A POLYNOMIAL-TIME SOLUTION TO THE REDUCIBILITY PROBLEM

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ABSTRACT. We propose an algorithm for deciding whether a given braid is pseudo-Anosov, reducible, or periodic. The algorithm is based on Garside’s weighted decomposition and is polynomial-time in the word-length of an input braid. Moreover, a reduction system of circles can be found completely if the input is a certain type of reducible braids.

1. Preliminaries and introduction

As a homeomorphism of a 2-dimensional disk that preserves \( n \) distinct interior points and fixes the boundary of the disk, an \( n \)-braid \( x \) is isotopic to one of the following three dynamic types known as the Nielsen-Thurston classification [16]:

(i) periodic if \( x^p \) is the identity for some nonnegative integer \( p \);
(ii) reducible if \( x \) preserves a set of disjointly embedded circles;
(iii) pseudo-Anosov if neither (i) nor (ii).

Obviously dynamic types are invariant under conjugation and taking a power. A set of disjointly embedded essential circles preserved by a reducible braid is called a reduction system. Suppose the distinct points lie on an axis. An essential circle, i.e. separating \( n \) distinct points, is standard if it intersects the axis exactly twice. A reduction system is standard if each circle in the system is standard. Up to conjugacy, every reducible braid has a standard reduction system. Standard reduction systems are especially nice in the sense that they can be recognized in polynomial time.

Recently, some evidence that the conjugacy problem could be easy for pseudo-Anosov braids has been found [12, 2]. In addition, the conjugacy problem for periodic braids is trivial once they are recognized. It is therefore important to know the dynamic type of a given braid to solve its conjugacy problem. If a braid is reducible, it is also important to know how the braid is decomposed into pseudo-Anosov braids or periodic braids. Thus the reducibility problem comes in two flavors depending on what is asked. Given an arbitrary braid, we may ask to determine its dynamic type or to find a reduction system if it is reducible. The latter problem will be called the reduction problem to distinguish them.

Our approach will be based on a weighted decomposition of braid words, invented by Garside [7], improved by Thurston [17] and El-Rifai and Morton [6]. We briefly review the idea together with necessary notations. The Artin presentation of the group \( B_n \) of \( n \)-strand braids has \( n - 1 \) generators \( \sigma_1, \ldots, \sigma_{n-1} \) and two types of defining relations: \( \sigma_j \sigma_i = \sigma_i \sigma_j \) for \( |i - j| > 1 \) and \( \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \) for \( |i - j| = 1 \). The monoid given by the same presentation is denoted by \( B^+_n \) whose elements will be called positive braids.

A partial order \( \prec \) on \( B^+_n \) can be given by saying \( x \prec y \) for \( x, y \in B^+_n \) if \( x \) is a (left) subword of \( y \), that is, \( xz = y \) for some \( z \in B^+_n \). Given \( x, y \in B^+_n \), the (left) join

\[ x \lor y \]
was used by Garside in [7] to solve the conjugacy problem in the braid group of the super summit set the conjugacy class Cx of C. The partial order "left" is our default choice, we sometimes need the corresponding right versions: the partial order <R of being a right subword, the right join ∨R, and the right meet ∧R. For example, x <R y if zx = y for some z ∈ Bn+.

The fundamental braid ∆ = (σ1 · · · σn−1)(σ1 · · · σn−2) · · · (σ1σ2)σ1 plays an important role in the study of Bn. Since it represents a half twist as a geometric braid, x∆ = ∆τ(x) for any braid x where τ denotes the involution of Bn sending σi to σn−i. It also has the property that σi ∼ ∆ for each i = 1, · · · , n − 1. Since the symmetric group Σn is obtained from Bn by adding the relations σi2 = 1, there is a quotient homomorphism q : Bn → Σn. For Sn = {x ∈ Bn+ | x ∼ ∆}, the restriction q : Sn → Σn becomes a 1:1 correspondence and an element in Sn is called a permutation braid.

A product ab of a permutation braid a and a positive braid b is (left) weighted, written a|b, if a* ∧ b = e where e denotes the empty word and a* = a−1∆ is the right complement of a. Each braid x ∈ Bn can be uniquely written as

\[ x = ∆^ux_1x_2\cdots x_k \]

where for each i = 1, · · · , k, xi ∈ S_{n+} \{e, ∆\} and xixi+1. This decomposition is called the (left) weighted form of x [7, 17, 6]. Sometimes the first and the last factors in a weighted form are called the head and the tail, denoted by H(x) and T(x), respectively. The weighted form provides a solution to the word problem in Bn and the integers u, u + k and k are well-defined and are called the supremum and the canonical length of x, denoted by inf(x), sup(x) and ℓ(x), respectively.

Given x = ∆^ux_1x_2\cdots x_k in its weighted form, there are two useful conjugations of x called the cycling c(x) and the decycling d(x) defined as follows:

\[ c(x) = ∆^ux_2\cdots x_kτ^u(x_1) = τ^u(H(x)^{-1})xτ^u(H(x)), \]

\[ d(x) = ∆^uτ^u(x_k)x_1\cdots x_{k-1} = T(x)xT(x)^{-1}. \]

A braid x = ∆^ux_1\cdots x_k in its weighted form is (left) i-rigid for 1 ≤ i ≤ k = ℓ(x) if the first i factors are identical in the weighted forms of x1· · · xk and x1· · · x_kτ^u(x_1), that is, x_1· · · x_i = y_1· · · y_i, where y_1· · · y_ky_{k+1} is the weighted form of x_1· · · x_kτ^u(x_1) and y_{k+1} may be empty. If a braid x is ℓ(x)-rigid, we simply say x is (left) rigid and this is equivalent to the fact that x_k | τ^u(x_1). We can also consider the corresponding right version.

Let infc(x) and supc(x) respectively denote the maximal infimum and the minimal supremum of all braids in the conjugacy class C(x) of x. A typical solution to the conjugacy problem in the braid group Bn is to generate a finite set uniquely determined by a conjugacy class. Historically the following four finite subsets of the conjugacy class C(x) of x ∈ Bn have been used in this purpose:

The summit set

\[ SS(x) = \{y ∈ C(x) | \inf(y) = \inf_c(x)\} \]

was used by Garside in [7] to solve the conjugacy problem in Bn for the first time. The super summit set

\[ SSS(x) = \{y ∈ C(x) | \inf(y) = \inf_c(x) \text{ and } \sup(y) = \sup_c(x)\} \]
was used by El-Rifai and Morton in [6] to improve Garside’s solution. The reduced super summit set

$$RSSS(x) = \{y \in C(x) \mid c^M(y) = y = d^N(y) \text{ for some positive integers } M, N\}$$

was used by Lee in his Ph.D. thesis [13] to give a polynomial-time solution to the conjugacy problem in $B_4$. Finally the ultra summit set

$$USS(x) = \{y \in SSS(x) \mid c^M(y) = y \text{ for some positive integer } M\}$$

was used by Gebhardt in [9] to propose a new algorithm together with experimental data demonstrating the efficiency of his algorithm. Clearly

$$RSSS(x) \subset USS(x) \subset SSS(x) \subset SS(x),$$

and $RSSS(x) = USS(x)$ if $x$ is rigid.

On the other hand, fewer researches have been done to solve the reducibility problem, perhaps due to lack of suitable tools. Bestvina and Handel [1] invented the “train track” algorithm and solved the reduction problem for any surface automorphism. Unfortunately, this algorithm is typically exponential for the length of input described as automorphisms of graphs. Bernardete, Nitecki and Gutíerrez [5] showed that a standard reduction system is preserved by cycling and decycling and so for any reducible braid $x$, some braid in $SSS(x)$ must have a standard reduction system and consequently the reduction problem can be solve as soon as $SSS(x)$ is generated. Humphries [11] solved the problem of recognizing split braids.

Recently there have two noticeable progresses. Ko and J. Lee [12], Birman, Gebhardt and González-Meneses [2] showed that some power of a pseudo-Anosov braid is rigid, up to conjugacy and in fact any braid in the ultra summit set of the power is rigid. E. Lee and S. Lee [14] showed that if the outermost component of a reducible braid $x$ is simpler than the whole braid $x$ up to conjugacy then any braid in $RSSS(x)$ has a standard reduction system. The difficulty of using the $USS$ or the $RSSS$ is that we do not know how long it takes to generate one element in the set, not to mention the whole set.

Our contribution in this article is two folds. We now know how fast we can obtain a rigid braid from a given pseudo-Anosov $n$-braid by taking powers and iterated cyclings. In fact the required power is at most $(n(n - 1)/2)^3$ and the required number of iterated cyclings is at most $2n!(n(n - 1)/2)^3\ell(x)$. The other contribution is a complete understanding of reducible braids that are rigid. In fact, if a reducible braid is rigid and no circles in its reduction system are standard, its conjugate by some permutation braid must preserve a set of standard circles.

We will start by introducing these contribution as two main theorems in the next section as well as a polynomial-time algorithm for the reducibility problem. Proofs for the main theorems will follow in the next couple of sections.

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2. Main theorems and an algorithm for the reducibility problem

We introduce two main theorems whose proofs constitute the next two sections. Then we present an algorithm for the reducibility problem based on the main theorems. Throughout this paper we will assume that $n \geq 3$ since every 2-braid is periodic.
2.1. Main theorems. The word length of the fundamental $n$-braid $\Delta$ is $\frac{n(n-1)}{2}$ and will be denoted by $D$.

**Theorem 2.1.** Let $x$ be a pseudo-Anosov $n$-braid. Then there are integers $1 \leq L \leq 2D$, $1 \leq M \leq D^2$, and $1 \leq N \leq n!\ell(x)LM$ such that the weighted form of $c^N(y^L)$ is rigid for any $y \in SSS(x^M)$.

**Theorem 2.2.** Let $x$ be a reducible, rigid $n$-braid. Then there exists a permutation $n$-braid $t$ such that $t^{-1}xt$ is rigid and has at least one orbit of standard reduction circles.

2.2. Reducibility Algorithm.

Input: An $n$-braid $x$ given as a word in the Artin generators

Output: The dynamical type of $x$, that is, whether $x$ is periodic, pseudo-Anosov, or reducible.

1. We first try to find a positive integer $M$ such that $\inf_c((x^M)^D) = D \inf_c(x^M)$ and $\sup_c((x^M)^D) = D \sup_c(x^M)$ and choose $y \in SSS(x^M)$. By [15], such an $M$ must exist between 1 and $D^2$ and so we can obtain it via the loop: For $1 \leq j \leq D^2$, we test whether

$$\inf(d^{D(x)^j}c^{D(x)^j}(x^j)) = D \inf(d^{D(x)^j}c^{D(x)^j}(x^j))$$

and

$$\sup(d^{D(x)^j}c^{D(x)^j}(x^j)) = D \sup(d^{D(x)^j}c^{D(x)^j}(x^j))$$

after computing necessary weighted forms, and then if the test answers affirmatively, return $M = j$.

2. If $d^{D(x)^M}c^{D(x)^M}(x^M) = \Delta^{2k}$ for some $k$, then conclude that $x$ is periodic and stop. Otherwise, set $y = (d^{D(x)^M}c^{D(x)^M}(x^M))^{2D}$.

3. Test whether there exists an integer $1 \leq N \leq n!\ell(y)$ such that $c^N(y)$ is rigid. If such an $N$ does not exist, then conclude that $x$ is reducible and stop. Or if $c^N(y)$ is rigid, set $z = c^N(y)$.

4. Test whether there exists a permutation braid $t \in S_n$ such that $t^{-1}zt \in RSSS(z)$ and $t^{-1}zt$ has at least one orbit of standard reduction circles. If such a $t$ exists, conclude that $x$ is reducible. Otherwise, conclude that $x$ is pseudo-Anosov.

We now explain why our algorithm works and analyze its complexity in step by step.

In Step (1), notice that $\inf_c(\beta) = \inf(c^{D(\beta)}(\beta))$ since if $\inf_c(\beta) > \inf(\beta)$, then $\inf_c(\beta) > \inf(\beta)$ by [4] and $\inf_c(\beta) \leq \inf(\beta) + \ell(\beta)$. Similarly, $\sup_c(\beta) = \sup(d^{D(\beta)}(\beta))$. Thus $d^{\ell(\beta)D}c^{D(\beta)}(\beta) \in SSS(\beta)$. The complexity of this step is dominated by $\ell(x)jD^2$ times of cyclings and decyclings on $x^jD$ for $j = 1, \ldots, D^2$. Thus it can be estimated as $\mathcal{O}(\ell(x)^3n^{21}\log n)$.

Step (2) is simple and its complexity can be dominated by other steps.

In Step (3), the existence of such an $N$ for any pseudo-Anosov braid and the upper bound for $N$ are a part of Theorem 2.1. The complexity of this step is $\mathcal{O}(n!\ell(y)\cdot \ell(y)^2n\log n) = \mathcal{O}(n!(2D)^3\ell(x)^3n\log n) = \mathcal{O}(\ell(x)^3n!n^{19}\log n)$.

Step (4) is the most complicated. One can check that $t^{-1}zt$ has at least one orbit of standard reduction circles by an easy algorithm such as one given in [14] since a standard reduction circle is preserved by each permutation braid that is a factor in the weighted form of $t^{-1}zt$. Indeed for each pair $(i, j)$ with $1 \leq i < j \leq n$,
let $L = \lfloor \frac{n^3}{4} \rfloor$ and $(t^{-1}zt)^L = \Delta^n z_1 z_2 \cdots z_L(z)$ be the weighted form and $\hat{z}_i$ denote the bijection on $\{1, \ldots, n\}$ corresponding to the permutation braid $z_i$. 

We need to check whether $\{\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_m(i), \hat{z}_1 \hat{z}_2 \cdots \hat{z}_m(i+1), \ldots, \hat{z}_1 \hat{z}_2 \cdots \hat{z}_m(j)\} = \{i, i+1, \ldots, j\}$ for some $1 \leq m \leq L\ell(z)$ that is a multiple of $\ell(z)$ and the set $\{\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_k(i), \hat{z}_1 \hat{z}_2 \cdots \hat{z}_k(i+1), \ldots, \hat{z}_1 \hat{z}_2 \cdots \hat{z}_k(j)\}$ is consisted of consecutive integers for each $1 \leq k \leq m$. The complexity of these two test is $O(\ell(z)^2 n \log n \cdot \ell(z) n^b) = O(\ell(x)^3 n^{25} \log n)$.

A naive algorithm would be to perform these two tests for each $t \in S_n$ and then the complexity of Step(4) is $O(\ell(x)^3 n! n^{25} \log n)$. Since $t^{-1}zt \in RSSS(z)$ iff $t^{-1}zt$ is rigid [2, 12], conjugators preserving rigid braids are closed under meet. Thus we may improve our algorithm in practice. Indeed find all minimal braids $t_1, t_2, \ldots, t_m$ for some $1 \leq m \leq n-1$ such that $t_i^{-1}zt_i \in RSSS(z)$. These $t_i$ can be found by a formula starting from generators. For each $t_i^{-1}zt_i$, perform the test for the possession of a standard circle. If nothing is found, inductively find the next larger minimal conjugators preserving $RSSS(z)$ by starting from a join of two minimal conjugators found in the previous steps. The latter method would be much faster in the average case but the complexity in the braid index $n$ is not clear yet. Thus the overall complexity is cubic in the canonical length of the input braid.

3. PROOF OF THEOREM [2, 12]

Lemma 3.1. Suppose that $x$ is an $n$-braid such that $x \in SSS(x)$ and $\inf(x^i) = \inf(x)$, $\sup(x^i) = \sup(x)$ for $i \geq 1$. If $x^K$ is rigid for some $K \geq 1$ then $x$ itself is rigid.

Proof. Under the hypotheses, neither new $\Delta$’s can be formed nor factors can be merged by taking powers. Thus $\tau^{(K-1)}(H(x)) \prec H(x^K)$ and $T(x) \succ R T(x^K)$. Consequently $T(x^K) [\tau^{(K-1)}(H(x^K))]$ implies $T(x) [\tau^{u}(H(x))]$.

Proposition 3.2 ([2, 12]). Let $x$ be a pseudo-Anosov braid in $B_n$. Then $x^M$ is conjugate to a rigid braid for some $1 \leq M \leq (\frac{n(n-1)}{2})^2$.

Proof. We refer to [2, 12] for the existence of such an $M$ and we add a comment on the upper bound for $M$. By Theorem 4.3 in [15], there exist a positive integer $M \leq (\frac{n(n-1)}{2})^2$ and $y \in C(x)$ such that $\inf((y^M)^i) = \inf(y^M)^i$ and $\sup((y^M)^i) = \inf(y^M)^i$ for $i > 0$. Let $z = y^M$. Since $z$ is pseudo-Anosov, $z^M$ is conjugate to a rigid braid for some $M' > 0$. Hence, $z$ is conjugate to a rigid braid by Lemma 3.1.

A braid $x$ is tame if $\inf(x^i) = \inf(x)$ and $\sup(x^i) = \sup(x)$ for all $i \geq 1$. For any $n$-braid $y$, $y^M$ becomes tame for some $1 \leq M \leq D^2$ by [15].

Lemma 3.3. If $x, y \in SSS(x)$ for a tame braid $x$, then $y$ is also tame.

Proof. It is clear that $\inf(x^i) \geq \inf(y^i) \geq \inf(y)$ for all $i \geq 1$. By the hypothesis, $x^i \in SSS(x^i)$ and so $\inf(x^i) = \inf(x^i) = \inf(x) = \inf(y)$ for all $i \geq 1$. Thus $\inf(y^i) = \inf(y)$. Similarly, $\sup(y^i) = \inf(y)$ for $i \geq 1$.

Lemma 3.4. Let $x$ be a tame braid and $i \geq 1$. Then $x^L$ is left (or right) $i$-rigid for some $1 \leq L \leq iD$. In particular, $x^D$ is left and right $i$-rigid.
Proof. For the simplicity of notations, assume that \( \inf(x) = 0 \) and prove the left version. Let \( H_i(y) \) denote the product of the first \( i \) factors in the weighted form of a tame braid \( y \). Since \( H_i(x^L) \) can not be strictly increasing without producing \( \Delta \) for all \( L \) that increases from 1 to \( iD \), \( H_i(x^L) = H_i(x^{L+1}) \) for some \( 1 \leq L \leq iD \). Then \( H_i(x^M) = H_i(x^{M-L-1}x_{L+1}) = H_i(x^{M-L-1}H_i(x^L)) = H_i(x^{M-1}) \) for all \( M \geq L+1 \). Thus \( H_i(x^M) = H_i(x^M) \) for all \( M \geq L \). In particular, \( H_i(x^M) = H_i(x^{2M}) \) and so \( x^M \) is \( i \)-rigid for all \( M \geq L \).

**Lemma 3.5.** Suppose that \( x \) is tame and \( x \in SSS(x) \). Then \( c^i(x^{iD}) \) is left and right \( i \)-rigid for all \( j \geq 0 \).

**Proof.** By Lemma 3.4, \( x^{iD} \) is left and right \( i \)-rigid. It is enough to show that \( c^i(x^{iD}) = y^{iD} \) for some tame braid \( y \in SSS(y) \). Then we ought to set \( y = \tau^u(H(x^{iD})^{-1})x\tau^u(H(x^{iD})) \) for \( u = \inf(x^{iD}) \). Since \( (x^{iD})^{-1}x(x^{iD}) = x \in SSS(x) \), \( y \in SSS(x) = SSS(y) \). By Lemma 3.3, \( y \) is also tame.

**Lemma 3.6.** If \( x \) is left 2-rigid, \( H(\tau^u(a^{-1})x) = \tau^u(H(\tau^u(a^{-1})x^2)) \) for any \( a \ll H(x) \) where \( u = \inf(x) \). The corresponding statement using right versions also holds.

**Proof.** Let \( x = \Delta^n x_1 \cdots x_k \) be the weighted form. Then \( a \ll x_1 \). Since the 2-rigidity implies \( H(x_2 \cdots x_k) = H(x_2 \cdots x_k \tau^u(x_1 \cdots x_k)) \), we have \( H((a^{-1}x_1)x_2 \cdots x_k) = H((a^{-1}x_1)x_2 \cdots x_k \tau^u(x_1 \cdots x_k)) \).

**Theorem 3.7.** Let an \( n \)-braid \( x \) be the \( 2D \)-th power of a tame braid that is in its super summit set. If \( x \) is conjugate to a rigid braid, then a rigid braid must be obtained from \( x \) by at most \( n! \ell(x) \) iterated cyclings.

**Proof.** Since we are assuming \( n \geq 3 \), \( \ell(x) \geq 6 \). It was proved in Theorem 3.3 in [12] and Theorem 3.15 in [2] that if \( USS(x) \) contains at least one rigid braid, then every braid in \( USS(x) \) is rigid. Thus iterated cyclings on \( x \) must produce a rigid braid. Let \( y = c^n(x) \) be the rigid braid obtained from \( x \) by the minimal number of iterated cyclings. Since \( \inf(x) \) is even, we can assume \( \inf(x) = 0 \) for the sake of simplicity without affecting the conclusion.

Let \( y = y_1y_2 \cdots y_k \) be the weighted form. We will prove by induction on \( i \geq 1 \) that for all \( 1 \leq i \leq N \)

\[
c^{N-i}(x) = a_i y_{[1-i]} z_i
\]

for some permutation braid \( a_i \) satisfying \( y_{[2-k-i]} \cdots y_{[-1-i]} y_{[-i]} \geq_R a_i \geq_R e \) and a positive braid \( z_i = y_{[2-k-i]} \cdots y_{[-1-i]} y_{[-i]} a_i^{-1} \) where \( |m| \) denotes the integer between 1 and \( k \) that equals \( m \mod k \). Let \( t_i = H(c^{N-i-1}(x)) \) for \( 1 \leq i \leq N \). Then \( c^{N-i-1}(x) = t_i c^{N-i}(x) T_i^{-1} \) and the properties

(i) \( t_i [c^{N-i}(x)] T_i^{-1} \);
(ii) \( c^{N-i}(x) \geq_R t_i \);
(iii) \( t_i \geq_R T(c^{N-i}(x)) \)

are clear from the definition of cycling, except the fact that \( t_i \neq T(c^i(x)) \). If \( t_i = T(c^i(x)) \), then \( c^{-1}(x) \) is already rigid and this violates the minimality of \( N \).
Let \( a_1 = t_1 y_k^{-1} \). Then \( a_1 \) is a permutation braid such that \( y_1 \cdots y_{k-1} \triangleright_R a_1 \triangleright_R e \) by (ii) and (iii). Thus our claim is proved for \( i = 1 \). Suppose that our claim holds for \( i \). Then \( c^{N-i-1}(x) = t_i a_i y_{1+i} y_i^{-1} \). We must have \( t_i a_i \triangleright_R y_{[i]} \), otherwise the factor \( y_{[i]} \) splits into two parts in \( c^{N-i-1}(x) \) so that \( \ell(c^{N-i-1}(x)) > \ell(x) \) by (i) which is a contradiction. Thus we can write \( t_i a_i = a_{i+1} y_{[i]} \) for some permutation braid \( a_{i+1} \) satisfying \( a_{i+1} \prec t_i \) and \( c^{N-i}(x) \triangleright_R a_{i+1} \). By the minimality of \( N \), \( a_{i+1} \neq e \). Let \( H_R(w) \) denote the right head of \( w \). Then

\[
H_R(c^{N-i}(x)) = H_R((c^{N-i}(x))^-^2) = H_R(a_i y_{1-i} y_i a_i y_{1-i} z_i) = H_R(a_i y_{1-i} \cdots y_{-i} y_{1-i} y_i a_i^{-1} y_{-i} a_i^{-1}) = H_R(y_{1-i} \cdots y_{-i} y_{1-i} a_i^{-1} a_i^{-1}) = H_R(y_{1-i} z_i).
\]

Here, the first equality holds since \( c^{N-i}(x) \) is right 1-rigid by Lemma 3.5 and the fourth equality follows from Lemma 3.6 since \( c^{N-i}(x) = y_{1-i} \cdots y_{-i} y_{1-i} y_{-i} \) is right 2-rigid by Lemma 3.5. Thus

\[
y_{2-i} \cdots y_{-i} y_{1-i} \triangleright_R y_{1-i} z_i \triangleright_R a_{i+1} \triangleright_R e
\]

and \( c^{N-i-1}(x) = a_{i+1} y_{[i]} z_{i+1} \) for \( z_{i+1} = y_{1-k-i} \cdots y_{-2-i} y_{1-i} a_{i+1}^{-1} \) and this completes the induction.

By our claim, \( c^{N-i}(x) \) is completely determined by choosing a nontrivial permutation braid \( a_i \) satisfying \( a_i \prec_R H_R(y_{[2-k-i]} \cdots y_{[1-i]} y_{[i]} \cdots y_{1-i} y_{-i}) \). For each \( 1 \leq i \leq \ell(x) \), there are at most \( n! \) such choices. Thus \( N \leq n! \ell(x) \).

We remark that the braid \( x \) in Theorem 3.7 need not be pseudo-Anosov.

4. Proof of Theorem 2.2

Theorem 2.2 is used in Step (4) of our algorithm in Section 2 and all braids dealt in Step (4) has the canonical length \( \geq 6 \). In this section, we will assume that the braid index is \( \geq 3 \) and the canonical length is \( \geq 2 \) unless stated otherwise in order to avoid any unnecessary nuisance.

Let \( P \) denote one of conjugacy invariant sets \( SS, SSS, USS, RSSS \) and let \( y \in P(x) \). If a nontrivial positive \( n \)-braid \( \gamma \) satisfies \( \gamma^{-1} y \gamma \in P(x) \), \( \gamma \) is called a \( P \)-conjugator of \( y \). A \( P \)-conjugator \( \gamma \) of \( y \) is \textit{minimal} if either \( \gamma \prec \beta \) or \( \gamma \wedge \beta = e \) for each positive braid \( \beta \) with \( \beta^{-1} y \beta \in P(x) \).

In fact it is not hard to see that a minimal \( P \)-conjugator satisfies \( \gamma \prec T(y) \) or \( \gamma \wedge T(y) = e \). For example, see [3, 12]. A conjugator \( \gamma \) satisfying \( \gamma \prec T(y) \) (or \( \gamma \wedge T(y) = e \)) will be called a \textit{cut-head} (or \textit{add-tail}) conjugator. In particular, if \( y \) is rigid and \( \gamma \) is its \( P \)-conjugator then it can not be both cut-head and add-tail since \( T(y) \wedge T(y) = e \). If \( \gamma \) is a \( USS \)-conjugator of a rigid braid \( y \), it is also a \( RSSS \)-conjugator and \( \gamma^{-1} y \gamma \) is rigid (For example, see [3, 12]).

Note that if \( \gamma \) is an add-tail conjugator of \( y \), then \( \gamma \) is a cut-head conjugator of \( y \) (For example, see [3, 12]).

Suppose that \( \gamma \) is a product of cut-head conjugators of \( y \), that is, \( \gamma = \gamma_1 \cdots \gamma_j \) such that \( \gamma_j \) is a cut head conjugator of

\[
(\gamma_1 \cdots \gamma_{j-1})^{-1} y (\gamma_1 \cdots \gamma_{j-1})
\]
where $\gamma_0 = e$. By the definition of cut head conjugators, $\gamma < \tau^{\inf(y^i)}(\Delta^{-\inf(y^i)}y^i)$ for some $1 \leq i \leq j$. Similarly, if $\gamma$ is a product of add-tail conjugators of $y$, then $\gamma < \tau^{\inf((y^{-1})^i)}(\Delta^{-\inf(y^{-1})^i})^j(y^{-1})^i$ for some $1 \leq i \leq j$.

**Lemma 4.1.** Suppose that an $n$-braid $x$ is rigid and $\ell(x) \geq 2$. If $\gamma$ is a positive $n$-braid such that $\gamma^{-1}x\gamma$ is rigid and $\ell(\gamma) \geq 2$ then there is a positive braid $\beta$ such that $\beta^{-1}x\beta$ is also rigid, $\ell(\beta) \geq 2$, and moreover $\beta$ is a left subword of either $\gamma \wedge \tau^{\inf(x^i)}(\Delta^{-\inf(x^i)}x^i)$ or $\gamma \wedge \tau^{\inf((x^{-1})^i)}(\Delta^{-\inf(x^{-1})^i})x^i$ for some $i \geq 1$.

**Proof.** If $h$ is a cut-head RSSS-conjugator of $x$ and $t$ is an add-tail RSSS-conjugator of $h^{-1}xh$, then it is shown in Proposition 3.23 in [3] that $ht$ is a permutation braid and there are an add-tail RSSS-conjugator $t'$ of $x$ and a cut-head RSSS-conjugator $h'$ of $t'^{-1}xt'$ such that $ht = t'h'$. Thus we can write $\gamma = HT = T'H'$ such that $H$ (or $H'$, respectively) is a product of cut-head RSSS-conjugators of $x$ (or $T'^{-1}xT'$) and $T$ (or $T'$, respectively) is a product of add-tail RSSS-conjugators of $H^{-1}xH$ (or $x$). If $\ell(\gamma) \geq 2$, at least one of $\ell(H)$ or $\ell(T')$ is $\geq 2$. Then the conclusion follows from the remark right before this lemma. \[\square\]

**Proposition 4.2.** Suppose that an $n$-braid $x$ is rigid and $\alpha^{-1}x\alpha \in SSS(x)$ for a permutation $n$-braid $\alpha$. Let $\mu$ be minimal among all $\beta$’s satisfying that $\alpha < \beta$ and $\beta^{-1}x\beta \in RSSS(x)$. Then $c^N(\alpha^{-1}x\alpha) = \mu^{-1}x\mu$ for some $N \geq 0$.

**Proof.** Since $x$ is rigid, $c^{2\ell(x)}(x) = x$ for $i \geq 1$. Note that there exists $\alpha'$ such that $c^{2\ell(x)}(\alpha^{-1}x\alpha) = \alpha'^{-1}x\alpha'$ and $\alpha < \alpha'$. Let $c^{2\ell(x)}(\alpha^{-1}x\alpha) = \alpha^{-1}_i x\alpha_i$ where $\alpha_i < \alpha_{i+1}$ and $\alpha_0 = \alpha$. Then $\alpha^{-1}_j x\alpha_j \in RSSS(x)$ for some $j > 0$ since $x$ is rigid. Let $\mu$ be minimal among all $\beta$’s satisfying that $\alpha < \beta$ and $\beta^{-1}x\beta \in RSSS(x)$. Since $\alpha < \mu$, $\alpha_i < \mu$ for $i \geq 0$ by Corollary 2.2 in [3]. Thus $\alpha_j < \mu$ and so $\alpha_j = \mu$ by the minimality of $\mu$. \[\square\]

**Figure 1.** The decomposition of a reducible braid

If a reducible $n$-braid $x$ has an orbit of standard reduction circles, we may assume that $x$ preserves a standard circle $C$ by replacing $x$ by $x^j$ for some $1 \leq j \leq n$ if necessary. $C$ spans a tube that does not intersect any strand of $x$. Then the standard reduction circle $C$ uniquely determines a decomposition $x = \hat{x}\hat{x} \hat{x}$ where all strands lying outside $C$ and the tube spanned by $C$ form a trivial braid in $\hat{x}$ and on the other hand all strands lying inside $C$ form a trivial braid in $\hat{x}$ as in Figure 1.
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Figure 2. A typical destroyer of $C$

For a standard circle $C$, a permutation $n$-braid $\beta$ is called a destroyer of $C$ if $\alpha(C)$ is not standard for all $\epsilon \preceq \alpha < \beta$. See Figure 2. Suppose the standard circle $C$ contains $\ell$ consecutive punctures from the $k$-th puncture where we must have either $k \geq 2$ or $k + \ell \leq n - 1$ to make $C$ a reduction circle. Let $\beta$ be a destroyer of $C$, then $\beta(C)$ is not standard, $\beta$ should ends with a nontrivial permutation $n$-braid $\delta$, i.e. $\beta \succ_R R \delta$ such that either $\delta \preceq \sigma_{k-1+s} \cdots \sigma_{k+\ell+s}$ or $\delta \preceq \sigma_{k+\ell+s} \cdots \sigma_{k-1+s}$ for some integer $s$ satisfying $k - 1 + s \geq 1$ and $k + \ell + s \leq n - 1$. Furthermore, since every left subword of $\beta$ must destroy the standard circle $C$, we have that $\beta = e$, there are no crossings among outer strands, and no outer strands pass through all of inner strands at once. Thus a typical destroyer of $C$ should look like one given in Figure 2 where $\beta$ is drawn by thicker strands.

Lemma 4.3. Given a standard circle $C$ and permutation braids $t$, $y$, and $s$, suppose that $ty$ sends $C$ to a standard circle, $(ty)[s]$, and $ty, ys \prec \Delta$. If $t$ is a destroyer of $C$ then $s$ is also a destroyer of $ty(C)$.

Proof. Let $C' = ty(C)$. Assume that $s$ is not a destroyer of $C'$, that is, $s'(C')$ is standard for some $\epsilon \preceq s' < s$. Suppose $s'$ is minimal among such braids. We have either $s'(C') = C'$ or $s'(C')$ is another standard circle. If $s'(C') = C'$ then $s'$ has one of two types in Figure 3. If $s'$ is of Type I, then $t$ must has crossings of strings in the tube spanned by $C$ to satisfy $(ty)[s]$, but this contradicts the assumption that $t$ is a destroyer of $C$. If $s'$ is of Type II, then there are three possibilities for $t$ to satisfy $(ty)[s]$ as in Figure 4. In all cases, $t$ must have a subword preserving a circle $C$ but this contradicts the assumption that $t$ is a destroyer of $C$.

If $s'(C')$ is another standard circle then $s'$ must be of the type in Figure 3. Thus $t$ should look like Figure 5 to satisfy $(ty)[s]$. Again $t$ must have a subword preserving a circle $C$ but this contradicts by the assumption that $t$ is a destroyer of $C$. Consequently, $s$ is a destroyer of $C'$.

□
Lemma 4.4. Suppose that a reducible $n$-braid $x$ is rigid and has a standard circle $C$. If $\beta$ is a permutation $n$-braid such that $\beta^{-1}x\beta$ is rigid and $\alpha(C)$ is not standard for any $e \not\leq \alpha < \beta$ such that $\alpha^{-1}x\alpha$ is rigid, then $\beta$ is a destroyer of $C$.

Proof. For the simplicity, assume $\inf(x) = 0$. Suppose that $\beta$ is not a destroyer of $C$. Then $\alpha(C)$ is standard for some $e \not\leq \alpha < \beta$. We choose a maximal such $\alpha$ so that $t = \alpha^{-1}\beta$ is a destroyer of $\alpha(C)$. We will show that $\alpha^{-1}x\alpha \in \text{SSS}(x)$. Then $c_N(\alpha^{-1}x\alpha) = \mu^{-1}x\mu$ for some $\alpha < \mu$ and $\mu$ is minimal among all braids $\gamma$ such that $\alpha < \gamma$ and $\gamma^{-1}x\gamma$ is rigid by Proposition 4.2. Thus $\epsilon \not\leq \mu < \beta$. Moreover since $\mu^{-1}x\mu$ is an iterated cycling of $\alpha^{-1}x\alpha$ that fixes a standard circle $\alpha(C)$, it fixes a standard circle $\mu(C)$ by [5]. However this contradicts the hypotheses.

Suppose $\alpha^{-1}x\alpha \not\in \text{SSS}(x)$. Then $\ell(\alpha^{-1}x\alpha) = \ell(x) + 1$. Let $k = \ell(x)$ and $\alpha^{-1}x\alpha = z_1 \cdots z_k z_{k+1}$ be the weighted form. Then the weighted form of $\beta^{-1}x\beta = t^{-1}z_1 \cdots z_k z_{k+1}$ can be written

$$(t_1^{-1}z_1 t_2) \cdots (t_{k-1}^{-1}z_{k-1} t_k)(t_k^{-1}z_k z_{k+1} t_1)$$

where $t_1 = t$ and $t_i < z_i$ for $1 \leq i \leq k$. Since $t_1 = t$ is a destroyer of $\alpha(C)$ by the choice of $\alpha$, $t_i$ is a destroyer of $C_i$ for $1 \leq i \leq k$, where $C_1 = \alpha(C)$ and $C_i = z_1 \cdots z_{i-1}(\alpha(C))$ for $2 \leq i \leq k+1$ by Lemma 4.3. Since $z_k[z_{k+1}$ and $t_k$ is a destroyer of $C_k$, $z_{k+1}$ is a destroyer of $C_{k+1}$ but this contradicts the fact that $z_{k+1}(C_{k+1})$ is standard. Hence $\alpha^{-1}x\alpha \in \text{SSS}(x)$. \qed

Theorem 4.5. Suppose that a reducible $n$-braid $x$ is rigid and has an orbit of standard circles starting with a standard circle $C$. If $\gamma$ is a positive $n$-braid such that $\gamma^{-1}x\gamma$ is rigid and $\beta(C)$ is not standard for any $e \not\leq \beta < \gamma$ such that $\beta^{-1}x\beta$ is rigid, then $\ell(\gamma) \leq 1$

Proof. For the sake of simplicity, we assume that $\inf(x) = 0$. If the conclusion holds for a power of $x$, so does it for $x$ itself. Thus we can further assume that $x(C) = C$ by replacing $x$ by its power if necessary. Recall the notations for inner braids and outer braids with respect to the standard curve $C$. 

![Possible t when s' is of type II](image1)

![Possible s' and t when s'(C') is another standard circle](image2)
Suppose $\ell(\gamma) \geq 2$. By Lemma 4.1, we may assume that $\gamma \prec x^i$ or $\gamma \prec (x^{-1})^i$ for some $i \geq 1$. Since $x^i(C) = C = (x^{-1})^i(C)$, we may work on either case. So we assume $\gamma \prec x^i$. Let $\gamma_1 \gamma_2$ be the first two factors in the weighted form of $\gamma$. By Lemma 4.4 $\gamma_1$ must be a destroyer of $C$ as in Figure 2.

Since $\gamma_1 \gamma_2$ is left weighted, $\gamma_2$ must start with one of two types of crossings given in Figure 6. A crossing of type I is formed by an inner strand and an outer strand. On the other hand, a crossing of type II is formed by two outer strands and is located between two inner strands. But we will show that both cases are impossible.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{typeI_typeII.png}
\caption{Possible $\gamma_1 \gamma_2$}
\end{figure}

For any reducible braid $y$ fixing a standard circle $C$, consider the two-component link $K_1 \cup K_2$ obtained from $y$ by either one of the following ways:

(i) $K_1$ is obtained by the plat closing of any two inner strands and $K_2$ is obtained by Markov’s closing of any outer strand.

(ii) $K_1$ is obtained by the plat construction of any two inner strands and $K_2$ is obtained by the plat closing of any two outer strands.

It is clear that the link $K_1 \cup K_2$ always splits, that is, there is an embedded 2-sphere separating two components. However two components of the link obtained from $\gamma_1 \gamma_2$ of type I via the construction (i) has the linking number 1. Also two components of the link obtained from $\gamma_1 \gamma_2$ of type II via the construction (ii) has the linking number 2. See Figure 7.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{typeI_typeII.png}
\caption{Two-component links obtained from $\gamma_1 \gamma_2$}
\end{figure}

Since $\gamma_1 \gamma_2 \prec \gamma \prec x^i$ and $x^i$ is a positive braid, this nontrivial linking can not be undone in $x^i$. This completes the proof. \hfill \Box

**Corollary 4.6.** Let $x$ be a reducible $n$-braid. Suppose that $x$ is rigid and $x$ has no standard reduction circle. If $\gamma^{-1} x \gamma$ is rigid and has a standard reduction circle $C$ for $\gamma \in B_n^+$ with $\ell(\gamma) > 1$, then there exists $\beta \preceq \gamma$ such that $\beta^{-1} x \beta$ is rigid and has at least one standard reduction circle.
Proof. Let $y = \gamma^{-1} \lambda \gamma$. Then $x = \gamma y \gamma^{-1}$. Since $\Delta_i(C)$ is a standard circle for all $i$ and $(\gamma \wedge R) \Delta^{-1}(C)$ is a standard circle, we can assume that $\inf(\gamma) = 0$ and $\gamma \wedge R \Delta = \epsilon$. Since $x = \gamma y \gamma^{-1}$, $x = \Delta^\ell(\gamma) \gamma^{*\ell} \gamma^{-*\ell} y \gamma \Delta^{-\ell}(\gamma)$, where $\gamma^{*\ell} = \Delta^{\ell}(\gamma)$. Let $\gamma' = \tau(\gamma) \gamma^*$ and $y' = \tau(y)$. If $\ell(\gamma') > 1$ then there exists $e \leq \beta' < \gamma'$ such that $\beta^{-\ell} y \beta'$ is rigid and $\beta^{-\ell}(C)$ is a standard circle by Theorem 4.5. Thus there exists $\gamma \wedge R \beta'' \geq_R e$ such that $\beta'' y \beta'^{-1}$ has a standard circle $C$. Thus $\beta^{-\ell} x \beta$ is rigid and has at least one reduction circle, where $\beta = \gamma \beta'^{-1}$. \hfill \Box

By inductively applying Step 4 of the algorithm in Section 2, one can find a whole reduction system for a reducible braid that is rigid after taking a power and iterated cyclings. The class of these reducible braids is disjoint from the class of reducible braids considered in [14]. In terms of (an) orbit of standard circles fixed by a reducible braid $x$, reducible braids considered in [14] roughly satisfy $\sup(\hat{x}) > \sup(\hat{x})$ and this property is not inherited to $\hat{x}$ or $\hat{x}$, while rigid reducible braids satisfy $\sup(\hat{x}) = \sup(\hat{x})$ and both $\hat{x}$ and $\hat{x}$ are also rigid. The class of reducible braids satisfying $\sup(\hat{x}) < \sup(\hat{x})$ will also need an attention.

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