Meissner superconductivity in itinerant ferromagnets

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Abstract

Recent results about the coexistence of ferromagnetism and unconventional superconductivity with spin-triplet Cooper pairing are reviewed on the basis of the quasi-phenomenological Ginzburg-Landau theory. The superconductivity in the mixed phase of coexistence of ferromagnetism and unconventional superconductivity is triggered by the spontaneous magnetization. The mixed phase is stable whereas the other superconducting phases that usually exist in unconventional superconductors are either unstable or metastable at relatively low temperatures in a quite narrow domain of the phase diagram and the stability properties are determined by the particular values of Landau parameters. The phase transitions from the normal phase to the phase of coexistence is of first order while the phase transition from the ferromagnetic phase to the coexistence phase can be either of first or second order depending on the concrete substance. The Cooper pair and crystal anisotropy are relevant to a more precise outline of the phase diagram shape and reduce the degeneration of the ground states of the system. The results are discussed in view of application to itinerant ferromagnetic compounds as UGe$_2$, ZrZn$_2$, URhGe.

1 Introduction

1.1 Notes about unconventional superconductivity

The phenomenon of unconventional Cooper pairing of fermions, i.e., the formation of Cooper pairs with nonzero angular momentum was theoretically predicted [1] in 1959 as a mechanism of superfluidity in Fermi liquids. In 1972 the same phenomenon - unconventional superfluidity due to a $p$-wave (spin triplet) Cooper pairing of $^3$He atoms, was experimentally discovered in the mK range of temperatures; for details and theoretical description see Refs. [2] [3] [4]. Note that, in contrast to the standard $s$-wave pairing in usual (conventional) superconductors where the electron pairs are formed by an attractive electron-electron interaction due to a virtual phonon exchange, the widely accepted mechanism of the Cooper pairing in superfluid $^3$He is based on an attractive interaction between the fermions ($^3$He atoms) as a result of a virtual exchange of spin fluctuations. Certain spin fluctuation mechanisms of unconventional Cooper pairing of
electrons are assumed also for the discovered in 1979 heavy fermion superconductors (see, e.g., Refs. [5, 6, 7]) as well as for some classes of high-temperature superconductors (see, e.g., Refs. [8, 9, 10, 11, 12, 13, 14, 15, 16]).

The possible superconducting phases in unconventional superconductors are described in the framework of the general Ginzburg-Landau (GL) effective free energy functional with the help of the symmetry groups theory. Thus a variety of possible superconducting orderings were predicted for different crystal structures [17, 18, 19, 20, 21, 22]. A detailed thermodynamic analysis of the homogeneous (Meissner) phases and a renormalization group investigation of the superconducting phase transition up to the two-loop approximation have been also performed (for a three-loop renormalization group analysis, see Ref. [23]; for effects of magnetic fluctuations and disorder, see [24, 25]). We shall essentially use these results in our present consideration.

1.2 Experimental predictions

In 2000, experiments [26] at low temperatures \(T \sim 1\) K and high pressure \(P \sim 1\) GPa demonstrated the existence of spin triplet superconducting states in the metallic compound UGe\(_2\). The superconductivity is triggered by the spontaneous magnetization of the ferromagnetic phase that occurs at much higher temperatures. It coexists with the superconducting phase in the whole domain of its existence below \(T \sim 1\) K; see also experiments from Refs. [27, 28], and the discussion in Ref. [29]. The same phenomenon of existence of superconductivity at low temperatures and high pressure in the domain of the \((T,P)\) phase diagram where the ferromagnetic order is present was observed in other ferromagnetic metallic compounds (ZrZn\(_2\) [30] and URhGe [31]) soon after the discovery [26] of superconductivity in UGe\(_2\).

In superconducting ternary and Chevrel compounds the influence of magnetic order on superconductivity is also substantial (see, e.g., [32, 33, 34, 35]) but in the newly found ferromagnetic substances the phase transition temperature \(T_f\) to the ferromagnetic state is much higher than the phase transition temperature \(T_{FS}\) from ferromagnetic to a mixed state of coexistence of ferromagnetism and superconductivity. For example, in UGe\(_2\), \(T_{FS} = 0.8\) K while the critical temperature of the phase transition from paramagnetic to ferromagnetic state in the same material is \(T_f = 35\) K [26, 27]. It can be assumed that in these substances the material parameter \(T_s\) defined as the usual critical temperature of the second order phase transition from normal to uniform (Meissner) superconducting state in a zero external magnetic field is much lower than the phase transition temperature \(T_{FS}\). The above mentioned experiments on the compounds UGe\(_2\), URhGe, and ZrZn\(_2\) do not give any evidence for the existence of a standard normal-to-superconducting phase transition in a zero external magnetic field.

It seems that the superconductivity in the metallic compounds mentioned above always coexists with the ferromagnetic order and is enhanced by it. In these systems, as claimed in Ref. [26], the superconductivity probably arises from the same electrons that create
the band magnetism and can be most naturally understood rather as a triplet than spin-singlet pairing phenomenon. Metallic compounds UGe$_2$, URhGe, and ZrZn$_2$, are itinerant ferromagnets. An unconventional superconductivity is also suggested as a possible outcome of recent experiments in Fe, in which a superconducting phase has been discovered at temperatures below 2 K and pressures between 15 and 30 GPa. There both vortex and Meissner superconductivity phases are found in the high-pressure crystal modification of Fe with a hexagonal close-packed lattice for which the strong ferromagnetism of the usual bcc iron crystal probably disappears. It can be hardly claimed that in hexagonal Fe the ferromagnetism and superconductivity coexist but the clear evidence for a superconductivity is also a remarkable achievement.

The reasonable question whether these examples of superconductivity and coexistence of superconductivity and ferromagnetism are bulk or surface effects can be stated. The earlier experiments performed before 2004 do not answer this question. Recent experiments show that surface superconductivity appears in ZrZn$_2$ and its presence depends essentially on the way of preparation of the sample. But in our study it is important that bulk superconductivity can be considered well established in this substance.

### 1.3 Ferromagnetism versus superconductivity

The important point in all discussions of the interplay of superconductivity and ferromagnetism is that a small amount of magnetic impurities can destroy superconductivity in conventional (s-wave) superconductors by breaking up the (s-wave) electron pairs with opposite spins (paramagnetic impurity effect). In this aspect the phenomenological arguments and the conclusions on the basis of the microscopic theory of magnetic impurities in s-wave superconductors are in a complete agreement with each other; see, e.g., Refs. [32, 33, 34, 35]. In fact, a total suppression of conventional (s-wave) superconductivity should occur in the presence of an uniform spontaneous magnetization $M$, i.e., in a standard ferromagnetic phase. The physical reason for this suppression is the same as in the case of magnetic impurities, namely, the opposite electron spins in the s-wave Cooper pair turn over along the vector $M$ in order to lower their Zeeman energy and, hence, the pairs break down. Therefore, the ferromagnetic order can hardly coexist with conventional superconducting states. Especially, this is valid for the coexistence of uniform superconducting and ferromagnetic states where the superconducting order parameter $\psi(x)$ and the magnetization $M$ do not depend on the spatial vector $x$.

But yet a coexistence of s-wave superconductivity and ferromagnetism may appear in uncommon materials and under quite special circumstances. Furthermore, let us emphasize that the conditions for the coexistence of nonuniform (“vertex”, “spiral”, “spin-sinosoidal” or “helical”) superconducting and ferromagnetic states are less restrictive than those for the coexistence of uniform superconducting and ferromagnetic orders. Coexistence of nonuniform phases has been discussed in details, both experiment and theory, in ternary and Chevrel-phase compounds where such a coexistence seems quite likely; for a
comprehensive review, see, for example, Refs. \cite{32, 33, 34, 35, 41}.

In fact the only two superconducting systems for which the experimental data allow assumptions in favor of a coexistence of superconductivity and ferromagnetism are the rare earth ternary boride compound ErRh$_4$B$_4$ and the Chervel phase compound HoMo$_6$S$_8$; for a more extended review, see Refs. \cite{33, 42}. In these compounds the phase of coexistence appears in a very narrow temperature region just below the Curie temperature $T_f$ of the ferromagnetic phase transition. At lower temperatures the magnetic moments of the rare earth 4$f$ electrons become better aligned, the magnetization increases and the $s$-wave superconductivity pairs formed by the conduction electrons disintegrate.

1.4 Unconventional superconductivity triggered by ferromagnetic order

We shall not consider all important aspects of the long standing problem of coexistence of superconductivity and ferromagnetism rather we shall concentrate our attention on the description of the newly discovered coexistence of ferromagnetism and unconventional (spin-triplet) superconductivity in the itinerant ferromagnets UGe$_2$, ZrZn$_2$, and URhGe. Here we wish to emphasize that the main object of our discussion is the superconductivity of these compounds and at a second place in the rate of importance we put the problem of coexistence. The reason is that the existence of superconductivity in itinerant ferromagnets is a highly nontrivial phenomenon. As noted in Ref. \cite{43} the superconductivity in these materials appears to be difficult to explain in terms of previous theories \cite{32, 33, 35} and requires new concepts to interpret the experimental data.

We have already mentioned that in ternary compounds the ferromagnetism comes from the localized 4$f$ electrons while the $s$-wave Cooper pairs are formed by conduction electrons. In UGe$_2$ and URhGe the 5$f$ electrons of U atoms form both superconductivity and ferromagnetic order \cite{26, 31}. In ZrZn$_2$ the same double role is played by the 4$d$ electrons of Zr. Therefore, the task is to describe this behavior of the band electrons at a microscopic level. One may speculate about a spin-fluctuation mediated unconventional Cooper pairing as in case of $^3$He and heavy fermion superconductors. These important issues have not yet a reliable answer and for this reason we shall confine our consideration to a phenomenological level.

In fact, a number of reliable experimental data as the coherence length and the superconducting gap measurements \cite{26, 27, 31, 30} are in favor of the conclusion about a spin-triplet Cooper pairing in these metallic compounds, although the mechanism of pairing remains unclear. We shall essentially use this reliable conclusion. This point of view is consistent with the experimental observation of coexistence of superconductivity only in a low temperature part of the ferromagnetic domain of the phase diagram ($T, P$), which means that a pure (non-ferromagnetic) superconducting phase is not observed, a circumstance, that is also in favor of the assumption of a spin-triplet superconductivity. Our investigation
leads to results which confirm this general picture.

On the basis of the experimental data and conclusions presented for the first time in Refs. [26, 29] and shortly afterwards confirmed in Refs. [27, 28, 30, 31] one may reliably accept that the superconductivity in these magnetic compounds is considerably enhanced by the ferromagnetic order parameter $M$ and, perhaps, it could not exist without this “mechanism of ferromagnetic trigger,” or, in short, “$M$-trigger”; see Refs. [44, 45, 46] where this concept has been introduced for the first time. The trigger phenomenon is possible for spin-triplet Cooper pairs where the electron spins point parallel to each other and their turn along the vector of the spontaneous magnetization $M$ does not produce a breakdown of the spin-triplet Cooper pairs but rather stabilizes them and, perhaps, stimulates their creation. We shall describe this phenomenon at a phenomenological level.

## 1.5 Phenomenological studies

A phenomenological theory that explains the coexistence of ferromagnetism and unconventional spin-triplet superconductivity of GL type has been developed recently in [43, 47] where possible low-order couplings between the superconducting and ferromagnetic order parameters are derived with the help of general symmetry group arguments. On this basis several important features of the superconducting vortex state of unconventional ferromagnetic superconductors were demonstrated [43, 47].

In our review we shall follow the approach from Refs. [43, 47] to investigate the conditions for the occurrence of the Meissner phase and to demonstrate that the presence of ferromagnetic order enhances the $p$-wave superconductivity. We also establish the phase diagram of ferromagnetic superconductors in a zero external magnetic field and show that the phase transition to the superconducting state can be either of first or second order depending on the particular substance. We confirm the predictions made in Refs. [43, 47] about the symmetry of the ordered phases.

In our study we use the mean-field approximation [13] and known results about the possible phases in nonmagnetic superconductors with triplet ($p$-wave) pairing [18, 11, 12, 6]. Our results [44, 45, 49] show that taking into account the anisotropy of the spin-triplet Cooper pairs modifies but does not drastically change the thermodynamic properties of the coexistence phase, especially in the temperature domain above the superconducting critical temperature $T_s$. The effect of crystal anisotropy is similar but we shall not make an overall thermodynamic analysis of this problem because we have to consider concrete systems and crystal structures [18, 6] for which there is no enough information from experiment to make conclusions about the parameters of the theory. Our results confirm the general concept that the anisotropy reduces the degree of ground state degeneration, and depending on the symmetry of the crystal, picks up a crystal direction for the ordering.

There exists a formal similarity between the phase diagram we obtain and the phase di-
agram of certain improper ferroelectrics. We shall make use of the concept in the theory of improper ferroelectrics, where the trigger of the primary order parameter by a secondary order parameter (the electric polarization) has been initially introduced and exploited; see Ref. The mechanism of the M-triggered superconductivity in itinerant ferromagnets is formally identical to the mechanism of appearance of structural order triggered by the electric polarization in improper ferroelectrics (see, e.g., Refs.).

Our investigation is based on the GL free energy functional of unconventional ferromagnetic superconductors presented in Sec. 2.1 and we shall establish the uniform phases which are described by it. More information about the justification of this investigation is presented in Sec. 2.2. We work with a quite general GL free energy and the problem is that there is no enough information about the values of the parameters of the model for concrete compounds where the ferromagnetic superconductivity has been discovered. On the one hand the lack of information makes impossible a detailed comparison of the theory to the available experimental data but on the other hand our results are not bound to one or more concrete substances and can be applied to any unconventional ferromagnetic superconductor. In Sec. 3 we discuss the phases in non-magnetic unconventional superconductors. In Sec. 4 the M-trigger effect will be described when only a linear coupling of the magnetization to the superconducting order parameter is considered; here the spatial dependence of order parameters and all anisotropy effects are ignored. In Sec. 5 we analyze the influence of quadratic coupling of magnetization to the superconducting order parameter on the thermodynamics of the ferromagnetic superconductors. The application of our results to experimental (T, P) phase diagrams is discussed in Sec. 5.3. In Sec. 6 the anisotropy effects are outlined. Our main attention is focussed on the Cooper-pair anisotropy. Note, that certain types of crystal anisotropy may produce more than one ferromagnetic phase but here we shall not dwell on this interesting topic. In Sec. 7 we summarize our conclusions.

2 Ginzburg-Landau free energy

Following Refs. in this Chapter we discuss the phenomenological theory of spin-triplet ferromagnetic superconductors and justify our consideration in Sections 3–6.

2.1 Model

The general GL free energy functional, we shall use in our analysis, is

$$F[\psi, M] = \int d^3 x f(\psi, M),$$ (1)
where the free energy density \( f(\psi, M) \) (hereafter called “free energy”) of a spin-triplet ferromagnetic superconductor is a sum of five terms [18, 47, 18], namely,

\[
f(\psi, M) = f_S(\psi) + f_F'(M) + f_1(\psi, M) + \frac{B^2}{8\pi} - B\cdot M.
\]  

(2)

In Eq. (2) the three dimensional complex vector \( \psi = \{\psi_j; j = 1, 2, 3\} \) represents the superconducting order parameter, \( B = (H + 4\pi M) = \nabla \times A \) is the magnetic induction; \( H \) is the external magnetic field, \( A = \{A_j; j = 1, 2, 3\} \) is the magnetic vector potential.

The last two terms on r.h.s. of Eq. (2) are related with the magnetic energy which includes both diamagnetic and paramagnetic effects in the superconductor; see, e.g., [32, 59].

The energy part \( f_S(\psi) \) in Eq. (2) describes the superconductivity for \( H = M \equiv 0 \). It can be written in the form

\[
f_S(\psi) = f_{\text{grad}}(\psi) + a_s|\psi|^2 + \frac{b_s}{2}|\psi|^4 + \frac{u_s}{2}|\psi^2|^2 + \frac{v_s}{2}\sum_{j=1}^{3} |\psi_j|^4.
\]  

(3)

Here

\[
f_{\text{grad}}(\psi) = K_1(D_i\psi_j)^*(D_i\psi_j) + K_2[(D_i\psi_j)^*(D_j\psi_j)] + (D_i\psi_j)^*(D_j\psi_i) + K_3(D_i\psi_i)^*(D_i\psi_i),
\]  

(4)

denotes a covariant differentiation. In Eq. (3), \( b_s > 0 \) and \( a_s = \alpha_s(T - T_s) \), where \( \alpha_s \) is a positive material parameter and \( T_s \) is the critical temperature of the standard second order phase transition which may occur at \( H = M = 0; H = |H|, \) and \( M = |M|. \) The quantities \( u_s \) and \( v_s \) describe the anisotropy of the spin-triplet Cooper pair and the crystal anisotropy, respectively, [13, 14]. Parameters \( K_j, (j = 1, 2, 3) \) in Eq. (4) are related with the effective mass tensor of anisotropic Cooper pairs [13].

The superconducting part (3) of the free energy \( f(\psi, M) \) is derived from symmetry group arguments and is independent of particular microscopic models; see, e.g., Refs. [13, 6].

According to classifications [13, 6] the \( p \)-wave superconductivity in the cubic point group \( O_h \) can be realized through one-, two-, and three-dimensional representations of the order parameter. The expressions (3) and (5) incorporate all three possible cases. The coefficients \( b_s, u_s, \) and \( v_s \) in Eq. (3) are different for weak and strong spin-orbit couplings but in our investigation they are considered as undetermined material parameters which depend on the particular substance.

The free energy of a standard isotropic ferromagnet is given by the term \( f_F'(M) \) in Eq. (2),

\[
f_F'(M) = c_f\sum_{j=1}^{3} |\nabla_j M_j|^2 + a_f(T'_f)M^2 + \frac{b_f}{2}M^4,
\]  

(6)
where $\nabla_j = \partial/\partial x_j$ and $b_f > 0$. The quantity $a_f(T'_f) = \alpha_f(T - T'_f)$ is expressed by the material parameter $\alpha_f > 0$ and the temperature $T'_f$ which is different from the critical temperature $T_f$ of the ferromagnet and this point will be discussed below. We have already added a negative term $(-2\pi M^2)$ to the total free energy $f(\psi, M)$ and that is obvious by setting $H = 0$ in Eq. (2). The negative energy $(-2\pi M^2)$ should be added to $f'_F(M)$. In this way one obtains the total free energy $f_F(M)$ of the ferromagnet in a zero external magnetic field that is given by a modification of Eq. (6) according to the rule

$$f_F(a_f) = f'_F \left[ a_f(T'_f) \to a_f(T_f) \right],$$

(7)

where $a_f = \alpha_f(T - T_f)$ and 

$$T_f = T'_f + \frac{2\pi}{\alpha_f}$$

(8)

is the critical temperature of a standard ferromagnetic phase transition of second order. This scheme was used in studies of rare earth ternary compounds [32, 59, 60, 61]. Alternatively [62], one may use from the beginning the total ferromagnetic free energy $f_F(a_f, M)$ as given by Eqs. (6) - (8) but in this case the magnetic energy included in the last two terms on r.h.s. of Eq. (2) should be replaced with $H^2/8\pi$. Both approaches are equivalent.

The interaction between the ferromagnetic order parameter $M$ and the superconducting order parameter $\psi$ is given by [43, 47]

$$f_1(\psi, M) = i\gamma_0 M \cdot (\psi \times \psi^*) + \delta M^2 |\psi|^2. $$

(9)

The $\gamma_0$-term in the above expression is the most substantial for the description of experimentally found ferromagnetic superconductors [47] and the $\delta M^2 |\psi|^2$ -term makes the model more realistic in the strong coupling limit as it gives the opportunity to enlarge the phase diagram including both positive and negative values of the parameter $a_s$. In this way the domain of the stable ferromagnetic order is extended down to zero temperatures for a wide range of values of material parameters and the pressure $P$, a situation that corresponds to the experiments in ferromagnetic superconductors.

In Eq. (9) the coupling constant $\gamma_0 > 0$ can be represented in the form $\gamma_0 = 4\pi J$, where $J > 0$ is the ferromagnetic exchange parameter [47]. In general, the parameter $\delta$ for ferromagnetic superconductors may take both positive and negative values. The values of the material parameters ($T_s, T_f, \alpha_s, \alpha_f, b_s, u_s, v_s, b_f, K_j, \gamma_0$ and $\delta$) depend on the choice of the concrete substance and on thermodynamic parameters as temperature $T$ and pressure $P$.

### 2.2 Way of treatment

The total free energy (2) is a quite complex object of theoretical investigation. The possible vortex and uniform phases of the model cannot be investigated within a single calculation, rather one should focus on concrete problems. In Ref. [47] the vortex phase was discussed with the help of the criterion [63] for a stability of this state near the phase
transition line $T_{c2}(H)$; see also, Ref. [63]. In case of $H = 0$ one should apply the same
criterion with respect to the magnetization $M$ for small values of $|\psi|$ near the phase
transition line $T_{c2}(M)$ as performed in Ref. [47].

Here we shall be interested in the uniform phases when the order parameters $\psi$ and $M$
do not depend on the spatial vector $x \in V$, where $V$ is the volume of the superconductor.
We shall restrict our analysis to the consideration of the coexistence of uniform (Meissner)
phases and ferromagnetic order. We shall make a detailed investigation in order to
show that the main properties of the uniform phases can be well determined within an
approximation when the crystal anisotropy is neglected. Even, some of the main features
of the uniform phases in unconventional ferromagnetic superconductors can be reliably
outlined when the Cooper pair anisotropy is neglected, too.

The assumption of a uniform magnetization $M$ is always reliable outside a quite close
vicinity of the magnetic phase transition and under the condition that the superconducting
order parameter $\psi$ is also uniform, i.e. that vortex phases are not present at the
respective temperature domain. This conditions are directly satisfied in type I superconductors
but in type II superconductors the temperature should be sufficiently low and
the external magnetic field should be zero. Nevertheless, the mentioned conditions for
type II superconductors may turn insufficient for the appearance of uniform superconducting
states in materials with quite high values of the spontaneous magnetization. In
such situation the uniform (Meissner) superconductivity and, hence, the coexistence of
this superconductivity with uniform ferromagnetic order may not appear even at zero
temperature. Up to now type I unconventional ferromagnetic superconductors have not
been found whereas the experimental data for the recently discovered compounds UGe$_2$,
URhGe, and ZrZn$_2$ are not enough to conclude definitely either about the lack or the
existence of uniform superconducting states at low and ultra-low temperatures.

If real materials can be modelled by the general GL free energy (1) - (9), the ground
state properties will be described by uniform states, which we shall investigate. The
problem about the availability of such states in real materials at finite temperatures is
quite subtle at the present stage of research when the experimental data are not enough.
Recently in Ref. [65] an experimental evidence was given for the coexistence of uniform
superconductivity and ferromagnetism in UGe$_2$. Thus we shall assume that uniform
phases may exist in some unconventional ferromagnetic superconductors. Moreover, we
have to emphasize that these phases appear as solutions of the GL equations corresponding
to the free energy (1) - (9). These arguments completely justify our study.

In case of a strong easy axis type of magnetic anisotropy, as is in UGe$_2$ [29], the overall
complexity of mean-field analysis of the free energy $f(\psi, M)$ can be avoided by performing
an “Ising-like” description: $M = (0, 0, M_Z)$, where $M = \pm |M|$ is the magnetization along
the “z-axis.” Because of the equivalence of the “up” and “down” physical states ($\pm M$)
the thermodynamic analysis can be performed within the “gauge” $M \geq 0$. When the
magnetic order has a continuous symmetry we can take advantage of the symmetry of the
total free energy $f(\psi, M)$ and avoid the consideration of equivalent thermodynamic
states that occur as a result of the respective symmetry breaking at the phase transition point but have no effect on thermodynamics of the system. In the isotropic system one may again choose a gauge, in which the magnetization vector has the same direction as $z$-axis ($|\mathbf{M}| = M_z = \mathcal{M}$) and this will not influence the generality of thermodynamic analysis. Here we shall prefer the alternative description within which the ferromagnetic state may appear through two equivalent up and down domains with magnetizations $\mathcal{M}$ and ($-\mathcal{M}$), respectively.

We shall make the mean-field analysis of the uniform phases and the possible phase transitions between such phases in a zero external magnetic field ($H = 0$), when the crystal anisotropy is neglected ($v_s \equiv 0$). The only exception will be the consideration in Sec. 3, where we briefly discuss the nonmagnetic superconductors ($\mathcal{M} \neq 0$).

We shall use notations in which the number of parameters is reduced. For this reason we introduce

$$b = (b_s + u_s + v_s)$$

and redefine the order parameters and the other quantities in the following way:

$$\varphi_j = b^{1/4}\psi_j = \phi_j e^{i\theta_j}, \quad M = b^{1/4}M, \quad r = \frac{a_s}{\sqrt{b}}, \quad t = \frac{a_f}{\sqrt{b_f}}, \quad w = \frac{u_s}{b}, \quad v = \frac{v_s}{b},$$

$$\gamma = \frac{\gamma_0}{b^{1/2}b_f^{1/4}}, \quad \gamma_1 = \frac{\delta}{(bb_f)^{1/2}}.$$  

Having in mind that the order parameters $\psi$ and $\mathbf{M}$ are considered uniform and using Eqs. (10) and (11), we can write the free energy density $f(\psi, M) = F(\psi, M)/V$ in the form

$$f(\psi, M) = r\phi^2 + \frac{1}{2}\phi^4 + 2\gamma\phi_1\phi_2M\sin(\theta_2 - \theta_1)$$

$$+ \gamma_1\phi^2M^2 + tM^2 + \frac{1}{2}M^4$$

$$- 2w[\phi_1^2\phi_2^2\sin^2(\theta_2 - \theta_1) + \phi_1^2\phi_3^2\sin^2(\theta_1 - \theta_3)]$$

$$+ \phi_2^2\phi_3^2\sin^2(\theta_2 - \theta_3))$$

$$- v[\phi_1^2\phi_2^2 + \phi_1^2\phi_3^2 + \phi_2^2\phi_3^2].$$

Note, that in the above expression the order parameters $\psi$ and $\mathbf{M}$ are defined per unit volume.

The equilibrium phases are obtained from the equations of state

$$\frac{\partial f(\mu_0)}{\partial \mu_\alpha} = 0,$$

where the series of symbols $\mu$ can be defined as, for example, $\mu = \{\mu_\alpha\} = (M, \phi_1, ..., \phi_3, \theta_1, ..., \theta_3); \mu_0$ denotes an equilibrium phase. The stability matrix $\tilde{F}$ of the phases $\mu_0$ is
defined by
\[ \hat{F}(\mu_0) = \{ F_{\alpha\beta}(\mu_0) \} = \frac{\partial^2 f(\mu_0)}{\partial \mu_\alpha \partial \mu_\beta}. \] (14)

An alternative treatment can be done in terms of real \( \psi'_j \) and imaginary \( \psi''_j \) parts of the complex numbers \( \psi_j = \psi'_j + i\psi''_j \). The calculation with moduli \( \phi_j \) and phase angles \( \theta_j \) of \( \psi_j \) is more simple but in cases of strongly degenerate phases some of the angles \( \theta_j \) remain unspecified. Then an alternative analysis with the help of the components \( \psi'_j \) and \( \psi''_j \) should be done.

The thermodynamic stability of the phases that are solutions of Eqs. (13) is checked with the help of the matrix (14). An additional stability analysis is done by the comparison of free energies of phases that satisfy (13) and render the stability matrix (14) positive in one and the same domain of parameters \( \{r, t, \gamma, \gamma_1, w, v\} \). This step is important because the complicated form of the free energy generates a great number of solutions of Eqs. (13) and we have to sift out the stable from metastable phases that correspond either to global or local minima of the free energy, respectively [13].

Some solutions of Eqs. (13) have a marginal stability, i.e., their stability matrix (14) is neither positively nor negatively definite. This is often a result of the degeneration of phases with broken continuous symmetry. If the reason for the lack of a clear positive definiteness of the stability matrix is precisely the mentioned degeneration of the ground state, one may reliably conclude that the respective phase is stable. If there is another reason, the analysis of the matrix (14) will be insufficient to determine the respective stability property. These cases are quite rare and occur for particular values of the parameters \( \{r, t, \gamma, \ldots\} \).

### 3 Pure superconductivity

Let us set \( M \equiv 0 \) in Eq. (12) and briefly summarize the known results [18, 11] for the “pure superconducting case” when the magnetic order cannot appear and magnetic effects do not affect the stability of the uniform (Meissner) superconducting phases. The possible phases can be classified by the structure of the complex vector order parameter \( \psi = (\psi_1, \psi_2, \psi_3) \). We shall often use the moduli vector \( (\phi_1, \phi_2, \phi_3) \) with magnitude \( \phi = (\phi_1^2 + \phi_2^2 + \phi_3^2)^{1/2} \) and the phase angles \( \theta_j \).

The normal phase \((0,0,0)\) is always a solution of the Eqs. (13). It is stable for \( r \geq 0 \), and corresponds to a free energy \( f = 0 \). Under certain conditions, six ordered phases [18, 11] occur for \( r < 0 \). Here we shall not repeat the detailed description of these phases [18, 11] but we shall briefly mention their structure.

The simplest ordered phase is of type \((\psi_1, 0, 0)\) with equivalent domains: \((0, \psi_2, 0)\) and \((0, 0, \psi_3)\). Multi-domain phases of more complex structure also occur, but we shall not always enumerate the possible domains. For example, the “two-dimensional” phases can
be fully represented by domains of type \((\psi_1, \psi_2, 0)\) but there are also other two types of
domains: \((\psi_1, 0, \psi_3)\) and \((0, \psi_2, \psi_3)\). As we consider the general case when the crystal
anisotropy is present \((v \neq 0)\), this type of phases possesses the property \(|\psi_i| = |\psi_j|\).

The two-dimensional phases are two and have different free energies. To clarify this point
let us consider, for example, the phase \((\psi_1, \psi_2, 0)\). The two complex numbers, \(\psi_1\) and
\(\psi_2\) can be represented either as two-component real vectors, or, equivalently, as rotating
vectors in the complex plane. One can easily show that Eq. (12) yields two phases: a
collinear phase, when \((\theta_2 - \theta_1) = \pi k (k = 0, \pm 1, \ldots)\), i.e. when the vectors \(\psi_1\) and \(\psi_2\) are
collinear, and another (non-collinear) phase when the same vectors are perpendicular to
each other: \((\theta_2 - \theta_1) = \pi (k + 1/2)\). Having in mind that \(|\phi_1| = |\phi_2| = \phi/\sqrt{2}\), the domain
\((\psi_1, \psi_2, 0)\) of the collinear phase is given by \((\pm 1, 1, 0)\phi/\sqrt{2}\) whereas the same domain for
the non-collinear phase will be \((\pm i, 1, 0)\phi/\sqrt{2}\). The two domains of these phases have
similar representations.

In addition to the mentioned three ordered phases, three other ordered phases exist. For
these phases all three components \(\psi_j\) have nonzero equilibrium values. Two of them
have equal to one another moduli \(\phi_j\), i.e., \(\phi_1 = \phi_2 = \phi_3\). The third phase is of the
type \(\phi_1 = \phi_2 \neq \phi_3\) and is unstable so it cannot occur in real systems. The two three-
dimensional phases with equal moduli of the order parameter components have different
phase angles and, hence, different structure. The difference between any couple of angles
\(\theta_j\) is given by \(\pm \pi/3\) or \(\pm 2\pi/3\). The characteristic vectors of this phase can be of the form
\((e^{i\pi/3}, e^{-i\pi/3}, 1)\phi/\sqrt{3}\) and \((e^{2i\pi/3}, e^{-2i\pi/3}, 1)\phi/\sqrt{3}\). The second stable three dimensional
phase is “real”, i.e. the components \(\psi_j\) lie on the real axis; \((\theta_j - \theta_j) = \pi k\) for any couple
of angles \(\theta_j\) and the characteristic vectors are \((\pm 1, \pm 1, 1)\phi/\sqrt{3}\). The stability properties
of these five stable ordered phases were presented in details in Refs. [18][11].

When the crystal anisotropy is not present \((v = 0)\) the picture changes. The increase
of the level of degeneracy of the ordered states leads to an instability of some phases
and to a lack of some noncollinear phases. Both two- and three-dimensional real phases,
where \((\theta_j - \theta_j) = \pi k\), are no more constrained by the condition \(\phi_1 = \phi_j\) but rather have
the freedom of a variation of the moduli \(\phi_j\) under the condition \(\phi^2 = -r > 0\). The two-dimensional noncollinear phase exists but has a marginal stability [11]. All other
noncollinear phases even in the presence of a crystal anisotropy \((v \neq 0)\) either vanish
or are unstable; for details, see Ref. [11]. This discussion demonstrates that the crystal
anisotropy stabilizes the ordering along the main crystallographic directions, lowers the
level of degeneracy of the ordered state related with the spontaneous breaking of the
continuous symmetry and favors the appearance of noncollinear phases.

The crystal field effects related to the unconventional superconducting order were estab-
lished for the first time in Ref. [18]. In our consideration of unconventional ferromag-
netic superconductors in Sec. 4–7 we shall take advantage of these effects of the crystal
anisotropy. In both cases \(v = 0\) and \(v \neq 0\) the matrix (14) indicates an instability of
three-dimensional phases (all \(\phi_j \neq 0\)) with an arbitrary ratios \(\phi_1/\phi_j\). As already
mentioned, for \(v \neq 0\) the phases of type \(\phi_1 = \phi_2 \neq \phi_3\) are also unstable whereas for \(v = 0\),
even the phase $\phi_1 = \phi_2 = \phi_3 > 0$ is unstable.

4 M-triggered superconductivity

We shall consider the Walker-Samokhin model [47] when only the $M\phi_1\phi_2$-coupling between the order parameters $\psi$ and $M$ is taken into account ($\gamma > 0$, $\gamma_1 = 0$) and the anisotropies ($w = v = 0$) are ignored. The uniform phases and the phase diagram in this case were investigated in Refs. [44, 45, 49].

Table 1: Phases and their existence and stability properties $[\theta = (\theta_2 - \theta_1), k = 0, \pm 1, \ldots]$.

|     | $\phi_j = M = 0$ | always | $t > 0, r > 0$ |
|-----|-----------------|--------|----------------|
| FM  | $\phi_j = 0, M^2 = -t$ | $t < 0$ | $r > 0, r > r_c(t)$ |
| SC1 | $\phi_1 = M = 0, \phi^2 = -r$ | $r < 0$ | unstable |
| SC2 | $\phi^2 = -r, \theta = \pi k, M = 0$ | $r < 0$ | $(t > 0)^*$ |
| SC3 | $\phi_1 = \phi_2 = M = 0, \phi^2 = -r$ | $r < 0$ | $r < 0, t > 0$ |
| CO1 | $\phi_1 = \phi_2 = 0, \phi^2 = -r$ | $r < 0$ | $r < 0$ |
|     | $\phi^2 = -r, M^2 = -t$ | $t < 0$ | $t < 0$ |
| CO2 | $\phi_1 = 0, \phi^2 = -r$ | $r < 0$ | unstable |
|     | $\theta = \pi k, M^2 = -t$ | $t < 0$ | $t < 0$ |
| FS  | $2\phi^2 = 2\phi^2_3 = \phi^2 = -r + \gamma M$ | $\gamma M > r$ | $3M^2 > (-t + \gamma^2/2)$ |
|     | $\phi_3 = 0, \theta = 2\pi(k - 1/4)$ | $M > 0$ | $\gamma r = (\gamma^2 - 2t)M - 2M^3$ |
| FS* | $2\phi^2 = 2\phi^2_3 = \phi^2 = -(r + \gamma M)$ | $-\gamma M > r$ | $3M^2 > (-t + \gamma^2/2)$ |
|     | $\phi_3 = 0, \theta = 2\pi(k + 1/4)$ | $M < 0$ | $\gamma r = (2t - \gamma^2)M + 2M^3$ |

Here we summarize the main results in order to make a clear comparison with the results presented in Sec. 5 and Sec. 6. Our main aim is the description of a trigger effect which consists of the appearance of a "compelled superconductivity" caused by the presence of ferromagnetic order (here, this is a standard uniform ferromagnetic order); see also Refs. [44, 45, 49] where this effect has been already established and briefly discussed. As mentioned in the Introduction, a similar trigger effect is known in the physics of improper ferroelectrics. We shall set $\theta_3 \equiv 0$ and use the notation $\theta \equiv \Delta \theta = (\theta_2 - \theta_1)$.

4.1 Phases

The possible (stable, metastable and unstable) phases are given in Table 1 together with the respective existence and stability conditions. The normal or disordered phase, denoted in Table 1 by $N$, always exists (for all temperatures $T \geq 0$) and is stable for $t > 0, r > 0$. The superconducting phase denoted in Table 1 by SC1 is unstable. The same is valid for the phase of
coexistence of ferromagnetism and superconductivity denoted in Table 1 by CO2. The N–phase, the ferromagnetic phase (FM), the superconducting phases (SC1–3) and two of the phases of coexistence (CO1–3) are generic phases because they appear also in the decoupled case (γ ≡ 0). When the $M\phi_1\phi_2$–coupling is not present, the phases SC1–3 are identical and represented by the order parameter $\phi$ with components $\phi_j$ that participate on equal footing. The asterisk attached to the stability condition of the second superconductivity phase (SC2) indicates that our analysis is insufficient to determine whether this phase corresponds to a minimum of the free energy.

It will be shown that the phase SC2, two other purely superconducting phases and the coexistence phase CO1, have no chance to become stable for $\gamma \neq 0$. This is so, because the phase of coexistence of superconductivity and ferromagnetism (FS in Table 1) that does not occur for $\gamma = 0$ is stable and has a lower free energy in their domain of stability. A second domain ($M < 0$) of the FS phase is denoted in Table 1 by FS*. Here we shall describe only the first domain FS. The domain FS* is considered in the same way.

The cubic equation for magnetization of FS-phase (see Table 1) is shown in Fig. 1 for $\gamma = 1.2$ and $t = -0.2$. For any $\gamma > 0$ and $t$, the stable FS thermodynamic states are given by $r(M) < r_m = r(M_m)$ for $M > M_m > 0$, where $M_m$ corresponds to the maximum of the function $r(M)$. The dependence of $M_m(t)$ and $M_0(t) = (-t + \gamma^2/2)^{1/2} = \sqrt{3}M_m(t)$ on $t$ is drawn in Fig. 2 for $\gamma = 1.2$. Functions $r_m(t) = 4M_m^3(t)/\gamma$ for $t < \gamma^2/2$ (depicted by the line of circles in Fig. 3) and

$$r_e(t) = \gamma|t|^{1/2},$$

for $t < 0$ define the borderlines of stability and existence of FS.
4.2 Phase diagram

We have outlined the domain in the \((t, r)\) plane where the FS phase exists and is a minimum of the free energy. For \(r < 0\) the cubic equation for \(M\) (see Table 1) and the existence and stability conditions are satisfied for any \(M \geq 0\) provided \(t \geq \gamma^2\). For \(t < \gamma^2\) the condition \(M \geq M_0\) have to be fulfilled, here the value \(M_0 = (-t + \gamma^2/2)^{1/2}\) of \(M\) is obtained from \(r(M_0) = 0\). Thus for \(r = 0\) the N-phase is stable for \(t \geq \gamma^2/2\), and FS is stable for \(t \leq \gamma^2/2\). For \(r > 0\), the requirement for the stability of FS leads to the inequalities

\[
max \left( \frac{r}{\gamma}, \frac{M_m}{\gamma} \right) < M < M_0, \tag{16}
\]

where \(M_m = (M_0/\sqrt{3})\) and \(M_0\) should be the positive solution of the cubic equation of state from Table 1; \(M_m > 0\) gives a maximum of the function \(r(M)\); see also Figs. 1 and 2.

The further analysis defines the existence and stability domain of FS below the line AB denoted by circles (see Fig. 3). In Fig. 3 the curve of circles starts from the point A with coordinates \((\gamma^2/2, 0)\) and touches two other (solid and dotted) curves at the point B with coordinates \((t_B = -\gamma^2/4, r_B = \gamma^2/2)\). Line of circles represents the function \(r(M_m) \equiv r_m(t)\) where

\[
r_m(t) = \frac{4}{3\sqrt{3} \gamma} \left( \frac{\gamma^2}{2} - t \right)^{3/2}. \tag{17}
\]

Dotted line represents \(r_e(t)\) defined by Eq. (15). The inequality \(r < r_m(t)\) is a condition for the stability of FS, whereas the inequality \(r \leq r_e(t)\) for \((-t) \geq \gamma^2/4\) is a condition for the existence of FS as a solution of the respective equation of state. This existence condition for FS is obtained from \(\gamma M > r\) (see Table 1).
Figure 3: The phase diagram in the plane \((t, r)\) with two tricritical points (A and B) and a triple point \(C\); \(\gamma = 1.2\). The parameters \(r \sim [T - T_s(P)]\) and \(t \sim [T - T_f(P)]\) are defined by Eq. (11). The domains of existence and stability of the phases N, FM and FS are shown. The line of circles represents the function \(r_m(t)\) given by Eq. (17). The dotted line represents the function \(r_e(t)\) given by Eq. (15). On the left of point \(B\), the same dotted curve corresponds to a FM-FS phase transition of second order. The equilibrium lines of N-FS and FM-FS phase transitions of first order are given by the solid lines \(AC\) and \(CB\), respectively.
In the region on the left of the point B in Fig. 3, the FS phase satisfies the existence condition $\gamma M > r$ only below the dotted line. In the domain confined between the lines of circles and the dotted line on the left of the point B the stability condition for FS is satisfied but the existence condition is broken. The inequality $r \geq r_e(t)$ is the stability condition of FM for $0 \leq (-t) \leq \gamma^2/4$. For $(-t) > \gamma^2/4$ the FM phase is stable for all $r \geq r_e(t)$.

In the region confined by the line of circles AB, the dotted line for $0 < (-t) < \gamma^2/4$, and the $t-$axis, the phases N, FS and FM have an overlap of stability domains. The same is valid for FS, the SC phases and CO1 in the third quadrant of the plane $(t, r)$. The comparison of the respective free energies for $r < 0$ shows that the stable phase is FS whereas the other phases are metastable within their domains of stability.

The part of the $t$-axis given by $r = 0$ and $t > \gamma^2/2$ is a phase transition line of second order which describes the N-FS transition. The same transition for $0 < t < \gamma^2/2$ is represented by the solid line AC which is the equilibrium transition line of a first order phase transition. The equilibrium transition curve is given by the function

$$r_{eq}(t) = \frac{1}{4} \left[ 3\gamma - (\gamma^2 + 16t)^{1/2} \right] M_{eq}(t). \tag{18}$$

Here

$$M_{eq}(t) = \frac{1}{2\sqrt{2}} \left[ \gamma^2 - 8t + \gamma (\gamma^2 + 16t)^{1/2} \right]^{1/2} \tag{19}$$

is the equilibrium jump of the magnetization. The order of the N-FS transition changes at the tricritical point A.

The domain above the solid line AC and below the line of circles for $t > 0$ is the region of a possible overheating of FS. The domain of overcooling of the N-phase is confined by the solid line AC and the axes ($t > 0, r > 0$). At the triple point C with coordinates $[0, r_{eq(0)} = \gamma^2/4]$ the phases N, FM, and FS coexist. For $t < 0$ the straight line

$$r_{eq}^*(t) = \frac{\gamma^2}{4} + |t|, \quad t_B < t < 0, \tag{20}$$

describes the extension of the equilibrium phase transition line of the N-FS first order transition to negative values of $t$. For $t < t_B$ the equilibrium phase transition FM-FS is of second order and is given by the dotted line on the left of the point B which is the second tricritical point in this phase diagram. Along the first order transition line $r_{eq}^*(t)$ given by Eq. (20) the equilibrium value of $M$ is $M_{eq} = \gamma/2$, which implies an equilibrium order parameter jump at the FM-FS transition equal to $(\gamma/2 - \sqrt{|t|})$. On the dotted line of the second order FM-FS transition the equilibrium value of $M$ is equal to that of the FM phase ($M_{eq} = \sqrt{|t|}$). The FM phase does not exist below $T_s$ and this is a shortcoming of the model (12) with $\gamma_1 = 0$.

The equilibrium FM-FS and N-FS phase transition lines in Fig. 3 can be expressed by the respective equilibrium phase transition temperatures $T_{eq}$ defined by the equations $r_e = r(T_{eq})$, $r_{eq} = r(T_{eq})$, $r_{eq}^* = r(T_{eq})$, and with the help of the relation $M_{eq} = M(T_{eq})$. This limits the possible variations of parameters of the theory. For example, the critical temperature ($T_{eq} \equiv T_c$) of the FM-FS second order transition $(\gamma^2/4 < -t)$ is obtained in the form $T_c = (T_s + 4\pi J \mathcal{M}/\alpha_s)$, or, using $\mathcal{M} = (-a_f/b_f)^{1/2}$,

$$T_c = T_s - \frac{T_s^*}{2} + \left[ \left( \frac{T_s^*}{2} \right)^2 + T^*(T_f - T_s) \right]^{1/2}. \tag{21}$$
Here $T_f > T_s$, and $T^* = (4\pi J)^2 \alpha_f / \alpha_s^2 b_f$ is a characteristic temperature of the model with $\gamma_1 = w = v = 0$. A discussion of Eq. (21) is given in Sec. 5.3.

The investigation of the conditions for the validity of Eq. (21) leads to the conclusion that the FM-FS continuous phase transition (at $\gamma^2 < -t$) will be possible only if the following condition is satisfied:

$$T_f - T_s > (\varsigma + \sqrt{\varsigma}) T^*,$$

where $\varsigma = b_f \alpha_s^2 / 4 b_s \alpha_f^2$. Therefore, the second order FM-FS transition should disappear for a sufficiently large $\gamma$-coupling. Such a condition does not exist for the first order transitions FM-FS and N-FS.

The inclusion of the gradient term (4) in the free energy should lead to a depression of the equilibrium transition temperature. As the magnetization increases with the decrease of the temperature, the vortex state should occur at temperatures which are lower than the equilibrium temperature $T_{eq}$ of the Meissner state. For example, the critical temperature ($\tilde{T}_c$) corresponding to the vortex phase of FS-type has been evaluated to be lower than the critical temperature (21): $T_c - \tilde{T}_c = 4\pi \mu B \mathcal{M} / \alpha_s$, where $\mu_B = |e| |\hbar| / 2 mc$ is the Bohr magneton. For $J \gg \mu_B$, we have $T_c \approx \tilde{T}_c$.

For $r > 0$, namely, for temperatures $T > T_s$ the superconductivity is triggered by the magnetic order through the $\gamma$-coupling. The superconducting phase for $T > T_s$ is entirely in the $(t, r)$ domain of the ferromagnetic phase. Therefore, the uniform superconducting phase can occur for $T > T_s$ only through a coexistence with the ferromagnetic order.

The properties of the magnetic susceptibility and the specific heat near the phase transition lines shown in Fig. 3 have been investigated in Refs. [49, 46] and here we shall not dwell on these topics. Note that the results for the thermodynamic quantities should be extended to include the physical effects considered in the next parts of this review. Such a consideration requires a numerical analysis.

In the next Sections we shall focus on the temperature range $T > T_s$ which seems to be of main practical interest. We shall not dwell on the superconductivity in the fourth quadrant ($t > 0, r < 0$) of the $(t, r)$ diagram where pure superconducting phases can occur for systems with $T_s > T_f$, but this is not the case for UGe$_2$, URhGe and ZrZn$_2$. Also we shall not discuss the possible metastable phases in the third quadrant ($t < 0, r < 0$) of the $(t, r)$ diagram.

### 4.3 Note about a simplified theory

The analysis in this Section can be done following an approximate scheme known from the theory of improper ferroelectrics; see, e.g., Ref. [55]. In this approximation the order parameter $M$ is considered small enough which makes possible to ignore $M^4$-term in the free energy. Then one easily obtains from the data for FS presented in Table 1 or by a direct calculation of the respective reduced free energy that the order parameters $\phi$ and $M$ of FS–phase are described by the simple equalities $r = (\gamma M - \phi^2)$ and $M = (\gamma / 2t) \phi^2$. For ferroelectrics working with oversimplified free energy gives a substantial departure of theory from experiment [55]. The same approximation has been recently applied to ferromagnetic Bose-Einstein condensates [66, 67].

For ferromagnetic superconductors the domain of reliability of this approximation could be
the close vicinity of the ferromagnetic phase transition, i.e., for temperatures near the critical temperature $T_f$. This discussion can be worthwhile if only the primary order parameter also exists in the same narrow temperature domain ($\phi > 0$). Therefore, the application of the simplified scheme can be useful in systems, where $T_s \geq T_f$.

For $T_s < T_f$, the analysis can be simplified if we suppose a relatively small value of the modulus $\phi$ of the superconducting order parameter. This approximation should be valid in some narrow temperature domain near the line of second order phase transition from FM to FS.

5 Effect of symmetry conserving coupling

Here we shall include in our consideration both linear and quadratic couplings of magnetization to the superconducting order parameter which means that both parameters $\gamma$ and $\gamma_1$ in free energy (12) are different from zero. In this way we shall investigate the effect of the symmetry conserving $\gamma_1$-term in the free energy on the thermodynamics of the system. When $\gamma$ is equal to zero but $\gamma_1 \neq 0$ the analysis is easy and the results are known from the theory of bicritical and tetracritical points \cite{13, 53, 68, 69}. For the problem of coexistence of conventional superconductivity and ferromagnetic order the analysis ($\gamma = 0, \gamma_1 \neq 0$) was made in Ref. \cite{32}.

At this stage we shall not take into account any anisotropy effects because we do not want to obscure the influence of quadratic interaction by considering too many parameters. For $\gamma, \gamma_1 \neq 0$ and $w = 0, v = 0$ the results again can be presented in an analytical form, only a small part of phase diagram should be calculated numerically.

5.1 Phases

The calculations show that for temperatures $T > T_s$, i.e., for $r > 0$, we have again three stable phases. Two of them are quite simple: the normal (N-) phase with existence and stability domains shown in Table 1, and the FM phase with the existence condition $t < 0$ as shown in Table 1, and a stability domain defined by the inequality $r_e^{(1)} \leq r$. Here

$$r_e^{(1)} = \gamma_1 t + \gamma \sqrt{-t},$$  \hspace{1cm} (23)

and one can compare it with the respective expression (15) for $\gamma_1 = 0$. In this paragraph we shall retain the same notations as in Sec. 4, but with a superscript (1) in order to distinguish them from the case $\gamma_1 = 0$ The third stable phase for $r > 0$ is a more complex variant of the mixed phase FS and its domain $\text{FS}^*$, discussed in Sec. 4. The symmetry of the FS phase coincides with that found in \cite{17}.

We have to mention that for $r < 0$ there are five pure superconducting ($M = 0, \phi > 0$) phases. Two of them, $(\phi_1 > 0, \phi_2 = \phi_3 = 0)$ and $(\phi_1 = 0, \phi_2 > 0, \phi_3 > 0)$ are unstable. Two other phases, $(\phi_1 > 0, \phi_2 > 0, \phi_3 = 0, \theta_2 = \theta_1 + \pi k)$ and $(\phi_1 > 0, \phi_2 > 0, \phi_3 > 0, \theta_2 = \theta_1 + \pi k, \theta_3 - \text{arbitrary}; k = 0, \pm 1, \ldots)$ show a marginal stability for $t > \gamma_1 r$.

Only one of the five pure superconducting phases, the phase $\text{SC3}$, given in Table 1, is stable. In case of $\gamma_1 \neq 0$ the values of $\phi_j$ and the existence domain of $\text{SC3}$ are the same as shown
in Table 1 for $\gamma_1 = 0$ but the stability domain is different and is given by $t > \gamma_1 r$. When the anisotropy effects are taken into account the phases exhibiting marginal stability within the present approximation may become stable. Besides, three other mixed phases ($M \neq 0, \phi > 0$) exist for $r < 0$ but one of them is metastable (for $\gamma_1^2 > 1, t < \gamma_1 r$, and $r < \gamma_1 t$) and the other two are absolutely unstable.

Here the thermodynamic behavior for $r < 0$ is much more abundant in phases than for improper ferroelectrics with two component primary order parameter [23]. However, at this stage of experimental needs about the properties of unconventional ferromagnetic superconductors the investigation of the phases for temperatures $T < T_s$ is not of primary interest and for this reason we shall focus our attention on the temperature domain $r > 0$.

The FS phase for $\gamma_1 \neq 0$ is described by the following equations:

$$\phi_1 = \phi_2 = \frac{\phi}{\sqrt{2}}, \quad \phi_3 = 0,$$

$$\phi^2 = (\pm \gamma M - r - \gamma_1 M^2),$$

$$(1 - \gamma_1^2)M^2 \pm 3\frac{1}{2} \gamma_1 M^2 + \left(t - \frac{\gamma_1^2}{2} - \gamma_1 r\right) M \pm \frac{\gamma r}{2} = 0,$$

and

$$(\theta_2 - \theta_1) = \pm \frac{\pi}{2} + 2\pi k,$$

($k = 0, \pm 1, \ldots$). The upper sign in Eqs. (24) - (27) corresponds to the FS domain where $\sin(\theta_2 - \theta_1) = -1$ and the lower sign corresponds to the FS* domain with $\sin(\theta_2 - \theta_1) = 1$. This is a generalization of the two-domain FS phase discussed in Sec. 3. The analysis of the stability matrix (14) for these phase domains shows that FS is stable for $M > 0$ and FS* is stable for $M < 0$, just like our result in Sec. 4. As these domains belong to the same phase, namely, have the same free energy and are thermodynamically equivalent, we shall consider one of them, for example, FS.

### 5.2 Phase stability and phase diagram

In order to outline the $(t, r)$ phase diagram we shall use the information given above for the other two phases which have their own domains of stability in the $(t, r)$ plane: N and FM. The FS stability conditions when $\gamma_1 \neq 0$ become

$$2\gamma M - r - \gamma_1 M^2 \geq 0,$$

$$\gamma M \geq 0,$$

$$3(1 - \gamma_1^2)M^2 + 3\gamma_1 M + t - \gamma_1 r - \gamma^2/2 \geq 0,$$

and we prefer to treat Eqs. (28) - (30) together with the existence condition $\phi^2 \geq 0$, with $\phi$ given by Eq. (25), with the help of the picture shown in Fig. 4.
The most direct approach to analyze the existence and stability of FS phase is to express \( r \) as a function of \((M, t)\) from the equation of state (26),

\[
 r_{\text{eq}}^{(1)}(t) = \frac{M_{\text{eq}}}{(\gamma_1 M_{\text{eq}} - \gamma/2)} \left( 1 - \gamma_1^2 M_{\text{eq}}^2 + \frac{3}{2} \gamma \gamma_1 M_{\text{eq}} + (t - \frac{\gamma^2}{2}) \right),
\]

and to substitute the above expression in the existence and stability conditions of FS-phase. It is obvious that there is a special value of \( M \)

\[
 M_{S1} = \frac{\gamma}{2 \gamma_1}
\]

that is a solution of Eq. (26) for any value of \( r \) and \( t \)

\[
 t_{S1} = -\frac{\gamma^2}{4 \gamma_1^2},
\]

for which this procedure cannot be applied and should be considered separately. Note, that \( M_{S1} \) is given by the respective horizontal dashed line in Fig. 4. The analysis shows that in the interval \( t_{B}^{(1)} < t < \gamma^2/2 \) the phase transition is again of first order; here

\[
 t_{B}^{(1)} = -\frac{\gamma^2}{4(1 + \gamma_1)^2}.
\]

To find the equilibrium magnetization of first order phase transition, depicted by the thick line \( ACB \) in Fig. 4 we need the expression for equilibrium free energy of FS-phase. It is obtained from Eq. (12) by setting \((w = 0, v = 0)\) and substituting \( r, \phi_i \) as given by Eqs. (24), (25) and (31). The result is

\[
 f_{FS}^{(1)} = -\frac{M^2}{2(M \gamma_1 - \gamma/2)^2} \times \left( (1 - \gamma_1^2) M^4 + \gamma \gamma_1 M^3 \right)
 + 2[t(1 - \gamma_1^2) - \frac{\gamma^2}{8}] M^2 - 2 \gamma \gamma_1 t M + t(t - \frac{\gamma^2}{2}) \right),
\]

where \( M \equiv M_{\text{eq}} \).

For the phase transition from N to FS phase \((0 < t < \gamma^2/2)\), \( M_{\text{eq}} \) is found by setting the FS free energy from the above expression equal to zero, as we have by convention that the free energy of the normal phase is zero. The value of \( M_{\text{eq}}^{(1)} \) for positive \( t \) is obtained numerically and is illustrated by thick black curve \( AC \) in Fig. 4. When \( t_{B}^{(1)} \leq t < 0 \) the transition is between FM and FS phases and we obtain \( M_{\text{eq}}^{(1)} \) from the equation \( f_{FS} = f_{FM} = (-t^2/2) \), where \( f_{FM} \) is the free energy of FM phase. The equilibrium magnetization in the above \( t \)-interval is given by the formula

\[
 M_{eq}^{(1)*} = \frac{\gamma}{2(1 + \gamma_1)},
\]

and is drawn by thick line \( CB \) in Fig. 4. The existence and stability analysis shows that for \( r > 0 \) the equilibrium magnetization of the first order phase transition should satisfy the condition \( M_m^{(1)} < M_{eq}^{(1)} < M_0^{(1)} \).
Figure 4: The dependence $M(t)$ as an illustration of stability analysis for $\gamma = 1.2, \gamma_1 = 0.8$ and $w = 0$. The parameters of the theory $(r, t, \gamma, \gamma_1, w, \ldots)$ are defined by Eq. (11). The horizontal dashed lines represent the quantities $M_{S1}$ given by Eq. (32) and $M_{S2} = 2M_{S1}$. The line of circles $AS_1S_2$ describes the positive solution of Eq. (31). The thick line $AC$ gives the equilibrium magnetization for $t > 0$. The thick line $BC$ represents the equilibrium magnetization for $t < 0$ as given by Eq. (35). The dotted curve is the smaller positive solution of the stability condition (30). The thin solid line $BS_1S_2$ is the magnetization $M = \sqrt{-t}$. The arrow indicates the triple point $C$. A and B are tricritical points of phase transition. The point $S_1$ corresponds to the maximum of the curve (23) for $t < 0$, and the point $S_2$ corresponds to $r_e^{(1)}(t) = 0$ in Eq. (23).

By $M_0^{(1)}$ we denote the positive solution of $r^{(1)}(M_{eq}) = 0$ and its $t$-dependence is drawn in Fig. 4 by the curve with circles. $M_m^{(1)}$ is the smaller positive root of stability condition (30) and also gives the maximum of the function $r_{eq}^{(1)}(M)$; see Eq. (31). The function $M_m^{(1)}$ is depicted by the dotted curve $AB$ in Fig. 4. When $t_{S1} < t < t_B^{(1)}$ the existence and stability conditions are fulfilled if $\sqrt{-t} < M < M_{S1}$, where $\sqrt{-t}$ is the magnetization of ferromagnetic phase and is drawn by a thin black line on the left of point B in Fig. (4). Here we have two possibilities: $r > 0$ for $\sqrt{-t} < M < M_0^{(1)}$ and $r < 0$ for $M_0^{(1)} < M < M_{S1}$. To the left of $t_{S1}$ and $t > t_{S2}$, where

$$t_{S2} = -\left(\frac{\gamma}{\gamma_1}\right)^2. \quad (37)$$

the FS phase is stable and exists for $M_{S1} < M < \sqrt{-t}$. Here $r$ will be positive when $M_0^{(1)} < M < \sqrt{-t}$ and $r < 0$ for $M_0^{(1)} > M > M_{S1}$. When $t < t_{S2}$, $M < \sqrt{-t}$ and $r$ is always negative.

On the basis of the existence and stability analysis we draw in Fig. 5 the $(t, r)$-phase diagram for concrete values of $\gamma$ and $\gamma_1$. As we have mentioned above the order of phase transitions is the same as for $\gamma_1 = 0$, see Fig. 3, Sec. 4. The phase transition between the normal and FS
Figure 5: The phase diagram in the \((t, r)\) plane for \(\gamma = 1.2\), \(\gamma_1 = 0.8\) and \(w = 0\). The parameters of the theory \((r, t, \gamma, \gamma_1, w, \ldots)\) are defined by Eq. (11). The domains of stability of the phases N, FM and FS are indicated. A and B are tricritical points of phase transitions separating the dashed lines (on the left of point B and on the right of point A) of second order phase transitions from the solid line ABC of first order phase transitions. The FS phase is stable in the whole domain of the \((t, r)\) below the solid and dashed lines. The vertical dashed line coinciding with the r-axis above the triple point C indicates the N-FM phase transition of second order.
phases is of first order and goes along the equilibrium line $AC$ in the interval $\left( t_A = \gamma^2 / 2 \text{ and } t_C = 0 \right)$. The function $r_{eq}^{(1)}(t)$ is given by Eq. (31) with $M_{eq}^{(1)}$ from Fig. 4.

N, FM, and FS phases coexist at the triple point $C$ with coordinates $t = 0$, and $r_{eq}^{(1)} = \gamma^2 / 4(\gamma_1 + 1)$. On the left of $C$ for $t_B^{(1)} < t < 0$ the phase transition line of first order $r_{eq}^{(1)}(t)$ is found by substituting in Eq. (31) the respective equilibrium magnetization, given by Eq. (35). In result we obtain

$$r_{eq}^{(1)}(t) = \frac{\gamma^2}{4(1 + \gamma_1)} - t. \quad (38)$$

This function is illustrated by the line $BC$ in Fig. 5 that terminates at the tricritical point $B$ with coordinates $t_B^{(1)}$ from Eq. (34), and

$$r_B^{(1)} = \frac{\gamma^2(2 + \gamma_1)}{4(1 + \gamma_1)^2}. \quad (39)$$

To the left of the tricritical point $B$ the second order phase transition curve is given by the relation (23). Here the magnetization is $M = \sqrt{-t}$ and the superconducting order parameter is equal to zero ($\phi = 0$). This line intersects $t$-axis at $t_{S2}$ and is well defined also for $r < 0$. The function $r_e^{(1)}(t)$ has a maximum at the point $(t_{S1}, \gamma^2 / 4\gamma_1)$; here $M = M_{S1}$. When this point is approached the second derivative of the free energy with respect to $M$ tends to infinity. The result for the curves $r_{eq}^{(1)}(t)$ of equilibrium phase transitions (N-FS and FM-FS) can be used to define the respective equilibrium phase transition temperatures $T_{FS}$.

We shall not discuss the region, $t > 0$, $r < 0$, because we have supposed from the very beginning that the transition temperature for the ferromagnetic ordering $T_f$ is higher then the superconducting transition temperature $T_s$, as is for the known unconventional ferromagnetic superconductors. But this case may become of substantial interest when, as one may expect, materials with $T_f < T_s$ may be discovered experimentally.

### 5.3 Discussion

The shape of the equilibrium phase transition lines corresponding to the phase transitions N-SC, N-FS, and FM-FS is similar to that of the more simple case $\gamma_1 = 0$ and we shall not dwell on the variation of the size of the phase domains with the variations of the parameter $\gamma_1$ from zero to values constrained by the condition $\gamma_1^2 < 1$. We shall draw the attention to the important qualitative difference between the equilibrium phase transition lines shown in Figs. 3 and 5. The second order phase transition line $r_e(t)$, shown by the dotted line on the left of point $B$ in Fig. 3, tends to large positive values of $r$ for large negative values of $t$ and remains in the second quadrant ($t < 0, r > 0$) of the plane ($t, r$) while the respective second order phase transition line $r_e^{(1)}(t)$ in Fig. 5 crosses the $t$-axis at the point $t_{S2}$ and is located in the third quadrant ($t < 0, r < 0$) for all possible values $t < t_{S2}$. This means that the ground state (at 0 K) of systems with $\gamma_1 = 0$ will be always the FS phase while two types of ground states, FM and FS, can exist for systems with $0 < \gamma_1^2 < 1$. The latter seems more realistic when we compare theory and experiment, especially, in ferromagnetic compounds like UGe$_2$, URhGe, and ZrZn$_2$ where the presence of FM phase is observed at very low temperatures and relatively low pressure $P$.

The final aim of the phase diagram investigation is the outline of the $(T, P)$ diagram. Important conclusions about the shape of the $(T, P)$ diagram can be made from the form of the $(t, r)$
diagram without an additional information about the values of the relevant material parameters \((a_s, a_f, \ldots)\) and their dependence on the pressure \(P\). One should know also the characteristic temperature \(T_s\), which has a lower value than the experimentally observed \((T_P \sim 1K)\) to the coexistence FS–phase. A supposition about the dependence of the parameters \(a_s\) and \(a_f\) on the pressure \(P\) was made in Ref. [17]. Our results for \(T_f > T_s\) show that the phase transition temperature \(T_{FS}\) varies with the variation of the system parameters \((a_s, a_f, \ldots)\) from values which are higher than the characteristic temperature \(T_s\) down to zero temperature. This is seen from Fig. 5.

In systems where a pure superconducting phase is not observed for temperatures \(T \sim T_f\) or \(T \sim T_{FS}\), we can set \(T_s \sim 0\) in Eq. (21). Neglecting \(T_s\) in Eq. (21) and assuming that \((T^*/T_f) \ll 1\) we obtain that \(T_c \equiv T_{FS} \sim (T^* T_f)^{1/2}\). Note that the first \((T^*/T_f)^{1/2}\)-correction to this result has a negative sign which means that a suitable dependence of the characteristic temperature \(T^*\) on the pressure \(P\) may be used in attempts to describe the experimental shape of the FM-FS phase transition line in the \((T, P)\) diagrams of UGe\(_2\) and ZrZn\(_2\); see, for example, Fig. 2 in Ref. [26], Fig. 3 in Ref. [27], Fig. 4 in Ref. [30]. The experimental phase diagrams indicate that \(T_f(P)\) is a smooth monotonically decreasing function of the pressure \(P\) and \(T_f(P)\) tends to zero when the pressure \(P\) exceeds some critical value \(P_c \sim 1\) GPa. Postulating the respective experimental shape of the function \(T_f(P)\) one may try to give a theoretical prediction for the shape of the curve \(T_{FS}\).

The lack of experimental data about important parameters of the theory forces us to make some suppositions about the behavior of the function \(T^*(P)\). The phase transition temperature \(T_{FS}\) will qualitatively follow the shape of \(T_f(P)\) provided the dependence \(T^*(P)\) is very smooth. This is in accord with the experimental shapes of these curves near the critical pressure \(P_c\) where both \(T_f\) and \(T_{FS}\) are very small. The substantial difference between \(T_f\) and \(T_{FS}\) at lower pressure \((P < P_c)\) can be explained with the negative sign of the correction term to the leading dependence \(T_{FS}(P) \sim [T^*(P)T_f(P)]^{1/2}\) mentioned above and a convenient supposition for the form of the function \(T^*(P)\).

Eq. (21) presents a rather simplified theoretical result for \(T_C = T_{FS}\) because the effect of \(M^2|\psi|^2\) coupling is not taken into account. But following the same ideas, used in our discussion of Eq. (21), a more reliable theoretical prediction of the shape of FM-FS phase transition line can be given on the basis of Eq. (23). With the help of the experimentally found shape of \(T_f(P)\) and the definition of the parameters \(r\) and \(t\) by Eq. (11) we can substitute \(T = T_{FS}(P)\) in Eq. (23). In doing this we have applied the following approximations, namely, that \(T_s \sim 0\) for any pressure \(P\), \(T_{FS}(P_c) \sim T_f(P_c) \sim 0\) and for substantially lower pressure \((P < P_c)\), \(T_f(P) \gg T_{FS}(P)\). Then near the critical pressure \(P_c\), we easily obtain the transition temperature \(T_{FS} \sim 0\), as should be. For substantially lower values of the pressure there exists an experimental requirement \((T_{FS} - T_s) \ll (T_f - T_{FS})\). Using the latter we establish the approximate formula

\[
(T_f - T_{FS}) = \gamma^2 b_f^{1/2}/\gamma_1^2 \alpha_f.
\]

The same formula for \((T_f - T_{FS})\) can be obtained from the parameter \(t_{S2}(T_{FS})\) given by Eq. (37). The pressure dependence of the parameters included in this formula defines two qualitatively different types of behavior of \(T_{FS}(P)\) at relatively low pressures \((P \ll P_c)\): (a) \(T_{FS}(P) \sim 0\) below some (second) critical value of the pressure \((P'_c < P_c)\), and (b) finite \(T_{FS}(P)\) up to \(P \sim 0\). Therefore, we can estimate the value of the pressure \(P'_c < P_c\) in UGe\(_2\), where \(T_{FS}(P'_c) \sim 0\). It
can be obtained from the equation

\[ T_f(P'_c) = \left( \gamma_2 b_j^{1/2}/\gamma_1 c_f \right) \]

provided the pressure dependence of the respective material parameters is known. So, the above consideration is consistent with the theoretical prediction that the dashed line in Fig. 5 crosses the axis \( r = 0 \) and for this reason we have the opportunity to describe two ordered phases at low temperatures and broad variations of the pressure. Our theory allows also a description of the shape of the transition line \( T_{FS}(P) \) in ZrZn\(_2\) and URhGe, where the transition temperature \( T_{FS} \) is finite at ambient pressure. To avoid a misunderstanding, let us note that the diagram in Fig. 5 is quite general and the domain containing the point \( r = 0 \) of the phase transition line for negative \( t \) may not be permitted in some ferromagnetic compounds.

Up to now we have discussed experimental curves of second order phase transitions. Our analysis gives the opportunity to describe also first order phase transition lines. Our investigation of the free energy (12) leads to the prediction of triple (\( C \)) and tricritical points (\( A \) and \( B \)); see Figs. 3 and 5. We shall not consider the possible application of these results to the phase diagrams of real substances, for which first order phase transitions and multicritical phenomena occur; see, e.g., Refs. [65, 70], where first order phase transitions and tricritical points have been observed. The explanation of the phase transition lines in Refs. [65, 70] requires further theoretical studies that can be done on the basis of a convenient extension of the free energy (12). For example, the investigation of vortex phases in Ref. [70] requires to take into account the gradient terms (4). Another generalization should be done in order to explain the observation of two FM phases [65, 70]. Note, that the experimentalists are not completely certain whether the FS phase is a uniform or a vortex phase, and this is a crucial point for the further investigations. But we find quite encouraging that our studies naturally lead to the prediction of the same variety of phase transition lines and multicritical points that has been observed in recent experiments [65, 70].

6 Anisotropy

Our analysis demonstrates that when the anisotropy of Cooper pairs is taken into account, there will be no drastic changes in the shape the phase diagram for \( r > 0 \) and the order of the respective phase transitions. Of course, there will be some changes in the size of the phase domains and the formulae for the thermodynamic quantities. It is readily seen from Figs. 6 and 7 that the temperature domain of first order phase transitions and the temperature domain of stability of FS above \( T_\theta \) essentially vary with the variations of the anisotropy parameter \( w \). The parameter \( w \) will also insert changes in the values of the thermodynamic quantities like the magnetic susceptibility and the entropy and specific heat jumps at the phase transition points.

Besides, and this seems to be the main anisotropy effect, the \( w- \) and \( \nu- \)terms in the free energy lead to a stabilization of the order along the main crystal directions which, in other words, means that the degeneration of the possible ground states (FM, SC, and FS) is considerably reduced. This means also a smaller number of marginally stable states.

The dimensionless anisotropy parameter \( w = u_s/(b_s + u_s) \) can be either positive or negative depending on the sign of \( u_s \). Obviously when \( u_s > 0 \), the parameter \( w \) will be positive too and
Figure 6: Phase diagram in the \((t, r)\) plane for \(\gamma = 1.2, \gamma_1 = 0.8,\) and \(w = 0.4.\) The meaning of lines and points is the same as given in Fig. 5.

Figure 7: Phase diagram in the \((t, r)\) plane for \(\gamma = 1.2, \gamma_1 = 0.8,\) and \(w = -2.\) The straight dotted line for \(r < 0\) indicates an instability of the FS phase. The meaning of other lines and notations is the same as given in Fig. 5.
will be in the interval $0 < w < 1$ to ensure the positiveness of parameter $b$ from Eq. (10). When $w < 0$, the latter condition is obeyed if the original parameters of free energy (3) satisfy the inequality $-b_s < u_s < 0$.

We should mention here that a new phase of coexistence of superconductivity and ferromagnetism occurs as a solution of Eqs. (13). It is defined in the following way:

$$
\phi_1^2 + \phi_2^2 = \frac{1}{1 - \gamma_1^2} \left[ \gamma_1 \left( t + \frac{\gamma^2}{2w} \right) - r \right],
$$

$$
M^2 = \frac{1}{1 - \gamma_1^2} \left[ \gamma_1 r - \left( t + \frac{\gamma^2}{2w} \right) \right],
$$

and

$$
2w \sin(\theta_2 - \theta_1) = \gamma M, \quad \cos(\theta_2 - \theta_1) \neq 0.
$$

In the present approximation the phase (42) - (44) is unstable, but this may be changed when the crystal anisotropy is taken into account.

We shall write the equations for order parameters $M$ and $\phi_j$ of FS phase in order to illustrate the changes when $w \neq 0$

$$
\phi_j^2 = \pm \gamma M - r - \gamma_1 M^2 \geq 0,
$$

and

$$
(1 - w - \gamma_1^2)M^3 = \pm \frac{3}{2} \gamma \gamma_1 M^2 + \left[ t(1 - w) - \frac{\gamma^2}{2} - \gamma_1 r \right]M \pm \frac{\gamma r}{2} = 0,
$$

where the meaning of the upper and lower sign is the same as explained just below Eq. (27). The difference in the stability conditions is more pronounced and gives new effects that will be explained further,

$$
\frac{(2 - w)\gamma M - r - \gamma_1 M^2}{1 - w} \geq 0,
$$

$$
\gamma M - wr - w\gamma_1 M^2 \geq 0,
$$

and

$$
\frac{3(1 - w - \gamma_1^2)M^2 + 3\gamma \gamma_1 M + t(1 - w) - \gamma^2/2 - \gamma_1 r}{1 - w} \geq 0.
$$

The calculations of the phase diagram in $(t, r)$ parameter space are done in the same way as in case of $w = 0$ and show that for $w > 0$ there is no qualitative change of the phase diagram. Quantitatively, the region of first order phase transition widens both with respect to $t$ and $r$ as illustrated in Fig. 6. On the contrary, when $w < 0$ the first order phase transition region becomes more narrow but the condition (47) limits the stability of FS for $r < 0$. This is seen from Fig. 7 where FS is stable above the straight dotted line for $r < 0$ and $t < 0$. So, purely superconducting (Meissner) phases occur also as ground states together with FS and FM phases.
7 Conclusion

We investigated the M-trigger effect in unconventional ferromagnetic superconductors. This effect arises from the $M\psi_1\psi_2$-coupling term in the GL free energy and provokes the appearance of superconductivity in a domain of the system’s phase diagram that is entirely occupied by the ferromagnetic phase. The coexistence of unconventional superconductivity and ferromagnetic order is possible for temperatures above and below the critical temperature $T_s$, that corresponds to the standard second-order phase transition from normal to Meissner phase – usual uniform superconductivity in a zero external magnetic field which occurs outside the domain of existence of ferromagnetic order. Our investigation is mainly intended to clarify the thermodynamic behavior at temperatures $T_s < T < T_f$ where the superconductivity cannot appear without the mechanism of M-triggering. We describe the possible ordered phases (FM and FS) in this most interesting temperature interval.

The Cooper pair and crystal anisotropies are investigated and their main effects on the thermodynamics of the triggered phase of coexistence is established. Of course, in discussions of concrete real materials the respective crystal symmetry should be considered. But the low symmetry and low order (in both $M$ and $\psi$) $\gamma$-term in the free energy determines the leading features of the coexistence phase and the dependence of essential thermodynamic properties on the type of crystal symmetry is not so considerable.

Below the superconducting critical temperature $T_s$ a variety of pure superconducting and mixed phases of coexistence of superconductivity and ferromagnetism exists and the thermodynamic behavior at these relatively low temperatures is more complex than in improper ferroelectrics. The case $T_f < T_s$ also needs a special investigation.

Our results are referred to the possible uniform superconducting and ferromagnetic states. Vortex and other nonuniform phases need a separate study.

The relation of the present investigation to properties of real ferromagnetic compounds, such as UGe$_2$, URhGe, and ZrZn$_2$, has been discussed throughout the text. In these compounds the ferromagnetic critical temperature is much larger than the superconducting critical temperature ($T_f \gg T_s$) and that is why the M-triggering of the spin-triplet superconductivity is very strong. Moreover, the $\gamma_1$-term is important to stabilize the FM order up to the absolute zero (0 K), as is in the known spin-triplet ferromagnetic superconductors. Ignoring the symmetry conserving $\gamma_1$-term does not allow a proper description of the real substances of this type. More experimental information about the values of the material parameters ($a_s, a_f, ...$) included in the free energy (12) is required in order to outline the thermodynamic behavior and the phase diagram in terms of thermodynamic parameters $T$ and $P$. In particular, a reliable knowledge about the dependence of the parameters $a_s$ and $a_f$ on the pressure $P$, the value of the characteristic temperature $T_s$ and the ratio $a_s/a_f$ at zero temperature are of primary interest.

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