Development of the analytic relations for the propellant grain geometrical characteristics required for a maximum pressure plateau feature

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Abstract. By using the fundamental interior ballistics relations, developed based on some simplification hypothesis, there was deduced a formula for the geometrical characteristics that allow the occurrence of a constant pressure plateau in a gun when a charge composed by a single type of underdeted propellant grain is used. The equation incorporates various factors as propellant ballistics properties, the gun/projectile/propellant gaboritical data and the targeted maximum pressure value. Nevertheless, the calculus done on a realistic set of data indicates the impossibility to reach a constant pressure plateau when a propellant characterised by a single set of geometrical characteristics over the entire web thickness is use.

1. Introduction
The internal ballistics community was always driven by the idea of an ideal ballistic cycle, with a flat pressure curve along the barrel length. Most of the improvements obtained along the time, related to the above mention goal, are related to the shape of grains (i.e. multiple channels) and to the specific burning rate (i.e. deterrent treatment).

Some notable results were obtained when layered propellant, characterized by the presence of an interior faster burning layer framed by two slow burning layers, was used [1]. The specific pressure time curve can be altered in this way to a more unusual shape of double hump [2]. This means a higher piezometric efficiency of the gun propulsion system. Still, the manufacture procedure of co-layered propellant proves to be time-consuming and more-over a difficult task [3].

A more attractive manufacturing technology was developed during the last years and consists in co-extrusion of co-layered propellants using a special die design [3]. The co-extrusion technology enables the achieving of some classical grain shapes, like single channel or multi-channel, and retains the ability to drastically change both evolutions, of pressure and temperature in a gun system [3]. The main disadvantage of this technology is the inability to obtained more complicated grain shapes than the classical ones.

In last years a more recently technology caught the ballistics community attention: the 3D printing of energetic materials and its applications [4]. The advantage of complex spatial design offered by 3D printing compared to old technologies, beside the control of the performance and the safety properties, is the ability to spatially combine two or more energetic materials in the most imaginative ways, that leads to a new or an optimized behaviour. The 3D printing technology is the ideal candidate to approach simultaneous some of the issues mentioned above: the optimization of the ballistic cycle, the
optimization of space arrangement and the utilization of new energetic materials like high explosives of the last generation.

Numerical calculus done by Dolman [5] for some undisclosed propellant geometries attainable through 3D printing technology indicates the real possibility to obtain a constant pressure plateau in a gun barrel. Also, the TNO scientists have done experimental tests with propellant obtained through 3D printing [6].

Despite the interest shown by community on the new technologies able to optimize the ballistic cycle there are no open literature references focused on the propellant geometrical characteristics values mandatory to get such features as a region of constant pressure at the maximum pressure value.

In followings we will show that using the fundamental interior ballistics relations, developed based on some simplification hypothesis, for a charge of a single type of undeterred propellant grain, there is possible to deduce a formula for such geometrical characteristics, that incorporates various factors as the propellant ballistics properties, the gun/projectile/propellant gaboritical data and the targeted maximum pressure value.

Also, was devised an algorithm for determination of such geometrical characteristics when the values of maximum pressure and the positions of the projectile in the bore at the beginning and the ending of such maximum pressure plateau are set. The application of the algorithm for one realistic gun system shows that there is not possible to reach a constant pressure plateau when a propellant characterized by a single set of geometrical characteristics ($\chi, \lambda$) over the entire web thickness is used.

2. Fundamental relations used in the interior ballistic calculus

2.1. The interior ballistic fundamental equation

The fundamental interior ballistic equation represent the energetic balance equation that express the interdependence between the mean pressure $p$, projectile velocity $v$ and space travelled by projectile $l$.

When the hypothesis of the instantaneous projectile forcing into the barrel is used, the resistance of the air displaced and the primer contribution are neglected and the heat losses are take in account through the modification of the propellant impetus or the adiabatic coefficient, the fundamental interior ballistic equation became

$$pW = f\omega\psi - \frac{\gamma^{-1}\varphi v^2}{2},$$

where

$$W = W_0 - \frac{\omega}{\delta}(1 - \psi) - \alpha\omega\psi + sl,$$  \hspace{1cm} (2)

$f$ is the propellant impetus, $\alpha$ the propellant covolume, $\gamma$ the adiabatic coefficient of burned gases, $\omega$ the total mass of propellant, $\psi$ the volume fraction of burned propellant, $q$ the projectile mass, $q$ the supra-unitary coefficient used to taking account the secondary losses proportional to the projectile kinetic energy $\frac{qv^2}{2}$, $W_0$ the initial volume of cartridge, $s$ the barrel cross section area, $l$ the space travelled by projectile inside of the barrel and $\delta$ the solid propellant density.

2.2. The burning rate law

The burning rate law gives the link between burning rate and pressure. There are several empirical relations proposed for this dependency but the most used is the Saint Robert's expression

$$u = Ap^v$$

where $A$ and $v$ are experimental coefficients depending on the propellant constituents and the burning conditions.
2.3. The propellant gases rate generation law

Propellant gases generation law expresses the dependency between the volume fraction of burned propellant $\psi$

$$\psi = \frac{\bar{\omega}}{\bar{\omega}}$$

and the zero dimensional relative burned propellant thickness $z$

$$z = \frac{\bar{e}}{\bar{e}_1},$$

where $e$ is the burned propellant thickness (space travelled by burning front) and $2\bar{e}_1$ is the web thickness.

Usually the law is given by a third degree polynomial expression [7]

$$\psi = \chi z (1 + \lambda z + \mu z^2),$$

where $\chi, \lambda$ and $\mu$ are the shape coefficients of the propellant grain.

2.4. The influence of grain geometry on the gases rate generation

The propellant gases rate generation is the variation of the volume fraction of burned propellant in the unit of time and is given by [7]

$$\frac{d\psi}{dt} = \frac{S_0}{A_0} \frac{S}{S_0} A_p^\psi,$$

where $S_0$ is the initial burning surface of the propellant grain, $S$ is the current burning surface of the propellant grain and $A_0$ is the initial volume of the propellant grain.

The term $\frac{S}{S_0}$ is the progressivivity coefficient of the propellant shape and is noted as $\sigma$.

Knowing that $\frac{dv}{dz} = \frac{d\psi}{dt} \frac{dz}{dv}$ and using the mathematical relations (3) and (5) can be demonstrated the equality

$$\frac{S_0}{A_0} e_1 = \chi,$$

that gives us

$$\sigma = 1 + 2\lambda z + 3\mu z^2.$$

2.5. The projectile translation equation

The projectile is pushed along the barrel by the force resulting from the pressure acting on the bottom of the projectile. When the projectile acceleration is expressed based on the current mean pressure value the following approximation expression can be used

$$sp = \varphi q \frac{dv}{dt},$$

where the fictive mass coefficient $\varphi$ can be expressed in terms of $\frac{\bar{\omega}}{q}$ ratio. As an example, for the artillery fired guns of medium caliber can be used the empirical relation proposed by Sluhoțki [8]

$$\varphi = 1.04 + \frac{1}{3} \frac{\bar{\omega}}{q}. $$
3. Deduction of the geometrical characteristics required for a constant pressure plateau

As long as we admit that after the \( p_{\text{max}} \) is reached the pressure remain constant for a certain period of time, the equality (1) is keep even when we apply the time derivate. For the same reason, the derivate \( \frac{dp}{dt} \) is nule for this period of time. Knowing the available volume expression (2), the derivate of (1) become

\[
p_{\text{max}} \left( Sv + \frac{\omega d\psi}{dt} - \alpha \omega \frac{d\psi}{dt} \right) = f \omega \frac{d\psi}{dt} - \theta \psi q v \frac{dv}{dt}.
\]  

(12)

In the same time the acceleration suffered by projectile based on relation (10) is

\[
\frac{dv}{dt} = \frac{sp_{\text{max}}}{\psi q}.
\]  

(13)

We can introduce (13) in (12) and we get

\[
p_{\text{max}} Sv (1 + \theta) = \omega \frac{d\psi}{dt} \left[ f + p_{\text{max}} \left( \alpha - \frac{1}{\delta} \right) \right].
\]  

(14)

The \( \omega \frac{d\psi}{dt} \) product represents the burning gases mass rate and based on (7) we have

\[
p_{\text{max}} Sv (1 + \theta) = \frac{S \sigma}{\lambda q} Ap_{\text{max}}^v \left[ f + p_{\text{max}} \left( \alpha - \frac{1}{\delta} \right) \right].
\]  

(15)

We may rearrange the equation (15) as

\[
\frac{v}{\sigma} = \frac{S \sigma}{\lambda \lambda q} Ap_{\text{max}}^v \left[ f + p_{\text{max}} \left( \alpha - \frac{1}{\delta} \right) \right] = \text{cst}.
\]  

(16)

As long as all parameters from the right member are constants, the ratio \( \frac{v}{\sigma} \) is constant while pressure remains constant.

Let assume that the \( p_{\text{max}} \) corresponds to the value \( z_{p_{\text{max}}} \) and that the pressure remain constant till the burning ends, respective when \( z = 1 \). Let us define \( v_{p_{\text{max}}} \) and \( v_k \) as the projectile velocities at the begining respective at the ending of such period of constant pressure. The values of \( \sigma \) for those two moments of time are given by the relation (9). We have then the equality

\[
\frac{v_{p_{\text{max}}}}{1 + 2 \lambda x_{p_{\text{max}}}^2 + 3 \mu_c x_{p_{\text{max}}}^2} = \frac{v_k}{1 + 2 \lambda x_c + 3 \mu_c}.
\]  

(17)

It is obvious that for a constant pressure the projectile velocity evolution in time is a linear one, see the (13) equation. That means a linear evolution of \( \sigma \) too. But in the same condition the burning velocity is constant, which lead to a linear evolution in time of \( z \) as long as, for that period of time, we have

\[
\frac{dz}{dt} = \frac{1}{e_1} \frac{de}{dt} = \frac{1}{e_1} u = \frac{1}{e_1} Ap_{\text{max}}^v.
\]  

(18)

Linear evolution in time for both \( \sigma \) and \( z \) parameters may be posible only if the geometrical characteristic \( \mu_c \) is nule. Being so, the (17) equality become

\[
\frac{v_{p_{\text{max}}}}{1 + 2 \lambda x_{p_{\text{max}}}^2} = \frac{v_k}{1 + 2 \lambda x_c}.
\]  

(19)
Let us assume that above mentioned period of time has the finite length $\Delta t = t_k - t_{p_{\text{max}}}$ Then the $v_k$ is

$$v_k = v_{p_{\text{max}}} + \frac{s p_{\text{max}}}{\varphi q} \Delta t.$$  

(20)

For the same period of time the burning front travels the distance

$$e_1 (1 - z_{p_{\text{max}}}) = \Delta t p_{\text{max}}^v.$$  

(21)

Then we may replace $\Delta t$ in expression (20) with equivalent expression given by (21) and we get

$$v_k = v_{p_{\text{max}}} + \frac{s p_{\text{max}} e_1 (1 - z_{p_{\text{max}}})}{\varphi q} p_{\text{max}}^v.$$  

(22)

Let us return to (19) which become

$$\frac{v_{p_{\text{max}}}}{1 + 2 \lambda_c z_{p_{\text{max}}}} = \frac{v_{p_{\text{max}}} + s p_{\text{max}} e_1 (1 - z_{p_{\text{max}}})}{\varphi q} p_{\text{max}}^v.$$  

Making the notation

$$B = \frac{S_0}{\Lambda_0} \frac{\omega A p_{\text{max}}^v}{s (1 + \theta) p_{\text{max}}} \left[ f + p_{\text{max}} \left( \alpha - \frac{1}{\delta} \right) \right] = c s t.,$$  

(24)

we have

$$\left( 1 + 2 \lambda_c \right) B = \left( 1 + 2 \lambda_c z_{p_{\text{max}}} \right) B + \frac{s p_{\text{max}} e_1 (1 - z_{p_{\text{max}}})}{\varphi q} p_{\text{max}}^v$$  

(25)

which become

$$2 \lambda_c B (1 - z_{p_{\text{max}}}) = \frac{s p_{\text{max}} e_1 (1 - z_{p_{\text{max}}})}{\varphi q}.$$  

(26)

respective

$$\lambda_c = \frac{(1 + \theta) e_1 \Lambda_0}{2 \varphi q} \frac{s p_{\text{max}}^2}{A p_{\text{max}}^v} \left[ f + p_{\text{max}} \left( \alpha - \frac{1}{\delta} \right) \right]^{-1}.$$  

(27)

or

$$\chi_s \lambda_c = \frac{\gamma}{2 \varphi q} \frac{e_1^2}{\omega} \left( \frac{s p_{\text{max}} (1 - \gamma)}{A} \right)^2 \left[ f + p_{\text{max}} \left( \alpha - \frac{1}{\delta} \right) \right]^{-1}.$$  

(28)

The above relation give us the required value of $\chi_s \lambda_c$ product in order to reach a plateau of $p_{\text{max}}$ value when the $\omega$ quantity of propellant characterised by the impetus $f$, the covolume $\alpha$, the adiabatic coefficient $\gamma$ and the density $\delta$ that burn over the half of the web thickness $2e_1$ with a velocity dependent on pressure, based on $A p^v$ relation, is used for the propulsion of a projectile of mass $q$ through a barrel of section $s$ and where the coefficient of fictive mass that take in account the
secondary losses is $\varphi$. Still, the value of $\chi_c \lambda_c$ given by relation (28) do not depend on charge chamber volume $W_0$ or forcing pressure $p_0$.

As long as $\chi_c + \chi_c \lambda_c = 1$ we have

$$\chi_c = 1 - \frac{\gamma \varepsilon_1^2}{2q_0 A} \left( \frac{\varepsilon_1}{\varepsilon_1 + \gamma} \right)^2 \left[ f + p_{\text{max}} \left( \alpha - \frac{1}{\beta} \right) \right]^{-1}$$

and

$$\lambda_c = \frac{\gamma \varepsilon_1^2}{2q_0 A} \left( \frac{\varepsilon_1}{\varepsilon_1 + \gamma} \right)^2 \left[ f + p_{\text{max}} \left( \alpha - \frac{1}{\beta} \right) \right]^{-1}.$$  


(29) and (30)

Beside the expression that defines the geometrical characteristics, the constant pressure gives additional limitation on the geometrical characteristic values. The constant pressure may be attained only if the shape of the element is a progressive one till the end of the burning. That means $\lambda_c \geq 0$ or $\chi_c \leq 1$.

As long as the $\chi_c > 0$, the condition for an initial burning surface greater than 0, the possible intervals for these two characteristics are $[0,1]$ and $[0,\infty)$. This limitation imply the following inequality

$$0 \leq \frac{\varepsilon_1^2}{\omega} < \frac{2q_0 A}{\gamma (\varepsilon_1 + \gamma)} \left[ f + p_{\text{max}} \left( \alpha - \frac{1}{\beta} \right) \right].$$

Further, if we integrate (16) over the time interval $\Delta t$ we get

$$\int_{t_{p_{\text{max}}}}^{t_k} vdt = \int_{t_{p_{\text{max}}}}^{t_k} B\sigma dt.$$  

(32)

We may change the variable of the right member from $t$ to $z$

$$\int_{t_{p_{\text{max}}}}^{t_k} B\sigma dt = \int_{t_{p_{\text{max}}}}^{t_k} B \frac{\varepsilon_1}{A P_{\text{max}}^2} (1 + 2\lambda z)dz$$

(33)

and after the integrals solving we have

$$W_k - W_{p_{\text{max}}} = \frac{\omega}{(1+\theta)}p_{\text{max}} \left( 1 - \psi_{p_{\text{max}}} \right),$$

(34)

where $W_{p_{\text{max}}}$ and $W_k$ are the volumes behind the projectile at times $t_{p_{\text{max}}}$ and $t_k$ and $\psi_{p_{\text{max}}}$ the burned propellant fraction at $t_{p_{\text{max}}}$.

As long as $W_k - W_{p_{\text{max}}} = s (l_k - l_{p_{\text{max}}})$ we get

$$\omega (1 - \psi_{p_{\text{max}}}) = \frac{s p_{\text{max}} (1+\theta) (l_k - l_{p_{\text{max}}})}{[f + p_{\text{max}} (\alpha - \frac{1}{\beta})]}.$$  

(35)

The above relation show that the fraction of propellant that burns after the reaching of the $P_{\text{max}}$ value depends only on the length $l_k - l_{p_{\text{max}}}$.

Starting from the

$$l_k - l_{p_{\text{max}}} = v_{p_{\text{max}}} \Delta t + \frac{s p_{\text{max}}}{2q_0} \Delta t^2,$$  

(36)
and relation (21) we get

$$\Delta e_{p_{\max}} = e_1(1 - z_{p_{\max}}) = \frac{-v_{p_{\max}} + \sqrt{v_{p_{\max}}^2 + \frac{2\partial p_{\max}}{\eta} (l_{p_{\max}} - l_{p_{max}})}}{\frac{\partial p_{\max}}{\eta}} A p_{p_{\max}}^\gamma.$$

(37)

Beside the dependency on the length $(l_k - l_{p_{\max}})$ the value of $\Delta e_{p_{\max}}$ is influenced by the value of $v_{p_{\max}}$.

Further we consider a known value for the $v_{p_{\max}}$. Writing the fundamental equation of interior ballistics for the $p_{\max}$ reaching moment we have

$$p_{\max} \left[ W_0 - \frac{\omega}{\delta} (1 - \psi_{p_{\max}}) - \alpha \omega \psi_{p_{\max}} + s l_{p_{\max}} \right] = f \omega \psi_{p_{\max}} - \frac{\theta \varphi v_{p_{\max}}^2}{2}.$$  

(38)

We may extract $\psi_{p_{\max}}$ from (35) and apply in above equation. In result we have an expression for $\omega$ that depend on $v_{p_{\max}}^2$

$$\omega = \frac{\theta \varphi v_{p_{\max}}^2 + p_{\max} (W_0 + s l_k) + \theta p_{\max} s (l_k - l_{p_{\max}})}{f + \alpha p_{\max}}.$$  

(39)

Thus we can calculate the $\psi_{p_{\max}}$ and $\psi_{p_0}$ as

$$\psi_{p_{\max}} = 1 - \frac{(1 + \theta) p_{\max} s (l_k - l_{p_{\max}}) (f + \alpha p_{\max})}{[\frac{\theta \varphi v_{p_{\max}}^2 + p_{\max} (W_0 + s l_k) + \theta p_{\max} s (l_k - l_{p_{\max}})}{f + \alpha p_{\max}}]^{1/\delta}}.$$  

(40)

$$\psi_{p_0} = \frac{\theta \varphi v_{p_{max}}^2 + p_{\max} (W_0 + s l_k) + \theta p_{max} s (l_k - l_{p_{\max}})}{f_{p_0} + \alpha - \frac{1}{\delta}}.$$  

(41)

Assuming a known value of $e_1 (z_{p_{\max}} - z_{p_0}) = \Delta e_{p_{\text{var}}}$, the product $e_1 (1 - z_{p_0}) = \Delta e_{p_0}$ is

$$\Delta e_{p_0} = \frac{-v_{p_{\max}} + \sqrt{v_{p_{\max}}^2 + \frac{2\partial p_{\max}}{\eta} (l_{p_{\max}} - l_{p_{max}})}}{\frac{\partial p_{\max}}{\eta}} A p_{p_{\max}}^\gamma + \Delta e_{p_{\text{var}}}.$$  

(42)

We may now write a system of three $\psi(z)$ equations, for the three specific moments: start of the projectile, the reach of maximum pressure and the end of the propellant burning,

$$\begin{align*}
\chi(e_1 - \Delta e_{p_0}) (e_1 + \lambda (e_1 - \Delta e_{p_0})) &= \psi_{p_0} e_1^2 \\
\chi(e_1 - \Delta e_{p_{\max}}) (e_1 + \lambda (e_1 - \Delta e_{p_{\max}})) &= \psi_{p_{\max}} e_1^2 \\
\chi(1 + \lambda) &= 1
\end{align*}$$  

(43)

where $\chi$, $\lambda$ and $e_1$ are the unknown values.

Using the above system we may express the geometrical characteristics as

$$\chi = \frac{\psi_{p_{\max}} (e_1 - \Delta e_{p_0})^2 - \psi_{p_0} (e_1 - \Delta e_{p_{\max}})^2}{\psi_{p_0} (e_1 - \Delta e_{p_{\max}}) \Delta e_{p_{\max}} - \psi_{p_{\max}} (e_1 - \Delta e_{p_0}) \Delta e_{p_0}}.$$  

(44)
The above equations are used to calculate the same geometrical characteristics as in (29) and (30) formula.

After \( \chi \) and \( \lambda \) replacement in the first equation of the (43) system we get the quadratic equation

\[
ae_1^2 + be_1 + c = 0
\]

with the solutions

\[
e_{11} = \frac{-b + \sqrt{b^2 - 4ac}}{2a}
\]

and

\[
e_{12} = \frac{-b - \sqrt{b^2 - 4ac}}{2a},
\]

where

\[
a = \Delta e_{p_{\text{max}}} (1 - \psi_{p_0}) - \Delta e_{p_0} (1 - \psi_{p_{\text{max}}}),
\]

\[
b = -\Delta e_{p_{\text{max}}}^2 (1 - \psi_{p_0}) - \Delta e_{p_0}^2 (1 - \psi_{p_{\text{max}}}),
\]

\[
c = \Delta e_{p_{\text{max}}} (\Delta e_{p_0} - \Delta e_{p_{\text{max}}}).
\]

In order to establish which of the two solutions has a physical meaning we firstly assume that the pair \((e_1, \chi, \lambda)\) is the solution of the system (43). Then, knowing that \(e_1 (1 - z_{p_0}) = \Delta e_{p_0}\) and \(e_1 (1 - z_{p_{\text{max}}}) = \Delta e_{p_{\text{max}}},\) we have

\[
\frac{a}{e_1} = (1 - z_{p_{\text{max}}}) (1 - \psi_{p_0}) - (1 - z_{p_0}) (1 - \psi_{p_{\text{max}}}),
\]

\[
\frac{b}{e_1} = -[(1 - z_{p_{\text{max}}})^2 (1 - \psi_{p_0}) - (1 - z_{p_0})^2 (1 - \psi_{p_{\text{max}}})],
\]

\[
\frac{c}{e_1} = (1 - z_{p_{\text{max}}}) (1 - z_{p_0}) (z_{p_0} - z_{p_{\text{max}}}).
\]

If we express the \(\psi_{p_0}\) and \(\psi_{p_{\text{max}}}\) as functions of \(z_{p_0}\) respectively \(z_{p_{\text{max}}}\) we get

\[
\frac{a}{e_1} = (1 - z_{p_{\text{max}}}) (1 - z_{p_0}) (z_{p_{\text{max}}} - z_{p_0}) (\chi - 1),
\]

\[
\frac{b}{e_1} = (1 - z_{p_{\text{max}}}) (1 - z_{p_0}) (z_{p_{\text{max}}} - z_{p_0}) (2 - \chi).
\]

When we introduce relations (51) – (53) in (47) we get

\[
e_1 = \frac{e_1^2 (1 - z_{p_{\text{max}}}) (1 - z_{p_0}) (z_{p_{\text{max}}} - z_{p_0}) (\chi - 2 + \sqrt{(2 - \chi)^2 + 4 (\chi - 1)})}{2 e_1 (1 - z_{p_{\text{max}}}) (1 - z_{p_0}) (z_{p_{\text{max}}} - z_{p_0}) (\chi - 1)},
\]

that is ultimately reduced to \(1 = 1\) and when we introduce the same relations in (48) we get

\[
e_1 = \frac{e_1^2 (1 - z_{p_{\text{max}}}) (1 - z_{p_0}) (z_{p_{\text{max}}} - z_{p_0}) (\chi - 2 - \sqrt{(2 - \chi)^2 + 4 (\chi - 1)})}{2 e_1 (1 - z_{p_{\text{max}}}) (1 - z_{p_0}) (z_{p_{\text{max}}} - z_{p_0}) (\chi - 1)},
\]
that is ultimately reduced to \(1 = \frac{1}{1-\chi}\), a relation that is true only when \(\chi = 0\), a value outside of the possible values interval.

Also, the physically relevant solution given by (47) must be positive and greater than \(\Delta e_{p_{v_{a}}r}\).

If we take an existing gun/projectile system characterised by the propellant mass, the propellant chamber volume, the pressure necessary to force projectile into the barrel, the determination of the propellant quantity, the web thickness and the geometrical characteristics necessary to achieve constant pressure when the projectile pass the interval \([l_{p_{m_{a}}x}, l_{k}]\) into the barrel, using a propellant of known ballistic properties, is reduced to the determination of the \((v_{p_{m_{a}}x}, \Delta e_{p_{v_{a}}r})\) pair that produce identical or very close results for the booth equations systems (29)-(30) and (44)-(45).

![Figure 1. The extreme limits of plausible pressure curves.](image)

The plausible values of \(v_{p_{m_{a}}x}\) and \(\Delta e_{p_{v_{a}}r}\) define the 2D space where the solution may be. For the given gun/amunition system the minimum and maximum values of \(v_{p_{m_{a}}x}\) and \(\Delta e_{p_{v_{a}}r}\) intervals are given by the two extrem/virtual evolutions of pressure on the \([0,l_{p_{m_{a}}x}]\) interval, figure 1:

- the minimum values are given when the pressure increases in a linear manner from \(p_{0}\) at 0 to \(p_{m_{a}}x\) at \(l_{p_{m_{a}}x}\) (case I);
- the maximum values are given when the pressure suffers a sudden increase from \(p_{0}\) to \(p_{m_{a}}x\) right at the beginning of the movement and then remain constant (this case correspond to a constant pressure on the entire \([0,l_{k}]\) interval) (case II).

These four limit values are then

\[
\begin{align*}
  v_{p_{m_{a}}x_{\text{min}}} &= \frac{2(l_{p_{m_{a}}x} (p_{m_{a}}x + p_{0}))}{q q} \\
  v_{p_{m_{a}}x_{\text{max}}} &= \frac{2(l_{p_{m_{a}}x} p_{m_{a}}x)}{q p_{m_{a}}x} \\
  \Delta e_{p_{v_{a}}r_{\text{min}}} &= \int_{0}^{p_{m_{a}}x} \frac{A (l_{p_{m_{a}}x} (p_{m_{a}}x - p_{0}) + p_{0})^\gamma}{q q} \frac{d l}{2(l_{p_{m_{a}}x} (p_{m_{a}}x - p_{0}) + p_{0})^\gamma} \\
  \Delta e_{p_{v_{a}}r_{\text{max}}} &= \int_{0}^{p_{m_{a}}x} \frac{A p_{m_{a}}x^\gamma}{q q} \frac{d l}{2(l_{p_{m_{a}}x} p_{m_{a}}x)} = A p_{m_{a}}x^\gamma \frac{2(q q) l_{p_{m_{a}}x}}{2 p_{m_{a}}x}
\end{align*}
\]

(56)  \(57)  \(58)  \(59)

4. Algorithm application on a realistic gun system

A search algorithm based on the generation of \((v_{p_{m_{a}}x}, \Delta e_{p_{v_{a}}r})\) pairs by uniform dividing of \([v_{p_{m_{a}}x_{\text{min}}}, v_{p_{m_{a}}x_{\text{max}}}]\) and \([\Delta e_{p_{v_{a}}r_{\text{min}}}, \Delta e_{p_{v_{a}}r_{\text{max}}}]\) intervals was applied on a 76 mm calibre gun/projectile system of constructive characteristics given in table 1. Instead of the projectile mass \(q\)
we define the fictive mass $\varphi q$ that allow us to eliminate some iterative steps necessary to estimate $\varphi$ as long as $\varphi$ depends on the unknown propellant mass $\omega$. In the same table the used propellant characteristics are provided.

**Table 1.** Gun, projectile and propellant characteristics.

| W₀ [m$^3$] | s [m$^2$] | $\varphi q$ [kg] | p₀ [Pa] | $\delta$ [kg/m$^3$] | f [J/kg] | $\alpha$ [m$^3$/kg] | A [m/s/Pa$^2$] | v [N/A] |
|------------|----------|------------------|--------|---------------------|---------|---------------------|------------|--------|
| 1.391·10$^{-3}$ | 4.35·10$^{-3}$ | 6.677 | 5·10$^7$ | 1650 | 93500 | 10$^{-3}$ | 1.1·10$^{-7}$ | 0.724 |

The search was done for several assumed values of pressure $p_{max}$ and lengths $l_k$ and $l_{p_{max}}$ as in table 2. The values were chosen in order to vary zero dimensional ratios $s/W_0$, $l_k/l_{p_{max}}$ and $p_{max}/p_0$ on wide ranges. For each studied case two strings of 101 points length were define for $v_{p_{max}}$ and $\Delta e_{p_{max}}$. The pairs generated in this way were used to calculate $\omega$, $\psi_{p_{max}}$, $\psi_{p_0}$, $\Delta e_{p_{max}}$, $\Delta e_{p_0}$ and $e_1$ with formula (47), values used then to calculate geometrical characteristics $\chi$ and $\lambda$ with formulas (29)-(30), respectively (44)-(45). For each studied case only the minimum and the maximum values for $\chi$ and $\chi_c$ were shown in table 2. When the $\Delta e_{p_0} > e_1$ inequality occurs the calculus of geometrical characteristics is skipped.

**Table 2.** The studied cases and $\chi$ and $\chi_c$ calculated intervals limits.

| No. | $p_{max}$ [Pa] | $l_{p_{max}}$ [m] | $l_k$ [m] | $\chi_{min}$ | $\chi_{max}$ | $\chi_{min}$ | $\chi_{max}$ |
|-----|----------------|------------------|----------|--------------|--------------|--------------|--------------|
| 1   | 10$^{8}$       | 1.99             | 2        | 2.375        | 2.451        | -1.152       | -0.811       |
| 2   | 10$^{8}$       | 0.199            | 0.2      | 2.324        | 2.387        | -0.334       | -0.144       |
| 3   | 10$^{8}$       | 0.0199           | 0.02     | 2.279        | 2.330        | 0.717        | 0.751        |
| 4   | 10$^{8}$       | 1.019            | 0.2      | 5.786        | 6.496        | -30.755      | -23.079      |
| 5   | 10$^{8}$       | 0.19             | 0.2      | 5.075        | 5.616        | -16.034      | -12.218      |
| 6   | 10$^{8}$       | 0.019            | 0.02     | 4.480        | 4.907        | -2.123       | -1.482       |
| 7   | 10$^{8}$       | 1.01             | 0.2      | 34.473       | 38.362       | -2.684·10$^4$ | -2.170·10$^4$ |
| 8   | 10$^{8}$       | 0.01             | 0.02     | 34.855       | 39.100       | -1.333·10$^4$ | -1.089·10$^4$ |
| 9   | 10$^{8}$       | 0.01             | 0.02     | 39.696       | 35.170       | -220.184     | -181.154     |
| 10  | 2.5·10$^8$     | 1.99             | 2        | $\Delta e_{p_0} > e_1$ |              |              |              |
| 11  | 2.5·10$^8$     | 0.199            | 0.2      | 2.476        | 2.644        | -0.580       | -0.194       |
| 12  | 2.5·10$^8$     | 0.0199           | 0.02     | 2.455        | 2.612        | 0.724        | 0.785        |
| 13  | 2.5·10$^8$     | 1.019            | 0.2      | $\Delta e_{p_0} > e_1$ |              |              |              |
| 14  | 2.5·10$^8$     | 0.19             | 0.2      | 6.706        | 8.280        | -30.700      | -18.935      |
| 15  | 2.5·10$^8$     | 0.019            | 0.02     | 6.438        | 7.913        | -4.252       | -2.398       |
| 16  | 2.5·10$^8$     | 1.01             | 0.2      | $\Delta e_{p_0} > e_1$ |              |              |              |
| 17  | 2.5·10$^8$     | 0.01             | 0.02     | 56.776       | 69.420       | -2.714·10$^3$ | -1.914·10$^3$ |
| 18  | 2.5·10$^8$     | 0.01             | 0.02     | 67.545       | 83.211       | -443.164     | -316.212     |
| 19  | 5·10$^8$       | 1.99             | 2        | $\Delta e_{p_0} > e_1$ |              |              |              |
| 20  | 5·10$^8$       | 0.199            | 0.2      | 2.599        | 2.847        | -1.042       | -0.453       |
| 21  | 5·10$^8$       | 0.0199           | 0.02     | 2.587        | 2.825        | 0.658        | 0.749        |
| 22  | 5·10$^8$       | 1.9              | 2        | $\Delta e_{p_0} > e_1$ |              |              |              |
| 23  | 5·10$^8$       | 0.19             | 0.2      | 7.993        | 10.354       | -51.232      | -29.080      |
| 24  | 5·10$^8$       | 0.019            | 0.02     | 7.877        | 10.184       | -7.504       | -4.043       |
| 25  | 5·10$^8$       | 1                | 2        | $\Delta e_{p_0} > e_1$ |              |              |              |
| 26  | 5·10$^8$       | 0.1              | 0.2      | 71.260       | 89.448       | -4.619·10$^3$ | -3.084·10$^3$ |
| 27  | 5·10$^8$       | 0.01             | 0.02     | 90.256       | 114.221      | -752.873     | -509.520     |
We show in table 2 only the calculated values for $\chi$ and $\chi_c$ characteristics because these data indicate without doubts that there is no possible match between these two characteristics as long as always the variation intervals do not overlap. The situation is the same even for values of $l_k/l_{p_{\text{max}}}$ no matter as big is $p_{\text{max}}$. In other words, for the analysed gun system, there is not possible to reach a constant pressure plateau when is used a propellant characterised by a single set of geometrical characteristics ($\chi, \lambda$) over the entire web thickness $e_1$.

Even if we are used only one gun in our study it seems reasonable that the above conclusion may be extended to all realistic gun systems as long as we varied zero dimensional ratios $sl_k/W_0, l_k/l_{p_{\text{max}}}$ and $p_{\text{max}}/p_0$ on wide ranges and the characteristics used for propellant are regular one.

5. Conclusions
In order to get a constant pressure the propellant should burn progressive and the required geometrical characteristics can be expressed through the analytic functions (29) and (30). The shape coefficient $\mu$ should be 0 too.

Despite the fact that grain thickness burned while the maximum pressure plateau is maintained depends on the velocity of projectile at the moment when maximum pressure is attained and the distance travelled by projectile while pressure is constant, the quantity of propellant burned depends only on such distance.

The proposed search algorithm for the required geometrical characteristics used on a specific realistic gun system reveal the impossibility to obtain a feature of maximum constant pressure.

It seems reasonable to believe that, in generally, there is not possible to reach a constant pressure plateau when in a realistic gun system a propellant characterised by a single set of geometrical characteristics over the entire web thickness is used. A rigorous mathematical demonstration of $\chi_{\text{min}} > \chi_{\text{max}}$ or other similar inequality may be usefull to clarify this matter. Nevertheless the deduced expression for the geometrical characteristics $\chi$ and $\lambda$ and the proposed search algorithm may represent a start point in the development of some similar formulas and methods that deal with more complicated grains geometries as such those obtained through co-extrusion or 3D printing, for which the constant maximum pressure plateau is feasible.

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