THE SIMPLICITY OF THE C*-ALGEBRAS ASSOCIATED TO ARBITRARY LABELED SPACES

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Abstract. In this paper, we consider the simplicity of the C*-algebra associated to an arbitrary weakly left-resolving labeled space (E, L, E), where E is the smallest non-degenerate accommodating set. We classify all gauge-invariant ideals of C*(E, L, E) and characterize minimality of (E, L, E) in terms of ideal structure of C*(E, L, E). Using these results, we prove that C*(E, L, E) is simple if and only if (E, L, E) is strongly cofinal and satisfies Condition (L), and for any A ∈ E \ {∅} and B ∈ E, there is C ∈ E_reg such that B \ C ∈ H(A), and if and only if (E, L, E) is minimal and satisfies Condition (L), and if and only if (E, L, E) is minimal and satisfies Condition (K).

1. INTRODUCTION

A class of C*-algebras C*(E) associated with directed graphs E was introduced in [9, 19] as a generalization of the Cuntz-Krieger algebras and there has been various generalizations of graph C*-algebras. The C*-algebras associated to ultragraphs [20], higher-rank graphs [16], subshifts, labeled spaces [3], Boolean dynamical systems [7] are those generalizations, and generalized Boolean dynamical systems was introduced in [6] to unify C*-algebras of labeled spaces and C*-algebras of Boolean dynamical systems. Among others, we focus on the C*-algebras associated to arbitrary weakly left-resolving normal labeled spaces. Throughout the paper, by a labeled space we always mean a weakly left-resolving normal labeled space.

The ideal structure of C*-algebras of set-finite, receiver set-finite labeled spaces (E, L, B) with E having no sinks is now well understood. It is known in [11, Theorem 5.2] that the gauge-invariant ideals of C*(E, L, B) are in one-to-one correspondence with the hereditary saturated subsets of B. We first generalize this results to C*-algebras of arbitrary labeled spaces. We show that there is a one-to-one correspondence between gauge-invariant ideals of C*(E, L, B) and pairs (H, S) where H is a hereditary saturated ideal of B and S is an ideal of \{A ∈ B : r(A, α) ∈ H for all but finitely many α\} such that H ∪ B_reg ⊆ S (Theorem 3.8). A quotient labeled space was also introduced in [11, Definition 3.2] to realize the quotient algebra C*(E, L, B)/I by a gauge-invariant ideal I as a C*-algebras of a quotient labeled space. But, a quotient labeled space is not a labeled space in general, but a Boolean dynamical system. So, in this paper we realize the quotient of C*(E, L, B).

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by $I_{(H, S)}$ as a $C^*$-algebra of a relative generalized Boolean dynamical system instead of newly defining $C^*$-algebras of relative quotient labeled spaces of arbitrary labeled spaces. Precisely, we show that the quotient of $C^*(E, \mathcal{L}, B)$ by the ideal $I_{(H, S)}$ is isomorphic to the $C^*$-algebra of relative generalized Boolean dynamical system $(B/\mathcal{H}, \mathcal{A}, \theta, [I_{r(\alpha)}]; [S])$ (Proposition 3.7). These will be easily done by viewing labeled graph $C^*$-algebras as $C^*$-algebras of generalized Boolean dynamical systems and applying results of [6].

The second goal of the paper is to investigate the question of when $C^*(E, \mathcal{L}, \mathcal{E})$ is simple, where $(E, \mathcal{L}, \mathcal{E})$ is an arbitrary weakly left-resolving labeled space with $\mathcal{E}$ is the smallest non-degenerate accommodating set. For an arbitrary graph $E$, we recall that $E$ is said to satisfy Condition $(L)$ if every loop has as exit, and is said to be cofinal if every vertex connects to every infinite path. Then the simplicity of $C^*(E)$ is characterized as follows.

**Theorem 1.1.** ([8, Corollary 2.15]) Let $E$ be a directed graph. Then the following are equivalent.

1. $C^*(E)$ is simple.
2. The following properties hold:
   a. $E$ is cofinal,
   b. $E$ satisfies Condition $(L)$, and
   c. for $w \in E^0$ and $v \in E^0_{\text{sing}}$, there is a path $\alpha \in E^*$ such that $s(\alpha) = w$ and $r(\alpha) = v$.
3. $E$ satisfies Condition $(L)$ and $E^0$ has no proper hereditary saturated subsets.
4. $E$ satisfies Condition $(K)$ and $E^0$ has no proper hereditary saturated subsets.

Many authors paid a great deal of attention to extend this result to the $C^*$-algebras associated to set-finite and receiver set-finite labeled spaces $(E, \mathcal{L}, \mathcal{E})$ with $E$ having no sinks or sources. In [4, Definition 6.1] Bates and Pask introduced a notion of cofinality appropriate for labeled spaces. In [10, Definition 3.1], a notion of strong cofinality of labeled spaces was given to modify minor mistake of results in [4]. Then again a modified version of strong cofinality of a labeled space was introduced in [15, Definition 2.10]. As an analogue of Condition $(L)$ of usual directed graph, the notion of a disagreeable labeled space was introduced in [4, Definition 5.1]. On the other hand, the notion of cycle was introduced in [7, Definition 9.5] to define condition $(L)$ for a labeled space (more generally for Boolean dynamical systems) which can be regarded as another condition analogous to Condition $(L)$ for usual directed graphs. It then is known in [14, Proposition 3.7] that if $(E, \mathcal{L}, B)$ is disagreeable, then $(E, \mathcal{L}, B)$ satisfies Condition $(L)$. But, the converse is not true, in general ([15, Proposition 3.2]). Based on these concepts, it is eventually known in [15, Theorem 3.17] that for a set-finite and receiver set-finite labeled space $(E, \mathcal{L}, \mathcal{E})$ with $E$ having no sinks or sources, $C^*(E, \mathcal{L}, \mathcal{E})$ is simple if and only if $(E, \mathcal{L}, \mathcal{E})$ is strongly cofinal in the sense of [15, Definition 2.10] and disagreeable. It is also prove in [15, Theorem 3.17] that for a labeled space whose Boolean dynamical system satisfies a sort of domain condition, $C^*(E, \mathcal{L}, \mathcal{E})$ is simple if and only if $(E, \mathcal{L}, \mathcal{E})$ satisfies Condition $(L)$ and there are no nonempty hereditary saturated subsets of $\mathcal{E}$.

We generalize these results to labeled graph $C^*$-algebras associated to arbitrary labeled spaces. We first give an example that shows why we need to change the
Theorem 1.2. (Theorem 4.17) Let \((E, \mathcal{L}, \mathcal{E})\) be a labeled space. Then the following are equivalent.

1. \(C^*(E, \mathcal{L}, \mathcal{E})\) is simple.
2. \((E, \mathcal{L}, \mathcal{E})\) is minimal and satisfies Condition (L).
3. \((E, \mathcal{L}, \mathcal{E})\) is minimal and satisfies Condition (K).
4. The following properties hold:
   a. \((E, \mathcal{L}, \mathcal{E})\) is strongly cofinal,
   b. \((E, \mathcal{L}, \mathcal{E})\) satisfies Condition (L), and
   c. for any \(A \in \mathcal{E}\) such that \(B \setminus C \in \mathcal{H}(A)\) if and only if \((E, \mathcal{L}, \mathcal{E})\) is minimal, in
   the sense that \(\{\emptyset\}\) and \(\mathcal{E}\) are the only hereditary saturated subsets of \(\mathcal{E}\).
5. The following properties hold:
   a. \((E, \mathcal{L}, \mathcal{E})\) is strongly cofinal,
   b. \((E, \mathcal{L}, \mathcal{E})\) is disagreeable, and
   c. for any \(A \in \mathcal{E}\) such that \(B \setminus C \in \mathcal{H}(A)\).

As a corollary, we show for a set-finite labeled space \((E, \mathcal{L}, \mathcal{E})\) with \(E\) having no
sinks, that \(C^*(E, \mathcal{L}, \mathcal{E})\) is simple if and only if \((E, \mathcal{L}, \mathcal{E})\) is strongly cofinal and is
disagreeable, if and only if \((E, \mathcal{L}, \mathcal{E})\) is strongly cofinal and satisfies Condition (L),
if and only if \((E, \mathcal{L}, \mathcal{E})\) is minimal and satisfies Condition (L), and if and only if
\((E, \mathcal{L}, \mathcal{E})\) is minimal and satisfies Condition (K). This generalizes [15, Theorem 3.7].

This paper is organized as follows. In Section 2 we review basic definitions and
terminologies needed for the rest of the paper. In Section 3 we classify the gauge-
invariant ideals in the \(C^*\)-algebras of arbitrary labeled spaces and describe the
quotients as \(C^*\)-algebras of relative generalized Boolean dynamical systems. In
Section 4 we examine strong cofinality, minimality and disagreeability for an arbit-
rary labeled space, and prove simplicity results for \(C^*\)-algebras of arbitrary labeled
spaces.

2. Preliminary

2.1. Directed graphs. A directed graph \(E = (E^0, E^1, r, s)\) consists of two count-
able sets of vertices \(E^0\) and edges \(E^1\), and the range, source maps \(r, s : E^1 \to E^0\).
A path of length \(n\) is a sequence \(\lambda = \lambda_1 \cdots \lambda_n\) of edges such that \(r(\lambda_i) = s(\lambda_{i+1})\)
for \(1 \leq i \leq n - 1\). We write \(|\lambda| = n\) for the length of \(\lambda\) and the vertices in \(E^0\) are
regarded as finite paths of length zero. By \(E^n\) we mean the set of all paths of length
\(n\). The maps \(r, s\) naturally extend to the set \(E^{\geq 0} := \bigcup_{n \geq 0} E^n\) of all finite paths,
where \(r(v) = s(v) = v\) for \(v \in E^0\). We denote by \(E^\infty\) the set of all infinite paths
\(x = \lambda_1 \lambda_2 \cdots, \lambda_i \in E^1\) with \(r(\lambda_i) = s(\lambda_{i+1})\) for \(i \geq 1\), and define \(s(x) := s(\lambda_1)\). We al-
so use notation like \(E^{\leq n}\) and \(E^{\geq n}\) which should have their obvious meaning.
A vertex \( v \in E^0 \) is called a source if \( |r^{-1}(v)| = 0 \) and \( v \) is called a sink if \( |s^{-1}(v)| = 0 \); and \( v \) is called an infinite-emitter if \( |s^{-1}(v)| = \infty \). We define \( E^0_{\text{sink}} \) to be the set of all sinks in \( E^0 \). We let \( E^0_{\text{reg}} = \{ v \in E^0 : 0 < |s^{-1}(v)| < \infty \} \) and let \( E^0_{\text{sing}} = E^0 \setminus E^0_{\text{reg}} \).

A finite path \( \lambda = \lambda_1 \cdots \lambda_{|\lambda|} \in E^{\geq 1} \) with \( r(\lambda) = s(\lambda) \) is called a loop, and an exit of a loop \( \lambda \) is a path \( \delta \in E^{\geq 1} \) such that \( |\delta| \leq |\lambda| \), \( s(\delta) = s(\lambda) \), and \( \delta \neq \lambda_1 \cdots \lambda_{|\delta|} \).

A graph \( E \) is said to satisfy Condition (L) if every loop has an exit and \( E \) is said to satisfy Condition (K) if every vertex \( v \in E^0 \) lies on no loops, or if there are two loops \( \alpha \) and \( \beta \) such that \( s(\alpha) = s(\beta) = v \) and neither \( \alpha \) nor \( \beta \) is an initial path of the other.

2.2. Labeled spaces. A labeled graph \((E, \mathcal{L})\) over a countable alphabet \( \mathcal{A} \) consists of a directed graph \( E \) and a labeling map \( \mathcal{L} : E^1 \to \mathcal{A} \). We assume that the map \( \mathcal{L} \) is onto. By \( \mathcal{A}^* \) and \( \mathcal{A}^\infty \), we denote respectively the sets of all finite words and infinite words in symbols of \( \mathcal{A} \). To each finite path \( \lambda = \lambda_1 \cdots \lambda_n \in E^n \), there corresponds a finite labeled path \( \mathcal{L}(\lambda) := \mathcal{L}(\lambda_1) \cdots \mathcal{L}(\lambda_n) \in \mathcal{L}(E^n) \subset \mathcal{A}^* \), and similarly an infinite labeled path \( \mathcal{L}(x) := \mathcal{L}(\lambda_1) \mathcal{L}(\lambda_2) \cdots \in \mathcal{L}(E^\infty) \subset \mathcal{A}^\infty \) to each infinite path \( x = \lambda_1 \lambda_2 \cdots \in E^\infty \). We use notation \( \mathcal{L}^*(E) := \mathcal{L}(E^{\geq 1}) \), where \( E^{\geq 1} = \bigcup_{n \geq 1} E^n \). We denote the subpath \( \alpha_i \cdots \alpha_j \) of \( \alpha = \alpha_1 \alpha_2 \cdots \alpha_{|\alpha|} \in \mathcal{L}(E^{\geq 1}) \) by \( \alpha_{[i,j]} \) for \( 1 \leq i \leq j \leq |\alpha| \). A subpath of the form \( \alpha_{[1,j]} \) is called an initial path of \( \alpha \). The range \( r(\alpha) \) of a labeled path \( \alpha \in \mathcal{L}^*(E) \) is a subset of \( E^0 \) defined by

\[
r(\alpha) = \{ r(\lambda) : \lambda \in E^{\geq 1}, \mathcal{L}(\lambda) = \alpha \}.
\]

The relative range of \( \alpha \in \mathcal{L}^*(E) \) with respect to \( A \subset E^0 \) is defined to be

\[
r(A, \alpha) = \{ r(\lambda) : \lambda \in E^{\geq 1}, \mathcal{L}(\lambda) = \alpha, s(\lambda) \in A \}.
\]

Let \( \mathcal{B} \subseteq 2^{E^0} \) be a collection of subsets of \( E^0 \). We say \( \mathcal{B} \) is closed under relative ranges for \((E, \mathcal{L})\) if \( r(A, \alpha) \in \mathcal{B} \) for all \( A \in \mathcal{B} \) and \( \alpha \in \mathcal{L}^*(E) \). We call \( \mathcal{B} \) an accommodating set for \((E, \mathcal{L})\) if it satisfies

(i) \( r(\alpha) \in \mathcal{B} \) for all \( \alpha \in \mathcal{L}^*(E) \),

(ii) it is closed under relative ranges,

(iii) it is closed under finite intersections and unions.

If, in addition, \( \mathcal{B} \) is closed under relative complements, then \( \mathcal{B} \) is said to be non-degenerate. The triple \((E, \mathcal{L}, \mathcal{B})\) is called a labeled space when \( \mathcal{B} \) is accommodating for \((E, \mathcal{L})\). Moreover, if \( \mathcal{B} \) is non-degenerate, then \((E, \mathcal{L}, \mathcal{B})\) is called normal as in [1]. By \( \mathcal{E} \), we denote the smallest non-degenerate accommodating set for a labeled graph \((E, \mathcal{L})\).

A labeled space \((E, \mathcal{L}, \mathcal{B})\) is said to be weakly left-resolving if it satisfies

\[
r(A, \alpha) \cap r(B, \alpha) = r(A \cap B, \alpha)
\]

for all \( A, B \in \mathcal{B} \) and \( \alpha \in \mathcal{L}^*(E) \). A labeled graph \((E, \mathcal{L})\) is left-resolving if \( \mathcal{L} : r^{-1}(v) \to \mathcal{A} \) is injective for each \( v \in E^0 \). Left-resolving labeled spaces are weakly left-resolving.

Assumptions. In this paper, \((E, \mathcal{L}, \mathcal{B})\) is always weakly left-resolving normal and \( \mathcal{L} : E^1 \to \mathcal{A} \) is onto.
For $A, B \subset E^0$ and $n \geq 1$, let

$$AE^n = \{ \lambda \in E^n : s(\lambda) \in A \}, \quad E^nB = \{ \lambda \in E^n : r(\lambda) \in B \}.$$  

A labeled space $(E, L, B)$ is said to be set-finite (receiver set-finite, respectively) if for every $A \in B$ and $n \geq 1$ the set $L(AE^n)$ ($L(E^nA)$, respectively) is finite. We also say that $A \in B$ is regular if $0 < |L(BE^1)| < \infty$ for any $\emptyset \neq B \in B$ with $B \subseteq A$. If $A \in B$ is not regular, then it is called a singular set. We write $B_{\text{reg}}$ for the set of all regular sets.

Note that if $E$ has no sinks, then $(E, L, B)$ is set-finite if and only if $B = B_{\text{reg}}$. A set $A \in B$ is called minimal (in $B$) if $A \cap B$ is either $A$ or $\emptyset$ for all $B \in B$.

Let $(E, L)$ be a labeled graph and let $\Omega_0(E)$ be the set of all vertices that are not sources, and let $l \geq 1$. The relation $\sim_l$ on $\Omega_0(E)$, given by $v \sim_l w$ if and only of $L(E^{\leq l}v) = L(E^{\leq l}w)$, is an equivalence relation, and the equivalence class $[v]_l$ of $v \in \Omega_0(E)$ is called a generalized vertex. If $k > l$, then $[v]_k \subseteq [v]_l$ is obvious and $[v]_l = \bigcup_{i=1}^{m} [v_i]_{l+1}$ for some vertices $v_1, \ldots, v_m \in [v]_l$ ([4, Proposition 2.4]). The generalized vertices of labeled graphs play the role of vertices in usual graphs.

### 2.3. $C^*$-algebras of labeled spaces

We review the definition of the $C^*$-algebras associated to labeled spaces from [1].

**Definition 2.1.** ([1, Definition 2.1]) Let $(E, L, B)$ be a labeled space. A representation of $(E, L, B)$ is a family of projections $\{ p_A : A \in B \}$ and partial isometries $\{ s_\alpha : \alpha \in A \}$ such that for $A, B \in B$ and $\alpha, \beta \in A$,

1. $p_\emptyset = 0$, $p_{A \cap B} = p_{AB}$, and $p_{A \cup B} = p_A + p_B - p_{A \cap B}$,
2. $p_A s_\alpha = s_{p\alpha}(A, \alpha)$,
3. $s_\alpha^* s_\alpha = p_{r(\alpha)}$ and $s_\alpha^* s_\beta = 0$ unless $\alpha = \beta$,
4. $p_A = \sum_{\alpha \in L(AE^1)} s_{p\alpha}(A, \alpha) s_\alpha^*$ for $A \in B_{\text{reg}}$.

**Remark 2.2.** Let $(E, L, B)$ be a labeled space.

1. For any $A \in B$, the condition $|L(AE^1)| < \infty$ and $A \cap B = \emptyset$ for all $B \in B$ satisfying $B \subseteq E_0^{\text{sink}}$ is equivalent to $A \in B_{\text{reg}}$. Thus, the condition (iv) in Definition 2.1 is equivalent to (iv) in [1, Definition 2.1].
2. If $E$ has no sinks, the condition (iv) in Definition 2.1 is equivalent to

$$p_A = \sum_{\alpha \in L(AE^1)} s_{p\alpha}(A, \alpha) s_\alpha^* \text{ for } A \in B \text{ with } |L(AE^1)| < \infty.$$ 

Given a labeled space $(E, L, B)$, it is known in [1, Theorem 3.8] that there exists a $C^*$-algebra $C^*(E, L, B)$ generated by a universal representation $\{ s_\alpha, p_A \}$ of $(E, L, B)$. We call $C^*(E, L, B)$ the labeled graph $C^*$-algebra of a labeled space $(E, L, B)$ and simply write $C^*(E, L, B) = C^*(s_\alpha, p_A)$ to indicate the generators.

Note that $s_\alpha \neq 0$ and $p_A \neq 0$ for $\alpha \in A$ and $A \in B$, $A \neq \emptyset$.

**Remark 2.3.** Let $(E, L, B)$ be a labeled space. For notational convenience, we use a symbol $\epsilon$ such that $r(\epsilon) = E_0^0$, $r(A, \epsilon) = A$ for all $A \subset E_0^0$, and $\alpha = \epsilon \alpha = \alpha \epsilon$ for all
α ∈ ℒ(ℰ^≥1), and write ℒ^#(ℰ) := ℒ(ℰ^≥1) ∪ {ε}. For ε ∈ ℒ^#(ℰ), let se denote the unit of the multiplier algebra of ℂ^*(ℰ, ℒ, ℬ).

1. We have the following equality

\[(s_α p_A s_β^*) (s_γ p_B s_δ^*) = \begin{cases} 
  s_α γ^* P_r(A, γ') ∩ B s_δ^*, & \text{if } γ = γ' \\
  s_α P_A r(B, β') s_β^*, & \text{if } β = β' \\
  s_α P_A r(B, β') s_δ^*, & \text{if } β = γ \\
  0, & \text{otherwise,}
\end{cases}\]

for α, β, γ, δ ∈ ℒ^#(ℰ) and A, B ∈ ℬ (see [3, Lemma 4.4]). Since s_α p_A s_β^* ≠ 0 if and only if A ∩ r(α) ∩ r(β) ≠ ∅, it follows that

\[C^*(ℰ, ℒ, ℬ) = \{s_α p_A s_β^* : \alpha, \beta ∈ ℒ^#(ℰ) \text{ and } A ⊆ r(α) ∩ r(β)\}.\]  

(1) By universal property of C^*(ℰ, ℒ, ℬ) = C^*(s_α, p_A), there is a strongly continuous action γ : T → Aut(C^*(ℰ, ℒ, ℬ)), called the gauge action, such that

\[γ_z(s_α) = z s_α \quad \text{and} \quad γ_z(p_A) = p_A\]

for α ∈ ℛ and A ∈ ℬ.

The notion of cycle was introduced ([7, Definition 9.5]) to define condition (L) for a labeled space (ℰ, ℒ, ℬ) (more generally for Boolean dynamical systems) which is an analogue to Condition (L) for usual directed graphs.

**Definition 2.4.** ([7, Definition 9.5]) Let (ℰ, ℒ, ℬ) be a labeled space.

1. A pair (α, A) with α ∈ ℒ^*(ℰ) and ∅ ≠ A ∈ ℬ is a cycle if B = r(β, α) for every B ∈ ℬ with B ⊆ A.

2. A cycle (α, A) has an exit if for there is a t ≤ |α| and a B ∈ ℬ such that ∅ ≠ B ⊆ r(A, α_1,t) and ℒ(BE^1) ≠ {α_{t+1}} (where α_{|α|+1} := α_1).

3. A cycle (α, A) has no exits if for all t ≤ |α| and all ∅ ≠ B ⊆ r(A, α_1,t), we have B ∈ ℬ_{reg} with ℒ(BE^1) = {α_{t+1}} (where α_{|α|+1} := α_1).

4. A labeled space (ℰ, ℒ, ℬ) satisfies Condition (L) if every cycle has an exit.

**Theorem 2.5.** (The Cuntz-Krieger Uniqueness theorem [3, Theorem 5.5], [7, Theorem 9.9]) Let {t_a, q_A} be a representation of a labeled space (ℰ, ℒ, ℬ) such that q_A ≠ 0 for all nonempty A ∈ ℬ. If (ℰ, ℒ, ℬ) satisfies Condition (L), then the canonical homomorphism φ : C^*(ℰ, ℒ, ℬ) → C^*(s_α, p_A) → C^*(t_a, q_A) such that φ(s_α) = t_a and φ(p_A) = q_A is an isomorphism.

2.4. Generalized Boolean dynamical systems and their C^*-algebras. For details of the following, we refer the reader to [5, 6, 7].

Let ℬ be a Boolean algebra. A non-empty subset ℐ of ℬ is called an ideal [7, Definition 2.4] if

1. if A, B ∈ ℐ, then A U B ∈ ℐ,
2. if A ∈ ℐ and B ∈ ℬ, then A ∩ B ∈ ℐ.

An ideal ℐ of a Boolean algebra ℬ is a Boolean subalgebra. For A ∈ ℬ, the ideal generated by A is defined by ℐ_A := {B ∈ ℬ : B ⊆ A}.

A Boolean dynamical system is a triple (ℬ, ℒ, θ) where ℬ is a Boolean algebra, ℒ is a set, and {θ_α}_{α ∈ ℒ} is a set of actions on ℬ such that for α = α_1 ··· α_n ∈ ℒ \ {∅}, the action \(θ_α : ℬ \to ℬ\) is defined as \(θ_α := θ_{α_n} ◦ ··· ◦ θ_{α_1}\). We also define \(θ_∅ := Id\).
For $B \in \mathcal{B}$, we define
\[ \Delta_B^{(\mathcal{B},\mathcal{C},\theta)} := \{ \alpha \in \mathcal{L} : \theta_\alpha(B) \neq \emptyset \} \text{ and } \lambda_B^{(\mathcal{B},\mathcal{C},\theta)} := |\Delta_B^{(\mathcal{B},\mathcal{C},\theta)}|.
\]

We will often just write $\Delta_B$ and $\lambda_B$ instead of $\Delta_B^{(\mathcal{B},\mathcal{C},\theta)}$ and $\lambda_B^{(\mathcal{B},\mathcal{C},\theta)}$. We say that $A \in B$ is regular ([7, Definition 3.5]) if for any $\emptyset \neq B \in \mathcal{I}_A$, we have $0 < \lambda_B < \infty$. If $A \in B$ is not regular, then it is called a singular set. We write $B^{(\mathcal{B},\mathcal{C},\theta)}_\text{reg}$ or just $B^{\text{reg}}$ for the set of all regular sets. Notice that $\emptyset \in B^{\text{reg}}$.

Let $\mathcal{R}_\alpha := \mathcal{R}_\alpha^{(\mathcal{B},\mathcal{C},\theta)} = \{ A \in \mathcal{B} : A \subseteq \theta_\alpha(B) \text{ for some } B \in \mathcal{B} \}$ for each $\alpha \in \mathcal{L}$. A generalized Boolean dynamical system is a quadruple $(B, \mathcal{L}, \theta, \mathcal{I}_\alpha)$ where $(B, \mathcal{L}, \theta)$ is a Boolean dynamical system and $\{ \mathcal{I}_\alpha : \alpha \in \mathcal{L} \}$ is a family of ideals in $B$ such that $\mathcal{R}_\alpha \subseteq \mathcal{I}_\alpha$ for each $\alpha \in \mathcal{L}$. A relative generalized Boolean dynamical system is a pentamous $(B, \mathcal{L}, \theta, \mathcal{I}_\alpha; \mathcal{J})$ where $(B, \mathcal{L}, \theta, \mathcal{I}_\alpha)$ is a generalized Boolean dynamical system and $\mathcal{J}$ is an ideal of $B^{\text{reg}}$ ([6, Definition 3.2]).

**Definition 2.6.** ([6, Definition 3.2]) Let $(B, \mathcal{L}, \theta, \mathcal{I}_\alpha; \mathcal{J})$ be a relative generalized Boolean dynamical system. A $(B, \mathcal{L}, \theta, \mathcal{I}_\alpha; \mathcal{J})$-representation is a family of projections $\{ P_A : A \in \mathcal{B} \}$ and a family of partial isometries $\{ S_{\alpha,B} : \alpha \in \mathcal{L}, \; B \in \mathcal{I}_\alpha \}$ such that for $A, A' \in \mathcal{B}$, $\alpha, \alpha' \in \mathcal{L}$, $B \in \mathcal{I}_\alpha$ and $B' \in \mathcal{I}_{\alpha'}$,

(i) $P_\emptyset = 0$, $P_{A\cap A'} = P_AP_{A'}$, and $P_{A\cup A'} = P_A + P_{A'} - P_{A\cap A'}$;
(ii) $P_A S_{\alpha,B} = S_{\alpha,B} P_{\theta_\alpha(A)}$;
(iii) $S_{\alpha,B}^* S_{\alpha',B'} = \delta_{\alpha,\alpha'} P_B|B'$;
(iv) $P_A = \sum_{\alpha \in \Delta_A} S_{\alpha,\theta_\alpha(A)} S_{\alpha,\theta_\alpha(A)}^*$ for all $A \in \mathcal{J}$.

Given a $(B, \mathcal{L}, \theta, \mathcal{I}_\alpha; \mathcal{J})$-representation $\{ P_A, S_{\alpha,B} \}$ in a $C^*$-algebra $A$, we denote by $C^*(P_A, S_{\alpha,B})$ the $C^*$-subalgebra of $A$ generated by $\{ P_A, S_{\alpha,B} \}$. It is shown in [6] that there exists a universal $(B, \mathcal{L}, \theta, \mathcal{I}_\alpha; \mathcal{J})$-representation $\{ P_A, S_{\alpha,B} : A \in \mathcal{B}, \alpha \in \mathcal{L} \}$ and $B \in \mathcal{I}_\alpha \}$ in a $C^*$-algebra. We write $C^*(B, \mathcal{L}, \theta, \mathcal{I}_\alpha; \mathcal{J})$ for $C^*(P_A, S_{\alpha,B})$ and call it the $C^*$-algebra of $(B, \mathcal{L}, \theta, \mathcal{I}_\alpha; \mathcal{J})$.

By a Cuntz–Krieger representation of $(B, \mathcal{L}, \theta, \mathcal{I}_\alpha; \mathcal{J})$ we mean a $(B, \mathcal{L}, \theta, \mathcal{I}_\alpha; \mathcal{B}^{\text{reg}})$-representation. We write $C^*(B, \mathcal{L}, \theta, \mathcal{I}_\alpha; \mathcal{J})$ for $C^*(B, \mathcal{L}, \theta, \mathcal{I}_\alpha; \mathcal{B}^{\text{reg}})$ and call it the $C^*$-algebra of $(B, \mathcal{L}, \theta, \mathcal{I}_\alpha)$. When $(B, \mathcal{L}, \theta)$ is a Boolean dynamical system, then we write $C^*(B, \mathcal{L}, \theta)$ for $C^*(B, \mathcal{L}, \theta, \mathcal{I}_\alpha)$ and call it the $C^*$-algebra of $(B, \mathcal{L}, \theta)$.

2.4.1. **Viewing labeled graph $C^*$-algebras as $C^*$-algebras of generalized Boolean dynamical systems.** We view labeled graph $C^*$-algebras as $C^*$-algebras of generalized Boolean dynamical systems. Let $(E, \mathcal{L}, B)$ be a labeled space where $\mathcal{L} : E^1 \to A$ is onto and put $C^*(E, \mathcal{L}, B) = C^*(p_A, s_{\alpha})$. Then $B$ is a Boolean algebra and for each $\alpha \in \mathcal{A}$, the map $\theta_\alpha : B \to B$ defined by $\theta_\alpha(A) := r(A, \alpha)$ is an action on $B$ ([7, Example 11.1]). Put $\mathcal{I}_\alpha = \{ A \in B : A \subseteq r(B, \alpha) \text{ for some } B \in \mathcal{B} \}$ and let

\[ \mathcal{I}_{r(\alpha)} = \{ A \in B : A \subseteq r(\alpha) \}. \]

It is clear that $\mathcal{R}_\alpha \subseteq \mathcal{I}_{r(\alpha)}$ for each $\alpha \in \mathcal{L}$. Then $(B, A, \theta, \mathcal{I}_{r(\alpha)})$ is a generalized Boolean dynamical system. We call it a generalized Boolean dynamical system associated to $(E, \mathcal{L}, B)$. It then is straightforward to check that

\[ \{ p_A, s_{\alpha}p_B : A \in B, \alpha \in A \text{ and } B \in \mathcal{I}_{\tau(\alpha)} \} \]
is a Cuntz–Krieger representation of \((B, A, \theta, I_r(\alpha))\). Then the universal property of \(C^*(B, A, \theta, I_r(\alpha))\) gives a \(*\)-homomorphism
\[
\phi : C^*(B, A, \theta, I_r(\alpha)) \to C^*(E, \mathcal{L}, B)
\]
defined by
\[
\phi(p_A) = p_A \text{ and } \phi(s_{\alpha, B}) = s_{\alpha}p_B
\]
for all \(A \in B, \alpha \in A\) and \(B \in I_r(\alpha)\). Since \(s_{\alpha} = s_{\alpha}p_{r(\alpha)}\), the family \(\{p_A, s_{\alpha}p_B : A \in B, \alpha \in A\) and \(B \in I_r(\alpha)\}\) generates \(C^*(E, \mathcal{L}, B)\), and hence, the map \(\phi\) is onto. Applying the gauge-invariant uniqueness theorem [6, Corollary 6.2], we conclude that \(\phi\) is an isomorphism.

We summarize this facts in the following.

**Proposition 2.7.** ([6, Example 4.2]) Let \((E, \mathcal{L}, B)\) be a labeled space, where \(\mathcal{L} : E^1 \to A\) is onto. Then \((B, A, \theta, I_r(\alpha))\) is a generalized Boolean dynamical system and \(C^*(E, \mathcal{L}, B)\) is isomorphic to \(C^*(B, A, \theta, I_r(\alpha))\).

### 3. Gauge-invariant ideals of \(C^*(E, \mathcal{L}, B)\)

In this section, we give a complete list of the gauge-invariant ideals of \(C^*\)-algebras of arbitrary labeled spaces \((E, \mathcal{L}, B)\) and describe the quotients as \(C^*\)-algebras of relative generalized Boolean dynamical systems. Most of results can be obtained by the same arguments used in [6]. So, we omit its proof.

We recall from [11, Definition 3.4] that a subset \(\mathcal{H}\) of \(B\) is said to be *hereditary* if

1. if \(A \in \mathcal{H}\), then \(r(A, \alpha) \in \mathcal{H}\) for all \(\alpha \in \mathcal{L}^*(E)\),
2. if \(A, B \in \mathcal{H}\), then \(\mathcal{H} \cap \mathcal{H}\) is hereditary,
3. if \(A \in \mathcal{H}\) and \(B \in \mathcal{B}\) with \(B \subseteq A\), then \(B \in \mathcal{H}\).

A hereditary set \(\mathcal{H}\) is said to be *saturated* if \(A \in \mathcal{H}\) whenever \(A \in \mathcal{B}_{\text{reg}}\) satisfies \(r(A, \alpha) \in \mathcal{H}\) for all \(\alpha \in \mathcal{L}^*(E)\) ([6, Section 2.3]).

The following lemma shows how to find a hereditary saturated subset.

**Lemma 3.1.** For a nonempty set \(A \in \mathcal{B}\), let
\[
\mathcal{H}(A) := \{ B \in \mathcal{B} : B \subseteq \bigcup_{i=1}^n r(A, \alpha_i) \text{ for some } \alpha_i \in \mathcal{L}^#(E) \},
\]
for some \(\alpha_i \in \mathcal{L}^#(E)\).
\[
(2)
\]
be the smallest hereditary set that contains \(A\), and
\[
\mathcal{S}(\mathcal{H}(A)) := \{ B \in \mathcal{B} : \text{there is an } n \geq 0 \text{ such that } r(B, \beta) \in \mathcal{H}(A) \text{ for all } \beta \in \mathcal{L}^n(E), \text{ and } r(B, \gamma) \in \mathcal{H}(A) \oplus \mathcal{B}_{\text{reg}} \text{ for all } \gamma \in \mathcal{L}^#(E) \text{ with } |\gamma| < n \}
\]
is the smallest saturated hereditary set that contains \(A\), where
\[
\mathcal{H}(A) \oplus \mathcal{B}_{\text{reg}} := \{ C \cup D : C \in \mathcal{H}(A) \text{ and } D \in \mathcal{B}_{\text{reg}} \}.
\]

**Proof.** It is straightforward to check that \(\mathcal{H}(A)\) is hereditary, and it is easy to see that if \(\mathcal{H}\) is a hereditary set and \(A \in \mathcal{H}\), then \(\mathcal{H}(A) \subseteq \mathcal{H}\).

It is rather obvious that \(\mathcal{S}(\mathcal{H}(A))\) is hereditary. To show that it is saturated, let \(B \in \mathcal{B}_{\text{reg}}\) satisfy \(r(B, \alpha) \in \mathcal{S}(\mathcal{H}(A))\) for all \(\alpha \in \mathcal{L}^*(E)\). Put \(\mathcal{L}(BE^1) = \{\alpha_1, \ldots, \alpha_n\}\) (this is a finite set since \(B \in \mathcal{B}_{\text{reg}}\)). Then \(r(B, \alpha_i) \in \mathcal{S}(\mathcal{H}(A))\) for each \(i\), and thus there is an \(n_i \geq 1\) such that \(r(r(B, \alpha_i), \beta) \in \mathcal{S}(\mathcal{H}(A))\) for all \(\beta \in \mathcal{L}^{n_i}(E)\) and \(r(r(B, \alpha_i), \gamma) \in \mathcal{H}(A) \oplus \mathcal{B}_{\text{reg}}\) for all \(\gamma \in \mathcal{L}^#(E)\) with \(|\gamma| < n_i\). Take \(n := \)
Example 3.2. For the labeled graph
\[
\begin{array}{cccccccc}
\vdots & v_4 & 3 & \cdots & 2 & \cdots & 1 & \vdots \\
& v_3 & & v_2 & & v_1 & & a \\
\end{array}
\]
where \( v_1 \) emits infinitely many labeled edges \((b_n)_{n \geq 1}\), consider the labeled space \((E, L, E)\). It is clear that \( H(r(a)) = \emptyset, \{w_1\}, \{w_2\}\). One sees that \( r(\{v_1\}, \beta) \in H(r(a)) \) for \( |\beta| \geq 1 \), but \( \{v_1\} \notin H(r(a)) \oplus \mathcal{E}_{\text{reg}} \). So, \( \{v_1\} \notin S(H(r(a))) \). It is also easy to see that \( \{v_i\} \notin S(H(r(a))) \) for each \( i > 1 \). In fact, \( S(H(r(a))) = H(r(a)) \).

3.1. A quotient Boolean dynamical system \((B/H, A, \theta)\) associated to \((E, L, B)\).

Let \((E, L, B)\) be a labeled space, where \( L : E^1 \to A \) is assumed to be onto. If \( H \) is a hereditary subset of \( B \), then the relation
\[
A \sim B \iff A \cup W = B \cup W \text{ for some } W \in H
\]
defines an equivalent relation on \( B \) ([11, Proposition 3.6]). We denote the equivalent class of \( A \in B \) with respect to \( \sim \) by \([A] \) (or \([A]_H \) if we need to specify the hereditary set \( H \)) and the set of all equivalent classes of \( B \) by \( B/H \). It is easy to check that \( B/H \) is a Boolean algebra with operations defined by
\[
[A] \cap [B] = [A \cap B], [A] \cup [B] = [A \cup B] \text{ and } [A] \setminus [B] = [A \setminus B].
\]
The partial order \( \subseteq \) on \( B/H \) is characterized by
\[
[A] \subseteq [B] \iff A \subseteq B \cup W \text{ for some } W \in H
\]
\[
\iff [A] \cap [B] = [A].
\]
If, in addition, we define \( \theta_\alpha : B/H \to B/H \) by \( \theta_\alpha([A]) = [r(A, \alpha)] \) for all \([A] \in B/H \) and \( \alpha \in A \), then \((B/H, A, \theta)\) becomes a Boolean dynamical system (see [11, Proposition 3.6]). We call it a quotient Boolean dynamical system associated to \((E, L, B)\).

Remark 3.3. Given a labeled space \((E, L, B)\) and hereditary set \( H \) of \( B \), there can be a \([A] \in B/H \) such that \([A] \neq [\emptyset] \), but \([r(A, \alpha)] = [\emptyset] \) for all \( \alpha \in A \). For example, for the labeled space \((E, L, E)\) in Example 3.2, we have \( H := H(r(a)) \) is a hereditary saturated subset of \( E \). It is easy to see that \( \{v_1\} \neq [\emptyset] \), but \([r(\{v_1\}, \alpha)] = [\emptyset] \) for all \( \alpha \in A \).

A filter [7, Definition 2.6] \( \xi \) in a Boolean algebra \( B \) is a non-empty subset \( \xi \subseteq B \) such that
\[
\begin{align*}
&\text{F0 } \emptyset \notin \xi, \\
&\text{F1 } \text{if } A \in \xi \text{ and } A \subseteq B, \text{ then } B \in \xi, \\
&\text{F2 } \text{if } A, B \in \xi, \text{ then } A \cap B \in \xi.
\end{align*}
\]
If in addition $\xi$ satisfies

1. $A \in \xi$ and $B, B' \in B$ with $A = B \cup B'$, then either $B \in \xi$ or $B' \in \xi$, then it is called an ultrafilter [7, Definition 2.6] of $B$. A filter is an ultrafilter if and only if it is a maximal element in the set of filters with respect to inclusion. We write $\widehat{B}$ for the set of all ultrafilters of $B$.

**Definition 3.4.** ([5, Definition 3.1 and Definition 5.1]) Let $(E, \mathcal{L}, \mathcal{B})$ be a labeled space and $\alpha \in \mathcal{L}^*(E) \setminus \{\emptyset\}$ and $\eta \in \widehat{B}$.

1. A pair $(\alpha, \eta)$ is an ultrafilter cycle if $r(A, \alpha) \in \eta$ for all $A \in \eta$.
2. $(E, \mathcal{L}, \mathcal{B})$ satisfies Condition (K) if there is no pair $((\alpha, \eta), A)$ where $(\alpha, \eta)$ is an ultrafilter cycle and $A \in \eta$ such that if $\beta \in \mathcal{L}^* \setminus \{\emptyset\}$, $B \subseteq A$, and $r(B, \beta) \in \eta$, then $B \in \eta$ and $\beta = \alpha^k$ for some $k \in \mathbb{N}$.

**Lemma 3.5.** ([5, Theorem 6.3]) Let $(E, \mathcal{L}, \mathcal{B})$ be a labeled space. Then $(E, \mathcal{L}, \mathcal{B})$ satisfies Condition (K) if and only if the quotient Boolean dynamical system $(\mathcal{B}/\mathcal{H}, A, \theta)$ associated to $(E, \mathcal{L}, \mathcal{B})$ satisfies Condition (L) for every hereditary saturated subset $\mathcal{H}$ of $\mathcal{B}$.

**Proof.** It follows by [5, Theorem 6.3].

### 3.2. Gauge-invariant ideals of $C^*(E, \mathcal{L}, \mathcal{B})$

Given a hereditary saturated subset $\mathcal{H}$ of $\mathcal{B}$, we define an ideal

$$\mathcal{B}_\mathcal{H} := \{ A \in \mathcal{B} : [A] \in (\mathcal{B}/\mathcal{H})_{\text{reg}} \}$$

of $\mathcal{B}$. Choose an ideal $\mathcal{S}$ of $\mathcal{B}_\mathcal{H}$ (and hence an ideal of $\mathcal{B}$) such that $\mathcal{H} \cup \mathcal{B}_{\text{reg}} \subseteq \mathcal{S}$. Let $I(\mathcal{H}, \mathcal{S})$ denote the ideal of $C^*(E, \mathcal{L}, \mathcal{B}) := C^*(p_A, s_\alpha)$ generated by the family of projections

$$\left\{ p_A - \sum_{\alpha \in \Delta[\alpha]} s_\alpha p_{r(A, \alpha)} s_\alpha^* : A \in \mathcal{S} \right\},$$

where $\Delta[\alpha] := \{ \alpha \in A : [r(A, \alpha)] \neq [\emptyset] \}$. Then the ideal $I(\mathcal{H}, \mathcal{S})$ is gauge-invariant ([6, Lemma 7.1]) and

$$I(\mathcal{H}, \mathcal{S}) = \text{span} \{ s_\alpha (p_A - p_{A, \mathcal{H}}) s_\beta^* : A \in \mathcal{S} \text{ and } \alpha, \beta \in \mathcal{L}^*(E) \},$$

where we put $p_{A, \mathcal{H}} := \sum_{\alpha \in \Delta[\alpha]} s_\alpha p_{r(A, \alpha)} s_\alpha^*$.

We shall prove in Theorem 3.7 that every gauge-invariant ideals of $C^*(E, \mathcal{L}, \mathcal{B})$ is of the form $I(\mathcal{H}, \mathcal{S})$ for a hereditary saturated subset $\mathcal{H}$ and an ideal $\mathcal{S}$ of $\mathcal{B}_\mathcal{H}$ with $\mathcal{H} \subseteq \mathcal{S}$ and $\mathcal{B}_{\text{reg}} \subseteq \mathcal{S}$. We first observe the following.

**Lemma 3.6.** ([6, Lemma 7.2]) Let $I$ be a nonzero ideal in $C^*(E, \mathcal{L}, \mathcal{B})$.

1. The set $\mathcal{H}_I := \{ A \in \mathcal{B} : p_A \in I \}$ is a hereditary saturated subset of $\mathcal{B}$.
2. The set

$$\mathcal{S}_I := \left\{ A \in \mathcal{B}_{\mathcal{H}_I} : p_A - \sum_{\alpha \in \Delta[\alpha]} s_\alpha p_{r(A, \alpha)} s_\alpha^* \in I \right\}$$

is an ideal of $\mathcal{B}_{\mathcal{H}_I}$ (and hence an ideal of $\mathcal{B}$) with $\mathcal{H}_I \subseteq \mathcal{S}_I$ and $\mathcal{B}_{\text{reg}} \subseteq \mathcal{S}_I$.

**Proof.** It follows by Proposition 2.7 and [6, Lemma 7.2].

□
A quotient labeled space was introduced in [11] to study the ideal structure of $C^*$-algebras of a set-finite and receiver set-finite labeled space $(E, \mathcal{L}, B)$ with $E$ having no sinks. Given a hereditary saturated subset $H$ of $B$, let $I_H$ be the ideal of $C^*(E, \mathcal{L}, B)$ generated by the projections $\{p_A : A \in H\}$. Then the quotient algebra $C^*(E, \mathcal{L}, B)/I_H$ by the gauge-invariant ideal $I_H$ as a $C^*$-algebras of a quotient labeled space $(E, \mathcal{L}, B/H)$ of a quotient labeled space $(E, \mathcal{L}, B)$ ([11, Theorem 5.2]). The quotient labeled space $(E, \mathcal{L}, B/H)$ is nothing but a Boolean dynamical system $(B/H, A, \theta)$. So, to generalize this result to $C^*$-algebras of arbitrary labeled spaces, we use relative generalized Boolean dynamical systems rather than we newly define relative quotient labeled spaces of arbitrary labeled spaces.

Proposition 3.7. ([6, Proposition 7.3]) Let $(E, \mathcal{L}, B)$ be a labeled space. Suppose that $I$ is an ideal of $C^*(E, \mathcal{L}, B)$. There is then a surjective $*$-homomorphism

$$\phi_I : C^*(B/H_I, \mathcal{A}, \theta, [I_{r(\alpha)}], [S_I]) \rightarrow C^*(E, \mathcal{L}, B)/I$$

such that

$$\phi_I(p_{[A]}) = p_A + I \text{ and } \phi_I(s_\alpha[B]) = s_\alpha p_B + I,$$

where $[I_{r(\alpha)}] := \{[A] \in B/H_I : A \in I_{r(\alpha)}\}$ and $[S_I] := \{[A] \in B/H_I : A \in S_I\}$. Moreover, the following are equivalent.

1. $I$ is gauge-invariant.
2. The map $\phi_I$ is an isomorphism.
3. $I = \mathcal{I}(H_I, S_I)$.

Proof. It follows by Proposition 2.7 and [6, Proposition 7.3]. \qed

We further say that the map $(H, S) \mapsto I_{(H, S)}$ is a lattice isomorphism. The set of pairs $(H, S)$, where $H$ is a hereditary saturated subset of $B$ and $S$ is an ideal of $B_H$ with $H \cup B_{reg} \subseteq S$ is a lattice with respect to the order relation defined by $(H_1, S_1) \leq (H_2, S_2) \iff (H_1 \subseteq H_2$ and $S_1 \subseteq S_2$). The set of gauge-invariant ideals of $C^*(E, \mathcal{L}, B)$ is a lattice with the order given by set inclusion.

Theorem 3.8. ([6, Theorem 7.4]) Let $(E, \mathcal{L}, B)$ be a labeled space. Then the map $(H, S) \mapsto I_{(H, S)}$ is a lattice isomorphism between the lattice of all pairs $(H, S)$, where $H$ is a hereditary saturated subset of $B$ and $S$ is an ideal of $B_H$ with $H \cup B_{reg} \subseteq S$, and the lattice of all gauge-invariant ideals of $C^*(E, \mathcal{L}, B)$.

Proof. It follows by Proposition 2.7 and [6, Theorem 7.4]. \qed

Example 3.9. Let $(E, \mathcal{L})$ be the following labeled graph

```
a -- v -- w -- a
```

where $v$ emits infinitely many labeled edges $(c_n)_{n \geq 1}$. Then $E = \{\emptyset, \{v\}, \{w\}, \{v, w\}\}$, $\mathcal{E}_{reg} = \{\emptyset, \{w\}\}$, and $H(\{w\}) = \emptyset$. Since $r(\{v\}, a^n) \notin H(\{w\})$ for all $n \geq 1$, it follows that $\{v\} \notin S(H(\{w\}))$. Thus, $S(H(\{w\})) = \emptyset$. 

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Let $\mathcal{H} := \mathcal{S}(\mathcal{H}(\{w\})) = \emptyset, \{w\}$. Then $\mathcal{E}/\mathcal{H} = \{[\emptyset], \{v\}\} = (\mathcal{E}/\mathcal{H})_{\text{reg}}$ and
\[ \mathcal{E}_r = \{ A \in \mathcal{E} : \{A\} \in (\mathcal{E}/\mathcal{H})_{\text{reg}} \} = \emptyset, \{v\}, \{v, w\} \} = \mathcal{E}. \quad (5) \]
Let $I_{(\mathcal{H}, \mathcal{E}_r)}$ denote the only one gauge-invariant ideal of $C^*(E, \mathcal{L}, \mathcal{E}) := C^*(p_A, s_\alpha)$ generated by the family of projections
\[ \left\{ p_A - \sum_{\alpha \in \Delta_{\{A\}}} s_\alpha p_{r(\alpha)} s_\alpha^* : A \in \mathcal{E}_r \right\}. \]
Note that $[\mathcal{E}_r] = \{[\emptyset], \{v\}\} = (\mathcal{E}/\mathcal{H})_{\text{reg}}$. Then, by Proposition 3.7, we have that
\[ C^*(E, \mathcal{L}, \mathcal{E})/I_{(\mathcal{H}, \mathcal{E}_r)} \cong C^*(\mathcal{E}/\mathcal{H}, \mathcal{A}, \theta, [I_{a(L)}]). \]
Since $C^*(\mathcal{E}/\mathcal{H}, \mathcal{A}, \theta, [I_{a(L)}])$ is generated by $\{p_{\{v\}}, s_{\alpha,\{v\}}\}$ satisfying $s_{\alpha,\{v\}}^* s_{\alpha,\{v\}} = p_{\{v\}} = s_{\alpha,\{v\}}^* s_{\alpha,\{v\}}$, we conclude that $C^*(E, \mathcal{L}, \mathcal{E})/I_{(\mathcal{H}, \mathcal{E}_r)}$ is isomorphic to $C(T)$.

4. Simple labeled graph $C^*$-algebras

In this section, we introduce strong cofinality, minimality and disagreeability for an arbitrary labeled space $(E, \mathcal{L}, \mathcal{E})$, and state our main result, Theorem 4.17.

4.1. Strong cofinality and minimality. We first recall the notion of strong cofinality given [10, Definition 3.1]. A set-finite and receiver set-finite labeled space $(E, \mathcal{L}, \mathcal{E})$ with $E$ having no sinks or sources was called strongly cofinal if for all $x \in \mathcal{L}(E^\infty)$, $[v]_l \in \mathcal{E}$ and $w \in s(x)$, there are $N \geq 1$ and a finite number of paths $\lambda_1, \cdots, \lambda_m \in \mathcal{L}^*(E)$ such that
\[ r([w]_1, x_{[1,N]}) \subset \bigcup_{i=1}^m r([v]_l, \lambda_i). \]
It is shown in [10, Theorem 3.16] that if $(E, \mathcal{L}, \mathcal{E})$ is strongly cofinal in the above sense and is disagreeable, then $C^*(E, \mathcal{L}, \mathcal{E})$ is simple. But, the following example shows that there is a strongly cofinal labeled space in the sense of [10, Definition 3.1] and is disagreeable, but its associated $C^*$-algebra is not simple.

Example 4.1. Consider the following labeled graph $(E, \mathcal{L})$:

\[ \begin{array}{c}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & v & 0 & v_0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \]

Then $(E, \mathcal{L}, \mathcal{E})$ is set-finite, receiver set-finite and left-resolving. For every vertex $v \in E^0$, there is $l > 0$ and $\alpha \in \mathcal{L}^*(E)$ such that $r(\alpha) = [v]_l = \{v\} \in \mathcal{E}$. Thus, one sees that $\mathcal{E} = \{\bigcup_{i=1}^n r(\alpha_i) : \alpha_i \in \mathcal{L}^*(E) \text{ and } n \in \mathbb{N}\}$. It then easy to see that $(E, \mathcal{L}, \mathcal{E})$ is strongly cofinal in the sense of [10, Definition 3.1].

On the other hand, consider the smallest hereditary saturated set $\mathcal{H}$ containing $r(1)$. One can see that $r(0) \notin \mathcal{H}$. So, $\mathcal{H}$ is a proper hereditary saturated subset...
of $\mathcal{E}$. Thus, by [11, Theorem 5.2], there is a nontrivial ideal of $C^*(E, \mathcal{L}, \mathcal{E})$, so that $C^*(E, \mathcal{L}, \mathcal{E})$ is not simple. In fact, $C^*(E, \mathcal{L}, \mathcal{E})$ contains many ideals that is not gauge-invariant. To see this, note that for each $\alpha \in \mathcal{L}(E)$, if $\alpha = \alpha'1$ for some $\alpha'$, then $r(\alpha) \in \mathcal{H}$. Observe also that $r(0) \sim r(0^n)$ for each $n \in \mathbb{N}$ since $r(0) \cup (r(10) \cup \cdots \cup r(10^{n-1})) = r(0^n) \cup (r(10) \cup \cdots \cup r(10^{n-1}))$ for each $n \in \mathbb{N}$. So, we have $\mathcal{E}/\mathcal{H} = \{ [r(0)], [0] \}$. One then can see that $C^*(E, \mathcal{L}, \mathcal{E})/\mathcal{I}_H \cong C^*(E/\mathcal{H}, \mathcal{A}, \theta) \cong C(\mathbb{T})$, where $\mathcal{A} = \{0, 1\}$ Thus, $C^*(E, \mathcal{L}, \mathcal{E})$ contains many ideals that is not gauge-invariant.

As we see in the above example, we need to modify the definition of strong cofinality given in [10, Definition 3.1]. To do that, let

$$\mathcal{L}(E^\infty) := \{ x \in \mathcal{L}(E)_{\mathbb{N}} \mid x_{[1, n]} \in \mathcal{L}(E^n) \text{ for all } n \geq 1 \}$$

be the set of all right infinite sequences $x$ such that all of its subpaths occurs as a labeled path in $(E, \mathcal{L})$. Clearly $\mathcal{L}(E^\infty) \subseteq \overline{\mathcal{L}(E)}(\mathbb{N})$, and in fact, $\mathcal{L}(E^\infty)$ is the closure of $\mathcal{L}(E^\infty)$ in the totally disconnected perfect space $\mathcal{A}^\mathbb{N}$ which has the topology with a countable basis of open-closed cylinder sets $Z(\alpha) := \{ x \in \mathcal{A}^\mathbb{N} : x_{[1, n]} = \alpha \}$, $\alpha \in \mathcal{A}^\mathbb{N}$, $n \geq 1$ (see Section 7.2 of [17]). In general, $\mathcal{L}(E^\infty) \subseteq \overline{\mathcal{L}(E)}(\mathbb{N})$. For example, in Example 4.1, we see that $1^\infty, 0^\infty \notin \mathcal{L}(E^\infty)$, but $1^\infty, 0^\infty \in \overline{\mathcal{L}(E^\infty)}$.

Adopting [15, Definition 2.10], we introduce strong cofinality of an arbitrary labeled space.

**Definition 4.2.** We say that a labeled space $(E, \mathcal{L}, \mathcal{E})$ is **strongly cofinal** if for each nonempty set $A \in \mathcal{E}$ and $x \in \overline{\mathcal{L}(E)}(\mathbb{N})$, there exist $N \in \mathbb{N}$ and a finite number of paths $\lambda_1, \ldots, \lambda_m \in \mathcal{L}^*(E)$ such that

$$r(x_{[1, N]}) \subseteq \bigcup_{i=1}^m r(A, \lambda_i).$$

Throughout the paper if we mention strong cofinality, we mean Definition 4.2.

**Example 4.3.** We continue Example 4.1. For $x := 0^\infty \in \overline{\mathcal{L}(E^\infty)} \setminus \mathcal{L}(E^\infty)$ and $\{v_0\} \in \mathcal{E}$, we see that $r(x_{[1, n]}) = r(0^n) \not\subseteq \bigcup_{i=1}^m r(\{v_0\}, \lambda_i)$ for any paths $\lambda_1, \ldots, \lambda_m \in \mathcal{L}^*(E)$ and any $n \in \mathbb{N}$. Thus, $(E, \mathcal{L}, \mathcal{E})$ is not strongly cofinal.

**Remark 4.4.** Let $(E, \mathcal{L}, \mathcal{E})$ be a labeled space.

1. $(E, \mathcal{L}, \mathcal{E})$ is strongly cofinal if and only if for all $\emptyset \neq A \in \mathcal{E}$, $B \in \mathcal{E}$ and $x \in \overline{\mathcal{L}(E^\infty)}$, there exists an $N \geq 1$ such that $r(B, x_{[1, N]}) \in \mathcal{H}(A)$.

2. If $E$ has no sources and $(E, \mathcal{L}, \mathcal{E})$ is set-finite and receiver set-finite, then $(E, \mathcal{L}, \mathcal{E})$ is strongly cofinal if and only if for each $[v]_l \in \mathcal{E}$, $x \in \overline{\mathcal{L}(E^\infty)}$ and $w \in s(x)$, there exist an $N \geq 1$ and a finite number of paths $\lambda_1, \ldots, \lambda_m \in \mathcal{L}^*(E)$ such that $r([w]_1, x_{[1, N]}) \subseteq \bigcup_{i=1}^m r([v]_l, \lambda_i)$.

**Proof.** 1.$(\Rightarrow)$ It is clear.

$$(\Leftarrow)$$ Let $A \in \mathcal{E}$ and $x = x_1 x_2 \cdots \in \overline{\mathcal{L}(E^\infty)}$. Then for $r(x_1) \in \mathcal{E}$ and $x_2 x_3 \cdots \in \overline{\mathcal{L}(E^\infty)}$, there exist $N \geq 1$ and a finite number of paths $\lambda_1, \ldots, \lambda_m \in \mathcal{L}^*(E)$ such that

$$r(x_1 \cdots x_N) = r(r(x_1), x_2 x_3 \cdots x_N) \subseteq \bigcup_{i=1}^m r(A, \lambda_i).$$

$$2.$$ It is clear since $r([w]_1, x_{[1, N]}) \subseteq r(x_{[1, N]})$ for any $w \in s(x)$. 

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\textbf{Definition 4.5.} We say that \((E, \mathcal{L}, \mathcal{E})\) is minimal if \(\{\emptyset\}\) and \(\mathcal{E}\) are the only hereditary saturated subsets of \(\mathcal{E}\).

We now describe a number of equivalent conditions of minimality. The ideal structure of \(C^*(E, \mathcal{L}, \mathcal{E})\) will be used to prove theorem.

\textbf{Theorem 4.6.} Let \((E, \mathcal{L}, \mathcal{E})\) be a labeled space. The following are equivalent.

1. \((E, \mathcal{L}, \mathcal{E})\) is minimal.
2. \(S(\mathcal{H}(A)) = \mathcal{E}\) for every \(A \in \mathcal{B} \setminus \{\emptyset\}\).
3. \((E, \mathcal{L}, \mathcal{E})\) is strongly cofinal and for any \(A \in \mathcal{E} \setminus \{\emptyset\}\) and \(B \in \mathcal{E}\), there exists \(C \in \mathcal{E}_{\text{reg}}\) such that \(B \setminus C \in \mathcal{H}(A)\).
4. The only ideal of \(C^*(E, \mathcal{L}, \mathcal{E})\) containing \(p_A\) for some \(A \in \mathcal{E} \setminus \{\emptyset\}\) is \(C^*(E, \mathcal{L}, \mathcal{E})\).
5. The only non-zero ideal of \(C^*(E, \mathcal{L}, \mathcal{E})\) which is gauge-invariant is \(C^*(E, \mathcal{L}, \mathcal{E})\).

\textbf{Proof.}\space (1) \implies (2): It is obvious.

(2) \implies (3): Choose \(\emptyset \neq A \in \mathcal{E}\), \(B \in \mathcal{E}\) and \(x = x_1x_2 \cdots \in \overline{\mathcal{L}(E)}\). Then \(B \in S(\mathcal{H}(A))\). It follows from the description of \(S(\mathcal{H}(A))\) given in Lemma 3.1 that there is an \(n \geq 0\) such that \(r(B, \beta) \in \mathcal{H}(A)\) for all \(\beta \in \mathcal{L}^n\), and \(r(B, \gamma) \in \mathcal{H}(A) \oplus \mathcal{E}_{\text{reg}}\) for all \(\gamma \in \mathcal{L}^\infty\) with \(|\gamma| < n\). If \(n = 0\) and we let \(C = \emptyset\), then \(C \in \mathcal{E}_{\text{reg}}\), \(B \setminus C = B \in \mathcal{H}(A)\). If \(n > 0\), then \(r(B, x_{[1,n]}) \in \mathcal{H}(A)\) and there is a \(C \in \mathcal{E}_{\text{reg}}\) such that \(B \setminus C \in \mathcal{H}(A)\). Thus, (3) holds (see Remark 4.4(1)).

(3) \implies (1): Choose a nonempty hereditary saturated subset \(\mathcal{H} \subseteq E\). Take \(\emptyset \neq A \in \mathcal{H}\). Suppose that \(B \notin \mathcal{H}\) for a set \(B \in \mathcal{E}\). Then there is a \(C_1 \in \mathcal{E}_{\text{reg}}\) such that \(B \setminus C_1 \in \mathcal{H}(A)\). Since \(B \notin \mathcal{H}\), we have \(C_1 \notin \mathcal{H}\). So, there is \(x_1 \in \mathcal{L}(E^1)\) such that \(r(C_1, x_1) \notin \mathcal{H}\) since \(\mathcal{H}\) is saturated. We can then choose \(C \in \mathcal{E}_{\text{reg}}\) such that \(r(C_1, x_1) \setminus C \in \mathcal{H}(A)\). Let \(C_2 := C \cap r(C_1, x_1)\). Since \(r(C_1, x_1) \notin \mathcal{H}\), it follows that \(C_2 \notin \mathcal{H}\). Since \(C_2 \in \mathcal{E}_{\text{reg}}\), we deduce that there is an \(x_2 \in \mathcal{L}(E^2)\) such that \(r(C_2, x_2) \notin \mathcal{H}\). Continuing this process, we can construct a sequence \((C_n, x_n)_{n \in \mathbb{N}}\) such that \(C_n \in \mathcal{E}_{\text{reg}} \setminus \mathcal{H}\), \(x_n \in \mathcal{L}(E^1)\), \(C_{n+1} \subseteq r(C_n, x_n)\), and \(r(C_n, x_n) \setminus C_{n+1} \in \mathcal{H}(A)\) for each \(n \in \mathbb{N}\). Let \(x := x_1x_2 \cdots \in \overline{\mathcal{L}(E)}\). Then \(C_{n+1} \subseteq r(C_1, x_{[1,n]})\) for each \(n \geq 1\). Since \(C_{n+1} \notin \mathcal{H}\) for all \(n \in \mathbb{N}\), \(r(C_1, x_{[1,n]}) \notin \mathcal{H}\) for all \(n \in \mathbb{N}\). Thus, \(r(x_{[1,n]}) \notin \mathcal{H}(A)\) for each \(n \geq 1\), which contradicts to strong cofinality of \((E, \mathcal{L}, \mathcal{E})\). Thus, we conclude that \(\mathcal{H} = \mathcal{E}\).
Choose a nonempty hereditary saturated subset of $E$. If $B \not\in H$ for a set $B \in E\setminus \mathcal{E}_{reg}$, then $B \in H(A)$ by assumption, a contradiction. So, if $B \not\in H$ for a set $B \in \mathcal{E}_{reg}$, there is $x_1 \in \mathcal{L}(E^1)$ such that $r(B, x_1) \notin H$ since $H$ is saturated. If $r(B, x_1) \in E\setminus \mathcal{E}_{reg}$, then again $r(B, x_1) \in H$, a contradiction. So, $r(B, x_1) \in \mathcal{E}_{reg}$. Repeating this process, we see that there is an infinite path $x = x_1x_2 \cdots \in \mathcal{L}(E^\infty)$ such that $r(B, x_{[1,n]}) \in \mathcal{E}_{reg} \setminus H$ for all $n \in \mathbb{N}$. Thus, $r(x_{[1,n]}) \notin H$ for all $n \in \mathbb{N}$ since $r(B, x_{[1,n]}) \subset r(x_{[1,n]})$. It then contradicts to strong cofinality of $(E, \mathcal{L}, \mathcal{E})$. Thus, we conclude that $H = \mathcal{E}$. □

The converse of Proposition 4.7 is not true.

Example 4.8. Consider the following labeled graph $(E, \mathcal{L})$:

![Diagram](image)

where $w$ emits infinitely many labeled edges $(\alpha_n)_{n \geq 1}$. Then $\mathcal{E} = \{\emptyset, \{v\}, \{w\}, \{v, w\}\}$ and $\mathcal{E}_{reg} = \{\emptyset, \{v\}\}$. It is rather obvious that $S(H(\{v\})) = \mathcal{E}$ and $S(H(\{v, w\})) = \mathcal{E}$. Consider $H(\{w\}) = \{\emptyset, \{w\}\}$ and clearly $\{v\} \in S(H(\{w\}))$. Since $r(\{v, w\}, \beta) \in H(\{w\})$ for all $\beta \in (E^1)$ and $\{v, w\} = \{v\} \cup \{w\} \in \mathcal{E}_{reg} \oplus H(\{w\})$, we have $\{v, w\} \in S(H(\{w\}))$. Thus, $S(H(\{w\})) = \mathcal{E}$. Thus, $(E, \mathcal{L}, \mathcal{E})$ is minimal by Theorem 4.6. It is also easy to see that $(E, \mathcal{L}, \mathcal{E})$ is strongly cofinal. But, $\{v, w\} \notin H(\{w\})$.

Corollary 4.9. If $E$ has no sinks and $(E, \mathcal{L}, \mathcal{E})$ is set-finite, then $(E, \mathcal{L}, \mathcal{E})$ is minimal if and only if $(E, \mathcal{L}, \mathcal{E})$ is strongly cofinal.

Proof. We only need to show "if" part: Note that $\mathcal{E} = \mathcal{E}_{reg}$. Choose $H$ is a nonempty hereditary saturated subset of $E$. If $B \notin H$ for a set $B \in E$, there is an infinite path $x = x_1x_2 \cdots \in \overline{L}(E^\infty)$ such that $r(B, x_{[1,n]}) \notin H$ for all $n \in \mathbb{N}$ since $H$ is saturated. Thus, $r(x_{[1,n]}) \notin H$ for all $n \in \mathbb{N}$ since $r(B, x_{[1,n]}) \subset r(x_{[1,n]})$. It contradicts to strong cofinality of $(E, \mathcal{L}, \mathcal{E})$. Thus, $H = \mathcal{E}$. □

The following lemma is proved in [15, Lemma 3.6] for a set-finite and receiver set-finite labeled space $(E, \mathcal{L}, \mathcal{E})$ with $E$ having no sinks or sources under the assumption that $C^*(E, \mathcal{L}, \mathcal{E})$ is simple. But, to prove it they only use minimality of $(E, \mathcal{L}, \mathcal{E})$.
as a property of simplicity of $C^*(E, \mathcal{L}, \mathcal{E})$. So, we can weaken the assumption as follows. The idea of proof is same with [15, Lemma 3.6]. We have only included the proof of Lemma 4.10(1) for completeness.

For a path $\beta := \beta_1 \cdots \beta_{|\beta|}$, let $\bar{\beta} = \beta \beta \beta \cdots$ denotes the infinite repetition of $\beta$ ([15, Notation 3.5]). we call a path $\beta \in \mathcal{L}^*(E)$ irreducible if it is not a repetition of its proper initial path.

**Lemma 4.10.** ([15, Lemma 3.6]) Let $(E, \mathcal{L}, \mathcal{E})$ be a minimal labeled space. If there are a nonempty set $A_0 \in \mathcal{E}$ and an irreducible path $\beta \in \mathcal{L}^*$ such that

$$\mathcal{L}(A_0 E^{n|\beta|}) = \{\beta^n\}$$

for all $n \geq 1$, then the following hold.

1. There is an $N \geq 1$ such that for all $n \geq N$,
   $$r(A_0, \bar{\beta}_{[1,n]}) \subset \bigcup_{j=1}^{n-1} r(A_0, \bar{\beta}_{[1,j]}) .$$

2. There is an $N \geq 1$ such that for all $k \geq 1$,
   $$r(A_0, \bar{\beta}_{[1,N+k]}) \subset \bigcup_{j=1}^{N} r(A_0, \bar{\beta}_{[1,j]}) .$$

3. There is an $N_0 \geq 1$ such that for all $k \geq 1$,
   $$r(A_0, \beta^{N+k}) \subset \bigcup_{i=1}^{N_0} r(A_0, \beta^i) .$$

Moreover, $A = r(A, \beta)$ for $A := \bigcup_{i=1}^{N_0} r(A_0, \beta^i)$.

**Proof.** (1): Note first that

$$r(A_0, \bar{\beta}_{[1,j]}) \cap r(A_0, \bar{\beta}_{[1,k]}) \neq \emptyset \iff j = k \pmod{|\beta|} .$$

Assume to the contrary that $r(A_0, \bar{\beta}_{[1,n]}) \setminus \bigcup_{j=1}^{n-1} r(A_0, \bar{\beta}_{[1,j]}) \neq \emptyset$ for infinitely many $n \geq 1$. Then by (6),

$$r(A_0, \beta^n) \notin \bigcup_{j=1}^{n-1} r(A_0, \beta^j)$$

for infinitely many $n \geq 1$. We then claim that $r(\beta^r) \notin \mathcal{H}(A_0)$ for all $r \geq 1$. If $r(\beta^r) \in \mathcal{H}(A_0)$ for some $r \geq 1$, then $r(\beta^r) \in \bigcup_{i=1}^{m} r(A_0, \bar{\beta}_{[1,m_i]})$ for some $m_1, \ldots, m_k \geq 1$. Then, for each $i$, $m_i = k_i |\beta|$ for some $k_i \geq 1$. Take $m := \max\{k_i\}$. Then we have

$$r(\beta^r) \subset \bigcup_{i=1}^{m} r(A_0, \beta^i) .$$

But, then for all sufficiently large $n > m |\beta|$, we have

$$r(A_0, \beta^n) = r(r(A_0, \beta^{n-r}), \beta^r) \subset r(\beta^r) \subset \bigcup_{i=1}^{m} r(A_0, \beta^i) \subset \bigcup_{j=1}^{n-1} r(A_0, \bar{\beta}_{[1,j]}) ,$$

which contradicts to (7). Thus, $r(\beta^r) \notin \mathcal{H}(A_0)$ for all $r \geq 1$. It then follows that

$$r(\beta^r) \notin S(\mathcal{H}(A_0))$$

for all $r \geq 1$. But, this is not the case since the minimility of $(E, \mathcal{L}, \mathcal{E})$ implies $S(\mathcal{H}(A_0)) = \mathcal{E}$.

(2): Follows by [15, Lemma 3.6 (i)].

(3): Follows by [15, Lemma 3.6 (ii)].

**Remark 4.11.** For a set-finite and receiver set-finite labeled space $(E, \mathcal{L}, \mathcal{E})$ with $E$ having no sinks or sources, it is shown in [15, Theorem 3.7] that if $C^*(E, \mathcal{L}, \mathcal{E})$ is simple, then $(E, \mathcal{L}, \mathcal{E})$ is disagreeable. Thus, if $C^*(E, \mathcal{L}, \mathcal{E})$ is simple, the labeled space $(E, \mathcal{L}, \mathcal{E})$ can not have a nonempty set $A \in \mathcal{E}$ and an irreducible path $\beta \in \mathcal{L}^*(E)$ such that $\mathcal{L}(AE^{n|\beta|}) = \{\beta^n\}$ for all $n \geq 1$. But, a minimal labeled space can
have a nonempty set $A \in \mathcal{E}$ and an irreducible path $\beta \in \mathcal{L}^*$ such that $\mathcal{L}(AE^n[\beta]) = \{\beta^n\}$ for all $n \geq 1$. See the following labeled graph $(E, \mathcal{L})$:

\[
\cdots \cdot a \rightarrow \cdot a \rightarrow \cdot a \rightarrow \cdot a \rightarrow \cdot a \rightarrow \cdot a \rightarrow \cdot a \rightarrow \cdot a \rightarrow \cdots.
\]

Then the smallest non-degenerate accommodating set is $\mathcal{E} = \{\emptyset, r(a)\}$. It is easy to see that $(E, \mathcal{L}, \mathcal{E})$ is minimal and $\mathcal{L}(r(a)E^n) = \{a^n\}$ for all $n \geq 1$.

We close this subsection with the following, which will be used to prove Theorem 4.17.

**Proposition 4.12.** Let $(E, \mathcal{L}, \mathcal{E})$ be a minimal labeled space. If $(\alpha, A)$ is a cycle with no exits, then $A$ is a minimal set.

**Proof.** Let $(\alpha, A)$ be a cycle with no exits. Say $\alpha = \alpha_1 \cdots \alpha_n \in \mathcal{L}^*(E)$. We show that if $B \in \mathcal{E}$ such that $\emptyset \neq B \subseteq A$, then $B = A$. Since $(E, \mathcal{L}, \mathcal{E})$ is minimal, we have $S(\mathcal{H}(B)) = \mathcal{E}$ by Theorem 4.6. Thus, $A \subseteq S(\mathcal{H}(B))$, and hence, for each $1 \leq i \leq n$, we have

$$r(A, \alpha_{[1,i]}) \subseteq B \cup r(B, \alpha_1) \cup r(B, \alpha_{[1,2]}) \cup \cdots \cup r(B, \alpha_{[1,n-1]})$$

since $(\alpha, A)$ and $(\alpha, B)$ are both cycle without exits. On the other hand, since $r(A, \alpha_{[1,i]}) \cap r(A, \alpha_{[1,j]}) = \emptyset$ for $i \neq j$ by [5, Lemma 6.1] and $r(B, \alpha_{[1,i]}) \subseteq r(A, \alpha_{[1,i]})$ for each $1 \leq j \leq n$, it follows that $r(A, \alpha_{[1,i]}) \cap r(B, \alpha_{[1,j]}) = \emptyset$ for $i \neq j$. Thus, we have $r(A, \alpha_{[1,i]}) \subseteq r(B, \alpha_{[1,i]})$ for each $1 \leq i \leq n$. It then follows that

$$A = r(r(A, \alpha_{[1,i]}), \alpha_{[i+1,n]}) \subseteq r(r(B, \alpha_{[1,i]}), \alpha_{[i+1,n]}) = B.$$

Hence, $B = A$. \hfill \Box

4.2. **Disagreeable labeled spaces.** A notion of disagreeability of set-finite and receiver set-finite labeled space $(E, \mathcal{L}, \mathcal{E})$ with $E$ having no sinks or sources was introduced in [4, Definition 5.1] as another analogue notion of Condition (L) of directed graphs. If we briefly recall it, a labeled path $\alpha \in \mathcal{L}([v]_lE^{\geq 1})$ is called agreeable for $[v]_l$ if $\alpha = \beta\alpha' = \alpha'\gamma$ for some $\alpha', \beta, \gamma \in \mathcal{L}^*(E)$ with $|\beta| = |\gamma| \leq l$. Otherwise $\alpha$ is called disagreeable. Note that any path $\alpha$ agreeable for $[v]_l$ must be of the form $\alpha = \beta^k\beta'$ for some $\beta \in \mathcal{L}(E^{\geq 1})$, $k \geq 0$, and an initial path $\beta'$ of $\beta$. We say that $[v]_l$ is disagreeable if there is an $N > 0$ such that for all $n > N$ there is an $\alpha \in \mathcal{L}(E^{\geq n})$ that is disagreeable for $[v]_l$. A labeled space $(E, \mathcal{L}, \mathcal{E})$ is disagreeable if $[v]_l$ is disagreeable for all $v \in E^0$ and $l \geq 1$.

In [11], they found its equivalent simpler conditions.

**Proposition 4.13.** ([14, Proposition 3.2]) For a set-finite and receiver set-finite labeled space $(E, \mathcal{L}, \mathcal{E})$ with $E$ having no sinks, the following are equivalent:

(a) $(E, \mathcal{L}, \mathcal{E})$ is disagreeable.
(b) $[v]_l$ is disagreeable for all $v \in E^0$ and $l \geq 1$.
(c) For each nonempty $A \in \mathcal{E}$ and a path $\beta \in \mathcal{L}^*(E)$, there is an $n \geq 1$ such that $\mathcal{L}(AE^{[\beta][n]}) \neq \{\beta^n\}$.

Motivated by Proposition 4.13(c), we define a notion of disagreeability of arbitrary labeled spaces as follows.
Definition 4.14. We say a labeled space \((E, \mathcal{L}, \mathcal{E})\) is disagreeable if for any nonempty set \(A \in \mathcal{E}\) and a path \(\beta \in \mathcal{L}^*(E)\), there is an \(n \geq 1\) such that \(\mathcal{L}(AE|\beta^n|) \neq \{\beta^n\}\).

Definition 4.15. \((12, \text{Definition 3.2})\) Let \((E, \mathcal{L}, \mathcal{E})\) be a labeled space and \(\alpha \in \mathcal{L}^*(E)\) and \(\emptyset \neq A \in \mathcal{E}\).

1. \(A\) is called a loop at \(A\) if \(A \subseteq r(A, \alpha)\).
2. A loop \((\alpha, A)\) has an exit if one of the following holds:
   - \(\{\alpha_{[1,k]} : 1 \leq k \leq |\alpha|\} \subseteq \mathcal{L}(AE|\alpha|^{|\alpha|})\).
   - \(A \subseteq r(A, \alpha)\).

By [14, Proposition 3.7], one can see that if \((E, \mathcal{L}, \mathcal{E})\) is disagreeable, then every loop in \((E, \mathcal{L}, \mathcal{E})\) has an exit, and hence, \((E, \mathcal{L}, \mathcal{E})\) satisfies Condition (L). It is shown in [15, Proposition 3.2] that the other implications are not true, in general. But, if \((E, \mathcal{L}, \mathcal{E})\) is minimal, these conditions are equivalent as we see in the following.

Lemma 4.16. \((15, \text{Proposition 3.2})\) Consider the following three conditions of a labeled space \((E, \mathcal{L}, \mathcal{E})\).

1. \((E, \mathcal{L}, \mathcal{E})\) is disagreeable.
2. Every loop in \((E, \mathcal{L}, \mathcal{E})\) has an exit.
3. \((E, \mathcal{L}, \mathcal{E})\) satisfies Condition (L), that is, every cycle has an exit.

Then we have \((1) \implies (2) \implies (3)\). If, in addition, \((E, \mathcal{L}, \mathcal{E})\) is minimal, \((1)-(3)\) are equivalent.

Proof. We only need to show that \((3) \implies (1)\) when \((E, \mathcal{L}, \mathcal{E})\) is minimal: Let \((E, \mathcal{L}, \mathcal{E})\) be minimal. Suppose that \((E, \mathcal{L}, \mathcal{E})\) is not disagreeable. Then there exist a nonempty set \(A_0 \in \mathcal{E}\) and a path \(\beta \in \mathcal{L}^*(E)\) such that for all \(n \geq 1\),

\[
\mathcal{L}(A_0E|\beta^n|) = \{\beta^n\},
\]

where we assume that \(\beta\) is irreducible. Then by Lemma 4.10, there is an \(N \geq 1\) such that for all \(k \geq 1\),

\[
r(A_0, \beta^{N+k}) \subseteq \cup_{j=1}^{N} r(A_0, \beta^j). \tag{8}
\]

Take \(A := \cup_{j=1}^{N} r(A_0, \beta^j)\). Then \(A = r(A, \beta)\). One then can show that \(A\) is a minimal set since \((E, \mathcal{L}, \mathcal{E})\) is minimal (see the proof of [15, Theorem 3.7]). Thus, \((\beta, A)\) is a cycle with no exits. Thus, \((E, \mathcal{L}, \mathcal{E})\) does not satisfy Condition (L). \(\square\)

4.3. Simplicity. We now characterize the simplicity of the \(C^*\)-algebra associated to an arbitrary labeled space \((E, \mathcal{L}, \mathcal{E})\).

Theorem 4.17. Let \((E, \mathcal{L}, \mathcal{E})\) be a labeled space. Then the following are equivalent.

1. \(C^*(E, \mathcal{L}, \mathcal{E})\) is simple.
2. \((E, \mathcal{L}, \mathcal{E})\) is minimal and satisfies Condition (L).
3. \((E, \mathcal{L}, \mathcal{E})\) is minimal and satisfies Condition (K).
4. The following properties hold:
   - (a) \((E, \mathcal{L}, \mathcal{E})\) is strongly cofinal,
   - (b) \((E, \mathcal{L}, \mathcal{E})\) satisfies Condition (L), and
   - (c) for any \(A \in \mathcal{E} \setminus \{\emptyset\}\) and \(B \in \mathcal{E}\), there is \(C \in \mathcal{E}_{\text{reg}}\) such that \(B \setminus C \in \mathcal{H}(A)\).
5. The following properties hold:
   - (a) \((E, \mathcal{L}, \mathcal{E})\) is strongly cofinal,
(b) \((E, \mathcal{L}, \mathcal{E})\) is disagreeable, and
(c) for any \(A \in \mathcal{E}\setminus\{\emptyset\}\) and \(B \in \mathcal{E}\), there is \(C \in \mathcal{E}_{\text{reg}}\) such that \(B \setminus C \in \mathcal{H}(A)\).

Proof. \((1) \implies (2)\): If \(C^*(E, \mathcal{L}, \mathcal{E})\) is simple, then the only gauge-invariant ideal of \(C^*(E, \mathcal{L}, \mathcal{E})\) is \(\{0\}\) and \(C^*(E, \mathcal{L}, \mathcal{E})\). Thus, \((E, \mathcal{L}, \mathcal{E})\) is minimal by Theorem 4.6. Suppose that \((E, \mathcal{L}, \mathcal{E})\) does not satisfy Condition (L). Then the labeled space has a cycle \((\alpha, A)\) with no exits. Since \((E, \mathcal{L}, \mathcal{E})\) is minimal, \(A\) is a minimal set by Proposition 4.12. Then, \(C^*(E, \mathcal{L}, \mathcal{E})\) has a hereditary subalgebra isomorphic to \(M_{|\alpha|}(C(T))\) by [11, Lemma 4.6]. It contradicts to \(C^*(E, \mathcal{L}, \mathcal{E})\) is simple.

\((2) \iff (1)\): Let \(I\) be a nonzero ideal of \(C^*(E, \mathcal{L}, \mathcal{E})\). Since \((E, \mathcal{L}, \mathcal{E})\) satisfies Condition (L), \(I\) contains a vertex projection \(p_A\) for some \(\emptyset \neq A \in \mathcal{E}\) by the Cuntz-Krieger Uniqueness Theorem 2.5. Then \(I = C^*(E, \mathcal{L}, \mathcal{E})\) by Theorem 4.6.

\((2) \iff (3)\): Follows by Lemma 3.5.
\((2) \iff (4)\): Follows by Theorem 4.6.
\((4) \iff (5)\): Follows by Lemma 4.16.

As a corollary, we have the following simplicity results of labeled graph \(C^*\)-algebras associated to set-finite labeled spaces with no sinks. It is an improvement on [15, Theorem 3.7].

**Corollary 4.18.** If \(E\) has no sinks and \((E, \mathcal{L}, \mathcal{E})\) is set-finite, then the following are equivalent.

\begin{enumerate}
\item \((E, \mathcal{L}, \mathcal{E})\) is simple.
\item \((E, \mathcal{L}, \mathcal{E})\) is minimal and satisfies Condition (L).
\item \((E, \mathcal{L}, \mathcal{E})\) is minimal and satisfies Condition (K).
\item \((E, \mathcal{L}, \mathcal{E})\) is strongly cofinal and disagreeable.
\item \((E, \mathcal{L}, \mathcal{E})\) is strongly cofinal and satisfies Condition (L).
\item \((E, \mathcal{L}, \mathcal{E})\) is strongly cofinal and satisfies Condition (K).
\end{enumerate}

**Proof.** It follows by Corollary 4.9 and Theorem 4.17.

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