SERIES EXPANSION OF A COTANGENT SUM RELATED TO
THE ESTERMANN ZETA FUNCTION

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ABSTRACT. In this paper, we study the cotangent sum $c_0\left(\frac{q}{p}\right)$ related to the Estermann zeta function for the special case when the numerator is equal to 1 and get two useful series expansions of $c_0\left(\frac{1}{p}\right)$.

1. INTRODUCTION

For a positive integer $p$ and $q = 1, 2, \ldots, p-1$, such that $(p, q) = 1$, let the cotangent sum (see [10])

$$c_0\left(\frac{q}{p}\right) = -\sum_{k=1}^{p-1} \frac{k}{p} \cot \frac{\pi kq}{p}.$$  

c_0\left(\frac{q}{p}\right) \text{ is the value at } s = 0,$

$$E_0\left(0, \frac{q}{p}\right) = \frac{1}{4} + \frac{i}{2} c_0\left(\frac{q}{p}\right)$$

of the Estermann zeta function

$$E_0\left(s, \frac{q}{p}\right) = \sum_{k \geq 1} \frac{d(k)}{k^s} \exp\left(\frac{2\pi ikq}{p}\right).$$

It is well-known that the sum $c_0\left(\frac{q}{p}\right)$ satisfies the reciprocity formula (see [2])

$$c_0\left(\frac{q}{p}\right) + \frac{p}{q} c_0\left(\frac{p}{q}\right) - \frac{1}{\pi q} = \frac{i}{2} \psi_0\left(\frac{q}{p}\right).$$

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The Vasyunin cotangent sum (see [11])

\[ V \left( \frac{q}{p} \right) = \sum_{r=1}^{p-1} \left\{ \frac{rq}{p} \right\} \cot \left( \frac{\pi r}{p} \right) = -c_0 \left( \frac{q}{p} \right) \]

arises in the study of the Riemann zeta function by virtue of the formula (see [2,9])

\[ \frac{1}{2\pi \sqrt{pq}} \int_{-\infty}^{+\infty} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \left( \frac{q}{p} \right)^{it} \frac{dt}{\frac{1}{4} + t^2} = \frac{\log 2\pi - \gamma}{2} \left( \frac{1}{p} + \frac{1}{q} \right) + \frac{p - q}{2pq} \log \frac{q}{p} - \frac{\pi}{2pq} \left( V \left( \frac{p}{q} \right) + V \left( \frac{q}{p} \right) \right). \]

This formula is connected to the approach of Nyman, Beurling and Báez-Duarte to the Riemann hypothesis (see [8]), which states that the Riemann hypothesis is true if and only if \( \lim_{n \to \infty} d_N = 0 \), where

\[ d_N^2 = \inf_{A_N} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| 1 - \zeta A \left( \frac{1}{2} + it \right) \right|^2 \frac{dt}{\frac{1}{2} + t^2}, \]

and the infimum is taken over all Dirichlet polynomials

\[ A_N(s) = \sum_{n=1}^{N} \frac{a_n}{n^s}. \]

In a recent work with A. Bayad [7], we have proved that the sum \( V \left( \frac{q}{p} \right) \) satisfies the reciprocity formula

(1.1) \[ V \left( \frac{q}{p} \right) + V \left( \frac{p}{q} \right) = \frac{1}{\pi} \left( G(p,p) + G(q,q) + G(p,q) + (q-p) \log \frac{q}{p} \right), \]

where

\[ G(p,q) = \sum_{k \geq 1} \frac{pq}{k(k+1)} \left\{ \frac{k}{p} \right\} \left\{ \frac{k}{q} \right\}. \]

Thereafter the restriction of the relationship (1.1) to \( q = 1 \) gives

\[ c_0 \left( \frac{1}{p} \right) = -\frac{1}{\pi} G(p,p) - (p-1) \log p. \]

Exactly our interest in this work is the case \( q = 1 \) in order to get two series expansions of \( c_0 \left( \frac{1}{p} \right) \). First we recall the different asymptotical writings of \( c_0 \left( \frac{1}{p} \right) \) in the literature. In [10, Theorem 1.2, Theorem 1.3] M. Th. Rassias proved that

\[ c_0 \left( \frac{1}{p} \right) = \frac{1}{\pi} \log p - \frac{p}{\pi} \left( \log 2\pi - \gamma \right) + \{ O (\log p) \} \text{ or } \{ O (1) \}. \]

In [9, Theorem 1.7] H. Maier and M. Th. Rassias provide the following improvement. Let \( b, n \in \mathbb{N}, b \geq 6N, \) with \( N = \left\lceil \frac{n}{2} \right\rceil + 1. \) There exist absolute real constants \( A_1, A_2 \geq 1 \)
and absolute real constants \( E_l, l \), with \( |E_l| \leq (A_l)^2 \), such that for each \( n \in \mathbb{N} \) we have

\[
c_0 \left( \frac{1}{p} \right) = \frac{1}{\pi} p \log p - \frac{p}{\pi} (\log 2\pi - \gamma) - \frac{1}{\pi} + \sum_{l=1}^{n} E_lp^{-l} + R^*_n(p),
\]

where \( |R^*_n(p)| \leq (A_2n)^{4n} p^{-(n+1)} \).

Only in [9, Theorem 1.9] H. Maier and M. Th. Rassias provide another improvement,

\[
c_0 \left( \frac{1}{p} \right) = \frac{1}{\pi} p \log p - \frac{p}{\pi} (\log 2\pi - \gamma) + C_1 p + O(1).
\]

We draw attention that S. Bettin finds other reformulations of \( c_0 \left( \frac{1}{p} \right) \) inspired from continued fraction theory (see [3]).

Finally from another point of view we show in [5] with A. Bayad and M. O. Hernane that

\[
c_0 \left( \frac{1}{p} \right) = -\frac{1}{\pi} \left( \log \frac{2\pi}{p} - \gamma \right) p + \frac{1}{\pi} + \frac{\pi}{36p} - \frac{1}{2} \sum_{k=2}^{\left\lfloor \frac{p}{2} \right\rfloor} (-1)^k \frac{4^k\pi^{2k-1}B_{2k}^2}{k(2k)!} \left( \frac{1}{p} \right)^{2k-1} + O \left( \frac{1}{p^N} \right).
\]

There is a misprint in the formula (1.22) Corollary 1.2 in [5] the correct one is in the formula (1.21) Corollary 1.2.

Otherwise in the same paper [5], an integral representation of \( c_0 \left( \frac{1}{p} \right) \) is given by

\[
(1.2) \quad c_0 \left( \frac{1}{p} \right) = \frac{1}{\pi} \int_0^1 \frac{(p-2)x^p - px^{p-1} + px - p + 2}{(x-1)^2 (x^p - 1)} \, dx.
\]

In this work we prove that

\[
(p-2)x^p - px^{p-1} + px - p + 2 = (x-1)^3 \sum_{r=1}^{p-1} (p-r-1) rx^{r-1}
\]

and we get another formulation that is

\[
c_0 \left( \frac{1}{p} \right) = \frac{1}{\pi} \int_0^1 \frac{\sum_{r=1}^{p-1} (p-r-1) rx^{r-1}}{1 + x + \cdots + x^{p-1}} \, dx.
\]

Applying some techniques from the generating function theory [4] to previous integrals; we find two series expansions of \( c_0 \left( \frac{1}{p} \right) \), as they are well explained in the next section.

2. Series Expansion of \( c_0 \left( \frac{1}{p} \right) \)

Let \( b_k \) be the integer sequence defined by \( b_0 = 1, b_1 = 2 \) and the recursive formulae:

\[
b_k - 2b_{k-1} + b_{k-2} = 0, \quad 2 \leq k \leq p - 1, \quad k = p + 1,
\]

\[
b_p - 2b_{p-1} + b_{p-2} = 1
\]
and
\[ b_k - 2b_{k-1} + b_{k-2} - b_{k-p} + 2b_{k-p-1} - b_{k-p-2} = 0, \quad k \geq p + 2. \]

According to the terms \( b_k \) we get the first series expansion in the following theorem.

**Theorem 2.1.**

\[(2.1) \quad c_0 \left( \frac{1}{p} \right) = \frac{1}{\pi} p (p - 1) (p - 2) \sum_{k \geq 0} \frac{b_k}{(k + 1) (k + p + 1) (k + 2) (k + p)}. \]

For \( p \geq 1 \) we define the arithmetic function \( a_p \) in the form
\[ a_p(k) = \begin{cases} 
1, & \text{if } p \mid k, \\
-1, & \text{if } k \equiv 1 \pmod{p}, \\
0, & \text{otherwise}. 
\end{cases} \]

This function is not multiplicative. In general the arithmetical functions are defined from the set of natural integers \( \mathbb{N} \) into \( \mathbb{C} \). We can extend this definition to \( \mathcal{F}(\mathbb{C}, \mathbb{C}) \); set of functions from \( \mathbb{C} \) to \( \mathbb{C} \). In that case the corresponding function is \( A : \mathbb{N} \to \mathcal{F}(\mathbb{C}, \mathbb{C}) \) with \( A(p) = a_p \). Furthermore, \( A(pq) = \pm A(p)A(q) \) and \( |A| \) is multiplicative.

Let the function \( M(p, k) \) defined by
\[ M(p, 0) = \frac{1}{2} p^2 - \frac{3}{2} p + 1 \]
and
\[ M(p, k) = (p - 1) \left( \frac{1}{2} p + k - 1 \right) - k (p + k - 1) (H_{p+k-1} - H_k), \quad k \geq 1, \]
where \( H_k \) is the Harmonic number
\[ H_k = \sum_{j=1}^{k} \frac{1}{j}. \]

Following this function a second series expansion of \( c_0 \left( \frac{1}{p} \right) \) is given in the following theorem.

**Theorem 2.2.**

\[(2.2) \quad c_0 \left( \frac{1}{p} \right) = \frac{1}{\pi} \sum_{k \geq 0} a_p(k) M(p, k). \]

2.1. **Proof of Theorem 2.1.** We take inspiration from the theory of generating functions [4,6], and prove that the sequence \( (b_k) \) is generated by the rational function:
\[ f(x) = \frac{1}{1 - 2x + x^2 - xp + 2xp^{p+1} - x^{p+2}}. \]

More precisely we get the following lemma.

**Lemma 2.1.**

\[(2.3) \quad \frac{1}{1 - 2x + x^2 - xp + 2xp^{p+1} - x^{p+2}} = \sum_{k \geq 0} b_k x^k, \quad |x| < 1. \]
Proof. It is well known that
\[ (2.4) \quad \frac{1}{1 - x} = \sum_{k \geq 0} x^k, \quad |x| < 1. \]

Since for \( 0 \leq x < 1 \)
\[ 0 < (x - 1)^2 (1 - x^p) < 1 \]
and
\[ (x - 1)^2 (1 - x^p) = 1 - \left(2x - x^2 + x^p - 2x^{p+1} + x^{p+2}\right), \]
then we have
\[ 0 < 2x - x^2 + x^p - 2x^{p+1} + x^{p+2} < 1. \]

Furthermore, \( f(x) \) is developable on entire series to get the result we have to take the quantity \( 2x - x^2 + x^p - 2x^{p+1} + x^{p+2} \) instead of \( x \) in the last formula (2.4). Now, writing
\[ \frac{1}{1 - 2x + x^2 - x^p + 2x^{p+1} - x^{p+2}} = \sum_{k \geq 0} d_k x^k \]
and then
\[ \left(1 - 2x + x^2 - x^p + 2x^{p+1} - x^{p+2}\right) \left(\sum_{k \geq 0} d_k x^k\right) = 1. \]

To compute this we use the well known Cauchy product of two entire series
\[ \left(\sum_{k \geq 0} a_k x^k\right) \left(\sum_{j \geq 0} d_j x^j\right) = \sum_{k \geq 0} \left(\sum_{j=0}^{k} a_j d_{k-j}\right) x^k, \]
which generates the product of a polynomial of degree \( n \) with an entire series that also gives an entire series as follows
\[ \left(\sum_{k=0}^{n} a_k x^k\right) \left(\sum_{j \geq 0} d_j x^j\right) = \sum_{k \geq 0} \left(\sum_{j=0}^{\min\{n,k\}} a_j d_{k-j}\right) x^k. \]

We return to \( f(x) \) in writing
\[ 1 - 2x + x^2 - x^p + 2x^{p+1} - x^{p+2} = \sum_{k=0}^{p+2} a_k x^k, \]
with \( a_0 = 1, \ a_1 = -2, \ a_2 = 1, \ a_p = -1, \ a_{p+1} = 2, \ a_{p+2} = -1, \) and the others are zero. We conclude that \( d_0 = 1, \ d_1 = 2. \) The formula
\[ \sum_{j=0}^{\min\{p+2,k\}} a_j d_{k-j} = 0 \]
states that
\[ d_k - 2d_{k-1} + d_{k-2} = 0, \quad 2 \leq k \leq p - 1, \ k = p + 1, \]
\[ d_p - 2d_{p-1} + d_{p-2} = 1 \]
and
\[ d_k - 2d_{k-1} + d_{k-2} - d_{k-p} + 2d_{k-p-1} - d_{k-p-2} = 0, \quad k \geq p + 2. \]
Finally, we see that $d_k$ and $b_k$ are identical for every integer $k \geq 0$. For more information on this approach we refer to [6].

To get the result (2.1) of Theorem 2.1 we must substitute the expression (2.3) in the identity (1.2) and one obtains

$$c_0 \left( \frac{1}{p} \right) = - \frac{1}{\pi} \sum_{k \geq 0} b_k \int_0^1 \left( (p - 2) x^{k+p} - p x^{k+p-1} + p x^{k+1} + (2 - p) x^k \right) dx.$$  

Furthermore,

$$c_0 \left( \frac{1}{p} \right) = - \frac{1}{\pi} \sum_{k \geq 0} b_k \left( \frac{p - 2}{k + p + 1} - \frac{p}{k + p} + \frac{p}{k + 2} - \frac{p - 2}{k + 1} \right).$$

Finally,

$$c_0 \left( \frac{1}{p} \right) = \frac{1}{\pi} \sum_{k \geq 0} b_k \frac{b_k}{(k + 1)(k + p + 1)(k + 2)(k + p)}$$

and $c_0 \left( \frac{1}{2} \right) = 0$ is compatible with the definition of $c_0$.

Regarding the identity (2.3) Lemma 2.1 we remark that

$$(1 - x)^2 \left( 1 - x^p \right) = \sum_{k \geq 0} b_k x^k, \quad |x| < 1.$$  

Furthermore, for $x = \frac{1}{2}$ we deduce that the coefficients $b_k$ satisfy the following statements

$$\sum_{k \geq 0} \frac{b_k}{2^k} = \frac{2^{p+2}}{2^p - 1} \quad \text{and} \quad \lim_{k \to \infty} \frac{b_k}{2^k} = 0.$$  

2.2. **Proof of Theorem 2.2.** First we began by proving another integral representation of $c_0 \left( \frac{1}{p} \right)$.

**Lemma 2.2.**

(2.5)  

$$c_0 \left( \frac{1}{p} \right) = \frac{1}{\pi} \int_0^1 \frac{\sum_{r=1}^{p-1} (p - r - 1) r x^{r-1}}{1 + x + \cdots + x^{p-1}} dx.$$  

**Proof.**

$$(x - 1)^3 \sum_{r=1}^{q-1} (q - r - 1) r x^{r-1} = \sum_{r=3}^{q} (q - r - 1)(r - 2) x^r - 3 \sum_{r=2}^{q-1} (q - r)(r - 1) x^r$$

$$+ 3 \sum_{r=1}^{q-2} (q - r - 1) r x^r - \sum_{r=0}^{q-3} (q - r - 2)(r + 1) x^r.$$  

It’s obvious to remark that

$$(q - r + 1)(r - 2) - 3 (q - r)(r - 1) + 3 (q - r - 1) r - (q - r - 2)(r + 1) = 0$$
and the quantity

\[ (t - 1)^3 \sum_{r=1}^{q-1} (q - r - 1) r x^{r-1} \]

is reduced to

\[ (q - 2) x^q + 2 (q - 3) x^{q-1} + 3(q - 4)x^{q-2} - 3(q - 2)x^{q-1} - 6(q - 3)x^{q-2} - 3(q - 2)x^2 + 3(q - 2)x^{q-2} + 3(q - 2)x + 6(q - 3)x^2 - q + 2 - 2(q - 3)x - 3(q - 4)x^2. \]

After simplification we obtain

\[ (t - 1)^3 \sum_{r=1}^{q-1} (q - r - 1) r x^{r-1} = (q - 2) x^q - qx^{q-1} + qx - q + 2. \]

The Theorem 2.2 is immediate from the Lemma 2.2 in the following way. Since

\[ \frac{1}{1 + x + \cdots + x^{p-1}} = \frac{1 - x}{1 - x^p} \]

and \(|x| < 1\), then

\[ \frac{1}{1 + x + \cdots + x^{p-1}} = \frac{1 - x}{1 - x^p} = \sum_{k \geq 0} (1 - x) x^{pk}. \]

Furthermore,

\[ \frac{1}{1 + x + \cdots + x^{p-1}} = \sum_{k \geq 0} a_p(k) x^k \]

and we have

\[ \sum_{r=1}^{p-1} \frac{(p - r - 1) r x^{r-1}}{1 + x + \cdots + x^{p-1}} = \sum_{k \geq 0} \sum_{r=1}^{p-1} a_p(k) (p - r - 1) r x^{k+r-1}. \]

The passage to the integral inducts

\[ c_0 \left( \frac{1}{p} \right) = \sum_{k \geq 0} \sum_{r=1}^{p-1} a_p(k) \frac{(p - r - 1) r}{k + r}. \]

But

\[ \sum_{r=1}^{p-1} \frac{(p - r - 1) r}{k + r} = (p - 1) \left( \frac{1}{2} p + k - 1 \right) - k (p + k - 1) \sum_{r=k+1}^{p+k-1} \frac{1}{r} \]

and the result (2.2) is deduced.
3. Connection to Digamma Function

We finish this work by revisiting the proof of the expression of \( c_0 \left( \frac{1}{p} \right) \) according to the function digamma and Bernoulli polynomials in the work [1] of L. Báez Duarte et al.

\[
c_0 \left( \frac{1}{p} \right) = \frac{2}{\pi} \sum_{r=1}^{p-1} B_1 \left( \frac{r}{p} \right) \psi \left( \frac{r}{p} \right),
\]

where \( B_1 \) is the reduced Bernoulli polynomial

\[
B_1(x) = \begin{cases} 
0, & \text{if } x \in \mathbb{Z}, \\
\{x\} - \frac{1}{2}, & \text{otherwise},
\end{cases}
\]

and \( \psi \) the digamma function defined by

\[
\psi(z) = -\gamma - \frac{1}{z} + \sum_{k \geq 1} \left( \frac{1}{k} - \frac{1}{k + z} \right).
\]

Starting with the demonstration of a property of \( \psi \) that will be used later.

**Proposition 3.1.**

(3.1) \[
\psi \left( \frac{r+1}{p} \right) - \psi \left( \frac{r}{p} \right) = p \int_0^1 \frac{x^{r-1}}{1 + x + \cdots + x^{p-1}} dx.
\]

**Proof.** We quote from [5] the formula

\[
\psi \left( \frac{r+1}{p} \right) - \psi \left( \frac{r}{p} \right) = p \sum_{k \geq 0} \frac{1}{(pk + r + 1)(pk + r)}.
\]

The general term \( \frac{1}{(pk + r + 1)(pk + r)} \) can be written as following

\[
\frac{1}{(pk + r + 1)(pk + r)} = \frac{1}{pk + r} - \frac{1}{pk + r + 1} = \int_0^1 \left( x^{pk+r-1} - x^{pk+r} \right) dx
\]

and the passage to the sum states that

\[
\sum_{k \geq 0} \frac{1}{(pk + r + 1)(pk + r)} = \int_0^1 \frac{x^{r-1} - x^r}{1 - x^p} dx.
\]

Finally,

\[
\sum_{k \geq 0} \frac{1}{(pk + r + 1)(pk + r)} = \int_0^1 \frac{x^{r-1}}{1 + x + \cdots + x^{p-1}} dx
\]

and we have (3.1). Proposition 3.1 follows. \( \square \)

In [5], it is shown that

\[
\log p = \frac{1}{p} \sum_{r=1}^{p-1} r \left( \psi \left( \frac{r+1}{p} \right) - \psi \left( \frac{r}{p} \right) \right).
\]

This identity conducts to the following interesting lemma.
Lemma 3.1.

(3.2) \[ \sum_{r=1}^{p} \psi \left( \frac{r}{p} \right) = -\gamma p - p \log p. \]

Proof. Since

\[ \sum_{r=1}^{p-1} r \left( \psi \left( \frac{r+1}{p} \right) - \psi \left( \frac{r}{p} \right) \right) = p \log p, \]

then

\[ -\sum_{r=1}^{p} \psi \left( \frac{r}{p} \right) + \psi \left( \frac{1}{p} \right) = p \log p. \]

Furthermore,

\[ \sum_{r=1}^{p} \psi \left( \frac{r}{p} \right) = -\gamma p - p \log p. \]

According to the identity (3.1) Proposition 3.1 and the integral representation (2.5) we conclude that

\[ c_{0} \left( \frac{1}{p} \right) = \frac{1}{\pi p} \sum_{r=1}^{p-1} (p - r - 1) r \left( \psi \left( \frac{r+1}{p} \right) - \psi \left( \frac{r}{p} \right) \right). \]

Furthermore combining this result with the identity (3.2) Lemma 3.1 we get

\[ c_{0} \left( \frac{1}{p} \right) = -\frac{1}{\pi} \log p + \frac{1}{\pi p} \sum_{r=1}^{p-1} (p - r) r \left( \psi \left( \frac{r+1}{p} \right) - \psi \left( \frac{r}{p} \right) \right) \]

and

\[ c_{0} \left( \frac{1}{p} \right) = -\frac{1}{\pi} \log p - \gamma \frac{p-1}{\pi p} + \frac{1}{\pi p} \sum_{r=1}^{p-1} (2r - p - 1) \psi \left( \frac{r}{p} \right), \]

then

\[ c_{0} \left( \frac{1}{p} \right) = \frac{1}{\pi p} \sum_{r=1}^{p-1} (2r - p) \psi \left( \frac{r}{p} \right). \]

But

\[ 2r - p = 2p \left( \frac{r}{p} - \frac{1}{2} \right) = 2p B_{1} \left( \frac{r}{p} \right), \]

which means that

\[ c_{0} \left( \frac{1}{p} \right) = \frac{2}{\pi} \sum_{r=1}^{p} B_{1} \left( \frac{r}{p} \right) \psi \left( \frac{r}{p} \right). \]
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