Corrigendum to ”Approximation by $C^p$-smooth, Lipschitz functions on Banach spaces” [J. Math. Anal. Appl., 315 (2006), 599-605]

R. Fry

ABSTRACT. In this erratum, we recover the results from an earlier paper of the author’s which contained a gap. Specifically, we prove that if $X$ is a Banach space with an unconditional basis and admits a $C^p$-smooth, Lipschitz bump function, and $Y$ is a convex subset of $X$, then any uniformly continuous function $f : Y \to \mathbb{R}$ can be uniformly approximated by Lipschitz, $C^p$-smooth functions $K : X \to \mathbb{R}$.

Also, if $Z$ is any Banach space and $f : X \to Z$ is $\eta$-Lipschitz, then the approximates $K : X \to Z$ can be chosen $C\eta$-Lipschitz and $C^p$-smooth, for some constant $C$ depending only on $X$.

1. Introduction

In this erratum to [8], we point out that there is a gap in the proof of Theorem 1 of that paper. Specifically, the estimate for $\sup_{x \in E_n} \left\| F_n(x) - F'_n(x) \right\|$ in [8] does not hold (as the inductive proof fails here), and as a consequence the conclusion of Theorem 1 does not follow. Nevertheless, using a construction from [11], techniques from [3], and employing a similar proof as originally, we are able to establish all the results of [8] under the additional assumption that the subset $Y \subset X$ is convex (see Theorem 1 below.)

We note that the main motivation for this work was to find an analogous result to that of [4] for not necessarily bounded functions. Let us recall in [4] it was shown, in particular, that for a separable Banach space $X$ admitting a Lipschitz, $C^p$ smooth bump function, that given $\varepsilon > 0$ and a bounded, uniformly continuous function $f : X \to \mathbb{R}$, there exists a Lipschitz, $C^p$ smooth function $K$ with $|f - K| < \varepsilon$ on $X$. We remark that to establish our result here, we need to further assume that our Banach space $X$ has an unconditional basis. However, in addition to relaxing the boundedness condition on $f$, when $f$ is also Lipschitz, unlike the result of [4], we are able to find Lipschitz, $C^p$ smooth approximates $K$ where the Lipschitz constants

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do not depend on the $\varepsilon$-degree of precision in the approximation. We also note that the results of [4] are restricted to real-valued maps.

The notation we employ is standard, with $X, Y$, etc. denoting Banach spaces. We write the closed unit ball of $X$ as $B_X$. The Gâteaux derivative of a function $f$ at $x$ in the direction $h$ will be denoted $D_h f(x)$, while the Fréchet derivative of $f$ at $x$ on $h$ is written $f'(x)(h)$. We note that a $C^p$-smooth function is necessarily Fréchet differentiable (see e.g., [5].)

A $C^p$-smooth bump function $b$ on $X$ is a $C^p$-smooth, real-valued function on $X$ with bounded, non-empty support, where

$$\text{support } (b) = \{x \in X : b(x) \neq 0\}.$$ 

If $f : X \to Y$ is Lipschitz with constant $\eta$, we will say that $f$ is $\eta$-Lipschitz. Most additional notation is explained as it is introduced in the sequel. For any unexplained terms we refer the reader to [6] and [7]. For further historical context see the introduction in [8].

2. Main Results

We first introduce some notation which will be used throughout the paper. Let $\{e_j, e_j^*\}_{j=1}^{\infty}$ be an unconditional Schauder basis on $X$, and $P_n : X \to X$ the canonical projections given by $P_n(x) = P_n(\sum_{j=1}^{\infty} x_j e_j) = \sum_{j=1}^{n} x_j e_j$, and where we set $P_0 = 0$. By renorming, we may assume that the unconditional basis constant is 1. In particular, $\|P_n\| \leq 1$ for all $n$. We put $E_n = P_n(X)$, and $E^\infty = \cup_n E_n$, noting that $\dim E_n = n$, $E_n \subset E_{n+1}$, and $E^\infty$ is a dense subspace of $X$. It will be convenient to denote the closed unit ball of $E_n$ by $B_{E_n}$.

The proof of our main theorem is a modification of some techniques found in [13] and [3], where $C^p$-fine approximation on Banach spaces is considered. We also rely on the main construction from [11]. We follow the original proof of [8] closely, and have decided to reproduce the details so that this note is self contained.

**Theorem 1.** Let $X$ be a Banach space with unconditional basis which admits a Lipschitz, $C^p$-smooth bump function. Let $Y \subset X$ be a convex subset and $f : Y \to \mathbb{R}$ a uniformly continuous map. Then for each $\varepsilon > 0$ there exists a Lipschitz, $C^p$-smooth function $K : X \to \mathbb{R}$ such that for all $y \in Y$,

$$|f(y) - K(y)| < \varepsilon.$$

If $Z$ is any Banach space, $Y \subset X$ is any subset, and $f : X \to Z$ (respectively $f : Y \to \mathbb{R}$) is Lipschitz with constant $\eta$, then we can choose
K : X \to Z (respectively K : X \to \mathbb{R}) to have Lipschitz constant no larger than C_0 \eta, where C_0 > 1 is a constant depending only on X (in particular, C_0 is independent of \epsilon.)

Proof As noted before, the main idea of the proof is a modification of the proof of \cite[Lemma 5]{3} using ideas from \cite{11}.

We will need to use the following result, and refer the reader to \cite[Proposition II.5.1]{6} and \cite{12} for a proof.

**Proposition 1.** Let Z be a Banach space. The following assertions are equivalent.

(a). Z admits a \(C^p\)-smooth, Lipschitz bump function.

(b). There exist numbers \(a, b > 0\) and a Lipschitz function \(\psi : Z \to [0, \infty)\) which is \(C^p\)-smooth on \(Z \setminus \{0\}\), homogeneous (that is \(\psi(tx) = |t|\psi(x)\) for all \(t \in \mathbb{R}, x \in Z\), and such that \(a \parallel \cdot \parallel \leq \psi \leq b \parallel \cdot \parallel\).

For such a function \(\psi\), the set \(A = \{z \in Z : \psi(z) \leq 1\}\) is what we call a \(C^p\)-smooth, Lipschitz starlike body, and the Minkowski functional of this body, \(\mu_A(z) = \inf\{t > 0 : (1/t)z \in A\}\), is precisely the function \(\psi\) (see \cite{1} and the references therein for further information on starlike bodies and their Minkowski functionals).

We will denote the open ball of center \(x\) and radius \(r\), with respect to the norm \(\parallel \cdot \parallel\) of \(X\), by \(B(x, r)\). If \(A\) is a bounded starlike body of \(X\), we define the open \(A\)-pseudoball of center \(x\) and radius \(r\) as

\[B_A(x, r) := \{y \in X : \mu_A(y - x) < r\}.\]

According to Proposition 1 and the preceding remarks, because \(X\) has a \(C^p\)-smooth, Lipschitz bump function, there is a bounded starlike body \(A \subset X\) (which we fix for the remainder of the proof) whose Minkowski functional \(\mu_A = \psi\) is Lipschitz and \(C^p\)-smooth on \(X \setminus \{0\}\), and there is a number \(M \geq 1\) such that \(\frac{1}{M} \parallel x \parallel \leq \mu_A(x) \leq M \parallel x \parallel\) for all \(x \in X\), and \(\parallel \mu_A'(x) \parallel \leq M\) for all \(x \in X \setminus \{0\}\). Notice that in this case we have,

\[B(x, \frac{r}{M}) \subseteq B_A(x, r) \subseteq B(x, Mr)\]

for every \(x \in X\), \(r > 0\). This fact will sometimes be used implicitly in what follows.

For the proof, we shall first define a function \(\overline{f} : E^\infty \to \mathbb{R}\), then a map \(\Psi : X \to E^\infty\), and finally our desired function \(K\) will be given by \(K = \overline{f} \circ \Psi\).

To begin the proof, first note that as \(f\) is real-valued and \(Y\) is convex, by \cite[Proposition 2.2.1 (i)]{5} \(f\) can be uniformly approximated by a Lipschitz map, and so we may and do suppose that \(f\) is Lipschitz with constant \(\eta\).

Using an infimal convolution, we extend \(f\) to a Lipschitz map \(F\) on \(X\) with the same constant \(\eta\) by defining, \(F(x) = \inf \{f(y) + \eta \parallel x - y \parallel : y \in Y\}\).
Let $\varepsilon > 0$ and $r \in (0, \varepsilon/3M\eta)$. We shall require the main construction from [11] (see also [9]), and for the sake of completeness we outline this here. Let $\{h_i\}_{i=1}^{\infty}$ be a dense sequence in $B_X$, and $\varphi_i \in C^\infty(\mathbb{R}, \mathbb{R}^+)$ with $\int_{\mathbb{R}} \varphi_i = 1$ and support($\varphi_i$) $\in \left[ -\frac{\varepsilon}{6\eta^2}, \frac{\varepsilon}{6\eta^2} \right]$.

Now we define functions $g_n : X \to \mathbb{R}$ by,

$$g_n(x) = \int_{\mathbb{R}^n} F \left( x - \sum_{i=1}^{n} t_i h_i \right) \prod_{i=1}^{n} \varphi_i(t_i) dt,$$

where the integral is $n$-dimensional Lebesgue measure.

It is proven in [11] that the following hold:

1. There exists $g$ with $g_n \to g$ uniformly on $X$,
2. $|g - F| < \varepsilon/3$ on $X$,
3. The map $g$ is $\eta$-Lipschitz,
4. The map $g$ is uniformly Gâteaux differentiable

Next, following [3, Lemma 5], let $\varphi : \mathbb{R} \to [0,1]$ be a $C^\infty$-smooth function such that $\varphi(t) = 1$ if $|t| < 1/2$, $\varphi(t) = 0$ if $|t| > 1$, $\varphi'([0,\infty)) \subseteq [-3,0]$, $\varphi(-t) = \varphi(t)$.

Let us define a function Gâteaux differentiable on $X$, and $C^p$-smooth on $E_n$, by

$$F_n(x) = \left( a_n \right)^n \int_{E_n} g(x - y) \varphi(a_n \mu_A(y)) dy$$

where

$$c_n = \int_{E_n} \varphi(\mu_A(y)) dy,$$

and (keeping in mind (2.1) and (3)) we have chosen the constants $a_n$ large enough that

$$\sup_{x \in E_n} |F_n(x) - g(x)| < \frac{\varepsilon}{6} 2^{-n}. \quad (2.2)$$

As pointed out to us by P. Hájek, since $g$ is Lipschitz and uniformly Gâteaux differentiable, by [10, Lemma 4] for each $h$ the map $x \to D_h g(x)$ is uniformly continuous. From this, the Lipschitzness of $g$, and compactness of $B_{E_n}$, we can choose the $a_n$ larger if need be so that for all $h \in B_{E_n}$ we have,

$$\sup_{x \in E_n} |D_h F_n(x) - D_h g(x)| < \frac{\eta}{2} 2^{-n}. \quad (2.3)$$

Note that for any $x, x' \in X$, 

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\[ |F_n(x) - F_n(x')| \leq \frac{(a_n)^n}{c_n} \int_{E_n} |g(x - y) - g(x' - y)| \varphi(a_n \mu_A(y)) dy \]

\[ \leq \eta \|x - x'\| \frac{(a_n)^n}{c_n} \int_{E_n} \varphi(a_n \mu_A(y)) dy = \eta \|x - x'\| , \]

that is, \( F_n \) is \( \eta \)-Lipschitz.

We next define a sequence of Gâteaux differentiable functions \( \tilde{f}_n : X \to \mathbb{R} \), \( C^p \)-smooth on \( E_n \), as follows. Put \( \tilde{f}_0 = f(0) \), and supposing that \( \tilde{f}_0, \ldots, \tilde{f}_{n-1} \) have been defined, we set

\[ \tilde{f}_n(x) = F_n(x) + \tilde{f}_{n-1}(P_{n-1}(x)) - F_n(P_{n-1}(x)) . \]

One can verify by induction, using \( \|P_n\| \leq 1 \), (2.2) and (2.3), that,

(i). The \( \tilde{f}_n \) are Gâteaux differentiable, the restriction of \( \tilde{f}_n \) to \( E_n \) is \( C^p \)-smooth, and \( \tilde{f}_n \) extends \( \tilde{f}_{n-1} \),

(ii). \( \sup_{x \in E_n} |\tilde{f}_n(x) - g(x)| < \frac{\varepsilon}{2} \left( 1 - \frac{1}{2^n} \right) \),

(iii). \( \sup_{x \in E_n} |D_h \tilde{f}_n(x) - D_h g(x)| \leq \eta \left( 1 - \frac{1}{2^n} \right) \), for all \( h \in B_{E_n} \).

We now define the map \( \bar{f} : E^\infty \to \mathbb{R} \) by

\[ \bar{f}(x) = \lim_{n \to \infty} \tilde{f}_n(x) . \]

For \( x \in E^\infty = \bigcup_n E_n \), define \( n_x \equiv \min \{ n : x \in E_n \} \), and note that we have for any \( m \geq n_x \),

(2.4) \[ \bar{f}(x) = \lim_{n \to \infty} \tilde{f}_n(x) = \tilde{f}_m(x) . \]

In particular, for any \( n \), \( \bar{f} \mid_{E_n} = \tilde{f}_n \). One can verify using (2.4), (i), (ii), and (iii) above that \( \bar{f} \) has the following properties:

(i)' The restriction of \( \bar{f} \) to every subspace \( E_n \) is \( C^p \)-smooth,

(ii)' \( \sup_{x \in E^\infty} |\bar{f}(x) - g(x)| \leq \frac{\varepsilon}{2} \).

(iii)' \( \sup_{x \in E_n} |D_h \bar{f}(x) - D_h g(x)| \leq \eta \), for all \( h \in B_{E_n} \).

The proof now closely follows [3] Lemma 5, and we provide some of the details for the sake of completeness.

Next let \( x = \sum_n x_n e_n \in X \) and define the maps

\[ \chi_n(x) = 1 - \varphi \left( \frac{\mu_A(x - P_{n-1}(x))}{r} \right) , \]

that is, \( F_n \) is \( \eta \)-Lipschitz.
and

\[ \Psi(x) = \sum_n \chi_n(x)x_ne_n. \]

For any \( x_0 \), because \( P_n(x_0) \to x_0 \) and the \( \|P_n\| \) are uniformly bounded, there exist a neighbourhood \( N_0 \) of \( x_0 \) and an \( n_0 = n_{x_0} \) so that \( \chi_n(x) = 0 \) for all \( x \in N_0 \) and \( n \geq n_0 \) and so \( \Psi(N_0) \subset E_{n_0} \). Thus, \( \Psi : X \to E^\infty \) is a \( C^p \)-smooth map whose range is locally contained in the finite dimensional subspaces \( E_n \). Using the fact that \( \{e_n\} \) is unconditional with constant \( C = 1 \), one can show that (see [3 Fact 7])

(2.5)

\[ \|x - \Psi(x)\| < Mr. \]

We next consider the derivative of \( \Psi \). A straightforward calculation, using the facts: \( |\varphi'(t)| \leq 3 \), \( |\mu'_A(x)| \leq M \) and \( \|(I - P_n)\)'(x)\| \leq 2 \) for all \( x, t \), verifies that \( \|\chi'_n(x)\| \leq 6Mr^{-1} \). Also, since \( (\chi_n(x)x_n)' = \chi'_n(x)x_n + \chi_n(x)e_n \), it follows that

(2.6)

\[ \Psi'(x)(\cdot) = \sum_n \chi'_n(x)(\cdot)x_ne_n + \sum_n \chi_n(x)e_ne^*_n(\cdot). \]

Now, using (2.7), the estimate for \( \|\chi'_n\| \) above, and again using the fact that \( \{e_n\} \) is unconditional with constant \( C = 1 \), it is shown in [3 Fact 7] that for all \( x \in X \),

\[ \|\Psi'(x)\| \leq 8M^2. \]

We define \( K(x) = \tilde{f}(\Psi(x)) \). Note that \( K \) is \( C^p \)-smooth on \( X \), being the composition of \( C^p \)-smooth maps, and noting that \( \Psi \) maps locally into some \( E_n \). Now, for \( x \in X \), by (ii)', choice of \( r \), using (2), (2.6) and that \( g \) is Lipschitz with constant \( \eta \), we have,

\[ |F(x) - K(x)| \leq |g(x) - F(x)| + |g(x) - g(\Psi(x))| \]

\[ + |\tilde{f}(\Psi(x)) - g(\Psi(x))| \]

\[ \leq \varepsilon/3 + \eta Mr + \varepsilon/3 < \varepsilon. \]

In particular, since \( F \mid_Y = f \), we have for \( y \in Y \) that \( |f(y) - K(y)| < \varepsilon \).

Finally we consider \( K'(x) = \tilde{f}'(\Psi(x))\Psi'(x) \). Fix \( x \in X \), and \( h \in X \) with \( \|h\| \leq \frac{1}{8M^2} \). Note that \( \|\Psi'(x)(h)\| \leq 1 \).

Now \( \Psi \) maps a neighbourhood of \( x \) into \( E_{n_x} \), and hence also \( \Psi'(x)(h) \in E_{n_x} \); in particular, \( \Psi'(x)(h) \in B_{E_{n_x}} \). Now using this fact, (3)', (iii)', and our estimates above, we have,
\[ |K'(x)(h)| = |\tilde{f}'(\Psi(x))(\Psi'(x)(h))| \]
\[ = |\tilde{f}'_n(\Psi(x))(\Psi'(x)(h))| \]
\[ < |D(\Psi'(x))g(\Psi(x)) + \eta| \]
\[ \leq \eta + \eta = 2\eta. \]

As \( K'(x) : X \rightarrow \mathbb{R} \) is continuous and linear, from the above estimate we have, \( |K'(x)(h)| \leq 16M^2\eta \) for all \( h \in B_X \), and hence \( \|K'(x)\| \leq 16M^2\eta \) for all \( x \in X \).

This proves the first statement of the theorem. For the second statement, we observe that for Lipschitz functions \( f : X \rightarrow Z \) into an arbitrary Banach space \( Z \), we do not require the real-valued assumption on \( f \) that was used to apply the result from [5], and the methods of [11] and [3] apply equally well to arbitrary Banach space valued maps. For Lipschitz functions \( f : Y \rightarrow \mathbb{R} \), we can directly extend \( f \) to a Lipschitz map on \( X \) via an infimal convolution as before. Hence the theorem follows with \( C_0 = 16M^2 \). ■

We have the following characterization slightly extending the one from [8].

**Corollary 1.** Let \( X \) have unconditional basis. Then the following are equivalent.

1. \( X \) admits a Lipschitz, \( C^p \)-smooth bump function.
2. For every convex subset \( Y \subset X \), uniformly continuous map \( f : Y \rightarrow \mathbb{R} \), and \( \varepsilon > 0 \), there exists a Lipschitz, \( C^p \)-smooth map \( K : X \rightarrow \mathbb{R} \) with \( |f - K| < \varepsilon \) on \( Y \).
3. For every subset \( Y \subset X \), Lipschitz function \( f : Y \rightarrow \mathbb{R} \), and \( \varepsilon > 0 \), there exists a Lipschitz, \( C^p \)-smooth map \( K : X \rightarrow \mathbb{R} \) with \( |f - K| < \varepsilon \) on \( Y \).
4. For every Banach space \( Z \), Lipschitz map \( f : X \rightarrow Z \), and \( \varepsilon > 0 \), there exists a Lipschitz, \( C^p \)-smooth map \( K : X \rightarrow Z \) with \( \|f - K\| < \varepsilon \) on \( X \).

**Proof.** That (1) \( \Rightarrow \) (2), (3), and (4) is Theorem 1. For (2) \( \Rightarrow \) (1), choose \( Y = X \), and \( f = \|\cdot\| \). Let \( K : X \rightarrow \mathbb{R} \) be a \( C^p \)-smooth, Lipschitz map with \( |f - K| < 1 \) on \( X \). Let \( \xi : \mathbb{R} \rightarrow \mathbb{R} \) be \( C^\infty \)-smooth and Lipschitz with, \( \xi(t) = 1 \) if \( t \leq 1 \) and \( \xi(t) = 0 \) if \( t \geq 2 \). Then \( b = \xi \circ K \) is a \( C^p \)-smooth, Lipschitz map with \( b(0) = 1 \) and \( b(x) = 0 \) when \( \|x\| \geq 3 \). The remaining implications are similar. \( \square \)

**Remark** The Lipschitz constant of \( K \) obtained for the second statement of Theorem 1 is not the best possible. By using better derivative
estimates, one can show that for any $\delta > 0$, we may arrange $\|K\| \leq (\eta + \delta) (2 (2 + \delta) M^2 + 1)$. This should be compared with the recent result in [2], where it is shown in particular that for separable Hilbert spaces $X$, any Lipschitz, real-valued function on $X$ can be uniformly approximated by $C^\infty$ smooth functions with Lipschitz constants arbitrarily close to the Lipschitz constant of $f$. It is open whether such a result holds outside the Hilbert space setting.

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