Bergman kernels and subadjunction

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Abstract. In this article our main result is a more complete version of the statements obtained in [6]. One of the important technical point of our proof is an $L^2/m$ extension theorem of Ohsawa-Takegoshi type, which is derived from the original result by a simple fixed point method. Moreover, we show that these techniques combined with an appropriate form of the “invariance of plurigenera” can be used in order to obtain a new proof of the celebrated Y. Kawamata subadjunction theorem.

§0 Introduction

Our starting point in this article is the following generalization of our work [6].

0.1 Theorem. Let $p : X \to Y$ be a surjective projective map between two non-singular manifolds, and let $L \to X$ be a line bundle endowed with a metric $h_L$ such that:

1) The curvature current of the bundle $(L, h_L)$ is semipositive, i.e. $\sqrt{-1} \Theta_{h_L}(L) \geq 0$;

2) There exists a generic point $z \in Y$ and a section $u \in H^0(X_z, mK_X + L)$ such that

$$\int_{X_z} |u|^2 m e^{-\varphi_L}/m < \infty.$$  

Then the line bundle $mK_{X/Y} + L$ admits a metric with positive curvature current. Moreover, this metric is equal to the fiberwise $m$–Bergman kernel metric on the generic fibers of $p$. □

In the statement above the meaning of the word generic is as follows: $w \in Y$ is generic if it is not a critical value of $p$ and if the sections of the bundle $mK_{X/Y} + L|_{X_w}$ extend locally near $w$.

This kind of positivity results (and their relevance for important problems in algebraic geometry) have been investigated since quite long time; we will only mention here a few contributors [8], [13], [14], [15], [16], [17], [19], [20], [21], [22], [23], [26], [27], [28], [31], [34], [35] (and we apologize to the ones we omit...). For example, one can see that if the metric of $L$ is given by a section of some of its multiples, then the qualitative part of the theorem above can be obtained as a consequence of E. Viehweg’s results on weak positivity of direct images (cf. [35] and the references therein, as well as the recent results of A. Höring [16]).

The difference between theorem 0.1 and the corresponding result we obtain in [6] is that here the map $p$ can be singular (in our previous work we have assumed that $p$ is a smooth fibration). Actually, the additional technical result allowing us to treat the general case is an $L^{2/m}$ extension theorem which we describe next.
The set-up is the following: let $\Omega \subset \mathbb{C}^n$ be a ball of radius $r$ and let $h : \Omega \to \mathbb{C}$ be a holomorphic function, such that $\sup_{\Omega} |h| \leq 1$; moreover, we assume that the gradient $\partial h$ of $h$ is nowhere zero on the set $V := (h = 0)$. We denote by $\varphi$ a plurisubharmonic function, such that its restriction to $V$ is well-defined (i.e., $\varphi|_V \neq -\infty$). Then the Ohsawa-Takegoshi extension theorem states that for any $f$ holomorphic on $V$, there exists a function $F$, holomorphic in all of $\Omega$, such that $F = f$ on $V$, and moreover

$$\int_{\Omega} |F|^2 \exp(-\varphi) d\lambda \leq C_0 \int_{V} |f|^2 \exp(-\varphi) \frac{d\lambda_V}{|\partial h|^2}.$$

Here, $C_0$ is an absolute constant. Our generalization is the following.

0.2 Proposition. For any holomorphic function $f : V \to \mathbb{C}$ with the property that

$$\int_{V} |f|^{2/m} \exp(-\varphi) \frac{d\lambda_V}{|\partial h|^2} \leq 1,$$

there exists a function $F \in \mathcal{O}(\Omega)$ such that:

(i) $F|_V = f$ i.e. the function $F$ is an extension of $f$;

(ii) The next $L^{2/m}$ bound holds

$$\int_{\Omega} |F|^{2/m} \exp(-\varphi) d\lambda \leq C_0 \int_{V} |f|^{2/m} \exp(-\varphi) \frac{d\lambda_V}{|\partial h|^2},$$

where $C_0$ is the same constant as in the Ohsawa-Takegoshi theorem.

Once this result is established, the proof of 0.1 runs as follows: we first restrict ourselves to the Zariski open set $Y_0 \subset Y$ which does not contain any critical value of $p$, and such that $\forall z \in Y_0$, all the sections of the bundle $mK_X + L$ extend locally near $z$ (the existence of such a set $Y_0$ is a consequence of standard semi-continuity arguments). Over $Y_0$, we can apply the results in [6] and therefore the $m$–Bergman kernel metric has a psh variation. We then use the $L^{2/m}$ extension result to estimate our metric from above by a uniform constant. Standard results of pluripotential theory then gives that the metric extends to a semipositively curved metric on all of $X$. At the end of this section, we derive some corollaries and further results; we equally provide some additional explanations concerning our previous work [6].

We show next that the techniques and results we obtain here can be used in order to provide a direct and transparent argument for the next statement 0.3, which appears to be the main part of the proof of the adjunction result originally due to Y. Kawamata (cf. [19]). We also refer to the recent presentation of J. Kollár (see [23]) and the references therein, especially the recent developments due to F. Ambro and O. Fujino. We will adopt the terminology in [19], [23], so that the relation between our arguments and the construction of the proof in these articles becomes as clear as possible.

Let $X$ be a normal projective variety, and let $\Delta$ be an effective Weil $\mathbb{Q}$-divisor on $X$, such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We assume for the moment that the pair $(X, \Delta)$ is log-canonical. This requirement means that there exists a log-resolution $\mu : X' \to X$ of the pair $(X, \Delta)$, such that we have

$$\mu^*(K_X + \Delta) = K_{X'} + \sum_{j \in J} a_j Y_j$$

where $a_j$ are integers. This equation is the adjunction formula.
where the coefficients \((a^j)\) above are rational numbers, such that \(\forall j \in J\) we have \(a^j \leq 1\). We take this opportunity to recall also that if the inequality \(a^j < 1\) is satisfied for all \(j \in J\), then the pair \((X, \Delta)\) is called Kawamata log terminal, or klt for short. The above properties are independent of the log-resolution \(\mu\) of the pair \((X, \Delta)\) (cf \[23\]).

A subvariety \(W \subset X\) is called a log canonical center if there exists a log resolution \(\mu\) of the pair \((X, \Delta)\) and a hypersurface say \(Y_1\) whose corresponding coefficient \(a^1\) is equal to 1, such that \(W = \mu(Y_1)\). Since each component of the intersection of two log canonical centers is a log canonical center (cf. \[18\], \[23\]), the notion of minimal log canonical center makes sense.

In this framework, the following important result was established by Y. Kawamata in \[19\].

**Theorem** (\[19\]). \textit{Let \((X, \Delta)\) be a log-canonical pair; we assume moreover the existence of an effective divisor \(\Delta_0 \leq \Delta\), such that the pair \((X, \Delta_0)\) is klt. Let \(H\) be an ample divisor on \(X\), and let \(\varepsilon > 0\) be a positive rational. We consider a minimal log-canonical center \(W\) of the pair \((X, \Delta)\). Then \(W\) is normal, and there exists an effective \(\mathbb{Q}\)-divisor \(\Delta_W\) on \(W\) such that:

i) We have the adjunction relation \(K_X + \Delta + \varepsilon H|_W \equiv K_W + \Delta_W\);

ii) The pair \((W, \Delta_W)\) is klt.}

We can distinguish two parts in the original proof of this result: one starts with a few important reductions, showing that under the hypothesis of the preceding theorem, \(W\) becomes an exceptional minimal center for some pair \((X, \tilde{\Delta})\) in the terminology used in \[23\]. By definition, this means that in the decomposition above corresponding to \(K_X + \tilde{\Delta}\) we have \(Y_j \cap Y_1 = \emptyset\), as soon as \(a^j = 1\); the technique used is the so-called tie-break method (cf. \[17\] \[18\], \[23\]). Also, the minimal center \(W\) is proved to be normal, as a consequence of the Kawamata-Viehweg vanishing theorem. The hypothesis \((X, \Delta_0)\) is klt is crucial for the proof of these results; we refer to the article by F. Ambro \[1\] for a generalization in the context of log-varieties (see also the work of O. Fujino \[13\]), and the references therein.

After these reductions, the heart of the matter is the construction of the divisor \(\Delta_W\). This is performed by using ramified cover tricks (in order to deal with \(\mathbb{Q}\)-divisors instead of line bundles), and very subtle properties derived from the existence of mixed Hodge structures, together with the work of P. Griffiths, W. Schmid, and many other contributors.

In the second part of the present article, our aim is to present a metric approach for the construction of the divisor \(\Delta_W\) above. Therefore, we will assume from the very beginning that \(W\) is an exceptional log canonical center of \((X, \Delta)\): starting from this, we will use the theorem 0.1 above in order to obtain \(\Delta_W\). We remark that no ramified covers or mixed Hodge structures are used in our arguments. Our proof is explicit and natural in the context of Bergman kernels, but on the other hand, the original argument provides a finer analysis of the singularities of \(\Delta_W\) on a birational model of \(W\).

Actually, we will work in the following local context. We assume that the pair \((X, \Delta)\) is log canonical at some point \(x_0 \in X\), that is to say that there exists a log-resolution
\[ \mu : X' \to X \] of the pair \((X, \Delta)\), such that we have

\begin{equation}
\mu^*(K_X + \Delta) = K_{X'} + \sum_{j \in J} a^j Y_j
\end{equation}

where the coefficients \((a^j)\) above are rational numbers, such that \(a^j \leq 1\) if \(x_0 \in \mu(Y_j)\). As before, the above property is independent of the log-resolution \(\mu\) of the pair \((X, \Delta)\).

By the preceding discussion, there is no loss of generality if we only focus on the properties of the exceptional centers \(W\), so we assume that there exists a log-resolution \(\mu : X' \to X\) of the pair \((X, \Delta)\) together with a decomposition of the inverse image of the \(\mathbb{Q}\)-divisor \(K_X + \Delta\) as follows

\begin{equation}
\mu^*(K_X + \Delta) = K_{X'} + S + \Delta' + R - \Xi,
\end{equation}

such that:

- \(S\) is an irreducible hypersurface, such that \(W = \mu(S)\);
- \(\Delta' := \sum_j a^j Y_j\), where \(x_0 \in \mu(Y_j)\) and \(a^j \in [0, 1]\);
- The divisor \(R\) is effective, and a hypersurface \(Y_j\) belongs to its support if either \(x_0 \not\in \mu(Y_j)\), or \(Y_j \cap S = \emptyset\) (so in particular the restriction \(R|_S\) is \(\mu|_S\)-vertical);
- The divisor \(\Xi\) is effective and \(\mu\)-contractible; in addition, we assume that the divisors appearing in the right hand side of the formula (†) have strictly normal crossings.

In general, the center \(W\) is singular, and we will assume that the restriction map \(\mu|_S : S \to W\) factors through the desingularization \(g : W' \to W\), so that we have

\[ \mu|_S = g \circ p \]

where \(p : S \to W'\) is a surjective projective map.

Before stating our next result, we introduce a last piece of notation: let \(A\) be an ample bundle on \(X\), and let \(F_1, ..., F_k\) be a set of smooth hypersurfaces of \(W'\) with strictly normal crossings, such that there exists positives rational numbers \((\delta_j)\) for which the \(\mathbb{Q}\)-bundle

\begin{equation}
g^*(A) - \sum \delta^j F_j + \varepsilon K_{W'}
\end{equation}

is semi-positive (in metric sense) for any \(\varepsilon\) small enough, and such that \(g(F_j) \subset W_{\text{sing}}\) for each \(j\). Indeed a set \((F_j)\) with the properties specified above does exists, as the variety \(X\) sits inside some projective space, and we can consider an embedded resolution of singularities of the set \(W\). It is routine to verify that the intersection of the exceptional divisors with \(W'\) will do. We remark that if the canonical bundle \(K_{W'}\) happens to be the inverse image of some bundle on \(W\), then we can take \(\delta^j = 0\) for all \(j\).

The family \((F_j)\) induces a decomposition of the divisor \(\Xi\) as follows

\begin{equation}
\Xi = \Xi_1 + \Xi_2
\end{equation}
where by definition $\Xi_1$ is the part of the divisor $\Xi$ whose support restricted to $S$ is mapped by $p$ into $\cup_j F_j$.

Our version of the result of Y. Kawamata is the following.

**0.3 Theorem.** Under the hypotheses above, there exists a closed positive current $\Theta_{W'}$ and an effective divisor $\Lambda$ on $W'$ such that:

(a) We have $K_{W'} + \Theta_{W'} \equiv \Lambda + g^*(K_X + \Delta)|_{W'}$;

(b) The (local) function

$$\exp \left((1 + \varepsilon_0)(\varphi_\Lambda - \varphi_{\Theta_{W'}})\right)$$

is in $L^1_{\text{loc}}$ each point of $W' \setminus p(\text{Supp} R|_S)$, where $\varepsilon_0$ is a positive real, and $\varphi_\Lambda$, respectively $\varphi_{\Theta_{W'}}$, are the local potentials of $\Lambda$, respectively $\Theta_{W'}$;

(c) The support of $\Lambda$ is contained in $\cup_j F_j$. $\square$

The point b) is the analogue of the klt property ii) in the result quoted above. Before explaining the main ideas involved in our arguments, we make here a few observations.

**Remarks.**

1) By the $L^2$ theory we can “convert” the current $\Theta_{W'}$ into an effective divisor $\Delta_{W'}$, as follows. For each $\varepsilon > 0$, there exists an effective divisor $\Delta_{W'}$ on $W'$ such that the part a) of the statement above becomes the familiar relation

$$K_{W'} + \Delta_{W'} \equiv \Lambda + g^*(K_X + \Delta + \varepsilon A)|_{W'};$$

and such that b) is still satisfied. The precise procedure will be detailed at the end of our article. As we have already mentioned, the minimal center $W$ is known to be normal, provided that the pair $(X, \Delta)$ belongs to an appropriate class of singularities (see e.g. the works of F. Ambro and O. Fujino in [1], [13]). Hence in this case the divisor $\Lambda$ is contractible with respect to the map $g$, which means that by projection to $W$, the relation above is the same as the point i) of the original theorem of Kawamata recalled above.

2) We can take $\Lambda = 0$ if the canonical bundle $K_{W'}$ happens to be the inverse image of some bundle on $W$, as a by-product of our proof.

3) In the approach by J. Kollár in [23], (the analogue of) the difference $\Theta_{W'} - \Lambda$ is further analyzed: he exhibits a nef part, as well as a klt and canonical part of it. Unfortunately, our methods as they stand by now do not allow to obtain such a decomposition (it is of course not impossible that one could get it by a more refined version of theorem 0.1). $\square$

Next we will give a brief account of our proof of 0.3. We first observe that we have

$$p^*(g^*(K_X + \Delta)|_{W'} - K_{W'}) + \Xi|_S \equiv K_{S/W'} + \Delta'|_S + R|_S.$$

Let $m_0$ be a positive and divisible enough integer; the main point of our strategy is to show that the current $m_0\Theta_{S/W'}$ associated to the $m_0$-Bergman kernel metric of the bundle

$$m_0(K_{S/W'} + \Delta'|_S)$$
is at least as singular as the divisor type current \( m_0|\Xi_2|_S \). Very roughly, this is proved to be so in two steps. We first observe that thanks to the numerical relation above together with (3), we can "move" the current \( m_0 \Theta_{S/W'} \) into the class \( m_0(K_S + \Delta'_S) + m_1 \mu^*(A)|_S \) (for some large enough positive integer \( m_1 \)) in such a way that the only additional singularities we get by this operation are concentrated along the support of \( \Xi_1|_S \). We approximate this new current with global sections of \( km_0(K_{X'} + S + \Delta'_S) + km_1 \mu^*(A) + A'|_S \), where \( k \gg 0 \) and \( A' \) is an ample line bundle on \( X' \) (which does not depends on \( k \)), by using the regularization of closed positive currents. The second step consists in showing that the approximating sections, twisted with \( km_0 R|_S \) extend to \( X' \); here we use the invariance of plurigenera technique originally due to Y.-T. Siu in [32], [33], in a slight variation of the version proved by L. Ein and M. Popa in [12]. This result is absolutely central for our proof, but nevertheless we have decided to include its proof in the sequel of this article [7], since the arguments needed for it are quite different from the topics presented here. The claim concerning the singularities acquired by \( m_0 \Theta_{S/W'} \) follows from the Hartogs principle.

The existence of this particular representative of the Chern class of the right hand side in the formula above shows that the divisor \( K_{S/W'} + \Delta'_S + R|_S - \Xi_2|_S \) is pseudoeffective. The current \( \Theta_{W'} \) and respectively the divisor \( \Lambda \) are obtained by taking the direct image of this difference. We would like to stress on the fact that the explicit expression of the \( m_0 \)-Bergman kernel metric in the theorem 0.1 is crucial for the proof of the integrability property b).

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§A. Proof of the main result

We start by establishing some technical tools which will be needed later on in our arguments. In the paragraph A.1 we deal with a local \( L^{2/m} \) extension result (proposition 0.2), whereas in A.2 we prove the theorem 0.1.

§A.1 The \( L^{2/m} \) version of the Ohsawa-Takegoshi theorem

In this section we establish an extension theorem with \( L^{2/m} \) bounds, analogous to the Ohsawa-Takegoshi result; as we have already mentioned, we will need it for the proof of the theorem 0.1. We recall that the setting is the following: let \( \Omega \subset \mathbb{C}^n \) be a ball of radius \( r \) and let \( h : \Omega \to \mathbb{C} \) be a holomorphic function, such that \( \sup_{\Omega} |h| \leq 1 \); moreover, we assume that the gradient \( \partial h \) of \( h \) is nowhere zero on the set \( V := (h = 0) \). We denote by \( \varphi \) a plurisubharmonic function, such that its restriction to \( V \) is well-defined (i.e., \( \varphi|_V \neq -\infty \)).
Proof of 0.3. Let $f$ be a holomorphic function $f : V \to \mathbb{C}$ with the property that
\[
\int_V |f|^{2/m} \exp(-\varphi) \frac{d\lambda_V}{\partial h} = 1.
\]

We begin with some reductions. In the first place we can assume that the function $\varphi$ is smooth, and that the functions $h$ (respectively $f$) can be extended in a neighbourhood of $\Omega$ (of $V$ inside $V \cap \overline{\Omega}$, respectively). Once the result is established under these additional assumptions, the general case follows by approximations and standard normal families arguments.

We can then clearly find some holomorphic $F_1$ in $\Omega$ that extends $f$ and satisfies
\[
\int_{\Omega} |F_1|^{2/m} \exp(-\varphi)d\lambda \leq A < \infty.
\]
We then apply the Ohsawa-Takegoshi theorem with weight
\[
\varphi_1 = \varphi + (1 - 1/m) \log |F_1|^2
\]
and obtain a new extension $F_2$ of $f$ satisfying
\[
\int \frac{|F_2|^2}{|F_1|^{2-2/m}} \exp(-\varphi) d\lambda \leq C_0 \int \frac{|f|^2}{|F_1|^{2-2/m}} \exp(-\varphi) d\lambda
\]
Hölder’s inequality gives that
\[
\int |F_2|^{2/m} \exp(-\varphi) d\lambda \int \frac{|F_2|^{2/m}}{|F_1|^{(2-2/m)/m}} |F_1|^{(2-2/m)/m} \exp(-\varphi) d\lambda \leq
\]
\[
\leq \left( \int \frac{|F_2|^2}{|F_1|^{2-2/m}} \exp(-\varphi) d\lambda \right)^{1/m} \left( \int |F_1|^{2/m} \exp(-\varphi) d\lambda \right)^{m/(m-1)}
\]
which is smaller than
\[
C_0^{1/m} A^{m/(m-1)} = A(C_0/A)^{1/m} =: A_1.
\]
If $A > C_0$, then $A_1 < A$. We can then repeat the same argument with $F_1$ replaced by $F_2$, etc, and get a decreasing sequence of constants $A_k$, such that
\[
A_{k+1} = A_k (C_0/A_k)^{1/m}
\]
for $k \geq 0$. It is easy to see that $A_k$ tends to $C_0$. Indeed, if $A_k > rC_0$ for some $r > 1$, then $A_k$ would tend to zero by the relation above. This completes the proof.

For further use, we reformulate the above result in terms of differential forms. One can find the details of the “conversion” of 0.2 into 0.2′ e.g. in the notes [5].

0.2′ Lemma. Given $u$ a holomorphic $m$-canonical form on $V$ such that
\[
\int_V |u|^{\frac{2}{m}} e^{-\varphi} < \infty,
\]
there exist a holomorphic $m$-canonical form $U$ on $\Omega$ such that:

(i) $U = u \wedge (dh)^{\otimes m}$ on $V$ i.e. the form $U$ is an extension of $u$.

(ii) The next $L^{2/m}$ bound holds

$$\int_{\Omega} |U|^{2/m} e^{-\varphi} \leq C_0 \int_{V} |u|^{2/m} e^{-\varphi},$$

where $C_0$ is the constant appearing in the Ohsawa-Takegoshi theorem.

\section*{A.2 Positivity properties of the $m$-Bergman kernel metric}

For the convenience of the reader, we first recall some facts concerning the variation of the fiberwise $m$–Bergman metric (also called NS metric in [6], [30], or just a pseudo-norm in [17]).

The general set-up is the following: let $p : X \to Y$ be a surjective projective map between two non-singular manifolds and consider a line bundle $L \to X$, endowed with a metric $h_L$, such that $\sqrt{-1} \Theta_{h}(L) \geq 0$. To start with, we will assume that $h_L$ is a genuine metric (i.e. non-singular) and that the map $p$ is a smooth fibration.

With this data, for each positive integer $m$ we can construct a metric $h^{(m)}_{X/Y}$ on the twisted relative bundle $mK_{X/Y} + L$ as follows: let us take a vector $\xi$ in the fiber $-(mK_{X/Y} + L)_x$; then we define its norm

$$\|\xi\|^2 := \sup \|\tilde{\sigma}(x) \cdot \xi\|^2$$

the "sup" being taken over all sections $\sigma$ to $mK_{X/y} + L$ such that

$$\|\sigma\|^{2/m}_{m,z} := \int_{X_y} |\sigma|^{2/m} e^{-\frac{1}{m} \varphi_L} \leq 1;$$

in the above notation $\varphi_L$ denotes as usual the metric of $L$ and $p(x) = y$; also, we have used the notation $\tilde{\sigma}$ to denote the section of the bundle $mK_{X/Y} + L|_{X_y}$ corresponding to $\sigma$ via the standard identification (see e.g. [6], section 1).

We give now the expression of the local weights of the metric $h^{(m)}_{X/Y}$; we denote by $(z^j)$ respectively $(t^i)$ some local coordinates centered at $x$, respectively $y$. Then we have

$$\exp \left( \varphi^{(m)}_{X/Y}(x) \right) = \sup_{\|u\|_{m,z} \leq 1} |u'(x)|^2$$

where $u \in H^0(X_y, mK_{X_y} + L)$ is a global section.

Finally, we denote by $\tilde{u} := u \wedge (p^* dt)^{\otimes m}$ and the above $u'$ is obtained as $\tilde{u} = u'(dz)^{\otimes m}$.

In the article [6], among other things we have established the fact that the metric $h^{(m)}_{X/Y}$ above has semi-positive curvature current, or it is identically $+\infty$. The latter
situation occurs precisely when there are no global $L$-twisted $m$-canonical forms on the fibers.

In fact, it turns out that the above construction has a meaning even if the metric $h_L$ we start with is allowed to be singular (but we still assume that the map $p$ is a non-singular fibration). We remark that in this case some fibers $X_y$ may be contained in the unbounded locus of $h_L$, i.e. $h_L|_{X_y} \equiv \infty$, but for such $y \in Y$ we adopt the convention that the metric $h_{X/Y}^{(m)}$ is identically $+\infty$ as well. As for the fibers in the complement of this set, the family of sections we consider in order to define the metric consists in twisted pluricanonical forms whose $m^{th}$ root is $L^2$.

We recall now the the result we have proved in the section 3 of [6] (see also [30] and the references therein).

A.2.1 Theorem ([6]). Let $p : X_0 \to Y_0$ be a proper projective non-singular fibration, and let $L \to X_0$ be a line bundle endowed with a metric $h_L$ such that:

1) The curvature current of the bundle $(L, h_L)$ is positive, i.e. $\sqrt{-1} \Theta_{h_L}(L) \geq 0$;
2) For each $y \in Y_0$, all the sections of the bundle $mK_{X_y} + L$ extend near $y$;
3) There exist $z \in Y_0$ and a section $u \in H^0(X_z, mK_{X_z} + L)$ such that

$$\int_{X_y} |u|^2 e^{-\frac{2}{m} \varphi_L} < \infty.$$  

Then the above fiberwise $m$–Bergman kernel metric $h_{X_0/Y_0}^{(m)}$ has semi-positive curvature current.

A.2.2 Remark. We take this opportunity to point out two imprecisions in the formulation of the corollary 4.2 from our previous work [6]. In the first place, it must be of course assumed that the metric $h_{X_0/Y_0}^{(m)}$ (called NS-metric in [6]) is not identically $+\infty$. Secondly, the assumption that $p_*(mK_{X/Y} + L)$ be locally free should be replaced by the assumption 2) of the preceding statement. Immediately after corollary 4.2 in [6] it is stated that these two conditions are equivalent, but this is not the case. In fact, the property of being locally free concerns the direct image of $p_*(mK_{X/Y} + L)$ as a sheaf: the stalks of this sheaf consists of sections over the fibers of $p$ that do extend to neighboring fibers, and local freeness means that this space of extendable sections has everywhere the same rank. We remark that the direct image $p_*(mK_{X/Y} + L)$ is torsion free, hence locally free if the dimension of the base is equal to one. For the proof of Corollary 4.2 we need the stronger property 2) above, saying that all sections extend locally to neighboring fibers.

This distinction between local freeness and the condition 2) is precisely the heart of the question of invariance of plurigenera for projective families over one dimensional base, where the local freeness is automatic, and 2) is exactly what is to be proved. We will continue this discussion at the end of the present subsection, after the proof of the corollaries of the Theorem 0.1.
Proof (of 0.1). By the usual semi-continuity arguments, there exist a Zariski open set $Y_0 \subset Y$ such that the condition (2) above is fulfilled; by restricting further the set $Y_0$ we can assume that if we denote by $X_0$ the inverse image of the set $Y_0$ via the map $p$, then $p : X_0 \to Y_0$ is a non-singular fibration.

Then we can use the above result and infer that our metric $h_0^{(m)}$ is explicitly given over the fibers $X_y$, as soon as $y \in Y_0$ and the restriction of $h_L$ to $X_y$ is well defined. We are going to show now that this metric admits an extension to the whole manifold $X$. The method of proof is borrowed from our previous work: we show that the local weights of the metric $h_0^{(m)}$ are locally bounded near $X \setminus X_0$ and then standard results in pluripotential theory imply the result.

Let us pick a point $x \in X$, such that $y := p(x) \in Y_0$; assume that the restriction of $h_L$ to $X_y$ is well defined. Let $u$ be some global section of the bundle $mK_{X_y} + L$. Locally near $x$ we consider a coordinates ball $\Omega$ and let $\Omega_y$ be its intersection with the fiber $X_y$. Thus on $\Omega_y$ our section $u$ is just the $m$’th tensor power of some $(n, 0)$ form, and let $\tilde{u} := u \wedge p^*(dt)^{\otimes m}$. With respect to the local coordinates $(z^j)$ on $\Omega$ we have $\tilde{u} = u' dz^{\otimes m}$ and as explained before, the local weight of the metric $h_0^{(m)}$ near $x$ is given by the supremum of $|u'|$ when $u$ is normalized by the condition

$$\int_{X_y} |u|^{\frac{2}{m}} e^{-\frac{1}{m} \phi} \leq 1.$$

But now we just invoke the lemma 0.2’, in order to obtain some $m$–form of maximal degree $\tilde{U}$ on $\Omega$ such that its restriction on $\Omega_y$ is $\tilde{u}$, and such that the next inequality hold

$$\int_{\Omega} |\tilde{U}|^{\frac{2}{m}} e^{-\frac{1}{m} \phi} \leq C_0 \int_{\Omega_y} |u|^{\frac{2}{m}} e^{-\frac{1}{m} \phi}.$$

The mean value inequality applied to the relation above show that we have

$$|u'(x)|^{2/m} \leq C$$

where moreover the bound $C$ above does not depend at all on the geometry of the fiber $X_y$, but on the ambient manifold $X$. In particular, the metric $h_0^{(m)}$ admits an extension to the whole manifold $X$ and its curvature current is semipositive, so our theorem 0.1 is proved.

A.2.3 Remark. The constant “$C$” above only depends on the sup of the $m^{th}$–root of the metric of the bundle $L$ and the geometry of the ambient manifold $X$ (and not at all on the geometry of the fiber $X_y$). As a consequence we see that if we apply the above arguments to a sequence $pK_{X/Y} + L_p$, then the constant in question will be uniformly bounded with respect to $p$, provided that $\frac{1}{p} \int \varphi_{L_p}$ is bounded from above. Hence we obtain the corollaries below (see also [11] and [30] for related statements).

A.2.4 Corollary. Let $p : X \to Y$ be a projective surjective map, and let $L \to X$ be a line bundle endowed with a metric $h_L$ such that:

1) The curvature current of the bundle $(L, h_L)$ is positive, i.e. $\sqrt{-1} \Theta_{h_L}(L) \geq 0$;
2) There exists a very general point \( z \in Y \) and a section \( u \in H^0(X_z, m_0K_{X/Y} + L|_{X_z}) \) such that
\[
\int_{X_z} |u|^2 m_0 e^{-\frac{1}{m_0} \varphi_L} < \infty.
\]
Then the bundle \( mK_{X/Y} + L \) admits a metric \( \varphi^{(\infty)}_{X/Y} \) with positive curvature current, whose restriction to any very general fiber \( X_w \) of \( p \) has the following (minimality) property: for any non-zero section \( v \in H^0(X_w, kmK_{X_z} + kL) \) we have
\[
|v|^2 km e^{-\frac{1}{m} \varphi_L} \leq \int_{X_w} |v|^2 km e^{-\frac{1}{m} \varphi_L}
\]
up to the identification of \( K_{X/Y}|_{X_w} \) with \( K_{X_w} \).

In the statement A.2.4 we call a point \( z \in Y \) very general if any holomorphic section of the bundle
\[
km_0K_{X/Y} + kL|_{X_z}
\]
extends locally near \( z \), for any positive integer \( k \). By the usual semi-continuity arguments we infer that the set of very general points is the complement of a countable union of Zariski closed sets of codimension at least one in \( Y \).

The proof of the above result is an immediate consequence of the arguments of 0.1. Indeed, we consider the \( km \)-Bergman metric on the bundle
\[
k(mK_{X/Y} + L)
\]
and we know that the weights of its \( 1/k^{th} \) root are uniformly bounded, independently of \( k \) (see the remark above). We obtain the metric \( \varphi^{(\infty)}_{X/Y} \) by the usual upper envelope construction, namely
\[
\varphi^{(\infty)}_{X/Y} := \sup_{k \geq 0} \left( \frac{1}{k} \varphi^{(km)}_{X/Y} \right).
\]

The corollary A.2.4 admits a metric version which we discuss next.

**A.2.5 Corollary.** Let \( p : X \to Y \) be a projective surjection, where \( X \) and \( Y \) are non-singular manifolds, and let \( L \to X \) be a line bundle, endowed with a positively curved metric \( h_L \). We assume that there exists a very general point \( w \in Y \) and a metric \( h_w \) of the bundle \( m_0K_{X_w} + L \) with positive curvature and such that
\[
e^{\varphi_w - \varphi_L} \in L^\frac{1+\varepsilon}{m_0}
\]
holds locally at each point of \( X_w \), where \( \varepsilon > 0 \) is a positive real.

Then the bundle \( m_0K_{X/Y} + L \) admits a positively curved metric whose restriction to \( X_w \) is less singular than the metric \( h_w \).

Let \( A = 2nH \) where \( n \) is the dimension of \( X \) and \( H \) is a very ample bundle on \( X \). In the theorem above, a point \( y \in Y \) is called generic if it is not a critical value of \( p \) and
if any section of the bundle $km_0K_{X/Y} + kL + A|_{X_w}$ extends over the nearby fibers, for all $k$.

We remark that the corollary A.2.5 is more coherent than A.2.4, in the sense that we start with a metric on a very general fiber $X_w$ of $p$, and we produce a metric defined on $X$. Even if we give ourselves a section on a very general fiber, the object we are able to produce (via A.2.4) will be in general a metric, which is not necessarily algebraic, as one would hope or guess.

We first write $e^{-\varphi_w}$ as limit of a sequence of algebraic metrics $e^{-\varphi_w^{(k)}}$ of $km_0K_{X/Y} + kL + A|_{X_w}$, and then we apply 0.1 to construct a sequence of global metrics on

$$km_0K_{X/Y} + kL + A,$$

whose restriction to $X_w$ is comparable with $e^{-\varphi_w^{(k)}}$. The metric we seek is obtained by a limit process. In order to complete this program we have to control several constants which are involved in the arguments, and this is possible thanks to the version of the Ohsawa-Takegoshi theorem 0.2. It would be very interesting to give a more direct, sections-less proof of A.2.5.

**Proof** (of A.2.5). The choice of the bundle $A \to X$ as above is explained by the following fact we borrow from the approximation theorem in [9].

Associated to the metric $\varphi_w$ on the bundle $m_0K_{X/Y} + L|_{X_w}$ given by hypothesis we consider the following space of sections

$$V_k \subset H^0(X_w, km_0K_{X_w} + kL + A)$$

defined by $u \in V_k$ if and only if

$$||u||^2_w := \int_{X_w} |u|^2 \exp(-k\varphi_w - \varphi_A)dV_\omega < \infty$$

We denote by $\varphi_w^{(k)}$ the metric on the bundle $km_0K_{X_w} + kL + A$ induced by an orthonormal basis $(u_j^{(k)})$ of $V_k$ endowed with the scalar product corresponding to (2). Then it is proved in [9] that there exists a constant $C_1$ independent of $k$ such that we have

$$k\varphi_w + \varphi_A \leq \varphi_w^{(k)} + C_1.$$

We remark that the ample bundle $A$ is independent of the particular point $w \in Y$ as well as on the metric $e^{-\varphi_w}$. This fact is very important for the following arguments. The Hölder inequality shows that the $km_0$th root of the sections $v \in V_k$ are integrable with respect to the metric $\frac{1}{m_0}\varphi_L$, as soon as $k$ is large enough. Indeed we have:

$$\int_{X_w} |v|^2 \frac{1}{km_0} e^{-k\varphi_L + \varphi_A} \leq c_k \left( \int_{X_w} |v|^2 e^{-k\varphi_w - \varphi_A} dV_\omega \right)^{\frac{1}{km_0}}$$
where we use the following notation
\[ \frac{c_k^{km_0}}{c_k^{km_0-1}} := \int_{X_w} e^{\frac{k}{km_0-1}(\varphi_w - \varphi_L)} dV_\omega < \infty. \]

We remark that the last integral is indeed convergent, by the integrability hypothesis in (5), provided that \( k \gg 0 \). In conclusion, we get
\[ (10) \int_{X_w} |v|^2 e^{-\frac{km_0}{k\varphi_L + k\varphi_A}} \leq C_2 \]
for every section \( v := u_j^{(k)} \in \mathcal{V}_k \). By definition, the expression of the constant in the relation (10) is the following
\[ C_2 := \sup_{k \gg 0} c_k < \infty \]
therefore it is independent of \( k \).

By theorem 0.1, for each \( k \) large enough we can construct the \( km_0 \)-Bergman metric \( h_{(km_0)}^{(km_0)} \) on the bundle \( km_0K_{X/Y} + kL + A \) with positive curvature current. Thanks to the relation (10), its restriction to \( X_w \) is less singular than the metric \( \varphi_w^{(k)} \), in the following precise way.
\[ (11) \varphi_{(km_0)}^{(km_0)}|_{X_w} \geq \varphi^{(k)} - km_0 \log C_2; \]
(we identify \( K_{X/Y} \) with \( K_{X_w} \), which is harmless in this context, since the point \( w \) is "far" from the critical loci of \( p \)) The inequality (11) combined with (8) shows furthermore that we have
\[ (12) \frac{1}{k} \varphi^{(km_0)}_{X/Y}|_{X_w} \geq \varphi_w + C_4 \]
where \( C_4 := -m_0 \log C_2 \).

On the other hand, by remark A.2.3 we have an a-priori upper bound
\[ (13) \frac{1}{k} \varphi^{(km_0)}_{X/Y} \leq C_5 \]
where \( C_5 \) is uniform with respect to \( k \).

The metric we seek is obtained by the usual upper envelope construction, namely
\[ (14) \varphi^{(\infty)}_{X/Y} := \limsup_{k \geq 1}^* \left( \frac{1}{k} \varphi^{(km_0)}_{X/Y} \right). \]
The local weights \( (\varphi^{(\infty)}_{X/Y}) \) glue together to give a metric for the bundle \( m_0K_{X/Y} + L \), since the auxiliary bundle \( A \) used in the approximation process is removed by the normalization factor \( 1/k \).

Moreover, the relation (12) shows that we have
\[ (15) \varphi^{(\infty)}_{X/Y}|_{X_w} \geq \varphi_w + O(1); \]
hence, the metric $\varphi_{X/Y}^{(\infty)}$ restricted to the fiber $X_w$ is less singular than the metric $e^{-\varphi_w}$ and the corollary is proved.

**A.2.6 Remark.** It is certainly worthwhile to understand the behavior of the metric constructed in Theorem 0.1 over the exceptional fibers, where it is just defined as the (unique) extension of $h_{X_0/Y_0}^{(m)}$. 

Let us here look at the fibers over a point $a$ where the condition 2) in 0.1 is not known to be satisfied, but such that $a$ is still a regular value of $p$. On the bundle $m_0K_{X_a} + L$ we can consider (at least) two natural extremal metrics: the one induced by all its global sections satisfying the $L^2/m$ integrability condition with respect to $h_L$ which we denote by $h_1$, and metric corresponding to the subspace of sections which extend locally near $a$, denoted by $h_2$. It is clear that we have $h_2 \geq h_1$, as one can see by the comparison between the corresponding $m$-Bergman kernels.

Our claim is that the metric $h_{X_0/Y_0}^{(m)}$ extends over $X_a$ to a metric which is less singular than $h_2$, at least if the singularities of the metric $\varphi_L$ are mild enough.

Indeed, as a consequence of the theorem 0.1 we have

$$\varphi_{X/Y}^{(m)}(x) = \limsup_{x' \to x} \varphi_{X_0/Y_0}^{(m)}(x')$$

where $x' \in X_0$ in the limit above. In order to establish the result we claim, let us consider a holomorphic section $u \in H^0(X_a, m_0K_{X_a} + L)$ which computes the local weight of $h_2$ at $x$, i.e. $\|u\|_{m,a} = 1$ and

$$\sup_{\|v\|_{m,a} = 1} |v'|(x)|^2 = |u'(x)|^2$$

in the notations/conventions at the beginning of the section A.2. If $x' \in X_0$ is close enough to $x$ and $y' := p(x')$, then we certainly have

$$\sup_{\|v\|_{m,y'} = 1} |v'(x')|^2 \geq \frac{|u'(x')|^2}{\|u\|_{m,y'}^2}$$

(here we use the fact that the section $u$ extends over the fibers near $X_a$). Hence in order to establish the claim above, all that we need is the relation

$$\liminf_{y' \to a} \|u\|_{m,y'} \leq \|u\|_{m,a}.$$

This last inequality clearly holds at least if $\varphi_L$ is continuous, and we actually believe that it always holds.

**§B. Adjunction type results**

The manner in which the technics developed in the first part of our article are used in order to prove 0.3 is explained in the subsection B.1 and B.2, together with comments about more general statements.
We use the notations in the introduction. A first step in the direction of our proof is to write the relation \((\dagger)\) as follows:

\[
p^* \left( g^* (K_X + \Delta_W) \right) + \Xi_{|S} \equiv K_{X'} + S + \Delta' + R_{|S}
\]

We subtract the inverse image of the canonical bundle of \(W'\) and we get

\[
p^* \left( g^* (K_X + \Delta_W) - K_{W'} \right) + \Xi_{|S} \equiv K_{S/W'} + \Delta' + R_{|S}
\]

by restriction to \(S\).

Let \(m_0\) be a positive integer, such that the multiples \(m_0(K_X + \Delta), m_0\Delta'\) and \(m_0R\) are integral. Given the formula (17) above and the statement we want to obtain, it is clear that our first task will be to analyze the positivity properties of the bundle

\[
m_0(K_{S/W'} + \Delta'_{|S});
\]

this will be done in the next paragraph.

\section*{§B.1 The Bergman metric and its singularities}

We state here the following direct consequence of the theorem 0.1.

\textbf{B.1.1 Corollary.} The bundle \(m_0(K_{S/W'} + \Delta'_{|S})\) is pseudoeffective. Moreover, it admits a positively curved metric \(h_{S/W'}\), whose restriction to the generic fiber \(S_{w'}\) of \(p\) is induced by the space of sections \(H^0(S_{w'}, m_0(K_{S/W'} + \Delta'_{|S_{w'}}))\) as explained in the paragraph A.2.

\textit{Proof.} Let \(L\) be a line bundle whose Chern class contains the Weil divisor \(m_0\Delta'_{|S}\), and such that the numerical relation (17) becomes linear when we replace \(\Delta'_{|S}\) with \(1/m_0L\). The bundle \(L\) admits a singular metric \(h_L\) whose curvature form is equal to \(m_0[\Delta'_{|S}]\). The multiplier ideal sheaf corresponding to the \(m_0^{th}\) root of this metric is equal to the structural sheaf, since the coefficients \(a^j\) belong to the interval \([0,1]\) by definition of \(\Delta'\). Therefore, given a generic point \(w' \in W'\), we infer that any section of the bundle \(m_0K_{S/W'} + L_{|S_{w'}}\) will automatically satisfy the condition 2 of 0.1.

Furthermore, we remark that we have

\[
H^0(S_{w'}, m_0K_{S/W'} + L_{|S_{w'}}) \neq 0
\]

since the support of the divisor \(R_{|S}\) does not intersects \(S_{w'}\), and since the bundle on the left hand side of (17) is effective when restricted to the generic fiber of \(p\). Thus we obtain a well-defined \(m_0\)-Bergman kernel metric \(h_{S/W'}\) on the bundle \(m_0K_{S/W'} + L\), which proves the corollary.

During the rest of this section, we use the symbol \(\Theta_{S/W'}\) to denote \(1/m_0\) times the curvature current associated to \(h_{S/W'}\). We remark that \(\Theta_{S/W'}\) is closed, positive, and it belongs to the cohomology class \(\{K_{S/W'} + \Delta'_{|S}\}\).
As a consequence of B.1.1, we note that by (17) and \( \dagger \dagger \) we have the next identity
\[
(19) \quad p^*(g^*(K_X + \Delta)|_{W'} - K_{W'}) + [\Xi_1|S] + [\Xi_2|S] \equiv \Theta_{S/W'} + [R|S].
\]

The existence of the current \( \Theta_{S/W'} \) is certainly excellent news in view of what is to be proved in 0.3; however, in order to conclude, we have to show that this current is at least as singular as the current \([\Xi_2|S]\). Once this is done, the objects we seek \((\Theta_{W'}, \Lambda)\) will be obtained by direct image.

The claim concerning the singularities of the current \( \Theta_{S/W'} \) is a consequence of the following statement.

**B.1.2 Theorem ([7]).** Let \( T \) be any closed positive \((1, 1)\)-current, for which there exists a line bundle \( F \) on \( W' \) with the property that \( T \equiv p^*(F) + m_0(K_{S/W'} + \Delta'|_S) \). Then we have
\[
T \geq [\Xi_2|S]
\]
in the sense of currents on \( S \). \( \square \)

Despite the fact that this result is absolutely central for what will follow, we have decided to include its proof in the sequel of this article [7], since the arguments needed for it are quite different from the topics presented here. \( \square \)

We now continue the proof of 0.3 by admitting B.1.2.

The relation (19) can be re-written as follows
\[
(20) \quad p^*(g^*(K_X + \Delta|_W) - K_{W'}) + [\Xi_1|S] \equiv \Theta_{S/W'} - [\Xi_2|S] + [R|S],
\]
where the right hand side term is a closed positive current. The positivity assertion is a consequence of the theorem B.1.2, where \( T := \Theta_{S/W'} \) and \( F \) is the trivial bundle. Next, we define
\[
(21) \quad \Lambda := \sum_j q_j F_j
\]
to be the smallest effective \( \mathbb{Q} \)-divisor such that
\[
(22) \quad p^*(\Lambda) \geq \Xi_1|S.
\]
Such an object indeed exists, by the definition of the decomposition \( \Xi = \Xi_1 + \Xi_2 \) in the introduction. We denote by \( D := p^*(\Lambda) - \Xi_1|S \); it is an effective \( \mathbb{Q} \)-divisor, and then the relation (20) reads as follows
\[
(23) \quad p^*(g^*(K_X + \Delta|_W) - K_{W'} + \Lambda) \equiv \Theta_{S/W'} - [\Xi_2|S] + [R|S] + [D].
\]

Again, we notice that the right hand side term of the preceding relation is a closed positive current. Since \( \Theta_{S/W'} - [\Xi_2|S] + [R|S] + [D] \) belongs to the \( p \)-inverse image of a \( \mathbb{Q} \)-bundle on \( W' \), we infer the existence of a closed positive current \( \Theta_{W'} \) on \( W' \), such that
\[
(24) \quad g^*(K_X + \Delta|_W) - K_{W'} + \Lambda \equiv \Theta_{W'},
\]
which is precisely the relation we are looking for. \( \square \)

In conclusion, the qualitative part of 0.3 is proved; we turn in the next subsection to the quantitative part of it, namely the integrability statement b).
§B.2 The critical exponent of $\Theta_{W'} - \Lambda$

Our main goal in this paragraph is to prove the point b) of 0.3. To this end, we will first provide a closer analysis of the structure of the curvature current $\Theta_{S/W'}$ when restricted to the generic fiber of $p$. Our first statement can be seen as a generalization of the result in [19], [23] corresponding to the case $m_0 = 1$.

B.2.1 Theorem. Let $w \in W'$ be a generic point and let $m \geq 1$ be a positive integer. Then the divisor $m m_0 \Xi_{|S_w}$ is contained in the zero set of any section $u$ of the bundle $m(m_0 K_{S_w} + L_{|S_w})$.

Moreover we have $h^0(S_w, m(m_0 K_{S_w} + L_{|S_w})) = 1$.

Remark. The bundle $L$ above is the one chosen in B.1.1. The idea of the following proof is quite simple: we first show that the sections above twist with inverse image of the section of some large enough ample line bundle on $W'$ extends to $S$. Then we apply the theorem B.1.2 above.

Proof (of B.2.1). As in the proof of the corollary B.1.1, theorem 0.1 provides us with a metric $h_{S/W'}^{(m)}$ on the bundle $m(m_0 K_{S/W'} + L)$ with positive curvature current. If $w \in W'$ is generic, then the restriction of the metric to the fiber $S_w$ has the same singularities as the algebraic metric given by the sections of the bundle $m(m_0 K_{S/W'} + L_{|S_w})$.

Let $A' \to W'$ be a positive enough line bundle; we want to apply the Ohsawa-Takegoshi theorem to the bundle $m(m_0 K_{S/W'} + p^* A')$ in order to extend $u \otimes p^*(\sigma_A)$ to $S$, where $\sigma_A$ is a generic section of $A'$.

To this end, we write it as an adjoint bundle as follows

$$m(m_0 K_{S/W'} + L) + p^* A' = K_S + \left(\frac{m m_0 - 1}{m_0}\right) (m_0 K_{S/W'} + L) + \frac{1}{m_0} L + p^* (A' - K_{W'})$$

The second bundle in the above decomposition is endowed with the appropriate multiple of the metric $h_{S/W'}^{(m_0)}$. If we denote by

$$F := \left(\frac{m m_0 - 1}{m_0}\right) (m_0 K_{S/W'} + L) + \frac{1}{m_0} L + p^* (A' - K_{W'})$$

then we have

$$u \otimes p^*(\sigma_A) \in H^0(S_w, K_S + F_{|S_w})$$

and let $h_F$ be the metric on $F$ induced by the $h_{S/W'}^{(m_0)}$ together with the natural metric on $\Delta'_{|S}$, and with the smooth, positively curved metric $h_A$ and on $p^* (A' - K_{W'})$. Then we claim that the curvature assumptions and the $L^2$ conditions required by the Ohsawa-Takegoshi extension theorem are satisfied.

To verify this claim, let $B \to W'$ be a very ample line bundle, such that there exists a family of sections $\rho_j \in H^0(W', B)$ with the property that $w = \cap_j (\rho_j = 0)$. The curvature conditions to be fulfilled are:
We see that both conditions will be satisfied if the curvature of $F$ is greater than say $2 \mu^* \Theta(B)$; this last condition can be easily satisfied if $A$ is positive enough.

Concerning the integrability of the section, we remark that we have

$$
\int_{S_w} |u|^2 e^{-(m-\frac{m_0}{m})\varphi_{S/W, s}^{(m_0)}-\varphi_{A'}-\varphi_A} \leq C \int_{S_{w'}} e^{-\varphi_{A'}} < \infty.
$$

The first inequality is satisfied because $|u|^{2/m}$ is pointwise bounded with respect to $h^{(m_0)}_{S/W}$ and the second one holds because the multiplier ideal sheaf of $\Delta_{s}'$ restricted to $S_w$ is trivial, provided that $w$ is generic (which is assumed to be the case).

Then the Ohsawa-Takegoshi theorem shows that $u$ admits an extension $U$ as section of the bundle $m(m_0 K_{S/W} + L) + p^* A'$. But if this is so, we just invoke the theorem B.1.2 and infer that the divisor of zeroes of the section $u$ is greater than $mm_0 \Xi_2|S_{w'}$.

In conclusion, we have just proved that the bundle $G := m(m_0 K_{S/W} + L|_{S_w})$ has the next property: it has a non-zero holomorphic section $v$ such that for any other section $u$ of $G$, the quotient $u/v$ is holomorphic. Since $u/v$ is a section of the trivial bundle, it is equal to a constant and thus theorem B.2.1 is completely proved.

If $D$ is an effective Weil divisor, we denote by $u_D$ the corresponding section of the bundle $\mathcal{O}(D)$. We consider a coordinate set $V \subset W'$; there exists a meromorphic section $u_V$ of the bundle

$$(25) \quad m_0 K_{S/W'} + L|_{p^{-1}(V)}$$

defined by the relation

$$(26) \quad u_V := \frac{u_{m_0 \Xi|S}}{u_{m_0 R|S}}. $$

Here we implicitly use the fact that the bundle

$$g^*(K_X + \Delta)|_{W'} - K_{W'}|_{V}$$

is trivial, in order to identify the quotient in (26) with a section of the bundle (25). We remark that the restriction of the meromorphic section above to the generic fiber $S_w$ is holomorphic, and it is the only section of the bundle $m_0 K_{S/W'} + L$ up to a multiple, by theorem B.2.1. The term “generic” in the sentence above means

$$w \in W_1 := W' \setminus (p|R_{S} \cup \Xi^{v})$$

In the above relation, we denote by $\Xi^{v}$ the vertical part (with respect to $p$) of the effective divisor $\Xi_{|S}$. By the discussion at the beginning of the paragraph A.2, the local
weight of the metric on $m_0K_{S/W'} + L$ is given in terms of the section $u_V$ normalized in a correct manner, as follows: for $w \in W_1$, let $\tau_w$ be the positive real number such that

$$\tau_w^2 \int_{S_w} \frac{|u_V|^2/m_0}{|\Lambda^r dp|^2} \exp(-\varphi_{\Delta'}) d\lambda_{S_w} = 1. \tag{27}$$

A-priori we need to fix metrics on $S$ and $W'$ in order to measure the norms above, but they are completely irrelevant, provided that we identify $u_V|_{S_w}$ with a section of $m_0K_{S_w} + L|_{S_w}$, therefore we skip this point, hoping that the confusion caused by it is not too big.

We denote by $\varphi_{S/W}^{(m_0)}$, the local weight of the $m_0$-Bergman kernel metric on the bundle $m_0K_{S/W} + L$ and then we have

$$\exp \left( \varphi_{S/W}^{(m_0)}(x) \right) = \tau_y^2 m_0 |f_V(x)|^2 \tag{28}$$

where $y = p(x)$ and $f_V$ is the local expression of the section $u_V$ near the point $x$ in the fiber $S_y$. It is understood that the equality (28) holds for $y \in V \setminus W_1$, and that the content of the theorem 0.1 is that the expression on the right hand side of (22) is uniformly bounded from above.

By definition of the current $\Theta_{W'}$, in (24), the exponential of the $m_0$ times its local potential satisfies the following relation

$$\exp \left( m_0 \varphi_{\Theta_{W'}} \circ p \right) = \exp \left( \varphi_{S/W'}^{(m_0)} \frac{|f_{m_0 R|_S}|^2}{|f_{m_0 \Xi_{2|S}|}^2 |f_{m_0 D}|^2} \right) \tag{29}$$

(cf. (23) and (24) a few lines above). On the other hand, by relations (26) and (28) we have the equality

$$\exp \left( \varphi_{S/W'}^{(m_0)} \frac{|f_{m_0 R|_S}|^2}{|f_{m_0 \Xi_{2|S}|}^2 |f_{m_0 D}|^2} \right) = \tau_p^{2m_0} |f_{m_0 \Lambda \circ p|^2} \tag{30}$$

We observe next that we have $|f_{m_0 \Xi_{1|S}}|^2 |f_{m_0 D}|^2 = |f_{m_0 \Lambda \circ p|^2$ by definition of the divisor $D$; hence we get

$$\exp \left( m_0 \varphi_{\Theta_{W'}} \circ p \right) = \tau_{p(x)}^{2m_0} |f_{m_0 \Lambda \circ p|^2 \tag{31}$$

and it is this expression which we will use in order to evaluate the critical exponent of $\Theta_{W'} - \Lambda$.

Let $\Omega \subset W'$ be a coordinate open set which does not intersect the $p$-direct image of the divisor $R|_S$; we will show next that the integral

$$\int_{w \in \Omega} \frac{d\lambda(w)}{\tau_w^{2+2\varepsilon_0}}$$

converges, provided that the positive real number $\varepsilon_0$ is small enough.
Subadjunction

To this end, we observe that we have

\[ \tau^2(w) \int_{S_w} \frac{|u_{m_0} \Xi|^{2/m_0}}{|\Lambda^r dp|^2} \exp(-\varphi_\Delta - \varphi_R) d\lambda_{S_w} = 1 \]

by the definition of the normalization factor \( \tau \) in (27), and the expression of the section \( u_V \) in (26).

Of course, it may happen that over some fiber \( S_w \) the metric induced by \( \Delta' \) is identically \( \infty \), or that the \( m_0 \) root of the section \( u_{m_0} \Xi \) does not belongs to the multiplier ideal of the restriction of the metric, but this kind of accidents can only happen for \( w \) in an analytic set. For such values of \( w \) we simply assign the value 0 to \( \tau(w) \).

We have

\[
\int_{\Omega} \frac{d\lambda(w)}{\tau^{2+2\varepsilon} w} = \int_{\Omega} d\lambda(w) \left( \int_{S_w} \frac{|u_{m_0} \Xi|^{2/m_0}}{|\Lambda^r dp|^2} \exp(-\varphi_\Delta - \varphi_R) d\lambda_{S_w} \right)^{1+\varepsilon_0} \leq C \int_{\Omega} d\lambda(w) \int_{S_w} \frac{|u_{m_0} \Xi|^{2+2\varepsilon_0}}{|\Lambda^r dp|^2} \exp \left( - (1 + \varepsilon_0) (\varphi_\Delta' + \varphi_R) \right) d\lambda_{S_w} \leq C \int_{p^{-1}(\Omega)} |u_{m_0} \Xi|^{2+2\varepsilon_0} \exp \left( - (1 + \varepsilon_0) (\varphi_\Delta' + \varphi_R) \right) d\lambda < \infty
\]

where the constant \( C \) above bounds the volume of the fibers \( S_w \), the first inequality is obtained by Hölder, and the last one is due to the fact that \( \Delta'|_S \) has trivial multiplier ideal sheaf. Therefore, the theorem 0.3 is proved.

We will show next that by adding an arbitrarily small ample part to \( K_X + \Delta \) we can “convert” the current \( \Theta_{W'} \) into a divisor, with the same integrability properties as in 0.3.

The main tool for this is the fact that a closed positive current of \((1,1)\)-type can be approximate in a very accurate way by a sequence of currents given by algebraic metrics. Let \( A \) be any ample line bundle on \( X \), and let \( F_j \) be the family of divisors considered in the formula (3) in the introduction. We consider the positive rational numbers \( a_j \) such that

\[ g^*(A) - \sum a_j F_j \]

is ample on \( W' \). For \( N \gg 0 \) and divisible enough, we denote by \( A'_N \) the bundle \( N(g^*(A) - \sum a_j F_j) \) on \( W' \). The regularization theorem in [9] shows that we have

\[ \varphi_{\Theta_{W'}} \leq \frac{1}{m} \log \left( \sum_{j=1}^{N_m} |h_j^{(m)}|^2 \right) + C \]

where the holomorphic functions \( h_j^{(m)} \) in the inequality above correspond to global sections of the bundle

\[ m(g^*(K_X + \Delta|_W) + \Lambda - K_{W'}) + A'_N. \]
We denote by $f_j$ the local equation of the hypersurface $F_j$, and then the inequality (33) implies

\[
\varphi \Theta_{W'} + \frac{N}{m} \sum_p a_p \log |f_p|^2 \leq \frac{1}{m} \log \left( \sum_{j=1}^{N_m} |l_j^{(m)}|^2 \right) + C
\]

where $l_j^{(m)} = h_j^{(m)} \prod P_p^{N_{ap}}$ are the local expression of sections of the holomorphic line bundle $m(g^*(K_X + \Delta + N/mA|_W) + \Lambda - K_{W'})$.

We denote by $\Theta_{W'}^{(m)}$ the current associated to the metric induced by the sections $(l_j^{(m)})$, and we observe that we have the numerical identity

\[
g^*(K_X + \Delta + N/mA|_W) + \Lambda \equiv K_{W'} + \Theta_{W'}^{(m)}.
\]

On the other hand, for $m \gg 0$ the relation

\[
ed^{\varphi \Lambda - \varphi \Theta_{W'}^{(m)}} \in L^{1+\epsilon_1}
\]

still holds on $W'$ minus the image of the support of $R_S$, thanks to the inequality (35). Modulo a blow-up of $W'$, we can assume that $\Theta_{W'}^{(m)}$ is the sum of an effective divisor and of a smooth, semi-positive $(1,1)$ form which morally should correspond to the decomposition “effective + nef ” in the article [23].

\[\square\]

**B.2.3 Remark.** The theorem 0.3 admits an immediate generalization as follows. Instead of the Weil divisor $\Delta$ we can work with a closed positive current with analytic singularities. The result obtained is absolutely the same, and the modifications one has to operate in the proof presented above are minimal, so we leave them to the interested reader.

\[\square\]

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