Two families of orthogonal polynomials on the unit circle from basic hypergeometric functions*

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Abstract

The sequence \( \{ \phi_1(q^{-k}, q^{k+1}; q^{-5-k+1}; q, q^{-5+1/2}z) \}_{k \geq 0} \) of basic hypergeometric polynomials is known to be orthogonal on the unit circle with respect to the weight function \(|(q^{1/2}e^{i\theta}; q)_\infty/(q^{b+1/2}e^{i\theta}; q)_\infty|^2 \). This result, where one must take the parameters \( q \) and \( b \) to be \( 0 < q < 1 \) and \( \text{Re}(b) > -1/2 \), is due to P.I. Pastro [18]. In the present manuscript we deal with the orthogonal polynomials on the unit circle with respect to the two parametric families of weight functions \( \hat{\omega}(b; \theta) = |(q^{1/2}e^{i\theta}; q)_\infty/(q^{b+1/2}e^{i\theta}; q)_\infty|^2 \) and \( \tilde{\omega}(b; \theta) = |(qe^{i\theta}; q)_\infty/(q^{b}e^{i\theta}; q)_\infty|^2 \), where \( 0 < q < 1 \) and \( \text{Re}(b) > 0 \). The orthogonal polynomials are given in terms of the sequence of basic hypergeometric polynomials \( \{ \phi_1(q^{-k}, q^{b}; q^{-5-k+1}; q, q^{-5+1/2}z) \}_{k \geq 0} \).

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1 Introduction

Orthogonal polynomials on the unit circle (in short, OPUC) are important objects of study in Classical Analysis. Like their counterpart on the real line, they have deep connections and applications in many areas of Mathematics and Engineering, their systematic study, started by Szegő and Geronimus (see [21] and [13]), still remains very active, especially because of their applications in spectral theory [21, 22].

Given a nontrivial probability measure \( \mu \) on the unit circle \( \mathbb{T} := \{ \zeta = e^{i\theta}; 0 \leq \theta \leq 2\pi \} \) the associated orthonormal OPUC \( \varphi_k(z) = \kappa_k z^k + \text{lower degree terms}, k \geq 0 \), are defined by \( \kappa_k > 0 \) and

\[
\int_{\mathbb{T}} \overline{\varphi_j(\zeta)} \varphi_k(\zeta) \, d\mu(\zeta) = \int_0^{2\pi} \overline{\varphi(je^{i\theta})} \varphi_k(e^{i\theta}) \, d\mu(e^{i\theta}) = \delta_{j,k}, \quad j, k = 0, 1, 2, \ldots ,
\]

where \( \delta_{j,k} \) stands for the Kronecker delta. Among their fundamental properties is that all their zeros belong to the open unit disk \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \), and that they satisfy the

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Szego recurrence,
\[ \Phi_k(z) = z\Phi_{k-1}(z) - \alpha_{k-1} \Phi_{k-1}^*(z), \quad k \geq 1, \]
given here in terms of the monic OPUC \( \Phi_k(z) = \varphi_k(z) / \kappa_k, \ k \geq 0 \). The coefficients \( \alpha_{k-1} = -\Phi_k(0) \) are known as the Verblunsky coefficients, and \( \Phi_k^*(z) = z^k \Phi_k(1/z) \). It is well known that \( |\alpha_k| < 1 \) for \( k \geq 0 \), and that the sequence \( \{\alpha_k\}_{k \geq 0} \) uniquely determines the measures \( \mu \) on \( \mathbb{T} \) (see e.g. \[20\], as well as \[11\]).

If the theory of OPUC still leaves something to be highly desired, then having only a very few concrete examples (examples with explicit formulas) of these polynomials has to be the one that comes to mind. Most of the known examples are related with orthogonal polynomials on the interval \([-1, 1]\) via the Szego-transformation (see \[24\]) or via the DG-transformation (see \[25\]) and, hence, the values of associated Verblunsky coefficients are restricted to be within \((-1, 1)\). Among the few known examples of monic OPUC with complex Verblunsky coefficients, the ones that we like to highlight are the following.

With \( \text{Re}(b) > -1/2 \), if \( \Phi_k(z) = \frac{(b+1)_k}{(b+1)_k} \) \(_2F_1(-k, b+1; b+1; 1-z) \), \( k \geq 0 \), then \( \{\Phi_k\}_{k \geq 0} \) is the sequence of monic OPUC such that
\[
\frac{(b-b^*+1)_k}{2\pi \Gamma(b+1)} \int_0^{2\pi} \Phi_j(e^{i\theta}) \Phi_k(e^{i\theta}) (e^{2i\theta} \sin^2(\theta/2))^{\text{Re}(b)} d\theta = \frac{(b+b+1)_k k!}{|(b+1)_k|^2} \delta_{j,k}.
\]
We have for the associated Verblunsky coefficients \( \alpha_{k-1} = -(b)_k/(\overline{b+1})_k, k \geq 1 \).

For information regarding the definitions and properties of the Pochhammer symbols \((.)_k\) and the hypergeometric functions \(_2F_1\), we refer to \[1\].

The above parametric family of OPUC came to public knowledge in \[23\]. However, it is important to mention that this family of OPUC is a subfamily of a family of biorthogonal polynomials presented by Askey in the Gabor Szego: Collected papers \[3, p. 304\].

The second parametric family of OPUC with complex Verblunsky coefficients, attributed to Pastro \[18\], that we like to mention here is
\[
\Phi_k^b(z) = \frac{(q^{b^*}; q)_k}{(q^{b+1}; q)_k} q^{k/2} \phi_1 \left( \frac{q^{-k}, q^{b+1}}{q^{-k+1}; q}; q, q^{-b+1/2}; z \right), \quad k \geq 0, \tag{1.2}
\]
again with \( \text{Re}(b) > -1/2 \). These monic polynomials satisfy the orthogonality
\[
\int_\mathbb{T} \overline{\Phi_j^b(\zeta)} \Phi_k^b(\zeta) d\mu^b(\zeta) = \rho_k^{(b,b+\overline{b}-1)} = \frac{(q; q)_k (q^{b+\overline{b}+1}; q)_k}{(q^{b+1}; q)_k (q^{\overline{b}+1}; q)_k} \delta_{j,k},
\]
where the probability measure \( \mu^b(\zeta) \) on the unit circle is such that
\[
d\mu^b(\zeta) = \frac{(q; q)_\infty (q^{b+\overline{b}+1}; q)_\infty}{(q^{b+1}; q)_\infty (q^{\overline{b}+1}; q)_\infty} \frac{1}{|q^{1/2}\zeta; q)_\infty|^2 |q^{(b+1)/2}\zeta; q)_\infty|^2 2\pi i d\zeta.
\]
For information on the definitions and properties of the \(q\)-Pochhammer symbols \((.; q)_k\) and the basic hypergeometric (or \(q\)-hypergeometric) functions \(_2\phi_1\) we refer, for example, to Gasper and Rahman \[12\] and Koekoek and Swarttouw \[16\].

Apart from the above two examples of OPUC with complex Verblunsky coefficients, it is important that we mention the system of OPUC with constant Verblunsky coefficients. This system of OPUC, also known as Geronimus polynomials, has been thoroughly studied.
by many including Geronimus [13], Golinskii, Nevai and Van Assche [14] and Simon [20, p. 83].

Our present aim is to consider the OPUC and other related objects associated with the positive measures \( \hat{\mu}^{(b)} \) and \( \hat{\mu}^{(b)} \) on the unit circle given by

\[
d\hat{\mu}^{(b)}(\zeta) = \hat{\rho}^{(b)} \frac{|q^b \zeta; q\infty|}{|q^{b+1} \zeta; q\infty|} \frac{1}{2\pi i} d\zeta
\]

and

\[
d\hat{\mu}^{(b)}(\zeta) = \hat{\rho}^{(b)} \frac{|q\zeta; q\infty|}{|q^{b+1} \zeta; q\infty|} \frac{1}{2\pi i} d\zeta,
\]

where \( \Re(b) > 0 \). The values of the positive constants \( \hat{\rho}^{(b)} \) and \( \hat{\rho}^{(b)} \), so that the measures \( \hat{\mu}^{(b)} \) and \( \hat{\mu}^{(b)} \) are probability measures, are given respectively in (4.1) and (5.1).

The elements that play important roles in this manuscript are the modified basic hypergeometric polynomials

\[
R_k(b; z) = \frac{(q^b; q)_k}{(q^b \cos(\eta q); q)_k} 2\phi_1 \left( \frac{q^{-b-1} z}{q^{-b-k+1}; q, q^{-b+1}} \right), \quad k \geq 0,
\]

and the sequence \( \{d^{(b)}_{k+1}\}_{k \geq 1} \) given by

\[
d^{(b)}_{k+1} = \frac{(1 - q^b) (1 - q^{2\lambda+k-1})}{4(1 - q^{\lambda+k-1} \cos(\eta q)) (1 - q^{\lambda+k} \cos(\eta q))}, \quad k \geq 0,
\]

where \( b = \lambda - i\eta \) and \( \eta q = \eta \ln(q) \). As detailed in Section 3 of this manuscript, the sequence \( \{d^{(b)}_{k+1}\}_{k \geq 1} \) is a positive chain sequence with the choice \( Re(b) = \lambda > 0 \). We cite Chihara [7] for a good source of information on positive chain sequences.

The multiplication (or modification) factor \( (q^b; q)_k/(q^b \cos(\eta q); q)_k \) in (1.3) is so that there hold \( R_k(b; z) = R_k^{(b)}(b; z) \), \( k \geq 0 \) and, further, the sequence \( \{R_k(b; .)\}_{k \geq 0} \) satisfies the nice three term recurrence formula (3.4).

Our main result with respect to the measure \( \hat{\mu}^{(b)} \) is the following theorem, the proof of which is given in Section 4.

**Theorem 1.1.** Let \( b = \lambda - i\eta, \eta q = \eta \ln(q) \) and \( \lambda > 0 \). Then the sequence \( \{\hat{\Phi}^{(b)}_k(b; z)\}_{k \geq 0} \) of monic OPUC with respect to the positive measure \( \hat{\mu}^{(b)} \) given by (1.3) is such that

\[
\hat{\Phi}^{(b)}_k(b; z) = \frac{(q^\lambda \cos(\eta q); q)_k}{(q^b; q)_k} R_{k+1}(b; z) - 2(1 - \ell^{(b)}_{k+1}) R_k(b; z), \quad k \geq 0.
\]

Here, \( \ell^{(b)}_{k+1}, k \geq 0 \) are such that \( \ell^{(b)}_{1+} = 0 \) and \( \ell^{(b)}_{k+1} = d^{(b)}_{k+1}/(1 - \ell^{(b)}_{k}), k \geq 1 \). In particular, the associated Verblunsky coefficients satisfy

\[
\hat{\alpha}^{(b)}_{k+1} = -\left[1 - 2\ell^{(b)}_{k+1} \frac{1 - q^{\lambda+k} \cos(\eta q)}{1 - q^{b+k}} \right] \frac{(q^b; q)_k}{(q^b; q)_k}, \quad k \geq 1.
\]

Moreover, if \( \hat{\mu}^{(b)} \) is a probability measure and if \( \hat{\phi}^{(b)}_k(b; z) = \hat{\kappa}^{(b)}_k \hat{\Phi}^{(b)}_k(b; z) \) are the associated orthonormal polynomials then

\[
[\hat{\kappa}^{(b)}_k]^{-2} = \frac{(q^b; q)_k (q^{2\lambda}; q)_k}{(q^{b+1}; q)_k (q^{b+1}; q)_k} \frac{1 - q^{\lambda+k} \cos(\eta q)}{1 - q^{\lambda+k} \cos(\eta q)} (1 - \ell^{(b)}_{k+1}), \quad k \geq 0.
\]
The sequence \( \{ \ell_{k+1}^{(b)} \}_{k \geq 0} \) is the minimal parameter sequence of the positive chain sequence \( \{ d_{k+1}^{(b)} \}_{k \geq 1} \). An explicit expression for \( \ell_{k+1}^{(b)} \) for any \( k \geq 1 \) is also given in Section 4. Finally, in this section the Szegő function associated with the measure \( \mu^{(b)} \) is also explicitly found.

Now with respect to the measure \( \mu^{(b)} \) our main result is the following theorem.

**Theorem 1.2.** Let \( b = \lambda - i \eta, \eta \neq 0 \) and \( \lambda > 0 \). Then the sequence \( \{ \Phi_k(b; z) \}_{k \geq 0} \) of monic OPUC with respect to the positive measure \( \hat{\mu}^{(b)} \) given by (1.4) is such that

\[
\Phi_k(b; z) = \frac{(q^\lambda \cos(\eta q); q)_k}{(q^b; q)_k} \left[ R_k(b; z) - 2(1 - M_k^{(b)}) R_{k-1}(b; z) \right], \quad k \geq 1.
\]

Here, \( M_k^{(b)}, k \geq 1, \) are such that \( M_{k+1}^{(b)} = d_{k+1}^{(b)}/ (1 - M_k^{(b)}), k \geq 1, \) with

\[
M_1^{(b)} = \frac{1}{2} \frac{1 - q^b}{1 - q^\lambda \cos(\eta q)} \frac{1}{2\phi_1(q, q^{-b+1}; q, q^b)}.
\]

In particular, the associated Verblunsky coefficients satisfy

\[
\alpha_{k-1}^{(b)} = \left[ 1 - 2M_k^{(b)} \frac{1 - q^\lambda + k - 1 \cos(\eta q)}{1 - q^{b+k-1}} \right] \frac{(q^b; q)_{k-1}}{(q^b q; q)_{k-1}}, \quad k \geq 1.
\]

Moreover, if \( \hat{\mu}^{(b)} \) is a probability measure and if \( \hat{\phi}_k(b; z) = \hat{\kappa}_k^{(b)} \Phi_k(b; z) \) are the associated orthonormal polynomials then

\[
[\hat{\kappa}_k^{(b)}]^{-2} = \frac{(q; q)_k (q^\lambda; q)_k}{(q^b q; q)_k (q^\lambda q^b; q)_k} \frac{1 - q^\lambda \cos(\eta q) M_1^{(b)}}{1 - q^{\lambda + k} \cos(\eta q) M_{k+1}^{(b)}}, \quad k \geq 0.
\]

The proof of Theorem 1.2 is given in Section 5 of this manuscript. As shown also in Section 5 the sequence \( \{ M_{k+1}^{(b)} \}_{k \geq 0} \) is the maximal parameter sequence of the positive chain sequence \( \{ d_{k+1}^{(b)} \}_{k \geq 1} \). Explicit expression for \( M_{k+1}^{(b)} \) for any \( k \geq 1 \) is also given in this section. Finally in this section, the Szegő function associated with the measure \( \mu^{(b)} \) is also explicitly found.

The manuscript is organized as follows. In Section 2 we briefly provide some results on a general family of basic hypergeometric biorthogonal polynomials that follow from [9]. The results given in Section 3 are with respect to the subfamily \( R_k(b; z) \) of these biorthogonal polynomials. Specifically, Section 4 provides information about the OPUC with respect to the measure \( \mu^{(b)} \) and Section 5 gives information about the OPUC with respect to the measure \( \mu^{(b)} \).

## 2 Basic hypergeometric biorthogonal polynomials

For \( 0 < q < 1, b \neq -1, -2, \ldots \) and \( c - b + 1 \neq 0, -1, -2, \ldots \), the basic hypergeometric polynomials \( 2\phi_1(q^{-k}; q, q^{b+1} \); \( q^{-c+b-k} ; q, q^{-c+d-1} z \) were the subject of study in [9]. To summarize some of the results presented in [9], let

\[
B_k^{(b,c,d)}(z) = \frac{(q^{-c+b-1}; q)_k}{(q^b+1; q)_k} q^{k(b-d+1)} 2\phi_1 \left( q^{-k}; q^{b+1}; q^{-c+b-k} ; q, q^{-c+d-1} z \right), \quad k \geq 1. \tag{2.1}
\]

Then the following can be stated with regard to the monic polynomials \( B_k^{(b,c,d)} \).
They satisfy the biorthogonality property
\[ \mathcal{L}^{(b,c,d)}[\zeta^{-j} B^{(b,c,d)}_k(\zeta)] = \delta_{k,j} \rho^{(b,c)}_k, \quad 0 \leq j \leq k, \quad k \geq 1, \] (2.2)
with respect to the (quasi-definite) moment functional
\[ \mathcal{L}^{(b,c,d)}[\zeta^{-j}] = \frac{(q^{-b}; q)_j}{(q^{c-b+2}; q)_j} q^{jd}, \quad j = 0, \pm 1, \pm 2, \ldots . \]

Here, \( \rho^{(b,c)}_k = \frac{(q^c q_k (q^{c+2}; q)_k)}{(q^{k+1}; q^c (q^{c-b+2}; q)_k)}. \)

Moreover, they satisfy the three term recurrence formula
\[ B_{k+1}^{(b,c,d)}(z) = (z + c_k^{(b,c,d)}) B_k^{(b,c,d)}(z) - \mathfrak{D}_{k+1}^{(b,c,d)} z B_{k-1}^{(b,c,d)}(z), \quad k \geq 1, \] (2.3)
with \( B_0^{(b,c,d)}(z) = 1 \) and \( B_1^{(b,c,d)}(z) = z + c_1^{(b,c,d)}, \)

where
\[ c_k^{(b,c,d)} = \frac{1 - q^{c-b+k}}{1 - q^{b+k}} q^{b-d+1}, \quad \mathfrak{D}_{k+1}^{(b,c,d)} = \frac{(1 - q^k)(1 - q^{c+k+1})}{(1 - q^{b+k+1})(1 - q^{b+k+1})} q^{b-d+1}, \quad k \geq 1. \]

To obtain the above results the following contiguous relation
\[ \frac{c_k^{(b,c,d)}}{f_k^{(b,c,d)}(z)} = z - \mathfrak{D}_{k+1}^{(b,c,d)} \frac{z}{f_k^{(b,c,d)}} \frac{B_k^{(b,c,d)}(z)}{c_k^{(b,c,d)}}, \quad k \geq 1, \] (2.4)
where
\[ f_k^{(b,c,d)}(z) = \frac{2 \phi_1(q^{k+1}, q^b; q^{c-b-k+2}; q, q^d z)}{2 \phi_1(q^k, q^b; q^{c-b-k+1}; q, q^d z)}, \quad k \geq 1, \]
and the continued fraction expansion resulting from it were part of the tools used in \[9\]. The relation (2.4) follows from results given by Heine (see \[12\], p. 22).

As shown also in \[9\], if one assumes the additional restriction that \( \Re (c+2) > \Re (d) > 0 \), then
\[ \mathcal{L}^{(b,c,d)}[\zeta^{-j}] = \rho^{(b,c)} \int_{\mathbb{T}} \zeta^{-j} \frac{(q^{-b-d}; \zeta; q)_\infty (q^{b-d+1}; \zeta; q)_\infty 1}{(q^d \zeta; q)_\infty (q^{c+2-d}; \zeta; q)_\infty 2 \pi i \zeta} d\zeta, \] (2.5)
for \( j = 0, \pm 1, \pm 2, \ldots \), where
\[ \rho^{(b,c)} = \frac{(q^c q; q)_\infty (q^{c+2}; q)_\infty}{(q^{b+1}; q)_\infty (q^{c-b+2}; q)_\infty}. \]

Here, \( \int_{\mathbb{T}} \) represents the integration along the unit circle \( z = e^{i \theta}, 0 \leq \theta \leq 2 \pi \).

Note that when (2.5) holds, then we also have for \( |z| < 1, \)
\[ \mathcal{L}^{(b,c,d)} \left[ \frac{\zeta}{\zeta - z} \right] = \sum_{j=0}^{\infty} \mathcal{L}^{(b,c,d)}[\zeta^{-j}] z^j = 2 \phi_1 \left( q, q^{-b}; q^{c-b+2}; q, q^d z \right). \] (2.6)

When the parameters \( b, c \) and \( d \) satisfy
\[ c = b + \overline{c} - 1, \quad d + \overline{d} = b + \overline{d} + 1 \quad \text{and} \quad \Re (b) > -1/2, \]
then, as also observed in \[9\], the respective moment functional is positive definite and the associated polynomials are OPUC with respect to a nontrivial probability measure supported on the unit circle. In particular, choosing \( c = b + \overline{c} - 1, d = b + 1/2 \) and
\( \Re(b) > -1/2 \), the resulting monic OPUC \( B_k^{(b, b+\bar{\tau}-1, b+1/2)}(z) \), \( k \geq 0 \), are the Pastro polynomials given by \([12]\). Moreover, by letting \( \Re(b) = \lambda \to \infty \), we also recover the monic Rogers-Szeg\'o polynomials \( \mathcal{H}_k(z \mid q) \) from \( B_k^{(b, b+\bar{\tau}-1, b+1/2)}(z) \). Precisely, we have
\[
\lim_{\lambda \to \infty} B_k^{(b, b+\bar{\tau}-1, b+1/2)}(z) = q^{k/2} \mathcal{H}_k(z \mid q), \quad k \geq 0.
\]

The Rogers-Szeg\'o polynomials (see \([15\text{ Chap.17}]\)) are orthogonal on the unit circle with respect to the weight function \( |(q^{1/2}e^{i\theta}; q)_\infty|^2 \).

### 3 Polynomials with zeros on the unit circle

We now consider the subfamily of biorthogonal polynomials given by
\[
P_k(b; z) = \frac{(q^b; q)_k}{(q^0; q)_k} 2\phi_1\left( \begin{array}{c} q^{-k}, q^b \\ q^{-b-k+1}; q, q^{-\bar{\tau}+1}z \end{array} \right), \quad k \geq 0,
\]

obtained as \( P_k(b; z) = B_k^{(b-1, b+\bar{\tau}-2, b)}(z) \), for \( b \neq 0, -1, -2, \ldots \).

It follows from \([2,3]\) that the sequence of monic polynomials \( \{P_k(b; \cdot)\}_{k \geq 0} \) satisfies the three term recurrence formula
\[
P_{k+1}(b; z) = (z + C_{k+1}^{(b-1, b+\bar{\tau}-2, b)}) P_k(b; z) - D_{k+1}^{(b-1, b+\bar{\tau}-2, b)} z P_{k-1}(b; z),
\]

for \( k \geq 1 \), with \( P_0(b; z) = 1 \) and \( P_1(b; z) = z + C_1^{(b)} \), where
\[
C_k^{(b-1, b+\bar{\tau}-2, b)} = \frac{1 - q^{\bar{\tau}+k-1}}{1 - q^{b+k-1}}, \quad D_k^{(b-1, b+\bar{\tau}-2, b)} = \frac{(1 - q^k)(1 - q^{b+\bar{\tau}+k-1})}{(1 - q^{b+k})(1 - q^{b+k})}, \quad k \geq 1.
\]

From \([2,2]\), they also satisfy the orthogonality
\[
\mathcal{L}^{(b-1, b+\bar{\tau}-2, b)}[\zeta^{-j} P_k(b; \zeta)] = \delta_{k, j} \rho_k^{(b-1, b+\bar{\tau}-2)}, \quad 0 \leq j \leq k, \quad k \geq 1,
\]

with respect to the quasi-definite moment functional
\[
\mathcal{L}^{(b-1, b+\bar{\tau}-2, b)}[\zeta^{-j}] = \frac{(q^{b+1}; q)_j}{(q^0; q)_j} q^j, \quad j = 0, \pm 1, \pm 2, \ldots .
\]

Here, \( \rho_k^{(b-1, b+\bar{\tau}-2)} = \frac{(q^0; q)_k}{(q^b; q)_k} \frac{(q^{b+1}; q)_k}{(q^0; q)_k} \).

Let \( b = \lambda - i\eta \) and \( \eta_q = \eta \ln(q) \). Then with the observation that \( 1 - q^{b+k} = 1 - q^{\lambda+k} \cos(\eta_q) + i q^{\lambda+k} \sin(\eta_q) \), let us consider the sequence of polynomials \( \{R_k(b; \cdot)\}_{k \geq 0} \) given by \( R_k(b; z) = \frac{(q^b; q)_k}{(q^\lambda; q)_k} P_k(b; z) \), \( k \geq 0 \). Then the polynomials \( R_k(b; z) \) takes the form \([1,3]\).

From \([3,2]\) one can easily verify that these polynomials satisfy the three term recurrence formula
\[
R_{k+1}(b; z) = [(1 + i c_{k+1}^{(b)}) z + (1 - i c_{k+1}^{(b)})] R_k(b; z) - 4 d_{k+1}^{(b)} R_{k-1}(b; z),
\]

for \( k \geq 1 \), with \( R_0(b; z) = 1 \) and \( R_1(b; z) = (1 + i c_1^{(b)}) z + (1 - i c_1^{(b)}) \), where
\[
c_k^{(b)} = \frac{q^{\lambda+k-1} \sin(\eta_q)}{1 - q^{\lambda+k-1} \cos(\eta_q)}, \quad k \geq 1.
\]
and the sequence \( \{d_{k+1}^{(b)}\}_{k \geq 1} \) is as in (1.6).

**Assumption on \( b \):** From now on we assume that the value of \( b \) be such that \( \Re e(b) > 0 \). Then the sequence \( \{d_{k+1}^{(b)}\}_{k \geq 1} \) satisfies

\[
d_{k+1}^{(b)} \leq d_{k+1}^{(\lambda)}, \quad k \geq 1.
\]

The sequence \( \{d_{k+1}^{(\lambda)}\}_{k \geq 1} \), for \( \lambda > 0 \), is also the sequence of coefficients that appear in the three term recurrence formula

\[
\hat{C}_{k+1}(x; q^{\lambda} | q) = x \hat{C}_k(x; q^{\lambda} | q) - d_{k+1}^{(\lambda)} \hat{C}_{k-1}(x; q^{\lambda} | q), \quad k \geq 1.
\]

of the monic continuous \( q \)-ultraspherical polynomials \( \{\hat{C}_k(x; q^{\lambda} | q)\}_{k \geq 0} \). These polynomials are symmetric and orthogonal on the interval \([-1, 1] \). The continuous \( q \)-ultraspherical polynomials were introduced by Rogers [19] and as more recent references to these polynomials we refer to [4] [15].

Thus, the sequence \( \{d_{k+1}^{(\lambda)}\}_{k \geq 1} \) is a positive chain sequence. This affirmation follows from a well known result regarding orthogonal polynomials defined on any finite interval of the real line, connecting the extreme points of the interval of orthogonality to positive chain sequences. Hence, by the comparison theorem for positive chain sequences [7] p. 97, the sequence \( \{d_{k+1}^{(b)}\}_{k \geq 1} \) is also confirmed to be a positive chain sequence for any \( b \) such that \( \Re e(b) = \lambda > 0 \).

By using the three term recurrence formula (3.4) together with results established in [10], it was shown in [2] that when \( \Re e(b) > 0 \) the zeros \( z_{k,j}^{(b)} \), \( j = 1, 2, \ldots, k \) of \( R_k(z) \) are all simple and lie on the unit circle \(|z| = 1 \). Moreover, with \( z_{k,j}^{(b)} = e^{i\theta_{k,j}^{(b)}} \), the interlacing property

\[
0 < \theta_{k+1,1}^{(b)} < \theta_{k+1,2}^{(b)} < \cdots < \theta_{k,k}^{(b)} < \theta_{k+1,k+1}^{(b)} < 2\pi, \quad k \geq 1.
\]

also holds.

Polynomials given by a three term recurrence formula of the form

\[
R_{k+1}(z) = \left[ (1 + ic_{k+1})z + (1 - ic_{k+1}) \right] R_k(z) - 4d_{k+1}z R_{k-1}(z), \quad k \geq 1,
\]

with \( R_0(z) = 1, \ R_1(z) = (1 + ic_1)z + (1 - ic_1) \), where \( \{c_k\}_{k=1}^{\infty} \) is any real sequence and \( \{d_{k+1}\}_{k=1}^{\infty} \) is a positive chain sequence, have been the subject of study in the recent publications [5] [6] [8] [10] [17]. These polynomials turn out to be para-orthogonal polynomials related to some associated orthogonal polynomials on the unit circle. The main results of the present manuscript are obtained as applications of results established in [5], [6] and [8].

Since \( \Re e(b) > 0 \), from (2.5)

\[
\mathcal{L}^{(b-1,b+\bar{b}-2)}(z) = \rho^{(b-1,b+\bar{b}-2)}(z) \int_{\mathbb{T}} z^d \left( \frac{q^{\lambda} \zeta \bar{q}^{\lambda} \zeta}{q^{\lambda} \zeta \bar{q}^{\lambda} \zeta} \right) \frac{1}{2\pi i \zeta} d\zeta,
\]

for \( j = 0, \pm 1, \pm 2, \ldots \), where

\[
\rho^{(b-1,b+\bar{b}-2)} = \frac{(q; q)_{\infty} (q^{b+\bar{b}}; q)_{\infty}}{(q^{b}; q)_{\infty} (q^{b+1}; q)_{\infty}}
\]
To be able to apply directly the results presented in [5], [6] and [8], we now introduce the moment functional $N^{(b)}$ given by

$$
\frac{1}{2d_1} N^{(b)}[\zeta^{-j}] = -\frac{1 - q^\lambda \cos(\eta q)}{(1 - q^\lambda)} L^{(b-1, b+1-2)}[\zeta^{-j}]
\left(1 - q^\lambda \cos(\eta q)\right) (q^b; q)_\infty (q^{b+1}; q)_\infty \int_\mathbb{R} \zeta^{-j} \frac{|(q \zeta, q)_{\infty}|^2}{|q^{b+1} \zeta, q)_{\infty}|^2} 2\pi i \zeta \, d\zeta.
$$

(3.7)

The non-zero real constant $d_1$ is arbitrary. However, in order to use results stated in [6], in Section 5 we will take $d_1 = (1 - t)M_1^{(b)}$ with $0 \leq t < 1$, where $\{M_k^{(b)}\}_{k \geq 0}$ is the maximal parameter sequence of the positive chain sequence $\{d_{k+1}\}_{k \geq 1}$.

From (3.3) and (3.7) we have

$$
N^{(b)}[\zeta^{-j}] = \nu_j^{(b)} = 2d_1 \frac{1 - q^\lambda \cos(\eta q)}{1 - q^\lambda} \left(\frac{(q^b; q)_j}{(q^{b+1}; q)_j}\right) q^{j}, \quad j = 0, \pm 1, \pm 2, \ldots
$$

(3.8)

The moments $\nu_j^{(b)}$ are such

$$
\nu_0^{(b)} = \frac{2d_1}{1 + i c_1^{(b)}} \quad \text{and} \quad \nu_j^{(b)} = -\nu_{-j+1}, \quad j = 1, 2, 3, \ldots
$$

Furthermore, from (2.2) and (3.4),

$$
N^{(b)}[\zeta^{-k+j} P_k(b; \zeta)] = \left\{ \begin{array}{ll}
-2d_1 \frac{1 - q^\lambda \cos(\eta q)}{1 - q^\lambda} \frac{p_k^{(b-1, b+1-2)}}{p_k^{(b-1, b+1-2)}}, & j = -1, \\
0, & 0 \leq j \leq k - 1,
\end{array} \right.
$$

(3.9)

and

$$
N^{(b)}[\zeta^{-k+j} R_k(b; \zeta)] = \left\{ \begin{array}{ll}
-\gamma_k^{(b)}, & j = -1, \\
0, & 0 \leq j \leq k - 1, \\
\gamma_k^{(b)}, & j = k,
\end{array} \right.
$$

for $k \geq 1$, where $\gamma_k^{(b)} = \frac{4d_k^{(b)}}{1 + i c_{k+1}^{(b)}} \gamma_{k-1}^{(b)}, k \geq 1$, with $\gamma_0^{(b)} = \nu_0^{(b)}$.

Now consider the polynomials $Q_k(b; z)$ defined by

$$
Q_k(b; z) = N^{(b)} \left[ \frac{R_k(b; z) - R_k(b; \zeta)}{z - \zeta} \right], \quad k \geq 0.
$$

It is not difficult to show that $\{Q_k(b; z)\}_{k \geq 0}$, where $Q_k(b; z)$ is of degree $k - 1$, satisfies the three term recurrence formula

$$
Q_{k+1}(b; z) = [(1 + i c_{k+1}^{(b)}) z + (1 - i c_{k+1}^{(b)})] Q_k(b; z) - 4d_k^{(b)} z Q_{k-1}(b; z), \quad k \geq 1,
$$

with $Q_0(b; z) = 0$ and $Q_1(b; z) = 2d_1$. Moreover,

$$
-\sum_{j=0}^\infty \nu_j^{(b)} z^j \frac{Q_k(b; z)}{R_k(b; z)} = \frac{\gamma_k^{(b)}}{R_k(b; 0)} z^k + O(z^{k+1}),
$$

$$
\sum_{j=1}^\infty \nu_j^{(b)} z^{-j} \frac{Q_k(b; z)}{R_k(b; z)} = \frac{\gamma_k^{(b)}}{R_k(b; 0)} \frac{1}{z^{k+1}} + O((1/z)^{k+2}).
$$
Since,
\[-\sum_{j=0}^{\infty} \nu_{j+1}^{(b)} z^j = 2d_1 \frac{1 - q^b \cos(\eta_q)}{1 - q^b} \sum_{j=0}^{\infty} \mathcal{L}_j^{(b-1,b+2,b)}(\zeta^{-j}) z^j,\]
from (2.6) we have for \(|z| < 1\),
\[2d_1 \frac{1 - q^b \cos(\eta_q)}{1 - q^b} 2\phi_1\left( q, q^{b+1} ; q, q^b z \right) - \frac{Q_k(b; z)}{R_k(b; z)} = 2d_1 \frac{\pi_k^{(b)}}{R_k(b; 0)} z^k + O(z^{k+1}).\]
Since the rational functions \(Q_k(b; z)/R_k(b; z)\) are analytic in \(|z| < 1\), an immediate consequence of the above result is that
\[\lim_{k \to \infty} Q_k(b; z) = \left( \frac{q^b; q}{q^b \cos(\eta_q); q} \right) \frac{(q^{b+1}; q)}{(q^b z; q)} \frac{(q; q)_{\infty}}{(q; q)_{\infty}} \frac{\pi_k^{(b)}}{R_k(b; 0)} z^k + O(k^{k+1}).\]
uniformly on compact subsets of \(|z| < 1\).

We can also state the asymptotic property,
\[\lim_{k \to \infty} R_k(b; z) = \frac{(q^b; q)_{\infty}}{(q^b \cos(\eta_q); q)_{\infty}} \frac{(q^{b+1}; q)}{(q^b z; q)} \frac{\pi_k^{(b)}}{R_k(b; 0)} z^k + O(k^{k+1}),\]
uniformly on compact subsets of \(|z| < 1\). Proof of this follows from (1.5) by using \(\lim_{k \to \infty} (q^{b+1}; q) = q^{(b-1)j}\) and the Lebesque’s dominated convergence theorem.

4 Orthogonal polynomials with respect to the measure \(\hat{\mu}^{(b)}\)

We now consider positive measure \(\hat{\mu}^{(b)}\) given by (1.3) for \(\Re(b) > 0\).

**Theorem 4.1.** With \(b = \lambda - i\eta, \eta_q = \eta \ln(q)\) and \(\lambda > 0\) let
\[\hat{\mu}^{(b)} = \frac{1}{2(1 - q^b \cos(\eta_q))} \frac{(q; q)_{\infty}}{(q^{b+1}; q)_{\infty}} \frac{(q^b; q)_{\infty}}{(q^b z; q)_{\infty}} \frac{\pi_k^{(b)}}{R_k(b; 0)} z^k + O(k^{k+1}).\]

Then the measure \(\hat{\mu}^{(b)}\) given by (1.3) is a probability measure on the unit circle.

**Proof.** To obtain information about this measure and the associated OPUC, we can apply the results obtained in [5] with the three term recurrence formula (3.1) and the associated moment functional \(\mathcal{N}^{(b)}\). From the integral representation given in (3.7) for \(\mathcal{N}^{(b)}\), we observe that \(\int_{\mathbb{T}} \zeta^{-j} d\hat{\mu}^{(b)}(\zeta) = \text{const} \mathcal{N}^{(b)}[\zeta^{-j}(1 - \zeta^{-1})]\).

Thus, using [5] Thm. 3.1 we obtain \(\hat{\mu}^{(b)}\) as a probability measure from
\[\int_{\mathbb{T}} \zeta^{-j} d\hat{\mu}^{(b)}(\zeta) = \frac{1 + (c_1^{(b)})^2}{4d_1} \mathcal{N}^{(b)}[\zeta^{-j}(1 - \zeta^{-1})], \quad j = 0, \pm 1, \pm 2, \ldots.
\]
Hence, from one hand
\[\int_{\mathbb{T}} \zeta^{-j} d\hat{\mu}^{(b)}(\zeta) = \frac{1 + (c_1^{(b)})^2}{4d_1} [\nu_j^{(b)} - \nu_{j+1}^{(b)}], \quad j = 0, \pm 1, \pm 2, \ldots,
\]
from which, using (3.8), one can confirm that \(\int_{\mathbb{T}} d\hat{\mu}^{(b)}(\zeta) = 1\).
On the other hand, using (3.4),
\[
\int_T \zeta^{-j} d\mu^{(b)}(\zeta) = \frac{1 + (c_1^{(b)})^2}{2} (1 - q^\lambda \cos(\eta_q)) \frac{(q; q)_\infty (q^{b+\tau}; q)_\infty (q^{b}; q)_\infty}{(q^{b}; q)_\infty (q^{\bar{\tau}}; q)_\infty} \int_T \zeta^{-j} \frac{|(\zeta; q)_\infty|^2}{|(q^b \zeta; q)_\infty|^2} 2\pi i \zeta \, d\zeta.
\]
Thus, from the expression for $c_1^{(b)}$ in (3.5) we arrive at the value of $\hat{\mu}^{(b)}$. \hfill \Box

To obtain the sequence of monic OPUC $\{\hat{\Phi}(b; z)\}_{k \geq 0}$ one needs the minimal parameter sequence of $\{d_{k+1}^{(b)}\}_{k \geq 1}$.

**Theorem 4.2.** With $b = \lambda - i\eta$, $\eta_q = \eta \ln(q)$ and $\lambda > 0$ let $\{d_{k+1}^{(b)}\}_{k \geq 1}$ be the positive chain sequence given by (1.6). Then the minimal parameter sequence $\{\ell_{k+1}^{(b)}\}_{k \geq 0}$ of $\{d_{k+1}^{(b)}\}_{k \geq 1}$ is such that
\[
1 - \ell_{k+1}^{(b)} = \frac{R_{k+1}(b; 1)}{2R_k(b; 1)} \left[ 1 - \frac{R_{k+1}(b; 1)}{2R_k(b; 1)} \right], \quad k \geq 1.
\]
Moreover, $\lim_{k \to \infty} \ell_{k+1}^{(b)} = 1/2$.

**Proof.** The proof of expression for $1 - \ell_{k+1}^{(b)}$ is easily obtained from rewriting the three term recurrence formula (3.4) for $z = 1$ in the form
\[
d_{k+1}^{(b)} = \frac{R_k(b; 1)}{2R_{k-1}(b; 1)} \left[ 1 - \frac{R_{k+1}(b; 1)}{2R_k(b; 1)} \right], \quad k \geq 1.
\]
Since $\lim_{k \to \infty} d_{k+1}^{(b)} = 1/4$, the asymptotic for $\ell_{k+1}^{(b)}$ follows from [7, p. 102]. \hfill \Box

**Proof of Theorem 1.1** If we apply the remaining results presented in [5, Thm. 3.1], we obtain
\[
\hat{\Phi}_k(b; z) = \frac{1}{R_k(b; 1) \prod_{j=1}^{k+1}(1 + i\ell_j^{(b)})} \frac{R_{k+1}(b; z)R_k(b; 1) - R_k(b; z)R_{k+1}(b; 1)}{z - 1}, \quad k \geq 0.
\]
Letting $z = 0$, the associated Verblunsky coefficients are found to be
\[
\alpha_{k-1}^{(b)} = -\frac{1 - 2\ell_{k+1}^{(b)} - i\ell_{k+1}^{(b)}}{1 - i\ell_{k+1}^{(b)}} \prod_{j=1}^{k} \frac{1 + i\ell_j^{(b)}}{1 - i\ell_j^{(b)}}, \quad k \geq 1, \tag{4.2}
\]
Hence, the first two parts of Theorem 1.1 follow from (3.5) and Theorem 4.2.

To obtain the last part of Theorem 1.1 we observe from (4.2) that
\[
1 - |\alpha_{k-1}^{(b)}|^2 = \frac{4\ell_{k+1}^{(b)}(1 - \ell_{k+1}^{(b)})}{1 + (\ell_{k+1}^{(b)})^2}, \quad k \geq 1.
\]
Hence, the result follows from $(\kappa_{k}^{(b)})^{-2} = \prod_{j=1}^{k}(1 - |\alpha_j^{(b)}|^2)$ and $(1 - \ell_{k}^{(b)})\ell_{k+1}^{(b)} = d_{k+1}^{(b)}$. \hfill \Box
Now from Theorem 4.2 and the reciprocal property of \( R_n(b; \cdot) \) we have

\[
\hat{\Phi}_k^+(b; z) = \frac{(q^k \cos(\eta_\ell); q)_{k+1}}{(q^k; q)_{k+1}} R_{k+1}(b; z) - \frac{2(1 - \hat{\ell}_{k+1}(b)); z R_k(b; z)}{1 - z}, \quad k \geq 0.
\]

Hence, from (3.11) \( \lim_{k \to \infty} \hat{\Phi}_k^+(b; z) = (q^b z; q)_\infty / (z; q)_\infty \), uniformly on compact subsets of \( |z| < 1 \). Thus, by considering the limit (see [20, p. 144]) of \( 1/(\hat{\ell}_k \hat{\Phi}_k^+(b; z)) \) we can state the following.

**Theorem 4.3.** The Szegő function associated with the probability measure \( \hat{\mu}^{(b)} \) given by (1.3) and (4.1) is

\[
\hat{D}(z) = \frac{1}{|(q^b+1; q)_\infty|} \sqrt{\frac{(q; q)_\infty (q^{2b}; q)_\infty (z; q)_\infty}{2(1 - q^b \cos(\eta_q)) (q^b z; q)_\infty}}.
\]

## 5 Orthogonal polynomials with respect to the measure \( \hat{\mu}^{(b)} \)

We now consider positive measure \( \hat{\mu}^{(b)} \) given by (1.4).

**Theorem 5.1.** With \( b = \lambda - i\eta \), \( \eta_q = \eta \ln(q) \) and \( \lambda > 0 \) let

\[
\rho^{(b)} = \frac{(1 - q^\eta)}{2 \phi_1(q, q^{-b+1}; q, q^b)} \frac{(q; q)_\infty (q^{b+\eta}; q)_\infty}{(q^b; q)_\infty (q^\eta; q)_\infty}.
\]

(5.1)

Then the measure \( \hat{\mu}^{(b)} \) given by (1.4) is a probability measure on the unit circle.

**Proof.** With \( \Re(b) > 0 \) we clearly have \( 2 \phi_1(q, q^{-b+1}; q^b+1; q, q^b) \) finite. Moreover, from (1.3) and (3.6), one can easily verify that for \( |z| < 1 \),

\[
\frac{1}{(1 - q^\eta)^2} 2 \phi_1 \left( q, q^{-b+1}; q^b+1, q^b z \right) = \frac{1}{(1 - q^\eta)^2} \mathcal{L}^{(b-1, b+\eta - 2, b)} \left[ \frac{\zeta}{\zeta - z} \right],
\]

\[
= \frac{(q; q)_\infty (q^{b+\eta}; q)_\infty}{(q^b; q)_\infty (q^\eta; q)_\infty} \int_T \frac{\zeta}{\zeta - z} \frac{(q^\eta \zeta; q)_\infty (1/\zeta; q)_\infty}{(q^b \zeta; q)_\infty (q^{\eta/\zeta}; q)_\infty} \frac{1}{2\pi i \zeta} d\zeta.
\]

Hence, taking the limit as \( z \to 1 \) from below, we have

\[
\frac{2 \phi_1(q, q^{-b+1}; q^b+1, q^b)}{(1 - q^\eta)^2} = \frac{(q; q)_\infty (q^{b+\eta}; q)_\infty}{(q^b; q)_\infty (q^\eta; q)_\infty} \int_T \frac{(q^\eta \zeta; q)_\infty (q/\zeta; q)_\infty}{(q^b \zeta; q)_\infty (q^{\eta/\zeta}; q)_\infty} \frac{1}{2\pi i \zeta} d\zeta > 0.
\]

The above result can be justified by Abel’s continuity theorem. Thus, with \( \rho^{(b)} \) given as in the theorem, we have

\[
\rho^{(b)} \int_T \frac{(q^\eta \zeta; q)_\infty (q/\zeta; q)_\infty}{(q^b \zeta; q)_\infty (q^{\eta/\zeta}; q)_\infty} \frac{1}{2\pi i \zeta} d\zeta = 1. \tag{5.2}
\]

From now on we will assume that \( \hat{\mu}^{(b)} \) is a probability measure. Hence, from (5.7) it is not difficult to verify that

\[
\frac{-1}{2d_1} \frac{(1 - q^\eta)}{1 - q^\lambda \cos(\eta_q)} \frac{\mathcal{N}^{(b)}(\zeta-j)}{2 \phi_1(q, q^{-b+1}; q^b+1, q^b)} = \frac{\mathcal{L}^{(b-1, b+\eta - 2, b)}(\zeta-j+1)}{2 \phi_1(q, q^{-b+1}; q^b+1, q^b)}
\]

\[
= \int_T (1 - \zeta^{-1}) \zeta^{-j+1} d\hat{\mu}^{(b)}(\zeta), \quad j = 0, \pm 1, \pm 2, \ldots.
\]
Now let us consider the monic polynomials
\[ A_k(z) = \frac{\hat{\Phi}_k(b,z) - \tilde{\tau}_k^{(b)} \Phi_k^*(b,z)}{z - 1}, \quad k \geq 1, \]
where
\[ \tilde{\tau}_k^{(b)} = \Phi_k(b,1) - \Phi_k^*(b,1), \quad k \geq 0. \]

The polynomial \( A_k(z) \) is a constant multiple of \( K_k(b;z,1) \), where
\[ \bar{K}_k(b;z,w) = \sum_{j=0}^{k} \phi_j(b,w) \phi_j(b;z), \quad k \geq 0, \]
are the associated Christoffel-Darboux (or CD) kernels. From the orthogonality of \( \{\Phi_k(b;z)\}_{k \geq 0} \) it is easy to see that
\[ \int \zeta^{−k+j} A_k(\zeta)(1 - \zeta) d\bar{\mu}^{(b)}(\zeta) = 0, \quad 0 \leq j \leq k - 1. \]

Hence, from (5.2) we also have
\[ \mathcal{N}^{(b)}[\zeta^{−k+j} A_k(\zeta)] = 0, \quad 0 \leq j \leq k - 1. \]

Thus, by considering the determinant representation for the monic polynomials \( A_k(z) \) from the above orthogonality conditions with the determinant representation for the monic polynomials \( P_k(b;z) \) from (3.9), it follows that
\[ A_k(z) = P_k(b;z), \quad k \geq 0. \]

Moreover, comparing the three term recurrence formula in [5, Thm. 2.2] for the polynomials
\[ \frac{\prod_{j=0}^{k-1} [1 - \tilde{\tau}_j^{(b)} \alpha_j^{(b)}]}{\prod_{j=0}^{k} [1 - \text{Re}(\tilde{\tau}_j^{(b)} \alpha_j^{(b)})]} A_k(z), \quad k \geq 1, \]
with the three term recurrence formula (3.4), we have
\[ \frac{-\text{Im}[\tilde{\tau}_k^{(b)} \alpha_k^{(b)}]}{1 - \text{Re}(\tilde{\tau}_k^{(b)} \alpha_k^{(b)})} = c_k^{(b)}, \quad k \geq 1 \]
and
\[ (1 - g_k^{(b)}) g_{k+1}^{(b)} = d_{k+1}^{(b)}, \quad k \geq 1, \]
with
\[ g_k^{(b)} = \frac{1}{2} \frac{|1 - \tilde{\tau}_k^{(b)} \alpha_k^{(b)}|^2}{1 - \text{Re}(\tilde{\tau}_k^{(b)} \alpha_k^{(b)})}. \]

The sequence \( \{g_{k+1}^{(b)}\}_{k \geq 0} \) is also shown in [5] to be a parameter sequence for the positive chain sequence \( \{d_{k+1}^{(b)}\}_{k \geq 1} \). Moreover, since the measure \( \bar{\mu}^{(b)} \) does not have a pure mass point (or pure point) at \( z = 1 \), the sequence \( \{g_{k+1}^{(b)}\}_{k \geq 0} \) is the maximal parameter sequence of \( \{d_{k+1}^{(b)}\}_{k \geq 1} \). In what follows, we denote \( g_k^{(b)} = M_k^{(b)} \), for \( k \geq 1 \). Hence,
\[ M_k^{(b)} = \frac{1}{2} \frac{|1 - \bar{\alpha}_0^{(b)}|^2}{1 - \text{Re}[\bar{\alpha}_0^{(b)}]}. \]
To find an explicit expression for $M_1^{(b)}$ we need to evaluate $\tilde{\alpha}_0^{(b)}$. Observe that $\tilde{\alpha}_0^{(b)} = \tilde{\mu}_1^{(b)} = \int_T \zeta^{-1} d\tilde{\mu}^{(b)}(\zeta)$. Hence, from $\zeta^{-1} = 1 - (1 - \zeta^{-1})$, we obtain

$$\tilde{\alpha}_0^{(b)} = 1 - \int_T (1 - \zeta^{-1}) d\tilde{\mu}^{(b)}(\zeta) = 1 - \mathcal{L}^{(b-1,b+\bar{b}-2,\bar{b})}[1]/2\phi_1(q, q^{-b+1}; q, q^b).$$

Since $\mathcal{L}^{(b-1,b+\bar{b}-2,\bar{b})}[1] = 1$ we then have

$$\tilde{\alpha}_0^{(b)} = 1 - [2\phi_1(q, q^{-b+1}; q, q^b)]^{-1}.$$

From this and by observing from (5.1) that $(1 - q^\bar{b})^{-1} 2\phi_1(q, q^{-b+1}; q, q^b)$ is also positive, we obtain

$$M_1^{(b)} = \frac{1}{2} \frac{1 - q^{\bar{b}}}{1 - q^b \cos(\eta q)} \frac{1}{2\phi_1(q, q^{-b+1}; q, q^b)}.$$

**Theorem 5.2.** With $b = \lambda - i\eta$, $\eta_q = \eta \ln(q)$ and $\lambda > 0$ let $\{d_k^{(b)}\}_{k \geq 1}$ be the positive chain sequence given by (1.6). Then if $\{M_k^{(b)}\}_{k \geq 0}$ is the maximal parameter sequences of $\{d_k^{(b)}\}_{k \geq 1}$ then

$$M_{k+1}^{(b)} = \frac{1}{2} \frac{1 - q^{\bar{b}+k}}{1 - q^\lambda + k \cos(\eta q)} \frac{2\phi_1(q^k, q^{-b+1}; q^{\bar{b}+k}; q, q^b)}{2\phi_1(q^{k+1}, q^{-b+1}; q^{\bar{b}+k+1}; q, q^b)}, \quad k \geq 0.$$

Moreover, $\lim_{k \to \infty} M_{k+1}^{(b)} = 1/2$.

**Proof.** The value of $M_1^{(b)}$ is confirmed above. Now from the contiguous relation (2.4) we have

$$\frac{c_k^{(b-1,b+\bar{b}-2,\bar{b})}}{f_k^{(b-1,b+\bar{b}-2,\bar{b})}(1)} = 1 + c_k^{(b-1,b+\bar{b}-2,\bar{b})} - D_k^{(b-1,b+\bar{b}-2,\bar{b})} \frac{f_k^{(b-1,b+\bar{b}-2,\bar{b})}(1)}{c_k^{(b-1,b+\bar{b}-2,\bar{b})}}, \quad k \geq 1,$$

where

$$c_k^{(b-1,b+\bar{b}-2,\bar{b})} = \frac{1 - q^{\bar{b}+k-1}}{1 - q^b + k - 1}, \quad D_k^{(b-1,b+\bar{b}-2,\bar{b})} = \frac{(1 - q^k)(1 - q^{b+\bar{b}+k-1})}{(1 - q^{b+k-1})(1 - q^{\bar{b}+k})}, \quad k \geq 1,$$

and

$$f_k^{(b-1,b+\bar{b}-2,\bar{b})}(z) = \frac{2\phi_1(q^{k+1}, q^{-b+1}; q^{\bar{b}+k+1}; q, q^b z)}{2\phi_1(q^k, q^{-b+1}; q^{\bar{b}+k}; q, q^b z)}, \quad k \geq 1.$$

Thus, we have

$$\left(1 - \frac{c_k^{(b-1,b+\bar{b}-2,\bar{b})}}{1 + c_k^{(b-1,b+\bar{b}-2,\bar{b})} f_k^{(b-1,b+\bar{b}-2,\bar{b})}(1)}\right) \frac{c_k^{(b-1,b+\bar{b}-2,\bar{b})}}{(1 + c_k^{(b-1,b+\bar{b}-2,\bar{b})}) f_k^{(b-1,b+\bar{b}-2,\bar{b})}(1)} = \frac{D_k^{(b-1,b+\bar{b}-2,\bar{b})}}{(1 + c_k^{(b-1,b+\bar{b}-2,\bar{b})}) (1 + c_k^{(b-1,b+\bar{b}-2,\bar{b})})}.$$
Hence, observing that

\[
\mathcal{D}_{k+1}^{(b-1,b+b-2,b)} = \Phi_{k+1} \left( \frac{1}{c_k^{(b-1,b+b-2,b)}}(1 + c_k^{(b-1,b+b-2,b)}) \right) = d_{k+1}^{(b)}, \quad k \geq 1,
\]

and

\[
\mathcal{C}_{k+1}^{(b-1,b+b-2,b)} = \Phi_{k+1} \left( \frac{1}{c_k^{(b-1,b+b-2,b)}} f_k^{(b-1,b+b-2,b)}(1) \right) = \frac{1}{2} \frac{1 - q^{\frac{1}{b}}}{1 - q^{\frac{1}{b}+k}} \cos(\eta_q) 2 \phi_1(q^k, q^{-b+1}; q^{b+k}; q, q^b) = 2 \phi_1(q^{k+1}, q^{-b+1}; q^{b+k+1}; q, q^b),
\]

for \( k \geq 0 \), we obtain \((1 - M_k^{(b)})M_{k+1}^{(b)} = d_{k+1}^{(b)}, k \geq 1 \), which gives the expression for \( M_{k+1}^{(b)} \).

Now to obtain the asymptotic for \( M_{k+1}^{(b)} \) we use the same argument that we used to obtain the asymptotic for \( \ell_{k+1}^{(b)} \) in Theorem 1.2.

From results given in [6], the sequence \( \{Q_k(b; 1)/[2d_1 R_k(b; 1)]\} \) is a positive and increasing sequence, and that

\[
\lim_{k \to \infty} \frac{Q_k(b; 1)}{R_k(b; 1)} = \frac{2d_1}{2M_1^{(b)}},
\]

where \( \{M_{k+1}^{(b)}\}_{k \geq 0} \) is the maximal parameter sequence of the positive chain sequence \( \{d_{k+1}^{(b)}\}_{k \geq 1} \). Thus, from (3.10), by continuity

\[
\frac{1 - q^{\frac{1}{b}} \cos(\eta_q)}{1 - q^{\frac{1}{b}}} 2 \phi_1(q, q^{-b+1}; q^{b+k}; q, q^b) = \frac{1}{2M_1^{(b)}},
\]

which again confirms (5.3).

With the choice \( d_1 = M_1^{(b)} \), we also have from (5.2)

\[
\mathcal{N}^{(b)}[\zeta] = \int_{\mathbb{T}} (1 - \zeta) \zeta^{-j} d\bar{\mu}^{(b)}(\zeta), \quad j = 0, \pm 1, \pm 2, \ldots.
\]

Now, with the three term recurrence formula (5.1) and the associated moment functional \( \mathcal{N}^{(b)} \), application of the results given in [6] and [8] gives the following.

For \( 0 \leq t < 1 \), let \( \bar{\mu}^{(b)}(t; .) \) be the probability measure given by \((1 - t)\bar{\mu}^{(b)}(.) + t\delta_1 \). Let \( \{m_k^{(b,t)}\}_{k \geq 0} \) be the minimal parameter sequence of the positive chain sequence \( \{d_1, d_2^{(b)}, d_3^{(b)}, d_4^{(b)}, \ldots\} \), where \( d_1 = (1-t)M_1^{(b)} \), then the monic OPUC \( \Phi_k(b, t; z) \) associated with \( \bar{\mu}^{(b)}(t; .) \) are given by

\[
\Phi_k^{(b)}(t; z) = \frac{1}{\prod_{j=1}^k (1 + ic_j^{(b)})} [R_k(b; z) - 2(1 - m_k^{(b,t)})R_{k-1}(b; z)], \quad k \geq 1.
\]

**Proof of Theorem 1.2.** To obtain the first part of Theorem 1.2 we simply let in the above formula \( t = 0 \) and then use (3.5). The result for the associated Verblunsky coefficients is obtained with the substitution \( z = 0 \). To obtain the last part of Theorem 1.2 observe that

\[
1 - |c_{k-1}^{(b)}|^2 = \frac{4M_k^{(b)}(1 - M_{k+1}^{(b)})}{1 + (c_k^{(b)})^2}, \quad k \geq 1.
\]

Hence, the result follows from \( (k_{k+1}^{(b)})^{-2} = \prod_{j=1}^k (1 - |c_{k-1}^{(b)}|^2) \) and \((1 - M_k^{(b)})M_{k+1}^{(b)} = d_{k+1}^{(b)} \) for \( k \geq 1 \).
Now from
\[
\Phi_k^*(b; z) = \frac{(q^\lambda \cos(\eta q); q)_k}{(q^n; q)_k} [R_k(b; z) - 2(1 - M^{(b)}_k) z R_{k-1}(b; z)], \quad k \geq 1,
\]
we find \( \lim_{k \to \infty} \Phi_k^*(b; z) = (q^b z; q)_\infty/(qz; q)_\infty \), uniformly on compact subsets of \(|z| < 1\).

Thus, by considering the limit (see [20, p. 144]) of \( \check{\kappa}_k \Phi_k^*(b; z) \) we can state the following.

**Theorem 5.3.** If \( \check{D}(z) \) is the Szegő function associated with the probability measure \( \check{\mu}^{(b)} \) given by (1.4) and (5.1) then
\[
\check{D}(z) = \sqrt{2 M^{(b)}_1(q; q)_\infty(q^{2\lambda}; q)_\infty(1 - q^\lambda \cos(\eta q))} \frac{(q^b z; q)_\infty}{(q^b; q)_\infty}.
\]

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