Accelerating vacua in Gauss-Bonnet gravity

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Abstract

Accelerating vacua with maximally symmetric, but not necessarily spherical, sections for Einstein and Gauss-Bonnet gravities in generic dimensions are obtained. The acceleration parameter has the effect of shifting the cosmological constants in Einstein gravity, whereas in Gauss-Bonnet gravity the effective cosmological constants remain the same in the presence of acceleration as in the case without acceleration.

1 Introduction

In a recent note [1], one of the authors found that, in $n$-dimensions, adding a conformal factor

$$\omega^2(r, \theta_1) = \frac{1}{(1 - \alpha r \cos \theta_1)^2}$$

(1)

in front of the standard de Sitter, Minkowski and/or anti-de Sitter metrics still makes Einstein vacua, with cosmological constant appropriately modified. The result can be summarized as follows. Let

$$ds^2_\Lambda = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega^2_{n-2}$$

(2)

with

$$f(r) = 1 - \frac{2\Lambda}{(n-1)(n-2)}r^2$$

(3)

representing an Einstein vacuum solution with cosmological constant $\Lambda$, where $d\Omega^2_{n-2}$ represents the line element of an $(n-2)$-dimensional sphere spanned by the angular coordinates $(\theta_1, \theta_2, ..., \theta_{n-2} = \phi)$. Then, the conformally transformed metric

$$ds^2_{\Lambda,\alpha} = \omega^2(r, \theta_1)ds^2_\Lambda$$

(4)

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is also an Einstein vacuum with cosmological constant

$$\Lambda_\alpha = \Lambda - \frac{(n-1)(n-2)\alpha^2}{2}. \tag{5}$$

For $\alpha \neq 0$, the cosmological constant $\Lambda_\alpha$ in the presence of the conformal factor is different from the original cosmological constant $\Lambda$. That the metric (4) represents an accelerating vacua was known before the work [1] for $n = 3$ [2], $n = 4$ [3, 4] and $n = 5$ [5].

Each of the constant $r = C \neq 0$ hyper surface in (2) represents a conicoid (or part of a conicoid) with eccentricity $\alpha C$, and one of the foci of the conicoid is located at the spatial origin $r = 0$. The physical interpretation of the parameter $\alpha$ is clear: it is the magnitude of the proper acceleration of the static observer located at the specified focus at the origin.

In another recent work [6], we found some accelerating vacua for Gauss-Bonnet gravity in 5 and 6 dimensions, with angular sections being either spherical or hyperbolic. For hyperbolic cases, the conformal factor in the metric is different from the one given in (1).

Physically, accelerating vacua are of interests because they are the vacua to be perceived by observers under relative accelerations with respect to the free-falling observers. In the study of gravitational theories, equivalence principle is often used to cancel out local gravitational effect by changing into the free-falling frames. However, in some cases, it might be necessary to make use of the equivalence principle in the converse way. Imagine that one day our human beings might be able to make intergalactic voyages. In such circumstances, it will be inevitable to encounter the case in which the observers undergo relative acceleration with respect to the free-falling observers, and equivalence principle predicts that the observed structure of spacetime for observers in the spaceship should be different from that for the free-falling observers. Though higher dimensional accelerating vacua are not directly related to such practical applications, they still constitute an essential part of gravity theories in the corresponding spacetime dimension and thus should be of academic interests.

The present work is aimed at extending the previous results to more general settings, e.g. allowing $d\Omega_{n-2}^2$ to be replaced by the metric of a maximally symmetric subspace and/or extending the results to higher curvature gravity theories such as Gauss-Bonnet gravity. We shall accomplish our goal via two steps: accelerating vacua for Einstein gravity will be considered in Section 2, and extensions to Gauss-Bonnet gravity will be presented in Section 3. Proper accelerations of the vacua are analyzed in Section 4. And finally we give the conclusion in Section 5.
2 Accelerating vacua in Einstein gravity

It is well known that a generalization of the metric (2) to the case with maximally symmetric sections exists, the metric is given by

$$ds^2_{\Lambda,k} = -f_k(r)dt^2 + f_k(r)^{-1}dr^2 + r^2d\Sigma^2_{n-2,k},$$

where

$$f_k(r) = k - \frac{2\Lambda}{(n-1)(n-2)}r^2.$$  \hspace{1cm} (7)

Here \(k = 0, \pm 1\) and \(d\Sigma^2_{n-2,k}\) represents the metric of an \((n-2)\)-dimensional maximally symmetric manifold and can be written explicitly as

$$d\Sigma^2_{n-2,k} = d\theta_1^2 + \rho_k^2(\theta_1)d\Omega^2_{n-3},$$

$$\rho_k(\theta_1) = \lim_{\kappa \to k} \frac{\sin(\sqrt{\kappa} \theta_1)}{\sqrt{\kappa}} = \begin{cases} 
\sin \theta_1 & (k = +1) 
\theta_1 & (k = 0) 
\sinh \theta_1 & (k = -1)
\end{cases}$$ \hspace{1cm} (9)

where \(d\Omega^2_{n-3}\) is the line element of an \((n-3)\)-sphere. The choice \(k = 1\) corresponds to the spherically symmetric case (2) and \(k = 0, -1\) correspond to flat and hyperbolic cases, respectively. The metric (6) with insertions (7)-(9) solves the vacuum Einstein equation

$$R_{MN} - \frac{1}{2}g_{MN}R + \Lambda g_{MN} = 0.$$ \hspace{1cm} (10)

We can rewrite the function \(f_k(r)\) in a more familiar form as

$$f_k(r) = k - \epsilon \frac{r^2}{\ell^2},$$ \hspace{1cm} (11)

where \(\ell\) represents the (A)dS radii and \(\epsilon\) is the sign of \(\Lambda\),

$$\frac{1}{\ell^2} = \frac{2|\Lambda|}{(n-1)(n-2)}, \quad \epsilon = \text{sign}(\Lambda).\hspace{1cm} (12)$$

One can read off the cosmological constant \(\Lambda\) from the metric function (11) using

$$\Lambda = \frac{(n-1)(n-2)\epsilon}{2\ell^2}. \hspace{1cm} (13)$$

\(^1\)The metric (6) with \(f_k(r)\) given in (7) is the zero mass zero charge limit of the so-called topological AdS black hole solution found in [7, 8, 9]. With nonzero, positive mass, the metric in [7, 8, 9] was called topological AdS black hole because \(k = 0, -1\) are allowed only for \(\Lambda < 0\). For \(\Lambda \geq 0\) and \(k = 0, -1\), the corresponding solution does not allow for the existence of a horizon due to Hawking’s black hole topology theorem [10] and its cousin in higher dimensions [11]. The restriction \(\Lambda < 0\) for \(k = 0, -1\) holds even in the zero mass limit, because we need a region in the spacetime in which \(f_k(r) > 0\) in order to interpret \(t\) as a timelike coordinate.
It is natural to ask what the $k = 0, -1$ analogues of (4) are. After some tedious tensor algebra we found the following to be the right answer:

$$ds^2_{\Lambda, k, \alpha} = \omega^2_k(r, \theta_1) ds^2_{\Lambda, k},$$

where

$$\omega^2_k(r, \theta_1) = \frac{1}{(1 - \alpha r \sigma_k(\theta_1))^2},$$

and

$$\sigma_k(\theta_1) = \cos \left( \sqrt{k} \theta_1 \right) = \begin{cases} 
\cos \theta_1 & (k = +1) \\
1 & (k = 0) \\
\cosh \theta_1 & (k = -1) 
\end{cases}$$

It can be easily checked that the metric (14) obeys the vacuum Einstein equation (10) with the cosmological constant $\Lambda$ replaced by

$$\Lambda_\alpha = \Lambda - \frac{(n - 1)(n - 2)k\alpha^2}{2}.$$ 

For $k = +1$, this reproduces the known result (5). Notice that for $k = \pm 1$, a nonzero cosmological constant $\Lambda_\alpha$ can be produced by the presence of the parameter $\alpha$ even if we start from $\Lambda = 0$.

## 3 Accelerating vacua in Gauss-Bonnet gravity

Gauss-Bonnet gravity has the equation of motion

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} + \xi H_{\mu\nu} = 0,$$ 

where

$$H_{\mu\nu} = 2 \left( R_{\mu\lambda\rho\sigma} R^{\lambda\rho\sigma} - 2 R_{\mu\rho\sigma} R^{\rho\sigma} - 2 R_{\mu\sigma} R^\sigma + R R_{\mu\nu} \right) - \frac{1}{2} \mathcal{L}_{GB} g_{\mu\nu}.$$ 

This is the consequence of variations of the action (dropping an appropriate boundary term) of Gauss-Bonnet gravity in $n$-dimensions

$$I = \frac{1}{16\pi G} \int d^n x \sqrt{-g} \left[ R - 2\Lambda + \xi \mathcal{L}_{GB} \right],$$

$$\mathcal{L}_{GB} = R_{\mu\nu\gamma\delta} R^{\mu\nu\gamma\delta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2,$$

where $G$ is the $n$-dimensional Newton constant, $\Lambda$ is a cosmological constant and $\xi$ is the Gauss-Bonnet parameter.
The standard vacua of the form (6) for Gauss-Bonnet gravity are given in terms of the metric function \( f_k(r) \) as

\[
(ds_{\text{GB}})^2_{\Lambda,k} = -f_k(r)dt^2 + f_k(r)^{-1}dr^2 + r^2d\Sigma^2_{n-2,k},
\]

(20)

\[
f_k(r) = k + \frac{1}{2(n-3)(n-4)\xi}(1 + \delta \sqrt{a_n})r^2,
\]

(21)

where

\[
a_n = 1 + \frac{8(n-3)(n-4)\Lambda}{(n-1)(n-2)}
\]

and \( \delta = \pm 1 \) indicate that there are two branches for the vacua which is the consequence of the \( R^2 \) form of the action. These vacua are obtained from the Gauss-Bonnet black hole solutions in the zero mass limit. The \( k = +1 \) version of these vacua was first found by Boulware and Deser in [12], and the general form (21) can be inferred from [13, 14]. The same solution can also be obtained from the works [15, 16] on general curvature squared gravity theory by taking the Gauss-Bonnet limit.

The function \( f_k(r) \) defined in (21) can also be written in the form (11), but now with \( \ell \) and \( \epsilon \) given as follows:

\[
\frac{1}{\ell^2} = \frac{|1 + \delta \sqrt{a_n}|}{2(n-3)(n-4)\xi}, \quad \epsilon = -\text{sign}(1 + \delta \sqrt{a_n}).
\]

(22)

This indicates that the solutions (20) are also Einstein vacua, with effective cosmological constants

\[
\Lambda_{\text{eff}} = -\frac{(n-1)(n-2)}{4(n-3)(n-4)\xi}(1 + \delta \sqrt{a_n}),
\]

(23)

thanks to the relation (13).

Now we would like to ask whether the accelerating vacua of the form similar to (14) exists for Gauss-Bonnet gravity. A direct substitution yields that the ansatz

\[
(ds_{\text{GB}})^2_{\Lambda,\alpha,k} = \omega_k^2(r, \theta_1)(ds_{\text{GB}})^2_{\Lambda,k}
\]

with \( (ds_{\text{GB}})^2_{\Lambda,k} \) given as in (20)-(21) fails to solve the Gauss-Bonnet field equations (18)-(19) unless \( k = 0 \) or \( \alpha = 0 \). However, one can check that the following metrics indeed solve the field equation for Gauss-Bonnet gravity:

\[
(ds_{\text{GB}})^2_{\Lambda,k,\alpha} = \omega_k^2(r, \theta_1)ds_{k,\alpha}^2,
\]

(24)

\[
ds_{k,\alpha}^2 = -f_{k,\alpha}(r)dt^2 + f_{k,\alpha}(r)^{-1}dr^2 + r^2d\Sigma^2_{n-2,k},
\]

(25)

\[
f_{k,\alpha}(r) = k - \left( ka^2 - \frac{1}{2(n-3)(n-4)\xi}(1 + \delta \sqrt{a_n}) \right) r^2.
\]

(26)
Here, again, $\omega_2^2(r, \theta_1)$ is given by (15). For $k = \pm 1$ and $n = 5, 6$, the above solution reproduces the results of [6]. We can also rewrite (26) as

$$f_{k,\alpha}(r) = k - \frac{r^2}{\ell^2},$$

but now with

$$\frac{1}{\ell^2} = \left| k\alpha^2 - \frac{1}{2(n-3)(n-4)\xi} \left( 1 + \delta \sqrt{a_n} \right) \right|,$$

$$\epsilon = \text{sign} \left( k\alpha^2 - \frac{1}{2(n-3)(n-4)\xi} \left( 1 + \delta \sqrt{a_n} \right) \right).$$

This shows that the metrics represented by $ds_{k,\alpha}^2$ are Einstein vacua (however, not Gauss-Bonnet vacua, which explains the absence of the suffix GB in the notation), with the corresponding cosmological constants (due to (13))

$$\Lambda_{k,\alpha} = \frac{(n-1)(n-2)}{2} \left( k\alpha^2 - \frac{1}{2(n-3)(n-4)\xi} \left( 1 + \delta \sqrt{a_n} \right) \right).$$

In the $\alpha \to 0$ limit, the metrics $ds_{k,\alpha}^2$ will approach the Gauss-Bonnet vacua (20).

The Gauss-Bonnet vacua (24) are also Einstein vacua, with cosmological constants given as

$$\Lambda_{\alpha} = \Lambda_{k,\alpha} - \frac{(n-1)(n-2)k\alpha^2}{2}$$

$$= -\frac{(n-1)(n-2)}{4(n-3)(n-4)\xi} \left( 1 + \delta \sqrt{a_n} \right),$$

which is identical to the effective cosmological constant $\Lambda_{\text{eff}}$ (eq.(23)) for the non-accelerating Gauss-Bonnet vacua (20). We stress that the effective cosmological constants for the accelerating vacua of Gauss-Bonnet gravity depend neither on $k$ nor on the parameter $\alpha$. This is in sharp contrast to the case of pure Einstein gravity.

### 4 Proper accelerations of the vacua

So far we have been referring to the spacetimes with conformal factor $\omega_2^2(r, \theta_1)$ with the term “accelerating vacua” Without justification for the appropriateness of this terminology. Now we are in a position to settle this problem.

Both the vacua (14) of Section 2 and (26) of Section 3 can be written in a unified way as

$$ds^2 = \omega_2^2(r, \theta_1) \left[ - \left( k - \frac{r^2}{\ell^2} \right) dt^2 + \left( k - \frac{r^2}{\ell^2} \right)^{-1} dr^2 + r^2 d\Sigma_{n-2,k}^2 \right].$$
This allows us to look at the static observers
\[ x^\mu = \left( \frac{1}{\sqrt{k - \epsilon r^2/\ell^2}} \right)^{\mu} r, \theta_1, \ldots, \theta_{n-2} \]
with proper velocity
\[ u^\mu = \left( \frac{1}{\sqrt{k - \epsilon r^2/\ell^2}} \right)^{\mu} 0, \ldots, 0, \]
where \( \tau \) is the proper time. Note that in writing (28) we have assumed that the observer lives in a region of the spacetime in which \( k - \epsilon r^2/\ell^2 > 0 \), otherwise the observer will not be static. This requires, in particular, \( \epsilon = -1 \) for \( k = 0, -1 \). Straightforward calculations show that the proper acceleration \( a^\mu = u^\nu \nabla_\nu u^\mu \) has only two nonzero components, i.e.
\[ a^r = \frac{1}{\ell^2} \left( 1 - \alpha r \sigma_k(\theta_1) \right) \left( \epsilon r + k \ell^2 \alpha \sigma_k(\theta_1) \right), \]
\[ a^{\theta_1} = (1 - \alpha r \sigma_k(\theta_1)) \frac{\alpha d\sigma_k(\theta_1)}{r} \frac{d\theta_1}{d\theta_1}. \]
Contracting \( a^\mu \) with itself, we get
\[ a^\mu a_\mu = \alpha^2 \left( \frac{d\sigma_k(\theta_1)}{d\theta_1} \right)^2 + \frac{1}{\ell^2} \frac{(\epsilon r/\ell + k \alpha \sigma_k(\theta_1))^2}{k - \epsilon r^2/\ell^2}. \tag{29} \]
For \( k = \pm 1 \), the squared norm \( a^\mu a_\mu \) at the origin reads
\[ a^\mu a_\mu = k \alpha^2, \]
which justifies our statement that \( \alpha \) is proportional to the magnitude of the proper acceleration of the observer at the origin. However, we should note that for \( (\epsilon = -1, k = 0) \), the point \( r = 0 \) is a singularity of the metric, so it is meaningless to talk about the proper acceleration at the origin in this case. Beyond the singularity at \( r = 0 \), the proper acceleration for \( (\epsilon = -1, k = 0) \) is uniform and independent of \( \alpha \):
\[ a^\mu a_\mu = \frac{1}{\ell^2}. \]

Let us remark that, since \( a^t = 0 \), the proper acceleration \( a^\mu \) is spacelike when \( t \) is timelike. So we ought to have \( a^\mu a_\mu > 0 \) in the static region. This is indeed the case as can be seen from (29), where the static region is determined by the condition \( k - \epsilon r^2/\ell^2 > 0 \). Notice that for \( k = -1, \epsilon = -1 \), the condition for the static region is reduced into \( r > \ell \), the origin \( r = 0 \) is clearly not in the static region. This explains why \( a^\mu a_\mu = -\alpha^2 < 0 \) at \( r = 0 \) for \( k = -1 \). Notice also that, for the two distinguished cases \( (\epsilon = +1, k = +1) \) and \( (\epsilon = -1, k = -1) \), \( a^\mu a_\mu \) becomes infinity at \( r = \ell \). This implies that the \( r = \ell \) hyper surfaces in these two cases are accelerating horizons. The
difference between the two cases lies in that, for \( (\epsilon = +1, k = +1) \), the condition for the static region is \( r < \ell \), so the acceleration horizon is like a cosmological horizon (i.e. static observers are located inside the horizon); while for \( (\epsilon = -1, k = -1) \), the condition for the static region is \( r > \ell \), and the acceleration horizon is like a black hole horizon (static observers are located outside the horizon)\(^2\).

5 Conclusions

We have thus addressed the problem of finding accelerating vacua for Einstein and Gauss-Bonnet gravities with maximally symmetric, but not necessarily spherical, sections. For Einstein gravity, the accelerating vacua can be obtained by conformally transforming some given, non accelerating, Einstein vacua with some prescribed cosmological constants. Then, the accelerating vacua will correspond to a different cosmological constant, i.e. acceleration has the effect of shifting cosmological constant for Einstein gravity.

For Gauss-Bonnet gravity, the accelerating vacua cannot be obtained by conformally transforming non accelerating Gauss-Bonnet vacua. Instead, they can be obtained by conformally transforming some non accelerating Einstein vacua, and the effective cosmological constants remain the same as those for non accelerating Gauss-Bonnet vacua.

It should be remarked that the solutions we obtained here are only accelerating Einstein vacua without black holes. It is more tempting to get the accelerating black hole solutions but this seems to be extremely difficult. The physical reason for why the accelerating black holes are difficult to find is because that it requires tremendous energy to accelerate black holes and hence the corresponding metrics would in general not vacua. However, just what kind of matter source can provide the energy necessary to accelerate black holes in Einstein and Gauss-Bonnet gravities remain an open problem, the only exceptions are the 4-dimensional C-metric black hole solutions found long ago by Levi-Civita (see [3, 4] for detailed description for versions with cosmological constant) for Einstein gravity, in which conical singularities on the horizons play the role of source for the acceleration.

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\(^2\) For \( k = 0, -1 \), the hyper surface with metric \( d\Sigma^{2-2k} \) are noncompact, so, at first sight, it is meaningless to talk about the inside and outside of the horizon. However, one can consider these cases modulo a discrete symmetry group, making the quotient compact. In this sense one can indeed talk about the inside and outside of the horizon for \( k = -1 \).
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