Effective matrix models for deconfinement in SU(N) and G(2) gauge theories

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Abstract. I present a simple matrix model for the deconfined phase of SU(N) theories at temperatures just above $T_c$. The model is designed to reproduce the anomaly, in particular the flatness of $(e - 3p)/T^2$ above $\sim 1.2T_c$ observed by lattice simulations with up to six colors. Furthermore, it predicts the existence of an adjoint Higgs phase where the masses of diagonal and off-diagonal gluons split and that the phase transition of SU(N) with three or more colors, and for exceptional groups with a trivial center such as G(2), is generically of first order. For G(2) gauge group, a small value of the Polyakov loop at $T - T_c$ can be obtained with a non-perturbative potential summed over the roots of SU(7) rather than G(2), which implements the principle of maximal eigenvalue repulsion.

1. Introduction

![Figure 1. SU(3) trace anomaly $e - 3p$ divided by $8T^2T_c^2$ [1].](image)

Lattice gauge theory simulations for $N = 2, 3, 4,$ and 6 colors find that the conformal anomaly $e - 3p$ scaled by $T^2$ is approximately constant from $\sim (1.2 - 4)T_c$, where $T_c$ is the critical temperature for deconfinement. Data from the WHOT collaboration for $N = 3$ is shown in fig. 1. If a bag constant were the dominant correction to perturbation theory in this range of temperature, this quantity would fall off as $\sim 1/T^2$. Rather, in this range, the pressure can be approximated as

$$p(T) \approx c_1 \left( T^4 - c_2 T_c^2 T^2 \right). \quad (1)$$
Thus, any non-perturbative terms which we introduce are assumed to be proportional to $\sim T^2$, $\sim T^0$, etc. [2, 3, 4, 5]. To describe the deconfined phase just above $T_c$ we use a matrix model, where the basic variables are the eigenvalues of the thermal Wilson line. The vacuum is found by minimizing an effective potential with respect to these $N$ matrix variables. Clearly the simplest way of obtaining a term $\sim T^2$ in the pressure is to introduce a similar term in the potential for the eigenvalues. We stress that this is merely an educated guess and that we have not performed an exhaustive search in the space of possible non-perturbative contributions.

With only the one-loop perturbative potential plus our model for non-perturbative contributions one can obtain decent fits of the pressure, energy density (incl. latent heat for $N \geq 3$) and the interaction measure of $SU(N)$ gauge theories up to a few times $T_c$ with only two parameters [4, 5]. One can also predict various interface tensions and the renormalized Polyakov loop; the latter approaches unity much more rapidly than indicated by lattice data [2, 3, 4, 5]. To describe the deconfined phase just above $T_c$, we find it necessary to add two more terms, $V_{npt} \sim T^2 T_c^2$, which mimics the confined vacuum.

The perturbative vacuum is $q = 0$, $L = 1$. The $Z(2)$ transform is $q = 1$, $L = -1$. As an angular variable, $q$ can be restricted to $[0, 1]$ so that a $Z(2)$ transformation corresponds to $q \rightarrow 1 - q$. In this model, $q_c = 1/2$ corresponds to maximal eigenvalue repulsion: $L_c = \text{diag}(i, -i)$, $\text{tr} L_c = 0$, which mimics the confined vacuum.

The free energy in the presence of the background field (2) at one loop order is

$$V_{\text{pt}}(q) = \pi^2 T^4 \left( -\frac{1}{15} + \frac{4}{3} q^2 (1 - q)^2 \right).$$

For $q = 0$, 1 this is the free energy of an ideal gas of three, massless gluons. The confined state at $q_c = 1/2$ is a maximum. To describe the transition to a confined phase we add non-perturbative terms to force the vacuum to go from the perturbative vacua, at $q = 0$ and 1, to $q_c = 1/2$. Given the behavior of the interaction measure, we assume that any such term is proportional to $T^2$. It must also be $Z(2)$ symmetric, $q \rightarrow 1 - q$, and so should be constructed in terms of $q(1 - q)$.

The simplest possibility, which also avoids unwanted additional phase transitions above $T_c$, is $V_{npt} \sim - T^2 T_c^2 c_1 q(1 - q)$. To fit to the lattice data, we find it necessary to add two more terms,

$$V_{npt} = -\frac{4\pi^2}{3} T^2 T_c^2 \left( \frac{1}{5} c_1 q(1 - q) + c_2 q^2(1 - q)^2 - c_3(t) \right), \quad (t = \frac{T}{T_c}) .$$

2. Model for $SU(2)$ and $SU(3)$

For two colors, we introduce a classical condensate [5]

$$A_0 = \frac{\pi T}{g} q \sigma_3, \quad L(\vec{x}) = \mathbf{P} \exp \left( ig \int_0^{1/T} A_0(\vec{x}, \tau) d\tau \right) = \begin{pmatrix} e^{i\pi q} & 0 \\ 0 & e^{-i\pi q} \end{pmatrix} . \quad (2)$$

The perturbative vacuum is $q = 0$, $L = 1$. The $Z(2)$ transform is $q = 1$, $L = -1$. As an angular variable, $q$ can be restricted to $[0, 1]$ so that a $Z(2)$ transformation corresponds to $q \rightarrow 1 - q$. In this model, $q_c = 1/2$ corresponds to maximal eigenvalue repulsion: $L_c = \text{diag}(i, -i)$, $\text{tr} L_c = 0$, which mimics the confined vacuum.
Note that we need to constrain the parameters so that the transition occurs at $T_c$, and we need to fix the pressure in the confined phase to zero. Thus $c_1$ and $c_2$ are not free. To generate terms $\sim T_0$ we let $c_3(t) = c_3(\infty) + \left[ c_3(1) - c_3(\infty) \right] / t^2$. The term $\sim c_3(1)$ in fact corresponds to a positive bag constant and is needed to fit the latent heat for $N \geq 3$. In all, this is then a model with two free parameters.

Viewing the $A_0$ as an adjoint Higgs condensate we observe that for $\langle q \rangle \neq 0$ there is a splitting of masses. Since $A_0 \sim \sigma_3$, the off diagonal components $\sim \sigma_1$, $\sigma_2$ develop a mass $\sim q$, while the diagonal ones do not [4]. Away from $T_c$ all components develop an equal mass from Debye screening. In our model the theory is in an adjoint Higgs phase for all $T > T_c$ though in practice, for the parameters of the model, we find that the condensate evaporates above $\sim 1.2 T_c$.

For three colors the background field formed by the Cartan subalgebra is

$$A_0 = \frac{\pi T}{3 g} \left( q_3 \lambda_3 + q_8 \lambda_8 \right) ; \quad \lambda_3 = \text{diag}(1,-1,0) ; \quad \lambda_8 = \text{diag}(1,1,-2) . \quad (5)$$

Moving along the $\lambda_8$ direction generates $Z(3)$ transformations: $L = 1$ when $q = 0$, and $L = \exp(2\pi i/3) 1$ when $q = 1$. The $\lambda_3$ direction takes one to the ground state $L_c = \text{diag} \left( e^{2\pi i/3}, e^{-2\pi i/3}, 1 \right)$ characterized by tr $L_c = 0$, like the confining vacuum. This path can be parametrized as $q_j(s) = (N - 2j + 1)/(2N) s$ and the potential is then minimized in $s$. In the limit of infinite $N$, this ansatz gives a uniform eigenvalue density to some maximum. We thus refer to it as the uniform eigenvalue ansatz.

![Figure 2](image-url)

**Figure 2.** Thermodynamics of $SU(3)$ (left) and $SU(4)$ (right): pressure $p/T^4$, energy density $e/(3T^4)$, and the rescaled interaction measure $\tilde{\Delta}$. All quantities are also scaled by $1/(N^2 - 1)$. Lattice data from ref. [7].

Numerical results for the thermodynamics of $SU(3)$ and $SU(4)$ are shown in fig. 2. Overall, the simple model with two free parameters performs quite well, especially near $T_c$. As already mentioned above, it will be interesting to see whether higher-loop perturbative corrections can be accommodated as well. The expectation value of the Polyakov loop approaches unity much more rapidly than on the lattice, see ref. [5], because in the current model the condensate is required to vanish quickly above $T_c$ to reproduce the pressure and energy density.

3. **$G(2)$ gauge group and the principle of maximal eigenvalue repulsion**

For $SU(N)$ gauge groups the Polyakov loop vanishes below $T_c$ which is a consequence of the global $Z(N)$ symmetry. There is maximal repulsion of the eigenvalues in the confined vacuum, as alluded to above. In fact, it may be more useful to think of confinement as arising not from
the center symmetry *per se*, but from eigenvalue repulsion. To see this consider groups with a trivial center such as $G(2)$, which is also simply connected. Thus, for $G(2)$ the Polyakov loop in the fundamental representation can be screened dynamically as there is no $Z(N)$ center symmetry. It may therefore appear similar to QCD with dynamical light quarks where there is no deconfining phase transition but just a cross-over. Rather strikingly, lattice simulations find that there is a first order transition [8] from a “confined phase” where the expectation value of the Polyakov loop in the fundamental representation is very small to a deconfined phase. Thus we need to consider effective potentials which give confinement, and a deconfining phase transition, without relying on any center symmetry.

The simplest approach is to note that in the nonperturbative potential there is nothing to forbid a term where we sum over powers of the fundamental $G(2)$ loop, the 7. If such a term is large, and of the right sign, one can drive the theory to a confined phase at low temperatures. Alternatively, we can construct a non-perturbative potential using both $SU(7)$ and $G(2)$ potentials since $G(2)$ is a subgroup of $SU(7)$; we must however restrict the $SU(7)$ potentials to the two dimensional Cartan space of $G(2)$. That way, it is possible to ensure that the stable vacuum is the $SU(7)$ confining vacuum at $T_c$. Confinement in $SU(7)$ then enforces the vanishing of the 7 loop in $G(2)$. In both cases, confinement is driven not by the center symmetry but through complete repulsion of eigenvalues.

The $SU(7)$ Wilson line is $L = \exp(2\pi i q_{SU7})$, where $q_{SU7} = \text{diag}(q_1, q_2, q_3, q_4, q_5, q_6, q_7)$ with $q_1 + \cdots + q_7 = 0$ so that $q_{SU7}$ is a tracless matrix. This constraint also defines the six dimensional Cartan space of $SU(7)$. $G(2)$ obeys the additional constraints $q_7 = q_1 + q_4 = q_3 + q_6 = 0$, which restricts to the Cartan space of $SO(7)$; and finally $q_1 + q_2 + q_3 = 0$, which leaves two degrees of freedom: $q_1$ and $q_2$.

Then, up to a factor of $4\pi^2 T^4/3$ and a shift by $-14\pi^2 T^4/45$, the one-loop perturbative $G(2)$ potential is given by

$$V^{G(2)}_2(q_{G2}) = B_4(q_1) + B_4(q_2) + B_4(q_1 + q_2) + B_4(q_1 - q_2) + B_4(2q_1 + q_2) + B_4(q_1 + 2q_2)$$

where $B_{2n}(q) \equiv q^n(1-q)^n$. Ref [5] considered a nonperturbative potential of the form

$$V_{\text{np}}(q_{G2}) = -\frac{4\pi^2}{3g^2} \left( c_1^{G(2)} V_1^{G(2)}(q_{G2}) + c_1^{SU(7)} V_1^{SU(7)}(q_{G2}) + c_2^{G(2)} V_2^{G(2)}(q_{G2}) + c_2^{SU(7)} V_2^{SU(7)}(q_{G2}) + d_2^{G(2)} V_2^7(q_{G2}) + c_3 \right).$$

(7)

where $V_1^{G(2)}$ is analogous to (6) with the replacement $B_4 \rightarrow B_2$. The $SU(7)$ potentials are

$$V_n^{SU(7)}(q_1, q_2) = B_{2n}(2q_1) + B_{2n}(2q_2) + B_{2n}(q_1 + 2q_2) + 2(B_{2n}(q_1 - q_2) + B_{2n}(2q_1 + q_2)) + 4(B_{2n}(q_1) + B_{2n}(q_2) + B_{2n}(q_1 + q_2)),$$

(8)

while

$$V_2^7(q_{G2}) = B_4(q_1) + B_4(q_2) + B_4(q_1 + q_2).$$

(9)

The difference between (8) and (6) is that the former sums over the roots of $SU(7)$ rather than those of $G(2)$.

To restrict the number of parameters, one can consider a minimal $G(2)$ model,

$$c_1^{G(2)}, c_3 \neq 0; \quad c_2^{G(2)} = c_1^{SU(7)} = c_2^{SU(7)} = d_2^{G(2)} = 0,$$

(10)

in analogy to the zero parameter $SU(N)$ model of ref. [2]. Alternatively, a model with a single fundamental loop,

$$d_2^{G(2)}, c_1^{G(2)}, c_2^{G(2)}, c_3 \neq 0; \quad c_1^{SU(7)} = c_2^{SU(7)} = 0.$$

(11)
Finally, a $SU(7)$ type model,

$$
c_2^{G(2)} = 1 \quad ; \quad c_1^{SU(7)}, c_2^{SU(7)}, c_3 \neq 0 \quad ; \quad c_1^{G(2)} = d_2^{G(2)} = 0 \, .
$$

(12)

Notice that here $c_2^{G(2)} = 1$ so that the $G(2)$ part of the nonperturbative potential cancels the perturbative $G(2)$ potential at $T_c$ fully. This ensures that the confining effects of the $SU(7)$ potential are maximized at $T_c$.

Figure 3. Left: Expectation value of the Polyakov loop in the fundamental representation of $G(2)$ for the minimal $G(2)$ model (10); the fundamental loop model (11); and two $SU(7)$ models (12). Right: Rescaled interaction measure for the four models.

Fig. 3 shows the fundamental Polyakov loop from the $G(2)$ models. The minimal $G(2)$ model appears to be excluded as it leads to a large (negative) value at $T_c^-$. The other models give fundamental loops which are small in the low temperature phase. In the fundamental loop model (11) this expectation value is positive at $T_c^-$ and the model is near a critical end point, with a small discontinuity for the Polyakov loop. The lattice measurements from [8] indicate a small and perhaps negative value of the fundamental loop at $T_c^-$ but correspond to bare expectation values. Lastly, the $SU(7)$ models automatically give a vanishing fundamental loop below $T_c$. The rescaled interaction measure for the minimal $G(2)$ and the fundamental loop models are flat and resemble those of $SU(N)$, while that for the $SU(7)$ model is not. Additional lattice simulations [9] for $G(2)$ could help understand the similarities and differences of its phase transition to that of $SU(N)$ and test the principle of eigenvalue repulsion.

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