Critical temperature of Bose–Einstein condensation
in trapped atomic Bose–Fermi mixtures

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Abstract

We calculate the shift in the critical temperature of Bose-Einstein condensation for a dilute Bose-Fermi mixture confined by a harmonic potential to lowest order in both the Bose-Bose and Bose-Fermi coupling constants. The relative importance of the effect on the critical temperature of the boson-boson and boson-fermion interactions is investigated as a function of the parameters of the mixture. The possible relevance of the shift of the transition temperature in current experiments on trapped Bose-Fermi mixtures is discussed.

I. INTRODUCTION

The achievement of Bose-Einstein condensation in ultracold, trapped dilute alkali gases, beyond realizing a striking and spectacular experimental confirmation of a long-standing,
fundamental prediction of quantum theory, has initiated and stimulated as well a whole new field of research in the physics of quantum gases in confined geometries [1,2]. Nowadays, many different atomic species and isotopes can be successfully cooled and trapped in the gaseous state, to investigate in exceptional conditions of purity and isolation the properties of interacting Bose and Fermi systems, or Bose-Fermi mixtures.

In particular, the experimental realizations of trapped gaseous mixtures of bosons and fermions are both an interesting new instance of a quantum many-body system and a very useful tool to reach the regime of quantum degeneracy for a Fermi gas via sympathetic cooling of the fermions by the bosons [3–7]. From a theoretical point of view, dilute Bose-Fermi mixtures have been the object of recent investigations addressing, for example, the determination of the density profiles of the two components in trapped systems [8], the problem of stability and phase separation [9], and the effect of boson-fermion interactions on the dynamics [10] and on the ground-state properties [11] of the mixture.

Boson-fermion interactions in a Bose-Fermi mixture can induce a net attractive interaction between the fermions, thus introducing a further mechanism toward the achievement of the BCS transition in trapped Fermi gases [12]. In this paper we address the reverse problem, i.e. how the transition temperature of Bose-Einstein condensation is affected by the presence of the fermions in a trapped mixture. The shift of the transition temperature $T_c$ due to interactions in a pure trapped Bose system has been calculated within the mean-field approximation in Ref. [13]. In the present work we extend the perturbative methods of Ref. [13] to obtain the shift of $T_c$ to lowest order in both the Bose-Bose and Bose-Fermi coupling constants. To this order the effect on $T_c$ of boson-boson and boson-fermion interactions are independent and add linearly. The relative importance of the two effects depends on the relevant parameters of the trapped mixture: number of bosons and fermions in the trap, ratio of the masses and of the oscillator frequencies for the two species and the ratio of the Bose-Bose and Bose-Fermi coupling constants. The calculation is carried out in local density approximation which is valid provided that the number of bosons and fermions in the trap is large. Finite size effects have not been included in the present treatment.
The plan of the paper is as follows: in Section II we generalize the scheme derived for pure Bose systems in Ref. [13], to include the effects of boson-fermion interactions. In Section III we derive analytical results for the shift of $T_c$ in the limits of a highly degenerate (Thomas-Fermi) and a classical (Boltzmann) Fermi gas, and we provide the full numerical solution for the intermediate regimes. We finally compare the theoretical predictions with the current experimental situations and we draw our conclusions.

II. THEORY

In a non-interacting Bose gas confined by the external harmonic potential $V^B_{\text{ext}}(r) = m_B(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)/2$, the critical temperature for Bose-Einstein condensation (BEC) is given by

$$k_B T_0^c = \hbar \omega_B \left( \frac{N_B}{\zeta(3)} \right)^{1/3} \approx 0.94 \hbar \omega_B N_B^{1/3},$$

where $\omega_B = (\omega_x \omega_y \omega_z)^{1/3}$ is the geometric mean of the oscillator frequencies and $m_B, N_B$ are respectively the particle mass and the number of bosons in the trap. The above result is obtained using the local density approximation (LDA), where the temperature of the gas is assumed to be much larger than the spacing between single particle levels: $k_B T \gg \hbar \omega_{x,y,z}$.

In this case the density of thermal atoms can be written as

$$n^0_B(r) = (\lambda^B_T)^{-3} g_{3/2}(\exp\{-[V^B_{\text{ext}}(r) - \mu_B]/k_B T\}),$$

where $\lambda^B_T = \hbar (2\pi/m_B k_B T)^{1/2}$ is the boson thermal wavelength, and $g_{3/2}(x) = \sum_{n=1}^{\infty} x^n / n^{3/2}$ is the standard Bose function of order 3/2. At $T = T_c^0$ the boson chemical potential takes the critical value $\mu_B = \mu^0_c = 0$, corresponding to the bottom of the external potential, and the density $n^0_B(0)$ in the center of the trap satisfies the critical condition $n^0_B(0)(\lambda^B_T)^3 = \zeta(3/2) \approx 2.61$ holding for a homogeneous system.

Finite size effects modify the prediction of the critical temperature (1) resulting in a reduction of $T_c^0$. The first correction due to the finite number of atoms in the trap is given by [14]:

3
\[
\left( \frac{\delta T_c}{T_c^0} \right)_{fs} = -\frac{1}{2} \frac{2\zeta(2)}{\zeta(3)\omega_B} N_B^{-1/3} \simeq -0.73 \frac{\bar{\omega}_B N_B^{-1/3}}{\bar{\omega}_B N_B^{-1/3}},
\]

where \( \bar{\omega}_B = (\omega_x + \omega_y + \omega_z)/3 \) is the arithmetic mean of the oscillator frequencies.

Interparticle interactions have an effect on the BEC transition temperature as well. The presence of repulsive interactions has the effect of expanding the atomic cloud, with a consequent decrease of the density. Lowering the peak density has then the effect of lowering the critical temperature. On the contrary, attractive interactions produce an increase of the density and thus an increase of \( T_c \). This effect, which is absent in the case of a uniform gas where the density is kept fixed, can be easily estimated within mean-field theory. For pure bosonic systems the shift \( \delta T_c = T_c - T_c^0 \) has been calculated in Ref. [13],

\[
\left( \frac{\delta T_c}{T_c^0} \right)_{BB} = -1.33 \frac{a_{BB}}{\ell_B} N_B^{1/6},
\]

to lowest order in the coupling constant \( g_{BB} = 4\pi\hbar^2 a_{BB}/m_B \). In the above equation \( a_{BB} \) is the boson-boson s-wave scattering length and \( \ell_B = \sqrt{\hbar/m_B \omega_B} \) is the harmonic oscillator length. Result (4) has been obtained within LDA and neglects finite size effects.

In the case of trapped Bose-Fermi mixtures the shift of \( T_c \), due to both Bose-Bose and Bose-Fermi couplings can be calculated in mean-field approximation using the methods of Ref. [13]. The transition temperature \( T_c \) of a trapped Bose gas is defined by the normalization condition

\[
N_B = \int d\mathbf{r} n_B(\mathbf{r}, T_c, \mu_c),
\]

where \( n_B \) is the thermal density of bosons and \( \mu_c \) is the critical value of the boson chemical potential. Within LDA the boson density above \( T_c \) is given by

\[
n_B(\mathbf{r}) = (\lambda^B_T)^{-3} g_{3/2}(\exp\{-[V^B_{\text{eff}}(\mathbf{r}) - \mu_B]/k_B T\}) ,
\]

where

\[
V^B_{\text{eff}}(\mathbf{r}) = V^B_{\text{ext}}(\mathbf{r}) + 2g_{BB}n_B(\mathbf{r}) + g_{BF}n_F(\mathbf{r}) ,
\]
is the effective potential acting on the bosons which is generated by the external field \( V_{\text{ext}}^B \) and by the mean field produced by interactions with the other bosons and with the fermions. Notice the factor 2 present in the Bose-Bose contribution and absent in the Bose-Fermi term due to exchange effects. In the above equation \( n_F(r) \) is the fermion density and \( g_{BF} = 2\pi \hbar^2 a_{BF}/m_R \) is the Bose-Fermi coupling constant, fixed by the boson-fermion s-wave scattering length \( a_{BF} \) and by the reduced mass \( m_R = m_B m_F/(m_B + m_F) \), where \( m_F \) is the fermion mass.

For a fixed value of the boson chemical potential \( \mu_B \) and a fixed temperature \( T \), the boson density (6) can be expanded to first order in \( g_{BB} \) and \( g_{BF} \) as

\[
n_B(r, T, \mu_B) = n_B^0(r, T, \mu_B) - [2g_{BB}n_B^0(r) + g_{BF}n_F^0(r)] \frac{\partial n_B^0}{\partial \mu_B},
\]

in terms of the non-interacting boson (2) and fermion density

\[
n_F^0(r) = (\lambda_T^F)^{-3} f_{3/2}(\exp\{-[V_{\text{ext}}^F(r) - \mu_F]/k_B T\}).
\]

In the above equation \( \lambda_T^F = \hbar(2\pi/m_F k_B T)^{1/2} \) is the fermion thermal wavelength, and \( f_{3/2}(x) \) is the Fermi function of order 3/2 defined as

\[
f_{3/2}(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty dz \frac{\sqrt{z}}{e^z/x + 1}.
\]

Result (9) has been obtained in LDA for a Fermi gas in the trapping potential \( V_{\text{ext}}^F(r) = m_F(\omega_x' x^2 + \omega_y' y^2 + \omega_z' z^2)/2 \). The fermion chemical potential \( \mu_F \) is fixed by the normalization condition

\[
N_F = \int d r \, n_F^0(r),
\]

where \( N_F \) is the total number of fermions in the trap. The condition of validity for LDA requires the Fermi temperature of the fermionic system to be much larger than the harmonic oscillator energies \( k_B T_F \gg \hbar \omega_{x,y,z} \). For a non-interacting trapped Fermi system the Fermi temperature, or equivalently the Fermi energy, is given by \( k_B T_F = \epsilon_F = \hbar \omega_F (6N_F)^{1/3} \), where \( \omega_F = (\omega_x' \omega_y' \omega_z')^{1/3} \) is the geometric mean of the fermion oscillator frequencies.
To first order in $g_{BB}$ and $g_{BF}$, the critical value $\mu_c$ of the boson chemical potential can be written as

$$
\mu_c = \mu_c^0 + 2g_{BB}n_B^0(r = 0) + g_{BF}n_F^0(r = 0).
$$

(12)

By writing $T_c = T_c^0 + \delta T_c$, one can expand Eq. (5) obtaining the following result for the total relative shift of the condensation temperature:

$$
\frac{\delta T_c}{T_c^0} = \left( \frac{\delta T_c}{T_c^0} \right)_{BB} + \left( \frac{\delta T_c}{T_c^0} \right)_{BF} = -\frac{2g_{BB} \int dr \frac{\partial n_B^0}{\partial \mu_B} [n_B^0(r = 0) - n_B^0(r)]}{\int dr \frac{\partial n_B^0}{\partial T}} - \frac{g_{BF} \int dr \frac{\partial n_B^0}{\partial \mu_B} [n_F^0(r = 0) - n_F^0(r)]}{\int dr \frac{\partial n_B^0}{\partial T}},
$$

(13)

where the derivatives of the non-interacting boson and fermion densities $n_B^0$ and $n_F^0$ are evaluated at the ideal critical point $\mu_c^0 = 0$, $T = T_c^0$. The first term $(\delta T_c/T_c^0)_{BB}$ in the above equation accounts for interaction effects among the bosons and coincides with the shift (4). The second term $(\delta T_c/T_c^0)_{BF}$ accounts instead for interaction effects between bosons and fermions, and its determination will constitute the main result of the present paper. Some comments are in order here. (i) The shift $\delta T_c$ derived above is a mean-field effect which originates from the fact that in a trapped Bose-Fermi mixture the total number of bosons and the total number of fermions are fixed, but not the density profiles of the two species. This effect is peculiar of trapped systems, since it vanishes identically in the case of uniform systems, and should not be confused with the shift of $T_c$ occurring in homogeneous Bose systems, which is instead due to many-body effects [15]. (ii) The shift originating from the Bose-Fermi coupling, similarly to the one arising from the Bose-Bose one, is negative if $g_{BF} > 0$ and is positive if $g_{BF} < 0$. If $a_{BB}$ and $a_{BF}$ have opposite sign, the corresponding shifts of $T_c$ go in opposite directions. (iii) Result (13) holds to lowest order in $g_{BB}$ and $g_{BF}$ and, since it has been obtained using LDA, is exact if the number of bosons and fermions is large. Finite-size corrections are not included in (13). For a finite system, a reasonable estimate of the total shift of the critical temperature can be obtained by adding to result (13) the finite-size correction (3) of the non-interacting model.
III. RESULTS

We now concentrate on the relative shift \( (\delta T_c/T_c)^{BF} \) due to the boson-fermion interaction. First of all we observe that

\[
\frac{\partial n^0_B(r)}{\partial \mu_B} = \frac{1}{(\lambda_{T_0}^B)^3 k_B T_0^c} g_{1/2}(\exp[-V_{ext}(r)/k_B T_0^c]),
\]

(14)

and \( T_c^0 \int d\mathbf{r} \partial n_B^0 / \partial T = 3N_B \), where the derivatives are evaluated at the condensation point of the non-interacting gas \( \mu_c^0 = 0, T = T_c^0 \). Using Eq. (9), the relative shift can then be rewritten as:

\[
\left( \frac{\delta T_c}{T_c^0} \right)^{BF} = \frac{-g_{BF}}{3N_B (\lambda_{T_0}^B)^3 (\lambda_{T_0}^F)^3 k_B T_0^c} \times \int d\mathbf{r} g_{1/2}(\exp[-V_{ext}(r)/k_B T_0^c])
\]

\[
\times \left[ f_{3/2}(\exp{\{\mu_F/k_B T_c\}}) - f_{3/2}(\exp{\{[\mu_F - V_{ext}(r)]/k_B T_0^c\}}) \right].
\]

(15)

In the following we shall assume that even if the trapping potentials of bosons and fermions can have different oscillator frequencies, nevertheless \( \omega_x/\omega'_x = \omega_y/\omega'_y = \omega_z/\omega'_z = \omega_B/\omega_F \), i.e. the anisotropy is the same for the bosonic and fermionic trapping potentials. This is always the case in today’s experiments, and assuming otherwise would introduce unnecessary complications. In fact, the assumption of equal anisotropies holds in general in magnetic traps since the confining potentials depend only on the (common) external magnetic field, the magnetic moments, and the masses of the atoms. Eq. (15) contains the fermion chemical potential \( \mu_F(N_F, T_c^0) \) which has to be determined from Eq. (11). Eqs. (15) and (11) have then to be solved simultaneously. We notice that Eq. (11) can be rewritten in dimensionless form as

\[
\tilde{T}_F^3 = 3 \int_0^\infty dt \frac{t^2}{\exp(t - \tilde{\mu}_F) + 1},
\]

(16)

where we have introduced the reduced chemical potential \( \tilde{\mu}_F = \mu_F/k_B T_0^c \) and the reduced Fermi temperature \( \tilde{T}_F = T_F/T_c^0 \). Eq. (16) reveals that \( \tilde{\mu}_F \) is only a function of \( \tilde{T}_F \), which in turn is a measure of the degeneracy of the Fermi gas at \( T = T_c^0 \). In terms of \( \tilde{\mu}_F \) and \( \tilde{T}_F \) Eq. (15) then becomes.
\[
\left( \frac{\delta T_c}{T_c^0} \right)_{BF} = -\frac{4\pi g_{BF} R_F^3}{3N_B (\lambda_{T_c^0}^B)^3 (\lambda_{T_c^0}^F)^3 k_B T_c^0} \times \int ds \, s^2 g_{1/2}(\exp\{-\tilde{T}_F \alpha s^2\}) \\
\times \left[ f_{3/2}(\exp\{\tilde{\mu}_F\}) - f_{3/2}(\exp\{\tilde{\mu}_F - \tilde{T}_F s^2\}) \right].
\]

(17)

In writing Eq. (17) we have rescaled each integration coordinate by the appropriate Thomas-Fermi radius of the fermion cloud \( R'_i = (2\epsilon_F/m_F \omega_i^2)^{1/2} \). We have then introduced the mean Fermi radius \( R_F = (R_x R_y R_z)^{1/3} \) and named \( \alpha = m_B \omega_B^2/m_F \omega_F^2 \). Since \( \tilde{\mu}_F \) depends only on \( \tilde{T}_F \) through Eq. (16), the integral in Eq. (17) above depends only on the values of the two parameters \( \tilde{T}_F \) and \( \alpha \).

The system of Eqs. (17) and (16) for general \( \tilde{T}_F \) and \( \alpha \) can only be solved numerically, and later we shall present the full numerical results for some specific choices of the parameters. However, analytical solutions exist in two limits: when \( \tilde{T}_F \gg 1 \) (i.e. \( T_F \gg T_c^0 \)) where the Fermi gas is completely degenerate at \( T = T_c^0 \) (Thomas-Fermi regime), and when \( \tilde{T}_F \ll 1 \) (i.e. \( T_F \ll T_c^0 \)) so that at \( T_c^0 \) fermions behave as a classical gas (Boltzmann regime).

In order to clarify the connection between the two limits and the general numerical solution, it is useful to further manipulate Eq. (17). By explicitly evaluating the prefactor, it can be finally recast in the convenient form

\[
\left( \frac{\delta T_c}{T_c^0} \right)_{BF} = -\frac{2^{5/3}}{3^{5/6} \pi \zeta(3)} \left( \frac{m_F}{m_B} + 1 \right) a_{BF} N_F^{1/6} \cdot F(\tilde{T}_F, \alpha),
\]

(18)

where

\[
F(\tilde{T}_F, \alpha) = \alpha^{3/2} \tilde{T}_F \int ds \, s^2 g_{1/2}(\exp\{-\tilde{T}_F \alpha s^2\}) \\
\times \left[ f_{3/2}(\exp\{\tilde{\mu}_F\}) - f_{3/2}(\exp\{\tilde{\mu}_F - \tilde{T}_F s^2\}) \right],
\]

(19)

and \( \ell_F = \sqrt{\hbar/m_F \omega_F} \) is the fermionic oscillator length. Notice the formal analogy between Eq. (18) and Eq. (4) for the shift \( (\delta T_c/T_c^0)_{BB} \) due to the boson-boson interactions alone.

Let us begin by considering the Thomas-Fermi limit \( (\tilde{T}_F \gg 1) \). In this limit the chemical potential of the fermions \( \mu_F \) tends to the Fermi energy \( \epsilon_F = k_B T_F \). Thus \( \tilde{\mu}_F \approx \tilde{T}_F \gg 1 \).

The limit of the Fermi functions in Eq. (10) for \( x \to \infty \) is \( f_{3/2}(x) \approx (\ln x)^{3/2}/3\sqrt{\pi} \). This
implies that the density profile of the fermion cloud takes the well known Thomas-Fermi shape

\[ n_F^0(r) = n_F^0(0) \left[ 1 - \left( \frac{x}{R'_x} \right)^2 - \left( \frac{y}{R'_y} \right)^2 - \left( \frac{z}{R'_z} \right)^2 \right]^{3/2}, \tag{20} \]

with \( n_F^0(0) = \frac{2\epsilon_F m_F}{\hbar^2}(6\pi^2) \), whenever the expression inside the square brackets is positive, and \( n_F^0(r) = 0 \) otherwise.

The function \( F(\tilde{T}_F, \alpha) \) then goes to the limiting form

\[ F(\tilde{T}_F, \alpha) \to \frac{4}{3\sqrt{\pi}} \alpha^{3/2}(\tilde{T}_F)^{5/2} \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \int_0^1 ds s^2 \times e^{-n\tilde{T}_F \alpha s^2}[1 - (1 - s^2)^{3/2}], \tag{21} \]

since \( g_{1/2}(x) = \sum_{n=1}^{\infty} x^n/n^{1/2} \).

We obtained Eq. (21) in the limit \( \tilde{T}_F \gg 1 \). Therefore, if \( \alpha \) is not too small (so that \( \tilde{T}_F \alpha \gg 1 \) still holds), then, for every \( n \) in the series, the exponential is non-vanishing only for values of \( s \ll 1 \), and we can adopt the expansion \( 1 - (1 - s^2)^{3/2} \simeq 3s^2/2 \). The integral in Eq. (21) becomes \( \int_0^1 ds s^4 e^{-n\tilde{T}_F \alpha s^2} \simeq 3\sqrt{\pi}/[8(n\tilde{T}_F \alpha)^{5/2}] \). Finally, therefore, \( F(\tilde{T}_F, \alpha) \to 3\zeta(3)/4\alpha \) and the Thomas-Fermi prediction for the relative shift reads

\[ \left( \frac{\delta T_c}{T_c} \right)_{BF} = -\frac{3^{1/6}}{2^{1/3}\pi} \left( \frac{m_F}{m_B} + 1 \right) \frac{m_F \omega_B^2}{m_B \omega_B^2} \frac{a_{BF}}{\ell_F} N_F^{1/6}, \tag{22} \]

where \( 3^{1/6}/(2^{1/3}\pi) \simeq 0.304 \). We notice that in the Thomas-Fermi regime the shift is independent of the number of bosons \( N_B \) and varies as the first inverse power of the parameter \( \alpha = m_B \omega_B^2/m_F \omega_F^2 \).

We now consider the Boltzmann limit for the Fermi gas (\( \tilde{T}_F \ll 1 \)). In this case the chemical potential \( \tilde{\mu}_F \) is large and negative and depends on \( \tilde{T}_F \) as: \( \tilde{\mu}_F \approx \ln\{(\tilde{T}_F)^3/6\} \). In the limit \( x \to 0, f_{3/2}(x) \approx x \), and then

\[ F(\tilde{T}_F, \alpha) \to \frac{\alpha^{3/2}(\tilde{T}_F)^4}{6} \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \int_0^\infty ds s^2 \times \left[ e^{-n\tilde{T}_F \alpha s^2} - e^{-(n\tilde{T}_F \alpha + \tilde{T}_F)s^2} \right]. \tag{23} \]

Evaluation of the integrals is straightforward and yields
\[ F(\tilde{T}_F, \alpha) \rightarrow \frac{\sqrt{\pi}}{24}(\tilde{T}_F)^{5/2} \cdot f(\alpha), \]  

(24)

with

\[ f(\alpha) = \sum_{n=1}^{\infty} \left( \frac{1}{n^2} - \frac{1}{n^{1/2}(n + \alpha^{-1})^{3/2}} \right), \]  

(25)

so that:

\[ \left( \frac{\delta T_c}{T_c^0} \right)_{BF} = -\frac{1}{24 \pi^{1/2} 3^{11/6} \zeta(3)} \left( \frac{m_F}{m_B} + 1 \right) \frac{a_{BF}}{\ell_F} N_F^{1/6}(\tilde{T}_F)^{5/2} \cdot f(\alpha), \]  

(26)

where the numerical prefactor is \( \simeq 0.025 \).

We notice that \( f(\alpha) \) in Eq. (25) is a monotonically decreasing function of \( \alpha \). In particular, one finds the following behaviours: \( f(\alpha \rightarrow 0) = \pi^2/6, f(1) \simeq 0.85, \) and \( f(\alpha \rightarrow \infty) = 3\zeta(3)/2\alpha \). As one should expect, in the Boltzmann limit \( (\tilde{T}_F \ll 1) \), the Bose-Fermi shift is negligible.

We now turn to the full numerical solution of Eqs. (18), (19), and (16) for more general values of the degeneracy parameter \( \tilde{T}_F \). In Fig. 1 we show the dimensionless function \( F(\tilde{T}_F, \alpha) \) as a function of \( \tilde{T}_F \) for three different values of the parameter \( \alpha = m_B \omega_B^2/m_F \omega_F^2 \), \( \alpha = 0.1, 1, \) and 10. For fixed \( \alpha \), \( F(\tilde{T}_F, \alpha) \) is a monotonically nondecreasing function of \( \tilde{T}_F \), which saturates for \( \tilde{T}_F \rightarrow \infty \) at the value predicted in the Thomas-Fermi regime \( 3\zeta(3)/4\alpha \simeq 0.9\alpha^{-1} \). For fixed \( \tilde{T}_F \), \( F(\tilde{T}_F, \alpha) \) increases by decreasing \( \alpha \). For the largest value of \( \alpha \) (\( \alpha = 10 \)) the function \( F \) reaches its asymptotic Thomas-Fermi value already at \( \tilde{T}_F \simeq 5 \). For \( \alpha = 1 \) and \( \alpha = 0.1 \) the function saturates for larger values of \( \tilde{T}_F \) not shown in the figure. The reason for this difference can be understood by recalling that the Thomas-Fermi result requires not only \( \tilde{T}_F \gg 1 \), but also \( \tilde{T}_F \gg \alpha^{-1} \) (see the discussion below Eq. (21)).
FIG. 1. Dimensionless function $F(\tilde{T}_F, \alpha)$ as a function of $\tilde{T}_F$ for the values $\alpha = 0.1$ (dotted line), $\alpha = 1.0$ (dashed line), and $\alpha = 10$ (solid line).

The physically relevant regimes in current experiments fall roughly around $\alpha \simeq 1$ and $\tilde{T}_F \simeq 1$. In this respect, a particularly interesting situation is the one realized in the Florence experiment [7], where a quantum degenerate trapped atomic mixture of fermionic $^{40}$K and bosonic $^{87}$Rb has been recently produced. One of the appealing features of this system is that the measured boson-fermion scattering length is large and negative: $a_{BF} = -22$ nm, giving rise to a fairly strong attractive boson-fermion interaction. The shift $(\delta T_c/T_c^0)_{BF}$ is thus positive and opposite to the shift $(\delta T_c/T_c^0)_{BB}$, since for pure $^{87}$Rb the boson-boson scattering length is $a_{BB} = 6$ nm, giving rise to a repulsive boson-boson interaction. In the Florence experiment the two atomic species are magnetically trapped, and are both prepared in their doubly polarized spin state. These states experience the same trapping potential so that $\alpha = m_B\omega_B^2/m_F\omega_F^2 = 1$, while the number of bosons and of fermions are respectively $N_B = 2 \times 10^4$, $N_F = 10^4$, so that $N_F/N_B = 0.5$, and $\tilde{T}_F = T_F/T_c^0 \simeq 2.3$. For the conditions of the Florence experiment the shift (4) due to the boson-boson coupling turns out to be:
\( \left( \frac{\delta T_c}{T_c^0} \right)_{BB} \simeq -0.037 \), and is comparable with the shift (3) due to finite size effects, which is given by: \( \left( \frac{\delta T_c}{T_c^0} \right)_{fs} = -0.044 \). For \( \alpha = 1 \) at \( \tilde{T}_F \simeq 2.3 \) the function \( F \) is at about 1/3 of its asymptotic value in the Thomas-Fermi regime, resulting in a Bose-Fermi shift considerably smaller than the Bose-Bose one: \( \left( \frac{\delta T_c}{T_c^0} \right)_{BF} \simeq 0.012 \). In Fig. 2 we show the shift \( \left( \frac{\delta T_c}{T_c^0} \right)_{BF} \) as a function of the ratio \( N_F/N_B \), with all the other parameters entering Eq. (18) fixed at the values of the Florence experiment [7]. In the same figure we include as a reference value the modulus of the boson-boson relative shift \( \left| \left( \frac{\delta T_c}{T_c^0} \right)_{BB} \right| \), calculated using the values of the parameters given by the Florence experiment.

![FIG. 2. Boson-fermion relative shift \( \left( \frac{\delta T_c}{T_c^0} \right)_{BF} \) from Eq. (18) (solid line) as a function of the ratio \( N_F/N_B \). Horizontal dashed line: value of the modulus \( \left| \left( \frac{\delta T_c}{T_c^0} \right)_{BB} \right| \) of the boson-boson shift (4). All other parameters, except the number of fermions \( N_F \), have been fixed at the values of the Florence experiment.](image)

From Fig. 2 we see that, while in the present experimental situation the boson-fermion shift is about 1/3 of the boson-boson one, by increasing the number of trapped fermions the two shifts become comparable at \( N_F \simeq 5N_B \). The boson-fermion shift is instead dominant at still larger values of \( N_F \). It is important to remark that, even if the Bose-Fermi shift of
the critical temperature is a small effect for the present experimental conditions, it might be observable. Since the fermions can be eliminated from the trap, one can look for the differences in the transition temperature with and without fermions.

In conclusion, we have determined the relative shift of the critical temperature of Bose-Einstein condensation in a trapped atomic Bose-Fermi mixture to lowest order in both the boson-boson and the boson-fermion coupling constants. We have determined numerically the general behaviour of the boson-fermion shift, and we have provided full analytical solutions in the quantum degenerate Thomas-Fermi regime and in the classical Boltzmann regime. We have applied our predictions to a specific experiment (the Florence experiment [7], chosen for the interesting value of the Bose-Fermi scattering length), and discussed the relative importance of the shifts due to boson-boson and boson-fermion interactions.
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