On maximum packings of $\lambda$-fold complete 3-uniform hypergraphs with triple-hyperstars of size 4

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Abstract

A symmetric triple-hyperstar is a connected, 3-uniform hypergraph where, for some edge $\{ a, b, c \}$, vertices $a$, $b$, and $c$ all have degree $k > 1$ and all other edges contain exactly 2 vertices of degree 1. Let $H$ denote the symmetric triple-hyperstar with 4 edges and, for positive integers $\lambda$ and $v$, let $\lambda K_{\lambda}(3)^{(v)}$ denote the $\lambda$-fold complete 3-uniform hypergraph on $v$ vertices. We find maximum packings of $\lambda K_{\lambda}(3)^{(v)}$ with copies of $H$.

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1. Introduction

A hypergraph $H$ consists of a finite, nonempty set $V$ of vertices and a finite collection $E = \{e_1, e_2, \ldots, e_m\}$ of nonempty subsets of $V$ called hyperedges or simply edges. For a given hypergraph $H$, we use $V(H)$ and $E(H)$ to denote the vertex set and the edge set (or multiset) of $H$, respectively. We call $|V(H)|$ and $|E(H)|$ the order and size of $H$, respectively. A hypergraph $H$ is simple if no edge appears more than once in $E(H)$. If for each $e \in E(H)$ we have $|e| = t$, then $H$ is said to be $t$-uniform. Thus $t$-uniform hypergraphs are generalizations of the concept...
of a graph (where \( t = 2 \)). Graphs with repeated edges are often called multigraphs. If \( H \) is a simple hypergraph and \( \lambda \) is a positive integer, then \( \lambda \)-fold \( H \), denoted \( \lambda H \), is the multi-hypergraph obtained from \( H \) by repeating each edge exactly \( \lambda \) times. The hypergraph with vertex set \( V \) and edge set the set of all \( t \)-element subsets of \( V \) is called the complete \( t \)-uniform hypergraph on \( V \) and is denoted by \( K^{(t)}_v \). If \( v = |V| \), then \( \lambda K^{(t)}_v \) is called the \( \lambda \)-fold complete \( t \)-uniform hypergraph of order \( v \) and is used to denote any hypergraph isomorphic to \( \lambda K^{(t)}_v \). When \( t = 2 \), we will use \( \lambda K_v \) in place of \( \lambda K^{(2)}_v \). Similarly, if \( \lambda = 1 \), then we will use \( K^{(t)}_v \) in place of \( 1 K^{(t)}_v \). If \( H' \) is a subhypergraph of \( H \), then \( H \setminus H' \) denotes the hypergraph obtained from \( H \) by deleting the edges of \( H' \). We may refer to \( H \setminus H' \) as the hypergraph \( H \) with a hole \( H' \). The vertices in \( H' \) may be referred to as the vertices in the hole.

A commonly studied problem in combinatorics concerns decompositions of graphs or multigraphs into edge-disjoint subgraphs. A decomposition of a multigraph \( K \) is a set \( \Delta = \{ G_1, G_2, \ldots, G_s \} \) of subgraphs of \( K \) such that \( \{ E(G_1), E(G_2), \ldots, E(G_s) \} \) is a partition of \( E(K) \). If each element of \( \Delta \) is isomorphic to a fixed graph \( G \), then \( \Delta \) is called a \( G \)-decomposition of \( K \). If exactly one element \( L \in \Delta \) is not isomorphic to \( G \), then \( \Delta \) is called a \( G \)-packing of \( K \) with leave \( L \). Such a \( G \)-packing is maximum if no other possible \( G \)-packing of \( K \) has a leave of a smaller size than that of \( L \). Clearly, if \( |E(L)| < |E(G)| \), then the \( G \)-packing is maximum. Moreover, a \( G \)-decomposition of \( K \) can be viewed as a maximum \( G \)-packing with an empty leave.

A \( G \)-decomposition of \( \lambda K_v \) is also known as a \( G \)-design of order \( v \) and index \( \lambda \). A \( K_v \)-design of order \( v \) and index \( \lambda \) is usually known as a \( 2-(v, k, \lambda) \) design or as a balanced incomplete block design of index \( \lambda \) or a \( (v, k, \lambda) \)-BIBD. The problem of determining all \( v \) for which there exists a \( G \)-design of order \( v \) is of special interest (see [1] for a survey).

The notion of decompositions of graphs naturally extends to hypergraphs. A decomposition of a hypergraph \( K \) is a set \( \Delta = \{ H_1, H_2, \ldots, H_s \} \) of subhypergraphs of \( K \) such that \( \{ E(H_1), E(H_2), \ldots, E(H_s) \} \) is a partition of \( E(K) \). Any element of \( \Delta \) isomorphic to a fixed hypergraph \( H \) is called an \( H \)-block. If all elements of \( \Delta \) are \( H \)-blocks, then \( \Delta \) is called an \( H \)-decomposition of \( K \). If exactly one element \( L \in \Delta \) is not an \( H \)-block, then \( \Delta \) is called an \( H \)-packing of \( K \) with leave \( L \), where we again define such a packing to be maximum if \( L \) has the fewest edges possible. An \( H \)-decomposition of \( \lambda K^{(t)}_v \) is called an \( H \)-design of order \( v \) and index \( \lambda \). The problem of determining all \( v \) for which there exists an \( H \)-design of order \( v \) and index \( \lambda \) is called the \( \lambda \)-fold spectrum problem for \( H \)-designs.

A \( K^{(t)}_k \)-design of order \( v \) and index \( \lambda \) is a generalization of \( 2-(v, k, \lambda) \) designs and is known as a \( t-(v, k, \lambda) \) design or simply as a \( t \)-design. A summary of results on \( t \)-designs appears in [16]. A \( t-(v, k, 1) \) design is also known as a Steiner system and is denoted by \( S(t, v, k) \) (see [9] for a summary of results on Steiner systems). Keevash [15] has recently shown that for all \( t \) and \( k \) the obvious necessary conditions for the existence of an \( S(t, k, v) \)-design are sufficient for sufficiently large values of \( v \). Similar results were obtained by Glock, Kühn, Lo, and Osthus [10, 11] and extended to include the corresponding asymptotic results for \( H \)-designs of order \( v \) for all uniform hypergraphs \( H \). These results for \( t \)-uniform hypergraphs mirror the celebrated results of Wilson [24] for graphs. Although these asymptotic results assure the existence of \( H \)-designs for sufficiently large values of \( v \) for any uniform hypergraph \( H \), the spectrum problem has been settled for very few hypergraphs of uniformity larger than 2.

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In the study of graph decompositions, a fair amount of the focus has been on $G$-decompositions of $K_v$, where $G$ is a graph with a relatively small number of edges (see [1] and [6] for known results). Some authors have investigated the corresponding problem for 3-uniform hypergraphs. For example, in [4], the 1-fold spectrum problem is settled for all 3-uniform hypergraphs on 4 or fewer vertices. More recently, the 1-fold spectrum problem was settled in [5] for all 3-uniform hypergraphs with at most 6 vertices and at most 3 edges. In [5], they also settle the 1-fold spectrum problem for the 3-uniform hypergraph of order 6 whose edges form the lines of the Pasch configuration. Authors have also considered $H$-designs where $H$ is a 3-uniform hypergraph whose edge set is defined by the faces of a regular polyhedron. Let $T$, $O$, and $I$ denote the tetrahedron, the octahedron, and the icosahedron hypergraphs, respectively. The hypergraph $T$ is the same as $K_4^{(3)}$, and its spectrum was settled in 1960 by Hanani [12]. In another paper [13], Hanani settled the spectrum problem for $O$-designs and gave necessary conditions for the existence of $I$-designs. The 1-fold spectrum problem is also settled for a type of 3-uniform hyperstars which is part of a larger class of hypergraphs known as delta-systems. For a positive integer $m$, let $S^{(3)}_m$ denote the 3-uniform hypergraph of size $m$ that consists of one vertex of degree $m$ and $2m$ vertices of degree one. Necessary and sufficient conditions for the existence of $S^{(3)}_m$-decompositions of $K^{(3)}_v$ are given in [22] for $m \in \{4, 5, 6\}$ and settled in [19] for any $m$. Some results on maximum $S^{(3)}_m$-packings of $K^{(3)}_v$ are given in [20]. Perhaps the best known general result on decompositions of complete $t$-uniform hypergraphs is Baranyai’s result [3] on the existence of 1-factorizations of $K^{(t)}_{mt}$ for all positive integers $m$. There are, however, several articles on decompositions of complete $t$-uniform hypergraphs (see [2] and [21]) and of $t$-uniform $t$-partite hypergraphs (see [17] and [23]) into variations on the concept of a Hamilton cycle. There are also several results on decompositions of 3-uniform hypergraphs into structures known as Berge cycles with a given number of edges (see for example [14] and [18]). We note however that the Berge cycles in these decompositions are not required to be isomorphic.

In this paper we are interested in maximum $H$-packings of $\lambda K^{(3)}_v$, where $H$ is a 3-uniform symmetric triple-hyperstar with 4 edges. A triple-hyperstar is a connected 3-uniform hypergraph where, for some edge $\{a, b, c\}$, vertices $a$, $b$, and $c$ all have degree greater than 1 and all other edges contain exactly two vertices of degree 1. That is, if the degrees of vertices $a$, $b$, and $c$ in the triple-hyperstar are $m_1 + 1$, $m_2 + 1$, and $m_3 + 1$, respectively, then the removal of edge $\{a, b, c\}$ would result in the hypergraph consisting of three components, namely $S^{(3)}_{m_1}$, $S^{(3)}_{m_2}$, and $S^{(3)}_{m_3}$. We call such a triple-hyperstar symmetric if $m_1 = m_2 = m_3 = m$. Thus a symmetric triple-hyperstar has $6m + 3$ vertices and $3m + 1$ edges. We are interested in the case $m = 1$.

Let $H[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9]$ denote the symmetric triple-hyperstar $H$ with vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ and edge set $\{\{v_1, v_2, v_3\}, \{v_1, v_4, v_5\}, \{v_2, v_6, v_7\}, \{v_3, v_8, v_9\}\}$ as seen Figure 1. Here we show that for all $v \geq 9$ and $\lambda \geq 1$, there exists a maximum $H$-packing of $\lambda K^{(3)}_v$ where the leave has fewer than 4 edges.

### 1.1. Additional Notation and Terminology

Let $\mathbb{Z}_n$ denote the group of integers modulo $n$. We next define some notation for certain types of 3-uniform hypergraphs.
Let $U_1, U_2, U_3$ be pairwise disjoint sets. The hypergraph with vertex set $U_1 \cup U_2 \cup U_3$ and edge set consisting of all 3-element sets having exactly one vertex in each of $U_1, U_2, U_3$ is denoted by $K^{(3)}_{U_1,U_2,U_3}$. The hypergraph with vertex set $U_1 \cup U_2$ and edge set consisting of all 3-element sets having at most 2 vertices in each of $U_1, U_2$ is denoted by $L^{(3)}_{U_1,U_2}$. If $|U_i| = u_i$ for $i \in \{1, 2, 3\}$, we may use $K^{(3)}_{u_1,u_2,u_3}$ or $L^{(3)}_{u_1,u_2}$ to denote any hypergraph that is isomorphic to $K^{(3)}_{U_1,U_2,U_3}$ or $L^{(3)}_{U_1,U_2}$, respectively.

2. Main Results

2.1. Decompositions and Packings of Simple Hypergraphs

We begin by giving necessary conditions for the existence of an $H$-decomposition of $K^{(3)}_v$. An obvious necessary condition is that 4 must divide the number of edges in $K^{(3)}_v$, and thus we must have $v \equiv 0, 1, 2, 4, \text{ or } 6 \pmod{8}$. Since $K^{(3)}_1$ and $K^{(3)}_2$ contain no edges, it is vacuously true that $H$ decomposes $K^{(3)}_1$ and $K^{(3)}_2$. Also, since $H$ has order 9, there is no $H$-decomposition of $K^{(3)}_4$, $K^{(3)}_6$, or $K^{(3)}_8$. Hence, we have the following.

**Lemma 1.** There exists an $H$-decomposition of $K^{(3)}_v$ only if $v \equiv 0, 1, 2, 4, \text{ or } 6 \pmod{8}$ and $v \not\in \{4, 6, 8\}$.

We intend to prove that the above conditions are sufficient by showing how to construct $H$-decompositions of $K^{(3)}_v$ for all $v \equiv 0, 1, 2, 4, \text{ or } 6 \pmod{8}$ with $v \geq 9$. Our constructions are dependent on the many small examples given in the Appendix. We begin by proving a lemma that is fundamental to our constructions.

**Lemma 2.** Let $n$, $x$, and $r$ be nonnegative integers such that $nx + r \geq 3$. There exists a decomposition of $K^{(3)}_{nx+r}$ that is comprised of isomorphic copies of each of the following under the given conditions:

- $K^{(3)}_r$ if $x = 0$,
- $K^{(3)}_{n+x}$ if $x \geq 1$,
• $K_{n+r}^{(3)} \setminus K_r^{(3)}$ if $x \geq 2$,
• $K_{r,n,n}^{(3)} \cup L_{n,n}^{(3)}$ if $x \geq 2$,
• $K_{n,n,n}^{(3)}$ if $x \geq 3$.

Furthermore, if $x \geq 1$ and $r \geq 3$, then the decomposition contains exactly one isomorphic copy of $K_{n+r}^{(3)}$.

Proof. If $x \in \{0, 1\}$, the decomposition is trivial. Similarly, if $n = 0$, then $r \geq 3$, and the result is trivial because $K_r^{(3)} = K_{n+r}^{(3)} = K_{nx+r}^{(3)}$ while $K_{n+r}^{(3)} \setminus K_r^{(3)}$, $K_{r,n,n}^{(3)} \cup L_{n,n}^{(3)}$, and $K_{n,n,n}^{(3)}$ are all empty (i.e., contain no edges). For the remainder of the proof, we assume that $x \geq 2$ and $n \geq 1$.

Let $V_0, V_1, \ldots, V_x$ be pairwise disjoint sets of vertices with $|V_0| = r$ and $|V_1| = |V_2| = \ldots = |V_x| = n$. Then, the decomposition of $K_{nx+r}^{(3)}$ results from the fact that the complete 3-uniform hypergraph on the vertex set $V_0 \cup V_1 \cup \cdots \cup V_x$, which is $nx + r$ vertices, can be viewed as the (edge-disjoint) union

$$K_{V_1 \cup V_0}^{(3)} \cup \bigcup_{2 \leq i \leq x} \left( K_{V_i \cup V_0}^{(3)} \setminus K_{V_0}^{(3)} \right) \cup \bigcup_{1 \leq i < j \leq x} \left( K_{V_0, V_i, V_j}^{(3)} \cup L_{V_i, V_j}^{(3)} \right) \cup \bigcup_{1 \leq i < j < k \leq x} \left( K_{V_i, V_j, V_k}^{(3)} \right).$$

In addition, if $r \geq 3$, the single isomorphic copy of $K_{n+r}^{(3)}$ in the decomposition is $K_{V_1 \cup V_0}^{(3)}$.

We now give our main results.

**Theorem 3.** There exists an $H$-decomposition of $K_v^{(3)}$ if and only if $v \equiv 0, 1, 2, 4, \text{ or } 6 \pmod 8$ and $v \not\in \{4, 6, 8\}$.

Proof. The necessary conditions for the existence of an $H$-decomposition of $K_v^{(3)}$ are established in Lemma 1. Thus we need only to establish their sufficiency. Let $v = 8x + r$ where $x \geq 1$ and $r \in \{1, 2, 4, 6, 8\}$. By Lemma 2 it suffices to find $H$-decompositions of $K_v^{(3)}$, $K_{v+2r}^{(3)} \setminus K_r^{(3)}$, $K_{r,8,8}^{(3)} \cup L_{8,8,8}^{(3)}$, and $K_{r,8,8,8}^{(3)}$. We note that if $r \in \{1, 2\}$ then $K_{8+r}^{(3)} \setminus K_r^{(3)}$ is isomorphic to $K_{8+r}^{(3)}$. Also, $K_{r,8,8,8}^{(3)}$ decomposes $K_{6,8,8}^{(3)}$ and $K_{4,8,8}^{(3)}$ decomposes $K_{8,8,8}^{(3)}$. Thus, it suffices to find $H$-decompositions of $K_{9}^{(3)}$, $K_{10}^{(3)}$, $K_{12}^{(3)}$, $K_{14}^{(3)}$, $K_{16}^{(3)}$, $K_{18}^{(3)} \setminus K_{4}^{(3)}$, $K_{14}^{(3)} \setminus K_{6}^{(3)}$, $K_{16}^{(3)} \setminus K_{8}^{(3)}$, $K_{18,8} \cup L_{8,8,8}^{(3)}$, $K_{2,8,8} \cup L_{8,8,8}^{(3)}$, $K_{3,8,8,8}^{(3)}$, $K_{4,8,8,8}^{(3)}$, and $L_{8,8,8}^{(3)}$, which are each shown to exist within Examples 1–16.

**Theorem 4.** If $v \geq 9$ is an integer, then there exists a maximum $H$-packing of $K_v^{(3)}$ where the leave has fewer than four edges.

Proof. If $v \equiv 0, 1, 2, 4, \text{ or } 6 \pmod 8$, then the result follows from the $H$-decomposition result in Theorem 3, which translates to a maximum $H$-packing with an empty leave. Hence, we need only consider when $v \equiv 3, 5, \text{ or } 7 \pmod 8$. Let $v = 8x + r$ where $x \geq 1$ and $r \in \{3, 5, 7\}$. By Lemma 2 it suffices to find

• a maximum $H$-packing of $K_{8+r}^{(3)}$, with a leave consisting of fewer than four edges and
• $H$-decompositions of $K_{8+r}^{(3)} \setminus K_r^{(3)}$, $K_{r,8,8}^{(3)} \cup L_{8,8,8}^{(3)}$, and $K_{8,8,8}^{(3)}$. 

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We note that an $H$-decomposition of $K_{11}^{(3)} \setminus K_3^{(3)}$ is a subset of an $H$-packing of $K_{11}^{(3)}$ with a leave consisting of the single edge in the hole, which is necessarily then a maximum $H$-packing of $K_{11}^{(3)}$. Also, $K_{7,8,8}^{(3)}$ is decomposable into copies of $K_{3,8,8}^{(3)}$ and $K_{4,8,8}^{(3)}$, and $K_{4,8,8}^{(3)}$ decomposes $K_{8,8,8}^{(3)}$. Thus, it suffices to find maximum $H$-packings (with leaves of fewer than four edges) of $K_{11}^{(3)}$, $K_{5}^{(3)}$, and $K_{13}^{(3)}$, which are each shown to exist in Examples 17–19, and $H$-decompositions of $K_{13}^{(3)} \setminus K_5^{(3)}$, $K_{15}^{(3)} \setminus K_7^{(3)}$, $K_{3,8,8}^{(3)}$, $K_{4,8,8}^{(3)}$, $K_{5,8,8}^{(3)}$, and $L_{8,8}^{(3)}$, which are each shown to exist within Examples 6–15.

2.2. Results for any Positive Index

We show here the necessary conditions for an $H$-decomposition of $\lambda$-fold $K_v^{(3)}$ for any positive integer $\lambda$. This will inform our choice on which combinations of $\lambda$ and $v$ we search for decompositions of $\lambda K_v^{(3)}$ versus finding maximum packings.

**Lemma 5.** Let $v \geq 9$ be an integer. There exists an $H$-decomposition of $\lambda$-fold $K_v^{(3)}$ only if the following hold:

- if $\gcd(\lambda, 4) = 1$, then $v \equiv 0, 1, 2, 4, \text{ or } 6 \pmod{8}$;
- if $\gcd(\lambda, 4) = 2$, then $v \equiv 0, 1, \text{ or } 2 \pmod{4}$;
- if $\gcd(\lambda, 4) = 4$, then $v \geq 9$.

**Proof.** Suppose there exists an $H$-decomposition of $\lambda K_v^{(3)}$. Since $|E(H)| = 4$, we must have $4 \mid \lambda(v^2 - 1)(v - 2)/6$, and thus $8 \mid \lambda v(v - 1)(v - 2)$. First, if $\gcd(\lambda, 4) = 1$, then $8 \mid v(v - 1)(v - 2)$, and thus $v \equiv 0, 1, 2, 4, \text{ or } 6 \pmod{8}$. Second, if $\gcd(\lambda, 4) = 2$, then $4 \mid v(v - 1)(v - 2)$, and thus $v \equiv 0, 1, \text{ or } 2 \pmod{4}$. Finally, if $\gcd(\lambda, 4) = 4$, then $2 \mid v(v - 1)(v - 2)$, which is true for any $v \geq 9$. 

Next, we settle the decomposition and maximum packing results for some small values of $\lambda$.

**Theorem 6.** Let $v \geq 9$ be an integer. There exists an $H$-decomposition of 2-fold $K_v^{(3)}$ if $v \equiv 0, 1, \text{ or } 2 \pmod{4}$.

**Proof.** If $v \equiv 0, 1, 2, 4, \text{ or } 6 \pmod{8}$, then the result follows from 2 copies of an $H$-decomposition of $K_v^{(3)}$, which exists by Theorem 3. Hence, we need only consider when $v \equiv 5 \pmod{8}$.

First, we consider when $v = 13$. Let $v_1, v_2, \ldots, v_9 \in V(K_v^{(3)})$. By Example 18, there exist both a maximum $H$-packing, say $\Delta_1$, of $K_{13}^{(3)}$ with a leave consisting of two edges that share a single vertex and a maximum $H$-packing, say $\Delta_2$, of $K_{13}^{(3)}$ with a leave consisting of two vertex-disjoint edges. Let $L_1$ and $L_2$ be the leaves of $\Delta_1$ and $\Delta_2$, respectively. Without loss of generality, we may assume that

$$E(L_1) = \{\{v_1, v_2, v_3\}, \{v_1, v_4, v_5\}\},$$
$$E(L_2) = \{\{v_2, v_6, v_7\}, \{v_3, v_8, v_9\}\}.$$
Now, let $L'$ be the hypergraph with edge set $E(L_1) \cup E(L_2)$. Hence, $L'$ is isomorphic to $H$, and the (multi-)set

$$(\Delta_1 \setminus \{L_1\}) \cup (\Delta_2 \setminus \{L_2\}) \cup \{L'\}$$

is a collection of $H$-blocks such that each edge of $K_v^{(3)}$ is represented exactly twice. Therefore, we have an $H$-decomposition of $2K_3^{(3)}$.

Now, let $v = 8x + 5$ where $x \geq 2$. By Lemma 2 it suffices to find $H$-decompositions of (2-fold) $K_{13}^{(3)}$, $K_{13}^{(3)} \setminus K_5^{(3)}$, $K_{5,8,8}^{(3)} \cup L_{8,8}^{(3)}$, and $K_{8,8,8}^{(3)}$. We note that $K_{4,8,8}^{(3)}$ decomposes $K_{8,8,8}^{(3)}$, and we already have that $H$ decomposes $2K_3^{(3)}$. Thus, we need only additionally find $H$-decompositions of $K_{13}^{(3)} \setminus K_5^{(3)}$, $K_{5,8,8}^{(3)}$, $L_{8,8}^{(3)}$, and $K_{8,8,8}^{(3)}$, which exist by Examples 13, 11, 6, and 10, respectively. \hfill \Box

**Theorem 7.** If $v \geq 9$ is an integer, then there exists a maximum $H$-packing of 2-fold $K_v^{(3)}$ where the leave has no edges or two vertex-disjoint edges.

**Proof.** If $v \equiv 0$, 1, or 2 (mod 4), then the result follows from the $H$-decomposition result in Theorem 6, which translates to a maximum $H$-packing with an empty leave. Hence, we need only consider when $v \equiv 3$ (mod 4).

First, we consider when $v = 11$. Let $\Delta_1$ and $\Delta_2$ be maximum $H$-packings of $K_{11}^{(3)}$ with leaves $L_1$ and $L_2$, respectively, which exist by Example 17. Now, let $L'$ be the hypergraph with edge (multi-)set $E(L_1) \cup E(L_2)$. Hence, $L'$ consists of two edges. In fact, we further note that $L'$ can be any hypergraph with two edges, including $2K_3^{(3)}$. Hence, the (multi-)set

$$(\Delta_1 \setminus \{L_1\}) \cup (\Delta_2 \setminus \{L_2\}) \cup \{L'\}$$

is a maximum $H$-packing of $2K_3^{(3)}$ with a leave, $L'$, consisting of two (possibly vertex-disjoint) edges.

Second, we consider when $v = 15$. Let $v_1, v_2, \ldots, v_9 \in V(K_v^{(3)})$. By Example 19, there exist maximum $H$-packings of $K_{15}^{(3)}$ where the leaves consist of three disjoint edges. Let $\Delta_1$ and $\Delta_2$ be such $H$-packings of $K_{15}^{(3)}$ with leaves $L_1$ and $L_2$, respectively. Without loss of generality, we may assume that

$$E(L_1) = \{\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}, \{v_7, v_8, v_9\}\},$$

$$E(L_2) = \{\{v_1, v_4, v_5\}, \{v_2, v_6, v_7\}, \{v_3, v_8, v_9\}\}.$$ 

Now, let $L'$ be the hypergraph with edge set $E(L_1) \cup E(L_2)$. We note that $L'$ is decomposable into copies of $K_3^{(3)}$ and $H$. That is, if we let $L''$ be the hypergraph with edge set $\{\{v_4, v_5, v_6\}, \{v_7, v_8, v_9\}\}$, then $L' \setminus L''$ is isomorphic to $H$, and the (multi-)set

$$(\Delta_1 \setminus \{L_1\}) \cup (\Delta_2 \setminus \{L_2\}) \cup \{L' \setminus L'', L''\}$$

is a maximum $H$-packing of $2K_{15}^{(3)}$ with a leave, $L''$, consisting of two (disjoint) edges.

Now, let $v = 8x + r$ where $x \geq 2$ and $r \in \{3, 7\}$. By Lemma 2 it suffices to find
• a maximum $H$-packing of (2-fold) $K_{8+r}^{(3)}$ with a leave consisting of fewer than four edges and
• $H$-decompositions of $K_{8+r}^{(3)} \setminus K_8^{(3)}, K_{r,8,8}^{(3)} \cup L_{8,8,8}^{(3)}$, and $K_{8,8,8}^{(3)}$.

We already have the maximum $H$-packing results. Also, we note that $K_{7,8,8}^{(3)}$ is decomposable into copies of $K_{3,8,8}^{(3)}$ and $K_{4,8,8}^{(3)}$, and $K_{4,8,8}^{(3)}$ decomposes $K_{8,8,8}^{(3)}$. Thus, we need only additionally find $H$-decompositions of $K_{11}^{(3)} \setminus K_3^{(3)}, K_{15}^{(3)} \setminus K_7^{(3)}, K_{3,8,8}^{(3)}, K_{4,8,8}^{(3)}$, and $L_{8,8,8}^{(3)}$, which exist by Examples 17, 15, 9, 10, and 6, respectively.

**Theorem 8.** If $v \geq 9$ is an integer, then there exists a maximum $H$-packing of 3-fold $K_v^{(3)}$ where the leave has fewer than four edges.

**Proof.** If $v \equiv 0, 1, 2, 4, \text{ or } 6 \pmod{8}$, then the result follows from the $H$-decomposition result in Theorem 3, which translates to a maximum $H$-packing with an empty leave. Hence, we need only consider when $v \equiv 3, 5, \text{ or } 7 \pmod{8}$.

First, we consider when $v = 11$. Let $\Delta_1$ be a maximum $H$-packing of $K_{11}^{(3)}$ with leave $L_1$ consisting of a single edge, which exists by Example 17, and let $\Delta_2$ be a maximum $H$-packing of $2K_{11}^{(3)}$ with leave $L_2$ consisting of two edges, which exists by Theorem 7. Now, let $L'$ be the hypergraph with edge (multi-)set $E(L_1) \cup E(L_2)$. Hence, $L'$ consists of three edges. In fact, we further note that $L'$ can be any hypergraph with three edges, including $3K_3^{(3)}$. Hence, the (multi-)set

$$((\Delta_1 \setminus \{L_1\}) \cup (\Delta_2 \setminus \{L_2\}) \cup \{L'\})$$

is a maximum $H$-packing of $3K_{11}^{(3)}$ with a leave, $L'$, consisting of three edges.

Second, we consider when $v = 13$. Let $\Delta_1$ be a maximum $H$-packing of $K_{13}^{(3)}$ with leave $L_1$ consisting of two edges, which exists by Example 18, and let $\Delta_2$ be an $H$-decomposition of $2K_{13}^{(3)}$, which exists by Theorem 6. Hence, the (multi-)set $\Delta_1 \cup \Delta_2$ is a maximum $H$-packing of $3K_{13}^{(3)}$ with a leave, $L_1$, consisting of two edges.

Third, we consider when $v = 15$. Let $v_1, v_2, \ldots, v_9 \in V(K_3^{(3)})$, let $\Delta_1$ be a maximum $H$-packing of $K_{15}^{(3)}$ with leave $L_1$ consisting of a three vertex-disjoint edges, which exists by Example 19, and let $\Delta_2$ be a maximum $H$-packing of $2K_{15}^{(3)}$ with leave $L_2$ consisting of two vertex-disjoint edges, which exists by Theorem 7. Without loss of generality, we may assume that

$$E(L_1) = \{\{v_1, v_4, v_5\}, \{v_2, v_6, v_7\}, \{v_3, v_8, v_9\}\},$$

$$E(L_2) = \{\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}\}.$$  

Now, let $L'$ be the hypergraph with edge set $E(L_1) \cup E(L_2)$. We note that $L'$ is decomposable into copies of $K_3^{(3)}$ and $H$. That is, if we let $L''$ be the hypergraph with the single edge $\{v_4, v_5, v_6\}$, then $L' \setminus L''$ is isomorphic to $H$, and the (multi-)set

$$((\Delta_1 \setminus \{L_1\}) \cup (\Delta_2 \setminus \{L_2\}) \cup \{L' \setminus L'', L''\})$$

is a maximum $H$-packing of $3K_{15}^{(3)}$ with a leave, $L''$, consisting of one edges.

Now, let $v = 8x + r$ where $x \geq 2$ and $r \in \{3, 5, 7\}$. By Lemma 2 it suffices to find
• a maximum $H$-packing of $(3$-fold) $K_{8+r}^{(3)}$ with a leave consisting of fewer than four edges and

• $H$-decompositions of $K_{8+r}^{(3)} \setminus K_r^{(3)}, K_r^{(3)} \setminus L_{8,8,8}$, and $K_{8,8,8}^{(3)}$.

We already have the maximum $H$-packing results. Also, we note that $K_{7,8,8}^{(3)}$ is decomposable into copies of $K_{3,8,8}^{(3)}$ and $K_{4,8,8}^{(3)}$, and $K_{r,8,8}^{(3)}$ decomposes $K_{8,8,8}^{(3)}$. Thus, we need only additionally find $H$-decompositions of $K_{11}^{(3)} \setminus K_3^{(3)}, K_{13}^{(3)} \setminus K_5^{(3)}, K_{15}^{(3)} \setminus K_7^{(3)}, K_3^{(3)}, K_{5,8,8}^{(3)}, K_{4,8,8}^{(3)}$, and $L_{8,8,8}$, which exist by Examples 17, 13, 15, 9, 11, 10, and 6, respectively.

Theorem 9. Let $v \geq 9$ be an integer. There exists an $H$-decomposition of $4$-fold $K_v^{(3)}$.

Proof. If $v \equiv 0, 1, \text{ or } 2 \pmod{4}$, then the result follows from $2$ copies of an $H$-decomposition of $2K_v^{(3)}$, which exists by Theorem 6. Hence, we need only consider when $v \equiv 3 \pmod{4}$. Let $v_1, v_2, \ldots, v_9 \in V(K_v^{(3)})$.

First, we consider when $v = 11$. For $i \in \{1, 2, 3, 4\}$, let $\Delta_i$ be a maximum $H$-packing of $K_{11}^{(3)}$ with leave $L_i$ consisting of a single edge, which exists by Example 17, Without loss of generality, we may assume that

\[
E(L_1) = \{\{v_1, v_2, v_3\}\}, \quad E(L_2) = \{\{v_1, v_4, v_5\}\},
E(L_3) = \{\{v_2, v_6, v_7\}\}, \quad E(L_4) = \{\{v_3, v_8, v_9\}\}.
\]

Now, let $L'$ be the hypergraph with edge set $E(L_1) \cup E(L_2) \cup E(L_3) \cup E(L_4)$. Hence, $L'$ is isomorphic to $H$, and the (multi-)set

\[
L' \cup \bigcup_{i=1}^{4}(\Delta_i \setminus \{L_i\})
\]

is a collection of $H$-blocks such that each edge of $K_{11}^{(3)}$ is represented exactly four times. Therefore, we have an $H$-decomposition of $4K_{11}^{(3)}$.

Second, we consider when $v = 15$. Let $\Delta_1$ be a maximum $H$-packing of $K_{15}^{(3)}$ with leave $L_1$ consisting of a three vertex-disjoint edges, which exists by Example 19, and let $\Delta_2$ be a maximum $H$-packing of $3K_{15}^{(3)}$ with leave $L_2$ consisting of a single edge, which exists by Theorem 8, Without loss of generality, we may assume that

\[
E(L_1) = \{\{v_1, v_4, v_5\}, \{v_2, v_6, v_7\}, \{v_3, v_8, v_9\}\},
E(L_2) = \{\{v_1, v_2, v_3\}\}.
\]

Now, let $L'$ be the hypergraph with edge set $E(L_1) \cup E(L_2)$. Hence, $L'$ is isomorphic to $H$, and the (multi-)set

\[(\Delta_1 \setminus \{L_1\}) \cup (\Delta_2 \setminus \{L_2\}) \cup \{L'\}
\]

is a collection of $H$-blocks such that each edge of $K_{15}^{(3)}$ is represented exactly four times. Therefore, we have an $H$-decomposition of $4K_{15}^{(3)}$. 

\[\square\]
Now, let \( v = 8x + r \) where \( x \geq 2 \) and \( r \in \{3, 7\} \). By Lemma 2 it suffices to find \( H \)-decompositions of (4-fold) \( K_{8r+7}^{(3)} \), \( K_{8r+3}^{(3)} \setminus K_r^{(3)} \), \( K_r^{(3)} \cup I_{8,8}^{(3)} \), and \( K_{8,8,8}^{(3)} \). We note that \( K_{8,8,8}^{(3)} \) is decomposable into copies of \( K_{3,8,8}^{(3)} \) and \( K_{4,8,8}^{(3)} \). Also, \( K_{8,8,8}^{(3)} \) decomposes \( K_{8,8,8}^{(3)} \), and we already have that \( H \) decomposes \( 4K_{11}^{(3)} \) and \( 4K_{15}^{(3)} \). Thus, we need only additionally find \( H \)-decompositions of \( K_{11}^{(3)} \setminus K_3^{(3)} \), \( K_{15}^{(3)} \setminus K_7^{(3)} \), \( K_{4,8,8}^{(3)} \), \( K_{4,8,8}^{(3)} \), and \( I_{8,8}^{(3)} \), which exist by Examples 17, 15, 9, 10, and 6, respectively.

Finally, we show that the necessary conditions for the existence of an \( H \)-decomposition of \( \lambda \)-fold \( K_v^{(3)} \) are sufficient.

**Theorem 10.** Let \( \lambda \) and \( v \) be positive integers with \( v \geq 9 \). There exists an \( H \)-decomposition of \( \lambda \)-fold \( K_v^{(3)} \) if and only if the following hold:

- if \( \gcd(\lambda, 4) = 1 \), then \( v \equiv 0, 1, 2, 4, \) or \( 6 \) (mod 8);
- if \( \gcd(\lambda, 4) = 2 \), then \( v \equiv 0, 1, \) or \( 2 \) (mod 4);
- if \( \gcd(\lambda, 4) = 4 \), then \( v \geq 9 \).

**Proof.** The necessary conditions are established in Lemma 5. For sufficiency, we consider the following cases.

**Case 1.** \( \lambda \equiv 0 \) (mod 4)
Let \( \lambda = 4t \) for some positive integer \( t \). Then the result follows from \( t \) copies of an \( H \)-decomposition of \( 4K_v^{(3)} \), which exists by Theorem 9.

**Case 2.** \( \lambda \equiv 1 \) or \( 3 \) (mod 4)
Since \( \gcd(\lambda, 4) = 1 \), we have that \( v \equiv 0, 1, 2, 4, \) or \( 6 \) (mod 8). Let \( \lambda = 4t + r \) for some integers \( t \geq 0 \) and \( r \in \{1, 3\} \). Then the result follows from \( t \) copies of an \( H \)-decomposition of \( 4K_v^{(3)} \), which exists by Theorem 9, and \( r \) copies of an \( H \)-decomposition of \( K_v^{(3)} \), which exists by Theorem 3.

**Case 3.** \( \lambda \equiv 2 \) (mod 4)
Since \( \gcd(\lambda, 4) = 2 \), we have that \( v \equiv 0, 1, \) or \( 2 \) (mod 4). Let \( \lambda = 4t + 2 \) for some nonnegative integer \( t \). Then the result follows from \( t \) copies of an \( H \)-decomposition of \( 4K_v^{(3)} \), which exists by Theorem 9, and 1 copy of an \( H \)-decomposition of \( 2K_v^{(3)} \), which exists by Theorem 6.

**Theorem 11.** If \( v \geq 9 \) is an integer, then there exists a maximum \( H \)-packing of \( \lambda \)-fold \( K_v^{(3)} \) where the leave has fewer than four edges.

**Proof.** If \( 1 \leq \lambda \leq 3 \), then the result follows from Theorems 4, 7, and 8. If \( \lambda = 4 \), then the result follows from the \( H \)-decomposition result in Theorem 9, which translates to a maximum \( H \)-packing with an empty leave. For the remainder of the proof, we assume that \( \lambda \geq 5 \). Let \( \lambda = 4t + r \) for some integers \( t \geq 1 \) and \( r \in \{1, 4\} \). Then the result follows from \( t \) copies of an \( H \)-decomposition of \( 4K_v^{(3)} \), which exists by Theorem 9, and 1 copy of a maximum \( H \)-packing of \( r \)-fold \( K_v^{(3)} \).
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References

[1] P. Adams, D. Bryant, and M. Buchanan, A survey on the existence of G-designs, *J. Combin. Des.* **16** (2008), 373–410.

[2] R.F. Bailey and B. Stevens, Hamiltonian decompositions of complete k-uniform hypergraphs, *Discrete Math.* **310** (2010), 3088–3095.

[3] Zs. Baranyai, On the factorization of the complete uniform hypergraph, in: Infinite and finite sets, *Colloq. Math. Soc. János Bolyai* **10**, North-Holland, Amsterdam, 1975, 91–108.

[4] J.C. Bermond, A. Germa, and D. Sotteau, Hypergraph-designs, *Ars Combinatoria* **3** (1977), 47–66.

[5] D. Bryant, S. Herke, B. Maenhaut, and W. Wannasit, Decompositions of complete 3-uniform hypergraphs into small 3-uniform hypergraphs, *Australas. J. Combin.* **60** (2014), 227–254.

[6] D.E. Bryant and T.A. McCourt, Existence results for G-designs, http://wiki.smp.uq.edu.au/G-designs/

[7] R.C. Bunge, S.I. El-Zanati, L. Haman, C. Hatzer, K. Koe, and K. Spornberger, On loose 4-cycle decompositions of complete 3-uniform hypergraphs, submitted.

[8] C.J. Colbourn and J.H. Dinitz (Editors), *Handbook of Combinatorial Designs*, 2nd ed., Chapman & Hall/CRC Press, Boca Raton, FL, 2007.

[9] C.J. Colbourn and R.Mathon, “Steiner systems,” in [8], pp. 102–110.

[10] S. Glock, D. Kühn, A. Lo, and D. Osthus, The existence of designs via iterative absorption, arXiv:1611.06827v2, (2017), 63 pages.

[11] S. Glock, D. Kühn, A. Lo, and D. Osthus, Hypergraph F-designs for arbitrary F, arXiv:1706.01800, (2017), 72 pages.

[12] H. Hanani, On quadruple systems, *Canad. J. Math.*, **12** (1960), 145–157.

[13] H. Hanani, Decomposition of hypergraphs into octahedra, Second International Conference on Combinatorial Mathematics (New York, 1978), pp. 260–264, *Ann. New York Acad. Sci.*, 319, New York Acad. Sci., New York, 1979.
[14] H. Jordon, and G. Newkirk, 4-cycle decompositions of complete 3-uniform hypergraphs, *Australas. J. Combin.* 71 (2018), 312–323.

[15] P. Keevash, The existence of designs, arXiv:1401.3665v2, (2018), 39 pages.

[16] G.B. Khosrovshahi, and R. Laue, “$t$-designs with $t \geq 3$,” in [8], pp. 79–101.

[17] J. Kuhl and M.W. Schroeder, Hamilton cycle decompositions of $k$-uniform $k$-partite hypergraphs, *Australas. J. Combin.* 56 (2013), 23–37.

[18] D. Kühn, and D. Osthus, Decompositions of complete uniform hypergraphs into Hamilton cycle graphs, *J. Combin. Theory Ser. A* 126 (2014), 128–135.

[19] Z. Lonc, Solution of a delta-system decomposition problem, *J. Combin. Theory, Ser. A* 55 (1990), 33–48.

[20] Z. Lonc, Packing, covering and decomposing of a complete uniform hypergraph into delta-systems, *Graphs Combin.* 8 (1992), 333–341.

[21] M. Meszka and A. Rosa, Decomposing complete 3-uniform hypergraphs into Hamiltonian cycles, *Australas. J. Combin.* 45 (2009), 291–302.

[22] A.F. Mouyart and F. Sterboul, Decomposition of the complete hypergraph into delta-systems II, *J. Combin. Theory, Ser. A* 41 (1986), 139–149.

[23] M.W. Schroeder, On Hamilton cycle decompositions of $r$-uniform $r$-partite hypergraphs, *Discrete Math.* 315 (2014), 1–8.

[24] R.M. Wilson, Decompositions of Complete Graphs into Subgraphs Isomorphic to a Given Graph, in “Proc. Fifth British Combinatorial Conference” (C. St. J. A. Nash-Williams and J. Sheehan, Eds.), pp. 647–659, *Congr. Numer.* XV, 1975.

**Appendix: Some Small Examples**

We give several examples of $H$-decompositions and $H$-packings that are used in proving our main result.

**Decomposition Examples**

**Example 1.** Let $V(K_9^{(3)}) = \mathbb{Z}_7 \cup \{\infty_1, \infty_2\}$ and let

$$B = \{H[0, 1, 4, 5, 6, \infty_1, 3, \infty_2, 2], H[\infty_1, \infty_2, 0, 3, 6, 1, 2, 4, 5], H[0, 2, 5, \infty_2, 4, \infty_1, 1, 6, 3]\}.$$

Then an $H$-decomposition of $K_9^{(3)}$ consists of the orbits of the $H$-blocks in $B$ under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1, 2\}$, and $j \mapsto j + 1 \pmod{7}$.
Example 2. Let $V(K_{10}^{(3)}) = \mathbb{Z}_{10}$ and let

$$B = \{ H[0, 2, 4, 8, 9, 3, 6, 5, 1], H[0, 2, 7, 1, 6, 5, 8, 9, 3], H[0, 1, 5, 7, 9, 2, 4, 8, 3] \}.$$ 

Then an $H$-decomposition of $K_{10}^{(3)}$ consists of the orbits of the $H$-blocks in $B$ under the action of the map $j \mapsto j + 1 \pmod{10}$.

Example 3. Let $V(K_{12}^{(3)}) = \mathbb{Z}_{11} \cup \{ \infty \}$ and let

$$B = \{ H[0, 1, 3, 8, 10, 2, 5, 6, 7], H[0, 1, 5, 7, 12, 2, 10, 6, 11], H[\infty, 4, 6, 0, 1, 2, 3, 5, 12],$$

$$H[\infty, 4, 8, 0, 3, 7, 12, 11, 1], H[\infty, 6, 11, 12, 5, 8, 7, 10, 2], H[0, 2, 7, 6, 10, 4, 11, 12, 1],$$

$$H[0, 2, 5, 8, 11, 12, 3, 9] \}.$$ 

Then an $H$-decomposition of $K_{12}^{(3)}$ consists of the orbits of the $H$-blocks in $B$ under the action of the map $\infty \mapsto \infty$ and $j \mapsto j + 1 \pmod{11}$.

Example 4. Let $V(K_{14}^{(3)}) = \mathbb{Z}_{13} \cup \{ \infty \}$ and let

$$B = \{ H[0, 1, 3, 10, 12, 2, 5, 6, 7], H[0, 1, 5, 7, 12, 2, 10, 6, 11], H[\infty, 4, 6, 0, 1, 2, 3, 5, 12],$$

$$H[\infty, 4, 8, 0, 3, 7, 12, 11, 1], H[\infty, 6, 11, 12, 5, 8, 7, 10, 2], H[0, 2, 7, 6, 10, 4, 11, 12, 1],$$

$$H[0, 2, 5, 8, 11, 12, 3, 9] \}.$$ 

Then an $H$-decomposition of $K_{14}^{(3)}$ consists of the orbits of the $H$-blocks in $B$ under the action of the map $\infty \mapsto \infty$ and $j \mapsto j + 1 \pmod{13}$.

Example 5. Let $V(K_{16}^{(3)}) = \mathbb{Z}_{15} \cup \{ \infty \}$ and let

$$B_1 = \{ H[0, 1, 3, 5, 6, 2, 14, 4, 9], H[0, 2, 5, 3, 11, 4, 14, 8, 12], H[0, 1, 4, \infty, 7, 2, 13, 8, 12],$$

$$H[0, 2, 6, 3, 9, 4, 13, \infty, 11], H[0, 2, 8, 7, 14, 4, 11, \infty, 10], H[0, 1, 7, 4, 9, 2, 10, \infty, 13],$$

$$H[0, 1, 5, 3, 6, 2, 12, \infty, 8], H[0, 2, 7, \infty, 1, 4, 12, 9, 11], H[0, 3, 8, 4, 10, 6, 13, \infty, 12] \},$$

$$B_2 = \{ H[0, 5, 10, 1, 2, 6, 7, 11, 12], H[1, 6, 11, 2, 3, 7, 8, 12, 13], H[2, 7, 12, 3, 4, 8, 9, 13, 14],$$

$$H[3, 8, 13, 4, 5, 9, 10, 14, 0], H[4, 9, 14, 5, 6, 10, 11, 0, 1] \}.$$ 

Then an $H$-decomposition of $K_{16}^{(3)}$ consists of the orbits of the $H$-blocks in $B_1$ under the action of the map $\infty \mapsto \infty$ and $j \mapsto j + 1 \pmod{15}$ along with the $H$-blocks in $B_2$.

Example 6. Let $V(L_{8,8}^{(3)}) = \mathbb{Z}_{16}$ with vertex partition $\{ \{ 0, 2, 4, 6, 8, 10, 12, 14 \}, \{ 1, 3, 5, 7, 9, 11, 13, 15 \} \}$ and let

$$B = \{ H[0, 1, 2, 7, 9, 4, 14, 8, 13], H[0, 1, 3, 5, 14, 2, 15, 8, 10], H[0, 4, 5, 7, 15, 8, 3, 12, 13],$$

$$H[0, 6, 13, 3, 7, 12, 15, 1, 4], H[0, 1, 7, 9, 15, 3, 14, 2, 4], H[0, 1, 6, 10, 15, 8, 12, 13, 2],$$

$$H[0, 1, 4, 3, 11, 2, 14, 12, 15] \}.$$ 

Then an $H$-decomposition of $L_{8,8}^{(3)}$ consists of the orbits of the $H$-blocks in $B$ under the action of the map $j \mapsto j + 1 \pmod{16}$.
Example 7. Let $V\left( L_{8,8}^{(3)} \cup K_{1,8,8}^{(3)} \right) = \mathbb{Z}_{16} \cup \{\infty\}$ with vertex partition $\{\infty\}, \{0, 2, 4, 6, 8, 10, 12, 14\}, \{1, 3, 5, 7, 9, 11, 13, 15\}$ and let

$$B = \{ H[0, 1, 3, 5, 14, 2, 15, 8, 10], H[0, 4, 5, 7, 15, 8, 3, 12, 13], H[0, 6, 13, 3, 7, 12, 15, 1, 4], H[0, 1, 7, 9, 15, 3, 14, 2, 4], H[0, 1, 6, 10, 15, 8, 12, 13, 2], H[0, 1, 4, 3, 11, 2, 14, 12, 15], H[\infty, 0, 9, 10, 13, 7, 14, 15, 4], H[\infty, 0, 11, 3, 4, 1, 2, 5, 8]\}.$$

Then an $H$-decomposition of $L_{8,8}^{(3)} \cup K_{1,8,8}^{(3)}$ consists of the orbits of the $H$-blocks in $B$ under the action of the map $\infty \mapsto \infty$ and $j \mapsto j + 1 \pmod{16}$.

Example 8. Let $V\left( L_{8,8}^{(3)} \cup K_{2,8,8}^{(3)} \right) = \mathbb{Z}_{16} \cup \{\infty_1, \infty_2\}$ with vertex partition $\{\infty_1, \infty_2\}, \{0, 2, 4, 6, 8, 10, 12, 14\}, \{1, 3, 5, 7, 9, 11, 13, 15\}$ and let

$$B = \{ H[0, 1, 3, 5, 14, 2, 15, 8, 10], H[0, 4, 5, 7, 15, 8, 3, 12, 13], H[\infty_1, 0, 15, 3, 10, 1, 4, 11, 14], H[0, 1, 7, 9, 15, 3, 14, 2, 4], H[0, 1, 6, 10, 15, 8, 12, 13, 2], H[\infty_1, 0, 3, 9, 14, 5, 13, 8, 11], H[0, 6, 13, 3, 7, 12, 15, 1, 4], H[\infty_2, 0, 9, 5, 6, 14, 15, 11, 2], H[\infty_2, 0, 13, 5, 10, 3, 6, 2, 7]\}.$$

Then an $H$-decomposition of $L_{8,8}^{(3)} \cup K_{2,8,8}^{(3)}$ consists of the orbits of the $H$-blocks in $B$ under the action of the map $\infty_1 \mapsto \infty_i$, for $i \in \{1, 2\}$, and $j \mapsto j + 1 \pmod{16}$.

Example 9. Let $V\left( K_{3,8,8}^{(3)} \right) = \mathbb{Z}_{16} \cup \{\infty_1, \infty_2, \infty_3\}$ with vertex partition $\{\infty_1, \infty_2, \infty_3\}, \{0, 2, 4, 6, 8, 10, 12, 14\}, \{1, 3, 5, 7, 9, 11, 13, 15\}$ and let

$$B = \{ H[\infty_1, 0, 1, 2, 5, \infty_2, 7, \infty_3, 6], H[\infty_2, 0, 1, 2, 5, \infty_3, 7, \infty_1, 6], H[\infty_3, 0, 1, 2, 5, \infty_1, 7, \infty_2, 6]\}.$$

Then an $H$-decomposition of $K_{3,8,8}^{(3)}$ consists of the orbits of the $H$-blocks in $B$ under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1, 2, 3\}$, and $j \mapsto j + 1 \pmod{16}$.

Example 10. Let $V\left( K_{4,8,8}^{(3)} \right) = \mathbb{Z}_{16} \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$ with vertex partition $\{\infty_1, \infty_2, \infty_3, \infty_4\}, \{0, 2, 4, 6, 8, 10, 12, 14\}, \{1, 3, 5, 7, 9, 11, 13, 15\}$ and let

$$B = \{ H[\infty_1, 0, 1, 2, 5, \infty_2, 7, \infty_3, 6], H[\infty_2, 0, 1, 2, 5, \infty_3, 7, \infty_4, 6], H[\infty_3, 0, 1, 2, 5, \infty_4, 7, \infty_1, 6], H[\infty_4, 0, 1, 2, 5, \infty_1, 7, \infty_2, 6]\}.$$

Then an $H$-decomposition of $K_{4,8,8}^{(3)}$ consists of the orbits of the $H$-blocks in $B$ under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1, \ldots, 4\}$, and $j \mapsto j + 1 \pmod{16}$.

Example 11. Let $V\left( K_{5,8,8}^{(3)} \right) = \mathbb{Z}_{16} \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$ with vertex partition $\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}, \{0, 2, 4, 6, 8, 10, 12, 14\}, \{1, 3, 5, 7, 9, 11, 13, 15\}$ and let

$$B = \{ H[\infty_1, 0, 1, 2, 5, \infty_2, 7, \infty_3, 6], H[\infty_2, 0, 1, 2, 5, \infty_3, 7, \infty_4, 6], H[\infty_3, 0, 1, 2, 5, \infty_4, 7, \infty_5, 6], H[\infty_4, 0, 1, 2, 5, \infty_5, 7, \infty_1, 6], H[\infty_5, 0, 1, 2, 5, \infty_1, 7, \infty_2, 6]\}.$$
Then an $H$-decomposition of $K_{5,8,8}^{(3)}$ consists of the orbits of the $H$-blocks in $B$ under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1, \ldots, 5\}$, and $j \mapsto j + 1 \pmod{16}$.

**Example 12.** Let $V \left( K_{12}^{(3)} \setminus K_4^{(3)} \right) = \mathbb{Z}_{12}$ with $0, 3, 6, 9$ being the vertices in the hole and let

$$B_1 = \{ H[0, 3, 7, 2, 5, 6, 11, 9, 4], H[0, 2, 6, 1, 11, 4, 10, 8, 9],$$

$$H[0, 1, 6, 7, 11, 2, 8, 10, 5], H[0, 1, 4, 8, 11, 3, 5, 9, 2] \}$$

$$B_2 = \{ H[7, 8, 10, 1, 4, 2, 5, 0, 9], H[1, 2, 4, 7, 10, 8, 11, 3, 6], H[8, 9, 11, 0, 4, 6, 7, 2, 5],$$

$$H[1, 10, 11, 5, 9, 4, 7, 0, 2], H[2, 3, 5, 6, 10, 0, 1, 8, 11], H[4, 5, 7, 1, 10, 6, 8, 11, 3] \}.$$

Then an $H$-decomposition of $K_{12}^{(3)} \setminus K_4^{(3)}$ consists of the orbits of the $H$-blocks in $B_1$ under the action of the map $j \mapsto j + 1 \pmod{12}$ along with the $H$-blocks in $B_2$.

**Example 13.** Let $V \left( K_{13}^{(3)} \setminus K_5^{(3)} \right) = \mathbb{Z}_8 \cup \{ \infty_1, \infty_2, \infty_3, \infty_4, \infty_5 \}$ with $\infty_1, \infty_2, \infty_3, \infty_4, \infty_5$ being the vertices in the hole and let

$$B_1 = \{ H[\infty_3, \infty_5, 0, 1, 3, \infty_2, 7, \infty_1, \infty_4], H[\infty_4, \infty_5, 0, \infty_3, 6, \infty_1, 7, 2, 3],$$

$$H[\infty_2, \infty_4, 0, 5, 7, 1, 4, \infty_5, 3], H[\infty_4, 0, 2, 4, 5, \infty_1, 7, \infty_5, 3] \},$$

$$B_2 = \{ H[\infty_1, \infty_3, 0, 3, 5, 1, 2, 4, 6], H[\infty_1, \infty_3, 1, 4, 6, 2, 3, 5, 7],$$

$$H[\infty_1, \infty_3, 2, 5, 7, 3, 4, 6, 0], H[\infty_1, \infty_3, 3, 6, 0, 4, 5, 7, 1],$$

$$H[\infty_1, \infty_3, 4, 7, 1, 5, 6, 0, 2], H[\infty_1, \infty_3, 5, 0, 2, 6, 7, 1, 3],$$

$$H[\infty_1, \infty_3, 6, 1, 3, 7, 0, 2, 4], H[2, 4, 7, \infty_2, \infty_3, \infty_1, 1, 5, 6],$$

$$H[3, 5, 0, \infty_2, \infty_3, \infty_1, 2, 6, 7], H[4, 6, 1, \infty_2, \infty_3, \infty_1, 3, 7, 0],$$

$$H[5, 7, 2, \infty_2, \infty_3, \infty_1, 4, 0, 1], H[6, 0, 3, \infty_2, \infty_3, \infty_1, 5, 1, 2],$$

$$H[7, 1, 4, \infty_2, \infty_3, \infty_1, 6, 2, 3], H[0, 2, 5, \infty_2, \infty_3, \infty_1, 7, 3, 4],$$

$$H[2, 3, 7, \infty_3, 5, \infty_1, \infty_2, 1, 6], H[3, 4, 0, \infty_3, 6, \infty_1, \infty_2, 2, 7],$$

$$H[4, 5, 1, \infty_3, 7, \infty_1, \infty_2, 3, 0], H[5, 6, 2, \infty_3, 0, \infty_1, \infty_2, 4, 1],$$

$$H[6, 7, 3, \infty_3, 1, \infty_1, \infty_2, 5, 2], H[7, 0, 4, \infty_3, 2, \infty_1, \infty_2, 6, 3],$$

$$H[0, 1, 5, \infty_3, 3, \infty_1, \infty_2, 7, 4], H[\infty_2, \infty_3, 1, 0, 4, \infty_1, 7, 2, 6],$$

$$H[\infty_2, 1, 5, \infty_1, 2, \infty_3, 0, 6, 4], H[1, 3, 6, \infty_3, 4, 5, 7, \infty_2, 2],$$

$$H[\infty_1, 0, 3, 2, 4, 5, 6, \infty_2, 7], H[\infty_2, 0, 1, 3, 6, \infty_1, 4, \infty_3, 5],$$

$$H[\infty_2, 1, 2, 4, 7, \infty_1, 5, \infty_3, 6], H[\infty_2, 2, 3, 5, 0, \infty_1, 6, \infty_3, 7],$$

$$H[\infty_2, 3, 4, 6, 1, \infty_1, 7, \infty_3, 0], H[\infty_2, 4, 5, 7, 2, \infty_4, 0, \infty_5, 1],$$

$$H[\infty_2, 5, 6, 0, 3, \infty_4, 1, \infty_5, 2], H[\infty_2, 6, 7, 1, 4, \infty_4, 2, \infty_5, 3],$$

$$H[\infty_2, 7, 0, 2, 5, \infty_4, 3, \infty_5, 4], H[\infty_5, 0, 2, 1, 3, 4, 5, 6, 7],$$

$$H[\infty_5, 3, 5, 2, 4, 7, 0, 6, 1], H[\infty_5, 4, 6, 5, 7, 0, 1, 2, 3], H[\infty_5, 1, 7, 0, 6, 2, 5, 3, 4] \}.$$

Then an $H$-decomposition of $K_{13}^{(3)} \setminus K_5^{(3)}$ consists of the orbits of the $H$-blocks in $B_1$ under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1, \ldots, 5\}$, and $j \mapsto j + 1 \pmod{8}$ along with the $H$-blocks in $B_2$. 

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Example 14. Let $V\left(K_{14}^{(3)} \setminus K_{6}^{(3)}\right) = \mathbb{Z}_{12} \cup \{\infty_1, \infty_2\}$ with $0, 3, 6, 9, \infty_1, \infty_2$ being the vertices in the hole and let

$$B_1 = \{ H[0, 1, 5, 7, 11, 2, 10, 6, 9], H[\infty_1, 0, 1, 6, 8, 10, 11, \infty_2, 2], H[\infty_2, 0, 4, 1, 3, 9, 11, \infty_1, 8], H[0, 1, 6, \infty_2, 7, 2, 8, \infty_1, 11], H[0, 2, 5, 7, 10, 4, 8, 9, 11], H[0, 2, 4, 3, 8, 5, 9, 6, 11]\},$$

$$B_2 = \{ H[\infty_1, 2, 8, 5, 11, 1, 4, 7, 10], H[\infty_2, 5, 11, 2, 8, 4, 7, 1, 10], H[0, 1, 3, 4, 8, \infty_1, 7, 2, 5], H[3, 4, 6, 7, 11, \infty_1, 10, 5, 8], H[6, 7, 9, 2, 10, \infty_2, 1, 8, 11], H[0, 9, 10, 2, 11, 1, 5, \infty_2, 4], H[\infty_1, \infty_2, 1, 2, 5, 8, 11, 4, 7], H[\infty_1, \infty_2, 2, 1, 4, 7, 10, 5, 8], H[\infty_1, \infty_2, 4, 5, 8, 2, 11, 7, 10], H[\infty_1, \infty_2, 5, 4, 7, 1, 10, 8, 11], H[\infty_1, \infty_2, 7, 8, 11, 2, 5, 1, 10], H[\infty_1, \infty_2, 8, 7, 10, 1, 4, 2, 11], H[\infty_1, \infty_2, 10, 2, 11, 5, 8, 1, 4], H[\infty_1, \infty_2, 11, 1, 10, 4, 7, 2, 5]\}.$$

Then an $H$-decomposition of $K_{14}^{(3)} \setminus K_{6}^{(3)}$ consists of the orbits of the $H$-blocks in $B_1$ under the action of the map $\infty_i \mapsto \infty_{i+j}$, for $i \in \{1, 2\}$, and $j \mapsto j + 1 \mod 12$ along with the $H$-blocks in $B_2$.

Example 15. Let $V\left(K_{15}^{(3)} \setminus K_{7}^{(3)}\right) = \mathbb{Z}_{8} \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7\}$ with $\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7$ being the vertices in the hole and let

$$B_1 = \{ H[\infty_1, \infty_4, 0, \infty_5, 5, 3, 4, \infty_6, \infty_7], H[\infty_2, \infty_4, 0, \infty_5, 5, 2, 4, \infty_7, 3], H[\infty_3, \infty_4, 0, \infty_5, 5, \infty_7, 4, \infty_6, 3], H[\infty_4, \infty_5, 0, 4, 7, \infty_6, 1, \infty_3, 2], H[\infty_5, \infty_6, 0, \infty_7, 6, 3, 4, \infty_1, 1], H[\infty_5, \infty_7, 0, 5, 6, \infty_1, 4, \infty_6, 2], H[\infty_6, \infty_7, 3, 5, 0, 1, \infty_1, \infty_6, \infty_2, 7]\},$$

$$B_2 = \{ H[\infty_1, \infty_3, 0, 3, 5, 1, 2, 4, 6], H[\infty_1, \infty_3, 1, 4, 6, 2, 3, 5, 7], H[\infty_1, \infty_3, 2, 5, 7, 3, 4, 6, 0], H[\infty_1, \infty_3, 3, 6, 0, 4, 5, 7, 1], H[\infty_1, \infty_3, 4, 7, 1, 5, 6, 0, 2], H[\infty_1, \infty_3, 5, 0, 2, 6, 7, 1, 3], H[\infty_1, \infty_3, 6, 1, 3, 7, 0, 2, 4], H[2, 4, 7, \infty_2, \infty_3, \infty_1, 1, 5, 6], H[3, 5, 0, \infty_2, \infty_3, \infty_1, 2, 6, 7], H[4, 6, 1, \infty_2, \infty_3, \infty_1, 3, 7, 0], H[5, 7, 2, \infty_2, \infty_3, \infty_1, 4, 0, 1], H[6, 0, 3, \infty_2, \infty_3, \infty_1, 5, 1, 2], H[7, 1, 4, \infty_2, \infty_3, \infty_1, 6, 2, 3], H[0, 2, 5, \infty_2, \infty_3, \infty_1, 7, 3, 4], H[2, 3, 7, \infty_3, 5, \infty_1, \infty_2, 1, 6], H[3, 4, 0, \infty_3, 6, \infty_1, \infty_2, 2, 7], H[4, 5, 1, \infty_3, 7, \infty_1, \infty_2, 3, 0], H[5, 6, 2, \infty_3, 0, \infty_1, \infty_2, 4, 1], H[6, 7, 3, \infty_3, 1, \infty_1, \infty_2, 5, 2], H[7, 0, 4, \infty_3, 2, \infty_1, \infty_2, 6, 3], H[0, 1, 5, \infty_3, 3, \infty_1, \infty_2, 7, 4], H[\infty_2, \infty_3, 1, 0, 4, \infty_1, \infty_2, 7, 2, 6], H[\infty_2, 1, 5, \infty_1, 2, \infty_3, 0, 6, 4], H[1, 3, 6, \infty_3, 4, 5, 7, \infty_2, 2], H[\infty_1, 0, 3, 2, 4, 5, 6, \infty_2, 7], H[\infty_2, 0, 1, 3, 6, \infty_1, 4, \infty_3, 5], H[\infty_2, 1, 2, 4, 7, \infty_1, 5, \infty_3, 6], H[\infty_2, 2, 3, 5, 0, \infty_1, 6, \infty_3, 7]\}.
Then an $H$-decomposition of $K_{15}^{(3)} \setminus K_{7}^{(3)}$ consists of the orbits of the $H$-blocks in $B_1$ under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1, \ldots, 7\}$, and $j \mapsto j + 1 \pmod{8}$ along with the $H$-blocks in $B_2$.

**Example 16.** Let $V \left( K_{16}^{(3)} \setminus K_{8}^{(3)} \right) = \mathbb{Z}_8 \cup \{ \infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7, \infty_8 \}$ with $\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7, \infty_8$ being the vertices in the hole and let

$$B_1 = \{ H[\infty_1, 2, 0, \infty_3, 3, \infty_4, \infty_5, 2, \infty_6, \infty_8] , H[\infty_2, \infty_3, 0, \infty_4, 1, \infty_5, 2, \infty_1, \infty_7] , H[\infty_4, \infty_5, 0, \infty_6, 1, \infty_8, 2, \infty_3, \infty_7] , H[\infty_5, \infty_6, 0, \infty_7, 1, \infty_1, 2, \infty_2, \infty_8] , H[\infty_6, \infty_7, 0, \infty_8, 1, \infty_2, 2, \infty_1, \infty_4] , H[\infty_7, \infty_8, 0, \infty_1, 1, \infty_3, 2, \infty_2, \infty_6] , H[\infty_8, 0, 1, \infty_1, 3, 2, \infty_2, 6, \infty_7, \infty_8, 4] , H[\infty_6, 0, 1, 2, 4, \infty_1, 6, \infty_7, 3] , H[\infty_7, 0, 1, 3, 6, \infty_3, 5, \infty_4, 4] , H[\infty_8, 0, 1, 2, 4, \infty_3, 6, \infty_3, 3] , H[0, 1, 4, \infty_1, 3, 2, \infty_2, 6, \infty_7, \infty_8, 4] , H[0, 2, 4, \infty_2, 5, 3, 7, 6, 1] \};$$

$$B_2 = \{ H[0, 1, 2, \infty_1, 4, \infty_5, 3, \infty_2, 6], H[1, 2, 3, \infty_1, 5, \infty_5, 4, \infty_2, 7], H[2, 3, 4, \infty_1, 6, \infty_5, 5, \infty_2, 0], H[3, 4, 5, \infty_1, 7, \infty_5, 6, \infty_2, 1], H[4, 5, 6, \infty_3, 0, \infty_5, 7, \infty_4, 2], H[5, 6, 7, \infty_3, 1, \infty_5, 0, \infty_4, 3], H[6, 7, 0, \infty_3, 2, \infty_5, 1, \infty_4, 4], H[7, 0, 1, \infty_3, 3, \infty_5, 2, \infty_4, 5], H[0, 1, 3, \infty_5, 4, \infty_6, 5, \infty_1, 2], H[1, 2, 4, \infty_5, 5, \infty_6, 6, \infty_1, 3], H[2, 3, 5, \infty_5, 6, \infty_6, 7, \infty_1, 4], H[3, 4, 6, \infty_5, 7, \infty_6, 0, \infty_1, 5], H[4, 5, 7, \infty_7, 0, \infty_8, 1, \infty_1, 6], H[5, 6, 0, \infty_7, 1, \infty_8, 2, \infty_1, 7], H[6, 7, 1, \infty_7, 2, \infty_8, 3, \infty_1, 0], H[7, 0, 2, \infty_7, 3, \infty_8, 4, \infty_1, 1], H[\infty_2, 0, 1, 5, 6, \infty_3, 7, \infty_4, 2], H[\infty_3, 0, 1, 5, 6, \infty_4, 7, \infty_2, 2], H[\infty_4, 0, 1, 5, 6, \infty_2, 7, \infty_3, 2], H[\infty_2, 3, 4, 6, 7, \infty_3, 2, \infty_4, 5], H[\infty_3, 3, 4, 6, 7, \infty_4, 2, \infty_2, 5], H[\infty_4, 3, 4, 6, 7, \infty_2, 2, \infty_3, 5] \}. $$

Then an $H$-decomposition of $K_{16}^{(3)} \setminus K_{8}^{(3)}$ consists of the orbits of the $H$-blocks in $B_1$ under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1, \ldots, 8\}$, and $j \mapsto j + 1 \pmod{8}$ along with the $H$-blocks in $B_2$.

**Maximum Packing Examples**

**Example 17.** Let $V \left( K_{11}^{(3)} \right) = \mathbb{Z}_{10} \cup \{ \infty \}$ and let

$$B_1 = \{ H[0, 2, 7, 1, 4, \infty, 9, 3, 6], H[0, 3, 6, 1, 5, \infty, 9, 7, 2], H[0, 2, 5, 1, 3, \infty, 4, 7, 8] \},$$

$$B_2 = \{ H[0, 1, \infty, 3, \infty_1, 4, \infty_2, 6], H[1, 2, \infty_1, 5, \infty_2, 4, \infty_3, 7], H[2, 3, \infty_2, 6, \infty_3, 7, \infty_1, 0], H[3, 4, \infty_3, 7, \infty_2, 1, \infty_1, 1], H[4, 5, \infty_1, 8, \infty_2, 2, \infty_3, 3], H[5, 6, \infty_2, 3, \infty_1, 4, \infty_3, 3], H[6, 7, \infty_3, 4, \infty_2, 5, \infty_1, 0], H[7, 0, \infty_3, 5, \infty_2, 1, \infty_1, 1], H[\infty_2, 0, 1, 5, 6, \infty_3, 7, \infty_4, 2], H[\infty_3, 0, 1, 5, 6, \infty_4, 7, \infty_2, 2], H[\infty_4, 0, 1, 5, 6, \infty_2, 7, \infty_3, 2], H[\infty_2, 3, 4, 6, 7, \infty_3, 2, \infty_4, 5], H[\infty_3, 3, 4, 6, 7, \infty_4, 2, \infty_2, 5], H[\infty_4, 3, 4, 6, 7, \infty_2, 2, \infty_3, 5] \}. $$
Then a maximum $H$-packing of $K_{11}^{(3)}$ consists of the orbits of the $H$-blocks in $B_1$ under the action of the map $j \mapsto j + 1 \pmod{10}$ along with the $H$-blocks in $B_2$ and a leave consisting of the edge $\{1, 3, 9\}$.

**Example 18.** Let $V(K_{13}^{(3)}) = \mathbb{Z}_{13}$ and let

\[
B_1 = \{H[0, 3, 7, 6, 10, 5, 11, 9, 1], H[0, 2, 11, 1, 7, 5, 12, 3, 8], H[0, 3, 5, 8, 10, 7, 1, 9, 11], \\
H[0, 1, 5, 8, 12, 2, 7, 10, 11], H[0, 1, 3, 10, 12, 2, 5, 6, 7]\};
\]

\[
B_2 = \{H[0, 4, 8, 1, 12, 5, 6, 9, 10], H[1, 5, 9, 2, 3, 6, 7, 10, 11], H[2, 6, 10, 3, 4, 7, 8, 11, 12], \\
H[3, 4, 5, 7, 11, 8, 12, 10, 1], H[7, 8, 9, 11, 2, 12, 3, 0, 4], H[11, 12, 0, 2, 6, 3, 7, 5, 9]\}. 
\]

Then a maximum $H$-packing of $K_{13}^{(3)}$ consists of the orbits of the $H$-blocks in $B_1$ under the action of the map $j \mapsto j + 1 \pmod{13}$ along with the $H$-blocks in $B_2$ and a leave consisting of the edges $\{0, 1, 2\}$ and $\{1, 6, 10\}$, which share a single vertex. Additionally, let

\[
B'_2 = (B_2 \setminus \{H[2, 6, 10, 0, 1, 7, 8, 11, 12]\}) \cup \{H[2, 6, 10, 0, 1, 7, 8, 11, 12]\}. 
\]

Then a maximum $H$-packing of $K_{13}^{(3)}$ consists of the orbits of the $H$-blocks in $B_1$ under the action of the map $j \mapsto j + 1 \pmod{13}$ along with the $H$-blocks in $B'_2$ and a leave consisting of the edges $\{1, 6, 10\}$ and $\{2, 3, 4\}$, which are vertex-disjoint.

**Example 19.** Let $V(K_{15}^{(3)}) = \mathbb{Z}_{15}$ and let

\[
B_1 = \{H[0, 4, 9, 6, 11, 7, 14, 12, 2], H[0, 4, 8, 3, 6, 7, 13, 10, 12], H[0, 1, 3, 12, 14, 2, 5, 6, 7], \\
H[0, 1, 6, 9, 14, 2, 12, 7, 11], H[0, 2, 8, 7, 13, 4, 12, 1, 3], H[0, 3, 7, 8, 12, 5, 14, 9, 13], \\
H[0, 2, 12, 7, 8, 10, 1, 3, 11]\};
\]

\[
B_2 = \{H[0, 5, 10, 1, 2, 6, 7, 11, 12], H[1, 6, 11, 2, 3, 7, 8, 12, 13], H[2, 7, 12, 3, 4, 8, 9, 13, 14], \\
H[3, 8, 13, 4, 5, 9, 10, 14, 0], H[4, 9, 14, 5, 6, 10, 11, 0, 1], H[0, 2, 5, 13, 3, 12, 14, 7, 10], \\
H[4, 6, 9, 14, 1, 8, 11, 12, 7], H[8, 10, 13, 3, 5, 12, 0, 11, 1]\}. 
\]

Then a maximum $H$-packing of $K_{15}^{(3)}$ consists of the orbits of the $H$-blocks in $B_1$ under the action of the map $j \mapsto j + 1 \pmod{15}$ along with the $H$-blocks in $B_2$ and a leave consisting of the edges $\{1, 3, 6\}$, $\{2, 4, 7\}$, and $\{9, 11, 14\}$, which are vertex-disjoint.