Research Article

Stability for a Non-Smooth Filippov Ratio-Dependent Predator-Prey System through a Smooth Lyapunov Function

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For nonsmooth Filippov systems, the stability of the system is assumed to be proved by nonsmooth Lyapunov functions, such as piecewise smooth Lyapunov functions. This extension was based on the Filippov solution and Clarke generalized gradient. However, it is difficult to estimate the gradient of a non-smooth Lyapunov function. In some cases, the nonsmooth system can be divided into continuous and discontinuous components. If the Lebesgue measure of the discontinuous components is zero, the smooth Lyapunov function can be utilized to prove the stability of the system owing to the inner product of the gradient of the Lyapunov function of the discontinuous components being zero. In this paper, we apply the smooth Lyapunov function to prove the stability of the nonsmooth ratio-dependent predator-prey system. In contrast to the existing literature, in this paper, although the system is divided into continuous and discontinuous components, the inner product of the gradient of the Lyapunov function of the discontinuous part is not zero but negative. In the proof of stability, the negative value condition is stricter than the zero-value condition. This proof method only needs to construct a smooth Lyapunov function, which is simpler than a non-smooth Lyapunov function or other methods.

1. Introduction

In the ecosystem, the relations among populations mainly include competition, predation, parasitism, neutrality, mutualism, partial benefit, partial harm, and so on [1]. The term “predation” is used to describe an interaction in which an individual of one species kills and is able to consume a large proportion of the individuals of another species. The species killed and consumed is called the prey, and the other is referred to as the predator [2]. The dynamical relationship between predators and their prey is one of the dominant issues in ecology. Lotka proposed the first model of predator-prey (trophic interaction) but failed to extend the logistic growth law to the two species, and soon it was assumed by Volterra that the response of the population is proportional to the product of its biomass density [3]. Afterwards, Leslie [4] introduced the “logistic law” to the predator in their model, where predator/prey ratios rather than products were considered. This new structure seems to be more reasonable on account of the equilibrium density of predators which is expected to depend on the number of preys. The next advance in the predator-prey systems was the addition of Holling functional response to a predator [5], where the prey death rate should be a nonlinear function of prey density because the predators can only capture a finite number of preys in a certain interval. It turned out to be identical to the Michaelis–Menten equation (or Holling-type II) [6]. Subsequently, the functional response was also considered for the prey equation [7]. Although it is intuitively attractive to include functional responses in the predator-prey model, there are still some paradoxical problems in this model. For example, predators capture
preys on a relatively fast-time scale, while population growth evolution acts on a slower time scale. Arditi and Ginzburg [8] assumed that the trophic function depends on the ratio of prey to predator abundances so as to solve the problem of inconsistent time scale. The ratio-dependent functional response gives rise to a parabolic prey isocline and a right slanting predator isocline; therefore, it can overcome the limitations of the classical predator-prey model such as the paradoxes of enrichment and biological control. LT he general ratio-dependent predator-prey with Michaelis–Menten-type system takes the following form:

\[
\begin{align*}
\frac{dx_1}{dt} &= x_1 \left(1 - \frac{x_1}{K}\right) - \frac{cx_1 x_2}{(mx_2 + x_1)} \\
\frac{dx_2}{dt} &= x_2 \left(\frac{-d + f x_1}{(mx_2 + x_1)}\right)
\end{align*}
\]  

where \(x_1\) represents the density of the prey at time \(t\) and \(x_2\) describes the density of the predator. \(a, K, c, m, f, d\) are positive constants that stand for the prey intrinsic growth rate, carrying capacity, capturing rate, half capturing saturation constant, conversion rate, and predator death rate, respectively. For simplicity, substituting \(b = a/K\) in system equation (1), then it can be written as

\[
\begin{align*}
\frac{dx_1}{dt} &= x_1 \left(a - bx_1\right) - \frac{cx_1 x_2}{(mx_2 + x_1)} \\
\frac{dx_2}{dt} &= x_2 \left(-d + f x_1\right)
\end{align*}
\]  

Both ecologists and mathematicians are absorbed in the ratio-dependent predator-prey model with Michaelis–Menten functional response [9–13]. One type of prey in the predator-prey system is called the pest. It is well known that pests have been one of the principal threats to crops, economic plants, animals, and humans around the world. In addition to their natural enemies, the pests can also be eradicated by artificial trapping, spraying insecticides, etc. It is expected that all pests will be eliminated. However, in practice, it is impossible to eradicate the pests completely nor is it biologically or economically desirable. Thus, in order to control pest outbreaks, we should adopt control strategies when the number of pests reaches or exceeds the economical threshold (ET), and the control strategies, especially spraying insecticides, which pollute the environment will be temporarily suspended once the density of pest population drops below the ET, which is referred to as threshold policy control (TPC) [14, 15]. Therefore, system equation (2) integrating TPC can be expressed by the following Filippov system:

\[
\begin{align*}
\frac{dx_1}{dt} &= x_1 \left(a - bx_1\right) - \frac{cx_1 x_2}{(mx_2 + x_1)} - \varepsilon px_1, \\
\frac{dx_2}{dt} &= x_2 \left(-d + f x_1\right)
\end{align*}
\]  

where,

\[
\varepsilon = \begin{cases} 
0, & x_1 < ET, \\
1, & x_1 \geq ET.
\end{cases}
\]

The Filippov system is continuous almost everywhere except at the switching line. The existence of a continuous differentiable solution is not guaranteed due to the discontinuity. Therefore, the classical theory of differential equations has been believed to be ineffective for dealing with the solutions of differential equations with discontinuous right-hand sides. Although the solution of discontinuous differential equations could be converted into a solution of differential inclusion by constructing the Filippov multimap [16], it is still difficult to prove the stability of the system’s equilibrium point. The most recent extension of Lyapunov’s second method to nonsmooth dynamic systems was stated by Shevitz and Paden [17], in which piecewise smooth Lyapunov functions were put forward. Their extension was based on Filippov’s solution concept and Clarke’s generalized gradient. The basis of these stability analysis methods of nonsmooth dynamic systems is that the nonsmooth Lyapunov function is natural for nonsmooth dynamic systems.

However, not all Filippov’s differential inclusions were easy to compute. For the majority of the Filippov systems with the right-hand side expressed as a product or composition of several functions, the direct calculation of the differential inclusion from the definition faces difficulties [18]. Instead, in some special cases, such as when the Lebesgue measure of the discontinuous components of the system is zero, the smooth Lyapunov function can be utilized to prove the stability of the non-smooth system [19]. The main contributions of this paper are the following two points: One is that this paper provides a calculation which makes the proof of stability for the Filippov ratio-dependent predator-prey system more easily because the proof only needs the construction of smooth Lyapunov functions rather than nonsmooth functions. The other is that we extend the premise that “the Lebesgue measure of the discontinuous component of the Filippov system is zero” to the condition that the gradient of the discontinuous part can be negative. For the proof of stability, this is reasonable because the negative value is stricter than zero.

The remaining part of this paper is organized as follows: the preliminary knowledge for Filippov’s solution concept and the bounded invariant set of system equation (2) are provided in Section 2. The main results are presented in Section 3. Finally, Section 4 is the conclusion.

2. Preliminaries

Consider the following vector differential equation as follows:

\[
\dot{x} = f(t, x),
\]

where \(f: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n\) is measurable and essentially locally bounded. The right-hand side of equation (5) is usually a piecewise smooth function. The solution of equation (5) was given by Filippov [20], which is stated in Definition 1.
Definition 1. [18] Vector function \( x(t) \), defined on the interval \((t_0, t_1)\), is a Filippov solution of (4) if it is absolutely continuous and for almost all \( t \in (t_0, t_1) \). For arbitrary \( \delta > 0 \), the vector \( \frac{dx(t)}{dt} \) belongs to the smallest convex closed set of \( n \) dimensional space containing all the values of the vector function \( f(t, x) \). \( x \) ranges over the entire \( \delta \) neighbourhood of the point \( x(t) \) in the space \( x \) except for a set of Lebesgue measure zeroes.

\[
\frac{dx(t)}{dt} \in K[f(t, x(t))],
\]

where

\[
K[f(t, x(t))] \equiv \bigcap_{\delta > 0} \bigcap_{\mu N = 0} \{ f(t, x(t), \delta) - N \},
\]

\( K[f(\cdot)] \) is called Filippov’s differential inclusion. \( \bigcap_{\mu N = 0} \) denotes the intersection over all sets \( N \) of Lebesgue measure zero. \( \sigma \) denotes the convex closure of a set, and \( B(x(t), \delta) \) represents the open ball of radius \( \delta \) centered at \( x \). \( N \) is the set of Lebesgue measure zeroes.

Note that system equation (2) always has equilibria \((0, 0), (a/b, 0)\) and has a unique positive equilibrium \( E^* = (x_1^*, x_2^*) \) if and only if any one of the following two conditions hold [9]:

(i) \( d < f < dc/(c - ma) \), if \( c > ma \),

(ii) \( f > d \), if \( c \leq ma \).

In both cases, there exists \( x_1^* = (cd - f(c - ma))/(bmf) \), \( x_2^* = x_1^*(f - d)/(dm) \).

Lemma 1 (see[21]). If \( f > d \) and \( ma > c \), then the bounded closed domain is as follows:

\[
D = \begin{cases} 
  \frac{ma - c}{mb} \leq x_1 \leq \frac{a}{b} - x_1, \\
  \frac{ma - c}{mb} \leq x_2 \leq \frac{a(f - d)}{bmd} - x_2,
\end{cases}
\]

then the bounded closed domain is eventually invariant set for system equation (2), that is, for any positive solution \((x_1, x_2)\),

\[
\frac{ma - c}{mb} \leq \lim_{t \to \infty} \inf x_1 \leq \lim_{t \to \infty} \sup x_1 \leq \frac{a}{b},
\]

\[
\frac{(f - d)(ma - c)}{m^2bd} \leq \lim_{t \to \infty} \inf x_2 \leq \lim_{t \to \infty} \sup x_2 \leq \frac{a(f - d)}{bmd}.
\]

(9)

(10)

3. Stability for a Filippov System

In this section, we will utilize the smooth Lyapunov function to prove the stability of the nonsmooth Filippov system. It is widely accepted that it is feasible to construct a smooth Lyapunov function to prove the stability of the smooth system, but it is arduous for its non-smooth counterparts. However, for some nonsmooth systems, if the Lebesgue measure of the discontinuous components (switching surface, etc.) is zero, the stability of such systems can be proved by constructing a smooth Lyapunov function [19].

3.1. Smooth Lyapunov Function. For some Filippov systems, we can calculate their differential inclusion directly, but it is difficult to calculate their differential inclusion for some nonsmooth systems that are expressed as a product or composition of several functions. Although the stability can also be proved by differential inclusion and other methods in this paper, it is simpler to construct a smooth Lyapunov function to prove the stability of the system. For a start, the right-hand sides of the Filippov system need to be decomposed into continuous and discontinuous components. We can reorganize equation (5) in the following form:

\[
\dot{x} = f(t, x) + g(t, x),
\]

(11)

where \( f(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) is bounded and contains all the continuous parts of the rate of the state vector. \( g(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) is Lebesgue measurable and locally bounded, \( g_i(t, x), i \in [1, n] \) is either zero or a function that is discontinuous on a surface \( S \). \( S \) represents one discontinuous surface or the intersection of several discontinuous surfaces. Then, if a Lyapunov function \( V \) is constructed for the nonsmooth system in the form of equation (11), the generalized gradient of \( V(t) \) is

\[
\nabla V \cdot (f(t, x) + g(t, x)) = \nabla V \cdot f(t, x) + \nabla V \cdot g(t, x),
\]

(12)

where \( \nabla V \) is the gradient of \( V \) function and \( f(t, x) \) is continuous function, while \( g(t, x) \) is function containing discontinuous components on \( S \). \( \nabla V \) is continuous, and \( \nabla V \cdot g(t, x) \) can further be rewritten in the following form:

\[
\nabla V \cdot g(t, x) = \nabla V^{(1)} \cdot g^{(1)}(t, x) + \nabla V^{(2)} \cdot g^{(2)}(t, x),
\]

(13)

where \( g^{(1)}(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m \) is the subvector \( g(t, x) \) containing discontinuous factors and \( g^{(2)}(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n-m} \) is a zero sub-vector of \( g(t, x) \). \( \nabla V(1) \) as well as \( \nabla V(2) \) are the components of \( \nabla V \) corresponding to \( g^{(1)}(t, x) \) and \( g^{(2)}(t, x) \) based on equation (13).

3.2. Stability for a Nonsmooth Ecosystem

Theorem 1. Suppose that the following conditions are satisfied: \( a = (mf_2/cx_2)^* > 0 \), \( f > d \), \( ma > c \), \( f^*(ma - c) > cd(f - d) \), and \( ET > x_1^* \), then the positive equilibrium \( E^* \) of system equation (3) is Lyapunov stable.

Proof. Lyapunov function is constructed for system equation (3) as follows:

\[
V(t) = a \left( x_1 - x_1^* - x_1^* \ln \frac{x_1}{x_1^*} \right) + x_2 - x_2^* - x_2^* \ln \frac{x_2}{x_2^*},
\]

(14)

where \( a \) will be derived later in the proof process. Thus,
\[ \dot{V}(t) = a \left( \dot{x}_1 - x_1^* \frac{1}{x_1} \dot{x}_1 - x_1^* \frac{1}{x_1} \right) + \left( \dot{x}_2 - x_2^* \frac{1}{x_2} \dot{x}_2 \right) \]

\[ = a \dot{x}_1 \left( 1 - x_1^* \frac{1}{x_1} \right) + \dot{x}_2 \left( 1 - x_2^* \frac{1}{x_2} \right) \]

\[ = a(x_1 - x_1^*) \frac{\dot{x}_1}{x_1} + (x_2 - x_2^*) \frac{\dot{x}_2}{x_2} \]

\[ = a\beta(x_1 - x_1^*) + (\omega + \tau - \omega)(x_2 - x_2^*) \]

\[ = -ba(x_1 - x_1^*)^2 + ca\mu (x_1 - x_1^*) \]

\[ - \epsilon p a(x_1 - x_1^*) + m f \mu (x_2 - x_2^*) \]

\[ = -ba(x_1 - x_1^*)^2 + ca\mu (x_1 - x_1^*) \]

\[ - \epsilon p a(x_1 - x_1^*) + m f \mu (x_2 - x_2^*) \]

\[ = -a \left( b - \frac{\rho}{m x_2 + x_1} \right) (x_1 - x_1^*)^2 \]

\[ - \frac{m \omega}{m x_2 + x_1} (x_2 - x_2^*)^2 - \epsilon p a(x_1 - x_1^*) \]

\[ - ca\mu x_1^* + m f \mu x_2^*, \]

where \( \beta = [a - bx_1^* + b - bx_1 + \rho - \eta - \epsilon p], \)

\( \gamma = [-d + \omega + \tau - \omega], \)

\( \nu = \frac{x_1 x_2^* - x_2 x_1^*}{(m x_2 + x_1)(m x_2 + x_1)}, \)

\( \mu = \frac{(x_1 - x_1^*)^2 + (x_1 x_2^* - x_2 x_1^*)}{(m x_2 + x_1)(m x_2 + x_1)}, \)

\( \rho = \frac{c x_1^*}{m x_2 + x_1}, \)

\( \omega = \frac{fx_1^*}{m x_2 + x_1}, \)

\( \eta = \frac{c x_2}{m x_2 + x_1}, \)

\( \tau = \frac{fx_1}{m x_2 + x_1}, \)

\( \varphi = \frac{(x_1 - x_1^*) (x_2 - x_2^*)}{(m x_2 + x_1)(m x_2 + x_1)}. \)

Obviously, \( \nabla V = \left[ a ((x_1 - x_1^*)/x_1), (x_2 - x_2^*)/x_2 \right]^T. \) In order to better elaborate the above concepts, we divide the right-hand sides of system equation (3) into continuous and discontinuous components.

\[ f(t, x) = \left[ x_1 (a - bx_1) - \eta x_1, x_2 (-d + \tau) \right]^T, \]

\[ g(t, x) = [-\epsilon p x_1, 0]^T. \]

Furthermore, according to equation (13), \( g(t, x) \) can be decomposed into \( g^{(1)}(t, x) = [-\epsilon p x_1] \) and \( g^{(2)}(t, x) = [0]. \) Accordingly, with reference to equation (13), \( \nabla V^{(1)} = \left[ a ((x_1 - x_1^*)/x_1) \right] \) and \( \nabla V^{(2)} = \left[ (x_2 - x_2^*)/x_2 \right]. \)

In the continuous region \( (x_1 < ET) \), \( \nabla V \cdot f(t, x) \) is continuous and negative as follows:

\[ \nabla V \cdot f(t, x) = \left\{ \begin{array}{c} a \left( \frac{(x_1 - x_1^*)}{x_1} \right) \dot{x}_1 + \left( \frac{(x_2 - x_2^*)}{x_2} \right) \dot{x}_2 \\ -a \left( b - \frac{\rho}{m x_2 + x_1} \right) (x_1 - x_1^*)^2 \\ - \frac{m \omega}{m x_2 + x_1} (x_2 - x_2^*)^2 \\ - ca\mu x_1^* + m f \mu x_2^* \\ \end{array} \right\} \]

assume \( -ca\mu x_1^* + m f \mu x_2^* = 0, \) then \( \alpha \) can be chosen as

\[ \alpha = \frac{m f \mu x_2^*}{c x_1^*} > 0. \]

Notice that \( f > d \) and \( f^2 (ma - c) > cd (f - d) \) indicate that

\[ b > \frac{c x_2^*}{(m x_2 + x_1)(m x_2 + x_1)}, \quad \forall (x_1, x_2) \in D, \]

where \( D \) is defined in Lemma 1. Combining formulas (18) and (19), it could derive that

\[ \nabla V \cdot f(t, x) = -a \left( b - \frac{\rho}{m x_2 + x_1} \right) (x_1 - x_1^*)^2 \]

\[ - \frac{m \omega}{m x_2 + x_1} (x_2 - x_2^*)^2 < 0. \]

Since \( \nabla V \cdot f(t, x) \) is continuous and negative, we continue to establish the requirements for the construction of smooth Lyapunov function for the nonsmooth system. Then, we consider \( \nabla V \cdot g(t, x). \) On the discontinuity surface \( S(x_1 = ET) \), there exists

\[ \nabla V \cdot g(t, x) \in \nabla V \tilde{K}[g(t, x)] \]

\[ = K \left\{ (\nabla V^{(1)} \cdot g^{(1)}) (t, x) + (\nabla V^{(2)} \cdot g^{(2)}) (t, x) \right\} \]

\[ c K \left\{ (\nabla V^{(1)} \cdot g^{(1)}) (t, x) \right\} + K \left\{ (\nabla V^{(2)} \cdot g^{(2)}) (t, x) \right\} \]

\[ = K \left\{ \alpha \frac{(x_1 - x_1^*)}{x_1} \cdot (-\epsilon p x_1) + K \left[ \frac{(x_2 - x_2^*)}{x_2} \cdot 0 \right] \right\}, \]

(21)
when \( ET > \) \( x_1^* ( x_1^* = [cd - f(c - ma)][bmf]) \), it is obvious that

\[ \nabla V \cdot g(t,x) < 0, \] (22)

integrating equations (20) and (22), \( \dot{V} < 0 \), this completes the proof.

4. Conclusion Remarks

In this paper, we construct a smooth Lyapunov function to prove the stability of the nonsmooth predator-prey model. For nonsmooth systems, it is more common to construct nonsmooth Lyapunov functions to prove the stability, but the proof process is more complicated. For nonsmooth systems, in some special cases, the system can be decomposed into continuous and discontinuous components. A smooth Lyapunov function can be constructed to prove the stability of nonsmooth systems, supposing that the Lebesgue measure of discontinuous components is zero. That is, as the trajectory of the system approaches the discontinuous surface infinitely, the inner product of the gradient of the Lyapunov function is zero. In this paper, the gradient of the discontinuous component is negative rather than zero, as distinct from previous studies, which is a more rigorous condition for proving stability. It can be concluded that when the economic threshold is greater than or equal to the density of the prey at the equilibrium point, that is, \( ET > x_1^* \), which does not affect the stability of the equilibrium point, note that this condition is sufficient but unnecessary. This phenomenon is obvious in the biological sense. We could see that if the economic threshold is relatively high and the stability of the equilibrium point is consistent with the one in the original ODE, the density of the prey can fluctuate in a large range. Other than releasing natural enemies, there is no need to adopt other control measures. However, when the economic threshold is low, that is, when the density of pests will be controlled in a lower range, the stability and type of the equilibrium point may vary; the regular equilibrium in ODE changes into virtual equilibrium in the Filippov system. In this case, other control measures such as spraying insecticides and manual capture need to be adopted together. In addition, this method of using the smooth Lyapunov function to prove the stability of nonsmooth systems can be extended to other nonsmooth ecosystems [22–24]. Some nonsmooth control systems can also be considered in the future [25–29].

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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