Heat flow in Riemannian manifolds with non-negative Ricci curvature

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Abstract
Let $\Omega$ be an open set in a geodesically complete, non-compact, $m$-dimensional Riemannian manifold $M$ with non-negative Ricci curvature, and without boundary. We study the heat flow from $\Omega$ into $M - \Omega$ if the initial temperature distribution is the characteristic function of $\Omega$. We obtain a necessary and sufficient condition which ensures that an open set $\Omega$ with infinite measure has finite heat content for all $t > 0$. We also obtain upper and lower bounds for the heat content of $\Omega$ in $M$. Two-sided bounds are obtained for the heat loss of $\Omega$ in $M$ if the measure of $\Omega$ is finite.

Keywords: Heat flow; Riemannian manifold; non-negative Ricci curvature.

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1 Introduction

Let \((M, g)\) be a geodesically complete, smooth, \(m\)-dimensional Riemannian manifold without boundary, and let \(\Delta\) be the Laplace-Beltrami operator on \(M\). It is well known (see [6], [7]) that the heat equation

\[\Delta u(x; t) = \frac{\partial u(x; t)}{\partial t}, \quad x \in M, \quad t > 0,\]  

(1)

has a unique, minimal, positive fundamental solution \(p_M(x, y; t)\) where \(x \in M, y \in M, t > 0\). This solution, the heat kernel for \(M\), is symmetric in \(x, y\), strictly positive, jointly smooth in \(x, y \in M\) and \(t > 0\), and it satisfies the semigroup property

\[p_M(x, y; s + t) = \int_M dz \, p_M(x, z; s)p_M(z, y; t),\]  

(2)

for all \(x, y \in M\) and \(t, s > 0\), where \(dz\) is the Riemannian measure on \(M\). See, for example, [13] for details.

We denote by \(d : M \times M \to \mathbb{R}^+\) the geodesic distance associated to \((M, g)\).

For \(x \in M, R > 0\), \(B(x; R) = \{y \in M : d(x, y) < R\}\). For a measurable set \(A \subset M\) we denote by \(|A|\) its Lebesgue measure. If \(A \subset M\) is a Cacciopoli set then we denote by \(P\) its perimeter.

In this paper, we obtain some results for the heat flow from an open set \(\Omega\) with boundary \(\partial \Omega\) into its complement \(M - \Omega\) with the characteristic function of \(\Omega\) as the initial temperature distribution. Define \(u_\Omega : \Omega \times (0, \infty) \to \mathbb{R}\) by

\[u_\Omega(x; t) = \int_\Omega dy \, p_M(x, y; t).\]  

(3)

Then \(u_\Omega\) is a solution of the heat equation (1) and satisfies

\[\lim_{t \to 0} u_\Omega(x; t) = 1_\Omega(x), \quad x \in M - \partial \Omega,\]  

(4)

where \(1_\Omega : M \to \{0, 1\}\) is the characteristic function of \(\Omega\), and where the convergence in (4) is locally uniform. It can be shown that if \(|\Omega| < \infty\), then the convergence is also in \(L^1(M)\). If \(\Omega\) has infinite measure and \(|\partial \Omega| = 0\), then the convergence is also in \(L^1_{\text{loc}}(M)\) (Section 7.4 in [8]). Furthermore, if for some \(p \in M, r_0 > 0, \int_0^{\infty} dr \, r (\log |B(p; r)|)^{-1} = +\infty\), then \(u_\Omega\) defined by (3) is the unique, bounded solution of (1) with the initial condition (4) in the sense of \(L^1_{\text{loc}}(M)\). We refer to Chapter 9 in [7].

We define the heat content of \(\Omega\) in \(M\) at time \(t\) by

\[H_\Omega(t) = \int_\Omega dx \, u_\Omega(x; t) = \int_\Omega dx \int_\Omega dy \, p_M(x, y; t).\]  

(5)

\(H_\Omega(t)\) is the amount of heat left in \(\Omega\) at time \(t\) if all physical constants, such as specific heat and thermal conductivity, have been put equal to 1. In general, the problem of finding \(H_\Omega(t)\) as a function of \(t\) is difficult as it requires an explicit formula for the heat kernel \(p_M(x, y; t)\). Even in the case of \(\mathbb{R}^m\), where such an explicit formula is available, it is not easy to calculate the asymptotic behaviour of \(H_\Omega(t)\) as \(t \downarrow 0\) for example. Similar problems have been studied in the Euclidean setting in [10], [11], [12] and subsequently in [1], [3] and [4].
The results in Theorem 2.4 of [11] imply that if $\Omega$ is an open, bounded subset of $\mathbb{R}^m$ with $C^1$-boundary $\partial \Omega$ then
\[
H_\Omega(t) = |\Omega| - \pi^{-1/2} P(\Omega) t^{1/2} + o(t^{1/2}), \ t \downarrow 0.
\] (6)

The uniform remainder estimates obtained in [3] improve upon (6), and imply an $O(t)$ remainder in (6).

In the Riemannian manifold setting, it was shown ([2]) that if $\Omega$ is non-empty, bounded, $\partial \Omega$ is of class $C^\infty$, and if $(M, g)$ satisfies exactly one of the following three conditions: (i) $M$ is compact and without boundary, (ii) $(M, g) = (\mathbb{R}^m, g_e)$ where $g_e$ is the usual Euclidean metric on $\mathbb{R}^m$, (iii) $M$ is a compact submanifold of $\mathbb{R}^m$ with smooth boundary and $g = g_e|_M$, then there exists a complete asymptotic series such that
\[
H_\Omega(t) = \sum_{j=0}^{J-1} \beta_j t^{j/2} + O(t^{J/2}), \ t \downarrow 0,
\] (7)

where $J \in \mathbb{N}$ is arbitrary, and where the $\beta_j : j = 0, 1, 2, \ldots$ are locally computable geometric invariants. For example, by [2], we have that $\beta_0 = |\Omega|$, $\beta_1 = -\pi^{-1/2} P(\Omega)$, $\beta_2 = 0$.

The main results of this paper concern the situation where $\Omega$ is an open set with infinite measure in a complete, non-compact, $m$-dimensional Riemannian manifold $M$ with non-negative Ricci curvature.

**Definition 1.** For $x \in M$, $\Omega \subset M$, and $R > 0$, 
\[
\mu_\Omega(x; R) = |B(x; R) \cap \Omega|.
\]

Our main result is the following.

**Theorem 2.** Let $M$ be a complete, non-compact, $m$-dimensional Riemannian manifold with non-negative Ricci curvature, and without boundary.

(i) Let $\Omega$ be an open subset of $M$. If $H_\Omega(T) < \infty$ for some $T > 0$ then
\[
\int_\Omega dx \frac{\mu_\Omega(x; t^{1/2})}{|B(x; t^{1/2})|} < \infty, \ \forall t > 0.
\] (8)

(ii) There exist $K_1 > 0$ and $K_2 < \infty$ such that if $\Omega$ satisfies (8) then
\[
K_1 \int_\Omega dx \frac{\mu_\Omega(x; t^{1/2})}{|B(x; t^{1/2})|} \leq H_\Omega(t) \leq K_2 \int_\Omega dx \frac{\mu_\Omega(x; t^{1/2})}{|B(x; t^{1/2})|}, \ \forall t > 0.
\] (9)

The heuristic interpretation of (8) and (9) is the following. Consider a partition of $M$ in open “boxes”, $B_1, B_2, \ldots$, centred around points $x_1, x_2, \ldots$ of diameter $t^{1/2}$ each. The amount of heat in box $B_i$ at $t = 0$ equals $|B_i \cap \Omega|$. The heat redistributes to a profile such that at time $t$ the maximum and minimum temperatures in $B_i$ are comparable. The average temperature $u_\Omega$ in $B_i$ is then of order $\frac{|B_i \cap \Omega|}{|B_i|}$. Integrating $u_\Omega$ over $\Omega$ then gives that $H_\Omega$ is of order (8). Here,
we have used that $|B_r| \asymp |B(x; t^{1/2})|$. The integrand in (8) is the fraction of space of the ball centred at $x$ with radius $t^{1/2}$ that is occupied by $\Omega$.

If $\Omega$ has finite Lebesgue measure, then it is natural to define the heat loss of $\Omega$ in $M$ at $t$ by

$$F_\Omega(t) = |\Omega| - H_\Omega(t).$$

(10)

By Lemma 5 and Corollary 3 below, we have that $t \mapsto H_\Omega(t)$ is decreasing and convex respectively. If $|\Omega| < \infty$, then the heat loss $t \mapsto F_\Omega(t)$ of $\Omega$ in $M$ is increasing from 0 to $|\Omega|$ and concave. If $\Omega$ is bounded and $\partial \Omega$ is smooth, then, by (7), there exists an asymptotic series of which the first few coefficients are known explicitly. Theorem 4 below deals with the case where $|\Omega| < \infty$ but where either the perimeter of $\Omega$ is infinite and/or $\partial \Omega$ is not smooth.

Definition 3. For $x \in M$, $\Omega \subset M$, and $R > 0$, 

$$\nu_\Omega(x; R) = |B(x; R) - \Omega|.$$  

Theorem 4. Let $M$ be a complete, non-compact, m-dimensional Riemannian manifold with non-negative Ricci curvature, and without boundary. There exist constants $L_1 > 0$ and $L_2 < \infty$ such that if $\Omega$ is open in $M$ with $|\Omega| < \infty$, then

$$L_1 \int_{\Omega} dx \frac{\nu_\Omega(x; t^{1/2})}{|B(x; t^{1/2})|} \leq F_\Omega(t) \leq L_2 \int_{\Omega} dx \frac{\nu_\Omega(x; t^{1/2})}{|B(x; t^{1/2})|}. $$

(11)

We give explicit numerical values for $K_1$, $K_2$ and $L_1$, $L_2$ in the proofs of Theorems 3 and 4 respectively, in terms of of the numerical constants that appear in the Li-Yau bounds for the heat kernel. These bounds, Corollary 3.1 and Theorem 4.1 in [9], are crucial ingredients in the proofs in Section 2 below. In Section 2 we give some examples of $\Omega$ in $\mathbb{R}^m$ where a precise analysis of $F_\Omega(t)$ is possible.

2 Proofs

There are several key ingredients of the proofs of Theorems 2 and 3 which we recall below. The Bishop-Gromov Theorem, [5], states that if $M$ is a complete, non-compact, m-dimensional, Riemannian manifold with non-negative Ricci curvature, then, for $p \in M$, the map $r \mapsto |B(p; r)|$ is monotone decreasing. In particular

$$\frac{|B(p; r_2)|}{|B(p; r_1)|} \leq \left( \frac{r_2}{r_1} \right)^m, \quad 0 < r_1 \leq r_2.$$  

(12)

The results of Li-Yau, Corollary 3.1 and Theorem 4.1 in [9], imply that if $M$ is complete with non-negative Ricci curvature, then for any $D_2 > 2$ and $0 < D_1 < 2$ there exist constants $0 < C_1 \leq C_2 < \infty$ such that for all $x \in M$, $y \in M$, $t > 0$,

$$C_1 \frac{e^{-d(x, y)^2/(2D_1 t)}}{|B(x; t^{1/2})||B(y; t^{1/2})|^{1/2}} \leq p_M(x, y; t) \leq C_2 \frac{e^{-d(x, y)^2/(2D_2 t)}}{|B(x; t^{1/2})||B(y; t^{1/2})|^{1/2}}. $$

(13)

Finally, since by (12) the measure of any geodesic ball with radius $r$ is bounded polynomially in $r$, the theorems of Grigor’yan in [7] imply stochastic completeness. That is, for all $x \in M$ and $t > 0$,

$$\int_M dy p_M(x, y; t) = 1.$$  

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We also recall the following contractivity property of the heat semigroup.

**Lemma 5.** Suppose $M$ is complete and with non-negative Ricci curvature. If $\Omega \subset M$ is open and $H_{\Omega}(t) < \infty$ for some fixed $t > 0$ then
\[
H_{\Omega}(t + s) \leq H_{\Omega}(t), \quad s \geq 0.
\]

**Proof.** By (2), (3), and Tonelli’s Theorem, we have that
\[
u_{\Omega}(x; t + s) = \int_M dy p_{M}(x, y; s)u_{\Omega}(y; t).
\]
Hence
\[
\int_M dx u_{\Omega}(x; t + s)^2
= \int_M dx \int_M dy_1 p_{M}(x, y_1; s)u_{\Omega}(y_1; t) \int_M dy_2 p_{M}(x, y_2; s)u_{\Omega}(y_2; t)
= \int_M dy_1 \int_M dy_2 p_{M}(y_1, y_2; 2s)u_{\Omega}(y_1; t)u_{\Omega}(y_2; t)
\leq \frac{1}{2} \int_M dy_1 \int_M dy_2 p_{M}(y_1, y_2; 2s)(u_{\Omega}(y_1; t)^2 + u_{\Omega}(y_2; t)^2)
= \int_M dy_1 \int_M dy_2 p_{M}(y_1, y_2; 2s)u_{\Omega}(y_1; t)^2
= \int_M dy u_{\Omega}(y; t)^2,
\]
where we have used the stochastic completeness of $M$ in the last equality. By (5) and (2) we have that
\[
H_{\Omega}(2t) = \int_M dz \int_\Omega dy \int_\Omega dx p_{M}(x, z; t)p_{M}(z, y; t)
= \int_M dz u_{\Omega}(z; t)^2.
\]
We conclude that the left-hand and right-hand sides of (14) equal $H_{\Omega}(2t + 2s)$ and $H_{\Omega}(2t)$ respectively. This proves the assertion.

We have a similar estimate in the proof of the following.

**Corollary 6.** Suppose $M$ is complete, $\Omega \subset M$ is open and $H_{\Omega}(s) < \infty$ for some $s > 0$. Then $t \mapsto H_{\Omega}(t)$ is convex on $(0, \infty)$.

**Proof.** Since $H_{\Omega}(s) < \infty$ for some $s > 0$, we have by Theorem 2 that $H_{\Omega}(s) < \infty$ for all $s > 0$. Hence it suffices to show that $t \mapsto H_{\Omega}(t)$ is mid-point convex. This follows from
\[
H_{\Omega}(t + s) = \int_M dz \int_\Omega dy \int_\Omega dx p_{M}(x, z; t)p_{M}(z, y; s)
= \int_M dz u_{\Omega}(z; t)u_{\Omega}(z; s)
\leq \frac{1}{2} \int_M dz (u_{\Omega}(z; t)^2 + u_{\Omega}(z; s)^2)
= \frac{1}{2} (H_{\Omega}(2t) + H_{\Omega}(2s)).
\]
Proof of Theorem 2. To prove part (i) of Theorem 2 we let $t > T > 0$ and suppose that $H_\Omega(T) < \infty$. Let $R > 0$. By Lemma 5, 5 and (13), we have that

$$H_\Omega(T) \geq H_\Omega(t)$$

$$\geq \int_\Omega dx \int_{\Omega \cap B(x,R)} dy p_M(x, y; t)$$

$$\geq C_1 e^{-R^2/(2D_2 t)} \int_\Omega dx \int_{\Omega \cap B(x,R)} dy \left( |B(x; t^{1/2})| |B(y; t^{1/2})| \right)^{-1/2}. \tag{15}$$

For $d(x, y) < R$, $B(y; t^{1/2}) \subset B(x; R + t^{1/2})$, so that by (12),

$$|B(y; t^{1/2})| \leq \left( \frac{R + t^{1/2}}{t^{1/2}} \right)^m |B(x; t^{1/2})|. \tag{16}$$

The choice $R = t^{1/2}$ implies, by (15) and (16), that

$$H_\Omega(T) \geq H_\Omega(t) \geq K_1 \int_\Omega dx \frac{\mu_\Omega(x; t^{1/2})}{|B(x; t^{1/2})|}, \ t \geq T, \tag{17}$$

with

$$K_1 = C_1 2^{-m/2} e^{-1/(2D_2 t)}.$$

Hence the integral in (8) is finite for all $t \geq T$.

Next suppose that $0 < t \leq T$. By (12) and (17), we have that for $0 < t \leq T$,

$$\int_\Omega dx \frac{\mu_\Omega(x; t^{1/2})}{|B(x; t^{1/2})|} \leq \left( \frac{T}{t} \right)^{m/2} \int_\Omega dx \frac{\mu_\Omega(x; T^{1/2})}{|B(x; T^{1/2})|} \leq \frac{1}{K_1} \left( \frac{T}{t} \right)^{m/2} H_\Omega(T).$$

This completes the proof of the assertion in part (i).

To prove part (ii) of Theorem 2 we let $n \geq 1$, $p \in \Omega$, and $\Omega_n = \Omega \cap B(p; n)$, and suppose that (3) holds. Then $|\Omega_n| \leq |B(p; n)| \leq n^m |B(p; 1)| < \infty$. Let $R > 0$. Reversing the roles of $x$ and $y$ in (10) we have that for $d(x, y) < R$,

$$|B(y; t^{1/2})| \geq \left( \frac{t^{1/2}}{R + t^{1/2}} \right)^m |B(x; t^{1/2})|. \tag{18}$$

We have that

$$\int_{\Omega_n} dx \int_{\Omega_n} dy p_M(x, y; t)$$

$$= \int_{\Omega_n} dx \int_{\Omega_n \cap B(x,R)} dy p_M(x, y; t) + \int_{\Omega_n} dx \int_{\Omega_n \cap B(x,R)} dy p_M(x, y; t). \tag{19}$$

Using (13) and (18), we see that

$$\int_{\Omega_n} dx \int_{\Omega_n \cap B(x,R)} dy p_M(x, y; t)$$

$$\leq C_2 \int_{\Omega_n} dx |B(x; t^{1/2})|^{-1} \int_{\Omega_n \cap B(x,R)} dy \left( \frac{|B(x; t^{1/2})|}{|B(y; t^{1/2})|} \right)^{1/2} e^{-d(x,y)^2/(2D_2 t)}$$

$$\leq C_2 \left( \frac{R + t^{1/2}}{t^{1/2}} \right)^{m/2} \int_{\Omega_n} dx \frac{\mu_\Omega_n(x; R)}{|B(x; t^{1/2})|}$$

$$\leq C_2 \left( \frac{R + t^{1/2}}{t^{1/2}} \right)^{m/2} \int_{\Omega_n} dx \frac{\mu_\Omega(x; R)}{|B(x; t^{1/2})|}. \tag{20}$$
To bound the second term in the right-hand side of (19), we let \( 0 < \alpha < 1 \), and note that
\[
d(x, y)^2/(2D_2t) \geq \alpha R^2/(2D_2t) + (1 - \alpha)d(x, y)^2/(2D_2t), \quad y \in \Omega_n - B(x; R).
\]
(21)

Hence, this second term is bounded by
\[
C_2e^{-\alpha R^2/(2D_2t)} \int_{\Omega_n} dx \int_{\Omega_n} dy (|B(x; t^{1/2})||B(y; t^{1/2})|)^{-1/2}e^{-(1-\alpha)d(x, y)^2/(2D_2t)}
\leq C_2e^{-\alpha R^2/(2D_2t)} \int_{\Omega_n} dx \int_{\Omega_n} dy \left( \frac{D_2}{(1-\alpha)D_1} \right)^{m/2}|B(x; t^{1/2})|^{1/2}e^{-(1-\alpha)d(x, y)^2/(2D_2t)}
\leq \frac{C_2}{C_1} \left( \frac{D_2}{(1-\alpha)D_1} \right)^{m/2}e^{-\alpha R^2/(2D_2t)}H_{\Omega_n}(\frac{D_2t}{(1-\alpha)D_1})
\leq \frac{C_2}{C_1} \left( \frac{D_2}{(1-\alpha)D_1} \right)^{m/2}e^{-\alpha R^2/(2D_2t)}H_{\Omega_n}(t),
\]
(22)

where we have used (21), (12), the lower bound in (13), and the monotonicity of \( t \mapsto H_{\Omega_n}(t) \). We now let \( R^2 = \beta m D_2t \), \( \beta > 1 \) and choose \( \alpha = 1 - \frac{mD_2}{R^2} \in (0, 1) \). This choice of \( \alpha \) minimises the right-hand side of (22), and gives the bound
\[
\int_{\Omega_n} dx \int_{\Omega_n - B(x; R)} dy p_M(x, y; t) \leq \frac{C_2}{C_1} \left( \frac{\beta e^{1-\beta} D_2}{D_1} \right)^{m/2}H_{\Omega_n}(t)
\leq \frac{C_2}{C_1} \left( \frac{2e^{-\beta/2} D_2}{D_1} \right)^{m/2}H_{\Omega_n}(t).
\]
(23)

We choose \( \beta \) such that the coefficient of \( H_{\Omega_n}(t) \) in the right-hand side of (23) is equal to \( \frac{1}{2} \). That is
\[
\beta = \frac{4}{m} \log \left( \frac{2C_2}{C_1} \left( \frac{2D_2}{D_1} \right)^{m/2} \right).
\]
(24)

Since \( C_2 \geq C_1 \) and \( D_2 \geq D_1 \), we have that the right-hand side of (24) is bounded from below by 1. By (19)–(24), we obtain that
\[
H_{\Omega_n}(t) \leq 2C_2((\beta m D_2)^{1/2} + 1)^{m/2} \int_{\Omega} dx \frac{\mu_0(x; (\beta m D_2)^{1/2})}{|B(x; t^{1/2})|}
\leq 2^{(2+\beta)/2}C_2(\beta m D_2)^{3m/4} \int_{\Omega} dx \frac{\mu_0(x; (\beta m D_2)^{1/2})}{|B(x; (\beta m D_2)^{1/2})|}.
\]

If we replace \( t \) by \( t/(\beta m D_2) \), and use the monotonicity of \( t \mapsto H_{\Omega}(t) \), then we obtain
\[
H_{\Omega_n}(t) \leq H_{\Omega_n}(t/(m \beta D_2)) \leq K_2 \int_{\Omega} dx \frac{\mu_0(x; t^{1/2})}{|B(x; t^{1/2})|},
\]
where
\[
K_2 = 2^{1+2m}C_2 \left( \frac{2C_2}{C_1} \left( \frac{2D_2}{D_1} \right)^{m/2} \right)^{3m/4}.
\]

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Finally letting $n \to \infty$ leads to the upper bound in (21) of Theorem 2. We now infer that the lower bound in (17) holds for all $t > 0$. This completes the proof of Theorem 2.

The following will be needed in the proof of Theorem 4.

**Lemma 7.** Let $M$ be a complete, non-compact, $m$-dimensional Riemannian manifold with non-negative Ricci curvature. If $\Omega \subset M$, $|\Omega| < \infty$, then for all $s > 0$ and all $t > 0$,

$$F_{\Omega}(s + t) \leq F_{\Omega}(s) + F_{\Omega}(t).$$

**Proof.** By the definition of $F_{\Omega}(t)$ in (10), and by stochastic completeness,

$$F_{\Omega}(t) = \int_M dx \int_\Omega dy p_M(x, y; t) - \int_\Omega dy p_M(x, y; t) = \int_{M - \Omega} dx \int_\Omega dy p_M(x, y; t). \quad (25)$$

By the heat semigroup property (2), (25) and Tonelli’s Theorem, we have that

$$F_{\Omega}(s + t) = \int_{M - \Omega} dx \int_\Omega dy p_M(x, y; s + t) = \int_M dz \int_{M - \Omega} dx \int_\Omega dy p_M(x, z; s) p_M(z, y; t) + \int_{M - \Omega} dz \int_M dx \int_\Omega dy p_M(x, z; s) p_M(z, y; t) \leq \int_\Omega dy p_M(z, y; t) \int_M dx p_M(x, z; s) + \int_{M - \Omega} dz \int_{M - \Omega} dx \int_\Omega dy p_M(x, z; s) = F_{\Omega}(s) + F_{\Omega}(t).$$

**Proof of Theorem 4**. To prove the lower bound in (11), we have by (25), (13) that for $R > 0$,

$$F_{\Omega}(t) \geq \int_\Omega dx \int_{B(x; R) - \Omega} dy p_M(x, y; t) \geq C_1 e^{-R^2/(2D_1 t)} \int_\Omega dx \int_{B(x; R) - \Omega} dy |B(x; t^{1/2})|^{-1/2} |B(y; t^{1/2})|^{1/2}.$$ 

Since $B(y; t^{1/2}) \subset B(x; R + t^{1/2})$, for $y \in B(x; R)$, we have by (16) that

$$F_{\Omega}(t) \geq C_1 \left( \frac{t^{1/2}}{R + t^{1/2}} \right)^{m/2} e^{-R^2/(2D_1 t)} \int_\Omega dx \frac{\nu_{\Omega}(x; R)}{|B(x; t^{1/2})|}.$$
The choice $R = t^{1/2}$ gives the lower bound in (11) with
$$L_1 = 2^{-m/2}e^{-1/(2D_t)} C_1.$$  

To prove the upper bound in (11), we let $R > 0$, and write (25) as
$$F_\Omega(t) = \int_\Omega dx \int_{(M-\Omega)\cap B(x;R)} dy p_M(x,y;t)$$
$$+ \int_\Omega dx \int_{(M-\Omega)\cap (M-B(x;R))} dy p_M(x,y;t). \quad (26)$$

By (13) and (18),
$$\int dx \int_{(M-\Omega)\cap B(x;R)} dy p_M(x,y;t)$$
$$\leq C_2 \int dx \int_{(M-\Omega)\cap B(x;R)} dy |B(x; t^{1/2})|^{-1} |B(y; t^{1/2})|^{1/2}$$
$$\leq C_2 \left( R + \frac{t^{1/2}}{D_t} \right)^{m/2} \int_\Omega dx \frac{\nu_\Omega(x;R)}{|B(x; t^{1/2})|}. \quad (27)$$

Furthermore,
$$\int_\Omega dx \int_{(M-\Omega)\cap (M-B(x;R))} dy p_M(x,y;t)$$
$$\leq C_2 e^{-R^2/(4D_t)} \int_\Omega dx \int_{M-\Omega} dy \frac{e^{-d(x,y)^2/(4D_t)}}{|B(x; t^{1/2})| |B(y; t^{1/2})|^{1/2}}$$
$$\leq C_2 e^{-R^2/(4D_t)} \left( \frac{2D_2}{D_1} \right)^{m/2} \int_\Omega dx \int_{M-\Omega} dy e^{-d(x,y)^2/(4D_t)}$$
$$\times |B(x; (2D_2 t/D_1)^{1/2})| |B(y; (2D_2 t/D_1)^{1/2})|^{1/2}$$
$$\leq e^{-R^2/(4D_t)} \frac{C_2}{C_1} \left( \frac{2D_2}{D_1} \right)^{m/2} \int_\Omega dx \int_{M-\Omega} dy p_M(x,y; 2D_2 t/D_1)$$
$$\leq e^{-R^2/(4D_t)} \frac{C_2}{C_1} \left( \frac{2D_2}{D_1} \right)^{m/2} F_\Omega(2D_2 t/D_1). \quad (28)$$

Let $n \in \mathbb{N}$ be such that $\frac{2D_2}{D_1} \in [n, n+1)$. Then $n \geq 2$ and $n + 1 \leq 3n/2 \leq 3D_2/D_1$, so that $F_\Omega(2D_2 t/D_1) \leq F_\Omega((n+1) t) \leq (n+1)F_\Omega(t) \leq 3D_2 F_\Omega(t)/D_1$ by Lemma 4. Using this, we see that the right-hand side of (28) is bounded by $3e^{-R^2/(4D_t)} \frac{C_2}{C_1} \left( \frac{2D_2}{D_1} \right)^{m/2} F_\Omega(t)$. We choose $R$ such that the coefficient of $F_\Omega(t)$ is equal to $\frac{1}{2}$. That is $R = \alpha t^{1/2}$, where
$$\alpha = \left( 4D_2 \log \left( \frac{6C_2D_2}{C_1D_1} \left( \frac{2D_2}{D_1} \right)^{m/2} \right) \right)^{1/2}.$$  

Rearranging terms in (26) via (27) and (28) with the above choices of $R$ and $\alpha$
gives that
\[
F_{\Omega}(t) \leq 2(\alpha + 1)^{m/2} C_2 \int_{\Omega} \frac{\nu_{\Omega}(x; \alpha t^{1/2})}{|B(x; t^{1/2})|} dx \leq 2(\alpha + 1)^{m/2} \alpha^m C_2 \int_{\Omega} \frac{\nu_{\Omega}(x; \alpha t^{1/2})}{|B(x; t^{1/2})|} \cdot (29)
\]
It follows that
\[
F_{\Omega}(t/\alpha^2) \leq 2(\alpha + 1)^{m/2} \alpha^m C_2 \int_{\Omega} \frac{\nu_{\Omega}(x; t^{1/2})}{|B(x; t^{1/2})|}.
\]
Let \(n \in \mathbb{N}\) be such that \(\frac{1}{\alpha^2} \in [\frac{1}{n+1}, \frac{1}{n})\). Since \(\alpha > 2\), \(n \geq 4\). Hence by Lemma 7,
\[
F_{\Omega}(t/\alpha^2) \geq F_{\Omega}(t/(n + 1)) \geq \frac{1}{n} F_{\Omega}(t) \geq \frac{4}{\alpha^2} F_{\Omega}(t). \text{ By (29)} \text{ we now conclude that}
\]
\[
F_{\Omega}(t) \leq \frac{5}{2} (\alpha + 1)^{m/2} \alpha^{m+2} C_2 \int_{\Omega} \frac{\nu_{\Omega}(x; t^{1/2})}{|B(x; t^{1/2})|}.
\]
We infer that the upper bound in (11) holds with
\[
L_2 = 5 \cdot 2^{1+m} \cdot 3^{m/2} C_2 \left( D_2 \log \left( \frac{6C_2 D_2}{C_1 D_1} \left( \frac{2D_2}{D_1} \right)^{m/2} \right) \right)^{(4+3m)/4}.
\]

3 Examples
In this section we give some examples of open sets \(\Omega\) in \(M = \mathbb{R}^m\), where a precise asymptotic analysis of the heat content for \(t \downarrow 0\) is possible. Recall that in Euclidean space,
\[
p(x, y; t) = (4\pi t)^{-m/2} e^{-|x - y|^2/(4t)}.
\]

Our first example is the following. Let
\[
\Omega = \bigcup_{i \in \mathbb{N}} B(z_i; r_i),
\]
where \((z_i)_{i \in \mathbb{N}}\) is an enumeration of \(\mathbb{Z}^m\), and where \(r_1 \geq r_2 \geq \ldots\). Furthermore, let
\[
\delta = 1 - 2r_1.
\]

**Theorem 8.** (i) If \(\delta > 0\), then \(H_{\Omega}(t) < \infty\) for all \(t > 0\) if and only if
\[
\sum_{i=1}^{\infty} r_i^{2m} < \infty.
\]

If \(\delta > 0\) and (31) holds, then
\[
\left| H_{\Omega}(t) - \sum_{i=1}^{\infty} H_{B(z_i, r_i)}(t) \right| \leq \omega_m^2 e^{-\delta^2/(16t)} \left( \frac{\delta}{(4\pi t)^{1/2}} + \frac{1}{(4\pi t)^{1/2}} \right)^m \sum_{i=1}^{\infty} r_i^{2m}, \quad (32)
\]
where \(\omega_m = |B(0; 1)|\).
(ii) Let
\[ r_i = a_i - \alpha, \quad i \in \mathbb{N}, \]
and let \( 0 < a < \frac{1}{4} \). If \( \frac{1}{2} < \alpha < \frac{1}{m} \) then
\[ H_\Omega(t) = c_{a,m} t^{(m \alpha - 1)/(2 \alpha)} + O(1), \quad t \downarrow 0, \]
where
\[ c_{a,m} = 2^{m-1} \pi^{-m/2} \alpha^{-1} \Gamma((2m \alpha - 1)/(2 \alpha)) a^{1/\alpha} \cdot \int_{B(0,1)} dx \int_{B(0,1)} dy \frac{1}{|x - y|^{(1-2m \alpha)/\alpha}}. \]
If \( \frac{1}{m} < \alpha < \frac{1}{m-1} \) then
\[ F_\Omega(t) = d_{a,m} t^{(m \alpha - 1)/(2 \alpha)} + O(t^{1/2}), \quad t \downarrow 0, \]
where
\[ d_{a,m} = 2^{m-1} \pi^{-m/2} \alpha^{-1} \Gamma((2m \alpha - 1)/(2 \alpha)) a^{1/\alpha} \cdot \int_{B(0,1)} dx \int_{\mathbb{R}^m - B(0,1)} dy \frac{1}{|x - y|^{(1-2m \alpha)/\alpha}}. \]
If \( m > 2 \) and \( \frac{1}{m-1} < \alpha < \frac{1}{m-2} \) or if \( m = 2 \) and \( \frac{1}{m-1} < \alpha \) then
\[ F_\Omega(t) = \pi^{-1/2} \mathcal{P}(\Omega) t^{1/2} + O(t^{(m \alpha - 1)/(2 \alpha)}), \quad t \downarrow 0. \]
If \( m > 2 \) and \( \alpha = \frac{1}{m-2} \) then
\[ F_\Omega(t) = \pi^{-1/2} \mathcal{P}(\Omega) t^{1/2} + O \left( t \log \frac{1}{t} \right), \quad t \downarrow 0. \]
If \( m > 2 \) and \( \frac{1}{m-2} < \alpha \) then
\[ F_\Omega(t) = \pi^{-1/2} \mathcal{P}(\Omega) t^{1/2} + O(t), \quad t \downarrow 0. \]

**Proof.** To prove the first assertion under (i) it suffices, by Theorem 2, to show that \( H_\Omega(\delta^2/4) < \infty \) if and only if \( \sum_{i=1}^{\infty} r_i^{2m} < \infty \). Suppose first that \( H_\Omega(\delta^2/4) < \infty \). By Theorem 2
\[
H_\Omega(\delta^2/4) \geq K_1 \int_\Omega dx \frac{\mu_\Omega(x; \delta/2)}{|B(x; \delta/2)|} \\
= K_1 \sum_{i \in \mathbb{N}} \int_{B(z_i; r_i)} dx \frac{\mu_\Omega(x; \delta/2)}{|B(x; \delta/2)|} \\
\geq K_1 \omega_m \left( \frac{2}{\delta} \right)^m \sum_{\{i: 2r_i < \delta/2\}} \int_{B(z_i; r_i)} dx |B(x; \delta/2) \cap \Omega| \\
= K_1 \omega_m \left( \frac{2}{\delta} \right)^m \sum_{\{i: 2r_i < \delta/2\}} r_i^{2m}, \]
where we have used that if \( r_i < \delta/4 \) and \( x \in B(z_i; r_i) \) then \( B(x; \delta/2) \cap \Omega = B(z_i; r_i) \).

Next suppose that \( \sum_{i \in \mathbb{N}} r_i^{2m} < \infty \). Then

\[
\int_{\Omega} \frac{\mu_\Omega(x; \delta/2)}{|B(x; \delta/2)|} \leq \sum_{\{i: r_i \geq \delta/4\}} \int_{B(z_i; r_i)} dx + \sum_{\{i: r_i < \delta/4\}} \int_{B(z_i; r_i)} dx \frac{\mu_\Omega(x; \delta/2)}{|B(x; \delta/2)|}
\]

\[
\leq \sum_{\{i: r_i \geq \delta/4\}} \int_{B(z_i; r_i)} dx (4r_i/\delta)^m + \omega_m \left( \frac{2}{\delta} \right)^m \sum_{\{i: r_i < \delta/4\}} r_i^{2m}
\]

which implies the reverse implication.

To prove (42), we first note that

\[
H_\Omega(t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{B(z_i; r_i)} dx \int_{B(z_j; r_j)} dy p_{x,y}(x, y; t)
\]

\[
\geq \sum_{i=1}^{\infty} \int_{B(z_i; r_i)} dx \int_{B(z_i; r_i)} dy p_{x,y}(x, y; t)
\]

\[
= \sum_{i=1}^{\infty} H_{B(z_i; r_i)}(t). \tag{41}
\]

To prove the upper bound it suffices to bound the double sum

\[
\sum_{i=1}^{\infty} \sum_{\{j \in \mathbb{N}: j \neq i\}} \int_{B(z_i; r_i)} dx \int_{B(z_j; r_j)} dy p_{x,y}(x, y; t) \tag{42}
\]

from above. We first observe that if \( x \in B(z_i; r_i), y \in B(z_j; r_j) \) and \( |z_i - z_j| = 1 \), then \( |x - y| \geq \delta \). For any other pair of points \( z_i, z_j \in \mathbb{R}^m \) with \( i \neq j \), we have that \( |z_i - z_j| \geq \sqrt{2} \). For such a pair and \( x \in B(z_i; r_i) \) and \( y \in B(z_j; r_j) \), we have by the definition of \( \delta \) that

\[
|z_i - z_j| \geq |z_i - z_j| - |z_i - x| - |y - z_j| \geq |z_i - z_j| + \delta - 1
\]

\[
\geq \frac{|z_i - z_j| + \delta - 1}{|z_i - z_j|} |z_i - z_j| \geq \frac{\sqrt{2} + \delta - 1}{\sqrt{2}} |z_i - z_j|
\]

\[
\geq \delta |z_i - z_j|.
\]

So, combining the estimate for \( |z_i - z_j| = 1 \) with the one for \( |z_i - z_j| \geq \sqrt{2} \), gives that

\[
|x - y|^2 \geq \frac{\delta^2}{2} |z_i - z_j|^2, \quad x \in B(z_i; r_i), \ y \in B(z_j; r_j), \ i \neq j.
\]

Hence, using (30), we have that the expression under (42) is bounded from
above by

\[
\omega_m^2 (4\pi t)^{-m/2} e^{-\delta^2/(16t)} \sum_{i=1}^{\infty} \sum_{\{j \in \mathbb{N} : j \neq i\}} r_i^m r_j^m e^{-|z_i - z_j|^2 \delta^2/(16t)} \\
\leq \frac{1}{2} \omega_m^2 (4\pi t)^{-m/2} e^{-\delta^2/(16t)} \sum_{i=1}^{\infty} \sum_{\{j \in \mathbb{N} : j \neq i\}} (r_i^m + r_j^m) e^{-|z_i - z_j|^2 \delta^2/(16t)} \\
= \omega_m^2 (4\pi t)^{-m/2} e^{-\delta^2/(16t)} \sum_{i=1}^{\infty} r_i^m \sum_{\{j \in \mathbb{N} : j \neq i\}} e^{-|z_i - z_j|^2 \delta^2/(16t)} \\
\leq \omega_m^2 (4\pi t)^{-m/2} e^{-\delta^2/(16t)} \sum_{i=1}^{\infty} r_i^m e^{-|z_i|^2 \delta^2/(16t)} \\
\leq \omega_m^2 (4\pi t)^{-m/2} e^{-\delta^2/(16t)} \sum_{i=1}^{\infty} r_i^m \left(1 + 2 \int_0^\infty dj e^{-j^2 \delta^2/(16t)} \right)^m,
\]

which gives the bound in (32).

To prove part (ii) of Theorem 8, we first consider the case where \( \Omega \) has infinite measure but finite heat content. That is \( \frac{1}{\omega} < \alpha < \frac{1}{m} \). By (32), it suffices to consider the sum in the left-hand side of (32). Since \( r \mapsto H_{B(0;r)}(t) \) is increasing, \( i \mapsto H_{B(0;ai^{-\alpha})}(t) \) is decreasing. Hence

\[
\sum_{i=1}^{\infty} H_{B(z_i;r_i)}(t) = \sum_{i=1}^{\infty} H_{B(0;ai^{-\alpha})}(t) \\
\leq \int_0^\infty di H_{B(0;ai^{-\alpha})}(t) \\
= \int_0^\infty di (ai^{-\alpha})^{2m} \int_{B(0;1)} dx \int_{B(0;1)} dy p_{R^m}(axi^{-\alpha}, ayi^{-\alpha}; t). \\
\]

(43)

A straightforward application of Tonelli’s Theorem gives the formulae under (34) and (35). To obtain a lower bound for the left-hand side of (43), we use the monotonicity of \( i \mapsto H_{B(0;ai^{-\alpha})}(t) \) once more, and obtain that

\[
\sum_{i=1}^{\infty} H_{B(z_i;r_i)}(t) \geq \int_1^\infty di H_{B(0;ai^{-\alpha})}(t) \\
= c_{a,m} t^{(m-1)/(2\alpha)} - \int_0^1 di H_{B(0;ai^{-\alpha})}(t). \\
\]

(44)

The last term in the right-hand side of (44) is bounded in absolute value by

\[
\int_0^1 di H_{B(0;ai^{-\alpha})}(t) \leq \int_0^1 di \int_{B(0;ai^{-\alpha})} dx \int_{\mathbb{R}^m} dy p_{R^m}(x, y; t) \\
= \omega_m \int_0^1 di (ai^{-\alpha})^m \\
= \omega_m a^m (1 - \alpha m)^{-1}.
\]

This completes the proof of the assertion under (33)–(35).
Next consider the case where $\Omega$ has finite measure but infinite perimeter. That is $\frac{1}{m} < \alpha < \frac{1}{m-1}$. We have, by (32) and scaling, that

$$
F_{ii}(t) = \sum_{i=1}^{\infty} F_{B(0, ai^{-\alpha})}(t) + O(e^{-\delta^2/(32t)})
$$

$$
= \sum_{i=1}^{\infty} (ai^{-\alpha})^m F_{B(0, 1)}(a^{-2i^{2\alpha}}t) + O(e^{-\delta^2/(32t)}).
$$

(46)

In a similar way to the proof of (34), (35), we approximate the sum with respect to $i$ by an integral. However, the heat loss $i \mapsto F_{B(0, 1)}(a^{-2i^{2\alpha}}t)$ is increasing in $i$, whereas $i \mapsto (ai^{-\alpha})^m$ is decreasing. Below we consider a decreasing function $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ and an increasing function $g : \mathbb{R}^+ \mapsto \mathbb{R}^+$. For $i \in \mathbb{N}$, and $fg$ summable, we have that

$$
\int_{i}^{i+1} dx f(x)g(x) \geq f(i+1)g(i) = f(i)g(i) + (f(i+1) - f(i))g(i).
$$

(47)

It follows that

$$
\sum_{i=1}^{\infty} f(i)g(i) \leq \int_{1}^{\infty} dx f(x)g(x) - \sum_{i=1}^{\infty} (f(i+1) - f(i))g(i)
$$

$$
\leq \int_{0}^{\infty} dx f(x)g(x) - \sum_{i=1}^{\infty} (f(i+1) - f(i))g(i).
$$

(48)

Similarly

$$
\int_{1}^{i+1} dx f(x)g(x) \leq f(i)g(i+1) = f(i+1)g(i+1) + (f(i) - f(i+1))g(i+1),
$$

(49)

and

$$
\sum_{i=1}^{\infty} f(i+1)g(i+1) \geq \int_{1}^{\infty} dx f(x)g(x) - \sum_{i=1}^{\infty} (f(i) - f(i+1))g(i+1).
$$

(50)

So

$$
\sum_{i=1}^{\infty} f(i)g(i) \geq \int_{0}^{\infty} dx f(x)g(x) - \int_{1}^{1} dx f(x)g(x) - \sum_{i=1}^{\infty} (f(i) - f(i+1))g(i+1).
$$

(51)

Let $f(x) = a^m x^{-m\alpha}$ and $g(x) = F_{B(0, 1)}(a^{-2x^{2\alpha}}t)$. Using $f(x) - f(x+1) \leq a^m max x^{-m\alpha-1}$, we obtain that

$$
0 \leq \sum_{i=1}^{\infty} (f(i) - f(i+1))g(i+1) \leq a^m m \sum_{i=1}^{\infty} \sum_{i=1}^{\infty} i^{-m\alpha-1} F_{B(0, 1)}(a^{-2(1+i)^{2\alpha}}t)
$$

$$
\leq a^m m \sum_{i=1}^{\infty} \sum_{i=1}^{\infty} i^{-m\alpha-1} F_{B(0, 1)}(a^{-2(2i)^{2\alpha}}t)
$$

$$
\leq a^m m^2 \omega_m \alpha \pi^{-1/2} \sum_{i=1}^{\infty} i^{-m\alpha-1} (a^{-2(2i)^{2\alpha}}t)^{1/2}
$$

$$
= O(t^{1/2}),
$$

(52)
where we have used that (Proposition 8 in [12])

\[ F_{B(0,1)}(t) \leq m\omega_m \pi^{-1/2} t^{1/2}. \]  

(53)

Furthermore, by (53), we have that

\[ \int_0^1 dx f(x)g(x) \leq a^{m-1} m\omega_m \pi^{-1/2} t^{1/2} \int_0^1 dx x^{-ma+\alpha} = O(t^{1/2}). \]

By (46)-(53) we conclude that

\[
F_\Omega(t) = \int_0^\infty dx a^m x^{-ma} F_{B(0,1)}(a^{-2}x^{2\alpha}t) + O(t^{1/2}) \\
= d_{a,m} t^{(ma-1)/(2\alpha)} + O(t^{1/2}),
\]  

(54)

where \( d_{a,m} \) is given by (37), and where the integral with respect to \( x \) in (53) has been evaluated with the change of variable \( a^{-2}tx^{2\alpha} = \theta \). This completes the proof of (36).

Finally, we consider the cases where \( \Omega \) had both finite measure and finite perimeter. Suppose \( m > 2 \) and \( \frac{1}{m} < \alpha < \frac{1}{m-2} \) or \( m = 2 \) and \( \frac{1}{m-1} < \alpha \). Let \( I \in \mathbb{N} \), and apply Theorem 2 from [3] to a ball of radius \( r \):

\[
|H_{B(0;r)} - |B(0;r)| + \pi^{-1/2} P(B(0;r))t^{1/2}| \leq c_m r^{m-2} t, \quad t > 0,
\]  

(55)

where \( c_m = m^3 \omega_m 2^{m+2} \). Then

\[
H_\Omega(t) \geq \sum_{i=1}^I H_{B(0;ai^{-\alpha})}(t) \\
\geq |\Omega| - \pi^{-1/2} P(\Omega)t^{1/2} - \sum_{i=1}^\infty \omega_m a^m i^{-\alpha m} - c_m \sum_{i=1}^I (ai^{-\alpha})^{m-2} t.
\]  

(56)

The third term in the right-hand side of (56) is bounded by \( I^{1-\alpha m} \) up to a multiplicative constant. The fourth term is bounded up to a multiplicative constant by \( I^{1-\alpha(m-2)}t \). Minimising the sum \( I^{1-\alpha m} + I^{1-\alpha(m-2)}t \), gives that \( I = \lfloor t^{-1/(2\alpha)} \rfloor \) up to a constant. This gives a remainder \( O(t^{(ma-1)/(2\alpha)}) \) for the lower bound.
To obtain an upper bound, we let $J \in \mathbb{N}$, and note that by \((32)\),

$$H_\Omega(t) \leq \sum_{i=1}^{\infty} H_{B(0; ai^{-\alpha})}(t) + O(e^{-\delta^2/(32t)})$$

$$\leq \sum_{i=1}^{J} H_{B(0; ai^{-\alpha})}(t) + \sum_{i=J+1}^{\infty} |B(0; ai^{-\alpha})| + O(e^{-\delta^2/(32t)})$$

$$\leq \sum_{i=1}^{J} \left( |B(0; ai^{-\alpha})| - \pi^{-1/2} \mathcal{P}(B(0; ai^{-\alpha})) t^{1/2} + c_m (ai^{-\alpha})^{-m-2} t \right)$$

$$+ \sum_{i=J+1}^{\infty} |B(0; ai^{-\alpha})| + O(e^{-\delta^2/(32t)})$$

$$\leq |\Omega| - \pi^{-1/2} \mathcal{P}(\Omega) t^{1/2} + \pi^{-1/2} m \omega_m \sum_{i=J+1}^{\infty} (ai^{-\alpha})^{-m-1} t^{1/2}$$

$$+ c_m \sum_{i=1}^{J} (ai^{-\alpha})^{-m-2} t + O(e^{-\delta^2/(32t)}). \quad (57)$$

The third term in the right-hand side of \((57)\) is bounded up to a multiplicative constant by $J^{1-\alpha(m-1)} t^{1/2}$. The fourth term in the right-hand side of \((57)\) is bounded up to a multiplicative constant by $J^{1-\alpha(m-2)} t$. Minimising $J^{1-\alpha(m-1)} t^{1/2} + J^{1-\alpha(m-2)} t$ gives that for $t \downarrow 0$, $J = [t^{-1/(2\alpha)}]$ up to a constant. This gives a remainder $O(t^{(m\alpha-1)/(2\alpha)})$ for the upper bound, and completes the proof of \((38)\).

Next consider the case $\alpha = \frac{1}{m-2}$. Then the sum of the third and fourth terms in the right-hand side of \((56)\) equals, up to constants, $I^{-2/(m-2)} + t \log I$. We now choose $I = \lceil t^{-(m-2)/2} \rceil$, and obtain the remainder in \((39)\). Similarly, the sum of the third and fourth terms in the right-hand side of \((57)\) is of order $J^{-1/(m-2)} t^{1/2} + t \log J$. We now choose $J = \lceil t^{-(m-2)/2} \rceil$ to obtain the same remainder.

Finally, consider the case $\alpha > (m - 2)^{-1}$. Then the uniform remainder in the right-hand side of \((55)\) is summable. Hence by \((32)\),

$$|H_\Omega(t) - |\Omega| + \pi^{-1/2} \mathcal{P}(\Omega) t^{1/2}| \leq c_m \sum_{i \in \mathbb{N}} (ai^{-\alpha})^{-m-2} t + O(e^{-\delta^2/(32t)}), \ t > 0,$$

and we obtain the remainder in \((40)\). This concludes the proof of Theorem \((8)\).

In our second example we take the same collection of radii as in \((33)\) but align the balls such that the centres are on the positive $x_1$ axis and such that the balls are pairwise disjoint, decreasing in size and touching. That is

$$\Lambda = \cup_{i \in \mathbb{Z}} B(v_i; r_i).$$

where $v_1 = (0, \ldots, 0)$, and

$$v_i = a(1^{-\alpha} + 2 \sum_{2 \leq j \leq i-1} j^{-\alpha} + i^{-\alpha}, 0, \ldots, 0), \ i \geq 2.$$
Theorem 9.  (i) \( H_\Lambda(t) < \infty \) for all \( t > 0 \) if and only if \( \alpha > \frac{1}{2m-1} \).

(ii) If \( \frac{1}{2m-1} < \alpha < \frac{1}{m} \), then

\[
H_\Lambda(t) \approx t^{(m\alpha - 1)/(2\alpha)}, \quad t \downarrow 0.
\]

We note that the range of \( \alpha \)'s for which \( H_\Lambda(t) \) is finite is different from those \( \alpha \)'s for which \( H_\Omega(t) \) is finite. Realigning the balls as in Theorem 9 reduces the heat flow out of \( \Lambda \). However, the leading order behaviour as \( t \downarrow 0 \) is the same for the common values of \( \alpha \).

Proof. A has infinite measure if and only if \( \alpha \leq \frac{1}{m} \), and so it suffices to consider this case. Let

\[
t \leq \frac{a^2}{64}, \quad i \geq \left( \frac{8a}{t^{1/2}} \right)^{1/\alpha}.
\]  \hspace{1cm} (58)

We note that (58) implies that \( i \geq 2^{12} \). We let \( x \in B_i := B(v_i; a i^{-\alpha}) \). We wish to find a lower bound for \( \mu_\Lambda(x; t^{1/2}) \), and consider the collection of balls \( B_j, \ j = i, i + 1, \ldots, I(i) \) that are contained in \( B(x; t^{1/2}) \). By (58), there exists \( i \in \mathbb{N} \) such that

\[
2a \sum_{j=1}^{I(i)} j^{-\alpha} \leq t^{1/2} < 2a \sum_{j=i}^{I(i)+1} j^{-\alpha}.
\]  \hspace{1cm} (59)

Then

\[
t^{1/2} < 2a(I(i) - 2 - i)a^{-\alpha}.
\]

So the number of balls \( B_j \) in \( B(x; t^{1/2}) \) with \( j \geq i \) is bounded from below by \( \frac{t^{1/2}}{2a^{-}\alpha} \geq \frac{t^{1/2}}{a^{-\alpha}} \). The smallest ball in this collection has measure \( \omega_m(a I(i)^{-\alpha})^m \).

We conclude that

\[
\mu_\Lambda(x; t^{1/2}) \geq \omega_m t^{1/2} a^{-\alpha} (a I(i)^{-\alpha})^m, \quad x \in B_i.
\]  \hspace{1cm} (60)

To obtain an upper bound for \( I(i) \), we bound the left-hand side of (59) from below by \( 2a \int_{i}^{I(i)} dx x^{-\alpha} \). This gives that

\[
I(i) \leq \left( \frac{1 - \alpha}{2a} t^{1/2} + i^{-1} \right)^{1/(1-\alpha)}.
\]  \hspace{1cm} (61)

By (60) and (61), we find that

\[
\int_{B_i} dx \mu_\Lambda(x; t^{1/2}) \geq \omega_m a^{-\alpha} (a I(i)^{-\alpha})^m \left( \frac{1 - \alpha}{2a} t^{1/2} + i^{-1} \right)^{-\alpha}.
\]  \hspace{1cm} (62)

The right-hand side of (62) is not summable if \( \alpha \leq (2m - 1)^{-1} \).

Next we show that \( H_\Lambda(t) \) is finite if \( \alpha > (2m - 1)^{-1} \). We consider all balls \( B_j \) that intersect \( B(x; t^{1/2}) \), \( x \in B_i \), and let \( B_{J(i)} \) be the largest of these balls. Then

\[
2a \sum_{j=J(i)+1}^{i-1} j^{-\alpha} < t^{1/2} \leq 2a \sum_{j=J(i)}^{i-1} j^{-\alpha}.
\]  \hspace{1cm} (63)
We have that \( \Lambda \cap B(x; t^{1/2}) \) is contained in a cylinder of height \( 2t^{1/2} \) and base an \((m-1)\)-dimensional ball with radius \( aJ(i)^{-\alpha} \). By monotonicity, we have that
\[
\mu_\Lambda(x; t^{1/2}) \leq 2\omega_{m-1}(aJ(i)^{-\alpha})^{m-1}t^{1/2}, \quad x \in B_t.
\]
Hence
\[
\int_{B_t} dx \mu_\Lambda(x; t^{1/2}) \leq 2\omega_{m-1}\omega_m(aJ(i)^{-\alpha})^m(aJ(i)^{-\alpha})^{m-1}t^{1/2}.
\]
(64)
By the first inequality in (63), we have that
\[
2a(i-2-J(i))(i-1)^{-\alpha} < t^{1/2}.
\]
So, by (63),
\[
J(i) \geq i - 2 - \frac{(i-1)^{\alpha}t^{1/2}}{2a} \geq i - 2 - \frac{i^{1/2}t^{1/2}}{2a} \geq i - 2 - \frac{i^{1/2}}{16} \geq \frac{i}{2}.
\]
(65)
since \(i \geq 2^{12} \). By (64) and (65), we have that
\[
\int_{B_t} dx \mu_\Lambda(x; t^{1/2}) \leq 2^{1+(m-1)\alpha}\omega_{m-1}\omega_m(aJ(i)^{-\alpha})^m(aJ(i)^{-\alpha})^{m-1}t^{1/2}.
\]
(66)
The right-hand side of (66) is summable for \( \alpha > (2m-1)^{-1} \). This concludes the proof of part (i).

To prove part (ii), we note by (41) that for any disjoint collection of balls with radii \( r_1 \geq r_2 \geq \cdots \),
\[
H_{\cup_{i \in \mathbb{N}} B(z_i, r_i)}(t) = \sum_{i \in \mathbb{N}} h_i(\sum_{i \in \mathbb{N}} r_i^m H_{B(0, 1)}(t/r_i^2)).
\]
It follows that
\[
H_\Lambda(t) \geq \sum_{i=1}^\infty a_i^{-\alpha} h_i(2^{\alpha}a^2).
\]
Since \( \frac{1}{\omega_m} < \alpha < \frac{1}{m} \), we just follow the lines of (11)-(15) to conclude that
\[
\lim_{t \to 0} t^{(1-m\alpha)/(2\alpha)} H_\Lambda(t) \geq c_{\alpha, m}.
\]
To prove the upper bound we use Theorem 2. For \( x \in \cup_{i: \omega_i < \frac{8a}{1-t^{1/2}}} B(v_i; aJ(i)^{-\alpha}) \), we bound the integrand in the right-hand side of (10) from above by 1. So these balls give a contribution
\[
K_2 \sum_{i: \omega_i t^{1/2} < a} |B(v_i; aJ(i)^{-\alpha})| \leq K_2\omega_m(1 - \alpha m)^{-1}a^m \left( \frac{8a}{t^{1/2}} \right)^{(1-\alpha m)/\alpha}.
\]
(67)
The contribution from the remaining balls in \( \Lambda \) that satisfy (58) can be estimated via (66). These give, for all \( t \) satisfying (58), a contribution that is bounded from above by
\[
K_2 2^{1+(m-1)\alpha}\omega_{m-1}\omega_m \sum_{i: \omega_i t^{1/2} \geq 8a} a_i^{-\alpha} m(aJ(i)^{-\alpha})^{m-1}(1-m)/2,
\]
that is non-negative and \( O(t^{(m-1)/(2\alpha)}) \). This, together with (67), completes the proof of Theorem 2(ii).
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