\textbf{L₁ COMPACTNESS OF BOUNDED BV SETS}

ISIDORE FLEISCHER

\textbf{Abstract.} Functions, uniformly bounded in BV norm in some bounded open set \( U \) in \( \mathbb{R}^n \), are compact in \( L₁(U) \). This result is known when \( U \) has Lipschitz boundary [EG Th. 4 p. 176], [G 1.19 Th. p. 17], [Z 5.34 Cor. p. 227]; the proof for general \( U \) here, after identifying the operator theoretic definition of bounded BV norm with that of the Tonelli variation, appeals to the standard compactness criterion in \( L₁ \) [DS 21 TH. p. 301] [Y, p. 275] (For completeness, these two auxiliary results are also presented).

\textbf{Proof}

For the purpose of establishing the compactness criterion, it will be necessary to bound integrals of the form \( \int_0^1 V(x+h) - V(x) \, dx \) for monotone non-decreasing \( V \) on \([0,1]\), constant on \([1,\infty)\), for \( h > 0 \). Since \( \int_0^1 V(x+h) \, dx = \int_0^{1+h} V(x) \, dx \), the difference integrates to \( \int_1^{1+h} V(x) \, dx - \int_0^1 V(x) \, dx \), bounded by \( 2h[V(1) - V(0)] \). It thus appears that the integral of the differences goes to 0 with \( h \), uniformly in the rise \( V(1) - V(0) \) of \( V \). If \( V \) is the variation of a \( BV \) function \( F \), then the same bound holds for \( \int |F(x+h) - F(x)| \, dx \). If the \( V \) are functions of other variables, one can also obtain this uniformity for the integrals extended over bounded (in measure) subsets of these.

\textbf{The Compactness Criterion}

A bounded subset \( \mathcal{K} \) of \( L₁(U) \) (\( U \) open bounded), satisfying 
\[ \lim_{t \to 0} \int |f(s+t) - f(s)| \, ds = 0 \] uniformly for \( f \in \mathcal{K} \) is precompact.

The hypothesis entails (by \( s \to s + t_0 \)) the formally stronger \( \int |f(s+t) - f(s+t_0)| \, ds \to 0 \) for every \( t_0 \) and uniformly in \( t_0 \) as well as in \( \mathcal{K} \).

\( (M_r f) s := f_{B_r}(s+t) \, dt \) \( (f_E := (1/|E|) f) \), average of integrand over measurable \( E; B_r := \text{centered ball of radius } r \) is uniformly equicontinuous in \( \mathcal{K} \) and \( r \geq r_0 > 0 \) and has the same \( L₁ \) norm as \( f \); hence, this set of \( M_r f \) is precompact (in \( C \), by Ascoli, hence) in \( L₁ \).

\( \|M_r f - f\| = \int |f_{B_r}(s+t) - f(s)| \, ds \leq f_{B_r} \int |f(s+t) - f(s)| \, ds \, dt \leq \sup_{t \in B_r} \int |f(s+t) - f(s)| \, ds \to 0 \) as \( r \downarrow 0 \), uniformly in \( \mathcal{K} \). It thus suffices to verify that \( \{M_r f : r > 0, f \in \mathcal{K}\} \) is precompact and for this, that every sequence with \( r \to 0 \) has a subsequence every two terms of which are \( < 1/n \) apart. For any such sequence, find an \( r_0 \) below which \( M_r f \) is within \( 1/2n \) of

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f uniformly in K and use the precompactness of the \( M_{r_0} f \) to extract such a subsequence. By diagonalizing, this will produce a Cauchy subsequence.

**BV Varieties**

The (more precisely, “A”) variation of an \( n \)-argument real-valued locally integrable function \( F \) in an open bounded set \( U \) is defined to be the sup of \( \int_U F \nabla \cdot \varphi \, dx \) over the \( C' \) \( \varphi \) to \([-1 + 1]\) with compact support in \( U \).

The integral is a sum of \( n \) terms \( \int_U F \partial \varphi_i / \partial x_i \, dx \, d\tilde{x}_i \) where \( d\tilde{x}_i \) is the element of \((n-1)\)-volume in the co-ordinate hyperplane orthogonal to the \( x_i \)-axis and \( \varphi_i \) is the \( i^{th} \) component of \( \varphi \). Since \( \varphi \) could have just this component different from 0, the sup of the sum dominates the sup of each term; and the sum of these non-negative sups dominates the sup of their sum. Hence bounded variation comes to finiteness of the \( n \) sups of \( \int_U F \partial \varphi_i / \partial x_i \, dx \, d\tilde{x}_i \) where \( d\tilde{x}_i \) is the distribution in \( x_i \) obtained from \( F \) by fixing the other co-ordinates; the minus sign can be dropped since the \( \varphi \)'s are closed for negation. The value of the integral is unchanged by making \( F \) right continuous in \( x_i \), so assume it so. The sup is dominated by the “Tonelli Variation” \( \int_{V_{x}} F \, d\tilde{x} \), the integrand being the classical variation of right continuous \( F \) with all the variables other than \( x \) fixed.\(^1\) This quantity is actually attained by the sup as we now show.

It would suffice to attain an arbitrary approximating sum whose terms are measures of finitely many disjoint \( \tilde{x} \)-measurable subsets with values of the integrand at points in the sets as coefficients: this integrand is itself a sup of approximating sums: \( \Sigma |F(t_{1+1}) - F(t_i)| \) with \( t_i \) points of continuity of \( F \).

It is easy to find a sequence of \([-1, +1]\)-valued \( C' \) functions on a real interval \([s,t]\) vanishing, along with their derivatives, at \( s \) and \( t \) which increase pointwise to \( 1_{(s,t)} \); the integral of this sequence against \( BV \, dF \) converges to \( F(t^-) - F(s^+) \). Glueing together translates of this sequence or its negative yields a \( C' \) sequence to \([-1, +1]\) whose integral against \( dF \) converges to \( \Sigma |F(t_{1+1}) - F(t_i)| \) provided the partition points \( t_i \) are restricted to points of continuity of \( F \).

One can also find a sequence of \([-1, +1]\)-valued \( C' \) functions on \( \tilde{X} \) which converges a.e. to the characteristic function of a given measurable set: represent the set and its complement a.e. as a union of a sequence of closed sets; for like-indexed pairs find (by Urysohn) a continuous function \( 1 \) on the contained closed set and \( 0 \) on the one disjoint; and approximate the resulting sequence (by Weierstrass-Stone) by a \( C' \) sequence with the same pointwise limit. The \( \tilde{x} \)-integral of these functions converges to the measure of the set (which the function characterizes) and so their pairwise product with the real \( (x-) \) variable sequence previously constructed yields a \( C' \) sequence on

\(^1\)The integrand is positive measurable as limit of a sequence of such; a priori the integral could be infinite.
\( \tilde{X} \times X \) to \([-1, +1]\) whose \( \int \varphi dF dx \) converges to \( \sum |F(t_{i+1}) - F(t_i)| \) multiplied by the measure of the \( \tilde{x} \)-subset, which is one term of the sought for approximating sum; adding these finitely many sequences pointwise yields a sequence converging to the full sum.

P.S. The same result is valid with the usual (Vitali) notion of BV: Over a bounded domain, a uniformly BV, \( L_1 \)-bounded set of functions is uniformly bounded (for if the values at some point were unbounded then by BV uniformity so would be their lower bounds uniformly over the domain; and then again by uniform BV, the pointwise bounded set would be uniformly bounded over the domain). By Helly’s Selection Theorem, some subsequence converges pointwise, whence by the uniform boundedness, in \( L_1 \).

References

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Centre de recherches mathématiques, C.P. 6128, succursale Centre-ville, Montréal, QC, H3C 3J7, Canada