Holomorphic boundary conditions for topological field theories via branes in twisted supergravity

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Motivation and Objective:

Twisting $3d\ N = 4$ theories with boundaries:

- **Signature:**
  - Study bulk TFT via boundary VOA data

- **Central issue:**
  - Compatibility of boundary with bulk topological twist via appropriate deformation

In this talk:

Show how topological twist is implemented as a deformation of a holomorphically twisted theory, by engineering the theories in brane world volumes in a particular background.
Motivation and Objective:

Topological Twisting:

- Choice of twisting homomorphism:
  \[ h : \text{Spin}(n) \to \text{Spin}'(n) \subset (\text{Spin}(n) \times G_R) \]
  \[ Q : \text{Spin}'(n) - \text{inv.}, \quad Q^2 = 0 \]

- Pass on \( Q \) - cohomology:
  - Restrict to \( Q \) -closed operators
  - \( Q \) - exact quantities vanish

- Focus on a protected (BPS) subsector of the theory

- \( P_\mu \) become \( Q \) - exact: Topological theory
  \[ \langle O_1 O_2 .. \rangle \neq f(\{x\}) \]

- Further twists?
  - Holomorphic twist: \( \partial / \partial \bar{z} \) become \( Q \) - exact
    \[ \langle O_1 O_2 .. \rangle = g(z) \]

\[ \text{Holomorphic} \quad \rightarrow \quad \text{Topological} \]
Outline:

- 3d $\mathcal{N}=4$ theories with boundaries:
  - Topological and Holomorphic twists
  - Boundaries, deformations and compatibility

- Brane engineering:
  - Bulk and boundary theory
  - Twisting and background geometry

- Twisted IIB supergravity:
  - Holomorphic brane engineering
  - Background deformations and compatibility

- Summary, future directions and ongoing work
3d $\mathcal{N} = 4$ with boundaries:

Superconformal algebra:

$$\mathfrak{g}(3, \mathcal{N} = 4) = \mathfrak{osp}(4|4) \supset \mathfrak{so}(3, 2) \times \mathfrak{so}(4)_R$$

$$Q \in [2]^{(2,2)} \quad SO(4)_R \simeq SU(2)_C \times SU(2)_H$$

$$\{ Q^A_{\alpha} \bar{Q}^{\dot{B}}_{\beta} \} = \epsilon^{AB} \epsilon^{\dot{A}\dot{B}} P_{\alpha\beta}$$

Twisting:

- $\mathfrak{h}_{C, H} : SU(2)_L \to SU(2)_L \times SU(2)_{C, H}$

- $Q : Q^2 = 0,$

- 3d N=4 Hypermultiplet

  Scalars: $(q, \bar{q}) \in [1]^{(1,2)}$

  Fermions: $(\psi, \bar{\psi}) \in [2]^{(2,1)}$}

Moduli space:

$$\mathcal{M} = \bigcup_n \mathcal{C}_n \times \mathcal{H}_n$$

- $\mathcal{M}_C \simeq \mathcal{C} \times \{0\}$

- $\mathcal{M}_H \simeq \{0\} \times \mathcal{H}$

- Three types of nilpotent supercharges:

  $$\{ Q_C, Q_H, Q_{\text{hol}} \}$$
Consider topological supercharges as *deformations* of the holomorphic one:

\[ Q_H = Q_{\text{hol.}} + \zeta Q_H \quad Q_C = Q_{\text{hol.}} + \zeta Q_C \]

Apply to construction of appropriate boundary conditions!

- \( \mathcal{N} = (0, 4) \): Compatible with bulk *holomorphic* but *not topological* twist

  \[ \underline{\text{Deformable to be } \text{compatible} \text{ with bulk topological twist}} \]

- \( \mathcal{N} = (2, 2) \): Compatible with both *holomorphic and topological* twist

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Gaiotto-Costello '18
3d $\mathcal{N} = 4$ with boundaries:

- (0,4) b.c for free hypermultiplet:
  
  Or how deformations of bdy conditions are induced by deformations of the preserved susy:

  $\mathcal{Q}_{\text{hol.}} + \zeta Q_{H,C} \rightarrow \mathcal{J}^{\text{hol.}}_{\perp} + \zeta \mathcal{J}^{H,C}_{\perp} = 0$

- $C$: $\tilde{B}_D \phi$: (0,4) Dirichlet b.c can be deformed to be compatible with C-twist
- $H$: $\tilde{B}_N \phi$: (0,4) Neumann b.c can be deformed to be compatible with H-twist

- $S_\partial \mapsto S_\partial + S$, $\delta_{Q_{\text{hol.}}} S = \mathcal{J}^{Q_{\text{hol.}}}_{\partial}$.

Defo preserving H,C-deformed susy if: $\delta_{Q_{\text{hol.}}} \tilde{S} = \mathcal{J}^{Q_{H,C}}_{\partial}$

\[ N: \text{H-twist} \quad \tilde{S} = S_{Sb} \]

\[ D: \text{C-twist} \quad \tilde{S} = S_{Fc} \]

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Symplectic Boson VOA

\[ X(z)Y(w) \sim \frac{1}{z-w} \]

Symplectic Fermion VOA

\[ x(z)y(w) \sim \frac{1}{(z-w)^2} \]
Brane engineering:

- 3d N=4 theories can be realized as brane world volume theories:

\[ \text{D5} \quad \text{NS5} \quad \text{NS5} \]

- Boundaries realized from extra IIB five-branes:

\[ \mathcal{N} = (0, 4) \]

Hanany-Witten '96

Hanany-Okazaki '18
Brane engineering:

Twisting homomorphisms:

- Appropriate choice of target space geometry:
  Bershadsky, Vafa, Sadov ‘96

\[ M_3 \hookrightarrow T^*_{H,C} M_3 \subset CY_3 \]

- Several points remain beyond reach:
  
  Q-cohomology?

  Engineering of holomorphic twist?

  Deformations of partial twists?

\[ \Gamma_{\mu\nu} \rightarrow \Gamma_{\mu\nu} + \Gamma_{(\mu+3)(\nu+3)}, \quad \mu, \nu = 0, 1, 2 \]

\[ \Gamma_{\mu\nu} \rightarrow \Gamma_{\mu\nu} + \Gamma_{(\mu+7)(\nu+7)} \]
Twisted IIB supergravity:

Definition:

Supergravity background where, the bosonic ghost of local supersymmetry acquires non-vanishing v.e.v

- Gauging local supersymmetry $\rightarrow$ bosonic ghost $q \in C^\infty(\mathbb{R}^{10}, S)$
- Supergravity EoM satisfied for: $q$ is cov. const. spinor: $q = Q$
- Twisted SQFT
  \[
  \Gamma(q, q) = 0 \quad \text{with} \quad \Gamma : S \otimes S \rightarrow \mathbb{R}^{10}
\]

SQFT coupled to twisted sugra background: $d_{\text{BRST}} \rightarrow d_{\text{BRST}} + Q$

Different choice of bosonic ghosts $\rightarrow$ different twists
Twisted IIB supergravity:

Conjecture:

Holomorphic twist of IIB supergravity: BCOV theory

- BCOV theory (Kodaira-Spencer): closed string field theory of the topological B-model

→ Complex structure deformations

- $X = \text{CY}_d : \quad PV^{i,j}(X) = \Omega^{0,j}(X, \wedge^i TX) \quad PV^{i,j}(X) \xrightarrow{\Omega_X} \Omega^{d-i,j}(X)$

- Space of fields: $\mathcal{E}_{BCOV} = \left( PV^{i,j}(X)[\ell][2], \bar{\partial} + t\partial \right)$

- Subspace of propagating fields: “Minimal BCOV”

$$\mathcal{E}_{BCOV} \supset \mathcal{E}_{mBCOV} = \left( \bigoplus_{i+j<d} t^i PV^{j,\bullet}(X)[2], \bar{\partial} + t\partial \right)$$

Bershadsky, Cecotti, Ooguri, Vafa '94
Costello, Li '16
Twisted IIB supergravity:

• Focus on flat space:

Holomorphically twisted IIB supergravity on $\mathbb{R}^{10}$ - $m$BCOV theory on $\mathbb{C}^{5}$

Supersymmetry and BCOV fields:

$10d \mathcal{N} = (2, 0)$ \hspace{1em} $\mathfrak{g}_{IIB} : \mathfrak{so}(10, \mathbb{C}) \oplus V \oplus \Pi(S_{+} \otimes \mathbb{C}^{2}) \oplus \{\ldots\}$ and $\psi \in S_{+} : \Gamma(\psi, \psi) = 0$

Twisted background: \hspace{1em} $Q_{\psi} \in \mathfrak{g}_{IIB}$, $Q^{2} = 0 \quad \longrightarrow \quad q = Q_{\psi}$

Residual supersymmetry: $Q_{\psi}$-cohomology of the IIB algebra: \hspace{1em} $\mathfrak{g}^{Q_{\psi}}_{IIB} : \mathfrak{sl}(5, \mathbb{C}) \oplus \mathbb{C}^{5} \oplus \Pi(S_{+} \otimes v) \oplus \{\ldots\}$

Susy present in $m$BCOV:

$\mathcal{R} : \mathcal{H}^{\bullet}(\mathfrak{g}_{IIB}, Q_{\psi}) \rightarrow \mathcal{E}_{mBCOV}$

$dz_{i} \rightarrow \mathcal{Z}_{i} \in \text{PV}^{0,0}$

$\partial_{z_{i}} \wedge \partial_{z_{j}} \rightarrow \partial_{z_{i}} \wedge \partial_{z_{j}} \in \text{PV}^{2,0}(\mathbb{C}^{5})$

• Odd elements of $\mathfrak{g}^{Q_{\psi}}_{IIB}$ transform as:

1-forms and bivectors under $\mathfrak{sl}(5, \mathbb{C})$

Costello, Li '16
Twisted IIB supergravity:

Holomorphic Hanany-Witten: the case of the 3d hypermultiplet

\[
\begin{align*}
\text{bulk:} & \quad \mathbb{C}_{z_1} \times \mathbb{C}_{w_2} \times \mathbb{C}_{z_2} \times \mathbb{C}_{w_1} \times \mathbb{C}_{z_3} \\
& \quad \simeq \simeq \simeq \simeq \simeq \\
& \quad \mathbb{R}^2_{01} \times \mathbb{R}^2_{26} \times \mathbb{R}^2_{34} \times \mathbb{R}^2_{59} \times \mathbb{R}^2_{78} \\
\text{D3:} & \quad \mathbb{C}_{z_1} \times \mathbb{C}_{w_2} \times 0 \times 0 \times 0.
\end{align*}
\]

✓ Holomorphic configuration on bulk and brane

- D3 world volume theory: holomorphic Chern-Simons on \( \mathbb{C}_{z_1} \times \mathbb{C}_{w_2} \subset \mathbb{C}^5 \)

\[
\mathcal{E}_{D3} = \Omega^{0,0} \cdot (\mathbb{C}^2)[d\bar{z}_2, d\bar{w}_1, d\bar{z}_3]
\]

- Next: need to add the D5s in order to get \(3d\, \mathcal{N} = 4\) configuration
Twisted IIB supergravity:

- The D5 support is not along a complex submanifold of $\mathbb{C}^5$
- Deform, using the bivector $\partial w_1 \wedge \partial w_2$

Theory topological along $w_1, w_2$ and holomorphic along $z$’s

Next step: Introduce $\mathcal{N} = (0, 4)$ boundary conditions:
Twisted IIB supergravity:

Dirichlet boundary conditions via additional D5:

| bulk: | $C_{z_1}$ $\times$ $C_{w_2}$ $\times$ $C_{z_2}$ $\times$ $C_{w_1}$ $\times$ $C_{z_3}$ |
|-------|-----------------------------------------------------------------|
|       | $\cong$ $\cong$ $\cong$ $\cong$ $\cong$ |
|       | $\mathbb{R}^2_{01}$ $\times$ $\mathbb{R}^2_{26}$ $\times$ $\mathbb{R}^2_{34}$ $\times$ $\mathbb{R}^2_{59}$ $\times$ $\mathbb{R}^2_{78}$ |
| D3:   | $C_{z_1}$ $\times$ $(\mathbb{R}_+)^2$ $\times$ $I_6$ $\times$ $0$ $\times$ $0$ $\times$ $0$. |
| D5:   | $C_{z_1}$ $\times$ $\mathbb{R}_2$ $\times$ $\partial I_6$ $\times$ $0$ $\times$ $\mathbb{R}_9$ $\times$ $C_{z_3}$ |
| D5':  | $C_{z_1}$ $\times$ $0$ $\times$ $\mathbb{R}_6$ $\times$ $C_{z_2}$ $\times$ $\mathbb{R}_5$ $\times$ $0$ |

- Which are further admissible deformations?

Look among residual supersymmetries and control which is compatible with the configuration:
Twisted IIB supergravity:

- Two possible deformations identified with odd elements of $3d \mathcal{N} = 4$:

  $Q_C = \partial z_1 \wedge \partial z_2$

  - Acts on the D-brane gauge theory as: $\partial z_1 \wedge \partial z_2 \rightarrow \epsilon_2 \partial z_1$

  - Deformation renders $z_2$ topological and keeps $z_3$ holomorphic:
    - Coulomb branch moduli become Q-exact $\rightarrow$ C-twist
    - Holomorphic d.o.f on the boundary $\rightarrow$ Compatible with Dirichlet b.c for the free hyper

  ✓ Holomorphic d.o.f on the boundary $\rightarrow$ Compatible with Dirichlet b.c for the free hyper

  $Q_H = z_2$

  - Acts on the D-brane gauge theory as: $z_2 \rightarrow \partial \epsilon_2$

  - Deformation renders $z_3$ topological and keeps $z_2$ holomorphic:
    - Higgs branch moduli become Q-exact $\rightarrow$ H-twist

  ✗ No solutions on the boundary $\rightarrow$ expected from field theory
Concluding remarks

• Construction of holomorphic boundary conditions for $3d \, \mathcal{N} = 4$ theories via holomorphic brane engineering:
  
  ✓ Embedding Hanany-Witten brane configurations in IIB twisted supergravity
  ✓ Deformation from holomorphic to topological twist
  ✓ Results in agreement with field theory and with purely holomorphic field theory construction

• Unexplored directions

  ➔ Study of interacting theories
  ➔ Bulk defect operators in 3d and 4d (work in progress with I. Brunner and I. Saberi)
  ➔ Twisted Holographic configurations
Thank you for your attention...!