DEGENERATE EISENSTEIN SERIES FOR $Sp(4)$

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to Steve Rallis, in memoriam

Abstract. In this paper we obtain a complete description of images and poles of degenerate Eisenstein series attached to maximal parabolic subgroups of $Sp_4(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of $\mathbb{Q}$.

1. Introduction

The degenerate Eisenstein series attached to Siegel parabolic subgroups of symplectic and metaplectic groups has been studied extensively for various applications such as example explicit construction of automorphic $L$-functions [3], and application in the Siegel–Weil formula [11] [12] [13] [8]. On the other hand, we have used a more general type of degenerate Eisenstein series to construct and prove unitarity of various significant classes of unitary representations of local (real or $p$-adic groups) and to construct various families of square–integrable automorphic forms [17] [18] [5]. In [6] we study degenerate Eisenstein series for $GL_n$, their restriction to archimedean place and images (this improves in part results of [10]).

The problem of getting complete information about poles of Eisenstein series and their images for classical groups is mostly related to our insufficient understanding of various types of degenerate principal series representations and standard intertwining operators especially for real groups that appear in the constant term of Eisenstein series. For Siegel Eisenstein series this problem is solved in [12]. In the current literature, there are also some other works which deal with various types of “Siegel–like” degenerate principal series (see for example [7] [14] [15] [16] [4]) but more complicated ones appear in the theory of automorphic forms (see the last section in [18] for simple examples or [5] for more sophisticated examples).

Date: November 17, 2021.
1991 Mathematics Subject Classification. 11F70, 22E50.
Key words and phrases. automorphic forms, degenerate Eisenstein series, normalized intertwining operators.
For the group $Sp_4(\mathbb{R})$, there is a description (see [19]) of all generalized and degenerate principal series in terms of the Langlands classification as well as some information about the images and poles of the local intertwining operators. This is used in the present paper along with the local information on $p$–adic places [21] to get the complete description of the images and poles of degenerate Eisenstein series for Siegel and Heisenberg parabolic subgroups of adelic $Sp_4$. The description of the residual spectrum and square–integrable non–cuspidal automorphic forms is well–known [9].

Now, we will describe the paper by sections. In Section 2 we state notation regarding the group $Sp_4$, we define degenerate Eisenstein series, and recall basic procedure of computing their poles thorough the constant term and local normalized intertwining operators. In Section 3 we deal with the Eisenstein series attached to the Heisenberg parabolic subgroup. The main results are Theorems 3.6 and 3.7. In Section 4 we deal with the Eisenstein series attached to the Siegel parabolic subgroup. The main results are Theorems 4.3 and 4.4. As we explain above, this last part has an overlap with the results of [12], but we have a different approach to the archimedean components [19] which gives us the answer in terms of the Langlands classification which is hard to see from [12].

In the future papers we plan to extend this work beyond $Sp_4$ (see also the last Section of [18]). One of the obstacles that we need to overcome is better understanding certain degenerate principal series which are induced from non–Siegel parabolic subgroups of $Sp_{2n}(\mathbb{R})$ (see also the last Section of [18]).

2. Preliminaries

For $n \in \mathbb{Z}_{\geq 1}$, we define $J_n$ as a $n \times n$ matrix with 1’s on the opposite diagonal, and zeroes everywhere else. We realize the group $Sp_4$ as a matrix group in the following way:

$$Sp_4(\mathbb{F}) = \left\{ g \in GL_4(\mathbb{F}) : g^t \begin{bmatrix} 0 & J_2 \\ -J_2 & 0 \end{bmatrix} g = \begin{bmatrix} 0 & J_2 \\ -J_2 & 0 \end{bmatrix} \right\}.$$  

For us $\mathbb{F} \in \{ \mathbb{Q}, \mathbb{Q}_p, \mathbb{R}, \mathbb{A} \}$, where $\mathbb{A}$ is the ring of adeles of $\mathbb{Q}$.

The upper triangular matrices in $Sp_4$ form a Borel subgroup $B$, which we fix. The standard parabolic subgroups are those containing this Borel subgroup. The diagonal matrices in the Borel subgroup form a maximal torus, which we denote by $T$. Thus

$$T(\mathbb{F}) = \{ diag(t_1, t_2, t_2^{-1}, t_1^{-1}) ; t_1, t_2 \in \mathbb{F}^* \}.$$
The unipotent matrices in $B$ form the unipotent radical of $B$. Let $W$ be the Weyl group of $Sp_4$ with respect to $T$. We define the action of the Weyl group elements with respect to the elementary reflections:

$$s(diag(t_1, t_2, t_2^{-1}, t_1^{-1})) = diag(t_2, t_1, t_1^{-1}, t_2^{-1})$$

and

$$c_2(diag(t_1, t_2, t_2^{-1}, t_1^{-1})) = diag(t_1, t_2^{-1}, t_2, t_1^{-1}).$$

All other elements of $W$ are generated by $s$ and $c_2$. The set of roots of $Sp_4$ with respect to $T$ is denoted by $\Sigma$, and $\Sigma^+$ denotes the set of positive roots with respect to the above choice of Borel subgroup. Let $\Delta$ denote the set of simple roots in $\Sigma^+$. Then $\Delta = \{e_1 - e_2, 2e_2\}$ with obvious meaning of $e_i$, $i = 1, 2$.

Let $\chi$ denote a unitary Grössencharacter of $\mathbb{Q} \times \mathbb{A} \rightarrow \mathbb{C}$. We study the degenerate Eisenstein series on $Sp_4$ acting on the holomorphic sections associated with the global representations of $Sp_4(\mathbb{A})$ (more precisely, of its Hecke algebra) induced from the characters of the maximal standard parabolic subgroups of $Sp_4$. Thus, we have the Heisenberg and the Siegel case. We denote by $P_1 = M_1U_1$ the Heisenberg parabolic subgroup of $Sp_4$ so that the standard Levi subgroup $M_1$ is isomorphic to $GL_1 \times SL_2$ and in the Siegel case, we denote the Siegel parabolic subgroups $P_2 = M_2U_2$, where now $M_2 = GL_2$. We describe the corresponding holomorphic sections $f_s$ in each of these cases more thoroughly in the second and the third section of this paper. In both cases, we form the degenerate Eisenstein series

$$(1) \quad E(f_s)(g) \overset{def}{=} \sum_{\gamma \in P_1(\mathbb{Q}) \backslash Sp_4(\mathbb{Q})} f_s(\gamma \cdot g)$$

which converges absolutely and uniformly in $(s, g)$ on compact sets when $s > 2$ in the Heisenberg case, and $s > 3$ in the Siegel case. This is proved by the restriction to $Sp_4(\mathbb{R})$ and then applying Godement’s theorem as in ([2], 11.1 Lemma). In particular, there are no poles for such $s$.

It continues to a function which is meromorphic in $s$. Outside of poles, it is an automorphic form. As usual and more convenient for computations, we write $E(s, f)$ instead of $E(f_s)$; in this notation $s$ signals that $f \in I(s)$.

We say that $s_0 \in \mathbb{C}$ is a pole of the degenerate Eisenstein series $E(s, \cdot)$ if there exists $f \in I(s)$ such that $E(s, f)$ has a pole at $s = s_0$ (for some choice of $g \in Sp_4(\mathbb{A})$). The order of pole at $s_0$ is denoted by $l$; it is supremum of all orders $E(s, f)$ at $s_0$ when $f$ ranges over $I(s)$.

It may happen that $l = \infty$ as it can be seen from our main theorems.
but if $0 \leq l < \infty$, then the map

$$(2) \quad \text{Ind}_{\text{Sp}^4(\mathbb{A})}^\mathbb{A} (\pi(\chi), i) \xrightarrow{f \rightarrow (s-s_0)^l E(s,f)} A (\text{Sp}^4(\mathbb{Q}) \setminus \text{Sp}^4(\mathbb{A}))$$

for $i = 1, 2$ is an intertwining operator for the action of $(\mathfrak{sp}(4), K_\infty) \times \prod_{p<\infty} \text{Sp}^4(\mathbb{Q}_p)$ in the space of automorphic forms. Here

$$(3) \quad \pi(\chi)_1 = \chi | \det |_{\text{GL}_1(\mathbb{A})} \otimes 1 \text{SL}_2(\mathbb{A})$$

and

$$(3) \quad \pi(\chi)_2 = \chi | \det |_{\text{GL}_2(\mathbb{A})}.$$
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$$r(\Lambda_{s,p}, w) = \prod_{\alpha \in \Sigma^+, w(\alpha) < 0} \frac{L(1, \Lambda_{s,p} \circ \alpha^\vee) \epsilon(1, \Lambda_{s,p} \circ \alpha^\vee, \psi_v)}{L(0, \Lambda_{s,p} \circ \alpha^\vee)},$$

where $\alpha^\vee$ denotes the coroot corresponding to the root $\alpha$, and $\psi_v$ is a non-degenerate additive character of $\mathbb{Q}_p$. We define the normalized intertwining operator by the following formula:

$$\mathcal{N}(\Lambda_{s,p}, \tilde{w}) = r(\Lambda_{s,p}, w) A(\Lambda_{s,p}, \tilde{w}).$$

Properties of normalized intertwining operators can be found in [22], [23]. Again, the summary can be found in ([18], Theorem 2-5).

Let us write $\beta$ for the simple root such that $\Delta - \{\beta\}$ determines $P_i$. Now, the constant term has the following expression ([17], Lemma 2.1):

$$E_{const}(s, f)(g) = \sum_{w \in W, w(\Delta \setminus \{\beta\}) > 0} M(\Lambda_s, w) f(g) = \sum_{w \in W, w(\Delta \setminus \{\beta\}) > 0} \int_{U(\mathbb{A}) \cap wU(\mathbb{A}) w^{-1}} f(\tilde{w}^{-1} u g) du,$$

where, by induction in stages, we identify

$$f \in \text{Ind}^{Sp_4(\mathbb{A})}_{P_i(\mathbb{A})} (\pi(\chi)_i) \subset \text{Ind}^{Sp_4(\mathbb{A})}_{T(\mathbb{A})U(\mathbb{A})} (\Lambda_s)$$

where $\pi(\chi)_i, i = 1, 2$ is defined in (3). This formula can be more refined up to its final form that we use. Let $S$ be the finite set of all places including $\infty$ such that for $p \not\in S$ we have that $\chi_p, \mu_p, \psi_p$, and $f_p$ are unramified. Then we have the following expression:

$$E_{const}(s, f)(g) = \sum_{w \in W, w(\Delta \setminus \{\beta\}) > 0} r(\Lambda_s, w)^{-1} (\otimes_{p \in S} \mathcal{N}(\Lambda_{s,p}, \tilde{w}) f_p) \otimes (\otimes_{p \not\in S} f_{w,p}),$$

where we let

$$r(\Lambda_s, w)^{-1} \overset{def}{=} \prod_{\alpha \in \Sigma^+, w(\alpha) < 0} \frac{L(0, \Lambda_s \circ \alpha^\vee)}{L(1, \Lambda_s \circ \alpha^\vee) \epsilon(1, \Lambda_s \circ \alpha^\vee)},$$

and we use a well-known property of normalization

$$\mathcal{N}(\Lambda_{s,p}, \tilde{w}) f_p = f_{w,p},$$

for unramified $f_p$ and $f_{w,p}$.

We note that, in the situation as above, taking the constant term is an isomorphism between different spaces of automorphic forms (well known fact, e.g. [6], Lemma 2-9.)

We use standard notation [21] (see also [19]) for the representation theory of classical groups (in local or global settings). In more detail,
if $\chi$ is a character of $GL(1)$ and $\pi$ is a representation of $SL_2$, then $\chi \times \pi$ denotes the representation unitarily induced from $P_1$ to $Sp_4$. If $\pi$ is a representation of $GL(2)$, then $\pi \times 1$ denotes the representation unitarily induced from $P_1$ to $Sp_4$. Also, if $\chi$ and $\mu$ are characters of $GL(1)$, then $\chi \times \mu \times 1$ is the associated principal series of $Sp_4$. Similar notation is used for $GL(2)$ and $SL_2$. We denote by $L(\ )$ the Langlands quotient whenever in parenthesis is an induced representation having a Langlands quotient.

We use repeatedly the following simple fact (and similarly for $GL(2)$):

**Lemma 2.1.** Let $p \leq \infty$. The complex number $s = s_0 \in \mathbb{C}$ is a pole of $\mathcal{N}(s, \mu_p, w)$ (a normalized intertwining operator) if and only if $Re(s_0) < 0$ and $Ind_{P(\mathbb{Q}_p)}^{SL_2(\mathbb{Q}_p)}(\mid \mid s_0^p \mu_p)$ is reducible.

**Proof.** Indeed, the assumption $Re(s_0) < 0$ is clear since we known that $\mathcal{N}(s, \mu_p, w)$ is holomorphic and non–trivial for $Re(s) \geq 0$. Assume $Re(s_0) < 0$. Then, if $Ind_{P(\mathbb{Q}_p)}^{SL_2(\mathbb{Q}_p)}(\mid \mid s_0^p \mu_p)$ is irreducible, then the functional equation

$$\mathcal{N}(s, \mu_p, w)\mathcal{N}(-s, \mu_p^{-1}, w^{-1}) = \mathcal{N}(-s, \mu_p^{-1}, w^{-1})\mathcal{N}(s, \mu_p, w) = id,$$

combined with holomorphy and non–triviality of $\mathcal{N}(-s_0, \mu_p^{-1}, w^{-1})$ imply that $\mathcal{N}(s, \mu_p, w)$ is holomorphic at $s = s_0$. Conversely, still assuming $Re(s_0) < 0$, if $\mathcal{N}(s, \mu_p, w)$ is holomorphic for $s = s_0$, then then the functional equation, combined with holomorphy and non–triviality of $\mathcal{N}(-s_0, \mu_p^{-1}, w^{-1})$, imply

$$Ind_{P(\mathbb{Q}_p)}^{SL_2(\mathbb{Q}_p)}(\mid \mid s_0^p \mu_p) \simeq Ind_{P(\mathbb{Q}_p)}^{SL_2(\mathbb{Q}_p)}(\mid \mid s_0^p \mu_p).$$

Then the argument with the Langlands quotient implies that $Ind_{P(\mathbb{Q}_p)}^{SL_2(\mathbb{Q}_p)}(\mid \mid s_0^p \mu_p)$ is irreducible. 

**3. The Heisenberg parabolic**

The standard Levi subgroup of the Heisenberg parabolic subgroup $P_1$ is isomorphic to $GL_1 \times SL_2$, thus we study the global induced representation

$$Ind_{GL_1(\mathbb{A}) \times SL_2(\mathbb{A})}^{GL_1(\mathbb{A}) \times SL_2(\mathbb{A})} (\chi \psi^s \otimes 1),$$

where $\chi$ denotes a Grossencharacter of $\mathbb{Q}$ and and $1$ is a trivial character of $SL_2(\mathbb{A})$. This space consists of all $C^\infty$ and right $K$–finite functions.
Lemma 3.1. 

Assume \( s \geq 0 \). Then, the denominators of all three of the above expressions are non-zero and holomorphic. Thus, the poles cannot come from the zeroes of the denominators. As for the numerators, they are all holomorphic if \( \chi \neq 1 \), and if \( \chi = 1 \) we have the following:

1. \( r(\Lambda_s, c_1)^{-1} \) has a pole of the first order for \( s = 1 \) and \( s = 2 \).
2. \( r(\Lambda_s, s)^{-1} \) has a pole of the first order for \( s = 0 \).
3. \( r(\Lambda_s, sc_1)^{-1} \) has a pole of the first order for \( s = 0 \) and \( s = 1 \).

Assume \( s < 0 \). We analyze the zeroes of the denominators. The denominators (for all three expressions) might have zeroes in the critical strip, i.e., \( 0 < s + 2 < 1 \), i.e., \( -2 < s < -1 \). The
numerators do not have poles if \( \chi \neq 1 \) and do have poles for \( r(\Lambda_s, s)^{-1} \) for \( s = -1 \) and \( \chi = 1 \).

Now we analyze local intertwining operators appearing in (6).

3.1. Local intertwining operators appearing in (6). We recall that for every \( p \), the trivial representation of \( SL_2(\mathbb{Q}_p) \) is embedded in the principal series representation \( \nu_p^{-1} \rtimes 1 \).

**Lemma 3.2.** The local intertwining operator \( \mathcal{N}(\Lambda_{s,p}, \tilde{c}_1) \) acting on \( \chi_p \nu^{s} \rtimes 1_p \) (where \( 1_p \) is the trivial representation of \( SL_2(\mathbb{Q}_p) \)) is holomorphic for every \( p < \infty \) and for every \( s \in \mathbb{R} \), except for \( s = -2 \), where it has a pole of the first order (for every \( p < \infty \)).

**Proof.** According to the decomposition \( c_1 = sc_2s \) we have the following decomposition of the local intertwining operator \( \mathcal{N}(\Lambda_{s,p}, \tilde{c}_1) : \)

\[
\begin{align*}
\chi_p \nu^s \times \nu_p^{-1} \rtimes 1 & \to \nu_p^{-1} \times \chi_p \nu^s \rtimes 1 \\
& \to \chi_p^{-1} \nu^{-s} \times \nu_p^{-1} \rtimes 1.
\end{align*}
\]

Then, the first (normalized) intertwining operator appearing in the above relation is induced from the \( GL_2 \)-case and is holomorphic unless \( \chi_p = 1 \) and \( s = -2 \). The second intertwining operator is induced from the intertwining operator \( \chi_p \nu^s \rtimes 1 \to \chi_p^{-1} \nu^{-s} \rtimes 1 \). This intertwining operator is holomorphic if \( s > 0 \), (from the Langlands’ condition), and for \( s \leq 0 \), this operator is holomorphic unless the induced representation \( \chi_p \nu^s \rtimes 1 \) is reducible, and this happens if \( \chi_p^2 = 1 \), \( \chi_p \neq 1 \) and \( s = 0 \) and if \( \chi_p = 1 \) and \( s = -1 \). If we examine the first case more closely, we see that then the unnormalized intertwining operator is holomorphic since the Plancherel measure does not have a zero for \( s = 0 \) in that case, and the normalizing factor is holomorphic, too, so that we actually have holomorphicity. On the other hand, if \( \chi_p = 1 \) and \( s = -1 \) the pole of the normalized intertwining operator occurs for the unique quotient of the representation \( \nu_p^{-1} \rtimes 1 \) and this is the Steinberg representation of \( SL_2(\mathbb{Q}_p) \). The third intertwining operator is holomorphic unless \( \chi_p = 1 \) and \( s = 0 \).

Now we examine the case \( \chi_p = 1 \) and \( s = 0 \). Note that in this case

\[
\mathcal{N}(\Lambda_{s,p}, \tilde{c}_1) : \nu_p^0 \times \nu_p^{-1} \rtimes 1 \to \nu_p^0 \times \nu_p^{-1} \rtimes 1.
\]

It is known that this induced representation is of length four ([21], Proposition 5.4 (ii)) and that \( \nu_p^0 \rtimes 1 \) is in irreducible tempered representation of \( SL_2(\mathbb{Q}_p) \). Then, \( \nu_p^0 \rtimes 1_{SL_2(\mathbb{Q}_p)} = L(\nu^1; \nu_p^0 \rtimes 1) \oplus L(\nu^{1/2}St_{GL_2(\mathbb{Q}_p)}; 1) \).

The (normalized) spherical vector belongs to (and generates) representation \( L(\nu^1; \nu_p^0 \rtimes 1) \), and according to (8), \( \mathcal{N}(\Lambda_{s,p}, \tilde{c}_1) \) acts on it as the
identity. Analogously, for $s = 0$ \( \mathcal{N}(\tilde{c}_1(\Lambda_{s,p}), \tilde{c}_1) \) acts as the identity on the spherical vector. We know that \( \mathcal{N}(\tilde{c}_1(\Lambda_{s,p}), \tilde{c}_1) \mathcal{N}(\Lambda_{s,p}, \tilde{c}_1) = \text{Id.} \)

Assume that \( \mathcal{N}(\tilde{c}_1(\Lambda_{s,p}), \tilde{c}_1) \) has a pole of order $n_2$ for $s = 0$, and that \( \mathcal{N}(\Lambda_{s,p}, \tilde{c}_1) \) has a pole of order $n_1$ for $s = 0$ on $L(\nu^{1/2}St_{GL_2(\mathbb{Q}_p)})$ (which is a subrepresentation of $\nu_p^0 \times \nu_p^{-1} \times 1$ and appears there with the multiplicity one). Let $N_1 = \lim_{s \to 0} s^n \mathcal{N}(\Lambda_{s,p}, \tilde{c}_1)$ and $N_2 = \lim_{s \to 0} s^n \mathcal{N}(\tilde{c}_1(\Lambda_{s,p}), \tilde{c}_1)$.

This means that $N_1$ is a holomorphic isomorphism on $L(\nu^{1/2}St_{GL_2(\mathbb{Q}_p)})$, and so is $N_2$. But if $n_1 > 0$ or $n_2 > 0$ the composition $N_2N_1|_{L(\nu^{1/2}St_{GL_2(\mathbb{Q}_p)})} = 0$, which is thus impossible. This means that \( \mathcal{N}(\Lambda_{s,p}, \tilde{c}_1) \) is holomorphic on $\nu^0_p \times 1_{SL_2(\mathbb{Q}_p)}$ and the dual of the commuting algebra theorem ([1]) says that on $L(\nu^{1/2}St_{GL_2(\mathbb{Q}_p)}; 1)$ it acts as $-\text{Id.}$

We know examine the case $\chi_p = 1$ and $s = -1$. In this case, both intertwining operators corresponding to $s$ (in the decomposition above corresponding to the decomposition $c_1 = sc_2S$) are holomorphic isomorphisms, and the second operator has a pole on the representation $\nu^{-1} \times \text{St}_{SL_2(\mathbb{Q}_p)}$ (which is an irreducible quotient of $\nu^{-1} \times \nu^{-1} \times 1$) but it is holomorphic on $\nu^{-1} \times 1_{SL_2(\mathbb{Q}_p)}$ (irreducible by Proposition 5.4 (ii) of [21]). The image $\mathcal{N}(\Lambda_{s,p}, \tilde{c}_1)(\nu^{-1} \times 1_{SL_2(\mathbb{Q}_p)})$ is thus irreducible subspace $\nu^1 \times 1_{SL_2(\mathbb{Q}_p)} = L(\nu^1, \nu^1; 1)$.

We discuss the case $\chi_p = 1$ and $s = -2$. In this case the last two operators corresponding to $s$ and $c_2$ are holomorphic isomorphisms, the pole occurs in the first operator, induced from the $GL_2$-case $\nu^{-2} \times \nu^{-1} \rightarrow \nu^{-1} \times \nu^{-2}$. Note that the representation $\nu^{-2} \times \nu^{-1} \times 1$ is of length four ([21] Proposition 5.4 (i)) and we have (in the appropriate Grothendieck group) $\nu^{-2} \times 1_{SL_2(\mathbb{Q}_p)} = L(\nu^{3/2}St_{GL_2(\mathbb{Q}_p)}; 1) + L(\nu^2, \nu^1; 1)$, from the Langlands classification it follows that $L(\nu^2, \nu^1; 1)$ is the unique subrepresentation of $\nu^{-2} \times 1_{SL_2(\mathbb{Q}_p)}$. On the other hand, the aforementioned pole is happening on the quotient $\nu^{-3/2}St_{GL_2(\mathbb{Q}_p)}$. We know that \( \mathcal{N}(\Lambda_{s,p}, \tilde{c}_1) \) is holomorphic on $L(\nu^2, \nu^1; 1)$ (this is the trivial character), and it has a pole on $L(\nu^{3/2}St_{GL_2(\mathbb{Q}_p)}; 1)$.

**Lemma 3.3.** Assume that $s \in R$. Intertwining operator $\mathcal{N}(\Lambda_{s,\infty}, \tilde{c}_1)$ acting on the representation $\nu_\infty^\infty \times 1_{SL_2(\mathbb{R})}$ has poles precisely if $s < -1$ is an even integer and $\chi_\infty = 1$ and if $s < -1$ is an odd integer and $\chi_\infty = \text{sgn}$.

**Proof.** As in [12], we see that we might have a pole if $\chi_\infty$ is trivial or $\text{sgn}$.

- The first operator in the decomposition has a pole if $s < -1$ is an integer, even if $\chi_\infty = 1$ and odd if $\chi_\infty = \text{sgn}$.
- The second operator has a pole if $s < 0$ is an integer, even if $\chi_\infty = \text{sgn}$ and odd if $\chi_\infty = 1$. 
• The third operator has a pole if \( s < 1 \) is an integer, even if \( \chi_\infty = 1 \) and odd if \( \chi_\infty = sgn \).

Note that if \( s \leq -1 \), the Langlands quotient \( L(\chi_\infty \nu^{-s}, \nu^1; 1) \) is the unique irreducible subrepresentation of \( \chi_\infty \nu^s \times 1_{SL_2(\mathbb{R})} \). If, in addition, \( \chi_\infty = 1 \) we know that \( \mathcal{N}(\Lambda_{s, \infty}, \tilde{c}_1) \) is holomorphic and non-zero on \( L(\nu^{-s}, \nu^1; 1) \) because this is the spherical subquotient of the principal series. So we firstly resolve the case of \( \chi_\infty = 1 \).

We first assume that \( s = 0 \) (and \( \chi_\infty = 1 \)) (so that the third operator has a pole). Then, the first two operators are holomorphic. We can now repeat the discussion from Lemma 3.2 where we had a similar situation of (13). We again have the decomposition \( \nu^0 \times 1_{SL_2(\mathbb{R})} = L(\nu^{1/2}_{\infty} St_{GL_2(\mathbb{R})}; 1) \oplus L(\nu^1_{\infty}; \nu^0_{\infty} \times 1) \). Indeed, cf. Theorem 2.5 (ii) of [19] for the essentially square-integrable representation of \( GL_2(\mathbb{R}) \) we denoted by \( \nu^{1/2}_{\infty} St_{GL_2(\mathbb{R})}; 1 \) on the other hand, the representation \( \nu^0_{\infty} \times 1 \) is irreducible (representation of \( SL_2(\mathbb{R}) \) (cf. Theorem 2.4.(i) of [19]). For the decomposition of \( \nu^0_{\infty} \times 1_{SL_2(\mathbb{R})} \) (cf. Theorem 10.7., equation (10.72) of [19]). On the other hand, the length of the representation \( \nu^0_{\infty} \times 1_{SL_2(\mathbb{R})} \) is six (Theorem 10.7), but the representation \( L(\nu^1_{\infty}; \nu^0_{\infty} \times 1) \) is again a subrepresentation on this principal series, and appears with the multiplicity one, so we again can conclude that \( \mathcal{N}(\Lambda_{s, \infty}, \tilde{c}_1) \) is holomorphic on \( \nu^0_{\infty} \times 1_{SL_2(\mathbb{R})} \) (and non-zero on each summand).

Now we examine the situation of \( s = -1 \) and \( \chi_\infty = 1 \). By [19], equation (9.31) and Lemma 9.5. we see that the representation \( \nu^1_{\infty} \times 1_{SL_2(\mathbb{R})} \) is irreducible. By our previous remark \( \mathcal{N}(\Lambda_{s, \infty}, \tilde{c}_1) \) is holomorphic and non-zero on that representation, moreover, the image in \( \nu^1_{\infty} \times 1_{SL_2(\mathbb{R})} \) generates an irreducible subrepresentation \( \nu^{-1}_{\infty} \times 1_{SL_2(\mathbb{R})} \) because there is no other subquotient (besides \( L(\chi_\infty \nu^{-s}, \nu^1; 1) \) of \( \nu^s_{\infty} \times 1_{SL_2(\mathbb{R})} \). On the other hand, the first intertwining operator has a pole, and the third operator does not vanish (on that subquotient). We conclude that \( \mathcal{N}(\Lambda_{s, \infty}, \tilde{c}_1) \) has a pole on that other subquotient (we could also argue as in the analogous non-archimedean case; namely it is easy to see that \( L(\chi_\infty \nu^{-s}, \nu^1; 1) \) cannot appear as a subrepresentation of \( \nu^{-s} \times 1_{SL_2(\mathbb{R})} \)).

We examine the case when \( s = -1 \) and \( \chi_\infty = sgn \). According to Theorem 10.4 (ii) of [19], the representation \( \nu^{-1}_{\infty} sgn \times 1_{SL_2(\mathbb{R})} \) is irreducible. The pole occurs for the third intertwining operator and it appears on the (induced) quotient of that intertwining operator,
namely on the representation $\delta(1, 2) \times 1$ (here we use the notation of [19] (cf. Theorem 2.5 and Lemma 8.1). The representation $\delta(1, 2) \times 1$ decomposes as a sum of two tempered representations, so $\mathcal{N}(\Lambda_{s, \infty}, \tilde{c}_1)$ is holomorphic on $\nu_{\infty}^{-1} \sgn \times 1_{SL_2(\mathbb{R})}$. The first and the second operator are holomorphic isomorphisms. Since $\mathcal{N}(\tilde{c}_1(\Lambda_{s, \infty}), \tilde{c}_1)\mathcal{N}(\Lambda_{s, \infty}, \tilde{c}_1) = 1d$, and $\mathcal{N}(\tilde{c}_1(\Lambda_{s, \infty}), \tilde{c}_1)$ is holomorphic on $\nu_{\infty}^{1} \sgn \times \nu^{-1} \times 1$ it follows that $\mathcal{N}(\Lambda_{s, \infty}, \tilde{c}_1)$ is non-zero on $\nu_{\infty}^{-1} \sgn \times 1_{SL_2(\mathbb{R})}$.

If $s < -1$ is odd integer and $\chi_{\infty} = \sgn$, the representation $\nu_{\infty}^{s}, \sgn \times 1_{SL_2(\mathbb{R})}$ is of the length two (Theorem 11.1.(i) of [19]). The first and the second operator have poles, and the second is holomorphic isomorphism. The first operator has a pole on the (induced) quotient $\delta(\nu^{-\frac{s+1}{2}}, p-t) \times 1$ and the third on $\delta(\nu^{\frac{s+1}{2}}, p+t) \times 1$. Note that the latter representation is a subquotient of the former (Theorem 10.3. of [19]). Note that the Langlands quotient $L(\nu^{-s} \sgn, \nu^{1}; 1)$ is a subrepresentation of $\nu_{\infty}^{s}, \sgn \times 1_{SL_2(\mathbb{R})}$ and does not appear in the composition series of these two induced representations, so $\mathcal{N}(\Lambda_{s, \infty}, \tilde{c}_1)$ is holomorphic on it. On the other hand, the “other” subquotient of $\nu_{\infty}^{s}, \sgn \times 1_{SL_2(\mathbb{R})}$ is $L(\delta(\nu^\frac{s+1}{2}, p-t); 1)$ and it appears with the multiplicity one in $\nu^{s} \sgn \times \nu^{1} \times 1$ as can be seen from Theorem 11.1. (i), (ii) and (iii) of [19], so that there is a pole on the “other” subquotient of $\nu_{\infty}^{s}, \sgn \times 1_{SL_2(\mathbb{R})}$ (of order one).

If $s < -1$ is even integer and $\chi_{\infty} = \sgn$, the representation $\nu_{\infty}^{s}, \sgn \times 1_{SL_2(\mathbb{R})}$ is irreducible by Lema 9.4 of [19]. The first and the third operator are holomorphic isomorphisms, and the second has a pole on the quotient $\nu^{-1} \times (X(-s, +) \oplus X(-s, -))$, (e.g., cf. Lemma 7.2 of [19]). The same lemma guarantees that $\nu^{-1} \times X(-s, \varepsilon), \varepsilon = \pm$, are irreducible (and are not isomorphic with $\nu_{\infty}^{s}, \sgn \times 1_{SL_2(\mathbb{R})}$), so the intertwining operator $\mathcal{N}(\Lambda_{s, \infty}, \tilde{c}_1)$ is holomorphic on $\nu_{\infty}^{s}, \sgn \times 1_{SL_2(\mathbb{R})}$, and similarly as above, we conclude that it is also non-zero there.

Now we study the intertwining operators for the other relevant element of the Weyl group-which is similar, but easier that the previous case of $c_1$.

**Proposition 3.4.** Let $p < \infty$. The intertwining operator $\mathcal{N}(\Lambda_{s, p}, \tilde{s})$, where $s \in \mathbb{R}$, is holomorphic on $\nu_{p}^{s} \chi_{p} \times 1_{SL_2(Q_{p})}$, unless $s = -2$ and $\chi_{p} = 1$; then it has a pole of the first order. If $p = \infty$ then we have poles precisely if $s < -1$ is even and $\chi_{\infty} = 1$ and if $s < -1$ is odd and $\chi_{\infty} = \sgn$.

**Proof.** Already proved in Lemma 3.2 and Lemma 3.3.

**Proposition 3.5.** Let $p < \infty$. The intertwining operator $\mathcal{N}(\Lambda_{s, p}, s \tilde{c}_1) = \mathcal{N}(\Lambda_{s, p}, \tilde{c}_2 s)$, where $s \mathbb{R}$, is holomorphic on $\chi_{p} \nu_{p}^{s} \times 1_{SL_2(Q_{p})}$ unless $\chi_{p} = 1$.
and $s = 2$. If $p = \infty$ then we have poles precisely if $s < 1$ is even and $\chi_\infty = 1$ and if $s < 1$ is odd and $\chi_\infty = \mathrm{sgn}$.

**Proof.** Already proved in Lemma 3.2 and Lemma 3.3 \(\square\)

Now we can explicitly describe the image of the constant term of the degenerate Eisenstein series.

**Theorem 3.6.** Assume $s \geq 0$. Then, the poles of the Eisenstein series can come only from the poles of the normalizing factors, and then only if $\chi = 1$ and $s \in \{0, 1, 2\}$ (by well-known general results we know that there are no poles for $s = 0$).

1. If $\chi = 1$ and $s = 0$ the global Eisenstein series is holomorphic (for any choice of $f = \otimes f_p \in \Ind_{\text{GL}_1(A) \times SL_2(A)}(\chi \nu^s \otimes 1)$). For $p \leq \infty$ we have $\nu_p^0 \times 1_{SL_2(\mathbb{Q}_p)} = L(\nu_p^1; \nu_p^0 \times 1) \oplus L(\nu_p^{1/2} \text{St}_{GL_2(\mathbb{Q}_p)}; 1)$. We choose a finite set of places $S'$ and we choose $f_p \in L(\nu_p^{1/2} \text{St}_{GL_2(\mathbb{Q}_p)}; 1)$, for $p \in S'$, and for some bigger finite set of places $S \supset S'$ we choose $f_p \in L(\nu_p^1; \nu_p^0 \times 1)$ for $p \in S \setminus S'$. For all the places outside of $S$ we choose $f_p$ to be the normalized spherical vector. Then, if $S'$ is even, this choice gives an automorphic realization of the corresponding (irreducible) global representation in the space of automorphic forms.

2. If $\chi = 1$ and $s = 1$ the global Eisenstein series is holomorphic, and for every $p \leq \infty$, the representation $\nu_p^1 \times 1_{SL_2(\mathbb{Q}_p)}$ is irreducible and \(\square\) gives the embedding of the irreducible global representation $\Ind_{\text{GL}_1(A) \times SL_2(A)}(\chi \nu^1 \otimes 1)$ in the space of automorphic forms.

3. If $\chi = 1$ and $s = 2$ the global Eisenstein series has a pole of the first order, and, after removing the poles, \(\square\) gives the automorphic realization of the trivial representation in the space of (square-integrable) automorphic forms.

4. Assume $s = 0$ and $\chi^2 = 1$ but $\chi \neq 1$ then for all $p \leq \infty$, $\chi_p \times 1_{SL_2(\mathbb{Q}_p)} = L(\nu_p^1; T_1) \oplus L(\nu_p^1; T_2)$, where $T_1$ and $T_2$ are non-isomorphic tempered representations (the limits of the discrete series if $p = \infty$) such that $\chi_p \times 1 = T_1 \oplus T_2$ in $SL_2(\mathbb{Q}_p)$. Here $T_1$ is such that $\mathcal{N}(s(\lambda_{s,p}), c_2)$ acts on $\nu_p^{-1} \times T_1$ as the identity, and on $\nu_p^{-1} \times T_2$ as minus identity (so for the spherical places, the spherical vector belongs to $L(\nu_p^1; T_1)$). Let $S$ be a finite set of places such that for $p \notin S$, $f_p$ is spherical. Let $S' \subset S$ be such that for $p \in S'$ we choose $f_p \in L(\nu_p^1; T_2)$ (and for $p \in S \setminus S'$ $f_p \in L(\nu_p^1; T_1)$). With these choices, the mapping \(\square\) gives an
automorphic realization of this irreducible global representation if $|S'|$ is even.

(5) In the rest of the cases (we still assume $s \geq 0$) which are not cover above, the embedding (3) is holomorphic and gives an automorphic realization of the whole representation $\text{Ind}_{GL_1(A) \times SL_2(A)}(\chi \nu^s \otimes 1)$.

Proof. Since $s = 0$, in the expression (6) all the local intertwining operators are holomorphic (by Lemma 3.2 Lemma 3.3, Proposition 3.4 and Proposition 3.5) and the poles can only come from the normalizing factors, and then, only if $\chi = 1$ (cf. Lemma 3.1). We now assume that $\chi = 1$.

First assume $s = 0$. We note that then $s(A_s) = c_2 s(A_s)$. We also claim that the intertwining operators $N(A_s, s_1, s_2)$ have the same effect on $\nu^0 \times 1_{SL_2(Q_p)}$ (for every $p$). Indeed, $\text{Ind}_{GL_1(A)}(\chi \nu^s)$ acts as the identity, and

$$\text{Ind}_{GL_1(A) \times SL_2(A)}(\chi \nu^s \otimes 1).$$

We also have that $c_1(A_s, s_1, s_2) = A_s$ for $s = 0$. Note that for every $p \leq \infty$, by Lemma 3.2 and Lemma 3.3 $\nu^0 \times 1_{SL_2(Q_p)} = L(\nu^0; \nu^0 \times 1)$, and on $L(\nu^0; \nu^0 \times 1)$ acts as the identity, and

on $L(\nu^0; \nu^0 \times 1)$ as minus identity.

We conclude that (6) for $s = 0$ becomes

\begin{equation}
E_{\text{const}}(f, \sigma) = f_s + r(A_s, c_1)^{-1}(\otimes_{p \in S} N(A_s, c_1))f_{p, s} \otimes (\otimes_{p \notin S} f_{c_1(s), p}) + (r(A_s, s_1)^{-1} + r(A_s, s_2)^{-1}) \otimes_{p \in S} N(A_s, s_1, s_2) f_{p, s} \otimes (\otimes_{p \notin S} f_{s, p}).
\end{equation}

Here we denote by $s$ a real number (in this case equal to 0) and an element of the Weyl group. Since we are dealing with the trivial global character over $Q$, all the global $c$ factors are trivial, and we get that $r(A_s, s_1)^{-1} + r(A_s, s_2)^{-1} = \frac{1}{L(s+2,1)}(L(s, 0) + L(s, 1))$, so this expression is holomorphic and non-zero. We conclude that $E_{\text{const}}(f, \sigma)$ is holomorphic for $s = 0$ for any choice of $f_s = \otimes f_{s, p}$ belonging to the global induced representation $\text{Ind}_{GL_1(A) \times SL_2(A)}(\chi \nu^0 \otimes 1)$, but this is well-known from the general results of Langlands. But we can now describe the image of Eisenstein series in the space of automorphic forms.

First we analyze the first row in (14). Let $S' \subset S$ be a finite set of primes such that for them we choose $f_p \in L(\nu^0; \nu^0 \times 1)$ and for rest of the places $S \setminus S'$ we chose $f_p \in L(\nu^0; \nu^0 \times 1)$. For $p \notin S$ we choose $f_p$ to be the normalized spherical vector. Then,
we have \( \lim_{s \to 0} r(\Lambda_s, c_1)^{-1} = 1 \), and \( \mathcal{N}(\Lambda_{s,p}, \tilde{c}_1) f_{p,s} = -f_{p,s}, \ p \in S' \), \( \mathcal{N}(\Lambda_{s,p}, \tilde{c}_1) f_{p,s} = f_{p,s}, \ p \notin S' \). Thus, the first line of (14) becomes \( f_0 + (-1)^{|S'|} f_0 \). We conclude that this is non-zero if \( |S'| \) is even and this means that the global representation (a subrepresentation of \( \text{Ind}_{GL_1(\mathbb{A}) \times SL_2(\mathbb{A})}^{\mathbb{A}}(\chi \nu^s \otimes 1) \)) with the local components consisting of \( L(\nu_p^{1/2} \text{St}_{GL_2(\mathbb{Q}_p)}; 1) \) on an even number of places and \( L(\nu_p^1; \nu_p^0 \times 1) \) on the rest of the places, is automorphic.

Now we analyze the second line. For \( p \in S \), the image \( \mathcal{N}(\Lambda_{s,p}, \tilde{s}) f_{p,s} \) is non-zero if \( f_{p,s} \) is not in the kernel of this operator (and the kernel is \( \nu^{-1/2} \text{St}_{GL_2(\mathbb{Q}_p)} \times 1 \)). Thus, if we pick \( f_{p,s} \) from \( L(\nu_p^1; \nu_p^0 \times 1) \) for every \( p \in S \), we’ll get a non-zero contribution. This now means that the projection of the constant term to part of the sum spanned by the images of the intertwining operators with respect to the Weyl group elements \( s \) and \( c_2 s \) gives an automorphic realization of the global representation whose every local component is \( L(\nu_p^1; \nu_p^0 \times 1) \) so we get only global representations which form a subset of the ones we obtained by analyzing the first line.

Now we analyze the case \( s = 1 \). Then, note that for each \( p \leq \infty \) the representation \( \nu_p^1 \times 1_{SL_2(\mathbb{Q}_p)} \) is irreducible (as was noted in Lemma 3.2 and Lemma 3.3), and thus spherical. In that case \( c_2 s(\Lambda_s) = s c_2 s(\Lambda_s) = c_1(\Lambda_s) \). Moreover, \( \mathcal{N}(\Lambda_{s,p}, s c_2 s) = \mathcal{N}(c_2 s(\Lambda_{s,p}), \tilde{s}) \mathcal{N}(\Lambda_{s,p}, \tilde{c}_2 s) \). Note that \( \mathcal{N}(c_2 s(\Lambda_{s,p}), \tilde{s}) \) is for \( s = 1 \) identity operator induced from the \( GL_2 \) operator \( \nu_p^{-1} \times \nu_p^{-1} = \nu_p^{-1} \times \nu_p^{-1} \). Then, we just sum \( r(\Lambda_s, s c_2 s)^{-1} + r(\Lambda_s, c_2 s)^{-1} \). Again we get that the poles cancel, and we obtain a non-zero holomorphic function for \( s = 1 \). Thus, the expression

\[
(r(\Lambda_s, s c_2 s)^{-1} + r(\Lambda_s, c_2 s)^{-1}) \otimes_{p \in S} \mathcal{N}(\Lambda_{s,p}, \tilde{c}_2 s) f_{p,s} \otimes (\otimes_{p \notin S} f_{c_2 s(s), p})
\]

gives a non-zero contribution to the constant term of Eisenstein series and this cannot cancel the identity contribution. This means that we have obtained an automorphic realization (through Eisenstein series) of the global (irreducible representation) \( Ind_{GL_1(\mathbb{A}) \times SL_2(\mathbb{A})}^{\mathbb{A}}(\nu^1 \otimes 1) \).

For \( s = 2 \) we have

\[
\lim_{s \to 2} (s - 2) E_{\text{const}}(f, s) = (\lim_{s \to 2} (s - 2) r(\Lambda_s, sc_2 s)^{-1}) \otimes_{p \in S} \mathcal{N}(\Lambda_{s,p}, sc_2 s) f_{p,s} \otimes (\otimes_{p \notin S} f_{sc_2 s(s), p})
\]

We know that \( \mathcal{N}(\Lambda_{s,p}, sc_2 s) \) is holomorphic (and non-zero) on \( \nu_p^2 \times 1_{SL_2(\mathbb{Q}_p)} \), for every \( p \leq \infty \). Note that, because of the Langlands classification, \( L(\nu_p^2, \nu_p^{1}; 1) \) is the unique quotient of \( \nu_p^2 \times 1_{SL_2(\mathbb{Q}_p)} \). On the other hand \( \mathcal{N}(\Lambda_{s,p}, sc_2 s)(\nu_p^2 \times 1_{SL_2(\mathbb{Q}_p)}) \subset \nu_p^{-2} \times \nu_p^{-1} \times 1 \) and this representation has a unique irreducible subrepresentation; namely \( L(\nu_p^2, \nu_p^{1}; 1) \).
This ensures that $\mathcal{N}(\Lambda_{s,p}, \tilde{sc}_2 s)(\nu_p^2 \otimes 1_{SL_2(\mathbb{Q}_p)}) = L(\nu_p^2, \nu_p^1; 1)$, for every $p \leq \infty$. Note that $L(\nu_p^2, \nu_p^1; 1)$ is actually a trivial representation, thus this (normalized) Eisenstein series gives a realization of the trivial representation as the irreducible subrepresentation in the space of square-integrable automorphic forms (since the constant term along the minimal parabolic has exponent $(-2, -1)$.

If $\chi \neq 1$, or $\chi = 1$ but $s \notin \{0, 1, 2\}$ we have the following situation. Let $w \in W$, $w(2c_2) > 0$. Then $w(\Lambda_s) \neq \Lambda_s$, unless $\chi^2 = 1$ and $s = 0$; then $sc_2 s(\Lambda_s) = \Lambda_s$. So if $\chi^2 \neq 1$ or $s \neq 0$, (and we are not in the situations already covered) nothing will cancel the identity contribution; this gives the automorphic realization of the whole representation $\text{Ind}_{GL_1(\mathbb{A}) \times SL_2(\mathbb{A})}(\chi, \nu^s \otimes 1)$.

Now assume $s = 0$ and $\chi^2 = 1$ but $\chi \neq 1$. Then, for every $p \leq \infty$ such that $\chi_p \neq 1$, the representation $\chi_p \times 1$ (of $SL_2(\mathbb{Q}_p)$) is reducible, and sum of two irreducible tempered representations, say $T_1$ and $T_2$ (the limits of the discrete series if $p = \infty$). Note that $\Lambda_s = sc_2 s(\Lambda_s)$ and $s(\Lambda_s) = c_2 s(\Lambda_s)$. Assume that on $T_1$ the normalized intertwining operator $\chi_p \times 1 \rightarrow \chi_p \times 1$ acts as the identity and on $T_2$ as identity. We also have that $\chi_p \times 1_{SL_2(\mathbb{Q}_p)} = L(\nu_p^1; T_1) \oplus L(\nu_p^1; T_2)$ for all $p$ (\cite{21}, Proposition 5.4 and \cite{19} Lema 9.6). We conclude that $\mathcal{N}(\Lambda_{s,p}, \tilde{s})$ acts on $\chi_p \times 1_{SL_2(\mathbb{Q}_p)}$ as a holomorphic isomorphism, and then $\mathcal{N}(s(\Lambda_{s,p}), \tilde{c}_2)$ acts on $\nu^1 \times T_1$ as identity, and on $\nu^1 \times T_2$ as minus identity. Now again $\mathcal{N}(c_2 s(\Lambda_{s,p}), \tilde{s})$ is a holomorphic isomorphism. Let $S$ be a finite set of places such that for $p \notin S$, $f_p$ is the normalized spherical vector. For a subset $S' \subset S$ such that for every $p \in S'$, $\chi_p \neq 1$, we choose $f_p$ to belong to $L(\nu_p^1; T_2)$ (and $f_p$ belongs to $L(\nu_p^1; T_1)$ for $p \in S \setminus S'$). We then have

\begin{equation}
E_{\text{const}}(f, s) = f_s + r(\Lambda_s, sc_2 s)^{-1}(1)^{|S'|} f_s + 
(r(\Lambda_s, s)^{-1} + (-1)^{|S'|} r(\Lambda_s, c_2 s)^{-1}) \otimes_{p \in S} \mathcal{N}(\Lambda_{s,p}, \tilde{s}) f_p s \otimes \otimes_{p \notin S} f_s(s, p).
\end{equation}

We use the functional equation (\cite{20}, p.279), $L(1-s, \chi^{-1}) = \varepsilon(s, \chi)L(s, \chi)$. Now, according to that the expression for $r(\Lambda_s, sc_2 s)^{-1}$ becomes

\[ r(\Lambda_s, sc_2 s)^{-1} = \frac{L(-s + 2, \chi)\varepsilon(-s + 2, \chi)}{L(s + 2, \chi)\varepsilon(s + 2, \chi)\varepsilon(s, \chi)\varepsilon(s + 1, \chi)}. \]

Since $L(s, \chi)$ is holomorphic for $s \in \mathbb{C}$ we have $\lim_{s \to 0} \frac{L(-s + 2, \chi)\varepsilon(-s + 2, \chi)}{L(s + 2, \chi)\varepsilon(s + 2, \chi)} = 1$ so that $\lim_{s \to 0} r(\Lambda_s, sc_2 s)^{-1} = 1$ so $r(\Lambda_s, sc_2 s)^{-1} = \frac{1}{\varepsilon(0, \chi)\varepsilon(1, \chi)}$. Further, we have $L(-s, \chi) = \varepsilon(1 + s, \chi) L(1 + s, \chi)$. If we multiply these two functional equations and let $s = 0$, we get $\varepsilon(s, \chi)\varepsilon(s + 1, \chi) = 1$ (since $L(0, \chi)L(1, \chi) \neq 0$). Then, the factor in the first line of (15) becomes $1 + (-1)^{|S'|}$. We conclude
that if $|S'|$ is odd that the first line of (15) vanishes. The numerical factor in the second line of (15) becomes

\[ \frac{1}{L(s+2,\chi)L(s+1,\chi)}(L(-s,\chi) + (-1)^{|S'|}L(s,\chi)). \]

We conclude that for $|S'|$ odd the second line also vanishes. So, we can get a non-zero contribution only if $|S'|$ is even. □

**Theorem 3.7.** Assume $s < 0$.

1. If $-1 < s < 0$ the Eisenstein series is holomorphic and (2) gives an embedding of the irreducible representation $\text{Ind}_{GL_1(A) \times SL_2(A)}(\chi^{s} \otimes 1)$ in the space of automorphic forms.

2. If $s = -1$ and $\chi \neq 1$ the result is analogous to the previous case.

3. If $s = -1$ and $\chi = 1$ the Eisenstein series is identically zero on $\text{Ind}_{GL_1(A) \times SL_2(A)}(\nu^{-1} \otimes 1)$.

4. If $-2 < s < -1$ all the (inverses) of the (non-trivial) normalizing factors can have a pole (of the same order) coming from the zero of the $L$–function in the denominator. After the normalization, we have an embedding of the representation $\text{Ind}_{GL_1(A) \times SL_2(A)}(\chi^{s} \otimes 1)$ in the space of automorphic forms.

5. If $s = -2$ and $\chi = 1$ then, if we pick $f_p \in L(\nu^{3/2}St_{GL_2(Q_p)}; 1)$ (the trivial representation) for all $p$, then $E_{\text{const}}(f, -2) = f_{-2}$ so (2) gives an embedding of the trivial representation in the space of (square-integrable) automorphic forms. If we pick at exactly one finite place $f_p \in L(\nu^{3/2}St_{GL_2(Q_p)}; 1)$ (and on the rest of the places the trivial representation), the Eisenstein series are holomorphic on this global representation and the image is, on that one place, an irreducible subrepresentation isomorphic to $L(\nu^{3/2}St_{GL_2(Q_p)}; 1)$, on the rest of the places the image spans a representation of the length two (in semisimplification $L(\nu^{3/2}St_{GL_2(Q_p)}; 1) + L(\nu^{3/2}St_{GL_2(Q_p)}; 1)$). Analogously, if we pick at the finite number of places, say $|S|$, a vector from $L(\nu^{3/2}St_{GL_2(Q_p)}; 1)$, the Eisenstein series have a pole of order $|S| - 1$ (so we can get a pole of any finite order), and after normalization, the local images are irreducible for the ramified choice of subquotient, and of the length equal to two for the choice of the unramified subquotient.

6. If $s = -2$ and $\chi \neq 1$ we have the following. For each $p$ such that $\chi_p = 1$ the local intertwining operators have a pole if $f_p \in L(\nu^{3/2}St_{GL_2(Q_p)}; 1)$, and if $\chi_p \neq 1$ there are no poles for the intertwining operators (all isomorphisms) and the image is isomorphic to the representation $\chi_p \nu_p^{-2} \otimes 1_{SL_2(Q_p)}$. The discussion about the image is analogous to the previous case when...
χ_p = 1, and we can obtain a pole of order |S| if, for every p ∈ S
χ_p = 1 and f_p ∈ L(ν_p^{3/2} St_{GL(2; Q_p)}; 1).

(7) Assume s < −2. Then unless s is even integer and χ_∞ = 1 or
s is odd integer and χ_∞ = sgn, the Eisenstein series is always
holomorphic and \( g \) gives an embedding of \( \text{Ind}_{GL_1(\mathbb{A}) \times SL_2(\mathbb{A})}(\chi^{s} \otimes 1) \)
in the space of automorphic forms. If s is even integer and
χ_∞ = 1 or s is odd integer and χ_∞ = sgn, and (in both of these
cases) we pick \( f_\infty \in L(\nu^{−s−1}, −s − 1) \) (notation \[19\] Theor-
em 11.1(i)), the Eisenstein series have a pole on \( \otimes f_p \) of the
first order. After the normalization, the image on the non-
archimedean place spans an irreducible representation, and on
the archimedean place it spans a representation \( \chi_\infty^{−s} \times 1_{SL_2(\mathbb{R})} \).

Proof. Assume −1 < s < 0. The claim is then obvious.
In the case s = −1 if χ ≠ 1 the Eisenstein series is holomorphic,
the other contributions in the constant term cannot cancel the identity
contribution, the global representation is again irreducible (Lemma 9.5.
or Theorem 10.4 of \[19\]), so again \( g \) gives an embedding in the space
of automorphic forms.

On the other hand, if s = −1 and χ = 1, the global representation
is again irreducible, and intertwining operators holomorphic) but the
normalizing factors \( r(\Lambda_s, c_1)^{-1} \) and \( r(\Lambda_s, c_2 s)^{-1} \) vanish. On the other
hand, \( \lim_{s \to -1} r(\Lambda_s, s)^{-1} = -\frac{1}{\varepsilon(1,1)} = -1 \) and \( N(\Lambda_{s,p}, \tilde{s}) f_{p,s} \) is the iden-
tity. In this way, \( f \) comes down to
\[ E_{\text{const}}(f; -1) = 0. \]
Now assume −2 < s < −1. Then, the normalized intertwining operators
are all holomorphic and the representation \( \text{Ind}_{GL_1(\mathbb{A}) \times SL_2(\mathbb{A})}(\chi^{s} \otimes 1) \)
is irreducible (obvious, e.g., Theorem 12.1. (ii) \[19\]). The (inverses
of) normalizing factors might have a pole (and the order of the possible
pole is equal to the order of the zero of \( L(s + 2, \chi) \), and this happens
for all the (non-trivial) normalizing factors. Note that, for examp le
\( sc_2 s(\Lambda_s) \neq s(\Lambda_s) \), so after the possible removing of the pole, the con-
tributions according to the different element of the Weyl group do not
cancel, so by \( f \) we have an embedding of this global representation
into the space of automorphic forms.

Assume s = −2 and χ = 1. Then all the nontrivial normalizing
factors vanish because of the poles in the denominator. On the other
hand, if we pick \( f_p \in L(\nu^2, \nu^1; 1) \) for all w appearing in the expression
for the constant term, \( N(\Lambda_{s,p}, \tilde{w}) f_p \) is holomorphic and non-zero, and if
we pick \( f_p \in L(\nu^{3/2} St_{GL_2(\mathbb{Q}_p)}; 1) \) we have a pole for each such w, ana-
gously for the archimedean places (Lemma \[32\] Lemma \[33\] Proposition
We conclude that by picking \( f_p \in L(\nu^2, \nu^1; 1) \) for each \( p \leq \infty \), the contributions in the constant term of Eisenstein series (from all the non-trivial elements of the Weyl group appearing there) will be zero, so that we would have \( E_{\text{const}}(f, -2) = f_{-2} \). This would imply that the global representation consisting of \( L(\nu_p^2, \nu_p^1; 1) \) for every \( p \leq \infty \) appears in the space of automorphic forms through \( \mathcal{A} \), moreover the appearance in \( \mathcal{A} \) of \( L(\nu^2, \nu^1; 1) \) is in the space of square–integrable automorphic forms (this is well-known); \( L(\nu_p^2, \nu_p^1; 1) \) is the trivial character. On the other hand, assume we pick \( f_p \in L(\nu^{3/2} \text{St}_{GL_2}(\mathbb{Q}_p); 1) \) at exactly one (say, finite) place from \( S \). Then, the contribution \( r(\Lambda, w)^{-1} \otimes \bigotimes_{p \in S} \mathcal{N}(\Lambda_{s,p}, \bar{w}) f_p \otimes \bigotimes_{p \notin S} f_{w(s), p} \) is holomorphic (for each \( w \neq 1 \)). To study the image, we examine what is happening on each place. On that specific place, \( f_p \) spans the representation \( L(\nu^{3/2} \text{St}_{GL_2}(\mathbb{Q}_p); 1) \) which is a subrepresentation in \( \text{Ind}_{B}^{\text{Sp}_4(\mathbb{Q}_p)}(w(\nu_p^{-2} \otimes \nu_p^{-1})) \). On the other places (in \( S \) and outside of \( S \)) it is enough to see what kind of space the spherical vector generates. It generates a subspace of length two in \( \text{Ind}_{B}^{\text{Sp}_4(\mathbb{Q}_p)}(w(\nu_p^{-2} \otimes \nu_p^{-1})) \) where \( w \in \{ s, c_2 s, sc_2 s \} : \) it cannot be an irreducible subrepresentation in any of those induced representations (a simple Jacquet module argument); on the other hand it generates a subrepresentation of length two in \( \text{Ind}_{B}^{\text{Sp}_4(\mathbb{Q}_p)}(sc_2 s(\nu_p^{-2} \otimes \nu_p^{-1})) \), and the other representations are isomorphic to it. The generated representation is (in semisimplification) \( L(\nu_p^2, \nu_p^1; 1) + L(\nu^{3/2} \text{St}_{GL_2}(\mathbb{Q}_p); 1) \). Analogously, by choosing an arbitrary but finite set of places \( p \) where we can pick \( f_p \in L(\nu^{3/2} \text{St}_{GL_2}(\mathbb{Q}_p); 1) \) we can make the pole for \( s = -2 \) of \( E_{\text{const}}(s, f) \) of the arbitrary high order. After removing the poles the local images are irreducible on the ramified choices and of length two on the unramified choices. Thus, the image of \( \mathcal{A} \) spans (highly) reducible representation. Similarly, if \( s = -2 \) but \( \chi \neq 1 \), all the normalizing factors are holomorphic and non-vanishing, and we have a pole for each place \( p \) for which \( \chi_p = 1 \), so the discussion about local images is similar to the discussion for \( \chi = 1 \)--case, modulo the number of places where we have \( \chi_p = 1 \). If \( \chi_p \neq 1 \), the representation \( \chi_p \nu_p^{-2} \otimes 1_{\text{SL}_2(\mathbb{Q}_p)} \) is irreducible, (e.g., Lemma 9.4. of [19], equally easy for the non-archimedean case) and all the intertwining operators are holomorphic isomorphisms.

If \( s < -2 \) all the normalizing factors are holomorphic and non-zero, and all the representations on the finite places are irreducible, and the intertwining operators acting on those representations are holomorphic isomorphisms. Then, the images generated locally are also irreducible. We can only get poles of the intertwining operators at the archimedean
place, when $s$ is odd and $\chi_\infty = \text{sgn}$ or when $s$ is even and $\chi_\infty = 1$ for all of them (attached to $s$, $c_2s$, $sc_2s$) as follows from Lemma 3.3. In all other cases the Eisenstein series are holomorphic and reducible for exceptional cases, we get a pole of the first order if we pick $f(2)$ gives the embedding in the space of automorphic forms. In these exceptional cases, we get a pole of the first order if we pick $f(2) \in L(\delta \nu^{\frac{s+1}{2}}, -s - 1; 1)$ in both of these cases (Theorem 11.1.(i) of [19]). By normalizing the Eisenstein series to eliminate this pole, we again get no cancelations between contributions attached to the different Weyl group elements. In the situation when we pick $f(2) \in L(\delta \nu^{\frac{s+1}{2}}, -s - 1; 1)$ (and normalize the Eisenstein series), the image will generate an irreducible representation in $\text{Ind}_B^{Sp(4)}(w(\nu^s \otimes \nu^{-1}), w \in \{s, c_2s\}$, but reducible for $w = sc_2s$; indeed, for $w = sc_2s$ $f(2)$ generates the whole representation $\chi_\infty^s \nu^s L_2(\nu^s) \leftrightarrow \nu^{-s} \times \nu^{-1} \times 1$. □

4. THE SIEGEL CASE

We now study the degenerate Eisenstein series of the representation

$$\text{Ind}_B^{Sp(4)(\mathbb{A})}(\chi^s \delta_{GL_2(\mathbb{A})}).$$

Thus $\Lambda_s = \chi^{s-1/2} \otimes \chi^{s+1/2}$, where $s \in \mathbb{R}$.

The terms in (16) which appear in the Siegel case are the following:

$$W' = \{w \in W : w(e_1 - e_2) > 0\} = \{id, c_2, sc_2, c_2sc_2\}.$$

The corresponding normalizing factors are

$$r(\Lambda_s, c_2)^{-1} = \frac{L(s + \frac{1}{2}, \chi)}{L(s + \frac{3}{2}, \chi)e(s + \frac{3}{2}, \chi)},$$

$$r(\Lambda_s, sc_2)^{-1} = \frac{L(s + \frac{1}{2}, \chi)L(2s, \chi^2)}{L(s + \frac{3}{2}, \chi)e(s + \frac{3}{2}, \chi)L(2s + 1, \chi^2)e(2s + 1, \chi^2)},$$

$$r(\Lambda_s, c_2sc_2)^{-1} = \frac{L(2s, \chi^2)L(s - \frac{1}{2}, \chi)}{L(s + \frac{3}{2}, \chi)L(2s + 1, \chi^2)e(s + \frac{3}{2}, \chi)e(s + \frac{1}{2}, \chi)e(2s + 1, \chi^2)}.$$

We discuss the poles of the (inverses) of the normalizing factors above, which appear only in the following situations.

**Proposition 4.1.**

1. Assume $s \geq 0$.
   (a) $r(\Lambda_s, c_2)^{-1}$ has a pole of the first order for $s = \frac{1}{2}$ if $\chi = 1$.
   (b) $r(\Lambda_s, sc_2)^{-1}$ has a pole for $s = \frac{1}{2}$ of the second order if $\chi = 1$ and of the first order if $\chi^2 = 1$, but $\chi \neq 1$.
   (c) For $r(\Lambda_s, c_2sc_2)^{-1}$ the same discussion as for $r(\Lambda_s, sc_2)^{-1}$.

2. Assume $s < 0$.
   (a) $r(\Lambda_s, c_2)^{-1}$ may have a pole for some $-\frac{3}{2} < s < -\frac{1}{2}$, where the denominator has a zero.
(b) \( r(\Lambda_s, sc_2)^{-1} \) may have a pole for some \(-\frac{3}{2} < s < -\frac{1}{2}\), or some \(-\frac{1}{2} < s < 0\) where the denominator has a zero.

c) for \( r(\Lambda_s, c_2 sc_2)^{-1} \) the same discussion as for \( r(\Lambda_s, sc_2)^{-1} \).

Now we discuss the local normalized intertwining operators corresponding to the elements of \( W' \) \([16]\). The following lemma follows straightforward from the \( GL_2 \) and \( SL_2 \) cases.

**Lemma 4.2.**

1. Assume \( p < \infty \). Then, the operator \( N(\Lambda_{s,p}, \tilde{c}_2) \) is holomorphic unless \( s = -\frac{3}{2} \) and \( \chi_p = 1 \). Assume \( p = \infty \). Then, the operator \( N(\Lambda_{s,\infty}, \tilde{c}_2) \) is holomorphic unless \( s + \frac{1}{2} \) is a negative odd integer and \( \chi_\infty = 1 \) and \( s + \frac{1}{2} \) is a negative even integer and \( \chi_\infty = \sgn \).

2. In addition to the poles from the case of \( N(\Lambda_{s,p}, \tilde{c}_2) \), the operator \( N(\Lambda_{s,p}, sc_2) \) may have additional poles as follows: if \( p < \infty \) then the pole might appear if \( \chi_\infty = 1 \) and \( s = -\frac{1}{2} \). If \( p = \infty \), then the pole appears if \( 2s \) is a negative odd integer, and \( \chi_\infty = \sgn \), \( l \in \{0,1\} \).

3. In addition to the poles from the case of \( N(\Lambda_{s,p}, \tilde{c}_2) \), the operator \( N(\Lambda_{s,p}, c_2 sc_2) \) may have additional poles as follows: if \( p < \infty \) and \( s = -\frac{1}{2} \) and \( \chi_p = 1 \). Assume \( p = \infty \). Then we may have a pole if \( s - \frac{1}{2} \) a negative odd integer and \( \chi_\infty = 1 \) and \( s - \frac{1}{2} \) is a negative even integer and \( \chi_\infty = \sgn \).

**Theorem 4.3.** Assume \( s \geq 0 \). Then all the local intertwining operators corresponding to \( w \in W' \) are holomorphic.

1. If \( s \neq \frac{1}{2} \), or \( \chi^2 \neq 1 \), then the Eisenstein series are holomorphic for every choice of \( f = \otimes f_p \), and \([4]\) gives and holomorphic embedding of \( \text{Ind}^{Sp(4)}_{P(2)}(\chi_{\nu^s}1_{GL_2}(\mathbb{A})) \) in the space of automorphic forms.

2. If \( s = \frac{1}{2} \) and \( \chi = 1 \), \( E_{\text{const}}(s, f) \) has a pole of the first order, and then the image of \([2]\) is the unique spherical subquotient of \( \text{Ind}^{Sp(4)}_{P(2)}(\chi_{\nu^{\frac{1}{2}}}1_{GL_2}(\mathbb{A})) \).

3. If \( s = \frac{1}{2} \) and \( \chi \neq 1 \), then for all \( p \leq \infty \) such that \( \chi_p \neq 1 \), let \( \chi_p \nu^{\frac{1}{2}} \times 1 = T_1 \oplus T_2 \) (of course, then \( T_1 \) and \( T_2 \) depend on \( p \)). Then, in the appropriate Grothendieck group we have

\[
\nu_p^{1/2} \chi_p 1_{GL_2(Q_p)} \times 1 = L(\chi_p \nu^{1} ; T_1) + L(\chi_p \nu^{\frac{1}{2}} ; T_2) + L(\chi_p \nu_p^{1/2} St_{GL_2(Q_p)} ; 1).
\]

Here \( T_2 \) is the representation on which \( SL_2(Q_p) \) operator on \( \chi_p \nu_p^{1/2} \times 1 \) attains value minus identity. Now, we pick \( f_p \) from the subquotient \( L(\chi_p \nu^{1} ; T_2) \) on the set of places \( S' \). If \( |S'| \) is even, then \( E_{\text{const}}(\frac{1}{2}, \cdot) \) has a pole of the first order, and \([2]\) gives,
in the space of automorphic forms, a realization of the global representation with \( L(\chi_p \nu^1; T_2) \) on the even number of places, and \( L(\chi_p \nu^1; T_1) \) on the rest of the places. If \( |S'| \) is odd, then \( E_{\text{const}}(\frac{5}{2}, \cdot) \) is holomorphic and (2) gives an embedding of the whole global representation \( \text{Ind}_{P_2(A)}^{\text{Sp}(4)}(\chi\nu^{\frac{1}{2}}_{GL_2(A)}) \) in the space of automorphic forms.

**Proof.** The claims for \( s \neq \frac{1}{2} \) or \( \chi^2 \neq 1 \) are obvious. Now assume that \( \chi^2 = 1 \) and \( s = \frac{1}{2} \). Since for every \( p \leq \infty \chi_p^2 = 1 \), we have that \( \mathcal{N}(\Lambda_{s, p}, \tilde{s}\tilde{c}_2) \) and \( \mathcal{N}(\Lambda_{s, p}, \tilde{c}_2) \) have the same codomain. Assume now \( p < \infty \). If \( \chi_p = 1 \) then \( \nu_p^{1/2}1_{GL_2(Q_p)} \times 1 \) has the unique irreducible quotient, namely \( L(\nu_p^1; \nu_p^0 \times 1) \) (which is spherical) (cf. [21] Proposition 5.4.(ii)) and a tempered subrepresentation, say \( T_2 \). Then, the spherical vector generates the whole representation \( \nu_p^{1/2}1_{GL_2(Q_p)} \times 1 \), and since \( \mathcal{N}(\Lambda_{s, p}, \tilde{s}\tilde{c}_2) \) and \( \mathcal{N}(\Lambda_{s, p}, \tilde{c}_2) \) agree on the spherical vector (and are holomorphic and non-zero on that vector), they agree on the whole representation \( \nu_p^{1/2}1_{GL_2(Q_p)} \times 1 \). The image of this spherical vector by \( \mathcal{N}(\Lambda_{s, p}, \tilde{s}\tilde{c}_2) \) (or by \( \mathcal{N}(\Lambda_{s, p}, \tilde{c}_2) \)) generates in \( \nu_p^{-1} \times \nu_p^0 \times 1 \) an irreducible subspace (thus isomorphic to \( L(\nu_p^1; \nu_p^0 \times 1) \). If \( \chi_p \neq 1 \), then \( \nu_p^{1/2} \chi_p 1_{GL_2(Q_p)} \times 1 \) is of length 3, and in the appropriate Grothendieck group we have

\[
\nu_p^{1/2} \chi_p 1_{GL_2(Q_p)} \times 1 = L(\chi_p \nu^1; T_1) + L(\chi_p \nu^1; T_2) + L(\chi_p \nu_p^{1/2}St_{GL_2(Q_p)}; 1).
\]

Here \( \chi_p \nu_p^0 \times 1 = T_1 \oplus T_2 \) and \( L(\chi_p \nu_p^{1/2}St_{GL_2(Q_p)}; 1) \) is an irreducible subrepresentation of \( \nu_p^{1/2} \chi_p 1_{GL_2(Q_p)} \times 1 \), moreover, we have an epimorphism \( \nu_p^{1/2} \chi_p 1_{GL_2(Q_p)} \times 1 \to L(\chi_p \nu^1; T_1) \oplus L(\chi_p \nu^1; T_2) \) (with the kernel \( L(\chi_p \nu_p^{1/2}St_{GL_2(Q_p)}; 1) \)). Since \( \mathcal{N}(\Lambda_{s, p}, \tilde{s}\tilde{c}_2) \) maps \( \nu_p^{1/2} \chi_p 1_{GL_2(Q_p)} \times 1 \) to the representation \( \chi_p \nu_p^{-1} \times \nu_p^0 \chi_p \times 1 \), which has two unique subrepresentations \( L(\chi_p \nu^1; T_1) \) and \( L(\chi_p \nu^1; T_2) \), we see that \( L(\chi_p \nu_p^{1/2}St_{GL_2(Q_p)}; 1) \) is in the kernel of the operator \( \mathcal{N}(\Lambda_{s, p}, \tilde{s}\tilde{c}_2) \) (restricted on \( \nu_p^{1/2} \chi_p 1_{GL_2(Q_p)} \times 1 \)) and it is easy to see that

\[
\mathcal{N}(\Lambda_{s, p}, \tilde{s}\tilde{c}_2)(\nu_p^{1/2} \chi_p 1_{GL_2(Q_p)} \times 1) = L(\chi_p \nu^1; T_1) \oplus L(\chi_p \nu^1; T_2).
\]

On the other hand

\[
\mathcal{N}(\Lambda_{s, p}, \tilde{c}_2) = \mathcal{N}(sc_2(\Lambda_{s, p}), c_2) \mathcal{N}(\Lambda_{s, p}, \tilde{s}\tilde{c}_2),
\]

and \( \chi_p \nu_p^{-1} \times \chi_p \nu_p^0 \times 1 = \chi_p \nu_p^{-1} \times T_1 \oplus \chi_p \nu_p^{-1} \times T_2 \), and \( \mathcal{N}(sc_2(\Lambda_{s, p}), \tilde{c}_2) \) acts on one of these two summands as identity and on the other as minus identity, say on the first one as identity. Assume now \( p = \infty \). If \( \chi_\infty = 1 \) then again as in the non-archimedean case, the operators \( \mathcal{N}(\Lambda_{s, p}, \tilde{s}\tilde{c}_2) \)
and \(N(\Lambda_{s,p}, \tilde{c}_2)\) have the same action on \(\nu_1^{1/2} \chi_\infty 1_{GL_2(\mathbb{R})} \times 1\) (which is again generated by the spherical vector), and the image is isomorphic to \(L(\nu_1^{1/2}, \rho_\infty \times 1)\). If \(\chi_\infty = sgn\) we have the same situation as in the non-archimedean case; cf. Theorem 11.2 of [19].

Now assume that \(\chi = 1\). Then \(E_{\text{const}}(1/2, \cdot)\) has a pole of the first order (due to the poles of the global normalizing factors) and (normalized) \((\mathfrak{G})\) becomes

\[
(17)
\lim_{s \to \frac{1}{2}} (s - \frac{1}{2}) (r(\Lambda_s, \chi_2)^{-1} + r(\Lambda_s, c_2 \chi_2)^{-1}) \otimes_{p \in S} N(\Lambda_{s,p}, \tilde{c}_2) f_{p,s} \otimes \otimes_{p \notin S} f_{p,s,\chi_2}(s) + \lim_{s \to \frac{1}{2}} (s - \frac{1}{2}) (r(\Lambda_s, \chi_2)^{-1} \otimes_{p \in S} N(\Lambda_{s,p}, \tilde{c}_2) f_{p,s} \otimes \otimes_{p \notin S} f_{p,s,\chi_2}(s).
\]

Note that for \(N(\Lambda_{s,p}, \tilde{c}_2)\), the image is again generated by the spherical vector, thus for \(N(\Lambda_{s,p}, \tilde{c}_2)\), all other subquotients of \(\nu_1^{1/2} \chi_p 1_{GL_2(\mathbb{Q}_p)} \times 1\), \(p \leq \infty\) are in the kernel. Because \(\chi = 1\), all the \(e\) factors are trivial, and \(\lim_{s \to \frac{1}{2}} (s - \frac{1}{2}) (r(\Lambda_s, \chi_2)^{-1} + r(\Lambda_s, c_2 \chi_2)^{-1})\) becomes \(\lim_{s \to \frac{1}{2}} (s - \frac{1}{2}) L(2s, 1) \lim_{t \to 0} (L(-t, 1) + L(t, 1))\). In the equation \((17)\) the contribution is zero if we pick a vector \(f_p\) which does not belong to the spherical quotient.

Assume \(\chi \neq 1\), but \(\chi^2 = 1\). Assume that we have picked \(S' \subset S\) such that for every \(p \in S'\) \(\chi_p \neq 1\) and \(f_p\) belongs to the subquotient \(L(\chi_p \nu^1; T_2)\). We denote \(A(s) = \frac{L(2s, \chi^2)}{L(s + \frac{1}{2}, \chi) L(2s + 1, \chi^2) L(2s + 1, \chi^2)}\). Then
\[
r(\Lambda_s, \chi_2)^{-1} = A(s) L(s + \frac{1}{2}, \chi) \quad \text{and} \quad r(\Lambda_s, c_2 \chi_2)^{-1} = A(s) \frac{L(s + \frac{1}{2}, \chi)}{\varepsilon(s + \frac{1}{2}, \chi)}.
\]
Then, it is easy to see that the contributions to \(E_{\text{const}}\) coming from \(s_2\) and \(c_2 \chi_2\) together give:

\[
\frac{A(s) (L(\frac{1}{2} - s, \chi) + L(s - \frac{1}{2}, \chi)(-1)^{|S'|})}{\varepsilon(s + \frac{1}{2}, \chi)} \otimes_{p \in S} N(\Lambda_{s,p}, \tilde{c}_2) f_{p,s} \otimes \otimes_{p \notin S} f_{p,s,\chi_2}(s).
\]

If \(|S'|\) is odd, since \(A(s)\) has a pole for \(s = \frac{1}{2}\), the expression
\[
A(s) (L(\frac{1}{2} - s, \chi) + L(s - \frac{1}{2}, \chi)(-1)^{|S'|})
\]

is then holomorphic and non-zero, and if \(|S'|\) is even it has a pole. Note that for \(\chi_p \neq 1\), the operator \(N(\Lambda_{s,p}, \tilde{c}_2)\) is a holomorphic isomorphism on \(\nu_1^{1/2} \chi_p 1_{GL_2(\mathbb{Q}_p)} \times 1\).

We conclude: if \(|S'|\) is even, then \(E_{\text{const}}(\frac{1}{2}, \cdot)\) has a pole of the first order, and \((\mathfrak{G})\) gives, in the space of automorphic forms, a realization of the global representation with \(L(\chi_p \nu^1; T_2)\) on the even number of
Degenerate Eisenstein series for $Sp(4)$ places, and $L(\chi_p \nu^1; T_1)$ on the rest of the places. Note that if $|S'|$ is odd, then $E_{\text{const}}(\frac{1}{2}; \cdot)$ is holomorphic and (2) gives an embedding of the whole global representation $\text{Ind}_{P_2(A)}^{Sp_4(A)}(\chi \nu_{GL_2(A)}^1)$ in the space of automorphic forms. □

Theorem 4.4. Assume now that $s < 0$. Then we have the following:

1. Assume $-\frac{1}{2} < s < 0$. Then either the Eisenstein series are holomorphic or they have a possible pole of the first order due to the pole of $r(\Lambda_s, s c_2)^{-1}$ and $r(\Lambda_s, c_2 s c_2)^{-1}$ coming from the zero of $L(2s + 1, \chi^2)$ for given $s$; in both cases the intertwining operators are holomorphic isomorphisms so that (2) gives an embedding of the whole global representation $\text{Ind}_{P_2(A)}^{Sp_4(A)}(\chi \nu_{GL_2(A)}^s)$ in the space of automorphic forms.

2. Assume $s = -\frac{1}{2}$. Then, if $\chi_p^2 \neq 1$ then all the local intertwining operators are holomorphic isomorphisms (for all $p \leq \infty$) and the image is isomorphic to $\chi_p \nu_p^{-1/2} \nu_{GL_2(\mathbb{Q}_p)} \times 1$. If $\chi_p^2 = 1$ but $\chi_p \neq 1$ all the intertwining operators are homomorphisms. If $\chi_p = 1$ the local intertwining operator $\mathcal{N}(\Lambda_{s, p}, \tilde{c} s c_2)$ where $p \leq \infty$ can have a pole for every $p$ with the choice of $f_p$ from the non-spherical subquotient. This gives us a pole of the Eisenstein series (for $\chi$ which allows this possibility, e.g., $\chi = 1$) of every possible order. In this case, the image (by (2) after the normalization) consists of the irreducible tempered representations (on these “problematic” places where $\chi_p = 1$ and a non-spherical (tempered) choice of a subquotient and of the representation generated by the spherical subquotient on the rest of the places (and the spherical subquotient generates the representation $\nu_p^{-1/2} \nu_{GL_2(\mathbb{Q}_p)} \times 1$).

3. Assume $-\frac{3}{2} < s < -\frac{1}{2}$. Then either the Eisenstein series are holomorphic or they have a possible pole of the first order due to the pole of $r(\Lambda_s, c_2)^{-1}$, $r(\Lambda_s, s c_2)^{-1}$ and $r(\Lambda_s, c_2 s c_2)^{-1}$ coming from the zero of $L(2s + \frac{3}{2}, \chi)$ for given $s$; in both cases the intertwining operators are holomorphic isomorphisms so that (2) gives an embedding of the whole global representation $\text{Ind}_{P_2(A)}^{Sp_4(A)}(\chi \nu_{GL_2(A)}^s)$ in the space of automorphic forms.

4. Assume $s = -\frac{3}{2}$. Then, if $\chi_p^2 \neq 1$ then all the local intertwining operators are holomorphic isomorphisms (for all $p \leq \infty$) and the image is isomorphic to $\chi_p \nu_p^{-3/2} \nu_{GL_2(\mathbb{Q}_p)} \times 1$. Assume $p < \infty$. If $\chi_p = 1$ all the nontrivial local intertwining operators have a pole for every $p$ with the choice of $f_p$ from the
non-spherical subquotient. On the spherical subrepresentation of \( \chi_p \nu_p^{-3/2} 1_{GL_2(\mathbb{Q}_p)} \times 1 \) (in this case just the trivial representation) all these operators are holomorphic, and the image generated in that way is isomorphic to the representation \( \nu_p^{3/2} 1_{GL_2(\mathbb{Q}_p)} \times 1 \) (thus reducible). For the choice of \( f_p \) which does not belong to the spherical subrepresentation, all these operators have a pole, and, after removing the pole by normalization, the image spans an irreducible subrepresentation \( L(\nu_2^3, St_{SL_2(\mathbb{Q})}) \).

For the choice of \( f_p \) which does not belong to the spherical subrepresentation, all these operators have a pole, and, after removing the pole by normalization, the image spans an irreducible subrepresentation \( L(\nu_2^3, St_{SL_2(\mathbb{Q})}) \).

If \( \chi_\infty \neq 1 \) all the local intertwining operators are isomorphisms.

At the archimedean place, all the choices of \( f_p \) different from \( L(\nu_2^3, St_{SL_2(\mathbb{Q})}) \) if \( \chi_\infty = 1 \) or \( L(\text{sgn} \nu_\infty^2, \text{sgn} \nu_\infty^1; 1) \) if \( \chi_\infty = \text{sgn} \) give us poles of the intertwining operators. This gives us a pole of the Eisenstein series (for \( \chi \) which allows this possibility, e.g., \( \chi = 1 \)) of every possible order.

(5) Assume \( s < -\frac{3}{2} \). Then, if \( \chi_\infty^2 \neq 1 \) or \( s \pm \frac{1}{2} \) is not an integer, the Eisenstein series is holomorphic and (2) gives an embedding of the whole global representation \( \text{Ind}_{Sp(1)}^{GL_2} (\chi \nu_\infty^s) \) in the space of automorphic forms. Assume that \( \chi_\infty^2 = 1 \) and \( s \pm \frac{1}{2} \) is an integer. Then, the Eisenstein series is holomorphic on \( \otimes p \neq \infty (\chi_p \nu_p^s 1_{GL_2(\mathbb{Q}_p)} \times 1) \otimes L(\chi_\infty \nu_\infty^{-(s-\frac{1}{2})}, \chi_\infty \nu_\infty^{-(s+\frac{1}{2})}; 1) \) (which is a subrepresentation of \( \chi_\infty \nu_\infty^s 1_{GL_2(\mathbb{R})} \times 1 \)). In this case, (2) gives an embedding of an irreducible global representation \( \otimes p < \infty (\chi_p \nu_p^s 1_{GL_2(\mathbb{Q}_p)} \times 1) \otimes L(\chi_\infty \nu_\infty^{-(s-\frac{1}{2})}, \chi_\infty \nu_\infty^{-(s+\frac{1}{2})}; 1) \) in the space of automorphic forms. If \( f_\infty \) is not chosen in that way, the Eisenstein series have a pole of the first order coming from the pole of the archimedean intertwining operators (in the constant term). After the normalization, the image of (2) in that case is a realization of a representation which has, on the archimedean place, the unique maximal proper subrepresentation of \( \nu_\infty^s 1_{GL_2(\mathbb{R})} \times 1 \).

Proof. The case of \( -\frac{1}{2} < s < 0 \) is obvious. Assume \( s = -\frac{1}{2} \). Then, all the inverses of the global normalizing factors are holomorphic. For \( p \leq \infty \), all the intertwining operators are holomorphic if \( \chi_p^2 \neq 1 \). Now assume \( p < \infty \). Assume that \( \chi_p = 1 \). From the discussion in Theorem 4.3 we know that we have

\[
L(\nu_p^1, \nu_p^0 \times 1) \hookrightarrow \nu^{-1/2} 1_{GL_2(\mathbb{Q}_p)} \times 1 \to T_2,
\]
where $T_2$ is a tempered representation, and $L(\nu_p^1;\nu_p^0 \times 1)$ the spherical subquotient. The operator $\mathcal{N}(\Lambda_{s,p}, \tilde{c}_2)$ is a holomorphic isomorphism on $\nu^{-1/2}1_{GL_2(\mathbb{Q}_p)} \times 1$ (actually, an identity operator). The operator $\mathcal{N}(\Lambda_{s,p}, \tilde{s})$ acting on $\nu^{-1/2}1_{GL_2(\mathbb{Q}_p)} \times 1$ (i.e., $\mathcal{N}(\Lambda_{s,p}, \tilde{s}\tilde{c}_2)$ acting on $\nu^{-1/2}1_{GL_2(\mathbb{Q}_p)} \times 1$ because of the previous remark) has a pole on the quotient $\nu^{-1/2}St_{GL_2(\mathbb{Q}_p)} \times 1$, but is holomorphic on $\nu^{-1/2}1_{GL_2(\mathbb{Q}_p)} \times 1$. The operator $\mathcal{N}(\Lambda_{s,p}, \tilde{c}_2)$ acting on $\nu^0 \times \nu^{-1} \times 1$ has a pole on $\nu^0 \times SL_2(\mathbb{Q}_p) = T_1 \otimes T_2$, where $T_1$ is the other tempered representation. We conclude that if we pick $f_p$ from $L(\nu_p^1;\nu_p^0 \times 1)$ then all the operators will be holomorphic $f_p$ (belongs to the spherical subquotient, generated by the spherical vector), but if we pick $f_p$ which belongs to the quotient $T_2$, the operator $\mathcal{N}(\Lambda_{s,p}, \tilde{c}_2\tilde{c}_2)$ has a pole on that $f_p$. Assume now that $\chi_p^2 = 1$, but $\chi_p \neq 1$. Then, we know by Theorem 4.3 that the length of $\nu^{-1/2}1_{GL_2(\mathbb{Q}_p)} \times 1$ is three, and that it has two subrepresentations. Now, the operator $\mathcal{N}(\Lambda_{s,p}, \tilde{c}_2)$ acts on $\chi_p \nu^{-1} \times T_1$ as the identity, and on $\chi_p \nu^{-1} \times T_2$ as minus identity. The operator $\mathcal{N}(\Lambda_{s,p}, \tilde{s})$ acting on $\chi_p \nu^{-1} \times \chi_p \nu^0 \times 1$ has a pole on $\chi_p \nu^{-1/2}St_{GL_2(\mathbb{Q}_p)} \times 1$, thus it does not have a pole on $\chi_p \nu^{-1/2}1_{GL_2(\mathbb{Q}_p)} \times 1$. The operator $\mathcal{N}(\Lambda_{s,p}, \tilde{c}_2)$ acts on $\chi_p \nu^0 \times \chi_p \nu^{-1} \times 1$ as a holomorphic isomorphism. We conclude that all the operators are holomorphic on $\chi_p \nu^{-1/2}1_{GL_2(\mathbb{Q}_p)} \times 1$. Now, assume that $p = \infty$. Then, again $\mathcal{N}(\Lambda_{s,\infty}, \tilde{c}_2)$ is holomorphic isomorphism, and $\mathcal{N}(\Lambda_{s,\infty}, \tilde{s}\tilde{c}_2)$ does not have a pole on $\chi_\infty \nu^{-1/2}1_{GL_2(\mathbb{R})} \times 1$. The operator $\mathcal{N}(\Lambda_{s,\infty}, \tilde{c}_2)$ acting on $\chi_\infty \nu^0 \times \chi_\infty \nu^{-1} \times 1$ can only have a pole if $\chi_\infty \nu^{-1} \times 1$ is reducible, and that is if $\chi_\infty = 1$. In that case, we have

$$V_1 = L(\nu^1_1; 1) \hookrightarrow \nu_\infty \times 1 \to X(1, +) \oplus X(1, -),$$

where $X(1, \pm)$ are discrete series of $SL_2(\mathbb{R})$ (cf. [19] Theorem 2.4). The operator $\mathcal{N}(\Lambda_{s,p}, \tilde{c}_2)$ has a pole on the quotient $\nu_\infty \times (X(1, +) \oplus X(1, -))$. This quotient decomposes as a sum of four tempered representations (cf. [19] Theorem 10.7), two of which are quotients of $\nu_\infty^{-1/2}1_{GL_2(\mathbb{R})} \times 1$. So, we conclude that the operator $\mathcal{N}(\Lambda_{s,\infty}, \tilde{s}\tilde{c}_2)$ is holomorphic on the spherical subrepresentation of $\nu_\infty^{-1/2}1_{GL_2(\mathbb{R})} \times 1$ (this also follows from the general properties of the normalized intertwining operators), but other choices of $f_p$ will give us a pole.

Now assume $s = -\frac{3}{2}$. Assume $p < \infty$ and $\chi_p = 1$ (this is the only case when poles might occur). Then, $\mathcal{N}(\Lambda_{s,p}, \tilde{c}_2)$ has a pole on the quotient $\nu_p^{-2} \times St_{SL_2(\mathbb{Q}_p)}$, which decomposes (in the appropriate Grothendick group) as $St_{Sp_4(\mathbb{Q}_p)} + L(\nu_p^2; St_{SL_2(\mathbb{Q}_p)})$. On the other hand, we have

$$L(\nu_p^1, \nu_p^0, 1) \hookrightarrow \nu_p^{-3/2}1_{GL_2(\mathbb{Q}_p)} \times 1 \to L(\nu_p^2, St_{SL_2(\mathbb{Q}_p)}).$$
Note that $\mathcal{N}(\Lambda_{s,p}, \tilde{s})$ acting on $\nu_p^2 \times \nu_p^1 \rtimes 1$ is a holomorphic isomorphism, and so is $\mathcal{N}(\Lambda_{s,p}, \tilde{c}_2)$ acting on $\nu_p^1 \times \nu_p^{-2} \rtimes 1$. We conclude that the images of $\mathcal{N}(\Lambda_{s,p}, \tilde{c}_2)$, $\mathcal{N}(\Lambda_{s,p}, \tilde{s}c_2)$ and $\mathcal{N}(\Lambda_{s,p}, c_2\tilde{s}c_2)$ will be mutually isomorphic on $\nu_p^{-3/2}1_{GL_2(\mathbb{Q}_p)} \rtimes 1$. We see that the poles may occur if an irreducible subrepresentation $\pi$ of the intertwining operators are isomorphisms. Now assume $\chi_\infty \neq 1$ all the local intertwining operators are isomorphisms. Now assume $p = \infty$. Then, by Lemma 4.2, we see that the poles may occur if $\chi_\infty = 1$ or $\chi_\infty = sgn$. We first deal with the case $\chi_\infty = 1$. Then, already the operator $\mathcal{N}(\Lambda_{s,\infty}, \tilde{c}_2)$ has a pole on each the subquotients of $\nu_p^{-3/2}1_{GL_2(\mathbb{R})} \rtimes 1$, except on it’s unique subrepresentation (which is again the trivial representation), cf. [19] Theorem 11.1.

Note that the action of the other intertwining operators cannot cancel this pole. Assume now that $\chi_\infty = sgn$. Then, $\mathcal{N}(\Lambda_{s,\infty}, \tilde{c}_2)$ acting on $sgn\nu^{-3/2}_\infty 1_{GL_2(\mathbb{R})} \rtimes 1$ is a holomorphic isomorphism. The pole of $\mathcal{N}(\Lambda_{s,\infty}, \tilde{s})$ acting on $sgn\nu^{1/2}_\infty \times sgn\nu^1_\infty \rtimes 1$ happens on the subquotients of $\delta(sgn\nu^{1/2}_\infty \rtimes 1)$ and $sgn\nu^{-3/2}_\infty 1_{GL_2(\mathbb{R})} \rtimes 1$ and this representation do not have common irreducible subquotients by [19], proof of Theorem 10.1 and (10.64) there. The pole of $\mathcal{N}(\Lambda_{s,\infty}, \tilde{c}_2)$ acting on $sgn\nu^1_\infty \times sgn\nu^{-2}_\infty \rtimes 1$ happens on the subquotients of $sgn\nu^1_\infty \rtimes X(2,+) \oplus sgn\nu^1_\infty \rtimes X(2,-)$. But the representation $sgn\nu^{-3/2}_\infty 1_{GL_2(\mathbb{R})} \rtimes 1$ is of length three and two of its subquotients (so the ones different from $L(sgn\nu^2_\infty, sgn\nu^1_\infty; 1)$) appear in this space where the poles occur. Note that the representation $sgn\nu^2_\infty \times sgn\nu^1_\infty \rtimes 1$ is multiplicity free.

From Lemma 4.2 we now that in the case $s < -\frac{3}{2}$ it is enough to see what is happening on the archimedean places (all the local intertwining operators at the non-archimedean places are holomorphic isomorphisms) the following situations:

1. $\chi_\infty = 1$, $s + \frac{1}{2}$ is odd integer,
2. $\chi_\infty = 1$, $s + \frac{1}{2}$ is even integer,
3. $\chi_\infty = sgn$, $s + \frac{1}{2}$ is an odd integer,
4. $\chi_\infty = sgn$, $s + \frac{1}{2}$ is an even integer.

Assume the first possibility. Then from Theorem 11.1 of [19], we see that $\mathcal{N}(\Lambda_{s,\infty}, \tilde{c}_2)$ acting on $\nu^s_\infty 1_{GL_2(\mathbb{R})} \rtimes 1$ has a pole of the first order for every choice $f_p$ which does not belong to the Langlands quotient $L(\nu^{s+1/2}_\infty, \nu^{s-1/2}_\infty; 1)$ (the representation $sgn\nu^s_\infty 1_{GL_2(\mathbb{R})} \rtimes 1$ is of
the length four). The second intertwining operator has a pole on \( \nu_{\infty}^{-s/2} \times \nu_{\infty}^{-(s+1/2)} \times 1 \), but it does not add a pole of the higher order on the image of \( N(\Lambda_{s,\infty}, \tilde{c}_2)(\nu_{\infty}^sGL_2(\mathbb{R}) \rtimes 1) \) (and is non-zero on the subquotients of \( \nu_{\infty}^cGL_2(\mathbb{R}) \rtimes 1 \)). The intertwining operator \( N(\Lambda_{s,\infty}, \tilde{c}_2) \) acting on \( \nu_{\infty}^{-(s+1/2)} \times \nu_{\infty}^{s-1/2} \times 1 \) is a holomorphic isomorphism by Lemma 4.2. In the second possibility, the first intertwining operator \( N(\Lambda_{s,\infty}, \tilde{c}_2) \) acting on \( \nu_{\infty}^cGL_2(\mathbb{R}) \rtimes 1 \) is a holomorphic isomorphism. The second intertwining operator has a pole on \( \delta(-1/2, 2s) \times 1 \) (again notation from \cite{19}, the proof of Theorem 10.1) which is a representation of the length three and does not have a common subquotient with the representation \( \nu_{\infty}^cGL_2(\mathbb{R}) \rtimes 1 \) (cf. Theorem 10.6 of \cite{19}), so it’s holomorphic and non-zero on it. The third intertwining operator \( N(\Lambda_{s,\infty}, \tilde{c}_2) \) acts on \( \nu_{\infty}^{(s+1/2)} \times \nu_{\infty}^{s-1/2} \times 1 \), and pole is obtained on the subquotients of \( \nu_{\infty}^{-(s+1/2)} \times (X(-(s - \frac{1}{2}), +) \oplus (X(-(s - \frac{1}{2}), -)) \). All the subquotients of \( \nu_{\infty}^cGL_2(\mathbb{R}) \rtimes 1 \), except \( L(\nu_{\infty}^{-(s-\frac{1}{2})}, \nu_{\infty}^{-(s+\frac{1}{2})}; 1) \) are among those on which the pole occurs.

The situation with the rest of the cases is totally symmetric (i.e., one uses Theorem 10.1, Theorem 10.6 or Theorem 11.1 of \cite{19}), and we conclude that in all of these four cases, at least one of the operators \( N(\Lambda_{s,\infty}, \tilde{c}_2), N(\Lambda_{s,\infty}, \tilde{s}\tilde{c}_2) \) and \( N(\Lambda_{s,\infty}, \tilde{c}_2\tilde{s}\tilde{c}_2) \) has a pole of the first order on each of the subquotients of \( sgn^\varepsilon\nu_{\infty}^cGL_2(\mathbb{R}) \rtimes 1 \), except \( L(sgn^\varepsilon\nu_{\infty}^{-(s-\frac{1}{2})}, sgn^\varepsilon\nu_{\infty}^{-(s+\frac{1}{2})}; 1) \) (here \( \varepsilon \in \{0, 1\} \)). If one removes the pole, the image spans the unique maximal proper subrepresentation of \( \nu_{\infty}^{-1}GL_2(\mathbb{R}) \rtimes 1 \). \( \square \)

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