$\mathcal{Z} [z_1, N] = \int_{-\infty}^{+\infty} \frac{ds}{2 \pi} \exp \left[ s z_2 N \right] \mathcal{N} (z_1, z_2),$$

(23)

with $z_2 = \mu + i s$, $\mu \in \mathbb{R}$, and the function $\mathcal{N} (z_1, z_2)$ is expressed as follows:

$$\mathcal{N} (z_1, z_2) = \int \mathcal{D} \varepsilon (r) \mathcal{D} \rho (r) \mathcal{D} \phi (r) \mathcal{D} J (r) \times$$

$$\times \exp \left[ - H (\varepsilon, \rho, \phi, \omega; z_1, z_2) \right].$$

(24)

The functional $H (\varepsilon, \rho, \phi, \omega; z_1, z_2)$ is expressed by:

$$H (\varepsilon, \rho, \phi, \omega; z_1, z_2) = \int d^3 r \ z_1 \mathcal{H} [\varepsilon (r), \rho (r), \phi (r)] +$$

$$+ z_2 \rho (r) - \omega (r) \left\{ \phi (r) - \mathcal{L} [r, \rho] \right\} - s [\varepsilon (r), \rho (r); \phi (r)],$$

(25)

where $\omega (r) = j (r) + i J (r)$. The auxiliary field $J (r)$ allows us the Fourier’s representation of the delta functional of the Newtonian potential, $\phi$, and $j (r)$ is an arbitrary real function.

As it was previously pointed out, the astrophysical systems can be considered as pseudoextensive [13]: they exhibit an exponential self-similarity scaling laws in the thermodynamic limit [12][3], the limit of many particles. As consequence of this behavior, they can be dealt with the usual Boltzmann-Gibbs’ Statistics, only if an appropriate representation of the integrals of motion determining their macroscopic state is chosen and if they do not present first-order phase transitions. In order to select a correct representation for the integrals of motion, the self-similarity scaling laws of the system must be found.

It is demanded the following symmetry:

$$(\mathcal{N} \rightarrow \alpha \mathcal{N}, \ \mu \rightarrow \mu, \ \beta E \rightarrow \alpha \beta E, \ \mathcal{P} \rightarrow \alpha \mathcal{P}) \Rightarrow S_B \rightarrow \alpha S_B,$$

(26)

in order to make equivalent the microcanonical with the canonical description when $N \rightarrow \infty$, where $\mathcal{P}$ is the Planck’s potential:

$$\mathcal{P} (\beta, N) = - \ln \mathcal{Z} [\beta, N],$$

(27)

and $S_B$, the Boltzmann’s entropy, Eq. (2). The above requests characterize the system as a whole. However, the description of the system is performed at a local level, and therefore, the scaling laws must be determined for the local fields: $\mathcal{E} (r), \rho (r), \phi (r), j (r), s [\varepsilon, \rho, \phi]$, the parameter $\beta$, as well as the scaling law of the spacial coordinate $r$.

Let us consider the following scaling laws:

$$r \rightarrow \alpha^c r, \ \beta \rightarrow \alpha^\varphi \beta, \ \varepsilon \rightarrow \alpha^{\chi \varepsilon}, \ \phi \rightarrow \alpha^{\eta \phi}, \ \mu \rightarrow \mu, \ \ j (r) \rightarrow j (r), \ \rho \rightarrow \alpha^\kappa \rho, \ s_0 \rightarrow \alpha^m s_0,$$

(28)

where $c, \varphi, \chi, \eta, \kappa$, and $m$ are certain real scaling exponent constants. From the analysis of the Eq. (25) it is demanded the following relations in order to satisfy the homogeneous scaling of the functional $H (\varepsilon, \rho, \phi, \omega; z_1, z_2)$:
The solution of the above equation system is given by:

\[ m = \kappa, \eta = \frac{\kappa^2}{3}, \pi = -\eta, \]

\[ c = \frac{1-\kappa}{3}, \chi = \frac{4\kappa+2}{3}. \]  

The scaling law for the total energy is given by:

\[ E \rightarrow \alpha^\tau E, \]  

where:

\[ \tau = \frac{\kappa + 5}{3}, \]  

and therefore, an appropriate selection of the representation of the integrals of motion is:

\[ I = (E, N) \text{ with } E \equiv E/N^\eta. \]  

In this case, the correct Legendre’s transformation between the thermodynamic potential is given by:

\[ S_B (E, N) \simeq \beta_o E - \mathcal{P} (\beta_o, N), \]  

with:

\[ \beta_o = \beta N^\eta \text{ and } E = \frac{\partial}{\partial \beta_o} \mathcal{P} (\beta_o, N). \]  

The above scaling laws depend on the parameter \( \kappa \), so that, they are only specified when the microscopic model for the fluid is assumed. When the thermodynamic limit is performed in the generalized canonical ensemble it will found that the system will carry out more probably those configurations minimizing the functional \( H (\varepsilon, \rho, \phi; z_1, z_2) \). Thus, it is arrived to an equilibrium mean field theory. These configurations are obtained by solving the following equations:

\[ \frac{\delta H (\varepsilon, \rho, \phi; j; \beta, \mu)}{\delta \varepsilon (r)} = 0, \quad \frac{\delta H (\varepsilon, \rho, \phi; j; \beta, \mu)}{\delta \rho (r)} = 0, \]

and

\[ \frac{\delta H (\varepsilon, \rho, \phi; j; \beta, \mu)}{\delta \phi (r)} = 0, \]  

imposing the contrains:

\[ \frac{\delta H (\varepsilon, \rho, \phi; j; \beta, \mu)}{\delta j (r)} = 0, \quad \frac{\delta H (\varepsilon, \rho, \phi; j; \beta, \mu)}{\delta \mu} = N, \]  

which are related with the conservation of the particles number and the consideration of the Poisson’s equation for the Newtonian potential \( \phi \). The maximization takes place when it is guarantied the positive definition of the functional matrix:

\[ D_{ij} (r', r) = \frac{\delta^2 H}{\delta f_i (r') \delta f_j (r)} |_{f=f_s}, \]  

with \( f_{i=1,2} = (\varepsilon, \rho) \), where the subindex \( s \) represents the solution for the minimization conditions, Eqs.\((36)\) and \((37)\). From the conditions given in the Eq.\((38)\) are derived the following relations:

\[ \beta = \frac{\partial}{\partial \varepsilon_s} s (\varepsilon_s, \rho_s; \phi_s), \]  

\[ \mu = \frac{\partial}{\partial \rho_s} s (\varepsilon_s, \rho_s; \phi_s) - \beta m \phi_s - \mathcal{L} (r, j_s), \]  

the conditions for thermodynamic equilibrium along the volume of the system, as well as the structure equation:

\[ \Delta \phi_s = 4\pi G \left[ m \rho_s - \beta^{-1} \left( j_s + \frac{\partial}{\partial \phi_s} s (\varepsilon_s, \rho_s; \phi_s) \right) \right]. \]  

From the validity of the Poisson’s equation:

\[ \Delta \phi_s = 4\pi G m \rho_s, \]  

the functional dependency for the auxiliary field \( j_s (r) \) is derived:

\[ j_s = -\frac{\partial}{\partial \phi_s} s (\varepsilon_s, \rho_s; \phi_s). \]  

It could be also used another alternative choice for the Hamiltonian of the system given in the Eq.\((27)\) :

\[ \mathcal{H} (\varepsilon, \rho, \phi) = \varepsilon + \frac{1}{2} m \rho \phi. \]  

In this case equilibrium conditions lead to the following relations:

\[ \beta = \frac{\partial}{\partial \varepsilon_s} s (\varepsilon_s, \rho_s; \phi_s), \]  

\[ \mu = \frac{\partial}{\partial \rho_s} s (\varepsilon_s, \rho_s; \phi_s) - \frac{1}{2} \beta m \phi_s - \mathcal{L} (r, j_s), \]  

\[ j_s = \frac{1}{2} \beta m \phi_s - \frac{\partial}{\partial \phi_s} s (\varepsilon_s, \rho_s; \phi_s). \]  

Taking into consideration the Green’s solution for the Poisson’s equation, the Eq.\((18)\), it is deduced the relation:

\[ \mu = \frac{\partial}{\partial \rho_s} s (\varepsilon_s, \rho_s; \phi_s) - \beta m \phi_s - \mathcal{L} \left( r, -\frac{\partial}{\partial \phi_s} s (\varepsilon_s, \rho_s; \phi_s) \right), \]  

which is the same obtained as using the first Hamiltonian, the Eq.\((23)\). Let us introduce the function \( C (r) \) as follows:

\[ C (r) = -\mathcal{L} \left( r, -\frac{\partial}{\partial \phi_s} s (\varepsilon_s, \rho_s; \phi_s) \right). \]
Summarizing: the equations that dictate the equilibrium of the system are the following:

\[ \beta = \frac{\partial}{\partial \varepsilon} s (\varepsilon, \rho_s; \phi_s), \quad \mu = \frac{\partial}{\partial \rho_s} s (\varepsilon, \rho_s; \phi) - \beta m \phi_s + C, \]  
\[ \Delta \phi_s = 4\pi G m \rho_s, \quad \Delta C = 4\pi G \frac{\partial}{\partial \phi_s} s (\varepsilon, \rho_s; \phi_s). \]  

which have to satisfy the minimum request: the non-negativity of the matrix functional of the Eq.(38). It is also necessary the exigency of the conservation of the number of particles:

\[ N = \int d^3r \rho_s (r). \]  

The Planck’s potential is expressed in terms of the local description as follows:

\[ \mathcal{P} (\beta, N) = \int d^3r \beta H [\varepsilon_s (r), \rho_s (r), \phi_s (r)] - s [\varepsilon_s (r), \rho_s (r); \phi_s (r)], \]  

where \( \beta \) is related with \( \beta_s \) by the first relation of the Eq.(55). Using the Eq.(53), the above relation is rewritten introducing the function \( p [\beta, \rho, \phi] \):

\[ \mathcal{P} (\beta, N) = \int d^3r \beta \frac{1}{2} m \rho_s (r) \phi_s (r) + p [\beta, \rho_s (r); \phi_s (r)], \]  

which is the Planck’s potential density at the point \( r \). Due to the scaling laws, it is convenient to set \( N = 1 \), and let the dependency on \( N \) to the scaling parameter \( \alpha \). That is to consider \( N \) as scaling parameter:

\[ \alpha = N, \]  

and set \( N = 1 \) in all the above relations. In this case it will be performed a scaling invariant description of the system. The equation system, the Eq.(51), must be solved under the constrain:

\[ \int d^3r \rho_s (r) = 1. \]  

Finally, the energy scaling invariant is given by:

\[ \epsilon (\beta) = \int d^3r \varepsilon_s (r) + \frac{1}{2} m \rho_s (r) \phi_s (r). \]  

In this way, when \( N \) dependency is taken into account in the scaling laws, the Eq.(28), it is obtained a suitable problem that could be solved by means of numerical computation. The microscopic model for the fluid must be specified in order to continue the calculation. As example of application, let us to apply this formalism in the next section to the analysis of the classical system of identical non-interacting particles (at the microscopic level).

As it can be seen, the consideration of the self-similarity scaling postulates do not alter the description in the scaling invariant equilibrium mean field theory \((N = 1)\). However, if the scaling laws are not correctly chosen, this fact must lead to some unphysical consequences, such as the non-proportionality of the Boltzmann’s entropy and the Planck’s potential with the particle number, similarly to the Gibbs’ paradox, and therefore, it there will be a trivial ensemble inequivalence.

### III. IDEAL GAS OF PARTICLES. RENORMALIZATION.

The main difficulty in the statistical description of the astrophysical systems is the existence of both, a short-range and long-range singularities due to the consideration of the Newtonian gravitational interaction. The first situation is the very-well known gravothermal catastrophe of the \( N \)-body self-gravitating system \([27,28]\). In such a system, there is no upper bound on the entropy and a state of arbitrarily large entropy can be constructed from a centrally concentrated density profile by shifting more of the mass towards the center (core-halo structures). It can be seen in \([29,30,31]\) for review. This situation can be easily avoided, since a new Physics appears at microscopic scales, i.e., the Quantum Physics, which constitutes a natural renormalization when the system is constituted by micro-particles: molecules, atoms and subatomic particles, or in general way, by the consideration of the particles size.

The second, the long-range singularity, has a different nature. It is very well-known that the gravitation is not able to confine the particles: it is always possible that some of them have the sufficient energy for escaping out from the system, so that, the system always undergoes an evaporation process. Therefore, the astrophysical systems will never be in thermodynamic equilibrium. However, there are intermediate stages where this behavior might be neglected and a quasi-equilibrium state might be reached (dynamical issues like ergodicity, mixing or “approach to equilibrium” \([32,33,34]\)). In principle, these quasi-stationary stages can be described dismissing the system evaporation. This last process could be considered as a secondary effect, which modifies the quasi-stationary equilibrium of the system. In the present approach, it will be only analyzed the quasi-stationary equilibrium.

Let \( K \) be the kinetic energy of a given particle, and \( \phi (r) \) its correspondent Newtonian potential energy at the point \( r \). That particle will be retained by the system gravity if the following condition is hold:

\[ K + m\phi < 0. \]
When the above condition is not satisfied, the particle will be able to escape out from the system if it does not lose its excessive energy. It will be only consider in our description those quasi-stationary stages in which the above condition is hold for each particle of the system. This assumption is the key of the so called Michie-King models (see refs. [27,28]). This exigency acts as a regularization procedure, since it is sufficient to confine the system, avoiding in this way the long-range singularity. No rigid boundaries are necessary in this case, so that, no artificial parameters like the box volume are introduced.

The local entropy density, that is, the function 

$$s_0 (\varepsilon, \rho, \phi),$$

is obtained from the following model: Let us consider a classical system of $N$ non-interacting particles which is confined by means of a rigid boundary with volume $V$. Taking into consideration the above renormalization prescription, the Eq. (58), the admissible stages of this system are those in which the kinetic energy of each particle satisfies the condition:

$$\frac{1}{2m} p^2 < U,$$  \hspace{1cm} (59)

where $m$ is the particle mass and $U$, the energy threshold. The renormalized partition function of the canonical ensemble, $Z_R (\lambda, N; U)$, is given by:

$$Z_R (\lambda, N; U) = \int_{\sum P_k < U, k=1...N} \frac{1}{N!} \frac{(2\pi h)^{3N}}{2} \exp \left( -\lambda \sum_{k=1}^{N} \frac{1}{2m} p_k^2 \right),$$  \hspace{1cm} (60)

where $\lambda$ is the canonical parameter. The calculation yields:

$$Z_R (\lambda, N; U) = \frac{V^N}{N!} \left( \frac{2m\pi}{h\lambda} \right)^{\frac{3}{2}N} \{ F \left[ (\lambda U)^\frac{3}{2} \right] \}^N.$$  \hspace{1cm} (61)

The function $F (z)$ in the above expression is defined by:

$$F (z) = \frac{4}{\sqrt{\pi}} \int_0^z x^2 \exp (-x^2) \, dx,$$  \hspace{1cm} (62)

that is:

$$F (z) = \text{erf} \left( z \right) - \frac{2}{\sqrt{\pi}} \exp \left( -z^2 \right).$$  \hspace{1cm} (63)

In the FIG. 1., it is shown the behavior of this function. The asymptotic dependency for low values of its argument is given by:

$$F (z) = \frac{4}{3\sqrt{\pi}} z^3 - \frac{4}{5\sqrt{\pi}} z^5 + O \left( z^7 \right), \text{ for } z \lesssim 0.5.$$  \hspace{1cm} (64)

The Planck’s potential in the thermodynamic limit is given by:

$$P (\lambda, N; V, U) =$$

$$N \ln \left( \frac{N}{V} \right) - \frac{3}{2} N \ln \left( \frac{2m\pi}{h^2\lambda} \right) - N \ln F \left[ (\lambda U)^\frac{3}{2} \right],$$  \hspace{1cm} (65)

and therefore, the energy is:

$$E = \frac{3N}{2\lambda} \left( 1 - \frac{1}{3} z \partial_z \ln F (z) \right)_{z=(\lambda U)^\frac{1}{2}}.$$  \hspace{1cm} (66)

The caloric curve is shown in the FIG. 2. It can be seen the linear behavior of the energy, at low values of the ‘temperature’, $T = \lambda^{-1}$, which corresponds with the usual Maxwell’s distribution. With the increasing of the parameter $T$, the divergency between the renormalized model with the ideal gas system becomes evident. This asymptotic dependency characterizes a uniform distribution function for the particles momenta. In the FIG. 3., it is shown the behavior of the distribution function of $p$ for different values of the parameter $z = (\lambda U)^{\frac{3}{2}}$. This graphic shows the transition from a gaussian distribution for high values of $z$, to the uniform distribution at value $z = 0$. Finally, the entropy function is obtained by means of the Legendre’s transformation:

$$S (E, N; V, U) = \lambda E - P (\lambda, N; V, U).$$  \hspace{1cm} (67)

FIG. 1. Behavior of the function $F (z)$. With the increasing of $z$ this function tends fastly to the unity.

FIG. 2. Caloric curve of the microscopic model. At low “temperatures”, $T = \lambda^{-1}$, the model behaves like the ordinary ideal gas, with a maxwellian velocity distribution function, but at high temperatures, the energy cutoff acts becoming homogeneous its velocity distribution function.

FIG. 3. Velocity distribution function at different values of $z$. All these functions were normalized as the unity at the origen.

Summarizing: as consequence of the renormalization procedure assumed in the Eq. (68), the local functions characterizing the local extensive subsystem are given by:

$$p_0 (\lambda, \rho, U) = \rho \ln \rho - \frac{3}{2} \rho \ln \left( \frac{2m\pi}{h^2\lambda} \right) - \rho \ln F \left[ (\lambda U)^\frac{3}{2} \right],$$  \hspace{1cm} (68)

$$\varepsilon = \frac{3\rho}{2\lambda} \left( 1 - \frac{1}{3} z \partial_z \ln F (z) \right)_{z=(\lambda U)^\frac{1}{2}},$$  \hspace{1cm} (69)

$$s_0 (\varepsilon, \rho; U) = \lambda \varepsilon - p_0 (\lambda, \rho, U).$$  \hspace{1cm} (70)

It is not difficult to predict some of the consequences of the above microscopic model. The energy threshold, $U$, is related with the Newtonian potential as follows:
and therefore, the value of $U$ decreases from the inner regions of the system to the outer ones. In the inner regions the local subsystems exhibit the greater values of the parameter $z$, and therefore, their local distribution functions for the particles momenta are almost gaussian. However, at the outer regions, the local distribution functions diverge from the gaussian shape, becoming asymptotically in a uniform distribution. It is seen that the gravity drives the behavior of the local distribution function.

IV. STRUCTURE EQUATIONS AND SYSTEM SCALING LAWS.

From the first equilibrium relation of the Eqs. (24) it is deduced that the parameter $\lambda$ in the renormalized model is constant along all the system and it is equal to $\beta$. The second equilibrium relation allows us to obtain the particle distribution, that is, the particle density:

$$\mu = -\ln \phi - 1 + \frac{3}{2} \ln \left(\frac{2m\pi}{\hbar^2 \beta}\right) - \ln F \left((-\beta m\phi)^\frac{1}{2}\right) - \beta m\phi + C,$$

that is:

$$\rho = N (\beta, \mu) \exp (-\beta m\phi + C) \exp (-\mu - 1), \quad \text{(72)}$$

where the normalization constant $N (\beta, \mu)$ is given by:

$$N (\beta, \mu) = \left(\frac{2m\pi}{\hbar^2 \beta}\right)^\frac{7}{2} \exp (-\mu - 1), \quad \text{(73)}$$

(It will be obviated the subindex $s$ in the mean field description).

This last result allows us to specify the scaling parameter $\kappa$ of the system scaling laws, Eqs. (28) and Eqs. (30), since from the Eq. (21) the following relation is valid:

$$\kappa = -\frac{3}{2} \pi = \frac{\pi + 2}{2}, \quad \text{(74)}$$

and therefore:

$$\kappa = 2. \quad \text{(75)}$$

Taking into consideration the relations of the Eq. (28), the scaling exponent constants are given by:

$$m = 2, \quad \eta = \frac{1}{3}, \quad \pi = -\frac{4}{3},$$

$$c = \frac{1}{3}, \quad \chi = 2, \quad \text{(76)}$$

and therefore, the energy scaling exponent $\tau$ in the Eq. (31) is:

$$\tau = 7 \quad \text{(77)}$$

This result differs from the proposed by Vega & Sanchez in the refs. 35,36. In these papers they studied the present model but this time considering a box renormalization. They claimed correctly that the thermodynamic limit for the astrophysical system must be invoked differently from the usual extensive systems. In the Eq. (1) of the ref. 35, they demand that the thermodynamic limit is performed when:

$$N \to \infty, \quad E \to \infty, \quad L \to \infty, \quad \frac{E}{N} = \text{const}, \quad \frac{N}{L} = \text{const}, \quad \text{(78)}$$

where $L$ is the characteristic dimension of the box. However, it is easy to see that this consideration leads to the non-proportionality of the Boltzmann’s entropy with the number of particles (see Eq. (8) or the expression before the Eq. (12) in the ref. 37), and therefore, ensemble inequivalence. In spite of this error, all the results obtained by them are still correct, since they performed correctly the scaling invariant description with the introduction of the $N$-independent parameters $\xi$ and $\eta$ defined as follows:

$$\xi = \frac{EL}{Gm^2 N^2} \quad \text{(in the microcanonical ensemble),} \quad \text{(79)}$$

$$\eta = \beta \frac{Gm^2 N}{L} \quad \text{(in the canonical ensemble),} \quad \text{(80)}$$

which are also in agreement with the scaling laws considered by us in the Eqs. (28), (30), (76), and (77). In their approach, they assumed incorrectly that the mechanical energy scales proportional to $N$. Our energy scaling law suggests that this model can not be applied for a number arbitrarily large of particles: an upper bound for $N$ appears for the validity of the non relativistic conditions (see in section VI).

As it can be easily seen, in the Eq. (72), when $\phi \to 0$, the density vanishes, $\rho \to 0$. Thus, our model is renormalized. In order to obtain the structure equations, the Eqs. (51), it is necessary to perform the following calculation:

$$-\beta \left(\frac{\partial}{\partial \phi} s_0 (\varepsilon, \rho; \phi)\right) = \beta \left(\frac{\partial}{\partial \phi} \rho_0 (\beta, \rho; \phi)\right) = \beta m\rho \frac{1}{2z} \frac{\partial}{\partial z} \ln F (z) \bigg|_{z = (-\beta m\phi)^\frac{1}{2}}, \quad \text{(81)}$$

Introducing the constant $K$:

$$K = Gm^2 \beta N (\beta, \mu), \quad \text{(82)}$$

the dimensionless function $\Phi$:

$$\Phi = -\beta m\phi, \quad \text{(83)}$$

and the coordinate $\xi$:
\[ \xi = K^{\frac{1}{2}} r, \quad \] (84)

the structure equations are finally written as follows:

\[ \Delta \Phi = -4\pi F_1 (C, \Phi), \quad \Delta C = -4\pi F_2 (C, \Phi), \quad \] (85)

where:

\[ F_1 (C, \Phi) = \exp (C + \Phi) F \left( \Phi^{\frac{1}{2}} \right), \]
\[ F_2 (C, \Phi) = \frac{2}{\sqrt{\pi}} \exp (C) \Phi^{\frac{1}{2}}. \quad \] (86)

Due to the scaling transformation, the constant \( K \) depends on \( N \) as follows:

\[ K = N^{\frac{4}{3}} \beta_o, \quad \] (87)

where \( \beta_o \) is obtained substituting \( \beta \) by \( \beta_o \) in the Eq.(82). The coordinate \( \xi \) is \( N \)-independent.

V. NUMERICAL ANALYSIS

In order to perform the numerical analysis of the above equations, it is convenient to introduce the following constant:

\[ c_0 = 2\pi \frac{G^2 m^5}{\hbar^2}, \quad \rho_0 = (2\pi)^3 \frac{G^3 m^9}{\hbar^6}, \quad l_0 = \frac{1}{2\pi} \frac{h^2}{G m^3}, \quad \] (88)

which are respectively the energy, density, and length characteristic units of the present analysis. The above consideration allows us to set \( \hbar = G = m = 1 \). Hereafter it will be imposed the condition \( N = 1 \).

In terms of the functions \( C \) and \( \Phi \), the particle number constrain is written as follows:

\[ \exp (-\mu - 1) \frac{K^2}{\beta_o^2} \int_0^{+\infty} 4\pi \xi^2 d\xi F_1 (C, \Phi) = 1, \quad \] (89)

which establishes a functional relation between the thermodynamic parameters \( \beta_o \) and \( \mu \). This exigency can be rewritten analyzing the behavior of the functions \( C \) and \( \Phi \) in the asymptotic region \( \xi \to +\infty \). The spherical solution of the Poisson's equation satisfy the following asymptotic relation:

\[ \lim_{r \to \infty} r^2 \nabla_r \phi (r) = GM, \quad \] (90)

where \( M \) is the total mass of the system. In the present problem, taking into consideration the characteristic units, the Eq.(88), the above relation is rewritten as follows:

\[ \lim_{\xi \to \infty} \xi^2 \frac{\partial}{\partial \xi} \Phi (\xi) = -\beta_o K^\frac{4}{3}. \quad \] (91)

In the asymptotic region \( \xi \to \infty \), the functions \( \Phi \) and \( C \) vanish identically:

\[ \lim_{\xi \to \infty} \Phi (\xi) = \lim_{\xi \to \infty} C (\xi) = 0, \quad \] (92)

due to the asymptotic behavior of the Green's function, Eq.(90), in their representations, Eq.(88) and Eq.(93). The numerical solution of the Poisson's like equations system, Eq.(88), is carried out imposing the boundary at the origin \( \xi = 0 \). This is performed demanding at the origin:

\[ \Phi (0) = \Phi_0 > 0, \quad \] (93)
\[ C (0) = C_0. \quad \] (94)

The value of \( C_0 \) must be selected appropriately since it must vanish when \( \xi \to \infty \), and therefore, \( C_0 \) depends on the parameter \( \Phi_0 \). This situation can be overcome redefining the problem as follows: firstly, displacing the function \( C (\xi) \):

\[ C (\xi) = -c (\infty) + c \left( \xi' \right), \quad \] (95)

where \( c \left( \xi' \right) \) is the solution of the Poisson's like equation system with the following boundary conditions:

\[ \Phi (0) = \Phi_o, \quad \] (96)
\[ c (0) = 0, \quad \] (97)

and the new coordinate \( \xi' \) is related with \( \xi \) throughout the relation:

\[ \xi' = \exp \left[ -\frac{1}{2} c (\infty) \right] \xi. \quad \] (98)

Hereafter it is obviated the punctuation in \( \xi \). The Eq.(91) is rewritten as follows:

\[ h (\Phi_0) \exp \left[ \frac{1}{2} c (\infty) \right] = \beta_o K^\frac{4}{3} = \beta_o \exp \left[ -\frac{1}{2} (\mu + 1) \right]. \quad \] (99)

where the function \( h (\Phi_0) \) was introduced:

\[ h (\Phi_0) = -\lim_{\xi \to \infty} \xi^2 \frac{\partial}{\partial \xi} \Phi (\xi). \quad \] (100)

So far, there are three parameters involved in the solution of the problem, \( \beta_o, \mu \) and \( \Phi_0 \). The above relation determines \( \mu \) as a function of \( \beta_o \) and \( \Phi_0 \). An additional relation is needed to specify the functional dependency between \( \beta_o \) and \( \Phi_0 \). The solution to this situation is obtained throughout the equivalency between the statistical ensembles when *this equivalency held*. From the Legendre's transformation is followed that the energy is obtained from the Planck's potential through the relation:

\[ c (\beta_o, \Phi_0) = \frac{d}{d\beta_o} \mathcal{P} (\beta_o, \Phi_0), \quad \] (101)
and therefore:

\[ \frac{d\beta_0 (\Phi_0)}{d\Phi_0} = \frac{\partial}{\partial\Phi_0} P (\beta_0, \Phi_0) \left[ \epsilon (\beta_0, \Phi_0) - \frac{\partial}{\partial\beta_0} P (\beta_0, \Phi_0) \right]^{-1}. \]  

(102)

This is the second relation that we have been looking for.

From the Eq. (103), it is deduced the functional dependency between the chemical potential \( \mu \) and the parameters \( \beta_0 \) and \( \Phi_0 \):

\[ \mu = -1 + \frac{3}{2} \ln \beta_0 - 2 \ln h (\Phi_0) - c (\infty). \]  

(103)

Using the Eq. (104), the energy is expressed by:

\[ \epsilon (\beta_0, \Phi_0) = \frac{1}{\beta_0} \left[ \frac{3}{2} - \frac{h_1 (\Phi_0)}{h (\Phi_0)} \right], \]  

(104)

where \( h_1 (\Phi_0) \) is defined as:

\[ h_1 (\Phi_0) = \int_0^{+\infty} 4\pi\xi^2 d\xi \Phi \left[ F_2 (c, \Phi) + \frac{1}{2} F_1 (c, \Phi) \right]. \]  

(105)

Similarly, the Planck’s potential in the Eq. (54) is rewritten as follows:

\[ P (\beta_0, \Phi_0) = -\frac{3}{2} \ln \beta_0 + 2 \ln h (\Phi_0) + \frac{h_2 (\Phi_0)}{h (\Phi_0)}, \]  

(106)

where \( h_2 (\Phi_0) \) is defined by the expression:

\[ h_2 (\Phi_0) = \int_0^{+\infty} 4\pi\xi^2 d\xi F_1 (c, \Phi) \left[ c + \frac{1}{2} \right]. \]  

(107)

Introducing the function \( H (\Phi_0) \):

\[ H (\Phi_0) = 2 \ln h (\Phi_0) + \frac{h_2 (\Phi_0)}{h (\Phi_0)}, \]  

(108)

the Eq. (102) is rewritten as follows:

\[ \frac{d\ln \beta_0 (\Phi_0)}{d\Phi_0} = \frac{\partial H (\Phi_0)}{\partial\Phi_0} \left[ 3 - \frac{h_1 (\Phi_0)}{h (\Phi_0)} \right]^{-1}. \]  

(109)

Finally, the entropy of the system is given by:

\[ S [\epsilon (\beta_0, \Phi_0)] = \frac{3}{2} + \frac{3}{2} \ln \beta_0 - H (\Phi_0) - \frac{h_1 (\Phi_0)}{h (\Phi_0)}. \]  

(110)

Of course, the validity of the Eq. (102) rests on the equivalency between the micro and canonical ensembles, which is satisfied wherever the parameter \( \beta_0 \) is a decreasing function of the energy:

\[ \frac{d\beta_0 (\Phi_0)}{d\epsilon (\Phi_0)} < 0. \]  

(111)

Introducing the function \( R (\Phi_0) \):

\[ R (\Phi_0) = \frac{\Phi_0}{\int_0^\infty \frac{\partial H (s)}{\partial s} \left[ 3 - \frac{h_1 (s)}{h (s)} \right]^{-1} ds}, \]  

(112)

the general solution of the Eq. (109) is expressed as follows:

\[ \beta_0 (\Phi_0) = C \exp [R (\Phi_0)], \]  

(113)

where the integration constant \( C \) could be fixed as the unity without loss of generality. Using the function \( R (\Phi_0) \) and the function \( K (\Phi_0) \):

\[ K (\Phi_0) = \frac{3}{2} - \frac{h_1 (\Phi_0)}{h (\Phi_0)}, \]  

(114)

the condition of the Eq. (111) is rewritten as:

\[ \frac{d}{d\Phi_0} K (\Phi_0) / \frac{d}{d\Phi_0} R (\Phi_0) - K (\Phi_0) < 0. \]  

(115)

The Eq. (109) will be justified when the above relation is satisfied, otherwise, at first sight, the results obtained from this methodology has apparently no sense. According to the results of the ref. [14], the above observation is correct: in the representation \((E, N)\) the equivalency of the ensemble demand the validity of the condition given in the Eq. (111). Nevertheless, it can be chosen another representation for the integrals of motion in which the equivalency between the ensemble takes place. For example, the canonical description could be performed introducing the following representation for the integrals of motion:

\[ (E, N) \rightarrow (\mathcal{U}_\varphi, N), \text{ where } \mathcal{U}_\varphi = N \varphi (E/N), \]  

where \( \varphi \) is an arbitrary piecewise monotonic function of \( E = E/N \) at least two times differentiable with exception of the juncture boundary. As it can be easily seen, \( \mathcal{U}_\varphi \) and \( \mathcal{E} \) possess the same scaling law in the thermodynamic limit \( N \rightarrow \infty \), that is the reason why both representations are adequate in the generalized canonical ensemble with probabilistic distribution function given by:

\[ \omega_\varphi (\mathcal{U}_\varphi; \beta_\varphi, N) = \frac{1}{Z_\varphi (\beta_\varphi, N)} \exp (-\beta_\varphi \cdot \mathcal{U}_\varphi), \]  

(116)

where \( Z_\varphi (\beta_\varphi, N) \) is the partition function in the present representation from which it is derived the corresponding Planck’s potential:

\[ \mathcal{P}_\varphi (\beta_\varphi, N) = -\ln Z_\varphi (\beta_\varphi, N). \]  

(117)

It is easy to see that in this case it is also valid the Legendre’s transformation between the thermodynamic potentials when the thermodynamic limit is performed:

\[ S (\mathcal{U}_\varphi, N) \sim \beta_\varphi \cdot \mathcal{U}_\varphi - \mathcal{P}_\varphi (\beta_\varphi, N), \]  

(118)
with:
\[ U_\varphi = \frac{\partial}{\partial \beta_\varphi} P_\varphi (\beta_\varphi, N). \]

(119)

During the representation changes the canonical parameter and the Planck' potential are transformed as follows:
\[ \beta_o \rightarrow \beta_\varphi = \left( \frac{d\varphi (\epsilon)}{d\epsilon} \right)^{-1} \beta_o, \]
\[ \mathcal{P} \rightarrow \mathcal{P}_\varphi = \mathcal{P} + \left( \frac{d\varphi (\epsilon)}{d\epsilon} \right)^{-1} \left( \varphi (\epsilon) - \epsilon \frac{d\varphi (\epsilon)}{d\epsilon} \right) \beta_o N, \]
\[ \text{in the way that the entropy remains unchanged:} \]
\[ S (\mathcal{E}, N) \rightarrow S (\mathcal{U}_\varphi, N) \equiv S (\mathcal{E}, N). \]

(122)

In the new representation the equivalence of the microcanonical and canonical ensembles takes place when it is satisfied the condition:
\[ \frac{d^2}{dU_\varphi^2} S (\mathcal{U}_\varphi, N) \equiv \beta_\varphi \frac{d}{d\varphi} \ln \beta_\varphi < 0, \]
\[ \text{that is, the negative definition of the curvature tensor associate to the new representation. This condition can be rewritten again as follows:} \]
\[ \left( \frac{d\varphi (\epsilon)}{d\epsilon} \right)^{-2} \beta_o \frac{d}{d\epsilon} \ln \left( \frac{d\varphi (\epsilon)}{d\epsilon} \right)^{-1} \beta_o < 0. \]
\[ \text{(124)} \]

Let us analyze a possible way for the selection of the function \( \varphi \). The Eq.(124) can be rewritten as:
\[ \left( \frac{d\varphi (\epsilon)}{d\epsilon} \right)^{-2} \left[ \frac{d\beta_o}{d\epsilon} - \beta_o \frac{d}{d\epsilon} \ln \left( \frac{d\varphi (\epsilon)}{d\epsilon} \right) \right] = - \left( \frac{d\varphi (\epsilon)}{d\epsilon} \right)^{-2} a (\epsilon), \]
\[ \text{(125)} \]

where the function \( a (\epsilon) \) > 0, from which is derived:
\[ \frac{d\varphi (\epsilon)}{d\epsilon} = C \beta_o (\epsilon) \exp \left( \int \frac{a (\epsilon)}{\beta_o (\epsilon)} d\epsilon \right), \]
\[ \text{(126)} \]

where \( C \) is a positive constant which could be set as unity. It is easy to see that a convenient choice for the function \( a (\epsilon) \), from the theoretical viewpoint, could be given by:
\[ a (\epsilon) = \begin{cases} 
- k (\epsilon) & \text{if } k (\epsilon) < 0, \\
\beta_c^2 (\epsilon) = \beta_c^2 \sim \text{const} & \text{if } k (\epsilon) = 0, \\
k (\epsilon) & \text{if } k (\epsilon) > 0, 
\end{cases} \]
\[ \text{(127)} \]

where it was introduced the nomenclature:
\[ k (\epsilon) = \frac{d\beta_o (\epsilon)}{d\epsilon} = \frac{d^2}{d\epsilon^2} s_B (\epsilon), \]
\[ \text{(128)} \]

where \( s_B (\epsilon) \) is the Boltzmann’s entropy per particle. In this representation change:
\[ \frac{d\varphi (\epsilon)}{d\epsilon} = \begin{cases} 
\beta_e \exp (\beta_e \epsilon) & \text{if } k (\epsilon) = 0, \\
\beta_e^2 (\epsilon) & \text{if } k (\epsilon) > 0. \end{cases} \]
\[ \text{(129)} \]

As it can be seen, in this representation \( \varphi (\epsilon) \) is an piecewise monotonous function of \( \epsilon \), and therefore, there is a bijective correspondence between \( \epsilon \) and \( \varphi (\epsilon) \). In this way it have been shown that it is always possible to chose a representation in which it is valid the correspondence among the ensembles allowing us to extend the applicability of the generalized canonical ensembles to those situations in which the ordinary heat capacity is negative.

The reparametrization freedom of the microcanonical ensemble allows us to extend the validity of the mean field approach with an appropriate selection of the function \( \varphi \). It can be easily shown that all the results of the canonical description in the different representations are only non-coincident when it is taken into account those observables involving second derivatives (or superior) of the thermodynamic potentials. That is the reason why it is considered that the Eq.(109) can be extended to those regions where there is an univocal dependence of the canonical parameter \( \beta_\varphi \) with the energy \( \epsilon \).

In order to solve the Eq.(109) it is necessary to known the asymptotic behavior of the functions \( h (\Phi_0) \), \( h_1 (\Phi_0) \) and \( h_2 (\Phi_0) \) for small values of \( \Phi_0 \). This is done developing the functions \( c \) and \( \Phi \) in a perturbative expansion of the parameter \( \Phi_0 \):
\[ \Phi = \Phi + \delta \Phi + ... \]
\[ \epsilon = \epsilon + \delta \epsilon + ... \]
\[ \Phi = \Phi + \delta \Phi + ... \]
\[ c = c + \delta c + ... \]
\[ \epsilon = \epsilon + \delta \epsilon \]
\[ \text{(130)} \]

as well as the functions \( F_1 (c, \Phi) \) and \( F_2 (c, \Phi) \):
\[ F_1 (c, \Phi) = f_1 (\epsilon, \Phi) + \delta F_1 \left( \delta c, \delta \Phi; \epsilon, \Phi \right) + ..., \]
\[ F_2 (c, \Phi) = f_2 (\epsilon, \Phi) + \delta F_2 \left( \delta c, \delta \Phi; \epsilon, \Phi \right) + ..., \]
\[ \text{(131)} \]

where the problem given in the Eq.(85) is rewritten as follows:
\[ \Delta \Phi = - 4 \pi f_1 (\epsilon, \Phi), \]
\[ \Delta \epsilon = - 4 \pi f_2 (\epsilon, \Phi), \]
\[ \text{(132)} \]

\[ \Delta (\delta \Phi) = - 4 \pi \delta F_1 \left( \delta c, \delta \Phi; \epsilon, \Phi \right), \]
\[ \Delta (\delta \epsilon) = - 4 \pi \delta F_2 \left( \delta c, \delta \Phi; \epsilon, \Phi \right), \]
\[ \text{(133)} \]

Using the power expansion of the function \( F (\epsilon) \), Eq.(144), it is very easy to obtain the following relations:
\[ f_1 (\epsilon, \Phi) = \exp (\epsilon) \frac{4}{3 \sqrt{\pi}} \Phi^\frac{3}{2}, \]
\[ f_2 (\epsilon, \Phi) = \exp (\epsilon) \frac{2}{\sqrt{\pi}} \Phi^\frac{1}{2}, \]
\[ \text{(135)} \]

\[ \delta F_1 \left( \delta c, \delta \Phi; \epsilon, \Phi \right) = \exp (\epsilon) \frac{2}{\sqrt{\pi}} \left[ - \frac{2}{\delta \Phi^\frac{3}{2}} + \frac{2}{3 \sqrt{\pi}} \delta c + \Phi^\frac{1}{2} \delta \Phi \right], \]
\[ \text{(136)} \]
\[ \delta F_2 (\delta c, \delta \Phi; \bar{\tau}, \bar{\Psi}) = \exp (\bar{\tau}) \frac{2}{\sqrt{\pi}} \left[ \bar{\Phi}^{-\frac{1}{2}} \delta c + \frac{1}{2} \bar{\Phi}^{-\frac{3}{2}} \delta \Phi \right]. \]

In this asymptotic region, the structure equations (the Eq. (55)) become in the following problem:

\[ \Delta \bar{\Phi} = -4\pi \exp (\bar{\tau}) \frac{4}{3\sqrt{\pi}} \bar{\Phi}^{-\frac{1}{2}}, \quad \Delta \bar{\tau} = -4\pi \exp (\bar{\tau}) \frac{2}{\sqrt{\pi}} \bar{\Phi}^{-\frac{3}{2}}, \]

(137)

\[ \bar{\Phi} (0) = \Phi_0, \quad \bar{\Phi} (0) = 0 \] and \( \bar{\tau} (0) = 0, \bar{\tau}' (0) = 0. \)

(138)

It is very easy to see that the above problem possesses a fractal characteristic and is quite similar to the polytropic model with polytropic index \( \gamma = \frac{3}{4} \), which allows us to express its general solution as follows:

\[ \bar{\Phi} = \Phi_0 \varphi \left( \Phi_0^{\frac{1}{4}} \bar{\xi} \right), \quad \bar{\tau} = \psi \left( \Phi_0^{\frac{1}{4}} \bar{\xi} \right), \]

(139)

where the functions \( \varphi (z) \) and \( \psi (z) \) are the solution of the problem:

\[ \Delta_z \varphi = -4\pi \exp (\psi) \frac{4}{3\sqrt{\pi}} \varphi^{\frac{1}{2}}, \quad \Delta_z \psi = -4\pi \exp (\psi) \frac{2}{\sqrt{\pi}} \varphi^{\frac{3}{2}}, \]

(140)

with boundary conditions:

\[ \varphi (0) = 1 \] and \( \varphi' (0) = 0, \psi (0) = 0 \) and \( \psi' (0) = 0. \)

(141)

It is interesting to point out that this polytropic index characterizes an adiabatic process of the ideal gas of particles, which in our case is the evaporation of the system in the vacuum. However, this equation system is not equivalent to polytropic model due to the presence of the term of the gravity driving, the presence of the function \( \bar{\tau} \). It is easy to understand that the maximum effects of this term appear in the outer region of the system, in the halo. In this region the results of the polytropic model and the pseudo-polytropic model given by the Eq. (68) are different. Moreover, this kind of behavior is also present for all the values of the parameter \( \Phi_0 \), and as a consequence of the renormalization prescription assumed in Eq. (58), it leads to configurations of the systems characterized by an isothermal core with a quasi-polytropic halo. The effect of the function \( \bar{\tau} \) is showed in the FIG.4. As it can be seen, the consideration of this term reduces the system size in comparison with the polytropic equation.

FIG. 4. Comparison between the polytropic model with \( \gamma = \frac{3}{4} \) and the quasi-polytropic model presented in the present analysis.

Similarly, the functions \( \delta c \) and \( \delta \Phi \) are expressed as follows:

\[ \delta \Phi = \Phi_0^2 \varphi_1 \left( \Phi_0^{\frac{1}{4}} \xi \right), \quad \delta c = \Phi_0 \psi_1 \left( \Phi_0^{\frac{1}{4}} \xi \right), \]

(142)

where the functions \( \varphi_1 (z) \) and \( \psi_1 (z) \) are obtained from the problem:

\[ \Delta_z \varphi_1 = -4\pi \exp (\psi) \frac{2}{\sqrt{\pi}} \left[ -\frac{2}{5} \varphi_1 \psi_1 + \frac{2}{3} \varphi_1 \psi_1 + \varphi_1 \psi_1 \right], \]

(143)

\[ \Delta_z \psi_1 = -4\pi \exp (\psi) \frac{4}{\sqrt{\pi}} \left[ \varphi_1 \psi_1 + \frac{1}{2} \varphi_1 \psi_1 \right], \]

(144)

with boundary conditions:

\[ \varphi_1 (0) = 0 \] and \( \varphi_1' (0) = 0, \psi_1 (0) = 0 \) and \( \psi_1' (0) = 0. \)

(145)

In this way it is obtained the asymptotic dependence of the functions \( h (s) \), \( h_1 (s) \) and \( h_2 (s) \):

\[ h (s) = a_1 s^2 + a_2 s^2 + O \left( s^2 \right), \]

(146)

\[ h_1 (s) = b_1 s^2 + b_2 s^2 + O \left( s^2 \right), \]

(147)

\[ h_2 (s) = c_1 s^2 + c_2 s^2 + O \left( s^2 \right). \]

(148)

The calculation yields:

\[ a_1 = 0.744, \quad a_2 = -0.143, \]

(149)

\[ b_1 = 1.5a_1 = 1.116, \quad b_2 = -0.242, \]

\[ c_1 = -0.622, \quad c_2 = 0.478. \]

Thus, in the asymptotic region, the relation between the canonical parameter \( \beta_0 \) and \( \Phi_0 \) is given by:

\[ \beta_0 (\Phi_0) = \Phi_0 + q \Phi_0^2 + O \left( \Phi_0^2 \right), \]

(150)

where:

\[ q \simeq 0.025. \]

(151)

Finally, the energy in this limit is given by:

\[ \epsilon \simeq 0.055 + O (\Phi_0). \]

(152)

VI. RESULTS AND DISCUSSIONS

Let us begin the discussion analyzing the scaling laws of our system. In general way, the scaling laws determine the specific form of the thermodynamic formalism. The astrophysical systems exhibit exponential self-similarity scaling laws in the thermodynamic limit, which is the reason why the Boltzmann-Gibbs’ Statistics is applicable to
the macroscopic description of this kind of systems in spite of that they are nonextensive. This analysis can be performed choosing an adequate selection of the representation of the integrals of motion. As it was shown in the section IV, the N-body Newtonian self-gravitating gas exhibits self-similarity properties under the following thermodynamic limit:

\[ N \to \infty, \quad E \to \infty, \quad L \to 0, \quad \frac{E}{N^{d+1}} = \text{const.}, \quad LN^{d+1} = \text{const.}, \]

(153)

where \( L \) is a characteristic linear dimension of the system. This result differed from the obtained by Vega & Sanchez in refs. [33][34].

Many investigators do not pay so much attention to the analysis of the system scaling laws when it is performed its macroscopic description. Fortunately, most of the systems studied so far belong to the class of the pseudoextensive systems. When it is performed the \( N \)-independent description, the results of the analysis do not depend on the scaling laws, and therefore, many of these results remain valid. However, the non or bad consideration of the scaling laws leads in many case to the trivial nonequivalence of the statistics ensembles.

As example of the above affirmation it can be seen the anomalies presented in the dynamical study of the self-gravitating systems performed by Cerruti-Sola & Pettini in the ref. [37]. In that paper, they observed a weakening of the system chaotic behavior with the increasing of the particles number \( N \). It is very well-known the consequence of this fact on the ergodicity of the system: the chaotic dynamics provides the mixing property in the phase space necessary for obtaining the equilibrium. In that example the chaoticity time grows with \( N \), and therefore, the systems could expend so much time to arrive to the equilibrium. Similar behavior has been seen in the dynamical study of the so called Hamiltonian Mean Field model (see for example in the ref. [38]).

However, our analysis allows us to understand the origin of this behavior, a least, for the self-gravitating gas. In that study it was considered that the energy is scaled proportional to \( N \) during the realization thermodynamic limit, which is a false assumption. It is very easy to see that the dependency of the stability exponent \( \lambda_H \) of the energy (Fig. 4 of the ref. [37]) is corrected when it is considered the right scaling law of the energy: its proportionality to \( N^{d+1} \). The anomalies disappear when it is assumed this scaling law for the energy. Anyway, in spite of they assumed a wrong scaling law for the energy, they obtained the correct dependency of the instability exponent \( \lambda_H \) with the energy per particle: the power law \( \epsilon^2 \). Our analysis suggests that the anomalies presented in the dynamical study of the Hamiltonian Mean Field model could possess the same origin. In future works we will consider this possibility.

Some other consequences of this scaling laws are found analyzing the limit of applicability of the model. In the present work it was considered a classical gas of identical particles with mass \( m \). Taking into account the rest energy of the particles, the nonrelativistic limit is valid when the absolute value of the mechanical energy of the system is much smaller than its rest energy:

\[ |\epsilon_0(\Phi_0) N^{d+1}| \ll mc^2 N, \]

(154)

where \( \epsilon_0 \) is the characteristic energy of the model, Eq.(38). However, this condition can not be satisfied for an arbitrary number of particles. In fact, when \( N \) tends to \( N_0 \):

\[ N_0 = \left( \frac{2\pi mc^2}{\epsilon_0} \right)^\frac{2}{d+1} = \left( \frac{hc}{G} \right)^\frac{2}{d+1} \frac{1}{m^2}, \]

(155)

to which corresponds to a characteristic mass of \( M_0 \):

\[ M_0 = \left( \frac{hc}{G} \right)^\frac{2}{d+1} \frac{1}{m^2}, \]

(156)

the model loses its validity. Everybody can recognize the fundamental constant of the stellar systems, which has much to do with the stability conditions of the stars (see in ref. [4]). Note that this constant appears as consequence of the energy scaling assumed, so that, it can not be obtained if a another scaling law had been adopted. A consequent analysis of these massive systems should be performed taking into account the relativity effects.

Finally, let us remember the well-known white dwarfs model based on the consideration of the Thomas-Fermi method to describe the state equation of the degenerate nonrelativistic electronic gas, whose pressure supports the hydrostatic equilibrium of the star. Using a simple dimensional analysis, from this model can be easily derived the same scaling laws obtained by us analysing the necessary conditions for the equivalence of the statistical ensembles. This coincidence is not casual: for the nonrelativistic particles these scaling laws only depend on the dimension of the physical space.

Let us discuss now the results of the numerical calculations. In the FIG. 5 it is shown the general dependency of the functions \( h(\Phi_0) \), \( h_1(\Phi_0) \) and \( h_2(\Phi_0) \). Observe the oscillatory character of these functions with the increasing of the parameter \( \Phi_0 \). This behavior is characteristic of the isothermal distribution. It can also be seen the quasi-polytropic asymptotic behavior for low values of \( \Phi_0 \), the power law \( \Phi_0^3 \).

FIG. 5. Behavior of the functions \( h(\Phi_0) \), \( h_1(\Phi_0) \) and \( h_2(\Phi_0) \). The oscillatory behavior is characteristic of the isothermal distribution.

Throughout the Eq.(109) it is obtained the dependence of the canonical parameter \( \beta_0 \), as well as the energy \( \epsilon \) from the parameter \( \Phi_0 \). Similarly, it is studied this dependence for the radio in which is contained the 80% of the total mass of the system, \( R_{80\%} \), as well as for the
central density $\rho_0$. All these dependencies are shown in the FIG.6, 7, 8 and 9 respectively.

FIG. 6. Canonical parameter $\beta$ vs $\Phi_0$. Observe the persistence of the oscillatory behavior.

FIG. 7. Scaling invariant energy, $\epsilon$ vs $\Phi_0$. It is observe again the oscillatory behavior.

FIG. 8. Radio at the 80% of the system total mass, $R_{80\%}$ vs $\Phi_0$.

FIG. 9. Central density $\rho_0$ vs $\Phi_0$. This function grows monotonically with the increasing of $\Phi_0$. This result allows us to understand that with the increasing of $\Phi_0$ the system develop a core-halo structure (a high dense core and a dilute halo).

Let us to comment briefly these results. At the first place it is noted the bounded character of the parameters $\beta_0$, $\epsilon$ and $R_{80\%}$ for all the values of the parameter $\Phi_0$: all of them are contained in the following intervals:

$$\beta_0 \in (0, 2.97); \quad \epsilon \in (-0.26, 0.055); \quad R_{80\%} \in (1.02, 1.33).$$

(157)

showing an oscillatory behavior for high values of $\Phi_0$ around the values $\beta_0 \simeq 2.12$, $\epsilon \simeq -0.18$, and $(R_{80\%}) \simeq 1.11$. However, the central density grows monotonic with the increasing of $\Phi_0$, exhibiting an exponential behavior for $\Phi_0 > 9.45$. It is very easy to understand that our solution exhibits the same features of the Antonov problem solution [27]. This is evident when analyzing the caloric curve: the dependency $\beta_0$ vs $\epsilon$, the FIG.10: it is found again the very well-known spiral. Additionally, it is shown in the FIG.11 the dependence of the radio in which is contained the 80% of the total mass of the system with the energy.

FIG. 10. Caloric curve of the astrophysical system. It is obtained again the classical spiral. In general, the results of the present model exhibit the same features of the Antonov’s problem.

FIG. 11. System size vs Energy. The system becomes smaller with the energy decreasing.

All these configurations contained between the points B and C can be described using the canonical description in the representation $(\mathcal{E}, N)$. Here the system exhibits a positive heat capacity. All the equilibrium configurations between the points C and D can not be accessed from the canonical description in the $(\mathcal{E}, N)$ representation. In this part of the spiral the systems exhibits a negative heat capacity. No equilibrium configurations exits for values of $\beta_0$ greater than $\beta_C = 2.97$. Ordinarily this is referred as the isothermal catastrophe. On the other hand, no equilibrium configurations exits for energies outside the interval $(-0.26, 0.055)$. For energies greater than $\epsilon_B = 0.055$ the system is extreme diluted and can not be confined. For energies below of $\epsilon_D = -0.26$, the systems collapse developing a core-halo structure. It is obtained again the gravothermal catastrophe.

All those configurations contained between the points B to D correspond to equilibrium situations, the other points of the spiral correspond to unstable saddle points. This is evident when analyzing the thermodynamic potentials of the ensembles, the FIG.12. and 13. For example, in the FIG.12. it is represented the dependence of the Planck potential from the canonical parameter $\beta_0$. Here it is evident that in the canonical ensemble (using the $(\mathcal{E}, N)$ representation) all those configurations between the points B to C are stable: in such points it is minimized this thermodynamic potential. Similarly, in the FIG.13. it is shown that in the microcanonical ensemble all those configurations between the points B to D are stable: in such points it is maximized the entropy for each accessible value of the energy.

FIG. 12. Planck potential vs canonical parameter $\beta$. Here is evident that in the canonical ensemble are stable all those configurations between the points B to C.

FIG. 13. In the microcanonical ensemble are stable all those configurations between the points B to D.

It has to be pointed out that the Gibbs’ argument, the equilibrium of a subsystem with a thermal bath, is not applicable to this situation, since it is based on the independence or weak correlation of the subsystem with the thermal bath. This is an invalid supposition for the nonextensive systems due to the long-range correlations existing in them as a consequence of the long-range interactions among the particles, which is the case that we are studying here. It is needed to remember that the use of generalized canonical description is supported by its equivalency with the microcanonical description in the thermodynamic limit, which is the description physically justified to this system. Moreover, no reasonable thermal bath exits for the astrophysical systems. Thus, the isothermal catastrophe is not a phenomenon with physical relevance since it can never be obtained in nature: the consideration of a thermal bath in the astrophysical system is out of context. A different significance possesses the gravothermal catastrophe. The gravitational collapse is the main engine of structure in astrophysics and it concerns almost all scales of the universe: the formation of planetesimals in the solar nebula, the formation of stars, the fractal nature of the interstellar medium, the evolution of globular clusters and galaxies and the formation of galactic clusters in cosmology [31].

Many authors claim that those regions characterized
VII. CONCLUDING REMARKS

In the present paper it has been shown how the Boltzmann-Gibbs' Statistics can be improved in order to extend its applicability to the study of the macroscopic description of the N-body self-gravitating gas, although this kind of systems are nonextensive. The Tsallis' Statistics is non applicable to this kind of system since this theory demands potential self-similarity scaling laws in the thermodynamic limit [13], which is not the case that is studying here: the astrophysical systems are pseudoextensive [13], they exhibit an exponential scaling laws when \( N \to \infty \).

The most important result obtained in the present analysis is the specification of the thermodynamic limit for this kind of systems. Although the \( N \)-independent picture of the traditional description of the astrophysical systems remains inalterable, the bad consideration of the scaling laws leads to a trivial ensemble inequivalency, which was shown in the dynamical analysis performed in the ref. [57].

The coincidences and connections of the results derived from this study with others results obtained in the past using both, thermodynamic and non thermodynamic methods, constitute important evidences of the validity of our considerations.

Finally, it was shown the importance of the geometrical aspects of the probabilistic distribution functions in the macroscopic study of systems. Specifically, it was used these geometrical properties to extend the validity of the generalized canonical description to those situations in which the traditional description limited to the consideration of an special representation for the space of the integrals of motion of the system fails. These geometric aspects could be used to enhance the possibilities of the Montecarlo method based on the canonical exponential weight during the study of those hamiltonian systems presenting anomalies in their heat capacity.

VIII. APPENDIX

A. Derivation of the decomposition formula.

The \( N \)-body phase space \( \mathcal{X}_N \) is an Cartesian external product of the generalized coordinates and momentum spaces of each particle, \( Q \) and \( P \), which can be represented in the following way:

\[
\mathcal{X}_N = \prod_{s=1}^{N} \left( Q_s \otimes P_s \right).
\]

Using the partition of the coordinates space \( Q \) in not overlapped cells \( \{ c_k \} \):

\[
Q = \bigcup_k c_k,
\]

the space \( \mathcal{X}_N \) can be decomposed in not overlapped subspaces \( \mathcal{X}_N^{(\sigma)} \):

\[
\mathcal{X}_N = \bigcup_\sigma \mathcal{X}_N^{(\sigma)},
\]

where

\[
\mathcal{X}_N^{(\sigma)} = \prod_{s=1}^{N} q_s \left( p_\sigma \left( s \right) \right) \otimes P_s.
\]

Here, \( p_\sigma \left( s \right) \) is a function which assigns each particle of the system to a determined cell. The index \( \sigma \) denotes all the possible ways to perform this correspondence. Moreover, \( q_s \left( k \right) \equiv c_k \). Using the Eq. (160), the \( N \)-body phase space integration can be expressed as follows:

\[
\frac{1}{N!} \int_{\mathcal{X}_N} dX_N = \sum_\sigma \frac{1}{N!} \int_{\mathcal{X}_N^{(\sigma)}} dX_N.
\]

Due to the identity of particles, all those terms corresponding to configurations with identical occupation number of particles at the cells are identical. Let \( \{ n_k \} \) be the occupation numbers of particles in the cells. For this this case, there are a total of:

\[
C_{\{ n_k \}}^N = \frac{N!}{\prod_k n_k!}
\]
identical terms in the sum of the Eq. (162). Let $\mathcal{X}_N^{(nk)}$ be a subspace of $\mathcal{X}_N$ with occupation numbers given by $\{nk\}$. Since $\mathcal{X}_N$ is a Cartesian external product of spaces, its volume element can be factorized in the volume elements of the different spaces. This can be conveniently done grouping all these particles belonging to the same cell. In this case the $N$-body phase space integration of the subspace $\mathcal{X}_N^{(nk)}$ can be decomposed as follows:

$$
\frac{1}{N!} C_N^{\{nk\}} \int d\mathcal{X}_N = \prod_k \hat{\sigma} \left[ \mathcal{X}_N^{(nk)} \right], \quad (164)
$$

where:

$$
\hat{\sigma} \left[ \mathcal{X}_N^{(nk)} \right] = \begin{cases} 
\frac{1}{\mathcal{X}_N^{(nk)}} \int d\mathcal{X}_N, & \text{if } n \neq 0, \\
1, & \text{if } n = 0,
\end{cases} \quad (165)
$$

and $\mathcal{X}_N^{(nk)}$ represents the $nk$-body phase space whose physical space of the particles is reduced to the cell $c_k$. Taking into account all the exposed above, it is easy to see that the $N$-body phase space integration can be finally expressed as:

$$
\frac{1}{N!} \int d\mathcal{X}_N = \prod_k \sum_{nk=0}^N \hat{\sigma} \left[ \mathcal{X}_N^{(nk)} \right] \delta(c) \left( N - \sum_k nk \right), \quad (166)
$$

where the $\delta(c)(n)$ is related with the Kronekel delta function as follows:

$$
\delta(c)(n) = \delta_{0n}. \quad (167)
$$

The presence of this function in the right hand of the Eq. (166), assures that the number of particles remain fixed and equals $N$.

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Fig. 1. L. Velazquez

Cubic Power Law

Function \( F(z) \)
Figure 2. Caloric Curve

- Renormalized Model
- Maxwell Distribution
- Asymptotic Convergence

E/U vs T/U
Figure 3. Distribution Function

- $z = 0$
- $z = 0.2$
- $z = 0.5$
- $z = 1$
- $z = 2$
- $z = 4$
- $z = 10$

$\frac{p}{p_0}$
The $\Phi(\xi)$ Function

by QUASI-POLYTROPIC EQUATIONS

by POLYTROPIC EQUATIONS
**ISO THERMAL BEHAVIOR**

**LINEAR DEPENDENCY**

**FIG. 6. L. VELAZQUEZ**
FIG. 8. L. VELAZQUEZ

RADIO 80 %
FIG. 9. L. VELAZQUEZ

\[ \ln \rho_0 \]

\[ \Phi \]

CENTRAL DENSITY

EXPONENTIAL BEHAVIOR
caloric curve

C \( \beta = 2.97 \)
\( E = -0.074 \)

B \( \beta = 0 \)
\( E = 0.055 \)

D \( \beta = 1.89 \)
\( E = -0.260 \)

S I N G U L A R E S P H E R E

FIG. 10. L. VELAZQUEZ
R = 1.06
E = -0.260
R = 1.06

B
E = 0.055
R = 1.33

 RADIO 80%
Figure 12. L. Velazquez
MICROCANONICAL ENSEMBLE

E = 0.055

B
E = 0.260

STABLE BRANCH

UNSTABLE BRANCHES

ENTROPY

ENERGY

FIG. 13. L. VELAZQUEZ