Title: Chiral Balls: Knotted Structures with Both Chirality and Three-dimensional Rotational Symmetry

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Abstract: Knots have been put forward to explain various physical phenomena because of their topological stability. Nevertheless, few works have reported on the exotic symmetry properties that certain knots possess. Here we reveal an exceptional form of symmetry for a family of knots that are both chiral and three-dimensional (3-D) rotationally symmetric about every axis of a standard Cartesian coordinate system. We call these unique knotted structures chiral balls. To demonstrate the unprecedented physical characteristics exhibited by these unique structures, we study the electromagnetic scattering properties of a representative conductive chiral ball. In particular, a characteristic mode analysis is performed to investigate the intrinsic scattering properties of this chiral ball. With both chirality and 3-D rotational symmetry, the chiral ball is shown to exhibit an extraordinary isotropic circularly polarized scattering property, which has not been previously reported for any known electromagnetic structures. Because of their unique properties, chiral balls are expected to not only have a profound impact on the fields of electromagnetics and optics but also far beyond.

One Sentence Summary: We discover an extraordinary form of symmetry for a family of knots, which are both chiral as well as 3-D rotationally symmetric, and show that they possess an unprecedented isotropic circularly polarized electromagnetic scattering property.

Main Text: Knots have played an important role throughout human history, for nets, traps, clothing, as well as a means of recording and communicating information. Mathematically, a knot is defined as a closed curve in three-dimensional space that does not intersect itself anywhere (1). Any deformations of the closed knotted curve through space that do not allow the curve to pass through itself are considered, from a topological point of view, to be the same knot. In an early scientific application, knots were considered as one possible explanation for the elementary particles in Kelvin’s atomic theory (2). More recently, knots have found a much wider range of interesting and useful applications in the fields of chemistry, physics, biology, and engineering. Examples of such diverse applications include fluid dynamics (3), photonics...
(4), molecular structure of DNA and other complex molecules (5–7), electrostatic and magnetostatic fields (8, 9), and electromagnetic scattering properties (10-16).

However, few works have been reported that exploit the exotic symmetry of knots. Symmetry is a universal and fundamentally important property in nature, art and science (17). In physics, geometrical symmetry is associated with physical conservation, which is one of the most powerful tools of theoretical physics, as it has become evident that all laws of nature originate from symmetries (18, 19). Chirality is a type of asymmetry with the absence of reflection symmetry. It widely exists in nature ranging from chemical molecules (20) to biological organisms (21). Conventional ball-shaped geometries, such as spheres and ellipsoids, have both reflection and rotational symmetry (Fig. 1 (A)). As their two-dimensional (2-D) projections, the conventional circle and ellipse are nonchiral. However, some 2-D chiral patterns, for example, a shape in the form of the letter “S” (Fig. 1 (B)), possess not only chirality, but also elliptic rotational symmetry (22). Some metamaterials have been studied that exploit the symmetry of 2-D chiral patterns as artificial atoms (23). However, their three-dimensional (3-D) counterparts, namely “chiral balls”, which possess both chirality and 3-D rotational symmetry along every axis of a Cartesian coordinate system (51), have not previously been explored.

Here we reveal that the simplest chiral ball with the desired symmetry is a (3, 2)-torus knot (i.e., a trefoil knot). Fig. 1 (C) depicts a perspective view of the knotted chiral ball. Fig. 1 (D), (E) and (F) are three views from the x, y, and z axis, respectively. The knot is evidently chiral for its absence of reflection symmetry. Moreover, it is two-fold rotationally symmetric around every axis in a Cartesian coordinate system.

Chirality determines the electromagnetic scattering properties of objects when they interact with circularly polarized (CP) waves. Here, for the first time, we investigate the electromagnetic response of a conductive chiral ball subject to illumination by a CP wave. As a comparison, the helix, which is one of the most common chiral structures, can be used to realize a circular polarizer (24, 25). However, due to the fact that helical structures have no rotational symmetry, their chiroptical responses are highly dependent on their orientation, as shown in Fig. 2 (A). On the other hand, the chirality of a knot depends only on the over and under crossings, and thus is independent of its orientation. Therefore, the 3-D rotational symmetry of the chiral ball facilitates its unique isotropic circularly polarized scattering property, as depicted in Fig. 2 (B).

In order to study the intrinsic electromagnetic properties of the chiral ball, excluding the influence of excitation and detection, we perform a characteristic mode analysis of the structure (26). The initial work on characteristic mode analysis was performed by Garbacz, who stated that any arbitrarily shaped perfectly conducting object will possess characteristic modes (27, 28). These modes are determined by the shapes and sizes of the object and are independent of any specific excitation or source. Given a specific object, characteristic modes can be used as a convenient set of basis functions for obtaining an expansion of the corresponding currents and scattered fields (29, 30). For example, the characteristic modes of a perfectly conducting sphere are identical to those based on the spherical wave functions, and the characteristic modes of an infinite circular cylinder are identical to the associated cylindrical wave functions. Similarly, through a characteristic mode analysis, intrinsic physical insight can be gained into the behavior of an arbitrarily-shaped conducting object, such as a chiral ball, by observing its characteristic scattered fields (26).
Here we employ the characteristic mode analysis solver available in the full-wave simulation software package CST (S2). In order to facilitate its isotropic circular polarized scattering property, the conventional torus knot is distorted with optimized parameters, and is idealized as a perfectly conducting filament (S3). The characteristic mode significance (25) is disclosed in Fig. 3 (A), where we observe that, at 2.91 GHz, mode 1 dominates with its associated current distribution shown in Fig. 3 (B). Fig. 3 (C) shows a perspective view of the electric far-field generated by the mode 1 characteristic current, which possesses a y-axis doughnut-shaped scattering pattern. In Fig. 3 (D), we observe a quasi-spherical axial ratio, which is less than 2 dB in all directions. The detailed response plots can be found in Fig. 3 (E), (F) and (G), depicting the axial ratio viewed from the x, y, and z axes, respectively. This quasi-spherical axial ratio suggests an inherent isotropic circular polarized scattering property. As the torus knot here is left-handed (S4), the scattered filed is also left-handed, as indicated in Fig. 3 (H).

For mode 2 at 3.225 GHz shown in Fig. 3 (A), the electric far-field is doughnut-shaped about the x axis. But it similarly generates a left-handed circularly polarized scattered field in almost every direction (S5).

As topologically stable objects, knots have attracted much attention in the fields of mathematics, physics, biology and engineering. In fact, in addition to their unique topological properties, this work reveals that knots also possess exotic symmetries that have not been previously investigated. Because of its chirality and 3-D rotational symmetry, we have demonstrated the remarkable property that a conductive chiral ball is able to scatter quasi-spherical circularly polarized waves in every direction. This work suggests that further investigation into the symmetry of knots is merited, as well as its potential relationship with various fundamental physical phenomena ranging from elementary particle theory and molecular structure to cosmic textures found in the universe.

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S1, S2, S3 and S4 are included in the Supplementary Material.

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Fig. 1. Geometry of a conventional ball, a 2-D chiral structure, and a 3-D chiral ball. (A) A conventional ball has both reflection and rotational symmetry. (B) A 2-D chiral pattern shaped like the letter “S” has both chirality and rotational symmetry. (C) Perspective view of a portion of a torus knot, functionalized as a 3-D chiral ball possessing both chirality and 3-D rotational symmetry. The knot (yellow) lives on the surface of a torus (purple). Three views of the knot from the (D) x, (E) y, and (F) z axes. The 3-D chiral ball is two-fold rotationally symmetric along the three axes of a Cartesian coordinate system.
Fig. 2. Comparison of the scattered field of (A) a conventional two-turn helical structure; and (B) the representative chiral ball. The helix has a circularly polarized scattering field in only one specific direction, while the chiral ball exhibits the remarkable property that it scatters circularly polarized waves in every direction.
Fig. 3. Characteristic mode study of the representative chiral ball with optimized parameters. (A) The modal significance; (B) the surface current of mode 1 at 2.91 GHz. (C) The characteristic mode scattering field magnitude of the chiral ball. (D) Perspective view of the far-field axial ratio. (E), (F), and (G) Axial ratio plots viewed from the x, y and z axes, respectively. (H) Axial ratio of the left-handed field component over the right-handed field component, indicating that the chiral ball scatters left-handed circular polarized waves in every direction.