I. INTRODUCTION

One of the most striking features of quantum entanglement is the existence of correlated measurement outcomes between spatially separated particles, which exceed expectations based on classical physics. These correlations are typically observed with carefully aligned local measurements. They get distorted if a common reference frame is lacking and especially in the presence of noise along the channels distributing the entangled particles. Yet, under a number of experimental conditions, for example in communication within multiparty quantum networks, noise along the channels or fluctuating orientations of reference frames may ruin the quality of the distributed states. Here we show that even for strong fluctuations one can still gain detailed information about the state and its entanglement using random measurements.

Multiplicity acts on the modes of orbital angular momentum, magnetic field fluctuations influence trapped ions, etc. Common sense tells that this renders the distributed quantum state useless and unrecognizable.

Here we provide a method for entanglement detection and analysis that is insensitive to local rotations and thus overcomes these difficulties. It requires neither reference frames nor alignment nor calibration of measuring devices. Still, it can both witness as well as classify multipartite entanglement in the presence of local unitary noise. The key to overcome the lack of control and knowledge regarding each single measurement is to harness uniform sampling of the entirety of all measurements. Especially without any prior knowledge about the state, the conceptually simple method of random sampling proves highly beneficial for entanglement detection and state analysis.

Previous work on entanglement detection relaxing the requirement of fully aligned reference frames first considered the absence of a shared reference frame, but still required the ability to choose or at least to repeat local measurement settings from a given set in order to detect, for example, the violation of a Bell inequality [1–6], or for tomographic reconstruction [7]. Under the same constraints, also adaptive methods for entanglement detection have been developed [8, 9]. In the absence of any reference frames Bell violations can be measured with some probability [10, 11] and entanglement can be detected by evaluating the second moment of the distribution of correlations obtained by measuring random observables on each subsystem [12–17]. Furthermore, it has been shown recently that higher-order moments of this distribution allow discrimination of very specific types of multipartite entanglement [18]. While these methods analyze full correlations, a recent experiment used second moments of subsets to deduce entanglement in systems of more than ten particles [19].

Our analysis method considers not only a specific moment of the distributions of full correlations, but all probability distributions of full as well as of marginal correlations taking into account their interdependencies. We show for specific pure states that this provides a detailed picture of the type of state and its entanglement structure. Furthermore, we derive general witnesses of genuine multipartite entanglement for pure and mixed states. We experimentally measure such distributions of various multiqubit states using reference frame free random measurements and show the applicability of the extended analysis methods. These methods are robust as they do not depend on the local unitary noise as long as the rate of generated entangled states is high enough to estimate the correlations for a momentarily constant noise.
II. SCENARIO

Consider a source producing copies of an unknown \( n \)-qubit state \( \rho \), which is transmitted through unstable quantum channels to \( n \) local observers (Fig. 1). During the \( j \)-th transmission the state \( \rho \) is transformed by \( n \) random local unitary operators \( U^{(j)}_i \) with \( i = 1, 2, \ldots, n \) according to

\[
\rho \rightarrow \rho^{(j)} = U^{(j)}_1 \otimes \cdots \otimes U^{(j)}_n \rho U^{(j)*}_1 \otimes \cdots \otimes U^{(j)*}_n. \tag{1}
\]

Additionally, each of the \( n \) observers is free to choose an arbitrary measurement setting \( \sigma^{(j)}_i \) to measure her qubit. If for each transmitted copy of \( \rho \) the transformations \( U^{(j)}_i \) change significantly, all information about the state is lost. However, in a very common scenario encountered by experimenters the unitary noise has a timescale which is sufficiently slow to obtain at least a few copies of \( \rho \) which have been affected by essentially the same noise, i.e., by the same set of local transformations \( U^{(j)}_i \). In this case the transformations are still much too fast to apply standard techniques of state analysis \([20]\), yet, it becomes possible to use the few equally transformed states to reliably record correlations

\[
E^{(j)}_{1...n} = \text{tr} \left( \sigma^{(j)}_1 \otimes \sigma^{(j)}_2 \otimes \cdots \otimes \sigma^{(j)}_n \rho^{(j)} \right) = \text{tr} \left( \tilde{\sigma}^{(j)}_1 \otimes \tilde{\sigma}^{(j)}_2 \otimes \cdots \otimes \tilde{\sigma}^{(j)}_n \rho \right), \tag{2}
\]

where each observer is keeping her local observable \( \sigma^{(j)}_i \) constant in the timescale of constant noise, which results in the effective random observable \( \tilde{\sigma}^{(j)}_i \equiv U^{(j)*}_i \sigma_i U^{(j)}_i \). Note that here and below the index \( j \) refers to a set of transmitted states which have all been affected by the same noise transformations and measured using the same settings.

We refer to \( E^{(j)}_{1...n} \) as “full correlation” or \( n \)-partite correlation because it involves measurement outcomes of all \( n \) observers. Besides full correlations, also “marginal correlations” can be measured, which are computed from the outcomes of a subset of observers. For example, the marginal correlation of all observers but the first one is

\[
E^{(j)}_{2...n} = \text{tr} \left( \mathbb{1} \otimes \tilde{\sigma}^{(j)}_2 \otimes \cdots \otimes \tilde{\sigma}^{(j)}_n \rho \right). \tag{3}
\]

The essential ingredient in our approach is to sample all local measurement directions \( \tilde{\sigma}^{(j)}_i \), for each observer randomly according to a Haar uniform distribution. This removes any dependence of the obtained information on the actual structure or time dependence of the various \( U^{(j)}_i \) and thus overcomes any bias in the random noise.

In our experiment we prepare four different four-qubit states using entangled photon pairs, where we encode two qubits in the polarization degree of freedom and two qubits in the path degree of freedom. To comprehensively demonstrate the informational content of distributions of random correlations, we consider four quantum states belonging to different entanglement classes, in particular a tri-separable, a bi-separable and two genuinely multipartite entangled states, namely a Greenberger-Horne-Zeilinger (GHZ) state and a cluster state, and a maximally entangled state of two qubits gives rise to a tri-separable, a bi-separable and two genuinely multipartite entangled states, namely a Greenberger-Horne-Zeilinger (GHZ) state and a cluster state,

\[
|\psi_{\text{trisep}}\rangle \propto (|00\rangle + |11\rangle) \otimes |0\rangle \otimes |0\rangle, \tag{4a}
\]

\[
|\psi_{\text{bisep}}\rangle \propto (|00\rangle + |11\rangle) \otimes (\cos \varphi |00\rangle + \sin \varphi |11\rangle), \tag{4b}
\]

\[
|\psi_{\text{GHZ}}\rangle \propto (|0000\rangle + |1111\rangle), \tag{4c}
\]

\[
|\psi_{\text{cl}}\rangle \propto (|0000\rangle + |0011\rangle - |1100\rangle + |1111\rangle). \tag{4d}
\]

We utilize the full experimental control over the choice of measurement settings to emulate the local unitary transformations due to noisy channels and the Haar random choices of measurement settings. Details of the setup can be found in [24]. It should be noted that while we clearly can deduce how characteristics of the state are reflected in the form of the distribution the other direction of deduction is in general much more difficult.

III. ANALYZING ENTANGLEMENT STRUCTURES

In the following we study distributions of random correlations for these four states, see Fig. 2. It is helpful to recall that for some particular states the distributions are known analytically. A pure product state of \( n \) qubits results in a distribution proportional to \(-\left(\ln |E|\right)^{n-1}\) \([16,17]\), which becomes uniform for \( n = 1 \), and a maximally entangled state of two qubits gives rise to a flat distribution \([22,23]\). In addition to this established knowledge, we use new criteria to show that the experimental data not only provide information about the amount of entanglement in the full state, but also give insight into how the entanglement is shared among the parties, allowing to reconstruct the whole multipartite entanglement structure. An important finding arises from the fact that for arbitrary product states of the subsystems \( A \) and \( B \) any full correlation value \( E_{AB} \) is the product of the corresponding marginal values with

\[
\varrho_{AB} = \varrho_A \otimes \varrho_B \Rightarrow E_{AB} = E_A E_B. \tag{5}
\]
This relation between single expectation values implies that the correlation distribution of parties $AB$ is a so-called product distribution of measurement results obtained on $A$ and $B$. Whenever this is not the case we can infer that, if the state is pure, it is entangled across the partition $AB$. Here, we first apply this criterion to product states, but later we will also generalize it for arbitrary mixed states.

Consider first the triseparable state in Fig. 2a. The bipartite distribution $E_{34}$, i.e., the distribution of the multiplication of outcomes for qubits 3 and 4, shows a logarithmic decay, which indicates a pure product state over these two parties. The bipartite distribution $E_{12}$ is uniform as it is characteristic for maximally entangled two-qubit states. The single qubit marginals confirm this observation. $E_3$ and $E_4$ are almost uniform (pure states), whereas $E_1$ and $E_2$ correspond to the maximally mixed state. Ideally, the correlation function for the maximally mixed state is equal to zero and results in a delta peak around 0. Finite statistics causes a broadening of this theoretical distribution and leads to the observed Gaussian shape. Several of the distributions are product distributions. For example we can verify that the full distribution $E_{1234}$ is the product distribution of multiplied results obtained on qubits 12 and on qubits 34, and that $E_{34}$ is the product distribution of the results on qubit 3 and on qubit 4. This is compatible with the state being separable across these partitions. On the other hand, clearly the distribution $E_{12}$ is not a product one for the outcomes on qubit 1 and on qubit 2, which indicates the presence of entanglement.

The distributions for the biseparable state are shown in Fig. 2b. As expected, the bipartite marginal $E_{12}$ is the same as for the triseparable state. The same also holds for the respective single qubit marginals of $E_1$ and $E_2$. In the bipartite distribution of $E_{34}$, however, one can nicely observe the signature of a pure state intermediate between a maximally entangled and a product state, as tuned by the parameter $\varphi$. For $\varphi \approx 0.2$, the bipartite distribution of $E_{34}$ is almost uniform until...
approximately 0.5 and decays logarithmically for larger values. Equally, the respective single qubit marginals also show an intermediate behavior between a uniform distribution (pure state) until approximately 0.8 and vanishing (white noise) for values above. Both the distributions $E_{12}$ and $E_{34}$ do not correspond to the product distributions from the constituent systems which implies entanglement across these partitions of the pure state.

The maximally entangled GHZ state (Fig. 2b) and the cluster state (Fig. 2d) are not distinguishable on the level of the four respective single qubit marginals. Also certain bipartite marginals are the same, e.g., when tracing out qubits 3 and 4. However, while for the permutationally invariant GHZ state all marginal distributions for the same number of qubits must be the same, a significantly different distribution (corresponding to the maximally mixed state) can be obtained for the cluster state, when tracing out for example qubits 1 and 4, i.e., for $E_{23}$. Finally, the cluster and GHZ state can be distinguished also via their distributions of the full correlations. From the plotted distributions for these two states only the distribution $E_{23}$ of the cluster state is (trivially) the product distribution for the results on qubits 2 and 3 (the same holds also for $E_{13}$, $E_{14}$, and $E_{24}$). All other distributions are not the product distributions and thus reveal entanglement.

While our data reflect the theoretical predictions based on Eqs. (4a-d) well, there are systematic differences which can be traced back chiefly to a broadening of the distributions due to finite statistics [24]. We used approximately 475 counts per estimated expectation value for $b$ distributions due to finite statistics [24]. We used approximations are not the product distributions and thus reveal the amount of purity and entanglement, respectively, already for four qubits we found by a numerical criterion can be generalized to the detection of genuine multipartite entanglement. Note that in the case of a single qubit, the purity parameterizes the spectrum of the density matrix and hence any function of the quantum state which is invariant under local unitary transformations.

Let us consider the simplest case of pure two-qubit states. The second moments of any product state satisfy $m_{12} = m_{1}m_{2}$. In consequence, the observation of $m_{12} > m_{1}m_{2}$ indicates entanglement for pure states. This reasoning cannot be easily extended to general states, since this inequality can also be satisfied for incoherent mixtures of product states. However, we have found a purity dependent tightening of the inequality such that any $m_{12}$ above a certain purity dependent threshold must be due to quantum entanglement. In Appendix A we derive the following entanglement witness condition:

\[ M_2 \equiv m_{12} - m_1m_2 \leq \left\{ \begin{array}{ll}
4(1 - \mathcal{P})\mathcal{P}/9 & \text{for } \mathcal{P} \geq \frac{1}{2}, \\
(4\mathcal{P} - 1)/9 & \text{for } \mathcal{P} < \frac{1}{2}.
\end{array} \right. \]

It holds for all separable states of two qubits with purity $\mathcal{P} \equiv \mathcal{P} (\rho)$. The bound is tight and achieved, e.g., by the state $\rho_{00}/\langle 00 \rangle + (1 - p)|11\rangle\langle 11|$. This powerful criterion can be generalized to the detection of genuine multipartite entanglement.

A state is genuinely multipartite entangled if it cannot be represented as a mixture of product states across any bipartition (such a mixture is called a bi-separable state). The left-hand side of Eq. (8) generalizes for an $n$-qubit state to

\[ M_n \equiv m_S - \frac{1}{2} \sum_{A \in (\mathcal{P}(S)) \setminus \{ S \}} m_A m_{S\setminus A}. \]

where the factor of $1/2$ resolves the issue of the double counting in the sum.

By numerical simulations, we find that the following condition holds for three-qubit bi-separable states

\[ M_3 = m_{123} - m_{1}m_{23} - m_{2}m_{13} - m_{3}m_{12} \leq \frac{8}{27}(1 - \mathcal{P})\mathcal{P}. \]

We have verified this inequality by extensive numerical search described in Appendix C. The bound is tight for $\mathcal{P} \geq \frac{1}{2}$ and is achieved by, e.g., the state $\rho_0|\phi^+\rangle\langle \phi^+| \otimes |0\rangle \langle 0| + (1 - p)|\phi^-\rangle\langle \phi^-| \otimes |1\rangle \langle 1| + \text{the Bell states}[|\psi^\pm\rangle \equiv \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)].$

The bounds of the last two inequalities give hope for a simple dependence on the number of qubits. Unfortunately, already for four qubits we found by a numerical...
study that the inequality satisfied by bi-separable states has a different functional dependence on the purity \( \mathcal{P} \):

\[
\mathcal{M}_4 \leq \frac{1}{86} (1 - \mathcal{P}^2).
\]  

(11)

This bound is also tight for \( \mathcal{P} \geq \frac{5}{8} \) and achieved by, e.g., the state \( p |\phi^+\rangle_{12} |\phi^+\rangle_{34} + (1-p) |\phi^+\rangle_{13} |\phi^+\rangle_{24} \).

Any violation of inequality (10) or (11) indicates genuine multipartite entanglement between three or four qubits respectively. We emphasize that these criteria require only the second moments of the observed distributions.

Application of the conditions of Eqs. (8), (10) and (11) to experimental data (Fig. 3) indeed enables detection of genuine \( n \)-partite entanglement for various subsets of particles. For the cluster and the GHZ state, genuine 4-partite entanglement is revealed with Eq. (11) using \( \mathcal{M}_4 \approx 0.0330 > 0.0076 \) and \( \mathcal{M}_4 \approx 0.0311 > 0.0099 \), respectively. The bi- and triseparable states do not violate their respective bound. Investigating the entanglement properties for their marginal states, one can now prove the entanglement for the 12-marginal and the 34-marginal of the biseparable state as well as the 12-marginal of the triseparable state. It is therefore possible to conclude that the biseparable state contains contributions of at least \( \varrho_{12} \otimes \varrho_{34} \), with entanglement between 1 and 2 and between 3 and 4, and the triseparable state contains \( \varrho_{13} \otimes \varrho_3 \otimes \varrho_4 \). Note that the state could also contain genuine 4-partite entanglement, which was not revealed by \( \mathcal{M}_4 \).

V. CONCLUSION

This work introduces a scheme to detect genuine multipartite entanglement and reveal its detailed structure in the absence of any reference frames and even for strongly fluctuating channels. Key to this method is to subject a multipartite quantum system to randomly chosen local measurements and to analyze full and marginal correlations between all local results using second moments of respective correlation distributions. Haar random sampling removes any bias of the noise and, provided that the generation rate of multqubit states is higher than the rate of fluctuations along the channel, neither the strength nor any characteristics of the noise matter. The power of our procedure is demonstrated here by reconstructing the entanglement structure of various, experimentally prepared photonic four-qubit states. From this, many more interesting questions arise, e.g., whether it is possible to - up to suitable transformations - tomographically reconstruct quantum states or characterize quantum processes in our scenario of fully randomized local measurement directions.

FIG. 3. Analyzing the entanglement structure using \( \mathcal{M}_4 \): (a) \( \mathcal{M}_4 \) of the GHZ state (4c) (red plus) and the cluster state (4d) (blue square) are violating the bound for biseparable states (11), clearly indicating genuine 4-partite entanglement. The negative values for \( \mathcal{M}_4 \) of the tri- and the biseparable states are not shown. (b) Evaluation of \( \mathcal{M}_4 \) for tripartite marginals for these states does not indicate any genuine tripartite entanglement as expected, as no point is found above the threshold given in Eq. (10). The filled and non-filled circles indicate the type of marginals giving rise to different values of \( \mathcal{M}_3 \). (c) \( \mathcal{M}_2 \) is shown for all bipartite marginals. The four-qubit biseparable state (4b) (green diamond) and the four-qubit triseparable state (4a) (purple cross) have two and one marginals, respectively, which themselves are shown to be two-qubit entangled. The shaded regions contain all types of quantum states, irrespective of their entanglement properties. All error bars are smaller than the markers.
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Appendix A: Finite sample size correction

In our experiment, two different types of statistical effects have to be taken into account. On one hand, for obtaining the distributions as in Fig. 2, a finite number $N_s$ of measurement settings ($N_s = 10,000$ in our case) is used. This leads to an uncertainty in estimating the second moments $m_{A} \equiv m_{A}^{(2)}$. This statistical error can be approximated by

$$
\Delta m_{A}^{(2)} = \frac{1}{N_s} \left[ m_{A}^{(4)} - \frac{N_s - 3}{N_s - 1} \left( m_{A}^{(2)} \right)^2 \right],
$$

which describes the variance of the sample variance.

On the other hand, each correlation $\mathcal{E}_{A}^{ij} \equiv E$ is obtained by performing $N_c$ measurements in the same setting. Due to this finite sample size, for each expectation value in general we do not obtain the ideal result $E_R$, but measure a value $E_M$ at random from a conditional probability distribution $p(E_M|E_R)$, approximately given by the Gaussian

$$
p(E_M|E_R) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left( -\frac{(E_M - E_R)^2}{2\sigma^2} \right)
$$

centered around $E_R$ with $\sigma = \sqrt{1 - E_R^2}/\sqrt{N_c}$, see, e.g., [24].

This statistical deviation leads to an overestimation of $m_A$. We mitigate this systematic inaccuracy by taking into account the well known statistical effect from Eq. [A2]. Employing Bayesian methods, we are able to obtain $p(E_R|E_M)$ from $p(E_M|E_R)$ allowing to calculate $m_A$ with reduced bias as

$$
m_A = \int_{-1}^{1} dE_R p(E_R) E_R^2
\begin{align*}
= \int_{-1}^{1} dE_R \int_{-1}^{1} dE_M p(E_R|E_M) p(E_M) E_R^2.
\end{align*}
$$

Bayes’ theorem provides $p(E_R|E_M)$ as

$$
p(E_R|E_M) = \frac{p(E_M|E_R)p(E_R)}{p(E_M)} = \frac{p(E_M|E_R)p(E_R)}{\int_{-1}^{1} dE_R p(E_M|E_R)p(E_R)},
$$

(A4)

where $p(E_R)$ represents the prior assumption about the unknown distribution $p(E_R)$. For our evaluation we use the measured distribution $p(E_M)$ as the prior guess about $p(E_R)$ and obtain an updated distribution according to the statistical analysis above. This distribution is used to evaluate the moments.

Appendix B: Two-qubit condition

Here we prove Eq. [5] of the main text. The problem is to maximize the value of $\mathcal{M}_2 = m_{12} - m_1 m_2$ over separable states of two qubits with fixed purity $\rho$. Any two-qubit state admits a decomposition

$$
\rho = \frac{1}{4} \sum_{\mu,\nu=0}^{3} T_{\mu\nu} \sigma_{\mu} \otimes \sigma_{\nu},
$$

(B1)

where $T_{\mu\nu} = \text{tr}(\sigma_{\mu} \otimes \sigma_{\nu})$. In order to simplify numerical factors, we note that the second moments satisfy [16, 17]:

$$
m_{12} = \frac{1}{9} \sum_{j,k=1}^{3} T_{jk}^2 = \frac{1}{9} \overline{m}_{12},
$$

(B2)

$$
m_1 = \frac{1}{3} \sum_{j=1}^{3} T_{j0}^2 = \frac{1}{3} \overline{m}_1,
$$

(B3)

$$
m_2 = \frac{1}{3} \sum_{k=1}^{3} T_{0k}^2 = \frac{1}{3} \overline{m}_2.
$$

(B4)

The problem is therefore to maximize $\overline{m}_{12} - \overline{m}_1 \overline{m}_2$ (and then multiply the result by $\frac{1}{9}$). Using the definition of the purity results in

$$
\overline{m}_{12} = 4\rho - 1 - \overline{m}_1 - \overline{m}_2.
$$

(B5)

For the figure of merit we obtain

$$
9\mathcal{M}_2 = 4\rho - 1 - (\overline{m}_1 + \overline{m}_2 + \overline{m}_1 \overline{m}_2).
$$

(B6)

Since the purity is fixed we solve the following optimization

minimize: $\overline{m}_1 + \overline{m}_2 + \overline{m}_1 \overline{m}_2$,

under condition: $\overline{m}_1 + \overline{m}_2 = 4\rho - 1 - \overline{m}_{12}$.

(B8)

By the Lagrange multiplier method one finds that the minimum is achieved for $\overline{m}_1 = \overline{m}_2 \equiv m$. Since the function to be minimized, $2m + m^2$, is increasing with $m$, the minimum is achieved for the smallest $m$ compatible with
the condition $2m = 4P - 1 - m_{12}$. All two-qubit separable states satisfy $m_{12} \leq 1$, and hence the minimum is for $m = \max(0, 2P - 1)$. There are therefore two cases. For $P \leq \frac{1}{2}$, the figure of merit equals $(4P - 1)/9$ obtained by putting $m_1 = m_2 = 0$ in Eq. (B6). For $P > \frac{1}{2}$ the figure of merit reads $4P(1 - P)/9$ obtained by putting $m_1 = m_2 = 2P - 1$ in Eq. (B6). Fig. 3 presents both of these bounds.

Appendix C: Numerical simulations

Here we give numerical evidence for the bounds of Eqs. (10) and (11) of the main text. We performed sampling of more than $10^6$ biseparable states and always found the bounds satisfied. Fig. 4 illustrates the results of numerical simulation.

For the case of three qubits we find the following improved boundary for small values of $P$:

$$M_3 \leq \begin{cases} (8P - 1)/27 & \text{for } P \in [\frac{1}{4}, \frac{1}{2}], \\ 4P/27 & \text{for } P \in (\frac{1}{4}, \frac{1}{2}], \\ 8(1 - P)P/27 & \text{for } P > \frac{1}{2}, \end{cases}$$

while the improved boundary for four qubits reads

$$M_4 \leq \begin{cases} (16P - 1)/81 & \text{for } P \in [\frac{1}{16}, \frac{1}{4}], \\ 2(-8P^2 + 16P + 1)/243 & \text{for } P \in (\frac{1}{4}, \mathcal{P}_0], \\ 8(1 - P^2)/81 & \text{for } P > \mathcal{P}_0, \end{cases}$$

where $\mathcal{P}_0 = \frac{-4 + 3\sqrt{2}}{2} \approx 0.60.$

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bound.