Barrier billiard and random matrices

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Abstract
The barrier billiard is the simplest example of pseudo-integrable models with interesting and intricate classical and quantum properties. Using the Wiener–Hopf method it is demonstrated that quantum mechanics of a rectangular billiard with a barrier in the centre can be reduced to the investigation of a certain unitary matrix. Under heuristic assumptions this matrix is substituted by a special low-complexity random unitary matrix of independent interest. The main results of the paper are (i) spectral statistics of such billiards is insensitive to the barrier height and (ii) it is well described by the semi-Poisson distributions.

Keywords: barrier billiards, random matrices, spectral statistics

1. Introduction

An implicit idea of quantum chaos studies is that quantum dynamics of even simple deterministic systems is so irregular and complex that the calculation of particular values of eigenenergies and eigenfunctions, though possible, leads to quasi-random quantities which may and have to be substituted by a statistical description of such quantum problems.

There are two big conjectures in quantum chaos:

- Local spectral statistics of generic quantum systems corresponding to classically integrable systems is well described by the Poisson statistics of independent random variables [1].
- Local spectral statistics of generic quantum systems corresponding to classically chaotic systems is described by eigenvalue statistics of standard ensembles of random matrices depended only on system symmetry [2].

These conjectures form a cornerstone of quantum chaos and have been checked in enormous number of examples.

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Nevertheless, they do not cover all possible types of dynamical systems. For simplicity, let us concentrate on two-dimensional Hamiltonian models. Classically integrable systems are characterised by the condition that a typical trajectory belongs to a torus (i.e. a two-dimensional surface of genus 1). For classically chaotic models typical trajectories cover the whole three-dimensional surface of constant energy. But there exist systems whose trajectories spread over two-dimensional surfaces of genus higher than 1. Such systems are neither integrable or chaotic and coined the name of pseudo-integrable models (see, e.g. [3]). A characteristic example of such systems is a plane polygonal billiard whose internal angles $\alpha_j$ are rational fractions of $\pi$:

$$\alpha_j = \frac{m_j}{n_j}$$

with co-prime integers $m_j$ and $n_j$. It has been proved [4] that in this case classical trajectories belong to a surface of genus

$$g = 1 + \frac{N}{2} \sum_j \frac{m_j - 1}{n_j},$$

where $N$ is the least common multiply of all denominators $n_j$. About classical dynamics of such billiards see, e.g. [5, 6] and references therein.

The knowledge of quantum properties of pseudo-integrable billiards is fragmentary and includes mainly numerical calculations of statistical properties of eigenenergies for billiards of simple shape: rhombus, right triangles, rectangular billiard with a barrier, etc, [7–15]. The only quantity accessible analytically in certain models is the spectral compressibility $\chi$ which determines the growth of the variance of number of levels in an interval of length $L$ [16]

$$\langle (N(L) - L)^2 \rangle \rightarrow \chi L,$$

where $N(L)$ is a number of levels in an interval $L$ normalised that its mean value equals $L$ and the averaging is taken over a small window of energies. The value of $\chi$ is of importance as for integrable models $\chi = 1$ and for chaotic ones $\chi = 0$ [16]. The calculation of the compressibility is done by the summation over classical periodic orbits in the diagonal approximation [16]. For pseudo-integrable billiards the description of periodic orbits is known analytically for special class of billiards called the Veech billiards [6, 17, 18]. In particular, for a right triangle with one angle $\pi/n$ in [13] has been proved that

$$\chi = \frac{n + \epsilon(n)}{3(n - 2)},$$

where $\epsilon(n) = 0$ for odd $n$, $\epsilon(n) = 3$ for even $n$ but $n \neq 0 \mod 3$, and $\epsilon(n) = 6$ for $n \equiv 0 \mod 6$.

For the barrier billiard discussed below it has been shown (see [15] for the barrier height equals one-half of the billiard length, $h/a = 1/2$, and appendix D of [19] for an arbitrary height) that independently of the barrier height

$$\chi = \frac{1}{2}.$$  

The fact that for these models $0 < \chi < 1$ is a clear-cut indication that spectral statistics of such billiards differ from both the Poisson distribution typical for integrable models and the random matrix statistics of chaotic systems.

Numerically, it has been confirmed (cf [13, 15]), that the spectral statistics of the above billiards is special and is characterised by following properties:
- Level repulsion at small distances as for the standard random matrix ensembles.
- Exponential decrease of nearest-neighbour spacing distributions as for the Poisson distribution.
- Non-trivial value of the spectral compressibility (cf, (4) and (5)).
- Multi-fractal dimensions of eigenfunctions [20, 21].

This type of statistics has been first observed in the Anderson model at the point of the metal–insulator transition [22, 23] and is called now an intermediate statistics.

A canonical model of such statistics is the critical power-law random banded matrix model [24] (see also [25, 26]) in which all matrix elements are independent Gaussian random variables with zero mean and the variances decreasing linearly from the main diagonal

$$\langle |H_{ij}|^2 \rangle = \left( 1 + \frac{|i-j|^2}{b^2} \right)^{-1}. \quad (6)$$

This model has been thoroughly investigated (see, e.g. [27] and references therein) but its universality remains questionable. There exist several examples of matrices with intermediate type spectral statistics [28–31] which clearly cannot be described by the above model. In a sense, the critical power-law random banded matrix model is a minimal mathematical model which leads to intermediate statistics but it does not corresponds to a physical problem.

The paper is devoted to the investigation of the simplest pseudo-integrable system, namely a rectangular billiard with a barrier at the centre of a side (see figure 1(a)). This billiard, called barrier billiard, has six \( \pi/2 \) angles plus one angle around the barrier tip equals \( 2\pi \). From (2) it follows that it corresponds to a genus-two surface. Classical motion in this model corresponds to the ray propagation with specular reflection when the ray hits either rectangular boundaries or the barrier. When unfolded by reflections in the rectangular sides the problem reduces to double periodic arrangement of barriers and classical ray motion is a kind of zigzag trajectories reflected only by barriers. Peculiarities of barrier billiard are simply seen from the fact that almost all such trajectories are fractals with non-trivial fractal dimensions [32]. (On rigorous calculation of anomalous diffusion for more general polygonal obstacles in two-dimensional lattices see, e.g. [33] and references therein.) The eminent property of any polygonal billiards with at least one singular angle \( \neq \pi/n \) with integer \( n \) (and, in particular, the barrier billiard) is discontinuous character of classical motion. If a bunch of parallel rays hits a singular wedge (e.g. the barrier tip) it splits into two different groups of rays. Though Lyapunov exponent for any polygonal billiard is zero [5] it is such discontinuities that are responsible for very sensitive dependence of initial conditions (i.e. irregular or erratic behaviour) of long classical rays. Quantum mechanical analog of classical discontinuous scattering on singular wedges is a strong diffraction on them [34]. An interesting properties of polygonal billiards is that multiple diffraction on singular wedges leads in the semiclassical limit to the formation of so-called superscar wave functions which are close to plane waves freely propagating in periodic orbit channels in such billiards [20, 21]. Many pictures of such superscars in the barrier billiard are presented in [20, 21].

The purpose of this work is twofold. First, in section 2 it is demonstrated that the investigation of the barrier billiard, can be reduced to the analysis of an unitary \( S \)-matrix corresponding to the scattering on the barrier multiplied by certain phases related on the barrier height. Using the Wiener–Hopf method, briefly reviewed in appendix A, this matrix is calculated analytically. Second, assuming that certain simple phases can be considered as random it is argued in section 3 that the exact \( S \)-matrix could be substituted by a random unitary matrix which belongs to a sub-class of low-complexity matrices with simple displacement structure [35, 36]. Using
the same method as for random Toeplitz and Hankel matrices [37] it is shown in section 4 that local spectral statistics of the resulting random unitary matrix is well described by the semi-Poisson distribution [28] which agrees well with numerical calculations. These results imply that eigenvalues of the barrier billiard are also statistically distributed by the same distribution. Section 5 gives a brief summary of the obtained results.

2. S-matrix approach to the quantisation of a barrier billiard

The quantisation of the barrier billiard shown in figure 1(a) consists in finding the eigenvalues $E_n$ and eigenfunctions $\Psi_n(x, y)$ of the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + E_n\right) \Psi_n(x, y) = 0$$  \hspace{1cm} (7)

which obey the Dirichlet boundary conditions on all sides of the rectangle and on the barrier

$$\Psi_n(x, y)|_{\text{sides}} = 0, \quad \Psi_n(x, y)|_{\text{barrier}} = 0.$$  \hspace{1cm} (8)

Due to the symmetry one set of solutions which equals zero at the whole line $y = b$ is evident

$$\Psi_n(x, y) = \sin \left(\frac{\pi n}{a}(x - h)\right) \sin \left(\frac{\pi m}{b} y\right), \quad n, m = 1, 2, \ldots$$  \hspace{1cm} (9)

We are interested in non-trivial solutions which are symmetric with respect to the inversion in the line passing through the barrier. In the coordinates as in figure 1(a) it means that these solutions have to obey two sets of boundary conditions

$$\Psi_n(x, b) = 0, \quad 0 < x < h,$$

$$\frac{\partial}{\partial y} \Psi_n(x, b) = 0, \quad h - a < x < 0,$$

$$\Psi_n(x, 0) = 0, \quad h - a < x < h,$$  \hspace{1cm} (10)
\[ \Psi_\alpha(h, y) = 0, \quad \Psi_\alpha(h - a, y) = 0, \quad 0 < y < b. \] 

No analytical solutions of the Helmholtz equation with such boundary conditions are known.

Let us disregard the vertical conditions (11) and find the scattering solutions of the infinite slab indicated in figure 1(b). It implies that we are now looking for the solutions of the equation

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2 \right) \Psi(x, y) = 0 \]  

inside the slab such that at horizontal boundaries they obey the following conditions

\[ \Psi(x, b) = 0, \quad 0 < x < \infty, \]
\[ \frac{\partial}{\partial y} \Psi_\alpha(x, b) = 0, \quad -\infty < x < 0, \]
\[ \Psi_\alpha(x, 0) = 0, \quad -\infty < x < \infty. \]

As it is well known, to uniquely define such solutions one has to fix the behaviour on the infinity.

The elementary solutions on negative and positive \( x \) with fixed energy have evidently the following forms (the normalisation of plane waves to the unit current is used)

\[ \phi_{2m}^{(\pm)}(x, y) = \frac{e^{\pm ip_{2m}x}}{\sqrt{2b p_{2m}}} \sin \left( \frac{\pi m}{b} y \right), \quad x > 0, \] 
\[ \phi_{2m-1}^{(\pm)}(x, y) = \frac{e^{\pm ip_{2m-1}x}}{\sqrt{2b p_{2m-1}}} \sin \left( \frac{\pi(2m - 1)}{2b} y \right), \quad x < 0, \]

where

\[ p_m = \sqrt{k^2 - \frac{\pi^2 m^2}{4b^2}}, \quad m = 1, 2, \ldots \] 

There exist two sets of standard solutions determined by fixing the incoming plane waves. Any of such solutions can be expanded into corresponding series of elementary waves (14) and (15).

For waves coming from the left one has the following expansion into reflected and transmitted waves

\[ \Phi_{2n-1}^{(+)}(x, y) = \begin{cases} 
\phi_{2n-1}^{(+)}(x, y) + \sum_{m=1}^{\infty} S_{2n-1,2m-1} \phi_{2m-1}^{(-)}(x, y), & x < 0 \\
\sum_{m=1}^{\infty} S_{2n-1,2m} \phi_{2m}^{(+)}(x, y), & x > 0 
\end{cases}. \]

For waves coming from the right such expansion is

\[ \Phi_{2n}^{(-)}(x, y) = \begin{cases} 
\sum_{m=1}^{\infty} S_{2n-2m-1} \phi_{2m}^{(-)}(x, y), & x < 0 \\
\phi_{2n}^{(-)}(x, y) + \sum_{m=1}^{\infty} S_{2n-2m+1} \phi_{2m}^{(+)}(x, y), & x > 0 
\end{cases}. \]
The matrix $S_{mn}$ is the $S$-matrix for the scattering inside the slab. In appendix A it is demonstrated that such matrix can be calculated analytically by the Wiener–Hopf method.

By construction, functions $\Phi_{2n-1}^-(x, y)$ and $\Phi_{2n}^+(x, y)$ obey boundary conditions on horizontal boundaries (13). To find functions obeying the vertical conditions (11) let us form the linear combinations of these functions

$$\Psi_n(x, y) = \sum_{n=1}^{\infty} a_{2n} \Phi_{2n}^-(x, y) + a_{2n-1} \Phi_{2n-1}^+(x, y). \quad (19)$$

Taking into account that functions (15) and (14) form complete set of functions at, respectively, negative and positive $x$, the requirements (11) signify that for $m = 1, 2, \ldots$

$$a_{2m} + e^{2ip_m h} \sum_{n=1}^{\infty} a_n S_{n, 2m} = 0, \quad \text{from } x = h, \quad (20)$$

and

$$a_{2m-1} + e^{2ip_m (a-h)} \sum_{n=1}^{\infty} a_n S_{n, 2m-1} = 0, \quad \text{from } x = h - a. \quad (21)$$

Notice that the summation in these expressions are done over both even and odd integers.

Finally these equations can be rewritten for all $m$ as follows

$$a_m + \sum_{n=1}^{\infty} a_n B_{n,m} = 0, \quad m = 1, 2, \ldots, \quad (22)$$

where matrix $B_{n,m}$ differs from $S_{n,m}$ only by special phases

$$B_{n,m} = e^{i\phi_m} S_{n,m}, \quad \phi_{2m} = 2p_m h, \quad \phi_{2m-1} = 2p_{2m-1} (a-h). \quad (23)$$

The existence of such solutions determines the eigenvalue of $k$ from the quantisation condition

$$\det (\delta_{n,m} + B_{n,m}) = 0. \quad (24)$$

Matrix $B$ contains the complete information about the quantisation of the barrier billiard. It constitutes of two parts: a specific $S$-matrix for the scattering on a barrier and additional phases related with the position of the barrier.

3. Random matrix description of the barrier billiard

As it is shown in appendix A (A27), the scattering $S$-matrix is a complex symmetric matrix of the form

$$S_{n,m} = \frac{L_n L_m}{x_n + x_m}, \quad (25)$$

where

$$x_m = (-1)^{m+1} h p_m \quad (26)$$

and vector $L_m$ is given by (A29).
By construction, matrix $B$ is

$$B_{n,m} = e^{i\phi_m} \frac{L_n L_m}{x_n + x_m},$$  

(27)

where $\phi_m$ are defined in (23). By conjugation this matrix can also be transformed into symmetric shape but it is not necessary.

For propagating modes with real $p_m$ matrix $S$ and, consequently, matrix $B$ are unitary

$$SS^\dagger = 1, \quad BB^\dagger = 1.$$  

(28)

Till now the calculations were exact. Below we discuss ‘natural’ simplifications appeared in the semiclassical limit $k \to \infty$. The first remark is that matrix $B$ includes both, propagating (with real $p_m$) and evanescent (corresponding to imaginary $p_m$) modes. As evanescent modes in the semiclassical limit decay exponentially quickly from the barrier tip one can neglect contributions of such modes provided that the tip is not very close to the boundaries, $\hbar k \gg 1$ and $(a - h) k \gg 1$.

Then the $B$-matrix becomes a finite dimensional unitary matrix of (large) dimension

$$N = [2kb/\pi]$$  

(29)

which corresponds, in a sense, to an exact semiclassical quantisation of a surface of section [38, 39] where only terms exponentially small in the semiclassical limit are disregarded.

Eigenvalues of $N \times N$ unitary matrix $B \equiv B(k)$ with fixed parameter $k$ are of the form $e^{i\epsilon_m(k)}$ with real $\epsilon_m(k)$. Assume that $\epsilon_j(k)$ with fixed $k$ are ordered and restricted to an interval $[0, 2\pi)$

$$0 \leq \epsilon_1(k) < \epsilon_2(k) < \ldots < \epsilon_N(k) < 2\pi.$$  

(30)

True eigenenergies of the barrier billiard correspond to such values of $k$ for which one of eigenvalues of $B$ equals $-1$

$$\epsilon_m(k_0) = \pi.$$  

(31)

Below we cite (slightly simplifying) heuristic arguments from [38, 39] which show that under reasonable conditions spectral statistics of eigenvalues of matrix $B$ and of barrier billiard eigenvalues are the same up to a rescaling.

- The motion of eigenvalues of $B(k)$ when $k$ is changed from $k = k_0$ to $k = k_0 + \delta k$ with small $\delta k$ (such that $N(k)$ in (29) remains constant) can be approximated as a sum of two terms, a smooth overall shift and a quasi-random contribution due to the scattering with other eigenvalues

$$\epsilon_m(k_0 + \delta k) = \epsilon_m + \tau \delta k,$$

$$\tau = \left\langle \frac{\partial \epsilon_m(k)}{\partial k} \right\rangle.$$  

(32)

- Quantities $\epsilon_m$ are supposed to be so erratic function of $k$ that their explicit form is irrelevant and they may be substituted by random numbers with certain correlation functions $R_m(x_1, \ldots, x_n)$ defined as the probability density that variables $\epsilon_m$ are in-between $x_n$ and $x_n + dx_n$ for all $n$.

- The values of true barrier billiard eigenmomenta $k_\alpha = k_0 + \delta k_\alpha$ are determined from (31)

$$\tau \delta k_\alpha = \pi - \epsilon_\alpha.$$  

(33)
The value of $\tau (= O(1)$ for billiards) can be estimated by the comparison the mean level density of unitary matrix eigenvalues, $d_B = N/(2\pi)$ where $N \approx 2bk/\pi$ as in (29), with the mean level density of barrier billiard eigenvalues in the momentum space given by the Weyl law, $\bar{d}(k) = abk/(2\pi)$,

$$\tau d_B = \bar{d}(k), \quad \tau \approx \frac{\pi a}{2}. \quad (34)$$

- If correlation functions $R_n(x_1, \ldots, x_n)$ are translation invariant, i.e. they depend only on the differences between eigenvalues, then spectral statistics of barrier billiard eigenvalues is (up to a rescaling) the same as spectral statistics of eigenvalues of matrix $B(k)$.

Matrix $B(k)$ has no explicit random parameters. As it is typical in quantum chaos, pseudo-randomness of its eigenvalues and eigenfunctions comes, supposedly, from erratic behaviour of its elements when parameter $k$ is changed. This statement, though physically natural, is difficult to prove rigorously (if any). To get a well defined random matrix we assume that in the semiclassical limit $N \to \infty$ deterministic exponential factors for propagating modes $e^{i\phi_m}$ where $\phi_m$ are as in (23) can be substituted by $e^{i\Phi_m}$ where $\Phi_m$ with $m = 1, \ldots, N$ are independent random variables distributed uniformly between 0 and $2\pi$.

Such assumption is also not easy to prove. It is similar to 'physical' statement that local spectral statistics of 'generic' integrable systems is well approximated by the Poisson statistics [1]. Here mathematically 'generic' usually means 'random' as for a fixed integrable deterministic system is notoriously difficult to prove the Poisson statistics due to the absence of explicitly random parameters. Even for a rectangular billiard only the two-point correlation function $R_2(s) = 1$ in a convenient normalisation) is accessible to analytical calculations [40]. In physical literature it is conjectured that if the square of the ratio of rectangle lengths is a 'good' irrational number (a Diophantine number?) then local spectral statistics of a rectangular billiard agrees with the Poisson statistics in accordance with the existing numerics. The proof or disproof of this conjecture seems to be beyond the known methods. On the other hand, the Poisson statistics had been rigorously proved for rotational invariant systems but only under the assumption that rotated curve is random [41].

Nevertheless, the combination of the following facts: (i) in the semiclassical limit $k \to \infty$ phases $\phi_m$ are large (except ones very close to the threshold of evanescent modes) and (ii) these phases are, in general, non-commensurable, permit to conjecture that quantities $\phi_m \mod 2\pi$ become pseudo-random (may be after an averaging over a small window of $k$). Though, in general, it may be true, there are proven counterexamples. In particular, the sequence $\sqrt{m} \mod 1$ with $m = 1, \ldots, N$ is uniformly distributed for large $N$ and its two-point correlation function agrees with the Poisson point process [42], but its nearest-neighbour spacing distribution differs from the Poisson expression [43].

As an example let us consider the phases $\phi_m$ in (23) for the case $a = 2h$. Denoting $k = \pi(N + \delta)/b$ and $\alpha = 2h\pi/b$, the phases $\phi_m$ take the form

$$\phi_m = \alpha \sqrt{(N + \delta)^2 - m^2}, \quad m = 1, \ldots, N \quad (35)$$

with a constant $\alpha$ and $0 < \delta < 1$. To check the validity of the above assumption for such phases numerical calculations of

$$\xi_m = \frac{N}{2\pi} (\phi_m \mod 2\pi) \quad (36)$$

were performed. The factor $N/2\pi$ ensures that the mean density of $\xi_m$ equals 1.
Figure 2. The \((n+1)\)th-nearest-neighbour spacing distributions with \(n = 0, 1, \ldots, 5\) computed numerically for pseudo-random phases \((35)\) mod \(2\pi\) with \(N = 10^5\), \(\alpha = 1\) and \(\delta = 1/2\) (black circles). Solid lines are the Poisson predictions for these quantities \((37)\). Insert: the difference between \(P_0(s)\) and the formula \(P_0^{(\text{Poisson})}(s) = e^{-s}\).

The standard way of investigation of local statistical properties of any series of numbers (e.g. phases or eigenvalue spectrum) consists in the calculation of the \((n+1)\)th-nearest-neighbour spacing distributions \(P_n(s)\) which determine the probability densities that between two numbers at a distance \(s\) there exist exactly \(n\) other numbers.

In figure 2 the numerical results for \(P_n(s)\) for unfolded phases \((36)\) are presented for \(n = 0, 1, \ldots, 5\) and \(N = 10^5\). In the calculations values \(\delta = 1/2\) and \(\alpha = 1\) were chosen but the results seems to be insensitive to specific choices of these parameters. Solid lines in this figure indicate the well-known Poisson expressions for independent identically distributed uniform random variables

\[
P_n^{(\text{Poisson})}(s) = \frac{s^n}{n!} e^{-s}.
\]

\[(37)\]

It is clearly seen that the random phase approximation works well for functions \((35)\). To see better the accuracy of such approximation the difference between the numerical nearest-neighbour spacing distribution \(P_0(s)\) and the Poisson value \(P_0^{(\text{Poisson})}(s) = e^{-s}\) is plotted in the insert of this figure.

Taking the above arguments as granted allow us to substitute the deterministic unitary matrix \(B\) by the random unitary matrix \(B_n\) defined as

\[
B_{n,m} = \frac{e^{i\Phi_n} I_{n,m}}{x_n + x_m}, \quad n, m = 1, \ldots, N,
\]

\[(38)\]
where $L_m$ by conjugation can be transformed into real quantities related with $x_j$ as follows (see equation (A31) from appendix A)

$$L_m = \sqrt{\frac{2x_m}{\prod_{j\neq m}^x x_m - x_j}}.$$  

$x_m$ are real quantities obeying the chain of inequalities (which is a consequence of the positivity of $L_m^2$)

$$x_1 > -x_2 > x_3 > -x_4, \ldots, > 0,$$

and $\Phi_m$ are independent random variables uniformly distributed between 0 and $2\pi$.

It is straightforward to check that any matrix as in (38) such that modulus $L_m$ is given by (39) is automatically unitary for arbitrary phases $\Phi_m$.

All information about the barrier height is contained in phases $\phi_m$ (23). After the replacement of these deterministic phases by random variables this information is dislodged which means that spectral statistics of the barrier billiard in semiclassical limit is independent on the barrier height. It concurs with the fact that the spectral compressibility (5) is the same for all barrier heights [19] and with the results [15] that numerically spectral statistics of the barrier billiard with $h/a = 1/2$ and with an irrational ratio $h/a$ look similar.

### 4. Properties of the main random matrix

Matrix (38) belongs to the class of low-complexity matrices [35] characterised by the following displacement operator [36]

$$\Delta_A(B) = AB + BA,$$  

where matrix $A$ is a diagonal matrix $A_{i,j} = x_j \delta_{i,j}$. From definition (38) it follows that

$$\Delta_A(B) = e^{i\phi_m} L_m L_m$$

which implies that the displacement operator of matrix $B$ is a rank-one matrix. According to a theorem proved in [36], principal matrix operations such as the matrix inversion and the calculation of matrix eigenvalues for matrices with finite displacement rank can be performed in $O(N^3)$ operations to compare with $O(N^5)$ operations needed for general matrices. Here $N$ is the matrix dimension.

It has been stressed in [37] that random low-complexity matrices are good candidates for matrices with intermediate spectral statistics discussed in introduction. The detailed investigation of statistical properties of matrix $B$ defined in (38) will be given elsewhere. Only main features of such matrix are discussed here.

The local statistical properties of the eigenvalue spectrum are encoded in the $(n + 1)$th-nearest-neighbour spacing distributions $P_n(s)$ discussed above. The exact expressions for correlation functions of matrices such as in (38) are unknown. To obtain simple approximate Wigner-type formulas for these quantities we use the method developed in [37] for random Toeplitz and Hankel matrices.

According to this method the $(n + 1)$th-nearest-neighbour spacing distributions are well approximated by the gamma-distributions

$$P_n(s) \approx a_n s^n \exp(-b_n s).$$
If $\gamma_n$ is known, constants $a_n$ and $b_n$ are determined from the standard normalisation conditions

$$
\int_0^\infty P_n(s)ds = 1, \quad \int_0^\infty sP_n(s)ds = n + 1.
$$

(44)

It has been argued in [37] that

$$
\gamma_n = q_n - 1,
$$

(45)

where $q_n$ is the minimal number of parameters (the co-dimension) needed to get $n + 2$ eigenvalues of the considered matrix equal to each other.

Matrix $B$ without random phases is also an unitary matrix

$$
B^{(0)*}B^{(0)} = 1.
$$

(46)

As this matrix is a real symmetric matrix, it implies that $B^{(0)*} = 1$. In other words, eigenvalues of matrix $B^{(0)}$ equal $\pm 1$.

It is straightforward to prove that

$$
\text{Tr } B^{(0)} = N \sum_{m=1}^N \frac{L_m^2}{2x_m} = \frac{1}{2} \left( 1 - (-1)^N \right).
$$

(47)

Therefore the minimum dimension matrix with $n + 2$ eigenvalues equal 1 (and $n + 1$ eigenvalues equal $-1$) is matrix $B^{(0)}$ of dimension $Nn = 2n^2 + 3$. When $Nn$ non-zero random phases $\exp(i\Phi_m)$ are added the degeneracy of eigenvalues is lifted. As an overall phase is unessential to us, the total number of independent (random) parameters is $q_n = N_n - 1 = 2n + 2$. In this way one comes to the prediction that for matrix (38)

$$
\gamma_n = 2n + 1
$$

(48)

which exactly corresponds to the semi-Poisson statistics discussed in [28] for which

$$
P_n(s) = \frac{2^{2n+2}}{(2n+1)!} s^{2n+1} e^{-2s}.
$$

(49)

Besides random phases matrix $B$ depend on coordinates $x_m$. In principle, for the barrier billiard these variables are related with the momenta as indicated in (26). As this matrix is independent on the over-all scale of $x_m$, such ‘natural’ $x_m$ can conveniently be expressed as follows (cf, (35))

$$
x_m = (-1)^m \sqrt{(N + \delta)^2 - m^2}, \quad m = 1, \ldots, N
$$

(50)

with $0 < \delta < 1$.

Nevertheless, the above conclusion that spectral statistics of matrix $B$ should be well described by the simple semi-Poisson distribution (49) is valid for any sequence of $x_m$ (but obeying (40)) which suggests that spectral statistics of this matrix is only weekly dependent of the choice of coordinates $x_m$.

To check these predictions numerical calculations of the $(n + 1)$th-nearest-neighbour spacing distributions for matrix $B$ with $n = 0, 1, \ldots, 5$ were performed for three different choices of $x_m$. The first corresponds to (50), the second is the linear $x_m$

$$
x_m = (-1)^m (N + 1 - m), \quad m = 1, \ldots, N
$$

(51)
Figure 3. The \((n + 1)\)th-nearest-neighbour spacing distributions with \(n = 0, \ldots, 5\) for \(x_m\) as in (50) (black circles). Solid lines are the semi-Poisson predictions for these quantities (49). Insert: the difference between \(P_0(s)\) and the semi-Poisson formula: \(P_0(s) = 4s e^{-2s}\).

Figure 4. The same as in figure 3 but for linear \(x_m\) as in (51).

and for the third one \(|x_m|\) are chosen independently and uniformly between 0 and \(N\), then arranged to obey (40), and remained fixed for different realisations of random phases.
The results of these calculations are presented in figures 3–5. The calculations were done for matrices of dimension \( N = 1000 \) averaged over 100 realisations of random phases \( \Phi_m \) chosen independently and uniformly between 0 and \( 2\pi \).

To see clearly the differences between the three different choices of variables \( x_m \) the corresponding data are indicated at different figures: figure 3 shows the data when \( x_m \) are chosen as in (50) with \( \delta = .5 \) (results seem to be insensitive to \( \delta > 0 \)), figure 4 displays the data for linear choice of \( x_m \) as in (51), and figure 5 exhibits the results for random choice of \( x_m \). In each figures small circles indicate numerical results for the \((n+1)\)th-nearest-neighbour spacing distributions \( P_n(s) \) with \( n = 0, 1, \ldots, 5 \). The solid lines are the semi-Poisson predictions (49). The differences between the nearest-neighbour spacing distribution \( P_0(s) \) and the semi-Poisson formula \( P_0(s) = 4s e^{-2s} \) are presented in the inserts of these figures.

The figures clearly demonstrate that simple approximate semi-Poisson formulas (49) agree quite well with numerical results for different local correlation functions of random matrix \( B \). As expected, the results for different choices of variables \( x_m \) are close to each others but the data for random \( x_m \) seems to have larger (and more regular) deviations from the semi-Poisson predictions.

5. Conclusion

The main result of the paper is the derivation of a random matrix associated with the pseudo-integrable barrier billiard. It is demonstrated that the quantisation of the barrier billiard can conveniently be performed by a two-steps procedure. First, two boundaries of the billiard are removed and the problem is reduced to the scattering inside of an infinite slab with different boundary conditions (the Dirichlet and the Newman ones) along one boundary. The exact solution for this configuration is done by the Wiener–Hopf method. Second, an eigenfunction of the closed billiard is represented as a linear combination of obtained scattering waves and the requirement that such eigenfunction obeys the correct boundary conditions on previously
removed boundaries leads to the quantisation condition that a certain unitary matrix has an

eigenvalue equals $-1$.

The resulting matrix differs from the $S$-matrix for the scattering inside the infinite slab only

by certain phases related with the position of the barrier. In principle, it could serve for numerical
calculations of quantum properties of the barrier billiard. But in the context of the paper,

its principal importance is due to the fact that under ‘physical’ assumptions the exact matrix
can be substituted by a random unitary matrix of a special form. An immediate consequence
of such replacement is that spectral statistics of the considered barrier billiard is independent
on the barrier height.

It seems that it is the first time that a random matrix has been extracted from the exact

quantum-mechanical description of a pseudo-integrable model. The resulting random unitary
matrix belongs to the so-called low-complexity matrices with interesting statistical properties

and is of independent interest. It is demonstrated that local spectral statistics of this matrix are

well approximated by the so-called semi-Poisson distribution in accordance with numerical
calculations of the $(n+1)$th-nearest-neighbour spacing distributions. As discussed in the text,
it implies that spectral statistics of the barrier billiard has to be also close to the semi-Poisson
statistics.

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Data availability statement

All data that support the findings of this study are included within the article.

Appendix A. Construction of the $S$-matrix for the slab by the Wiener–Hopf

method

The purpose of this appendix is to calculate explicitly the $S$-matrix for the scattering inside the

slab indicated in figure 1(b). Due to the special geometry of the slab the Wiener–Hopf method

[44] seems to be ideally suited for this purpose. Though this old method is well known (see
e.g. [44]), for completeness, the main steps of the solution of this problem are briefly indicated

below.

Consider the incident plane wave $\phi_{2n-1}^{(+)}(x,y)$ as in (15) entering the slab from the left in

figure 1(b). The total field inside the slab is the sum of the incident field and the reflected field

$\psi(x,y)$

\[
\psi(x,y) = e^{ip_{2n-1}x} \sin \left( \frac{\pi (2n-1)}{2b} y \right) + \psi(x,y). \tag{A1}
\]

As it is inherent in the Wiener–Hopf method [44] one assumes that the momentum $k$ has a small
positive imaginary part so $\text{Im} k > 0$ and the reflected field is determined by requirement

that $\psi(x,y) \rightarrow 0$ as $|x| \rightarrow \infty$. By construction the total field has to obey boundary conditions indicated

in (13).
To obtain the Wiener–Hopf equation we follow closely the method of [44]. Define

$$
\Phi_+(\alpha, y) = \int_0^\infty \psi(x, y) e^{i\alpha x} \, dx, \quad \Phi_-(\alpha, y) = \int_{-\infty}^0 \psi(x, y) e^{i\alpha x} \, dx.
$$  \hspace{1cm} (A2)

Here $\alpha$ is a complex variable such that

$$
-\text{Im} \, k < \text{Im} \, \alpha < \text{Im} \, k.
$$  \hspace{1cm} (A3)

From boundary conditions (13) one gets the boundary values of $\Phi_\pm(\alpha, b) \equiv \Phi_\pm(\alpha)$

$$
\Phi_+(\alpha) + \int_0^\infty e^{i\alpha x} \sin \left( \frac{\pi(2n - 1)}{2} \right) \, dx = 0 \rightarrow \Phi_+(\alpha) = \frac{i(-1)^n}{\alpha + P_{2n-1}}.
$$  \hspace{1cm} (A4)

As the total and the incident fields obey the Neumann boundary conditions at negative $x$ and $y = b$ (see (13) and (15)) it follows that

$$
\frac{\partial}{\partial y} \Phi_-(\alpha) = 0.
$$  \hspace{1cm} (A5)

It is plain that $\Phi(\alpha, y) = \Phi_+(\alpha, y) + \Phi_-(\alpha, y)$ obeys the equation

$$
\left( \frac{\partial^2}{\partial y^2} + q^2(\alpha) \right) \Phi(\alpha, y) = 0, \quad q(\alpha) = \sqrt{k^2 - \alpha^2}.
$$  \hspace{1cm} (A6)

Its solution equal zero at $y = 0$ is

$$
\Phi(\alpha, y) = A(\alpha) \sin(q(\alpha)y),
$$  \hspace{1cm} (A7)

where $A(\alpha)$ is a certain function.

Evaluating this expression at $y = b$ gets two equations

$$
\Phi_-(\alpha) + \frac{i(-1)^n}{\alpha + P_{2n-1}} = A(\alpha) \sin(q(\alpha)b)
$$  \hspace{1cm} (A8)

$$
\frac{\partial}{\partial y} \Phi_+(\alpha) = qA(\alpha) \cos(q(\alpha)b).
$$  \hspace{1cm}

Removing $A(\alpha)$ from these equations leads to the standard Wiener–Hopf equation

$$
\Phi_-(\alpha) + \frac{i(-1)^n}{\alpha + P_{2n-1}} = bK(\alpha) \frac{\partial}{\partial y} \Phi_+(\alpha), \quad K(\alpha) = \frac{\tan(q(\alpha)b)}{q(\alpha)b}.
$$  \hspace{1cm} (A9)

The principal step in the Wiener–Hopf method is the factorisation of $K(\alpha)$

$$
K(\alpha) = K_+(\alpha)K_-(\alpha),
$$  \hspace{1cm} (A10)

where $K_+(\alpha)$ has no zero and singularities in the upper half-plane $\text{Im} \, \alpha > -\text{Im} \, k$ and $K_-(\alpha)$ is free of zero and singularities in the lower half-plane $\text{Im} \, \alpha < \text{Im} \, k$.

Using well known formulas

$$
\sin x = x \prod_{n=1}^\infty \left( 1 - \frac{x^2}{\pi^2 n^2} \right), \quad \cos x = \prod_{n=1}^\infty \left( 1 - \frac{x^2}{\pi^2 (n - 1/2)^2} \right)
$$  \hspace{1cm} (A11)
it is plain that
\[
K_+(\alpha) = \prod_{n=1}^{\infty} \frac{\sqrt{1 - k^2 \beta_n^2} + i \alpha \beta_n}{\sqrt{1 - k^2 \beta_{n-1/2}^2} + i \alpha \beta_{n-1/2}}. \quad K_-(\alpha) = K_+(-\alpha).
\] (A12)

Here
\[
b_n = \frac{b}{\pi n}, \quad b_{n-1/2} = \frac{b}{\pi(n - 1/2)}.
\] (A13)

Dividing (A9) by $K_-(\alpha)$ and separating the pole at $\alpha = -p_{2n-1}$ one obtains
\[
\frac{\Phi_-(\alpha)}{K_-(\alpha)} + \frac{i(-1)^n}{(\alpha + p_{2n-1})} \left( \frac{1}{K_-(\alpha)} - \frac{1}{K_-(\alpha)} \right)
= bK_+(\alpha) \frac{\partial}{\partial y} \Phi_+(\alpha) - \frac{i(-1)^n}{(\alpha + p_{2n-1})K_-(\alpha)}.
\] (A14)

The left-hand side of this equation is free of singularities in the lower half-plane of $\alpha$ and the right-hand side is regular in the upper half-plane. These half-planes have a common part (A3), thus the both sides have to be analytic in the whole plane of complex variable $\alpha$, i.e. equal to a certain polynomial. From boundary conditions it follows that this polynomial is zero. Therefore
\[
\frac{\Phi_-(\alpha)}{K_-(\alpha)} + \frac{i(-1)^n}{(\alpha + p_{2n-1})} \left( \frac{1}{K_-(\alpha)} - \frac{1}{K_-(\alpha)} \right) = 0
\] (A15)

and
\[
bK_+(\alpha) \frac{\partial}{\partial y} \Phi_+(\alpha) - \frac{i(-1)^n}{(\alpha + p_{2n-1})K_-(\alpha)} = 0.
\] (A16)

From (A8) it follows that
\[
A(\alpha) = \frac{i(-1)^n}{\sin(qb)(\alpha + p_{2n-1})K_-(\alpha)} \frac{K_-(\alpha)}{K_-(\alpha)}
= \frac{i(-1)^n \text{sgn}(q)}{qb \cos(qb)(\alpha + p_{2n-1})K_+(\alpha)K_-(\alpha)}.
\] (A17)

The first expression is convenient for $x > 0$ and the second one for $x < 0$.

The knowledge of this function permits to calculate the reflected field by the inverse Fourier transform
\[
\psi(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\alpha) \sin(q\alpha)y e^{-i\alpha x} \, d\alpha.
\] (A18)

For $x > 0$ one can shift the integration contour into the lower half-plane of $\alpha$. As $K_-(\alpha)$ has no singularity here the poles come only from zeros of $\sin(q(\alpha)b)$ plus a pole at $\alpha = -p_{2n-1}$. One has
\[
\sin(qb) = 0 \quad \text{ implies } \quad q = \frac{\pi m}{b}, \quad \alpha_n = -p_{2n} = -\sqrt{k^2 - \frac{\pi^2 m^2}{b^2}}, \quad m = 1, 2, \ldots
\] (A19)
The residue at this point is
\[
\frac{\partial}{\partial \alpha} \sin(qb) \bigg|_{\alpha = -p_{2m}} = b \frac{\partial q(\alpha)}{\partial \alpha} \cos(q(\alpha)b) \bigg|_{\alpha = -p_{2m}} = \frac{b^2(-1)^m p_{2m}}{\pi m}. \tag{A20}
\]

The contribution from the pole at \(\alpha = -p_{2n-1}\) cancels the incident field and in the end one gets that for \(x > 0\) the total transmitted field is as in (17) with
\[
S_{2n-1,2m} = \frac{(-1)^{m+n} \pi m}{b^2 \sqrt{p_{2n-1}p_{2m}} K_+(p_{2m}) K_+(p_{2n-1})}.	ag{A21}
\]

For \(x < 0\) one can shift the integration contour in the upper half-plane. The only singularities of the second expression in (A17) are poles at points where \(\cos(q(\alpha)b) = 0\) or
\[
q = \frac{\pi}{b}(m - 1/2), \quad \alpha = p_{2m-1}, \quad m = 1, 2, \ldots, \tag{A22}
\]

and
\[
\frac{\partial}{\partial \alpha} \cos(qb) \bigg|_{\alpha = p_{2m-1}} = -b \frac{\partial q(\alpha)}{\partial \alpha} \sin(q(\alpha)b) \bigg|_{\alpha = p_{2m-1}} = \frac{b^2(-1)^m p_{2m-1}}{\pi (m - 1/2)}. \tag{A23}
\]

Combining all terms together one concludes that the reflected field has the form as in (17) with
\[
S_{2n-1,2m-1} = \frac{(-1)^{m+n}}{b^2 \sqrt{p_{2n-1}p_{2m-1}} (p_{2n-1} + p_{2m-1}) K_+(p_{2m}) K_+(p_{2n-1})}.	ag{A24}
\]

In these expressions the relation \(K_-(\alpha) = K_+(\alpha)\) was used.

Exactly the same method can be used to find the scattering field for the incoming wave from \(+\infty\) (18) and the corresponding coefficients are
\[
S_{2n,2m-1} = \frac{(-1)^{m+n} \pi m}{b^2 \sqrt{p_{2n}p_{2m-1}} (p_{2m-1} - p_{2n}) K_+(p_{2m}) K_+(p_{2n-1})}	ag{A25}
\]

and
\[
S_{2n,2m} = \frac{(-1)^{m+n} \pi m n}{b^2 \sqrt{p_{2n}p_{2m}} (p_{2n} + p_{2m})} K_+(p_{2n}) K_+(p_{2m}). \tag{A26}
\]

The above expressions for the \(S\)-matrix can conveniently be rewritten in the following compact form
\[
S_{n,m} = \frac{L_n L_m}{x_n + x_m}, \tag{A27}
\]

where
\[
x_{2m-1} = bp_{2m-1}, \quad x_{2m} = -bp_{2m} \tag{A28}
\]

and
\[
L_{2n-1} = \frac{(-1)^n}{\sqrt{bp_{2n-1} K_+(p_{2n-1})}}, \quad L_{2n} = \frac{(-1)^n \pi n K_+(p_{2n})}{\sqrt{bp_{2n}}}. \tag{A29}
\]
In general, there exist two types of waves, propagating and evanescent corresponding, respectively, to real and imaginary values of momenta, \( p_m = \sqrt{k^2 - \pi^2 m^2 / 4b^2} \). There are \( N_e \) propagating modes with even \( m \) and \( N_o \) with odd \( m \)

\[
N_e = \left[ \frac{kb}{\pi} \right], \quad N_o = \left[ \frac{kb}{\pi} + \frac{1}{2} \right].
\] (A30)

The modulus of \( L_m \) is determined by propagating modes. One has an important relation

\[
|L_m|^2 = 2x_m \prod_{n \neq m} \frac{x_m + x_n}{x_m - x_n}.
\] (A31)

Indeed, from (A12) by separating propagating and evanescent modes it follows that

\[
K_+(p_{2n}) = \frac{\prod_{m=1}^{N_e} (p_{2m} + p_{2n})}{\prod_{m=1}^{N_o} (p_{2m} + p_{2n-1})} W_{2m},
\] (A32)

where

\[
W_{2m} = \prod_{n=1}^{N_e} \prod_{k>N_o} (\sqrt{\pi^2 n^2 / b^2 - k^2} + i p_{2n}) \prod_{n=1}^{N_o} \left( 1 - \frac{1}{2n} \right). \] (A33)

In \( W_{2m} \) quantities \( \sqrt{\pi^2 n^2 / 4b^2 - k^2} \) are real. Therefore

\[
|W_{2m}|^2 = \left( \frac{\pi}{b} \right)^{N_e-N_o} \prod_{n=1}^{\infty} \prod_{n=1}^{\infty} ((n - 1/2)^2 - m^2) \prod_{n=1}^{\infty} (1 - 1/(2n))^2.
\] (A34)

This expression can be rewritten as follows

\[
|W_{2m}|^2 = \prod_{n=1}^{\infty} \prod_{n=1}^{\infty} (n^2 - m^2) (n - 1/2)^2 - m^2 \left(1 - 1/(2n)^2\right)^2 \left( \frac{b^2}{\pi^2} \right) \prod_{n=1}^{N_e} (p_{2n}^2 - p_{2n-1}^2).
\] (A35)

The first product is equal

\[
\lim_{x \to m} \frac{1}{m^2 - x^2} \prod_{n=1}^{\infty} \frac{1 - x^2/n^2}{1 - x^2/(n - 1/2)^2} = \lim_{x \to m} \frac{1}{m^2 - x^2} \tan \frac{\pi x}{\pi x} = -\frac{1}{2m^2}.
\] (A36)

Using the definition (A28) one gets (A31) for even indices. Similar arguments prove (A31) for odd indices.

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