Quantum tunneling effect and quantum Zeno effect in a Topological system

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(Dated: December 13, 2010)

Abstract

A spin interaction Hamiltonian for topological basis is constructed in this paper. When we select proper parameters, this Hamiltonian system can be simulated by a quantum double well potential system. If the parameter $\Delta = 0$, the topological system is equivalent to two independent quantum wells. If the parameter $\Delta \neq 0$, the system is equivalent to a double well potential with finite potential barrier. The quantum tunneling effect and quantum Zeno effect for this topological system are investigated in detail.

PACS numbers: 03.65.Xp, 03.65.Fd, 02.10.Kn

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I. INTRODUCTION

Topological quantum computing (TQC) is one of the most important approaches to achieve a fault-tolerant quantum computer [1–4]. The computation scheme relies on the existence of topological states of matter whose quasi-particle excitations are anyons, and they obey the braiding statistics. Quantum information is stored in the states with multiple quasi-particles with a topological degeneracy. Quantum gate operations are implemented by braiding the quasi-particles. Thus a quantum computer arises from the nonlocal encoding of the multi-quasiparticle states, which makes them immune to errors caused by local perturbations.

In Refs. [5–7] Kauffman et al. use the CAP-CUP language reveal the relations between quantum mechanics and topology. They said, “the connection of quantum mechanics and topology is an amplification of Dirac notation.” For the following convenience, we will introduce the Kauffman’s CAP-CUP language. In this paper, we refer to lines connecting lattice points in the top row as $|\text{cup}\rangle$ states, those connecting lattice points in the bottom row as $\langle\text{cap}\mid$ states. We use $\langle\text{CAP}\mid$ and $|\text{CUP}\rangle$ to denote the composition of caps and the composition of caps, respectively. For example, the $|\text{CAP}\rangle$ state $\begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array}$ is composition of two $\langle\text{cap}\mid$ states (i.e. $\begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array}$), and the $\langle\text{CUP}\mid$ state $\begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array}$ is composition of two $|\text{cup}\rangle$ states (i.e. $\begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array}$). By means of CAP-CUP states, we can define operators and inner-product as following,

\[
\hat{O} = |\phi\rangle\langle\psi| = \begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array} \cdot \begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array} = \begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array},
\]

\[
Z(\psi, \phi) = \langle\psi|\phi\rangle = \begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array} \cdot \begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array} = \begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array} = \bigcirc.
\]

On the other hand, an experimental results for a small-scale approximate evaluation of the Jones polynomial by nuclear magnetic resonance (NMR) was presented in Ref. [9]. The authors could obtain the value of the Jones polynomial via measuring the nuclear spin state of the molecule.

Recently, in Ref. [10] Ge et al. constructed a set of spin realization topological basis. By means of Temperley-Lieb algebra and Topological basis, the authors reduced the four-dimensional Yang-Baxter Equation (YBE) into its two-dimensional form. Then they point out that YBE can be tested in terms of quantum optics. On our knowledge, there is not a Hamiltonian system describing the spin realization of topological basis. Motivated by this, we will construct a spin interaction Hamiltonian for this topological system. Consequently, we can study physical properties of this topological system, such as quantum tunneling effect (QTE) and quantum Zeno effect (QZE).
This paper is organized as follows: in Sec. III we recall the Temperley-Lieb algebra and topological basis. In Sec. III we construct a topological Hamiltonian and obtain the eigen-system for the topological system. In Sec. IV we study the quantum tunneling effect and quantum Zeno effect for this topological system. We end with a summary.

II. TEMPERLEY-LIEB ALGEBRA AND TOPOLOGICAL BASIS

We first briefly review the theory of Temperley-Lieb (T-L) algebra [12]. Given a choice of topological parameter \(d\) and a natural number \(m\), the T-L algebra \(TL_m(d)\) is generated by \(\{I, U_1, U_2 \cdots U_{m-1}\}\) with the T-L algebra relations:

\[
\begin{align*}
U_i^2 &= dU_i & 1 \leq i \leq m - 1, \\
U_iU_{i+1}U_i &= U_i & 1 \leq i \leq m, \\
U_iU_j &= U_jU_i & |i-j| \geq 2,
\end{align*}
\]

where the notation \(U_i \equiv U_{i,i+1}\) is used. The \(U_i\) represents \(1_1 \otimes 1_2 \otimes 1_3 \cdots \otimes 1_{i-1} \otimes U \otimes 1_{i+2} \otimes \cdots \otimes 1_m\), and \(1_j\) represents the unit matrix in the \(j\)-th space \(\mathcal{V}_j\). In addition, the T-L algebra is easily understood in terms of knot diagrams, please refer to Refs. [5, 9, 11] and references therein.

The \(4 \times 4\) T-L matrix \(U\) with \(d = \sqrt{2}\) which satisfies T-L algebra in Eqs. (1) takes the representation [13, 14],

\[
U = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & iq^{-1} \\
0 & 1 & i\epsilon & 0 \\
0 & -i\epsilon & 1 & 0 \\
-iq & 0 & 0 & 1
\end{pmatrix},
\]

where \(\epsilon = \pm\) and \(q = e^{i\phi}\) with real \(\phi\). For the following convenience, we use the single solid lines, double solid lines, single dash lines and double dash lines to distinguish different topological states. Then we can introduce a set of \(|cup\rangle\) and \(|cap\rangle\) states and their spin realization as,

\[
\begin{align*}
|\uparrow\uparrow\rangle &= \sqrt{d}|\psi_d^{(1)}\rangle_{ij} = \sqrt{\frac{2}{d^3}}(|\uparrow\uparrow\rangle_{ij} + e^{-i\phi}|\downarrow\downarrow\rangle_{ij}) = \sqrt{d}|\langle\psi_d^{(1)}|\rangle_{ij}^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
|\downarrow\downarrow\rangle &= \sqrt{d}|\psi_d^{(2)}\rangle_{ij} = \sqrt{\frac{2}{d^3}}(|\downarrow\downarrow\rangle_{ij} - ie^{i\phi}|\uparrow\uparrow\rangle_{ij}) = \sqrt{d}|\langle\psi_d^{(2)}|\rangle_{ij}^\dagger = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\
|\uparrow\downarrow\rangle &= \sqrt{d}|\psi_d^{(3)}\rangle_{ij} = \sqrt{\frac{2}{d^3}}(|\uparrow\downarrow\rangle_{ij} - e^{-i\phi}|\downarrow\uparrow\rangle_{ij}) = \sqrt{d}|\langle\psi_d^{(3)}|\rangle_{ij}^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
|\downarrow\uparrow\rangle &= \sqrt{d}|\psi_d^{(4)}\rangle_{ij} = \sqrt{\frac{2}{d^3}}(|\downarrow\uparrow\rangle_{ij} + ie^{i\phi}|\uparrow\downarrow\rangle_{ij}) = \sqrt{d}|\langle\psi_d^{(4)}|\rangle_{ij}^\dagger = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},
\end{align*}
\]
where the notation $\uparrow(\downarrow)$ denotes spin-up(spin-down), and the notation $|\alpha\beta\rangle_{ij}$ is the abbreviated form of $|\alpha\rangle_i \otimes |\beta\rangle_j (\alpha, \beta = \uparrow, \downarrow)$. The topological parameter (the single loop) $d = \bigcirc = \bigotimes = \bigotimes = \bigcirc = \sqrt{2}$ in this paper. In terms of CAP-CUP language, the T-L matrix in Eq. (11) can be recast as following,

$$U_{ij} = \begin{pmatrix} i & j \\ j & i \end{pmatrix} + \begin{pmatrix} i & j \\ j & i \end{pmatrix}.$$  \hfill (4)

Following Ge et al. in Ref. [8], to reduce the $4 \times 4$ Temperley-Lieb matrix, we can introduce a set of topological basis states with four quasi-particles. The topological basis states take the following form,

$$|e_1\rangle = \frac{1}{d \sqrt{2}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix},$$

$$|e_2\rangle = \frac{1}{d \sqrt{2}} \begin{pmatrix} 1 & i e^{-\phi} & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} + (i e + e^{-\phi}) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix},$$

$$|e_3\rangle = \frac{1}{d \sqrt{2}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix},$$

$$|e_4\rangle = \frac{1}{d \sqrt{2}} \begin{pmatrix} 1 & i e^{\phi} & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} - i e (1 - i e^{-\phi}) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$ \hfill (5)

We can verify that this set of basis states are orthonormal basis, \textit{i.e.} $\langle e_i | e_j \rangle = \delta_{ij}$. In fact, topological bases $|e_2\rangle$ and $|e_4\rangle$ are equivalent to the following simply form,

$$|e_2\rangle = \frac{-i e}{d \sqrt{2}} \begin{pmatrix} e^{\phi} & 1 & 2 & 3 \\ 2 & 3 & 4 & 1 \end{pmatrix} + e^{-\phi} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix},$$

$$|e_4\rangle = \frac{i e}{d \sqrt{2}} \begin{pmatrix} e^{\phi} & 1 & 2 & 3 \\ 2 & 3 & 4 & 1 \end{pmatrix} - e^{-\phi} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}. $$ \hfill (6)

By means of Eqs. (5), Eqs. (4) and Eqs. (5), we can verify that the T-L matrix can be reduced to two same 2D representations. The bases of subspace are $\{|e_1\rangle, |e_2\rangle\}$ and $\{|e_3\rangle, |e_4\rangle\}$. The 2D representations on the subspace $\{|e_1\rangle, |e_2\rangle\}$ take the following form,

$$U_A = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}; \quad U_B = \begin{pmatrix} d^{-1} & \sqrt{1 - d^{-2}} \\ \sqrt{1 - d^{-2}} & d^{-1} \end{pmatrix},$$ \hfill (7)

where $(U_A)_{ij} = \langle e_i | U_{12} | e_j \rangle$ and $(U_B)_{ij} = \langle e_i | U_{23} | e_j \rangle (i, j = 1, 2)$. We can verify $U_A$ and $U_B$ satisfy the 2D T-L relations, $U_A^2 = d U_A$, $U_B^2 = d U_B$, $U_A U_B U_A = U_A$ and $U_B U_A U_B = U_B$. This 2D form T-L matrices can be constructed by using the conformal field theory, and have been applied to the fractional quantum Hall effect. Please see Refs. [8, 15] and references therein.
III. HAMILTONIAN AND EIGEN-SYSTEM FOR THE TOPOLOGICAL SYSTEM

For investigating the physical effect of the spin realization of the topological basis, we should construct a Hamiltonian for the topological system. The Hamiltonian for our system reads,

\[ \hat{H} = Jd^{-2}\left[(1 + \Delta)(U_{12}^{(1)}U_{34}^{(1)} + 4\tilde{U}_{12}^{(1)}\tilde{U}_{34}^{(1)}) + (1 - \Delta)(U_{12}^{(2)}U_{34}^{(2)} + 4\tilde{U}_{12}^{(2)}\tilde{U}_{34}^{(2)})\right] \]

(8)

where \(U_{ij}^{(1)} = \begin{pmatrix} i & j \end{pmatrix}, \ U_{ij}^{(2)} = \begin{pmatrix} i & j \end{pmatrix}, \ \tilde{U}_{ij}^{(1)} = \begin{pmatrix} i & j \end{pmatrix} \) and \(\tilde{U}_{ij}^{(2)} = \begin{pmatrix} i & j \end{pmatrix}.\) If \(\Delta = 0,\) we can verify that \(|e_i(i = 1, 2, 3, 4)|\) are eigenstates of the Hamiltonian. The corresponding eigen-energy are \(E_1 = E_3 = J\) and \(E_2 = E_4 = 4J.\) The energy levels of the Hamiltonian \(\hat{H}\) in Eq.(8) when \(\Delta = 0\) are doubly degenerate. If the parameter \(\Delta\) is a finite real number, then the degenerate energy levels \(J\) and \(4J\) will split into two non-degenerate energy levels. Then the eigen-states for the Hamiltonian \(\hat{H}\) in Eq.(8) are found to be,

\[ |E_{1}^{\pm}\rangle = \frac{1}{\sqrt{2}}(|e_{1}\rangle \pm |e_{3}\rangle) \]
\[ |E_{2}^{\pm}\rangle = \frac{1}{\sqrt{2}}(|e_{2}\rangle \pm |e_{4}\rangle), \]

with the corresponding eigen-values \(E_{1}^{\pm} = J(1 \pm \Delta)\) and \(E_{2}^{\pm} = 4J(1 \pm \Delta).\) If we set \(J = \frac{\hbar^2g^2}{2m(L-a)^2}\) and \(\Delta = \frac{g}{\xi(L-a)}\) with \(\xi = \frac{\sqrt{2mV_0}}{\hbar},\) then the Hamiltonian \(H\) in Eq.(8) is equivalent to a double well system with the following potential function,

\[ V(x) = \begin{cases} \ V_0 & |x| \leq a \\ 0 & a < |x| < L \\ +\infty & |x| > L. \end{cases} \]

(10)

If potential barrier is infinity (i.e. \(V_0 \to +\infty\)), then \(\Delta \to 0.\) In this case, the system is equivalent to two independent potential wells. The topological bases \(|e_{1}\rangle\) and \(|e_{3}\rangle\) are equivalent to the ground states for the left and right potential well, correspondingly. The topological bases \(|e_{2}\rangle\) and \(|e_{4}\rangle\) are equivalent to the first excited states for the left and right potential well, correspondingly.

IV. QUANTUM TUNNELING EFFECT AND QUANTUM ZENO EFFECT

In this section, we will consider two physical effects, Quantum tunneling effect\[16, 17] and quantum Zeno effect\[18] in the topological system.
A. Quantum tunneling effect

For our convenience, we will focus on the subspace \( \{|e_1\rangle, |e_3\rangle\} \). Quantum mechanics predict that even if the system has an energy less than the barrier height, it has a finite probability to tunnel the barrier to the other side of the potential barrier. By means of the topological system, we can study the process of quantum dynamical tunneling. At \( t = 0 \), we suppose that the initial state is \( |e_1\rangle \), and \( |e_1\rangle \) can be recast as a superposition of the eigenstates of \( H \) in Eq.(8),

\[
|e_1\rangle = \frac{1}{\sqrt{2}}(|E^+_1\rangle + |E^-_1\rangle).
\]

Then the evolution of the system at any time \( t \) is given by,

\[
|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-i\omega_+ t}(|E^+_1\rangle + e^{i\delta t}|E^-_1\rangle),
\]

where \( \omega_+ = J(1 + \Delta)/\hbar, \omega_- = J(1 - \Delta)/\hbar \) and \( \delta = \omega_+ - \omega_- \). So that,

\[
P(|e_1\rangle) = \cos^2(\delta t/2), \ P(|e_3\rangle) = \sin^2(\delta t/2),
\]

with \( P(|e_1\rangle) \) and \( P(|e_3\rangle) \) being probability for the topological bases \( |e_1\rangle \) and \( |e_3\rangle \), correspondingly. So, the \( |\psi(t)\rangle \) will oscillate between topological bases \( |e_1\rangle \) and \( |e_3\rangle \). According to the Eqs.(12), then we can obtain the tunneling time is \( \tau = \pi/\delta \).

B. Quantum Zeno effect

Quantum measurement will alter the dynamics of the system. In 1977, Misra et.al. show that an unstable particle will never be found to decay if it is continuously observed. They called it quantum Zeno effect(QZE). Now, we show that in our topological system, the QZE can be observed. In Sec.IVA, we obtain the tunneling time is \( \tau = \pi/\delta \). Assume the time \( \tau \) is divided into \( n \) equal parts. After time \( \tau/n \), the probability of founding the topological basis \( |e_1\rangle \) is,

\[
P_{|e_1\rangle}(\tau/n) = \cos^2 \frac{\pi}{2n}.
\]

After \( n \) times measurements, the probability reads,

\[
P_{|e_1\rangle}^n = (\cos^2 \frac{\pi}{2n})^n.
\]

As \( n \) is large enough(i.e. \( n \gg 1 \)), the probability approximately is,

\[
P_{|e_1\rangle}^n \approx e^{-\frac{\pi^2}{4n}} \to 1.
\]

Then this is just the QZE.
V. SUMMARY

In summary, we have presented the Hamiltonian to the topological system, and we obtain the eigen-system for this system. When we select proper parameters, this topological system can be simulated by a quantum double well potential system. Based on which, quantum tunneling effect and quantum Zeno effect have been studied in detail. Let us make a summary to end this paper. Firstly, quantum tunneling effect can be observed in this topological system, and the tunneling time is $\tau = \pi/\delta$. Secondly, if we divide the tunneling time $\tau$ into $n$ equal parts (i.e., each time interval is $\tau/n$), after $n$ times measurements, the quantum Zeno effect can occur in this topological system.

Eventually, people have currently found that topological basis has some important physical applications in topological quantum computation, quantum entanglement and topological quantum teleportation, how to reveal the role of the topological parameter $d$ is an interesting and significant topic. We shall investigate this subject subsequently.

Acknowledgments

This work was supported by NSF of China (Grants No. 10875026) and the Fundamental Research Funds for the Central Universities (Grants No. 09SSXT026)

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